ON THE MONOTONICITY OF THE GENERALIZED MARKOV NUMBERS

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Abstract. Using the Markov distance and Ptolemy inequality introduced by Lee-Li-Rabideau-Schiffler [10], we completely determine the monotonicity of the generalized Markov numbers along the lines of a given slope.

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1. Introduction

A Markov number is any number in the triple \((x, y, z)\) of positive integer solutions to the Diophantine equation

\[ x^2 + y^2 + z^2 = 3xyz, \]

known as the Markov equation.

Every Markov number appears as the maximum of some Markov triple, The Markov Uniqueness Conjecture by Frobenius from 1913 asserts that each Markov number appears as the maximum of a unique Markov triple [5, 1, 10] [6, 16].

As an approach to studying the Uniqueness Conjecture, Aigner [1] proposed three conjectures, called fixed numerator, fixed denominator, and fixed sum conjectures, which say that the Markov numbers increase along the lines \(y\)-axis, \(x\)-axis, and \(y = x\), respectively.

Propp [15] and Beineke-Brüstle-Hille [2] found that the Markov numbers are the specialized cluster variables of the once-punctured torus cluster algebras. The family of cluster algebras from surfaces, introduced by Fomin-Shapiro-Thurston [4]

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is a class of important and special cluster algebras. The cluster variables can be computed in terms of the perfect matching of certain snake graphs \([11, 12, 8]\), see also for the quantum case in \([3, 7, 9]\). In \([13]\), for a given cluster algebra \(A\) from a marked surface \((S, M)\), Musiker-Schiffler-Williams associate any generalized curve \(\gamma\) on \((S, M)\) with an element \(x_\gamma\).

To study the ordering of the Markov numbers and the Uniqueness Conjecture, Lee-Li-Rabideau-Schiffler \([10]\) introduced the Markov distance on the plane. By comparing the Markov tree and the Farey tree, one sees that the Markov numbers can be indexed by the rational numbers between zero and one, equivalently, the set \(\{(q, p) \in \mathbb{Z}^2_+ | p < q, \text{g.c.d.}(p, q) = 1\}\), see \([1]\). Using the Markov distance, Lee-Li-Rabideau-Schiffler extended the Markov numbers to the numbers indexed by all \((q, p) \in \mathbb{Z}^2_+\) with \(p < q\), we call them generalized Markov numbers in this paper. They show that the Markov numbers increase and decrease along the line with some special slopes. They also show that there are some lines such that the Markov numbers are not monotonic along these lines.

Using hyperbolic geometry, Gaster provides the boundary slopes for which the Markov numbers decrease and increase, respectively. In this paper, we consider the monotonicity of generalized Markov numbers and give a parallel result to that of Gaster.

Our main result is the following, which was conjectured by Lee-Li-Rabideau-Schiffler \([10]\).

1.1. Theorem. \((\text{Theorem 4.11, Proposition 3.13 (2)})\)

(1) For \(k \in \mathbb{Q}\) with \(k \geq -\frac{\ln \left(\frac{\sqrt{2} + \sqrt{7}}{2}\right)}{\ln \left(\frac{\sqrt{2} + \sqrt{7}}{2}\right)} \approx -1.1432\), the generalized Markov numbers increase with \(x\) along any line \(l: y = kx + b\);

(2) For \(k \in \mathbb{Q}\) with \(k \leq -\frac{2\ln \left(\frac{1 + \sqrt{5}}{2}\right)}{\ln \left(\frac{1 + \sqrt{5}}{2}\right)} \approx -1.2417\), the generalized Markov numbers decrease with \(x\) along any line \(y = kx + b\);

(3) For any \(k \in \mathbb{Q}\) with \(-\frac{2\ln \left(\frac{1 + \sqrt{5}}{2}\right)}{\ln \left(\frac{1 + \sqrt{5}}{2}\right)} < k < -\frac{\ln \left(\frac{\sqrt{2} + \sqrt{7}}{2}\right)}{\ln \left(\frac{\sqrt{2} + \sqrt{7}}{2}\right)}\), then for almost all \(b \in \mathbb{Q}\), the generalized Markov numbers are not monotonic along the line \(y = kx + b\).

(4) For the lines along which the generalized Markov numbers are not monotonic, when the \(x\)-coordinate increases the generalized Markov numbers first decrease and then increase.

(5) The generalized Markov numbers are not monotonic along the line \(l\) if and only if \(r_1(u_1, v_1) < 1\) and \(r_1(z_1, w_1) > 1\).

1.2. Remark. Theorem 1.1(1) solves \([10, \text{Conjecture 6.11}]\), Theorem 1.1(2) solves \([10, \text{Conjecture 6.8}]\) and Theorem 1.1(3) (4) solve \([10, \text{Conjecture 6.12}]\).

Note that the Markov distance does not have triangle inequality, we find the following interesting inequality, as modified triangle inequality.

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1Thanks to the authors of \([10]\) for sharing their revised manuscript in private communication.

See also \([6]\).
1.3. Proposition.  \((Proposition \ 3.14)\) Let \((x, y), (x', y'), (x'', y'') \in \mathbb{Z}^2_{\geq 0}\) be three points with \(x \geq y, x' \geq y'\) and \(x'' \geq y''\). If \((x', y') = \frac{(x+y)+(x''+y'')}{2}\) then
\[
m_{x,y} + m_{x'',y''} \geq 2m_{x',y'}.
\]

The structure of this paper is the following. Section 2 is preliminary, we review the Markov distance, generalized Markov numbers defined by Lee-Li-Rabideau-Schiffler, and some properties herein. We introduce the ratios between two generalized Markov numbers and study the monotonicity in Section 3. We study the monotonicity of the generalized Markov numbers and give the proof of the main result in Section 4.

Convention: (i) The points that appeared in the paper are always assumed to lie in the area \(\{(x, y) \in \mathbb{Z}^2_{\geq 0} \mid x > y\}\) unless otherwise stated. (ii) When we consider the monotonicity of the generalized Markov numbers along a line \(l\), we always assume that \(l \cap \{(x, y) \in \mathbb{Z}^2_{\geq 0} \mid x > y\} \neq \emptyset\).

2. Preliminary

In this section, we review the Markov distance, generalized Markov numbers defined in [10], and some properties.

Let \(\gamma\) be a curve connecting two points but not passing through a third one. Musiker-Schiffler-Williams [13] associated \(\gamma\) with an element \(x_{\gamma}\) in the once-punctured torus cluster algebra. Denote \(|\gamma| = x_{\gamma}|x_1,x_2,x_3| = 1\) the positive integer obtained from \(x_{\gamma}\) by specializing the initial cluster algebras \(x_1, x_2\) and \(x_3\) to 1.

2.1. Definition. \([10]\) For any points \(A, B \in \mathbb{Z}^2\), the Markov distance \(|AB|\) between \(A\) and \(B\) is defined as \(|\gamma^L_{AB}|\), where \(\gamma^L_{AB}\) is the left deformation of \(AB\).

By the skein relation [14] of cluster algebras from surfaces, the Markov distance has the following important property.

2.2. Theorem. \([10, Corollary \ 3.6]\) (Ptolemy inequality) Given four points \(A, B, C, D\) in the plane such that the straight line segments \(AB, BC, CD, DA\) form a convex quadrilateral with diagonals \(AC\) and \(BD\), we have
\[
|AC| \cdot |BD| \geq |AB| \cdot |CD| + |AD| \cdot |BC|.
\]

For any \(A = (x, y) \in \mathbb{Z}^2\), denote \(m_{x,y} = |OA|\).

