Factorization of tree QCD amplitudes in the high-energy limit and in the collinear limit

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Abstract

In the high-energy limit, we compute the gauge-invariant three-parton forward clusters, which in the BFKL theory constitute the tree parts of the NNLO impact factors. In the triple collinear limit, we obtain the polarized double-splitting functions. For the unpolarized and the spin-correlated double-splitting functions, our results agree with the ones obtained by Campbell-Glover and Catani-Grazzini, respectively. In addition, we compute the four-gluon forward cluster, which in the BFKL theory forms the tree part of the NNNLO gluonic impact factor. In the quadruple collinear limit we obtain the unpolarized triple-splitting functions, while in the limit of a three-parton central cluster we derive the Lipatov vertex for the production of three gluons, relevant for the calculation of a BFKL ladder at NNLL accuracy. Finally, motivated by the reorganization of the color in the high-energy limit, we introduce a color decomposition of the purely gluonic tree amplitudes in terms of the linearly independent subamplitudes only.
1 Introduction

QCD calculations of multijet rates beyond the leading order (LO) in the strong coupling constant $\alpha_s$ are generally quite involved. However, in recent years it has become clear how to construct general-purpose algorithms for the calculation of multijet rates at next-to-leading order (NLO) accuracy [1, 2, 3, 4, 5]. The crucial point is to organise the cancellation of the infrared (i.e. collinear and soft) singularities in a universal, i.e. process-independent, way. The universal pieces in a NLO calculation are given by the tree-level splitting [6] and eikonal [7, 8] functions, and by the universal structure of the poles of the one-loop amplitudes [1, 4, 5].

Eventually, the same procedure will permit the construction of general-purpose algorithms at next-to-next-to-leading order (NNLO) accuracy. It is mandatory then to fully investigate the infrared structure of the phase space at NNLO. The universal pieces needed to organise the cancellation of the infrared singularities are given by the tree-level double-splitting [10, 11], double-eikonal [8, 12] and splitting-eikonal [10, 12] functions, by the one-loop splitting [13, 14] and eikonal [13] functions, and by the universal structure of the poles of the two-loop amplitudes [15].

Another outstanding issue in QCD, at first sight unrelated to the topics discussed above, is the calculation of the higher-order corrections to the BFKL equation [16, 17]. In scattering processes characterised by two large and disparate scales, like $s$, the squared parton center-of-mass energy, and $t$, a typical momentum transfer, the BFKL equation resums the large logarithms of type $\ln(s/t)$. The LO term of the resummation requires gluon exchange in the cross channel, which for a given scattering occurs at $O(\alpha_s^2)$. The corresponding QCD amplitude factorizes then into a gauge-invariant effective amplitude formed by two scattering centers, the LO impact factors, connected by the gluon exchanged in the cross channel. The LO impact factors are characteristic of the scattering process at hand. The BFKL equation resums then the universal leading-logarithmic (LL) corrections, of $O(\alpha_s^n \ln^{n-1}(s/t))$, to the gluon exchange in the cross channel. The building blocks of the BFKL resummation are the Lipatov vertex [18], i.e. the effective gauge-invariant emission of a gluon along the gluon ladder in the cross channel, and the gluon reggeization [16], i.e. the LL part of the one-loop corrections to the gluon exchange in the cross channel.

The accuracy of the BFKL equation is improved by computing the next-to-leading logarithmic (NLL) corrections [19, 20], i.e. the corrections of $O(\alpha_s^n \ln^{n-1}(s/t))$, to the gluon exchange in the cross channel. In order to do that, the universal building blocks of the BFKL ladder must be computed to NLL accuracy. These are given by the tree corrections to the Lipatov vertex, i.e. the emission of two gluons [21, 22, 23] or of a $q\bar{q}$ pair [24, 25] along the gluon ladder, by the one-loop corrections to the Lipatov vertex [27, 26, 27], and finally by the NLL gluon reggeization [28], i.e. the NLL part of the two-loop corrections to the gluon ladder. However, to compute jet production rates at NLL accuracy, the impact factors must be computed at NLO [29, 30]. For jet production at
large rapidity intervals, they are given by the one-loop corrections [29] to the LO impact factors, and by the tree corrections [21, 22, 24, 31], i.e. the emission of two partons in the forward-rapidity region. In the collinear or soft limits, the latter reduce to the tree splitting or eikonal functions [32].

To further improve the accuracy of the BFKL ladder one needs to compute the next-to-next-to-leading logarithmic (NNLL) corrections, i.e. the corrections of $O(\alpha_s^n \ln^{n-2}(s/t))$, to the gluon ladder. At present it is not known whether such corrections can be resummed. If that is the case, the universal building blocks of a BFKL ladder at NNLL would be: the emission of three partons along the gluon ladder, the one-loop corrections to the emission of two partons along the ladder, the two-loop corrections to the Lipatov vertex, and the gluon reggeization at NNLL accuracy. None of them is known at present. In this paper we compute the gluonic NNLO Lipatov vertex, i.e. the emission of three gluons along the ladder.

In addition, to compute jet production rates at NNLL accuracy, the BFKL ladder should be supplemented by impact factors at NNLO. They are not known either. In this paper we compute their tree components, i.e. the emission of three partons in the forward-rapidity region. By taking then the triple collinear limit of the tree NNLO impact factors, we obtain the polarized double-splitting functions. Summing over the parton polarizations, we obtain the unpolarized and the spin-correlated double-splitting functions, previously computed in Ref. [10] and [11], respectively, in the conventional dimensional regularization (CDR) scheme. Since we sum over two helicity states of the external partons, as it is done in the dimensional reduction (DR) scheme [33, 34], our results agree with the ones in the CDR scheme by setting there the dimensional regularization scheme (RS) parameter $\epsilon = 0$.

For a scattering with production of $m$ partons, we define the $n$-parton cluster, with $m > n$, as the set of $n$ final-state partons where the distance in rapidity between any two partons in the cluster is much smaller than the rapidity distance between a parton inside the cluster and a parton outside. In the BFKL theory, $(n + 1)$-parton forward clusters provide the tree parts of $N^n$LO impact factors, while $(n + 1)$-parton central clusters provide the tree parts of the $N^n$LO Lipatov vertex. $n$-parton clusters were given also a field-theoretical basis in terms of an effective action describing the interaction between physical partons grouped into gauge-invariant clusters and the gluons exchanged in the cross channel [35]. In addition to computing the three-parton forward clusters and the three-gluon central cluster mentioned above, we compute the four-gluon forward cluster, i.e. the purely gluonic tree part of the NNNLO impact factor. By taking then the quadruple collinear limit, we obtain the polarized triple-splitting functions. They could be used in a gauge-invariant evaluation of the Altarelli-Parisi evolution at three loops [36].

The outline of the paper is: in Section 2 we review the standard color decompositions of the $n$-parton tree amplitudes, and we present a color decomposition of the gluon amplitudes in terms of the linearly independent subamplitudes only. In Section 3 we review
the elastic scattering of two partons in the high-energy limit, which allows for the extraction of the LO impact factors. In Section 4 we review the amplitudes for the production of three partons, with a gauge-invariant two-parton forward cluster; from these, we can extract the tree parts of the NLO impact factors; by taking the collinear limit, we obtain the LO splitting functions. In Section 5 we compute the amplitudes for the production of four partons, with a three-parton forward cluster; then we extract the tree parts of the NNLO impact factors, and by taking the triple collinear limit we obtain the polarized and unpolarized double-splitting functions. In Section 6.1 we compute the amplitude for the production of five gluons, with a four-gluon forward cluster. We extract the tree part of the gluonic NNNLO impact factor, and by taking the quadruple collinear limit we obtain the polarized triple-splitting functions. In addition, by taking the limit in which three gluons are emitted in the central-rapidity region, we obtain the gauge-invariant three-gluon central cluster, i.e. the tree part of the NNLO Lipatov vertex. In Section 7 we draw our conclusions.

2 Tree Amplitudes

In this section we review the color decomposition of purely gluonic and quark-gluon tree amplitudes. For the purely gluonic tree amplitudes, we introduce a color decomposition in terms of the linearly independent subamplitudes, Eq. (2.1).

2.1 Gluon amplitudes

For an amplitude with $n$ gluons the usual color decomposition at tree level reads [37, 38, 39, 40, 41],

$$iA(g_1, \ldots, g_n) = ig^{n-2} \sum_{\sigma \in S_n/Z_n} \text{tr}(\lambda^{\sigma_1} \cdots \lambda^{\sigma_n})A(g_{\sigma_1}, \ldots, g_{\sigma_n})$$  

(2.1)

where $S_n/Z_n$ are the non-cyclic permutations of $n$ elements. The dependence on the particle helicities and momenta in the subamplitude, and on the gluon colors in the trace, is implicit in labelling each leg with the index $i$. Helicities and momenta are defined as if all particles were outgoing.

The gauge invariant subamplitudes $A$ satisfy the relations [10, 41], proven for arbitrary $n$ in Ref. [12],

$$A(1, 2, \ldots, n-1, n) = A(n, 1, 2, \ldots, n-1)$$  cyclic
$$A(1, 2, \ldots, n) = (-1)^n A(n, \ldots, 2, 1)$$  reflection  

(2.2)

$$A(1, 2, 3, \ldots, n) + A(2, 1, \ldots, n) + \ldots + A(2, 3, \ldots, 1, n) = 0$$  dual Ward identity

The above relations are sufficient to show that, for $n \leq 6$ the number of independent subamplitudes can be reduced from $(n-1)!$ to $(n-2)!$. For $n \geq 7$ it is still possible to
introduce a basis of \((n - 2)\) elements by using Kleiss-Kuijf’s relation \[43\]

\[
A(1, x_1, \ldots, x_p, 2, y_1, \ldots, y_q) = (-1)^p \sum_{\sigma \in OP\{\alpha\}\{\beta\}} A(1, 2, \{\alpha\}\{\beta\})
\] (2.3)

where \(\alpha_i \in \{\alpha\} \equiv \{x_p, x_{p-1}, \ldots, x_1\}, \beta_i \in \{\beta\} \equiv \{y_1, \ldots, y_q\}\) and \(OP\{\alpha\}\{\beta\}\) is the set of permutations of the \((n - 2)\) objects \(\{x_1, \ldots, x_p, y_1, \ldots, y_q\}\) that preserve the ordering of the \(\alpha_i\) within \(\{\alpha\}\) and of the \(\beta_i\) within \(\{\beta\}\), while allowing for all possible relative orderings of the \(\alpha_i\) with respect to the \(\beta_i\). The above relation has been checked up to \(n = 8\) in Ref. \[42\], and proven for arbitrary \(n\) in Ref. \[44\]. Accordingly, the expression for the summed amplitude squared can be written as

\[
\sum_{i_1, \ldots, i_n} |A(1, \ldots, n)|^2 = \sum_{i,j=1}^{(n-1)!} c_{ij} A_i A_j^* \] (2.4)

\[
= C_n(N_c) \sum_{\sigma \in S_{n-1}} \left[ |A(1, \sigma_2, \ldots, \sigma_n)|^2 + \mathcal{O}\left( \frac{1}{N_c^2} \right) \right] \] (2.5)

\[
= \sum_{i,j=1}^{(n-2)!} \tilde{c}_{ij} A_i A_j^*, \] (2.6)

where \(c_{ij}\) in Eq. (2.4) is

\[
c_{ij} = (g^2)^{n-2} \sum_{\text{colors}} \text{tr}(P_i(\lambda^d_1, \ldots, \lambda^d_n)) \text{tr}(P_j(\lambda^d_1, \ldots, \lambda^d_n))^* \] (2.7)

with \(P_i\) the \(i\)th permutation in \(S_n/\mathbb{Z}_n\). In Eq. (2.5), the coefficient \(C_n(N_c)\) is

\[
C_n(N_c) = \frac{(g^2 N_c)^{n-2}}{2^n} (N_c^2 - 1). \] (2.8)

The first term in Eq. (2.3) constitutes the Leading Color Approximation (LCA). Up to \(n = 5\), the \(1/N_c^2\) corrections in Eq. (2.5) vanish and LCA is exact. The reduced color matrix \(\tilde{c}_{ij}\) in Eq. (2.6), has been obtained from \(c_{ij}\) applying the linear transformations of Eq. (2.3), thus the labels \(i, j\) in Eq. (2.4) run only on the permutations of the linearly independent subamplitudes.

Motivated by the reorganization of the color in the high-energy limit \[22, 31, 46, 47\], and using Eqs. (2.2) and (2.3) we rewrite Eq. (2.1) as

\[
iA(g_1, \ldots, g_n) = i \frac{(ig)^{n-2}}{2} \sum_{\sigma \in S_{n-2}} f_{a_1a_2x_1} f_{x_1a_3x_2} \cdots f_{x_n-3a_{n-1}a_n} A(g_1, g_{\sigma_2}, \ldots, g_{\sigma_{n-1}}, g_n), \]

\[
= i \frac{g^{n-2}}{2} \sum_{\sigma \in S_{n-2}} (F_{a_2} \cdots F_{a_{n-1}})_{a_1a_n} A(g_1, g_{\sigma_2}, \ldots, g_{\sigma_{n-1}}, g_n), \] (2.9)

where \((F^a)_{bc} \equiv if^{bac}\). We have checked Eq. (2.9) up to \(n = 7\). Eq. (2.9) enjoys several remarkable properties. Firstly, it shows explicitly which is the color decomposition that
allows us to write the full amplitude $iA$ in terms of the $(n-2)!$ linearly independent subamplitudes only. In the following we shall refer to it as to a color ladder. Hence the color matrix obtained squaring Eq. (2.9) yields directly the $c_{ij}$ matrix in Eq. (2.6). We have checked it against the explicit results of Ref. [45], up to $n = 5$. Moreover, it is quite suggestive to note the formal correspondence with the amplitudes with a quark-antiquark pair and $(n-2)$ gluons, Eq. (2.11), where the only difference between the two is the appropriate representation for the color matrices, namely the adjoint for the $n$-gluon amplitude and the fundamental for the one with the $\bar{q}q$ pair. Finally, the most relevant applications of Eq. (2.9) for this work are to the study of the multi-gluon amplitudes in the high-energy limit. As discussed in the following, the color ladder naturally arises in the configurations where the gluons are strongly ordered in rapidity, i.e. in the multi-Regge kinematics. Indeed in the strong-rapidity ordering only the subamplitude with the corresponding order in the color coefficient contributes to Eq. (2.9). At NLO, where the strong ordering is relaxed for two adjacent gluons, the leading subamplitudes are the two which differ just by the exchange of the gluon labels in the color ladders [31]. As we shall see this result generalizes at NNLO and beyond. Nonetheless, in the following we have chosen to derive our results starting from Eq. (2.1) instead of using directly Eq. (2.9). The former, though more laborious, shows explicitly how the color traces must be recombined to obtain the color ladder and, more importantly, allows us to find the relations necessary to prove the factorization in the multi-collinear limits.

For the maximally helicity-violating configurations, $(-,-,+,\ldots,+)$, in Eq. (2.1) or Eq. (2.9), there is only one independent color/helicity subamplitude, the Parke-Taylor (PT) subamplitude

$$A(g_1,\ldots,g_n) = 2^{n/2} \frac{\langle ij \rangle^4}{\langle 12 \rangle \cdots \langle (n-1)\, n \rangle \langle n 1 \rangle},$$

(2.10)

where the $i^{th}$ and the $j^{th}$ gluons have negative helicity. All other color/helicity amplitudes can be obtained by relabelling and by use of reflection symmetry, Eq. (2.2), and parity inversion. Parity inversion flips the helicities of all particles, and it is accomplished by the substitution $\langle i \, j \rangle \leftrightarrow [j \, i]$. Subamplitudes of non-PT type, i.e. with three or more gluons of $-$ helicity have a more complicated structure.

### 2.2 Quark-gluon amplitudes

For an amplitude with two quarks and $(n-2)$ gluons the color decomposition at tree-level is

$$iA(\bar{q}, q; g_1,\ldots,g_{n-2}) = ig^{n-2} \sum_{\sigma \in S_{n-2}} (\lambda^{\sigma_1} \cdots \lambda^{\sigma_{n-2}}) \bar{\lambda} \ A(\bar{q}, q; g_{\sigma_1},\ldots,g_{\sigma_{n-2}}),$$

(2.11)

*The factor 1/2 in front of Eq. (2.9) is due to our choice for the normalization of the fundamental representation matrices, i.e. $\text{tr}(\lambda^a \lambda^b) = \delta^{ab}/2$. 
where $S_{n-2}$ is the permutation group on $n - 2$ elements.

For the maximally helicity-violating configurations, $(-, -+,\ldots, +)$, there is one independent color/helicity subamplitude, the Parke-Taylor (PT) subamplitude

$$A(q^+, q^-; g_1,\ldots, g_{n-2}) = 2^{(n-2)/2} \frac{\langle \vec{q} i \rangle \langle qi \rangle^3}{\langle \vec{q} q \rangle \langle q1 \rangle\cdots \langle(n-2)q \rangle},$$

where gluon $g_i$ has negative helicity. Helicity is conserved along the massless-fermion line. All other color/helicity amplitudes can be obtained by relabelling and by use of parity inversion, reflection symmetry and charge conjugation. In performing parity inversion, there is a factor of $-1$ for each pair of quarks participating in the amplitude. Reflection symmetry is like in Eq. (2.2), for gluons and/or quarks alike. Charge conjugation swaps quarks and antiquarks without inverting helicities. In particular, using reflection symmetry and charge conjugation on Eq. (2.12) we obtain

$$A(q^-, q^+; g_1,\ldots, g_{n-2}) = 2^{(n-2)/2} \frac{\langle \vec{q} i \rangle \langle qi \rangle^3}{\langle \vec{q} q \rangle \langle q1 \rangle\cdots \langle(n-2)q \rangle},$$

where gluon $g_i$ has negative helicity.

For an amplitude with four quarks and $(n - 4)$ gluons the color decomposition at tree-level is [37]

$$iA(q_1, q_1; q_2, q_2, g_1,\ldots, g_{n-4}) = ig^{n-2} \sum_{k=0}^{n-4} \sum_{\sigma \in S_{k}} \sum_{\rho \in S_{l}} (\lambda^{\sigma_1}\ldots\lambda^{\sigma_{k}})_{j_1}^{\bar{i}_1} (\lambda^{\rho_1}\ldots\lambda^{\rho_{l}})_{j_2}^{\bar{i}_2} A(q_1, q_1; q_2, q_2, g_{\sigma_1},\ldots, g_{\sigma_{k}}; g_{\rho_1},\ldots, g_{\rho_{l}})$$

$$- \frac{1}{N_c} (\lambda^{\sigma_1}\ldots\lambda^{\sigma_{k}})_{j_1}^{\bar{i}_1} (\lambda^{\rho_1}\ldots\lambda^{\rho_{l}})_{j_2}^{\bar{i}_2} B(q_1, q_1; q_2, q_2, g_{\sigma_1},\ldots, g_{\sigma_{k}}; g_{\rho_1},\ldots, g_{\rho_{l}}),$$

with $k + l = n - 4$, and where we suppose that the two quark pairs have distinct flavor. The sums are over the partitions of $(n - 4)$ gluons between the two quark lines, and over the permutations of the gluons within each partition. For $k = 0$ or $l = 0$, the color strings reduce to Kronecker delta’s. For identical quarks, we must subtract from Eq. (2.14) the same term with the exchange of the quarks $(q_1 \leftrightarrow q_2)$.

For the maximally helicity-violating configurations, $(-, -, +,\ldots, +)$, with like-helicity for all of the gluons, the $A$ and $B$ subamplitudes factorize into distinct contributions for the two quark antennae [37, 38, 40, 41]. However, as we shall see in Sect. 5.4, we need the helicity configurations with two gluons of opposite helicity. For these the above mentioned factorization does not occur.

### 3 The Leading Impact Factors

We consider the elastic scattering of two partons of momenta $p_a$ and $p_b$ into two partons of momenta $p_a'$ and $p_b'$, in the high-energy limit, $s \gg |t|$. Firstly, we consider the amplitude
To an impact factor, the other half being obtained by parity.

For gluon-gluon scattering (Fig. 1a). Using Eqs. (2.1), (2.2), or Eq. (2.3), and Eq. (2.10), and Appendix B, we obtain [47]

in Eq. (3.1) four helicity configurations are leading, two for each impact factor $g g \rightarrow g g$, with $g^*$ an off-shell gluon, are

They conserve helicity along the on-shell gluon line and transform under parity into their complex conjugates,

With $q = p_{\nu'} + p_b$ and $t \simeq -|q_\perp|^2$. The LO impact factors $g^* g \rightarrow g$, with $g^*$ an off-shell gluon, are

They conserve helicity along the on-shell gluon line and transform under parity into their complex conjugates,

In Eq. (3.4) four helicity configurations are leading, two for each impact factor $q q$. The helicity-flip impact factor $C^g g (p^+; p^+)$ is subleading in the high-energy limit.

