Internal Time Superoperator for Quantum Systems with Diagonal Singularity

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(September 10th., 1996.)

We generalize the concepts of Internal Time Superoperator, its associated non unitary similarity transformations and Liapounov variables, to quantum systems with diagonal singularity, and we give a constructive proof of the existence of these superoperators for systems with purely diagonal Hamiltonian having uniform absolutely continuous spectrum on \([0, \infty)\).

I. INTRODUCTION

Recently, I. Antoniou et al. [1] [2] [3] have shown that there is a natural formalism to deal with quantum mechanical systems with continuous spectrum, as it is the case for decaying processes and non equilibrium statistical mechanics. This formalism is in the line of an old paper of Segal [4] who assumed that a state \(\rho\) is a positive normal functional over the space of observables. The mean value of an observable \(O\) in the state \(\rho\) is given by \(\langle O \rangle_\rho = \rho(O)\). This assumption implies an extension of the set of possible states. This set is ”too small” in the usual model, formulated in a Hilbert space of density operators, to include the ”final” state in the case of continuous spectrum [1] [2] [3]. The new formalism introduces the concept of ”diagonal” and ”off diagonal” states and observables, with its corresponding projectors. For this reason, the natural generalization consists of considering a \(\ast\)-algebra of observables splitted from the very beginning in two separated direct summands. This, in turn, generates a corresponding splitting of the dual space of states, as well as two projectors on the direct summands. What kind of \(\ast\)-algebra of observables shall we take is not clear ”a priori”. It could be a von Neumann or other C*-algebra, or a Banach algebra with involution, or some other structure [1] [2] [3].

The aim of this work is twofold. First we are going to give a natural generalization of the concept of Internal Time Superoperator which measures the ”dynamical age” or ”degree of evolution” of the system [5] [6] [7], its associated non unitary similarity transformations and Liapunov variables. Then we shall prove the existence of these generalized superoperators for any quantum systems with a diagonal Hamiltonian having uniform absolutely continuous spectrum on \([0, \infty)\).

In section 2, we begin with a simple introduction to the formalism of quantum systems with diagonal singularity and introduce the necessary notation.

The generalization of the time superoperator is given in section 3, together with its spectral decomposition.

In section 4 we obtain a set of Liapounov variables.

In section 5 we show that our results prove the quantum analog of a well known result: the existence of an Internal Time for classical K-fluxes [5] [6] [7] [8].

II. STATES AND OBSERVABLES WITH DIAGONAL SINGULARITY.

Let us consider a quantum system such that, in the ordinary formalism, has a Hamiltonian with a uniform absolutely continuous spectrum \([6]\) on \([0, \infty)\). For simplicity, we only consider the non degenerate case. Then:

\[
H = \int_0^\infty dE E |E\rangle\langle E|
\]  

being \(|E\rangle\langle E|\) generalized right (left) eigenvectors of \(H\) with eigenvalue \(E\) [9] [11], and \(dE\) the Lebesgue measure on the real line.
The time evolution of a pure state is given by

\[ |\Psi_t\rangle = e^{-iHt} |\Psi_0\rangle = \int_0^\infty dE |E\rangle \Psi_t(E) \]

\[ \Psi_t(E) = e^{-iEt} \langle E|\Psi_0\rangle. \] (2)

The wave function \( \Psi_t(E) \) has an oscillating time dependence with no well defined limit for \( t \to \infty \). However, it is possible to obtain a well defined limit for the mean value of any observable represented by an operator of the form

\[ O = O^d + O^c \]

\[ O^d = \int_0^\infty dE O_E |E\rangle\langle E| \]

\[ O^c = \int_0^\infty dE \int_0^\infty dE' O_{EE'} |E\rangle\langle E'|. \] (3)

We shall assume that \( O^c \) (where the letter \( c \) relates to correlations), belongs to \( \mathcal{O}^c \), the Hilbert-Schmidt class (H-S) of operators, including its own Hilbertian topology. This implies that \( O^c \) is an integral operator with a square integrable kernel function \( \mathcal{U}(\mathcal{O}) \). Alternatively, we could choose its weak closure in the space of all bounded operators of the Hilbert space of states, \( \mathcal{U}(\mathcal{O})' \) which is a von Neumann algebra (a factor of type I, having a countable relative dimensional \( [7] \) containing \( H-S \) \( [10] \). But here our aim is to deal with the internal time superoperator, whose classical theory is formalized in a Hilbertian language, and so we prefer to choose the H-S class.

In addition, we shall explicitly include a "diagonal part" \( O^d \), of the form indicated by \( (2) \). \( O^d \) will belong to \( \mathcal{O}^d \), the maximal Abelian von Neumann algebra \( [16] \) generated by the spectral projections of a complete set of commuting observables of the system. This send us outside of the H-S class \( [3] \ \) \( [1] \ \) \( [2] \).

