Ergodicity for Stochastic Conservation Laws with Multiplicative Noise

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Abstract: We proved that there exists a unique invariant measure for solutions of stochastic conservation laws with Dirichlet boundary condition driven by multiplicative noise. Moreover, a polynomial mixing property is established. This is done in the setting of kinetic solutions taking values in an $L^1$-weighted space.

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1. Introduction

In this paper, we investigate the long time behaviour of stochastic scalar conservation laws with multiplicative noise. The (deterministic) conservation laws are fundamental to our understanding of the space-time evolution laws of interesting physical quantities. Mathematically or statistically, such physical laws should incorporate with noise influences, due to the lack of knowledge of certain physical parameters as well as bias
or incomplete measurements arising in experiments or modeling. More precisely, fix any $T > 0$ and let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0,T]}, \{(\beta_k(t))_{t \in [0,T]}\}_{k \in \mathbb{N}})$ be a stochastic basis. Without loss of generality, here the filtration $\{\mathcal{F}_t\}_{t \in [0,T]}$ is assumed to be complete and $\{\beta_k(t)\}_{t \in [0,T]}$, $k \in \mathbb{N}$, are independent (one-dimensional) $\{\mathcal{F}_t\}_{t \in [0,T]}$–Wiener processes. We use $\mathbb{E}$ to denote the expectation with respect to the probability measure $\mathbb{P}$. Let $D \subset \mathbb{R}^d$ denote a bounded convex open set whose boundary $\partial D$ is $C^2$. We are concerned with the following initial-Dirichlet boundary valued problem of the scalar conservation law with stochastic forcing, denoted byb $\mathcal{E}(\Phi, \vartheta)$:

$$
\text{div}(A(u)) dt = \Phi(u) dW(t) \quad \text{in} \quad D \times (0, T),
$$

with the initial condition

$$
u(0, \cdot) = \vartheta \quad \text{in} \quad D,
$$

and the boundary condition

$$
u = u_b = 0 \quad \text{on} \quad \Sigma.
$$

Here, $\Sigma = (0, T) \times \partial D$, $u : (\omega, t, x) \in \Omega \times [0, T] \times D \mapsto u(\omega, t, x) := u(t, x) \in \mathbb{R}$ is a random field, the flux function $A : \mathbb{R} \rightarrow \mathbb{R}^d$ and the coefficient $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ are measurable and fulfill certain conditions specified later, and $W$ is a cylindrical Wiener process on a given (separable) Hilbert space $U$ with the form $W(t) = \sum_{k \geq 1} \beta_k(t)e_k$, $t \in [0, T]$, where $(e_k)_{k \geq 1}$ is a complete orthonormal basis of the Hilbert space $U$. Set $Q = (0, T) \times D$.

The deterministic conservation laws (i.e., $\Phi \equiv 0$ in (1.1)) are well studied in the PDEs literature, see e.g. the monograph [7] and the most recent reference Ammar, Willbold and Carrillo [1] (and references therein). As well known, the Cauchy problem for the deterministic first-order PDE (1.1) does not admit any (global) smooth solutions, but there exist infinitely many weak solutions and an additional entropy condition has to be added to get the uniqueness and further to identify the physical weak solution. The notion of entropy solutions for the deterministic problem in the $L^\infty$ framework was initiated by Otto in [24]. Porretta and Vovelle [25] studied the problem in the $L^1$ setting. To deal with unbounded solutions, the authors of [25] defined a notion of renormalized entropy solutions which generalized Otto’s original definition of entropy solutions. The kinetic formulation of weak entropy solution of the Cauchy problem for a general multidimensional scalar conservation law was derived by Lions, Perthame and Tadmor in [22]. Concerning the initial-boundary value problem for deterministic conservation laws, it is crucial to give an interpretation of the boundary condition (1.3). In the setting of functions of bounded variation, Bardos, Le Roux and Nédélec [2] considered the boundary condition (1.3) as an “entropy” inequality on the boundary $\Sigma$ and obtained the global well-posedness of entropy solutions to (1.1)–(1.3). Later, Otto [24] extended it to the $L^\infty$ setting by introducing the notion of boundary entropy-flux pairs. Imbert and Vovelle [17] derived a kinetic formulation of weak entropy solutions of the initial-boundary value problem and proved the uniqueness of such a kinetic solution.

