A MONGE NORMAL FORM FOR THE ROLLING DISTRIBUTION

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Abstract. Using a parametrisation of sl_2 given by the second prolongation of the group action of unimodular fractional linear transformations as presented in an article of Clarkson and Olver [6], we find a Monge normal form describing the rolling of two hyperboloid surfaces over each other.

1. Introduction

Let \( D \) be a maximally non-integrable rank 2 distribution on a 5-manifold \( M \). The maximally non-integrable condition of \( D \) determines a filtration of the tangent bundle \( TM \) given by
\[
D \subset [D, D] \subset [D, [D, D]] \cong TM.
\]
The distribution \([D, D]\) has rank 3 while the full tangent space \( TM \) has rank 5, hence such a geometry is also known as a \((2, 3, 5)\)-distribution. Let \( M_{\text{xyzpq}} \) denote the 5-dimensional mixed order jet space \( J^{3,0}(\mathbb{R}, \mathbb{R}^2) \cong J^2(\mathbb{R}, \mathbb{R}) \times \mathbb{R} \) with local coordinates given by \((x, y, z, p, q) = (x, y, z, y', y'')\) (see also [10], [11]). Let \( D_{F(x,y,z,y',y'')} \) denote the maximally non-integrable rank 2 distribution on \( M_{\text{xyzpq}} \) associated to the underdetermined differential equation \( z' = F(x, y, z, y', y'') \). This means that the distribution is annihilated by the following three 1-forms
\[
\omega_1 = dy - pdx, \quad \omega_2 = dp - qdx, \quad \omega_3 = dF(x, y, z, p, q)dx.
\]
Such a distribution \( D_{F(x,y,z,y',y'')} \) is said to be in Monge normal form (see page 90 of [11]). In Section 5 of [7], it is shown how to associate canonically to such a \((2, 3, 5)\)-distribution a conformal class of metrics of split signature \((2, 3)\) (henceforth known as Nurowski’s conformal structure or Nurowski’s conformal metrics) such that the rank 2 distribution is isotropic with respect to any metric in the conformal class. The method of equivalence [5] (also see the introduction to [2], Section 5 of [7] and [9]) produces the 1-forms \((\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)\) that gives a coframing for Nurowski’s

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metric. These 1-forms satisfy the structure equations
\[ \begin{align*}
    d\theta_1 &= \theta_1 \wedge (2\Omega_1 + \Omega_4) + \theta_2 \wedge \Omega_2 + \theta_3 \wedge \theta_4, \\
    d\theta_2 &= \theta_1 \wedge \Omega_3 + \theta_2 \wedge (\Omega_1 + 2\Omega_4) + \theta_3 \wedge \theta_5, \\
    d\theta_3 &= \theta_1 \wedge \Omega_5 + \theta_2 \wedge \Omega_6 + \theta_3 \wedge (\Omega_1 + \Omega_4) + \theta_4 \wedge \theta_5, \\
    d\theta_4 &= \theta_1 \wedge \Omega_7 + \frac{4}{3} \theta_3 \wedge \Omega_6 + \theta_4 \wedge \Omega_1 + \theta_5 \wedge \Omega_2, \\
    d\theta_5 &= \theta_2 \wedge \Omega_7 - \frac{4}{3} \theta_3 \wedge \Omega_5 + \theta_4 \wedge \Omega_3 + \theta_5 \wedge \Omega_4,
\end{align*} \]
(1.1)
where \((\Omega_1, \ldots, \Omega_7)\) and two additional 1-forms \((\Omega_8, \Omega_9)\) together define a rank 14 principal bundle over the 5-manifold \(M\) (see [5] and Section 5 of [7]). A representative metric in Nurowski’s conformal class [7] is given by
\[ g = 2\theta_1 \theta_5 - 2\theta_2 \theta_4 + \frac{4}{3} \theta_3 \theta_3. \]
(1.2)
When \(g\) has vanishing Weyl tensor, the distribution is called maximally symmetric and has split \(G_2\) as its group of local symmetries. For further details, see the introduction to [2] and Section 5 of [7]. For further discussion on the relationship between maximally symmetric \((2, 3, 5)\)-distributions and the automorphism group of the split octonions, see Section 2 of [10].

\((2, 3, 5)\)-distributions also arise from the study of the configuration space of two surfaces rolling without slipping or twisting over each other [1], [3] and [4]. The configuration space can be realised as the An-Nurowski circle twistor distribution [1] and in the case of two spheres with radii in the ratio 1 : 3 rolling without slipping or twisting over each other, there is again maximal \(G_2\) symmetry.

