LOCAL SOLUTION AND EXTENSION TO THE CALABI FLOW

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1. Introduction

Let \( M \) be a compact Kähler manifold of complex dimension \( n \) and let \( g \) be a smooth Kähler metric on \( M \). A Kähler metric \( g \) can be written as a Hermitian matrix valued function in holomorphic coordinates \((z^1, \cdots, z^n)\),

\[
g = g_{i\bar{j}}dz^i \otimes d\bar{z}^j.
\]

The corresponding Kähler form \( \omega = \sqrt{-1} g_{i\bar{j}}dz^i \wedge d\bar{z}^j \) is a closed \((1,1)\) form. So a Kähler class \([\omega]\) defines a nontrivial cohomology class. For simplicity, we use both \( g \) and \( \omega \) to denote the same Kähler metric. For any Kähler metric \( \omega_\varphi \) in the same Kähler class \([\omega]\), we can write

\[
\omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi,
\]

where \( \varphi \) is a real valued function and it is called a Kähler potential. The Calabi flow is defined by

\[
\frac{\partial \varphi}{\partial t} = R_\varphi - \overline{R},
\]

\( \varphi(0) = \varphi_0 \),

where \( R_\varphi \) is the scalar curvature of \( \omega_\varphi \), \( \overline{R} \) is the average of the scalar curvature and it is a constant depending only on \((M,[\omega])\). In local coordinates, the Ricci curvature of a Kähler metric \( g \) is given by

\[
R_{i\bar{j}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{k\bar{l}}),
\]

and the scalar curvature is given by

\[
R = -g^{i\bar{j}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{k\bar{l}}).
\]

Equation (1.1) is the gradient flow of the Calabi energy, aiming to find an extremal metric or a constant scalar curvature metric on a compact Kähler manifold within a fixed Kähler class \([\omega]\), which is an important problem in Kähler geometry. In the case of Riemann surfaces \((n=1)\), P. Chrusciel [11] showed that the Calabi flow exists for all time and converges to a constant Gaussian curvature metric, making use of the existence of such a metric and the Bondi mass in general relativity. Without using the Bondi mass, Chen [6], Struwe [16] reproved Chrusciel’s theorem. Recently Chen-Zhu [10] (for two sphere) removed the assumption that there is a constant Gaussian curvature metric. While in higher dimensions...
(n \geq 2), the nonlinearity of the Calabi flow becomes more acute and the analytic difficulty becomes more daunting. But recently some progress has been made \cite{8} \cite{19} \cite{17} \cite{9} \cite{14} etc...

From (1.2), we can see that (1.1) is a fourth order quasi-linear parabolic equation. A long standing problem about the Calabi flow (n \geq 2) is whether it exists for all time or not. X. Chen conjectured that the Calabi flow always exists for all time on a compact Kähler manifold with smooth initial data.

**Conjecture 1.1** (X. Chen). The Calabi flow exists for all time for any smooth Kähler metric on a compact Kähler manifold (M, [\omega]).

The motivation of this conjecture relies in part on the fact that the Calabi flow is a distance contracting flow over the space of Kähler metrics \cite{4}, \cite{7}. X. Chen then suggested to seek the optimal condition on initial data for the local solution to the Calabi flow. In particular, the distance over the space of Kähler metrics makes sense for L^\infty Kähler metrics. It motivates to consider the local solution to the Calabi flow for L^\infty initial metrics. X. Chen asked

**Problem 1.2** (X. Chen). Can the Calabi flow start with a L^\infty Kähler metric?

In a joint work with X. Chen \cite{8}, by using standard quasilinear parabolic method, we proved that the short time solution to the Calabi flow exists for initial potential \varphi_0 \in C^{3,\alpha}(M), namely when the initial metric

\[ g_{\varphi_0} = g + \frac{\partial^2 \varphi_0}{\partial z^i \partial \bar{z}^j} dz^i \otimes d\bar{z}^j \]

is in C^{1,\alpha}. The quasilinear parabolic nature of the equation allows the solution with a singularity at time t = 0 (the curvature is not pointwise well defined for \varphi_0 \in C^{3,\alpha}). In this note we consider the initial value in C^{2,\alpha} (the initial metric is in C^\alpha). We prove that for any smooth metric g \in (M, [\omega]), there exists a small C^\alpha neighborhood around g (0 < \alpha < 1), such that the short time solution to the Calabi flow exists for all C^\alpha metric in this neighborhood (see Theorem 2.1). As an application, we can prove that if g itself is a constant scalar curvature metric, there exists a small C^\alpha neighborhood of g, such that for any initial metric in this neighborhood, the Calabi flow exists for all time and converges to a constant scalar curvature metric in the same class. This result is proved previously in \cite{8} for C^{1,\alpha} small neighborhood.

