CROSSED PRODUCTS AND CLEFT EXTENSIONS FOR COQUASI-HOPF ALGEBRAS

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Abstract. The notion of crossed product by a coquasi-bialgebra $H$ is introduced and studied. The resulting crossed product is an algebra in the monoidal category of right $H$-comodules. We give an interpretation of the crossed product as an action of a monoidal category. In particular, necessary and sufficient conditions for two crossed products to be equivalent are provided. Then, two structure theorems for coquasi Hopf modules are given. First, these are relative Hopf modules over the crossed product. Second, the category of coquasi-Hopf modules is trivial, namely equivalent to the category of modules over the starting associative algebra. In connection the crossed product, we recall from \cite{1} the notion of a cleft extension over a coquasi-Hopf algebra. A Morita context of Hom spaces is constructed in order to explain these extensions, which are shown to be equivalent with crossed product with invertible cocycle. At the end, we give a complete description of all cleft extensions by the non-trivial coquasi-Hopf algebras of dimension two and three.

1. Introduction

The notion of a crossed product by a bialgebra was first introduced by Sweedler, in his study on cohomology over bialgebras (\cite{2}). Later it was generalized and intensively studied in relation with the theory of Hopf-Galois algebra extensions (\cite{3}, \cite{4}, \cite{5}, \cite{6}). As it was noticed in \cite{7}, a crossed product by a bialgebra $H$ can be interpreted as an action of the monoidal category of comodules over the category of modules over a ring. Using this point of view, it is natural to try to define crossed product by a coquasi-bialgebra instead of a bialgebra. This is motivated by the fact that coquasi-bialgebras generalize bialgebras, preserving the monoidality of the category of comodules. One of the aims of this paper is to introduce the coquasi-algebraic version of crossed product by a bialgebra, including interpretations in terms of monoidal categories, and also to study some properties of such crossed products.

In the Hopf algebra theory, crossed products are the same as cleft extensions (\cite{6}, \cite{4}). Motivated by this correspondence, we also investigate the notion of a cleft extension over a coquasi-Hopf algebra (which was already introduced by the author in the previous paper \cite{1}).

The paper is organized as follows. It begins with a short review of the known results about coquasi-bialgebras and coquasi-Hopf algebras, their categories of comodules and about algebras and modules within these monoidal categories mentioned above. In Section 3.1, we shall see that given a coquasi-bialgebra $H$ and an associative algebra $R$, endowed with a weak left $H$-action, the existence of a $R$-valued 2-cocycle allows us to define a multiplication on $R \otimes H$, by the same formula as in the Hopf algebra situation. But the similarity stops here: the conditions that we have to impose on the cocycle are modified because of the reassociator $\omega$ of $H$. The resulting crossed product will no longer be an associative algebra, but a right $H$-comodule algebra $R^\#_{\sigma}H$. Also it is interesting to notice that the crossed product can be built on the base field $k$ if and only if the coquasi-bialgebra is a deformation of a bialgebra. An example of crossed product is provided by the associative algebra $R = H^*$, with regular left weak action and cocycle given by the reassociator $\omega$. Then $H^*\#_{\sigma}H$ can be interpreted as an analogue of the Heisenberg double for

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coquasi-bialgebras (the quasi-bialgebra case was studied by Panaite and Van Oystaeyen in [8]). This can be
generalized by taking \( R = \text{Hom}(H, A) \), for any \( A \) a right \( H \)-comodule algebra. Then there is an associative
multiplication on this space, which generalize Doi’s smash product and allows us to construct the crossed
product \( \text{Hom}(H, A) \#_\sigma H \). Another example is obtained in the finite dimensional case. Namely, we show
that giving an associative algebra \( R \) together with a weak action and a two cocycle such that relations (3.1),
(3.2), (3.3), (3.4) hold (a crossed system), is the same as giving a right \( H^\ast \)-comodule algebra (as it
was defined by Hausser and Nill in [9]). In particular, the crossed product in this case coincides with the
quasi-smash product from [10].

The next Section is devoted to find a categorical explanation of the conditions imposed on the cocycle
and weak action. Namely, the monoidal category of \( H \)-comodules (or bicomodules) acts on the category of
\( R \)-modules (or \( R \)-modules, \( H \)-comodules) by usual tensor product if the conditions for the crossed product
are fulfilled. Changing the monoidal category by twisting the coquasi-bialgebra implies changing the crossed
system. Two structures of crossed product on the same algebra \( R \) with same coquasi-bialgebra \( H \) (meaning
we change the action of the monoidal category) are equivalent if and only if the corresponding cocycles differ
by a coboundary.

In Section 3.3, the category of coquasi-Hopf modules \( (\mathcal{M}_R^H)_H \) is introduced and studied, again by monoidal
category arguments. Namely, the category of \( H \)-bicomodules \( \mathcal{M}_R^H \) acts on the category of right \( R \)-modules,
\( H \)-comodules \( \mathcal{M}_R^H \) by usual tensor product and \( H \) is an algebra in this monoidal category. Hence it makes
sense to construct right \( H \)-modules within \( \mathcal{M}_R^H \). These will be called right coquasi-Hopf modules. Now
the crossed product algebra comes in: the category of right coquasi-Hopf modules \( (\mathcal{M}_R^H)_H \) is isomorphic to
the category of relative Hopf modules over \( R \#_{\sigma} H \). It is interesting to notice that a similar category, but
for the finite dimensional dual case, was defined in [10] and called the category of two-sided Hopf modules.
This is isomorphic to our category of coquasi-Hopf modules, but the isomorphism seems to do not have a
monoidal category explanation (Remark 2). This Section ends with a structure theorem for the category
of coquasi-Hopf modules: we show that this category is trivial, if the coquasi-bialgebra is endowed with an
antipode and the cocycle is invertible: namely, there is a special projection on the coinvariants space, which
induces an equivalence with the category of right modules over the starting associative algebra \( R \). Combining
this with Theorem [19] it follows that Hopf modules over the crossed product algebra are trivial. In the Hopf
algebra case, this holds because crossed products with invertible cocycle are the same as cleft extensions. The
second main part of the paper is devoted to find a similar result in the context of coquasi-Hopf algebras. But
this requires an appropriate notion of cleft extension for coquasi-Hopf algebras. Given a right \( H \)-comodule
algebra \( A \), this is a cleft extension of the subalgebra of coinvariants \( B = A^{coH} \) if conditions (4.1)-(4.3)
hold. This definition was introduced in author’s previous paper [1] and is significantly different from that of
cleft Hopf algebra extensions. As this involves the convolution product (which is no longer associative), the
invertibility of the cleaving map has to be translated now in relations (4.2), (4.3) involving the antipode and
the linear maps \( \alpha, \beta \). We shall give an interpretation of these relations. Namely, a Morita context involving
four different \( \text{Hom} \) spaces is constructed, similar to the one used in [11] for the coring case. The strictness
of the context is deeply connected with the notion of Galois extension (as it was defined in [1]), and the
cleftness is equivalent to the existence of two elements in the connecting bimodules which are mapped by
the Morita homomorphisms to the units elements of the involved algebras, in particular the Morita context
is strict.

In a previous paper ([11]), we have shown the equivalence between cleft extensions and Galois extensions
with normal basis property. It is then natural to pursue the characterization of cleft extensions in terms of
crossed products with coquasi-Hopf algebras. We generalize in Section 4.2 the result of Doi and Takeuchi ([6]),
respectively of Blattner and Montgomery ([4]) about the equivalence between the two structures mentioned
previously. As an application of this, in the Appendix we give a full characterization of all cleft extensions
by certain coquasi-Hopf algebras, namely the unique non-trivial coquasi-Hopf algebras of dimension 2 and
3, as they were described in [12].
As we shall see, the theory of coquasi-Hopf algebras is technically more complicated than the classical Hopf algebra theory. This happens because of the appearance of the reassociator $\omega$ and of the elements $\alpha$ and $\beta$ in the definition of the antipode. All these things increase the complexity of formulas, and therefore of computations and proofs.

2. Preliminaries

In this Section we recall some definitions, results and fix notations. Throughout the paper we work over some base field $\mathbb{k}$. Tensor products, algebras, linear spaces, etc. will be over $\mathbb{k}$. Unadorned $\otimes$ means $\otimes_{\mathbb{k}}$. An introduction to the study of quasi-bialgebras and quasi-Hopf algebras and their duals (coquasi-bialgebras, respectively coquasi-Hopf algebras) can be found in [13]. A good reference for monoidal categories is [14], while actions of monoidal categories are exposed in [17].

**Definition 1.** A coquasi-bialgebra $(H, m, u, \omega, \Delta, \varepsilon)$ is a coassociative coalgebra $(H, \Delta, \varepsilon)$ together with coalgebra morphisms: the multiplication $m : H \otimes H \rightarrow H$ (denoted $m(h \otimes g) = hg$), the unit $u : \mathbb{k} \rightarrow H$ (denoted $u(1) = 1_H$), and a convolution invertible element $\omega \in (H \otimes H \otimes H)^*$ such that:

\[
\begin{align*}
(2.1) \quad h_1(g_1k_1)\omega(h_2g_2,k_2) &= \omega(h_1,g_1,k_1)(h_2g_2)k_2 \\
(2.2) \quad 1_H h &= h_1 1_H = h \\
(2.3) \quad \omega(h_1g_1k_1l_1)\omega(h_2g_2k_2l_2) &= \omega(g_1,k_1l_1)\omega(h_1,g_2k_2l_2)\omega(h_2,g_3,k_3) \\
(2.4) \quad \omega(h,1_H,g) &= \varepsilon(h)\varepsilon(g)
\end{align*}
\]

hold for all $h, g, k, l \in H$.

As a consequence, we have also $\omega(1_H, h, g) = \omega(h, g, 1_H) = \varepsilon(h)\varepsilon(g)$ for each $g, h \in H$.

**Definition 2.** A coquasi-Hopf algebra is a coquasi-bialgebra $H$ endowed with a coalgebra antihomomorphism $S : H \rightarrow H$ (the antipode) and with elements $\alpha, \beta \in H^*$ satisfying

\[
\begin{align*}
(2.5) \quad S(h_1)\alpha(h_2)h_3 &= \alpha(h)1_H \\
(2.6) \quad h_1\beta(h_2)S(h_3) &= \beta(h)1_H \\
(2.7) \quad \omega(h_1\beta(h_2), S(h_3), \alpha(h_4)h_5) &= \omega^{-1}(S(h_1), \alpha(h_2)h_3\beta(h_4), S(h_5)) = \varepsilon(h)
\end{align*}
\]

for all $h \in H$.

These relations imply also $S(1_H) = 1_H$ and $\alpha(1_H)\beta(1_H) = 1$, so by rescaling $\alpha$ and $\beta$, we may assume that $\alpha(1_H) = 1$ and $\beta(1_H) = 1$. The antipode is unique up to a convolution invertible element $U \in H^*$: if $(S', \alpha', \beta')$ is another triple with the above properties, then according to [13] we have

\[
\begin{align*}
(2.8) \quad S'(h) &= U(h_1)S(h_2)U^{-1}(h_3), \quad \alpha'(h) = U(h_1)\alpha(h_2), \quad \beta'(h) = \beta(h_1)U^{-1}(h_2)
\end{align*}
\]

for all $h \in H$.

We shall use in this paper the monoidal structure of the category of right (left) $H$-comodules and of the category of $H$-bicomodules: the tensor product is over the base field and the comodule structure (left or right) of the tensor product is the codiagonal one. The reassociators are

\[\Phi_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)\]

for $u \in U$, $v \in V$, $w \in W$ and $U, V, W \in \mathcal{M}^H$, respectively

\[\Phi_{U,V,W}((u \otimes v) \otimes w) = u_0 \otimes (v_0 \otimes w_0)\omega(u_1, v_1, w_1)\]

\[\Phi_{U,V,W}((u \otimes v) \otimes w) = \omega^{-1}(u_{-1}, v_{-1}, w_{-1})u_0 \otimes (v_0 \otimes w_0)\]
for $u \in U, v \in V, w \in W$ and $U, V, W \in \mathcal{H} \mathcal{M}$. For the category of $H$-bicomodules, one can obtain the reassociator by combining the above two, namely by multiplication to the left by $\omega^{-1}$, respectively to the right by $\omega$.

For $H$ a coquasi-bialgebra, the linear dual $H^* = \text{Hom}(H, k)$ becomes an associative algebra with multiplication given by the usual convolution product
\[(h^* g^*)(h) = h^*(h_1)g^*(h_2) \quad \forall h \in H \text{ and } h^*, g^* \in H^*\]
and unit $\varepsilon$. This algebra is acting on $H$ by the formulas:
\[(h \mapsto h^*)(g) = h^*(gh), \quad (h^* \mapsto h)(g) = h^*(hg)\]
for any $h^* \in H^*$, $g, h \in H$.

Even though $H$ is not an associative algebra, we keep the notation from the Hopf algebra case for the weak action of $H$ on $H^*$
\[(h \mapsto h^*)(g) = h^*(gh), \quad (h^* \mapsto h)(g) = h^*(hg)\]
for any $h^* \in H^*$, $g, h \in H$.

If $H$ is a finite dimensional coquasi-bialgebra, then it is easy to check that $H^*$ is a quasi-bialgebra with the induced dual operations and conversely, the linear dual of any finite dimensional quasi-bialgebra becomes a coquasi-bialgebra, which justifies some common notations and definitions. An immediate consequence is the identification between the category of right $H$-comodules $\mathcal{M}^H$ and the category of left $H^*$-modules $H^* \mathcal{M}$. A right $H$-comodule $V$ becomes a left $H^*$-module by $h^* v = h^*(v_1)v_0, \forall \ h^* \in H^*, \ v \in V$. Conversely, to any left $H^*$-module $V$ we may associate an $H$-coaction by $\rho_V(v) = \sum_{i=1}^{\dim H} e_i^* v \otimes e_i$, where again $(e_i)_{i=1, \dim H}$ and $(e_i^*)_{i=1, \dim H}$ are dual bases for $H$, respectively $H^*$.

Now, recall from [15] the following: for $\tau \in (H \otimes H)^*$ a convolution invertible map such that $\tau(1, h) = \tau(h, 1) = \varepsilon(h)$ for all $h \in H$ ($\tau$ is called a twist or a gauge transformation), one can define a new structure of coquasi-bialgebra (or coquasi-Hopf algebra) on $H$, denoted $H_\tau$, by taking
\[(h \mapsto h^*) = \tau(h_1, g_1)h_2g_2\tau^{-1}(h_3, g_3)\]
\[\omega_\tau(h, g, k) = \tau(g_1, k_1)\tau(h_1, g_2k_2)\omega(h_2, g_3, k_3)\tau^{-1}(h_3g_4, k_4)\tau^{-1}(h_4, g_5)\]
\[\alpha_\tau(h) = \tau^{-1}(S(h_1), \alpha(h_2)h_3)\]
\[\beta_\tau(h) = \tau(h_1, \beta(h_2), S(h_3))\]
for all $h, g, k \in H$, and keeping the unit, the comultiplication, the counit and the antipode unchanged.

**Remark 3.** There is a monoidal isomorphism $\mathcal{M}^H \simeq \mathcal{M}^{H^*}$, which is the identity on objects and on morphisms, with monoidal structure given by $V \otimes W \longrightarrow V \otimes W, \ v \otimes w \longrightarrow v_0 \otimes w_0 \tau(v_1, w_1)$, where $v \in V, w \in W$ and $V, W \in \mathcal{M}^H$.

