Primordial Trispectrum from Isocurvature Fluctuations

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Abstract

We study non-Gaussianity generated by adiabatic and isocurvature primordial perturbations. We first obtain, in a very general setting, the non-linear perturbations, up to third order, for an arbitrary number of cosmological fluids, going through one or several decay transitions. We then apply this formalism to the mixed curvaton and inflaton model, allowing for several decay channels. We compute the various contributions to the bispectrum and trispectrum resulting from adiabatic and isocurvature perturbations, which are correlated in general. By investigating some hybrid decay scenario, we show that significant non-Gaussianity of adiabatic and isocurvature types can be generated without conflicting with the present isocurvature constraints from the power spectrum. In particular, we find cases where non-Gaussianity of isocurvature origin can dominate its adiabatic counterpart, both in the bispectrum and in the trispectrum.
1 Introduction

Observations of cosmic density fluctuations, such as cosmic microwave background (CMB) and large scale structure, are our only accessible windows onto phenomena that occurred in the early Universe. It is therefore crucial to extract as much information as possible from these observations, as a way to test or constrain various early universe scenarios (see e.g. [1] for recent lecture notes).

This is the reason why primordial non-Gaussianity has been the subject of an intense study recently, as it would provide invaluable information beyond the power spectrum. In particular, a detection of primordial non-Gaussianity would rule out the simplest models of inflation (with a single field), which generate only very small non-Gaussianity.

Observations of the CMB anisotropies by the WMAP satellite [2] have set the present limit \( f_{\text{local}}^\text{NL} = 32 \pm 21 \) (68% CL) \([\text{or} \ -10 < f_{\text{local}}^\text{NL} < 74 \ (95 \% \ CL)]\) on the non-linearity parameter \( f_{\text{local}}^\text{NL} \), which characterizes the amplitude of the bispectrum, for the local shape\(^\#1\). Although Gaussian fluctuations are still consistent with current observations, this constraint suggests that primordial fluctuations may deviate from Gaussianity since the central value of this range is somewhat away from zero. The Planck satellite should be able to confirm or infirm such level of non-Gaussianity. If Planck detects significant local non-Gaussianity, then early Universe scenarios with additional fields, such as a curvaton [3] or a modulaton [4–9], would become natural alternatives to the simplest inflation scenarios.

As soon as one considers models with multiple scalar fields (several inflatons or spectator fields beside the inflaton), one must take into account the possibility that some (baryon or dark matter) isocurvature perturbations may be generated, in addition to the usual adiabatic fluctuations. The amplitude of such isocurvature modes, which could be partially or fully correlated with the adiabatic one, is now severely constrained by observations of the CMB power spectrum. However, complementary information, or constraints, on isocurvature modes can be obtained from non-Gaussianity.

Non-Gaussianity in models with isocurvature fluctuations has been studied in several works [10–15]. In particular, a general treatment for the primordial bispectrum which allows various decay scenarios has been given in [15] recently. Although the works mentioned above have mainly studied the bispectrum (except [14]), the trispectrum will also become, in the near future, a major target since combined information from the bispectrum and the

\(^\#1\)Three types of \( f_{\text{NL}} \) have been discussed in the literatures. Other two types and their constraints are \(-214 < f_{\text{eq}}^\text{NL} < 266 \) for the equilateral type and \(-410 < f_{\text{orthog}}^\text{NL} < 6 \) for the orthogonal type (95 % C.L.) [2].
trispectrum is a powerful way to discriminate between scenarios that generate local non-Gaussianity, as discussed in detail in [16]. In light of this, a general treatment of non-Gaussianity in models with isocurvature fluctuations for the trispectrum is worth investigating and is the main topic of this paper. More specifically, in the present work, we extend the general formalism given in [15] to third order, then apply it to the mixed curvaton and inflaton scenario [17–21].

The organization of this paper is as follows. In the next section, we present the general formalism to describe the curvature and isocurvature perturbations, up to third order. Then, in Section 3, we apply this formalism to the mixed curvaton inflaton scenarios and compute the bispectra and trispectra generated by the adiabatic and isocurvature modes in these scenarios. Some quantitative discussions are also given. The final section is devoted to the conclusion. Throughout this paper, we set the reduced Planck energy scale $M_{\text{pl}}$ to be unity.

2 General formalism

In this section, we review and extend the general formalism introduced in [15] to describe the adiabatic and isocurvature perturbations generated in scenarios with a cosmological transition due to the decay of some species of particles, which we call $\sigma$. Explicit results up to second order were given in [15], whereas we extend here the analysis up third order, in order to be able to compute the relevant trispectra.

The purpose of the present analysis is to determine, at the non-linear level, the perturbations of an arbitrary number of fluids, after a cosmological transition where one of the fluid, labelled $\sigma$, decays. For any fluid $A$, one can define, in a covariant way [22,23] (see also the recent review [24]), a fully non-linear curvature perturbation covector as

$$\zeta^A_\mu = \nabla_\mu \mathcal{N} + \frac{\nabla_\mu \rho_A}{3(\rho_A + P_A)},$$  \hspace{1cm} (1)

where $\rho_A$ and $P_A$ are energy density and pressure for the fluid A and $\mathcal{N} \equiv \int d\tau \nabla_\mu u^\mu/3$ is the number of e-folds along the fluid worldlines (with proper time $\tau$) with $u^\mu$ being the four-velocity of the fluid.

If $w_A \equiv P_A/\rho_A$ is constant, which will be assumed in the following as we will later consider only relativistic species ($w = 1/3$) and nonrelativistic species ($w = 0$), the covector (1) can be written as the total gradient of

$$\zeta_A = \delta \mathcal{N} + \frac{1}{3(1 + w_A)} \ln \left( \frac{\rho_A}{\bar{\rho}_A} \right),$$  \hspace{1cm} (2)
where $\delta N \equiv N - \bar{N}$ and the barred quantities are defined in a reference spacetime, which is strictly homogeneous and isotropic. The fully non-linear perturbation $\zeta_A$, which coincides with the definition of [25], is conserved on large scales for a non-interacting fluid.

We also introduce the non-linear isocurvature perturbation of any fluid $A$, with respect to the radiation fluid, as

$$ S_A \equiv 3(\zeta_A - \zeta_r), \quad (3) $$

where $\zeta_r$ is the curvature perturbation of the radiation fluid.

Let us now focus on the transition due to the decay. In the sudden decay approximation, which we adopt here, the decay takes place on the hypersurface characterized by the condition

$$ H_d = \Gamma_\sigma, \quad (4) $$

where $H_d$ is the Hubble parameter at the decay and $\Gamma_\sigma$ is the decay rate of $\sigma$. Since $H$ depends only on the total energy density, the decay hypersurface is a hypersurface of uniform total energy density, with $\delta N_d = \zeta$, where $\zeta$ is the global curvature perturbation. The equality between the sum of all energy densities, before the decay and after the decay, thus reads

$$ \sum_A \bar{\rho}_A e^{3(1+w_A)(\zeta_A - \zeta)} = \bar{\rho}_{\text{decay}} = \sum_B \bar{\rho}_B e^{3(1+w_B)(\zeta_B + \zeta)}, \quad (5) $$

where the subscripts $-$ and $+$ denote quantities defined, respectively, before and after the transition. In the above formula, we have used the non-linear energy densities of the individual fluids, which can be expressed in terms of their curvature perturbation $\zeta_A$ by inverting the expression (2).

