Quasi-Compact Finite Difference Schemes for Space Fractional Diffusion Equations

Han Zhou · WenYi Tian · Weihua Deng

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Abstract In this paper, a compact difference operator, termed CWSGD, is designed to establish the quasi-compact finite difference schemes for approximating the space fractional diffusion equations in one and two dimensions. The method improves the spatial accuracy order of the weighted and shifted Grünwald difference (WSGD) scheme (Tian et al., arXiv:1201.5949) from 2 to 3. The numerical stability and convergence with respect to the discrete $L^2$ norm are theoretically analyzed. Numerical examples illustrate the effectiveness of the quasi-compact schemes and confirm the theoretical estimations.

Keywords Quasi-compact difference approximation · Riemann-Liouville fractional derivatives · Stability and convergence · Space fractional diffusion equation

1 Introduction

Diffusion is a fundamental phenomena in real world. The classical diffusion equation can be derived from the conservation law of particles, when assuming the particles’ diffusion satisfies the Fick’s law. Based on the CTRW model of statistical physics, it can also be easily derived. Nowadays, anomalous diffusion is widely recognized in scientific community, its basic feature is that the classical Fick’s law doesn’t hold again. In fact, anomalous diffusion is usually characterized by the mean square displacement of the particles, given by $\langle x^2(t) \rangle \sim t^{\alpha}$. The exponent $\alpha$ classifies the type of diffusion: for $\alpha = 1$, we have normal diffusion; for $0 < \alpha < 1$, subdiffusion; and $\alpha > 1$, superdiffusion. The anomalous diffusion equations can still be derived from two ways: using the conservation law and fractional Fick’s law; and from the CTRW model with power law waiting time and/or power law jump length distribution. Fractional calculus plays an important role in obtaining the anomalous diffusion equations [1–4, 8, 9, 16, 20].

The equation describing the superdiffusion is the space fractional diffusion equation, and its concrete form is to replace the second order derivative of the classical diffusion equation...
by the Riemann-Liouville fractional derivative of order \( \alpha \) and \( 1 < \alpha < 2 \) \([8, 9]\). This paper concerns the high accurate numerical difference method for the space fractional diffusion equations. Meerschaert and his collaborators first introduce the shifted Grünwald formula and successfully get the stable finite difference scheme for numerically solving the space fractional equation based on this formula \([10]\). Later more research works are appeared by using the shifted Grünwald formula to discretize the space fractional derivatives \([10–12]\) with first order accuracy in space, and possibly obtaining second order accuracy after extrapolation \([17, 18]\). In \([19]\), the weighted and shifted Grünwald difference (WSGD) operators are introduced to discretize the Riemann-Liouville fractional derivative with second order accuracy, and the second order finite difference schemes by the WSGD operators are established to numerically solve the space fractional diffusion equations. It is also verified in \([19]\) that the 3-WSGD operator can approximate the Riemann-Liouville fractional derivative with third order accuracy, but the 3-WSGD operator fails to numerically solve the time dependent space fractional diffusion equations with unconditional stability.

As the sequel to \([19]\), based on the WSGD operators we construct the CWSGD operator for the fractional problems. When the order of fractional derivative \( \alpha \) equals to 1 or 2, it becomes the compact difference operators for the first or second order spatial derivatives with fourth order accuracy, which have been widely used in numerically solving the linear and nonlinear, steady and evolution equations to achieve high order accuracy. By using the WSGD operators \([19]\) and the CWSGD operator, we get the third order numerical schemes for the space fractional diffusion equations. We theoretically make the numerical stability and error analysis, and the detailed numerical experiments confirm the theoretical results. In performing the theoretical analysis, the negative definite property of the matrix generated by the WSGD discretization to the fractional diffusion operator still plays a key role.

The remainder of this paper is organized as follows. In Sect. 2, we introduce the CWSGD operator. Based on the CWSGD operator, in Sect. 3, we build the quasi-compact finite difference schemes for one and two dimensional space fractional diffusion equations. The stability and third order convergence with respect to the discrete \( L^2 \) norm of the difference schemes are proven in Sect. 4. In Sect. 5, the extensive numerical experiments are performed to verify the accuracy and convergent order. And some concluding remarks are made in the last section.

## 2 Compact WSGD Operator for the Riemann-Liouville Fractional Derivatives

Now from the WSGD operators and the Taylor’s expansions of the shifted Grünwald finite difference formulae, we derive the CWSGD operator for the Riemann-Liouville fractional derivatives, the definition of which is introduced as follows.

**Definition 1** (\([13]\)) The \( 0 \leq n – 1 < \alpha < n \) order left and right Riemann-Liouville fractional derivatives of the function \( u(x) \) on \((a, b)\) are, respectively, defined as

\[
(1) \text{left Riemann-Liouville fractional derivative:} \quad aD_x^\alpha u(x) = \frac{1}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dx^n} \int_a^x \frac{u(\xi)}{(x-\xi)^{n-\alpha+1}} \, d\xi;
\]

\[
(2) \text{right Riemann-Liouville fractional derivative:} \quad bD_x^\alpha u(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \cdot \frac{d^n}{dx^n} \int_x^b \frac{u(\xi)}{(\xi-x)^{n-\alpha+1}} \, d\xi,
\]
where \( n \) is an integer. If \( \alpha = n \), then \( D_x^n u(x) = \frac{d^n}{dx^n} u(x) \) and \( D_x^n u(x) = (-1)^n \frac{d^n}{dx^n} u(x) \).

The second order finite difference approximations (the WSGD operators) for Riemann-Liouville fractional derivatives are given as follows.

**Lemma 1** ([19]) Let \( u \in L^1(\mathbb{R}) \), \( -\infty D_x^{\alpha+2} u, x D_x^{\alpha+2} u \) and their Fourier transformations belong to \( L^1(\mathbb{R}) \), and define the weighted and shifted Grünwald difference (WSGD) operators by

\[
L_D^{\alpha}_{h,p,q} u(x) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} \left( \frac{\alpha - 2q}{2(p-q)} u(x - (k-p)h) + \frac{2p-\alpha}{2(p-q)} u(x - (k-q)h) \right),
\]

\[
R_D^{\alpha}_{h,p,q} u(x) = \frac{1}{h^\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} \left( \frac{\alpha - 2q}{2(p-q)} u(x + (k-p)h) + \frac{2p-\alpha}{2(p-q)} u(x + (k-q)h) \right),
\]

(2.1a)

where \( g_k^{(\alpha)} = (-1)^k \binom{\alpha}{k} \) are the coefficients of the power series of the function \((1 - z)^\alpha\), and \( p, q \) are integers, \( p > q \). Then we have

\[
L_D^{\alpha}_{h,p,q} u(x) = -\infty D_x^{\alpha} u(x) + O(h^2),
\]

\[
R_D^{\alpha}_{h,p,q} u(x) = x D_x^{\alpha} u(x) + O(h^2),
\]

(2.2a)

uniformly for \( x \in \mathbb{R} \).

From [17], we know the Taylor’s expansions of the shifted Grünwald finite difference formulae, which are necessary for establishing the CWSGD operator for Riemann-Liouville fractional derivatives.

**Lemma 2** ([17]) Assuming that \( u \in C^{n+3}(\mathbb{R}) \) such that all derivatives of \( u \) up to order \( n + 3 \) belong to \( L^1(\mathbb{R}) \), then for any nonnegative integer \( p \), we can obtain that

\[
\frac{1}{h^\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} u(x - (k-p)h) = -\infty D_x^{\alpha} u(x) + \sum_{l=1}^{n-1} \left( a_{p,l}^{\alpha} -\infty D_x^{\alpha+l} u(x) \right) h^l + O(h^n),
\]

(2.3a)

\[
\frac{1}{h^\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} u(x + (k-p)h) = x D_x^{\alpha} u(x) + \sum_{l=1}^{n-1} \left( a_{p,l}^{\alpha} x D_x^{\alpha+l} u(x) \right) h^l + O(h^n),
\]

(2.3b)

uniformly for \( x \in \mathbb{R} \), where \( a_{p,l}^{\alpha} \) are the coefficients of the power series expansion of function

\[
w_{a,p}(z) = (1 - e^{-z})^\alpha e^{pz}, \quad \text{and} \quad w_{a,p}(z) = \sum_{k=0}^{\infty} a_{p,k}^\alpha z^k = 1 + (p - \frac{\alpha}{2})z + \frac{1}{24}(\alpha + 3\alpha^2 - 12\alpha p + 12p^2)z^2 + O(z^3).
\]

For any function \( u(x) \), denoting by \( \delta_x^2 \) the second order central difference operator, that is \( \delta_x^2 u(x) = (u(x-h) - 2u(x) + u(x+h))/h^2 \), we introduce the following finite difference operator

\[
C_x u = (1 + c_{p,q,2}^\alpha h^2 \delta_x^2) u,
\]

(2.4)
where \(c_{\alpha,p,q}^a = \frac{a^{-2q}}{2(p-q)} a_{\alpha}^a + \frac{2p-a}{2(p-q)} a_{\alpha}^q\). We call \(C_x\) the compact WSGD (CWSGD) operator of Riemann-Liouville fractional derivatives, of which the detailed construction is described in the following proposition.

**Proposition 1** Under the conditions of Lemmas 1 and 2, there exist

\[
L D_{h,p,q}^\alpha u(x) = C_x (-\infty D_x^\alpha u(x)) + c_{\alpha,p,q,3}^a \infty D_x^{\alpha+3} u(x) h^3 + O(h^4),
\]

\[
R D_{h,p,q}^\alpha u(x) = C_x (\infty D_x^\alpha u(x)) + c_{\alpha,p,q,3}^b \infty D_x^{\alpha+3} u(x) h^3 + O(h^4),
\]

uniformly for \(x \in \mathbb{R}\), where \(p, q\) are integers and \(p \neq q\). The operator \(C_x\) is defined in (2.4).