We shall also need a particular twist $f \in (H \otimes H)^*$, which appears in [16] and controls how far is the antipode $S$ from a anti-algebra morphism:
\[f(h_1, g_1)S(h_2g_2) = S(g_1)S(h_1)f(h_2, g_2) \quad \text{for all } h, g \in H\]
We have also from [17] that
\[\beta(h_1g_1)f^{-1}(h_2, g_2) = \omega(h_1g_1, S(g_3), S(h_4))\omega^{-1}(h_2, g_2, S(g_4))\beta(h_3)\beta(g_3)\]

**Definition 4.** ([18]) A right comodule algebra $A$ over a coquasi-bialgebra $H$ is an algebra in the monoidal category $\mathcal{M}^H$. This means $(A, \rho_A)$ is a right $H$-comodule with a multiplication map $\mu_A : A \otimes A \longrightarrow A$, denoted $\mu_A(a \otimes b) = ab$, for $a, b \in A$, and a unit map $u_A : k \longrightarrow A$, where we put $u_A(1) = 1_A$, which are both $H$-colinear, such that
\[(ab)c = a_0(b_0c_0)\omega(a_1, b_1, c_1)\]
holds for any $a,b,c \in A$.

**Definition 5.** ([18]) For $H$ a coquasi-bialgebra and $A$ a right $H$-comodule algebra, we may define the notion of right module over $A$ in the category $M^H$. Explicitly, this is a right $H$-comodule $(M, \rho_M)$, endowed with a right $A$-action $\mu_M : M \otimes A \to M$, denoted $\mu_M(m,a) = ma$, such that

\[
\begin{align*}
(ma)b &= m_0(a_0b_0)\omega(m_1, a_1, b_1) \\
m_{1A} &= m \\
\rho_M(ma) &= m_0a_0 \otimes m_1a_1
\end{align*}
\]

hold for all $m \in M$, $a,b \in A$. The category of such objects, with morphisms the right $H$-colinear maps which respect the $A$-action, is called the category of relative right $(H,A)$-Hopf modules and denoted $M^H_A$.

**Remark 6.** It was proven in [18] that if $\tau$ is a twist on $H$, then the formula

\[
a \cdot \tau b = a_0b_0\tau^{-1}(a_1, b_1)
\]

for all $a,b \in A$ defines a new multiplication such that $A$, with this new multiplication (denoted $A^{\tau^{-1}}$) becomes a right $H_\tau$-comodule algebra. It is easy to see that the isomorphism of Remark 5 sends the algebra $A$ of the monoidal category $M^H$ exactly to the algebra $A^{\tau^{-1}}$ in $M^{H_\tau}$. Therefore the categories of right relative Hopf modules $M^H_A$ and $M^{H_\tau}_A^{\tau^{-1}}$ will also be isomorphic.

Let $A$ be a right $H$-comodule algebra. Consider the space of coinvariants

\[
B = A^{coH} = \{ a \in A | \rho_A(a) = a \otimes 1_H \}
\]

It is immediate that this is an associative $k$-algebra with unit and multiplication induced by the unit and the multiplication of $A$. There is a pair of adjoint functors which arises naturally, namely the induced and the coinvariant functors

\[
M^B = (-) \otimes_{B^A} M_A^H 
\]

where $N \otimes_B A$ is a relative Hopf module with action and coaction induced by $A$, and $M^{coH}$ becomes naturally a right $B$-module by restricting the scalars, for any $N \in M_B^A$ and $M \in M^{H_A}_A$. Notice also the natural isomorphism

\[
Hom^H_A(A, M) \simeq M^{coH}
\]

for any $M \in M^{H_A}_A$. Finally, we recall from [1] the notion of a Galois extension:

**Definition 7.** ([1]) Let $H$ a coquasi-Hopf algebra and $A$ a right $H$-comodule algebra with coinvariants $B = A^{coH}$. The extension $B \subseteq A$ is Galois if the map can : $A \otimes_B A \to A \otimes H$, given by

\[
a \otimes_B b \mapsto a_0b_0 \otimes b_4\omega^{-1}(a_1, b_1\beta(b_2), S(b_3))
\]

is bijective.

Although this definition implies the existence of the antipode, unlike the classical associative case, it is deeply connected with the above mentioned adjunction of functors, exactly as for Hopf algebras (see [1]).

**3. Crossed products by coquasi-bialgebras**

**3.1. Definition of a crossed product.** We start by developing a suitable theory of crossed products, generalizing that of [3] and [6]. Let $H$ be a coquasi-bialgebra and $R$ an associative algebra. On $R$ we consider the following structures:
It is obvious that (3.5) and (3.2) (r).

Definition 8. The crossed product algebra $R\otimes H$ is $R \otimes H$ as vector space with multiplication

\[(r\otimes H)(s\otimes g) = r(h_1 \cdot s)\sigma(h_2, g_1)\]

And the following Theorem explains what is this new structure:

Theorem 9. $R\otimes H$ is a right $H$-comodule algebra, with unit $1_R \otimes 1_H$ and coaction $\Delta$ if and only if the following relations are satisfied:

\[(1_H \cdot r) = r, \quad \sigma(h_1, g_1)\]

\[= \sigma(1_H, r) = \epsilon(h)1_R \quad \text{for all } h \in H \quad \text{and } r, s \in R;
\]

\[\text{a linear map } \sigma : H \otimes H \rightarrow R.\]

Proof. It is obvious that $R\otimes H$ becomes an $H$-comodule via $I_R \otimes \Delta$. Let’s check the coassociativity of the multiplication:

\[\rho_{l\otimes H}((r\otimes H)(s\otimes g)) = \rho_{l\otimes H}(r(h_1 \cdot s)\sigma(h_2, g_1)\otimes h_3g_2) = (r(h_1 \cdot s)\sigma(h_2, g_1)\otimes h_3g_2 \otimes h_4g_3) = (\sigma(r\otimes g_1)(s\otimes k) \otimes h_2g_2)\]

for any $r\otimes h, s\otimes g \in R\otimes H$. Next, if the above conditions are fulfilled, then

\[\frac{(r\otimes H)(s\otimes g)}{(t\otimes k)} = (r(h_1 \cdot s)\sigma(h_2, g_1)\otimes h_3g_2)\]

\[= r(h_1 \cdot s)\sigma(h_2, g_1)\sigma(h_3, g_2)\sigma(h_4, g_3, k_1)\]

\[= r(h_1 \cdot s)\sigma(h_2, g_1)\sigma(h_3, g_2)\sigma(h_4, g_3, k_1)\]

\[= r(h_1 \cdot s)\sigma(h_2, g_1)\sigma(h_3, g_2)\sigma(h_4, g_3, k_1)\]

\[= r(h_1 \cdot s)\sigma(h_2, g_1)\sigma(h_3, g_2)\sigma(h_4, g_3, k_1)\]

\[= r(h_1 \cdot s)\sigma(h_2, g_1)\sigma(h_3, g_2)\sigma(h_4, g_3, k_1)\]

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\[= r(h_1 \cdot s)\sigma(h_2, g_1)\sigma(h_3, g_2)\sigma(h_4, g_3, k_1)\]

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\[= r(h_1 \cdot s)\sigma(h_2, g_1)\sigma(h_3, g_2)\sigma(h_4, g_3, k_1)\]

\[= r(h_1 \cdot s)\sigma(h_2, g_1)\sigma(h_3, g_2)\sigma(h_4, g_3, k_1)\]

for all $r\otimes h, s\otimes g, t\otimes k \in R\otimes H$. Finally, we have

\[1_R \otimes 1_H \cdot r) = r(1_H \cdot r)\sigma(1_H, h_1)\otimes h_2\]

\[= r\otimes h, \quad (3.3), \quad (3.4)\]

\[= r\otimes h, \quad (3.5), \quad (3.6)\]

so we obtain an algebra in the monoidal category of right $H$-comodules. Conversely, $(1_R \otimes h)(r\otimes 1_H) = r\otimes 1_H$ gives $1_H \cdot r = r$. Also, $(1_R \otimes h)(1_R \otimes h) = 1_R \otimes h$ implies $\sigma(1_H, h_1)\otimes h_2 = 1_R \otimes h$ and applying...
ε on the second component gives us σ(1_H, l) = ε(l). Similarly, (1_R#_σ h)(1_R#_σ 1_H) = 1_R#_σ h implies σ(h, 1_H) = ε(h). For the last identity, write down the associativity of the crossed product algebra in the monoidal category of comodules, and compute the product [(1_R#_σ h)(1_R#_σ g)](1_R#_σ l) in the two possible ways. At the end, apply ε on the second tensorand. Finally, in order to obtain relation (3.4), repeat this procedure for the product [(1_R#_σ h)(1_R#_σ g)](R#_1_H).

**Remark 10.**

1. If H is a bialgebra, then we recover the usual definition of the crossed product of an algebra by the bialgebra H. Therefore all known examples for the associative case fit in our picture.
2. If the cocycle σ is trivial (i.e. σ(h, g) = ε(h)ε(g)1_R, for h, g ∈ H), then by relation (3.6) it follows that ω is also trivial. Hence H is a bialgebra and R is a left H-module algebra. The result is the usual smash product R#H.
3. For the trivial weak action (i.e. h · r = ε(h)r, where h ∈ H, r ∈ R), relation (3.4) implies Im σ ⊆ Z(R) and by (3.6) we have

\[\sigma(g_1, l_1)σ(h, g_2 l_2) = σ(h_1, g_1)σ(h_2 g_2, l_1)ω^{-1}(h_3, g_3, l_2)\]

for all h, g, l ∈ H.
4. If R = k, then the weak action must be trivial, according to (3.7) and σ is a twist on H. Hence by (3.7) it follows that H is a deformation of a bialgebra by the twist σ. Therefore there are no crossed products of the base field by a nontrivial coquasi-bialgebra.

Before ending this Section, we shall notice the following relation, which will be used later on:

**Proposition 11.** If σ is convolution invertible and the relations (3.1), (3.3), (3.4), (3.5), (3.6) are satisfied, then

\[h · σ^{-1}(g, l) = σ(h_1, g_1 l_1)ω(h_2, g_2, l_2)σ^{-1}(h_3 g_3, l_1)σ^{-1}(h_4, g_4)\]

for all h, g, l ∈ H.

**Proof.** By (3.1), it follows that the map H ⊗ H ⊗ H → R, (h, g, l) → h · σ(g, l) is convolution invertible, with convolution inverse \((h, g, l) → h · σ^{-1}(g, l)\). As σ is invertible, relation (3.6) implies

\[h · σ(g, l) = σ(h_1, g_1)σ(h_2 g_2, l_1)ω^{-1}(h_3, g_3, l_2)σ^{-1}(h_4, g_4 l_3)\]

But the map \((h, g, l) → σ(h_1, g_1)σ(h_2 g_2, l_1)ω^{-1}(h_3, g_3, l_2)σ^{-1}(h_4, g_4 l_3)\) is easily checked to be convolution invertible, with inverse \((h, g, l) → σ(h_1, g_1 l_1)ω(h_2, g_2, l_2)σ^{-1}(h_3 g_3, l_1)σ^{-1}(h_4, g_4)\). By the uniqueness of the inverse of an element in the convolution algebra Hom(H ⊗ H ⊗ H, R) we get the desired formula. □

### 3.2. Examples of crossed products

We shall now provide some examples of crossed products by coquasi-bialgebras.

1. By Remark 10.(4), if we start with a bialgebra H and a twist τ : H ⊗ H → k, then we may form the crossed product k#_τ H of the base field by the deformed coquasi-bialgebra H_τ, having as cocycle the convolution inverse of the twist. We apply now this construction to the group algebra of a finite group G, in particular to C[G]. It follows from [11] that all Cayley and Clifford algebras can be obtained in this way, as crossed products by some coquasi-bialgebras.

2. Take H a coquasi-bialgebra. Then H^* is an associative algebra with multiplication given by (2.4). The formula (2.11) defines a weak action of H on H^*. It is easy to check that (3.1) holds. Define now σ : H ⊗ H → H^*, by σ(h, g)(k) = ω^{-1}(k, h, g). Then all the relations (3.1), (3.3), (3.4), (3.6), (3.9) hold. We obtain thus the crossed product H^#_σ H, which in Hopf algebra case reduces to the Heisenberg double.

3. The previous example can be generalized as follows: let H be a coquasi-bialgebra and A a right H-comodule algebra. On the vector space Hom(H, A) we define the following multiplication:

\[\langle ϕ ⊙ ψ⟩(h) = ϕ(ψ(h_3) h_2) ω^{-1}(ϕ(ψ(h_3) h_2)_1, ψ(h_3)_1, h_1)\]
for all \( \varphi, \psi \in \text{Hom}(H, A) \) and \( h \in H \). Then \((\text{Hom}(H, A), \otimes)\) becomes an associative algebra with unit \( e_1 \):

\[
((\varphi \otimes \psi) \circ \lambda)(h) = (\varphi \otimes \psi)(\lambda(h_3)h_2)\lambda(h_3)\omega^{-1}((\varphi \otimes \psi)(\lambda(h_3)h_2)_0, \lambda(h_3)_1, h_1)
\]

\[
= [\varphi(\psi(\lambda(h_4)_4)3(\lambda(h_5)_3h_3))_0\psi(\lambda(h_5)_4h_4)_0\lambda(h_5)_0
\]

\[
\omega^{-1}(\varphi(\psi(\lambda(h_4)_4)3(\lambda(h_5)_3h_3))_2, \psi(\lambda(h_5)_4h_4)_2, \lambda(h_5)_2h_2)
\]

\[
= \omega^{-1}(\varphi(\psi(\lambda(h_5)_4h_4)_4(\lambda(h_5)_3h_3))_1, \psi(\lambda(h_5)_4h_4)_1, \lambda(h_5)_1, h_1)
\]

\[
\omega^{-1}(\varphi(\psi(\lambda(h_5)_4h_4)4(\lambda(h_5)_3h_3))_2, \psi(\lambda(h_5)_4h_4)_2, \lambda(h_5)_2h_1)
\]

\[
= \omega^{-1}(\varphi(\psi(\lambda(h_5)_4h_4)_3, \lambda(h_5)_3, h_2)
\]

\[
\varphi(\psi(\lambda(h_5)_4h_4)_4(\lambda(h_5)_3h_3))_0\psi(\lambda(h_5)_4h_4)_0\lambda(h_5)_0
\]

\[
\omega^{-1}(\varphi(\psi(\lambda(h_5)_4h_4)_4(\lambda(h_5)_3h_3))_1, \psi(\lambda(h_5)_4h_4)_1, \lambda(h_5)_1, h_1)
\]

\[
= \varphi(\psi(\lambda(h_5)_4h_4)_2, \lambda(h_5)_2h_2)
\]

\[
\omega^{-1}(\varphi(\psi(\lambda(h_5)_4h_4)_2, \lambda(h_5)_2h_2)
\]

\[
= \varphi(\psi(\lambda(h_5)_4h_4)_2, \lambda(h_5)_2h_2)
\]

This algebra may be seen as a generalization of Doi’s smash product \(#(H, A)\) \([21]\), where \( H \) is a Hopf algebra and \( A \) a comodule algebra. Consider now the maps

\[
(h \cdot \varphi)(g) = \varphi(hg)
\]

\[
\sigma(h, g)(k) = \omega^{-1}(k, h, g)1_A
\]

Then we may form the crossed product \( \text{Hom}(H, A)\#_\phi H \). The particular case \( A = k \) reduces to the previous example. If \( H \) is a co-Frobenius Hopf algebra, restricting the crossed product to the subalgebra (with local units) \( A\# H^{*\text{rat}} \subseteq \text{Hom}(H, A) \), gives the isomorphism \((A\# H^{*\text{rat}})\# H \approx M_H^A(A)\), where \( M_H^A(A) \) is the ring of matrices with rows and columns indexed by a basis of \( H \), with only finitely many non-zero entries in \( A \) \([21]\), Theorem 6.5.11). It remains an open question if such a duality result holds also for co-Frobenius quasi-Hopf algebras.

