Steady-state flow of a power-law fluid between two coaxial cylinders taking into account the temperature dependence of the rheological parameters and the viscous dissipation

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Abstract. The non-isothermal power-law fluid flow between two coaxial cylinders under the pressure drop providing a specified flow rate is studied with allowance for the viscous dissipation of the mechanical energy and dependence of the rheological parameters on temperature. The flow regime is assumed to be laminar. Mathematical formulation of the problem includes the dimensionless equations of motion and energy. The no-slip conditions are used on the channel walls while the temperature and the heat flux are assigned on the outer and inner walls, respectively. The stated problem is solved numerically with the use of the finite-difference method to discretize the equations. A tridiagonal matrix algorithm is applied to solve the obtained system of algebraic equations. The parametric investigation of the kinematic, dynamical, and thermo-physical characteristics of the flow is carried out.

1. Introduction
The flow between coaxial cylinders is encountered in various technological equipments, for example, in the heat exchangers, liquid-cooled reactors, gas turbines, extruders, and bore wells. Non-isothermality of the process could be associated with a viscous dissipation of the mechanical energy and heat exchange boundary conditions.

Numerous of works are focused on the investigation of the flow between coaxial cylinders. However, these studies are limited to the consideration of the Newtonian fluid behavior [1,2], or they leave the viscous dissipation out of account [3]. The research data on the power-law and viscoplastic fluid flow with allowance for the dissipation are presented in [4-6].

The aim of this work is to study the influence of the temperature dependence of the rheological parameters on the power-law fluid flow pattern, which is formed when the fluid flows between two cylinders, taking into account the viscous dissipation.

2. Mathematical formulation of the problem
A laminar steady-state non-Newtonian fluid flow between two coaxial cylinders under the pressure drop providing the assigned flow rate is considered with allowance for the viscous dissipation and dependence of the rheological parameters on temperature. Mathematical formulation of the problem includes the motion and energy equations written using dimensionless variables as follows:

\[ \frac{1}{r} \frac{d}{dr} \left( r \eta \frac{du}{dr} \right) = \delta \rho \frac{1}{r} \frac{d}{dr} \left( r \frac{d\theta}{dr} \right) + \eta \left( \frac{du}{dr} \right)^2 = 0. \]
The rheological behavior of the medium is described by the Ostwald de Waele power law with a temperature dependence of the consistency defined by exponential law. The apparent viscosity $\eta$ corresponding to the rheological model is determined by formula

$$\eta = e^{c_0\theta} \frac{du}{dr} r^{n-1}.$$  \hspace{1cm} (2)

Here $r$ is the radial coordinate, $u$ is the velocity, $\delta p$ is the pressure drop per unit length, $\theta$ is the temperature, and $n$ is the power-law index. A typical length scale and velocity scale are the outer radius of the channel $R_0$ and the average velocity $U_0$, respectively. The dimensionless temperature is given as

$$\theta = \frac{T - T_w}{\eta_0 U_0^2},$$  \hspace{1cm} (3)

where $\lambda$ is the thermal conductivity, $T$ and $T_w$ are the dimensional temperatures of the fluid in the flow and on the outer wall, respectively, and $\eta_0$ is the typical scale for viscosity. The no-slip conditions are used on the channel walls while the temperature is assigned on the outer wall, and the heat flux on the inner wall is

$$r = \alpha : u = 0, \frac{d\theta}{dr} = \frac{1}{2Br(1-\alpha)};$$

$$r = 1 : u = 0, \theta = 0.$$  \hspace{1cm} (4)

The dimensionless coordinate of the inner wall is determined by the ratio $\alpha = R_i / R_0$, where $R_i$ is the dimensional radius of the inner channel. Formulation of the problem includes two dimensionless criteria

$$C = \frac{Br \eta_0 U_0^2}{\lambda}, \quad Br = \frac{\eta_0 U_0^2}{q_w D}.$$  \hspace{1cm} (5)

Here $Br$ is the Brinkman number characterizing the ratio of the energy transferred from the heated wall to the fluid and the energy losses due to the viscous friction, $\beta$ is the rheological parameter, $q_w$ is the dimensional heat flux through the inner wall, $\eta_0 = k_0''(U_0 / R_i)^{n-1}$, $k_0''$ is the fluid consistency at a temperature of the inner wall, and $D = 2(R_0 - R_i)$. The value of $\delta p$ is chosen so as to provide the fluid volumetric flow rate per unit area equal to unity

$$2 \int_a^1 urdr = 1 - \alpha^2.$$  \hspace{1cm} (6)