In recent years, there has been a growing interest in the study of conservation laws driven by stochastic forcing. Having a stochastic forcing term in (1.1) is very natural and important for various modeling problems arising in a wide variety of fields, e.g., physics, engineering, biology and so on. The Cauchy problem (1.1) driven by additive noise has been studied by Kim in [18] wherein the author proposed a method of compensated compactness to prove the existence of a stochastic weak entropy solution.
via vanishing viscosity approximation. Concerning the case with multiplicative noise, Feng and Nualart [15] introduced a notion of strong entropy solutions and established the existence and uniqueness in the one-dimensional case. Using a kinetic formulation, Debussche and Vovelle [11] solved the stochastic Cauchy problem (1.1) with periodic boundary condition in any dimension. Based on [11], Dong et al. [13] established small noise large deviations for kinetic solutions of periodic stochastic conservation laws with multiplicative noise. On the other hand, Vallet and Wittbold in [27] studied the multi-dimensional Dirichlet boundary value problem for stochastic conservation laws driven by additive noise. For the initial-Dirichlet boundary value problem with multiplicative noise, Bauzet, Vallet and Wittbold [3] established the existence and uniqueness of stochastic entropy solutions when the flux function is assumed to be globally Lipschitz. Recently, Kobayasi and Noboriguchi [20] relaxed the condition on the flux function to be of polynomial growth by using kinetic formulation for stochastic conservation laws with nonhomogeneous Dirichlet boundary conditions.

We remark that there are not many works on the long time behavior/ergodicity of stochastic scalar conservation laws. In the space dimension one, E et al. [14] proved the existence and uniqueness of invariant measures for the periodic stochastic inviscid Burgers equation with additive forcing. Debussche and Vovelle [12] studied scalar conservation laws with additive stochastic forcing on toruses of any dimension and proved the existence and uniqueness of an invariant measure for sub-cubic fluxes and sub-quadratic fluxes, respectively. Later, Chen and Pang [4] extend the result of [12] to degenerate second-order parabolic-hyperbolic conservation laws driven by additive noise. We want to stress that in the above papers, only additive noise was considered and no convergence rate to the invariant measure was obtained.

The purpose of this paper is to obtain the ergodicity and further to establish the polynomial mixing property for stochastic conservation laws (1.1)–(1.3) driven by multiplicative noise. As far as we know, this is the first result for the case of multiplicative noise. Our method is inspired by the work [9] where the authors proved the ergodicity for entropy solutions of stochastic porous media equations on smooth bounded domain with Dirichlet boundary conditions.

However, we will work on the setting of kinetic formulation of the solutions. As in [9], in order to obtain a polynomial rate of convergence to the invariant measure, we choose to work on a weighted \( L_{w; x}^1 \) space for a suitable weight function \( w \). As an important part of the proof, we apply the doubling variables method in \( L_{w; x}^1 \) to obtain a “super \( L_{w; x}^1 \)-contraction principle” for the solutions, that is, there exists an extra strictly negative term on the right hand side of the \( L_{x}^1 \)-contraction principle (see (4.7)), which is the key to obtain the polynomial decay rate. As the invariant measure is living in the \( L_{x}^1 \)-space, we need to show that the kinetic solution to (1.1)–(1.3) has a continuous extension (with respect to the time) in the space \( L_{w; x}^2 L_{x}^1 \). To do so, we use the vanishing viscosity method to introduce approximating equations and to overcome difficulties caused by the unboundedness of the flux function. This is quite different from the work [9] where the authors used smooth approximation of the coefficients. The Markov semigroup associated with the kinetic solution is defined in the \( L_{x}^1 \)-space, which is further proved to be Feller. The final step is to show that the solutions of the stochastic conservation laws converges to a unique stationary solution with a polynomial convergence rate. The “super \( L_{w; x}^1 \)-contraction principle” plays a key role.

