Supersymmetry of a spin $\frac{1}{2}$ particle on the real line

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Abstract

We study one dimensional supersymmetric (SUSY) quantum mechanics of a spin $\frac{1}{2}$ particle moving in a rotating magnetic field and scalar potential. We also discuss SUSY breaking and it is shown that SUSY breaking essentially depends on the strength and period of the magnetic field. For a purely rotating magnetic field the eigenvalue problem is solved exactly and two band energy spectrum is found.

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1 Introduction

Supersymmetric (SUSY) quantum mechanics was introduced by Witten as a laboratory for investigating SUSY breaking which is one of the fundamental issue in SUSY quantum field theory [1]. Prior to Witten’s paper Nicolai
had shown that SUSY could also be a useful tool in nonrelativistic quantum mechanics \[2\]. Subsequently SUSY quantum mechanics has proved to be interesting on its own merit and has been studied from different points of view \[3, 4\].

In the present paper we shall study a generalised one dimensional SUSY quantum mechanical problem concerning the motion of a spin \(1/2\) particle in the presence of a scalar potential as well as a magnetic field. In this connection we would like to point out that supersymmetry based methods have previously been used to study various systems involving coupled channel problems \[5\], matrix Hamiltonians \[6, 7\] as well as models involving spin-orbit coupling \[8\]. Quasi exactly solvable matrix models have also been studied \[9\]. In the present case we shall obtain exact solutions of the eigenvalue problem when a spin \(1/2\) particle moves in the presence of a rotating magnetic field. In particular it will be shown that supersymmetry breaking depends nontrivially on the strength and period of the magnetic field. Finally we shall also indicate briefly how supersymmetry is affected when apart from the rotative magnetic field a scalar potential is also present.

\section*{2 SUSY of a spin \(1/2\) particle on the real line}

In Witten’s model of SUSY quantum mechanics the Hamiltonian consists of two factorized Schrödinger operators

\[ H_- = A^+ A^-, \quad H_+ = A^- A^+, \]

where the operators \(A^+\) and \(A^-\) are given by

\[ A^\pm = \mp \frac{d}{dz} + W(z), \]

and \(W(z)\) is the superpotential.

The pair of Hamiltonians in (1) are called SUSY partner Hamiltonians and each of these Hamiltonians describe motion of a spinless particle in one dimension. We shall now generalize Witten’s model of SUSY quantum mechanics in such way that each of the Hamiltonians \(H_-\), \(H_+\) will describe the motion of a spin \(1/2\) particle in a magnetic field and scalar potential. In order
to do this we generalise the operators $A^\pm$ in the following way:

$$A^\pm = \mp \frac{d}{dz} + W(z) + V(z)S, \quad (3)$$

It may be noted that here we consider motion of the particle along $z$-axis and components of the spin operator $S$ are $S_\alpha = \sigma_\alpha/2$ ($\alpha = x, y, z$), $\sigma_\alpha$ being the Pauli matrices. Then SUSY partner Hamiltonians can be obtained as in (1) and are given by

$$H^\pm = -\frac{d^2}{dz^2} + V^\pm(z) + B^\pm(z)S, \quad (4)$$

where

$$V^\pm(z) = W^2 \pm W' + V^2/4, \quad (5)$$

$$B^\pm(z) = 2WV \pm V'. \quad (6)$$

The Hamiltonians $H^\pm$ in (4) describe a spin $1/2$ particle moving along the $z$-axis in a scalar potential $V^\pm$ and a magnetic field $B^\pm(z)$. In the case $V = 0$ we obtain standard Witten model of SUSY quantum mechanics.

### 3 Spin $\frac{1}{2}$ particle in a rotating magnetic field and constant scalar potential

Let us now consider the motion of spin $1/2$ particle in a constant scalar potential and rotating magnetic field in the $x - y$-plane:

$$(B^\pm)_x = \mp B_0 \cos kz, \quad (B^\pm)_y = \mp B_0 \sin kz, \quad (B^\pm)_z = 0. \quad (7)$$

In this case without any loss of generality we can choose $W = 0$ and thus

$$V_x = -\frac{B_0}{k} \sin kz, \quad V_y = \frac{B_0}{k} \cos kz, \quad V_z = 0. \quad (8)$$

