The Existence of Radial Solutions to the Schrödinger System Containing a Nonlinear Operator

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1. Introduction

Osgood-type condition is of great significance in the field of mathematics and has been widely applied to different equations or systems by many authors. In 1898, under the Osgood type condition

$$\int_0^U \frac{ds}{\psi(s)} = \infty, \forall U > 0,$$

Osgood [1] presented the existence result of solutions for the following equation without the Cauchy–Lipschitz condition

$$\frac{dy}{d\chi} = \psi(\chi, y),$$

where $\psi(s)$ is a continuous function satisfying $|\psi(\chi, y) - \psi(\chi, y')| \leq \phi(|y - y'|).$ Then, lots of authors began to consider applying the Osgood-type condition to other problems and gained many excellent results such as the comparison result of viscosity upper and lower solutions for fully nonlinear parabolic equations [2], the existence result of solutions for backward stochastic differential equations (BSDEs) [3], and the nonexistence result of the local solution for semilinear fractional heat Equation [4]. For more results, see [5–9].

The Schrödinger equation was derived from mathematical physics and closely related to several physical phenomena. In [10], Kurihura used it to model the superfluid film equation in plasma physics. In [11,12], it was used to model the phenomena of the self-channeling of a high-power ultrashort laser in matter. More examples and details of applications can be found in [13–16].

In 2017, by employing the analysis technique and weighted norm method, Sun [17] established the existence result of solutions to the following Schrödinger equation...
\[ \Delta \mu + \psi(|\chi|)b(\mu) = 0, \]

where \(|\chi| \in E_D, \psi(|\chi|) \in C^1_{\text{loc}}(E_D, R), \lambda \in (0, 1), b(\mu) \in C^1_{\text{loc}}(R, R) \] (locally Hölder continuous), \(E_D = \{ \chi \in R^2 : |\chi| > D \}, S_D = \{ \chi \in R^2 : |\chi| = D \}, \) for \( D > 0. \)

In 2018, by introducing a growth condition and employing the iterative technique, Zhang, Wu and Cui [18] established the nonexistence and existence results of the entire blow-up solutions to the following Schrödinger equation

\[ \text{div} (\Lambda(|\nabla \mu| \nabla \mu) = b(|\chi|)\psi(\mu), \quad \chi \in R^n, \]

where \( n \geq 2, \Lambda \) is a nonlinear operator belonging to the set \( \Lambda \in C^2([0, \infty), (0, \infty)]) \exists \beta \in (0, \infty) : \Lambda(mS) \leq m^\beta \Lambda(s), 0 < m < 1 \} . \)

In 2020, by employing the iterative technique, Wang et al. [19] established the existence result of the entire radial solutions for the following Schrödinger system

\[
\begin{cases}
\Delta \mu = b(|\chi|)\psi(\mu), \quad \chi \in R^n, \\
n(b(|\chi|)\psi(\mu) ) = h(|\chi|)\varphi(\mu), \quad \chi \in R^n,
\end{cases}
\]

where \( n \geq 3, b, h, \psi, \varphi \in C([0, \infty), [0, \infty]), \) and \( \Lambda \) is a nonlinear operator belonging to \( \theta = \{ \Lambda \in C^2((0, \infty), (0, \infty)) \exists \beta \in (0, \infty) : \Lambda(mS) \leq m^\beta \Lambda(s), 0 < m < 1 \} . \)

Motivated by the above work, we studied the existence of entire positive radial solutions to the following Schrödinger system

\[
\begin{cases}
\Delta \mu = b(|\chi|)\psi(\mu, v), \quad \chi \in R^n, \\
n(b(|\chi|)\psi(\mu, v) ) = h(|\chi|)\varphi(\mu, v), \quad \chi \in R^n,
\end{cases}
\]  

where \( n \geq 3, b, h \) are continuous functions, \( \Lambda \) is a nonlinear operator belonging to \( \theta \) and \( \psi, \varphi \) are continuous functions satisfying Osgood-type conditions

\[
\int_1^\infty \frac{1}{\left( \psi(t, (\psi(t, t))^\frac{1}{p-1}) + 1 \right)^{\frac{p}{p-1}}} dt = \infty, \quad \forall i > 0 
\]

and

\[
\int_1^\infty \frac{1}{\left( \varphi((\psi(t, t))^\frac{1}{p-1}, t) + 1 \right)^{\frac{p}{p-1}}} dt = \infty, \quad \forall j > 0. 
\]

By employing the monotone iterative method, we give the existence results of positive entire radial solutions to the Schrödinger system (1) under the cases where the components \( \mu \) and \( v \) are bounded, \( \mu \) and \( v \) are blow-up, and one of the components is bounded while the other is blow-up. The monotone iterative method plays a significant role in the study of nonlinear problem, as can be seen in [18–28] and the references therein. To the best of our knowledge, there is no work about the existence of the positive radial solutions of the Schrödinger system (1) under the Osgood-type conditions. In addition, our results extended the work of authors in [18,28–33].

2. Preliminaries

In this section, we give a definition, some notations, assumptions and Lemmas that are subsequently needed in the proof.

Firstly, we present the definition about the classification of solutions.

**Definition 1** ([34]). A solution \((\mu, v) \in C^2[0, \infty) \times C^2[0, \infty)\) of system (1) is called an entire bounded solution if condition (2) is established; it is called an entire blow-up solution if condition (3) is established; it is called a semifinite entire blow-up solution if condition (4) or (5) is established.
Finite case: both components $\mu$ and $\nu$ are bounded, that is
\[
\lim_{|x| \to \infty} \mu(|x|) < \infty \quad \text{and} \quad \lim_{|x| \to \infty} \nu(|x|) < \infty. \tag{2}
\]

Infinite case: both components $\mu$, $\nu$ are blow-up, that is
\[
\lim_{|x| \to \infty} \mu(|x|) = \infty \quad \text{and} \quad \lim_{|x| \to \infty} \nu(|x|) = \infty. \tag{3}
\]

Semifinite Case: one of the components is bounded, while the other is blow-up, that is
\[
\lim_{|x| \to \infty} \mu(|x|) < \infty \quad \text{and} \quad \lim_{|x| \to \infty} \nu(|x|) = \infty, \tag{4}
\]
or
\[
\lim_{|x| \to \infty} \mu(|x|) = \infty \quad \text{and} \quad \lim_{|x| \to \infty} \nu(|x|) < \infty. \tag{5}
\]

We then present the notations as follows: $\tau = |x|$, $i, j, c_1, c_2 \in (0, \infty)$ are suitably chosen,
\[
G_1(\tau) = \int_0^\tau \psi(t) \left( \int_0^t b(s)ds \right) dt, \quad G_2(\tau) = \int_0^\tau \psi(t) \left( \int_0^t h(s)ds \right) dt,
\]
\[
L(\tau) = \int_{i+j}^\tau \left( \frac{1}{\psi(i,j)} \right) dt, \quad L(\infty) := \lim_{\tau \to \infty} L(\tau),
\]
\[
G(\tau) = G_1(\tau) + G_2(\tau), \quad G_k(\infty) := \lim_{\tau \to \infty} G_k(\tau), \quad k = 1, 2,
\]
\[
\omega_1(\tau) = \psi \left( \int_0^\tau \left( \frac{1}{\psi(i,j)} \right) dt \right), \quad \omega_2(\tau) = \psi \left( \int_0^\tau \left( \frac{i}{\psi(i,j)} \right) dt \right),
\]
\[
U_k(\tau) = \int_0^\tau \psi(t) \left( \int_0^t b(s)ds \right) dt, \quad V_k(\tau) = \int_0^\tau \psi(t) \left( \int_0^t h(s)ds \right) dt,
\]
\[
U_k(\infty) := \lim_{\tau \to \infty} U_k(\tau), \quad V_k(\infty) := \lim_{\tau \to \infty} V_k(\tau), \quad \text{for } k = 1, 2,
\]
\[
F_1(\tau) = \int_0^\tau \frac{1}{\psi(t, (\psi(t, t))^{1/\tau} + 1)^{1/\tau}} dt, \quad F_1(\infty) := \lim_{\tau \to \infty} F_1(\tau),
\]
\[
F_2(\tau) = \int_0^\tau \frac{1}{\psi((\psi(t, t))^{1/\tau}, t + 1)^{1/\tau}} dt, \quad F_2(\infty) := \lim_{\tau \to \infty} F_2(\tau).
\]