In the work of [8], a description of maximally symmetric \((2, 3, 5)\)-distributions obtained from Pfaffian systems with \(SU(2)\) symmetry was discussed and its relationship with the rolling distribution was investigated. In particular, the An-Nurowski circle twistor bundle can be realised by considering the Riemannian surface element of the unit sphere arising from one copy of \(SU(2)\) and the other Riemannian surface element with Gaussian curvature 9 or \(\frac{1}{9}\) from another copy of \(SU(2)\). Both Lie algebras of \(su(2)\) are parametrised by the left-invariant vector fields. See [8] for further details.

In the \(SL(2)\) case, the spheres are replaced by hyperboloids equipped with Lorentzian signature metrics, but the ratios of the Gaussian curvatures of the surfaces remain unchanged for there to be maximal split \(G_2\) symmetry. It is in this context where the Lorentzian surface elements of the hyperboloids arise from two different copies of \(SL(2)\) that we consider the rolling distribution. We base the computations here from that already obtained in [8].

Using a different parametrisation of \(sl_2\) given in [6] obtained from the second prolongation of the group action of unimodular fractional linear transformations, we show a change of coordinates that bring the 1-forms annihilating the rolling
A Monge normal form for the rolling distribution

The distribution to the 1-forms encoding the Monge equation

\begin{align*}
\omega_1 &= dy - pdx, \\
\omega_2 &= dp - qdx, \\
\omega_3 &= dz - \left(\frac{qz^2}{\alpha^2 - 1} + \frac{1}{\alpha^2 - 1} \left(\sqrt{\frac{qz}{\alpha^2 - 1}} - \frac{1}{2\sqrt{x}}\right)^2\right) dx.
\end{align*}

In Monge normal form, \( F(x, y, z, p, q) = \left(\frac{qz^2}{\alpha^2 - 1} + \frac{1}{\alpha^2 - 1} \left(\sqrt{\frac{qz}{\alpha^2 - 1}} - \frac{1}{2\sqrt{x}}\right)^2\right) \). This is the content of Theorem 3.1 in the paper. The distribution has maximal split \( G_2 \) symmetry whenever \( \alpha^2 = 9 \) or \( \alpha^2 = \frac{1}{9} \). The Monge normal form we obtain here is of course not unique and is defined up to symmetries of the distribution \( D_F \). Some more work would be needed to see if we can find a further change of coordinates to eliminate the \( z^2 \) term in the expression of \( F(x, y, z, p, q) \). The computations here are done with the aid of the DifferentialGeometry package in MAPLE 2018.

2. A Hyperboloid Rolling Distribution

The real \( SL(2) \) analogue of Section 2 in [8] is given as follows. We can parametrise the Lie algebra \( sl_2 \) using the left-invariant vector fields, from which we get

\begin{align*}
\sigma_1 &= \sinh(y) \cosh(z) dp - \sinh(z) dy, \\
\sigma_2 &= -\sinh(y) \sinh(z) dp + \cosh(z) dy, \\
\sigma_3 &= -dz - \cosh(y) dp.
\end{align*}

These 1-forms satisfy the relations

\begin{align*}
d\sigma_1 &= -\sigma_2 \wedge \sigma_3, \\
d\sigma_2 &= -\sigma_1 \wedge \sigma_3, \\
d\sigma_3 &= \sigma_1 \wedge \sigma_2,
\end{align*}

and the Lorentzian hyperbolic surface element with Gauss curvature \(-1\) is given by

\[ \sigma_2 \sigma_2 - \sigma_1 \sigma_1 = dy^2 - \sinh^2(y) dp^2. \]

We find that the distribution annihilated by the the Pfaffian system spanned by the 1-forms

\[ \omega_1 = -(\sigma_1 + \exp(\alpha x) dq), \quad \omega_2 = \sigma_2 + dx, \quad \omega_3 = -(\sigma_3 + \alpha \exp(\alpha x) dq) \]

is a \((2, 3, 5)\)-distribution whenever \( \alpha^2 \neq 1 \) and has maximal symmetry whenever \( \alpha^2 = \frac{1}{9} \) or \( \alpha^2 = 9 \). The 1-forms can be completed into a coframing by taking
ω₁ = −dx and ω₅ = exp(α x) dq. To satisfy Cartan’s structure equations, we take

