The generalized Giambelli formula and polynomial CKP tau-functions

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Abstract
We find all polynomial tau-functions of the CKP hierarchy and its n-reductions. In particular, for n = 3 we find all polynomial tau-functions of the Kaup-Kupershmidt hierarchy.

1 Introduction
The concept of a tau-function of a hierarchy of soliton equations, developed by the Kyoto school in early 80’s (see [18],[3],[6]) is very useful for construction of solutions of these equations.

The geometric meaning of a tau-function is very simple: it is, up to a constant factor, a non-zero element of the orbit of a highest weight vector of a representation of an infinite-dimensional group [18],[3],[6],[12].

The first and the most famous example is the KP hierarchy, constructed as follows. Let $C\ell$ be the associative algebra on generators $\psi^+_j$ and $\psi^-_j$, $j \in \frac{1}{2} + \mathbb{Z}$, subject to the relations

$$[\psi^+(z), \psi^-(w)]_+ = \delta(z - w), \quad [\psi^\pm(z), \psi^\pm(w)]_+ = 0,$$

where

$$\psi^\pm(z) = \sum_{i \in \frac{1}{2} + \mathbb{Z}} \psi^\pm_i z^{-i - \frac{1}{2}}$$

are the generating series, called the charged free fermionic fields, and $\delta(z - w) = z^{-1} \sum_{n \in \mathbb{Z}} (\frac{w}{z})^n$ is the formal delta function. Let $F$ be an irreducible representation of the algebra $C\ell$, which admits a non-zero vector $|0\rangle$, such that

$$\psi^\pm_j |0\rangle = 0, \text{ for } j > 0.$$ (3)

Let $GL_\infty$ be the group of matrices $(g_{ij})_{i,j \in \frac{1}{2} + \mathbb{Z}}$ with entries in $\mathbb{C}$, which are invertible and all, but a finite number of $g_{ij} - \delta_{ij}$, are 0. We obtain a representation $R$ of this group on $F$ by letting

$$R(I + aE_{ij}) = 1 + a\psi^+_i \psi^-_j, \quad i, j \in \frac{1}{2} + \mathbb{Z}, \quad a \in \mathbb{C}.$$ (4)
Defining the charge decomposition

\[ F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}, \]  

(5)

by letting

\[
\text{charge}(\left|0\right\rangle) = 0 \text{ and } \text{charge}(\psi_j^\pm) = \pm 1,
\]

we see that each \( F^{(m)} \) is an irreducible highest weight module over \( GL_\infty \), and

\[
\left| \pm m \right\rangle = \psi_{-2m-1}^+ \cdots \psi_{-\frac{3}{2}}^+ \psi_{-\frac{1}{2}}^+ \left|0\right\rangle, \quad m \in \mathbb{Z}_{\geq 0},
\]

(6)

is a highest weight vector for \( F^{(\pm m)} \).

The KP hierarchy in the fermionic picture is defined as the following equation:

\[
\text{Res}_z \psi^+(z) \tau \otimes \psi^-(z) \tau = 0, \quad \tau \in F^{(0)},
\]

(7)

where \( \text{Res}_z \sum_i f_i z^i = f_{-1} \). It is easy to show \([12]\) that equation (7) holds for a non-zero \( \tau \in F^{(0)} \) if and only if \( \tau \) lies in the \( R(GL_\infty) \)-orbit of \( \left|0\right\rangle \).

A remarkable fact is that equation (7) can be converted in a collection of PDE’s, using bosonization of \( F \). For this one introduces the free bosonic field

\[
\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1} = : \psi^+(z) \psi^-(z) :,
\]

(8)

where, as usual, \( : \psi_i^+ \psi_j^- := \psi_i^+ \psi_j^- \) if \( i \leq j \), and \( = -\psi_j^- \psi_i^+ \) if \( i > j \). Then the \( \alpha_n \) satisfy the commutation relations of the infinite Heisenberg Lie algebra

\[
[\alpha_m, \alpha_n] = m \delta_{m,-n},
\]

(9)

and since \( \alpha_i |0\rangle = 0 \) for \( i \geq 0 \), there exists an isomorphism

\[ \sigma : F \rightarrow \mathbb{C}[q, q^{-1}, t_1, t_2, \ldots], \]

called the bosonization of \( F \), which is uniquely determined by the following properties

\[
\sigma(\left|m\right\rangle) = q^m, \quad \sigma \alpha_0 \sigma^{-1} = q \frac{\partial}{\partial q}, \quad \sigma \alpha_{-i} \sigma^{-1} = it_i \text{ and } \sigma \alpha_i \sigma^{-1} = \frac{\partial}{\partial t_i} \text{ for } i > 0.
\]

(10)

Furthermore, since \( [\alpha_k, \psi^\pm(z)] = \pm z^k \psi^\pm(z) \), one can identify, under the isomorphism \( \sigma \), the charged free fermions with the vertex operator

\[
\sigma \psi^\pm(z) \sigma^{-1} = q^{\pm 1} z^{\pm 1} \frac{\partial}{\partial m} \exp \left( \pm \sum_{i=1}^{\infty} t_i z^i \right) \exp \left( \mp \sum_{i=1}^{\infty} \frac{\partial}{\partial t_i} \frac{z^i}{i} \right).
\]

(11)

Then equation (7) gets converted to the KP hierarchy of bilinear PDE’s on \( \tau(t) \in \mathbb{C}[t_1, t_2, \ldots] \):

\[
\text{Res}_z \exp \left( \sum_{i=1}^{\infty} (t_i - t'_i) z^i \right) \exp \left( \sum_{i=1}^{\infty} \left( \frac{\partial}{\partial t'_i} - \frac{\partial}{\partial t_i} \right) \frac{z^{-i}}{i} \right) \tau(t) \tau(t') = 0.
\]

(12)
Here and further $t'$ denotes another copy of $t = (t_1, t_2, \ldots)$.

Next, equation (12) can be rewritten in terms of Lax type equations via the dressing operators $P(t, \partial)$, where $\partial = \frac{\partial}{\partial t_1}$ [18]. This is a monic pseudodifferential operator, whose symbol is

$$P(t, z) = \exp(- \sum_{i \geq 1} \frac{z^{-i}}{i} \frac{\partial}{\partial t_i}) \tau(t).$$

The associated to $\tau(t)$ Lax operator $L(t, \partial)$ is defined as the pseudodifferential operator

$$L(t, \partial) = P(t, \partial) \circ \partial \circ P(t, \partial)^{-1}.$$  \hspace{1cm} (13)

Then equation (12) on the tau-function $\tau(t)$ is equivalent to the following hierarchy of Lax-Sato evolution PDE’s on $L(t, \partial) = \partial + \sum_{j>0} u_j(t) \partial^{-j}$:

$$\frac{\partial L(t, \partial)}{\partial t_k} = [(L(t, \partial)^k)_+, L(t, \partial)], \quad k = 1, 2, 3, \ldots, \hspace{1cm} (14)$$

where the subscript $+$, as usual, denotes the differential part of $L(t, \partial)^k$.

A famous result of Sato [18] is that all Schur polynomials $s_\lambda(t)$ are tau-functions of the KP hierarchy. Recall that the Schur polynomial $s_\lambda(t)$, associated to a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell > 0)$ is defined by the Jacobi-Trudi formula (see e.g. [17], Section I.3):

$$s_\lambda(t) = \det (s_{\lambda_i+j-i}(t))_{1 \leq i,j \leq \ell}, \hspace{1cm} (15)$$

where the elementary Schur polynomials $s_j(t)$ are defined by the generating series

$$\sum_{j=0}^{\infty} s_j(t) z^j = \exp \sum_{i=1}^{\infty} t_i z^i. \hspace{1cm} (16)$$

In our paper [9] we proved that all polynomial tau-functions of the KP hierarchy are, up to a constant factor, of the form

$$\tau_{\lambda,c}(t) = \det (s_{\lambda_i+j-i}(t_1 + c_{1i}, t_2 + c_{2i}, t_3 + c_{3i}, \ldots))_{1 \leq i,j \leq \ell}, \hspace{1cm} (17)$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, and $c = (c_{ij})$ is a $(\lambda_1 + \ell - 1) \times \ell$ matrix over $\mathbb{C}$. We call equation (17) the generalized Jacobi-Trudi formula for polynomial KP tau-functions.

It is well known that, using the Frobenius notation $\lambda = (a_1, a_2, \ldots, a_k | b_1, b_2, \ldots, b_k)$ for the partition $\lambda$, one can write the Schur polynomial $s_\lambda(t)$ in the Giambelli form (see e.g. [17], Section I.3):

$$s_\lambda(t) = \det \left( \chi(a_i | b_j)(t; t) \right)_{1 \leq i,j \leq k}, \hspace{1cm} (18)$$

where

$$\chi(a | b)(t; t') := (-1)^b \sum_{n=0}^{b} s_{n+a+1}(t)s_{b-n}(-t'), \quad a, b \in \mathbb{Z}_{\geq 0}. \hspace{1cm} (19)$$
The first new result of the paper is Theorem 5 in Section 3, describing all polynomial KP tau-functions by the generalized Giambelli formula:

\[ \tau_{\lambda,c,d}(t) = \det \left( \chi_{(a_i|b_j)}(t_1 + c_{11}, t_2 + c_{22}, \ldots; t_1 + d_{11}, t_2 + d_{22}, \ldots) \right)_{1 \leq i, j \leq k}, \quad (20) \]

where \( c = (c_{ij}) \) is a \((a_1 + b_1 + 1) \times k\) matrix over \( \mathbb{C} \) and \( d = (d_{ij}) \) is a \( b_1 \times k\) matrix over \( \mathbb{C} \).

In Section 4, using the Jacobi-Trudi formalism, as in [13], we construct more general KP tau-functions, and find a Jacobi-Trudi type formula for the wave function \( w^+(t, z) = P(t, \partial) \exp \sum_{i=1}^{\infty} t_i z^i \). We also find analogous formulas in the framework of the Giambelli formalism.

Our Theorem 9 describes, in particular, the polynomial tau-functions for the CKP hierarchy. Recall that the CKP hierarchy in the Lax-Sato form is the following hierarchy of evolution equations on the skew-adjoint pseudodifferential operator \( L(t_o, \partial) = \partial + \sum_{j>0} a_j(t_o) \partial^{-j} \), where \( t_o = (t_1, 0, t_3, 0, t_5, \ldots) \):

\[ \frac{\partial L(t_o, \partial)}{\partial t_k} = [\left( (L(t_o, \partial)^k)_+, L(t_o, \partial) \right], \quad k = 1, 3, 5, \ldots. \quad (21) \]

There are at least two ways to construct the corresponding tau-function. One is to use the construction of the metaplectic representation of the infinite symplectic group \( SP_{\infty} \) via symplectic bosons, as in [10] and [11]. However, in this paper we use another way, via the reduction of the representation of \( GL_{\infty} \) in \( F^{(0)} \) to \( SP_{\infty} \) (see also [14], [11]).

Let \( \mathbb{C}^{\infty} = \bigoplus_{i \in \mathbb{Z}+1} \mathbb{C} e_i \), so that \( GL_{\infty} \) is the group of automorphism of this vector space, leaving all but a finite number of the \( e_j \) fixed. Define a skew-symmetric bilinear form \((\cdot, \cdot)_C \) on \( \mathbb{C}^{\infty} \) by

\[ (e_i, e_j)_C = (-1)^{i+j} \delta_{i,-j}. \quad (22) \]

Then \( SP_{\infty} \) is the subgroup of \( GL_{\infty} \), leaving this bilinear form invariant.

Define the automorphism \( \iota_C \) of the algebra \( C\ell \) by [6]

\[ \iota_C(\psi^+_j) = (-1)^{j+\frac{1}{2}} \psi^+_j. \quad (23) \]

This automorphism induces an automorphism of the vector space \( F \), which we again denote by \( \iota_C \), by letting \( \iota_C([0]) = [0] \). The subspace \( F^{(0)} \) of \( F \) is \( \iota_C \)-invariant, and we denote by \( F^{(0)}_C \) the fixed point set of \( \iota_C \) in \( F^{(0)} \). An element \( \tau \) in the orbit \( R(SP_{\infty})[0] \) then satisfies the following equation (cf. [21])

\[ \text{Res}_z \psi^+(z) \tau \otimes \psi^+(-z) \tau = 0, \quad (24) \]

which is called the CKP hierarchy in the fermionic picture. After bosonization equation (24) becomes (cf. [12]):

\[ \text{Res}_{z=0} \exp \left( \sum_{i=1}^{\infty} (t_i + (-1)^i t'_i) z^i \right) \exp \left( -\sum_{i=1}^{\infty} \left( \frac{\partial}{\partial t_i} + (-1)^i \frac{\partial}{\partial t'_i} \right) \right) \tau(t) \tau(t') = 0. \quad (25) \]
A non-zero element $\tau(t) \in \mathbb{C}[t_1, t_2, \ldots]$, satisfying (25), is called a tau-function of the CKP hierarchy, if it satisfies
\[
\tau(t_1, t_2, t_3, t_4, \ldots) = \iota_C(\tau(t_1, t_2, t_3, t_4, \ldots)) = \tau(t_1, -t_2, t_3, -t_4, \ldots),
\]
(26)
since, under the bosonization we have
\[
\sigma : F_C^{(0)} \overset{\sim}{\rightarrow} B_C := \{f \in \mathbb{C}[t_1, t_2, \ldots] | \iota_C(f) = f\}.
\]
In order to obtain a skew-adjoint Lax operator $L(t_0, \partial)$ from $L(t, \partial)$, satisfying (13) (14), we substitute $t_{2j} = 0$, $j = 1, 2, 3, \ldots$, in the CKP tau-function $\tau(t_1)$.

Our Theorem 9 describes, in particular, all polynomial tau-functions of the CKP hierarchy in the Giambelli form. Namely, they correspond to self-conjugate partitions, which in the Frobenius notation are $\lambda = (a_1, a_2, \ldots, a_k | a_1, a_2, \ldots, a_k)$, and are of the form (20), where $d = \iota_C(\overline{c})$, $\overline{c}$ consists of the first $b_1$ rows of $c$, and $\iota_C$ stands for changing the sign of even numbered rows of the matrix $c$; in addition, the matrix $c$ must satisfy the constraint (75) in Section 5 (which holds for $c = 0$).