Assume that $\psi$ and $\varphi$ satisfy the following assumptions.
(N1) $\psi, \varphi \in C([0, \infty) \times [0, \infty), [0, \infty))$ are increasing for every variable and $\psi(\mu, \nu) > 0, \varphi(\mu, \nu) > 0$ for all $\mu, \nu > 0$;
(N2) for fixed constants $i, j \in (0, \infty)$, there exist $c_1, c_2 \in (0, \infty)$ such that
\[
\psi(t_1s_1, t_2s_2) \leq c_1 \psi(t_1, t_2) \psi(s_1, s_2),
\]
\[ \varphi(t_1s_1, t_3s_3) \leq c_2 \varphi(t_1, t_3) \varphi(s_1, s_3), \]

\[ \varphi(i, j) \geq \frac{\sqrt{3} - 1}{2} \text{ and } \varphi(i, j) \geq \frac{\sqrt{3} - 1}{2}, \]

where \( t_1 \geq \min \{ i, j, \varphi^{1/3}(j, i) \}, \ s_1 \geq \min \{ i, j, \varphi^{1/3}(j, i) \}, \ t_2 \geq \min \{ j, \varphi^{1/3}(i, i) \}, \ s_2 \geq \min \{ 1, j \varphi^{1/3}(i, i) \}, \ t_3 \geq \min \{ i, j \}, \ s_3 \geq 1; \]

(1) \( U_2(\infty) < F_1(\infty) < \infty, V_2(\infty) < F_2(\infty) < \infty; \)

(2) \( U_1(\infty) < \infty, V_1(\infty) < \infty; \)

(3) \( F_1(\infty) = F_2(\infty) = \infty, U_1(\infty) = V_2(\infty) = \infty; \)

(4) \( V_1(\infty) < \infty, V_2(\infty) < F_2(\infty) < \infty; \)

(5) \( F_2(\infty) = \infty, V_1(\infty) = \infty, V_2(\infty) = \infty; \)

(6) \( U_1(\infty) < \infty, U_2(\infty) < F_1(\infty) < \infty. \)

Lemma 1 ([18]). If \( \Lambda \in \theta, \) let \( \Im(s) = s \Lambda(s^{-2}). \) We have

(1) \( \Im(s) \) has a nonnegative increasing inverse mapping \( \Im^{-1}(s); \)

(2) If \( 0 < q < 1, \) we have

\[ \Im^{-1}(qs) \geq q^{1/\tau} \Im^{-1}(s); \]

(3) If \( q \geq 1, \) we have

\[ \Im^{-1}(qs) \leq q^{1/\tau} \Im^{-1}(s). \]

Through the similar proof as in [19], we can obtain the following Lemma.

Lemma 2. \((\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty)\) is a radial solution of the Schrödinger system (1) if and only if it is a solution of the following ordinary differential system

\[
\begin{cases}
(\Lambda(|\mu'|^{p-2})\mu')' + \frac{n-1}{\tau} \Lambda(|\mu'|^{p-2})\mu' = b(\tau)\psi(\mu, \nu), & \tau > 0, \\
(\Lambda(|\nu'|^{p-2})\nu')' + \frac{n-1}{\tau} \Lambda(|\nu'|^{p-2})\nu' = h(\tau)\varphi(\mu, \nu), & \tau > 0.
\end{cases}
\]  \( (6) \)

3. The Entire Positive Bounded Radial Solutions

In this section, we prove Theorems 1 and 2.

Theorem 1. Assume that (N1), (N2) hold, then the system (1) has an entire positive radial solution \((\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty).\)

Proof. Through an operation on system (6), we obtain

\[
\begin{cases}
(\Im(\mu'))' + \frac{n-1}{\tau} \Im(\mu') = b(\tau)\psi(\mu(\tau), \nu(\tau)), & \tau > 0, \\
(\Im(\nu'))' + \frac{n-1}{\tau} \Im(\nu') = h(\tau)\varphi(\mu(\tau), \nu(\tau)), & \tau > 0.
\end{cases}
\]

Obviously, the above system can be transformed into the following system

\[
\begin{cases}
\mu(\tau) = \mu(0) + \int_0^\tau \Im^{-1}\left(\frac{1}{\mu^{-1}-1}\int_0^t s^{\mu-1} b(s)\psi(\mu(s), \nu(s))ds\right)dt, & \tau \geq 0, \\
\nu(\tau) = \nu(0) + \int_0^\tau \Im^{-1}\left(\frac{1}{\mu^{-1}-1}\int_0^t s^{\mu-1} h(s)\varphi(\mu(s), \nu(s))ds\right)dt, & \tau \geq 0.
\end{cases}
\]
Define the sequences \( \{\mu_m(\tau)\}_{m \geq 0} \) and \( \{v_m(\tau)\}_{m \geq 0} \) on \([0, \infty)\) by
\[
\begin{align*}
\mu_0(\tau) &= \mu(0) = i, \quad v_0(\tau) = v(0) = j, \quad \tau \geq 0, \\
\mu_m(\tau) &= \mu(0) + \int_0^\tau \Im^{-1}\left( \frac{1}{\mu - \tau} \int_0^\tau s^{\mu-1} b(s) \psi(\mu_{m-1}(s), v_{m-1}(s)) ds \right) d\tau, \quad \tau \geq 0, \\
v_m(\tau) &= v(0) + \int_0^\tau \Im^{-1}\left( \frac{1}{\mu - \tau} \int_0^\tau s^{\mu-1} h(s) \phi(\mu_{m-1}(s), v_{m-1}(s)) ds \right) d\tau, \quad \tau \geq 0.
\end{align*}
\] (7)

Using the similar arguments as in [19], we obtain the sequences \( \{\mu_m(\tau)\}_{m \geq 0} \) and \( \{v_m(\tau)\}_{m \geq 0} \) are increasing and
\[
\frac{(\mu_m(\tau) + v_m(\tau))'}{(\psi + \phi)(\mu_m(\tau) + v_m(\tau), \mu_m(\tau) + v_m(\tau)) + 1} \leq G_1'(\tau) + G_2'(\tau).
\]

We then arrive at
\[
\int_{i+j}^{\mu_m(\tau) + v_m(\tau)} \frac{dt}{[(\psi + \phi)(t, t) + 1]^\frac{1}{p-1}} \leq G(\tau).
\]

Therefore,
\[
L(\mu_m(\tau) + v_m(\tau)) \leq G(\tau).
\]

By (N1), we can obtain that \( L'(\tau) > 0 \) and \( L(\tau) \) is a bijection. Clearly, the inverse function \( L^{-1} \) is strictly increasing on \([0, L(\infty))\) and
\[
\mu_m(\tau) + v_m(\tau) \leq L^{-1}(G(\tau)).
\] (8)