The rest of the paper is organized as follows. In Sect. 2, the mathematical formulation of stochastic scalar conservation laws and some known results are presented. In Sect. 3,
we state our main results. Section 4 is devoted to proving a “super $L^1_{w;x}$—contraction principle” for the kinetic solutions. The existence of a continuity extension of kinetic solutions is proved in Sect. 5. In Sect. 6, we prove that the kinetic solution of the initial-boundary value problem admits a unique invariant measure and satisfies the polynomial mixing property. In the sequel, we use the letter $C$ to denote a generic constant whose values may change from one line to another. Sometimes, we precise its dependence on parameters.

2. Preliminaries

Let $L(K_1, K_2)$ (resp. $L_2(K_1, K_2)$) be the space of bounded (resp. Hilbert-Schmidt) linear operators from a Hilbert space $K_1$ to another Hilbert space $K_2$, whose norm is denoted by $\|\cdot\|_{L(K_1, K_2)}$ (resp. $\|\cdot\|_{L_2(K_1, K_2)}$). Further, $C_b$ represents the space of bounded, continuous functions.

Let $\|\cdot\|_{L^p}$ denote the norm of Lebesgue space $L^p(D)$ for $p \in [1, \infty]$, where $x$ indicates the name of the variable. In particular, set $H = L^2(D)$ with the corresponding norm $\|\cdot\|_H$. For all $a \in \mathbb{R}$ and $p \in [1, \infty]$, let $W^{a,p}(D)$ be the usual Sobolev space, whose norm is denoted by $\|\cdot\|_{W^{a,p}}$. When $p = 2$, set $H^a(D) = W^{a,2}(D)$. Moreover, we use the brackets $\langle \cdot, \cdot \rangle$ to denote the duality between $C^\infty_c(D \times \mathbb{R})$ and the space of distributions over $D \times \mathbb{R}$. With a slight abuse of the notation $\langle \cdot, \cdot \rangle$, we set

$$\langle F, G \rangle := \int_D \int_{\mathbb{R}} F(x, \xi)G(x, \xi)dx d\xi, \quad F \in L^p(D \times \mathbb{R}), G \in L^q(D \times \mathbb{R}).$$

for $1 \leq p < \infty$ and $q := \frac{p}{p-1}$, the conjugate exponent of $p$. In particular, when $p = 1$, we set $q = \infty$ by convention.

For a measure $m$ on the Borel measurable space $D \times [0, T] \times \mathbb{R}$, the shorthand $m(\phi)$ is defined by

$$m(\phi) := \langle m, \phi \rangle([0, T])$$

$$:= \int_{D \times [0, T] \times \mathbb{R}} \phi(x, t, \xi) dm(x, t, \xi), \quad \phi \in C_b(D \times [0, T] \times \mathbb{R}).$$

Define

$$w(x) = -(x_1 + x_2 + \cdots + x_d) + C_0, \quad x = (x_1, \ldots, x_d),$$

(2.1)

where $C_0$ is a constant bigger than $\max_{x \in D}(x_1 + x_2 + \cdots + x_d)$ so that $w(x) > 0$ in $D$. Let $L^1_{w;x}$ be the space of all measurable functions $f : D \rightarrow \mathbb{R}$ such that

$$\|f\|_{L^1_{w;x}} := \int_D |f(x)| w(x) dx < \infty.$$ 

Clearly, $\|\cdot\|_{L^1_{w;x}}$ is equivalent to $\|\cdot\|_{L^1_x}$.

To end this subsection, we mention some notations related to the predictability. For a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$ and $p \in [1, \infty)$, we denote by $L^p_\omega$ the space of $p$—integrable random variables in $\omega \in \Omega$. For $T > 0$, let $\mathcal{B}([0, T])$ be the Borel $\sigma$—algebra on $[0, T]$ and denote by $\mathcal{P}_T \subset \mathcal{B}([0, T]) \otimes \mathcal{F}$ the predictable $\sigma$—algebra (see [6], Section 2.2). Let $L^p_\omega L^q_t$ stand for the space of $p$—integrable random variables taking values in $L^q_t$ and $L^p_\omega L^q_t$ represent the set of functions $v \in L^p(\Omega \times [0, T]; L^q_t)$ which are equal $\mathbb{P} \times dt$—almost everywhere to a predictable process $u$, where $dt$ is the Lebesgue measure on $[0, T]$. 
2.1. Hypotheses. For the initial value \( \vartheta \), the flux function \( A \), and the coefficient \( \Phi \) of (1.1)–(1.3), we introduce the following hypotheses.

