The Symmetry Groups of Noncommutative Quantum Mechanics and Coherent State Quantization

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Abstract

We explore the group theoretical underpinning of noncommutative quantum mechanics for a system moving on the two-dimensional plane. We show that the pertinent groups for the system are the two-fold central extension of the Galilei group in $(2+1)$-space-time dimensions and the two-fold extension of the group of translations of $\mathbb{R}^4$. This latter group is just the standard Weyl-Heisenberg group of standard quantum mechanics with an additional central extension. We also look at a further extension of this group and discuss its significance to noncommutative quantum mechanics. We build unitary irreducible representations of these various groups and construct the associated families of coherent states. A coherent state quantization of the underlying phase space is then carried out, which is shown to lead to exactly the same commutation relations as usually postulated for this model of noncommutative quantum mechanics.

I Introduction

Noncommutative quantum mechanics is a much frequented topic of research these days. The expectation here is that a modification, or rather an extension, of standard quantum mechanics is needed to model physical space-time at very short distances. In this paper we restrict ourselves to the version of non-commutative quantum mechanics which describes a quantum system with two degrees of freedom and in which, in addition to having the usual canonical commutation relations, one also imposes an additional non-commutativity
between the two position coordinates, i.e.,

\[ [Q_i, P_j] = i \hbar \delta_{ij} I, \quad i, j = 1, 2, \quad [Q_1, Q_2] = i \vartheta I, \quad (1.1) \]

Here the \( Q_i, P_j \) are the quantum mechanical position and momentum observables, respectively, and \( \vartheta \) is a (small, positive) parameter which measures the additionally introduced noncommutativity between the observables of the two spatial coordinates. The limit \( \vartheta = 0 \) then corresponds to standard (two-dimensional) quantum mechanics. The literature, even on this rather focused and simple model, is already extensive. We refer to [8, 14] and the many references cited therein for a review of the background and motivation behind the model. One could continue with this game of increasing noncommutativity between the observables by augmenting the above system by an additional commutator between the two momentum operators:

\[ [P_i, P_j] = i \gamma \delta_{ij} I, \quad i, j = 1, 2, \quad (1.2) \]

where \( \gamma \) is yet another positive parameter. Physically, such a commutator would signal, for example, the presence of a magnetic field in the system [8].

The purpose of this paper is two-fold. First, we undertake a group theoretical analysis of the above sets of commutation relations, i.e., to find the groups behind noncommutative quantum mechanics, in the same way as a centrally extended Galilei group [11] or the Weyl-Heisenberg group underlies ordinary quantum mechanics. The second objective of this paper is to arrive at the commutation relations (1.1) by the method of coherent state quantization (see, for example, [3] for a discussion of this method). This will involve constructing appropriate families of coherent states, emanating from the groups underlying noncommutative quantum mechanics, using standard techniques (see, for example, [2]). It will turn out however, that the coherent states that we shall be using here are very different from the ones introduced in [14], in that ours come from the representations of the related groups and satisfy standard resolutions of the identity condition.

II Noncommutative quantum mechanics in the two-plane and the (2+1)-Galilei group

The (2+1)-Galilei group \( G_{\text{Gal}} \) is a six-parameter Lie group. It is the kinematical group of a classical, non-relativistic space-time having two spatial and one time dimensions. It consists of translations of time and space, rotations in the two dimensional space and velocity boosts. As is well-known [11], non-relativistic quantum mechanics can be seen as arising from representations of central extensions of the Galilei group. We will thus be concerned here with the centrally extended (2+1)-Galilei group. The Lie algebra \( \mathfrak{g}_{\text{Gal}} \) of the group \( G_{\text{Gal}} \) has a three dimensional vector space of central extensions. This *extended*
algebra has the following Lie bracket structure (see, for example, \[5\,6\]),

\[
[M, N_i] = \epsilon_{ij}N_j \quad [M, P_i] = \epsilon_{ij}P_j \\
[H, P_i] = 0 \quad [M, H] = \hbar \\
[N_i, N_j] = \epsilon_{ij}\phi \quad [P_i, P_j] = 0 \\
[N_i, P_j] = \delta_{ij}\mathfrak{m} \quad [N_i, H] = P_i, 
\]

(i, j = 1, 2 and $\epsilon_{ij}$ is the totally antisymmetric tensor with $\epsilon_{12} = -\epsilon_{21}$). The three central extensions are characterized by the three central generators $\mathfrak{h}, \phi$ and $\mathfrak{m}$ (they commute with each other and all the other generators). The $P_i$ generate space translations, $N_i$ velocity boosts, $H$ time translations and $M$ is the generator of angular momentum.

Passing to the group level, the universal covering group $\mathcal{G}_{\text{Gal}}$, of $G_{\text{Gal}}$, has three central extensions, as expected. However, $G_{\text{Gal}}$ itself has only two central extensions (i.e., $\mathfrak{h} = 0$, identically \([5]\)). We shall denote this 2-fold centrally extended $(2 + 1)$-Galilei group by $G_{\text{Gal}}^{\text{ext}}$ and its Lie algebra by $\mathfrak{g}_{\text{Gal}}^{\text{ext}}$.

A generic element of $G_{\text{Gal}}^{\text{ext}}$ may be written as $g = (\theta, \phi, R, b, v, a) = (\theta, \phi, r)$, where $\theta, \phi \in \mathbb{R}$, are phase terms corresponding to the two central extensions, $b \in \mathbb{R}$ a time-translation, $R$ is a $2 \times 2$ rotation matrix, $v \in \mathbb{R}^2$ a 2-velocity boost, $a \in \mathbb{R}^2$ a 2-dimensional space translation and $r = (R, b, v, a)$. The two central extensions are given by two cocycles, $\xi_m^1$ and $\xi_\lambda^2$, depending on the two real parameters $m$ and $\lambda$. Explicitly, these are,

\[
\xi_m^1(r; r') = e^{\frac{i\hbar}{2}(a \cdot Rv' - v \cdot Ra' + b \cdot v - Rv')}, \\
\xi_\lambda^2(r; r') = e^{\frac{i\hbar}{2}v \cdot Rv'}, \quad \text{where} \quad q \wedge p = q_1p_2 - q_2p_1, \tag{2.2}
\]

$$(q = (q_1, q_2), \; p = (p_1, p_2)).$$

The group multiplication rule is given by

\[
gg' = (\theta, \phi, R, b, v, a)(\theta', \phi', R', b', v', a') \\
= (\theta + \theta', \phi + \phi' + \xi_m^1(r; r'), \phi + \phi' + \xi_\lambda^2(r; r'), \\
RR', b + b', v + Rv', a + Ra' + vb'). \tag{2.3}
\]

The projective unitary irreducible representations (PURs) of $G_{\text{Gal}}^{\text{ext}}$, from which we can obtain its unitary irreducible representations, have all been computed in, e.g., \([6]\). In this paper we shall only consider the case where $m \neq 0$ and $\lambda \neq 0$. These representations, realized on the Hilbert space $L^2(\mathbb{R}^2, d\mathbf{k})$ (see \([3,4]\) below), are characterized by ordered pairs $(m, \vartheta)$ of reals and by the number $s$, expressed as an integral multiple of $\frac{\hbar}{2}$. Here, $m$ is to be interpreted as the mass of the nonrelativistic system under study, while $\lambda$ will be seen to be related to the parameter $\vartheta$ appearing in \([1,1]\). The quantity $s$ is the eigenvalue of the intrinsic angular momentum operator $S$ (representing rotations in the
rest-frame). The physical significance of these quantities have been studied extensively in [5, 6, 11] and [15].

Recall that we should take $\hbar = 0$ in (2.1), to get the Lie algebra $\mathfrak{g}^{\text{ext}}_{\text{Gal}}$. In the representation Hilbert space of the PURs of the group $G^{\text{ext}}_{\text{Gal}}$, the basis elements of the algebra are realized as self-adjoint operators, the two central elements appearing as multiples of the identity operator. Thus, the operator representation of $\mathfrak{g}^{\text{ext}}_{\text{Gal}}$ looks like

$$
\begin{align*}
[\hat{M}, \hat{N}_i] &= i\epsilon_{ij} \hat{N}_j \\
[\hat{M}, \hat{P}_i] &= i\epsilon_{ij} \hat{P}_j \\
[\hat{H}, \hat{P}_i] &= 0 \\
&[\hat{M}, \hat{H}] = 0 \\
[\hat{N}_i, \hat{N}_j] &= i\epsilon_{ij} \lambda \hat{I} \\
[\hat{P}_i, \hat{P}_j] &= 0 \\
[\hat{N}_i, \hat{P}_j] &= i\delta_{ij} m \hat{I} \\
[\hat{N}_i, \hat{H}] &= i\hat{P}_i. 
\end{align*}
$$

(2.4)

Here the operators $\hat{N}_i$ generate velocity shifts. The other operators $\hat{P}_i$, $\hat{M}$, $\hat{H}$, and $\hat{I}$ are just the linear momentum, angular momentum, energy and the identity operators, respectively, acting on $L^2(\mathbb{R}^2, d\mathbf{k})$, the representation space of the PUIRs of $G^{\text{ext}}_{\text{Gal}}$.