2.3. Definition. For any \((x, y) \in \mathbb{Z}^2_{>0}\) such that \(x > y\), we call \(m_{x,y}\) a generalized Markov numbers.

Note that if \(x\) and \(y\) are coprime then \(m_{x,y} = m_{x,y}\) is the usual Markov number.

We now list a few results on the generalized Markov numbers that we will need later.

Recall that the Fibonacci numbers are \(\{F_n \mid n \geq 0\}\) which satisfies \(F_0 = 0, F_1 = F_2 = 1, F_n = F_{n-1} + F_{n-2}\) for \(n \geq 2\). It is well-known that the Markov numbers indexed by \((q, 1), q > 1\) are the odd Fibonacci numbers \(F_{2q+1}, q > 1\).

\begin{equation}
m_{q,1} = F_{2q+1} = (\phi^{2q+1} + \phi^{-2q-1})/\sqrt{5} \sim \phi^{2q+1}/\sqrt{5},
\end{equation}

where \(\phi = \frac{\sqrt{5} + 1}{2}\).
Recall that the Pell numbers are \( \{P_n \mid n \geq 0\} \) which satisfies \( P_0 = 0, P_1 = 1, P_2 = 1, P_n = 2P_{n-1} + P_{n-2} \) for \( n \geq 2 \). It is well-known that the Markov numbers indexed by \( (q, q-1) \), \( q > 1 \) are the odd Pell numbers \( P_{2q-1}, q > 1 \).

\[
(2) \quad m_{q,q-1} = P_{2q-1} = \frac{(1 + \sqrt{2})^{2q-1} - (1 - \sqrt{2})^{2q-1}}{2\sqrt{2}} \sim \frac{(1 + \sqrt{2})^{2q-1}}{2\sqrt{2}}.
\]

The following useful lemma is given in [10].

2.4. Lemma. [10, Lemma 6.2] Let \( p, q \) be coprime positive integers and let \( f_n = m_{nq, np} \). Thus \( f_0 = 0 \) and \( f_1 = m_{q,p} \) is the Markov number. Then, for \( n \geq 2 \),

\[
f_n = 3f_1f_{n-1} - f_{n-2}.
\]

As a consequence, we have

\[
f_n = \frac{f_1}{\sqrt{9f_1^2 - 4}}(\alpha^n - \alpha^{-n}), \text{ where } \alpha = \frac{3f_1 + \sqrt{9f_1^2 - 4}}{2}.
\]

In particular, we have

2.5. Proposition. [10] For any \( q, n > 1 \), we have

\[
(1) \quad m_{qn,n} = \frac{m_{q,1}}{\sqrt{9m_{q,1}^2 - 4}}(\alpha^n - \alpha^{-n}) \sim 3^{n-1}m_{q,1} \quad (q \to \infty),
\]

\[
(2) \quad m_{qn,(q-1)n} = \frac{m_{q,q-1}}{\sqrt{9m_{q,q-1}^2 - 4}}(\alpha^n - \alpha^{-n}) \sim 3^{n-1}m_{q,q-1} \quad (q \to \infty),
\]

where \( \alpha = \frac{3m_{q,1} + \sqrt{9m_{q,1}^2 - 4}}{2} \).

As a corollary of Lemma 2.4, we have the following observation.

2.6. Lemma. For any \( q \in \mathbb{Z}_{\geq 0}, \) we have

(1) \( m_{p,0} = \mathcal{F}_{2q} \),

(2) \( m_{p,p} = P_{2q} \).

Proof. We shall only prove (1) as (2) can be proved similarly. For any \( q \geq 2 \), as \( \mathcal{F}_{2q-2} = \mathcal{F}_{2q-3} + \mathcal{F}_{2q-4} \), we have that \( \mathcal{F}_{2q-2} + \mathcal{F}_{2q-3} = 2\mathcal{F}_{2q-2} - \mathcal{F}_{2q-4} \), this implies

\[
\mathcal{F}_{2q-1} = 2\mathcal{F}_{2q-2} - \mathcal{F}_{2q-4},
\]

Thus we have

\[
\mathcal{F}_{2q} = 3\mathcal{F}_{2q-2} - \mathcal{F}_{2q-4}.
\]

Moreover as \( m_{0,0} = 0 = \mathcal{F}_0 \) and \( m_{1,0} = 1 = \mathcal{F}_2 \), by Lemma 2.4 the result follows.

\[\square\]

3. Ratio between two generalized Markov numbers

3.1. Horizontal and vertical ratios.

3.1. Definition. For any \( (q, p) \in \mathbb{Z}_{\geq 0}^2 \) with \( p \leq q \), the horizontal ratio at \( (q, p) \) is the ratio \( \frac{m_{q,p+1}}{m_{q,p}} \), denote by \( h(q, p) \). If \( p < q \), the vertical ratio at \( (q, p) \) is the ratio \( \frac{m_{q,p+1}}{m_{q,p}} \), denote by \( v(q, p) \).
Thus $m_{q+1,p} > m_{q,p}$ if and only if $h(q, p) > 1$, $m_{q,p+1} > m_{q,p}$ if and only if $v(q, p) > 1$.

We first investigate the monotonicity of $h(q, p)$ and $v(q, p)$.

3.2. Lemma. For any $(q, p) \in \mathbb{Z}_+^2$ with $p \leq q$, we have

(i) $h(q+1, p) > h(q, p),$

(ii) if $p + 1 \leq q$ then $h(q, p) > h(q, p + 1)$.

Proof. (1) Let $O = (0, 0), A_1 = (q, p), A_2 = (q + 1, p), A_3 = (q + 2, p)$. Let $O' = (1, 0)$. By the Ptolemy inequality, we have $|OA_3||O'A_2| > |OA_2||O'A_3|$, that is $|OA_3||O'A_1| > |OA_2|^2$.

It follows that $\frac{|OA_3|}{|OA_2|} > \frac{|OA_2|}{|OA_1|}$, that is $h(q+1, p) > h(q, p)$.

(2) Let $O = (0, 0), A_1 = (q, p), A_2 = (q + 1, p), A'_1 = (q, p + 1), A'_2 = (q + 1, p + 1)$. Let $O' = (0, 1)$. By the Ptolemy inequality, we have $|OA'_1||O'A'_2| > |OA'_2||OA'_1|$, that is $|OA'_1||O'A_2| > |OA_1||OA'_2|$.

It follows that $\frac{|OA_2|}{|OA_1|} > \frac{|OA'_2|}{|OA'_1|}$, that is $h(q, p) > h(q, p + 1)$.

3.3. Remark. Lemma 3.2 implies the function $h(x, y)$ is increasing along the $x$-axis and decreasing along the line $y$-axis.