### 2.1 Before the decay

The first equality in (5) implies

$$ \sum_A \Omega_A e^{\beta_A(\zeta_A - \zeta)} = 1, \quad \beta_A \equiv 3(1 + w_A), \quad (6) $$

where we have introduced the relative abundances $\Omega_A \equiv \bar{\rho}_A / \bar{\rho}_{\text{decay}}$, defined just before the decay.

The above relation determines $\zeta$ as a function of the $\zeta_A$‘s. Expanding it up to third order, one finds

$$ \zeta = \sum_A \lambda_A \left[ \zeta_A - \frac{\beta_A}{2} (\zeta_A - \zeta)^2 + \frac{\beta_A^2}{6} (\zeta_A - \zeta)^3 \right], \quad (7) $$
with the coefficients
\[ \lambda_A \equiv \frac{\tilde{\Omega}_A}{\Omega}, \quad \tilde{\Omega}_A \equiv (1 + w_A) \Omega_A, \quad \tilde{\Omega} \equiv \sum_A \tilde{\Omega}_A. \] (8)

The global perturbation \( \zeta \) appears on both sides of the relation (7), but one can use it iteratively to determine, order by order, the expression of \( \zeta \) in terms of all the \( \zeta_{A-} \), up to third order.

### 2.2 After the decay

In general, the fluid \( \sigma \) can decay into various species, and it is convenient to introduce the relative branching ratio \( \gamma_{A\sigma} \), defined as
\[ \gamma_{A\sigma} \equiv \frac{\Gamma_{A\sigma}}{\Gamma_\sigma}, \quad \Gamma_\sigma \equiv \sum_A \Gamma_{A\sigma}, \] (9)
where \( \Gamma_{A\sigma} \) is the decay rate to \( A \). Therefore, the energy density for any species \( A \), just after the decay of \( \sigma \), is simply given by
\[ \rho_{A+} = \rho_{A-} + \gamma_{A\sigma} \rho_\sigma. \] (10)

This relation, which is fully non-linear, can be reexpressed, upon using (2), in the form
\[ e^{\beta_A (\zeta_{A+} - \zeta)} = (1 - f_A) e^{\beta_A (\zeta_{A-} - \zeta)} + f_A e^{\beta_\sigma (\zeta_{\sigma-} - \zeta)}, \] (11)
where the parameter
\[ f_A \equiv \frac{\gamma_{A\sigma} \Omega_\sigma}{\Omega_A + \gamma_{A\sigma} \Omega_\sigma} \] (12)
represents the fraction of the fluid \( A \) that has been created by the decay.

Expanding the above expression (11) up to third order, and using (7), one gets
\[ \zeta_{A+} = \sum_B T_A^B \left[ \zeta_{B-} + \frac{\beta_B}{2} \left( \zeta_{B-} - \zeta \right)^2 + \frac{\beta_B^2}{6} \left( \zeta_{B-} - \zeta \right)^3 \right] \]
\[ -\frac{\beta_A}{2} \left( \zeta_{A+} - \zeta \right)^2 - \frac{\beta_A^2}{6} \left( \zeta_{A+} - \zeta \right)^3, \] (13)
with the coefficients
\[ T_A^A = f_A \left( 1 - \frac{\beta_\sigma}{\beta_A} \right) \lambda_A + (1 - f_A), \] (14)
\[ T_A^\sigma = f_A \left( 1 - \frac{\beta_\sigma}{\beta_A} \right) \lambda_\sigma + f_A \frac{\beta_\sigma}{\beta_A}, \] (15)
\[ T_A^C = f_A \left( 1 - \frac{\beta_\sigma}{\beta_A} \right) \lambda_C, \quad C \neq A, \sigma. \] (16)
At second order, after substituting the first order expression for $\zeta$ that follows from (7), Equation (13) yields

$$\zeta_{A+} = \sum_B T_A^B \zeta_{B-} + \sum_{B,C} U_A^{BC} \zeta_{B-}\zeta_{C-},$$

with

$$U_A^{BC} \equiv \frac{1}{2} \left[ \sum_E \beta_E T_A^E (\delta_{EB} - \lambda_B)(\delta_{EC} - \lambda_C) - \beta_A (T_{AB} - \lambda_B)(T_{AC} - \lambda_C) \right].$$

Note that the above expression for $U_A^{BC}$ corresponds to the symmetrized version (with respect to the two indices $B$ and $C$) of the expression given in [15]. This does not change the expansion (17) since $U$ is contracted with a symmetric term.

At third order, (13) yields, after using (7) up to second order,

$$\zeta_{A+} = \sum_B T_A^B \zeta_{B-} + \sum_{B,C} U_A^{BC} \zeta_{B-}\zeta_{C-} + \sum_{B,C,D} V_A^{BCD} \zeta_{B-}\zeta_{C-}\zeta_{D-},$$

with

$$V_A^{BCD} \equiv -\frac{1}{2} \sum_{E,F} \beta_E T_A^E (\delta_{EB} - \lambda_B)\lambda_F \beta_F (\delta_{FC} - \lambda_C) (\delta_{FD} - \lambda_D)$$

$$+ \frac{1}{6} \sum_E \beta_E^2 T_A^E (\delta_{EB} - \lambda_B)(\delta_{EC} - \lambda_C)(\delta_{ED} - \lambda_D)$$

$$- \beta_A (T_{AB} - \lambda_B) \left[ U_A^{CD} - \frac{1}{2} \sum_E \beta_E \lambda_E (\delta_{EC} - \lambda_C)(\delta_{ED} - \lambda_D) \right]$$

$$- \frac{1}{6} \beta_A^2 (T_{AB} - \lambda_B)(T_{AC} - \lambda_C)(T_{AD} - \lambda_D).$$

The above expression is the main result of this section. It provides a systematic computation of the post-decay curvature perturbations for all fluids in a very general setting. For scenarios with several decay transitions, the perturbations can be obtained by combining the various expressions of the type (18) for each transition. This was done explicitly, up to second order, for a scenario with two curvaton in [15]. In the present work, we will consider only single curvaton scenarios and thus use (18) only once.

### 3 Mixed curvaton and inflaton scenario

We now apply the general formalism given in the previous section to the mixed curvaton scenario, where fluctuations of both the curvaton and the
inflaton are taken into account\textsuperscript{#2}.

Although CDM isocurvature fluctuations and their non-Gaussianity have been discussed in several works \cite{11,31–35}, it is usually assumed that all of the CDM (or baryons for baryon isocurvature perturbations) is produced either before the curvaton decay or as a product of the curvaton decay. However, one can also envisage (as in \cite{12,15}) intermediate situations, where the CDM is produced both before and during the curvaton decay. Such cases can be treated in the formalism discussed in the previous section and lead, as we will see, to interesting observational consequences.