From Eqs. (2.11)-(2.12), we obtain the quark-gluon $q g \rightarrow q g$ scattering amplitude in the high-energy limit [3],

where we have labelled the incoming quarks as outgoing antiquarks with negative momentum, e.g. the antiquark is $p_a$ in Eq. (3.4) (Fig. 1b), and $p_b$ in Eq. (3.5) (Fig. 1c). The LO impact factors $g^* q \rightarrow q$ are,

$\dagger$ All throughout this paper, we shall always write only half of the helicity configurations contributing to an impact factor, the other half being obtained by parity.
Under parity, the functions \( \langle 3.4 \rangle \) transform as

\[
[C^{q\bar{q}}(\{k^\nu\})]^* = S C^{\bar{q}q}(\{k^-\nu\}) \quad \text{with} \quad S = -\text{sign}(\bar{q}^0 q^0), \tag{3.7}
\]

and in general an impact factor acquires a coefficient \( S \) for each pair of quarks (see Sect. \( \| \)). Analogously, the antiquark-gluon \( \bar{q} g \to \bar{q} g \) amplitude is

\[
\mathcal{A}^{q\bar{g} \to g}(p_a^{\nu_a}; p_a'^{\nu_a'} \mid p_{b'}^{\nu_b'} p_{b}^{\nu_b}) = 2s \left[ g \lambda_{a\bar{a}}^c C^{q\bar{q}}(p_a^{-\nu_a'; p_a'^{\nu_a'}}) \right] \frac{1}{t} \left[ ig f^{bblc} C^{g\bar{g}}(p_{b'}^{\nu_b'} p_{b}^{\nu_b}) \right] \tag{3.8}
\]

\[
\mathcal{A}^{q\bar{g} \to g}(p_a^{\nu_a}; p_a'^{\nu_a'} \mid p_{b'}^{\nu_b'} p_{b}^{\nu_b}) = 2s \left[ ig f^{a\nu'c} C^{g\bar{g}}(p_a^{\nu_a'} p_a'^{\nu_a}) \right] \frac{1}{t} \left[ g \lambda_{b\bar{b}}^c C^{q\bar{q}}(p_b^{-\nu_b'; p_b'^{\nu_b'}}) \right] \tag{3.9}
\]

where the antiquark is \( p_a' \) in Eq. \( \langle 3.8 \rangle \) and \( p_{b'} \) in Eq. \( \langle 3.9 \rangle \), and the LO impact factors \( g \to \bar{q} g \) are

\[
C^{q\bar{q}}(p_a^-; p_a'^+) = i; \quad C^{q\bar{q}}(p_b^-; p_b'^+) = -i \left( \frac{p_{b'}^{\perp}}{p_{b'}^{\perp}} \right)^{1/2}. \tag{3.10}
\]

In the amplitudes \( \langle 3.1 \rangle, \langle 3.4 \rangle, \langle 3.5 \rangle, \langle 3.8 \rangle, \langle 3.9 \rangle \), the leading contributions from all the Feynman diagrams have been included. However, the amplitudes have the effective form of a gluon exchange in the \( t \) channel (Fig. \( 1 \)), and differ only for the relative color strength in the production vertices \( \langle 18 \rangle \). This allows us to replace an incoming gluon with a quark, for instance on the upper line, via the simple substitution

\[
ig f^{a\nu'c} C^{g\bar{g}}(p_a^{\nu_a'} p_a'^{\nu_a}) \leftrightarrow g \lambda_{a\bar{a}}^c C^{q\bar{q}}(p_a^{-\nu_a'; p_a'^{\nu_a'}}), \tag{3.11}
\]

and similar ones for an antiquark and/or for the lower line. For example, the quark-quark \( q q \to q g \) scattering amplitude in the high-energy limit is

\[
\mathcal{A}^{q\bar{g} \to g}(p_a^{\nu_a}; p_a'^{\nu_a'} \mid p_{b'}^{\nu_b'} p_{b}^{\nu_b}) = 2s \left[ g \lambda_{a\bar{a}}^c C^{q\bar{q}}(p_a^{-\nu_a'; p_a'^{\nu_a'}}) \right] \frac{1}{t} \left[ g \lambda_{b\bar{b}}^c C^{q\bar{q}}(p_b^{-\nu_b'; p_b'^{\nu_b'}}) \right]. \tag{3.12}
\]

### 4 The Next-to-leading Impact Factors

Let three partons be produced with momenta \( k_1, k_2 \) and \( p_{b'}, p_{b} \) in the scattering between two partons of momenta \( p_a \) and \( p_b \), and to be specific, we shall take partons \( k_1 \) and \( k_2 \) in the forward-rapidity region of parton \( p_a \), the analysis for \( k_1 \) and \( k_2 \) in the forward-rapidity region of \( p_b \) being similar. Parametrizing the momenta as in Eq. \( \langle 4.1 \rangle \), we have

\[
y_1 \simeq y_2 \gg y_0; \quad |k_{1\perp}| \simeq |k_{2\perp}| \simeq |p_{b'} \perp| \tag{4.1}
\]

#### 4.1 The NLO impact factor \( g g^* \to g g \)

We consider the amplitude for the scattering \( gg \to gg g \) (Fig. \( 2a \)). Only PT subamplitudes contribute, thus using Eqs. \( \langle 2.1 \rangle, \langle 2.2 \rangle \) and \( \langle 2.10 \rangle \), and Appendix \( \| \), we obtain \( \langle 21 \rangle, \langle 22 \rangle \)

\[
\mathcal{A}^{g g^* g}(p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2} \mid p_{b'}^{\nu_b'}, p_{b}^{\nu_b})
\]
The momentum fractions are defined as
\[ x_i = \frac{k_i^+}{k_1^+ + k_2^+} \quad i = 1, 2 \quad (x_1 + x_2 = 1), \]
and the function \( A^+ \) as follows:
\[ A^+(k_1, k_2) = -\sqrt{2} \frac{q_\perp}{k_1^+} \sqrt{\frac{x_1}{x_2}} \frac{1}{\langle 12 \rangle}, \]
with \( \langle 12 \rangle \) a shorthand for \( \langle k_1 k_2 \rangle \). Using the dual Ward identity \( \mathcal{B}_7 \), or U(1) decoupling equations \( \mathcal{B}_5, \mathcal{B}_8 \), the function \( B^\varphi \) in Eq. (4.3), and thus the function \( B^{g:gg} \), can be written as
\[ B^\varphi(k_1, k_2) = -[A^\varphi(k_1, k_2) + A^\varphi(k_2, k_1)]. \]
The function $C^{gg\rightarrow gg}(p_1^+, k_1^+, k_2^+)$ is subleading to the required accuracy. The function $A^\rho$ has a collinear divergence as $2k_1 \cdot k_2 \to 0$, but the divergence cancels out in the function $B^\rho$ where gluons 1 and 2 are not adjacent in color ordering \cite{24}.

Using Eq. (4.7), and fixing $t \simeq -|q_\perp|^2$, the amplitude (4.2) may be rewritten as,

$$A^{gg\rightarrow 3g}(p_1^{\nu_1}, k_1^{\nu_2}, k_2^{\nu_3} | P_b^{\nu'}, P_b^{\nu''})$$

$$= 2s \left\{ (ig)^2 \sum_{a=2}^A \int f_{q_1a} f_{q_2a} A^{gg\rightarrow gg}(p_1^{\nu_1}; k_1^{\nu_2}, k_2^{\nu_3}) \right\} \frac{1}{t_1} \left[ ig f^{\mu'\nu''} C^{gg\rightarrow g}(p_1^{\nu_1}; p_2^{\nu''}, p_2^{\nu''}) \right],$$

where the NLO impact factor for $g^* g \to g g$ is enclosed in curly brackets, and includes six helicity configurations.

In the multi-Regge limit $y_1 \gg y_2$,

$$\lim_{y_1 \gg y_2} A^{gg\rightarrow gg}(p_1^{\nu_1}, k_1^{\nu_2}, k_2^{\nu_3}) = C^{gg\rightarrow g}(p_1^{\nu_1}; k_1^{\nu_2}, q),$$

with $q_1 = -(p_a + k_1)$, and $t_1 \simeq -|q_{1\perp}|^2$, and with LO Lipatov vertex, $g^* g \to g$ \cite{17, 18},

$$C^g(q_1, k^+, q_2) = \sqrt{2} \frac{q_{1\perp} q_{2\perp}}{k_{\perp}}.$$

Accordingly, the amplitude (4.8) is reduced to an amplitude in multi-Regge kinematics \cite{13, 17}, with the effective form of a gluon-ladder exchange in the $t$ channel,

$$A^{gg\rightarrow 3g}(p_1^{\nu_1}, k_1^{\nu_2}, k_2^{\nu_3} | P_b^{\nu'}, P_b^{\nu''}) =$$

$$2s \left[ ig f^{\mu'\nu''} C^{gg\rightarrow g}(p_1^{\nu_1}; k_1^{\nu_2}) \right] \frac{1}{t_1} \left[ ig f^{\mu'\nu''} C^g(q_1, k_2^{\nu_3}, q_2) \right] \frac{1}{t_2} \left[ ig f^{\mu'\nu''} C^{gg\rightarrow g}(p_1^{\nu_1}; p_2^{\nu''}, p_2^{\nu''}) \right],$$

with $q_2 = p_b + p_{\nu'}$ and $t_2 \simeq -|q_{2\perp}|^2$.

### 4.2 The NLO impact factor $g g^* \to \bar{q}q$

The amplitude $g g \to \bar{q}q g$ for the production of a $q\bar{q}$ pair in the forward-rapidity region of gluon $a$ (Fig. 2c) is obtained by taking the amplitudes (2.11)-(2.13) in the kinematics (1.1) \cite{24},

$$A^{g g \rightarrow g q q}(p_1^{\nu_1}, k_1^{\nu_2}, k_2^{\nu_3} | P_b^{\nu'}, P_b^{\nu''})$$

$$= 2s \left\{ g^2 \left[ \left( \lambda^e \lambda^a \right)_{d_2d_1} A^{g \bar{q}q}(p_1^{\nu_1}; k_1^{\nu_2}, k_2^{\nu_3}) + \left( \lambda^a \lambda^c \right)_{d_2d_1} A^{g q q}(p_1^{\nu_1}; k_2^{\nu_3}, k_1^{\nu_1}) \right] \right\}$$

$$\times \frac{1}{t} \left[ ig f^{\mu'\nu''} C^{gg\rightarrow g}(p_1^{\nu_1}; p_2^{\nu''}, p_2^{\nu''}) \right],$$

with $k_1$ the antiquark, the NLO impact factor $g^* g \to \bar{q}q$ in curly brackets, and with

$$A^{g \bar{q}q}(p_1^{\nu_1}, k_1^{\nu_2}, k_2^{\nu_3}) = C^{g \bar{q}q}(p_1^{\nu_1}; k_1^{\nu_2}, k_2^{\nu_3}) A^\rho(k_1, k_2)$$

$$C^{g \bar{q}q}(p_1^{\nu_1}, k_1^{\nu_2}, k_2^{\nu_3}) = \sqrt{x_1 x_2^3}$$

$$C^{g \bar{q}q}(p_1^{\nu_1}, k_1^{\nu_2}, k_2^{\nu_3}) = \sqrt{x_1^3 x_2}.$$

10
with momentum fractions as in Eq. (1.5), \( A^\rho \) in Eq. (1.6) and \( \nu = \nu_a \). The NLO impact factor \( g^* g \rightarrow \bar{q}q \) allows for four helicity configurations.

In the multi-Regge limit \( k_1^+ \gg k_2^+ \), the NLO impact factor \( g^* g \rightarrow \bar{q}q \) vanishes, since quark production along the multi-Regge ladder is suppressed.

### 4.3 The NLO impact factor \( qg^* \rightarrow qg \)

The amplitude \( qg \rightarrow qg \) for the production of a \( qg \) pair in the forward-rapidity region of quark \( a \) (Fig. 2b) is obtained by taking the amplitudes (2.11)-(2.13) in the kinematics (4.14) 31

\[
A^{qg \rightarrow qg} (p_{a}^{\nu_{1}}, k_{1}^{\nu_{1}}, k_{2}^{\nu_{2}} | p_{b}, p_{b}') = 2s \left\{ \left[ \lambda d_{2} \lambda c \right]_{d_{1}a} A^{qg} (p_{a}^{\nu_{1}}, k_{1}^{\nu_{1}}, k_{2}^{\nu_{2}}) + \left[ \lambda c' \lambda d_{2} \right]_{d_{1}a} B^{qg} (p_{a}^{\nu_{1}}, k_{1}^{\nu_{1}}, k_{2}^{\nu_{2}}) \right\}
\]

\[ \times \frac{1}{t} \left[ i g \; f_{b b'} C^{qg} (p_{b}^{\nu_{b}}; p_{b'}^{\nu_{b}'}) \right], \]

with \( k_1 \) the final-state quark, and the NLO impact factor \( qg^* \rightarrow qg \) in curly brackets. As above, the NLO impact factor includes four helicity configurations,

\[
\begin{align*}
A^{qg} (p_{a}^{\nu_{1}}, k_{1}^{\nu_{1}}, k_{2}^{\nu_{2}}) &= C^{qg} (p_{a}^{\nu_{1}}, k_{1}^{\nu_{1}}, k_{2}^{\nu_{2}}) A^\rho (k_1, k_2) \\
B^{qg} (p_{a}^{\nu_{1}}, k_{1}^{\nu_{1}}, k_{2}^{\nu_{2}}) &= C^{qg} (p_{a}^{\nu_{1}}, k_{1}^{\nu_{1}}, k_{2}^{\nu_{2}}) B^\rho (k_1, k_2) \\
C^{qg} (p_{a}^{\nu_{1}}, k_{1}^{\nu_{1}}, k_{2}^{\nu_{2}}) &= -i \sqrt{x_1} \\
C^{qg} (p_{a}^{\nu_{1}}, k_{1}^{\nu_{1}}, k_{2}^{\nu_{2}}) &= i \sqrt{x_1},
\end{align*}
\]

with \( A^\rho \) in Eq. (4.10), and \( B^\rho \) given by Eq. (4.7), with \( \bar{\nu} = \nu_2 \). As in Section 4.1, the function \( B^{qg} \) vanishes in the collinear limit.

In the multi-Regge limit \( k_1^+ \gg k_2^+ \) the amplitude (4.14) reduces to Eq. (1.11), with the substitution (3.11) for the upper line, and the LO impact factor \( C^{qg} \) in Eq. (3.6).

The treatment of the amplitude \( \bar{q}g \rightarrow \bar{q}g \) for the production of a \( \bar{q}g \) pair in the forward-rapidity region of antiquark \( a \) is identical to the former, thus the NLO impact factor \( \bar{q}g^* \rightarrow \bar{q}g \) is the same as in Eq. (4.14) up to inverting the color flow on the quark line [31]. The corresponding functions \( A \) and \( B \) are the same as in Eq. (4.16).

### 4.4 NLO impact factors in the collinear limit

The collinear factorization for a generic amplitude occurs both on the subamplitude and on the full amplitude [37], since in Eqs. (2.11), (2.14) and (2.14) color orderings where the collinear partons are not adjacent do not have a collinear divergence. Hence in the

\[\text{In this context, Eq. (4.7) is only a bookkeeping, since the U(1) decoupling equation is valid only for the gluino-gluon subamplitudes corresponding to the quark-gluon subamplitudes used in Eq. (4.14).}\]
collinear limit for partons \(i\) and \(j\), with \(k_i = z P\) and \(k_j = (1 - z) P\), a generic amplitude (2.1) can be written as
\[
\lim_{k_i \parallel k_j} A^{-d_1d_2\cdots}(...) \left[ k_i^\nu, k_j^\nu \right] = \sum_\nu A^{-c_1\cdots}(...) \text{Split}^{-f_{\mu_1}f_{\mu_2}} (k_i^{\nu_1}, k_j^{\nu_2}),
\]
with \(f\) denoting the parent species. Accordingly, for \(k_1 = z P\) and \(k_2 = (1 - z) P\), we can write the amplitudes (4.8), (4.12) and (4.14) as
\[
\lim_{k_1 \parallel k_2} A^{f_g \rightarrow f_1f_2g} (P_{\mu_1}^\nu, k_1^{\nu_1}, k_2^{\nu_2}, p_1^\nu, p_2^\nu)
\]
\[
= A^{f_g \rightarrow f_1g} (P_{\mu_1}^\nu, P_{\nu}^\nu, p_1^\nu, p_2^\nu) \cdot \text{Split}^{-g_{\lambda_1}g_{\lambda_2}} (k_1^{\nu_1}, k_2^{\nu_2}),
\]
with \(A^{f_g \rightarrow f_1g}\) as in Eq. (3.1), (3.4) and (3.8), respectively, and where we have used helicity conservation in the s channel (Section 3). For the collinear factors, \(\text{Split}^{-f_{\mu_1}f_{\mu_2}}\), we obtain
\[
\text{Split}^{-g_{\nu}g_{\nu}} (k_1^{\nu_1}, k_2^{\nu_2}) = ig f^{cd_1d_2} \text{split}^{-g_{\nu}g_{\nu}} (k_1^{\nu_1}, k_2^{\nu_2})
\]
\[
\text{Split}^{-g_{\nu}g_{\nu}} (k_1^{\nu_1}, k_2^{\nu_2}) = g (\lambda)_{d_1d_2} \text{split}^{-g_{\nu}g_{\nu}} (k_1^{\nu_1}, k_2^{\nu_2})
\]
\[
\text{Split}^{-g_{\nu}g_{\nu}} (k_1^{\nu_1}, k_2^{\nu_2}) = g (\lambda_{d_2})_{d_1} \text{split}^{-g_{\nu}g_{\nu}} (k_1^{\nu_1}, k_2^{\nu_2})
\]
\[
\text{Split}^{-g_{\nu}g_{\nu}} (k_1^{\nu_1}, k_2^{\nu_2}) = g (\lambda_{d_2})_{d_1} \text{split}^{-g_{\nu}g_{\nu}} (k_1^{\nu_1}, k_2^{\nu_2}),
\]
with splitting factors (37), (44),
\[
\text{split}^{-g_{\nu}g_{\nu}} (k_1^+, k_2^+) = \sqrt{2} \frac{1}{\sqrt{z(1 - z)}} (12)
\]
\[
\text{split}^{-g_{\nu}g_{\nu}} (k_1^-, k_2^-) = \sqrt{2} \frac{z^2}{\sqrt{z(1 - z)(12)}}
\]
\[
\text{split}^{-g_{\nu}g_{\nu}} (k_1^+, k_2^-) = \sqrt{2} \frac{(1 - z)^2}{\sqrt{z(1 - z)(12)}}
\]
\[
\text{split}^{-g_{\nu}g_{\nu}} (k_1^-, k_2^+) = \sqrt{2} \frac{1 - z}{(12)}
\]
\[
\text{split}^{-g_{\nu}g_{\nu}} (k_1^+, k_2^-) = \sqrt{2} \frac{z}{(12)}
\]
\[
\text{split}^{-g_{\nu}g_{\nu}} (k_1^-, k_2^-) = \sqrt{2} \frac{1}{\sqrt{1 - z(12)}}
\]
\[
\text{split}^{-g_{\nu}g_{\nu}} (k_1^+, k_2^+) = \sqrt{2} \frac{1}{\sqrt{1 - z(12)}}
\]
\[
\text{split}^{-g_{\nu}g_{\nu}} (k_1^-, k_2^+) = \sqrt{2} \frac{z}{\sqrt{1 - z(12)}}
\]
and \(\text{split}^{-f_{\mu_1}f_{\mu_2}} (k_1^{\nu_1}, k_2^{\nu_2})\) obtained from \(\text{split}^{-f_{\mu_1}f_{\mu_2}} (k_1^{\nu_1}, k_2^{\nu_2})\) by exchanging \(\langle k_1 k_2 \rangle\) with \([k_2 k_1]\), and multiplying by the coefficient \(S\), Eq. (3.7), if the splitting factor includes a quark pair.

