Geometric quantum phase in the spacetime of topological defects

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Abstract. In this contribution, we study the quantum dynamics of a neutral particle in the presence of a topological defect. We investigate the appearance of a geometric phase in the relativistic quantum dynamics of a neutral particle which possesses permanent magnetic and electric dipole moments in the presence of an electromagnetic field in this curved background. We also study the influence of noninertial effects of a rotating frame and obtain several contributions to the relativistic geometric phase due to the noninertial effects and the topology of spacetime. The analogous Aharonov-Casher and He-Mckellar-Wilkens effects are investigated in the nonrelativistic dynamics with the presence of a topological defect and under the influence of noninertial effects. We also obtain effects analogous to the Sagnac effect and Mashhoon effect due to the presence of the topological defect.

1. Introduction
After the discovery of the Aharonov-Bohm effect [1] fifty years ago and the publication of Berry’s work [2] twenty-five years ago, the study of geometric phases is still a subject of a great interest in several branches of Physics. In the classic paper of phase shifts in non-quantal polarized light, Pancharatnam [3] anticipated the quantal geometric phases. It is worth mentioning the related effects to the Aharonov-Bohm effect have been anticipated by Franz [4] and by Ehrenberg and Siday [5]. The Aharonov-Bohm effect is a topological effect given by a phase shift on the wave function of a charged particle moving around a infinite thin solenoid that contains a magnetic flux inside it. Berry [2] showed that this phase shift constitutes a particular class of quantum phases that it is termed geometric phases. These quantum phases have been generalized to the case of a non-adiabatic evolution of a quantum system by Aharonov and Anandan [6]. In any case, the phase depends on the geometric nature of the path where the system evolves dynamically. Aharonov and Casher [7] studied the quantum behavior of a neutral particle with a permanent magnetic dipole moment interacting with an external electric field and obtained a topological quantum phase due to this interaction. This topological effect is called the Aharonov-Casher effect. The effect dual to the Aharonov-Casher effect has been proposed by He and MacKellar [8] and, independently, by Wilkens [9] by investigating the quantum dynamics of a neutral particle with a permanent electric dipole moment in the presence of an external magnetic field.
The phase shift acquired by the wave function of the neutral particle due to the interaction between the electric dipole moment and the magnetic field is called the He-McKellar-Wilkens effect. Following this way, Dowling et al. [10] and Furtado et al. [11] studied the dual effect of the Aharonov-Bohm effect by using the Maxwell duality transformation and the quantum dynamics of a magnetic monopole in the presence of an electric solenoid, respectively. A general study of the quantum phases for neutral particles with permanent dipoles has been done by Anandan [12] and Silenko [13]. Ribeiro and Furtado [14] demonstrated the Anandan quantum phase is a case of the Berry quantum phase. Recently, the dual Aharonov-Bohm effect in the dynamics of a composite particles has been discussed by Horsley et al. [15].

The gravitational analogue of the Aharonov-Bohm effect has been investigated and discussed by several authors. Cai and Papini [16] obtained the covariant generalization of the Berry phase and applied their results by using the approach of weak gravitational fields. Corichi and Pierri [17] investigated the behavior of a scalar particle in a class of stationary spacetime backgrounds and studied the emergence of geometric phases in the dynamics of a particle in the presence of a rotating cosmic string. Mazur [18] obtained the gravitational analogue of the Aharonov-Bohm effect in the spinning cosmic string spacetime background. Mostafazadeh [19] investigated the relativistic geometric phase for a scalar particle in the background of a rotating cosmic string by employing the two-component formalism. Recently, one of us [22, 23] has investigated the Berry quantum phase for a scalar particle in the presence of a magnetic chiral cosmic string background. Many authors have investigated the geometric phase in a series of gravitational backgrounds [24, 25]. The gravitational analogue of the Aharonov-Casher effect has been obtained by Anandan [20] and Resnik [21].

