Long rainbow path in properly edge-colored complete graphs*

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Abstract

Let $G$ be an edge-colored graph. A rainbow (heterochromatic, or multicolored) path of $G$ is such a path in which no two edges have the same color. Let the color degree of a vertex $v$ be the number of different colors that are used on the edges incident to $v$, and denote it to be $d_c(v)$. It was shown that if $d_c(v) \geq k$ for every vertex $v$ of $G$, then $G$ has a rainbow path of length at least $\min\{\lceil\frac{2k+1}{3}\rceil, k-1\}$. In the present paper, we consider the properly edge-colored complete graph $K_n$ only and improve the lower bound of the length of the longest rainbow path by showing that if $n \geq 20$, there must have a rainbow path of length no less than $\frac{3n-1}{4} - \frac{n}{4} \sqrt{\frac{n}{2} - \frac{39}{11} - \frac{11}{16}}$.

Keywords: properly edge-colored graph, complete graph, rainbow (heterochromatic, or multicolored) path.

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1. Introduction

We use Bondy and Murty [3] for terminology and notation not defined here and consider simple graphs only.

Let $G = (V, E)$ be a graph. By an edge-coloring of $G$ we mean a function $C : E \rightarrow \mathbb{N}$, the set of natural numbers. If $G$ is assigned such a coloring, then we

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say that $G$ is an edge-colored graph. Denote the edge-colored graph by $(G, C)$, and call $C(e)$ the color of the edge $e \in E$. We say that $C(uv) = \emptyset$ if $uv \notin E(G)$ for $u, v \in V(G)$. For a subgraph $H$ of $G$, we denote $C(H) = \{ C(e) \mid e \in E(H) \}$ and $c(H) = |C(H)|$. For a vertex $v$ of $G$, the color neighborhood $CN(v)$ of $v$ is defined as the set $\{ C(e) \mid e$ is incident with $v \}$, the color degree $d^c(v) = |CN(v)|$. A subgraph of $G$ is called rainbow (heterochromatic, or multicolored) if any two edges of it have different colors. If $u$ and $v$ are two vertices on a path $P$, $uPv$ denotes the segment of $P$ from $u$ to $v$, whereas $vP^{-1}u$ denotes the same segment but from $v$ to $u$.

There are many existing publications dealing with the existence of paths and cycles with special properties in edge-colored graphs. The heterochromatic Hamiltonian cycle or path problem was studied by Hahn and Thomassen [14], Rödl and Winkler (see [11]), Frieze and Reed [11], and Albert, Frieze and Reed [1]. In [2], Axenovich, Jiang and Tuza studied the local variation of anti-Ramsey problem, namely, they studied the maximum $k$-good edge-coloring of $K_n$ containing no heterochromatic copy of a given graph $H$, and denote it by $g(n, H)$. They showed that for a fixed integer $k \geq 2$, $k - 1 \leq g(n, P_{k+1}) \leq 2k - 3$, i.e., if $K_n$ is edge-colored by a $(2k - 2)$-good coloring, then there must exist a heterochromatic path $P_{k+1}$, and there exists an a $(k - 1)$-good coloring of $K_n$ such that no heterochromatic path $P_{k+1}$ exists.

In [4], the authors considered long heterochromatic paths in general graphs with a $k$-good coloring and showed that if $G$ is an edge-colored graph with $d^c(v) \geq k$ (color degree condition) for every vertex $v$ of $G$, then $G$ has a heterochromatic
path of length at least $\lceil \frac{4k + 1}{2} \rceil$. In [5, 6], we got some better bound of the length of longest heterochromatic paths in general graphs with a $k$-good coloring.

In [7], we showed that if $|CN(u) \cup CN(v)| \geq s$ (color neighborhood union condition) for every pair of vertices $u$ and $v$ of $G$, then $G$ has a heterochromatic path of length at least $\lceil \frac{4k + 1}{2} \rceil$, and gave examples to show that the lower bound is best possible in some sense.

In [12], Gyárfás and Mhalla showed that in any properly edge-colored complete graph $K_n$, there is a rainbow path with no less than $(2n + 1)/3$ vertices. In [6] we got a better result, showing that in any edge-colored graph $G$, if for every vertex of $G$ there are at least $k$ colors appear on it, then the longest rainbow path in $G$ is no shorter than $\lceil \frac{2k}{3} \rceil + 1$.

**Theorem 1.1** [6] Let $G$ be an edge-colored graph. If $d^c(v) \geq k$ for every vertex $v \in V(G)$, then $G$ has a heterochromatic path of length at least $\min\{\lceil \frac{2k}{3} \rceil + 1, k - 1\}$.