Here we consider a simplified scenario with three initial species: radiation \((r)\), CDM \((c)\) and a curvaton \((\sigma)\). We also assume that the potential for the curvaton is quadratic\textsuperscript{#3}, and we thus treat \(\sigma\) as a pressureless fluid, which then decays into radiation and CDM.

### 3.1 Perturbations after the decay

#### 3.1.1 Transfer matrix

In the present case, we find that the linear transfer matrix, whose coefficients are defined in (14)–(16), is given by

\[
T = \begin{pmatrix}
1 - r & x_c & r - x_c \\
0 & 1 - f_c & f_c \\
0 & 0 & 0
\end{pmatrix}, \quad r \equiv f_r \frac{\Omega}{\tilde{\Omega}}, \quad x_c \equiv \frac{1}{4} \Omega_c r,
\]

where the order of the species is \((r,c,\sigma)\). We have used the definitions (8) for the coefficients \(\lambda_A\), which leads in particular to \(\lambda_r = 4(1 - \tilde{\Omega}^{-1})\) (since \(\tilde{\Omega} = 1 + \Omega_r/3\) in our scenario with two nonrelativistic species).

In the following, we will assume \(\Omega_c \ll 1\), since the decay must occur deep in the radiation dominated era\textsuperscript{#4}. As a consequence, we assume \(\lambda_c = 0\) and, therefore, \(x_c = 0\). This also implies \(\tilde{\Omega} = (4 - \Omega_\sigma)/3\).

In order to compare easily our results with the existing literature, it will be convenient to rewrite the parameter \(r\) as

\[
r \equiv \xi \tilde{r},
\]

\textsuperscript{#2}The adiabatic (curvature) perturbation in such a scenario has been investigated in [17–21].

\textsuperscript{#3}The self-interacting curvaton model which include some non-quadratic potential has been studied in [26–30].

\textsuperscript{#4}Note that, although \(\Omega_c\) is assumed to be very small, it cannot be neglected in the expression for \(f_c\) because \(\gamma_c\) or \(\Omega_\sigma\) can be very small, and \(f_c\) can take any value between 0 and 1.
where the quantity
\[
\xi \equiv \frac{f_r}{\Omega_\sigma} = \frac{\gamma_r \sigma}{1 - (1 - \gamma_r \sigma) \Omega_\sigma}
\] (21)
can be interpreted as the efficiency of the energy transfer from the curvaton into radiation (\(\xi = 1\) if the curvaton decays only into radiation, i.e. \(\gamma_r \sigma = 1\)), and
\[
\tilde{r} = \frac{3 \Omega_\sigma}{4 - \Omega_\sigma}
\] (22)
is the familiar coefficient that appears in the literature on the curvaton, which coincides with our \(r\) only if \(\xi = 1\).

### 3.1.2 Nonlinear perturbations

It is now straightforward to obtain the post-decay curvature perturbations for each of the remaining fluid, up to third order, by using our general result (18).

The expression for the CDM curvature perturbation reads
\[
\zeta_{c+} = \zeta_c + \frac{1}{3} f_c (S_\sigma - S_c) + \frac{1}{6} f_c (1 - f_c) (S_\sigma - S_c)^2 + \frac{1}{18} f_c (1 - 3 f_c + 2 f_c^2) (S_\sigma - S_c)^3,
\] (23)
where, on the right-hand side, we have substituted \(\zeta_\sigma - \zeta_c = (S_\sigma - S_c)/3\) and the pre-decay subscript is implicit for all quantities.

The above expression, which turns out to be valid also for an arbitrary value of \(x_c\), can in fact be derived much more rapidly by using directly (11) for the CDM fluid. Indeed, since \(\beta_c = \beta_\sigma = 3\), one gets
\[
\zeta_{c+} = \zeta + \frac{1}{3} \ln \left[ (1 - f_c) e^{3(\zeta_c - \zeta)} + f_c e^{3(\zeta_\sigma - \zeta)} \right] = \zeta + \frac{1}{3} \ln \left[ 1 + f_c \left( e^{S_\sigma - S_c} - 1 \right) \right],
\] (24)
and expanding the last expression gives immediately (23).

The curvature perturbation for radiation is much more complicated in the general case. In the limit \(x_c = 0\), the full expression reduces to
\[
\zeta_{r+} = \zeta_{r-} + \frac{r}{3} \frac{S_{\sigma-}}{S_{\sigma-}} + \frac{r}{18} \left[ 3 - 4r + \frac{2r}{\xi} - \frac{r^2}{\xi^2} \right] \frac{S_{\sigma-}^2}{S_{\sigma-}}
+ \frac{r}{162} \left[ 9 + 18(1 - 2\xi) \frac{r}{\xi} + 4 \left( 8\xi^2 - 6\xi - 3 \right) \frac{r^2}{\xi^2} + 2(6\xi - 1) \frac{r^3}{\xi^3} + 3 \frac{r^4}{\xi^4} \right] S_{\sigma-}^3
\] (25)
In the limit \(\xi = 1\), one recovers the expression given in [36].
3.2 Primordial adiabatic and isocurvature perturbations

In order to determine the statistical properties of the primordial adiabatic and isocurvature perturbations, defined at the onset of the standard cosmological era, i.e. after the curvaton decay, one needs to relate the perturbation of the curvaton fluid to the fluctuations of the curvaton field, generated during inflation.

For a massive curvaton without self-interaction, the relation between the isocurvature perturbation $S_\sigma$ and the fluctuation $\delta \sigma$ is simply given by the non-linear expression

$$e^{S_\sigma} = \left(1 + \frac{\delta \sigma}{\bar{\sigma}}\right)^2.$$  

(26)

This result can be obtained by writing the (non-linear) energy density of the oscillating curvaton defined on the spatially flat hypersurfaces, characterized by $\delta N = \zeta_r$ when the curvaton is still subdominant:

$$\rho_\sigma = m^2 \sigma^2 = m^2 (\bar{\sigma} + \delta \sigma)^2 = \bar{\rho}_\sigma e^{3(\zeta_r - \zeta_\sigma)} = \bar{\rho}_\sigma e^{S_\sigma}.$$  

(27)

Expanding the expression (26) up to third order, and using the conservation of $\delta \sigma / \sigma$ in a quadratic potential, we obtain

$$S_\sigma = \hat{S} - \frac{1}{4} \hat{S}^2 + \frac{1}{12} \hat{S}^3,$$  

(28)

where the quantity

$$\hat{S} \equiv 2 \frac{\delta \sigma}{\bar{\sigma}}$$  

(29)

is Gaussian.

For simplicity, we now restrict our analysis to the situation where

$$\zeta_{c-} = \zeta_{r-} \equiv \zeta_{\text{inf}},$$  

(30)

by assuming that the CDM and radiation perturbations, before the curvaton decay, depend only on the inflaton fluctuations. This means that, effectively, there are only two independent degrees of freedom from the inflationary epoch, $\zeta_{\text{inf}}$ and $\hat{S}$.