Topological defects is a topic widely studied in various areas of the modern Physics. An excellent review of the applications and the problems involving topological defects in condensed matter physics, cosmology and gravitation can be found in the books of Kleinert [26]. The presence of topological defects changes the physical properties of the medium in which it is present. The study of the influence of topological defects in various physical systems have been analyzed, for instance, in superfluids [27], superconductors [26], graphene [28, 29, 30], liquid crystals [31], elastic solids [32, 33], effects on the electromagnetic field [34], the Casimir effect [35], the Landau quantization [36], etc. The study of the Berry quantum phase in media with dislocations has been done in Ref. [37]. Ribeiro et al. [38] investigated the geometric phase for an induced electric dipole by using the field configuration proposed by He, McKellar and Wilkens [8, 9]. Recently, geometric quantum phases for neutral particles with permanent magnetic dipole moment have been investigated in curved spacetime backgrounds [39], and in the presence of noninertial references frames [40, 41]. In this paper, we briefly review the results of the studies of geometric quantum phases in the presence of topological defects obtained by the authors in the articles mentioned above.

2. The cosmic string spacetime and the field configuration in a rotating frame

In this section, our focus is to present the curved spacetime background and show how we build a rotating frame in such a way that we can obtain a field configuration that produces no torque on the dipole moment of the neutral particle. The chosen curved spacetime background in this work is called the cosmic string spacetime. The cosmic string is a topological defect described by the line element:

\[ ds^2 = -dT^2 + dR^2 + \eta^2 R^2 d\Phi^2 + dZ^2. \]  

(1)

where \( \eta = 1 - 4\nu \) is a parameter associated with the deficit angle of the cosmic string spacetime. It is defined in the range \( 0 < \eta < 1 \), and \( \nu \) is the linear mass density. In this work, we consider the units where \( h = c = 1 \). The azimuthal angle varies in the interval: \( 0 \leq \varphi < 2\pi \). The parameter \( \eta \) in the cosmic string spacetime can assume only values \( \eta < 1 \). Values for \( \eta \) greater
than 1 correspond to a spacetime with negative curvature, and it does make sense only in the
description of linear topological defects in solids [32, 42]. The geometry of the cosmic string
possesses a conical singularity [43] which gives rise to the curvature concentrated on the cosmic
string axis. This conical singularity is represented by the curvature tensor $R^{\rho}_{\mu \nu \phi} = \frac{1-\eta}{4\eta} \delta_2(\vec{r})$,
where $\delta_2(\vec{r})$ is the two-dimensional delta function. Since we are interested in discussing the
influence of the noninertial effects and the topology of the cosmic string spacetime on the
geometrical quantum phases of a neutral particle with a permanent dipole moment, is it possible
to build a noninertial frame in such a way that the field configuration induced by the noninertial
effects does not produce any torque on the dipole moment? The answer for this question is yes.
By using the approach of the quantum field theory in the curved spacetime background [44],
we can build local reference frames for the observers and study the influence of the noninertial
effects on the geometric quantum phases. This study starts when we carry out the coordinate
transformation $T = t, \ R = \rho, \ \Phi = \phi + \omega t, \ Z = z$, where $\omega$ is the constant angular velocity
of the rotating frame and we consider $\omega \rho << 1$. With this transformation, the line element (1)
becomes

$$ds^2 = -(1 - \omega^2 \eta^2 \rho^2) dt^2 + 2 \omega \eta \rho^2 d\phi dt + d\rho^2 + \eta^2 \rho^2 d\phi^2 + dz^2.$$  (2)

Noninertial effects have been discussed in the literature for a long time. The Sagnac effect [45]
is the best famous effect related to the appearance of a phase shift in the wave function of a
quantum particle. Another well-known noninertial effect is the Mashhoon effect [46], a quantum
phase given by the coupling of the spin of the particles with the angular velocity of the rotating
frame. More discussions about the noninertial effects above can be found in [47, 48]. Recently ,
noninertial effects have been investigated in different contexts, for instance, in the studies of
holonomies [49], Landau quantization for neutral particles [51, 50] and for bound states for a
neutral particle analogous to a quantum dot [52]. How exactly can we build a rotating frame
based on the line element (2) and no torque is produced on the magnetic dipole moment of a
neutral particle? Since the spinors are defined locally in the curved spacetime background [44],
we can build a local reference for the observer through a noncoordinate basis $\hat{e}_a = e^a_\mu(x) dx^\mu$,
where the components $e^a_\mu(x)$ satisfy the relation $g_{\mu \nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab}$. The components
$e^a_\mu(x)$ are called tetrads or vierbein, $g_{\mu \nu}(x)$ is the metric tensor and $\eta_{ab} = \text{diag} (- + + +)$ is
the Minkowsky tensor [53, 54]. The tetrads have an inverse defined as $dx^\mu = e^a_\mu(x) \bar{e}^a_\nu$, where
$e^a_\mu(x) \bar{e}^b_\nu(x) = \delta^a_b$ and $e^a_\mu(x) e^\nu_\alpha(x) = \delta^a_\nu$. In that way, we choose the tetrads in the form
[40]:

