Hyperhamiltonian dynamics

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Abstract. We introduce an extension of hamiltonian dynamics, defined on
hyperkahler manifolds, which we call “hyperhamiltonian dynamics”. We show
that this has many of the attractive features of standard hamiltonian dynam-
ics. We also discuss the prototypical integrable hyperhamiltonian systems, i.e.
quaternionic oscillators.

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Introduction

The description provided by Hamiltonian dynamics applies to many fields of physics; due to its rich geometrical structure, it is also very convenient and indeed widely used whenever possible.

Hamiltonian formalism is based on symplectic structures; a special but relevant class of symplectic manifolds is provided by Kahler manifolds. Actually, in any symplectic manifold $M$ we can give locally (and globally if $M$ is contractible, e.g. $M = \mathbb{R}^{2n}$) a complex structure associated to the symplectic one and, by the introduction of a suitable metric, a Kahler structure.

In relatively recent years, mathematicians on the one hand, and (theoretical and mathematical) physicists on the other, have become interested in a special kind of Kahler structures, i.e. hyperkahler ones \[8\]. These are Kahler with respect to three different complex structure, having such relations among them so that, roughly speaking, they can be seen as the complex structures associated to the three independent imaginary units of the quaternions field.

A riemannian manifold (necessarily of dimension $4n$) equipped with a hyperkahler structure is called a hyperkahler manifold. These turn out to be very interesting from the point of view of Geometry \[20, 31, 33\], and also relevant in the description of (non-abelian) monopoles \[1, 19, 27\]; they also bear a close connection, which we will not discuss here, with twistors \[8, 30, 35\] and thus in particular to interesting classes of integrable systems \[17, 28\]. It has also been realized that hyperkahler (and quaternionic-kahler \[3\]) manifolds are relevant in supersymmetry and supergravity theories and related to sigma-models, see e.g. \[14, 18, 27\]. Canonical examples of hyperkahler manifolds are quaternion linear spaces $H^n \approx \mathbb{R}^{4n}$ and cotangent bundles of (special) Kahler manifolds \[13, 34\]; a specially fruitful method of constructing nontrivial hyperkahler manifolds is through a generalization of Marsden-Weinstein momentum map \[27\].

For an overview of recent results in quaternionic and hyperkahler geometry (not needed in the present work), the reader is referred to \[20\].

It is obvious that a hyperkahler structure can be described in symplectic terms; we speak then of a hypersymplectic structure. It is remarkable, and came much to our surprise, that it is possible to provide a generalization of Hamilton mechanics based on such a hypersymplectic structure, which we do here. What is more relevant is that this hyperhamiltonian dynamics retains most of the appealing features of standard hamiltonian mechanics, as we show in the present note.

Our initial motivation was provided by integrable systems. We will consider a special class of these, i.e. quaternionic oscillators (see sect. 7), which we expect to be the paradigm of a nontrivial integrable hyperhamiltonian system.

Limiting to consider systems with compact energy manifolds, a $2m$-dimensional hamiltonian integrable system can be described by means of suitable real (action-angle) coordinates $(I_a, \varphi_a)$, or more compactly complex coordinates $z_a = I_a \exp[i \varphi_a]$ so that its evolution is described by $m$ constant complex rotations, $\dot{z}_a(t) = i \omega_a z(t)$. 
In slightly different (but equivalent) terms, we can use coordinates \((I_a, g_a)\) where \(g_a \in G = U(1)\); here of course \(g_a = \exp[i \varphi_a]\) so that we are just describing action angle coordinates in group-theoretical terms, using the isomorphism \(S^1 \simeq \mathbb{C}_1 \simeq U(1)\) (here \(\mathbb{C}_1\) are the complex numbers of unit norm). In this language, the evolution is described by \(dI_a/dt = 0\), \(dg_a/dt = \gamma_a\), constant elements of the Lie algebra \(u(1)\); indeed, as well known, the isomorphism \(S^1 \simeq \mathbb{C}_1 \simeq U(1)\) identifies the Lie algebra of \(U(1)\) with imaginary numbers.

As discussed below, a \(4n\)-dimensional hyperhamiltonian integrable system can be described by real (spin) coordinates \((I_a; g_a)\), where \(g_a \in G = SU(2)\). Its evolution is accordingly described by \(dI_a/dt = 0\), \(dg_a/dt = \gamma_a\), constant elements of the Lie algebra \(su(2)\).

As well known, an integrable hamiltonian system in a \(2m\) dimensional phase space \(M\) is associated to a fibration of \(M\) in tori \(T^m\). In the case of integrable hyperhamiltonian systems in a \(4k\) dimensional phase space, we will not have a fibration in tori \(T^{2k} = S^1 \times ... \times S^1 \equiv U(1) \times ... \times U(1)\), but rather in manifolds \(V^k = S^3 \times ... \times S^3 \equiv SU(2) \times ... \times SU(2)\).

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1 Quaternionic symplectic structures.

In this section we recall the basic geometric definitions to be used in the following. These will mainly concern hyperkahler and quaternionic geometry; for a short introductions to these the reader is referred to [8, 13], while more details on these are provided e.g. by [1, 3, 6, 7, 8, 9, 12, 17, 26, 31, 33]. Symplectic geometry is discussed e.g. in [5, 14, 18, 24]; for hamiltonian mechanics see e.g. [1, 3, 6].

We preliminarly recall that if \(M\) is a \(2m\) dimensional manifold equipped with a (riemannian) metric \(g\), a complex structure on \(M\) is a \((1, 1)\) type tensor field \(Y\) such that \(Y^2 = -I\) which is covariant constant; a symplectic form \(\omega\) on \(M\) is a non-degenerate and closed two-form. Then \(\omega(v, w)\) can be written as \(g(v, Jw)\) for some \((1, 1)\)-type antisymmetric tensor field on \(M\). If this \(J\) is orthogonal for \(g\), we say that \(\omega\) is compatible with the metric (or briefly \(g\)-compatible), or equivalently that \(\omega\) is unimodular. If this is the case, then \((1/m!)\omega \wedge ... \wedge \omega = s\Omega\), with \(s = \pm 1\) and \(\Omega\) the volume form on \(M\). We say accordingly that unimodular symplectic forms are of positive or negative type.

We also recall that \(M\) is a Kahler manifold if it is equipped with a metric \(g\), a complex structure \(Y\) and a symplectic form \(\omega\), satisfying the Kahler relation \(\omega(v, w) := g(v, Yw)\), or equivalently \(g(v, w) = \omega(Yv, w)\).

In local coordinates, if \(Y_{ij}(x)\) describes the complex structure, the associated symplectic form is given by \(\omega = (1/2)K_{ij}(x)dx^i \wedge dx^j\) with \(K_{ij}(x) = \).
We can pass to consider hyperkahler manifolds. Now – and always in the following – \( M \) will be a smooth \( 4n \)-dimensional real manifold endowed with a riemannian metric \( g \), and \( \epsilon \) will denote the completely antisymmetric (Levi-Civita) symbol.

**Definition 1.** A hypercomplex structure on \( M \) is an ordered triple \( Y = (Y_1, Y_2, Y_3) \) of complex structures on \( M \) satisfying 
\[ Y_\alpha Y_\beta = \epsilon_{\alpha\beta\gamma} Y_\gamma - \delta_{\alpha\beta} I. \]
If the \( Y_\alpha \) are orthogonal complex structures on \((M, g)\), we say that \( Y \) is orthogonal.

**Remark 1.** The \( Y_\alpha \) making up a hypercomplex structure satisfy the quaternionic relations. In Lie algebraic terms, the \( \tilde{Y}_\alpha = (1/2) Y_\alpha \), which satisfy 
\[ [\tilde{Y}_\alpha, \tilde{Y}_\beta] = \epsilon_{\alpha\beta\gamma} \tilde{Y}_\gamma, \]
realize the \( su(2) \) algebra. \( \sphericalangle \)

**Definition 2.** A hyperkahler structure on \( M \) is a quadruple \( (g, Y_1, Y_2, Y_3) \) where: \( g \) is a metric on \( M \); the \( Y_\alpha \) are an orthogonal hypercomplex structure on \( (M, g) \); and the two-forms \( \omega_\alpha \) defined by the complex structures \( Y_\alpha \) via the Kahler relation are closed and nondegenerate on \( M \). Notice that the forms \( \omega_\alpha \) are therefore (independent) symplectic forms on \( M \).