**Proof** Substituting formulae (2.3a) and (2.3b) into (2.1a) and (2.1b), respectively, and taking \(n = 4\), we arrive at

\[
L D_{h,p,q}^\alpha u(x) = \left(1 + c_{\alpha,p,q,2}^a h^2 \frac{d^2}{dx^2}\right) (-\infty D_x^\alpha u(x)) + c_{\alpha,p,q,3}^a \infty D_x^{\alpha+3} u(x) h^3 + O(h^4),
\]

\[
R D_{h,p,q}^\alpha u(x) = \left(1 + c_{\alpha,p,q,2}^b h^2 \frac{d^2}{dx^2}\right) (\infty D_x^\alpha u(x)) + c_{\alpha,p,q,3}^b \infty D_x^{\alpha+3} u(x) h^3 + O(h^4).
\]

Since \(\delta^2 u = \frac{d^2}{dx^2} u + O(h^2)\), it yields

\[
C_x u = \left(1 + c_{\alpha,p,q,2}^a h^2 \frac{d^2}{dx^2}\right) u + O(h^4).\]

Then we obtain the needed results by substituting (2.7) into (2.6). \(\Box\)

**Remark 1** If the function \(u(x)\) is defined on the bounded interval \([a, b]\) with boundary condition \(u(a) = 0\) or \(u(b) = 0\), then the WSGD formulae approximating the \(\alpha\) order left and right Riemann-Liouville fractional derivatives of \(u(x)\) at each point \(x\) are written as

\[
L D_{h,p,q}^\alpha u(x) = \frac{\mu_1}{h^a} \sum_{k=0}^{\lfloor \frac{a}{p} \rfloor + p} g_k^{(a)} u(x - (k - p)h) + \frac{\mu_2}{h^a} \sum_{k=0}^{\lfloor \frac{b-a}{p} \rfloor + q} g_k^{(a)} u(x - (k - q)h),
\]

\[
R D_{h,p,q}^\alpha u(x) = \frac{\mu_1}{h^a} \sum_{k=0}^{\lfloor \frac{b-a}{p} \rfloor + p} g_k^{(a)} u(x + (k - p)h) + \frac{\mu_2}{h^a} \sum_{k=0}^{\lfloor \frac{b-a}{p} \rfloor + q} g_k^{(a)} u(x + (k - q)h),
\]

where \(\mu_1 = \frac{a^{-2q}}{2(p-q)}, \mu_2 = \frac{2p-a}{2(p-q)}\). After applying Proposition 1, we have

\[
L D_{h,p,q}^\alpha u(x) = C_x (a D_x^\alpha u(x)) + c_{\alpha,p,q,3}^a u D_x^{\alpha+3} u(x) h^3 + O(h^4),
\]

\[
R D_{h,p,q}^\alpha u(x) = C_x (b D_x^\alpha u(x)) + c_{\alpha,p,q,3}^b b D_x^{\alpha+3} u(x) h^3 + O(h^4).
\]

**Remark 2** The integers \(p, q\) in (2.8) should be chosen satisfying \(|p| \leq 1, |q| \leq 1\) to ensure that the nodes at which the values of \(u\) needed in (2.8) are within the bounded interval, when employing the finite difference method with WSGD formulae for numerically solving non-periodic fractional differential equations on bounded interval; otherwise, we need to use another way to discretize the fractional derivative when \(x\) is close to the right/left boundary. It
was indicated in [19] that the approximation by formula (2.8) with \((p, q) = (0, -1)\) turns out to be unstable for time dependent problems. Then only two sets of \((p, q) = (1, 0), (1, -1)\) can be selected to establish the difference scheme for fractional diffusion equations, which is also appropriate for the quasi-compact difference approximations (2.5). The coefficients \(c_{p,q,i}^\alpha\) in (2.4) with \((p, q) = (1, 0), (1, -1)\) are

\[
\begin{align*}
    c_{1,0,2}^\alpha &= \frac{1}{24}(7\alpha - 3\alpha^2), \\
    c_{1,0,3}^\alpha &= \frac{1}{24}(\alpha^3 - 3\alpha^2 + 2\alpha), \\
    c_{1,-1,2}^\alpha &= \frac{1}{24}(\alpha - 3\alpha^2 + 12), \\
    c_{1,-1,3}^\alpha &= \frac{1}{24}(\alpha^3 - 4\alpha).
\end{align*}
\]

(2.10)

For \(\alpha = 2\), the WSGD operators (2.8) are the centered difference approximation of second order derivative when \((p, q)\) equals to \((1, 0)\) or \((1, -1)\), and the approximations (2.5) behave as the compact difference operators of second order derivative as \(c_{1,0,2}^2 = c_{1,0,1}^2 = \frac{1}{12}\) and \(c_{1,0,3}^2 = c_{1,-1,3}^2 = 0;\) for \(\alpha = 1\) and \((p, q) = (1, 0), c_{1,0,2}^1 = \frac{1}{6}\) and \(c_{1,0,3} = 0\), then the centered and compact difference scheme for first order derivative is recovered.

For the cases of \((p, q) = (1, 0)\) and \((p, q) = (1, -1)\), the WSGD formulae at grid point \(x_i = a + i \ h (h = \frac{b-a}{N}, 1 \leq i \leq N - 1)\) are denoted as

\[
\begin{align*}
    \alpha D_{h,p,q}^\alpha u(x_i) &= \frac{1}{h^\alpha} \sum_{k=0}^{i+1} w_k^<(u(x_{i-k+1})), \\
    \alpha D_{h,p,q}^\alpha u(x_i) &= \frac{1}{h^\alpha} \sum_{k=0}^{N-i+1} w_k^>(u(x_{i+k-1})),
\end{align*}
\]

(2.11)

where

\[
\begin{align*}
    (p, q) = (1, 0), & \quad w_0^<(a) = \frac{\alpha}{2} g_0^\alpha, \quad w_k^<(a) = \frac{\alpha}{2} g_k^\alpha + \frac{2 - \alpha}{2} g_{k-1}^\alpha, \quad k \geq 1; \\
    (p, q) = (1, -1), & \quad w_0^>(a) = \frac{2 + \alpha}{4} g_0^\alpha, \quad w_1^>(a) = \frac{2 + \alpha}{4} g_1^\alpha, \quad w_k^>(a) = \frac{2 + \alpha}{4} g_k^\alpha + \frac{2 - \alpha}{4} g_{k-2}^\alpha, \quad k \geq 2.
\end{align*}
\]

(2.12)

**Remark 3** Let \(S_{N-1}\) be a symmetric tri-diagonal matrix of \((N - 1)\)-square, denoted by \(\text{tridiag}(1, -2, 1)\). And we have the eigenvalues of the matrix \(S_{N-1}\) in decreasing order (see [6]),

\[
\lambda_k(S_{N-1}) = -4 \sin^2\left(\frac{k\pi}{2N}\right), \quad k = 1, 2, \ldots, N - 1.
\]

(2.13)

Define

\[
C_{\alpha} = I_{N-1} + c_{p,q,2}^\alpha S_{N-1},
\]

(2.14)

where \(I_{N-1}\) is the unit matrix of \((N - 1)\)-square. Then the eigenvalues of \(C_{\alpha}\) are given by

\[
\lambda_k(C_{\alpha}) = 1 - 4c_{p,q,2}^\alpha \sin^2\left(\frac{k\pi}{2N}\right), \quad k = 1, 2, \ldots, N - 1.
\]

(2.15)
For the case of \((p, q) = (1, 0)\), we have

\[
\lambda_k(C_{\alpha}) > 1 - 4c_{1,0,2}^\alpha \geq \frac{23}{72} > 0,
\]

(2.16)

when \(0 < \alpha < \frac{7}{3}\); and \(\lambda_k(C_{\alpha}) \geq 1\) when \(\alpha \geq \frac{7}{3}\).

Then, from the Rayleigh-Ritz Theorem (see Theorem 8.8 in [21]), we know that the matrix \(C_{\alpha} = (I_{N-1} + c_{1,0,2}^\alpha S_{N-1})\) is positive definite. And for the case of \((p, q) = (1, -1)\) and \(\alpha > 0\), we have

\[
\lambda_k(C_{\alpha}) > 1 - 4c_{1,-1,2}^\alpha > 0 \quad \text{iff} \quad \frac{1 + \sqrt{73}}{6} < \alpha < \frac{1 + \sqrt{145}}{6},
\]

(2.17)

and \(\lambda_k(C_{\alpha}) \geq 1\) when \(\alpha \geq \frac{1 + \sqrt{145}}{6}\); and \(1 - 4c_{1,-1,2}^\alpha = 0\) when \(\alpha = \frac{1 + \sqrt{73}}{6}\), thus, the matrix \(C_{\alpha} = (I_{N-1} + c_{1,-1,2}^\alpha S_{N-1})\) is positive definite for any natural number \(N\) if and only if \(\alpha \in \left[\frac{1 + \sqrt{73}}{6}, \infty\right)\).

**Lemma 3 ([19])** Let the matrix \(A_{\alpha}\) be of the following form

\[
A_{\alpha} = \begin{bmatrix}
  u_1^{(\alpha)} & u_0^{(\alpha)} \\
  w_2^{(\alpha)} & u_1^{(\alpha)} & u_0^{(\alpha)} \\
  \vdots & \ddots & \ddots & \ddots \\
  w_{N-2}^{(\alpha)} & \cdots & \cdots & \cdots & u_0^{(\alpha)} \\
  w_N^{(\alpha)} & \cdots & \cdots & \cdots & w_1^{(\alpha)}
\end{bmatrix},
\]

(2.18)

where the diagonals \(\{w_k^{(\alpha)}\}_{k=0}^{N-1}\) are the coefficients given in (2.12) corresponding to \((p, q) = (1, 0), (1, -1)\). Then we have that any eigenvalue \(\lambda\) of \(A_{\alpha}\) satisfies

1. \(\text{Re}(\lambda) = 0\), for \((p, q) = (1, 0), \alpha = 1\),
2. \(\text{Re}(\lambda) < 0\), for \((p, q) = (1, 0), 1 < \alpha \leq 2\),
3. \(\text{Re}(\lambda) < 0\), for \((p, q) = (1, -1), 1 \leq \alpha \leq 2\).

Moreover, when \(1 < \alpha \leq 2\), the matrix \(A_{\alpha}\) is negative definite, and the real parts of the eigenvalues \(\lambda\) of matrix \(c_1A_{\alpha} + c_2A_{\alpha}^T\) are less than 0, where \(c_1, c_2 \geq 0, c_1^2 + c_2^2 \neq 0\).

**Lemma 4 ([5, 14])** The matrix \(A \in \mathbb{R}^{n \times n}\) is asymptotically stable if and only if there exists a symmetric and positive (or negative) definite solution \(X \in \mathbb{R}^{n \times n}\) to the Lyapunov equation

\[
AX + XA^T = C,
\]

(2.19)

where \(C = C^T \in \mathbb{R}^{n \times n}\) is a negative (or positive) definite matrix. And a matrix \(A\) is called asymptotically stable if all its eigenvalues have real parts in the open left half-plane, i.e., \(\text{Re}(\lambda(A)) < 0\).