By Lemma 1, (N1), (N2), (7) and (8), the monotonicity of \( \{\mu_m(\tau)\}_{m \geq 0} \) and \( \{v_m(\tau)\}_{m \geq 0} \), we obtain
\[
\mu_m(\tau) \leq i + \int_0^\tau \Im^{-1}\left( \frac{1}{\mu - \tau} \int_0^\tau s^{\mu-1} b(s) \psi(\mu_m(s), v_m(s)) ds \right) d\tau
\]
\[
\leq i + \left( \psi(\mu_m(\tau), v_m(\tau)) + 1 \right)^{\frac{1}{p-1}} \int_0^\tau \Im^{-1}\left( \frac{1}{\mu - \tau} \int_0^\tau s^{\mu-1} b(s) ds \right) d\tau
\]
\[
\leq i + \left( \psi(v_m(\tau) + L^{-1}(G(\tau)), v_m(\tau)) + 1 \right)^{\frac{1}{p-1}} G_1(\tau)
\]
\[
= i + \left( \psi(v_m(\tau)(1 + \frac{L^{-1}(G(\tau))}{v_m(\tau)}), v_m(\tau)) + 1 \right)^{\frac{1}{p-1}} G_1(\tau)
\]
\[
\leq i + \left( \psi(v_m(\tau)(1 + \frac{L^{-1}(G(\tau))}{v_m(\tau)}), v_m(\tau)) + 1 \right)^{\frac{1}{p-1}} G_1(\tau)
\]
\[
\leq i + \left( \psi(1 + \frac{L^{-1}(G(\tau))}{v_m(\tau)}), v_m(\tau)) + 1 \right)^{\frac{1}{p-1}} G_1(\tau)
\]
\[
= \left( \psi(v_m(\tau), v_m(\tau)) \right)^{\frac{1}{p-1}} + \left( \psi(v_m(\tau), v_m(\tau)) \right)^{\frac{1}{p-1}} G_1(\tau)
\]
\[
= \left( \psi(v_m(\tau), v_m(\tau)) \right)^{\frac{1}{p-1}} + \left( \psi(v_m(\tau), v_m(\tau)) \right)^{\frac{1}{p-1}} G_1(\tau)
\]
\[
\leq \left( \psi(v_m(\tau), v_m(\tau)) \right)^{\frac{1}{p-1}} + \left( \psi(v_m(\tau), v_m(\tau)) \right)^{\frac{1}{p-1}} G_1(\tau)
\]
\[
\leq \left( \psi(v_m(\tau), v_m(\tau)) \right)^{\frac{1}{p-1}} + \left( \psi(v_m(\tau), v_m(\tau)) \right)^{\frac{1}{p-1}} G_1(\tau)
\]
and
\[
\begin{align*}
\nu_m(\tau) &\leq j + \int_0^\tau \Im^{-1} \left( \frac{1}{\mu-1} \int_0^t s^{\mu-1} h(s) \varphi(\mu_m(s), \nu_m(s)) \, ds \right) \, dt \\
&\leq j + \left( \varphi(\mu_m(\tau), \nu_m(\tau)) + 1 \right) \frac{1}{\mu-1} \int_0^\tau \Im^{-1} \left( \frac{1}{\mu-1} \int_0^t s^{\mu-1} h(s) \, ds \right) \, dt \\
&\leq j + \left( \varphi(\mu_m(\tau), \mu_m(\tau) + L^{-1}(G(\tau))) + 1 \right) \frac{1}{\mu-1} G_2(\tau) \\
&= j + \left( \varphi(\mu_m(\tau), \mu_m(\tau)(1 + \frac{L^{-1}(G(\tau))}{\mu_m(\tau)}) + 1 \right) \frac{1}{\mu-1} G_2(\tau) \\
&\leq j + \left( \varphi(\mu_m(\tau), \mu_m(\tau)(1 + \frac{L^{-1}(G(\tau))}{i}) + 1 \right) \frac{1}{\mu-1} G_2(\tau) \\
&\leq j + \left( c_2 \varphi(\mu_m(\tau), \mu_m(\tau)) \varphi(1, 1 + \frac{L^{-1}(G(\tau))}{i}) + 1 \right) \frac{1}{\mu-1} G_2(\tau) \\
&= \left( \varphi(\mu_m(\tau), \mu_m(\tau)) \right)^{\frac{1}{\mu-1}} \left( \frac{i}{\varphi(i, \mu_m(\tau))} + (c_2 \varphi(1, 1 + \frac{L^{-1}(G(\tau))}{i}) + 1 \frac{1}{\varphi(i, \mu_m(\tau))} \right) \frac{1}{\mu-1} G_2(\tau) \\
&\leq \left( \varphi(\mu_m(\tau), \mu_m(\tau)) \right)^{\frac{1}{\mu-1}} \left( \frac{i}{\varphi(i, \mu_m(\tau))} + (c_2 \varphi(1, 1 + \frac{L^{-1}(G(\tau))}{i}) + 1 \frac{1}{\varphi(i, \mu_m(\tau))} \right) \frac{1}{\mu-1} G_2(\tau).
\end{align*}
\]

By (N1), (N2), (9) and (10) and the monotonicity of \{\mu_m(\tau)\}_{m \geq 0} and \{\nu_m(\tau)\}_{m \geq 0}, we obtain

\[
\left( \Im(\mu_m(\tau)) \right)' + \frac{n-1}{\tau} \Im(\nu_m(\tau)) = b(\tau) \varphi(\mu_{m-1}(\tau), \nu_{m-1}(\tau)) \\
\leq b(\tau) \psi(\mu_m(\tau), \nu_m(\tau)) \\
\leq b(\tau) \varphi\left( \mu_m(\tau), \left( \varphi(\mu_m(\tau), \mu_m(\tau)) \right)^{\frac{1}{\mu-1}} \left( \frac{i}{\varphi(i, \mu_m(\tau))} + (c_2 \varphi(1, 1 + \frac{L^{-1}(G(\tau))}{i}) + 1 \frac{1}{\varphi(i, \mu_m(\tau))} \right) \frac{1}{\mu-1} G_2(\tau) \right) \\
\leq b(\tau) c_1 \varphi\left( \mu_m(\tau), \left( \varphi(\mu_m(\tau), \mu_m(\tau)) \right)^{\frac{1}{\mu-1}} \right) \omega_1(\tau)
\]

and

\[
\left( \Im(\nu_m(\tau)) \right)' + \frac{n-1}{\tau} \Im(\nu_m(\tau)) = b(\tau) \varphi(\mu_{m-1}(\tau), \nu_{m-1}(\tau)) \\
\leq b(\tau) \psi(\mu_m(\tau), \nu_m(\tau)) \\
\leq b(\tau) \varphi\left( \left( \varphi(\nu_m(\tau), \nu_m(\tau)) \right)^{\frac{1}{\psi(j, \nu_m(\tau))} \frac{i}{\psi(i, j) \nu_m(\tau)} + (c_2 \varphi(1 + \frac{L^{-1}(G(\tau))}{j}, 1) + 1 \frac{1}{\psi(i, j) \nu_m(\tau)} \right) \frac{1}{\mu-1} G_1(\tau), \nu_m(\tau) \right) \\
\leq b(\tau) c_2 \varphi\left( \left( \varphi(\nu_m(\tau), \nu_m(\tau)) \right)^{\frac{1}{\mu-1}}, \nu_m(\tau) \right) \omega_2(\tau).
\]
From the above inequalities, we obtain

\[
\left( \mathfrak{H}(\mu_m(\tau))' \right) \leq \left( \mathfrak{H}(\mu_m(\tau))' \right) + \frac{n-1}{\tau} \mathfrak{H}^{-1} \left( (\mu_m(\tau))' \right)
\]

\[
\leq b(\tau) c_1 \psi \left( \mu_m(\tau), \left( \varphi(\mu_m(\tau), \mu_m(\tau)) \right) \right) \omega_1(\tau)
\]

