Spectral Distance on the Circle

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Vienna, Preprint ESI 2006 (2008)  
February 14, 2008

Supported by the Austrian Federal Ministry of Education, Science and Culture  
Available via http://www.esi.ac.at
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February 29, 2008

Abstract

A building block of noncommutative geometry is the observation that most of the geometric information of a compact riemannian spin manifold $M$ is encoded within its Dirac operator $D$. Especially via Connes’ distance formula one is able to extract from the spectral properties of $D$ the geodesic distance on $M$. In this paper we investigate the distance $d$ encoded within a covariant Dirac operator on a trivial $U(n)$-fiber bundle over the circle with arbitrary connection. It turns out that the connected components of $d$ are tori whose dimension is given by the holonomy of the connection. For $n = 2$ we explicitly compute $d$ on all the connected components. For $n \geq 2$ we restrict to a given fiber and find that the distance is given by the trace of the module of a matrix. The latest is defined by the holonomy and the coordinate of the points under consideration. This paper extend to arbitrary $n$ and arbitrary connection the results obtained in a previous work for $U(2)$-bundle with constant connection. It confirms interesting properties of the spectral distance with respect to another distance naturally associated to connection, namely the horizontal or Carnot-Carathéodory distance $d_H$. Especially in case the connection has irrational components, the connected components for $d$ are the closure of the connected components of $d_H$ within the euclidean topology on the torus.

I Introduction

In Connes framework a noncommutative geometry consists in a spectral triple $\mathcal{A}, \mathcal{H}, D$ ($\mathcal{A}$ is an involutive algebra commutative or not, $\mathcal{H}$ an Hilbert space carrying a representation $\Pi$ of $\mathcal{A}$ and $D$ a selfadjoint operator on $\mathcal{H}$) together with a graduation $\Gamma$ (called chirality in physicists’ words) and a real structure $J$ both acting on $\mathcal{H}$. In analogy with the commutative case, points are recovered as pure states $\mathcal{P}(\mathcal{A})$ of $\mathcal{A}$ and the distance $d$ between two states $\omega, \omega'$ is defined as

$$d(\omega, \omega') = \sup_{a \in A} \{ |\omega(a) - \omega'(a)| ; \| [D, \Pi(a)] \| \leq 1 \}$$

(1)

where the norm is the operator norm on $\mathcal{H}$. In the commutative case,

$$\mathcal{A}_E = C^\infty(M), \mathcal{H}_E = L_2(M, S), D_E = -i\gamma^\mu \partial_\mu$$

(2)

with $\mathcal{H}_E$ the space of square integrable spinors over a compact, riemannian, spin manifold $M$ and $D_E$ the ordinary Dirac operator of quantum field theory ($\gamma^\mu$ is the representation of the Clifford algebra on $\mathcal{H}_E$ and we omit the spin connection for it commutes with $\Pi$),
the distance \(d\) coincides with the geodesic distance defined by the riemannian structure of \(M\). Thus (1) appears as a natural extension to the noncommutative realm of the classical definition of the distance as the length of the shortest path, all the more as it does not involve any notion ill-defined in a quantum framework such as trajectory between points. Since (1) only involves the spectral properties of the algebra and of the Dirac operator, we shall in the following refer to \(d\) as the spectral distance associated to \((\mathcal{A}, \mathcal{H}, D)\).

Connections are implemented within a geometry \((\mathcal{A}, \mathcal{H}, D)\) by substituting \(D\) with a covariant operator \(D_A \doteq D + A + JAJ^{-1}\)

\[ \Omega^1 \doteq \{ a^i[D, b_i] ; a^i, b_i \in \mathcal{A} \}. \]  

Only the part of \(D_A\) that does not obviously commute with the representation, namely

\[ D \doteq D + A, \]

enters in the distance formula (1) and induces a so called fluctuation of the metric. In the following we consider almost commutative geometries obtained as the product of the continuous - also called external - geometry (2) by a so called internal geometry \(\mathcal{A}_I, \mathcal{H}_I, D_I\).

The product of two spectral triples,

\[ \mathcal{A} \doteq \mathcal{A}_E \otimes \mathcal{A}_I, \mathcal{H} \doteq \mathcal{H}_E \otimes \mathcal{H}_I, D \doteq D_E \otimes I_I + \gamma^5 \otimes D_I \]

where \(I_I\) is the identity operator of \(\mathcal{H}_I\) and \(\gamma^5\) the graduation of the external geometry, is again a spectral triple. The corresponding 1-forms are

\[ -i\gamma^\mu f^i_\mu \otimes m_i + \gamma^5 h^j \otimes n_j \]

where \(m_i \in \mathcal{A}_I, h^j, f^i_\mu \in C^\infty (M), n_j \in \Omega^1_I\). Selfadjoint 1-forms decompose in an \(\mathcal{A}_I\)-valued skew-adjoint 1-form field over \(M\),

\[ A_\mu \doteq f^i_\mu m_i, \]

and an \(\Omega^1_I\)-valued selfadjoint scalar field

\[ H \doteq h^j n_j. \]

When the internal algebra \(\mathcal{A}_I\) has finite dimension, \(A_\mu\) has value in the Lie algebra of the unitaries of \(\mathcal{A}\) and is called the gauge part of the fluctuation.

In [12] we have computed the distance (1) for scalar fluctuations only \((A_\mu = 0)\) with various choices of finite dimensional \(\mathcal{A}_I\). In [10] we have studied pure gauge fluctuations \((H = 0)\) for \(\mathcal{A}_I = M_n(\mathbb{C})\). Then the set of pure states \(\mathcal{P}(\mathcal{A})\) is a trivial \(U(n)\)-bundle \(P\) on \(M\) with fiber \(\mathbb{C}P^{n-1}\) and a connection whose corresponding 1-form is given by the

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*In previous papers we used the terminology ”noncommutative distance” that might be confusing, suggesting \(d\) was not reflexive. Of course \(d\) is reflexive and is a distance in the strict mathematical sense.

†Most of the time we omit the symbol \(\Pi\) and it should be clear from the context whether \(a\) means an element of \(\mathcal{A}\) or its representation on \(\mathcal{H}\). We also use Einstein summation over repeated indices in alternate positions (up/down).

‡\(M_n(\mathbb{C})\) denotes the algebra of square \(n \times n\) complex matrices.
gauge fluctuation $A_\mu$. It turns out that the spectral distance $d$ is always smaller or equal to the Carnot-Carathéodory distance defined by this connection,

$$d_H(p, q) = \inf_{\dot{c}(t) \in \mathcal{H} \cap P} \int_0^1 \|\dot{c}(t)\| \, dt \quad \forall p, q \in P$$

where the horizontal distribution $\mathcal{H} P$ is the kernel of the connection 1-form $A_\mu$. The two distances are equal when the holonomy of the connection is trivial (i.e. the connection is flat and $M$ is simply connected) otherwise $d$ has properties rather different from $d_H$. In [10] these differences have been illustrated on a simple ($A_\mu = \text{constant}$) low dimensional ($\mathcal{A}_I = M_2(\mathbb{C})$) example on the circle ($M = S^1$), focusing on equatorial states (see definition, equation (44) below). The present paper aims at generalizing this result to arbitrary $A_\mu$ (non constant and of dimension $n \geq 2$) on the circle, taking into account non-equatorial states. Although self-contained, this work is better understood in relation with [10].

Our work organizes as follows: in section II we find that the connected components $\text{Con}(\xi_x)$ of the spectral distance on the $U(n)$-bundle $P \xrightarrow{\pi} S^1$ with arbitrary connection are tori $U_\xi$ with dimension $n_c \leq n$ given by the holonomy. The connected component $\text{Acc}(\xi_x)$ for the horizontal distance is a dense or a discrete subset of $U_\xi$, depending on the irrationality of the connection. The main result is stated in proposition II.4. In section III we compute all the distances in the $n = 2$ case, extending the results of [10] to non-equatorial states. Especially it turns out that the dependence of the distance on the "altitude" $z_\xi$ of the states is far from trivial, as explained in proposition III.5 together with its two related corollaries III.4 and III.6. These rather technical results are discussed in section III.3. Section IV deals with the $n > 2$ case for which we explicitly compute the spectral distance between states on the same fiber. Quite nicely we find a simple expression in terms of the trace of the module of a matrice $S$ whose components are given by the holonomy and the components of the two pure states under consideration.

