PERIODIC POINT AND FIXED POINT RESULTS FOR MONOTONE MAPPINGS IN COMPLETE ORDERED LOCALLY CONVEX SPACES WITH APPLICATION TO DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we establish some new periodic point and fixed point theorems of single-valued mapping operating between complete ordered locally convex spaces under weaker assumptions. As an application, we prove the existence of lower and upper solutions of differential equations.

Keywords: periodic point; fixed point; measure of noncompactness; ordered locally convex spaces; differential equations.

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1. INTRODUCTION

A lot of research has been devoted to the study of the existence of a fixed and periodic points of single-valued and multivalued mappings in ordered Banach spaces and Metric spaces,[16],[17], [1], [9], and in complete locally convex spaces [7], [4], [5]. In the present work, we discuss an analogue of a periodic and a fixed point theorems proved in [1] in the setting of a complete ordered locally convex spaces.

The aim of this paper is to investigate the notion of order in a complete ordered locally convex
spaces which will give us a new periodic and a new fixed point results for a monotone mappings in the case of single-valued mapping.

The concept of measure of noncompactness in locally convex spaces [3, p 90] is used to define condensing operators in this new setting. Hence, we prove in Theorem 3.2 the equivalent of [1, Theorem 2.1.1] in complete ordered locally convex spaces and use it to prove the existence of a periodic and a fixed point in the theorem 3.5.

It is well known that fixed point theorems play an important role in differential equations, game theory and mathematical economics..., Toshio Yuasa [7], D. Guo, V. Lakshmikantham [1], S. Reich [15].

In Section 4, we prove the existence of lower and upper solutions of differential equations in a new framework.

2. Notations and Preliminaries

Let $E$ be a real vector space. A cone $K$ in $E$ is a subset of $E$ with $K + K \subset K$, $\alpha K \subset K$ for all $\alpha \geq 0$, and $K \cap (-K) = \{0\}$. As usual $E$ will be ordered by the (partial) order relation

$$x \leq y \iff y - x \in K$$

and the cone $K$ will be denoted by $E^+$. $E$ is said to be an ordered topological vector space, if $E$ is an ordered vector space equipped with a linear topology for which the positive cone $E^+$ is closed. For two vectors $x, y \in E$ the order interval $[x, y]$ is the set defined by

$$[x, y] = \{z \in E : x \leq z \leq y\}.$$

Note that if $x \not\leq y$ then $[x, y] = \emptyset$.

A cone $E^+$ of an ordered topological vector space $E$ is said to be normal whenever the topology of $E$ has a base at zero consisting of order convex sets. If the topology of $E$ is also locally convex, then $E$ is said to be an ordered locally convex space, and in this case the topology of $E$ has a base at zero consisting of open, circled, convex, and order convex neighborhoods.

The following two lemmas will be useful in the proofs of our results.

Lemma 2.1 ([2, Lemma 2.3]). If $E$ is an ordered topological vector space, then $E$ is Hausdorff and the order intervals of $E$ are closed.
Lemma 2.2 ([2, Lemma 2.22 and Theorem 2.23]). If the cone $E^+$ of an ordered topological vector space $(E, \tau)$ is normal, then the following assertions hold:

1. Every order interval is $\tau$–bounded.
2. For every two nets $(x_\alpha), (y_\alpha) \subset E$, (with the same index set $I$) satisfy $0 \leq x_\alpha \leq y_\alpha$ for each $\alpha$ and $y_\alpha \xrightarrow{\tau} 0$ imply $x_\alpha \xrightarrow{\tau} 0$.

Let $E$ be an ordered locally convex space whose topology is defined by a family $\mathcal{P}$ of continuous semi-norms on $E$, $\mathcal{B}$ is the family of all bounded subsets of $E$, and $\Phi$ is the space of all functions $\varphi : \mathcal{B} \rightarrow \mathbb{R}^+$ with the usual partial ordering $\varphi_1 \leq \varphi_2$ if $\varphi_1(p) \leq \varphi_2(p)$ for all $p \in \mathcal{P}$. The measure of noncompactness on $E$ is the function $\alpha : \mathcal{B} \rightarrow \Phi$ such that for every $B \in \mathcal{B}$, $\alpha(B)$ is the function from $\mathcal{P}$ into $\mathbb{R}^+$ defined by

$$\alpha(B)(p) = \inf \{d > 0 : \sup \{p(x-y) : x,y \in B_i \} \leq d \ \forall i\}$$

where the infimum is taken on all subsets $B_i$ such that $B$ is finite union of $B_i$. Properties of measure of noncompactness in locally convex spaces are presented in [4, Proposition 1.4].