Substituting (28) into (25) and (23) then yields the primordial curvature perturbation

$$\zeta_r = \zeta_{\text{inf}} + z_1 \hat{S} + \frac{1}{2} z_2 \hat{S}^2 + \frac{1}{6} z_3 \hat{S}^3,$$  

(31)
with
\[ z_1 = \frac{r}{3}, \quad z_2 = \frac{r}{18} \left( 3 - 8r + \frac{4r}{\xi} - 2\frac{r^2}{\xi^2} \right), \]  \hspace{1cm} (32)
\[ z_3 = \frac{r^2}{54} \left( \frac{6r^3}{\xi^4} + \frac{24r^2}{\xi^2} - \frac{4r^2}{\xi^3} - \frac{48r}{\xi} - \frac{15r}{\xi^2} + 64r + \frac{18}{\xi} - 36 \right), \]  \hspace{1cm} (33)
and the primordial isocurvature perturbation
\[ S_c = s_1 \dot{S} + \frac{1}{2} s_2 \dot{S}^2 + \frac{1}{6} s_3 \dot{S}^3, \]  \hspace{1cm} (34)

with
\[ s_1 = f_c - r, \quad s_2 = \frac{1}{6} \left( 3f_c(1 - 2f_c) + \frac{2r^3}{\xi^2} - \frac{4r^2}{\xi} + 8r^2 - 3r \right), \]  \hspace{1cm} (35)
\[ s_3 = -\frac{1}{2} f_c^2(3 - 4f_c) - \frac{r^2}{18} \left( \frac{6r^3}{\xi^4} + \frac{24r^2}{\xi^2} - \frac{4r^2}{\xi^3} - \frac{48r}{\xi} - \frac{15r}{\xi^2} + 64r + \frac{18}{\xi} - 36 \right). \]  \hspace{1cm} (36)

These expressions will be useful to determine the power spectrum, the bispectrum and the trispectrum, which can be probed by observations.

### 3.3 Power spectrum

The power spectrum for the total curvature perturbation is given, according to (31), by
\[ P_\zeta = P_{\zeta_{\text{inf}}} + \frac{r^2}{9} P_{\dot{S}} \equiv (1 + \lambda) P_{\zeta_{\text{inf}}} \equiv \Xi^{-1} \frac{r^2}{9} P_{\dot{S}}, \]  \hspace{1cm} (37)
where \( \lambda \) is the ratio between the curvaton and inflaton contributions,
\[ \lambda \equiv \frac{(r^2/9) P_{\dot{S}}}{P_{\zeta_{\text{inf}}}}, \]  \hspace{1cm} (38)
and \( \Xi = 1/(1+\lambda^{-1}) \) represents the fraction of the power spectrum due to the curvaton contribution. Since the contribution from the inflaton (for standard slow-roll single field inflation) can be written as
\[ P_{\zeta_{\text{inf}}} = \frac{1}{2\epsilon} P_{\delta \phi}, \]  \hspace{1cm} (39)
with \( P_{\delta \phi} \) being the power spectrum of the inflaton fluctuations \( \delta \phi \) and \( \epsilon = (1/2)(dV/d\phi)^2/V^2 \) being the slow-roll parameter. The parameter \( \lambda \) is explicitly given by
\[ \lambda = \frac{8r^2\epsilon}{9\sigma^2_{\phi}}, \]  \hspace{1cm} (40)
Thus $\lambda$ depends on the curvaton model parameters, as well as on the inflation model.

We can also write down the power spectrum for the isocurvature fluctuations. Using Eq. (34), we find

$$P_{S_\epsilon} = (f_c - r)^2 P_\zeta. \quad (41)$$

In our model, both curvature and isocurvature perturbations depend on the curvaton fluctuations. Therefore, the two types of perturbations are correlated, as quantified by the correlation coefficient:

$$C = \frac{P_{S_\epsilon, \zeta}}{\sqrt{P_{S_\epsilon} P_{\zeta}}} = \varepsilon_f \sqrt{\Xi}, \quad \varepsilon_f \equiv \text{sgn}(f_c - r). \quad (42)$$

In the pure curvaton limit ($\Xi \approx 1$), adiabatic and isocurvature perturbations are either fully correlated, if $\varepsilon_f > 0$, or fully anti-correlated, if $\varepsilon_f < 0$. In the opposite limit ($\Xi \ll 1$), the correlation vanishes. For intermediate values of $\Xi$, the correlation is only partial, as can also be obtained in multifield inflation [37].

Finally, it is convenient to introduce the ratio of the isocurvature power spectrum with respect to the adiabatic one

$$\alpha = \frac{P_{S_\epsilon}}{P_{\zeta}} = 9 \left(1 - \frac{f_c}{r}\right)^2 \Xi. \quad (43)$$

This ratio can be strongly constrained by cosmological observations, but the precise limits depend on the assumed level of correlation between the isocurvature and adiabatic perturbations (since the impact of isocurvature perturbations on the observable power spectrum depends crucially on this correlation, as illustrated in [38]). Constraints on $\alpha$ from the latest data have been published only for the uncorrelated and fully correlated cases. In terms of the parameter $a \equiv \alpha/(1 + \alpha)$, the limits (based on WMAP+BAO+SN data) given in [2] are\(^5\)

$$a_0 < 0.064 \text{ (95\%CL)}, \quad a_1 < 0.0037 \text{ (95\%CL)}, \quad (44)$$

respectively for the uncorrelated case ($\Xi = 0$) and for the fully correlated case ($\Xi = 1$).

According to the expression (43), the observational constraint $\alpha \ll 1$ can be satisfied if $|f_c - r| \ll r$ (which includes the case $f_c = 1$ with $r \approx 1$) or if $\Xi \ll 1$, i.e. the curvaton contribution to the observed power spectrum is very small.

\(^5\)Our notations differ from those of [2]: our $a$ corresponds to their $\alpha$ and our fully correlated limit corresponds to their fully anti-correlated limit, because their definition of the correlation has the opposite sign (see also [39] for a more detailed discussion).
3.4 Bispectrum

We now discuss the three-point functions for our curvature and isocurvature perturbations, summarizing the results obtained in [15].

