Information matrix equivalence in the presence of censoring: A goodness-of-fit test for semiparametric copula models with multivariate survival data

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Abstract
Various goodness-of-fit tests are designed based on the so-called information matrix equivalence: if the assumed model is correctly specified, two information matrices that are derived from the likelihood function are equivalent. In the literature, this principle has been established for the likelihood function with fully observed data, but it has not been verified under the likelihood for censored data. In this manuscript, we prove the information matrix equivalence in the framework of semiparametric copula models for multivariate censored survival data. Based on this equivalence, we propose an information ratio (IR) test for the specification of the copula function. The IR statistic is constructed via comparing consistent estimates of the two information matrices. We derive the asymptotic distribution of the IR statistic and propose a parametric bootstrap procedure for the finite-sample \( P \)-value calculation. The performance of the IR test is investigated via a simulation study and a real data example.

Key words: blanket test, copula selection, in-and-out-of-sample pseudo likelihood ratio test, omnibus test, parametric bootstrap.

1 Introduction

When we were graduate students, we learned an important derivation about the likelihood method: assume a random variable \( X \) has a distribution function \( f(x; \theta) \) with a \( p \)-dimensional parameter \( \theta \), and we have the following equation:

\[
- \int \frac{\partial^2 \log f(x; \theta)}{\partial \theta \partial \theta'} f(x; \theta) dx = \int \left[ \frac{\partial \log f(x; \theta)}{\partial \theta} \right] \left[ \frac{\partial \log f(x; \theta)}{\partial \theta} \right]' f(x; \theta) dx. \tag{1}
\]

When \( f(x; \theta) \) is the true data generating mechanism of \( X \), the left-side of equation (1) can be expressed as a \( p \times p \) matrix

\[
- \mathbb{E} \left[ -\frac{\partial^2 \ell(\theta; x)}{\partial \theta \partial \theta'} \right] = S^*(\theta),
\]

where \( \ell(\theta; x) = \log f(x; \theta) \) is
the log-likelihood function, and \( \mathbb{E}^* \) denotes the expectation with respect to (w.r.t.) the true distribution of \( X \). This matrix is called the \textit{sensitivity matrix}. The right-side of equation (1) can be expressed as another \( p \times p \) matrix \( \mathbb{E}^* \left\{ \left[ \frac{\partial (\ell(\theta, X))}{\partial \theta} \right] \left[ \frac{\partial (\ell(\theta, X))}{\partial \theta} \right]^T \right\} \) \( \triangleq \mathbf{V}^*(\theta) \), called the \textit{variability matrix}. Equation (1) becomes \( S^*(\theta) = \mathbf{V}^*(\theta) \), which is referred to as the second Bartlett identity (Bartlett 1953a,b) and information matrix equivalence (White 1982).

Several papers have utilized this equality to design goodness-of-fit (GoF) tests for model misspecification. White (1982) proposed an information matrix (IM) test based on the elements of \( \mathbf{V}^*(\theta) - S^*(\theta) \) for univariate data. Huang and Prokhorov (2013) extended this test to semi-parametric copula models for multivariate data. Zhou et al. (2012) proposed an information ratio (IR) test by comparing \( S^*(\theta)^{-1}\mathbf{V}^*(\theta) \) with a \( p \)-dimensional identity matrix. Both the IM and IR tests are variations of a general GoF testing framework proposed in Golden et al. (2013) and Golden et al. (2016), called generalized IM test, based on the comparison between these two information matrices.

All the above tests were developed for complete data: variable values are observed for all the individuals in the sample. What about censored data? For example, survival times (also called event times) are often subject to censoring, and thus, the survival times for some individuals might not be observed. No work has investigated whether the information matrix equivalence still holds under the likelihood for censored data. In this manuscript, we will prove this equivalence under a class of semiparametric copula models for multivariate censored survival times. In addition, following Zhou et al. (2012)’s work, we propose an IR test for the specification of the copula function.

Copulas are a popular tool for modeling the dependence among multiple event times. Let \( (T_1, T_2, \cdots, T_d) \) denote a \( d \)-dimensional multivariate event times with the joint survival function \( H(t_1, \cdots, t_d) = \Pr(T_1 > t_1, \cdots, T_d > t_d) \) and marginal survival functions of individual event times \( H_r(t) = \Pr(T_r > t), r = 1, \cdots, d \). According to Sklar (1959), there exists a unique copula function \( C(u_1, u_2, \cdots, u_d) : [0, 1]^d \rightarrow [0, 1] \) such that \( H(t_1, \cdots, t_d) = C(H_1(t_1), \cdots, H_d(t_d)) \). This expression implies that the joint distribution is completely determined by two independent components: marginal distributions and the copula function. Thus, compared to another well-known method, the frailty model, copulas enjoy the flexibility in coupling different marginal distribution models with a wide variety of copula families that exhibit different dependence structures.

Misspecification of the marginal distributions can lead to poor estimation (Kim et al. 2007). Thus, Shih and Louis (1995) considered a class of semiparametric copula models, which assume a parametric form for the copula function but leave the marginal distributions unspecified. Regarding the copula function, Archimedean families (Nelsen 2006), such as Clayton, Frank, and Joe, are the most popular choices. Li et al. (2008) and Othus and Li (2010) considered the Gaussian copula, which belongs to the elliptical families (including Gaussian and t copulas). Different copulas families display different features. For example, in terms of the tail dependence, Clayton has a lower-tail dependence; Joe has an upper-tail dependence; both Gaussian and Frank have no dependence for either lower-tail or upper-tail. Misspecification of the copula function can lead to incorrect estimation of the joint distribution as well as its derivatives, such as conditional distributions.

A number of GoF tests have been proposed for the specification of the copula function. Genest et al. (2009) reviewed and compared a class of so-called \textit{blanket} tests “whose
implementation requires neither an arbitrary categorization of the data nor any strategic choice of smoothing parameter, weight function, kernel, window, etc. All the blanket tests reviewed in Genest et al. (2009) were designed for complete data. Several tests were proposed specifically for censored data, and some of them targeted Archimedean families by using their unique properties. For example, Shih (1998) and Emura et al. (2010) designed their test statistics using the cross-ratio function expressed as a function of the joint survival. The test statistics in Wang and Wells (2000), Wang (2010), and Lakhal-Chaieb (2010) used the Kendall distribution, expressed in terms of the generator function. All these tests cannot be applied beyond Archimedean. Yilmaz and Lawless (2011) and Lin and Wu (2020) proposed tests that are not limited to Archimedean families. However, they imposed assumptions on the form of copulas under the alternative hypothesis. For example, in Yilmaz and Lawless (2011), the null and alternative hypotheses are nested, i.e., the null is embedded in the alternative. Lin and Wu (2020) assumes a particular form for the alternative copulas. In addition, several existing tests, such as Shih (1998), Emura et al. (2010), and Andersen et al. (2005), involve choosing weight function or bandwidth, or partition of the data, and thus, they are not blanket tests.

Our proposed IR test is a blanket test. First, it can be applied to any copula families, Archimedean or elliptical. Second, it is likelihood-based and depends solely on the parametric form of the null copula. Thus, it does not impose any assumptions on the alternative copulas. Third, it does not require any smoothing parameter, weight function, kernel, or partition of the data. The IR test was first proposed in the framework of quasi-likelihood inference for cross-sectional or longitudinal data (Zhou et al. 2012). Later, this test was extended to various models for univariate and multivariate time series, such as stochastic diffusion models and semi-parametric copula models (Zhang et al. 2012; 2016; 2019; 2021). All the data that the IR test has been applied are fully observed. However, for censored data, the asymptotic properties of the IR statistic have not been investigated.

In this paper, we will show that under the null hypothesis, the IR statistic is asymptotically distributed as a normal random variable. Since the expression of the asymptotic variance is complicated, the $P$-value calculation with an analytical variance estimate is not feasible. Thus, we suggest a parametric bootstrap procedure to approximate the IR’s null distribution via generating replications of the multivariate censored data from the copula under the null hypothesis. In addition, we examine the finite-sample size and power of the proposed test via a simulation study.

Zhang et al. (2016) established the asymptotic equivalence between the IR statistic and another class of test statistics, called the in-and-out-of-sample pseudo (PIOS) likelihood ratio test statistic. The PIOS statistic is based on the comparison between two types of pseudo likelihood: the in-sample likelihood, which is the full likelihood, and the out-of-sample likelihood, which is a “leave-one-out” cross-validated likelihood. In this manuscript, we will prove the asymptotic equivalence between these two classes of test statistics with censored data. We created an R package that implements the IR test and PIOS test under the semiparametric copula model for bivariate censored data, and it is available at https://github.com/michellezhou2009/IRtests.
2 Information Matrix Equivalence under Semiparametric Copula Models for Censored Data

As introduced earlier, \((T_1, \cdots, T_d)\) denotes the multivariate event times. They can be times to different types of events collected on each subject, such as time to relapse or second cancer and time to cardiovascular disease among breast cancer survivors \cite{Li et al. 2020}. Or they are times to the same type of event from different individuals within a cluster, such as the survival times of acute lymphoblastic leukemia patients from 104 institutions \cite{Othus and Li 2010}. Although our test can be applied for multivariate event times with \(d \geq 2\), for the ease of illustration, we present the proposed method in the context of bivariate event times \((T_1, T_2)\).

2.1 Semiparametric copula model

As described earlier, a semi-parametric copula model assume that (i) the marginal survival functions \(H_1(t)\) and \(H_2(t)\) are unspecified, and (ii) a parametric copula function \(C(u_1, u_2; \theta)\) with a \(p\)-dimensional parameter vector \(\theta \in \Theta \subset R^p\). To distinguish from this assumed parametric copula, we use \(C^*(u_1, u_2)\) to denote the true copula, i.e., \(Pr(T_1 > t_1, T_2 > t_2) = C^*(H_1(t_1), H_2(t_2))\) for any \((t_1, t_2)\). The objective of this paper is to test the null hypothesis that the assumed copula function is correctly specified, expressed as

\[
H_0 : C^*(u_1, u_2) \in C_\theta = \{ C(u_1, u_2; \theta), \theta \in \Theta \}. \tag{2}
\]

That is, there exists \(\theta^*\) such that \(C(u_1, u_2; \theta^*) = C^*(u_1, u_2)\) for almost all \((u_1, u_2) \in [0, 1]^2\). The alternative hypothesis is \(H_a : C^*(u_1, u_2) \notin C_\theta\). Note that no assumptions are imposed on the forms of alternative copulas. For example, if we test Clayton family for the null hypothesis, the alternative copulas include other Achimedean copulas and non-Achimedean copulas such as the elliptical families.

2.2 Parameter estimation in the presence of censoring

Let \((C_1, C_2)\) denote the bivariate censoring times. We assume independent censoring, i.e., \((C_1, C_2)\) are independent of \((T_1, T_2)\). Due to censoring, one can observe only

\[
X_r = \min \{T_r, C_r\}, \text{ and } \delta_r = I(T_r \leq C_r), \ r = 1, 2, \tag{3}
\]

where \(I(\cdot)\) is the identity function. Let \(\mathcal{D} = \{ (X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}), i = 1, \cdots, n \}\) be \(n\) independent realizations from the underlying distributions of \((T_1, T_2)\) and \((C_1, C_2)\).

