Quantum entropy inequalities are studied. Some quantum entropy inequalities are obtained by several methods. For entanglement breaking channel, we show that the entanglement-assisted classical capacity is upper bounded by $\log d$. A relationship between entanglement-assisted and one-shot unassisted capacities is obtained. This relationship shows the entanglement-assisted channel capacity is upper bounded by the sum of $\log d$ and the one-shot unassisted classical capacity.

Index Terms: Channel capacity, entanglement, quantum information, quantum entropy.

I. INTRODUCTION

Quantum information theory has been attracting a great deal of interests. Several capacities of quantum channels are proposed and studied, such as the Holevo-Schumacher-Westmoreland channel capacity and the recent proposed entanglement-assisted classical capacity, adaptive classical capacity. In studying capacities of quantum channels, the quantum entropy inequalities are very important. In ref., a survey of quantum entropy inequalities are presented. Some of these quantum inequalities are independent but equivalent, i.e. they are necessary and sufficient conditions to each other. In some cases, the results can be obtained much easier from one quantum entropy inequality than from others. So, all of these inequalities are necessary. It will be better if we can find more inequalities. In this paper, we try to study some of these quantum entropy inequalities and to find their applications in channel capacities. In particular, we propose the strong concavity of von Neumann entropy inequality and prove it by several methods.

The additivity of classical capacity of quantum channels is one of the fundamental problems in studying the quantum information theory. The additivity of classical capacity of several special channels is proved, such as unital qubit channels, depolarizing channels and entanglement breaking channels. By using directly the strong concavity of von Neumann entropy, we give a simple proof of the additivity of classical capacity of entanglement breaking channels.

It is known that the classical capacity of quantum channels may be enhanced with prior entanglement such as the super-dense coding protocol. A general theorem called entanglement-assisted classical capacity was proposed and proved recently concerning about the classical capacity of quantum channels with the help of shared entanglement between the sender and the receiver. It can be expected that if the channel itself is entanglement breaking, its entanglement-assisted classical capacity has less advantage than other channels. Really, we show in this paper that entanglement breaking channel, the entanglement-assisted classical capacity is upper bounded by $\log d$ while generally we have an extra term $\chi$.

A simple proof of the entanglement-assisted channel capacity was given by Holevo, he also found the entanglement-assisted channel capacity is upper bounded by the sum of $\log d$ and the unassisted classical capacity. We shall show further in this paper that the entanglement-assisted channel capacity is upper bounded by the sum of $\log d$ and the one-shot unassisted classical capacity. This result also eliminates one of the possible ways in which one might prove non-additivity of the classical capacity.

II. EQUIVALENT QUANTUM ENTROPY INEQUALITIES

There are 4 equivalent quantum entropy inequalities as reviewed by Ruskai. In this section, we point out that we can add another equivalent entropy inequality.

First, let us introduce some definitions. The von Neumann entropy is defined as:

$$S(\rho) \equiv -\text{Tr}(\rho \log \rho),$$

where $\rho$ is the density operator. The relative entropy is defined as:
In a recent review paper, Ruskai listed the first 4 equivalent quantum entropy inequalities as presented as follows, see [13] and the references therein:

1. Monotonicity of relative entropy under completely positive, trace preserving maps:
   \[ S(\Phi(\rho)||\Phi(\sigma)) \leq S(\rho||\sigma). \]  

2. Monotonicity of relative entropy under partial trace:
   \[ S(\rho_A||\sigma_A) \leq S(\rho_{AC}||\sigma_{AC}). \]

3. Strong subadditivity of von Neumann entropy I and II, where I and II are equivalent:
   \[ I), \ S(\rho_A) + S(\rho_B) \leq S(\rho_{AC}) + S(\rho_{BC}); \]
   \[ II), \ S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}). \]

4. Joint convexity of relative entropy:
   \[ S\left(\sum_i p_i \rho_A^i ||\sum_i p_i \sigma_A^i\right) \leq \sum_i p_i S(\rho_A^i ||\sigma_A^i). \]

5. Actually, we can add another equivalent inequality, concavity of conditional entropy:
   \[ S\left(\sum_i p_i \rho_{AB}^i\right) - S\left(\sum_i p_i \rho_B^i\right) \geq \sum_i p_i [S(\rho_{AB}^i) - S(\rho_B^i)]. \]

The last inequality was deduced from 4, joint convexity of relative entropy in Ref. [12]. Then it was used to deduce the inequality 3, strong subadditivity. So, inequality 5, the concavity of conditional entropy is an equivalent inequality with the other 4 inequalities.