As dealing with differential forms is equivalent to – but rather more convenient in practice – than dealing with \((1,1)\) tensor fields, we will generally find more convenient to focus on these.

**Definition 3.** A hypersymplectic structure \( \mathcal{O} = \{\omega_1, \omega_2, \omega_3\} \) on the riemannian manifold \( (M, g) \) is an ordered triple of \( g \)-compatible symplectic structures on \( M \), such that the complex structures \( Y_\alpha \) defined by the \( \omega_\alpha \) via the Kahler relation are a hypercomplex structure on \( M \).

If \( \mathcal{O} \) is a hypersymplectic structure on \((M, g)\), the linear span (with real coefficients) of the \( \omega_\alpha \) is the real linear space \( Q := \{\sum c_\alpha \omega_\alpha\} \subset \Lambda^2(M) \). This is called the *quaternionic symplectic structure* generated by the hypersymplectic structure \( \mathcal{O} \), and \( \mathcal{O} \) is an admissible basis for \( Q \) \( \sphericalangle \). The unit sphere in \( Q \) (with the natural metric, see below) will be denoted as \( S \) \( (S \approx S^2) \).

**Remark 2.** The \( S \) defined above is related to the *twistor space* on \( M \) \( \sphericalangle \). \( \sphericalangle \)

The natural scalar product in \( Q \) (seen as a linear space) between \( q_1 = a_\alpha \omega_\alpha \) and \( q_2 = b_\alpha \omega_\alpha \) is \( (q_1, q_2) := a_\alpha b_\alpha \). If we choose a local coordinates system, and we associate to \( q_i \) the matrices \( Q_i \), this coincides with the natural scalar product in the linear space \( Q \) generated by the complex structures \( \{Y_\alpha\} \), i.e. 
\[ (Q_1, Q_2) := (4n)^{-1} \text{Tr}(Q_1^\dagger Q_2). \]

**Lemma 1.** Any nonzero \( \omega \in Q \) is a symplectic structure on \( M \). If \( \omega \in Q \), then \( \omega \) is unimodular (and thus defines a Kahler structure in \( M \)) if and only if \( \omega \in S \).

**Proof.** The first part is trivial. As for the second, if \( \omega = c_\alpha \omega_\alpha \in Q \), then \( \omega \) yields the complex structure \( Y = c_\alpha Y_\alpha \), where the \( \{Y_\alpha\} \) are the hyperkahler
structure (so that, in particular, \(\{Y_\alpha, Y_\beta\} = -2\delta_{\alpha\beta}\)). We have therefore \(Y^2 = -Y^TY = -(\sum \alpha c_\alpha^2)I\), and hence the statement. \(\triangle\)

The three symplectic structures \(\omega_\alpha\) can be seen as associated to the imaginary units of the quaternions; it is thus natural that if we operate a (pure imaginary) rotation in the quaternions, we obtain three different symplectic structures which still generate the same quaternionic structure. In other words, we can change the basis in \(Q\) preserving the quaternionic relations, i.e. passing to a different admissible basis. Notice that in this case the sphere \(S \subset Q\) is invariant.

**Definition 4.** Two hypersymplectic structures \((O, g)\) and \((\hat{O}, g)\) on \(M\), spanning the same quaternionic symplectic structure are said to be equivalent.

**Remark 3.** More generally, consider a map \(\Phi : M \to M\) and let \(\Phi^*\) be its pullback; if we consider local coordinate \(\{x^i\}\) on \(M\), we can write \(\omega = (1/2)K_{ij}(x)dx^i \wedge dx^j\); then \(\Phi^*(\omega) = (1/2)(A^*KA)_{ij}(x)dx^i \wedge dx^j\), where \(A = (D\Phi)\) is the jacobian of \(\Phi\). Thus if \(A(x) \in O(4n, \mathbb{R})\), then \(\Phi^*\) is a morphism of hyperkahler structures of \((M, g)\); if \(A^*Q\mathbb{A} = Q\), then \(\Phi^*\) maps \(Q\) into itself (and necessarily preserves \(S\)), i.e. maps \(O\) to an equivalent hypersymplectic structure. \(\odot\)

### 2 Equations of motion.

In this section we define a class of equations of motion in a hyperkähler manifold; these are associated to the hyperkähler structure and define a Liouville dynamics on the manifold.

Let us consider a hyperkähler manifold \((M, g, Y_1, Y_2, Y_3)\) of real dimension \(4n\). This can be equivalently seen as a hypersymplectic manifold \((M, g, \omega_\alpha)\) with \(\omega_\alpha\) the symplectic forms associated to \(Y_\alpha\) via \(g\); in the following we will refer to the hyperkähler structure even when we will focus on the symplectic aspect. The symbol \(s\) will have value \(\pm 1\), depending if we are considering positive or negative type symplectic forms \(\omega_\alpha\). We define, for ease of notation, \(\zeta_\alpha = \omega_\alpha \wedge \ldots \wedge \omega_\alpha\) (with \(2n - 1\) factors).

Any triple of smooth functions \(\mathcal{H}_\alpha : M \to \mathbb{R}\) \((\alpha = 1, 2, 3)\), defines a vector field \(X : M \to TM\) by the equations of motion

\[
X \mathcal{I} \Omega = \frac{1}{(2n-1)!} \sum_{\alpha=1}^{3} d\mathcal{H}_\alpha \wedge \zeta_\alpha . \tag{1}
\]

We call this the hyperhamiltonian vector field on \((M, g, O)\) associated to the triple \(\mathcal{H}_\alpha\).

We stress that the vector field \(X\) is uniquely defined by this. Also note that, for any \(\alpha\), one gets \(\omega_\alpha \wedge \ldots \wedge \omega_\alpha = [(2n)!]s\Omega\) (the \(\omega_\alpha\) involved in the wedge product are \(2n\)). Using this relation, the equations of motion can also be
Lemma 2. Equation (1) defines a Liouville vector field on $M$, i.e. $\mathcal{L}_X(\Omega) = 0$.

Proof. By the general definition of Lie derivative, $\mathcal{L}_X(\Omega) = d(X \lrcorner \Omega) + X \lrcorner (d\Omega)$, and obviously $d\Omega = 0$. If $X$ satisfies (1) we have therefore $\mathcal{L}_X(\Omega) = \sum_\alpha d(d\mathcal{H}^\alpha \wedge \zeta^\alpha)$, which is zero since the forms $\zeta^\alpha$ are closed. △

Lemma 3. The vector field $X$ defined by (1) can be rewritten as $X = X_1 + X_2 + X_3$, where $X_\alpha$ satisfies $X_\alpha \lrcorner \omega^\alpha = d\mathcal{H}^\alpha$ for $\alpha = 1, 2, 3$.

Proof. It is immediate to check that $X = \sum_\alpha X_\alpha$ with $X_\alpha$ defined as above yields a solution to (1). As $X$ is uniquely defined by (1), this proves the statement. △

Remark 4. It follows from these that if $\omega^\alpha = (1/2)K^{(\alpha)}_{ij} dx^i \wedge dx^j$, then the hyperkahler vector field (1) can be written as $X = f^i \partial_i$ where $f^i = \sum_\alpha K^{ij}_\alpha \nabla_j \mathcal{H}^\alpha$.

Remark 5. If we have a hyperkahler manifold $(M,g)$ and an hamiltonian vector field $X$ (with respect to a symplectic structure $\omega$ being part of a hypersymplectic structure $\mathbf{O}$), this can obviously be seen as a hyperhamiltonian vector field just by choosing two of the $\mathcal{H}^\alpha$ to be constant. One could wonder if all the hyperhamiltonian vector fields on $M$ can be hamiltonian by a suitable choice of a symplectic structure; this is not the case even in the simplest setting $(M = \mathbb{R}^4)$, as we show by explicit example in lemma 5 (see section 6).

The hyperhamiltonian vector field in $M$ induces a vector field (which we also call hyperhamiltonian) in extended phase space, i.e. in $M \times \mathbb{R}$, where the $\mathbb{R}$ space has coordinate $t$ (and represents the time).

