Supersymmetric $t$-$J$ models with long-range interactions: thermodynamics and criticality

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Received 29 March 2019
Accepted for publication 7 May 2019
Published 25 July 2019

Abstract. We analyze the thermodynamics and the critical behavior of the supersymmetric $\text{su}(m)$ $t$-$J$ model with long-range interactions. Using the transfer matrix formalism, we obtain a closed-form expression for the free energy per site both for a finite number of sites and in the thermodynamic limit. Our approach, which is different from the usual ones based on the asymptotic Bethe ansatz and generalized exclusion statistics, can in fact be applied to a large class of models whose spectrum is described in terms of supersymmetric Young tableaux and their associated Haldane motifs. In the simplest and most interesting $\text{su}(2)$ case, we identify the five ground state phases of the model and derive the complete low-temperature asymptotic series of the free energy per site, the magnetization and charge densities, and their susceptibilities. We verify the model’s characteristic spin-charge separation at low temperatures, and show that it holds to all orders in the asymptotic expansion. Using the low-temperature asymptotic expansions of the free energy, we also analyze the critical behavior of the model in each of its ground state phases. While the standard $\text{su}(1|2)$ phase is described by two independent conformal field theories...
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(CFTs) with central charge $c = 1$ in correspondence with the spin and charge sectors, we find that the low-energy behavior of the $\text{su}(2)$ and $\text{su}(1|1)$ phases is that of a single $c = 1$ CFT. We show that the model exhibits an even richer behavior on the boundary between zero-temperature phases, where it can be non-critical but gapless, critical in the spin sector but not in the charge one, or critical with central charge $c = 3/2$.

**Keywords:** integrable spin chains and vertex models, quantum criticality, solvable lattice models

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1. Introduction

The $\text{su}(m)$ $t$-$J$ model is one of the most intensively studied lattice models of strongly correlated fermions, due to its relevance for the theoretical understanding of high-temperature superconductivity and as one of the simplest quantum systems exhibiting spin-charge separation \cite{1, 2, 3, 4}. The sites of this model can be occupied by at most one charged fermion with $m$ internal degrees of freedom, which can hop between contiguous lattice sites and interacts with its nearest neighbors through spin exchange and charge repulsion. The one-dimensional $t$-$J$ model is of particular interest, since it is supersymmetric and exactly solvable through the nested Bethe ansatz when its two parameters are suitably related \cite{1, 5, 6, 7, 8, 9}.

In a recent paper \cite{10} we have computed in closed form the partition function of the supersymmetric $\text{su}(m)$ $t$-$J$ model with long-range interactions introduced by Kuramoto and Yokoyama \cite{11, 12}. The lattice sites of the latter model are equispaced on a circle, and each fermion can now interact with any other and hop among any two sites. Moreover, both the interaction strength and the hopping amplitude are inversely proportional to the square of the chord distance between the corresponding sites. The supersymmetric character of the $\text{su}(m)$ Kuramoto–Yokoyama (KY) model can be established by mapping it to a suitable modification of the $\text{su}(1|m)$ Haldane–Shastry (HS) spin chain \cite{13}. This connection can in fact be exploited to fully determine the spectrum of the former model in terms of supersymmetric motifs and their corresponding Young tableaux \cite{10, 14}.

The thermodynamics of the supersymmetric KY model have been actively investigated ever since their introduction. In fact, in the original reference \cite{11} the low-temperature asymptotic behavior of the magnetic and charge susceptibilities was determined by means of the asymptotic Bethe ansatz (see, e.g. \cite{15}). A few years later, the thermodynamics of the $\text{su}(m)$ KY model at arbitrary temperature in the $N \to \infty$ limit was studied by Kato and Kuramoto \cite{16} applying Polychronakos’s freezing trick \cite{17} to the $\text{su}(1|m)$ supersymmetric spin Sutherland model \cite{18, 19}. This method, which is rather involved, requires first establishing the equivalence of the latter model to a system of non-interacting $\text{su}(1|m)$ particles and then modding out the contribution of the dynamical degrees of freedom. Moreover, it essentially relies on specific properties of the HS chain such as its equivalence to a model of free particles with generalized momenta obeying fractional statistics. On a more practical level, the formula for the grand potential obtained by Kato and Kuramoto depends on a function which must be determined by solving an implicit equation with an appropriate choice of branch.

In this paper we propose a novel direct method for analyzing the thermodynamics of the supersymmetric KY model, which can be applied to a wide range of models with (complete or broken) Yangian symmetry. We shall show how a formula for the grand potential of these models, akin to Kato and Kuramoto’s for the long-range supersymmetric $t$-$J$ model, emerges in a transparent way from their partition function without requiring that they be described by generalized pseudo-momenta or fractional statistics. In the simplest and most interesting case $m = 2$, the corresponding implicit equation is quadratic and can therefore be explicitly solved, which leads to a new closed-form expression for the grand potential of the spin 1/2 KY model.
The starting point in our method is the explicit formula for the partition function of the KY model with an arbitrary (finite) number of sites $N$ obtained in [10], which can be recast into that of a related inhomogeneous vertex model. This key observation makes it feasible to apply the transfer matrix method in [20, 21] to derive a closed-form expression for the grand potential of the su$(m)$ KY model in the thermodynamic limit in terms of the largest eigenvalue (in modulus) of a site-dependent transfer matrix. The characteristic equation of this matrix, when expressed in an appropriate variable, is precisely the implicit equation deduced by Kato and Kuramoto (henceforth referred to as the KK equation). In fact, our method can be applied to any model described by an effective inhomogeneous vertex model, whose energy function is expressible in terms of a dispersion relation and the supersymmetric Young tableaux associated with finite-dimensional representations of the Yangian acting on tame modules [22–24]. By varying the dispersion relation we can derive the thermodynamics of a large class of (partially or totally) Yangian-invariant systems, which includes not only the KY model (or, equivalently, the su$(1|m)$ supersymmetric HS chain) but other well-known lattice models like the Polychronakos–Frahm (PF) [17, 25] or the Frahm–Inozemtsev (FI) [26, 27] spin chains. The grand potential of all of these models can again be expressed in terms of the largest eigenvalue of a suitable transfer matrix. In the su$(1|m)$ case, we explicitly show that the characteristic equation of this transfer matrix is equivalent to a generalized KK equation for a system of one boson and $m$ fermions. This strongly suggests that the models in this class can be reformulated as systems of ‘free’ su$(1|m)$ particles (holons and spinons) interacting via appropriate fractional statistics. So far, this conjecture has only been proved by an ad hoc method for the su$(1|2)$ case in [16].

A further aim of this paper is to take advantage of the explicit formula for the grand potential of the spin 1/2 supersymmetric KY model in order to analyze in detail the low-temperature behavior of its main thermodynamic functions, going beyond the first-order calculations in [16, 28]. To this end, we first determine the zero-temperature limit of the magnetization and charge densities for all values of the magnetic field $h$ and the charge chemical potential $\mu$. In this way we identify the model’s five ground state phases in $(h, \mu)$ space, characterized by their content of holes and fermions of both species. We then compute the asymptotic expansion of the grand potential to all orders in $T$, which turns out to be different in each of these ground state phases. From the asymptotic series of the grand potential we derive analogous infinite asymptotic expansions for the main thermodynamic functions of interest, namely the magnetization and charge densities and their respective susceptibilities. Apart from recovering the lowest-order results of [28], we show that the strong spin-charge separation characteristic of the model under consideration is a non-perturbative property, in the sense that it persists at all orders in the low-temperature asymptotic expansion of both susceptibilities. We also use our low-temperature asymptotic expansions to briefly analyze the critical behavior of the spin 1/2 KY model for arbitrary values of the magnetic field and the charge chemical potential. In the genuinely su$(1|2)$ phase we confirm the well-known result that the model is described by two independent $c = 1$ conformal field theories (CFTs), one for each of the spin and charge sectors [29]. On the other hand, in the su$(2)$ and su$(1|1)$ phases we interestingly find that the model, while still critical, is instead described by a single $c = 1$ CFT. The situation is even more complex on the boundary.
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between ground state phases, where the model can be non-critical but gapless, critical in the spin sector but not in the charge one, or have fractional central charge \( c = \frac{3}{2} \).

The paper is organized as follows. In section 2 we introduce the model and recall from [10] its precise equivalence to (a modification of) the \( \text{su}(1|m) \) supersymmetric HS chain. We then exploit this equivalence to obtain explicit formulas for the free energy per site and the main thermodynamic functions in the thermodynamic limit. In section 3 we discuss the derivation of the KK equations and their generalizations by means of the transfer matrix formalism. In the remaining sections we focus on the simplest and most interesting case, namely the spin 1/2 supersymmetric KY model. More precisely, in section 4 we obtain exact expressions for the zero-temperature magnetization and charge densities, and apply them to identify the different ground state phases in terms of the magnetic field strength and the charge chemical potential. Section 5 is devoted to the derivation of the complete asymptotic series of the free energy per site in each of the ground state phases, which we then use to analyze in detail the model’s critical behavior. In section 6 we compute the corresponding series for the magnetization per site, the charge density and their susceptibilities, and discuss the spin-charge separation characteristic of the model under study. We present our conclusions and outline some future developments in section 7. The paper ends with three appendices in which we deal with several technical questions arising in the derivation of the asymptotic series in section 5.

2. Free energy of the \( \text{su}(m) \) Kuramoto–Yokoyama model

2.1. The model

As shown in our previous paper [10], the Hamiltonian of the supersymmetric \( \text{su}(m) \) KY model can be written as

\[
H_0 = \frac{t \pi^2}{N^2} \sum_{i<j} \sin^{-2} \left( \frac{\pi}{N} (i - j) \right) \mathcal{P} \left[ -\sum_{\sigma} \left( c_{i\sigma}^\dagger c_{j\sigma} + c_{j\sigma}^\dagger c_{i\sigma} \right) + 2 \mathbf{T}_i \cdot \mathbf{T}_j - \left( 1 - \frac{1}{m} \right) n_i n_j \right] \mathcal{P}.
\]

(2.1)

In the latter equations \( c_{i\sigma}^\dagger \) (respectively \( c_{i\sigma} \)) denotes the operator creating (respectively destroying) a fermion of type \( \sigma \in \{1, \ldots, m\} \) at site \( i \) and \( n_i = \sum_{\sigma} n_{i\sigma} \), where \( n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma} \) is the total number of fermions at site \( i \). The operator \( \mathcal{P} \) is the projector onto single-occupancy states, in which each site is occupied by at most one fermion. Finally, \( \mathbf{T}_i \equiv (T_{i1}^1, \ldots, T_{im}^{m^2-1}) \), where \( T_{ir}^r \) is the \( r \)th \( \text{su}(m) \) Hermitian generator in the fundamental representation acting on the \( \sigma \)th site. More precisely,

\[
T_{ir}^r = \sum_{\sigma, \sigma'} T_{r\sigma\sigma'}^r c_{i\sigma}^\dagger c_{i\sigma'},
\]

(2.2)

where the complex numbers \( T_{r\sigma\sigma'}^r \) are the matrix elements of the \( r \)th (Hermitian) generator of \( \text{su}(m) \) in the fundamental representation with the normalization

\[5\]

Here and in what follows, unless otherwise stated, sums and products over Latin indexes run over the set \( 1, \ldots, N \) while Greek indices range from 1 to \( m \).

https://doi.org/10.1088/1742-5468/ab25e0

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\[
\text{tr}(T^r T^s) = \frac{1}{2} \delta_{rs}.
\]

Thus the first term between braces in equation (2.1) accounts for the hopping of fermions between sites \(i\) and \(j\), while the last two terms respectively model the spin (exchange) and charge interaction between the latter sites.

As shown in [10], the KY Hamiltonian (2.1) can be mapped to (a suitable modification of) the \(\text{su}(1|m)\) HS spin chain Hamiltonian by identifying the holes of the KY model with the bosons of the HS spin chain. Indeed, the Hilbert space of the latter chain is \(\mathcal{H}_i = \bigotimes_{i=1}^N \hat{\mathcal{H}}_i\), where \(\hat{\mathcal{H}}_i\) is the linear span of the one-particle states \(b_\sigma^i |\Omega\rangle_i, f_\sigma^i |\Omega\rangle_i\) \((\sigma = 1, \ldots, m)\), \(b_\sigma^i\) and \(f_\sigma^i\) are the operators creating respectively a boson and a fermion of type \(\sigma\) at the \(i\)th site and \(|\Omega\rangle_i\) is the vacuum in \(\hat{\mathcal{H}}_i\). Similarly, let \(\mathcal{H} = \bigotimes_{i=1}^N \hat{\mathcal{H}}_i\), denote the Hilbert space of the original model (2.1), where \(\hat{\mathcal{H}}_i\) is the space spanned by its vacuum \(|\Omega\rangle_i\) and the one-particle states \(c_\sigma^i |\Omega\rangle_i\). The unitary mapping \(\varphi: \mathcal{H} \rightarrow \hat{\mathcal{H}}\) defined by

\[
\varphi |\Omega\rangle_i = b_\sigma^i |\Omega\rangle_i, \quad \varphi \big( c_\sigma^i |\Omega\rangle_i \big) = f_\sigma^i |\Omega\rangle_i
\]

induces a natural way of associating to each linear operator \(A: \mathcal{H} \rightarrow \mathcal{H}\) a corresponding linear operator \(\hat{A} = \varphi A \varphi^{-1} = \varphi A \varphi^\dagger\) acting on \(\hat{\mathcal{H}}\). It is shown in [10] that under this correspondence the Hamiltonian \(H_0\) in equation (2.1) is transformed into

\[
\hat{H}_0 = \frac{t \pi^2}{N^2} \left\{ \sum_{i<j} \sin^{-2} \left( \frac{\pi}{N} (i - j) \right) (1 - P_{ij}^{(1|m)}) - \frac{1}{3} (N^2 - 1) \mathcal{F} \right\},
\]

where \(\mathcal{F} \equiv \sum_i \hat{n}_i\) is the total number of \(\text{su}(1|m)\) fermions and \(P_{ij}^{(1|m)}\) denotes the \(\text{su}(1|m)\) supersymmetric permutation operator. Recall that the action of \(P_{ij}^{(1|m)}\) on the canonical basis of \(\hat{\mathcal{H}}\) is given by

\[
P_{ij}^{(1|m)} |\cdots \sigma_i \cdots \sigma_j \cdots \rangle = \epsilon(\sigma) |\cdots \sigma_j \cdots \sigma_i \cdots \rangle,
\]

where \(\epsilon(\sigma)\) is 1 (respectively \(-1\)) if \(\sigma_i = \sigma_j = 0\) (respectively \(\sigma_i, \sigma_j > 1\)), while for \(\sigma_i, \sigma_j = 0\) and \(\sigma_i \neq \sigma_j\), it is equal to the number of fermionic spins \(\sigma_k\) with \(i + 1 \leq k \leq j - 1\). Thus the first term in \(\hat{H}_0\) coincides with the Hamiltonian of the \(\text{su}(1|m)\) supersymmetric HS chain [13, 30–33], as we had anticipated.