$$\dot{\theta}^0 = \sqrt{1 - \beta^2} dt - \frac{\omega \eta^2 \rho^2}{\sqrt{1 - \beta^2}} d\phi; \quad \dot{\theta}^1 = d\rho; \quad \dot{\theta}^2 = \frac{\eta \rho}{\sqrt{1 - \beta^2}} d\phi; \quad \dot{\theta}^3 = dz,$$  (3)

where we have called $\beta = \omega \eta \rho$ in (3). Considering an electric charge density $\lambda$ concentrated on
the z axis in the inertial frame, we have that this distribution of charges creates a cylindrically
symmetric electric field in the rest frame of the observer $E^\rho_{(r)} = \frac{\lambda}{\sqrt{-g}}$, where $g = \det(g_{\mu \nu})$. Since
we have chosen the 1-axis of the local reference frame parallel to the $\rho$ axis of the spacetime, we
can write $E^1 = \sqrt{1 - \beta^2} E^\rho_{(r)} = \lambda/\sqrt{-g}$. Thus, in the corotating frame of the observers (3), we have
only one non-null components of the electric field [40]:

$$E^\rho = \frac{1}{\sqrt{1 - \beta^2}} \frac{\lambda}{\eta \rho},$$  (4)

that is, the noninertial effects of the rotating frame (3) induce only one component of the electric
field and no components of the magnetic field are induced in the rotating frame (3). It has been
shown in [50, 52] that non-null components of the magnetic field are induced by the noninertial
effects of a nonrotating frame called the Fermi-Walker reference. Our next step in this work is
to discuss the dynamics of a neutral particle in this background.
3. Relativistic dynamics of a neutral particle

In this section, we discuss the appearance of relativistic geometric phases in the quantum dynamics of a neutral particle with a permanent magnetic dipole moment. The Dirac equation in the curved spacetime background with the interaction of the magnetic dipole moment of the neutral particle with an external electric field is given by the following expression:

\[ i \gamma^a e^\mu_a (x) \partial_\mu \psi + i \gamma^a \Gamma^a_\mu (x) \psi + \frac{\mu}{2} F_{\mu \nu} \Sigma^{\mu \nu} \psi = m \psi, \quad (5) \]

where \( \Gamma^a_\mu (x) = \frac{i}{2} \omega_{\mu ab} (x) \Sigma^{ab} \) is the spinorial connection, \( F_{\mu \nu} \) is the electromagnetic tensor \( (F_{0i} = E_i \) and \( F_{ij} = -\epsilon_{ijk} B^k) \) and \( \Sigma^{ab} = \frac{i}{2} \begin{pmatrix} \gamma^a, \gamma^b \end{pmatrix} \). The indices \( (a, b, c = 0, 1, 2, 3) \) indicate the local reference frame, while the greek indices indicate the spacetime coordinates. The \( \gamma^a \) matrices are defined in the local reference frame and they are identical to the Dirac matrices in the flat spacetime, i.e.,

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}; \quad \Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}, \quad (6) \]

with \( \vec{\Sigma} \) being the spin vector and \( \sigma^i \) are the Pauli matrices satisfying the relation \( (\sigma^i \sigma^j + \sigma^j \sigma^i) = 2 \eta^{ij} \). The spin connections \( \omega_{\mu ab} (x) \) can be obtained by solving the Maurer-Cartan structure equations \( d \theta^a + \omega_{\mu ab} (x) dx^\mu \wedge \hat{\theta}^b = 0 \) [54], and they provide us the following spinorial connections [40]

\[ \Gamma_\theta = -\frac{1}{2} \frac{\omega^2 \eta \rho}{\sqrt{1 - \beta^2}} \hat{\alpha}^1 - i \frac{1}{2} \frac{\omega \eta}{\sqrt{1 - \beta^2}} \Sigma^3; \quad \Gamma_\rho = \frac{1}{2} \frac{\omega \eta}{(1 - \beta^2)} \hat{\alpha}^2; \quad (7) \]

\[ \Gamma_\varphi = -\frac{1}{2} \frac{\omega \eta^2 \rho}{\sqrt{1 - \beta^2}} \hat{\alpha}^1 - i \frac{1}{2} \frac{\eta}{\sqrt{1 - \beta^2}} \Sigma^3. \quad (8) \]

