BOUNDEDNESS OF SUBLINEAR OPERATORS ON WEIGHTED MORREY SPACES AND APPLICATIONS

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ABSTRACT. We study the boundedness of some sublinear operators on weighted Morrey spaces under certain size conditions. These conditions are satisfied by most of the operators in harmonic analysis, such as the Hardy-Littlewood maximal operator, Calderón-Zygmund singular integral operator, Bochner-Riesz means at the critical index, oscillatory singular operators, singular integral operators with oscillating kernels and so on. As applications, the regularity in weighted Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients are established.

1. INTRODUCTION AND MAIN RESULTS

As is well known that Morrey [31] introduced the classical Morrey spaces to investigate the local behavior of solutions to second order elliptic partial differential equations (PDE). We recall its definition as

\[ M_{p,q}(\mathbb{R}^n) = \left\{ f : \|f\|_{M_{p,q}(\mathbb{R}^n)} = \sup_{B \subset \mathbb{R}^n} \left( \frac{1}{|B|^{1-\frac{q}{p}}} \int_B |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty \right\}, \]

where \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) and \( 1 \leq p \leq q < \infty \). Here and after, \( B \) denotes any balls in \( \mathbb{R}^n \). \( M_{p,q}(\mathbb{R}^n) \) was an expansion of \( L^p(\mathbb{R}^n) \) in the sense that \( M_{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \). Morrey found that many properties of solutions to PDE can be attributed to the boundedness of some operators on Morrey spaces.

Maximal functions and singular integrals play a key role in harmonic analysis since maximal functions could control crucial quantitative information concerning the given functions, despite their larger size, while singular integrals, Hilbert transform as it’s prototype, nowadays intimately connected with PDE, operator theory and other fields.

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Let $f \in L_{loc}^{1}({\mathbb R}^{n})$. The Hardy-Littlewood (H-L) maximal function of $f$ is defined by
\[
Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_{B} |f(y)|dy.
\]
The Calderón-Zygmund (C-Z) singular integral operator is defined by
\[
Tf(x) = \text{p.v.} \int_{\mathbb R^{n}} K(x-y)f(y)dy,
\]
where $K$ is a C-Z kernel \[18\]. Chiarenza and Frasca \[11\] obtained the boundedness of H-L maximal function $Mf(x)$ and C-Z singular integral operator $T$ on $M_{p,q}({\mathbb R}^{n})$. For some works on the boundedness for the multilinear C-Z singular integral operators on Morrey type spaces, see e.g \[19\].

Let $0 < \alpha < n$. The fractional integral is defined by
\[
I_{\alpha}f(x) = \int_{\mathbb R^{n}} \frac{f(y)}{|y-x|^{n-\alpha}}dy.
\]
An early impetus to the study of fractional integrals originated from the problem of fractional derivation, see e.g. \[3\] and \[38\]. Besides it’s contributions to harmonic analysis, fractional integrals also play an essential role in many other fields. The Hardy-Littlewood-Soblev inequality about fractional integral is still an indispensable tool to establish time-space estimates for the heat semigroup of nonlinear evolution equations, for some of this work, see e.g. \[22\]. In recent times, the applications to Chaos and Fractal have become another motivation to study fractional integrals, see e.g. \[25\] and \[28\]. The boundedness of $I_{\alpha}$ on $M_{p,q}({\mathbb R}^{n})$ was first established by Adams in \[1\].

On the other hand, it is very important to study weighted estimates for these operators in harmonic analysis. It is well known that $M$ is a bounded operator on $L^{p}(w)$ \[35\] with $w \in A_{p}, 1 < p < \infty$. For any non-negative locally functions $w$ and any Lebesgue measurable function $f$, we set
\[
\|f\|_{L^{p}(w)} = \left( \int_{\mathbb R^{n}} |f(x)|^{p}w(x)dx \right)^{1/p}
\]
and if $w \equiv 1$, we denote $\|f\|_{L^{p}(w)}$ simply by $\|f\|_{L^{p}({\mathbb R}^{n})}$. For the weighted $L^{p}(w)$ estimates and weighted weak $(1,1)$ type estimates for $T$, see \[18\]. In \[36\], the authors obtained the corresponding weighted boundedness on weighted $L^{p}$ spaces for $I_{\alpha}$ with $w \in A_{p,q}(1 \leq p, q < \infty)$. Here and after, $A_{p}(1 \leq p < \infty)$ and $A_{p,q}(1 \leq p, q < \infty)$ denote the Muckenhoupt classes \[33\].

In \[26\], Komori and Shirai introduced a weighted Morrey space, which is a natural generalization of weighted Lebesgue space, and investigated the boundedness of classical operators in harmonic analysis, that is, the H-L maximal operator $M$, the C-Z singular integral operator $T$ and the fractional integral $I_{\alpha}$. Let $1 \leq p < \infty$, $0 < \lambda < 1$ and $w$ be a function. Then the weighted Morrey space $M^{p,\lambda}(w)$ is defined by
\[
M_{p,\lambda}(w) = \left\{ f : \|f\|_{M_{p,\lambda}(w)} = \sup_{B} \left( \frac{1}{w(B)^{\lambda}} \int_{B} |f(x)|^{p}w(x)dx \right)^{\frac{1}{p}} < \infty \right\}.
\]
It is obviously that if $w = 1, \lambda = 1 - \frac{p}{q}$, then $M_{p,\lambda}(w) = M_{p,q}(\mathbb{R}^n)$. For $w \in A_p(1 \leq p < \infty)$, if $\lambda = 0$, then $M_{p,0}(w) = L^p(w)$ and if $\lambda = 1$, $M_{p,1}(w) = L^\infty(w)$.

