COMPRESSIBLE EULER EQUATION WITH DAMPING ON TORUS IN ARBITRARY DIMENSIONS

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ABSTRACT. We study the exponential stability of constant steady state of isentropic compressible Euler equation with damping on $T^n$. The local existence of solutions is based on semigroup theory and some commutator estimates. We propose a new method instead of energy estimates to study the stability, which works equally well for any spatial dimensions.

1. Introduction

We consider the compressible Euler equation with frictional damping in a periodic box $[0,1]^n$, which takes the form

\[
\begin{cases}
\rho_t + \nabla \cdot (\rho U) = 0, \\
(\rho U)_t + \nabla \cdot (\rho U \otimes U) + \nabla P = -\alpha \rho U, \\
(\rho, U)(x, 0) = (\rho_0, U_0)(x), \\
(\rho, U)(t, \cdots, x_i, \cdots) = (\rho, U)(t, \cdots, x_i + 1, \cdots) \in R^{n+1}, t \geq 0,
\end{cases}
\]

Our goal is show the exponential stability of the steady state $(\bar{\rho}, 0)$.

Such a system occurs in the mathematical modeling of compressible flow through a porous medium. Here the unknowns $\rho, U$ and $P$ denote the density, velocity and pressure, respectively. The constant $\alpha > 0$ models friction. We assume the flow is a polytropic perfect gas, then $P(\rho) = P_0 \rho^\gamma$, where $\gamma > 1$ is the adiabatic gas exponent. To keep our exposition clean (but not loss of generality), we will take $P_0 = \frac{1}{\gamma}, \alpha = 1$ and $\bar{\rho} = 1$ starting from section 2.

There exist extensive literatures in past decades about compressible Euler equation subject to various initial and initial-boundary conditions. Both classical and weak solutions are constructed and their long time behavior are investigated. For the Cauchy problem, we refer the readers to [7, 18, 20, 21] and references therein for the existence of small smooth solutions; to [3, 4, 19] for $L^\infty$ solutions. For large time behavior of solutions, we refer to [11, 13, 22] for small smooth solutions and [5, 23] and references therein for weak solutions. In the direction of initial-boundary value problems, we refer to [8, 9, 12] for small solutions and [16] for $L^\infty$ solutions. For non-isentropic...
flows, see [14, 15] and references therein. It is also worth to mention some recent work [1, 2, 10] for hyperbolic systems with partial dissipation in several spatial dimensions.

With the notation introduced in section 2, our main result is the following:

**Main Theorem.** Let \( s > \frac{n}{2} \). There exists \( \epsilon_* > 0 \) and \( \delta > 0 \) such that if 
\[
\|\rho_0 - \overline{\rho}\|_{s+1} + \|U_0\|_{s+1} < \epsilon_*,
\]
there exists a unique solution with initial value \((\rho_0, U_0)\) such that
\[
(\rho, U) \in C^0([0, \infty), X_{s+1}) \cap C^1([0, \infty), X_s),
\]
which also satisfies
\[
\|\rho(t, \cdot) - \overline{\rho}\|_{s+1} + \|U(t, \cdot)\|_{s+1} \leq C(\|\rho_0 - \overline{\rho}\|_{s+1} + \|U_0\|_{s+1})e^{-\delta t}.
\]

The result in this paper is not new (especially in low dimensions) for researchers working on compressible Euler equations. However, the method we use is new in the literature. Moreover, the proof is simpler and shorter compared to earlier works and applies to any spatial dimensions equally well.

Energy estimates is the standard approach for analyzing global existence and asymptotic behavior of partial differential equations. However, such method gets a little tediously long when the spatial dimension increases, because the function spaces need to be more regular in order to make the equation well-posed. The main contribution of this paper is contained in subsection 3.3, where we propose a new method to obtain the decay property of small solutions. Our strategy is to make a change of coordinate so that the solution lives on a subspace under such coordinate system. Unlike the usual coordinate system using a fixed frame, we define ours by using a moving frame which depends on the solution itself. Under such moving frame, we can decompose the solution into two parts and discover the decay property.

The rest of this paper is organized as follows. In Section 2, we introduce some notations and rewrite (1.1) in a dynamical system form. In Section 3, we present the proof of the Main Theorem.

### 2. Set Up

Throughout this paper, we use \( H^s \) to denote Sobolev space of periodic functions equipped with norm \( \| \cdot \|_s \), i.e.,
\[
\|f\|_s \triangleq \left( \sum_{|\alpha| \leq s} \|D^\alpha f\|_{L^2}^2 \right)^{\frac{1}{2}},
\]
where \( \alpha \) is a multi-index.

