Algebraic Cuntz–Pimsner rings

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Abstract

From a system consisting of a ring $R$, a pair of $R$-bimodules $Q$ and $P$ and an $R$-bimodule homomorphism $\psi : P \otimes Q \rightarrow R$, we construct a $\mathbb{Z}$-graded ring $T_{(P,Q,\psi)}$ called the Toeplitz ring and (for certain systems) a $\mathbb{Z}$-graded quotient $O_{(P,Q,\psi)}$ of $T_{(P,Q,\psi)}$ called the Cuntz–Pimsner ring. These rings are the algebraic analogues of the Toeplitz $C^*$-algebra and the Cuntz–Pimsner $C^*$-algebra associated to a $C^*$-correspondence (also called a Hilbert bimodule). This new construction generalizes, for example, the algebraic crossed product by a single automorphism, fractional skew monoid rings by a single corner automorphism and Leavitt path algebras. We also describe the structure of the graded ideals of our graded rings in terms of pairs of ideals of the coefficient ring and show that our Cuntz–Pimsner rings satisfy the Graded Uniqueness Theorem.

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Introduction

In [24] Pimsner introduced a way to construct a $C^*$-algebra $O_X$ from a $C^*$-correspondence (which Pimsner calls a Hilbert bimodule) $X$ over a $C^*$-algebra $A$. These so-called Cuntz–Pimsner algebras have been found to be a class of $C^*$-algebras that is extraordinarily rich and with numerous examples included in the literature: crossed products by automorphisms, Cuntz algebras, Cuntz–Krieger algebras, $C^*$-algebras associated to graphs without sinks and Exel–Laca algebras. Later on Katsura [16] improved the construction of Pimsner in the case that the left action on the correspondence is not injective; this, for example, allows us to include the class of $C^*$-algebras associated with any graph in the Cuntz–Pimsner algebras class. Consequently the study of the Cuntz–Pimsner algebras has received a lot of attention in recent years, and because information about $O_X$ is densely codified in $X$ and $A$, determining how to extract it has been the focus of considerable interest. This has, for example, resulted in methods for computing the $K$- and $KK$-theory of Cuntz–Pimsner algebras (see [16, 24]).
a gauge-invariant uniqueness theorem for Cuntz–Pimsner algebras (see [16, Theorem 6.4]), the finding of necessary and sufficient conditions for Cuntz–Pimsner algebras to be nuclear and exact [16, Theorems 7.1 and 7.3], and a description of the gauge-invariant ideals of a Cuntz–Pimsner algebra (see [17, Theorem 8.6]).

Many of the $C^*$-algebras which can be constructed as Cuntz–Pimsner algebras have algebraic analogues. For example, the crossed product of a ring by an automorphism is the obvious analogue of the crossed product of a $C^*$-algebra of an automorphism. In [5] Ara, González-Barroso, Goodearl and Pardo, inspired by a construction in $C^*$-algebra, constructed fractional skew monoid rings from actions of monoids on rings by endomorphisms. In [20] Leavitt described a class of $F$-algebras $L_F(m, n)$ (where $F$ is an arbitrary field) which are universal with respect to an isomorphism property between finite-rank modules, that is, $L_F(m, n)^n \cong L_F(m, n)^m$. Later Cuntz [8] (independently) constructed and investigated the $C^*$-algebra $O_n$, called the Cuntz algebras. When $F$ is the field of complex numbers then $O_n$ can be viewed as a completion, in an appropriate norm, of $L_C(1, n)$. Soon after the appearance of [8], Cuntz and Krieger [9] described the significantly more general notion of the $C^*$-algebra of a (finite) matrix $A$, denoted by $O_A$. In [18] Cuntz–Krieger algebras were generalized to $C^*$-algebras of locally finite directed graphs, and this construction has later been generalized several times and now applied to arbitrary directed graphs (cf. [11, 25]). Inspired by the fractional skew monoid rings and by the graph $C^*$-algebras, Abrams and Aranda Pino [2] constructed the Leavitt path algebra of a row-finite directed graph. This construction was later generalized to apply to arbitrary directed graphs (cf. [3]). The Leavitt path algebras provide a generalization of Leavitt algebras of type $(1, n)$ just in the same way as graph $C^*$-algebras $C^*(E)$ provide a generalization of Cuntz algebras, and they have recently attracted a great deal of interest (see, for example [1, 3, 6, 27]).

It would be interesting and useful to put these rings and algebras in a larger category of rings whose properties can be studied and analysed from more simple objects, just as it has been done in the $C^*$-algebraic setting with Cuntz–Pimsner algebras. This is the purpose of this paper.

From a ring $R$ and a triple $(P, Q, \psi)$, called an $R$-system, consisting of two $R$-bimodules $P$ and $Q$ and a $R$-bimodule homomorphism $\psi : P \otimes Q \to R$ we construct a universal $\mathbb{Z}$-graded ring $T_{(P, Q, \psi)}$, called the Toeplitz ring associated with $(P, Q, \psi)$, which contains copies of $R$, $P$ and $Q$ and which implements the $R$-bimodule structure of $P$ and $Q$ and the $R$-bimodule homomorphism $\psi$. We then, for $R$-systems satisfying a certain condition which we call (FS), carefully study quotients of $T_{(P, Q, \psi)}$ which preserve the $\mathbb{Z}$-grading of $T_{(P, Q, \psi)}$. We show that under a mild assumption about the $R$-system $(P, Q, \psi)$, there exists a smallest quotient of $T_{(P, Q, \psi)}$ which preserves the $\mathbb{Z}$-grading of $T_{(P, Q, \psi)}$ and which leaves the embedded copy of $R$ intact. We define the Cuntz–Pimsner ring $O_{(P, Q, \psi)}$ of $(P, Q, \psi)$ to be this quotient.

We show that the construction of Cuntz–Pimsner rings is a generalization of, for example, the crossed product of a ring by an automorphism, the Leavitt path algebra of a directed graph and of the fractional skew monoid ring of a corner isomorphism. We also generalize the Graded Uniqueness Theorem known from Leavitt path algebras to our class of Cuntz–Pimsner rings, and describe the structure of the graded ideals of $T_{(P, Q, \psi)}$ (and thus of $O_{(P, Q, \psi)}$, if it exists), in terms of pairs of ideals of $R$.

We believe that our construction is interesting both from the point of view of algebra and from the point of view of operator algebra. Our construction unifies many interesting classes of rings, and we believe it will provide us with the right frame for studying properties, such as the ideal structure, the $K$-theory, pure infiniteness, and the real and stable rank of these rings. It is also worth mentioning that the construction of Cuntz–Pimsner algebras has been generalized in several ways in $C^*$-algebra theory (see, for example, [10, 14, 26]), and there is no reason to believe that the same cannot be done in the algebraic setting. We also expect that other examples of classes of $C^*$-algebras which can be obtained through the Cuntz–Pimsner
construction, such as $C^*$-algebras associated with subshifts (cf. [7]) and $C^*$-algebras of self-similar groups (cf. [22]), can be adapted to the algebraic setting through our construction. So this paper is hopefully only the first step on the way of what we hope to be a fruitful adaption of work done in operator algebra to the algebraic setting.

We also believe that if one is only interested in the $C^*$-algebraic case, then there is some insight to be gained by reading this paper. One reason is that $C^*$-algebras have some nice properties not shared by all rings. For example, a $C^*$-algebra is always non-degenerate and semiprime. This means that things which automatically work in the $C^*$-algebraic setting do not necessarily work in the algebraic setting, and we believe that by studying the algebraic case, one gains some insight into why things work the way they do in the operator algebraic case. Here are some of the specific differences between the $C^*$-algebraic case and the purely algebraic case.

(1) In the algebraic case we are not just working with a single bimodule equipped with an inner product, but with more general systems consisting of two $R$-bimodules $Q$ and $P$ connected by a bimodule homomorphism $\psi : P \otimes Q \to R$.

(2) If we are working with a right degenerate ring, then the Fock space representation does not have the universal property the Toeplitz representation should have. We therefore have to construct the Toeplitz representation in a different way.

(3) Unlike in the $C^*$-algebraic case, we do not in the algebraic case automatically have that every representation will induce a representation of the finite-rank operators (which correspond to the compact operators) of the $R$-system in question. We therefore have to introduce a condition on the $R$-systems we are working with which ensures that every representation will induce a representation of the finite-rank operators. We do that by introducing the condition we call (FS). This is probably not the optimal condition, but it is quite natural and satisfied by all the interesting examples we consider in this paper.

(4) Unlike the Toeplitz and Cuntz–Pimsner $C^*$-algebras, the algebraic Toeplitz and Cuntz–Pimsner rings do not in general carry a gauge action. Instead, we have to work with $\mathbb{Z}$-gradings.

(5) In the algebraic case, it is not always the case that a representation with all the properties the Cuntz–Pimsner representation should have exists (that it always exists in the $C^*$-algebraic case is because every $C^*$-algebra is semiprime). We think this is an interesting fact on its own, but it means that we, in general, have to work with relative Cuntz–Pimsner rings instead of Cuntz–Pimsner rings.

Another reason why we believe that our construction is interesting from the point of view of operator algebra is that since we do not have to worry about any norms or topology, our arguments become more tangible than in the $C^*$-algebraic setting. This allows us, for example, to put everything into a frame of category theory, which we think makes this whole construction more transparent. We believe that something similar can, and ought to, be done in the $C^*$-algebraic setting.

The contents of the paper

The contents of this paper can be summarized as follows.

In Section 1 we give some basic definitions and introduce $R$-systems $(P, Q, \psi)$ (Definition 1.1). We define the category $\mathcal{C}_{(P, Q, \psi)}$ of surjective covariant representations of an $R$-system $(P, Q, \psi)$ (Definition 1.4), and we prove that this category has an initial object which we call the Toeplitz representation (Theorem 1.7). We then introduce some essential examples of this construction, namely $R$-systems associated with ring automorphisms (Example 1.9) and with directed graphs (Example 1.10), and we study their Toeplitz representations.

Section 2 defines the ring of adjointable homomorphisms $\mathcal{L}_P(Q)$ (Definition 2.1) as well as its ideal of the finite-rank adjointable homomorphisms $\mathcal{F}_P(Q)$ (Definition 2.2) and gives us the
Fock space representation (Proposition 2.5) which we later show is isomorphic to the Toeplitz representation under certain conditions (Proposition 4.2).

In Section 3 we show that the Toeplitz ring $T(P,Q,\psi)$, on which the Toeplitz representation of an $R$-system $(P,Q,\psi)$ lives, comes with a $\mathbb{Z}$-grading (Proposition 3.1). We then go on to study graded and injective representations of $(P,Q,\psi)$; that is, representations which are compatible with the $\mathbb{Z}$-grading of $T(P,Q,\psi)$ (Definition 3.3) and for which the representation of $R$ is injective (Definition 1.2). To do this we need that every representation of $(P,Q,\psi)$ induces a representation of $F_P(Q)$. In contrast to the $C^*$-algebraic case where a representation of a Hilbert bimodule always induces a representation of the compact operators of the bimodule, a representation of $(P,Q,\psi)$ does not automatically induce a representation of $F_P(Q)$. We introduce a condition called (FS) on $(P,Q,\psi)$ (Definition 3.4) which guarantees that every representation of $(P,Q,\psi)$ induces a representation of $F_P(Q)$ (Proposition 3.11). Under this condition we define the relative Cuntz–Pimsner ring $O(P,Q,\psi)(J)$ of an $R$-system $(P,Q,\psi)$ with respect to an ideal $J$ as a certain quotient of the Toeplitz ring $T(P,Q,\psi)$ (Definition 3.16), and we show that the representations of $(P,Q,\psi)$ corresponding to these relative Cuntz–Pimsner rings include, up to isomorphism, all graded and injective representations of $(P,Q,\psi)$ (Remark 3.30).

In Section 4 we use the classification of graded and injective representations obtained in Section 3 to first show that under certain conditions the Fock representation of an $R$-system is isomorphic to the Toeplitz representation (Proposition 4.2), and we then show that a relative Cuntz–Pimsner ring $O(P,Q,\psi)(J)$ satisfies the Graded Uniqueness Theorem (Definition 4.5) if and only if the ideal $J$ is maximal among the ideals of $R$ for which the corresponding relative Cuntz–Pimsner representation is injective (Theorem 4.7). We also show by example that there can be more than one such maximal ideal (Example 4.11). This is in contrast to the $C^*$-algebraic case where there always exists a unique such maximal ideal.

If such a unique maximal ideal exists, then we define the Cuntz–Pimsner representation of the $R$-system in question to be the relative Cuntz–Pimsner representation corresponding to this maximal ideal (Definition 5.1). We do this in Section 5 where we also give conditions under which such a unique maximal ideal exists (Lemmas 5.2 and 5.3) and show that several interesting examples satisfy these conditions (Examples 5.5, 5.6, 5.7 and 5.8). We then show that the Cuntz–Pimsner ring, the ring on which the Cuntz–Pimsner representation lives, automatically satisfies the Graded Uniqueness Theorem (Corollary 5.4) and use this to show that we can construct the Leavitt path algebras (Example 5.8), the crossed product of a ring $R$ by an automorphism (Example 5.5) and the fractional skew monoid ring of a corner isomorphism (Example 5.7) as Cuntz–Pimsner rings.

In Section 6 we generalize the Algebraic Gauge-Invariant Uniqueness Theorem of [1] to our Cuntz–Pimsner rings (Corollary 6.3), and thereby to all Leavitt path algebras (Corollary 6.4).

Finally in Section 7 we extend the classification of graded and injective representations obtained in Section 3 to graded representations which are not necessarily injective (Remark 7.12) and use this classification to give a complete description of the graded ideals of relative Cuntz–Pimsner rings (and thereby of Toeplitz rings, and of Cuntz–Pimsner rings) in terms of certain pairs of ideals of $R$ (Theorem 7.27 and Corollaries 7.28 and 7.29).

1. The Toeplitz ring

First we establish the basic definitions for our setting. Throughout the paper we let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0$ the set of non-negative integers.

A ring $R$ is said to be right (left) non-degenerate if $rR = 0$ ($Rr = 0$) implies $r = 0$. A ring $R$ is said to be non-degenerate if it is both right and left non-degenerate. A non-degenerate ring has local units if for every finite set $\{r_1, \ldots, r_n\} \subseteq R$ there exists an idempotent $e \in R$ such that $r_i \in eRe$ for every $i = 1, \ldots, n$. 
Let $R$ be a ring. Given two $R$-bimodules $P$ and $Q$ we will by $P \otimes Q$ denote the $R$-balanced tensor product.

1.1. $R$-systems, covariant representations and the Toeplitz representation

**Definition 1.1.** Let $R$ be a ring. An $R$-system is a triple $(P, Q, \psi)$ where $P$ and $Q$ are $R$-bimodules, and $\psi$ is a $R$-bimodule homomorphism from $P \otimes Q$ to $R$.

**Definition 1.2** (cf. [21, Definition 2.11]). Let $R$ be a ring and $(P, Q, \psi)$ an $R$-system. We say that a quadruple $(S, T, \sigma, B)$ is a covariant representation of $(P, Q, \psi)$ on $B$ if:

(i) $B$ is a ring;
(ii) $S : P \to B$ and $T : Q \to B$ are additive maps;
(iii) $\sigma : R \to B$ is a ring homomorphism;
(iv) $S(pr) = S(p)\sigma(r)$, $S(rp) = \sigma(r) S(p)$, $T(qr) = T(q)\sigma(r)$ and $T(rq) = \sigma(r) T(q)$ for $r \in R$, $p \in P$ and $q \in Q$;
(v) $\sigma(\psi(p \otimes q)) = S(p) T(q)$ for $p \in P$ and $q \in Q$.

We denote by $\mathcal{R}(S, T, \sigma)$ the subring of $B$ generated by $\sigma(R) \cup T(Q) \cup S(P)$. If $\mathcal{R}(S, T, \sigma) = B$, then we say that the covariant representation $(S, T, \sigma, B)$ is surjective, and if the ring homomorphism $\sigma$ is injective, then we say that the covariant representation $(S, T, \sigma, B)$ is injective.

**Examples 1.3.** (i) Let $R$ be any ring and let $P = Q = R$ be the regular $R$-bimodules. Define $\psi : P \otimes Q \to R$ by $\psi(p \otimes q) = pq$. We then have that $(P, Q, \psi)$ is an $R$-system. We can define a covariant representation $(S, T, \sigma, R[t, t^{-1}])$, where $R[t, t^{-1}]$ is the Laurent polynomial ring with coefficients in $R$, by letting $T(q) = qt$, $S(p) = pt^{-1}$ and $\sigma(r) = r$ for every $p \in P$, $q \in Q$ and $r \in R$. It is easy to check that $(S, T, \sigma, R[t, t^{-1}])$ is indeed a covariant representation of $(P, Q, \psi)$. Observe that this representation is injective and surjective.

(ii) Let $P = Q = \mathbb{R}$ be the $\mathbb{R}$-module $\mathbb{R}$. Define $\psi : \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ by $\psi(p \otimes q) = -pq$. We then have that $(P, Q, \psi)$ is an $\mathbb{R}$-system. We can then define a covariant representation $(S, T, \sigma, \mathbb{C})$ by letting $T(q) = qi$, $S(p) = pi$ and $\sigma(r) = r$ for every $p \in P$, $q \in Q$ and $r \in \mathbb{R}$. This representation is injective and surjective.

(iii) Let $P = Q$ be the $\mathbb{Z}$-module $\mathbb{Z}$. Then if given any $a \in \mathbb{Z}$ we define $\psi_a : \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ by $\psi_a(p \otimes q) = apq$. Then $(P, Q, \psi_a)$ is a $\mathbb{Z}$-system. We can then define a covariant representation $(S, T, \sigma, \mathbb{C})$ by letting $T(q) = q\sqrt{a}$, $S(p) = p\sqrt{a}$ and $\sigma(r) = r$ for every $p \in P$, $q \in Q$ and $r \in \mathbb{R}$. Note that the representation $(S, T, \sigma, \mathbb{C})$ is injective but not surjective.

(iv) Let $V$ be a $K$-vector space and let $Q(-, -) : V \times V \to K$ be a non-degenerate quadratic form. Then $V$ is a $K$-module, and if we let $P = Q = V$ and define $\psi_V : V \otimes V \to K$ by $\psi_V(p \otimes q) = Q(p, q)$, then $(P, Q, \psi_V)$ is a $K$-system. Recall that the Clifford algebra $\mathcal{Cl}(V, Q)$ is the universal unital $K$-algebra generated by $V$ and with the relation $v^2 = Q(v, v)1$ for every $v \in V$. Therefore we can define a covariant representation $(S, T, \sigma, \mathcal{Cl}(V, Q))$ of $(P, Q, \psi_V)$ by letting $T(v) = v$, $S(v) = v$ and $\sigma(k) = k1$. This representation is surjective.

**Definition 1.4.** Let $R$ be a ring and $(P, Q, \psi)$ an $R$-system. We denote by $\mathcal{C}(P, Q, \psi)$ the category whose objects are surjective covariant representations $(S, T, \sigma, B)$ of $(P, Q, \psi)$, and where the class of morphisms between two representations $(S_1, T_1, \sigma_1, B_1)$ and $(S_2, T_2, \sigma_2, B_2)$ is the class of ring homomorphisms $\phi : B_1 \to B_2$ such that $\phi \circ T_1 = T_2$, $\phi \circ S_1 = S_2$ and $\phi \circ \sigma_1 = \sigma_2$. 


The main purpose of this paper is, for a given $R$-system $(P,Q,\psi)$, to study the category $\mathcal{C}_{(P,Q,\psi)}$. First we will show that $\mathcal{C}_{(P,Q,\psi)}$ has an initial object, but we begin with some more definitions and an easy lemma.

Given an $R$-system $(P,Q,\psi)$ we define recursively the $R$-bimodules $P^\otimes n$ and $Q^\otimes n$ by letting $P^1 = P$ and $Q^1 = Q$, and letting $P^\otimes n = P^\otimes (n-1) \otimes P$ and $Q^\otimes n = Q^\otimes (n-1) \otimes Q$ for $n > 1$. We also let $P^\otimes 0 = Q^\otimes 0 = R$. Then we define $\psi : P^k \otimes Q^0 \to R$ by

$$r_1 \otimes r_2 \mapsto r_1 r_2$$

for $r_1, r_2 \in R$, and we let $\psi = \psi$ and define recursively $\psi_n : P^\otimes n \otimes Q^\otimes n \to R$ for $n > 1$ by

$$(p_1 \otimes p_2) \otimes (q_1 \otimes q_2) \mapsto \psi(p_1 \cdot \psi_{n-1}(p_2 \otimes q_1) \otimes q_2)$$

for $p_2 \in P^\otimes n$, $p_1 \in P$, $q_1 \in Q^\otimes n-1$ and $q_2 \in Q$.

\textbf{Lemma 1.5.} Let $R$ be a ring and $(P,Q,\psi)$ an $R$-system, and let $(S,T,\sigma,B)$ be a covariant representation of $(P,Q,\psi)$. For each $n \in \mathbb{N}$ there exist maps $T^n : Q^\otimes n \to B$ and $S^n : P^\otimes n \to B$ such that $T^n(q_1 \otimes q_2 \otimes \ldots \otimes q_n) = T(q_1)T(q_2)\ldots T(q_n)$ and $S^n(p_1 \otimes p_2 \otimes \ldots \otimes p_n) = S(p_1)S(p_2)\ldots S(p_n)$ for $q_1,\ldots,q_n \in Q$ and $p_1,\ldots,p_n \in P$, and such that $(S^n,T^n,\sigma,B)$ is a covariant representation of the $R$-system $(P^\otimes n,Q^\otimes n,\psi_n)$.

\textbf{Proof.} Let $n \in \mathbb{N}$.

The existence of additive maps $T^n : Q^\otimes n \to B$ and $S^n : P^\otimes n \to B$ satisfying that $T^n(q_1 \otimes q_2 \otimes \ldots \otimes q_n) = T(q_1)T(q_2)\ldots T(q_n)$ and $S^n(p_1 \otimes p_2 \otimes \ldots \otimes p_n) = S(p_1)S(p_2)\ldots S(p_n)$ for $q_1,\ldots,q_n \in Q$ and $p_1,\ldots,p_n \in P$, and property (iv) of Definition 1.2, easily follows from the universal property of tensor products.

That property (v) of Definition 1.2 also holds can easily been shown by induction over $n$. 

Gradings by the following semigroup will play an important role in this paper.

\textbf{Definition 1.6.} We define $S$ to be the semigroup $\mathbb{N}_0^2$ with multiplication defined by

$$(m,n)(k,l) = \begin{cases} (m,n-k+l) & \text{if } n \geq k, \\ (m+k-n,l) & \text{if } k \geq n. \end{cases}$$

We are now ready to show that the category $\mathcal{C}_{(P,Q,\psi)}$ has an initial object.

\textbf{Theorem 1.7 (cf. [24]).} Let $R$ be a ring and $(P,Q,\psi)$ an $R$-system. Then there exists an injective and surjective covariant representation $(\iota_P,\iota_Q,\iota_R,\mathcal{T}_{(P,Q,\psi)})$ with the following property:

If $(S,T,\sigma,B)$ is a covariant representation of $(P,Q,\psi)$, then there exists a unique (TP)

ring homomorphism $\eta_{(S,T,\sigma,B)} : \mathcal{T}_{(P,Q,\psi)} \to B$ such that $\eta_{(S,T,\sigma,B)} \circ \iota_R = \sigma$, $\eta_{(S,T,\sigma,B)} \circ \iota_Q = T$ and $\eta_{(S,T,\sigma,B)} \circ \iota_P = S$.

Moreover, $(\iota_P,\iota_Q,\iota_R,\mathcal{T}_{(P,Q,\psi)})$ is the, up to isomorphism in $\mathcal{C}_{(P,Q,\psi)}$, unique surjective covariant representation of $(P,Q,\psi)$ which possesses the property (TP); in fact, if $(S,T,\sigma,B)$ is a surjective covariant representation of $(P,Q,\psi)$ and $\phi : B \to \mathcal{T}_{(P,Q,\psi)}$ is a ring homomorphism such that $\phi \circ \sigma = \iota_R$, $\phi \circ S = \iota_P$ and $\phi \circ T = \iota_Q$, then $\phi$ is an isomorphism.

If we, for $m,n \in \mathbb{N}$, let $\mathcal{T}_{(m,n)} := \text{span}\{\iota_Q^n(q)\iota_P^m(p) \mid q \in Q^\otimes m, p \in P^\otimes n\}$, and for $k \in \mathbb{N}$ let $\mathcal{T}_{(k,0)} := \iota_Q^k(Q^\otimes k)$ and $\mathcal{T}_{(0,k)} := \iota_P^k(P^\otimes k)$, and $\mathcal{T}_{(0,0)} := \iota_R(R)$, then $\oplus_{(m,n) \in S} \mathcal{T}_{(m,n)}$ is a $S$-grading of $\mathcal{T}_{(P,Q,\psi)}$. The grading $\oplus_{(m,n) \in S} \mathcal{Y}_{(m,n)}$ of $\mathcal{T}_{(P,Q,\psi)}$ such that $\iota_R(R) \subseteq \mathcal{Y}_{(0,0)}$, $\iota_Q(Q) \subseteq \mathcal{Y}_{(1,0)}$, and $\iota_P(P) \subseteq \mathcal{Y}_{(0,1)}$.  

We call \((\iota_P, \iota_Q, \iota_R, T_{(P,Q,\psi)})\) the Toeplitz representation of \((P,Q,\psi)\), and \(T_{(P,Q,\psi)}\) for the Toeplitz ring of \((P,Q,\psi)\).

**Proof.** Let \(T_{(P,Q,\psi)} := \bigoplus_{(m,n) \in S} T_{(m,n)}\) where \(T_{(m,n)} := Q^\otimes m \otimes P^\otimes n\) for \((m,n) \in \mathbb{N}^2\), \(T_{(k,0)} = Q^\otimes k\) and \(T_{(0,k)} = P^\otimes k\) for \(k \in \mathbb{N}\), and \(T_{(0,0)} = R\). Since each \(T_{(m,n)}\) is an \(R\)-bimodule, \(T_{(P,Q,\psi)}\) becomes an \(R\)-bimodule. It is not difficult (but a bit tedious) to show that there exists a (unique) multiplication on \(T_{(P,Q,\psi)}\) which respects the \(R\)-bimodule structure of each \(T_{(m,n)}\) and which satisfies that \(q_1 q_m = q_1 \otimes q_m\), \(q_m q_1 = q_m \otimes q_1\), \(p_1 p_n = p_1 \otimes p_n\), \(p_n p_1 = p_n \otimes p_1\), \(q_m p_n = q_m \otimes p_n\) and \(p_1 q_1 = \psi(p_1 \otimes q_1)\) for \(q_1 \in Q, p_1 \in P, q_m \in Q^\otimes m, p_n \in P^\otimes n\) and \(m, n \in \mathbb{N}\) . With this multiplication and the addition coming from the \(R\)-bimodule structure, \(T_{(P,Q,\psi)}\) becomes a ring.

Let \(\iota_R : R \to T_{(0,0)} \subseteq T_{(P,Q,\psi)}\) be the canonical inclusion maps. Then \((\iota_P, \iota_Q, \iota_R, T_{(P,Q,\psi)})\) is an injective and surjective covariant representation of \((P,Q,\psi)\).

Let \((S, T, \sigma, B)\) be a covariant representation of \((P,Q,\psi)\). Since \(T_{(P,Q,\psi)}\) is generated by \(\iota_R(R) \cup \iota_Q(Q) \cup \iota_P(P)\), there can at most be one ring homomorphism \(\eta_{(S,T,\sigma,B)} : T_{(P,Q,\psi)} \to B\) such that \(\eta_{(S,T,\sigma,B)} \circ \iota_R = \sigma, \eta_{(S,T,\sigma,B)} \circ \iota_Q = T\) and \(\eta_{(S,T,\sigma,B)} \circ \iota_P = S\). For \((m,n) \in \mathbb{N}^2\) the set span\(\{T^m(q)S^n(p) \mid q \in Q^\otimes m, p \in P^\otimes n\}\) is a subgroup of \(B\) in which the relations

1. \(T^m(q)S^n(p_1) + T^m(q)S^n(p_2) = T^m(q)(S^n(p_1) + p_2)\) for \(q \in Q^\otimes m\) and \(p_1, p_2 \in P^\otimes n\),
2. \(T^m(q_1)S^n(p) + T^m(q_2)S^n(p) = T^m(q_1 + q_2)(S^n(p))\) for \(q_1, q_2 \in Q^\otimes m\) and \(p \in P^\otimes n\);
3. \(T^m(q)S^n(p) = T^m(q)S^n(rp)\) for \(r \in R, q \in Q^\otimes m\) and \(p \in P^\otimes n\);

are satisfied. It follows from the universal property of \(Q^\otimes m \otimes P^\otimes n\) that there exists an \(R\)-bimodule homomorphism \(\eta_{(m,n)}\) from \(T_{(m,n)} = Q^\otimes m \otimes P^\otimes n\) to \(B\) such that \(\eta_{(m,n)}(\iota_R(p_{(m,n)})) = T^m(q)S^n(p)\) for \(q \in Q^\otimes m\) and \(p \in P^\otimes n\). For \(k \in \mathbb{N}\) let \(\iota_{(k,0)}(p)\) denote the map \(T^k\), and let \(\iota_{(0,k)}(p)\) denote the map \(S^k\). Finally, let \(\iota_{(0,0)}(p)\) denote the map \(\sigma\). Then there exists a linear map \(\eta_{(S,T,\sigma,B)} : T_{(P,Q,\psi)} \to B\) such that for each \((m,n) \in S\) the restriction of \(\eta_{(S,T,\sigma,B)}\) to \(T_{(m,n)}\) is equal to \(\eta_{(m,n)}\). It is not difficult to check that \(\eta_{(S,T,\sigma,B)}\) is multiplicative, and thus a ring homomorphism. It is clear that \(\eta_{(S,T,\sigma,B)} \circ \iota_R = \sigma, \eta_{(S,T,\sigma,B)} \circ \iota_Q = T\) and \(\eta_{(S,T,\sigma,B)} \circ \iota_P = S\). Thus the representation \((\iota_P, \iota_Q, \iota_R, T_{(P,Q,\psi)})\) possesses property (TP).

If \((S,T,\sigma,B)\) is a covariant representation of \((P,Q,\psi)\) and \(\phi : B \to T_{(P,Q,\psi)}\) is a ring homomorphism such that \(\phi \circ \sigma = \iota_R, \phi \circ S = \iota_S\) and \(\phi \circ T = \iota_T\), then \(\eta_{(S,T,\sigma,B)} \circ \phi(S(p)) = S(p)\) for all \(p \in P\) and \(\eta_{(S,T,\sigma,B)} \circ \phi(T(q)) = T(q)\) for all \(q \in Q\). Since \(B\) is generated by \(\sigma(R) \cup S(P) \cup T(Q)\), it follows that \(\eta_{(S,T,\sigma,B)} \circ \phi\) is equal to the identity map of \(B\). One can in a similar way show that \(\phi \circ \eta_{(S,T,\sigma,B)}\) is equal to the identity map of \(T_{(P,Q,\psi)}\). Thus \(\phi\) and \(\eta_{(S,T,\sigma,B)}\) are inverse of each other, and \(\phi\) is an isomorphism.

It is clear that \(T_{(m,n)} = \text{span}\{(\iota_Q^m(q)\iota_P^n(p) \mid q \in Q^\otimes m, p \in P^\otimes n)\}\) for \(m,n \in \mathbb{N}\), that \(T_{(k,0)} = \iota_Q^k(Q^\otimes k)\) and \(T_{(0,k)} = \iota_P^k(P^\otimes k)\) for \(k \in \mathbb{N}\), that \(T_{(0,0)} = \iota_R(R)\), and that \(\otimes (m,n) \in S\) is a \(S\)-grading of \(T_{(P,Q,\psi)}\).

If \(\otimes (m,n) \in S\) is another \(S\)-grading of \(T_{(P,Q,\psi)}\) such that \(\iota_R(R) \subseteq \gamma_{(0,0)}, \iota_Q(Q) \subseteq \gamma_{(1,0)}\) and \(\iota_P(P) \subseteq \gamma_{(0,1)}\), then it follows that \(T_{(m,n)} \subseteq \gamma_{(m,n)}\) for each \((m,n) \in S\), and thus that \(T_{(m,n)} = \gamma_{(m,n)}\) for each \((m,n) \in S\).

**Remark 1.8.** Let \(R\) be a ring and \((P,Q,\psi)\) an \(R\)-system. It follows from Theorem 1.7 that the Toeplitz representation \((\iota_P, \iota_Q, \iota_R, T_{(P,Q,\psi)})\) is an initial object of \(C_{(P,Q,\psi)}\). It also follows that there is a bijective correspondence between covariant representations of \((P,Q,\psi)\) and ring homomorphisms defined on \(T_{(P,Q,\psi)}\).
1.2. Examples

We end this section by looking at some examples. We will return to these examples later in the paper.

Example 1.9. Let $R$ be a ring which has local units and let $\varphi \in \text{Aut}(R)$ be a ring automorphism. Let $P := R_\varphi$ be the $R$-bimodule with the right action defined by $p \cdot r = p\varphi(r)$ and the left action defined by $r \cdot p = rp$ for $p \in P$ and $r \in R$. Likewise, let $Q := R_{\varphi^{-1}}$ be the $R$-bimodule with the right action defined by $q \cdot r = q\varphi^{-1}(r)$ and the left action defined by $r \cdot q = rq$ for $q \in Q$ and $r \in R$. Then we can define the following bimodule homomorphism:

$$\psi : P \otimes_R Q \to R$$

$$p \otimes q \mapsto p\varphi(q).$$

Note that we, for every $n \in \mathbb{N}$, have that $P^{\otimes n}$ is isomorphic to $R_{\varphi^n}$ and that $Q^{\otimes n}$ is isomorphic to $R_{\varphi^{-n}}$. We will in the following, for every $n \in \mathbb{N}_0$, identify $P^{\otimes n}$ and $Q^{\otimes n}$ with $R$. We then have that $p_1 \otimes p_2 = p_1\varphi^{n_1}(p_2)$ for $p_1 \in P^{\otimes n_1}$ and $p_2 \in P^{\otimes n_2}$, and that $q_1 \otimes q_2 = q_1\varphi^{-n_1}(q_2)$ for $q_1 \in Q^{\otimes n_1}$ and $q_2 \in Q^{\otimes n_2}$.

