A class of Finsler metrics admitting first integrals

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A CLASS OF FINSLER METRICS ADMITTING FIRST INTEGRALS

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Abstract. We use two non-Riemannian curvature tensors, the $\chi$-curvature and the mean Berwald curvature to characterise a class of Finsler metrics admitting first integrals. This class includes Finsler metrics of constant flag curvature.

1. Introduction

Finsler geometry is a natural extension of Riemannian geometry and, while many geometric structures can be extended from the Riemannian to the Finslerian setting, within the Finslerian context there are many non-Riemannian geometric quantities, [17, Ch. 6].

Existence of first integrals (conservation laws, or constants of motion) is of great importance, they provide a lot of information about the corresponding geometry, including some rigidity results, [6], [7], [21].

In Riemannian geometry, Topalov and Matveev obtained in [21], for two projectively equivalent metrics on an $n$-dimensional manifold, a set of $n$ first integrals. An extension of this result to the Finslerian context has been proposed by Sarlet in [16]. In [7], Foulon and Ruggiero have shown the existence of a first integral for $k$-basic (of isotropic curvature) Finsler surfaces.

It has been proven by Li and Shen in [11], that Finsler metrics of isotropic curvature can be characterised using the $\chi$-curvature tensor. The $\chi$-curvature has been introduced by Shen in [18], in terms of another important non-Riemannian quantity, the Shen-function (S-function) [17, §5.2]. Since then, a lot of effort has been devoted to study the $\chi$-curvature, [10, 14, 19].

In this work we extend the result of Foulon and Ruggiero from [7] to Finsler manifolds of dimension $n \geq 2$, by providing a class of Finsler metrics that admit first integrals. This class of Finsler metrics is characterised using the $\chi$-curvature tensor and the mean Berwald curvature, $E_{jk} = \frac{1}{2}B_{ijk}$, where $B_{ijkl}$ is the Berwald curvature, [17, §6.1]. Very important in our work is the fact that the mean Berwald curvature can be expressed also in terms of the $S$-function. The $S$-function is a Finsler function if and only if the mean Berwald curvature has maximal possible rank, $n - 1$. For a Finsler function $F$, we denote by $\text{det} g$, the determinant of its metric tensor $g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$, where $y^i$ are the fiber coordinates in the tangent bundle $TM$.

The main result of this paper provides a class of Finsler metrics that admit a first integral.

Theorem 1.1. Consider $F$ a Finsler metric that satisfies the following two conditions:

i) the $\chi$-curvature vanishes;

ii) the mean Berwald curvature has rank $n - 1$.
Then,

$$
\lambda = -\frac{1}{\det g} \begin{vmatrix}
2F E_{ij} & \frac{\partial F}{\partial y^i} \\
\frac{\partial F}{\partial y^j} & 0
\end{vmatrix}
$$

is a first integral for the geodesic spray $G$ of the Finsler metric $F$, which means that $G(\lambda) = 0$.

Finsler metrics of constant flag curvature have vanishing $\chi$-curvature. Therefore, Finsler metrics of constant curvature that satisfy also the algebraic condition regarding the mean Berwald curvature represent an important class admitting the first integral (1.1).

For a Finsler surface, the first condition of Theorem 1.1, $\chi = 0$, is equivalent to the fact that the Finsler metric has isotropic curvature (it is a $k$-basic Finsler metric). Also, in dimension 2, the mean Berwald curvature is proportional to the vertical Hessian of the Finsler metric, the proportionality factor, the function $\lambda$, was known since Berwald, [1, (8.7)]. Hence for Finsler surfaces, the second condition of Theorem 1.1 is automatically satisfied. Moreover, the first integral $\lambda$, given by formula (1.1), reduces in the 2-dimensional case to the first integral $f$ obtained by Foulon and Ruggiero in [7, Theorem B].

For the proof of Theorem 1.1, the two conditions $\chi = 0$ and $\text{rank}(E_{ij}) = n - 1$ tell us that the $S$-function is a Finsler metric, projectively related to $F$. Then, we will obtain the first integral $\lambda$, given by (1.1), using the Painlevé first integral, associated to the two projectively related Finsler metrics $F$ and $S$.

Next theorem deals with a concrete class of Finsler metrics that satisfy the second assumption of Theorem 1.1. We say that a Finsler metric $F$ has scalar mean Berwald curvature $f$ if the mean Berwald curvature is proportional to the vertical Hessian of $F$, $2E_{ij} = f F_{y^i y^j}$.

**Theorem 1.2.** Consider $F$ a Finsler metric that satisfies the following two conditions:

i) the $\chi$-curvature vanishes;

ii) the Finsler metric has scalar mean Berwald curvature $f$.

Then, the scalar mean Berwald curvature satisfies:

1) $f$ is a first integral of the Finsler metric $F$.

