Signatures of non-Gaussianity in the curvaton model

Kari Enqvist\textsuperscript{1,2} and Tomo Takahashi\textsuperscript{3}

\textsuperscript{1} Helsinki Institute of Physics, University of Helsinki, PO Box 64, FIN-00014, Finland
\textsuperscript{2} Department of Physical Science, University of Helsinki, PO Box 64, FIN-00014, Finland
\textsuperscript{3} Department of Physics, Saga University, Saga 840-8502, Japan
E-mail: kari.enqvist@helsinki.fi and tomo@cc.saga-u.ac.jp

Received 28 July 2008
Accepted 25 August 2008
Published 18 September 2008

Abstract. We discuss the signatures of non-Gaussianity in the curvaton model where the potential includes also a non-quadratic term. In such a case the non-linearity parameter $f_{NL}$ can become very small, and we show that non-Gaussianity is then encoded in the non-reducible non-linearity parameter $g_{NL}$ of the trispectrum, which can be very large. Thus the place to look for the non-Gaussianity in the curvaton model may be the trispectrum rather than the bispectrum. We also show that $g_{NL}$ measures directly the deviation of the curvaton potential from the purely quadratic form. While $g_{NL}$ depends on the strength of the non-quadratic terms relative to the quadratic one, we find that for reasonable cases roughly $g_{NL} \sim \mathcal{O}(10^4)-\mathcal{O}(10^5)$, which are values that may well be accessible by future observations.

Keywords: cosmological perturbation theory, cosmology of theories beyond the SM, physics of the early universe

ArXiv ePrint: 0807.3069
1. Introduction

The high precision of cosmological observations such as WMAP [1, 2] and the forthcoming Planck Surveyor Mission [3] will soon make it possible to probe the actual physics of the primordial perturbation rather than merely describing it. As is well known, the primordial scalar and tensor power spectra, characterized by spectral indices and the tensor-to-scalar ratio, can be used to test both models of inflation and other mechanisms for the generation of the primordial perturbation such as the curvaton [4]–[6] and the modulated reheating scenario [7, 8]. However, many models can imprint similar features on the primordial power spectrum. Therefore the possible non-Gaussianity of the photon temperature fluctuation, which may provide invaluable implications for the physics of the early universe, has been the focus of much attention recently.

The simplest inflation models generate an almost Gaussian fluctuation. In contrast, in the curvaton scenario there can arise a large non-Gaussianity [9]–[18]. However, in most studies on the curvaton so far, one simply assumes a quadratic curvaton potential. Since the curvaton cannot be completely non-interacting (it has to decay), it is of interest to consider the implications of the deviations from the exactly quadratic potential, which represent curvaton self-interactions. Such self-interactions would arise e.g. in curvaton models based on the flat directions of the minimally supersymmetric standard model (MSSM) [19]. Even small deviations could be important for phenomenology, as was pointed out in [11] where it was shown that the non-Gaussianity predicted by the curvaton model can be sensitive to the shape of the potential. In particular, the non-linearity parameter $f_{\text{NL}}$ which quantifies the bispectrum of primordial fluctuation can, in contrast to the case for the quadratic curvaton potential, be very small in some cases.

However, the signatures of non-Gaussianity can be probed not only with the bispectrum but also with the trispectrum. Thus even if $f_{\text{NL}}$ is very small, it does not necessarily imply that the fluctuation is almost Gaussian; rather, in that case the imprint of non-Gaussianity may be detected only in higher order statistics. The situation is then more complicated since the trispectrum is characterized by two numbers, conventionally denoted as $\tau_{\text{NL}}$ and $g_{\text{NL}}$. Roughly, $\tau_{\text{NL}} \sim f_{\text{NL}}^2$ for single-field models including the curvaton, whereas $g_{\text{NL}}$ measures the four-point correlator that is not reducible to the three-point
Signatures of non-Gaussianity in the curvaton model
correlator. In general, one can expect that when \( f_{\text{NL}} \) is large, also \( g_{\text{NL}} \) is large, although this is not a necessity, as will be discussed below.

The main purpose of this paper is to investigate non-Gaussianity in the curvaton model by considering both the bispectrum and the trispectrum assuming a potential which includes also a non-quadratic term. We will show that even when \( f_{\text{NL}} \) is negligibly small, \( g_{\text{NL}} \) can be very large, indicating that the first signature of the curvaton-induced primordial non-Gaussianity may not come from the bispectrum but rather from the trispectrum.

