ABSTRACT

This paper considers so called $1/\epsilon$ problem, the divergent behavior of the ground state energy of asymmetric potential, which is calculated with resurgence technique. Using resolvent method, I show that including not only one complex bion but multi-complex bions and multi-bounces contributions solves this problem. This result indicates the importance of summing all possible saddle points contribution and also the relationship between Exact WKB and path integral formalism.

Keywords resurgence · SUSY QM

1 Introduction

For many quantum theories, perturbative expansion is used to obtain physical quantities. Nevertheless, generally the radius of convergence is zero – it means the series is not convergent but asymptotic. This result was first suggested by Dyson\cite{1} in QED and it proved that the statement comes from the number of Feynman diagrams grows factorially in higher order terms\cite{2} \cite{3} \cite{4}. Resurgence theory gives a systematic treatment to solve this problem and also hidden relationship between perturbative and nonperturbative effects as follows:

In quantum theory, perturbative expansion around classical solutions gives this type of series.

$$Z(h) = \int_{\text{periodic}} \mathcal{D}\phi \, e^{-S_{\text{cl}}}$$

$$= \sum_{n} a_{n} h^{n} + e^{-\frac{2\pi i}{\alpha}} \sum_{n} b_{n} h^{n} + e^{-\frac{2\pi i}{\alpha}} \sum_{n} c_{n} h^{n} + ...$$

(2)

$S_{b}$ is an action of nonperturbative saddle which satisfies periodic boundary condition for imaginary time, like bion(instanton and anti-instanton) configuration. This type of series is called trans-series and each of series is asymptotic. A method to make sense of factorially divergent series is the Borel transform, which makes the series convergent. The Laplace transform of the function is called Borel sum, which has the same asymptotic expansion as the original series but the convergent function. i.e. For a series:

$$Z(h) = e^{-\frac{4}{\alpha}} \sum_{n=0}^{\infty} a_{n} h^{n+\alpha} \quad \alpha \notin \{-1, -2, -3, ...\}$$

(3)

The Borel transform of this series is

$$B[Z](z) \equiv \sum_{n=0}^{\infty} \frac{a_{n}}{\Gamma(n+\alpha)} (z - A)^{n+\alpha-1}$$

If a perturbative series is convergent, it indicates there are some symmetries which cancel a large number of diagrams, e.g. supersymmetry.
The Borel summation: $S[Z]$ is defined as

$$S[Z](h) \equiv \int_{\lambda}^{\infty} e^{-\frac{\theta}{\hbar} B[Z](z)} \, dz \quad \theta = \text{arg}(h)$$

(5)

Borel summation often has ambiguities, so called Borel ambiguity when poles exist on the integral path of the Laplace transform. The resurgence theory claims that even though the Borel summation of each term can have ambiguities, these ambiguities are cancelled when all perturbation series (i.e. trans-series) are taken into account. From this cancellation, we can obtain the information of other saddle points, which mean the nonperturbative effects. [5][6][7][8]

The resurgence method is used in various systems, not only quantum mechanics (for QFT, [9][10][11] and for string, [12][13]) because perturbative expansion is quite general method in physics. Further, the relationship between resurgence theory and Picard-Lefschetz theory[14] tells us a new perspective in path integral formalism – complex classical solutions, which look naively unphysical configurations – can contribute to physical quantity[15][16][17][18][19][20]. This statement is precisely related to the work of Witten on Chern-Simons theory [21], which claims the complexification of the phase space formalism of path integral.

In this paper I show a method to solve the problem and the reason behind the prescription. It indicates the relationship between path integral formalism and Exact WKB[23][24][25], which is a resurgence method to analyze the structure of differential equation[2].

This paper is organized as follows: In the rest of this section I explain SUSY QM briefly and what $1/\epsilon$ problem is. Sections 2 and 3 are dedicated to the detailed calculation of the partition function and the leading nonperturbative contribution of the ground state energy. In 4 I discuss the relation between Fredholm determinant and Exact WKB calculous, while in 5 I give conclusions and summary.

### 1.1 $1/\epsilon$ problem

Consider this ($0+1$) dimension supersymmetric quantum mechanics (Witten model):

$$S = \int_{-\infty}^{\infty} dt \left( \frac{1}{2} \dot{x}^2 - \frac{1}{2} \left( W'(x)^2 + i \eta^\dagger \dot{\eta} + W''(x) \eta^\dagger \eta \right) \right)$$

(6)

where $W(x)$ is a superpotential. The Hamiltonian of this system can be written in terms of only bosonic variable after projecting to fermion number eigenstates:

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}$$

(7)

$$H_\pm = \frac{1}{2} \dot{p}^2 + V_\pm(x)$$

$$= \frac{1}{2} \dot{p}^2 + \frac{1}{2} W'(x)^2 \mp \frac{1}{2} \hbar W''(x)$$

(8)