3.4. Corollary. For any $(q, p) \in \mathbb{Z}_+^2$ with $p \leq q$, we have

\[
\frac{(1 + \sqrt{2})^{4q+2} + 1}{(1 + \sqrt{2})(1 + \sqrt{2})^{4q} - 1} \leq h(q, p) \leq \frac{\phi^{4q+6} + 1}{\phi^2(\phi^{4q+2} + 1)},
\]

where $\phi = \frac{\sqrt{5} + 1}{2}$. In particular, we have

\[
(1 + \sqrt{2})m_{q,p} < m_{q,p+1} < \frac{3 + \sqrt{3}}{2}m_{q,p+1}.
\]
Proof. By Lemma 3.2, we have $h(q, q) \leq h(q, p) \leq h(q, 1)$. By Lemma 2.6, we have

$$h(q, q) = \frac{m_{q+1, q}}{m_{q, q}} = \frac{P_{2q+1}}{P_{2q}} = \frac{(1 + \sqrt{2})^{4q+2} + 1}{(1 + \sqrt{2})(1 + \sqrt{2})^{4q} - 1},$$

$$h(q, 1) = \frac{m_{q+1, 1}}{m_{q, 1}} = \frac{F_{2q+3}}{F_{2q+1}} = \frac{\phi^{4q+6} + 1}{\phi^2(\phi^{4q+2} + 1)}.$$

As $\frac{(1 + \sqrt{2})^{4q+2} + 1}{(1 + \sqrt{2})(1 + \sqrt{2})^{4q} - 1} > 1 + \sqrt{2}$ and $\frac{\phi^{4q+6} + 1}{\phi^2(\phi^{4q+2} + 1)} < \phi^2 = \frac{3 + \sqrt{5}}{2}$, we have

$$(1 + \sqrt{2})m(q, p) < m(q, p+1) < \frac{3 + \sqrt{5}}{2}m(q, p+1).$$

\[\square\]

3.5.Lemma. For any $(q, p) \in \mathbb{Z}_{\geq 0}$ with $p < q$, we have

(1) if $p + 1 < q$ then
$$v(q, p + 1) > v(q, p),$$

(2)
$$v(q, p) < v(q + 1, p).$$

Proof. (1) Let $O = (0, 0), A_1 = (q, p), A_2 = (q, p + 1), A_3 = (q, p + 2)$. Let $O' = (0, 1)$. By the Ptolemy inequality, we have $|OA_3||O'A_2| > |OA_2||O'A_3|$, that is
$$|OA_3||OA_1| > |OA_2|^2.$$ It follows that $$\frac{|OA_3|}{|OA_2|} > \frac{|OA_2|}{|OA_1|},$$ that is
$$v(q, p + 1) > v(q, p).$$

(2) Let $O = (0, 0), A_1 = (q, p), A_2 = (q + 1, p), A'_1 = (q, p + 1), A'_2 = (q + 1, p + 1)$. Let $O' = (0, 1)$. By the Ptolemy inequality, we have $|OA_2||O'A'_2| > |O'A_2||OA'_2|$, that is
$$|OA_2||OA'_1| > |OA_1||OA'_2|.$$ It follows that $$\frac{|OA_2|}{|OA_1|} > \frac{|OA'_2|}{|OA'_1|},$$ that is
$$v(q, p) > v(q + 1, p)$$

\[\square\]

3.6. Remark. Lemma 3.5 implies the function $v(x, y)$ is increasing along the x-axis and decreasing along the y-axis.
3.7. COROLLARY. For any \((q, p) \in \mathbb{Z}_{\geq 0}^2\) with \(p < q\), we have

\[
\frac{(1 + \sqrt{2})^{4p+4} - 1}{(1 + \sqrt{2})((1 + \sqrt{2})^{4p+2} + 1)} \leq v(q, p) \leq \frac{\phi^{4q+2} + 1}{\phi^{q+1}},
\]

where \(\phi = \frac{\sqrt{5} + 1}{2}\). In particular,

\[m_{q,p} < m_{q,p+1} < \phi m_{q,p}.\]

Proof. By Lemma 3.5, we have \(v(p+1, p), v(q, q-1) \leq v(q, p) \leq v(q, 0)\). By Lemma 2.6 we have

\[
v(p + 1, p) = \frac{m_{p+1,p+1}}{m_{p+1,p}} = \frac{P_{2p+2}}{P_{2p+1}} = \frac{(1 + \sqrt{2})^{4p+4} - 1}{(1 + \sqrt{2})((1 + \sqrt{2})^{4p+2} + 1)}.
\]

\[
v(q, 0) = \frac{m_{q,1}}{m_{q,0}} = \frac{F_{2q+1}}{F_{2q}} = \frac{\phi^{4q+2} + 1}{\phi^{q+1}}.
\]

The result follows. \(\square\)

3.2. Ratio along any line. Inspired by the horizontal and vertical ratio, we now define the ratio along any line \(l : y = kx + b\) where \(k, b \in \mathbb{Q}\).

For any \(t \in \mathbb{R}\) denote by \(l[t] : y = k(x - t) + b\) the line obtained from \(l\) by shift along \(x\)-axis by \(t\), by \(l(t) : (y - t) = k(x - t) + b\) the line obtained from \(l\) by shift along \(y = x\) by \(\sqrt{2}t\). We always assume that \(t \in \mathbb{Z}_{\geq 0}\) unless otherwise stated. Note that \(l(t) = l[t - t/k]\) when \(k \neq 0\).

List the integral points in \(\{(q, p) \in \mathbb{Z}_{\geq 0}^2 \mid p < q\} \cap l\) as \((x_1, y_1), (x_2, y_2), \cdots\) in order such that \(x_1 < x_2 < \cdots\).

3.8. NOTATION. Denote by \((u_1, v_1), (u_1', v_1')\) the first two integral points in \(\{(q, p) \in \mathbb{Z}_{\geq 0}^2 \mid p < q\} \cap l\). If \(k < 0\), denote by \((z_1, w_1), (z_1', w_1')\) the last two integral points in \(\{(q, p) \in \mathbb{Z}_{\geq 0}^2 \mid p < q\} \cap l\).

The following follows immediately by the construction of \(l[t]\) and \(l(t)\).

3.9. LEMMA. Assume that the line \(l\) has negative slope. Then for any \(t \in \mathbb{Z}_{\geq 0}\), we have

1. \((u_{i(t)}, v_{i(t)}) = (u_i + t, v_i + t)\) and \((u_{i(t)}, v_{i(t)}') = (u_i' + t, v_i' + t)\);
2. \((z_{i(t)}, w_{i(t)}) = (z_i + t, w_i)\) and \((z_{i(t)}, w_{i(t)}') = (z_i' + t, w_i')\).

3.10. DEFINITION. Let \(k, b \in \mathbb{Q}\) and \(l : y = kx + b\) be a line in \(\mathbb{R}^2\). Let \((x_i, y_i), (x_{i+1}, y_{i+1})\) be consecutive points on \(l\). The ratio \(r_l(x_{i}, y_{i})\) along \(l\) at \((x_i, y_i)\) is defined to be

\[r_l(x_i, y_i) := \frac{m_{(x_{i+1}, y_{i+1})}}{m_{(x_i, y_i)}}.\]

3.11. REMARK. By the definition,

1. the generalized Markov numbers increase with \(x\) along the line \(l\) if and only if \(r_l(x_i, y_i) \geq 1\) for any \(i\);
2. the generalized Markov numbers decrease with \(x\) along the line \(l\) if and only if \(r_l(x_i, y_i) \leq 1\) for any \(i\).
3.12. Proposition. With the foregoing notation. Let \( k, b \in \mathbb{Q} \) and \( l : y = kx + b \). Let \((x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})\) be three consecutive points on \( l \).

(1) If \( b \neq 0 \) then
\[
r_l(x_{i+1}, y_{i+1}) > r_l(x_i, y_i),
\]
that is, the ratios along \( l \) are increase with \( x \).

(2) If \( b = 0 \) then
\[
r_l(x_{i+1}, y_{i+1}) < r_l(x_i, y_i),
\]
that is, the ratios along \( l \) are decrease with \( x \).

Proof. Let \( O = (0, 0), A_1 = (x_1, y_1), A_2 = (x_{i+1}, y_{i+1}) \) and \( A_3 = (x_{i+2}, y_{i+2}) \). Let \( O' = (x_1 + x_{i+1} - y_{i+1} - y_1) \).

