Analysis of an Il’in Scheme for a System of Singularly Perturbed Convection-Diffusion Equations

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Abstract

In this paper, a numerical solution for a system of singularly perturbed convection-diffusion equations is studied. The system is discretized by the Il’in scheme on a uniform mesh. It is proved that the numerical scheme has first order accuracy, which is uniform with respect to the perturbation parameters. We show that the condition number of the discrete linear system obtained from applying the Il’in scheme for a system of singularly perturbed convection-diffusion equations is O(N) and the relevant coefficient matrix is well conditioned in comparison with the matrices obtained from applying upwind finite difference schemes on this problem. Numerical results confirm the theory of the method.

Keywords: Convection-Diffusion, Il’in Scheme, Uniform Convergence, Singular Perturbation, Condition Number

1. Introduction

Consider the following system of $l$ coupled singularly perturbed convection-diffusion equations: Find $u = (u_1, \ldots, u_l) \in (C^2(0,1) \cap C[0,1])$ such that

$$ Lu := -Eu^* - Bu + Au = f, $$

$$ x \in \Omega = (0,1), u(0) = u(1) = 0, \quad (1.1) $$

With $E = \text{diag}\{\epsilon_1, \epsilon_2, \ldots, \epsilon_l\}$, where $\epsilon_i$s for $i = 1, \ldots, l$ are known small positive diffusion coefficients, $A = (a_{ij}(x))$ is an $l \times l$ matrix, and $f = f_i(x)$ for $i = 1, \ldots, l$ is a vector-valued right hand side. Furthermore, we shall assume that $B$ is diagonal with diagonal elements $b_i(x)$ and define

$$ \beta_k = \min_{x \in [0,1]} |b_k(x)| > 0 \quad (k = 1, \ldots, l). \quad (1.2) $$

Some results for systems of singularly perturbed of differential equations can be found in: Linss and Madden [1], Madden and Stynes [2] and Gracia and Lisbona [3].

Bellew and O’Riordan [4], Cen [5], Amiraliyev [6] and Andreev [7] used the finite difference method for a coupled system of two singularly perturbed convection–diffusion equations.

T. Linss [8] considered an upwind finite difference scheme on special layer adapted Shishkin and Bakhvalov meshes. He showed that the error in the discrete maximum norm is bounded by $CN^{-1} \ln N$ and $CN^{-1}$ for Shishkin and Bakhvalov meshes respectively, where $C$ is independent of the perturbation parameters $\epsilon_k$ for $k = 1, \ldots, l$, and $N$ is the number of mesh points used.

The discrete linear systems that arise from Shishkin or Bakhvalov meshes do not have a good condition number. H. G. Roos in [9] showed that the condition number of the discrete linear system associated with the upwind schemes on Shishkin meshes for a single equation is $O(\epsilon^{-2}N^2 \ln^2 N)$, which is not good when $\epsilon$ is small. In fact, if $A$ is the coefficient matrix of the linear system associated with the upwind schemes on Shishkin meshes, then $\|A\| \leq C \frac{N^2}{\epsilon^2} \ln^2 N$ and $\|A^{-1}\| \leq C$, where $C$ is a constant independent of $E$ and the norm is the discrete maximum norm for matrices. Nevertheless, he proposed a precondition which has reduced this condition number to $O(N^3 \ln^{-1} N)$.

In this paper, we study the Il’in scheme (see [10]), for problem (1.1). We show that for this method the error in the discrete maximum norm is bounded by $CN^{-1} \ln N$ and $CN^{-1}$, where $C$ is independent of the perturbation parameters $\epsilon_k$ ($k = 1, \ldots, l$), and we prove that if $A$ is the coefficients...
matrix of the linear system associated with the II’in scheme on uniform meshes, then \( \| u \| \leq C N \) and \( \| A \| < C \). So the condition number of the discrete linear system associated with the II’in scheme on uniform meshes for single equation is \( O(N) \), which is better in comparison with the precondition of upwind schemes on Shishkin meshes.