θ₁ = ω₁,  θ₂ = ω₂,  θ₃ = K¹/₂ω₃,
θ₄ = K⁻¹/₂ω₁ + Pθ₁ + Qθ₂ + Rθ₃,
θ₅ = K⁻¹/₂ω₅ + Sθ₁ + Tθ₂ + Uθ₃,

where K = \frac{1}{α² - 1}, P = T = 0, R = U = 0 and Q = S = \frac{3α² - 7}{10(α² - 1)²}. This gives the conformal metric

\[ K^{1/2}g = 2ω₁ω₅ - 2ω₂ω₄ + \frac{3α² - 7}{5}K(ω₁ω₁ - ω₂ω₂) + \frac{4}{3}Kω₃ω₃ \]
\[ = 2ω₁² - 2ω₅² + \frac{1}{2}(σ₁ - ω₅)² - \frac{1}{2}ω₁² - \frac{1}{2}(σ₂ + ω₄)² + \frac{1}{2}ω₂² \]
\[ + \frac{3α² - 7}{5(α² - 1)}(ω₁² - ω₂²) + \frac{4}{3(α² - 1)}ω₃². \]

Let

\[ \bar{ω}_1 = -σ₁ + ω₅ \quad \text{and} \quad \bar{ω}_2 = σ₂ + ω₄. \]

Using the fact that

\[ σ₁² - σ₂² + ω₅² - ω₄² = \frac{1}{2}(ω₁² - ω₂²) + \frac{1}{2}(ω₅² - ω₄²), \]

we find

\[ K^{1/2}g = 2ω₁² - 2ω₅² + \frac{1}{2}ω₁² - \frac{1}{2}ω₅² + \frac{1}{2}ω₂² + \frac{3α² - 7}{5(α² - 1)}(ω₁² - ω₂²) + \frac{4}{3(α² - 1)}ω₃² \]
\[ = 2ω₁² - 2ω₅² + \frac{1}{2}ω₁² + \frac{1}{2}ω₅² + \frac{1}{2}ω₂² - \frac{1}{2}ω₅² + (\frac{3α² - 7}{5(α² - 1)} - 1)(ω₁² - ω₂²) + \frac{4}{3(α² - 1)}ω₃² \]
\[ = ω₁² - ω₅² - (σ₁² - ω₅²) + \frac{3α² - 7}{5(α² - 1)} - 1)(ω₁² - ω₂²) + \frac{4}{3(α² - 1)}ω₃². \]

In the terminology of [8], the 1-forms \{ω₁, ω₂, ω₃\} form a “sign-reversed” Pfaffian system corresponding to the symmetry \((α, x, q) \mapsto (-α, -x, -q)\). Observe that the metric we obtain in (2.2) is diagonal. Each hyperboloid surface \(Σ\) and \(Σ'\) is equipped with a Lorentzian surface element. The Gauss curvature of the surface element \(ω₁² - ω₅²\) over \(Σ\) is given by \(-α²\), and the Gauss curvature of \(σ₂² - σ₁²\) over \(Σ'\) is given by \(-1\). The maximally symmetric \((2, 3, 5)\)-distribution is obtained when the ratios of the Gauss curvature of the two surfaces are 9 or \(\frac{1}{5}\). In these cases, the metric (2.2) has vanishing Weyl tensor. See also [1].

We call this distribution rolling because the 1-forms \{ω₁, ω₂, ω₃\} given in (2.1) define a connection on the An-Nurowski circle twistor bundle over the product of the two surfaces \(Σ\) and \(Σ'\) with elements (or metric tensor) given by ω₄² − ω₅² and
\[\sigma^2 - \sigma_1^2\] respectively. Furthermore, the An-Nurowski circle twistor bundle realises the system of two surfaces rolling without slipping or twisting over each other \([1]\). See \([8]\) for details in the \(SU(2)\) case. We shall also refer to the Pfaffian system spanned by the 1-forms in (2.1) as a rolling system.

We now reparametrise the Lie algebra of \(\mathfrak{sl}_2\) using the vector fields arising from the second prolongation of the group action of unimodular fractional linear transformations. From this reparametrisation, we are able to reduce the distribution to a Monge normal form after a change of coordinates.