(4) Let \( \mathcal{B} \) be a finite dimensional quasi-bialgebra and \((R, \rho, \phi_R)\) a right \( \mathcal{B} \)-comodule algebra (that is, an associative algebra endowed with an algebra morphism \( \rho : R \rightarrow R \otimes \mathcal{B} \) and an invertible element \( \phi_R \in R \otimes \mathcal{B} \otimes \mathcal{B} \), satisfying some compatibility conditions), as it was defined in \([10]\) and \([9]\). Denote \( \phi^{-1}_R = x^1_R \otimes x^2_R \otimes x^3_R \), summation understood. Then \( H = \mathcal{B}^* \) is a coquasi-bialgebra, and if we put

\[
\sigma(\varphi, \psi) = x^1_R \varphi(x^2_R)\psi(x^3_R), \text{ for } \varphi, \psi \in H,
\]

we obtain a cocycle as before. Taking also the weak action given by \( \varphi \cdot r = r_0 \varphi(r_1) \), where we have denoted \( r = r_0 \otimes r_1 \), \( r \in R \), we get the trivial example of a crossed product algebra \( R^\# \mathcal{B}^* = R^\# \mathcal{B}^* \), namely the quasi-smash product, as it was defined in \([10]\). Conversely, if \( H \) is a coquasi-bialgebra and \( R \) an associative algebra endowed with a weak action \( \cdot \) and a convolution invertible 2-cocycle \( \sigma \), which fulfill relations \((1.1), (1.3), (3.1), (3.2), (3.5)\), then \( H^* \) is a quasi-bialgebra which coact weakly on \( R \) by \( \rho(r) = \sum_{i=1}^{\dim H} e_i \cdot r \otimes e^i \), where \( (e_i)_{i=1,\dim H} \) and \( (e^j)_{j=i=1,\dim H} \) are dual bases for \( H \), respectively for \( H^* \). If we denote by \( \phi^R \in R \otimes H^* \otimes H^* \) the invertible element \( \phi^R(h \otimes g) = \sigma^{-1}(h, g) \) (here we have used the vector space isomorphism \( R \otimes H^* \otimes H^* \approx \text{Hom}(H \otimes H, R) \)), we obtain that \( R \) is a right \( H^* \)-comodule algebra.

3.3. Crossed products viewed towards monoidal categories. Let \( H \) be a coquasi-bialgebra and \( R \) an associative algebra, endowed with a weak action \( \cdot \) and a linear map \( \sigma \), as in the beginning of the previous
Section. Remember that the category of left $H$-comodules $H\mathcal{M}$ is monoidal. Now, for each right $R$-module $M \in \mathcal{M}_R$ and left $H$-comodule $V \in H\mathcal{M}$, we define on $M \otimes V$ the following structure:

$$(m \otimes v)r = m(v_{-1} \cdot r) \otimes v_0$$

for any $m \in M$, $v \in V$, $r \in R$.

**Proposition 12.**

1. With the previous notations, $M \otimes V$ is a right $R$-module if and only if the conditions (3.7) are fulfilled.

2. For $M \in \mathcal{M}_R$ and $V,W \in H\mathcal{M}$, consider the map

$$\Psi_{M,V,W} : (M \otimes V) \otimes W \rightarrow M \otimes (V \otimes W)$$

$$(m \otimes v) \otimes w \rightarrow m\sigma(v_{-1},w_{-1}) \otimes (v_0 \otimes w_0)$$

Then $\Psi_{M,V,W}$ is right $R$-linear if and only if (3.4) holds.

3. $\mathcal{M}_R$ becomes a right $H\mathcal{M}$-category with the above structures if and only if (3.1), (3.3), (3.4), (3.6), (3.7) hold.

**Proof.** Straightforward. Verifications are left to the reader.

Hence we have obtained a categorical explanation for the conditions imposed on the weak action and on the cocycle $\sigma$.

We make the observation that a right action of $H\mathcal{M}$ on $R\mathcal{M}$ can be constructed similarly. We can go even further, by considering bicomodules instead of one-sided comodules. All we need is to change properly the $H$ as an algebra in the monoidal category of right $R$-modules. Therefore we may consider the category of right $R$-modules $M^H_R$ within this monoidal category (an object $M$ is a right $R$-module, right $H$-comodule with comodule map $\rho_M(m) = m_0 \otimes m_1$ such that $\rho_M(mr) = m_0r \otimes m_1$, $\forall m \in M$, $r \in R$). Now, for each $M \in M^H_R$ and $V \in H\mathcal{M}^H$, we define on $M \otimes V$ the following structures:

$$(m \otimes v)r = m(v_{-1} \cdot r) \otimes v_0$$

$$\rho(m \otimes v) = m_0 \otimes v_0 \otimes v_1v_1$$

for any $m \in M$, $v \in V$, $r \in R$. Also, for $M \in M^H_R$ and $V,W \in H\mathcal{M}^H$, consider the map

$$\Psi_{M,V,W} : (M \otimes V) \otimes W \rightarrow M \otimes (V \otimes W)$$

$$(m \otimes v) \otimes w \rightarrow m\sigma(v_{-1},w_{-1}) \otimes (v_0 \otimes w_0)\omega(m_1, v_1, w_1)$$

As in the previous proposition, $M^H_R$ becomes a right $H\mathcal{M}^H$-category with the above structures if and only if (3.1), (3.3), (3.4), (3.6), (3.7) hold. Again, this construction can be performed also for left $R$-modules left $H$-comodules, obtaining a right action of $H\mathcal{M}^H$ on $H^R_R\mathcal{M}$. We can summarize all these in the following Theorem:

**Theorem 13.** Let $H$ be a coquasi-bialgebra and $(R, \cdot, \sigma)$ an $H$-crossed system. Then we have the following:

1. $\mathcal{M}_R$ is a right $H\mathcal{M}$-category;
2. $\mathcal{M}^H_R$ is a right $H\mathcal{M}^H$-category;
3. $H^R_R\mathcal{M}$ is a right $H\mathcal{M}^H$-category;
4. $H^R_R\mathcal{M}$ is a right $H\mathcal{M}^H$-category.

In all four cases, the action is given by the usual tensor product.

As constructing a crossed system means giving an action of the monoidal category, we want to see what is happening if we change the monoidal category or the action. We shall treat only the case $\mathcal{M}_R$ is a right $H\mathcal{M}$-category, but obvious similar arguments work for the other cases. First, consider an equivalence of monoidal categories. The easiest way to do this is using a twist $\tau$ on $H$. Then from Remark (3) (left version) it follows that deforming the multiplication of $H$ by conjugation gives us a new coquasi-bialgebra, having
category of comodules monoidal isomorphic with the starting category of comodules $^H\mathcal{M} \simeq ^{H^*}\mathcal{M}$. Via this isomorphism, we obtain an action of the deformed category of comodules over the category of $R$-modules. Moreover, the crossed system is changed and it follows that the resulting crossed product comodule algebra is precisely the image of the initial one via this monoidal category isomorphism (right version, as we deal with right comodule algebras). Explicitly, we have:

**Proposition 14.** Let $H$ be a quasi-bialgebra, $\tau \in (H \otimes H)^*$ a twist on $H$ and $(R, \cdot, \sigma)$ a crossed system. Then:

1. There is a right action of $^{H^*}\mathcal{M}$ on $\mathcal{M}_R$;
2. $(R, \cdot, \sigma\tau^{-1})$ is a crossed $H^*$-system;
3. $R\#_{\sigma\tau^{-1}}H_\tau = (R\#_{\sigma}H)_{\tau^{-1}}$.

**Proof.** (1) Remember that the monoidal category isomorphism $^{H^*}\mathcal{M} \cong ^{H^*}\mathcal{M}$ is the identity on objects and morphisms, but with monoidal structure $V \otimes W \rightarrow V \otimes W$, $v \otimes w \rightarrow v_0 \otimes w_0 \tau(v_{-1}, w_{-1})$, $v \in V$, $w \in W$ and $V, W \in ^{H^*}\mathcal{M}$. Then simply a transport of structures gives us the action, whose changes are reflected only in the reassociator’s formula

$$\Psi_{M,V,W} : (M \otimes V) \otimes W \rightarrow M \otimes (V \otimes W)$$

$$(m \otimes v) \otimes w \mapsto m\sigma(v_1, w_1)\tau^{-1}(v_2, w_2) \otimes (v_0 \otimes w_0)$$

for $M \in \mathcal{M}_R$, $V, W \in ^{H^*}\mathcal{M}$.

(2) It follows from the relations (2.12), (2.13) defining the multiplication and the reassociator of the deformed coquasi-bialgebra. Here we have denoted by $\sigma\tau^{-1}$ the convolution product.

(3) It is enough to write down explicitly the multiplication formulas for both $(R\#_{\sigma}H)_{\tau^{-1}}$ and $R\#_{\sigma\tau^{-1}}H_\tau$ to conclude that they coincide.

Now we want to change the action, by keeping both categories involved unchanged. Let $H$ be a coquasi-bialgebra, $(R, \cdot, \sigma)$ a crossed $H$-system and $\varphi : H \rightarrow R$ a convolution invertible map. Define

$$h \cdot \varphi = a^{-1}(h_1)(h_2 \cdot r)a(h_3)$$

$$\sigma_a(h, g) = a^{-1}(h_1)[h_2 \cdot a^{-1}(g_1)]\sigma(h_3, g_2)a(h_4g_3)$$

for all $h, g \in H$, $r \in R$. Then we can easily check that (3.1), (3.3), (3.4), (3.6), (3.5) hold for “$\cdot \varphi$” and $\sigma_a$, therefore we have:

**Proposition 15.** For any $\varphi : H \rightarrow R$ convolution invertible map, $(R, \cdot \varphi, \sigma_a)$ is again a crossed $H$-system.

The resulting crossed product $R\#_{\sigma_a}H$ will be called a twist deformation of $R\#_{\sigma}H$. As before, we expect to obtain a connection between these two structures. But we can say more in this case: first, the twist transformation of the crossed product is isomorphic to the initial one as left $R$-modules right $H$-comodules and second, any such isomorphism is given by a twist transformation:

**Theorem 16.** Let $H$ be a coquasi-bialgebra and $R$ an associative algebra. Consider on $R$ two crossed systems $(R, \cdot_1, \sigma_1)$ and $(R, \cdot_2, \sigma_2)$. For any algebra isomorphism $\theta : R\#_{\sigma_1}H \rightarrow R\#_{\sigma_2}H$ left $R$-linear, right $H$-colinear, there is a convolution invertible map $\varphi : H \rightarrow R$ such that:

1. $\theta(r\#_{\sigma_1}h) = ra(h_1)\#_{\sigma_2}h_2$;
2. $h \cdot_2 r = a^{-1}(h_1)(h_2 \cdot_1 r)a(h_3)$;
3. $\sigma_2(h, g) = a^{-1}(h_1)[h_2 \cdot_1 a^{-1}(g_1)]\sigma_1(h_3, g_2)a(h_4g_3)$.

Conversely, for each invertible map $\varphi : H \rightarrow R$ with properties (2) and (3), the map $\theta$ given by (1) is an $H$-comodule algebra isomorphism which is left $R$-linear, right $H$-colinear.

**Proof.** As in the Hopf case, see [22], Theorem 7.3.4. $\square$
3.4. Coquasi-Hopf modules. Let $H$ be a coquasi-bialgebra and $R$ an associative algebra, such that $(R, \cdot, \sigma)$ is a crossed system. By the previous results, $\mathcal{M}_R^H$ is a right $H\mathcal{M}^H$-category. But $H$ is an algebra in $H\mathcal{M}^H$, thus we may consider right modules over $H$ in the category $\mathcal{M}_R^H$. This means

**Definition 17.** A right $(R,H)$-coquasi-Hopf module $M$ is a right $R$-module, right $H$-comodule equipped with a map $\circ : M \otimes H \rightarrow M$, such that

\begin{align*}
\rho_M(m \circ h) &= m_0 \circ h_1 \otimes m_1 h_2 \quad (R\text{-linearity}) \\
(m \circ h)r &= [m(h_1 \cdot r)] \circ h_2 \quad (H\text{-colinearity}) \\
(m \circ h) \circ g &= [m_0 \sigma(h_1, g_1)] \circ (h_2 g_2) \omega(m_1, h_3, g_3) \\
m \circ 1_H &= m
\end{align*}

for all $m \in M$, $h, g \in H$, $r \in R$. The category of right coquasi-Hopf modules will be denoted $(\mathcal{M}_R^H)_H$.

Similarly, we may define the category of left coquasi-Hopf modules $(\mathcal{M}_R^H)_H$ as the category of right $H$-modules within $H\mathcal{M}$.

We shall see now the connection between the crossed product and the category defined above. We start with a lemma.

**Lemma 18.**

1. There is a functor

$$F : (\mathcal{M}_R^H)_H \rightarrow (\mathcal{M}_{R\mathcal{M}^H})^H$$

defined as follows: it is identity on morphisms, and for each $(M, \rho_M) \in (\mathcal{M}_R^H)_H$, we take $F(M) = M$, with structure maps

\begin{align*}
\tilde{\rho}_M(m) &= \rho_M(m) \\
m \ast (r \mathcal{M}^H \pi h) &= (m r) \circ h
\end{align*}

where $m \in M$, $r \in R$, $h \in H$.

2. We have a functor

$$G : (\mathcal{M}_{R\mathcal{M}^H})^H \rightarrow (\mathcal{M}_R^H)_H$$

which acts as identity on morphisms, and for each $(M, \tilde{\rho}_M) \in (\mathcal{M}_{R\mathcal{M}^H})^H$, we put $G(M) = M$, with structure maps

\begin{align*}
\rho_M(m) &= \tilde{\rho}_M(m) \\
m r \ast (r \mathcal{M}^H \pi h) &= m \ast (1_{R\mathcal{M}^H} h)
\end{align*}

where $m \in M$, $r \in R$, $h \in H$.

*Proof.* It is easy to check that the above formulas define indeed two functors. \hfill \square

**Theorem 19.** The category of right coquasi-Hopf modules $(\mathcal{M}_R^H)_H$ is isomorphic to the category of right $(H, R\mathcal{M}^H)$-Hopf modules $\mathcal{M}^H_{R\mathcal{M}^H}$.