The paper is organized as follows: in section 2, we give some properties of the solution of (1.1) and in section 3, we state the difference approximation II’in scheme. In section 4, we analyze the error of II’in scheme applied to (1.1), and some numerical examples are presented in section 5.

2. Properties of the Exact Solutions

In this section, we analyze the exact solution of (1.1). Assume that \( v \in L^\infty ([0,1]) \) and 
\[
\omega = \{ x_j | 0 = x_0 < x_1 < \cdots < x_N = 1 \}, \quad \text{where} \ x_i = ih \quad \text{and} \quad h = 1/N.
\]
Consider the following norms:
\[
\| v \|_\infty = \max_{x \in (0,1)} |v(x)|, \quad \| v \|_{1,\omega} = \max_{x_i \in \omega} |v(x)|,
\]
\[
\| v \|_l = \int_0^1 |v(x)| dx, \quad \| v \|_{l,\omega} = h \sum_{j=0}^N |v(x_j)|,
\]
where \( v_i = v(x_i) \) for \( i = 0, \cdots, N \). Consider the following discrete norm:
\[
\| v \|_{h,1} = \| v \|_{1,\omega} + h \sum_{j=1}^N |Dv|.
\]

In this case according to (2.1) since \( x_i \in (0,1) \), the space \( W_{h,1} \) contains \( v \in R^{N+1} \), such that \( \| v \|_{h,1} < \infty \). Difference operators are defined as follows:
\[
D^+ u_i = \frac{u_{i+1} - u_i}{h}, \quad D^- u_i = \frac{u_{i-1} - u_i}{h}, \quad D^0 u_i = \frac{u_{i+1} - u_{i-1}}{2h}.
\]

For the vector-valued function \( v(x) = [v_1(x), \cdots, v_l(x)]^T \), consider the infinity norm as
\[
\| v(x) \|_\infty = \max_{i=1,\cdots,l} |v_i|.
\]

To estimate the error in our difference approximation, we shall require some bounds for the derivatives of the solution of (1.1), under the assumptions
\[
\sum_{j=1}^l a_{ij}(x) \geq 0, \quad a_{ij}(x) > 0 \quad \text{and} \quad a_{ij}(x) \leq 0
\]
for \( i \neq j, \ x \in (0,1) \) and \( i, j = 1, \cdots, l \) and strict inequality hold at least for one \( k \) i.e.,
\[
\sum_{j=1}^l a_{ij}(x) > 0
\]

Lemma 2.1 If \( y = (y_1(x), \cdots, y_l(x))^T \), \( Ly \geq 0 \) in \( \Omega \) and \( y(0) \geq 0, y(1) \geq 0 \) then \( y(x) \geq 0 \) in \( \Omega \).

Proof. Let \( y_j(x) \) be minimum at \( t_i \) for \( i = 1,2,\cdots,l \), i.e., \( y_j(t_i) = \min_{t \in (0,1)} y_i(x) \) and also assuming \( y_j(t_i) = \min_{t \in (0,1)} y_i(x) \) and strictly hold at least for one \( k \) i.e.,

\[
\sum_{j=1}^l a_{ij}(x) > 0
\]

Lemma 2.2 If \( y_j(t_j) \geq 0 \) the lemma is proved. So let \( y_j(t_j) < 0 \). If \( y_j(t_j) = y_k(t_j) \) for \( k = 1,2,\cdots,l \), then it follows that \( y_j(t_j) > 0 \) and \( y_k(t_j) \geq 0 \). By (1.1)
\[
Ly_j(t_j) = -Ey_j(t_j) - By_j(t_j) + Ay_j(t_j)
\]
\[
\leq Ay_j(t_j) = y_j(t_j)\ A 1.
\]

