Maximum Likelihood Estimation and Bayesian Estimation of three-parameter Weibull Distribution Based on Interval-Censored Data

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Abstract. The interval-censored data is that represents adjacent inspection times that surround an unknown failure time. This paper gives a review of the classical approach of the maximum likelihood estimating method to parameters of three-parameters Weibull distribution with interval-censored data. It also considers the Bayes’ estimators under asymmetric three loss functions squared error loss (SEL), linear-exponential (LINDEX), and generalized entropy loss (GEL) functions. For the unknown parameters of three-parameters Weibull distributions with interval-censored data. We use Lindley’s approximation to compute the Bayes estimates. Then we will apply a Monte Carlo simulation study is carried out to compare the performances of the methods using the R programming language to compute and compare the performance of the proposed estimators. A real data application is also presented. The study observed that the Bayesian estimator under generalized entropy loss (GEL) functions is preferred over the classical maximum likelihood estimator for all parameters of scale, shape, and location.

1. Introduction

The three-parameters Weibull distribution is a very important life testing model. The three-parameters Weibull distribution provides the generalization of many other life testing distributions like as the two-parameter exponential distribution (when the value of shape parameter is one), the Weibull two-parameter distribution (when the value of location/shift parameter is zero) it turns into one-parameter exponential distribution (when the shape parameter is one and the location parameter is zero) and it turns into a Rayleigh distribution (when the shape parameter is two). The Weibull distribution with non-zero location parameter has three parameters, denoted by: \( \alpha > 0 \), the scale parameter; \( \beta > 0 \), the shape parameter responsible for the skew of the distribution; finally \( \gamma \), the location parameter which is a lower bound. The location parameter also called” the threshold of the parameter represents the time below which no failure occurs or the minimal survival time of all the items in a specific population”. (Hiros and Lai 1993) the technical report states that the inclusion of the threshold/location parameter introduces many additional difficulties and make a guess at on the parameter or the very useful result and therefore better to battle with the difficulties associated with its estimation for better interpretation of the properties of a particular dataset. Interval censoring has to do with a test or study subject of interest that is not under ordinary observation. Therefore, generally conceivable to notice the failure or survival time of the subject. With interval censoring, one only knows a range, that is, an interval, inside of which one can say the survival event has occurred. Left- or right-censored failure times are special
cases of interval-censored failure times. (Turnbull 1976), an interval-censored observation could be defined as a union of several non-overlapping windows or intervals. The many areas where the interval-censored failure time data occur, according to (Sun 2007), include demographical, epidemiological, financial, medical, sociological, and engineering studies. Medical or health studies that entail periodic follow-ups are where a typical example of interval-censored data occurs and there are Many clinical studies and longitudinal studies that fall into this category. In such situations, interval-censored data can appear in several ways. For example, a person may miss one or more scheduled observation times to clinically observe possible changes in the disease state, and then return with an altered state determined by (Sun 2007). The essence of this study is the use of the Bayesian estimation approach concerning the three-parameter Weibull distribution, and so far none in the literature, to our knowledge, has data censored by interval using this approach. The ability to provide reasonably accurate analysis and prediction with extremely small samples is one of the main advantages of Weibull three-parameter analysis. (Abernethy 2004). Cost-effectiveness is allowed by small samples. The technical report states that the inclusion of the threshold parameter introduces many additional difficulties and makes valid inferences on the parameter or the functions thereof particularly difficult (Hirose and Lai 1993).

The paper is coordinated as follows: In Section Two, we introduce the model and discussed survival in the character of distribution. And in Section three, we had obtained the maximum likelihood estimates (MLE) of unknown parameters and survival characteristics. Section four, derived these parameters by sing Bayes estimators and also for survival characteristic. Also, we have obtained Bayes estimators by using the Lindley approximation approach. In Section Five, the Monte Carlo simulation study and numerical contemplation are performed between the methods of estimates in terms of their mean square error and bias values, and one data set is analyzed. In Section six, the purpose of clarification of the result.

2. Model

The probability density function (PDF) and cumulative distribution function of a random variable with three-parameters Weibull distribution are given by

\[ f(t; \alpha, \beta, \gamma) = \frac{\beta}{\alpha} \left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} \exp\left[-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}\right], \quad t \geq 0, \ t > \gamma \]

\[ F(t; \alpha, \beta, \gamma) = 1 - \exp\left[-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}\right] \]

Where, \( \alpha \) is the scale parameter, \( \beta \) is the shape parameter and \( \gamma \) is the location parameter.

The survival function of the three-parameters Weibull distribution is of the form

\[ S(t) = \exp\left[-\left(\frac{t-\gamma}{\alpha}\right)^{\beta}\right] \]

while its hazard function is given by

\[ h(t) = \frac{\beta}{\alpha} \left(\frac{t-\gamma}{\alpha}\right)^{\beta-1} \]

properties of the 3-parameter Weibull distribution are below:

| Mean (\( \mu \)) | \( \gamma + \alpha \frac{1}{\beta} \left(1 + \frac{1}{\beta}\right) \) |
|------------------|--------------------------------------------------|
| Median           | \( \gamma + \alpha \ln 2 \frac{1}{\beta} \)              |
| Mode             | \( \gamma + \left(1 - \frac{1}{\beta}\right)^{\frac{1}{\beta}} \) if \( \beta > 1 \) |
| Variance ($\sigma^2$) |
|----------------------|
| $\alpha^2 \Gamma\left(1 + \frac{2}{\beta}\right) - \mu^2$ |

In the next section we derive the maximum likelihood estimator for the parameters $\alpha, \beta$ and $\gamma$

### 3. The maximum likelihood estimator

Let $t_1, \ldots, t_n$ be the lifetimes from a random sample of size $n$ be a random sample from three-parameters Weibull distribution. Let us assume that in that sample, we have $n_1$ complete (uncensored) observations and $n_2$ censored observations, where $n = n_1 + n_2$.

Define $T_i$; $i = 1 \ldots n_1$ as complete observations and $[L_i; R_i]$; $i = 1 \ldots n_2$ as interval censored observations as stated by Gómez et al. [17], which is given by

$$L(\alpha, \beta, \gamma) = \prod_{i=1}^{n_1} 1 \left(\frac{t_i - \gamma}{\alpha}\right)^{\beta - 1} \exp\left[-\left(\frac{t_i - \gamma}{\alpha}\right)^{\beta}\right] \prod_{i=1}^{n_1 + n_2} \left[\exp\left[-\left(\frac{r_i - \gamma}{\alpha}\right)^{\beta}\right] - \exp\left[-\left(\frac{l_i - \gamma}{\alpha}\right)^{\beta}\right]\right]$$

(6)

Where $t_i$, $l_i$, $r_i$ are observed values of $T_i$, $L_i$, $R_i$ respectively.