If we introduce local coordinates $\{x^1, \ldots, x^{4n}, t\}$ in $M \times \mathbb{R}$, the dynamics in $M \times \mathbb{R}$ will be described by a vector field $Z = z^0(x,t) \partial_t + z^i(x,t) \partial_i$ (here and in the following we write $\partial_i$ for $\partial/\partial x^i$, $\partial_t = \partial/\partial t$). The equations of motion given above are equivalent to defining the vector field $Z$ to be $Z = \partial_t + X$, where obviously $X$ is defined by (1).

Due to the closeness of $\omega_\alpha$, we can locally find one-forms $\sigma_\alpha$ such that $\omega_\alpha = d\sigma_\alpha$, and locally define forms $\varphi, \vartheta$ in $\Lambda^{(4n-1)}(M \times \mathbb{R})$ given by

$$\varphi = \sum_{\alpha=1}^3 \sigma_\alpha \wedge \zeta_\alpha \ , \ \vartheta = \varphi + (6sn) \sum_{\alpha=1}^3 \mathcal{H}^\alpha \zeta_\alpha \wedge dt .$$

Note, for later use, that the $(4n)$-form $d\vartheta$ is nonsingular, and that $d\varphi$ is proportional to the volume form $\Omega$.

When $\omega_\alpha$ is exact (in particular if $M$ has vanishing second cohomology group, e.g. for $M = \mathbb{R}^{4n}$), the $\sigma_\alpha$ and related forms are globally defined. In order to
avoid repeating too frequently that the considerations to be presented are local, we will assume from now on that the $\omega_\alpha$ are exact.

**Theorem 1.** Let $M$ be a hyperkahler manifold, and let $\mathcal{H}^\alpha : M \to \mathbb{R}$ ($\alpha = 1, 2, 3$) be assigned smooth functions; let $\vartheta$ be the form defined by (2). Then the equations of motion (1) are equivalent to

$$Z \lhd d\vartheta = 0 \ , \ Z \lhd dt = 1 \quad (3)$$

where $Z$ is a vector field on $M \times \mathbb{R}$.

**Proof.** The equation $Z \lhd dt = 1$ means that we can write $Z$ in the form $Z = \partial_t + Y$; the other equation $Z \lhd d\vartheta = 0$ yields then, with simple algebra and separating forms with and without a $dt$ factor, two equations:

$$Y \lhd \sum_{\alpha=1}^3 \omega_\alpha \wedge \zeta_\alpha = (6sn)^3 \sum_{\alpha=1}^3 d\mathcal{H}^\alpha \wedge \zeta_\alpha \quad \text{and} \quad Y \lhd \sum_{\alpha=1}^3 d\mathcal{H}^\alpha \wedge \zeta_\alpha = 0 .$$

The second of these is a trivial consequence of the first one; but the first is just (1). Thus, as (1) uniquely determines $X$, we have $Y \equiv X$. $\triangle$

### 3 Conservation laws and Poisson-like brackets

For the class of systems defined above we have a natural conserved $(4n-1)$-form $\Theta$, canonically associated to the triple $\{\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3\}$ and defined as

$$\Theta := \sum_{\alpha=1}^3 d\mathcal{H}^\alpha \wedge \zeta_\alpha . \quad (4)$$

**Theorem 2.** Let $(M, g, \omega_\alpha)$ be a hyperkahler manifold, $\{\mathcal{H}^\alpha\}$ be any triple of functions $\mathcal{H}^\alpha : M \to \mathbb{R}$, $X$ be the hyperhamiltonian flow defined by (1), and $\Theta$ defined by (4). Then $\mathcal{L}_X(\Theta) = 0$.

**Proof.** The form $\Theta$ is closed, hence $\mathcal{L}_X(\Theta) = d(X \lhd \Theta)$; the explicit expression of $\Theta$ and the equations of motion (1) give

$$(X \lhd \Theta) = (2n-1)! [X \lhd (X \lhd \Omega)] ,$$

which is identically zero since we are contracting twice an alternating form with the same vector. $\triangle$

Notice that $(4n-1)$ forms $\chi$ on $M$ are canonically associated to vector fields $Y$ on $M$ via $Y \lhd \Omega = \chi$; we write $Y = F(\chi) = Y_\chi$. There is a natural operation $\{.,.\} : \Lambda^{(4n-1)}(M) \times \Lambda^{(4n-1)}(M) \to \Lambda^{(4n-1)}(M)$, defined as follows. Given forms $\chi, \Psi \in \Lambda^{(4n-1)}(M)$, we consider the associated vector fields $Y_\chi, Y_\Psi$; take the commutator $Y_\Gamma := [Y_\chi, Y_\Psi]$. This defines an associated form $\Gamma \in \Lambda^{(4n-1)}(M)$, and we define $\{\chi, \Psi\}$ to be just $\Gamma$. In other words,

$$\{\chi, \Psi\} := F^{-1} ([F(\chi), F(\Psi)] ) . \quad (5)$$
We stress that with this notation, $\Theta = F(X)$ with $X$ the hyperhamiltonian vector field. Note also that if we have two conserved $(4n - 1)$-forms $\Theta_i$, we can generate another (possibly not independent from these, or zero) conserved form $\{\Theta_1, \Theta_2\}$. In this respect, $\{\ldots\}$ is reminiscent of the Poisson brackets of standard Hamiltonian mechanics; however we have to remark that the situation differs substantially from the standard Poisson brackets, because to define our brackets we don’t use the hyperkahler structure of the manifold, but only the isomorphism between vector fields and $(4n - 1)$-forms induced by the volume form on $M$.

4 Variational formulation

In this section we will formulate a local variational principle related to the hyperhamiltonian equations of motion introduced in section 2.

In order to state in a geometrical framework our principle, we need to consider a local fiber bundle structure on the hyperkahler manifold $M$. This local fibration allows to describe a particular class of variations (“vertical” with respect to the fibration, as we precise in a moment) that generalize isochronous variations considered in the variational principle for standard hamiltonian mechanics.

Let us consider a hyperkahler manifold $(M, g, \omega_\alpha)$ of real dimension $4n$, and a triple $\{H_\alpha\}$ of hamiltonian functions; we will consider the extended phase space $M \times \mathbb{R}$, which we see as a trivial fiber bundle $t : M \times \mathbb{R} \to \mathbb{R}$.

In order to properly set the local variational problem in a chart $M_i$ of $M$, we will need to consider a double fibration

$M_i \times \mathbb{R} \overset{\pi_i}{\longrightarrow} B_i \overset{\tau_i}{\longrightarrow} \mathbb{R}$

where the base manifold $B_i$ of the fiber bundle $\pi_i : M_i \times \mathbb{R} \to B_i$ is a manifold of dimension $(4n - 1)$, fibered itself over $\mathbb{R}$ with projection $\tau_i$. We also require, obviously, that $\tau_i \circ \pi_i = t$ on $M_i \times \mathbb{R}$.

For ease of notation, we will from now on just write $M$ for $M_i$ and $B$ for $B_i$, i.e. use a “global” notation. We stress that the double fibration considered here is not a general global construction associated to the geometrical structure of an hyperkahler manifold; anyway, this double fibration can be considered locally in $M_i$ for a generic hyperkahler manifold $M$, and the choice of the local base manifold $B_i$ is widely arbitrary. Note that in the simple but relevant case $M = \mathbb{R}^{4n}$ the double fibration exists globally.

We denote the sets of sections of the bundles introduced above, respectively, by $\Gamma(\pi)$ and $\Gamma(\tau)$; and similarly for $\Gamma(t)$. We denote by $\mathcal{V}(\pi)$ the set of vertical vector fields for the fibration $\pi : M \times \mathbb{R} \to B$.

For $V \in \mathcal{V}(\pi)$, we denote by $\psi_V : M \times \mathbb{R} \to M \times \mathbb{R}$ the flow generated by $V$. We want to consider variations of sections $\Phi \in \Gamma(\pi)$ under the action of $V \in \mathcal{V}(\pi)$.