Let $(S, T, \sigma, B)$ be a covariant representation of $(P, Q, \psi)$. For $r \in R$ and $n \in \mathbb{N}$ let $[r, n] := S^n(r)$, $[r, -n] := T^n(r)$ and $[r, 0] := \sigma(r) = T^0(r) = S^0(r)$. If $r_1, r_2 \in R$ and $n_1, n_2 \in \mathbb{N}_0$, then we have

$$[r_1, n_1][r_2, n_2] = S^{n_1}(r_1)S^{n_2}(r_2) = S^{n_1+n_2}(r_1 \otimes r_2),$$

$$[r_1, -n_1][r_2, -n_2] = T^{n_1}(r_1)T^{n_2}(r_2) = T^{n_1+n_2}(r_1 \otimes r_2),$$

$$[r_1, n_1][r_2, -n_2] = [r_1\varphi^{n_1}(r_2), n_1 + n_2],$$

$$[r_1, -n_1][r_2, n_2] = [r_1\varphi^{-n_1}(r_2), -n_1 - n_2],$$

$$[r_1, n_1][r_2, -n_2] = \sigma(\psi_1(r_1 \otimes r_2)) = \sigma(\varphi^{n_1}(r_2), 0),$$

$$[r_1, n_1][r_2, n_2] = [u_1r_1, n_1 + n_2][r_2, -n_2]$$

$$= [u_1, n_1][\varphi^{-n_1}(r_1), n_2][r_2, -n_2]$$

$$= [u_1, n_1][\varphi^{-n_1}(r_1)\varphi^{n_2}(r_2), 0]$$

$$= [u_1\varphi^{n_1+n_2}(r_2), n_1] = [r_1\varphi^{n_1+n_2}(r_2), n_1]$$

and

$$[r_1, n_1][r_2, -n_1 - n_2] = [r_1, n_1][r_2, -n_1][\varphi^{n_1}(u_2), -n_2]$$

$$= [r_1\varphi^{n_1}(r_2), 0][\varphi^{n_2}(u_2), -n_2]$$

$$= [r_1\varphi^{n_1}(r_2), -n_2].$$

Thus $[r_1, k_1][r_2, k_2] = [r_1\varphi^{k_1}(r_2), k_1 + k_2]$ for $r_1, r_2 \in R$ and $k_1, k_2 \in \mathbb{Z}$ if $k_1$ and $k_2$ both are non-positive, or both are non-negative or if $k_1$ is non-negative and $k_2$ is non-positive. We also have that $[r_1, k] + [r_2, k] = [r_1 + r_2, k]$ for $r_1, r_2 \in R$ and $k \in \mathbb{Z}$.

If on the other hand we have a ring $B$ which contains a set of elements $\{[r, k] : r \in R, k \in \mathbb{Z}\}$ satisfying $[r_1, k] + [r_2, k] = [r_1 + r_2, k]$ and $[r_1, k_1][r_2, k_2] = [r_1\varphi^{k_1}(r_2), k_1 + k_2]$ if $k_1$ and $k_2$ both are non-positive, or both are non-negative or if $k_1$ is non-negative and $k_2$ is non-positive, and we define $\sigma : R \to B$ by $\sigma(r) = [r, 0]$, $S : P \to B$ by $S(p) = [p, 1]$ and $T : Q \to B$ by $T(q) = [q, -1]$, then $(S, T, \sigma, B)$ is a covariant representation of $(P, Q, \psi)$.

Thus $T_{(P, Q, \psi)}$ is the universal ring generated by elements $\{[r, k] : r \in R, k \in \mathbb{Z}\}$ satisfying $[r_1, k] + [r_2, k] = [r_1 + r_2, k]$ and $[r_1, k_1][r_2, k_2] = [r_1\varphi^{k_1}(r_2), k_1 + k_2]$ if $k_1$ and $k_2$ both are non-positive, or both are non-negative or if $k_1$ is non-negative and $k_2$ is non-positive. We will, in Example 5.5, see that a certain quotient of $T_{(P, Q, \psi)}$ (the Cuntz–Pimsner ring $O_{(P, Q, \psi)}$ of $(P, Q, \psi)$) is isomorphic to the crossed product $R \times_{\varphi} \mathbb{Z}$. 
Example 1.10. Let $E = (E^0, E^1)$ be a directed graph and let $F$ be a commutative unital ring. We define the ring $R := \bigoplus_{e \in E^0} F_e$, where every $F_e$ is a copy of $F$, and we denote for each $v \in E^0$ by $1_v$ the unit of $F_v$. Observe that $R$ is non-degenerate with local units. We also define $Q := \bigoplus_{e \in E^1} F_e$ and $P := \bigoplus_{e \in E^0} F_e$, where every $F_e$ and $F_e$ are copies of $F$ with units $1_e$ and $1_e$, respectively, with the following $R$-bimodule operations:

\[
\begin{align*}
\left( \sum_{e \in E^1} \lambda_e 1_e \right) \cdot \left( \sum_{v \in E^0} s_v 1_v \right) &= \sum_{e \in E^1} \left( \sum_{r(e) = v} \lambda_e s_v \right) 1_e, \\
\left( \sum_{v \in E^0} s_v 1_v \right) \cdot \left( \sum_{e \in E^1} \lambda_e 1_e \right) &= \sum_{e \in E^1} \left( \sum_{s(e) = v} s_v \lambda_e \right) 1_e, \\
\left( \sum_{e \in E^1} \lambda_e 1_e \right) \cdot \left( \sum_{v \in E^0} s_v 1_v \right) &= \sum_{e \in E^1} \left( \sum_{r(e) = v} s_v \lambda_e \right) 1_e, \\
\left( \sum_{v \in E^0} s_v 1_v \right) \cdot \left( \sum_{e \in E^1} \lambda_e 1_e \right) &= \sum_{e \in E^1} \left( \sum_{s(e) = v} s_v \lambda_e \right) 1_e,
\end{align*}
\]

for $\{s_v\}_{v \in E^0} \subseteq F$ and $\{\lambda_e\}_{e \in E^1} \subseteq F$.

Now if we define the following $R$-bimodule homomorphism:

\[
\psi : P \otimes_R Q \longrightarrow R
\]

\[
\left( \sum_{e \in E^1} p_e 1_e \right) \otimes \left( \sum_{e \in E^1} q_e 1_e \right) \longmapsto \left( \sum_{v \in E^0} \sum_{r(e) = v} p_e q_v \right) 1_v,
\]

then $(P, Q, \psi)$ is an $R$-system.

Let $(S, T, \sigma, B)$ be a covariant representation of $(P, Q, \psi)$ and let $p_v := \sigma(1_v)$ for $v \in E^0$, and let $x_e = T(1_e)$ and $y_e = S(1_e)$ for $e \in E^1$. It is easy to check that $\{p_v\}_{v \in E^0}$ is a family of pairwise orthogonal idempotents, and that for all $e, f \in E^1$ we have $p_{s(e)} x_e = x_e = x_e p_{r(e)}$, $p_{r(e)} y_e = y_e = y_e p_{s(e)}$ and $y_e x_f = \delta_{e,f} p_{r(e)}$. Since $R$ is an $F$-algebra and $P$ and $Q$ are $F$-modules, the ring $R\langle S, T, \sigma \rangle$ becomes an $F$-algebra when we equip it with an $F$-multiplication of $F$ defined by $\lambda \sigma(r) = \sigma(\lambda r)$, $\lambda S(p) = S(\lambda p)$ and $\lambda T(q) = T(\lambda q)$ for $\lambda \in F$, $r \in R$, $p \in P$ and $q \in Q$.

If on the other hand $B$ is an $F$-algebra which contains a family $\{p_v\}_{v \in E^0}$ of pairwise orthogonal idempotents and families $\{x_e\}_{e \in E^1}$ and $\{y_e\}_{e \in E^1}$ satisfying for all $e, f \in E^1$ that $p_{s(e)} x_e = x_e = x_e p_{r(e)}$, $p_{r(e)} y_e = y_e = y_e p_{s(e)}$ and $y_e x_f = \delta_{e,f} p_{r(e)}$, and we, for $r = \sum_{v \in E^0} s_v 1_v \in R$ let $\sigma(r) := \sum_{e \in E^1} \lambda_e 1_e p_v$, for $p = \sum_{e \in E^1} \lambda_e 1_e \in P$ let $S(p) := \sum_{e \in E^1} \lambda_e 1_e y_e$ and for $q = \sum_{e \in E^1} \lambda_e 1_e \in Q$ let $T(q) := \sum_{e \in E^1} \lambda_e 1_e x_e$, then $(S, T, \sigma, B)$ is a covariant representation of $(P, Q, \psi)$.

Thus $T_E := T_{\sigma(P, Q, \psi)}$ is the universal $F$-algebra generated by a set $\{p_v : v \in E^0\}$ of pairwise orthogonal idempotents, together with a set $\{x_e, y_e : e \in E^1\}$ of elements satisfying for $e, f \in E^1$

\begin{enumerate}
  \item $p_{s(e)} x_e = x_e = x_e p_{r(e)}$;
  \item $p_{r(e)} y_e = y_e = y_e p_{s(e)}$;
  \item $y_e x_f = \delta_{e,f} p_{r(e)}$.
\end{enumerate}

We will, in Example 5.8, see that a certain quotient of $T_E$ (the Cuntz–Pimsner ring $O_{(P, Q, \psi)}$ of $(P, Q, \psi)$) is isomorphic to the Leavitt path algebra $L_F(E)$ associated with the graph $E$; cf. [1, 3, 6, 27].
2. The Fock space representation

We will in this section for an arbitrary ring $R$ and an arbitrary $R$-system $(P, Q, \psi)$, construct a representation which we call the Fock space representation. This construction is inspired by a similar construction in the $C^*$-algebra setting; cf. \[16, 24\]. We will later show (see Proposition 4.2) that the Fock space representation under certain conditions is isomorphic to the Toeplitz representation.

We begin by establishing some notation which will be used in the rest of the paper.

**Definition 2.1.** Let $R$ be a ring and $(P, Q, \psi)$ an $R$-system. Then a right $R$-module homomorphism $T : Q_R \to Q_R$ is called adjointable with respect to $\psi$ if there exists a left $R$-module homomorphism $S : R_P \to R_P$ such that
\[
\psi(p \otimes T(q)) = \psi(S(p) \otimes q) \quad \forall p \in P, \quad \forall q \in Q.
\]
We call $S$ an adjoint of $T$ with respect to $\psi$. We write $L_P(Q)$ for the set of all the adjointable homomorphisms (with respect to $\psi$). Note that without further conditions imposed on $(P, Q, \psi)$ the adjoint can be non-unique. We denote by $L_Q(P)$ the set of all the adjoints.

Observe that $L_P(Q)$ and $L_Q(P)$ are subrings of $\text{End}(Q_R)$ and $\text{End}(R_P)$, respectively.

**Definition 2.2.** Let $R$ bear an and $(P, Q, \psi)$ an $R$-system. For every $p \in P$ and $q \in Q$ we define the following homomorphisms:
\[
\theta_{q,p} : Q_R \longrightarrow Q_R, \quad \theta_{p,q} : R_P \longrightarrow R_P,
\]
\[
x \mapsto q\psi(p \otimes x), \quad y \mapsto \psi(y \otimes q)p.
\]
Then $\theta_{q,p} \in L_P(Q)$ and has $\theta_{p,q}$ as an adjoint.

We call these homomorphisms rank 1 adjointable homomorphisms, and we denote by $F_P(Q)$ the linear span of all the rank 1 adjointable homomorphisms. Similarly, we denote by $F_Q(P)$ the set of all rank 1 adjoints.

**Lemma 2.3.** Let $R$ be a ring and $(P, Q, \psi)$ an $R$-system. If $T \in L_P(Q)$ (with an adjoint $S$), $p \in P$ and $q \in Q$, then we have
\[
T\theta_{q,p} = \theta_{T(q),p} \quad \text{and} \quad \Theta_{q,p}T = \Theta_{q,S(p)}.
\]
Thus $F_P(Q)$ is a two-sided ideal of $L_P(Q)$.

**Proof.** It is easy to check that the lemma follows directly from the definitions.

Note that the above result does not depend on the choice of the adjoint. Note also that by a dual argument we have $F_Q(P)$ is a two-sided ideal of $L_Q(P)$.

**Definition 2.4 (cf. \[21, Section 2.2; 24\]).** Given a ring $R$ and an $R$-bimodule $Q$ we define the tensor ring or Fock ring $F(Q)$ by
\[
F(Q) = \bigoplus_{n=0}^{\infty} Q^\otimes n.
\]

Despite the inherited ring structure of $F(Q)$ (see \[13\] for more information about tensor rings) we are only interested in the $R$-bimodule structure of $F(Q)$. If $(P, Q, \psi)$ is an $R$-system,
then we can define an $R$-balanced $R$-bilinear form

$$
\langle \cdot, \cdot \rangle : F(P) \times F(Q) \to R,
$$

$$(\{p_n\}, \{q_n\}) \mapsto \sum_{n \in \mathbb{N}_0} \psi_n(p_n \otimes q_n)
$$

that one can extend to an $R$-bimodule homomorphism $\psi : F(P) \otimes F(Q) \to R$ by the universal property of the tensor product.

Define the ring homomorphism $\phi_{\infty} : R \to \mathcal{L}_F(P)(F(Q))$ assigning to $r \in R$ the adjointable homomorphism $\phi_{\infty}(r)$ of $F(Q)$ defined by $\phi_{\infty}(r)(\{q_n\}) = \{rq_n\}$. Note that $\phi_{\infty}(r)$ defined by $\varphi_{\infty}(r)(\{p_n\}) = \{p_n r\}$ is an adjoint of $\phi_{\infty}(r)$.

If for every $n \in \mathbb{N}_0$ we define $\phi_{\infty}^n : R \to \mathcal{L}_{P\otimes n}(Q\otimes n)$ by $\phi_{\infty}^n(r)(q_n) = rq_n$, then we can write $\phi_{\infty}(r)$ in the following matrix form:

$$
\phi_{\infty}(r)(\{q_n\}) = \begin{pmatrix}
\phi_{\infty}^0(r) & 0 \\
0 & \phi_{\infty}^1(r) \\
& \phi_{\infty}^2(r) \\
& & \ddots \\
& & & \ddots \\
& & & & \ddots
\end{pmatrix}
\begin{pmatrix}
q_0 \\
n_1 \\
n_2 \\
\vdots \\
\vdots
\end{pmatrix}.
$$

Given an $R$-system $(P, Q, \psi)$, for every $n, m \in \mathbb{N}_0$ with $n \leq m$ and $q \in Q^{\otimes m-n}$, we define the following right $R$-module homomorphism:

$$
T_{q}^{(n,m)} : Q^{\otimes n} \longrightarrow Q^{\otimes m},
q_n \mapsto q \otimes q_n
$$

and the left $R$-module homomorphism

$$
U_{q}^{(m,n)} : P^{\otimes m} \longrightarrow P^{\otimes n},
p_1 \otimes p_2 \mapsto p_1 \psi_{m-n}(p_2 \otimes q),
$$

where $p_1 \in P^{\otimes n}$ and $p_2 \in P^{\otimes m-n}$.

For $q \in Q$ let $T_{q}^{(n)} := T_{q}^{(n,n+1)}$ and $U_{q}^{(n)} := U_{q}^{(n+1,n)}$. We define the creator homomorphism $T_q : F(Q) \rightarrow F(Q)$ by

$$
T_q(\{q_n\}) := \{0, T_{q}^{(0)}(q_0), T_{q}^{(1)}(q_1), \ldots\} = \{0, q_0, q \otimes q_1, \ldots\}.
$$

Observe that we can write $T_q$ in the following matrix form:

$$
T_q(\{q_n\}) = \begin{pmatrix}
0 & T_{q}^{(0)} & 0 \\
T_{q}^{(1)} & 0 & T_{q}^{(2)} \\
& \ddots & \ddots \\
& & \ddots
\end{pmatrix}
\begin{pmatrix}
q_0 \\
n_1 \\
n_2 \\
\vdots
\end{pmatrix}.
$$

One gets that $T_q \in \mathcal{L}_F(P)(F(Q))$ with an adjoint homomorphism $U_q : F(P) \rightarrow F(P)$ defined by $U_q(\{p_n\}) = \{U_q^{(0)}(p_1), U_q^{(1)}(p_2), \ldots\}$ and which can be written in the matrix form

$$
U_q(\{p_n\}) = \begin{pmatrix}
0 & U_q^{(0)} & 0 \\
0 & U_q^{(1)} & 0 \\
& \ddots & \ddots \\
& & \ddots
\end{pmatrix}
\begin{pmatrix}
p_0 \\
p_1 \\
p_2 \\
p_3
\end{pmatrix}.
Similarly, for every \( n, m \in \mathbb{N}_0 \) with \( n \leq m \) and given any \( p \in P^{\otimes m-n} \) we define the following right \( R \)-module homomorphism

\[
S_p^{(n,m)} : Q^{\otimes m} \rightarrow Q^{\otimes n},
q_1 \otimes q_2 \mapsto \psi_{m-n}(p \otimes q_1)q_2,
\]

where \( q_1 \in Q^{\otimes m-n} \) and \( q_2 \in Q^{\otimes n} \), and the left \( R \)-module homomorphism

\[
V_p^{(n,m)} : P^{\otimes n} \rightarrow P^{\otimes m},
p_n \mapsto p_n \otimes p.
\]

We denote \( S_p^{(n)} := S_p^{(n,n+1)} \) and \( V_p^{(n)} := V_p^{(n+1,n)} \), where \( p \in P \), and we then define the right \( R \)-module homomorphism \( S_p : F(Q) \rightarrow F(Q) \) by \( S_p(\{q_n\}) := \{S_p(0)(q_1), S_p(1)(q_2), \ldots\} \) which can be written in the following matrix form:

\[
S_p(\{q_n\}) = \begin{pmatrix}
0 & S_p^{(0)} & S_p^{(1)} & S_p^{(2)} & \cdots \\
0 & 0 & S_p^{(1)} & S_p^{(2)} & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}
\begin{pmatrix}
q_0 \\
q_1 \\
q_2 \\
q_3 \\
\vdots
\end{pmatrix}.
\]

One gets that \( S_p \in \mathcal{L}_F(P)(F(Q)) \) with an adjoint homomorphism \( V_p : F(P) \rightarrow F(P) \) given by \( V_p(\{p_n\}) := \{0, V_p^{(0)}(p_0), V_p^{(1)}(p_1), \ldots\} \) and with matrix form

\[
V_p(\{p_n\}) = \begin{pmatrix}
0 & V_p^{(0)} & V_p^{(1)} & V_p^{(2)} & \cdots \\
0 & 0 & V_p^{(1)} & V_p^{(2)} & \cdots \\
\vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}
\begin{pmatrix}
p_0 \\
p_1 \\
p_2 \\
p_3 \\
\vdots
\end{pmatrix}.
\]

**Proposition 2.5.** Let \( R \) be a ring and \((P, Q, \psi)\) an \( R \)-system. Denote by \( T_F \) the map from \( Q \) to \( \mathcal{L}_F(P)(F(Q)) \) given by \( q \mapsto T_q \), by \( S_F \) the map from \( P \) to \( \mathcal{L}_F(P)(F(Q)) \) given by \( p \mapsto S_p \) and by \( \sigma_F \) the map from \( R \) to \( \mathcal{L}_F(P)(F(Q)) \) given by \( r \mapsto \phi_\infty(r) \), and let \( \mathcal{F}(P,Q,\psi) \) be the subring of \( \mathcal{L}_F(P)(F(Q)) \) generated by \( T_F(Q) \cup S_F(P) \cup \sigma_F(R) \). Then \((S_F, T_F, \sigma_F, \mathcal{F}(P,Q,\psi))\) is a surjective covariant representation of \((P, Q, \psi)\). This representation is injective if and only if \( R \) is right non-degenerate.

We call \((S_F, T_F, \sigma_F, \mathcal{F}(P,Q,\psi))\) for the Fock space representation of \((P, Q, \psi)\).

**Proof.** It is clear that the maps \( T_F, S_F \) and \( \sigma_F \) are additive, and that for every \( r \in R, p \in P \) and \( q \in Q \) we have

\[
\phi_\infty(r)T_q = T_{rq}, \quad T_q\phi_\infty(r) = T_{qr}, \quad \phi_\infty(r)S_p = S_{rp}, \quad S_p\phi_\infty(r) = S_{pr},
\]

from which it follows that \( \sigma_F \) is a ring homomorphism and that

\[
S_F(pr) = S_F(p)\sigma_F(r), \quad S_F(rp) = \sigma_F(r)S_F(p),
\]

\[
T_F(qr) = \sigma_F(r)T_F(q), \quad T_F(qr) = T_F(q)\sigma_F(r)
\]

for every \( r \in R, p \in P \) and \( q \in Q \).
Given any \( p \in P \) and \( q \in Q \) we have for every \( n \in \mathbb{N}_0 \) that \( S_p^{(n)} T_q^{(n)}(q_n) = \psi(p \otimes q)q_n \) for \( q_n \in Q^\otimes n \), and hence the composition homomorphism \( S_p T_q \) gives

\[
S_p T_q(\{q_n\}) = \begin{pmatrix}
0 & S_p^{(0)} & 0 & \\
0 & 0 & S_p^{(1)} & 0 \\
0 & 0 & 0 & S_p^{(2)} \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
0 & T_q^{(0)} & 0 & \\
0 & 0 & T_q^{(1)} & 0 \\
0 & 0 & 0 & T_q^{(2)} \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
q_0 \\
q_1 \\
q_2 \\
\vdots
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\psi(p \otimes q)q_0 \\
\psi(p \otimes q)q_1 \\
\psi(p \otimes q)q_2 \\
\vdots
\end{pmatrix} = \varphi_\infty (\psi(p \otimes q))(\{q_n\}),
\]

from which it follows that \( \sigma_F(\psi(p \otimes q)) = S_F(p) T_F(q) \) for every \( p \in P \) and \( q \in Q \). Thus \( (S_F, T_F, \sigma_F, F(P,Q,\psi)) \) is a surjective covariant representation of \((P,Q,\psi)\).

Finally it is clear that \( \sigma_F \), and thus \((S_F, T_F, \sigma_F, F(P,Q,\psi))\), is injective if and only if \( R \) is right non-degenerate.

**Notation 2.6.** Let us denote by \( \text{B}^{\text{op}} \) the opposite ring of \( B \). Given \( a, b \in \text{B}^{\text{op}} \) we write \( a \cdot b \) for the product of \( a \) and \( b \) in \( \text{B}^{\text{op}} \). Thus \( a \cdot b = ba \).

**Remark 2.7.** Let \( R \) be a ring and let \((P,Q,\psi)\) be an \( R \)-system. We could define an anti-representation of \((P,Q,\psi)\) to be a quadruple \((V,U,\eta,\text{B}^{\text{op}})\), where \( B \) is a ring, \( \eta : R \to \text{B}^{\text{op}} \) is a ring homomorphism, \( U : Q \to \text{B}^{\text{op}} \) and \( V : P \to \text{B}^{\text{op}} \) are linear maps, and \( U(qr) = U(q) \cdot \eta(r) \), \( U(rp) = \eta(r) \cdot V(p) \), \( U(pr) = \psi(p \otimes q) \) and \( V(p) \cdot U(q) = \eta(\psi(p \otimes q)) \) for every \( r \in R, q \in Q \) and \( p \in P \). If we then denoted by \( U_{\varphi_1} \) the map from \( Q \) to \( L_F(Q)(F(P)) \) given by \( q \mapsto U_q \), by \( V_{\varphi_1} \) the map from \( P \) to \( L_F(Q)(F(P)) \) given by \( p \mapsto V_p \), and by \( \eta_{\varphi_1} \) the map from \( R \) to \( L_F(Q)(F(P)) \) given by \( r \mapsto \varphi_\infty(r) \), then \((V_{\varphi_1}, U_{\varphi_1}, \eta_{\varphi_1}, (F_2(P,Q,\psi))^{\text{op}}) \) would be an anti-representation of \((P,Q,\psi)\), where \( F_2(P,Q,\psi) \) is the subring of \( L_F(Q)(F(P)) \) generated by \( U_{\varphi_1}(Q) \cup V_{\varphi_1}(P) \cup \eta_{\varphi_1}(R) \).

Note that in general the rings \( F(P,Q,\psi) \) and \( F_2(P,Q,\psi) \) are not isomorphic. For example, if \( R \) is a right non-degenerate ring, but not a left non-degenerate ring, then if we consider the \( R \)-system \((P,Q,\psi)\), where \( P = Q = 0 \) and \( \psi \) is the zero homomorphism, we have \( F(P,Q,\psi) \cong R \) and \( F_2(P,Q,\psi) \cong R/I \), where \( I = \{ r \in R | \text{Ir} = 0 \} \).

### 3. Relative Cuntz–Pimsner rings

The Toeplitz representation of an \( R \)-system \((P,Q,\psi)\) is in general too big to be an attractive representation of \((P,Q,\psi)\). We will in this section study a certain subclass of covariant representations of \((P,Q,\psi)\) and, for \( R \)-systems satisfying the condition \( (FS) \) defined below, completely classify these representations up to isomorphism in \( \mathcal{C}(P,Q,\psi) \). We begin by describing this class of representations.
Remember (cf. Theorem 1.7) that $T_{(P,Q,\psi)}$ comes with an $S$-grading $\oplus_{(m,n)}T_{(m,n)}$, where $S$ is the semigroup defined in Definition 1.6. It will often be more convenient to work with a $\mathbb{Z}$-grading instead of this $S$-grading.

**Proposition 3.1.** Let $R$ be a ring and let $(P,Q,\psi)$ be an $R$-system. If we, for $k \in \mathbb{Z}$, let

$$T_{(P,Q,\psi)}^{(k)} := \bigoplus_{(m,n) \in S, m-n = k} T_{(m,n)},$$

then $\oplus_{n \in \mathbb{Z}} T_{(P,Q,\psi)}^{(n)}$ is a $\mathbb{Z}$-grading of $T_{(P,Q,\psi)}$.

The grading $\oplus_{n \in \mathbb{Z}} T_{(P,Q,\psi)}^{(n)}$ is the only $\mathbb{Z}$-grading $\oplus_{n \in \mathbb{Z}} \mathcal{Y}^{(n)}$ of $T_{(P,Q,\psi)}$ for which $\iota_R(R) \subseteq \mathcal{Y}^{(0)}$, $\iota_Q(Q) \subseteq \mathcal{Y}^{(1)}$ and $\iota_P(P) \subseteq \mathcal{Y}^{(1)}$.

**Proof.** It easily follows from Theorem 1.7 that $\oplus_{n \in \mathbb{Z}} T_{(P,Q,\psi)}^{(n)}$ is a $\mathbb{Z}$-grading of $T_{(P,Q,\psi)}$ and that $\iota_R(R) \subseteq T_{(P,Q,\psi)}^{(0)}$, $\iota_Q(Q) \subseteq T_{(P,Q,\psi)}^{(1)}$ and $\iota_P(P) \subseteq T_{(P,Q,\psi)}^{(-1)}$.

Suppose $\oplus_{n \in \mathbb{Z}} \mathcal{Y}^{(n)}$ is another $\mathbb{Z}$-grading of $T_{(P,Q,\psi)}$ and that $\iota_R(R) \subseteq \mathcal{Y}^{(0)}$, $\iota_Q(Q) \subseteq \mathcal{Y}^{(1)}$ and $\iota_P(P) \subseteq \mathcal{Y}^{(-1)}$. Then $T_{(P,Q,\psi)}^{(n)} \subseteq \mathcal{Y}^{(n)}$ for each $n \in \mathbb{Z}$ from which it follows that $T_{(P,Q,\psi)}^{(n)} = \mathcal{Y}^{(n)}$ for each $n \in \mathbb{Z}$. \[\square\]

**Proposition 3.2.** Let $R$ be a ring, $(P,Q,\psi)$ an $R$-system, $(S,T,\sigma,B)$ a surjective covariant representation of $(P,Q,\psi)$ and let $\eta_{(S,T,\sigma,B)} : T_{(P,Q,\psi)} \rightarrow B$ be the ring homomorphism from Theorem 1.7. If $\oplus_{n \in \mathbb{Z}} B^{(n)}$ is a $\mathbb{Z}$-grading of $B$ such that $\sigma(R) \subseteq B^{(0)}$, $T(Q) \subseteq B^{(1)}$ and $S(P) \subseteq B^{(-1)}$, then $\eta_{(S,T,\sigma,B)}(T_{(P,Q,\psi)}^{(n)}) = B^{(n)}$ for every $n \in \mathbb{Z}$.

**Proof.** If $\oplus_{n \in \mathbb{Z}} B^{(n)}$ is a $\mathbb{Z}$-grading of $B$ such that $\sigma(R) \subseteq B^{(0)}$, $T(Q) \subseteq B^{(1)}$ and $S(P) \subseteq B^{(-1)}$, then $\eta_{(S,T,\sigma,B)}(T_{(P,Q,\psi)}^{(n)}) \subseteq B^{(n)}$ for every $n \in \mathbb{Z}$. It follows that $\oplus_{n \in \mathbb{Z}} \eta_{(S,T,\sigma,B)}(T_{(P,Q,\psi)}^{(n)})$ is a $\mathbb{Z}$-grading of $B$, and thus that $\eta_{(S,T,\sigma,B)}(T_{(P,Q,\psi)}^{(n)}) = B^{(n)}$ for every $n \in \mathbb{Z}$. \[\square\]

**Definition 3.3.** Let $R$ be a ring and $(P,Q,\psi)$ an $R$-system. A surjective covariant representation $(S,T,\sigma,B)$ of $(P,Q,\psi)$ is graded if there exists a $\mathbb{Z}$-grading $\oplus_{n \in \mathbb{Z}} B^{(n)}$ of $B$ such that $\sigma(R) \subseteq B^{(0)}$, $T(Q) \subseteq B^{(1)}$ and $S(P) \subseteq B^{(-1)}$.

The aim of this section is to classify all surjective, injective and graded representations of an $R$-system. Unfortunately, we do not know how to do that for general $R$-systems, but only for $R$-systems satisfying a condition we have chosen to call (FS) and which is defined below. This condition is probably not the optimal one, but many interesting examples do satisfy this condition (cf. Examples 4.11, 5.5, 5.6 and 5.8).

### 3.1. Condition (FS)

We will now introduce the condition (FS) and show some fundamental results for $R$-systems satisfying this condition.

**Definition 3.4.** Let $R$ be a ring. An $R$-system $(P,Q,\psi)$ is said to satisfy condition (FS) if for all finite sets $\{q_1, \ldots, q_n\} \subseteq Q$ and $\{p_1, \ldots, p_m\} \subseteq P$ there exist $\Theta \in \mathcal{F}_P(Q)$ and $\Delta \in \mathcal{F}_Q(P)$ such that $\Theta(q_i) = q_i$ and $\Delta(p_j) = p_j$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$, respectively.
EXAMPLE 3.5. Observe that condition (FS) appears in a natural context. Let $R$ be a unital ring and let $Q$ be an $R$-bimodule such that $Q_R$ is a finitely generated projective right $R$-module. Then define $P := Q^* = \text{Hom}_R(Q_R, R)$. We then have that $P$ is an $R$-bimodule such that $pP$ is a finitely generated projective left $R$-module with $P^* = Q^{**} = Q$. Therefore we can define

$$\psi : P \otimes_R Q \rightarrow R,$$

$$f \otimes q \mapsto f(q).$$

Observe that by the Dual Basis Lemma (see, for example, [19, 2.9]) there exist $q_1, \ldots, q_n \in Q$ and $f_1, \ldots, f_n \in P$ such that $\sum_{i=1}^n q_i f_i(q) = q$ for every $q \in Q$. Dually and since $P^* = Q$, there exist $p_1, \ldots, p_m \in Q$ and $g_1, \ldots, g_m \in P^*$ such that $\sum_{j=1}^m g_j(p)p_j = p$ for every $p \in P$, from which condition (FS) follows.

DEFINITION 3.6. Let $R$ be a ring. An $R$-system $(P, Q, \psi)$ is non-degenerate if whenever $\psi(p \otimes q) = 0$ for every $p \in P$ we have $q = 0$, and whenever $\psi(p \otimes q) = 0$ for every $q \in Q$ we have $p = 0$.

Notice that if $(P, Q, \psi)$ is non-degenerate then every $T \in \mathcal{L}_P(Q)$ has a unique adjoint.

LEMMA 3.7. Let $R$ be a ring let $(P, Q, \psi)$ an $R$-system satisfying condition (FS). Then $(P, Q, \psi)$ is non-degenerate.

Proof. Let $\psi(p \otimes q) = 0$ for every $p \in P$. Then by condition (FS) there exists $\Theta = \sum_{i=1}^n \theta_{q_i, p_i} \in F_P(Q)$ such that

$$q = \Theta(q) = \sum_{i=1}^n \theta_{q_i, p_i}(q) = \sum_{i=1}^n q_i \psi(p_i \otimes q) = 0.$$

Thus $(P, Q, \psi)$ is non-degenerate. \hfill \Box

Observe that if $R$ is right non-degenerate then $\psi_0 : P^{\otimes_0} \otimes Q^{\otimes_0} \rightarrow R$ is non-degenerate. For general $n \in \mathbb{N}$ we need the condition (FS).

LEMMA 3.8. Let $R$ be a ring and $(P, Q, \psi)$ an $R$-system satisfying condition (FS). For every $n \in \mathbb{N}$ we have the $R$-system $(P^{\otimes n}, Q^{\otimes n}, \psi_n)$ satisfies condition (FS).

