Rényi entropy flows from quantum heat engines

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(Dated: Aug 6, 2014)

We evaluate Rényi entropy flows from generic quantum heat engines (QHE) to a weakly-coupled probe environment kept in thermal equilibrium. We show the flows are determined by two quantities: heat flow and fictitious dissipation that manifest the quantum coherence in the engine. This pertains also the common Shannon entropy flow. The results appeal for revision of the concept of entropy flows in quantum non-equilibrium thermodynamics.

Entropy production in heat engines has been a key concept to establish the fundamental laws of thermodynamics \cite{1}. Recently, the laws have been reconsidered for small systems in the context of fluctuation relations \cite{2}, this gave rise to much of experimental \cite{3} and theoretical \cite{4} research. It is worth noting that the fluctuation relations are traditionally formulated in terms of entropy production that is computed with the definition described in \cite{5}. While the definition is perfect for classical states, its validity needs to be revisited in quantum mechanics, where Shannon entropy is non-linear in density matrix and its change is not necessarily related to the expectation value of any operator, therefore its measurability is questionable.\cite{6}

A quantum heat engine is a system of several discrete quantum states that, similar to a common heat engine, is connected to several environments kept at different temperatures. The motivation for research in QHE comes from studying models of photocells and photosynthesis \cite{7}. It has been demonstrated that quantum effects can dramatically change the thermodynamics of QHEs \cite{8} and their fluctuations \cite{9} manifesting the role of quantum coherence. We need to stress that the mere presence of discrete quantum states in the engine is not enough to reveal the coherence. The effects come from non-diagonal elements of the engine density matrix that require a coherent drive and/or degeneracy of the engine states to facilitate the formation of the quantum superpositions \cite{9}.

A generalization of Shannon entropy is the Rényi entropy \cite{10} defined here as $S_M \equiv \text{Tr}[\rho^M]$, $\rho$ being density matrix of a quantum system. Shannon entropy $S$ is obtained from $S_M$ by taking a formal limit $S = \lim_{M \to 1} \partial S_M / \partial M$. Much theoretical research addresses Rényi entropies in strongly interacting systems \cite{11}, in particular spin chains \cite{12}. Since $S_M$ is not linear in density matrix, its observability is not evident: some tricks \cite{13} may help in certain situations.

Recently, one of the authors has proposed a method for consistent quantum entropy flows - R-flows \cite{14}, defined as $F_M \equiv -d \ln S_M / dt = -(dS_M / dt) / S_M$. The Shannon entropy flow $F_S$ is obtained by taking limit $F_S = \lim_{M \to 1} \partial F_M / \partial M$. In this Letter, we will adjust and apply this method for QHE.

We evaluate R-flows from a generic heat engine to a probe environment that is weakly coupled with the engine thus undisturbing its workings. We find that R-flows can be naturally separated onto incoherent and coherent parts: this also pertains in the flow of Shannon entropy. Incoherent part is related to the heat flow $Q$ to the environment: for Shannon entropy flow we recover the textbook formula $F_S = Q / T$, $T$ being the temperature of the probing environment. Coherent part is specific for coherent drive and is proportional to the second power of the density matrix of the engine. This rises concerns about its observability. Surprisingly, we show that both parts can be extracted from the measurements.

We consider a quantum system with discrete states $|n\rangle$ separated into two sets $\{|u\rangle, \{d\}$. All states within a set have approximately same energy $E_u(E_d)$, the splitting $\epsilon_n$ within a set being much smaller than $E_u - E_d > 0$. The system is subject to the external field with the frequency $\omega \approx E_u - E_d$ (we set $\hbar, k_B = 1$ where appropriate) described by the Hamiltonian $H_d = \sum_{m,n} \sum_{\pm} \Omega_{mn} |m\rangle \langle n| e^{-i \omega t \pm} + h.c.$, the relevant matrix elements are between the states of two sets.