*Proof.* It follows from the previous lemma. One has only to check that the above correspondences are inverse to each other, which is almost immediate. \hfill \square

**Remark 20.**

1. In [7], Corollary 3.6 states the following:

Let $B$ a strict monoidal category, $C$ a monoidal category, $R$ a coflat algebra in $B$, $(B_R, \otimes)$ a right $C$-category compatible with its natural $B$-category structure, $A$ an algebra in $C$ such that the functor $A \otimes - : B_R \rightarrow B_R$ preserves coequalizers. Then $R \otimes A$ is a $R$-ring in $B$ and we have a category equivalence $(B_R)_A \simeq B_{R \otimes A}$.

Notice that if we relax the condition that $B$ is strict (as any monoidal category is equivalent to a strict one, by [23]), and take the particular case $B = \mathcal{M}^H$, $C = H\mathcal{M}^H$, $A = H$, then we get exactly
our crossed product $R \otimes A = R\#_{\alpha} H$ as an algebra in $\mathcal{M}^H$, and recover the category isomorphism $(\mathcal{M}^H_{\mathcal{R}})_H \simeq \mathcal{M}^H_{R\#_{\alpha} H}$ of Theorem \[17\].

(2) In [10], a similar notion of Hopf module was defined, but for $\mathcal{H}$ a finite dimensional quasi-Hopf algebra, and $R$ a right $\mathcal{H}$-comodule algebra, as in Section 3.2, example (4). Then the category of $(\mathcal{H}, R)$-bimodules $\mathcal{H}\mathcal{M}_R$ is a right category over the monoidal category (with tensor product over the base field) of $\mathcal{H}$-bimodules $\mathcal{H}\mathcal{M}_\mathcal{H}$. As $\mathcal{H}$ is a coalgebra in $\mathcal{H}\mathcal{M}_\mathcal{H}$, the category of two sided Hopf modules $\mathcal{H}\mathcal{M}_R$ was defined as having the objects the right $\mathcal{H}$-comodules in this category and the morphisms the left $\mathcal{H}$-linear, right $R$-linear, right $\mathcal{H}$-colinear maps. Moreover, it was proved that there is an isomorphism of categories $\mathcal{H}\mathcal{M}_R \simeq \mathcal{H}\mathcal{M}_{\#_{\alpha} H}$, (where $R\#_{\alpha} H^*$ is the quasi-smash product mentioned also in Example (4) from Section 3.2). Now, take $H=\mathcal{H}^*$, as in the mentioned example. Then the acting monoidal category is the same $\mathcal{H}\mathcal{M}_\mathcal{H} = H^H$, the right $\mathcal{H}\mathcal{M}_\mathcal{H}$-category is the same $\mathcal{H}\mathcal{M}_R = \mathcal{M}^H_R$ and the resulting categories are isomorphic, as they are both isomorphic to $\mathcal{M}^H_{R\#_{\alpha} H}$. We do not write down explicitly this isomorphism, as it implies some very complicated notations not needed in this paper, but only remark that this isomorphism is preserving the $R$-module structure, the $\mathcal{H}$-module structure being modified by $S^2$. Hence we cannot say that this category isomorphism is induced by a duality between the coalgebra $\mathcal{H}$ and the algebra $H$ in the category $\mathcal{H}\mathcal{M}_\mathcal{H}$ (notice that the right dual of $H$ - which is $\mathcal{H}$ as vector space - inherits a coalgebra structure within this monoidal category, with comultiplication $\Delta_h(h) = (S^{-1} \otimes S^{-1})(f)\Delta^\text{cop}(h)f(-1)^{21}$, where $f \in \mathcal{H} \otimes \mathcal{H}$ is the twist introduced by Drinfeld in [24]). The counit is $\varepsilon$, while the bimodule structure of the dual is given by $h \rightarrow g = gS(h)$, $g \leftarrow h = S^{-1}(h)g$ for $g, h \in \mathcal{H}$, and

$$
\begin{align*}
\text{db} & : \quad k \rightarrow \mathcal{H}^* \otimes \mathcal{H} \\
\text{db}(1) & = \sum_{i=1, \dim \mathcal{H}} \epsilon_i \otimes S^{-1}(\varepsilon_i \beta)
\end{align*}
$$

are the rigidity morphisms, with $(\epsilon_i)_{i=1, \dim \mathcal{H}}$ and $(\epsilon_i^*)_{i=1, \dim \mathcal{H}}$ dual bases for $\mathcal{H}$, respectively $\mathcal{H}^*$. It would be interesting to find a categorical explanation of this isomorphism of categories. Moreover, remark that an advantage of the formulas used in the present paper is their naturality, compared with the ones in [10], and they do not involve any hard computations.

From now on we consider $H$ a coquasi-Hopf algebra and $(R, \cdot, \sigma)$ a crossed system. Let $M$ a right coquasi-Hopf module and denote by $M^{co}$ the subspace of coinvariants, namely $M^{co} = \{m \in M | \rho(m) = m \otimes 1_H\}$. Consider also the map $\Pi : M \rightarrow M$, $\Pi(m) = m_0 \circ S(m_1 \leftarrow \beta)$. This map enjoys the following properties:

**Proposition 21.** Under the above assumptions, we have the following:

1. $\Pi^2 = \Pi$;
2. $\rho_M \Pi(m) = m \otimes 1_H$;
3. $\Pi(m)r = \Pi(m_0(S(m_1) \cdot r))$, for any $m \in M$, $r \in R$.

**Proof.** (1) Take $m \in M$. Then

$$
\Pi^2(m) = \Pi(m_0 \circ S(m_1 \leftarrow \beta))
$$

$$
= [m_0 \circ \beta(m_3)]S(m_0) \circ \beta(m_1S(m_3))S(m_2S(m_4))
$$

$$
= m_0 \circ \beta(m_2)S(m_4)\beta(m_1S(m_3))
$$

$$
= m_0 \circ \beta(m_1)S(m_2)
$$

$$
= \Pi(m)
$$

proving that $\Pi$ is a projection.
(2) Again, for \( m \in M \) we have
\[
\rho \Pi(m) = \rho(m_0 \circ \beta(m_1) S(m_2)) = m_0 \circ \beta(m_2) S(m_4) \otimes m_1 S(m_3) = m_0 \circ \beta(m_1) S(m_2) \otimes 1_H = \Pi(m) \otimes 1_H
\]

(3) We compute
\[
\Pi(m)r = (m_0 \circ \beta(m_1) S(m_2))r = [m_0(S(m_3) \cdot r) \circ S(m_2) \beta(m_1)] = \Pi(m_0(S(m_1) \cdot r))
\]
for any \( m \in M \) and \( r \in R \).

\[\square\]

**Corollary 22.** \( M^{\text{co}H} = \Pi(M) \).

**Proof.** If \( m \in M^{\text{co}H} \), then by the defining formula for \( \Pi \) we have \( \Pi(m) = m \). The converse results from Proposition 21(2).

Let \( M \) a right coquasi-Hopf module. By Theorem 19, \( M \) has a natural structure of \( (H, R_{\#} \sigma H) \)-Hopf module and from (2.20) it follows that \( M^{\text{co}H} \) inherits a structure of right module over \( (R_{\#} \sigma H)^{\text{co}H} = R_{\#} \sigma k1_H \simeq R \).

**Corollary 23.** We have thus obtained a functor \((-)^{\text{co}H} : (M^H_R)^H_R \longrightarrow M_R \).

For any \( R \)-module \( N \), we may consider on \( N \otimes H \) the structure of a coquasi-Hopf module, by the following formulas:
\[
(n \otimes h)r = n(h_1 \cdot r) \otimes h_2 \\
\rho(n \otimes h) = n \otimes h_1 \otimes h_2 \\
(n \otimes h) \circ g = n \sigma(h_1, g_1) \otimes h_2 g_2
\]

**Theorem 24.** Let \( H \) be a coquasi-Hopf algebra and \( (R, \cdot, \sigma) \) a crossed system with invertible cocycle. Then the functors \(- \otimes H, (-)^{\text{co}H}\) define a pair of inverse equivalences \( M_R \overset{\otimes H}{\longrightarrow} (M^H_R)^H_R \).

**Proof.** For \( M \) a coquasi-Hopf module, define the map \( \varepsilon_M : M^{\text{co}H} \otimes H \longrightarrow M, \varepsilon_M(m \otimes h) = m \circ h \). Then \( \varepsilon_M \) is a morphism in \( (M^H_R)^H_R \), and it is natural in \( M \). We need an inverse for \( \varepsilon_M \). Define \( \eta_M : M \longrightarrow M^{\text{co}H} \otimes H, \)
by \( \varkappa_M(m) = \Pi(m_0\sigma^{-1}(S(m_1), m_2 \rightarrow \alpha)) \otimes m_3 \). Then for \( m \in M^{coH}, h \in H \) we have:

\[
\varkappa_M \varepsilon_M(m \otimes h) = \varkappa_M(m \circ h) = \Pi((m_0 \circ h_1)\sigma^{-1}(S(h_2), (m_2 \circ h_3) \rightarrow \alpha)) \otimes m_3 h_4
\]

\[
= \Pi((m \circ h_1)\sigma^{-1}(S(h_2), h_3 \rightarrow \alpha)) \otimes h_4
\]

\[
= [(m \circ h_1)\sigma^{-1}(S(h_2), h_3 \rightarrow \alpha) \circ \beta((m \circ h_1)_1)S((m \circ h_1)_2) \otimes h_4
\]

\[
= [(m \circ h_1)\sigma^{-1}(S(h_2), h_6)] \circ \beta(h_2)S(h_3)\alpha(h_5) \otimes h_7
\]

\[
= \{(m_1 \cdot \sigma^{-1}(S(h_5), h_7)) \circ h_2 \circ \beta(h_3)S(\alpha(h_6) \otimes h_8
\]

\[
= m(h_1 \cdot \sigma^{-1}(S(h_3), h_7))\sigma(h_2, S(h_8)) \circ (h_3S(h_7))\beta(h_5)\alpha(h_10)\omega(1, h_4, S(h_6)) \otimes h_12
\]

\[
= m(h_1 \cdot \sigma^{-1}(S(h_3), h_7))\sigma(h_2, S(h_4))\beta(h_3)\alpha(h_6) \otimes h_8
\]

\[
= m \circ \varepsilon(h_1)h_2
\]

Conversely, for each \( m \in M \), we compute

\[
\varepsilon_M \varkappa_M(m) = \Pi(m_0\sigma^{-1}(S(m_1), m_2 \rightarrow \alpha)) \circ m_3
\]

\[
= \{(m_0\sigma^{-1}(S(m_3), \alpha(m_4)m_5)) \circ \beta(m_1)S(m_2)\} \circ m_6
\]

\[
= m_0\sigma^{-1}(S(m_3), \alpha(m_7)m_8)\sigma(S(m_5), m_9) \circ \beta(m_2)(S(m_4)m_{10})\omega(m_1, S(m_3), m_{11})
\]

\[
= m_0\circ \alpha(m_4)\beta(m_2)(S(m_4)m_6)\omega(m_1, S(m_3), m_7)
\]

\[
= m_0\alpha(m_4)\beta(m_2)\omega(m_1, S(m_3), m_5)
\]

\[
= m_0\varepsilon(m_1)
\]

\[
= m
\]

Next, for any right \( R \)-module \( N \), define \( u_N : N \rightarrow (N \otimes H)^{coH} \), by \( u_N(n) = n \otimes 1_H \). It is easy to see that \( u_N \) is well defined, \( R \)-linear and natural in \( N \). As in the Hopf case, we take \( v_N : (N \otimes H)^{coH} \rightarrow N \), \( v_N(\sum_i n_i \otimes h_i) = \sum_i n_i \varepsilon(h_i) \). Then \( u_N \) and \( v_N \) are inverses to each other.

We still need to check that \( \varepsilon_M \) and \( u_N \) make \( F \) and \( G \) a pair of adjoint factors.

\[
\varepsilon_N \otimes_H (u_N \otimes 1_H)(\sum_i n_i \otimes h_i) = \sum_i (n_i \otimes 1_H) \circ h_i
\]

\[
= \sum_i n_i \sigma(1_H, h_{i1}) \otimes h_{i2}
\]

\[
= \sum_i n_i \otimes h_i
\]

and

\[
\varepsilon_M u_{M^{coH}}(m) = m \circ 1_H = m
\]

Therefore, we have obtained the equivalence between the two categories.

If we compose the previous equivalence with the isomorphism from Theorem 19 we obtain exactly the adjunction between the induced and the coinvariant functor from [1]. Therefore, if the coquasi-bialgebra admits an antipode and the comodule algebra is a crossed product with invertible cocycle, the two above categories are equivalent. We shall see in Section 12 why this happening.
4. Cleft extensions for coquasi-Hopf algebras

4.1. Cleft extensions from the Morita theory point of view. One of the main results in the theory of Hopf algebras is the equivalent characterization of cleft extensions as crossed product algebras with invertible cocycle (cf. [4, 6]). In order to derive such a characterization for coquasi-Hopf algebras, we need an appropriate definition of a cleft extension.

Let $H$ be a coquasi-Hopf algebra and $A$ a right $H$-comodule algebra. Denote $B = A^\text{co}H$. We recall from [1] the following definition:

**Definition 25.** Let $A$ a right $H$-comodule algebra and $\gamma : H \to A$ a colinear map. The extension $B \subseteq A$ is cleft with respect to the cleaving map $\gamma$ if there is a linear map $\delta : H \to A$ such that

\begin{align*}
(4.1) & \quad \rho \delta(h) = \delta(h_2) \otimes S(h_1) \\
(4.2) & \quad \delta(h_1)\gamma(h_2) = \alpha(h)1_A \\
(4.3) & \quad \gamma(h_1)\delta(h_2)\delta(h_3) = \varepsilon(h)1_A
\end{align*}

In this case, we call the pair $(\gamma, \delta)$ a cleaving system for the extension $B \subseteq A$.

**Remark 26.**

1. This definition of cleftness is slightly different from the classical one, where it is only required that $\gamma$ is convolution invertible (denote by $\delta$ the convolution inverse of $\gamma$) and $H$-colinear. The property (4.1) appears naturally by passing from a bialgebra to a Hopf algebra. Unfortunately, in our case the convolution product on $\text{Hom}(H, A)$ is no longer associative, therefore a left inverse for $\gamma$ is not necessarily a right inverse and the property (4.1) does not seem to result from the other properties of $\gamma$. So we had to state it separately.

2. For a cleft comodule algebra $A$, the application $\delta$ and relations (4.2), (4.3) depend on the antipode, again unlike the classical case. But if we change the antipode and the linear maps $\alpha, \beta$ to $(S', \alpha', \beta')$ as in (2.8) and define $\delta'(h) = U(h_1)\delta(h_2)$, then it follows immediately that $A$ is also $H$-cleft, but with respect to the new antipode. In the sequel, we shall suppose the antipode and the elements $\alpha, \beta$ fixed once for all.

In [1], the above conditions imposed on the cleaving map $\gamma$ were stated without further explanations, the only motivation being the equivalence with Galois extensions with normal basis property. We shall see now that the relations (4.1)-(4.3) come from a Morita context and that this is the reason for their non-symmetry. The following construction was inspired from [11], where the coring case was treated.

