The factorization problem for Jordan algebras: applications

A. L. Agore¹,² · G. Militaru¹,³

Received: 24 February 2022 / Accepted: 8 July 2022 / Published online: 27 July 2022
© The Author(s), under exclusive licence to Universitat de Barcelona 2022

Abstract
We investigate the factorization problem as well as the classifying complements problem in the setting of Jordan algebras. Matched pairs of Jordan algebras and the corresponding bicrossed products are introduced. It is shown that any Jordan algebra which factorizes through two given Jordan algebras is isomorphic to a bicrossed product associated to a certain matched pair between the same two Jordan algebras. Furthermore, a new type of deformation of a Jordan algebra is proposed as the main step towards solving the classifying complements problem.

Keywords Matched pair of Jordan algebras · Bicrossed products of Jordan algebras · The factorization problem for Jordan algebras · Deformations of a Jordan algebra

Mathematics Subject Classification 17C10 · 17C50 · 17C55

1 Introduction

The factorization problem is an old and notoriously difficult problem which stems in group theory. A weaker version of the factorization problem was first considered in [23, 27] where groups $G$ which admit two subgroups $A$ and $B$ such that $G = AB$, are studied under the name of permutably groups. If we assume, in addition, that $A$ and $B$ have trivial intersection then we say that $G$ factorizes through $A$ and $B$. The factorization problem for groups asks for the description and classification of all groups which factorize through two given groups. As explained in [3, 4], despite its elementary statement, the factorization problem is far from being an easy question. An important turning point for the factorization problem was the introduction of the matched pairs of groups [28] which consist of

---

¹ Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1‑764, 014700 Bucharest, Romania
² Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussels, Belgium
³ Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, 010014 Bucharest 1, Romania
two groups that act on each other in a compatible way. To each matched pair of groups one can associate a so-called bicrossed product, i.e. a group structure on the direct product of the underlying sets constructed from the two actions. This leads to a new and more computational approach to the factorization problem. Consequently, the factorization problem comes down to finding all matched pairs between two given groups and classifying the corresponding bicrossed products. Therefore, the same strategy relying on matched pairs and bicrossed products was used to approach the factorization problem for various mathematical objects such as: (co)algebras [11–13], Lie algebras and Lie groups [21, 24], Leibniz algebras [6], Hopf algebras [25], fusion categories [15] and so on. Furthermore, this new approach has also the advantage of opening the way to new classification methods as evidenced in [2, 3, 8, 19] for Hopf algebras. Since their introduction by P. Jordan in 1933, Jordan algebras have appeared in various different fields of mathematics and mathematical physics such us the theory of superstrings, supersymmetry, projective geometry, Lie algebras and algebraic groups, representation theory or functional analysis [26]. In the setting of Jordan algebras, the factorization problem can be stated as follows:

**Factorization problem** Let $A$ and $V$ be two given Jordan algebras. Describe and classify up to an isomorphism all Jordan algebras $E$ that factorize through $A$ and $V$, i.e. $E$ contains $A$ and $V$ as Jordan subalgebras such that $E = A + V$ and $A \cap V = \{0\}$.

Another closely-related problem which we will consider here was first introduced in [5] at the level of Lie/Hopf algebras as a converse of the factorization problem. Corresponding theories have been developed for associative algebras [1] or groups [7] where an important connection with the problem of classifying all groups of a given finite order was highlighted. For Jordan algebras it comes down to the following:

**Classifying complements problem** Let $A \subseteq E$ be a Jordan subalgebra of a Jordan algebra $E$. If a complement of $A$ in $E$ exists (i.e. a Jordan subalgebra $V \subseteq E$ such that $E = A + V$ and $A \cap V = \{0\}$), describe and classify up to an isomorphism all others complements of $A$ in $E$.

The paper is devoted to the two problems listed above and is organized as follows: in Sect. 2 we set the notation and recall some basic definitions and properties of Jordan algebras. Sect. 3 focuses on the factorization problem for Jordan algebras. Definition 3.1 introduces matched pairs of Jordan algebras and Theorem 3.2 connects them to the bicrossed product construction. These are the Jordan algebra counterparts of similar constructions performed for groups [28] or Lie algebras [22, 25]. Theorem 3.6 proves that the bicrossed product of two Jordan algebras is the object responsible for the factorization problem and is the Jordan algebra version of [22, Theorem 3.9]. As the examples in this section show, most low dimensional Jordan algebras can be written as bicrossed products of certain Jordan subalgebras. This suggests that achieving the classification of bicrossed products can lead to a successful strategy for classifying Jordan algebras. To this end, a general classification result for bicrossed products is presented in Theorem 3.7.

The last section of the paper deals with the classifying complements problem for Jordan algebras following the general strategy we developed previously in [1, 5, 7] for groups and associative/Lie/Hopf algebras. Consequently, a new type of deformation of a Jordan algebra arising from a matched pair is introduced in Theorem 4.3. More precisely, these deformations involve a certain linear map $r : B \to A$ associated to the canonical matched pair $(A, B, \langle, \rangle)$, called a deformation map. It is shown that all $A$-complements of a Jordan algebra $E$ are obtained from a given complement $B$ by this new kind of deformation. More precisely, for each deformation map $r : B \to A$ we can define a new $A$-complement denoted
by $B$ and, conversely, for any $A$-complement $B'$ we can find a deformation map of the canonical matched pair $(A, B, \prec, \triangleright)$ such that $B' \cong B$. Relevant examples are presented. In particular, it is proved that 4 out of the 6 isomorphism types of real Jordan algebras of dimension 2 ([10, Theorem 1]) appear as deformations of a given 2-dimensional Jordan algebra. This further substantiate the importance of these objects for the classification problem of Jordan algebras.

2 Preliminaries

All vector spaces, (bi)linear maps, tensor products are over a field $k$ of characteristic $\neq 2$. A Jordan algebra is a vector space $A$ together with a bilinear map $\cdot : A \times A \to A$, called multiplication, which is commutative and satisfies the Jordan identity, i.e. for any $a, b \in A$ we have:

$$a \cdot b = b \cdot a, \quad (a^2 \cdot b) \cdot a = a^2 \cdot (b \cdot a)$$

A Jordan algebra $A$ with trivial multiplication, i.e. $a \cdot b := 0$ for all $a, b \in A$, will be called abelian and a vector space $V$ endowed with the abelian multiplication will be denoted by $V_0$. Throughout, when there is no fear of confusion, we will denote the multiplication of a Jordan algebra just by juxtaposition. As in the case of Lie algebras, any associative algebra induces a Jordan algebra. More precisely, given an associative algebra $A$ one can define a Jordan algebra structure on the underlying vector space by

$$x \cdot y := 2^{-1}(xy + yx),$$

for all $x, y \in A$. The automorphism group of the Jordan algebra $A$ will be denoted by $\text{Aut}_J(A)$. We recall from [9] the following concept.