## II Connected components

Consider the product of the finite dimensional spectral triple

$$\mathcal{A}_I = M_n(\mathbb{C}), \quad \mathcal{H}_I = M_n(\mathbb{C}), \quad D_I = 0,$$

by a riemannian compact spin manifold $M$. The vanishing of $D_I$ guarantees that the scalar part $H$ of the fluctuation is zero, so that the part of the fluctuated Dirac operator that plays a role in the computation of the distance is the usual covariant operator,

$$\mathcal{D} = -i\gamma^\mu (\partial_\mu \otimes \mathbb{I}_I + \mathbb{I}_E \otimes A_\mu).$$

$\mathcal{A}_E$ being nuclear the set of pure states of

$$\mathcal{A} = \mathcal{C}^\infty (M) \otimes M_n(\mathbb{C}) = \mathcal{C}^\infty (M, M_n(\mathbb{C}))$$

is $\mathcal{P}(\mathcal{A}) \cong \mathcal{P}(\mathcal{A}_E) \times \mathcal{P}(\mathcal{A}_I)$ with $\mathcal{P}(\mathcal{A}_I)$ the projective space $\mathbb{C}P^{n-1}$. The evaluation of

$$\xi_x \doteq (\omega_x \in \mathcal{P}(\mathcal{C}^\infty (M)), \omega_\xi \in \mathcal{P}(\mathcal{A}_I))$$

on $a = f^i \otimes m_i \in \mathcal{A}$ reads

$$\xi_x(a) = \text{Tr}_\xi s_\xi a(x)$$

(10)
where
\[ a(x) \doteq f^1(x) \otimes m_i \]  
(11)
and \( s_\xi \) is the support of \( \omega_\xi \) (the density matrix in physics),
\[ \omega_\xi(m) = \langle \xi, m\xi \rangle = \text{Tr } s_\xi m \]  
(12)
for \( m \in A_I, \xi \in \mathbb{C}P^{n-1} \). We write
\[ P \xrightarrow{\pi} M \]
the trivial bundle \( P(A) \) equipped with the connection (1-form) \( A_\mu \). The latest defines both a noncommutative distance \( d \) (formula (1) with \( \mathcal{D} \) instead of \( D_E \)) and a Carnot-Caratheodory - or horizontal - distance \( d_H \) (given in (7)).

Two states in \( P \) are at infinite horizontal distance if and only if there is no horizontal path between them. All pure states at finite horizontal distance from \( \xi_x \) are said accessible,
\[ \text{Acc}(\xi_x) \doteq \{ q \in P; d_H(\xi_x, q) < +\infty \}. \]  
(13)
Two pure states \( \xi_x, \zeta_y \) are said connected for \( d \) if and only if \( d(\xi_x, \zeta_y) \) is finite and we write
\[ \text{Con}(\xi_x) \doteq \{ q \in P; d(\xi_x, q) < +\infty \} \]  
(14)
the connected component of \( \xi_x \) for \( d \). One easily checks\(^{10}\) that \( \text{Acc}(\xi_x) \) is connected for \( d \),
\[ \text{Acc}(\xi_x) \subset \text{Con}(\xi_x). \]  
(15)
However there is no reason for \( \text{Con}(\xi_x) \) to equal \( \text{Acc}(\xi_x) \). \( d \) can remains finite although \( d_h \) is infinite. What matters is the holonomy of the connection: for \( d \) to equal \( d_H \) one needs a minimal horizontal curve \( c \) whose projection \( c_* \) on \( M \) does not selfintersects too many times. This seems an open question for subriemannian geometry\(^{13}\) (can one deform \( c \) keeping its length fixed and reducing the number of times \( c_* \) selfintersects ?). Here we escape the problem assuming there is only one possible minimal horizontal curve, namely
\[ M = S^1. \]

In low dimension \((A_I = M_2(\mathbb{C}))\) with constant connection \((A_\mu = \text{diag}(0, -i\theta \in i\mathbb{R}))\) we showed in [10] that \( \text{Con}(\xi_x) \) is a 2-torus and \( \text{Acc}(\xi_x) \) a dense subset if \( \theta \) is irrational. In the rest of this section we generalize this result to non constant gauge fields of dimension \( n \geq 2 \).

Let us begin by \( \text{Acc}(\xi_x) \), which is nothing but the horizontal lift of the circle with initial condition given by \( \xi \). It is interesting to work it out explicitly in order to introduce the main notations of the paper. On \( M = S^1 \) the gauge field has only one component \( A_\mu = -A_\mu^* = A \in \mathfrak{u}(n) \). Once for all we fix on \( \mathcal{H} \) a basis of real eigenvectors of \( iA \) such that
\[ A = i \begin{pmatrix} \theta_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \theta_n \end{pmatrix} \]  
(16)
where the \( \theta_j \)'s are real functions on \( S^1 \). Let \([0, 2\pi]\) parameterize the circle and \( x \) be the point with coordinate 0. Within a trivialization \((\pi, V)\) the horizontal lift \( c \) of the curve
\[ c_*(\tau) = \tau \mod [2\pi], \quad \tau \in ]-\infty, +\infty[ \]  
(17)
with initial condition
\[ V(c(0)) = \xi = \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} \in \mathbb{C}P^{n-1} \]
is \( c(\tau) = (c_s(\tau), V(\tau)) \) where \( V(\tau) \) has components
\[ V_j(\tau) = V_j e^{-i\Theta_j(\tau)} \quad (18) \]
with
\[ \Theta_j(\tau) = \int_0^\tau \theta_j(t)dt. \quad (19) \]
The points of \( P \) accessible from \( \xi_x = \xi_0 \doteq (\omega_{c_s(0)}, \omega_\xi) \) are the pure states
\[ \xi_\tau \doteq (\omega_{c_s(\tau)}, \omega_{V(\tau)}), \quad \tau \in \mathbb{R}. \quad (20) \]
On a given fiber \( \pi^{-1}(c_s(\tau)) \), \( \text{Acc}(\xi_x) \) reduces to
\[ \xi^k_\tau \doteq \xi_{\tau+2k\pi}, \quad k \in \mathbb{Z}, \quad (21) \]
with components
\[ V_j(\tau + 2k\pi) = V_j(\tau)e^{-i(k\Theta_j)(2\pi)}. \]
Dividing each \( \xi^k_\tau \) by the irrelevant phase \( e^{-i(k\Theta_i)(2\pi)} \) and writing
\[ \Theta_{ij} \doteq \Theta_i - \Theta_j, \quad (22) \]
one obtains that
\[ F^\xi_\tau \doteq \text{Acc}(\xi_x) \cap \pi^{-1}(c_s(\tau)) = \left\{ e^{i(k\Theta_{1j})2\pi(V_1(\tau))} : k \in \mathbb{Z}; j = 2, \ldots, n \right\}, \]
is a subset of the \((n-1)\)-torus
\[ T_\xi \doteq \left\{ e^{i\varphi_j}V_j : \varphi_j \in \mathbb{R}, \ j = 2, \ldots, n \right\}. \quad (23) \]
Hence as an immediate result

**Proposition II.1** \( \text{Acc}(\xi_x) \) is a subset of the \( n \)-torus (see figure 1)
\[ T_\xi \doteq S^1 \times T_\xi. \quad (24) \]

It is important to note that \( \text{Acc}(\xi_x) \) cannot be the whole torus,
\[ \text{Acc}(\xi_x) \subsetneq T_\xi. \quad (25) \]
At best, when all the \( \Theta_{1j}(2\pi) \)'s are distinct and irrational, \( T_\xi \) is the completion of \( \text{Acc}(\xi_x) \) with respect to the euclidean norm on \( T_\xi \). But as soon as one of the \( \Theta_{1j}(2\pi) \)'s is rational, \( \text{Acc}(\xi_x) \) is no longer dense in \( T_\xi \). For instance when all the \( \Theta_{1j}(2\pi) \)'s are rational, \( F^\xi_\tau \) is a discrete subset of \( T_\xi \). Say differently, the horizontal distance "forgets" about the fiber bundle structure of the set of pure states; and it forgets it twice:
1. whatever the connection two pure states on the same fiber belonging to distinct tori $\mathbb{T}_\xi, \mathbb{T}_\eta$ - namely $\xi_x$ and $\eta_x$ with components $|\eta_j| \neq |\nu_j|$ for at least one value $j$ - are infinitely Carnot-Carathéodory far from each other although they might be close in a suitable topology of $\mathbb{C}P^{n-1}$.

2. on a given component $\mathbb{T}_\xi$, $d_H(\xi_\tau, \xi_{k\tau}) = 2k\pi$ so in case the $\Theta_{1j}(2\pi)$ are irrational one can find close to $\xi_\tau$ (in the euclidean topology of $\mathbb{T}_\xi$) some $\xi^{k\tau}$ arbitrarily Carnot-Carathéodory far from $\xi_\tau$.

On the contrary the spectral distance keeps better in mind the fibre structure of $P$. For instance in the case $n = 2$ (i.e. $T_\xi = S^1$) and a constant non-trivial connection (i.e. $\theta_1 \neq \theta_2$) we found in [10] that $\text{Con}(\xi_x)$ is the whole 2-torus $\mathbb{T}_\xi$. Specifically on each $F_\xi^\tau$ the spectral distance appears as a "smoothing" of the euclidean distance $d_E$ on the circle (figure 2).

The situation is slightly more complicated with some arbitrary connection. Indeed less restriction on the connection gives more freedom for states to be at infinite distance from one another. As shown in proposition II.4 below, $\text{Con}(\xi_x)$ is still a subset of $\mathbb{T}_\xi$ but not necessarily equal to it. In particular when $n > 2$ some "directions" (with respect to the diagonalisation of $A$, eq.(16)) in the torus can be at infinite distance from one another.

