Research Article

On an Integral Transform of a Class of Analytic Functions

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For \( f \) denote the class of analytic functions \( \mathcal{A} \), the Hadamard product, Fournier and Ruscheweyh (1934 and 1935) that functions in \( \mathcal{A}(1,0) \) are close-to-convex and hence univalent for \( 0 < \beta < 1 \). Let \( \mathcal{V}(f) \) be the subclass of \( \mathcal{A}(1,0) \) whenever \( f \in \mathcal{A}(1,0) \) and \( \mathcal{V}(g) \in \mathcal{A}(1,0) \) whenever \( f \in \mathcal{A}(1,0) \).

1. Introduction

Let \( \mathcal{A} \) denote the class of analytic functions \( f \) defined in the open unit disc \( E = \{ z : |z| < 1 \} \) with the normalizations \( f(0) = f'(0) - 1 = 0 \), and let \( \mathcal{S} \) be the subclass of \( \mathcal{A} \) consisting of functions univalent in \( E \). For any two functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \) in \( \mathcal{A} \), the Hadamard product (or convolution) of \( f \) and \( g \) is the function \( f * g \) defined by

\[
(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.
\]

For \( f \in \mathcal{A} \), Fournier and Ruscheweyh [1] introduced the integral operator

\[
F(z) = V_1(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt,
\]

\[
\lambda(t) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt.
\]
where $\lambda$ is a nonnegative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t) dt = 1$. This operator contains some well-known operators such as Libera, Bernardi, and Komatu as its special cases. Fournier and Ruscheweyh [1] applied the famous duality theory to show that for a function $f$ in the class

$$D(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi} \left( f'(z) - \beta \right) > 0, \ z \in E \right\},$$

(1.3)

the linear integral operator $V_\lambda(f)$ is univalent in $E$. Since then, this operator has been studied by a number of authors for various choices of $\lambda(t)$. In another remarkable paper, Barnard et al. in [2] obtained conditions such that $V_\lambda(f) \in D(\beta)$ whenever $f$ is in the class

$$D_1(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi} \left( (1 - \gamma) \frac{f(z)}{z} + \gamma f'(z) - \beta \right) > 0, \ z \in E \right\},$$

(1.4)

with $\beta < 1$, $\gamma \geq 0$. Note that for $0 \leq \beta < 1$, functions in $D_1(\beta) \equiv D(\beta)$ satisfy the condition $\Re f'(z) > \beta$ in $E$ and thus are close-to-convex in $E$. A domain $D$ in $\mathbb{C}$ is close-to-convex if its compliment in $\mathbb{C}$ can be written as union of nonintersecting half lines.

In 2008, Ponnusamy and Rønning [3] discussed the univalence of $V_\lambda(f)$ for the functions in the class

$$R_\gamma(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi} \left( f'(z) + \gamma zf''(z) - \beta \right) > 0, \ z \in E \right\}.$$  

(1.5)

In a very recent paper, Ali et al. [4] studied the class

$$\mathcal{K}_\beta(\alpha, \gamma) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \mid \Re e^{i\phi} \left( (1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma zf''(z) - \beta \right) > 0, \ z \in E \right\},$$

(1.6)

where $\alpha, \gamma \geq 0$ and $\beta < 1$. In this paper, they obtained sufficient conditions so that the integral transform $V_\lambda(f)$ maps normalized analytic functions $f \in \mathcal{K}_\beta(\alpha, \gamma)$ into the class of starlike functions. It is evident that $\mathcal{K}_\beta(1,0) \equiv D(\beta)$, $\mathcal{K}_\beta(\alpha,0) \equiv D_1(\beta)$ and $\mathcal{K}_\beta(1+2\gamma,\gamma) \equiv R_\gamma(\beta)$.

In the present paper, we shall mainly tackle the following problems.

1. For given $\delta < 1$, find sharp values of $\beta = \beta(\delta, \alpha)$ such that $V_\lambda(f) \in \mathcal{K}_\beta(1,0)$ whenever $f \in \mathcal{K}_\beta(\alpha, \gamma)$.

2. For given $\delta < 1$, find sharp values of $\beta = \beta(\delta)$ such that $V_\lambda(f) \in \mathcal{K}_\beta(\alpha, \gamma)$ whenever $f \in \mathcal{K}_\beta(\alpha, \gamma)$.