Thus in this case the operators $A^\pm$ are given by

$$A^\pm = \mp \frac{d}{dz} + \frac{B_0}{k} (- \sin kzS_x + \cos kzS_y). \quad (9)$$
Then from (4) we can obtain the explicit form of the SUSY partner Hamiltonians:

\[ H_\pm = -\frac{d^2}{dz^2} \mp B_0(\cos kzS_x + \sin kzS_y) + \frac{B_0^2}{4k^2}. \]  

(10)

From the form of the Hamiltonians in (10) it is seen that the term coupling spin and magnetic field is dependent on z. In order to remove this dependence we now perform the following unitary transformation:

\[ \tilde{\psi} = e^{ikzS_z}\psi. \]  

(11)

As a result of this transformation we obtain the following set of Hamiltonians

\[ \tilde{H}_\pm = e^{ikzS_z}He^{-ikzS_z} = \tilde{A}^\pm \tilde{A}^\pm \]

\[ = \left(-i\frac{d}{dz} - kS_z\right)^2 \mp B_0S_x + \frac{B_0^2}{4k^2}, \]  

(12)

where

\[ \tilde{A}^\pm = e^{ikzS_z}A^\pm e^{-ikzS_z} = \mp \frac{d}{dz} \pm ikS_z + \frac{B_0}{k}S_y. \]  

(13)

In order to determine whether or not supersymmetry is broken it is necessary to investigate if there are normalisable zero energy ground state wave functions (it may be recalled that for SUSY to be unbroken the ground state energy must be zero while if SUSY is broken the ground state energy is positive). So we seek solutions of the equations

\[ \tilde{A}^\pm \tilde{\psi}_0^\pm = \left(\mp \frac{d}{dz} \pm ikS_z + \frac{B_0}{k}S_y\right)\tilde{\psi}_0^\pm = 0. \]  

(14)

Thus SUSY is unbroken if at least one of wave functions \(\tilde{\psi}_0^\pm\) is a true zero mode. We now seek solutions of the above equations in the form

\[ \tilde{\psi}_0^\pm = \tilde{\chi}_0^\pm e^{iqz}, \]  

where \(q\) is wave vector of the particle, \(\tilde{\chi}_0^\pm\) is spin part of the wave function and it satisfies the following equation

\[ \left(\mp iq \pm ikS_z + \frac{B_0}{k}S_y\right)\tilde{\chi}_0^\pm = 0. \]  

(16)
It can be shown that non zero solutions of equation (16) i.e., \( \tilde{\chi}_0^+ \) or \( \tilde{\chi}_0^- \) exists for the same value of the wave vector \( q \)

\[
q = \pm \frac{k}{2} \sqrt{1 - \frac{B_0^2}{k^4}}.
\]  

This implies that true zero modes exist both for \( H_+ \) and \( H_- \) only when \( q \) is real and from (17) it follows that \( q \) is real if

\[
\frac{B_0^2}{k^4} < 1.
\]

Thus when \( B_0^2 < k^4 \) zero modes exist both for \( H_\pm \) and SUSY is unbroken. In the other case when \( q \) is complex we do not have normalisable zero energy solutions and so SUSY is broken. It is interesting to note that when \( q \) is real zero energy states exist in both the sectors \( H_+ \) and \( H_- \) and thus are strictly isospectral. It may be noted that a similar situation arises when spinless particles move in periodic potentials [10, 11].

Finally we note that eigenvalue problem for the Hamiltonians in (14) and thus for those in (10) can be solved exactly for the entire energy spectrum. After performing the unitary transformation the Hamiltonian (14) is transformed to (12) and the corresponding eigenfunctions can be written in the form

\[
\tilde{\psi}^\pm = \tilde{\chi}^\pm e^{iqz}.
\]

Then the eigenvalue problem becomes

\[
[(q - kS_z)^2 \mp B_0S_x] \tilde{\chi}^\pm = E \tilde{\chi}^\pm,
\]

from which we obtain a two band energy spectrum

\[
E_{1,2}(q) = q^2 + (k/2)^2 \pm \sqrt{q^2k^2 + (B_0/2)^2 + \frac{B_0^2}{4k^2}}.
\]

We note that the energy spectrum in (21) is the same for both \( H_\pm \). The lowest energy \( E = 0 \) for the first band \( E_1(q) \) (with ”+” in (21)) is at wave vector given by (17) where \( B_0^2/k^4 < 1 \). If however \( B_0^2/k^4 > 1 \) the lowest energy is at \( q = 0 \) and \( E_1(q = 0) = (k/2 - |B_0|/2k)^2 \) which is greater than zero. Thus in this case the SUSY is broken.
4 Ground state in the case of a rotating magnetic field and non constant scalar potential