Given the data \(\mathcal{D}\) and the above parametric copula function \(C(u_1, u_2, \theta)\), the log-likelihood function is defined as \(\ell_n(\theta; \mathcal{D}) = \sum_{i=1}^n \ell(\theta, U_{i1}, U_{i2}, \delta_{i1}, \delta_{i2})\), where \(U_{ir} = H_r(X_{ir})\), \(r = 1, 2\), are referred to as the pseudo-observations, and the \(\ell(\cdot)\) function is expressed as

\[
\ell(\theta, u_1, u_2, \delta_1, \delta_2) = \delta_1 \delta_2 \log C(u_1, u_2; \theta) + \delta_1 (1 - \delta_2) \log C_1(u_1, u_2; \theta) + (1 - \delta_1) \delta_2 \log C_2(u_1, u_2; \theta) + (1 - \delta_1)(1 - \delta_2) \log C(u_1, u_2; \theta), \tag{4}
\]
where $\ell_r(u_1, u_2; \theta) = \partial \mathcal{C}(u_1, u_2; \theta) / \partial \theta$ for $r = 1, 2$ and $\hat{\ell}(u_1, u_2; \theta) = \partial^2 \mathcal{C}(u_1, u_2; \theta) / \partial \theta^2$.

For this class of semiparametric models, Shih and Louis (1995) proposed the following two-step estimation procedure. At the first step, the marginal survival functions are estimated by $\hat{H}_r(t)$, $r = 1, 2$. Under the assumption that the censoring times are independent of the event times, $\hat{H}_r(t)$ can be a nonparametric estimator such as the Kaplan-Meier estimator (Kaplan and Meier, 1958). At the second step, the copula parameter $\theta$ is estimated via maximizing the pseudo log-likelihood function, which is the log-likelihood function with estimated pseudo-observations $\hat{U}_r = \hat{H}_r(X_i)$, $r = 1, 2$. Specifically, the pseudo maximum likelihood estimate (PMLE) of $\theta$ is given as

$$
\hat{\theta}_n = \arg \max_{\theta} \sum_{i=1}^n \ell(\theta, \hat{U}_{i1}, \hat{U}_{i2}, \delta_{i1}, \delta_{i2}).
$$

Chen et al. (2010) discussed the consistency and asymptotic normality of $\hat{\theta}_n$ under mis-specification of the copula function. They showed that regardless of whether the assumed copula function is correctly specified or not, under certain conditions, the PMLE $\hat{\theta}_n$ converges to a value $\theta^*$ in probability as $n \to \infty$. This limiting value $\theta^* = \arg \max_\theta \mathbb{E}^* [\ell(\theta, U_1, U_2, \delta_1, \delta_2)]$, where $U_r = H_r(X_r)$ for $r = 1, 2$, and $\mathbb{E}^*$ denotes the expectation w.r.t. the underlying true distributions of $(T_1, T_2)$ and $(C_1, C_2)$. Furthermore, if $\mathcal{C}(u_1, u_2; \theta)$ is correctly specified, $\mathcal{C}(u_1, u_2; \theta^*) = \mathcal{C}^*(u_1, u_2)$ for almost all $(u_1, u_2) \in [0, 1]^2$.

### 2.3 Information matrix equivalence

Under the log-likelihood function $\ell(\theta, U_1, U_2, \delta_1, \delta_2)$ in equation (4) with censored data, in Appendix A, we prove the information matrix equivalence in the following Theorem 1.

**Theorem 1 (Information Matrix Equivalence)** Under the log-likelihood function $\ell(\theta, U_1, U_2, \delta_1, \delta_2)$ in equation (4), define the sensitivity matrix as

$$
S^*(\theta) = \mathbb{E}^* [-\ell_{\theta\theta}(\theta, U_1, U_2, \delta_1, \delta_2)],
$$

where $\ell_{\theta\theta} = \partial^2 \ell / \partial \theta \partial \theta'$ is a $p \times p$ matrix, and define the variability matrix as

$$
V^*(\theta) = \mathbb{E}^* [\ell_{\theta}(\theta, U_1, U_2, \delta_1, \delta_2) \ell_{\theta}(\theta, U_1, U_2, \delta_1, \delta_2)'],
$$

where $\ell_{\theta} = \partial \ell / \partial \theta$ is a $p \times 1$ vector. Let $\theta^*$ be the limiting value of the PMLE $\hat{\theta}_n$. If the assumed copula is correctly specified, $S^*(\theta^*) = V^*(\theta^*)$.

### 3 Information Ratio Statistic

Based on the information matrix equivalence in Theorem 1, we propose an information ratio (IR) statistic by comparing consistent estimates of these two information matrices. With the data $\mathcal{D}$, the two information matrices can be estimated as

$$
\hat{S}_n(\hat{\theta}_n) = - \frac{1}{n} \sum_{i=1}^n \ell_{\theta \theta}(\hat{\theta}_n; \hat{U}_{i1}, \hat{U}_{i2}, \delta_{i1}, \delta_{i2})
$$

$\frac{1}{n} \sum_{i=1}^n \mathcal{M}(u_1, u_2; \theta) = \mathcal{M}(u_1, u_2; \theta)$
and
\[ \hat{V}_n(\theta_n) = \frac{1}{n} \sum_{i=1}^{n} \ell(\hat{\theta}_n; \hat{U}_i, \hat{V}_i, \hat{\delta}_i) \ell(\theta_n; \hat{U}_i, \hat{V}_i, \hat{\delta}_i)' . \]

The IR statistic is defined as
\[ R_n = tr \left[ S_n(\theta_n)^{-1} \hat{V}_n(\theta_n) \right] \]
where \( tr(A) \) denotes the trace of a matrix \( A \).

In Theorem 2 below, we prove that \( R_n \) is asymptotically equivalent to \( tr[S^*(\theta^*)^{-1}V^*(\theta^*)] \), regardless of whether the assumed copula is correctly specified or not. If correctly specified, i.e., the null hypothesis \( H_0 \) in equation (2) is true, the information matrix equivalence (Theorem 1) holds, leading to \( S^*(\theta^*)^{-1}V^*(\theta^*) = I_p \), a \( p \times p \)-dimensional identity matrix, and consequently, \( tr[S^*(\theta^*)^{-1}V^*(\theta^*)] = p \). Thus, under the null hypothesis \( H_0 \), \( R_n \to p \) in probability as \( n \to \infty \). We further derived the asymptotic normality of \( R_n \) under the null hypothesis in Theorem 3.

### 3.1 Asymptotic properties of IR statistic

Before proceeding to the theorems regarding the asymptotic distribution of the IR statistic, we list all the required notation and conditions. Let \( \|x\| \) denote the usual Euclidean metric of any \( p \)-dimensional vector \( x = (x_1, \ldots, x_p) \), i.e., \( \|x\| = \sqrt{x_1^2 + \cdots + x_p^2} \). For a \( p \times p \) matrix \( A \), define \( \|A\| = \sqrt{\sum_{j,k=1}^{p} a_{jk}^2} \), where \( a_{jk} \) is the \((j,k)\)-th element of \( A \).

For the simplicity of notation, in the remaining of the manuscript, we suppress \( \delta_1 \) and \( \delta_2 \) from the log-likelihood function \( \ell(\theta, u_1, u_2, \delta_1, \delta_2) \) as well as its partial derivatives. For \( j, k = 1, \ldots, p \), let \( \ell(\theta, u_1, u_2) = \ell(\theta) \) denote the \( j \)-th element of the \( p \times 1 \) vector \( \ell(\theta, u_1, u_2) \), and let \( \ell_{\theta_{jk}}(\theta, u_1, u_2) = \partial^2 \ell/\partial \theta_j \partial \theta_k \) denote the \((j,k)\)-th element of the \( p \times p \) matrix \( \ell_{\theta_{jk}}(\theta, u_1, u_2) \). Define \( \ell_{\theta_{jk}}(\theta, u_1, u_2) = \partial \ell_{\theta_j}/\partial \theta_k \) and \( \ell_{\theta_{jk}}(\theta, u_1, u_2) = \partial \ell_{\theta_j}/\partial \theta_k \), both \( p \times 1 \) vectors. For \( r = 1,2 \), let \( \ell_{\theta_{ij},u_r}(\theta, u_1, u_2) = \partial \ell_{\theta_j}/\partial u_r \) and \( \ell_{\theta_{ij},u_r}(\theta, u_1, u_2) = \partial \ell_{\theta_j}/\partial u_r \). Let \( \ell_{\theta_{ij},u_r}(\theta, u_1, u_2) \) denote a \( p \times 1 \) vector with the \( j \)-th element \( \ell_{\theta_{ij},u_r}(\theta, u_1, u_2) \). Let \( \ell_{\theta_{ijk}}(\theta, u_1, u_2) \) denote a \( p \times p \) matrix with the \((j,k)\)-th element \( \ell_{\theta_{ijk}}(\theta, u_1, u_2) \).

**Condition I** Conditions C1 - C5 and A1 - A4 in [Chen et al. (2010)](https://doi.org/10.1007/s10973-010-0004-8) that ensures the consistency and asymptotic normality of the PMLE \( \theta_n \), except that in this paper the censoring times \( \{C_{i1}, C_{i2}\}, i = 1, \cdots, n \) are independent replicates from a common continuous joint distribution. More comments on this assumption are given in Section 7.

**Condition II** Function \( \ell_{\theta}(\theta, u_1, u_2) \) is well-defined and continuous in \( (\theta, u_1, u_2) \in \Theta \times (0,1)^2 \).

**Condition III** (1) For \( j, k = 1, \ldots, p \), \( r = 1,2 \), let
\[
\begin{align*}
LL_{\theta_{ijk},u_r} &= \sup_{\theta \in \Theta} |\ell_{\theta,i}(\theta, U_1, U_2)\ell_{\theta,j}(\theta, U_1, U_2) + \ell_{\theta,k}(\theta, U_1, U_2)\ell_{\theta,j}(\theta, U_1, U_2)|, \\
LL_{\theta_{ijk}} &= \sup_{\theta \in \Theta} \|\ell_{\theta,i}(\theta, U_1, U_2)\ell_{\theta,j}(\theta, U_1, U_2) + \ell_{\theta,k}(\theta, U_1, U_2)\ell_{\theta,j}(\theta, U_1, U_2)\|.
\end{align*}
\]
Then, \( \lim_{K \to \infty} \sup_{(j,k)} E^* [ LL_{\theta_k, \theta_k} I ( LL_{\theta_k, \theta_k} \geq K ) ] = 0; \) For \( j, k = 1, \ldots , p, \) let

\[
Q_{1, \theta_k} ( \theta, u_1, u_2 ) = \left| \ell_{\theta_k} ( \theta, u_1, u_2 ) \right| \ell_{\theta_k} ( \theta, u_1, u_2 ) + \left| \ell_{\theta_k, \theta} ( \theta, u_1, u_2 ) \ell_{\theta_k} ( \theta, u_1, u_2 ) \right|.
\]

For any \( \eta > 0 \) and any \( \epsilon > 0, \) there is \( K > 0, \) such that \( Q_{1, \theta_k} ( \theta, u_1, u_2 ) \leq K Q_{1, \theta_k} ( \theta, u_1, u_2 ) \) for all \( \theta \in \Theta \) and all \( u_r \in [\eta, 1) \) such that \( 1 - u_r \geq \epsilon (1 - u_r), \) \( r = 1, 2. \)

