UNITRIANGULAR FACTORISATIONS OF CHEVALLEY GROUPS

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In the present paper we show that comparing results of Hyman Bass [15] with those of Oleg Tavgen [10], one immediately gets the following result, which is both more general, and more precise than all recent results pertaining to unitriangular factorisations.

**Theorem 1.** Let $\Phi$ be a reduced irreducible root system and $R$ be a commutative ring such that $sr(R) = 1$. Then the elementary Chevalley group $G(\Phi, R)$ admits unitriangular factorisation

$$E(\Phi, R) = U(\Phi, R)U^-(\Phi, R)U(\Phi, R)U^-(\Phi, R).$$

of length 4.

On the other hand, since $U^-(\Phi, R) \cap B(\Phi, R) = 1$, one has

$$T(\Phi, R) \cap U(\Phi, R)U^-(\Phi, R)U(\Phi, R) = 1.$$ 

In other words, 1 is the only element of the torus $T(\Phi, R)$, that admits a unitriangular factorisation of length $< 4$. Thus, if the ring $R$ has at least one non-trivial unit, factorisation obtained in Theorem 1 is the best possible.

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What is truly amazing here, is that the usual linear stable rank condition works for groups of all types! Under somewhat stronger stability conditions, a similar result holds also for twisted Chevalley groups. However, the analysis of twisted groups requires much more detailed calculations for groups of Lie ranks 1 and 2, where one cannot simply invoke known results. The proofs for this case are relegated to the next paper by the authors.

Clearly, for rings of dimension $\geq 1$ in general there is no way to obtain unitriangular factorisations of length 4. However, for some particularly nice rings of dimension 1 one can obtain unitriangular factorisations of length 5 or 6. The second main result of the present paper gives a simplest such example.

**Theorem 2.** Let $\Phi$ be a reduced irreducible root system and $p \in \mathbb{Z}$ be a rational prime. Under assumption of the Generalised Riemann’s Hypothesis the simply connected Chevalley group $G(\Phi, \mathbb{Z}[\frac{1}{p}])$ admits unitriangular factorisation

$$G(\Phi, \mathbb{Z}[\frac{1}{p}]) = (U(\Phi, \mathbb{Z}[\frac{1}{p}])U^-(\Phi, \mathbb{Z}[\frac{1}{p}]))^3$$

of length 6.

Presently, we are working towards generalisation of this result to other Hasse domains and plan to return to this topic in a separate paper.

The first part of the present paper is the term paper of the second author, under the supervision of the first author, whereas the second part is a by-product of our joint work on arithmetic problems our cooperative Russian–Indian project “Higher composition laws, algebraic groups, K-theory and exceptional groups” at the Saint Petersburg State University, Tata Institute of Fundamental Research (Mumbai) and Indian Statistical Institute (Bangalore).

Since there are a considerable number of papers addressing some aspects of unitriangular factorisations, and many of these papers give no reference to previous works, where similar — or even stronger! — results are established, we start our paper with a brief survey of the known results, as we understand them. This is done in §§1–4. After that in §§5 and 6 we introduce basic notation. In §7 we prove a version of Tavgen’s theorem on rank reduction, which immediately implies Theorem 1. In §§8 we discuss connection of unitriangular factorisations with Bruhat and Gauss decompositions. In §9 we consider Chevalley groups over arithmetic rings and prove Theorem 2. Finally, in §10 we mention some further related topics and state several unsolved problems.
§1. Existence of unitriangular factorisations

Recently, the following problem cropped up anew, in several independent contexts: find the shortest factorisation \( G = UU^-UU^- \ldots U^\pm \) of a Chevalley group \( G = (\Phi, R) \), in terms of the unipotent radical \( U = U(\Phi, R) \) of the standard Borel subgroup \( B = B(\Phi, R) \), and the unipotent radical \( U^- = U^-(\Phi, R) \) of the opposite Borel subgroup \( B^- = B^-(\Phi, R) \).

Let us mention three such extensive clusters of various subjects, where it occurred. In the following sections we see that although information between various subjects within one cluster is transmitted with serious delays, still communication is incredibly faster and more effective, than information transfer between clusters themselves.

- Algebraic K-theory, structure theory of algebraic groups, theory of arithmetic groups.
- Theory of finite and profinite groups, asymptotic group theory, finite geometries.
- Computational linear algebra, wavelet theory, computer graphics, control theory.

Let us list some existing results concerning factorisations of the form \( G = UU^-UU^- \ldots U^\pm \). Firstly, one has to establish existence of such factorisations, and, secondly, estimate their length.

Clearly, existence of some unitriangular factorisation is equivalent to two following conditions.

- The Chevalley group \( G(\Phi, R) \) coincides with its elementary subgroup \( E(\Phi, R) \), spanned by elementary generators.
- The width of the elementary Chevalley group \( E(\Phi, R) \) with respect to elementary generators is bounded.

The answer to both of these questions is, in general, negative beyond retrieve, so that one can only expect existence of unitriangular factorisations for some very special classes of rings. There is vast literature, dedicated to both of these problems, and we will not even attempt to address them in full. Instead, we confine ourselves to a brief excerpt of the introduction to [61], referring to [63, 48] for a broader picture and further references.

- In the case when the ring \( R \) is semilocal — for instance, when it is of finite dimension over a field — for simple connected Chevalley groups one has the equality \( E(\Phi, R) = G(\Phi, R) \). Bounded width in terms of elementary generators immediately follows from Gauss decomposition.

- It is classically known that expressions of a matrix in the group \( SL(2, R) \) over a Euclidean ring \( R \) as a product of elementaries correspond to continued fractions. Existence of arbitrary long division chains in \( \mathbb{Z} \) shows that the group \( SL(2, \mathbb{Z}) \) cannot have bounded width in elementary generators. Further such results are discussed in [23, 24].
On the other hand, David Carter and Gordon Keller [16, 17] and Oleg Tavgen [9, 10, 63] demonstrated that for a Dedekind ring of arithmetic type $R$, the simply connected Chevalley groups $G(\Phi, R) = E(\Phi, R)$ of rank $\geq 2$ have bounded width in elementary generators.

Wilberd van der Kallen [32] made a striking discovery that in general even Chevalley groups of rank $\geq 2$ over a Euclidean ring may have infinite width in elementary generators. More precisely, he proved that $\text{SL}(3, \mathbb{C}[t])$ – and, since $sr(\mathbb{C}[t]) = 2$, all groups $\text{SL}(n, \mathbb{C}[t])$ for $n \geq 4$ – do not have bounded width in elementary generators.

Thus, in general one can hope to establish existence of unitriangular factorisations only over some very special rings of dimension $\leq 1$.

In the next three sections we list some works addressing the length of unitriangular factorisations. Already this short summary shows that experts in one field are usually completely unaware of the standard notions and results in another field. It is hard to imagine, how much time and energy could have been saved, should millions of programmers and engineers learn the words parabolic subgroup and Levi decomposition, rather than persisting in retarded matrix manipulations.

The same remark applies to Application 1.1 of the work [14], where, at least for the linear case, the result was known for quite some time, in a much larger generality, and with better bound, to experts in other subjects, both in algebraic $K$-theory [25] and in computational linear algebra [62]. Obviously, after its driving forces are understood for the case of $\text{SL}(n, R)$, its generalisation to Chevalley groups does not require any serious intellectual effort, just the mastery of the appropriate techniques.

§2. The length of factorisations: linear groups

For the linear case a systematic study of this problem was started in a paper by Keith Dennis and Leonid Vaserstein [25]. Originally, their interest to this problem was motivated by the following observation.

Any element of the group $U(n, R)$, $n \geq 3$, over an arbitrary associative ring $R$ is a product of not more than two commutators in the elementary group $E(n, R)$, Lemma 13. Earlier, van der Kallen citedvK82 noticed that the elements of $U(n, R)$ are products of not more than three commutators.