The likelihood function for three-parameters Weibull distribution with interval censored data become:

$$L(\alpha, \beta, \gamma) = \prod_{i=1}^{n_1} \left[\frac{\beta}{\alpha} \left(\frac{t_i - \gamma}{\alpha}\right)^{\beta - 1} \exp\left[-\left(\frac{t_i - \gamma}{\alpha}\right)^{\beta}\right]\right] \prod_{i=1}^{n_1 + n_2} \left[\exp\left[-\left(\frac{r_i - \gamma}{\alpha}\right)^{\beta}\right] - \exp\left[-\left(\frac{l_i - \gamma}{\alpha}\right)^{\beta}\right]\right]$$

(6)

Where $S(\theta)$ denote a survival function.

Then taking the natural log of above equation (6) we get:

$$l_\alpha(\alpha, \beta, \gamma) = n_1 \ln(\beta) - n_1 \beta \ln(\alpha) \sum_{i=1}^{n_1} \left[(\beta - 1) \ln(t_i - \gamma) - \left(\frac{t_i - \gamma}{\alpha}\right)^{\beta}\right] + \sum_{i=n_1+1}^{n_1+n_2} \ln(CS_i)$$

(7)

When we compute derivative of equation (7) for three parameters $(\alpha, \beta, \gamma)$, we will at first compute derivative of censored parts for $(\alpha, \beta, \gamma)$ then we get:

the first partial derivative $CS_i$ with respect $\alpha$ is:

$$CS_{i\alpha} = \frac{\partial CS_i}{\partial \alpha} = \left(\frac{l_i - \gamma}{\alpha}\right)^{\beta} \beta \exp\left[-\left(\frac{l_i - \gamma}{\alpha}\right)^{\beta}\right] - \left(\frac{r_i - \gamma}{\alpha}\right)^{\beta} \beta \exp\left[-\left(\frac{r_i - \gamma}{\alpha}\right)^{\beta}\right]$$

(8)

the first partial derivative $CS_i$ with respect $\beta$ is:

$$CS_{i\beta} = \frac{\partial CS_i}{\partial \beta} = \left(\frac{l_i - \gamma}{\alpha}\right)^{\beta} \ln\left(\frac{l_i - \gamma}{\alpha}\right) \exp\left[-\left(\frac{l_i - \gamma}{\alpha}\right)^{\beta}\right] + \left(\frac{r_i - \gamma}{\alpha}\right)^{\beta} \ln\left(\frac{r_i - \gamma}{\alpha}\right) \exp\left[-\left(\frac{r_i - \gamma}{\alpha}\right)^{\beta}\right]$$

(9)

the first partial derivative $CS_i$ with respect $\gamma$ is:

$$CS_{i\gamma} = \frac{\partial CS_i}{\partial \gamma} = \left(\frac{l_i - \gamma}{\alpha}\right)^{\beta} \beta \exp\left[-\left(\frac{l_i - \gamma}{\alpha}\right)^{\beta}\right] - \left(\frac{r_i - \gamma}{\alpha}\right)^{\beta} \beta \exp\left[-\left(\frac{r_i - \gamma}{\alpha}\right)^{\beta}\right]$$

(10)
Now, take the first derivative (7) with respect to \( \alpha, \beta \) and \( \gamma \) and equating each equation to zero, and from (8), (9) and (10) then we get three nonlinear respectively as follows:

Hence the score functions of \( \alpha, \gamma \) is given by:

\[
\frac{\partial l_n(\alpha,\beta,\gamma)}{\partial \alpha} = -\frac{n_1 \beta}{\alpha} + \sum_{i=1}^{n_1} \left[ \frac{\beta (t_i - \gamma)^\beta}{\alpha} \right] + \sum_{i=n_1+1}^{n_1+n_2} \frac{CS_{i\alpha}}{CS_i} = 0
\]

(11)

\[
\frac{\partial l_n(\alpha,\beta,\gamma)}{\partial \beta} = -n_1 \ln(\alpha) + \frac{n_1 \beta}{\alpha} + \sum_{i=1}^{n_1} \left[ \ln(\alpha) \left( \frac{t_i - \gamma}{\alpha} \right)^\beta - \ln(t_i - \gamma) \left( \frac{t_i - \gamma}{\alpha} \right)^\beta + \ln(t_i - \gamma) \right] + \sum_{i=n_1+1}^{n_1+n_2} \frac{CS_{i\beta}}{CS_i} = 0
\]

(12)

\[
\frac{\partial l_n(\alpha,\beta,\gamma)}{\partial \gamma} = \sum_{i=1}^{n_1} \left[ \frac{\beta (t_i - \gamma)^\beta - \beta + 1}{t_i - \gamma} \right] + \sum_{i=n_1+1}^{n_1+n_2} \frac{CS_{i\gamma}}{CS_i} = 0
\]

(13)

To find the maximum likelihood estimations for the parameters \((\alpha, \beta, \gamma)\), we must solve the system of three nonlinear equations (11), (12) and (13) by using numerical approximation method. Such as Newton – Raphson method to obtain the solution.

The steps of this method are as follows:

\[
\begin{bmatrix}
\alpha_{K+1} \\
\beta_{K+1} \\
\gamma_{K+1}
\end{bmatrix}
= \begin{bmatrix}
\alpha_K \\
\beta_K \\
\gamma_K
\end{bmatrix} - J_{K_{-1}}^{-1}
\begin{bmatrix}
f_1(\alpha) \\
f_2(\beta) \\
f_3(\gamma)
\end{bmatrix}
\]

\]

(14)

Where, \( J_{K_{-1}} \) is Jacobian Matrix.

\[
f_1(\alpha) = -\frac{n_1 \beta}{\alpha} + \sum_{i=1}^{n_1} \left[ \frac{\beta (t_i - \gamma)^\beta}{\alpha} \right] + \sum_{i=n_1+1}^{n_1+n_2} \frac{CS_{i\alpha}}{CS_i}
\]

(15)

\[
f_2(\beta) = -n_1 \ln(\alpha) + \frac{n_1 \beta}{\alpha} + \sum_{i=1}^{n_1} \left[ \ln(\alpha) \left( \frac{t_i - \gamma}{\alpha} \right)^\beta - \ln(t_i - \gamma) \left( \frac{t_i - \gamma}{\alpha} \right)^\beta + \ln(t_i - \gamma) \right] + \sum_{i=n_1+1}^{n_1+n_2} \frac{CS_{i\beta}}{CS_i}
\]

(16)

\[
f_3(\gamma) = \sum_{i=1}^{n_1} \left[ \frac{\beta (t_i - \gamma)^\beta - \beta + 1}{t_i - \gamma} \right] + \sum_{i=n_1+1}^{n_1+n_2} \frac{CS_{i\gamma}}{CS_i}
\]

(17)