Summing over the two helicity states of partons 1 and 2, we obtain a two-dimensional matrix, whose entries are the Altarelli-Parisi splitting functions at fixed color and helicity of the parent parton
\[
\sum_{\nu_1 \nu_2} \text{Split}^{-f_{\mu_1}f_{\mu_2}} (k_1^{\nu_1}, k_2^{\nu_2}) \left[ \text{split}^{-f_{\mu_1}f_{\mu_2}} (k_1^{\nu_1}, k_2^{\nu_2}) \right]^* = \delta_{\nu_1 \nu_2} 2 g^2 \sqrt{s_{12}} e^{i (\phi_{\lambda_1} - \phi_{\nu})} P_{\nu}^{-f_{\mu_1}f_{\mu_2}},
\]
with $e^{i(\phi_\lambda - \phi_\rho)}$ a phase, where $e^{i(\phi_\lambda - \phi_\rho)} = [21]/[12]$, and where by definition $P_{++}^{f \to f_1 f_2} = P_{-+}^{f \to f_1 f_2}$, and $P_{++}^{f \to f_1 f_2} = P_{-+}^{f \to f_1 f_2}$, and

$$
P_{++}^{g \to g} = 2C_A \left[ \frac{z}{1 - z} + \frac{1 - z}{z} + z(1 - z) \right],
$$

$$
P_{+-}^{g \to g} = 2C_A z(1 - z),
$$

$$
P_{++}^{q \to q} = \frac{1}{2} \left[ z^2 + (1 - z)^2 \right],
$$

$$
P_{+-}^{g \to q} = z(1 - z),
$$

$$
P_{++}^{g \to q} = P_{++}^{q \to g} = C_F \frac{1 + z^2}{1 - z},
$$

$$
P_{+-}^{g \to q} = P_{+-}^{q \to g} = 0. \tag{4.21}
$$

For $P_{++}^{q \to g}$ helicity conservation on the quark line sets the off-diagonal elements equal to zero. $P_{+-}^{q \to g}$ is obtained from $P_{++}^{q \to g}$ by exchanging $(z \leftrightarrow 1 - z)$. Since we sum over two helicity states of the external partons, Eq. (4.21) is valid in the dimensional reduction (DR) scheme \[33, 34\]. Eq. (4.21) agrees with the corresponding spin-correlated splitting functions of Ref. \[49\] in the DR scheme, after contracting the ones of type $P_{++}^{g \to f_1 f_2}$ with a parent-gluon polarization as in Appendix E. The connection of Eq. (4.21) with other regularization schemes (RS) is also given in Ref. \[49\].

Averaging over the trace of $P_{++}^{f \to f_1 f_2}$ in Eq. (4.20), i.e. over color and helicity of the parent parton on the left hand side of Eq. (4.20), we obtain the unpolarized Altarelli-Parisi splitting functions \[§\]

$$
\frac{1}{2C} \sum_{\nu_1 \nu_2} |\text{Split}_{-\nu}^{f \to f_1 f_2}(k_{\nu_1}, k_{\nu_2})|^2 = \frac{2g_s^2}{s_{12}} \langle P_{++}^{f \to f_1 f_2} \rangle, \tag{4.22}
$$

with $C = N_c^2 - 1$ for a parent gluon and $C = N_c$ for a parent quark, and where the averaged trace of $P_{++}^{f \to f_1 f_2}$ is $\langle P_{++}^{f \to f_1 f_2} \rangle = \text{tr} P_{++}^{f \to f_1 f_2} / 2 = P_{++}^{f \to f_1 f_2}$. \[§\]

5 The Next-to-next-to-leading Impact Factors

In order to derive the next-to-next-to-leading (NNLO) impact factors, we repeat the analysis of Sect. 4 with one more final-state parton. Let four partons be produced with momenta $k_1$, $k_2$, $k_3$ and $p_{b'}$ in the scattering between two partons of momenta $p_a$ and $p_b$, with a cluster of three partons, $k_1$, $k_2$ and $k_3$, in the forward-rapidity region of parton $p_a$,

$$
y_1 \simeq y_2 \simeq y_3 \gg y_{b'}; \quad |k_{1\perp}| \simeq |k_{2\perp}| \simeq |k_{3\perp}| \simeq |p_{b'\perp}|. \tag{5.1}
$$

---

\[§\]Note that in the DR scheme the unpolarized splitting functions do not coincide with the azimuthally-averaged ones. The latter are given in any RS in Ref. \[49\].
Figure 3: Amplitudes for the production of four partons, with partons $k_1$, $k_2$ and $k_3$ in the forward-rapidity region of parton $p_a$.

5.1 The NNLO impact factor $g g^* \rightarrow g g g$

We begin with the amplitude for the scattering $g g \rightarrow g g g$ (Fig. 3a) in the kinematics (5.1). Using Eqs. (2.1), (2.3) and (2.10), and the subamplitudes of non-PT type, with three gluons of $+$ helicity and three gluons of $-$ helicity, (37), and Appendix D, we obtain

$$A^{gg\rightarrow 4g}(p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}|P_b^{\nu_b}, P_b^{\nu_b})$$

$$= 4g^4 \left| q_\perp \right|^2 C^{gg}(p_b^{\nu_b}, P_b^{\nu_b}) \sum_{\sigma \in S_3} A^{g3g}(p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) \sum_{\sigma \in S_3} \left[ B^{g3g}(p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) \right]$$

$$+ C^{gg}(p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) A^{g}(k_1, k_2, k_3),$$

with the sum over the permutations of the three gluons 1, 2 and 3, and the LO impact factor, $C^{gg}(p_b^{\nu_b}, P_b^{\nu_b})$, as in Eq. (1.2). From the PT subamplitudes (2.10) we obtain the function of $(-+++)$ helicities

$$A^{g3g}(p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) = C^{g3g}(p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) A^{g}(k_1, k_2, k_3),$$

where $\bar{\nu} = \text{sign}(\nu_a + \nu_1 + \nu_2 + \nu_3)$ and

$$A^{+}(k_1, k_2, k_3) = -2 \frac{q_\perp}{k_{1\perp}} \sqrt{\frac{x_1}{x_3}} \frac{1}{\langle 1\rangle \langle 2\rangle \langle 3\rangle},$$

and

$$x_i = \frac{k_i^+}{k_1^+ + k_2^+ + k_3^+} \quad i = 1, 2, 3 \quad (x_1 + x_2 + x_3 = 1).$$

The functions $C^{g3g}$ are a straightforward generalization of the functions $C^{ggg}$ defined in Eq. (1.4) and read,

$$C^{g3g}(p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) = \begin{cases} 
1 \quad \nu_a = - \\
\nu_i = - \quad i = 1, 2, 3 
\end{cases}$$

with $\bar{\nu} = +$, (5.6)
From the non-PT subamplitudes \[37\] we obtain the function of \((- - ++)\) helicities

\[
A^{g\rightarrow 3g}(p_a^-, k_1^+, k_2^+, k_3^-) = \frac{2}{s_{12}|k_{1\perp}|^2} \times \\
\left[ -\frac{s_{12}s_{123}}{s_{23}} \left( \frac{\beta(k_1, k_2, k_3)x_1 + \gamma(k_1, k_2, k_3)(x_1x_2 + \beta(k_1, k_2, k_3)(x_2 + x_3))}{x_2} \right) \right. \\
-\frac{\beta(k_1, k_2, k_3)^2s_{12}}{x_2} + \frac{\gamma(k_1, k_2, k_3)^2s_{123}|k_{1\perp}|^2}{s_{23}x_1x_2x_3} + \frac{s_{12}x_1^2x_2^2|q_{\perp}|^2}{s_{23}(x_2 + x_3)} \right] \\
\left. -\alpha(k_1, k_3, k_2)^2s_{12} \right) \\
\left. -\alpha(k_1, k_3, k_2)^2s_{12} \right) \\
\left. + \frac{\alpha(k_2, k_3, k_1)s_{123}x_3}{s_{23}x_2x_3|k_{3\perp}|^2} \right],
\]

with \(s_{ijk} = (p_i + p_j + p_k)^2\) the three-particle invariant, and

\[
\alpha(k_1, k_2, k_3) \equiv \sqrt{x_1}k_{1\perp}(\sqrt{x_1}q_{\perp}^1 + \sqrt{x_2}[1, 2]) \\
\beta(k_1, k_2, k_3) \equiv \sqrt{(k_{1\perp} + k_{2\perp})[1, 2]x_1x_2} \\
\gamma(k_1, k_2, k_3) \equiv \sqrt{x_1x_2x_3[12](\sqrt{x_1}[12] + \sqrt{x_2}[23])}.
\]

Using the U(1) decoupling equations \[8, 39\], the function \(B\) in Eq. (5.2) can be written as

\[
B^{g\rightarrow 3g}(p_a^\nu; k_1^\nu, k_2^\nu, k_3^\nu) = \\
- \left[ A^{g\rightarrow 3g}(p_a^\nu; k_1^\nu, k_2^\nu, k_3^\nu) + A^{g\rightarrow 3g}(p_a^\nu; k_1^\nu, k_3^\nu, k_2^\nu) + A^{g\rightarrow 3g}(p_a^\nu; k_3^\nu, k_1^\nu, k_2^\nu) \right].
\]

In the triple collinear limit, \(k_1||k_2||k_3\), Sect. 5.6 the function \(A\) has a double collinear divergence, while the function \(B\), whose gluon 3 is not color adjacent to gluons 1 and 2, has only a single collinear divergence.

Using Eq. (5.3), we can rewrite Eq. (5.2) as

\[
A^{g\rightarrow g\rightarrow 4g}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu | p_b^\nu, p_b^\nu) = \\
2s \left\{ (ig)^3 \sum_{\sigma \in S_3} f_{\alpha\delta\sigma_1}^a c_{\alpha\delta\sigma_2} f_{\alpha'\delta'\sigma_3} c_{\alpha'\delta'\sigma_3'} A^{g\rightarrow 3g}(p_a^\nu; k_{\sigma_1}^\nu, k_{\sigma_2}^\nu, k_{\sigma_3}^\nu) \right\} \frac{1}{t} \left[ ig f^{\beta\gamma'\epsilon} C^{g\rightarrow g}(p_b^\nu; p_b^\nu) \right],
\]

(5.10)
where the NNLO impact factor $g^* g \to g g g$ is enclosed in curly brackets, and includes 14 helicity configurations.

### 5.2 The NNLO impact factor $g g^* \to g \bar{q} q$

We consider the amplitude for the scattering $g g \to g \bar{q} q g$ (Fig. 3b), in the kinematics (5.1). Using Eqs. (2.11)-(2.13) and the subamplitudes of non-PT type, with two gluons of $+$ helicity and two gluons of $-$ helicity [37], we obtain

$$A_{ggg}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu | p_b^\nu, p_c^\nu) = 2s I_{ggg}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu) \frac{1}{t} \left[ i g \int d^4 \gamma C_{ggg}(p_b^\nu, p_c^\nu) \right], \quad (5.11)$$

with $k_3$ the quark, and with NNLO impact factor $g g^* \to g \bar{q} q$,

$$I_{ggg}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu) =$$

$$g^3 \left[ \left( \lambda^d \lambda a \lambda d_i \right)_{d_3 d_2} A_{1}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu) + \left( \lambda^a \lambda c \lambda d_i \right)_{d_3 d_2} A_{2}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu) \
+ \left( \lambda^d \lambda c \lambda a \right)_{d_3 d_2} A_{3}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu) + \left( \lambda^a \lambda c \lambda d \right)_{d_3 d_2} A_{4}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu) \right]$$

The NNLO impact factor allows for eight helicity configurations. From the PT subamplitudes (2.12)-(2.13) we obtain

$$A_{1}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu) = -2 \frac{q}{k_{1\perp}} \sqrt{x_1 x_2} \frac{1}{(12)(23)}$$

$$A_{2}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu) = 2 \frac{q}{k_{3\perp}} \sqrt{x_3 x_1} \frac{1}{(12)(23)}$$

$$A_{3}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu) = -2 \frac{q}{k_{2\perp}} \sqrt{x_2 x_1} \frac{1}{(13)(32)}$$

$$A_{4}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu) = 2 \frac{q}{k_{1\perp}} \sqrt{x_1 x_2 x_3} \frac{1}{(13)(32)}$$

$$B_{1}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu) = -2 \frac{q}{k_{3\perp}} x_2 x_3 \frac{1}{k_{1\perp}(23)}$$

$$B_{2}^{ggg}(p_a^\nu, k_1^\nu, k_2^\nu, k_3^\nu) = -2 \frac{q}{k_{2\perp}} x_1 x_3 \frac{1}{k_{1\perp}(23)}$$

with momentum fractions as in Eq. (5.3). The impact factors from the non-PT subamplitudes [37] are,

$$A_{1}^{ggg}(p_a^-; k_1^\perp, k_2^-; k_3^-) =$$

$$2 \left\{ \gamma(k_1, k_3, k_2) s_{123} + \sqrt{x_1} \gamma(k_1, k_3, k_2) s_{123} a(k_1, k_3, k_2) s_{12} \right\}$$

with \( s_{123} = x_1 x_2 x_3 \) and \( s_{12} = k_{1\perp} x_3 \).
\[
\begin{align*}
&\frac{-(k_{3\perp} + q_{\perp})\alpha(k_1, k_3, k_2)^2 s_{3\perp}}{x_2 s_{123} k_1 k_2} \quad - \quad \frac{|q_{\perp}|^2 x_1^3/2}{x_1 s_{3\perp}}
- \quad \frac{s_{123} \sqrt{x_2 x_3} \alpha(k_1, k_3, k_2)}{x_1 s_{23} k_1 |s_{3\perp}|^2 (x_2 + x_3)}
+ \frac{|q_{\perp}|^2 x_2^3}{x_1^2 |s_{3\perp}|^2 (x_2 + x_3)}

A_2^{g\perp\eta}(p_0^-, k_1^+, k_2^-, k_3^+) =
2 \left\{ -\frac{\gamma(k_1, k_3, k_2)^2 s_{123}}{[13] \langle 12 \rangle s_{23} x_1 x_2 x_3} - \frac{\sqrt{x_2 x_3}}{x_1} \left( -[23] \sqrt{x_1} + [13] \sqrt{x_2} \right) \right. \\
&+ \frac{k_{2\perp} k_{3\perp} - [23] \sqrt{x_2 x_3}^2}{s_{23} k_1 k_2 s_{3\perp} \sqrt{x_2 x_3}} + \frac{\sqrt{x_2 x_3}}{s_{23}} + \frac{\sqrt{x_2 x_3} \gamma(k_1, k_3, k_2) s_{123}}{x_1 \langle 12 \rangle \langle 23 \rangle [13] k_{3\perp}^*} \right\} \\
A_3^{g\perp\eta}(p_0^-, k_1^+, k_2^-, k_3^+) =
2 \left\{ \frac{k_{3\perp} \sqrt{x_3} \alpha(k_3, k_1, k_2)^2 s_{1\perp}}{k_{2\perp} k_{3\perp}^* s_{3\perp}} - \frac{\gamma(k_1, k_3, k_2)^2 s_{123}}{[13] \langle 12 \rangle s_{23} x_1 x_2 x_3} \right. \\
&+ \frac{x_1 x_2 x_3}{x_1 \langle 12 \rangle \langle 23 \rangle [13] s_{123} s_{3\perp}} + \frac{x_2^2 q_{\perp}^2 \sqrt{x_2 x_3}}{[13] s_{123} x_1 x_2 x_3} + \frac{\sqrt{x_2 x_3}}{s_{23}} \left. + \frac{\sqrt{x_2 x_3} \gamma(k_1, k_3, k_2) s_{123}}{x_1 \langle 12 \rangle \langle 23 \rangle [13] k_{2\perp}^*} \right\} \\
A_4^{g\perp\eta}(p_0^-, k_1^+, k_2^-, k_3^+) =
2 \left\{ \frac{\langle 13 \rangle k_{3\perp}^* k_{3\perp}^* \sqrt{x_2}}{[13] |k_{3\perp}|^2 s_{23} x_1 x_2 x_3} \right. \\
&+ \frac{\langle 13 \rangle \gamma(k_1, k_3, k_2)^2 s_{123}}{[13] s_{123} x_1 x_2 x_3} + \frac{x_1^2 q_{\perp}^2 \sqrt{x_2 x_3}}{[13] s_{123} x_1 x_2 x_3} + \frac{\sqrt{x_2 x_3}}{s_{23}} \left. + \frac{\sqrt{x_2 x_3} \gamma(k_1, k_3, k_2) s_{123}}{x_1 \langle 12 \rangle \langle 23 \rangle [13] k_{2\perp}^*} \right\} \\
B_1^{g\perp\eta}(p_0^-, k_1^+, k_2^-, k_3^+) =
2 \left\{ -\frac{\alpha(k_3, k_1, k_2)^2 s_{1\perp}}{k_{2\perp} k_{3\perp}^* s_{23} \sqrt{x_2 x_3}} + \frac{(k_{2\perp} + q_{\perp})^2}{[13] \sqrt{x_2 x_3} \alpha(k_1, k_2, k_3)} \right. \\
&- \frac{|q_{\perp}|^2 x_2^3}{x_2 |k_{2\perp}|^2 s_{3\perp} s_{23}} - \frac{\sqrt{x_2} |k_{1\perp}|^2 |k_{3\perp}|^2 s_{23}}{s_{23} x_1 x_3 x_2} \left. - \frac{\sqrt{x_2} |k_{1\perp}|^2 |k_{3\perp}|^2 s_{23}}{s_{23} x_1 x_3 x_2} \right\} \\
&- \frac{\sqrt{x_1}}{|k_{1\perp}|^2 s_{23} x_2} \left[ -[13] k_{1\perp} x_2 + q_{3\perp}^* \sqrt{x_1 x_3} \left( \frac{q_{\perp} x_2}{x_2 + x_3} + k_{2\perp} \right) \right] \\
\right\} (5.14)
\]

\[
\right\} (5.15)
\]

\[
\right\} (5.16)
\]

\[
\right\} (5.17)
\]

\[
\right\} (5.18)
\]
The functions $A$ and $B$ for the remaining helicity configurations are derived using the relations,

\[
A_i^{g g q}(p_a^\nu; k_1^+, k_2^+, k_3^-) = -A_{5-i}^{g g q}(p_b^\nu; k_1^+, k_3^-, k_2^-) \quad i = 1, 2, 3, 4 \\
B_i^{g g q}(p_a^\nu; k_1^+, k_2^+, k_3^-) = -B_{3-i}^{g g q}(p_a^\nu; k_1^+, k_3^-, k_2^-) \quad i = 1, 2
\]

### 5.3 The NNLO impact factor $q g^* \to q g g$

We consider the amplitude $q g \to q g g g$ for the production of a quark and two gluons in the forward-rapidity region of quark $a$ (Fig. 3c) in the kinematics (5.11). Using Eqs. (2.11)-(2.13) and the subamplitudes of non-PT type, with two gluons of $+ \h$ and two gluons of $- \h$ (37), we obtain

\[
\mathcal{A}^{q g \to 3g}(p_a^{-\nu_1}; k_1^+, k_2^+, k_3^-) = 2s I^\nu_1^\nu_2^\nu_3(p_a^{-\nu_1}; k_1^+, k_2^+, k_3^-) \frac{1}{t} \left[ i g f^{b c d} C_{b c d}^{q g}(p_b^\nu_1; p_b^\nu_2^\nu_3) \right],
\]

with $k_1$ the final-state quark, and the NNLO impact factor $q g^* \to q g g$,

\[
I^{\nu_1^\nu_2^\nu_3}(p_a^{-\nu_1}; k_1^+, k_2^+, k_3^-) = g^3 \sum_{\sigma \in S_2} \left( \lambda^{\sigma_2} \lambda^{\sigma_3} \lambda^{\nu_3} \right)_{d_1 d_2} A^{q g q}(p_a^{-\nu_1}; k_1^+, k_2^+, k_3^-) \\
+ \left( \lambda^{\nu_1} \lambda^{\sigma_2} \lambda^{\nu_3} \right)_{d_1 d_2} B^1(p_a^{-\nu_1}; k_1^+, k_2^+, k_3^-) \\
+ \left( \lambda^{\sigma_2} \lambda^{\nu_1} \lambda^{\nu_3} \right)_{d_1 d_2} B^2(p_a^{-\nu_1}; k_1^+, k_2^+, k_3^-) .
\]

The NNLO impact factor allows for eight helicity configurations. From the PT subamplitudes (2.12)-(2.13) we obtain

\[
A^{q g q}(p_a^{-\nu_1}; k_1^+, k_2^+, k_3^-) = 2i \frac{q_1}{k_{1\perp}} \frac{1}{\sqrt{x_3}} \left( \frac{1}{2} \right) (23) \\
B^1(p_a^{-\nu_1}; k_1^+, k_2^+, k_3^-) = 2i \frac{q_1}{k_{3\perp}} \sqrt{\frac{x_1 x_3}{x_2}} \frac{1}{k_{1\perp}} (23)
\]

**Note:** The expressions for $B^1$ and $B^2$ in the original text seem to be incomplete or contain errors, as they do not match with the general form for the NNLO impact factor. The corrected expressions for $B^1$ and $B^2$ are provided above.
\[ B_2^{qgg}(p^-_a; k_1^+, k_2^+, k_3^+) = 2i \frac{q_{\perp} x_1}{k_{1\perp} \sqrt{x_2} k_{3\perp}} \frac{1}{(12)} \]

and

\[ A_1^{qgg}(p^+_a, k^-_1, k_2^+, k_3^-) = -x_1 A_1^{qgg}(p^-_a; k_1^+, k_2^+, k_3^+) \]

\[ B_i^{qgg}(p^+_a; k^-_1, k_2^+, k_3^-) = -x_1 B_i^{qgg}(p^-_a; k_1^+, k_2^+, k_3^+) \quad i = 1, 2 \quad (5.24) \]