The quantum system is coupled to a number of environments labelled by $a$ and kept at different temperatures $T_a$. The interaction with the environment is described by $H_{int} = \sum_{m,n} |m\rangle \langle n| X^n_{ma}$, $X^n_{ma}$ being the op-
erators in the space of environment $a$. We assume linear response of each environment on the state of quantum system. In this case, each environment is completely characterized by the set of frequency-dependent generalized susceptibilities $\chi_{ij,kl}(\nu)$ that are related to the correlators of $\hat{X}$, $S_{ij,kl}(t) \equiv \langle \hat{X}_{ij}(0)\hat{X}_{kl}(t) \rangle$. The fluctuation-dissipation theorem yields the relations in frequency domain: $S_{ij,kl}(\nu) = 2\eta_{ij,kl}(\nu)$ where $\eta_{ij,kl}(\nu) = (\chi_{ij,kl}(\nu) - \chi_{ij,kl}(-\nu))/2i$, and the Bose distribution $\eta_{ij,kl}(\nu) = 1/(e^{\beta\nu} - 1)$.

The environments produce the transition rates between the states of quantum system and affect the coherence of its density matrix $\rho_{nm}$. The dynamics are expressed by Bloch-master equation in the rotating wave approximation:

$$\frac{d\rho_{nm}}{dt} = -i \sum_p (H_{mp}\rho_{pn} - \rho_{mp}H_{pn}) + \frac{1}{2} \sum_p (\Gamma_{mp}\rho_{pn} + \rho_{mp}\Gamma_{pn}) + \sum_{p,k} \Gamma_{mn,pk}\rho_{pq}$$

To distinguish the sets, let us introduce a matrix $\eta_{nm}$,

$\eta_{nm} = 1$ if $n \in \{u\}$ and $m \in \{d\}$, $\eta_{nm} = -1$ if $n \in \{d\}$ and $m \in \{u\}$, $\eta_{nm} = 0$ otherwise. The residual Hamiltonian is composed of three groups of terms

$$H_{nm} = \epsilon_n\delta_{nm} + \Re\Omega_{nm}\eta_{nm}^2 + i\eta_{nm}\Im\Omega_{nm} + \sum_{a,k} \frac{d\nu}{2\pi} \frac{S_{nk,km}(\nu)}{\nu - E_k + E_m}\rho_{nm}$$

First term is the original small splitting of the states, second and third represent the coherent drive and the last term is the renormalization due to the interaction with the environments.

The dissipative terms $\Gamma$ are sums over the contributions, of each environment,

$$\Gamma_{nm,kp} = \sum_a S_{pm,nk}(E_n - E_k); \Gamma_{nm} = \sum_k \Gamma_{nk,km}$$

The relevant terms satisfy $E_k - E_l \approx E_m - E_n$. In the absence of the drive and for the non-degenerate states the only relevant terms $\Gamma_{nm,nm}$ are the transition rates from $m$ to $n$, the density matrix is diagonal and the equation reduces to the master equation. In rotating wave approximation we can replace $E_m - E_n$ with $\omega\eta_{nm}$.

The Bloch equation (1) can be obtained by time-dependent perturbation theory for density matrix in time interval $-\infty < \tau < t$ where evolution operators for bra and ket are expanded in terms of $\hat{X}$, this sets the time ordering along the Keldysh contour that has opposite time directions for bra and ket (left diagram in Fig. 2). For relevant diagrams the $\hat{X}^a$ are pairwise grouped and the result of tracing over the environment is readily expressed in terms of $S_{ij,kl}(t)$. The density matrix $\dot{\rho}(t)$ is obtained by summation over all such diagrams. The compact way to achieve the summation is to take the sum of diagrams ending at $\tau = t$ and thus contributing to $d\rho/dt$ at $\tau = t$ and replace $\dot{\rho}(-\infty)$ with $\dot{\rho}(t)$: this reproduces Eq. (1).

To evaluate the Rényi entropy flow of $M$-th order to an environment $b$ we need to use the perturbation theory for the $M$-th power of its density matrix $\{\rho_b(t)\}^M$. To this end, we consider $M$ copies of the world consisting of the quantum system and the environments [13], each world bringing its own double Keldysh contour. The closing of the contours at $\tau = t$ is different for different degrees of freedom (right diagram in Fig. 2). For those of the environment $b$, the contour that defines the ordering of $\hat{X}_b$ encompasses all the worlds supplying the matrix multiplication of $\rho_b$ and the final trace. For all other degrees of freedom, the bra and ket parts of the contours are closed within each world providing the partial trace over these degrees of freedom: that yields $\rho_b$ for each world. The relevant diagrams are pairwise-grouped. For those arising from the environments other than $b$, both operators are within the same world. Summation over these diagrams reproduces evolution equation (1). The operators in diagrams from environment $b$ can be either in the same world, or in different worlds. The same-world diagrams have already been considered in [14]. The different-world diagrams though contain non-diagonal elements of the system density matrix and are thus specific for the case of coherent drive and degeneracies.