**H1** The flux function \( A = (A_1, \ldots, A_d) \in C^2(\mathbb{R}; \mathbb{R}^d) \). Each component \( A_j \) is differentiable, strictly increasing and odd. The derivative \( a_j = A'_j \geq 0 \) has at most polynomial growth. That is, there exist constants \( C > 0 \) and \( q_0 \geq 1 \) such that

\[
\sum_{j=1}^{d} |a_j(\xi) - a_j(\zeta)| \leq C(1 + |\xi|^{q_0-1} + |\zeta|^{q_0-1})|\xi - \zeta|. \tag{2.2}
\]

Moreover, assume that there exists \( C_{q_0} \) such that

\[
\sum_{j=1}^{d} \left| A_j(u) - A_j(v) \right| \geq C_{q_0} |u - v|^{q_0+1} \quad \text{for } u, v \in \mathbb{R}. \tag{2.3}
\]

**H2** The initial value \( \vartheta \in L^q_0 L^q_\omega \) for all \( q \geq 1 \) and is an \( \mathcal{F}_0 \otimes B(D) \)-measurable random variable.

**H3** For each \( u \in \mathbb{R} \), the map \( \Phi(u) : U \to H \) is defined by \( \Phi(u)e_k = g_k(\cdot, u) \), where \( g_k(\cdot, u) \) is a regular function on \( D \). More precisely, we assume that \( g_k \in C(D \times \mathbb{R}) \) satisfying

\[
G^2(x, u) = \sum_{k \geq 1} |g_k(x, u)|^2 \leq K_0(1 + |u|^2), \tag{2.4}
\]

\[
\sum_{k \geq 1} |g_k(x, u) - g_k(y, v)|^2 \leq K_0 \left( |x - y|^2 + |u - v|^2 \right), \tag{2.5}
\]

for some constant \( K_0 > 0 \) and \( x, y \in D, u, v \in \mathbb{R} \). Since \( \|g_k(\cdot, u)\|_H \leq C \|g_k(\cdot, u)\|_{C(D)} \), we deduce that \( \Phi(u) \in L_2(U, H) \), for each \( u \in \mathbb{R} \).

To deduce the \( L^1 \)-theory of (1.1)–(1.3), we need a stronger condition than (H3) on \( \Phi \):

**H4** There exist constants \( C_k \geq 0 \) such that

\[
|g_k(x, u)| \leq C_k(1 + |u|), \quad \sum_{k \geq 1} C_k^2 < \infty, \tag{2.6}
\]

and (2.5) remains unchanged.

Remark 1. The set of \( A_i \) satisfying Hypothesis (H1) is not empty, e.g. taking \( A_i(u) = |u|^{q_0} u \) with an even integer \( q_0 \). Moreover, the condition (2.3) shows that there exists at least one non-zero component of \( A(u) \), which satisfies the non-degeneracy condition required by Theorem 1 in [12].
2.2. Kinetic solution. We follow closely the framework of [19–21]. Firstly, the domain $D$ can be localized by the following method: choosing a finite open cover $\{U_i\}_{i=0,\ldots,M}$ of $\overline{D}$ and a partition of unity $\{\lambda_i\}_{i=0,\ldots,M}$ on $\overline{D}$ subject to $\{U_i\}_{i=0,\ldots,M}$ such that $U_0 \cap \partial D = \emptyset$, for $i = 1, \ldots, M$, up to a change of coordinates represented by an orthogonal matrix $A_i$, the set $D \cap U_i$ is the epigraph of a $C^2$ function $h_i : \mathbb{R}^{d-1} \to \mathbb{R}$, i.e.,

$$D^{\lambda_i} := D \cap U_i = \left\{ x \in U_i; (A_i x)_{d} > h_i(A_i x) \right\}, \quad \partial D^{\lambda_i} := \partial D \cap U_i = \left\{ x \in U_i; (A_i x)_{d} = h_i(A_i x) \right\},$$

where $x = (\bar{x}, x_d) \in \mathbb{R}^d$ and $\bar{x} = (x_1, \ldots, x_{d-1})$. For simplicity, we will drop the index $i$ and suppose that the change of coordinates is trivial: $A_i = I_d$.