The impact factors from the non-PT subamplitudes \([31]\) are,

\[ A_1^{qgg}(p^-_a; k_1^+, k_2^+, k_3^-) = \]

\[ 2i \left\{ \frac{k_{2\perp} (k_{1\perp}^* + [12] \sqrt{x_1 x_2})^2}{k_{1\perp} |[12]| x_2} + \frac{\gamma(k_1, k_2, k_3)^* \gamma(k_1, k_3, k_2)^2 s_{12}^2}{s_{12} s_{23}} \frac{k_{3\perp}}{k_{1\perp}} + \frac{13}{12} x_1 (\sqrt{x_2} (23) - k_{1\perp} \sqrt{x_2}) + k_{1\perp} \sqrt{x_3} \gamma(k_1, k_3, k_2) s_{123} \right. \]

\[ + \left. \frac{x_1 (q_{\perp} x_1 x_2 + k_{2\perp} (x_2 + x_3)) - |q_{\perp}|^2 \frac{x_1 x_2 x_3}{x_2 + x_3} + 1. \right) \}

\[ (5.25) \]

\[ A_2^{qgg}(p^-_a; k_1^+, k_2^+, k_3^-) = \]

\[ 2i \left\{ \frac{\beta(k_1, k_2, k_3)^2 s_{34}}{s_{12}^2 k_{1\perp}^2 x_2^{3/2}} + \beta(k_1, k_2, k_3) s_{34}^2 k_{3\perp}^* \sqrt{x_1} (x_3 + x_2) \right\} \]

\[ - \frac{k_{2\perp} x_1^{3/2}}{s_{12} k_{1\perp}^2 \sqrt{x_1 x_3} (23)} \frac{q_{\perp} \gamma(k_1, k_2, k_3)^* s_{132}}{x_2 \beta(k_1, k_2, k_3)^2 s_{34}^2} \frac{k_{3\perp}}{k_{1\perp}} + \frac{x_1 (q_{\perp} x_1 x_2 + \langle 13 \rangle \beta(k_1, k_2, k_3)^2 s_{34}^2) \times (x_3 + x_2)^2}{s_{12} s_{32} k_{1\perp}^2 x_2 x_3 (x_3 + x_2)} \]

\[ + \frac{x_1 s_{34}^2 [12] \langle 13 \rangle k_{2\perp} \sqrt{x_3^2 x_2} + \beta(k_1, k_2, k_3)^2 k_{3\perp}^* s_{132} \sqrt{x_1 x_3} - \langle 13 \rangle k_{1\perp} q_{\perp} x_2)}{s_{12} s_{32} k_{1\perp}^2 \sqrt{x_2 x_3}} \}

\[ (5.26) \]

\[ B_1^{qgg}(p^-_a; k_1^+, k_2^+, k_3^-) = \]

\[ 2i \left\{ \frac{[k_{2\perp} + q_{\perp}]^2 [13] x_1 \sqrt{x_3}}{k_{1\perp}^2 |k_{3\perp}| s_{34}^2} + \frac{k_{2\perp} [13] x_1 x_3}{(23) k_{1\perp}^* |k_{3\perp}|^2 \sqrt{x_2}} - \frac{\alpha(k_2, k_3)^2 s_{12}^2}{k_{3\perp}^2 k_{2\perp} s_{23} \sqrt{x_1 x_2}} \right. \]

\[ + \left. \frac{x_3}{k_{1\perp}^2 |k_{3\perp}|^2 s_{23} x_2} \left[ k_{2\perp} + q_{\perp} x_1 x_2 + |k_{1\perp}|^2 k_{2\perp} (x_1 + x_2) x_3 \right] \right. \]

\[ + \left. \frac{k_{3\perp}^* k_{3\perp}^2 s_{23} x_2 \sqrt{x_1}}{x_1 \left[ k_{2\perp} k_{3\perp}^* |k_{3\perp}|^2 x_1 + k_{2\perp} |k_{3\perp}|^2 [12] x_1^{3/2} \sqrt{x_2} \right. \right. \]

\[ - q_{\perp} \left( \frac{q_{\perp} |k_{3\perp}|^2 x_2 x_2}{x_2 + x_3} + |k_{1\perp}|^2 k_{2\perp} (x_1 + x_2) x_3 \right) \}

\[ (5.27) \]
\[ B_1^{q\bar{q}g}(p_a^-; k_1^+, k_2^+, k_3^-) = \\
2i \left\{ -\frac{\alpha(k_3, k_1, k_2) \alpha(k_2, k_1, k_3) s_{1\bar{b}b}}{k_{3\perp} k_{2\perp}^2 |k_{3\perp}|^2 s_{32\perp}} \left( \frac{13}{\sqrt{x_3}} \frac{\sqrt{x_3 x_1}}{k_{3\perp}} - \frac{\sqrt{x_3 x_1}}{k_{3\perp}^2} \frac{1}{s_{32\perp}} \right) \right. \\
\left. + \frac{\sqrt{x_3}}{|k_{3\perp}|^2 s_{32\perp}} \left[ 12 \right] (13) \frac{\sqrt{x_1 x_3}}{x_2} (s_{2\bar{b}b}/x_1 - s_{1\bar{b}b}/x_2) \\
+ \sqrt{x_1} (k_{1\perp} q_{\perp} x_{3/2}^2 + (13) k_{3\perp} \sqrt{x_1}(x_3 + x_2)) \left( q_{\perp} x_{3/2}^2 - [12]^2 \frac{x_1 - x_2}{\sqrt{x_1 x_2}} q_{\perp} [12] \right) \\
\right. \\
\left. - q_{\perp} x_{3/2} \left( \frac{k_{3\perp}^2 q_{\perp} \sqrt{x_1}}{x_3 + x_2} + (13) (s_{1\bar{b}b} + s_{2\bar{b}b}) \sqrt{x_3} \right) \right\} \right) \\
(5.28)

\[ B_2^{q\bar{q}g}(p_a^-; k_1^+, k_2^-, k_3^-) = \\
2i \left\{ \frac{(13) \sqrt{x_3} \alpha(k_1, k_2, k_3) s_{2\bar{b}b}}{k_{2\perp} k_{3\perp}^2 s_{2\bar{b}b}} + \frac{\beta(k_1, k_2, k_3) s_{3\bar{b}b}/k_{2\perp}}{s_{2\bar{b}b}/x_2} \\
\right. \\
\left. - \frac{1}{(12) k_{2\perp}^2} \left( \frac{12}{3} \alpha(k_1, k_2, k_3) s_{2\bar{b}b}/x_2 \right) \right\} \\
(5.29)

5.4 The NNLO impact factor \( q g^* \rightarrow q \bar{Q} Q \)

We consider the amplitude \( q g \rightarrow q \bar{Q} Q g \) for the production of three quarks in the forward-rapidity region of quark \( a \) (Fig. 3d) in the kinematics (5.1). Using Eq. (2.14) and the subamplitudes of non-PT type, with two gluons of opposite helicities [5], we obtain

\[ A^{qg\rightarrow q\bar{Q}Qg}(p_a^{-\nu_1}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{-\nu_2} \mid p_b^{\nu_3}, p_b^{\nu_4}, p_b^{\nu_5}) = \\
2i \frac{\lambda_{\alpha_1\beta_1\gamma_1}^c}{t} \left[ i g f^{\bar{b}b'c} C^{g\gamma g}(p_b^{-\nu_3}, p_b^{\nu_4}) \right] , \]

with NNLO impact factor \( q g^* \rightarrow q \bar{Q} Q \)

\[ I^{q\bar{q}gQ}(p_a^{-\nu_1}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{-\nu_2}; p_b^{\nu_3}, p_b^{\nu_4}) = \\
g^3 \left[ \lambda_{d_1a}^{\alpha_1} \delta_{d_1d_2}^{\alpha_2} A_1^{q\bar{q}gQ}(p_a^{-\nu_1}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{-\nu_2}) \right. \\
\left. - \frac{1}{N_c} \lambda_{d_1a}^{\alpha_1} \delta_{d_3d_4}^{\alpha_2} A_2^{q\bar{q}gQ}(p_a^{-\nu_1}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{-\nu_2}) \\
+ \frac{1}{N_c} \lambda_{d_1a}^{\alpha_1} \delta_{d_3d_4}^{\alpha_2} B_1^{q\bar{q}gQ}(p_a^{-\nu_1}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{-\nu_2}) \right] \\
- \delta_{\bar{Q}Q}(1 \leftrightarrow 3) . \]

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The term proportional to $\delta_{qQ}$ is due to the interference of identical quarks (i.e. with the same flavour and helicity) in the final state. The NNLO impact factor allows for four helicity configurations. From the non-PT subamplitudes [50] we obtain,

$$A_{1}^{q\bar{q}QQ}(p_{1}^{-}; k_{1}^{+}, k_{2}^{-}, k_{3}^{+}) = i \left\{ \frac{\sqrt{x_{1}x_{3}x_{2}}\alpha(k_{1}, k_{3}, k_{2})}{|k_{1}||^{2}} + \frac{\sqrt{x_{1}x_{2}x_{3}}|q_{1}|^{2}}{1-x_{1}s_{23}|k_{1}||^{2}} + \frac{\gamma(k_{1}, k_{3}, k_{2})}{\sqrt{x_{1}x_{2}x_{3}s_{23}}} \right\}$$

$$B_{1}^{q\bar{q}QQ}(p_{1}^{-}; k_{1}^{+}, k_{2}^{-}, k_{3}^{+}) = \frac{i}{\sqrt{x_{2}x_{3}s_{23}}} \left( \frac{\gamma(k_{1}, k_{3}, k_{2})}{\sqrt{x_{1}}} + \frac{\sqrt{x_{1}x_{2}x_{3}}}{3} \right)$$

$$B_{2}^{q\bar{q}QQ}(p_{1}^{-}; k_{1}^{+}, k_{2}^{-}, k_{3}^{+}) = \frac{i\sqrt{x_{1}}}{|k_{1}||^{2}} \left( \frac{\sqrt{x_{2}}x_{3}}{x_{2}} \frac{\beta(k_{1}, k_{3}, k_{2})}{x_{1}} + \frac{\sqrt{x_{1}x_{2}x_{3}s_{23}}}{3} \right)$$

$$A_{2}^{q\bar{q}QQ}(p_{1}^{-}; k_{1}^{+}, k_{2}^{-}, k_{3}^{+}) = i \left\{ \frac{\sqrt{x_{1}x_{3}x_{2}}\alpha(k_{1}, k_{3}, k_{2})}{|k_{1}||^{2}} + \frac{\sqrt{x_{1}x_{2}x_{3}}|q_{1}|^{2}}{1-x_{1}s_{23}|k_{1}||^{2}} + \frac{\gamma(k_{1}, k_{3}, k_{2})}{\sqrt{x_{1}x_{2}x_{3}s_{23}}} \right\}$$

with $\alpha, \beta, \gamma$ defined in Eq. (5.8) and

$$A_{1}^{q\bar{q}QQ}(p_{1}^{+}; k_{1}^{-}, k_{2}^{-}, k_{3}^{+}) = A_{1}^{q\bar{q}QQ}(p_{1}^{-}; k_{1}^{+}, k_{3}^{-}, k_{2}^{+}) \quad i = 1, 2$$

$$B_{1}^{q\bar{q}QQ}(p_{1}^{+}; k_{1}^{-}, k_{2}^{-}, k_{3}^{+}) = B_{1}^{q\bar{q}QQ}(p_{1}^{-}; k_{1}^{+}, k_{3}^{-}, k_{2}^{+}) \quad i = 1, 2. \quad (5.34)$$

Note that for each helicity configuration, we have the following relation between the functions $A$ and $B$

$$A_{1}^{q\bar{q}QQ} + B_{1}^{q\bar{q}QQ} = A_{2}^{q\bar{q}QQ} + B_{2}^{q\bar{q}QQ}, \quad (5.35)$$

### 5.5 NNLO impact factors in the high-energy limit

The amplitudes (5.10), (5.11), (5.21) and (5.31) have been computed in the kinematic limit (3.1), in which they factorize into an effective amplitude with a ladder structure, made of a three-parton forward cluster and a LO impact factor connected by a gluon exchanged in the crossed channel (Fig. 3). In the limits $y_{1} \simeq y_{2} \gg y_{3}$ or $y_{1} \gg y_{2} \simeq y_{3}$, the amplitudes must factorize further into NLO impact factors or into NLO Lipatov vertices for the production of two partons along the ladder. Such limits constitute then necessary consistency checks, and we display them in this section.

In the limit, $y_{1} \gg y_{2} \simeq y_{3}$, the NNLO impact factor, $g^{*} g \rightarrow g g g$, Eq. (5.10), factorizes into a NLO Lipatov vertex for the production of two gluons convoluted with a multi-Regge ladder (Fig. 3a)

$$\lim_{y_{1} \gg y_{2} \simeq y_{3}} \left\{ (ig)^{3} \sum_{\sigma \in S_{3}} f^{a_{1}c} f^{a_{2}c'} f^{c'd_{3}c''} A^{a_{3}g}(p_{a_{1}1}^{\nu_{a_{1}1}}, k_{a_{1}2}^{\nu_{a_{2}2}}, k_{a_{1}3}^{\nu_{a_{3}3}}) \right\} \quad (5.36)$$

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Figure 4: Limits of the amplitude for the production of three gluons in the forward-rapidity region of gluon \( p_a \), for \( y_1 \gg y_2 \approx y_3 \) (a) and \( y_1 \approx y_2 \gg y_3 \) (b).

\[
\mathcal{A}^{gg}(q_1, k_2^+, k_3^+, q_2) = 2 \frac{q_1^2}{k_2^\perp} \frac{q_2^2}{k_3^\perp} \sqrt{\frac{x_2}{x_3}} \left( \frac{1}{(23)} \right) \sum_{\sigma \in S_2} \mathcal{A}^{gg}(q_1, k_2^\sigma, k_3^\sigma, q_2)
\]

with exchanged momenta in the \( t \) channel \( q_1 = -(p_a + p_c) \), \( q_2 = p_v + p_b \), three-particle invariant \( s_{3b^\prime} = (k_3 + q_2)^2 \approx -\left((q_2 + k_3^\perp)^2 + k_2^\perp k_3^\perp\right) \), and with the mass-shell conditions \( k_i^\perp = |k_i|^2/k_i^\perp \) for \( i = 2, 3 \).

In the collinear limit, \( k_2 = zP \) and \( k_3 = (1 - z)P \), the NLO Lipatov vertex (5.37) reduces to the splitting factor (4.19), and amplitude (5.10) factorizes into a multi-Regge amplitude (4.11) times a collinear factor (4.18).

\[
\lim_{k_2 \parallel k_3} \mathcal{A}^{gg \rightarrow A_g}(p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}, p_b^{\nu_b}) = \sum_{\nu} \mathcal{A}^{gg \rightarrow 3g}(p_a^{\nu_a}, k_1^{\nu_1}, P^{\nu_2}, P^{\nu_3}, p_b^{\nu_b}) \cdot \text{Split}^{gg \rightarrow gg}(k_2^{\nu_2}, k_3^{\nu_3}),
\]
In the limit, $y_1 \simeq y_2 \gg y_3$, the NNLO impact factor in Eq. (5.10) factorizes into a NLO impact factor, $g^* g \to g g$, Eq. (4.8), convoluted with a multi-Regge ladder (Fig. 4b)

$$
\lim_{y_1 \simeq y_2 \gg y_3} \left\{ (ig)^3 \sum_{\sigma \in S_3} f^{a_1,c} f^{c_2,c_3} f^{c_3,c_4} A^{g^* g}(p_a^{\nu_a}; k_{\sigma_1}, k_{\sigma_2}, k_{\sigma_3}) \right\} = \\
\left\{ (ig)^2 \sum_{\sigma \in S_2} f^{a_1,c} f^{c_2,c_3} f^{c_3,c_4} A^{g^* g}(p_a^{\nu_a}; k_{\sigma_1}, k_{\sigma_2}) \right\} \frac{1}{t_1} \left[ i g f^{c_3,c_4} C^g(q_1, k_{\nu_3}^{\nu}) \right],
$$

with $q_1 = -(p_a + k_1 + k_2)$, and with LO Lipatov vertex $C^g(q_1, k_{\nu_3}^{\nu}, q_2)$, Eq. (4.10).

In the limit, $y_1 \gg y_2 \simeq y_3$, the functions $A$ and $B$ in Eq. (5.13)-(5.20) fulfill the relations $A_2^{qgq} = A_3^{qgq} = 0$, $B_2^{gqg} = -A_1^{gqg}$, and $B_1^{gqg} = -A_2^{gqg}$, thus the NNLO impact factor, $g^* g \to g \bar{q} q$, Eq. (5.12), factorizes into a NLO Lipatov vertex for the production of a $\bar{q} q$ pair convoluted with a multi-Regge ladder (Fig. 4b),

$$
\lim_{y_1 \gg y_2 \simeq y_3} J^{g:qg}(p_a^{\nu_a}; k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_2}) = \\
\left[ ig f^{a_1,c} C^g(p_a^{\nu_a}; k_{\nu_1}) \right] \frac{1}{t_1} \left\{ g^2 \left[ (\lambda^c \lambda^c)_{a_3,d_2} A^{qg q}(q_1, k_2^{\nu_2}, k_3^{\nu_2}, q_2) + (\lambda^c \lambda^c)_{a_3,d_2} A^{qg q}(q_1, k_3^{\nu_2}, k_2^{\nu_2}, q_2) \right] \right\},
$$

with the NLO Lipatov vertex, $g^* g \to \bar{q} q$, for the production of a $\bar{q} q$ pair

$$
A^{qg q}(q_1, k_2^{\nu_2}, k_3^{\nu_2}, q_2) = -2 \sqrt{\frac{k_2^{\nu_2} k_2^{\nu_2}}{k_3^{\nu_2}}} \left\{ \frac{k_2^{\nu_2} |q_3|}{k_3^{\nu_2}} \right\} + \frac{k_3^{\nu_2} |q_3|}{k_2^{\nu_2} + k_3^{\nu_2}} + \frac{k_2^{\nu_2} k_3^{\nu_2} (q_3 + k_3^{\nu_2})}{k_2^{\nu_2} k_3^{\nu_2}}
$$
Figure 6: Same as Fig. 4 for the production of a quark and two gluons in the forward-rapidity region of quark $p_a$.

$$\frac{(q_{2\perp} + k_{3\perp})[k_2^- k_3^+ - k_2^+ k_{3\perp} - (q_{2\perp}^+ + k_{3\perp}^+) k_{3\perp}]}{k_{2\perp} s_{23}} - \frac{|k_{3\perp}|^2}{s_{23}} \right\}, \quad (5.40)$$

with $q_1$, $q_2$, and $s_{3b'}$ as in Eq. (5.37).

In the collinear limit, $k_2 = z P$ and $k_3 = (1 - z) P$, the NLO Lipatov vertex (5.40) reduces to the splitting factor (4.19), and amplitude (5.11) factorizes into a multi-Regge amplitude (4.11) times a collinear factor (4.18).

$$\lim_{k_2 \rightarrow |k_3|} A^{g g \rightarrow q q g} (p_a^{\nu_a}, k_1^{\nu_1} \parallel k_2^{\nu_2}, k_3^{-\nu_2} \parallel p_{b'}^{\nu_{b'}}, p_{b''}^{\nu_{b''}})$$

$$= \sum_{\nu} A^{g g \rightarrow q g} (p_a^{\nu_a}, k_1^{\nu_1} \parallel P^{\nu} \parallel p_{b'}^{\nu_{b'}}, p_{b''}^{\nu_{b''}}) \cdot \text{Split}^{q_2 \rightarrow q g} (k_2^{\nu_2}, k_3^{-\nu_2}).$$

In the limit, $y_2 \simeq y_3 \gg y_1$, the functions $A$ and $B$ in Eq. (5.13)-(5.20) fulfill the relations $A_1^{q g q q} = A_4^{q g q q} = 0$, $B_2^{g g q q} = -A_3^{g g q q}$, and $B_4^{g g q q} = -A_2^{g g q q}$ thus the NNLO impact factor, Eq. (5.12), factorizes into a NLO impact factor, $g^* g \rightarrow \bar{q} q$, Eq. (4.12), convoluted with a multi-Regge ladder (Fig. 5b),

$$\lim_{y_2 \simeq y_3 \gg y_1} I^{g g q q} (p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{-\nu_2})$$

$$= \left\{ g^2 \left[ (\lambda^c \lambda^a)_{d_3 d_2} A^{g q q} (p_a^{\nu_a}, k_2^{\nu_2}, k_3^{-\nu_2}) + (\lambda^a \lambda^c)_{d_3 d_2} A^{g q q} (p_a^{\nu_a}, k_3^{\nu_3}, k_2^{\nu_2}) \right] \right\}$$

$$\times \frac{1}{t_1} \left[ ig f^{a c d} C^g (q_1, k_1^{\nu_1}, q_2) \right], \quad (5.41)$$

with $q_1 = -(p_a + k_2 + k_3)$. 