### 2.2. Free energy

We shall next explain how to compute in closed form the grand potential of the \(\text{su}(m)\) KY model (2.1) by exploiting its equivalence with the \(\text{su}(1|m)\) HS spin chain Hamiltonian (2.4). In the thermodynamic limit, this is equivalent to computing the free energy of the Hamiltonian

\[
H = H_0 - \frac{1}{2} \sum_{\sigma=1}^{m-1} h_\sigma (n^\sigma - n^{m}) - \mu_c \sum_\sigma n^\sigma \equiv H_0 + H_1,
\]

where \(n^\sigma \equiv \sum_i n_{i\sigma}\) denotes the total number of fermions of type \(\sigma\). The last term in \(H_1\) is the chemical potential of the fermions (or, equivalently, of the electric charge),

<https://doi.org/10.1088/1742-5468/ab25e0>
while the first one can be interpreted as arising from the interaction with an external \( su(m) \) magnetic field with strengths \( h_1, \ldots, h_{m-1} \) along each (Hermitian) generator of the standard Cartan subalgebra of \( su(m) \) (see [10] for more details). In particular, for \( m = 2 \) the term \( -(h_1/2)(n^1 - n^2) \) equals \( -h_1 S^z \), where \( S^z \) is the \( z \) component of the total spin operator. The \( su(1|m) \) spin chain Hamiltonian \( \hat{H} \) equivalent to \( H \) under the mapping (A.2) is \( \hat{H} = \hat{H}_0 + \hat{H}_1 \), where

\[
\hat{H}_1 = -\frac{1}{2} \sum_{\sigma=1}^{m-1} h_\sigma (N_\sigma - N_m) - \mu_c \mathcal{F}
\]

and \( N_\sigma \equiv \hat{n}^\sigma \) is the total number of \( su(1|m) \) fermions of type \( \sigma \). We can thus write

\[
\hat{H} = J H_{\text{HS}}^{(1|m)} - \frac{1}{2} \sum_{\sigma=1}^{m-1} h_\sigma (N_\sigma - N_m) - (t_0 + \mu_c) \mathcal{F},
\]

where

\[
H_{\text{HS}}^{(1|m)} = \frac{1}{2} \sum_{i<j} \sin^{-2}\left(\frac{\pi}{N} (i - j)\right) (1 - P_{ij}^{(1|m)})
\]

and

\[
J = \frac{2t \pi^2}{N^2}, \quad t_0 = \frac{t \pi^2}{3N^2} (N^2 - 1).
\]

The Hamiltonian \( \hat{H} \) can be more concisely expressed as

\[
\hat{H} = J H_{\text{HS}}^{(1|m)} - \sum_\sigma \mu_\sigma N_\sigma,
\]

where \( \mu_\sigma \) is the chemical potential of the fermion of type \( \sigma \), given by

\[
\mu_\sigma = \frac{1}{2} h_\sigma + \mu_c + t_0, \quad 1 \leq \sigma \leq m - 1;
\]

\[
\mu_m = -\frac{1}{2} \sum_{\sigma=1}^{m-1} h_\sigma + \mu_c + t_0.
\]

As shown in [10] and [21], the spectrum of the \( su(1|m) \) spin chain (2.8), and hence of the equivalent Hamiltonian (2.5), can be generated from the formula

\[
E(s) = J \sum_{i=1}^{N-1} \delta(s_i, s_{i+1}) i (N - i) - \sum_i \mu_{s_i},
\]

where \( \mu_0 \equiv 0, \ s \in \{0, \ldots, m\}^N \) and \( \delta(s, s') \) is defined by

\[
\delta(s, s') = \begin{cases} 
1, & s > s' \text{ or } s = s' > 0 \\
0, & s < s' \text{ or } s = s' = 0.
\end{cases}
\]

The vectors \( \delta(s) \) with components \( \delta(s_i, s_{i+1}) \) \( 1 \leq i \leq N - 1 \) in equation (2.11) are \( su(1|m) \) motifs [13, 14, 34]. In fact, the first sum in equation (2.11) can be interpreted
as the energy of a one-dimensional vertex model with \( N + 1 \) vertices \( 0, \ldots, N \) joined by \( N \) bonds with values \( s_1, \ldots, s_N \in \{0, \ldots, m\} \), the energy associated with the \( i \)th vertex being equal to \( \delta(s_i, s_{i+1})i(N - i) \) [14].

Equation (2.11) is the key ingredient in the exact computation of the free energy in the thermodynamic limit through the (site-dependent) transfer matrix method developed in [21]. Indeed, from equation (2.11) it follows that the partition function can be expressed as

\[
Z = \text{tr}(A(x_0) \cdots A(x_{N-1})),
\]

where \( x_k \equiv k/N \) and the \( (m + 1) \times (m + 1) \) transfer matrix \( A(x) \) has matrix elements

\[
A_{\alpha\beta}(x) = q^{K\varepsilon(x)\delta(\alpha, \beta) - \frac{1}{2}(\mu_{\alpha} + \mu_{\beta})}, \quad 0 \leq \alpha, \beta \leq m.
\]

In the latter equation we have defined

\[
q = e^{-1/T}, \quad \varepsilon(x) = x(1 - x), \quad K = 2t\pi^2 > 0,
\]

and as above we have taken (without loss of generality) \( \mu_0 \equiv 0 \). It can then be shown that in the thermodynamic limit \( N \to \infty \) equation (2.13) yields the following closed-form expression for the free energy per site of the Hamiltonian (2.5):

\[
f = -2T \int_0^{1/2} \log \lambda_1(x) \, dx,
\]

where \( \lambda_1(x) \) is the largest eigenvalue in modulus of the matrix \( A(x) \) (simple and positive, by the Perron–Frobenius theorem). In fact, in the next section we shall explain how the latter formula leads to the expression for the grand potential derived by a more laborious method in [16].

From now on we shall restrict ourselves to the su(2) case, for which the eigenvalue \( \lambda_1(x) \) can be computed in closed form. Indeed, in this case the matrix \( A(x) \) is given by

\[
A(x) = \begin{pmatrix}
1 & q^{-\mu_1/2} & q^{-\mu_2/2} \\
q^{K\varepsilon(x) - \mu_1/2} & q^{K\varepsilon(x) - \mu_1} & q^{-(\mu_1 + \mu_2)/2} \\
q^{K\varepsilon(x) - \mu_2/2} & q^{K\varepsilon(x) - (\mu_1 + \mu_2)/2} & q^{K\varepsilon(x) - \mu_2}
\end{pmatrix}.
\]

By equations (2.7), (2.9) and (2.10), the chemical potentials \( \mu_\sigma \) are given (in the thermodynamic limit) by

\[
\mu_\sigma = (-1)^{\sigma+1} \frac{h}{2} + \mu,
\]

where \( h \equiv h_1 \) and \( \mu \equiv \mu_c + K/6 \). Taking these relations into account, the Perron–Frobenius eigenvalue of the matrix \( A(x) \) reads

\[
\lambda_1(x) = e^{\beta(\mu - K\varepsilon(x))} \left[ b(x) + \sqrt{b^2(x) + e^{K\beta\varepsilon(x)} - 1} \right],
\]

where \( \beta \equiv 1/T \) and

\[
b(x) = \frac{1}{2} e^{\beta(K\varepsilon(x) - \mu)} + \cosh \left( \frac{\beta h}{2} \right).
\]

https://doi.org/10.1088/1742-5468/ab25e0
The previous equations yield the following exact explicit formula for the free energy per site of the su(2) KY model in the presence of a magnetic field $h$ and charge chemical potential $\mu_c$:

$$
f = -\mu_c - 2T \int_0^{1/2} \log \left[ b(x) + \sqrt{b^2(x) + e^{K\beta \varepsilon(x)}} - 1 \right] dx.
$$

(2.18)

The magnetization (per site) $m_s = \langle n^1 - n^2 \rangle / (2N)$ and the charge (fermion) density $n_c = \langle n^1 + n^2 \rangle / N$ (where $\langle \cdot \rangle$ denotes thermal average) are easily computed in closed form differentiating the latter equation, namely

$$
m_s = -\frac{\partial f}{\partial h} = \sinh\left( \frac{\beta h}{2} \right) \int_0^{1/2} D(x)^{-1/2} dx,
$$

(2.19)

$$
n_c = -\frac{\partial f}{\partial \mu_c} = 1 - \int_0^{1/2} D(x)^{-1/2} e^{\beta (K \varepsilon(x) - \mu)} dx,
$$

(2.20)

with

$$
D(x) \equiv b^2(x) + e^{K\beta \varepsilon(x)} - 1.
$$

The corresponding susceptibilities $\chi_s$ and $\chi_m$ are then given by

$$
\chi_s = \frac{\partial m_s}{\partial h} = \frac{\beta}{2} \int_0^{1/2} e^{\beta (K \varepsilon(x) - \mu)} D(x)^{-3/2} \left[ 1 + \frac{1}{2} \sinh^2 \left( \frac{\beta h}{2} \right) \right. \\
+ \cosh \left( \frac{\beta h}{2} \right) \left( e^{\beta \mu} + \frac{1}{4} e^{\beta (K \varepsilon(x) - \mu)} \right) \right] dx,
$$

(2.21)

$$
\chi_m = \frac{\partial n_c}{\partial \mu_c} = \beta \int_0^{1/2} e^{\beta (K \varepsilon(x) - \mu)} D(x)^{-3/2} \left[ \sinh^2 \left( \frac{\beta h}{2} \right) \\
+ \frac{1}{2} e^{\beta (K \varepsilon(x) - \mu)} \cosh \left( \frac{\beta h}{2} \right) + e^{K \beta \varepsilon(x)} \right] dx.
$$

(2.22)

Other thermodynamic functions, like the energy $u$, entropy $s = \beta (u - f)$ and specific heat (per site) $c_V = (\partial u) / (\partial T)$, are easily computed from equation (2.18). For instance,

$$
u = \frac{\partial}{\partial \beta} \langle \beta f \rangle = -\mu_c - \int_0^{1/2} D(x)^{-1/2} \left[ h \sinh \left( \frac{\beta h}{2} \right) + (K \varepsilon(x) - \mu) e^{\beta (K \varepsilon(x) - \mu)} \right] dx.
$$

(2.23)

### 3. Derivation of a generalized Kato–Kuramoto equation through the transfer matrix formalism

As explained in the Introduction, Kato and Kuramoto [16] obtained an expression for the grand potential per spin $\omega$ of the KY Hamiltonian $H_0$ in equation (2.1) in terms of a function implicitly determined by an algebraic equation that we have named the Kato–Kuramoto equation. It should be noted that the deduction of this equation in [16] requires that (in our notation) the dispersion relation $\varepsilon(x)$ of the model be given by

https://doi.org/10.1088/1742-5468/ab25e0
equation (2.15). In this section we shall first of all show how the KK equation emerges in a transparent way from equation (2.16) for the free energy per spin of the su(m) KY model with the general chemical potential term H1 in equation (2.5). More importantly, we shall outline how this equation can be generalized to a large class of solvable lattice models with (complete or partial) Yangian invariance, including the supersymmetric PF and FI spin chains.

To this end, let us denote by \( P_m(\lambda) = \det(\lambda - A(x)) \) the characteristic polynomial of the transfer matrix \( A(x) \) in equation (2.14), where we have suppressed the dependence of \( P_m \) on \( x \) for the sake of conciseness. As remarked in [21], \( P_m(\lambda) \) is divisible by \( \lambda \), so that

\[ Q_m(\lambda) = \frac{P_m(\lambda)}{\lambda} \]

is an \( m \)th degree monic polynomial. We shall next show that \( Q_m \) satisfies the recursion relation

\[ Q_m(\lambda) = \lambda Q_{m-1}(\lambda) - \eta a_m^2 \prod_{\sigma=1}^{m-1} \left( \lambda + (1 - \eta) a_\sigma^2 \right), \quad m \geq 2, \tag{3.1} \]

where we have set

\[ \eta = e^{-K\beta \epsilon(x)}, \quad a_\sigma = e^{\beta \mu_\sigma / 2}, \quad 1 \leq \sigma \leq m. \]

To see this, note that we can write

\[ (-1)^{m+1} P_m(\lambda) = \begin{vmatrix}
1 - \lambda & a_1 & a_2 & \cdots & a_{m-1} & a_m \\
\eta a_1 & \eta a_1^2 - \lambda & a_1 a_2 & \cdots & a_1 a_{m-1} & a_1 a_m \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\eta a_{m-1} & \eta a_{m-1} a_1 & \eta a_{m-1} a_2 & \cdots & \eta a_{m-1}^2 - \lambda & a_{m-1} a_m \\
\eta a_m & \eta a_m a_1 & \eta a_m a_2 & \cdots & \eta a_m a_{m-1} & \eta a_m^2 - \lambda \\
\end{vmatrix}. \]

Multiplying the first row of the latter determinant by \( \eta a_m \) and subtracting it from the last one, after a straightforward calculation we obtain

\[ Q_m(\lambda) = \lambda Q_{m-1}(\lambda) - \eta a_m^2 \begin{vmatrix}
a_1 & a_2 & \cdots & a_{m-1} & 1 \\
\eta a_1^2 - \lambda & a_1 a_2 & \cdots & a_1 a_{m-1} & a_1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\eta a_{m-1} a_1 & \eta a_{m-1} a_2 & \cdots & \eta a_{m-1}^2 - \lambda & a_{m-1} a_m \\
\eta a_m a_1 & \eta a_m a_2 & \cdots & \eta a_m a_{m-1} & \eta a_m^2 - \lambda \\
\end{vmatrix}. \]

The determinant in the previous formula can be easily evaluated by subtracting the last column multiplied by \( a_i \) from the \( i \)th column for \( i = 1, \ldots, m - 1 \), which yields equation (3.1). From the latter recursion relation and the initial condition

\[ Q_1(\lambda) = \lambda - \eta a_1^2 - 1 \]

\[ \text{In point of fact, in [16] there is a more general derivation of the KK equation based on the equivalence of the KY model to a system of q-ons with an appropriately chosen statistical matrix, but this derivation is valid only for the su(1|2) case.} \]
we readily obtain the following explicit formula for $Q_m(\lambda)$:

$$Q_m(\lambda) = \lambda^m - \lambda^{m-1} - \eta \sum_{k=0}^{m-1} \lambda^k a_{m-k}^2 \prod_{\sigma=1}^{m-k} \left( \lambda + (1 - \eta) a_\sigma^2 \right).$$

This expression can be somewhat simplified with the help of the identities

$$\prod_{\sigma=1}^{p} \left( \lambda + (1 - \eta) a_\sigma^2 \right) = \sum_{l=0}^{p} \lambda^l (1 - \eta)^{p-l} e_{p-l}(a_1^2, \ldots, a_p^2)$$

and

$$e_k(x_1, \ldots, x_{k+q}) = \sum_{l=0}^{q} x_{k+l} e_{k-1}(x_1, \ldots, x_{k+l-1}),$$

where

$$e_k(x_1, \ldots, x_r) \equiv \sum_{1 \leq \sigma_1 < \cdots < \sigma_k \leq r} x_{\sigma_1} \cdots x_{\sigma_k}$$

denotes the elementary symmetric polynomial of degree $k$ in $r \geq k$ variables. Indeed, after a lengthy but straightforward calculation we obtain

$$(1 - \eta)Q_m(\lambda) = \lambda^m - (1 - \eta) \lambda^{m-1} - \eta \prod_{\sigma=1}^{m} \left( \lambda + a_\sigma^2 (1 - \eta) \right). \quad (3.2)$$

Performing the change of variable

$$X = \frac{1 - \eta}{\lambda}, \quad \lambda \neq 0,$$

we thus arrive at the fundamental identity

$$\lambda^{-m} (1 - \eta)Q_m(\lambda) = 1 - X - \eta \prod_{\sigma=1}^{m} \left( 1 + a_\sigma^2 X \right), \quad \lambda \neq 0. \quad (3.4)$$