For any vector valued function \( F = (f_1, \ldots, f_m) : [0, 1]^n \rightarrow \mathbb{R}^m \),
\[
\|F\|_s \triangleq \sum_{i=1}^n \|f_i\|_s.
\]
For any \( s \in \mathbb{Z} \), we define Banach spaces
\[
X_s \triangleq \{(f, g)\mid f \in H^s, g \in \oplus^n H^s\}, \quad \tilde{X}_s \triangleq \{(f, g) \in X_s \mid \int_{[0,1]^n} f(x) \, dx = 0\},
\]
which are equipped with norms
\[
\|(f, g)\|_s \triangleq \|f\|_s + \|g\|_s.
\]
From now on, we will assume \( P_0 = \frac{1}{\gamma}, \alpha = 1 \) and \( \overline{\rho} = 1 \). Let \( \theta = \frac{2\gamma - 1}{2} \) and \( \sigma = \frac{\theta - 1}{\theta} \). Then the system (1.1) can be written as
\[
\begin{aligned}
\left\{
\begin{array}{l}
(\sigma, U)_t = 
\begin{pmatrix}
0 & -\nabla & \cdot
\end{pmatrix}
\begin{pmatrix}
\sigma

\end{pmatrix} - 
\begin{pmatrix}
U \cdot \nabla & \theta \sigma \nabla.
\end{pmatrix}
\begin{pmatrix}
\sigma

\end{pmatrix},

(\sigma, U)(x, 0) = (\sigma_0, U_0)(x),
\\
\int_{[0,1]^n} \sigma_0 \, dx = \overline{\sigma}.
\end{array}
\end{aligned}
\]

(2.1)

It is clear that classical solutions of (1.1) and (2.1) are equivalent through the transformation \( \sigma = \frac{\theta - 1}{\theta} \). Let
\[
A \triangleq \begin{pmatrix}
0 & -\nabla.
\end{pmatrix}, \quad B_{\sigma, U} \triangleq \begin{pmatrix}
U \cdot \nabla & \theta \sigma \nabla.
\end{pmatrix}
\]
and (2.1) can be written abstractly as
\[
(\sigma, U)_t = (A + B_{\sigma, U})(\sigma, U).
\]

For any bounded linear operator \( T \in L(X, Y) \), we use \( |T|_{L(X,Y)} \) to denote the operator norm. We will drop the subscript if the context causes no confusion. For any Banach space \( Z \), we use \( B_\delta(Z) \) to denote the ball centered at the origin of radius \( \delta \) in \( Z \).

### 3. Proof

In this section, we give the proof of the Main Theorem. We split this section into three subsections. In subsection 3.1, we show that \( A \) generates a semigroup \( T(t) \) and \( A + B_{\sigma, U} \) generates an evolutionary operator \( M_{\sigma, U} \) for suitably chosen \((\sigma, U)\), respectively. To make our presentation self-contained, we prove local well-posedness in subsection 3.2. In subsection 3.3, we analyze the asymptotic behavior of local solutions.

#### 3.1. Semigroup and Evolutionary Operator

In this subsection, we provide estimates on the semigroup and the evolutionary operator generated by \( A \) and \( A + B_{\sigma, U} \) on \( X_s \). We begin with some properties of \( A \).

**Lemma 3.1.** The linear operator \( A \) generates a strongly continuous semigroup \( T(t) \) on \( X_s \) for every \( s \). Moreover, there exists \( K \geq 1 \) such that for \( t \geq 0 \),
\[
|T(t)|_{L(X_s, X_s)} \leq K, \quad |T(t)|_{L(\tilde{X}_s, \tilde{X}_s)} \leq Ke^{-\frac{1}{2}t}.
\]

(3.1)
Proof. Since $A$ is a closed, densely defined linear operator and is the sum of an anti-selfadjoint and a bounded linear operator, the standard semigroup theory implies $A$ generates a $C_0$-semigroup $T(t)$ on $X_s$. Let $\hat{A}(\xi)$ be the symbol of $A$, namely,

$$\hat{A}(\xi) = \begin{pmatrix} 0 & -i\xi^T \\ -i\xi & -I \end{pmatrix}, \quad \xi \in \mathbb{Z}^n.$$ 