(9)

The term $\mp \frac{1}{2} \hbar W''$ comes from the fermion terms. The Euclidean path integral is $Z = \int Dx \, e^{-\frac{S}{\hbar}}$ in this notation. Now, set the superpotential to $W(x) = \frac{1}{4} x^3 - a^2 x$. It gives

$$H_\pm = \frac{1}{2} \dot{p}^2 + \frac{1}{2} (x^2 - a^2)^2 \mp \hbar x$$

(11)

The zero energy eigenstate is

$$\langle x|0 \rangle = e^{-\frac{W(x)}{\hbar}} = e^{-\frac{\hbar}{4} \left( \frac{1}{4} x^3 - a^2 x \right)}$$

(12)

2This claim is coming from the equivalence of Borel summation and integration on Lefschetz thimble.

3This is a non-BPS periodic solution satisfying complexified Newton’s equation.

4Originally resurgence by Ecalle was used for Stokes phenomena of differential equations.
This state is not normalizable in the real axis: \((-\infty, \infty)\). Therefore the supersymmetry of this system is dynamically broken. However, the effect of SUSY breaking is nonperturbative from non-renormalizable theorem. It means the perturbative expansion of the ground state energy is zero in all order. To examine the resurgence structure, we need to introduce deforming parameter \(\epsilon\) here:

\[
H_\pm = \frac{1}{2} p^2 + \frac{1}{2} (x^2 - a^2)^2 \mp \epsilon \hbar x
\]  

(13)

When \(\epsilon = 1\) goes back to the original SUSY Hamiltonian.

Where the parameter \(\epsilon\) is originally introduced to study resurgence structure because the perturbative expansion of the ground state energy is zero when SUSY exists (\(\epsilon = \pm 1\)). Thanks to this parameter we can see a nontrivial relation between vacuum saddle point \((x = \pm a)\) and complex bion solution [22]:

\[
x_{cb}(t) = x_1 - \frac{x_1 - x_T}{2} \coth \left( \frac{\omega_{cb} t_0}{2} \right) \left( \tanh \left( \frac{\omega_{cb} (t + t_0)}{2} \right) - \tanh \left( \frac{\omega_{cb} (t - t_0)}{2} \right) \right)
\]

(14)

Where \(x_1\) is a vacuum, \(x_T = -x_1 + i \sqrt{\frac{\hbar}{x_1}}\), \(\omega_{cb} = \sqrt{V''(x_1)}\), \(t_0 = \frac{2}{\omega_{cb}} \arccosh \left( \frac{\sqrt{3}}{1 - \sqrt{1 - V''(x_T)/\omega_{cb}^2}} \right)\).

When we consider both saddle points and calculate the Borel summation of the ground state energy, it gives the unambiguous expression.

From this paper[26], the perturbative expansion of ground state energy\(^5\) is

\[
E_{0,\text{pert}} = \sum_{n=0}^{\infty} a_n \hbar^n
\]

(15)

\[
a_n = -\frac{6^{-\epsilon+1} \Gamma(n - \epsilon + 2) \hbar}{2\pi \Gamma(1 - \epsilon)} \left( \frac{1}{2 S_I} \right)^n
\]

(16)

Where \(S_I = \frac{3 \hbar}{4a^2}\) is the Action of one instanton. The Borel summation of this series is

\[
-\frac{6^{-\epsilon+1}}{2\pi} \frac{1}{\Gamma(1 - \epsilon)} \int_0^\infty e^{-z} \frac{z^{-\epsilon+1}}{1 - \frac{z}{2 S_I}} dz
\]

(17)

It has this Borel ambiguity,

\[
\text{Im} S[E_{0,\text{pert}}](\hbar) = \pm \frac{1}{2} 6^{-\epsilon+1} \frac{1}{\Gamma(1 - \epsilon)} 2 S_I^{\epsilon - \epsilon} e^{-2 S_I}
\]

(18)

\(\pm\) is corresponded to the sign of \(\text{Im}(\hbar)\). Also The nonperturbative effect for the ground state energy was calculated at\(^6\)

\[
E_0 = -\frac{1}{\beta} \log Z = -\frac{1}{\beta} \log (Z_0 + Z_{\text{bion}} + Z_{2-\text{bion}} + ...)
\]

(19)

\[
= -\frac{1}{\beta} \log Z_0 - \frac{1}{\beta} \frac{Z_{\text{bion}}}{Z_0} + ... \quad \left( \text{when } \frac{Z_{\text{bion}}(s)}{Z_0(s)} < 1 \right)
\]

(20)