Moreover, we set

$$Q^\lambda = (0, T) \times D^\lambda, \quad \Sigma^\lambda = (0, T) \times \partial D^\lambda, \quad \Pi^\lambda = \{\bar{x}; x \in U_\lambda\}.$$  

We denote by $n(\bar{x})$ the outward unit normal to $D^\lambda$ at a point $(\bar{x}, h_\lambda(\bar{x}))$ of $\partial D^\lambda$ and by $d\sigma(\bar{x})$ the $(d - 1)$—dimensional area element in $\partial D^\lambda$:

$$n(\bar{x}) = (1 + |\nabla \bar{x}h_\lambda(\bar{x})|^2)^{-1/2}(\nabla \bar{x}h_\lambda(\bar{x}), -1), \quad d\sigma(\bar{x}) = (1 + |\nabla \bar{x}h_\lambda(\bar{x})|^2)^{1/2}d\bar{x}. \quad (2.7)$$

Denote by $L_\lambda$ the Lipschitz constant of $h_\lambda$ on $\Pi^\lambda$, $\bar{L}_\lambda := \sqrt{d - 1}L_\lambda$ and set

$$L := \left( \sum_{i=0}^{M} L_{\lambda_i} \right) \lor \left( \max_{0 \leq i \leq M} \| \Delta \bar{x}h_i(\bar{x}) \|_{L^\infty(\Sigma^\lambda_i)} \right). \quad (2.8)$$

To regularize functions that are defined on $D^\lambda$ and $\mathbb{R}$, let us consider a standard modifier $\psi$ on $\mathbb{R}$, that is, $\psi$ is a nonnegative and even function in $C^\infty_c((-1, 1))$ with $\int_{\mathbb{R}} \psi(x)dx = 1$. We set

$$\rho^\lambda(x) = \Pi_{i=1}^{d-1} \psi(x_i) \psi(x_d - (\bar{L}_\lambda + 1)), \quad (2.9)$$

for $x = (x_1, \ldots, x_d)$. For $\gamma, \delta > 0$, we set $\rho^\lambda_\gamma(x) = \frac{1}{\gamma^d} \rho^\lambda \left( \frac{x}{\gamma} \right)$ and $\psi_\delta(\xi) = \frac{1}{\delta} \psi \left( \frac{\xi}{\delta} \right)$.

Finally, let $\psi^\lambda$ stand for $\psi \lambda$ and $\tilde{\psi}$ denote the restriction of $\psi$ to $\Sigma \times \mathbb{R}_\xi$, i.e., $\tilde{\psi}(t, \bar{x}, h(\bar{x}), \xi) = \psi(t, \bar{x}, h(\bar{x}), \xi)$, where $\psi$ is a function on $(0, T) \times \mathbb{R}^{d+1}$ and $\lambda$ is an element of the partition of unity $\{\lambda_i\}_{i=0}^{M}$.

Recall that we are working on the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \in [0, T]}, (\beta_k(t))_{k \in \mathbb{N}})$.

**Definition 2.1** (Kinetic measure). A map $m$ from $\Omega$ to $\mathcal{M}_b^c(D \times [0, T) \times \mathbb{R})$, the set of non-negative finite measures over $D \times [0, T) \times \mathbb{R}$, is said to be a kinetic measure if

1. $m$ is weakly measurable, that is, for each $\phi \in C_b(D \times [0, T) \times \mathbb{R})$, $\langle m, \phi \rangle : \Omega \to \mathbb{R}$ is measurable,
2. $m$ vanishes at infinity, i.e.,

$$\lim_{R \to +\infty} \mathbb{E}[m(D \times [0, T) \times B^c_R)] = 0, \quad B^c_R := \{\xi \in \mathbb{R}; |\xi| \geq R\}. \quad (2.9)$$
3. for every $\phi \in C_b(D \times \mathbb{R})$, the process

$$(\omega, t) \in \Omega \times [0, T) \rightarrow \int_{D \times [0, t] \times \mathbb{R}} \phi(x, \xi) \, dm(x, s, \xi) \in \mathbb{R}$$

is predictable.