Let us start by considering, in the general case of an arbitrary number of observable quantities $X^I$, all possible bispectra, defined by

$$\langle X^I_{k_1} X^J_{k_2} X^K_{k_3} \rangle = (2\pi)^3 \delta(\Sigma_i k_i) B^{IJK}(k_1, k_2, k_3). \quad (45)$$

We then assume that the $X^I$ can be written, up to second order, in the form

$$X^I = N^I_a \phi^a + \frac{1}{2} N^I_{ab} \delta \phi^a \delta \phi^b + \ldots \quad (46)$$

(with implicit summation over the indices $a$ and $b$), where the $N^I_a$ and $N^I_{ab}$ are arbitrary background-dependent coefficients and the $\delta \phi^a$ are Gaussian fluctuations, generated during inflation and characterized by their two-point correlation functions

$$\langle \delta \phi^a(k) \delta \phi^b(k') \rangle = (2\pi)^3 P^{ab}(k) \delta(k + k'). \quad (47)$$

Substituting the decomposition (46) into the left hand side, and using (47), one finds [40]

$$B^{IJK}(k_1, k_2, k_3) = N^I_a N^J_b N^K_{cd} P^{ae}(k_1) P^{bd}(k_2) + N^I_a N^J_{bc} N^K_d P^{ab}(k_1) P^{cd}(k_3) + N^I_{ab} N^J_c N^K_d P^{ac}(k_2) P^{bd}(k_3). \quad (48)$$

In our particular case, we have only two observables $X^I = \{\zeta, S_c\}$, corresponding to two indices which we denote $I = \{\zeta, S\}$. Moreover, since there is only one Gaussian degree of freedom, $\hat{S}$, in the non-linear expansions for $\zeta$ and $S_c$, respectively (31) and (34), the bispectra reduce to

$$B^{IJK}(k_1, k_2, k_3) = b^I_{NL} P_S(k_2) P_S(k_3) + b^{IJK}_{NL} P_S(k_1) P_S(k_3) + b^{KIJ}_{NL} P_S(k_1) P_S(k_2), \quad (49)$$

with

$$b^I_{NL} \equiv N^I_{(2)} N^I_{(1)}, \quad (50)$$

where $N^\zeta_{(2)} = z_2$, $N^S_{(2)} = s_2$, $N^\zeta_{(1)} = z_1$, $N^S_{(1)} = s_1$, respectively.

In order to compare these coefficients with the usual parameter $f_{NL}$ defined in the purely adiabatic case, one must remember that $f_{NL}$ is proportional to the bispectrum of $\zeta$ divided by the square of the power spectrum.

---

#6 All the nonlinear coefficients that we introduce are of the local type and we drop the superscript “local” for simplicity.
The analogs of \( f_{NL} \) can therefore be defined by dividing the coefficient \( b_{NL}^{IJK} \) by the square of the ratio \( P_\zeta / P_\hat{S} = z_1^2 / \Xi \):

\[
\tilde{f}_{NL}^{IJK} \equiv \frac{6}{5} f_{NL}^{IJK} \equiv \frac{\Xi^2}{z_1^4} b_{NL}^{IJK}.
\]

(51)

Taking into account the fact that the last two indices can be permuted, this leads to six different coefficients, explicitly given by the expressions

\[
\tilde{f}_{NL}^{\zeta,\zeta\zeta} = \frac{s_2}{z_1^2} \Xi^2, \quad \tilde{f}_{NL}^{\zeta,\zeta S} = \frac{s_1 s_2}{z_1^2} \Xi^2, \quad \tilde{f}_{NL}^{S,\zeta\zeta} = \frac{s_2^2}{z_1^2} \Xi^2, \quad \tilde{f}_{NL}^{\zeta,SS} = \frac{s_1^2}{z_1^2} \Xi^2, \quad f_{NL}^{S,\zeta S} = \frac{s_2}{z_1^2} \Xi^2, \quad f_{NL}^{S,SS} = \frac{s_2^2}{z_1^2} \Xi^2.
\]

(52)

(53)

It is worth noting that all the coefficients are related via the two rules

\[
f_{NL}^{I,JS} = \frac{s_1}{z_1} f_{NL}^{I.J\zeta}, \quad f_{NL}^{S,II} = \frac{s_2}{z_2} f_{NL}^{\zeta,II}.
\]

(54)

Therefore, the hierarchy between the parameters can be deduced from the value of the ratios \( s_1/z_1 \) and \( s_2/z_2 \), which are given in the small \( r \) limit (assuming \( \xi = 1 \)) by the simple expressions

\[
\frac{s_1}{z_1} = \sqrt{\frac{\alpha}{\Xi}} = 3 \left( \frac{f_c}{r} - 1 \right), \quad \frac{s_2}{z_2} \simeq \frac{3 f_c (1 - 2f_c)}{r} - 3, \quad (r \ll 1, \xi = 1).
\]

(55)

Observational constraints on isocurvature non-Gaussianities are given in [13], for an isocurvature perturbation of the form \( S = \hat{S} + f_{NL}^{(iso)} \hat{S}^2 \), where \( \hat{S} \) is Gaussian. The relations between the non-linear parameter \( f_{NL}^{(iso)} \) and the parameters defined above are the following: \( \tilde{f}_{NL}^{S,SS} = 2 f_{NL}^{(iso)} \alpha^2 \), \( \tilde{f}_{NL}^{S,\zeta S} = 2 f_{NL}^{(iso)} \alpha^{3/2} |C| \) and \( f_{NL}^{\zeta,\zeta\zeta} = 2 f_{NL}^{(iso)} \alpha C^2 \), where \( \alpha \) and \( C \) are respectively defined in (43) and (42).

3.5 Trispectrum

We now turn to the novel part of this work, which consists of the study of the trispectrum.

Let us first consider the general case of several observables \( X^I \) and let us introduce the trispectra \( T^{IJKL} \) defined from the connected four-point correlation functions as

\[
\langle X^I_{k_1} X^J_{k_2} X^K_{k_3} X^L_{k_4} \rangle_c \equiv (2\pi)^3 \delta(\sum_i k_i) T^{IJKL}(k_1, k_2, k_3, k_4).
\]

(56)
If the observables can be written, up to third order, in the form
\[ X^I = N^I_a \delta \phi^a + \frac{1}{2} N^I_{ab} \delta \phi^a \delta \phi^b + \frac{1}{6} N^I_{abc} \delta \phi^a \delta \phi^b \delta \phi^c + \ldots \] (57)

one finds that the trispectra are given by [41]
\[ T_{IJKL}(k_1, k_2, k_3, k_4) = N^I_a N^J_b N^K_c N^L_d P^{a_1b}(k_2)P^{a_2c}(k_3)P^{a_3d}(k_4) + 3 \text{ perms} \]
\[ + N^I_{a_1a_2} N^J_{b_1b_2} N^K_c N^L_d [P^{a_1c}(k_3)P^{b_1d}(k_4)P^{a_2b_2}(k_13) + P^{b_2c}(k_3)P^{a_1d}(k_4)P^{a_2b_2}(k_14)] + 5 \text{ perms}, \] (58)

with the notation \( k_{13} \equiv |k_1 + k_3| \), etc.

In our particular case, where there is only one Gaussian degree of freedom at the non-linear level, the trispectra reduce to
\[ T_{IJKL}(k_1, k_2, k_3, k_4) = t_{NL}^{IJKL} P_S(k_2)P_S(k_3)P_S(k_4) + 3 \text{ perms} \]
\[ + \hat{t}_{NL}^{IJKL} [P_S(k_3)P_S(k_4)P_S(k_{13}) + P_S(k_3)P_S(k_4)P_S(k_{14})] + 5 \text{ perms}, \] (59)

with
\[ t_{NL}^{IJKL} \equiv N^I_{(3)} N^J_{(1)} N^K_{(1)} N^L_{(1)}, \quad \hat{t}_{NL}^{IJKL} = N^I_{(2)} N^J_{(2)} N^K_{(1)} N^L_{(1)}, \] (60)

where \( N^\zeta_{(3)} = z_3 \) and \( N^S_{(3)} = s_3 \), in analogy with the notations introduced previously. Taking into account the symmetries under permutations of the indices, one finds, for two observables (\( I = \{ \zeta, S \} \)), 8 different parameters \( t_{NL}^{IJKL} \) and 9 parameters \( \hat{t}_{NL}^{IJKL} \).