Let us state several typical results of [25], pertaining to the length of unitriangular factorisations.

If for a ring $R$ the group $E(m, R)$, $m \geq 2$, is represented as a finite product $UU^-UU^-\ldots U^\pm$ with $L$ factors, then all groups $E(n, R)$, $n > m$, can be presented as the product of the same form, with the same number of factors, Lemma 7.

If the stable rank of the ring $R$ equals 1, then, as is classically known from the work of Hyman Bass [15], the group $E(2, R)$ – and
thus by the previous item all groups $E(n, R)$ admit unitriangular factorisation

$$E(n, R) = U(n, R)U^-(n, R)U(n, R)U^-(n, R)$$

of length 4.

To the authors of [25], as to all experts in algebraic $K$-theory, this fact is obvious to such an extent, that in [25] it is not even separately stated, and just casually mentioned inside the proof of Theorem 6. Nevertheless, since this is one of the key steps in the proof of Theorem 1, at that the only one, which invokes stability condition, we reproduce its proof, which is an adaptation of a more general similar calculation, implemented in [15].

Recall that a ring $R$ has stable rank 1, if for all $x, y \in R$, which generate $R$ as a right ideal, there exists a $z \in R$ such that $x + yz$ is right invertible. In this case we write $sr(R) = 1$.

The simplest and the most characteristic example of a ring of stable rank 1 are semilocal rings. However, there are many far less trivial examples, such as the ring of all algebraic integers. Many further examples and further references can be found in [66].

It is classically known that rings of stable rank 1 are actually weakly finite (Kaplansky—Lenstra theorem), so that in their definition one could from the very start require that $x + yz \in R^*$. Since for the linear case the result is well known, and Chevalley groups of other types only exist over commutative rings, from here on we assume that the ring $R$ is commutative, in which case the proof below at the same time demonstrates that $SL(2, R) = E(2, R)$.

In the following proof, and henceforth, in discussing linear case we use standard matrix notation. In particular, $e$ denotes identity matrix, $e_{ij}, 1 \leq i, j \leq n$, denotes a standard matrix unit, i.e. the matrix which has 1 in the position $(i, j)$ and zeroes elsewhere. Further, for $1 \leq i \neq j \leq n$ and $\xi \in R$ one denotes by $t_{ij}(\xi)$ the elementary transvection $e + \xi e_{ij}$. The matrix entry of $g$ in position $(i, j)$ will be denoted by $g_{ij}$, whereas the corresponding entry of the inverse matrix $g^{-1}$ will be denoted by $g_{ij}'$.

**Lemma 1.** Let $R$ be a commutative ring of stable rank 1. Then

$$SL(2, R) = U(2, R)U^-(2, R)U(2, R)U^-(2, R).$$

**Proof.** Let us trace, how many elementary transformations one needs to bring an arbitrary matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, R)$ to the form $e$.

We will not introduce new notation at each step, but rather replace the matrix $g$ by its current value, as is common in computer science. Obviously, its entries $a, b, c, d$ should be also reset to their current values at each step.
Step 1. Multiplication by a single lower elementary matrix on the right allows to make the element in the South-West corner invertible. Indeed, since the rows of the matrix are unimodular, one has \( cR + dR = R \). Since \( \text{sr}(R) = 1 \), there exists such an \( z \in R \), that \( c + dz \in R^* \). Thus,

\[
\begin{pmatrix}
    a + bz & b \\
    c + dz & d
\end{pmatrix},
\]

where \( c + dz \in R^* \).

Step 2. Thus, we can assume that \( c \in R^* \). Multiplication by a single upper elementary matrix on the right allows to make the element in the South-East corner equal to 1. Indeed,

\[
\begin{pmatrix}
    a & b + ac^{-1}(1 - d) \\
    c & 1
\end{pmatrix},
\]

Step 3. Thus, we can now assume that \( d = 1 \). Multiplication by a single lower elementary matrix on the right allows to make the element in the South-West corner equal to 0. Indeed,

\[
\begin{pmatrix}
    a - bc & b \\
    0 & 1
\end{pmatrix}.
\]

Since \( \det(g) = 1 \), the matrix on the right hand side is equal to \( t_{12}(b) \). Bringing all elementary factors to the right hand side, we see that any matrix \( g \) with determinant 1 can be expressed as a product of the form \( t_{12}(*)t_{21}(*)t_{12}(*) \), as claimed. \( \Box \)

Now, let us return to the Dennis–Vaserstein paper [25].

• If \( R \) is a Boolean ring, in other words, if \( x^2 = x \) for all \( x \in R \), then the elementary group \( E(n, R) \), \( n \geq 2 \), admits the unitriangular factorisation

\[
E(n, R) = U(n, R)U^-(n, R)U(n, R)
\]

of length 3. For commutative rings the converse is obviously true: if for some \( n \geq 2 \) the group \( E(n, R) \) admits unitriangular factorisation of length 3, then the ring \( R \) is Boolean, Remark 15.

• Let \( d = \text{diag}(\varepsilon_1, \ldots, \varepsilon_n) \), \( \varepsilon_i \in R^* \), be a diagonal matrix such that \( \varepsilon_1 \ldots \varepsilon_n = 1 \). Then \( d \in U(n, R)U^-(n, R)U(n, R)U^-(n, R) \), Lemma 18. The proof in [25] is based on general position arguments. In §8 we give another proof of a slightly stronger fact, based on induction on rank, which can be easily generalised to all Chevalley groups.

• If the stable rank of the ring \( R \) is finite for some \( m \geq 2 \), the group \( E(m, R) \) has finite width with respect to the elementary generators, then for all sufficiently large \( n \) one has the unitriangular factorisation

\[
E(n, R) = (U(n, R)U^-(n, R))^3
\]

of length 6.
It is extremely suggestive to compare this result with Sharpe decomposition. Recall that it is established in [56] that for any associative ring $R$ the finitary elementary group $E(R) = \lim \rightarrow E(n, R)$, $g \mapsto g \oplus 1$, admits decomposition

$$E(R) = B(R)N(R)U(R)U^-(R) = B(R)U^-(R)U(R)U^-(R) = U(R)D(R)U^-(R)U(R)U^-(R),$$

where $B(R)$, $N(R)$, $U(R)$, $U^-(R)$, $D(R)$ denote inductive limits of the corresponding groups of finite degree, with respect to the same embedding. Since $D(R) \subseteq U(R)U^-(R)U(R)U^-(R)$, it follows that for an arbitrary associative ring the finitary elementary group $E(R)$ admits unitriangular factorisation

$$E(R) = (U(R)U^-(R))^3$$

of length 6.

• Recently, motivated by applications in factorisation of integer matrices, Thomas Laffey citelaffey-a,laffey-b explicitly calculated the bound in Dennis—Vaserstein theorem for the ring $R = \mathbb{Z}$. In particular, unitriangular factorisation

$$SL(n, \mathbb{Z}) = (U(n, \mathbb{Z})U^-(n, \mathbb{Z}))^3$$

of length 6 holds for any $n \geq 82$.

§3. THE LENGTH OF FACTORISATIONS: CHEVALLEY GROUPS

The proof that Chevalley groups over Dedekind rings of arithmetic type have finite width in elementary generators, given by Oleg Tavgen [10], relied on reduction to groups of rank 2. The base of this reduction is the following fact, see [10], Proposition 1.

• If all Chevalley groups of a certain rank $l$ admit unitriangular factorisation $G = (UU^-)^L$ of certain length, then all groups of larger ranks also admit unitriangular factorisations of the same length. Actually, our Theorem 3 is a minor elaboration of Tavgen’s idea.