For operators representing observables, we shall assume the reasonable non-Hilbertian condition of "self adjointness", i.e. \( O_E = O_E^* \) and \( O_{EE'} = O_{EE'}^* \). The involutive Banach algebra with identity \( [10] \) of all possible operators \( O = O^d + O^c \) will be denoted by \( O = O^d \oplus O^c \), and its involution by \( \dagger \). (The sum is a direct one, because a non null diagonal operator cannot be compact \( [3] \), and all the H-S operators are compact \( [2] \). If we choose \( \mathcal{U}(\mathcal{O}) \), as it contains the identity, the sum is not direct)

With the pure state \( (3) \) we can construct the corresponding nuclear (and then, of Hilbert-Smith class \( [3] \ \) \( [10] \)) density operator

\[ \hat{\rho}_t = |\Psi_t\rangle\langle \Psi_t| = \int \int dE dE' |E\rangle\langle E'| e^{-i(E-E')t} \langle E|\Psi_0\rangle\langle \Psi_0|E'\rangle \] (4)

\[ Tr(\hat{\rho}_t) = Tr(\hat{\rho}_0) = \langle \Psi_0|\Psi_0\rangle = 1. \] (5)

In the usual quantum formalism, the mean value of an observable \( O \) is

\[ \langle O \rangle_t = Tr(\hat{\rho}_t O). \]

For an observable \( O \) of the form given in \( (3) \) we obtain:

\[ \langle O \rangle_t = \int dE \langle E|\Psi_0\rangle\langle \Psi_0|E\rangle O_E + \]

\[ + \int \int dE dE' e^{-i(E-E')t} \langle E|\Psi_0\rangle\langle \Psi_0|E'\rangle O_{EE'} \] (6)

If the function \( f(E, E') := \langle E|\Psi_0\rangle\langle \Psi_0|E'\rangle \) \( O_{EE'} \) is integrable on \( [0, \infty) \times [0, \infty) \), the second term in \( (3) \) goes to zero when \( t \to \infty \), as a consequence of the Riemann-Lebesgue Lemma, and we obtain

\[ \lim_{t \to \infty} \langle O \rangle_t = \int dE \langle E|\Psi_0\rangle\langle \Psi_0|E\rangle O_E. \] (7)

We may try to find a density operator \( \hat{\rho}_\infty = \int dE dE' (\hat{\rho}_\infty)_{EE'}|E\rangle\langle E'| \) such that

\[ \lim_{t \to \infty} \langle O \rangle_t = Tr(\hat{\rho}_\infty O) \int dE (\hat{\rho}_\infty)_{EE} O_E + \int \int dE dE' (\hat{\rho}_\infty)_{EE'} O_{EE'}. \]
which would imply
\[
\int dE \langle E | \Psi_0 \rangle \langle \Psi_0 | E \rangle \rho_E = \int dE \langle \hat{\rho}_\infty \rangle_{E,E} \rho_E + \int \int dE dE' \langle \hat{\rho}_\infty \rangle_{E,E'} \rho_{E,E'}.
\]

But there is no function \( \langle \hat{\rho}_\infty \rangle_{E,E'} \) satisfying the previous equation for arbitrary functions \( O_E \) and \( O_{E,E'} \), and therefore \( \rho_\infty \) do not exist.

Moreover, for the pure state corresponding to a well defined value \( E \) of the energy and represented by the generalized vector \( |E\rangle \), expressions like \( \langle E | \hat{E} | \rangle \) or \( \langle E | H | E \rangle \) are not defined.

These difficulties can be overcome by an extended definition of states as functionals acting on the operators representing observables. This approach was developed by I.Antoniou et al. If the observable is represented by a "self adjoint" operator \( O \) having diagonal part, as it is the case for expression (13), the state \( \rho \) of the system is represented by two functions \( \rho_E \) and \( \rho_{E,E'} \) such that the mean value \( \langle O \rangle_\rho \) of the observable \( O \) in the state \( \rho \) is given by
\[
\langle O \rangle_\rho = \int dE \rho_E^* O_E + \int \int dE dE' \rho_{E,E'}^* O_{E,E'}.
\]  
(8)

The mean value \( \langle O \rangle_\rho \) is real if
\[
(\rho_E)^* = \rho_E, \quad (\rho_{E,E'})^* = \rho_{E',E}.
\]  
(9)

If \( I \) denotes the identity operator, we also have: \( \langle I \rangle_\rho = \langle \int dE |E\rangle \langle E| \rangle_\rho = 1 \) if
\[
\int_0^\infty dE \rho_E = 1.
\]  
(10)

Therefore, the states \( \rho \) are represented by functionals acting on the space of operators representing observables. The states can be expressed in terms of the functionals \( |E\rangle \) and \( \langle E| \rangle \), defined by the following relations
\[
|E\rangle := O_E, \quad \langle E'| \rangle := O_{E,E'}.
\]  
(11)

