Path Integrals in Lattice Quantum Chromodynamics

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Abstract: I discuss the use of path integrals to study strong-interaction physics from first principles. The underlying theory is cast into path integrals which are evaluated numerically using Monte Carlo methods on a space-time lattice. Examples are given on progress related to nuclear physics.

Keywords: path integrals, lattice QCD, strong interaction, Monte Carlo

1. Introduction

Quantum chromodynamics (or QCD) is the underlying theory of the strong interaction (one of the four fundamental interactions in nature besides gravity, electromagnetism and the weak interaction). It is the force that binds quarks and gluons as the nucleus in the heart of the atom. Since all matter in the universe is built that way, unraveling the structure of matter at its deepest level as governed by QCD is key to our understanding of the physical world, and presents one of the most challenging tasks in contemporary nuclear and particle physics. While it is easy to write down the basic equation of QCD, it is very difficult to obtain quantitative solutions in view of the complex quark-gluon dynamics. At present, the only known way to solve QCD directly is by numerically evaluating the path integrals in the theory on a discrete space-time lattice using supercomputers. For textbooks on the subject, see Refs.1–3

2. Formalism

All physics can be computed by path integrals in QCD. Take the calculation of the proton mass for example. It requires the fully-interacting quark propagator defined by the Euclidean-space path integrals over gluon field

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\( A_\mu \) and quarks fields \( \psi \) and \( \bar{\psi} \),

\[
< M^{-1} > \equiv < 0 | \psi(x) \bar{\psi}(0) | 0 > = \frac{\int D\psi D\bar{\psi} D A_\mu e^{-S_{QCD}} M^{-1}}{\int D\psi D\bar{\psi} D A_\mu e^{-S_{QCD}}}.
\] (1)

The action is the sum of a gluon part and a quark part

\[
S_{QCD} = S_G + S_q = \frac{1}{2} \int dx^4 Tr F_{\mu\nu} F^{\mu\nu} + \int dx^4 \bar{\psi} M \psi,
\] (2)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g [A_\mu, A_\nu] \) is the gluon field strength tensor and \( M = \gamma^\mu D_\mu + m_q \) is the Dirac operator for quarks and \( \gamma^\mu \) the 4x4 gamma matrices. The interaction between the two is provided by the covariant derivative \( D_\mu = \partial_\mu + g A_\mu \). The basic parameters of the theory are the coupling constant \( g \) and quark mass \( m_q \). The quark part of the integration can be done analytically using Grassmann variable integration, leading to a path integral over only gluon fields

\[
< M^{-1} > \equiv \frac{\int DA_\mu \det(M) e^{-S_G} M^{-1}}{\int DA_\mu \det(M) e^{-S_G}}
\] (3)

which is evaluated numerically by Monte-Carlo methods. In the quenched approximation, the quark determinant \( \det(M) \) is ignored (equivalent to suppressing vacuum polarization), making the numerical calculation significantly faster.

To compute the proton mass, one considers the time-ordered, two-point correlation function in the QCD vacuum, projected to zero momentum:

\[
G(t) = \sum_\vec{x} \langle 0 | T \{ \eta(x) \bar{\eta}(0) \} | 0 \rangle.
\] (4)

Here \( \eta \) is called the interpolating field which is built from quark fields with the quantum numbers of the proton (spin-1/2, isospin-1/2, quark content uud, charge +e)

\[
\eta(x) = \epsilon^{abc} [u^T(x)C\gamma_5 d^b(x)] u^c(x)
\] (5)

where \( C \) is the charge conjugation operator and the superscript \( T \) means transpose. Sum over color indeces \( a,b,c \) is implied and the \( \epsilon^{abc} \) ensures the proton is color-singlet.

The calculation of \( G(t) \) at the quark level proceeds by contracting out all the quark pairs, resulting in

\[
G(t) = \sum_\vec{x} \epsilon^{abc} \epsilon^{a'b'c'} \left[ S_a^{a'} \gamma_5 CS_{d'}^{c} C_{\gamma_5} S_u^{b'} + S_u^{a'} \text{Tr}(CS_{d'}^{c} C_{\gamma_5} S_u^{b'}) \right],
\] (6)
where $S^{ij}_q$ denotes the fully-interacting quark propagator $< M^{-1} >$ in Eq. (3).

On the hadronic level, the correlation function is saturated by the complete spectrum of intermediate states (after a parity projection) with the proton as the ground state:

$$G(t) = \sum_i \lambda_i^2 e^{-m_i t}$$  \hspace{1cm} (7)

where $m_i$ are the masses and $\lambda_i^2$ are the 'amplitudes' which are a measure of the ability of the interpolating field to excite or annihilate the states from the QCD vacuum. At large time, the ground state proton, $\lambda_1^2 e^{-m_1 t}$, dominates $G(t)$, with the excited states exponentially suppressed. Other physics quantities are computed in a similar way.

3. Some examples

It is impossible to give a full account of the achievements made in lattice QCD in the space given here. A good place to gauge the progress in the field (which is not limited to QCD) is the annual gathering by active practitioners. Here I select a few examples relevant to nuclear physics.

Example 1. Fig. 1 shows that the computed light hadron mass spectrum which comes within 10% of the observed spectrum. The remaining discrepancy is attributed to the quenched approximation. Intense efforts are under way to extend the success to the excited sectors of the mass spectrum.\(^6\)

Example 2. Fig. 2 shows the high-precision lattice QCD calculations for selected physics quantities which are within 3\% of the observed values. The effects of the quenched approximation are clearly demonstrated.

Example 3. Fig. 3 shows a calculation of the magnetic moment of the proton and neutron which find good agreement between lattice QCD and experiment. Results from a different method can be found in Ref.\(^9\) as well as precise determinations of the strangeness magnetic ($G_M^s$)\(^10\) and electric form factors ($G_E^s$)\(^11\) of the proton.

Example 4. Fig. 4 shows the good agreement between lattice QCD and experiment for pion-pion scattering length,\(^12\) using Lüscher’s method.\(^13\) Other hadron-hadron scattering channels ($\pi^+ K^+$, $n\Lambda$, $n\Sigma^-$, etc.) have been studied by the same group.

Example 5. Fig. 5 shows a direct lattice QCD calculation of the nucleon-nucleon potential\(^14\) whose features are consistent with the known phenomenological features of the nuclear force.
Fig. 1. The computed light hadron mass spectrum in the quenched approximation. See Ref.\textsuperscript{5} for details.

Fig. 2. Lattice QCD results compared to experiment as ratios for selected physics quantities (the dashed line is perfect agreement), in the quenched approximation of QCD (left panel) and full QCD (right panel). See Ref.\textsuperscript{7} for details.
Fig. 3. Magnetic moment of the proton and neutron as a function of the pion mass squared. The line is a chiral extrapolation. See Ref.\textsuperscript{8} for details.

Fig. 4. Lattice QCD calculation of the $\pi - \pi$ scattering length in the isospin 2 channel. The dashed line is a chiral fit. See Ref.\textsuperscript{12} for details.

4. Conclusion

The path-integral formulation of QCD, coupled with large-scale numerical simulations, has played a crucial role in the progress of nuclear and particle physics. Its continued prominence as the only way to solve QCD with controlled errors is expected to play out with better results in the foreseeable future. This work is supported in part by U.S. Department of Energy under grant DE-FG02-95ER40907.
Fig. 5. The lattice QCD result of the central (effective central) part of the NN potential in the $^1S_0$ ($^3S_1$) channels. The inset shows its enlargement. The solid lines correspond to the one-pion exchange potential (Yukawa potential). See Ref. 14 for details.

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