In the fractional case, we need to consider a weighted Morrey space with two weights which also introduced by Komori and Shirai in [26]. Let $1 \leq p < \infty$,  $0 < \lambda < 1$. For two weights $w_1$ and $w_2$,

$$M_{p,\lambda}(w_1, w_2) = \left\{ f : \|f\|_{M_{p,\lambda}(w_1,w_2)} = \sup_{B} \left( \frac{1}{w_2(B)^\lambda} \int_{B} |f(x)|^pw_1(x)dx \right)^{\frac{1}{p}} < \infty \right\}.$$  

If $w_1 = w_2 = w$, then we denote $M_{p,\lambda}(w_1, w_1) = M_{p,\lambda}(w_2, w_2) = M_{p,\lambda}(w)$.

In [49], Wang obtained some estimates for Bochner-Riesz means operators on $M_{p,\lambda}(w)$ by the similar method as in [26]. In this paper, we shall establish some boundedness for some sublinear operators on $M_{p,\lambda}(w)$ and $M_{p,\lambda}(w_1, w_2)$, which includes, as particular cases, the known results in [26] and [49]. Applications to the strong solutions of nondivergence elliptic equations with VMO coefficients are also given.

Let $D_k = \{x \in \mathbb{R}^n : |x| \leq 2^k \}$ and $A_k = D_k/D_{k-1}$ for $k \in \mathbb{Z}$. Let $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$, where $\chi_E$ is the characteristic function of the set $E$. Our mean results are as follows:

**Theorem 1.1.** Suppose that a sublinear operator $\mathcal{T}$ satisfies the size conditions

$$|\mathcal{T}f(x)| \leq C\|f\|_{L^1(\mathbb{R}^n)}/|x|^n, \quad (1.1)$$

when supp $f \subseteq A_k$ and $|x| \geq 2^{k+1}$ with $k \in \mathbb{Z}$ and

$$|\mathcal{T}f(x)| \leq C2^{-kn}\|f\|_{L^1(\mathbb{R}^n)}, \quad (1.2)$$

when supp $f \subseteq A_k$ and $|x| \leq 2^{k-1}$ with $k \in \mathbb{Z}$. Then we have

(a) If $\mathcal{T}$ is bounded on $L^p(w)$ with $w \in A_p(1 < p < \infty)$, then $\mathcal{T}$ is bounded on $M_{p,\lambda}(w)$.

(b) If $\mathcal{T}$ is bounded from $L^1(w)$ to $L^{1,\infty}(w)$ with $w \in A_1$, then there exist constant $C > 0$ such that for all $\mu > 0$ and all $B$,

$$w(\{x \in B : \mathcal{T}f(x) > \mu\}) \leq C/\mu\|f\|_{M_{1,\lambda}(w)}w(B)^\lambda.$$

It is easy to check that the H-L maximal function $M(f)$ satisfies the hypotheses of Theorem 1.1.

We say $b$ is a $BMO$ function, which means $\|b\|_{BMO} = \|b^\sharp\|_{L^\infty} < \infty$, where $b^\sharp(x)$ is sharp maximal function

$$b^\sharp(x) = \sup_{B} \frac{1}{|B|} \int_{B} |f(y) - f_B| dy,$$

where the supreme is taken over all balls $B \subset \mathbb{R}^n$ and $f_B = \frac{1}{|B|} \int_{B} f(y)dy$. For $1 < p < \infty$, there is a close relation between $BMO$ and $A_p$ weights

$$BMO = \{\alpha \log w : w \in A_p, \alpha \geq 0\}.$$

Given a operator $N$ acting on functions and given a function $b$, the commutator $[b, N]$ is formally defined as

$$N_b f = [b, N] f = bN(f) - N(bf).$$
There is a great amount of works that deal with the topic of commutators of different operators with $BMO$ functions on Lebesgue spaces. The first results on this commutator were obtained by Coifman, Rochberg and Weiss [15] in their study of certain factorization theorems for generalized Hardy spaces. They show that $N_b f$ is bounded on $L^p(\mathbb{R}^n), 1 < p < \infty,$ if and only if $b \in BMO$ when $N$ is a classical singular integral operator with smooth kernel. For some classical weighted boundedness of $N_b f$ on $L^p(w)$ with $w \in A_p, 1 < p < \infty,$ see e.g. [6], [7]. It is well known that the commutators formed by $BMO$ functions and the fractional integral $I_\alpha$, the C-Z singular integral $T$ are all bounded on weighted $L^p$ spaces [44]. In [26], the authors also established the weighted boundedness for $T_b$ and $I_{\alpha,b}$ on weighted Morrey spaces.

In this paper, we extend the results of [26] and obtain

**Theorem 1.2.** Let $1 < p < \infty$, $w \in A_p$ and a sublinear operator $\mathcal{T}$ satisfies the conditions

\begin{equation}
|\mathcal{T} f(x)| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^n} dy, \quad x \notin \text{supp } f
\end{equation}

for any integral function $f$ with compact support. Then we have

(a) If $\mathcal{T}$ is bounded on $L^p(w)$, then $\mathcal{T}$ is bounded on $M_{p,\lambda}(w)$.

(b) If $\mathcal{T}_b$ is bounded $L^p(w)$ with $b \in BMO(\mathbb{R}^n)$, then $\mathcal{T}_b$ is bounded on $M_{p,\lambda}(w)$.

It is worth pointing out that (1.3), which implies the size conditions in Theorem 1.1, is satisfied by many operators in harmonic analysis, such as C-Z singular integral operator, the Carleson maximal operator, C. Fefferman’s singular multiplier operator, R. Fefferman’s singular integral operator and so on, see e.g. [40] and [45].

Besides the H-L maximal operators and C-Z singular integral operators, oscillatory integral operators have been an essential part of harmonic analysis; three chapters are devoted to them in the celebrated Stein’s book [46]. Many important operators in harmonic analysis are some versions of oscillatory integrals, such as the Fourier transform, the Bochner-Riesz means, the Radon transform [39] in CT technology and so on. For a more complete account on oscillatory integrals in classical harmonic analysis, we would like to refer the interested reader to [29], [30], [32] and references therein. Another early impetus for the study of oscillatory integrals came with their application to number theory [5]. In more recent times, the operators fashioned from oscillatory integrals, such as pseudo-differential operator in PDE become another motivation to study them. Based on the estimates of some kinds of oscillatory integrals, one can establish the well-posedness theory of a class of dispersive equations, for some of this works, we refer to [14], [23] and [24].