Figure 2: from left to right, $d_H(\xi_\tau, \xi_{k\tau})$, $d(\xi_\tau, \xi_{k\tau})$, $d_E(\xi_\tau, \xi_{k\tau})$ with $k$ running from 0 to 60. Horizontal unit is $\pi$. 
Definition II.2 We say that two directions \( i, j \) of \( T_\xi \) are far from each other if the components \( i \) and \( j \) of the holonomy at \( x \) are equal, and we write \( \text{Far}(\cdot) \) the equivalence classes,

\[
\text{Far}(i) = \{ j \in [1, n] \text{ such that } \Theta_j(2\pi) = \Theta_i(2\pi) \mod[2\pi] \}. \tag{26}
\]

Two directions belonging to distinct equivalence classes are said close to each other. We denote \( n_c \) the numbers of such equivalence classes and we label them as

\[
\text{Far}_1 = \text{Far}(1), \text{Far}_p = \text{Far}(j_p) \quad p = 2, \ldots, n_c
\]

where \( j_p \neq 0 \) is the smallest integer that does not belong to \( n+1 \sum_{q=1}^{p-1} \text{Far}_q \).

Of course our terminology is transparent, and we show below that two states are connected if and only if they belong to close directions. Before establishing this result, let us recall two simple lemmas: first [5, lemma 1] the supremum in the distance formula can be searched on selfadjoint elements of \( A \). Second [10, lemma III.2]

Lemme II.3 \( d(\xi_x, \zeta_y) \) is infinite if and only if there is a sequence \( a_n \in \mathcal{A} \) such that

\[
\lim_{n \to +\infty} \| [D, a_n] \| \to 0, \quad \lim_{n \to +\infty} | \xi_x(a_n) - \zeta_y(a_n) | = +\infty. \tag{27}
\]

This allows to prove the main result of this section:

Proposition II.4 \( \text{Con}(\xi_x) \) is the \( n_c \) torus

\[
\mathcal{U}_\xi = \bigcup_{\tau \in S^1} U^\tau_\xi \tag{28}
\]

where \( U^\tau_\xi \subset T_\xi \) is the \( (n_c-1) \) torus defined by \( (V_i(\tau) \text{ is given in } (18))

\[
U^\tau_\xi = \left\{ \begin{array}{c}
V_i(\tau) \quad \forall i \in \text{Far}_1 \\
e^{i\varphi_2} V_i(\tau) \quad \forall i \in \text{Far}_2 \\
e^{i\varphi_\infty} V_i(\tau) \quad \forall i \in \text{Far}_{n_c}
\end{array} \right\}, \varphi_j \in \mathbb{R}, j \in [2, n_c]. \tag{29}
\]

Proof. Let \( a_{ij} \in \mathcal{A}_E \) be the components of \( a = a* \in \mathcal{A} \), identified as \( 2\pi \)-periodic complex functions on \( \mathbb{R} \),

\[
a_{ij}(\tau) = a_{ij}(c_\tau(\tau)) = a_{ij}(\tau + 2k\pi) \quad k \in \mathbb{Z} \tag{30}
\]

with initial condition \( a_{ij}(0) = a_{ij}(x) \). Let dot denote the derivative. The Clifford action reduces to the multiplication by \( 1 \ (\gamma^0 = \gamma^1 = 1) \) so that \( [D, a_{ij}] = -ia_{ij} \) and

\[
i[D, a] = \begin{pmatrix}
a_{11} & a_{12} + ia_{12} \theta_{12} & \cdots & a_{1n} + ia_{1n} \theta_{1n} \\
a_{21} - ia_{21} \theta_{12} & a_{22} & \ddots & \\
\vdots & \ddots & \ddots & \ddots \\
a_{n1} - ia_{n1} \theta_{1n} & \cdots & \cdots & a_{nn}
\end{pmatrix} \tag{31}
\]

where \( \theta_{ij} \) is defined similarly as in (22). The commutator (31) is zero if and only if

\[
a_{ii} = C_i = \text{constant}, \quad a_{ij}(\tau) = C_{ij} e^{-i\Theta_{ij} \tau}. \tag{32}
\]
Under these conditions, for any $\xi \in \mathbb{C}P^{n-1}$ with components $V^i$ and $y = c_s^{-1}(\tau)$,

$$\xi_y(a) = \sum_{i=1}^n C_i |V_i|^2 + \sum_{i<j} 2 \text{Re} \left( \bar{V}_i V_j C_{ij} e^{-i \Theta_{ij}(\tau)} \right). \quad (33)$$

Consider now $\zeta \in \mathbb{C}P^{n-1}$ with components $W_i$ such that, for at least one value $i_0 \in [1, n]$,

$$|V_{i_0}| \neq |W_{i_0}|. \quad (34)$$

One easily finds some $a_0$ commuting with $\mathcal{D}$ such that $\xi_x(a_0) \neq \zeta_y(a_0)$, for instance

$$C_{ij} = C_i = 0 \text{ for all } i, j \text{ except } C_{i_0} \neq 0.$$

Thus $d(\xi_x, \zeta_y)$ is infinite by lemma II.3 (consider $a_n = na_0$). Hence $\text{Con}(\xi_x) \cap \pi^{-1}(c_s(\tau))$ is included within the set of pure states $\zeta_y$ that do not satisfy (34), i.e. those with components

$$W_i = e^{i \varphi_i} V_i, \quad \varphi_i \in \mathbb{R}, \; i = 1, ..., n. \quad (35)$$

Note that dividing by an irrelevant phase $e^{i \varphi_1}$, one is back to the torus $T_\xi$ defined in (23), hence $\text{Con}(\xi_x) \subset T_\xi$ as expected. Let us equivalently rewrite (35) as

$$W_i = e^{i \varphi_i} V_i(\tau) \quad (36)$$

and assume $\zeta$ has such components. For $a$ commuting with the Dirac operator (33) yields

$$\zeta_y(a) - \xi_x(a) = \sum_{i<j} 2 \text{Re} \left( \bar{V}_i V_j C_{ij} (e^{-i(\varphi_i - \varphi_j)} - 1) \right). \quad (37)$$

Since $a_{ij}$ is $2\pi$-periodic, the complex constant $C_{ij}$ defined in (32) can be non zero only if

$$\Theta_{ij}(\tau + 2k\pi) = \Theta_{ij}(\tau) \mod[2\pi], \quad (38)$$

that is to say, since by periodicity of $\theta_{ij}$

$$\Theta_{ij}(\tau + 2k\pi) = \Theta_{ij}(\tau) + k\Theta_{ij}(2\pi), \quad (39)$$

only if the directions $i$ and $j$ are far from each other in the sense of definition II.2. In other terms $C_{ij}$ is zero for any couples of close directions $i, j$. Therefore requiring (37) to vanish puts some conditions only on couples of directions far from each other. Explicitly (37) vanishes for any $a$ commuting with $\mathcal{D}$ if and only if

$$\varphi_i = \varphi_j \mod[2\pi] \text{ for all } j \text{ in } \text{Far}(i).$$

Therefore, by lemma II.3, the only $\zeta_y$ at finite distance from $\xi_x$ are those with components

$$W_i = e^{i \varphi_i} V_i(\tau) \quad \forall i \in \text{Far}(j)$$

which, up to an irrelevant phase $e^{i \varphi_1}$, is nothing but $U^\xi_\tau$. \hfill \blacksquare

Let us close this section with few comments. Obviously $e^{i \Theta_{ij}(2k\pi)} = e^{i \Theta_{ij}(2k\pi)}$ for any $j \in \text{Far}(i)$, hence $F^\xi_\tau \subset U^\xi_\tau$ as expected from (15). Also obvious is the inclusion of $U_\xi$ within $T_\xi$. So, to summarize,

$$\text{Acc}(\xi_x) \subset U_\xi \subset T_\xi. \quad (40)$$
The difference between $\text{Acc}(\xi x)$ and $U_\xi$ is governed by the irrationality of the connection - as explained below eq. (25) - whereas the difference between $\text{Con}(\xi x)$ and $T_\xi$ is governed by the number of close directions. More specifically

$$T_\xi = \bigcup_{\zeta \in T_\xi} \text{Acc}(\zeta x)$$  \hspace{1cm} (41)

while

$$U_\xi = \bigcup_{\zeta \in U_\xi} \text{Acc}(\zeta x)$$  \hspace{1cm} (42)

where we write $U_\xi = U_0^\xi$. When all the directions are close to each other (e.g. when the functions $\theta_i$’s are constant and distinct from one another) then $U_\xi = T_\xi$ so that

$$U_\xi = T_\xi.$$

On the contrary when all the directions are far from each other, that is to say when the holonomy is trivial, $n_c = 1$ and in agreement with [10, Prop. IV.1] one obtains

$$U_\xi = S^1 = \text{Acc}(\xi x).$$

Note that none of the distance is able to ”see” between different tori $T_\xi$, $T_\eta$. However within a given $T_\xi$ eq.(42) indicates that the spectral distance is able to see between the horizontal components. In this sense the spectral distance keeps ”better in mind” the bundle structure of the set of pure states $P$. This indicates that in a more general framework of foliation, the spectral distance could be relevant to study some transverse metric structure.

### III Spectral distance on the circle with a $\mathbb{C}P^1$ fiber

In the rest of the paper we investigate the spectral distance $d$ on $\text{Con}(\xi x)$ for arbitrary connection and any choice of $\xi x$. In the case $n = 2$ we explicitly compute $d$ on all $\text{Con}(\xi x)$, thus generalizing the result of [10] to non-constant connection and non-equatorial $\xi$ (see definition below, eq.(44)). For $n \geq 2$ we compute $d$ on the fiber.