• Another elaboration of the same idea can be found in a recent paper by Andrei Rapinchuk and Igor Rapinchuk [53]. There, the same idea is used to establish that Chevalley groups over a local ring $R$ can be factored as

$$G(\Phi, R) = (U(\Phi, R)U^-(\Phi, R))^4.$$  

• For symplectic group our Theorem 1 was established by You Hong [71]. For orthogonal groups a similar, but weaker result, under stronger stability conditions was established by Frank Arlinghaus, Leonid V Aserstein and You Hong [13].

Over fields, especially over finite fields, there was a vivid interest in explicit calculation of length, over the last decade. Let $\mathbb{F}_q$, $q = p^m$, be a finite field of characteristic $p$. In this case the group $U(\Phi, q) = U(\Phi, \mathbb{F}_q)$ is a Sylow $p$-subgroup of the Chevalley group $G(\Phi, q) = G(\Phi, \mathbb{F}_q)$. 
Thus, in this case calculation of the minimal length of unitriangular factorisation is essentially equivalent to the calculation of the minimal length of factorisation of a finite simple group of Lie type in terms of its Sylow $p$-subgroups, in the defining characteristic.

- Martin Liebeck and Laszlo Pyber [43, Theorem D], prove that finite Chevalley groups admit unitriangular factorisation

\[ G(\Phi, q) = (U(\Phi, q)U^-(\Phi, q))^6U(\Phi, q) \]

of length 13. Actually, they also consider twisted Chevalley groups and obtain for them a similar bound, with the only exception of the senior Ree groups $^2F_4(q)$, for which they prove existence of factorisation $G = (UU^-)^{12}U$.

- Laszlo Babai, Nikolay Nikolov and Laszlo Pyber [14, Application 1.1], prove that finite Chevalley groups admit unitriangular factorisation

\[ G(\Phi, q) = U(\Phi, q)U^-(\Phi, q)U(\Phi, q)U^-(\Phi, q)U(\Phi, q)U(\Phi, q) \]

of length 5. They also obtain a similar result with the same bound for twisted groups.

As opposed to [43], the proofs in the paper [14] are of extremely mysterious nature. In the final count, they rely on the product growth estimates of the following sort. Let $X, Y \subseteq G$ be two nonempty subsets of a finite group $G$. Then

\[ |XY| \geq \min\left( \frac{|G|}{2}, \frac{m|X| \cdot |Y|}{2|G|} \right), \]

where $m$ denotes the smallest dimension of a nontrivial real representation of the group $G$. In particular, in the case when the group $G$ is simple, and the orders $|X|, |Y|$ are large enough, but still much smaller than $|G|$, products rapidly grow: the order $|XY|$ is much larger, than $\max(|X|, |Y|)$.

In this connection, the authors of [14] make the following extravagant claim: “For the most part we can argue by the size of certain subsets, ignoring the structure and thus greatly simplifying the proofs and at the same time obtaining considerably better, nearly optimal bounds.”

Frankly, the proposal to completely ignore the structure of the objects in question, seems to us excessive. Asymptotic methods are in their place in the study of profinite groups, or groups of infinite rank. On the other hand, when applied to the finite groups proper, or to groups of Lie type of finite rank, asymptotic methods should be considered as surrogate of structural algebraic methods, as a method to verify the result we are interested in, which is identical to a real proof.

Certainly, in many situations such a genuine algebraic proof does not exist. What is worse, for many interesting problems such algebraic solutions are not in sight, since they would either require enormous case by case analysis, or cannot be obtained by the present day methods.

As an example of the first situation, we can cite the remarkable paper by Martin Liebeck and Aner Shalev [38, where it is proven that — apart
from three exceptional series, Suzuki groups and groups \( \text{Sp}(4, 2^m) \) and \( \text{Sp}(4, 3^m) \) — almost all finite simple groups are \((2, 3)\)-generated. For each individual group of Lie type, and at that not only over a finite field, but also over finitely generated rings, it is basically clear how to prove its \((2, 3)\)-generation, or the lack thereof. In fact, for important classes of groups, including all groups of sufficiently large Lie ranks (in the order of magnitude of several dozens, for finite simple groups) such constructive algebraic proofs have been obtained. However, a complete analysis of groups of small ranks requires computational effort so extensive that it has not been completed yet, despite enormous work by many authors, over more than two decades.

As an example of the second situation, we could cite recent papers on the verbal width of finite simple groups. An amazing general result by Aner Shalev asserts that for any nontrivial word \( w \neq 1 \) the verbal width of almost all finite simple groups \( G \) equals 3. In other words, every element of the group \( G \) can be presented as a product of not more than three values of the word \( w \). See also [30, 36, 39, 43, 54], where one can find further results in the same style, improvement of the estimate for verbal width to 2, in some cases, and references to preceding works. As far as we know, similar results are not available not to say over rings, but even over infinite fields. What is worse, such similar results are not available not just for all words, but for simplest specific words, such as powers. What is still worse, there is no clear understanding, how one could prove such results at all.

Nevertheless, we believe that whenever one can apply algebraic methods, coming from the structure theory and representation theory of algebraic groups, they would invariably give stronger and more general results. For those results that hold over arbitrary fields they should also give better bounds, and hopefully, have simpler proofs.

§4. The length of factorisations: computational linear algebra

Definitely, unitriangular factorisations for the group \( \text{SL}(n, K) \) are so natural and obvious, that there is little doubt that they should have been known to experts in linear algebra for quite some time. This is however the earliest reference we could trace.

- Gilbert Strang noticed that all groups \( \text{SL}(n, K) \) over a field \( K \) admit unitriangular factorisation

\[
\text{SL}(n, K) = U^-(n, K)U(n, K)U^-(n, K)U(n, K)
\]

of length 4. This is what experts in computational linear algebra call LULU-factorisation, where mnemonically L should be interpreted as the first letter of the word ‘lower’, whereas U should be interpreted as the first letter of the word ‘upper’.
Observe, that this fact got to the household of linear algebra decades after it became standard in algebraic $K$-theory, and at that in a much larger generality. However, as we know, the walls between different branches of mathematics are high.

As another amusing circumstance, we could mention that, as far as we know, this result was only observed in linear algebra due to applications in computer graphics! Let us explain in somewhat more detail, how it happened. It is hard for us to construe the precise sense in which experts in computer algebra use the term shear. As a first approximation one can think of shear as an arbitrary unipotent element of $\text{SL}(n, K)$. Or, at least, majority of authors in this field call arbitrary elements of the groups $\text{U}(n, K)$ and $\text{U}^-(n, K)$ shears.

However, there is no doubt how one should interpret the term one-dimensional shears. These are precisely transvections, not necessarily elementary. An exclusive role is played by transvections concentrated in one row — they are called beam shears — or in one column — they are called slice shears. The point is that at the level of pixels such transvections correspond to string copy with offset, which admits extremely efficient hardware implementations.

Alain Paeth proposed the following cute method to implement a 2D rotation:

$$
\begin{pmatrix}
\cos(\varphi) & \sin(\varphi) \\
-sin(\varphi) & \cos(\varphi)
\end{pmatrix} = \begin{pmatrix} 1 & \tan(\varphi/2) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\sin(\varphi) & 1 \end{pmatrix} \begin{pmatrix} 1 & \tan(\varphi/2) \\ 0 & 1 \end{pmatrix}.
$$

As we already mentioned, the shears themselves are implemented at the level of data transfer, this method turned out to be much more efficient than coordinate conversion.

At this point it is natural to ask, whether 3D rotations admit a similar efficient hardware implementation? Initially, one attempted to first decompose a 3D rotation into a product of three 2D rotations, and then each of them into a product of three transvections. However, shortly thereafter Tommaso Toffoli and Jason Quick [65] proposed a scheme based upon decomposition of a 3D rotation into a product of three unipotent matrices, and designed corresponding hardware.