**Definition 2.2** (Kinetic solution). Let $\vartheta \in L^q_\omega L^q_T$ for all $q \geq 1$. A measurable function $u : \Omega \times [0, T] \times D \to \mathbb{R}$ is called a kinetic solution to (1.1)–(1.3) with initial data $\vartheta$ if

1. for all $q \geq 1$, $u \in L^q_\omega L^q_T$ and there exists $C_q \geq 0$ such that

$$\mathbb{E} \text{ ess sup} \|u(t)\|_{L^q_T} \leq C_q, \quad (2.10)$$

2. there exists a kinetic measure $m$ and for any $N > 0$, there exist nonnegative functions $\tilde{m}^\pm_N \in L^1(\Omega \times \Sigma \times (-N, N))$ such that \{\tilde{m}^\pm_N(t)\} are predictable,

$$\lim_{\xi \uparrow N} \tilde{m}^+_N(t, x, \xi) = \lim_{\xi \downarrow -N} \tilde{m}^-_N(t, x, \xi) = 0,$$

for all $\varphi \in C^\infty_c([0, T) \times \overline{D} \times (-N, N))$, $f := I_{u > \xi}$ satisfies

$$\int_0^T \langle f(t), \partial_t \varphi(t) \rangle \, dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f(t), a(\xi) \cdot \nabla \varphi(t) \rangle \, dt$$

$$+ M_N \int_{\Sigma \times \mathbb{R}} f_b \varphi d\xi d\sigma(\bar{\xi}) \, dt$$

$$= - \sum_{k \geq 1} \int_0^T \int_D g_k(x, u(t, x)) \varphi(t, x, u(t, x)) \, dx \, d\beta_k(t)$$

$$- \frac{1}{2} \int_0^T \int_D \partial_\xi \varphi(t, x, u(t, x)) G^2(x, u(t, x)) \, dx \, dt$$

$$+ m(\partial_\xi \varphi) + \int_{\Sigma \times \mathbb{R}} \partial_\xi \varphi \tilde{m}^+_N \, d\xi d\sigma(\bar{\xi}) \, dt, \quad a.s.,$$

(2.11)

and $\tilde{f} := 1 - f = I_{u \leq \xi}$ satisfies

$$\int_0^T \langle \tilde{f}(t), \partial_t \varphi(t) \rangle \, dt + \langle \tilde{f}_0, \varphi(0) \rangle + \int_0^T \langle \tilde{f}(t), a(\xi) \cdot \nabla \varphi(t) \rangle \, dt$$

$$+ M_N \int_{\Sigma \times \mathbb{R}} \tilde{f}_b \varphi d\xi d\sigma(\bar{\xi}) \, dt$$

$$= \sum_{k \geq 1} \int_0^T \int_D g_k(x, u(t, x)) \varphi(t, x, u(t, x)) \, dx \, d\beta_k(t)$$

$$+ \frac{1}{2} \int_0^T \int_D \partial_\xi \varphi(t, x, u(t, x)) G^2(x, u(t, x)) \, dx \, dt$$

$$- m(\partial_\xi \varphi) - \int_{\Sigma \times \mathbb{R}} \partial_\xi \varphi \tilde{m}^-_N \, d\xi d\sigma(\bar{\xi}) \, dt, \quad a.s.,$$

(2.12)

where $a(\xi) := A'(\xi)$, $G^2 = \sum_{k=1}^\infty |g_k|^2$, $M_N = \max_{-N \leq \xi \leq N} |a(\xi)|$, $f_0 = I_{\vartheta \gtrless \xi}$ and $f_b = I_{0 \gtrless \xi}$. 
Remark 2. The boundary function $\bar{m}^\pm$ does not appear in the case of the periodic boundary condition, hence it is enough to consider the equality (2.11) for $f$ (the equation satisfied by $\bar{f}$ can be derived from (2.11)). However, in the case of the Dirichlet boundary conditions, the boundary functions $\bar{m}^+$ and $\bar{m}^-$ are different from each other, thus, we need to consider both (2.11) and (2.12).