Proof. We will prove by induction that $\psi_n : P^{\otimes n} \otimes Q^{\otimes n} \rightarrow R$ satisfies condition (FS) for every $n \in \mathbb{N}$. By hypothesis $(P, Q, \psi)$ satisfies (FS). Now suppose that $(P^{\otimes n-1}, Q^{\otimes n-1}, \psi_{n-1})$ satisfies condition (FS). Let $q_1^1 \otimes q_1^2, \ldots, q_m^i \otimes q_m^i \in Q^{\otimes n}$, where $q_1^i, \ldots, q_m^i \in Q$ and where $q_1^{i-1}, \ldots, q_m^{i-1} \in Q^{\otimes n-1}$. Since $(P, Q, \psi)$ satisfies condition (FS) there exists $\Theta_1 = \sum_{j=1}^l \theta_{a_j/b_j} \in F_P(Q)$ with $a_j \in Q$ and $b_j \in P$ for every $j = 1, \ldots, l$ such that $\Theta_1(q_1^i) = q_1^i$ for every $i = 1, \ldots, m$, and since $(P^{\otimes n-1}, Q^{\otimes n-1}, \psi_{n-1})$ also satisfies condition (FS), by the induction hypothesis, there exists $\Theta_2 = \sum_{k=1}^t \theta_{c_k/d_k} \in F_{P^{\otimes n-1}}(Q^{\otimes n-1})$ with $c_k \in Q^{\otimes n-1}$ and $d_k \in P^{\otimes n-1}$ for every $k = 1, \ldots, t$ such that $\Theta_2(\psi(b_j \otimes q_1^i)q_1^{i-1}) = \psi(b_j \otimes q_1^i)q_1^{i-1}$ for every $i = 1, \ldots, m$ and $j = 1, \ldots, l$. Let

$$\Theta := \sum_{j=1}^l \sum_{k=1}^t \theta_{a_j/b_j, c_k/d_k} \in F_{P^{\otimes n}}(Q^{\otimes n}).$$
It is then straightforward to check that \( \Theta(q^i_1 \otimes q^{n-1}_i) = q^i_1 \otimes q^{n-1}_i \) for every \( i = 1, \ldots, m \). It follows that for every finite set \( \{q^i_n, \ldots, q^i_m\} \) of \( Q^{\otimes n} \), there exists a \( \Theta \in \mathcal{F}_{P^{\otimes n}}(Q^{\otimes n}) \) such that \( \Theta(q^i_n) = q^i_n \) for every \( i = 1, \ldots, m \). One can in a similar way show that for every finite set \( \{p^i_n, \ldots, p^i_m\} \) of \( P^{\otimes n} \), there exists a \( \Delta \in \mathcal{F}_{P^{\otimes n}}(Q^{\otimes n}) \) such that \( \Delta(p^i_n) = p^i_n \) for every \( i = 1, \ldots, m \). Thus \( (P^{\otimes n}, Q^{\otimes n}, \psi_n) \) satisfies condition (FS)

\[ \square \]

**Lemma 3.9.** Let \( R \) be a ring and let \( (S, T, \sigma, B) \) be a covariant representation of an \( R \)-system \( (P, Q, \psi) \) satisfying condition (FS). If \( \sigma \) is injective, then so are \( T^n \) and \( S^n \) for every \( n \in \mathbb{N} \).

**Proof.** Let \( q \in Q^{\otimes n} \) such that \( T^n(q) = 0 \). For every \( p \in P^{\otimes n} \) we then have that \( 0 = S^n(p)T^n(q) = \sigma(\psi_n(p \otimes q)) \), and since \( \sigma \) is injective, it follows that \( \psi_n(p \otimes q) = 0 \) for every \( p \in P^{\otimes n} \). It therefore follows from the non-degeneracy of \( \psi_n \) (cf. Lemma 3.7 and 3.8) that \( q = 0 \). Similarly one can check that \( S^n \) is injective.

**Definition 3.10.** Let \( R \) be a ring and \( (P, Q, \psi) \) an \( R \)-system. We define the ring homomorphism \( \Delta : R \to \text{End}_R(Q_R) \) and the ring homomorphism \( \Gamma : R \to \text{End}_R(RP)^{op} \) by

\[
\Delta(r)(q) = rq, \quad \Gamma(r)(p) = pr
\]

for \( r \in R \), \( p \in P \) and \( q \in Q \).

Notice that for every \( r \in R \) we have \( \Gamma(r) \) is the adjoint of \( \Delta(r) \), and thus \( \Delta(r) \in \mathcal{L}_P(Q) \) and \( \Gamma(r) \in \mathcal{L}_Q(P) \).

**Proposition 3.11** (cf. [15, Lemma 2.2; 24]). Let \( R \) be a ring and \( (P, Q, \psi) \) an \( R \)-system satisfying condition (FS) and let \( (S, T, \sigma, B) \) be a covariant representation of \( (P, Q, \psi) \). Then there exist a unique ring homomorphism \( \pi_{T, S} : \mathcal{F}_P(Q) \to B \) such that \( \pi_{T, S}(\theta_{q,p}) = T(q)S(p) \) for \( p \in P \) and \( q \in Q \), and a unique ring homomorphism \( \chi_{S, T} : \mathcal{F}_Q(P) \to B^{op} \) such that \( \chi_{S, T}(\theta_{p,q}) = S(p) \cdot T(q) \) for \( p \in P \) and \( q \in Q \). These maps satisfy

\[
\begin{align*}
\pi_{T, S}(\Delta(r)\Theta) &= \sigma(r)\pi_{T, S}(\Theta), \\
\pi_{T, S}(\Theta \Delta(r)) &= \pi_{T, S}(\Theta)\sigma(r), \\
\chi_{S, T}(\Gamma(r)\Omega) &= \sigma(r) \cdot \chi_{S, T}(\Omega), \\
\chi_{S, T}(\Omega \Gamma(r)) &= \chi_{S, T}(\Omega) \cdot \sigma(r), \\
\pi_{T, S}(\Theta)T(q) &= T(\Theta(q)), \\
S(p) \cdot \chi_{S, T}(\Omega) &= S(\Omega(p))
\end{align*}
\]

for \( r \in R \), \( p \in P \), \( q \in Q \), \( \Omega \in \mathcal{F}_Q(P) \) and \( \Theta \in \mathcal{F}_P(Q) \).

If \( \Omega \in \mathcal{F}_Q(P) \) is the adjoint of \( \Theta \in \mathcal{F}_P(Q) \), then \( \pi_{T, S}(\Theta) = \chi_{S, T}(\Omega) \). Moreover we have \( \pi_{T, S}(\mathcal{F}_P(Q)) = \chi_{S, T}(\mathcal{F}_Q(P)) = \text{span}\{T(q)S(p) \mid q \in Q, p \in P\} \), and if \( \sigma \) is injective, then \( \pi_{T, S} \) and \( \chi_{S, T} \) are injective too.

**Proof.** Since \( \mathcal{F}_P(Q) = \text{span}\{\theta_{q,p} \mid p \in P, q \in Q\} \), there can at most be one ring homomorphism from \( \mathcal{F}_P(Q) \) to \( B \) which for all \( p \in P \) and \( q \in Q \) sends \( \theta_{q,p} \) to \( T(q)S(p) \).

Assume \( p_1, p_2, \ldots, p_n \in P \), \( q_1, q_2, \ldots, q_n \in Q \) and \( \sum_{i=1}^n \theta_{q_i, p_i} = 0 \). Then \( \sum_{i=1}^n q_i \psi(p_i \otimes z) = 0 \) for every \( z \in Q \). By condition (FS) there exists \( \Theta = \sum_{j=1}^k \theta_{e_j, f_j} \in \mathcal{F}_Q(P) \) such that

\[
\Theta(p_i) = \sum_{j=1}^k \theta_{e_j, f_j}(p_i) = \sum_{j=1}^k \psi(p_i \otimes f_j)e_j = p_i
\]
for every \( i = 1, \ldots, n \). We then have that
\[
\sum_{i=1}^{n} T(q_i)S(p_i) = \sum_{i=1}^{n} T(q_i)S(\Theta(p_i)) = \sum_{i=1}^{n} T(q_i)S \left( \sum_{j=1}^{k} \psi(p_i \otimes f_j)e_j \right) \\
= \sum_{i=1}^{n} \sum_{j=1}^{k} T(q_i \psi(p_i \otimes f_j))S(e_j) = \sum_{j=1}^{k} T \left( \sum_{i=1}^{n} q_i \psi(p_i \otimes f_j) \right) S(e_j) = 0,
\]

since \( \sum_{i=1}^{n} q_i \psi(p_i \otimes f_j) = 0 \) for every \( j = 1, \ldots, k \). Thus there exists a linear map \( \pi_{T,S} : \mathcal{F}_P(Q) \to B \) which for \( p \in P \) and \( q \in Q \) sends \( \theta_{q,p} \) to \( T(q)S(p) \).

Let \( r \in R, p \in P \) and \( q \in Q \). Then we have
\[
\pi_{T,S}(\Delta(r)\theta_{q,p}) = \pi_{T,S}(\partial_{q,p}) = T(r)S(p) = \sigma(r)T(q)S(p) = \sigma(r)\pi_{T,S}(\theta_{q,p}),
\]
from which it follows that \( \pi_{T,S}(\Delta(r)\Theta) = \sigma(r)\pi_{T,S}(\Theta) \) for every \( \Theta \in \mathcal{F}_P(Q) \). One can in a similar way show that \( \pi_{T,S}(\Theta \Delta(r)) = \pi_{T,S}(\Theta)\sigma(r) \) for every \( \Theta \in \mathcal{F}_P(Q) \).

Now suppose that \( \sigma : R \to B \) is injective and let \( \sum_{i=1}^{n} \theta_{q_i,p_i} \in \mathcal{F}_P(Q) \) with \( \pi_{T,S}(\sum_{i=1}^{n} \theta_{q_i,p_i}) = \sum_{i=1}^{n} T(q_i)S(p_i) = 0 \). For every \( p \in P \) and \( q \in Q \) we then have that
\[
0 = S(p) \left( \sum_{i=1}^{n} T(q_i)S(p_i) \right) T(q) = \sigma \left( \sum_{i=1}^{n} \psi(p \otimes q_i) \psi(p_i \otimes q) \right).
\]

Since \( \sigma \) is injective it follows that \( \sum_{i=1}^{n} \psi(p \otimes q_i) \psi(p_i \otimes q) = \psi(p \otimes \sum_{i=1}^{n} q_i \psi(p_i \otimes q)) = 0 \) for every \( p \in P \) and \( q \in Q \). By Lemma 3.7 the map \( \psi \) is non-degenerate, so it follows that \( \sum_{i=1}^{n} q_i \psi(p_i \otimes q) = 0 \) for every \( q \in Q \). Thus \( \sum_{i=1}^{n} \theta_{q_i,p_i} = 0 \) which proves that \( \pi_{T,S} \) is injective.

The existence and uniqueness of \( \chi_{S,T} \) and that \( \chi_{S,T} \) is a ring homomorphism and has the properties \( \chi_{S,T}(T(\tau)\Theta) = \sigma(\tau) \cdot \chi_{S,T}(\Theta) \), \( \chi_{S,T}(\Theta T(\tau)) = \chi_{S,T}(\Theta \cdot \sigma(\tau)) \), and \( S(\phi(p) \cdot \chi_{S,T}(\Theta)) = S(\Omega(p)) \) for \( r \in R, p \in P \) and \( \Theta \in \mathcal{F}_Q(P) \). That \( \chi_{S,T} \) is injective if \( \sigma \) is injective, can be proved in a similar way.

If \( p \in P \) and \( q \in Q \), then \( \theta_{p,q} \) is the adjoint of \( \theta_{q,p} \) and \( \pi_{T,S}(\theta_{q,p}) = T(q)S(p) = S(p) \cdot T(q) = \chi_{S,T}(\theta_{p,q}) \). It follows that if \( \Omega \in \mathcal{F}_Q(P) \) is the adjoint of \( \Theta \in \mathcal{F}_P(Q) \), then \( \pi_{T,S}(\Theta) = \chi_{S,T}(\Omega) \).

Finally we see that \( \pi_{T,S}(\mathcal{F}_P(Q)) = \text{span}\{T(q)S(p) \mid p \in P, q \in Q\} = \chi_{S,T}(\mathcal{F}_Q(P)) \).

**Notation 3.12.** To avoid too heavy notation, we will often, when working with a given \( R \)-system \((P,Q,\psi)\) satisfying condition (FS), let \( \pi \) denote \( \pi_{Q^n,Q^n}^{op} \) and let \( \chi \) denote \( \chi_{Q^n,Q^n}^{op} \) for any \( n \in \mathbb{N} \). We will then view \( \pi \) as a map from \( \bigcup_{n \in \mathbb{N}} \mathcal{F}_{P^n}(Q^{\otimes n}) \) to \( T_{(P,Q,\psi)} \) and \( \chi \) as a map from \( \bigcup_{n \in \mathbb{N}} \mathcal{F}_{Q^n}(P^{\otimes n}) \) to \( T_{(P,Q,\psi)}^{op} \).

**Remark 3.13.** Let \( R \) be a ring and \((P,Q,\psi)\) an \( R \)-system satisfying condition (FS). If \((S_1,T_1,\sigma_1,B_1)\) and \((S_2,T_2,\sigma_2,B_2)\) are two covariant representations of \((P,Q,\psi)\) and \( \phi : B_1 \to B_2 \) is a ring homomorphism such that \( \phi \circ T_1 = T_2, \phi \circ S_1 = S_2 \) and \( \phi \circ \sigma_1 = \sigma_2 \), then \( \phi \circ \pi_{T_1,S_1} = \pi_{T_2,S_2} \) and \( \phi \circ \chi_{S_1,T_1} = \chi_{S_2,T_2} \).
3.2. Cuntz–Pimsner invariant representations

As already mentioned, the aim of this section is to classify all injective and graded representations of an $R$-system satisfying condition (FS). We will now, for a given $R$-system $(P, Q, \psi)$ satisfying condition (FS), construct a family of surjective, injective and graded representations of $(P, Q, \psi)$. We will then later show that up to isomorphism this family of surjective, injective and graded representation of $(P, Q, \psi)$ contains all surjective, injective and graded representation of $(P, Q, \psi)$.

**Definition 3.14.** Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). We say that a two-sided ideal $J$ of $R$ is $\psi$-compatible if $J \subseteq \Delta^{-1}(\mathcal{F}_P(Q))$, and we say that a $\psi$-compatible two-sided ideal $J$ of $R$ is faithful if $J \cap \ker \Delta = \{0\}$.

**Definition 3.15.** Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). For a $\psi$-compatible two-sided ideal $J$ of $R$, we define $T(J)$ to be the minimal two-sided ideal of $T_{(P, Q, \psi)}$ that contains $\{ \iota_R(x) - \pi(\Delta(x)) \mid x \in J \}$.

**Definition 3.16** (cf. [12, Proposition 1.3; 21, Proposition 2.18]). Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS) and let $J$ be a $\psi$-compatible two-sided ideal of $R$. We define the Cuntz–Pimsner ring relative to the ideal $J$ to be the quotient ring $\mathcal{O}_{(P, Q, \psi)}(J) := T_{(P, Q, \psi)}/T(J)$. We denote by $\rho_J$ the quotient map $\rho_J : T_{(P, Q, \psi)} \to \mathcal{O}_{(P, Q, \psi)}(J)$.

**Definition 3.17** (cf. [12, Definition 1.1]). Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS) and let $J$ be a $\psi$-compatible two-sided ideal of $R$. A covariant representation $(S, T, \sigma, B)$ of $(P, Q, \psi)$ is said to be Cuntz–Pimsner invariant relative to $J$ if $\pi_T, S(\Delta(x)) = \sigma(x)$ for every $x \in J$.

The following theorem gives a complete characterization of $\mathcal{O}_{(P, Q, \psi)}(J)$.

**Theorem 3.18** (cf. [12, Proposition 1.3]). Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS) and let $J$ be a $\psi$-compatible two-sided ideal of $R$. Let $\iota_R^J := \rho_J \circ \iota_R$, $\iota_Q^J := \rho_J \circ \iota_Q$ and $\iota_P^J := \rho_J \circ \iota_P$. Then $(\iota_P^J, \iota_Q^J, \iota_R^J, \mathcal{O}_{(P, Q, \psi)}(J))$ is a surjective covariant representation of $(P, Q, \psi)$ which is Cuntz–Pimsner invariant relative to $J$ and has the following property:

If $(S, T, \sigma, B)$ is a covariant representation of $(P, Q, \psi)$ which is Cuntz–Pimsner invariant relative to $J$, then there exists a unique ring homomorphism

$$(CP) \quad \eta^J_{(S, T, \sigma, B)} : \mathcal{O}_{(P, Q, \psi)}(J) \to B$$

such that $\eta^J_{(S, T, \sigma, B)} \circ \iota_R^J = \sigma$, $\eta^J_{(S, T, \sigma, B)} \circ \iota_Q^J = T$ and $\eta^J_{(S, T, \sigma, B)} \circ \iota_P^J = S$.

The representation $(\iota_P^J, \iota_Q^J, \iota_R^J, \mathcal{O}_{(P, Q, \psi)}(J))$ is the, up to isomorphism in $\mathcal{C}_{(P, Q, \psi)}$, unique surjective covariant representation of $(P, Q, \psi)$ which is Cuntz–Pimsner invariant relative to $J$ and which possesses the property (CP); in fact if $(S, T, \sigma, B)$ is a surjective covariant representation of $(P, Q, \psi)$ which is Cuntz–Pimsner invariant relative to $J$ and $\phi : B \to \mathcal{O}_{(P, Q, \psi)}(J)$ is a ring homomorphism such that $\phi \circ \sigma = \iota_R^J$, $\phi \circ S = \iota_P^J$ and $\phi \circ T = \iota_Q^J$, then $\phi$ is an isomorphism.

We have moreover that the ring homomorphism $\iota_R^J$ is injective if and only if $J$ is faithful, and that the representation $(\iota_P^J, \iota_Q^J, \iota_R^J, \mathcal{O}_{(P, Q, \psi)}(J))$ is graded.

We call $(\iota_P^J, \iota_Q^J, \iota_R^J, \mathcal{O}_{(P, Q, \psi)}(J))$ the Cuntz–Pimsner representation of $(P, Q, \psi)$ relative to $J$. 
Remark 3.19. For a ring \( R \), an \( R \)-system \((P, Q, \psi)\) satisfying condition (FS), and a \( \psi \)-compatible two-sided ideal \( J \) of \( R \), let \( C^J_{(P, Q, \psi)} \) denote the subcategory of \( C_{(P, Q, \psi)} \) consisting of all surjective covariant representations of \((P, Q, \psi)\) which are Cuntz–Pimsner invariant relative to \( J \). It follows from Theorem 3.18 that \((\iota^J_R, \iota^J_Q, C^J_{(P, Q, \psi)})\) is an initial object in \( C^J_{(P, Q, \psi)} \).

To prove Theorem 3.18 we need a definition, a lemma and a proposition.

Definition 3.20. Let \( R \) be a ring, let \((P, Q, \psi)\) be an \( R \)-system and let \((S, T, \sigma, B)\) be a surjective and graded covariant representation of \((P, Q, \psi)\). It follows from Proposition 3.2 and Definition 3.3 that there is a unique \( \mathbb{Z} \)-grading \( \bigoplus_{n \in \mathbb{Z}} B^{(n)} \) of \( B \) such that \( \sigma(R) \subseteq B^{(0)} \), \( T(Q) \subseteq B^{(1)} \) and \( S(P) \subseteq B^{(-1)} \).

A two-sided ideal \( I \) of \( B \) is said to be graded if \( \bigoplus_{n \in \mathbb{Z}} I^{(n)} \) is a \( \mathbb{Z} \)-grading of \( I \), where \( I^{(n)} := I \cap B^{(n)} \) for each \( n \in \mathbb{Z} \). It is not difficult to show that in this case \( \bigoplus_{n \in \mathbb{Z}} \sigma(B^{(n)}) \) is a \( \mathbb{Z} \)-grading of the quotient ring \( B/I \), where \( \varphi_I \) denotes the quotient map from \( B \) to \( B/I \), and that the covariant representation \((S_I, T_I, \sigma_I, B/I)\), where \( T_I := \varphi_I \circ T, S_I = \varphi_I \circ S \) and \( \sigma_I = \varphi_I \circ \sigma \), is graded.

For \((m, n) \in \mathcal{S} \), let \( \mathcal{P}_{(m, n)} \) denote the projection of \( T_{(P, Q, \psi)} \) onto \( \mathcal{T}_{(m, n)} \) given by the \( \mathcal{S} \)-grading \( \bigoplus_{(k,l) \in \mathcal{S}} T_{(k,l)} \) (cf. Theorem 1.7).

Lemma 3.21 (cf. [21, Lemma 2.20]). Let \( R \) be a ring, let \((P, Q, \psi)\) be an \( R \)-system satisfying condition (FS) and let \( J \) be a \( \psi \)-compatible two-sided ideal of \( R \). For \( n \in \mathbb{N} \) let

\[
T^{(n)}(J) = \text{span} \left\{ (\iota^k_Q(q)(\iota_R(x) - \pi(\Delta(x)))\iota^l_P(p) \mid x \in J, q \in Q^{\otimes k}, p \in P^{\otimes l}, k, l \in \mathbb{N} \text{ with } k - l = n \} \cup \{ \iota^k_Q(q)(\iota_R(x) - \pi(\Delta(x))) \mid x \in J, q \in Q^{\otimes n} \} \right\}
\]

and

\[
T^{(-n)}(J) = \text{span} \left\{ (\iota^k_Q(q)(\iota_R(x) - \pi(\Delta(x)))\iota^l_P(p) \mid x \in J, q \in Q^{\otimes k}, p \in P^{\otimes l}, k, l \in \mathbb{N} \text{ with } l - k = n \} \cup \{ (\iota_R(x) - \pi(\Delta(x)))\iota^l_P(p) \mid x \in J, p \in P^{\otimes n} \} \right\},
\]

and let

\[
T^{(0)}(J) = \text{span} \left\{ (\iota^k_Q(q)(\iota_R(x) - \pi(\Delta(x))) \iota^l_P(p) \mid x \in J, q \in Q^{\otimes k}, p \in P^{\otimes k}, k \in \mathbb{N} \} \cup \{ (\iota_R(x) - \pi(\Delta(x))) \mid x \in J \} \right\}.
\]

Then we have that \( T^{(m)}(J) = T^{(m)}_{(P, Q, \psi)} \cap T(J) \) for each \( m \in \mathbb{Z} \), and that \( \bigoplus_{m \in \mathbb{Z}} T^{(m)}(J) \) is a \( \mathbb{Z} \)-grading of \( T(J) \). Thus \( T(J) \) is a graded two-sided ideal of \( T_{(P, Q, \psi)} \).

We furthermore have that the following holds for every \( x \in T(J) \):

(i) \( T_{(0, 0)}(x) \in T(R) \);

(ii) there exists an \( n \in \mathbb{N} \) such that \( x\iota^m_Q(q) = 0 \) for every \( m \geq n \) and every \( q \in Q^{\otimes m} \).

Proof. It is clear that \( T^{(m)}(J) \subseteq T^{(m)}_{(P, Q, \psi)} \cap T(J) \) for each \( m \in \mathbb{Z} \). It is also clear that \( \bigoplus_{m \in \mathbb{Z}} T^{(m)}_{(P, Q, \psi)} \cap T(J) \subseteq T(J) \).

If \( x \in J, q \in Q \) and \( p \in P \), then we have that

\[
(\iota_R(x) - \pi(\Delta(x)))\iota_Q(q) = \iota_Q(xq) - \iota_Q(\Delta(x)q) = \iota_Q(xq) - \iota_Q(x) = 0,
\]

(3.1)
and that
\[ \iota_P(p)(\iota_R(x) - \pi(\Delta(x))) = \iota_P(p)(\iota_R(x) - \chi(\Gamma(x))) = \iota_P(px) - \iota_P(\Gamma(x)p) = \iota_P(px) - \iota_P(px) = 0, \]
from which it follows that \( \oplus_{m \in \mathbb{Z}} T^{(m)}(J) \) is a two-sided ideal of \( T(P,Q,\psi) \). Since
\[ \{\iota_R(x) - \pi(\Delta(x)) \mid x \in J\} \subseteq T^{(0)}(J), \]
it follows that \( T(J) \subseteq \oplus_{m \in \mathbb{Z}} T^{(m)}(J) \). Thus we have that
\[
\bigoplus_{m \in \mathbb{Z}} T^{(m)}(J) = T(J) \tag{3.2}
\]
and that \( T^{(m)}(J) = T^{(m)}(P,Q,\psi) \cap T(J) \) for each \( m \in \mathbb{Z}. \)

Let \( x \in T(J) \). That (i) holds directly follows from equation (3.2), and that (ii) holds directly follows from equations (3.1) and (3.2). \( \square \)

**Proposition 3.22** (cf. [21, Proposition 2.21]). Let \( R \) be a ring, let \( (P,Q,\psi) \) be an \( R \)-system satisfying condition (FS) and let \( J \) be a faithful \( \psi \)-compatible two-sided ideal of \( R \). Then the ring homomorphism \( \rho : R \to T(P,Q,\psi)/T(J) \) given by \( \rho(r) = \iota_R(r) + T(J) \) is injective.

**Proof.** Assume that \( r \in R \) and that \( \iota_R(r) \in T(J) \). It follows from Lemma 3.21 that there exists an \( n \in \mathbb{N} \) such that \( \iota_R(r)\iota_Q^n(q) = 0 \) for every \( m \geq n \) and every \( q \in Q^{\otimes m} \). We will show that we can choose \( n \) to be equal to 1. We will do that by showing that if \( n > 1 \) and \( \iota_R(r)\iota_Q^n(q) = 0 \) for every \( q \in Q^{\otimes n} \), then \( \iota_R(r)\iota_Q^{n-1}(q) = 0 \) for every \( q \in Q^{\otimes n-1} \). Assume that \( n > 1 \) and \( \iota_R(r)\iota_Q^n(q) = 0 \) for every \( q \in Q^{\otimes n} \). Let \( q \in Q^{\otimes n-1} \). Then we have for every \( q' \in Q \) that
\[ \iota_Q^n(rq \otimes q') = \iota_R(r)\iota_Q^n(q \otimes q') = 0. \]
Since \( \iota_Q^n \) is injective (cf. Lemma 3.9), it follows that \( rq \otimes q' = 0 \). Hence for every \( p \in P^{\otimes n-1} \) and every \( p' \in P \) we have that
\[ \psi(p' \otimes \psi_{n-1}(p \otimes rq)q') = \psi_n((p' \otimes p) \otimes (rq \otimes q')) = 0. \]
The above holds for every \( p' \in P \), so by Lemma 3.7 we have that
\[ \psi_{n-1}(p \otimes rq)q' = 0. \]
Since the last equation holds for every \( q' \in Q \), it follows that \( \psi_{n-1}(p \otimes rq) \in \ker \Delta \) for every \( p \in P^{\otimes n-1} \). We have that \( \iota_{P,-1}(p)\iota_R(r)\iota_Q^{n-1}(q) \in T(J) \), so it follows from Lemma 3.21 that
\[ \iota_R(\psi_{n-1}(p \otimes rq)) = P_{(0,0)}(\iota_{P,-1}(p)\iota_R(r)\iota_Q^{n-1}(q)) \in \iota_R(J). \]
Thus \( \psi_{n-1}(p \otimes rq) \in J \cap \ker \Delta = \{0\} \) for all \( p \in P^{\otimes (n-1)} \), so by Lemma 3.7 and 3.8 we have that \( rq = 0 \). Hence \( \iota_R(r)\iota_Q^{n-1}(q) = 0 \).

Thus \( \iota_Q(\Delta(r)q) = \iota_R(r)\iota_Q(q) = 0 \) for every \( q \in Q \). From the injectivity of \( \iota_Q \) (cf. Lemma 3.9) it follows that \( r \in \ker \Delta \). Then by Lemma 3.21 we have that \( \iota_R(r) = P_{(0,0)}(\iota_R(r)) \in \iota_R(J) \). Therefore \( r \in J \cap \ker \Delta = \{0\} \), which shows that \( r = 0 \) as desired. \( \square \)

It follows from Lemma 3.21 and Proposition 3.22 that if \( R \) is a ring, \( (P,Q,\psi) \) is an \( R \)-system satisfying condition (FS) and \( J \) is a faithful \( \psi \)-compatible two-sided ideal of \( R \), then \( T(J) \) is a graded two-sided ideal of \( T(P,Q,\psi) \) which satisfies that \( \iota_R(R) \cap T(J) = \{0\} \). We will show (see Remark 4.1) that every graded two-sided ideal \( I \) of \( T(P,Q,\psi) \) for which \( \iota_R(R) \cap I = \{0\} \), of this form.
Proof of Theorem 3.18. It is clear that $(ι_P^J, ι_Q^J, ι_R^J, O^J_{(P,Q,ψ)})$ is a covariant representation of $(P,Q,ψ)$ which is Cuntz–Pimsner invariant relative to $J$, and that it possesses property (CP) follows from Theorem 1.7 and the definition of $T(J)$ and $(ι_P^J, ι_Q^J, ι_R^J, O^J_{(P,Q,ψ)})$.

If $(S,T,σ,B)$ is a surjective covariant representation of $(P,Q,ψ)$ which is Cuntz–Pimsner invariant relative to $J$ and $φ: B → O^J_{(P,Q,ψ)}(J)$ is a ring homomorphism such that $φ ◦ σ = ι_R^J$, $φ ◦ S = ι_P^J$ and $φ ◦ T = ι_Q^J$, then $η^J_{(S,T,σ,B)} ◦ φ(σ(r)) = φ(r)$ for all $r ∈ R$, $η^J_{(S,T,σ,B)} ◦ φ(S(p)) = S(p)$ for all $p ∈ P$ and $η^J_{(S,T,σ,B)} ◦ φ(T(q)) = T(q)$ for all $q ∈ Q$, and since $B$ is generated by $σ(R) ∪ S(P) ∪ T(Q)$, it follows that $η^J_{(S,T,σ,B)} ◦ φ$ is equal to the identity map of $B$. One can in a similar way show that $φ ◦ η^J_{(S,T,σ,B)}$ is equal to the identity map of $O^J_{(P,Q,ψ)}(J)$. Thus $φ$ and $η^J_{(S,T,σ,B)}$ are inverse of each other, and $φ$ is an isomorphism.

If $J$ is faithful, then it follows from Proposition 3.22 that $ι_R^J$ is injective. If $x ∈ J ∩ ker Δ$, then $ι_R^J(x) = 0$; so if $J$ is not faithful, then $ι_R^J$ is not injective.

It follows directly from Lemma 3.21 that $(ι_P^J, ι_Q^J, ι_R^J, O^J_{(P,Q,ψ)}(J))$ is a surjective and graded covariant representation of $(S,T,σ,B)$ which is Cuntz–Pimsner invariant relative to $S, T, σ, B$.

3.3. Injective and graded covariant representations

Let $R$ be a ring and $(P,Q,ψ)$ an $R$-system which satisfies condition (FS). As mentioned before, we will show that every surjective, injective and graded covariant representations of $(P,Q,ψ)$ is isomorphic to $(ι_P^J, ι_Q^J, ι_R^J, O^J_{(P,Q,ψ)}(J))$ for some faithful $ψ$-compatible two-sided ideal $J$ of $R$.

Definition 3.23. Let $R$ be a ring, let $(P,Q,ψ)$ be an $R$-system satisfying condition (FS) and let $(S,T,σ,B)$ be a covariant representation of $(P,Q,ψ)$. We define $J_{(S,T,σ,B)} := \{ r ∈ R \mid σ(r) ∈ π_{T,S}(F_P(Q)) \}$.

Lemma 3.24 (cf. [16, Proposition 3.3]). Let $R$ be a ring and let $(S,T,σ,B)$ be an injective covariant representation of an $R$-system $(P,Q,ψ)$ that satisfies condition (FS). Then $r ∈ R$ is in $J_{(S,T,σ,B)}$ if and only if $r ∈ Δ^{-1}(F_P(Q))$ and $σ(r) = π_{T,S}(Δ(r))$.

Proof. It is obvious that if $r ∈ Δ^{-1}(F_P(Q))$ and $σ(r) = π_{T,S}(Δ(r))$, then $r ∈ J_{(S,T,σ,B)}$. If $Θ ∈ F_P(Q)$ and $σ(r) = π_{T,S}(Θ)$, then we have for every $q ∈ Q$ that $T(q) = π_{T,S}(Θ)T(q) = T(Θ(q))$, and since $T$ is injective (cf. Lemmas 3.7 and 3.9), it follows that $rq = Θ(q)$. Hence $Δ(r) = Θ$.

Remark 3.25. Let $R$ be a ring, $(P,Q,ψ)$ be an $R$-system satisfying condition (FS), let $J$ be a $ψ$-compatible two-sided ideal of $R$ and let $(S,T,σ,B)$ be an injective covariant representation of $(P,Q,ψ)$. Then it follows from Lemma 3.24 that $(S,T,σ,B)$ is Cuntz–Pimsner invariant with respect to $J$ if and only if $J ⊆ J_{(S,T,σ,B)}$.

Lemma 3.26. Let $R$ be a ring, $(P,Q,ψ)$ an $R$-system satisfying condition (FS) and let $(S,T,σ,B)$ be a covariant representation of $(P,Q,ψ)$. Then $J_{(S,T,σ,B)}$ is a $ψ$-compatible two-sided ideal of $R$. If $(S,T,σ,B)$ is injective, then $J_{(S,T,σ,B)}$ is faithful.

Proof. It easily follows from Proposition 3.11 that $J_{(S,T,σ,B)}$ is a two-sided ideal of $R$ and it is $ψ$-compatible by construction. If $x ∈ J_{(S,T,σ,B)} ∩ ker Δ$ and $(S,T,σ,B)$ is injective, then it
follows from Lemma 3.24 that $\sigma(x) = \pi_{T,S}(\Delta(x)) = 0$, and since $\sigma$ is injective, it follows that $x = 0$. Thus $J_{(S,T,\sigma,B)}$ is faithful if $(S,T,\sigma,B)$ is injective.

**Notation 3.27.** To avoid too heavy notation, we will, often when working with a given $R$-system $(P,Q,\psi)$ satisfying condition (FS) and a faithful $\psi$-compatible two-sided ideal $J$ of $R$, let $\pi^J$ denote $\pi_{(\psi^J)^n,(\psi^J)^n}$ for any $n \in \mathbb{N}$. We will then view $\pi^J$ as a map from $\bigcup_{n \in \mathbb{N}} \mathcal{F}_{P \otimes n}(Q^{\otimes n})$ to $\mathcal{O}_{(P,Q,\psi)}(J)$.

**Proposition 3.28.** Let $R$ be a ring, let $(P,Q,\psi)$ be an $R$-system satisfying condition (FS) and let $J$ be a faithful $\psi$-compatible two-sided ideal of $R$. Then $J = J_{(\iota_P,\iota_Q,\iota_R,\mathcal{O}_{(P,Q,\psi)}(J))}$.