To prove one of our results, we shall need the generalized hypergeometric function $\ _pF_q$, so we define it here.
Let \( \alpha_j (j = 1, 2, \ldots, p) \) and \( \beta_j (j = 1, 2, \ldots, q) \) be complex numbers with \( \beta_j \neq 0, -1, -2, \ldots (j = 1, 2, \ldots, q) \). Then the generalized hypergeometric function \( pF_q \) is defined by

\[
pF_q(z) = pF_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} \quad (p \leq q + 1),
\]

(1.7)

where \((a)_n\) is the Pochhammer symbol, defined in terms of the Gamma function, by

\[
(a)_n := \frac{\Gamma(a + n)}{\Gamma(a)} = \begin{cases} 
1, & n = 0, \\
(a + 1) \cdots (a + n - 1), & n \in \mathbb{N}.
\end{cases}
\]

(1.8)

In particular, \( zF_1 \) is called the Gaussian hypergeometric function. We note that the \( pF_q \) series in (1.7) converges absolutely for \(|z| < \infty\) if \( p < q + 1 \) and for \( z \in E \) if \( p = q + 1 \).

We shall also need the following lemma.

**Lemma 1.1** (see [5]). Let \( \beta_1 < 1, \beta_2 < 1, \) and \( \eta \in \mathbb{R} \). Then, for \( p, q \) analytic in \( E \) with \( p(0) = q(0) = 1 \), the conditions \( \Re p(z) > \beta_1 \) and \( \Re e^\eta(q(z) - \beta_2) > 0 \) imply \( \Re e^\eta((p \ast q)(z) - \delta) > 0 \), where \( 1 - \delta = 2(1 - \beta_1)(1 - \beta_2) \).

### 2. Main Results

We use the notations introduced in [4]. Let \( \mu \geq 0 \) and \( \nu \geq 0 \) satisfy

\[
\mu + \nu = \alpha - \gamma, \quad \mu \nu = \gamma.
\]

(2.1)

When \( \gamma = 0 \), then \( \mu \) is chosen to be 0, in which case, \( \nu = \alpha \geq 0 \). When \( \alpha = 1 + 2\gamma \), (2.1) yields \( \mu + \nu = 1 + \gamma = 1 + \mu \nu \) or \((\mu - 1)(1 - \nu) = 0\).

(i) For \( \gamma > 0 \), then choosing \( \mu = 1 \) gives \( \nu = \gamma \).

(ii) For \( \gamma = 0 \), then \( \mu = 0 \) and \( \nu = \alpha = 1 \).

**Theorem 2.1.** Let \( \mu \geq 0, \nu \geq 0 \) satisfy (2.1). Further, let \( \delta < 1 \) be given, and define \( \beta = \beta(\delta, \mu, \nu) \) by

\[
1 - \frac{1 - \delta}{2} \left( 1 - \frac{1}{\nu} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1 + ts^\mu} \right) dt + \left( \frac{1}{\nu} - 1 \right) \int_0^1 \lambda(t) \left( \int_0^1 \frac{d\eta d\xi}{1 + t\eta^\nu} \right) dt \right)^{-1}, \quad \gamma \neq 0,
\]

\[
1 - \frac{1 - \delta}{2} \left( 1 - \frac{1}{\alpha} \int_0^1 \frac{1}{1 + t^\alpha} dt + \left( \frac{1}{\alpha} - 1 \right) \int_0^1 \frac{d\eta}{1 + t\eta^\nu} dt \right)^{-1}, \quad \gamma = 0 \quad (\mu = 0, \nu = \alpha > 0).
\]

(2.2)

If \( f \in \mathcal{W}_p(\alpha, \gamma) \), then \( F = V_1(f) \in \mathcal{W}_\delta(1, 0) \subset S \). The value of \( \beta \) is sharp.