**Condition IV** (1) Functions \( \ell_{\theta_k, \theta} ( \theta, u_1, u_2 ), \ell_{\theta_k, \theta} ( \theta, u_1, u_2 ), \) and \( \ell_{\theta_k, \theta} ( \theta, u_1, u_2 ), \) \( j = 1, \ldots , p, \) \( r = 1, 2, \) are well-defined and continuous in \( ( \theta, u_1, u_2 ) \in \Theta \times (0,1)^2. \) (2) For \( j, k = 1, \ldots , p, \) \( r = 1, 2, \)

\[
L_{\theta_k, \theta} = \sup_{\theta \in \Theta} \| \ell_{\theta_k, \theta} ( \theta, U_1, U_2 ) \|, \text{ and } \sup_{\theta \in \Theta} \| \ell_{\theta_k, \theta} ( \theta, U_1, U_2 ) \|.
\]

**Condition V** (1) \( \| \ell_{\theta_k} ( \theta^*, u_1, u_2 ) \| \leq \text{constant} \times \{ u_1 (1 - u_1) \}^{-a_1} \{ u_2 (1 - u_2) \}^{-a_2} \) for some \( a_1 \geq 0 \) and \( a_2 \geq 0; \) \( \text{sup} \{ U_1 (1 - U_1)^{-2a_1} U_2 (1 - U_2)^{-2a_2} \} < \infty; \) (2) \( \| \ell_{\theta, \theta} ( \theta^*, u_1, u_2 ) \| \leq \text{constant} \times \{ u_1 (1 - u_1) \}^{-b_1} \{ u_2 (1 - u_2) \}^{-b_2} \) for some \( b_1 \geq 0 \) and \( b_2 \geq 0; \) \( \text{sup} \{ U_1 (1 - U_1)^{2b_1} U_2 (1 - U_2)^{2b_2} \} < \infty; \) (3) \( \| \ell_{\theta, \theta} ( \theta^*, u_1, u_2 ) \| \leq \text{constant} \times \{ u_1 (1 - u_1) \}^{-b_1} \{ u_2 (1 - u_2) \}^{-b_2} \) for some \( b_1 \geq 0 \) and \( b_2 \geq 0; \) \( \text{sup} \{ U_1 (1 - U_1)^{2b_1} U_2 (1 - U_2)^{2b_2} \} < \infty; \) (4) \( \| \ell_{\theta, \theta} ( \theta^*, u_1, u_2 ) \| \leq \text{constant} \times \{ u_1 (1 - u_1) \}^{-b_1} \{ u_2 (1 - u_2) \}^{-b_2} \) for some \( b_1 \geq 0 \) and \( b_2 \geq 0; \) \( \text{sup} \{ U_1 (1 - U_1)^{2b_1} U_2 (1 - U_2)^{2b_2} \} < \infty. \)

**Theorem 2** Under Conditions I - III, we have \( R_n \to \text{tr} [ S^* ( \theta^* )^{-1} V^* ( \theta^* ) ] . \)

The proof of this theorem (Appendix B) requires the consistency of \( \hat{\theta}_n ( \theta_n ) \) and \( \hat{V}_n ( \theta_n ) . \) Chen et al. (2010) has shown the consistency of \( \hat{S}_n ( \theta_n ) , \) which requires their Conditions A2 and A4. We follow their arguments to prove the consistency of \( \hat{V}_n ( \theta_n ) , \) where our Conditions II is analogous to Chen et al’s Condition A2, and our Conditions III is analogous to Chen et al’s Condition A4.

**Theorem 3** Assume Conditions I - V hold. If the null hypothesis is true, \( \sqrt{n} ( R_n - p ) \) converges in distribution to a normal random variable with mean 0 and variance \( \sigma_R^2 = \text{Var}[ h_R ( X_{11}, X_{12}, \delta_{11}, \delta_{12}, \theta ) ] , \) where \( h_R ( X_{11}, X_{12}, \delta_{11}, \delta_{12}, \theta ) \) is given by equation (13) in Appendix C.

The proof of this theorem (Appendix C) utilizes the Taylor expansion of \( \hat{S}_n ( \theta_n ) \) and \( \hat{V}_n ( \theta_n ) . \) One of the steps requires the consistency of the first-order derivative of these two
matrices w.r.t. $\theta$. Again, we follow the arguments of Chen et al. (2010) for proving the consistency of $\hat{S}_n(\hat{\theta}_n)$, where our Condition IV (1) is analogous to Chen et. al’s Condition A2, and our Conditions IV (2) & (3) together are analogous to Chen et. al’s Condition A4. Another component in the proof involves the expansion of the estimated pseudo-observations $\hat{U}_{i1}$ and $\hat{U}_{i2}$. Our Condition V (1) is analogous to Chen et. al’s Condition A3 (i), and so is our Condition V (3). Our Condition V (2) is analogous to Chen et. al’s Condition A3 (ii), and so is our Condition V(4).

3.2 Asymptotic equivalence to the In-and-out-of-sample pseudo likelihood ratio statistic

For semiparametric copula models with fully observed data, Zhang et al. (2016) showed that the IR statistic $R_n$ is asymptotically equivalent to a class of in-and-out-of-sample pseudo (PIOS) likelihood ratio test statistic. Theorem 4 below demonstrates this asymptotic equivalence still holds in the present of right censoring. The PIOS statistic is defined as a difference between two types of pseudo log-likelihood functions: in-sample and out-of-sample. The in-sample pseudo log-likelihood is defined as $\ell_{in}^n = \sum_{i=1}^{n} \ell(\hat{\theta}, \hat{U}_{i1}, \hat{U}_{i2}, \delta_{i1}, \delta_{i2})$, where $\hat{\theta}$ is obtained from equation (5) using all the observations. The out-of-sample pseudo log-likelihood employs the leave-one-out technique, and is defined as $\ell_{out}^n = \sum_{i=1}^{n} \ell(\hat{\theta}_{(-i)}, \hat{U}_{i1}, \hat{U}_{i2}, \delta_{i1}, \delta_{i2})$, where $\hat{\theta}_{(-i)} = \arg \max_{\theta} \sum_{s=1, s \neq i}^{n} \ell(\theta, \hat{U}_{s1}, \hat{U}_{s2}, \delta_{s1}, \delta_{s2})$ is the PMLE obtained from the data with the $i$-th observation deleted. The PIOS test statistic is defined as $T_n = \ell_{in}^n - \ell_{out}^n$. A large value of $T_n$ suggests that the assumed copula model is a poor fit to the data since it is sensitive to the deletion of individual observations.

**Theorem 4** Under Condition I, $|R_n - T_n| = o_p(1)$.

Because of this asymptotic equivalence, if the null hypothesis $H_0$ is true, the PIOS statistic $T_n$ also converges to $p$ in probability, and $\sqrt{n}(T_n - p)$ also converges in distribution to a normal random variable with mean 0 and the same variance $\sigma^2_R$.

4 Information Ratio Test and Copula Selection

In practice, it is challenging to calculate $P$-values using an analytical estimate of the asymptotic variance $\sigma^2_R$ because its expression is complicated. Thus, we suggest a parametric bootstrap resampling procedure for the $P$-value calculation.

4.1 $P$-value calculation via bootstrap resampling

The key idea is to generate a large number of data replicates, referred to as the bootstrapped data, from the copula under the null hypothesis. To differentiate from the original data $D$, we use $D^{(b)}$ denote a bootstrapped data. The null distribution of $R_n$ is approximated by the test statistics values calculated from $D^{(b)}$. Specifically, the resampling procedure includes the following steps:
Step 1: Generate a bootstrapped data, \( \mathcal{D}^{(b)} = \{(X_{1i}^{(b)}, X_{2i}^{(b)}, \delta_{1i}^{(b)}, \delta_{2i}^{(b)}), i = 1, \cdots, n\} \), with the same sample size of the original data, from the copula \( C_{\theta} \) under the null hypothesis using \( \hat{\theta}_n \) (the PMLE obtained from the original data) as the parameter value. The details of the data generation are provided in Section 4.2 below.

Step 2: Based on the bootstrapped data \( \mathcal{D}^{(b)} \), calculate the test statistic, denoted as \( R_{n}^{(b)} \), referred to as a bootstrap resample of \( R_n \).

Step 3: Repeat Steps 1 and 2 \( B \) times, producing \( B \) bootstrap resamples \( \{R_n^{(b)}, b = 1, \cdots, B\} \).

Let \( \bar{R}_n^{(b)} = R_n^{(b)} - \bar{R}_n^{b} \) be the centered bootstrap resamples, where \( \bar{R}_n^{b} \) is the average of \( \{R_n^{(b)}, b = 1, \cdots, B\} \). The \( \{\bar{R}_n^{(b)}, b = 1, \cdots, B\} \) approximates the finite-sample distribution of \( R_n - p \) under the null hypothesis. We can calculate a two-sided \( P \)-value as

\[
B^{-1} \sum_{b=1}^{B} I\left(\bar{R}_n^{(b)} > |R_n - p|\right) + I\left(\bar{R}_n^{(b)} < -|R_n - p|\right).
\] (6)

4.2 Generate bivariate censored survival data from a copula

The generation process consists of two parts: generate the bivariate event times \((T_{1i}, T_{2i})\) and generate the bivariate censoring times \((C_{1i}, C_{2i})\). The observed survival data \((X_{1i}, X_{2i}, \delta_{1i}, \delta_{2i})\) can be obtained following equation (3).

Generate \((T_{1i}, T_{2i})\). Given a copula family \( C_{\theta} \) and some parameter value \( \theta_0 \), we first generate \((U_{1i}, U_{2i})\) from \( C(u_1, u_2; \theta_0) \). This step can be implemented using the R package copula. Next, we obtain \( T_{ir} = H_r^{-1}(U_{ir}), r = 1, 2 \), where \( H_r^{-1} \) are the inverse of the marginal survival functions. To generate a bootstrap resample from a given data, neither \( \theta_0 \) or \( H_r(t) \) is known. Thus, we use the estimates from the original data: \( \theta_0 = \hat{\theta}_n \), the PMLE, and \( T_{ir} = \tilde{H}_r^{-1}(U_{ir}) \), where \( \tilde{H}_r \) is the Kaplan-Meier estimator.

Generate \((C_{1i}, C_{2i})\). The censoring times \( C_{i1} \) and \( C_{i2} \) might be correlated, but our method does not rely on their joint distribution. Thus, we can simulate them separately from their own marginal distributions. Let \( G_r(t), r = 1, 2 \), denote the survival function of \( C_{ir} \). We first generate a random number \( v_{ir} \) from a uniform distribution between 0 and 1, and then obtain \( C_{ir} = G_r^{-1}(v_{ir}) \).

Again, to generate a bootstrap resample from a given data, \( G_r(t) \) are unknown, and thus, we use the estimated distributions from the original data. Under the assumption of independent censoring, we can obtain a Kaplan-Meier estimate \( \tilde{G}_r(t) \) using the data \( \{(C_{ir}, 1 - \delta_{ir}), i = 1, \cdots, n\} \). In some applications, both event times are subject to the same censoring time, i.e., \( C_{i1} = C_{i2} = C_i \), such as due to the end of study or lost to follow up. In this case, its sole survival distribution \( G(t) \) can be estimated using the data \( \{(X_{1i} \wedge X_{2i}, 1 - \delta_{i1} \delta_{i2}), i = 1, \cdots, n\} \).
4.3 Select the “best” copula family

For some data, a GoF test would fail to reject several copula families. It might be because the sample size is small or the censoring rate is high or both, and thus, the data do not contain sufficient information to reject the null hypothesis. In addition, if the level of dependence is not strong, several families appear similar so it is more difficult for a test to tell them apart. In some situations, the underlying true dependence structure might be complicated, and any parametric copula family is merely an approximation. For these cases, we are more concerned with selecting the “best” copula family from several candidates in the sense that the data exhibit the weakest evidence against it, i.e., showing the highest agreement between the assumed copula and the data. Here, we propose using the $P$-value of the IR test as the selection criteria: the “best” copula family is the one with the largest $P$-value.