#### III.1 From $\mathbb{C}P^1$ to $S^2$

The set of pure states of $A_I = M_2(\mathbb{C})$ identifies to the sphere $S^2$ via the traditional correspondence

$$\xi = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \in \mathbb{C}P^1 \longleftrightarrow \begin{pmatrix} x_\xi \equiv 2\Re(V_1 \overline{V_2}) \\ y_\xi \equiv 2\Im(V_1 \overline{V_2}) \\ z_\xi \equiv |V_1|^2 - |V_2|^2 \end{pmatrix}. \hspace{1cm} (43)$$

The torus $T_\xi$ defined in (23) is now a circle $S_R$ of radius $R \doteq (1 - z_\xi^2)^{1/2}$ located inside $S^2$ at the "altitude" $z_\xi$. Thus $P = \mathcal{P}(A)$ decomposes in 2-tori $^8$

$$T_\xi = S^1 \times S_R,$$

each of them containing the connected component $U_\xi$ associated to the pure states of altitude $z_\xi$. In [10] we assumed the connection $A$ was constant and not a multiple of the

\footnote{To each value of $R$ correspond two tori, labeled by the sign of $z_\xi$.}
identity so that the two directions of $T_\xi$ were closed to each other in the sense of definition II.2, hence $T_\xi = U_\xi$. We computed $d(\xi_x, \zeta_y)$ for any $\zeta_y \in T_\xi$ assuming $\xi$ was an equatorial state, namely

$$z_\xi = 0.$$  \hspace{1cm} (44)

In the following we investigate the case of non-vanishing $z_\xi$ with arbitrary connection $A$.

Let us first parameterize $T_\xi$ in a most suitable way regarding distance's calculation.

**Definition III.1** Given $\xi_x$ in $P$, any $\zeta_y \in T_\xi$ is in one-to-one correspondence with an equivalence class (see figure 3)

$$(k \in \mathbb{N}, 0 \leq \tau_0 \leq 2\pi, 0 \leq \varphi \leq 2\pi) \sim (k + \mathbb{Z}, \tau_0, \varphi - 2\mathbb{Z}\omega\pi)$$  \hspace{1cm} (45)

with\footnote{In order to deal with positive $k$ and $\tau_0$, we assume that the coordinate $\tau$ of $y$ in the parameterization (17) of $S^1$ is positive. This is always possible up to a permutation of $\xi_x$ and $\zeta_y$.}

$$\tau = 2k\pi + \tau_0, \quad \omega = \frac{\Theta_1(2\pi) - \Theta_2(2\pi)}{2\pi}$$  \hspace{1cm} (46)

such that

$$\zeta_y = \begin{pmatrix} V_1(\tau) \\ e^{i\varphi}V_2(\tau) \end{pmatrix}$$  \hspace{1cm} (47)

where $V(\tau)$ are the components of $\xi_\tau$, parallel transport of $\xi_x$ from $x$ to $y$, given in (18).

Figure 3: $\zeta_y$ parameterized by $\tau_0$ and an equivalence class $(k, \varphi)$. 
III.2 Detail of the computation in the case \( n = 2 \)

Let us begin by some preliminary results, before stating the main result of this section, proposition III.5. The following subsection is technical, interpretation and link with the result of [10] is postponed to the next subsection.

**Lemme III.2** The search for the supremum in computing \( d(\xi_x, \zeta_y) \) can be restricted to the set of selfadjoint \( a \in A \) whose diagonal \( a_1 \) and off-diagonal \( a_0 = a - a_1 \) are such that

\[
\begin{align*}
a_1(x) &= 0, \quad \text{Tr} a_1(y) \geq 0, \\
\Delta_\xi(a_1) &\geq 0, \quad \Delta_\xi(a_0) \geq 0 \\
z_\xi \delta(y) &\geq 0
\end{align*}
\]

where \( z_\xi \) is defined in (43) and

\[
\Delta_\xi \hat{=} \zeta_y - \xi_x, \quad \delta \hat{=} a_{11} - a_{22}
\]

with \( a_{ij} \) the components of \( a \).

**Proof.** That \( a \) is selfadjoint comes from a general result according to which the supremum in the distance formula can searched on positive elements (see [5], lemma 1). The diagonal part \( a_1 \) enters the commutator condition,

\[
||[\mathcal{D}, a]|| = \sup_{x \in S^1} \left( \frac{1}{2}|\text{Tr} a_1(x)| + \sqrt{\left(\frac{\delta^2(x)}{4} + ||[\mathcal{D}, a_0](x)||^2\right)} \right)
\]

only via its derivative. Moreover for any constant \( K_i, i = 1, 2 \),

\[
\Delta_\xi(a_1 - K_i e_i) = \Delta_\xi(a_1)
\]

where \( e_i \) is the projector on the \( i^{th} \) term of the diagonal. Therefore if \( a \) attained the supremum, so does \( a - K_i e_i \) whose diagonal vanishes at \( x \) as soon as one chooses \( K_i = a_{ii}(x) \). Hence the first term of (48).

Let \( G \) be the group, acting on the components of \( a \), generated by the replacements

\[
\begin{align*}
a_1 &\rightarrow -a_1, \\
a_0 &\rightarrow -a_0
\end{align*}
\]

and the permutation

\[
a_{11} \leftrightarrow a_{22}.
\]

Since (52) is invariant under the action of \( G \) and \( a \rightarrow -a \) belongs to \( G \), \( \Delta_\xi(a) \) can be assumed positive. Moreover \( a \) attaining the supremum means that

\[
\Delta_\xi(a) \geq \Delta_\xi(g(a))
\]

for any \( g \in G \). In particular one gets (49) from \( g \) given by (53) and (54). (50) comes from (55). \( \text{Tr} a_1(y) \) being positive comes from (53) followed by (55). \( \blacksquare \)
Other simplifications come from the choice of $S^1$ as the base manifold.

**Lemme III.3** Let $a = a_0 + a_1$ with components $a_{ij}$ as in lemma above. If $a$ satisfies the commutator norm condition, then for $i = 1, 2$

$$ |a_{ii}(\tau)| \leq F(\tau) \tag{57} $$

where $F(\tau)$ is the $2\pi$-periodic function defined on $[0, 2\pi]$ by

$$ F(\tau) = \min(\tau, 2\pi - \tau). \tag{58} $$

Meanwhile

$$ a_0 = \begin{pmatrix} 0 & ge^{-i\Theta} \\ ge^{i\Theta} & 0 \end{pmatrix} \tag{59} $$

where

$$ \Theta = \Theta_1 - \Theta_2 $$

($\Theta_1$ is defined in (19)) and $g$ is a smooth function on $\mathbb{R}$ given by

$$ g(\tau) = g_0 + \int_0^\tau \rho(u)e^{i\phi(u)}du \tag{60} $$

with $\rho \in C^\infty(\mathbb{R}, \mathbb{R}^+)$, $\|\rho\| \leq 1$, and $\phi \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying

$$ \rho(u + 2\pi)e^{i\phi(u + 2\pi)} = \rho(u)e^{i(\phi(u) + \Theta(2\pi))} \tag{61} $$

while the integration constant $g_0 \in \mathbb{C}$ is fixed by the equation

$$ g_0(e^{i\Theta(2\pi)} - 1) = \int_0^{2\pi} \rho(u)e^{i\phi(u)}du. \tag{62} $$

**Proof.** (57) comes from the commutator norm condition

$$ \|\dot{a}_{ii}\| = \|e_i[D, a]e_i\| \leq \|[D, a]\| $$

where $e_i$ is the projector on the $i$th component on the diagonal, together with the $2\pi$-periodicity of $a_{ii}$ (30) namely

$$ a_{ii} = \int_0^\tau \dot{a}_{ii}(u)du = -\int_{\tau}^{2\pi} \dot{a}_{ii}(u)du. $$

The explicit form of $a_0$ is obtained by noting that any complex smooth function $a_{12} \in A_E$ can be written $ge^{-i\Theta}$ where, using (39), $g = a_{12}e^{i\Theta} \in C^\infty(\mathbb{R})$ satisfies

$$ g(\tau + 2\pi) = g(\tau)e^{i\Theta(2\pi)}. \tag{63} $$

Hence any selfadjoint $a_0$ writes as in (59), which yields by (31)

$$ i[D, a_0] = \begin{pmatrix} 0 & \dot{g}e^{-i\Theta} \\ \dot{g}e^{i\Theta} & 0 \end{pmatrix}. \tag{64} $$

By (52) the commutator norm condition implies

$$ \|[D, a_0]\| = \|\dot{g}\| \leq 1, $$

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that is to say
\[ g(\tau) = g(0) + \int_0^\tau \rho(u)e^{i\phi(u)}du \quad (65) \]

where \( \rho \in C^\infty(\mathbb{R}, \mathbb{R}^+) \), \( \|\rho\| \leq 1 \), \( \phi \in C^\infty(\mathbb{R}, \mathbb{R}) \). (62) is obtained from (60) inserted in (63) for \( \tau = 0 \). Finally (62) inserted back in (63) gives
\[
\int_0^\tau \rho(u + 2\pi)e^{i\phi(u+2\pi)}du = \int_0^\tau \rho(u)e^{i(\phi(u)+\Theta(2\pi))},
\]
hence (61) by derivation with respect to \( \tau \).