2) If $\dim M > 2$ then the first integral $f$ is constant.

3) If $M$ is compact and $\dim M > 2$ then the first integral $f$ vanishes identically.

The proof of Theorem 1.2 is a direct extension, to the $n$-dimensional case, of the techniques used by Foulon and Ruggiero in [7] to prove the existence of a first integral for $k$-basic Finsler surfaces. These techniques allow to provide more information about the first integral and one can further use [6] to obtain a rigidity result for the class of Finsler metrics with vanishing $\chi$-curvature and scalar mean Berwald curvature.

2. FINSLER METRICS: A GEOMETRIC SETTING AND SOME NON-RIEMANNIAN QUANTITIES

In this work, we assume that $M$ is an $n$-dimensional $C^\infty$ manifold, of dimension $n > 1$. We consider $TM$ its tangent bundle and $T_0M = TM \setminus \{0\}$ the tangent bundle with the zero section removed. Local coordinates on $M$, denoted by $(x^i)$, induce canonical coordinates on $TM$ (and $T_0M$), denoted by $(x^i, y^i)$. On $TM$ there are two canonical structures that we will use: the Liouville (dilation) vector field, $C = y^i \frac{\partial}{\partial y^i}$, and the tangent structure (vertical endomorphism), $J = \frac{\partial}{\partial y^i} \otimes dx^i$. 
2.1. A geometric setting for Finsler metrics. We will use the Fr"olicher-Nijenhuis theory to describe the geometric setting we follow in this work. For a vector-valued $l$-form $L$ on $T_0M$, we denote by $i_L$ the induced $i_*$-derivation of degree $(l - 1)$ and by $d_L := [i_L, d]$ the $d_*$ derivation of degree $l$, [3, 8, 9, 20, 22]. For two vector valued forms, an $l$-form $L$ and a $k$-form $K$, consider the $(k + l)$-form $[K, L]$, uniquely determined by

$$d_{[K, L]} d_L - (-1)^k d_L d_K.$$  

A spray is a second order vector field $G \in \mathfrak{X}(T_0M)$ such that $JG = C$ and $[C, G] = G$. Locally, a spray can be expressed as

$$G = y^i \partial_i - 2G^i \partial_{y^i},$$  

with the functions $G^i(x, y)$ positively $2$-homogeneous in $y$ ($2^+$-homogeneous). A geodesic of a spray $G$ is a smooth curve $c$ on $M$ whose velocity $\ddot{c}$ is an integral curve of $G$, $G(c(t)) = \ddot{c}(t)$. For a given spray $G$, an orientation preserving reparameterization $t \to t(t)$, of its geodesics, leads to a new spray $\tilde{G} = G - 2PC$, where $P$ is a $1^+$-homogeneous function on $T_0M$. We say that the two sprays $G$ and $\tilde{G}$ are projectively related, while $P$ is called the projective factor.

**Definition 2.1.** A Finsler structure on $M$ is a continuous function $F : TM \to [0, +\infty)$ that satisfies the following conditions:

i) $F$ is smooth on $T_0M$;

ii) $F$ is $1^+$-homogenous, $F(x, \lambda y) = \lambda F(x, y)$, $\forall \lambda > 0$, $\forall (x, y) \in T_0M$;

iii) the metric tensor

$$g_{ij} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

is non-degenerate on $T_0M$.

A Finsler manifold is a pair $(M, F)$, with $F$ a Finsler structure on the manifold $M$. For a Finsler manifold, one can identify the sphere bundle $SM$ with the indicatrix bundle $IM = \{(x, y) \in TM, F(x, y) = 1\}$. Geometric objects on $T_0M$ that are invariant under positive rescaling ($0^+$-homogenous) can be restricted to the sphere bundle $SM$.

For a Finsler structure $F$, the metric tensor $g_{ij}$ can be expressed in terms of the angular metric $h_{ij}$ as follows:

$$g_{ij} = h_{ij} + \frac{1}{F^2} y_i y_j = h_{ij} + \frac{\partial F}{\partial y^j} \frac{\partial F}{\partial y^i}, \quad h_{ij} = F \frac{\partial^2 F}{\partial y^i \partial y^j} = FF_{y^i y^j},$$

where $y_i = g_{ik} y^k = F \frac{\partial F}{\partial y^i}$. The regularity condition iii) from Definition 2.1 is equivalent to the fact that the angular metric $h_{ij}$ has rank $n - 1$, [12, Proposition 16.2].

For a spray $G$ and a function $L$ on $T_0M$, we consider the Euler-Lagrange 1-form

$$\delta_G L := \mathcal{L}_G d_J L - d L = \left\{ G \left( \frac{\partial L}{\partial y^j} \right) - \frac{\partial L}{\partial x^i} \right\} dx^i.$$  

Every Finsler metric uniquely determines a geodesics spray, solution of the Euler-Lagrange equation $\delta_G F^2 = 0$.