Consider next the so-called two-dimensional noncommutative Weyl-Heisenberg group, or the group of noncommutative quantum mechanics. The group generators are the operators $\hat{Q}_i, \hat{P}_j$ and $\hat{I}$, obeying the commutation relations (1.1). The resulting algebra of operators is also referred to as the the noncommutative two-oscillator algebra. Realized on the Hilbert space $L^2(\mathbb{R}^2, d\mathbf{x})$ (coordinate representation) these operators can be brought into the form

$$
\begin{align*}
\tilde{Q}_1 &= x + \frac{i\hbar}{2} \frac{\partial}{\partial y} \\
\tilde{Q}_2 &= y - \frac{i\hbar}{2} \frac{\partial}{\partial x} \\
\tilde{P}_1 &= -i\hbar \frac{\partial}{\partial x} \\
\tilde{P}_2 &= -i\hbar \frac{\partial}{\partial y}.
\end{align*}
$$

(2.5)

If we add to this set the the Hamiltonian (corresponding to a mass $m$)

$$
\tilde{H} = -\frac{\hbar^2}{2m} \nabla^2 = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right),
$$

(2.6)

the angular momentum operator,

$$
\hat{M} = -i\hbar \left( x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right),
$$

(2.7)

and furthermore, define $\tilde{N}_i = m \tilde{Q}_i, i = 1, 2$, then the resulting set of seven operators is easily seen to obey the commutation relations

$$
\begin{align*}
[\tilde{M}, \tilde{N}_i] &= i\hbar \epsilon_{ij} \tilde{N}_j \\
[\tilde{M}, \tilde{P}_i] &= i\hbar \epsilon_{ij} \tilde{P}_j \\
[\tilde{H}, \tilde{P}_i] &= 0 \\
&[\tilde{M}, \tilde{H}] = 0 \\
[\tilde{N}_i, \tilde{N}_j] &= i\epsilon_{ij} m^2 \hat{I} \\
&[\tilde{P}_i, \tilde{P}_j] = 0 \\
[\tilde{N}_i, \tilde{P}_j] &= i\hbar \delta_{ij} m \hat{I} \\
[\tilde{N}_i, \tilde{H}] &= i\hbar \tilde{P}_i.
\end{align*}
$$

(2.8)
Taking \( h = 1 \) and writing \( \lambda = m^2 \vartheta \) this becomes exactly the same set of commutation relations as that in (2.4) of the Lie algebra \( \mathfrak{g}_{\text{Gal}}^{\text{ext}} \) of the extended Galilei group. This tells us that the kinematical group of non-relativistic, noncommutative quantum mechanics is the \((2+1)\)-Galilei \( \mathfrak{g}_{\text{Gal}}^{\text{ext}} \), with two extensions, a fact which has already been noted and exploited in [9].

At this point we note that in terms of \( Q_1, Q_2 \) and \( P_1, P_2 \), the usual quantum mechanical position and momentum operators defined on \( L^2(\mathbb{R}^2, dx \, dy) \), the noncommutative position operators \( \hat{Q}_i \) can be written as

\[
\hat{Q}_1 = Q_1 - \frac{\vartheta}{2h} P_2 \\
\hat{Q}_2 = Q_2 + \frac{\vartheta}{2h} P_1.
\] (2.9)

The above transformation is linear and invertible and may be thought of as giving a non-canonical transformation on the underlying phase space. Since \( \hat{Q}_i = Q_i \leftrightarrow \vartheta = 0 \), the noncommutativity of the two-plane is lost if the parameter \( \vartheta \) is turned off. However, from the group theoretical discussion above we see that the noncommutativity of the two spatial coordinates should not just be looked upon as a result of this non-canonical transformation. Rather, it is also the two-fold central extension of the \((2+1)\)-Galilei group, governing nonrelativistic mechanics, which is responsible for it. The extent to which the two spatial coordinates fail to be commutative is encoded in of the representation parameters of the underlying group, namely, \( \vartheta \). It is noteworthy in this context that had we centrally extended the \((2+1)\)-Galilei group using only the cocycle \( \xi_1^m \) in (2.2) (i.e., set \( \vartheta = 0 \)), we would have just obtained standard quantum mechanics. In this sense we claim that the group underlying noncommutative quantum mechanics, as governed by the commutation relations (2.9), is the doubly centrally extended \((2 + 1)\)-Galilei group. (It is also worth mentioning in this context that the noncommuting position operators \( \hat{Q}_i \), arising from the \((2+1)\)-centrally extended Galilei group, also describe the position of the center of mass of the underlying non-relativistic system.) (see [15]).

### III Quantization using coherent states associated to non-commutative quantum mechanics

In this section we first write down the unitary irreducible representations of the extended Galilei group \( \mathfrak{g}_{\text{Gal}}^{\text{ext}} \). Next we construct coherent states for these representations, which we identify as being the \textit{coherent states of noncommutative quantum mechanics}. We then carry out a quantization of the underlying phase space using these coherent states, obtaining thereby the operators \( Q_i, P_i \) (see (1.1)) of non-commutative quantum mechanics. In the literature other coherent states have been defined for noncommutative quantum
mechanics – see, for example, [14]. These latter coherent states are basically the one-
dimensional projection operators, \([z]|z\rangle\langle z|\), \(z \in \mathbb{C}\), where \(|z\rangle\) is the well-known canonical
coherent state, familiar from quantum mechanics (see, for example, [2]). These coher-
ent states have been shown to satisfy a sort of an “operator resolution of the identity”
and have been used to study localization properties of systems obeying noncommutative
quantum mechanics. By contrast, the coherent states which we obtain (see (3.5) below),
using the representations of the group \(G_{\text{Gal}}^{\text{ext}}\), i.e., the kinematical group of noncommuta-
tive quantum mechanics, satisfy a standard resolution of the identity (see (3.6)). We shall
also discuss the relationship of these coherent states to the canonical coherent states (in
this case arising from the Weyl-Heisenberg group), for two degrees of freedom, and the
fact that these latter can be recovered from the coherent states (3.5) of noncommutative
quantum mechanics in the limit of \(\vartheta = 0\).

III.1 UIRs of the group \(G_{\text{Gal}}^{\text{ext}}\)

The unitary irreducible representations of the extended Galilei group \(G_{\text{Gal}}^{\text{ext}}\) can be ob-
tained from its projective unitary irreducible representations worked out in [13]. We take,
as mentioned earlier, both extension parameters \(m\) and \(\lambda\) to be non-zero. The repre-
sentation space is \(L^2(\mathbb{R}^2, d\mathbf{k})\) (momentum space representation). Denoting the unitary
representation operators by \(\hat{U}_{m,\lambda}\), we have,

\[
(\hat{U}_{m,\lambda}(\theta, \phi, R, b, v, a)\hat{f})(\mathbf{k}) = e^{i(\theta + \phi)} e^{i[a \cdot (\mathbf{k} - \frac{1}{2}m\mathbf{v}) + \frac{b}{2m} \mathbf{k} \cdot \mathbf{k} + \frac{\lambda}{2m} \mathbf{v} \wedge \mathbf{k}s(R)\hat{f}(R^{-1}(\mathbf{k} - m\mathbf{v}))},
\]

(3.1)

for any \(\hat{f} \in L^2(\mathbb{R}^2, d\mathbf{k})\). Here, \(s\) denotes the irreducible representation of the rotation
group in the rest frame (spin). It is useful to Fourier transform the above representation
to get its configuration space version (on \(L^2(\mathbb{R}^2, d\mathbf{x})\)). A straightforward computation,
using Fourier transforms, leads to:

**Lemma III.1.** The unitary irreducible representations of \(G_{\text{Gal}}^{\text{ext}}\) in the (two-dimensional)
configuration space are given by

\[
(U_{m,\lambda}(\theta, \phi, R, b, v, a)f)(\mathbf{x}) = e^{i(\theta + \phi)} e^{im(\mathbf{x} + \frac{1}{2}a) \cdot v} e^{-i\frac{b}{2m} \nabla^2}s(R)f\left(R^{-1}\left(\mathbf{x} + a - \frac{\lambda}{2m}Jv\right)\right),
\]

(3.2)

where \(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\), \(J\) is the 2 \(\times\) 2 skew-symmetric matrix \(J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\) and
\(f \in L^2(\mathbb{R}^2, d\mathbf{x})\).
III.2 Coherent states of the centrally extended (2+1)-Galilei group

It is easy to see from (3.2) that the representation $U_{m,\lambda}$ is not square-integrable. This means that there is no non-zero vector $\eta$ in the representation space for which the function $f_\eta(g) = \langle \eta | U_{m,\lambda}(g)\eta \rangle$ has finite $L^2$-norm, i.e., for all non-zero $\eta \in L^2(\mathbb{R}^2, dx)$,

$$\int_{G_{\text{Gal}}^{\text{ext}}} |f_\eta(g)|^2 \, d\mu(g) = \infty ,$$

$d\mu$ being the Haar measure.