In this paper, we will improve the bound in [12], and show that a longest rainbow path in a properly edge-colored $K_n$ is not shorter than $\left(\frac{3}{4} - o(1)\right) n$.

### 2. Propositions of a longest rainbow path

Suppose $G$ is a properly edge-colored $K_n$, $P = v_0v_1v_2 \cdots v_l$ is one of the longest rainbow paths in $G$, and $C(v_{i-1}v_i) = C_i$ $(i = 1, 2, \cdots, l)$.

Suppose $l < n - 2$ and $u$ is an arbitrary vertex which does not belong to the path $P$. Then we can easily get the following proposition.

**Proposition 2.1** $C(v_0u) \in C(P)$, $C(v_lu) \in C(P)$.

*Proof.* Otherwise, $uv_0Pu_l$ or $uv_lP^{-1}v_0$ is a rainbow path of length $l + 1$, a contradiction. ■

**Proposition 2.2** If $C(uv_i) \notin C(P)$, then $C(uv_{i-1}) \in C(P)$, $C(uv_{i+1}) \in C(P)$.

*Proof.* Otherwise, $v_0Pv_{i-1}uv_iPu_l$ or $v_0Pv_iuv_{i+1}Pu_l$ is a rainbow path of length $l + 1$, a contradiction. ■

**Proposition 2.3** If $C(uv_i) \notin C(P)$, then $\{C(v_0v_{i+1}), C(v_lv_{i-1})\} \subset C(P) \cup C(uv_i)$.

*Proof.* Otherwise, $uv_iP^{-1}v_0v_{i+1}Pv_l$ or $uv_lPv_{i-1}P^{-1}v_0$ is a rainbow path of length $l + 1$, a contradiction. ■
Proposition 2.4 If $C( uv_i) \notin CP$, then $C( v_0v_l) \in C(P) \setminus \{C_{i-1}, C_i\}$.

Proof. Otherwise, $uv_iPv_0v_lPv_{i-1}$ or $uv_iP^{-1}v_0v_lP^{-1}v_{i+1}$ is a rainbow path of length $l + 1$, a contradiction. \hfill \blacksquare

Proposition 2.5 If $C(v_0v_l) \notin C(P)$, then $C(v_{i-1}u) \in C(P) \setminus C(v_iP_{i+1})$; if $C(v_iP_{i+1}) \notin C(P)$, then $C(v_0u) \in C(P) \setminus C(v_iP_{i+1})$.

Proof. Otherwise, $v_{i-1}P^{-1}v_0v_lPv_i$ or $v_{i+1}Pv_lP^{-1}v_0u$ is a rainbow of length $l + 1$, a contradiction. \hfill \blacksquare

Proposition 2.6 If $C(v_0v_l) \notin C(P)$, then $C(v_{i-1}u) \in C(P) \cup C(v_0v_l)$; if $C(v_iP_{i+1}) \notin C(P)$, then $C(v_0u) \in C(P) \cup C(v_0v_l)$.

Proof. Otherwise, $uv_{i-1}P^{-1}v_0v_lPv_i$ or $uv_{i+1}Pv_lP^{-1}v_0$ is a rainbow of length $l + 1$, a contradiction. \hfill \blacksquare

With these propositions, we can give new lower bound of a longest rainbow path. And we will do that separately in the following two situations: the biggest rainbow cycle is of length $l + 1$, and the biggest rainbow cycle is of length less than $l + 1$.

3. A longest rainbow path has the same number of vertices as a biggest rainbow cycle

If the longest rainbow path has the same number of vertices as the biggest rainbow cycle, then the biggest rainbow cycle is of length $l + 1$, and there exists a rainbow path $P = v_0v_1 \cdots v_l$ such that $C(v_0v_l) \notin C(P)$.

Then, we can easily get the following conclusion from Proposition 2.4.

Lemma 3.1 If $C(v_0v_l) \notin C(P)$, then for an arbitrary $u \in V(G \setminus P)$, $C(u, P) \in C(P) \cup C(v_0v_l)$.

By using this Lemma, we can get one of our main conclusions.

Theorem 3.2 If $n \geq 20$ and $C(v_0v_l) \notin C(P)$, then $l \geq \frac{3}{4}n - 1$.

Proof. We will prove it by contradiction. Suppose a longest rainbow path in $G$ is of length $l < \frac{3}{4}n - 1$. Then $|V(G) \setminus V(P)| = n - l - 1 > \frac{n}{4} \geq 5$. 