In the textbook of Nielsen and Chuang [12], inequality 5 is obtained from 4, the joint convexity. Next, we show two other methods to obtain the concavity of conditional entropy. First, we use 2, monotonicity of relative entropy under partial trace. Suppose \( \rho_{AB} = \sum_i p_i \rho_{AB}^i \), so \( \rho_B = \sum_i p_i \rho_B^i \). Since inequality 2, we have
\[ S(\rho_B||\rho_B) \leq S(\rho_{AB}||\rho_{AB}) . \]

So, the average of relative entropies has inequality
\[ \sum_i p_i S(\rho_B^i||\rho_B) \leq \sum_i p_i S(\rho_{AB}^i||\rho_{AB}) . \]

From the definition of relative entropy, we obtain 5,
\[ S(\rho_{AB}) - S(\rho_B) \geq \sum_i p_i [S(\rho_{AB}^i) - S(\rho_B^i)]. \]

Secondly, we also use the joint convexity to deduce 5, but by a different method. The joint convexity of relative entropy means,
\[ S\left(\sum_i p_i \rho_{AB}^i ||\sum_i p_i \rho_B^i\right) \leq \sum_i p_i S(\rho_{AB}^i ||\rho_B^i). \]

By definition, we have
\[ -S(\rho_{AB}) + S(\rho_B) \leq -\sum_i p_i [S(\rho_{AB}^i) - S(\rho_B^i)]. \]

This is exactly 5, the concavity of conditional entropy.

Since these 5 inequalities are equivalent, we can obtain any of them by one of the other 4 inequalities. Recently, Bennett et al proposed and proved the entanglement-assisted channel capacity [3,4]. Holevo subsequently gave a modified proof [4], and one of the simplifications is due to the replacement of strong subadditivity by concavity of conditional entropy, i.e., the fifth inequality was used directly in Ref. [4] rather than the third inequality used in Ref. [4], though they are equivalent.
III. STRONG CONCAVITY OF VON NEUMANN ENTROPY

In this section, we propose the following quantum entropy inequality: strong concavity of von Neumann entropy,

\[
S \left( \sum_i p_i \rho_A^i \otimes \rho_B^i \right) \geq \max \left\{ \sum_i p_i S(\rho_A^i) + S \left( \sum_i p_i \rho_B^i \right), \sum_i p_i S(\rho_B^i) + S \left( \sum_i p_i \rho_A^i \right) \right\}.
\]

(13)

To prove this inequality, we need to show that both of the following two inequalities hold,

\[
S \left( \sum_i p_i \rho_A^i \otimes \rho_B^i \right) \geq \sum_i p_i S(\rho_A^i) + S \left( \sum_i p_i \rho_B^i \right).
\]

(14)

and

\[
S \left( \sum_i p_i \rho_A^i \otimes \rho_B^i \right) \geq \sum_i p_i S(\rho_B^i) + S \left( \sum_i p_i \rho_A^i \right).
\]

(15)

We denote \( \rho_A = \sum_i p_i \rho_A^i, \rho_B = \sum_i p_i \rho_B^i. \)

In the following, we present several methods to derive the strong concavity of quantum entropy.

A, Due to 2, monotonicity of relative entropy, we have

\[
S(\rho_A^i | \rho_A) \leq S(\rho_A^i \otimes \rho_B^i, \sum_i p_i \rho_A^i \otimes \rho_B^i).
\]

(16)

Take average with probability \( p_i \), we have

\[
\sum_i p_i S(\rho_A^i | \rho_A) \leq \sum_i p_i \left( S(\rho_A^i \otimes \rho_B^i, \sum_j p_j \rho_A^j \otimes \rho_B^j) \right).
\]

(17)

So, we have

\[
- \sum_i p_i S(\rho_A^i) + S(\rho_A) \leq - \sum_i p_i [S(\rho_A^i) + S(\rho_B^i)] + S \left( \sum_i p_i \rho_A^i \otimes \rho_B^i \right).
\]

(18)

Thus we obtain the strong concavity.

C, Due to 4, joint convexity of relative entropy,

\[
S \left( \sum_i p_i \rho_A^i \otimes \rho_B^i \right) \leq \sum_i p_i S(\rho_A^i \otimes \rho_B^i | \rho_A^i)
\]

(19)

\[
= - \sum_i p_i S(\rho_B^i).
\]

We have the strong concavity of von Neumann entropy.