**Proof.** If $x \in J$, then $\iota_P^J(x) = \pi^J(\Delta(x)) \in \pi^J(\mathcal{F}_P(Q))$, and so $x \in J_{(\iota_P,\iota_Q,\iota_R,\mathcal{O}_{(P,Q,\psi)}(J))}$. If $x \in J_{(\iota_P,\iota_Q,\iota_R,\mathcal{O}_{(P,Q,\psi)}(J))}$, then it follows from Lemma 3.24 that $x \in \Delta^{-1}(\mathcal{F}_P(Q))$ and $\iota_P^J(x) = \pi^J(\Delta(x))$. So $\iota_R(x) - \pi(\Delta(x)) \in T(J)$, and we then get from Lemma 3.21 that $\iota_R(x) = \mathcal{P}_{(0,0)}(\iota_R(x) - \pi(\Delta(x))) \in \iota_R(J)$, and thus that $x \in J$.

We are now ready to show that every surjective, injective and graded covariant representation of an $R$-system $(P,Q,\psi)$ satisfying condition (FS) is isomorphic to $(\iota_P^J,\iota_Q^J,\iota_R^J,\mathcal{O}_{(P,Q,\psi)}(J))$ for some faithful $\psi$-compatible two-sided ideal $J$ of $R$.

**Theorem 3.29.** Let $R$ be a ring, let $(P,Q,\psi)$ be an $R$-system satisfying condition (FS), let $J$ be a faithful $\psi$-compatible two-sided ideal of $R$ and let $(S,T,\sigma,B)$ be a covariant representation of $(P,Q,\psi)$. Then we have:

(i) If there exists a ring homomorphism $\eta : \mathcal{O}_{(P,Q,\psi)}(J) \to B$ such that $\eta \circ \iota_P^J = T$, $\eta \circ \iota_Q^J = S$ and $\eta \circ \iota_R^J = \sigma$, then the representation $(S,T,\sigma,B)$ is Cuntz–Pimsner invariant with respect to $J$.

(ii) If the representation $(S,T,\sigma,B)$ is Cuntz–Pimsner invariant with respect to $J$, then there exists a unique ring homomorphism $\eta_{(S,T,\sigma,B)}^J : \mathcal{O}_{(P,Q,\psi)}(J) \to B$ such that $\eta_{(S,T,\sigma,B)}^J \circ \iota_P^J = T$, $\eta_{(S,T,\sigma,B)}^J \circ \iota_Q^J = S$ and $\eta_{(S,T,\sigma,B)}^J \circ \iota_R^J = \sigma$.

(iii) If the representation $(S,T,\sigma,B)$ is Cuntz–Pimsner invariant with respect to $J$, then the ring homomorphism $\eta_{(S,T,\sigma,B)}$ is an isomorphism if and only if $(S,T,\sigma,B)$ is surjective, injective and graded and $J = J_{(S,T,\sigma,B)}$.

For the proof of Theorem 3.29 we need some lemmas, but before we introduce them, let us note that the promised classification of all surjective, injective and graded covariant representations of a given $R$-system $(P,Q,\psi)$ satisfying condition (FS) follows from Lemma 3.26 and Theorem 3.29.

**Remark 3.30.** Let $R$ be a ring and let $(P,Q,\psi)$ be an $R$-system satisfying condition (FS). It follows from Lemma 3.26 and Theorem 3.29 that every surjective, injective and graded covariant representation of $(P,Q,\psi)$ is isomorphic to $(\iota_P^J,\iota_Q^J,\iota_R^J,\mathcal{O}_{(P,Q,\psi)}(J))$ for some faithful $\psi$-compatible two-sided ideal $J$ of $R$. And it follows from Remark 3.13 and Proposition 3.28 that if $J_1$ and $J_2$ are two faithful $\psi$-compatible two-sided ideals of $R$, then there exists a ring homomorphism $\phi$ from $\mathcal{O}_{(P,Q,\psi)}(J_1)$ to $\mathcal{O}_{(P,Q,\psi)}(J_2)$ satisfying $\phi \circ \iota_{Q,1}^J = \iota_{Q,2}^J$, $\phi \circ \iota_{P,1}^J = \iota_{P,2}^J$ and $\phi \circ \iota_{R,1}^J = \iota_{R,2}^J$ if and only if $J_1 \subseteq J_2$. 
We will now introduce and prove the lemmas which we will use in the proof of Theorem 3.29.

**Lemma 3.31.** Let $R$ be a ring and $(P, Q, \psi)$ an $R$-system. If $n, m \in \mathbb{N}$, $\Theta \in \mathcal{F}_{P \otimes (n+m)}(Q \otimes n+m)$, $q \in Q^\otimes n$ and $p \in P^\otimes n$, then there is a $\Theta_{p,q} \in \mathcal{F}_{P \otimes m}(Q^\otimes m)$ such that

$$
\pi_{T^m, S^m}(\Theta_{p,q}) = S^n(p)\pi_{T(n+m), S(n+m)}(\Theta)T^n(q)
$$

(3.3)
for all covariant representations $(S, T, \sigma, B)$ of $(P, Q, \psi)$.

**Proof.** Choose $q_i \in Q^\otimes n$, $q'_i \in Q^\otimes m$, $p_i \in P^\otimes n$ and $p'_i \in P^\otimes m$ for $i \in \{1, 2, \ldots, k\}$ such that $\sum_{i=1}^k \theta_{q_i \otimes q'_i, p_i \otimes p'_i} = \Theta$, and let $\Theta_{p,q} = \sum_{i=1}^k \theta_{\psi_n(p \otimes q), \psi_n(p' \otimes q')}$, Then $\Theta_{p,q} \in \mathcal{F}_{P \otimes m}(Q^\otimes m)$ and it is straightforward to check that equation (3.3) is satisfied by all covariant representations $(S, T, \sigma, B)$ of $(P, Q, \psi)$. $\square$

**Lemma 3.32.** Let $R$ be a ring and $(P, Q, \psi)$ an $R$-system. Let $n \in \mathbb{N}$ and $T \in \mathcal{L}_{P \otimes n}(Q^\otimes n)$. Then there is a unique $T \otimes 1_Q \in \mathcal{L}_{P \otimes n+1}(Q^\otimes n+1)$ such that $(T \otimes 1_Q)(q \otimes q') = T(q) \otimes q'$ for $q \in Q^\otimes n$ and $q' \in Q$.

**Proof.** It easily follows from the universal property of tensor products that there exists a unique map $T \otimes 1_Q : Q^\otimes n+1 \to Q^\otimes n+1$ which for all $q \in Q^\otimes n$ and $q' \in Q$ maps $q \otimes q'$ to $T(q) \otimes q'$. Likewise, if $S$ denotes an adjoin of $T$, then there is a unique map $1_P \otimes S : P^\otimes n+1 \to P^\otimes n+1$ which for all $p \in P^\otimes n$ and $p' \in P$ maps $p' \otimes p$ to $p' \otimes S(p)$. We have

$$
\psi_{n+1}((p' \otimes p) \otimes (T(q) \otimes q')) = \psi(p' \psi_n(p \otimes T(q)) \otimes q') = \psi(p' \psi_n(S(p) \otimes q) \otimes q')
$$

$$
= \psi_{n+1}((p' \otimes S(p)) \otimes (q \otimes q'))
$$

for $p' \in P$, $p \in P^\otimes n$, $q' \in Q$ and $q \in Q^\otimes n$, from which it follows that $1_P \otimes S$ is an adjoin of $T \otimes 1_Q$ and thus that $T \otimes 1_Q \in \mathcal{L}_{P \otimes n+1}(Q^\otimes n+1)$ (and $1_P \otimes S \in \mathcal{L}_{Q \otimes n+1}(P^\otimes n+1)$). $\square$

The following abuse of notation will be convenient in what follows.

**Notation 3.33.** Let $R$ be a ring and $(P, Q, \psi)$ an $R$-system. If $n = 0$, then we will, on occasion, let $\mathcal{F}_{P \otimes n}(Q^\otimes n)$ denote $R$, and we will, for $T \in \mathcal{L}_{P \otimes n}(Q^\otimes n)$, use $T \otimes 1_Q$ to denote $\Delta(T)$.

**Lemma 3.34.** Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS), let $(S, T, \sigma, B)$ be a covariant representation and let $n \in \mathbb{N}_0$. Then

$$
\pi_{T^m, S^m}(\Theta_1 \otimes 1_Q) = \pi_{T^m, S^m}(\Theta_1)\pi_{T^{n+1}, S^{n+1}}(\Theta_2)
$$

for $\Theta_1 \in \mathcal{F}_{P \otimes n}(Q^\otimes n)$ and $\Theta_2 \in \mathcal{F}_{P \otimes n+1}(Q^\otimes n+1)$.

**Proof.** If $n = 0$, then the result follows directly from Proposition 3.11. Assume that $n \in \mathbb{N}$. It suffices to prove the lemma in the case where $\Theta_2 = \theta_{q \otimes q', p}$ and $q \in Q^\otimes n$, $q' \in Q$ and $p \in P^\otimes n+1$. In that case $(\Theta_1 \otimes 1_Q)\theta_{q \otimes q', p} = \theta_{\Theta_1(q) \otimes q', p}$, so it follows from Proposition 3.11 that

$$
\pi_{T^{n+1}, S^{n+1}}((\Theta_1 \otimes 1_Q)\theta_{q \otimes q', p}) = \pi_{T^{n+1}, S^{n+1}}(\Theta_1(q) \otimes q')S(p)
$$

$$
= T(\Theta_1(q))T(q')S(p) = \pi_{T^n, S^n}(\Theta_1)T(q)T(q')S(p)
$$

$$
= \pi_{T^n, S^n}(\Theta_1)T(q \otimes q')S(p) = \pi_{T^n, S^n}(\Theta_1)\pi_{T^{n+1}, S^{n+1}}(\theta_{q \otimes q', p}).
$$

$\square$
Lemma 3.35. Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system, let $(S, T, \sigma, B)$ be a surjective and graded covariant representation of $(P, Q, \psi)$ and let $H$ be a two-sided ideal of $B$. If $H$ is generated as a two-sided ideal of $B$ by $H \cap B^{(0)}$, then $H$ is graded. If $(P, Q, \psi)$ satisfies condition (FS) and $H$ is graded, then $H$ is generated as a two-sided ideal of $B$ by $H \cap B^{(0)}$.

Proof. For each $n \in \mathbb{Z} \setminus \{0\}$ let

$$H^{(n)} = \text{span} \left( \bigcup_{m \in \mathbb{Z}} \left\{ yxz \mid y \in B^{(m)}, x \in H \cap B^{(0)}, z \in B^{(n-m)} \right\} \right)$$

and let

$$H^{(0)} = H \cap B^{(0)}.$$ 

Then $H^{(n)} \subseteq B^{(n)}$ for all $n \in \mathbb{Z}$, and it is not difficult to show that $\oplus_{n \in \mathbb{Z}} H^{(n)}$ is a graded two-sided ideal of $B$ which contains $H \cap B^{(0)}$, and that every two-sided ideal of $B$ which contains $H \cap B^{(0)}$ also contains $\oplus_{n \in \mathbb{Z}} H^{(n)}$. So if $H$ is generated by $H \cap B^{(0)}$, then it is equal to $\oplus_{n \in \mathbb{Z}} H^{(n)}$ and thus graded.

For the last assertion assume that $H$ is graded and that $(P, Q, \psi)$ satisfies condition (FS). We will show that $H = \oplus_{n \in \mathbb{Z}} H^{(n)}$. Since $H$ is graded it is enough to show that if $n \in \mathbb{Z}$ and $x \in H \cap B^{(n)}$ then $x \in H^{(n)}$. If $n > 0$ and $x \in H \cap B^{(n)}$, then there exist $q_0, q_1, q_2, \ldots, q_k \in Q^{\otimes n}$ and $y_1, y_2, \ldots, y_k \in B^{(0)}$ such that $x = T^n(q_0) + \sum_{i=1}^{k} T^n(q_i)y_i$. It follows from Lemma 3.8 that there exist $q'_1, q'_2, \ldots, q'_l \in Q^{\otimes n}$ and $p'_1, p'_2, \ldots, p'_l \in P^{\otimes n}$ such that $\sum_{j=1}^{l} q'_j \psi_n(p'_j \otimes q_i) = q_i$ for $i \in \{0, 1, 2, \ldots, k\}$. We then have that

$$\sum_{j=1}^{l} T^n(q'_j)S^n(p'_j)x = \sum_{j=1}^{l} T^n(q'_j)S^n(p'_j)T^n(q_0) + \sum_{i=1}^{k} \sum_{j=1}^{l} T^n(q'_j)S^n(p'_j)T^n(q_i)y_i$$

$$= T^n(q_0) + \sum_{i=1}^{k} T^n(q_i)y_i = x,$$

and that $S^n(p'_j)x \in H^{(0)}$ for every $j \in \{1, 2, \ldots, l\}$, from which it follows that $x \in H^{(n)}$. One can in a similar way show that if $n < 0$ and $x \in H \cap B^{(n)}$ then $x \in H^{(n)}$. Thus we have for all $n \in \mathbb{Z}$ that if $x \in H \cap B^{(n)}$ then $x \in H^{(n)}$, from which it follows that $H = \oplus_{n \in \mathbb{Z}} H^{(n)}$. \hfill $\square$

Lemma 3.36. Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS) and let $H$ be a two-sided ideal of $\mathcal{T}_{(P, Q, \psi)}$. Then we have that

$$J_H := \{ r \in \Delta^{-1}(F_P(Q)) \mid \iota_R(r) - \pi(\Delta(r)) \in H \}$$

is a $\psi$-compatible two-sided ideal of $R$ and $T(J_H) \subseteq H$. If in addition $H$ is graded and $H \cap \iota_R(R) = \{0\}$, then $J_H$ is faithful and $T(J_H) = H$.

Proof. It directly follows from Proposition 3.11 that $J_H$ is a two-sided ideal of $R$, and it is $\psi$-compatible by construction. It follows directly from the definition of $T(J_H)$ that $T(J_H) \subseteq H$.

Assume that $H$ is graded and $H \cap \iota_R(R) = \{0\}$. If $x \in J_H \cap \ker \Delta$, then $\iota_R(x) = \iota_R(x) - \pi(\Delta(r)) \in T(J_H) \subseteq H$ and so $x = 0$ proving that $J_H \cap \ker \Delta = \{0\}$.

We will then prove that $H \subseteq T(J_H)$. By Lemma 3.35 it suffices to show that $H \cap T^{(0)}_{(P, Q, \psi)} \subseteq T(J_H)$. It follows from Theorem 1.7 and Propositions 3.1 and 3.11 that $T^{(0)}_{(P, Q, \psi)} = \oplus_{i=0}^{\infty} \pi(F_{P^{\otimes i}}(Q^{\otimes i}))$ (where we let $F_{P^{\otimes i}}(Q^{\otimes i}) = R$ and $\pi : F_{P^{\otimes i}}(Q^{\otimes i}) \to T_{(P, Q, \psi)} = \iota_R$), so it
suffices to prove that the following inclusion holds for every $n \in \mathbb{N}$:
\[
H \cap \left( \bigoplus_{i=0}^{n} \pi(F_{P \odot i}(Q^\odot i)) \right) \subseteq T(J_H).
\] (3.4)

We will prove that (3.4) holds by induction over $n$.

First we notice that $H \cap (\pi(F_{P \odot 0}(Q^\odot 0))) = H \cap \iota_R(R) = \{0\} \subseteq T(J_H)$, proving that (3.4) holds for $n = 0$.

Assume now that $n \in \mathbb{N}_0$ and that (3.4) holds. Let $\Theta_i \in F_{P \odot i}(Q^\odot i)$ for $i \in \{0,1,\ldots,n+1\}$ such that $\sum_{i=0}^{n+1} \pi(\Theta_i) \in H$. We want to prove that $\sum_{i=0}^{n+1} \pi(\Theta_i) \in T(J_H)$. Let $\varphi_H : T(P,Q,\psi) \to T(Q,\psi)/H$ denote the quotient map, and let $\sigma_H := \varphi_H \circ \iota_P$, $T_H := \varphi_H \circ \iota_Q$ and $S_H := \varphi_H \circ \iota_P$. Then $(S_H, T_H, \sigma_H, T(Q,\psi)/H)$ is an injective covariant representation of $(P,Q,\psi)$ and $\varphi_H \circ \iota = \pi_{T_H,S_H}$. We then have that
\[
\sum_{i=0}^{n+1} \pi_{T_H,S_H}(\Theta_i) = \varphi_H \left( \sum_{i=0}^{n+1} \pi(\Theta_i) \right) = 0.
\]

Choose $q_j \in Q^\odot n$, $p_j \in P^\odot n$, $q_j' \in P$, $\pi_j' \in P$ for $j \in \{1,\ldots,m\}$ such that $\Theta_{n+1} = \sum_{j=1}^{m} \theta_{q_j \odot q_j' \odot \pi_j' \odot p_j}$, and $a_h \in Q^\odot n$, $b_h \in P^\odot n$ for $h \in \{1,\ldots,l\}$ such that $\sum_{h=1}^{l} \theta_{a_h \odot b_h}(q_j') = q_j$ for every $j \in \{1,\ldots,m\}$. We then have that $\sum_{h=1}^{l} (\theta_{a_h \odot b_h} \odot 1_Q) \Theta_{n+1} = \Theta_{n+1}$. Let $\Theta = \left( \sum_{h=1}^{l} \theta_{a_h \odot b_h} \right) (\sum_{i=0}^{n} \Theta_i \odot 1_{Q^\odot n-1}) \in F_{P \odot n}(Q^\odot n)$. It follows from Lemma 3.34 that we then have
\[
\pi_{T_H,S_H}(\Theta) = \pi_{T_H,S_H} \left( \left( \sum_{h=1}^{l} \theta_{a_h \odot b_h} \right) \left( \sum_{i=0}^{n} \Theta_i \odot 1_{Q^\odot n-1} \right) \right)
\]
\[
= \pi_{T_H,S_H} \left( \sum_{h=1}^{l} \theta_{a_h \odot b_h} \right) \sum_{i=0}^{n} \pi_{T_H,S_H}(\Theta_i)
\]
\[
= -\pi_{T_H,S_H} \left( \sum_{h=1}^{l} \theta_{a_h \odot b_h} \right) \pi_{T_H,S_H}(\Theta_{n+1}) = -\pi_{T_H,S_H} \left( \sum_{h=1}^{l} (\theta_{a_h \odot b_h} \odot 1_Q) \Theta_{n+1} \right)
\]
\[
= -\pi_{T_H,S_H} \left( \sum_{h=1}^{l} (\theta_{a_h \odot b_h} \odot 1_Q) \Theta_{n+1} \right),
\]
so $\sum_{i=0}^{n} \pi_{T_H,S_H}(\Theta_i) - \pi_{T_H,S_H}(\Theta) = \sum_{i=0}^{n+1} \pi_{T_H,S_H}(\Theta_i) = 0$, and therefore $\sum_{i=0}^{n} \pi(\Theta_i) - \pi(\Theta) \in H$. Thus it follows from the induction assumption that $\sum_{i=0}^{n} \pi(\Theta_i) - \pi(\Theta) \in T(J_H)$. Hence it suffices to prove that $\pi(\Theta) + \pi(\Theta_{n+1}) \in T(J_H)$.

Choose $q_k' \in Q^\odot n$, $p_k' \in P^\odot n$ for $k \in \{1,\ldots,g\}$ such that $\Theta = \sum_{k=1}^{g} \theta_{q_k' \odot p_k'}$. Since $(P^\odot n, Q^\odot n, \psi_a)$ satisfies condition (FS) there exist $a_t \in Q^\odot n$ and $d_t \in P^\odot n$ for $t \in \{1,\ldots,s\}$ such that $\sum_{t=1}^{s} \theta_{c_t \odot d_t}(q_j) = q_j$ for all $j \in \{1,\ldots,m\}$, and $\sum_{t=1}^{s} \theta_{c_t \odot d_t}(q_k') = q_k'$ for all $k \in \{1,\ldots,g\}$. There also exist $e_u \in P^\odot n$ and $f_u \in Q^\odot n$ for $u \in \{1,\ldots,v\}$ such that $\sum_{u=1}^{v} \theta_{e_u \odot f_u}(p_j) = p_j$ for all $j \in \{1,\ldots,m\}$, and $\sum_{u=1}^{v} \theta_{e_u \odot f_u}(p_k') = p_k'$ for all $k \in \{1,\ldots,g\}$.

We have that
\[
\sum_{t=1}^{s} \iota_{Q}(c_t) \iota_{P}(d_t) (\pi(\Theta) + \pi(\Theta_{n+1})) + \sum_{u=1}^{v} \iota_{Q}(f_u) \iota_{P}(e_u) = \pi(\Theta) + \pi(\Theta_{n+1}),
\]
so it suffices to prove that $\iota_{P}(d) (\pi(\Theta) + \pi(\Theta_{n+1})) \iota_{Q}(f) \in T(J_H)$ for every $d \in P^\odot n$ and $f \in Q^\odot n$. Let $r = \psi_n (d \odot \Theta(f)) \in R$. We then have that
\[
\sigma_H(r) = S_H(d) \pi_{T_H,S_H} \Theta_{T_H(f)} = -S_H(d) \pi_{T_H,S_H} \Theta_{n+1} T_H(f).
\]

It follows from Lemma 3.31 that there is a $(\Theta_{n+1})_{d,f} \in F_P(Q)$ such that $\pi_{T,S}((\Theta_{n+1})_{d,f}) = S(d) \pi_{T_{n+1},S_{n+1}} \Theta_{n+1} T(f)$ for every covariant representation $(S,T,A,B)$ of $(P,Q,\psi)$. Hence $\sigma_H(r) \in \pi_{T,S}(F_P(Q))$, so it follows from Lemma 3.24 that $r \in \Delta^{-1}(F_P(Q))$ and
existence and uniqueness of \( \eta \) with respect to \( \iota \). Hence \((\sigma, 626)\) is a \( \mathrm{H} \)-TS-Q grading of \( x \in \mathbb{T} \), which proves that the representation \((S, T, \sigma, B)\) is isomorphic to the Toeplitz representation if \( H \) is right non-degenerate and \( (P, Q, \psi) \) satisfies condition \( \mathrm{FS} \). Thus we have that ker \( \eta(S,T,\sigma,B) = H = \mathcal{T}(J_H) = \mathcal{T}(J) \) as desired.

\[ \eta_H(r) = \pi_{T,S,H}(\Delta(r)) \]  
Thus \( r \in J_H \). Since \( \pi_{T,S} \) is injective (Proposition 3.11), it follows that \( \Delta(r) = -\Theta_{n+1}(d_f) \). Thus we have that
\[
\iota_{P}^d(\pi(\Theta) + \pi(\Theta_{n+1}))\iota_{Q}^n(f) = \iota_{P}^d(\pi(\Theta)\iota_{Q}^n(f) + \iota_{P}^d(\pi(\Theta_{n+1})\iota_{Q}^n(f) ) = \iota_{R}(r) - \pi(\Delta(r)) \in \mathcal{T}(J_H)
\]
as wanted.

Proof of Theorem 3.29. (i) If there exists a ring homomorphism \( \eta : \mathcal{O}_{(P,Q,\psi)}(J) \to B \) such that \( \eta \circ \iota_J^J = T \), \( \eta \circ \iota_J^J = S \) and \( \eta \circ \iota_J^J = \sigma \), and \( x \in J \), then \( \sigma(x) = \eta(\iota_J^J(x)) = \eta(\pi^{J}(\Delta(x))) = \eta^{J}(\Delta(x)) \), which proves that the representation \((S, T, \sigma, B)\) is Cuntz–Pimsner invariant with respect to \( J \).

(ii) If the representation \((S, T, \sigma, B)\) is Cuntz–Pimsner invariant with respect to \( J \), then the existence and uniqueness of \( \eta^{J}(S,T,\sigma,B) \) follows from Proposition 3.18.

(iii) Assume that \( \eta^{J}(S,T,\sigma,B) \) is an isomorphism. Then \( \sigma = \eta^{J}(S,T,\sigma,B) \circ \iota_J^J \) is injective, and
\[
\bigoplus_{n \in \mathbb{Z}} \eta^{J}(S,T,\sigma,B)(\mathcal{O}_{(P,Q,\psi)}^{(n)}(J))
\]
is a \( \mathbb{Z} \)-grading of \( B \) such that
\[
\sigma(R) \leq \eta^{J}(S,T,\sigma,B)(\mathcal{O}_{(P,Q,\psi)}^{(0)}(J)),
\]
\[
T(Q) \leq \eta^{J}(S,T,\sigma,B)(\mathcal{O}_{(P,Q,\psi)}^{(1)}(J)) \quad \text{and}
\]
\[
S(P) \leq \eta^{J}(S,T,\sigma,B)(\mathcal{O}_{(P,Q,\psi)}^{(-1)}(J)).
\]
Hence \((S, T, \sigma, B)\) is injective, surjective and graded. If \( x \in J \), then we have that
\[
\sigma(x) = \eta^{J}(S,T,\sigma,B)(\iota_J^J(x)) = \eta^{J}(S,T,\sigma,B)(\pi^{J}(\Delta(x))) = \eta^{J}(S,T,\sigma,B)(\pi^{J}(\Delta(x))),
\]
and thus \( x \in J(S,T,\sigma,B) \). If \( x \in J(S,T,\sigma,B) \), then it follows from Lemma 3.24 that \( x \in \Delta^{-1}(\mathcal{F}_{P}(Q)) \) and
\[
\eta^{J}(S,T,\sigma,B)(\iota_J^J(x)) = \sigma(x) = \pi_{T,S}(\Delta(x)) = \eta^{J}(S,T,\sigma,B)(\pi^{J}(\Delta(x))),
\]
and since \( \eta^{J}(S,T,\sigma,B) \) is injective, it follows that \( \iota_J^J(x) = \pi^{J}(\Delta(x)) \). It follows that \( \iota_{J}(x) - \pi(\Delta(x)) \in \mathcal{T}(J) \), and we then get from Lemma 3.21 that \( \iota_{J}(x) = \mathcal{P}_{(0,0)}(\iota_{J}(x) - \pi(\Delta(x))) \in \iota_{J}(J) \), and thus that \( x \in J \). Hence \( J = J(S,T,\sigma,B) \).

Assume then that \((S, T, \sigma, B)\) is surjective, injective and graded and that \( J = J(S,T,\sigma,B) \). Then \( \eta^{J}(S,T,\sigma,B) \) is surjective. Let \( \eta^{J}(S,T,\sigma,B) : \mathcal{T}_{(P,Q,\psi)} \to B \) be as in Theorem 1.7. Then \( \eta^{J}(S,T,\sigma,B) = \eta^{J}(S,T,\sigma,B) \circ \rho J \), so \( \eta^{J}(S,T,\sigma,B) \) is injective if ker \( \eta(S,T,\sigma,B) = \ker \rho J = \mathcal{T}(J) \). Let \( H = \ker \eta(S,T,\sigma,B) \). Then \( H \) is a graded two-sided ideal of \( \mathcal{T}_{(P,Q,\psi)} \) and \( H \cap \iota_{R}(R) = \{0\} \), so it follows from Lemma 3.36 that \( \mathcal{T}(J_H) = H \). It easily follows from Lemma 3.24 that \( J = J(S,T,\sigma,B) = J_H \), so we have that ker \( \eta(S,T,\sigma,B) = H = \mathcal{T}(J_H) = \mathcal{T}(J) \) as desired.

4. The Graded Uniqueness Theorem

We will in this section look at some consequences of the classification of the surjective, injective and graded representations of an \( R \)-system \( (P, Q, \psi) \) satisfying condition \( \mathrm{FS} \), obtained in the last section.

We begin by noticing that we get a description of all graded two-sided ideals \( H \) of \( \mathcal{T}_{(P,Q,\psi)} \) satisfying \( \iota_{R}(R) \cap H = \{0\} \), and then that the Fock space representation of \( (P, Q, \psi) \) is isomorphic to the Toeplitz representation if \( R \) is right non-degenerate and \( (P, Q, \psi) \) satisfies condition \( \mathrm{FS} \). Finally we will characterize the faithful \( \psi \)-compatible two-sided ideals \( J \) of \( R \) for which \( \mathcal{O}_{(P,Q,\psi)}(J) \) satisfies the Graded Uniqueness Theorem; cf. [27, Theorem 4.8].
Remark 4.1. Let $R$ be a right non-degenerate ring and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). It easily follows from Lemma 3.24 that if $H$ is a graded two-sided ideal of $T_{(P, Q, \psi)}$ satisfying $\iota_R(R) \cap H = \{0\}$, then $J_H = J_{(\iota^H_P, \iota^H_Q, \iota^H_R, \phi_{(P, Q, \psi)}(J_H))}$. Thus it follows from Proposition 3.28 and Lemma 3.36 that

$$H \mapsto J_H \quad J \mapsto T(J)$$

is an order preserving bijective correspondence between the set of graded two-sided ideal $H$ of $T_{(P, Q, \psi)}$ satisfying $\iota_R(R) \cap H = \{0\}$, and the set of faithful $\psi$-compatible two-sided ideals $J$ of $R$.

We will later (cf. Corollary 7.28) classify all graded two-sided ideals of $T_{(P, Q, \psi)}$.

We will now show that the Fock space representation of an $R$-system $(P, Q, \psi)$ is isomorphic to the Toeplitz representation if $R$ is right non-degenerate and $(P, Q, \psi)$ satisfies condition (FS).

Proposition 4.2. Let $R$ be a right non-degenerate ring and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). Then the Fock space representation $(S_F, T_F, \sigma_F, F_{(P, Q, \psi)})$ of $(P, Q, \psi)$ is isomorphic to the Toeplitz representation $(\iota_P, \iota_Q, \iota_R, T_{(P, Q, \psi)})$.

Proof. To ease the notation let $T = T_F$, $S = S_F$, $\sigma = \sigma_F$, and $B = F_{(P, Q, \psi)}$. It follows from Theorem 1.7 that there exists a unique ring homomorphism $\eta(S, T, \sigma, B) : T_{(P, Q, \psi)} \rightarrow B$ such that $\eta(S, T, \sigma, B) \circ \iota_R = \sigma$, $\eta(S, T, \sigma, B) \circ \iota_Q = T$, and $\eta(S, T, \sigma, B) \circ \iota_P = S$.

For each $m \in \mathbb{N}_0$ let $\iota_m$ denote the inclusion of $Q^{\otimes m}$ into $F(Q)$. It is easy to check that if $x \in T_{(P, Q, \psi)}^{(n)}$, where $n \geq -m$, then $\eta(S, T, \sigma, B)(x) \iota_m(Q^{\otimes m}) \subseteq Q^{\otimes n + m}$. It follows that $(S, T, \sigma, B)$ is graded. It follows from the right non-degeneracy of $R$ that the covariant representation $(S, T, \sigma, B)$ is injective.

Let $q \in Q$ and $p \in P$. Then $\pi_{T, S}(\theta_{q, p}) = T(q)S(p)$ acts as the zero map on $\iota_0(R)$. Thus it follows that if $\Theta \in F_P(Q)$, then $\pi_{T, S}(\Theta)$ acts as the zero map on $\iota_0(R)$. If $r \in R$, then it follows from the right non-degeneracy of $R$ that if $\sigma(r) = \phi_{\infty}(r)$ acts as the zero map on $\iota_0(R)$, then $r = 0$. Thus $J_{(S, T, \sigma, B)} = 0$, and it follows from Theorem 3.29 that $\eta(S, T, \sigma, B)$ is an isomorphism from $T_{(P, Q, \psi)}$ to $F_{(P, Q, \psi)}$.

Remark 4.3. Let $R$ be a ring and $(P, Q, \psi)$ an $R$-system. It is clear that it is a necessary condition for the Fock space representation of $(P, Q, \psi)$ to be isomorphic to the Toeplitz representation that $R$ is right non-degenerate. The following example shows that it is not in general sufficient. This is in contrast to the $C^*$-algebraic case where the Fock representation is always isomorphic to the universal Toeplitz representation; cf. [16, Proposition 6.5].

Example 4.4. Let $R = Q = P = \mathbb{Z}$, let $R$ act on the left and the right on $Q$ and $P$ by multiplication, and let $\psi : P \otimes Q \rightarrow R$ be the zero map. Then $R$ is a non-degenerate ring, and $(P, Q, \psi)$ is an $R$-system. It is easy to check that $S_F$ is the zero map.

Let $B = \oplus_{n \in \mathbb{Z}} \mathbb{Z}$, and for each $n \in \mathbb{Z}$ let $e_n$ be the element of $B$ given by $e_n(m)$ is 1 if and only if $n = m$ and 0 otherwise. We turn $B$ into a ring by using the usual addition and defining a multiplication by

$$e_m e_n = \begin{cases} e_{m+n} & \text{if } nm \geq 0, \\ 0 & \text{if } nm < 0. \end{cases}$$
We define maps $\sigma : R \to B$ by $\sigma(r) = re_0$, $S : P \to B$ by $S(p) = pe_{-1}$ and $T : Q \to B$ by $T(q) = qe_1$. It is easy to check that $(S, T, \sigma, B)$ is a covariant representation of $(P, Q, \psi)$. Since $S \neq 0$, it follows that $\iota P \neq 0$ (in fact, it is not difficult to show that $(S, T, \sigma, B)$ is isomorphic to the Toeplitz representation of $(P, Q, \psi)$). Thus, the Fock space representation cannot be isomorphic to the Toeplitz representation in this example.

We now define what it means for a relative Cuntz–Pimsner ring of an $R$-system to satisfy the Graded Uniqueness Theorem, and then characterize when it does that.

**Definition 4.5** (cf. [27, Theorem 4.8]). Let $R$ be a ring, $(P, Q, \psi)$ an $R$-system satisfying condition (FS) and let $J$ be a faithful $\psi$-compatible two-sided ideal of $R$. We say that the relative Cuntz–Pimsner ring $O_{(P, Q, \psi)}(J)$ satisfies the Graded Uniqueness Theorem if and only if the following holds:

1. If $B$ is a $\mathbb{Z}$-graded ring and $\eta : O_{(P, Q, \psi)}(J) \to B$ is a graded ring homomorphism such that $\eta \circ \iota^J_B$ is injective, then $\eta$ is injective.