Lemma 3 and 4 are used to analyze the stability and convergence of the quasi-compact difference approximations in the sequel.
3 Application of CWSGD Operator to Space Fractional Diffusion Equations

In this section, taking \((p, q) = (1, 0)\) and \((p, q) = (1, -1)\), we establish high order finite difference schemes of the two-sided space fractional diffusion equations in one and two dimensions by using CWSGD operator and the WSGD formulae.

3.1 Quasi-Compact Difference Scheme of One Dimensional Problem

We consider the space fractional diffusion equation in one dimension as follows

\[
\begin{align*}
\frac{\partial u(x, t)}{\partial t} &= K_1 D_x^\alpha u(x, t) + K_2 D_b^\alpha u(x, t) + f(x, t), \quad (x, t) \in (a, b) \times (0, T], \\
u(x, 0) &= u_0(x), \quad x \in [a, b], \\
u(a, t) &= \phi_a(t), \quad \nu(b, t) = \phi_b(t), \quad t \in [0, T],
\end{align*}
\]

where \(K_1 D_x^\alpha u\) and \(K_2 D_b^\alpha u\) are given by Definition 1 with \(1 < \alpha \leq 2\). The diffusion coefficients \(K_1\) and \(K_2\) are nonnegative constants with \(K_1^2 + K_2^2 \neq 0\). If \(K_1 \neq 0\), then \(\phi_a(t) \equiv 0\), and if \(K_2 \neq 0\), then \(\phi_b(t) \equiv 0\). We assume that the solution of (3.1) is unique and sufficiently smooth to ensure the feasability of establishing the finite difference scheme and achieving its accuracy.

We introduce a uniform mesh with the space step size \(h = (b - a)/N\) on interval \([a, b]\) and the time step size \(\tau = T/M\),

\[
\{(x_i, t_n) \mid x_i = a + ih, \quad i = 0, \ldots, N; \quad t_n = n\tau, \quad n = 0, \ldots, M\},
\]

where \(N, M\) are two positive integers. Denoting by \(t_{n+1/2} = (t_n + t_{n+1})/2\) for \(0 \leq n \leq M - 1\), we introduce the following notations

\[
u^n_i = \nu(x_i, t_n), \quad f^{n+1/2}_i = f(x_i, t_{n+1/2}), \quad \delta_t u_i^n = (u_i^{n+1} - u_i^n)/\tau.
\]

In time discretization, using the Crank-Nicolson technique, we obtain

\[
\delta_t u_i^n - \frac{1}{2}(K_1 (a D_x^\alpha u_i^n)^{n+1} + K_1 (b D_b^\alpha u_i^n)^{n+1} + K_2 (a D_x^\alpha u_i^n)^n + K_2 (b D_b^\alpha u_i^n)^{n+1}) = f_i^{n+1/2} + O(\tau^2).
\]

Recalling the definition of operator \(C_x\) and (2.9), we have

\[
\begin{align*}
C_x (a D_x^\alpha u_i^n) &= \mathcal{L} D_{h,p,q}^\alpha u_i^n - (c_{p,q,3} a D_x^{\alpha+3} u_i^n) h^3 + O(h^4), \\
C_x (b D_b^\alpha u_i^n) &= \mathcal{R} D_{h,p,q}^\alpha u_i^n - (c_{p,q,3} b D_b^{\alpha+3} u_i^n) h^3 + O(h^4),
\end{align*}
\]

where \(\mathcal{L} D_{h,p,q}^\alpha\) and \(\mathcal{R} D_{h,p,q}^\alpha\) are given in (2.11).

Acting the invertible operator \(\tau C_x\) on both sides of (3.2) and substituting (3.3) into it, we get

\[
\begin{align*}
C_x u_i^{n+1} - \frac{K_1}{2} \mathcal{L} D_{h,p,q}^\alpha u_i^{n+1} - \frac{K_2}{2} \mathcal{R} D_{h,p,q}^\alpha u_i^{n+1} \\
= C_x u_i^n + \frac{K_1}{2} \mathcal{L} D_{h,p,q}^\alpha u_i^n + \frac{K_2}{2} \mathcal{R} D_{h,p,q}^\alpha u_i^n + \tau C_x f_i^{n+1/2} + \tau \epsilon_i^{n+1/2},
\end{align*}
\]

where

\[
\epsilon_i^{n+1/2} = - (K_1 c_{p,q,3} a D_x^{\alpha+3} u_i^{n+1/2} + K_2 c_{p,q,3} b D_b^{\alpha+3} u_i^{n+1/2}) h^3 + O(\tau^2 + h^4).
\]
Thus, the quasi-compact difference scheme for (3.1) is followed by replacing $u^n_i$ by its numerical approximation $U^n_i$ 

\[ C_x U^{n+1}_i = \frac{K_1 \tau}{2h^\alpha} \sum_{k=0}^{i+1} w_k^{(\alpha)} U^{n+1}_{i-k+1} - \frac{K_2 \tau}{2h^\alpha} \sum_{k=0}^{N-i+1} w_k^{(\alpha)} U^{n+1}_{i+k-1} \]

\[ = C_x U^n_i + \frac{K_1 \tau}{2h^\alpha} \sum_{k=0}^{i+1} w_k^{(\alpha)} U^n_{i-k+1} + \frac{K_2 \tau}{2h^\alpha} \sum_{k=0}^{N-i+1} w_k^{(\alpha)} U^n_{i+k-1} + \tau C_x f_i^{n+1/2}. \]  

(3.6)

And the corresponding matrix form of (3.6) are given by

\[ \left( C_\alpha - \frac{\tau}{2h^\alpha} (K_1 A_\alpha + K_2 A_\alpha^T) \right) U^{n+1} = \left( C_\alpha + \frac{\tau}{2h^\alpha} (K_1 A_\alpha + K_2 A_\alpha^T) \right) U^n + \tau C_\alpha F^n + H^n, \]  

(3.7)

where $U^n$ and $F^n$ are

\[ U^n = (U^n_1, U^n_2, \ldots, U^n_{N-1})^T, \quad F^n = (f^n_{1+1/2}, f^n_{2+1/2}, \ldots, f^n_{N-1+1/2})^T, \]

$C_\alpha$ and $A_\alpha$ are given in (2.14) and (2.18), respectively, and

\[ H^n = \begin{bmatrix} c_{p,q,2}^\alpha & 0 & \cdots & 0 \\ 0 & c_{p,q,2}^\alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_{p,q,2}^\alpha \end{bmatrix} \begin{bmatrix} \frac{K_1 w_2^{(\alpha)} + K_2 w_0^{(\alpha)}}{K_1 w_3^{(\alpha)}} \\ \vdots \\ \frac{K_1 w_{N-1}^{(\alpha)} + K_2 w_0^{(\alpha)}}{K_1 w_N^{(\alpha)}} \\ 0 \end{bmatrix} (U^n_0 + U^n_{0+1}) \]

\[ + \frac{\tau}{2h^\alpha} \begin{bmatrix} \frac{K_1 w_2^{(\alpha)}}{K_2 w_3^{(\alpha)}} \\ \vdots \\ \frac{K_1 w_{N-1}^{(\alpha)}}{K_2 w_N^{(\alpha)}} \\ 0 \end{bmatrix} (U^n_0 + U^n_{0+1}) \]

\[ + \frac{\tau}{2h^\alpha} \begin{bmatrix} \frac{K_2 w_N^{(\alpha)}}{K_1 w_0^{(\alpha)} + K_2 w_2^{(\alpha)}} \\ \vdots \\ \frac{K_2 w_{N-1}^{(\alpha)}}{K_1 w_0^{(\alpha)} + K_2 w_2^{(\alpha)}} \end{bmatrix} (U^n_0 + U^n_{0+1}). \]  

(3.8)
3.2 Quasi-Compact Difference Scheme of Two Dimensional Problem

In the following, we discuss the quasi-compact difference scheme for the following space fractional diffusion equation in two dimensions

\[
\begin{bmatrix}
\frac{\partial u(x,y,t)}{\partial t} = (K_1^+ aD_x^\alpha u(x,y,t) + K_2^+ bD_y^\beta u(x,y,t)) \\
+ (K_1^- cD_x^\alpha u(x,y,t) + K_2^- dD_y^\beta u(x,y,t)) + f(x,y,t),
\end{bmatrix}
\]

\[(x,y,t) \in \Omega \times (0, T),\]

\[u(x,y,0) = u_0(x,y), \quad (x,y) \in \Omega,\]

\[u(x,y,t) = \varphi(x,y,t), \quad (x,y) \in \partial \Omega \times [0, T),\]

(3.9)

where \( \Omega = (a,b) \times (c,d), aD_x^\alpha, bD_y^\beta \) and \( cD_x^\alpha, dD_y^\beta \) are Riemann-Liouville fractional derivatives with \( 1 < \alpha, \beta \leq 2 \). The diffusion coefficients \( K_i^+, K_i^-, i = 1, 2, \) are nonnegative and at least one of them is positive for each \( i \). And the boundary condition satisfies, \( \varphi(a,y,t) \equiv 0 \) if \( K_1^+ \neq 0 \); \( \varphi(b,y,t) \equiv 0 \) if \( K_1^- \neq 0 \); \( \varphi(x,c,t) \equiv 0 \) if \( K_2^+ \neq 0 \); \( \varphi(x,d,t) \equiv 0 \) if \( K_2^- \neq 0 \). The solution of (3.9) is assumed to be unique and sufficiently smooth.