The structure of the paper is as follows. In section 2, we summarize the formalism, describe the set-up of the curvaton model, and give some key formulae for the study of non-Gaussianity in the curvaton model. Then in section 3, we investigate non-Gaussianity in curvaton models, presenting the predictions for the non-linearity parameters \( f_{\text{NL}} \) and \( g_{\text{NL}} \). We also discuss how the predictions are affected by the size of the coupling and the power of a non-quadratic term. The final section is devoted to the conclusions.

2. Non-linearity parameters

Our starting point is the \( \delta N \) formalism [20]–[23] that can be used to calculate the primordial curvature perturbation \( \zeta \). In this approach, the final amplitude of the primordial curvature perturbation on the uniform energy density hypersurface is given by \( \delta N(t, \vec{x}) \) where \( \delta N \) is the perturbation of the number of e-folds on a uniform energy density hypersurface during the final radiation dominated epoch\(^4\) with respect to the initial flat time-slice at horizon crossing during inflation. In the following, we are interested in the fluctuations generated from the curvaton field; thus we can write \( \zeta \) as

\[
\zeta = N_\sigma \delta \sigma_* + \frac{1}{2} N_{\sigma\sigma} (\delta \sigma_*)^2 + \frac{1}{6} N_{\sigma\sigma\sigma} (\delta \sigma_*)^3 + \cdots \tag{1}
\]

where \( \sigma \) is the curvaton field and the derivatives of \( N \) with respect to \( \sigma \) are represented as \( N_\sigma \equiv dN/d\sigma, N_{\sigma\sigma} \equiv d^2N/d\sigma^2 \) and \( N_{\sigma\sigma\sigma} \equiv d^3N/d\sigma^3 \).

To discuss non-Gaussianity in the scenario, we make use of the non-linearity parameters \( f_{\text{NL}} \) and \( g_{\text{NL}} \) defined by the expansion\(^5\)

\[
\zeta = \zeta_1 + \frac{3}{5} f_{\text{NL}} \zeta_1^2 + \frac{9}{25} g_{\text{NL}} \zeta_1^3 + O(\zeta_1^4). \tag{2}
\]

Writing the power spectrum as

\[
\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle = (2\pi)^3 P_\zeta(k_1) \delta(\vec{k}_1 + \vec{k}_2), \tag{3}
\]

the bispectrum and trispectrum are given by

\[
\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \zeta_{\vec{k}_4} \rangle = (2\pi)^3 B_\zeta(k_1, k_2, k_3) \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3), \tag{4}
\]

\[
\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \zeta_{\vec{k}_4} \rangle = (2\pi)^3 T_\zeta(k_1, k_2, k_3, k_4) \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4), \tag{5}
\]

\(^4\) In the curvaton scenario, there may arise two radiation dominated epochs. The final radiation epoch here means the one after the curvaton decay.

\(^5\) The following definitions hold only when a single source of perturbation is present. For a multi-field case, see e.g. [24].
where $B_\zeta$ and $T_\zeta$ are products of the power spectra and can be written as

$$B_\zeta(k_1, k_2, k_3) = \frac{6}{5} f_{NL} \left( P_\zeta(k_1) P_\zeta(k_2) + P_\zeta(k_2) P_\zeta(k_3) + P_\zeta(k_3) P_\zeta(k_1) \right),$$

$$T_\zeta(k_1, k_2, k_3, k_4) = \tau_{NL} \left( P_\zeta(k_{13}) P_\zeta(k_3) P_\zeta(k_4) + 11 \text{ permutations} \right)$$

$$+ \frac{54}{25} \xi_{NL} \left( P_\zeta(k_2) P_\zeta(k_3) P_\zeta(k_4) + 3 \text{ permutations} \right).$$

Note the appearance of the independent non-linearity parameter $\tau_{NL}$. However, in the scenario that we are considering in the following, $\tau_{NL}$ is related to $f_{NL}$ by

$$\tau_{NL} = \frac{36}{25} f_{NL}^2.$$  

To evaluate the primordial curvature fluctuation in the curvaton model, we need to specify a potential for the curvaton. Here we go beyond the usual quadratic approximation and consider the following potential:

$$V(\sigma) = \frac{1}{2} m_\sigma^2 \sigma^2 + \lambda m_\sigma^4 \left( \frac{\sigma}{m_\sigma} \right)^n,$$

which contains a higher polynomial term in addition to the quadratic term. For later discussion, we define a parameter $s$ which represents the size of the non-quadratic term relative to the quadratic one:

$$s \equiv 2\lambda \left( \frac{\sigma_s}{m_\sigma} \right)^{n-2}.$$  

Thus the larger $s$ is, the larger the contribution from the non-quadratic term is.