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We can conclude by Lemma 3.1 that for any vertex \( u \in V(G) \setminus V(P) \), \( C(u, P) \subseteq C(P) \cup C(v_0v_l) \). On the other hand, \(|V(P)| = |C(P) \cup C(v_0v_l)| = l + 1 \) and \( G \) is a properly edge-colored \( K_n \). Therefore, \( C(u, P) = C(P) \cup C(v_0v_l) \), \( C(G \setminus P) \cap (C(P) \cup C(v_0v_l)) = \emptyset \).

Since \( P \) is one of the longest rainbow paths, by Proposition 2.1 there exist \( 2 \leq i_1 < i_2 < \cdots < i_{n-2-l} < l \), \( 1 \leq j_1 < j_2 < \cdots < j_{n-2-l} < l - 1 \), such that
\[
\begin{align*}
\mathcal{F}(C(v_0v_{i_1}), (C(v_0v_{i_2}), \cdots , C(v_0v_{i_{n-2-l}})) \setminus (C(P) \cup \{ C(v_0v_l) \}) \\
= \mathcal{F}\left( C(v_0v_{j_1}), (C(v_0v_{j_2}), \cdots , C(v_0v_{j_{n-2-l}})) \setminus (C(P) \cup \{ C(v_0v_l) \}) \right) \\
= n - l - 2
\end{align*}
\]
Additionally, \( C(w_{v_j-1}) \neq C(v_0v_l), C(w_{v_j+1}) \neq C(v_0v_l), k = 1, 2, \cdots , n - l - 2 \).

Let \( I = \{ i-1|C(v_0v_i) \notin C(P) \cup C(v_0v_l), 2 \leq i \leq l-1 \}, J = \{ j+1|C(v_0v_j) \notin C(P) \cup C(v_0v_l), 1 \leq j \leq l-1 \}. \)

Now we distinguish the following two cases:

Case 1. \( I \cap J \neq \emptyset \).

This implies that there exists some \( t \) in \( I \cap J \), i.e.,
\[
\{ C(v_0v_{i+t}), C(v_0v_{i-t}) \} \cap (C(P) \cup C(v_0v_l)) = \emptyset
\]

Case 1.1. \( C(v_0v_{i+t}) \neq C(v_0v_{i-t}) \).

Since \( n-l \geq 4 \) and \( C(u, P) = C(P) \cup C(v_0v_l) \), there are no less than 3 colors which is not in \( C(P) \cup C(v_0v_l) \) such that they belong to the color set \( C(u, V(G) \setminus V(P)) \). Therefore, there exist \( u_1, u_2 \in V(G) \setminus V(P) \) such that \( C(u_1u_2) \notin C(P) \setminus \{ C(v_0v_l), C(v_0v_{i+t}), C(v_0v_{i-t}) \} \).

By Lemma 3.1 there exists some vertex \( v \in V(P) \) such that \( C(u_1v) = C(v_0v_l) \), denote it by \( v_{i_0} \). We can conclude from Proposition 2.1 that \( i_0 \neq t \). Since \( C' = v_0v_{i_0+1}Pv_{i_0-1}P^{-1}v_0 \) is a rainbow cycle of length \( l \) in which the color \( C(v_0v_l) \) does not appear on it. Therefore, \( u_2u_{i_0}v_0C \) contains a rainbow path of length \( l + 1 \), a contradiction.

Case 1.2. \( C(v_0v_{i+t}) = C(v_0v_{i-t}) \notin C(P) \cup C(v_0v_l) \).

First, we can conclude that \( C(v_{i-1}u) \neq C(v_0v_l) \) for any vertex \( u \in V(G) \setminus V(P) \). Otherwise, suppose there exists some \( u \in V(G) \setminus V(P) \) such that \( C(v_{i-1}u) = C(v_0v_l) \). Since \( |V(G) \setminus V(P)| > 5 \), there exists a vertex \( u_1 \in V(G) \setminus (V(P) \cup \{ u \}) \) such that \( C(uu_1) \notin C(P) \cup \{ C(v_{0v_l}), C(v_0v_{i+t}) \} \). Therefore, \( u_1u_2u_{i-1}P^{-1}v_0v_{i+t}Pv_l \) is a rainbow path of length \( l + 1 \), a contradiction.

Then, we will show that \( t-1 \notin I \cup J \).