Let us recall, that a three-dimensional rotation $g$ is completely determined by its Euler angles $(\alpha, \beta, \gamma)$, for instance, as follows:

$$
g = \begin{pmatrix}
c(\alpha) c(\beta) c(\gamma) - s(\alpha) s(\gamma) & -c(\alpha) c(\beta) s(\gamma) - s(\alpha) c(\gamma) & c(\alpha) s(\beta) \\
s(\alpha) c(\beta) c(\gamma) + c(\alpha) s(\gamma) & -s(\alpha) c(\beta) s(\gamma) + c(\alpha) c(\gamma) & s(\alpha) s(\beta) \\
-s(\beta) c(\gamma) & s(\beta) s(\gamma) & c(\beta)
\end{pmatrix},
$$

where $c(\varphi)$ and $s(\varphi)$ denote $\cos(\varphi)$ and $\sin(\varphi)$, respectively. It is easy to notice — this was the starting point of [65] — that $g$ admits the
following unitriangular factorisation of length 3:

\[
g = \begin{pmatrix}
1 & -\tan \left(\frac{\alpha + \gamma}{2}\right) & \cos(\alpha) \tan \left(\frac{\beta}{2}\right) \\
0 & 1 & \sin(\alpha) \tan \left(\frac{\beta}{2}\right) \\
0 & 0 & 1
\end{pmatrix}
\times
\begin{pmatrix}
1 & 0 & 0 \\
\sin(\alpha + \gamma) & 1 & 0 \\
-\cos(\gamma) \sin(\beta) & -\frac{\sin(\alpha - \gamma)}{\cos(\alpha + \gamma)} & \cos(\alpha + \gamma) \sin(\beta) \ 1
\end{pmatrix}
\times
\begin{pmatrix}
1 & \tan \left(\frac{\alpha + \gamma}{2}\right) & \cos \left(\frac{\alpha - \gamma}{2}\right) \ tan \left(\frac{\beta}{2}\right) \\
0 & 1 & -\sin(\gamma) \tan \left(\frac{\beta}{2}\right) \\
0 & 0 & 1
\end{pmatrix}.
\]

The work by Strang [62], cited at the beginning of the section, emerged as an attempt to generalise this formula to the case of an arbitrary \( n \). After that many subsequent papers appeared, which discussed various aspects of such decompositions. To convey some flavour of this activity, let us state a couple of typical results from the next paper by Toffoli [64].

- Almost all elements \( SL(n, \mathbb{R}) \) admit unitriangular factorisations of length 3 — ULU-factorisation, as it is called by the experts in computational linear algebra. Obviously, as opposed to the previous section, here the expression ‘almost all’ should be interpreted in terms of Lebesgue measure.

- For any matrix \( g \in SL(n, K) \) there exists such a permutation matrix \( (\pi) \), \( \pi \in S_n \), that at least one of the matrices \( (\pi)g \) or \( (\pi)^{-1}g \) admits unitriangular factorisation of length 3

We will make no attempts to systematically cover subsequent literature in the field, and limit ourselves to several typical works [20, 31, 35, 57], where one can find further references.
§5. SOME SUBGROUPS OF CHEVALLEY GROUPS

Our notation pertaining to Chevalley groups are utterly standard and coincide with the ones used in [69, 70], where one can find many further references.

Let \( \Phi \) be a reduced irreducible root system of rank \( l \), \( W = W(\Phi) \) be its Weyl group and \( P \) be a weight lattice intermediate between the root lattice \( Q(\Phi) \) and the weight lattice \( P(\Phi) \). Further, we fix an order on \( \Phi \) and denote by \( \Pi = \{\alpha_1, \ldots, \alpha_l\} \), \( \Phi^+ \) and \( \Phi^- \) the corresponding sets of fundamental, positive and negative roots, respectively. Our numbering of the fundamental roots follows Bourbaki. Finally, let \( R \) be a commutative ring with 1, as usual, \( R^* \) denotes its multiplicative group.

It is classically known that with these data one can associate the Chevalley group \( G = G_P(\Phi, R) \), which is the group of \( R \)-points of an affine groups scheme \( G_P(\Phi, -) \), known as the Chevalley–Demazure group scheme. Since mostly our results do not depend on the choice of the lattice \( P \), in the sequel we usually assume that \( P = P(\Phi) \) and omit any reference to \( P \) in the notation. Thus, \( G(\Phi, R) \) will denote the simply connected Chevalley group of type \( \Phi \) over \( R \).

In what follows, we fix a split maximal torus \( T(\Phi, -) \) of the group scheme \( G(\Phi, -) \) and set \( T = T(\Phi, R) \). As usual, \( X_\alpha, \alpha \in \Phi \), denotes a unipotent root subgroup in \( G \), elementary with respect to \( T \). We fix isomorphisms \( x_\alpha : R \rightarrow X_\alpha \), so that \( X_\alpha = \{x_\alpha(\xi) \mid \xi \in R\} \), which are interrelated by the Chevalley commutator formula, see [8, 19, 70]. Further, \( E(\Phi, R) \) denotes the elementary subgroup of \( G(\Phi, R) \), generated by all root subgroups \( X_\alpha, \alpha \in \Phi \).

In the sequel the elements \( x_\alpha(\xi) \) are called root unipotents. Now, let \( \alpha \in \Phi \) and \( \varepsilon \in R^* \). As usual, we set \( h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(1)^{-1} \), where \( w_\alpha(\varepsilon) = x_\alpha(\varepsilon)x_\alpha(\varepsilon^{-1})x_\alpha(\varepsilon) \). The elements \( h_\alpha(\varepsilon) \) are called semisimple root elements. For a simply connected group one has

\[
T = T(\Phi, R) = \langle h_\alpha(\varepsilon), \alpha \in \Phi, \varepsilon \in R^* \rangle.
\]

Finally, let \( N = N(\Phi, R) \) be the algebraic normaliser of the torus \( T = T(\Phi, R) \), i.e. the subgroup, generated by \( T = T(\Phi, R) \) and all elements \( w_\alpha(1), \alpha \in \Phi \). The factor-group \( N/T \) is canonically isomorphic to the Weyl group \( W \), and for each \( w \in W \) we fix its preimage \( n_w \) in \( N \).

The following result is obvious, well known, and very useful.

**Lemma 2.** The elementary Chevalley group \( E(\Phi, R) \) is generated by unipotent root elements \( x_\alpha(\xi), \alpha \in \pm \Pi, \xi \in R \), corresponding to the fundamental and negative fundamental roots.

**Proof.** Indeed, every root is conjugate to a fundamental root by an element of the Weyl group, while the Weyl group itself is generated by the fundamental reflections \( w_\alpha, \alpha \in \Pi \). Thus, the elementary group \( E(\Phi, R) \) is generated by the root unipotents \( x_\alpha(\xi), \alpha \in \Pi, \xi \in R \), and
the elements \(w_\alpha(1), \alpha \in \Pi\). It remains only to observe that
\[w_\alpha(1) = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1).\]

Further, let \(B = B(\Phi, R)\) and \(B^- = B^-(\Phi, R)\) be a pair of opposite Borel subgroups containing \(T = T(\Phi, R)\), standard with respect to the given order. Recall that \(B\) and \(B^-\) are semidirect products \(B = TU\) and \(B^- = T \rtimes U^-\), of the torus \(T\) and their unipotent radicals
\[U = U(\Phi, R) = \langle x_\alpha(\xi), \alpha \in \Phi^+, \xi \in R \rangle,\]
\[U^- = U^-(\Phi, R) = \langle x_\alpha(\xi), \alpha \in \Phi^-, \xi \in R \rangle.\]

Here, as usual, for a subset \(X\) of a group \(G\) one denotes by \(\langle X \rangle\) the subgroup in \(G\) generated by \(X\). Semidirect product decomposition of \(B\) amounts to saying that \(B = TU = UT\), and at that \(U \leq B\) and \(T \cap U = 1\). Similar facts hold with \(B\) and \(U\) replaced by \(B^-\) and \(U^-\).