Thus, is Jacobian matrix which is defined as follows:

\[
J_{K_{-1}}^{-1} = \begin{bmatrix}
\frac{\partial f_1(\alpha)}{\partial \alpha} & \frac{\partial f_1(\beta)}{\partial \beta} & \frac{\partial f_1(\gamma)}{\partial \gamma} \\
\frac{\partial f_2(\alpha)}{\partial \alpha} & \frac{\partial f_2(\beta)}{\partial \beta} & \frac{\partial f_2(\gamma)}{\partial \gamma} \\
\frac{\partial f_3(\alpha)}{\partial \alpha} & \frac{\partial f_3(\beta)}{\partial \beta} & \frac{\partial f_3(\gamma)}{\partial \gamma}
\end{bmatrix}^{-1}
\]

(18)

Where:

\[
\frac{\partial f_1(\alpha)}{\partial \alpha} = -\frac{n_1 \beta}{\alpha^2} + \sum_{i=1}^{n_1} \left[ -\beta^2 \left( \frac{t_i - \gamma}{\alpha} \right)^\beta \right] + \sum_{i=n_1+1}^{n_1+n_2} \frac{CS_{i,\alpha,\alpha} - (CS_{i,\alpha})^2}{(CS_i)^2}
\]

(19)

\[
\frac{\partial f_1(\beta)}{\partial \beta} = \frac{\partial f_1(\gamma)}{\partial \gamma} = -\frac{n_1 \alpha}{\beta} + \sum_{i=1}^{n_1} \left[ \frac{1}{\alpha} \left( \frac{t_i - \gamma}{\alpha} \right)^\beta \right] \left( -\ln(\alpha) \beta + 1 + \beta \ln(t_i - \gamma) \right) + \sum_{i=n_1+1}^{n_1+n_2} \frac{CS_{i,\beta,\alpha,\alpha} - CS_{i,\alpha,\beta}}{(CS_i)^2}
\]

(20)
The absolute value for the difference of parameters between the new founded values with the initial value, is the error term, it must be a symbol by $\epsilon$, where which is a very small value and assumed.

Then, error term is formulated as:

$$
\begin{bmatrix}
\epsilon_{k+1}(\alpha) \\
\epsilon_{k+1}(\beta) \\
\epsilon_{k+1}(\gamma)
\end{bmatrix} =
\begin{bmatrix}
\alpha_{k+1} \\
\beta_{k+1} \\
\gamma_{k+1}
\end{bmatrix} -
\begin{bmatrix}
\alpha_k \\
\beta_k \\
\gamma_k
\end{bmatrix}
$$

(25)

So, the maximum likelihood estimators for the parameters $\alpha$, $\beta$ and $\gamma$ are $\hat{\alpha}_{ML}$, $\hat{\beta}_{ML}$ and $\hat{\gamma}_{ML}$ respectively.

Hence the standard deviation of the MLEs are:

$$
\text{sd}(\hat{\alpha}_{ML}) = \sqrt{\frac{\partial f_\alpha(\alpha)}{\partial \alpha}}, \text{sd}(\hat{\beta}_{ML}) = \sqrt{\frac{\partial f_\beta(\beta)}{\partial \beta}} \text{ and } \text{sd}(\hat{\gamma}_{ML}) = \sqrt{\frac{\partial f_\gamma(\gamma)}{\partial \gamma}},
$$

the 95% asymptotic intervals of $\alpha$, $\beta$ and $\gamma$ are:

$$
[\hat{\alpha}_{ML} - 1.96 \text{sd}(\hat{\alpha}_{ML}), \hat{\alpha}_{ML} + 1.96 \text{sd}(\hat{\alpha}_{ML})]
$$

$$
[\hat{\beta}_{ML} - 1.96 \text{sd}(\hat{\beta}_{ML}), \hat{\beta}_{ML} + 1.96 \text{sd}(\hat{\beta}_{ML})]
$$

$$
[\hat{\gamma}_{ML} - 1.96 \text{sd}(\hat{\gamma}_{ML}), \hat{\gamma}_{ML} + 1.96 \text{sd}(\hat{\gamma}_{ML})]
$$

4. Bayesian Inference

The Bayes' rule is employed by the Bayesian inference approach to update the probability estimate of hypothesis taking into account new evidence as it becomes available. One of the essential techniques used in modern statistics is Bayesian updating, especially important in mathematical statistics. Bayesian updating is particularly important in analyzing data that is progressive. There are many other fields where the Bayesian inference can be applied like engineering, medicine, and accounting. Our prior beliefs, known as the prior distribution, are used in the Bayesian approach. The above is a distribution of the parameters before any data is observed and is reported as $p(\theta)$. It also takes into account the observed data, obtained using the probability function and reported as $p(\theta)$. 

In Bayesian estimations, the Bayes estimator is considered under important three loss functions. They are asymmetric symmetric (squared error) loss function and (LINEX and general entropy) loss functions. The unknown parameters, which need to be assumed for the Bayesian inference, can be given the following prior distribution:

Let the parameters take on a Gamma($\alpha$, $b$), Gamma($p$, $q$) and U(0, $h$) for $\alpha$, $\beta$ and $g$ respectively. We assume that the priors are independent.

\[ g_1(\alpha) = \frac{b^\alpha \alpha^{-1} \exp(-b\alpha)}{\Gamma(\alpha)}, \alpha > 0, a > 0, b > 0 \]

\[ g_2(\beta) = \frac{q^p \beta^{-p} \exp(-q\beta)}{\Gamma(p)}, \beta > 0, p > 0, q > 0 \]

\[ g_3(\gamma) = \frac{1}{h}, h > 0 \]

Thus the joint prior distribution for $\alpha$, $\beta$ and $\gamma$ is

\[ g(\alpha, \beta, \gamma) = \frac{b^\alpha \alpha^{-1} q^p \beta^{-p} \exp(-b\alpha - q\beta)}{h \Gamma(\alpha) \Gamma(p)}, \text{ all } \alpha > 0, a > 0, b > 0, \beta > 0, p > 0, q > 0, h > 0 \quad (26) \]

The posterior distribution is what the Bayesian inference based on, We compute Bayesian estimation of the three parameters by Substituting likelihood $L(\alpha, \beta, g)$ and the joint prior distribution for $\alpha, \beta$ and $g$ $g(\alpha, \beta, g)$ from (6) and (26) respectively we get the correspond joint posterior $P(\alpha, \beta, g)$ which is simply the ratio of the joint density function of prior distribution to the marginal distribution function.

\[ P(\alpha, \beta, \gamma) = \frac{g(\alpha, \beta, \gamma) L(\alpha, \beta, \gamma)}{\int \int \int g(\alpha, \beta, \gamma) \times L(\alpha, \beta, \gamma) d(\alpha, \beta, \gamma)} \quad (27) \]

1. The definition of the squared error loss function is: (Varian 1975)

\[ L(\theta, \tilde{\theta}) = (\tilde{\theta} - \theta)^2 \]

Where $\tilde{\theta}$ is estimate of $\theta$.