If \( t-1 \in I \), i.e., \( C(v_0v_{i-1}) \notin C(P) \cup \{ C(v_0v_l), C(v_0v_{i+t}) \} \), \( C' = v_0Pv_{i-1}v_{i}P^{-1}v_{i-1}v_0 \) is a rainbow cycle of length \( l + 1 \) without color \( C(v_0v_l) \). On the other hand, by 3.1 there exists a vertex \( u \in V(G) \setminus V(P) \) and a vertex \( v_{i_0} \in V(P) \) such that \( C(uv_{i_0}) = C(v_0v_l) \). Then \( uv_{i_0}C \) contains a rainbow path of length \( l + 1 \), a contradiction.
If \( t - 1 \in J \), i.e., \( C(v_{t-2}v_t) \notin C(P) \cup \{C(v_0v_t), C(v_{t-1}v_t)\} \), \( C' = v_0Pv_{t-2}v_tP^{-1}v_{t+1}v_0 \) is a rainbow cycle of length \( l \) without color \( C(v_0v_t) \). Since \( |V(G) \setminus V(P)| > 5 \), for any vertex \( u \in V(G) \setminus V(P) \), \( d^e_{G \setminus P}(u) \geq 5 \). So, by Theorem 5.1 there exists a rainbow path \( u_1u_2u_3 \in G \setminus P \) with no colors in \( C(P) \cup \{C(v_0v_t), C(v_0v_{t+1}), C(v_{t-2}v_t)\} \). Since \( G \) is properly edge-colored, at least one edge in \( \{v_1u_1, v_1u_3\} \) does not have color \( C(v_0v_t) \), W.O.L.G., assume \( C(v_1u_1) \neq C(v_0v_t) \). Then, because \( C(v_{t-1}u_1) \neq C(v_0v_t), C(u_1, P) = C(P) \cup C(v_0v_t) \). So, by Lemma 3.1 there exists some \( i_0, 0 \leq i_0 \leq l, i_0 \neq t - 1, t \) such that \( C(u_1v_{i_0}) = C(v_0v_t) \). Then \( u_3u_2u_1v_{i_0}C' \) contains a rainbow path of length \( l + 1 \), a contradiction.

So, we have \( t - 1 \notin I \cup J \).

Let \( K = I \cap J, I' = (I \setminus K) \cup \{t - 1 | t \in K\} \). Then \( |I'| = |I| \) and \( I' \cap J = \emptyset \).

Additionally, for any \( t \in I' \cup J \) and any \( u \in V(G) \setminus V(P) \), \( C(v_tu) \neq C(v_0v_t) \). Otherwise, there exist some \( i_0 \in K \) and some vertex \( u \in V(G) \setminus V(P) \), such that \( C(v_{i_0}v_t) = C(v_0v_t) \). Since \( |V(G) \setminus V(P)| \geq 6 \), there exists some vertex \( u_1 \in V(G) \setminus V(P) \) such that \( C(uu_1) \notin C(P) \cup \{C(v_0v_t), C(v_0v_{i_0+1})\} \). Then \( u_1uv_{i_0-1}P^{-1}v_0v_{i_0+1}Pv_t \) is a rainbow path of length \( l + 1 \), a contradiction.

On the other hand,

\[ |I' \cup J| = |I'| + |J| = |I| + |J| \geq 2[(n - 1) - (l + 1)] = 2(n - l - 2), \]

and \( |V(G) \setminus V(P)| = n - (l + 1) = n - l - 1 \). So there are at least \( n - l - 1 \) \( i \)'s \( (1 \leq i \leq l - 1) \) such that \( C(uv_i) = C(v_0v_t) \) for some \( u \in V(G) \setminus V(P) \). So we have \( |I' \cup J| + (n - l - 1) \leq l - 1 \), and then \( 2(n - l - 2) + n - l - 1 \leq l - 1 \), which implies \( l \geq \frac{3}{4}n - 1 \), a contradiction.

**Case 2.** \( I \cap J = \emptyset \).

By Proposition 2.3, we have that for any \( t \in I \cup J \) and any \( u \in V(G) \setminus V(P) \), \( C(v_tu) \neq C(v_0v_t) \). On the other hand, there are at least \( |V(G) \setminus V(P)| = n - l - 1 \) \( i \)'s \((1 \leq i \leq l - 1)\) such that \( C(uv_i) = C(v_0v_t) \) for some \( u \in V(G) \setminus V(P) \). So we have \( |I \cup J| + (n - l - 1) \leq l - 1 \), and then \( 2(n - l - 2) + n - l - 1 \leq l - 1 \), which implies \( l \geq \frac{3}{4}n - 1 \), a contradiction.

This complete the proof.