In this case according to (2.4) since \( y_j(t_j) < 0 \), the \( k \)th component of \( Ay_j(t_j) \) is negative, which is a contradiction to the assumption of the lemma. If there is a \( k \) with \( 1 \leq k \leq l \) such that \( y_j(t_j) < y_k(t_j) \) then
\[
-e_jy_j(t_j) - b_jy_j(t_j) + \sum_{n=1}^l s_{jm}(t_j)y_m(t_j)
\]
\[
= -e_jy_j(t_j) + \sum_{n=1}^l s_{jm}(t_j)y_m(t_j)
\]
\[
+ \sum_{n=1}^l a_{jm}(t_j)(y_m(t_j) - y_j(t_j))
\]
\[
\leq \sum_{n=1}^l a_{jm}(t_j)(y_m(t_j) - y_j(t_j)) + \sum_{n=1}^l a_{jm}(t_j) + y_j(t_j)\sum_{n=1}^l a_{jm}(t_j).
\]

If \( \sum_{n=1}^l a_{jm}(t_j) < 0 \), it is obvious that the right hand side of the above inequality is negative. If \( \sum_{n=1}^l a_{jm}(t_j) = 0 \), then since \( a_{ij} > 0 \) we have
\[
\sum_{n=1}^l a_{jm}(t_j) > 0 \quad \text{so the right hand side is negative and again we reach a contradiction. So the lemma is proved.}
Theorem 2.2 Suppose $u$ solves (1.1) and assume that $a_{ij}$ for $i, j = 1, \cdots, l$ satisfies (2.3) and (2.4), then
\[ \|u\| \leq C\|f\|. \]
By theorem 2.2, T. Linss [10] has proved the following lemma.

Lemma 2.3 Let $u$ be the solution of (1.1) and suppose (1.2), (2.3) and (2.4) hold. Then for $x \in [0, 1]$ and $n = 0, 1$
\[ |u^{(n)}(x)| \leq C. \]

By lemma (2.3) and the application of the technique mentioned in [12], the following lemma can be proved in a similar way.

Remark 2.5 By lemma 2.4 and direct computation, without loss of generality, assume that $b_k \geq b_k$; hence, we have
\[ L_x z = -e_k z_k - b_k (x) z_k + \sum_{m=1}^k a_{km} z_m (x) \]
\[ = f_k - \sum_{m=1}^k a_{km} v_m (x) + e_k b_k (0) z_k (x) - b_k (x) v_k (x) \]
\[ = g_k (x) \]
and for $n = 0, 1$
\[ |g^{(n)}(x)| \leq C. \]

3. Discretization

In this section, we deal with the discretization of prob-em (1.1) by the Il'in scheme. We apply the Il'in scheme: Find $U \in (R^{1 \times l})$ such that:
\[ L_x U = -e_k \sigma_{e_k} D^+ D^U - b_{k,j} D^q U_j + \sum_{m=1}^k a_{km} U_{m,j} = f_{k,j}, \]
for $k = 1, \cdots, l$ and $j = 1, \cdots, N-1$, where
\[ \sigma_k (x) = q_k (x) \text{coth} q_k (x) \text{ with } q_k (x) = \frac{h b_k (x)}{2 \epsilon_k}, \quad U_{k,j} \]
is the approximate value for $u_k (x)$ that is obtained by the Il'in scheme, $b_{k,j} = b_k (x_j)$ and $a_{km,j} = a_{km} (x_j)$.
Consider the diagonal matrices
\[ Q(x) = \text{diag} \{ q_1 (x), \cdots, q_l (x) \} \]
and
\[ \text{coth} Q(x) = \text{diag} \{ \text{coth} q_1 (x), \cdots, \text{coth} q_l (x) \}. \]
We define
\[ \Sigma = Q(x) \text{coth} Q(x) = \frac{h \epsilon^{-1} B}{2} \text{coth} \frac{h \epsilon^{-1} B}{2}. \]
We rewrite (1.1) in matrix form as follows:
\[ -E \Sigma J^+ D^+ U_j - B_j D^q U_j + A_j U_j = f_j \]
for $j = 1, \cdots, N-1,$
where $U_j = (U_{1,j}, \cdots, U_{l,j})^T$, $D^+ U_j = (D^+ U_{1,j}, \cdots, D^+ U_{l,j})^T$ and
\[ D^q U_j = (D^q U_{1,j}, \cdots, D^q U_{l,j})^T \]
The corresponding coefficient matrix in (3.1) is:
\[ A^b = \begin{pmatrix} D_{11} & D_{12} & \cdots & D_{1l} \\ D_{21} & D_{22} & \cdots & D_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ D_{l1} & D_{l2} & \cdots & D_{ll} \end{pmatrix} \]
where blocks $D_{kj}$ in $A^b$ are $(N-1) \times (N-1)$ and diagonal blocks $D_{kk}$ for $k = 1, \cdots, l$ are tridiagonal matrices as follows
\[ D_{kk} = \begin{pmatrix} \frac{2 \epsilon \sigma_{k,1}}{h^2} + a_{kk,1} & -\epsilon \sigma_{k,1} & b_{k,1} \epsilon \sigma_{k,1} & \cdots & 0 \\ -\epsilon \sigma_{k,1} & \frac{2 \epsilon \sigma_{k,2}}{h^2} + a_{kk,2} & -\epsilon \sigma_{k,2} & \cdots & 0 \\ b_{k,1} \epsilon \sigma_{k,1} & -\epsilon \sigma_{k,2} & \frac{2 \epsilon \sigma_{k,3}}{h^2} + a_{kk,3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \frac{2 \epsilon \sigma_{k,N-1}}{h^2} + a_{kk,N-1} \end{pmatrix} \]
and $D_q$ for $i, j = 1, \cdots, l$ where $i \neq j$ are diagonal matrix as
\[
D_q = \text{diag} \{a_{i,j}, \cdots, a_{i,N-i}\}.
\]