In this paper, we restrict ourselves to a simple case when the transition rates induced by environment $b$ are smaller than those induced by others. The environment $b$ is thus probe one and hardly affect density matrix of the system. In this case, Rényi entropy flow to the environment $b$ is directly given by the second-order diagrams encompassing two operators $\hat{X}^{(b)}$. The diagrams are expressed in terms of the generalized correlators of two $\hat{X}^{(b)}$ that contain multiple powers of $\rho_b$,

$$S_{ij,kl}^{N,M}(\tau) \equiv \text{tr}_b \left[ \hat{X}_{ij}(t)\rho_b^N\hat{X}_{kl}(t + \tau)\rho_b^M \right] / \text{tr}_b \left[ \rho_b^M \right]$$

and, for general $\rho_b$, do not correspond to any physical quantities. However, we derive that for the probe environment in the state of thermal equilibrium the correlators obey the generalized KMS (see Supplementary Materials and for more details [15]) relations

$$S_{ij,kl}^{N,M}(\nu) = 2n_B(M\nu)e^{\beta\nu N}\chi_{ij,kl}(\nu)$$

and therefore are all expressed in terms of the dissipative susceptibilities. In derivation, we assume that $\hat{\chi}$ does not depend on temperature. If this is not so, $\hat{\chi}$ is taken at $\beta^* = \beta M$.

Collecting all diagrams (see Supplementary Materials),
would cause energy dissipation to the probe environment \( \frac{\beta}{\omega} \). Involving and interesting. To proceed, let us replace in the R-flows is related to full counting statistics of energy transfers are not restricted to \( \hbar \) text-book equation for the entropy flow, \( M \) taking limit 

\[ Q_i = 2\omega \left\{ \sum_{i,j,k,l} \chi_{ij,kl}(\omega)\rho_{ij}(1 + n_B(\omega)) \right\} \]

\[ Q_c = 4\omega \sum_{i,j,k,l} \rho_{ij}\chi_{ij,kl}(\omega)\rho_{kl} \]

The R-flow is naturally separated onto two parts, which we name incoherent and coherent. The same-world diagrams contribute to the incoherent part that is proportional to \( Q_i \). \( Q_i \) is linear in \( \rho \) so that is an observable. The different-world diagrams form the coherent part \( \propto Q_c \), that is quadratic in \( \rho \) and in principle would be no observable. The \( M \) dependence is identical for both parts.

Let us interpret the parts and the quantities \( Q_{i,c} \). Inspection of the rates in Eq. \( (1) \) unambiguously identifies \( Q_i \) with an observable: the total energy flow to the probe environment. The terms \( \propto (1 + n_B(\omega)) \) describe absorption of energy quanta \( \hbar \omega \) by the environment, while those \( \propto n_B(\omega) \) correspond to the emission to the system. Upon taking limit \( M \to 1 \), the incoherent part reproduces text-book equation for the entropy flow, \( F_S = Q_i/T_b \). We prove that for a general situation where elementary energy transfers are not restricted to \( \pm \hbar \omega \), this part of the R-flows is related to full counting statistics of energy transfers and therefore can be measured.

The interpretation of the coherent part is more involved and interesting. To proceed, let us replace in \( H_{\text{int}} \) the operators \( |m\rangle\langle n| \) with classical external forces \( f_{ij} \) that are numerically equal to the elements of the system density matrix. The time-dependence of these forces is given by \( f_{ij} \propto \exp(-i\omega \eta_{ij}) \). These classical forces would cause energy dissipation to the probe environment that is determined from the forces and dissipative part of the susceptibility \( \xi \). This fictitious energy dissipation is precisely \( Q_c \). We stress that this is not the physical dissipation occurring in the probe environment: that is given by \( Q_i \neq Q_c \). However, \( Q_c \) can be extracted from the measurement results: for this, one can characterize the susceptibilities involved, measure \( \rho_{ij} \) (or corresponding \( \langle X_{ij}(t) \rangle \)) and compute \( Q_c \).