In Section 6 we prove Theorem 13 on polynomial tau-functions for the $n$-reduced CKP hierarchy, using the results on the polynomial tau-functions of the KP hierarchy and the $n$-reduced KP hierarchies. The 2-reduced CKP hierarchy (90) is just the KdV hierarchy. The 3-reduced CKP hierarchy is called the Kaup-Kupershmidt hierarchy. It is a hierarchy of evolution PDE’s on the function
\[
u(t_0) = 3\frac{\partial^2 \log \tau(t_0)}{\partial t_1^2},
\]
(27)
where $t_0 = (t_1, t_3, t_5, \ldots)$, written as Lax equations on the differential operator
\[
L(t_0, \partial) = \partial^3 + \nu \partial + \frac{1}{2} \nu_x, \quad \text{where } x = t_1,
\]
(28)
namely
\[
\frac{\partial L(t_0, \partial)}{\partial t_k} = [(L(t_0, \partial)^{1/2})_+, L(t_0, \partial)], \quad k = 1, 3, 5, \ldots
\]
(29)
If $k$ is a multiple of 3, then $\frac{\partial \nu}{\partial t_k} = 0$, hence the first non-trivial equation is (29) for $k = 5$, and it is the Kaup-Kupershmidt equation (93) in section 6.

In the conclusion of the paper we compare our results on polynomial tau-functions of the CKP hierarchy with that of the BKP hierarchy, found in [10], [13], [15].

2 The KP hierarchy

First, we briefly recall the basics of the theory of the KP hierarchy, see [3], [6], [9], [12]. Consider the infinite matrix group $GL_\infty$ (resp. its Lie algebra $gl_\infty$) consisting of all infinite matrices $G = (g_{ij})_{i,j \in \mathbb{Z} + \mathbb{Z}}$ with entries in $\mathbb{C}$, which are invertible and all but a finite number of $g_{ij} - \delta_{ij}$ are 0 (resp. consisting of all matrices $g = (g_{ij})_{i,j \in \mathbb{Z} + \mathbb{Z}}$ for which are all but a finite number of $g_{ij}$ are 0). Both act on the vector space $\mathbb{C}^\infty = \bigoplus_{j \in \mathbb{Z} + \mathbb{Z}} \mathbb{C} e_j$ via the usual formula $E_{ij}(e_k) = \delta_{jk} e_i$. 
The semi-infinite wedge representation \([12, 9]\) \(F = \Lambda^{\frac{1}{2}}C^\infty\) is the vector space with a basis consisting of all semi-infinite monomials of the form \(e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \ldots\), where \(i_0 > i_1 > i_2 > \ldots\) and \(i_{t+1} = i_t - 1\) for \(t > 0\). One defines the representation \(R\) of \(GL_\infty\) (resp. \(r\) of \(gl_\infty\)) on \(F\) by

\[
R(G)(e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots) = Ge_{i_0} \wedge Ge_{i_1} \wedge Ge_{i_2} \wedge \cdots,\quad G \in GL_\infty,
\]

\[
r(g)(e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots) = \sum_{j=0}^\infty e_{i_0} \wedge \cdots \wedge e_{i_{j-1}} \wedge ge_{i_j} \wedge e_{i_{j+1}} \wedge \cdots,\quad g \in gl_\infty,
\]

assuming the usual rules of the product \(\wedge\).

The representation \(r\) of the Lie algebra \(gl_\infty\) can be given in terms of a Clifford algebra as follows. Define the wedging and contracting operators \(\psi^+_j\) and \(\psi^-_j\) \((j \in \frac{1}{2} + \mathbb{Z})\) on \(F\) by

\[
\psi^+_j(e_{i_0} \wedge e_{i_1} \wedge \cdots) = e_{-j} \wedge e_{i_0} \wedge e_{i_1} \cdots,
\]

\[
\psi^-_j(e_{i_0} \wedge e_{i_1} \wedge \cdots) = \begin{cases} 0 & \text{if } j \neq i_s \text{ for all } s \\ (-1)^s e_{i_0} \wedge e_{i_1} \wedge \cdots \wedge e_{i_{s-1}} \wedge e_{i_{s+1}} \wedge \cdots & \text{if } j = i_s. \end{cases}
\]

Then \(r(E_{ij}) = \psi^+_i \psi^-_j\). These operators satisfy the relations (cf.(1)) \((i, j, s, \ell) \in \frac{1}{2} + \mathbb{Z}, \lambda, \mu = +, -)\):

\[
\psi^\lambda_j \psi^\mu_j + \psi^\mu_j \psi^\lambda_j = \delta_{\lambda, -\mu} \delta_{i, -j},
\]

hence they generate a Clifford algebra, which we denote by \(C\ell\). Introduce the following elements of \(F\) \((m \in \mathbb{Z})\):

\[
|m\rangle = e_{m-\frac{3}{2}} \wedge e_{m-\frac{1}{2}} \wedge e_{m-\frac{5}{2}} \wedge \cdots.
\]

It is clear that \(F\) is an irreducible \(C\ell\)-module such that the relations \((3)\) hold.

It will be convenient to define also the opposite spin module with vacuum vector \(\langle 0|\), where

\[
\langle 0| \psi^+_j = 0, \quad \text{for } j < 0,
\]

and for \(m > 0\) one defines

\[
\langle \pm m| = \langle 0| \psi^+_\pm \psi^+_\pm \cdots \psi^+_\pm m-\frac{1}{2}.
\]

The vacuum expectation value is defined on \(C\ell\) as \((a) = \langle 0|a|0\rangle\) and \(\langle 0|1|0\rangle = 1\). Recall the charge decomposition \((5)\) the space \(F^{(m)}\) is an irreducible highest weight \(g_\infty\)-module, where \(|m\rangle\) is its highest weight vector, i.e.

\[
r(E_{ij})|m\rangle = 0 \quad \text{for } i < j, \quad r(E_{ii})|m\rangle = 0 \quad (\text{resp. } = |m\rangle) \quad \text{if } i > m \quad (\text{resp. if } i < m).
\]

Let \(S\) be the following operator on \(F \otimes F\)

\[
S = \sum_{i \in \frac{1}{2} + \mathbb{Z}} \psi^+_i \otimes \psi^-_i
\]

and let \(\mathcal{O}_m = R(GL_\infty)|m\rangle \subset F^{(m)}\) be the \(GL_\infty\)-orbit of the highest weight vector \(|m\rangle\), then the following simple result holds
We obtain a representation \( \hat{\sigma} \) Heisenberg Lie algebra. Using this, one constructs the isomorphism (bosonization) has \([3],[6],[12]\)

\[
\alpha \text{ extension by a central element } K
\]

\( \text{Proposition 2} \ [8] \) Let \( f_m \in \mathcal{O}_m \) be an integer and let 0 \( \neq f_m \in F^{(m)} \). Then \( f_m \in \mathcal{O}_m \) if and only if

\[
S(f_m \otimes f_m) = 0. \tag{32}
\]

Equation (32) is called the KP hierarchy in the fermionic picture.

To each \( f_m = R(G)|m \rangle \in \mathcal{O}_m \) one associates a point in the Sato infinite Grassmannian which is the linear span of \( \{ Ge_i \mid i < m \} \subset \mathbb{C}^\infty \). Another way to describe this subspace is as a subspace of \( \Psi^+ \), where \( \Psi^\pm = \bigoplus_{i \in \frac{1}{2} + \mathbb{Z}} \mathbb{C} \psi_i^\pm \), defined as the annihilation space \( \text{Ann}_+ f_m \), where

\[
\text{Ann}_\pm f_m = \{ v^\pm \in \Psi^\pm | v^\pm f_m = 0 \}.
\]

The connection between the two subspaces \( \text{Ann}_+ f_m \) and \( \text{Ann}_- f_m \) is as follows (cf. (30)):

If \( Ge_j = \sum_{i \in \frac{1}{2} + \mathbb{Z}} G_{ij} e_i \), then \( G_{i,j} \psi_j^\pm = \sum_{i \in \frac{1}{2} + \mathbb{Z}} G_{ij} \psi_j^\pm \), and \( \text{Ann}_+ f_m \) is the linear span of \( \{ G \psi_j^\pm | i < m \} \). We find \( \text{Ann}_- f_m \) as follows (see e.g. [8], Lemma 2.4):

\( \text{Ann}_- f_m \) is the linear span of \( \{ G \psi_j^- | i > -m \} \), and letting \( G \psi_k^- = \sum_{i \in \frac{1}{2} + Z} H_{-i,k} \psi_i^- \), since \( (G \psi_j^+, G \psi_k^-) = \delta_{jk} \), we find that the matrix \( (H_{ij}) \) is the inverse transpose of \( G \).

Note that \( \text{Ann}_+ f_m = \text{Ann}_+ f_m \oplus \text{Ann}_- f_m \) is a maximal isotropic subspace of \( \Psi = \Psi^+ \oplus \Psi^- \), with respect to the symmetric bilinear form \((, )\), which defines the Clifford algebra \( C \ell = C \ell(\Psi) \),

\[
(\psi_i^+, \psi_j^-) = \delta_{i,-j}, \quad (\psi_i^+, \psi_j^+) = 0. \tag{33}
\]

\( \text{Proposition 2} \ [8] \) Let \( f_m \in \mathcal{O}_m \), and \( v^\pm \in \Psi^\pm \), such that \( v^\pm f_m \neq 0 \), then \( v^\pm f_m \in \mathcal{O}_{m \pm 1} \).

We can extend the above description to the Lie algebra \( a_\infty \), which is the central extension by a central element \( K \) of the Lie algebra of infinite matrices \( (g_{ij}) \) such that \( g_{ij} = 0 \) if \( |i - j| > 0 \). The Lie bracket is given by \( [g + \lambda K, h + \mu K] = gh - hg + C(g, h) K \), where \( C \) is the 2-cocycle given by

\[
C(E_{ij}, E_{ji}) = 1 = -C(E_{ji}, E_{ij}) \text{ if } i < 0 < j, \quad \text{and } C(E_{ij}, E_{kl}) = 0 \text{ otherwise.}
\]

We obtain a representation \( \hat{\sigma} \) of \( a_\infty \) on \( F^{(m)} \), by the formula

\[
\hat{\sigma}(E_{ij}) := \psi_i^+ \psi_j^-, \quad \hat{\sigma}(K) = 1.
\]

Recall the free bosonic field \( \alpha(z) = \sum_{n \in \mathbb{Z}} \alpha_n z^{-n-1} \), defined by \([8]\). Then the operators \( \alpha_n \) lie in \( \hat{\sigma}(a_\infty) \), and satisfy the commutation relations \([9]\) of the infinite Heisenberg Lie algebra. Using this, one constructs the isomorphism (bosonization) \( \sigma : F \rightarrow \mathbb{C}[q, q^{-1}, t_1, t_2, \ldots] \), uniquely defined by the properties \([10]\). Recall that one has \([3],[6],[12]\)

\[
\sigma \psi^\pm(z) \sigma^{-1} = q^{\pm z} z^{\pm \frac{\partial}{\partial q^\mp}} \exp \left( \pm \sum_{i=1}^\infty t_i z^i \right) \exp \left( \mp \sum_{i=1}^\infty \frac{\partial}{\partial t_i} z^i \right). \tag{34}
\]
Let \( f_m \) satisfy the KP hierarchy in the fermionic picture \( \{32\} \), and let \( \tau_m = \sigma(f_m) \), then \( \tau_m \) satisfies the following equation, called the KP hierarchy of bilinear equations on \( \tau_m(t) \in \mathbb{C}[t_1, t_2, \ldots] \):

\[
\text{Res}_{z=0} \exp \left( \sum_{i=1}^{\infty} (t_i - t'_i) z^i \right) \exp \left( \sum_{i=1}^{\infty} \left( \frac{\partial}{\partial t'_i} - \frac{\partial}{\partial t_i} \right) z^{-i} \right) \tau_m(t) \tau_m(t') = 0. \tag{35}
\]

A solution \( \tau_m(t) \) of (35) is called a tau-function of the KP hierarchy. A beautiful formula for the tau-function, corresponding to the point \( R(G)|m \), where \( G \in GL_{\infty} \), was given in \( \{3\} \):

\[
\tau_m(t) = \langle m | (\exp H(t)) G|m \rangle, \tag{36}
\]

where \( H(t) = \sum_{i=1}^{\infty} t_i \alpha_i \).

**Remark 3** The totality of tau-functions is independent of \( m \). Namely if \( G \in GL_{\infty} \) and \( \Lambda = \sum_{i \in \mathbb{Z}} E_{i,i+1} \), then \( \Lambda^{-m} G \Lambda^m \in GL_{\infty} \) for any \( m \in \mathbb{Z} \), and

\[
\tau_m(t) = \langle m | \exp (H(t)) G|m \rangle = \langle 0 | \exp (H(t)) \Lambda^{-m} G \Lambda^m |0 \rangle.
\]

Let \( m = 0 \), and \( \tau(t) = \tau_0(t) \). Define the wave function \( w^+(t,z) \) and the adjoint wave function \( w^-(t,z) \) by

\[
w^\pm(t,z) = \frac{\langle \pm 1 | (\exp H(t)) \psi^\pm(z) G|0 \rangle}{\langle 0 | (\exp H(t)) G|0 \rangle}. \tag{37}
\]

Then

\[
w^\pm(t,z) = \frac{\sigma^1(\psi^\pm(z) \sigma^{-1}) \tau(t)}{\tau(t)} = \frac{\tau(t + [z^{-1}])}{\tau(t)} \exp \left( \pm \sum_{n=1}^{\infty} t_n z^n \right), \tag{38}
\]

where \([z^{-1}] = \left( \frac{z^{-1}}{1}, \frac{z^{-2}}{2}, \frac{z^{-3}}{3}, \ldots \right)\), and equation \( \{35\} \) is equivalent to the equation \( \{3\} \)

\[
\text{Res}_z w^+(t,z) w^-(s,z) = 0. \tag{39}
\]

One can write this in terms of monic pseudo-differential operators \( P^\pm(t, \partial) \) in \( \partial = \frac{\partial}{\partial t_1} \). Namely write \( \{3\} \)

\[
w^\pm(t,z) = P^\pm(t, \pm z) \exp \left( \pm \sum_{n=1}^{\infty} t_n z^n \right) \tag{40}
\]

then it is straightforward, see e.g. \( \{9\} \), Sections 3 and 4, to prove that \( (k = 1, 2, 3, \ldots) \):

\[
P^-(t, \partial)^* = P^+(t, \partial)^{-1} \quad \text{and} \quad \frac{\partial P^+(t, \partial)}{\partial t_k} = -(P^+(t, \partial) \circ \partial^k \circ P^+(t, \partial)^{-1}) \circ P^+(t, \partial).
\]

Using these equations, one finds that the KP Lax-Sato pseudodifferential operator \( L(t, \partial) = P^+(t, \partial) \circ \partial \circ P^+(t, \partial)^{-1} \) satisfies the Lax-Sato evolution equations \( \{14\} \).
3 Polynomial KP tau-functions and the generalized Jacobi-Trudi and Giambelli formulas

Let $\text{Par}_\ell$ denote the set of partitions consisting of $\ell \in \mathbb{Z}_{\geq 0}$ non-zero parts, i.e. sequences of integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$, and let $\lambda = (a_1, \ldots, a_k|b_1, \ldots, b_k)$ be the Frobenius notation for $\lambda$ (see e.g. [17], Section I.1). Let $\text{Par} = \bigcup_{\ell=0}^\infty \text{Par}_\ell$.