**Corollary III.4** The search for the supremum in the computation of \( d(\xi, \zeta) \) can be restricted to elements \( a \) whose diagonal components at \( y \),
\[
\Delta = \frac{1}{2}\delta(y), \quad T = \frac{1}{2} \text{Tr}(a(y)) \quad (66)
\]
lie within the triangle
\[
T_\pm = T \pm \Delta \leq \min(\tau_0, 2\pi - \tau_0) \quad (67)
\]
where the sign is the one of \( z_\xi \) (fig. 4).

**Proof.** When \( z_\xi \geq 0 \), \( \Delta \) is positive by (50) (i.e. \( a_{11}(y) \geq a_{22}(y) \)) so that at least \( a_{11}(y) \) is positive otherwise (48) does not hold. Therefore
\[
T + \Delta = a_{11}(y) \leq F(\tau_0)
\]
by (57) and the parameterization (45) of \( \zeta_y \). Similarly for negative \( z_\xi \) at least \( a_{22}(y) \) is positive, hence \( T - \Delta = a_{22}(y) \leq F(\tau_0) \).

In [10] we obtained by simple algebraic manipulations that for \( z_\xi = 0 \) the supremum was reached by \( \Delta = 0 \). **Corollary III.4** indicates that the \( z_\xi \) dependence of the distance is in fact far from trivial.

Thanks to these preliminary results it is easy to come to the main result of this section, namely the computation of all the distances on \( \mathbb{T}_\xi \).

![Figure 4: The diagonal part of a. Unit is min(\( \tau_0, 2\pi - \tau_0 \)).](image-url)
Proposition III.5 Let $\xi_x$ be a pure state in $P$ and $\zeta_y$ a pure state in $T_\xi$ parameterized according to (45) by a triple $(k, \tau_0, \varphi)$. Then either the two directions are far from each other so that $\text{Con}(\xi_x) = \text{Acc}(\xi_x)$ and

$$d(\xi_x, \zeta_y) = \begin{cases} \min(\tau_0, 2\pi - \tau_0) & \text{when } \varphi = 0 \\ +\infty & \text{when } \varphi \neq 0 \end{cases} ;$$

(68)

or the directions are close to each other so that $\text{Con}(\xi_x) = T_\xi(\xi_x)$ and

$$d(\xi_x, \zeta_y) = \max_{T_\xi} H_\xi(T_\Delta)$$

(69)

where the sign is the one of $z_\xi$ and

$$H_\xi(T_\Delta) = T + z_\xi \Delta + RW_{k+1} \sqrt{(\tau_0 - T)^2 - \Delta^2} + RW_k \sqrt{(2\pi - \tau_0 - T)^2 - \Delta^2}$$

(70)

with

$$W_k = \frac{|\sin(k\omega\pi + \varphi)|}{|\sin \omega\pi|}.$$  

(71)

Proof. (68) comes from a general result [10, Prop. 3] according to which for trivial holonomy the spectral distance coincides with the horizontal distance, which in turn is either infinite or coincides with the geodesic distance on the basis. In the present context one can retrieve this result by explicit calculation and we comment on this point in the paragraph next to this proof. For the time being, we assume $\omega \neq 0$. Let $a$ be an element of $A$ given by lemma III.2. By (49),

$$|\Delta_\xi(a)| = |\Delta_\xi(a_0) + \Delta_\xi(a_1)|$$

(72)

so that a good strategy to find the supremum is to begin with constraining $\Delta_\xi(a_0)$ on the one side (eq.(108) below), then adding $\Delta_\xi(a_1)$. Writing

$$2V_1V_2 = Re^{i\theta_0},$$

(73)

Eqs. (10) and (12) written for $\xi_x$ and $\zeta_y$ (the lastest given by (47)) yield

$$\Delta_\xi(a_0) = \Re \left( Re^{-i\theta_0} g(\tau)e^{i\varphi} - g(0) \right),$$

that we rewrite, using (60),

$$\Delta_\xi(a_0) = R \int_0^\tau \rho(u) \cos \phi'(u) + \Re \left( Re^{-i\theta_0} g_0(e^{i\varphi} - 1) \right),$$

(74)

where

$$\phi'(u) = \phi(u) - \theta_0 + \varphi.$$  

(75)

By the definition (46) of $\tau$, using (61), the integral term of (74) splits into

$$\Re \int_0^{2k\pi} \rho(u)e^{i\phi'(u)} du = \Re \left( \sum_{n=0}^{k-1} e^{2in\omega\pi} \int_0^{2\pi} \rho(u)e^{i\phi'(u)} du \right)$$

(76)

and

$$\Re \int_{2k\pi}^{2k\pi+\tau_0} \rho(u)e^{i\phi'(u)} du = \Re \left( e^{2ik\omega\pi} \int_0^{\tau_0} \rho(u)e^{i\phi'(u)} du \right)$$

(77)

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that recombine as
\[ S_{k+1} \int_{0}^{\tau_0} \rho(u) \cos \phi_k(u) \, du + S_k \int_{\tau_0}^{2\pi} \rho(u) \cos \phi_{k-1}(u) \, du \tag{78} \]

where
\[ S_k = \frac{\sin k\omega \pi}{\sin \omega \pi} \quad \text{and} \quad \phi_k = \phi' + k\omega \pi. \tag{79} \]

Meanwhile, after a few calculations with (62), the real-part term of (74) writes
\[ S_{1/2} \int_{0}^{2\pi} \rho(u) \cos \phi_{1/2}(u) \, du \tag{80} \]

where
\[ S_{1/2} = \frac{\sin \varphi/2}{\sin \omega \pi} \quad \text{and} \quad \phi_{1/2} = \phi' - \frac{\varphi}{2} - \omega \pi. \tag{81} \]

Combining (78) and (80) one finally obtains
\[ \Delta_\xi(a_0) \leq RW_{k+1} \int_{0}^{\tau_0} \rho(u) \, du + RW_{k} \int_{\tau_0}^{2\pi} \rho(u) \, du \tag{82} \]

where \( W_k \) is the maximum on \([0, 2\pi]\) of \(|G_k(u)|\) defined by
\[ G_k \doteq S_k \cos \phi_{k-1} + S_{1/2} \cos \phi_{1/2}. \tag{83} \]

So far we used the commutator norm condition on \( a_0 \) and \( a_1 \) independently, via lemmas III.2 and III.3. To obtain a global constraint on \( a \), first note that
\[ \|[D, a_0]\| = \left\| -i \begin{pmatrix} 0 & \hat{g} e^{-i\Theta} \\ \hat{g} e^{i\Theta} \end{pmatrix} \right\| = \rho \tag{84} \]

by (60), so that (52) yields for any \( a \) satisfying the commutator norm condition
\[ \rho \leq \sqrt{\left(1 - \frac{|\text{Tr} a_1|}{2}\right)^2 - \left(\frac{\delta}{2}\right)^2} \tag{85} \]

that is, by Jensen inequality,\(^\dagger\)
\[ \int_{a}^{b} \rho(t) \, dt \leq \sqrt{\left(b - a - \frac{|\text{Tr} a_1(a) - \text{Tr} a_1(b)|}{2}\right)^2 - \left(\frac{\delta(a) - \delta(b)}{2}\right)^2}. \tag{89} \]

\(^\dagger\)For \( a \leq b \) and any \( f \in L_1([a, b]) \) (\( f \) positive in the first equation) one has\(^7\)
\[ \int_{a}^{b} \sqrt{f(t)} \, dt \leq \sqrt{b - a} \sqrt{\int_{a}^{b} f(t) \, dt}, \tag{86} \]
\[ \int_{a}^{b} f^2(t) \, dt \geq \frac{1}{b - a} \left( \int_{a}^{b} f(t) \, dt \right)^2, \tag{87} \]
\[ \int_{a}^{b} (1 - f(t))^2 \, dt \leq (b - a) \left(1 - \frac{1}{b - a} \int_{a}^{b} f(t) \, dt \right)^2. \tag{88} \]
Therefore (82) together with the vanishing of $a_1$ at $x$ and the positivity of $\text{Tr} \ a_1(y)$ yields the announced bound

$$\Delta_\xi(a_0) \leq RW_{k+1} \sqrt{(\tau_0 - T)^2 - \Delta^2} + RW_k \sqrt{((2\pi - \tau_0) - T)^2 - \Delta^2}$$

(90)

where we use the notations (66). Adding

$$\Delta_\xi(a_1) = T + z_\xi \Delta$$

(91)

one obtains that $\Delta_\xi(a)$ is inferior or equal to the right hand side of (69).