(1) If \( b \neq 0 \), then \( O \) is not on \( l \), and thus \( OA_3 \) crosses \( O'A_2 \). Then by the Ptolemy inequality, we have \(|OA_3||O'A_2| > |OA_2||O'A_3|\), that is,
\[
|OA_3||OA_1| > |OA_2||OA_2|.
\]
Thus,
\[
r_l(x_{i+1}, y_{i+1}) = \frac{|OA_3|}{|OA_2|} > \frac{|OA_2|}{|OA_1|} = r_l(x_i, y_i).
\]

(2) If \( b = 0 \) we have \( k > 0 \). Then \( O \in l \) and \( \gamma_{O,A_2}^L \) crosses \( \gamma_{O'A_3}^L \), where \( \gamma_{O,A_2}^L \) and \( \gamma_{O,A_3}^L \) are the left deformations of \( OA_2 \) and \( O'A_3 \), respectively, given in \([10]\). By the Ptolemy inequality, we have \(|OA_2||O'A_3| > |OA_3||O'A_2|\), that is
\[
|OA_2||OA_2| > |OA_3||OA_1|.
\]
Thus,
\[
r_l(x_i, y_i) = \frac{|OA_2|}{|OA_1|} > \frac{|OA_3|}{|OA_2|} = r_l(x_{i+1}, y_{i+1}).
\]

![Figure for Proposition 3.12](image-url)

3.13. Proposition. With the foregoing notation. Let \( k, b \in \mathbb{Q} \) and \( l : y = kx + b \).

(1) If \( k \geq 0 \) then the generalized Markov numbers increase with \( x \) along \( l \);

(2) If \( k < 0 \) then the generalized Markov numbers first decrease with \( x \) to some point \((x_i(t), y_i(t))\) then increase with \( x \) along \( l \); moreover,

(2.1) the generalized Markov numbers increase with \( x \) along \( l \) if and only if
\[
m_{u_1,v_1} \leq m_{u'_1,v'_1},
\]
in terms of Notation 3.8, that is \( r_l(u_1, v_1) \geq 1 \).
(2.2) the generalized Markov numbers decrease with $x$ along $l$ if and only if

$$m_{z_l,w_l} \geq m_{z'_l,w'_l},$$

in terms of Notation 3.8, that is $r_l(z_l, w_l) \leq 1$.

(2.3) the generalized Markov numbers are not monotonic along the line $l$ if and only if $r_l(u_l, v_l) < 1$ and $r_l(z_l, w_l) > 1$

Proof. It follows immediately by Proposition 3.12 and Remark 3.11.

As a corollary of Proposition 3.12, the following can be viewed as modified triangle inequality for the Markov distance.

3.14. Proposition. Let $(x, y), (x', y'), (x'', y'') \in \mathbb{Z}_+^2$ be three points with $x \geq x', x'' \geq y'$ and $x' \geq y''$. If $(x', y') = \frac{(x, y) + (x'', y'')}{2}$ then

$$m_{x,y} + m_{x'',y''} \geq 2m_{x',y'}.$$  

Proof. If $x = x' = x''$ we may assume that $y < y' < y''$, by Lemma 3.5 we have $v(x, y) < v(x', y')$. Thus

$$m_{x,y} + m_{x'',y''} = (\frac{1}{v(x,y)} + v(x', y'))m_{x',y'}$$

$$> (\frac{1}{v(x,y)} + v(x, y))m_{x',y'}$$

$$\geq 2m_{x',y'}.$$  

If $x \neq x'$ we may assume that $(x, y), (x', y'), (x'', y'')$ are on the line $l : y = kx + b$. We may further assume that $x < x' < x''$.

If $b \neq 0$, by Proposition 3.12 repeatedly, we have $\frac{m_{x,y}}{m_{x',y'}} > \frac{m_{x',y'}}{m_{x,y}}$. Thus

$$m_{x,y} + m_{x'',y''} = (\frac{m_{x,y}}{m_{x',y'}} + \frac{m_{x'',y''}}{m_{x',y'}})m_{x',y'}$$

$$> (\frac{m_{x,y}}{m_{x',y'}} + \frac{m_{x'',y''}}{m_{x,y}})m_{x',y'}$$

$$\geq 2m_{x',y'}.$$  

If $b = 0$ then $k \geq 0$. By Corollaries 3.4 and 3.7, we have $r_l(x, y) > 1$ for all $(x, y) \in l \cap \mathbb{Z}_+^2$. Assume that $q, p$ are coprime and $(q, p) \in l$. By Lemma 2.4 we have $\lim_{x \to \infty} r_l(x, y) + \frac{1}{m_{x,y}} = 3m_{q,p} > 3$. It follows that $\lim_{x \to \infty} r_l(x, y) > 2$. By Proposition 3.12, $r_l(x, y)$ is decreasing along $x$, it follows that $r_l(x_i, y_i) > \lim_{x \to \infty} r_l(x, y) > 2$ for any integral point $(x_i, y_i)$ on $l$. It follows that $\frac{m_{x_i,y_i}}{m_{x,y}} > 2$. Thus

$$m_{x,y} + m_{x'',y''} > 2m_{x',y'}.$$  

The following result generalizes Lemmas 3.2 and 3.5 to the lines with a negative slope.

3.15. Proposition. Let $l, l'$ be two lines with same negative slope. Let $(x_i, y_i)$ and $(x_i', y_i')$ be integral points on $l$ and $l'$, respectively.

(1) If $x_i < x'_i$ and $y_i = y'_i$ then

$$r_l(x_i, y_i) < r_{l'}(x'_i, y'_i).$$
(2) If \( x_i = x_i' \) and \( y_i < y_i' \) then
\[ r_l(x_i, y_i) > r_{l'}(x_i', y_i'). \]

(3) If \( y_i - x_i = y_i' - x_i' \) and \( x_i < x_i' \), then
\[ r_l(x_i, y_i) > r_{l'}(x_i', y_i'). \]

Proof. Let \( O = (0,0), A_1 = (x_i, y_i), A_2 = (x_{i+1}, y_{i+1}), A'_1 = (x_i', y_i'), A'_2 = (x_{i+1}', y_{i+1}'). \)

Let \( O' = (x_2 - x_1, y_2 - y_1) \).

(1) As \( x_i < x_i' \) and \( y_i = y_i' \), \( OA'_2 \) crosses \( OA_2 \). By the Ptolemy inequality, we have \( |OA'_2||OA_2| > |OA_2||OA'_2| \), that is
\[ |OA'_2||OA_1| > |OA_2||OA'_1|. \]

It follows that \( \frac{|OA'_2|}{|OA_2|} > \frac{|OA'_1|}{|OA_1|} \), that is
\[ r_l(x_i, y_i) < r_{l'}(x_i', y_i'). \]

(2) As \( x_i = x_i' \) and \( y_i < y_i' \), \( OA_2 \) crosses \( OA'_2 \). By the Ptolemy inequality, we have \( |OA_2||OA'_2| > |OA'_2||OA_2| \), that is
\[ |OA_2||OA'_1| > |OA'_2||OA_1|. \]

It follows that \( \frac{|OA_2|}{|OA'_1|} > \frac{|OA'_2|}{|OA_1|} \), that is
\[ r_l(x_i, y_i) > r_{l'}(x_i', y_i'). \]

(3) As \( x_i < x_i' \) and \( OO' = A_1A_2 = A'_1A'_2 \), we have \( OA_2 \) crosses \( OA'_2 \). By the Ptolemy inequality, we have \( |OA_2||OA'_2| > |OA'_2||OA_2| \), that is
\[ |OA_2||OA'_1| > |OA'_2||OA_1|. \]

It follows that \( \frac{|OA_2|}{|OA_1|} > \frac{|OA'_2|}{|OA'_1|} \), that is
\[ r_l(x_i, y_i) > r_{l'}(x_i', y_i'). \]

\[ \square \]

Proposition 3.15 implies the ratios along the lines with a given negative slope are increase with \( x \), decrease with \( y \).