Therefore we show that both parts of R-flows can be extracted from the measurement results, although in a different way: R-flows are physical. In addition, we show that the entropy flow is not directly related to the energy flow. Rather,

\[ F_S = (Q_i - Q_c)/T_b \]

The difference is due to quantum coherent effects in our heat engine.

Let us discuss \( M \)-dependence of the R-flows. In Fig. 3 (left pane) we plot \( F_M/F_S = M n_B(M\omega)/(n_B(M - 1)\omega n_B(\omega)\beta) \) that conveniently depends on \( \beta \omega \) only. We see that \( M \gg 1 \) \( F \propto M \), that is, to the number of worlds involved, this is seen already for moderate \( M \). The proportionality coefficient \((1 - e^{-\beta \omega})/\beta \omega \) drops down with decreasing temperature. From the other hand, at \( M \to 1 \)

\[ F_M/F_S \approx (M - 1) \] with a coefficient not depending on temperature. This sets qualitative behavior of the curves plotted in Fig. 3. The low-temperature limit of R-flows reads

\[ F_M = M(Q_i - Q_c)/\omega \]

(this limit does not commute with \( M \to 1 \) since \( F_S \) diverges at low temperatures). In the absence of coherent effects, low-temperature R-flow is readily interpreted semiclassically \( \equiv \) as number of events (in our case, \( \hbar \omega \) quant absorptions) per second in \( M \) parallel worlds. With coherencies, such simple interpretation does not work since \( F_M \) can be negative.

Let us illustrate the behaviors of \( Q_{i,c} \) for the simplest quantum heat engine possible. It has only two states, \( |0 \rangle \) and \( |1 \rangle \) coupled by coherent drive amplitude \( \Omega \), with driving frequency exactly matching the energy difference \( E_1 - E_0 = \omega \). The relevant susceptibilities are \( \chi_{01,10}(\omega) \). The main environment kept at temperature \( T^* \) produces the transition rates \( \Gamma_0 = \Gamma_1 = \Gamma(1 + n_B(\omega/T^*)) \) while the probing environment produces similar rates \( \Gamma^b_0 = \Gamma^b_1 = \Gamma^b(1 + n_B(\omega/T_b)) \) with \( T_b \ll \Gamma \).

The \( Q_{i,c} \) in this case are expressed as

\[ Q_i/\omega = \Gamma^b_{i1}\rho_{11} - \Gamma^b_{i0}\rho_{00}; \quad Q_c/\omega = 2\Gamma^b_{i1}|\rho_{01}|^2 \]

where the elements of the density matrix are determined from Eq. \( (1) \) and read

\[ \rho_{00} = \frac{\Gamma_1(\Gamma_\downarrow + \Gamma_\uparrow) + \Omega^2}{(\Gamma_\downarrow + \Gamma_\uparrow)^2 + 4\Omega^2}, \quad \rho_{11} = 1 - \rho_{00}, \]

\[ \rho_{10} = -\frac{i\Omega(\rho_{11} - \rho_{00})}{\Gamma_\downarrow + \Gamma_\uparrow} \]
The plots of the $Q_{t,c}$ versus drive strength are given in Fig. 3 (right panel) for zero $T_c$ and different $\omega/T^*$. The coherent dissipation $Q_c$ is absent in the absence of the drive, reaches maximum $\Gamma^6/2$ at $T^* = 0$ and vanishes upon increasing $\Omega$ since non-diagonal elements of $\rho$ vanish in this limit. Finite $T^*$ suppresses the coherence and $Q_t$. The heat flow $Q_t$ at $T^* = 0$ is absent at $\Omega = 0$ since the system is not excited. It increases and saturates at $\Gamma^6/2$ for $\Omega \gg \Gamma$ when the states $|0\rangle$ and $|1\rangle$ are equally populated. At finite $T^*$, $Q_t$ is present in the absence of the drive as well. At $T^* \to 0$ and $|\Omega| < \Gamma/\sqrt{2}$ we have $Q_c > Q_t$. This implies that Shannon entropy of the probe environment decreases despite the positive heat flow to the environment. The interval of $\Omega$ where this interesting situation occurs shrinks with increasing $T^*$ and disappears at $T^* \approx 0.56\omega$.