B, Due to 5, concavity of conditional entropy, we have

\[
S \left( \sum_i p_i \rho_A^i \otimes \rho_B^i \right) - S \left( \sum_i p_i \rho_A^i \right) \geq \sum_i p_i [S(\rho_A^i \otimes \rho_B^i) - S(\rho_A^i)]
\]

(20)

\[
= \sum_i p_i S(\rho_A^i).
\]

Then we arrive at the strong concavity. Since the situations for \( \rho_A \) and \( \rho_B \) are the same, we know that Eq.\((13)\) holds. To the author’s knowledge, this inequality has not been explicit proposed previously. Though it is implied in the study of the additivity of entanglement breaking channel \((14)\) and the additivity of entanglement of formation in some cases \((18)\). Next, we would like to show some applications of this inequality.
IV. APPLICATION OF STRONG CONCAVITY IN CHANNEL CAPACITY OF ENTANGLEMENT BREAKING CHANNEL

Recently, Shor proved the additivity of the classical capacity of the entanglement breaking quantum channel [16]. Both c-q (classical-quantum) and q-c (quantum-classical) channels are special cases of entanglement breaking channels. And the entanglement breaking channel can be expressed as a q-c-q channel. Other properties and conjectures of entanglement breaking channel can be found in Ref. [14]. We next give a simple proof of the additivity of channel capacity of the entanglement breaking channel by directly use the strong concavity inequality though there are no essential differences from Shor’s original proof.

An entanglement breaking channel $\Phi$ means $(I \otimes \Phi) \rho_{AB}$ is always a separable state, which can be written as [19],

$$(I \otimes \Phi) \rho_{AB} = \sum_i p_i \rho^i_A \otimes \rho^i_B.$$  

(21)

So we know $\Phi(\rho_B) = \sum_i p_i \rho^i_B$. Suppose $\sum_j q_j \rho^j_{AB} = \rho_{AB}$ are the optimal signal states for channel $\Psi \otimes \Phi$, where $\Psi$ is an arbitrary quantum channel. The Holevo-Schumacher-Westmoreland channel capacity $\chi^*(\Psi \otimes \Phi)$ takes the following form

$$\chi^*(\Psi \otimes \Phi) = - \sum_j q_j S\left(\sum_i p_{ji} \Psi(\rho^j_i) \otimes \rho^i_B\right) + S(\Psi(\rho_A)) + S(\Phi(\rho_B)) + S((\Phi \otimes I)(|\Psi_{AB}\rangle\langle\Psi_{AB}|)).$$  

(22)

Then using the strong concavity inequality to the first term and subadditivity to the second term, we have

$$\chi^*(\Psi \otimes \Phi) \leq - \sum_j q_j p_{ji} S\left(\Psi(\rho^j_i)\right) - \sum_j q_j S\left(\sum_i p_{ji} \rho^i_B\right) + S(\Psi(\rho_A)) + S(\Phi(\rho_B))$$

$$= \sum_j q_j p_{ji} S\left(\Psi(\rho^j_i)\right) + \sum_j q_j S\left(\sum_i p_{ji} \rho^i_B\right) + S(\Phi(\rho_B))$$

$$\leq \chi^*(\Psi) + \chi^*(\Phi).$$  

(23)

Since the classical capacity of quantum channel is strong additive, we thus know the capacity of entanglement breaking channel is additive,

$$\chi^*(\Psi \otimes \Phi) = \chi^*(\Psi) + \chi^*(\Phi).$$  

(24)

V. APPLICATION OF STRONG CONCAVITY OF VON NEUMANN ENTROPY IN ENTANGLEMENT-ASSISTED CHANNEL CAPACITY FOR AN ENTANGLEMENT BREAKING CHANNEL

Recently, Bennett et al [3,4] (BSST theorem) proposed and proved the entanglement-assisted channel capacity in terms of quantum mutual information. Holevo [8] then gave a simple proof. The BSST theorem states that the classical capacity of the entanglement-assisted channel is written as the form

$$C_E(\Phi) = \max_{\rho_A \in \mathcal{H}_A} S(\rho_A) + S(\Phi(\rho_A)) - S((\Phi \otimes I)(|\Psi_{AB}\rangle\langle\Psi_{AB}|)),$$

(25)

where $|\Psi_{AB}\rangle$ is a purification of $\rho_A$.