We recall now the geometric structures induced by a Finsler metric, and its geodesic spray $G$. We first have the canonical nonlinear connection, characterised by a horizontal and a vertical projector on $T_0M$

$$h = \frac{1}{2} (\text{Id} - [G, J]), \quad v = \frac{1}{2} (\text{Id} + [G, J]).$$
In induced local charts on $T_0M$, the two projectors can be expressed as:

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \quad v = \frac{\partial}{\partial y^i} \otimes dy^i,$$

where $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$, $\delta y^i = dy^i + N^i_j dx^j$ and $N^i_j = \frac{\partial G^i}{\partial y^j}$.

**Lemma 2.2.** Consider $F$ a Finsler metric and $\tilde{F}$ a $1^+$-homogeneous function, nowhere vanishing on $T_0M$. Then, we can express the determinant of the metric tensor $g_{ij}$ as follows:

$$\det g = -\frac{F^{n+1}}{F^2} \begin{vmatrix} \frac{\partial^2 F}{\partial y^i \partial y^j} & \frac{\partial \tilde{F}}{\partial y^i} \\ \frac{\partial \tilde{F}}{\partial y^j} & 0 \end{vmatrix}. \tag{2.2}$$

**Proof.** First, we recall a formula that connects the determinant of the metric tensor $g_{ij}$ in terms of the angular metric $h_{ij}$, [15, (1.26)]:

$$\det g = -F^{n-1} \begin{vmatrix} \frac{\partial^2 F}{\partial y^i \partial y^j} & \frac{\partial F}{\partial y^i} \\ \frac{\partial F}{\partial y^j} & 0 \end{vmatrix}. \tag{2.3}$$

For the metric tensor $g_{ij}$, consider $\{h_1 = \frac{\partial}{\partial y^i}, h_2, \ldots, h_n\}$ an orthonormal horizontal basis and $\{h^i, i = 1, \ldots, n\}$, the dual frame. Since, for $\alpha \geq 2$, $h^\alpha(h_1) = 0$, and

$$h^\alpha = h^\alpha_i dx^i, \quad h_1 = \frac{\partial}{\partial y^i} \frac{\delta}{\delta x^i},$$

we obtain $h^\alpha y^i = 0$. Using also $h_{ij} y^j = 0$, we obtain, for each $\alpha \geq 2$, that on $T_0M$,

$$\begin{pmatrix} h_{ij} & h_1^\alpha \\ h_1^\alpha & 0 \end{pmatrix} \begin{pmatrix} \gamma^1 \\ \vdots \\ \gamma^n \\ 0 \end{pmatrix} = 0$$

and consequently,

$$\begin{vmatrix} h_{ij} & h_1^\alpha \\ h_1^\alpha & 0 \end{vmatrix} = 0. \tag{2.4}$$

The semi-basic 1-form $d_J \tilde{F}$ can be expressed as follows

$$d_J \tilde{F} = \frac{\partial \tilde{F}}{\partial y^j} dx^j = d_J \tilde{F}(h_1) h^1 + \sum_{\alpha=2}^n d_J \tilde{F}(h_\alpha) h^\alpha = \left\{ \frac{\tilde{F}}{F} \frac{\partial F}{\partial y^i} + \sum_{\alpha=2}^n J(h_\alpha)(\tilde{F}) h_\alpha \right\} dx^i.$$
In the determinant from (2.3), we replace \( \tilde{F} \frac{\partial F}{\partial y^i} \) and obtain
\[
\begin{vmatrix}
\frac{\partial^2 F}{\partial y^i \partial y^j} & \frac{\partial F}{\partial y^j} & \frac{\partial^2 F}{\partial y^i \partial y^j} & \tilde{F} \frac{\partial F}{\partial y^j} \\
\frac{\partial F}{\partial y^j} & 0 & \frac{\partial^2 F}{\partial y^j \partial y^j} & \tilde{F} \frac{\partial F}{\partial y^j}
\end{vmatrix} = \left( \frac{F}{\tilde{F}} \right)^2 \begin{vmatrix}
\frac{\partial^2 F}{\partial y^i \partial y^j} & \frac{\partial F}{\partial y^j} & \frac{\partial^2 F}{\partial y^i \partial y^j} & \tilde{F} \frac{\partial F}{\partial y^j} \\
\frac{\partial F}{\partial y^j} & 0 & \frac{\partial^2 F}{\partial y^j \partial y^j} & \tilde{F} \frac{\partial F}{\partial y^j}
\end{vmatrix} = \left( \frac{F}{\tilde{F}} \right)^2 \begin{vmatrix}
\frac{\partial^2 F}{\partial y^i \partial y^j} & \frac{\partial F}{\partial y^j} & \frac{\partial^2 F}{\partial y^i \partial y^j} & \tilde{F} \frac{\partial F}{\partial y^j} \\
\frac{\partial F}{\partial y^j} & 0 & \frac{\partial^2 F}{\partial y^j \partial y^j} & \tilde{F} \frac{\partial F}{\partial y^j}
\end{vmatrix}.
\]

We replace this in formula (2.3) to obtain (2.2) and complete the proof.