**4. A biggest rainbow cycle has less vertices than a longest rainbow path**

Since a biggest rainbow cycle have less vertices than a longest rainbow path, then \( C(v_0v_t) \in C(P) \).

For any longest rainbow path \( P \), by Proposition 2.1 and Theorem 3.2, there
exist 2 \leq i_1 < i_2 < \cdots < i_{t_1} < l \ (t_1 \geq n - 1 - l) such that
\[ |\{C(v_0v_{i_1}), C(v_0v_{i_2}), \cdots, C(v_0v_{i_{t_1}})\}| = |CN(v_0) \setminus C(P)| = t_1. \]

Now we will distinguish two cases: the case when there is a vertex \( u \in V(G) \setminus V(P) \) such that \( C(v_u) = C_1 \), and the case when there is no such vertex.

We first consider the case when there is a vertex \( u \in V(G) \setminus V(P) \) such that \( C(v_u) = C_1 \).

**Theorem 4.1** If \( C(v_0v_t) \in C(P) \) and there is a vertex \( u \in V(G) \) such that \( C(v_u) = C_1 \), then \( l \geq \frac{3}{4}n - \frac{1}{4} \sqrt{n^2 - \frac{39}{11} - \frac{11}{16}} \).

**Proof.** Suppose \( P \) is a longest rainbow path that has the minimized \( t_1 \).

We can conclude from Proposition 2.5 that \( C_{i_k} \notin C(v_t, C(G) \setminus V(P)) \), \( k = 1, 2, \cdots, t_1 \).

Let \( C^1_j = \{C_{i_k}|k = 1, 2, \cdots, t_1\} \), \( C^0_j = CN(v_{i_{t-1}}) \setminus (C(P) \cup C(v_0v_t)) \). Let the color set \( C^1_j, C^*_j \) \( (j = 1, 2, \cdots, t_1) \) be defined by the following procedure.

For \( j = 1 \) to \( t_1 \) do
\[ C^*_j = \emptyset, \]
for \( s = 1 \) to \( i_j - 3 \)
\[ \text{if } C(v_{i_{j-1}}v_s) \in C^0_j, \text{ let } C^*_j = C^*_j \cup C_{s+1}; \]
for \( s = i_j + 1 \) to \( l - 1 \)
\[ \text{if } C(v_{i_{j-1}}v_s) \in C^0_j, \text{ let } C^*_j = C^*_j \cup C_s, \]
\[ C^1_j = C^1_j \cup C^*_j. \]
Then we can conclude that \( |C^*_j| = |C^0_j| \geq t_1 - 1 \) by Proposition 2.1.

Suppose \( |C^1_{t_1}| - |C^0| = j_0 \) and \( j \geq j_0 + 2 \).

Let \( C_{j,1} = \{C(v_{i_{j-1}}v_t)|t > j \text{ and } C(v_{i_{j-1}}v_{i_t}) \in C^0_j\} \),
\[ C_{j,2} = \{C(v_{i_t-1}v_i)|t < j \text{ and } C(v_{i_{j-1}}v_{i_{t-1}}) = C(v_0v_{i_t})\}, \]
\[ C_{j,3} = \{C(v_{i_{t-1}})|t < j \text{ and } C(v_{i_{j-1}}v_{i_{t-1}}) \notin C^1_j \setminus C(v_0v_{i_t})\}. \]

Then \( C_{j,1}, C_{j,2}, C_{j,3} \) are mutually independent and \( C^*_j \cap C^0 = C_{j,1} \cup C_{j,2} \cup C_{j,3} \). By the definition \( |C_{j,1}| \leq t_1 - j, \bigcup_{j = j_0 + 2}^{t_1} C_{j,2} \subseteq \{C_{i_1}, C_{i_2}, \cdots, C_{i_{t-1}}\} \) and \( C_{j,2} \cap C_{j',2} = \emptyset \) since \( G \) is properly edge-colored.

Since \( C(v_tu) = C_1 \), we have \( C_{j,3} = \emptyset \); otherwise, \( v_2Pv_{i_{t-1}}v_{i_{t-1}}P^{-1}v_{i_t}v_0v_{i_t}Pv_t \) is a rainbow path of length \( l + 1 \), a contradiction. Therefore, \( C^*_j \cap C^0 = C_{j,1} \cup C_{j,2} \).
On the other hand, \(|C_j \setminus C^0| \leq |C_j^1 \setminus C^0| \leq j_0\). So, \(|C_{j,2}| = |C_j^* \cap C^0| - |C_{j,1}| \geq (t_1 - 1 - j_0) - (t_1 - j) = j - j_0 - 1\). Notice that \(\sum_{j=j_0+2}^{t_1} |C_{j,2}| = \bigcup_{j=j_0+2}^{t_1} C_{j,2} \leq t_1 - 1\).