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In the limit, $y_1 \gg y_2 \approx y_3$, the functions $A$ and $B$ in (5.23)-(5.30) fulfill the relations,

$$B_1^{qgq}(p_a^{-\nu}; k_1^1, k_2^2, k_3^3) = A_1^{qgq}(p_a^{-\nu}; k_1^1, k_2^2, k_3^3) = 0$$

$$B_2^{qgq}(p_a^{-\nu}; k_1^1, k_2^2, k_3^3) = B_2^{qgq}(p_a^{-\nu}; k_1^1, k_2^2, k_3^3) = 0$$

$$B_2^{qgq}(p_a^{-\nu}; k_1^1, k_2^2, k_3^3) = B_1^{qgq}(p_a^{-\nu}; k_1^1, k_2^2, k_3^3),$$

thus the NNLO impact factor, $q g^* \rightarrow q g g$, Eq. (5.22), factorizes into a NLO Lipatov vertex for the production of two gluons (5.34) convoluted with a multi-Regge ladder (Fig. 6a)

$$\lim_{y_1 \gg y_2 \approx y_3} I^{qgq}(p_a^{-\nu}; k_1^1, k_2^2, k_3^3) = \left[ g \lambda\frac{c}{d_1 a} C^{qg}(p_a^{-\nu}; k_1^1) \right] \frac{1}{t_1} \left\{ \left( ig \right)^2 \sum_{\sigma \in S_2} f^{c_{d_2 c'} f_{c' d_{\sigma} c''}} A^{qg}(q_1, k_2^2, k_3^3, q_2) \right\}.$$

In the limit, $y_1 \approx y_2 \gg y_3$, the functions $A$ and $B$ in (5.23)-(5.30) fulfill the relations,

$$A_1^{qgq}(p_a^{-\nu}; k_1^1, k_2^2, k_3^3) = 0$$

$$B_2^{qgq}(p_a^{-\nu}; k_1^1, k_2^2, k_3^3) = 0$$

$$B_2^{qgq}(p_a^{-\nu}; k_1^1, k_2^2, k_3^3) = B_2^{qgq}(p_a^{-\nu}; k_1^1, k_2^2, k_3^3),$$

thus the NNLO impact factor, Eq. (5.22), factorizes into a NLO impact factor, $q g^* \rightarrow q g g$, Eq. (1.14), convoluted with a multi-Regge ladder (Fig. 6b)

$$\lim_{y_1 \approx y_2 \gg y_3} I^{qgq}(p_a^{-\nu}; k_1^1, k_2^2, k_3^3) = \left\{ g^2 \left[ \left( \lambda d_2 \lambda c' \right)_{d_1 a} A^{qg}(p_a^{-\nu}; k_1^1, k_2^2) \right] + \left( \lambda c' \lambda d_2 \right)_{d_1 a} B^{qgq}(p_a^{-\nu}; k_1^1, k_2^2) \right\} \times \frac{1}{t_1} \left\{ ig f^{c_{d_3 c'} c''} C^q (q_1, k_3^3, q_2) \right\},$$

with $q_1 = -(p_a + k_1 + k_2)$.

In the limit, $y_1 \gg y_2 \approx y_3$, the function $A_2$ in Eq. (5.33) vanishes, $A_2^{qgQ} = 0$, and using Eqs. (5.33)-(5.35) the NNLO impact factor, $q g^* \rightarrow q Q$, Eq. (5.32), factorizes into a NLO Lipatov vertex for the production of a $q\bar{q}$ pair (5.40) convoluted with a multi-Regge ladder (Fig. 6)

$$\lim_{y_1 \gg y_2 \approx y_3} I^{qgQ}(p_a^{-\nu}; k_1^1, k_2^2, k_3^{-\nu}) = \left[ g \lambda\frac{c}{d_1 a} C^{qg}(p_a^{-\nu}; k_1^1) \right] \times \frac{1}{t_1} \left\{ g^2 \left[ \left( \lambda c' \lambda c \right)_{d_3 d_2} A^{qg}(q_1, k_2^2, k_3^3, q_2) + \left( \lambda c' \lambda c \right)_{d_3 d_2} A^{qg}(q, k_3^{-\nu}, k_2^2, q_2) \right\}. $$
5.6 NNLO impact factors in the triple collinear limit

In the triple collinear limit, \( k_i = z_i P \), with \( z_1 + z_2 + z_3 = 1 \) a generic amplitude must factorize as \([10, 11]\)

\[
\lim_{k_1||k_2||k_3} A^{...d_1d_2d_3...}(..., k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}, ...) = \sum_\nu A^{...}(..., P^\nu, ...) \cdot \text{Split}^{f \rightarrow f_1 f_2 f_3}(k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}).
\]  

(5.45)

Accordingly, we must show that taking the triple collinear limit of the NNLO impact factors, we can write the amplitudes (5.10), (5.11), (5.21) and (5.31) as

\[
\lim_{k_1||k_2||k_3} A^{f \rightarrow f_1 f_2 f_3 g}(p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3} | p_b^{-\nu_b}, p_b^{\nu_b}) \\
= A^{f \rightarrow f g}(p_a^{\nu_a}, P^{-\nu_a} | p_b^{-\nu_b}, p_b^{\nu_b}) \cdot \text{Split}^{f \rightarrow f_1 f_2 f_3}(k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}),
\]  

(5.46)

with \( f \) denoting the parton species, \( A^{f \rightarrow f g} \) given in Eqs. (3.1), (3.4) and (3.8), and with \( \text{Split}^{f \rightarrow f_1 f_2 f_3} \) the polarized double-splitting functions.

In the triple collinear limit, the functions \( A \) of Sect. 5.1, 5.2, 5.3 and 5.4 yield a quadratic divergence as \( s_{123} \rightarrow 0 \) or \( s_{ij} \rightarrow 0 \) with \( i, j = 1, 2, 3 \). In the same limit, the functions \( B \) have a single collinear divergence since only two out of the three partons are color adjacent. However, terms with a single divergence when integrated over the triple collinear region of phase space yield a negligible contribution \([10]\), thus we ignore them.

It is easy to show that a function \( A^{g;3g} \), Eqs. (3.3)-(5.7), differs from its reflection by a term which contains only a single divergence. Using this property and Eq. (5.9), we obtain a dual Ward identity and a reflection identity for the functions \( A^{g;3g} \), up to singly divergent terms,

\[
A^{g;3g}(p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) + A^{g;3g}(p_a^{\nu_a}, k_1^{\nu_1}, k_3^{\nu_3}, k_2^{\nu_2}) + A^{g;3g}(p_a^{\nu_a}, k_2^{\nu_2}, k_3^{\nu_3}, k_1^{\nu_1}) = 0
\]

\[
A^{g;3g}(p_a^{\nu_a}, k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) = A^{g;3g}(p_a^{\nu_a}, k_3^{\nu_3}, k_2^{\nu_2}, k_1^{\nu_1}).
\]  

(5.47)

Using the identities (5.47) in the impact factor in Eq. (5.10), we can factorize the color...
structure on a leg

\[
(i g)^3 \sum_{\sigma \in S_3} f_{a d e_1 c} f_{c d e_2 c'} f_{c' d e_3 c''} A^{g-3g}(p_{a}^{\nu_a}; k_{\sigma_1}^{\nu_{e_1}}, k_{\sigma_2}^{\nu_{e_2}}, k_{\sigma_3}^{\nu_{e_3}}) = \frac{(i g)^3}{3} f_{a c''} \sum_{\sigma \in S_3} f_{c d e_1 c'} f_{c' d e_2 d_3} A^{g-3g}(p_{a}^{\nu_a}; k_{\sigma_1}^{\nu_{e_1}}, k_{\sigma_2}^{\nu_{e_2}}, k_{\sigma_3}^{\nu_{e_3}}) = i g f_{a c''} \left\{ g^2 \sum_{\sigma \in S_2} (F_{d_1}^{d_2} F_{d_2}^{d_3})_{cd_3} A^{g-3g}(p_{a}^{\nu_a}; k_{\sigma_1}^{\nu_{e_1}}, k_{\sigma_2}^{\nu_{e_2}}, k_{\sigma_3}^{\nu_{e_3}}) \right\}, \quad (5.48)
\]

where \((F^n)_{bc} \equiv i f^{bac}\). Thus amplitude (5.10) can be put in the form of Eq. (5.46) with collinear factor

\[
\text{Split}^{g-3g}_{-\nu}(k_{1}^{\nu_1}, k_{2}^{\nu_2}, k_{3}^{\nu_3}) = g^2 \sum_{\sigma \in S_2} (F_{d_1}^{d_2} F_{d_2}^{d_3})_{cd_3} \text{split}^{g-3g}_{-\nu}(k_{1}^{\nu_1}, k_{2}^{\nu_2}, k_{3}^{\nu_3}). \quad (5.49)
\]

The splitting factors \text{split}^{g-3g}_{-\nu} are the functions \(A\), Eqs. (5.3)-(5.7), in the triple collinear limit, up to singly divergent terms, and thus they fulfill the identities, Eq. (5.47). The splitting factors of non-PT type can be soon read off from Eqs. (5.3)-(5.4), while for the ones of non-PT type we note that the coefficients of Eq. (5.8) reduce to

\[
\alpha(k_{1}, k_{2}, k_{3}) \rightarrow \frac{z_{1}z_{3}}{z_{1} + z_{3}}
\]

\[
\beta(k_{1}, k_{2}, k_{3}) \rightarrow -\frac{\sqrt{z_{1}z_{2}}}{P_{\perp}}[1 2]
\]

\[
\gamma(k_{1}, k_{2}, k_{3}) \rightarrow \frac{\sqrt{z_{2}z_{3}z_{1}}}{s_{123}}\delta(1, 2, 3)
\]

with

\[
\delta(1, 2, 3) \equiv [1 2] (\sqrt{z_{1}} \langle 1 3 \rangle + \sqrt{z_{2}} \langle 2 3 \rangle).
\]

Thus we obtain

\[
\text{split}^{g-3g}_{-\nu}(k_{1}^{+}, k_{2}^{+}, k_{3}^{+}) = 2 \frac{1}{\sqrt{z_{1}z_{3}}} \frac{1}{(1 2) \langle 2 3 \rangle}
\]

\[
\text{split}^{g-3g}_{+}(k_{1}^{-}, k_{2}^{+}, k_{3}^{+}) = 2 \frac{z_{1}^{2}}{\sqrt{z_{1}z_{3}}} \frac{1}{(1 2) \langle 2 3 \rangle}
\]

\[
\text{split}^{g-3g}_{+}(k_{1}^{+}, k_{2}^{-}, k_{3}^{+}) = 2 \frac{z_{2}^{2}}{\sqrt{z_{1}z_{3}}} \frac{1}{(1 2) \langle 2 3 \rangle}
\]

\[
\text{split}^{g-3g}_{+}(k_{1}^{+}, k_{2}^{+}, k_{3}^{-}) = 2 \frac{z_{3}^{2}}{\sqrt{z_{1}z_{3}}} \frac{1}{(1 2) \langle 2 3 \rangle}
\]

\[
\text{split}^{g-3g}_{-\nu}(k_{1}^{+}, k_{2}^{+}, k_{3}^{-}) = 2 \frac{1}{s_{12}s_{23}} \left[ \frac{s_{12}z_{2}}{(1 - z_{1})} + \frac{(1, 2, 3)^{2}}{s_{123}} + \sqrt{\frac{z_{2}}{z_{1}z_{3}}} (1 - z_{3})\delta(1, 2, 3) \right]
\]

\[
\text{split}^{g-3g}_{-\nu}(k_{1}^{-}, k_{2}^{+}, k_{3}^{+}) = \text{split}^{g-3g}_{-\nu}(k_{3}^{+}, k_{2}^{+}, k_{1}^{-})
\]

\[
\text{split}^{g-3g}_{-\nu}(k_{1}^{+}, k_{2}^{-}, k_{3}^{+}) = \text{split}^{g-3g}_{-\nu}(k_{3}^{+}, k_{1}^{+}, k_{2}^{-}) - \text{split}^{g-3g}_{-\nu}(k_{1}^{+}, k_{3}^{+}, k_{2}^{-}).
\]
In the triple collinear limit of the NNLO impact factor $q\, g^* \rightarrow \bar{q}\, g\, g$, the functions $A_{g,g,g}^{g,q,q}$, Eqs. (5.13)-(5.20), fulfill the relations $A_2^{g,q,q} = -A_1^{g,q,q}$ and $A_4^{g,q,q} = -A_3^{g,q,q}$, and $A_3^{g,q,q}(p_{\nu}^d, k_1, k_2, k_3, k_{\bar{v}}) = A_1^{g,q,q}(p_{\nu}^d, k_1, k_3, k_{\bar{v}}, k_{\bar{v}})$. Thus amplitude (5.11) can be put in the form of Eq. (5.46) with collinear factor

$$
\text{Split}_{g\rightarrow g,g}(k_1^+, k_2^-, k_3^+) = g^2 \left( \lambda^c \lambda^d \right)_{d_3 d_2} \text{split}_{g\rightarrow g,g}(k_1^+, k_2^-, k_3^+) + \left( \lambda^d \lambda^c \right)_{d_3 d_2} \text{split}_{g\rightarrow g,g}(k_1^+, k_2^-, k_3^+) \right),
$$

with

$$
\text{split}_{g\rightarrow g,g}(k_1^+, k_2^-, k_3^+) = 2 \sqrt{z_2 \left( \frac{1}{2}, \frac{1}{3} \right)} z_1 \langle 2 \rangle \langle 3 \rangle
$$

$$
\text{split}_{g\rightarrow g,g}(k_1^+, k_2^+, k_3^+) = 2 \sqrt{z_2 \left( \frac{1}{2}, \frac{1}{3} \right)} z_1 \langle 2 \rangle \langle 3 \rangle
$$

$$
\text{split}_{g\rightarrow g,g}(k_1^-, k_2^+, k_3^+) = -\frac{2}{s_{12} s_{23}} \left[ \frac{\delta(1, 3, 2)^2 [1, 2]}{[1, 3], s_{123}} \right]
$$

$$
\text{split}_{g\rightarrow g,g}(k_1^-, k_2^-, k_3^+) = -\frac{2}{s_{12} s_{23}} \left[ \frac{\delta(1, 3, 2)^2 [1, 2]}{[1, 3], s_{123}} \right] + \frac{\delta(1, 2, 3)(1 - z_3)}{\sqrt{z_1}} + \frac{\sqrt{z_2 s_2 s_3 s_1}}{(1 - z_1)}
$$

Writing the functions $A$, Eqs. (5.23)-(5.30), in the triple collinear limit of the NNLO impact factor $q\, g^* \rightarrow q\, g\, g$, the amplitude (5.21) can be put in the form of Eq. (5.46) with collinear factor

$$
\text{Split}_{\nu\rightarrow q,g}(k_1^+, k_2^-, k_3^+) = g^2 \sum_{\sigma \in S_2} \left( \lambda^d \lambda^c \right)_{d_3 d_2} \text{split}_{\nu\rightarrow q,g}(k_1^+, k_2^-, k_3^+) \right),
$$

with

$$
\text{split}_{\nu\rightarrow g,g}(k_1^+, k_2^-, k_3^+) = -\frac{2i}{\sqrt{z_3}} \frac{1}{\langle 2 \rangle \langle 3 \rangle}
$$

$$
\text{split}_{\nu\rightarrow g,g}(k_1^+, k_2^+, k_3^+) = \frac{2i z_1}{\sqrt{z_3}} \langle 2 \rangle \langle 3 \rangle
$$

$$
\text{split}_{\nu\rightarrow g,g}(k_1^-, k_2^+, k_3^+) = \frac{2i}{s_{12} s_{23}} \left[ \frac{\delta(1, 3, 2)^2 (\sqrt{z_1} [1, 3] + \sqrt{z_2} [2, 3])}{[1, 3], s_{123}} \right] + \frac{\sqrt{z_2} \delta(1, 3, 2)(1 - z_3)}{\sqrt{z_3}} + \frac{\sqrt{z_2 s_2 s_3 s_1}}{1 - z_1}
$$

$$
\text{split}_{\nu\rightarrow g,g}(k_1^-, k_2^-, k_3^+) = \frac{2i}{s_{12} s_{23}} \left[ \frac{\delta(1, 3, 2)^2 (\sqrt{z_1} [1, 2] - \sqrt{z_3} [2, 3])}{[1, 2], s_{123}} \right] + \frac{\sqrt{z_2} \delta(1, 3, 2)(1 - z_3)}{\sqrt{z_3}} + \frac{\sqrt{z_2 s_2 s_3 s_1}}{1 - z_1}
$$

In the triple collinear limit of the NNLO impact factor $q\, g^* \rightarrow \bar{q}\, \bar{Q}\, Q$, the functions $A_{\bar{q},Q,Q}^{q,q,Q}$ fulfill the relation $A_1^{\bar{q},q,Q} = A_2^{q,q,Q}$. Thus the amplitude (5.31) can be put in the
form of Eq. (5.46), with collinear factor

\[
\text{Split}_{\nu}^{q \rightarrow q Q}(k_1^\nu, k_2^{\nu_2}, k_3^{\nu_3}) = 2g^2 \left\{ \lambda^a_{d_1c} \lambda^a_{d_3d_2} \cdot \text{split}_{\nu}^{q \rightarrow q Q}(k_1^\nu, k_2^{\nu_2}, k_3^{\nu_3}) - \delta_{Q} \lambda^a_{d_3c} \lambda^a_{d_1d_2} \cdot \text{split}_{\nu}^{q \rightarrow q Q}(k_3^{\nu_3}, k_2^{\nu_2}, k_1^{\nu_1}) \right\},
\]

(5.57)

where the second term occurs for the case of identical quarks, and \( \bar{c} \) is the color index of the parent quark. The splitting factors are

\[
\text{split}_{-\nu}^{q \rightarrow q Q}(k_1^+, k_2^-, k_3^+) = \frac{i}{s_{23}} \left( \frac{\sqrt{z_1 z_2 z_3}}{1 - z_1} + \delta(1, 3, 2) \right)
\]

(5.58)

The factor \( \text{split}_{-\nu}^{f \rightarrow f_1 f_2 f_3}(k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) \) can be obtained from \( \text{split}_{-\nu}^{f \rightarrow f_1 f_2 f_3}(k_1^\nu, k_2^{\nu_2}, k_3^{\nu_3}) \) in Eqs. (5.52), (5.54), (5.56) and (5.58) by exchanging \( \langle i j \rangle \) with \( [ij] \), and multiplying by a coefficient \( S \), Eq. (3.7), for each quark pair the splitting factor includes.

Using Eq. (5.49) and Eqs. (5.52)-(5.58), and summing over the two helicity states of partons 1, 2 and 3, we obtain, as in Section 4.4, the two-dimensional Altarelli-Parisi splitting functions [10]

\[
\sum_{\nu_1 \nu_2 \nu_3} \text{Split}_{-\nu}^{f \rightarrow f_1 f_2 f_3}(k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) \left[ \text{Split}_{\rho}^{f \rightarrow f_1 f_2 f_3}(k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) \right]^* = \delta_{cc'} \frac{4g^4}{s_{123}} P_{\lambda \rho}^{f \rightarrow f_1 f_2 f_3},
\]

(5.59)

where \( P_{++}^{f \rightarrow f_1 f_2 f_3} = P_{--}^{f \rightarrow f_1 f_2 f_3} \), and \( P_{-+}^{f \rightarrow f_1 f_2 f_3} = (P_{+--}^{f \rightarrow f_1 f_2 f_3})^* \). For splitting functions of type \( P_{q \rightarrow q g f_3} \), namely for \( P_{q \rightarrow q g g} \), \( P_{q \rightarrow q Q Q} \) and \( P_{q \rightarrow q g q} \), where the last splitting function is for identical quarks, helicity conservation on the quark line sets the off-diagonal elements equal to zero.

Averaging over the trace of matrix (5.59), i.e. over color and helicity of the parent parton, we obtain the unpolarized Altarelli-Parisi splitting functions [10]

\[
\frac{1}{2C} \sum_{\nu_1 \nu_2 \nu_3} \left| \text{Split}_{-\nu}^{f \rightarrow f_1 f_2 f_3}(k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) \right|^2 = \frac{4g^4}{s_{123}} \langle P_{f \rightarrow f_1 f_2 f_3} \rangle,
\]

(5.60)

with \( C \) defined below Eq. (1.22). For \( \langle P_{g \rightarrow g g g} \rangle \), the sum over colors can be immediately done using Eq. (2.5), and it yields

\[
\left| \text{Split}_{-\nu}^{g \rightarrow g g g}(k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}) \right|^2 = 4C_4(N_c) \sum_{\sigma \in S_3} \left| \text{split}_{-\nu}^{g \rightarrow g gg}(k_{\sigma_1}^{\nu_1}, k_{\sigma_2}^{\nu_2}, k_{\sigma_3}^{\nu_3}) \right|^2,
\]

(5.61)

with \( C_4(N_c) \) as in Eq. (2.8). Eq. (5.61) shows that for the purely gluonic unpolarized splitting function the color factorizes.