The above matrix has eigenvalues $-1$ with multiplicity $n-1$ and $\lambda_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4|\xi|^2}$, $\lambda_2 = -\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4|\xi|^2}$. For $\xi \in \mathbb{Z}^n/\{0\}$, we choose an orthonormal basis as follows

$$V_1 = \frac{1}{\sqrt{|\xi|^2 + |\lambda_2|^2}} \begin{pmatrix} -i\lambda_2 \\ \xi \end{pmatrix}, \quad V_2 = \begin{pmatrix} h_0 \\ h_1 \end{pmatrix} = \begin{pmatrix} \frac{i|\xi|^2}{\sqrt{|\xi|^2 + |\lambda_2|^2}} \\ \frac{i|\xi|^2}{\sqrt{|\xi|^2 + |\lambda_2|^2}} \xi \end{pmatrix},$$

$$V_i = \begin{pmatrix} 0 \\ \eta_i \end{pmatrix}, \quad \eta_i \in \{\mathbb{C}\xi\}^\perp, \quad 1 \leq i \leq n-1.$$ 

Let $R(\xi) = (V_1, \cdots, V_{n+1})$ and we have $R(\xi)$ is unitary, namely, $R^{-1}(\xi) = R^*(\xi) = R^t(\xi)$. Consequently,

$$R^{-1}(\xi)\hat{A}(\xi)R(\xi) = \begin{pmatrix} h_0 & h_1 & \cdots & h_n \\ \lambda_1 & \eta_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} \end{pmatrix}
\begin{pmatrix} 0 & -i\xi^T \\ -i\xi & -1 \end{pmatrix}
\begin{pmatrix} h_0 & \cdots & h_n \\ \frac{i\lambda_2}{\sqrt{|\xi|^2 + |\lambda_2|^2}} & \cdots & 0 \\ \frac{\eta_1}{\sqrt{|\xi|^2 + |\lambda_2|^2}} \xi & \cdots & \eta_{n-1} \end{pmatrix}
= \begin{pmatrix} \lambda_2 & 0 & \cdots & 0 \\ 0 & \lambda_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -I_{n-1} \end{pmatrix} \triangleq B(\xi).$$

It follows that

$$e^{tB(\xi)} = \begin{pmatrix} e^{\lambda_2 t} & 0 & e^{\lambda_1 t} \\ \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} & e^{\lambda_1 t} \\ 0 & e^{-tI_{n-1}} \end{pmatrix}.$$ 

We note that for $\xi \in \mathbb{Z}^n/\{0\}$, there exists $K \geq 1$ independent of $\xi$ such that

$$\left| \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \right| \leq K e^{-\frac{1}{2}t},$$
which implies
\[
|e^{t \hat{A}(\xi)}|_{L^1([0,T],C^{n+1})} = |e^{tR^{-1}(\xi)B(\xi)R(\xi)}| = |R^{-1}(\xi) e^{tB(\xi)} R(\xi)| \leq K e^{-\frac{1}{2}t}.
\]

For \( \xi = 0 \), we have \( \hat{A}(0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \). Therefore, for \( t \geq 0 \),
\[
|T(t)|_{L(X_s,X_s)} \leq K , \quad |T(t)|_{L(\tilde{X}_s,\tilde{X}_s)} \leq K e^{-\frac{1}{2}t}.
\]

\[ \square \]

**Proposition 3.2.** Let \( s > \frac{n}{2} \) and \( \epsilon_* > 0 \) be a sufficiently small constant. For any \((\sigma,U)\) such that \( \| (\sigma,U) \|_{C_0([0,T],X_{s+1})} < \epsilon_* \), \( A + B_{\sigma,U} \) generates an evolutionary system \( M_{\sigma,U}(t,\tau) \) on \( X_s \), which satisfies
\[
|M_{\sigma,U}(t,\tau)|_{L(X_s,X_s)} \leq K e^{C\epsilon_* (t-\tau)} \quad \text{for} \quad 0 \leq \tau \leq t \leq T,
\]
where \( C \) only depends on \( s \) and \( n \).