First of all, notice that if $C$ is a $k$-linear monoidal category, $(A, \mu_A, u_A)$ an algebra and $(C, \Delta_C, \varepsilon_C)$ a coalgebra in this monoidal category, then $\text{Hom}_C(C, A)$ becomes an associative $k$-algebra, with multiplication $\varphi \ast \psi = \mu_A(\varphi \otimes \psi)\Delta_C$ and unit $u_A \varepsilon_C$. Take now $C = M^H$ the category of comodules over a coquasi-Hopf algebra and $A$ a right $H$-comodule algebra. We also need a coalgebra in this monoidal category. In [11], the authors deformed the comultiplication on $H$ in order to obtain a left $H$-comodule coalgebra (actually, they showed that for any coalgebra $C$ with coalgebra map $C \to H$, there is a structure of left $H$-comodule coalgebra on $C$). Repeating their argument, but for $H^{\text{op,cop}}$ this time, we obtain that $H$ is a coalgebra in $M^H$, with the following structures: right adjoint coaction $\overline{\rho}(h) = h_2 \otimes S(h_1)h_3$, comultiplication

$$\overline{\Delta}(h) = h_3 \otimes h_0\omega(s(h_2)h_4, S(h_8), h_{10})\beta(h_6)\omega^{-1}(S(h_1), h_5, S(h_7))$$

and counit $\overline{\varepsilon}(h) = \alpha(h)$, for any $h \in H$. We shall denote by $\overline{\text{P}}$ this new structure. Hence $\text{Hom}^H(\overline{\text{P}}, A)$ becomes an associative algebra, with multiplication and unit as described above. As $B = A^{\text{co}H}$ is an associative algebra, $\text{Hom}(H, B)$ will also be with the usual convolution product. We have the two rings for the Morita context, we need the connecting bimodules. One of them will be $\text{Hom}^H(H, A)$, where $H$ is a comodule via $\Delta$, and the other $\text{Hom}^H(H^S, A)$. Here we have denoted by $H^S$ the comodule structure of $H$ twisted by the antipode, namely $h \mapsto h_2 \otimes S(h_1)$. 
Lemma 27. \( \text{Hom}^H(H, A) \) becomes a \((\text{Hom}(H, B), \text{Hom}^H(\overline{H}, A))\)-bimodule with the following structures:

\[
(4.4) \quad (\tau p)(h) = \tau(h_1)p(h_2) \\
(4.5) \quad (ps)(h) = p(h_1)s(h_5)\beta(h_3)\omega(h_2, S(h_4), h_6)
\]
for \( h \in H, \ p \in \text{Hom}^H(H, A), \ s \in \text{Hom}^H(\overline{H}, A) \) and \( \tau \in \text{Hom}(H, B) \).

Proof. It is easy to see that the formula (4.4) defines a structure of \( \text{Hom}(H, B) \)-module on \( \text{Hom}^H(H, A) \). We need now to verify that \( ps \) is \( H \)-colinear, for any \( s \in \text{Hom}^H(\overline{H}, A) \) and \( p \in \text{Hom}^H(H, A) \):

\[
(\text{ps})(h_0) \otimes (\text{ps})(h_1) = p(h_1)s(h_5) \otimes p(h_1)s(h_5)h_3\omega(h_2, S(h_4), h_6) \\
= p(h_1)s(h_7) \otimes h_2(S(h_6)h_8)\beta(h_4)\omega(h_3, S(h_5), h_9) \\
= (ps)(h_1) \otimes h_2
\]

For any \( s, \overline{s} \in \text{Hom}^H(\overline{H}, A) \), \( p \in \text{Hom}^H(H, A) \) and \( h \in H \), we compute that

\[
(p(s + \overline{s}))(h) = p(h_1)(s + \overline{s})(h_3)\beta(h_3)\omega(h_2, S(h_4), h_6) \\
= p(h_1)(s(h_7)\overline{s}(h_3))\omega(S(h_6)h_8, S(h_12), h_{14})\beta(h_10)\omega^{-1}(S(h_5), h_9, S(h_11)) \\
\beta(h_3)\omega(h_2, S(h_4), h_15) \\
= p(h_1)\beta(h_13)\omega^{-1}(S(h_6), h_{12}, S(h_{14}))\beta(h_3)\omega(h_3, S(h_5), h_20) \\
= (p(h_1)s(h_5)\overline{s}(h_10)\omega^{-1}(h_2, S(h_7)h_9, S(h_{15})h_{17})\omega^{-1}(S(h_6), h_{10}, S(h_4)h_{18}) \\
\beta(h_{12})\omega(h_{11}, S(h_{13}, h_{19}))\beta(h_4)\omega(h_3, S(h_5), h_20) \\
= p(h_1)\beta(h_11)\omega^{-1}(h_2, S(h_{10}), h_{12})\omega^{-1}(S(h_6)h_9, h_{13}, S(h_{17})h_{19}) \\
\omega^{-1}(h_4, S(h_8), h_{20})\beta(h_15)\omega(h_{14}, S(h_{16}, h_{20}))\beta(h_5)\omega(h_5, S(h_7), h_{22}) \\
= (p(h_1)s(h_5)\overline{s}(h_10)\omega(h_2, S(h_4), h_6)\omega(h_7, S(h_9), h_{11})\beta(h_3)\beta(h_8) \\
= (ps)(h_1) \otimes h_2
\]

and \( (p\alpha)(h) = p(h_1)\alpha(h_5)\beta(h_3)\omega(h_2, S(h_4), h_6) = p(h) \). Hence \( \text{Hom}^H(H, A) \) is a right \( \text{Hom}^H(\overline{H}, A) \)-module, and it is easy to check now the compatibility between the two module structures. \( \square \)

Lemma 28. \( \text{Hom}^H(H^S, A) \) becomes a \((\text{Hom}(H, B), \text{Hom}^H(\overline{H}, A))\)-bimodule with the following structures:

\[
(4.6) \quad (sq)(h) = s(h_2)q(h_6)\beta(h_4)\omega^{-1}(S(h_1), h_3, S(h_5)) \\
(4.7) \quad (qv)(h) = q(h_1)\tau(h_2)
\]
for \( h \in H, \ q \in \text{Hom}^H(H^S, A), \ s \in \text{Hom}^H(\overline{H}, A) \) and \( \tau \in \text{Hom}(H, B) \).

Proof. As in the previous Lemma, the only difficult part to check is the left \( \text{Hom}^H(\overline{H}, A) \)-module structure. For this, let \( h \in H, \ q \in \text{Hom}^H(H^S, A), \ s \in \text{Hom}^H(\overline{H}, A) \) and compute

\[
(sq)_0(h) \otimes (sq)_0(h) = s(h_2)q(h_6) \otimes s(h_2)q(h_6)\beta(h_4)\omega^{-1}(S(h_1), h_3, S(h_5)) \\
= s(h_3)q(h_9) \otimes (S(h_2)h_4)S(h_8)\beta(h_6)\omega^{-1}(S(h_1), h_5, S(h_7)) \\
= s(h_3)q(h_7) \otimes (S(h_1)\beta(h_5)\omega^{-1}(S(h_2), h_4, S(h_6)) \\
= (sq)(h_2) \otimes S(h_1)
\]

Therefore the action of \( \text{Hom}^H(\overline{H}, A) \) on \( q \in \text{Hom}^H(H^S, A) \) is well defined. Take now \( s, \overline{s} \in \text{Hom}^H(\overline{H}, A), \ q \in \text{Hom}^H(H^S, A) \) and \( h \in H \). Then

\[
((s + \overline{s})q)(h) = (s + \overline{s})(h_2)q(h_6)\beta(h_4)\omega^{-1}(S(h_1), h_3, S(h_5))
\]
From (2.7) it follows that (2.1)

$$\beta(h_9)\omega^{-1}(S(h_2), h_8, S(h_{10}))\beta(h_{17})\omega^{-1}(S(h_3), h_7, S(h_9))\omega^{-1}(S(h_1), h_{16}, S(h_{18}))$$

(2.3), (2.1), (2.4)

$$= s(h_3)(\overline{\tau}(h_{13})q(h_{21}))\omega^{-1}(S(h_{11}), h_{13}, S(h_{21}))\omega(S(h_4)h_6, S(h_{10}), h_{14}S(h_{20}))$$

(2.3)

$$\beta(h_8)\omega(S(h_2), h_{15}, S(h_{19}))\beta(h_{17})\omega^{-1}(S(h_3), h_7, S(h_9))\omega^{-1}(S(h_1), h_{16}, S(h_{18}))$$

(2.4)

$$= s(h_2)(\overline{\tau}(h_7)q(h_{11}))\omega^{-1}(S(h_6), h_8, S(h_{10}))\beta(h_4)\omega^{-1}(S(h_1), h_3, S(h_5))\beta(h_9)$$

$$= s(h_2)(\overline{\tau}(q)(h_6))\beta(h_4)\omega^{-1}(S(h_1), h_3, S(h_5))$$

$$= (s(\overline{\tau}q)(h))$$

From (2.7) it follows that (sα)(h) = s(h), hence $Hom^H(H^S, A)$ is a left $Hom^H(\overline{\Pi}, A)$-module.

Proposition 29. We have a Morita context

$$\mathcal{M}(A) = (Hom^H(\overline{\Pi}, A), Hom(H, B), Hom^H(H, A), Hom^H(H^S, A), (-, -), [-, -])$$

with connecting morphisms

$$(-, -) : Hom^H(H, A) \otimes_{Hom^H(\overline{\Pi}, A)} Hom^H(H^S, A) \to Hom(H, B) \quad (4.8)$$

$$(p, q)(h) = p(h_1)\beta(h_2)q(h_3)$$

$$[-, -] : Hom^H(H^S, A) \otimes_{Hom(H, B)} Hom^H(H, A) \to Hom^H(\overline{\Pi}, A) \quad (4.9)$$

$$[q, p](h) = q(h_1)p(h_2)$$

Proof. It is not difficult to see that $(p, q) \in Hom(H, B)$, $[q, p] \in Hom^H(\overline{\Pi}, A)$, $[-, -]$ is $Hom(H, B)$-balanced and $(-, -)$ is $Hom(H, B)$ bilinear. All the remaining verifications involve the algebra $Hom^H(\overline{\Pi}, A)$, and we shall do them in detail, for the convenience of the reader.

We show first that $(-, -)$ is $Hom^H(\overline{\Pi}, A)$-balanced:

$$(ps, q)(h) = (ps)(h_1)\beta(h_2)q(h_3)$$

$$= (p(h_1)s(h_5)q(h_6))\beta(h_3)\omega(h_2, S(h_4), h_6)\beta(h_7) \quad (2.1)$$

$$= p(h_1)(s(h_7)q(h_{12}))\omega(h_2, S(h_6)h_8, S(h_{11}))\beta(h_4)\omega(h_3, S(h_5), S(h_9))\beta(h_{10})$$

$$= p(h_1)(s(h_4)q(h_8))\beta(h_2)\omega^{-1}(S(h_3), h_5, S(h_7))\beta(h_6) \quad (2.3), (2.6), (2.6)$$

$$= p(h_1)\beta(h_2)(sq)(h_3)$$

$$= (p, sq)(h)$$

Next, we check the $Hom^H(\overline{\Pi}, A)$-bilinearity of $[-, -]$:

$$[sq, p](h) = (sq)(h_1)p(h_2)$$

$$= (s(h_2)q(h_6))p(h_7)\beta(h_4)\omega^{-1}(S(h_1), h_3, S(h_5)) \quad (2.1)$$

$$= s(h_3)(q(h_9)p(h_{10}))\omega(S(h_2)h_4, S(h_8), S(h_{11}))\beta(h_6)\omega^{-1}(S(h_1), h_5, S(h_7))$$

$$= s(h_3)p(h_4)(h_9)\omega(S(h_2)h_4, S(h_8), h_{10})\beta(h_6)\omega^{-1}(S(h_1), h_5, S(h_7))$$

$$= (s[q, p])(h)$$
and

\[ [q, ps](h) = q(h_1)(ps)(h_2) = q(h_1)(p(h_2)σ(h_6))β(h_4)ω(h_3, S(h_5), h_7) \]

\[ (2.1) = (q(h_2)p(h_3)σ(h_9)w^{-1}(S(h_1), h_4, S(h_8)h_{10})β(h_6)ω(h_5, S(h_7), h_{11}) \]

\[ (2.6) = (q(h_3)p(h_4)σ(h_{10})ω(S(h_2)h_5, S(h_9), h_{11})β(h_7)ω^{-1}(S(h_1), h_6, S(h_8)) \]

\[ = [q, p][h_3]σ(h_9)ω(S(h_2)h_4, S(h_8), h_{10})β(h_6)ω^{-1}(S(h_1), h_5, S(h_7)) \]

\[ = ([q, p]s)(h) \]

for any \( s ∈ \text{Hom}^H(\overline{H}, A) \), \( p ∈ \text{Hom}^H(H, A) \), \( q ∈ \text{Hom}^H(H^S, A) \) and \( h ∈ H \). Finally, we compute

\[ ((p, q)\overline{p})(h) = (p, q)(h_1)\overline{p}(h_2) = (p(h_1)q(h_3))\overline{p}(h_4)β(h_2) \]

\[ (2.1) = p(h_1)(q(h_5)\overline{p}(h_6))ω(h_2, S(h_4), h_7)β(h_3) \]

\[ = p(h_1)q(h_5)\overline{p}(h_6)ω(h_2, S(h_4), h_6)β(h_3) \]

\[ = (p(q, \overline{p}))(h) \]

and

\[ ([q, p]\overline{q})(h) = [q, p](h_2)\overline{q}(h_6)β(h_4)ω^{-1}(S(h_1), h_3, S(h_5)) \]

\[ (2.1) = q(h_2)p(h_3)\overline{q}(h_7)β(h_5)ω^{-1}(S(h_1), h_4, S(h_6)) \]

\[ (2.6) = q(h_3)p(h_4)\overline{q}(h_{10})ω(S(h_2), h_5, S(h_9))β(h_7)ω^{-1}(S(h_1), h_6, S(h_8)) \]

\[ = q(h_1)p(h_2)\overline{q}(h_4)β(h_3) \]

\[ = q(h_1)(p, \overline{q})(h_2) \]

\[ = (q(p, \overline{q}))(h) \]

where \( p, \overline{p} ∈ \text{Hom}^H(H, A) \), \( q, \overline{q} ∈ \text{Hom}^H(H^S, A) \) and \( h ∈ H \).

\[ □ \]

**Remark 30.** We shall denote by \( B\text{-}M^H \) the category of left \( B \)-modules, right \( H \)-comodules \((M, ρ_M) \) such that \( ρ_M(bm) = bm_0 ⊗ m_1 \) for all \( b ∈ B \), \( m ∈ M \). The morphisms are the left \( B \)-linear, right \( H \)-colinear maps. Two objects in this category are \( A \) and \( B ⊗ H \) with obvious structures. Then \( B\text{-}M^H \) can be seen as a category of entwined modules, with trivial left-right entwining structure. Therefore \( B^{\text{opp}} ⊗ H \) is \( B^{\text{opp}} \)-coring, and \( B\text{-}M^H \) is precisely the category of right comodules over this coring. We shall use in the proof of the next theorem the Lemma 3.5 from [11], applied to our situation.

We are now able to see the relationship between the above Morita context and cleft extensions. The following theorem is the coquasi-Hopf version of Theorem 3.6 from [11]:

**Theorem 31.** Let \( H \) a coquasi-Hopf algebra and \( A \) a right \( H \)-comodule algebra. Then:

1. The map \([,] \) is surjective if and only if \( B ⊆ A \) is Galois and there is a nonnegative integer \( n \) such that \( A \) is a direct summand in \((B ⊗ H)^n \) as left \( B \)-module, right \( H \)-comodule.