In a joint work with X. Chen \cite{8}, we proved that the flow can be extended once the Ricci curvature is uniformly bounded along the Calabi flow. In this direction we can prove that on a Kähler surface, the Calabi flow can be extended once the metrics are bounded in L^\infty along the Calabi flow (see Theorem 3.1), by using a blowing up argument as in \cite{9}. In particular, it implies that if we can obtain the L^\infty estimate of the metric g (or the C^{1,1} bound of the potential \phi), we can get that all higher derivatives of g (or the higher derivatives of \phi respectively) are bounded. We can recall the seminal Evans-Krylov theory for second order fully nonlinear equations (elliptic or parabolic type). The theory asserts that when the (elliptic) operator is concave, then one can obtain the Hölder estimates of the second order derivatives, and then higher regularity follows from the standard boot-strapping argument. Our result can be viewed as obtaining higher order estimates only assuming C^{1,1} bound of
Kähler potential \( \varphi \), by a geometric method for the Calabi flow, a fourth order nonlinear parabolic equation, on Kähler surfaces.

2. Local solution for \( C^\alpha \) initial metrics

To state and prove our theorem, we need to define some Banach spaces. The setting is similar to that in [12], [8]. In the following we assume \( \theta \in (0, \frac{1}{2}] \), and \( E \) is a Banach space, \( J = [0, T] \) for some \( T > 0 \). We consider functions defined on \( \tilde{J} = (0, T] \) with a prescribed singularity at 0. Set

\[
C_{1-\theta}(J, E) := \left\{ u \in C(\tilde{J}, E); [t \to t^{1-\theta} u] \in C(J, E), \lim_{t \to 0^+} t^{1-\theta} |u(t)|_E = 0 \right\},
\]

and

\[
|u|_{C_{1-\theta}(J, E)} := \sup_{t \in \tilde{J}} t^{1-\theta} |u|_E,
\]

One can verify that \( C_{1-\theta}(J, E) \), equipped with the norm \( \| . \|_{C_{1-\theta}(J, E)} \), is a Banach space and \( C_{1-\theta}(J, E) \) is a subspace. We also set

\[
C_{1}(J, E) := C_{1-\theta}(J, E),
\]

\[
C_{0}(J, E) = C(0, T), \quad C_{0}(J, E) = C(J, E).
\]

Given \( x \in E \), we use the notation \( B_E(x, r) \) to denote the ball in the Banach space \( E \) with the center \( x \) and the radius \( r \).

Let \( E_1, E_0 \) be two Banach spaces such that \( E_1 \) is continuously, densely embedded into \( E_0 \). Set

\[
E_0(J) := C_{1-\theta}(J, E_0),
\]

and

\[
E_1(J) := C_{1-\theta}(J, E_0) \cap C_{1-\theta}(J, E_1),
\]

where

\[
|u|_{E_1(J)} := \sup_{t \in J} t^{1-\theta} (|\dot{u}|_{E_0} + |u|_{E_1}).
\]

We also use the notation

\[
E_\mu := (E_0, E_1)_\mu, \mu \in (0, 1),
\]

for the continuous interpolation spaces of DaPrato and Grisvard [13].

Recall that if \( k \in \mathbb{N}, \alpha \in (0, 1) \), the Hölder space \( C^{k,\alpha} \) is the Banach space of all \( C^k \) functions \( f : \mathbb{R}^n \to \mathbb{R} \) which have finite Hölder norm. The subspace of \( C^\infty \) function in \( C^{k,\alpha} \) is not dense under the Hölder norm. One defines the little Hölder space \( c^{k,\alpha} \) to be the closure of smooth functions in the usual Hölder space \( C^{k,\alpha} \). And one can verify \( c^{k,\alpha} \) is a Banach space and that \( c^{l,\beta} \hookrightarrow c^{k,\alpha} \) is a continuous and dense imbedding for \( k \leq l \) and \( 0 < \alpha < \beta < 1 \). These definitions can be extended to functions on a smooth manifold \( M \).
naturally [2]. For our purpose, the key fact [13] about the continuous interpolation spaces is that for \( k \leq l \) and \( 0 < \alpha < \beta < 1 \), and \( 0 < \mu < 1 \), there is a Banach space isomorphism

\[
(c^{k,\alpha}, c^{l,\beta})_\mu \cong c^{l+(1-\mu)k+\mu\beta+(1-\mu)\alpha},
\]

provided that the exponent \( \mu l + (1-\mu)k + \mu\beta + (1-\mu)\alpha \) is not an integer.