In 1987, Ricci and Stein [40] introduced one class of oscillatory integrals

\[ T_O f(x) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x,y)} K(x - y) f(y) dy, \]

which initially defined for smooth function $f$ with compact support. Here $P(x, y)$ is a real valued polynomial defined on $\mathbb{R}^n \times \mathbb{R}^n$, and $K \in C^1(\mathbb{R}^n \setminus \{0\})$ is a C-Z kernel. In [40], Ricci and Stein established the boundedness of $T_O$ on $L^p(\mathbb{R}^n)(1 < p < \infty)$. In 1992, Lu and Zhang [33] gave the boundedness of $T_O$ on $L^p(w)(1 < p < \infty)$ with
$w \in A_p$. For the case $p = 1$, Chanillo and Christ \cite{8} gave a weak $(1,1)$ type estimates while Sato obtained the weighted version of \cite{8} in \cite{12}. It is easy to see that $T_M$ satisfies Theorem 1.2.

The Bochner-Riesz mean operators of order $\delta > 0$ in $\mathbb{R}^n (n \geq 2)$ are defined initially for Schwartz functions in terms of Fourier transforms by

$$(T_R^\delta f)(\xi) = (1 - \frac{|\xi|^2}{R^2})^\delta \hat{f}(\xi),$$

where $\hat{f}$ denotes the Fourier transform of $f$. These operators were first introduced by Bochner \cite{4} in connection with summation of multiple Fourier series and played an important role in harmonic analysis. $T_R^\delta$ can be expressed as convolution operators $T_R^\delta f(x) = (f * \phi_x)(x)$, where $\phi(x) = [(1 + |x|^2)^\delta](x)$. It is well know that the kernel $\phi$ can be represented as \cite{31}:

$$\phi(x) = \pi^{-\delta} \Gamma(\delta + 1)|x|^{-(n/2 + \delta)} J_{n/2 + \delta}(2\pi|x|),$$

where $J_\mu(t)$ is the Bessel function. In \cite{43}, Shi and Sun obtained the weighted $L^p$ boundedness of $T_R^\delta$ while Vargas given the weighted weak $(1,1)$ type estimates in \cite{48}. By the well known boundedness criterion for the commutators of linear operators, which was obtained by Alvarez, Bagby, Kurtz and Pérez \cite{2}, we see that the commutator $T_{R,b}^\delta$ is also bounded on $L^p(w)$ with $w \in A_p$ for all $1 < p < \infty$. From the asymptotic properties of the Bessel function, we can deduce that when $\delta = (n - 1)/2$, the critical index, $|\phi(x)| \leq \frac{C}{(1 + |x|)^n}$, which implies that $T_R^\delta$ satisfies the condition of Theorem 1.2.

Given a positive real number $0 < a \neq 1$, the oscillating kernel $K_a$ is defined by \cite{10}:

$$K_a(x) = (1 + |x|)^{-1} e^{j|x|^a}.$$ The convolution operator $T_a = K_a * f$ and closely related weakly singular operators and multiplier operators have been studied by many authors, see e.g. \cite{9}, \cite{10} and \cite{21}. In \cite{10}, Chanillo, Kurtz and Sampson had given the weighted weak $(1,1)$ type estimates and weighted $L^p(1 < p < \infty)$ estimates for $T_a$. It is easy to see that $T_a$ satisfies Theorem 1.2.

**Theorem 1.3.** Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $1 < p \leq q \leq \infty$. Suppose that a sublinear operator $T_a$ satisfies the size conditions

$$|T_a f(x)| \leq C|x|^{-(n - \alpha)} \|f\|_{L^1(\mathbb{R}^n)} \quad (1.4)$$

when supp $f \subseteq A_k$ and $|x| \geq 2^{k+1}$ with $k \in Z$ and

$$|T_a f(x)| \leq C2^{-k(n - \alpha)} \|f\|_{L^1(\mathbb{R}^n)} \quad (1.5)$$

when supp $f \subseteq A_k$ and $|x| \leq 2^{k-1}$ with $k \in Z$. Then we have

(a) If $T_a$ maps $L^p(w^p)$ into $L^q(w^q)$ with $w \in A(p, q)$, then $T_a$ is bounded from $M_{p, \lambda}(w^p, w^q)$ to $M_{q, \lambda/p}(w^q)$.

(b) If $T_a$ is bounded from $L^1(w)$ to $L^{q, \infty}(w^q)$ with $w \in A(1, q)$, then there exist constant $C > 0$ such that for all $\mu > 0$ and all ball $B$,

$$w^q(\{x \in B : T_a f(x) > \mu\}) \leq C/\mu^q \|f\|_{M_{1,\lambda}(w^q)}^q w^q(B)^{\delta \lambda}.$$
The fractional maximal operator $M_\alpha$ is defined by

$$M_\alpha f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\alpha/n}} \int_B |f(y)|dy, \quad 0 < \alpha < n.$$ 

$M_\alpha$ satisfies the hypotheses of Theorem 1.3 since the pointwise inequality $M_\alpha f(x) \leq I_\alpha(|f|)(x)$ for $0 < \alpha < n$.

**Theorem 1.4.** Let $p, q, \alpha, w$ be as the same as that of Theorem 1.3 and a sublinear operator $T_\alpha$ satisfies the conditions

$$|T_\alpha f(x)| \leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^{n-\alpha}}dy, \quad x / \in \text{supp} f$$

for any integral function $f$ with compact support.

(a) If $T_\alpha$ maps $L^p(w^p)$ into $L^q(w^q)$, then $T_\alpha$ is bounded from $M_{p,\lambda}(w^p, w^q)$ to $M_{q,\lambda/p}(w^q)$.

(b) If $T_{\alpha,b}$ maps $L^p(w^p)$ into $L^q(w^q)$ with $b \in \text{BMO}(\mathbb{R}^n)$, then $T_{\alpha,b}$ is bounded from $M_{p,\lambda}(w^p, w^q)$ to $M_{q,\lambda/p}(w^q)$.