\footnote{Should the reader be misled by our “global” notation, we note these are actually local sections $\Phi_i \in \Gamma(\pi_i)$, i.e. $\Phi_i : M_i \to M_i \times \mathbb{R}$.}
Definition 5. Let $\Phi \in \Gamma(\pi)$. The variation of $\Phi$ under the vertical vector field $V$ is the section $\tilde{\psi}_s(\Phi) := \psi_s \circ \Phi \in \Gamma(\pi)$.

Remark 6. The nature of the double fibration $M \times \mathbb{R} \overset{\pi}{\longrightarrow} B \overset{\tau}{\longrightarrow} \mathbb{R}$, where $\tau \circ \pi = t$, ensures that vertical vector fields $V \in \mathcal{V}$ cannot have components along $\partial_t$; that is, we are actually considering isochronous variations. We also recall that, in order to consider variation of the section $\Phi$, we don’t need a vertical vector field defined on all $M \times \mathbb{R}$, but just a vertical vector field defined along $\Phi$. ◯

The main object to be considered is the $(4n-1)$-form $\vartheta$ on $M \times \mathbb{R}$, defined by (2). We recall that $\vartheta := \sum_{\alpha=1}^3 \alpha \sigma_\alpha \wedge \zeta_\alpha + (6ns)H^\alpha \zeta_\alpha \wedge dt$, where $d\sigma_\alpha = \omega_\alpha$.

Let us consider a compact $(4n-1)$ dimensional submanifold with boundary $C \subseteq B$. We define a functional $I : \Gamma(\pi) \rightarrow \mathbb{R}$ given by

$$I(\Phi) := \int_C \Phi^*(\vartheta)$$

where $\Phi^*(\vartheta)$ denotes, as customary, the pullback of $\vartheta$ by $\Phi$.

In the following we will consider only vertical vector fields $V \in \mathcal{V}(\pi)$ such that $V$ vanish on $\pi^{-1}(\partial C)$, where $\partial C$ is the boundary of $C$. This is just the familiar condition of zero variation on the boundary of the integration region. We denote these as $\mathcal{V}_C(\pi)$.

Definition 6. A section $\Phi \in \Gamma(\pi)$ is extremal for $I$ if and only if

$$\frac{d}{ds} \left[ \int_C \left( \tilde{\psi}_s(\Phi) \right)^*(\vartheta) \right]_{s=0} = 0$$

whenever $V \in \mathcal{V}(\pi)$. In this case we write $\langle \delta I \rangle(\Phi) = 0$.

Theorem 3a. A section $\Phi \in \Gamma(\pi)$ is extremal for $I$ defined by (6) if and only if $\Phi^*(V \lrcorner d\vartheta) = 0$ for all $V \in \mathcal{V}_C(\pi)$.

Proof. This is a standard theorem of variational analysis, see e.g. chapter XII of [25]. △

Remark 7. Note that $\mathcal{V}(\pi)$ is two dimensional as a module over the algebra of smooth functions $F : M \times \mathbb{R} \rightarrow \mathbb{R}$. With $V_1, V_2$ a pair of generators for $\mathcal{V}(\pi)$, the condition $\Phi^*(V \lrcorner d\vartheta) = 0 \ \forall V \in \mathcal{V}_C(\pi)$ can be written as $\Phi^*(V_1 \lrcorner d\vartheta) = 0 = \Phi^*(V_2 \lrcorner d\vartheta)$. This is independent of $C$. ◯

Sections $\Phi$ which are extremal for $I$ are related to the hyperhamiltonian vector field $Z$ in that $Z$ is the characteristic vector field for $\Phi$, as discussed below.

The relation between $I$ and $Z$ is better understood in the language of ideals of differential forms [14, 22] (some basic definitions used here are recalled in the appendix), which we will call just ideals for short. With this language, and recalling remark 7, theorem 3a above can be restated as follows:
Theorem 3b. Let $V_1, V_2$ generate $V(\pi)$. A section $\Phi \in \Gamma(\pi)$ is extremal for $I$ defined by (6) if and only if $\Phi$ is an integral manifold of the ideal $J$ generated by $V_1 \lhd d\vartheta$ and $V_2 \lhd d\vartheta$.

In view of this fact, we will say that $J$ is the ideal associated to the variational principle $\delta I = 0$.

We can now discuss the relation between the vector field $Z$ introduced in section 2 and the variational principle based on $I$. We will first establish a simple lemma and an immediate corollary thereof.

Lemma 4. Let $\alpha$ be a nonzero $N$-form in the $(N+1)$-dimensional manifold $M$. Let $X, V_1, V_2$ be three independent and nonzero vector fields on $M$. Then $V_1 \lhd (X \lhd \alpha) = 0 = V_2 \lhd (X \lhd \alpha) = 0$ implies (and is thus equivalent to) $X \lhd \alpha = 0$. Moreover, the space of vector fields $Y$ satisfying $Y \lhd \alpha = 0$ is a one-dimensional module over $\Lambda^0(M)$.

Proof. Choose local coordinates $\{x^0, x^1, ..., x^N\}$ in $M$; we can always take $X = \partial_0, V_1 = \partial_1, V_2 = \partial_2$. We write $\Omega = dx^0 \wedge ... \wedge dx^N$; then, in full generality, $\alpha = \sum_{k=0}^{N} c_k (\partial_k \lhd \Omega)$. Now $\partial_1 \lhd (\partial_0 \lhd \alpha) = 0$ implies $c_j = 0$ for $j \neq 0, 1$; and $\partial_2 \lhd (\partial_0 \lhd \alpha) = 0$ implies $c_j = 0$ for $j \neq 0, 2$. Imposing both equations yields $\alpha = c_0 (\partial_0 \lhd \Omega) \equiv c_0 (X \lhd \Omega)$. This satisfies, of course, $X \lhd \alpha = 0$; conversely $Y \lhd \alpha = 0$ implies $Y = fX$. $\triangle$

Corollary. Let $\alpha, V_1, V_2$ be as above. Then the ideal $J$ generated by $\{\Psi_1 = (V_1 \lhd \alpha), \Psi_2 = (V_2 \lhd \alpha)\}$ is nonsingular and admits a one-dimensional characteristic distribution $D(J)$; this is given by vector fields satisfying $X \lhd \alpha = 0$.

Proof. As $\Psi_1, \Psi_2$ are both $N-1$ forms, $(X \lhd \Psi_j) \in J$ is equivalent to $X \lhd \Psi_j = 0$. Thus the corollary is merely a restatement of lemma 4; notice this implies that the space of vector fields satisfying $X \lhd \alpha = 0$ has constant dimension, i.e. $J$ is nonsingular. $\triangle$

Theorem 4. Let $(M, g, \omega_\alpha)$ be a hyperkahler manifold of real dimension $4n$; let $\{H^\alpha : M \to \mathbb{R}\}$ be three smooth functions. Let $\vartheta$ be the $(4n-1)$-form defined by (2), and let $J$ be the nonsingular ideal associated to the variational principle defined by $I$. Then the characteristic distribution $D(J)$ for $J$ is one-dimensional and is generated by the hyperhamiltonian vector field $Z$ defined by (3).

Proof. Specialize lemma 4 and its corollary to the case $\alpha = d\vartheta$, and use theorem 1. $\triangle$

Remark 8. It follows from this that the vector field $Z$ is everywhere tangent to integral manifolds of $J$, i.e. to extremal sections for $I$. Moreover, by proposition A1 (see the appendix), it also shows that the $(4n-1)$-dimensional extremal sections $\Phi$ for $I$ can be described by assigning their value on a suitable $(4n-2)$-dimensional manifold and pulling them along integral curves of $Z$. $\odot$
5 Integral invariants.

The Poincaré invariants (Poincaré form and Poincaré-Cartan integral invariant) play a central role in the canonical structure of Hamiltonian mechanics. In hyperhamiltonian mechanics, we have objects enjoying the same properties; these turn out to be, respectively, the forms $\vartheta$ and $\varphi$ introduced above, see (2).

We consider as usual a differentiable manifold $M$ of dimension $4n$ equipped with a hypersymplectic structure $\{\omega_\alpha\}$, and the extended phase space $M \times \mathbb{R}$.