Recall that the Schur polynomials are given by the Jacobi-Trudi formula (15). It is natural to call (41) the generalized Jacobi-Trudi formula for polynomial KP tau-functions. Similarly, the polynomial KP tau-functions can be given in the Giambelli form. Namely, we can restate Theorem 4 in the following form.

**Theorem 4** All polynomial tau-functions of the KP hierarchy are, up to a constant factor, of the form

$$
\tau_{\lambda;c}(t) = \det (s_{\lambda_i+j-i}(t_1 + c_i t_2 + c_{i-1} t_3 + \ldots))_{1 \leq i,j \leq \ell},
$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \text{Par}_\ell$, $\ell \geq 0$, and $c_i = (c_{1i}, c_{2i}, c_{3i}, \ldots, c_{\lambda_i-1+i,1})$ for $i = 1, \ldots, \ell$, with $c_{ji} \in \mathbb{C}$ arbitrary.

It is natural to call (41) the generalized Jacobi-Trudi formula for polynomial KP tau-functions. Similarly, the polynomial KP tau-functions can be given in the Giambelli form. Namely, we can restate Theorem 4 in the following form.

**Theorem 5** All polynomial tau-functions of the KP hierarchy are, up to a constant factor, of the form

$$
\tau_{\lambda;c,d}(t) = \det (\chi(a_{|b_i})(t_1 + c_1 t_2 + c_{2i} t_3 + \ldots, t_1 + d_1 t_2 + d_{2i} t_3 + \ldots))_{1 \leq i,j \leq k},
$$

where $\chi(a_{|b})$ is given by [19]. Here $\lambda = (a_1, a_2, \ldots, a_k|b_1, b_2, \ldots, b_k)$ is the Frobenius notation for $\lambda \in \text{Par}$, and $c_i = (c_{1i}, c_{2i}, c_{3i}, \ldots, c_{a_i+b_i+1,1})$, $d_i = (d_{1i}, d_{2i}, d_{3i}, \ldots, d_{b_i,i})$ for $i = 1, \ldots, k$, with $c_{ji}, d_{ji} \in \mathbb{C}$ arbitrary.

Since the Giambelli formula for Schur polynomials [18] is obtained from (42) by substituting $c_{ij} = d_{ij} = 0$ for all $1 \leq i \leq k$ and $j = 1, 2, \ldots$, we call formula (42) the generalized Giambelli formula for polynomial KP tau-functions.

Before giving the proof of Theorem 5, we will first state and prove a lemma, and discuss the ideas of the proof of Theorem 4 ([9], Theorem 16).

**Lemma 6**

(a) $\exp(H(t))|m\rangle = |m\rangle$, $m \in \mathbb{Z}$,

(b) $(\exp H(t))\psi^\pm(z) \exp(-H(t)) = \psi^\pm(z) \exp \left( \pm \sum_{k>0} t_k z^k \right)$,

(c) $\langle (\exp H(t)) \psi^+(y) \psi^-(z) \rangle = i_{y,z} \frac{1}{y-z} \exp \left( \sum_{k>0} t_k (y^k - z^k) \right)$,

(d) $\langle (\exp H(t)) \psi^+_{-1+i \frac{1}{2}} \psi^-_{-1-j \frac{1}{2}} \rangle = \sum_{\ell=0}^{j} s_{i+\ell+1}(t) s_{j-\ell}(-t)$,

(e) $\langle (\exp H(t)) \psi^+_{-1+i \frac{1}{2}} \psi^-_{-1-j \frac{1}{2}} \rangle = (-1)^j \chi(a|b)(t; t) = (-1)^j s_{(i+1,j)}(t)$. 

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Proof. (a) follows from the fact that all \( \alpha_k|0\) = 0 for all \( k > 0 \).
(b) follows from the fact that \( \alpha_k, \psi^\pm(z) \) = \( \pm z^k \psi^\pm(z) \).
(c) follows from (a), (b) and the fact that \( \langle \psi^+(y)\psi^-(z) \rangle = i_y z y^{-z} \).
(d) follows by taking the coefficient of \( y^i z^j \) in (c).
Finally, (e) follows from (d), the equality

\[
\psi^+_{-i-\frac{1}{2}} \psi^-_{-j-\frac{1}{2}} |0\rangle = (-1)^j e_{i+\frac{1}{2}} \wedge e_{-\frac{1}{2}} \wedge e_{-\frac{3}{2}} \wedge \cdots \wedge e_{-j-\frac{1}{2}} \wedge e_{-j-\frac{3}{2}} \wedge \cdots,
\]
and the fact that (cf. [2], Section 6)

\[
\sigma(e_{i+\frac{1}{2}} \wedge e_{-\frac{1}{2}} \wedge e_{-\frac{3}{2}} \wedge \cdots \wedge e_{-j-\frac{1}{2}} \wedge e_{-j-\frac{3}{2}} \wedge \cdots) = s_{(i+1,1)}(t).
\]

Since a tau-function is independent of the charge \( m \), see Remark [3] we may assume that \( m = 0 \). Then by the Bruhat decomposition of \( GL_\infty \), the \( GL_\infty \) orbit \( O_0 \) of \( |0\rangle \) is the disjoined union of Schubert cells:

\[
O_0 = \bigcup_{\lambda \in Par} R(U)f_\lambda,
\]
where

\[
f_\lambda = e_{\lambda_1-\frac{1}{2}} \wedge e_{\lambda_2-\frac{3}{2}} \wedge \cdots \wedge e_{\lambda_\ell-\ell+\frac{1}{2}} \wedge e_{-\ell-\frac{1}{2}} \wedge e_{-\ell-\frac{3}{2}} \wedge \cdots, \quad \lambda \in Par_\ell,
\]
and \( U \) is the subgroup of \( GL_\infty \) consisting of all upper-triangular matrices, with 1 on the diagonal.

Note that the elements \( f_\lambda \) form a basis of \( F^{(0)} \), and (see [12], Theorem 4.1) \( \sigma(f_\lambda) = s_\lambda(t) \).

Now let \( A = (a_{ij}) \in U \), then \( R(A)|-\ell\rangle = |-\ell\rangle \), for any \( \ell \in \mathbb{Z} \), since \( |-\ell\rangle \) is the highest weight vector of \( F^{(-\ell)} \). Thus

\[
R(A)f_\lambda = Ae_{\lambda_1-\frac{1}{2}} \wedge Ae_{\lambda_2-\frac{3}{2}} \wedge \cdots \wedge Ae_{\lambda_\ell-\ell+\frac{1}{2}} \wedge e_{-\ell-\frac{1}{2}} \wedge e_{-\ell-\frac{3}{2}} \wedge \cdots
\]

\[
= w_{\lambda_1-\frac{1}{2}} \wedge w_{\lambda_2-\frac{3}{2}} \wedge \cdots \wedge w_{\lambda_\ell-\ell+\frac{1}{2}} \wedge e_{-\ell-\frac{1}{2}} \wedge e_{-\ell-\frac{3}{2}} \wedge \cdots,
\]
where

\[
w_{\lambda_j-\frac{1}{2}} = e_{\lambda_j-\frac{1}{2}} + \sum_{i \leq \lambda_j-\frac{1}{2}} a_{ij} e_i.
\]
In fact we may obviously assume that

\[
w_{\lambda_j-\frac{1}{2}} = e_{\lambda_j-\frac{1}{2}} + \sum_{i = \frac{1}{2}-\ell}^{\lambda_j-\frac{1}{2}} a_{i,\lambda_j-\frac{1}{2}} e_i.
\]
Hence

\[
R(A)f_\lambda = w_{\lambda_1-\frac{1}{2}}^+ w_{\lambda_2-\frac{3}{2}}^+ \cdots w_{\lambda_\ell-\ell+\frac{1}{2}}^+ |-\ell\rangle,
\]
where
\[ w^+_{\lambda_j-j+\frac{1}{2}} = \psi^+_{\lambda_j-1} - \lambda_j + \sum_{i \geq j+\frac{1}{2}-\lambda_j} a_{-i,\lambda_j-j+\frac{1}{2}} \psi^+_i, \quad (47) \]
and
\[ w^+_{\lambda_j-j+\frac{1}{2}} = \text{Res}_z a_j(z) \psi^+(z), \quad \text{with} \quad a_j(z) = z^{j-\lambda_j-1} + \sum_{i \geq j+\frac{1}{2}-\lambda_j} a_{-i,\lambda_j-j+\frac{1}{2}} z^{i-\frac{1}{2}}. \]

Now we can find constants \( c_{\lambda_j-j+\frac{1}{2}} = (c_{1,\lambda_j-j+\frac{1}{2}}, c_{2,\lambda_j-j+\frac{1}{2}}, \ldots) \) such that
\[ a_{\lambda_j-j+\frac{1}{2}}(z) = z^{j-\lambda_j-1} \exp \sum_{i=1}^{\infty} c_{i,\lambda_j-j+\frac{1}{2}} z^i, \]
thus
\[ w^+_{\lambda_j-j+\frac{1}{2}} = \text{Res}_z z^{j-\lambda_j-1} \exp \left( \sum_{i=1}^{\infty} c_{i,\lambda_j-j+\frac{1}{2}} z^i \right) \psi^+(z). \quad (48) \]

Finally, using \([36]\), we find that (see \([13]\), Section 3)
\[
\sigma(w^+_{\lambda_1-\frac{1}{2}} w^+_{\lambda_2-\frac{1}{2}} \cdots w^+_{\lambda_\ell-\frac{1}{2}}| - \ell) = \langle 0 | \exp(H(t)) w^+_{\lambda_1-\frac{1}{2}} w^+_{\lambda_2-\frac{1}{2}} \cdots w^+_{\lambda_\ell-\frac{1}{2}} | - \ell \rangle
\]
\[ = \text{Res}_{z_1} \cdots \text{Res}_{z_\ell} z_1^{\lambda_1} \cdots z_\ell^{\lambda_\ell-1} \exp \left( \sum_{j=1}^{\ell} \sum_{i=1}^{\infty} c_{i,j-\frac{1}{2}} z_j^i \right) \langle 0 | \exp(H(t)) z_1^\ell \cdots z_\ell^\ell | - \ell \rangle
\]
\[ = \text{Res}_{z_1} \cdots \text{Res}_{z_\ell} z_1^{\lambda_1-\ell} z_2^{\lambda_2-\ell} \cdots z_\ell^{\lambda_\ell-\ell} \prod_{1 \leq j < k \leq \ell} (z_j - z_k) \exp \left( \sum_{j=1}^{\ell} \sum_{i=1}^{\infty} (t_i + c_{i,j-\frac{1}{2}}) z_j^i \right), \]
which is equal to \((41)\), where one has to replace \( c_j \) by \( c_{\lambda_j-j+\frac{1}{2}} \). In the above calculations we assume that \(|z_1| > |z_2| > \cdots > |z_\ell|\). It is clear from the above generalized Jacobi-Trudi formula \((41)\), that the constants \( c_{ij} \) for \( i > \lambda_j - j + \ell \) do not appear there, so we can choose them all equal to 0, which gives the restriction that \( c_j \in \mathbb{C}^{\lambda_j-j+\ell} \).

**Proof of Theorem 5.** We first rewrite \((41)\). Since
\[ \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_\ell) = (a_1, a_2, \ldots, a_k | b_1, b_2, \ldots, b_k), \]
we find that
\[ f_\lambda = \psi^+_{\frac{1}{2}-\lambda_1} \psi^+_{\frac{1}{2}-\lambda_2} \cdots \psi^+_{\frac{1}{2}-\lambda_\ell} \psi^-_{\frac{1}{2}-\epsilon} \psi^-_{\frac{1}{2}-\ell} \cdots \psi^-_{\frac{1}{2}} |0\rangle
\]
\[ = \psi^+_{-a_1-\frac{1}{2}} \psi^+_{-a_2-\frac{1}{2}} \cdots \psi^+_{-a_k-\frac{1}{2}} |k+\frac{1}{2}-\lambda_{k+1} \cdots \psi^+_{\frac{1}{2}-\lambda_{\ell-1}} \psi^-_{\frac{1}{2}-\ell} \cdots \psi^-_{\frac{1}{2}-\frac{1}{2}} |0\rangle. \]

Observe that \( a_j = \lambda_j - j \geq 0 \), for \( j = 1, \ldots, k \) and \( \lambda_j - j < 0 \) for \( j > k \). We now move all \( \psi^+_{\frac{1}{2}-\lambda_j} \) with \( j > k \) to the right of all \( \psi^-_i \) with \( 0 < i < \ell \). This has the effect that one removes all \( \psi^-_i \), for which \( i \) is equal to one of the \( j - \frac{1}{2} - \lambda_j \) for \( j > k \). This gives that \( f_\lambda \) is equal, up to a sign, which we may ignore, to
\[ f_\lambda = \psi^+_{-a_1-\frac{1}{2}} \psi^+_{-a_2-\frac{1}{2}} \cdots \psi^+_{-a_k-\frac{1}{2}} \psi^-_{-b_1-\frac{1}{2}} \psi^-_{-b_2-\frac{1}{2}} \cdots \psi^-_{-b_k-\frac{1}{2}} |0\rangle. \quad (49) \]
Next we calculate $R(A)f_{\lambda}$ for $A \in U$. Clearly $A\psi_{-a_j-\frac{1}{2}} = A\psi_{-a_j-\frac{1}{2} - \lambda_j} = w^{+}_{\lambda_j-j+\frac{1}{2}}$ for $j = 1, \ldots, k$, where $w^{+}_{\lambda_j-j+\frac{1}{2}} = w^{+}_{a_j+\frac{1}{2}}$ is as in (17), and thus is equal (18).