Sometimes, to speak of both subgroups \(U\) and \(U^-\) simultaneously, we denote \(U = U(\Phi, R)\) by \(U^+ = U^+(\Phi, R)\).

Generally speaking, one can associate a subgroup \(E(S) = E(S, R)\) to any closed subset \(S\) in \(\Phi\). Recall that a subset \(S\) in \(\Phi\) is called closed, if for any two roots \(\alpha, \beta \in S\) the fact that \(\alpha + \beta \in \Phi\), implies that already \(\alpha + \beta \in S\). Now, we define \(E(S) = E(S, R)\) as the subgroup generated by all elementary root unipotent subgroups \(X_\alpha, \alpha \in S:\)
\[E(S, R) = \langle x_\alpha(\xi), \alpha \in S, \xi \in R \rangle.\]

In this notation, \(U\) and \(U^-\) coincide with \(E(\Phi^+, R)\) and \(E(\Phi^-, R)\), respectively. The groups \(E(S, R)\) are particularly important in the case where \(S\) is a special = unipotent set of roots, in other words, where \(S \cap (-S) = \emptyset\). In this case \(E(S, R)\) coincides with the product of root subgroups \(X_\alpha, \alpha \in S\), in some/any fixed order.

Let again \(S \subseteq \Phi\) be a closed set of roots. Then \(S\) can be decomposed into a disjoint union of its reductive = symmetric part \(S^r\), consisting of those \(\alpha \in S\), for which \(-\alpha \in S\), and its unipotent part \(S^u\), consisting of those \(\alpha \in S\), for which \(-\alpha \not\in S\). The set \(S^r\) is a closed root subsystem, whereas the set \(S^u\) is special. Moreover, \(S^u\) is an ideal of \(S\), in other words, if \(\alpha \in S, \beta \in S^u\) and \(\alpha + \beta \in \Phi\), then \(\alpha + \beta \in S^u\).

Levi decomposition asserts that the group \(E(S, R)\) decomposes into semidirect product \(E(S, R) = E(S^r, R) \rtimes E(S^u, R)\) of its Levi subgroup \(E(S^r, R)\) and its unipotent radical \(E(S^u, R)\).

§6. Elementary parabolic subgroups

The main role in the proof of Theorem 1 is played by Levi decomposition for elementary parabolic subgroups. Denote by \(m_k(\alpha)\) the coefficient of \(\alpha_k\) in the expansion of \(\alpha\) with respect to the fundamental roots:
\[\alpha = \sum m_k(\alpha)\alpha_k, \quad 1 \leq k \leq l.\]
Now, fix an \( r = 1, \ldots, l \) — in fact, in the reduction to smaller rank it suffices to employ only terminal parabolic subgroups, even only the ones corresponding to the first and the last fundamental roots, \( r = 1, l \).

Denote by

\[
S = S_r = \{ \alpha \in \Phi, \ m_r(\alpha) \geq 0 \}
\]

the \( r \)-th standard parabolic subset in \( \Phi \). As usual, the reductive part \( \Delta = \Delta_r \) and the special part \( \Sigma = \Sigma_r \) of the set \( S = S_r \) are defined as

\[
\Delta = \{ \alpha \in \Phi, \ m_r(\alpha) = 0 \}, \quad \Sigma = \{ \alpha \in \Phi, \ m_r(\alpha) > 0 \}.
\]

The opposite parabolic subset and its special part are defined similarly

\[
S^- = S_r^- = \{ \alpha \in \Phi, \ m_r(\alpha) \leq 0 \}, \quad \Sigma^- = \{ \alpha \in \Phi, \ m_r(\alpha) < 0 \}.
\]

Obviously, the reductive part \( S^- \) equals \( \Delta \).

Denote by \( P_r \) the elementary maximal parabolic subgroup of the elementary group \( E(\Phi, R) \).

By definition,

\[
P_r = E(S_r, R) = \langle x_\alpha(\xi), \ \alpha \in S_r, \ \xi \in R \rangle.
\]

Now Levi decomposition asserts that the group \( P_r \) can be represented as the semidirect product

\[
P_r = L_r \rtimes U_r = E(\Delta, R) \rtimes E(\Sigma, R)
\]

of the elementary Levi subgroup \( L_r = E(\Delta, R) \) and the unipotent radical \( U_r = E(\Sigma, R) \). Recall that

\[
L_r = E(\Delta, R) = \langle x_\alpha(\xi), \ \alpha \in \Delta, \ \xi \in R \rangle,
\]

Whereas

\[
U_r = E(\Sigma, R) = \langle x_\alpha(\xi), \ \alpha \in \Sigma, \ \xi \in R \rangle.
\]

A similar decomposition holds for the opposite parabolic subgroup \( P_r^- \), whereby the Levi subgroup is the same as for \( P_r \), but the unipotent radical \( U_r \) is replaced by the opposite unipotent radical \( U_r^- = E(-\Sigma, R) \).

As a matter of fact, we use Levi decomposition in the following form. It will be convenient to slightly change the notation and write \( U(\Sigma, R) = E(\Sigma, R) \) and \( U^-(\Sigma, R) = E(-\Sigma, R) \).

**Lemma 3.** The group \( \langle U^\sigma(\Delta, R), U^\rho(\Sigma, R) \rangle \), where \( \sigma, \rho = \pm 1 \), is the semidirect product of its normal subgroup \( U^\rho(\Sigma, R) \) and the complementary subgroup \( U^\sigma(\Delta, R) \).

In other words, it is asserted here that the subgroup \( U^\pm(\Delta, R) \) normalised each of the groups \( U^\pm(\Sigma, R) \), so that, in particular, one has the following four equalities for products

\[
U^\pm(\Delta, R)U^\pm(\Sigma, R) = U^\pm(\Sigma, R)U^\pm(\Delta, R),
\]

and, furthermore, the following four obvious equalities for intersections hold:

\[
U^\pm(\Delta, R) \cap U^\pm(\Sigma, R) = 1.
\]
In particular, one has the following decompositions:
\[ U(\Phi, R) = U(\Delta, R) \ltimes U(\Sigma, R), \quad U^-(\Phi, R) = U^-(\Delta, R) \ltimes U^-(\Sigma, R). \]

§7. Reduction to groups of smaller rank

The following result is a minor elaboration of Proposition 1 from the paper by Oleg Tavgen [10]. Tavgen states a slightly weaker, but more general result, in terms of existence of factorisations for all irreducible root systems of a certain rank. On the other hand, he considers also twisted groups, apart from those of type $^2A_2$.

**Theorem 3.** Let $\Phi$ be a reduced irreducible root system of rank $l \geq 2$, and $R$ be a commutative ring. Suppose that for subsystems $\Delta = \Delta_1, \Delta_2$ the elementary Chevalley group $E(\Delta, R)$ admits unitriangular factorisation
\[ E(\Delta, R) = (U(\Delta, R)U^-(\Delta, R))^L. \]
Then the elementary Chevalley group $E(\Phi, R)$ admits unitriangular factorisation
\[ E(\Phi, R) = (U(\Phi, R)U^-(\Phi, R))^L. \]
of the same length $2L$.

Clearly, Theorem 1 immediately follows from Lemma 1 and Theorem 3, so that it only remains to prove Theorem 3.

The leading idea of Tavgen’s proof is so general and beautiful that it works in many other similar contexts. It relies on the fact that for systems of rank $\geq 2$ every fundamental root falls into the subsystem of smaller rank obtained by dropping either the first or the last fundamental root. Eiichi Abe and Kazuo Suzuki [11] and [12] used the same argument in their description of normal subgroups, to extract root unipotents. A similar consideration, in conjunction with a general position argument, was used by Vladimir Chernousov, Erich Ellers, and Nikolai Gordeev in their simplified proof of the Gauss decomposition with prescribed semisimple part [22].