We remarks that fractional integral $I_\alpha$ and oscillatory fractional integral of Ricci and Stein’s [40] are all examples of operators which satisfies (1.6). For the corresponding boundedness in unweighted cases of the sublinear operators on Herz space, we refer to [20] and [27].

We end this section with the outline of this paper. In Section 2 we give the proofs of Theorem 1.1-Theorem 1.4. Section 3 contains some applications of Theorem 1.1-Theorem 1.4. Throughout this paper, the letter $C$ is used for various constants, and may change from one occurrence to another. All balls are assumed to have their sides parallel to the coordinate axes. $B = B(x_0, r)$ denotes the ball centered at $x_0$ and with radius $r$ and $\lambda B = B(x_0, \lambda r)$.

## 2. Proofs of the main results

Our methods are adopted from [17] in the case of the Lebesgue measure and from [26] dealing with the classical operators. Before the proof of Theorem 1.1, we give some properties of $A_p$ weights.

**Lemma 2.1.** [18] Let $1 \leq p < \infty$, and $w \in A_p$. Then the following statements are true

(a) There exists a constant $C$ such that

$$w(2B) \leq Cw(B).$$

When $w$ satisfies this condition, we say $w$ satisfies doubling condition.

(b) There exists a constant $C > 1$ such that

$$w(2B) \geq Cw(B).$$

When $w$ satisfies this condition, we say $w$ satisfies reverse doubling condition.

(c) There exist two constant $C$ and $r > 1$ such that the following reverse Hölder inequality holds for every ball $B \subset \mathbb{R}^n$

$$\left(\frac{1}{|B|} \int_B w(x)^r dx \right)^{1/r} \leq C \left(\frac{1}{|B|} \int_B w(x) dx \right).$$
(d) For all $\lambda > 1$, we have
\begin{equation}
(2.4) \quad w(\lambda B) \leq C\lambda^{np}w(B).
\end{equation}

(e) There exist two constant $C$ and $\delta > 0$ such that for any measurable set $Q \subset B$
\begin{equation}
(2.5) \quad \frac{w(Q)}{w(B)} \leq C\left(\frac{|Q|}{|B|}\right)^\delta.
\end{equation}

If $w$ satisfies (2.5), we say $w \in A_\infty$.

2.1. Proof of Theorem 1.1. Let $1 < p < \infty$, $w \in A_p$ and $0 < \lambda < 1$. We first give the proof of (a), which suffices to show that
\begin{equation}
(2.6) \quad \frac{1}{w(B)^\lambda} \int_B |Tf(x)|^pw(x)dx \leq C\|f\|_{M_{p,\lambda}(w)}^p.
\end{equation}

For a fixed ball $B = B(x_0, r)$, there is no loss of generality in assuming $r = 1$. We decompose $f = f\chi_{2B} + f\chi_{(2B)^c} := f_1 + f_2$. Since $T$ is a sublinear operator, so we get
\begin{equation}
\frac{1}{w(B)^\lambda} \int_B |Tf(x)|^pw(x)dx \leq \frac{1}{w(B)^\lambda} \int_B (|Tf_1(x)|^p + |Tf_2(x)|^p)w(x)dx := I + II.
\end{equation}

Using the fact that $T$ is bounded on $L^p(w)$, we can easily get
\begin{equation}
(2.7) \quad I \leq \frac{1}{w(B)^\lambda} \int_{\mathbb{R}^n} |Tf_1(x)|^pw(x)dx \leq C\frac{1}{w(B)^\lambda} \int_{2B} |f(x)|^pw(x)dx \leq C\|f\|_{M_{p,\lambda}(w)}^p.
\end{equation}

We are now in a position to estimate the term $II$. We conclude from $w \in A_p$ that
\begin{align*}
\int_{(2B)^c} |f(y)|dy &\leq C\sum_{k=1}^\infty \int_{2^{k+1}B/2^kB} |f(y)|dy \\
&\leq C\sum_{k=1}^\infty \left( \int_{2^{k+1}B} |f(y)|^p w(y)dy \right)^{1/p} \left( \frac{1}{w(B)^{\lambda/p'}} \int_{2^{k+1}B} w(y)^{-p'/p}dy \right)^{1/p'} \\
&\leq C\|f\|_{M_{p,\lambda}(w)} \sum_{k=1}^\infty \frac{|2^{k+1}B|}{w(2^{k+1}B)^{1-\lambda/p'}}.
\end{align*}

By (1.2), we have
\begin{align*}
II &\leq \frac{C}{w(B)^\lambda} 2^{-knp} \int_B \|f_2\|^p_{L^1(\mathbb{R}^n)}w(x)dx \\
&\leq \frac{C}{w(B)^\lambda-1} 2^{-knp} \left( \int_{(2B)^c} |f(y)|dy \right)^p \\
&\leq C\|f\|^p_{M_{p,\lambda}(w)} \left( \sum_{k=1}^\infty \frac{w(B)^{(1-\lambda)/p}}{w(2^{k+1}B)^{(1-\lambda)/p}} \right)^p \\
&\leq C\|f\|^p_{M_{p,\lambda}(w)}.
\end{align*}

Here we use (2.1) in the last inequality above. Combing (2.7) with (2.8), we get (2.6), which yields the proof of (a).
Now, we have a position to give the proof of (b), which is similar to that of (a). We want to set up the following inequality:

$$\sup_{\mu > 0} \frac{\mu}{w(B)} \omega \{x \in B : |Tf(x)| > \mu\} \leq C \|f\|_{M_{1,\lambda}(w)}^p.$$ 

Decompose $f = f\chi_{2B} + f\chi_{(2B)^c} := f_1 + f_2$ with $B$ as that of (a). For any given $\mu > 0$, we write

$$w \{x \in B : |Tf(x)| > \lambda\} \leq w \{x \in B : |Tf_1(x)| > \mu/2\} + w \{x \in B : |Tf_2(x)| > \mu/2\} := J + JJ.$$ 