Let $w^{+}_{-b_j-\frac{1}{2}} = A\psi_{-b_j-\frac{1}{2}}$, then there exist constants $c_{-b_j-\frac{1}{2}} = (c_{1,-b_j-\frac{1}{2}}, c_{2,-b_j-\frac{1}{2}}, \ldots)$, such that

$$w^{+}_{-b_j-\frac{1}{2}} = \psi_{-b_j-\frac{1}{2}} + \sum_{i \geq \frac{1}{2}-b_j} c_{i,-b_j-\frac{1}{2}} \psi^{+}_{i}$$

$$= \text{Res}_{z}(z^{-b_j-1} + \sum_{i \geq \frac{1}{2}-b_j} c_{i,-b_j-\frac{1}{2}} z^{i-\frac{1}{2}})\psi^{+}(z).$$

In a similar way as for the $w^{+}_{a_i+\frac{1}{2}}$, there exist constants $d_{-b_j-\frac{1}{2}} = (d_{1,-b_j-\frac{1}{2}}, d_{2,-b_j-\frac{1}{2}}, \ldots)$, such that

$$w^{-}_{-b_j-\frac{1}{2}} = \text{Res}_{z}z^{-b_j-1} \exp(-\sum_{i=1}^{\infty} d_{i,-b_j-\frac{1}{2}} z^{i})\psi^{-}(z). \quad (50)$$

Using (48) and (50), the relation between the constants $c_i$ and $d_{-j}$ is as follows. Since $(w^{+}_{i}, w^{-}_{j}) = \delta_{i,-j}$, we find that

$$\delta_{i,-j} = \text{Res}_{y} \text{Res}_{z} y^{-i-\frac{1}{2}} z^{-j-\frac{1}{2}} \exp\left(\sum_{a=1}^{\infty} c_{a,j} y^{a}\right) \exp\left(-\sum_{a=1}^{\infty} d_{a,-j} z^{a}\right) \psi^{+}(y) \psi^{-}(z)$$

$$= \text{Res}_{y} \text{Res}_{z} y^{-i-\frac{1}{2}} z^{-j-\frac{1}{2}} \exp\left(\sum_{a=1}^{\infty} c_{a,j} y^{a} - d_{a,-k} z^{a}\right) \delta(y - z)$$

$$= \text{Res}_{y} \text{Res}_{z} z^{-i-j-1} \exp\left(\sum_{a=1}^{\infty} (c_{a,j} - d_{a,-k}) z^{a}\right) \delta(y - z)$$

$$= \text{Res}_{z} z^{-i-j-1} \exp\left(\sum_{a=1}^{\infty} (c_{a,j} - d_{a,-k}) z^{a}\right)$$

$$= s_{i+j}(c_{i} - d_{-j}).$$

This means that we have a possible restriction on the constants $d_{-b_j-\frac{1}{2}}$ for $j = 1, \ldots, k$, viz.

$$s_{a_i+b_j+1}(c_{a_i+\frac{1}{2}} - d_{-b_j-\frac{1}{2}}) = 0 \quad \text{for } 1 \leq i, j \leq k.$$ 

Stated differently,

$$d_{a_i+b_j+1,-b_j-\frac{1}{2}} = s_{a_i+b_j+1}(c_{1,a_i+\frac{1}{2}} - d_{1,-b_j-\frac{1}{2}}, \ldots, c_{a_i+b_j,a_i+\frac{1}{2}} - d_{a_i+b_j,-b_j-\frac{1}{2}}, c_{a_i+b_j+1,a_i+\frac{1}{2}}). \quad (51)$$

Finally we calculate

$$\sigma(R(A)f_{\lambda}) = \langle 0 | (\exp H(t)) w^{+}_{a_1+\frac{1}{2}} \cdots w^{+}_{a_k+\frac{1}{2}} w^{-}_{-b_1-\frac{1}{2}} \cdots w^{-}_{-b_k-\frac{1}{2}} | 0 \rangle.$$ 

Using Wick’s theorem, this is equal, up to a sign, to

$$\det(\langle (\exp H(t)) w^{+}_{a_i+\frac{1}{2}} w^{-}_{-b_j-\frac{1}{2}} \rangle)_{1 \leq i,j \leq k}.$$
Now, using (48), (50), and Lemma 6 we find that in the domain $|y| > |z|$ we have

$$\langle (\exp H(t))w^+_{a_i+j_\frac{1}{2}}w^-_{b_j-\frac{1}{2}} \rangle =$$

Res$_y$Res$_z$ $y^{-a_i-1}z^{-b_j-1}$ $\exp(\sum_{r=1}^{\infty} c_{r,a_i+\frac{1}{2}}y^r - d_{r,-b_j-\frac{1}{2}}z^r) \langle (\exp H(t))\psi^+(y)\psi^-(z) \rangle =$

Res$_y$Res$_z$ $y^{-a_i-1}z^{-b_j-1} \frac{1}{y - z} \exp(\sum_{r>0} (t_r + c_{r,a_i+\frac{1}{2}})y^r - ((t_r + d_{r,-b_j-\frac{1}{2}})z^r) =$

Res$_y$Res$_z$ $\sum_{p,q,r=0}^{\infty} y^{p-r-a_i-2}z^{q+r-b_j-1} s_p(t + c_{a_i+\frac{1}{2}}) s_q(\psi^+(t + d_{b_j-\frac{1}{2}}) =$

$$\sum_{r=0}^{\infty} s_{r+a_i+1}(t + c_{a_i+1}) s_{b_j-r}(\psi^+(t + d_{b_j-\frac{1}{2}}) =$$

$$(-1)^{b_j} \chi_{a_i,b_j}(t + c_{a_i+\frac{1}{2}}; t + d_{b_j-\frac{1}{2}}).$$

Hence the tau-function $\sigma(R(A) f_\lambda)$ is equal to (12), where we replace the constants $c_i$ (resp. $d_i$) by $c_{a_i+\frac{1}{2}}$ (resp. $d_{b_j-\frac{1}{2}}$).

Note that the constants $d_{m,-b_j-\frac{1}{2}}$ that appear in $\chi_{a_i,b_j}(t + c_{a_i+\frac{1}{2}}; t + d_{b_j-\frac{1}{2}})$ are $d_{1,-b_j-\frac{1}{2}}, \ldots, d_{b_j,-b_j-\frac{1}{2}}$, and the $d_{m,-b_j-\frac{1}{2}}$ with $m > b_j$ do not appear. However, looking at the restriction (51), we see that the first dependence of the constants $d_{m,-b_j-\frac{1}{2}}$ on $d_{r,-b_j-\frac{1}{2}}$ with $r < m$, and on the $c_{n,a_i+\frac{1}{2}}$ with $i = 1, \ldots, k$, $n = 1, 2, \ldots$, is for $m = a_k + b_j + 1 > b_j$. But since the only coefficients that appear in the tau-function are the $d_{m,-b_j-\frac{1}{2}}$ with $m \leq b_j$, the restriction (51) is void.

If we look at which elementary Schur polynomials appear in $\chi_{a_i,b_j}(t; t')$ we see that the constants $c_{n_j}$ (resp. $d_{n_j}$) with $n > a_j + b_i + 1 = \lambda_j - j + \ell$ (resp. $n > b_j$) do not appear.  

\[ \square \]

### 4 More general tau-functions and the wave function of the KP hierarchy

Following [13], we can construct generating functions of tau-functions using the Jacobi-Trudi formalism. Namely, we consider, instead of $R(A) f_\lambda$, the element

$$\psi^+(z_1)\psi^+(z_2)\ldots\psi^+(z_\ell)\psi^-_{\frac{1}{2}-\ell}\psi^-_{\frac{1}{2}-\ell}\ldots\psi^-_{\frac{1}{2}} |0\rangle,$$

where we assume $|z_i| > |z_{i+1}|$ for all $i = 1, \ldots, \ell - 1$. Let

$$T(z_1, \ldots, z_\ell) = \prod_{1 \leq j < k \leq \ell} (z_j - z_k) \exp \left( \sum_{j=1}^{\ell} \sum_{n=1}^{\infty} t_n z_j^n \right)$$

$$= \det \left( z_i^{j-1} \exp \left( \sum_{n=1}^{\infty} t_n z_i^n \right) \right)_{1 \leq i, j \leq \ell}.$$  

(52)
Then the corresponding tau-function is equal to

$$\langle 0 | (\exp H(t)) \psi^+(z_1) \cdots \psi^+(z_\ell) |-\ell \rangle = z_1^{-\ell} \cdots z_\ell^{-\ell} T(z_1, \ldots, z_\ell).$$

(53)

If we now take consecutive residues of $T(z_1, \ldots, z_\ell) \prod_{i=1}^\ell a_i(z_i)$, where $a_i(z)$ are some Laurent series in $z$, we obtain a tau-function:

$$\tau(t) = \text{Res}_{z_1} \cdots \text{Res}_{z_\ell} T(z_1, \ldots, z_\ell) \prod_{i=1}^\ell a_i(z_i).$$

(54)

These expressions are not polynomial in general. To obtain the polynomial tau-function of the previous section, we take

$$a_j(z) = z^{j-\ell-\lambda_j-1} \exp(\sum_{i=1}^\infty c_{i,\lambda_j-j+i/2} z^i).$$

(55)

But $T(y_1, \ldots, y_\ell)$ is also a tau-function, for this one chooses $a_i(z_i) = \delta(z_i - y_i)$. One can even construct the wave function $\langle 55 \rangle$ in this way. Assume $\tau(t)$ is given by $\langle 54 \rangle$, then

$$\tau(t) w^+(t, z) = \langle 1 | (\exp H(t)) \psi^+(z) w_1^+ \cdots w_\ell^+ | -\ell \rangle,$n

(56)

where, by $\langle 17 \rangle$, $w_j^+ = \text{Res}_{z_j} a_j(z_j) \psi^+(z_j)$, so that

$$\langle 1 | (\exp H(t)) \psi^+(z) \psi^+(z_1) \cdots \psi^+(z_\ell) | -\ell \rangle = z^{-\ell} z_1^{-\ell} \cdots z_\ell^{-\ell} T(z, z_1, \ldots, z_\ell).$$

Thus

$$\tau(t) w^+(t, z) = z^{-\ell} \text{Res}_{z_1} \cdots \text{Res}_{z_\ell} T(z, z_1, \ldots, z_\ell) \prod_{i=1}^\ell a_i(z_i),$$

(57)

and, using $\langle 54 \rangle$, the wave function is equal to

$$w^+(t, z) = z^{-\ell} \frac{\text{Res}_{z_1} \cdots \text{Res}_{z_\ell} T(z, z_1, \ldots, z_\ell) \prod_{i=1}^\ell a_i(z_i)}{\text{Res}_{z_1} \cdots \text{Res}_{z_\ell} T(z_1, \ldots, z_\ell) \prod_{i=1}^\ell a_i(z_i)}.$$

(58)

Taking all $a_j(z)$ as in $\langle 55 \rangle$, with $c_{\lambda_j-j+i/2}$ replaced by $c_j$, we obtain, as denominator of $w^+(t, z)$, the polynomial $\tau_\lambda(t)$ given by $\langle 41 \rangle$. Using $\langle 53 \rangle$ and $\langle 56 \rangle$, by Wick’s theorem we obtain the numerator of $\langle 58 \rangle$. Thus

$$w^+(t, z) = \frac{1}{\tau_\lambda(t)} \det \left( \begin{array}{cccc} e^{\sum_n t_n z^n} & z_1^{-\ell} e^{\sum_n t_n z^n} & \cdots & z_\ell^{-\ell} e^{\sum_n t_n z^n} \\ s_{\lambda_1-1}(t + c_1) & s_{\lambda_1}(t + c_1) & \cdots & s_{\lambda_1+\ell}(t + c_1) \\ s_{\lambda_2-2}(t + c_2) & s_{\lambda_2-1}(t + c_2) & \cdots & s_{\lambda_2+\ell-1}(t + c_1) \\ \vdots & \vdots & \ddots & \vdots \\ s_{\lambda_\ell-\ell}(t + c_\ell) & s_{\lambda_\ell-\ell+1}(t + c_\ell) & \cdots & s_{\lambda_\ell}(t + c_\ell) \end{array} \right).$$

(59)

We call this the Jacobi-Trudi formula for the wave function related to a polynomial tau-functions. Note that this implies that $\tau(t) w^+(t, z)$ is also a tau-function.
We now want to do a similar thing using the Giambelli formalism. For this we consider for \(|y_i| > |y_{i+1}|, |z_i| > |z_{i+1}|\) for all \(i = 1, \ldots, k - 1\), and \(|y_i| > |z_j|\) for \(1 \leq i, j \leq k\). Consider the element

\[
\psi^+(y_1)\psi^+(y_2) \ldots \psi^+(y_k)\psi^-(z_1)\psi^-(z_2) \ldots \psi^-(z_k)|0\rangle.
\]

We want to calculate the corresponding tau-function, i.e.

\[
\langle 0 | \exp((H(t))\psi^+(y_1)\psi^+(y_2) \ldots \psi^+(y_k)\psi^-(z_1)\psi^-(z_2) \ldots \psi^-(z_k)|0\rangle.
\]

Using Wick’s theorem, since we have fermions, we obtain a Pfaffian,

\[
Pf \begin{pmatrix} 0 & \langle 0 | \exp((H(t))\psi^+(y_i)\psi^-(z_j)|0\rangle)_{ij} \\ -\langle 0 | \exp((H(t))\psi^+(y_j)\psi^-(z_i)|0\rangle)_{ij} & 0 \end{pmatrix}_{1 \leq i, j \leq k},
\]

which is equal, up to sign, to

\[
S(y_1, \ldots, y_k; z_1, \ldots, z_k) = \det \left( \frac{\exp \left( \sum_{n=1}^{\infty} t_n (y^n_i - z^n_j) \right)}{y_i - z_j} \right)_{1 \leq i, j \leq k}. \tag{60}
\]

To obtain the polynomial tau-functions of the previous section, we take

\[
a_i(y) = y^{-a_i-1} \exp \left( \sum_{r=1}^{\infty} c_{r, a_i+\frac{1}{2}} y^r \right), \quad b_i(z) = z^{-b_i-1} \exp \left( -\sum_{r=1}^{\infty} d_{r, -b_i-\frac{1}{2}} z^r \right), \tag{61}
\]

and let

\[
\tau(t) = \text{Res}_{y_1} \cdots \text{Res}_{y_k} \text{Res}_{z_1} \cdots \text{Res}_{z_k} S(y_1, \ldots, y_k; z_1, \ldots, z_k) \prod_{i=1}^{k} a_i(y_i) b_j(z_j). \tag{62}
\]