**Theorem 4.7.** Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). Let $C^\text{inj, grad}_{(P, Q, \psi)}$ be the subcategory of $C_{(P, Q, \psi)}$ consisting of all surjective, injective and graded covariant representation of $(P, Q, \psi)$. Let $J$ be a faithful $\psi$-compatible two-sided ideal of $R$. Then the following three statements are equivalent.

(i) The Cuntz–Pimsner ring $O_{(P, Q, \psi)}(J)$ of $(P, Q, \psi)$ relative to $J$ satisfies the Graded Uniqueness Theorem.

(ii) The Cuntz–Pimsner representation $(\iota^J_P, \iota^J_Q, \iota^J_R, O_{(P, Q, \psi)}(J))$ of $(P, Q, \psi)$ relative to $J$ is minimal in $C^\text{inj, grad}_{(P, Q, \psi)}$ in the sense that if $(S, T, \sigma, B)$ is a surjective, injective and graded representation of $(P, Q, \psi)$ and $\eta : O_{(P, Q, \psi)}(J) \to B$ is a homomorphism such that $\eta \circ \iota^J_Q = T$, $\eta \circ \iota^J_R = S$ and $\eta \circ \iota^J_P = \sigma$, then $\eta$ is an isomorphism.

(iii) The ideal $J$ is maximal.

**Proof.** If $B$ is a $\mathbb{Z}$-graded ring and $\eta : O_{(P, Q, \psi)}(J) \to B$ is a graded ring homomorphism such that $\eta \circ \iota^J_B$ is injective, and we let $T = \eta \circ \iota^J_Q$, $S = \eta \circ \iota^J_P$ and $\sigma = \eta \circ \iota^J_R$, then $(S, T, \sigma, B)$ is a surjective, injective and graded representation of $(P, Q, \psi)$. The equivalence of (i) and (ii) easily follows from this.

The equivalence of (ii) and (iii) follows from Remark 3.30. □

**Definition 4.8.** Let $R$ be a ring, and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). A faithful $\psi$-compatible two-sided ideal $J$ of $R$ is called maximal if $J = J'$ for any faithful $\psi$-compatible two-sided ideal $J'$ of $R$ satisfying $J \subseteq J'$.

**Remark 4.9.** Let $R$ be a ring, and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). It is clear that if $J$ is a uniquely maximal faithful $\psi$-compatible two-sided ideal of $R$, then it is the only maximal faithful $\psi$-compatible two-sided ideal of $R$. The standard argument using Zorn’s lemma shows that every faithful $\psi$-compatible two-sided ideal of $R$ is contained in a
maximal faithful ψ-compatible two-sided ideal of R. Thus if there is only one maximal faithful ψ-compatible two-sided ideal of R, then this ideal is automatically uniquely maximal.

Remark 4.10. Let R be a ring and let (P, Q, ψ) be an R-system satisfying condition (FS). It follows from Remark 3.30 that if J is a faithful ψ-compatible two-sided ideal of R, then \((i_J^P, i_J^Q, i_J^R, O_{(P, Q, ψ)}(J))\) is a final object of \(\mathcal{C}_{\text{inj}, \text{grad}}\) if and only if J is uniquely maximal. If such a J exists, then it would be natural (cf. [17, Proposition 7.14]) to define the Cuntz–Pimsner ring of the R-system \((P, Q, ψ)\) to be \(O_{(P, Q, ψ)}(J)\) (and we will do that in Definition 5.1), however, as the following example shows, such a J does not in general exist (in contrast to the \(C^*\)-algebraic case where one always can use the analogue of the ideal \((\ker Δ)^+ \cap Δ^{-1}(\mathcal{F}_P(Q))\) cf. [16]).

Example 4.11. Let \(R = \mathbb{Z} \times \mathbb{R} \times \mathbb{Z}\) be a ring with multiplication defined by

\[
(x, y, z) \cdot (x', y', z') := (xx', xy' + yx', xz' + zx').
\]

Notice that R is a unital ring with unit \((1, 0, 0)\).

Let \(\delta : R \to R\) be a map defined as \(\delta(x, y, z) = (x, y - z, 0)\). We claim that \(\delta\) is a ring homomorphism. Indeed, let \((x, y, z), (x', y', z') \in R\). Then we have

\[
\delta(x, y, z)\delta(x', y', z') = (x, y - z, 0)(x', y' - z', 0) = (xx', x(y' - z') + x'y - z, 0)
\]

\[
= (xx', xy' + yx' - xz' + zx', 0) = \delta(xx', xy' + yx', xz' + zx')
\]

\[
= \delta((x, y, z)(x', y', z')).
\]

Let \(P = Q = \{ (x, y, 0) \mid x \in \mathbb{Z}, y \in \mathbb{R} \} \subseteq R\), and endow \(P = Q\) with the following \(R\)-bimodule structure: Given \(p \in P, q \in Q\) and \(r \in R\) let

\[
p \cdot r = p\delta(r), \quad r \cdot p = \delta(r)p,
\]

\[
q \cdot r = q\delta(r), \quad r \cdot q = \delta(r)q.
\]

Finally let \(ψ : P \otimes_R Q \to R\) be defined by \(ψ(p \otimes q) = pq\). We will now check that the \(R\)-system \((P, Q, ψ)\) satisfies property (FS). Indeed, if \(q \in Q\) then

\[
(1, 0, 0) \cdot ψ((1, 0, 0) \otimes q) = (1, 0, 0) \cdot q = q,
\]

and if \(p \in P\) then

\[
ψ(p \otimes (1, 0, 0)) \cdot (1, 0, 0) = p \cdot (1, 0, 0) = p.
\]

It easy to check that

\[
Δ^{-1}(\mathcal{F}_P(Q)) = R \quad \text{and} \quad \ker Δ = \{ (0, z, z) \mid z \in \mathbb{Z} \}.
\]

Now we define

\[
J_1 := \{ (0, y, 0) \mid y \in \mathbb{R} \} \quad \text{and} \quad J_2 := \{ (0, 0, z) \mid z \in \mathbb{Z} \}.
\]

We will prove that both \(J_1\) and \(J_2\) are maximal faithful ψ-compatible two-sided ideals of R. Let J be a faithful ψ-compatible two-sided ideal of R such that \(J_1 \subseteq J\) and assume that there exists \(0 \neq (x, y, z) \in J \setminus J_1\). Then \((x, 0, z) \in J\), with either x or z being non-zero. If \(x = 0\), then \(z \neq 0\) and then \((0, z, z) \in J \cap \ker Δ\), but if \(x \neq 0\) then \((0, 0, 1)(x, 0, z) = (0, 0, x) \in J\) and hence \(0 \neq (0, x, x) \in J \cap \ker Δ\), a contradiction. Thus \(J_1\) is maximal. We can do the same to prove that \(J_2\) is also maximal.

Note that \(J_1\) and \(J_2\) are clearly non-isomorphic, however we cannot deduce from this that their associated relative Cuntz–Pimsner rings are non-isomorphic.
5. Cuntz–Pimsner rings

Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS), and let $J$ be a uniquely maximal faithful $\psi$-compatible two-sided ideal of $R$. In view of Remark 4.10 it is natural to define $\mathcal{O}_{(P, Q, \psi)}(J)$ to be the Cuntz–Pimsner ring of $(P, Q, \psi)$. We will do that now.

Definition 5.1 (Cf. [16, Definition 3.5; 24, Definition 1.1]). Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). If there exists a uniquely maximal faithful $\psi$-compatible two-sided ideal $J$ of $R$, then we define the Cuntz–Pimsner ring of $(P, Q, \psi)$ to be the ring

$$\mathcal{O}_{(P, Q, \psi)} := \mathcal{O}_{(P, Q, \psi)}(J)$$

and we let

$$(\iota_{P}^\text{CP}, \iota_{Q}^\text{CP}, \iota_{R}^\text{CP}, \mathcal{O}_{(P, Q, \psi)})$$

denote the covariant representation $(\iota_{P}^{\text{CP}}, \iota_{Q}^{\text{CP}}, \iota_{R}^{\text{CP}}, \mathcal{O}_{(P, Q, \psi)}(J))$ and call it the Cuntz–Pimsner representation of $(P, Q, \psi)$. We let $p_{r} := \iota_{R}^{\text{CP}}(r)$ for $r \in R$, $y_{p} := \iota_{P}^{\text{CP}}(p)$ for $p \in P$ and $x_{q} := \iota_{Q}^{\text{CP}}(q)$ for $q \in Q$.

It follows from Remark 4.10 that $\mathcal{O}_{(P, Q, \psi)}$, if it exists, is the (up to isomorphism) unique final object of $\text{c}^{\text{inj, grad}}_{(P, Q, \psi)}$. It can also be described as the smallest quotient of $\mathcal{T}_{(P, Q, \psi)}$ which preserves the $\mathbb{Z}$-grading of $\mathcal{T}_{(P, Q, \psi)}$ and which leaves the embedded copy of $R$ intact.

It follows from Example 4.11 that it is not always the case that there exists a uniquely maximal faithful $\psi$-compatible two-sided ideal of $R$. We will now describe a condition which will guarantee the existence of such an ideal. This condition is satisfied by many interesting examples; cf. Examples 5.5–5.8.

If $J$ is a two-sided ideal of a ring $R$, then we let $J^\perp$ denote the two-sided ideal $\{x \in R \mid \forall y \in J \mid xy = yx = 0\}$. The following lemma is then obvious.

Lemma 5.2. Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system which satisfies condition (FS). If $(\Delta^{-1}(\mathcal{F}_{P}(Q)) \cap (\ker \Delta)^\perp) \cap \ker \Delta = \{0\}$, then $J := \Delta^{-1}(\mathcal{F}_{P}(Q)) \cap (\ker \Delta)^\perp$ is a uniquely maximal faithful $\psi$-compatible two-sided ideal of $R$. Thus the Cuntz–Pimsner ring of $(P, Q, \psi)$ is defined in this case.

A ring $R$ is said to be semiprime if whenever $I$ is a two-sided ideal of $R$ such that $I^{2} = \{0\}$, then $I = \{0\}$. A two-sided ideal $I$ is said to be semiprime if whenever there exists a two-sided ideal $J$ with $J^{2} \subseteq I$, then $J \subseteq I$. Equivalently $I$ is a semiprime ideal if and only if $R/I$ is a semiprime ring. Observe that in particular every $C^*$-algebra $A$ is semiprime and every closed two-sided ideal $I$ of $A$ is also semiprime (since $A/I$ is a $C^*$-algebra itself).

Lemma 5.3. Let $R$ be a ring which is semiprime, and let $(P, Q, \psi)$ be an $R$-system which satisfies condition (FS). Then $(\ker \Delta)^\perp \cap \ker \Delta = \{0\}$.

Proof. It is clear that $(\ker \Delta)^\perp \cap \ker \Delta$ is a two-sided ideal of $R$ satisfying $((\ker \Delta)^\perp \cap \ker \Delta)^{2} = \{0\}$. Thus $(\ker \Delta)^\perp \cap \ker \Delta = \{0\}$.

Thus when $R$ is semiprime, then $\Delta^{-1}(\mathcal{F}_{P}(Q)) \cap (\ker \Delta)^\perp$ is a uniquely maximal faithful $\psi$-compatible two-sided ideal of $R$ for every $R$-system $(P, Q, \psi)$ and the Cuntz–Pimsner ring $\mathcal{O}_{(P, Q, \psi)}$ is defined.
Before we look at some examples where the Cuntz–Pimsner ring is defined, we notice that it directly follows from Theorem 4.7 that if the Cuntz–Pimsner ring of an $R$-system is defined, then it satisfies the Graded Uniqueness Theorem.

**Corollary 5.4 (The Graded Uniqueness Theorem, cf. [16, Theorem 6.4]).** Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system which satisfies condition (FS), and assume that there exists a uniquely maximal faithful $\psi$-compatible two-sided ideal of $R$. If $A$ is a $\mathbb{Z}$-graded ring and $\eta : \mathcal{O}(P, Q, \psi) \to A$ is a graded ring homomorphism with $\eta(p_r) \neq 0$ for every $r \in R \setminus \{0\}$, then $\eta$ is injective.

**Example 5.5.** Let us return to Example 1.9. We saw that if $R$ is a ring which has local units, $\varphi \in \text{Aut}(R)$, $P = R_\varphi$, $Q = R_{\varphi^{-1}}$ and

$$\psi : P \otimes_R Q \longrightarrow R, \quad p \otimes q \longrightarrow p\varphi(q),$$

then $(P, Q, \psi)$ is an $R$-system.

Since we assume that $R$ has local units, it follows that if $q_1, q_2, \ldots, q_n \in Q$ and $p_1, p_2, \ldots, p_m \in P$ then there exists an idempotent $e \in R$ such that $e q_i = q_i$ for all $i \in \{1, 2, \ldots, n\}$ and $p_j e = p_j$ for all $j \in \{1, 2, \ldots, m\}$ (we are here viewing $q_i$ and $p_j$ as elements of $R$ and are using the multiplication of $R$). We then have that $\theta_{e, \varphi^{-1}}(q_i) = e \varphi^{-1}(\varphi(e)\varphi(q_i)) = eeq_i = q_i$ for all $i \in \{1, 2, \ldots, n\}$ and $\theta_{e, \varphi^{-1}}(p_j) = p_j \varphi(\varphi^{-1}(e)) = p_j e e p_j = p_j$ for all $j \in \{1, 2, \ldots, m\}$. Thus $(P, Q, \psi)$ satisfies condition (FS). Observe that we may assume that $\Delta^{-1}(\mathcal{F}_P(Q)) = R$ because $\Delta(r) = \theta_{u, \varphi}(r)$ for every $r \in R$ and $u \in R$ with $ur = ru = r$. Notice also that $\Delta$ is injective, so $R$ is a uniquely maximal, faithful, $\psi$-compatible, two-sided ideal. Thus the Cuntz–Pimsner ring of the $R$-system $(P, Q, \psi)$ exists and is equal to $\mathcal{O}(P, Q, \psi)(R)$.

We saw in Example 1.9 that if $(S, T, \sigma, B)$ is a covariant representation of $(P, Q, \psi)$ and we, for every $r \in R$ and $n \in \mathbb{N}_0$, let $[r, n] = S^t(r)$, $[r, -n] = T^n(r)$ and $[r, 0] = \sigma(r)$, then $[r_1 + r_2, k] + [r_2, k] = [r_1 + r_2, k]$ for $r_1, r_2 \in R$ and $k \in \mathbb{Z}$, and $[r_1, k_1][r_2, k_2] = [r_1 \varphi^{k_1}(r_2), k_1 + k_2]$ for $r_1, r_2 \in R$ and $k_1, k_2 \in \mathbb{Z}$ if $k_1$ and $k_2$ are both non-negative, or both are non-negative, or if $k_1$ is non-negative and $k_2$ is non-positive. If in addition $(S, T, \sigma, B)$ is Cuntz–Pimsner invariant relative to $R$, then we have for $r_1, r_2, u_1, u_2 \in R$ where $r_2 u_1 = r_2$ and $u_2 r_1 = r_1$, and $n_1, n_2 \in \mathbb{N}_0$, that

$$[r_1, -n_1][r_2, n_1] = T^{n_1}(r_1)S^{n_1}(r_2) = \pi S^{n_1}, T^{n_1}(\theta_{r_1, r_2}),$$

$$[r_1, -n_1][r_2, n_1 + n_2] = [r_1, -n_1][r_2, n_1][\varphi^{-n_1}(u_1), n_2]$$

$$= [r_1 \varphi^{-n_1}(r_2), 0][\varphi^{-n_1}(u_1), n_2]$$

$$= [r_1 \varphi^{-n_1}(r_2), \varphi^{-n_1}(u_1), n_2]$$

$$= [r_1 \varphi^{-n_1}(r_2) \varphi^{-n_1}(u_1), n_2] = [r_1 \varphi^{-n_1}(r_2), n_2],$$

$$[r_1, -n_1 - n_2][r_2, n_1] = [u_2, -n_2][\varphi^{n_2}(r_1), -n_1][r_2, n_1]$$

$$= [u_2, -n_2][\varphi^{n_2}(r_1), n_1, n_2]$$

$$= [u_2, -n_2][\varphi^{n_2}(r_1), \varphi^{n_1}(r_2), 0]$$

$$= [u_2, -n_2][\varphi^{n_2}(r_1) \varphi^{n_1}(r_2), 0]$$

$$= [u_2, -n_2][\varphi^{n_2}(r_1) \varphi^{n_1}(r_2), -n_2] = [r_1 \varphi^{-n_1}(r_2), n_2].$$

Thus $[r_1, k_1][r_2, k_2] = [r_1 \varphi^{k_1}(r_2), k_1 + k_2]$ for $r_1, r_2 \in R$ and $k_1, k_2 \in \mathbb{Z}$.

If on the other hand we have a ring $B$ which contains a set of elements $\{[r, k] \mid r \in R, k \in \mathbb{Z}\}$ satisfying $[r_1, k] + [r_2, k] = [r_1 + r_2, k]$ and $[r_1, k_1][r_2, k_2] = [r_1 \varphi^{k_1}(r_2), k_1 + k_2]$, and we define $\sigma : R \to B$ by $\sigma(r) = [r, 0]$, $S : P \to B$ by $S(p) = [p, 1]$ and $T : Q \to B$ by $T(q) = [q, -1]$, then $(S, T, \sigma, B)$ is a covariant representation of $(P, Q, \psi)$ which is Cuntz–Pimsner invariant relative to $R$. 
Thus $\mathcal{O}_{(P,Q,\psi)}$ is the universal ring generated by elements \{\([r,k] \mid r \in R, \ k \in \mathbb{Z}\}\} satisfying \([r_1,k] + [r_2,k] = [r_1 + r_2,k]\) and \([r_1,k_1][r_2,k_2] = [r_1\varphi^{k_1}(r_2),k_1 + k_2]\); that is, $\mathcal{O}_{(P,Q,\psi)}$ is isomorphic to the crossed product $R \times_\varphi \mathbb{Z}$.

We will return to this example in Example 7.30.

**Example 5.6.** Let $R$ be a ring and let $\alpha : R \to R$ be a ring homomorphism. Let $P := \text{span}\{r_1,\alpha(r_2) \mid r_1, r_2 \in R\}$ be the $R$-module with left action defined by $r \cdot p = rp$ and right action defined by $p \cdot r = p\alpha(r)$ for $r \in R$ and $p \in P$, and let $Q := \text{span}\{\alpha(r_1)r_2 \mid r_1, r_2 \in R\}$ be the $R$-module with left action defined by $r \cdot q = \alpha(r)q$ and right action defined by $q \cdot r = qr$ for $r \in R$ and $q \in Q$. Finally let $\psi : P \otimes Q \to R$ be the bimodule homomorphism defined by $\psi(p \otimes q) = pq$. Then $(P, Q, \psi)$ is an $R$-system.

If $(S, T, \sigma, B)$ is a covariant representation of $(P,Q,\psi)$, then $S(p)\sigma(r) = S(p\alpha(r))$, $(\psi)S(p) = S(rp)$, $T(q)\sigma(r) = T(q\alpha(r))$, $\sigma(r)T(q) = T(\alpha(r)q)$ and $S(p)T(q) = \sigma(q)p$ for $p \in P$, $q \in Q$ and $r \in R$ where we view $p$ and $q$ as elements of $R$ and use the multiplication of $R$.

It is not difficult to show that if $R$ has local units, then $(P,Q,\psi)$ satisfies condition (FS), $\Delta^{-1}(FP(Q)) = R$ and that ker $\Delta = \{0\}$. Thus the Cuntz–Pimsner ring of $(P,Q,\psi)$ is defined in this case and is equal to $\mathcal{O}_{(P, Q, \psi)}(R)$. If in addition $\alpha$ is injective and $\alpha(r_1)r_2 \alpha(r_3) \in \alpha(R)$ for all $r_1, r_2, r_3 \in R$, then a covariant representation $(S, T, \sigma, B)$ of $(P, Q, \psi)$ is Cuntz–Pimsner invariant relative to $R$ if and only if $T(q)S(p) = \sigma(\alpha^{-1}(qp))$ for all $p \in P$ and $q \in Q$.

It is not difficult to see that if $\alpha$ is an automorphism and $R$ has local units, then $\mathcal{O}_{(P,Q,\psi)}$ is isomorphic to the crossed product $R \times_\alpha \mathbb{Z}$; cf. Example 5.5.

**Example 5.7.** Given a unital ring $R$ and a ring isomorphism $\alpha : R \to eRe$, where $e$ is an idempotent of $R$, Ara, González-Barroso, Goodearl and Pardo have, in [5], defined the fractional skew monoid ring of the system $(R, \alpha)$ to be the universal unital ring $R[t_+,t_-;\alpha]$ generated by elements $t_+, t_-$ and $\{\phi(r) \mid r \in R\}$ satisfying that $\phi : R \to R[t_+,t_-;\alpha]$ is a unital ring homomorphism and that the relations

$$t_-t_+ = 1, \quad t_+t_- = \phi(e), \quad rt_- = t_-\alpha(r), \quad t_+r = \alpha(r)t_+$$

hold for all $r \in R$. This construction is an exact algebraic analogue of the construction of the crossed product of a $C^*$-algebra by an endomorphism introduced by Paschke [23]. In fact, if $A$ is a $C^*$-algebra and the corner isomorphism $\alpha$ is a $*$-homomorphism, then Paschke’s $C^*$-crossed product, which he denotes by $A \rtimes_\alpha \mathbb{N}$, is just the completion of $A[t_+,t_-;\alpha]$ in a suitable norm. The Cuntz–Krieger rings, crossed products by automorphisms and Leavitt path algebras of finite graphs without sinks are examples of fractional skew monoid rings among many others (see [5]). As an important advance in the study of this class of rings, in [5, Theorem 5.3] conditions for $R[t_+,t_-;\alpha]$ being a simple and purely infinite ring are given, and in [4] the $K_1$ of fractional skew monoid rings is computed.

We will now show that the fractional skew monoid ring $R[t_+,t_-;\alpha]$ is isomorphic, as a $\mathbb{Z}$-graded ring, to $\mathcal{O}_{(P,Q,\psi)}$, where $(P,Q,\psi)$ is the $R$-system considered in Example 5.6. First we notice that if $r_1, r_2, r_3 \in R$, then $\alpha(r_1)r_2 \alpha(r_3) \in eReReRe \subseteq eRe = \alpha(R)$. Define $S : P \to R[t_+,t_-;\alpha]$ and $T : Q \to R[t_+,t_-;\alpha]$ by $S(p) = \phi(p)t_+$ and $T(q) = t_-\phi(q)$. It is then easy to check that $(S, T, \phi, [t_+,t_-;\alpha])$ is a surjective covariant representation of $(P,Q,\psi)$ which is Cuntz–Pimsner invariant relative to $R$; cf. Example 5.6. Thus it follows from Theorem 3.18 that there exists a ring homomorphism $\eta : \mathcal{O}_{(P,Q,\psi)} \to R[t_+,t_-;\alpha]$ such that $\eta(p_1t_+ + p_2t_-) = \phi(p_1)t_+$ and $\eta(q) = t_-\phi(q)$ for $r \in R$, $p \in P$ and $q \in Q$. It follows from [5, Proposition 1.6 and Corollary 1.11] that $\eta$ is graded and that $\eta(p_r) \neq 0$ for $r \neq 0$, so $\eta$ is injective and thus an isomorphism according to Corollary 5.4.
Example 5.8. Let us return to Example 1.10. Given \( q = \left( \sum_{e \in E^1} \lambda_e 1_e \right) \in Q \) we let
\[
\text{Supp}(q) := \{ e \in E^1 \mid \lambda_e \neq 0 \}.
\]
Notice that \(|\text{Supp}(q)| < \infty\). Given \( q_1, \ldots, q_n \in Q \) we have that the homomorphism
\[
\Theta = \sum_{e \in \text{Supp}(q_i) \cup \cdots \cup \text{Supp}(q_n)} \theta_{1_e} : \mathcal{F}_P(Q)
\]
satisfies \( \Theta(q_i) = q_i \) for every \( i \in \{1, 2, \ldots, n\} \). Similarly, we have that there exists for \( p_1, p_2, \ldots, p_n \in P \) a homomorphism \( \Delta \in \mathcal{F}_Q(P) \) such that \( \Delta(p_i) = p_i \) for every \( i \in \{1, 2, \ldots, n\} \). Thus the \( R \)-system \((P, Q, \psi)\) satisfies the condition (FS).

Now it is easy to see that
\[
\Delta^{-1}(\mathcal{F}_P(Q)) = \text{span}_F \{ 1_e \mid |s^{-1}(v)| < \infty \}, \quad \text{ker} \Delta = \text{span}_F \{ 1_e \mid |s^{-1}(v)| = 0 \}.
\]
It follows that \((\text{ker} \Delta)^\perp = \text{span}_F \{ 1_e \mid |s^{-1}(v)| > 0 \}\), and thus that \((\text{ker} \Delta)^\perp \cap \text{ker} \Delta = \{ 0 \}\). Hence the Cuntz–Pimsner ring of \((P, Q, \psi)\) is defined in this case and is equal to \( \mathcal{O}_{(P, Q, \psi)}(\Delta^{-1}(\mathcal{F}_P(Q)) \cap (\text{ker} \Delta)^\perp) \).

We saw in Example 1.10 that if \((S, T, \sigma, B)\) is a covariant representation of \((P, Q, \psi)\) and we let \( p_v := \sigma(1_e) \) for \( v \in E^0 \), and \( x_e = T(1_e) \) and \( y_e = S(1_e) \) for \( e \in E^1 \), then \( R(S, T, \sigma) \) becomes an \( F \)-algebra when we equip it with an \( F \)-multiplication defined by \( \lambda \sigma(r) = \sigma(\lambda r) \), \( \lambda S(p) = S(\lambda p) \) and \( \lambda T(q) = T(\lambda q) \) for \( \lambda \in F \). If in addition \((S, T, \sigma, B)\) is Cuntz–Pimsner invariant relative to \( \Delta^{-1}(\mathcal{F}_P(Q)) \cap (\text{ker} \Delta)^\perp \), then we have for \( v \in E^0 \) with \( 0 < |s^{-1}(v)| < \infty \) that
\[
p_v = \sigma(1_e) = \pi_{T, S} (\Delta(1_e)) = \pi_{T, S} \left( \sum_{e \in s^{-1}(v)} \theta_{1_e} \right) = \sum_{e \in s^{-1}(v)} T(1_e) S(1_e) = \sum_{e \in s^{-1}(v)} x_{e} y_{e}.
\]
On the other hand, let \( B \) be an \( F \)-algebra which contains a family \( \{p_v\}_{v \in E^0} \) of pairwise orthogonal idempotents and families \( \{x_e\}_{e \in E^1} \) and \( \{y_e\}_{e \in E^1} \) satisfying \( p_v x_e = x_e \), \( x_e p_v = p_v \), \( y_e p_r = y_e \), \( p_r y_r = y_r \sigma(p_r) \), \( y_e x_f = \delta_{e,f} \sigma(p_r) \), for all \( e, f \in E^1 \). For \( r = \sum_{v \in E^0} s_v 1_e \in R \) let \( \sigma(r) := \sum_{e \in E^1} \lambda_e 1_e \), and \( B \) is an \( F \)-algebra generated by a set \( \{p_v \mid v \in E^0\} \) of pairwise orthogonal idempotents, together with a set \( \{x_e, y_e \mid e \in E^1\} \) of elements satisfying
\[
\text{(i)} \quad p_s x_e = x_e = x_e p_r \quad \text{for } e \in E^1;
\]
\[
\text{(ii)} \quad p_r y_e = y_e = y_e p_s \quad \text{for } e \in E^1;
\]
\[
\text{(iii)} \quad y_e x_f = \delta_{e,f} \sigma(p_r) \quad \text{for } e, f \in E^1;
\]
\[
\text{(iv)} \quad p_v = \sum_{e \in s^{-1}(v)} x_{e} y_{e} \quad \text{for } v \in E^0 \text{ with } 0 < |s^{-1}(v)| < \infty.
\]
That is, \( \mathcal{O}_{(P, Q, \psi)} \) is isomorphic to the Leavitt path algebra \( L_E(E) \) associated with \( E \); cf. [1–3, 6, 27]. Thus we recover from Corollary 5.4 the Graded uniqueness theorem [27, Theorem 4.8] for Leavitt path algebras.

We will return to this example in Example 7.31.

6. The Algebraic Gauge-Invariant Theorem

We saw in Example 5.5 that our Graded Uniqueness Theorem (Corollary 5.4) is a generalization of the Graded Uniqueness Theorem for Leavitt path algebras (see [27, Theorem 4.8]). We will
now generalize the Algebraic Gauge-Invariant Uniqueness Theorem for row-finite graphs (see [13, Theorem 1.8]) to Cuntz–Pimsner rings and thereby to all directed graphs.

**Proposition 6.1** (cf. [12, Proposition 1.3; 24, Remark 1.2(2)]). Let $R$ be an (associative) $F$-algebra, where $F$ is a field, let $(P,Q,\psi)$ be an $R$-system satisfying condition (FS) and let $J$ be a $\psi$-compatible two-sided ideal of $R$. Then there exists for every $t \in F^*$ ($F^*$ denotes the multiplicative group of $F$) a unique automorphism $\tau^t_J$ on $\mathcal{O}(P,Q,\psi)(J)$ satisfying $\tau^t_J((v^r_J(r)) = \tau^t_J(v^r_J(p)) = tv^r_J(p)$ and $\tau^t_J((v^q_J(q))) = t^{-1}v^q_J(q)$ for $r \in R$, $p \in P$ and $q \in Q$.

The action

$$\tau^t_J : F^* \to \text{Aut}_F(\mathcal{O}(P,Q,\psi)(J)), \quad t \mapsto \tau^t_J$$

is called the gauge action of $F$ on $\mathcal{O}(P,Q,\psi)(J)$.

**Proof.** Since $\mathcal{O}(P,Q,\psi)(J)$ is generated by $\{v^r_J(r) \mid r \in R\} \cup \{v^r_J(p) \mid p \in P\} \cup \{v^q_J(q) \mid q \in Q\}$, it follows that a ring homomorphism defined on $\mathcal{O}(P,Q,\psi)(J)$ is uniquely determined by its values on $\{v^r_J(r) \mid r \in R\} \cup \{v^r_J(p) \mid p \in P\} \cup \{v^q_J(q) \mid q \in Q\}$. Let $t \in F^*$. For $r \in R$, $p \in P$ and $q \in Q$ let $\sigma(r) = v^r_J(r)$, $S(p) = tv^r_J(p)$ and $T(q) = t^{-1}v^q_J(q)$. Then $(S,T,\sigma,\mathcal{O}(P,Q,\psi)(J))$ is a covariant representation of $(P,Q,\psi)$, which is Cuntz–Pimsner invariant relative to $J$. Thus there exists a homomorphism $\tau^t_J : \mathcal{O}(P,Q,\psi)(J) \to \mathcal{O}(P,Q,\psi)(J)$ such that $\tau^t_J((v^r_J(r))) = \tau^t_J(v^r_J(r))$, $\tau^t_J((v^r_J(p))) = tv^r_J(p)$ and $\tau^t_J((v^q_J(q))) = t^{-1}v^q_J(q)$ for $r \in R$, $p \in P$ and $q \in Q$. If $t_1,t_2 \in F^*$ and $r \in R$, $p \in P$ and $q \in Q$, then $\tau^t_{t_1} \circ \tau^t_{t_2}((v^r_J(r))) = \tau^t_{t_1t_2}(v^r_J(r))$, $\tau^t_{t_1} \circ \tau^t_{t_2}(v^r_J(p))) = tv^r_J(p)$ and $\tau^t_{t_1} \circ \tau^t_{t_2}(v^q_J(q)) = t^{-1}v^q_J(q)$, so $\tau^t_J \circ \tau^t_J = \tau^t_{t_1t_2}$. We have, in particular, that $\tau^t_J \circ \tau^t_{t^{-1}} = \text{Id}_{\mathcal{O}(P,Q,\psi)(J)}$, so $\tau^t_J$ is an automorphism. 

**Theorem 6.2.** Let $F$ be an infinite field, $R$ an (associative) $F$-algebra, and let $(P,Q,\psi)$ be an $R$-system satisfying condition (FS). Assume that $J$ is a maximal faithful $\psi$-compatible two-sided ideal of $R$, and let $A$ be an $F$-algebra. Suppose that

$$\phi : \mathcal{O}(P,Q,\psi)(J) \to A$$

is an $F$-algebra homomorphism such that $\phi((v^r_J(r))) \neq 0$ for every $r \in R \setminus \{0\}$. If there exists a group action $\sigma : F^* \to \text{Aut}_F(A)$ such that $\phi \circ \tau^t_J = \sigma_t \circ \phi$ for every $t \in F^*$, then $\phi$ is injective.

**Proof.** Let $B := \phi(\mathcal{O}(P,Q,\psi)(J))$. By Theorem 4.7 it is enough to check that $\oplus_{n \in \mathbb{Z}} \phi(\rho_J(T_{(P,Q,\psi)}^{(n)}))$ is a grading of $B$. We will do that by showing that $\phi$ is a graded ideal. Assume that $\phi(z_{n_1} + \ldots + z_{n_r}) = 0$, $n_1, \ldots, n_r \in \mathbb{Z}$, $n_i \neq n_j$ for $i \neq j$ and $z_{n_i} \in \rho_J(T_{(P,Q,\psi)}^{(n_i)})$ for every $i = 1,\ldots,r$. We then have for $t \in F^*$ that

$$0 = \sigma_t(\phi(z_{n_1} + \ldots + z_{n_r})) = \phi(\tau^t_J(z_{n_1} + \ldots + z_{n_r})) = \phi(t^{n_1}z_{n_1} + \ldots + t^{n_r}z_{n_r}).$$

On the other hand we have that $0 = t^{n_r} \phi(z_{n_1} + \ldots + z_{n_r}) = \phi(t^{n_r}z_{n_1} + \ldots + t^{n_r}z_{n_r})$. It follows that

$$0 = \phi((t^{n_r} - t^{n_i})z_{n_1} + \ldots + (t^{n_r} - t^{n_{r-1}})z_{n_{r-1}}),$$

and since $F$ is an infinite field there is a $t \in F^*$ such that $t^{n_r} - t^{n_i} \neq 0$ for every $i = 1,\ldots,r - 1$. Repeating this process $r - 1$ times we get that $\phi(z_{n_i}) = 0$ as desired. Repeating the same argument we get that $\phi(z_{n_i}) = 0$ for every $i = 1,\ldots,r$. This shows that $\ker \phi$ is a graded ideal and thus that $\oplus_{n \in \mathbb{Z}} \phi(\rho_J(T_{(P,Q,\psi)}^{(n)}))$ is a grading of $B$. 