3. A Monge normal form for the rolling distribution

Let
\[\tau_1 = dy + ydz, \quad \tau_2 = -(dp - pdz), \quad \tau_3 = -dz.\]

The set of 1-forms
\[s_1 = \tau_1 + y^2\tau_2, \quad s_2 = \tau_2, \quad s_3 = \tau_3 - 2y\tau_2,\]
forms a basis dual to the \(\mathfrak{sl}_2\) Lie algebra of vector fields
\[\partial_y, \quad y^2\partial_y - 2y\partial_z - (2yp + 1)\partial_p, \quad y\partial_y - \partial_z - p\partial_p,\]
(see \([6]\)) that arise from the second prolongation of the group action of unimodular fractional linear transformations. We have
\[ds_1 = -s_1 \wedge s_3,\]
\[ds_2 = s_2 \wedge s_3,\]
\[ds_3 = -2s_1 \wedge s_2.\]

We now consider the rolling distribution annihilated by the 1-forms
\[\theta_1 = s_1 + s_2 - \exp(\alpha x) dq = dy + ydz - (y^2 + 1)(dp - pdz) - \exp(\alpha x) dq,\]
\[\theta_2 = s_1 - s_2 + dx = (1 - y^2)(dp - pdz) + (dy + ydz) + dx,\]
\[\theta_3 = -s_3 + \alpha \exp(\alpha x) dq = -2y(dp - pdz) + dz + \alpha \exp(\alpha x) dq.\]

The Pfaffian system is equivalent to the Pfaffian system (2.1) above
\[\omega_1 = - (\sigma_1 + \exp(\alpha x) dq), \quad \omega_2 = \sigma_2 + dx, \quad \omega_3 = - (\sigma_3 + \alpha \exp(\alpha x) dq)\]
through the isomorphism
\[\{\sigma_1, \sigma_2, \sigma_3\} \cong \{-(s_1 + s_2), s_1 - s_2, -s_3\}.\]

The distribution is equivalently annihilated by the 1-forms
\[\theta_1 + \frac{1}{\alpha} \theta_3 = dy + ydz - (y^2 + 1 + \frac{2}{\alpha} y)(dp - pdz) + \frac{1}{\alpha} dz,\]
\[\theta_1 - \theta_2 = -2(dp - pdz) - dx - \exp(\alpha x) dq,\]
\[\theta_3 = \frac{1}{\alpha} \theta_1 - \theta_2 = dz + ydx + (y + \alpha) \exp(\alpha x) dq.\]
From the equations (3.1) determined by the ideal \( \{ \theta_1, \theta_2, \theta_3 \} \), we are now going to find the change of coordinates that bring it to Monge normal form. We define

\[
\hat{y} = \exp(z)(y + \frac{1}{\alpha}) \quad \hat{p} = \exp(-z)p \quad \hat{q} = \frac{1}{\alpha} \exp(-\alpha x),
\]

\[
\hat{z} = \exp(z - \alpha x) \quad \hat{x} = q - \frac{1}{\alpha} \exp(-\alpha x).
\]

Under this change of coordinates, the system spanned by the 1-forms given in (3.1) is equivalently spanned by the 1-forms

\[
d\hat{y} - p \, dx = d\hat{x} + 2\hat{p} \, d\hat{z} + 2\hat{z} \, d\hat{p} - 2\hat{p} \, d\hat{z} = d\hat{x} + 2\hat{z} \, d\hat{p},
\]

\[
d\hat{p} - q \, dx = d\hat{p} - \frac{2}{4(\hat{z} \hat{y} + (1 - \frac{1}{\alpha^2}) \frac{\hat{z}^2}{q})} d\hat{z} = \frac{1}{2\hat{z}} (d\hat{x} + \frac{1}{\hat{y} + (1 - \frac{1}{\alpha^2}) \frac{\hat{z}^2}{q}} d\hat{z}),
\]

\[
dz - F \, dx = d\hat{y} - 2F d\hat{z} = d\hat{y} - 4F(\hat{z} \hat{y} + (1 - \frac{1}{\alpha^2}) \frac{\hat{z}^2}{q}) d\hat{p},
\]

where \( F = F(x, y, z, p, q) \) is to be determined. It follows that

\[
F = qz^2 + (1 - \frac{1}{\alpha^2})q \exp(2z).
\]

