Matrix Rearrangement Inequalities Revisited

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Abstract. Let \( \|X\|_p = \text{Tr}((X^*X)^{p/2})^{1/p} \) denote the \( p \)-Schatten norm of a matrix \( X \in M_{n \times n}(\mathbb{C}) \), and \( \sigma(X) \) the singular values with \( \uparrow \downarrow \) indicating its increasing or decreasing rearrangements. We wish to examine inequalities between \( \|A+B\|_p^p + \|A-B\|_p^p, \|\sigma_1(A) + \sigma_1(B)\|_p^p + \|\sigma_1(A) - \sigma_1(B)\|_p^p \) and \( \|\sigma_1(A) + \sigma_1(B)\|_p^p + \|\sigma_1(A) - \sigma_1(B)\|_p^p \) for various values of \( 1 \leq p < \infty \). It was conjectured in [6] that a universal inequality \( \|\sigma_1(A) + \sigma_1(B)\|_p^p + \|\sigma_1(A) - \sigma_1(B)\|_p^p \leq \|A+B\|_p^p + \|A-B\|_p^p \) might hold for \( 1 \leq p < 2 \), potentially providing a stronger inequality to the generalization of Hanner’s Inequality to complex matrices \( \|A+B\|_p^p + \|A-B\|_p^p \geq (\|A\|_p + \|B\|_p\|p^p + ||A\|_p - \|B\|_p\|p^p \). We extend some of the cases in which the inequalities of [6] hold, but offer counterexamples to any general rearrangement inequality holding. We simplify the original proofs of [6] with the technique of majorization. This also allows us to characterize the equality cases of all of the self-adjoint matrices to the \( \{A,B\} = 0 \) case for all ranges of \( p \).

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