An operator $T : Q \subset E \rightarrow E$ is called to be countably condensing if $T(Q)$ is bounded and if for any countably bounded set $A$ of $Q$ with $\alpha(A)(p) > 0$ we have

$$\alpha(T(A))(p) < \alpha(A)(p)$$

Definition 2.3. Let $E$ be a complete ordered locally convex space with a normal cone $E^+$. An element $x \in E$ is said to be a fixed point of a mapping $T : E \rightarrow E$ if $x = T(x)$.

Definition 2.4. Let $E$ be a complete ordered locally convex space with a normal cone $E^+$. An element $x \in E$ is said to be a periodic point of a mapping $T : E \rightarrow E$ if $T^n(x) = x$ the smallest such positive integer $n$ is called the period of $x$ (with respect to $T$). We denote the set of all periodic points of $T$ by $\text{Per}(T)$.

For each integer $n \geq 1$, $T^n$ denotes the $n^{th}$ iterate of $T$, that is, the composition $T \circ T \circ ... \circ T$ of $T$ with itself $n-1$ times ($T^1 = T, T^2 = T \circ T...$). Also, $T^0$ is the identity map of $E$.

Definition 2.5. Let $E$ be a complete ordered locally convex space with a normal cone $E^+$. A map $T : E \rightarrow E$ is said to be nondecreasing if for $x,y \in E$ and $x \leq y$ we have $Tx \leq Ty$.

A map $T : E \rightarrow E$ is said to be nonincreasing if for $x,y \in E$ and $x \leq y$ we have $Tx \geq Ty$. 
Definition 2.6. Let $E$ be an ordered locally convex spaces and let $x \in E$. A mapping $f : E \to E$ is said to be order continuous in $x$ if $f(x_\alpha) \to f(x)$ for each increasing or decreasing net $\{x_\alpha\}$ that converges to $x$.

It is evident that continuity implies order continuity.

3. MAIN RESULTS

The following results generalize the results of [1] in complete ordered locally convex spaces, and we add another results with low conditions.

Lemma 3.1. Let $E$ be an ordered topological vector space with a normal cone $E^+$. Then a monotone net $(u_\alpha) \subset E$ is convergent if and only if it has a weakly convergent subnet.

Proof. The “only if” part is obvious. For the ” if ” part, assume that $(u_\alpha)_{\alpha \in (\alpha)}$ is nondecreasing and let $(u_{\alpha_i})_{i \in (i)} \subset (u_\alpha)$ be a subnet such that $u_{\alpha_i} \to u$ weakly for some $u \in E$, where $(\alpha)$ stands for the indexed set of the net $(u_\alpha)$. Let $\beta \in (\alpha)$ be fixed. For each $\alpha \geq \beta$, let $i_0 \in (i)$ such that $\alpha_{i_0} \geq \alpha$. Thus, for each $i \geq i_0$ we have

\[ u_\beta \leq u_\alpha \leq u_{\alpha_i}. \]  

(3.1)

Thus, since $u_{\alpha_i} \to u$ weakly and the cone $E^+$ is weakly closed (being a closed and convex set) we see that $u_\beta \leq u$ for each $\beta \in (\alpha)$. Thus, it follows from [2, Lemma 2.28] that $\lim u_{\alpha_i} = u$.

Now, let $V \in V(0)$ be arbitrary. Since the cone $E^+$ is normal we may assume that $V$ is an order convex set. Let $j \in (i)$ such that $u - u_{\alpha_i} \in V$ for each $i \geq j$. If $\beta \geq \alpha_j$ then $0 \leq u - u_{\beta} \leq u - u_{\alpha_j}$, and hence $u - u_{\beta} \in V$. That is $\lim u_{\beta} = u$ as required. The desired conclusion is proved similarly when $(u_\alpha)$ is nonincreasing. \(\Box\)

in the following theorem, a Hausdorff locally convex space is regular, [8, see Chapter VI, Section 1]

Theorem 3.2. Let $E$ be a complete ordered locally convex space with a normal cone $E^+$. Let $\Omega$ be an order convex subset of $E$, and let $u_0, v_0 \in \Omega$, $u_0 \leq v_0$ and $T : \Omega \to \Omega$ be a continuous and nondecreasing mappings such that:
Suppose that $T$ is condensing from $\Omega$ in to itself.