(11)

and

\[
\left( \mathfrak{H}(\nu_m(\tau))' \right) \leq \left( \mathfrak{H}(\nu_m(\tau))' \right) + \frac{n-1}{\tau} \mathfrak{H}^{-1} \left( (\nu_m(\tau))' \right)
\]

\[
\leq h(\tau) c_2 \varphi \left( \left( \varphi(\nu_m(\tau), \nu_m(\tau)) \right) \right) \omega_2(\tau).
\]

(12)

We then arrive at

\[
\mathfrak{H}(\mu_m(\tau))' \leq \int_0^T b(s) c_1 \psi \left( \mu_m(s), \left( \varphi(\mu_m(s), \mu_m(s)) \right) \right) \omega_1(s) ds
\]

(13)

and

\[
\mathfrak{H}(\nu_m(\tau))' \leq \int_0^T h(s) c_2 \varphi \left( \left( \varphi(\nu_m(s), \nu_m(s)) \right) \right) \omega_2(s) ds.
\]

(14)

By Lemma 1, (N1), (13) and (14), we obtain

\[
(\mu_m(\tau))' \leq \mathfrak{H}^{-1} \left( \int_0^T b(s) c_1 \psi \left( \mu_m(s), \left( \varphi(\mu_m(s), \mu_m(s)) \right) \right) \omega_1(s) ds \right)
\]

\[
\leq \mathfrak{H}^{-1} \left( c_1\omega_1(\tau) \int_0^T b(s) \varphi \left( \mu_m(s), \left( \varphi(\mu_m(s), \mu_m(s)) \right) \right) \omega_1(s) ds \right)
\]

\[
\leq \left( c_1\omega_1(\tau) + 1 \right) \mathfrak{H}^{-1} \left( \int_0^T b(s) \psi \left( \mu_m(s), \left( \varphi(\mu_m(s), \mu_m(s)) \right) \right) \omega_1(s) ds \right)
\]

(15)

and

\[
(\nu_m(\tau))' \leq \mathfrak{H}^{-1} \left( \int_0^T h(s) c_2 \varphi \left( \left( \varphi(\nu_m(s), \nu_m(s)) \right) \right) \omega_2(s) ds \right)
\]

\[
\leq \mathfrak{H}^{-1} \left( c_2\omega_2(\tau) \int_0^T h(s) \varphi \left( \left( \varphi(\nu_m(s), \nu_m(s)) \right) \right) \omega_2(s) ds \right)
\]

\[
\leq \left( c_2\omega_2(\tau) + 1 \right) \mathfrak{H}^{-1} \left( \int_0^T h(s) \psi \left( \left( \varphi(\nu_m(s), \nu_m(s)) \right) \right) \omega_2(s) ds \right)
\]

(16)

From the above two inequalities, we easily deduce that

\[
\frac{(\mu_m(\tau))'}{\left( \varphi(\mu_m(\tau), \left( \varphi(\mu_m(\tau), \mu_m(\tau)) \right) \right) \omega_1(\tau) + 1} \mathfrak{H}^{-1} \left( \int_0^T b(s) ds \right)
\]

\[
\leq \left( c_1\omega_1(\tau) + 1 \right) \mathfrak{H}^{-1} \left( \int_0^T b(s) ds \right)
\]

(17)
and
\[
\left(\frac{(v_m(\tau))'}{\left(\varphi((\psi(v_m(\tau), v_m(\tau)))^{1/n}, v_m(\tau)) + 1\right)^{1/m}}\right)^n -\leq \left(\varepsilon_2\psi(\tau) + 1\right)^{1/m-n} \left(\int_0^{\tau} h(s)ds\right).
\] (18)

We then arrive at
\[
\int_{i}^{\mu_m(\tau)} \frac{1}{(\psi((\varphi(t, (\varphi(t, t)))^{1/n}), t) + 1)^{1/m}} dt
\]
\[
\leq \int_0^{\tau} \left(\varepsilon_1\psi(\tau) + 1\right)^{1/m-n} \left(\int_0^{\tau} h(s)ds\right) dt
\]
and
\[
\int_{i}^{\nu_m(\tau)} \frac{1}{(\varphi((\psi(t, t)))^{1/n}, t) + 1)^{1/m}} dt
\]
\[
\leq \int_0^{\tau} \left(\varepsilon_2\psi(\tau) + 1\right)^{1/m-n} \left(\int_0^{\tau} h(s)ds\right) dt.
\]
Now the above two inequalities can be expressed as
\[
F_1(\mu_m(\tau)) \leq U_2(\tau), \quad \forall \tau \geq 0
\] (19)
and
\[
F_2(\nu_m(\tau)) \leq V_2(\tau), \quad \forall \tau \geq 0.
\] (20)

It follows from the (N1) that \(F_1^{-1}\) and \(F_2^{-1}\) are strictly increasing on \([0, F_1(\infty))\) and \([0, F_2(\infty))\) separately, we obtain
\[
\mu_m(\tau) \leq F_1^{-1}(U_2(\tau)), \quad \forall \tau \geq 0
\]
and
\[
\nu_m(\tau) \leq F_2^{-1}(V_2(\tau)), \quad \forall \tau \geq 0.
\]
Since
\[
(\mu_m(\tau))' \geq 0 \quad \text{and} \quad (\nu_m(\tau))' \geq 0, \quad \forall \tau \geq 0,
\]
we obtain
\[
\mu_m(\tau) \leq \mu_m(\tau_0) \leq W_1 \quad \text{and} \quad \nu_m(\tau) \leq \nu_m(\tau_0) \leq W_2, \quad \text{on} \ [0, \tau_0],
\]
where \(W_1 = F_1^{-1}(U_2(\tau_0)) \) and \(W_2 = F_2^{-1}(V_2(\tau_0)) \) are positive constants. Moreover, from (15) and (16), we can deduce that \{(\mu_m(\tau))'\} and \{(\nu_m(\tau))'\} are bounded on \([0, \tau_0]\) for arbitrary \(\tau_0 > 0\). Therefore, the monotone sequences \{\mu_m(\tau)\} and \{\nu_m(\tau)\} are bounded and equicontinuous on \([0, \tau_0]\). By employing the Arzela–Ascoli theorem, we obtain the subsequences of \{\mu_m(\tau)\} and \{\nu_m(\tau)\} uniformly converging towards \(\mu(\tau)\) and \(\nu(\tau)\) on \([0, \tau_0]\). According to the arbitrariness of \(\tau_0\), we obtain that \((\mu, \nu)\) is an entire positive solution of the system (6). Thus, from Lemma 2, we obtain that \((\mu, \nu)\) is an entire positive radial solution of the system (1). \(\Box\)
Theorem 2. Assuming that (N1), (N2), (S1) and (S2) hold, then the system (1) has an entire positive bounded radial solution \((\mu, \nu)\) such that

\[
\begin{aligned}
i + U_1(\tau) &\leq \mu(\tau) \leq F_1^{-1}(U_2(\tau)), \\
j + V_1(\tau) &\leq \nu(\tau) \leq F_2^{-1}(V_2(\tau)).
\end{aligned}
\]