\(Z_0\) and \(Z_{\text{bion}}\) are partition functions around a vaum and one bion solution, respectively. It suggests one bion is enough to obtain the leading nonperturbative contribution for ground state energy. Although this method is quite common in many literatures, it leads to incorrect result as I explain in this paper. Using this method, the nonperturbative contribution, which is from one (complex) bion is

\[
I_{cb} = -\frac{1}{\beta} \frac{Z_{cb}}{Z_0} = \frac{1}{2\pi} \left( \frac{\hbar}{16a^2} \right)^{\epsilon-1} \left( -\cos(\pi) \Gamma(\epsilon) \pm i \frac{\pi}{\Gamma(1 - \epsilon)} \right) e^{-2 S_I}
\]

(21)

Again, \(\pm\) is corresponded to the sign of \(\text{Im}(\hbar)\)\(^7\) Combining the two calculations, we can show

\[
\text{Im}(S[E_{0,\text{pert}}] + I_{cb}) = 0
\]

(22)

\(^5\)Of course when we set \(\epsilon = 1\), the all coefficients are vanished by supersymmetry.

\(^6\)Here I denote any periodic solution like real bion, bounce, complex-bion as “bion”

\(^7\)This ambiguity is not Borel ambiguity but coming from Stokes phenomena of quasi-moduli integral. However, It does not conflict with the resurgence claim because of the equivalence of Borel summation and integral on the Lefschetz thimble.
Therefore the ground state energy is
\begin{equation}
E_0 = -\frac{1}{2\pi} \left( -\frac{\hbar}{16a^2} \right)^{\epsilon^{-1}} \Gamma(\epsilon) e^{-\frac{\epsilon a}{2\pi} \cos(\epsilon \pi)}
\end{equation}
\begin{equation}
= -\frac{1}{2\pi} \left( -\frac{\hbar}{16a^2} \right)^{\epsilon^{-1}} \left( \frac{1}{\epsilon - \gamma + \mathcal{O}(\epsilon)} \right) e^{-2S_I}
\end{equation}

As we can assume resurgence theory, the Borel ambiguity from perturbative expansion is exactly cancelled by the other ambiguity from nonperturbative (bion) saddle. However, this expression has two strange facts: 1. This is singular at $\epsilon \to 0$, which is the case of symmetric double well potential. 2. In the case of symmetric double well, the nonperturbative contribution of the ground state energy is coming from one-instanton but not bion. [27]

This is the $1/\epsilon$ problem in deformed SUSY quantum system. Actually the similar problem occurs in $CP^n$ and sine-Gordon system. [20] [28] [29]

2 The calculation and result

2.1 The prescription

The method to calculate the ground state energy is based on Euclidean path integral with periodic boundary condition.

\begin{equation}
Z(\beta) = \int_{\text{periodic,}\beta} \mathcal{D}xe^{-\beta E[\pi(x)]}
\end{equation}

For the partition function $Z(\beta)$, all classical solutions whose period are $\beta$ should contributes. Also the partition function should be invariant under $\epsilon \to -\epsilon$ because the spectrum doesn’t change by this reflection. Although the calculation in [26], they considered only one complex bion and $\beta \to \infty$ limit first. These procedures does not treat the symmetry properly and lead $1/\epsilon$ problem.

Therefore the partition function $Z(\beta)$ must satisfy these conditions.

- Including not only one (complex) bion but multi-complex bions. Especially we have to consider finite $\beta$ to calculate these contributions correctly.
- Including also bounce solutions because all periodic classical solution can contribute to the partition function from path integral view.

2.2 The resolvent method

To calculate the ground state energy, I used a resolvent method [30]

\begin{equation}
\int_0^{\infty} Z(\beta) e^{\beta E} d\beta = \int_0^{\infty} \sum_n e^{-\beta(E_n-E)} d\beta
\end{equation}
\begin{equation}
= \sum_n \frac{1}{E_n - E}
\end{equation}
\begin{equation}
= \text{tr} \frac{1}{H - E} = G(E)
\end{equation}

The trace of resolvent $G(E)$ can be expressed as

\begin{equation}
-\frac{\partial}{\partial E} \log D = G(E)
\end{equation}

Where $D(E) = \det(H - E)$ is the Fredholm determinant. The poles of $G(E)$ or zeros of $D(E)$ is the spectrum of $H$. 