In order to facilitate the comparison with the parameters \( \tau_{NL} \) and \( g_{NL} \), which have been defined in the purely adiabatic case by dividing the trispectrum by the cube of the power spectrum, it is convenient to rescale the above parameters and to introduce the coefficients
\[ \tau_{NL}^{IJKL} \equiv \frac{N^I_{(3)} N^J_{(1)} N^K_{(1)} N^L_{(1)}}{z_1^6} \Xi^3, \] (61)

which depend only on the second order non-linearities, and the coefficients
\[ \hat{g}_{NL}^{IJKL} \equiv \frac{54}{25} g_{NL}^{IJKL} \equiv \frac{N^I_{(3)} N^J_{(1)} N^K_{(1)} N^L_{(1)}}{z_1^6} \Xi^3, \] (62)

which depend on the third order non-linearities.

For the purely adiabatic parameters, we obtain
\[ \tau_{NL}^{\zeta,\zeta} = \frac{z_2^2}{z_1^3} \Xi^3, \] (63)
\[ \hat{g}_{NL}^{\zeta,\zeta} = \frac{z_3^2}{z_1^3} \Xi^3, \] (64)
which exactly coincide with the usual $\tau_{NL}$ and $(54/25)g_{NL}$. The present constraints on these parameters, assuming the data do not contain any isocurvature contribution, are [42]

$$-7.4 < 10^{-5}g_{NL} < 8.2 \text{ (95\% CL)}, \quad -0.6 < 10^{-4}\tau_{NL} < 3.3 \text{ (95\% CL)}.$$

Similarly for the (purely) isocurvature mode, we have

$$\tau_{NLSS} = \frac{s^2_1 s^2_2}{z^6_1} \Xi^3, \quad g_{NL} = \frac{s^3_2}{z^6_1} \Xi^3. \quad (66)$$

In addition to the purely adiabatic or isocurvature nonlinear coefficients, we also find the following cross-correlated nonlinear coefficients:

$$\tau_{NL\zeta\zeta} = \frac{s_1 z^3_1}{z^5_1} \Xi^3, \quad \tau_{NLSS} = \frac{s_1 s^2_2}{z^5_1} \Xi^3, \quad \tau_{NL\zeta S} = \frac{s^3_2}{z^4_1} \Xi^3, \quad (71)$$

$$g_{NL} = \frac{s_3}{z^3_1} \Xi^3, \quad g_{NL\zeta \zeta} = \frac{s_3}{z^4_1} \Xi^3, \quad (72)$$

These nonlinear coefficients can be related by very simple rules, in analogy with the rules obtained for the bispectrum coefficients. The first rule is that the replacement of an index $\zeta$ by $S$ on the right hand side of the comma corresponds to a rescaling by $s_1/z_1$:

$$g_{NL} = \frac{s_1}{z_1} g_{NL}, \quad \tau_{NL} = \frac{s_1}{z_1} \tau_{NL}. \quad (73)$$

The replacement of $\zeta$ by $S$ on the left hand side of the comma leads to a rescaling by $s_2/z_2$ for the $\tau_{NL}^{JKL}$ and a rescaling by $s_3/z_3$ for the $g_{NL}^{JKL}$:

$$g_{NL} = \frac{s_1}{z_1} g_{NL}, \quad \tau_{NL} = \frac{s_1}{z_1} \tau_{NL}. \quad (74)$$
These rules will be extremely useful in the subsequent discussion, enabling us to establish hierarchies between the 17 nonlinear coefficients defined above.

Moreover, there are consistency relations between the trispectrum parameters $\tau_{NL}^{I,J,KL}$ and the bispectrum parameters $f_{NL}^{IJK}$. For example, one finds

$$\tau_{NL}^{I,J,\zeta\zeta} = \Xi^{-1} f_{NL}^{I,J,\zeta\zeta} f_{NL}^{J,\zeta\zeta},$$

relating the trispectrum parameters to the bispectrum ones, which generalizes the purely adiabatic consistency relation $\tau_{NL} = (6f_{NL}/5)^2/\Xi$, discussed in [16].

### 3.6 Quantitative discussion

In the pure curvaton scenario, when CDM is created before the curvaton decay (i.e. $f_c = 0$), large isocurvature fluctuations (correlated with the adiabatic ones) are generated and they turn out to be too large to be compatible with current data. One way out is to consider the mixed inflaton and curvaton scenario, where the inflaton fluctuations also contribute to the observed power spectrum, in addition to the curvaton ones. The isocurvature fluctuations are then “diluted,” (by the $\Xi$ factor) and can thus be made consistent with observations, as studied in [11, 31, 34, 35].

At the same time, since non-Gaussianity in these models originates from the curvaton sector, one can naively expect that non-Gaussianity will also be small if the curvaton contribution to the power spectrum is small. Thus it seems that large non-Gaussianity is difficult to realize without conflicting with the isocurvature constraint, although $f_{NL} > O(10)$ is still possible while satisfying isocurvature constraint [34, 35]. However, in the models investigated in the present work, we also include the possibility for CDM to be created both from the curvaton decay and from some pre-decay epoch. As we will see explicitly below, this leads to very interesting consequences in the parameter space.

#### 3.6.1 Bispectrum parameters

In Fig. 1, we have plotted the contours of $\tilde{f}_{NL}^{\zeta,\zeta\zeta}$ and $\tilde{f}_{NL}^{S,SS}$ in the $\lambda-r$ parameter plane, for two values of $f_c$ and assuming $\xi = 1$.

We recall that the parameter $\lambda$, defined in (40) as the ratio between the curvaton and inflaton contributions to the power spectrum, is directly related to the parameter $\Xi$, which we have preferred to use in the analytical expressions for the nonlinear coefficients, by the relation

$$\lambda = \frac{\Xi}{1 - \Xi}.$$
In the limit $\lambda \ll 1$, one gets $\Xi \simeq \lambda$, whereas $\Xi \simeq 1 - \lambda^{-1}$ in the limit $\lambda \gg 1$.

The constraint ($a_0 < 0.064$ at 95% C.L.) on uncorrelated isocurvature mode from WMAP7 is also used to identify regions still allowed by the current data. For small values of $r$, the purely adiabatic parameter is given by

$$f_{NL}^{\zeta,\zeta} \simeq \frac{3}{2r} \Xi^2,$$

and the other $f_{NL}^{I,JK}$ can be deduced from it by appropriate factors of $s_1/z_1$ or $s_2/z_2$, according to (54).

In Fig. 2, we have plotted all the parameters $f_{NL}^{I,JK}$ as a function of $f_c$, for fixed (and small) values of $\Xi$ and $r$. The figure shows how the parameters evolve from the region $f_c \ll r$ to the region $f_c \gg r$. The figure clearly illustrates the various hierarchies between the non-linearity coefficients, which we discuss below.