To prove the r.h.s. of (69) is the lowest upper bound, we need a sequence of elements

$$a_n = \left( \begin{array}{cc} f_+^n & gn e^{-i\Theta} \\ \bar{g}_n e^{i\Theta} & f_-^n \end{array} \right)$$

(92)

such that

$$\lim_{n \to +\infty} \Delta_\xi(a_n) = \max_{T^\pm} H_\xi(T, \Delta).$$

(93)

The diagonal part of $a_n$ is defined by the values $(T_0, \Delta_0) \in T^\pm$ for which $H$ attains its maximum (see corollary III.6 below) in the following way: assuming $\tau_0 \neq 0$, then $f_n^\pm$ is a sequence of smooth functions approximating from below the $2\pi$-periodic continuous function,

$$f_n^\pm(t) = \begin{cases} C_n^\pm t & \text{for } 0 \leq t \leq \tau_0 \\ C_n^\pm \tau_0 - \frac{C_n^\pm \tau_0}{2\pi - \tau_0} (t - \tau_0) & \text{for } \tau_0 \leq t \leq 2\pi \end{cases} \text{ with } C_n^\pm = \frac{T_0 \pm \Delta_0}{\tau_0}. $$

(94)

In other words we build $a_n$ in such a way that

$$\lim_{n \to +\infty} \frac{1}{2} \text{Tr} \ a_n(t) = \begin{cases} \frac{T_0 t}{\tau_0} & \text{for } 0 \leq t \leq \tau_0 \\ \frac{T_0}{\tau_0} - \frac{T_0}{2\pi - \tau_0} (t - \tau_0) & \text{for } \tau_0 \leq t \leq 2\pi \end{cases}$$

(95)

while, writing $\delta_n = f_n^+ - f_n^-$,

$$\lim_{n \to +\infty} \frac{1}{2} \delta_n(t) = \begin{cases} \frac{\Delta_0 t}{\tau_0} & \text{for } 0 \leq t \leq \tau_0 \\ \frac{\Delta_0}{2\pi - \tau_0} (t - \tau_0) & \text{for } \tau_0 \leq t \leq 2\pi \end{cases}. $$

(96)

Thus the diagonal part $a_{1,n}$ of $a_n$ does satisfy lemmas III.2,III.3 and yields

$$\lim_{n \to +\infty} \Delta_\xi(a_{n,1}) = T_0 + z_\xi \Delta_0.$$

(97)

In case of vanishing $\tau_0 = 0$, one simply choose $a_{n,1} = 0$. The off-diagonal part $a_{n,0}$ of $a_n$ is defined by substituting $\rho$ and $\phi$ in the definition (65) with sequences $\rho_n, \phi_n$ approximating from below the $2\pi$-periodic step functions $\Gamma, \Phi$ defined as follow: $\Gamma$ is given by (85) with $\text{Tr} \ a_1$ and $\delta$ replaced by their limit values above, namely

$$\Gamma \equiv \begin{cases} \sqrt{\left(1 - \frac{T_0}{\tau_0}\right)^2 - \left(\frac{\Delta_0}{\tau_0}\right)^2} & \text{for } 0 \leq t \leq \tau_0 \\ \sqrt{\left(1 - \frac{T_0}{2\pi - \tau_0}\right)^2 - \left(\frac{\Delta_0}{2\pi - \tau_0}\right)^2} & \text{for } \tau_0 \leq t \leq 2\pi \end{cases}. $$

(98)
In case \( \tau_0 = 0 \), one takes \( \Gamma = 1 \). This guarantees that \( a_n \) satisfy the commutator norm condition. In order to define \( \Phi \), one needs to work out \( W_k \) explicitly. Easy calculations from (83) yields
\[
G_k = A_k \cos \phi' + B_k \sin \phi' \quad \text{with} \quad \begin{cases} 
A_k &= S_k^2 \cos \left( \frac{\varphi}{2} + \omega \pi \right) + S_k \cos (k-1) \omega \pi, \\
B_k &= S_k^2 \sin \left( \frac{\varphi}{2} + \omega \pi \right) - S_k \sin (k-1) \omega \pi,
\end{cases}
\]
so that \( G_k \) attains its maximum value
\[
W_k = \sqrt{A_k^2 + B_k^2} = \frac{\left| \sin(k \omega \pi + \frac{\varphi}{2}) \right|}{\sin(\omega \pi)}
\]
when
\[
\tan \phi' = \frac{B_k}{A_k}
\]
that is to say, after a few algebraic manipulations, when
\[
\cos^2 \phi' = \frac{A_k^2}{W_k^2} = \cos^2 \left( (k-1) \omega \pi - \frac{\varphi}{2} \right).
\]
Thus, remembering (75), \( G_k[u] \) is constantly maximum on \([0, 2\pi]\) as soon as \( \phi(u) = \Phi_k \)

where
\[
\Phi_k \triangleq (1 - k) \omega \pi - \frac{\varphi}{2} + \theta_0 \quad \text{or} \quad \Phi_k \triangleq (1 - k) \omega \pi - \frac{\varphi}{2} + \theta_0 + \pi
\]
depending on the sign of \( A_k \) and \( B_k \). Hence the required \( 2\pi \)-periodic step function \( \Phi \) defined on \([0, 2\pi]\) by
\[
\Phi(u) = \begin{cases} 
\Phi_{k+1} & \text{for } 0 \leq u < \tau_0 \\
\Phi_k & \text{for } \tau_0 < u < 2\pi.
\end{cases}
\]
Repeating the procedure that leads to (82), on easily checks that
\[
\lim_{n \to +\infty} \Delta_\xi(a, n, 0) = RW_{k+1} \int_0^{\tau_0} \Gamma(u) du + RW_k \int_{\tau_0}^{2\pi} \Gamma(u) du
\]
\[= RW_{k+1} \sqrt{(\tau_0 - T_0)^2 - \Delta_0^2} + RW_k \sqrt{(2\pi - \tau_0 - T_0)^2 - \Delta_0^2} \]
Adding (97), one finally gets \( \lim_{n \to +\infty} \Delta_\xi(a, n) = H_i(T_0, \Delta_0) \) as expected in (93). Hence (69).

One could be tempted to search a more explicit formula of \( d(\xi_x, \zeta_y) \) by determining the maximum of the function \( H \). This is not clear whether this maximum has such a simpler form. To see where is the difficulty, let us first examine the easy case of vanishing \( \omega \). The integral part of (74) is still given by (78) assuming \( S_k = k \). Regarding the real part term of (74), either it can be made infinite when \( \varphi \) does not vanish (since \( g_0 \) does not constrain \( g_0 \) any longer) thus yielding the second line of (68), or it vanishes when \( \varphi = 0 \) and (80) makes sense assuming \( S_{\frac{1}{2}} = 0 \). Practically one obtains
\[
\Delta_\xi(a_0) \leq R(k + 1) \int_0^{\tau_0} \rho(u) \cos \phi'(u) du + Rk \int_{\tau_0}^{2\pi} \rho(u) \cos \phi'(u) du.
\]
Now the vanishing of $\omega \pi = 0$ makes the function $g$ defined in (63) $2\pi$-periodic, therefore
\[ \int_0^{2\pi} \rho(u) e^{i\phi(u)} = 0 \] (105)
and (108) writes
\[ \Delta_\xi(a_0) \leq R \int_0^{\tau_0} \rho(u) \cos \phi'(u) \, du = -R \int_{\tau_0}^{2\pi} \rho(u) \cos \phi'(u) \, du \] (106)
\[ \leq R \min \left( \int_0^{\tau_0} \rho(u) \, du, \int_{\tau_0}^{2\pi} \rho(u) \, du \right) . \] (107)

By Jensen inequalities one has, instead of the r.h.s. of (69),
\[ \Delta_\xi(a) \leq \min \left( \max_{T_\pm} H_{\gamma_0}, \max_{T_\pm} H_{2\pi-\tau_0} \right) \] (108)
where
\[ H_\lambda = T + z_\xi \Delta + R \sqrt{(\lambda - T)^2 - \Delta^2} . \] (109)

The maximum of $H_\lambda$ is easily found by noting that for a fixed value of $T$, $H_\lambda$ is maximum for $\Delta^2 = z^2_\xi (\delta - T)^2$, which yields $H_\lambda = \lambda$ and (109) = (68) as expected.

For non vanishing $\omega$ the situation is more complicated since $H_\xi$, unlike $H_\lambda$, involve two square roots. In fact there does not seem to be a simple form for the maximum of $H_\xi$. So far the best we managed to do is summarized in the following corollary.

**Corollary III.6** $H_\xi$ reaches its maximum either on the segment $T = 0$ or on the segment $T \pm \Delta = \min(\tau_0, 2\pi - \tau_0)$.