Domain \( \Omega \) is divided into a uniform mesh with the space step sizes \( h_x = (b - a)/N_x, h_y = (d - c)/N_y \) in each direction, and the time step size is \( \tau = T/M \), where \( N_x, N_y, M \) being positive integers. Then we can denote the grid points by \( x_i = a + ih_x, y_j = c + jh_y \) and \( t_n = n\tau \) for \( 0 \leq i \leq N_x, 0 \leq j \leq N_y \) and \( 0 \leq n \leq M \). For \( 0 \leq n \leq M - 1 \), some notations are given as follows

\[u_{i,j}^n = u(x_i,y_j,t_n), \quad \delta_t u_{i,j}^n = (u_{i,j}^{n+1} - u_{i,j}^n)/\tau, \quad f_{i,j}^{n+1/2} = f(x_i,y_j,t_{n+1/2}).\]

Using the Crank-Nicolson technique, we discrete (3.9) in the time direction as

\[
\begin{align*}
\delta_t u_{i,j}^n &= \frac{1}{2} \left( K_1^+ (aD_x^\alpha u_{i,j})^{n+1}_{i,j} + K_2^+ (bD_y^\beta u_{i,j})^{n+1}_{i,j} + K_1^- (cD_x^\alpha u_{i,j})^{n+1}_{i,j} + K_2^- (dD_y^\beta u_{i,j})^{n+1}_{i,j} \\
&\quad + K_1^+ (aD_x^\alpha u_{i,j})^{n}_{i,j} + K_2^+ (bD_y^\beta u_{i,j})^{n}_{i,j} + K_1^- (cD_x^\alpha u_{i,j})^{n}_{i,j} + K_2^- (dD_y^\beta u_{i,j})^{n}_{i,j} \\
&\quad + f_{i,j}^{n+1/2} + O(\tau^2). \right)
\end{align*}
\]

(3.10)

Now we introduce the finite difference operators

\[
C_x u_{i,j} = (1 + c^\alpha_{p,q,3}h_x^2\delta_x^2)u_{i,j}, \quad C_y u_{i,j} = (1 + c^\beta_{p,q,3}h_y^2\delta_y^2)u_{i,j},
\]

(3.11)

and it yields from (2.9) that

\[
\begin{align*}
C_x(aD_x^\alpha u_{i,j}) &= \ell D_x^\alpha u_{i,j} - c^\alpha_{p,q,3}h_x^3D_x^{\alpha+3}u_{i,j} + O(h_x^4), \\
C_x(bD_y^\beta u_{i,j}) &= \ell D_y^\beta u_{i,j} - c^\beta_{p,q,3}h_y^3D_y^{\beta+3}u_{i,j} + O(h_y^4), \\
C_y(cD_x^\alpha u_{i,j}) &= \ell D_x^\alpha u_{i,j} - c^\alpha_{p,q,3}h_x^3D_x^{\alpha+3}u_{i,j} + O(h_x^4), \\
C_y(dD_y^\beta u_{i,j}) &= \ell D_y^\beta u_{i,j} - c^\beta_{p,q,3}h_y^3D_y^{\beta+3}u_{i,j} + O(h_y^4).
\end{align*}
\]

(3.12a-d)
Acting the invertible operator $\tau C_x C_y$ on both sides of (3.10) and adding (3.12a)–(3.12d), we obtain

$$
\left( C_x C_y - \frac{K_1^+ \tau}{2} C_y L D^\alpha_{x,p,q} - \frac{K_2^+ \tau}{2} C_y R D^\alpha_{x,p,q} - \frac{K_1^- \tau}{2} C_x L D^\beta_{y,p,q} - \frac{K_2^- \tau}{2} C_x R D^\beta_{y,p,q} \right) u_{i,j}^{n+1} \\
= \left( C_x C_y + \frac{K_1^+ \tau}{2} C_y L D^\alpha_{x,p,q} + \frac{K_2^+ \tau}{2} C_y R D^\alpha_{x,p,q} + \frac{K_1^- \tau}{2} C_x L D^\beta_{y,p,q} + \frac{K_2^- \tau}{2} C_x R D^\beta_{y,p,q} \right) u_{i,j}^n + \frac{K_2 \tau}{2} C_x R D^\beta_{y,p,q} u_{i,j}^n + \frac{K_2 \tau}{2} C_x R D^\beta_{y,p,q} u_{i,j}^n + \frac{K_2 \tau}{2} C_x R D^\beta_{y,p,q} u_{i,j}^n
$$

(3.13)

where

$$
\epsilon_{i,j}^{n+1/2} = -h^3 \left( K_1^+ c_{p,q,3} D_x^{a+3} u + K_2^+ c_{p,q,3} D_b^{a+3} u \right)_{i,j} + h^3 \left( K_1^- c_{p,q,3} D_y^{b+3} u + K_2^- c_{p,q,3} D_d^{b+3} u \right)_{i,j} + O \left( \tau^2 + h^4 + h^4_y \right). \quad (3.14)
$$

Taylor’s expansion shows that

$$
\frac{\tau^2}{4} \delta_x^\alpha \delta_y^\beta \left( u_{i,j}^{n+1} - u_{i,j}^n \right) = \frac{\tau^3}{4} \left( K_1^+ a D_x^a + K_2^+ b D_b^a \right) \left( K_1^- c D_y^b + K_2^- d D_d^b \right) \frac{\partial u}{\partial t}_{i,j} + \tau^3 O \left( \tau^2 + h^2_x + h^2_y \right),
$$

(3.15)

$$
\frac{\tau^3}{4} \delta_x^\alpha \delta_y^\beta \left( f_{i,j}^{n+1/2} - f_{i,j}^n \right) = \frac{\tau^3}{4} \left( K_1^+ a D_x^a + K_2^+ b D_b^a \right) \left( K_1^- c D_y^b + K_2^- d D_d^b \right) \frac{\partial u}{\partial t}_{i,j} + \tau^3 O \left( h^2_x + h^2_y \right).
$$

(3.16)

For notational convenience, we denote

$$
\delta_x^\alpha = K_1^+ \alpha L D^\alpha_{x,p,q} + K_2^+ R D^\beta_{x,p,q}, \quad \delta_y^\beta = K_1^- \alpha L D^\beta_{y,p,q} + K_2^- R D^\beta_{y,p,q}.
$$

Then formula (3.13), adding (3.15), can be factorized as

$$
\left( C_x - \frac{\tau}{2} \delta_x^\alpha \right) \left( C_y - \frac{\tau}{2} \delta_y^\beta \right) u_{i,j}^{n+1} = \left( C_x + \frac{\tau}{2} \delta_x^\alpha \right) \left( C_y + \frac{\tau}{2} \delta_y^\beta \right) u_{i,j}^n + \tau C_x C_y f_{i,j}^{n+1/2} + \tau \epsilon_{i,j}^{n+1/2},
$$

(3.17)

with

$$
\epsilon_{i,j}^{n+1/2} = \epsilon_{i,j}^{n+1/2} + O \left( \tau^2 + \tau^2 h^2_x + \tau^2 h^2_y + h^4_x + h^4_y \right),
$$

(3.18)

Thus the quasi-compact finite difference scheme for (3.9) is

$$
\left( C_x - \frac{\tau}{2} \delta_x^\alpha \right) \left( C_y - \frac{\tau}{2} \delta_y^\beta \right) U_{i,j}^{n+1} = \left( C_x + \frac{\tau}{2} \delta_x^\alpha \right) \left( C_y + \frac{\tau}{2} \delta_y^\beta \right) U_{i,j}^n + \tau C_x C_y f_{i,j}^{n+1/2}. \quad (3.19)
$$

And as an efficient way to implementation, we derive the quasi-compact Peaceman-Rachford ADI scheme:
\[
(C_x - \frac{\tau}{2} \delta_x^a) V_{i,j}^n = \left( C_y + \frac{\tau}{2} \delta_y^b \right) U_{i,j}^n + \frac{\tau}{2} C_y f_{i,j}^{n+1/2}, \quad (3.20a)
\]
\[
(C_y - \frac{\tau}{2} \delta_y^b) U_{i,j}^{n+1} = \left( C_x + \frac{\tau}{2} \delta_x^a \right) V_{i,j}^n + \frac{\tau}{2} C_x f_{i,j}^{n+1/2}, \quad (3.20b)
\]
the quasi-compact Douglas ADI scheme:
\[
\left( C_x - \frac{\tau}{2} \delta_x^a \right) V_{i,j}^n = \left( C_x C_y + \frac{\tau}{2} C_y \delta_x^a + \tau C_x \delta_y^b \right) U_{i,j}^n + \tau C_x C_y f_{i,j}^{n+1/2}, \quad (3.21a)
\]
\[
\left( C_y - \frac{\tau}{2} \delta_y^b \right) U_{i,j}^{n+1} = V_{i,j}^n - \frac{\tau}{2} \delta_y^b U_{i,j}^n, \quad (3.21b)
\]
and the quasi-compact D’yakonov ADI scheme:
\[
\left( C_x - \frac{\tau}{2} \delta_x^a \right) V_{i,j}^n = \left( C_x + \frac{\tau}{2} \delta_x^a \right) \left( C_y + \frac{\tau}{2} \delta_y^b \right) U_{i,j}^n + \tau C_x C_y f_{i,j}^{n+1/2}, \quad (3.22a)
\]
\[
\left( C_y - \frac{\tau}{2} \delta_y^b \right) U_{i,j}^{n+1} = V_{i,j}^n. \quad (3.22b)
\]
Adding (3.16) to (3.17) yields that
\[
\left( C_x - \frac{\tau}{2} \delta_x^a \right) \left( C_y - \frac{\tau}{2} \delta_y^b \right) U_{i,j}^{n+1} = \left( C_x + \frac{\tau}{2} \delta_x^a \right) \left( C_y + \frac{\tau}{2} \delta_y^b \right) U_{i,j}^n + \tau C_x C_y f_{i,j}^{n+1/2} + \frac{\tau^3}{4} \delta_x^a \delta_y^b f_{i,j}^{n+1/2} + \tau \tilde{\varepsilon}_{i,j}^{n+1/2} \quad (3.23)
\]
with
\[
\tilde{\varepsilon}_{i,j}^{n+1/2} = \tilde{\varepsilon}_{i,j}^{n+1/2} + O(\tau^2 + \tau^2 h_x^2 + \tau^2 h_y^2). \quad (3.24)
\]
Eliminating the truncating error, we get the other quasi-compact scheme of (3.9)
\[
\left( C_x - \frac{\tau}{2} \delta_x^a \right) \left( C_y - \frac{\tau}{2} \delta_y^b \right) U_{i,j}^{n+1} = \left( C_x + \frac{\tau}{2} \delta_x^a \right) \left( C_y + \frac{\tau}{2} \delta_y^b \right) U_{i,j}^n + \tau C_x C_y f_{i,j}^{n+1/2} + \frac{\tau^3}{4} \delta_x^a \delta_y^b f_{i,j}^{n+1/2} \quad (3.25)
\]
and the corresponding quasi-compact locally one-dimensional (LOD) scheme:
\[
\left( C_x - \frac{\tau}{2} \delta_x^a \right) V_{i,j}^n = \left( C_x + \frac{\tau}{2} \delta_x^a \right) U_{i,j}^n + \frac{\tau}{2} \left( C_x + \frac{\tau}{2} \delta_x^a \right) f_{i,j}^{n+1/2}, \quad (3.26a)
\]
\[
\left( C_y - \frac{\tau}{2} \delta_y^b \right) U_{i,j}^{n+1} = \left( C_y + \frac{\tau}{2} \delta_y^b \right) V_{i,j}^n + \frac{\tau}{2} \left( C_y - \frac{\tau}{2} \delta_y^b \right) f_{i,j}^{n+1/2}. \quad (3.26b)
\]
4 Stability and Convergence

We first give some auxiliary lemmas necessary to theoretical analyses.