Using the inverse map

\[
(\hat{x}, \hat{y}, \hat{z}, \hat{p}, \hat{q}) = \left( y - xp, \frac{x}{2}, p, \left( 1 - \frac{1}{\alpha^2} \right) \frac{q x^2}{1 - 2z qx} \right),
\]

we deduce that

\[
\frac{\hat{z}}{\hat{q}} = \frac{x}{2\hat{q}} = \left( 1 - \frac{1}{\alpha^2} \right)^{-1} \left( \frac{1}{2q x} - \frac{z}{q} \right).
\]
This gives
\[ \exp(2z) = \frac{x^2}{4\alpha^2q^2} = \frac{1}{\alpha^2} \left( \frac{1}{1 - \frac{1}{\alpha^2}} \right)^2 \left( \frac{1}{2qx} - z \right)^2. \]

We therefore obtain
\[ F = qz^2 + \frac{1}{\alpha^2 - 1} \left( \sqrt{qz} - \frac{1}{2\sqrt{qx}} \right)^2 \]
\[ = qz^2 + \frac{1}{\alpha^2 - 1} q \left( z^2 - \frac{z}{q} + \frac{1}{4q^2x^2} \right). \]

When \( \alpha^2 = 9 \), we obtain
\[ F = \frac{9}{8} qz^2 - \frac{1}{8} z + \frac{1}{32q^2x^2}. \]

When \( \alpha^2 = \frac{1}{9} \), we obtain
\[ F = -\frac{1}{8} qz^2 + \frac{9}{8} z - \frac{9}{32q^2x^2}. \]

In both of these cases they give the Monge normal form of a maximally symmetric \((2, 3, 5)\)-distribution. We have the following theorem

**Theorem 3.1.** Let \( \mathcal{D} \) be the \((2, 3, 5)\)-distribution associated to the rolling system spanned by the 1-forms \( \{\omega_1, \omega_2, \omega_3\} \) given by (2.1). Then by the change of coordinates given above, this Pfaffian system can be brought into the Monge normal form given by

\[ \omega_1 = dy - p\,dx, \]
\[ \omega_2 = dp - q\,dx, \]
\[ \omega_3 = dz - \left( \frac{qz^2}{\alpha^2 - 1} \left( \sqrt{qz} - \frac{1}{2\sqrt{qx}} \right)^2 \right) \, dx. \]

The \((2, 3, 5)\)-distribution has split \( G_2 \) symmetry whenever \( \alpha = \pm\frac{1}{3}, \alpha = \pm 3. \)

### 4. \( SL(2) \) Pfaffian systems

In this Section, we give an example of a homogeneous bracket-generating \((2, 3, 5)\)-distribution that can be reduced to the above Monge normal form given by the rolling distribution. This \((2, 3, 5)\)-distribution can be seen as a generalisation of the rolling distribution, where we assume that each hyperboloid surface comes from a copy of \( SL(2) \). This is also the real analogue of the \( SU(2) \) picture discussed in Section 5 of [8]. However, because our parametrisation of \( sl_2 \) is different and does not use the left-invariant vector fields on \( SL(2) \), the geometric relationship with the rolling distribution is not immediately apparent.
On $M^5$ with local coordinates given by $(x, y, z, p, q)$, consider the 1-forms
\[
\begin{align*}
\omega_1 &= dy + ydz, \quad \omega_2 = -(dp - pdz), \quad \omega_3 = -dz, \\
\omega_4 &= dq + qdz, \quad \omega_5 = -(dx - xdz).
\end{align*}
\]
The vector fields dual to the set of 1-forms
\[
\begin{align*}
s_1 &= \omega_1 + y^2 \omega_2, \quad s_2 = \omega_2, \quad s_3 = \omega_3 - 2y \omega_2, \\
\end{align*}
\]
form a copy of the Lie algebra $sl_2$, and the vector fields dual to the set of 1-forms
\[
\begin{align*}
s_4 &= \omega_4 + q^2 \omega_5, \quad s_5 = \omega_5, \quad s_6 = \omega_3 - 2q \omega_5,
\end{align*}
\]
form a second copy of $sl_2$. We form the following set of 1-forms
\[
\begin{align*}
s_1 &= \omega_1 + y^2 \omega_2, \quad s_2 = \omega_2, \quad s_3 = \omega_3 - 2y \omega_2, \\
\bar{s}_3 &= \omega_3 - 2y \omega_2 - 2q \omega_5 = s_3 - 2q \omega_5, \quad \bar{s}_4 = \frac{1}{q}(\omega_4 + q^2 \omega_5), \quad \bar{s}_5 = q \omega_5.
\end{align*}
\]
These 1-forms satisfy the equations
\[
\begin{align*}
ds_1 &= -s_1 \wedge \bar{s}_3 - 2s_1 \wedge \bar{s}_5, \\
\end{align*}
\]
\[
\begin{align*}
ds_2 &= s_2 \wedge \bar{s}_3 + 2s_2 \wedge \bar{s}_5, \\
\end{align*}
\]
\[
\begin{align*}
d\bar{s}_3 &= -2s_1 \wedge s_2 - 2\bar{s}_4 \wedge \bar{s}_5, \\
\end{align*}
\]
\[
\begin{align*}
d\bar{s}_4 &= \bar{s}_4 \wedge \bar{s}_5, \\
\end{align*}
\]
\[
\begin{align*}
d\bar{s}_5 &= \bar{s}_4 \wedge \bar{s}_5.
\end{align*}
\]
Let
\[
\begin{align*}
\theta_1 &= s_1 - \beta \bar{s}_4, \\
\theta_2 &= s_2 - \gamma \bar{s}_5, \\
\theta_3 &= s_3 + \bar{s}_4 + \bar{s}_5.
\end{align*}
\]