From (11) we can deduce the following formal relations
\[
\langle E | E' \rangle = \delta (E - E') \]
\[
\langle E' | E'' E''' \rangle = \delta (E - E'') \delta (E' - E''') \]
\[
\langle E | E' E'' \rangle = 0, \]
\[
\langle E | E' | E'' \rangle = 0, \]
(12)

where
\[
|E\rangle := |E\rangle \langle E| \quad |E E'\rangle := |E\rangle \langle E'| \rangle.
\]  
(13)

Using (13), we can give the following expression for an operator \( |O\rangle \) representing an observable with diagonal part
\[
|O\rangle = \int dE O_E |E\rangle + \int \int dE dE' O_{E,E'} |E E'\rangle.
\]  
(14)

Using (12), the following expression can be given for a functional representing a state \( \rho \)
\[
\langle \rho | \in S = S^d \bigoplus S^c \subset (as a convex subset) \mathcal{O}' = (O^d)' \bigoplus (O^c)' \]
\[
\langle \rho | = \langle \rho^d | + \langle \rho^c | \in S
\]
\[
\langle \rho^d | = \int dE \rho_E^* |E\rangle \in S^d
\]  
(15)

\[
\langle \rho^c | = \int \int dE dE' \rho_{E,E'}^* (E E') \in S^c.
\]

Using (12), (14) and (13) we can easily prove that
\[ \langle O \rangle_\rho = (\rho|O). \]  

(16)

Conditions (9) and (10) can be written as

\[ (\rho|O) = (\rho|O)^* \text{ if } O^\dagger = O \]  

(17)

\[ (\rho|I) = (\rho|\int dE|E) = 1. \]  

(18)

Expression (18) is a generalization of the concept of trace.

It is interesting to point out that the formalism defined above already contain the usual approach of quantum mechanics. In fact, consider a pure state, represented by a normalized vector

\[ |\Psi\rangle = \int dE \Psi(E)|E\rangle \]  

(19)

\[ \langle \Psi|\Psi\rangle = \int dE \Psi(E)^* \Psi(E) = 1. \]  

(20)

For an observable \( O \) with diagonal singularity, as in (3), using the standard formalism we obtain

\[ \langle O\rangle_\Psi = \langle \Psi|O|\Psi\rangle = \int dE \Psi(E)^* \Psi(E) O_E + \int \int dE dE' \Psi(E)^* \Psi(E') O_{EE'}. \]  

(21)

In the new formalism, a pure state is represented by the functional

\[ (\rho_{\text{pure}}) := \int dE \Psi(E)^* \Psi(E) |E\rangle\langle E| + \int \int dE dE' \Psi(E)^* \Psi(E') (E E'). \]  

(22)

It is easy to verify from the definition (22) that \( (\rho_{\text{pure}})_E^* = (\rho_{\text{pure}})_E \text{ and } (\rho_{\text{pure}})_{EE'} = (\rho_{\text{pure}})_{EE'}, \) and therefore \( (\rho_{\text{pure}}) \) satisfies (17). Condition (18) is also verified by \( (\rho_{\text{pure}}) \) as a consequence of the normalization (20) of the vector \( |\Psi\rangle \). By acting with the functional \( (\rho_{\text{pure}}) \) on \( |O\rangle \) we obtain

\[ (\rho_{\text{pure}})|O\rangle = (\Psi|O|\Psi). \]

If the state is a mixture, represented in the standard formalism by a density operator \( \hat{\rho} \) satisfying

\[ \hat{\rho} = \int \int dE dE' (\hat{\rho})_{EE'} |E\rangle\langle E'| \]

\[ (\hat{\rho})_{EE'} = (\hat{\rho})_{E'E} \]

\[ Tr(\hat{\rho}) = \int dE (\hat{\rho})_{EE} = 1, \]

the mean value of an observable \( O \) is given by

\[ \langle O\rangle_{\hat{\rho}} = Tr(\hat{\rho} O) = \int dE (\hat{\rho})_{EE} O_E + \int \int dE dE' (\hat{\rho})_{EE'} O_{EE'}. \]

The mixture can also be described using the extended formalism, provided we define

\[ (\rho_{\text{mix}}) := \int dE (\hat{\rho})_{EE} |E\rangle\langle E| + \int \int dE dE' (\hat{\rho})_{EE'} (E E'). \]  

(23)

It is easy to verify

\[ (\rho_{\text{mix}})|O\rangle = Tr(\hat{\rho} O) \quad (\rho_{\text{mix}})|I\rangle = 1 \quad (\rho_{\text{mix}})|O\rangle = (\rho_{\text{mix}})|O\rangle^*. \]

As we see from (22) and (23), ordinary states (pure or mixtures) satisfy \( \rho_E = \rho_{EE} \), and therefore the diagonal and the off-diagonal parts of \( (\rho_{\text{pure}}) \) or \( (\rho_{\text{mix}}) \) are not independent. However, the generalized formalism allows more
general states represented by functionals (ρ| satisfying [17] and [18] for which ρE ≠ ρE, i.e. states which cannot be represented by normalized vectors or by density operators.