**Proof.** For any \((\sigma_1,U_1) \in X_{s+1}\), we have
\[
< B_{\sigma,U} \begin{pmatrix} \sigma_1 \\ U_1 \end{pmatrix}, \begin{pmatrix} \sigma_1 \\ U_1 \end{pmatrix} >_{H^s}
= - \sum_{|\alpha| \leq s} \int_{[0,1]^n} D^\alpha (U \cdot \nabla \sigma_1 + \theta \sigma \nabla \cdot U_1) \cdot D^\alpha \begin{pmatrix} \sigma_1 \\ U_1 \end{pmatrix} \, dx
= - \int_{[0,1]^n} (U \cdot D^s \nabla \sigma_1 + \theta \sigma D^s \nabla \cdot U_1) D^s \sigma_1 + (\theta \sigma D^s \nabla \sigma_1 + U \cdot D^s \nabla U_1) \cdot D^s U_1 \, dx + l.o.t
\]
\[
\leq C \| D(\sigma,U) \|_{L^\infty} \| (\sigma_1,U_1) \|_{s}^2 \leq C \epsilon_* \| (\sigma_1,U_1) \|_{s},
\]
where \( C \) only depends on \( s \) and \( n \). This means \( B_{\rho,U} - C \epsilon_* \) is dissipative.

Next we introduce the following norm on \( X_s \),
\[
|x|_s = \sup_{t \geq 0} \| T(t)x \|_s,
\]
where \( T(t) \) is defined in Lemma 3.1. It is clear that for all \( x \in X_s \)
\[
\| x \|_s \leq |x|_s \leq K \| x \|_s , \quad |T(t)x|_s \leq |x|_s.
\]
Therefore, the new norm \( | \cdot |_s \) is equivalent to \( \| \cdot \|_s \) on \( X_s \) and \( A \) generates a contraction on \( X_s \) under \( | \cdot |_s \). Note that
\[
\| (B_{\rho,U} - C \epsilon_*)(\sigma_1,U_1) \|_s 
\leq \| U \cdot \nabla \sigma_1 + \theta \sigma \nabla \cdot U_1 \|_s + \| \theta \sigma \nabla \sigma_1 + U \cdot \nabla U_1 \|_s + C \epsilon_* \| (\sigma_1,U_1) \|_s
\]
\[
\leq C \| (\sigma,U) \|_{L^\infty} + \| D(\sigma,U) \|_{L^\infty} \| \nabla \sigma_1 \|_s + \| \nabla \cdot U_1 \|_s
\]
\[
\leq C \epsilon_* \| \nabla \sigma_1 + U_1 \|_s + \| \nabla \cdot U_1 \|_s + \| (\sigma_1,U_1) \|_s
\]
\[
\leq C \epsilon_* \| A(\sigma_1,U_1) \|_s + C \epsilon_* \| (\sigma_1,U_1) \|_s.
\]
From (3.5) and (3.4), we have

\[ |(B_{\sigma,U} - C\epsilon_*)(\sigma_1, U_1)| \leq CK\epsilon_*(|A(\sigma_1, U_1)|_s + |(\sigma_1, U_1)|_s). \]

Moreover, \( B_{\sigma,U} - C\epsilon_* \) is still dissipative under the inner product induced by the norm \( |\cdot|_s \). Therefore, by Corollary 3.3.3 in [17], \( A + B_{\rho,U} - C\epsilon_* \) generates a family of contractions on \( X_s \) under the norm \( |\cdot|_s \). Switching back to the original norm, we obtain (3.3). □

3.2. Local existence. In this subsection, we prove the local existence of small initial data though a contraction mapping argument. Recall that we obtain the evolutionary system \( M_{\sigma,U}(t, \tau) \in L(X_s, X_s) \) for small \( (\sigma, U) \in X_{s+1} \). Therefore, we need to show such operator also maps \( X_{s+1} \) to \( X_{s+1} \). This fact can be proved by the following commutator estimate. The main idea is developed by Kato in [6], which has wide applications in studying local well-posedness of quasi-linear equations. Let

\[ Sf(x) \triangleq \sum_{\xi \in \mathbb{Z}^n} e^{2\pi i x \cdot \xi} (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi), \]

which is an isomorphism between \( X_{s+1} \) and \( X_s \) such that \( |S| = |S^{-1}| = 1 \).