Since the averaged trace of \( P_{f \rightarrow f_1 f_2 f_3} \) is \( \langle P_{f \rightarrow f_1 f_2 f_3} \rangle = \text{tr} P_{f \rightarrow f_1 f_2 f_3} / 2 = P_{++}^{f \rightarrow f_1 f_2 f_3} \), we have checked that for the diagonal elements, \( P_{++}^{f \rightarrow f_1 f_2 f_3} \), our expressions agree with the
unpolarized splitting functions of Ref. [10] by setting there the RS parameter $\epsilon = 0$. Finally, for the off-diagonal elements of the splitting functions of type $P_{+-}^{g\rightarrow g f_2 f_3}$ we obtain

$$P_{+-}^{g\rightarrow g f_2 f_3} = C_A^2 \sum_{\sigma_1, \sigma_2 \in S_3} \frac{s_{123}}{s_{\sigma_1 \sigma_2}} \left\{ -\frac{2(\sigma_1 \sigma_2)^2}{s_{\sigma_1 \sigma_2}} \frac{z_{\sigma_1} z_{\sigma_2}}{(1 - z_{\sigma_1}) z_{\sigma_2}} \frac{D_{\sigma_2} D_{\sigma_3}}{z_{\sigma_1} z_{\sigma_2}} \left[ 3 - 2 \frac{(1 - z_{\sigma_2}) z_{\sigma_2}}{(1 - z_{\sigma_3}) z_{\sigma_3}} \right] \right\} ,$$

(5.62)

with

$$D_i = [i j] \sqrt{z_j} + [i k] \sqrt{z_k} \quad \text{with } i, j, k = 1, 2, 3 \text{ and } j, k \neq i ,$$

(5.63)

and

$$P_{+-}^{g\rightarrow g f_2 f_3} = \frac{1}{2} \left( C_F P_{+-}^{g\rightarrow g f_2 f_3 (ab)} + C_A P_{+-}^{g\rightarrow g f_2 f_3 (nab)} \right)$$

(5.64)

where the abelian and non-abelian terms are,

$$P_{+-}^{g\rightarrow g f_2 f_3 (ab)} = \frac{2 s_{123}}{s_{12} s_{13}} \left\{ z_1 D_1^2 - 2 \sqrt{z_2} z_3 D_2 D_3 \right\}$$

$$P_{+-}^{g\rightarrow g f_2 f_3 (nab)} = \sum_{\sigma_1, \sigma_2 \in S_2} \frac{s_{123}}{s_{1 \sigma_2} s_{1 \sigma_3}}$$

$$\left\{ -\frac{1}{2} z_1 D_1^2 + \sqrt{z_{\sigma_2} z_{\sigma_3}} D_{\sigma_2} D_{\sigma_3} + \frac{2(\sigma_2 \sigma_3)^2}{s_{\sigma_2 \sigma_3}} \frac{z_{\sigma_2} z_{\sigma_3}}{(1 - z_1) z_1} \frac{s_{1 \sigma_2} s_{1 \sigma_3}}{s_{1 \sigma_3} (1 - z_1) z_1} \right\} .$$

(5.65)

We have checked that Eqs. (5.62)-(5.65) agree with the corresponding spin-correlated splitting functions of Ref. [11] after contracting them with a parent-gluon polarization as in Appendix E, and after setting the RS parameter $\epsilon = 0$.

### 6 Four-Parton Forward Clusters

The procedure of Sects. 4 and 5 can be clearly extended to $n$-parton forward clusters. In a forward cluster there are one incoming and $n$ outgoing partons. Thus, for purely gluonic clusters there are $2^{n+1}$ helicity configurations. However, in the high-energy limit two of these are subleading, thus an $n$-gluon forward cluster contains $2(2^n - 1)$ helicity configurations. For $n$-parton forward clusters including $\bar{q} q$ pairs, all the helicity configurations are leading; then an easy counting yields $2^n$ helicity configurations for the one including a $\bar{q} q$ pair, $2^{n-1}$ for the one including two $\bar{q} q$ pairs, and so on. For $n = 3$, we obtain the helicity configurations dealt with in Section 5.
Figure 8: Amplitude for the production of five gluons, with gluons \( k_1, k_2, k_3 \) and \( k_4 \) in the forward-rapidity region of gluon \( p_a \).

### 6.1 The NNNLO impact factor \( gg^* \rightarrow gggg \)

Here we analyse in detail the four-gluon forward cluster. We take the production of five gluons with momenta \( k_1, k_2, k_3, k_4 \) and \( p_{b'} \) in the scattering between two partons of momenta \( p_a \) and \( p_b \), and we take partons \( k_1, k_2, k_3 \) and \( k_4 \) in the forward-rapidity region of parton \( p_a \) (Fig. 8a),

\[
y_1 \simeq y_2 \simeq y_3 \simeq y_4 \gg y_{b'}; \quad |k_{1\perp}| \simeq |k_{2\perp}| \simeq |k_{3\perp}| \simeq |k_{4\perp}| \simeq |p_{b'\perp}|. \tag{6.1}
\]

Using Eqs. (2.1), (2.2) and (2.10) and the subamplitudes of non-PT type, with four gluons of \( + \) helicity and three gluons of \( - \) helicity \([45]\), we obtain

\[
A^{gg \rightarrow 5g}(p_a^{\nu_a}; k_1^{\nu_1}, k_2^{\nu_2}, k_3^{\nu_3}, k_4^{\nu_4} | p_{b'}^{\nu_{b'}}, p_{b''}^{\nu_{b''}}) =
\]

\[
= 4 g^5 \frac{s}{|q_\perp|^2} C^{ggg}(p_b^{\nu_b}; p_{b''}^{\nu_{b''}}) \sum_{\sigma \in S_4} [A^{4g}(p_a^{\nu_a}; k_{\sigma_1}^{\nu_{\sigma_1}}, k_{\sigma_2}^{\nu_{\sigma_2}}, k_{\sigma_3}^{\nu_{\sigma_3}}, k_{\sigma_4}^{\nu_{\sigma_4}})] \tag{6.2}
\]
\begin{equation}
\text{tr} \left( \lambda^a \lambda_{d_1} \lambda_{d_2} \lambda_{d_3} \lambda_{d_4} \lambda^b \lambda^c \right) - \lambda^b \lambda^c \lambda_{d_1} \lambda_{d_2} \lambda_{d_3} \lambda_{d_4} + 2 \lambda^b \lambda^c \lambda_{d_1} \lambda_{d_2} \lambda_{d_3} \lambda_{d_4}
\end{equation}

with the sum over the permutations of the four gluons 1, 2, 3 and 4. From the PT subamplitudes (2.10) we obtain the functions of \((-++++)\) helicities

\begin{equation}
A^{g:4g}(p^{\nu_1}_{a}, k_1^+, k_2^+, k_3^+, k_4^+) = C^{g:4g}(p^{\nu_1}_{a}, k_1^+, k_2^+, k_3^+, k_4^+) A^\bar{\nu}(k_1, k_2, k_3, k_4),
\end{equation}

where \(\bar{\nu} = \text{sign}(\nu_a + \nu_1 + \nu_2 + \nu_3 + \nu_4)\) and

\begin{equation}
A^+(k_1, k_2, k_3, k_4) = -2\sqrt{2} \frac{q^+}{k_{1+}} \sqrt{\frac{x_1}{x_4}} \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle},
\end{equation}

and

\begin{equation}
x_i = \frac{k_i^+}{k_1^++k_2^++k_3^++k_4^+} \quad i = 1, 2, 3, 4 \quad (x_1 + x_2 + x_3 + x_4 = 1).
\end{equation}

As in Eq. (6.6), the functions \(C^{g:4g}\) are

\begin{equation}
C^{g:4g}(p^{\nu_a}_{a}, k_1^+, k_2^+, k_3^+, k_4^+) = \begin{cases} 1 & \nu_a = - \\
\frac{1}{x_i^2} & \nu_i = - \quad i = 1, 2, 3, 4 \quad \text{with } \bar{\nu} = +,\end{cases}
\end{equation}

From the non-PT subamplitudes we have obtained the functions of \((----++)\) helicities. We do not reproduce them here because they are quite lengthy. They are available from the authors upon request.

Using the U(1) decoupling equations for one and two photons, the functions \(B\) and \(D\) in Eq. (6.2) can be written as

\begin{equation}
B^{g:4g}(p^{\nu_a}_{a}, k_1^+, k_2^+, k_3^+, k_4^+) =
\end{equation}

\begin{equation}
D^{g:4g}(p^{\nu_a}_{a}, k_1^+, k_2^+, k_3^+, k_4^+) =
\end{equation}

\begin{equation}
\frac{2}{\sqrt{2}} \sqrt{\frac{1}{x_4}} \frac{1}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle}.
\end{equation}
In the quadruple collinear limit, $k_1\|k_2\|k_3\|k_4$, Sect. 6.4, the function $A$ has a triple collinear divergence; the function $B$, whose gluon 4 is not color adjacent to gluons 1, 2 and 3, has only a double collinear divergence; the function $D$, where gluon 1 is adjacent to 2 and gluon 3 is adjacent to 4 but the pairs are not adjacent one to another, has two single collinear divergences.

Using Eqs. (6.6) and (6.7), we can rewrite Eq. (6.2) as

$$
\mathcal{A}^{g g g g g g}(p'_a, k_1; k_2, k_3; k_4) = 2s \left\{ (ig)^4 \sum_{\sigma \in S_4} f_{a d s_1 c} f_{c d s_2 c'} f_{c' d s_3 c''} f_{c'' d s_4 c''' m} A^{g g g g g}(p'_a, k_{\sigma_1}, k_{\sigma_2}, k_{\sigma_3}, k_{\sigma_4}) \right\}
$$

$$
\times \frac{1}{t_1} \left[ ig f^{b b' c' c''} C^{g g}(p'_a, p'_b) \right],
$$

where the NNLO impact factor $g^* g \to g g g g g$ is enclosed in curly brackets, and includes 30 helicity configurations, in agreement with the counting above.

### 6.2 NNNLO impact factors in the high-energy limit

The amplitude (6.8) has been computed in the kinematic limit (6.1), in which it factorizes into a four-gluon cluster and a LO impact factor connected by a gluon exchanged in the cross channel. In the limits $y_1 \approx y_2 \approx y_3 \gg y_4$ or $y_1 \approx y_2 \approx y_3 \approx y_4$, or $y_1 \approx y_2 \approx y_3 \approx y_4$, Eq. (6.8) must factorize further into a NNLO impact factor or into a NLO impact factor times a NLO Lipatov vertex, or into a NNLO Lipatov vertex (Fig. 8), respectively. While the first two limits constitute necessary consistency checks, the last one allows us to derive the so far unknown NNLO Lipatov vertex for the production of three gluons along the ladder.

In the limit, $y_1 \approx y_2 \approx y_3 \gg y_4$, the NNLO impact factor, $g^* g \to g g g g g$, in Eq. (6.8) factorizes into a NNLO impact factor, $g^* g \to g g g g$, in Eq. (5.10), convoluted with a multi-Regge ladder (Fig. 8a)

$$
\lim_{y_1 \approx y_2 \approx y_3 \gg y_4} \left\{ (ig)^3 \sum_{\sigma \in S_3} f_{a d s_1 c} f_{c d s_2 c'} f_{c' d s_3 c''} A^{g g g g g}(p'_a, k_{\sigma_1}, k_{\sigma_2}, k_{\sigma_3}) \right\}
$$

$$
\times \frac{1}{t_1} \left[ ig f^{b b' c' c''} C^g(q_1, k_{\sigma_3}, q_2) \right],
$$

with $q_1 = -(p_a + k_1 + k_2 + k_3)$, $q_2 = p'_b + p_b$, and with LO Lipatov vertex $C^g(q_1, k_3, q_2)$, Eq. (1.10).

In the limit $y_1 \approx y_2 \gg y_3 \approx y_4$, the NNLO impact factor in Eq. (6.8) factorizes into a NLO impact factor, $g^* g \to g g$, Eq. (1.8), times a NLO Lipatov vertex for production of
two gluons $g^* g^* \rightarrow g g$ (5.37), convoluted with a multi-Regge ladder (Fig. 8b):

$$\lim_{y_1 \gg y_2 \approx y_3 \approx y_4} \left\{ (ig)^4 \sum_{\sigma \in S_4} f^{a_1 a_2 c} f^{c d_2 c'} f^{c' d_3 c''} f^{c'' d_4 c''' \prime} A^{A_4 g}(p^{\nu}_{\alpha}; k_{\sigma_1}^{\nu}, k_{\sigma_2}^{\nu}, k_{\sigma_3}^{\nu}, k_{\sigma_4}^{\nu}) \right\}$$

$$= \left\{ (ig)^2 \sum_{\sigma \in S_2} f^{a_1 a_2 c} C^{g g}(p^{\nu}_{\alpha}; k_{\sigma_1}^{\nu}, k_{\sigma_2}^{\nu}) \right\}$$

$$\times \frac{1}{t_1} \left\{ (ig)^2 \sum_{\sigma \in S_2} f^{c' d_3 c''} f^{c'' d_4 c''' \prime} A^{gg}(q_1, k_{\sigma_3}^{\nu}, k_{\sigma_4}^{\nu}, q_2) \right\}, \quad (6.10)$$

with $q_1 = -(p_{a} + k_1 + k_2)$.

### 6.3 The NNLO Lipatov vertex

In the limit $y_1 \gg y_2 \approx y_3 \approx y_4$, the NNLO impact factor in Eq. (6.8) factorizes into a NNLO Lipatov vertex convoluted with a multi-Regge ladder (Fig. 8c):

$$\lim_{y_1 \gg y_2 \approx y_3 \approx y_4} \left\{ (ig)^4 \sum_{\sigma \in S_4} f^{a_1 c} f^{c d_2 c'} f^{c' d_3 c''} f^{c'' d_4 c''' \prime} A^{A_4 g}(p^{\nu}_{\alpha}; k_{\sigma_1}^{\nu}, k_{\sigma_2}^{\nu}, k_{\sigma_3}^{\nu}, k_{\sigma_4}^{\nu}) \right\}$$

$$= \left[ ig f^{a_1 c} C^{g g}(p^{\nu}_{\alpha}; k_{\sigma_1}^{\nu}, k_{\sigma_2}^{\nu}) \right]$$

$$\times \frac{1}{t_1} \left\{ (ig)^3 \sum_{\sigma \in S_3} f^{c d_2 c'} f^{c' d_3 c''} f^{c'' d_4 c''' \prime} A^{gg}(q_1, k_{\sigma_2}^{\nu}, k_{\sigma_3}^{\nu}, k_{\sigma_4}^{\nu}, q_2) \right\}, \quad (6.11)$$

with the NNLO Lipatov vertex, $g^* g^* \rightarrow g g g$, for the production of three gluons $k_2$, $k_3$ and $k_4$ enclosed in curly brackets in the right hand side, with

$$A^{gg}(q_1, k_2^+, k_3^+, k_4^+, q_2) = -2\sqrt{2} \sqrt{\frac{x_2}{x_4}} \frac{1}{(23) (34)} \frac{q_1^* q_2^*}{k_{2\perp}^*} \quad (6.12)$$

$$A^{gg}(q_1, k_2^-, k_3^+, k_4^+, q_2) = 2\sqrt{2} \left\{ -\frac{|q_{1\perp}|^2 k_{2\perp}^* (k_{3\perp}^*)^2 (q_{2\perp}^* + k_{4\perp}^*) x_2}{s_{4bb'} k_{2\perp}^* s_{23} (|k_{3\perp}|^2 x_2 + |k_{2\perp}|^2 x_3)} + \right.$$  

$$+ \frac{k_{2\perp}^*}{s_{4bb'} (34) k_{2\perp}^* s_{23} \sqrt{x_3}} \left[ (q_{1\perp}^* k_{3\perp}^* (34) + \langle 24 \rangle (q_{1\perp}^* - k_{3\perp}^* q_{2\perp}^* + k_{4\perp}^* \rangle [23] \sqrt{x_3} \right. \right.$$  

$$- k_{3\perp}^* (|q_{1\perp}^* - k_{2\perp}^*| s_{23} + q_{1\perp}^* s_{34}) \sqrt{x_4} \right. \right.$$  

$$+ \frac{|q_{1\perp}|^2 k_{2\perp}^* (k_{3\perp}^*)^2 x_2 \sqrt{x_3}}{(34) k_{2\perp}^* s_{23} (|k_{3\perp}|^2 x_2 + |k_{2\perp}|^2 x_3) \sqrt{x_4}} + \right.$$  

$$+ \frac{1}{(34) k_{2\perp}^* s_{23} \sqrt{x_3 x_4}} \left( -|k_{2\perp}|^2 k_{3\perp}^* + \sqrt{x_2} (-q_{1\perp}^* (q_{1\perp}^* - k_{2\perp}^* k_{3\perp}^* \sqrt{x_2} \right.$$  

$$- q_{1\perp}^* (-k_{2\perp}^* k_{3\perp}^* \sqrt{x_2} + (23) k_{3\perp}^* \sqrt{x_3} + k_{2\perp} [23] \sqrt{x_3} + \langle 24 \rangle k_{3\perp}^* \sqrt{x_4}) \right)$$
\[ A^{3q}(q_1, k^+_2, k^+_3, k^{+1}_4, q_2) = 2\sqrt{2} \times \]

\[
\begin{align*}
&\frac{|q_{1\perp}|^2 k_{3\perp} \sqrt{x_2} \sqrt{x_3}}{(23)k_{2\perp}s_{34} (1 - x_2)} + \frac{|q_{2\perp}|^2 (q_{1\perp} - k_{2\perp}) k^{*}_{3\perp} x_3}{s_{2aa'}k_{2\perp}s_{34} (1 - x_2)} - \frac{|q_{1\perp}|^2 (k_{3\perp})^2 k^{*}_{3\perp} (q_{2\perp} + k^{*}_{3\perp}) x_2}{s_{4bb'}k_{2\perp}s_{34} (1 - x_2)} \\
&\quad + \frac{(k_{3\perp})^2 \sqrt{x_2} (- (q_{1\perp} + k^{*}_{4\perp}) [23] \sqrt{x_3} + q_{1\perp} (q_{2\perp} \sqrt{x_2} + [24] \sqrt{x_4}))}{s_{4bb'}k_{2\perp}s_{34} x_3} \\
&\quad + \frac{k_{3\perp} k^{*}_{3\perp} (q_{2\perp} k_{2\perp} \sqrt{x_2} - q_{1\perp} (q_{2\perp} + k^{*}_{3\perp}) \sqrt{x_2} + k_{2\perp}[24] \sqrt{x_4})}{s_{2aa'}k_{2\perp}s_{34} \sqrt{x_3} \sqrt{x_2}} \\
&\quad + \frac{k^{*}_{3\perp} \sqrt{x_3} (-q_{1\perp} q_{2\perp} k^{*}_{4\perp} \sqrt{x_2} x_4 - (q_{1\perp} - k_{2\perp}) k_{3\perp} [34] \sqrt{x_2} x_4 + q_{2\perp} k_{2\perp}[24] \sqrt{x_3} x_4)}{s_{2aa'}k_{2\perp}s_{34} \sqrt{x_2} x_4} \\
&\quad + \frac{|q_{1\perp}|^2 (k_{3\perp})^2 k^{*}_{3\perp} x_2 \sqrt{x_3}}{(34)k_{2\perp}s_{34} (|k_{3\perp}|^2 x_2 + |k_{2\perp}|^2 x_3) \sqrt{x_4}} + \frac{|q_{2\perp}|^2 [24] x_3 (- (23) \sqrt{x_2} + [34] \sqrt{x_4})}{(23)s_{34} s_{234} \sqrt{x_3} \sqrt{x_2}} \\
&\quad + \frac{q_{2\perp} [24]^2 \sqrt{x_3} ((34) (q_{2\perp} + k^{*}_{4\perp}) \sqrt{x_2} - (23) k^{*}_{3\perp} \sqrt{x_4})}{s_{34} s_{234} \sqrt{x_2} x_4} \\
&\quad - \frac{|q_{1\perp}|^2 (k_{3\perp})^3 k^{*}_{3\perp} k^{*}_{4\perp} x_2}{\sqrt{x_3} x_4 (34)k_{2\perp}s_{34} (|q_{1\perp} - q_{2\perp}|^2 + s_{34} (|k_{3\perp}|^2 x_2 + |k_{2\perp}|^2 x_3))}
\end{align*}\]
\[
\frac{|q_{1\perp}|^2 (k_{3\perp}^* k_{2\perp})^2 \sqrt{x_2} - (23) k_{2\perp}^* k_{1\perp}}{x_3 \sqrt{x_4} (34) k_{2\perp} s_{234} (|q_{1\perp} - q_{2\perp}|^2 + s_{234})} \\
+ \frac{1}{k_{2\perp} s_{234}} \sqrt{x_2} x_3 (x_3 \left( -x_2 (k_{3\perp} q_{2\perp} k_{3\perp}^* [24] + k_{3\perp} [23][34]) \\
+ q_{2\perp} k_{3\perp} k_{2\perp}^* [34] \sqrt{x_2} x_3 - q_{2\perp} |k_{2\perp}|^2 [24] x_3) \\
+ q_{1\perp} k_{3\perp} x_2 (q_{2\perp} [24] x_3 + k_{3\perp} [34] \sqrt{x_2} x_3 + k_{3\perp} [24] x_4)) \right]
\]