**Proof.** The case \( \gamma = 0 \) (\( \mu = 0, \nu = \alpha > 0 \)) corresponds to Theorem 1.5 in [2]. So we assume that \( \gamma > 0 \).
Define

\[
(1 - a + 2\gamma) \frac{f(z)}{z} + (a - 2\gamma) f'(z) + \gamma z f''(z) = H(z). \tag{2.3}
\]

Writing \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \), it follows that

\[
H(z) = 1 + \sum_{n=1}^{\infty} a_{n+1} (nv + 1)(n\mu + 1) z^n. \tag{2.4}
\]

It is a simple exercise to see that

\[
f'(z) = H(z) * 3F_2 \left( 2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu+1}, \frac{1}{\mu+1}; z \right). \tag{2.5}
\]

Let \( F(z) = V_1(f)(z) \), where \( V_1(f) \) is defined by (1.2). Then for \( \gamma \neq 0 \), we can write

\[
F'(z) = f'(z) * \int_0^1 \frac{\lambda(t)}{1 - tz} dt = H(z) * 3F_2 \left( 2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu+1}, \frac{1}{\mu+1}; z \right) * \int_0^1 \frac{\lambda(t)}{1 - tz} dt = H(z) * \int_0^1 \lambda(t) 3F_2 \left( 2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu+1}, \frac{1}{\mu+1}; tz \right) dt. \tag{2.6}
\]

Since \( f \in \mathcal{H}_d(a, \gamma) \), it follows that \( \Re \{ e^{i\phi}(H(z) - \beta) \} > 0 \) for some \( \phi \in \mathbb{R} \). Now, for each \( \gamma > 0 \), we first claim that

\[
\Re \left[ \int_0^1 \lambda(t) 3F_2 \left( 2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu+1}, \frac{1}{\mu+1}; tz \right) dt \right] > 1 - \frac{1 - \delta}{2(1 - \beta)}, \quad z \in E, \tag{2.7}
\]

which, by Lemma 1.1, implies that \( F \in \mathcal{H}_d(1, 0) \). Therefore, it suffices to verify the inequality (2.7). Using the identity (which can be checked by comparing the coefficients of \( z^n \) on both sides)

\[
3F_2(2, b, c; d, e; z) = (d - 1) 3F_2(1, b, c; d - 1, e; z) - (d - 2) 3F_2(1, b, c; d, e; z), \tag{2.8}
\]

it follows that

\[
3F_2 \left( 2, \frac{1}{\nu}, \frac{1}{\mu}; \frac{1}{\nu+1}, \frac{1}{\mu+1}; z \right) = \frac{1}{\nu} \int_0^1 \frac{ds}{1 - zs^\mu} + \left( 1 - \frac{1}{\nu} \right) \int_0^1 \frac{d\eta d\zeta}{1 - z\eta^\nu \zeta^\mu}. \tag{2.9}
\]
Thus,

\[
\int_{0}^{1} \left( \frac{\lambda}{t} \right)^{3} F_{2} \left( 2, 1 ; \frac{1}{\mu}, \frac{1}{\nu + 1}, \frac{1}{\mu + 1}; tz \right) dt
\]

\[
= \int_{0}^{1} \lambda(t) \left\{ \frac{1}{\nu} \int_{0}^{1} \frac{ds}{1 - ts^{\mu}} + \left( 1 - \frac{1}{\nu} \right) \int_{0}^{1} \int_{0}^{1} \frac{d\eta d\zeta}{1 - t\eta^{\nu}\zeta^{\mu}} \right\} dt.
\]

(2.10)

Therefore, for \( \gamma > 0 \), we have

\[
\Re \left[ \int_{0}^{1} \lambda(t) \left( \frac{1}{\nu} \int_{0}^{1} \frac{ds}{1 - ts^{\mu}} \right) dt \right]
\]

\[
= \frac{1}{\nu} \int_{0}^{1} \lambda(t) \left( \int_{0}^{1} \frac{ds}{1 + ts^{\mu}} \right) dt + \left( \frac{1}{\nu} - 1 \right) \int_{0}^{1} \lambda(t) \left( \int_{0}^{1} \int_{0}^{1} \frac{d\eta d\zeta}{1 + t\eta^{\nu}\zeta^{\mu}} \right) dt
\]

(2.11)

in the view of (2.2).

To prove the sharpness, let \( f \in K_{\beta}(\alpha, \gamma) \) be the function determined by

\[
(1 - \alpha + 2\gamma) \frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma zf''(z) = \beta + (1 - \beta) \frac{1 + z}{1 - z}.
\]

(2.12)

Using a series expansion, we see that we can write

\[
f(z) = z + \sum_{n=2}^{\infty} \frac{2(1 - \beta)}{(nv + 1 - \nu)(n\mu + 1 - \mu)} z^{n}.
\]

(2.13)

Then,

\[
F(z) = V_{1}(f)(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{q_{n}}{(nv + 1 - \nu)(n\mu + 1 - \mu)} z^{n},
\]