2. The Morita context is strict if and only if \( B ⊆ A \) is Galois and there are nonnegative integers \( n, π \) such that \( A \) is a direct summand in \((B ⊗ H)^n \) and \( B ⊗ H \) is direct summand in \( \mathbb{A}^\circ \) as left \( B \)-modules, right \( H \)-comodules.

3. If \( B ⊆ A \) is cleft, then the above Morita context is strict.

**Proof.** (1) If \([,] \) is surjective, choose \( p_i ∈ \text{Hom}^H(H, A), q_i ∈ \text{Hom}^H(H^S, A) \), for \( i ∈ I \) a finite index set such that \( ∑_{i∈I} [q_i, p_i] = α1_A \). Take the map \( Υ : A ⊗ H → A ⊗_B A, Υ(a ⊗ h) = ∑_{i∈I} aq_i(h_1) ⊗_B p_i(h_2) \). We
claim that \( \Upsilon \) is the inverse of \( \text{can} \). Indeed, for any \( a \in A \) and \( h \in H \), we have

\[
\text{can}( \sum_{i \in I} a_i q_i(h_1) \otimes_B p_i(h_2) ) = \sum_{i \in I} [a_0 q_i(h_1)_0] p_i(h_2)_0 \otimes p_i(h_2)_4 \]

\[
(\rho_i \in \text{Hom}^H(H,A), \ q_i \in \text{Hom}^H(H^S,A)) = \sum_{i \in I} [a_0 q_i(h_2)_0] p_i(h_3) \otimes h_7 \omega^{-1}(a_1 S(h_1), h_4, \beta(h_5) S(h_6))
\]

\[
(\sum_{i \in I} [q_i, p_i] = \alpha 1_A) = \sum_{i \in I} a_0 [q_i(h_3) p_i(h_4)] \otimes h_8 \omega(a_1, S(h_2), h_5)
\]

\[
\omega^{-1}(a_2 S(h_1), h_6, \beta(h_7) S(h_8))
\]

\[
\omega^{-1}(a_2 S(h_1), h_5, \beta(h_6) S(h_7))
\]

\[
\omega^{-1}(a_2 S(h_1), h_5, \beta(h_6) S(h_6))
\]

\[
\omega^{-1}(a_2 S(h_1), h_5, \beta(h_6) S(h_5))
\]

\[
\omega^{-1}(a_2 S(h_1), h_5, \beta(h_6) S(h_7))
\]

On the other hand, for any \( a, b \in A \),

\[
\Upsilon \text{can}(a \otimes_B b) = \Upsilon(a_0 b_0 \otimes b_4 \omega^{-1}(a_1, b_1, \beta(b_2), S(b_3)))
\]

\[
(\rho_i \in \text{Hom}^H(H^S, A)) = \sum_{i \in I} [a_0 b_0 q_i(h_0)] \otimes_B p_i(h_7) \omega(a_1, b_1, S(b_5)) \omega^{-1}(a_2, b_2 \beta(b_3), S(b_4))
\]

\[
(\sum_{i \in I} [q_i, p_i] = \alpha 1_A) = \sum_{i \in I} a_0 [b_0 q_i(h_3)] \otimes_B p_i(h_4) \beta(b_1)
\]

But for any \( b \in A \),

\[
\rho_A(\sum_{i \in I} b_0 \beta(b_1) q_i(h_2)) = \sum_{i \in I} b_0 q_i(h_1)_0 \otimes b_1 \beta(b_2) q_i(h_1)_1
\]

\[
(\sum_{i \in I} [q_i, p_i] = \alpha 1_A) = \sum_{i \in I} b_0 q_i(h_1)_1 \otimes b_1 \beta(b_2) S(b_3)
\]

\[
(4.10)
\]

\[
\text{can}(a \otimes_B b) = \sum_{i \in I} [a_0 b_0 q_i(h_2)] \otimes_B p_i(h_3) \beta(b_1)
\]

\[
(\rho_i \in \text{Hom}^H(H, A), \ q_i \in \text{Hom}^H(H^S, A)) = \sum_{i \in I} a_0 \otimes_B b_0 [q_i(h_3) p_i(h_4)] \beta(b_1) \omega(b_1, S(b_2), b_5)
\]

\[
(\sum_{i \in I} [q_i, p_i] = \alpha 1_A) = a_0 \otimes_B b_0 A \beta(b_1) \omega(b_1, S(b_2), b_4)
\]
\[\begin{align*}
\{p_i, q_i\} &= a \otimes_B b
\end{align*}\]

Hence the extension is Galois.

We want now to prove the second statement. Consider the following maps, for every \(i \in I:\)

\[
(4.12) \quad \xi_i : A \to B \otimes H, \quad \xi_i(a) = \sum_{i \in I} a_0 \beta(a_1) q_i(a_2) \otimes a_3
\]

\[
(4.13) \quad \xi_i : B \otimes H \to A, \quad \xi_i(b \otimes h) = b p_i(h)
\]

The maps \(\xi_i\) are well defined from (4.10), left \(B\)-linear and right \(H\)-colinear, as it can be easily checked. Also \(\xi_i\) are left \(B\)-linear and right \(H\)-colinear and \(\sum_{i \in I} \xi_i(\xi_i(a)) = \sum_{i \in I} [a_0 \beta(a_1) q_i(a_2)] p_i(a_3) = a\), after a similar computation as in (4.11). It follows from Remark [39] that \(A\) is a direct summand in \((B \otimes H)^n\) as left \(B\)-module, right \(H\)-comodule, where \(n\) is the cardinal of the index set \(I\).

Conversely, if \(A\) is a direct summand in \((B \otimes H)^n\) then again by Remark [39] there exist some morphisms \(\xi_i \in \text{Hom}_{\mathbf{H}}^H(A, B \otimes H), \, \xi_i \in \text{Hom}_{\mathbf{H}}^H(B \otimes H, A), \, i \in I\) with \(I\) a finite index set, \(|I| = n\), such that \(\sum_{i \in I} \xi_i = I_A\). Define then

\[
(4.14) \quad p_i : H \to A, \quad p_i(h) = \xi_i(1_A \otimes h)
\]

\[
(4.15) \quad q_i : H \to A, \quad q_i(h) = (I_A \otimes_B (I_B \otimes \varepsilon))(I_A \otimes_B \xi_i) \text{can}^{-1}(1_A \otimes h)
\]

It is immediate that \(p_i \in \text{Hom}^H(H, A), \forall i \in I\). In order to show that \(q_i \in \text{Hom}^H(H^S, A)\), we start by recalling from [11] the notation \(\text{can}^{-1}(1_A \otimes h) = \sum_{j} l_j(h) \otimes_B r_j(h)\) and the following properties:

\[
(4.16) \quad \sum_{j} l_j(h_0) \otimes_B r_j(h) \otimes l_j(h_1) = \sum_{j} l_j(h_2) \otimes_B r_j(h_2) \otimes S(h_1)
\]

\[
(4.17) \quad \sum_{j} l_j(h_1) \otimes_B r_j(h_1) \otimes h_2 = \sum_{j} l_j(h) \otimes_B r_j(h_0) \otimes r_j(h_1)
\]

\[
(4.18) \quad \sum_{j} l_j(h) r_j(h) = a(h) 1_A
\]

\[
(4.19) \quad \sum_{j} a l_j(h) \otimes_B r_j(h) = \text{can}^{-1}(a \otimes h)
\]

\[
(4.20) \quad a_0 \otimes \beta(a_2) a = \text{can}(1_A \otimes_B a)
\]

for any \(h \in H, \, a \in A\). Using the identification \(A \otimes_B B \simeq A\), we may write \(q_i(h) = \sum_{j} l_j(h)(I_B \otimes \varepsilon)\xi_i(r_j(h))\) and the \(H\)-colinearity of \(q_i\) follows from (4.10).

We compute now, for \(h \in H\)

\[
\sum_{i} \{q_i, p_i\}(h) = \sum_{i} q_i(h_1) p_i(h_2)
\]

\[
= \sum_{i,j} l_j(h_1) (I_B \otimes \varepsilon) \xi_i(r_j(h_1)) \xi_i(1_A \otimes h_2)
\]

\[
= \sum_{i,j} l_j(h_1) \xi_i((I_B \otimes \varepsilon) \xi_i(r_j(h_1)) \otimes h_2)
\]

\[
= \sum_{i,j} l_j(h_1) [\xi_i(I_B \otimes \varepsilon \otimes I_H)(\xi_i(r_j(h_1)) \otimes h_2)]
\]

\[
(4.17) \quad = \sum_{i,j} l_j(h) [\xi_i(I_B \otimes \varepsilon \otimes I_H)(\xi_i(r_j(h_0)) \otimes r_j(h_1))]
\]

\[
= \sum_{i,j} l_j(h) [\xi_i(I_B \otimes \varepsilon \otimes I_H)(I_B \otimes \Delta)\xi_i(r_j(h))]
\]
\[\sum_{i,j} I_j(h)[\xi_i \zeta_i(r_j(h))]
= \sum_j I_j(h)r_j(h) \overset{\text{(1.18)}}{=} \alpha(h)1_A\]

This proves the surjectivity of (\).

(2) Suppose that the Morita context is strict. By (1), we have only to show that \(B \otimes H\) is direct summand in \(A^\pi\) for some integer \(\pi\). In order to do this, we repeat the arguments from (1), but now for the Morita map (\(\cdot\)). Therefore we may find \(\overline{p}_i \in \text{Hom}^H(H, A), \overline{\xi}_i \in \text{Hom}^H(H^S, A)\), for \(i \in T\) a finite index set such that \(\sum_i (\overline{p}_i, \overline{\xi}_i) = \varepsilon 1_A\). We define \(\xi_i \in \text{Hom}^H_H(A, B \otimes H), \xi_i \in \text{Hom}^H_H(B \otimes H, A)\) by similar formulas to (4.12)-(4.13), but with \(\overline{p}_i, \overline{\xi}_i\) instead of \(p_i, q_i\). The linearity and colinearity of them follow easily. Then we find that, for any \(b \in B\) and \(h \in H\), we have

\[\sum_{i \in I} \zeta_i(\xi_i(b \otimes h)) = \sum_{i \in I} \zeta_i(b \overline{\xi}_i(h))\]
\[= \sum_{i \in I} b \zeta_i(\overline{p}_i(h)) = b \sum_{i \in I} \overline{p}_i(h)0 \beta(\overline{p}_i(h)1)\overline{p}_i(h)2 \otimes \overline{p}_i(h)3\]
\[= b \sum_{i \in I} \overline{p}_i(h1) \beta(h2)\overline{p}_i(h3) \otimes h4\]
\[= b \otimes h\]

As in (1), it follows that \(B \otimes H\) is direct summand in \(A^\pi\) for \(\pi = |T|\).

For the converse statement, again by (1) we need only to check the surjectivity of the Morita map (\(\cdot\)). Similar to (1), from the fact that \(B \otimes H\) is direct summand in \(A^\pi\), it follows the existence of a finite index set \(T\) with \(|T| = \pi\) and of two families of morphisms \(\zeta_i \in \text{Hom}^H_A(A, B \otimes H), \xi_i \in \text{Hom}^H_B(B \otimes H, A)\), \(i \in \mathcal{T}\) such that \(\sum_i \zeta_i \xi_i = I_B \otimes I_H\). Again use formulas (4.14)-(4.15) to define \(\overline{p}_i \in \text{Hom}^H_H(H, A), \overline{\xi}_i \in \text{Hom}^H_H(H^S, A)\) by means of \(\zeta_i\) and \(\xi_i\). Then

\[\sum_i (\overline{p}_i, \overline{\xi}_i)(h) = \sum_i \overline{p}_i(h1) \beta(h2)\overline{\xi}_i(h3)\]
\[= \sum_{i,j} \overline{p}_i(h1) \beta(h2)l_j(h3)(I_B \otimes \varepsilon)\overline{\xi}_i(r_j(h3))\]

But relation (4.19) implies
\[\sum_{i,j} \overline{p}_i(h1) \beta(h2)l_j(h3) \otimes_B r_j(h3) = \sum_i \text{can}^{-1}(\overline{p}_i(h1) \otimes \beta(h2)h3)\]
\[\overset{(4.20)}{=} 1_A \otimes_B \overline{p}_i(h)\]

where the last equality uses the fact that \(\overline{p}_i\) are \(H\)-colinear, \(\forall i \in \mathcal{T}\). Therefore

\[\sum_i (\overline{p}_i, \overline{\xi}_i)(h) = \sum_i (I_B \otimes \varepsilon)\overline{p}_i(h)\]
\[= \sum_i (I_B \otimes \varepsilon)\overline{\xi}_i(1_A \otimes h)\]
\[= \varepsilon(h)1_A\]

and the surjectivity of (\(\cdot\)) follows from its \(\text{Hom}(H, B)\)-bilinearity.

(3) Suppose \(B \subseteq A\) is cleft with maps \(\gamma, \delta : H \rightarrow A\), where \(\gamma \in \text{Hom}^H(H, A)\). But (4.11) implies that \(\delta \in \text{Hom}^H(H^S, A)\), while properties (4.2)-(4.3) are equivalent to \([\delta, \gamma] = \alpha 1_A\), respectively to \((\gamma, \delta) = \varepsilon 1_A\).
Therefore the cleaving maps are sent by the Morita morphisms \([18], (4.9)\) to the unit elements of the corresponding algebras in the Morita context. It follows that the Morita context is strict. \(\square\)

**Remark 32.** (1) Notice that in the Hopf algebra case, all algebras and bimodules involved in the above Morita context are included in \(\text{Hom}(H, A)\), and all structure maps and connecting homomorphisms are precisely the convolution product. This was observed in \([20]\) for the coring case, but it remains true for coquasi-Hopf algebras. We can say even more in this case. There is a second Morita context that we may build generalizing Doi’s construction \([20]\). We plan to investigate the relationship between these two Morita contexts in a forthcoming paper.

(2) The Morita context which inspired us \([25]\) has a very natural conceptual meaning, it is simply given by the natural transformations of two functors between comodule categories over some corings. It is unclear for the moment how this should be applied to the present situation, mainly because \(A \otimes H\) is no longer an \(A\)-coring in the usual sense, as \(A\) is not an associative algebra.

### 4.2. Crossed products are the same as cleft extensions by a coquasi-Hopf algebra.

Recall that in \([1]\), the notion of a Galois extension for a coquasi-Hopf algebra was introduced, and it was also proven the equivalence between cleft extensions and Galois extensions with the normal basis property, under the additional hypothesis of the bijectivity of the antipode. This is a generalization of a well-known result for Hopf algebras. Also, in the Hopf algebra theory, cleft extensions of Hopf algebras can be characterized as crossed products with invertible cocycle by Hopf algebras. We shall see that this identification remains true for the coquasi-Hopf algebras.

**Theorem 33.** Let \(H\) a coquasi-Hopf algebra, \(A\) a right \(H\)-comodule algebra and \(B = A^{\text{co}H}\) the subalgebra of coinvariants. The following statements are equivalent:

1. The extension \(B \subseteq A\) is cleft;
2. There exist an invertible cocycle \(\sigma\) and a weak action of \(H\) on \(B\) such that \(A\) is isomorphic as left \(B\)-module, right \(H\)-comodule algebra with the crossed product \(B^\text{co}_\sigma H\).