**Remark 3.1** When $B = \text{diag} \{b_i, \cdots, b_j\}$ is constant and $A \equiv 0$, the system (1.1) is reduced to
\[
-Eu' = Bu = f.
\]
Then its general solution for homogeneous equation, which is also boundary layer function, is
\[
u_i(x) = C_i \exp \left( -\frac{b_i x}{\varepsilon_k} \right) \quad \text{for } b_i > 0 \quad \text{and}
\nu_i(x) = C_i \exp \left( -\frac{b_i (1-x)}{\varepsilon_k} \right) \quad \text{for } b_i < 0,
\]
where $C_i$ for $k = 1, \cdots, l$ are arbitrary constants. The difference operator (3.2) is exact for these boundary layer functions. i.e.,
\[
-E\Sigma_j D_j U_j - B_i D_j U_j = -Eu_j - Bu_j 
\]
for $j = 1, \cdots, N-1$, where index $j$ indicates the evaluation at point $x_j$, $u_j = (u_{i,j}, \cdots, u_{l,j})$ and $u_{k,j} = u_k(x_j)$.

In what follows, we try to estimate the condition number of the discrete linear system associated with the Il'in scheme presented. Then we apply M-criterion [13, 14] to obtain the condition number.

**Theorem 3.2** (M-criterion). Let a matrix $A$ satisfy $a_{ij} \leq 0$ for $i \neq j$. Then $A$ is an M-matrix if and only if there exists vector $e > 0$ such that $Ae > 0$. Furthermore, we have
\[
\|A^{-1}\| \leq \frac{1}{\min_k (Ae)_k}. \tag{3.5}
\]

**Theorem 3.3** The corresponding coefficient matrix in (3.1) is an M-matrix and satisfies
\[
\|A^t\| \leq CN
\]
and
\[
\|(A^t)^{-1}\| \leq \frac{2}{\beta}
\]
where $C$ is independent of $N$ and $e_k$ for $k = 1, \cdots, l$ and $\beta = \min_{k=1, \cdots l} (\beta_k)$

**Proof.** To evaluate the maximum norm of the matrix $A^t$ for the $i$th row of the $k$th row of the block matrix $A^t$, where $i = 1, \cdots, N-1$ and $k = 1, \cdots, l$, we have
\[
\left( q_k b_{ii} - 1 \right) \leq C \frac{q_k^2}{q_k b_{ii} + 1},
\]

See [14], we conclude that
\[
\sum_{j=1}^{l(N-1)} |a_{ij}^h| \leq \frac{4\varepsilon_k}{h^2} \left( q_{k,j} \coth q_{k,j} - 1 + 1 \right) + \sum_{m=1}^{l} |a_{kmj}| \leq C \frac{\varepsilon_k}{h^2} \left( q_{k,j}^2 + 1 \right).
\]