Definition 2.1 A right action of a Jordan algebra $A$ on a vector space $M$ is a bilinear map $\triangleright : M \times A \to M$ such that for any $a \in A$ and $x \in M$ we have:

$$(x \triangleright a^2) \triangleright a = (x \triangleright a) \triangleright a^2$$

Similarly, a left action of $A$ on $M$ is a bilinear map $\rhd : A \times M \to M$ such that for any $a \in A$ and $x \in M$ we have:

$$a \rhd (a^2 \rhd x) = a^2 \rhd (a \rhd x)$$

The canonical maps $\rhd : A \times A \to A$ and $\triangleright : A \times A^* \to A^*$ given for any $a, b \in A$ and $a^* \in A^*$ by:

$$a \rhd b := a \cdot b, \quad (a \triangleright a^*)(b) := a^*(a \cdot b)$$

are left actions of $A$ on $A$ and $A^* = \text{Hom}_k(A, k)$, respectively.

Definition 2.2 A Jordan bimodule [14] over a Jordan algebra $A$ is a vector space $M$ together with two linear maps:

$$M \otimes A \to M, \quad x \otimes a \mapsto xa$$

$$A \otimes M \to M, \quad a \otimes x \mapsto ax$$

subject to the following compatibilities:
\[ xa = ax \]  
(4)

\[ a(x^2a) = a^2(xa) \]  
(5)

\[ (a^2b)x - a^2(bx) = 2[(ab)(ax) - a(b(ax))] \]  
(6)

For all unexplained notions pertaining to Jordan algebra theory we refer the reader to [16, 20, 26].

### 3 Matched pairs and bicrossed products of Jordan algebras

In this section we introduce matched pairs of Jordan algebras and the corresponding bicrossed product in order to approach the factorization problem. Part of our results regarding matched pairs will be derived as a special case from the more general theory developed in [9, Sect. 2] in connection to the extending structures problem.

**Definition 3.1** Let \( A \) and \( V \) be two Jordan algebras. Then the quadruple \((A, V, \triangleleft, \triangleright)\) is called a matched pair of Jordan algebras if \( \triangleleft : V \times A \rightarrow V \) is a right action of the Jordan algebra \( A \) on \( V \), \( \triangleright : V \times A \rightarrow A \) is a left action of the Jordan algebra \( V \) on \( A \) and the following compatibility conditions hold for all \( a, b \in A \) and \( x, y \in V \):

\[
\begin{align*}
\text{(MP1)} & \quad a(x \triangleright a^2) + (x \triangleleft a^2) \triangleright a = a^2(x \triangleright a) + (x \triangleleft a) \triangleright a^2; \\
\text{(MP2)} & \quad x \triangleleft (x^2 \triangleright a) + (x^2 \triangleleft a) x = x^2 \triangleleft (x \triangleright a) + x^2 (x \triangleleft a); \\
\text{(MP3)} & \quad 2[(y \triangleleft (x \triangleright a)b) \triangleleft a + x \triangleleft ((x \triangleright a)b) + x \triangleleft ((x \triangleleft a) \triangleright b) + (x \triangleleft a) \triangleright b] + (x^2 \triangleleft b) \triangleleft a \\
& \quad + x \triangleleft (ba^2) = 2[(x \triangleleft a) \triangleleft ba + (x \triangleleft a) \triangleleft (x \triangleright b) + (x \triangleleft b) \triangleleft (x \triangleright a) + (x \triangleleft a)(x \triangleleft b)] \\
& \quad + x^2 \triangleleft ba + (x \triangleleft b) \triangleleft a^2; \\
\text{(MP4)} & \quad 2[y \triangleright (x \triangleright a) + x \triangleright (y \triangleright (x \triangleright a)) + (y \triangleleft (x \triangleright a) \triangleright b) + (x \triangleleft a) \triangleright y + (x^2) \triangleright a \\
& \quad + x \triangleright (y \triangleright a^2) = 2[(y \triangleright a)(x \triangleright a) + xy \triangleright (x \triangleright a) + (y \triangleleft a) \triangleright (x \triangleright a) + (x \triangleleft a) \triangleright (y \triangleright a)] \\
& \quad + x^2 \triangleright (y \triangleright a) + xy \triangleright a^2; \\
\text{(MP5)} & \quad 2[(y \triangleleft (x \triangleright a) \triangleleft a + (x \triangleleft a)y) \triangleleft a + x \triangleleft (y \triangleright (x \triangleright a)) + (y \triangleleft (x \triangleright a)) x + ((x \triangleleft a)y) \triangleleft a] \\
& \quad + x^2y \triangleleft a + x \triangleleft (y \triangleright a^2) + (y \triangleleft a^2)x = 2[(x \triangleleft a) \triangleleft (y \triangleright a) + (y \triangleleft a) \triangleleft (x \triangleright a) \\
& \quad + xy \triangleleft (x \triangleright a) + (x \triangleleft a)xy + (x \triangleleft a)(y \triangleleft a)] + x^2 \triangleleft (y \triangleright a) + xy \triangleleft a^2 + x^2(y \triangleleft a); \\
\text{(MP6)} & \quad 2[(x \triangleright a)b + (x \triangleleft a) \triangleleft b) \triangleright a + x \triangleright ((x \triangleleft a) \triangleright b) + (x \triangleright a) \triangleright b + b x \triangleright (a^2b) = 2[(x \triangleright a)(ab) + (x \triangleleft a) \triangleright (ba) + (x \triangleright a)(x \triangleright b) \\
& \quad + (x \triangleleft b) \triangleright (x \triangleright a) + (x \triangleleft a) \triangleright (x \triangleright b) + x \triangleleft b + (x \triangleleft b) \triangleleft a^2 + a^2(x \triangleright b)].
\end{align*}
\]

We denote by \( A \bowtie V \) the vector space \( A \times V \) together with the bilinear map \( \circ : (A \times V) \times (A \times V) \rightarrow A \times V \) defined by:

\[
(a, x) \circ (b, y) := (ab + x \triangleright b + y \triangleright a, x \triangleleft b + y \triangleleft a + xy)
\]  
(7)

The next result provides a motivation for the matched pair axioms as introduced in Definition 3.1 and is the Jordan algebra version of [25, Theorem 4.1]:

[Note: The content continues with further mathematical derivations and proofs.]
Theorem 3.2 Let $A$ and $V$ be two Jordan algebras and $\triangleleft : V \times A \to V$, $\triangleright : V \times A \to A$ two bilinear maps. The multiplication defined in (7) is a Jordan algebra structure on $A \times V$ if and only if $(A, V, \triangleleft, \triangleright)$ is a matched pair of Jordan algebras. In this case, the Jordan algebra $A \bowtie V$ will be called the bicrossed product associated to the matched pair $(A, V, \triangleleft, \triangleright)$.