(2.14)
where \( q_n = \int_0^1 \lambda(t)t^{n-1} \, dt \). Equation (2.2) can be restated as

\[
\frac{1}{1-\beta} = \frac{2}{1-\delta} \left\{ 1 - \frac{1}{\nu} \int_0^1 \lambda(t) \left( \int_0^1 \frac{ds}{1+ts^\mu} \right) \, dt + \left( \frac{1}{\nu} - 1 \right) \int_0^1 \lambda(t) \left( \int_0^1 \frac{d\eta d\zeta}{1+tn^{\nu}\zeta^{\mu}} \right) \, dt \right\}
\]

\[
= \frac{2}{1-\delta} \left\{ 1 + \int_0^1 \lambda(t) \left( -\frac{1}{\nu} \int_0^1 \frac{ds}{1+ts^\mu} + \left( \frac{1}{\nu} - 1 \right) \int_0^1 \frac{d\eta d\zeta}{1+tn^{\nu}\zeta^{\mu}} \right) \, dt \right\}
\]

\[
= \frac{2}{1-\delta} \int_0^1 \lambda(t) \left\{ \sum_{n=2}^{\infty} \frac{(-1)^{n-1}n^{\nu}}{(n\mu + 1 - \mu)} \left( -\frac{1}{\nu} + \left( \frac{1}{\nu} - 1 \right) \frac{1}{(n\nu + 1 - \nu)} \right) \right\} \, dt
\]

\[
= -\frac{2}{1-\delta} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}n^{\nu}q_n}{(n\nu + 1 - \nu)(n\mu + 1 - \mu)}.
\]

Finally,

\[
F'(z) = 1 + 2(1-\beta) \sum_{n=2}^{\infty} \frac{n^{\nu}q_n}{(n\nu + 1 - \nu)(n\mu + 1 - \mu)} z^{n-1},
\]

which for \( z = -1 \) takes the value

\[
F'(-1) = 1 + 2(1-\beta) \sum_{n=2}^{\infty} \frac{(-1)^{n-1}n^{\nu}q_n}{(n\nu + 1 - \nu)(n\mu + 1 - \mu)} = 1 + 2(1-\beta) \left\{ \frac{1-\delta}{2(1-\beta)} \right\} = \delta.
\]

This shows that the result is sharp.

Letting \( \gamma = 0 \) and \( \alpha = 1 \) in Theorem 1.1, we obtain the following result of Ruscheweyh [6].

**Corollary 2.2.** Let \( \delta < 1 \), and define \( \beta = \beta(\delta, 1) < 1 \) by

\[
\beta(\delta) = 1 - \frac{1-\delta}{2} \left\{ 1 - \int_0^1 \frac{\lambda(t)}{1+t} \, dt \right\}^{-1}.
\]

If \( f \in \mathcal{V}_p(1, 0) \equiv \mathcal{D}_1(\beta) \), then \( F = V_\lambda(f) \in \mathcal{V}_\delta(1, 0) \subset S \). The value of \( \beta \) is sharp.

**Theorem 2.3.** Let \( \delta < 1 \) and \( \alpha, \gamma \geq 0 \), and define \( \beta = \beta(\delta) < 1 \) by

\[
\frac{\beta}{1-\beta} = \int_0^1 \frac{\lambda(t) \left( 1 - ((1+\delta)/(1-\delta))t \right)}{(1+t)} \, dt.
\]

If \( f \in \mathcal{V}_p(\alpha, \gamma) \), then \( V_\lambda(f) \in \mathcal{V}_\delta(\alpha, \gamma) \). The value of \( \beta \) is sharp.
Proof. The idea of the proof is similar to the one used to prove Theorem 2 in [1].

Let $F(z) = V_{\lambda}(f)(z) = \int_0^1 \lambda(t)(f(tz)/t)\,dt$. Clearly,

$$F'(z) = \int_0^1 \frac{\lambda(t)}{1-tz} \, dt \ast f'(z).$$  \hfill (2.20)

Since, $f \in \mathcal{K}_\beta(\alpha, \gamma)$, so with

$g(z) = \frac{(1-\alpha + 2\gamma)(f(z)/z) + (\alpha - 2\gamma)f'(z) + \gamma zf''(z) - \beta}{1-\beta}$,  \hfill (2.21)

we have $\Re[e^{i\phi}g(z)] > 0$, where $\phi \in \mathbb{R}$.