The question we want to solve is: can we also express the wave function in this way using formula (60)? Recall that the wave function, multiplied by \(\tau(t)\) can be calculated, by multiplying \(\sigma^{-1}(\tau(t))\) by \(\psi^+(z)\) and then calculating \(\sigma\) of this. In other words

\[
\tau(t)w^+(t, z) = \langle -1 | (\exp H(t))\psi^+(z)R(G)|0\rangle.
\]
So we to calculate, for $|z| > |y_1| > \ldots > |y_k| > |z_1| > \ldots > |z_k|$: 

\begin{align*}
\langle 1 | (\exp H(t)) \psi^+(z) \psi^+(y_1) \psi^+(y_2) \ldots \psi^+(y_k) \psi^-(z_1) \psi^-(z_2) \ldots \psi^-(z_k) | 0 \rangle \\
= \langle 0 | \psi^-_t (\exp H(t)) \psi^+(z) \psi^+(y_1) \psi^+(y_2) \ldots \psi^+(y_k) \psi^-(z_1) \psi^-(z_2) \ldots \psi^-(z_k) | 0 \rangle \\
= \exp \left( \sum_{n=1}^{\infty} t_n z^n \right) \exp \left( \sum_{i=1}^{k} \sum_{n=1}^{\infty} t_n (y_i^n - z_i^n) \right) \times \\
\langle 0 | \psi^-_t \psi^+(z) \psi^+(y_1) \psi^+(y_2) \ldots \psi^+(y_k) \psi^-(z_1) \psi^-(z_2) \ldots \psi^-(z_k) | 0 \rangle \\
= \pm \det \left( 1 + \frac{1}{z - \frac{1}{y_1 - 1}} \cdot \frac{1}{z - \frac{1}{y_1 - 2}} \cdot \ldots \cdot \frac{1}{z - \frac{1}{y_1 - k}} \right) \exp \left( \sum_{i=1}^{k} \sum_{n=1}^{\infty} t_n (y_i^n - z_i^n) \right) \times \\
\langle 0 | \psi^-_t \psi^+(z) \psi^+(y_1) \psi^+(y_2) \ldots \psi^+(y_k) \psi^-(z_1) \psi^-(z_2) \ldots \psi^-(z_k) | 0 \rangle \\
= \pm z_0 S(z, y_1, \ldots, y_k; z_0, z_1, \ldots, z_k) |_{z_0 = \infty} \quad \text{for} \ |z_0| > |z|.
\end{align*}

In the 3th equality, we used Wick’s theorem and the observation that the Pfaffian, we obtain in this way, is of the form

$$
Pf \begin{pmatrix}
0 & A \\
A^T & 0
\end{pmatrix},
$$

and is equal, up to a sign, to det $A$. Hence, the wave function is equal to

$$
w^+(t, z) = -\frac{\text{Res}_{y_1} \cdots \text{Res}_{y_k} \text{Res}_{z_1} \cdots \text{Res}_{z_k} z_0 S(z, y_1, \ldots, y_k; z_0, z_1, \ldots, z_k) |_{z_0 = \infty} \prod_{i=1}^{k} \alpha_i(y_i) b_j(z_j)}{	ext{Res}_{y_1} \cdots \text{Res}_{y_k} \text{Res}_{z_1} \cdots \text{Res}_{z_k} S(y_1, \ldots, y_k; z_1, \ldots, z_k) \prod_{i=1}^{k} \alpha_i(y_i) b_j(z_j)} \\
= \frac{1}{\tau(t)} \exp \left( \sum_{n=1}^{\infty} t_n z^n \right) \times \\
\left( 1 \cdot \chi(0 | b_1) ([z^{-1}] | t + d_1) \cdot \chi(0 | b_2) ([z^{-1}] | t + d_2) \cdots \chi(0 | b_k) ([z^{-1}] | t + d_k) \\
\frac{s_{a_1} (t + c_1)}{\chi(a_1 | b_1) (t + c_1; t + d_1) \cdot \chi(a_1 | b_2) (t + c_1; t + d_2) \cdots \chi(a_1 | b_k) (t + c_1; t + d_k)} \\
\vdots \\
\frac{s_{a_k} (t + c_k)}{\chi(a_k | b_1) (t + c_k; t + d_1) \cdot \chi(a_k | b_2) (t + c_k; t + d_2) \cdots \chi(a_k | b_k) (t + c_k; t + d_k)}
\right),
$$

(63)

Now, taking $a_i(z)$ and $b_i(z)$ as in (62), thus replacing $c_{a_i} + \frac{1}{b_i}$ by $c_i$ and $d_{a_i} - \frac{1}{b_i}$ by $d_i$, we obtain the wave function corresponding to the tau-function given by (62):

$$
w^+(t, z) = \frac{1}{\tau(t)} \exp \left( \sum_{n=1}^{\infty} t_n z^n \right) \times \\
\left( 1 \cdot \chi(0 | b_1) ([z^{-1}] | t + d_1) \cdot \chi(0 | b_2) ([z^{-1}] | t + d_2) \cdots \chi(0 | b_k) ([z^{-1}] | t + d_k) \\
\frac{s_{a_1} (t + c_1)}{\chi(a_1 | b_1) (t + c_1; t + d_1) \cdot \chi(a_1 | b_2) (t + c_1; t + d_2) \cdots \chi(a_1 | b_k) (t + c_1; t + d_k)} \\
\vdots \\
\frac{s_{a_k} (t + c_k)}{\chi(a_k | b_1) (t + c_k; t + d_1) \cdot \chi(a_k | b_2) (t + c_k; t + d_2) \cdots \chi(a_k | b_k) (t + c_k; t + d_k)}
\right),
$$

(64)
Here \( z^{-1} = (\frac{z^{-1}}{1}, \frac{z^{-2}}{2}, \frac{z^{-3}}{3}, \ldots) \), \( \chi_{(a|b)} \) is defined by (19), and \( \chi_{(0|b)}([z^{-1}]; t) = (-1)^b \sum_{j=0}^b z^j s_{b-j}(-t) \).

## 5 The formulation of the CKP hierarchy

The group \( SP_\infty \), the corresponding Lie algebra \( sp_\infty \) and its central extension \( c_\infty \) can be defined using the following bilinear form on \( \mathbb{C}^\infty \), see e.g. [1], Section 7.11:

\[
(e_i, e_j)_C = (-1)^{i+j} \delta_{i,-j}.
\]

Then

\[
SP_\infty = \{ G \in GL_\infty | (G(v), G(w))_C = (v, w)_C \text{ for all } v, w \in \mathbb{C}^\infty \},
\]

\[
sp_\infty = \{ g \in gl_\infty | (g(v), w)_C + (v, g(w))_C = 0 \text{ for all } v, w \in \mathbb{C}^\infty \},
\]

\[
c_\infty = \{ g + \lambda K \in a_\infty | (g(v), w)_C + (v, g(w))_C = 0 \text{ for all } v, w \in \mathbb{C}^\infty \},
\]

The elements \( C_{jk} = E_{-j,k} - (-1)^{j+k} E_{-k,j} \), with \( j \geq k \) form a basis of \( sp_\infty \).

Note that

\[
r(C_{jk}) = \psi_j^+ \psi_k^- - (-1)^{j+k} \psi_k^+ \psi_j^- \text{ and } \hat{r}(C_{jk}) =: \psi_j^+ \psi_k^- + (-1)^{j+k} : \psi_k^+ \psi_j^- :.
\]

This suggests to define an automorphism of the Clifford algebra \( C\ell \):

\[
\iota_C(\psi_j^+) = (-1)^{j+1} \psi_j^+.
\]

This induces via \( \hat{r} \) and the observation that \( \iota_C(\psi_j^+ \psi_k^-) = (-1)^{j+k} : \psi_j^+ \psi_k^- : = -(-1)^{j+k} : \psi_k^+ \psi_j^- : \) the following involution on \( a_\infty \):

\[
\iota_C(E_{-j,k}) = -(-1)^{j+k} E_{-k,j}, \quad \iota_C(K) = K,
\]

so that

\[
c_\infty = \{ g + \lambda K \in a_\infty | \iota_C(g) = g \}.
\]

Let us study this automorphism and its consequences a bit better. First of all \( \iota_C(\text{Ann}_\pm \{0\}) = \text{Ann}_\mp \{0\} \) and thus \( \iota_C(\text{Ann} \{0\}) = \text{Ann} \{0\} \), which makes it possible to define the automorphism also on the module \( F \), namely define \( \iota_C(\{0\}) = \{0\} \), and the rest is induced by (66). Since \( \iota_C(\psi^\pm(z)) = \pm \psi^\mp(-z) \), we see that \( \iota_C(F(m)) = F(-m) \) and we deduce that

\[
\iota_C(\alpha(z)) = \iota_C(\psi^+(z))\psi^-(z) =: \psi^+(z)\psi^-(z) =: \alpha(-z),
\]

and hence that \( \iota_C(\alpha_k) = -(-1)^k \alpha_k \). Moreover, \( \iota_C(\{m\}) = (-1)^{\frac{m(m-1)}{2}} \{ -m \} \). Using the bosonization \( \sigma : F \to B \), the automorphism \( \iota_C \) defines an automorphism, which we also denote by \( \iota_C \), of \( B \). Clearly \( \iota_C(1) = 1 \), and we get that the operators on \( B \) satisfy

\[
\iota_C(t_i) = -(-1)^i t_i, \quad \iota_C(\frac{\partial}{\partial t_i}) = -(-1)^i \frac{\partial}{\partial t_i}, \quad \iota_C(q \frac{\partial}{\partial q}) = -q \frac{\partial}{\partial q}, \quad \iota_C(g) = q^{-1}(-1)^{\frac{q}{2} \frac{\partial}{\partial q}}.
\]
In particular, \( F^{(0)} \) is \( \iota_C \) invariant, and

\[
F_C^{(0)} = \{ f \in F^{(0)} | \iota_C(f) = f \}
\]

is an invariant subspace for the representation \( \hat{\tau} \), restricted to \( c_\infty \). Hence we want to look at \( f \in \mathcal{O}_0 \) that satisfies the condition \( \iota_C(f) = f \). Let \( v^\pm \in \text{Ann}_f \) for such an \( f \), then \( v^\pm f = 0 \) and thus \( \iota_C(v^\pm f) = \iota_C(v^\pm) f = 0 \). Hence \( \iota_C(\text{Ann}_f) = \text{Ann}_f \). This makes it possible to define a skew-symmetric form on \( \Psi \), given by \( \omega(v, w) = (v, \iota_C(w)) \); more explicitly (cf. (65))

\[
\omega(\psi_i^+, \psi_j^+) = (\psi_i^+,-(1)^{i\pm\frac{1}{2}}\psi_j^+) = (1)^{i\pm\frac{1}{2}}\delta_{i,-j}, \quad \omega(\psi_i^+, \psi_j^-) = 0. \quad (67)
\]

So we find that \( \text{Ann}_f \) is a maximal isotropic subspace of \( \Psi = \Psi^+ \oplus \Psi^- \), with respect to both the symmetric bilinear form \( (\cdot, \cdot) \) and the skew-symmetric \( \omega(\cdot, \cdot) \). Since \( f = G|0\rangle \in F^{(0)}_C \), we find that \( \text{Ann}_f|0\rangle \subset \Psi^\pm \) is maximal isotropic, and therefore also \( \text{Ann}_f \subset \Psi^\pm \) is maximal isotropic. Thus \( \text{Ann}_f \) (resp. \( \text{Ann}_f \)) is a Lagrangian subspace of \( \Psi \) (resp. \( \Psi^\pm \)).

Moreover, when applying the bosonization \( \sigma \) to such an element \( f \), we obtain a tau-function \( \tau(t) \), which satisfies

\[
\tau(t_1, t_2, t_3, t_4, \ldots) = \iota_C(\tau(t_1, t_2, t_3, t_4, \ldots)) = \tau(t_1, -t_2, t_3, -t_4, \ldots). \quad (68)
\]

**Remark 7** Using formula (e) of Lemma 6 and (66), we find that

\[
\iota_C(\chi_{(ij)}(t, t)) = (1, t)(\exp H(t))\iota_C(\psi_i^+, \psi_j^- |0\rangle) = (1, t)(\exp H(t))\psi_i^+, \psi_j^- |0\rangle = \chi_{(ij)}(t, t).
\]

Hence, using the Giambelli formula (18), we find that \( \iota_C(s_\lambda(t)) = s_{\lambda'}(t) \), where \( \lambda' \) is the conjugate partition of \( \lambda \). Thus the only Schur polynomials that are invariant under \( \iota_C \) are the ones for which \( \lambda \) is self-conjugate.

Next apply \( 1 \otimes \iota_C \) to (32) for \( m = 0 \), and assume that \( \iota_C(f_0) = (f_0) \). Then equation (32) turns into

\[
\text{Res}_z \psi^+(z)f_0 \otimes \psi^+(-z)f_0 = 0, \quad f_0 \in F^{(0)}_C. \quad (69)
\]

Letting \( \tau(t) = \sigma(f_0) \), gives, by (65), the following CKP hierarchy of equations on the tau-function:

\[
\text{Res}_z=0 \exp \left( \sum_{i=1}^{\infty} (t_j + (1)^{i}t'_j) z^i \right) \exp \left( -\sum_{i=1}^{\infty} \left( \frac{\partial}{\partial t_i} + (1)^{i} \frac{\partial}{\partial t'_i} \right) \frac{z^{-i}}{i} \right) \tau(t)\tau(t') dz = 0. \quad (70)
\]

A polynomial KdV tau-function is a KP tau-function that is independent of the even times \( t_{2j} \) for \( j = 1, 2, \ldots \), hence the automorphism \( \iota_C \) fixes this tau-function and we have the following
Corollary 8 A polynomial KdV tau-function is also a CKP tau-function.

This corollary does not hold for the non-polynomial KdV tau-functions, which can depend exponentially on the even times; such a tau-function is not fixed by \( \iota_C \).

Let \( a(z) = \sum_{i=-M}^N a_i z^{i-1} \) and \( v^\pm = \text{Res}_z a(z) \psi^\pm (\pm z) = \sum_{i=-M}^N (\pm 1)^i c_i \psi_{i+\frac{1}{2}}^\pm \).

Since \( \iota_C(v^\pm) = \pm v^\mp \), we have \( \iota_C(v^+ v^-) = -v^-v^+ = -(v^-, v^+) + v^+v^- = v^+v^- \), because \( (v^-, v^+) = 0 \). Using this observation, we are now ready to prove the main result of this section.