Proof. On the basis of (N1) and (N2), by Theorem 1, we see that the system (1) has an entire positive radial solution \((\mu, \nu)\). Moreover, it follows from (19), (20) and (S1) that

\[
F_1(\mu_\ast(\tau)) \leq U_2(\infty) < F_1(\infty), \quad \forall \tau \geq 0
\]

and

\[
F_2(\nu_\ast(\tau)) \leq V_2(\infty) < F_2(\infty), \quad \forall \tau \geq 0.
\]

Since \(F_1^{-1}\) and \(F_2^{-1}\) are strictly increasing on \([0, F_1(\infty))\) and \([0, F_2(\infty))\) separately, we obtain

\[
\mu_\ast(\tau) \leq F_1^{-1}(U_2(\infty)) < \infty, \quad \forall \tau \geq 0
\]

and

\[
\nu_\ast(\tau) \leq F_2^{-1}(V_2(\infty)) < \infty, \quad \forall \tau \geq 0.
\]

Letting \(m \to \infty\) into the above two inequalities, we obtain

\[
\mu(\tau) \leq F_1^{-1}(U_2(\infty)) < \infty, \quad \forall \tau \geq 0
\]

and

\[
\nu(\tau) \leq F_2^{-1}(V_2(\infty)) < \infty, \quad \forall \tau \geq 0.
\]

Letting \(m \to \infty\) in (7), we obtain

\[
\mu(\tau) = i + \int_0^\tau \mathcal{A}^{-1} \left( \frac{1}{\eta^{\alpha-1}} \int_0^t s^{\alpha-1} b(s) \psi(\mu(s), \nu(s)) ds \right) dt
\]

and

\[
\nu(\tau) = j + \int_0^\tau \mathcal{A}^{-1} \left( \frac{1}{\eta^{\alpha-1}} \int_0^t s^{\alpha-1} h(s) \varphi(\mu(s), \nu(s)) ds \right) dt.
\]

Then, it follows from Lemma 1, (N1), (N2) and (S2) that

\[
\begin{aligned}
\mu(\tau) &= i + \int_0^\tau \mathcal{A}^{-1} \left( \frac{1}{\eta^{\alpha-1}} \int_0^t s^{\alpha-1} b(s) \psi(\mu(s), \nu(s)) ds \right) dt \\
&\geq i + \int_0^\tau \mathcal{A}^{-1} \left( \frac{1}{\eta^{\alpha-1}} \int_0^t s^{\alpha-1} b(t) \psi(i, j + \int_0^t \mathcal{A}^{-1} \left( \frac{1}{\eta^{\alpha-1}} \int_0^\tau s^{\alpha-1} h(s) \varphi(\mu(s), \nu(s)) ds \right) dt) \right) ds \\
&\geq i + \int_0^\tau \mathcal{A}^{-1} \left( \frac{1}{\eta^{\alpha-1}} \int_0^t s^{\alpha-1} b(t) \psi(i, j + \frac{1}{\phi(i,j) + 1}) \mathcal{A}^{-1} G_2(t) \right) dt \\
&= i + U_1(\tau).
\end{aligned}
\]

As with the above proof, we can prove that

\[
\nu(\tau) \geq j + V_1(\tau).
\]
4. The Entire Positive Blow-Up Radial Solutions

In this section, we prove Theorem 3.

**Theorem 3.** Assume that (N1), (N2), (S3) and (S4) hold, then the system (1) has an entire positive blow-up radial solution $(\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty)$.

**Proof.** On the basis of (N1), (N2), by Theorem 1, we see that the system (1) has an entire positive radial solution $(\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty)$. Moreover, it follows from (19) and (20) that

$$F_1(\mu_m(\tau)) \leq U_2(\infty), \quad \forall \tau \geq 0$$

and

$$F_2(\nu_m(\tau)) \leq V_2(\infty), \quad \forall \tau \geq 0.$$ 

Since $F_1^{-1}$ and $F_2^{-1}$ are strictly increasing on $[0, F_1(\infty))$ and $[0, F_2(\infty))$ separately, we arrive at

$$\mu_m(\tau) \leq F_1^{-1}(U_2(\infty)), \quad \forall \tau \geq 0$$

and

$$\nu_m(\tau) \leq F_2^{-1}(V_2(\infty)), \quad \forall \tau \geq 0.$$ 

When (S3) holds, we see that $F_1^{-1}(\infty) = F_2^{-1}(\infty) = \infty$. Letting $m \to \infty$ into the above two inequalities, we have

$$\mu(\tau) \leq F_1^{-1}(U_2(\infty)), \quad \forall \tau \geq 0$$

and

$$\nu(\tau) \leq F_2^{-1}(V_2(\infty)), \quad \forall \tau \geq 0.$$ 

By condition (S3), letting $\tau \to \infty$ into the above two inequalities, we obtain

$$\lim_{\tau \to \infty} \mu(\tau) \leq F_1^{-1}(U_2(\infty)) = \infty, \quad \forall \tau \geq 0$$

and

$$\lim_{\tau \to \infty} \nu(\tau) \leq F_2^{-1}(V_2(\infty)) = \infty, \quad \forall \tau \geq 0.$$ 

Then, it follows from (S4), (23) and (24) that

$$\lim_{\tau \to \infty} \mu(\tau) \geq i + \lim_{\tau \to \infty} U_1(\tau) > U_1(\infty) = \infty$$

and

$$\lim_{\tau \to \infty} \nu(\tau) \geq j + \lim_{\tau \to \infty} V_1(\tau) > V_1(\infty) = \infty.$$ 

Consequently, $\lim_{\tau \to \infty} \mu(\tau) = \infty$ and $\lim_{\tau \to \infty} \nu(\tau) = \infty$, which imply that the system (6) has an entire positive blow-up solution $(\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty)$. From Lemma 2, the system (1) has an entire positive blow-up radial solution $(\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty)$.
5. The Semifinite Entire Positive Blow-Up Radial Solutions

In this section, we prove Theorems 4 and 5.

**Theorem 4.** Assuming that \( (N1), (N2), (S5) \) and \( (S6) \) hold, the system (1) then has a semifinite entire positive blow-up radial solution \( (\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty) \).

**Proof.** In view of \( (N1), (N2) \), by Theorem 1, we see that system (1) has an entire positive radial solution \( (\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty) \). By \( (S5) \), \( (25) \) and \( (27) \), we obtain

\[
\lim_{\tau \to \infty} \mu(\tau) \leq F_1^{-1}(U_2(\infty)) = \infty
\]

and

\[
\lim_{\tau \to \infty} \nu(\tau) \geq i + \lim_{\tau \to \infty} U_1(\tau) > U_1(\infty) = \infty,
\]

which imply that

\[
\lim_{\tau \to \infty} \mu(\tau) = \infty.
\]

Moreover, by \( (S6), (22) \) and \( (24) \), we obtain

\[
\lim_{\tau \to \infty} \nu(\tau) \leq F_2^{-1}(V_2(\infty)) < \infty
\]

and

\[
\lim_{\tau \to \infty} \nu(\tau) \geq j + \lim_{\tau \to \infty} V_1(\tau) > V_1(\infty), \quad V_1(\infty) < \infty,
\]

which imply that

\[
\lim_{\tau \to \infty} \nu(\tau) < \infty.
\]

Therefore, system (6) has a semifinite entire positive blow-up solution \( (\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty) \). From Lemma 2, the system (1) has a semifinite entire positive blow-up radial solution \( (\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty) \). \( \square \)

**Theorem 5.** Assume that \( (N1), (N2), (S7) \) and \( (S8) \) hold, then the system (1) has a semifinite entire positive blow-up radial solution \( (\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty) \).