Then, $T$ has a minimal periodic point $u$ and a maximal periodic point $v$ in $\Omega$.

Proof. We pose : $S = T^k$. Consider the sequences $(u_n)$ and $(v_n)$ defined by:

$$(3.2) \quad u_n = Su_{n-1}, \quad v_n = Sv_{n-1}, \quad n \in \mathbb{N}$$

Since $T$ is nondecreasing and fixes the interval $[u_0, v_0]$. Then from (3.2) it follows that

$$(3.3) \quad u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0$$

And $[u_0, v_0] \subset \Omega$ because $\Omega$ is a order convex subset of $E$.

Let $A = \{u_0, u_1, \ldots\}$, we have $A = \{u_0\} \cup S(A)$ and the set $A$ is bounded since $S$ is condensing (because $T$ is condensing and in $T(\Omega)$ is bounded).

So $\overline{A}$ is compact, by [3, p 89],

$\{u_n\}$ has a convergent subnet which converges to $u \in [u_0, v_0]$, and by (3.3), $\{u_n\}$ is nondecreasing, so by lemma 3.1, the original sequence $\{u_n\}$ converges to $u \in [u_0, v_0] \subset \Omega$. Also we have

$$u = \lim_{n \to \infty} u_n$$

Since $S$ is continuous mapping, so, $u = Su \Leftrightarrow u = T^k u$

Similarly, we can prove that $\{v_n\}$ converges to some $v \in E$ and $v = T^k v$.

Finally, we prove that $u$ and $v$ are the maximal and minimal periodic points of $T$ in $[u_0, v_0] \subset \Omega$.

Indeed, let $x \in [u_0, v_0]$ and $x = T^k x$. Since $T$ is nondecreasing, we have $u_n \leq x \leq v_n$, taking limit $n \to \infty$, we obtain $u \leq x \leq v$. 

Remark 1. this theorem remains true if continuity is replaced by ordered continuity.

Corollary 3.3. Let $E$ be a complete ordered locally convex space with a normal cone $E^+$. Let $\Omega$ be an order convex subset of $E$, 

and let $u_0, v_0 \in \Omega$, $u_0 \leq v_0$ and $T : \Omega \to \Omega$ be a order continuous and nondecreasing mappings such that : $u_0 \leq T(u_0)$ and $T(v_0) \leq v_0$.

Suppose that $T$ is condensing from $\Omega$ in to itself.

Then, $T$ has a minimal fixed point $u$ and a maximal fixed point $v$ in $\Omega$.  

Proof. It is obtained by taking \( k = 1 \) in Theorem 3.2.

**Corollary 3.4.** Let \( E \) be a complete ordered locally convex space with a normal cone \( E^+ \).

Let \( u_0, v_0 \in E \) such that \( u_0 \leq v_0 \) and \( T : [u_0, v_0] \rightarrow [u_0, v_0] \) be a continuous and nondecreasing mapping such.

Suppose that \( T \) is condensing from \([u_0, v_0]\) in to itself.

Then, \( T \) has a minimal fixed point \( u \) and a maximal fixed point \( v \) in \([u_0, v_0]\).

**Proof.** It is obtained by taking \( k = 1 \) and \([u_0, v_0] = \Omega\) in Theorem 3.2.

**Theorem 3.5.** Let \( E \) be a complete ordered locally convex space with a normal cone \( E^+ \). Let \( \Omega \) be an order convex subset of \( E \), and let \( u_0, v_0 \in \Omega \), \( u_0 \leq v_0 \) and let \( T : \Omega \rightarrow \Omega \) be a continuous nonincreasing mappings such that \( u_0 \leq T^{2k}(u_0) \) and \( T^{2k}(v_0) \leq v_0 \) where \( k \) is a positive integer.

Suppose that \( T \) is condensing mapping from \( \Omega \) in to itself.

Then, the set \( \text{Per}(T) = \{ x \in \Omega : T^k x = x \} \) is nonempty and compact.

**Proof.** Since \( T \) is condensing and continuous, then so is \( T^2 \), also \( T^2 \) is nondecreasing and fixes the interval \([u_0, v_0]\).