On the other hand, the group composition law (2.3) reflects the fact that the subgroup $H := \Theta \times \Phi \times SO(2) \times T$, with generic group elements $(\theta, \phi, R, b)$, is an abelian subgroup of $G_{\text{Gal}}^{\text{ext}}$. The left coset space $X := G_{\text{Gal}}^{\text{ext}}/H$ is easily seen to be homeomorphic to $\mathbb{R}^4$, corresponding to the left coset decomposition,

$$(\theta, \phi, R, b, v, a) = (0, 0, \mathbb{I}_2, 0, v, a)(\theta, \phi, R, 0, 0) , \quad (\mathbb{I}_2 = 2 \times 2 \text{ unit matrix}).$$

Writing $q$ for $a$ and replacing $v$ by $p := mv$, we identify $X$ with the phase space of the quantum system corresponding to the UIR $\hat{U}_{m,\lambda}$ and write its elements as $(q, p)$. The homogeneous space carries an invariant measure under the natural action of $G_{\text{Gal}}^{\text{ext}}$, which in these coordinates is just the Lebesgue measure $dq \, dp$ on $\mathbb{R}^4$. Also we define a section $\beta : X \mapsto G_{\text{Gal}}^{\text{ext}},$

$$\beta(q, p) = (0, 0, \mathbb{I}_2, 0, \frac{P}{m}, q). \quad (3.3)$$

We show next that the representation $U_{m,\lambda}$ is square-integrable mod $(\beta, H)$ in the sense of [2] and hence construct coherent states on the homogeneous space (phase space) $X$. Let $\chi \in L^2(\mathbb{R}^2, dx)$ be a fixed vector. At a later stage (see Theorem III.2) we shall need to impose a symmetry condition on this vector, but at the moment we leave it arbitrary. For each phase space point $(q, p)$ define the vector,

$$\chi_{q, p} = U_{m,\lambda}(\beta(q, p))\chi . \quad (3.4)$$

so that from (3.2) and (3.3),

$$\chi_{q, p}(x) = e^{i(x \cdot \frac{q}{2})} \, p_x \left( x + q - \frac{\lambda}{2m^2} J p \right). \quad (3.5)$$

**Lemma III.2.** For all $f, g \in L^2(\mathbb{R}^2)$, the vectors $\chi_{q, p}$ satisfy the square integrability condition

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle f | \chi_{q, p} \rangle \langle \chi_{q, p} | g \rangle \, dq \, dp = (2\pi)^2 \|\chi\|^2 \langle f | g \rangle . \quad (3.6)$$

The proof is given in the Appendix. Additionally, in the course of the proof we have also established that the operator integral

$$T = \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\chi_{q, p}\rangle \langle \chi_{q, p}| \, dq \, dp ,$$
in (5.3) converges weakly to $T = 2\pi \|\chi\|^2 I$. Let us now define the vectors
\[
\eta = \frac{1}{\sqrt{2\pi}\|\chi\|} \chi, \quad \text{and} \quad \eta_{q,p} = U(\beta(q,p))\eta, \quad (q,p) \in X.
\] (3.7)

Then, as a consequence of the above lemma, we have proved the following theorem.

**Theorem III.1.** The representation $U_{m,\lambda}$ in (3.2), of the extended Galilei group $G^\text{ext}_{\text{Gal}}$, is square integrable mod $(\beta, H)$ and the vectors $\eta_{q,p}$ in (3.7) form a set of coherent states defined on the homogeneous space $X = G^\text{ext}_{\text{Gal}}/H$, satisfying the resolution of the identity
\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} |\eta_{q,p}\rangle\langle \eta_{q,p}| \, dq \, dp = I,
\] (3.8)
on $L^2(\mathbb{R}^2, dx)$.

Note that
\[
\eta_{q,p}(x) = e^{i(x + \frac{1}{2}q) \cdot p} \eta \left( x + q - \frac{\lambda}{2m^2} Jp \right).
\] (3.9)

We shall consider these coherent states to be the ones associated with non-commutative quantum mechanics. Note that writing $\vartheta = \frac{\lambda}{m^2}$ as before, and letting $\vartheta \to 0$, we recover the standard or canonical coherent states of ordinary quantum mechanics, if $\eta$ is chosen to be the gaussian wave function. Since this also corresponds to setting $\lambda = 0$, it is consistent with constructing the coherent states of the $(2+1)$-Galilei group with one central extension (using only the first of the two cocycles in (2.2), with mass parameter $m$).

Let us emphasize again that the coherent states (3.9) are rooted in the underlying symmetry group of noncommutative quantum mechanics and they are very different from the ones introduced, for example, in [14] and often used in the literature. These latter coherent states are defined as $|z\rangle = |z\rangle\langle z|$, $z \in \mathbb{C}$, where $|z\rangle$ is the usual canonical coherent state of ordinary quantum mechanics. If $\mathcal{H}$ denotes the Hilbert space of a one dimensional quantum system moving on the line, then $|z\rangle$, is an element of the space $B_2(\mathcal{H})$ of Hilbert-Schmidt operators on $\mathcal{H}$ and this space is then taken to be the state space of noncommutative quantum mechanics. The coherent states $|z\rangle$ satisfy a resolution of the identity which is also of a very different nature from (3.8). On $B_2(\mathcal{H})$ the algebra of operators in (2.3) is realized by the operators $\hat{Q}_i, \hat{P}_i$, $i = 1, 2$. These have the actions
\[
\hat{Q}_1 X = QX, \quad \hat{Q}_2 X = \vartheta PX,
\]
\[
\hat{P}_1 X = h[P, X], \quad \hat{P}_2 X = -\frac{h}{\vartheta} (Q, X),
\] (3.10)
on elements $X$ of $B_2(\mathcal{H})$. The $Q$ and $P$ are two operators on $\mathcal{H}$, satisfying the commutation relation $[Q, P] = i\mathcal{H}$. The state space with which we are working here and on which the
operators (2.5) are realized, is \( L^2(\mathbb{R}^2, dx) \). It is not hard to see that the unitary Wigner map, \( \mathcal{W} : \mathcal{B}_2(\mathfrak{g}) \to L^2(\mathbb{R}^2, dx) \), given by

\[
(\mathcal{W}X)(x, y) = \frac{1}{\sqrt{2\pi}} \text{Tr}[e^{-i(xQ + yP)}X],
\]

transforms the set \( \{\hat{Q}_i, \hat{P}_i\} \) to the set \( \{\hat{Q}_i, \hat{P}_i\} \) in (2.5). In other words, the formulation of noncommutative quantum mechanics on these two state spaces are completely equivalent.

### III.3 Coherent state quantization on phase space leading to the noncommutative plane

It has been already noted that we are identifying the homogeneous space \( X = G^\text{ext}_{\text{Gal}}/H \) with the phase space of the system. We shall now carry out a coherent state quantization of functions on this phase space, using the above coherent states of the extended Galilei group. It will turn out that such a quantization of the phase space variables of position and momentum will lead precisely to the operators (2.5).

Recall that given a (sufficiently well behaved) function \( f(\mathbf{q}, \mathbf{p}) \), its quantized version \( \hat{O}_f \), obtained via coherent state quantization, is the operator (on \( L^2(\mathbb{R}^2, dx) \)) given by the prescription (see, for example, [3]),

\[
\hat{O}_f = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(\mathbf{q}, \mathbf{p}) \eta_{\mathbf{q}, \mathbf{p}}(x) \eta_{\mathbf{q}, \mathbf{p}}(x') \, d\mathbf{q} \, d\mathbf{p}
\]

provided this operator is well-defined (again the integral being weakly defined). The operators \( \hat{O}_f \) act on a \( g \in L^2(\mathbb{R}^2, dx) \) in the following manner

\[
(\hat{O}_f g)(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(\mathbf{q}, \mathbf{p}) \eta_{\mathbf{q}, \mathbf{p}}(x) \left[ \int_{\mathbb{R}^2} \overline{\eta_{\mathbf{q}, \mathbf{p}}(x')} g(x') \, dx' \right] \, d\mathbf{q} \, d\mathbf{p}.
\]

If we now take the function \( f \) to be one of the coordinate functions, \( f(\mathbf{q}, \mathbf{p}) = q_i, \ i = 1, 2, \) or one of the momentum functions, \( f(\mathbf{q}, \mathbf{p}) = p_i, \ i = 1, 2, \) then the following theorem shows that the resulting quantized operators \( \hat{O}_{q_i} \) and \( \hat{O}_{p_i} \) are exactly the ones given in [11] for noncommutative quantum mechanics (with \( \hbar = 1 \)) or the ones in (2.5), for the generators of the UIRs of \( G^\text{ext}_{\text{Gal}} \) or of the noncommutative Weyl-Heisenberg group.

**Theorem III.2.** Let \( \eta \) be a smooth function which satisfies the rotational invariance condition, \( \eta(x) = \eta(||x||) \), for all \( x \in \mathbb{R}^2 \). Then, the operators \( \hat{O}_{q_i}, \hat{O}_{p_i}, \ i = 1, 2, \) obtained by a quantization of the phase space functions \( q_i, p_i, \ i = 1, 2, \) using the coherent states \( |\mathbf{x}\rangle \) of the \( (2+1) \)-centrally extended Galilei group, \( G^\text{ext}_{\text{Gal}} \), are given by

\[
(\hat{O}_{q_i} g)(x) = \left( x_1 + \frac{i\lambda}{2m^2} \frac{\partial}{\partial x_2} \right) g(x) \quad (\hat{O}_{q_2} g)(x) = \left( x_2 - \frac{i\lambda}{2m^2} \frac{\partial}{\partial x_1} \right) g(x) \]

\[
(\hat{O}_{p_1} g)(x) = -i \frac{\partial}{\partial x_1} g(x) \quad (\hat{O}_{p_2} g)(x) = -i \frac{\partial}{\partial x_2} g(x),
\]

for \( g \in L^2(\mathbb{R}^2, dx) \), in the domain of these operators.
In (3.14) if we make the substitution \( \vartheta = \frac{\lambda}{m^2} \), we get the operators (2.5) and the commutation relations of non-commutative quantum mechanics (with \( \hbar = 1 \)):

\[
[\hat{O}_{q_1}, \hat{O}_{q_2}] = i\vartheta I, \quad [\hat{O}_{q_i}, \hat{O}_{p_j}] = i\delta_{ij} I, \quad [\hat{O}_{p_i}, \hat{O}_{p_j}] = 0.
\] (3.15)

Moreover, it ought to be emphasized here that the rotational invariance of \( \eta \), in the sense that \( \eta(x) = \eta(\|x\|) \) was essential in deriving (3.14).