An application of (2.1) and the weighted weak (1,1) type estimates for $T$ yield that

$$J \leq w \{x \in \mathbb{R}^n : |Tf_1(x)| > \mu/2\} \leq C/\mu \|f\|_{M_{1,\lambda}(w)} w(B)^\lambda.$$ 

Next we turn to deal with the term $JJ$. An elementary estimate shows

$$JJ \leq C \int_{\{x \in B : |Tf(x)| > \frac{\mu}{2}\}} |Tf_2(x)| w(x) dx.$$ 

Applying (1.2), we conclude that

$$|Tf_2(x)| \leq C 2^{-kn} \int_{(2B)^c} |f(y)| dy \leq C \sum_{k=1}^{\infty} 2^{-kn} \int_{2^{k+1}B} |f(y)| dy.$$ 

Hölder’s inequality and the $A_1$ condition imply that

$$JJ \leq C \sum_{k=1}^{\infty} 2^{-kn} \int_{2^{k+1}B} |f(y)| w(y) dy \leq C \|f\|_{M_{1,k}(w)} \sum_{k=1}^{\infty} 2^{kn(\lambda-1)} w(B)^\lambda \leq C \|f\|_{M_{1,k}(w)} w(B)^\lambda.$$ 

Then, we have completed the proof of (b). \qed

2.2. Proof of Theorem 1.2

The proof of Theorem 1.2 depend heavily on the following remarks about $BMO$ functions.

**Lemma 2.2.** [47] Let $1 \leq p < \infty$, $b \in BMO(\mathbb{R}^n)$. Then for any ball $B \subset \mathbb{R}^n$, the following statements are true

(a) There exist constants $C_1$, $C_2$ such that for all $\alpha > 0$

$$|\{x \in B : |b(x) - b_B| > \alpha\}| \leq C_1 |B| e^{-C_2 \alpha/\|b\|_{BMO(\mathbb{R}^n)}}.$$ 

The inequality (2.9) is also called John-Nirenberg inequality.

(b)

$$|b_{2^k B} - b_B| \leq 2^n \lambda \|b\|_{BMO(\mathbb{R}^n)}.$$
Lemma 2.3. Let $w \in A_\infty$. Then the following statements are equivalent:

(a)  
$$
\|b\|_{BMO(\mathbb{R}^n)} \sim \sup_B \left( \frac{1}{|B|} \int_B |b(x) - b_B|^p dx \right)^{\frac{1}{p}}.
$$

(b)  
$$
\|b\|_{BMO(\mathbb{R}^n)} \sim \sup_B \inf_{a \in \mathbb{R}} \frac{1}{|B|} \int_B |b(x) - a|^p dx.
$$

(c)  
$$
\|b\|_{BMO(w)} = \sup_B \frac{1}{w(B)} \int_B |b(x) - b_B| w(x) dx.
$$

where $BMO(w) = \{ b : \|b\|_{BMO(w)} < \infty \}$ and $b_B = \frac{1}{w(B)} \int_B b(y) w(y) dy$.

Lemma 2.4. Let $b \in BMO(\mathbb{R}^n)$, $B = B(x_0, r)$, $0 < \lambda < 1$ and $1 < p < \infty$. Then the inequality

$$
\left( \int_{|x_0 - y| > 2r} \frac{|f(y)|}{|x_0 - y|^n} |b_{B,w} - b(y)| dy \right)^p w(B)^{1-\lambda} \leq C \|f\|_{M_{p,\lambda}(w)}^p \|b\|_{BMO(\mathbb{R}^n)}^p.
$$

holds for every $y \in (2B)^c$, where $(2B)^c = \mathbb{R}^n/2B$.

Proof. Using Hölder’s inequality to the left-hand-side of (2.14), we have

$$
\left( \int_{|x_0 - y| > 2r} \frac{|f(y)|}{|x_0 - y|^n} |b_{B,w} - b(y)| dy \right)^p w(B)^{1-\lambda}
\leq \left( \sum_{j=1}^\infty \int_{2^{j+1}B < |x_0 - y| < 2^j B} \frac{|f(y)|}{|x_0 - y|^n} |b_{B,w} - b(y)| dy \right)^p w(B)^{1-\lambda}
\leq \left( \sum_{j=1}^\infty \frac{1}{|2^j B|} \int_{2^{j+1}B} |f(y)||b_{B,w} - b(y)| dy \right)^p w(B)^{1-\lambda}
\leq C \left[ \sum_{j=1}^\infty \frac{1}{|2^j B|} \left( \int_{2^{j+1}B} |f(y)|^p w(y) dy \right)^{\frac{1}{p}} \left( \int_{2^{j+1}B} |b_{B,w} - b(y)|^{p'} w(y)^{1-p'} dy \right)^{\frac{1}{p'}} \right]^p w(B)^{1-\lambda}
\leq C \|f\|_{M_{p,\lambda}(w)}^p \left[ \sum_{j=1}^\infty \frac{w(2^{j+1}B)^{\frac{1}{p}}}{|2^j B|} \left( \int_{2^{j+1}B} |b_{B,w} - b(y)|^{p'} w(y)^{1-p'} dy \right)^{\frac{1}{p'}} \right]^p w(B)^{1-\lambda}.
$$

For the simplicity of analysis, we denote $A$ as

$$
\left( \int_{2^{j+1}B} |b_{B,w} - b(y)|^{p'} w(y)^{1-p'} dy \right)^{\frac{1}{p'}}.
$$
By an elementary estimate, we have

\[
A \leq \left( \int_{2^{j+1}B} |b_{2^{j+1}B,w^1-w'} - b(y)| + |b_{2^{j+1}B,w^1-w'} - b_{B,w}||w(y)^{1-p'} dy \right)^{\frac{1}{p'}} \\
\leq \left( \int_{2^{j+1}B} \frac{|b_{2^{j+1}B,w^1-w'} - b(\cdot)| + |b_{2^{j+1}B,w^1-w'} - b_{B,w}|}{w(\cdot)} \right)_{L^p(w)}^{\frac{1}{p'}} \\
\leq \left( \int_{2^{j+1}B} |b_{2^{j+1}B,w^1-w'} - b(y)|w(y)^{1-p'} dy \right)^{\frac{1}{p'}} + |b_{2^{j+1}B,w^1-w'} - b_{B,w}|w^{1-p'}(2^{j+1}B)^{\frac{1}{p'}} \\
=: A_1 + A_2.
\]