**Lemma 3.3.** Let \( s > \frac{n}{2} \) and \( (\sigma, U) \in X_{s+1} \), then

\[ S(A + B_{\sigma,U})S^{-1} - (A + B_{\sigma,U}) \in L(X_s, X_s). \]

**Proof.** It is clear that \( A \) commutes with \( S \). Thus, we only need to show \( SB_{\sigma,U}S^{-1} - B_{\sigma,U} \in L(X_s, X_s) \). The Fourier transform of the difference operator has kernel

\[
(SB_{\sigma,U}S^{-1} - B_{\sigma,U})(\xi, \eta)
= (1 + |\xi|^2)^{\frac{s}{2}}(1 + |\eta|^2)^{\frac{s}{2}} \begin{pmatrix}
\hat{\sigma}(\xi - \eta) \cdot (i\eta) & \theta \hat{\sigma}(\xi - \eta)(i\eta^T) \\
\theta \hat{\sigma}(\xi - \eta)^T (i\eta) & \hat{\sigma}(\xi - \eta) \cdot (i\eta)
\end{pmatrix} (1 + |\eta|^2)^{-\frac{s}{2}}.
\]

Note that \( \frac{|\eta|}{\sqrt{1 + |\eta|^2}} \leq 1 \) and

\[
|(1 + |\xi|^2)^{\frac{s}{2}} - (1 + |\eta|^2)^{\frac{s}{2}}| = |\int_0^1 \frac{d}{dp}(1 + |p\xi + (1 - p)\eta|^2)^{\frac{s}{2}} dp|
\leq |\xi - \eta| \int_0^1 \frac{|p\xi + (1 - p)\eta|}{(1 + |p\xi + (1 - p)\eta|^2)^{\frac{s}{2}}} dp \leq |\xi - \eta|.
\]

Therefore, we have

\[
|SB_{\sigma,U}S^{-1} - B_{\sigma,U}| \leq C \sum_{\eta} |\xi - \eta|(|\hat{\sigma}(\xi - \eta)| + |\hat{\sigma}(\xi - \eta)|)
\leq C||(|\sigma, U||_{s+1},
\]

which completes the proof. □
Consequently, for any \( y \in X_{s+1} \), let \( x = Sy \) and we have
\[
\|M_{\sigma,U}(t,\tau)y\|_{s+1} = \|S^{-1}SM_{\sigma,U}S^{-1}x\|_{s+1} \leq \|SM_{\sigma,U}(t,\tau)S^{-1}|_{L(X_s,X_s)}\|y\|_{s+1}.
\]
Therefore, if \( \|(\sigma,U)\|_{C^0([0,T],X_{s+1})} < \epsilon_\star \), by (3.6), we have
\[
(3.7) \quad |M_{\sigma,U}(t,\tau)|_{L(X_{s+1},X_{s+1})} \leq Ke^{2CK_{\epsilon_\star}(t-\tau)},
\]
where \( C, K \) are the same as in (3.3).

Define the following space
\[
Z \triangleq \{(\sigma,U) \in C([0,T],X_s) \mid (\sigma(0),U(0)) = (\sigma_0,U_0), \|(\sigma, U)\|_{C^0([0,T],X_{s+1})} \leq \epsilon_\star \},
\]
equipped with the norm \( \|(\sigma, U)\|_Z \triangleq \|(\sigma, U)\|_{C^0([0,T],X_s)} \). For any \( (\sigma_0,U_0) \in X_{s+1} \) and \( (\sigma, U) \in Z \), we let
\[
F(\sigma, U, \sigma_0, U_0)(t) = M_{\sigma,U}(t,0)(\sigma_0,U_0).
\]

**Proposition 3.4.** Let \( s > \frac{n}{2} \). There exists \( \epsilon_\star > 0 \) such that for any \( \|(\sigma_0,U_0)\|_{s+1} \leq \frac{\epsilon_\star}{K^2} \), where \( K \) is defined in Lemma 3.1, the mapping \( F(\cdot, \cdot, \rho_0, U_0) \) has a unique fixed point in \( Z \). Consequently, for small initial data, (2.1) has a unique solution
\[
(3.8) \quad (\sigma,U) \in C([0,T],X_{s+1}) \cap C^1([0,T],X_s) \quad \text{for some} \ T > 0.
\]