---

Figure 1: Contours for the bispectrum coefficients $f_{NL}^{\zeta,\zeta}$ and $f_{NL}^{S,SS}$ in the $\lambda$--$r$ plane in the cases with $f_c = 0$ (left) and $10^{-4}$ (right). The regions $a > 0.064$ are shaded.

---

#7 Although this limit, strictly speaking, applies only for the region $\lambda \ll 1$, we have used the same limit for the intermediate region $\lambda \sim 1$ as well as for the region $\lambda \gg 1$. In the latter region, we know that the real limit is more stringent when $s_f = 1$ since it is given by the constraint on $a_1$. But constraints from the current data on $a_{-1}$ or $a_{1/2}$ with intermediate values for $\Xi$ are not given in the literature, so we have preferred to use the limit on $a_0$ everywhere as indicative constraints on the parameter space.
Figure 2: Plots of the coefficients $\tilde{f}^{IJK}_{NL}$ as functions of $f_c$. The indices $I, J, K$ are specified in the figure for each curve. The other parameters are $\xi = 1$, $\lambda = 10^{-3} \simeq \Xi$ and $r = 10^{-5}$.

In the case of $f_c = 0$, illustrated in the first plot of Fig. 1, both factors $s_1/z_1$ and $s_2/z_2$ reduce to $(-3)$. As a consequence, the $f^{IJK}_{NL}$ are slightly enhanced with respect to $f^{\zeta \zeta \zeta}_{NL}$ by a factor $(-3)^{I_S}$ where $I_S$ is the number of indices $S$. In particular the pure isocurvature parameter is $(-27)$ times $f_{NL}$ as confirmed by the figures. The constraint on the isocurvature power spectrum imposes $\Xi \ll 1$ and the non-Gaussianity can therefore be significant only if $r$ is sufficiently small to compensate the $\Xi^2$ suppression.

When $f_c$ does not vanish, as illustrated in the second plot of Fig. 1, a new region appears in the parameter space, where the isocurvature constraint is satisfied even if $\Xi \simeq 1$. This requires a fine-tuning between $f_c$ and $r$, more precisely, that of

$$f_c - r \simeq \varepsilon_f \frac{\sqrt{\alpha}}{3} r \quad (\Xi \simeq 1), \quad (78)$$

due to (43). With respect to the purely adiabatic non-Gaussianity, the other types of non-Gaussianities are suppressed in general (except $f^{S \zeta \zeta}_{NL}$, which could be of the same order of magnitude in the particular case $\xi \ll 1$ and $\tilde{r} \equiv r/\xi \sim 1$).
In the case $r \ll f_c \ll 1$, another interesting feature appears in the region $\Xi \ll 1$. Indeed, the isocurvature bispectrum amplitude $f_{NL}^{S,SS}$ becomes larger than $f_{NL}^{\zeta,\zeta\zeta}$, as pointed out in [15]. The reason is that the factors $s_1/z_1$ and $s_2/z_2$ become $3f_c/r$ in this region so that the $f_{NL}$ with adiabatic indices are suppressed with respect to the purely isocurvature non-Gaussianity. In this region, $f_{NL}^{\zeta,\zeta\zeta}$ and $f_{NL}^{S,SS}$ are respectively given by

$$f_{NL}^{\zeta,\zeta\zeta} \approx \frac{3}{2r} \Xi^2 \approx \frac{r^3}{54 f_c^4} \alpha^2, \quad f_{NL}^{S,SS} \approx \frac{81 f_c^3}{2 r^4} \Xi^2 \approx \frac{\alpha^2}{2 f_c}, \quad (r \ll f_c \ll 1), \quad (79)$$

from which we can see that $f_{NL}^{S,SS}$ is enhanced by the factor $(3f_c/r)^3$ compared to $f_{NL}^{\zeta,\zeta\zeta}$ and can be significant if $f_c$ is well below $\alpha^2$. In such a scenario, future observations would thus detect non-Gaussianity from the isocurvature perturbations rather than from the adiabatic ones. For the correlated nonlinear coefficients $f_{NL}^{I,J,K}$, we obtain

$$f_{NL}^{\zeta,\zeta S} \approx f_{NL}^{S,\zeta \zeta} \approx \frac{9 f_c}{2 r^2} \Xi^2, \quad f_{NL}^{\zeta,SS} \approx f_{NL}^{S,\zeta S} \approx \frac{27 f_c^2}{2 r^3} \Xi^2. \quad (80)$$

Their value depends only on the total number of adiabatic indices. The hierarchy between the non-linearity parameters depends only on the ratio $f_c/r$.

In the opposite limit where $f_c$ is much smaller than $r$ ($f_c \ll r \ll 1$), all the non-linearity coefficients become independent of $f_c$. Once again, the amplitude of the purely isocurvature coefficient is enhanced (now by a factor 27) with respect to the purely adiabatic one, but the sign is changed:

$$f_{NL}^{\zeta,\zeta\zeta} \approx \frac{3}{2r} \Xi^2 \approx \frac{\alpha^2}{54r}, \quad f_{NL}^{S,SS} \approx -\frac{81 f_c^3}{2 r^4} \Xi^2. \quad (81)$$

The correlated nonlinear coefficients have intermediate values,

$$f_{NL}^{\zeta,\zeta S} \approx f_{NL}^{\zeta,\zeta\zeta} \approx -\frac{9 f_c}{2 r} \Xi^2, \quad f_{NL}^{\zeta,SS} \approx f_{NL}^{S,\zeta \zeta} \approx \frac{27 f_c^2}{2 r} \Xi^2. \quad (82)$$

All these values differ simply by powers of $(-3)$, since $s_1/z_1 \simeq s_2/z_2 \simeq -3$ in this limit. An interesting consequence is that the nonlinear coefficients with an odd number of isocurvature indices are negative.

### 3.6.2 Trispectrum parameters

The contours of the purely adiabatic and isocurvature nonlinear coefficients $\tilde{r}_{NL}^{\zeta,\zeta\zeta}$ and $\tilde{r}_{NL}^{S,SS,SS}$ have been plotted in Fig. 3, while the coefficients $\tilde{g}_{NL}^{\zeta,\zeta\zeta}$ and
$g^{SS,SS}_{NL}$ are depicted in Fig. 4. In these figures, the region excluded by current observations of the isocurvature mode is again shaded.