4
2.3 The partition function

The expression of the partition function is this form:

\[ Z = 1e^{\beta \alpha e} + e^{-\beta \alpha e} + \sum_{n=1}^{\infty} \left( e^{-2S_I} \frac{S_I}{2\pi} \left( \frac{\det M_I}{\det M_0} \right)^{-1} \right)^n \beta QMI^n(\epsilon) \]

\[ + \sum_{n=1}^{\infty} \left( e^{-2S_I} \frac{S_I}{2\pi} \left( \frac{\det M_I}{\det M_0} \right)^{-1} \right)^n \beta QMI^n(-\epsilon) \quad \text{(bounce)} \]  

(30)

Two \( Z \) are from stationary classical solutions (vacuum and false vacuum). The factors \( e^{\pm \beta \alpha e} \) come from \( e^{-S_{\text{vac}}/\hbar} = e^{-V(x_{\text{vac}})/\hbar} = e^{\beta \alpha e} \). The rest of summations come from the nonperturbative contributions, which are complex bions and real bounces, respectively. The linear factor \( \beta \) is from the translation symmetry of (imaginary) time dependent solutions. \( B = e^{-2S_I} \frac{S_I}{2\pi} \left( \frac{\det M_I}{\det M_0} \right)^{-1} \) is the square of one instanton contribution: bion contribution. The terms in \( B \) are

\[ x_I(\tau) = a \tan h a (\tau - \tau_c) \quad \text{(31)} \]

\[ S_I = \frac{S[x_I, \epsilon = 0]}{\hbar} = \frac{4a^3}{3\hbar} \quad \text{(32)} \]

\[ \frac{\det M_I}{\det M_0} = \frac{1}{12} \quad \text{(33)} \]

The exact classical solution is not this instanton but complex bions and bounces (These solutions are interchanged by \( \epsilon \rightarrow -\epsilon \)). To calculate the contribution from these solutions for path integral, we have to consider quasi-moduli integral, which comes from a nearly flat direction in the configuration space.

\( QMI \), so called quasi-moduli integral(QMI) comes from the nearly flat direction in complex bion solution. i.e. The separation between instanton and anti-instanton in a complex bion is

\[ \tau = 2t_0 \approx \frac{1}{2a} \log \left( \frac{16a^3}{\epsilon h} \pm i\pi \right) \quad \text{(34)} \]

This can be infinite under \( \hbar \rightarrow 0 \), which leads to quasi-zero modes\(^8\). Therefore we have to consider the interaction potential \( V \) whose variable is \( \tau \). Actually the complex bion itself is understood as the saddle point of \( V \).

The form of quasi-moduli integral for \( n \)-complex bions is\(^9\)

\[ QMI^n(\epsilon) = e^{\beta \alpha e} \frac{1}{2n} \left( \prod_{i=1}^{2n} \int_{0}^{\infty} d\tau_i e^{-V_i(\tau_i)} \right) \delta \left( \sum_{k=1}^{2n} \tau_k - \beta \right) \]

(35)

\[ V_i(\tau) = \begin{cases} -\frac{16a^3}{\hbar} e^{-2\alpha \tau} + 2a\epsilon \tau & (i = \text{odd}) \\ -\frac{16a^3}{\hbar} e^{-2\alpha \tau} - a\epsilon \tau & (i = \text{even}) \end{cases} \]

(36)

The factor \( \frac{1}{2n} \) in front of the integral arises because the configuration is invariant under cyclic permutation of the \( \tau_i \). The factor \( e^{\beta \alpha e} \) comes from changing the off-set because the interaction potential \( V_i \) is determined from the true vacuum (the minimum point) but the two vacua have the potential \( e^{\beta \alpha e} \). Actually this procedure is equivalent to

\[ V_i(\tau) = \begin{cases} -\frac{16a^3}{\hbar} e^{-2\alpha \tau} + a\epsilon \tau & (i = \text{odd}) \\ \frac{16a^3}{\hbar} e^{-2\alpha \tau} - a\epsilon \tau & (i = \text{even}) \end{cases} \]

(37)

and omitting the factor \( e^{\beta \alpha e} \). However the integral for \( i = \text{even} \) is ill-defined in this case, using analytic continuation of \( \Gamma \) function, both give the same resolvent. From this view, we can see the contribution of multi-complex bions and multi-bounces are equivalent.

Also if we set \( V_i = 0 \) and \( \epsilon = 0 \) This integral becomes

\[ \left( \prod_{i=1}^{2n} \int_{0}^{\infty} d\tau_i \right) \delta \left( \sum_{k=1}^{2n} \tau_k - \beta \right) = \frac{1}{(2n-1)!} \beta^{2n-1} \]

(38)

\(^8\)The existence of this direction comes from the \( \hbar \) dependence of our potential.

\(^9\)There are two quasi-moduli integrals for one complex bion because we consider finite \( \beta \) now.
Therefore the partition function becomes
\[
Z = 2 \sum_{n=0}^{\infty} \frac{B^n \beta^{2n}}{(2n)!} = 2 \cosh(\sqrt{B\beta})
\] (39)

This is usual dilute instanton gas approximation of symmetric double well potential. (Actually we can see the importance of including multi-bion configurations in this approximation. See Appendix 5.)