When the potential is quadratic, the fluctuation evolves exactly as the homogeneous mode. However, when the curvaton field evolves under a non-quadratic potential, the fluctuation of the curvaton evolves non-linearly on large scales. In that case the curvature fluctuation can be written, up to the third order, as

$$\zeta = \delta N = \frac{2}{3} \sigma_{\text{osc}}' \delta \sigma_s + \frac{1}{9} \left[ \frac{3}{2} \rho_{\text{osc}}'' \rho_{\text{osc}}^2 \sigma_{\text{osc}}'' \right] \left( \frac{\sigma_{\text{osc}}'}{\sigma_{\text{osc}}^2} \right)^2 (\delta \sigma_s)^2$$

$$+ \frac{4}{81} \left[ \frac{9}{4} \frac{\sigma_{\text{osc}}^2}{\sigma_{\text{osc}}'^3} + 3 \frac{\sigma_{\text{osc}}^2}{\sigma_{\text{osc}}'^2} \right] \left( \frac{\sigma_{\text{osc}}'}{\sigma_{\text{osc}}^2} \right)^2 (\delta \sigma_s)^3$$

$$+ \frac{r^3}{2} \left( 1 - 9 \frac{\sigma_{\text{osc}}'' \sigma_{\text{osc}}'''}{\sigma_{\text{osc}}'^2} \right) + 10 r^4 + 3 r^5 \right] \left( \frac{\sigma_{\text{osc}}'}{\sigma_{\text{osc}}} \right)^3 (\delta \sigma_s)^3,$$

where $\sigma_{\text{osc}}$ is the value of the curvaton at the onset of its oscillation while $r$ roughly represents the ratio of the energy density of the curvaton to the total density at the time of the curvaton decay. The exact definition is given by

$$r \equiv \frac{3 \rho_{\sigma}}{4 \rho_{\text{rad}} + 3 \rho_{\sigma}} \bigg|_{\text{decay}}.$$  

$$JCAP09(2008)012$$

$$\text{Journal of Cosmology and Astroparticle Physics 09 (2008) 012}$$

(stacks.iop.org/JCAP/2008/i=09/a=012)
Notice that $\sigma'_{\text{osc}}/\sigma_{\text{osc}} = 1/\sigma_*$ for the case of the quadratic potential. With this expression, we can write down the non-linearity parameter $f_{\text{NL}}$ as

$$f_{\text{NL}} = \frac{5}{4r} \left( 1 + \frac{\sigma'_{\text{osc}} \sigma''_{\text{osc}}}{\sigma_{\text{osc}}^2} \right) - \frac{5}{3} - \frac{5r}{6}. \tag{13}$$

Also notice that, although the curvaton scenario generally generates large non-Gaussianity with $f_{\text{NL}} \gtrsim \mathcal{O}(1)$, the non-linearity parameter $f_{\text{NL}}$ can be very small in the presence of the non-linear evolution of the curvaton field which can render the term $1 + (\sigma'_{\text{osc}} \sigma''_{\text{osc}})/\sigma_{\text{osc}}^2 \simeq 0$ [11,13].

In the curvaton model, the non-linearity parameter $g_{\text{NL}}$ can be written as

$$g_{\text{NL}} = \frac{25}{54} \frac{9}{4r^2} \left( \frac{\sigma'_{\text{osc}} \sigma''_{\text{osc}}}{\sigma_{\text{osc}}^2} + 3 \frac{\sigma'_{\text{osc}} \sigma''_{\text{osc}}}{\sigma_{\text{osc}}^2} \right) - \frac{9}{r} \left( 1 + \frac{\sigma'_{\text{osc}} \sigma''_{\text{osc}}}{\sigma_{\text{osc}}^2} \right) + \frac{1}{2} \left( 1 - 9 \frac{\sigma'_{\text{osc}} \sigma''_{\text{osc}}}{\sigma_{\text{osc}}^2} \right) + 10r + 3r^2 \right]. \tag{14}$$

As one can easily see, even if the non-linear evolution of $\sigma$ cancels to give a very small $f_{\text{NL}}$, such a cancellation does not necessarily occur for $g_{\text{NL}}$. This indicates an interesting possibility where the non-Gaussian signature of the curvaton may come from the trispectrum rather than from the bispectrum\(^6\). In the next section, we discuss the ramifications of this possibility in detail.

**3. Signatures of non-Gaussianity**

Let us now consider non-Gaussianity in curvaton models, paying particular attention to the non-linearity parameters $f_{\text{NL}}$ and $g_{\text{NL}}$. We have studied their behaviour numerically and plotted the results in figures 1–4.