Before summarizing our results for probe environment, let us shortly outline how to compute $R$-flows to an environment that essentially disturbs the dynamics of the system: for the example considered this implies $\Gamma \approx \Gamma^*$.

This equation is for a matrix $R$ that is an analogue of density matrix of $M$ copies of the system and is indexed by a compound $I \equiv \{i_1, \ldots, i_M\}$ encompassing all the worlds. The linear equation has a set of eigensolutions $R(t) \simeq \exp(-\Lambda_0 t)$. In distinction from a usual equation for density matrix, there is no solution with $\Lambda = 0$. The R-flow is shown to be given by $F_M = \Lambda_0$, $\Lambda_0$ being the eigenvalue which is closest to 0. The eigenvalues for a given number of worlds $M$ and concrete situation can be readily solved numerically. However, the analytical continuation to arbitrary $M$ is not evident for this moment and requires further research.

To conclude, we have computed Rényi entropy flows from a generic quantum heat engine to a probe environment and obtained Shannon entropy flows by taking limit $M \to 1$. The flows are expressed in terms of two quantities $Q_{t,c}$, $Q_t$ being the heat flow and $Q_c$ being an energy dissipation for the situation where the driven heat engine is replaced by fictitious coherent time-dependent classical forces. Both quantities are measurable. The entropy flow is proportional to $Q_t - Q_c$ and there are situations where it is opposite to the heat flow. This is in contrast with frequently used relations for entropy production along classical stochastic trajectory and implies that the concept of (Rényi) entropy flows requires revision and clarification in quantum case.

The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement n 308850 (INFERNOS).

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Supplementary materials: Rényi entropy flows from quantum heat engines

Diagrammatic Rényi entropy flows

Let us compute the R-flows from expansion of Bloch equation in the second order of interaction Hamiltonian. Considering that far in the past the coupling between system and environment is absent, the evolution is formally:

\[ \rho(t) = T \exp \left( i \int_{-\infty}^{t} d\tau H_{\text{int}}(\tau) \right) \rho \exp \left( -i \int_{-\infty}^{t} d\tau H_{\text{int}}(\tau) \right) \]  

(S1)

\( T \exp (\hat{T} \exp) \) refer to forward time ordering (backward time ordering). Without loss of generality, the system-bath Hamiltonian can be taken \( H_{\text{int}} = H_{s} H_{b} \) where \( H_{s(b)} \) acts on system (bath). Given a Gaussian correlations of the bath: \( \langle H_{b}(t_{2}) H_{b}(t_{1}) \rangle = tr_{b}(H_{b}(t_{1}) H_{b}(t_{2}) \rho_{b}) \). In the second order expansion, we place one \( H_{\text{int}} \) at \( t \) and the second one at any time before it, say \( t - \tau \) for \( 0 \leq \tau < \infty \). Without loss of generality we can set the global time to \( t = 0 \). The system density matrix in interaction picture \( \rho_{s} = U_{s}(0,t) \rho'_{s}(t) U_{s}(t,0) \) evolves according to

\[ \frac{d\rho_{s}}{dt} = \int_{0}^{\infty} d\tau \left( \langle H_{b}(-\tau) H_{b}(0) \rangle H_{s}(0) \rho_{s}(-\tau) + \langle H_{b}(0) H_{b}(-\tau) \rangle H_{s}(-\tau) \rho_{s} H_{s}(0) \right) \]

(S2)

where \( \rho_{s} \) acts on system (bath). Given that Renyi entropy is \( S_{M} = tr \rho^{M} \), its flux \( dS_{M}/dt \) can be determined directly from the generalization of \( S_{2} \). Evolution of \( M \) copies of \( \rho_{1}^{M} \) can influence more than one copy of the worlds. In this sense the evolution of Renyi entropy is more complex than \( S_{2} \) because different worlds may exchange energies. For this aim calculating a generalized correlator \( \langle H_{b}(0) \rho_{env}^{N}(t) H_{b}(0) \rangle \) with \( 0 \leq N \leq M \) is required. We use the following diagrams to evaluate the partial evolution. In the diagrams the solid (black) line denotes evolution of the system and narrow (white) line the rest of a world except its system.