From here we set \( a = \frac{1}{2} \) for simplicity. This integral can be evaluated as
\[
QMI^\alpha(\epsilon) = e^{\beta_\pm \frac{1}{2n}} \prod_{i=1}^{2n} \left( \int_{0}^{\infty} d\tau e^{-V_i(\tau)} \right) \delta \left( \sum_{k=1}^{2n} \tau_k - \beta \right)
\] (40)
\[
= e^{\beta_\pm \frac{1}{2n}} \prod_{i=1}^{2n} \left( \int_{0}^{\infty} d\tau e^{-V_i(\tau)} \right) \frac{1}{2\pi} \int_{-\infty}^{\infty} de^{i \sum_{k=1}^{2n} \tau_k - \beta}
\] (41)
\[
= e^{\beta_\pm \frac{1}{2n}} \frac{1}{2\pi} \int_{-\infty}^{\infty} de^{i \beta} \left( \left( \int_{0}^{\infty} d\tau e^{i(\beta + \frac{2i\pi}{\beta} e^{-\tau})} \right) \left( \int_{0}^{\infty} d\tau e^{i(\beta - \epsilon - \epsilon + \frac{2i\pi}{\beta} e^{-\tau})} \right) \right)^n
\] (42)
\[
= e^{\beta_\pm \frac{1}{2n}} \frac{1}{2\pi} \int_{-\infty}^{\infty} ds e^{-s\beta} \left( \left( \int_{0}^{\infty} d\tau e^{i(\epsilon - \epsilon + \frac{2i\pi}{\beta} e^{-\tau})} \right) \left( \int_{0}^{\infty} d\tau e^{i(\epsilon - \epsilon + \frac{2i\pi}{\beta} e^{-\tau})} \right) \right)^n
\] (43)

We can show the integral is evaluated as
\[
\int_{0}^{\infty} d\tau e^{(\epsilon - \epsilon + \frac{2i\pi}{\beta} e^{-\tau})} = e^{\pm i\pi(\epsilon - s)} \left( \frac{\beta}{2} \right)^{\epsilon - s} \Gamma(\epsilon - s)
\] (44)

Here the \( \pm \) is corresponded to the sign of \( \Im(\beta) \), which comes from Stokes phenomenon of this quasi-moduli integral (See Appendix F in [29]).

After all, the form of quasi-moduli integral in finite \( \beta \) is
\[
QMI^\alpha(\epsilon) = \frac{1}{4\pi i n} e^{\beta_\pm \frac{1}{2n}} \int_{-\infty}^{\infty} ds e^{-s\beta} \left( \left( e^{\pm i\pi(\epsilon - s)} \left( \frac{\beta}{2} \right)^{\epsilon - s} \Gamma(\epsilon - s) \right) \left( e^{\pm i\pi(\epsilon - s)} \left( \frac{\beta}{2} \right)^{\epsilon - s} \Gamma(\epsilon - s) \right) \right)^n
\] (45)

2.4 Calculating the resolvent

Using \( Z_0 = \sum_{k=0}^{\infty} e^{-\beta(k+1/2)} \), (30) can be written as
\[
Z = \left\{ Z_0 e^{\beta_\pm \frac{1}{2n}} + \sum_{n=1}^{\infty} B^n \sum_{k=0}^{\infty} e^{-\beta(k+1/2)} e^{\beta_\pm \frac{1}{2n}} \frac{1}{4\pi i n} \int_{-\infty}^{\infty} ds e^{-s\beta} (I(s, \epsilon I(s, 0))^n \right\} + \left\{ (\epsilon \to -\epsilon) \right\}
\] (46)
\[
B = e^{-2S I} \left( \frac{\det M_I}{\det M_0} \right)^{-1} = e^{-2S I} \left( \frac{2\pi}{2\pi} \right)^{-1}
\] (47)
\[
I(s, \epsilon) = e^{\pm i\pi(\epsilon - s)} \left( \frac{\beta}{2} \right)^{\epsilon - s} \Gamma(\epsilon - s)
\] (48)

The Laplace transform gives the trace of resolvent \( G(\epsilon) \).
\[
G(\epsilon) = \left\{ G_0(\epsilon + \epsilon/2) + \sum_{n=1}^{\infty} B^n e^{\beta_\pm \frac{1}{2n}} \int_{0}^{\beta} d\beta \int_{-\infty}^{\infty} ds \sum_{k=0}^{\infty} e^{(E-s-1/2-k+\epsilon/2)\beta} (I(s, \epsilon I(s, 0))^n \right\} + \left\{ (\epsilon \to -\epsilon) \right\}
\] (49)
Where $G_0(E) = \frac{\partial}{\partial E} \log \Gamma(1/2 - E)$ is the resolvent of harmonic oscillator. (See Appendix B)
Using $-\log (1 + x) = \sum_{n=1}^{\infty} \frac{(-x)^n}{n}$, finally

$$G(E) = -\frac{\partial}{\partial E} \left\{ \log \frac{1}{2} - E - \frac{\epsilon}{2} \right\} + \frac{1}{2} \sum_{k=0}^{\infty} \log (1 - BI(s = E - 1/2 - k + \epsilon/2, \epsilon) I(s = E - 1/2 - k + \epsilon/2, 0))\right\} + \frac{\partial}{\partial E} \left\{ (\epsilon \rightarrow -\epsilon) \right\} \quad (50)$$