**Theorem 5.** Let $\gamma_0$ be a closed and oriented $(4n-1)$ submanifold of the extended phase space $M \times \mathbb{R}$; let $\gamma_t$ be the manifold obtained by transporting $\gamma$ along the flow of the hyperhamiltonian vector field $Z$ defined in (3). Then

$$\frac{d}{dt} \int_{\gamma_t} \vartheta = 0 .$$  

(8)

**Proof.** Let $Z_t$ denote the flow of $Z$. We have

$$\frac{d}{dt} \int_{\gamma_t} \vartheta = \frac{d}{dt} \int_{\gamma_0} Z_t^* \vartheta = \int_{\gamma_0} \frac{d}{dt} (Z_t^* \vartheta) = \int_{\gamma_0} Z_t^* [d(Z \lrcorner \vartheta) + Z \lrcorner d\vartheta] = 0 ,$$

(9)

where the last integral vanishes because $Z \lrcorner d\vartheta = 0$ and (by Stokes’ theorem) because $\gamma_0$ is closed. $\triangle$

We consider the special case of the construction considered above (leading to the Poincaré-Cartan invariant) in which the manifold $\gamma_0$ lie on a hyperplane at $t$ constant.

This gives the Poincaré relative invariant, that should also be reinterpreted as the conservation of the volume form under the hyperhamiltonian flow.

**Theorem 6.** Let $\gamma_0$ be a closed and oriented $(4n-1)$ submanifold of the extended phase space $M \times \mathbb{R}$ lying in the fiber over $t_0$ of the fibration $t : M \times \mathbb{R} \to \mathbb{R}$. let $\gamma_t$ be the manifold obtained by transporting $\gamma$ along the flow of the hyperhamiltonian vector field $Z$ defined in (3). Then

$$\frac{d}{dt} \int_{\gamma_t} \varphi = 0 .$$

(10)

**Proof.** In this case $dt = 0$ on $\gamma_t$ and the integration of $\vartheta$ on $\gamma_t$ reduces to the integration of $\varphi$ on the same manifold $\gamma_t$. Therefore (8) means that the integral of $\varphi$ over $\gamma_t$ is constant, i.e. (10). $\triangle$

6 Hypersymplectic structures in $\mathbb{R}^4$.  

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6.1 Standard structures

After developing the general theory in abstract terms, it will be useful to consider the simplest nontrivial example of hypersymplectic manifold. This is provided by \( M = \mathbb{R}^4 \) with standard euclidean metric \( g_{ij}(x) = \delta_{ij} \).

Despite the fact this is just a (simple) exercise of linear algebra, we will explicitly write the hyperhamiltonian equations, and this for three reasons: (a) this is the simplest case in which our construction applies, and having a fully explicit example can only help our understanding; (b) the explicit formulation of the equation of motion in the standard case will be useful in Section 7 when we discuss quaternionic oscillator; (c) discussion of this simple case will clarify some points which were not fully discussed above, referring instead to this explicit example.

The latter were: first, the reason why we left the possibility that the orientation of the metric and the orientation of the symplectic structure disagree; and second, we use the standard structure to give an explicit example of a hyperhamiltonian vector field that is not hamiltonian, whatever symplectic structure we define on \( M \), showing that the dynamic that we propose is a real extension of Hamiltonian dynamics.

We use cartesian coordinates \( x^i \) in \( \mathbb{R}^4 \), and the volume form will be \( \Omega = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \). The space \( \Lambda^2(M) \) is six dimensional, and is spanned by

\[
\mu_1 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4, \quad \eta_1 = dx^1 \wedge dx^3 + dx^2 \wedge dx^4,
\]
\[
\mu_2 = dx^1 \wedge dx^4 + dx^2 \wedge dx^3, \quad \eta_2 = dx^4 \wedge dx^1 + dx^2 \wedge dx^3,
\]
\[
\mu_3 = dx^1 \wedge dx^3 + dx^4 \wedge dx^2, \quad \eta_3 = dx^2 \wedge dx^1 + dx^3 \wedge dx^4.
\]

Note that the \( \mu \) span the space \( \Lambda^2_+ (M) \) of self-dual forms, the \( \eta \) span the space \( \Lambda^2_- (M) \) of anti-self-dual forms.

Note also that the \( \mu_\alpha \wedge \mu_\alpha = \Omega, \eta_\alpha \wedge \eta_\alpha = -\Omega \). We will refer to these as standard hypersymplectic structures of positive and negative type respectively. We also denote as \( Q_\pm \) the quaternionic structures spanned by these, and \( S_\pm \) their unit spheres. Obviously we have \( \Lambda^2_\pm (M) = Q_\pm \).

We write two-forms on \( \mathbb{R}^4 \) as \( \omega = (1/2)(J)_{im} dx^i \wedge dx^m \), with \( J \) an anti-symmetric tensor. We write the tensors corresponding to the \( \mu_\alpha \) as \( K_\alpha \), those corresponding to \( \eta_\alpha \) as \( H_\alpha \) (and their triples as \( K \) and \( H \)).

Explicit expressions of these are as follows:

\[
K_1 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix},
K_2 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
K_3 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix},
\]
\[
H_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
H_2 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
H_3 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\]
We consider the positive-type hypersymplectic structure and derive explicit expressions for the associated hyperhamiltonian dynamics.

Obviously, with \( H \), we have
\[ X \cdot \Omega = f^1 dx^2 \wedge dx^3 \wedge dx^4 + f^2 dx^1 \wedge dx^3 \wedge dx^4 + f^3 dx^1 \wedge dx^2 \wedge dx^4. \]

The computation of \( dH^\alpha \wedge \omega_\alpha \) is also immediate, and we get
\[
\sum_{\alpha=1}^{3} dH^\alpha \wedge \omega_\alpha = (\partial_3 H^1 + \partial_1 H^2 - \partial_2 H^3) dx^1 \wedge dx^2 \wedge dx^3 + (\partial_1 H^2 + \partial_2 H^3 - \partial_3 H^1) dx^1 \wedge dx^2 \wedge dx^4 + (\partial_2 H^1 + \partial_3 H^2 - \partial_1 H^3) dx^1 \wedge dx^3 \wedge dx^4 + (\partial_3 H^1 + \partial_2 H^3 - \partial_1 H^2) dx^2 \wedge dx^3 \wedge dx^4.
\]
The equations of motion are immediately obtained by comparing this and the previous expression; these are

\[
\begin{align*}
\dot{x}_1 &= \left( \partial H^1 / \partial x^2 \right) + \left( \partial H^2 / \partial x^4 \right) + \left( \partial H^3 / \partial x^3 \right) \\
\dot{x}_2 &= -\left( \partial H^1 / \partial x^1 \right) + \left( \partial H^2 / \partial x^3 \right) - \left( \partial H^3 / \partial x^4 \right) \\
\dot{x}_3 &= \left( \partial H^1 / \partial x^4 \right) - \left( \partial H^2 / \partial x^2 \right) - \left( \partial H^3 / \partial x^3 \right) \\
\dot{x}_4 &= -\left( \partial H^1 / \partial x^3 \right) - \left( \partial H^2 / \partial x^1 \right) + \left( \partial H^3 / \partial x^2 \right).
\end{align*}
\]

### 6.2 General structures

Let us now consider a general hypersymplectic structure $O$; we denote a symplectic structure by $\omega$ and a point in $M = \mathbb{R}^4$ by $x$; let $\omega_x$ be the evaluation of $\omega$ in $x \in M$. The two lemmas below show that $O$ is always equivalent to one of the two structures considered above.

**Lemma 6.** If $\omega_x$ is unimodular, it belongs either to $S_+$ or to $S_-$.

**Proof.** Let $Y$ be the complex structure associated to $\omega$. We work in coordinates, and write in full generality $Y(x) = (a \cdot K) + (b \cdot H)$. Requiring $\omega$ to be unimodular, i.e. $Y^T Y = I$, we obtain two conditions: the vanishing of off-diagonal terms reads $a_\alpha b_\beta = 0 \forall \alpha, \beta = 1, 2, 3$, so that $|a| \cdot |b| = 0$; setting diagonal terms equal to one (together with the previous condition) yields moreover $|a|^2 + |b|^2 = 1$. \(\triangle\)

**Lemma 7.** A hypersymplectic structure in $M = \mathbb{R}^4$ is made either of positive type symplectic structures, or of negative type symplectic structures, but in no case by symplectic structures of the two kinds.