Two final remarks, before leaving this section, are in order. First, the operators \( Q_i, P_i \), appearing in (2.9), together with the identity operator \( I \), generate a representation of the Lie algebra of the Weyl-Heisenberg group. Thus, it would seem that the operators \( \hat{Q}_i, \hat{P}_i \) are just a different basis in this same algebra. However, this only appears to be so at the representation level, in which both central elements of the extended Galilei group are mapped to the identity operator. The two sets of operators, \( Q_i, P_i \) and \( \hat{Q}_i, \hat{P}_i \), in fact refer to the Lie algebras of two different groups namely, the \((2 + 1)\)-Galilei groups with one and two extensions, respectively. Moreover, the set of commutation relations (1.1), governing noncommutative quantum mechanics, is not unitary equivalent to that of standard quantum mechanics (where \( \vartheta = 0 \)). In the following section we look at extensions of the Weyl-Heisenberg group which will throw more light on this issue. As a second point, we note that the first commutation relation, between \( \hat{O}_{q_1} \) and \( \hat{O}_{q_2} \) in (3.15) above, also implies that the two dimensional plane \( \mathbb{R}^2 \) becomes noncommutative as a result of quantization.

IV Central extensions of the abelian group of translations in \( \mathbb{R}^4 \) and noncommutative quantum mechanics

We start out with the abelian group of translations \( G_T \) in \( \mathbb{R}^4 \), a generic element of which, denoted \((q, p)\), obeys the group composition rule

\[
(q, p)(q', p') = (q + q', p + p').
\] (4.1)

At the level of the Lie algebra, all the generators commute with each other. In order to arrive at quantum mechanics out of this abelian Lie group, and to go further to obtain noncommutative quantum mechanics, we need to centrally extend this group of translations by some other abelian group, say by \( \mathbb{R} \). In this section we will first discuss the double central extension of \( G_T \) and see that the double central extension by \( \mathbb{R} \) yields the commutation relations (1.1) of noncommutative quantum mechanics. We will, next go a step further and extend \( G_T \) triply by \( \mathbb{R} \). The Lie algebra basis will be found to satisfy the additional commutation relation (1.2) between the momentum operators. We start by recalling some facts about central extensions, following closely the treatment of Bargmann in [4].
Given a connected and simply connected Lie group $G$, the local exponents $\xi$ giving its central extensions are functions $\xi : G \times G \to \mathbb{R}$, obeying the following properties:

\[
\begin{align*}
\xi(g''g', g'') + \xi(g''g', g) &= \xi(g''g', g') + \xi(g', g) \quad (4.2) \\
\xi(g, e) &= 0 = \xi(e, g), \quad \xi(g, g^{-1}) = \xi(g^{-1}, g). \quad (4.3)
\end{align*}
\]

We call the central extension trivial when the corresponding local exponent is simply a coboundary term, in other words, when there exists a continuous function $\zeta : G \to \mathbb{R}$ such that

\[
\xi(g', g) = \xi_{\text{cob}}(g', g) := \zeta(g') + \zeta(g) - \zeta(g'g). \quad (4.4)
\]

Two local exponents $\xi$ and $\xi'$ are equivalent if they differ by a coboundary term, i.e. $\xi'(g', g) = \xi(g', g) + \xi_{\text{cob}}(g', g)$. A local exponent which is itself a coboundary is said to be trivial and the corresponding extension of the group is called a trivial extension. Such an extension is isomorphic to the direct product group $\mathbb{U}(1) \times G$. Exponentiating the inequivalent local exponents yields the $\mathbb{U}(1)$ local factors or the familiar group multipliers, and the set of all such inequivalent multipliers form the well known second cohomology group $H^2(G, \mathbb{U}(1))$ of $G$.

**Theorem IV.1.** The three real valued functions $\xi$, $\xi'$ and $\xi''$ on $G_T \times G_T$ given by

\[
\begin{align*}
\xi((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) &= \frac{1}{2}[q_1p_1' + q_2p_2' - p_1q_1' - p_2q_2'], \quad (4.5) \\
\xi'((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) &= \frac{1}{2}[p_1p_2' - p_2p_1'], \quad (4.6) \\
\xi''((q_1, q_2, p_1, p_2), (q_1', q_2', p_1', p_2')) &= \frac{1}{2}[q_1q_2' - q_2q_1'], \quad (4.7)
\end{align*}
\]

are inequivalent local exponents for the group, $G_T$, of translations in $\mathbb{R}^4$ in the sense of (4.4).

The proof is given in the Appendix.

**IV.1 Double central extension of $G_T$**

In this section, we study the doubly (centrally) extended group $\overline{G_T}$ where the extension is achieved by means of the two multipliers $\xi$ and $\xi'$ enumerated in Theorem [IV.1]. The group composition rule for the extended group $\overline{G_T}$ reads

\[
(\theta, \phi, q, p)(\theta', \phi', q', p') = (\theta + \theta' + \alpha\langle q, p' \rangle - \langle q', p \rangle, \phi + \phi' + \beta[q \wedge p', q + q', p + p']), \quad (4.8)
\]

where $q = (q_1, q_2)$ and $p = (p_1, p_2)$. Also, $\langle ., . \rangle$ and $\wedge$ are defined as $\langle q, p \rangle := q_1p_1 + q_2p_2$ and $q \wedge p := q_1p_2 - q_2p_1$ respectively.
A matrix representation for the group $G_T$ obeying the group multiplication rule (4.8) is given by the following $7 \times 7$ upper triangular matrix

\[
\begin{pmatrix}
1 & 0 & -\frac{\alpha}{2}p_1 & -\frac{\alpha}{2}p_2 & \frac{\beta}{2}q_1 & \frac{\beta}{2}q_2 & \theta \\
0 & 1 & 0 & 0 & -\frac{\beta}{2}p_2 & \frac{\beta}{2}p_1 & \phi \\
0 & 0 & 1 & 0 & 0 & 0 & q_1 \\
0 & 0 & 0 & 1 & 0 & 0 & p_1 \\
0 & 0 & 0 & 0 & 1 & 0 & p_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\] (4.9)

Let us denote the generators of the Lie group $G_T$, or equivalently the basis of the associated Lie algebra, $\mathfrak{g}_T$ by $\Theta, \Phi, Q_1, Q_2, P_1$ and $P_2$. These generate the one-parameter subgroups corresponding to the group parameters $\theta, \phi, p_1, p_2, q_1$ and $q_2$, respectively. The bilinear Lie brackets between the basis elements of $\mathfrak{g}_T$ are given by

\[
[P_i, Q_j] = \alpha \delta_{i,j} \Theta, \quad [Q_1, Q_2] = \beta \Phi, \quad [P_1, P_2] = 0, \quad [P_i, \Theta] = 0, \\
[Q_i, \Theta] = 0, \quad [P_i, \Phi] = 0, \quad [Q_i, \Phi] = 0, \quad [\Theta, \Phi] = 0, \quad i, j = 1, 2.
\] (4.10)

It is easily seen from (4.10) that $\Theta$ and $\Phi$ form the center of the algebra $\mathfrak{g}_T$. It is also noteworthy that, unlike in standard quantum mechanics, the two generators of space translation, $Q_1, Q_2$, no longer commute, the noncommutativity of these two generators being controlled by the central extension parameter $\beta$. It is in this context that it is reasonable to call the Lie group $G_T$ the noncommutative Weyl-Heisenberg group and the corresponding Lie algebra the noncommutative Weyl-Heisenberg algebra.

We now proceed to find a unitary irreducible representation of $G_T$. From the matrix representation (4.9) we see that $G_T$ is a nilpotent Lie group. Hence, it is convenient to apply the orbit method of Kirillov (see [10]) for finding the unitary dual of the group.

Switching to a slightly different notation, for computational convenience, we replace the group parameters $p_1, p_2, q_1, q_2, \theta$ and $\phi$ by $a_1, a_2, a_3, a_4, a_5$ and $a_6$, respectively, then a generic group element $g(a_1, a_2, a_3, a_4, a_5, a_6)$ is represented by the following matrix (compare with (4.9)):

\[
g(a_1, a_2, a_3, a_4, a_5, a_6) =
\begin{pmatrix}
1 & 0 & -\frac{\alpha}{2}a_1 & -\frac{\alpha}{2}a_2 & \frac{\beta}{2}a_3 & \frac{\beta}{2}a_4 & a_5 \\
0 & 1 & 0 & 0 & -\frac{\beta}{2}a_2 & \frac{\beta}{2}a_1 & a_6 \\
0 & 0 & 1 & 0 & 0 & 0 & x_3 \\
0 & 0 & 0 & 1 & 0 & 0 & a_4 \\
0 & 0 & 0 & 0 & 1 & 0 & a_1 \\
0 & 0 & 0 & 0 & 0 & 1 & a_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\] (4.11)
If $X_1, X_2, X_3, X_4, X_5$ and $X_6$ stand for the respective group generators, a generic Lie algebra element can be written as $X = x^1 X_1 + x^2 X_2 + x^3 X_3 + x^4 X_4 + x^5 X_5 + x^6 X_6$. In matrix notation, $X$ can be read off as

$$X = \begin{bmatrix}
0 & 0 & -\frac{\alpha}{2} x^1 & -\frac{\alpha}{2} x^2 & -\frac{\alpha}{2} x^3 & \frac{\alpha}{2} x^4 & x^5 \\
0 & 0 & 0 & -\frac{\beta}{2} x^2 & \frac{\beta}{2} x^1 & x^6 \\
0 & 0 & 0 & 0 & 0 & 0 & x^3 \\
0 & 0 & 0 & 0 & 0 & 0 & x^4 \\
0 & 0 & 0 & 0 & 0 & 0 & x^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
X_5 & X_6 & X_3 & X_4 & X_1 & X_2 & 0
\end{bmatrix}. \tag{4.12}$$

