Constraint on the Squeeze Parameter of Inflaton from Cosmological Constant

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Abstract

The inflaton is highly likely to settle in a squeezed vacuum state after inflation. The relic inflaton after inflation and reheating undergoes a damped oscillatory motion and contributes to the effective cosmological constant. We interpret the renormalized energy density from the squeezed vacuum state as an effective cosmological constant. Using the recent observational data on the cosmological constant, we find the constraint on the squeeze parameter of the inflaton in the early universe.

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I. INTRODUCTION

Recent observations indicate that our present universe is dominated by the dark energy with the pressure-to-density ratio $p/\rho \lesssim -1/3$, and that our universe is nearly flat ($\Omega_0 \approx 1$). One of the promising candidates for the dark energy is the cosmological constant with $p = -\rho$. From the field theory point of view, the vacuum energy density of fields contributes to an effective cosmological constant and would have been created through various phase transition processes such as the inflation, electro-weak, and QCD phase transitions. The effective vacuum energy density, $\rho_V$, defined as

$$\rho_V = \frac{\Lambda_{\text{eff}}}{8\pi G} = \langle \hat{\rho} \rangle + \frac{\Lambda}{8\pi G},$$

(1)

can be constrained by the present dark energy observation

$$\rho_V \leq \Omega_{DE}\rho_c,$$

(2)

where $\Omega_{DE}$ is the fraction of the dark energy density in the present universe and $\rho_c$ is the present critical energy density.

Inflation scenarios are a successful model of the early universe that solves many puzzles of the standard cosmological model. As the inflaton, a scalar field, rolled slowly over any nearly flat potential, the universe would have undergone a phase of accelerated expansion. The energy density of the inflaton during the slow-roll over played the role of a cosmological constant. Any initial vacuum state of the inflaton would have evolved to a squeezed quantum state during an inflation period through the parametric amplification. When inflation ended, the inflaton began to oscillate around the global minimum of its effective potential and created particles and thus reheated the universe. A more efficient mechanism for a cornucopia of particles would be a preheating process where another scalar field couples to the oscillating inflaton and undergoes a parametric resonance. In either case, the inflaton evolved to a squeezed vacuum state.

Recently there has been an attempt to understand the cosmological constant as a relic of vacuum fluctuations of the inflaton. The energy density of the inflaton in a vacuum state after the reheating process constitutes a significant fraction of the cosmological constant. Even with the correct theory of particle physics, there is still an ambiguity in choosing the vacuum state. In this paper we study how a general squeezed vacuum state affects the
cosmological constant problem. The renormalized energy density of the squeezed vacuum state that is obtained by subtracting the vacuum energy density \[5\] is interpreted as an effective cosmological constant. Further, we find a constraint on the squeeze parameter of the inflaton from the present observational data.

The organization of this paper is as follows. In Sec. II, we employ a massive inflaton model to find the squeezed vacuum state after the reheating period. The equation is found for the evolution of the energy density of the squeezed vacuum state from the reheating period to the present epoch. In Sec. III, using the energy density and the recent observational data, we put the constraint on the squeeze parameter of the inflaton. We compare our result with the squeeze parameter of gravitational waves and trans-Planckian physics. In Sec. IV, we discuss the physical implication of the squeeze parameter of the inflaton.

II. SQUEEZED VACUUM STATES OF INFLATON

We consider a simple inflation model which is described by a single inflaton, a scalar field, and has the action

\[ S_\phi = -\int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right], \] (3)

in the Friedmann-Robertson-Walker universe

\[ ds^2 = -dt^2 + a^2(t) dx^i dx_i. \] (4)

After Fourier transforming the inflaton into the momentum space and varying the action with respect to \( \phi \), we get the equation of motion for the \( k \)-th mode

\[ \ddot{\phi}_k + 3H \dot{\phi}_k + \frac{k^2}{a^2} \phi_k + V_\phi = 0, \] (5)

where \( H = \dot{a}/a \) is a Hubble parameter, dots denote derivatives with respect to \( t \), and the subscript \( \phi \) denotes a derivative with respect to \( \phi \). When the inflaton stops slow-rolling over the potential, inflation ends and the inflaton settles down and oscillates around the potential minimum. In this regime the potential can be approximated up to quadratic order of \( \phi \) as

\[ V \simeq V_0 + \frac{1}{2} m^2 \phi^2, \] (6)
where \( m^2 \) is the curvature of the potential with the dimension of mass. Then the equation of motion becomes

\[
\ddot{\phi}_k + 3H\dot{\phi}_k + \omega_k^2 \phi_k = 0, \quad \omega_k^2 \equiv m^2 + \frac{k^2}{a^2}.
\] (7)