Consider for example a pure state corresponding to a well defined value E of the energy. If we represent this state by the generalized eigenvector |E⟩ of the Hamiltonian, ⟨E|E⟩ and ⟨E|H|E⟩ are not defined. The standard trick is to make the spectrum of the Hamiltonian discrete by putting the system in a box, and to make the volume of the box very big after all the relevant calculations.

This trick is not necessary in the generalized formalism. Consider for example the state |E⟩, for which the generalized trace is well defined

\[(E|I) = (E| \int dE'|E') = \int dE' \delta(E - E') = 1.\]

This state satisfy

\[\langle H^n \rangle = (E|H^n) = (E| \int dE'|E')^n = \int dE' \delta(E - E') (E')^n = E^n\]

from which we easily deduce that ⟨E⟩ has a well defined value of the energy

\[\langle H \rangle = (E|H) = E \quad ((H - \langle H \rangle)^n) = (E|(H - \langle H \rangle)^n) = 0.\]

Therefore ⟨E⟩ represent a ”generalized pure state” with energy E.

Concerning the time evolution of the states, it is determined by

\[\mathbb{U}_t \text{ acting on } \rho \text{ by } \langle \rho_t|O\rangle = (\mathbb{U}_t \rho_0|O) := (\rho_0|\mathbb{U}_t^+ O) := (\rho_0|e^{iHt}Oe^{-iHt}),\]

which also gives the relation between Schrödinger and Heisemberg pictures. The last equation is a special case of the general rule (Mρ|O) := (ρ|M^+ O), which defines M acting on ρ in terms of a certain M^+ (that will be given in each case) acting on O. Notice that this amount to give a generalized non-Hilbertian ”adjoint” relation +, a kind of ”duality”.

The generalized Liouville-Von Neumann equation is

\[-i\frac{d}{dt}(\rho_t) = (\mathbb{L}_t \rho_t) = (\rho_t|\mathbb{L}_t^+ O,\]

with the general solution

\[\langle \rho_t| = \int dE (\rho_t)_E^* (E) + \int dE dE' (\rho_t)_{E E'}^* (E E') = \int dE (\rho_0)_E^* (E) + \int dE dE' (\rho_0)_{E E'}^* e^{i(E-E')t} (E E').\]

Therefore

\[\langle O | \rho_t \rangle = \int dE (\rho_0)_E^* O_E + \int dE dE' (\rho_0)_{E E'}^* e^{i(E-E')t} O_{E E'} .\]

For integrable functions f(E, E') = (\rho_0)_{E E'} O_E O_{E'}, the second term of the previous expression vanishes when t → ∞, and therefore we have the weak limit

\[w- \lim_{t \to \infty} (\rho_t) = \int dE (\rho_0)_E^* (E)\]

The existence of weak limits when t → ∞ makes this formalism specially suitable to describe time evolution of decaying quantum systems and the approach to equilibrium in statistical mechanics.

Up to now we expressed the spectral resolution of the operators representing observables in terms of |E⟩ := |E⟩⟨E| and |EE'⟩ := |E⟩⟨E'|, being |E⟩ ⟨⟨E|⟩ generalized right (left) eigenvectors of the total Hamiltonian H of the system. Equations (2.11) define the corresponding functionals ⟨E| and ⟨EE'| to expand the states. These generalized states and observables are left and right generalized eigenvectors of the Liouville-Von Neumann superoperator \(\mathbb{L}_t^+\).
\[ (E|L^+ = 0, \quad (E E'|L^+ = (E - E')(E E') \]
\[ L^+|E) = 0, \quad L^+|EE') = (E - E')(EE') \]