5 Simulation

In the following simulation study, we investigate (i) how close the finite-sample null distribution of the IR statistic to the asymptotic normality, (ii) how close the null distributions between the IR and PIOS statistics, (iii) the empirical type I error rate and test power of the IR test, and (iv) the performance of using the IR $P$-values for copula selection.

5.1 Simulation setting

We follow the procedure in Section 4.2 to generate $(T_{i1}, T_{i2})$ and $(C_{i1}, C_{i2})$, except that the copula parameter $\theta$, the marginal distributions $H_r(t)$ of the event times, and the marginal distributions $G_r(t)$ of censoring times are specified in this simulation.

We consider four copula families: Clayton, Frank, Joe, and Gaussian. The marginal distributions of $T_{i1}$ and $T_{i2}$ are both exponential distribution with mean 1. Both event times are subject to a common censoring time $C_i$, generated from an exponential distribution with mean 4 or $3/2$ that correspond to 20% and 40% of censoring rate for individual event times. In addition, we include a no-censoring setting, i.e., $T_{i1}$ and $T_{i2}$ are fully observed, to investigate the effect of censoring on the performance of the IR test. Thus, there are three censoring scenarios, denoted as “no-censoring”, “20%-censored”, “40%-censored”.

5.2 Null distributions of IR and PIOS statistics

To examine the finite-sample null distributions of the IR and PIOS statistics, we generate data from a copula family, say Clayton, and test for the same family, also Clayton. Figures 1 - 4 in Supplementary Material are the normal quantile-quantile (QQ) plots of 500 replications of these two statistics under each of four copula families with Kendall’s $\tau = 0.5$ at sample size $n = 100, 300, 600$.

First, we focus on the null distribution of the IR statistic. When the sample size is small, the distribution appears skewed to the right. As the sample size increases, it is getting closer to the normal distribution, which confirms the asymptotic normality of the IR
statistic (Theorem 3). Second, let us compare IR and PIOS. The QQ plots clearly show that their distributions are close, and they get more similar as the sample size increases. It confirms the asymptotic equivalence between IR and PIOS (Theorem 4). However, their computational times are substantially different. The PIOS statistic requires repeated ($n$ times) estimation of the copula parameter, $\hat{\theta}(-i)$, when obtaining the out-of-sample pseudo log-likelihood. Thus, its computational burden is more intensive than IR, and their gap increases as the sample size increases. For example, using a MacBook Air with 1.2 GHz Quad-core Intel Core i7, the computational time with sample size $n = 100$ is 0.0059 seconds for calculating the IR statistic and 0.3 seconds for PIOS (about 50 times of IR’s time). When the sample size increases to 600, the computational time is 0.013 seconds for IR and 3.2 seconds for PIOS (about 250 times of IR’s time).

5.3 Test size and power of IR test

In this experiment, $(T_{i1}, T_{i2})$ are generated from each of the four copula families; given the true copula, we test each of four copula families as the null hypothesis. For example, in one scenario, $(T_{i1}, T_{i2})$ is generated from Clayton, i.e., the true copula is Clayton, and we test four different null hypotheses: Clayton, Frank, Joe, and Gaussian. We consider three different dependence levels, characterized by Kendall’s $\tau = 0.3, 0.5, 0.7$ and three sample sizes $n = 100, 300, 600$. Figures 1 - 4 in Supplementary Material are the scatter plots of the estimated pseudo-observations $(\hat{U}_{i1}, \hat{U}_{i2})$, obtained from a bivariate censored data generated under each of the four copula families with Kendall’s $\tau = 0.3$ or $0.7$ and sample size $n = 100$ or 600.

For each generated data, the $P$-values are calculated from $B = 500$ bootstrap resamples. We summarize the proportion of rejecting the null hypothesis at the significance level 0.05 among 500 replications. Figures 1-4 plot these rejection proportions, and each figure corresponds to each true copula family. When the null copula is the same as the true copula, the rejection proportions are the empirical type I error rates, also extracted in Table 1. In general, the empirical type I error rates are close to the significance level 0.05, which indicates that the IR test performs adequately in terms of the test size. However, when the true copula is Clayton and the event times are censored, the IR test tends to be more conservative, i.e., having a smaller type I error rate.

When the null copula is different from the true copula, the rejection proportions are the empirical test power. The results indicate that Kendall’s $\tau$, sample size, and censoring rate all affect the power. First, Kendall’s $\tau$ reflects the strength of the dependence between the bivariate event times. When $\tau$ is large, i.e., the event times are highly correlated, the true copula’s distinct features such as tail dependence are more pronounced, and thus, our IR test is more powerful to detect deviations from the null copula. However, when the correlation is weak, copula families appear similar to each other (See Figures 1 and 3 in Supplementary Material). Thus, the IR test has a lower power with a smaller Kendall’s $\tau$. Similar patterns are observed in Genest et al. (2009) and Zhang et al. (2016). Second, as expected, when the sample size is larger or the censoring rate is lower or both, the data provide more information of the underlying true copula, and consequently, the IR test is more powerful.

We want to point out two scenarios with a low power. The first scenario is when the
true copula is Clayton, the power for rejecting Frank is low for the 40%-censored scenario, despite of a large sample size and a strong dependence. A possible explanation could be that the distinction between Clayton and Frank concentrates on the lower-tail dependence. However, since the censoring time follows an exponential distribution, it is more likely to censor smaller event times, resulting in losing information on the lower-tail dependence. Thus, when the censoring rate is high, it is difficult to differentiate these two families.

The second scenario is when the true copula is Gaussian, the power for rejecting Frank is low for all sample sizes, Kendall’s $\tau$ values, and censoring settings. It could be because both families are symmetric with no dependence on either tails. However, when the true copula is Frank and the null copula is Gaussian, the test performs better, although the power for rejecting Gaussian is still lower than rejecting Clayton or Joe. It calls for more investigations.

### 5.4 Copula selection

In this study, we also examine how well using the $P$-value of the IR test as the criterion can correctly select the true copula as the best among the four families. With each simulated data, we obtain the $P$-value for testing each of Clayton, Frank, Joe, and Gaussian as the null hypothesis. Following Section 4.3, we select the copula family with the largest $P$-value as the best. Figures 9 - 12 in Supplementary Material report the percentage of choosing each family as the best among the 500 replications. Consistent with our findings on the test power, when the sample size is larger or the dependence is stronger or the censoring rate is lower, the proportion of selecting the true copula as the best is higher. Copulas with similar properties are more difficult to tell apart. For example, when the true copula is Gaussian, Frank copula is a strong competitor, even when the sample size is 600, Kendall’s $\tau = 0.7$, and the event times are fully observed.

### 6 Data Example

The data example is from the Australian NHMRC Twin Registry (Duffy et al. 1990), and the bivariate event times $(T_1, T_2)$ are the ages at appendicectomy measured for each twin pair. We are interested in four different groups of same-sex twin pairs: 567 monozygotic (MZ) male pairs, 1231 MZ female pairs, 350 dizygotic (DZ) male pairs, and 748 DZ female pairs. The data can be found at [https://genepi.qimr.edu.au/staff/davidD/Appendix/](https://genepi.qimr.edu.au/staff/davidD/Appendix/). Figure 5 shows the scatter plots of the estimated pseudo-observations $\hat{U}_1$ and $\hat{U}_2$ for each group of twin pairs. We test for five copula families: Clayton, Frank, Gumbel, Joe, and Gaussian, and the $P$-values are calculated using $B = 1000$ bootstrapped resamples.

#### 6.1 MZ male pairs

In this group of 567 pairs, 37 (7%) pairs have both event times observed, 129 (23%) have one event time observed and the other censored, and 401 (70%) have both event times censored. The censoring rates for individual event times are 82% for $T_1$ and 83% for
$T_2$. The Kendall's $\tau$ of $(\hat{U}_1, \hat{U}_2)$ is estimated to be 0.53. In addition, we examine their upper and lower tail dependence coefficients (TDCs) \cite{SchmidtStadtmuller2006}, denoted as $\lambda_{\text{upper}}$ and $\lambda_{\text{lower}}$. A bivariate distribution is said to be upper (or lower) tail independent if $\lambda_{\text{upper}} = 0$ (or $\lambda_{\text{lower}} = 0$), and it is said to be upper (or lower) tail dependent if $0 < \lambda_{\text{upper}} \leq 1$ (or $0 < \lambda_{\text{lower}} \leq 1$). The magnitude of the TDC indicates the dependence strength. We estimate these two TDCs using the non-parametric estimation method proposed by \cite{SchmidtStadtmuller2006}, which can be implemented via the R function `tdc` of the package FRAPo. The upper 5% quantile TDC is estimated to be 0.25, and the lower 5% quantile TDC is estimated to be 0.93. These estimates indicate that the bivariate survival distribution of this group has a stronger lower-tail dependence, compare to the upper-tail. Using the proposed IR test, the $P$-value is 0.573 for Clayton (with IR statistic $R_n = 1.06$), 0.075 for Frank (with $R_n = 1.08$), 0.026 for Gumbel (with $R_n = 1.15$), 0.021 for Joe (with $R_n = 1.19$), and 0.123 for Gaussian (with $R_n = 1.10$). At the significance level 0.05, we reject Gumbel and Joe (both have an upper-tail dependence). Among the families that are not rejected, Clayton is the obvious winner: it has a much larger $P$-value than Frank and Gaussian. In addition, Clayton has a lower-tail dependence, consistent with the estimated TDCs described earlier.

### 6.2 MZ female pairs

In this group of 1231 pairs, 166 (13%) pairs have both event times observed, 322 (26%) have one event time observed and the other censored, and 743 (60%) have both event times censored. The censoring rates for individual event times are 73% for $T_1$ and 74% for $T_2$. The Kendall’s $\tau$ of $(\hat{U}_1, \hat{U}_2)$ is estimated to be 0.57, the upper 5% quantile TDC is estimated to be 0.26, and the lower 5% quantile TDC is estimated to be 0.81. For this bivariate survival distribution, the lower-tail dependence appears stronger than the upper-tail dependence. Using the proposed IR test, the $P$-value is 0.050 for Clayton (with $R_n = 1.13$), 0.009 for Frank (with $R_n = 1.08$), 0 for Gumbel (with $R_n = 1.22$), 0 for Joe (with $R_n = 1.31$), and 0 for Gaussian (with $R_n = 1.20$). All the $P$-values are small: this group is an example of data with a large sample size, for which the test would detect even a slight deviation from the null copula. For this group, the data provide the weakest evidence against Clayton, and at the significance level 0.05, we only fail to reject Clayton. Thus, Clayton can be regarded as the best among these five candidate families.