Unlike in the last section here we consider the motion of the particle in a rotating magnetic field and a non constant superpotential \( W(z) \). In this case the equation for the ground state of \( H_- \), after the unitary transformation (11) is given by

\[
\tilde{A}^- \tilde{\psi}_0^- = \left( \frac{d}{dz} - ikS_z + \frac{B_0}{k}S_y + W(z) \right) \tilde{\psi}_0^- = 0.
\] (22)

As before spin and coordinate parts of the wave function can be separated and the solution can be written in the form

\[
\tilde{\psi}_0^- = \tilde{\chi}_0^- \exp(-\int^z W(z)dz - \lambda z)
\] (23)

where \( \tilde{\chi}_0^- \) satisfies the equation

\[
\left( -ikS_z + \frac{B_0}{k}S_y \right) \tilde{\chi}_0^- = \lambda \tilde{\chi}_0^-.
\] (24)

Eigenvalues of this equation are easily obtained and are given by

\[
\lambda = \pm \frac{k}{2} \sqrt{\frac{B_0^2}{k^4} - 1}.
\] (25)

Thus equation (22) has two solutions (23) which correspond to the two eigenvalues (25).

An interesting feature which emerges from this scenario is that even in the presence of a non constant scalar potential the rotating magnetic field can lead to SUSY breaking if it is sufficiently strong. To see this let us choose a superpotential \( W(z) \) such that \( W(z) \to \pm W_0 \) when \( z \to \pm \infty \). Then it follows from (23) that in the case when

\[
\frac{k}{2} \sqrt{\frac{B_0^2}{k^4} - 1} > W_0
\] (26)
the wave function becomes non square integrable. Thus sufficiently strong magnetic field destroys zero energy ground state and leads to SUSY breaking.

For the purpose of illustration let us consider an explicit example. We choose \( W(z) = \alpha \tanh z, \alpha > 0 \) so that \( W(z) \to \pm \alpha \) as \( z \to \pm \infty \). Then the ground state wave function corresponding to \( H_- \) is given by

\[
\tilde{\psi}^-(0) = \tilde{\chi}^-(0) (\cosh z)^{-\alpha} e^{-\lambda z},
\]

where \( \lambda \) is given by (25). This wave function is square integrable, when \( \alpha > \lambda \). Thus in this case SUSY is unbroken. In the other case when \( \alpha < \lambda \) the ground state wave function is nonnormalisable so that the magnetic field leads to the breaking of SUSY. We would like to point out that if \( \lambda \) as given by (25) is imaginary (in other words if the magnetic field is small enough) then SUSY is always unbroken irrespective of the value of \( \alpha \).

Finally let us point out about the zero energy solution corresponding to \( H_+ \). We note that in this case

\[
\tilde{\psi}^+(0) = \tilde{\chi}^+(0) (\cosh z)^{\alpha} e^{\lambda z},
\]

so that it is non square integrable. Thus \( H_+ \) posses no zero energy solution. This is in contrast to the case considered in the previous section where both \( H_\pm \) had zero energy states.

5 Conclusions

In the present paper we have studied motion of a spin \( \frac{1}{2} \) particle in rotating magnetic field and a scalar potential within the framework of SUSY quantum mechanics. The eigenvalue problem in the case of a purely magnetic field (scalar potential is constant) is solved exactly and two band energy spectrum is obtained. An interesting feature of the free motion of spin \( \frac{1}{2} \) particle in a rotating magnetic field is that both Hamiltonians \( H_+ \) and \( H_- \) can have zero energy states simultaneously. It may be noted that existence of zero modes and thus exact SUSY depends on parameters of magnetic field and are given by (18). For sufficiently strong magnetic field we have broken SUSY.

We have also studied SUSY breaking when the particle is moving in a rotating magnetic field and non constant superpotential \( W(z) \). In contrast to the free motion (\( W(z) = 0 \)) for non constant \( W(z) \) only one of the Hamiltonians \( H_- \) or \( H_+ \) has a zero energy ground state. In this case if the magnetic
field is sufficiently strong then SUSY can be broken. The condition for this is given by (26).

It may be noted that in addition to the magnetic field inclusion of a non constant superpotential \( W(z) \) leads to the appearance of discrete energy levels. Investigation of complete discrete energy spectrum in the presence of a magnetic field and different scalar potentials will be the subject of a future publication.

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