From now on we set \( E_0 = c^\alpha(M, g) \), \( E_1 = c^{1,\alpha}(M, g) \), and take \( \theta = \mu = 1/2 \), then \( E_{1/2} = c^{2,\alpha}(M, g) \), where \( g \) is a fixed Kähler metric. For simplicity we will use \( c^\alpha(M) \) etc if there is no confusion. One can write

\[
R_\varphi = -A(\nabla^2 \varphi)\varphi + f(\nabla \varphi, \nabla^2 \varphi, \nabla^3 \varphi),
\]

where the operator \( A \) is given by

\[
A(\nabla^2 \varphi)w = g^i_j g^k_l \nabla_i \nabla_j \nabla_k \nabla_l w.
\]

Note that \( A(\nabla^2 \varphi) \) involves only the second derivative of \( \varphi \). This fact is important for the initial potential in \( c^{2,\alpha} \). As a quasi-linear equation, the equation (1.1) has lower order term \( f(\nabla \varphi, \nabla^2 \varphi, \nabla^3 \varphi) \) which involves the third derivatives of \( \varphi \). The standard quasi-linear theory can apply for \( \varphi_0 \in c^{3,\alpha} \). However, computation shows that \( f(\nabla \varphi, \nabla^2 \varphi, \nabla^3 \varphi) \) involves only quadratic terms of \( \nabla^3 \varphi \), which allows us to obtain

**Theorem 2.1.** For any smooth Kähler metric \( g \) on \( M \), there exist positive constants \( T = T(g), \varepsilon = \varepsilon(g) \) and \( c = c(g) \) such that for any initial value \( x \in E_{1/2} = c^{2,\alpha}, |x|_{c^{2,\alpha}} \leq \varepsilon \) the Calabi flow equation (1.1) has a unique solution

\[
\varphi(\cdot, x) \in C_{1/2}^1([0, T], E_0) \cap C_{1/2}(0, T), E_1)
\]

on \([0, T]\). Moreover \( \varphi \in C([0, T], E_{1/2}) \) and for \( x, y \in B_{E_{1/2}}(0, \varepsilon) \), we have

\[
\|\varphi(\cdot, x) - \varphi(\cdot, y)\|_{C([0, T], E_{1/2})} \leq c\|x - y\|_{E_{1/2}}
\]

\[
\|\varphi(\cdot, x) - \varphi(\cdot, y)\|_{E_1([0, T])} \leq c\|x - y\|_{E_{1/2}}.
\]

As an application, we can improve the stability theorem in \([8]\) as follows,

**Theorem 2.2.** Suppose \( g \) is a constant scalar curvature metric in \([\omega]\) on \( M \). If the initial metric \( g_{\varphi_0} = g + \partial\partial^c \varphi_0 \) satisfies \( |\varphi_0|_{c^{2,\alpha}(M)} < \varepsilon \), where \( \varepsilon = \varepsilon(g) \) is small enough, then the Calabi flow exists for all time and \( g(t) = g_{\varphi(t)} \) converges to a constant scalar curvature metric \( g_\infty \) exponentially fast in the same class \([\omega]\) in \( C^\infty \) sense.

Let \( x \in B_{E_{1/2}}(0, \varepsilon) \) for a fixed small \( \varepsilon > 0 \). Recall

\[
E_1(J) = C_{1/2}^1(J, E_0) \cap C_{1/2}(J, E_1).
\]

For any \( x \in B_{E_{1/2}}(0, \varepsilon) \), set

\[
V_x(J) = \left\{ v \in E_1(J); v(0) = x, \|v\|_{C(J, E_{1/2})} \leq \varepsilon_0 \right\} \cap B_{E_1(J)}(0, \varepsilon_0)
\]

and equip this set with the topology of \( E_1(J) \), where \( \varepsilon_0 = C_0\varepsilon \) and \( C_0 \) is a fixed constant depending only on \( g \) and will be determined later. We will show that \( V_x(J) \) is not empty if
$T$ is small enough. Let $v \in V_x(J)$ be given. Consider the following linear parabolic equation for $u$ such that for $t \in J = (0, T]$,

$$
\begin{align*}
\frac{\partial u}{\partial t} + \Delta_g^2 u &= f(v), \\
u(0) &= x,
\end{align*}
$$

(2.2)

where

$$
f(v) = \Delta_g^2 v + R_v - R_v.
$$

We first show that the linear equation (2.2) has a unique solution $u \in V_x(J)$ for any $v \in V_x(J)$ provided $T, \varepsilon$ sufficiently small. So we can define a map

$$
\Pi_x : v \rightarrow u
$$

from $V_x(J)$ to $V_x(J)$. Note $V_x(J)$ is a close subset of the Banach space $E_1(J)$. Then we show that the map $\Pi_x$ is a contracting map when $T, \varepsilon$ are both small. It follows that $\Pi_x$ has a fixed point in $V_x(J)$ by contract mapping theorem and the fixed point is the desired solution. The smoothing property will follow simultaneously. We will need some facts of the linear parabolic theory, in particular the analytic semigroup generated by $\Delta_g^2$. The results hold for more general linear operators. One can find a nice reference such as in [12], [15].