### 6.3 DZ male pairs

In this group of 350 pairs, 19 (5%) pairs have both event times observed, 67 (19%) have one event time observed and the other censored, and 264 (76%) have both event times censored. The censoring rates for individual event times are 85% for $T_1$ and 85% for $T_2$. The Kendall’s $\tau$ of $(\hat{U}_1, \hat{U}_2)$ is estimated to be 0.6, the upper 5% quantile TDC is estimated to be 0.22, and the lower 5% quantile TDC is estimated to be 0.81. Using the proposed IR test, the $P$-value is 0.874 for Clayton (with $R_n = 1.04$), 0.597 for Frank (with $R_n = 1.04$), 0.360 for Gumbel (with $R_n = 1.09$), 0.259 for Joe (with $R_n = 1.12$), and 0.323 for Gaussian (with $R_n = 1.10$). At the significance level 0.05, we cannot reject any copula families, probably due to the small sample size and high censoring rate. However, the
evidence against Clayton is the weakest, while the evidence against Gumbel and Joe, two copulas with an upper-tail dependence, is the strongest. Thus, Clayton is the best among the five candidates.

6.4 DZ female pairs

The 748 DZ female pairs was analyzed in Emura et al. (2010), which concluded that Gumbel provides the best fit among Clayton, Frank, Log-copula, and Gumbel. In this group of 748 pairs, 82 (11%) pairs have both event times observed, 222 (30%) have one event time observed and the other censored, and 444 (59%) have both event times censored. The censoring rates for individual event times are 75% for $T_1$ and 73% for $T_2$. The Kendall’s $\tau$ of $(\hat{U}_1, \hat{U}_2)$ is estimated to be 0.49, the upper 5% quantile TDC is estimated to be 0.27, and the lower 5% quantile TDC is estimated to be 0.35. Using the proposed IR test, the $P$-value is 0.234 for Clayton (with $R_n = 1.08$), 0.044 for Frank (with $R_n = 1.07$), 0.296 for Gumbel (with $R_n = 1.06$), 0.216 for Joe (with $R_n = 1.08$), and 0.169 for Gaussian (with $R_n = 1.08$). Gumbel has the largest $P$-value, and thus, it is the best among the five candidates, which is consistent with the result of Emura et al. (2010).

7 Concluding Remarks

Information matrix equivalence plays an important role in model diagnosis, and a number of GoF tests have been established based on this principle. However, this equivalence has not been verified for censored data. Thus, one major contribution of this work is to prove the equivalence of the two information matrices under a class of semi-parametric copula models for multivariate data in the presence of right censoring. The proof provides a framework which might be extended to other types of censoring. In addition, based on this equivalence, we propose an IR test for the specification of the copula function via comparing consistent estimates of the two information matrices. We derive the asymptotic properties of the test statistic and layout the necessary conditions.

The IR test is likelihood-based and depends on only the parametric form of the assumed copula function. Thus, it can be applied to all copula families, and do not rely on choices of weight functions, bandwidth, or smoothing parameters. In addition, the IR statistic is asymptotically equivalent to a class of PIOS test statistics, which provides a global measure of how the assumed model fits the data via the leave-one-out cross-validation. Furthermore, the IR test does not assume any parametric form of alternative copulas. It can be regarded as an omnibus test.

In this manuscript, we derived the asymptotic distribution of the IR statistic following similar arguments in Chen et al. (2010). They considered a more general distributional assumption for censoring: the joint distribution of the bivariate censoring times could be different across subjects. Under this relaxed assumption, the limiting value of the PMLE $\hat{\theta}_n$, is defined

$$\theta^*_n = \arg\max_{\theta} n^{-1} \sum_{i=1}^n \mathbb{E}^*[\ell(\theta, U_{i1}, U_{i2})].$$

This limiting value depends on the sample size since the “observed” survival times might...
not be identically distributed due to non-identically distributed bivariate censoring times. Thus, the definitions of the sensitivity and variability matrices become $S^\ast(\theta) = n^{-1} \sum_{i=1}^{n} E^\ast [-\ell_{\theta\theta}(\theta, U_{i1}, U_{i2})]$ and $V^\ast(\theta) = n^{-1} \sum_{i=1}^{n} E^\ast [\ell_{\theta\theta}(\theta, U_{i1}, U_{i2}) \ell_{\theta\theta}(\theta, U_{i1}, U_{i2})']$, which also depends on the sample size. We want to point out that under this relaxed assumption, the proof of Theorem 1 is still valid, and thus, the information matrix equivalence still holds.

In general, if testing within Archimedean families, the GoF tests that target these families are expected to be more powerful than our proposed IR test because they utilize their distinct properties such as cross-ratio functions or Kendall distribution. On the other hand, our proposed IR test can compare copula families beyond Archimedean. In Section 4.3, we demonstrate how to use the $P$-value of the IR test to select the optimal copula family among several candidates. Although in the simulation and data example, the copulas we considered all have only one parameter, our method can be applied to copulas with more than one parameters such as $t$-copula.

In the simulation study, we observe the low power in the IR test for differentiating between Gaussian and Frank, which are both symmetric and with no dependence on either tails. As described in [Golden et al. (2016)], there are a number of ways to compare the two information matrices and our IR test is just one of them. It worths further research to search for other comparison forms that can better differentiate copula families with similar tail-dependence properties.

8 Supplementary Material

The supplementary material includes (i) QQ plots (Figures 1 - 4) of the IR and PIOS statistics under the null copula of Clayton, Frank, Joe, and Gaussian with Kendall’s $\tau = 0.5$ with different sample sizes and censoring settings, (ii) scatter plots (Figures 5 - 8) of estimated pseudo-observations $(\hat{U}_{i1}, \hat{U}_{i2})$ from a bivariate censored data of size $n = 100$ or 600 generated from each of the following four copula families: Clayton, Frank, Joe, and Gaussian with Kendall’s $\tau = 0.3$ or 0.7 under three censoring settings, and (iii) barplots (Figures 9 - 12) of proportions of selecting different copula families as the best copula when the true copula is Clayton, Frank, Joe, or Gaussian with different Kendall’s $\tau$, sample sizes, and censoring settings.

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Figure 1: Simulation results: Proportion of rejecting four null copulas: Clayton, Frank, Joe, and Gaussian, when the true copula is Clayton.
Table 1: Simulation results: Empirical type I errors.

| True & Null Copula | \( n = 100 \) | \( n = 300 \) | \( n = 600 \) |
|--------------------|----------------|----------------|----------------|
| **Clayton**        |                |                |                |
| \( \tau = 0.3 \)   | 0.036          | 0.040          | 0.040          |
| \( \tau = 0.5 \)   | 0.044          | 0.040          | 0.048          |
| \( \tau = 0.7 \)   | 0.054          | 0.044          | 0.046          |
| **Frank**          |                |                |                |
| \( \tau = 0.3 \)   | 0.036          | 0.046          | 0.038          |
| \( \tau = 0.5 \)   | 0.046          | 0.044          | 0.034          |
| \( \tau = 0.7 \)   | 0.056          | 0.062          | 0.042          |
| **Joe**            |                |                |                |
| \( \tau = 0.3 \)   | 0.058          | 0.066          | 0.052          |
| \( \tau = 0.5 \)   | 0.068          | 0.066          | 0.050          |
| \( \tau = 0.7 \)   | 0.058          | 0.066          | 0.070          |
| **Gaussian**       |                |                |                |
| \( \tau = 0.3 \)   | 0.050          | 0.048          | 0.056          |
| \( \tau = 0.5 \)   | 0.034          | 0.050          | 0.048          |
| \( \tau = 0.7 \)   | 0.040          | 0.050          | 0.052          |
Figure 2: Simulation results: Proportion of rejecting four null copulas: Clayton, Frank, Joe, and Gaussian, when the true copula is Frank.
Figure 3: Simulation results: Proportion of rejecting rejecting four null copulas: Clayton, Frank, Joe, and Gaussian, when the true copula is Joe.

No Censoring 20% Censored 40% Censored
\( \tau = 0.3 \) \( \tau = 0.5 \) \( \tau = 0.7 \)
n=100 n=300 n=600 n=100 n=300 n=600 n=100 n=300 ...

Proportion of rejecting null hypothesis

\( H_0 \)
- Clayton
- Frank
- Joe
- Gaussian

n=100 n=300 n=600 n=100 n=300 n=600 n=100 n=300...
Figure 4: Simulation results: Proportion of rejecting four null copulas: Clayton, Frank, Joe, and Gaussian, when the true copula is Gaussian.
Figure 5: Scatter plots of estimated pseudo-observations $\hat{U}_1$ and $\hat{U}_2$ with the estimate Kendall’s $\tau$ for each group of twin pairs.

|                  | MZ – male : $\tau = 0.53$ | MZ – female : $\tau = 0.57$ |
|------------------|---------------------------|-----------------------------|
| $\hat{U}_2$     |                           |                             |
| $\hat{U}_1$     |                           |                             |

MZ – male : $\tau = 0.53$

MZ – female : $\tau = 0.57$

DZ – male : $\tau = 0.6$

DZ – female : $\tau = 0.49$
Appendix

A Proof of Theorem 1

Let $g_\theta(\theta)$ and $g_{\theta\theta}(\theta)$ denote the first-order and second-order derivatives of a function $g(\theta)$ such as $C$, $c$, $c_1$, and $c_2$ w.r.t. $\theta$. Let $f_1(t_1)$ and $f_2(t_2)$ be the marginal probability distribution functions (pdfs) of $T_1$ and $T_2$.

The outline of the prove is as follows. First, we will show that for a given $\theta$, $S^*(\theta) = E^*[A(\theta)] + V^*(\theta)$, where $A(\theta)$ is a $p \times p$ matrix. Second, we show that $E^*[A(\theta^*)] = 0_{p \times p}$. To this end, we invoke the double expectation theorem. Given the censoring times $C_1$ and $C_2$, the conditional expectation

$$E^*[A(\theta^*) | C_1, C_2] = \int_0^\infty \int_0^\infty c_{\theta\theta}(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2.$$

By interchanging the second-order derivative w.r.t. $\theta$ with the double integral w.r.t. $t_1$ and $t_2$, we have

$$\int_0^\infty \int_0^\infty c_{\theta\theta}(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2 = \frac{\partial^2}{\partial \theta \partial \theta} \int_0^\infty \int_0^\infty c(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2.$$

If the assumed copula function $C(u_1, u_2; \theta)$ is correctly specified, $c(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2)$ is the joint pdf of $(T_1, T_2)$, and thus,

$$\int_0^\infty \int_0^\infty c(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2 = 1.$$

It implies that the conditional expectation $E^*[A(\theta^*) | C_1, C_2] = 0_{p \times p}$ for any given $C_1$ and $C_2$, and consequently, we can show that $E^*[A(\theta^*)] = 0_{p \times p}$.