Then, we have \(\sum_{j=j_0+2}^{t_1} (j - j_0 - 1) \leq t_1 - 1\), i.e., \(\frac{1}{2} (t_1^2 - 2j_0t_1 - t_1 + j_0^2 + j_0) \leq t_1 - 1\).

Therefore, \(j_0 \geq t_1 - \frac{1}{2} - \sqrt{2t_1 - \frac{7}{4}}\), \(|C_{t,1}| = t_1 + j_0 \geq 2t_1 - \frac{1}{2} - \sqrt{2t_1 - \frac{7}{4}}\).

Since \(C(v_i, V(G) \setminus V(P)) \subseteq C(P) \setminus (C_i^1 \cup \{C_i\})\) and \(G\) is properly edge-colored, \(|V(G) \setminus V(P)| \leq l - \left(2t_1 - \frac{1}{2} - \sqrt{2t_1 - \frac{7}{4}}\right) - 1\), i.e., \(n - (l + 1) \leq l - \left(2t_1 - \frac{1}{2} - \sqrt{2t_1 - \frac{7}{4}}\right) - 1\). So, \(2t_1 - \sqrt{2t_1 - \frac{7}{4}} \leq 2l - n + \frac{1}{2}\). Since \(f(x) = 2x - \sqrt{2x - \frac{7}{4}}\) increases when \(x > 2\) and \(t_1 \geq n - l - 1 > 2\), we have

\[2(n - l - 1) - \sqrt{2(n - l - 1) - \frac{7}{4}} \leq 2l - n + \frac{1}{2},\]

Therefore, \(l \geq \frac{3}{4}n - \frac{1}{4}\sqrt{\frac{n}{2} - \frac{39}{16}} - \frac{11}{16}\).

This completes the proof.

Now we consider the case when for any longest rainbow path \(P = v_0v_1v_2 \cdots v_t\) and any \(u \in V(G) \setminus V(P)\), \(C(v_iu) \neq C_1\).

**Lemma 4.2** If for any longest rainbow path \(P = v_0v_1v_2 \cdots v_t\) and any \(u \in V(G) \setminus V(P)\), \(C(v_iu) \neq C_1\) and there are at most two \(j\)’s satisfying \(2 \leq j \leq t_1\), \(i_j - i_{j-1} \geq 2\), then \(l \geq \frac{3n-4}{4}\).

**Proof.** For any \(j\) (\(1 \leq j \leq t_1\)), \(v_{i_j-1}P^{-1}v_0v_i\) \(P\) is a rainbow path. So we can get by Proposition 2.3 and the condition of this lemma that \(\{C_{i_j-1}, C_i\} \cap C(v_i, V(G) \setminus V(P)) = \emptyset\).

Let \(C^* = \bigcup_{j=1}^{t_1} \{C_{i_j-1}, C_i\}\). Then \(|C^*| \geq 2t_1 - 2\) since there are at most two \(j\)’s satisfying \(2 \leq j \leq t_1\), \(i_j - i_{j-1} \geq 2\). On the other side, \(C(v_i, V(G) \setminus V(P)) \subseteq C(P) \setminus (C^* \cup \{C_i\})\). So we have \(n - l - 1 \leq l - (2t_1 - 2) - 1 = l - 2t_1 + 1 \leq l - 2(n - l - 1) + 1\).

This implies that \(l \geq \frac{3n-4}{4}\) and completes the proof.

Then we can get the following conclusion.
Theorem 4.3 If $C(v_0v_l) \in C(P)$ and for any vertex $u \in V(G)$, $C(v_lu) \neq C_1$, then $l \geq \frac{3}{4} n - \frac{1}{4} \sqrt{\frac{n}{2} - \frac{39}{11}} - \frac{11}{16}$.

Proof. Let $i_0 = \min\{i | \exists u \not\in V(P) \text{ s.t. } C(v_lu) = C_i\}$. Suppose $P$ is one of the longest rainbow paths such that $i_0$ is the smallest.

Let $j^* = \max\{j | i_j - i_{j-1} = 1\}$. Then we have $i_0 > i_{j^*}$; otherwise, $v_lPv_{i_{j^*}}v_0v_{i_{j^*}}Pv_l$ is also a rainbow path of length $l$, but $C_{i_0}$ appears on the $(i_0 - 1)$-th edge of the path, a contradiction.