Theorem 9 (a) Let \( v^+_1 \in \Psi^+ \) for \( i = 1, \ldots, k \), then

\[
\tau^{v^+_1,\ldots,v^+_k}(t) := \langle (\exp H(t)) v^+_1 \iota_C(v^+_1) v^+_2 \iota_C(v^+_2) \cdots v^+_k \iota_C(v^+_k) \rangle
\]

is a CKP tau-function. For

\[
v^+_j = \text{Res}_z z^{-a_j-1} \exp(\sum_{i=1}^\infty c_{i,j} z^i) \psi^+(z), \quad j = 1, \ldots, k,
\]

this tau-function is, up to a sign, equal to

\[
\det \left( T_{i,j}(t) \right)_{1 \leq i, j \leq k}, \quad \text{where } T_{i,j}(t) = \begin{cases} 
\chi(a_i|a_j)(t + c_i; t + \iota_C(c_j)) & i \leq j, \\
\iota_C(\chi(a_j|a_i)(t + \iota_C(c_j); t + c_i)) & i > j,
\end{cases}
\]

for certain constants \( c_{a_j} \in \mathbb{C}, \ j = 1, \ldots, k, \ n = 1, 2, \ldots \). Here \( \chi(a|b)(s; t) \) is given by (19) \( c_j = (c_{1j}, c_{2j}, c_{3j}, \ldots, c_{a_j+a_1+1,j}) \) and \( \iota_C(c_j) = (c_{1j}, -c_{2j}, c_{3j}, \ldots, (-1)^{a_j+a_1} c_{a_j+a_1+1,j}) \) with \( j = 1, \ldots, k \).

(b) Any polynomial CKP tau-function is of the form (71) and hence, up to a constant factor, equal to (73).

(c) Any polynomial CKP tau-function is, up to a constant factor, equal to

\[
\tau(a_1,\ldots,a_k|a_1,\ldots,a_k)(t) = \det \left( \chi(a_i|a_j)(t + c_i; t + \iota_C(c_j)) \right)_{1 \leq i, j \leq k},
\]

where the constants \( c_j = (c_{1j}, c_{2j}, c_{3j}, \ldots, c_{a_j+a_1+1,j}) \) for \( 1 \leq i < j \leq k \), must satisfy the following constraints:

\[
c_{a_i+a_j+1,j} = (-1)^{a_i+a_j} c_{a_i+a_j+1}(c_i, -c_{1j}, c_{2j}, c_{3j}, \ldots, c_{a_i+a_j+1}(a_i, \ldots), c_{a_i+a_j+1}(a_{i+1}, a_{i+2}, \ldots))
\]

(d) Any polynomial CKP tau-function satisfies the following equation:

\[
\sum_{k=0}^\infty s_k(\pm 2t_e) s_{k+1} T_{\tilde{\iota}_e} \tau(t) = 0,
\]

were \( t_e = (t_2, t_4, t_6, \ldots) \) and \( \tilde{\iota}_e = (\frac{\partial}{\partial t_2}, \frac{1}{2} \frac{\partial}{\partial t_4}, \frac{1}{3} \frac{\partial}{\partial t_6}, \ldots) \).
Proof. (a) It is obvious that $\tau^{v_1^+, \ldots, v_k^+}(t)$ is a CKP tau function. Let us calculate its explicit form for the $v_j^+$ given by (72). If

$$v_j^+ = \text{Res}_z b_j(z) \psi^+(z),$$

then $\iota_C(v_j^+) = \text{Res}_z b_j(z) \psi^-(z) = -\text{Res}_z b_j(-z) \psi^-(z),$

where we take

$$b_j(z) = z^{-a_j-1} \exp(\sum_{i=1}^{\infty} c_{i,j} z^i).$$

Then

$$b_j(-z) = (-z)^{-a_j-1} \exp(-\sum_{i=1}^{\infty} (-1)^i c_{i,j} z^i)$$

$$= (-1)^{a_j+1} z^{-a_j-1} \exp(-\sum_{i=1}^{\infty} \iota_C(c_{i,j}) z^i).$$

Since $\text{Res}_z b_j(z) \psi^\pm(z) = \psi^\pm|_{a_j}^{a_j} + \sum_{i>a_j} g_i \psi_i^\pm$ for some $g_i \in \mathbb{C}$ (cf. Lemma 0(e)), we find that $\lambda = (a_1, \ldots, a_k | a_1, \ldots, a_k).$ Using Wick's theorem, in the second equality below, we find that $\tau^{v_1^+, \ldots, v_k^+}(t)$, with $v_j^+$ given by (72), is equal, up to a possible sign, to

$$\langle (\exp H(t)) v_1^+ \iota_C(v_1^+) \cdots v_k + \iota_C(v_1^+) \rangle$$

$$= \text{Res}_{y_1} \text{Res}_{z_1} \cdots \text{Res}_{y_k} \text{Res}_{z_1} \langle \psi^+(y_1) \psi^-(z_1) \cdots \psi^+(y_k) \psi^-(z_k) \rangle \prod_{j=1}^{k} b_j(y_j) b_j(-z_j)$$

$$= \text{Res}_{y_1} \text{Res}_{z_1} \cdots \text{Res}_{y_k} \prod_{j=1}^{k} b_j(y_j) b_j(-z_j) \times$$

$$Pf \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
\frac{\sum_n t_n(y_1^n - z_1^n)}{y_1 - z_1} & 0 & \cdots & 0 & \frac{\sum_n t_n(y_1^n - z_k^n)}{y_1 - z_k} \\
0 & \frac{\sum_n t_n(y_2^n - z_1^n)}{y_2 - z_1} & \cdots & 0 & \frac{\sum_n t_n(y_2^n - z_k^n)}{y_2 - z_k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \frac{\sum_n t_n(y_k^n - z_k^n)}{y_k - z_k} \\
0 & 0 & \cdots & 0 & 0 \\
\end{pmatrix},$$

for $|y_1| > |z_1| > |y_2| > \cdots > |z_k|$. Next permuting rows and columns in the Pfaffian, this is equal to

$$\pm \text{Res}_{y_1} \text{Res}_{z_1} \cdots \text{Res}_{y_k} \prod_{j=1}^{k} b_j(y_j) b_j(-z_j) \det \left( \frac{\sum_n t_n(y_i^n - z_j^n)}{y_i - z_j} \right)_{1 \leq i,j \leq k}. \quad (77)$$
Using that for \( i \leq j \), thus \( |y_i| > |z_j| \), we have

\[
\text{Res}_{y_i} \text{Res}_{z_j} b_i(y_i)b_j(-z_j) \frac{e^{\sum_n t_n(y_i^n - z_j^n)}}{y_i - z_j} =
\]

\[
\text{Res}_{y_i} \text{Res}_{z_j} y_i^{-a_j-1}(-z_j)^{-a_j-1} \exp \sum_{k=1}^{\infty} c_{k,i}y_i^k \exp \sum_{l=1}^{\infty} c_{l,j}(-z_j)^l e^{\sum_n t_n(y_i^n - z_j^n)} \frac{e^{\sum_n t_n(y_i^n - z_j^n)}}{y_i - z_j} =
\]

\[
= (-1)^{a_j+1} \chi(a_{j|a_j})(t + c_i; t + \iota_C(c_j)),
\]

and that for \( i > j \), thus \( |y_i| < |z_j| \),

\[
\text{Res}_{z_j} \text{Res}_{y_i} b_i(y_i)b_j(-z_j) \frac{e^{\sum_n t_n(y_i^n - z_j^n)}}{y_i - z_j} = \pm \iota_C \chi(a_{j|a_i})(t + \iota_C(c_j); t + c_i)),
\]

we find that (77) is equal, up to some sign, to (73).

(b) Any polynomial CKP tau-function corresponds to a Lagrangian subspace
\( L \subset \Psi^+ \) with respect to \( \omega \); it is of the form

\[
L = \text{span}\{w^+_1, w^+_2, \ldots, w^+_k\} \oplus \bigoplus_{j > k} \mathbb{C} \psi^+_j.
\]

Since 0 = \( \omega(w^+_i, w^+_j) = (w^+_i, \iota_C(w^+_j)) \), we find that \( w^+_i \) and \( \iota_C(w^+_j) \) anticommute for all \( 1 \leq i, j \leq k \). Hence, the corresponding tau-function is then equal, up to a constant factor, to \( \tau_{w^+_1, \ldots, w^+_k}(t) \). This proves part (b).

(c) Let \( \tau(t) \) be a polynomial CKP tau-function, i.e. \( \tau(t) \) is a KP tau-function that satisfies (68). This \( \tau \) is invariant under \( \iota_C \). We know from Section 3 that

\[
\sigma^{-1}(\tau(t)) = f = R(A)f_\lambda \in R(U)f_\lambda, \text{ for some partition } \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) = (a_1, \ldots, a_k|b_1, \ldots, b_k) \in \text{Par}_\ell. \text{ Thus } f \text{ is of the form (cf. (46))}
\]

\[
f = w^{\lambda_1 - 1} w^{\lambda_2 - 1} \cdots w^{\lambda_\ell - 1} - \ell), \tag{78}
\]

where

\[
w^{\lambda_j - j + \frac{1}{2}} = \psi^{\lambda_j - j + \frac{1}{2}} - \lambda_j + \sum_{j + \frac{1}{2} - \lambda_j \leq i < \ell} a_{-i, \lambda_j - j + \frac{1}{2}} \psi^+_i \tag{79}
\]

and

\[
a_1 = \lambda_1 - 1 > \cdots > a_k = \lambda_k - k \geq 0 > \lambda_{k+1} - k - 1 > \cdots > \lambda_\ell - \ell. \tag{80}
\]

Recall that

\[
\iota_C(w^{\lambda_j - j + \frac{1}{2}}) = (-1)^{\lambda_j - j} \psi^{\lambda_j - j + \frac{1}{2}} - \lambda_j + \sum_{j + \frac{1}{2} - \lambda_j \leq i < \ell} (-1)^{i + \frac{1}{2}} a_{-i, \lambda_j - j + \frac{1}{2}} \psi^+_i. \tag{81}
\]
Now let us study $\text{Ann} f$, which is a maximal isotropic subspace of $\Psi$. From (78), (51), and the fact that $\iota_C(\text{Ann} f) = \text{Ann} f$, we deduce that

$$\text{Ann}_+ f = \text{span}\{w^+_{\lambda_1-\frac{1}{2}}, w^+_{\lambda_2-\frac{1}{2}}, \ldots, w^+_{\lambda_i-\frac{1}{2}}, \psi^+_{\ell_{1}+\frac{1}{2}}, \psi^+_{\ell_{2}+\frac{1}{2}}, \psi^+_{\ell_{3}+\frac{1}{2}}, \ldots\},$$

$$\text{Ann}_- f = \text{span}\{\iota_C(w^+_{\lambda_1-\frac{1}{2}}), \iota_C(w^+_{\lambda_2-\frac{1}{2}}), \ldots, \iota_C(w^+_{\lambda_i-\frac{1}{2}}), \psi^-_{\ell_{1}+\frac{1}{2}}, \psi^-_{\ell_{2}+\frac{1}{2}}, \psi^-_{\ell_{3}+\frac{1}{2}}, \ldots\}.$$

Thus $(\psi^-_i, w^+_{\lambda_j-j+\frac{1}{2}}) = 0$ for all $i > \ell$, and $j = 1, \ldots, \ell$, which means that $a_{-\ell, \lambda_j} = 0$ for all $i < -\ell$. Therefore all $\lambda_j - j \leq \ell$, for $j = 1, \ldots, \ell$. In particular $a_1 \leq \ell$.

Next, consider the element

$$g = w^+_{\lambda_1-\frac{1}{2}} \iota_C(w^+_{\lambda_1-\frac{1}{2}})w^+_{\lambda_2-\frac{1}{2}} \iota_C(w^+_{\lambda_2-\frac{1}{2}}) \cdots w^+_{\lambda_k-\frac{1}{2}} \iota_C(w^+_{\lambda_k-\frac{1}{2}}) |0\rangle, \quad (82)$$

where $k$ is determined by the Frobenius notation of $\lambda$. Note that this element is not equal to 0, because the subspace spanned by $w^+_{\lambda_1}, \iota_C(w^+_{\lambda_1}), \ldots, w^+_{\lambda_k}, \iota_C(w^+_{\lambda_k})$ is isotropic with respect to $\omega(\cdot, \cdot)$ and all elements $w^+_{\lambda_j-j+\frac{1}{2}}$ and $\iota_C(w^+_{\lambda_j-j+\frac{1}{2}})$ for $j = 1, \ldots, k$ do not annihilate $|0\rangle$. First of all consider $\psi^+_i g$ for $i > \ell$. Since $\psi^+_i \in \text{Ann}_+ f$, this element anticommutes with all $w^+_{\lambda_j-j+\ell}$ and $\iota_C(w^+_{\lambda_j-j+\ell})$ for $j = 1, \ldots, \ell$.

Thus

$$\psi^+_i g = w^+_{\lambda_1-\frac{1}{2}} \iota_C(w^+_{\lambda_1-\frac{1}{2}})w^+_{\lambda_2-\frac{1}{2}} \iota_C(w^+_{\lambda_2-\frac{1}{2}}) \cdots w^+_{\lambda_k-\frac{1}{2}} \iota_C(w^+_{\lambda_k-\frac{1}{2}}) \psi^+_i |0\rangle = 0.$$

Also $w^+_{\lambda_j-j+\frac{1}{2}}$ (resp. $\iota_C(w^+_{\lambda_j-j+\frac{1}{2}})$) anticommutes with $\iota_C(w^+_{\lambda_j-i+\frac{1}{2}})$ (resp. $w^+_{\lambda_j-i+\frac{1}{2}}$).