Since in singularly perturbed equations, $\varepsilon \leq h$ is considered, hence
\[
\sum_{j=1}^{N+1} |a_{ij}| \leq C \left( \frac{h+2}{2h} \right) \leq C \frac{1}{h} = CN.
\]

Therefore,
\[
\|A^h\| = \max_i \left\{ \sum_{j=1}^{N-1} |a_{ij}| \right\} \leq CN.
\]

If we choose \( e = [e^1, \ldots, e^q]^T \) where \( e^k_1 = 1 + x_i \) for \( b_k(x) < 0 \) and \( e^k_2 = 1 - x_i \) for \( b_k(x) > 0 \), then \( A^h e \) for the \( i \)th row of the \( k \)th row of the block matrix \( A^h \) for case \( b_k(x) < 0 \) we have
\[
(A^h e)_k = - \frac{b_{kj}}{2h} \coth \left( \frac{h b_{kj}}{2e_k^j} \right) \left( 1 + x_{i-1} \right) + \frac{b_{kj}}{h} \coth \left( \frac{h b_{kj}}{2e_k^j} \right) \left( 1 + x_i \right) + \frac{b_{kj}}{2h} \coth \left( \frac{h b_{kj}}{2e_k^j} \right) - \frac{b_{kj}}{2h} \left( 1 + x_{i+1} \right) + \sum_{m=1, m \neq k}^l a_{kmj}^m e_m^j \geq -b_{kj} > \beta_k > \beta
\]

and from (3.1), for \( i \neq j \), we have \( d_{ij}^0 \leq 0 \). Hence the M-criterion yields that \( A^h \) is an M-matrix and
\[
\left\| A^h \right\| \leq \frac{\|e\|}{\min_k (A^h e)_k} \leq \frac{2}{\beta}.
\]

A similar proof is for the case \( b_k(x) > 0 \).

**Notation.** Throughout the above theorem we let \( C \) denote positive constant that may take different values in different formulas but that are always independent of both the perturbation parameters \( \varepsilon_k \) and of \( N \), the number of mesh intervals. By theorem 3.3, the condition number of \( A \) is of order \( O(N) \). We recall that, in theorem 3.2 and theorem 3.3, \( \|A\| \) is the usual maximum norm of matrices. This shows that the matrices arising from the II’ in scheme for discretization of a singularly perturbed equation are well conditioned respect to the upwind finite difference method applied on Shiskin or Bakhvalov meshes.

For arbitrary \( U = (U_1, \ldots, U_l) \in (R^N)^l \), we define
\[
\|U\|_{\infty, \omega} = \max_{i=1, \ldots, l} \|U_i\|_{\infty, \omega}.
\]

A single equation of (3.1) can be written as follows:
\[
\begin{align*}
[L_k U_k]_j &= -\varepsilon_k \sigma_k (q_j) D^+ D^j U_k^j - b_{kj} D^j U_k^j \\
+ a_{kkj}^j U_k^j &= f_{kj} - \sum_{m=1, m \neq k}^l a_{kmj}^m U_{mj}.
\end{align*}
\]

(3.6)

Suppose \( U \) solves (3.1). We propose the following lemma to obtain a bound for \( U \).

**Lemma 3.4** Suppose \( A \) is a matrix such that \( a_{ii} > 0 \) and \( a_{ij} \leq 0 \) for \( i \neq j \) and \( i, j = 1, \ldots, n \). Also assume \( \sum_{j=1}^n a_{jj} \geq 1 \) for \( j = 1, \ldots, n \). Then for every arbitrary vector \( \eta = (\eta_1, \ldots, \eta_n)^T \) we have
\[
\|\eta\|_{\infty, \omega} \leq \|A\|_{\infty, \omega} \|\eta\|_{\infty, \omega}.
\]