2. the LINEX loss function (LINEX) for can be expressed as: (Varian 1975)

\[ L(\theta, \tilde{\theta}) = e^c(\tilde{\theta} - \theta) - c (\tilde{\theta} - \theta) - 1, c \neq 1 \]

with $\tilde{\theta}$ being the estimate of $\theta$. According to Zellner, the posterior expectation of the LINEX loss function is

\[ E_{\tilde{\theta}}[L(\theta, \tilde{\theta})] = e^c \tilde{\theta} E_{\tilde{\theta}}[e^{-c\theta}] - E_{\tilde{\theta}}[c (\tilde{\theta} - \theta) - 1] \]

the value of $\tilde{\theta}$ that minimizes the above equation is

\[ \tilde{\theta} = -\frac{1}{c} \ln[E_{\tilde{\theta}}(e^{-c\theta})] \]

provided $E_{\tilde{\theta}}(\cdot)$ exists and is finite.

3. the General Entropy loss (GEL) can be expressed as:

\[ L(\theta, \tilde{\theta}) = \left( \frac{\theta}{\tilde{\theta}} \right)^c - c \ln \left( \frac{\theta}{\tilde{\theta}} \right) - 1, c \neq 0 \]

It is called The modified Linex loss.

It may be noted that (27) contains the triple integral which has no analytical solution due to the complex form of the likelihood function given in (6). To compute the integrals in (27) we propose to use Lindley approximation method.
by Lye et al. the posterior Bayes estimator of an arbitrary function of parameters given by Lindley is:

\[
E\{u(\hat{\alpha}, \hat{\beta}, \hat{\gamma})|x\} = \frac{\int \int u(\alpha, \beta, \gamma)g(\alpha, \beta, \gamma) \times \pi_p(\alpha, \beta, \gamma) \times g(\alpha, \beta, \gamma) d\alpha d\beta d\gamma}{\int \int g(\alpha, \beta, \gamma) \times \pi_p'(\alpha, \beta, \gamma) \times g(\alpha, \beta, \gamma) d\alpha d\beta d\gamma}
\]

(28)

where \(l_n\) is the log likelihood and \(g(\alpha, \beta, \gamma)\) are arbitrary functions of \((\alpha, \beta, \gamma)\).

Now we need compute logarithm of the joint prior distribution for \(\alpha, \beta\) and \(g\) say \(p(\alpha, \beta, g)\) then we get :

\[
\rho(\alpha, \beta, g) = \log(g(\alpha, \beta, g)) = a \ln b + (a - 1) \ln \alpha + a \ln q + (p - 1) \ln \beta - b \alpha - q \beta - \ln h - \ln \Gamma(a) - \ln \Gamma(p)
\]

(29)

the partial derivatives of (29) with respect \(\alpha, \beta\) and \(g\) are given by

\[
\rho_\alpha = \frac{\partial \rho(\alpha, \beta, g)}{\partial \alpha} = \frac{a - 1}{a} - b
\]

(30)

\[
\rho_\beta = \frac{\partial \rho(\alpha, \beta, g)}{\partial \beta} = \frac{p - 1}{p} - q
\]

(31)

\[
\rho_g = \frac{\partial \rho(\alpha, \beta, g)}{\partial g} = 0
\]

(32)

The integral in (28) can be

\[
+u_1 a_1 + u_2 a_2 + u_3 + \frac{1}{2} [A(u_1 \sigma_{uu} + u_2 \sigma_{ug} + u_3 \sigma_{ug}) + B(u_2 \sigma_{uu} + u_2 \sigma_{ug} + u_3 \sigma_{ug}) + C(u_3 \sigma_{uu} + u_3 \sigma_{ug} + u_3 \sigma_{ug})] + 4
\]

where

\[
\hat{\alpha}, \hat{\beta} \text{ and } \hat{\gamma} \text{ are the MLE } \alpha, \beta \text{ and } g \text{ respectively.}
\]
Lindly’s approximation under GEL function is given: 

\[ \alpha_{BS} = \alpha + \left( \frac{a - 1 - b}{\alpha} \right) \sigma_{\alpha} + \left( \frac{p - 1 - q \beta}{\beta} \right) \sigma_{\beta} + \frac{1}{2} \left( A \sigma_{\alpha+B} + B \sigma_{\beta+C} \sigma_{\alpha} \right) \]

II) If \( u(\alpha, \beta, \gamma) = \beta \) 

Then we get 

\[ \beta_{BS} = \beta + \left( \frac{a - 1 - b}{\alpha} \right) \sigma_{\beta} + \left( \frac{p - 1 - q \beta}{\beta} \right) \sigma_{\gamma} + \frac{1}{2} \left( A \sigma_{\alpha+B} + B \sigma_{\beta+C} \sigma_{\gamma} \right) \]

III) If \( u(\alpha, \beta, \gamma) = \gamma \) 

Then we get 

\[ \gamma_{BS} = \gamma + \frac{1}{2} \left( A \sigma_{\alpha+B} + B \sigma_{\beta+C} \sigma_{\gamma} \right) \]

Under LINEX Loss Function.

I) If \( u(\alpha, \beta, \gamma) = e^{-c\alpha} \) 

Then we get 

\[ u_{\alpha} = -c e^{-c\alpha}, u_{\alpha} = c^2 e^{-c\alpha} \text{ and } u_{\beta} = u_{\beta} = u_{\gamma} = u_{\beta} = u_{\beta} = u_{\gamma} = 0 \]

Hence, the Bayesian estimator by using Lindly’s approximation under LINEX function is given: 

\[ \alpha_{BL} = - \frac{1}{c} \log \left[ e^{-c\alpha} + \left( \frac{a - 1 - b}{\alpha} \right) \sigma_{\alpha} + \left( \frac{p - 1 - q \beta}{\beta} \right) \sigma_{\beta} + \frac{1}{2} c^2 e^{-c\alpha} \sigma_{\alpha} - \frac{c}{2} e^{-c\alpha} \left( A \sigma_{\alpha+B} + B \sigma_{\beta+C} \sigma_{\alpha} \right) \right] \]

II) If \( u(\alpha, \beta, \gamma) = e^{-c\beta} \) 

Then we get 

\[ u_{\beta} = -c e^{-c\beta}, u_{\gamma} = c^2 e^{-c\beta} \text{ and } u_{\alpha} = u_{\beta} = u_{\alpha} = u_{\beta} = u_{\beta} = u_{\alpha} = u_{\gamma} = 0 \]

Hence, the Bayesian estimator by using Lindly’s approximation under LINEX function is given: 

\[ \beta_{BL} = - \frac{1}{c} \log \left[ e^{-c\beta} + \left( \frac{a - 1 - b}{\alpha} \right) \sigma_{\beta} + \left( \frac{p - 1 - q \beta}{\beta} \right) \sigma_{\gamma} + \frac{1}{2} c^2 e^{-c\beta} \sigma_{\beta} - \frac{c}{2} e^{-c\beta} \left( A \sigma_{\alpha+B} + B \sigma_{\beta+C} \sigma_{\beta} \right) \right] \]