If $j \leq k$, then

$$w^+_{\lambda_j-j+\frac{1}{2}} = w^+_{\lambda_1-\frac{1}{2}} \iota_C(w^+_{\lambda_1-\frac{1}{2}}) \cdots w^+_{\lambda_j-j+\frac{1}{2}} \iota_C(w^+_{\lambda_j-j+\frac{1}{2}}) \cdots w^+_{\lambda_k-\frac{1}{2}} \iota_C(w^+_{\lambda_k-\frac{1}{2}}) |0\rangle = 0,$$

and similarly $\iota_C(w^+_{\lambda_j-j+\frac{1}{2}}) g = 0$ for $j \leq k$. Next, let $j > k$, then also

$$w^+_{\lambda_j-j+\frac{1}{2}} = w^+_{\lambda_1-\frac{1}{2}} \iota_C(w^+_{\lambda_1-\frac{1}{2}})w^+_{\lambda_2-\frac{1}{2}} \iota_C(w^+_{\lambda_2-\frac{1}{2}}) \cdots w^+_{\lambda_k-\frac{1}{2}} \iota_C(w^+_{\lambda_k-\frac{1}{2}}) w^+_{\lambda_j-j+\frac{1}{2}} |0\rangle = 0,$$

since $\lambda_j - j < 0$ and $w^+_{\lambda_j-j+\frac{1}{2}} = \psi^+_j - \psi^-_j + \sum_{i>j} a_{i, \lambda_j-j+\frac{1}{2}} \psi^+_i$, i.e. a linear combination of $\psi^+_i$ with $i > 0$. This also holds for $\iota_C(w^+_{\lambda_j-j+\frac{1}{2}})$ for $j > k$. Hence $\text{Ann} f = \text{Ann} g$ and $g$ must be a multiple of $f$. Therefore, $\tau(t) = \text{const} \tau w^+_{\lambda_1-\frac{1}{2}} w^+_{\lambda_2-\frac{1}{2}} \cdots w^+_{\lambda_k-\frac{1}{2}} (t)$, and $b_j = a_j$ for all $1 \leq j \leq k$. Hence $\lambda$ is self-conjugate and equal to $(a_1, \ldots, a_k | a_1, \ldots, a_k)$, in the Frobenius notation.

Note that in this construction, the span of $w^+_{\lambda_1-\frac{1}{2}}, \ldots, w^+_{\lambda_k-\frac{1}{2}}$ is isotropic with respect to $\omega(\cdot, \cdot)$. This gives some restriction on the $a_{i, \lambda_j-j+\frac{1}{2}}$. For arbitrary vectors $v^+_i, v^+_k$, as in part (a) of the theorem, the linear span of the $v^+_i$ does not have to be isotropic with respect to $\omega(\cdot, \cdot)$.
As in the previous section, we can write
\[
  w_{\lambda_j-j+\frac{1}{2}}^+ = v_{j-\frac{1}{2}-\lambda_j}^+ + \sum_{j+\frac{1}{2}-\lambda_j \leq i < \ell} a_{-i,\lambda_j-j+\frac{1}{2}} v_i^+
\]
\[
  = \text{Res}_z z^{j-\lambda_j-1} \psi^+(z) \exp\left(\sum_{i=1}^{\infty} c_{i,j} z^i\right),
\]
\[
  \iota_C(w_{\lambda_j-j+\frac{1}{2}}^+) = \text{Res}_z (-1)^{j-\lambda_j} z^{j-\lambda_j-1} \psi^-(z) \exp\left(-\sum_{i=1}^{\infty} \iota_C(c_{i,j}) z^i\right).
\]

For these isotropic \(w_{\lambda_j-j+\frac{1}{2}}^+\), we have that
\[
  \tau(t) = \langle (\exp H(t)) w_{\lambda_1-\frac{1}{2}}^+ \cdots w_{\lambda_k-\frac{1}{2}}^+ \iota_C(w_{\lambda_1-\frac{1}{2}}^+) \cdots \iota_C(w_{\lambda_k-\frac{1}{2}}^+) \rangle
\]
so we can apply Theorem \[5\] so that \(\tau(t)\) is given, up to a constant factor, by \[42\], with \(b_j = a_j = \lambda_j - j\) and \(d_j = \iota_C(c_j)\). But there are restrictions on the constants, namely, since \(\omega(w_{\lambda_i-i+\frac{1}{2}}^+, w_{\lambda_j-j+\frac{1}{2}}^+) = 0\) for \(1 \leq i < j \leq k\), we find that
\[
  0 = \omega(w_{\lambda_i-i+\frac{1}{2}}^+, w_{\lambda_j-j+\frac{1}{2}}^+)
\]
\[
  = \text{Res}_y \text{Res}_z y^{-a_i-1} z^{-a_j-1} \exp\left(\sum_{n=1}^{\infty} (c_{n,i} y^n - \iota_C(c_{n,j}) z^n)\right)(\psi^+(y), \psi^-(z))
\]
\[
  = \text{Res}_y \text{Res}_z y^{-a_i-a_j-2} \exp\left(\sum_{n=1}^{\infty} (c_{n,i} - \iota_C(c_{n,j})) y^n\right) \delta(y-z)
\]
\[
  = \text{Res}_y y^{-a_i-a_j-2} \exp\left(\sum_{n=1}^{\infty} (c_{n,i} - \iota_C(c_{n,j})) y^n\right)
\]
\[
  = s_{a_i+a_j+1}(c_i - \iota_C(c_j)).
\]

In other words, we obtain the restriction \[11\] for \(b_j = a_j\) and \(d_j = \iota_C(c_j)\), which means that we have to choose \(c_{a_i+a_j+1,j}\) for \(j > i\) as in \[15\]. (d) follows from Remark \[10\] below.

**Remark 10** Following \[7\], \[2\], we can define \(\iota_C\) on \(\Psi = \Psi^+ \oplus \Psi^-\) as \(\text{ad } \Omega\), i.e. \(\iota_C(\psi_j^\pm) = \text{ad } \Omega(\psi_j^\pm)\) for
\[
  \Omega = \frac{1}{2}(\Omega_+ + \Omega_-), \quad \text{where } \Omega_\pm = \sum_{i \in \frac{1}{2} + \mathbb{Z}} (-1)^{i+\frac{1}{2}} \psi_i^\pm \psi_{-i}^\pm.
\]

Since \(\Omega|0\) = 0, we find that a CKP element \(f \in F^{(0)}\) of the form
\[
  f = v_1^+ \iota_C(v_1^+) v_2^+ \iota_C(v_2^+) \cdots v_k^+ \iota_C(v_k^+)|0\]
satisfies \(\Omega f = 0\). Since \(\Omega_\pm f \in F^{(\pm 2)}\), we must have that.
\[
  \Omega_\pm f = 0.
\]

(84)
Now applying $\sigma$ to the above equations (84), and using that
\[
\Omega_+ = \text{Res}_z \psi^+(-z)\psi^+(z), \quad \Omega_- = \text{Res}_z \psi^-(z)\psi^-(z),
\]
we find that a polynomial CKP tau function $\tau(t)$ satisfies
\[
\text{Res}_z z \exp \left( \pm 2 \sum_{i=1}^{\infty} t_{2i} z^{2i} \right) \exp \left( \mp \sum_{i=1}^{\infty} \frac{\partial}{\partial t_{2i}} z^{-2i} \right) \tau(t) = 0,
\]
which is, in terms of the elementary Schur polynomials, equation (76).

**Example 11** To show that the constraints (75) are non-trivial, we calculate an example explicitly. The smallest example where this constraint occurs, is for $\lambda = (2,2) = (1,0)|1,0)$. We have
\[
\tau_{(1,0)|1,0};c(t) = 1/12(12c_{21}c_{22} - 12c_{31}t_1 - 6c_{21}t_1^2 + 6c_{22}t_1^2 + t_1^4 - 2c_{11}^2(t_2 + t_1) + 6c_{11}(c_{12}^2 + 2c_{22} - c_{21})t_1 + c_{12}(t_1^3 - 2c_{21} - 2t_2)) + 12c_{21}t_2 + 12c_{22}t_2 + 12t_2^2 + 3c_{11}(c_{12}^2 + 2c_{22} - t_1^2 - 2t_2) + 3c_{12}(2c_{21} + t_1^2 + 2t_2) - 12t_1t_3 - 4c_{12}(3c_{31} - t_1^3 + 3t_3)).
\]
Calculating $\tau_{(1,0)|1,0};c(t) - \tau_{(1,0)|1,0};c(t_C(t))$, we find that
\[
\tau_{(1,0)|1,0};c(t) - \tau_{(1,0)|1,0};c(t_C(t)) = (c_{11}^2 - 2c_{11}c_{12} + c_{12}^2 + 2(c_{21} + c_{22}))t_2,
\]
and this term has to be 0 for a CKP tau-function. This happens if we choose
\[
c_{22} = -\left(\frac{1}{2}c_{11}^2 - c_{11}c_{12} + \frac{1}{2}c_{12}^2 + c_{21}\right) = -s_2(c_{11} - c_{12}, c_{21})
\]
which is exactly the constraint (75).

Let $\tau(t)$ be as in (73). We find in a similar way as we obtained (64), that the corresponding CKP wave function is given by
\[
w^+(t, z) = \frac{1}{\tau(t)} \exp \left( \sum_{n=1}^{\infty} t_n z^n \right) \times
\[
\det \begin{pmatrix}
1 & \chi_{(0)a_1}([z^{-1}]; t + \iota C(c_1)) & \cdots & \chi_{(0)a_k}([z^{-1}]; t + \iota C(c_k)) \\
\text{Res}_z(t + c_1) & T_{11}(t) & \cdots & T_{1k}(t) \\
\text{Res}_z(t + c_2) & T_{21}(t) & \cdots & T_{2k}(t) \\
\vdots & \vdots & & \vdots \\
\text{Res}_z(t + c_k) & T_{k1}(t) & \cdots & T_{kk}(t)
\end{pmatrix},
\]
where the $T_{ij}(t)$ are given by (73).

It follows from the bilinear identity (35) on the tau-function, that the wave function satisfies
\[
\text{Res}_z w^+(t, z) w^+(t', -z) = 0.
\]
Writing \( w^+(t, z) \) as in [40], we obtain as in [86]:

\[
0 = \text{Res}_z P^+(t, z)P^+(t', -z) \exp \left( \sum_{n=1}^{\infty} t_n z^n + t'_n (-z)^n \right)
\]

\[
= \text{Res}_z P^+(t, z)P^+(t' - z) \exp \left( \sum_{n=1}^{\infty} (t_n - \nu_C(t'_n))z^n \right),
\]

from which we deduce that \( P^+(t, \partial)^{-1} = P^+(\nu_C(t), \partial)^* \). This implies that, besides the Lax equations (14), the pseudodifferential operator \( L(t, \partial) = P^+(t, \partial) \circ \partial \circ P^+(t, \partial)^{-1} \) satisfies the condition

\[
L(t, \partial)^* = (P^+(t, \partial) \circ \partial \circ P^+(\nu_C(t), \partial)^*)^* = -L(\nu_C(t), \partial).
\]

It follows from (87) that the pseudodifferential operator \( L(t, \partial) \) is skew-adjoint if and only if \( L(t, \partial) \) satisfies the Krichever-Zabrodin condition [14]

\[
\left[ \frac{\partial}{\partial t_{2n}}, L(t, \partial) \right] \bigg|_{t_{2n} = 0} = 0, \quad n = 1, 2, \ldots;
\]

in particular, if \( L(t, \partial) = L(t_o, \partial) \), where \( t_o = (t_1, 0, t_3, 0, t_5, \ldots) \). In the latter case the hierarchy (14) of Lax-Sato equations becomes

\[
\frac{\partial L(t_o, \partial)}{\partial t_k} = [(L(t_o, \partial)^k)_+, L(t_o, \partial)], \quad k = 1, 3, 5, \ldots,
\]

### 6 Reductions of the CKP hierarchy

Let \( n \) be a positive integer \( \geq 2 \). We want to study \( n \)-reductions of the CKP hierarchy, which means that we restrict \( c_\infty \) to the Lie algebra \( c_\infty \cap \hat{gl}_n \), where \( \hat{gl}_n = gl_n(\mathbb{C}[t, t^{-1}]) \oplus K \) is the subalgebra of \( a_\infty \), consisting of all \( n \)-periodic matrices \( g = (g_{ij})_{i,j \in \frac{1}{2} + \mathbb{Z}} \), i.e. \( g_{i+n,j+n} = g_{ij} \), together with \( K \). This intersection is equal to the affine Lie algebra \( sp_n \) if \( n \) is even, and to the twisted affine Lie algebra \( \hat{gl}_n^{(2)} \) if \( n \) is odd (see [8], page 977).

Let \( G \) be an element in the group \( G_n \), corresponding to this affine Lie algebra. Then \( G\psi_j^+G^{-1} = \sum_{j \in \frac{1}{2} + \mathbb{Z}} a_{ij} \psi_i^+ \), where \( (a_{ij})_{i,j \in \frac{1}{2} + \mathbb{Z}} \) is periodic, i.e. \( a_{i+n,j+n} = a_{ij} \), and satisfies

\[
(-1)^{j+\frac{3}{2}} \delta_{j,-\ell} = \omega(\psi_j^+, \psi_{\ell}^+) = \sum_{i,k} a_{ij} a_{k\ell} \omega(\psi_i^+, \psi_k^+).
\]

Hence \( \sum_{i \in \frac{1}{2} + \mathbb{Z}} (-1)^{-i} a_{ij} a_{-i,\ell} = \) and thus also \( \sum_{j \in \frac{1}{2} + \mathbb{Z}} (-1)^{j+\frac{1}{2}} a_{ij} a_{\ell,-j} = \)
$(-1)^{i+\frac{1}{2}\delta_{i,-\ell}}$. Now let $p$ be an arbitrary positive integer, then

\[
(G \otimes G) \sum_{j \in \frac{1}{i} + \mathbb{Z}} \psi_j^+ \otimes (-1)^{j-pn-\frac{1}{2}}\psi_{pn-j}^+
\]

\[
= \sum_{j \in \frac{1}{i} + \mathbb{Z}} (-1)^{j-pn-\frac{1}{2}}(G\psi_j^+ G^{-1})G \otimes (G\psi_{pn-j}^+ G^{-1})G
\]

\[
= \sum_{i,j,k \in \frac{1}{i} + \mathbb{Z}} (-1)^{j-pn-\frac{1}{2}}a_{ij}a_{k,pn-j}\psi_i^+ G \otimes \psi_k^+ G
\]

\[
= (-)^{pm-1} \sum_{i,k \in \frac{1}{i} + \mathbb{Z}} (-1)^{i+\frac{1}{2}\delta_{i,pn-k}}\psi_i^+ G \otimes \psi_k^+ G
\]

\[
= \sum_{i,k \in \frac{1}{i} + \mathbb{Z}} (-1)^{i-pn-\frac{1}{2}}\psi_i^+ G \otimes \psi_{pn-i}^+ G.
\]

Thus $G \otimes G$ commutes with the operator

\[
\sum_{j \in \frac{1}{i} + \mathbb{Z}} \psi_j^+ \otimes (-1)^{j-pn-\frac{1}{2}}\psi_{pn-j}^+.
\]

Since

\[
\sum_{j \in \frac{1}{i} + \mathbb{Z}} \psi_j^+ |0\rangle \otimes (-1)^{j-pn-\frac{1}{2}}\psi_{pn-j}^+ |0\rangle = 0,
\]

we find that

\[
\sum_{j \in \frac{1}{i} + \mathbb{Z}} \psi_j^+ G|0\rangle \otimes (-1)^{j-pn-\frac{1}{2}}\psi_{pn-j}^+ G|0\rangle = 0.
\]

Stated differently, we find that

\[
\text{Res}_z z^m \psi^+(z)f \otimes \psi^+(-z)f = 0 \quad \text{for} \quad f \in \mathcal{G}_n|0\rangle, \quad p = 0, 1, \ldots.
\]

The case $p = 0$ is the CKP hierarchy. Using the isomorphism $\sigma$, we obtain the $n$-reduced CKP hierarchy of bilinear equations on the tau-function ($p = 0, 1, 2, \ldots$):

\[
\text{Res}_{z=0} z^m \exp \left( \sum_{i} (t_i + (-1)^i t'_i) z^i \right) \exp \left( -\sum_{i} \left( \frac{\partial}{\partial t_i} + (-1)^i \frac{\partial}{\partial t'_i} \right) \frac{z^{-i}}{i} \right) \tau(t)\tau(t') = 0.
\]

(89)

**Remark 12**

(a) Since $\tau(t)$ is also an $n$-reduced KP tau function we find that $\frac{\partial \tau(t)}{\partial t_{pn}} = \text{const} \tau(t)$, which gives for polynomial tau-functions that $\frac{\partial \tau(t)}{\partial t_{pn}} = 0$.