4.1 Preliminary

**Lemma 5** (Discrete Gronwall Lemma [15]) Assume that \( \{k_n\} \) and \( \{p_n\} \) are nonnegative sequences, and the sequence \( \{\phi_n\} \) satisfies

\[
\phi_0 \leq g_0, \quad \phi_n \leq g_0 + \sum_{l=0}^{n-1} p_l + \sum_{l=0}^{n-1} k_l \phi_l, \quad n \geq 1,
\]

where \( g_0 \geq 0 \). Then the sequence \( \{\phi_n\} \) satisfies

\[
\phi_n \leq \left( g_0 + \sum_{l=0}^{n-1} p_l \right) \exp \left( \sum_{l=0}^{n-1} k_l \right), \quad n \geq 1. \tag{4.1}
\]

In the following, we first list some properties of Kronecker products of matrices.

**Lemma 6** ([5]) Let \( A \in \mathbb{R}^{n \times n} \) have eigenvalues \( \{\lambda_i\}_{i=1}^n \) and \( B \in \mathbb{R}^{m \times m} \) have eigenvalues \( \{\mu_j\}_{j=1}^m \). Then the \( mn \) eigenvalues of \( A \otimes B \), which represents the Kronecker product of matrix \( A \) and \( B \), are

\[
\lambda_1 \mu_1, \ldots, \lambda_1 \mu_m, \lambda_2 \mu_1, \ldots, \lambda_2 \mu_m, \ldots, \lambda_n \mu_1, \ldots, \lambda_n \mu_m.
\]

**Lemma 7** ([5]) Let \( A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{r \times s}, C \in \mathbb{R}^{n \times p}, D \in \mathbb{R}^{s \times t} \). Then

\[
(A \otimes B)(C \otimes D) = AC \otimes BD \quad (\in \mathbb{R}^{mr \times pt}). \tag{4.2}
\]

Moreover, if \( A, B \in \mathbb{R}^{n \times n} \), \( I \) is a unit matrix of order \( n \), then matrices \( I \otimes A \) and \( B \otimes I \) commute.

**Lemma 8** ([5]) For all \( A \) and \( B \), \( (A \otimes B)^T = A^T \otimes B^T \) and \( (A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \) if \( A \) and \( B \) are invertible.

**Lemma 9** ([21]) Let \( A, B \) be two symmetric and positive semi-definite matrices, symbolized \( A \geq 0 \) and \( B \geq 0 \). Then \( A \otimes B \geq 0 \).

**Lemma 10** ([21]) Let \( A \) be an \( n \)-square symmetric and positive semi-definite matrix. Then there is a unique \( n \)-square symmetric and positive semi-definite matrix \( B \) such that \( B^2 = A \). Such a matrix \( B \) is called the square root of \( A \), denoted by \( A^{1/2} \).

4.2 One Dimensional Case

Next we consider the stability and convergence analysis for the scheme (3.6). Let

\[
V_h = \{v : v = \{v_i\} \text{ is a grid function in } \{x_i = a + ih\}_{i=0}^N \text{ and } v_0 = v_N = 0\}.
\]
For any \( v = \{v_i\} \in V_h \), we define its pointwise maximum norm and the discrete \( L^2 \) norm as
\[
\|v\|_{\infty} = \max_{1 \leq i \leq N-1} |v_i|, \quad \|v\|^2 = h \sum_{i=1}^{N-1} v_i^2. \tag{4.3}
\]

**Theorem 1** For the case of \((p, q) = (1, 0)\), the difference scheme (3.6) is unconditionally stable for all \( 1 < \alpha \leq 2 \), and for the case of \((p, q) = (1, -1)\), the difference scheme (3.6) is also unconditionally stable for \( \frac{1+\sqrt{73}}{6} \leq \alpha \leq 2 \).

**Proof** Denoting \( D_\alpha = \frac{\tau}{2h^\alpha}(K_1 A_\alpha + K_2 A_\alpha^T) \), we rewrite (3.7) as
\[
(C_\alpha - D_\alpha)U^{n+1} = (C_\alpha + D_\alpha)U^n + \tau C_\alpha F^n + H^n. \tag{4.4}
\]
From Remark 3, we know that \( C_\alpha \) is a symmetric and positive definite matrix when \((p, q) = (1, 0)\) with \( 1 < \alpha \leq 2 \) and \((p, q) = (1, -1)\) with \( \frac{1+\sqrt{73}}{6} \leq \alpha \leq 2 \), which follows that \( C_\alpha^{-1} \) is also symmetric and positive definite. On the other hand, Lemma 3 shows that the eigenvalues of the matrix \( D_\alpha + D_\alpha^T = \tau(K_1 + K_2)(A_\alpha + A_\alpha^T) \) are all negative for \( 1 < \alpha \leq 2 \), thus \( (D_\alpha + D_\alpha^T) \) is a symmetric and negative definite matrix. Then, for any \( v = (v_1, v_2, \ldots, v_{N-1})^T \in \mathbb{R}^{N-1} \setminus 0 \), there exists
\[
v^T((C_\alpha^{-1} D_\alpha) C_\alpha^{-1} + C_\alpha^{-1} (C_\alpha^{-1} D_\alpha)^T) v = v^T C_\alpha^{-1} (D_\alpha + D_\alpha^T) C_\alpha^{-1} v < 0, \tag{4.5}
\]
which means that the matrix \( (C_\alpha^{-1} D_\alpha) C_\alpha^{-1} + C_\alpha^{-1} (C_\alpha^{-1} D_\alpha)^T \) is negative definite. Then it yields from Lemma 4 that all the eigenvalues of \( C_\alpha^{-1} D_\alpha \) have negative real parts. In addition, \( \lambda \) is an eigenvalue of \( (C_\alpha^{-1} D_\alpha) \) if and only if \( \frac{1+\sqrt{73}}{1-\lambda} < 1 \) holds. Hence, the spectral radius of the matrix \( (C_\alpha - D_\alpha)^{-1}(C_\alpha + D_\alpha)(I - C_\alpha^{-1} D_\alpha)^{-1}(I + C_\alpha^{-1} D_\alpha) \) is less than one, and the difference scheme (3.6) is stable. \( \square \)

**Theorem 2** Let \( u_i^n \) be the exact solution of problem (3.1), and \( U_i^n \) be the solution of difference scheme (3.6) at grid point \((x_i, t_n)\). Then the following estimate
\[
\|u^n - U^n\| \leq c(\tau^2 + h^3), \quad 1 \leq n \leq M, \tag{4.6}
\]
holds for all \( 1 < \alpha < 2 \) with \((p, q) = (1, 0)\) and \( \frac{1+\sqrt{73}}{6} < \alpha < 2 \) with \((p, q) = (1, -1)\).

**Proof** Denoting \( e_i^n = u_i^n - U_i^n \), from formulae (3.4) and (3.6) we have
\[
C_\alpha (e_i^{n+1} - e^n) = \frac{K_1 \tau}{2h^\alpha} A_\alpha (e_i^{n+1} + e^n) - \frac{K_2 \tau}{2h^\alpha} A_\alpha^T (e_i^{n+1} + e^n) = \tau e_i^{n+1/2}, \tag{4.7}
\]
where
\[
e^n = (e_1^n, e_2^n, \ldots, e_{N-1}^n)^T, \quad e_i^{n+1/2} = (e_1^{n+1/2}, e_2^{n+1/2}, \ldots, e_{N-1}^{n+1/2})^T.
\]
Multiplying (4.7) by \( h(e_i^{n+1} + e^n)^T \), we obtain that
\[
h(e_i^{n+1} + e^n)^T C_\alpha (e_i^{n+1} - e^n) = \frac{K_1 \tau}{2h^{\alpha-1}} (e_i^{n+1} + e^n)^T A_\alpha (e_i^{n+1} + e^n) \]
\[
- \frac{K_2 \tau}{2h^{\alpha-1}} (e_i^{n+1} + e^n)^T A_\alpha^T (e_i^{n+1} + e^n) = \tau h(e_i^{n+1} + e^n)^T e_i^{n+1/2}. \tag{4.8}
\]
By Lemma 3, $A_{\alpha}$ and its transpose $A_{\alpha}^T$ are both negative semi-definite matrices for $1 \leq \alpha \leq 2$, thus
\[(e^{n+1} + e^n)^T A_{\alpha} (e^{n+1} + e^n) \leq 0, \quad (e^{n+1} + e^n)^T A_{\alpha}^T (e^{n+1} + e^n) \leq 0. \quad (4.9)\]

Then (4.8) leads to
\[h(e^{n+1} + e^n)^T C_{\alpha} (e^{n+1} - e^n) \leq \tau h(e^{n+1} + e^n)^T e^{n+1/2}. \quad (4.10)\]

As the matrix $C_{\alpha}$ is symmetric, we derive that
\[h(e^{n+1} + e^n)^T C_{\alpha} (e^{n+1} - e^n) = E^{n+1} - E^n, \quad (4.11)\]

where
\[E^n = h(e^n)^T C_{\alpha} (e^n) \geq (1 - 4 c_{\alpha}^{p,q,2}) \| e^n \|^2. \quad (4.12)\]

From (2.16) and (2.17), it yields $E^n \geq \lambda \| e^n \|^2$, where $\lambda = 1 - 4 c_{i-1,2}^{\alpha} > 0$ if $\frac{1+176}{6} < \alpha \leq 2$ and $(p, q) = (1, -1)$; and $\lambda = \frac{23}{72}$ if $1 \leq \alpha \leq 2$ and $(p, q) = (1, 0)$. Together with (4.10), it yields that
\[E^{k+1} - E^k \leq \tau h(e^{k+1} + e^k)^T e^{k+1/2} \leq \frac{\tau \lambda}{2} \| e^{k+1} \|^2 + \frac{\tau}{\lambda} \| e^{k+1/2} \|^2. \quad (4.13)\]