In view of the previous discussion, we next rewrite equation (2.16) for the free energy per site of the supersymmetric su($m$) KY model with a chemical potential term as

$$f = 2T \int_0^{1/2} \log \left( \frac{X_1(x)}{1 - e^{-K \beta \varepsilon(x)}} \right) dx, \quad (3.5)$$

where

$$X_1(x) \equiv \frac{1 - e^{-K \beta \varepsilon(x)}}{\lambda_1(x)}.$$

Since $\lambda_1(x)$ is a nonzero root of the characteristic equation of the matrix $A(x)$, from equations (3.3) and (3.4) and the definition of $\eta$ we deduce that the function $X_1(x)$ satisfies

$$1 - X_1 = \eta \prod_{\sigma=1}^{m} \left( 1 + a_\sigma^2 X_1 \right).$$
or equivalently
\[ K \beta \varepsilon(x) = -\log(1 - X_1(x)) + \sum_{\sigma} \log(1 + e^{\beta \mu_{\sigma} X_1(x)}). \] (3.6)

Note that, since \( \lambda_1 \) does not vanish and \( \varepsilon(0) = 0 \), we must have
\[ X_1(0) = 0. \] (3.7)

Equations (3.5)–(3.7) are equivalent to the expression for the grand potential per site \( \omega \) of the \( \text{su}(m) \) KY model in [16]. To see this in more detail, note that in the thermodynamic limit the Hamiltonian \( \hat{H}_0 \) in equation (2.4) (with \( t = 1 \)) is related to the analogous Hamiltonian
\[ \mathcal{H}_{t-J} = -\frac{\pi^2}{N^2} \sum_{i<j} \sin^{-2}\left(\frac{\pi}{N} (i-j)\right) P_{ij}^{(1|m)} \] (3.8)
in [16] by
\[ \mathcal{H}_{t-J} = \hat{H}_0 + \frac{\pi^2}{3} \mathcal{F} - \frac{N \pi^2}{6} , \]
where we have used the identity
\[ \sum_{i \neq j} \sin^{-2}(\pi(i-j)/N) = \frac{N}{3} (N^2 - 1). \] (3.9)

Thus in the thermodynamic limit the grand potential of \( \mathcal{H}_{t-J} \) should be equal to the free energy of the right-hand side of equation (3.8) with the addition of a chemical potential term. Since we have absorbed the term proportional to \( \mathcal{F} \) in the definition of the fermion chemical potentials \( \mu_{\sigma} \) (see equations (2.9) and (2.10)), we must then show that
\[ \omega = f - \frac{\pi^2}{6} = 2T \int_0^{1/2} \log\left(\frac{X_1(x)}{1 - e^{-K \beta \varepsilon(x)}}\right) \, dx - \frac{\pi^2}{6} . \] (3.10)

Performing the change of variable \( x = (\pi - p)/(2\pi) \) in the integral, under which
\[ K \varepsilon(x) = 2\pi^2 x(1 - x) = \frac{1}{2} (\pi^2 - p^2) \equiv \varepsilon_0(p) , \]
we see that it suffices to show that
\[ \omega = \frac{T}{\pi} \int_0^{\pi} \log\left(\frac{\tilde{X}_1(p)}{1 - e^{-\beta \varepsilon_0(p)}}\right) \, dp - \frac{\pi^2}{6} , \] (3.11)
where by equations (3.6) and (3.7) \( \tilde{X}_1(p) \equiv X_1(x(p)) \) satisfies
\[ \beta \varepsilon_0(p) = -\log\left(1 - \tilde{X}_1(p)\right) + \sum_{\sigma} \log\left(1 + e^{\beta \mu_{\sigma} \tilde{X}_1(p)}\right) \] (3.12)
and
\[ \tilde{X}_1(\pi) = X_1(0) = 0. \] (3.13)
Our claim now follows from the fact that equations (3.11) and (3.12) are nothing but equations (2.33) and (2.22) in [16] (taking into account that we have set, without loss of generality, $\mu_0 = 0$), while equation (3.13) is the condition that according to the latter reference determines the appropriate branch $\hat{X}_1(p)$ of the algebraic function defined by equation (3.12).

A few remarks on the equivalence of equation (2.16) to (3.11)–(3.13)—or, more generally, (3.5)–(3.7)—are now in order. The approach of [16] is based on the derivation of the thermodynamics of the $su(1|m)$ spin Sutherland model in the $N \to \infty$ limit, which yields the thermodynamics of the $su(1|m)$ HS chain in the strong coupling limit through Polychronakos’s freezing trick. An essential ingredient in this approach is the equivalence of the $su(1|m)$ spin Sutherland model to a system of non-interacting $su(1|m)$ particles whose spectrum can be effectively described in terms of generalized momenta obeying appropriate exclusion statistics (see also [35]). From this description follows an integral relation satisfied by the one-particle energy $\hat{\varepsilon}(p)$ (defined by $\hat{X}_1(p) = e^{-\beta\hat{\varepsilon}(p)}$), which in turn yields equation (3.12). Equation (3.11) is then derived through a fairly elaborate argument, by first expressing the grand potential of the spin Sutherland model in terms of the one-particle energy and then subtracting the phonon contribution. By contrast, in our approach equation (2.16) follows in a straightforward way from equations (2.11) and (2.12) applying the transfer matrix method, as summarized in the previous section (see [21] for full details). That the spectrum of the $su(1|m)$ HS chain is completely described by equations analogous to (2.11) and (2.12) is in fact a fundamental property of all $su(n|m)$ spin chains of HS type, stemming directly from the structure of their partition function [14]. This description does not rely at all on the properties of the associated spin Sutherland model, such as the existence of generalized quasimomenta satisfying an appropriate exclusion statistics, and may thus apply even to other types of Yangian-invariant models not necessarily derived from a dynamical spin model. In view of the above argument, the free energy of all these models, including the three families of $su(n|m)$ spin chains of HS type, should also be described by equations analogous to (3.5)–(3.7). This is indeed remarkable, and it underscores the fact that the range of applicability of the latter equations is much wider than could naively be expected from their original derivation in [16].

As a simple example of the last assertion, consider the $su(n|m)$ PF and FI chains with a chemical potential term $-\sum_{\alpha=1}^{m+n-1}\mu_{\alpha}N_{\alpha}$. When $n = 1$, the spectra of these models can be obtained replacing the HS dispersion relation $i(N - i)$ in equation (2.11) respectively by $i$ and $i(i + N\gamma)$ (with $N\gamma > -1$) [20, 21]. It is then straightforward to show that if we set

$$K = \begin{cases} N J, & \text{for the PF chain} \\ N^2 J, & \text{for the FI chain}, \end{cases}$$

the partition function of these models is still given by equations (2.13) and (2.14), but with $\varepsilon(x)$ replaced by

$$\varepsilon(x) = \begin{cases} x, & \text{for the PF chain} \\ x(x + \gamma), & \text{for the FI chain} \end{cases}$$

(3.14)
(γ being now a non-negative parameter). Consequently, in the thermodynamic limit their free energy can be expressed in the form\(^7\)

\[
f = - T \int_0^1 \log \lambda_1(x) \, dx,
\]

(3.15)

where \(\lambda_1(x)\) is the Perron–Frobenius eigenvalue of the transfer matrix (2.14) with \(\varepsilon(x)\) as in equation (3.14). In the \(n = 1\) case, from the latter equation we deduce reasoning as above that the grand potential of the PF and FI chains can be written as

\[
\omega = T \int_0^1 \log \left( \frac{X_1(x)}{1 - e^{-K\beta\varepsilon(x)}} \right) \, dx,
\]

(3.16)

where \(X_1(x)\) satisfies the generalized KK equation (3.6) and (3.7) with \(\varepsilon(x)\) given by (3.14). By the same token, in the general su\((n|m)\) case with \(n > 1\) we conjecture that (3.15) is equivalent to (3.16), where now \(X_1(x)\) should satisfy the generalized KK equation

\[
K\beta\varepsilon(x) = -\sum_{\alpha=0}^{n-1} \log(1 - e^{\beta\mu_\alpha} X_1(x)) + \sum_{\alpha=n}^{m+n-1} \log(1 + e^{\beta\mu_\alpha} X_1(x)) \quad (\text{with } \mu_0 \equiv 0)
\]

(3.17)

with the condition \(X_1(0) = 0\). More generally, this should be true for any model whose spectrum be given by an equation of the form

\[
E(s) = \frac{K}{N^a} \sum_{i=1}^{N-1} \mathcal{E}_N(i) \delta(s, s_{i+1}) - \sum_i \mu_{s_i},
\]

(3.18)

together with the su\((n|m)\) analogue of equation (2.12):

\[
\delta(s, s') = \begin{cases} 1, & s > s' \text{ or } s = s' \geq n \\ 0, & s < s' \text{ or } s = s' < n, \end{cases}
\]

(3.19)

provided that \(\lim_{N \to \infty} \mathcal{E}_N(Nx) \equiv \varepsilon(x)\) exists for all \(x \in [0, 1]\). In fact, an equation akin to (3.17) has been proposed in [35] for the supersymmetric Sutherland model.

On a more technical level, our approach also helps clarify an important concern concerning the definition of the function \(\hat{X}_1(p)\) through equations (3.12)–(3.13), or equivalently \(X_1(x)\) through equation (3.6) and the condition \(X_1(0) = 0\). Indeed, from the previous discussion it follows that the latter equation is equivalent to the algebraic equation (with coefficients depending on the parameter \(x \in [0, 1/2]\))

\[
\hat{Q}(X) \equiv 1 - X - e^{-K\beta\varepsilon(x)} \prod_{\sigma=1}^m \left( 1 + e^{\beta\mu_\sigma} X \right) = 0
\]

(3.20)

for the variable \(X\). When \(x = 0\) it is clear that \(X = 0\) is a simple root of this equation, since \(\hat{Q}\) vanishes at the origin and the coefficient of \(X\) in the latter polynomial is \(-1 - \sum_{\sigma=1}^m e^{\beta\mu_\sigma} \neq 0\). By the implicit function theorem, the condition \(X_1(0) = 0\) uniquely defines a branch of the algebraic function (3.20) near \(x = 0\). However, it is not clear whether this is still the case—i.e. whether there is no branch crossing—as \(x\) increases. From a practical standpoint, the actual computation of \(X_1(x)\) through equations (3.20)

\(^7\) The missing factor of 2 and the different range of integration in equation (3.15) compared to the analogous equation (2.16) are due to the lack of symmetry about \(x = 1/2\) of \(\varepsilon(x)\) in equation (3.14).
and (3.7) at a point \( x > 0 \) is arduous at best, since it requires following the appropriate branch of the algebraic function (3.20) all the way from \( x = 0 \). Both problems are solved by our approach, since it is now clear from equation (3.3) that \( X_1(x) \) is simply given by \( (1 - e^{-K\beta\varepsilon(x)})/\lambda_1(x) \), where \( \lambda_1(x) \) is the eigenvalue of the matrix \( A(x) \) with the largest modulus (whose uniqueness for arbitrary \( x \) is guaranteed by the Perron–Frobenius theorem).

### 4. Ground state phases

In this section we shall use equations (2.19) and (2.20) to derive exact expressions for the zero-temperature magnetization and charge densities of the spin 1/2 KY model for arbitrary values of the magnetic field strength \( h \) and charge chemical potential \( \mu \). In this way we shall identify the model’s five ground state phases, which in turn determine the form of the low-temperature asymptotic series that we shall compute in the next section.

In order to simplify the calculations, in the rest of the paper we shall take \( K \) as the unit of energy and hence of temperature (since \( k_B = 1 \) from the outset). We can also suppose without loss of generality that \( h \geq 0 \), since changing the sign of \( h \) is equivalent to exchanging \( n_1 \) with \( n_2 \), or equivalently replacing \( (m_s, n_c) \) by \( (-m_s, n_c) \). The magnetization per site \( m_s \) clearly vanishes for \( h = 0 \). On the other hand, for \( h > 0 \) and \( T \to 0 \) we can replace \( \sinh(\beta h/2) \) and \( \cosh(\beta h/2) \) by \( e^{\beta h/2} / 2 \) up to an exponentially small term \( O(e^{-\beta h/2}) \). Hence at low temperatures we can write

\[
m_s \simeq \int_0^{1/2} \left[ 4e^{-\beta h} D(x) \right]^{-1/2} \, dx,
\]

with

\[
4e^{-\beta h} D(x) \simeq 1 + 4e^{\beta(\varepsilon(x)-h)} + 2e^{\beta(\varepsilon(x)-\mu - \frac{h}{2})} + e^{2\beta(\varepsilon(x)-\mu - \frac{h}{2})},
\]

where we have dropped several exponentially small terms \( O(e^{-\beta h}) \). From the previous equations it immediately follows that the zero-temperature magnetization is given by

\[
m_s = \left\{ x \in [0, \frac{1}{2}] : \varepsilon(x) < h, \varepsilon(x) < \mu + \frac{h}{2} \right\},
\]

where \(|A|\) denotes the measure of the set \( A \). Thus, at \( T = 0 \) we have

\[
m_s = \begin{cases} 
  x_0(h), & (h, \mu) \in B_1 \cup T \\
  \frac{1}{2}, & (h, \mu) \in W_1 \\
x_0(\mu + \frac{h}{2}), & (h, \mu) \in B_0 \\
0, & (h, \mu) \in W_0,
\end{cases}
\]

where

\[
x_0(t) = \frac{1}{2} \left( 1 - \sqrt{1 - 4t} \right)
\]
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Table 1. Definitions of the regions $B_i$, $T$ and $W_i$ in the half-plane $h \geq 0$ (see figure 1 left) and their species content.

| Region | Equation | Species content |
|--------|----------|-----------------|
| $B_0$  | $0 < \mu + \frac{h^2}{2} < \frac{1}{4}$, $\mu < \frac{h}{2}$ | Bosons and ‘up’ fermions |
| $B_1$  | $h < \frac{1}{4}$, $\mu > \frac{1}{8}$ | Fermions |
| $T$    | $\frac{h}{2} < \mu < \frac{1}{8}$ | Bosons and fermions |
| $W_0$  | $\mu < -\frac{h}{2}$ | Bosons |
| $W_1$  | $h > \frac{1}{4}$, $\mu > -\frac{h}{2} + \frac{1}{4}$ | ‘Up’ fermions |

is the unique root of the equation $\varepsilon(x) = t$ in the interval $[0, 1/2]$, and the regions $B_i$, $T$ and $W_i$ are defined in table 1 (see figure 1 left). It is also straightforward to check that $m_s$ is continuous on the boundaries of the latter sets, and hence everywhere.

Similarly,

$$1 - n_c = 2 \int_0^{1/2} [4e^{-2\beta(\varepsilon(x) - \mu)}D(x)]^{-1/2} dx,$$

where the term in brackets can be approximated at low temperatures by

$$4e^{-2\beta(\varepsilon(x) - \mu)}(e^{\beta\varepsilon(x)} - 1) + (1 + e^{-\beta(\varepsilon(x) - \mu - \frac{1}{2})})^2 = 1 + e^{-2\beta(\varepsilon(x) - \mu - \frac{1}{2})} + 2e^{-\beta(\varepsilon(x) - \mu - \frac{1}{2})} + 4e^{-\beta(\varepsilon(x) - 2\mu)} - 4e^{-2\beta(\varepsilon(x) - \mu)}$$

Proceeding as before we obtain the following expression for the zero-temperature charge density:

$$\frac{1}{2}(1 - n_c) = \left| \{x \in [0, \frac{1}{2}] : \varepsilon(x) > 2\mu, \varepsilon(x) > \mu + \frac{h}{2} \} \right|,$$

or equivalently

$$n_c = 2 \left| \{x \in [0, \frac{1}{2}] : \varepsilon(x) < 2\mu \text{ or } \varepsilon(x) < \mu + \frac{h}{2} \} \right|.$$

Thus the zero-temperature charge density is given by

$$n_c = \begin{cases} 
1, & (h, \mu) \in B_1 \cup W_1 \\
2x_0(2\mu), & (h, \mu) \in T \\
2x_0(\mu + \frac{h}{2}), & (h, \mu) \in B_0 \\
0, & (h, \mu) \in W_0.
\end{cases} 
(4.4)$$

As before, it is easily verified that $n_c$ is continuous across the boundaries of the regions $B_i$, $W_i$, $T$. It should also be noted that equations (4.2)–(4.4) were derived in [10] by a more laborious method, based on determining the magnon content of the ground state for arbitrary values of the parameters $h$ and $\mu$ using equations (2.11)–(2.12) for the energies of the equivalent vertex model.