$\square$
If $F$ is a field, $R$ is an $F$-algebra, $(P,Q,\psi)$ is an $R$-system satisfying condition (FS) and $J$ is a uniquely maximal faithful $\psi$-compatible two-sided ideal of $R$, then we denote by $\tau^J$ the gauge action $\tau^J_{\wp} = \wp(J)$. We then get as a corollary to the previous theorem the following Gauge-Invariant Uniqueness Theorem for Cuntz–Pimsner rings.

**Corollary 6.3** (The Gauge-Invariant Uniqueness Theorem for Cuntz–Pimsner Rings, cf. [12, Theorem 4.1]). Let $F$ be an infinite field, $R$ an (associative) $F$-algebra and let $(P,Q,\psi)$ be an $R$-system satisfying condition (FS). Assume that there exists a uniquely maximal faithful $\psi$-compatible two-sided ideal of $R$. Let $A$ be an $F$-algebra. Suppose that

$$\phi : O_{(P,Q,\psi)} \to A$$

is an $F$-algebra homomorphism such that $\phi(p_r) \neq 0$ for every $r \in R \setminus \{0\}$. If there exists a group action $\sigma : F^* \to \text{Aut}_F(A)$ such that $\phi \circ \tau^E_{\sigma} = \sigma_t \circ \phi$ for every $t \in F^*$, then $\phi$ is injective.

When we specialize to directed graphs, we get a generalization of the Algebraic Gauge-Invariant Uniqueness Theorem [1, Theorem 1.8.] from row finite graphs to all directed graphs.

**Corollary 6.4.** Let $E$ be a directed graph, let $F$ be an infinite field and let $A$ be an $F$-algebra. Suppose that

$$\phi : L_F(E) \to A$$

is a $F$-algebra homomorphism such that $\phi(p_v) \neq 0$ for every $v \in E^0$. If there exists a group action $\sigma : F^* \to \text{Aut}_F(A)$ such that $\phi \circ \tau^E_{\sigma} = \sigma_t \circ \phi$ for every $t \in F^*$, then $\phi$ is injective.

**Proof.** This follows from Example 5.8 and Corollary 6.3. □

7. **Graded covariant representations**

In Section 3 we classified all surjective, injective and graded covariant representations of an $R$-system satisfying condition (FS). We will in this section extend this classification to all surjective and graded covariant representations. As a corollary we get a description of all graded two-sided ideals of a relative Cuntz–Pimsner algebra (and therefore of the Toeplitz ring and the Cuntz–Pimsner ring whenever it is defined) of an $R$-system satisfying condition (FS).

We will proceed as in Section 3 and first describe a family of surjective and graded covariant representations of a given $R$-system which satisfies condition (FS), and then show that this family contains up to isomorphism all surjective and graded covariant representations. This approach is inspired by the work of Katsura in [17] (notice however that our definition of a $T$-pair (see Definition 7.5) is different from Katsura’s definition).

At the end of the section we will see how our description of the graded two-sided ideals of a Cuntz–Pimsner ring agrees with Tomforde’s characterization of the graded two-sided ideals of a Leavitt path algebra. We will also show (cf. Proposition 7.26) that if the $R$-system $(P,Q,\psi)$ satisfies condition (FS), then any quotient of a relative Cuntz–Pimsner ring of $(P,Q,\psi)$ by a graded two-sided ideal is again a relative Cuntz–Pimsner ring (but of a different system).

7.1. **The classification of graded covariant representations of an R-system**

We begin with some definitions and some notation.
Definition 7.1. Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system. A two-sided ideal $I$ of $R$ is said to be $\psi$-invariant if $\psi(p \otimes xq) \in I$ for every $p \in P$, $q \in Q$ and $x \in I$.

If $I$ is a two-sided ideal of $R$, then $QI := \text{span}\{qx \mid q \in Q, x \in I\}$ and $IQ := \text{span}\{xq \mid q \in Q, x \in I\}$ are $I$-bimodules. Similarly we define $IP := \text{span}\{xp \mid p \in P, x \in I\}$ and $PI := \text{span}\{pr \mid p \in P, x \in I\}$ which are also $I$-bimodules.

Remark 7.2. Observe that if $R$ is a ring, $(P, Q, \psi)$ is an $R$-system which satisfies condition (FS), and $I$ is a $\psi$-invariant two-sided ideal of $R$, then $IQ \subseteq QI$ and $PI \subseteq IP$. Indeed, let $x \in I$ and $q \in Q$, then by the (FS) condition there exists $\Theta = \sum_{i=1}^{n} \theta_{q,p} \in \mathcal{F}_{P}(Q)$ such that $xq = \Theta(xq) = \sum_{i=1}^{n} \theta_{q,p}(xq) = \sum_{i=1}^{n} q_i \psi(p_i \otimes xq) \in QI$ since $\psi(p_i \otimes xq) \in I$ for every $i \in \{1, \ldots, n\}$. Similarly one can prove that $PI \subseteq IP$.

Definition 7.3. Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). For a two-sided ideal $I$ of $R$ we define $R_I := R/I$, $Q_I := Q/PI$ and $I_P := P/IP$. We let $\varphi_I$ be their respective projections.

It follows from Remark 7.2 that if $I$ is a $\psi$-invariant two-sided ideal of $R$, then $Q_I$ and $I_P$ are $R_I$-bimodules. We can, in this case, define a $R_I$-bimodule homomorphism $\psi_I : I_P \otimes Q_I \rightarrow R_I$ by $\psi_I(\varphi_I(p) \otimes \varphi_I(q)) = \varphi_I(\psi(p \otimes q))$.

Observe that we can also define a projection $\varphi_I : L_P(Q) \rightarrow L_P(QI)$ such that $\varphi_I(T)(\varphi_I(q)) = \varphi_I(T(q))$ for every $T \in L_P(Q)$ and $q \in Q$, and then we have that $\varphi_I(\mathcal{F}_P(Q)) = \mathcal{F}_{P_I}(QI)$. We also define a ring homomorphism $\Delta_I : R_I \rightarrow \text{End}(Q_I)$ by $\Delta_I(\varphi_I(r))\varphi_I(q) = \varphi_I(rq)$ for $r \in R$ and $q \in Q$. We then have that $\Delta_I(\varphi_I(r)) = \varphi_I(\Delta(r))$ for every $r \in R$.

We then have the following straightforward lemma:

Lemma 7.4. Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS), and let $I$ be a $\psi$-invariant two-sided ideal of $R$. Then the $R_I$-system $(I_P, Q_I, \psi_I)$ satisfies condition (FS).

Definition 7.5 (cf. [17, Definition 5.6]). Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). A pair $\omega = (I, J)$ of two-sided ideals of $R$ such that $I \subseteq J$ is said to be a $T$-pair of $(P, Q, \psi)$ if $I$ is a $\psi$-invariant ideal and $J_I := \varphi_I(J)$ is a faithful, $\psi_I$-compatible, two-sided ideal of $R_I$.

Note that since $I \subseteq J$, we have that $\varphi_I^{-1}(J_I) = J$.

Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS), and let $\omega = (I, J)$ be a $T$-pair. Then we define the following maps:

\[
\begin{align*}
\iota_{P,I}^J &:= \iota_{R_I}^J \circ \varphi_I : R \longrightarrow \mathcal{O}_{(I,P,Q_I,\psi_I)}(J_I), \\
\iota_{Q,I}^J &:= \iota_{Q_I}^J \circ \varphi_I : Q \longrightarrow \mathcal{O}_{(I,P,Q_I,\psi_I)}(J_I), \\
\iota_{P}^J &:= \iota_{P_I}^J \circ \varphi_I : P \longrightarrow \mathcal{O}_{(I,P,Q_I,\psi_I)}(J_I),
\end{align*}
\]

where $(\iota_{P,I}^J, \iota_{Q,I}^J, \iota_{R,I}^J, \mathcal{O}_{(I,P,Q_I,\psi_I)}(J_I))$ is the universal Cuntz–Pimsner invariant representation of $(I_P, Q_I, \psi_I)$ relative to $J_I$. It is easy to check that $(\iota_{P}^J, \iota_{Q}^J, \iota_{P}^J, \mathcal{O}_{(I,P,Q_I,\psi_I)}(J_I))$ is a surjective and graded covariant representation of $(P, Q, \psi)$. We will in this section show that the family $\{(\iota_{P}^J, \iota_{Q}^J, \iota_{P}^J, \mathcal{O}_{(I,P,Q_I,\psi_I)}(J_I)) \mid \omega \text{ is a $T$-pair of $(P, Q, \psi)$}\}$ up to isomorphism contains all surjective and graded covariant representations of $(P, Q, \psi)$. 


**Definition 7.6.** Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system that satisfies condition (FS) and let $(S, T, \sigma, B)$ be a covariant representation of $(P, Q, \psi)$. Then we define $I_{(S, T, \sigma, B)}$ as the two-sided ideal $\ker \sigma$ of $R$.

**Lemma 7.7.** Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). If $(S, T, \sigma, B)$ is a covariant representation of $(P, Q, \psi)$, then $T = Q I_{(S, T, \sigma, B)}$ and $S = I_{(S, T, \sigma, B)} P$.

**Proof.** Clearly $Q I_{(S, T, \sigma, B)} \subseteq \ker T$. Now let $q \in \ker T$, then for every $p \in P$ we have $0 = S(p) T(q) = \sigma(\psi(p \otimes q))$ and hence $\psi(p \otimes q) \in \ker \sigma = I_{(S, T, \sigma, B)}$ for every $p \in P$. By condition (FS) there exists $\Theta = \sum_{i=1}^{n} \theta_{i}, p_{i}$ such that $\Theta(q) = \sum_{i=1}^{n} \theta_{q_{i}p_{i}} = \sum_{i=1}^{n} q_{i} \psi(p_{i} \otimes q) \in Q I_{(S, T, \sigma, B)}$, as desired.

That $\ker S = I_{(S, T, \sigma, B)} P$ can be proved in a similar way. \hfill \Box

**Proposition 7.8.** Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). Let $(S, T, \sigma, B)$ be a covariant representation of $(P, Q, \psi)$, and let $I_{(S, T, \sigma, B)}$ be as defined in Definition 7.6, and let $J_{(S, T, \sigma, B)}$ be as defined in Definition 3.23. Then the pair $\omega_{(S, T, \sigma, B)} := (I_{(S, T, \sigma, B)}, J_{(S, T, \sigma, B)})$ is a $T$-pair of $(P, Q, \psi)$.

**Proof.** We let $I := I_{(S, T, \sigma, B)}$ and $J := J_{(S, T, \sigma, B)}$. It is clear that $I$ is a two-sided ideal of $R$, and it follows from Lemma 3.26 that also $J$ is a two-sided ideal of $R$. It is clear that $I \subseteq J$.

First we prove that $I$ is $\psi$-invariant. Indeed, let $x \in I$, $p \in P$ and $q \in Q$. Then $\sigma(\psi(p \otimes xq)) = S(p) \sigma(x) T(q) = 0$, so $\psi(p \otimes xq) \in \ker \sigma = I$.

Now let $x \in J = \sigma^{-1}(\pi_{T, S}(F_{P}(Q)))$. Then there exists a $\Theta \in F_{P}(Q)$ with $\sigma(x) = \pi_{T, S}(\Theta)$. Thus we have for every $q \in Q$ that

$$T(xq) = \sigma(x) T(q) = \pi_{T, S}(\Theta) T(q) = T(\Theta(q)),$$

and it follows from Lemma 7.7 that $xq - \Theta(q) \in \ker T = QI$. Hence $\varphi_{I}(xq) - \varphi_{I}(\Theta(q)) = 0$, so $\varphi_{I}(x) \varphi_{I}(q) = \varphi_{I}(\Theta)(\varphi_{I}(q)).$ Since $\varphi_{I}(\Theta) \in F_{P}(Q)$, it follows that $\Delta_{I}(\varphi_{I}(x)) \in F_{P}(Q)$.

Now we check that $J_{I} \cap \ker \Delta_{I} = 0$. Let $x \in J$ and assume that $\varphi_{I}(x) \in \ker \Delta_{I}$. Then $xq \in QI$ for every $q \in Q$. But since $x \in J$, there exists $\Theta = \sum_{i=1}^{n} \theta_{i}, p_{i} \in F_{P}(Q)$ such that $\sigma(x) = \pi_{T, S}(\Theta) = \sum_{i=1}^{n} T(q_{i}) S(p_{i})$. It then follows from Lemma 7.7 that $xq - \sum_{i=1}^{n} q_{i} \psi(p_{i} \otimes q) \in \ker T = QI$, so $\sum_{i=1}^{n} q_{i} \psi(p_{i} \otimes q) \in QI$ for every $q \in Q$. Now by condition (FS) there exist $\Theta_{1} = \sum_{i=1}^{n} \theta_{j_{i}, b_{i}} \in F_{P}(Q)$ and $\Theta_{2} = \sum_{i=1}^{n} \theta_{c_{i}, d_{k}} \in F_{Q}(P)$ such that $\Theta_{1}(q_{i}) = q_{i}$ and $\Theta_{2}(p_{i}) = p_{i}$ for every $i = 1, \ldots, n$. Then we have

$$\sigma(x) = \sum_{i=1}^{n} T(q_{i}) S(p_{i}) = \sum_{i=1}^{n} T(\Theta_{1}(q_{i})), S(\Theta_{2}(p_{i}))$$

$$= \sum_{i=1}^{n} T \left( \sum_{j=1}^{m} \theta_{j_{i}, b_{j}} (q_{i}) \right) S \left( \sum_{k=1}^{l} \theta_{c_{i}, d_{k}} (p_{i}) \right)$$

$$= \sum_{i=1}^{n} T \left( \sum_{j=1}^{m} a_{j} \psi(b_{j} \otimes q_{i}) \right) S \left( \sum_{k=1}^{l} \psi(p_{i} \otimes d_{k}) c_{k} \right)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{l} T(a_{j}) \sigma(\psi(b_{j} \otimes q_{i}) \psi(p_{i} \otimes d_{k})) S(c_{k})$$
\[= \sum_{j=1}^{m} \sum_{k=1}^{l} T(\alpha_j) \sigma \left( \psi(\beta_j \otimes \sum_{i=1}^{n} q_i \psi(p_i \otimes d_k)) \right) S(c_k) \]
\[= \sum_{j=1}^{m} \sum_{k=1}^{l} T(\alpha_j) \sigma(\psi(\beta_j \otimes \Theta(d_k))) S(c_k), \]

but \( \Theta(d_k) \in QI \) for every \( k = 1, \ldots, l \), and hence \( \psi(\beta_j \otimes \Theta(d_k)) \in I \). So \( \sigma(\psi(\beta_j \otimes \Theta(d_k))) = 0 \), from which it follows that \( 0 = \sum_{i=1}^{n} T(q_i) S(p_i) = \sigma(x) \), and therefore \( x \in \ker \sigma = I \). Thus \( \varphi_I(x) = 0 \).

**Proposition 7.9.** Let \( R \) be a ring and let \((P, Q, \psi)\) an \( R \)-system satisfying condition (FS). If \( \omega = (I, J) \) is a \( T \)-pair, then \( \omega = \omega(I, q) = \omega(I, q) = \omega(I, q) = I \) by injectivity of \( \psi_I \).

First note that \( I(\psi, \psi, \psi, \Theta)_{(P, Q, \psi)}(J_I) = \ker \psi = \ker \psi = \ker \psi = I \) by injectivity of \( \psi_I \).

Let \( x \in J \). Then we have that \( \varphi_I(x) \in J_I \) and thus that
\[\psi_I(x) = \psi_I(\varphi_I(x)) = \pi^{J_I}(\Delta_I(\varphi_I(x))) \in \pi^{J_I}(F_{I, P}(Q_I)) = \pi^{J_I}(\varphi_I(F_{I, P}(Q_I))) = \pi^{J_I}(\varphi_I(F_{I, P}(Q_I))), \]
and therefore \( x \in \psi_I^{-1}(\pi^{J_I}(F_{I, P}(Q_I))). \) This shows that \( J \subseteq J(\psi, \psi, \psi, \Theta)_{(P, Q, \psi)}(J_I) \).

Assume now that \( x \in J(\psi, \psi, \psi, \Theta)_{(P, Q, \psi)}(J_I) \). Then we have
\[\psi_I(x) = \psi_I(\varphi_I(x)) = \pi^{\psi_I}(\varphi_I(F_{I, P}(Q_I))) = \pi^{\psi_I}(\varphi_I(F_{I, P}(Q))), \]

Since \( J_I \subseteq \Delta_I^{-1}(F_{I, P}(Q_I)) \) and \( J_I \cap \ker I = 0 \), it follows from Proposition 3.28 that \( \varphi_I(x) \in J_I \). Thus \( x \in J \) which shows that \( J(\psi, \psi, \psi, \Theta)_{(P, Q, \psi)}(J_I) \subseteq J \).

**Lemma 7.10.** Let \( R \) be a ring and \((P, Q, \psi)\) an \( R \)-system. Let \((S, T, \sigma, B)\) be a covariant representation and let \( I \) be a \( \psi \)-invariant two-sided ideal of \( R \). Then we have:

(i) If there is a covariant representation \((S_I, T_I, \sigma_I, B)\) of \((P, Q, \psi_I)\) such that \( T = T_I \circ \varphi_I \), \( S = S_I \circ \varphi_I \) and \( \sigma = \sigma_I \circ \varphi_I \), then \( I \subseteq I(S, T, \sigma, B) \).

(ii) If \( I \subseteq I(S, T, \sigma, B) \), then there exists a unique covariant representation \((S_I, T_I, \sigma_I, B)\) of \((P, Q, \psi_I)\) such that \( T = T_I \circ \varphi_I \), \( S = S_I \circ \varphi_I \) and \( \sigma = \sigma_I \circ \varphi_I \).

(iii) If \( I \subseteq I(S, T, \sigma, B) \), then the covariant representation \((S_I, T_I, \sigma_I, B)\) is injective if and only if \( I = I(S, T, \sigma, B) \).

(iv) If \( I \subseteq I(S, T, \sigma, B) \), then the covariant representation \((S_I, T_I, \sigma_I, B)\) is surjective and graded if and only if \((S, T, \sigma, B)\) is.

(v) If \( I \subseteq I(S, T, \sigma, B) \) and \((I, J)\) is a \( T \)-pair of \((P, Q, \psi)\), then the covariant representation \((S_I, T_I, \sigma_I, B)\) is Cuntz–Pimsner invariant relative to \( J_I \) if and only if \( J \subseteq J(S, T, \sigma, B) \).

**Proof.** If there is a covariant representation \((S_I, T_I, \sigma_I, B)\) of \((P, Q, \psi_I)\) such that \( T = T_I \circ \varphi_I \), \( S = S_I \circ \varphi_I \) and \( \sigma = \sigma_I \circ \varphi_I \), then \( I \subseteq I(S, T, \sigma, B) \).

Assume now that \( I \subseteq I(S, T, \sigma, B) \). It follows from Lemma 7.7 that we can define maps \( \sigma_I : R_I \rightarrow B \) by letting \( \sigma_I(r + I) = \sigma(r) \) for every \( r \in R \), \( T_I : Q_I \rightarrow B \) by letting \( T_I(q + QI) = T(q) \) for every \( q \in Q \) and \( S_I : P_I \rightarrow B \) by letting \( S_I(p + PI) = S(p) \) for every \( p \in P \). It is then clear that \((S_I, T_I, \sigma_I, B)\) is a covariant representation of \((P, Q_I, \psi_I)\) satisfying \( T = T_I \circ \varphi_I \), \( S = S_I \circ \varphi_I \) and \( \sigma = \sigma_I \circ \varphi_I \). It is also clear that \((S_I, T_I, \sigma_I, B)\) is the unique covariant representation of \((P, Q_I, \psi_I)\) with this property. Finally it is straightforward to check that \((S_I, T_I, \sigma_I, B)\) is injective if and only if \( I = I(S, T, \sigma, B) \), that \((S_I, T_I, \sigma_I, B)\) is surjective and graded if and only if
Let $R$ be a ring and $(P, Q, ψ)$ be an $R$-system that satisfies condition (FS). Let $(S, T, σ, B)$ be a covariant representation of $(P, Q, ψ)$ and let $ω = (I, J)$ be a $T$-pair of $(P, Q, ψ)$. Then we have:

(i) If there is a ring homomorphism $η : O_{(1, P, Q, ψ)}(J_1) → B$ such that $η ◦ i^ω_R = σ, η ◦ i^ω_Q = T$ and $η ◦ i^ω_P = S$, then $I ⊆ I_{(S, T, σ, B)}$ and $J ⊆ J_{(S, T, σ, B)}$.

(ii) If $I ⊆ I_{(S, T, σ, B)}$ and $J ⊆ J_{(S, T, σ, B)}$, then there exists a unique ring homomorphism $η^ω_{(S, T, σ, B)} : O_{(1, P, Q, ψ)}(J_1) → B$ such that $η^ω_{(S, T, σ, B)} ◦ i^ω_R = σ, η^ω_{(S, T, σ, B)} ◦ i^ω_Q = T$ and $η^ω_{(S, T, σ, B)} ◦ i^ω_P = S$.

(iii) If $I ⊆ I_{(S, T, σ, B)}$ and $J ⊆ J_{(S, T, σ, B)}$, then $η^ω_{(S, T, σ, B)}$ is an isomorphism if and only if $(S, T, σ, B)$ is a surjective and graded representation and $ω = ω_{(S, T, σ, B)}$.

Proof. It is easy to check that if there exists a ring homomorphism $η : O_{(1, P, Q, ψ)}(J_1) → B$ such that $η ◦ i^ω_R = σ, η ◦ i^ω_Q = T$ and $η ◦ i^ω_P = S$, then $I ⊆ I_{(S, T, σ, B)}$ and $J ⊆ J_{(S, T, σ, B)}$.

Assume now that $I ⊆ I_{(S, T, σ, B)}$ and $J ⊆ J_{(S, T, σ, B)}$. It follows from Lemma 7.10 that there exists a covariant representation $(S_1, T_1, σ_1, B)$ of $(1, P, Q, ψ_{1})$ which is Cuntz–Pimsner invariant relative to $J_1$ such that $T = T_1 ◦ ψ_{1}$, $S = S_1 ◦ ψ_{1}$ and $σ = σ_1 ◦ ψ_{1}$. It then follows from Theorem 3.18 that there exists a ring homomorphism $η^ω_{(S, T, σ, B)} : O_{(1, P, Q, ψ)}(J_1) → B$ such that $η^ω_{(S, T, σ, B)} ◦ i^ω_{R_1} = σ_1, η^ω_{(S, T, σ, B)} ◦ i^ω_{Q_1} = T_1$ and $η^ω_{(S, T, σ, B)} ◦ i^ω_{P_1} = S_1$. It follows that $η^ω_{(S, T, σ, B)} ◦ i^ω_{R} = σ, η^ω_{(S, T, σ, B)} ◦ i^ω_{Q} = T$ and $η^ω_{(S, T, σ, B)} ◦ i^ω_{P} = S$. Since $O_{(1, P, Q, ψ)}(J_1)$ is generated by $i^ω_{P}(R), i^ω_{Q}(Q)$ and $i^ω_{P}(P)$, the uniqueness of $η^ω_{(S, T, σ, B)}$ follows.

Assume that $η^ω_{(S, T, σ, B)}$ is an isomorphism. Then $(S, T, σ, B)$ is surjective and graded, and $ω(η^ω_{S, T, σ, B}, i^ω_{P, Q, ψ}, ω_{(1, P, Q, ψ)}(J_1)) = ω_{(S, T, σ, B)}$. It therefore follows from Proposition 7.9 that $ω = ω_{(S, T, σ, B)}$.

Finally assume that $(S, T, σ, B)$ is surjective and graded, and that $ω = ω_{(S, T, σ, B)}$. Then it follows from Lemma 7.10 that $(S_1, T_1, σ_1, B)$ is surjective, injective and graded, and it is easy to check that $J_1 = φ_{1}(J_{(S, T, σ, B)}) = J_{(S_1, T_1, σ_1, B)}$. It then follows from Theorem 3.29 that $η^ω_{(S, T, σ, B)}$ is an isomorphism.

We now have the promised classification of all surjective and graded covariant representations of a given $R$-system satisfying condition (FS).

Remark 7.12. Let $R$ be a ring and let $(P, Q, ψ)$ be an $R$-system satisfying condition (FS). Then it follows from Proposition 7.8 and Theorem 7.11 that every surjective and graded covariant representation of $(P, Q, ψ)$ is isomorphic to $(i^ω_{P}, i^ω_{Q}, i^ω_{R}, O_{(1, P, Q, ψ)}(J_1))$ for some $T$-pair $ω = (I, J)$ of $(P, Q, ψ)$. It also follows that if $ω_1 = (I_1, J_1)$ and $ω_2 = (I_2, J_2)$ are two $T$-pairs of $(P, Q, ψ)$, then there is a ring homomorphism $φ : O_{(1, P, Q, ψ)}(J_1) → O_{(1, P, Q, ψ)}(J_2)$ such that $φ ◦ i^ω_{R_1} = i^ω_{R_2}, φ ◦ i^ω_{Q_1} = i^ω_{Q_2}$ and $φ ◦ i^ω_{P_1} = i^ω_{P_2}$ if and only if $I_1 ⊆ I_2$ and $J_1 ⊆ J_2$.

Let $R$ be a ring and let $(P, Q, ψ)$ be an $R$-system satisfying condition (FS). If $(I, J)$ is a pair of two-sided ideals of $R$ such that $I ⊆ J$, the ideal $I$ is $ψ$-invariant and $ψ_{1}(J) ⊆ Δ_{I}^{-1}(F_{P}(Q_{I}))$, then $(i^ω_{P_1} ◦ ψ_{1}, i^ω_{Q_1} ◦ ψ_{1}, i^ω_{R_1} ◦ ψ_{1}, O_{(1, P, Q, ψ)}(J_1))$ is a surjective and graded covariant representation of $(P, Q, ψ)$, even though $ψ_{1}(J) ∩ ker Δ_{I} ≠ 0$, and it then follows from the previous remark that this representation is isomorphic to $(i^ω_{P}, i^ω_{Q}, i^ω_{R}, O_{(1, P, Q, ψ)}(J_1))$. 

\[ (S, T, σ, B) \] is, and that $(S_1, T_1, σ_1, B)$ is Cuntz–Pimsner invariant relative to $J_1$ if and only if $J ⊆ J_{(S, T, σ, B)}$. 

\[ \square \]
for some $T$-pair $\omega' = (I', J')$. We will now describe this $T$-pair in terms of the pair $(I, J)$. We will begin with the case where $I = \{0\}$, but first a lemma.

**Lemma 7.13.** Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). If $x \in \Delta^{-1}(F_P(Q))$, $q \in Q_{\otimes^0}$ and $p \in P_{\otimes^0}$, then $\theta_{q, x, p} \otimes 1_Q \in F_{P_{\otimes^0+1}}(Q_{\otimes^{n+1}})$ and

$$
\pi(\theta_{q, x, p} \otimes 1_Q) = \iota_{q}^{n}(q)\pi(\Delta(x))\iota_{p}^{n}(p).
$$

(7.1)

**Proof.** Choose $q_1, q_2, \ldots, q_k \in Q$ and $p_1, p_2, \ldots, p_k \in P$ such that $\Delta(x) = \sum_{i=1}^{k} \theta_{q_i, p_i}$. Then we have for $q^n \in Q_{\otimes^n}$ and $q^1 \in Q$ that

$$
\theta_{q, x} \otimes 1_Q(q_{\otimes^n} \otimes q^1) = q \otimes x_{\psi}(p \otimes q^n)q^1 = \sum_{i=1}^{k} q \otimes q_i \psi(p_i \otimes x_{\psi}(p \otimes q^n)q^1)
$$

$$
= \sum_{i=1}^{k} q \otimes q_i \psi_{n+1}((p_i \otimes p) \otimes (q^n \otimes q^1)) = \sum_{i=1}^{k} \theta_{q \otimes q_i, p \otimes q_i}(q^n \otimes q^1).
$$

It follows that $\theta_{q, x} \otimes 1_Q = \sum_{i=1}^{k} \theta_{q \otimes q_i, p \otimes q_i} \in F_{P_{\otimes^0+1}}(Q_{\otimes^{n+1}})$ and that

$$
\pi(\theta_{q, x} \otimes 1_Q) = \sum_{i=1}^{k} \iota_{q}^{n}(q)\iota_{q_i}^{n}(q_i)\iota_{p}^{n}(p_i)\iota_{p}^{n}(p) = \iota_{q}^{n}(q)\pi(\Delta(x))\iota_{p}^{n}(p).
$$

Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system. For every $x \in R$ we define $\Delta^n(x) \in \mathcal{L}_{P_{\otimes^n}}(Q_{\otimes^n})$ inductively by letting $\Delta^1(x) = \Delta(x)$ and $\Delta^n(x) = \Delta^{n-1}(x) \otimes 1_Q$ for $n \geq 2$.

**Lemma 7.14.** Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS), and let $J$ be a $\psi$-invariant two-sided ideal of $R$. If we let

$$
I = \{ x \in J \mid \forall m \in \mathbb{N} : \Delta^m(x)(Q_{\otimes^m}) \subseteq Q_{\otimes^m} J \land \exists n \in \mathbb{N} : \Delta^n(x) = 0 \},
$$

then $I = I_{(\iota_{P}, \iota_{Q}, \iota_{\psi}, \mathcal{O}(P, Q, \psi)(J))}$ and $J = J_{(\iota_{P}, \iota_{Q}, \iota_{\psi}, \mathcal{O}(P, Q, \psi)(J))}$.

**Proof.** Let $x \in I_{(\iota_{P}, \iota_{Q}, \iota_{\psi}, \mathcal{O}(P, Q, \psi)(J))}$. Then $\iota_{R}(x) \in T(J)$. It follows from Lemma 3.21 that $\iota_{R}(x) = P_{(0)}(\iota_{R}(x)) \subseteq \iota_{R}(J)$ and that there is an $n \in \mathbb{N}$ such that $\iota_{Q}^{n}(xq) = \iota_{R}(x)\iota_{Q}^{n}(q) = 0$ for every $q \in Q_{\otimes^n}$. Since $\iota_{R}$ and $\iota_{Q}^{n}$ are injective (cf. Theorem 1.7 and Lemma 3.9) it follows that $x \in J$ and that $\Delta^n(x) = 0$. It also follows from Lemma 3.21 that

$$
\iota_{R}(x) = \iota_{R}(x) - \pi(\Delta(x)) + \sum_{i=1}^{m_i} \iota_{Q}^{n}(q_j^i)(\iota_{R}(x_j^i) - \pi(\Delta(x_j^i)))\iota_{P}^{n}(p_j^i)
$$

for some $x_j^i \in J$, $q_j^i \in Q_{\otimes^i}$, $p_j^i \in P_{\otimes^i}$. We will by induction show that

$$
\Delta^i(x) = \sum_{j=1}^{m_i} \theta_{q_j^i, x_j^i, p_j^i}
$$

(7.2)

for every $i \in \{1, 2, \ldots, n - 1\}$. It will then follow that $\Delta^i(x)(q) = \sum_{j=1}^{m_i} q_j^i x_j^i \psi_j(p_j^i \otimes q) \in Q_{\otimes^i} J$ for every $i \in \{1, 2, \ldots, n - 1\}$ and every $q \in Q_{\otimes^i}$, and thus that $x \in I$. For $i = 1$ we have

$$
0 = P_{(1)}(\iota_{R}(x)) = -\pi(\Delta(x)) + \sum_{j=1}^{m_1} \iota_{Q}^{n}(q_j^1)\iota_{R}(x_j^1)\iota_{P}^{n}(p_j^1).
$$
Thus we have
\[ \pi(\Delta(x)) = \sum_{j=1}^{m_1} \iota_Q(q_j^1)\iota_R(x_j^1)p_j^1 = \pi \left( \sum_{j=1}^{m_1} \theta_j x_j^1 p_j^1 \right) \]
and since \( \pi \) is injective (cf. Proposition 3.11), it follows that equation (7.2) holds for \( i = 1 \).

Let \( k \in \{1, 2, \ldots, n - 2\} \) and assume that equation (7.2) holds for \( i = k \). We have that
\[ 0 = P_{(k+1,k+1)}(R(x)) = -\sum_{j=1}^{m_k} \iota_Q(q_j^k)\pi(\Delta(x_j^k))\iota_P(p_j^k) + \sum_{j=1}^{m_{k+1}} \iota_Q(q_j^{k+1})\iota_R(x_j^{k+1})\iota_P(p_j^{k+1}) \]

It follows that if \( q_k \in Q^\otimes k \) and \( q_1 \in Q \), then we have that
\[ \iota_Q^{k+1}(\Delta^{k+1}(q_k \otimes q_1)) = \pi(\Delta(x_k))\iota_Q(q_k)\iota_Q(q_1) = \sum_{j=1}^{m_k} \iota_Q(q_j^k)\iota_R(x_j^k)\iota_P(p_j^k)\iota_Q(q_k)\iota_Q(q_1) \]
\[ = \sum_{j=1}^{m_k} \iota_Q(q_j^k)\pi(\Delta(x_j^k))\iota_P(p_j^k)\iota_Q(q_k)\iota_Q(q_1) \]
\[ = \sum_{j=1}^{m_{k+1}} \iota_Q(q_j^{k+1})\iota_R(x_j^{k+1})\iota_P(p_j^{k+1})\iota_Q(q_k)\iota_Q(q_1) \]
\[ = \iota_Q^{k+1} \left( \sum_{j=1}^{m_{k+1}} \theta_{q_j^{k+1}x_j^{k+1}p_j^{k+1}}(q_k \otimes q_1) \right), \]
and since \( \iota_Q^{k+1} \) is injective and \( Q^{k+1} = \text{span}\{q_k \otimes q_1 \mid q_k \in Q^\otimes k, \ q_1 \in Q\} \), it follows that equation (7.2) holds for \( i = k + 1 \). Hence equation (7.2) holds for every \( i \in \{1, 2, \ldots, n - 1\} \).

We have thus proved that \( I(\iota_Q, \iota_Q^F, \iota_R, Q, \wp(J)) \subseteq I \).