(6.14)

\[
A^{3g}(q_1, k_2^+, k_3^+, k_4^-, q_2) = 2 \sqrt{2} \left[ \frac{k_{2\perp}^*}{s_{234}} (q_{1\perp} q_{2\perp} (23) k_{2\perp}^* \sqrt{x_2} \\
- (24) [34] \sqrt{x_2} (q_{1\perp} - k_{2\perp}) (q_{2\perp} + k_{4\perp}) + k_{2\perp} \sqrt{x_3} (q_{2\perp} s_{234} + (q_{2\perp} + k_{4\perp}) s_{34})) \\
+ \frac{q_{1\perp} (q_{2\perp} + k_{4\perp})^2 (q_{1\perp} k_{3\perp} \sqrt{x_2} - k_{2\perp} [23] \sqrt{x_3})}{s_{234} (23) k_{2\perp} \sqrt{x_3}} + \frac{|q_{2\perp}|^2 k_{3\perp} \sqrt{x_2} x_3}{(23) k_{2\perp} s_{34} (1 - x_2)} \\
+ \frac{|q_{1\perp}|^2 (k_{4\perp}^2 (34) k_{3\perp}^* \sqrt{x_2} + (24) k_{2\perp}^* \sqrt{x_3})}{x_4 \sqrt{x_3} (23) (34) k_{2\perp} s_{34} (|q_{1\perp} - q_{2\perp}|^2 + s_{234})} + \frac{|q_{2\perp}|^2 (q_{1\perp} - k_{2\perp}) k_{2\perp}^* x_3}{s_{234} s_{34} (1 - x_2)} \\
+ \frac{s_{34} k_{4\perp} \sqrt{x_2} x_3 + (q_{2\perp} + k_{4\perp}) ((34) k_{3\perp}^* + k_{4\perp} [34]) \sqrt{x_2} + (24) k_{2\perp}^* \sqrt{x_3}) \sqrt{x_4}}{23) k_{2\perp} s_{34} x_4 \\
+ \frac{|q_{2\perp}|^2 [23] (24) \sqrt{x_2} + (34) \sqrt{x_3}) \sqrt{x_4}}{23) s_{34} s_{234}} + \frac{|q_{1\perp}|^2 \sqrt{x_2} x_3 x_4}{(23) s_{34} (1 - x_2)} \\
+ \frac{[23]}{(23) k_{2\perp} s_{34} s_{234} x_4} \left( |q_{1\perp}|^2 (k_{4\perp})^2 x_2 + q_{2\perp} \left( \frac{(24) |k_{2\perp}|^2}{\sqrt{x_2}} + (34) k_{2\perp} k_{2\perp}^* \sqrt{x_3} \right) \sqrt{x_4} \\
- q_{1\perp}^* k_{2\perp} k_{4\perp} (k_{4\perp} + q_{2\perp} x_4) \right],
\]

(6.15)

where in Eqs. (6.13)-(6.15) we have used the three-particle invariants, \( s_{2aa'} = (k_2 - q_1)^2 \) and \( s_{ab} = (k_4 + q_2)^2 \).

Eq. (6.11) must not diverge more rapidly than \( 1/|q_{1\perp}| \) for \( |q_{1\perp}| \to 0 \), with \( i = 1, 2 \), in order for the related cross section not to diverge more than logarithmically. Since Eq. (6.11) is proportional to \( 1/|q_{1\perp}|^2 \), the NNLO Lipatov vertex must be at least linear in \( |q_{1\perp}| \),

\[
\lim_{|q_{1\perp}| \to 0} A^{3g}(q_1, k_1^\mu, k_2^\nu, k_3^\rho, q_2) = O(|q_{1\perp}|),
\]

(6.16)

which is fulfilled by Eqs. (6.12)-(6.15).

As a consistency check on Eq. (6.11), in the further limits \( y_2 \gg y_3 \simeq y_4 \) or \( y_2 \simeq y_3 \gg y_4 \), the NNLO Lipatov vertex in Eq. (6.11) must factorize into a NLO Lipatov vertex.
convoluted with a multi-Regge ladder,

$$\lim_{y_2 \gg y_3 \gg y_4} \left\{ (ig)^3 \sum_{\sigma \in S_3} f^{cd\sigma_{2}c} f^{d\sigma_{3}c} f^{c'' d_{d_{4}} d''} A^{3\sigma}(q_1, k_{\sigma_2}^{\nu}, k_{\sigma_3}^{\nu}, k_{\sigma_4}^{\nu}, q_2) \right\}$$

(6.17)

$$= \left[ ig f^{cd\sigma_{2}c} C^g(q_1, k_{\sigma_2}^{\nu}, q_{12}) \right] \frac{1}{t_{12}} \left\{ (ig)^2 \sum_{\sigma \in S_2} f^{c d\sigma_{3} c} f^{c'' d_{d_4} d''} A^{gg}(q_{12}, k_{\sigma_3}^{\nu}, k_{\sigma_4}^{\nu}, q_2) \right\},$$

with $q_{12} = q_1 - k_2$, and

$$\lim_{y_2 \gg y_3 \gg y_4} \left\{ (ig)^3 \sum_{\sigma \in S_3} f^{cd\sigma_{2}c} f^{d\sigma_{3}c} f^{c'' d_{d_{4}} d''} A^{3\sigma}(q_1, k_{\sigma_2}^{\nu}, k_{\sigma_3}^{\nu}, k_{\sigma_4}^{\nu}, q_2) \right\}$$

(6.18)

$$= \left\{ (ig)^2 \sum_{\sigma \in S_2} f^{cd\sigma_{2}c} f^{d\sigma_{3}c} f^{c'' d_{d_4} d''} A^{gg}(q_1, k_{\sigma_2}^{\nu}, k_{\sigma_3}^{\nu}, q_{12}) \right\} \frac{1}{t_{12}} \left[ ig f^{c d_{d_4} d''} C^g(q_{12}, k_{4}^{\nu}, q_2) \right],$$

with $q_{12} = q_2 + k_4$.

In the triple collinear limit, $k_2 = z_2 P$, $k_3 = z_3 P$ and $k_4 = z_4 P$, with $z_2 + z_3 + z_4 = 1$, the coefficients of the NNLO Lipatov vertex (6.12)-(6.13) reduce to the splitting functions (5.52), and amplitude (6.11) factorizes into a multi-Regge amplitude (4.11) times a double-collinear factor (5.49)

$$\lim_{k_2 || k_3 || k_4} A^{gg \rightarrow 5g}(p_1^{\nu}, k_1^{\nu}, k_2^{\nu}, k_3^{\nu}, k_4^{\nu}, \nu^{\nu}, p_1^{\nu}, p_3^{\nu}) =$$

$$\sum_{\nu} A^{gg \rightarrow 3g}(p_1^{\nu}, k_1^{\nu}, P^{\nu}, p_1^{\nu}, p_3^{\nu}) \cdot \text{Split}_{\nu}^{g \rightarrow 3g}(k_1^{\nu}, k_2^{\nu}, k_3^{\nu}).$$

6.4 NNNLO impact factors in the quadruple collinear limit

In the quadruple collinear limit, $k_i = z_i P$, with $z_1 + z_2 + z_3 + z_4 = 1$ a generic amplitude is expected to factorize as

$$\lim_{k_1 || k_2 || k_3 || k_4} A^{abcd_{1}d_{2}d_{3}d_{4}...}(\ldots, k_1^{\nu}, k_2^{\nu}, k_3^{\nu}, k_4^{\nu}, \ldots)$$

$$= \sum_{\nu} A^{...c...}(\ldots, P^{\nu}, \ldots) \cdot \text{Split}_{\nu}^{f \rightarrow f_{1}f_{2}f_{3}f_{4}}(k_1^{\nu}, k_2^{\nu}, k_3^{\nu}, k_4^{\nu}).$$

(6.19)

Accordingly, we show that we can write Eq. (6.8) as

$$\lim_{k_1 || k_2 || k_3 || k_4} A^{gg \rightarrow 5g}(p_1^{\nu}, k_1^{\nu}, k_2^{\nu}, k_3^{\nu}, k_4^{\nu}, \nu^{\nu}, p_1^{\nu}, p_3^{\nu})$$

$$= A^{gg \rightarrow 4g}(p_1^{\nu}, P^{\nu}, p_1^{\nu}, p_3^{\nu}) \cdot \text{Split}_{\nu}^{g \rightarrow 4g}(k_1^{\nu}, k_2^{\nu}, k_3^{\nu}, k_4^{\nu}),$$

(6.20)

by taking the quadruple collinear limit of the NNLO impact factor.

In the quadruple collinear limit, the functions $A^{g4g}$ of Eq. (6.3) yield a cubic divergence as $s_{1234} = (k_1 + k_2 + k_3 + k_4)^2 \rightarrow 0$ or $s_{ijk} \rightarrow 0$, or $s_{ij} \rightarrow 0$ with $i, j, k = 1, 2, 3, 4$. 

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Analogously to Sect. 5.6, a function $A^{g:4g}$ differs from its reflection by a term which contains only a quadratic divergence in the vanishing invariants. Using this property and Eqs. (6.6) and (6.7), we obtain a reflection identity and dual Ward identities, up to quadratically divergent terms,

$$A^{g:4g}(p_1^{\nu_1}, k_1^{\nu_1}, k_2^{\nu_1}, k_3^{\nu_1}, k_4^{\nu_1}) = -A^{g:4g}(p_1^{\nu_1}, k_4^{\nu_1}, k_3^{\nu_1}, k_2^{\nu_1}, k_1^{\nu_1}),$$

$$A^{g:4g}(p_1^{\nu_1}, k_1^{\nu_1}, k_2^{\nu_1}, k_3^{\nu_1}, k_4^{\nu_1}) + A^{g:4g}(p_1^{\nu_1}, k_1^{\nu_1}, k_2^{\nu_1}, k_3^{\nu_1}, k_4^{\nu_1}) + A^{g:4g}(p_1^{\nu_1}, k_4^{\nu_1}, k_3^{\nu_1}, k_2^{\nu_1}, k_1^{\nu_1}) = 0,$$

$$A^{g:4g}(p_1^{\nu_1}, k_1^{\nu_1}, k_2^{\nu_1}, k_3^{\nu_1}, k_4^{\nu_1}) + A^{g:4g}(p_1^{\nu_1}, k_1^{\nu_1}, k_2^{\nu_1}, k_3^{\nu_1}, k_4^{\nu_1}) + A^{g:4g}(p_1^{\nu_1}, k_1^{\nu_1}, k_2^{\nu_1}, k_3^{\nu_1}, k_4^{\nu_1}) = 0.$$
\[
\text{split}^g_{k_1^+, k_2^+, k_3^+, k_4^+} = -B_1(4, 3, 2, 1)
\]
\[
\text{split}^g_{k_1^+, k_2^+, k_3^+, k_4^+} = \mathcal{B}_1(4, 3, 1, 2) + \mathcal{B}_1(4, 1, 3, 2) + \mathcal{B}_1(1, 4, 3, 2)
\]
\[
\text{split}^g_{k_1^+, k_2^+, k_3^+, k_4^+} = -B_1(1, 2, 4, 3) - B_1(1, 4, 2, 3) - B_1(4, 1, 2, 3)
\]
\[
\text{split}^g_{k_1^+, k_2^+, k_3^+, k_4^+} = \mathcal{B}_1(1, 2, 3, 4)
\]
\[
\text{split}^g_{k_1^+, k_2^+, k_3^+, k_4^+} = -B_2(4, 3, 2, 1)
\]
\[
\text{split}^g_{k_1^+, k_2^+, k_3^+, k_4^+} = \mathcal{B}_3(1, 2, 3, 4)
\]
\[
\text{split}^g_{k_1^+, k_2^+, k_3^+, k_4^+} = -B_3(1, 2, 4, 3) + \mathcal{B}_2(3, 2, 4, 1) + \mathcal{B}_2(3, 2, 1, 4)
\]
\[
\text{split}^g_{k_1^+, k_2^+, k_3^+, k_4^+} = -B_2(1, 4, 2, 3) + \mathcal{B}_3(3, 4, 2, 1) - \mathcal{B}_2(4, 1, 2, 3)
\]
\[
\text{split}^g_{k_1^+, k_2^+, k_3^+, k_4^+} = -B_3(4, 3, 2, 1)
\]
\[
\text{split}^g_{k_1^+, k_2^+, k_3^+, k_4^+} = \mathcal{B}_2(1, 2, 3, 4)
\]

with

\[
\mathcal{B}_1(1, 2, 3, 4) = \frac{2 \sqrt{2}}{(12) (23) s_{34}} \left[ -\frac{\delta(1, 2, 3)}{\sqrt{z_1 (z_3 + z_4)}} - \frac{\sqrt{z_2 z_3 (z_4) (12)}}{(1 - z_1) (z_3 + z_4)} + \frac{z_3}{s_{1234} s_{234}} (1 - z_1) \epsilon(1, 2, 3, 4) \right] \tag{6.27}
\]

\[
\mathcal{B}_2(1, 2, 3, 4) = \frac{2 \sqrt{2}}{(12) (23) s_{34}} \left[ \frac{z_2 \sqrt{z_3}}{(1 - z_1) \sqrt{z_4}} + \frac{\sqrt{z_2 (z_1 + z_2) \delta(1, 2, 3)}}{\sqrt{z_1 z_4} s_{12}} - \frac{\delta(1, 2, 3) \epsilon(1, 2, 3, 4)}{s_{1234} \sqrt{z_4 z_1} s_{12} s_{13}} \right] \tag{6.28}
\]

\[
\mathcal{B}_3(1, 2, 3, 4) = \frac{2 \sqrt{2}}{(12) (23) s_{34}} \left[ \frac{z_2 \sqrt{z_3}}{s_{12} s_{23} s_{34}} + \frac{\sqrt{z_1 (z_1 + z_2) \delta(1, 2, 3)}}{1 - z_2} \left( \frac{\sqrt{z_1 (12) s_{34}}}{1 - z_4} - \frac{\sqrt{z_3 (12) s_{23}}}{z_3 + z_4} \right) \right]
\]

\[
\left( \frac{z_2}{z_1 z_4} \right) \delta(2, 4, 3) - \frac{z_2}{(1 + z_2) \epsilon(1, 2, 4, 3) s_{23}} - (23) \langle 34 \rangle \left( \frac{z_2}{z_2 (13)} \right) \delta(2, 4, 3) - \frac{z_2}{z_2 (13) \langle 34 \rangle} \right] \tag{6.29}
\]

\[
\left( \frac{z_2}{z_1 z_4} \right) \delta(2, 4, 3) - \frac{z_2}{(1 + z_2) \epsilon(1, 2, 4, 3) s_{23}} - (23) \langle 34 \rangle \left( \frac{z_2}{z_2 (13)} \right) \delta(2, 4, 3) - \frac{z_2}{z_2 (13) \langle 34 \rangle} \right] \tag{6.29}
\]

\[
\left( \frac{z_2}{z_1 z_4} \right) \delta(2, 4, 3) - \frac{z_2}{(1 + z_2) \epsilon(1, 2, 4, 3) s_{23}} - (23) \langle 34 \rangle \left( \frac{z_2}{z_2 (13)} \right) \delta(2, 4, 3) - \frac{z_2}{z_2 (13) \langle 34 \rangle} \right] \tag{6.29}
\]

\[
\left( \frac{z_2}{z_1 z_4} \right) \delta(2, 4, 3) - \frac{z_2}{(1 + z_2) \epsilon(1, 2, 4, 3) s_{23}} - (23) \langle 34 \rangle \left( \frac{z_2}{z_2 (13)} \right) \delta(2, 4, 3) - \frac{z_2}{z_2 (13) \langle 34 \rangle} \right] \tag{6.29}
\]

\[
\left( \frac{z_2}{z_1 z_4} \right) \delta(2, 4, 3) - \frac{z_2}{(1 + z_2) \epsilon(1, 2, 4, 3) s_{23}} - (23) \langle 34 \rangle \left( \frac{z_2}{z_2 (13)} \right) \delta(2, 4, 3) - \frac{z_2}{z_2 (13) \langle 34 \rangle} \right] \tag{6.29}
\]

\[
\left( \frac{z_2}{z_1 z_4} \right) \delta(2, 4, 3) - \frac{z_2}{(1 + z_2) \epsilon(1, 2, 4, 3) s_{23}} - (23) \langle 34 \rangle \left( \frac{z_2}{z_2 (13)} \right) \delta(2, 4, 3) - \frac{z_2}{z_2 (13) \langle 34 \rangle} \right] \tag{6.29}
\]

\[
\left( \frac{z_2}{z_1 z_4} \right) \delta(2, 4, 3) - \frac{z_2}{(1 + z_2) \epsilon(1, 2, 4, 3) s_{23}} - (23) \langle 34 \rangle \left( \frac{z_2}{z_2 (13)} \right) \delta(2, 4, 3) - \frac{z_2}{z_2 (13) \langle 34 \rangle} \right] \tag{6.29}
\]
with \( \delta(1,2,3) \) as in Eq. (5.51), and

\[
e^{(1,2,3,4)} = \sqrt{z_1} \langle 1 4 \rangle + \sqrt{z_2} \langle 2 4 \rangle + \sqrt{z_3} \langle 3 4 \rangle. \tag{6.30}
\]

As in Section 5.6 summing over the helicities of gluons 1, 2, 3 and 4, one can obtain the two-dimensional polarization matrix,

\[
\sum_{\nu_1 \nu_2 \nu_3 \nu_4} \text{Split}_{\nu}^{g \rightarrow 4g}(k_{1}^{\nu_1}, k_{2}^{\nu_2}, k_{3}^{\nu_3}, k_{4}^{\nu_4}) \{ \text{Split}_{-\nu}^{g \rightarrow 4g}(k_{1}^{\nu_1}, k_{2}^{\nu_2}, k_{3}^{\nu_3}, k_{4}^{\nu_4}) \}^* = \delta^{\nu \nu'} \frac{8g^6}{s_{1234}^3} P_{\lambda \rho}^{g \rightarrow 4g}, \tag{6.31}
\]

where \( P_{++}^{g \rightarrow 4g} = P_{-+}^{g \rightarrow 4g} \), and \( P_{-+}^{g \rightarrow 4g} = (P_{++}^{g \rightarrow 4g})^* \). Averaging then over the trace of matrix (6.31), i.e. over color and helicity of the parent gluon, one can obtain the unpolarized Altarelli-Parisi gluon triple-splitting function

\[
\frac{1}{2(N_c^2 - 1)} \sum_{\nu_1 \nu_2 \nu_3 \nu_4} |\text{Split}_{\nu}^{g \rightarrow 4g}(k_{1}^{\nu_1}, k_{2}^{\nu_2}, k_{3}^{\nu_3}, k_{4}^{\nu_4})|^2 = \frac{8g^6}{s_{1234}^3} \langle P_{g \rightarrow 4g} \rangle, \tag{6.32}
\]

with \( \langle P_{g \rightarrow 4g} \rangle = P_{++}^{g \rightarrow 4g} \). As in Section 5.6 the sum over colors can be done using Eq. (2.5), and we obtain,

\[
|\text{Split}_{\nu}^{g \rightarrow 4g}(k_{1}^{\nu_1}, k_{2}^{\nu_2}, k_{3}^{\nu_3}, k_{4}^{\nu_4})|^2 = 4C_{5}(N_c) \sum_{\sigma \in S_4} |\text{split}_{-\nu}^{g \rightarrow 4g}(k_{\sigma_1}^{\nu_1}, k_{\sigma_2}^{\nu_2}, k_{\sigma_3}^{\nu_3}, k_{\sigma_4}^{\nu_4})|^2, \tag{6.33}
\]

with \( C_{5}(N_c) \) as in Eq. (2.8). It is then clear that for the splitting functions \( P_{g \rightarrow n g} \), with \( n > 4 \), the color will not factorize since LCA, Eq. (2.5), is not exact any more. We do not compute here \( P_{++}^{g \rightarrow 4g} \) and \( P_{+}^{g \rightarrow 4g} \), all the information about them being already contained in Eqs. (6.20)-(6.30).