For the term \(A_1\), Lemma 2.3 implies

\[
(2.15) \quad A_1 \leq C \|b\|_{BMO(w^1-w')}w^{1-p'}(2^{j+1}B)^{\frac{1}{p'}} \leq C w^{1-p'}(2^{j+1}B)^{\frac{1}{p'}}.
\]

To deal with \(A_2\), by (2.10), we have

\[
|b_{2^{j+1}B,w^1-w'} - b_{B,w}| \leq |b_{2^{j+1}B,w^1-w'} - b_{2^{j+1}B}| + |b_{2^{j+1}B} - b_B| + |b_B - b_{B,w}| \\
\leq \frac{1}{w^{1-p'}(2^{j+1}B)} \int_{2^{j+1}B} |b(y) - b_{2^{j+1}B}|w(y)^{1-p'} dy + 2^n(j + 1)\|b\|_{BMO(\mathbb{R}^n)} \\
+ \frac{1}{w(B)} \int_B |b(y) - b_B|w(y)dy \\
=: A_{21} + A_{22} + A_{23}.
\]

Combining (2.3) with (2.9), we have

\[
A_{23} = \frac{1}{w(B)} \int_0^\infty w(\{x \in B : |b(y) - b_{B}| > \alpha\})d\alpha \\
\leq C \int_0^\infty e^{-C_2\alpha\delta/\|b\|_{BMO(\mathbb{R}^n)}}d\alpha \\
\leq C.
\]

In the same manner we can see that

\[
A_{21} \leq C.
\]

It follows immediately that

\[
(2.16) \quad A_2 \leq C(2^n(j + 1) + 2)w^{1-p'}(2^{j+1}B)^{\frac{1}{p'}}.
\]

As a by-product of (2.15) and (2.16), we have

\[
A \leq C(j + 1)w^{1-p'}(2^{j+1}B)^{\frac{1}{p'}}.
\]
Then, applying (2.2), the proof of (2.14) based on the following observation
\[
\left[ \sum_{j=1}^{\infty} \frac{\mu(2^{j+1}B)}{|2^jB|^p} \left( \int_{2^{j+1}B} |b(y) - b_{B,w}|^{p'}w(y)^{1-p'}dy \right)^\frac{p}{p'} \right]^{\frac{1}{p}} w(B)^{1-k} \leq C \left[ \sum_{j=1}^{\infty} \frac{w(B)^{\frac{1-k}{p}(j+1)}}{w(2^{j+1}B)^{\frac{1-k}{p}}} \right]^p = C.
\]

Now, we come back to the proof of Theorem 1.2. (a) is trivial since (1.3) satisfies Theorem 1.1. We only need to give the proof of (b). The task is now to find a constant $C$ such that for fixed ball $B = B(x_0, 1)$, we can obtain
\[
\left( \frac{1}{w(B)^{\lambda}} \right)^p \int_B |\mathcal{T}_b f(x)|^p w(x)dx \leq C\|f\|^p_{M_{p,\lambda}(w)}.
\]

(2.17) We decompose $f = f\chi_{2B} + f\chi_{(2B)^c} := f_1 + f_2$, and consider the corresponding splitting
\[
\int_B |\mathcal{T}_b f(x)|^p w(x)dx \leq C \left( \int_B |\mathcal{T}_b f_1(x)|^p w(x)dx + \int_B |\mathcal{T}_b f_2(x)|^p w(x)dx \right) =: K + KK.
\]

It follows from the $L^p(w)$ boundedness of $\mathcal{T}_b$ and $w \in A_p$ that
\[
K \leq C \int_{2B} |f(x)|^p w(x)dx \leq C\|f\|^p_{M_{p,\lambda}(w)} w(B)^{\lambda}.
\]

Then a further use of (1.3) derives that
\[
|\mathcal{T}_b f_2(x)|^p \leq C \left( \int_{\mathbb{R}^n} \frac{|f_2(y)| |b(x) - b(y)|}{|x - y|^n} dy \right)^p \leq C \left( \int_{|x_0 - y| > 2} \frac{|f(y)|}{|x_0 - y|^n} \left( |b(x) - b_{B,w}| + |b_{B,w} - b(y)| \right) dy \right)^p.
\]

where $b_{B,w} = \frac{1}{w(B)} \int_B b(x)w(x)dx$. Then, we have
\[
KK \leq C \left( \int_{|x_0 - y| > 2} \frac{|f(y)|}{|x_0 - y|^n} dy \right)^p \int_B |b(x) - b_{B,w}|^p w(x)dx + C \left( \int_{|x_0 - y| > 2} \frac{|f(y)|}{|x_0 - y|^n} |b(y) - b_{B,w}| dy \right)^p w(B)
\]
\[
:= KK_1 + KK_2.
\]

A further use of Lemma 2.3, we get
\[
KK_2 \leq C\|f\|^p_{M_{p,\lambda}(w)} w(B)^{\lambda}.
\]