The Hamiltonian density for the action (3) is a collection of time-dependent harmonic oscillators

\[
\hat{\mathcal{H}} = \sum_k \hat{\mathcal{H}}_k = \sum_k \left[ \frac{1}{2} \pi_k^2 + \frac{a^3}{2} \omega_k^2 \phi_k^2 \right],
\] (8)

where \( \pi_k = a^3 \dot{\phi}_k \) is a conjugate momentum.

For the Hamiltonian density (8), we may introduce the time-dependent annihilation and creation operators (in units of \( \hbar = c = 1 \)) [6, 7, 8]

\[
\hat{a}_k(t) = i[\varphi_k(t)\hat{\pi}_k - a^3\dot{\varphi}_k(t)\hat{\phi}_k],
\]

\[
\hat{a}^\dagger_k(t) = -i[\varphi_k(t)\hat{\pi}_k - a^3\dot{\varphi}_k(t)\hat{\phi}_k],
\] (9)

where \( \varphi_k(t) \) is a complex solution to the classical equation of motion

\[
\ddot{\varphi}_k + 3H\dot{\varphi}_k + \omega_k^2 \varphi_k = 0.
\] (10)

These are invariant operators satisfying the quantum Liouville-von Neumann equation

\[
i \frac{\partial}{\partial t} \hat{a}_k + [\hat{a}_k, \hat{\mathcal{H}}_k] = 0,
\] (11)

and so does \( \hat{a}^\dagger_k \). Also these operators satisfy the equal-time commutation relation

\[
[\hat{a}_k, \hat{a}^\dagger_{k'}] = \delta_{kk'},
\] (12)

provided that the Wronskian condition

\[
a^3(\varphi_k\dot{\varphi}_k^* - \varphi_k^*\dot{\varphi}_k) = i
\] (13)

be imposed. Further, we may select a solution \( \varphi_k \) that has the minimum uncertainty [6]. Then the most general complex solution to Eq. (10) is given by

\[
\varphi_{k\nu} = \mu_k^* \varphi_k + \nu_k^* \varphi_k^*,
\] (14)

where

\[
|\mu_k|^2 - |\nu_k|^2 = 1.
\] (15)
Here we set the parameters as

\[ \mu_k = \cosh r_k, \quad \nu_k = e^{i\theta_k} \sinh r_k, \]  

(16)

where \( r_k \) and \( \theta_k \) are two real squeeze parameters for the \( k \)-th mode.

The time-dependent annihilation and creation operators defined in terms of \( \varphi_{k\nu} \) and \( \varphi_k \) are related through the Bogoliubov transformation

\[ \hat{a}_{k\nu}(t) = \mu_k \hat{a}_k(t) - \nu_k \hat{a}^*_k(t) = \hat{S}_k(z_k, t) \hat{a}_k(t) \hat{S}^*_k(z_k, t), \]

\[ \hat{a}^*_{k\nu}(t) = \mu_k^* \hat{a}^*_k(t) - \nu_k^* \hat{a}_k(t) = \hat{S}_k(z_k, t) \hat{a}^*_k(t) \hat{S}^*_k(z_k, t), \]

(17)

where the squeeze operator \( \hat{S}_k(z_k, t) \) is defined by

\[ \hat{S}_k(z_k, t) = \exp \left[ \frac{1}{2}(z_k \hat{a}_{k}^{12}(t) - z_k^* \hat{a}^{12}_k(t)) \right]. \]