As remarked in [10], the regions $B_i$, $T$ and $W_i$ defined above have a clear interpretation as different zero-temperature phases of the model. Indeed, taking into account that
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Equation (4.2) for $m_s$ can be expressed in terms of the independent variables $h \geq 0$ and $0 \leq n_c \leq 1$ by taking into account how the regions $B_i$, $T$, and $W_i$ transform under the (non-invertible) mapping $(h, \mu) \mapsto (h, n_c)$ determined by equation (4.4) (see figure 1 right). To begin with, it is obvious that the wedge $W_0$ collapses into the line $n_c = 0$. Similarly, the wedge $W_1$ is transformed into the horizontal half-line $n_c = 1$, $h > 1/4$, while the vertical band $B_1$ goes into the segment $n_c = 1$, $h < 1/4$. On the other hand, the triangle $T$ is mapped into the bounded region to the left of the parabola $h = h_0(n_c)$, where

$$h_0(n_c) = \varepsilon(n_c/2) = \frac{n_c}{4}(2 - n_c).$$

Indeed, in the triangle $T$ we have

$$n_c = 2x_0(2\mu) \iff \varepsilon(n_c/2) = 2\mu > h.$$

Likewise, the oblique band $B_0$ is transformed into the unbounded region to the right of the parabola $h = h_0(n_c)$, since in this band

$$n_c = 2x_0(\mu + \frac{h}{2}) \iff \varepsilon(n_c/2) = \mu + \frac{h}{2} < h.$$
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From these considerations it readily follows that the zero-temperature magnetization is expressed in terms of the variables \((h,n_c)\) by

\[
m_s = \begin{cases} 
  x_0(h), & 0 \leq h \leq h_0(n_c) \\
  \frac{1}{2} n_c, & h \geq h_0(n_c) 
\end{cases}
\]

(see [29]). Note that \(m_s\) should be continuous everywhere in \((h,n_c)\) space, since it is continuous when expressed in terms of the variables \((h,\mu)\). In particular, the previous expression for the critical magnetic field \(h_0\) is recovered by imposing the continuity of \(m_s\) across the parabola \(h = h_0(n_c)\).

From equations (4.2)–(4.4) it is straightforward to compute the zero-temperature magnetic and charge susceptibilities by differentiation. To begin with, the magnetic susceptibility vanishes in the wedges \(W_0\) and \(W_1\). On the other hand, in the region \(B_1 \cup T\) (including the segment \(\mu = 1/8, h < 1/4\)) we have

\[
\chi_s = \frac{\partial m_s}{\partial h} = x'_0(h) = \frac{1}{1-2x_0(h)} = \frac{1}{1-2m_s}, \quad (h, \mu) \in B_1 \cup T.
\]

Likewise, in the band \(B_0\) the magnetic susceptibility is given by

\[
\chi_s = \frac{1}{2} x'_0(\mu + h/2) = \frac{1}{2(1-2m_s)} = \frac{1}{2(1-n_c)}, \quad (h, \mu) \in B_0.
\]

A similar analysis for the charge susceptibility \(\chi_c \equiv \partial n_c/\partial \mu\mu = \partial n_c/\partial \mu\) yields the result

\[
\chi_c = \begin{cases} 
  0, & (h, \mu) \in B_1 \cup W_0 \cup W_1 \\
  1 \frac{1}{1-n_c}, & (h, \mu) \in T \\
  \frac{1}{1-n_c}, & (h, \mu) \in B_0.
\end{cases}
\]

(In fact, \(\chi_c\) vanishes also on the segment \(h = 1/4, \mu > 1/8\).) Comparing with the previous expressions for \(m_s\) and \(n_c\) at zero temperature, we deduce that \(\chi_s\) diverges on the half-lines \(\{h = 1/4, \mu \geq 1/8\}\) and \(\{\mu + h/2 = 1/4, \mu \leq 1/8\}\), while \(\chi_c\) is divergent on the half-lines \(\{\mu = 1/8, h < 1/4\}\) and \(\{\mu + h/2 = 1/4, \mu \leq 1/8\}\). This is a well-known fact (see, e.g. [28]). More surprising is the behavior of the magnetic and charge susceptibilities at the boundaries of the su(1|1) phase with the su(1|2) phase and the vacuum. Indeed, these functions present jump discontinuities on the segment \(\{\mu = h/2, 0 \leq h \leq 1/4\}\) and the half-line \(\{\mu + h/2 = 0, h \geq 0\}\). The discontinuity on the latter segment (which, to the best of our knowledge, had not been previously pointed out in the literature) is particularly interesting, since it is due to the fact that \((1-2m_s)\chi_s\) and \((1-n_c)\chi_c\) are different constants on each side of this segment. It is also worth mentioning that \(\chi_c = 4\chi_s\) on the union of the half-planes \(\mu < h/2\) and \(h > 1/4\). Note, finally, that our formulas for \(\chi_s\) and \(\chi_m\) agree with those of [28] in the triangle \(T\), which is the only region considered in the latter reference. In any case, the dependence of \(\chi_s\) (respectively \(\chi_c\)) exclusively on \(m_s\) (respectively on \(n_c\)) at zero temperature is a manifestation of the spin-charge separation characteristic of the t-J model.

It is also straightforward to express the zero-temperature susceptibilities as functions of the variables \((h, n_c)\). The key fact in this respect is that the segment \(\mu = h/2\),

---

Note that the magnetic field and the magnetic moment in [28] are respectively \(h/2\) and \(2m\), in our notation. Indeed, in the latter reference the magnetic field interaction in the Hamiltonian is taken as \(h(n^1 - n^2)\), while the magnetization is defined as \((n^1 - n^2)\).

https://doi.org/10.1088/1742-5468/ab25e0
0 \leq h \leq 1/4 is mapped to the arc of the parabola \( n_c = 2x_0(h) \equiv 1 - \sqrt{1 - 4h} \) (or, equivalently, \( h = h_0(n_c) \)) with \( 0 \leq h \leq 1/4 \). The susceptibilities are then given by

\[
\chi_s = \begin{cases} 
0, & (h, n_c) \in W_0' \cup W_1' \\
\frac{1}{1-2m_s}, & (h, n_c) \in W_0' \cup W_1' \\
\frac{1}{2(1-2m_s)}, & (h, n_c) \in B_0'
\end{cases}
\]

and

\[
\chi_c = \begin{cases} 
0, & (h, n_c) \in B_0' \\
\frac{4}{1-n_c}, & (h, n_c) \in B_0' \\
\frac{2}{1-n_c}, & (h, n_c) \in B_0'.
\end{cases}
\]

In particular, we see that \( \chi_c = 4\chi_s \) in the infinite region \( \sqrt{1 - 4h} < 1 - n_c \).

### 5. Asymptotic series for the free energy and criticality

Starting from the exact formula \( \text{(2.18)} \), in this section we shall derive the complete asymptotic series of the free energy of the su(2) KY model\(^9\) at \( T = 0 \) for arbitrary values of the parameters \( h \) and \( \mu \). We shall also use this asymptotic series to analyze the model’s criticality properties and the low-temperature behavior of its main thermodynamic functions.

#### 5.1. Wedges \( W_0 \) and \( W_1 \)

To begin with, it is straightforward to show that in the wedges \( W_{0,1} \) the free energy is exponentially small in \( \beta \) as \( T \to 0 \). Indeed, we can rewrite equation \( \text{(2.18)} \) as

\[
f(T) = -\mu_c - \frac{h}{2} - 2T \int_0^{1/2} \log \left[ \tilde{b} + \sqrt{\tilde{b}^2 + e^{-\beta(h-x)} - e^{-\beta h}} \right] dx,
\]

with

\[
\tilde{b}(x) \equiv e^{-\beta h} b(x) = \frac{1}{2} \left[ 1 + e^{-\beta(x/2-\varepsilon)} + e^{-\beta h} \right].
\]

Clearly all the exponents in the previous formula for \( f \) are strictly negative in the region \( W_1 \), so that \( f(0) = -\mu_c - h/2 \). Taking into account that \( \varepsilon(x) \leq \varepsilon(1/2) = 1/4 \) we easily obtain

\[
|f(T) - f(0)| \leq T \log \left[ a + \sqrt{a^2 + e^{-\beta(h-1/4)}} \right],
\]

with \( a \equiv \tilde{b}(1/2) > 1/2 \). From the elementary inequality \( \sqrt{a^2 + x} \leq a + x/(2a) \) (where \( x > 0 \)) it then follows that

\[
a + \sqrt{a^2 + e^{-\beta(h-1/4)}} \leq 2a + e^{-\beta(h-1/4)},
\]

\(^9\) From now on, by ‘su(m) KY model’ we shall understand the full Hamiltonian \( \text{(2.5)} \), whose free energy (in the thermodynamic limit) is therefore the grand potential of the original KY Hamiltonian \( H_0 \) in equation \( \text{(2.1)} \).
which easily yields the estimate
\[ |f(T) - f(0)| = O\left(\frac{\pi cT^2}{6v}\right), \; (h, \mu) \in \mathcal{W}_1. \]

A similar analysis in the region \( \mathcal{W}_0 \) shows that
\[ |f(T) - f(0)| = O\left(\frac{\pi cT^2}{6v}\right), \; (h, \mu) \in \mathcal{W}_0. \]

Recall that at low temperatures the free energy per unit length of a \((1 + 1)\)-dimensional CFT (in natural units \(\hbar = k_B = 1\)) behaves as \([37, 38]\)
\[ f(T) \approx f(0) - \frac{\pi cT^2}{6v}, \quad (5.3) \]
where \(c\) is the central charge and \(v\) is the Fermi velocity (effective speed of light). From the previous estimates for \(f(T) - f(0)\) at low temperatures it then follows that the \(\text{su}(2)\) KY model is not critical when \((h, \mu)\) lies on the wedges \(\mathcal{W}_0\) and \(\mathcal{W}_1\). In fact, the exponentially small bounds in \(\beta\) for \(|f(T) - f(0)|\) found above show that the spin 1/2 KY model is gapped on the wedges \(\mathcal{W}_{0,1}\), with the energy gap given by \(|\mu + \beta/2|\) in \(\mathcal{W}_0\) and \(\min(h - 1/4, \mu + \beta/2 - 1/4)\) in \(\mathcal{W}_1\).

### 5.2. Vertical band \(\mathcal{B}_1\)

Splitting the integration interval in equation (5.1) into \([0, x_0(h)]\) and \([x_0(h), 1/2]\), and setting
\[ \hat{b}(x) \equiv e^{-\beta \tilde{\varepsilon}(x)} b(x) = \frac{1}{2} \left[ e^{-\beta (\varepsilon - h)} + e^{-\beta (\varepsilon + h)} + e^{-\beta (\mu - \frac{\varepsilon}{2})} \right], \quad (5.4) \]
we can write
\[ f(T) = f_0 - 2T \int_0^{x_0(h)} \log \left[ \hat{b} + \sqrt{\hat{b}^2 + e^{-\beta (\varepsilon - h)} - e^{-\beta h}} \right] dx \]
\[ - 2T \int_{x_0(h)}^{1/2} \log \left[ \hat{b} + \sqrt{\hat{b}^2 + 1 - e^{-\beta \varepsilon}} \right] dx, \quad (5.5) \]

where
\[ f_0 = -\mu c - hx_0(h) - \int_{x_0(h)}^{1/2} \varepsilon(x) dx. \quad (5.6) \]

When \((h, \mu) \in \mathcal{B}_1\) all the exponents in the previous formulas for \(\hat{b}\) and \(\tilde{b}\) are negative for \(0 \leq x < x_0(h)\) and \(x_0(h) < x \leq 1/2\), respectively. Thus both integrals in equation (5.5) vanish at \(T = 0\), and hence \(f_0 = f(0)\). Furthermore, we have
\[ \int_0^{x_0(h)} \log \left[ \hat{b} + \sqrt{\hat{b}^2 + e^{-\beta (\varepsilon - h)} - e^{-\beta h}} \right] dx = I_1 + O(e^{-\beta \min(h, \mu - \frac{\varepsilon}{2})}), \quad (5.7) \]
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\[
\int_{x_0(h)}^{1/2} \log \left[ \hat{b} + \sqrt{\hat{b}^2 + 1 - e^{-\beta \epsilon}} \right] \, dx = I_2 + O(e^{-\beta \min(h, \mu - \frac{1}{8})}), \tag{5.8}
\]

where

\[
I_1 \equiv \int_0^{x_0(h)} \log \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + e^{-\beta(h - \epsilon)}} \right] \, dx, \tag{5.9}
\]

\[
I_2 \equiv \int_{x_0(h)}^{1/2} \log \left[ \frac{1}{2} e^{-\frac{\beta}{2} (\varepsilon - h)} + \sqrt{1 + \frac{1}{4} e^{-\beta(\varepsilon - h)}} \right] \, dx \tag{5.10}
\]

(see appendix A). We shall next derive the full asymptotic series of the integrals \( I_k \) in powers of \( T \). To this end, let us perform in the integral \( I_1 \) the change of variable \( y = \beta(h - \varepsilon(x)) \), or equivalently

\[
x = x_0(h - Ty), \tag{5.11}
\]

obtaining

\[
I_1 = T \int_0^{\beta h} \log \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + e^{-y}} \right] x_0'(h - Ty) \, dy. \tag{5.12}
\]

We next expand the last term in the previous equation around \( T = 0 \) taking into account the identity \( x_0'(h - Ty) = \sum_{l=0}^{\infty} a_l(h)(-Ty)^l, \) with the result

\[
x_0'(h - Ty) = \sum_{l=0}^{\infty} a_l(h)(-Ty)^l, \tag{5.13}
\]

where

\[
a_l(s) = \frac{2^l(2l - 1)!!}{l! [1 - 2x_0(s)]^{2l+1}} = \frac{2^l(2l - 1)!!}{l! (1 - 4s)^{l+\frac{1}{2}}} \tag{5.14}
\]

and \((-1)!! \equiv 1\). As shown in appendix B, the asymptotic series of \( I_1 \) is then given by

\[
I_1 \sim \sum_{l=0}^{\infty} (-1)^l a_l(h) T^{l+1} \int_0^{\infty} y^l \log \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + e^{-y}} \right] \, dy. \tag{5.15}
\]

Consider next the integral \( I_2 \). Since the natural change of variable

\[
x = x_0(h + Ty) \tag{5.16}
\]

is singular at the endpoint \( y = \beta(1/4 - h) \), we first subdivide the integration range into the intervals \( x_0(h) \leq x \leq x_0(\frac{1}{8} + \frac{3}{2}) \) and \( x_0(\frac{1}{8} + \frac{3}{2}) \leq x \leq 1/2 \). The integral over the second interval is clearly \( O(e^{-\beta(1/4 - h)}) \), so that

\[
I_2 = \int_{x_0(h)}^{x_0(\frac{1}{8} + \frac{3}{2})} \log \left[ \frac{1}{2} e^{-\frac{\beta}{2} (\varepsilon - h)} + \sqrt{1 + \frac{1}{4} e^{-\beta(\varepsilon - h)}} \right] \, dx + O(e^{-\beta(1/4 - h)}). \tag{5.17}
\]