Taking the induced field configuration (4), it is easy to check that there is no torque \( \vec{\tau} = \vec{\mu} \times \vec{B} \) on the magnetic dipole moment of the neutral particle. In that way, we are able to study the influence of the noninertial effects of the rotating frames (3) on the geometric quantum phases for neutral particles with permanent magnetic dipole moment. The Dirac equation in the cosmic string spacetime background with rotating frames has the form

\[ m \psi = \frac{i}{\sqrt{1 - \beta^2}} \gamma^0 \frac{\partial \psi}{\partial t} + i \gamma^2 \frac{\omega \eta \rho}{\sqrt{1 - \beta^2}} \frac{\partial \psi}{\partial t} + i \gamma^1 \left( \frac{\partial _\rho - \mu \hat{\beta} E_\rho + \frac{1}{2 \rho}}{2} \right) \psi \]

\[ + \frac{i}{\sqrt{1 - \beta^2} \eta \rho} \frac{\partial \psi}{\partial \varphi} - i \gamma^2 \frac{\omega^2 \eta \rho}{\sqrt{1 - \beta^2}} \frac{\partial \psi}{\partial \varphi} + i \gamma^3 \frac{\partial \psi}{\partial z} - \frac{\gamma^0}{2} \frac{\eta \vec{\Sigma} \cdot \vec{E}}{(1 - \beta^2)^{3/2}} \psi. \quad (9) \]

The relativistic geometric quantum phase acquired by the wave function of the particle is given by using the Dirac phase factor method [55]. The Dirac phase factor method consists in writing the wave function in the form \( \psi = e^{i \phi} \psi_0 \), where \( \phi \) is the phase shift and \( \psi_0 \) is the solution of the Dirac equation in the absence of fields. Substituting \( \psi = e^{i \phi} \psi_0 \) into (9), we find that the wave function acquires a phase shift given by

\[ \phi = -\mu \gamma^0 \oint \frac{\vec{\Sigma} \times \vec{E}}{\sqrt{1 - \beta^2}} d\varphi + \oint \frac{\eta}{2} \sqrt{1 - \beta^2} \Sigma^3 d\varphi + \frac{1}{2} \oint \frac{(\vec{\Sigma} \times \vec{E})}{(1 - \beta^2)^{3/2}} d\varphi \]

\[ - \frac{\eta}{2} \oint \frac{\vec{\omega} \cdot \vec{\Sigma}}{\sqrt{1 - \beta^2}} dt, \quad (10) \]

\[ \text{306 (2011) 012069 doi:10.1088/1742-6596/306/1/012069} \]
where we have defined the vector $\vec{E} = (\omega^2 \eta^2 \rho) \hat{\rho}$, with $\hat{\rho}$ being a unit vector in the $\rho$-direction. The expression for the relativistic quantum phase (10) is the relativistic analog of the Anandan quantum phase [12] given in a curved spacetime background and in a noninertial frame. This relativistic quantum phase has four independent contributions [40]: the first contribution in (10) is given by the interaction between the permanent magnetic moment and the external electric field; the second contribution is given by the topology of the cosmic string spacetime; the third contribution is given by the noninertial effects of the rotating frame (3), where the vector $\vec{E}$ plays the role analogous to the electric field [48]; the last contribution to the relativistic phase is given by spin-rotation coupling. We can see that every term in (10) has the presence of the term $(1 - \beta^2)^{3/2}$. This term corresponds to a correction term due to rotation of the local reference frames of the observers. In particular, integrating the last contribution for the relativistic quantum phase (10), we have

$$\phi' = \frac{\omega^2 \eta^2}{(1 - \beta^2)^{3/2}} \vec{E} \cdot \vec{\Sigma},$$

where $\vec{A} = A \hat{n}$, with $A$ being the area perpendicular to the symmetry axis of the cosmic string and $\hat{n}$ the unitary vector perpendicular to the area $A$. Taking the limit $\eta \rightarrow 1$ in $\phi'$, we obtain the relativistic phase shift induced by the noninertial effects analogous to that one obtained by Anandan and Suzuki in [48] without using the weak field approximation, up to the correction term $(1 - \beta^2)^{3/2}$ due to the rotation of the local reference frame of the observers [40].