The superoperator \( L^+ \) can be written as
\[ L^+ = \int \int dE dE' (E - E')(EE')(E E') \tag{29} \]
We can make the change of variables
\[ \nu = E - E', \quad \lambda = \frac{E + E'}{2}, \tag{30} \]
and define
\[ |\lambda\nu) : = |EE') = |\lambda + \frac{\nu}{2}\rangle\langle\lambda - \frac{\nu}{2}| \]
\[ |\lambda) : = |E) = |\lambda\rangle\langle\lambda| \]
\[ \langle\lambda\nu| : = (EE') \]
\[ \langle\lambda| : = (E). \tag{31} \]
Using (12) and (31) we obtain
\[ (\lambda|\lambda') = \delta(\lambda - \lambda') \]
\[ (\lambda|\nu') = 0 \]
\[ (\lambda\nu|\lambda') = \delta(\lambda - \lambda')\delta(\nu - \nu') \]
\[ (\lambda\nu|\lambda) = 0. \tag{32} \]
For the superoperators \( L^+ \) and \( I^+ \) (defined by \( I^+ O = O \) for all \( O \in \mathcal{O} \)), we obtain
\[ L^+ = \int_0^\infty d\lambda \int_{-2\lambda}^{2\lambda} d\nu \nu |\lambda\nu)(\lambda\nu| \tag{33} \]
\[ I^+ = I^+_d + I^+_c \]
\[ I^+_d = \int_0^\infty dE |E) (E| = \int_0^\infty d\lambda |\lambda)(\lambda| \]
\[ I^+_c = \int \int dE dE' |EE') (E E') = \int_0^\infty d\lambda \int_{-2\lambda}^{2\lambda} d\nu |\lambda\nu)(\lambda\nu| \tag{34} \]

III. THE TIME SUPEROPERATOR

In classical context, the time superoperator \( T \) is defined through \([T, L] = i I\), where \( I \) stand for the identity superoperator acting on fluctuations of the equilibrium state \( \rho_\infty \) \[3 \] \[4 \] \[5 \]. This amount of projecting on the off diagonal part of the state, because in that case the "diagonal part" is trivial: \( \mathbb{C}\rho_\infty \), being \( \mathbb{C} \) the complex number field. Therefore, in the quantum case, where we have a more general diagonal part, we propose the following definition:
\[ [T, L] = i I_c, \tag{35} \]
where \( T, L \) and \( I_c \) are superoperators acting on the states. \( L \) is the generator of time evolution for the states, and \( I_c \) is the projection onto the off diagonal part of the states.
Our hypothesis on the spectrum of \( H \), determines the corresponding spectral properties of \( L \) acting on \((\mathcal{O}^c)' \equiv \mathcal{O}^c \supset \mathcal{S}^c \). In fact, \( L \) has uniform Lebesgue spectrum on all the real line. It is a well known fact \[12 \] \[13 \] \[14 \] that this implies the existence of a spectral measure \( E \) from the Lebesgue \( \sigma \)-algebra of \( \mathbb{R} \) to the projection superoperators on the Hilbert-Schmidt class \( \mathcal{O}^c \), such that \((U_t |(\mathcal{O}^c)', E)\) is a system of imprimitivity based on \( \mathbb{R} \), that is to say, for every Lebesgue-measurable set \( \Delta \), we have:
\[ U_{-t} E(\Delta) U_t \big|_{(O')^c} = E(\Delta + t) \] (36)

Consequently there exists \( T \big|_{(O')^c} = \int_{-\infty}^{+\infty} s \, dE \). Then we can define \( T \) over \( O' \) by requiring: \( T \big|_{(O')^c} = 0 \). If we also define \( E(\Delta) \big|_{(O')^c} = 0 \), then \( \int_{-\infty}^{+\infty} s \, dE \big|_{(O')^c} = 0 \), and \( \int_{-\infty}^{+\infty} dE \big|_{(O')^c} = 0 \), and we can write

\[ T = \int_{-\infty}^{+\infty} s \, dE \] (37)

\[ I_c = \int_{-\infty}^{+\infty} dE \]

\[ U_{-t} E(\Delta) U_t = E(\Delta + t) \] (38)

If we define \( E_s := E(-\infty, s] \), the previous expressions give:

\[ T = \int_{-\infty}^{+\infty} s \, dE_s \] (39)

\[ I_c = \int_{-\infty}^{+\infty} dE_s \]

\[ U_{-t} E_s U_t = E_{s-t} \] (40)

In what follows we are going to obtain explicit expressions for \( T \) and \( E \) starting from the generalized spectral decomposition of the Hamiltonian given in equation (1).

Using the relations

\[
(L\rho \mid O) = (\rho \mid L^+ O) := (\rho \mid H, O) \\
(L_c \rho \mid O) = (\rho \mid L_{c}^+ O) := (\rho \mid O^c) \\
(T\rho \mid O) = (\rho \mid T^+ O)
\]

we obtain

\[
[T^+, L^+] = i I_c^+,
\]

(41)

where \( L^+ \) is given by (33) or (29), and

\[ I_c^+ = \int_{0}^{\infty} d\lambda \int_{-2\lambda}^{2\lambda} d\nu \, |\lambda\nu\rangle \langle \lambda\nu| \]

From (41) we obtain

\[
(\nu - \nu')(\lambda'\nu' | T^+ | \lambda\nu) = i \delta(\lambda' - \lambda) \delta(\nu' - \nu) \\
\nu'(\lambda' | T^+ | \lambda\nu) = 0 \\
\nu'(\lambda'\nu' | T^+ | \lambda) = 0
\]