Proposition 4.1. The distribution given by the kernel of the 1-forms $\{\theta_1, \theta_2, \theta_3\}$ in (4.1) is a bracket-generating $(2, 3, 5)$-distribution whenever $\beta \gamma \neq 1$.

We shall call the Pfaffian system given by the 1-forms $\{\theta_1, \theta_2, \theta_3\}$ an $SL(2)$ Pfaffian system. It can be seen that these three 1-forms can be completed into a coframing that satisfies Cartan’s structure equations. Let
\[
\begin{align*}
\theta_1 &= s_1 - \beta \bar{s}_4, \\
\theta_2 &= s_2 - \gamma \bar{s}_5, \\
\bar{\theta}_3 &= K^{-1}\bar{s}_4(s_3 + \bar{s}_4 + \bar{s}_5), \\
\theta_4 &= K^{-1}\bar{s}_4(\beta \bar{s}_4), \\
\theta_5 &= K^{-1}\bar{s}_5(-\gamma \bar{s}_5) + T\theta_2,
\end{align*}
\]
where
\[ K = \frac{\beta \gamma}{2(\beta \gamma - 1)} \text{ and } T = \frac{-9\beta \gamma \pm 3 \pm \sqrt{3(3\beta^2 \gamma^2 - 26\beta \gamma + 3)}}{6(\beta \gamma - 1)(K)^{\frac{3}{2}}} \]
are constants. Then Cartan’s structure equations (1.1) are satisfied for the 1-forms \((\theta_1, \theta_2, \bar{\theta}_3, \theta_4, \theta_5)\). We find that Nurowski’s conformal metric given by
\[ g = 2\theta_1 \theta_5 - 2\theta_2 \theta_4 + \frac{4}{3} \bar{\theta}_3 \bar{\theta}_3 \]
is conformally flat whenever \(\beta \gamma = 9\) or \(\beta \gamma = \frac{1}{9}\). We have
\[ K^{\frac{1}{3}} g = -2\gamma (1 + \lambda)s_1 \bar{s}_5 - 2\beta (1 + \lambda)s_2 \bar{s}_4 + 2\beta \gamma (2 + \lambda)s_4 s_5 + 2\lambda s_1 s_2 + \frac{4}{3} K(s_3 + \bar{s}_4 + \bar{s}_5)^2 \]
where
\[ \lambda = K^{\frac{1}{3}} T = \frac{-9\beta \gamma \pm 3 \pm \sqrt{3(3\beta^2 \gamma^2 - 26\beta \gamma + 3)}}{6(\beta \gamma - 1)}. \]
Also note that \(\bar{s}_4 \bar{s}_5 = s_4 s_5\).

**Theorem 4.2.** The \((2, 3, 5)\)-distribution given by the kernel of the 1-forms \(\{\theta_1, \theta_2, \theta_3\}\) in (4.1) has split \(G_2\) symmetry whenever \(\beta \gamma = 9\) or \(\beta \gamma = \frac{1}{9}\).