Consider the linear equation for some $x \in E_0$

$$
\begin{align*}
\frac{\partial u}{\partial t} &= -\Delta_g^2 u, \\
u(0) &= x.
\end{align*}
$$

(2.3)

Then $\Delta_g^2 : E_1 \rightarrow E_0$ defines an analytic semigroup $e^{-t\Delta_g^2} : E_0 \rightarrow E_1$. The solution of (2.3) is given by $u(t) = e^{-t\Delta_g^2} x$ for $t \in J$. For $x \in E_{1/2}$, we have the following equivalent norm

$$
|x|_{E_{1/2}} := \sup_{s > 0} s^{1/2} \left| \Delta_g^2 \left( e^{-t\Delta_g^2} x \right) \right|_{E_0}.
$$

(2.4)

Moreover $[t \mapsto e^{-t\Delta_g^2} x] \in E_1(J)$ and there exists a constant $c_1 > 0$ independent of $J$ such that for any $t \in J$

$$
\left| e^{-t\Delta_g^2} x \right|_{E_1(J)} \leq c_1 |x|_{E_{1/2}}.
$$

(2.5)

If $u \in E_1(J)$ with $u(0) = 0$, then there exists a constant $c_2$ independent of $J$ such that

$$
|u|_{C_1(J, E_{1/2})} \leq c_2 |u|_{E_1(J)}.
$$

(2.6)

The inequalities (2.4) — (2.6) hold for more general settings and one can find the proof in [12] (see Remark 2.1, Lemma 2.2 in [12]). Also if $T$ is small, by the strong continuity of the semigroup $\{e^{-t\Delta_g^2}, t \geq 0\}$, we can get that

$$
\left| e^{-t\Delta_g^2} x - x \right|_{E_{1/2}} \leq \frac{1}{4} \varepsilon_0.
$$

(2.7)

In particular, for any $x \in B_{E_{1/2}}(0, \varepsilon)$, by (2.5),

$$
\left| e^{-t\Delta_g^2} x \right|_{E_1(J)} \leq c_1 |x|_{E_{1/2}}.
$$
We choose \( C_0 = 2c_1 + 1 \), then we have
\[
|e^{-t\Delta^2_y}x|_{E_1(J)} \leq \varepsilon_0/2.
\]
Also by (2.7) we get
\[
|e^{-t\Delta^2_y}x|_{E_1/2} \leq |x|_{E_1/2} + |e^{-t\Delta^2_y}x - x|_{E_1/2} < \varepsilon_0
\]
provided \( T \) is small. It follows that \( V_x(J) \) is not empty.

Now we are in the position to prove Theorem 2.1.

**Proof.** For any \( v \in V_x(J) \), the linear parabolic equation (2.2) has the unique solution which takes the formula
\[
u(t) = \left( e^{-t\Delta^2_y} \right) x + (Kf)(t),\]
where
\[(Kf)(t) = \int_0^t e^{-(t-s)\Delta^2_y} f(v(s)) ds.\]
It is clear that \( K : E_0(J) \to E_1(J) \) is a linear operator. We can define the norm of \( K \) by \( \|K\| \). We can get for some \( c_3 \) depending only on \( g \),
\[(2.8) \quad |Kf|_{E_1(J)} \leq \|K\| |f|_{E_0(J)} \leq c_3 |f|_{E_0(J)}.\]
First we need to show \( u \in V_x(J) \). We compute
\[
f(v) = \Delta^2_g v + R_v - R
\]
\[
= - g_v^j g_i^k \partial_i \partial_j (\log \det (g_{kl} + v_{kl})) + g_v^j g_i^k \partial_i \partial_j (g^{kl}_{g_{kl}}) \partial_k \partial_l v - R
\]
\[
= g_v^j g_i^k g_v^l \partial_i \partial_j g_v^l (\partial_i \partial_j g_{kl} + \partial_j \partial_i g_{kl}) + g_v^j g_i^k \partial_i \partial_j \partial_k \partial_l v
\]
\[
+ g_v^j g_i^k \partial_i \partial_j \partial_k \partial_l v - g_v^j g_i^k g_j^l \partial_i \partial_j v_{kl} - g_v^j g_i^k \partial_i \partial_j g^{kl} v_{kl}
\]
\[
+ g_v^j \partial_i \partial_j (g^{kl}_{g_{kl}}) v_{kl} - R
\]
(2.9)