The following follows immediately from Proposition 3.15.

3.16. COROLLARY. With the foregoing notation. Let \( l \) be a line with a negative slope. Then the sequence
\[ r_l(t(z_{i[t]}, w_{i[t]})), \quad t \in \mathbb{Z}_{>0} \]

is a strictly decreasing sequence, where \( (z_{i[t]}, w_{i[t]}) \) is given in Notation 3.8.

4. Monotonicity of the generalized Markov numbers

4.1. On the last ratio. Recall that we denote by \( (z_l, w_l) \), \( (z'_l, w'_l) \) the last two integral points on \( l \) for any line \( l \) with a negative slope. In this subsection, we study the ratio \( r_l(z_l, w_l) \).

For a given \( k = -\frac{a_2}{a_1} \in \mathbb{Q}_{<0} \) with \( a_1, a_2 \in \mathbb{Z}_{>0} \) and \( g.c.d.(a_1, a_2) = 1 \), denote by \( l_n : y = k(x - n) + 1, n \in \mathbb{Z} \) the sequence of lines through \((n, 1)\) with slope \( k \). The last two integral points on \( l_n \) are \((n - a_2, 1 + a_1)\) and \((n, 1)\). For any line \( l : y = kx + b \) with slope \( k \), assume that \( l \neq l_n \) for any \( n \in \mathbb{Z} \), then there exists \( N^+ \in \mathbb{Z} \) such that \( l \) lies between \( l_{N^+ - 1} \) and \( l_{N^+} \). Thus \( z_l < N^+ - 1 \). Note that to
ensure there are at least two integral points in \(l \cap \{(x, y) \in \mathbb{Z}_2^2 \mid x > y\}\), we may assume that \(N^+ \geq 4\).

4.1. Lemma. With the foregoing notation. We have

\[ r_l(z_l, w_l) < r_{N^+}(z_{N^+}, w_{N^+}). \]

Proof. Let \(O = (0, 0), A_1 = (z_l, w_l), A_2 = (z'_l, w'_l), A'_1 = (N^+ - a_2, 1 + a_1), A'_2 = (N^+, 1)\). As the slopes of \(l\) and \(l_n\) are \(k\), \(A_1, A_2, A'_2\) and \(A'_1\) form a parallelogram. Let \(O' = (a_2, -a_1)\).

As \(z_l, a_2 < N^+\) and \(w_l > 1\), we see that \(OA'_2\) crosses \(O'A_2\). By the Ptolemy inequality, we have \(|OA'_2||O'A_2| > |OA_2||O'A'_2|\), that is

\[ |OA'_2||OA_1| > |OA_2||OA'_1|. \]

It follows that \(\frac{|OA'_2|}{|OA_1|} > \frac{|OA_2|}{|OA'_1|}\), that is

\[ r_l(z_l, w_l) < r_{N^+}(z_{N^+}, w_{N^+}). \]

4.2. Lemma. Let \(l\) be a line with negative slope \(k = -\frac{a_1}{a_2}\) with \(a_1, a_2 \in \mathbb{Z}_{>0}\) and \(\text{g.c.d.}(a_1, a_2) = 1\). For a larger enough \(t \in \mathbb{Z}_{>0}\), there exists \(N^- \in \mathbb{N}\) such that

\[ r_{l[t]}(z_{l[t]}, w_{l[t]}) > r_{N^-}(z_{N^-}, w_{N^-}), \]

moreover, \(N^-\) can be chosen such that \(\lim_{t \to +\infty} N^- = +\infty\).

Proof. Let \(O = (0, 0), O' = (a_2, -a_1), A_1 = (z_l, w_l), A_2 = (z'_l, w'_l)\). For any \(t \in \mathbb{Z}_{>0}\), denote \(A_1[t] = (w + t, w)\) and \(A_2[t] = (w'_l + t, v'_l)\). According to Lemma 3.9, \(A_1[t]\) and \(A_2[t]\) are the last two integral points on \(l[t]\). We see that \(A_2[t]\) lies on the line
\[ y = w'_t. \] On the other hand, the last integral points on \( b_n, n \in \mathbb{Z}_{>0} \) lies on the line \( y = 1 \).

Note that if \( t \) approximates to \(+\infty\), the \( x \)-coordinate of the crossing point of the segment \( OA_2[t] \) and the line \( y = 1 \) approximate to \(+\infty\). Therefore, for larger enough \( t \), we can find \( N_- \in \mathbb{N} \) such that \( OA_2[t] \) crosses \( OA'_2 \), where \( A'_2 \) is the last integral points on \( l_{N_-} \).

By the Ptolemy inequality, we have \(|OA_2[t]| \cdot |OA'_2| > |OA'_2| \cdot |OA_2[t]|\), that is
\[ |OA_2[t]| \cdot |OA'_2| > |OA'_2| \cdot |OA_1[t]|. \]

It follows that \(|OA_2[t]| > \frac{|OA'_2|}{|OA'_1|}\), that is
\[ r_{l[t]}(z_{l[t]}, w_{l[t]}) > r_{l_{N_-}}(z_{l_{N_-}}, w_{l_{N_-}}). \]

Moreover, when \( t \) approximates to \(+\infty\), the \( x \)-coordinate of the crossing point of the segment \( OA'_2[t] \) and the line \( y = 1 \) approximates to \(+\infty\), so we can choose \( N_- \in \mathbb{N} \) such that \( N_- \) approximates to \(+\infty\).

From the proof of Lemma 4.2, we have the following observation.

4.3. **Lemma.** Assume that \( k = -\frac{a_1}{a_2} \) with \( a_1, a_2 \in \mathbb{Z}_{>0} \) and \( \gcd(a_1, a_2) = 1 \). Let \( \gamma : y = kx \) be the line passing through \( O \) with slope \( k \). Suppose that \( n > 1 + a_1 + a_2 \)

(1) Let \( l \) be a line with slope \( k \). Then there exists \( \delta(n, l) \in \mathbb{R} \) such that
\[ r_{l[t]}(z_{l[t]}, w_{l[t]}) > r_{l_n}(z_{l_n}, w_{l_n}) \]
for all \( t \in \mathbb{Z}_{>0} \) such that \( l[t] \) lies on the right side of \( \gamma[\delta(n, l)] \).

(2) Let \( l' \) be another line with slope \( k \). If \( w'_l < w'_l \) then \( \gamma[\delta(n, l)] < \gamma[\delta(n, l')] \).
(3) There exists $\delta(n) \in \mathbb{R}$ such that for any line $l$ with slope $k$,

$$ r_n(t)(z_{t|l}, w_{t|l}) > r_{t|n}(z_{t|n}, w_{t|n}) $$

for all $t \in \mathbb{Z}_{>0}$ such that $l[t]$ lies on the right side of $\gamma[\delta(n)]$.