An element $F \in (\mathcal{G}_T)^*$ with coordinates $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ is now represented by the following $7 \times 7$ lower triangular matrix

$$F = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
X_5 & X_6 & X_3 & X_4 & X_1 & X_2 & 0
\end{bmatrix}, \tag{4.13}$$

with the dual pairing being given as $\langle F, X \rangle = \text{tr}(FX) = \sum_{i=1}^{6} x^i X_i$. Hence the coadjoint action $K$ of the underlying group $\mathcal{G}_T$ on the dual Lie algebra $(\mathcal{G}_T)^*$ can be computed as

$$g(a_1, a_2, a_3, a_4, a_5, a_6) F g(a_1, a_2, a_3, a_4, a_5, a_6)^{-1} = \begin{bmatrix}
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
* & * & * & * & * & * & * \\
X'_5 & X'_6 & X'_3 & X'_4 & X'_1 & X'_2 & *
\end{bmatrix}, \tag{4.14}$$

with

$$X'_1 = X_1 - \frac{\alpha}{2} a_3 X_5 + \frac{\beta}{2} a_2 X_6, \quad X'_2 = X_2 - \frac{\alpha}{2} a_4 X_5 - \frac{\beta}{2} a_1 X_6, \quad X'_3 = X_3 + \frac{\alpha}{2} a_1 X_5, \quad X'_4 = X_4 + \frac{\alpha}{2} a_2 X_5, \quad X'_5 = X_5, \quad X'_6 = X_6. \tag{4.15}$$
The required coadjoint action $K$ of the group on the dual algebra is therefore given by

$$Kg(a_1, a_2, a_3, a_4, a_5, a_6)(X_1, X_2, X_3, X_4, X_5, X_6) = (X_1 - \frac{\alpha}{2} a_3 X_5 + \frac{\beta}{2} a_2 X_6, X_2 - \frac{\alpha}{2} a_4 X_5 - \frac{\beta}{2} a_1 X_6, X_3 + \frac{\alpha}{2} a_1 X_5,$$

$$X_4 + \frac{\alpha}{2} a_2 X_5, X_5, X_6).$$  \hspace{1cm} (4.16)

The entries denoted by *'s in (4.14) are some nonzero values that we are not interested in. From (4.16) one observes that the two coordinates $X_5$ and $X_6$ remain unchanged under the coadjoint action. This is expected since they correspond to the center of the underlying algebra. The only two polynomial invariants in this case are just $P(F) = X_5$ and $Q(F) = X_6$. The coadjoint orbits are given by the set $S_{\rho, \sigma}$, for some fixed real numbers $\rho, \sigma$, with

$$S_{\rho, \sigma} = \{ F \in \mathcal{G}_T | P(F) = \rho, Q(F) = \sigma \}. \hspace{1cm} (4.17)$$

Now, the first four coordinates of the vector on the right hand side of (4.16) can be made zero by a suitable choice of the group parameters $a_1, a_2, a_3, a_4, a_5$ and $a_6$. Therefore, for nonzero values of $\rho$ and $\sigma$ in (4.17), the vector $(0, 0, 0, 0, \rho, \sigma)$ will lie in a coadjoint orbit $S_{\rho, \sigma}$ of codimension 2. Since the dual algebra space is six dimensional, i.e., the coadjoint orbit in question is 4 dimensional and it passes through the point $(0, 0, 0, 0, \rho, \sigma)$.

We next have to find the subalgebra, of correct dimension, subordinate to $F$ (see (4.13)). If we work with this appropriate polarizing subalgebra and solve the master equation (see [10]), the representation we end up with will be irreducible and unitary. The correct dimension of the polarizing subalgebra in this context turns out to be $2 + 6 = 4$.

The maximal abelian subalgebra $\mathfrak{h}$ of the underlying algebra $\overline{\mathcal{G}_T}$ serves as the appropriate polarizing subalgebra in this case, i.e. $\mathfrak{h}$ is the maximal subalgebra with $F|_{[\mathfrak{h}, \mathfrak{h}]} = 0$. A generic element of $\mathfrak{h}$ can be obtained from (4.12) just by putting $x^1 = x^2 = 0$ in there. A generic element of the corresponding abelian subgroup $H$ can be represented by the following matrix

$$h(\theta, \phi, q) = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{\alpha}{2} q_1 & \frac{\alpha}{2} q_2 & \theta \\ 0 & 1 & 0 & 0 & 0 & 0 & \phi \\ 0 & 0 & 1 & 0 & 0 & 0 & q_1 \\ 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (4.18)

We now choose a section $\delta : S = H/\overline{\mathcal{G}_T} \to \overline{\mathcal{G}_T}$ with $\delta(s) = \delta(s_1, s_2)$ being given by the
following $7 \times 7$ matrix

$$
\delta(s_1, s_2) = \begin{bmatrix}
1 & 0 & -\frac{\alpha}{2} s_1 & -\frac{\beta}{2} s_2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -\frac{\beta}{2} s_2 & \frac{\beta}{2} s_1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & s_1 & 0 \\
0 & 0 & 0 & 0 & 1 & s_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\tag{4.19}
$$

With all the relevant matrices at our disposal, we move on to solving the master equation, which in this case involves solving the matrix equation

$$
\begin{bmatrix}
1 & 0 & -\frac{\alpha}{2} (p_1 + s_1) & -\frac{\alpha}{2} (p_2 + s_2) & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -\frac{\alpha}{2} (p_2 + s_2) & \frac{\beta}{2} (p_1 + s_1) & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -\frac{\alpha}{2} (p_1 + s_1) & -\frac{\alpha}{2} (p_2 + s_2) & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -\frac{\alpha}{2} (p_2 + s_2) & \frac{\beta}{2} (p_1 + s_1) & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\tag{4.20}
$$

for the unknowns $A, B, C, D, E$ and $F$. Comparing (4.20) with (4.21), one gets

$$
A = q_1, \quad B = q_2, \quad E = p_1 + s_1, \quad F = p_2 + s_2,
\tag{4.22}
$$

$$
C = \theta - \alpha \langle q, s + \frac{1}{2} p \rangle, \quad D = \phi - \frac{\beta}{2} p \wedge s.
$$

We recall that the coadjoint orbit vector, associated to which we found the polarizing algebra, was of the form $(0, 0, 0, 0, \rho, \sigma)$. In view of (4.22), we therefore have the following theorem

**Theorem IV.2.** The noncommutative Weyl-Heisenberg group $\mathcal{G_T}$ admits a unitary irreducible representation realized on $L^2(\mathbb{R}^2, ds)$ by the operators $U(\theta, \phi, q, p)$:

$$
(U(\theta, \phi, q, p)f)(s) = \exp i \left( \theta + \phi - \alpha \langle q, s + \frac{1}{2} p \rangle - \frac{\beta}{2} p \wedge s \right) f(s + p),
\tag{4.23}
$$

where $f \in L^2(\mathbb{R}^2, ds)$. 

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From the one-parameter unitary groups $U(\theta, 0, 0, 0, 0)$, $U(0, 0, q_1, 0, 0, 0)$, etc, we obtain their self-adjoint generators (on $L^2(\mathbb{R}^2, ds)$), $\hat{\Theta}$, $\hat{\Phi}$, $\hat{P}_1$, $\hat{P}_2$, $\hat{Q}_1$ and $\hat{Q}_2$, using the general formula
\[ \hat{X}_\phi = i \frac{dU(\phi)}{d\phi} \bigg|_{\phi=0}. \]
Thus, we have the following Hilbert space representation of the noncentral group generators
\[ \hat{P}_1 = \alpha s_1, \quad \hat{Q}_1 = \frac{\beta}{2} s_2 + i \frac{\partial}{\partial s_1}, \]
\[ \hat{P}_2 = \alpha s_2, \quad \hat{Q}_2 = -\frac{\beta}{2} s_1 + i \frac{\partial}{\partial s_2}, \] (4.24)
while the two central generators $\hat{\Theta}$ and $\hat{\Phi}$ are both mapped to the Identity operator $\mathbb{I}_H$ of $\mathfrak{h} = L^2(\mathbb{R}^2, ds)$. An inverse Fourier transformation leads to the expressions, (on the coordinate Hilbert space $L^2(\mathbb{R}^2, dx)$)
\[ \hat{P}_1 = -i\alpha \frac{\partial}{\partial x}, \quad \hat{P}_2 = -i\alpha \frac{\partial}{\partial y}, \]
\[ \hat{Q}_1 = x - \frac{i\beta}{2} \frac{\partial}{\partial y}, \quad \hat{Q}_2 = y + \frac{i\beta}{2} \frac{\partial}{\partial x}. \] (4.25)
which coincide with (2.5) if we identify $\alpha$ with $\hbar$ and $-\beta$ with $\vartheta$.

The commutation relations are now
\[ [\hat{Q}_1, \hat{P}_2] = i\alpha \delta_{ij} \mathbb{I}_H, \quad [\hat{Q}_1, \hat{Q}_2] = -i\beta \mathbb{I}_H, \quad [\hat{P}_1, \hat{P}_2] = 0. \] (4.26)

If we now set $\alpha = \hbar$ and $-\beta = \vartheta$, we again retrieve the commutation relations (1.1) of noncommutative quantum mechanics. This means, that as in the case of the Galilei group, an additional central extension of the Weyl-Heisenberg group leads to non-commutative quantum mechanics.