The existence and uniqueness of kinetic solutions to (1.1)–(1.3) with initial datum $\vartheta \in L^\infty(\Omega \times D)$ has been proved by [20]. Later, the condition imposed on the initial data was relaxed to $\vartheta \in L^q(\Omega \times D)$ for all $q \in [1, \infty)$ by [21]. Precisely, the following result was stated by Theorem 2.4 in [21].

**Theorem 2.1.** Under Hypotheses (H1)–(H3), there exists a unique kinetic solution to (1.1)–(1.3), which has almost surely continuous trajectories in $L^q_\Omega$ for all $q \in [1, \infty)$.

Note that the equations (2.11)–(2.12) satisfied by the kinetic solution in Definition 2.2 imply that the solution is weak with respect to $t, x$ and $\xi$. In fact, as stated in Proposition 1 of [20], (2.11)–(2.12) can be strengthened to be weak only respect to $t$ and $\xi$. In order to state it, we need the following definition.

**Definition 2.3** (Young measure). Let $(X, \lambda)$ be a finite measure space and $\mathcal{M}_1(\mathbb{R})$ be the set of all (Borel) probability measures on $\mathbb{R}$. A map $\nu : X \to \mathcal{M}_1(\mathbb{R})$ is said to be a Young measure on $X$, if for each $\phi \in C_b(\mathbb{R})$, the map $z \in X \mapsto \nu_z(\phi) \in \mathbb{R}$ is measurable. We say that a Young measure $\nu$ vanishes at infinity if, for each $p \geq 1$,

$$
\int_X \int_{\mathbb{R}} |\xi|^p d \nu_z(\xi) d \lambda(z) < +\infty. \quad (2.13)
$$

Let $(X, \lambda)$ be a finite measure space. For some measurable function $u : X \to \mathbb{R}$, define $f : X \times \mathbb{R} \to [0, 1]$ by $f(z, \xi) = I_{u(z) > \xi}$ a.e. and we use $\bar{f} := 1 - f$ to denote its conjugate function. Define $\Lambda f(z, \xi) := f(z, \xi) - I_{0 > \xi}$, which can be viewed as a correction to $f$. Note that $\Lambda f$ is integrable on $X \times \mathbb{R}$ if $u$ is.

**Proposition 2.2** (Left and right weak limits). Let $u$ be a kinetic solution to (1.1)–(1.3). Then $f = I_{u > \xi}$ admits, almost surely, left and right limits respectively at every point $t \in [0, T]$. More precisely, for any $t \in [0, T]$, there exist functions $f^{t^\pm}$ on $\Omega \times D \times \mathbb{R}$ such that $\mathbb{P}$–a.s.

$$
\langle f(t - \varepsilon), \varphi \rangle \to \langle f^{t^-}, \varphi \rangle, \quad \langle f(t + \varepsilon), \varphi \rangle \to \langle f^{t^+}, \varphi \rangle,
$$

as $\varepsilon \to 0$ for all $\varphi \in C^1_c(D \times \mathbb{R})$. Moreover, almost surely,

$$
\langle f^{t^+} - f^{t^-}, \varphi \rangle = -\int_{\Omega \times [0, T] \times \mathbb{R}} \partial_\xi \varphi(x, \xi) I_{\{t\}}(s) dm(x, s, \xi).
$$

In particular, almost surely, the set of $t \in [0, T]$ fulfilling that $f^{t^+} \neq f^{t^-}$ is countable.