Therefore, the Fredholm determinant is

$$D(E) = \frac{1}{\Gamma(\frac{1}{2} - E - \frac{s}{2}) \Gamma(\frac{1}{2} - E + \frac{s}{2})} \prod_{k=0}^{\infty} \sqrt{1 - BI(s_+, \epsilon) I(s_+, 0)} \sqrt{1 - BI(s_-, \epsilon) I(s_-, 0)} \quad (51)$$

(51)

(with $s_{\pm} = E - 1/2 - k \pm \epsilon/2$)

Therefore, $D(E) = 0$ gives this equation:

$$\frac{1}{\Gamma(\frac{1}{2} - E - \frac{s}{2}) \Gamma(\frac{1}{2} - E + \frac{s}{2})} - B e^{\pm i\pi(1-2E)} \left( \frac{\hbar}{2} \right)^{(1-2E)} = 0 \quad (53)$$

where $B = \frac{1}{2\pi} \frac{\hbar}{2}$. If we only consider the finite number of bions, there partition function is still singular like $\sim 1/\epsilon^n$. However, using the analytic continuation $(-\log (1 + x) = \sum_{n=1}^{\infty} \frac{(-x)^n}{n})$ and two different quasi-moduli integrals give analytic function around $\epsilon = 0$. Therefore the summation of all classical periodic solutions is necessary to obtain the correct result. This statement is relevant to the case of “fixed singularities”, which is the infinite number of Borel singularities of wave function in terms of Exact WKB analysis.

3 Calculating the energy

3.1 When $\epsilon = 0$

$$\frac{1}{\Gamma(\frac{1}{2} - E) \Gamma(\frac{1}{2} - E)} - B e^{\mp i\pi(1-2E)} \left( \frac{\hbar}{2} \right)^{(1-2E)} = 0 \quad (54)$$

Setting $E = \frac{1}{2} + x$, $(x$ is the nonperturbative effect$)$ it gives

$$\frac{1}{\Gamma(-x)} = \sqrt{B} e^{\mp \pi i x} \left( \frac{\hbar}{2} \right)^{-x} \quad \text{(55)}$$

$$\frac{1}{\Gamma(-x)} = -\sqrt{B} e^{\mp \pi i x} \left( \frac{\hbar}{2} \right)^{-x} \quad \text{(56)}$$

Using the reflection formula: $\Gamma(x) \Gamma(1 - x) = \frac{\pi}{\sin \pi x}$,

$$\frac{\sin \pi x}{\pi} = \sqrt{B} e^{(\mp \pi i - \log \frac{\hbar}{2})} \frac{1}{\Gamma(1 + x)} \quad \text{(57)}$$

$$\frac{\sin \pi x}{\pi} = -\sqrt{B} e^{(\mp \pi i - \log \frac{\hbar}{2})} \frac{1}{\Gamma(1 + x)} \quad \text{(58)}$$

expand around $x = 0$, it shows

$$x = \sqrt{B} \left( 1 + \left( \mp \pi i - \log \frac{\hbar}{2} + \gamma \right) x + O(x^2) \right) \quad \text{(59)}$$

$$= \sqrt{B} + \left( \mp \pi i - \log \frac{\hbar}{2} + \gamma \right) B + O(B^{3/2}) \quad \text{(60)}$$

and

$$x = -\sqrt{B} - \left( \mp \pi i - \log \frac{\hbar}{2} + \gamma \right) B + O(B^{3/2}) \quad \text{(62)}$$

\footnote{I dropped $k$ here. This integer $k$ is just changing the argument of $\Gamma$ function and we can ignore it when we consider the zeros of $D(E)$.}
Therefore, the result gives indeed energy splitting by one instanton: $\sqrt{B}$, and the imaginary ambiguity is proportional to $B$ (bion) for the symmetric double well. The ambiguity of ground state energy \((62)\) is exactly cancelled by Borel ambiguity from the perturbative expansion around the vacuum. (set $\epsilon = 0$ in \((18)\))