**Proof.** Suppose for the element \( j \) of \( \eta \), \( \|\eta\|_{\infty, \omega} = |\eta_j| \). Without lose of generality, let \( |\eta_j| = |\eta| \) (otherwise we consider \( |\eta_j| = |\eta| \)).

\[
(A\eta)_j \geq a_{jj} |\eta_j| + \sum_{k=1, k \neq j}^n a_{kj} |\eta_k| - |\eta_j| + \sum_{k=1, k \neq j}^n a_{kj} |\eta_k| - |\eta_j|
\]

(since \( a_{kk} \leq 0 \) for \( k \neq k \)). Therefore
\[
(A\eta)_j - |\eta_j| \geq a_{jj} |\eta_j| + \sum_{k=1, k \neq j}^n a_{kj} |\eta_k| - |\eta_j|
\]

\[
\geq a_{jj} |\eta_j| + \sum_{k=1, k \neq j}^n a_{kj} |\eta_k| - |\eta_j| = \sum_{k=1, k \neq j}^n a_{kj} (|\eta_k| - |\eta_j|) \geq 0.
\]

So
\[
(A\eta)_j \geq |\eta_j| > 0.
\]

Hence
\[
\|\eta\|_{\infty, \omega} = \max_{j=1, \ldots, l} |(A\eta)_j| \geq \|\eta\|_{\infty, \omega}.
\]

Similar to theorem 2.2, we have the following theorem in the discrete case.

**Theorem 3.5** Suppose \( U \) solves (3.1). Assume \( b_k \) and \( a_{kk} \) satisfy (1.2), (2.3) and (2.4). Then
\[
\|U\|_{\infty, \omega} \leq C \|U\|_{\infty, \omega}.
\]

**Proof.** Dividing (3.6) by \( a_{kkj} \), we have
\[
\left[ \tilde{L}_k U_k \right]_j = [L_k U_k]_j / a_{kkj} = \frac{f_{kj}}{a_{kkj}} - \sum_{m=1, m \neq k}^l \frac{a_{kmj}}{a_{kkj}} U_{mj}.
\]

(3.7)

The matrix associated with operator \( \tilde{L}_k \) is a matrix that satisfy lemma 3.4. We have
\[
\|U_k\|_{\infty, \omega} \leq \|\tilde{L}_k U_k\|_{\infty, \omega} \leq \frac{f_{kj}}{a_{kkj}} \|U_k\|_{\infty, \omega} + \sum_{m=1, m \neq k}^l \frac{a_{kmj}}{a_{kkj}} \|U_{mj}\|_{\infty, \omega}.
\]

Therefore
By (2.3) and after some manipulation, we obtain
\[ \| u - U \|_{e,0} \leq C \| L (u - U) \|_{e,0}. \] (3.8)

4. Error Analysis

For the error analysis of II’in scheme applied to the system of singularly perturbed differential equations (1.1), suppose that \( b_i(x) > 0 \) for \( i = 1, \ldots, l \) and \( u \) satisfies (1.1). Similar to lemma 2.4, we split \( u(x) = v(x) + z(x) \), where \( v = (v_1(x), \ldots, v_l(x))^T \), \( z(x) = (z_1(x), \ldots, z_l(x))^T \). So we have \( u(x) = z_k(x) + v_k(x) \), and for \( n = 0, 1 \)
\[ \left\| z_k^{(n)}(x) \right\|_{e,0} \leq C \]
and
\[ v_k(x) = \frac{e_i^* u_i^*(0)}{b_i(0)} \exp(-b_i(0)e_i^{-1}x). \]

First we analyze the truncation error in a single equation. We introduce the continuous and discrete operators
\[ \langle A v \rangle(x) = e_i v_i'(x) + b_i v_i + \int_j b_j'(s) v_j(s) ds \]
and
\[ \sum_{j} a_{ij} v_j(s) ds, \quad x \in (0, 1), i = 1, \ldots, l \] (4.1)
and
\[ [A_j V] = e_i^* D^+ V_j(x) + b_i^2 V_j \]
\[ + \sum_{j=1}^{N-1} h V_j + b_i V_j + \sum_{j=0}^{N-1} a_{i} V_m V_n, \] (4.2)
where
\[ \tilde{D} v_i = \frac{v_i - v_i^+}{h} \]
and \( h = q_i \left( \coth(q_i) - 1 \right) \).