(18)

with \( z_k = r_k e^{i\theta_k} \). The squeezed number state, which is an eigenstate of the squeezed number operator, \( \hat{a}^*_k \hat{a}_k \), is defined by the squeeze number operator as

\[ |n_k, z_k, t\rangle = \hat{S}_k(z_k, t) |n_k, t\rangle, \quad \hat{a}^*_{k\nu}(t) \hat{a}_{k\nu}(t) |n_k, z_k, t\rangle = n_k |n_k, z_k, t\rangle. \]

(19)

Then the Hamiltonian density for the \( k \)-th mode in Eq. (8) has the representation

\[ \hat{H}_k = \frac{\hbar}{2} \{(\dot{\varphi}_k^2 + \omega_k^2 \varphi_k^2) \hat{a}^2_{k\nu} + (\dot{\varphi}_k^* \varphi_k^* + \omega_k^2 \varphi_k^* \varphi_k) (\hat{a}_{k\nu} \hat{a}^*_k + \hat{a}^*_{k\nu} \hat{a}_k) + (\dot{\varphi}_k^2 + \omega_k^2 \varphi_k^2) \hat{a}^{12}_k \}, \]

(20)

from which follows the expectation value of the \( k \)-th mode energy density

\[ \langle \hat{\rho}_k \rangle_n = \frac{1}{\hbar^3} \langle n_k, z_k, t | \hat{H}_k | n_k, z_k, t \rangle = (n_k + 1/2) \left\{ (|\mu_k|^2 + |\nu_k|^2)(\dot{\varphi}_k \varphi_k^* + \omega_k^2 \varphi_k \varphi_k^*) + 2 \Re \left[ (\dot{\varphi}_k^2 + \omega_k^2 \varphi_k^2) \mu_k^* \nu_k \right] \right\}. \]

(21)

We can safely drop the second term of the right hand side in the last line by employing, for instance, the random phase approximation [9]. The squeezed vacuum state result, obtained by setting \( n_k = 0 \), is given by

\[ \langle \hat{\rho}_k \rangle_{sv} = \frac{1}{2} (|\mu_k|^2 + |\nu_k|^2)(\dot{\varphi}_k \varphi_k^* + \omega_k^2 \varphi_k \varphi_k^*). \]

(22)

As the adiabatic vacuum state corresponds to \( \mu_k = 1 \) and \( \nu_k = 0 \), the vacuum expectation value of the \( k \)-th mode is

\[ \langle \hat{\rho}_k \rangle_{\text{vac}} = \frac{1}{2} (\dot{\varphi}_k \varphi_k^* + \omega_k^2 \varphi_k \varphi_k^*). \]

(23)
To calculate the $\langle \hat{\rho}_k \rangle_{\text{vac}}$, we need to solve the equation of motion (10). Changing the variable

$$\varphi_k = a^{-3/2} u_k,$$  

(24)

Eq. (10) can be written in a canonical form

$$\ddot{u}_k + \left( \omega_k^2 - \frac{9}{4} H^2 - \frac{3}{2} \dot{H} \right) u_k = 0.$$  

(25)

After slow-rolling over the potential, the inflaton begins to oscillate around the minimum and has an adiabatic solution of the form \cite{10}

$$u_k(t) = \frac{1}{\sqrt{2\Omega_k(t)}} e^{-i \int \Omega_k(t) dt},$$  

(26)

where

$$\Omega_k^2 = \left[ \omega_k^2 - \frac{9}{4} H^2 - \frac{3}{2} \dot{H} + \frac{3}{4} \dot{\Omega}_k^2 - \frac{1}{2} \ddot{\Omega}_k \right].$$  

(27)

As in Ref. \cite{10}, we shall assume $\omega_k \gg H$, which can be justified for the inflaton inside the horizon, and also assume $\Omega_k \gg |\dot{\Omega}_k|, |\ddot{\Omega}_k|$ so that $\Omega_k \approx \omega_k$. The vacuum energy density of the inflaton itself is the sum of the vacuum energy of all the modes

$$\langle \hat{\rho} \rangle_{\text{vac}} = \sum_k \langle \hat{\rho}_k \rangle_{\text{vac}} = \frac{1}{(2\pi)^3} \int \frac{\omega_k}{2a^3} d^3 k.$$  

(28)