Let us reproduce the details of the argument. By definition
\[ Y = (U(\Phi, R)U^-(\Phi, R))^L \]
is a subset in $E(\Phi, R)$. Usually, the easiest way to prove that a subset $Y \subseteq G$ coincides with the whole group $G$ consists in the following.

**Lemma 4.** Assume that $Y \subseteq G$, $Y \neq \emptyset$, and $X \subseteq G$ be a symmetric generating set. If $XY \subseteq Y$, then $Y = G$.

**Proof of Theorem 3.** By Lemma 2 the group $G$ is generated by the fundamental root elements
\[ X = \{x_\alpha(\xi) \mid \alpha \in \pm \Pi, \xi \in R\}. \]

---

1Observe, that one should read the first equality in the statement of Proposition 1 in [10] as $\text{rk}(\sigma \Phi_0) = m$. 
Thus, by Lemma 4 is suffices to prove that \( X Y \subseteq Y \).

Let us fix a fundamental root unipotent \( x_\alpha(\xi) \). Since \( \text{rk}(\Phi) \geq 2 \), the root \( \alpha \) belongs to at least one of the subsystems \( \Delta = \Delta_r \), where \( r = 1 \) or \( r = l \), generated by all fundamental roots, except for the first or the last one, respectively. Set \( \Sigma = \Sigma_r \) and express \( U^\pm(\Phi, R) \) in the form

\[
U(\Phi, R) = U(\Delta, R)U(\Sigma, R), \quad U^-(\Phi, R) = U^-\Delta, R^-U^-\Sigma, R^-.
\]

Using Lemma 3 we see that

\[
Y = (U(\Delta, R)U^-\Delta, R))U^-(\Sigma, R)U^-\Sigma, R^-)U^-\Delta, R^-U^-\Sigma, R^-).
\]

Since \( \alpha \in \Delta \), one has \( x_\alpha(\xi) \in E(\Delta, R) \), so that the inclusion \( x_\alpha(\xi)Y \subseteq Y \) immediately follows from the assumption. \( \square \)

§8. The use of Bruhat and Gauss decomposition

Observe that, as opposed to the proofs in \[14\], the proof in the paper by Liebeck and Pyber \[43\] is natural and transparent, and is based on the Bruhat decomposition \( g = udnwv \), where \( u, v \in U \), \( d \in T \), \( w \in W \).

Excessive number of factors in their result is due to the fact that they decompose \( d \) and \( n \) separately, and, on top of that, the way they do it is far from being optimal. For instance, to decompose an element \( h_\alpha(\varepsilon) \) they rely directly on the definition \( h_\alpha(\varepsilon) = w_\alpha(\varepsilon)w_\alpha(-1) \), rather than perform the actual calculation in \( \text{SL}(2, R) \). Substituting the expression of \( w_\alpha(\varepsilon) \) in terms of root unipotents, in the definition of \( h_\alpha(\varepsilon) \) we express \( h_\alpha(\varepsilon) \) as a product of 5 root unipotents as follows:

\[
\begin{pmatrix}
\varepsilon & 0 \\
0 & \varepsilon^{-1}
\end{pmatrix} = \begin{pmatrix}
1 & \varepsilon \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-\varepsilon^{-1} & 1
\end{pmatrix} \begin{pmatrix}
1 & \varepsilon - 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix} \begin{pmatrix}
1 & -1 \\
0 & 1
\end{pmatrix}.
\]

Factually, it is well known that \( h_\alpha(\varepsilon) \) is already a product of 4 root unipotents:

\[
\begin{pmatrix}
\varepsilon & 0 \\
0 & \varepsilon^{-1}
\end{pmatrix} = \begin{pmatrix}
1 & -1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
1 - \varepsilon & 1
\end{pmatrix} \begin{pmatrix}
1 & \varepsilon^{-1} \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
\varepsilon(\varepsilon - 1) & 1
\end{pmatrix}.
\]

As a matter of fact, the same idea allows to it immediately obtain for Chevalley groups over semilocal rings unitriangular factorisation of the same length 5, as in \[14\]. The proof is based on Gauss decomposition and the following toy version of Theorem 1.

Recall, that the following analogue of Gauss decomposition was established by Eiichi Abe and Kazuo Suzuki \[11, 12\] and by Michael Stein \[60\].

\[2\]In terms of computational linear algebra this is the ULU-decomposition, whereas Bruhat decomposition is the UPU-decomposition, where \( P \) should be interpreted as the first letter of the word ‘permutation’. Though, Stein himself called this decomposition a Bruhat type decomposition. Erich Ellers and Nikolai Gordeev usually intend by Gauss decomposition the LPU-decomposition, which is normally called Birkhoff decomposition. At the same time, in the works on computational linear algebra the name ‘Gauss decomposition’ often refers either to the PLU-decomposition, or the LUP-decomposition.
Lemma 5. Let $R$ be a semilocal ring. Then for a simply connected Chevalley group one has the following decomposition

$$G(\Phi, R) = U(\Phi, R)T(\Phi, R)U^{-}(\Phi, R)U(\Phi, R).$$

As we know, Gauss decomposition for the group $\text{SL}(n, R)$ holds under a weaker assumption $\text{sr}(R) = 1$, see, for instance, [68]. First author [69] noticed that in fact condition $\text{sr}(R) = 1$ is necessary for a Chevalley group $G(\Phi, R)$ to admit Gauss decomposition, for all types. For the linear case this obvious circumstance was rediscovered some 20+ later in [49, 21]. On the other hand, for all types except $\Phi = A_l, C_l$ all known sufficient conditions are somewhat stronger, say, something like $\text{asr}(R) = 1$. Thus, there is still some gap between the necessary and sufficient conditions.

The following result is a generalisation of Lemma 18 of [25], where a slightly weaker fact is proven for the case $\Phi = A_l$.

Theorem 4. Let $\Phi$ be a reduced irreducible root system and $R$ be an arbitrary commutative ring. Then one has the following inclusion

$$N(\Phi, R) \subseteq U(\Phi, R)U^{-}(\Phi, R)U(\Phi, R)U^{-}(\Phi, R).$$

Corollary. Let $\Phi$ be a reduced irreducible root system and $R$ be a commutative semilocal ring. Then the simply connected Chevalley group $G(\Phi, R)$ admits unitriangular factorisation

$$G(\Phi, R) = U(\Phi, R)U^{-}(\Phi, R)U(\Phi, R)U^{-}(\Phi, R)U(\Phi, R)$$

of length 5.

Proof. Indeed,

$$G = UTU^{-}U \leq U(UU^{-}UU^{-})U^{-}U = UU^{-}UU^{-}U.$$

Since this result is both less general in terms of condition on the ground ring and weaker in terms of the resulting length than Theorem 1, we do not present the proof of Theorem 4 in general. However, for the purpose of illustration, let us reproduce its proof in the linear case. Recall that in this case the group $N(\text{A}_{n-1}, R)$ coincides with the group $N = \text{SN}(n, R)$ of monomial matrices with determinant 1.

Proof of Theorem 4 for $\Phi = \text{A}_{n-1}$. Let $g = (g_{ij}) \in \text{SN}(n, R)$. Let us argue by induction on $n$. In the case $n = 1$ there is nothing to prove. Thus, let $n \geq 2$.

Case 1. First, let $g_{nn} = 0$. Then there exists a unique $1 \leq r \leq n - 1$ such that $a = g_{rn} \neq 0$ and a unique $1 \leq s \leq n - 1$ such that $b = g_{ns} \neq 0$, all other entries in the $s$-th and the $n$-th columns are equal to 0. Since $g$ is invertible, automatically $a, b \in R^*$. The matrix $gt_{sn}(b^{-1})$ differs from $g$ only in the position $(n, n)$, where now we have 1 instead of 0. Consecutively multiplying the resulting matrix on the right by $t_{ns}(-b)$
and then by $t_{sn}(b^{-1})$, we get the matrix $h$, which differs from $g$ only at the intersection of the $r$-th and the $n$-th rows with the $s$-th and the $n$-th columns, where now instead of \(\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}\) one has \(\begin{pmatrix} -ab & 0 \\ 0 & 1 \end{pmatrix}\).