Proof Note that the proof can be performed in a direct manner by a rather long and laborious computation. Instead, we will apply the results from [9, Sect. 2]. Indeed, by [9, Definition 2.2], a Jordan extending datum of $A$ through $V$ with $f := 0$ comes down precisely to the definition of a matched pair while the multiplication of the corresponding unified product reduces to the one in (7). The conclusion now follows from [9, Theorem 2.4] and [9, Example 2.6].

Our first examples of a bicrossed product are generic ones, namely the semidirect product of Jordan algebras [16] and the null split extension of a Jordan algebra by a bimodule [18, Sect. 2.2].

Example 3.3 1. Let $A$, $V$ be two Jordan algebras and $\triangleright : V \times A \to A$ a left action such that the following compatibilities hold for all $a, b \in A$ and $x, y \in V$:

(L1) \[ a (x \triangleright a^2) = a^2 (x \triangleright a) \]
(L2) \[ 2[y \triangleright (x \triangleright a) - (y \triangleright a)(x \triangleright a) + x \triangleright (y \triangleright (x \triangleright a)) - xy \triangleright (x \triangleright a)] = x^2 \triangleright (y \triangleright a) - (x^2 y) \triangleright a + (xy) \triangleright a^2 - x \triangleright (y \triangleright a^2) \]
(L3) \[ 2[(x \triangleright a)b)a - (x \triangleright a)(ab) + x \triangleright ((x \triangleright a)b) - (x \triangleright a)(x \triangleright b)] = x^2 \triangleright (ba) - (x^2 \triangleright b)a + a^2 (x \triangleright b) - x \triangleright (a^2 b) \]

Then the direct product $A \times V$ together with the multiplication given as follows $a, b \in A$ and $x, y \in V$:

\[ (a, x) \circ (b, y) := (ab + x \triangleright b + y \triangleright a, xy) \]

is a Jordan algebra called the left semidirect product and will be denoted by $A \ltimes V$.

It can be easily seen that the left semidirect product can be recovered from Definition 3.1 by considering a matched pair $(A, V, \triangleleft, \triangleright)$ with $\triangleleft := 0$. Then (MP2), (MP3) and (MP5) are trivially fulfilled while (MP1), (MP4) and (MP6) reduce to the compatibilities (L1), (L2) and (L3) respectively.

Similarly, the right semidirect product also appears as a special case of the bicrossed product. To this end, consider $A$, $V$ to be two Jordan algebras and $\triangleleft : V \times A \to V$ a right action such that the following compatibilities hold for all $a, b \in A$ and $x, y \in V$:

(R1) \[ x (x^2 \triangleleft a) = x^2 (x \triangleleft a) \]
(R2) \[ 2[(x \triangleleft a) \triangleleft b) \triangleleft a - (x \triangleleft a) \triangleleft ba + ((x \triangleleft a) \triangleleft b)x - (x \triangleleft a) (x \triangleleft b)] = x^2 \triangleleft ba - (x^2 \triangleleft b) \triangleleft a + (x \triangleleft b) \triangleleft a^2 - x \triangleleft (ba^2) \]
(R3) \[ 2[(x \triangleleft a)y) \triangleleft a - (x \triangleleft a)(y \triangleleft a) + ((x \triangleleft a)y \triangleleft x - (x \triangleleft a)xy] = xy \triangleleft a^2 - (y \triangleleft a^2)x + x^2 (y \triangleleft a) - x^2 y \triangleleft a \]

Then the direct product $A \times V$ together with the multiplication given as follows $a, b \in A$ and $x, y \in V$:  

$\text{ Springer}$
\[(a, x) \circ (b, y) := (ab, \, x \triangleleft b + y \triangleleft a + xy)\]  
\hspace{1cm} (9)

is a Jordan algebra called the right semidirect product and will be denoted by \(A \rtimes V\).

The right semidirect product can be recovered from Definition 3.1 by considering a matched pair \((A, V, \triangleleft, \triangleright)\) with \(\triangleright := 0\). Then (MP1), (MP4) and (MP6) are trivially fulfilled while (MP2), (MP3) and (MP5) reduce to the compatibilities (R1), (R2) and (R3) respectively.

2. Let \(A\) be a Jordan algebra and \(M\) a Jordan bimodule over \(A\). We will see \(M\) as an abelian Jordan algebra, i.e. \(xy = 0\) for all \(x, y \in M\). It is straightforward to see that \(\triangleleft : M \otimes A \to M\) defined by \(x \triangleleft a = xa\) is a right action of \(A\) on \(M\) satisfying (R1)–(R3), where we denote by juxtaposition the bimodule structure on \(M\). Indeed, to start with, \(\triangleleft\) is a right action by virtue of (5). Furthermore, (R1) and (R3) hold trivially since we assumed \(M\) to be an abelian Jordan algebra and by the same argument (R2) follows from (6). The multiplication on the corresponding right semidirect product \(A \rtimes M\) is given as follows for all \(a, b \in A\) and \(x, y \in M\) by:

\[ (a, x) \circ (b, y) := (ab, \, xb + ya) \]

which is precisely the null split extension of a Jordan algebra by a bimodule.

**Remark 3.4** A straightforward computation shows that \(V \cong \{0\} \times V\) is an ideal of \(A \rtimes V\), \(A \cong A \times \{0\}\) is a subalgebra of \(A \rtimes V\) and the canonical inclusion \(i_A : A \to A \rtimes V\), \(i_A(a) = (a, 0)\) is a split monomorphism of Jordan algebras: its retraction \(\pi_A : A \rtimes V \to A\), \(\pi_A(a, x) := a\), for all \(a \in A\) and \(x \in V\) is a Jordan algebra map.