### 3.2 When $\epsilon = 1$ (SUSY)

\[
\frac{1}{\Gamma(-E)\Gamma(1-E)} - B e^{\pm i\pi(1-2E)} \left(\frac{\hbar}{2}\right)^{(1-2E)} = 0 \tag{63}
\]

Setting $E = k + x$, it gives

\[
\frac{1}{\Gamma(-x)\Gamma(1-x)} - B e^{\pm i\pi(1-2x)} \left(\frac{\hbar}{2}\right)^{(1-2x)} = 0 \tag{64}
\]

Therefore

\[
\frac{\sin \pi x}{\pi} = B \frac{\hbar}{2} + 2\pi i x \left(\frac{\hbar}{2}\right)^{-2x} \tag{65}
\]

Finally,

\[
x = B \frac{\hbar}{2} + O(x) \tag{66}
\]

\[
x = \frac{e^{-1/3h}}{2\pi} + O(B^2) \tag{67}
\]

The leading nonperturbative contribution is one bion.

### 4 Future works (Relation between Exact WKB method)

I conjecture the form of the Fredholm determinant is equivalent to the quantization condition derived from Exact WKB method.

At leading order of the Exact WKB calculation, we can evaluate the quantization condition from the connection formula of symmetric double well

\[
\sim (1 - e^{2\pi i x})^2 \left(1 - \frac{\sqrt{B}}{1 - e^{2\pi i x}}\right) \left(1 + \frac{\sqrt{B}}{1 - e^{2\pi i x}}\right) = 0 \quad (x = 1/2 - E) \tag{68}
\]

\((1 - e^{2\pi i x})\) comes from Voros coefficient related to harmonic oscillator(2 nondegenerate Stokes curves with a simple turning point) and the other part \((1 - \sqrt{B} / (1 - e^{2\pi i x}))\) comes from the infinite number of Borel singularities (At $z = 2\pi n E$ in the Borel plane), so called fixed singularities in Exact WKB literatures. The factor $B$ is also a nonperturbative term \((\sim e^{-\frac{A g}{\hbar}})\) from the other Voros coefficient.

When we compare the distribution of zeros, we can assume \(\frac{1}{1(x)} \sim 1 - e^{2\pi i x}\) and \(1 - \Gamma(x) \sim 1 - \frac{1}{1 - e^{2\pi i x}}\). If this identification is verified, we can say the Fredholm determinant via path integral is equivalent to the quantization condition from Exact WKB. Furthermore, it suggests the reason why $1/\epsilon$ problem is solved considering multi-bions configuration. To obtain the correct quantization condition, we have to take into account the infinite number of Borel singularities, which are corresponded to multi-bions.

Also this correspondence suggests a method to calculate the index of Lefschetz thimble (intersection number) with Exact WKB:

\[
Z = \text{tr} e^{-\beta H} \tag{69}
\]

\[
= \int D x \ e^{-S[x]} \tag{70}
\]

\[
= S[e^{-S[x]}] \sum a_n \hbar^n + S[e^{-S[x]}] \sum b_n \hbar^n + ... \tag{71}
\]

\[
= \sum n_{\sigma} \int J_{\sigma} D x \ e^{-S[x]} = \sum n_{\sigma} Z_{\sigma}(\beta) \tag{72}
\]
$S$ denotes the Borel summation of each series and $x_\sigma$ is a saddle point. $J_\sigma$ and $n_\sigma$ are called Lefschetz thimble and intersection number in Picard Lefschetz theory. The Laplace transform of $Z(\beta)$ gives the trace of resolvent $G(E)$ but this is linear transform, therefore we can write

$$\text{tr} \frac{1}{H - E} = G(E) = \int_0^\infty Z(\beta) e^{\beta E} d\beta$$

(73)

$$= \sum_\sigma n_\sigma \int_0^\infty Z_\sigma(\beta) e^{\beta E} d\beta$$

(74)

$$= \sum_\sigma n_\sigma G_\sigma(E)$$

(75)

The trace of resolvent $G(E)$ can be expressed as $-\frac{\partial}{\partial E} \log D = G(E)$, it means

$$D(E) = \prod_\sigma D_\sigma^\sigma(E)$$

(76)

Therefore if this $D(E)$

is equivalent to the quantization condition derived from Exact WKB, we can calculate the index $n_\sigma$ with this method.

5 Conclusion

In this paper, I showed the $1/\epsilon$ problem in the tilted double well potential is solved when including multi complex bion and multi bounce contributions. The problem is a thorn in our side to compare the complex bion calculation used nowadays to classic instanton gas calculation. Therefore to show the divergence disappeared with this prescription is important in resurgence calculous based on path integral, also this problem may suggest the relationship between Exact WKB and path integral formalism.