Another point of interest is when we take $\omega = 0$. In this way, we recover the relativistic Anandan quantum phase in the cosmic string spacetime background obtained in [39] in an inertial frame. It is interesting to note that, taking $\omega = 0$ in the second term of (10), this second term corresponds to the relativistic Berry phase proposed by Cai and Papini [16] for a spin-half particle without using the weak field approximation [39]. Moreover, substituting the value of the defect parameter $\eta$ into the second term of (10) (with $\omega = 0$), we obtain the phase shift $\phi'' \propto (8\pi G) \nu \Sigma^3$. This phase shift $\phi''$ corresponds to the phase shift obtained by Resnik [21] for the gravitational effect analogue of the Aharonov-Casher effect [39].

4. The nonrelativistic dynamics of the neutral particle

In this section, we discuss the appearance of geometric quantum phases in the nonrelativistic dynamics of the neutral particle with a permanent dipole moment interacting with external fields. Taking the nonrelativistic of the Dirac equation (9), we obtain the following the Schrödinger-Pauli equation [40]

$$i \frac{\partial \psi}{\partial t} = \frac{1}{2m} (\vec{p} + \vec{\Xi})^2 \psi + \frac{\mu}{2m} \vec{\nabla} \cdot \vec{E} \psi - \frac{\mu^2 E^2}{2m} \psi - m A_0 \psi$$

$$- \frac{\eta}{2} \vec{\omega} \cdot \vec{\sigma} \psi + \frac{\mu \lambda \eta \omega^2}{4m} \psi + O \left( \frac{\beta^2}{m^2} \right),$$

where we have considered $\omega \eta \rho \ll 1$ to obtain the equation (11). The vector $\vec{\Xi}$ has the following components

$$\Xi_k = \mu \left( \vec{\sigma} \times \vec{E} \right)_k - \frac{1}{2\rho} \sigma^3 \delta_{k2} - m A_k - \frac{1}{2} \left( \vec{\sigma} \times \vec{E} \right)_k.$$  (12)

We have defined in the expressions (11) and (12) the vector components: $A_0 = \frac{1}{2} \eta^2 (\vec{\omega} \times \vec{r})^2 = \frac{1}{8} \omega^2 \eta^2 \rho^2$, $A_\varphi = \eta (\vec{\omega} \times \vec{r})_\varphi$, and $E^\rho = \omega^2 \eta^2 \rho$. The nonrelativistic geometric phases acquired by the wave function of the neutral particle can be obtained by using again the Dirac phase factor method [55], where $\psi = e^{i\phi} \psi_0$. Since the terms proportional to $\vec{\nabla} \cdot \vec{E}$, $E^2$ and $A_0$ are local terms, they do not contribute to the geometric phase [39, 40]. Hence, the terms which contribute to
the nonrelativistic geometric phase are the vector $\vec{E}$ and the spin-rotation coupling $\vec{\omega} \cdot \vec{\sigma}$. Here, we have that $\psi_0$ is the solution of the equation

$$\frac{-1}{2m} \nabla^2 \psi_0 - \frac{\mu^2 \vec{E}^2}{2m} + m A^0 \psi_0 = 0. \tag{13}$$

Taking the radial electric field given in (4) (with $\omega \eta \rho \ll 1$), we have that the nonrelativistic Anandan quantum phase corresponds to the effect analogous to the Aharonov-Casher effect [7] given in a noninertial frame in the presence of a topological defect. The expression for the nonrelativistic geometric phase is [40]:

$$\phi_{AC} = 2\pi \mu \lambda \sigma^3 - \eta \pi \sigma^3 - 2m \eta \vec{\omega} \cdot \vec{A} - \omega^2 \eta^2 \vec{A} \cdot \vec{\sigma} + \frac{\eta}{2} \vec{\omega} \cdot \vec{\sigma} T. \tag{14}$$

Thus, we can see that the nonrelativistic geometric phase is given by five independents contributions. In particular, the third contribution for the geometric phase (14) is given by the gauge field $m \vec{A}$. This result comes from $\phi^\prime_{NR} = -m \int \vec{A}_k \vec{e}^k _\varphi d\varphi = -2m \eta \vec{\omega} \cdot \vec{A}$, where $A = \pi \eta \rho^2$ is the area enclosed by the path of the neutral particle, perpendicular to the angular velocity $\omega$. This phase shift appears due to the rotation frame and give us an effect analogous to the Sagnac effect [45, 47, 48] in the presence of the topological defect.