IV.2 Triple central extension of $G_T$

In this section we study the triple central extension of $G_T$ by $\mathbb{R}$ and compute a unitary irreducible representation of the extended group $G_T$. We will make use of all the three local exponents $\xi$, $\xi'$ and $\xi''$ enumerated in Theorem IV.1 to do this triple extension. The group composition rule for the resulting triply extended Lie group $G_T$ then reads
\[ (\theta, \phi, \psi, q, p)(\theta', \phi', \psi', q', p') \]
\[ = (\theta + \theta' + \frac{\alpha}{2} |q, p'\rangle - |q', p\rangle, \phi + \phi' + \frac{\beta}{2} [p, q], \psi + \psi' + \frac{\gamma}{2} [q, q'] \]
\[ , q + q', p + p'). \] (4.27)
The matrix representation of $G_T$, consistent with the above group law, is then seen to be

$$(\theta, \phi, \psi, q, p)_{\alpha, \beta, \gamma} = \begin{bmatrix}
1 & 0 & 0 & -\frac{\alpha}{2} p_1 & -\frac{\alpha}{2} p_2 & \frac{\alpha}{2} q_1 & \frac{\alpha}{2} q_2 & \theta \\
0 & 1 & 0 & 0 & -\frac{\beta}{2} p_2 & \frac{\beta}{2} p_1 & \phi \\
0 & 0 & 1 & -\frac{\gamma}{2} q_2 & \frac{\gamma}{2} q_1 & 0 & 0 & \psi \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & q_1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & p_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}. \tag{4.28}$$

Let us denote the Lie algebra of $G_T$ by $G_T$. Denoting the basis elements of $G_T$ by $\Theta, \Phi, \Psi, Q_1, Q_2, P_1$ and $P_2$, corresponding to the group parameters $\theta, \phi, \psi, p_1, p_2, q_1$ and $q_2$, respectively, we have the following Lie bracket relations between them

$$[P_i, Q_j] = \alpha \delta_{i,j} \Theta, \quad [Q_1, Q_2] = \beta \Phi, \quad [P_1, P_2] = \gamma \Psi, \quad [P_1, \Theta] = 0,$$
$$[Q_i, \Theta] = 0, \quad [P_1, \Phi] = 0, \quad [Q_i, \Phi] = 0, \quad [P_1, \Psi] = 0,$$
$$[Q_i, \Psi] = 0, \quad [\Theta, \Phi] = 0, \quad [\Phi, \Psi] = 0, \quad [\Theta, \Psi] = 0, \quad i,j = 1, 2. \tag{4.29}$$

In addition to the two central elements $\Theta$ and $\Phi$ appearing in the double extension case (see (4.10)), we have a third central element $\Psi$ in (4.29), which makes the two generators $P_1$ and $P_2$ noncommutative as well, with the noncommutativity being controlled by the extension parameter $\gamma$. We shall call this centrally extended Lie group $G_T$ the *triply extended group of translations* and the corresponding Lie algebra $G_T$ the *triply extended algebra of translations*.

It remains now to find a unitary irreducible representation of the group $G_T$. In doing so we will be following exactly the same course as for the UIR of the $G_T$ in Section [IV.1]. Since $G_T$ is also a nilpotent Lie group, (see (4.28)), we again apply the orbit method of Kirillov.

We again change notations and replace the group parameters $p_1, p_2, q_1, q_2, \theta, \phi$ and $\psi$ by $a_1, a_2, a_3, a_4, a_5, a_6$ and $a_7$, respectively. Then, a generic group element has the matrix representation

$$g(a_1, a_2, a_3, a_4, a_5, a_6, a_7) = \begin{bmatrix}
1 & 0 & 0 & -\frac{\alpha}{2} a_1 & -\frac{\alpha}{2} a_2 & \frac{\alpha}{2} a_3 & \frac{\alpha}{2} a_4 & a_5 \\
0 & 1 & 0 & 0 & -\frac{\beta}{2} a_2 & \frac{\beta}{2} a_1 & a_6 \\
0 & 0 & 1 & -\frac{\gamma}{2} a_4 & \frac{\gamma}{2} a_3 & 0 & 0 & a_7 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & a_3 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & a_4 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & a_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & a_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 
\end{bmatrix}. \tag{4.30}$$
Denoting by \( X_1, X_2, X_3, X_4, X_5, X_6 \) and \( X_7 \), the respective group generators, and writing a generic Lie algebra element as \( X = x^1 X_1 + x^2 X_2 + x^3 X_3 + x^4 X_4 + x^5 X_5 + x^6 X_6 + x^7 X_7 \), we have the matrix

\[
X = \begin{bmatrix}
0 & 0 & 0 & -\frac{4}{3} x^1 & -\frac{4}{3} x^3 & \frac{4}{3} x^4 & x^5 \\
0 & 0 & 0 & 0 & 0 & -\frac{4}{3} x^2 & x^6 \\
0 & 0 & -\gamma x^4 & \gamma x^3 & 0 & 0 & x^7 \\
0 & 0 & 0 & 0 & 0 & 0 & x^3 \\
0 & 0 & 0 & 0 & 0 & 0 & x^1 \\
0 & 0 & 0 & 0 & 0 & 0 & x^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]  
(4.31)

We represent an element \( F \in (G_T)^* \) with the following \( 8 \times 8 \) lower triangular matrix

\[
F = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{a} X_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{2}{a} X_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
X_5 & X_6 & X_7 & X_3 & X_4 & 0 & 0 & 0
\end{bmatrix},
\]  
(4.32)

where the dual pairing is given by \( \langle F, X \rangle = \text{tr}(FX) = \sum_{i=1}^{7} x^i X_i \). Therefore, the coadjoint action of the underlying Lie group \( G_T \) on the corresponding dual Lie algebra \( (G_T)^* \) follows from the following computation

\[
g(a_1, a_2, a_3, a_4, a_5, a_6, a_7) F g(a_1, a_2, a_3, a_4, a_5, a_6, a_7)^{-1} = \begin{bmatrix}
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
-\frac{2}{a} X'_1 & * & * & * & * & * & * & * \\
-\frac{2}{a} X'_2 & * & * & * & * & * & * & * \\
* & * & * & * & * & * & * & * \\
X'_5 & X'_6 & X'_7 & X'_3 & X'_4 & * & * & *
\end{bmatrix},
\]  
(4.33)
The required coadjoint action of the group on the dual algebra is therefore given by

\[ X_1' = -\frac{2}{\alpha} X_1 + a_3 X_5, \quad X_2' = -\frac{2}{\alpha} X_2 + a_4 X_5, \quad X_3' = \frac{\alpha}{2} a_1 X_5 + \frac{\gamma}{2} a_4 X_7 + X_3, \]

\[ X_4' = \frac{\alpha}{2} a_2 X_5 - \frac{\gamma}{2} a_3 X_7 + X_4, \quad X_5' = X_5, \quad X_6' = X_6, \quad X_7' = X_7. \]  

(4.34)

The required coadjoint action of the group on the dual algebra is therefore given by

\[
Kg(a_1, a_2, a_3, a_4, a_5, a_6, a_7)(X_1, X_2, X_3, X_4, X_5, X_6, X_7) \\
= (-\frac{2}{\alpha} X_1 + a_3 X_5, -\frac{2}{\alpha} X_2 + a_4 X_5, \frac{\alpha}{2} a_1 X_5 + \frac{\gamma}{2} a_4 X_7 + X_3 \\
\quad, \frac{\alpha}{2} a_2 X_5 - \frac{\gamma}{2} a_3 X_7 + X_4, X_5, X_6, X_7).
\]  

(4.35)

The nonzero entries denoted by *’s in (4.33) are of no interest to us. From (4.35), one observes that the three dual algebra coordinates \(X_5, X_6\) and \(X_7\) remain unaltered under the coadjoint action of the underlying group element, coming as they do from the center of the Lie algebra. We therefore have three polynomial invariants in our theory given by \(P(F) = X_5, Q(F) = X_6\) and \(R(F) = X_7\). The coadjoint orbits in this case are given by the sets \(S_{\rho,\sigma,\tau}\) with

\[
S_{\rho,\sigma,\tau} = \{ F \in (\mathfrak{g}_T)^* \mid P(F) = \rho, Q(F) = \sigma, R(F) = \tau \}. \label{eq:4.36}
\]

It is also obvious from (4.35) that by choosing \(a_1, a_2, a_3\) and \(a_4\) in a suitable manner, we can make all of \(X_1', X_2', X_3'\) and \(X_4'\) vanishing at the same time. Therefore, for nonzero values of \(\rho, \sigma\) and \(\tau\), the vector \((0, 0, 0, \rho, \sigma, \tau)\) will always lie in the coadjoint orbit \(S_{\rho,\sigma,\tau}\) of codimension 3. In other words, the underlying coadjoint orbit \(S_{\rho,\sigma,\tau}\) turns out to be 4 dimensional which passes through the point \((0, 0, 0, \rho, \sigma, \tau)\) of the dual algebra space.

We now have to find the maximal subalgebra subordinate to \(F\) given by (4.32). This maximal subalgebra or the polarizing subalgebra turns out to be of the correct dimension \(\frac{3q+7}{2} = 5\) and hence, the representation for \(\mathfrak{g}_T\) that we end up with using the orbit method will be irreducible and unitary. As in the case of \(\mathfrak{g}_T\), the maximal abelian subalgebra \(\mathfrak{h}\) of the Lie algebra \(\mathfrak{g}_T\) serves as the polarizing subalgebra. A generic element of the corresponding abelian subgroup \(H\) can be represented by the following \(8 \times 8\) matrix

\[
h(\theta, \phi, \psi, p_1, q_2) = \begin{bmatrix}
1 & 0 & 0 & -\frac{\alpha}{2} p_1 & 0 & 0 & \frac{\alpha}{2} q_2 & \theta \\
0 & 1 & 0 & 0 & 0 & \frac{\beta}{2} p_1 & \phi \\
0 & 0 & 1 & \frac{-\gamma}{2} q_2 & 0 & 0 & 0 & \psi \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & q_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & p_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. \label{eq:4.37}
\]
Then the section $\delta : H_{\overline{G_T}} \to \overline{G_T}$ will be represented by the following matrix

$$
\delta(r_1, s_2) = \begin{bmatrix}
1 & 0 & 0 & -\frac{\alpha}{2} s_2 & \frac{\beta}{2} r_1 & 0 & 0 \\
0 & 1 & 0 & 0 & -\frac{\beta}{2} s_2 & 0 & 0 \\
0 & 0 & 1 & 0 & \frac{\beta}{2} r_1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & s_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}.
$$