**Proof.** A hypersymplectic structure corresponds to a $su(2)$ algebra in the way discussed above. As also mentioned above, the spans of the $K$ and of the $H$ correspond to $su(2)_\pm$ algebras, but no $su(2)$ algebra is generated by a mixture of matrices belonging to $su(2)_+$ and $su(2)_-$ algebras. \(\triangle\)

### 6.3 Extension to 4n dimensions

We stress that the standard structures are immediately extended to structures in higher dimension. In the case $M = \mathbb{R}^{4n}$ (with euclidean metric), take block reducible structures. By this we mean that $\omega_{\alpha} = (1/2)(J_{\alpha})_{ij} dx^i \wedge dx^m$, and the $Y_{\alpha} = g^{-1} J_{\alpha}$ generate a representation of the $su(2)$ algebra in $\mathbb{R}^{4n}$, which is the direct sum of irreducible representations on four-dimensional subspaces.

In this case the matrices acting on each four dimensional block will be either $K$ or $H$. We thus have, in block notation, $J_{\alpha} = L_{\alpha}^{s_1} \oplus \ldots \oplus L_{\alpha}^{s_n}$, where $s_k = \pm$, and $L_{\alpha}^{(+)} = K_{\alpha}$, $L_{\alpha}^{(-)} = H_{\alpha}$ (notice that we could get equivalent hypersymplectic structures by orthogonal changes of variables on each $\mathbb{R}^4$ block). The analysis conducted in $\mathbb{R}^4$ does apply on each block.
7 Quaternionic oscillators.

There is no need to stress the relevance and ubiquitous role of (harmonic and nonlinear) oscillators in standard Hamiltonian mechanics; we want to discuss here the hyperHamiltonian oscillators, with a view at the problem of integrable hyperHamiltonian systems (we assume the reader is familiar with Hamiltonian integrable systems).

Our intuitive understanding of hyperHamiltonian integrable systems will be that of systems which can be mapped to a system of hyperHamiltonian oscillators.

We will consider systems in $M = \mathbb{R}^{4n}$ (with standard Euclidean metric), and standard (say positive type) hypersymplectic structure; as the hyperkahler structure induces in this case a quaternionic structure, we will speak of quaternionic oscillators.

We will consider nontrivial systems with compact invariant manifolds, and start by discussing the case $n = 1$.

7.1 The standard four dimensional case.

The simplest nontrivial case of hyperHamiltonian dynamics is the one where we have quadratic Hamiltonians $H^\alpha$, i.e. $H^\alpha(x) = (1/2)c_\alpha|x|^2$, with $c_\alpha$ real constants; in this case we get $\dot{x}^i = c_\alpha K^\alpha_{ij} x^j$, which is easily integrated (see the more general discussion below).

Let us actually write $\rho \equiv (1/2)||x||^2$, and consider the class of nonlinear systems where $H^\alpha(x) = H^\alpha(\rho)$, i.e. assume the $H^\alpha$ are arbitrary smooth functions of $\rho$. We call these quaternionic oscillators.

In this case, write $A^\alpha = dH^\alpha/d\rho$; we have $\nabla H^\alpha = A^\alpha(\rho)x$, and the equations of motion (1) read simply (see section 6)

$$\dot{x}^i = \sum_{\alpha=1}^3 A_\alpha(\rho) (K^\alpha)^i_j x^j .$$

Notice that $d\rho/dt = \sum_{\alpha=1}^3 A_\alpha(\rho) [x^i (K^\alpha)^i_j x^j] = 0$; the last equality follows from $K^\alpha = -K^{\alpha T}$. Therefore $\rho$ (and hence $|x(t)|$) is a constant of motion under any hyperHamiltonian flow for Hamiltonians which are functions of $\rho$ alone.

As $\rho(t) = \rho_0$, we can on any trajectory rewrite (9) as

$$\dot{x}^i = \sum_{\alpha=1}^3 c_\alpha^0 (K^\alpha)^i_j x^j = \nu_0 (K_\alpha)^i_j x^j ,$$

where $c_\alpha^0 = A_\alpha(\rho_0)$, and

$$\nu_0 := \sqrt{(c_1^0)^2 + (c_2^0)^2 + (c_3^0)^2} , \quad K^i_j = \frac{1}{\nu_0} \sum_{\alpha=1}^3 c_\alpha^0 (K^\alpha)^i_j .$$
The solution to (10) is obviously \( x(t) = \exp[K\nu_0 t]x(0) \); expanding this in a power series in \( t \) and using \( K^2 = -I \), we obtain immediately
\[
x(t) = \left[ \cos(\nu_0 t) I + \sin(\nu_0 t) K \right] x(0) .
\]
This represents a uniform motion on a great circle – identified by the vectors \( x_0 = x(0) \) and \( x_1 = Kx(0) \) – of the sphere \( S^3 \) of radius \( r_0 = |x(0)| \). The frequency \( \nu_0 \) of such motions will be the same for motions on the same sphere: it depends only on the radius \( r_0 \) (note the \( c_{\alpha}^\rho \) also depend on \( r_0 \)).

Therefore any sphere \( S^3 \) of radius \( r_0 \neq 0 \) is covered by periodic circular motions, unless \( \nu_0 (r_0) = 0 \), all of them with the same period \( T_0 = 2\pi/\nu_0 \); in this way the hyperhamiltonian flow (9) partitions \( S^3 \) into \( S^1 \) equivalence classes (the dynamical orbits) and thus realizes a Hopf fibration \( S^3/S^1 = S^2 \) of the three-sphere \([1]\).

### 7.2 The (4n)-dimensional case.

Let us pass to consider \( M = \mathbb{R}^{4n} \), again with standard euclidean metric; we will use cartesian coordinates \( \{x^1, ..., x^{4n}\} \). We also define block variables \( \{\xi_1, ..., \xi_n\} \) with \( \xi_p \in \mathbb{R}^4 \) corresponding to \( x \) coordinates in the \( p \)-th block, \( \xi^i_p = x^{(p-1)+i} \) (where \( i = 1, ..., 4 \) and \( p = 1, ..., n \)); we also write \( \rho_p = (1/2)|\xi_p|^2 \).

We will consider the case where \( \mathbb{R}^{4n} \) is equipped with a standard block reducible hypersymplectic structure (see section 6), \( J_\alpha = L_{\alpha}^+ \oplus ... \oplus L_{\alpha}^- \). We have \( L_{\alpha}^+ = K_{\alpha}, L_{\alpha}^- = H_{\alpha} \).

We will now assume the hamiltonians depend only on the \( \rho_p \) (we say we have a quaternionic \( n \)-oscillator):
\[
H^\alpha(x) = H^\alpha(\rho_1, ..., \rho_n) ;
\]
we write the jacobian of the \( H \) with respect to the \( \rho \) variables as \( A_p^\alpha := \partial H^\alpha/\partial \rho_p \).

In this case the equations of motion are (sum on repeated greek and latin indices will be implied, except for the block index \( p \))
\[
\dot{x}^i = (J_\alpha)^{ik} \partial_k H^\alpha
\]
and can be written as (no sum on \( p \))
\[
\dot{\xi}^i_p = A_p^\alpha(\rho_1, ..., \rho_n) (L_{\alpha}^\sigma)^i_k \xi^k_p .
\]

Again the \( \rho_p \) are constants of motion (no sum on \( p \)):
\[
\frac{d\rho_p}{dt} = \frac{d\xi^i_p}{dt} \partial_{\rho_p} = A_p^\alpha(\rho_1, ..., \rho_n) [\xi^i_p (L_{\alpha}^\sigma)^i_k \xi^k_p] = 0 ,
\]
where the last equality follows from the antisymmetry of the \( L_{\alpha}^\sigma \).