(42)

A possible solution for equations (42) is

\[
(\lambda'\nu' | T^+ | \lambda\nu) = i \delta(\lambda' - \lambda) \delta'(\nu' - \nu) \\
(\lambda' | T^+ | \lambda\nu) = 0 \\
(\lambda'\nu' | T^+ | \lambda) = 0 \\
(\lambda' | T^+ | \lambda) = 0
\]

(43)

and therefore

\[
(\rho | T^+ | O) = i \int_{0}^{\infty} d\lambda \int_{-2\lambda}^{2\lambda} d\nu \, (\rho | \lambda\nu\rangle \langle \lambda\nu| \partial_{\nu'}(\lambda\nu|O)).
\]

To obtain the right eigenvectors \( |\varphi\rangle \) of the time superoperator we make a Fourier expansion on the \( \nu \) variable.
\[ |\varphi| = \int_0^\infty d\lambda |\lambda| (\lambda |\varphi|) + \int_0^\infty d\lambda \int_{-2\lambda}^{2\lambda} d\nu (\lambda \nu |\varphi|) \]
\[ = \int_0^\infty d\lambda |\varphi| (\lambda) + \int_0^\infty d\lambda \int_{-2\lambda}^{2\lambda} d\nu (\lambda \nu) \sum_{n=-\infty}^{+\infty} \varphi_n(\lambda) \exp(-in\pi \nu/2\lambda). \]

Replacing this last expression in \( T^+ |\varphi| = s |\varphi| \) we obtain the following generalized eigenvalues and eigenvectors

\[ s = 0 \quad |\varphi_\lambda| := |\lambda| \]
\[ s = 0 \quad |\varphi_{0\lambda}| := \frac{1}{2\sqrt{|s|}} \int_{-2\lambda}^{2\lambda} d\nu (\lambda \nu) \exp(-i\nu). \]
\[ s \neq 0 \quad |\varphi_{sn}| := \frac{1}{2\sqrt{|s|}} \int_{-2\lambda}^{2\lambda} d\nu (\frac{\nu}{2s}, \nu) \exp(-is\nu). \] (44)

The generalized left eigenvectors, satisfying \( (\varphi|T^+ = s (\varphi|, \) are

\[ s = 0 \quad (\varphi_\lambda) := (\lambda) \]
\[ s = 0 \quad (\varphi_{0\lambda}) := \frac{1}{2\sqrt{|s|}} \int_{-2\lambda}^{2\lambda} d\nu (\lambda \nu) \]
\[ s \neq 0 \quad (\varphi_{sn}) := \frac{1}{2\sqrt{|s|}} \int_{-2\lambda}^{2\lambda} d\nu (\frac{\nu}{2s}, \nu) \exp(is\nu). \] (45)

In the expressions (44) and (45), \( n \) is an integer positive (negative) number if \( s > 0 \) (\( s < 0 \)). These generalized eigenvectors form a biorthonormal system

\[ (\varphi_\lambda |\varphi_{\lambda'}) = \delta(\lambda - \lambda') \]
\[ (\varphi_{0\lambda} |\varphi_{0\lambda'}) = \delta(\lambda - \lambda') \]
\[ (\varphi_{sn} |\varphi_{s'n'}) = \delta(s - s')\delta_{nn'} \]
\[ (\varphi_{sn} |\varphi_{0\lambda'}) = (\varphi_{sn} |\varphi_{\lambda'}) = (\varphi_{0\lambda} |\varphi_{\lambda'}) = 0 \] (46)

This system is also complete

\[ \mathbb{I}^+ = \int_0^\infty d\lambda |\varphi_\lambda| (\varphi_\lambda) + \int_0^\infty ds \sum_{n=-\infty}^{-1} |\varphi_{sn}| (\varphi_{sn}) + \int_0^\infty d\lambda |\varphi_{0\lambda}| (\varphi_{0\lambda}) + \int_0^\infty ds \sum_{n=1}^{+\infty} |\varphi_{sn}| (\varphi_{sn}). \] (47)

The spectral measure \( E_s \), in weak sense, is:

\[ E_s := \int_{-\infty}^{s} ds' \sum_{n=-\infty}^{-1} |\varphi_{sn}| (\varphi_{sn}) \quad s < 0 \]
\[ E_s := \int_{-\infty}^{s} ds' \sum_{n=-\infty}^{-1} |\varphi_{sn}| (\varphi_{sn}) + \int_0^{+\infty} d\lambda |\varphi_{0\lambda}| (\varphi_{0\lambda}) + \int_0^{+\infty} ds \sum_{n=1}^{+\infty} |\varphi_{sn}| (\varphi_{sn}) \quad s > 0 \] (48)

and therefore

\[ \mathbb{I}^+ = \int_0^\infty d\lambda |\varphi_\lambda| (\varphi_\lambda) + \int_0^{+\infty} dE_s, \quad \mathbb{I}^+ = \int_0^{+\infty} dE_s, \quad T^+ = \int_0^{+\infty} s dE_s. \] (49)
IV. LYAPOUNOV VARIABLES