**Prove $S^*(\theta) = E^*[A(\theta)] + V^*(\theta)$**. Based on the log-likelihood function in equation [4], the expressions of $\ell_\theta(\theta, U_1, U_2, \delta_1, \delta_2)\ell_\theta(\theta, U_1, U_2, \delta_1, \delta_2)'$ and $\ell_{\theta\theta}(\theta, U_1, U_2, \delta_1, \delta_2)$ are given as

$$\ell_\theta(\theta, U_1, U_2, \delta_1, \delta_2)\ell_\theta(\theta, U_1, U_2, \delta_1, \delta_2)' = \delta_1 \delta_2 w_\theta(U_1, U_2; \theta)w_\theta(U_1, U_2; \theta)'$$

$$+ \delta_1(1 - \delta_2) w_{1,0}(U_1, U_2; \theta)w_1(0, U_2; \theta)'$$

$$+ (1 - \delta_1) \delta_2 w_{2,0}(U_1, U_2; \theta)w_2(0, U_2; \theta)'$$

$$+ (1 - \delta_1)(1 - \delta_2) \frac{C_\theta(U_1, U_2; \theta)}{C(U_1, U_2; \theta)^2}.$$

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We will show that

\[ \ell_{\theta \theta}(\theta, U_1, U_2, \delta_1, \delta_2) \ell_{\theta}(\theta, U_1, U_2, \delta_1, \delta_2)' \]

\[ = \delta_1 \delta_2 \left[ \frac{c_{\theta \theta}(U_1, U_2; \theta)}{c(U_1, U_2; \theta)} - \frac{c_{\theta}(U_1, U_2; \theta) c_{\theta}(U_1, U_2; \theta)'}{c(U_1, U_2; \theta)^2} \right] + \delta_1(1 - \delta_2) \left[ \frac{c_{1,\theta \theta}(U_1, U_2; \theta)}{c_1(U_1, U_2; \theta)} - \frac{c_{1,\theta}(U_1, U_2; \theta) c_{1,\theta}(U_1, U_2; \theta)'}{c_1(U_1, U_2; \theta)^2} \right] + (1 - \delta_1) \delta_2 \left[ \frac{c_{2,\theta \theta}(U_1, U_2; \theta)}{c_2(U_1, U_2; \theta)} - \frac{c_{2,\theta}(U_1, U_2; \theta) c_{2,\theta}(U_1, U_2; \theta)'}{c_2(U_1, U_2; \theta)^2} \right] + (1 - \delta_1)(1 - \delta_2) \left[ \frac{C_{\theta \theta}(U_1, U_2; \theta)}{C(U_1, U_2; \theta)} - \frac{C_{\theta}(U_1, U_2; \theta) C_{\theta}(U_1, U_2; \theta)'}{C(U_1, U_2; \theta)^2} \right]. \]

Thus, \( S^*(\theta) = \mathbb{E}^*[A(\theta)] + V^*(\theta) \), where

\[ A(\theta) = \delta_1 \delta_2 \frac{c_{\theta \theta}(U_1, U_2; \theta)}{c(U_1, U_2; \theta)} + \delta_1(1 - \delta_2) \frac{c_{1,\theta \theta}(U_1, U_2; \theta)}{c_1(U_1, U_2; \theta)} \]

\[ + (1 - \delta_1) \delta_2 \frac{c_{2,\theta \theta}(U_1, U_2; \theta)}{c_2(U_1, U_2; \theta)} + (1 - \delta_1)(1 - \delta_2) \frac{C_{\theta \theta}(U_1, U_2; \theta)}{C(U_1, U_2; \theta)}. \]

Next, we show that if the assumed copula is the true copula, for any given \( C_1 \) and \( C_2 \),

\[ \mathbb{E}^*[A(\theta^*)] \mid C_1, C_2 = \int_0^\infty \int_0^\infty c_{\theta \theta}(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2. \quad (7) \]

Prove equation \( (7) \). Recall that if the assumed copula is correctly specified, i.e., \( \mathcal{C}^*(u_1, u_2) = \mathcal{C}(u_1, u_2; \theta^*) \), all the distribution functions of \((T_1, T_2)\) can be expressed as the assumed parametric copula function with \( \theta^* \). For example, the joint probability density function of \((T_1, T_2)\) can be expressed as \( c(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) \).

We express the conditional expectation \( \mathbb{E}^*[A(\theta^*)] \mid C_1, C_2 \) by further conditioning on four categories of different censoring status in the following:

\[ \mathbb{E}^*[A(\theta^*)] \mid C_1, C_2 \]

\[ = \mathbb{E}^* \left[ \frac{c_{\theta \theta}(U_1, U_2; \theta^*)}{c(U_1, U_2; \theta^*)} \mid \delta_1 = 1, \delta_2 = 1, C_1, C_2 \right] Pr(\delta_1 = 1, \delta_2 = 1 \mid C_1, C_2) \quad (8) \]

\[ + \mathbb{E}^* \left[ \frac{c_{1,\theta \theta}(U_1, U_2; \theta^*)}{c_1(U_1, U_2; \theta^*)} \mid \delta_1 = 1, \delta_2 = 0, C_1, C_2 \right] Pr(\delta_1 = 1, \delta_2 = 0 \mid C_1, C_2) \quad (9) \]

\[ + \mathbb{E}^* \left[ \frac{c_{2,\theta \theta}(U_1, U_2; \theta^*)}{c_2(U_1, U_2; \theta^*)} \mid \delta_1 = 0, \delta_2 = 1, C_1, C_2 \right] Pr(\delta_1 = 0, \delta_2 = 1 \mid C_1, C_2) \quad (10) \]

\[ + \mathbb{E}^* \left[ \frac{C_{\theta \theta}(U_1, U_2; \theta^*)}{C(U_1, U_2; \theta^*)} \mid \delta_1 = 0, \delta_2 = 0, C_1, C_2 \right] Pr(\delta_1 = 0, \delta_2 = 0 \mid C_1, C_2). \quad (11) \]

We will show that

- term \( (8) \) is equivalent to \( \int_0^{C_1} \int_0^{C_2} c_{\theta \theta}(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2; \)
- term \( (9) \) is equivalent to \( \int_0^{C_1} \int_{C_2} c_{\theta \theta}(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2; \)
- term \( (10) \) is equivalent to \( \int_0^{C_1} \int_0^{C_2} c_{\theta \theta}(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2; \)
- term \( (11) \) is equivalent to \( \int_0^{C_1} \int_{C_2} c_{\theta \theta}(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2; \)
• term (10) is equivalent to $\int_{C_1}^{\infty} \int_0^{C_2} \epsilon_{\theta \theta}(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2$;

• term (11) is equivalent to $\int_{C_1}^{\infty} \int_0^{C_2} \epsilon_{\theta \theta}(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2$.

Consequently, the sum of these four terms is $\int_0^{\infty} \int_0^{\infty} \epsilon_{\theta \theta}(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2$.

• For term (8), given $\delta_1 = 1$ and $\delta_2 = 1$, we have $T_1 \leq C_1$, $T_2 \leq C_2$, $X_1 = T_1$, and $X_2 = T_2$. In this case, both $U_1 = H_1(T_1)$ and $U_2 = H_2(T_2)$ are random variables. Thus, the conditional expectation in term (8) is taken w.r.t. the conditional distribution of $X_1, X_2$. However, for this case, $\epsilon(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2)$ is given. The conditional expectation in term (9) is taken w.r.t. the conditional distribution of $(T_1, T_2)$ given $T_1 \leq C_1, T_2 \leq C_2$, which can be expressed as $\frac{\epsilon(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2)}{Pr(T_1 \leq C_1, T_1 \leq C_2 | C_1, C_2)}$. It gives

$$\mathbb{E}^\ast \left[ \frac{\epsilon_{\theta \theta}(U_1, U_2; \theta^*)}{\epsilon(U_1, U_2; \theta^*)} \mid \delta_1 = 1, \delta_2 = 1, C_1, C_2 \right] = \int_0^{C_1} \int_0^{C_2} \epsilon_{\theta \theta}(H_1(t_1), H_2(t_2); \theta^*) \epsilon(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) \Pr(T_1 \leq C_1, T_1 \leq C_2 | C_1, C_2) dt_1 dt_2.$$ 

Thus, (8) = $\int_0^{C_1} \int_0^{C_2} \epsilon_{\theta \theta}(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2$.

• For term (9), given $\delta_1 = 1$ and $\delta_2 = 0$, we have $T_1 \leq C_1, T_2 > C_2$, $X_1 = T_1$, and $X_2 = C_2$. Note that in this case, $U_1 = H_1(T_1)$ is a random variable, but $U_2 = H_2(C_2)$ is a given constant since $C_2$ is given. The conditional expectation in term (9) is taken w.r.t. the conditional distribution of $T_1$ given $T_1 \leq C_1$ and $T_2 > C_2$, expressed as $-\epsilon_1(H_1(t_1), U_2; \theta^*) f_1(t_1)$, derived from $\frac{\partial \Pr(T_1 \leq C_1, T_1 > C_2)}{\partial t_1}$. Thus, the conditional expectation is given as

$$\mathbb{E}^\ast \left[ \epsilon_{\theta \theta}(U_1, U_2; \theta^*) \mid \delta_1 = 1, \delta_2 = 0, C_1, C_2 \right] = \int_0^{C_1} \int_0^{C_2} \epsilon_{\theta \theta}(H_1(t_1), U_2; \theta^*) -\epsilon_1(H_1(t_1), U_2; \theta^*) f_1(t_1) \Pr(T_1 \leq C_1, T_1 > C_2 | C_1, C_2) dt_1.$$

In addition, $\epsilon_1(H_1(t), U_2; \theta^*)$ can be expressed as $\int_{C_2}^{\infty} -\epsilon(H_1(t), H_2(t_2); \theta^*) f_2(t_2) dt_2$. Consequently, $\epsilon_{\theta \theta}(H_1(t), U_2; \theta^*) = \frac{\partial^2 \epsilon(H_1(t), U_2; \theta^*)}{\partial \theta^2}$ can be expressed as

$$\epsilon_1(H_1(t), U_2; \theta^*) = \int_{C_2}^{\infty} -\epsilon(H_1(t), H_2(t_2); \theta^*) f_2(t_2) dt_2$$

by interchanging the second-order derivative w.r.t. $\theta$ with the integral w.r.t. $t_2$. Thus, (9) = $\int_0^{C_1} \int_0^{\infty} \epsilon_{\theta \theta}(u_1, u_2; \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2$.

• For term (10), given $\delta_1 = 0$ and $\delta_2 = 1$, we can use similar arguments for term (9). However, for this case, $U_2 = H_2(T_2)$ is the only random variable, and $U_1 = H_1(C_1)$ is a constant. The conditional expectation in this term is taken w.r.t. the conditional distribution of $T_2$ given $T_1 > C_1$ and $T_2 < C_2$. Using the similar arguments with switching $T_1$ with $T_2$, we have (10) = $\int_{C_1}^{\infty} \int_0^{C_2} \epsilon_{\theta \theta}(u_1, u_2; \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2$. 

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• For term \( \{1\} \), given \( \delta_1 = 0 \) and \( \delta_2 = 0 \), we have \( T_1 > C_1, T_2 > C_2, X_1 = C_1, \) and \( X_2 = C_2. \) In this case, both \( U_1 = H_1(C_1) \) and \( U_2 = H_2(C_2) \) are constant, and thus, \( \frac{\psi(U_1, U_2; \theta^*)}{\psi(U_1, U_2; \theta^*)} \) is a constant. Since

\[
\psi(U_1, U_2; \theta^*) = Pr(T_1 > C_1, T_2 > C_2 \mid C_1, C_2)
= \int_{C_1}^{\infty} \int_{C_2}^{\infty} \psi(H_1(t_1), H_2(t); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2,
\]

we have

\[
\psi(U_1, U_2; \theta^*) = \frac{\partial^2}{\partial \theta \partial \theta'} \int_{C_1}^{\infty} \int_{C_2}^{\infty} \psi(H_1(t_1), H_2(t); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2
= \int_{C_1}^{\infty} \int_{C_2}^{\infty} \psi(U_1, U_2; \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2.
\]

Thus, \( \{1\} = \int_{C_1}^{\infty} \int_{C_2}^{\infty} \psi(U_1, U_2; \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2. \)

Gathering all the four terms together,

\[
\mathbb{E}^* [A(\theta^*) \mid C_1, C_2] = \int_0^\infty \int_0^\infty \psi(U_1, U_2; \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2
= \frac{\partial^2}{\partial \theta \partial \theta'} \int_0^\infty \int_0^\infty \psi(H_1(t_1), H_2(t_2); \theta^*) f_1(t_1) f_2(t_2) dt_1 dt_2.
\]

Since \( \int_0^\infty \int_0^\infty \psi(H_1(t_1), H_2(t_2); \theta) f_1(t_1) f_2(t_2) dt_1 dt_2 = 1, \mathbb{E}^* [A(\theta^*) \mid C_1, C_2] = 0, \) and thus, \( \mathbb{E}^* [\theta^*] = 0, \) i.e., \( \theta^* = \theta^* \).