Hence the matrices \( A_p^\alpha \) are constant under the flow. If we are given an initial datum \( x(0), 0 \), and thus the value of the constants of motion \( \rho_1 = b_1, ..., \rho_n = b_n \), we can write
\[
\dot{\xi}(p) = \sum_{\alpha=1}^3 c_{(p)}^\alpha (L_{\alpha}^\sigma) \xi(p) = \nu(p) L(p) \xi(p) ,
\]

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where $c_\alpha^\nu(p) = A_\alpha^\nu(b_1, ..., b_n)$, and

$$
\nu_p = \sqrt{(c_1^\nu(p))^2 + (c_2^\nu(p))^2 + (c_3^\nu(p))^2}, \quad L(p) = \frac{1}{\nu_p} \sum_\alpha c_\alpha^\nu(p) L_\alpha^\nu p.
$$

That is, on each block we have the same situation discussed in subsection 1; notice that the frequencies $\nu_p$ depend not only on the value $b_p$ of $\rho_p$, but also on the values $b_q$ of the other variables $\rho_q$ ($p \neq q$).

8 Discussion: the relation between hyperhamiltonian and standard hamiltonian integrability.

We would like to discuss the relation between hyperhamiltonian integrability and standard hamiltonian integrability for the class of systems considered here.

8.1 Dimension four.

Let us first of all focus on the case given by $H_1 = |x|^2/2$, $H_2 = H_3 = 0$; this corresponds to two uncoupled and identical harmonic oscillators with conserved energies $E_a = (1/2)|x|^2 + (x^2)^2$ and $E_b = (1/2)(x^3)^2 + (x^4)^2$.

The solutions of nonzero energy $E = E_a + E_b = r_0^2/2$ describe a circle $S^1$ lying on the sphere $S^3$ of radius $r_0$. When $E_a$ and $E_b$ are both nonzero (i.e. both oscillators are actually excited) these also lie on a torus $T^2 \subset S^3$, and the circle $S^1$ corresponding to the solution is a combination of the two fundamental cycles of the torus.

The cases $E_a = 0, E_b \neq 0$ and $E_a \neq 0, E_b = 0$ correspond to degenerate situations in which the common level set of $E_a$ and $E_b$ is not a torus $T^2$, but is reduced to a circle $T^1 = S^1$, which is just the trajectory of the solution.

It should be recalled that the Hopf $S^3$ fibration can indeed be described as a singular fibration of $S^3$ in $T^2$ tori, with two singular fibers, which correspond to the special cases in which all the energy is on one oscillator and the other is not excited; thus these two ways (hyperhamiltonian and standard hamiltonian) of describing the situation are immediately related, as it should be.

Let us now come back to the general (nonlinear) integrable case described by (9), whose solutions are given by (11); on each $S^3$ sphere of radius $r_0 \neq 0$, i.e. on each nonzero level manifold for the energy $E = \rho$ we can indeed reduce to a two-oscillators description; see (9) and (10) above. Such a system is integrable in the Arnold-Liouville sense, since the set on which the fibration in tori is singular is of zero measure in the phase space.

However, it should be noticed that in considering this system as an integrable two-oscillator system, we are completely overlooking the quaternionic structure of the system and of the whole class to which it belongs. Also, this system is

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\[\text{In this case we can speak of energy level manifolds as the three hamiltonians depend on a single scalar function } \rho.\]
strongly degenerate if seen in terms of two oscillators: indeed the two oscillators are in 1:1 resonance for all values of $H$, i.e. all values of the action variables $I_1 = E_1$ and $I_2 = E_2$. Such a degeneration is of course enforced by the quaternionic structure, and thus generic in the frame of “quaternionic oscillators”.

On the other hand, if we recognize the quaternionic structure and the fact that we need therefore only the global constant of motion $\rho$ to guarantee integrability (see the above discussion, and the remarks below in this section), we have at once a much stronger information on the structure of the system and also need an easier construction to guarantee integrability.

The situation is similar to the one met when we represent a quaternion by a pair of complex numbers (or a complex number by a pair of real ones): this is possible and correct, but in this way we are overlooking an additional and relevant structure, which we must then introduce by suitable relations between complex (or real) quantities.

Thus, in order to guarantee integrability in the sense of standard hamiltonian mechanics we need two constants of motion and we have to construct a system of two action and two angle coordinates; using the quaternionic structure we only need one constant of motion, i.e. $\rho$, and we have to construct a system of coordinates in which to the “action” coordinate $\rho$ are associated three “angle-like” coordinates. By “angle-like” we mean coordinates on the sphere $S^3$ which can be seen as a generalization, from $S^1 \simeq \mathbb{C}^1$ to $S^3$ of the angular coordinates of standard hamiltonian mechanics; as $S^3 \simeq H^1 \simeq SU(2)$ (here $H$ is the quaternion field and $H^1$ the set of quaternions of unit norm), these are of quaternionic nature. We call them spin coordinates.

Notice that the evolutions along spin coordinates do not commute; thus the equivalent of the familiar integrable hamiltonian evolution equations $\dot{I}_k = 0$, $\dot{\varphi}_k = \omega_k(I)$, related to the abelian group $T^2$, is now given by (9), (10) or, more intrinsically, by $\dot{I} = 0 (I \equiv \rho)$, $\dot{\psi} = \alpha(I)$, where $\psi$ represents coordinates on the group $SU(2) \simeq S^3$, and $\alpha(I) \in su(2)$ is an element of the algebra $su(2)$, constant on each level set of $I \equiv \rho$. This more involved (and not separable) structure is unavoidable, due to the non-abelian nature of $SU(2)$.

### 8.2 Higher dimension.

In the standard hamiltonian integrable case with $m$ degrees of freedom, i.e. for $m$ hamiltonian oscillators (say all of them excited) we have invariant $T^m$ tori, and the solutions will cover densely $T^k \subset T^m$ tori, with $k \leq m$ depending on the rational relations between the frequencies of different degrees of freedom on the given $T^m$; in the hyperhamiltonian integrable case (for $n$ quaternionic oscillators) we have a similar situation, as we now discuss.

First of all we remark that, since $\rho = (\rho_1, ..., \rho_n)$ are constants of motion, the common level sets of the $\rho_p$ are invariant manifolds under the dynamics we are considering; these level sets $\rho^{-1}(b_1, ..., b_n)$ will be, when all the $b_p$ are nonzero, manifolds

$$S^3 \times ... \times S^3 = V^n$$

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notice that these \( V^n \) represent a generalization of tori: in the same way as \( T^n \) is the topological product of \( n \) (distinct) \( S^1 \) factors, \( V^n \) is the topological product of \( n \) (distinct) \( S^3 \) factors. If \( k \) out of the \( n \) numbers \( b_p \) are zero, the level set \( \rho^{-1}(b_1, ..., b_n) \) will be a \( V^{n-k} \) manifold.

We will denote the trajectory with initial datum \( x(0) \) as \( \gamma \subset \mathbb{R}^{4n} \). The previous discussion shows that the projection of \( \gamma \) to each \( \mathbb{R}^4 \) block, given by \( \zeta(t) \), will be periodic.

If all the frequencies \( \{\nu_1, ..., \nu_n\} \) are rational with respect to each other, the full solution in \( \mathbb{R}^{4n} \) will also be periodic, i.e. \( \gamma \approx S^3 \); if \( m \) degrees of freedom are excited (i.e. there are \( m \) nonzero \( b_p \)'s), this \( \gamma \) will also be a submanifold of the invariant manifold \( V^m = \rho^{-1}(b_1, ..., b_n) \).

If \( m \) degrees of freedom are excited and the frequencies corresponding to \( b_p \neq 0 \) split in \( k \leq m \) sets, each \( \nu_p \) being rational with respect to frequencies in the same set and irrational with respect to frequencies in different sets, the solutions \( \gamma \) will densely cover \( T^k \) tori.

These \( T^k \) will be submanifolds of \( V^m \), and we can always choose the generators \( S^3 \) of \( T^k \) so that each generator lies in a different generator \( S^3 \) of \( V^m \). Indeed each generator of \( T^k \) will be a linear combinations of the projections of \( \gamma \) to the blocks corresponding to each rational subset of frequencies; we can choose it to be just a single projection and thus to be in a \( S^3 \) factor of \( V^m \).

9 Final remarks

In this final section we briefly present some additional remarks to put our work into perspective and mention directions of future development. We thank an unknown referee for raising the problem discussed in point 4 below.