Performing the change of variable \( \text{(5.16)} \) in the integral in equation \( \text{(5.15)} \) and proceeding as before we obtain

https://doi.org/10.1088/1742-5468/ab25e0
\[ I_2 \sim \sum_{l=0}^{\infty} a_l(h) T^{l+1} \int_0^{\infty} y^l \log \left[ \frac{1}{2} e^{-y/2} + \sqrt{1 + \frac{1}{4} e^{-y}} \right] \, dy. \]  

(5.16)

From equations (5.5)–(5.8) and (5.13)–(5.16) we finally obtain the asymptotic series of the free energy per site in the open band \( B_1 \):

\[ f(T) - f(0) \sim -2 \sum_{l=0}^{\infty} a_l(h) I_l T^{l+2}, \quad (h, \mu) \in B_1, \]  

(5.17)

where

\[ I_l = \int_0^{\infty} y^l \left\{ \log \left[ \frac{1}{2} e^{-y/2} + \sqrt{1 + \frac{1}{4} e^{-y}} \right] + (-1)^l \log \left[ \frac{1}{2} + \sqrt{1 + e^{-y}} \right] \right\} \, dy. \]  

(5.18)

As explained in appendix C, the latter integrals can be expressed in several alternative ways, to wit

\[ I_l = \frac{1}{2(l+1)} \int_{-\infty}^{\infty} x^l \left[ \frac{1}{\sqrt{1 + 4e^x}} - \theta(-x) \right] \, dx = \frac{1}{(l+1)(l+2)} \int_{-\infty}^{\infty} x^{l+2} e^{x} \, dx, \]  

(5.19)

where \( \theta(t) = (1 + \text{sgn} t)/2 \) is Heaviside’s step function. The integrals \( I_l \) can actually be computed in closed form for low values of \( l \), namely

\[ I_0 = \frac{\pi^2}{6}, \quad I_1 = \zeta(3), \quad I_2 = \frac{\pi^4}{10}, \quad I_3 = 2 \left[ \pi^2 \zeta(3) + 9 \zeta(5) \right], \]  

where \( \zeta(z) \) denotes Riemann’s zeta function. We thus obtain the low-temperature expansion

\[ f(T) - f(0) = -\frac{\pi^2 T^2}{3(1 - 4h)^{1/2}} - \frac{4\zeta(3)T^3}{(1 - 4h)^{3/2}} - \frac{6\pi^4 T^4}{5(1 - 4h)^{5/2}} + O(T^5), \quad (h, \mu) \in B_1. \]  

(5.20)

Equation (5.20) strongly suggests that the su(2) KY model is critical in the vertical band \( B_1 \). To ascertain this fact and compute the central charge, however, we first need to determine the Fermi velocity \( v \) of the low-energy excitations above the ground state. To this end, recall first of all that in the limit \( N \to \infty \) the ground state energy of the spin 1/2 KY model is approximately given by

\[ E_0 \simeq \sum_{k=1}^{N_{m_s}} \left( \varepsilon(x_k) - h \right) + \sum_{k=1}^{N_{n_c}/2} \left( \varepsilon(x_k) - 2\mu \right), \quad x_k \equiv k/N, \]  

(5.21)

while its momentum (mod. \( 2\pi \)) can be written as

\[ P \simeq 2\pi \sum_{k=1}^{N_{m_s}} x_k + 2\pi \sum_{k=1}^{N_{n_c}/2} x_k \]  

(5.22)

(see [10] and, e.g. [39, 40]). As shown in section 4, in the vertical band \( B_1 \) we have

\[ m_s = x_0(h), \quad n_c = 1, \]
and hence
\[ E_0 \approx \sum_{k=1}^{N/2} \left( \varepsilon(x_k) - h \right) + \sum_{k=1}^{N/2} \left( \varepsilon(x_k) - 2\mu \right), \quad P \approx 2\pi \sum_{k=1}^{N/2} x_k + 2\pi \sum_{k=1}^{N/2} x_k. \]

Low-energy excitations above the ground state are obtained by adding a ‘mode’ with \( k = N\bar{x}_0(h) + 1 \) or removing one with \( k = N\bar{x}_0(h) - 1 \). The energy of these excitations is thus \( E_0 + \Delta E \), with
\[ \Delta E = \pm [\varepsilon(x_0(h)) \pm \frac{1}{N} - h] \approx \pm \left[ \frac{\varepsilon'(x_0(h))}{N} - h \right] = \frac{\varepsilon'(x_0(h))}{N}. \]

On the other hand, the momentum carried by the mode added (respectively removed) is
\[ p = 2\pi x_k \equiv p_0 \pm \Delta p, \]
where \( p_0 = 2\pi x_0(h) \) is the Fermi momentum and \( \Delta p = 2\pi/N \). The Fermi velocity of the low-energy excitations is therefore given by
\[ v = \frac{\Delta E}{\Delta p} = \frac{\varepsilon'(x_0(h))}{2\pi} = \frac{1 - 2x_0(h)}{2\pi} = \frac{\sqrt{1 - 4h}}{2\pi}. \tag{5.23} \]

From equations (5.20) and (5.23) it then follows that in this case the model is critical\(^{10} \) with central charge \( c = 1 \). It should also be noted that in the limit \( T \to 0 \) the only contribution to the integrals \( I_{1,2} \) in equations (5.7) and (5.8), in terms of which
\[ f - f(0) \sim -2T(I_1 + I_2), \]
comes from an arbitrarily small neighborhood of the point \( x_0(h) = p_0/(2\pi) \) up to exponentially small terms in \( \beta \). We shall express this relationship by saying that these integrals are critical at \( x = x_0(h) \). Thus the Fermi velocity (5.23) is proportional to the derivative of the dispersion relation \( \varepsilon(x) \) at the unique critical point of the integrals \( I_{1,2} \).

5.3. Oblique band \( B_0 \)

In this case equation (5.5) becomes
\[ f(T) = f_0 - 2T \int_0^{x_0(\frac{h}{2} + \mu)} \log \left[ b + \sqrt{b^2 + e^{-\beta(h-x)} - e^{-\beta h}} \right] \, dx \]
\[ - 2T \int_{x_0(\frac{h}{2} + \mu)}^{1/2} \log \left[ b + \sqrt{b^2 + e^{-\beta(x-\mu)} - e^{-2\beta(x-\mu)}} \right] \, dx, \tag{5.24} \]
where now
\[ f_0 = -\mu_c - h x_0(\frac{h}{2} + \mu) - 2 \int_{x_0(\frac{h}{2} + \mu)}^{1/2} (\varepsilon(x) - \mu) \, dx \tag{5.25} \]

\(^{10} \) It is also important to mention in this respect that the ground state of the spin 1/2 KY model has finite degeneracy (at most 4) [40].
and

$$\tilde{b}(x) \equiv e^{\beta(x - \varepsilon(x))} b(x) = \frac{1}{2} \left[1 + e^{\beta(\varepsilon - \frac{h}{2})} + e^{\beta(\varepsilon + \frac{h}{2})}\right].$$  \hfill (5.26)

Again, all the exponents appearing in equation (5.24) are non-positive, so that \(f_0 = f(0)\). Moreover, proceeding as above we obtain the estimates

$$\int_{x_0(\frac{h}{2} + \mu)}^{x_0(\frac{h}{2} + \mu + \mu)} \log \left[1 + e^{-\beta(x - \mu - \varepsilon(x))}\right] dx = I_3 + O(e^{-\beta \min(h, \frac{h}{2} - \mu)}),$$  \hfill (5.27)

$$\int_{x_0(\frac{h}{2} + \mu)}^{1/2} \log \left[1 + e^{-\beta(x - \mu - \varepsilon(x))}\right] dx = I_4 + O(e^{-\beta \min(h, \frac{h}{2} - \mu)}),$$  \hfill (5.28)

where

$$I_3 = \int_{0}^{x_0(\frac{h}{2} + \mu)} \log \left[1 + e^{-\beta(x - \mu - \varepsilon(x))}\right] dx,$$

$$I_4 = \int_{x_0(\frac{h}{2} + \mu)}^{1/2} \log \left[1 + e^{-\beta(x - \mu - \varepsilon(x))}\right] dx = \int_{x_0(\frac{h}{2} + \mu)}^{x_0(\frac{h}{2} + \mu + \mu)} \log \left[1 + e^{-\beta(x - \mu - \varepsilon(x))}\right] dx$$

$$+ O(e^{-\beta \min(h, \frac{h}{2} - \mu)}).$$  \hfill (5.30)

The asymptotic series of the integral \(I_5\) is obtained as above through the change of variable \(y = \beta(\varepsilon(x) - \frac{h}{2} - \mu)\), namely

$$I_3 \sim \sum_{l=0}^{\infty} (-1)^{l} a_l \left(\frac{h}{2} + \mu\right) T^{l+1} \int_{0}^{\infty} y^{l} \log(1 + e^{-y}) dy$$

$$= \sum_{l=0}^{\infty} (-1)^{l} a_l \left(\frac{h}{2} + \mu\right) l!(1 - 2^{-l-1}) \zeta(l + 2) T^{l+1}.$$  \hfill (5.31)

Likewise, performing the analogous change of variable \(y = \beta(\varepsilon(x) - \frac{h}{2} - \mu)\) in the RHS of equation (5.30) we obtain the asymptotic series

$$I_4 \sim \sum_{l=0}^{\infty} a_l \left(\frac{h}{2} + \mu\right) l!(1 - 2^{-l-1}) \zeta(l + 2) T^{l+1}.$$  \hfill (5.32)

Combining equations (5.31)–(5.32) we finally arrive at the following asymptotic series for the free energy per site in the oblique band \(B_0\):

$$f(T) - f(0) \sim -2 \sum_{l=0}^{\infty} \frac{(2^{l+1} - 1)(4l - 1)!!}{1 - 2(h + 2\mu)^{2l+\frac{h}{2}}} \zeta(2l + 2) T^{2l+2}.$$  \hfill (5.33)
In particular, the first few terms in the latter series are explicitly given by\textsuperscript{11}
\[
f(T) - f(0) = -\frac{\pi^2 T^2}{3[1 - 2(h + 2\mu)]^{1/2}} - \frac{7\pi^4 T^4}{15[1 - 2(h + 2\mu)]^{5/2}} + O(T^6), \quad (h, \mu) \in B_0.
\]

As before, the previous asymptotic expansion indicates that the model is critical when \((h, \mu)\) lie in the oblique band \(B_0\). In this case we have
\[
m_s = \frac{n_c}{2} = x_0(\mu + \frac{h}{2}),
\]
so that the ground state energy, momentum and Fermi momentum are given by
\[
E_0 \approx 2\sum_{k=1}^{N_0(\mu + \frac{h}{2})} (\varepsilon(x_k) - \mu - \frac{h}{2}), \quad P \approx 2\cdot 2\pi \sum_{k=1}^{N_0(\mu + \frac{h}{2})} x_k, \quad p_0 = 4\pi x_0(\mu + \frac{h}{2}),
\]
and therefore the low-energy excitations satisfy
\[
\Delta E = \frac{2}{N} \varepsilon'(x_0(\mu + \frac{h}{2})), \quad \Delta p = \frac{4\pi}{N}.
\]
We conclude that the Fermi velocity is given in this case by
\[
v = \frac{\varepsilon'(x_0(\mu + \frac{h}{2}))}{2\pi} = \sqrt{1 - 2(h + 2\mu)},
\]
and thus the central charge is again \(c = 1\). Note that, as in the previous case, \(x_0(\mu + \frac{h}{2})\) is the unique critical point of the integrals \(I_{3,4}\) determining the asymptotic expansion of \(f(T) - f(0)\) in the oblique band \(B_0\).

5.4. Triangle \(T\)

We now have
\[
f(T) - f(0) = -2T \int_0^{x_0(h)} \log \left[ \tilde{b} + \sqrt{\tilde{b}^2 + e^{-\beta(h-x)}} - e^{-\beta h} \right] dx
- 2T \int_{x_0(h)}^{x_0(2\mu)} \log \left[ \tilde{b} + \sqrt{\tilde{b}^2 + 1 - e^{-\beta x}} \right] dx
- 2T \int_{x_0(2\mu)}^{\sqrt{1/2}} \log \left[ \tilde{b} + \sqrt{\tilde{b}^2 + e^{-\beta(\varepsilon-2\mu)} - e^{-2\beta(\varepsilon-\mu)}} \right] dx
\]
\[(5.34)\]

\textsuperscript{11} The coefficients \(\zeta(2l + 2)\) can be expressed as
\[
\zeta(2l + 2) = \frac{(2\pi)^{2l+2}}{2(2l + 2)!} |B_{2l+2}|,
\]
where \(B_k\) is the \(k\)th Bernoulli number defined by \(x/(e^x - 1) = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \).
with
\[
f(0) = -\mu_c - \hbar x_0(h) - \int_{x_0(h)}^{x_0(2\mu)} \varepsilon(x) \, dx - 2 \int_{x_0(2\mu)}^{1/2} \left( \varepsilon(x) - \mu \right) \, dx.
\] (5.35)

The first integral in equation (5.34) coincides with the LHS of equation (5.7), so that its asymptotic series is given by equation (5.13). The last integral in equation (5.34) differs from
\[
\int_{x_0(2\mu)}^{x_0(3/2+\mu)} \log \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + e^{-\beta(\varepsilon-\mu)}} \right] \, dx
\]
by terms \( O(e^{-\beta(\frac{3}{2}-\mu)}) \), and thus its asymptotic series can be computed performing the change of variable \( y = \beta(\varepsilon(x) - 2\mu) \) in the latter integral, with the result
\[
\sum_{l=0}^{\infty} a_l(2\mu) T^{l+1} \int_{0}^{\infty} y^l \log \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + e^{-y}} \right] \, dy.
\] (5.36)

Finally, the second integral in equation (5.34) is dominated by the term \( \frac{1}{2}e^{-\beta(\varepsilon-h)} \) in \( \hat{b} \) for \( x(h) \leq x \leq x_0(\frac{h}{2} + \mu) \), while in the interval \( [x_0(\frac{h}{2} + \mu), x_0(2\mu)] \) the dominant term is instead \( \frac{1}{2}e^{-\beta(\mu-\frac{h}{2})} \). More precisely, we have
\[
\int_{x_0(h)}^{x_0(3/2+\mu)} \log \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + e^{-\beta(\varepsilon-h)}} \right] \, dx
\]
and
\[
\int_{x_0(3/2+\mu)}^{x_0(2\mu)} \log \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + e^{-\beta(\varepsilon-h)}} \right] \, dx
\]
\[
= \int_{x_0(3/2+\mu)}^{x_0(2\mu)} \log \left[ \frac{1}{2} e^{-2(\mu-\varepsilon)} + \sqrt{1 + \frac{1}{4} e^{-\beta(2\mu-\varepsilon)}} \right] \, dx + O(e^{-\beta \min(\frac{3}{2}, \mu-\frac{h}{2})}).
\] (5.37)

Comparing with equation (5.15) we conclude that the asymptotic series of the LHS of equation (5.37) is given by equation (5.16). On the other hand, the asymptotic series of the LHS of equation (5.38) is easily derived performing the change of variable \( y = \beta(2\mu - \varepsilon(x)) \) in the RHS, with the result
\[
\sum_{l=0}^{\infty} \int_{0}^{\infty} y^l \log \left[ \frac{1}{2} e^{-y/2} + \sqrt{1 + \frac{1}{4} e^{-y}} \right] \, dy.
\] (5.39)

Putting all of the above together we obtain the following asymptotic series for the free energy per site in the triangle \( T \):
\[
f(T) - f(0) \sim -2 \sum_{l=0}^{\infty} \left[ a_l(h) + (-1)^l a_l(2\mu) \right] I_l T^{l+2},
\] (5.40)
where the integral $I_l$ is given by equation (5.18). In particular, in this case
\[
f(T) - f(0) \sim \psi(T, h) + \psi(-T, 2\mu),
\]
where
\[
\psi(T, h) = -2 \sum_{l=0}^{\infty} a_l(h) I_l T^{l+2}
\]
is the asymptotic series for $f(T) - f(0)$ in the region $B_1$ (see equation (5.17)).