For small values of $r$, the purely adiabatic trispectrum coefficients are (for $\xi = 1$)

$$
\tau^{\zeta\zeta,\zeta\zeta}_{NL} \simeq \frac{9}{4r^2} \Xi^3, \quad \tilde{g}^{\zeta,\zeta\zeta}_{NL} \simeq -\frac{9}{r} \Xi^3, \quad (r \ll 1, \xi = 1).
$$

They can be also related to the purely adiabatic nonlinear coefficient $\tilde{f}_{NL} \equiv \tilde{f}_{NL}^{\zeta\zeta}$ by the following expressions [16]:

$$
\tau^{\zeta\zeta,\zeta\zeta}_{NL} = \frac{\tilde{f}_{NL}^2}{\Xi}, \quad g^{\zeta,\zeta\zeta}_{NL} \simeq -\frac{10}{3} f_{NL} \Xi.
$$

This implies that, when $\Xi \ll 1$, $\tau^{\zeta\zeta,\zeta\zeta}_{NL}$ is enhanced with respect to $\tilde{f}_{NL}^2$ whereas, by contrast, $g^{\zeta,\zeta\zeta}_{NL}$ is suppressed with respect to $f_{NL}$.

For $f_c = 0$, all $\tau^{IJ,KL}_{NL}$ are slightly enhanced with respect to the adiabatic $\tau_{NL}$ by factors $(-3)^{l_3}$. In particular, $\tau^{SS,SS}_{NL} = 81 \tau_{NL}$. The same conclusion applies to the $g^{IJ,KL}_{NL}$ since $s_3/z_3 = -3$ for $f_c = 0$.

In the fine-tuned region $f_c \sim r$, the purely isocurvature nonlinear coefficients and most of the correlated coefficients are suppressed with respect to the purely adiabatic ones (except in the case $\xi \ll 1$, as discussed earlier).
However, since \( s_3/z_3 \simeq 3/2 \), one finds

\[
\tilde{g}_{NL}^{S,\zeta\zeta\zeta} \simeq \frac{3}{2} \tilde{g}_{NL}^{\xi\zeta\zeta\zeta}, \quad (f_c \sim r, \ \Xi \simeq 1, \ \xi = 1). \tag{85}
\]

In the region \( r \ll f_c \ll 1 \) and \( \Xi \ll 1 \), where the bispectrum is dominated by its purely isocurvature component, we find the same conclusion for the trispectrum coefficients since \( s_3/z_3 \sim f^2/r^2 \). The largest coefficients are thus (for \( \xi = 1 \))

\[
\tau_{NL}^{SS,SS} \simeq \frac{729 f_c^4}{4 r^6} \Xi^3 \simeq \frac{\alpha^3}{4 f_c^2}, \quad \tilde{\tau}_{NL}^{S,SS} \simeq \frac{2187 f_c^5}{2 r^6} \Xi^3 \simeq \frac{3 \alpha^3}{2 f_c} \quad (r \ll f_c \ll 1, \xi = 1), \tag{86}
\]

and one finds the following relations with \( f_{NL}^{S,SS} \):

\[
\tau_{NL}^{SS,SS} \simeq \left( f_{NL}^{S,SS} \right)^2 \alpha^{-1}, \quad \tilde{\tau}_{NL}^{S,SS} \simeq -3 \alpha f_{NL}^{S,SS}, \tag{87}
\]

which are very similar to their adiabatic counterparts (84).

In Fig. 5, we have plotted the evolution of the hierarchy between all the parameters \( \tau_{NL}^{IJ, KL} \) as the parameter \( f_c \) varies, \( r \) being kept fixed (with a small value). In the limit \( r \ll f_c \ll 1 \), one sees clearly that the purely isocurvature
Figure 5: Plots of the coefficients $\tau_{NL}^{IJKL}$ as functions of $f_c$. The indices $I, J, K, L$ are specified in the figure for each curve. The other parameters are $\xi = 1, \lambda = 10^{-3}$ and $r = 10^{-5}$.

coefficient dominates, while the other coefficients vary according to

$$
\tau_{NL}^{IJKL} \simeq \left( \frac{r}{3f_c} \right)^{I_\zeta} \tau_{NL}^{SS,SS}, \quad (r \ll f_c \ll 1), \tag{88}
$$

where $I_\zeta$ is the number of adiabatic indices among the four indices of the coefficient. In this opposite limit, $f_c \ll r \ll 1$, we have

$$
\tau_{NL}^{IJKL} \simeq \left( -\frac{1}{3} \right)^{I_\zeta} \tau_{NL}^{SS,SS}, \quad (f_c \ll r \ll 1). \tag{89}
$$

In Fig. 6, we have plotted the evolution of all the $\bar{g}_{NL}^{IJKL}$ as $f_c$ varies. Once again, the two opposite limits have very distinctive hierarchies between the nonlinear coefficients. In the limit $r \ll f_c \ll 1$, $\bar{g}_{NL}^{S,SS}$ dominates. Moreover,

$$
\bar{g}_{NL}^{\zeta,SS} \simeq \frac{243f_c^3}{r^4} \Xi^3 \simeq \frac{2r^2}{9f_c^2} \bar{g}_{NL}^{S,SS}, \tag{90}
$$

and all the other coefficients can be deduced by using the relations

$$
\bar{g}_{NL}^{S,JKL} \simeq \left( \frac{r}{3f_c} \right)^{I_\zeta} \bar{g}_{NL}^{S,SS}, \quad \bar{g}_{NL}^{\zeta,JKL} \simeq \left( \frac{r}{3f_c} \right)^{I_\zeta} \bar{g}_{NL}^{\zeta,SS}, \quad (r \ll f_c \ll 1), \tag{91}
$$
where $I_{\xi}^3$ is the number of adiabatic indices among the three indices after the comma. In the opposite limit, $f_c \ll r \ll 1$, the hierarchy is simply

$$g_{NL}^{IJKL} \approx \left(-\frac{1}{3}\right)^{I_{\xi}} \tilde{g}_{NL}^{S,SSS}, \quad (f_c \ll r \ll 1),$$

(92)

where $I_{\xi}$ is the number of adiabatic indices among the four indices.

4 Conclusion

In this paper, we have investigated non-Gaussianity in models with isocurvature fluctuations, paying particular attention to the trispectrum. After presenting a general formalism to calculate bi- and tri-spectra of adiabatic, isocurvature and correlated types, allowing various decay scenarios, we have applied the formalism to the mixed curvaton and inflaton model.

We have studied how the amplitude of the non-linearity parameters, which consist of six $f_{NL}^{IJK}$, nine $\tau_{NL}^{IJKL}$ and eight $g_{NL}^{IJKL}$, depend on the parameters $r$, the curvaton contribution to the power spectrum $\Xi$ (or, equivalently, $\lambda$) and $f_c$, which represents the fraction of dark matter produced from the
curvaton decay. We have found that large non-Gaussianity (in the bispectrum and trispectrum) from both of adiabatic and isocurvature modes is possible in some cases without conflicting the isocurvature constraint from the measured CMB power spectrum. We have also compared the relative size of the non-linearity coefficients of various types as depicted in Figs. 2, 5 and 6, and found that different regions in the parameter space correspond to very distinctive hierarchies between the non-Gaussianity coefficients.

Observations of non-Gaussianity, and in particular of the trispectrum have become an important goal in cosmology. From another perspective, isocurvature fluctuations are associated with the generation of dark matter and baryon asymmetry in the Universe. Thus, non-Gaussianity from isocurvature fluctuations, if detected in the future, would give us a lot of insight into the nature of dark matter, the mechanism of baryogenesis, and therefore into high energy physics.

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