**Proof.** In view of \( (N1), (N2) \), by Theorem 1, we see that system (1) has an entire positive radial solution \( (\mu, \nu) \in C^2[0, \infty) \times C^2[0, \infty) \). By \( (S7), (26) \) and \( (28) \), we obtain

\[
\lim_{\tau \to \infty} \nu(\tau) \leq F_2^{-1}(V_2(\infty)) = \infty
\]

and

\[
\lim_{\tau \to \infty} \nu(\tau) \geq j + \lim_{\tau \to \infty} V_1(\tau) > V_1(\infty) = \infty,
\]

which imply that

\[
\lim_{\tau \to \infty} \nu(\tau) = \infty.
\]

Moreover, by \( (S8), (21) \) and \( (23) \), we obtain

\[
\lim_{\tau \to \infty} \mu(\tau) \leq F_1^{-1}(U_2(\infty)) < \infty
\]

and

\[
\lim_{\tau \to \infty} \mu(\tau) \geq i + \lim_{\tau \to \infty} U_1(\tau) > U_1(\infty), \quad U_1(\infty) < \infty,
\]

which imply that

\[
\lim_{\tau \to \infty} \mu(\tau) < \infty.
\]
Therefore, system (6) has a semifinite entire positive blow-up solution \((\mu, \nu) \in C^2([0, \infty) \times C^2([0, \infty)\). From Lemma 2, system (1) has a semifinite entire positive blow-up radial solution \((\mu, \nu) \in C^2([0, \infty) \times C^2([0, \infty)\). \(\square\)

6. Example

Example 1. Consider the following Schrödinger system

\[
\begin{align*}
\text{div}(\Lambda(|\nabla \mu|^3)\nabla \mu) &= \frac{3}{4} |\chi|^{-\frac{1}{2}} \nu^3, \quad \chi \in \mathbb{R}^6, \\
\text{div}(\Lambda(|\nabla \nu|^3)\nabla \nu) &= \frac{1}{4} |\nu|^{-\frac{1}{2}} \mu^3 \nu^3, \quad \chi \in \mathbb{R}^6.
\end{align*}
\]

(29)

Let \(\Lambda(s) = s^5\), \(p = 7\), then \(\Lambda \in \theta\). Here \(b(s) = \frac{3-s}{9s^2}\), \(h(s) = \frac{3-\omega}{3s^2}\), \(\psi(\mu, \nu) = \frac{3}{4} \mu^2 \nu^2\), \(\varphi(\mu, \nu) = \frac{1}{4} \mu^3 \nu^3\), then \(\psi\) and \(\varphi\) are increasing for each variable and (N1) holds. Obviously, when \(i = j = 4\), we have \(t_1 \geq \sqrt{3} s_1 \geq 1, t_2 \geq s_2 \geq 1, t_3 \geq s_3 \geq 1\),

\[
\begin{align*}
\psi(t_1 s_1, t_2 s_2) &= \frac{3}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \leq c_1 \frac{3}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \leq c_1 \psi(t_1, t_2) \psi(s_1, s_2), \quad \forall c_1 \geq \frac{4}{3}, \\
\varphi(t_1 s_1, t_2 s_2) &= \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} s_1 s_2 \leq c_2 \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \leq c_2 \varphi(t_1, t_2) \varphi(s_1, s_2), \quad \forall c_2 \geq 4,
\end{align*}
\]

\[
\varphi(i, j) \geq \sqrt{\frac{5-1}{2}} \quad \text{and} \quad \varphi(i, j) \geq \sqrt{\frac{5-1}{2}},
\]

meaning that (N2) is established. From Theorem 1, the Schrödinger system (29) has an entire positive radial solution \((\mu, \nu) \in C^2([0, \infty) \times C^2([0, \infty)\).

Example 2. Consider the following Schrödinger system

\[
\begin{align*}
\text{div}(\Lambda(|\nabla \mu|^3)\nabla \mu) &= |\chi|^{3} (\mu^4 + \nu^3), \quad \chi \in \mathbb{R}^4, \\
\text{div}(\Lambda(|\nabla \nu|^3)\nabla \nu) &= (3|\chi|^{-1} e|\chi| + e|\chi|^3) \mu \nu^3, \quad \chi \in \mathbb{R}^4.
\end{align*}
\]

(30)

Let \(\Lambda(s) = s^5\), \(p = 5\), then \(\Lambda \in \theta\). Here, \(b(s) = s^3\), \(h(s) = 3 s^{-1} e^s + e^s\), \(\psi(\mu, \nu) = \mu^4 + \nu^3\), \(\varphi(\mu, \nu) = \mu \nu^3\), then \(\varphi\) and \(\varphi\) are increasing for each variable and (N1) holds. Obviously, when \(i = j = 1\), we have \(t_1 \geq s_1 \geq \frac{1}{\sqrt{2}} t_2 \geq 1, s_2 \geq 1, t_3 \geq s_3 \geq 1\),

\[
\begin{align*}
\psi(t_1 s_1, t_2 s_2) &= \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} s_1 s_2 \leq c_1 \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \leq c_1 \psi(t_1, t_2) \psi(s_1, s_2), \quad \forall c_1 \geq 1,
\end{align*}
\]

\[
\varphi(t_1 s_1, t_2 s_2) = \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \leq c_2 \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \frac{1}{4} \leq c_2 \varphi(t_1, t_2) \varphi(s_1, s_2), \quad \forall c_2 \geq 1,
\]

\[
\varphi(i, j) \geq \sqrt{\frac{5-1}{2}} \quad \text{and} \quad \varphi(i, j) \geq \sqrt{\frac{5-1}{2}},
\]

meaning that (N2) is established. After a simple calculation, one has

\[
\begin{align*}
U_2(\infty) &= \int_0^\infty \left( c_1 \omega_1(t) + 1 \right) \frac{1}{2} t^{-3} \left( \int_0^t b(s) ds \right) dt > \int_0^\infty \sqrt{\frac{1}{4}} t^4 dt \\
&= \int_0^\infty \frac{1}{4} t^2 dt = \infty, \\
V_2(\infty) &= \int_0^\infty \left( c_2 \omega_2(t) + 1 \right) \frac{1}{2} t^{-3} \left( \int_0^t h(s) ds \right) dt > \int_0^\infty \sqrt{2} e^t dt \\
&= \int_0^\infty e^t dt = \infty, \\
F_1(\infty) &= \int_i^\infty \frac{1}{\left( \psi(t, \varphi(t, t)) \right)^{\frac{1}{2}}} dt = \int_i^\infty \frac{1}{\sqrt{t^4 + t^3 + 1}} dt = \infty.
\end{align*}
\]
and
\[ F_2(\infty) = \int_j^\infty \frac{1}{\varphi((\varphi(t,t))^{1/2},t)} dt = \int_j^\infty \frac{1}{\sqrt{(\mu^4 + \nu^4)^{1/2}} dr} dt = \infty, \]
meaning that (S3) is established. We then have
\[ G_1(\tau) = \int_0^\tau 3^{-1} \left( \frac{1}{\tau^2} \int_0^\tau s^3 b(s) ds \right) dt = \int_0^\tau \left( \frac{1}{\tau^2} \int_0^\tau s^6 ds \right) \frac{1}{\tau} dt = \frac{10}{7} \int_0^\tau \frac{r^2}{\tau^2} dr = \frac{5}{7} \int_0^\tau r^2 dr, \]
\[ G_2(\tau) = \int_0^\tau 3^{-1} \left( \frac{1}{\tau^3} \int_0^\tau s^3 h(s) ds \right) dt = \int_0^\tau \left( \frac{1}{\tau^3} \int_0^\tau s^2 \psi(3 + s) ds \right) \frac{1}{\tau} dt = \int_0^\tau e^{\frac{3}{\tau}} dt = 10 e^{\frac{3}{\tau}}, \]
\[ U_1(\infty) = \int_0^\infty 3^{-1} \left( \frac{1}{\varphi^3} \int_0^\infty \tau^3 b(t) \varphi(i,j + \frac{1}{\varphi(i,j)} \tau^{1/2} G_2(t)) dt \right) \varphi \]
\[ = \int_0^\infty 3^{-1} \left( \frac{1}{\varphi^3} \int_0^\infty \tau^3 b(t) \varphi(i,j + \frac{1}{\varphi(i,j)} \tau^{1/2} G_2(t)) dt \right) \varphi \]
\[ = \frac{1}{2+1} \frac{1}{\tau} G_1(\infty) = \infty, \]
and
\[ V_1(\infty) = \int_0^\infty 3^{-1} \left( \frac{1}{\varphi^3} \int_0^\infty \tau^3 h(t) \varphi(i,j + \frac{1}{\varphi(i,j)} \tau^{1/2} G_2(t)) dt \right) \varphi \]
\[ = \int_0^\infty 3^{-1} \left( \frac{1}{\varphi^3} \int_0^\infty \tau^3 h(t) \varphi(i,j + \frac{1}{\varphi(i,j)} \tau^{1/2} G_2(t)) dt \right) \varphi \]
\[ = \frac{1}{2+1} \frac{1}{\tau} G_2(\infty) = \infty, \]
meaning that (S4) is established. From Theorem 3, the Schrödinger system (30) has an entire positive blow-up radial solution \((u,v) \in C^2[0,\infty) \times C^2[0,\infty)\).