For simplicity, from now on we use \( C \) to denote a uniformly bounded constant independent of \( \varepsilon, T \) and can vary line by line. We can also compute that
\[
g_v^j g_i^k \partial_i \partial_j \partial_k \partial_l v - g_v^j g_i^k \partial_i \partial_j \partial_k \partial_l v = \left( (g_v^j - g_v^j_{g_{kl}}) g_i^k + g_v^j g_i^k_{g_{kl}} \right) \partial_i \partial_j \partial_k \partial_l v
\]
(2.10)
\[
= \left( g_i^k g_v^j g_v^p q_j + g_v^j g_i^k g_v^p \partial_q \partial_r \partial_i \partial_j \partial_k \partial_l v.\right)
\]

It follow from (2.9) and (2.10) that
\[
f(v) = \left( g_i^k g_v^j g_v^p q_j + g_v^j g_i^k g_v^p \partial_q \partial_r \partial_i \partial_j \partial_k \partial_l v - g_v^j g_i^k g_j^l \partial_i \partial_j v_{kl} - g_v^j g_i^k \partial_i \partial_j g^{kl} \partial_i v_{kl}
\]
\[
+ g_v^j g_i^k g_v^p \partial_q \partial_r \partial_i \partial_j \partial_k \partial_l v) - R
\]
(2.11)
By (2.11), we can get that
\[
\sqrt{t}|f(v(t))|_{E_0} \leq C \sqrt{t}(|v|_{c,.\alpha}^2 + |v|_{c,\alpha} |v|_{c,\alpha} + 1) \\
\leq C \sqrt{t}(|v|_{c,\alpha} |v|_{c,\alpha} + 1) \\
\leq C \varepsilon_0^2 + C \sqrt{T}, \quad t \in (0, T],
\]
where we have used the convexity of Hölder norms,
\[
|v|_{c,\alpha}^2 \leq C|v|_{c,\alpha} |v|_{c,\alpha}.
\]
It follows that \( f(v) \in E_0(J) \). By (2.7) and (2.8), we can get that
\[
|u|_{C(J,E_{1/2})} \leq \left| \left( e^{-t\Delta^2} \right) x \right|_{C(J,E_{1/2})} + \left| (Kf)(t) \right|_{C(J,E_{1/2})} \\
\leq |x|_{E_{1/2}} + \frac{\varepsilon_0}{4} + C|f|_{E_0(J)} \\
\leq \frac{\varepsilon_0}{2} + C \left( C \varepsilon_0^2 + C \sqrt{T} \right) \\
\leq \frac{1}{2} \varepsilon_0 + C \varepsilon_0^2 + C \sqrt{T} \\
\leq \varepsilon_0,
\]
provided that \( T, \varepsilon \) are both small enough. Also by (2.5) and (2.8) we get that
\[
|u|_{E_1(J)} \leq \left| \left( e^{-t\Delta^2} \right) x \right|_{E_1(J)} + \left| (Kf)(t) \right|_{E_1(J)} \\
\leq c_1 |x|_{E_{1/2}} + C|f|_{E_0(J)} \\
\leq \varepsilon_0,
\]
if \( \varepsilon, T \) are both small enough. By (2.12), (2.13), \( u \in V_2(J) \).

Now let \( x_1, x_2 \in B_{E_{1/2}}(0, \varepsilon) \) be given and \( v_1 \in V_{x_1}(J), v_2 \in V_{x_2}(J) \). We can get solutions \( u_1, u_2 \) to (2.2) with initial value \( x_1, x_2 \) and \( f(v_1), f(v_2) \) respectively. It is clear that
\[
|u_1 - u_2|_{E_1(J)} \leq \left| e^{-t\Delta^2} (x_1 - x_2) \right|_{E_1(J)} + |K(f(v_1) - f(v_2))|_{E_1(J)}.
\]
Note
\[
f(v_1) - f(v_2) = \Delta^2(v_1 - v_2) + R_{v_1} - R_{v_2}.
\]
For \( v_1, v_2 \) fixed, we compute similarly as in (2.9) that
\[
f(v_1) - f(v_2) = g^{ij}_{v_1} g^{kl}_{v_2} \partial_i \partial_j \partial_k \partial_l v_2 - g^{ij}_{v_1} g^{kl}_{v_2} \partial_i \partial_j \partial_k \partial_l v_1 + g^{ij}_{v_1} g^{kl}_{v_2} \partial_i \partial_j \partial_k \partial_l (v_1 - v_2) \\
+ g^{ij}_{v_1} g^{kl}_{v_2} \partial_i \partial_j \partial_k \partial_l v_1 \left( \partial_l g_{ki} + \partial_k g_{li} \right) \\
- g^{ij}_{v_1} g^{kl}_{v_2} \partial_i \partial_j \partial_k \partial_l v_1 \left( \partial_l g_{ki} + \partial_k g_{li} \right) \\
+ g^{ij}_{v_1} g^{kl}_{v_2} \partial_i \partial_j \partial_k \partial_l (v_1 - v_2) + g^{ij}_{v_1} g^{kl}_{v_2} \partial_i \partial_j \partial_k \partial_l (v_1 - v_2) \\
+ \left( g^{ij}_{v_1} g^{kl}_{v_2} - g^{ij}_{v_1} g^{kl}_{v_1} \right) \partial_i \partial_j g_{kl}.
\]
Note that we also have for each $x$ a unique fixed point (2.17)