Let \( x \in J \) and assume that \( \Delta^m(x)(Q^\otimes m) \subseteq Q^\otimes mJ \) for all \( m \in \mathbb{N} \) and that there is an \( n \in \mathbb{N} \) such that \( \Delta^n(x) = 0 \). We will by induction show that for every \( i \in \{1, 2, \ldots, n - 1\} \) there exist \( x_j^i \in J, q_j^i \in Q^\otimes i, p_j^i \in P^\otimes i \) such that
\[ \Delta^i(x) = \sum_{j=1}^{m_i} \theta_{q_j^ix_j^ip_j^i} \]
and such that \( \Delta^{i+1}(x) \in F_{P^\otimes i+1}(Q^\otimes i+1) \) and
\[ \pi(\Delta^{i+1}(x)) = \sum_{j=1}^{m_{i+1}} \iota_Q(q_j^i)\pi(\Delta(x_j^i))\iota_P(p_j^i). \]

It will then follow that we have
\[ \iota_R(x) = \iota_R(x) - \pi(\Delta(x)) + \sum_{j=1}^{m-1} \sum_{j=1}^{m_i} \iota_Q(q_j^i) \left( \iota_R(x_j^i) - \pi(\Delta(x_j^i)) \right) \iota_P(p_j^i) \in T(J), \]
and thus that \( x \in I(\iota_Q, \iota_Q^F, \iota_R, Q, \wp(J)) \).

Choose \( q_1, q_2, \ldots, q_k \in Q, \ p_1, p_2, \ldots, p_k \in P \) such that \( \Delta(x) = \sum_{j=1}^{k} \theta_{q_j p_j} \). It follows from condition (FS) that there exist \( q_1', q_2', \ldots, q_h' \in Q \) and \( p_1', p_2', \ldots, p_h' \in P \) such that \( \sum_{l=1}^{h} \theta_{p_l' q_l'}(p_j) = p_j \) for every \( j \in \{1, 2, \ldots, k\} \). We then have that
\[ \Delta(x) = \sum_{j=1}^{k} \theta_{q_j p_j} = \sum_{j=1}^{k} \theta_{q_j} \sum_{l=1}^{h} \psi(p_l \otimes q_l')p_l' = \sum_{l=1}^{h} \theta_{\Delta(x) q_l' p_l'}. \]
Since $\Delta(x)q'_l \in QJ$ for each $l \in \{1, 2, \ldots, h\}$, it follows that there exist $x^1_j \in J$, $q^1_j \in Q$, $p^1_j \in P$ such that equation (7.3) holds for $i = 1$. It then follows from Lemma 7.13 that also equation (7.4) holds for $i = 1$.

Assume then that $k \in \{1, 2, \ldots, n - 1\}$ and that there exist $x^k_j \in J$, $q^k_j \in Q \otimes^k$, $p^k_j \in P \otimes^k$ such that Equation (7.4) holds for $i = k$. For each $j \in \{1, 2, \ldots, m_k\}$ choose $q_{(j,1)}, q_{(j,2)}, \ldots, q_{(j,n_j)} \in Q$ and $p_{(j,1)}, p_{(j,2)}, \ldots, p_{(j,n_j)} \in P$ such that $\Delta(x^k_j) = \sum_{h=1}^{n_j} \theta_{q_{(j,h)}, p_{(j,h)}}$. If $q^k \in Q \otimes^k$ and $q^1 \in Q$, then we have

$$
t^{k+1}_Q(\Delta^{k+1}(x)(q^k \otimes q^1)) = \sum_{j=1}^{m_k} \sum_{h=1}^{n_j} \theta_{q^k_j \otimes q_{(j,h)}, p^k_j \otimes p_{(j,h)}}.
$$

It follows that $\Delta^{k+1}(x) = \sum_{j=1}^{m_k} \sum_{h=1}^{n_j} \theta_{q^k_j \otimes q_{(j,h)}, p^k_j \otimes p_{(j,h)}}$. An application of condition (FS) gives us $q'_1, q'_2, \ldots, q'_r \in Q \otimes^{k+1}$ and $p'_1, p'_2, \ldots, p'_r \in P \otimes^{k+1}$ such that $\sum_{l=1}^{r} \theta_{q^k_j \otimes q'_l, p^k_j \otimes p'_l} = p_{(j,h)} \otimes p^1_j$ for every $j \in \{1, 2, \ldots, m_k\}$ and every $h \in \{1, 2, \ldots, n_j\}$. We then have that

$$
\Delta^{k+1}(x) = \sum_{j=1}^{m_k} \sum_{h=1}^{n_j} \sum_{l=1}^{n_j} \theta_{q^k_j \otimes q, p^k_j \otimes p'_l}.
$$

Since $\Delta^{k+1}(x)q'_l \in Q^{k+1}J$ for each $l \in \{1, 2, \ldots, r\}$, it follows that there exist $x^{k+1}_j \in J$, $q^{k+1}_j \in Q^{\otimes^{k+1}}$, $p^{k+1}_j \in P^{\otimes^{k+1}}$ such that equation (7.3) holds for $i = k + 1$. It then follows from Lemma 7.13 that also equation (7.4) holds for $i = k + 1$.

Thus there exist for every $i \in \{1, 2, \ldots, n - 1\}$ elements $x^i_j \in J$, $q^i_j \in Q^{\otimes^i}$, $p^i_j \in P^{\otimes^i}$ such that equations (7.3) and (7.4) hold, and $x \in I(\cap_{i=1}^{\infty} I_{x^i_j, q^i_j, p^i_j, \mathcal{O}(P, Q, \psi)(J)})$. This shows that $I \subseteq \bigcap_{i=1}^{\infty} I_{x^i_j, q^i_j, p^i_j, \mathcal{O}(P, Q, \psi)(J)}$ and so we have proved that $I = \bigcap_{i=1}^{\infty} I_{x^i_j, q^i_j, p^i_j, \mathcal{O}(P, Q, \psi)(J)}$.

We will now show that $J = J_{x^i_j, q^i_j, p^i_j, \mathcal{O}(P, Q, \psi)(J)}$. If $x \in J$, then $\iota_R(x) - \pi(\Delta(x)) \in \mathcal{T}(J)$, so $\iota^R_R(x) = \pi^{\mathcal{T}}(\Delta(x))$ and $x \in J_{x^i_j, q^i_j, p^i_j, \mathcal{O}(P, Q, \psi)(J)}$. In the other direction, if $x \in J_{x^i_j, q^i_j, p^i_j, \mathcal{O}(P, Q, \psi)(J)}$, then it follows from Lemma 3.24 that $\iota^R_R(x) = \pi^{\mathcal{T}}(\Delta(x))$ and so $\iota_R(x) - \pi(\Delta(x)) \in \mathcal{T}(J)$. It then follows from Lemma 3.21 that $\iota_R(x) = \mathcal{P}_{(0,0)}(\iota_R(x) - \pi(\Delta(x))) \in \iota_R(J)$, and since $\iota_R$ is injective, we have $x \in J$.

PROPOSITION 7.15. Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). Let $(I, J)$ be a pair of two-sided ideals of $R$ such that $I \subseteq J$, the ideal $I$ is $\psi$-invariant and
\( \varphi_I(J) \subseteq \Delta_I^{-1}(F_{I,P}(Q_I)). \) If we let

\[ I' = \{ x \in I \mid \forall m \in \mathbb{N} : \Delta_I^m(x)(Q_I^{\otimes m}) \subseteq Q_I^{\otimes m} \} \]

then \( I' = I' \cup \bigcup_{i \in I} j'_i \) and \( J = j'_i \cup \bigcup_{i \in I} j'_i \).

Proof. It is clear that we have

\[ I' = \sum_{i \in I} j'_i \cup \bigcup_{i \in I} j'_i \]

and the result then follows from Lemma 7.14.

7.2. Products and coproducts in \( \mathcal{C}_{(P,Q,\psi)} \)

We will show that if \( R \) is a ring and \( (P,Q,\psi) \) is an \( R \)-system, then \( \mathcal{C}_{(P,Q,\psi)} \) has products and coproducts, and we will, in the case where \( (P,Q,\psi) \) satisfies condition (FS), show how the product and coproduct are related to \( T \)-pairs of \( (P,Q,\psi) \).

Proposition 7.16. Let \( R \) be a ring, let \( (P,Q,\psi) \) be an \( R \)-system and let \( \{(S,\lambda,\tau,\varphi,\psi)\}_{\lambda \in \Lambda} \) be a family of surjective covariant representations of \( (P,Q,\psi) \).

Then the product of \( (S,\lambda,\tau,\varphi,\psi) \) in \( \mathcal{C}_{(P,Q,\psi)} \) exists; that is, there exists a surjective covariant representation

\[
(S_{\lambda \in \Lambda}(S,\lambda,\tau,\varphi,\psi), T_{\lambda \in \Lambda}(S,\lambda,\tau,\varphi,\psi)), \sigma_{\lambda \in \Lambda}(S,\lambda,\tau,\varphi,\psi), B_{\lambda \in \Lambda}(S,\lambda,\tau,\varphi,\psi))
\]

of \( (P,Q,\psi) \) and a family \( \{\phi_{\lambda}\}_{\lambda \in \Lambda} \) of ring homomorphisms \( \phi_{\lambda} : B_{\lambda \in \Lambda}(S,\lambda,\tau,\varphi,\psi) \to B_{\lambda} \) satisfying

\[
\phi_{\lambda} \circ \sigma_{\lambda}(S,\lambda,\tau,\varphi,\psi) = S_{\lambda}, \phi_{\lambda} \circ T_{\lambda}(S,\lambda,\tau,\varphi,\psi) = T_{\lambda}, \phi_{\lambda} \circ \sigma_{\lambda}(S,\lambda,\tau,\varphi,\psi) = \tau_{\lambda}, \phi_{\lambda} \circ \sigma_{\lambda}(S,\lambda,\tau,\varphi,\psi) = \sigma_{\lambda}
\]

for all \( \lambda \in \Lambda \), with the following properties:

1. If \( (S,\lambda,\tau,\varphi,\psi) \) is a surjective covariant representation of \( (P,Q,\psi) \) and for each \( \lambda \in \Lambda \) there exists a ring homomorphism \( \psi_{\lambda} : B_{\lambda} \to B_{\lambda} \) such that \( \psi_{\lambda} \circ \sigma_{\lambda} = \tau_{\lambda} \) and \( \psi_{\lambda} \circ \sigma_{\lambda} = \sigma_{\lambda} \) for all \( \lambda \in \Lambda \), then there exists a unique ring homomorphism \( \tau : B \to B_{\lambda \in \Lambda}(S,\lambda,\tau,\varphi,\psi) \) such that \( \tau \circ S_{\lambda} = S_{\lambda}, \psi_{\lambda} \circ T_{\lambda} = T_{\lambda}, \psi_{\lambda} \circ \sigma_{\lambda} = \sigma_{\lambda} \) for all \( \lambda \in \Lambda \).

2. \( \psi_{\lambda} \circ \sigma_{\lambda} = \psi_{\lambda} \circ \sigma_{\lambda} = \psi_{\lambda} \circ \sigma_{\lambda} = \psi_{\lambda} \circ \sigma_{\lambda} = \psi_{\lambda} \circ \sigma_{\lambda} = 0 \) if and only if \( \phi_{\lambda}(x) = 0 \) for all \( \lambda \in \Lambda \).

Proof. Let \( H = \cap_{\lambda \in \Lambda} \ker \eta_{\lambda}(S,\lambda,\tau,\varphi,\psi) \) where for each \( \lambda \in \Lambda \) the homomorphism

\[
\eta_{\lambda}(S,\lambda,\tau,\varphi,\psi) : T_{\lambda}(P,Q,\psi) \to B_{\lambda} \]

is the homomorphism given by Theorem 1.7. If the family \( \{(S,\lambda,\tau,\varphi,\psi)\}_{\lambda \in \Lambda} \) is empty, then we let \( H = T_{\lambda}(P,Q,\psi) \) and \( \Gamma_{H} : T_{\lambda}(P,Q,\psi) \to T_{\lambda}(P,Q,\psi) \) be the corresponding quotient map, and let \( S_{\lambda \in \Lambda}(S,\lambda,\tau,\varphi,\psi) = \eta_{\lambda} \circ \eta_{\lambda}, T_{\lambda}(S,\lambda,\tau,\varphi,\psi) = \eta_{\lambda} \circ \eta_{\lambda}, \sigma_{\lambda}(S,\lambda,\tau,\varphi,\psi) = \sigma_{\lambda} \circ \sigma_{\lambda}, B_{\lambda}(S,\lambda,\tau,\varphi,\psi) = B_{\lambda} \).

We then have that

\[
(S_{\lambda \in \Lambda}(S,\lambda,\tau,\varphi,\psi), T_{\lambda \in \Lambda}(S,\lambda,\tau,\varphi,\psi), \sigma_{\lambda \in \Lambda}(S,\lambda,\tau,\varphi,\psi), B_{\lambda \in \Lambda}(S,\lambda,\tau,\varphi,\psi))
\]
is a surjective covariant representation of \((P,Q,\psi)\). We also have that for each \(\lambda \in \Lambda\) there is a ring homomorphism \(\phi_\lambda : B_{\Pi_{\lambda \in A}(S,T_j,\sigma_j,B_j)} \to B_{\lambda}\) satisfying \(\phi_\lambda \circ \sigma_{\Pi_{\lambda \in A}(S,T_j,\sigma_j,B_j)} = S_\lambda\), \(\phi_\lambda \circ T_{\Pi_{\lambda \in A}(S,T_j,\sigma_j,B_j)} = T_\lambda\) and \(\phi_\lambda \circ \sigma_{\Pi_{\lambda \in A}(S,T_j,\sigma_j,B_j)} = \sigma_\lambda\), and we have that \(x \in B_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}\) is zero if and only if \(\phi_\lambda(x) = 0\) for all \(\lambda \in \Lambda\).

If \((T,S,\sigma,B)\) is a surjective covariant representation of \((P,Q,\psi)\) and for each \(\lambda \in \Lambda\) there exists a ring homomorphism \(\psi_\lambda : B \to B_{\lambda}\) such that \(\psi_\lambda \circ S = S_\lambda\), \(\psi_\lambda \circ T = T_\lambda\) and \(\psi_\lambda \circ \sigma = \sigma_\lambda\), then \(\ker \eta(S,T,B) \subseteq H\) where \(\eta(S,T,B) : T_{(P,Q,\psi)} \to B\) is the homomorphism given by Theorem 1.7, and it follows that there is a unique ring homomorphism \(\tau : B \to B_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}\) such that \(\tau \circ S = S_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}\), \(\tau \circ T = T_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}\) and \(\tau \circ \sigma = \sigma_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}\), and such that \(\phi_\lambda \circ \tau = \psi_\lambda\) for each \(\lambda \in \Lambda\). If there, in addition, is a ring homomorphism \(\varphi : B_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} \to B\) such that \(\varphi \circ S_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = S\), \(\varphi \circ T_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = T\) and \(\varphi \circ \sigma_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = \sigma\), then \(\tau\) is an inverse of \(\varphi\), and it follows that \(\varphi\) is an isomorphism and \(\psi_\lambda \circ \varphi = \phi_\lambda\) for each \(\lambda \in \Lambda\).

\[\text{PROPOSITION 7.17.} \text{Let } R \text{ be a ring, } (P,Q,\psi) \text{ be an } R\text{-system and let } ((S_\lambda,T_\lambda,\sigma_\lambda,B_\lambda))_{\lambda \in \Lambda} \text{ be a family of surjective covariant representations of } (P,Q,\psi). \]

Then the coproduct of \(((S_\lambda,T_\lambda,\sigma_\lambda,B_\lambda))_{\lambda \in \Lambda}\) in \(C(P,Q,\psi)\) exists; that is, there exists a surjective covariant representation

\[
(S_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}, T_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}, \sigma_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}, B_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)})
\]

of \((P,Q,\psi)\) and a family \((\phi_\lambda)_{\lambda \in \Lambda}\) of ring homomorphisms \(\phi_\lambda : B_\lambda \to B_{\Pi_{\lambda \in A}(S,T_j,\sigma_j,B_j)}\) satisfying \(\phi_\lambda \circ S_\lambda = S_{\Pi_{\lambda \in A}(S,T_j,\sigma_j,B_j)}\), \(\phi_\lambda \circ T_\lambda = T_{\Pi_{\lambda \in A}(S,T_j,\sigma_j,B_j)}\) and \(\phi_\lambda \circ \sigma_\lambda = \sigma_{\Pi_{\lambda \in A}(S,T_j,\sigma_j,B_j)}\) for all \(\lambda \in \Lambda\), with the following property:

If \((S,T,\sigma,B)\) is a surjective covariant representation of \((P,Q,\psi)\) and \(\psi_\lambda : B_\lambda \to B\) such that \(\psi_\lambda \circ S_\lambda = S\), 

\[\psi_\lambda \circ T_\lambda = T \text{ and } \psi_\lambda \circ \sigma_\lambda = \sigma, \text{ then there exists a unique ring homomorphism } \tau : B_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} \to B \text{ such that } \tau \circ S_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = S, \tau \circ T_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = T \text{ and } \tau \circ \sigma_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = \sigma, \text{ and such that } \tau \circ \phi_\lambda = \psi_\lambda \text{ for each } \lambda \in \Lambda.\]

We furthermore have that the pair consisting of the surjective covariant representation

\[
(S_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}, T_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}, \sigma_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}, B_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)})
\]

and the family \((\phi_\lambda)_{\lambda \in \Lambda}\) is, up to isomorphism, the unique pair which possesses property 7.17; in fact if \((S,T,\sigma,B)\) is a surjective covariant representation of \((P,Q,\psi)\) and \((\psi_\lambda)_{\lambda \in \Lambda}\) is a family of ring homomorphisms \(\psi_\lambda : B_\lambda \to B\) satisfying \(\psi_\lambda \circ S_\lambda = S, \psi_\lambda \circ T_\lambda = T \text{ and } \psi_\lambda \circ \sigma_\lambda = \sigma\) for each \(\lambda \in \Lambda\), and \(\varphi : B \to B_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}\) is a ring homomorphism such that \(\varphi \circ S_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = S, \varphi \circ T_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = T \text{ and } \varphi \circ \sigma_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = \sigma,\) then \(\varphi\) is an isomorphism and \(\psi_\lambda \circ \varphi = \phi_\lambda\) for each \(\lambda \in \Lambda\).

Moreover, if each \((S_\lambda,T_\lambda,\sigma_\lambda,B_\lambda)\) is graded, then the surjective covariant representation \((S_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}, T_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}, \sigma_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}, B_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)})\) is also graded.

\[\text{Proof.}\] Let \(H\) be the smallest two-sided ideal of \(T_{(P,Q,\psi)}\) which contains \(\bigcup_{\lambda \in \Lambda} \ker \eta(S_\lambda,T_\lambda,\sigma_\lambda,B_\lambda)\), where for each \(\lambda \in \Lambda\) the homomorphism \(\eta(S_\lambda,T_\lambda,\sigma_\lambda,B_\lambda) : T_{(P,Q,\psi)} \to B_\lambda\) is the homomorphism given by Theorem 1.7. Let \(\varphi_H : T_{(P,Q,\psi)} \to T_{(P,Q,\psi)}/H\) be the corresponding quotient map, and let \(\tilde{S}_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = \varphi_H \circ t_P, \tilde{\sigma}_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = \varphi_H \circ t_Q, \tilde{\sigma}_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = \varphi_H \circ t_R\) and \(\tilde{B}_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)} = T_{(P,Q,\psi)}/H\). We then have that

\[
(S_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}, T_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}, \sigma_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)}, B_{\Pi_{\lambda \in A}(S,T_\lambda,\sigma_\lambda,B_\lambda)})
\]
is a surjective covariant representation of \((P, Q, \psi)\). We also have that for each \(\lambda \in \Lambda\) there is a ring homomorphism \(\phi_\lambda : B_\lambda \to B_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)}\) satisfying \(\phi_\lambda \circ S_\lambda = S_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)}\), \(\phi_\lambda \circ T_\lambda = T_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)}\) and \(\phi_\lambda \circ \sigma_\lambda = \sigma_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)}\).

If \((S_\lambda, T_\lambda, \sigma_\lambda, B_\lambda)\) is graded, then \(\eta(S_\lambda, T_\lambda, \sigma_\lambda, B_\lambda)\) is a graded two-sided ideal of \(T(P, Q, \psi)\). It easily follows that if each \((S_\lambda, T_\lambda, \sigma_\lambda, B_\lambda)\) is graded, then \(H\) is a graded two-sided ideal of \(T(P, Q, \psi)\), and thus that

\[
(S_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)}, T_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)}, \sigma_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)}, B_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)})
\]

is also graded.

If \((S, T, \sigma, B)\) is a surjective covariant representation of \((P, Q, \psi)\) and for each \(\lambda \in \Lambda\) there exists a ring homomorphism \(\psi_\lambda : B_\lambda \to B\) such that \(\psi_\lambda \circ S_\lambda = S, \psi_\lambda \circ T_\lambda = T\) and \(\psi_\lambda \circ \sigma_\lambda = \sigma\), then \(H \subseteq \ker(\eta(S, T, \sigma, B))\) where \(\eta(S, T, \sigma, B) : T(P, Q, \psi) \to B\) is the homomorphism given by Theorem 1.7, and it follows that there is a unique ring homomorphism \(\tau : B_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)} \to B\) such that \(\tau \circ S_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)} = S, \tau \circ T_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)} = T\) and \(\tau \circ \sigma_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)} = \sigma\), and such that \(\tau \circ \phi_\lambda = \psi_\lambda\) for each \(\lambda \in \Lambda\). In addition, there is a ring homomorphism \(\varphi : B \to B_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)}\) such that \(\varphi \circ S = S_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)}\), \(\varphi \circ T = T_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)}\) and \(\varphi \circ \sigma = \sigma_{\Pi_{\lambda \in \Lambda}(S, T_\lambda, \sigma_\lambda, B_\lambda)}\), then \(\tau\) is an inverse of \(\varphi\), and it follows that \(\varphi\) is an isomorphism and \(\varphi \circ \psi_\lambda = \phi_\lambda\) for each \(\lambda \in \Lambda\).

**Lemma 7.18.** Let \(R\) be a ring and let \((P, Q, \psi)\) be an \(R\)-system. If \((S_1, T_1, \sigma_1, B_1)\) and \((S_2, T_2, \sigma_2, B_2)\) are two covariant representations of \((P, Q, \psi)\) and \(\phi : B_1 \to B_2\) is a ring homomorphism satisfying \(\phi \circ T_1 = T_2, \phi \circ S_1 = S_2\) and \(\phi \circ \sigma_1 = \sigma_2\), then the following holds:

(i) If \((S_2, T_2, \sigma_2, B_2)\) is injective, then so is \((S_1, T_1, \sigma_1, B_1)\).

(ii) If \(\phi\) is surjective and \((S_2, T_2, \sigma_2, B_2)\) is surjective and graded, then so is \((S_1, T_1, \sigma_1, B_1)\).

**Proof.** That (i) holds is obvious. By Proposition 3.2 that \(\oplus_{n \in \mathbb{Z}}\eta(S_1, T_1, \sigma_1, B_1)(T^{(n)}_{(P, Q, \psi)})\) is a grading of \(B_2\). It follows that \(\oplus_{n \in \mathbb{Z}}\eta(S_1, T_1, \sigma_1, B_1)(T^{(n)}_{(P, Q, \psi)})\) is a grading of \(B_1\), and thus that \((S_1, T_1, \sigma_1, B_1)\) is graded.

**Proposition 7.19.** Let \(R\) be a ring, let \((P, Q, \psi)\) be an \(R\)-system satisfying condition (FS) and let \(\Omega = (\omega_\lambda)_{\lambda \in \Lambda} = ((I_\lambda, J_\lambda))_{\lambda \in \Lambda}\) be a non-empty family of \(P\)-tuples of \((P, Q, \psi)\). For each \(\lambda \in \Lambda\) denote by \(\Gamma_\lambda\) the covariant representation \((\iota_P^{\omega_\lambda}, \iota_Q^{\omega_\lambda}, \iota_R^{\omega_\lambda}, O_{(\iota_P, \iota_Q, \iota_R, \psi)}((J_\lambda)_{I_\lambda}))\). Then we have:

(i) If we let \(I_{\Pi_\Omega} = \cap_{\lambda \in \Lambda} I_\lambda\) and \(J_{\Pi_\Omega} = \cap_{\lambda \in \Lambda} J_\lambda\), then the pair \(\omega_{\Pi_\Omega} = (I_{\Pi_\Omega}, J_{\Pi_\Omega})\) is a \(T\)-pair of \((P, Q, \psi)\), and the covariant representation

\[
(S_{\Pi_{\lambda \in \Lambda} \Gamma_\lambda}, T_{\Pi_{\lambda \in \Lambda} \Gamma_\lambda}, \sigma_{\Pi_{\lambda \in \Lambda} \Gamma_\lambda}, B_{\Pi_{\lambda \in \Lambda} \Gamma_\lambda})
\]

is surjective and graded, and isomorphic to

\[
((\iota_P^{\Pi_\Omega}, \iota_Q^{\Pi_\Omega}, \iota_R^{\Pi_\Omega}, O_{(\iota_P, \iota_Q, \iota_R, \psi)}((J_{\Pi_\Omega})_{I_{\Pi_\Omega}})) \).
\]

(ii) If we let \(I\) be the smallest two-sided ideal of \(R\) containing \(\cup_{\lambda \in \Lambda} I_\lambda\), \(J_{\Pi_\Omega}\) be the smallest two-sided ideal of \(R\) containing \(\cup_{\lambda \in \Lambda} J_\lambda\) and

\[
I_{\Pi_\Omega} = \{ x \in J_{\Pi_\Omega} | \forall m \in \mathbb{N} : \Delta^m_{\Pi_\Omega}(x)(Q_1^{\leq m}) \subseteq Q_1^{\leq m}(J_{\Pi_\Omega})_I \land \exists n \in \mathbb{N} : \Delta^m_{\Pi_\Omega}(x) = 0 \},
\]

then the pair \(\omega_{\Pi_\Omega} = (I_{\Pi_\Omega}, J_{\Pi_\Omega})\) is a \(T\)-pair of \((P, Q, \psi)\), and the covariant representation

\[
(S_{\Pi_{\lambda \in \Lambda} \Gamma_\lambda}, T_{\Pi_{\lambda \in \Lambda} \Gamma_\lambda}, \sigma_{\Pi_{\lambda \in \Lambda} \Gamma_\lambda}, B_{\Pi_{\lambda \in \Lambda} \Gamma_\lambda})
\]
is surjective and graded, and isomorphic to

\[
\left( i_P^{\omega_{\Pi\Omega}}, i_Q^{\omega_{\Pi\Omega}}, i_R^{\omega_{\Pi\Omega}}, \mathcal{O}_{(\Pi\Omega, P, Q, \psi_{\Pi\Omega})}\left((J_{\Pi\Omega})_{\Pi\Omega}\right) \right).
\]

**Proof.** (i) It follows from Lemma 7.18 that the surjective covariant representation

\[
\left( S_{\Pi\lambda, \Gamma_{\lambda}} \bigoplus \Sigma_{\Pi\lambda, \Gamma_{\lambda}}, B_{\Pi\lambda, \Gamma_{\lambda}} \right)
\]

is graded. It therefore follows from Propositions 7.8 and Theorem 7.11 that

\[
\left( S_{\Pi\lambda, \Gamma_{\lambda}} \bigoplus \Sigma_{\Pi\lambda, \Gamma_{\lambda}}, B_{\Pi\lambda, \Gamma_{\lambda}} \right)
\]

is isomorphic to \((i_{\Pi\lambda}, \psi_{\Pi\lambda}, \psi_{\Pi\lambda}, \mathcal{O}_{(i_{\Pi\lambda}, P, Q, \psi_{\Pi\lambda})}(J_{\Pi\lambda}))\) for some \(T\)-pair \(\omega = (I, J)\) of \((P, Q, \psi)\). It follows from Lemma 3.24 and Proposition 7.16 that we have

\[
x \in I \iff \Sigma_{\Pi\lambda, \Gamma_{\lambda}}(x) = 0 \iff \forall \lambda \in \Lambda : i_{\Pi\lambda}(x) = 0 \iff x \in \bigcap_{\lambda \in \Lambda} I_{\lambda} = I_{\Pi\Omega}
\]

and

\[
x \in J \iff \Sigma_{\Pi\lambda, \Gamma_{\lambda}}(x) = \pi_{\Pi\lambda, \Gamma_{\lambda}} S_{\Pi\lambda, \Gamma_{\lambda}} \Gamma_{\lambda}(\Delta(x)) \iff \forall \lambda \in \Lambda : i_{\Pi\lambda}(x) = \pi_{\Pi\lambda, \Gamma_{\lambda}}(\Delta(x)) \iff x \in \bigcap_{\lambda \in \Lambda} J_{\lambda} = J_{\Pi\Omega}
\]

from which (i) follows.

(ii) It follows from Proposition 7.17 that the representation

\[
\left( S_{\Pi\lambda, \Gamma_{\lambda}} \bigoplus \Sigma_{\Pi\lambda, \Gamma_{\lambda}}, B_{\Pi\lambda, \Gamma_{\lambda}} \right)
\]

is surjective and graded.

It is easy to check that \(I \subseteq J_{\Pi\Omega}\), that \(I\) is \(\psi\)-invariant and that \(\psi_I(J_{\Pi\Omega}) \subseteq \Delta^{-1}(\mathcal{F}_P(Q_I))\). It therefore follows from Propositions 7.8 and 7.15 that \((I_{\Pi\Omega}, J_{\Pi\Omega})\) is a \(T\)-pair of \((P, Q, \psi)\).

We have, for each \(\lambda \in \Lambda\), that \(I_{\lambda} \subseteq I_{\Pi\Omega}\), and \(J_{\lambda} \subseteq J_{\Pi\Omega}\), so it follows from Proposition 7.9 and Theorem 7.11 (ii) that there exists a ring homomorphism

\[
\psi : \mathcal{O}(i_{\Pi\Omega}, P, Q_{\Pi\lambda}, \psi_{\Pi\lambda})(I_{\Pi\lambda}) \rightarrow \mathcal{O}(i_{\Pi\Omega}, P, Q_{\Pi\lambda}, \psi_{\Pi\lambda})(I_{\Pi\Omega})
\]

such that \(\psi \circ i_{\Pi\lambda}^{\psi_{\Pi\lambda}} = i_{\Pi\lambda}^{\psi_{\Pi\lambda}}\), \(\psi \circ i_{\Pi\lambda}^{\psi_{\Pi\lambda}} = i_{\Pi\lambda}^{\psi_{\Pi\lambda}}\), and \(\psi \circ i_{\Pi\lambda}^{\psi_{\Pi\lambda}} = i_{\Pi\lambda}^{\psi_{\Pi\lambda}}\).

We will show that there exists a ring homomorphism

\[
\phi : \mathcal{O}(i_{\Pi\Omega}, P, Q_{\Pi\lambda}, \psi_{\Pi\lambda})(I_{\Pi\Omega}) \rightarrow B_{\Pi\lambda, \Gamma_{\lambda}}
\]

such that \(\phi \circ i_{\Pi\lambda}^{\psi_{\Pi\lambda}} = \Sigma_{\Pi\lambda, \Gamma_{\lambda}} \Gamma_{\lambda}, \phi \circ i_{\Pi\lambda}^{\psi_{\Pi\lambda}} = T_{\Pi\lambda, \Gamma_{\lambda}} \Gamma_{\lambda}\), and \(\phi \circ i_{\Pi\lambda}^{\psi_{\Pi\lambda}} = S_{\Pi\lambda, \Gamma_{\lambda}} \Gamma_{\lambda}\). It will then follow from Proposition 7.17 that the two representations

\[
\left( S_{\Pi\lambda, \Gamma_{\lambda}} \bigoplus \Sigma_{\Pi\lambda, \Gamma_{\lambda}}, B_{\Pi\lambda, \Gamma_{\lambda}} \right)
\]

and

\[
\left( i_P^{\psi_{\Pi\lambda}}, i_Q^{\psi_{\Pi\lambda}}, i_R^{\psi_{\Pi\lambda}}, \mathcal{O}(i_{\Pi\lambda}, P, Q_{\Pi\lambda}, \psi_{\Pi\lambda})(I_{\Pi\lambda}) \right)
\]

are isomorphic.