\(^6\) The possibility of having a large $g_{\text{NL}}$ while $f_{\text{NL}}$ vanishes was pointed out for some general set-up in [13,24].
Figure 2. Left: plot of the combination $\sigma_{osc}' \sigma_{osc}'' / \sigma_{osc}'^2$ as a function of $n$ for several values of $s$. Right: plot of the combination $\sigma_{osc}^2 \sigma_{osc}' / \sigma_{osc}'^3$ as a function of $n$ for several values of $s$.

Figure 3. Contours of $f_{NL}$ are shown for the case $r = 0.01$ in the $n$–$s$ plane. Note that the line of $f_{NL} = 0$ is not changed for small values of $r$.

In the left panel of figure 1, the value of $f_{NL}$ is plotted as a function of the power $n$ for several values of $s$. There we have fixed the value of $r$ to $r = 0.01$. As can be read off from equation (13), when $r$ is small, $f_{NL}$ can be well approximated as

$$f_{NL} \simeq \frac{5}{4r} \left( 1 + \frac{\sigma_{osc}' \sigma_{osc}''}{\sigma_{osc}'^2} \right).$$

(15)

Notice that the combination $\sigma_{osc}' \sigma_{osc}'' / \sigma_{osc}'^2$ is zero when $n = 2$. As the value of $n$ becomes larger, the above combination yields a negative contribution, which is depicted in the left panel of figure 2. Thus at some point, $(\sigma_{osc}' \sigma_{osc}'' / \sigma_{osc}'^2)$ becomes $\sim -1$ and gives $f_{NL} \sim 0$. In other words, $f_{NL}$ decreases to zero as the potential deviates away from a quadratic form and then becomes zero for some values of $n$ and $s$ (see also [11]). To see in what cases we obtain $f_{NL} = 0$, we show the contours for $f_{NL}$ in figure 3. As seen from the figure, for small values of $s$, which correspond to the cases where the non-quadratic term is relatively
small compared to the quadratic one, the power \( n \) should be large to make \( f_{\text{NL}} \) very small. It should also be mentioned that, for a fixed \( s \), if we take larger values of \( n \) beyond the 'cancellation point' \( f_{\text{NL}} = 0 \), \( f_{\text{NL}} \) becomes negative.

However, as already discussed, even if we obtain very small values for \( f_{\text{NL}} \), it does not necessarily indicate that non-Gaussianity is small in the model but may show up in the higher order statistics. Indeed, this appears to be a generic feature of the curvaton model: the trispectrum cannot be suppressed and \( g_{\text{NL}} \) can be quite large and is always negative for small values of \( r \). This can be seen in the right panel of figure 1, where we plot the value of \( g_{\text{NL}} \) as a function of \( f_{\text{NL}} \). Note that once we choose the value of \( n \), \( f_{\text{NL}} \) is given for fixed \( s \) and \( r \). Thus a function of \( f_{\text{NL}} \) can be regarded as a function of \( n \). The corresponding value of \( n \) for \( f_{\text{NL}} \) can be read off from the left panel of figure 1. (We also show the plot of \( g_{\text{NL}} \) as a function of \( n \) in the top left panel of figure 4.)

Interestingly, even if \( f_{\text{NL}} \) is zero, the value of \( |g_{\text{NL}}| \) can be very large, as can be seen in figure 1. This is because a cancellation which can occur for \( f_{\text{NL}} \) does not take place for \( g_{\text{NL}} \), which is a smooth function of \( n \). When \( r \) is small, \( g_{\text{NL}} \) is mainly determined by the
first term in equation (14):

\[
g_{NL} \simeq \frac{25}{54} \left[ \frac{9}{4r^2} \left( \frac{\sigma_{\text{osc}}^2 \sigma_{\text{osc}}'''}{\sigma_{\text{osc}}''} + 3 \frac{\sigma_{\text{osc}}^2 \sigma_{\text{osc}}'''}{\sigma_{\text{osc}}''} \right) \right].
\] (16)

In the right panel of figure 2, we plot the combination of \( \sigma_{\text{osc}}^2 \sigma_{\text{osc}}''' / \sigma_{\text{osc}}'' \) as a function of \( n \) for several values of \( s \). There one sees that both terms in equation (16) give negative contributions, which indicates that the cancellation between these terms never occurs. Therefore \( g_{NL} \) is always negative for small values of \( r \) even when \( f_{NL} \) is very small. Thus, it may turn out that the best place to look for non-Gaussianity in curvaton models is in the trispectrum and \( g_{NL} \) in particular. Moreover, since for the quadratic potential with small \( r \), \( f_{NL} \simeq 5/4r \) and \( g_{NL} \simeq -10/3r \), in the purely quadratic case there is a relation between these parameters given by

\[
g_{NL} \simeq -\frac{10}{3} f_{NL}.
\] (17)