**Proof.** Let us first show that $H_\xi$ attains it maximum on the border of $T_\pm$. A local maximum $p = (T, \Delta)$ of $H_\xi$ inside $T_\pm$ is a solution of the system
\[ \frac{\partial H_\xi}{\partial T} \bigg|_p = 0, \quad \frac{\partial H_\xi}{\partial \Delta} \bigg|_p = 0 . \] (110)

Seen as a system in the non vanishing variables
\[ V_1 = \sqrt{(\tau_0 - T)^2 - \Delta^2}, \quad V_2 = \sqrt{(2\pi - \tau_0 - T)^2 - \Delta^2} , \] (111)
(110 has solution
\[ \left\{ \begin{array}{l}
\Delta - Z(\tau_0 - T) = 2RW_k \Delta V_2^{-1}(\pi - \tau_0), \\
Z(2\pi - \tau_0 - T) - \Delta = 2RW_{k+1} \Delta V_2^{-1}(\pi - \tau_0) .
\end{array} \right. \] (112)

For $\pi = \tau_0$, the above system in $T, \Delta$ is solved by the straight line $l$
\[ \Delta = 0 \quad \text{when } Z = 0, \] (113)
\[ T = \pi - \frac{\Delta}{Z} \quad \text{when } Z \neq 0. \] (114)

The latest yields
\[ H_\xi(T, \Delta) = \pi + \frac{\Delta R^2}{Z} (W_k + W_{k+1} - 1) \] (115)
whose derivative with respect to $\Delta$ is constant. Therefore whatever $Z$, $H_\xi$ attains its maximum on one of the end point of $l \cap T_\pm$, i.e. $H_\xi$ is maximum on the border of $T_\pm$. For $\pi \neq \tau_0$, equaling $V_1$ and $V_2$ expressed by (112) to their definition (111), one finds that $p$ is a common root of two polynomials

\[
P_1 = \Delta^4 - 2Z\Delta^3(2\pi - \tau_0 - T) + \Delta^2(4R^2W_k^2(\pi - \tau_0)^2 + Z^2(2\pi - \tau_0 - T)^2 - (\tau_0 - T)^2)
+ 2Z\Delta(2\pi - \tau_0 - T)(\tau_0 - T)^2 - Z^2(2\pi - \tau_0 - T)^2(\tau_0 - T)^2,
\]

\[
P_2 = \Delta^4 - 2Z\Delta^3(\pi - T) + \Delta^2(4R^2W_k^2(\pi - \tau_0)^2 - (2\pi - \tau_0 - T)^2 + Z^2(\tau_0 - T)^2)
+ 2Z\Delta(2\pi - \tau_0 - T)^2(\tau_0 - T) - Z^2(2\pi - \tau_0 - T)^2(\tau_0 - T)^2.
\]

Since two polynomials have common roots if and only if their discriminant vanishes, the coordinate $\Delta$ of $p$ is a root of the resultant of $P_1$, $P_2$ viewed as polynomials in $T$. Thanks to formal computation programs\(^{11}\) one finds that this resultant is the product of $256Z^8\Delta^8(\pi - \tau_0)^6$ by

\[
P_3 = R^4\Delta^2 - 4R^2Z\Delta(\tau_0 - \pi)\frac{\sin((2k + 1)\omega\pi + \varphi)}{\sin\omega\pi}
+ 4(\tau_0 - \pi)^2\frac{(R^2\cos(k\omega\pi + \varphi)^2 - 1)(R^2\cos((k + 1)\omega\pi + \varphi)^2 + 1)}{\sin^2\omega\pi}.
\]

The discriminant of $P_3$ viewed as an equation in $\Delta$ of the second degree is

\[-\frac{16R^4(\pi - \tau_0)^2}{\sin^2\omega\pi} \left(Z^2\cos(k\omega\pi + \varphi)\cos((k + 1)\omega\pi + \varphi) - \sin(k\omega\pi + \varphi)\sin((k + 1)\omega\pi + \varphi)\right)^2\]

hence $P_3$ has no real solution. In other terms, the function $H_\xi$ does not have local extrema inside $T_\pm$, hence it reaches its maximum on the border of $T_\pm$.

Now observing that

\[
\frac{\partial H_\xi(T, 0)}{\partial T} = 1 - RW_{k+1} - RW_k \quad (116)
\]

is a constant, one deduces that $H_\xi(T, 0)$ is maximum at one of the end points of the segment $\Delta = 0$.

Let us underline that the derivative of $H_\xi$ is constant neither on the segment $T = 0$ nor on the hypotenuse $T \pm \Delta = \min(\tau_0, 2\pi - \tau_0)$. For instance one can show that $H_\xi(0, \Delta)$ has a maximum $h_1$ for a value $\Delta_0$ which is a root of a polynomial of degree 4. We did not manage to find a simpler expression for $\Delta_0$, neither did we for the maximum $h_2$ of $H_\xi$ on the hypotenuse. Hence, unless numerically specifying $k, \tau_0, z_\xi$ and $\varphi$, it seems difficult to compare $h_2$ and $h_1$ and go beyond corollary III.6.

### III.3 Discussion

**Equatorial versus non equatorial states**

In case $\xi$ is an equatorial state, i.e. $z_{\xi_0} = 0$, corollary III.4 indicates that $H_\xi$ attains its maximum on the segment $\Delta = 0$. Thus

\[
d(\xi_x, \xi_y) = \max_{(T, \Delta) \in T_\pm} (X + TY) \quad (117)
\]

\(^{11}\)Here we used Mathematica
where we put

\[ X \doteq H_{\xi}(0,0) = RW_{k+1}\tau_0 + RW_k(2\pi - \tau_0), \quad Y \doteq \frac{\partial H_{\xi}(T,0)}{\partial T} \]

given by (116). With these notations one retrieves proposition V.4 of [10]. Note that in the statement of this proposition we did not insist on the restriction to equatorial states\(^\dagger\) (specifically we did not replace \( R \) by its value 1) because we hoped the result would still be true for non vanishing \( z_\xi \). On the contrary the present work shows that the spectral distance depends on \( z_\xi \) on a non-trivial manner.

Now putting \( R = 1 \) one observes that \( Y \leq 0 \) since \( |\sin \omega \pi| \leq |\sin(k+1)\omega \pi| + |\sin k\omega \pi| \).

Thus for equatorial states \( d(\xi_x, \zeta_y) = X \). For non equatorial states \( Y \) can be positive and there is no further conclusion than corollary III.6.

**The shape of the fiber**

When \( \tau_0 = 0 \), i.e. when \( \zeta_y = \zeta_x \) belongs to the fiber of \( \xi_x \), \( T_\pm \) reduces to the single point 0 and

\[ d(\xi_x, \xi_y) = 2\pi RW_k = \frac{2\pi R}{|\sin \omega \pi|} \sin \frac{\Xi}{2} \]  

(118)

where we parameterize the fiber over \( x \) by

\[ \Xi \doteq 2\omega \pi + \phi. \]

We retrieve formula (134) of [10] which, as already mentioned there, is valid also for non equatorial state. In [11] we give an interpretation of this result saying that for an intrinsic metric point of view, the \( S^1 \) fiber over \( x \) equipped with the spectral distance has the shape of a cardioid. In fact (118) can also be viewed as the length of the straight segment inside the circle\(^*\). Therefore the spectral distance appears as the geodesic distance \textit{inside} the disk, in the same way that in the two-sheets mode\(^\dagger\) the spectral distance coincides with

\(^\dagger\)In the published version this point is however clearly indicated at the beginning of the section, (eq. 61, op.cite). On the arXiv version the restriction is recalled in the statement of the proposition.

\(^*\)This has been pointed out by the audience, during a talk given at Toulouse university.
a geodesic distance within the two sheets. Say differently, the spectral distance coincides with the geodesic distance of a manifold $M'$ whose boundary is the set of pure states of $\mathcal{A}$. Of course it is appealing to identify $M'$ to the set of all states of $\mathcal{A}$ (i.e. all convex combinations of pure states) but this idea does not survive here since the convex combination $\rho = \lambda \xi + (1 - \lambda) \zeta$. We deserve to further work a better comprehension of this observation.

IV Spectral distance on the circle with a $\mathbb{C}P^{n-1}$ fiber

Let us now investigate the general case

$$\mathcal{A} = C^\infty(M, M_n(\mathbb{C}))$$

for arbitrary integer $n \in \mathbb{N}$. $T_\xi$ is now a $n$-torus and instead of (45) one deals with equivalence classes of $(n+1)$-tuples

$$(k \in \mathbb{N}, 0 \leq \tau_0 \leq 2\pi, 0 \leq \varphi_i \leq 2\pi) \sim (k + \mathbb{Z}, \tau_0, \varphi_j - 2\mathbb{Z}\omega_j\pi)$$

with

$$\omega_j = \Theta_1(2\pi) - \Theta_j(2\pi) \quad \forall j \in [2, n]$$

such that $\zeta_y$ in $T_\xi$ writes

$$\zeta_y = \begin{pmatrix} V_1(\tau) \\ e^{i\varphi_j}V_j(\tau) \end{pmatrix}$$

where $\tau = 2k\pi + \tau_0$. As soon as $n > 2$ there is no longer correspondence between the fiber of $P$ and a sphere, however in analogy with (73) we write

$$V_j = \sqrt{\frac{R_j}{2}} e^{i\theta_j^0}$$

where $R_j \in \mathbb{R}^+$, $\theta_j^0 \in [0, 2\pi]$.

The simplifications relying on the choice of $S^1$ as a basis, namely lemma III.3, still hold for $n \geq 2$:

**Lemme IV.1** Whatever $\zeta_y \in T_\xi$, the supremum in computing $d(\xi_x, \zeta_y)$ can be searched on selfadjoint elements $a$ whose diagonal part $a_1$ vanishes as $x$ and whose components $a_{ii}$ satisfy (57). The non-diagonal components $a_{ij}$ can be written

$$a_{ij} = g_{ij}e^{-i\Theta_{ij}}$$

with $\Theta_{ij}$ defined in (22) and

$$g_{ij}(\tau) = g_{ij}^0 + \int_0^\tau \rho_{ij}(u)e^{i\phi_{ij}(u)}du$$

where $\rho_{ij}$ is a smooth function on $[0, 2\pi]$. The expression (45) is more general than (19) since it provides a way to compute the distance between $\xi_x$ and $\zeta_y$. For $n = 2$, the formula (20) coincides with the expression (19). However, for $n > 2$ the formula (20) is more general, providing a way to compute the distance between $\xi_x$ and $\zeta_y$ in a $n$-dimensional circle with a $\mathbb{C}P^{n-1}$ fiber. The simplifications relying on the choice of $S^1$ as a basis, namely lemma III.3, still hold for $n \geq 2$.
where - for each couple of indices \((i, j)\) - the real positive normalized smooth function \(\rho_{ij}\), the real smooth function \(\phi_{ij}\) and the complex constant \(g_{ij}^0\) satisfy equations (61) and (62).