The fourth contribution to the geometric phase (14) is given by the noninertial effects of the rotating frames, where we have the interaction between the effective electric field $\vec{E}$ and the spin of the neutral particle, that is, $\phi^\prime_{NR} = -\frac{1}{2} \int (\vec{\sigma} \times \vec{E})_\varphi d\varphi = -\omega^2 \eta^2 \vec{A} \cdot \vec{\sigma}$. Note that, taking the limit $\eta \rightarrow 1$ in the phase shift $\phi^\prime_{NR}$, we recover the results obtained in [48] without making the weak field approximation. The last contribution to the nonrelativistic geometric phase (14) is given by the rotation-spin coupling, that is, $\phi^\prime_{N\rho''} = \frac{\eta}{2} \int \vec{\omega} \cdot \vec{\sigma} dt = \frac{T}{\eta} \vec{\omega} \cdot \vec{\sigma}$, where $T$ is the time which the particle spends moving along a closed path around the symmetry axis of the topological defect.

We can observe that taking $\omega = 0$ in (14), we recover the effect analogous to the Aharonov-casher effect obtained by the authors in [39], that is, we have that $\phi^\prime_{AC} = 2\pi \mu \lambda \sigma^3 - \eta \pi \sigma^3$. The term $\eta \pi \sigma^3$ corresponds to the contribution given by the presence of the topological defect. In the limit $\eta \rightarrow 1$, that is, the absence of the topological defect, we recover the original Aharonov-Casher effect [7], up to a constant phase factor. In the same way of the relativistic case, the contribution given by the presence of the topological defect in the nonrelativistic geometric phase provide us the gravitational effect analogous to the Aharonov-Casher effect $\phi'' \propto (8\pi G) \nu \sigma^3$ obtained by Resnik in [21].

Our last comment is related to the He-McKellar-Wilkens effect [8, 9]. In the same way of the effect analogue of the Aharonov-Casher effect that we have obtained in (14), the effect analogous to the He-McKellar-Wilkens effect can be achieve by making the transformation: $\mu \rightarrow d$, $\vec{E} \rightarrow -\vec{B}$ and $\lambda \rightarrow \lambda_m$, where $d$ is the permanent electric dipole moment of the neutral particle and $\lambda_m$ is a linear distribution of magnetic charges. In that way, the geometric phase (14) becomes $\phi_{HMW}$ and corresponds to the effect analogue of the He-McKellar-Wilkens effect.

5. Conclusions
We have discussed the appearance of geometric quantum phases for neutral particles with a permanent magnetic and electric dipole moments in the relativistic and in the nonrelativistic regime. We have seen that we can build a rotating frame in the cosmic string spacetime background in such a way that the field configuration induced by the noninertial effects produces no torque on the magnetic dipole moment of the neutral particle. In the relativistic regime, we have obtained that the Anandan quantum phase was given by four independent contributions: one from the interaction between the electric field and the magnetic dipole moment of the neutral
particle, one from the topology of the cosmic string spacetime background and two from the noninertial effects of the rotating frames of the observers. We have also seen that in the limit $\omega = 0$, we can recover the relativistic Anandan quantum phase in the cosmic string background obtained in [39], where the contribution given by the topology of the spacetime provide us a phase shift analogous to the gravitational Aharonov-Casher effect [21], without making the weak field approximation [39]. In the nonrelativistic dynamics of the neutral particle, we have obtained the effect analogous to the Aharonov-Casher effect in the presence of a topological defect in a rotating frame. This geometric phase was given by five independent contributions, where we could observe effects analogous to the Sagnac effect and the Mashhoon effect. We have also seen that the topology of the defect provided a flux equivalent to the gravitational Aharonov-Casher effect [21]. At the end, we have seen that, taking $\omega = 0$, the analogous Aharonov-Casher effect given by the presence of a topological defect [39] has been recovered.

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