Proof. (1) Let $O = (0,0), O' = (a_2, -a_1), A'_1 = (n - a_1, a_2 + 1), A'_2 = (n, 1)$. Assume that the line connecting $O$ and $A'_2$ crosses $y = w'_1$ at some point $A$. Then the line through $A$ with slope $k$ equals to $l[\delta(n,l)]$ for some $\delta(n,l) \in \mathbb{R}$. Thus, for any $t \in \mathbb{Z}_{>0}$ such that $l[t]$ lies on the right side of $\gamma[\delta(n,l)]$, we have $OA_2[t]$ crosses $O'A'_2$, where $A_2[t]$ is the last integral points on $l[t]$. From the proof of Lemma 4.3 we have

$$ r_n[t](z_{t|l}, w_{t|l}) > r_{t|n}(z_{t|n}, w_{t|n}). $$

(2) It follows immediately from the construction of $\delta(n,l)$.

(3) From the construction of $\delta(n,l)$, we see that $\delta(n,l)$ only depends on $w'_1$. Note that for any $l$ we have $w'_1 \leq a_2$ as $(z'_1, w'_1)$ is the last integral point. Thus we may let $\delta(n) = \delta(n,l)$, where $l$ is the line with slope $k$ such that $w'_1$ is maximal.

As a corollary of Lemmas 4.1 and 4.2 we have the following.

4.4. Corollary. Let $k \in \mathbb{Q}_{<0}$ and $l : y = kx + b$ be a line with slope $k$. Let $l_n : y = k(x - n) + 1$ be the lines through $(n,1)$ with slope $k$. Then

$$ \lim_{t \to \infty} r_{[t]}(z_{t|l}, w_{t|l}) = \lim_{n \to \infty} r_{n}(z_{t|n}, w_{t|n}). $$

4.5. Proposition. With the foregoing notation. Assume that $k = -\frac{a_1}{a_2} \in \mathbb{Q}_{<0}$ with $a_1, a_2 \in \mathbb{Z}_{>0}$. Let $l$ be a line with slope $k$. Then

(1) 

$$ \lim_{n \to \infty} r_{n}(z_{t|n}, w_{t|n}) = \left( \frac{2\sqrt{5}}{3(1 + \sqrt{5})} \right)^{a_1} \left( \frac{1 + \sqrt{5}}{2} \right)^{2a_2}. $$

(2) 

$$ \lim_{t \to \infty} r_{t|n}(z_{t|l}, w_{t|l}) = \left( \frac{2\sqrt{5}}{3(1 + \sqrt{5})} \right)^{a_1} \left( \frac{1 + \sqrt{5}}{2} \right)^{2a_2}. $$

Proof. We shall only prove (1) as (2) follows by Corollary 4.4.

For each $n$, we have $(z_{t|n}, w_{t|n}) = (n - a_2, 1 + a_1)$ and $(z'_{t|n}, w'_{t|n}) = (n, 1)$. By Proposition 3.15 (1), the sequence $r_{n}(z_{t|n}, w_{t|n}), n \in \mathbb{Z}_{>0}$ is strictly increasing. It suffices
to consider the subsequence indexed by \( q(1 + a_1) + a_2, q \in \mathbb{Z}_>0 \). By Proposition 2.5 we have

\[
\lim_{q \to \infty} r_{q(1 + a_1) + a_2} (z_{q(1 + a_1) + a_2}, w_{q(1 + a_1) + a_2}) = \frac{m_{q(1 + a_1) + a_2 - 1}}{m_{q(1 + a_1) + a_4}} = \frac{(1 + \sqrt{5}/2)^{2(q(1 + a_1) + a_2 + 1)}}{3^{q+1} q + 1} = \frac{(1 + \sqrt{5}/2)^{2(q(1 + a_1) + a_2 + 1)}}{3^{q+1} (1 + \sqrt{5}/2)^{2q+1} + a_1} = \left( \frac{\sqrt{5}}{3} \right) a_1 \left( \frac{1 + \sqrt{5}}{2} \right) - a_1 + 2a_2 = \left( \frac{2\sqrt{5}}{3(1 + \sqrt{5})} \right) a_1 \left( \frac{1 + \sqrt{5}}{2} \right)^{2a_2}.
\]

\( \square \)

4.2. On the first ratio. Recall that we denote by \((u_1, v_1), (u_1', v_1')\) the first two integral points on \( l \) for any line \( l \) with a negative slope. In this subsection, we study the ratio \( r_l(u_1, v_1) \).

For a given \( k = -\frac{a_1}{a_2} \in \mathbb{Q}_{<}0 \) with \( a_1, a_2 \in \mathbb{Z}_>0 \) and \( \text{g.c.d.}(a_1, a_2) = 1 \), denote by \( L_n : y = k(x - n) + n - 1, n \in \mathbb{Z} \) the sequence of lines through \((n, n - 1)\) with slope \( k \). The first two integral points on \( L_n \) are \((n, n - 1)\) and \((n + a_2, n - 1 - a_1)\). For any line \( l : y = kx + b \) with slope \( k \), assume that \( l \neq L_n \) for any \( n \in \mathbb{Z} \), there exists \( \tilde{N}_+ \in \mathbb{Z} \) such that \( l \) lies between \( l_{\tilde{N}_+ - 1} \) and \( l_{\tilde{N}_+} \).

4.6. Lemma. With the foregoing notation. We have

\[ r_l(u_1, v_1) > r_{L_{\tilde{N}_+}'} (u_{L_{\tilde{N}_+}'}, v_{L_{\tilde{N}_+}'}) \]

Proof. Let \( O = (0, 0) \), \( A_1 = (u_1, v_1), A_2 = (u_1', v_1'), A_1' = (u_{L_{\tilde{N}_+}'}, v_{L_{\tilde{N}_+}'}, A_2' = (u_{L_{\tilde{N}_+}'}, v_{L_{\tilde{N}_+}'}) \).

As the slopes of \( l \) and \( l_n \) are \( k, A_1, A_2, A_2' \) and \( A_1' \) form a parallelogram. Let \( O' = (a_2, -a_1) \).

As \( l \) lies between \( l_{\tilde{N}_+ - 1} \) and \( l_{\tilde{N}_+} \), we see that \( O'A_2' \) crosses \( O'A_2 \). By the Ptolemy inequality, we have \(|O'A_2||O'A_2'| > |O'A_2||O'A_1|\), that is

\[ |O'A_2| |O'A_1'| > |O'A_2||O'A_1|. \]

It follows that \( \frac{|O'A_2|}{|O'A_1|} > \frac{|O'A_1'|}{|O'A_1|} \), that is

\[ r_l(u_1, v_1) > r_{L_{\tilde{N}_+}} (u_{L_{\tilde{N}_+}}, v_{L_{\tilde{N}_+}}) \]

\( \square \)

4.7. Lemma. Let \( l \) be a line with negative slope \( k = -\frac{a_1}{a_2} \) with \( a_1, a_2 \in \mathbb{Z}_>0 \) and \( \text{g.c.d.}(a_1, a_2) = 1 \). For a larger enough \( t \in \mathbb{Z}_>0 \), there exists \( \tilde{N}_- \in \mathbb{N} \) such that

\[ r_{l(t)}(u_{l(t)}, v_{l(t)}) < r_{L_{\tilde{N}_-}} (u_{L_{\tilde{N}_-}}, v_{L_{\tilde{N}_-}}) \]

moreover, \( \tilde{N}_- \) can be chosen such that \( \lim_{t \to \infty} \tilde{N}_- = +\infty \).
Proof. Let \( O = (0, 0), O' = (a_2, -a_1), A_1 = (u, v), A_2 = (u', v'). \) For any \( t \in \mathbb{Z}_{>0}, \) denote \( A_1(t) = (u + t, v + t) \) and \( A_2(t) = (u' + t, v' + t) \). According to Lemma 3.9, \( A_1(t) \) and \( A_2(t) \) are the first two integral points on \( l(t) \). We see that \( A_2(t) \) lies on the line \( y = x - u' + v' \). On the other hand, the second integral points on \( L_n, n \in \mathbb{Z}_{>0} \) lies on the line \( y = x - a_1 - a_2 - 1 \).