Conversely, the right semidirect product of Jordan algebras describes split monomorphisms in the category: more precisely, if \(i : A \to E\) is a split monomorphism of Jordan algebras with splitting map \(p : E \to A\), then \(E \cong A \rtimes V\), where \(V = \ker p\) is obviously a Jordan subalgebra of \(E\). Indeed, \(\psi : A \rtimes V \to E\) defined by \(\psi(a, x) = a + x\) for all \((a, x) \in A \rtimes V\) is a Jordan algebra isomorphism, where \(A \rtimes V\) is the right semidirect product corresponding to the right action \(\triangleleft : V \times A \to V\) given by \(x \triangleleft a = xa\). To this end, for all \(a, b \in A\) and \(x, y \in V\) we have:

\[
\psi(a, x)\psi(b, y) = (a + x)(b + y) = ab + ay + xb + xy = \psi(ab, xb + ya + xy)
\]

\[
= \psi((a, x) \circ (b, y))
\]

In what follows, when describing a matched pair of Jordan algebras, we only indicate the non-zero values of the two actions.

**Examples 3.5** 1. [17, Sect. 4.1] lists the 7 isomorphism classes of non-associative unitary Jordan algebras of dimension 4. It can be easily seen that all those Jordan algebras can be written as bicrossed products of certain subalgebras. We include here one example; \(J_5\) is the 4-dimensional real Jordan algebra with multiplication table defined as follows:

|    | \(a\) | \(b\) | \(u\) | \(v\) |
|----|------|------|------|------|
| \(a\) | \(a\) | 0    | \(\frac{1}{2}u\) | \(v\) |
| \(b\) | 0    | \(b\) | \(\frac{1}{2}u\) | 0    |
| \(u\) | \(\frac{1}{2}u\) | \(\frac{1}{2}u\) | 0    | 0    |
$J_5$ is a right semidirect product between the 2-dimensional Jordan algebras $A = \langle a, b \mid a^2 = a, b^2 = b, ab = 0 \rangle$ and the abelian 2-dimensional Jordan algebra $V$ generated by $u$ and $v$ corresponding to the right action $\lhd : V \times A \to V$ defined as follows:

$$v \lhd a = v, \quad u \lhd a = u \lhd b = \frac{1}{2}u.$$

2. Similarly, among the 19 isomorphism classes of non-associative unitary Jordan algebras of dimension 5 listed in [17, Sect. 4.2] there is only one which can not be written as a bicrossed product, namely $J_3$. We include below two examples: the Jordan algebras denoted in [17] by $J_7$ and $J_{17}$. $J_7$ is the 5-dimensional real Jordan algebra with multiplication table defined as follows:

$$J_7$$

|   | $a$ | $b$ | $c$ | $u$ | $v$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $0$ | $\frac{1}{2}c$ | $\frac{1}{2}u$ | $\frac{1}{2}v$ |
| $b$ | $0$ | $b$ | $\frac{1}{2}c$ | $\frac{1}{2}u$ | $\frac{1}{2}v$ |
| $c$ | $\frac{1}{2}c$ | $\frac{1}{2}c$ | $0$ | $\frac{1}{2}(a + b)$ | $0$ |
| $u$ | $\frac{1}{2}u$ | $\frac{1}{2}u$ | $\frac{1}{2}(a + b)$ | $0$ | $0$ |
| $v$ | $\frac{1}{2}v$ | $\frac{1}{2}v$ | $0$ | $0$ | $0$ |

$J_7$ is a bicrossed product between the 3-dimensional Jordan algebra $A = \langle a, b, c \mid a^2 = a, b^2 = b, c^2 = 0, ab = 0, ac = \frac{1}{2}c, bc = \frac{1}{2}c \rangle$ and the abelian 2-dimensional Jordan algebra $V$ generated by $u$ and $v$. The actions ($\lhd, \rhd$) of the associated matched pair are defined as follows:

$$u \lhd a = u \lhd b = \frac{1}{2}u, \quad v \lhd a = v \lhd b = \frac{1}{2}v, \quad u \rhd c = \frac{1}{2}(a + b)$$

$J_{17}$ is the 5-dimensional real Jordan algebra with the following multiplication table:

$$J_{17}$$

|   | $a$ | $b$ | $c$ | $u$ | $v$ |
|---|---|---|---|---|---|
| $a$ | $a$ | $b$ | $\frac{1}{2}c$ | $0$ | $\frac{1}{2}v$ |
| $b$ | $b$ | $0$ | $0$ | $0$ | $0$ |
| $c$ | $\frac{1}{2}c$ | $0$ | $0$ | $\frac{1}{2}c$ | $0$ |
| $u$ | $0$ | $0$ | $\frac{1}{2}c$ | $u$ | $\frac{1}{2}v$ |
| $v$ | $\frac{1}{2}v$ | $0$ | $0$ | $\frac{1}{2}v$ | $0$ |

$J_{17}$ is a bicrossed product between the Jordan algebras $A = \langle a, b, c \mid a^2 = a, b^2 = 0, c^2 = 0, ab = b, ac = \frac{1}{2}c, bc = 0 \rangle$ and $V = \langle u, v \mid u^2 = u, v^2 = 0, uv = \frac{1}{2}v \rangle$. The actions ($\lhd, \rhd$) of the associated matched pair are defined as follows:
The bicrossed product of two Jordan algebras is the construction responsible for the factorization problem for Jordan algebras as formulated in the introduction. Indeed, we can prove the Jordan algebra version of [22, Theorem 3.9]:

**Theorem 3.6** A Jordan algebra $E$ factorizes through two given Jordan algebras $A$ and $V$ if and only if there exists a matched pair of Jordan algebras $(A, V, \triangleright, \triangleleft)$ such that $E \cong A \bowtie V$.

**Proof** First observe that $A \cong A \times \{0\}$ and $V \cong \{0\} \times V$ are Jordan subalgebras of $A \bowtie V$ and obviously $A \bowtie V$ factorizes through $A \times \{0\}$ and $\{0\} \times V$.