All terms except $|v_1 - v_2|_{c,\alpha}$ in the above inequality are easy to control. We can estimate

$$\sqrt{t} |v_1 - v_2|_{c,\alpha} \leq \sqrt{t} |v_1 - v_2|_{c,\alpha}^{1/2} |v_1 - v_2|_{c,\alpha}^{1/2} \leq t^{1/4} (|v_1 - v_2|_{c,\alpha} + \sqrt{t} |v_1 - v_2|_{c,\alpha})$$

It implies that

$$|f(v_1) - f(v_2)|_{E_0(J)} \leq C(\varepsilon_0 + T^{1/4}) |v_1 - v_2|_{E_1(J)} + C(\varepsilon_0 + T^{1/4}) |v_1 - v_2|_{C(J,E_{1/2})}$$

We estimate

$$|v_1 - v_2|_{C(J,E_{1/2})} \leq |(v_1 - v_2) - e^{-t\Delta^2_{b}}(x_1 - x_2)|_{C(J,E_{1/2})} + |e^{-t\Delta^2_{b}}(x_1 - x_2)|_{C(J,E_{1/2})},$$

By (2.15), it follows that we get

$$|v_1 - v_2|_{C(J,E_{1/2})} \leq C|(v_1 - v_2) - e^{-t\Delta^2_{b}}(x_1 - x_2)|_{E_1(J)} + C|x_1 - x_2|_{E_{1/2}}$$

(2.17)

By (2.14), (2.16) and (2.17), we can get that

$$|u_1 - u_2|_{E_1(J)} \leq c_1 |x_1 - x_2|_{E_{1/2}} + ||K|| |f(v_1) - f(v_2)|_{E_0(J)} \leq C|x_1 - x_2|_{E_{1/2}} + C(\varepsilon_0 + T^{1/4}) |v_1 - v_2|_{E_1(J)} \leq C|x_1 - x_2|_{E_{1/2}} + \frac{1}{2} |v_1 - v_2|_{E_1(J)},$$

(2.18)

provided that $\varepsilon, T$ are both small enough. In particular taking $x_1 = x_2 = x$, we obtain from (2.18) that

$$|u_1 - u_2|_{E_1(J)} \leq \frac{1}{2} |v_1 - v_2|_{E_1(J)}$$

for $v_1, v_2 \in V_x(J)$. The fixed point theorem in Banach space implies that the map $\Pi_x$ has a unique fixed point

$$\varphi(\cdot, x) \in V_x(J) \subset C^1_{1/2}(J, E_0) \cap C_{1/2}(J, E_1)$$

for each $x \in B_{E_{1/2}}(0, \varepsilon)$. By (2.18), it is clear that

$$|\varphi(\cdot, x) - \varphi(\cdot, y)|_{E_1(J)} \leq C|x - y|_{E_{1/2}}.$$ 

Note that we also have

$$||\varphi(\cdot, x) - \varphi(\cdot, y)||_{C([0,T],E_{1/2})} \leq C|\varphi(\cdot, x) - \varphi(\cdot, y)|_{E_1(J)}.$$
Lastly, we need to show that the solution of (1.1) is unique. Suppose that \( u_1, u_2 \) are two solutions of (1.1). Let

\[
T_1 = \sup\{ t \in [0, T]; u_1(s) = u_2(s) \quad \forall 0 \leq s < t \}.
\]

Since \( u_1, u_2 \in C^1_{1/2}(J, E_0) \cap C_{1/2}(J, E_1) \) we conclude that both belong to the set \( V_x(J^*) \) for \( J^* = [0, T^*] \) provided that \( T^* \) small enough, where

\[
V_x(J^*) = \left\{ v \in E_1(J^*); \ v(0) = x, \| v \|_{C(J^*, E_{1/2})} \leq \varepsilon_0 \right\} \cap B_{E_1(J^*)}(0, \varepsilon_0).
\]

The fixed point theorem in Banach space provides a unique solution in \( V_x(J^*) \) and we conclude that \( T_1 > 0 \). Assume that \( T_1 < T \). It is clear that \( u_1(T_1) = u_2(T_1) = y \).