**Example 3.** Consider the following Schrödinger system
\[
\left\{
\begin{array}{ll}
\text{div}(\Lambda(|\nabla \mu|^{p}) \nabla \mu) = |\chi|^2 (\mu^4 + \nu), & \chi \in \mathbb{R}^5, \\
\text{div}(\Lambda(|\nabla v|^{q}) \nabla v) = (6|\chi|^{-1} e^{5|\chi|} + 2 e^{5|\chi|}) \mu^2 v^2, & \chi \in \mathbb{R}^5.
\end{array}
\right.
\]
Let \(\Lambda(s) = s^4, p = 6, s \in \theta.\) Here, \(b(s) = s^5, h(s) = 6s^{-1} e^s + 2 e^s, \varphi(u,v) = u^4 + v, \varphi(u,v) = \mu^2 v^2,\) then \(\varphi\) and \(\psi\) are increasing for each variable and \((N1)\) holds. Obviously, when \(i = j = 1,\) we have \(t_1 \geq 1, s_1 \geq \frac{1}{\sqrt{2}}, t_2 \geq 1, s_2 \geq 1, t_3 \geq 1, s_3 \geq 1,\)
\[ \varphi(t_1 s_1, t_2 s_2) = t_1^4 s_1^4 + t_2 s_2 \leq c_1 (t_1^4 + t_2) (s_1^4 + s_2) = c_1 \psi(t_1, t_2) \psi(s_1, s_2), \quad \forall c_1 \geq 1, \]
\[ \varphi(t_1 s_1, t_3 s_3) = t_1^2 s_1^2 s_3^2 \leq c_2 t_1^2 s_1^2 s_3^2 = c_2 \psi(t_1, t_3) \psi(s_1, s_3), \quad \forall c_2 \geq 1, \]
\[ \psi(i, j) \geq \frac{\sqrt{3} - 1}{2} \quad \text{and} \quad \varphi(i, j) \geq \frac{\sqrt{3} - 1}{2}, \]

meaning that (N2) is established. After a simple calculation, one has

\[ U_2(\infty) = \int_0^\infty \left( c_1 \omega_1(t) + 1 \right) \frac{1}{t} \Xi^{-1} \left( \int_0^t b(s)ds \right) dt > \int_0^\infty \frac{1}{\sqrt{6}} \frac{1}{t} dt = \infty, \]

\[ V_2(\infty) = \int_0^\infty \left( c_2 \omega_2(t) + 1 \right) \frac{t}{2} \Xi^{-1} \left( \int_0^t h(s)ds \right) dt > \int_0^\infty \frac{\sqrt{6}}{\sqrt{23}} \frac{1}{t} dt = \infty, \]

\[ F_1(\infty) = \int_i^\infty \frac{1}{\left( \psi(t, (\varphi(t, t))^\frac{1}{2}, t) + 1 \right)^\frac{1}{2}} dt = \int_i^\infty \frac{1}{\sqrt{t^2 + t + 1}} dt = \infty \]

and

\[ F_2(\infty) = \int_j^\infty \frac{1}{\left( \varphi((\psi(t, t))^\frac{1}{2}, t) + 1 \right)^\frac{1}{2}} dt = \int_j^\infty \frac{1}{\sqrt{t^2(t^2 + t)^\frac{1}{2} + 1}} dt = \infty, \]

meaning that (S3) is established. We then have

\[ G_1(\tau) = \int_0^\tau \Xi^{-1} \left( \frac{1}{\tau} \int_0^\tau s^2 b(s)ds \right) dt = \int_0^\tau \left( \frac{1}{13} \int_0^\tau s^6 ds \right)^\frac{1}{2} dt = \frac{17}{23} \frac{1}{\sqrt{9} \tau^\frac{23}{2}}, \]

\[ G_2(\tau) = \int_0^\tau \Xi^{-1} \left( \frac{1}{\tau} \int_0^\tau s^3 h(s)ds \right) dt = \int_0^\tau \left( \frac{1}{\tau} \int_0^\tau s^2 \varphi(6 + 2s)ds \right)^\frac{1}{2} dt = \frac{17}{23} \frac{1}{\sqrt{9} \tau^\frac{23}{2}}, \]

\[ U_1(\infty) = \int_0^\infty \Xi^{-1} \left( \int_0^\infty t^4 b(t) \psi \left(i, j + \frac{1}{\varphi(i, j) + 1} \right)^\frac{1}{p-1} G_2(t) dt \right) dq \]

\[ > \int_0^\infty \Xi^{-1} \left( \int_0^\infty t^4 b(t) \psi(i, j) dt \right) dq \]

\[ > \int_0^\infty \Xi^{-1} \left( \int_0^\infty t^4 b(t) \left( \frac{1}{\varphi(i, j) + 1} \right) dt \right) dq \]

\[ > \left( \frac{1}{\varphi(i, j) + 1} \right)^\frac{1}{p-1} \int_0^\infty \Xi^{-1} \left( \frac{1}{\varphi(i, j) + 1} \right)^\frac{1}{p-1} t^4 b(t) dt \right) dq \]

\[ = \left( \frac{1}{2 + 1} \right)^\frac{1}{2} G_1(\infty) = \infty \]

and

\[ V_1(\infty) = \int_0^\infty \Xi^{-1} \left( \frac{1}{\tau} \int_0^\tau t^4 h(t) \varphi \left(i + \frac{1}{\varphi(i, j) + 1} \right)^\frac{1}{p-1} G_1(t, j) dt \right) dq \]

\[ > \int_0^\infty \Xi^{-1} \left( \int_0^\tau t^4 h(t) \varphi(i, j) dt \right) dq \]

\[ > \int_0^\infty \Xi^{-1} \left( \int_0^\tau t^4 h(t) \left( \frac{1}{\varphi(i, j) + 1} \right) dt \right) dq \]

\[ > \left( \frac{1}{\varphi(i, j) + 1} \right)^\frac{1}{p-1} \int_0^\infty \Xi^{-1} \left( \frac{1}{\varphi(i, j) + 1} \right)^\frac{1}{p-1} t^4 h(t) dt \right) dq \]

\[ = \left( \frac{1}{1 + 1} \right)^\frac{1}{2} G_2(\infty) = \infty, \]
which mean that (S4) is established. From Theorem 3, the Schrödinger system (31) has an entire positive blow-up radial solution \((\mu, v) \in C^2[0, \infty) \times C^2[0, \infty)\).