THE END.

\section{Proof of Theorem 2}

To show \( R_n - tr \left[ \theta^* \right] = o_p(1), \) we need to first prove the consistency of \( \theta_n(\theta) \) and \( \hat{V}_n(\theta_n) \). Chen et al. (2010) has shown the consistency of \( \hat{V}_n(\theta_n) \) which requires Conditions A2 and A4 (i) & (ii) listed in their paper. To prove the consistency of \( \hat{V}_n(\theta_n) \), our Conditions II and III are analogous to Chen et al.’s those two conditions, respectively. Thus, following the same arguments in their paper, we can show that \( \sup_{\theta \in \Theta} n^{-1} \sum_{i=1}^n \| \ell_\theta (\theta, \hat{U}_i) \| = o_p(1). \) This together with the continuity of \( \ell_\theta (\theta, u_1, u_2) \) (our Condition II), and the consistency of the Kaplan-Meier estimator \( \hat{H}_t \) and the PMLE \( \hat{\theta}_n, \) leads to \( \| \hat{V}_n(\theta_n) - V^*(\theta^*) \| = o_p(1). \)

Condition A1 (i) in Chen et al. (2010) ensures that \( \theta^* \) is finite and non-singular. Thus, by Slutsky’s Theorem, it implies \( \theta^* \) converges in probability as \( n \to \infty. \)

THE END.
C Proof of Theorem 3

To prove this theorem, we need to prove the following lemma:

Lemma 1 Under Conditions I - IV,

\( 1 \) \( \sqrt{n} \left\{ \hat{S}_n(\theta_n) - S^*(\theta^*) \right\} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h_S(\theta^*, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}) + o_p(1), \) where \( h_S(\theta^*, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}) \) is a \( p \times p \) matrix with the \((j,k)\)-th element \( h_{S,j,k}(\theta^*, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}) \) given in equation (16) being independent random variables with mean 0.

\( 2 \) \( \sqrt{n} \left\{ \hat{V}_n(\theta_n) - V^*(\theta^*) \right\} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h_V(\theta^*, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}), \) where \( h_V(\theta^*, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}) \) is a \( p \times p \) matrix with the \((j,k)\)-th element \( h_{V,j,k}(\theta^*, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}) \) given in equation (17) being independent random variables with mean 0.

Proof of Lemma 1 Let \( S_{jk}(\theta^*) = \mathbb{E}^* \left[ -\ell_{\theta,\ell_{i1}}(\theta^*, U_{i1}, U_{i2}) \right] \) denote the \((j,k)\)-th element of \( S^*(\theta^*) \). Similarly, let \( \hat{S}_{n,jk}(\theta_n) = -n^{-1} \sum_{i=1}^{n} \ell_{\theta,\ell_{i1}}(\theta_n, U_{i1}, U_{i2}) \) denote the \((j,k)\)-th element of \( \hat{S}_n(\theta_n) \), \( j,k = 1, \ldots, p \). By the mean-value theorem, we have

\[
\hat{S}_{n,jk}(\theta_n) = -n^{-1} \sum_{i=1}^{n} \ell_{\theta,\ell_{i1}}(\theta^*, U_{i1}, U_{i2}) + \left[ -n^{-1} \sum_{i=1}^{n} \ell_{\theta,\ell_{i1}}(\theta, U_{i1}, U_{i2}) \right]' (\theta_n - \theta^*),
\]

where \( \theta \) lies on the line segment between \( \theta^* \) and \( \theta_n \).

Using the same arguments for the consistency of \( \hat{S}_n(\theta_n) \), by Condition IV (1) (analogous to Chen et. al’s Condition A2) and Condition IV (2) & (3) (analogous to Chen et. al’s Condition A4), we can show \( \sup_{\theta \in \Theta} n^{-1} \sum_{i=1}^{n} \| \ell_{\theta,\ell_{i1}}(\theta, U_{i1}, U_{i2}) I(U_{i1} \leq \eta) \| \) is asymptotically ignorable as \( \eta \to 0 \). This together with the continuity of \( \ell_{\theta,\ell_{i1}} \) (in our Condition IV (1)) and the consistency of the Kaplan-Meier estimate and PMLE \( \hat{\theta}_n \), we can show that \( \| n^{-1} \sum_{i=1}^{n} \ell_{\theta,\ell_{i1}}(\theta, U_{i1}, U_{i2}) - \mathbb{E}^* \left[ \ell_{\theta,\ell_{i1}}(\theta^*, U_{i1}, U_{i2}) \right] \| = o_p(1) \). Let \( M_{jk}(\theta^*) = \mathbb{E}^* \left[ \ell_{\theta,\ell_{i1}}(\theta^*, U_{i1}, U_{i2}) \right] \) (a \( p \times 1 \) vector). Because \( \hat{\theta}_n \) is \( \sqrt{n} \)-consistent, we have

\[
\hat{S}_n(\theta_n)_{jk} = -n^{-1} \sum_{i=1}^{n} \ell_{\theta,\ell_{i1}}(\theta^*, U_{i1}, U_{i2}) - M_{jk}(\theta^*)'(\hat{\theta}_n - \theta^*) + o_p(n^{-1/2}).
\]

Again applying the mean-value theorem on equation (12), we have

\[
\hat{S}_n(\theta_n)_{jk} - S^*_j(\theta^*) = \sum_{i=1}^{n} \left[ -\ell_{\theta,\ell_{i1}}(\theta^*, U_{i1}, U_{i2}) - S^*_j(\theta^*) \right] - n^{-1} \sum_{r=1}^{2} \sum_{i=1}^{n} \ell_{\theta,\ell_{i1},u_r}(\theta^*, U_{i1}, U_{i2})(U_{ir} - U_{i0}) - M_{jk}(\theta^*)'(\hat{\theta}_n - \theta^*) + o_p(n^{-1/2})
\]

where \( (U_{i1}, U_{i2}) \) lies on the linear segment between \( (\hat{U}_{i1}, \hat{U}_{i2}) \) and \( (U_{i1}, U_{i2}) \).
Based on the expansion of $\hat{\theta}_n$ around $\theta^*$ in \cite{Chen2010}, we have

$$\hat{\theta}_n - \theta^* = S^*(\theta^*)^{-1} \sum_{i=1}^n \left[ \ell_\theta(\theta^*, U_{i1}, U_{i2}) + W_1(\theta^*, X_{i1}, \delta_{i1}) + W_2(\theta^*, X_{i2}, \delta_{i2}) \right] + o_p(n^{-1/2}) \tag{14}$$

where for $r = 1, 2$, $W_r(\theta^*, X_{ir}, \delta_{ir}) = E^* \left[ \ell_{\theta,ir}(\theta^*, U_1, U_2) I_{ir}(X_r) \mid X_{ir}, \delta_{ir} \right]$ and

$$I_{ir}(X_r) = -H_r(X_r) \left[ \int_{-\infty}^{X_r} \frac{dN_{ir}(u)}{P_{n,r}(u)} - \int_{-\infty}^{X_r} \frac{I(X_{ir} \geq u) d\Lambda^*_n(u)}{P_{n,r}(u)} \right]$$

with $\Lambda_r(u) = -\log H_r(u)$ the cumulative hazard function of $T_{ir}$, $N_{ir}(u) = \delta_{ir} I(X_{ir} \leq u)$, $dN_{ir}(u) = N_{ir}(u) - N_{ir}(u-)$, and $P_{n,r}(u) = n^{-1} \sum_{k=1}^n Pr(X_{kr} \geq u)$. Using similar arguments for obtaining equation (14), under our Condition V (1) & (2) (analogous to Chen et. al’s Condition A3 (i) & (ii)) and Condition IV (2) (analogous to Chen et. al’s Condition A4 (i)), we can show that

$$n^{-1} \sum_{i=1}^n \ell_{\theta,ir}(\theta^*, \hat{U}_{i1}, \hat{U}_{i2})(\hat{U}_{ir} - U_{ir}) = n^{-1} \sum_{i=1}^n h_{S_{jk},r}(\theta^*, X_{ir}, \delta_{ir}) + o_p(n^{-1/2}), \tag{15}$$

where $h_{S_{jk},r}(\theta^*, X_{ir}, \delta_{ir}) = E^* \left[ \ell_{\theta,ir}(\theta^*, U_1, U_2) I_{ir}(X_r) \mid X_{ir}, \delta_{ir} \right]$. From equations (13), (14), and (15), we have

$$\sqrt{n} \left\{ \hat{S}_{n,jk}(\hat{\theta}_n) - S^*_{jk}(\theta^*) \right\} = \frac{1}{\sqrt{n}} \sum_{i=1}^n h_{S_{jk}}(\theta^*, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}) + o_p(1),$$

where

$$h_{S_{jk}}(\theta^*, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}) = \left[ -\ell_{\theta,1}(\theta^*, U_{i1}, U_{i2}) - S^*_{jk}(\theta^*) \right] - h_{S_{jk},1}(\theta^*, X_{i1}, \delta_{i1}) - h_{S_{jk},2}(\theta^*, X_{i2}, \delta_{i2}) - M_{jk}(\theta^*, S^*(\theta^*))^{-1} \left[ \ell_{\theta}(\theta^*, U_1, U_2) + W_1(\theta^*, X_{i1}, \delta_{i1}) + W_2(\theta^*, X_{i2}, \delta_{i2}) \right] \tag{16}$$

Let $V_{jk}^*(\theta^*)$ and $\hat{V}_{n,jk}(\hat{\theta}_n)$ denote the $(j,k)$-th element of $V^*(\theta^*)$ and $\hat{V}_n(\hat{\theta}_n)$. We apply the same techniques above, with our Condition III (1) (analogous to Chen et. al’s Condition A4 (ii)) and Condition V (3) & (4) (analogous to Chen et. al’s Condition A3 (i) & (ii)), we can show

$$\sqrt{n} \left\{ \hat{V}_{n,jk}(\hat{\theta}_n) - V_{jk}^*(\theta^*) \right\} = \frac{1}{\sqrt{n}} \sum_{i=1}^n h_{V_{jk}}(\theta^*, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}) + o_p(1),$$

where

$$h_{V_{jk}}(\theta^*, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}) = \left[ \ell_{\theta,1}(\theta^*, U_{i1}, U_{i2}) \ell_{\theta,2}(\theta^*, U_{i1}, U_{i2}) - V_{jk}^*(\theta^*) \right] + h_{V_{jk},1}(\theta^*, X_{i1}, \delta_{i1}) + h_{V_{jk},2}(\theta^*, X_{i2}, \delta_{i2}) + P_{jk}(\theta^*) S^*(\theta^*)^{-1} \left[ \ell_{\theta}(\theta^*, U_1, U_2) + W_1(\theta^*, X_{i1}, \delta_{i1}) + W_2(\theta^*, X_{i2}, \delta_{i2}) \right] \tag{17}$$
with \( h_{\nu_p}^{*}(\theta^*, X_{ir}, \delta_{ir}) = \mathbb{E}^* \left\{ \ell_{\theta_p,\phi} (\theta^*, U_1, U_2) \ell_{\theta_l,\phi}(\theta^*, U_1, U_2) + \ell_{\theta,\phi} (\theta^*, U_1, U_2) \ell_{\theta_l}(\theta^*, U_1, U_2) \right\} \star I_p(X_r) | X_{ir}, \delta_{ir} \) and \( P_{\theta^*}^{p} = \mathbb{E}^* \left[ \ell_{\theta_p,\phi} (\theta^*, U_1, U_2) \ell_{\theta_l,\phi}(\theta^*, U_1, U_2) + \ell_{\theta,\phi} (\theta^*, U_1, U_2) \ell_{\theta_l}(\theta^*, U_1, U_2) \right] \).