**Proof.** (1) \(\Rightarrow\) (2) In \([1]\), an isomorphism \(\nu : B \otimes H \rightarrow A\), left \(B\)-linear and right \(H\)-colinear was constructed by the formulas

\[
\begin{align*}
\nu(b \otimes h) &= b \gamma(h) \\
\nu^{-1}(a) &= a_0 \delta(a_1 \leftarrow \beta) \odot a_2
\end{align*}
\]

Via this isomorphism, \(B \otimes H\) becomes an algebra in \(\mathcal{M}^H\), with multiplication

\[
(b \otimes h) \odot (c \otimes g) = \nu^{-1}(\nu(b \otimes h)\nu(c \otimes g))
\]

and unit \(\nu^{-1}(1_A)\). But relations \([1.2], (4.3)\) imply \(\gamma(1_H) \in B\) and invertible in \(B\), hence we may assume \(\gamma(1_H) = 1_A\) (if not, replace \(\gamma\) by \(\tilde{\gamma}(h) = \gamma(h)\gamma(1_H)^{-1}\) and \(\delta\) by \(\tilde{\delta}(h) = \gamma(1_H)\delta(h)\)). It implies \(\nu^{-1}(1_A) = 1_A \otimes 1_H\) and \(\nu(b \otimes 1_H) = b, \forall b \in B\). Now the rest of the proof follows as in \([20]\). Define

\[
\begin{align*}
h \cdot b &= (I_B \otimes \varepsilon)\nu^{-1}(\nu(1_A \otimes h)\nu(b \otimes 1_H)) \\
\sigma(h, g) &= (I_B \otimes \varepsilon)\nu^{-1}(\nu(1_A \otimes h)\nu(1_R \otimes g))
\end{align*}
\]

for any \(h, g \in H, b \in B\). Then

\[
\begin{align*}
h_1 \cdot b \otimes h_2 &= (I_B \otimes \varepsilon)\nu^{-1}(\nu(1_A \otimes h_1)\nu(b \otimes 1_H)) \otimes h_2 \\
&= (I_B \otimes \varepsilon \otimes I_H)(\nu^{-1} \otimes I_H)(\nu(1_A \otimes h_1)\nu(b \otimes 1_H) \otimes h_2)
\end{align*}
\]

\[
\begin{align*}
(\nu \text{ is colinear}) &= (I_B \otimes \varepsilon \otimes I_H)(\nu^{-1} \otimes I_H)(\nu(1_A \otimes h)\nu(b \otimes 1_H) \otimes \nu(1_A \otimes h)1) \\
(\nu^{-1} \text{ is colinear}) &= (I_B \otimes \varepsilon \otimes I_H)(I_B \otimes \Delta)\nu^{-1}(\nu(1_A \otimes h)\nu(b \otimes 1_H))
\end{align*}
\]

\[
\begin{align*}
&= \nu^{-1}(\nu(1_A \otimes h)\nu(b \otimes 1_H)) \\
&= (1_A \otimes h) \odot (b \otimes 1_H)
\end{align*}
\]
and

\[
\sigma(h_1, g_1) \otimes h_{2g_2} = (I_B \otimes \varepsilon)\nu^{-1}(\nu(1_A \otimes h_1)\nu(1_R \otimes g_1)) \otimes h_{2g_2}
\]

\[
= (I_B \otimes \varepsilon \otimes I_H)((\nu^{-1} \otimes I_H)(\nu(1_A \otimes h_1)\nu(1_R \otimes g_1) \otimes h_{2g_2})
\]

(\nu \text{ is colinear})

\[
= (I_B \otimes \varepsilon \otimes I_H)((\nu^{-1} \otimes I_H)(\nu(1_A \otimes h_1)\nu(1_R \otimes g_1) \otimes \nu(1_A \otimes h_1))
\]

\[
\nu(1_R \otimes g_1)
\]

(\nu^{-1} \text{ is colinear})

Then we may compute that

\[
b(h_1 \cdot c)\sigma(h_2, g_1) \otimes h_{3g_2} = b(h_1 \cdot c)[\sigma(h_2, g_1) \otimes h_{3g_2}]
\]

(\nu \text{ is } B-\text{linear})

\[
= \nu^{-1}(b\nu(1_A \otimes h_2)\nu(c \otimes 1_H))\nu(1_A \otimes g_2)
\]

= \nu^{-1}(b\nu(1_A \otimes h_2)\nu(c \otimes 1_H))\nu(1_A \otimes g_2)

= \nu^{-1}(b\nu(1_A \otimes h_2)\nu(c \otimes 1_H))\nu(1_A \otimes g_2)

= \nu^{-1}(\nu(1_A \otimes h_2)\nu(1_A \otimes g_2))

= (1_A \otimes h_2) \otimes (1_A \otimes g_2)

It results that the multiplication formula for \(B \otimes H\) is precisely the one for the crossed product. As we have defined the multiplication on \(B \otimes H\) such that it is comodule algebra, Theorem 9 implies that \(\sigma\) and \(\sigma\) verify the requested relations (4.1, 3.3, 3.4, 3.5, 3.6).

If we write down explicitly the morphisms \(\nu\) and \(\nu^{-1}\) from relations (4.21) and (4.22), we obtain that the weak \(H\)-action on \(B\) and the cocycle \(\sigma\) are given by the formulas

\[
h \cdot b = \gamma(h_1)b\delta(h_2) \leftarrow \beta
\]

\[
\sigma(h, g) = [\gamma(h_1)\gamma(g_1)]\delta((h_2g_2) \leftarrow \beta)
\]

which are, up to the element \(\beta \in H^*\), precisely as in the Hopf algebra case. For the proof to be complete, we have to show that \(\sigma\) is convolution invertible. Define for any \(h, g \in H\),

\[
\sigma^{-1}(h, g) = \gamma(\beta \rightarrow (h_1g_1))f^{-1}(h_2g_2)[\delta(g_3)\delta(h_3)]
\]

Then this is coinvariant with respect to the \(H\)-coaction:

\[
\rho_A(\sigma^{-1}(h, g)) = \gamma(h_1g_1)\beta(h_2g_2)f^{-1}(h_3g_3)[\delta(g_4)\delta(h_4)] \otimes \gamma(h_1g_1)\beta(g_1h_1)\delta(h_4)\delta(h_3)
\]

\[
1 = \gamma(h_1g_1)\beta(h_3g_3)f^{-1}(h_4g_4)[\delta(g_5)\delta(h_6)] \otimes \gamma(h_3g_3)\beta(h_3g_3)\delta(h_4)\delta(h_3)
\]

\[
\sigma^{-1}(h, g) \otimes 1_H
\]

We shall now compute the convolution product

\[
\sigma(h_1, g_1)\sigma^{-1}(h_2, g_2) = \{[\gamma(h_1)\gamma(g_1)]\delta(h_3g_3)\} \{\gamma(h_4g_4)[\delta(g_5)\delta(h_7)]\}
\]

\[
\beta(h_2g_2)\beta(h_3g_3)f^{-1}(h_6, g_6)
\]

\[
\sigma^{-1}(h, g) \otimes 1_H
\]
\[\begin{align*}
\text{(2.10)}, \text{(2.11)} &= [\gamma(h_1)\gamma(g_1)\{\delta(h_3g_3)[\gamma(h_4g_4)(\delta(g_7)\delta(h_7))]\} \\
\beta(h_2g_2)\beta(h_5g_5)f^{-1}(h_6, g_6) \\
\text{(2.18)}, \text{(4.1)}, \text{(4.2)}, \text{(2.10)} &= [\gamma(h_1)\gamma(g_1)\{\delta(g_9)\delta(h_9)\}\omega^{-1}(S(h_3g_3, h_5g_5, S(h_7g_7)) \\
\beta(h_2g_2)\alpha(h_4g_4)\beta(h_6g_6)f^{-1}(h_8, g_8) \\
\text{(2.17)} &= [\gamma(h_1)\gamma(g_1)\{\delta(g_4)\delta(h_4)\}\beta(h_2g_2)f^{-1}(h_5, g_3) \\
\text{(2.18), (2.18)} &= \gamma(h_1)\{[\gamma(g_1)\delta(g_8)]\delta(h_7)\}\beta(h_3g_4)\beta^{-1}(h_4, g_3) \\
\omega(h_2, g_3, S(g_9)S(h_5))\omega^{-1}(g_2, S(g_7), S(h_6)) \\
\text{(2.17)} &= \gamma(h_1)\{[\gamma(g_1)\delta(g_9)]\delta(h_7)\}\beta(h_5)\beta(g_6)\omega(h_3g_4, S(g_8), S(h_6)) \\
\omega^{-1}(h_4, g_5, S(g_7))\omega(h_2, g_3, S(g_9)S(h_7))\omega^{-1}(g_2, g_4, S(g_4), S(h_4)) \\
\text{(2.6), (4.3), (4.3)} &= \varepsilon(g)\varepsilon(h)1_A
\end{align*}\]

The similar computations for showing that \(\sigma^{-1}(h_1, g_1)\sigma(h_2, g_2) = \varepsilon(g)\varepsilon(h)1_A\), for \(h, g \in H\) are left to the reader. Therefore \(\sigma\) is invertible with respect to the convolution product.

(2) \(\iff\) (1) Define \(\gamma : H \longrightarrow \mathcal{B}_{\#\sigma}^{\#\sigma}H\) by \(\gamma(h) = 1_A\mathcal{B}_{\#\sigma}^{\#\sigma}h\). It is \(H\)-colinear, because the comodule structure on the crossed product is \(I_A \otimes \Delta\). We need an inverse for \(\gamma\) (in the sense of Definition 25). If we denote \(\delta(h) = \sigma^{-1}(S(h_2), h_3 \leftarrow \alpha)\mathcal{B}_{\#\sigma}S(h_1)\) for \(h \in H\), then
\[\rho_A(\delta(h)) = \sigma^{-1}(S(h_2), h_3 \leftarrow \alpha)\mathcal{B}_{\#\sigma}S(h_1)1 \otimes S(h_2) = \sigma^{-1}(S(h_3), h_4 \leftarrow \alpha)\mathcal{B}_{\#\sigma}S(h_2) \otimes S(h_1) = \delta(h_2) \otimes S(h_1)\]
We can now compute that:
\[\delta(h_1)\gamma(h_2) = [\sigma^{-1}(S(h_2), h_3 \leftarrow \alpha)\mathcal{B}_{\#\sigma}S(h_1)](1_A\mathcal{B}_{\#\sigma}h_4) = \sigma^{-1}(S(h_2), h_3 \leftarrow \alpha)\sigma(S(h_1), h_4)\mathcal{B}_{\#\sigma}S(h_2)\alpha(h_3) = 1_A\mathcal{B}_{\#\sigma}\delta(h_2)\alpha(h_3) = 1_A\mathcal{B}_{\#\sigma}1_H\alpha(h)\]
and
\[\gamma(h_1)\beta(h_2)\delta(h_4) = (1_{\mathcal{B}_{\#\sigma}}h_1\beta(h_2))([\sigma^{-1}(S(h_1), h_5 \leftarrow \alpha)\mathcal{B}_{\#\sigma}S(h_3)]) = [h_1, \beta(h_2) \cdot \sigma^{-1}(S(h_1), h_5 \leftarrow \alpha)]\sigma(h_1, S(h_3))\mathcal{B}_{\#\sigma}h_1S(h_3) = [h_1\beta(h_4) \cdot \sigma^{-1}(S(h_7), h_8 \leftarrow \alpha)]\sigma(h_2, S(h_6))\mathcal{B}_{\#\sigma}h_3S(h_5) = [h_1\beta(h_3) \cdot \sigma^{-1}(S(h_5), h_6 \leftarrow \alpha)]\sigma(h_2, S(h_4))\mathcal{B}_{\#\sigma}1_H = \sigma(h_2, S(h_4))\mathcal{B}_{\#\sigma}1_H\alpha(h_6)\beta(h_3) = \omega(h_1, S(h_5), h_7)\sigma^{-1}(h_2S(h_4), h_8)\mathcal{B}_{\#\sigma}1_H\alpha(h_6)\beta(h_3) = \omega(h_1, S(h_3), h_5)\mathcal{B}_{\#\sigma}1_H\alpha(h_4)\beta(h_2) = \varepsilon(h)1_A\mathcal{B}_{\#\sigma}1_H\]
for all \(h \in H\). \qed
Lemma 37. The extension \( \beta \) trivial.

For any Hopf algebra

Remark 36.

□

are convolution inverse to each other.

two and three constructed in [12]. These are the smallest and simplest known examples of coquasi-Hopf algebras. However, we shall see that characterizing cleft extensions is not simple, even if we work only with generators and relations, and the amount of data increases even when passing from dimension 2 to dimension 3. The method we use is inspired from [27].

Let \( k = \mathbb{C} \) be a field of characteristic different from 2. Following [12] or [28] (where the dual case was considered), the 2-dimensional coquasi-Hopf algebra \( H(2) \) is generated by a grouplike element \( x \) such that \( x^2 = 1 \). It is isomorphic as an algebra and as a coalgebra with \( k[C_2] \), but with associator given by \( \omega(x, x, x) = -1 \) and trivial elsewhere. It is not twist equivalent to a Hopf algebra, and any 2-dimensional coquasi-Hopf algebra is known to be twist equivalent either to \( k[C_2] \) or to \( H(2) \). The antipode is the identity, the linear map \( \beta \) is trivial \( \beta(1) = \beta(x) = 1 \), but the map \( \alpha \) is not: \( \alpha(1) = 1, \alpha(x) = -1 \).

Let \( A \) be a right \( H(2) \)-comodule algebra and denote \( B = A^{coH(2)} \).

Lemma 37. The extension \( B \subseteq A \) is \( H(2) \)-cleft if and only if there exist elements \( a, b \in A \) such that

\[
ab = 1_A, \quad ba = -1_A \\
\rho_A(a) = a \otimes x, \quad \rho_A(b) = b \otimes x
\]

In this case, the following hold:

(1) The maps \( \gamma : H(2) \rightarrow A, \gamma(1) = 1_A, \gamma(x) = a \) and \( \delta : H(2) \rightarrow A, \delta(1) = 1_A, \delta(x) = b \) form the cleaving system;

(2) \( A \) is free as left \( B \)-module with basis \( \{1_A, a\} \);

(3) The elements \( c = a^2, d = -b^2 \) are invertible in \( B \) and \( c^{-1} = d \);
(4) The $H(2)$-weak action on $B$ and the cocycle corresponding to the crossed product structure are

$$1 \cdot e = e, \quad x \cdot e = aeb$$

$$\sigma(x, x) = c \text{ and trivial elsewhere}$$

for all $e \in B$.

Proof. (1) For the first statement, it is enough to take $a = \gamma(x)$ and $b = \delta(x)$. Conversely, for any elements $a$ and $b$ with such properties, the maps given by (1) define the cleaving system.

(2) Follows from the normal basis property (Theorem 33).

(3) We have $\rho_A(c) = \rho_A(a^2) = a^2 \otimes x^2 = c \otimes 1_H$. In the same way it follows that $d$ is a coinvariant element. The second relation is easy to get from the properties of $a$ and $b$.