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**References**

1. Osgood, W.F. Beweis der Existenz einer Lösung der Differentialgleichung \(\frac{du}{dx} = f(x, y)\) ohne Hinzunahme der Cauchy-Lipschitz'schen Bedingung. *Monatsh. Math. Phys.* **1898**, 9, 331–345. [CrossRef]

2. Papi, M. A generalized Osgood condition for viscosity solutions to fully nonlinear parabolic degenerate equations. *Adv. Differ. Equ.* **2002**, 7, 1125–1151.

3. Fan, S.; Jiang, L.; Davison, M. Existence and uniqueness result for multidimensional BSDEs with generators of Osgood type. *Front. Math. China* **2013**, 8, 811–824. [CrossRef]

4. Li, K. No local \(L^1\) solutions for semilinear fractional heat equations. *Fract. Calc. Appl. Anal.* **2017**, 20, 1328–1337. [CrossRef]

5. Villa-Morales, J. An Osgood condition for a semilinear reaction-diffusion equation with time-dependent generator. *Arab J. Math. Sci.* **2016**, 22, 86–95. [CrossRef]

6. Zhang, L.; Hou, W. Standing waves of nonlinear fractional \(p\)-Laplacian Schrödinger equation involving logarithmic nonlinearity. *Appl. Math. Lett.* **2020**, 102, 106149. [CrossRef]

7. Laister, R.; Robinson, J.C.; Sierżega, M. Non-existence of local solutions of semilinear heat equations of Osgood type in bounded domains. *Comptes Rendus Math.* **2014**, 352, 621–626. [CrossRef]

8. Laister, R.; Robinson, J.C.; Sierżega, M. A necessary and sufficient condition for uniqueness of the trivial solution in semilinear parabolic equations. *J. Differ. Equ.* **2017**, 262, 4979–4987. [CrossRef]

9. Fan, S. Existence, uniqueness and approximation for \(L^p\) solutions of reflected BSDEs with generators of one-sided Osgood type. *Acta Math. Sin.* **2017**, 33, 1–32. [CrossRef]

10. Kurihura, S. Large-amplitude quasi-solitons in superfluid films. *J. Phys. Soc. Jpn.* **1981**, 50, 3262–3267. [CrossRef]

11. Kosevich, A.; Ivanov, B.; Kovalev, A. Magnetic solitons. *Phys. Rep.* **1990**, 194, 117–238. [CrossRef]

12. Quispel, G.; Capel, H. Equation of motion for the Heisenberg spin chain. *Phys. Lett. A* **1982**, 110, 41–80. [CrossRef]

13. Severo, J. Quasilinear Schrödinger equations involving concave and convex nonlinearities. *Commun. Pure Appl. Anal.* **2009**, 8, 621–644. [CrossRef]

14. Miyagaki, O.H.; Soares, S.H. Soliton solutions for quasilinear Schrödinger equations with critical growth. *J. Differ. Equ.* **2010**, 248, 722–744.

15. Floer, A.; Weinstein, A. Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential. *J. Funct. Anal.* **1986**, 69, 397–408. [CrossRef]

16. Litvak, A.; Sergeev, A. One-dimensional collapse of plasma waves. *JETP Lett.* **1978**, 27, 517–520.

17. Sun, Y.; Liu, L.; Wu, Y. The existence and uniqueness of positive monotone solutions for a class of nonlinear Schrödinger equations on infinite domains. *J. Comput. Appl. Math.* **2017**, 321, 478–486. [CrossRef]

18. Zhang, X.; Wu, Y.; Cui, Y. Existence and nonexistence of blow-up solutions for a Schrödinger equation involving a nonlinear operator. *Appl. Math. Lett.* **2018**, 82, 85–91. [CrossRef]

19. Wang, G.; Yang, Z.; Agarwal, R.P.; Zhang, L. Study on a class of Schrödinger elliptic system involving a nonlinear operator. *Nonlinear Anal. Model. Control* **2020**, 25, 846–859. [CrossRef]

20. Zhang, Z.; Zhou, S. Existence of entire positive \(k\)-convex radial solutions to Hessian equations and systems with weights. *Appl. Math. Lett.* **2015**, 50, 48–55. [CrossRef]

21. Zhang, Z. Optimal global and boundary asymptotic behavior of large solutions to the Monge-Ampère equation. *J. Funct. Anal.* **2020**, 278, 108512. [CrossRef]

22. Qin, J.; Wang, G.; Zhang, L.; Ahmad, B. Monotone iterative method for a \(p\)-Laplacian boundary value problem with fractional conformable derivatives. *Bound. Value Probl.* **2019**, 2019, 145. [CrossRef]

23. Wang, G. Twin iterative positive solutions of fractional \(q\)-difference Schrödinger equations. *Appl. Math. Lett.* **2018**, 76, 103–109. [CrossRef]
24. Wang, G.; Ren, X.; Zhang, L.; Ahmad, B. Explicit iteration and unique positive solution for a Caputo-Hadamard fractional turbulent flow model. *IEEE Access* 2019, 7, 109833–109839. [CrossRef]
25. Pei, K.; Wang, G.; Sun, Y. Successive iterations and positive extremal solutions for a Hadamard type fractional integro-differential equations on infinite domain. *Appl. Math. Comput.* 2017, 312, 158–168. [CrossRef]
26. Zhang, L.; Ahmad, B.; Wang, G. Explicit iterations and extremal solutions for fractional differential equations with nonlinear integral boundary conditions. *Appl. Math. Comput.* 2015, 268, 388–392. [CrossRef]
27. Peterson, J.D.; Wood, A.W. Large solutions to non-monotone semilinear elliptic systems. *J. Math. Anal. Appl.* 2011, 384, 284–292. [CrossRef]
28. Zhang, X.; Liu, L. The existence and nonexistence of entire positive solutions of semilinear elliptic systems with gradient term. *J. Math. Anal. Appl.* 2010, 371, 300–308. [CrossRef]
29. Covei, D.-P. Radial and nonradial solutions for a semilinear elliptic system of Schrödinger type. *Funkcial. Ekvac.* 2011, 54, 439–449. [CrossRef]
30. Lair, A.V. A necessary and sufficient condition for the existence of large solutions to sublinear elliptic systems. *J. Math. Anal. Appl.* 2010, 365, 103–108. [CrossRef]
31. Lair, A.V.; Wood, A.W. Existence of entire large positive solutions of semilinear elliptic systems. *J. Differ. Equ.* 2000, 164, 380–394. [CrossRef]
32. Li, H.; Zhang, P.; Zhang, Z. A remark on the existence of entire positive solutions for a class of semilinear elliptic system. *J. Math. Anal. Appl.* 2010, 365, 338–341. [CrossRef]
33. Ghanmi, A.; Maagli, H.; Radulescu, V.; Zeddini, N. Large and bounded solutions for a class of nonlinear Schrödinger stationary systems. *Anal. Appl.* 2009, 7, 391–404. [CrossRef]
34. Covei, D.-P. The Keller-Osserman-type conditions for the study of a semilinear elliptic system. *Bound. Value Probl.* 2019, 2019, 104. [CrossRef]