For a Finsler metric \( F \), the regularity condition iii) of Definition 2.1 can be reformulated in terms of the Hilbert 1-form \( dF \). Since \( dF \) is 0\(^{\circ} \)-homogeneous, we can view it as a 1-form on \( SM \). Due to the fact that the Hilbert 2-form can be expressed as
\[
\omega_{SM} = dF \wedge ddF^{(n-1)},
\]

it follows that \( dF \) is a contact structure on \( SM \) and hence the \((2n-1)\)-form \( \omega_{SM} = dF \wedge ddF^{(n-1)} \) is a volume form on \( SM \).

### 2.2. Non-Riemannian structures in the Finslerian setting

The first non-Riemannian structures, associated to a Finsler metric \( F \), are the Cartan torsion and the mean Cartan torsion,
\[
C_{ijk} = \frac{1}{4} \frac{\partial^2 F^2}{\partial y^i \partial y^j \partial y^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}, \quad \bar{C}_k = g^{ij} C_{ijk}.
\]

A Finsler metric reduces to a Riemannian metric if and only if the (mean) Cartan torsion vanishes.

The curvature of the nonlinear connection determined by the geodesic spray \( G \) is defined by
\[
R = \frac{1}{2} [h, h] = \frac{1}{2} R^i_{jk} \frac{\partial}{\partial y^i} \otimes dx^j \wedge dx^k = \frac{1}{2} \left( \frac{\delta N_j^i}{\delta x^k} - \frac{\delta N_k^i}{\delta x^j} \right) \frac{\partial}{\partial y^i} \otimes dx^j \wedge dx^k.
\]

The canonical nonlinear connection provides a tensor derivation on \( T_0 M \), the dynamical covariant derivative \( \nabla \), whose action on functions and vector fields is given by [3, (21)]:
\[
\nabla(f) = G(f), \forall f \in C^\infty(TM), \quad \nabla X = h[G, hX] + v[G, vX], \forall X \in \mathfrak{X}(TM).
\]

The geodesic spray \( G \) of a Finsler metric induces a linear connection on \( T_0 M \), the Berwald connection, [20, §8.1.1], with two curvature components, the Berwald curvature and the Riemannian curvature, [17, §6.1.§, 8.1]:
\[
B_{ij}^k = \frac{\partial^2 G^i}{\partial y^j \partial y^k}, \quad R_{ijkl} = \frac{\partial R^i_{jk}}{\partial y^l}.
\]

The mean Berwald curvature of a spray \( G \) is defined as [17, Def. 6.1.2]
\[
E_{ijk} = \frac{1}{2} B_{ijk} = \frac{1}{2} \frac{\partial^2 G^i}{\partial y^j \partial y^k}.
\]
Definition 2.3. A Finsler metric has scalar mean Berwald curvature if the mean Berwald curvature is proportional to the angular metric, which means that there exists a $0^+\text{-}-$homogeneous function $f$ on $T_0M$ such that

$$(2.7) E_{ij} = \frac{1}{2} f \frac{\partial^2 F}{\partial y^i \partial y^j}. $$

In [5], Chen and Shen study Finsler metrics of isotropic mean Berwald curvature, with a similar definition as above, with $f$ being a scalar function on $M$.

In the next lemma we prove that in dimension $n > 2$, Finsler metrics of scalar mean Berwald curvature have isotropic mean Berwald curvature. In other words, the scalar mean Berwald curvature $f$ is constant along the fibres of $T_0M$.

Lemma 2.4. Consider $F$ a Finsler metric of scalar mean Berwald curvature $f$. If $n > 2$, then $f$ is constant along the fibres of $T_0M$.

Proof. From the definition of the mean Berwald curvature (2.6) we obtain that its vertical derivative is a $(0,3)$-type tensor symmetric in all three arguments, therefore for a Finsler metric of scalar mean Berwald curvature we have

$$\frac{\partial E_{ij}}{\partial y^k} = \frac{\partial E_{ik}}{\partial y^j} \stackrel{(2.7)}{=} \frac{\partial f}{\partial y^k} \frac{\partial^2 F}{\partial y^i \partial y^j} = \frac{\partial f}{\partial y^i} \frac{\partial^2 F}{\partial y^j \partial y^k} \Rightarrow \frac{\partial f}{\partial y^k} h_{ij} = \frac{\partial f}{\partial y^i} h_{ik}.$$