As in the previous two cases, the asymptotic behavior of the free energy near $T = 0$ spelled out in equation (5.40) is a strong indication that the model is critical when $(h, \mu)$ belongs to the triangle $\mathcal{T}$. To confirm this indication and compute the central charge, note first that in this case
\[
m_s = x_0(h), \quad n_c = 2x_0(2\mu),
\]
and therefore the ground state energy and momentum are given by
\[
E_0 \simeq \sum_{k=1}^{N x_0(h)} (\varepsilon(x_k) - h) + \sum_{k=1}^{N x_0(2\mu)} (\varepsilon(x_k) - 2\mu), \quad P \simeq 2\pi \sum_{k=1}^{N x_0(h)} x_k + 2\pi \sum_{k=1}^{N x_0(2\mu)} x_k.
\]
We thus have two types of low-energy excitations associated with spin and charge, with $\Delta E = \varepsilon'(x_0(h))/N$ and $\Delta E = \varepsilon'(x_0(2\mu))/N$ respectively for the spin and charge excitations. Since in both cases $\Delta P = 2\pi/N$, the Fermi velocities of the spin and charge excitations are respectively given by
\[
v_s = \frac{\varepsilon'(x_0(h))}{2\pi} = \frac{\sqrt{1 - 4h}}{2\pi}, \quad v_c = \frac{\varepsilon'(x_0(2\mu))}{2\pi} = \frac{\sqrt{1 - 8\mu}}{2\pi}.
\]
The leading term in the expansion (5.40) can therefore be expressed as
\[
f(T) - f(0) \simeq -2[a_0(h) + a_0(2\mu)] I_0 T^2 = -\frac{\pi^2 T^2}{3} \left( \frac{1}{\sqrt{1 - 4h}} + \frac{1}{\sqrt{1 - 8\mu}} \right)
\]
\[
= -\frac{\pi T^2}{6} \left( \frac{1}{v_s} + \frac{1}{v_c} \right).
\]
Hence both the charge and the spin sectors of the model are described at low energies by a CFT with central charge $c = 1$. This is indeed known to be the case, as first shown in [29] using the asymptotic Bethe ansatz. Note finally that, as in the previous cases, the points $x_0(h)$ and $x_0(\mu)$ appearing in equation (5.43) for the Fermi velocities are nothing but the critical points of the integrals in the RHS of equation (5.34) for $f(T) - f(0)$.

5.5. Critical behavior on the boundaries

The asymptotic behavior of the free energy on the boundaries of the five ground state phases $\mathcal{W}_i$, $\mathcal{B}_i$, $\mathcal{T}$ can be analyzed in much the same way as above. In particular, it is not difficult, and is certainly of interest, to examine the criticality properties of the model on these boundaries. Consider, as an example, the segment $\mu = 1/8$, $0 < h < 1/4$ separating the triangle $\mathcal{T}$ (su(1|2) phase) from the vertical band $\mathcal{B}_1$ (su(2) phase). From equations (5.34) and (5.38) with $\mu = 1/8$ we readily obtain

\[
https://doi.org/10.1088/1742-5468/ab25e0
\]
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\[ f(T) - f(0) \sim -\frac{\pi T^2}{6v_s} - 2T \int_{x_0(\frac{T}{2} + \theta)}^{1/2} \log \left[ \frac{1}{2} e^{-\frac{\theta}{2}(\frac{1}{4} - \varepsilon)} + \sqrt{1 + \frac{1}{4} e^{-\beta(\frac{1}{4} - \varepsilon)}} \right] dx, \]

where the Fermi (spin) velocity \( v_s \) is given by equation (5.43). Although the last integral is critical at \( x = 1/2 \), it is not asymptotic to \( -\pi T^2 / (6v_c) \) as the Fermi velocity \( v_c \) vanishes at \( \mu = 1/8 \). In fact, performing the usual change of variable

\[ y = \beta(1/4 - \varepsilon(x)) = \left( \frac{1}{2} - x \right)^2 \]

we have

\[ 2T \int_{x_0(\frac{T}{2} + \theta)}^{1/2} \log \left[ \frac{1}{2} e^{-\frac{\theta}{2}(\frac{1}{4} - \varepsilon)} + \sqrt{1 + \frac{1}{4} e^{-\beta(\frac{1}{4} - \varepsilon)}} \right] dx \]

\[ = T^{3/2} \int_0^{\frac{\beta}{2}(1 - h)} \log \left[ \frac{1}{2} e^{-\frac{y}{2}} + \sqrt{1 + \frac{1}{4} e^{-y}} \right] \frac{dy}{\sqrt{y}} \sim \kappa_c T^{3/2}, \]

with

\[ \kappa_c \equiv \int_0^\infty \log \left[ \frac{1}{2} e^{-\frac{y}{2}} + \sqrt{1 + \frac{1}{4} e^{-y}} \right] \frac{dy}{\sqrt{y}} \simeq 1.2255036. \]

Thus the first two nonvanishing terms in the asymptotic expansion of \( f(T) - f(0) \) on the segment \( \mu = 1/8, 0 < h < 1/4 \) are\(^{12}\)

\[ f(T) - f(0) \sim -\kappa_c T^{3/2} - \frac{\pi T^2}{6v_s}. \]

We conclude that the model is not critical on this segment, due to the term in the expansion proportional to \( T^{3/2} \). However, the second term (proportional to \( T^2 \)) can be interpreted as signaling that on the spin sector the model is critical with central charge \( c = 1. \) This conclusion is borne out by the behavior of the ground state energy and momentum, which in this case are given by

\[ E_0 \simeq \sum_{k=1}^{N/2} (\varepsilon(x_k) - h) + \sum_{k=1}^{N/2} (\varepsilon(x_k) - \frac{1}{4}), \quad P \simeq 2\pi \sum_{k=1}^{N/2} x_k + 2\pi \sum_{k=1}^{N/2} x_k. \]

Since now \( \varepsilon'(1/2) = 0 \), the low-energy excitations obtained by removing the mode with \( k = N/2 - 1 \) (or adding the one with \( k = N/2 + 1 \)) carry an energy

\[ \Delta E = \frac{1}{4} - \varepsilon(\frac{1}{2}) - \frac{1}{N} = -\frac{\varepsilon''(\frac{1}{2})}{2N^2} = \frac{1}{N^2}, \]

which is quadratic in \( \Delta p = 2\pi/N. \) These are in fact the excitations responsible for the \( T^{3/2} \) asymptotic behavior of \( f(T) - f(0) \) at low temperatures. On the other hand,

\(^{12}\) In fact, it is straightforward to obtain the full asymptotic series

\[ f(T) - f(0) \sim -\kappa_c T^{3/2} - 2 \sum_{l=0}^{\infty} a_l(h) T^{l+2}. \]

https://doi.org/10.1088/1742-5468/ab25e0
exciting the mode with \( k = N x_0(h) + 1 \) (or suppressing the one with \( k = N x_0(h) - 1 \)) increases the energy by
\[
\Delta E = \frac{\varepsilon'(x_0(h))}{N} = v_s \Delta p.
\]

These excitations are therefore described by a CFT with Fermi velocity \( v_s \).

Proceeding in an analogous way we can determine the critical behavior (including, where appropriate, the value of the central charge) in the remaining parts of the boundary. In general, the vanishing of the spin or charge Fermi velocity implies that the model is not critical in the corresponding sector (and, hence, as a whole), although it can still be critical in the other sector provided that its Fermi velocity is nonzero. Since \( v_s \) (respectively \( v_c \)) vanishes only for \( h = 1/4 \) (respectively \( \mu = 1/8 \)), we conclude that the su(2) KY model should be critical at the vertical segment \( h = 0, 0 \leq \mu < 1/8 \), non-critical (but critical in the spin sector) at the point \( (0,1/8) \) and non-critical (in both sectors) at the other vertex \( (1/4,1/8) \). This is indeed confirmed by a detailed calculation (see table 2 for a summary of the results). This calculation also shows that in all the non-critical parts of the boundary the model is gapless, with \( f(T) - f(0) \) growing as \( T^{3/2} \) at low temperatures.

An interesting situation presents itself on the oblique segment \( \mu = h/2, 0 < h < 1/4 \). Indeed, by equation (5.34)—or (5.24)—with \( \mu = h/2 \) we have
\[
f(T) - f(0) = -2 T \int_0^{x_0(h)} \log \left[ \tilde{b} + \sqrt{\tilde{b}^2 + e^{-\beta(h - \varepsilon(x))}} - e^{-\beta h} \right] dx
\]
\[
-2 T \int_{x_0(h)}^{1/2} \log \left[ \tilde{b} + \sqrt{\tilde{b}^2 + e^{-\beta(x-h)} - e^{-\beta(2\varepsilon-h)}} \right] dx \equiv -2T(J_1 + J_2),
\]
where
\[
\tilde{b} = \frac{1}{2} \left( 1 + e^{-\beta(h - \varepsilon(x))} \right) + e^{-\beta h}, \quad \tilde{b} = \frac{1}{2} \left( 1 + e^{-\beta(\varepsilon(x)-h)} + e^{-\beta \varepsilon(x)} \right).
\]
Since both integrals are critical at \( x_0(h) \equiv x_0(2\mu) \), the Fermi velocity is expected to be
\[
v = \frac{\varepsilon'(x_0(h))}{2\pi} = \frac{\sqrt{1 - 4h}}{2\pi} > 0.
\]
This is indeed the case, since the continuity of the ground state energy, momentum, magnetization and charge density implies that when \( h = 2\mu \) we have
\[
m_s = \frac{n_c}{2} = x_0(h), \quad E_0 \simeq 2 \sum_{k=1}^{N x_0(h)} (\varepsilon(x_k) - h), \quad P \simeq 2 \cdot 2\pi \sum_{k=1}^{N x_0(h)} x_k,
\]
and therefore the low-energy excitations satisfy
\[
\Delta E = \frac{2}{N} \varepsilon'(x_0(h)), \quad \Delta p = \frac{4\pi}{N}.
\]
Performing the changes of variable \( y = \beta(h - \varepsilon(x)) \) and \( y = \beta(\varepsilon(x) - h) \) respectively in the integrals \( J_1 \) and \( J_2 \), and proceeding as above, we easily obtain
\[ J_{1,2} \sim -\frac{T}{2\pi v} \int_0^\infty \log \left[ \frac{1}{2} (1 + e^{-y}) + \sqrt{\frac{1}{4} (1 + e^{-y})^2 + e^{-y}} \right] dy = -\frac{\pi T}{16 v}, \]

and therefore

\[ f(T) - f(0) \sim -\frac{\pi T^2}{4 v}. \]

Thus in the segment \( \mu = h/2, 0 < h < 1/4 \) the low-temperature behavior of the model is described by a single CFT with \( c = 3/2 \). It is also interesting to note that, although both Fermi velocities are nonzero in this case, this result cannot be obtained setting \( v_s = v_c \) in equation (5.44) for the triangle \( T \). The reason is of course that the terms \( e^{-\beta (\mu + h/2 - \epsilon(x))} \) and \( e^{-\beta (\epsilon(x) - \mu - h/2)} \) appearing respectively in \( \bar{b} \) and \( \bar{b} \) are exponentially small throughout their whole integration ranges \([0, x_0(h)]\) and \([x_0(2\mu), 1/2]\) (and can therefore be discarded) only if \( \mu \) is strictly greater than \( h/2 \).

### 5.6. Discussion

It should be noted that the expansions (5.17), (5.33) and (5.40) are all true asymptotic series, i.e. their radius of convergence vanishes. Indeed, for equations (5.17) and (5.40) this stems from the following bound on the integral \( \mathcal{I}_t \) in equation (5.18):

\[ \mathcal{I}_t \gtrless \int_0^\infty [y^t (c e^{-y/2} - e^{-y})] dy = (2^{t+1} c - 1)!, \]

where \( c \equiv \log (1 + \sqrt{5}/2) \). To derive the latter bound, simply observe that the function

\[ \phi(y) \equiv e^{y/2} \log \left[ \frac{1}{2} e^{-y/2} + \sqrt{1 + \frac{1}{4} e^{-y}} \right] \]

is monotonically increasing on \([0, \infty]\), so that \( \phi(y) \gtrless \phi(0) = c \). As for equation (5.33), from the elementary identity

\[ \log(1 + e^{-y}) \gtrsim \log 2 \cdot e^{-y}, \quad y \geq 0, \]
it follows that

$$\int_0^\infty y^k \log (1 + e^{-y}) dy \geq \log 2 \int_0^\infty y^k e^{-y} dy = k! \log 2.$$ 

Our claim then follows from the fact that the coefficient of $T^{l+2}$ in equation (5.33) is proportional to

$$a_2 \left( \frac{h}{2} + \mu \right) \int_0^\infty y^{2l} \log (1 + e^{-y}) dy$$

(see (5.31) and its analog

$$I_4 \sim \sum_{l=0}^{\infty} a_l \left( \frac{h}{2} + \mu \right) T^{l+1} \int_0^\infty y^l \log (1 + e^{-y}) dy$$

for $I_4$).

In figure 2 we compare the free energy per site numerically computed through equation (2.18) with its asymptotic expansions up to four terms at the points $h = \mu = 1/12$ in the triangle $T$ and $h = 1/4, \mu = 0$ in the oblique band $B_0$ (see equations (5.40) and (5.33)). We see from this figure that the agreement between the exact value of $f$ and its expansions is quite good at sufficiently low temperature. To be more precise, from the estimates for each of the integrals in equation (5.34) it can be checked that the exponentially small terms discarded to obtain the asymptotic series (5.40) for $f$ in the triangle $T$ are $O(T e^{-\beta/48})$ at the point $h = \mu = 1/12$. Thus the latter series should not be expected to provide a good approximation for $f$ unless $T \lesssim 0.02$. This is clearly in agreement with the numerical results represented in figure 2 (left). In particular, from the inset in the latter figure it is apparent that the absolute value of the error of the asymptotic expansions up to four terms does not exceed $1.5 \times 10^{-5}$ for $T \leq 0.01$. A similar remark can be made for the band $B_0$ (see figure 2, right). Note also that, although the absolute value of the error of the three asymptotic expansions considered diminishes with their order at sufficiently low temperature (see the insets in figure 2), this need not be the case at higher temperatures. In fact, it is a well-known feature of divergent asymptotic series that the optimum order varies with the range of the independent variable considered.