Then, from 3.2, \( T^2 \) has a minimal periodic point \( u \) and a maximal periodic point \( v \) in \([u_0, v_0]\). It is easy to see that \( Tu \) and \( Tv \) are likewise a periodic points of \( T^2 \). Therefore, we have :

\[
u \leq Tv \leq Tu \leq v
\]

Now, if \( x \in [u, v] \), then :

\[
u \leq Tv \leq Tx \leq Tu \leq v
\]

It follows that \( T \) fixes the interval \([u, v]\), we pose \( S = T^{k'} \)

with \( k' \in \mathbb{N}^* \), so, \( S \) also fixes the interval \([u, v]\), then \( S[u, v] \) is bounded. Now, because the cone \( E^+ \) is normal, the interval \([u, v]\) is a convex, closed, and bounded subset of \( E \).

Then applying [4, Theorem 2.7] for the set \([u, v]\) in the case where \( T_i = Id_E \), it follows that \( S \) has a fixed point in \([u, v] \subset \Omega\).
Then, $T$ has a periodic point in $[u, v] \subset \Omega$.

For the compacity of $Per(T)$, note that $Per(T) \subset [u, v]$. Therefore, $Per(T)$ is a bounded set. If $\alpha(Per(T))(p) \neq 0$ for all $p \in \mathcal{P}$.

Then we have:

$$\alpha(Per(T))(p) = \alpha(T^k(Per(T)))(p) < \alpha(T^{k-1}(Per(T)))(p) < \ldots < \alpha(Per(T))(p),$$

which is a contradiction. Therefore $\alpha(Per(T))(p) = 0$,

that is by [4, Proposition 1.4] and by continuity of $T$, $Per(T)$ is a compact set in $\Omega$. □

**Corollary 3.6.** Let $E$ be a complete ordered locally convex space with a normal cone $E^+$. Let $u_0, v_0 \in E$, $u_0 \leq v_0$ and let $T : E \to E$ be a continuous nonincreasing mappings such that $u_0 \leq T^{2k}(u_0)$ and $T^{2k}(v_0) \leq v_0$ where $k$ is a positive integer. Suppose that $T$ is condensing mapping from $E$ in to itself.

Then, the set $PerT = \{x \in E : T^kx = x\}$ is nonempty and compact.

**Proof.** It is obtained by taking $[u_0, v_0] = \Omega$ in Theorem 3.5 since $[u_0, v_0]$ is order convex. □

**Corollary 3.7.** Let $E$ be a complete ordered locally convex space with a normal cone $E^+$. Let $\Omega$ be an order convex subset of $E$, and let $u_0, v_0 \in \Omega$, $u_0 \leq v_0$ and let $T : \Omega \to \Omega$ be a continuous nonincreasing mappings such that $u_0 \leq T(u_0)$ and $T(v_0) \leq v_0$.

Suppose that $T$ is condensing mapping from $\Omega$ in to itself.

Then, the set $FixT = \{x \in \Omega : Tx = x\}$ is nonempty and compact.

**Proof.** It is obtained by taking $k = 1$ in Theorem 3.5. □

4. **Application to Differential Equations**

In this section we will give an application of Corollary 3.4 to the following equation differential:

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad x_0 \in A.$$

Where $X$ be a complete ordered Hausdorff locally convex space and $A \subset X$ be open, $J = [t_0, t_0 + a] \subset \mathbb{R}$ be an interval, $C(J, X)$ be the space of continuous functions from $J$ to $X$, $f(t, x) \in C(J \times$
In this section, \( \leq \) and \( < \) mean the total order relation of \( \mathbb{R} \).

We define an order relation \( \preceq \) in \( C(J, X) \) by the order cone \( P \) in \( C(J, X) \) defined by the cone \( P = \{ x \in C(J, X) / x(t) \in X^+, \forall t \in J \} \) where \( X^+ \) is a normal cone in \( X \).

\( C(J, X) \) is a complete ordered Hausdorff locally convex space with a normal cone \( P \).

The equation (4.1) is equivalent to the integral equation:

\[
(4.2) \quad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds
\]

where the integral is in the Riemann sense. see [12, II,p.29.Theorem 21].

**Proposition 4.1.**

\[
\int_{t_0}^{t_1} f(s, x(s))ds \in \overline{co}(\{ f(s, x(s)) / s \in [t_0, t_1] \})
\]

This proposition directly follows from the definition of the Riemann integral.

The following proposition characterizes the measure of nonprecompactness of a bounded, equicontinuous subset \( H \) of \( C(J, X) \). Similar results are obtained for \( \alpha(A) \) and \( \omega(A) \) by [13, Ambrosetti] and [13, Mitchell, Smith] respectively.