To get the desired estimate, we are led to estimate the term $KK_1$. This estimate will be done via (2.1), (2.3) and Lemma 2.3. In fact,
\[ KK_1 = \left( \sum_{j=1}^{\infty} \int_{2^j < |x-y| < 2^{j+1}} \frac{|f(y)|}{|x_0 - y|^\alpha} \, dy \right)^p \int_B |b(x) - b_{B,w}|^pw(x) \, dx \]
\[ \leq \left( \sum_{j=1}^{\infty} \frac{1}{|2^j B|} \int_{2^j B} |f(y)| \, dy \right)^p \int_B |b(x) - b_{B,w}|^pw(x) \, dx \]
\[ \leq C \sum_{j=1}^{\infty} \frac{1}{|2^j B|} \left( \frac{1}{w(2^{j+1} B)^\lambda} \int_{2^{j+1} B} |f(y)|^p w(y) \, dy \right)^{1/p} \]
\[ \times w(2^{j+1} B)^{\lambda/p} \left( \int_{2^{j+1} B} (w(y)^{-1/p-1}) \, dy \right)^{\frac{p-1}{p}} \int_B |b(x) - b_{B,w}|^pw(x) \, dx \]
\[ \leq C \| f \|_{M_{p,\lambda}^{(w)}} \left( \sum_{j=1}^{\infty} \frac{2^{j+1} B}{|2^j B|} \left( \frac{1}{|2^{j+1} B|} \int_{2^{j+1} B} w(y) \, dy \right)^{-1/p} w(2^{j+1} B)^{\lambda/p} \right)^p \]
\[ \times \int_B |b(x) - b_{B,w}|^pw(x) \, dx \]
\[ \leq C \| f \|^p_{M_{p,\lambda}^{(w)}} \| b \|^p_{BMO(\mathbb{R}^n)} \sum_{j=1}^{\infty} \left( \frac{w(B)^{(1-\lambda)/p}}{w(2^{j+1} B)^{(1-k)/p}} \right)^p w(B)^\lambda \]
\[ \leq C \| f \|^p_{M_{p,\lambda}^{(w)}} w(B)^\lambda \]

Hence
\[ (2.19) \quad KK \leq C \| f \|^p_{M_{p,\lambda}^{(w)}} w(B)^\lambda. \]

Combing (2.18), (2.19), we obtain (2.17), which is the desired conclusion. \qed

2.3. Proof of Theorem 1.3. We can use the similar argument as the proof of Theorem 1.1. For the proof of (a), it suffices to show that

\[ (2.20) \quad \frac{1}{w^q(B)^{\lambda/p}} \int_B |\mathcal{T}_\alpha f(x)|^q w(x)^q \, dx \leq C \| f \|^q_{M_{p,\lambda}(w^p, w^q)}. \]

For a fixed ball \( B = B(x_0, 1) \), we decompose \( f = f_1 + f_2 \). Since \( \mathcal{T}_\alpha \) is a sublinear operator, so we get
\[ \frac{1}{w^q(B)^{\lambda/p}} \int_B |\mathcal{T}_\alpha f(x)|^q w(x)^q \, dx \leq \frac{1}{w^q(B)^{\lambda/p}} \int_B (|\mathcal{T}_\alpha f_1(x)|^q + |\mathcal{T}_\alpha f_2(x)|^q) w^q(x) \, dx := L + LL. \]

To estimate the term \( L \), using the fact that \( \mathcal{T}_\alpha \) is bounded from \( L^p(w^p) \) to \( L^q(w^q) \) with \( w \in A_{(p, q)} \), we can get
\[ \int_B |\mathcal{T}_\alpha f_1(x)|^q w^q(x) \, dx \leq C \| f \|^q_{M_{p,\lambda}(w^p, w^q)} w^q(B)^{\lambda/p}, \]
which implies that
\[ L \leq C \| f \|^q_{M_{p,\lambda}(w^p, w^q)}. \]
For the term $LL$. By the similar argument as that of Theorem 1.1, we obtain

$$LL \leq C \sum_k \left(2^{-k(n-\alpha)} \int_{A_k} |f(y)|dy \right)^q w^q(B)^{1-q\lambda/p}$$

$$\leq C \sum_k \left(2^{-k(n-\alpha)} \|f\|_{M_{p,\lambda}(w^p,w^q)} |2k+1B|^{1-\alpha/n} \frac{1}{w^q(2k+1B)^{1/q-\lambda/p}} \right)^q w^q(B)^{1-q\lambda/p}$$

$$\leq C \left(\sum_{k=1}^{\infty} w^q(B)^{(1/q-\lambda/p)} \right)^q$$

$$\leq C \left(\sum_{k=1}^{\infty} w^q(B)^{1/q-\lambda/p} \right)^q.$$

We have completed the proof of (a).

We shall omit the proof of (b) since we can prove by using $A_{(1,q)}$ condition and the weak type estimates of $T_\alpha$. \hfill \Box

2.4. Proof of Theorem 1.4. The proof of Theorem 1.4 is similar to that of Theorem 1.2, except using $w \in A_{p,q}$.

3. Applications

In this section, we shall give some applications of our main results to nondivergence elliptic equations. Dirichlet problem on the second order elliptic equation in nondivergence form is

$$\begin{align*}
\left\{ \begin{array}{c}
Lu = \sum_{i,j=1}^{n} a_{ij}(x) u_{i} u_{j} = f \quad \text{a.e in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega.
\end{array} \right.
\end{align*}$$

(3.1)

Here $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $\Omega$ is a bounded domain of $\mathbb{R}^n$. The coefficients $(a_{ij})_{i,j=1}^{n}$ of $L$ are symmetric and uniformly elliptic, i.e., for some $\nu \geq 1$ and every $\xi \in \mathbb{R}^n$, $a_{ij}(x) = a_{ji}(x)$ and $\nu^{-1} |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \nu |\xi|^2$ with a.e. $x \in \Omega$. In [17], Fan, Lu and Yang investigate the regularity in $M_{p,\lambda}(\Omega)$ of the strong solution to (3.1) with $a_{ij} \in VMO(\Omega)$, the space of the functions of vanishing mean oscillation introduced by Sarason in [41]. The main method of [17] is based on integral representation formulas established in [12] and [13] for the second derivatives of the solution $u$ to (3.1), and on the theories of singular integrals and sublinear commutators in Morrey spaces.

By extending some theorems of [17] to weighted versions, we can establish regularity in weighted Morrey spaces of strong solutions to nondivergence elliptic equations with $VMO$ coefficients. Theorem 1.2 is just the weighted version of the important theorem-Theorem 2.1 in [17]. For the complement of our paper, we take another important theorem-Theorem 2.3 of [17] to state this. All other proofs of the corresponding theorems are straightforward.