The proof of Lemma 1 ends.

**Proof of Theorem 3** Under the null hypothesis that the assumed copula function is the true copula, by Theorem 1, \( S_{\theta^*}^{*} = V^{*}(\theta^*) \) and thus
\[
tr \left[ S_{\theta^*}^{*} - V^{*}(\theta^*) \right] = tr \left[ I_p \right] = p,
\]
where \( I_p \) is a \( p \)-dimensional identity matrix. Thus, \( R_n - p \) can be expressed as
\[
R_n - p = tr \left[ \hat{S}_n(\hat{\theta}_n) - \hat{V}_n(\hat{\theta}_n) - S_{\theta^*}^{*} - V^{*}(\theta^*) \right].
\]
With algebraic derivations, we have
\[
\sqrt{n}(R_n - p) = \sqrt{n} tr \left[ \hat{S}_n(\hat{\theta}_n) - \hat{V}_n(\hat{\theta}_n) - S_{\theta^*}^{*} - V^{*}(\theta^*) \right] = tr \left[ S_{\theta^*}^{*} - V^{*}(\theta^*) \right] \approx \sqrt{n} tr \left[ \hat{S}_n(\hat{\theta}_n) - \hat{V}_n(\hat{\theta}_n) \right]
\]

Under the null hypothesis, \( S_{\theta^*}^{*} = V^{*}(\theta^*) \), the second term in equation (18) becomes
\[
tr \left[ S_{\theta^*}^{*} - V^{*}(\theta^*) \right] \approx \sqrt{n} \left\{ S_{\theta^*}^{*}(\hat{\theta}_n) - \hat{S}_n(\hat{\theta}_n) \right\}.
\]
The third term in equation (18) is \( O_p(1) \) because
\[
\left\| \hat{S}_n(\hat{\theta}_n) - S_{\theta^*}^{*} \right\| = O_p(1) \quad \text{shown in the proof of Theorem 2 (Appendix B)}
\]
and \( \left\| \hat{S}_n(\hat{\theta}_n) - S_{\theta^*}^{*} \right\| = O_p(n^{-1/2}) \) by Lemma 1.
Thus, we can write
\[
\sqrt{n}(R_n - p) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} h_R(X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}, \theta) + o_p(1),
\]
where
\[
h_R(\theta^*, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}) = tr \left[ S_{\theta^*}^{*} - \{ h_{\nu}^{*}(\theta^*, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}) - h_\delta(\theta, X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}) \} \right].
\]

By Central Limit Theorem for independent random variables, we can show that \( \sqrt{n}(R_n - p) \) converges in distribution to a normal random variable with mean 0 and variance \( \sigma^2 = Var[h_R(X_{i1}, X_{i2}, \delta_{i1}, \delta_{i2}, \theta)] \).

THE END.

**D Proof of Theorem 4**

To prove this theorem, we need to first prove the following lemma:

**Lemma 2** Under Condition I, \( \sup_{1 \leq i \leq n} || \hat{\theta}_n - \hat{\theta}_{(-i)} || = O_p(n^{-1}). \)
Proof of Lemma 2. The "out-of-sample" PMLE \( \hat{\theta}_{(-i)} \) is obtained by maximizing \( \sum_{s=1,s \neq i}^{n} \ell(\theta, \hat{U}_{s1}, \hat{U}_{s2}) \), i.e., \( \sum_{s=1,s \neq i}^{n} \ell_{\theta}(\hat{\theta}_{(-i)}, \hat{U}_{s1}, \hat{U}_{s2}) = 0 \). Apply the mean-value theorem, we have

\[
0 = \sum_{s=1,s \neq i}^{n} \ell_{\theta}(\hat{\theta}_{(-i)}, \hat{U}_{s1}, \hat{U}_{s2}) = \sum_{s=1,s \neq i}^{n} \ell_{\theta}(\hat{\theta}_{n}, \hat{U}_{s1}, \hat{U}_{s2}) + \sum_{s=1,s \neq i}^{n} \ell_{\theta}(\hat{\theta}, \hat{U}_{s1}, \hat{U}_{s2})(\hat{\theta}_{(-i)} - \hat{\theta}_{n})
\]

where \( \hat{\theta} \) lies in the linear segment between \( \hat{\theta}_{(-i)} \) and \( \hat{\theta}_{n} \). Since \( \sum_{s=1}^{n} \ell_{\theta}(\hat{\theta}_{n}, \hat{U}_{s1}, \hat{U}_{s2}) = 0 \) (\( \hat{\theta}_{n} \) is the PMLE using all the observations), we have

\[
\hat{\theta}_{n} - \hat{\theta}_{(-i)} = \mathbf{S}_{(-i)}(\hat{\theta})^{-1}n^{-1}\ell_{\theta}(\hat{\theta}_{n}, \hat{U}_{1}, \hat{U}_{2})
\]

where \( \mathbf{S}_{(-i)}(\hat{\theta}) = -n^{-1} \sum_{s=1,s \neq i}^{n} \ell_{\theta}(\hat{\theta}, \hat{U}_{s1}, \hat{U}_{s2}) \). Thus,

\[
\sup_{1 \leq i \leq n} \| \hat{\theta}_{n} - \hat{\theta}_{(-i)} \| \leq n^{-1} \sup_{1 \leq i \leq n} \| \mathbf{S}_{(-i)}(\hat{\theta})^{-1} \| \times \sup_{1 \leq i \leq n} \| \ell_{\theta}(\hat{\theta}_{n}, \hat{U}_{1}, \hat{U}_{2}) \|.
\]

Using the same arguments for proving the consistency of \( \mathbf{S}_{n}(\hat{\theta}) \), we can prove that as \( n \to \infty \), \( \mathbf{S}_{(-i)}(\hat{\theta}) \to \mathbf{S}^{*}(\theta^{*}) \) in probability. Chen et. al’s Condition A1 (ii) (boundedness for the eigenvalues of \( \mathbf{S}^{*}(\theta^{*}) \)) ensures that \( \sup_{1 \leq i \leq n} \| \mathbf{S}_{(-i)}(\hat{\theta})^{-1} \| < \infty \). In addition, Chen et. al’s Condition A3 ensures that \( \sup_{1 \leq i \leq n} \| \ell_{\theta}(\hat{\theta}_{n}, \hat{U}_{1}, \hat{U}_{2}) \| = O_{p}(1) \), and thus, \( \sup_{1 \leq i \leq n} n^{-1} \ell_{\theta}(\hat{\theta}_{n}, \hat{U}_{1}, \hat{U}_{2}) = O_{p}(1) \). It leads to \( \sup_{1 \leq i \leq n} \| \hat{\theta}_{n} - \hat{\theta}_{(-i)} \| = O_{p}(n^{-1}) \).

The proof of Lemma 2 ends.

Proof of Theorem 4. Recall that the PIOS test statistic is defined as

\[
T_{n} = \sum_{i=1}^{n} \ell_{\theta}(\hat{\theta}_{n}, \hat{U}_{1i}, \hat{U}_{2i}) - \sum_{i=1}^{n} \ell_{\theta}(\hat{\theta}_{(-i)}, \hat{U}_{1i}, \hat{U}_{2i}).
\]

Applying the mean value theorem on \( \ell_{\theta}(\hat{\theta}_{(-i)}, \hat{U}_{1i}, \hat{U}_{2i}) \), we have

\[
T_{n} = -n \sum_{i=1}^{n} \ell_{\theta}(\hat{\theta}_{n}, \hat{U}_{1i}, \hat{U}_{2i})'(\hat{\theta}_{(-i)} - \hat{\theta}_{n}) - \frac{1}{2} n \sum_{i=1}^{n} \ell_{\theta}((\hat{\theta}, \hat{U}_{1i}, \hat{U}_{2i}) (\hat{\theta}_{(-i)} - \hat{\theta}_{n})^{2},
\]

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where $\hat{\theta}$ lies on the linear segment between $\hat{\theta}_{(-i)}$ and $\hat{\theta}_n$. Plugging in equation (20), we have

$$T_n = n^{-1} \sum_{i=1}^{n} \ell_\theta(\hat{\theta}_n, \hat{U}_i, \hat{U}_2) \left\{ \hat{S}_{(-i)}(\theta) \right\}^{-1} \ell_\theta(\hat{\theta}_n, \hat{U}_i, \hat{U}_2) - \frac{1}{2} \sum_{i=1}^{n} \ell_\theta(\theta, \hat{U}_i, \hat{U}_2) \left( \hat{\theta}_{(-i)} - \hat{\theta}_n \right)^2$$

$$= tr \left[ \hat{S}_{(-i)}(\theta)^{-1} \left\{ n^{-1} \sum_{i=1}^{n} \ell_\theta(\theta, \hat{U}_i, \hat{U}_2) \ell_\theta(\hat{\theta}_n, \hat{U}_i, \hat{U}_2) \right\} \right] - \frac{1}{2} \left\{ n^{-1} \sum_{i=1}^{n} \ell_\theta(\theta, \hat{U}_i, \hat{U}_2) \right\} n \left( \hat{\theta}_{(-i)} - \hat{\theta}_n \right)^2. $$

Thus,

$$T_n - R_n$$

$$= tr \left[ \left\{ \hat{S}_{(-i)}(\theta)^{-1} - \hat{S}_n(\theta)^{-1} \right\} V_n(\theta) \right] - \frac{1}{2} \left\{ n^{-1} \sum_{i=1}^{n} \ell_\theta(\theta, \hat{U}_i, \hat{U}_2) \right\} n \left( \hat{\theta}_{(-i)} - \hat{\theta}_n \right)^2. $$

(21)

In the proof of Lemma 2, we have shown that $\|\hat{S}_{(-i)}(\theta) - S^*(\theta^*)\| = o_p(1)$. In addition, because $\|\hat{S}_n(\theta) - S^*(\theta^*)\| = o_p(1)$, we have $\|\hat{S}_{(-i)}(\theta) - \hat{S}_n(\theta)\| = o_p(1)$, and consequently, the first term in equation (21) is $o_p(1)$. For the second term, following similar arguments, we can show $\|n^{-1} \sum_{i=1}^{n} \ell_\theta(\theta, \hat{U}_i, \hat{U}_2) - S^*(\theta^*)\| = o_p(1)$. Together with $\sup_{1 \leq i \leq n} \|\hat{\theta}_n - \hat{\theta}_{(-i)}\| = O_p(n^{-1})$, the second term is $O_p(n^{-1})$. Combining the two terms, we have $|T_n - R_n| = o_p(1)$.

THE END.