In the last formula above, we multiply with $g^{il}$, the inverse of the metric tensor and obtain:

$$\frac{\partial f}{\partial y^k} \left( \delta^l_j - \frac{1}{F^2} y^l y^j \right) = \frac{\partial f}{\partial y^i} \left( \delta^l_k - \frac{1}{F^2} y^l y^k \right).$$

Now, if we use that $f$ is $0^+$-homogeneous and take the trace, $j = l$, we obtain $(n - 2) \partial f / \partial y^k = 0$. Since $n > 2$, we obtain that the function $f$ is constant along the fibres of $T_0M$. \[\square\]

Due to the $2^+$-homogeneity of the spray coefficients $G^i$, it follows that $E_{ij} y^j = 0$, hence rank$(E_{ij}) \leq n - 1$. In the 2-dimensional case, we obtain that the mean Berwald curvature has rank 1, it is proportional to the angular metric $h_{ij}$ (of rank 1 as well), and hence all 2-dimensional Finsler manifolds have scalar mean Berwald curvature. The proportionality factor has been known since Berwald, [1, (8.7)], but it has been shown only recently that it is a first integral for $k$-basic Finsler surfaces, [7, Theorem B].

The Berwald connection is not a metric connection, with respect to the metric tensor of a Finsler structure. Due to this non-metricity property of the Berwald connection, it follows that the $(0,4)$-type Riemann curvature tensor $R_{ijkl} = g_{ir} R^r_{kl}$ is not skew-symmetric in the first two indices, [17, (10.6)], and hence $R^i_{kl} \neq 0$. A measure of this failure is given by the $\chi$-curvature, [19, Lemma 3.1]:

$$\chi_j = -\frac{1}{2} R^i_{ijk} y^k.$$

This non-Riemannian quantity has been introduced by Shen in [18].

The key ingredients we will use in this work are the $\chi$-curvature, the mean Berwald curvature, and the fact that both curvature tensors can be expressed in terms of yet another non-Riemannian quantity, the $S$-function.
For a fixed vertically invariant volume form $\sigma(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n$ on $TM$, [20, p. 490], we consider the Shen-function ($S$-function) and the distortion $\tau$, [17, §5.2],

$$S = G(\tau), \quad \tau = \frac{1}{2} \ln \frac{\det g}{\sigma}. \quad (2.8)$$

From the various expressions of the $\chi$-curvature, we will use its expression in terms of the $S$-function, [18, (1.10)],

$$\chi = \frac{1}{2} \delta_G S = \frac{1}{2} \left\{ \nabla \left( \frac{\partial S}{\partial y^i} \right) - \frac{\delta S}{\delta x^j} \right\} dx^i. \quad (2.9)$$

The mean Berwald curvature can also be expressed in terms of the $S$-function as follows, [17, (6.13)]:

$$E_{ij} = \frac{1}{2} \frac{\partial^2 S}{\partial y^i \partial y^j}. \quad (2.10)$$

In view of formula (2.10), the second assumption of Theorem 1.1 or 1.2, assures that the vertical Hessian of the $S$-function has maximal rank $(n - 1)$. Therefore, we can interpret the $S$-function as a Finsler metric on its own.

3. Proof of Theorem 1.1

For the proof of Theorem 1.1, we proceed with the following steps. We show first that the two assumptions of Theorem 1.1 assure that the $S$-function is a Finsler metric, projectively related to $F$. Then, we obtain the first integral (1.1) using the Painlevé first integral associated to the two projectively related Finsler metrics $S$ and $F$.

Two Finsler metrics $F$ and $\tilde{F}$ are projectively related if their geodesic sprays $G$ and $\tilde{G}$ are projectively related. One can characterise projective equivalence of two Finsler metrics $F$ and $\tilde{F}$ using the following equivalent forms of Rapcsák equations, [20, §9.2.3]:

$$(R_1) \quad \delta_G \tilde{F} = 0;$$

$$(R_2) \quad d_\alpha d_j \tilde{F} = 0.$$  

In Riemannian geometry, Topalov and Matveev [21, Theorem 1] associate to each pair of geodesically equivalent metrics a set of $n$ first integrals. An extension of this result, to the Finslerian setting, has been proposed by Sarlet in [16] and his Ph.D student Vermeire [23].

In the next lemma, we show that two projectively related Finsler metrics $F$ and $\tilde{F}$ induce a first integral (Painlevé first integral). This first integral, given by formula (3.1), is the Finslerian extension of the first integral determined by two projectively equivalent Riemannian metrics, [13, Theorem 2].

**Lemma 3.1.** Consider $F$ and $\tilde{F}$, two projectively related Finsler metrics. Then,

$$I_0 = \frac{\tilde{F}}{F} \left( \frac{\det g}{\det \tilde{g}} \right)^{\frac{1}{n+1}} \quad (3.1)$$

is a first integral for $F$.