### 7 Conclusions

In this paper, the structure of QCD amplitudes in the high-energy limit and in the collinear limit has been explored beyond NLO. We have computed forward clusters of three partons and four gluons, which in the BFKL theory constitute the tree parts of NNLO and NNNLO impact factors for jet production. In the BFKL theory the NNLO impact factors could be used to compute jet rates at NNLL accuracy. In Sect. 5.1, 5.2, 5.3 and 5.4, we have computed the tree parts of the NNLO impact factors for all the parton flavors. On these we have performed in Sect. 5.5 a set of consistency checks in the high-energy limit, and we have obtained in the triple collinear limit (Sect. 5.6) the polarized, the spin-correlated and the unpolarized double-splitting functions. The last two agree with previous calculations by Catani-Grazzini and Campbell-Glover, respectively. They can be used to set up general algorithms to compute jet rates at NNLO.
From the four-gluon forward cluster we have obtained in Sect. 6.1 the tree part of the purely gluonic NNNLO impact factor. In the quadruple collinear limit, this yields (Sect. 6.4) the purely gluonic unpolarized triple-splitting functions. They could be used to compute the three-loop Altarelli-Parisi evolution, or to compute jet rates at NNNLO. In addition, by separating a central cluster of three gluons out of the four-gluon forward cluster, we have computed the emission of three gluons along the ladder, Eqs. (6.11)-(6.15), which contributes to the NNLO Lipatov vertex. This constitutes one of the universal building blocks in an eventual construction of a BFKL resummation at NNLL accuracy.

Finally, inspired by the color structure in the high-energy limit, we have found a compact color decomposition of the tree multigluon amplitudes in terms of the linearly independent subamplitudes only, Eq. (2.9). It would be interesting to analyse whether this structure generalizes to multigluon amplitudes at one loop, and beyond.

The decomposition in rapidity of amplitudes in terms of gauge-invariant parton clusters performed in this work suggests naturally a modular decomposition of a generic multiparton amplitude, where each module is an $n$-parton cluster. Such an approximation could be tested against existing approximations of multiparton amplitudes [51, 52]. In the high-energy limit, the cluster decomposition seems superior, in that it does not use only PT-type subamplitudes, like the Kunszt-Stirling approximation [51], and within a cluster it is not limited to collinear kinematics, like the Maxwell approximation [52].

A Multiparton kinematics

We consider the production of $n$ partons of momentum $p_i$, with $i = 1, ..., n$ and $n \geq 2$, in the scattering between two partons of momenta $p_a$ and $p_b$.

Using light-cone coordinates $p^\pm = p_0 \pm p_z$, and complex transverse coordinates $p_\perp = p^x + ip^y$, with scalar product $2p \cdot q = p^+ q^- + p^- q^+ - p_\perp q_\perp - p_\perp^* q_\perp$, the 4-momenta are,

\begin{align}
    p_a &= \left( p_a^+ / 2, 0, 0, p_a^+ / 2 \right) \equiv \left( p_a^+, 0; 0, 0 \right), \\
    p_b &= \left( p_b^- / 2, 0, 0, -p_b^- / 2 \right) \equiv \left( 0, p_b^-; 0, 0 \right), \\
    p_i &= \left( (p_i^+ + p_i^-) / 2, \text{Re}[p_i], \text{Im}[p_i], (p_i^+ - p_i^-) / 2 \right) \\
    &\equiv \left( |p_i| e^{\mu_i}, |p_i| e^{-\mu_i}, |p_i| \cos \phi_i, |p_i| \sin \phi_i \right),
\end{align}

where $\mu$ is the rapidity. The first notation in Eq. (A.1) is the standard representation $p^\mu = (p^0, p^x, p^y, p^z)$, while in the second we have the + and - components on the left.

By convention we consider the scattering in the unphysical region where all momenta are taken as outgoing, and then we analytically continue to the physical region where $p_a^0 < 0$ and $p_b^0 < 0$. Thus partons are ingoing or outgoing depending on the sign of their energy. Since the helicity of a positive-energy (negative-energy) massless spinor has the same (opposite) sign as its chirality, the helicities assigned to the partons depend on whether they are incoming or outgoing. Our convention is to label outgoing (positive-energy) particles with their helicity; so if they are incoming the actual helicity and charge is reversed.
of the semicolon, and on the right the transverse components. In the following, if not
differently stated, $p_i$ and $p_j$ are always understood for $1 \leq i, j \leq n$. From the momentum
conservation,

$$
0 = \sum_{i=1}^{n} p_{i\perp},
$$

$$
p_{a}^+ = -\sum_{i=1}^{n} p_{i}^+,
$$

$$
p_{b}^- = -\sum_{i=1}^{n} p_{i}^-,
$$

the Mandelstam invariants may be written as,

$$
s_{ij} = 2p_i \cdot p_j = p_{i}^+ p_{j}^- + p_{i}^- p_{j}^+ - p_{i\perp} p_{j\perp}^* - p_{i\perp}^* p_{j\perp},
$$

so that

$$
s = 2p_a \cdot p_b = \sum_{i,j=1}^{n} p_{i}^+ p_{j}^-
$$

$$
s_{ai} = 2p_a \cdot p_i = -\sum_{j=1}^{n} p_{i}^- p_{j}^+,
$$

$$
s_{bi} = 2p_b \cdot p_i = -\sum_{j=1}^{n} p_{i}^+ p_{j}^-.
$$

Massless Dirac spinors $\psi_{\pm}(p)$ of fixed helicity are defined by the projection,

$$
\psi_{\pm}(p) = \frac{1 \pm \gamma_5}{2} \psi(p),
$$

with the shorthand notation,

$$
\psi_{\pm}(p) = |p\pm>, \quad \overline{\psi_{\pm}(p)} = \langle p \pm |,
$$

$$
\langle pk \rangle = \langle p - |k+\rangle = \overline{\psi_{-}(p)}\psi_{+}(k),
$$

$$
[pk] = \langle p + |k-\rangle = \psi_{+}(p)\overline{\psi_{-}(k)}.
$$

Using the chiral representation of the $\gamma$-matrices,

$$
\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix},
$$

and the normalization condition:

$$
\langle p \pm |\gamma_\mu|p\pm\rangle = 2p_\mu,
$$

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and the complex notation $p_\perp = |p_\perp|e^{i\phi}$, the spinors for the momenta (A.1) are

\[
\psi_+(p_i) = \begin{pmatrix} \sqrt{p_i^+} \\ \sqrt{p_i^+} e^{i\phi_i} \\ 0 \\ 0 \end{pmatrix}, \quad \psi_-(p_i) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{p_i^-} e^{-i\phi_i} \\ -\sqrt{p_i^-} \end{pmatrix},
\]

\[
\psi_+(p_a) = i \begin{pmatrix} \sqrt{-p_a^+} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \psi_-(p_a) = i \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\sqrt{-p_a^-} \end{pmatrix}.
\]

\[
\psi_+(p_b) = -i \begin{pmatrix} 0 \\ \sqrt{-p_b^+} \\ 0 \\ 0 \end{pmatrix}, \quad \psi_-(p_b) = -i \begin{pmatrix} 0 \\ 0 \\ \sqrt{-p_b^-} \\ 0 \end{pmatrix}.
\]

Using the above spinor representation, the spinor products for the momenta (A.1) are

\[
\langle p_i p_j \rangle = p_i \perp \sqrt{\frac{p_j^+}{p_i^+}} - p_j \perp \sqrt{\frac{p_i^+}{p_j^+}},
\]

\[
\langle p_a p_i \rangle = -i \sqrt{\frac{-p_a^+}{p_i^+}} p_i \perp ,
\]

\[
\langle p_i p_b \rangle = i \sqrt{\frac{-p_b^+}{p_i^+}} p_i \perp ,
\]

\[
\langle p_a p_b \rangle = -\sqrt{s},
\]

where we have used the mass-shell condition $|p_i \perp|^2 = p_i^+ p_i^-$. Note that in the present convention the spinors (A.8) and the spinor products (A.9) differ by phases with respect to the same in Ref. [47].

We consider also the spinor products $\langle p_i + |\gamma \cdot p_k|p_j+ \rangle$, which in the spinor representation (A.8) take the form,

\[
\langle p_i + |\gamma \cdot p_k|p_j+ \rangle = \frac{1}{\sqrt{p_i^+ p_j^+}} \left( p_i^+ p_j^+ p_k^- - p_i^+ p_j p_k^\ast - p_i^\ast p_j^+ p_k^- + p_i^\ast p_j p_k^\ast \right), \quad \forall k
\]

\[
\langle p_i + |\gamma \cdot p_j|p_a+ \rangle = i \sqrt{\frac{-p_a^+}{p_i^+}} \left( p_i^+ p_j^- - p_i^\ast p_j p_i^\ast \right), \quad \forall j
\]

\[
\langle p_i + |\gamma \cdot p_j|p_b+ \rangle = -i \sqrt{\frac{-p_b^+}{p_i^+}} \left( -p_i^+ p_j^\ast - p_i^\ast p_j p_i^\ast \right), \quad \forall j.
\]

\[\text{\footnote{The spinors of the incoming partons must be continued to negative energy after the complex conjugation. For instance, } \psi_+(p_a) = i \left( \sqrt{-p_a^+}, 0, 0 \right)\]
The spinor products fulfill the identities \((i \equiv p_i, j \equiv p_j)\),
\[
\langle ij \rangle = -\langle ji \rangle, \\
[ij] = -[ji], \\
\langle ij \rangle^* = \text{sign}(p_i^0 p_j^0) [ji] \\
(\langle i + |\gamma^\mu|j+\rangle)^* = \text{sign}(p_i^0 p_j^0) \langle j + |\gamma^\mu|i+ \rangle \\
\langle ij \rangle [ji] = 2 p_i \cdot p_j = s_{ij} \\
\langle i + |\gamma^\mu|j+\rangle \langle ij \rangle = \text{sign}(p_i^0 p_j^0) \langle j + |\gamma^\mu|i+ \rangle \\
[ij][kl] = \langle ik \rangle \langle jk \rangle \\
\langle ij \rangle \langle kl \rangle = \langle ik \rangle \langle jk \rangle + \langle il \rangle \langle jk \rangle
\]
and if \(\sum_{i=1}^n p_i = 0\) then
\[
\sum_{i=1}^n [ji] \langle ik \rangle = 0 .
\]

Throughout the paper the following representation for the gluon polarization is used,
\[
\epsilon_{\mu}^\pm(p, k) = \pm \frac{\langle p \pm |\gamma_\mu|k \pm \rangle}{\sqrt{2} \langle k \mp |p \pm \rangle} ,
\]
which enjoys the properties
\[
\epsilon_{\mu}^\pm(p, k) = \epsilon_{-\mu}^\mp(p, k) , \\
\epsilon_{-\mu}^\pm(p, k) \cdot k = 0 , \\
\sum_{\nu=\pm} \epsilon_{\mu}^{-\nu}(p, k) \epsilon_{\nu}^{-\nu}(p, k) = -g_{\mu\nu} + \frac{p_\mu k_\nu + p_\nu k_\mu}{p \cdot k} ,
\]
where \(k\) is an arbitrary light-like momentum. The sum in Eq. \((A.14)\) is equivalent to use
an axial, or physical, gauge.

**B Multi-Regge kinematics**

In the multi-Regge kinematics, we require that the gluons are strongly ordered in rapidity
and have comparable transverse momentum,
\[
y_1 \gg \cdots \gg y_n; \quad |p_{1\perp}| \simeq \cdots \simeq |p_{n\perp}|.
\]
Momentum conservation \((A.2)\) then becomes
\[
0 = \sum_{i=1}^n p_{i\perp} , \\
p_{a\perp}^+ \simeq -p_{1\perp}^+ , \\
p_{b\perp}^- \simeq -p_{n\perp}^- .
\]
The Mandelstam invariants (A.3) are reduced to,

\[ s = 2p_a \cdot p_b \simeq p_1^+ p_n^- \]
\[ s_{ai} = 2p_a \cdot p_i \simeq -p_1^+ p_i^- \]
\[ s_{bi} = 2p_b \cdot p_i \simeq -p_i^+ p_n^- \]
\[ s_{ij} = 2p_i \cdot p_j \simeq |p_{i\perp}| |p_{j\perp}| e^{y_i-y_j} \]

(C.1)

to leading accuracy. The spinor products (A.9) become,

\[ \langle p_i p_j \rangle \simeq -\sqrt{\frac{p_i^+}{p_j^+}} p_{j\perp} \quad \text{for } y_i > y_j , \]
\[ \langle p_a p_i \rangle \simeq -i \sqrt{\frac{p_a^+}{p_i^+}} p_{i\perp} , \]
\[ \langle p_a p_b \rangle \simeq i \sqrt{p_i^+ p_n^-} , \]
\[ \langle p_a p_b \rangle \simeq -\sqrt{p_i^+ p_n^-} . \]

(C.2)

C NLO Multi-Regge kinematics

We consider the production of \( n \) partons of momenta \( p_1, \ldots, p_n \), with partons 1 and 2 in the forward-rapidity region of parton \( p_a \),

\[ y_1 \simeq y_2 \gg y_3 \gg \cdots \gg y_n ; \quad |p_{1\perp}| \simeq |p_{2\perp}| \simeq \cdots \simeq |p_{n\perp}| . \]

(C.1)

Momentum conservation (A.2) becomes

\[ 0 = \sum_{i=1}^n p_{i\perp} , \]
\[ p_a^+ \simeq -(p_1^+ + p_2^+) , \]
\[ p_b^- \simeq -p_n^- . \]

(C.2)

The spinor products (A.9) become

\[ \langle p_a p_b \rangle = -\sqrt{s} \simeq -\sqrt{(p_1^+ + p_2^+) p_n^-} , \]
\[ \langle p_a p_i \rangle = -i \sqrt{-\frac{p_a^+}{p_i^+}} p_{i\perp} \simeq i \frac{p_{n\perp}}{|p_{n\perp}|} \langle p_a p_b \rangle , \]
\[ \langle p_a p_k \rangle = -i \sqrt{-\frac{p_a^+}{p_k^+}} p_{k\perp} \simeq -i \sqrt{\frac{p_1^+ + p_2^+}{p_k^+}} p_{k\perp} , \quad k = 1, \ldots, n-1 \]
\[ \langle p_k p_b \rangle = i \sqrt{-p_b^- p_k^+} \simeq i \sqrt{p_k^- p_n^-} , \quad k = 1, \ldots, n-1 \]
\[ \langle p_n p_b \rangle = i \sqrt{-p_b^{-} p_n^{+}} \simeq i |p_{n\perp}|, \]  
\[ \langle p_k p_n \rangle = p_k \perp \sqrt{-\frac{p_k^{+} p_n^{+}}{p_k^{+} p_n^{+}}} - p_n \perp \sqrt{-\frac{p_k^{+} p_n^{+}}{p_k^{+} p_n^{+}}}, \quad k = 1, \ldots, n - 1 \]
\[ \langle p_1 p_2 \rangle = p_1 \perp \sqrt{-\frac{p_2^{+} p_1^{+}}{p_2^{+} p_1^{+}}} - p_2 \perp \sqrt{-\frac{p_1^{+} p_2^{+}}{p_1^{+} p_2^{+}}}, \]
\[ \langle p_k p_i \rangle = p_k \perp \sqrt{-\frac{p_i^{+} p_k^{+}}{p_i^{+} p_k^{+}}} - p_i \perp \sqrt{-\frac{p_k^{+} p_i^{+}}{p_k^{+} p_i^{+}}}, \quad k = 1, 2; i = 3, \ldots, n - 1. \]

which differ by phases with respect to the same spinor products in Ref. [23] because of the convention for the spinor representation we use in Sect. [4].

D  NNLO Multi-Regge kinematics

The extension to the production of \( n \) partons of momenta \( p_1, \ldots, p_n \), with partons 1, 2 and 3 in the forward-rapidity region of parton \( p_a \),

\[ y_1 \simeq y_2 \simeq y_3 \gg y_4 \gg \ldots \gg y_n; \quad |p_{1 \perp}| \simeq |p_{2 \perp}| \simeq \ldots \simeq |p_{n \perp}|, \]  
(D.1)

is straightforward. We mention it here because by taking the further limit \( y_1 \gg y_2 \simeq y_3 \), one obtains the kinematics of the NLO Lipatov vertex (sect. 5.5).

With Eq. (D.1), momentum conservation (A.2) becomes

\[ 0 = \sum_{i=1}^{n} p_{i \perp}, \]
\[ p_{a \perp} \simeq -(p_{1 \perp}^{+} + p_{2 \perp}^{+} + p_{3 \perp}^{+}), \]
\[ p_{a -} \simeq -p_{n -}. \]  
(D.2)

The spinor products (A.9) become

\[ \langle p_a p_b \rangle = -\sqrt{s} \simeq -\sqrt{(p_1^{+} + p_2^{+} + p_3^{+}) p_n^{-}}, \]
\[ \langle p_a p_n \rangle = -i \sqrt{-\frac{p_n^{-}}{p_n^{+}}} p_{n \perp} \simeq i \frac{p_{n \perp}}{|p_{n \perp}|} \langle p_a p_b \rangle, \]
\[ \langle p_a p_k \rangle = -i \sqrt{-\frac{p_k^{+}}{p_k^{+}}} p_{k \perp} \simeq -i \sqrt{\frac{p_k^{+} + p_2^{+} + p_3^{+}}{p_k^{+}}} p_{k \perp}, \quad k = 1, \ldots, n - 1 \]
\[ \langle p_k p_b \rangle = i \sqrt{-p_b^{-} p_k^{+}} \simeq i \sqrt{p_k^{+} p_b^{-}}, \quad k = 1, \ldots, n - 1 \]
\[ \langle p_n p_b \rangle = i \sqrt{-p_b^{-} p_n^{+}} \simeq i |p_{n \perp}|, \]
\[ \langle p_k p_n \rangle = p_k \perp \sqrt{-\frac{p_n^{+} p_k^{+}}{p_n^{+} p_k^{+}}} - p_n \perp \sqrt{-\frac{p_k^{+} p_n^{+}}{p_k^{+} p_n^{+}}} \simeq -p_{n \perp} \sqrt{\frac{p_k^{+}}{p_n^{+}}}, \quad k = 1, \ldots, n - 1 \]
\begin{align*}
\langle p_k p_i \rangle &= p_k \sqrt{\frac{p_{k+}^2}{p_k^2}} - p_{i\perp} \sqrt{\frac{p_{i+}^2}{p_i^2}} \simeq -p_{i\perp} \sqrt{\frac{p_{i+}^2}{p_i^2}}, \quad k = 1, 2, 3 ; i = 4, \ldots, n - 1,
\end{align*}
while the others spinor products remain unchanged. The spinor products (D.3) generalize straightforwardly to the kinematics (6.1).

### E The Sudakov parametrization

We want to elucidate the relationship between our parametrization of the momenta and the one of Ref. [11]. Recalling the last of Eqs. (A.1), we can write,

\begin{equation}
p_i = \frac{x_i P^+}{2} (1, 0, 0, 1) + (0, \text{Re}[p_{i\perp}], \text{Im}[p_{i\perp}], 0) + \frac{|p_{i\perp}|^2}{2x_i P^+} (1, 0, 0, -1) \quad \text{(E.1)}
\end{equation}

where \( P^\mu \) is the sum of the three momenta, the \( x_i \) are the momentum fractions and we used the mass-shell condition \( p_i^+ p_i^- = |p_{i\perp}|^2 \). This is exactly what is obtained from the general Sudakov parametrization of Ref. [11],

\begin{equation}
p_i^\mu = x_i p^\mu + k^\mu_{\perp i} - \frac{k^2_{\perp i}}{2x_i} n^\mu / (n \cdot n) \quad \text{(E.2)}
\end{equation}

through the following choices for the lightlike vectors,

\begin{equation}
p^\mu = \frac{P^+}{2} (1, 0, 0, 1) \quad \text{and} \quad n^\mu = (1, 0, 0, -1), \quad \text{(E.3)}
\end{equation}

and the identification,

\begin{equation}
k^\mu_{\perp i} = (0, \text{Re}[p_{i\perp}], \text{Im}[p_{i\perp}], 0). \quad \text{(E.4)}
\end{equation}

The spin-correlated splitting functions of Ref. [11] are expressed in terms of the vectors \( \tilde{k}^\mu_i \) defined as \( \tilde{k}^\mu_i = k^\mu_{\perp i} - \frac{k^2_{\perp i}}{x_i} n^\mu \), where, as in our case, the \( z_i \) variables represent the momentum fractions in the collinear limit. In order to compare Eq. (5.59) with the spin-correlated splitting functions of Ref. [11], we must project the latter onto the helicity basis, namely to contract them with the polarization vector, Eq. (A.13),

\begin{equation}\epsilon^\pm_{\mu}(P, n) = \frac{1}{\sqrt{2}} (0, 1, \mp i, 0). \quad \text{(E.5)}\end{equation}

The contraction of the \( \tilde{k}^\mu_i \) vectors with \( \epsilon^+ \) is,

\begin{equation}\tilde{k}_i \cdot \epsilon^+ = \sqrt{\frac{2}{2}} [i j] \sqrt{z_j} + [i l] \sqrt{z_l}, \quad \text{(E.6)} \end{equation}

with \( i, j, l = 1, 2, 3 \) and \( j, l \neq i \), with the analogous expressions for \( \epsilon^- \) obtained by complex conjugation.

For the off-diagonal terms, \( P^g \rightarrow g f_2 f_3 \), we find a relative minus sign between the results of Ref. [11] and ours, which, however, has no physical relevance.
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