Conversely, assume that a Jordan algebra $E$ factorizes through two Jordan subalgebras $A$ and $V$. Consider $\pi_A : E \to A$ be the canonical projection, i.e. $\pi_A(a) = a$ for all $a \in A$ and $\ker \pi_A = V$. We can now define the following:

$$\triangleright : V \times A \to A, \quad x \triangleright a := \pi_A(xa) = \pi_A(ax)$$

$$\triangleleft : V \times A \to V, \quad x \triangleleft a := xa - \pi_A(xa)$$

We will show that $(A, V, \triangleleft, \triangleright)$ is a matched pair of Jordan algebras and $\varphi : A \bowtie V \to E$, $\varphi(a, x) := a + x$ is an isomorphism of Jordan algebras. To this end note that $\varphi$ is a linear isomorphism between $E$ and the direct product of vector spaces $A \times V$ with the inverse given by $\varphi^{-1}(y) := (\pi_A(y), y - \pi_A(y))$, for all $y \in E$. Therefore, there exists a unique Jordan algebra structure on $A \times V$ such that $\varphi$ becomes an isomorphism of Jordan algebras and this unique multiplication on $A \times V$ is given for any $a, b \in A$ and $x, y \in V$ by:

$$(a, x) \circ (b, y) := \varphi^{-1}(\varphi(a, x) \varphi(b, y))$$

The following straightforward computation shows that the above multiplication indeed coincides with the one associated to the matched pair $(A, V, \triangleleft, \triangleright)$ as defined in (7):

$$(a, x) \circ (b, y) = \varphi^{-1}(\varphi(a, x) \varphi(b, y)) = \varphi^{-1}((a + x)(b + y))$$

$$= \varphi^{-1}(ab + ay + xb + xy)$$

$$= (\pi_A(ab) + \pi_A(ay) + \pi_A(xb) + \pi_A(xy)),$$

$$ab + ay + xb + xy - \pi_A(ab) - \pi_A(ay) - \pi_A(xb) - \pi_A(xy))$$

$$= \left(ab + x \triangleright b + y \triangleright a, \ x \triangleleft b + y \triangleleft a + x y\right)$$

\[\square\]

Based on Theorem 3.6 we can restate the factorization problem as follows:

*Let $A$ and $V$ be two given Jordan algebras. Describe the set of all matched pairs $(A, V, \triangleleft, \triangleright)$ and classify up to an isomorphism all bicrossed products $A \bowtie V$.*

In this way, the factorization problem can be divided into two parts: the first one is a computational one which requires the explicit description of all matched pairs $(\triangleleft, \triangleright)$ between Jordan algebras $A$ and $V$ while the second one deals with the classification of all bicrossed products $A \bowtie V$ associated to the matched pairs previously described.
There is no general strategy for approaching the first part of the problem which needs to be treated ‘case by case’ for all specific examples of Jordan algebras (in the spirit of [2, 3]). The tool for approaching the second part of the problem is the next result which can be also used to describe the automorphisms group $\text{Aut}_r(A \Join V)$ of a given bicrossed product. In light of Example 3.5, the classification of bicrossed products is an important step in the classification of all Jordan algebras of a given dimension.

**Theorem 3.7** Let $(A, V, \langle, \rangle)$ and $(A', V', \langle', \rangle')$ be two matched pairs of Jordan algebras. Then there exists a bijective correspondence between the set of all morphisms of Jordan algebras $\psi : A \Join V \to A' \Join' V'$ and the set of all quadruples $(r, s, t, q)$, where $r : A \to A'$, $s : A \to V'$, $t : V \to A'$, $q : V \to V'$ are linear maps satisfying the following compatibility conditions for any $a, b \in A$ and $x, y \in V$:

\begin{align*}
  r(ab) - r(a)r(b) &= s(a)\langle' r(b) + s(b)\langle' r(a) \quad (12) \\
  s(ab) - s(a)s(b) &= s(a)\langle r(b) + s(b)\langle r(a) \quad (13) \\
  t(xy) - t(x)t(y) &= q(x)\langle' t(y) + q(y)\langle' t(x) \quad (14) \\
  q(xy) - q(x)q(y) &= q(x)\langle t(y) + q(y)\langle t(x) \quad (15) \\
  r(x \langle a) + t(x \langle a) &= r(a)t(x) + s(a)\langle' t(x) + q(x)\langle' r(a) \quad (16) \\
  s(x \langle b) + q(x \langle b) &= s(a)q(x) + s(a)\langle' t(x) + q(x)\langle r(a) \quad (17)
\end{align*}

Under the above bijection the Jordan algebra map $\psi = \psi_{(r,s,t,q)} : A \Join V \to A' \Join' V'$ corresponding to $(r, s, t, q)$ is given by:

$$\psi\left((a, x)\right) = \left(r(a) + t(x), s(a) + q(x)\right)$$

for all $a \in A$ and $x \in V$.

**Proof** By the universal property of the direct product of vector spaces we obtain that for any linear map $\psi : A \Join V \to A' \Join' V'$ there exists a unique quadruple $(r, s, t, q)$ of linear maps $r : A \to A'$, $s : A \to V'$, $t : V \to A'$, $q : V \to V'$ such that $\psi\left((a, x)\right) = \left(r(a) + t(x), s(a) + q(x)\right)$, for all $a \in A$ and $x \in V$. It remains to investigate when such a map $\psi = \psi_{(r,s,t,q)}$ is a morphism of Jordan algebras between the two bicrossed products, i.e. the following holds for all $(a, x), (b, y) \in A \times V$:

$$\psi\left((a, x)\circ_{A \Join V} (b, y)\right) = \psi(a, x)\circ_{A' \Join' V'}\psi(b, y)$$

(19)