**Proof.** Let \( \epsilon_\star = \frac{1}{2CK} \) and \( T = \frac{\ln K}{4CK^2} \), where \( C \) is some Sobolev embedding constant appearing in Proposition 3.2. For \( \|(\rho,U)\|_Z \leq \epsilon_\star \), by (3.3) and (3.7), we have
\[
\|F(\sigma, U, \sigma_0, U_0)\|_Z \leq K^2 \cdot \epsilon_\star \leq \epsilon_\star, \quad \|F(\sigma, U, \sigma_0, U_0)\|_{C^0([0,T],X_{s+1})} \leq \epsilon_\star,
\]
which means \( F(\cdot, \cdot, \rho_0, U_0) \) maps \( B_{\epsilon_\star}(Z) \) into itself. For any \( (\sigma_1,U_1) \) and \( (\sigma_2,U_2) \) in \( Z \), we have
\[
\|F(\sigma_1,U_1,\sigma_0,U_0) - F(\sigma_2,U_2,\sigma_0,U_0)\|_Z
\]
\[
= \sup_{t \in [0,T]} \| \int_0^t M_{\sigma_1,U_1}(t,\tau)(B_{\sigma_1,U_1}(\tau) - B_{\sigma_2,U_2}(\tau))M_{\sigma_2,U_2}(\tau,0)(\rho_0,U_0) \, d\tau \|_s
\]
\[
\leq TK^2e^{2CKKT}C\|(\sigma_0,M_0)\|_{s+1}\|(\sigma_1 - \sigma_2, U_1 - U_2)\|_Z
\]
\[
\leq \frac{\ln K}{4CK^2}K^2\frac{1}{K^2}\epsilon_\star \|(\sigma_1 - \sigma_2, U_1 - U_2)\|_Z < \|(\sigma_1 - \sigma_2, U_1 - U_2)\|_Z,
\]
which implies \( F \) is a contraction. Therefore, (2.1) has a unique local solution in \( Z \) for all \( \|(\rho_0,U_0)\|_{s+1} \leq \frac{\epsilon_\star}{K} \) and (3.8) follows from the definition of \( Z \). \( \Box \)

### 3.3. Asymptotic Behavior

In this subsection, we introduce a new method to study the asymptotic behavior of local solutions obtained in the previous subsection. The method is motivated by the following observation. From Lemma 3.1, we see that the phase space \( X_s \) can be decomposed into two subspaces \( \bar{X}_s \) and its orthogonal complement generated by the vector \( (\sigma, U) = (1, 0) \), which are invariant under the semigroup \( S(t) \). Moreover, the semigroup has exponential decay on \( \bar{X}_s \). Intuitively, for small \( (\sigma, U) \),
the linear operator $B_{\sigma,U}$ is a small perturbation relative to $A$ and thus we expect a similar dynamical picture for $M_{\sigma,U}$. It turns out given any local solution $(\sigma(t), U(t))$, we can define a codimension one linear subspace $E_2(t)$ for each $t$, which is isomorphic and close to $\tilde{X}_s$, such that

$$X_s = E_1 \oplus E_2(t), \quad M_{\sigma,U}(t, \tau) \bigg|_{E_1} = I, \quad M_{\sigma,U}(t, \tau)E_2(\tau) \subset E_2(t),$$

for $0 \leq \tau \leq t \leq T$. Here $E_1 = \{(\sigma, U) = (c, 0), c \in \mathbb{R}\}$. By conjugating the flow from $E_2(\tau)$ onto $\tilde{X}_s$, we discover the exponential decay property of the solution. We complete the proof of the Main Theorem by noting $\sigma$ satisfies the Poincare’s inequality, which forces its projection on $E_1$ to be 0 and thus the solution lies entirely on $E_2$.

We begin to construct $E_2(t)$ in the following Lemma. We will use $Pf$ to denote the average of $f$ over $[0,1]^n$, i.e., $Pf = \int_{[0,1]^n} f \, dx$.

**Lemma 3.5.** Let $s > \frac{n}{2}$. There exists $\delta > 0$ such that for all $\| (\sigma, U) \|_s < \delta$, there exist operators $(L_1, L_2) : B_\delta(X_0) \to L(X_s, \mathbb{R})$ such that for all $(\sigma_1, U_1) \in \tilde{X}_{s+1}$,

$$\begin{align*}
(A + B_{\sigma,U}) & \left( \frac{L_1 \sigma_1 + L_2 U_1 + \sigma_1}{U_1} \right) = \left( \frac{L_1 \bar{\sigma}_1 + L_2 \bar{U}_1 + \bar{\sigma}_1}{\bar{U}_1} \right),
\end{align*}$$

where

$$\begin{align*}
\bar{\sigma}_1 &= (I - P)(U \cdot \nabla (L_1 \sigma_1 + L_2 U_1 + \sigma_1) + (1 + \theta \sigma) \nabla \cdot U_1), \\
\bar{U}_1 &= (1 + \theta \sigma) \nabla (L_1 \sigma_1 + L_2 U_1 + \sigma_1) + U_1 + U \cdot \nabla U_1.
\end{align*}$$