To remove the ultraviolet divergences, we may introduce a cut-off momentum $k_\Lambda$

$$\langle \hat{\rho} \rangle_{\text{vac}} = \frac{4\pi^2}{2(2\pi)^3a^3} \int_0^{k_\Lambda} \sqrt{m^2 + \frac{k^2}{a^2}} dk$$

$$= \frac{1}{16\pi^2a} \left[ k_\Lambda \left( \frac{k_\Lambda^2}{a^2} + m^2 \right)^{3/2} - \frac{m^2}{2} k_\Lambda \left( \frac{k_\Lambda^2}{a^2} + m^2 \right)^{1/2} \cdots \right],$$  

(29)

and regularize the infinite quantities with the cosmological constant, gravitational constant, mass and \textit{etc} \cite{11}. There is a simple method to get the renormalized value \cite{5}

$$\langle \hat{\rho} \rangle_{\text{ren}} = \langle \hat{\rho} \rangle_{\text{sv}} - \langle \hat{\rho} \rangle_{\text{vac}} = \frac{1}{a^3} \sum_k |\nu_k|^2 \omega_k.$$  

(30)

Solving the semiclassical Einstein equation

$$H^2 = \frac{8\pi G}{3} \langle \hat{\rho} \rangle_{\text{ren}},$$  

(31)
we get the leading term of the power-law expansion
\[ a(t) = \left( 3\pi G_{\text{ren}} \sum_k |\nu_k|^2 \omega_k \right)^{1/3} t^{2/3}. \] (32)
where \( G_{\text{ren}} \) denotes the renormalized gravitational constant. This shows that as the inflaton settles down to the damped oscillatory motion, the universe enters the matter-dominated era. Using the power-law expansion (32), the squeezed vacuum expectation value of the energy density evolves as
\[ \langle \hat{\rho}(t) \rangle_{\text{ren}} = \langle \hat{\rho}(t_r) \rangle_{\text{ren}} \left( \frac{t_r}{t} \right)^2 \] (33)
where \( \langle \hat{\rho}(t_r) \rangle_{\text{ren}} \) is the value at \( t = t_r \).

III. CONSTRAINT ON THE SQUEEZE PARAMETER FROM OBSERVATIONS

Assuming inflation to occur at GUT scales, the energy and time scales at the reheating time are approximately \( E_r \sim 10^{14} \text{GeV} \) and \( t_r \simeq 10^{-34} \text{sec} \), respectively. Hence the vacuum energy density at the reheating,
\[ \langle \hat{\rho}_k(t_r) \rangle \geq (10^{14} \text{GeV})^4, \] (34)
evolves, according to Eq. (33), to the present value
\[ \langle \hat{\rho}(t_0) \rangle_{\text{ren}} \simeq \sum_k |\nu_k|^2 \omega_k(t_r) \times \left( \frac{10^{-34}}{4 \times 10^{17}} \right)^2 \]
\[ \gg \sum_k |\nu_k|^2 \times 6.25 \times 10^{-48} (\text{GeV})^4. \] (35)
Here we used the age of the universe, \( t_0 \simeq 4 \times 10^{17} \text{sec} \), from the recent observational data \[2\]. In the last line the inequality \( \langle \hat{\rho}_k(t_r) \rangle \gg (10^{14} \text{GeV})^4 \) was used for high momentum mode. Again using the data from observations
\[ \rho_{DE}(t_0) = \Omega_{DE}\rho_c \simeq 2.94 \times 10^{-47} (\text{GeV})^4 \] (36)
we put a constraint on the squeeze parameter
\[ \sum_k |\nu_k|^2 \ll 4.7. \] (37)
A few comments are in order. First, note that the constraint on the squeeze parameter applies only to the period after reheating and should be distinguished from another constraint from the inflation period. Second, the number of created pairs

\[ n_k = |\nu_k|^2 = \sinh^2 r_k \]  