Now we distinguish the following two cases.

Case 1. $i_0 < i_{t_1}$.

Let the integer $j_0$ and the color sets $C^0_j$, $C^*_j$, $C_{j,1}$, $C_{j,2}$, $C_{j,3}$ be defined as in Theorem 4.1.

Suppose $i_{j_1-1} < i_0 < i_{j_1}$. Then we have that for any $j_1 \leq j_2 \leq t_1$, $\{C(v_{j_2-1}v_{t_1-1}) | 1 \leq t < j_1\} \cap C^0_{j_2} = \emptyset$. Otherwise, there exists $j_3 < j_1 \leq j_2$, such that $C(v_{i_{j_3-1}v_{i_{j_2-1}}}) \not\in C^0_{j_2}$. Then, $v_{i_{j_3}}Pv_{i_{j_2}v_{i_{j_3}-1}P^{-1}v_0v_{i_{j_2}}Pv_l}$ is a rainbow path of length $l$, but the color $C_{i_0}$ appears on the $(i_0 - i_{j_3})$-th edge of this path, a contradiction to the choice of $P$.

If there exists $j_1 \leq j_2 < j_3$ such that $C(v_{i_{j_3-1}v_{i_{j_2-1}}}) \not\in \{C^0_{j_3} \cup C(v_0v_{i_2})\}$, then $v_lPv_{i_{j_2}v_{i_{j_3}-1}P^{-1}v_{i_{j_2}}v_0v_{i_{j_3}}Pv_l$ is a rainbow path of length $l$, but $C_{i_0}$ appears on the $(i_0 - 1)$-th edge of this path, a contradiction.

Therefore, for any $j \geq j_1$, $C_{j,3} = \emptyset$, $C_{j,2} \subseteq \{C_i | j_1 \leq t < t_1\}$.

Case 1.1. $j_1 > j_0$.

As in Theorem 4.1, we can get that $\sum_{j=j_1}^{t_1} (j - j_0 - 1) = \sum_{j=j_1}^{t_1} |C_{j,2}| = \left| \bigcup_{j=j_1}^{t_1} C_{j,2} \right| \leq t_1 - j_1$. This implies that $(t_1 - j_1 + 1)(j_1 + t_1 - 2j_0 - 2) \leq 2(t_1 - j_1)$. Therefore, $j_0 \geq \frac{1}{2}[(j_1^2 - 3j_1) - (j_1^2 - 3j_1) + 2j_1 - 2] > \frac{1}{2}(2j_1 - 2) = j_1 - 1$, a contradiction.

Case 1.2. $j_1 \geq j_0$.

By the same calculation we did in Theorem 4.1, we can conclude that $l \geq \frac{3}{4} n - \frac{1}{4} \sqrt{\frac{n}{2} - \frac{39}{11}} - \frac{11}{16}$.

Case 2. $i_0 > i_{t_1}$.

If there are at most two $j$'s satisfying $2 \leq j \leq t_1$, $i_j - i_{j-1} \geq 2$, then by Lemma 4.2 $l \geq \frac{3n - 4}{4} \geq \frac{3}{4} n - \frac{1}{4} \sqrt{\frac{n}{2} - \frac{39}{11}} - \frac{11}{16}$.

So we will only consider the case when there are at least three $j$'s satisfying $2 \leq j \leq t_1$, $i_j - i_{j-1} \geq 2$. Then $l \geq \frac{3n - 4}{4}$. Suppose there are exactly $k$ ($k \geq 3$) such $j$'s satisfying $s_1 < s_2 < \cdots < s_k$. Then for any integer $p$ ($1 \leq p \leq k$)
$v_1 P v_{is_p-1}, v_0 v_{is_p} P v_l$ is a rainbow path of length $l$. Therefore,

$$C(v_1, V(G) \setminus V(P)) \subseteq (C(P) \setminus \{C_{is_p}\}) \cup \{C(v_0 v_{is_p-1}), C(v_0 v_{is_p})\}.$$ 

Notice that $k \geq 3$, and so $\bigcap_{p=1}^{k} \{C(v_0 v_{is_p-1}), C(v_0 v_{is_p})\} = \emptyset$, and then $C(v_1, V(G) \setminus V(P)) \subseteq (C(P) \setminus \{C_{is_p}\})$. Let $C^* = C(P) \setminus \left(\bigcup_{p=1}^{s} C_{is_p} \cup \{C_1, C_2\}\right)$. Then $C(v_1, V(G) \setminus V(P)) \subseteq C^*$.