This is again a rather cumbersome but straightforward computation which will be skipped. We only indicate the main strategy for the proof: first, we can prove that (19) holds for the pairs $(a, 0)$ and $(b, 0)$ if and only if (12) and (13) are fulfilled. Secondly, it can be shown that (19) holds for the pairs $(0, x)$ and $(0, y)$ if and only if (14) and (15) hold. Finally, (19) holds for the pairs $(a, 0)$ and $(0, x)$ if and only if (16) and (17) hold and this finishes the proof. □
**Example 3.8** Let $A_0$ be an abelian Jordan algebra and consider $k_0$ to be the abelian Jordan algebra of dimension 1. Then there exists a bijection between the set of all matched pairs of Jordan algebras $(A_0, k_0, \prec, \succ)$ and the set of all linear endomorphisms $D \in \text{End}_k(A)$ such that $D^3 = 0$ and the matched pair $(\prec, \succ)$ associated to $D$ is given as follows for all $a \in k$ and $a \in A$:

$$
\alpha \prec a := 0, \quad \alpha \succ a := \alpha D(a)
$$

In particular, any bicrossed product $A_0 \rtimes k_0$ of Jordan algebras is a left semidirect product $A_0 \bowtie k_0$, which we denote by $A(D) := A \times k$ and whose multiplication is given for all $a, b \in A$ and $\alpha, \beta \in k$ by:

$$(a, \alpha) \circ (b, \beta) = (\alpha D(b) + \beta D(a), 0)$$

Indeed, since $V = k$, there exists a bijection between the set of all bilinear maps $(\prec, \succ)$, $\prec : k \times A \rightarrow k$ and $\succ : k \times A \rightarrow A$ and the set of all linear maps $(\lambda, D) \in A^* \times \text{End}_k(A)$ given such that

$$
\alpha \prec a := \alpha \lambda(a), \quad \alpha \succ a := \alpha D(a)
$$

for all $\alpha \in k$, $a \in A$. Now since both $A$ and $k_0$ are abelian Jordan algebras, a laborious but straightforward computation shows that the axioms (MP1)–(MP6) from Definition 3.1 are reduced to the following two conditions: $\lambda = 0$ and $D^3(a) = 0$, for all $a \in A$.

### 4 Application: classifying complements for Jordan algebras

Let $A \subseteq E$ be a Jordan subalgebra of $E$. A Jordan subalgebra $B$ of $E$ is called a *complement* of $A$ in $E$ (or an $A$-complement of $E$) if $E$ factorizes through $A$ and $B$, i.e. $E = A \oplus B$ and $A \cap B = \{0\}$. Theorem 3.6 implies that the Jordan algebra $B \cong \{0\} \times B$ is a complement of $A \cong A \times \{0\}$ in the bicrossed product $A \bowtie B$. Moreover, if $B$ is an $A$-complement of $E$, then there exists a matched pair of Jordan algebras $(A, B, \triangleright, \prec)$ such that the corresponding bicrossed product $A \bowtie B$ is isomorphic as a Jordan algebra with $E$. The actions of the matched pair $(A, B, \triangleright, \prec)$ arise from the unique decomposition:

$$(0, b) \circ (a, 0) = (b \triangleright a, b \prec a)$$

for all $a \in A, b \in B$. The matched pair constructed in (21) will be called the *canonical matched pair* associated to the $A$-complement $B$ of $E$.

For a Jordan subalgebra $A$ of $E$ we denote by $\mathcal{F}(A, E)$ the isomorphism classes of $A$-complements of $E$. The *factorization index* of $A$ in $E$ is defined as $[E : A] := | \mathcal{F}(A, E) |$.

**Definition 4.1** Let $(A, B, \triangleright, \prec)$ be a matched pair of Jordan algebras. A $k$-linear map $r : B \rightarrow A$ is called a *deformation map* of the matched pair $(A, B, \triangleright, \prec)$ if the following compatibility holds for all $x, y \in B$:

$$
|r(xy) - r(x)r(y)| = x \triangleright r(y) + y \triangleright r(x) - r(x \prec r(y) + y \prec r(x))
$$

We denote by $\mathcal{DM}(B, A | (\triangleright, \prec))$ the set of all deformation maps of the matched pair $(A, B, \triangleright, \prec)$. 

\[\text{Springer}\]
**Example 4.2** Consider $J$ to be the 4-dimensional real Jordan algebra with multiplication table defined as follows:

|   | $a$ | $b$ | $u$ | $v$ |
|---|-----|-----|-----|-----|
| $a$ | $a$ | 0   | 0   | $\frac{1}{2}v$ |
| $b$ | 0   | $b$ | 0   | $\frac{1}{2}v$ |
| $u$ | 0   | 0   | $u$ | 0   |
| $v$ | $\frac{1}{2}v$ | $\frac{1}{2}v$ | 0   | 0   |

$J$ is a semidirect product between the 2-dimensional Jordan algebras $A = \langle a, b \mid a^2 = a, b^2 = b, ab = 0 \rangle$ and $V = \langle u, v \mid u^2 = u, v^2 = 0, uv = 0 \rangle$. Furthermore, the associated matched pair $(\langle, \rangle, \rangle)$ consists of the trivial left action $\langle : V \times A \to A$ and the right action $\rangle : V \times A \to V$ defined as follows:

$$v \langle a = v < b = \frac{1}{2}v.$$  

It can be easily seen by a straightforward computation that the deformation maps $r : V \to A$ associated to this matched pair are given as follows for some $\alpha \in \mathbb{R}$:

$$r(u) = 0, \quad r(v) = \alpha b$$  

(23)  

$$r(u) = 0, \quad r(v) = \alpha a$$  

(24)  

$$r(u) = a + b, \quad r(v) = \alpha a$$  

(25)  

$$r(u) = a + b, \quad r(v) = \alpha a$$  

(26)  

$$r(u) = a, \quad r(v) = 0$$  

(27)  

$$r(u) = b, \quad r(v) = 0$$  

(28)  

Using this concept the following deformation of a Jordan algebra is proposed:

**Theorem 4.3** Let $A$ be a Jordan subalgebra of $E$, $B$ a given $A$-complement of $E$ and $r : B \to A$ a deformation map of the associated canonical matched pair $(A, B, \rangle, \langle)$.  

1. Let $f_r : B \to E = A \bowtie B$ be the $k$-linear map defined for all $x \in B$ by:

$$f_r(x) = (r(x), x)$$  

Then $\widetilde{B} := \text{Im}(f_r)$ is an $A$-complement of $E$.  

2. $B_r := B$, as a $k$-vector space, with the new multiplication defined for all $x, y \in B$ by:

$$x *_r y := xy + x \langle r(y) + y \langle r(x)$$  

(29)
is a Jordan algebra called the \( r \)-deformation of \( B \). Furthermore, \( B, \cong \widetilde{B} \), as Jordan algebras.