(4.38)

Thus, we again have to solve the master equation,

$$
\begin{align*}
\begin{bmatrix}
& & & & & & \\
1 & 0 & 0 & -\frac{\alpha}{2} s_2 & \frac{\beta}{2} r_1 & 0 & 0 \\
0 & 1 & 0 & 0 & -\frac{\beta}{2} s_2 & 0 & 0 \\
0 & 0 & 1 & 0 & \frac{\beta}{2} r_1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & s_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
& \begin{bmatrix}
\frac{\alpha}{2} s_2 & \frac{\beta}{2} r_1 & 0 & 0 \\
\frac{\beta}{2} s_2 & 0 & 0 & 0 \\
\frac{\beta}{2} r_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\alpha}{2} s_2 & \frac{\beta}{2} r_1 & 0 & 0 \\
\frac{\beta}{2} s_2 & 0 & 0 & 0 \\
\frac{\beta}{2} r_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\alpha}{2} s_2 & \frac{\beta}{2} r_1 & 0 & 0 \\
\frac{\beta}{2} s_2 & 0 & 0 & 0 \\
\frac{\beta}{2} r_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{align*}
= \begin{bmatrix}
1 & 0 & 0 & -\frac{\alpha}{2} s_2 & \frac{\beta}{2} r_1 & 0 & 0 \\
0 & 1 & 0 & 0 & -\frac{\beta}{2} s_2 & 0 & 0 \\
0 & 0 & 1 & 0 & \frac{\beta}{2} r_1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & s_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
$$

(4.39)

$$
\begin{align*}
& \begin{bmatrix}
1 & 0 & 0 & -\frac{\alpha}{2} s_2 & \frac{\beta}{2} r_1 & 0 & 0 \\
0 & 1 & 0 & 0 & -\frac{\beta}{2} s_2 & 0 & 0 \\
0 & 0 & 1 & 0 & \frac{\beta}{2} r_1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & s_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\frac{\alpha}{2} s_2 & \frac{\beta}{2} r_1 & 0 & 0 \\
\frac{\beta}{2} s_2 & 0 & 0 & 0 \\
\frac{\beta}{2} r_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\alpha}{2} s_2 & \frac{\beta}{2} r_1 & 0 & 0 \\
\frac{\beta}{2} s_2 & 0 & 0 & 0 \\
\frac{\beta}{2} r_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\alpha}{2} s_2 & \frac{\beta}{2} r_1 & 0 & 0 \\
\frac{\beta}{2} s_2 & 0 & 0 & 0 \\
\frac{\beta}{2} r_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\alpha}{2} s_2 & \frac{\beta}{2} r_1 & 0 & 0 \\
\frac{\beta}{2} s_2 & 0 & 0 & 0 \\
\frac{\beta}{2} r_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{align*}
= \begin{bmatrix}
1 & 0 & 0 & -\frac{\alpha}{2} s_2 & \frac{\beta}{2} r_1 & 0 & 0 \\
0 & 1 & 0 & 0 & -\frac{\beta}{2} s_2 & 0 & 0 \\
0 & 0 & 1 & 0 & \frac{\beta}{2} r_1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & s_2 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
$$

(4.40)

The unknowns $A, B, C, D, E, F$ and $G$ can easily be computed by comparing (4.39) with (4.40). We get

$$
A = p_1, \quad B = q_2, \quad G = r_1 + q_1, \quad F = s_2 + p_2,
$$

$$
C = \theta - \alpha q_2 s_2 + \alpha p_1 r_1 + \frac{\alpha}{2} q_1 p_1 - \frac{\alpha}{2} q_2 p_2,
$$

$$
D = \phi - \beta p_1 s_2 - \frac{\beta}{2} p_1 p_2, \quad E = \psi + \gamma q_2 r_1 + \frac{\gamma}{2} q_1 q_2.
$$

(4.41)

Now, the dual algebra vector lying in the underlying four dimensional coadjoint orbit was found to be $(0, 0, 0, 0, \rho, \sigma, \tau)$. In light of (4.41), we therefore have the following theorem
Theorem IV.3. The triply extended group of translations $\hat{G}_T$ admits a unitary irreducible representation realized on $L^2(\mathbb{R}^2)$. The explicit form of the representation is given by

$$
(U(\theta, \phi, \psi, q_1, q_2, p_1, p_2)f)(r_1, s_2) = e^{i(\theta - \alpha q_2 s_2 + \alpha p_1 r_1 + \frac{\theta}{2} q_1 p_1 - \frac{\phi}{2} q_2 p_2)} e^{i(\phi - \beta p_1 s_2 - \frac{\theta}{2} p_1 p_2)} \times e^{i(\psi + \gamma q_1 + \frac{\gamma}{2} q_2) f(r_1 + q_1, s_2 + p_2)},
$$

(4.42)

where $f \in L^2(\mathbb{R}^2, dr_1 ds_2)$.

Now, let us take the Fourier transform of (4.42) with respect to the first coordinate $r_1$ and call the transformed coordinate $s_1$. The noncentral generators of $\hat{G}_T$ can be represented by self adjoint operators defined on $L^2(\mathbb{R}^2, ds)$ in the following manner

$$
\hat{P}_1 = -s_1, \quad \hat{Q}_1 = \beta s_2 - i\alpha \frac{\partial}{\partial s_1},
\hat{P}_2 = \alpha s_2 - i\gamma \frac{\partial}{\partial s_1}, \quad \hat{Q}_2 = i \frac{\partial}{\partial s_2},
$$

(4.43)

while the three central elements $\Theta, \Phi$ and $\Psi$ of the corresponding Lie algebra $\hat{G}_T$ are all mapped to the identity operator $I_H$ of the underlying Hilbert space $\mathcal{H} = L^2(\mathbb{R}^2, ds)$. The corresponding commutation relations now read

$$
[\hat{Q}_i, \hat{P}_j] = i\alpha \delta_{i,j} I_H, \quad [\hat{Q}_1, \hat{Q}_2] = -i\beta I_H, \quad [\hat{P}_1, \hat{P}_2] = -i\gamma I_H.
$$

(4.44)

Once again, if we write $\alpha = \hbar, -\beta = \vartheta$ and replace $\gamma$ by $-\gamma$ we recover (1.1) together with (1.2), the additional central extension making the two momentum operators noncommuting.

V Conclusion and outlook

We have derived the commutation relations between the position and momentum operators of noncommutative quantum mechanics by three different means: using the appropriate unitary irreducible representations of the centrally extended (2+1)-Galilei group $G^\text{ext}_{\text{Gal}}$, of the doubly extended group $\hat{G}_T$, of translations of $\mathbb{R}^4$, and by a coherent state quantization of the classical phase space variables of position and momentum, using the coherent states of $G^\text{ext}_{\text{Gal}}$. It is not hard to see, from the expressions for the unitary representations (3.1) and (4.23), that the same commutation relations could also be obtained by a coherent state quantization, using the coherent states of $\hat{G}_T$ (which could be similarly constructed). There is, as usual a positive operator valued (POV) measure naturally associated to the coherent states (3.7). Indeed, for any measurable set $\Delta$ of $\mathbb{R}^2$ (phase
space), we can associate the positive operator

\[ a(\Delta) = \int_{\Delta} |\eta_{q,p}\rangle\langle\eta_{q,p}| \, dq \, dp \, . \]

These define localization operators on phase space, whose marginals in \( q \) and \( p \) should then give localization operators in configuration and momentum spaces, respectively. For the canonical coherent states and standard quantum mechanics, such operators have been studied extensively, in e.g., [1, 7]. There, one understands these localization operators in an extended or unsharp sense. It would be interesting to do a similar study for the present case.

VI Appendix

In this Appendix we collect the proofs of some of the results quoted in the paper.

Proof of Lemma III.2

We start out by taking two compactly supported and infinitely differentiable functions \( f, g \in L^2(\mathbb{R}^2, dx) \). Then,

\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle f | \chi_{q,p} \rangle \langle \chi_{q,p} | g \rangle \, dq \, dp \\
= \int_{\mathbb{R}^2 \times \mathbb{R}^2} dq \, dp \left[ \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i(x-x') \cdot p} \chi(x + q - \frac{\lambda}{2m^2} Jp) \chi(x' + q - \frac{\lambda}{2m^2} Jp) \chi(x + q') \chi(x + q') \right] \\
= (2\pi)^2 \int_{\mathbb{R}^2} dq' \left[ \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i(x-x') \cdot p} \chi(x + q') \chi(x + q') g(x') f(x) \, dx \, dx' \right] \\
= (2\pi)^2 \|\chi\|^2 \langle f | g \rangle ,
\]

the change in the order of integration and the introduction of the delta measure being easily justified in view of the compact supports and smoothness property of the functions.
f and g. Thus, introducing the formal operator

$$T = \int_{\mathbb{R}^2 \times \mathbb{R}^2} |\chi_{q,p}\rangle \langle \chi_{q,p}| \, dq \, dp \, , \tag{6.3}$$

we see that for functions f, g of the chosen type,

$$\langle f | T g \rangle = 2\pi \|\chi\|^2 \langle f | g \rangle \, ,$$

I being the identity operator on $L^2(\mathbb{R}^2, dx)$. But since the compactly supported and infinitely differentiable functions are dense in $L^2(\mathbb{R}^2, dx)$, we use the continuity of the scalar product to extend the above equality to arbitrary pairs of functions f, g in $L^2(\mathbb{R}^2, dx)$, thus proving the lemma. \qed