In a typical diagrams with \( M \) worlds, given that there are \( N \) worlds between the operators \( A(t) \) and \( B(t + \tau) \), the Fourier transformed correlations consist of two parts: \( S_{A,B}^{N,M} \) and \( \Pi_{A,B}^{N,M} \). These two are related through a generalized Kramers-Kronig relation. The forward correlator is

\[ \int_{0}^{\infty} d\tau e^{i\omega \tau} tr \left( A(0) \rho_{env}^{N} B(\pm \tau) \rho_{env}^{M-N} \right) / tr \left( \rho_{env}^{M} \right) = \frac{1}{2} S_{A,B}^{N,M} (\pm \omega) \pm \Pi_{A,B}^{N,M} (\pm \omega) \]  

(S3)

where \( S_{A,B}^{N,M} (-\omega) = S_{A,B}^{M-N,M} (\omega) \), \( \Pi_{A,B}^{N,M} (-\omega) = -\Pi_{A,B}^{M-N,M} (\omega) \) and \( \Pi_{A,B}^{N,M} (\omega) = -(1/2\pi) \int dz S_{A,B}^{N,M} (z) / (z - \omega) \).

Single world diagrams

(a) \hspace{1cm} (b) \hspace{1cm} (c) \hspace{1cm} (d)

Summation over the diagram (a-d), respectively, results into:

\[ \sum_{x' y' y} \rho_{x' y'} \left( -\frac{1}{2} S_{y' y, g z'}^{0,M} (\omega \eta_{g x'}) + i \Pi_{y' y, g z'}^{0,M} (\omega \eta_{g x'}) + \frac{1}{2} S_{y z', y' y}^{1,M} (\omega \eta_{g y'}) - i \Pi_{y z', y' y}^{1,M} (\omega \eta_{g y'}) \right) \]

\[ -\frac{1}{2} S_{y' y, g z'}^{0,M} (-\omega \eta_{g y'}) - i \Pi_{y' y, g z'}^{0,M} (-\omega \eta_{g y'}) + \frac{1}{2} S_{y z', y' y}^{1,M} (-\omega \eta_{g y'}) + i \Pi_{y z', y' y}^{1,M} (-\omega \eta_{g y'}) \]  

(S4)
Due to the conservation of energy in these diagrams $E_y' - E_y = E_{x'} - E_x$ which means $\eta_{yx} = -\eta_{yx'}$. Substituting this in eq. (S4)

$$\sum_{x'y'y} \rho_{x'y'} \left( -S^{M,M}_{y',y} \omega \eta_{y'y'} + S^{1,M}_{y',y} \omega \eta_{y'y'} \right)$$

Multi-world diagrams

The forward propagation diagrams (e-h) and backward ones diagrams (i-l) are summed to

$$\sum_{xx'y'y} \rho_{x'y'} \left( -S^{n-1,M}_{yy',xx} \omega \eta_{yy'} - 2i\Pi^{n-1,M}_{yy',xx} \omega \eta_{yy'} + \frac{1}{2} S^{n,M}_{yy',xx} \omega \eta_{yy'} + i\Pi^{n,M}_{yy',xx} \omega \eta_{yy'} \right)$$

$$+ \sum_{xx'y'y} \rho_{x'y'} \left( -S^{n-1,M}_{xx',yy} \omega \eta_{yy'} + 2i\Pi^{n-1,M}_{xx',yy} \omega \eta_{yy'} + \frac{1}{2} S^{n,M}_{xx',yy} \omega \eta_{yy'} - i\Pi^{n,M}_{xx',yy} \omega \eta_{yy'} \right)$$

(S5)

where $n = 2$ to $n = M$. 
\[ \sum_{xx',yy'} \rho_{x'\rho_{y'y}}^{M-1} \left( -2S_{xx',yy'}^{n,M} (\omega_{y'y}) + S_{xx',yy'}^{n+1,M} (\omega_{y'y}) + S_{xx',yy'}^{n-1,M} (\omega_{y'y}) \right) + \text{terms} \]

\[ = \sum_{xx',yy'} \rho_{x'\rho_{y'y}} \left( S_{xx',yy'}^{0,M} (\omega_{y'y}) - S_{xx',yy'}^{1,M} (\omega_{y'y}) - S_{xx',yy'}^{M-1,M} (\omega_{y'y}) + S_{xx',yy'}^{M,M} (\omega_{y'y}) \right) \]