**Proof.** For a Finsler metric $F$, the dynamical covariant derivative of its metric tensor vanishes, [2], hence:

$$\nabla (g_{ij}) = G(g_{ij}) - g_{im} N_{j}^{m} - g_{mj} N_{i}^{m} = 0.$$
Contracting with $g^{ij}$, we obtain
\[ g^{ij}G(g_{ij}) = g^{ij}(g_{im}N^m_i + g_{mj}N^m_j) = 2N^i_i, \quad \text{and hence} \quad N^i_i = \frac{1}{2}G(\ln(\det g)). \]

The two Finsler metrics $F$ and $\tilde{F}$ being projectively related, their geodesic sprays and nonlinear connections are connected through

\[ \tilde{G} = G - 2PC, \quad \tilde{G}^i = G^i + Py^i, \quad \tilde{N}^i_j = N^i_j + \frac{\partial P}{\partial y^j}y^i + P\delta^i_j. \]

If in the last formula above we take the trace $i = j$, it follows that the projective factor $P$ is given by

\[ P = \frac{1}{n+1}(\tilde{N}^i_i - N^i_i) = \frac{1}{2(n+1)}G\left(\ln\left(\frac{\det \tilde{g}}{\det g}\right)\right). \]

We also use the alternative expression of the projective factor $P$,

\[ P = \frac{G(\tilde{F})}{2F} = \frac{1}{2}G\left(\ln \tilde{F}\right). \]

By comparing the two expressions of the projective factor $P$, we obtain $G(I_0) = 0$, which concludes the proof of our lemma.

We will give the proof of Theorem 1.1 now. The second assumption ii) on Theorem 1.1 together with formula (2.10) assure that the angular metric of the $S$-function has rank $n - 1$ and therefore $S$ is a Finsler metric. The vanishing of the $\chi$-curvature (2.9) assures that the Finsler metric $S$ is projectively related to $F$. In view of Lemma 3.1 we obtain that

\[ I_0 = \frac{S}{F} \left(\frac{\det g}{\det s}\right)^{\frac{n+1}{n}} \]

is a first integral for the Finsler metric $F$.

We will use Lemma 2.2 for the Finsler metric $S$ and the $1^+$-homogenous function $F$. According to formula (2.2), we can express the determinant of the metric tensor $s_{ij}$ as follows:

\[ \det s = -\frac{S^{n+1}}{F^2} \begin{vmatrix} \frac{\partial^2 S}{\partial y^i \partial y^j} & \frac{\partial F}{\partial y^j} \\ \frac{\partial F}{\partial y^i} & 0 \end{vmatrix} = -\frac{S^{n+1}}{F^2} \begin{vmatrix} 2F_{ij} & \frac{\partial F}{\partial y^j} \\ \frac{\partial F}{\partial y^i} & 0 \end{vmatrix} = -\frac{S^{n+1}}{F^{n+1}} \begin{vmatrix} 2F_{ij} & \frac{\partial F}{\partial y^j} \\ \frac{\partial F}{\partial y^i} & 0 \end{vmatrix}. \]

Since $I_0$ is a first integral for the Finsler metric $F$, it follows that

\[ \frac{1}{I_0^{n+1}} = \frac{F^{n+1}}{S^{n+1}} \frac{\det s}{\det g} = -\frac{1}{\det g} \begin{vmatrix} \frac{\partial F}{\partial y^j} \\ \frac{\partial F}{\partial y^i} \end{vmatrix} \]

is also a first integral for $F$ that coincides with $\lambda$ given by formula (1.1).

The first two assumptions of Theorem 1.1 tell us that $S$ is a Finsler metric projectively related to $F$. One can use this and [21, Theorem 1] and [16, Theorem 2] to provide a set of $n$ first integrals for Finsler metric with vanishing $\chi$-curvature and mean Berwald curvature of maximal rank.
4. Proof of Theorem 1.2

4.1. Partial proof of Theorem 1.2. First we prove the first two conclusions of Theorem 1.2, using Theorem 1.1. For this proof it is essential that the scalar mean Berwald curvature \( f \) is nowhere vanishing, hence we cannot reach the third conclusion of Theorem 1.2 using these techniques.

In view of the equivalence of the two Rapcsák equations \( R_1 \) and \( R_2 \), we can reformulate the vanishing of the \( \chi \)-curvature (2.9) as \( d_h d_J S = 0 \). Using also the assumption that \( F \) has scalar mean Berwald curvature, we obtain that the Hilbert 2-form of the \( S \)-function can be written as follows

\[
\dd d_J S = d_v d_J S = \frac{\partial^2 F}{\partial y^i \partial y^j} \delta y^i \wedge dx^j = 2E_{ij} \delta y^i \wedge dx^j = f d h d_J S = f d d_J F.
\]

(4.1)

For a non-vanishing scalar mean Berwald curvature \( f \), it follows from (4.1) that \( \text{rank} \left( \frac{\partial^2 S}{\partial y^i \partial y^j} \right) = n - 1 \) and hence \( S \) is a Finsler metric.