Observe, that the determinant of the leading submatrix of order $n - 1$ of the matrix $h$ equals 1, and thus we can apply induction hypothesis and obtain for that last matrix the desired factorisation in the group $SL(n - 1, R)$. This factorisation does not affect the last row and the last column. Thus, all factors of the above factorisation of the submatrix $h$ lie in $L_{n-1}$. It only remains to notice that $t_{sn}(b^{-1}) \in U_{n-1}$, and $t_{ns}(-b) \in U_{n-1}$, and then to invoke Lemma 3.

**Case 2.** Now, let $b = g_{nn} \neq 0$. Take arbitrary $1 \leq r, s \leq n - 1$ for which $a = g_{rs} \neq 0$. Again, automatically $a, b \in R^*$. As in the previous case, let us concentrate on the $r$-th and the $n$-th rows and the $s$-th and the $n$-th columns. Since there are no further non-zero entries in these rows and columns, any additions between them do not change other entries of the matrix, and only affect the submatrix at the intersection of the $r$-th and the $n$-th rows with the $s$-th and the $n$-th columns. Now, multiplying $g$ by $t_{ns}(b^{-1})t_{sn}(1-b)t_{ns}(-1)t_{ns}(-b^{-1}(1-b))$ on the right, we obtain the matrix $h$, where this submatrix, which was initially equal to $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, will be replaced by $\begin{pmatrix} ab & 0 \\ 0 & 1 \end{pmatrix}$. At this point the proof can be finished in exactly the same way as in the previous case. \(\square\)

§9. Dedekind rings of arithmetic type

Let $K$ be a global field, i.e. either a finite algebraic extension of the field $\mathbb{Q}$, or a field of algebraic functions in one variable over a finite field of constants $\mathbb{F}_q$. Further, let $S$ be a finite set of non-equivalent valuations of $K$, nonempty in the functional case and containing all Archimedian valuations in the number case. For a non-Archimedian valuation $p$ of the field $K$ we denote by $v_p$ the corresponding exponent.

As usual, $R = \mathcal{O}_S$ denotes the ring consisting of all $x \in K$ such that $v_p(x) \geq 0$ for all valuations $p$ of the field $K$, which do not belong to $S$. The ring $\mathcal{O}_S$ is called the Dedekind ring of arithmetic type determined by the set of valuations $S$ of the field $K$, or, otherwise, a Hasse domain, see, for instance, [1]. We will be mostly interested in the case $|S| \geq 2$, where by Dirichlet unit theorem the ring $\mathcal{O}_S$ has a unit of infinite order.

As another immediate corollary of reduction to smaller ranks, of the positive solution for the congruence subgroup problem [1, 4, 5, 7] and of the Cooke—Weinberger paper [24], one can state the following result. Observe, that Cooke—Weinberger calculations depend on the infinity of primes in arithmetic progressions, subject to additional multiplicative restrictions, and thus depend upon GRH = Generalised Riemann’s Hypothesis.
Theorem 5. Let $\Phi$ be a reduced irreducible root system and $R = \mathcal{O}_S$ be a Dedekind ring of arithmetic type with an infinite multiplicative group. Under assumption of the Generalised Riemann’s Hypothesis the simply connected Chevalley group $G(\Phi, R)$ admits unitriangular factorisation

$$G(\Phi, R) = (U(\Phi, R) U^{-}(\Phi, R))^4 U(\Phi, R)$$

of length 9.

One can prove similar results also in the absence of Generalised Riemann’s Hypothesis. However, in that case the known length estimates would be much worse, and depend on the number of classes of the ring $R$, the number of prime divisors of its discriminant, or something of the sort. Nevertheless, we are convinced that 9 here could be actually replaced by 5 or 6.

Proof of Theorem 2. Theorem 3 implies that one only has to prove the existence of a unitriangular factorisation of length 6 for the group $\text{SL}(2, \mathbb{Z}[\frac{1}{p}])$. In fact, we will prove the following slightly more precise result, where $U = U(2, \mathbb{Z}[\frac{1}{p}])$ and $U^- = U^{-}(2, \mathbb{Z}[\frac{1}{p}])$, respectively.

Lemma 6. Let $p \in \mathbb{Z}$ be a rational prime. Then under assumption of the Generalised Riemann’s Hypothesis one has

$$\text{SL}(2, \mathbb{Z}[\frac{1}{p}]) = U^- U U^- U^- \cup U U^- U^- U.$$

Our proof critically depends on the recent results on Artin’s conjecture, specifically, on the following result which was first stated in the papers by Pieter Moree [46, 47]. A complete proof can be found in [37], Corollary 5.4.

Lemma 7. Let $a \in \mathbb{Z}$, $a \neq 0, 1, -1$, is a square-free integer and $c, d \in \mathbb{Z}$, $c \perp d$, be coprime integers. Then under assumption of the Generalised Riemann’s Hypothesis density of the set of primes $q$ in the residue class $c$ (mod $d$), for which $q$ is a primitive root mod $d$, exists and is positive, with the only exception of the case, where the discriminant of $\mathbb{Q}(\sqrt{a})$ divides $d$, and every prime in the residue class $c$ (mod $d$) completely decomposes in $\mathbb{Q}(\sqrt{a})$.

Let us state the following immediate corollary of this result.

Corollary. Let $p \in \mathbb{Z}$ be a rational prime and $c \perp d$ be two coprime integers such that $p \perp d$. Then under assumption of the Generalised Riemann’s Hypothesis there are infinitely many primes $q$ in the residue class $c$ (mod $d$), for which $p$ is a primitive root modulo $q$.

Now, we are all set to prove Lemma 6, and thus also Theorem 2.
Proof. Let \( g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \text{SL}(2, \mathbb{Z}[\frac{1}{p}]) \). Since the case, where at least one of the matrix entries \( x, y, z, w \) equals 0 was already considered in the above proof of Theorem 4 for the linear case, in the sequel we can assume that \( xyzw \neq 0 \). Now, set
\[
g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} p^\alpha a & p^\beta b \\ * & * \end{pmatrix} \in \text{SL}(2, \mathbb{Z}[\frac{1}{p}]),
\]
where \( a, b \in \mathbb{Z} \) are not divisible by \( p \), and \( \alpha, \beta \in \mathbb{Z} \).

Case 1: \( \alpha \geq \beta \). By corollary of Lemma 7, there are infinitely many primes \( q \) of the form \( p^\alpha a + bk \) such that \( p \) is a primitive root modulo \( q \). Then
\[
g t_{21}(k) = \begin{pmatrix} p^\beta q & p^\beta b \\ * & * \end{pmatrix}.
\]
Since \( p \) is a primitive root \( \mod q \), there exists a \( u \geq 1 \) such that \( p^u = b \mod q \). Let, say, \( p^u = b + lq \). In this case
\[
g t_{21}(k) t_{12}(l) = \begin{pmatrix} p^\beta q & p^{\beta+u} \\ * & * \end{pmatrix}.
\]
Then for \( \theta = (1 - p^\alpha q) / p^{\beta+u} \), we get
\[
g t_{21}(k) t_{12}(l) t_{21}(\theta) = \begin{pmatrix} 1 & p^{\beta+u} \\ * & * \end{pmatrix}
\]
and, finally,
\[
g t_{21}(k) t_{12}(l) t_{21}(\theta) t_{12}(-p^{\beta+u}) = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}.
\]
Thus, in this case \( g \in U^-UU^-UU^- \).