Holevo [8] pointed out that there is a relationship between the entanglement-assisted and unassisted capacities,

$$C_E(\Phi) \leq C(\Phi) + \log d,$$

(26)

where $d$ is the dimension of the Hilbert space $\mathcal{H}_A$. If the additivity of the classical capacity holds, we can replace $C(\Phi)$ by one-shot classical capacity $\chi^*(\Phi)$. Since Shor [16] already proved that the classical capacity of the entanglement breaking channel is additive. So, for entanglement breaking channel $\Phi$, we have
Next, we show that a tighter upper bound can be obtained for entanglement breaking channel. Because Φ is an entanglement breaking channel, so we have

\[(\Phi \otimes I)(|\Psi_{AB}\rangle\langle \Psi_{AB}|) = \sum_i p_i \rho_A^i \otimes \rho_B^i, \tag{28}\]

where both \(\rho_A^i\) and \(\rho_B^i\) are pure states. By strong concavity of von Neumann entropy, we know

\[S((\Phi \otimes I)(|\Psi_{AB}\rangle\langle \Psi_{AB}|)) \geq \sum_i p_i S(\rho_A^i) + \sum_i p_i S(\rho_B^i) = S(\Phi(\rho_A)), \tag{29}\]

Substitute these relations to BSST theorem (25), we have

\[C_E(\Phi) \leq \max_{\rho_A \in H} S(\rho_A) \leq \log d, \tag{30}\]

or,

\[C_E(\Phi) \leq \max_{\rho_A \in H} S(\Phi(\rho_A)) \leq \log d. \tag{31}\]

So, we know for entanglement breaking channel, the entanglement-assisted classical capacity has an upper bound

\[C_E(\Phi) \leq \log d. \tag{32}\]

Comparing this relation with the general relation (27), we find that the term \(\chi^*(\Phi)\) does not appear here though it is not always zero. So, we show there is an upper bound for \(C_E(\Phi)\) when Φ is an entanglement breaking channel. It might be interpreted as, since the channel itself is entanglement-breaking, the prior entanglement may not help much to increase the classical capacity.

**VI. RELATIONSHIP BETWEEN ENTANGLEMENT-ASSISTED AND ONE-SHOT UNASSISTED CAPACITIES**

As already pointed out in last section, Holevo [8] found entanglement-assisted channel capacity is upper bounded by the sum of \(\log d\) and the unassisted classical capacity as relation (27). If the classical channel capacity is additive which is a long-standing conjecture, then we have the inequality

\[C_E(\Phi) \leq \chi^*(\Phi) + \log d. \tag{33}\]

For an arbitrary quantum channel Φ, if this relation does not hold, that means \(C(\Phi) > \chi^*(\Phi)\), thus the additivity conjecture of classical channel capacity does not hold. So, (33) may provide a criterion to test the additivity problem of classical capacity. However, we show in this section, relation (33) always holds for an arbitrary quantum channel Φ.

We assume that \(\rho_A\) have the following pure state decomposition

\[\rho_A = \sum_j q_j |\tilde{\Psi}_A^j\rangle \langle \tilde{\Psi}_A^j|. \tag{34}\]

Using the same technique as that of Ref. [16], we define

\[|\tilde{\Psi}_{ABC}\rangle = \sum_j \sqrt{q_j} |\tilde{\Psi}_A^j\rangle |j\rangle_B |j\rangle_C. \tag{35}\]

So, we have

\[(\Phi \otimes I_{BC})(|\tilde{\Psi}_{ABC}\rangle\langle \tilde{\Psi}_{ABC}|) = \sum_{jj'} \sqrt{q_j q_j'} \Phi(|\tilde{\Psi}_A^j\rangle\langle \tilde{\Psi}_A^{j'}|) \otimes |j\rangle_B |j\rangle_C |j\rangle. \tag{36}\]

With the help of the quantum entropy inequality, we obtain
\[
S \left( (\Phi \otimes I_B) (|\tilde{\Psi}_{ABC}\rangle \langle \tilde{\Psi}_{ABC}|) \right) \geq S \left( (\Phi \otimes I_B) (\tilde{\rho}_{AB}) \right) - S(\tilde{\rho}_C) \\
= S \left( \sum_j q_j \Phi (|\tilde{\Psi}_j\rangle \langle \tilde{\Psi}_j|) \otimes |j\rangle_B \langle j| \right) - S \left( \sum_j q_j |j\rangle_C \langle j| \right) \\
= \sum_j q_j S \left( \Phi (|\tilde{\Psi}_j\rangle \langle \tilde{\Psi}_j|) \right). 
\] (37)