**Proof** (1) To start with, we will prove that \( \widetilde{B} = \{ (r(x), x) \mid x \in B \} \) is a Jordan subalgebra of \( E = A \bowtie B \). Indeed, for all \( x, y \in B \) we have:

\[
(r(x), x) \odot (r(y), y) = \begin{cases} 
(r(x)r(y) + x \triangleright r(y) + y \triangleright r(x), \ x \lhd r(y) + y \lhd r(x) + xy) \\
(r(y), y) \odot (r(x), x) \end{cases}
\]

and the latter term obviously belongs to \( \widetilde{B} \). Moreover, it is straightforward to see that \( A \cap \widetilde{B} = \{0\} \) and \( (a, b) = (a - r(b), 0) \oplus (r(b), b) \in A + \widetilde{B} \). Therefore, \( \widetilde{B} \) is an \( A \)-complement of \( E \).

(2) We denote by \( \widetilde{f}_r \), the \( k \)-linear isomorphism from \( B \) to \( \widetilde{B} \) induced by \( f_r \). We will prove that \( \widetilde{f}_r \) is also a Jordan algebra map if we consider on \( B = B_r \), the multiplication given by (29). Indeed, for any \( x, y \in B \) we have:

\[
\widetilde{f}_r (x \cdot_r y) \equiv \begin{cases} 
\widetilde{f}_r (xy + x \lhd r(y) + y \lhd r(x)) \\
= \begin{cases} 
(r(x)r(y) + x \triangleright r(y) + y \triangleright r(x)), \ xy + x \lhd r(y) + y \lhd r(x) \\
(29)
\end{cases}
\end{cases}
\]

This shows that \( B_r \) is a Jordan algebra and the proof is now finished.

We are now able to describe all complements of a Jordan subalgebra \( A \) of \( E \).

**Theorem 4.4** Let \( A \) be a Jordan subalgebra of \( E \), \( B \) a given \( A \)-complement of \( E \) with the associated canonical matched pair of Jordan algebras \( (A, B, \triangleright, \lhd) \). Then \( \widetilde{B} \) is an \( A \)-complement of \( E \) if and only if there exists an isomorphism of Jordan algebras \( \widetilde{B} \cong B_r \), for some deformation map \( r : B \to A \) of the matched pair \( (A, B, \triangleright, \lhd) \).

**Proof** Let \( \widetilde{B} \) be an arbitrary \( A \)-complement of \( E \). Since \( E = A \bowtie B = A \bowtie \widetilde{B} \) we can find four \( k \)-linear maps:

\[
u : B \to A, \ \ v : B \to \widetilde{B}, \ \ t : \widetilde{B} \to A, \ \ w : \widetilde{B} \to B
\]

such that for all \( x \in B \) and \( y \in \widetilde{B} \) we have:

\[
(0, x) = (u(x), v(x)), \quad (0, y) = (t(y), w(y))
\]

(30)

It can be easily proved that \( v \) and \( w \) are inverses to each other and, in particular, that \( v : B \to \widetilde{B} \) is a \( k \)-linear isomorphism. We denote by \( \widetilde{v} : B \to A \bowtie B \) the composition:

\[
\widetilde{v} : B \overset{v}{\longrightarrow} \widetilde{B} \overset{i}{\hookrightarrow} E = A \bowtie B
\]
More precisely, we have \((0, \bar{v}(x)) = (0, v(x))^{(30)} = (-u(x), x)\), for all \(x \in B\). Consider now \(r := -u\); we will prove that \(r\) is a deformation map and \(\bar{B} \cong B_r\). Indeed, \(\bar{B} = \text{Im}(v) = \text{Im}(\bar{v})\) is a Jordan subalgebra of \(E = A \triangleright B\) and therefore we have:

\[
(r(x), x) \triangleright (r(y), y) = (r(x)r(y) + x \triangleright r(y) + y \triangleright r(x), \ x \triangleleft r(y) + y \triangleleft r(x) + xy)
\]

for some \(z \in B\). Thus, we obtain:

\[
r(z) = r(x) r(y) + x \triangleright r(y) + y \triangleright r(x), \quad z = x \triangleleft r(y) + y \triangleleft r(x) + xy
\]

(31)

Now by applying \(r\) to the second part of (31) we obtain:

\[
r(z) = r(x) r(y) + y \triangleleft r(x) + xy
\]

which combined with the first part of (31) shows that \(r\) is a deformation map of the matched pair \((A, B, \triangleright, \triangleleft)\). Moreover, for all \(x, y \in B\) we have:

\[
(0, v(x)v(y)) = (0, v(x)) \triangleright (0, v(y)) = (r(x), x) \triangleright (r(y), y) = (r(z), z)
\]

\[
= (0, v(z)) = (0, v(x \triangleleft r(y) + y \triangleleft r(x) + xy)) = (0, v(x \cdot_r y))
\]

that is, \(v : B_r \rightarrow \bar{B}\) is a Jordan algebra map and the proof is now finished. \(\square\)

In order to classify all complements we need to introduce the following:

**Definition 4.5** Let \((A, B, \triangleright, \triangleleft)\) be a matched pair of Jordan algebras. Two deformation maps \(r, s : B \rightarrow A\) are called *equivalent* and we denote this by \(r \sim s\) if there exists \(\sigma : B \rightarrow B\) a \(k\)-linear automorphism of \(B\) such that for all \(x, y \in B\) we have:

\[
\sigma(xy) - \sigma(x) \sigma(y) = \sigma(x) \triangleleft s(\sigma(y)) - \sigma(x \triangleleft r(y)) + \sigma(y) \triangleleft s(\sigma(x)) - \sigma(y \triangleleft r(x))
\]

(32)

We conclude this section with the following classification result for complements of Jordan algebras:

**Theorem 4.6** Let \(A\) be a Jordan subalgebra of \(E\), \(B\) an \(A\)-complement of \(E\) and \((A, B, \triangleright, \triangleleft)\) the associated canonical matched pair. Then:

1. \(\sim\) is an equivalence relation on \(\mathcal{DM}(B, A \mid (\triangleright, \triangleleft))\). We denote by \(\mathcal{HA}^2(B, A \mid (\triangleright, \triangleleft))\) the quotient set \(\mathcal{DM}(B, A \mid (\triangleright, \triangleleft))/\sim\).
2. There exists a bijection between the isomorphism classes of all \(A\)-complements of \(E\) and \(\mathcal{HA}^2(B, A \mid (\triangleright, \triangleleft))\). In particular, the factorization index of \(A\) in \(E\) is computed by the formula:

\[
[E : A]^f = |\mathcal{HA}^2(B, A \mid (\triangleright, \triangleleft))|
\]

where \(|X|\) denotes the cardinal of the set \(X\).