III) If \( u(\alpha, \beta, \gamma) = e^{-c\gamma} \) 

Then we get 

\[ u_{\gamma} = -c e^{-c\gamma}, u_{\gamma} = c^2 e^{-c\gamma} \text{ and } u_{\alpha} = u_{\beta} = u_{\gamma} = u_{\gamma} = u_{\beta} = u_{\beta} = u_{\alpha} = 0 \]

Hence, the Bayesian estimator by using Lindly’s approximation under LINEX function is given: 

\[ \gamma_{BL} = - \frac{1}{c} \log \left[ e^{-c\gamma} + \frac{1}{2} c^2 e^{-c\gamma} \sigma_{\gamma} - \frac{c}{2} e^{-c\gamma} \left( A \sigma_{\alpha+B} + B \sigma_{\beta+C} \sigma_{\gamma} \right) \right] \]

Under GEL Loss Function.

I) If \( u(\alpha, \beta, \gamma) = \alpha^{c} \) 

Then we get 

\[ u_{\alpha} = c \alpha^{-c+1}, u_{\alpha} = (c^2 + c) \alpha^{-c+2} \text{ and } u_{\beta} = u_{\beta} = u_{\alpha} = u_{\beta} = u_{\beta} = u_{\gamma} = 0 \]

Hence, the Bayesian estimator by using Lindly’s approximation under GEL function is given: 

\[ \alpha_{BL} = \left[ \alpha^{c} + \left( \frac{a - 1 - b}{\alpha} \right) \sigma_{\alpha} + \left( \frac{p - 1 - q \beta}{\beta} \right) \sigma_{\alpha} + \frac{1}{2} \left( c^2 + c \right) \sigma_{\alpha} + \frac{1}{2} \left( A \sigma_{\alpha+B} + B \sigma_{\beta+C} \sigma_{\gamma} \right) \right] \]
II) If \( u(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) = \hat{\beta}^e \)
Then we get
\[
\begin{align*}
\hat{\beta} = -c + \frac{\alpha - 1 - b \hat{\alpha}}{\alpha} \sigma_{\beta \alpha} + \frac{c^2 + c - 2}{2} \hat{\beta}^2 \sigma_{\beta \beta} + \frac{1}{2} (A \sigma_{\alpha \beta} + B \sigma_{\beta \theta} + C \sigma_{\theta \theta})
\end{align*}
\]
Hence, the Bayesian estimator by using Lindley’s approximation under GEL function is given:
\[
\beta_{\text{BE}} = \left[ \frac{\hat{\beta}^e + \frac{\alpha - 1 - b \hat{\alpha}}{\alpha} \sigma_{\beta \alpha} + \frac{c^2 + c - 2}{2} \hat{\beta}^2 \sigma_{\beta \beta} + \frac{1}{2} (A \sigma_{\alpha \beta} + B \sigma_{\beta \theta} + C \sigma_{\theta \theta})}{1 - c} \right]
\]

III) If \( u(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) = \hat{\gamma}^e \)
Then we get
\[
\begin{align*}
\hat{\gamma} = -c + \frac{\gamma^2 + c}{2} \gamma \hat{\gamma}^2 \sigma_{\gamma \gamma} + \frac{1}{2} (A \sigma_{\alpha \gamma} + B \sigma_{\beta \gamma} + C \sigma_{\gamma \gamma})
\end{align*}
\]
Hence, the Bayesian estimator by using Lindley’s approximation under GEL function is given:
\[
\gamma_{\text{BE}} = \left[ \frac{\hat{\gamma}^e + \frac{\gamma^2 + c}{2} \gamma \hat{\gamma}^2 \sigma_{\gamma \gamma} + \frac{1}{2} (A \sigma_{\alpha \gamma} + B \sigma_{\beta \gamma} + C \sigma_{\gamma \gamma})}{1 - c} \right]
\]

5. Monte Carlo simulation study
We carried out a simulation study to determine the best estimator for the three-parameters Weibull distribution with interval censoring.

The steps of generate values of X which are distributed according to this distribution. By using the inverse transform sampling method works as follows:

Step1:- Generate a random number U from the standard uniform distribution in the interval [0,1], e.g. from U~Unif[0,1].

Step2:- Find the inverse of the desired CDF, e.g. \( F^{-1}(x) \).

Step3:- Compute \( X = F^{-1}(U) \). The computed random variable X has distribution \( f(x) \).

The procedure for generating

Step4:- The random censoring intervals is generate as follows. Let L and Q be independent exponential random variables have means \((1/\theta_1)\) and \((1/\theta_2)\), respectively. Then compute R as \( R = L + Q \). For exact observations, we generate n1 from the distribution while for censored observations, n2. For each observation, if \( T_i < L_i \) then we have left-censored \( L_i \), if \( L_i < T_i < R_i \) then we have interval-censored \([L_i;R_i]\) and otherwise we have right-censored \( R_i \).

First, explain how to generate data from three-parameters Weibull distribution using Bayessian method. three loss functions are considered; The first is the squared error loss function (SEL). The second is the linear-exponential loss function (LINEX) and the third is General Entropy loss (GEL).

A Monte Carlo simulation study is conducted to compare the performance of the estimators of the three-parameter Weibull model parameters discussed in the previous section before making application to the internally displaced person’s return time. Progressively censored samples from three-parameter Weibull distribution with \( \alpha = 1, \beta = 1.2 \) and \( \gamma = 0.3 \) were generated using the algorithm described in R program command. The sample used was sizes \( n= 200, 300, 400 \) and \( 600 \) and \( c=0.5 \); with 1000 number of iteration.
Table 1: Maximum likelihood Estimation (MLE) method with interval censoring. \( \alpha = 1, \beta = 1.2, \gamma = 0.3 \)