(b) Note that the above equations (89) on the tau-function induce the following equations on the wave function $w^+(t,z)$:

\[
\text{Res}_z z^m w^+(t,z)w^+(t',-z) = 0, \quad p = 1, 2, \ldots.
\]
Taking $p = 1$, one deduces that a reduced Lax operator $\mathcal{L}$, which we define as $\mathcal{L}(t, \partial) = L(t, \partial)^n$, is an $n$-th order monic differential operator, satisfying

$$\frac{\partial \mathcal{L}(t, \partial)}{\partial t_k} = [\mathcal{L}(t, \partial)^n, \mathcal{L}(t, \partial)] \quad \text{and} \quad \mathcal{L}(t, \partial)^* = (-1)^n \mathcal{L}(iC(t), \partial), \quad (90)$$

where $k$ is a positive integer. If $k$ is divisible by $n$, the right-hand side of the first equation is 0. The second equation follows from $[\mathcal{S}_n]$. We prove this along the lines of the proof of Theorem 5 and Theorem 9, assuming that $\lambda \in \text{Par}_\ell$ is $n$-periodic and self-conjugate, lead to the following two conditions on the set $A^{(n)}_\lambda := \{a_1, a_2, \ldots, a_k\}$:

- If $a_j \in A^{(n)}_\lambda$, then either $a_j - n \in A^{(n)}_\lambda$ or $a_j - n < 0$.

- For all $a_i, a_j \in A^{(n)}_\lambda$, the integer $a_i + a_j + 1$ is not a multiple of $n$.

Note that if $a_i + a_j + 1 = kn$, for some positive integer $k$, then $a_i \in V^{(n)}_\lambda$ but $a_i - kn = -a_j - 1 \notin V^{(n)}_\lambda$.

We prove this along the lines of the proof of Theorem 5 and Theorem 9, assuming that $\lambda \in \text{Par}_\ell$ is $n$-periodic and self-conjugate. Then $\lambda = (a_1, \ldots, a_k|a_1, \ldots, a_k)$ in the Frobenius notation, and

$$f_\lambda = \pm \prod_{i=1}^{k} \psi_{-a_i - 1/2}^+ \psi_{-a_i - 1/2}^C(\psi_{-a_i - 1/2}^+ | 0).$$
Now let $G \in U$, such that $G\psi^+_jG^{-1} = \sum_i a_{ij}\psi^+_i$, where $A = (a_{ij})_{i,j \in \frac{1}{2}+\mathbb{Z}}$, satisfying the condition that $\sum_{i,j \in \frac{1}{2}+\mathbb{Z}} (-1)^{j-i}a_{i,j}a_{-i,-j} = (-1)^{j-i}\delta_{j,-\ell}$ (and thus also $\sum_{j \in \frac{1}{2}+\mathbb{Z}} (-1)^{j+\frac{1}{2}}a_{ij}a_{\ell,-j} = (-1)^{j+\frac{1}{2}}\delta_{i,-\ell}$) and $a_{i+n,j+n} = a_{ij}$. The first condition gives that the vectors

$$w^+_{-a_j-\frac{1}{2}} = G\psi^+_{-a_j-\frac{1}{2}}G^{-1} = \sum_{i \neq -a_j} a_{i,-a_j-\frac{1}{2}}\psi_i^+$$

for $j = 1, \ldots, k$ form an isotropic subspace of $\Psi^+$ with respect to $\omega(\cdot, \cdot)$. Next, we investigate

$$Gf_\lambda = w^+_{-a_1-\frac{1}{2}} \cdots w^+_{-a_k-\frac{1}{2}} t_C(w^+_{-a_1-\frac{1}{2}}) \cdots t_C(w^+_{-a_k-\frac{1}{2}})[0].$$

The $n$-periodicity of an element in $U$ gives for $a_j - n \geq 0$, that $a_j - n = a_r$ for an $r > j$, and that $a_{i,-a_j-\frac{1}{2}}$ is equal to $a_{i+n,-a_r-\frac{1}{2}}$. Hence

$$w^+_{-a_j-\frac{1}{2}} = \psi^+_{-a_j-\frac{1}{2}} + \sum_{i \neq -a_j} a_{i,-a_j-\frac{1}{2}}\psi_i^+ = \psi^+_{-a_j-\frac{1}{2}} + \sum_{i \neq -a_j} a_{i+n,-a_r-\frac{1}{2}}\psi_i^+ = \psi^+_{-a_{i+n}-\frac{1}{2}} + \sum_{i \neq -a_r} a_{i,-a_r-\frac{1}{2}}\psi_{i+n}^+$$

So if we assume, as in the proof of Theorem 5 that

$$w^+_{-a_j-\frac{1}{2}} = \text{Res}_z z^{-a_j-1}\psi^+_j(z) \exp \left( \sum_{i=1}^{\infty} c_{i,a_j} z^i \right),$$

then

$$w^+_{-a_r-\frac{1}{2}} = w^+_{-a_j+n-\frac{1}{2}} = \text{Res}_z z^{-a_j+n-1}\psi^+_j(z) \exp \left( \sum_{i=1}^{\infty} c_{i,a_j} z^i \right).$$

This gives that there are at most $m$ different vectors $c_{a_j} = (c_{1,a_j}, c_{2,a_j}, \ldots)$. So, instead of $c_{a_j}$ we will write $c_{\overline{a_j}}$, where $\overline{a_j}$ stands for the congruence class of $a_j$ modulo $n$. In a similar way as in Theorem 9, we obtain the restrictions on the constants. This gives

**Theorem 13** Any polynomial $n$-reduced CKP tau-function is, up to a constant factor, equal to

$$\tau_{a_1, \ldots, a_k|a_1, \ldots, a_k; c}(t) = \det \left( \chi_{a_1|a_j}(t + c_{\overline{a_j}}; t + \tau_C(c_{\overline{a_j}})) \right)_{1 \leq i,j \leq k},$$

(91)

where $\lambda = (a_1, \ldots, a_k|a_1, \ldots, a_k)$ is $n$-periodic and $c_{\overline{a_j}} = (c_{1,\overline{a_j}}, c_{2,\overline{a_j}}, \ldots, c_{m_j,\overline{a_j}}) \in \mathbb{C}^{m_j+a_j+1}$. Here $m_j$ is the largest integer among all $a_1, \ldots, a_k$, such that $m_j = \overline{a_j}$. We have the following restrictions on the constants for $1 \leq i < j \leq k$:

$$s_{a_i+a_j+1}(c_{1,\overline{a_i}} - c_{1,\overline{a_j}}, c_{2,\overline{a_i}} - c_{2,\overline{a_j}}, \ldots, c_{a_i+a_j,\overline{a_i}} - c_{a_i+a_j,\overline{a_j}} + (-1)^{a_i+a_j+1}c_{a_i+a_j+1,\overline{a_j}}) = 0.$$  

(92)
It is easy to see that the $n = 2$-reduced CKP hierarchy \([90]\) coincides with the KdV hierarchy on the differential operator $L = \partial^2 + u$.

The next case, the $n = 3$-reduced CKP hierarchy, is called the Kaup-Kupershmidt hierarchy. It is the hierarchy \([29]\) of Lax equations on the differential operator $L$ given by \([28]\). Since $(L^2_+)^+ = \partial^6 + \frac{5}{3}u\partial^3 + \frac{5}{2}u u_x \partial^2 + \frac{5}{18}(2u^2 + 7u_{xx})\partial + \frac{5}{9}(uu_x + u_{xxx})$, the first non-trivial such equation occurs for $k = 5$, and it gives

$$
\frac{\partial u}{\partial t_5} = -\frac{1}{18}(10u^2 u_x + 25u u_{xx} + 10uu_{xxx} + 2u_{xxxxx}), \quad (93)
$$

which is the Kaup-Kupershmidt equation (see e.g. \([5]\), Subsec. 11.3).

In this case there are, besides $\lambda = \emptyset$, two possible sets of self-conjugate partitions, which are 3-periodic, viz. $(m \in \mathbb{Z}_{\geq 0})$:

(1) $\lambda = (3m, 3m - 3, 3m - 6, \ldots, 3, 0|3m, 3m - 3, 3m - 6, \ldots, 3, 0),$

(2) $\lambda = (3m + 2, 3m - 1, 3m - 4, \ldots, 5, 2|3m + 2, 3m - 1, 3m - 4, \ldots, 5, 2). \quad (94)$

The corresponding CKP tau-functions are equal, up to a constant factor, to, respectively,

(1) $\det \left( \chi_{(3i|3j)}(t + c; t + \iota C(c)) \right)_{0 \leq i,j \leq m},$

(2) $\det \left( \chi_{(3i+2|3j+2)}(t + c; t + \iota C(c)) \right)_{0 \leq i,j \leq m},$

with the following constraints on the vector of constants $c = (c_1, c_2, c_3, \ldots)$

$$
c_{2k} = -\frac{1}{2} s_k(2c_2, 2c_4, 2c_6, \ldots, 2c_{2k-4}, 2c_{2k-2}, 0),
$$

for $k = 2, 5, 8, \ldots, 3m - 4, 3m - 1$, and $k = 4, 7, 10, \ldots, 3m - 2, 3m + 1$, respectively.

Recall that, by the second equation in \([91]\), $L(t, \partial)^* = -L(\iota(t), \partial)$. Hence, in order to obtain a skew-adjoint differential operator, one has to let all $t_{2i} = 0$, $i = 1, 2, 3, \ldots$. Also, there is only one vector of constants, viz. $c \in \mathbb{C}^{6m+1}$, and $c = \in \mathbb{C}^{6m+5}$, respectively.

Note that due to the equation \([27]\), which expresses $u$ in terms of the tau-functions, the tau-functions (1) and (2) with $t = t_o$ produce rational solutions of the Kaup-Kupershmidt hierarchy.

7 Comparison with the polynomial solutions of BKP

It is interesting to compare the polynomial tau-functions of the CKP hierarchy and that of the BKP hierarchy. The first observation is that both tau-functions are parametrized more or less by the same kind of permutations. The ones of the CKP are parametrized by the self-conjugate partitions $\lambda = (a_1, \ldots, a_k|a_1, \ldots, a_k)$, where $a_1 > a_2 > \cdots > a_k \geq 0$. Hence $\mu := (a_1, a_2, \ldots, a_k)$ is an extended strict partition.

We use the word extended because we allow $a_k$ to be zero. The polynomial solutions
of the BKP hierarchy are parametrized by the same set of extended strict partitions, with the only additional restriction that $k$ has to be even.

The second observation is that the solutions are expressed in terms of the polynomials:

$$\chi_{M,N}(t, t') = (-1)^N \left( \frac{1}{2} s_M(t) s_N(-t') + \sum_{k=1}^{N} s_{M+k}(t) s_{N-k}(-t') \right)$$

$$= \chi_{(M,N)}(t, t') - (-1)^M \frac{1}{2} s_M(t) s_N(-t').$$

(95)

Namely, one has (see [10], [13] and [15])

**Theorem 14** (a) All polynomial tau-functions of the BKP hierarchy are, up to a scalar factor of the form

$$\tau_{B\lambda}(t_o) = Pf \left( \chi_{\lambda,\lambda}(t_o + c_i, t_o + \iota_C(c_j)) \right)_{1 \leq i,j \leq 2^n},$$

(96)

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2n})$ is an extended strict partition, i.e. $\lambda_1 > \lambda_2 > \cdots \lambda_{2n} \geq 0$, where $t_o = (t_1, 0, t_3, 0, t_5, 0, \ldots)$ and $c_i = (c_{i_1}, c_{2i}, c_{3i}, \ldots)$ are arbitrary constants.

(b) This tau-function is the square root of a KP tau-function $\tau_{\lambda}(t_o)$, where

$$\chi = \begin{cases} 
(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{2n} - 1 | \lambda_1, \lambda_2, \ldots, \lambda_{2n}), & \text{if } \lambda_{2n} \neq 0 \text{ and } \\
(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_{2n-1} - 1 | \lambda_1, \lambda_2, \ldots, \lambda_{2n-1}), & \text{if } \lambda_{2n} = 0.
\end{cases}$$

Note the following:

- In the above BKP tau-function, the even times do not appear, but the even constants $c_{2k,i}$ do appear.

- The formulas look different from the ones for the BKP tau-function in e.g. [10], but here we have used the fact that

$$\sum_{j=0}^{\infty} (-1)^j s_j(t) z^j = \exp\left( \sum_{i=1}^{\infty} t_i (-z)^i \right) = \exp\left( - \sum_{i=1}^{\infty} \iota_C(t) z^i \right) = \sum_{j=0}^{\infty} s_j(-\iota_C(t)) z^j$$

- The square of $\tau_{B\lambda}(t_o)$ is equal to

$$\tau_{B\lambda}^2(t_o) = \pm \det \left( \chi_{\lambda,\lambda}(t_o + c_i, t_o + \iota_C(c_j)) \right)_{1 \leq i,j \leq 2^n}.$$

But since $\chi$ is not self-conjugate, this is never equal to a CKP tau-function where one puts the even times equal to 0, except when $\lambda = 0$, in that case $\tau_{0\lambda}^2(t_o) =$constant.
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