**Case 2.1.** $|C^* \cap \{C_1, C_2, \ldots, C_{i_{11}}\}| < t_1$.

$|C^* \cap \{C_1, C_2, \ldots, C_{i_{11}}\}| < t_1$ implies that $i_{t_1} - k - 2 < t_1$ and there exists a vertex $u \in V(G) \setminus V(P)$ such that $C(v_1 u) = C_t$, where $t \geq i_{t_1} - [t_1 - (i_{t_1} - k - 2)] = t_1 + k + 2$ and it appears on the $(l - t + 1)$-th edge of the rainbow path $v_1 P^{-1} v_{is_p} v_0 v_{is_p-1} P^{-1} v_1$ of length $l + 1$. By the choice of $P$, we can conclude that $l - t + 1 \geq i_0 > i_{t_1}$, i.e., $t \leq l - i_{t_1} + 1$. Remember that $i_{t_1} \geq 2t_1 - k$ and $t_1 \geq n - l - 1$, and so we have $t_1 + k + 2 \leq t \leq l - i_{t_1} + 1 \leq l - 2t_1 + k + 1$, i.e., $l \geq 3t_1 + 1 \geq 3n - 3l - 2$, and therefore $l \geq \frac{3n - 2}{4} = \frac{3}{4} n - \frac{1}{4} \sqrt{\frac{n}{2} - \frac{39}{11} - \frac{11}{16}}$.

**Case 2.2.** $|C^* \cap \{C_1, C_2, \ldots, C_{i_{11}}\}| \geq t_1$.

Suppose $C_t$ is the $t_1$-th color in $C^*$, $i_{j_0 - 1} < t \leq i_{j_0}$ and there are $k_1$ $j$’s in the set $\{2, \ldots, j_0 - 1\}$ satisfying $i_j - i_{j-1} = 1$. Then we can conclude that $t = t_1 + k_1 + 2$ and $t > 2(j_0 - 1) - k_1 - 2 = 2j_0 - k_1 - 4$. Since if $i_{p} - i_{p-1} > 1$ then $|C^* \cap \{C_{i_{p-1}+1}, \ldots, C_{i_p}\}| \leq i_{p} - i_{p-1}$, we have

$$t_1 = i_{1} - 2 + \sum_{p \leq j_0-1}^{p \geq j_0-1} (i_{p} - i_{p-1}) + t - i_{j_0} = t_1 - k_1 - 2.$$ 

On the other hand, $i_{t_1} \geq i_{j_0} + 2(t_1 - j_0) - (k - k_1) = i_{j_0} + 2t_1 - 2j_0 - k + k_1 \geq t + 2t_1 - 2j_0 - k + k_1$. By Lemma 1.2 there is some integer $j$ satisfying $i_j - i_{j-1} = 1$, and so $v_1 P^{-1} v_{i_j} v_0 v_{i_{j-1}} P^{-1} v_1$ is a rainbow path of length $l + 1$ and $C_t$ appears on the $(l - t)$-th or $(l - t + 1)$-the edge. Therefore, we have $i_0 \leq l - t$ by the choice of $P$. Then we have $l - t \geq i_0 > i_{t_1} \geq t + 2t_1 - 2j_0 - k + k_1$, i.e.,

$$l - t_1 - k - 2 \geq 3t_1 + 2k_1 - 2j_0 + 2 > 3t_1 + 2k_1 + (-t - k_1 - 4) + 2 = 3t_1 - t + k_1 - 2 = 3t_1 - (t_1 + k_1 + 2) + k_1 - 2 = 2t_1 - 4$$

So, $l \geq 3t_1 + k - 2 \geq 3t_1 - 2 \geq 3(n - l - 1) - 2$, which implies that

$$l \geq \frac{3n - 5}{4} \geq \frac{3}{4} n - \frac{1}{4} \sqrt{\frac{n}{2} - \frac{39}{11} - \frac{11}{16}}.$$ 

This completes the proof. □
5. Conclusion

By Theorems 3.2, 4.1 and 4.3 we can easily get the following conclusions.

**Theorem 5.1** For any properly edge-colored complete graph $K_n$ ($n \geq 20$), there is a rainbow path of length no less than $\frac{\sqrt{n}}{2} - \frac{39}{11} - \frac{11}{16}$. 

**Corollary 5.2** For any properly edge-colored complete graph $K_n$ ($n \geq 20$), there is a rainbow path of length no less than $(\frac{3}{4} - o(1))n$.

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