| \( \theta_1 \) | \( \theta_2 \) | N | (\( \alpha, \beta, \gamma \)) | Bias | SSD | ESD | Total Deviation TD | Ratio of Censored data | C.I | CP% |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.75 | 0.3 | 200 | 1, 2, 0.3 | -0.1243 | 0.0658 | 0.0560 | 0.199633 | 15% | 0.7658, 0.9855 | 90 |
| 0.75 | 0.3 | 300 | 1, 2, 0.3 | -0.1272 | 0.0514 | 0.0511 | 0.196867 | 17% | 0.7727, 0.9728 | 91 |
| 0.75 | 0.3 | 400 | 1, 2, 0.3 | -0.1243 | 0.0394 | 0.0372 | 0.191050 | 20% | 0.8028, 0.9486 | 94 |
| 0.75 | 0.3 | 600 | 1, 2, 0.3 | -0.1281 | 0.0365 | 0.0301 | 0.178600 | 24% | 0.8129, 0.9309 | 90 |
| 1 | 0.5 | 200 | 1, 2, 0.3 | -0.1489 | 0.0646 | 0.0636 | 0.212733 | 22% | 0.7265, 0.9758 | 96 |
| 1 | 0.5 | 300 | 1, 2, 0.3 | -0.1517 | 0.0504 | 0.0543 | 0.208367 | 26% | 0.7418, 0.9548 | 93 |
| 1 | 0.5 | 400 | 1, 2, 0.3 | -0.1517 | 0.0430 | 0.0362 | 0.208283 | 28% | 0.7772, 0.9193 | 90 |
| 1 | 0.5 | 600 | 1, 2, 0.3 | -0.1548 | 0.0347 | 0.0282 | 0.203467 | 25% | 0.7899, 0.9005 | 92 |
| 1.25 | 0.3 | 200 | 1, 2, 0.3 | -0.2358 | 0.0569 | 0.0728 | 0.278633 | 38% | 0.6216, 0.9069 | 99 |
| 1.25 | 0.3 | 300 | 1, 2, 0.3 | -0.2403 | 0.0474 | 0.0903 | 0.278633 | 38% | 0.6172, 0.9023 | 99 |
| 1.25 | 0.3 | 400 | 1, 2, 0.3 | -0.2426 | 0.0400 | 0.0344 | 0.283683 | 46% | 0.6899, 0.8249 | 91 |
| 1.25 | 0.3 | 600 | 1, 2, 0.3 | -0.2435 | 0.0321 | 0.0261 | 0.281000 | 43% | 0.7053, 0.8077 | 90 |

**Figure:**
- **MLE of Alpha (\( \alpha \))**
- **MLE of Beta (\( \beta \))**
- **MLE of Gamma (\( \gamma \))**

The graphs show the frequency distribution for each parameter with different values for \( \theta_1 \) and \( \theta_2 \).
Table 2 Bayes estimators squared error loss function (SEL) with interval-censoring

| $\theta_1$ | $\theta_2$ | N | $(\alpha, \beta, \gamma)$ | Bias | SSD | ESD | Total Deviation TD | Ratio of Censored data | C.I | CP% |
|------------|------------|---|-----------------|------|-----|-----|-------------------|----------------------|-----|-----|
| 0.75       | 0.3        | 200| 1               | -0.1006 | 0.0585 | 0.0554 | 0.199633 | 15% | (0.7843, 1.0144) | 89 |
|            |            |    | 1.2             | 0.0758  | 0.0698 | 0.0693 |             |                      |     |     |
|            |            |    | 0.3             | -0.0085 | 0.0333 | 0.0167 |             |                      |     |     |
| 0.75       | 0.3        | 300| 1               | -0.1016 | 0.0439 | 0.0447 | 0.196867 | 17% | (0.8090, 0.9877) | 96 |
|            |            |    | 1.2             | 0.0680  | 0.0559 | 0.0555 |             |                      |     |     |
|            |            |    | 0.3             | -0.0060 | 0.0021 | 0.0118 |             |                      |     |     |
| 0.75       | 0.3        | 400| 1               | -0.1196 | 0.0395 | 0.0365 | 0.192100 | 20% | (0.8109, 0.9500) | 92 |
|            |            |    | 1.2             | 0.1011  | 0.0455 | 0.0484 |             |                      |     |     |
|            |            |    | 0.3             | -0.0061 | 0.0016 | 0.0092 |             |                      |     |     |
| 0.75       | 0.3        | 600| 1               | -0.1222 | 0.0362 | 0.0302 | 0.178267 | 24% | (0.8213, 0.9344) | 92 |
|            |            |    | 1.2             | 0.0662  | 0.0413 | 0.0382 |             |                      |     |     |
|            |            |    | 0.3             | -0.0041 | 0.0011 | 0.0064 |             |                      |     |     |
| 1          | 0.5        | 200| 1               | -0.1322 | 0.0641 | 0.0538 | 0.224183 | 22% | (0.7505, 0.9851) | 93 |
|            |            |    | 1.2             | 0.0906  | 0.0688 | 0.0711 |             |                      |     |     |
|            |            |    | 0.3             | -0.0072 | 0.0039 | 0.0164 |             |                      |     |     |
| 1          | 0.5        | 300| 1               | -0.1384 | 0.0535 | 0.0417 | 0.191033 | 26% | (0.7823, 0.9409) | 93 |
|            |            |    | 1.2             | 0.1093  | 0.0517 | 0.0564 |             |                      |     |     |
|            |            |    | 0.3             | -0.0070 | 0.0016 | 0.0113 |             |                      |     |     |
| 1          | 0.5        | 400| 1               | -0.1365 | 0.0403 | 0.0368 | 0.231700 | 28% | (0.7911, 0.9359) | 94 |
|            |            |    | 1.2             | 0.0769  | 0.0456 | 0.0481 |             |                      |     |     |
|            |            |    | 0.3             | -0.0038 | 0.0020 | 0.0089 |             |                      |     |     |
| 1          | 0.5        | 600| 1               | -0.1474 | 0.0337 | 0.0294 | 0.252817 | 25% | (0.7951, 0.9101) | 93 |
|            |            |    | 1.2             | 0.0337  | 0.0386 | 0.0384 |             |                      |     |     |
|            |            |    | 0.3             | -0.0028 | 0.0012 | 0.0061 |             |                      |     |     |
| 1          | 1.25       | 200| 1               | -0.1567 | 0.0499 | 0.0557 | 0.213250 | 38% | (0.7401, 0.9464) | 93 |
|            |            |    | 1.2             | 0.0301  | 0.0491 | 0.0665 |             |                      |     |     |
|            |            |    | 0.3             | -0.0050 | 0.0027 | 0.0154 |             |                      |     |     |
| 1          | 1.25       | 300| 1               | -0.2045 | 0.0436 | 0.0412 | 0.184817 | 39% | (0.7125, 0.8785) | 97 |
|            |            |    | 1.2             | 0.0454  | 0.0518 | 0.0528 |             |                      |     |     |
|            |            |    | 0.3             | -0.0035 | 0.0020 | 0.0098 |             |                      |     |     |
| 1          | 1.25       | 400| 1               | -0.2051 | 0.0352 | 0.0355 | 0.198450 | 46% | (0.7259, 0.8639) | 97 |
|            |            |    | 1.2             | 0.0305  | 0.0423 | 0.0444 |             |                      |     |     |
|            |            |    | 0.3             | -0.0025 | 0.0015 | 0.0075 |             |                      |     |     |
| 1          | 1.25       | 600| 1               | -0.2125 | 0.0265 | 0.0284 | 0.254000 | 43% | (0.7325, 0.8426) | 98 |
|            |            |    | 1.2             | 0.0281  | 0.0334 | 0.0358 |             |                      |     |     |
|            |            |    | 0.3             | -0.0020 | 0.0010 | 0.0052 |             |                      |     |     |
Table 3: Bayes estimators linear-exponential loss function (LINEX) With interval censoring