**Proof.** We note (3.9) is equivalent to

$$\begin{align*}
0 &= \mathcal{G}(L_1, L_2; \sigma, U)(\sigma_1, U_1) \\
&= P(U \cdot \nabla (L_1 \sigma_1 + L_2 U_1 + \sigma_1) + (1 + \theta \sigma) \nabla \cdot U_1) \\
&\quad - L_1((I - P)U \cdot \nabla (L_1 \sigma_1 + L_2 U_1 + \sigma_1) + (1 + \theta \sigma) \nabla \cdot U_1) \\
&\quad - L_2((1 + \theta \sigma) \nabla (L_1 \sigma_1 + L_2 U_1 + \sigma_1) + U_1 + U \cdot \nabla U_1) \\
&= P(U \cdot \nabla \sigma_1 + (1 + \theta \sigma) \nabla \cdot U_1) - L_1((I - P)U \cdot \nabla \sigma_1 \\
&\quad + (1 + \theta \sigma) \nabla \cdot U_1) - L_2((1 + \theta \sigma) \nabla \sigma_1 + U_1 + U \cdot \nabla U_1).
\end{align*}$$

Consider $\mathcal{G}$ as a mapping from $\tilde{X}_{-s} \times X_s$ to $\tilde{X}_{-(s+1)}$. It is obvious that $\mathcal{G}(0,0; 0, 0) = 0$ and

$$D_{L_1, L_2} \mathcal{G}(0, 0; 0, 0)(\tilde{L}_1, \tilde{L}_2) = (\tilde{L}_1, \tilde{L}_2) \left( \begin{array}{cc} 0 & -\nabla \\ -\nabla & -I \end{array} \right).$$

Since $
\begin{pmatrix}
0 & -\nabla \\
-\nabla & -I
\end{pmatrix}$

is an isomorphism between $\tilde{X}_r$ and $\tilde{X}_{r-1}$ for any $r$, the implicit function theorem implies there exists $\delta > 0$ such that for any $\| (\rho, U) \|_s < \delta$, there exists a unique pair of operators $(L_1, L_2) \in L(\tilde{X}_s, \mathbb{R})$
such that (3.10) (or equivalently (3.11)) holds. Moreover, \((L_1, L_2)\) are smooth in \((\sigma, U)\) and

\[(3.12) \quad L_1(0, 0) = 0, \quad L_2(0, 0) = 0, \quad |D(L_1, L_2)|_{C^0(B_\delta(\tilde{X}_s), L(\tilde{X}_s, \tilde{X}_s))} \leq C_1\delta,
\]

where \(C_1\) is independent of \((\sigma, U)\). □

We extend \(L_1\) to a linear functional on \(H^s\) (still denoted by \(L_1\)) by setting

\[(3.13) \quad L_1 f = L_1(I - P)f, \quad f \in H^s.
\]

Consequently, we have

\[(3.14) \quad L_1^2 = 0, \quad L_1L_2 = 0, \quad (I + L_1)^{-1} = I - L_1,
\]

and

\[(3.15) \quad \left(\begin{array}{cc} I + L_1 & L_2 \\ 0 & I \end{array}\right)^{-1} = \left(\begin{array}{cc} I - L_1 & -L_2 \\ 0 & I \end{array}\right).
\]

Let \((\sigma(t), U(t))\) be a local solution of (2.1) and we define

\[(3.16) \quad E_2(t) = \{ \left(\begin{array}{cc} I + L_1(\sigma(t), U(t)) & L_2(\sigma(t), U(t)) \\ 0 & I \end{array}\right) \left(\begin{array}{c} \sigma_1 \\ U_1 \end{array}\right) \mid (\sigma_1, U_1) \in \tilde{X}_s\}.
\]

Recall that \(E_1 = \{(\sigma, U) = (c, 0), c \in \mathbb{R}\}\). It is clear that

\[X_s = E_1 \oplus E_2(t).
\]

We then decompose \((\sigma(t), U(t))\) dynamically as

\[(3.17) \quad c_t(t) = 0 \implies c(t) = c,
\]

and

\[(3.18) \quad \left(\begin{array}{cc} I + L_1 & L_2 \\ 0 & I \end{array}\right) \left(\begin{array}{c} (I - P)(\sigma) \\ U \end{array}\right)_t = (A + B_{\sigma, U}) \left(\begin{array}{cc} I + L_1 & L_2 \\ 0 & I \end{array}\right) \left(\begin{array}{c} (I - P)(\sigma) \\ U \end{array}\right).
\]

Therefore, (3.9) follows from (3.18) and (3.10).