We have for each \(\lambda \in \Lambda\) that there is a ring homomorphism \(\phi : \mathcal{O}(i_{\lambda}, P, Q_{\lambda}, \psi_{\lambda})(I_{\lambda}) \rightarrow B_{\Pi\lambda, \Gamma_{\lambda}}\) such that \(\phi \circ i_{\lambda}^{\psi_{\lambda}} = \Sigma_{\Pi\lambda, \Gamma_{\lambda}} \Gamma_{\lambda}, \phi \circ i_{\lambda}^{\psi_{\lambda}} = T_{\Pi\lambda, \Gamma_{\lambda}} \Gamma_{\lambda}\), and \(\phi \circ i_{\lambda}^{\psi_{\lambda}} = S_{\Pi\lambda, \Gamma_{\lambda}} \Gamma_{\lambda}\). It follows from Theorem 7.11 that we have

\[
I_{\lambda} \subseteq I_{\Pi\lambda, \Gamma_{\lambda}} \bigoplus \Sigma_{\Pi\lambda, \Gamma_{\lambda}} \Gamma_{\lambda}, B_{\Pi\lambda, \Gamma_{\lambda}}
\]

and

\[
J_{\lambda} \subseteq J_{\Pi\lambda, \Gamma_{\lambda}} \bigoplus \Sigma_{\Pi\lambda, \Gamma_{\lambda}} \Gamma_{\lambda}, B_{\Pi\lambda, \Gamma_{\lambda}}.
\]
We therefore have that

$$I \subseteq I(S_{\|A \rightarrow \Lambda} \varphi, T_{\|A \rightarrow \Lambda} \varphi, \sigma_{\|A \rightarrow \Lambda} \varphi, B_{\|A \rightarrow \Lambda} \varphi)$$

and

$$J_{J[I]} \subseteq J(S_{\|A \rightarrow \Lambda} \varphi, T_{\|A \rightarrow \Lambda} \varphi, \sigma_{\|A \rightarrow \Lambda} \varphi, B_{\|A \rightarrow \Lambda} \varphi).$$

(7.5)

It then follows from Lemma 7.10 that there exists a covariant representation $(S, T, \sigma, B_{\|A \rightarrow \Lambda} \varphi)$ of $(\nu P, \nu Q, \psi J)$ such that $S \circ \nu J = S_{\|A \rightarrow \Lambda} \varphi$, $T \circ \nu J = T_{\|A \rightarrow \Lambda} \varphi$ and $\sigma \circ \nu J = \sigma_{\|A \rightarrow \Lambda} \varphi$. It follows from equation (7.5) that this representation is Cuntz–Pimsner invariant relative to $(J[I])_I$, and it then follows from Theorem 3.18 that there is a ring homomorphism $\eta : O((\nu P, \nu Q, \psi J)) \rightarrow B_{\|A \rightarrow \Lambda} \varphi$ such that $\eta \circ \nu J = \sigma = \eta \circ \nu Q = T$ and $\eta \circ \nu P = S$. It follows from Proposition 7.15 that the two representations

$$(\nu J[I], \nu P, \nu Q, \psi J, \nu I) \rightarrow B_{\|A \rightarrow \Lambda} \varphi$$

are isomorphic, and it follows that there exists a ring homomorphism

$$\phi : O((\nu P, \nu Q, \psi J, \nu I)) \rightarrow B_{\|A \rightarrow \Lambda} \varphi$$

such that $\phi \circ \nu J[I] = \sigma_{\|A \rightarrow \Lambda} \varphi$, $\phi \circ \nu P = T_{\|A \rightarrow \Lambda} \varphi$ and $\phi \circ \nu P[I] = S_{\|A \rightarrow \Lambda} \varphi$. \(\square\)

**Remark 7.20.** Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system and let $((S_{\|A \rightarrow \Lambda} \varphi, T_{\|A \rightarrow \Lambda} \varphi, \sigma_{\|A \rightarrow \Lambda} \varphi, B_{\|A \rightarrow \Lambda} \varphi))$ be a family of injective and surjective covariant representations of $(P, Q, \psi)$. Then the product

$$(S_{\|A \rightarrow \Lambda} \varphi(S_{\|A \rightarrow \Lambda} \varphi, T_{\|A \rightarrow \Lambda} \varphi, \sigma_{\|A \rightarrow \Lambda} \varphi, B_{\|A \rightarrow \Lambda} \varphi))$$

is injective and surjective, but the coproduct

$$\left(S_{\|A \rightarrow \Lambda} \varphi(S_{\|A \rightarrow \Lambda} \varphi, T_{\|A \rightarrow \Lambda} \varphi, \sigma_{\|A \rightarrow \Lambda} \varphi, B_{\|A \rightarrow \Lambda} \varphi))ight)$$

is not necessarily injective. Example 4.11 gives us an example of this phenomenon.

### 7.3. Graded ideals of $O(P, Q, \psi)(J)$

Let $R$ be a ring and $(P, Q, \psi)$ an $R$-system satisfying condition $(FS)$. We will now show how the classification of surjective and graded representations of $(P, Q, \psi)$ can be used to describe the graded two-sided ideals of $O(P, Q, \psi)(J)$ for any faithful $\psi$-compatible two-sided ideal $J$ of $R$, and in particular of $T(P, Q, \psi)$ and $O(P, Q, \psi)$ (if it exists).

**Definition 7.21.** Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition $(FS)$ and let $K$ be a faithful $\psi$-compatible two-sided ideal of $R$. For a two-sided ideal $H$ of $O(P, Q, \psi)(K)$ we define two two-sided ideals $I_H^K$ and $J_H^K$ of $R$ by

$$I_H^K := \{x \in R \mid \nu J[I]_H^K(x) \in H\} \quad \text{and} \quad J_H^K := \{x \in R \mid \nu J[I]_H^K(x) \in H + F_P(Q)\}.$$  

We set $\omega_H^K = (I_H^K, J_H^K)$.

**Proposition 7.22.** Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition $(FS)$ and let $K$ be a faithful $\psi$-compatible two-sided ideal of $R$. For a two-sided ideal $H$ of $O(P, Q, \psi)(K)$, denote by $\nu J[I]_H^K \psi$ the projection from $O(P, Q, \psi)(K)$ to $O(P, Q, \psi)(K)/H$. If we consider the covariant representation

$$(S_H, T_H, \sigma_H, O(P, Q, \psi)(K)/H) := (\nu J[I]_H^K, \nu J[I]_H^K, \nu J[I]_H^K, O(P, Q, \psi)(K)/H),$$
then we have that \( \omega^K_H = \omega(\sigma_H, \pi^K_H) \). Hence \( \omega^K_H \) is a T-pair satisfying \( K \subseteq J^K_H \).

We furthermore have that the representation \((\sigma_H, \pi^K_H) \) is graded if and only if \( H \) is graded.

**Proof.** By using that \( \varphi_H \circ \iota^K_R = \sigma_H \) and \( \varphi_H \circ \pi^K = \pi_T \), it is straightforward to check that \( I^K_H = I(\sigma_H, \pi^K_H, O_{(P,Q)}(K)/H) \) and \( J^K_H = J(\sigma_H, \pi^K_H, O_{(P,Q)}(K)/H) \), and thus that \( \omega^K_H = \omega(\sigma_H, \pi^K_H, O_{(P,Q)}(K)/H) \). It is also easy to check that \( K \subseteq J^K_H \). That \( \omega(\sigma_H, \pi^K_H, O_{(P,Q)}(K)/H) \), and thus \( \omega^K_H \), is a T-pair follows from Proposition 7.8.

Assume that \( H \) is graded. If \( x = \sum_{i=1}^m x^n_i \in H \) where each \( x^n_i \in \varphi_K(T^{(n)}_{(P,Q)}) \), then each \( x^n_i \in H \). This shows that \( \oplus_{n \in Z} \varphi_K(T^{(n)}_{(P,Q)}) \) is a grading of \( O_{(P,Q)}(K)/H \), and it follows that \( (\sigma_H, \pi^K_H, O_{(P,Q)}(K)/H) \) is graded.

If \( (\sigma_H, \pi^K_H, O_{(P,Q)}(K)/H) \) is graded and \( x = \sum_{i=1}^m x^n_i \in H \), where each \( x^n_i \in \varphi_K(T^{(n)}_{(P,Q)}) \), then each \( \varphi_H(x^{(n)}) = 0 \) which shows that \( H = \oplus_{n \in Z} \varphi_K(T^{(n)}_{(P,Q)}) \cap H \), and thus that \( H \) is graded.

**Lemma 7.23.** Let \( R \) be a ring, let \((P,Q,\psi)\) be an R-system satisfying condition (FS) and let \( K \) be a faithful \( \psi \)-compatible two-sided ideal of \( R \). If \( \omega = (I,J) \) is a T-pair such that \( K \subseteq J \), then there exists a unique surjective and graded ring homomorphism \( \Psi^K_\omega : O_{(P,Q)}(K) \to O_{(I,P,Q)}(J) \) such that \( \Psi^K_\omega \circ \iota^K_R = \psi^K_R, \Psi^K_\omega \circ \iota^K_Q = \psi^K_Q \) and \( \Psi^K_\omega \circ \iota^K_H = \iota^K_T \).

**Definition 7.24.** Let \( R \) be a ring, let \((P,Q,\psi)\) be an R-system satisfying condition (FS) and let \( K \) be a faithful \( \psi \)-compatible two-sided ideal of \( R \). Given a T-pair \( \omega = (I,J) \) such that \( K \subseteq J \), we define \( H^K_\omega \) to be the two-sided ideal ker \( \Psi^K_\omega \) of \( R \) where \( \Psi^K_\omega \) is as in Lemma 7.23.

**Lemma 7.25.** Let \( R \) be a ring, let \((P,Q,\psi)\) be an R-system satisfying condition (FS) and let \( K \) be a faithful \( \psi \)-compatible two-sided ideal of \( R \). If \( \omega = (I,J) \) is a T-pair such that \( K \subseteq J \), then \( H^K_\omega \) is a graded two-sided ideal of \( O_{(P,Q)}(K) \) and \( \omega_H^K = \omega \).

**Proof.** Let \( \Psi^K_\omega \) be the homomorphism from Lemma 7.23. That \( H^K_\omega \) is a graded two-sided ideal follows from the fact that \( \Psi^K_\omega \) is graded.

To show \( \omega_H^K = \omega \) we have to show that

\[ I = (\iota^K_R)^{-1}(\ker \Psi^K_\omega), \]

and that

\[ J = (\iota^K_R)^{-1}(\ker \Psi^K_\omega + \pi^K(F_P(Q))). \]

If \( x \in I \), then \( \Psi^K_\omega(\iota^K_R(x)) = \iota^K_R(\varphi_I(x)) = 0 \). Thus \( I \subseteq (\iota^K_R)^{-1}(\ker \Psi^K_\omega) \). If \( x \in R \) and \( \Psi^K_\omega(\iota^K_R(x)) = 0 \), then \( \iota^K_R(\varphi_I(x)) = 0 \), and since \( \iota^K_R \) is injective, it follows that \( x \in \ker \varphi_I = I \). Thus \( I = (\iota^K_R)^{-1}(\ker \Psi^K_\omega) \).

Let \( x \in J \). Then \( \varphi_I(x) \in J_I \), so we have

\[ \Psi^K_\omega(\iota^K_R(x)) = \iota^K_R(\varphi_I(x)) = \pi^K(\Delta_I(\varphi_I(x))). \]
Thus there exist $q_1, q_2, \ldots, q_n \in Q$ and $p_1, p_2, \ldots, p_n \in P$ such that
\[
\Psi^K_K(i_R^K(x)) = \sum_{i=1}^n i_{Q_i}^J(j_{Q_i}(q_i))i_J^P(p_i).
\]
We then have that $i_R^K(x) - \sum_{i=1}^n i_{Q_i}^K(q_i)i_J^P(p_i) \in \ker \Psi^K_K$, which shows that $J \subseteq (i_R^K)^{-1}(\ker \Psi^K_K + \pi^K(F_P(Q)))$.

Let $\sigma_\omega := i_{Q_1}^J \circ \varphi_I$, $T_\omega := i_{Q_1}^J \circ \varphi_I$ and $S_\omega := i_{J_1}^P \circ \varphi_I$. It follows from Proposition 7.9 that $\sigma_\omega^{-1}(\pi_{\tau_\omega,S_\omega}(F_P(Q))) = J$. If $x \in N$, $y \in \ker \Psi^K_K$, $q_1, q_2, \ldots, q_n \in Q$, $p_1, p_2, \ldots, p_n \in P$ and $i_R^K(x) = y + \sum_{i=1}^n i_{Q_i}^K(q_i)i_J^P(p_i)$, then
\[
\sigma_\omega(x) = i_{Q_1}^J(j_{Q_1}(q_I))(i_R^K(x)) = \Psi^K_K(i_R^K(x)) = \Psi^K_K\left(\sum_{i=1}^n i_{Q_i}^K(q_i)i_J^P(p_i)\right)
\]
\[
= \sum_{i=1}^n i_{Q_i}^J(j_{Q_i}(q_i))i_J^P(p_i) = \pi_{\tau_\omega,S_\omega}\left(\sum_{i=1}^n (\theta_{\omega,p_i})\right) \in \pi_{\tau_\omega,S_\omega}(F_P(Q)),
\]
so $x \in J$. Thus $J = (i_R^K)^{-1}(\ker \Psi^K_K + \pi^K(F_P(Q)))$. \hfill \square

**Proposition 7.26.** Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS) and let $K$ be a faithful $\psi$-compatible two-sided ideal of $R$. Let $H$ be a two-sided ideal of $O_{(P, Q, \psi)}(K)$ and let $\omega = (I, J)$ be a $T$-pair of $(P, Q, \psi)$. Let $\varphi_H$ denote the quotient map from $O_{(P, Q, \psi)}(K)$ to $O_{(P, Q, \psi)}(K)/H$. Then we have:

(i) If there exists a ring homomorphism $\Upsilon : O_{(I, P, Q, \psi)}(J_I) \to O_{(P, Q, \psi)}(K)/H$ such that $\Upsilon \circ i_{K_R} = \varphi_H \circ i_{K_R}$, $\Upsilon \circ i_{K_Q} = \varphi_H \circ i_{K_Q}$ and $\Upsilon \circ i_{K_P} = \varphi_H \circ i_{K_P}$, then $I \subseteq I^K_H$ and $J \subseteq J^K_H$.

(ii) If $I \subseteq I^K_H$ and $J \subseteq J^K_H$, then there exists a unique ring homomorphism $\Upsilon : O_{(I, P, Q, \psi)}(J_I) \to O_{(P, Q, \psi)}(K)/H$ such that $\Upsilon \circ i_{K_R} = \varphi_H \circ i_{K_R}$, $\Upsilon \circ i_{K_Q} = \varphi_H \circ i_{K_Q}$ and $\Upsilon \circ i_{K_P} = \varphi_H \circ i_{K_P}$.

(iii) If $I \subseteq I^K_H$ and $J \subseteq J^K_H$, then the ring homomorphism $\Upsilon$ is an isomorphism if and only if $H$ is graded and $\omega^K_H = \omega$.

**Proof.** (i) Assume that there exists a ring homomorphism $\Upsilon : O_{(I, P, Q, \psi)}(J_I) \to O_{(P, Q, \psi)}(K)/H$ such that $\Upsilon \circ i_{K_R} = \varphi_H \circ i_{K_R}$, $\Upsilon \circ i_{K_Q} = \varphi_H \circ i_{K_Q}$ and $\Upsilon \circ i_{K_P} = \varphi_H \circ i_{K_P}$. If $x \in I$, then it follows from Proposition 7.9 that $\varphi_H(i_R^K(x)) = \Upsilon(i_R^K(x)) = \Upsilon(\pi_{\Delta}(\Delta(x))) = \varphi_H(\pi^K(\Delta(x)))$, so $x \in J^K_H$.

(ii) Assume that $I \subseteq I^K_H$ and $J \subseteq J^K_H$. Let $(S_H, T_H, \sigma_H, O_{(P, Q, \psi)}(K)/H)$ be as in Proposition 7.22. Then we have $(I, J) \subseteq \omega^K_H = \omega(S_H, T_H, \sigma_H)$, so the existence and uniqueness of $\Upsilon$ follows from Theorem 7.11.

(iii) It also follows from Theorem 7.11 that $\Upsilon$ is an isomorphism if and only if the representation $(S_H, T_H, \sigma_H, O_{(P, Q, \psi)}(K)/H)$ is surjective and graded and $\omega = \omega(S_H, T_H, \sigma_H) = \omega^K_H$. The representation $(S_H, T_H, \sigma_H, O_{(P, Q, \psi)}(K)/H)$ is always surjective, and it follows from Proposition 7.22 that it is graded if and only if $H$ is graded, and the desired result follows.

**Theorem 7.27.** Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system satisfying condition (FS). Let $K$ be a faithful $\psi$-compatible two-sided ideal of $R$. Then
\[
H \mapsto \omega^K_H, \quad \omega \mapsto H^K_w
\]
is a bijective correspondence between the set of all the graded two-sided ideals \( H \) of \( \mathcal{O}(P,Q,\psi)(K) \) and the set of all \( T \)-pairs \( \omega = (I,J) \) of \( (P,Q,\psi) \) satisfying \( K \subseteq J \). This bijection preserves inclusion, and if \( (H_\lambda)_{\lambda \in \Lambda} \) is a non-empty family of graded two-sided ideals of \( \mathcal{O}(P,Q,\psi)(K) \) and \( \Omega = (\omega^K_{H_\lambda})_{\lambda \in \Lambda} \), then \( H^K_{\omega^K_{\Pi \Omega}} = \cap_{\lambda \in \Lambda} H_\lambda \) and \( H^K_{\omega^K_{\Pi \Omega}} \) is the smallest two-sided ideal of \( \mathcal{O}(P,Q,\psi)(K) \) containing \( \cup_{\lambda \in \Lambda} H_\lambda \).

**Proof.** If \( \omega = (I,J) \) is a \( T \)-pair of \( (P,Q,\psi) \) satisfying \( K \subseteq J \), then it follows from Lemma 7.25 that \( H^K_\omega \) is a graded two-sided ideal of \( \mathcal{O}(P,Q,\psi)(K) \), and that \( \omega H^K_\omega = \omega \).

If \( H \) is a graded two-sided ideal of \( \mathcal{O}(P,Q,\psi)(K) \), then it follows from Proposition 7.22 that \( \omega^K_H \) is a \( T \)-pair of \( (P,Q,\psi) \) satisfying \( K \subseteq J^K_H \). Let \( \Psi^K_H \) be the unique ring homomorphism \( \Psi^K_H : \mathcal{O}(P,Q,\psi)(K) \to \mathcal{O}(P,Q,\psi)(J^K_H) \) satisfying \( \Psi^K_H \circ \iota^K_H = \iota^K_{J}, \Psi^K_H \circ \iota^K_Q = \iota^K_{J} \) and \( \Psi^K_H \circ \iota^K_P = \iota^K_{J} \). Let \( (I,J) = \omega = \omega^K_H \). Then it follows from Proposition 7.26 that there is a ring isomorphism \( \Upsilon : \mathcal{O}(P,Q,\psi)(J^K_H) \to \mathcal{O}(P,Q,\psi)(K)/H \) such that \( \Upsilon \circ \iota^K_H = \varphi_H \circ \iota^K_{J}, \Upsilon \circ \iota^K_Q = \varphi_H \circ \iota^K_{J} \) and \( \Upsilon \circ \iota^K_P = \varphi_H \circ \iota^K_{J} \). We then have that \( \Upsilon \circ \Psi^K_H \) is the quotient map from \( \mathcal{O}(P,Q,\psi)(K) \) to \( \mathcal{O}(P,Q,\psi)(K)/H \), and it follows that \( H^K_{\omega^K_H} = \ker \Psi_H = H \).

Thus \( \omega^K_{\omega^K_H} \) and \( \omega \mapsto H^K_{\omega} \) is a bijective correspondence between the set of all the graded two-sided ideals of \( \mathcal{O}(P,Q,\psi)(K) \) and the set of all the \( T \)-pairs \( \omega = (I,J) \) of \( (P,Q,\psi) \) satisfying \( K \subseteq J \). It is easy to check that the correspondence preserves inclusion.

Let \( (H_\lambda)_{\lambda \in \Lambda} \) be a non-empty family of graded two-sided ideals of \( \mathcal{O}(P,Q,\psi)(K) \) and let \( \Omega = (\omega^K_{H_\lambda})_{\lambda \in \Lambda} \). For each \( \lambda \in \Lambda \) let \( (S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},\mathcal{O}(P,Q,\psi)(K)/H_\lambda) \) be as in Proposition 7.22. It follows from Proposition 7.26 that \( (S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},\mathcal{O}(P,Q,\psi)(K)/H_\lambda) \) is isomorphic to the covariant representation

\[
\begin{pmatrix}
\omega^K_{H_\lambda} & \omega^K_Q & \omega^K_R \\
\iota^K_P & \iota^K_Q & \iota^K_R
\end{pmatrix}
\mathcal{O}(P,Q,\psi)(K),
\begin{pmatrix}
\iota^K_{H_\lambda} & \iota^K_{P,Q,\psi} & \iota^K_{R_\lambda}
\end{pmatrix}
\]

It therefore follows from Proposition 7.19 that there exists a ring isomorphism

\[
\phi : B_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda) \to \mathcal{O}(I_{\Pi \Omega}P,Q,\psi(I_{\Pi \Omega}),\Omega(I_{\Pi \Omega}))(J_{\Pi \Omega}I_{\Pi \Omega},\Omega(I_{\Pi \Omega}))(J_{\Pi \Omega}I_{\Pi \Omega})
\]

satisfying

\[
\phi \circ \sigma_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda) = \iota_{\Omega}^{\Pi \Omega}, \quad \phi \circ T_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda) = \iota_{\Omega}^{\Pi \Omega}
\]

and

\[
\phi \circ S_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda) = \iota_{\Omega}^{\Pi \Omega}.
\]

If \( x \in K \), then we have for all \( \lambda \in \Lambda \) that \( \sigma_{H_\lambda}(x) - \pi_{TH,S_H}(\Delta(x)) = 0 \), and it thus follows from Proposition 7.16 that

\[
\sigma_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda)(x) = \pi_{T_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda)S_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda)}(\Delta(x)).
\]

Thus the covariant representation

\[
\begin{pmatrix}
S_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda), T_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda), \sigma_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda), B_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda)
\end{pmatrix}
\]

of \( (P,Q,\psi) \) is Cuntz–Pimsner invariant relative to \( K \). It therefore follows from Theorem 3.18 that there exists a ring homomorphism \( \eta : \mathcal{O}(P,Q,\psi)(K) \to B_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda) \) such that

\[
\eta \circ \iota^K_{H} = \sigma_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda), \quad \eta \circ \iota^K_{Q} = T_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda) \quad \text{and} \quad \eta \circ \iota^K_{P} = S_{\Pi \lambda \in \Lambda}(S_{H_\lambda},T_{H_\lambda},\sigma_{H_\lambda},B_\lambda).
\]

We then have that

\[
\phi \circ \eta : \mathcal{O}(P,Q,\psi)(K) \to \mathcal{O}(I_{\Pi \Omega}P,Q,\psi(I_{\Pi \Omega}),\Omega(I_{\Pi \Omega}))(J_{\Pi \Omega}I_{\Pi \Omega}),
\]

is a ring homomorphism satisfying \( \phi \circ \eta \circ \iota^K_{H} = \iota_{\Omega}^{\Pi \Omega}, \phi \circ \eta \circ \iota^K_{Q} = \iota_{\Omega}^{\Pi \Omega} \) and \( \phi \circ \eta \circ \iota^K_{P} = \iota_{\Omega}^{\Pi \Omega} \).

It follows that \( H^K_{\omega^K_{\Pi \Omega}} = \ker(\phi \circ \eta) = \ker \eta \), and since it follows from Proposition 7.16 that \( \ker \eta = \cap_{\lambda \in \Lambda} H_\lambda \), we can conclude that \( H^K_{\omega^K_{\Pi \Omega}} = \cap_{\lambda \in \Lambda} H_\lambda \).
It follows from Propositions 7.17, 7.19 and 7.26 that for each \( \lambda \in \Lambda \) there exists a ring homomorphism \( \psi_\lambda: \mathcal{O}(P,Q,\psi)(K)/H_\lambda \to \mathcal{O}(P,Q,\psi)(K)/H^K_{\omega^0_k} \) such that \( \psi_\lambda \circ \sigma_\lambda = \sigma_{H^K_{\omega^0_k}} \). 

\( \psi_\lambda \circ T_{H_\lambda} = T_{H^K_{\omega^0_k}} \) and \( \psi_\lambda \circ S_{H_\lambda} = S_{H^K_{\omega^0_k}} \). It follows that \( \psi_\lambda \circ \psi_{H_\lambda} = \varphi_{H^K_{\omega^0_k}} \), and thus that \( H_\lambda \subseteq H^K_{\omega^0_k} \).

Let \( H \) be a two-sided ideal of \( \mathcal{O}(P,Q,\psi)(K) \) containing \( \bigcup_{\lambda \in \Lambda} H_\lambda \). Then we have for each \( \lambda \in \Lambda \) that there exists a ring homomorphism \( \psi_\lambda: \mathcal{O}(P,Q,\psi)(K)/H_\lambda \to \mathcal{O}(P,Q,\psi)(K)/H \) such that \( \psi_\lambda \circ \sigma_\lambda = \sigma_H \), \( \psi_\lambda \circ T_{H_\lambda} = T_H \) and \( \psi_\lambda \circ S_{H_\lambda} = S_H \). It therefore follows from Propositions 7.17 and 7.19 that there exists a ring homomorphism

\[
\tau: \mathcal{O}_I(\lambda P,Q,\psi) \mathcal{O}(P,Q,\psi)(K) (J_{\lambda}^{\lambda P,Q,\psi}) \to \mathcal{O}(P,Q,\psi)(K)/H
\]
satisfying \( \tau \circ t^R_{\lambda P,Q,\psi} = \varphi_H \circ t^R_{\lambda P,Q,\psi} \) and \( \tau \circ t^Q_{\lambda P,Q,\psi} = \varphi_H \circ t^Q_{\lambda P,Q,\psi} \). It then follows that \( \tau \circ \Psi_{\omega^0_k} = \varphi_H \), and thus that \( H^K_{\omega^0_k} = \ker \Psi_{\omega^0_k} \subseteq H \). Hence \( H^K_{\omega^0_k} \) is the smallest two-sided ideal of \( \mathcal{O}(P,Q,\psi)(K) \) containing \( \bigcup_{\lambda \in \Lambda} H_\lambda \).

**Corollary 7.28.** Let \( R \) be a ring, and let \( (P,Q,\psi) \) be an \( R \)-system satisfying condition (FS). Then

\[
H \mapsto \omega^0_H, \quad \omega \mapsto H^0_{\omega^0_k}
\]
is a bijective correspondence between the set of all the graded two-sided ideals \( H \) of \( T(P,Q,\psi) \) and the set of all \( T \)-pairs \( \omega = (I,J) \) of \( (P,Q,\psi) \). This bijection preserves inclusion, and if \( (H_\lambda)_{\lambda \in \Lambda} \) is a non-empty family of graded two-sided ideals of \( T(P,Q,\psi) \) and \( \Omega = (\omega^0_{H_\lambda})_{\lambda \in \Lambda} \), then \( H^0_{\omega^0_k} = \bigcap_{\lambda \in \Lambda} H_\lambda \) and \( H^0_{\omega^0_k} \) is the smallest two-sided ideal of \( T(P,Q,\psi) \) containing \( \bigcup_{\lambda \in \Lambda} H_\lambda \).

**Corollary 7.29** (cf. [17, Theorem 8.6]). Let \( R \) be a ring, let \( (P,Q,\psi) \) be an \( R \)-system satisfying condition (FS) and assume that there exists a uniquely maximal faithful \( \psi \)-compatible two-sided ideal \( K \) of \( R \). Then

\[
H \mapsto \omega^K_H, \quad \omega \mapsto H^K_{\omega^0_k}
\]
is a bijective correspondence between the set of all the graded two-sided ideals \( H \) of \( \mathcal{O}(P,Q,\psi) \) and the set of all \( T \)-pairs \( \omega = (I,J) \) of \( (P,Q,\psi) \) satisfying \( K \subseteq J \). This bijection preserves inclusion, and if \( (H_\lambda)_{\lambda \in \Lambda} \) is a non-empty family of graded two-sided ideals of \( \mathcal{O}(P,Q,\psi) \) and \( \Omega = (\omega^K_{H_\lambda})_{\lambda \in \Lambda} \), then \( H^K_{\omega^0_k} = \bigcap_{\lambda \in \Lambda} H_\lambda \) and \( H^K_{\omega^0_k} \) is the smallest two-sided ideal of \( \mathcal{O}(P,Q,\psi) \) containing \( \bigcup_{\lambda \in \Lambda} H_\lambda \).

**Example 7.30.** Let us once again return to Example 1.9. We saw in Example 5.5 that if \( R \) is a ring with local units, \( \varphi \in \text{Aut}(R) \), \( P = R_{\varphi} \), \( Q = R_{\varphi^{-1}} \) and

\[
\psi: P \otimes_R Q \to R, \quad p \otimes q \mapsto p \varphi(q),
\]
then \( (P,Q,\psi) \) is an \( R \)-system which satisfies condition (FS), \( \ker \Delta = \{0\} \), \( \Delta^{-1}(F_P(P)) = R \), and \( \mathcal{O}(P,Q,\psi)(R) \) is the universal ring generated by elements \( \{[r,k] \mid r \in R, k \in \mathbb{Z}\} \) satisfying \( [r_1,k] + [r_2,k] = [r_1 + r_2,k] \) and \( [r_1,k_1][r_2,k_2] = [r_1 \varphi_{k_2}(r_2), k_1 + k_2] \).

It is easy to see that a two-sided ideal \( I \) of \( R \) is \( \psi \)-invariant if and only if \( \varphi(I) \subseteq I \). It is also easy to see that if \( I \) is a \( \psi \)-invariant two-sided ideal, then \( \ker \Delta = \varphi^{-1}(I) + I \). Thus \( (I,R) \) is a \( T \)-pair if and only if \( I \) is a two-sided ideal of \( R \) such that \( \varphi(I) = I \). It therefore follows from Corollary 7.29 that we have a bijective correspondence between \( \varphi \)-invariant two-sided ideals of \( R \) and graded two-sided ideals of \( \mathcal{O}(P,Q,\psi) \) which takes a \( \psi \)-invariant two-sided ideal \( I \) to the
graded two-sided ideal \( \{[x,k] \in \mathcal{O}_{(P,Q,\psi)} \mid x \in I, \ k \in \mathbb{Z} \} \), which is isomorphic to the crossed product \( I \times_{\varphi} \mathbb{Z} \).

It is easy to see that if we denote by \( \varphi_I \) the automorphism of \( R/I \) induced by \( \varphi \), then \( IP = (R/I)\varphi_I \) and \( QI = (R/I)\varphi^{-1}_I \). It follows from Proposition 7.26 that the quotient of \( \mathcal{O}_{(P,Q,\psi)} \) by the ideal \( \{[x,k] \in \mathcal{O}_{(P,Q,\psi)} \mid x \in I, \ k \in \mathbb{Z} \} \) is isomorphic to \( \mathcal{O}_{(I,P,Q,\psi_1,\psi)}(R/I) = \mathcal{O}_{(I,P,Q,\psi_1)} \) and thus to the crossed product \( (R/I) \times_{\varphi_I} \mathbb{Z} \).

**Example 7.31.** Let \( E = (E^0, E^1) \) be a directed graph and \( F \) a commutative unital ring. Let \( R \) be the ring and \( (P,Q,\psi) \) the \( R \)-system associated with \( E \) in Examples 1.10 and 5.8. For a two-sided ideal \( I \) of \( R \), let \( H = \{v \in E^0 \mid 1_v \in I \} \). We then have that \( I = \text{span}_F \{1_v \mid v \in I \} \).

We may define \( R_H \) with \( \text{span}_F \{\varphi_I(1_v) \mid v \in E^0 \setminus H \} \). It is easy to see that \( I \) is \( \psi \)-invariant if and only if the set of vertices \( H \) is hereditary, that is, whenever \( e \in E^1 \) with \( s(e) \in H \), we have \( r(e) \in H \). In that case we have

\[
IP = \text{span}_F \{1_v \mid e \in E^1, r(e) \in H \} \quad \text{and} \quad QI = \text{span}_F \{1_v \mid e \in E^1, r(e) \notin H \},
\]

so we may, and will, identify \( IP \) with \( \text{span}_F \{\varphi_I(1_v) \mid e \in E^1, r(e) \notin H \} \) and \( QI \) with \( \text{span}_F \{\varphi_I(1_v) \mid e \in E^1, r(e) \notin H \} \). We then have that

\[
\text{ker} \Delta_I = \text{span} \{\varphi_I(1_v) \mid v \in \partial H \text{ or } s^{-1}(v) = \emptyset \} \subseteq \text{ker} \Delta,
\]

where \( \partial H := \{v \in E^0 \mid 0 < |s^{-1}(v)| < \infty \text{ and } r(s^{-1}(v)) \subseteq H \} \). The set \( H \) is called saturated if \( \partial H \subseteq H \). We define the set of breaking vertices of \( H \) to be

\[
B_H := \{v \in E^0_\text{int} \setminus H \mid 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus H)| < \infty \},
\]

where \( E^0_\text{int} = \{v \in E^0 \mid |s^{-1}(v)| = \infty \} \). We then have that

\[
\Delta^{-1}_I(\mathcal{F}_P(Q_I)) = \text{span} \{\varphi_I(1_v) \mid v \in E^0_\text{reg} \setminus H \text{ or } v \in B_H \},
\]

where \( E^0_\text{reg} := \{v \in E^0 \mid 0 \leq |s^{-1}(v)| < \infty \} \).

Let \( J \) be a two-sided ideal of \( R \). Then \( I \cup \Delta^{-1}(\mathcal{F}_P(Q)) \subseteq J \) if and only if for all \( v \in H \) and all \( v \in E^0 \) with \( 0 < |s^{-1}(v)| < \infty \) we have that \( 1_v \in J \). Similarly \( \varphi_I(1_v) \subseteq \Delta^{-1}_I(\mathcal{F}_P(Q_I)) \cap (\text{ker} \Delta_I)^+ \) if and only if \( v \in E^0_\text{reg} \cup B_H, v \notin \partial H \) and \( s^{-1}(v) = \emptyset \) for all \( v \in E^0 \setminus H \) for which \( 1_v \in J \). Thus if \( H \) is not saturated, then there does not exist any two-sided ideal \( J \) of \( R \) such that \( I \cup \Delta^{-1}(\mathcal{F}_P(Q)) \subseteq J \) and \( \varphi_I(1_v) \subseteq \Delta^{-1}_I(\mathcal{F}_P(Q_I)) \cap (\text{ker} \Delta_I)^+ \) and if \( H \) is saturated, then there is a bijective correspondence between two-sided ideals \( J \) of \( R \), for which \( I \cup \Delta^{-1}(\mathcal{F}_P(Q)) \subseteq J \) and \( \varphi_I(1_v) \subseteq \Delta^{-1}_I(\mathcal{F}_P(Q_I)) \cap (\text{ker} \Delta_I)^+ \), and subsets of \( B_H \). This correspondence takes a subset \( S \) of \( B_H \) to the ideal \( \text{span}_F \{1_v \mid v \in H \cup S \text{ or } 0 < |s^{-1}(v)| < \infty \} \).

Hence it follows from Corollary 7.29 that there is a bijective correspondence between pairs \( (H,S) \), where \( H \) is a hereditary and saturated subset of \( E^0 \) and \( S \) is a subset of \( B_H \), and graded two-sided ideals of \( \mathcal{O}_{(P,Q,\psi)} \). This correspondence takes a graded two-sided ideal \( K \) to \( (H,S) \), where

\[
H = \{v \in E^0 \mid p_v \in K \} \quad \text{and} \quad S = \left\{v \in B_H \mid p_v - \sum_{e \in s^{-1}(v) \cap r^{-1}(E^0 \setminus H)} x_e y_e \in K \right\},
\]

and it takes a pair \( (H,S) \) to the graded two-sided ideal generated by

\[
\left\{p_v \mid v \in H \right\} \cup \left\{p_v - \sum_{e \in s^{-1}(v), r(e) \notin H} x_e y_e \mid v \in S \right\}.
\]

Thus we recover [27, Theorem 5.7(1)].
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