Let’s look at a typical Π-term: \[ \sum_{xx',yy'} \rho_{x'\rho_{y'y}} \Pi_{xx',yy'}^{n,M} (\omega_{y'y}) \] with \( a = 0, 1, M - 1, M \). In the energy eigenbasis of four states \( n, m, k, l \) with the property \( E_n - E_m = E_l - E_k > 0 \) the series summation is expanded into \( \rho_{nm}\rho_{kl}[\Pi_{mn,kl}^{n,M}(\omega) - \Pi_{mn,kl}^{M-a,M}(\omega)] + \rho_{mn}\rho_{lk}[\Pi_{kl,mn}^{M-a,M}(\omega) + \Pi_{kl,nn}^{n,M}(\omega)] \). Substituting in eq. (S6) the Π-terms vanish.

As a result the Renyi entropy flow becomes

\[ \frac{1}{S_M} \frac{dS_M}{dt} = M \left( \sum_{x'\hat{y}_y} \rho_{x'\rho_{y'y}} \left( -S_{xx',yy'}^{M,M} (\omega_{y'y}) + S_{xx',yy'}^{1,M} (\omega_{y'y}) \right) + \sum_{xx',yy'} \rho_{x'\rho_{y'y}} \left( S_{xx',yy'}^{0,M} (\omega_{y'y}) - S_{xx',yy'}^{1,M} (\omega_{y'y}) - S_{xx',yy'}^{M-1,M} (\omega_{y'y}) + S_{xx',yy'}^{M,M} (\omega_{y'y}) \right) \right) \] (S7)

**Generalized KMS**

The generalized correlator of two operators \( A \) and \( B \) is defined (see eq. (4)):

\[ S_{AB}^{N,M}(\omega) = \int d\tau e^{i\nu \tau} \text{tr} \left( A(0) \rho_{B}^{N} B(\tau) \rho_{B}^{M-N} / \text{tr}(\rho_{B}^{M}) \right) \]

This correlator in the energy eigenbasis can be rewritten in matrix form

\[ S_{nm,mm}^{N,M} = \int d\tau e^{i\nu \tau} \left( A_{nm} \frac{e^{-E_{n}E_{m}}}{Z(\beta)^N} B_{mn} e^{i(E_{n} - E_{m})\tau} \frac{e^{-E_{n}(M-N)}}{Z(\beta)^{M-N}} \right) \frac{Z(\beta)^{M-N}}{Z(\beta)} \]

\[ = 2\pi \delta (E_{m} - E_{n} + \nu) \frac{A_{nm}B_{mn}e^{-E_{n}E_{m}}}{Z(\beta)} e^{\beta N \nu} \] (S8)

\( Z(\beta) \) is the partition function defined as \( Z(\beta) = \sum e^{-\beta E_{i}} \).

The standard correlator is \( S_{AB}(\omega) = \int d\tau e^{i\nu \tau} \text{tr} \left( A(0)B(\tau)\rho_{B}/\text{tr}(\rho_{B}) \right) = 2\pi \delta (E_{m} - E_{n} + \nu) A_{nm}B_{mn}e^{-E_{n}/Z(\beta)} \), where KMS relation links this to dynamical susceptibility: \( S_{AB}(\nu) = 2\tilde{\chi}_{AB}(\nu)n_{B}(\nu) \). Substituting this in (S8) a generalized KMS relation is obtained:

\[ S_{AB}^{N,M}(\omega) = 2n_{B}(M\omega) e^{\beta \omega N} \tilde{\chi}_{AB}(\omega) \] (S9)

**Renyi entropy flow**

By substituting the generalized KMS relation (S9) in (S7) the Renyi entropy flow is determined based on susceptibility:

\[ \frac{1}{S_M} \frac{dS_M}{dt} = -2M \sum_{xx',yy'} \rho_{x'\rho_{y'y}} \tilde{\chi}_{yy',y')(\omega_{y'y'} \omega_{y'y}) \frac{n_{B}(M\omega_{y'y})}{n_{B}((M-1)\omega_{y'y})} e^{\beta \omega_{y'y'}} + 2M \sum_{xx',yy'} \rho_{x'\rho_{y'y}} \tilde{\chi}_{yy',y')(\omega_{y'y'}) \frac{n_{B}(M\omega_{y'y})}{n_{B}((M-1)\omega_{y'y})} \left( e^{\beta \omega_{y'y'}} - 1 \right) \] (S10)