(38)
is exponentially suppressed for high momentum modes when the inflaton evolves adiabatically, that is, keeps the adiabatic vacuum state throughout the evolution [12]. For instance, the squeeze parameter of gravitational waves (with \( m = 0 \)) which were produced from quantum fluctuations and got amplified, was calculated by Grishchuk and Sidorov during inflation period [3]. It was shown that \( r_k \) goes to zero for wavelengths smaller than the horizon size at the end of inflation but increases for wavelengths larger than the horizon size. The horizon-sized wavelength has \( r_k \simeq 1 \) for \( f \approx 10^8 \text{Hz} \) where \( f \) corresponds to the current frequency whereas the superhorizon-sized wavelength has \( r_k \simeq 10^2 \) for \( f \approx 10^{-16} \text{Hz} \). This means that the high momentum modes with wavelength smaller than the horizon remain in the adiabatic vacuum state whereas the superhorizon-sized low momentum modes are highly squeezed due to the parametric interaction with the gravitational field during inflation.

Finally, we compare our result with trans-Planckian physics. In the very early universe before inflation, the inflaton belonged to a ultra-relativistic regime and behaved like a massless field. A large number of particles would have been created for wavelengths smaller than the horizon size. Thus the squeeze parameter \( \nu_k \) should be constrained to protect overproduction of particle pairs at the end of the inflation [3, 13]

\[ |\nu_k|^2 \ll \left( \frac{H}{M_{pl}} \right)^2. \]  

(39)
The present CMB data restricts to \( H/M_{pl} < 10^{-5} \). Whereas the constraint gives bounds for high momentum modes

\[ |\nu_k|^2 \ll \frac{C}{k^{3+\epsilon}} \]  

(40)
for a positive \( \epsilon \) and \( C \) independent of the momentum.

**IV. SUMMARY AND DISCUSSION**

The inflaton provides a necessary energy not only to drive a quasi-exponential expansion and but also to reheat the universe during and after inflation. The inflaton would be properly
described by quantum theory (semiclassical gravity) in the early universe. The initial vacuum state of the inflaton is dynamically squeezed by the expansion of the universe and remains a squeezed vacuum during and after reheating. Also it is likely that the inflaton may survive as a relic after the reheating period and undergo a damped oscillatory motion. Then the renormalized energy density of the squeezed vacuum state contributes to the effective cosmological constant and evolves according to the matter-dominated power-law. The recent observational data on the dark energy or cosmological constant may put a constraint on the energy density at the reheating time. It is thus possible to find the constraint on the squeeze parameter of the squeezed vacuum state of the inflaton. It is important to know what quantum state of the inflaton would be in the early universe. The initial quantum state may affect on the temperature anisotropy [14]. This result of this paper may help tackle some of discussions on the early universe.

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[1] S. Weinberg, Rev. Mod Phys. 61, 1 (1989).
[2] D.N. Spergel *et al.*, Astrophys. J. Supp. 148, 175 (2003).
[3] L.P. Grishchuk and Y.V. Sidorov, Phys. Rev. D 42, 3413 (1990).
[4] J.R. Choi, C.I. Um, C.W. Kim, and S.P. Kim, “Quantum Evolution of Cosmological Constant after Reheating”, (2004).
[5] T. Tanaka, “A comment on trans-Planckian physics in inflationary universe”, astro-ph/0012431 (2000).
[6] J.K. Kim and S.P. Kim, J. Phys. A 32, 2711 (1999).
[7] S.P. Kim and D.N. Page, Phys. Rev. A 64, 012104 (2001).
[8] S.P. Kim, J. Korean Phys. Soc. 44, 446 (2004); *ibid.* 43, 11 (2003).
[9] T. Prokopec, Class. Quantum. Grav. 10, 2295 (1993).
[10] S.P. Kim and D.N. Page, J. Korean Phys. Soc. 35, S660 (1999).
[11] S.P. Kim, S.K. Kim, K.-S. Soh, and J.H. Yee, Phys. Rev. D 55, 2159 (1997).

[12] N.D. Birrell and P.C.W. Davies, Quantum Fields in Curved Spacetime (Cambridge University Press, Cambridge, 1982).

[13] A.A. Starobinsky, Pisma Zh. Eksp. Teor. Fiz. 73, 415 (2001) [JETP Lett. 73, 371 (2001)].

[14] S. Koh, S.P. Kim, and D.J. Song, “Gravitational Wave Spectrum in Inflation with Nonclassical States”, gr-qc/0402065, submitted to JHEP (2004).