(1) First of all, it should be stressed that here we were mostly interested in the local structure of this hyperhamiltonian dynamics, and we have not considered problems arising from the global structure of the hyperkahler manifold \( M \) on which it is defined. Locally, any such \( M \) is isomorphic to \( \mathbb{R}^{4n} \), so that we could have limited to consider these spaces (as in sections 6–8).

However, as in the standard hamiltonian case, most of our construction will extend to more general hyperhamiltonian manifolds, so that in our general discussion (sections 1–5) we preferred to deal with a generic hyperkahler manifold, pointing out where our discussion requires to work chart by chart.

Focusing on local properties means, of course, that we are not concerned with the geometrically most interesting recent results on hyperkahler manifolds and their global structure (which is also relevant in connection with Physics); see the references given in the introduction, and in particular ??, for an overview of these.

A fortiori we are not providing any new insight into hyperkahler geometry nor are we providing any new nontrivial hyperkahler manifold. We actually needed only the very basic definitions of hyperkahler geometry; we supposed that a hyperkahler manifold \( M \) is given, and we defined a dynamics on \( M \) related to the choice of three hamiltonian functions.
(2) We should also notice that no attempts to generalize hamiltonian dynamics in the direction proposed here seems to be present in the rapidly growing literature on hyperkahler manifolds (mostly devoted to their geometry and construction of nontrivial examples). A somehow orthogonal approach to a hyperkahler generalization of the structure of standard hamiltonian mechanics, focusing on Poisson structures, was suggested by Xu [36].

(3) We also mention that in field theory considerations of multiple symplectic structures is suggested by covariance requirements and lies at the basis of the de Donder - Weyl formalism, as discussed in detail in [23], who call this “multisymplectic field theory”; however our approach (limited to mechanics) seems – at least at the present stage – not related to this theory.

(4) As mentioned in the introduction, a most important result in hyperkahler geometry concerns the construction of nontrivial hyperkahler manifolds via a moment map procedure starting from a (possibly trivial) hyperkahler manifold equipped with a Lie group action [10, 27]. It is natural to ask what happens when a (covariant) hyperhamiltonian dynamics is defined on the first manifold, i.e. how the dynamics descends to the quotient.

Let \((M, g; \omega_\alpha)\) be a hyperkahler manifold of dimension \(m\). Assume that there is a compact Lie group \(G\) (we denote by \(G^\ast\) the Lie algebra of \(G\) and by \(G^\ast\) its dual) acting freely on \(M\) and preserving its metric \(g\) and the three forms \(\omega_\alpha\) (thus acting triholomorphically); this defines three moment maps \(\mu_\alpha : M \to G^\ast\otimes \mathbb{R}^3\). It is known [27] that the quotient metric on \(N = \mu^{-1}(0)/G\) is hyperkahler. We denote by \(\beta_\alpha\) the reduction of \(\omega_\alpha\) on \(N\).

Let \(H^\alpha\) be three \(G\)-invariant smooth functions \(H^\alpha : M \to \mathbb{R}\), \(H^\alpha(x) = H^\alpha(gx)\) for all \(g \in G\) and \(x \in M\), and \(X\) be the hyperhamiltonian vector field on \(M\) corresponding to these; we recall that \(X_{\alpha}\) is given by \(X_{\alpha} = \sum X_{\alpha}\) with \(X_{\alpha}\) identified by \(X_{\alpha} \omega_{\alpha} = dH_{\alpha}\) (no sum on \(\alpha\)).

However, each \(X_{\alpha}\) is a hamiltonian vector field, generated by the hamiltonian \(H^\alpha\), with respect to the symplectic structure \(\omega_{\alpha}\). Thus, each \(X_{\alpha}\) descends to a hamiltonian vector field \(W_{\alpha}\) on \(N\), by standard symplectic reduction. In other words, for each \(\alpha\) there is a smooth function \(K^\alpha : N \to \mathbb{R}\) such that \(W_{\alpha} \omega_{\alpha} = dK^\alpha\) (no sum on \(\alpha\)). This shows at once that \(X\) descends to a hyperhamiltonian vector field \(W = \sum W_{\alpha}\) on the hyperkahler quotient \(N\).

(5) Physically, one should consider generalizations of the present approach in at least two directions: on the one hand, one would like to consider pseudoriemannian rather than riemannian manifolds; and on the other hand one should consider quantum version of the theory. It appears that both of these are feasible, and we will report on these matters in a separate note.

(6) Finally, we would like to point out that the dynamics introduced here can be obtained in a completely different and quite interesting way. One can look at standard Hamilton equations in terms of complex analysis, and extend them from the complex to the quaternionic case; one obtains then exactly the equations introduced here, as discussed in [29].
Appendix. Ideals of differential forms.

In our discussion of the variational formulation of the hyperHamiltonian equations of motion, we used some concepts from the theory of ideals of differential forms (here called just ideals, for short). This is maybe less widely known than the other tools used in the paper, so we collect here some definitions for convenience of the reader; see \cite{15,22} or \cite{4} for further detail.

**Definition A1.** Let $M$ be a smooth $N$-dimensional manifold, and $\mathcal{J}_k \subset \Lambda^k(M)$, for $k = 0,\ldots,N$. The subset $\mathcal{J} = \bigcup_{k=0}^N \mathcal{J}_k \subset \Lambda(M)$ is said to be an ideal of differential forms iff: (i) $\eta \in \mathcal{J}$, $\psi \in \Lambda(M)$, $\Rightarrow \eta \wedge \psi \in \mathcal{J}$; (ii) $\beta_1, \beta_2 \in \mathcal{J}_k$, $f_1, f_2 \in \Lambda^0(M)$, $\Rightarrow f_1 \beta_1 + f_2 \beta_2 \in \mathcal{J}_k$.

**Definition A2.** Let $i : S \to M$ be a smooth submanifold of $M$; $S$ is said to be an integral manifold of the ideal $\mathcal{J}$ iff $i^*(\eta) = 0$ for all $\eta \in \mathcal{J}$.

The ideal $\mathcal{J}$ is said to be generated by the forms $\{\eta^{(\alpha)}, \alpha = 1,\ldots,r\}$ (with $\eta^{(\alpha)} \in \mathcal{J}$) if each $\varphi \in \mathcal{J}$ can be written as $\varphi = \sum_\alpha \rho^{(\alpha)}_\eta \wedge \eta^{(\alpha)}$ for a suitable choice of $\rho^{(\alpha)} \in \Lambda(M)$, $\alpha = 1,\ldots,r$. If $\mathcal{J}$ is generated by $\{\eta^{(\alpha)}, \alpha = 1,\ldots,r\}$, then $i : S \to M$ is an integral manifold for $\mathcal{J}$ iff $i^*(\eta^{(\alpha)}) = 0$ for all $\alpha = 1,\ldots,r$.

Given an ideal $\mathcal{J}$, we associate to any point $x \in M$ the subspace $D_x(\mathcal{J}) \subset T_xM$ defined by $D_x(\mathcal{J}) := \{\xi \in T_xM : \xi \wedge \mathcal{J}_x \subset \mathcal{J}_x\}$.

**Definition A3.** If $D_x(\mathcal{J})$ has constant dimension, the ideal $\mathcal{J}$ is said to be non-singular, and the distribution $D(\mathcal{J}) = \{D_x(\mathcal{J}), x \in M\}$ is its characteristic distribution; any vector field $X \in D(\mathcal{J})$ is said to be a characteristic field for $\mathcal{J}$.

The following proposition is easy to prove (e.g. using the local coordinates introduced in section 44 of \cite{15}, or directly from definitions above) and is used in remark 8.

**Proposition A1.** Let $\mathcal{J}$ be generated by forms of degree $k$, and let $i : S \to M$ be an $r$-dimensional integral manifold of $\mathcal{J}$, with $r < k$. Let $X$ be a characteristic vector field for $\mathcal{J}$, and let $X$ be nowhere tangent to $i(S)$. Let $\varphi_t$ be the local one-parameter group of diffeomorphisms generated by $X$. Then the $(r+1)$-dimensional manifold $(-\varepsilon,\varepsilon) \times S \ni (t,x) \mapsto \varphi_t(x) \in M$ is an integral manifold of $\mathcal{J}$.
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