| $\theta_1$ | $\theta_2$ | N | (α, β, γ) | Bias | SSD | ESD | Total Deviation TD | Ratio of Censored data | C.I | CP% |
|------------|------------|---|-----------|-------|-----|-----|-------------------|----------------------|-----|-----|
| 0.75       | 0.3        | 200 | 1, 1.2, 0.3 | -0.1113 | 0.0423 | 0.0554 | 0.0523 | 0.199633 | 15% | (0.7826, 1.9948) | 94 |
| 0.75       | 0.3        | 300 | 1, 1.2, 0.3 | -0.1243 | 0.0794 | 0.0017 | 0.0687 | 0.196867 | 17% | (0.7960, 0.9554) | 89 |
| 0.75       | 0.3        | 400 | 1, 1.2, 0.3 | -0.1304 | 0.0786 | 0.0011 | 0.0394 | 0.164550 | 20% | (0.8005, 0.9387) | 89 |
| 0.75       | 0.3        | 600 | 1, 1.2, 0.3 | -0.1303 | 0.0563 | 0.0008 | 0.0289 | 0.196133 | 24% | (0.8143, 0.9250) | 94 |
| 1          | 0.5        | 200 | 1, 1.2, 0.3 | -0.1288 | 0.0368 | 0.0062 | 0.0541 | 0.199567 | 22% | (0.7711, 0.9713) | 96 |
| 1          | 0.5        | 300 | 1, 1.2, 0.3 | -0.1509 | 0.0891 | 0.0013 | 0.0528 | 0.179883 | 26% | (0.7763, 0.9220) | 91 |
| 1          | 0.5        | 400 | 1, 1.2, 0.3 | -0.1567 | 0.0730 | 0.0015 | 0.0360 | 0.180133 | 28% | (0.7760, 0.9107) | 95 |
| 1          | 0.5        | 600 | 1, 1.2, 0.3 | -0.1558 | 0.0570 | 0.0010 | 0.0390 | 0.229483 | 25% | (0.7915, 0.8969) | 90 |
| 1          | 1.25       | 200 | 1, 1.2, 0.3 | -0.1728 | 0.0061 | 0.0009 | 0.0409 | 0.222533 | 38% | (0.7334, 0.9210) | 96 |
| 1          | 1.25       | 300 | 1, 1.2, 0.3 | -0.2186 | 0.0254 | 0.0034 | 0.0439 | 0.206633 | 39% | (0.7078, 0.8551) | 94 |
| 1          | 1.25       | 400 | 1, 1.2, 0.3 | -0.2159 | 0.0152 | 0.0007 | 0.0352 | 0.190883 | 46% | (0.7177, 0.8505) | 95 |
| 1          | 1.25       | 600 | 1, 1.2, 0.3 | -0.2225 | 0.0193 | 0.0017 | 0.0252 | 0.251100 | 43% | (0.7252, 0.8297) | 96 |

**Graphs:**
- Histograms of MLE of α, β, γ for each scenario.
### Table 4: Bayes estimators generalization of the Entropy loss (GEL) with interval-censoring

| $\theta_1$ | $\theta_2$ | N  | ($\alpha, \beta, \gamma$) | Bias       | SSD   | ESD   | Total Deviation TD | Ratio of Censored data | C.I              | CP% |
|------------|------------|----|---------------------------|------------|-------|-------|--------------------|------------------------|-------------------|-----|
| 0.75       | 0.3        | 200| 1, 0.3                    | -0.0898    | 0.0558| 0.0567| 0.165467           | 15 %                  | (0.7952, 1.0251) | 96  |
|            |            |    | 1, 0.2                    | 0.0616     | 0.0614| 0.0686|                    |                        | (1.1538, 1.3693) | 94  |
|            |            |    | 1, 0.1                    | 0.0045     | 0.0045| 0.0169|                    |                        | (0.1829, 0.4025) | 100 |
| 0.75       | 0.3        | 300| 1, 0.3                    | -0.1138    | 0.0495| 0.0432| 0.214883           | 17 %                  | (0.8014, 0.9711) | 91  |
|            |            |    | 1, 0.2                    | 0.0917     | 0.0528| 0.0560|                    |                        | (1.1986, 1.3847) | 91  |
|            |            |    | 1, 0.1                    | -0.0074    | 0.0019| 0.0118|                    |                        | (0.2121, 0.3732) | 91  |
| 0.75       | 0.3        | 400| 1, 0.3                    | -0.1221    | 0.0394| 0.0365| 0.223767           | 20 %                  | (0.8084, 0.9475) | 94  |
|            |            |    | 1, 0.2                    | 0.0980     | 0.0480| 0.0483|                    |                        | (1.2225, 1.3735) | 92  |
|            |            |    | 1, 0.1                    | -0.0060    | 0.0016| 0.0091|                    |                        | (0.2196, 0.3684) | 100 |
| 0.75       | 0.3        | 600| 1, 0.3                    | -0.1229    | 0.0362| 0.0302| 0.191067           | 24 %                  | (0.8206, 0.9337) | 92  |
|            |            |    | 1, 0.2                    | 0.0654     | 0.0413| 0.0382|                    |                        | (1.2058, 1.3249) | 91  |
|            |            |    | 1, 0.1                    | -0.0041    | 0.0011| 0.0064|                    |                        | (0.2378, 0.3541) | 100 |
| 1          | 0.5        | 200| 1, 0.3                    | -0.0906    | 0.0508| 0.0578| 0.163017           | 20 %                  | (0.7922, 1.0266) | 98  |
|            |            |    | 1, 0.2                    | 0.0601     | 0.0601| 0.0704|                    |                        | (1.1468, 1.3734) | 97  |
|            |            |    | 1, 0.1                    | -0.0067    | 0.0040| 0.0172|                    |                        | (0.1851, 0.4015) | 100 |
| 1          | 0.5        | 300| 1, 0.3                    | -0.1271    | 0.0461| 0.0436| 0.205683           | 26%                   | (0.7855, 0.9602) | 95  |
|            |            |    | 1, 0.2                    | 0.0747     | 0.0533| 0.0560|                    |                        | (1.1870, 1.3624) | 95  |
|            |            |    | 1, 0.1                    | -0.0049    | 0.0025| 0.0115|                    |                        | (0.2051, 0.3852) | 100 |
| 1          | 0.5        | 400| 1, 0.3                    | -0.1348    | 0.0396| 0.0370| 0.209300           | 28%                   | (0.7929, 0.9376) | 93  |
|            |            |    | 1, 0.2                    | 0.0742     | 0.0452| 0.0480|                    |                        | (1.2014, 1.3470) | 93  |
|            |            |    | 1, 0.1                    | -0.0038    | 0.0020| 0.0089|                    |                        | (0.2223, 0.3701) | 100 |
| 1          | 0.5        | 600| 1, 0.3                    | -0.1483    | 0.0400| 0.0294| 0.218133           | 25%                   | (0.7950, 0.9084) | 90  |
|            |            |    | 1, 0.2                    | 0.0690     | 0.0331| 0.0385|                    |                        | (1.2081, 1.3299) | 96  |
|            |            |    | 1, 0.1                    | -0.0037    | 0.0009| 0.0061|                    |                        | (0.2361, 0.3565) | 100 |
| 1          | 1.25       | 200| 1, 0.3                    | -0.1589    | 0.0499| 0.0557| 0.198650           | 38%                   | (0.7380, 0.9442) | 93  |
|            |            |    | 1, 0.2                    | 0.0281     | 0.0490| 0.0664|                    |                        | (1.1180, 1.3381) | 98  |
|            |            |    | 1, 0.1                    | -0.0049    | 0.0027| 0.0154|                    |                        | (0.1820, 0.4082) | 100 |
| 1          | 1.25       | 300| 1, 0.3                    | -0.2128    | 0.0439| 0.0407| 0.263883           | 39%                   | (0.7061, 0.8683) | 95  |
|            |            |    | 1, 0.2                    | 0.0477     | 0.0499| 0.0528|                    |                        | (1.1648, 1.3306) | 90  |
|            |            |    | 1, 0.1                    | -0.0034    | 0.0020| 0.0096|                    |                        | (0.2179, 0.3745) | 100 |
| 1          | 1.25       | 400| 1, 0.3                    | -0.2061    | 0.0352| 0.0355| 0.239017           | 46%                   | (0.7249, 0.8628) | 97  |
|            |            |    | 1, 0.2                    | 0.0295     | 0.0423| 0.0444|                    |                        | (1.1569, 1.3020) | 94  |
|            |            |    | 1, 0.1                    | -0.0025    | 0.0015| 0.0075|                    |                        | (0.2289, 0.3661) | 100 |
| 1          | 1.25       | 600| 1, 0.3                    | -0.2262    | 0.0274| 0.0278| 0.259200           | 43%                   | (0.7188, 0.8288) | 94  |
|            |            |    | 1, 0.2                    | 0.0320     | 0.0349| 0.0357|                    |                        | (1.1789, 1.2851) | 89  |
|            |            |    | 1, 0.1                    | -0.0019    | 0.0009| 0.0050|                    |                        | (0.2429, 0.3533) | 100 |
6. Description of Real Data