**Proof** In light of Theorem 4.4, in order to classify all \(A\)-complements of \(E\) it suffices to consider only \(r\)-deformations of \(B\), for various deformation maps \(r : B \rightarrow A\). Now let \(r\),

\[ Springer \]
Consider again $(A, V, \triangleright, \vartriangleleft)$ to be the matched pair depicted in Example 4.2. We will prove that the factorization index $[J : A]' = 4$, where $J = A \Join V$. We start by describing the Jordan algebras $V_r$ corresponding to the deformation maps listed in Example 4.2. First note that the Jordan algebras $V_r$ corresponding to the $r$-deformations given in (23) and (24) are equal and defined by $V_r = \{ u, v \mid u^2 = u, v^2 = av, uv = 0 \}$ for some $a \in \mathbb{K}$. If $a = 0$ we obtain the Jordan algebra $V_r$ while if $a \neq 0$ we have an isomorphism between $V_r$ and the Jordan algebra $V_r = \{ u, v \mid u^2 = u, v^2 = 0, uv = 0 \}$. Furthermore, it can be easily seen that $V_r$ is not isomorphic to $V_j$. Next, the Jordan algebras $V_r$ corresponding to the $r$-deformations given in (25) and (26) are equal and defined by $V_r = \{ u, v \mid u^2 = u, v^2 = \beta v, uv = v \}$ for some $\beta \in \mathbb{K}$. If $\beta = 0$ we obtain the Jordan algebra $V_r = \{ u, v \mid u^2 = u, v^2 = 0, uv = v \}$ while if $\beta \neq 0$ we have an isomorphism between $V_r$ and the Jordan algebra $V_r = \{ u, v \mid u^2 = u, v^2 = v, uv = v \}$. It can be easily seen that $V_r$ is isomorphic to $V_j$ and that $V_r$ is not isomorphic to $V$ nor to $V_j$. Similarly, the Jordan algebras $V_r$ corresponding to the $r$-deformations given in (27) and (28) are equal and defined by $V_r = \{ u, v \mid u^2 = u, v^2 = 0, uv = \frac{1}{2}v \}$. As $V_r$ is not isomorphic to any of the Jordan algebras $V, V_j$ or $V_r$ we can conclude that $[J : A]' = 4$, as desired.

Acknowledgements This work was supported by a grant of the Ministry of Research, Innovation and Digitalization, CNCS/CCCDI–UEFISCDI, project number PN-III-P4-ID-PCE-2020-0458, within PNCDI III.

References

1. Agore, A.L.: Classifying complements for associative algebras. Linear Algebra Appl. 446, 345–355 (2014)
2. Agore, A.L.: Classifying bicrossed products of two Taft algebras. J. Pure Appl. Algebra 222, 914–930 (2018)
3. Agore, A.L., Bontea, C.G., Militaru, G.: Classifying bicrossed products of Hopf algebras. Algebr. Represent. Theory 17, 227–264 (2014)
4. Agore, A.L., Chirvăștiu, A., Ion, B., Militaru, G.: Bicrossed products for finite groups. Algebr. Represent. Theory 12, 481–488 (2009)
5. Agore, A.L., Militaru, G.: Classifying complements for Hopf algebras and Lie algebras. J. Algebra 391, 193–208 (2013)
6. Agore, A.L., Militaru, G.: Unified products for Leibniz algebras. Applications. Linear Algebra Appl. 439, 2609–2633 (2013)
7. Agore, A.L., Militaru, G.: Classifying complements for groups. Ann. Inst. Fourier Appl. 65, 1349–1365 (2015)
8. Agore, A.L., Năstăsescu, L.: Bicrossed products with the Taft algebra. Arch. Math. 113, 21–36 (2019)
9. Agore, A.L., Militaru, G.: Unified products for Jordan algebras. Applications, arXiv:2107.04970, submitted for publication
10. Ancochea Bermúdez, J.M., Campoamor-Stursberg, R., Garcia Vergnolle, L., Sanchez Hernandez, J.: Contractions de algebres de Jordan en dimension 2. J. Algebra 319, 2395–2409 (2008)
11. Brzeziński, T.: Deformation of algebra factorisations. Commun. Algebra 29, 737–748 (2001)
12. Carotenuto, A., Dabrowski, L., Dubois-Violette, M.: Differential calculus on Jordan algebras and Jordan modules. Lett. Math. Phys. 109, 113–133 (2019)
13. Gelaki, S.: Exact factorizations and extensions of fusion categories. J. Algebra 480, 505–518 (2017)
14. Jacobson, N.: Structure and representations of Jordan Algebras. Am. Math. Soc. (1968)
15. Kashuba, I.: Variety of Jordan algebras in small dimension. Algebra Discrete Math. 2, 62–76 (2006)
16. Kashuba, I., Ovsienko, O., Shestakov, I.: Representation type of Jordan algebras. Adv. Math. 226, 385–418 (2011)
17. Keilberg, M.: Automorphisms of the doubles of purely non-abelian finite groups. Algebr. Represent. Theory 18, 1267–1297 (2015)
18. Koecher, M.: The Minnesota Notes on Jordan Algebras and their Applications. Lecture Notes in Math, vol. 1710. Springer, Berlin (1999)
19. Kosmann-Schwarzbach, Y., Magri, F.: Poisson-Lie groups and complete integrability. I. Drinfel’d bialgebras, dual extensions and their canonical representations. Ann. Inst. H. Poincare Phys. Theor 49, 433–460 (1988)
20. Lu, J.H., Weinstein, A.: Poisson Lie groups, dressing transformations and Bruhat decompositions. J. Differ. Geom. 31, 501–526 (1990)
21. Maillet, E.: Sur les groupes échangeables et les groupes décomposables. Bull. Soc. Math. France 28, 7–16 (1900)
22. Majid, S.: Matched pairs of Lie groups and Hopf algebra bicrossproducts. Nuclear Phys. B 6, 422–424 (1989)
23. Majid, S.: Physics for algebraists: non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction. J. Algebra 130, 17–64 (1990)
24. McCrimmon, K.: A Taste of Jordan Algebras. Universitext, Springer (2004)
25. Ore, O.: Structures and group theory. I. Duke Math. J. 3(2), 149–174 (1937)
26. Takeuchi, M.: Matched pairs of groups and bismash products of Hopf algebras. Commun. Algebra 9, 841–882 (1981)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.