We will express now, the first integral \( \lambda \), (1.1), using the assumption that \( F \) has scalar mean Berwald curvature \( f \). We have

\[
\lambda = -\frac{1}{\det g} \begin{vmatrix} 2FE_{ij} & \frac{\partial F}{\partial y^i} & 0 \\ \frac{\partial F}{\partial y^j} & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = -F^{n-1} \begin{vmatrix} f \frac{\partial^2 F}{\partial y^i \partial y^j} & \frac{\partial F}{\partial y^i} \\ \frac{\partial F}{\partial y^j} & 0 \end{vmatrix} (2.3) f^{n-1}.
\]

(4.2)

Since \( \lambda \) is a first integral, it follows that \( f \) is also a first integral for \( F \), and this proves the first conclusion of Theorem 1.2.

According to Lemma 2.4, the scalar mean Berwald curvature \( f \) is a scalar function on \( M \), which means that \( d_J f = 0 \). We use now that \( G(f) = 0 \), which can be written as \( \nabla f = 0 \). If we apply \( d_J \) to this formula and use the commutation rule for \( \nabla \) and \( d_J \), \([4, (2.11)]\), we obtain

\[
0 = d_J \nabla f = \nabla d_J f + d_v f.
\]

Therefore, \( d_v f = 0 \) and hence \( f \) is a constant, which proves the second conclusion of Theorem 1.2.

4.2. Complete proof of Theorem 1.2. In this section we present a proof of Theorem 1.2, independent of the results of Theorem 1.1, by extending to the \( n \)-dimensional case, the techniques of \([7]\). This method allows to provide more information about the first integral, when the base manifold is compact.

The mean Cartan torsion can be expressed in terms of the distortion \( \tau \), and it does not depend on the fixed volume form on \( M \),

\[
I_k = \frac{1}{2} \partial g_{ij}(\ln \sqrt{\det g}) = \frac{\partial \tau}{\partial y^k}, \quad I = I_k dx^k = d_J (\ln \sqrt{\det g}) = d_J \tau.
\]

The key ingredient in this proof is the following 1-form

\[
\alpha = i_{[J,G]} \mathcal{L}_G I = \nabla I_k dx^k - I_k \delta y^k = \nabla d_J \tau - d_v \tau = d_J \nabla \tau - d_v \tau = d_J S - d_s \tau.
\]

(4.3)

In the 2-dimensional case, this form reduces to the form \( \alpha \) from \([7, \S2]\).

We will use the last expression from (4.3) of the form \( \alpha \) to connect it with the \( \chi \)-curvature:

\[
\mathcal{L}_G \alpha = \mathcal{L}_G d_J S - \mathcal{L}_G d\tau = \mathcal{L}_G d_J S - d\tau = \delta_G S = 2\chi.
\]

(4.4)
In view of this formula, the \( \chi \)-curvature vanishes if and only if the form \( \alpha \) is invariant by the geodesic flow. Moreover, the \( \chi \)-curvature vanishes if and only if the \( S \)-function satisfies the Rapcsák equation \( \delta G S = 0 \), which is equivalent to \( d_h d_J S = 0 \).

Therefore, we can express the 2-form \( d\alpha \) as follows:

\[
d\alpha = d d_J S = d_h d_J S + d_v d_J S = \frac{\partial^2 S}{\partial y^i \partial y^j} \delta y^i \wedge dx^j = 2E_{ij} \delta y^i \wedge dx^j.
\]

If we consider now the assumption that the Finsler metric has scalar mean Berwald curvature, then the 2-form \( d\alpha \) is proportional to the Hilbert 2-form \( dd_J F \):

\[
d\alpha = 2E_{ij} \delta y^i \wedge dx^j = \frac{f}{F} h_{ij} \delta y^i \wedge dx^j = f dd_J F.
\]

From formula (4.4) we obtain that \( \chi = 0 \) implies \( L_G \alpha = 0 \) and therefore \( L_G d\alpha = 0 \). In view of formula (4.5) and using the fact that \( L_G dd_J F = 0 \) we obtain \( G(f) = 0 \), which means that the scalar mean Berwald curvature \( f \) is a first integral for the geodesic flow \( G \).