Summing up for all $0 \leq k \leq n - 1$, we have
\[\lambda \| e^n \|^2 \leq \tau h(e^n + e^{n-1})^T e^{n-1/2} + \frac{\tau \lambda}{2} \sum_{k=0}^{n-2} \| e^{k+1} \|^2 + \frac{\tau}{\lambda} \sum_{k=1}^{n-2} \| e^{k+1/2} \|^2 \leq \frac{\lambda}{2} \| e^n \|^2 + \frac{\tau^2}{2 \lambda} \| e^{n-1/2} \|^2 + \tau \lambda \sum_{k=1}^{n-1} \| e^k \|^2 + \frac{\tau}{\lambda} \sum_{k=1}^{n-1} \| e^{k+1/2} \|^2. \quad (4.14)\]

Since $|e_i^{k+1/2}| \leq \tilde{c}(\tau^2 + h^2)$ for any $0 \leq k \leq n - 1$, then it leads to
\[\| e^n \|^2 \leq 2 \tau \sum_{k=1}^{n-1} \| e^k \|^2 + \frac{2 \tau}{\lambda^2} \sum_{k=1}^{n-1} \| e^{k+1/2} \|^2 + \frac{\tau^2}{\lambda^2} \| e^{n-1/2} \|^2 \leq 2 \tau \sum_{k=1}^{n-1} \| e^k \|^2 + c(\tau^2 + h^3)^2, \quad (4.15)\]

which completes the proof by Lemma 5. \[\square\]

**Remark 4** The truncation error in (3.5) becomes $\varepsilon_i^{n+1/2} = O(\tau^2 + h^4)$ when $\alpha = 1, 2$ with $(p, q) = (1, 0)$ and $\alpha = 2$ with $(p, q) = (1, -1)$, so when taking $\alpha = 1, 2$ the compact finite difference schemes for the classical diffusion equations are recovered and the corresponding error estimate of the difference scheme (3.6) satisfies
\[\| u^n - U^n \| \leq c(\tau^2 + h^4), \quad 1 \leq n \leq M. \quad (4.16)\]
4.3 Two Dimensional Case

Define the following grid function space

\[ V_h = \{ v : v = \{v_{i,j}\} \text{ is a grid function in } \Omega_h \text{ and } v_{i,j} = 0 \text{ on } \Gamma_h \}, \]

where

\[ \Omega_h = \{(i, j) : 1 \leq i \leq N_x - 1, 1 \leq j \leq N_y - 1\}, \]

\[ \Gamma_h = \{(i, j) : i = 0, N_x; 0 \leq j \leq N_y \} \cup \{(i, j) : 0 \leq i \leq N_x; j = 0, N_y \}. \]

For any \( v = \{v_l\} \in V_h \), its pointwise maximum norm and discrete \( L^2 \) norm are given by

\[ \|v\|_\infty = \max_{(i,j) \in \Omega_h} |v_{i,j}|, \quad \|v\| = \sqrt{\sum_{i=1}^{N_x-1} \sum_{j=1}^{N_y-1} v_{i,j}^2}. \quad (4.17) \]

**Theorem 3** For the case of \((p, q) = (1, 0)\), the difference schemes (3.19) and (3.25) are unconditionally stable for \(1 < \alpha, \beta \leq 2\). And for the case of \((p, q) = (1, -1)\), the difference schemes (3.19) and (3.25) are also unconditionally stable when \(\frac{1+\sqrt{13}}{2} \leq \alpha, \beta \leq 2\).

**Proof** We express grid function \( U^n_{i,j} \) in the vector form as

\[ U^n = (u^n_{1,1}, u^n_{2,1}, \ldots, u^n_{N_x-1,1}, u^n_{1,2}, u^n_{2,2}, \ldots, u^n_{N_x-1,2}, \ldots, u^n_{1,N_y-1}, \ldots, u^n_{N_x-1,N_y-1})^T, \]

and denote

\[ C_x = I_y \otimes C_\alpha, \quad C_y = C_\beta \otimes I_x, \quad (4.18) \]

\[ D_x = \frac{K_{x,1}}{2h^2_x} I_y \otimes A_\alpha + \frac{K_{x,2}}{2h^2_x} I_y \otimes A_\alpha^T, \quad D_y = \frac{K_{y,1}}{2h^2_y} A_\beta \otimes I_x + \frac{K_{y,2}}{2h^2_y} A_\beta^T \otimes I_x, \quad (4.19) \]

where the symbol \( \otimes \) denotes the Kronecker product, \( I_x \) and \( I_y \) are unit matrices of \((N_x - 1)\) and \((N_y - 1)\) squares, respectively, and the matrices \( A_\alpha \) and \( A_\beta \) are defined in (2.18) corresponding to \( \alpha \) and \( \beta \). \( C_\alpha \) and \( C_\beta \) are given in (2.14) with coefficients \( \alpha \) and \( \beta \). Therefore, denoting the disturbances of \( U^{n+1} \) and \( U^n \) by \( \delta U^{n+1} \) and \( \delta U^n \), respectively, we have from (3.19) and (3.25) that

\[ \delta U^{n+1} = (C_y - D_y)^{-1}(C_x - D_x)^{-1}(C_x + D_x)(C_y + D_y)\delta U^n. \quad (4.20) \]

Using Lemma 7, we can check that \( C_x \) and \( D_x \) commute with \( C_y \) and \( D_y \), which deduces that \((C_y - D_y)^{-1}\) and \((C_y + D_y)\) commute with \((C_x - D_x)^{-1}\) and \((C_x + D_x)\). Then it obtains from (4.20) that

\[ \delta U^n = ((C_y - D_y)^{-1}(C_y + D_y))^n ((C_x - D_x)^{-1}(C_x + D_x))^n \delta U^0. \quad (4.21) \]

From Remark 3, Lemma 6 and Lemma 7, we have that \( C_x \) and \( C_y \) are symmetric and positive definite matrices in the cases of \((p, q) = (1, 0)\) with \(1 < \alpha, \beta \leq 2\) and \((p, q) = (1, -1)\) with
\[
\frac{1 + \sqrt{73}}{6} \leq \alpha, \beta \leq 2, \text{ which yields that } C^{-1}_x \text{ and } C^{-1}_y \text{ are also symmetric and positive definite.}
\]

On the other hand, Lemmas 3 and 8 indicate that the eigenvalues of \( \frac{A_{\alpha + A_{\beta}^T}}{2} \) and \( \frac{A_{\beta} + A_{\alpha}^T}{2} \) are all negative when \( 1 < \alpha, \beta \leq 2 \), then employing Lemma 6, we obtain that \( (D_x + D_y^T) \) and \( (D_y + D_x^T) \) are both symmetric and negative definite matrices. Then it yields that \( v^T(D_x + D_y^T)v < 0 \) and \( v^T(D_y + D_x^T)v < 0 \) hold for any non-zero vector \( v \in \mathbb{R}^{(N_x - 1)(N_y - 1)} \), and

\[
v^T((C_y^{-1}D_y)C_y^{-1} + C_y^{-1}(C_y^{-1}D_y)^T)v = v^TC_y^{-1}(D_y + D_y^T)C_y^{-1}v < 0, \quad y = x, y, \quad (4.22)
\]

which means that \( (C_y^{-1}D_y)C_y^{-1} + C_y^{-1}(C_y^{-1}D_y)^T \) for \( y = x, y \) are symmetric and negative definite matrices, then it implies from Lemma 4 that the real parts of all the eigenvalues \( \lambda_y \) of \( C_y^{-1}D_y \) for \( y = x, y \) are negative, and \( |\frac{1 + j\sqrt{3}}{2} - \lambda_y| < 1 \). Additionally, \( \lambda_y \) is an eigenvalue of \( C_y^{-1}D_y \) if and only if \( \frac{1 + j\sqrt{3}}{2} - \lambda_y \) is an eigenvalue of \( (I - C_y^{-1}D_y)^{-1}(I + C_y^{-1}D_y) \), thus the spectral radius of each matrix is less than 1, which concludes that \( ((I - C_y^{-1}D_y)^{-1}(I + C_y^{-1}D_y))^n \) and \( ((I - C_y^{-1}D_y)^{-1}(I + C_y^{-1}D_y))^n \) converge to zero matrix, therefore, the difference schemes (3.19) and (3.25) are stable.

**Theorem 4** Let \( u_{i,j}^n \) be the exact solution of (3.9), and \( U_{i,j}^n \) be the solution of the difference schemes (3.19) or (3.25), then in the cases of \( (p, q) = (1, 0) \) with \( 1 < \alpha, \beta < 2 \) and \( (p, q) = (1, -1) \) with \( \frac{1 + \sqrt{73}}{6} \leq \alpha, \beta < 2 \), we have

\[
\|u^n - U^n\| \leq c(\tau^2 + h_x^3 + h_y^3), \quad 1 \leq n \leq M, \quad (4.23)
\]

where \( c \) denotes a positive constant and \( \| \cdot \| \) stands for the discrete \( L^2 \) norm.

**Proof** Let \( e_{i,j}^n = u_{i,j}^n - U_{i,j}^n \), subtracting (3.17) from (3.19) leads to

\[
(C_x - D_x)(C_y - D_y)e^{n+1} = (C_x + D_x)(C_y + D_y)e^n + \tau E^{n+1/2}, \quad (4.24)
\]

where

\[
e = (e_{1,1}, e_{2,1}, \ldots, e_{N_x-1,1}, e_{1,2}, e_{2,2}, \ldots, e_{N_x-1,2}, \ldots, e_{1,N_y-1}, e_{2,N_y-1}, \ldots, e_{N_x-1,N_y-1})^T,
\]

\[
E = (\hat{e}_{1,1}, \hat{e}_{2,1}, \ldots, \hat{e}_{N_x-1,1}, \hat{e}_{1,2}, \hat{e}_{2,2}, \ldots, \hat{e}_{N_x-1,2}, \ldots, \hat{e}_{1,N_y-1}, \hat{e}_{2,N_y-1}, \ldots, \hat{e}_{N_x-1,N_y-1})^T,
\]

and the matrices \( C_x, C_y, D_x, D_y \) are given by (4.18) and (4.19), respectively.