We know
\[
S \left( (\Phi \otimes I_B) (|\Psi_{AB}\rangle \langle \Psi_{AB}|) \right) = S \left( (\Phi \otimes I_B) (|\tilde{\Psi}_{ABC}\rangle \langle \tilde{\Psi}_{ABC}|) \right), 
\] (38)

where both \(|\Psi_{AB}\rangle\) and \(|\tilde{\Psi}_{ABC}\rangle\) are purifications of \(\rho_A\). From BSST theorem (25), we have
\[
C_E(\Phi) = \max_{\rho_A \in \mathcal{H}_A} S(\rho_A) + S(\Phi(\rho_A)) - S \left( (\Phi \otimes I) (|\Psi_{AB}\rangle \langle \Psi_{AB}|) \right) \\
\leq \max_{\rho_A \in \mathcal{H}_A} S(\rho_A) + S(\Phi(\rho_A)) - \sum_j q_j S \left( \Phi (|\tilde{\Psi}_j\rangle \langle \tilde{\Psi}_j|) \right) \\
\leq \log d + \chi^*(\Phi). 
\] (39)

Thus, we conclude, for an arbitrary quantum channel \(\Phi\), the entanglement-assisted and one-shot unassisted capacities have the relationship
\[
C_E(\Phi) \leq \chi^*(\Phi) + \log d. 
\] (40)

If the additivity of classical capacity holds, this relation is the same as the relation (26). If the additivity does not hold for classical capacity, this relation is tighter than (26).

VII. SUMMARY

In summary, we pointed out that there are another quantum entropy inequality, the concavity of conditional entropy inequality, to be equivalent to other 4 equivalent quantum entropy inequalities. We proposed the strong concavity of von Neumann entropy and proved it by several methods. Using directly this inequality, the additivity of capacity of entanglement breaking channels can be proved simply. We also showed for entanglement breaking channel, the entanglement-assisted channel capacity is upper bounded by \(\log d\) which is tighter than the general case. A new upper bound is obtained for the entanglement-assisted classical capacity, the entanglement-assisted classical capacity is upper bounded by the sum of \(\log d\) and the one-shot unassisted capacity. This result also eliminates one possible way to test the non-additivity of classical capacity.

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[1] G.Adami and N.Cerf, Phys.Rev.A 56, 3470 (1997).
[2] H.Barnum, M.A.Nielsen, and B.Schumacher, Phys.Rev.A 57, 4153 (1998).
[3] C.H. Bennett, P.W.Shor, J.A. Smolin, and A.V. Thapliyal, Phys.Rev.Lett. 83, 3081 (1999).
[4] C.H. Bennett, P.W.Shor, J.A. Smolin, and A.V. Thapliyal, IEEE Trans. Info. Theory, 48, 2637 (2002).
[5] C.H. Bennett, S.J.Wiesner, Phys.Rev.Lett. 69, 2881 (1992).
[6] H.Fan, Additivity of entanglement of formation for some special cases, quant-ph/0210169.
[7] A.S.Holevo, IEEE Trans. Info. Theory, 44, 269 (1998).
[8] A.S.Holevo, J.Math.Phys. 43, 4326 (2002).
[9] A.S.Holevo, R.F.Werner, Phys.Rev.A 63, 032313 (2000).
[10] C.King, Additivity for unital qubit channels, quant-ph/0103156.
[11] C.King, The capacity of the quantum depolarizing channel, quant-ph/0204172.
[12] M.A.Nielsen,I.L.Chuang, Quantum Computation and Quantum Information, Cambridge University Press (2000).
[13] M.B.Ruskai, J.Math.Phys. 43, 4358 (2002), quant-ph/0205064.
[14] M.B.Ruskai, *Entanglement breaking channels*, quant-ph/0207100.
[15] B.W.Schumacher, M.D.Westmoreland, Phys.Rev.A 56, 131 (1997).
[16] P.Shor, J.Math.Phys.43, 4334 (2002) quant-ph/0201144.
[17] P.W.Shor, *The adaptive classical capacity of a quantum channel, Information capacities of three symmetric pure states in three dimensions*, quant-ph/0206055.
[18] A.Wehrl, Rev.Mod.Phys.50, 221 (1978).
[19] R.F.Werner, Phys.Rev.A40, 4277 (1989).