**Proof of Theorem III.2.**

We only work out the derivation of the first of the above equations, the others being obtained in similar ways. By (3.9) and (3.13)

$$(\hat{O}_q, g)(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} q_1 \left[ \int_{\mathbb{R}^2} e^{i(x-x') \cdot p} \eta \left( x + q - \frac{\lambda}{2m^2} J^p \right) \times \eta \left( x' + q - \frac{\lambda}{2m^2} J^p \right) g(x') \, dx' \right] dq \, dp \, . \tag{6.4}$$

Making the change of variables $q - \frac{\lambda}{2m^2} J^p = q'$, and noting the form of the skew-symmetric matrix J from Lemma (III.1), we have

$$q'_1 = q_1 + \frac{\lambda}{2m^2} p_2, \quad q'_2 = q_2 - \frac{\lambda}{2m^2} p_1 \, ,$$

using which (6.4) becomes

$$(\hat{O}_q, g)(x) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} q'_1 \left[ \int_{\mathbb{R}^2} e^{i(x-x') \cdot p} \eta(x + q'') \eta(x' + q') g(x') \, dx' \right] \, dq' \, dp \, . \tag{6.5}$$

Let us consider the first integral in (6.5). Assuming $\eta$ to be sufficiently smooth functions, we have

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} q'_1 \left[ \int_{\mathbb{R}^2} e^{i(x-x') \cdot p} \eta(x + q'') \eta(x' + q') g(x') \, dx' \right] \, dq' \, dp$$

$$= (2\pi)^2 \int_{\mathbb{R}^2} q'_1 \left[ \int_{\mathbb{R}^2} \delta(x - x') \eta(x + q'') \eta(x' + q') g(x') \, dx' \right] \, dq'$$

$$= (2\pi)^2 \int_{\mathbb{R}^2} q'_1 |\eta(x + q')|^2 g(x) \, dq' \, . \tag{6.6}$$
Making a second change of variables, \(x + q' = -u\), the last term in (6.6) becomes

\[
(2\pi)^2 \int_{\mathbb{R}^2} q'_1 |\eta(x + q')|^2 g(x) \, dq'
= (2\pi)^2 x_1 g(x) \int_{\mathbb{R}^2} |\eta(u)|^2 \, du + (2\pi)^2 g(x) \int_{\mathbb{R}^2} u_1 |\eta(u)|^2 g(x) \, du .
\]

The second integral in the last line vanishes since, in view of the imposed symmetry, \(\eta\) is an even function of \(u\). Thus, noting the normalization of \(\eta\) in (3.7),

\[
\int_{\mathbb{R}^2 \times \mathbb{R}^2} q'_1 \left[ \int_{\mathbb{R}^2} e^{i(x-x') \cdot p} \eta(x + q') \overline{\eta(x' + q')} g(x') \, dx' \right] \, dq' \, dp = x_1 g(x).
\]

Next, we observe that,

\[
- i \frac{\partial}{\partial x_2} \left( e^{i(x-x') \cdot p} \eta(x + q') \overline{\eta(x' + q')} g(x') \right)
= p_2 e^{i(x-x') \cdot p} \eta(x' + q') g(x') \eta(x + q')
\]

so that the second integral in (6.5) becomes

\[
\frac{\lambda}{2m^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} p_2 \left[ \int_{\mathbb{R}^2} e^{i(x-x') \cdot p} \eta(x + q') \overline{\eta(x' + q')} g(x') \, dx' \right] \, dq' \, dp
\]

\[
\frac{\lambda}{2m^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ \int_{\mathbb{R}^2} \left( -i \frac{\partial}{\partial x_2} \right) \left( e^{i(x-x') \cdot p} \eta(x + q') \overline{\eta(x' + q')} g(x') \right) \, dx' \right] \, dq' \, dp
\]

\[
- \frac{\lambda}{2m^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ \int_{\mathbb{R}^2} \left\{ e^{i(x-x') \cdot p} \eta(x + q') g(x') \right\} \, dx' \right] \, dq' \, dp.
\]

Assuming the usual smoothness condition on \(\eta\) and again introducing a delta-distribution in \(x, x'\), the first integral on the right hand side of (6.9) gives

\[
\frac{\lambda}{2m^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left( -i \frac{\partial}{\partial x_2} \right) \left[ \left\{ e^{i(x-x') \cdot p} \eta(x + q') \eta(x' + q') g(x') \right\} \, dx' \right] \, dq' \, dp
\]

\[
= \frac{(2\pi)^2 \lambda}{2m^2} \int_{\mathbb{R}^2} \left( -i \frac{\partial}{\partial x_2} \right) \left[ \left\{ \int_{\mathbb{R}^2} \delta(x - x') \eta(x + q') \eta(x' + q') g(x') \, dx' \right\} \, dq' \right]
\]

\[
= - \frac{i\lambda}{2m^2} \frac{\partial}{\partial x_2} g(x) .
\]

Similarly, the second integral (6.9) yields

\[
- \frac{\lambda}{2m^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ \int_{\mathbb{R}^2} \left\{ e^{i(x-x') \cdot p} \eta(x' + q') g(x') \right\} \left( -i \frac{\partial}{\partial x_2} \right) \eta(x + q') \, dx' \right] \, dq' \, dp
\]

\[
= - \frac{\lambda}{2m^2} \int_{\mathbb{R}^2} \left[ \int_{\mathbb{R}^2} \delta(x - x') \eta(x' + q') g(x') \left( -i \frac{\partial}{\partial x_2} \right) \eta(x + q') \, dx' \right] \, dq'
\]

\[
= - \frac{\lambda}{2m^2} g(x) \int_{\mathbb{R}^2} \eta(x + q') \left( -i \frac{\partial}{\partial x_2} \right) \eta(x + q') \, dq'.
\]
Introducing another change of variables, \( x + q' = u \), this becomes
\[
-\frac{\lambda}{2m^2} g(x) \int_{\mathbb{R}^2} \eta(u) \left( -i \frac{\partial}{\partial u_2} \right) \eta(u) \, du = i \frac{\lambda}{2m^2} g(x) \int_{\mathbb{R}^2} \frac{\partial}{\partial u_2} \eta(u) \, du = 0,
\]
the last equality following since, in view of the evenness of \( \eta \), the derivative term, \( \frac{\partial}{\partial u_2} \eta(u) \), is an odd function.

Thus finally, combining (6.11) with (6.5), (6.8), and (6.10), we obtain
\[
(\hat{O}_{q_1}g)(x) = \left( x_1 - i \frac{\lambda}{2m^2} \frac{\partial}{\partial x_2} \right) g(x).
\]

\[\square\]

**Proof of Theorem IV.1.**

Using (4.2) and (4.3), it can easily be verified that \( \xi, \xi', \) and \( \xi'' \) given in Proposition IV.1 are local exponents for the group of translations \( G_T \) in \( \mathbb{R}^4 \).

It remains to prove the inequivalence of the given multipliers. Let us first prove the fact that \( \xi_1 := \xi - \xi' \) is not trivial. Indeed we have,
\[
\xi_1\left( (q_1, q_2, p_1, p_2), (q'_1, q'_2, p'_1, p'_2) \right) = \xi\left( (q_1, q_2, p_1, p_2), (q'_1, q'_2, p'_1, p'_2) \right) - \xi\left( (q_1, q_2, p_1, p_2), (q'_1, q'_2, p'_1, p'_2) \right)
\]
\[
= \frac{1}{2} q_1 p'_1 + \frac{1}{2} q_2 p'_2 + \frac{1}{2} p_2 p'_1 - \frac{1}{2} q_1 p'_1 - \frac{1}{2} q_2 p'_2 - \frac{1}{2} p_1 p'_2.
\]
Now from (4.4), it follows immediately that triviality of a multiplier \( \eta \) for some abelian group in terms of a suitable continuous function implies the fact that \( \eta(g, g') = \eta(g', g) \) holds for any two group elements of the given abelian group. By contraposition, \( \eta(g, g') \neq \eta(g', g) \) guarantees the nontriviality of the multiplier in question.

In other words, to prove the nontriviality of \( \xi_1 \), it suffices to show that
\[
\xi_1\left( (q_1, q_2, p_1, p_2), (q'_1, q'_2, p'_1, p'_2) \right) \neq \xi_1\left( (q'_1, q'_2, p'_1, p'_2), (q_1, q_2, p_1, p_2) \right)
\]
always holds. Indeed,
\[
\xi_1\left( (q'_1, q'_2, p'_1, p'_2), (q_1, q_2, p_1, p_2) \right)
\]
\[
= \frac{1}{2} q'_1 + \frac{1}{2} q'_2 p_2 + \frac{1}{2} p'_2 p_1 - \frac{1}{2} p'_1 q_1 - \frac{1}{2} p'_2 q_2 - \frac{1}{2} p'_1 p_2,
\]
\[
= -\xi_1\left( (q_1, q_2, p_1, p_2), (q'_1, q'_2, p'_1, p'_2) \right).
\]
Let us now prove that \( \xi_2 := \xi' - \xi'' \) is nontrivial. We have,
\[
\xi_2\left( (q_1, q_2, p_1, p_2), (q'_1, q'_2, p'_1, p'_2) \right)
\]
\[
= \xi'\left( (q_1, q_2, p_1, p_2), (q'_1, q'_2, p'_1, p'_2) \right) - \xi''\left( (q_1, q_2, p_1, p_2), (q'_1, q'_2, p'_1, p'_2) \right)
\]
\[
= \frac{1}{2} \left[ p_1 p'_2 + q_1 q'_2 - p_2 p'_1 - q_2 q'_1 \right]
\]
\[
= -\xi_2\left( (q'_1, q'_2, p'_1, p'_2), (q_1, q_2, p_1, p_2) \right).
\]
The above equation reflects the fact that $\xi_2$ is indeed nontrivial which in turn implies that $\xi'$ and $\xi''$ are inequivalent. Hence it follows that $\xi$, $\xi'$ and $\xi''$ are three inequivalent local exponents of $G_T$. 

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