Next we conjugate flows from \(E_2(t)\) to \(\tilde{X}_s\). By (3.18), we have

\[(3.19) \quad \left(\begin{array}{c} \sigma_1 \\ U_1 \end{array}\right)_t = \left(\begin{array}{cc} I - L_1 & -L_2 \\ 0 & I \end{array}\right)(A + B_{\sigma, U}) \left(\begin{array}{cc} I + L_1 & L_2 \\ 0 & I \end{array}\right) \left(\begin{array}{c} \sigma_1 \\ U_1 \end{array}\right) - \left(\begin{array}{cc} I - L_1 & -L_2 \\ 0 & I \end{array}\right) \left(\begin{array}{cc} \partial_t L_1 & \partial_t L_2 \\ 0 & 0 \end{array}\right) \left(\begin{array}{c} \sigma_1 \\ U_1 \end{array}\right),
\]
where \( \begin{pmatrix} \sigma_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} I - L_1 & -L_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} \sigma - c \\ U \end{pmatrix} \). By \((3.13)\), we have
\[
\begin{pmatrix} I - L_1 & -L_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} \partial_t L_1 & \partial_t L_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} \partial_t L_1 \sigma_1 + \partial_t L_2 U_1 \\ 0 \end{pmatrix} \in E_1 \cap E_2(t) = 0.
\]

Consequently, one has
\[
\begin{pmatrix} \sigma_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} I - L_1 & -L_2 \\ 0 & I \end{pmatrix} \left( A + B_{c+(I+L_1)\sigma_1+L_2U_1,U_1} \right) \left( I + L_1 \begin{pmatrix} L_2 \\ 0 \end{pmatrix} \right) \begin{pmatrix} \sigma_1 \\ U_1 \end{pmatrix} = A + \tilde{B}_{\sigma_1,U_1},
\]
where \( \tilde{B}_{\sigma_1,U_1} = (\tilde{B}_{i,j}) \) for \( i, j = 1, 2 \) and
\[
\begin{align*}
\tilde{B}_{1,1} &= - (I - L_1) U_1 \cdot \nabla + L_2 (1 + \theta (c + (I + L_1) \sigma_1 + L_2 U_1)) \nabla, \\
\tilde{B}_{1,2} &= - (I - L_1) \theta (c + (I + L_1) \sigma_1 + L_2 U_1) \nabla \cdot + L_1 \nabla \cdot - L_2 (I + U_1) \nabla, \\
\tilde{B}_{2,1} &= - \theta (c + (I + L_1) \sigma_1 + L_2 U_1) \nabla, \\
\tilde{B}_{2,2} &= - U_1 \cdot \nabla.
\end{align*}
\]

By repeating a similar argument as in Proposition \(3.2\) and \(3.5\), we can show there exists \( C \) only depending on \( s \) and \( n \) such that
\[
\| \tilde{M}_{\sigma_1,U_1}(t, \tau) \|_{L(\tilde{X}_{s+1},\tilde{X}_s)} \leq Ke^{-\frac{1}{2}(t-\tau)}.
\]
The commutator estimates as in Lemma \(3.3\) also holds for \( A + \tilde{B}_{\sigma_1,U_1} \) and thus for small \( \epsilon_* \) (also implies \( c \) is small), we have
\[
\| \tilde{M}_{\sigma_1,U_1}(t, \tau) \|_{L(\tilde{X}_{s+1},\tilde{X}_s)} \leq Ke^{-\frac{1}{2}(t-\tau)}.
\]
The proof is completed. \(\square\)

The above lemma and \((3.17)\) implies the local solution can be extended globally, namely, \( T = \infty \). We conclude the Main Theorem by observing that if \( \rho \) satisfies \((3.11)\), then \( \rho - 1 \) satisfies the Poincare's inequality,
\[
\| \rho - 1 \|_{L^2} \leq C \| \nabla \rho \|_{L^2}.
\]
According to the definition of $\sigma = \frac{\rho \theta - 1}{\theta}$, we also have

$$\|\sigma\|_{L^2} \leq C\|\nabla \sigma\|_{L^2}.$$ 

Recall that we can write $\sigma = (I + L_1)(I - P)c + c$. By (3.20), we have $\|(I + L_1)(I - P)c(t)\|_{L^2}$ and $\|\nabla c(t)\|_{L^2}$ decay exponentially, which forces $c = 0$. Therefore, $\sigma = (I + L_1)(I - P)c$, whose $H^{s+1}$ norm decays exponentially again by (3.20). The proof is completed.

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