For $\gamma \neq \alpha/2$,

$$f'(z) = \frac{1}{\alpha - 2\gamma}(\beta + (1-\beta)g(z)) - \frac{1-\alpha + 2\gamma f(z)}{\alpha - 2\gamma} - \frac{\gamma}{\alpha - 2\gamma}zf''(z).$$  \hfill (2.22)

Putting this value in (2.20),

$$F'(z) = \int_0^1 \frac{\lambda(t)}{1-tz} \, dt \ast \left( \frac{1}{\alpha - 2\gamma}(\beta + (1-\beta)g(z)) - \frac{1-\alpha + 2\gamma f(z)}{\alpha - 2\gamma} - \frac{\gamma}{\alpha - 2\gamma}zf''(z) \right).$$  \hfill (2.23)

Equivalently,

$$F'(z) = \frac{1}{\alpha - 2\gamma}g(z) \ast \left[ \beta + (1-\beta) \int_0^1 \frac{\lambda(t)}{1-tz} \, dt \right] - \frac{1-\alpha + 2\gamma f(z)}{\alpha - 2\gamma} - \frac{\gamma}{\alpha - 2\gamma}zf''(z).$$  \hfill (2.24)

Thus

$$(1-\alpha + 2\gamma)(F(z)/z) + (\alpha - 2\gamma)F'(z) + \gamma zf''(z) = g(z) \ast \left[ \beta + (1-\beta) \int_0^1 \frac{\lambda(t)}{1-tz} \, dt \right].$$  \hfill (2.25)

In the case when $\gamma = \alpha/2$,

$$g(z) = \frac{f(z)/z + \gamma zf''(z) - \beta}{1-\beta}.$$  \hfill (2.26)

Since

$$\frac{f(z)}{z} = \beta + (1-\beta)g(z) - \gamma zf''(z),$$  \hfill (2.27)
This leads to,

\[
\frac{F(z)}{z} + \gamma z F''(z) = g(z) * \left[ \beta + (1 - \beta) \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right],
\]

(2.28)

which is clearly (2.25) with \( \gamma = \alpha/2 \).

Further \( F \in \mathcal{K}_0(\alpha, \gamma) \) if and only if \( G(z) := (F(z) - \delta z) / (1 - \delta) \in \mathcal{K}_0(\alpha, \gamma) \). Now using (2.25), we obtain

\[
(1 - \alpha + 2\gamma) \frac{G(z)}{z} + (\alpha - \gamma) G'(z) + \gamma z G''(z) = g(z) * \left[ \frac{\beta - \delta}{1 - \delta} + \frac{1 - \beta}{1 - \delta} \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right].
\]

(2.29)

Since \( \Re e^{\phi} g(z) > 0 \) for some \( \phi \in \mathbb{R} \), it follows by duality principle [8, page 23] that

\[
(1 - \alpha + 2\gamma) \frac{G(z)}{z} + (\alpha - 2\gamma) G'(z) + \gamma z G''(z) \neq 0
\]

(2.30)

if, and only if,

\[
\Re \left[ \frac{\beta - \delta}{1 - \delta} + \frac{1 - \beta}{1 - \delta} \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right] > \frac{1}{2}.
\]

(2.31)

Using \( \Re (1/(1 - tz)) > 1/(1 + t) \), we get

\[
\Re \left[ \frac{\beta - \delta}{1 - \delta} + \frac{1 - \beta}{1 - \delta} \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right] > \frac{1 - \beta}{1 - \delta} \left[ \frac{\beta - \delta}{1 - \beta} + \int_0^1 \frac{\lambda(t)}{1 + t} dt \right].
\]

(2.32)

By using (2.19), we have

\[
\frac{\beta - (1 + \delta)/2}{1 - \beta} = \int_0^1 \frac{\lambda(t)}{(1 + t)} dt.
\]

(2.33)

Thus,

\[
\frac{\beta - \delta}{1 - \beta} + \int_0^1 \frac{\lambda(t)}{1 + t} dt = \frac{1 - \delta}{2 (1 - \beta)}
\]

(2.34)

which implies that

\[
\Re \left[ \frac{\beta - \delta}{1 - \delta} + \frac{1 - \beta}{1 - \delta} \int_0^1 \frac{\lambda(t)}{1 - tz} dt \right] > \frac{1 - \beta}{1 - \delta} \left[ \frac{\beta - \delta}{1 - \beta} + \int_0^1 \frac{\lambda(t)}{1 + t} dt \right] = \frac{1}{2}.
\]

(2.35)
Thus, we deduce, using duality principle, that \((1 - \alpha + 2\gamma)(G(z)/z) + (\alpha - \gamma)G'(z) + \gamma z G''(z)\) is contained in a half plane not containing the origin. So, \(G \in \mathcal{K}_0(\alpha, \gamma)\) and hence \(F \in \mathcal{K}_0(\alpha, \gamma)\).