Thus any deviation from the relation equation (17) can indicate that the potential is not quadratic. Therefore the non-linearity parameter \( g_{NL} \) can be a direct measure for the deviation of the curvaton potential from the quadratic one and hence probes directly the underlying physics of the model. The larger the deviation from the purely quadratic case, the more negative \( g_{NL} \) is. Note that the right panel of figure 1 is somewhat confusing in this respect. To see how \( g_{NL} \) depends on the parameters, we show several plots for \( g_{NL} \) in figure 4. In the top left and right panels of figure 4 we plot \( g_{NL} \) for different values of \( s \) and \( n \) for a fixed \( r = 0.01 \), respectively, whereas in the bottom panel of the figure, we plot contours of \( g_{NL} \) in the \( n-r \) plane for fixed \( s = 0.1 \) (bottom left) \( s = 0.01 \) (bottom right).

4. Conclusion

We have discussed the signatures of non-Gaussianity in the curvaton model with the potential including a non-quadratic term in addition to the usual quadratic term. When the curvaton potential is not purely quadratic, fluctuations of the curvaton field evolve non-linearly on superhorizon scales. This gives rise to predictions for the bispectrum, characterized by the non-linearity parameter \( f_{NL} \), which can deviate considerably from the quadratic case. In particular, depending on the power and the relative strength of a non-quadratic term, the value of \( f_{NL} \) can become zero, as was already pointed out in [11]. In this paper, we investigated this issue by considering also the trispectrum, paying particular attention to the non-linearity parameter \( g_{NL} \) which quantifies the non-reducible part of the trispectrum. The second non-linearity parameter describing the trispectrum, denoted as \( \tau_{NL} \), is proportional to \( f_{NL}^2 \) and hence is small whenever \( f_{NL} \) is small. In contrast, by studying \( f_{NL} \) and \( g_{NL} \) simultaneously, we find that even when \( f_{NL} \) is negligibly small, the absolute value of \( g_{NL} \) can be very large. Thus the signature of non-Gaussianity in the curvaton model may come from the trispectrum rather than from the bispectrum if its potential deviates from a purely quadratic form, as one would expect in realistic particle physics models.

Moreover, we have shown that the deviation of \( g_{NL} \) from the relation equation (17), valid for the purely quadratic case in the small \( r \) limit, is a direct measure of the deviation from a quadratic curvaton potential. In the small \( f_{NL} \) limit, typical values of \( g_{NL} \) depend on the strength of the non-quadratic terms relative to the quadratic one, denoted as \( s \).
in equation (10), but roughly we find that \( g_{NL} \sim \mathcal{O}(-10^4) - \mathcal{O}(-10^5) \), as can be seen in figures 1 and 4. Such values may well be accessible in future experiments such as the Planck Surveyor Mission. Hence we may soon be in a position to test curvaton models in a meaningful way by exploring physics beyond the simple phenomenological quadratic potential.

Acknowledgments

TT would like to thank the Helsinki Institute of Physics for hospitality during the visit in which this work was initiated. This work was supported in part by the Sumitomo Foundation (TT) and the Grant-in-Aid for Scientific Research from the Ministry of Education, Science, Sports, and Culture of Japan No. 19740145 (TT), and the Academy of Finland Finnish–Japanese Core Programme grant 112420.

References

[1] Komatsu E et al (WMAP Collaboration), 2008 arXiv:0803.0547 [astro-ph]
[2] Dunkley J et al (WMAP Collaboration), 2008 arXiv:0803.0586 [astro-ph]
[3] (Planck Collaboration), 2006 arXiv:astro-ph/0604069
Planck webpage http://www.rssd.esa.int/index.php?project=planck
[4] Enqvist K and Sloth M S, 2002 Nucl. Phys. B 626 395 [SPIRES] [arXiv:hep-ph/0109214]
[5] Lyth D H and Wands D, 2002 Phys. Lett. B 524 5 [SPIRES] [arXiv:hep-ph/0110002]
[6] Moroi T and Takahashi T, 2001 Phys. Lett. B 522 215 [SPIRES] [arXiv:hep-ph/0110006]
[7] Dvali G, Gruzinov A and Zaldarriaga M, 2004 Phys. Rev. D 69 023505 [SPIRES] [arXiv:hep-ph/0403035]
[8] Komatsu E et al (WMAP Collaboration), 2008 arXiv:0803.0547 [astro-ph]
[9] Dunkley J et al (WMAP Collaboration), 2008 arXiv:0803.0586 [astro-ph]