Let \( v_j(t) = u_j(t + T_1), \ j = 1, 2 \) with \( t \in J_2 = [0, T_2] \) for some \( T_2 \in (0, T - T_1) \). Then \( v_1, v_2 \in C^1(J_2, E_0) \cap C(J_2, E_1) \) and \( v_1, v_2 \) solve

\[
\begin{align*}
\frac{\partial \varphi}{\partial t} &= R\varphi - \mathcal{R}, \\
\varphi(0) &= y.
\end{align*}
\]

If \( T_2 \) is small enough, then \( v_1, v_2 \) belong to the set \( V_y(J_2) \) and we conclude again that \( v_1 = v_2 \) in the set \( V_x(J_2) \). Therefore \( u_1 = u_2 \) for \( t \in [0, T_1 + T_2] \), contradiction. \( \square \)

Theorem 2.2 is a direct consequence of Theorem 2.1 and Theorem 4.1 in [8].

**Proof.** Suppose \( g \) is a constant scalar curvature metric on \( M \). By Theorem 2.1 there exists \( T = T(g), \ c = c(g), \ \varepsilon = \varepsilon(g) \) such that for \( \| \varphi_0 \|_{C^2, \alpha} \leq \varepsilon \), there exists a short time solution \( \varphi(t) \) in \([0, T]\). Also we have estimate

\[
t^{1/2}\| \varphi(t) \|_{C^{1, \alpha}} \leq c\| \varphi_0 \|_{C^2, \alpha}.
\]

By choosing \( \varepsilon \) small enough, we can get that \( \| \varphi(T) \|_{C^{1, \alpha}} \leq T^{-1/2}c\| \varphi_0 \|_{C^2, \alpha} \) is still small enough and then we can apply Theorem 4.1 in [8] to finish the proof. \( \square \)

3. Extension for \( L^\infty \) metrics

In this section, we prove that on Kähler surfaces one can extend the flow once the metrics are bounded in \( L^\infty \), using a blowing up argument.

**Theorem 3.1.** Suppose the Calabi flow exists for time \([0, T]\) on a compact Kähler surface \((M, [\omega])\). If there exist constants \( c_1, C_2 > 0 \) such that \( c_1\omega_0 \leq \omega(t) \leq C_2\omega_0 \) for any \( t \in [0, T) \), then \( T = \infty \) and the curvature is uniformly bounded in \([0, T]\). Also the Calabi flow converges to an extremal metric in Cheeger-Gromov sense when \( t \to \infty \).

**Proof.** For simplicity, we can assume that the initial metric is \( \omega(0) = \omega_0 \). Suppose the curvature blows up when \( t \to T \), then we can pick up \( x_i, t_i \) such that \( Q_i = \max_{t \leq t_i} |Rm| = |Rm(x_i, t_i)| \to \infty \). Consider the blowing up solution of the Calabi flow

\[
g_i(t) = Q_i(g(t_i + tQ_i^2))
\]

and the pointed manifold \( \{ M, x_i, g_i(t) \} \). We consider volume ratio for the metric \( \omega(t) \) before blowing up. Note that the volume ratio is scaling invariant. Denote \( B_p(r, t) \) to be the geodesic ball with respect to \( \omega(t) \) at \( p \) with radius \( r \). We assume that \( r \leq r_0 \) for some constant \( r_0 > 0 \). We can pick up \( r_0 \) such that for any point \( p \in M, B_p(r_0, 0) \) is in normal
neighborhood with respect to $\omega_0$, and so $B_p(r,0)$ is diffeomorphic to a Euclidean ball for any $r \leq r_0$. For $r \leq r_0$, we can assume that the volume ratio of $\omega_0$ is bounded away from zero by $l_0$,
\[
\frac{Vol_0(B_p(r,0))}{r^n} \geq l_0.
\]
Since $c_1\omega_0 \leq \omega(t) \leq C_2\omega_0$, it follows that the volume ratio of $\omega(t)$ is also bounded from below for $r \leq r_0$,
\[
\frac{Vol_t(B_p(r,t))}{r^n} \geq \frac{l_0c_1^n}{C_2^n}.
\]
For the sequence $g_i(t)$, the curvature is uniformly bounded and the volume ratio is also bounded from below, then the injectivity radius is also bounded away from zero for $g_i(t)$. Also it is clear that the Sobolev constant for $\omega(t)$ is uniformly bounded away from zero since $c_1\omega_0 \leq \omega(t) \leq C_2\omega_0$. It follows that the pointed sequence of the Calabi flow solutions $\{M, x_i, g_i(t)\}$ will converge to a Calabi flow solution $\{M_\infty, x_\infty, g_\infty(t)\}$ by the result in [9]. Moreover, the limit metric $g_\infty(t)$ is a scalar flat ALE Kähler metric on $M_\infty$. Now we show that $M_\infty$ is diffeomorphic to $\mathbb{R}^4$. Roughly speaking, $M_\infty$ is just the limit of the geodesic balls $B_{x_i}(r_0,0)$ after blowing up. But all these balls are actually diffeomorphic to a Euclidean ball, and it follows that $M_\infty$ itself has to be diffeomorphic to $\mathbb{R}^4$.