Using Lemma 2.4 we obtain that the scalar mean Berwald curvature \( f \) is a scalar function on \( M \), hence \( df = d_h f \). From formula (4.5), we obtain

\[
0 = df \wedge dd_J F = d_h f \wedge d_v d_J F = \frac{1}{2} \left( \frac{\partial f}{\partial x^j} h_{kj} - \frac{\partial f}{\partial x^j} h_{ki} \right) dx^i \wedge dx^j \wedge \delta y^k.
\]

It follows that

\[
\frac{\partial f}{\partial x^j} h_{kj} = \frac{\partial f}{\partial x^j} h_{ki} \implies \frac{\partial f}{\partial x^j} \left( g_{kj} - \frac{1}{F^2} y_h y_j \right) = \frac{\partial f}{\partial x^j} \left( g_{ki} - \frac{1}{F^2} y_h y_i \right).
\]

In the last formula above, we multiply with \( g^{il} \) and obtain:

\[
\frac{\partial f}{\partial x^j} \left( \delta^l_j - \frac{1}{F^2} y^l y_j \right) = \frac{\partial f}{\partial x^j} \left( \delta^l_j - \frac{1}{F^2} y^l y_i \right).
\]

If we take the trace \( l = j \), we obtain

\[
(n - 2) \frac{\partial f}{\partial x^j} = -\frac{1}{F^2} G(f) y_i.
\]

Now using that \( G(f) = 0 \), we obtain that the scalar function \( f \) is constant if \( \dim M > 2 \).

To complete the proof of Theorem 1.2, we need the following lemma that gives new properties for the first integral \( f \) and can be useful for some rigidity results.

**Lemma 4.1.** Let \((M, F)\) be a compact Finsler manifold with vanishing \( \chi \)-curvature and of scalar mean Berwald curvature \( f \). Then,

\[
\int_{SM} f \omega_{SM} = 0.
\]

**Proof.** By Stokes Theorem we have that

\[
0 = \int_{SM} d \left( \alpha \wedge d_J F \wedge (dd_J F)^{n-2} \right) = \int_{SM} d\alpha \wedge d_J F \wedge (dd_J F)^{n-2} - \int_{SM} \alpha \wedge (dd_J F)^{n-1}.
\]

We will prove now that on \( SM, \alpha \wedge (dd_J F)^{n-1} = 0 \).

Let \( \lambda_1, ..., \lambda_{n-1} \) be the non-zero eigenvalues of the angular metric \( h_{ij}, h_1, ..., h_{n-1} \) the corresponding horizontal eigenvectors and \( v_i = J h_{ii}, i \in \{1, ..., n-1\} \). Then, \( \{h_1, ..., h_{n-1}, v_1, ..., v_{n-1}\} \) is a local frame of the \((2n - 2)\)-dimensional distribution \( \text{Ker}(d_J F) \) on \( SM \). We consider also the
local co-frame $\{h^1, ..., h^{n-1}, v^1, ..., v^{n-1}\}$. Using the expression (2.5) of the Hilbert 2-form, $dd_JF$, it follows that

$$(dd_JF)^{n-1} = \lambda_1 \cdots \lambda_{n-1} h^1 \wedge \cdots \wedge h^{n-1} \wedge v^1 \cdots v^{n-1}.$$ 

Since $i_G \alpha = 0$, it follows that $\alpha \in \text{span}\{h^1, ..., h^{n-1}, v^1, ..., v^{n-1}\} = \text{Ker}(d_JF)$, we obtain that $\alpha \wedge (dd_JF)^{n-1} = 0$ on $SM$. Now, using (4.5), we obtain

$$0 = \int_{SM} d\alpha \wedge d_JF \wedge (dd_JF)^{n-2} = \int_{SM} J^d d_JF \wedge d_JF \wedge (dd_JF)^{n-2} = \int_{SM} J^d \omega_{SM}.$$

If $\dim M > 2$ then $f$ is constant and using formula (4.6) we obtain that $f = 0$, which completes the proof of Theorem 1.2.

Existence of first integrals for Finsler manifolds can be used to provide rigidity results under some topological restrictions:

- compact surface, without conjugate points and of genus greater than one, [7, Theorem A];
- compact manifold, without conjugate points and of uniform visibility, for dimension $n > 2$, [6, Theorem A].

If $M$ is a compact manifold of dimension $n > 2$, with vanishing $\chi$-curvature and of scalar mean Berwald curvature $f$, we obtain that $f = 0$. Using formula (4.5), it follows that the form $\alpha$, given by (4.3), is closed. Using the assumptions of [6, Theorem A] we can conclude that the form $\alpha$ is exact. Assume $\alpha = dh$, for some function $h$ on $T_0M$. Since $i_G \alpha = 0$, it follows that $G(h) = 0$ and $h$ is a first integral for the geodesic flow. Using again [6, Theorem A] we obtain that the function $h$ is constant, then $\alpha = 0$. The expression (4.3) of the form $\alpha$ allows to conclude that the mean Cartan tensor vanishes, $I = 0$, and hence $(M, F)$ is a Riemannian manifold.

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