Let $\mathbb{R}^n_+ = \{ x = (x', x_n) : x' = (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0 \}$, $L^p_+(w) = L^p(w, \mathbb{R}^n_+)$ and $M^p_{\alpha} = M_{p,\lambda}(w, \mathbb{R}^n_+)$. To establish the boundary estimates of the solutions to (3.1), we need the following general theorem for sublinear operators.
Theorem 3.1. Let $1 < p < \infty$, $0 < \lambda < 1$, $w \in A_p$, $\tilde{x} = (x', -x_n)$ for $x = (x', x_n) \in \mathbb{R}^n_+$. If a sublinear operator $\mathfrak{T}$ is bounded on $L^p_+(w)$ for any $f \in L^1_+(w)$ with compact support and satisfies

\begin{equation}
|\mathfrak{T}f(x)| \leq C \int_{\mathbb{R}^n_+} \frac{|f(y)|}{|\tilde{x} - y|^{n}} dy,
\end{equation}

then $\mathfrak{T}$ is bounded on $M^{p,\lambda}_+(w)$.

Proof. Let $z \in \mathbb{R}^n_+$ and $\delta > 0$. Set $B^+_\delta(z) = B_\delta(z) \cap \mathbb{R}^n_+$, where $B_\delta(z) = \{y \in \mathbb{R}^n : |z - y| < \delta\}$. We consider two cases.

Case 1. $0 \leq z_n < 2\delta$. In this case, we write

$$f(y) = f(y)\chi_{B^+_\delta(z)}(y) + \sum_{i=4}^{\infty} f(y)\chi_{B^+_{2^i+1,\delta}(z)/B^+_{2^i,\delta}(z)}(y) \equiv \sum_{i=3}^{\infty} f_i(y).$$

Therefore, by the $L^p_+(w)$ boundedness of $\mathfrak{T}$ and (3.2), we obtain

$$\frac{1}{w(B^+_\delta)^{\lambda/p}} \left( \int_{B^+_\delta} |\mathfrak{T}f(x)|^p w(x) dx \right)^{1/p} \leq \frac{1}{w(B^+_\delta)^{\lambda/p}} \sum_{i=3}^{\infty} \left( \int_{B^+_\delta} |\mathfrak{T}f_i(x)|^p w(x) dx \right)^{1/p} \leq C \frac{\|f\|_{M^{p,\lambda}_+(w)}}{w(B^+_\delta)^{\lambda/p}} \sum_{i=4}^{\infty} \left( \int_{B^+_\delta} \left( \int_{B^+_{2^i+1,\delta}(z)/B^+_{2^i,\delta}(z)} |f(y)| |\tilde{x} - y|^{n} dy \right)^p w(x) dx \right)^{1/p} \leq C \|f\|_{M^{p,\lambda}_+(w)} + C \sum_{i=4}^{\infty} \frac{1}{(2^i\delta)^n} \left( \int_{B^+_\delta} |f(y)| dy \right) w(B^+_\delta)^{1-\lambda/p} \leq C \|f\|_{M^{p,\lambda}_+(w)} \left( 1 + \sum_{i=4}^{\infty} \frac{w(B^+_\delta)^{1-\lambda/p}}{w(B^+_{2^i+1,\delta})^{1-\lambda/p}} \right) \leq C \|f\|_{M^{p,\lambda}_+(w)}.$$

In the last inequality, we use Lemma 2.1 in Section 2.

Case 2. There exists $i \in \mathbb{N}$ such that $2^i\delta \leq z_n < 2^{i+1}\delta$. In this case, we write

$$f(y) = f(y)\chi_{B^+_{2^i+1,\delta}(z)}(y) + \sum_{j=1}^{\infty} f(y)\chi_{B^+_{2^i+j+\delta}(z)}(y) \equiv \sum_{j=0}^{\infty} f_j(y).$$
By (3.2) and Lemma 2.1, we have

\[
\frac{1}{w(B^+_\delta)^{\lambda/p}} \left( \int_{B^+_\delta} |\mathcal{T} f(x)|^p w(x) \, dx \right)^{1/p} \\
\leq \frac{C}{w(B^+_\delta)^{\lambda/p}} \left( \int_{B^+_\delta} \left( \int_{B^+_{2i+4\delta}(z)} \frac{|f(y)|}{|x-y|^n} \, dy \right)^p w(x) \, dx \right)^{1/p} \\
+ \frac{C}{w(B^+_\delta)^{\lambda/p}} \sum_{j=1}^\infty \left( \int_{B^+_\delta} \left( \int_{B^+_{2i+j+4\delta}(z)/B^+_{2i+j+3\delta}(z)} \frac{|f(y)|}{|x-y|^n} \, dy \right)^p w(x) \, dx \right)^{1/p} \\
\leq \frac{C}{w(B^+_\delta)^{\lambda/p}(2\delta)^n} \left( \int_{B^+_\delta} \left( \int_{B^+_{2i+j+4\delta}(z)} |f(y)| \, dy \right)^p w(x) \, dx \right)^{1/p} \\
+ \frac{C}{w(B^+_\delta)^{\lambda/p}(2^{i+j}\delta)^n} \sum_{j=1}^\infty \left( \int_{B^+_\delta} \left( \int_{B^+_{2i+j+4\delta}(z)} |f(y)| \, dy \right)^p w(x) \, dx \right)^{1/p} \\
\leq C \|f\|_{M^+_{p,\lambda}(w)} \left( \frac{w(B^+_\delta)^{1-\lambda/p}}{w(B^+_{2i+j+4\delta})^{1-\lambda/p}} + \sum_{j=1}^\infty \frac{w(B^+_\delta)^{1-\lambda/p}}{w(B^+_{2i+j+3\delta})^{1-\lambda/p}} \right) \\
\leq C \|f\|_{M^+_{p,\lambda}(w)}.
\]

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