The Nusselt number is given as

$$Nu = \frac{Dk}{\lambda} = \frac{2(1-\alpha)\frac{d\theta}{dr}}{(\theta_w - \theta_0)},$$  \hspace{1cm} (7)

where $k$ is the heat-transfer coefficient of the wall, $\theta_0 = \frac{2}{1-\alpha^2} \int_a^1 \theta urdr$ is the dimensionless bulk temperature. Lower index $w$ corresponds to the outer or inner wall of the channel.
3. Computational method
The formulated problem is solved numerically using the finite-difference method described in [7]. To test the approximation convergence of the computational algorithm, a set of calculations was carried out. The following flow characteristics are examined: the values of Nusselt number on the walls, the maximum velocity, and the temperature of the inner wall. The data presented in Table 1 demonstrate a convergence of the results when reducing the grid step $h$.

| $h$     | $\text{Nu}_i$ | $\text{Nu}_o$ | $U_{\text{max}}$ | $T_0$  |
|---------|---------------|---------------|------------------|--------|
| 1/100   | 12.55611      | 1.876144      | 5.265697         | 13.97200 |
| 1/200   | 12.57716      | 1.875215      | 5.263897         | 13.97183 |
| 1/400   | 12.58244      | 1.874982      | 5.263447         | 13.97179 |
| 1/800   | 12.58376      | 1.874924      | 5.263334         | 13.97178 |
| 1/1600  | 12.58409      | 1.87491       | 5.263305         | 13.97177 |

The analytical solution to the problem of a Newtonian fluid flow between coaxial cylinders is shown in [2], where the authors leave the dependence of the consistency on temperature out of account. A comparison of the analytical dependency $Nu(\alpha)$ from [2] (the dotted line) with calculated results (the solid line) is presented in figure 1.

![Figure 1. Nusselt number as a function of $\alpha$ (C=0, n=1, and Br=0.2)](image)

4. Calculation results
Let us consider a non-isothermal fluid flow when the dependence of the rheological parameters on temperature is left out of account (C=0). In this case, the solution to the hydrodynamic problem is independent of the temperature distribution. Figure 2 illustrates the flow characteristics at various power-law indexes $n$. For pseudoplastic fluid the values of the dissipative function $\Phi$ are less than that for Newtonian and dilatant fluids, and, as a consequence, the temperature of the pseudoplastic fluid flow is lower.
Figure 2. Distributions of the velocity $u$, temperature $\theta$, and dissipative function $\Phi$ ($Br=-0.025$, $C=0$, and $\alpha=0.5$)

Figure 3. Distributions of the velocity and viscosity ($Br=-0.025$, $\alpha=0.5$: (a), (b) – $C=0.5$; and (c) – $C=0$)

The effect of the temperature dependence of the consistency on the flow characteristics at various $n$ is reflected in figure 3. For $n=0.5$, the maximum velocity is shifted towards the outer wall, while in the vicinity of the inner wall, a “dead” zone is observed, which is characterized by a high viscosity. In the range of the considered values of the power-law index, the apparent viscosity increases near the inner wall.

Let us consider the effect of the Brinkman number on the flow structure (figure 4). For $Br=0.025$, the maximum velocity is located near the inner wall, while, in the vicinity of the outer wall, a “dead” zone is formed. The Nusselt numbers calculated on the inner and outer walls for various $Br$ are presented in figure 4c. $Nu$ tends to infinity when the bulk temperature tends to the wall temperature. If a zero heat flux is set as a boundary condition on the inner wall, i.e. if $Br \to \infty$, then $Nu_{\infty}=6.68$ and $Nu_0=0$. It is revealed that for absolute values of $Br$ greater than 2, the velocity profile almost coincides with that for $C=0$, i.e. for the case of not taking into account the dependence of the consistency on temperature.
Figure 4. Distributions of the velocity (a) and temperature (b), and the dependence of the Nusselt number on the Brinkman number (c) ($\alpha=0.5$, $C=0.5$, and $n=0.5$)

Conclusion

The mathematical modeling of non-isothermal steady-state power-law fluid flow between two coaxial cylinders is carried out taking into account the dependence of the consistency on temperature. The characteristic distributions of the velocity, temperature, apparent viscosity, and dissipative function are shown. A parametric investigation of the Nusselt number on the inner and outer walls is implemented with respect to the Brinkman number.

Acknowledgment

This work was supported by the Russian Foundation for Basic Research (project № 18-08-00412).

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