Note that if \( t \) approximates to \( +\infty \), the \( x \)-coordinate of the crossing point of the segment \( O'A_2(t) \) and the line \( y = x - a_1 - a_2 - 1 \) approximate to \( +\infty \). Therefore, for larger enough \( t \), we can find \( \widetilde{N}_- \in \mathbb{N} \) such that \( OA_2(t) \) crosses \( O'A_2(t) \), where \( A_2' \) is the second integral points on \( L_{\widetilde{N}_-} \).

By the Ptolemy inequality, we have \( |O'A_2(t)||OA_2'| > |OA_2(t)||O'A_2| \), that is

\[
|O'A_1(t)||OA_1'| > |OA_2(t)||O'A_2|.
\]

It follows that \( \frac{|O'A_1'|}{|O'A_1|} > \frac{|O'A_2(t)|}{|O'A_2(t)|} \), that is

\[
\frac{r_{l(t)}(u_1(t), v_1(t))}{r_{L_{\widetilde{N}_-}}(u_{L_{\widetilde{N}_-}}, v_{L_{\widetilde{N}_-}})}.
\]

Moreover, when \( t \) approximates to \( +\infty \), the \( x \)-coordinate of the crossing point of the segment \( O'A_2(t) \) and the line \( y = x - a_1 - a_2 - 1 \) approximates to \( +\infty \), so we can choose \( \widetilde{N}_- \in \mathbb{N} \) such that \( \widetilde{N}_- \) approximates to \( +\infty \).
4.8. **Lemma.** Assume that $k = -\frac{a_1}{a_2}$ with $a_1, a_2 \in \mathbb{Z}_{>0}$ and $\text{g.c.d.}(a_1, a_2) = 1$. Let $\gamma : y = kx$ be the line passing through $O$ with slope $k$. Suppose that $n > 1 + a_1$.

(1) Let $l$ be a line with slope $k$. Then there exists $\sigma(n,l) \in \mathbb{R}$ such that

$$r_l(u_l(t), v_l(t)) < r_{\sigma(n,l)}(u_{\sigma(n,l)}, v_{\sigma(n,l)})$$

for all $t \in \mathbb{Z}_{>0}$ such that $l(t)$ lies on the right side of $\gamma(\sigma(n,l))$.

(2) Let $l'$ be another line with slope $k$. If $u'_1 - v'_1 < u'_2 - v'_2$, then $\sigma(n,l) < \sigma(n,l')$.

(3) There exists $\sigma(n) \in \mathbb{R}$ such that for any line $l$ with slope $k$,

$$r_l(u_l(t), v_l(t)) < r_{\sigma(n,l)}(u_{\sigma(n,l)}, v_{\sigma(n,l)})$$

for all $t \in \mathbb{Z}_{>0}$ such that $l(t)$ lies on the right side of $\gamma(\sigma(n))$.

**Proof.** (1) Let $O = (0,0)$, $O' = (a_2, -a_1)$, $A'_1 = (n, n-1)$, $A'_2 = (n+a_2, n-1-a_1)$. Assume that the line connecting $O$ and $A'_2$ crosses $y = x - u'_1 + v'_1$ at some point $A$. Then the line through $A$ with slope $k$ equals to $l(\sigma(n,l))$ for some $\sigma(n,l) \in \mathbb{R}$. Thus, for any $t \in \mathbb{Z}_{>0}$ such that $l(t)$ lies on the right side of $\gamma(\sigma(n,l))$, we have $OA'_1$ crosses $O'A_2(t)$, where $A_2(t)$ is the second integral points on $l(t)$. From the proof of Lemma 4.3, we have

$$r_l(u_l(t), v_l(t)) < r_{\sigma(n,l)}(u_{\sigma(n,l)}, v_{\sigma(n,l)})$$

(2) It follows immediately from the construction of $\sigma(n,l)$.

(3) From the construction of $\sigma(n,l)$, we see that $\sigma(n,l)$ only depends on $u'_1 - v'_1$. Note that for any $l$ we have $u'_1 - v'_1 \leq 2a_1 + 1$ as $(u'_1, v'_1)$ is the second integral point. Thus we may let $\sigma(n) = \sigma(n,l)$, where $l$ is the line with slope $k$ such that $u'_1 - v'_1$ is maximal.

![Figure for Lemma 4.8](image)

As a corollary of Lemmas 4.8 and 4.7, we have the following.

4.9. **Corollary.** With the foregoing notation. Let $l$ be a line with a negative slope. Then

$$\lim_{t \to \infty} r_l(u_l(t), v_l(t)) = \lim_{n \to \infty} r_{\sigma(n,l)}(u_{\sigma(n,l)}, v_{\sigma(n,l)})$$
4.10. Proposition. With the foregoing notation. Assume that $k = -\frac{a_2}{a_1} \in \mathbb{Q}_{<0}$ with $a_1, a_2 \in \mathbb{Z}_{>0}$ and $\gcd(a_1, a_2) = 1$. Let $l$ be a line with slope $k$. Then

(1)
$$\lim_{n \to \infty} r_L(ul_n, vl_n) = \left(\frac{3}{2\sqrt{2}}\right)^{a_1 + a_2} \left(1 + \sqrt{2}\right)^{-a_1 + a_2}.$$  

(2)
$$\lim_{t \to \infty} r_{l(t)}(u_{l(t)}, v_{l(t)}) = \left(\frac{3}{2\sqrt{2}}\right)^{a_1 + a_2} \left(1 + \sqrt{2}\right)^{-a_1 + a_2}.$$  

Proof. We shall only prove (1) as (2) follows by Corollary 4.9.

For each $n$, we have $(u_{l_n}, v_{l_n}) = (n, n - 1)$ and $(u'_{l_n}, v'_{l_n}) = (n + a_2, n - 1 - a_1)$. By Proposition 3.15 (3), the sequence $r_{l_n}(ul_n, vl_n), n \in \mathbb{Z}_{>0}$ is strictly decreasing. It suffices to consider the subsequence indexed by $q(1 + a_1 + a_2) + 1 + a_1, q \in \mathbb{Z}_{>0}$. By Proposition 2.5 we have

$$\lim_{q \to \infty} r_{l(q(1+a_1+a_2)+1+a_1)}(ul_{q(1+a_1+a_2)+1+a_1}, vl_{q(1+a_1+a_2)+1+a_1})$$

$$= \frac{m(q(1+a_1+a_2)+1+a_1,a(q(1+a_1+a_2)+1+a_1)}{m(q(1+a_1+a_2)+1+a_1,q(1+a_1+a_2)+1+a_1)}$$

$$= \frac{3^{a_1+a_2}(m(q+1),q+1)}{(1+\sqrt{2}g.c.d(1+a_1+a_2)+2a_1+1)}$$

$$= \frac{3^{a_1+a_2}(1+\sqrt{2}q)}{(1+\sqrt{2}g.c.d(1+a_1+a_2)+2a_1+1)}$$

$$= \left(\frac{3}{2\sqrt{2}}\right)^{a_1+a_2} \left(1 + \sqrt{2}\right)^{-a_1 + a_2}.$$

4.3. On the monotonicity.

4.11. Theorem. (1) For $k \in \mathbb{Q}$ with $k \geq \frac{\ln(3(1+\sqrt{5})}{\ln(2+\sqrt{2})} \approx -1.1432$, the generalized Markov numbers increase with $x$ along any line $l : y = kx + b$;

(2) For $k \in \mathbb{Q}$ with $k \leq -\frac{2\ln(1+\sqrt{5})}{\ln(3(1+\sqrt{5}))} \approx -1.2417$, the generalized Markov numbers decrease with $x$ along any line $y = kx + b$;

(3) For any $k \in \mathbb{Q}$ with $-\frac{2\ln(1+\sqrt{5})}{\ln(3(1+\sqrt{5}))} < k < -\frac{3(1+\sqrt{5})}{2\sqrt{2}}$, then for almost all $b \in \mathbb{Q}$, the generalized Markov numbers are not monotonic along the line $y = kx + b$.