As stated in Theorem 3, under the cases of \( (p, q) = (1, 0) \) with \( 1 < \alpha, \beta \leq 2 \) and \( (p, q) = (1, -1) \) with \( \frac{1 + \sqrt{73}}{6} \leq \alpha, \beta < 2 \), the matrices \( C_x \) and \( C_y \) and their inverse are symmetric and positive definite. And from Lemmas 8 and 10, we know that \( (C_x^{-1})^{1/2} = I_x \otimes (C_{\alpha}^{-1})^{1/2} \) and \( (C_y^{-1})^{1/2} = (C_{\beta}^{-1})^{1/2} \otimes I_y \) uniquely exist and are symmetric and positive semi-definite matrices. Then multiplying (4.24) by \( (C_x^{-1})^{1/2}(C_y^{-1})^{1/2} \), and making the discrete \( L^2 \)-norm on both sides, we have

\[
\left\|((C_x^{-1})^{1/2}(C_y^{-1})^{1/2}(C_x - D_x)(C_y - D_y)e^{n+1})\right\| \\
\leq \left\|((C_x^{-1})^{1/2}(C_y^{-1})^{1/2}(C_x + D_x)(C_y + D_y)e^n)\right\| + \tau \left\|((C_x^{-1})^{1/2}(C_y^{-1})^{1/2}E^{n+1/2})\right\|. \quad (4.25)
\]

Simple calculations show that \( (C_x - D_x) \) commutes with \( (C_x - D_x), (C_x^{-1})^{1/2}, (C_x - D_x^T); (C_y + D_y) \) commutes with \( (C_x + D_x), (C_x^{-1})^{1/2}, (C_x + D_x^T); \) and \( (C_y^{-1})^{1/2} \) commutes with...
\((C_x^{-1})^{1/2}, (C_y^{1/2})\). From Lemmas 3 and 9, we have that \(D_x + D_x^T (\nu = x, y)\) are symmetric and negative definite matrices. Thus, using Lemmas 7 and 9, we obtain that

\[
\begin{align*}
((C_x^{-1})^{1/2} (C_y^{-1})^{1/2} (C_x - D_x)(C_y - D_y))^T ((C_x^{-1})^{1/2} (C_y^{-1})^{1/2} (C_x - D_x)(C_y - D_y)) & \\
= (C_y - D_y - D_y + D_y^T C_y^{-1} D_y)(C_x - D_x + D_x^T C_x^{-1} D_x) & \\
\geq (C_y + D_y^T C_y^{-1} D_y)(C_x + D_x^T C_x^{-1} D_x) & \\
+ (D_y^T + D_y)(D_x^T + D_x),
\end{align*}
\]

(4.26)

and

\[
\begin{align*}
((C_x^{-1})^{1/2} (C_y^{-1})^{1/2} (C_x + D_x)(C_y + D_y))^T ((C_x^{-1})^{1/2} (C_y^{-1})^{1/2} (C_x + D_x)(C_y + D_y)) & \\
= (C_y + D_y + D_y^T C_y^{-1} D_y)(C_x + D_x + D_x^T C_x^{-1} D_x) & \\
\leq (C_y + D_y^T C_y^{-1} D_y)(C_x + D_x^T C_x^{-1} D_x) & \\
+ (D_y^T + D_y)(D_x^T + D_x),
\end{align*}
\]

(4.27)

where the matrices \(A \geq B\) means that \((A - B)\) is positive semi-definite. And define

\[
E^n = \sqrt{h_x h_y (e^n)^T ((C_y + D_y^T C_y^{-1} D_y)(C_x + D_x^T C_x^{-1} D_x) + (D_y^T + D_y)(D_x^T + D_x)) (e^n)},
\]

(4.28)

it concludes from (4.18), (4.19), Lemma 7, Lemma 9, and Lemma 3 that the matrices \(C_x D_x^T C_x^{-1} D_x, C_x D_x^T C_x^{-1} D_x, D_x D_x^T C_x^{-1} D_x, D_x^T C_x^{-1} D_x, (D_y^T + D_y)(D_x^T + D_x)\) are all symmetric and positive definite, which follows that

\[
E^n \geq \sqrt{h_x h_y (e^n)^T C_x C_y (e^n)} = \sqrt{h_x h_y (e^n)^T (C_{\beta} \otimes C_{\alpha})(e^n)} \geq \sqrt{\lambda_{\min}(C_{\alpha}) \lambda_{\min}(C_{\beta})} \| e^n \|,
\]

(4.29)

where \(\lambda_{\min}(C_{\alpha})\) and \(\lambda_{\min}(C_{\beta})\) are the minimum eigenvalue of matrix \(C_{\alpha}\) and \(C_{\beta}\), respectively. As stated in Remark 3, \(\lambda_{\min}(C_{\alpha}) > 1 - 4c_{1,-1,2} > 0, \lambda_{\min}(C_{\beta}) > 1 - 4c_{1,-1,2} > 0\) if \(1 + \frac{\sqrt{75}}{6} < \alpha, \beta \leq 2\) and \((p, q) = (1, 1); \lambda_{\min}(C_{\alpha}), \lambda_{\min}(C_{\beta}) > \frac{23}{72}\) if \(1 \leq \alpha, \beta \leq 2\) and \((p, q) = (1, 0)\). Together with (4.26) and (4.27), then (4.25) becomes as

\[
E^{k+1} - E^k \leq \tau \left\| (C_x^{-1})^{1/2} (C_y^{-1})^{1/2} E^{k+1/2} \right\|.
\]

(4.30)

From the Rayleigh-Ritz Theorem (see Theorem 8.8 in [21]) and Lemma 6, we have for \(k = 0, \ldots, n - 1\) that

\[
\begin{align*}
\left\| (C_x^{-1})^{1/2} (C_y^{-1})^{1/2} E^{k+1/2} \right\| & = \sqrt{h_x h_y (E^{k+1/2})^T (C_x^{-1} C_y^{-1}) E^{k+1/2}} \\
& \leq \sqrt{\lambda_{\max}(C_x^{-1} C_y^{-1})} \| E^{k+1/2} \| = \frac{1}{\sqrt{\lambda_{\min}(C_{\alpha}) \lambda_{\min}(C_{\beta})}} \| E^{k+1/2} \|,
\end{align*}
\]

(4.31)

Summing up (4.30) for all \(0 \leq k \leq n - 1\) shows that

\[
E^n \leq \tau \sum_{k=0}^{n-1} \left\| (C_x^{-1})^{1/2} (C_y^{-1})^{1/2} E^{k+1/2} \right\| \leq \frac{\tau}{\sqrt{\lambda_{\min}(C_{\alpha}) \lambda_{\min}(C_{\beta})}} \sum_{k=0}^{n-1} \| E^{k+1/2} \|.
\]

(4.32)
Combining (4.32) and (4.29), and noticing $|\hat{\varepsilon}_{i,j}^{k+1/2}| \leq \tilde{c}(\tau^2 + h_x^3 + h_y^3)$ for $1 \leq i \leq N_x - 1$ and $1 \leq j \leq N_y - 1$, we obtain
\[ \| e^n \| \leq c \frac{T}{\lambda_{\min}(C_\alpha)\lambda_{\min}(C_\beta)} (\tau^2 + h_x^3 + h_y^3). \] (4.33)

The estimate for scheme (3.25) can also be obtained by the similar approach used above. □

**Remark 5** If $\alpha, \beta = 1, 2$ with $(p,q) = (1,0)$ and $\alpha, \beta = 2$ with $(p,q) = (1,-1)$, then by the same reasoning of Theorem 4, we obtain the following error estimate for the difference scheme (3.19) and (3.25)
\[ \| u^n - U^n \| \leq c (\tau^2 + h_x^4 + h_y^4), \quad 1 \leq n \leq M. \] (4.34)

## 5 Numerical Experiments

In this section, the numerical results of one and two dimensional cases are presented to show the effectiveness and convergence orders of the schemes.

For saving computational time, we do one extrapolation to increase the accuracy to the third order in time (see [7]). The detailed extrapolation algorithm is described as follows.

**Step 1.** Calculate $\zeta_1, \zeta_2$ from the following linear algebraic equations,
\[ \begin{align*}
\zeta_1 + \zeta_2 &= 1, \\
\zeta_1 + \frac{\zeta_2}{4} &= 0;
\end{align*} \]

**Step 2.** Compute the solution $U^n$ of the quasi-compact difference schemes with two time step sizes $\tau$ and $\tau/2$;

**Step 3.** Evaluate the extrapolation solution $W^n(\tau)$ by
\[ W^n(\tau) = \zeta_1 U^n(\tau) + \zeta_2 U^n(\tau/2). \]

**Example 1** Consider the following one dimensional space fractional diffusion equation
\[ \frac{\partial u(x,t)}{\partial t} = D_0^\alpha u(x,t) + \frac{\Gamma(4)}{\Gamma(4 - \alpha)} (x^3 - (1 - x)^3) \]
\[ u(0,t) = u(1,t) = 0, \quad t \in [0,1], \]
\[ u(x,0) = x^3(1-x)^3, \quad x \in [0,1], \] (5.1)

with the source term
\[ f(x,t) = -e^{-t} \left( x^3(1-x)^3 + \frac{\Gamma(4)}{\Gamma(4 - \alpha)} (x^3 - (1 - x)^3) \right) - 3 \frac{\Gamma(5)}{\Gamma(5 - \alpha)} (x^{4-\alpha} + (1 - x)^{4-\alpha}) + 3 \frac{\Gamma(6)}{\Gamma(6 - \alpha)} (x^{5-\alpha} + (1 - x)^{5-\alpha}) - \frac{\Gamma(7)}{\Gamma(7 - \alpha)} (x^{6-\alpha} + (1 - x)^{6-\alpha}) \].

And the exact solution of (5.1) is given by $u(x,t) = e^{-t}x^3(1-x)^3$. 