Case 2: \( \alpha < \beta \). In this case exactly the same argument shows that \( g \in UU^-UU^-U \).

\[ \square \]

§10. Some Related Unsolved Problems

Let us list some further problems which are intimately related to unitriangular factorisations. Our Theorem 1 allows to slightly improve known bounds in some of them.

- Let us mention another problem similar to the one discussed in the present paper. Estimate the width of the elementary group \( E(\Phi, R) \) with respect to all unipotent elements. From Gauss decomposition with prescribed semisimple part, as obtained by Erich Ellers and Nikolai Gordeev, it follows that every noncentral element of a Chevalley group \( G(\Phi, K) \) over a field \( K \) is a product of two unipotent elements [26].

Theorem 1 immediately implies the following result.
Corollary 1. Let $\Phi$ be a reduced irreducible root system and $R$ be a commutative ring such that $sr(R) = 1$. Then every element of the simply connected Chevalley group $G(\Phi, R)$ is a product of three unipotent elements.

For rings of dimension $\geq 1$, the situation is much more delicate, see, for example, the paper by Fritz Grunewald, Jens Mennicke and Leonid Vaserstein [29].

- Another related problem, which was recently considered in the paper by Martin Liebeck, Nikolay Nikolov and Aner Shalev [40], is the width of a Chevalley group in fundamental $SL(2, R)$’s. For Chevalley groups over a [finite] field that paper gives the estimate $5|\Phi^+|$, which follows from [13]. Our Theorem 1 immediately implies a better bound in a much more general situation.

Corollary 2. Let $\Phi$ be a reduced irreducible root system and $R$ be a commutative ring such that $sr(R) = 1$. Then the simply connected Chevalley group $G(\Phi, R)$ can be written as the product of $4|\Phi^+|$ copies of the fundamental $SL(2, R)$.

As a matter of fact, even this estimate is far from the real one. Bruhat decomposition immediately implies the bound $3|\Phi^+|$, and somewhat more precise calculations allow to get the estimate $2|\Phi^+|$, for all Bezout rings. This is done in the paper by the first author and Evdokim Kovach, currently under way.

- A large number of papers are dedicated to the width of Chevalley groups in commutators, see in particular [71, 68, 61, 13, 26, 41, 59, 61]. For Chevalley groups over a ring of stable rank 1 our Theorem 1 implies the following obvious estimate,

Corollary 3. Let $\Phi$ be a reduced irreducible root system and $R$ be a commutative ring such that $sr(R) = 1$. Then the commutator width of the elementary Chevalley groups $E(\Phi, R)$ does not exceed 5.

Proof. For any $g \in E(\Phi, R)$ there exist $u, v \in U(\Phi, R)$ and $x, y \in U^-(\Phi, R)$ such that $g = u x v y = [^ux, ^uv] \cdot uv \cdot xy$.

Since $uv \in U(\Phi, R)$ and $xy \in U^-(\Phi, R)$, the claim follows. \qed

We believe that similarly to [71, 13] the commutator width of elementary Chevalley groups over a ring of stable rank 1 does not exceed 2 or 3.

In conclusion, let us state some further unsolved problems in the field.

Problem 1. Find for a Chevalley group of rank $\geq 2$ the minimal $L$ such that $G(\Phi, \mathbb{Z}) = (U(\Phi, \mathbb{Z})U^-(\Phi, \mathbb{Z}))^L$. 

In fact, the precise estimate is not even known in the following special case. We believe that the estimate by Thomas Laffey [33, 34], which follows from [25], is grossly exaggerated.

**Problem 2.** Find the minimal \( n \), starting from which one has the unitriangular factorisation

\[
\text{SL}(n, \mathbb{Z}) = (U(n, \mathbb{Z})U^-(n, \mathbb{Z}))^3.
\]

The following question is of obvious relevance, in connection with Lemma 6.

**Problem 3.** Is it true that \( U^-UU^-U = UUU^-U^-U \)?

Most probably, in general this is not the case, but it would be very interesting to see an explicit counter-example.

Generally speaking, no unitriangular factorisations exist over rings of dimension \( > 1 \), and no such factorisations can possibly exist. Their role is taken by parabolic factorisations, see [3, 4, 58], and the references therein. These factorisations allow to efficiently reduce problems regarding the group itself, to similar problems for groups of smaller ranks.

**Problem 4.** Estimate the width of Chevalley groups over commutative rings of small stable ranks in terms of classical subgroups/subgroups of type \( \text{A}_l \).

Even for fields such estimates should be much better than the ones obtained in [50, 51].

The following problem can be easily solved by the methods of the present paper.

**Problem 5.** Calculate the width of the elementary Chevalley group \( E(\Phi, R) \) over a semilocal ring \( R \) in terms of unipotent radicals \( U_P \) and \( U^-_P \) of two opposite parabolic subgroups.

Let us mention yet another direction in which it would be natural to generalise the results of the present paper. In [6] Victor Petrov and Anastasia Stavrova constructed the elementary subgroup \( E(R) \) in an isotropic reductive group \( G(R) \) and, for groups of relative rank \( \geq 2 \), proved normality of \( E(R) \) in \( G(R) \). See also the recent paper [5] by Alexander Luzgarev and Anastasia Stavrova, where it is proven that \( E(R) \) is perfect, with the known exceptions for Chevalley groups of rank 2.

Despite the apparent similarity of the statements, the following problem is not at all trivial, and, to the best of our knowledge, presently there are no non-trivial estimates at all.

**Problem 6.** Calculate the width of the elementary subgroup \( E(R) \) of an isotropic reductive group \( G(R) \) over a semilocal ring \( R \), in terms of unipotent radicals \( U_P \) and \( U^-_P \) of two opposite parabolic subgroups.
Even less than that is known in the arithmetic context.

**Problem 7.** Prove that the elementary subgroup $E(R)$ of an isotropic reductive group $G(R)$ of relative rank $\geq 2$ has bounded width with respect to the unipotent radicals $U_P$ and $U_{-P}$ of two opposite parabolic subgroups, in the case where $R = O_S$ is a Dedekind ring of arithmetic type.

The only result we are aware of, regarding this problem, is the paper by Igor Erovenko and Andrei Rapinchuk [27], which addresses the case of orthogonal groups, corresponding to a form of Witt index $\geq 2$.

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Vavilov N. A., Smolensky A. V., Sury B. Unitriangular factorisations of Chevalley groups.

Lately, the following problem attracted a lot of attention in various contexts: find the shortest factorisation $G = UU^−UU−...U^±$ of a Chevalley group $G = G(Φ, R)$ in terms of the unipotent radical $U = U(Φ, R)$ of the standard Borel subgroup $B = B(Φ, R)$ and the unipotent radical $U^− = U^−(Φ, R)$ of the opposite Borel subgroup $B^− = B^−(Φ, R)$.

So far, the record over a finite field was established in a 2010 paper by Babai, Nikolov, and Pyber, where they prove that a group of Lie type admits unitriangular factorisation $G = UU^−UU−U$ of length 5. Their proof invokes deep analytic and combinatorial tools. In the present paper we notice that from the work of Bass and Tavgen one immediately gets a much more general result, asserting that over any ring of stable rank 1 one has unitriangular factorisation $G = UU^−UU−U$ of length 4. Moreover, we give a detailed survey of triangular factorisations, prove some related results, discuss prospects of generalisation to other classes of rings, and state several unsolved problems. Another main result of the present paper asserts that, in the assumption of the Generalised Riemann’s Hypothesis, Chevalley groups over the ring $\mathbb{Z}[^1_p]$ admit unitriangular factorisation $G = UU^−UU^−UU^−$ of length 6. Otherwise, the best length estimate for Hasse domains with infinite multiplicative groups that follows from the work of Cooke and Weinberger, gives 9 factors.