Integrating the identity function $s$ in equation (40) with respect to $s$, we have

$$U_{-t} \left( \int_{-\infty}^{+\infty} s \, dE_s \right) U_t = \int_{-\infty}^{+\infty} s \, dE_{s-t}$$

With the change of variable $u = s - t$, we obtain

$$U_{-t} T U_t = T + t I_c, \quad (50)$$

or the dual equation

$$U_{t}^+ T^+ U_{-t}^+ = T^+ + t I_c^+. \quad (51)$$

The superoperator $A(T^+)$ (which will define the non unitary similarity transformation, the "Λ" of the Brussels group), is defined by

$$A(T^+) := I_d^+ + A(T^+) = \int_{0}^{+\infty} d\lambda |\varphi_\lambda\rangle \langle \varphi_\lambda| + \int_{-\infty}^{+\infty} A(s) \, dE_s, \quad (52)$$

where $A(s)$ is a positive real function satisfying $A(s_1) > A(s_2)$ for $s_1 < s_2$. From (50) and (52)

$$U_{-t}^+ A(T^+) U_{-t} = \int_{0}^{+\infty} d\lambda |\varphi_\lambda\rangle \langle \varphi_\lambda| + \int_{-\infty}^{+\infty} A(s + t) \, dE_s. \quad (53)$$

We also define the transformed states

$$(\tilde{\rho}) := (\rho |A(T^+)) \quad (54)$$

and the Lyapounov variable

$$L(t) := (\tilde{\rho}_t^+ |\tilde{\rho}_t)_{H-S} = Tr(\tilde{\rho}_t^+ |\tilde{\rho}_t) \quad (55)$$

Using (53) we deduce

$$L(t) = (\rho_0^+ |U_{-t}^+ A^2(T^+) U_{-t}^+ |\rho_0^+)_{H-S} = $$

$$= (\rho_0^+ | \int_{-\infty}^{+\infty} A^2(s + t) \, dE_s |\rho_0^+)_{H-S} \quad (56)$$

Being $A(s)$ a decreasing function, $L(t)$ is a positive decreasing function of the time (measuring the decay of correlations), reaching the minimum value for $t \to \infty$

\[1\text{In section 2 we defined the correlation part of the states as functionals of the H-S class, which is a Hilbert space, of the form} \]

$$\langle \rho^c | = \int dE dE' \rho_{EE'}^c (EE') \]

By the Riesz representation theorem, there is a corresponding 'dual operator' defined by

$$\langle \rho^c | = \int dE dE' \rho_{EE'}^c (EE') \]

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\[ L_{\text{min}} = \lim_{t \to \infty} L(t) = 0. \] (57)

The minimum value of \( L \) is independent of the initial state energy distribution. However the final state \( (t \to \infty) \) is not unique. This is a consequence of the fact that we are considering the time evolution of an isolated system in a state which is a mixture of different energies. In this case there is no interaction capable to rearrange the energy distribution, and the system keeps memory of the initial condition.

However, the existence of the time superoperator and the Liapounov variable reflects a special kind of instability and intrinsic irreversibility of quantum systems with continuum spectrum.

V. GENERALIZED K-FLOWS

The classical concept of K-flow is generalized in \[8\] to cover situations encountered in nonequilibrium quantum statistical mechanics.

It is known that, given a faithful (injective) normal \((18 \ 16)\) state \( \rho_\infty \) on a von Neumann algebra \( \mathcal{U} \) (for example a thermal equilibrium state), there exists a unique continuous one parameter group \( \sigma(\mathbb{R}) \) of automorphism of \( \mathcal{U} \) with respect to which \( \rho_\infty \) satisfies the KMS (Kubo-Martin-Schwinger) boundary condition. This group is called the modular group canonically associated to \( \rho_\infty \). Now, a generalized K-flow is defined as a tetrad \((\mathcal{U}, \rho_\infty, \alpha(\mathbb{R}), A)\), where \( \mathcal{U} \) is a von Neumann algebra acting on a separable Hilbert space \( \mathcal{H} \); \( \rho_\infty \) is a faithful normal state on \( \mathcal{U} \); \( \alpha(\mathbb{R}) \) is a continuous one parameter group of automorphism of \( \mathcal{U} \) such that \( \rho_\infty \circ \alpha(t) = \rho_\infty \) for every \( t \in \mathbb{R} \); and \( A \) is a von Neumann subalgebra of \( \mathcal{U} \) that is stable under the modular group, and has "the K-property" \([13 \ 20]\), that is to say:

i) \( A \subseteq \alpha(t)[A] \) for every \( t \in \mathbb{R}^+ \)

ii) the von Neumann algebra generated by \\{\( \alpha(t)[A] : t \in \mathbb{R} \)\} coincides with \( \mathcal{U} \)

iii) the largest von Neumann algebra contained in all \( \alpha(t)[A] \) for every \( t \in \mathbb{R} \), is CI