We now encode the Pfaffian system given by (4.1) by a Monge equation equivalent to (1.3). The 1-forms annihilating the vector fields can be expressed as
\[
\begin{align*}
\theta_1 &= (dy + ydz) - y^2(dp - pdz) - \frac{\beta}{q}((dq + qdz) - q^2(dx - xdz)), \\
\theta_2 &= (pdz - dp) + \gamma q(dx - xdz), \\
\theta_3 &= 2y(dp - pdz) + \frac{1}{q} dq - 2q(dx - xdz).
\end{align*}
\]
Equivalently, taking linear combinations, the vector fields are annihilated by the 1-forms
\[
\begin{align*}
\theta_1 + \beta \theta_3 + \frac{\beta}{\gamma} \theta_2 &= (d\bar{y} + \bar{y}dz) - \left(\frac{\bar{y}}{\gamma} + (\bar{y} + (\beta - \frac{1}{\gamma}))(\bar{y} - \beta)\right)(dp - pdz), \\
\theta_2 &= (pdz - dp) + \gamma q(dx - xdz), \\
\frac{1}{2q(\gamma y - 1)} (\theta_3 + 2y \theta_2) &= (dx - xdz) + \frac{dq}{2(\gamma y + (\beta - \frac{1}{\gamma})) \gamma q^2}.
\end{align*}
\]
where $\bar{y} = y - \beta$. We now take

$$\dot{y} = \frac{1}{c} \exp(z) \bar{y},$$
$$\dot{p} = c \exp(-z) p,$$
$$\dot{x} = -c \exp(-z) x,$$
$$\dot{q} = -\frac{1}{\gamma q},$$
$$\dot{z} = (\beta - \frac{1}{\gamma}) \exp(z).$$

This reduces the system (4.1) to the ideal spanned by the 1-forms

$$d\dot{x} - \dot{q} d\dot{p},$$

$$d\dot{y} - \left( \frac{\bar{y}^2}{\bar{y} + \beta - \frac{1}{\gamma}} \right) d\dot{p} = d\dot{y} - (\bar{y}^2 - \frac{\beta}{\beta - \frac{1}{\gamma}} \bar{z}^2) d\dot{p},$$

$$d\dot{x} - \frac{c \exp(-z)}{2(\bar{y} + \beta - \frac{1}{\gamma})} d\dot{q} = d\dot{x} - \frac{1}{2(\bar{y} + \bar{z})} d\dot{q} = d\dot{x} - \frac{1}{2(\bar{y} + \bar{z})} d\dot{q}. $$

We now map into more familiar coordinates on the mixed jet space by taking $(x, y, z, p, q) = (\dot{q}, \dot{x} - \dot{p} \dot{q}, -\dot{y}, -\dot{p}, -\frac{1}{2(\bar{y} + \bar{z})})$, from which we obtain

$$dy - p dx = d\dot{x} - \dot{q} d\dot{p} - \dot{p} d\dot{q} + \dot{q} d\dot{p} = d\dot{x} - \dot{q} d\dot{p},$$

$$dp - q dx = -d\dot{p} + \frac{1}{2(\bar{y} + \bar{z})} d\dot{q} = -\frac{1}{q} (d\dot{x} - \frac{1}{2(\bar{y} + \bar{z})} d\dot{q}),$$

$$dz - F dx = -d\dot{y} - 2(\bar{y} + \bar{z}) \dot{q} F d\dot{p} = -d\dot{y} + (\bar{y}^2 - \mu \bar{z}^2) d\dot{p},$$

with $\mu = \frac{\beta \gamma}{\beta \gamma - 1}$ and $F = F(x, y, z, p, q)$ again to be determined. This means

$$F = -\frac{1}{2(\bar{y} + \bar{z})} \left( \bar{y}^2 - \mu \bar{z}^2 \right)$$

$$= q(z^2 - \mu (z - \frac{1}{2qz})^2)$$

$$= \left( 1 - \frac{\beta \gamma}{\beta \gamma - 1} \right) z^2 q - \left( \frac{\beta \gamma}{\beta \gamma - 1} \right) \frac{1}{4qz^2} + \left( \frac{\beta \gamma}{\beta \gamma - 1} \right) \frac{1}{x}$$

$$= z^2 q - \frac{\beta \gamma}{\beta \gamma - 1} (z^2 q + \frac{1}{4qz^2} - \frac{z}{x})$$

$$= z^2 q + \frac{1}{\beta \gamma - 1} \left( \sqrt{qz} - \frac{1}{2\sqrt{qz}} \right)^2.$$