Recall that for any $p \in M, r \leq r_0$, the geodesic ball $B_p(r,0)$ is diffeomorphic to a Euclidean ball. Since $c_1\omega_0 \leq \omega(t) \leq C_2\omega_0$, then
\[
B_p(r/\sqrt{C_2},t) \subset B_p(r,0) \subset B_p(r/\sqrt{c_1},t).
\]
Let us recall the construction of $(M_\infty, x_\infty, g_\infty)$, which is the geometric limit in the Cheeger-Gromov sense of a subsequence of $(M, x_i, g_i(0) = Q_i g(t_i))$ for $i \to \infty$. Without loss of generality, we can assume that $(M, x_i, Q_i g_i(t_i))$ converges to $(M_\infty, x_\infty, g_\infty)$. Denote $g_i = Q_i g(t_i)$. By the construction of $M_\infty$, there exists open subsets $U_i$ of $M_\infty$ for each $i$ such that $x_i \in U_i$, $U_i \subset U_{i+1}$ and $M_\infty = \cup U_i$. Also for each $i$, there exists a diffeomorphism $\varphi_i : U_i \to \varphi_i(U_i) \subset \{M, x_i, g_i\}$, $\varphi_i(x_\infty) = x_i$ and $\varphi_i^* g_i \to g_\infty$ on any compact subset $K$ of $M_\infty$. Now for any fixed $R \in (0, \infty)$, consider $B_{x_\infty}(R) \subset M_\infty$. For $i$ large enough, we can assume that $B_{x_\infty}(R) \subset U_i$.

We consider $\varphi_i(B_{x_\infty}(R)) \subset \{M, x_i, g_i\}$. Since $\varphi_i$ is almost an isometry when $i$ large enough, we can assume that $\varphi_i(B_{x_\infty}(R))$ contains the geodesic ball centered at $x_i$ with radius $R - 1$ and is contained in the geodesic ball centered at $x_i$ with radius $R + 1$ with respect to metric $g_i$. Before scaling, we can get
\[
B_{x_i}(Q_i^{-1}(R - 1), t_i) \subset \varphi_i(B_{x_\infty}(R)) \subset B_{x_i}(Q_i^{-1}(R + 1), t_i).
\]
By (3.1), we can get that
\[
B_{x_i}(Q_i^{-1}(R - 1)\sqrt{c_1}/\sqrt{C_2}, t_i) \subset B_{x_i}(Q_i^{-1}(R - 1)\sqrt{c_1}, t_i) \subset B_{x_i}(Q_i^{-1}(R - 1), t_i).
\]
It is clear that for $i$ large enough, $B_{x_i}(Q_i^{-1}(R - 1)\sqrt{c_1}, t_i)$ is diffeomorphic $\mathbb{R}^4$. Now we fix $i = i_R$ large enough. Let $V_R = \varphi_i^{-1}(B_{x_i}(Q_i^{-1}(R - 1)\sqrt{c_1}, 0))$. Then $V_R$ is an open set on $M_\infty$ and it is diffeomorphic to $\mathbb{R}^4$. In particular, by (3.2),
\[
B_{x_\infty}((R - 1)\sqrt{c_1}/\sqrt{C_2} - 1) \subset V_R \subset B_{x_\infty}(R).
\]
By (3.3), we can pick up an increasing sequence $R_k \to \infty$ when $k \to \infty$ and correspondingly we can construct $V_{R_k}$ such that $V_{R_k} \subset V_{R_{k+1}}$. It is also clear that 

$$M_\infty = \cup_k V_{R_k}.$$

Hence $M_\infty$ is diffeomorphic to $\mathbb{R}^4$. But a scalar flat ALE (asymptotically locally Euclidean) Kähler metric on $\mathbb{R}^4$ has to be flat [1]. This contradicts that $(M_\infty, g_\infty)$ is non-flat. This implies that $T = \infty$ and the curvature has to be bounded for all $[0, \infty)$. Also it is clear that the flow converges to an extremal metric in Cheeger-Gromov sense. □

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