As it is well known, every von Neumann algebra is a C*-algebra, and therefore they admit an essentially unique cyclic representation \( \pi : \mathcal{U} \to \mathcal{B}(\mathcal{H}) \) into the bounded operators of a Hilbert space. This is the so called "GNS (Gelfand-Naimark-Segal) Construction" \[21\]. Let \( \mathbb{L} \) be the generator of the strongly continuous one parameter, unitary group \( \mathcal{U}(\mathbb{R}) \) implementing (by the representation \( \pi \)) \( \alpha(\mathbb{R}) \) on \( \mathcal{H} \), and let \( \Phi_\infty \in \mathcal{H} \) denotes the cyclic and separating element for \( \mathcal{U} \) such that

\[ \rho_\infty[A] = (\Phi_\infty | \pi(\mathcal{O}) \Phi_\infty)_{\mathcal{H}} \text{ for every } \mathcal{O} \in \mathcal{U} \]
\[ \mathcal{U}(t) \Phi_\infty = \Phi_\infty \text{ for every } t \in \mathbb{R} \] (58)

Therefore 0 is a simple eigenvalue of \( \mathbb{L} \), and it has uniform Lebesgue spectrum on \((\mathbb{C} \Phi)^\perp \mathbb{R}\).

There is a generalized Kolmogoroff entropy, and a proof that every non singular generalized K-flow has strictly positive entropy, and is strongly mixing, in the sense that, for every \( \mathcal{O} \) and \( Q \) in \( \mathcal{U} \):

\[ \lim_{|t| \to \infty} \rho_\infty[Q \cdot \alpha(t)[O]] = \rho_\infty[Q] \rho_\infty[O] \]

Thus, the dynamical evolution in a von Neumann algebra of operators belonging to a separable Hilbert space \( \mathcal{H} \), can be split into a 1-dimensional "diagonal part", equal (or isomorphic) to \( \mathbb{C} \), plus an "off diagonal" part. In this last one, the flow can be represented as a strongly continuous unitary group of \( \mathcal{H} \), whose generator has uniform Lebesgue spectrum.

The Brussels group has shown the existence of an Internal Time Superoperator, as well as its associated similarity transformations and Liapounov variables, for every classical K-flow. Those elements were used to prove the intrinsic irreversibility of these dynamical systems.

Now, what we have done can be seen as the corresponding generalization for the quantum case. In fact, we can assume, in a "Heisemberg picture", that our generalized evolution is taking place in the von Neumann algebra \( \mathcal{U} = \mathbb{C}(\mathcal{O}_d \bigoplus \mathcal{O}_c) + \mathcal{U}(\mathcal{O}_c) \), because the canonical inclusion

\[ i : \mathbb{C}(\mathcal{O}_d \bigoplus \mathcal{O}_c) \to \mathbb{C}(\mathcal{O}_d \bigoplus \mathcal{O}_c) + \mathcal{U}(\mathcal{O}_c) \]

is continuous. This, in turn, can be demonstrated by the following argument. Let \{\( \{a_n, T_n\} \}\, with

\[ T_n(\psi) = \sum_{k=1}^\infty \tau_{nk} < \Omega_k | \psi > \Omega_k \text{ where } \sum_{k=1}^\infty | \tau_{nk} |^2 < \infty \]

be a sequence of the domain, covering to \( (a, 0) \). This implies that \( a_n \to a \) in \( \mathbb{C} \), and \( T_n \to 0 \) in the H-S topology, i.e.:

\[ Tr(T_n^* T_n) \to 0 \]

be a sequence of the weak operator topology, which is the topology of \( \mathcal{U}(\mathcal{O}_c) \).
VI. CONCLUSIONS

We have shown the intrinsic irreversibility of a class of quantum systems with certain spectral properties, that includes some Large Poincare Systems, the generalized quantum K-flows, and certainly, the free quantum particle, whose Hamiltonian satisfies the spectral hypothesis of our result, and therefore goes to a diagonal final state, has an internal time operator, etc., as we have demonstrated. This is not difficult to understand, because this system is very different to its classical analog. In fact, free wave packets dispersion is a well known result. Additionally, in David Bohm’s "ontological interpretation" of quantum mechanics, that frequently provides an heuristic picture of what is implied in the Schrödinger equation (independently of its own validity or not), even "free" wave packets interact with a "quantum potential", so their equation of motion is nonlinear and "chaotic".

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