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Table 1 The maximum and $L^2$ errors and corresponding convergence rates to (5.1) approximated by the quasi-compact difference scheme at $t = 1$ for different $\alpha$ with $\tau = h$

| $\alpha$ | $N$ | $(p, q) = (1, 0)$ | Rate | $\|u^n - W^n\|_\infty$ | Rate | $\|u^n - W^n\|$ | Rate |
|-------|-----|----------------|-------|-----------------|-------|----------------|-------|
| 1.2   | 8   | 7.24249E–05    | –     | 4.12739E–05     | –     | 1.67141E–01   | –     |
| 16    | 9.86726E–06 | 2.88 | 6.00551E–06     | 2.78  | 6.57473E–01    | –1.98 | 4.14981E–01   | –1.82 |
| 32    | 1.41964E–06 | 2.80 | 8.38665E–07     | 2.84  | 1.20582E+04    | –14.16| 7.00900E+03   | –14.04|
| 64    | 1.91577E–07 | 2.89 | 1.13698E–07     | 2.88  | 6.67837E+11    | –25.72| 3.06749E+11   | –25.38|
| 128   | 2.52254E–08 | 2.92 | 1.50548E+04     | 2.84  | 6.55394E+25    | –46.48| 2.42775E+25   | –46.17|
| 256   | 3.26935E–09 | 2.95 | 1.95986E+04     | 2.86  | 7.87651E+50    | –83.31| 2.34722E+50   | –83.00|
| $1+\sqrt{73}/6$ | 8   | 5.48070E–05    | –     | 3.18317E–05     | –     | 1.26032E–04   | –     |
| 16    | 6.71566E–06 | 3.03 | 4.15788E–06     | 2.94  | 4.38053E–05    | –2.60 | 2.50890E–05   | –2.69 |
| 32    | 8.92036E–07 | 2.91 | 5.69588E–07     | 2.87  | 6.07692E–06    | –2.85 | 3.69218E–06   | –2.76 |
| 64    | 1.19966E–07 | 2.89 | 7.81948E–08     | 2.86  | 8.00832E–07    | –2.92 | 5.16694E–07   | –2.84 |
| 128   | 1.60292E–08 | 2.90 | 1.06194E–08     | 2.88  | 1.05299E–07    | –2.93 | 7.00251E–08   | –2.88 |
| 256   | 2.11675E–09 | 2.92 | 1.42376E–09     | 2.90  | 1.37085E–08    | –2.94 | 9.35050E–09   | –2.91 |

In Table 1, we present the errors $\|u^n - W^n\|$, $\|u^n - W^n\|_\infty$ and corresponding convergence orders with different space step sizes, where $W^n_i(\tau) = -\frac{1}{3} U^n_i(\tau) + \frac{4}{3} U^n_i(\tau/2)$ is the extrapolation solution, and $U^n_i$ satisfies the quasi-compact scheme (3.6). It can be noted that for the case of $(p, q) = (1, -1)$, the numerical results are neither stable nor convergent when the order $\alpha$ is less than the critical value $\frac{1+\sqrt{73}}{6} \approx 1.59$, which coincides with the theoretical results. Figure 1 shows that the convergence rates of the maximum and $L^2$ errors to (5.1) approximated by the quasi-compact difference scheme at $t = 1$ with $N = 128$ for different $\alpha$, where the convergence rates fall from 4 to 3 near $\alpha = 1$ and increase gradually with $\alpha$ from 1.9 to 2.

Example 2 The following fractional diffusion problem

$$\frac{\partial u(x, y, t)}{\partial t} =_0 D^\alpha_x u(x, y, t) + _x D^\alpha_x u(x, y, t) + _y D^\beta_y u(x, y, t) + f(x, y, t)$$

(5.2)

is considered in the domain $\Omega = (0, 1)^2$ and $t > 0$ with the boundary conditions

$$u(x, y, t) = 0, \quad (x, y) \in \partial \Omega, \quad t \in [0, 1],$$

and initial value

$$u(x, y, 0) = u(x, y, 0) = x^3 (1 - x)^3 y^3 (1 - y)^3, \quad (x, y) \in [0, 1]^2.$$
The source term is
\[
f(x, y, t) = -e^{-t}\left[ x^3(1-x)^3y^3(1-y)^3 + \frac{\Gamma(4)}{\Gamma(4-\alpha)}(x^{3-\alpha} + (1-x)^{3-\alpha}) - \frac{3\Gamma(5)}{\Gamma(5-\alpha)}(x^{4-\alpha} + (1-x)^{4-\alpha}) \right.
\]
\[
+ \frac{3\Gamma(6)}{\Gamma(6-\alpha)}(x^{5-\alpha} + (1-x)^{5-\alpha}) - \frac{\Gamma(7)}{\Gamma(7-\alpha)}(x^{6-\alpha} + (1-x)^{6-\alpha}) \right] y^3(1-y)^3
\]
\[
+ \left. \left( \frac{\Gamma(4)}{\Gamma(4-\beta)}(y^{3-\beta} + (1-y)^{3-\beta}) - \frac{3\Gamma(5)}{\Gamma(5-\beta)}(y^{4-\beta} + (1-y)^{4-\beta}) \right) \right]
\]
\[
+ \frac{3\Gamma(6)}{\Gamma(6-\beta)}(y^{5-\beta} + (1-y)^{5-\beta}) - \frac{\Gamma(7)}{\Gamma(7-\beta)}(y^{6-\beta} + (1-y)^{6-\beta}) \right] x^3(1-x)^3 \right].
\]

And the exact solution of (5.1) is given by \( u(x, t) = e^{-t}x^3(1-x)^3y^3(1-y)^3 \).

In Table 2, the errors \( \|u^n - W^n\|_\infty \), \( \|u^n - W^n\|_\infty \) and their respective convergence rates are presented for different space step sizes, where \( W_{n,j}(\tau) = -\frac{1}{3}U_{n,j}(\tau) + \frac{4}{3}U_{n,j}(\tau/2) \) is the numerical solution by extrapolation in time, and \( U_{n,j} \) satisfies the quasi-compact LOD (Q-CLOD) scheme (3.26a) and (3.26b), quasi-compact Peaceman-Richardson (Q-CPR) scheme (3.20a) and (3.20b), quasi-compact Douglas (Q-C Douglas) scheme (3.21a) and (3.21b) and quasi-compact D’yakonov (Q-C D’yakonov) scheme (3.22a) and (3.22b), respectively. The third order accuracy both in time and space is verified, and in the computational process, the time costs are largely reduced.

6 Conclusion

The compact finite difference operator for Riemann-Liouville fractional derivatives has been proposed and applied to establishing the quasi-compact difference schemes for one and two
Table 2 The maximum and $L^2$ errors and corresponding convergence rates to (5.1) approximated by the quasi-compact difference splitting schemes at $t = 1$ with $\tau = h_x = h_y$

| Scheme     | $N$ | $(p, q) = (1, 0), (\alpha, \beta) = (1.1, 1.7)$ | Rate | $(p, q) = (1, -1), (\alpha, \beta) = (1.6, 1.9)$ | Rate |
|------------|-----|-----------------------------------------------|------|-----------------------------------------------|------|
| Q-CLOD     |     | $\|u^n - W^n\|_\infty$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|_\infty$ | $\|u^n - W^n\|_{\infty}$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ |
| 8          | 5.18337E–07 | – | 2.35962E–07 | – | 1.27051E–05 | – | 5.20284E–06 | – | 5.09460E–07 | 4.07 |
| 16         | 7.47852E–08 | 2.79 | 3.12736E–08 | 2.92 | 8.45409E–07 | 3.91 | 3.09460E–08 | 4.07 |
| 32         | 9.61857E–09 | 2.96 | 4.13976E–09 | 2.92 | 8.71092E–08 | 3.28 | 2.87346E–08 | 3.43 |
| 64         | 1.28508E–09 | 2.90 | 5.56054E–10 | 2.90 | 9.67026E–09 | 3.17 | 3.35453E–09 | 3.10 |
| 128        | 1.71211E–10 | 2.91 | 7.47594E–11 | 2.90 | 1.17863E–09 | 3.04 | 4.22753E–010 | 2.99 |
| 256        | 2.26863E–11 | 2.92 | 9.77266E–12 | 2.91 | 1.49079E–10 | 2.98 | 5.43715E–011 | 2.96 |
| Q-CPR      |     | $\|u^n - W^n\|_\infty$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|_\infty$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ |
| 8          | 6.30467E–007 | – | 2.63036E–007 | – | 3.59941E–006 | – | 1.33801E–006 | – | 1.33801E–006 |
| 16         | 7.67884E–008 | 3.04 | 3.20134E–008 | 3.04 | 5.12851E–007 | 2.81 | 1.88287E–007 | 2.83 |
| 32         | 9.54417E–009 | 3.01 | 4.19273E–009 | 2.93 | 7.24724E–008 | 2.82 | 2.48958E–008 | 2.92 |
| 64         | 1.28241E–009 | 2.90 | 5.60865E–10 | 2.90 | 9.12654E–009 | 2.99 | 3.25063E–009 | 2.94 |
| 128        | 1.71042E–010 | 2.91 | 7.51181E–11 | 2.90 | 1.15503E–009 | 2.98 | 4.22913E–010 | 2.94 |
| 256        | 2.26759E–011 | 2.92 | 1.00029E–011 | 2.91 | 1.49079E–10 | 2.95 | 5.48530E–011 | 2.95 |
| Q-C Douglas |     | $\|u^n - W^n\|_\infty$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|_\infty$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ |
| 8          | 6.28561E–007 | – | 2.54619E–007 | – | 3.01132E–006 | – | 1.15694E–006 | – | 1.15694E–006 |
| 16         | 7.67367E–008 | 3.03 | 3.12227E–008 | 3.03 | 4.37299E–007 | 2.78 | 1.67892E–007 | 2.78 |
| 32         | 9.54200E–009 | 3.01 | 4.12678E–009 | 2.92 | 6.67372E–008 | 2.71 | 2.30579E–008 | 2.86 |
| 64         | 1.28220E–009 | 2.90 | 5.55710E–10 | 2.89 | 8.61113E–009 | 2.95 | 3.09798E–009 | 2.90 |
| 128        | 1.71028E–010 | 2.91 | 7.47144E–11 | 2.89 | 1.11175E–009 | 2.95 | 4.09795E–010 | 2.92 |
| 256        | 2.26748E–011 | 2.92 | 9.97005E–12 | 2.91 | 1.45187E–10 | 2.94 | 5.36563E–011 | 2.93 |
| Q-C D’yakonov |    | $\|u^n - W^n\|_\infty$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|_\infty$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ | $\|u^n - W^n\|$ |
| 8          | 6.28561E–007 | – | 2.54619E–007 | – | 3.01132E–006 | – | 1.15694E–006 | – | 1.15694E–006 |
| 16         | 7.67367E–008 | 3.03 | 3.12227E–008 | 3.03 | 4.37299E–007 | 2.78 | 1.67892E–007 | 2.78 |
| 32         | 9.54200E–009 | 3.01 | 4.12678E–009 | 2.92 | 6.67372E–008 | 2.71 | 2.30579E–008 | 2.86 |
| 64         | 1.28220E–009 | 2.90 | 5.55710E–10 | 2.89 | 8.61113E–009 | 2.95 | 3.09798E–009 | 2.90 |
| 128        | 1.71028E–010 | 2.91 | 7.47144E–11 | 2.89 | 1.11175E–009 | 2.95 | 4.09795E–010 | 2.92 |
| 256        | 2.26748E–011 | 2.92 | 9.97005E–12 | 2.91 | 1.45187E–10 | 2.94 | 5.36563E–011 | 2.93 |

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