5.7. Comparison with the $su(1|1)$ and $su(2)$ HS chains

It should be clear from the above results that the asymptotic series of the free energy per site exhibits a different qualitative behavior in each of the regions $B_i, W_i$ and $T$ identified in the previous section (see table 1 and figure 1, left). Moreover, we shall next show that in the bands $B_0$ and $B_i$, corresponding respectively to the $su(1|1)$ and $su(2)$ zero-temperature phases, the asymptotic series for $f$ coincides term by term with those for the free energy of the $su(1|1)$ and $su(2)$ HS chains (with a chemical potential and a magnetic field term, respectively). This does not mean that the supersymmetric KY model is equivalent to the $su(1|1)$ and $su(2)$ HS chains in these regions, since their respective free energies differ by exponentially small terms in $\beta$ which become significant as $T$ increases (see figure 3).

Consider, to begin with, the $su(1|1)$ HS chain with a chemical potential term, whose Hamiltonian shall be taken in accordance with [36] as

https://doi.org/10.1088/1742-5468/ab25e0
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\[
\hat{H}^{(1|1)} = \frac{\pi^2}{N^2} \sum_{i<j} \frac{1 - P_{ij}^{(1|1)}}{\sin^2(\pi(i - j)/N)} - \lambda \mathcal{F}. \tag{5.45}
\]

In the oblique band \(B_0\), the number of ‘down’ fermions at \(T = 0\) is \(n^2 = 0\). We should then compare (5.45) with the (spin chain version of) the Hamiltonian of the supersymmetric KY model in the sector \(n^2 = 0\), given by

\[
\hat{H}|_{n^2=0} = \frac{1}{2N^2} \sum_{i<j} \frac{1 - P_{ij}^{(1|1)}}{\sin^2(\pi(i - j)/N)} - (\mu + \frac{h}{2})N_1 \tag{5.46}
\]

(see equation (2.6)). We thus see that \(\hat{H}^{(1|1)} = 2\pi^2 \hat{H}|_{n^2=0}\) provided that \(\lambda = 2\pi^2(\mu + h/2)\).

The free energy of the Hamiltonian \(\hat{H}^{(1|1)}\) was computed in [36], namely

\[
f^{(1|1)}(T) = -\frac{T}{\pi} \int_0^{\pi} \log \left[ 1 + e^{-2\pi^2 \beta(\epsilon(x) - \frac{h}{2} - \mu)} \right] \, dx. \tag{5.47}
\]

Taking into account the connection between \(\hat{H}^{(1|1)}\) and \(\hat{H}|_{n^2=0}\), we should then expect that in the oblique band \(B_0\), and at sufficiently low temperature,

\[
f(T) \simeq \frac{1}{2\pi^2} f^{(1|1)}(2\pi^2 T) = -2T \int_0^{1/2} \log \left[ 1 + e^{-\beta(\epsilon(x) - \frac{h}{2} - \mu)} \right] \, dx. \tag{5.47}
\]

In fact, from equations (5.25)–(5.28) it readily follows that

\[
f(T) = f_0 - 2T(I_3 + I_4) + O(e^{-\beta \min(h, \frac{h}{2} - \mu)})
\]

\[
= -2T \int_0^{1/2} \log \left[ 1 + e^{-\beta(\epsilon(x) - \frac{h}{2} - \mu)} \right] \, dx + O(e^{-\beta \min(h, \frac{h}{2} - \mu)}).
\]

Thus the low-temperature asymptotic series of \(f^{(1|1)}(2\pi^2 T)/(2\pi^2)\) coincides term by term with that of \(f\) in the band \(B_0\), as claimed.

https://doi.org/10.1088/1742-5468/ab25e0
Similarly, in the vertical band $B_1$ the number of bosons at $T = 0$ is $n_0 = 0$. The (spin chain version of the) Hamiltonian of the supersymmetric KY model in the subspace $n_0 = 0$ is given by

$$\hat{H}|_{n_0=0} = \frac{1}{2N^2} \sum_{i<j} \frac{1 + P_{ij}}{\sin^2(\pi(i-j)/N)} - \frac{h}{2}(N_1 - N_2) - \mu N,$$

where $P_{ij}$ is the ordinary permutation operator. This should be compared with the Hamiltonian of the (antiferromagnetic) $su(2)$ HS chain in an external magnetic field from [20] (with $K = -1$ and $2B = h$), namely

$$\hat{H}^{(2)} = \frac{1}{2N^2} \sum_{i<j} \frac{P_{ij} - 1}{\sin^2(\pi(i-j)/N)} - \frac{h}{2}(N_1 - N_2).$$

From equation (3.9) it follows that in the thermodynamic limit $\hat{H}|_{n_0=0} = \hat{H}^{(2)} - \mu_c N$. It should therefore be expected that in the vertical band $B_1$, and at sufficiently low temperatures,

$$f(T) \simeq f^{(2)}(T) - \mu_c = -\mu_c - 2T \int_0^{1/2} \log \left[ \cosh \left( \frac{\beta h}{2} \right) + \sqrt{\sinh^2 \left( \frac{\beta h}{2} \right) + e^{\beta \epsilon(x)}} \right] dx \equiv g(T),$$

where we have used the exact formula for $f^{(2)}$ from [20]. The asymptotic series of $g$ around $T = 0$ can be obtained following the above procedure. More precisely, we first write

$$g(T) = g(0) - 2T \int_{x_0(h)}^{x_0(\infty)} \log \left[ \frac{1}{2} (1 + e^{-\beta h}) + \sqrt{\frac{1}{4} (1 - e^{-\beta h})^2 + e^{-\beta (\epsilon(x) - h)}} \right] dx$$

$$- 2T \int_{x_0(h)}^{1/2} \log \left[ \frac{1}{2} (1 + e^{-\beta h}) e^{-\frac{1}{2} \epsilon(x) - h} + \sqrt{1 + \frac{1}{4} (1 - e^{-\beta h})^2 e^{-\beta (\epsilon(x) - h)}} \right] dx,$$
where

\[ g(0) = -\mu_c - hx_0(h) - \int_{x_0(h)}^{1/2} \varepsilon(x) \, dx = f(0) \]

(see equation (5.6)). Discarding the exponentially small term \( e^{-\beta h} \), and comparing with equations (5.9) and (5.10), we thus see that

\[ g(T) \sim f(0) - 2T(I_1 + I_2). \]

From equations (5.5), (5.7) and (5.8) we conclude that \( g \) has the same asymptotic series as \( f \) in the vertical band \( B_1 \), as stated.

6. Low-temperature asymptotic expansions of densities and susceptibilities

Differentiating the asymptotic series for the free energy per site obtained in the previous section with respect to the parameters \( h \) and \( \mu \), we shall next derive the analogous series for the magnetization per site, the charge density and their corresponding susceptibilities. Since the asymptotic series of all these quantities are trivially equal to their zero-temperature values on the wedges \( W_0 \) and \( W_1 \), we shall concentrate in what follows on the remaining regions \( B_0, B_1 \) and \( T \).

6.1. \( B_1 \cup T \)

In the region \( B_1 \cup T \), by equation (5.41) the asymptotic series for the magnetization is given by

\[ m_s \sim -\frac{\partial f(0)}{\partial h} - \frac{\partial \psi(T, h)}{\partial h}. \]

From equations (5.6) and (5.35) it follows that

\[ -\frac{\partial f(0)}{\partial h} = x_0(h), \]

which coincides with the value of \( m_s(0) \) computed in section 4. Using equation (5.42) we thus obtain

\[ m_s \sim x_0(h) + \sum_{l=0}^{\infty} \frac{2^{l+2}(2l + 1)!!}{l! (1 - 4h)^{l+2}} I_l T^{l+2}, \tag{6.1} \]

where \( I_l \) is given by equation (5.18). The first few terms in this series are thus

\[ m_s = x_0(h) + \frac{2\pi^2 T^2}{3(1 - 4h)^{3/2}} + \frac{24\zeta(3)T^3}{(1 - 4h)^{5/2}} + \frac{12\pi^4 T^4}{(1 - 4h)^{7/2}} + O(T^5). \]

Differentiating equation (6.1) with respect to \( h \) we obtain the corresponding asymptotic series for the magnetic susceptibility \( \chi_s \):
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\[ \chi_s \sim (1 - 4h)^{-1/2} + \sum_{l=0}^{\infty} \frac{2^{l+3}(2l + 3)!!}{l! (1 - 4h)^{l+\frac{3}{2}}} \mathcal{I}_l T^{l+2}. \]  

(6.2)

To the best of our knowledge, for \((h, \mu)\) lying on the vertical band \(B_1\) the asymptotic expansions (6.1)–(6.2) are new.

Consider next the charge density \(n_c\) and its susceptibility \(\chi_c\). To begin with, it is obvious that \(n_c \sim 1\) and \(\chi_c \sim 0\) in the vertical band \(B_1\), since in this region \(f(0) + \mu\) and the asymptotic series for \(f(T) - f(0)\) depend only on \(h\) (see equations (5.6) and (5.17)). On the other hand, in the triangle \(\mathcal{T}\) the asymptotic series of \(n_c\) and \(\chi_c\) follow immediately from those of \(m_s\) and \(\chi_s\) taking into account equation (5.41), namely

\[ n_c \sim 2x_0(2\mu) + \sum_{l=0}^{\infty} \frac{2^{l+3}(2l + 1)!!}{l! (1 - 8\mu)^{l+\frac{3}{2}}} \mathcal{I}_l (-T)^{l+2}, \]  

(6.3)

\[ \chi_c \sim 4(1 - 8\mu)^{-1/2} + \sum_{l=0}^{\infty} \frac{2^{l+5}(2l + 3)!!}{l! (1 - 8\mu)^{l+\frac{3}{2}}} \mathcal{I}_l (-T)^{l+2}, \quad (h, \mu) \in \mathcal{T}. \]  

(6.4)

In fact, equations (6.2) and (6.4) agree to all orders in \(T\) with the asymptotic series which can be obtained from the low-temperature approximations (up to exponentially small terms) to \(m_s\) and \(n_c\) in [28]. To see this, note first that in our notation the latter approximations read

\[ 1 - n_c \sim 2 \int_{-\infty}^{\infty} \frac{\sqrt{1 - 8\mu + 4Tx}}{(1 + 4e^x)^{3/2}} e^x \, dx, \quad 1 - 2m_s \sim 2 \int_{-\infty}^{\infty} \frac{\sqrt{1 - 4h - 4Tx}}{(1 + 4e^x)^{3/2}} e^x \, dx. \]

Expanding the square root in either integral in powers of \(T\) and using the alternative expression (5.19) for the integrals \(\mathcal{I}_l\) we readily obtain equations (6.1) and (6.3), from which the corresponding asymptotic series for \(\chi_s\) and \(\chi_c\) follow by term-by-term differentiation.

From the previous formulas it follows that in the triangle \(\mathcal{T}\) the magnetization \(m_s\) and its susceptibility \(\chi_s\) depend only on \(h\), while \(n_c\) and \(\chi_c\) depend only on \(\mu\), up to terms \(O(T^k e^{-\beta})\) (with \(c > 0\)). This is an indication that spin-charge separation is valid at low temperatures up to terms exponentially small in \(\beta\), as we shall more explicitly show in what follows.

Indeed, inverting the asymptotic expansion (6.1) of \(m_s\) to obtain an expansion of \(1 - 2m_s(0) \equiv (1 - 4h)^{1/2}\) to a given order in \(T\) and substituting into equation (6.2) we can derive a corresponding asymptotic expansion of \(\chi_s\) in terms of \(m_c\). For instance, up to third order in \(T\) we have

\[ \chi_s = \frac{1}{1 - 2m_s} \left[ 1 + \frac{8\pi^2T^2}{3(1 - 2m_s)^4} + \frac{192\zeta(3)T^3}{(1 - 2m_s)^6} \right] + O(T^4). \]  

(6.5)

Comparing the asymptotic series (6.3)–(6.4) for \(n_c\) and \(\chi_c\) with the corresponding series (6.1)–(6.2) for \(m_s\) and \(\chi_s\) we conclude that the asymptotic series of \(\chi_c\) in the triangle \(\mathcal{T}\) is given by \(\chi_c \sim 4X(\frac{h}{2}, -T)\), where \(X(m_s, T)\) is the asymptotic series for \(\chi_s\) in terms of \(m_s\) and \(T\). From equation (6.5) we thus obtain

https://doi.org/10.1088/1742-5468/ab25e0
\[ \chi_c = 4 \left[ 1 + \frac{8\pi^2T^2}{3(1-n_c)^4} - \frac{192\zeta(3)T^3}{(1-n_c)^6} \right] + O(T^4), \quad (h, \mu) \in \mathcal{T}. \]  

From the asymptotic expansions (6.5)–(6.6) it is clear that strong spin-charge separation holds at sufficiently low temperatures if we discard exponentially small terms. Previously, this result had been numerically checked only for \( h = 0 \) (i.e. \( m_s = 0 \)) and within the triangle \( \mathcal{T} \) [16, 28]. With the help of the exact formulas (2.21)–(2.22) for the susceptibilities derived in section 2, we have been able to numerically verify the strong spin-charge separation for nonzero magnetic fields and several charge densities (see figures 4 and 5). As is customary, we have taken \((m_s, n_c)\) instead of \((h, \mu)\) as independent variables, which requires solving for the latter variables in terms of the former by means of equations (2.19) and (2.20). For this to be possible, the mapping \((h, \mu) \mapsto (m_s, n_c)\) should be invertible in the temperature range considered. At zero temperature, this is the case provided that \((h, \mu)\) lies in the triangle \( \mathcal{T} \), which is mapped to the triangle \( 0 < 2m_s < n_c < 1 \). For this reason, in the previous plots we have taken magnetizations not greater than \( n_c/2 \).
Again, when \((h, \mu) \in \mathcal{B}_1\) the asymptotic expansion (6.5) appears to be new, while in
the triangle \(\mathcal{T}\) equations (6.5)–(6.6) coincide with those derived in [28] to order \(T^2\). (To
verify this assertion one should first replace the dimensionless quantities \(T, \chi_s\) and \(\chi_c\)
in the latter equations by their true values \(T/K, K\chi_s\) and \(K\chi_c\), with \(K = 2\pi^2 t\), and
take into account that in the latter reference \(t\) has been set to 1.) Note also that the
asymptotic expansions of \(\chi_s\) and \(\chi_c\) do \textit{not} have the same functional form, due to the
different sign of the coefficients of the odd powers of \(T\). This fact had not been previously
noted, since it can only be detected at order \(T^3\) or higher.