This paper, depending on real data for the children patient of the cancer disease, we choosing this type data because it is diffusion and deadly in current time in Iraq and this type of diseases has failure time (death time) occurs which is interesting phenomenon in this paper.

To collect data for the children patient of the cancer disease, returning the child spatial Hospital, and center of cancerous diseases, Basrah, Iraq.

The time of study point determined from two years (2018, 2019), that means the duration time of this study is constant and fixed for (24) months or (730) days.

The number of patients in the experiment for the above duration time is (134), and (100) from them follow-up and they were dead during the time of the study with know time, that means the data became complete.

And (34) of the study could not be done for them, they were dead during the time of the study without know time, that means the data became interval censored data.

We used the software program Easy Fit Professional to fit the curve to demonstrate the good match of data to the specific probability distribution. This software program uses a variety of tests under consideration, such as the Kolmogorov–Smirnov test, Anderson–Darling test and chi-square test. Through this program, concluded that the real-time data follow the three parameters Weibull distribution.

Now, we compute the estimated values for three-parameter Weibull distribution at real data we will used to find estimation values for probability death density function \( \hat{f}(t) \), cumulative distribution function \( \hat{F}(t) \), survival function \( \hat{S}(t) \), and hazard function \( \hat{h}(t) \) at MLE, BSEL, BLINX and BGEL respectively at table below:

| Parameter estimation | Estimate by MLE | Estimate by BSEL | Estimate by BLINX | Estimate by BGEL |
|----------------------|-----------------|------------------|-------------------|------------------|
| \( \hat{\alpha} \)   | 0.99209680000   | 1.01566069       | 0.9808223         | 1.01251840       |
| \( \hat{\beta} \)    | 1.3484333000    | 1.38059050       | 1.3331812         | 1.3768391        |
| \( \hat{\gamma} \)   | 0.0840846642    | 0.06944383       | 0.09139019        | 0.070501227      |
we can saw that a all types of four methods:

The values of death density function \( \hat{f}(t) \) were increasing slightly and then the values decreased slightly until the end of failure.

The values of cumulative distribution function \( \hat{F}(t) \) increased with increasing failure times because it collected the probability values for all previous observations step by step, meaning

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**Figure (1)** \( f(t) \) using MLE parametric estimation method

**Figure (2)** \( F(t) \) using MLE parametric estimation method

**Figure (3)** \( h(t) \) using MLE parametric estimation method

**Figure (4)** \( S(t) \) using MLE parametric estimation method

**Figure (5)** \( f(t) \) using BSEL parametric estimation method

**Figure (6)** \( F(t) \) using BSEL parametric estimation method

**Figure (7)** \( S(t) \) using BSEL parametric estimation method

**Figure (8)** \( h(t) \) using BSEL parametric estimation method

**Figure (9)** \( f(t) \) using BLinx parametric estimation method

**Figure (10)** \( F(t) \) using BLinx parametric estimation method

**Figure (11)** \( h(t) \) using BLinx parametric estimation method

**Figure (12)** \( S(t) \) using BLinx parametric estimation method

**Figure (13)** \( f(t) \) using BGEL parametric estimation method

**Figure (14)** \( F(t) \) using BGEL parametric estimation method

**Figure (15)** \( h(t) \) using BGEL parametric estimation method

**Figure (16)** \( S(t) \) using BGEL parametric estimation method
there was a direct relationship between failure times and cumulative death distribution function.
The values of survival function $\hat{S}(t)$ decreased gradually with increasing failure times for the observations, meaning there was an opposite relationship between failure times and survival function.
The values of hazard function $\hat{h}(t)$ increased gradually with increasing failure times.
We can notice that all estimation methods have close values and close to the real values, and the accuracy increases as the sample size increases.

7. Conclusion
The estimation problem of the unknown parameters of the three-parameters Weibull distribution based on interval censoring scheme was discussed. A comparison between the methods of estimators (maximum likelihood estimator, and Bayesian estimation under three loss functions are squares error loss function, linear exponential loss function and general entropy loss functions) on the essential of Monte Carlo Simulation study.
The conducting of the different estimator's interval censoring schemas is compared based on simulation study to determine the interval censoring schemas by using Bias and total deviation (TD). Finally, a real data set has been considered to the practical and show how the schemas works in practice. According to comparisons among the MLE, BSEL, BLINX and BGEL methods. It was observed that Bayesian estimation under GEL behave quite better for three parameters Weibull distribution, where Bias and Total Deviation (TD) decrease than another method.

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