We note that \( h > h \). We introduce the continuous and discrete functions
\[ z_i(x) = \int_{s}^{x} [g_i(s) ds, \quad i = 1, \ldots, l \] (4.3)
and
\[ \left| h \phi_{ij} - \int_{t}^{t^{k+1}} \phi_i(t) ds \right| \leq \left| \int_{t}^{t^{k+1}} \phi_i(t) ds \right| \leq C \left| h z_{ij} D^+ b_i \right|, \]
where
\[ G_j = \sum_{i=1}^{N-1} h g_i, \quad i = 1, \ldots, l, \quad j = 1, \ldots, N - 1, \] (4.4)
where \( g_i(x) \) for \( i = 1, \ldots, l \) is defined in remark 2.5. By (4.1)-(4.4), we can establish the following theorem.

**Theorem 4.1** Let \( u \) be the solution of (1.1) and \( U \) be the solution obtained by the II’in scheme (3.1). Suppose the data \( g_i, b_i, a_{ij}, f_j \in C^1[0, 1] \) for \( k, m = 1, \ldots, l \) satisfy (1.2), (2.3) and (4.4). Then
\[ \| u - U \|_{e,0} \leq C \] (4.5)

**Proof.** From lemma 2.4, we can split \( u(x) \) as \( u(x) = v(x) + z(x) \), where \( v(x) \) is a boundary layer function. Hence,
\[ \| u - U \|_{e,0} = \| v - z + v - v \|_{e,0} \leq \| v - v \|_{e,0} \]
and
\[ (L z) = -D^+ (A z), \quad \text{and} \quad g_{ij} = -D^+ G_{ij} \]
for \( j = 0, \ldots, N \).

Thus, by \( (L z) = g_j \) and \( (L z) = g_j \),
\[ (A z) - z_j(x) = 0, \quad x \in (0, 1) \] and
\[ (A z) - G_j = 0, \quad j = 0, \ldots, N \] (4.6)
with constants \( \alpha \) and \( \gamma \).

From (2.1),
\[ \| u - U \|_{e,0} = \min_{t \in K} \| u(t) - U(t) \|_{e,0}. \]
Taking \( c = \gamma - \alpha \), from (4.6) we get
\[ \| z - z \|_{e,0} \leq \| A z - z \|_{e,0} + \| z \|_{e,0}. \]

Furthermore,
\[ (A z - A z - G_i + z_j(x), \quad z - G_i + z_j(x) \|_{e,0}. \]

We have
\[ \| z - z \|_{e,0} \leq \| A z - z \|_{e,0} + \| z \|_{e,0}. \]

Applying Taylor expansion with the integral remainder form and using lemma 2.4 and remark 2.5, we have
\[ \| u - U \|_{e,0} \leq C h. \]
and
\[ e_i \left( \Delta z_i - z_i' \right) = \left[ e_i \left( \frac{h}{h} \right) z_i' + \int_{v=1}^{\nu} e_i \int_{v}^{z_i'(s)} ds dr \right] \leq \frac{e_i}{h} \left( \frac{h}{h} - h \right) \left| z_i' \right| + \frac{e_i}{h} \int_{v}^{z_i'(s)} ds dr \].

On the other hand, we have
\[ e_i \left( \Delta z_i - z_i' \right) = e_i \left( 1 - q_i (\coth(q_i) - 1) \right) \leq e_i q_i = \frac{bh}{2} \leq Ch. \]

Hence
\[ e_i \left( \Delta z_i - z_i' \right) \leq Ch + \frac{1}{h} \left[ \int_{v=1}^{\nu} \int_{v}^{z_i'(s)} g_i + b_i z_i - \sum_{m=1}^{n} a_m e_m \right] ds dr \leq Ch. \]

Thus
\[ \left\| Ax - \Delta z - G + q \right\|_{e,\omega} \leq Ch. \]