This is equivalent to the form given in (1.3) after identifying $\beta \gamma = \alpha^{-2}$. When $\beta \gamma = \frac{1}{6}$, this gives (3.2), and when $\beta \gamma = 9$, this gives (3.3).
Theorem 4.3. Let $\mathcal{D}$ be the $(2, 3, 5)$-distribution associated to the $SL(2)$ Pfaffian system spanned by the 1-forms $\{\theta_1, \theta_2, \theta_3\}$ in (4.1). Then by the change of coordinates given above, this Pfaffian system can be brought into the Monge normal form equivalent to the rolling distribution given by

$$\omega_1 = dy - pdx,$$
$$\omega_2 = dp - qdx,$$
$$\omega_3 = dz - \left(qz^2 + \frac{\beta \gamma}{1 - \beta \gamma}(\sqrt{qz} - \frac{1}{2\sqrt{qx}})^2\right)dx.$$

This has split $G_2$ symmetry whenever $\beta \gamma = \frac{1}{9} \text{ or } 9$.

5. Conformal metric from Monge normal form

In this section, for the sake of completeness, we give the description of the Nurowski’s conformal class associated to the above Monge normal form (1.3). The distribution $\mathcal{D}_F$ where $F = qz^2 + \frac{1}{\alpha^2 - 1}(qz^2 - \frac{z}{x} + \frac{1}{4qx^2}) = qz^2 + \frac{1}{\alpha^2 - 1}(\sqrt{qz} - \frac{1}{2\sqrt{qx}})^2$ is annihilated by the three 1-forms

$$\omega_1 = dy - pdx, \quad \omega_2 = dp - qdx, \quad \omega_3 = dz - \left(qz^2 + \frac{1}{\alpha^2 - 1}(qz^2 - \frac{z}{x} + \frac{1}{4qx^2})\right)dx.$$

These three 1-forms are completed to a coframing on $M_{xyzpq}$ by the additional 1-forms

$$\omega_4 = \frac{1}{2q^3x^2(\alpha^2 - 1)}dq - \frac{4\alpha^2q^2x^2z^2 - 4\alpha^2qxz + (3 - 2\alpha^2)}{4(\alpha^2 - 1)x^3q^2}dx, \quad \omega_5 = -dx.$$

If we take

$$\theta_1 = \omega_3 - \frac{4\alpha^2q^2x^2z^2 - 1}{4q^2x^2(\alpha^2 - 1)}\omega_2, \quad \theta_2 = \omega_1, \quad \theta_3 = K^{\frac{1}{2}}\omega_2,$$

$$\theta_4 = K^{-\frac{1}{2}}\omega_4 + a_{41}\theta_1 + a_{42}\theta_2 + a_{43}\theta_3,$$

$$\theta_5 = K^{-\frac{1}{2}}\omega_5 + a_{51}\theta_1 + a_{52}\theta_2 + a_{53}\theta_3,$$

where

$$K = \frac{1}{2q^3x^2(\alpha^2 - 1)}, \quad a_{41} = 0,$$

$$a_{42} = \frac{2^{\frac{3}{2}}\alpha^2(\alpha^2 - 9)(4qxz(qxz - 1) + 1)}{60x^{\frac{9}{2}}(\alpha^2 - 1)^{\frac{5}{2}}q^2},$$

$$a_{43} = -\frac{2^{\frac{7}{2}}(12\alpha^2q^2x^2z^2 - 8\alpha^2qxz - 2\alpha^2 + 3)}{12x^{\frac{7}{2}}(\alpha^2 - 1)^{\frac{5}{2}}q},$$

$$a_{51} = 0, \quad a_{52} = \frac{2^{\frac{3}{2}}(2\alpha^2 - 3)}{5(\alpha^2 - 1)^{\frac{5}{2}}x^\frac{3}{2}}, \quad a_{53} = -2^{\frac{3}{2}}qx^\frac{3}{2}(\alpha^2 - 1)^{\frac{3}{2}},$$
then we find Cartan’s structure equations are satisfied for \((\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)\), and the coframing gives a representative metric from Nurowski’s conformal class with
\[
g = 2\theta_1\theta_5 - 2\theta_2\theta_4 + \frac{4}{3}\theta_3\theta_3.
\]
The metric is conformally flat when \(\alpha^2 = 9\) or \(\alpha^2 = \frac{1}{9}\), in which case the distribution has maximal split \(G_2\) symmetry. In comparison to the metric obtained in (2.2), the metric here is no longer in diagonal form.

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