Proof. If $a_1 \leq a_2$ then we have $\left(\frac{3}{2\sqrt{2}}\right)^{a_1+a_2} (1 + \sqrt{2})^{-a_1+a_2} > 1$. If $a_1 < a_2$, then $\left(\frac{3}{2\sqrt{2}}\right)^{a_1+a_2} (1 + \sqrt{2})^{-a_1+a_2} = 1$ if and only if $\left(\frac{3}{2\sqrt{2}}\right)^{a_1+a_2} \geq (1 + \sqrt{2})^{a_1-a_2} \iff (a_1 + a_2)\ln(\frac{3}{2\sqrt{2}}) \geq (a_1 - a_2)\ln(1 + \sqrt{2}) \iff (\ln(\frac{3}{2\sqrt{2}}) + \ln(1 + \sqrt{2}))a_2 \geq (\ln(1 + \sqrt{2}) - \ln(\frac{3}{2\sqrt{2}}))a_1 \iff \frac{a_1}{a_2} \leq \frac{\ln(1 + \sqrt{2})}{\ln(\frac{3}{2\sqrt{2}})} + \frac{1}{1 - \frac{\ln(1 + \sqrt{2})}{\ln(\frac{3}{2\sqrt{2}})}}$. As $\frac{\ln(1 + \sqrt{2})}{\ln(\frac{3}{2\sqrt{2}})} > 1$, we have $\left(\frac{3}{2\sqrt{2}}\right)^{a_1+a_2} (1 + \sqrt{2})^{-a_1+a_2} \geq 1$ if and only if $\frac{a_1}{a_2} \leq \frac{\ln(1 + \sqrt{2})}{\ln(\frac{3}{2\sqrt{2}})} - \frac{\ln(\frac{3}{2\sqrt{2}})}{\ln(1 + \sqrt{2})}$. We can also have $\frac{a_1}{a_2} \geq \frac{\ln(1 + \sqrt{2})}{\ln(\frac{3}{2\sqrt{2}})} - \frac{\ln(\frac{3}{2\sqrt{2}})}{\ln(1 + \sqrt{2})}$.

On the other hand, $\left(\frac{3\sqrt{5}}{4}\right)^{a_1} \leq a_1 \ln(\frac{3(1 + \sqrt{5})}{2\sqrt{2}}) \iff \frac{a_1}{a_2} \geq \frac{2\ln(1 + \sqrt{2})}{\ln(\frac{3(1 + \sqrt{5})}{2\sqrt{2}})} \iff \frac{a_1}{a_2} \leq -\frac{2\ln(1 + \sqrt{2})}{\ln(\frac{3(1 + \sqrt{5})}{2\sqrt{2}})}$. 


(1) For any \( l \), by Proposition \ref{3.15} (3), the sequence \( r_{l(t)}(u_{l(t)}, v_{l(t)}) \), \( t \in \mathbb{Z}_{>0} \) is decreasing. If \( k \geq \frac{\lg(1+\sqrt{2})+\lg(3\sqrt{2}/4)}{\lg(1+\sqrt{2})-\lg(3\sqrt{2}/4)} \), we have \( \left( \frac{3}{2\sqrt{2}} \right)^{a_1+a_2}(1+\sqrt{2})^{-a_1+a_2} > 1 \). Thus by proposition \ref{4.10} we see that \( r_l(u_l, v_l) > 1 \). Then the result follows by Proposition \ref{3.13}.

(2) The proof is similar to the proof of (1).

(3) If \( -\frac{2\sqrt{n}+\sqrt{3}}{\sqrt{n}+\sqrt{3}} < k < -\frac{2\sqrt{n}+\sqrt{3}}{\sqrt{n}+\sqrt{3}} \), assume that \( k = -\frac{a_1}{a_2} \) with \( a_1, a_2 \in \mathbb{Z}_{>0} \) and \( g.c.d.(a_1, a_2) = 1 \), then we have \( \left( \frac{3}{2\sqrt{2}} \right)^{a_1+a_2}(1+\sqrt{2})^{-a_1+a_2} < 1 \) and \( \left( \frac{2\sqrt{5}}{3(1+\sqrt{5})} \right)^{a_1+1}(1+\sqrt{3})^2 > 1 \).

As \( \left( \frac{3}{2\sqrt{2}} \right)^{a_1+a_2}(1+\sqrt{2})^{-a_1+a_2} < 1 \), by Proposition \ref{4.10} (1), there exists \( n \) such that \( r_{L_n}(u_{L_n}, v_{L_n}) < 1 \). By Lemma \ref{4.8} there exists \( \sigma(n) \) such that for any line \( l \) with slope \( k \), we have \( r_{l(t)}(u_{l(t)}, v_{l(t)}) < r_{L_n}(u_{L_n}, v_{L_n}) < 1 \) for all \( t \in \mathbb{Z}_{>0} \) such that \( l(t) \) lies on the right side of \( \gamma(\sigma(n)) \).

As \( \left( \frac{2\sqrt{5}}{3(1+\sqrt{5})} \right)^{a_1+1}(1+\sqrt{3})^2 > 1 \), by Proposition \ref{4.5} (1), there exists \( n' \) such that \( r_{l_{n'}}(z_{l_{n'}}, w_{l_{n'}}) > 1 \). By Lemma \ref{4.3} there exists \( \delta(n') \in \mathbb{R} \) such that for any line \( l \) with slope \( k \), we have \( r_{l_{[t]}}(z_{l_{[t]}}, w_{l_{[t]}}) > r_{l_{n'}}(z_{l_{n'}}, w_{l_{n'}}) > 1 \) for all \( t \in \mathbb{Z}_{>0} \) such that \( l_{[t]} \) lies on the right side of \( \gamma(\delta(n')) \).

Let \( \eta = \max(\sigma(n) - \frac{\sigma(n)}{\sqrt{3}}, \delta(n')) \). Then for any line \( l \) with slope \( k \) which lies on the right side of \( \gamma(\eta) \), we have \( r_{l}(u_{l}, v_{l}) < 1 \) and \( r_{l}(z_{l}, w_{l}) > 1 \). By Proposition \ref{3.13} the generalized Markov numbers neither increase nor decrease with \( x \) along \( l \).

\( \square \)

In view of \[ \text{Theorem 1.1} \] on the monotonicity of the usual Markov numbers and Theorem \ref{1.1} compare with the uniqueness conjecture of Markov numbers, we propose the following uniqueness conjecture for the generalized Markov numbers.

4.12. Conjecture. For any \((q, p), (q', p') \in \mathbb{Z}_{>0}^2 \) with \( q \geq p, q' \geq p' \), if \((q, p) \neq (q', p')\) then we have \( m_{q, p} \neq m_{q', p'} \).

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