In [15] it has been shown that
\[ \left\| (Z - z) \right\|_{e,\omega} \leq \frac{2}{R} \left\| L_i (Z - z) \right\|_{e,\omega}. \]

Hence we have
\[ \left\| Z - z \right\|_{e,\omega} \leq Ch. \quad (4.7) \]

To bound the boundary layer function \( v(x) = (v_1(x), \ldots, v_5(x)) \), a direct computation gives
\[ LV = \frac{2}{h} \sinh^{-1}(Q(x)) B(x) \sinh(Q(0)) \sinh(Q(x) - Q(0)) V + AV. \]

By a technique used in [9], we can show that
\[ \left\| v - V \right\|_{e,\omega} \leq Ch. \quad (4.8) \]

So by (4.7) and (4.8) we have
\[ \left\| u - U \right\|_{e,\omega} \leq Ch. \]

This completes the proof of the theorem.

5. Numerical Experiments

In this section, we compare the Il’in scheme with the upwind finite difference scheme (see [8,15]) for the following two examples.

Example 1. Consider

\[ -e_i u_i'' - u_i' + 2u_i - u_i = e^i \quad u_i(0) = u_i(1) = 0, \]
\[ -e_i u_i'' - 2u_i' + u_i - 4u_i = \cos(x) \quad u_i(0) = u_i(1) = 0, \]

In this example, we expect two layers at \( x = 0 \) which behave like \( \exp \left( \frac{-x}{e_i} \right) \) for \( h_i = 1 \) and like \( \exp \left( \frac{-2x}{e_i} \right) \) for \( h_i = 2 \). Let \( e_i = 10^{-8} \) and \( e_i = 10^{-6} \), which are sufficiently small values to bring out the singularly perturbed nature of the problem. The exact solution to the test problem is not available, so we estimate the accuracy of the numerical solution by comparing it to the numerical solution computed on the finer mesh. Let \( U^N \) be a numerical solution in \( N \) grid point. We estimate the error by

\[ \left\| U^N - U^{2N} \right\|_{e,\omega} \]

The rates of convergence \( r^N \) are computed using the following formula:

\[ r^N = \log_{2} \left( \frac{\left\| U^N - U^{2N} \right\|_{e,\omega}}{\left\| U^{2N} - U^{4N} \right\|_{e,\omega}} \right). \]

For the upwind scheme we use Shishkin mesh with \( r_1 = m \) and \( r_2 = \min \left( \frac{1}{2}, e_2 \log N \right) \). We divide the intervals \([0, r_1]\) and \([r_2, 1]\) into \( N/4 \) subintervals and \([r_2, 1]\) into \( N/2 \) subintervals of equal length. For Il’in scheme, we use the uniform mesh and divide the interval \([0, 1]\) into \( N \) subintervals of equal length. Numerical results are contained in Table 1. From this Table, we observe that the Il’in scheme is a first order uniformly convergent method.

Example 2. Let

\[ -e_1 u_1'' + 3u_1' + u_1 - u_2 = e^i \]
\[ -e_2 u_2'' + 0.5u_2' + u_2 - u_1 = \cos(x) \]
\[ -e_2 u_1'' + (5 - x)u_1' - u_1 + 2u_1 = 1 + x^3. \]

In this example, we expect layers \( \exp \left( -3x/e_i \right) \).

| N   | Error   | Rate \( r^N \) | Error   | Rate \( r^N \) |
|-----|---------|----------------|---------|----------------|
| 128 | 4.62e-2 | 0.8019         | 5.00e-3 | 1              |
| 256 | 2.65e-2 | 0.8461         | 2.50e-3 | 1              |
| 512 | 1.48e-2 | 0.8945         | 1.25e-3 | 1              |
| 1024| 7.90e-3 | 0.9462         | 6.30e-4 | 1              |
| 2048| 4.10e-3 |               |         |                |

Table 1. Numerical results for example 1.
From Table 1 and Table 2 we see that the rate of convergence for II’in scheme is close to 1, which agree with the convergence estimate of theorem 4.1. Numerical results confirm the theoretical results.

11. References

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