To prove the sharpness, let \(f(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} (\omega_n z^n/(n\mu + 1 - \mu)(n\nu + 1 - \nu))\).

\[
F(z) = V_n(f)(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} \frac{\omega_n z^n}{(n\mu + 1 - \mu)(n\nu + 1 - \nu)}, \quad \text{where} \quad \omega_n = \int_0^1 \lambda(t) t^{n-1} dt. 
\]  
(2.36)

Further,

\[
\frac{\beta}{1 - \beta} = -\int_0^1 \lambda(t) \frac{(1 - ((1 + \delta)/(1 - \delta))t)}{(1 + t)} dt 
\]  
(2.37)

gives

\[
\frac{\beta}{1 - \beta} = -1 + \int_0^1 \lambda(t) \frac{(1 + (1 + \delta)/(1 - \delta))t}{(1 + t)} dt, 
\]  
(2.38)

or

\[
\frac{1}{1 - \beta} = \frac{2}{1 - \delta} \int_0^1 \frac{t\lambda(t)}{1 + t} dt = \frac{2}{1 - \delta} \sum_{n=2}^{\infty} (-1)^n \omega_n. 
\]  
(2.39)

Further, assume that

\[
H(z) = (1 - \alpha + 2\gamma) \frac{F(z)}{z} + (\alpha - \gamma)F'(z) + \gamma z F''(z). 
\]  
(2.40)

Since \(F(z) = z + 2(1 - \beta) \sum_{n=2}^{\infty} (\omega_n z^n/(n\mu + 1 - \mu)(n\nu + 1 - \nu))\),

so,

\[
H(z) = 1 + 2(1 - \beta) \sum_{n=2}^{\infty} \omega_n z^{n-1}. 
\]  
(2.41)

Therefore, for \(z = -1\),

\[
H(-1) = 1 - 2(1 - \beta) \sum_{n=2}^{\infty} \omega_n (-1)^n = 1 - 2(1 - \beta) \frac{1 - \delta}{2(1 - \beta)} = \delta. 
\]  
(2.42)

This shows that the result is sharp.

Letting \(\gamma = 0\) in Theorem 2.3 above, we obtain the following result of Kim and Rønning [9].
Corollary 2.4. Let $\delta < 1$ and $\alpha \geq 0$, and define $\beta = \beta(\delta)$ by

\[
\frac{\beta}{1 - \beta} = -\int_0^1 \lambda(t) \left( 1 - \frac{(1 + \delta)/(1 - \delta))t}{1 + t} \right) dt.
\] (2.43)

If $f \in \mathcal{W}_\rho(\alpha, 0) \equiv \mathcal{P}_\alpha(\beta)$, then $V_\lambda(f) \in \mathcal{W}_\rho(\alpha, 0) \equiv \mathcal{P}_\alpha(\beta)$. The value of $\beta$ is sharp.

Upon setting $\lambda(t) = (1 + c)t^\epsilon$ with $-1 < c$, we have the following corollary.

Corollary 2.5. Let $\delta < 1$, $\alpha, \gamma \geq 0$, and $-1 < c \leq 0$ be given, and let $G(z)$ be defined by

\[
G(z) = \frac{(1 + c)}{z^\epsilon} \int_0^z u^{c-1} f(u) du.
\] (2.44)

Suppose that $f \in \mathcal{W}_\rho(\alpha, \gamma)$, then $G \in \mathcal{W}_\rho(\alpha, 0)$, where

\[
\beta = \frac{2(1 + c) \, 2F_1(1,2 + c;3 + c,-1) - (2 + c)}{2(1 + c) \, 2F_1(1,2 + c;3 + c,-1)}.
\] (2.45)

The constant $\beta$ is sharp.

The special case of Corollary 2.5 (with $\gamma = 0$) has been obtained by Aghalary et al. [11].

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