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Appendices

A About the leading contribution of symmetric double well potential in dilute gas approximation

The partition function must satisfy with periodic boundary condition. Therefore

$$Z = 2(Z_0 + Z_{\text{bion}} + Z_{2-\text{bion}} + ... )$$

(77)

The factor 2 comes from the two vacua. The bion contributions are nonperturbative effects. So we often consider $Z_0 > Z_{\text{bion}(s)}$ in weak coupling limit. Under this assumption we can calculate the ground state energy like this:

$$E_0 = -\frac{1}{\beta} \log Z = -\frac{1}{\beta} \log(Z_0 + Z_{\text{bion}} + ...)$$

(78)

$$= -\frac{1}{\beta} \log \left( Z_0 \left( 1 + \frac{Z_{\text{bion}}}{Z_0} + ... \right) \right)$$

(79)

$$= -\frac{1}{\beta} \log Z_0 - \frac{1}{\beta} \log \left( 1 + \frac{Z_{\text{bion}}}{Z_0} + ... \right)$$

(80)

$$= -\frac{1}{\beta} \log Z_0 - \frac{1}{\beta} \left( \frac{Z_{\text{bion}}}{Z_0} \right) + ... \quad \text{when } \frac{Z_{\text{bion}(s)}}{Z_0} < 1$$

(81)

The definition of Fredholm determinant (or resolvent) needs a regularization, e.g. $G_{\text{reg.}} \equiv G(E) - G(0)$ or $D_{\text{reg.}} \equiv \frac{D(E)}{D(0)}$ or zeta function regularization for $D(E)$.
Therefore the leading nonperturbative contribution is coming from one bion in this calculation. In fact, this method is quite ordinal in many literatures. Although, it gives a wrong answer in this case. As well known, the leading non-perturbative contribution is one instanton but not one bion. This contradiction is because of the incorrect assumption: \( \frac{Z_{\text{bion}(s)}}{Z_0} < 1 \).

As I showed in (39), the partition function of this system is \( \frac{Z}{Z_0} = 2 \sum_{n=0}^{\infty} \frac{B^n \beta^{2n}}{(2n)!} = 2 \cosh(\sqrt{B} \beta) \). This expression contains the multi-bions. It gives

\[
E_0 = -\frac{1}{\beta} \log Z
\]

\[
= -\frac{1}{\beta} \log Z_0 - \frac{1}{\beta} \log \left( e^{\sqrt{B} \beta} + e^{-\sqrt{B} \beta} \right) - \frac{1}{\beta} \log 2
\]

\[
= -\frac{1}{\beta} \log Z_0 - \frac{1}{\beta} \log \left( e^{\sqrt{B} \beta} (1 + e^{-2\sqrt{B} \beta}) \right) - \frac{1}{\beta} \log 2
\]

\[
= -\frac{1}{\beta} \log Z_0 - \sqrt{B} \quad \text{(in } \beta \rightarrow \infty)\]

Therefore the leading contribution is one instanton. This puzzle comes from the \( \beta \), which comes from the translation symmetry of classical solution. This \( \beta \) is infinity to calculate the ground state energy, so the assumption \( \frac{Z_{\text{bion}(s)}}{Z_0} < 1 \) is invalid generally.

### B Resolvent of harmonic oscillator

Consider harmonic oscillator. i.e. the eigenvalues are \( \frac{1}{2} + n \quad (n = 0, 1, 2...) \) Then the Fredholm determinant is

\[
D(E) = \det(H - E) = \prod_{n=0}^{\infty} \left( n + \frac{1}{2} - E \right)
\]

This infinite product is ill-defined though, we can define this quantity with zeta function regularization (zeta regularized product).

In this case, the spectral zeta function is

\[
\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{(n + \frac{1}{2} - E)^s}
\]

(87)

This is Hurwitz zeta function. Also we get

\[
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}
\]

(88)

\[
\frac{\partial}{\partial s} \zeta(s, a) \bigg|_{s=0} = \log \Gamma(a) - \frac{1}{2} \log(2\pi)
\]

(89)

Therefore the determinant is evaluated in terms of zeta regularized product,

\[
D(E) = \det(H - E) = e^{-\zeta(0, \frac{1}{2} - E)}
\]

(90)

\[
= \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} - E\right)}
\]

(91)

The trace of the resolvent is

\[
G(E) = -\frac{\partial}{\partial E} \log D = \frac{\partial}{\partial E} \log \Gamma\left(1 + \frac{1}{2} - E\right)
\]

\[
= -\frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} - E\right)}
\]

(92)

(The constant term \( \sqrt{2\pi} \) is dropped by the derivative)

The zeros of \( D(E) \) are the poles of \( \Gamma\left(\frac{1}{2} - E\right) \). i.e. the eigenvalue of harmonic oscillator: \( \frac{1}{2} + n \quad (n = 0, 1, 2...) \)
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