**Proof.** The vanishing of \(a_1\) and the boundary values of the \(a_{ii}\)'s are obtained like in lemmas III.2 and III.3. The properties of the non-diagonal components are obtained like in III.3, except that one has to begin with \((e_i + e_j)a(e_i + e_j)\) instead of \(a\), where \(e_i\) denotes the \(i\)th eigenprojector of the connection 1-form \(A\).  

For explicit computation, we restrict in the following to a given fiber, say the one over \(x \in S^1\) corresponding to \(\tau = 0\) within the trivialization (17). In other words we consider only those \(\zeta_y = \zeta_x\) given by (119) with \(\tau_0 = 0\).

**Proposition IV.2** Given a pure state \(\zeta_x = (k, 0, \varphi_j) \in T_\xi\), either \(\zeta_x\) does not belongs to the connected component \(U_\xi\) and

\[
d(\xi_x, \zeta_x) = +\infty
\]

or \(\zeta_x \in U_\xi\) and

\[
d(\xi_x, \zeta_x) = \pi \text{Tr} |S_k|
\]

where \(|S_k| = \sqrt{S_k^* S_k}\) and \(S_k\) is the matrix with components

\[
S^k_{ij} \doteq \sqrt{R_i R_j} \frac{\sin \left( k\pi (\omega_j - \omega_i) + \frac{\phi_j - \phi_i}{2} \right)}{\sin \pi (\omega_j - \omega_i)}
\]

where \(\omega_i\) is defined in (120).

**Proof.** Staying within the same fiber, by lemma IV.1 the supremum can be searched on elements \(a\) whose diagonal part \(a_1\) vanishes. Therefore (51) with \(\zeta_y\) given by (121), together with (122) and (120) yield

\[
\Delta_\xi (a) = \sum_{i,j=1}^n \sqrt{R_i R_j} \left( a_{ij}(x) e^{i(\theta^0_j - \theta^0_i)} \left( e^{2ik\pi(\omega_j - \omega_i)} e^{i(\varphi_j - \varphi_i)} - 1 \right) \right) .
\]

For any couple of directions \(i, j\) far from each other, \(\omega_i = \omega_j\) so that the corresponding term in (128) either vanishes if \(\varphi_i = \varphi_j\), or can be made infinite if \(\varphi_i \neq \varphi_j\) since \(g_{ij}^0\) is no longer constrained by (62). In this case \(\zeta_x \notin U_\xi\) by definition (see (29), hence (125). Consequently (128) reduces to a sum on couples of close directions. Then (124) together with (62) yield

\[
a_{ij}(x) = g_{ij}^0 = \frac{1}{2i} \frac{e^{-i\pi(\omega_j - \omega_i)}}{\sin \pi (\omega_j - \omega_i)} \int_0^{2\pi} \rho_{ij}(u)e^{i\phi_{ij}(u)}du
\]

and (128) rewrites

\[
\Delta_\xi (a) = \frac{1}{2} \sum_{i,j=1}^n S^k_{ij} \int_0^{2\pi} \rho_{ij}(u)e^{i\phi^k_{ij}(u)}du
\]

where \(S^k_{ij}\) is defined in (127) and

\[
\phi^k_{ij} \doteq \phi_{ij} + k\pi(\omega_j - \omega_i) + \theta^0_j - \theta^0_i + \frac{\varphi_j - \varphi_i}{2}.
\]
Calling $\Gamma$ the matrix with components $\rho_{ij} e^{i\phi_{ij}}$, 
\[
\Delta_\xi(a) = \frac{1}{2} \int_0^{2\pi} \text{Tr} \left( S^k \Gamma(u) \right) \, \text{d}u. \tag{132}
\]

For $a$ whose vanishing diagonal and off-diagonal given by (123) the commutator norm condition is easily computed thanks to (31), 
\[
\|\|[D, a]\|\| = \left\| \begin{pmatrix} 0 & \rho_{ij} e^{i\phi_{ij}} e^{i(\Theta_j - \Theta_i)} \\ \rho_{ij} e^{-i\phi_{ij}} e^{-i(\Theta_j - \Theta_i)} & 0 \end{pmatrix} \right\| \leq 1. \tag{133}
\]

Introducing the diagonal matrix $E$ with components 
\[
E_{ii} = e^{i(\Theta_i - (k-1)\pi\omega_i - \theta_0^i + \varphi_i)},
\]
one observes that 
\[
\Gamma = E[D, a]E^*,
\]
hence $\|\Gamma(u)\| \leq 1$ for any $u \in [0, 2\pi]$. Back to (132),
\[
\text{Tr} \left( S_k \Gamma(u) \right) \leq \text{Tr} |S_k|.
\]

Hence 
\[
\Delta_\xi(a) \leq \pi \text{Tr} |S_k|. \tag{134}
\]

This upper bound is reached by the diagonal constant matrix $\Gamma$ whose components in the basis that diagonalizes $S_k$ (with corresponding indices $\mu, \nu$) are 
\[
\Gamma_{\mu\mu} = \text{sign} S_{\mu\mu}^k
\]
and $\phi_{ij}$ the constant function 
\[
\phi_{ij}(u) = (k-1)\pi(\omega_i - \omega_j) + \theta_0^i - \theta_0^j + \frac{\varphi_i - \varphi_j}{2}. \tag{135}
\]

To check consistency with the $n = 2$ case, note the following correspondence
\[
\omega \text{ in } (46) \text{ reads } \omega_2 \text{ in } (120), \tag{136}
\]
\[
\varphi \text{ in } (45) \text{ reads } \varphi_2 \text{ in } (121), \tag{137}
\]
\[
\theta_0 \text{ in } (116) \text{ reads } \theta_0^1 - \theta_0^2 \text{ in } (122), \tag{138}
\]
\[
\omega_1 = \varphi_1 = 0 \text{ and } R_1 = R_2 = R, \tag{139}
\]
the latest coming from the requirement that the directions in the $n = 2$ case are close to each other. Within this correspondence, all the quantities without indices used in the proof of of proposition III.5 corresponds to the quantities with indices 12 in the above

\[\text{Tr} \left( S_k \Gamma(u) \right) = \sum_{\alpha} S_{\alpha\alpha} S_{\alpha\alpha}^k \leq \sum_{\alpha} |S_{\alpha\alpha}| \|\Gamma\| = \text{Tr} |S_k| \]
proof. For instance \( \phi_{12} \) in (135) equals \( \Phi_k \) in (100) as expected from (101). This allows to check the coherence of our results: for \( n = 2 \)
\[
\text{Tr} |S_k| = 2 |S_{12}^k| = 2 R \frac{\sin k \pi \omega + \frac{\omega}{2}}{|\sin \omega \pi|}
\]
so that
\[
d(\xi_x, \zeta_x) = 2 \pi R \frac{\sin k \pi \omega + \frac{\omega}{2}}{|\sin \omega \pi|}
\]
as expected from proposition III.5. Indeed when \( \tau_0 = 0 \) the domains \( T_\pm \) reduce to the origin and (70) reduces to (140).

V Conclusion and outlook

Let us conclude on the three generalizations of [10] presented in this paper:

- the connected component of the spectral distance for arbitrary dimension \( n \) and arbitrary connection maintains the interesting features of the \( n = 2 \) case. With respect to the horizontal distance, the spectral distance seems to have some transverse properties that could be relevant in a foliation framework.

- the explicit formula of the distance on all the connected component when \( n = 2 \) shows a non-trivial dependence on \( \zeta_x \). This is quite unexpected.

- the explicit formula on the fiber for arbitrary \( n \) is remarkably simple. We deserve its interpretation to further work. Especially it would be interesting to see whether or not the trace of \( |S| \) corresponds to some riemannian distance within a torus, as in the \( n = 2 \) case. In any case the link between the circle and the disk from the spectral metric point of view has to be studied further. Remember that the spectral distance associated to \( M_2(\mathbb{C}) \) only is the euclidean distance on the circle\(^5\), while here we find the euclidean distance on the disk. Why considering the circle rather than a point allows the distance to see through the disk?

On the circle the difference between the horizontal and the spectral distances is due to the topology of the basis. It would be interesting to study local properties in order to see how the curvature enters the game. The question for sub-riemannian geometry (how to minimize the number of selfintersecting points on a minimal horizontal curve \( \)?) may prevent to go very far in this direction. Nevertheless one can hope to obtain interesting non trivial results for basis more complicated than \( S^1 \) but for which the horizontal distance is still tractable, for instance tori.

Finally, from a more physicist point of view, simple examples of the spectral distance for finite dimensional algebra have been quantized\(^{14,1} \). A similar procedure for covariant Dirac operator would be of great interest.

Acknowledgements: Work partially supported by EU Marie Curie fellowship EIF-025947-QGNC.

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