(4) It is enough to use Remark 54. The convolution inverse of $\sigma$ is given by $\sigma^{-1}(x, x) = d$. In particular, notice that $\rho_A(aeb) = ac_0b \otimes x_c1x = aeb \otimes 1_H$, for any $e \in B$. \qed

As $H(2)$ has the coalgebra structure of $k[C_2]$, it is natural that $A$ admits a $C_2$-grading: $A = A_1 \oplus A_x$, with $A_1 = B$. For a clef extension, notice that the grading is strong, by Prop. 11 and Thm. 27 from [1], and $A_x$ is free cyclic left $B$-module.

Let $B \subseteq A$ a clef extension and define $F : B \longrightarrow B$, $F(e) = aeb$, for all $e \in B$, where $a, b \in A$ are the elements given by the previous Lemma. Then by the above, $F$ is a linear endomorphism of $B$. From the commutation relations for $a$ and $b$ it follows that $F$ is even an algebra morphism. Statement (3) of the Lemma implies that the square of $F$ is inner: $F^2(c) = ccc^{-1}$, for all $e \in B$ and that $F(c) = -c$. We shall see now that this is precisely what we need to built a crossed product by $H(2)$.

**Proposition 38.** Let $B$ be an associative algebra. For each $F \in \text{End}(B)$ and invertible element $c \in B$, define the following:

$$1 \cdot e = e, \quad x \cdot e = F(e)$$

$$\sigma(x, x) = c \text{ and 1 elsewhere}$$

Then $(B, \cdot, \sigma)$ form a crossed system if and only if

1. $F$ is an algebra morphism;
2. $F^2(e) = ccc^{-1}, \forall e \in B$;
3. $F(c) = -c$.

Proof. Notice first that $\sigma$ is convolution invertible regardless the above conditions, with inverse $\sigma^{-1}(x, x) = c^{-1}$. It is easy to see that (3.1) is equivalent with $F$ being an algebra endomorphism, relation (3.4) with property (2) and (3.6) is equivalent with (3). \qed

For an associative algebra $B$, we call the pair $(F, c)$ (where $F \in \text{End}_B(B)$ and $c \in U(B)$) a clef $H(2)$-datum for $B$ if conditions (1)-(3) from the previous proposition are fulfilled. The crossed product $B \#_{\sigma}H(2)$ will be denoted by $(F, c)$. In particular, the morphism $B \longrightarrow B \#_{\sigma}H(2), e \longrightarrow e \otimes 1_H$ is injective and preserves multiplication (can be seen as a comodule algebra morphism if we endow $B$ with the trivial comodule structure). Via this morphism, we shall be able to identify elements like $e \otimes 1_H$ with $e$, for any $e \in B$. If we also denote $a = 1_B \#_{\sigma}x$ and $b = -c^{-1}a = -c^{-1}1_B \#_{\sigma}x$, then $\{1_B \otimes 1_H, a\}$ is a left $B$-basis for the crossed product with relations $a^2 = c, b^2 = -c^{-1}, ab = 1_B \#_{\sigma}1, ba = -1_B \#_{\sigma}1$. The comodule structure is as follows: $\rho(a) = a \otimes x$ and $\rho(b) = b \otimes x$.

Hence we have obtained the following:

**Corollary 39.** Any clef $H(2)$-extension $B \subseteq A$ is of the form $(F, c)$. 

Remark 40.  
(1) If \( B = \mathbb{K} \supseteq k \) is a field extension, then Proposition 38 implies \( F^2 = 1_k \) and \( F(c) = -c, \) \( c \in \mathbb{K} \setminus \{0\} \). In particular, \( F(e) = e^{(1)} - e^{(2)}c \) where \( e = e^{(1)} + e^{(2)}c \) is the decomposition of an element \( e \in \mathbb{K} \) over the subfield \( \mathbb{K}^F \) fixed by \( F \) \( (e^{(1)}, e^{(2)} \in \mathbb{K}^F) \). For all \( c \neq 1_k \), the resulting product (as an algebra in the monoidal category of comodules) is a \( \mathbb{K} \)-vector space, with basis \( \{1_k \otimes 1, 1_k \otimes x\} \), multiplication (nonassociative, noncommutative) \( (\lambda \otimes x)(\kappa \otimes x) = \lambda F(\kappa)c \otimes 1 \) and neutral element \( 1_k \otimes 1 \).

(2) For a \( k \)-finite dimensional central simple algebra, conditions (1) and (2) of the previous Proposition imply that \( F \) is an algebra automorphism, so there is an invertible element \( c \in B \) such that \( F(e) = \overline{c}ec^{-1} \), for all \( e \in B \). In particular, it means that \( c^{-1}\overline{c}^2 \in Z(B) = k \), therefore \( c^{-1}\overline{c}^2 \) is a nonzero scalar. But condition (3) implies \( \overline{c}c = -c\overline{c} \), contradiction with \( \text{char}(k) \neq 2 \). Hence for a central finite simple dimensional algebra there are no \( H(2) \)-cleft extensions.

Let see now when two such crossed products are isomorphic.

Proposition 41. Let \((F,c)\) and \((F',c')\) be two \( H(2) \)-data. Then \( \frac{F,c}{B} \simeq \frac{F',c'}{B} \) as right \( H(2) \)-comodule algebras and left \( B \)-modules if and only if it exists an invertible element \( s \in B \) such that \( c' = s^{-1}F(s^{-1})c \) and \( F'(e) = s^{-1}F(e)s \), for all \( e \in B \).

Proof. It is a direct application of Theorem 16. Let \( \theta : \left( \frac{F,c}{B} \right) \longrightarrow \left( \frac{F',c'}{B} \right) \) be such an isomorphism. Then \( \theta(e\# h) = ea(h_1)\# a(h_2) \) for all \( e \in B \) and \( h \in H(2) \), where \( a : H(2) \longrightarrow B \) is convolution invertible. As \( \eta(1_B \# 1) = 1_B \# 1 \), it follows that \( a(1_B) = 1 \). Denote \( a(x) = s \). Then \( s \) is a unit because \( a \) is convolution invertible. Moreover, by simply applying Theorem 16 (2) and (3) we obtain the relations \( F(e) = s^{-1}F(e)s \) and \( c' = s^{-1}F(s^{-1})c \). Conversely for \( s \in B \) invertible with the above properties, it is easy to check that the mapping \( a : H(2) \longrightarrow B, a(1) = 1, a(x) = s \) fulfills the required conditions for the existence of a crossed product isomorphism. \( \square \)

We consider now another example of a coquasi-Hopf algebra, but this time of dimension 3. Start with a field \( k \) containing a root \( q \neq 1 \) of order 3 of the unit. We denote by \( H(3) \) the coquasi-bialgebra of basis \( \{1,x,x^2\} \) with algebra and coalgebra structures as for \( k[C_3] \), but with cocycle \( \omega \) given by the formulas:

\[
\begin{align*}
\omega(x, x^2, x) &= \omega(x^2, x, x) = \omega(x^2, x^2, x) = q^{-1} \\
\omega(x, x^2, x^2) &= \omega(x^2, x, x^2) = \omega(x^2, x^2, x^2) = q
\end{align*}
\]

and trivial in rest. Then in [12] it is shown that \( \omega \) is not a coboundary (cannot be obtained from a twist). The antipode is the same as for the group algebra, the linear map \( \alpha \) is trivial, but \( \beta \) is not:

\[
\beta(1) = 1, \beta(x) = q, \beta(x^2) = q^{-1}
\]

Consider \( A \) a right \( H(3) \)-comodule algebra and \( B = A^{co H(3)} \). We obtain then the following results (that we present without proof, as they follow the same arguments as for dimension 2):

Lemma 42. The extension \( B \subseteq A \) is \( H(3) \)-cleft if and only if there exist elements \( a, b, c, d \in A \) such that

\[
\begin{align*}
ac &= q^{-1}1_A, \quad ca = 1_A \\
bd &= q1_A, \quad db = 1_A \\
\rho_A(a) &= a \otimes x, \quad \rho_A(c) = c \otimes x^2 \\
\rho_A(b) &= b \otimes x^2, \quad \rho_A(d) = d \otimes x
\end{align*}
\]

In this case, the following hold:

(1) The maps \( \gamma : H(3) \longrightarrow A, \gamma(1) = 1_A, \gamma(x) = a, \gamma(x^2) = b \) and \( \delta : H(3) \longrightarrow A, \delta(1) = 1_A, \delta(x) = c, \delta(x^2) = d \) provide the cleft extension;
(2) A is free left $B$-module with basis $\{1_A, a, b\}$;
(3) If we denote
\[
\begin{align*}
  u^{(1)} &= (a^2)dq^{-1}, & u^{(2)} &= (b^2)cq \\
  v^{(1)} &= ab, & v^{(2)} &= ba
\end{align*}
\]
Then these are invertible $B$-elements, with inverses
\[
\begin{align*}
  u^{(1)}^{-1} &= b(c^2)q^{-1}, & u^{(2)}^{-1} &= a(d^2)q \\
  v^{(1)}^{-1} &= dcq, & v^{(2)}^{-1} &= cdq^{-1}
\end{align*}
\]
(4) The $H(3)$-weak action on $B$ and the cocycle are given by
\[
1 \cdot e = e, \quad x \cdot e = acq, \quad x^2 \cdot e = bedq^{-1}
\]
for all $e \in B$, respectively

| $\sigma(-, -)$ | 1 | $x$ | $x^2$ |
|----------------|---|-----|-------|
| 1              | 1 | 1   | 1     |
| $x$            | 1 | $u^{(1)}$ | $v^{(1)}$ |
| $x^2$          | 1 | $u^{(2)}$ | $v^{(2)}$ |

Proposition 43. Let $B$ be an associative algebra. For any $F, G \in \text{End}_k(B)$ and $u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)} \in U(B)$, define an $H(3)$-weak action on $B$ by
\[
1 \cdot e = e, \quad x \cdot e = F(e), \quad x^2 \cdot e = G(e)
\]
Consider also the linear map $\sigma : H(3) \otimes H(3) \to B$, given by

| $\sigma(-, -)$ | 1 | $x$ | $x^2$ |
|----------------|---|-----|-------|
| 1              | 1 | 1   | 1     |
| $x$            | 1 | $u^{(1)}$ | $v^{(1)}$ |
| $x^2$          | 1 | $u^{(2)}$ | $v^{(2)}$ |

Then $(B, \cdot, \sigma)$ form a crossed system if and only if
(1) $F$ and $G$ are algebra endomorphisms;
(2) The composition rules for these two endomorphisms are

(5.1)

| $\sigma(-, -)$ | 1 | $x$ | $x^2$ |
|----------------|---|-----|-------|
| 1              | 1 | 1   | 1     |
| $x$            | 1 | $u^{(1)}$ | $v^{(1)}$ |
| $x^2$          | 1 | $u^{(2)}$ | $v^{(2)}$ |

(3) The actions of $F$ and $G$ on the above invertible elements are

$\begin{array}{|c|c|c|c|c|}
\hline
& u^{(1)} & u^{(2)} & v^{(1)} & v^{(2)} \\
\hline
\mathcal{F} & u^{(1)} & u^{(2)} & v^{(1)} & v^{(2)} \\
\hline
\mathcal{G} & u^{(2)} & u^{(2)} & v^{(1)} & v^{(2)} \\
\hline
\end{array}$

Let $B$ be again an associative algebra. We shall call $(\mathcal{F}, \mathcal{G}, u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)})$ (where $\mathcal{F}, \mathcal{G} \in \text{End}_k(B)$ and $u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)} \in U(B)$) a cleft $H(3)$-datum for $B$ if conditions (1)-(3) from the previous Proposition are fulfilled. The resulting crossed product $B \#_{\sigma} H(3)$ will be denoted by $(B, \#_{\sigma} H(3))$. In particular, the morphism $B \to B \#_{\sigma} H(3)$, $e \to e \otimes 1_H$ is injective and preserves multiplication. Via this morphism, we may identify elements $e \otimes 1_H$ with $e$, for $e \in B$. If denote $a = 1_B \#_{\sigma} x, b = 1_B \#_{\sigma} x^2,$
c = v^{(2)}-1\sigma_1 x^2, d = v^{(1)}-1\sigma_1 x, then \{1_B \otimes 1_{H(2)}, a, b\} forms a left B-basis for \((F, G, u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}) / B\) with multiplication table

|   | a          | b          | c           | d           |
|---|------------|------------|-------------|-------------|
| a | u^{(1)}\#_1 x^2 | v^{(1)}\#_1 H(3) | q^{-1}(1_B \#_1 H(3)) | u^{(2)}-1\#_1 x |
| b | v^{(2)}\#_1 H(3) | u^{(2)}\#_1 x | q^{-1}(u^{(1)}-1\#_1 x) | q(1_B \#_1 H(3)) |
| c | 1_B \#_1 H(3) | v^{(2)}-1\#_1 x | q^{-1}(v^{(2)}-1\#_1 x) | q(v^{(2)}-1\#_1 H(3)) |
| d | v^{(1)}-1\#_1 H(3) | 1_B \#_1 H(3) | q(v^{(1)}-1\#_1 H(3)) | v^{(1)}-1\#_1 x^2 |

The comodule structure is given by \(\rho(a) = \rho(b) = b \otimes x^2, \rho(c) = c \otimes x^2, \rho(d) = d \otimes x\).

We have then obtained the following:

**Corollary 44.** Any \(H(3)\)-cleft extension \(B \subseteq A\) is isomorphic to a crossed product \((F, G, u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}) / B\).

**Remark 45.** Relations \([5.1]\) imply that \(F\) and \(G\) are algebra automorphisms, with \(FG, GF, F^3\) and \(G^3\) inner. In particular, for \(B\) commutative, it follows that \(F^3 = I_B\) and \(F^2 = G\).

Finally we shall see when two such crossed products are isomorphic:

**Proposition 46.** Let \((F, G, u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)})\) and \((F', G', u'^{(1)}, u'^{(2)}, v'^{(1)}, v'^{(2)})\) two \(H(3)\)-data. Then

\[
\left(\frac{F, G, u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}}{B}\right) \simeq \left(\frac{F', G', u'^{(1)}, u'^{(2)}, v'^{(1)}, v'^{(2)}}{B}\right)
\]

as right \(H(3)\)-comodule algebras and left \(B\)-modules if and only if there exist invertible elements \(s^{(1)}, s^{(2)} \in B\) such that

\[
\begin{align*}
F'(e) &= s^{(1)}-1 F(e)s^{(1)} \\
G'(e) &= s^{(2)}-1 G(e)s^{(2)}
\end{align*}
\]

for all \(e \in B\), and

\[
\begin{align*}
u'^{(1)} &= s^{(1)}-1 F(s^{(1)}-1)u^{(1)}s^{(2)} \\
u'^{(2)} &= s^{(2)}-1 G(s^{(2)}-1)u^{(2)}s^{(1)} \\
v'^{(1)} &= s^{(1)}-1 F(s^{(2)}-1)v^{(1)} \\
v'^{(2)} &= s^{(2)}-1 G(s^{(2)}-1)v^{(2)}
\end{align*}
\]

**Proof.** Denote \(a(x) = s^{(1)}\) and \(a(x^2) = s^{(2)}\), where \(a : H(3) \rightarrow B\) is the convolution invertible map given by Theorem \([16]\). \(\Box\)

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