6.2. \(\mathcal{B}_0\)

Consider, finally, the oblique band \(\mathcal{B}_0\). The asymptotic series for \(m_s\) and \(\chi_s\) are easily
obtained differentiating equations (5.25) and (5.33), i.e.

\[
m_s \sim x_0 \left( \frac{h}{2} + \mu \right) + 2 \sum_{l=0}^{\infty} \frac{ (2l+1)(4l+1)!! }{ [1 - 2(h + 2\mu)]^{2l+2} } \zeta(2l+2) T^{2l+2},
\]

\[
\chi_s \sim \frac{1}{2} \left[ 1 - 2(h + 2\mu) \right]^{-1/2} + \sum_{l=0}^{\infty} \frac{ (2l+3)(4l+3)!! }{ [1 - 2(h + 2\mu)]^{2l+2} } \zeta(2l+2) T^{2l+2}.
\]

Proceeding as above, it is straightforward to derive from the previous series an
asymptotic expansion of \(\chi_c\) in terms of the variables \((m_s, T)\) in the oblique band \(\mathcal{B}_0\) to any
desired order. The first few terms in this expansion are

\[
\chi_s \sim \frac{1}{2(1 - 2m_s)} \left[ 1 + \frac{4\pi^2 T^2}{3(1 - 2m_s)^4} + \frac{28\pi^4 T^4}{9(1 - 2m_s)^8} + \frac{3352\pi^6 T^6}{27(1 - 2m_s)^{12}} + O(T^8) \right].
\]

From equations (5.25) and (5.33) it follows that the asymptotic series of the free energy is a function of \(h + 2\mu\). Thus the asymptotic series of \(n_c\) and \(\chi_c\) are proportional to those of \(m_s\) and \(\chi_s\), namely,

\[
n_c \sim 2m_s, \quad \chi_c \sim 4\chi_s.
\]

Thus in the band \(\mathcal{B}_0\) the magnetic and the charge quantities are proportional, up to
exponentially small terms in \(\beta\). In particular, from equation (6.9) we obtain the following
asymptotic expansion of \(\chi_c\) in terms of \(1 - n_c\) in the band \(\mathcal{B}_0\):

\[
\chi_c \sim \frac{2}{1 - n_c} \left[ 1 + \frac{4\pi^2 T^2}{3(1 - n_c)^4} + \frac{28\pi^4 T^4}{9(1 - n_c)^8} + \frac{3352\pi^6 T^6}{27(1 - n_c)^{12}} + O(T^8) \right].
\]

7. Conclusions

In this paper we analyze the thermodynamics of the supersymmetric \(su(m)\) \(t-J\) model
with long-range interactions through a novel approach based on the transfer matrix
method. This method exploits the equivalence of the latter model to a modification of
the \(su(1|m)\) supersymmetric HS spin chain, whose spectrum coincides with that of an

https://doi.org/10.1088/1742-5468/ab25e0
inhomogeneous vertex model with a simple dispersion function. The energy function of this vertex model is related to suitable representations of the Yangian associated with supersymmetric Young tableaux and their corresponding Haldane motifs. This makes it possible to express the partition function by means of an appropriate site-dependent transfer matrix, which in the thermodynamic limit yields a simple closed-form expression for the free energy per site in terms of the largest eigenvalue (in modulus) of the latter matrix. One of the main advantages of our method is the fact that it can be applied to a wide range of models with (broken or unbroken) Yangian symmetry and arbitrary dispersion relations, including the supersymmetric PF and FI spin chains. In the su(1|m) case analyzed in the paper, we explicitly show that the free energy per site of all of these models can be expressed in terms of a function of one variable obeying an algebraic equation which generalizes the one derived by Kato and Kuramoto for multi-component boson–fermion systems [16, 35]. We conjecture that this is still the case for more general su(n|m) models with n > 1.

In the spin 1/2 case, we apply the explicit expression for the free energy to analyze in detail the thermodynamic and criticality properties of the model. To this end, we first determine all the ground state phases by computing the zero-temperature values of the magnetization and charge densities for arbitrary values of the magnetic field strength and the charge chemical potential. In particular, we show that the magnetic and charge susceptibilities present hitherto unnoticed jump discontinuities along the common boundary of the su(1|2) and su(1|1) phases. We then derive the complete asymptotic series of the free energy per site, showing that it takes different forms on each of the ground state phases. From the lowest-order term in the asymptotic series we determine the regions in parameter space in which the model is described at low energies by an effective CFT, and compute its corresponding central charge. Our results confirm that in the su(1|2) phase the model is described by a CFT with conformal charge c = 1 in both the spin and the charge sectors. However, in the su(2) and su(1|1) phases we find that the model is equivalent to a single CFT with c = 1. We also analyze in detail the critical behavior on the boundary between zero-temperature phases, finding that the system can be critical, gapless but not critical or even critical in the spin sector but not in the charge one. Using the asymptotic series for the free energy, we also derive the complete asymptotic series of the magnetization and charge densities and their corresponding susceptibilities. We numerically verify the strong spin-charge separation characteristic of the model for different (nonzero) values of the magnetization and the charge density, and show that it persists at all orders in the asymptotic expansion. This can be regarded as an analytic confirmation of spin-charge separation in a su±iciently small range of temperatures near T = 0, where the asymptotic expansions provide an excellent approximation for the thermodynamic functions.

Although in this paper we have concentrated on the su(2) case, it would be of interest to apply the transfer matrix method to investigate the ground state phases, thermodynamics and criticality properties of the general su(m) KY model with m > 2. In fact, as explained above, this method could in principle be extended to more general models with partial or total Yangian symmetry provided that their spectrum coincides with that of the inhomogeneous vertex model (3.18) and (3.19) for a suitable dispersion relation $\mathcal{E}_N$. 

https://doi.org/10.1088/1742-5468/ab25e0
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Acknowledgments

This work was partially supported by Spain’s MINECO grant FIS2015-63966-P and the Universidad Complutense de Madrid through a G/6400100/3000 Grant. JAC would also like to acknowledge the financial support of the Universidad Complutense de Madrid through a 2015 predoctoral scholarship.

Appendix A. Derivation of equations (5.7) and (5.8)

In this appendix we shall establish the validity of the estimates (5.7) and (5.8) for the integrals appearing in equation (5.5). To begin with, the difference between the first of these integrals and the integral $I_1$ in equation (5.9) is given by

$$\Delta_1 \equiv \int_0^{x_0(h)} \log \left[ \frac{b + \sqrt{b^2 + e^{-\beta(h-\varepsilon)} - e^{-\beta h}}}{\frac{1}{2} + \sqrt{\frac{1}{4} + e^{-\beta(h-\varepsilon)}}} \right] dx = \int_0^{x_0(h)} \log (1 + \phi_1(x)) dx,$$

where

$$\phi_1(x) \equiv \frac{1}{2} (e^{-\beta(\mu + \frac{1}{2} - \varepsilon)} + e^{-\beta h}) + \Delta R_1 \quad \text{(A.2)}$$

and

$$\Delta R_1 \equiv \sqrt{\frac{1}{4} + e^{-\beta(h-\varepsilon)}} - e^{-\beta h} - \sqrt{b^2 + e^{-\beta(h-\varepsilon)}} = \frac{\sqrt{b^2 - \frac{1}{4}} - e^{-\beta h}}{\sqrt{b^2 + e^{-\beta(h-\varepsilon)}} - e^{-\beta h} + \sqrt{\frac{1}{4} + e^{-\beta(h-\varepsilon)}}}.$$

Since $\tilde{b} > 1/2$ the denominator in $\Delta R_1$ is $\geq 1$, and hence

$$|\Delta R_1| \leq e^{-\beta h} + \tilde{b}^2 - \frac{1}{4} - e^{-\beta h} + \frac{1}{4} (e^{-\beta(\mu + \frac{1}{2} - \varepsilon)} + e^{-\beta h}) (2 + e^{-\beta(\mu + \frac{1}{2} - \varepsilon)} + e^{-\beta h})$$

$$\leq e^{-\beta(\mu - \frac{1}{2})} + 2e^{-\beta h},$$

where we have used the inequality $\varepsilon(x) \leq \varepsilon$ valid in the interval $[0, x_0(h)]$. From equation (A.2) we thus obtain

$$|\phi_1(x)| \leq \frac{3}{2} e^{-\beta(\mu - \frac{1}{2})} + \frac{5}{2} e^{-\beta h} = O\left(e^{-\beta \min(\mu - \frac{1}{2}, h)}\right),$$

which in particular shows that when $(h, \mu) \in B_1$ the function $\phi_1(x)$ tends to zero as $T \to 0$ uniformly in $x \in [0, x_0(h)]$. Since $|\log(1 + \phi_1(x))| = O(\phi_1(x))$ when $\phi_1(x) \to 0$, from equation (A.1) it immediately follows that

$$\Delta_1 = O\left(e^{-\beta \min(\mu - \frac{1}{2}, h)}\right).$$

Similarly, the difference between the LHS of equation (5.8) and the integral (5.10) can be written as

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\[ \Delta_2 = \int_{x_0(h)}^{1/2} \log(1 + \phi_2(x)) \, dx, \quad (A.3) \]

where

\[ \phi_2(x) \equiv \frac{\frac{1}{2} (e^{-\frac{\beta}{2}(x+h)} + e^{-\beta(\mu - \frac{1}{4})}) + \Delta R_2}{\frac{1}{2} e^{-\frac{\beta}{2}(e-h)} + \sqrt{1 + \frac{1}{4} e^{-\beta(e-h)}}} \]

\[ \Delta R_2 \equiv \sqrt{\hat{b}^2 + 1 - e^{-\beta \varepsilon}} - \sqrt{1 + \frac{1}{4} e^{-\beta(e-h)}} = \frac{\hat{b}^2 - \frac{1}{2} e^{-\beta(e-h)} - e^{-\beta \varepsilon}}{\sqrt{\hat{b}^2 + 1 - e^{-\beta \varepsilon} + \sqrt{1 + \frac{1}{4} e^{-\beta(e-h)}}}}. \]

Proceeding as before, and taking into account that in the interval \([x_0(h), 1/2]\) we have

\[ \varepsilon(x) \geq h, \quad \mu - \frac{\varepsilon(x)}{2} \geq \mu - \frac{1}{8}, \]

after a straightforward calculation we obtain the estimate

\[ \left| \hat{b}^2 - \frac{1}{4} e^{-\beta(e-h)} - e^{-\beta \varepsilon} \right| \leq e^{-\frac{\beta}{2}(e+h)} + e^{-\beta(\mu - \frac{1}{4})} + e^{-\beta \varepsilon} \leq 2e^{-\beta h} + e^{-\beta(\mu - \frac{1}{8})}. \]

From the definition of \(\phi_2(x)\) it immediately follows that

\[ |\phi_2(x)| \leq \frac{3}{2} e^{-\beta(\mu - \frac{1}{8})} + \frac{5}{2} e^{-\beta h} = O(e^{-\beta \min(\mu - \frac{1}{8}, h)}), \]

which easily yields (5.8) on account of equation (A.3).

Appendix B. Asymptotic series for the integral \(I_1\)

In this appendix we derive the asymptotic series (5.13) for the integral \(I_1\). Calling for simplicity \(a_l \equiv a_l(h)\) and setting

\[ g(y) \equiv \log \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + e^{-y}} \right], \quad \phi(z) \equiv \sum_{l=0}^{\infty} (-1)^l a_l z^l, \]

we need to show that

\[ \int_0^{\beta h} g(y) \phi(Ty) \, dy \sim \sum_{l=0}^{\infty} (-1)^l a_l T^l \int_0^{\infty} y^l g(y) \, dy. \]

Note first of all that the power series \(\phi(z)\) converges for \(|z| < 1/4\). Since \(h < 1/4\) when \((h, \mu) \in B_1\), it follows that \(Ty \) lies inside the convergence disc of \(\phi(z)\) for fixed \(h\) and all \(y \in [0, \beta h]\). We must check that for all \(n \in \mathbb{N}\)

\[ \sum_{l=0}^{n} (-1)^l a_l T^l \int_0^{\infty} y^l g(y) \, dy - \int_0^{\beta h} g(y) \phi(Ty) \, dy = o(T^n), \]

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\[ \sum_{l=0}^{n} (-1)^{l} a_{l} T^{l} \int_{\beta h}^{\infty} y^{l} g(y) dy - \int_{0}^{\beta h} dy g(y) \sum_{l=n+1}^{\infty} (-1)^{l} a_{l} (T y)^{l} = o(T^{n}). \]  

(B.1)

Since

\[ 0 \leq \int_{\beta h}^{\infty} y^{l} g(y) dy \leq \int_{0}^{\infty} y^{l} e^{-y} dy = O(e^{-\beta h}), \]

the first sum in equation (B.1) is \( O(e^{-\beta h}) \). As for the second term, note that

\[ \sum_{l=n+1}^{\infty} (-1)^{l} a_{l} (T y)^{l} = (T y)^{n+1} \tilde{\phi}(T y), \]

where \( \tilde{\phi}(z) \) is a convergent power series and hence analytic for \( |z| < 1/4 \). Since \( T y \in [0, h] \subset [0, 1/4) \) when \( y \in [0, \beta h] \), it follows that \( |\phi(T y)| < M(h) \) independently of \( \beta \). Hence

\[ \left| \int_{0}^{\beta h} dy g(y) \sum_{l=n+1}^{\infty} (-1)^{l} a_{l} (T y)^{l} \right| \leq M(h) T^{n+1} \int_{0}^{\beta h} y^{n+1} g(y) dy \]

\[ \leq \left( M(h) \int_{0}^{\infty} y^{n+1} e^{-y} dy \right) T^{n+1} = (n+1)! M(h) T^{n+1}, \]

so that both terms in the LHS of equation (B.1) are indeed \( o(T^{n}) \).

Appendix C. Alternative expression for the integrals \( I_{l} \)

In this appendix we will derive the alternative expressions (5.19) for the integrals \( I_{l} \) appearing in the asymptotic series for the free energy in the triangle \( T \) and the vertical band \( B_{1} \) (see equation (5.18)). To begin with, consider the integral

\[ I_{l,1} = \int_{0}^{\infty} y^{l} \{ \log \left[ \frac{1}{2} e^{-y/2} + \sqrt{1 + \frac{1}{4} e^{-y}} \right] \} dy = \int_{0}^{\infty} y^{l} \text{arc sinh}(\frac{1}{2} e^{-y/2}) dy. \]

Writing \( \text{arc sinh}(\frac{1}{2} e^{-y/2}) \) as the integral of its derivative, namely

\[ \text{arc sinh}(\frac{1}{2} e^{-y/2}) = \frac{1}{2} \int_{y}^{\infty} \frac{dx}{\sqrt{1 + 4e^{-x}}}, \]

we can express \( I_{l,1} \) as a double integral as

\[ I_{l,1} = \frac{1}{2} \int_{0}^{\infty} dy \int_{y}^{\infty} dx \frac{y^{l}}{\sqrt{1 + 4e^{-x}}}. \]

Reversing the order of integration we obtain

\[ I_{l,1} = \frac{1}{2} \int_{0}^{\infty} \frac{dx}{\sqrt{1 + 4e^{-x}}} \int_{0}^{x} dy y^{l} = \frac{1}{2(l+1)} \int_{0}^{\infty} \frac{x^{l+1}}{\sqrt{1 + 4e^{-x}}} dx. \]  

(C.1)
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Consider next the second integral in equation (5.18), namely

$$ I_{l,2} = \int_0^{\infty} y^l \left\{ \log \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + e^{-y}} \right] \right\} dy = \int_0^{\infty} y^l \left[ -\frac{y}{2} + \text{arcsinh} \left( \frac{1}{2} e^{y/2} \right) \right] dy. $$

Proceeding as before we write

$$ -\frac{y}{2} + \text{arcsinh} \left( \frac{1}{2} e^{y/2} \right) = \frac{1}{2} \int_y^{\infty} \left( 1 - \frac{1}{\sqrt{1 + 4e^{-x}}} \right) dx $$

and therefore

$$ I_{l,2} = \frac{1}{2} \int_0^{\infty} dy \int_y^{\infty} dx \, y^l \left( 1 - \frac{1}{\sqrt{1 + 4e^{-x}}} \right) = \frac{1}{2} \int_0^{\infty} dx \left( 1 - \frac{1}{\sqrt{1 + 4e^{-x}}} \right) \int_0^x dy \, y^l $$

$$ = \frac{(-1)^l}{2(l + 1)} \int_{-\infty}^{0} x^{l+1} \left( \frac{1}{\sqrt{1 + 4e^{x}}} - 1 \right) dx. $$

Combining equations (C.1) and (C.2) we obtain the first equality in equation (5.19). The second equality in the latter equation easily follows by integrating by parts in equations (C.1) and (C.2).

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