**Proposition 4.2.** [7, proposition 2]

Let \( X \) be a complete Kausdorff locally convex space and let \( J = [t_0, t_0 + a] \subset \mathbb{R} \) be a interval. Let \( H \subset C(J, X) \) be a bounded equicontinuous set. Then we have:

\[
\alpha(H)(p) = \alpha(H(J))(p) = \bigcup_{t \in J} \alpha(H(t))(p)
\]

for all \( p \in \mathcal{P} \)

**Definition 4.3.** A function \( f(t, x) \) is said to be nondecreasing with respect to \( x \) if for any \( x, y \in X \) with \( x \preceq y \) we have that \( f(t, x) \preceq f(t, y) \) for all \( t \in J \).

**Theorem 4.4.** Assume the following hypotheses:

1. \( f(t, x) \) increasing in \( x \).
2. There exists a order convex set \( F \) such as \( x_0 \in F \subset A \) and \( B_0 = \overline{co}((f(J \times F) \cup \{0\}) \) is bounded and \( x_0 + \alpha_0 B_0 \subset F \) for some \( \alpha_0 > 0 \).
(3) For any bounded set \( B_1 \subset \overline{B}_1 \subset A \) there exist an interval \( J' = [t_0, t_0 + a'] \subset J \) and a constant \( \lambda > 0 \) such that for any countably bounded set \( B \subset B_1 \) with \( \alpha(B)(p) > 0 \) we have:

\[
\alpha(f(J' \times B))(p) < \alpha(B)(p)
\]

(4) there exists \( \gamma, \delta \in C(J', X) \) such that \( \gamma \leq \delta \):

\[
\gamma(t) \leq x_0 + \int_{t_0}^{t} f(s, x(s))ds \leq \delta(t)
\]

Then, \( \exists \beta \in [0, a] \) such that the equation (4.1) has a lower and upper solution in the order interval \( [\gamma, \delta] \subset C(I, X) \forall t \in I = [t_0, t_0 + \beta] \).

Proof. Let \( \beta = \inf \{\alpha_0, a'\} \) and let \( I = [t_0, t_0 + \beta] \).

Since \( I \subset J' \), it follows that:

\[
\alpha(f(I \times B))(p) < \alpha(B)(p)
\]

for any countably bounded set \( B \subset B_1 \) with \( \alpha(B)(p) > 0 \).

By hypotheses (4), we have:

\[
[\gamma, \delta] = \{x \in C(I, X)/x(t_0) = x_0, x(t) - x(t') \in (t - t')B_0, \gamma(t) \leq x(t) \leq \delta(t), \forall t, t' \in I\}
\]

Clearly, \( [\gamma, \delta] \) is a nonempty, order convex, equicontinuous set in \( C(I, F) \subset C(J, X) \).

We define the operator \( T : [\gamma, \delta] \rightarrow [\gamma, \delta] \) by:

\[
Tx(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds
\]

\( T \) is well defined, it remains to show that the operator \( T \) satisfies the conditions of Corollary 3.4.

First, the proof of the continuity of \( T \) is similar to that of [7, p543].
Second, for any countably bounded set $B \subset H$ with $\alpha(B)(p) > 0$, we have:

$$\alpha(T(B))(p) = \alpha\left(\bigcup_{t \in I} T(B(t))\right)(p)$$

$$= \alpha\left(\bigcup_{t \in I} \left\{x_0 + \int_{t_0}^{t_1} f(s, x(s))ds : x \in B\right\}\right)(p)$$

$$= \alpha\left(\bigcup_{t \in I} \left\{\int_{t_0}^{t_1} f(s, x(s))ds : x \in B\right\}\right)(p)$$

$$\leq \alpha\left(\bigcup_{t \in I} \left\{(t - t_0)\text{conv} f(I \times B(I))\right\}\right)(p)$$

$$\leq \alpha(\text{conv} f(I \times B(I)))(p)$$

$$= \alpha(f(I \times B(I)))(p)$$

$$< \alpha(B(I))(p)$$

$$= \alpha(B)(p)$$

Finally, by hypotheses (1) and the monotonicity of integral, we have $T$ is nondecreasing. Thus the conditions of Corollary 3.4 are satisfied. Consequently, $T$ has a minimal fixed point $u$ and a maximal fixed point $v$ in $[\omega, \delta]$. This further implies that differential equation (4.1) has a lower and upper solution in the order interval $[\gamma, \delta]$. This completes the proof.

Conflict of Interests
The authors declare that there is no conflict of interests.

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