The proof of Proposition 2.2 can be done by using entirely the similar arguments as in the proof of Proposition 10 in [11], as the additional terms generated by the boundary conditions can be incorporated into $J_\vartheta$ there which will not cause any difficulties. In addition, for the function $f = I_{u > \xi}$, we set $f^{\pm}(t) = f^{t^\pm}, t \in [0, T]$. Since we are dealing with the filtration associated to Brownian motion, both $f^{\pm}$ are clearly predictable as well. Also $f = f^+ = f^-$ almost everywhere in time and we can take any of them in an integral with respect to the Lebesgue measure or in a stochastic integral.
Define two non-increasing functions $\mu_m(\xi)$ and $\mu_v(\xi)$ on $\mathbb{R}$ by
\[
\mu_m(\xi) = E m([0, T) \times D \times (\xi, \infty)), \quad \mu_v(\xi) = E \int_{(0,T) \times D \times (\xi, \infty)} d\nu_{t,v}(\xi) dx dt.
\]
(2.14)
where $m$ is a kinetic measure and $v$ is a Young measure on $(0, T) \times D$ satisfying (2.13). Let $\mathbb{D}$ be the set of $\xi \in (0, \infty)$ such that both of $\mu_m$ and $\mu_v$ are differentiable at $-\xi$ and $\xi$. Since $\mu_m(\xi)$ and $\mu_v(\xi)$ are two non-increasing functions with respect to $\xi \in \mathbb{R}$, $\mathbb{D}$ is a full set in $(0, +\infty)$, which means that the Lebesgue measure of $\mathbb{D}$ is equal to 0. Denote by $\mu'_m$ and $\mu'_v$ the derivatives of $\mu_m$ and $\mu_v$, respectively. It was shown in Lemma 2 of [20] that

Lemma 2.1 (i) For any $p \geq 1$,
\[
\lim_{\xi \to \infty, \xi \in \mathbb{D}} \mu'_m(\pm \xi) = 0, \quad \lim_{\xi \to \infty, \xi \in \mathbb{D}} \xi^p \mu'_v(\pm \xi) = 0.
\]
(ii) If $N \in \mathbb{D}$, then as $\delta \downarrow 0$,
\[
\int_{\mathbb{R}} \psi_\delta(N \pm \xi) d\mu_m(\xi) \to \mu'_m(\mp N),
\]
\[
\int_{\mathbb{R}} \psi_\delta(N \pm \xi)(1 + |\xi|^2) d\mu_v(\xi) \to (1 + N^2) \mu'_v(\mp N).
\]

We also need to introduce the following cutoff function. For any $0 < \eta < N$, let
\[
\Psi_\eta(\xi) = \int_{-\infty}^\xi (\psi_\eta(\xi + N - \eta) - \psi_\eta(\xi - N + \eta)) d\xi.
\]
Clearly, $\Psi_\eta(\xi) \geq 0$ and $\Psi_\eta(\xi) = 0$ for $|\xi| \geq N$. With the help of the cutoff function, the test functions in (2.11) and (2.12) can be extended to the class of functions in $C^\infty_c((0, T) \times \mathbb{R}^d \times \mathbb{R})$.

The following result was proved by Proposition 1 in [20]. It states that the weak form (2.11)–(2.12) satisfied by a kinetic solution can be strengthened to be weak only respect to $x$ and $\xi$.

Proposition 2.3. Assume that $u$ is a kinetic solution of (1.1)–(1.3). Set $f(t, x, \xi) := I_{u(t,x) > \xi}$ and define
\[
f^{\lambda,\gamma}(t, x, \xi) = \int_{D^\lambda} f(t, y, \xi) \rho^\lambda_y(y - x) dy, \quad (t, x, \xi) \in \Sigma^\lambda \times \mathbb{R}, \quad (2.15)
\]
for any element $\lambda$ of the partition of unity $\{\lambda_i\}$ on $\overline{D}$. Let $\tilde{f}^{(\lambda)}$ be any weak star limit of $\{f^{\lambda,\gamma}\}$ as $\gamma \to 0$ in $L^\infty(\Omega \times \Sigma^\lambda \times \mathbb{R})$ and define $\tilde{f} := \sum_{i=0}^M \lambda_i \tilde{f}^{(\lambda_i)}$. Then

(i) For any $\varphi \in C^\infty_c(\mathbb{R}^d \times \mathbb{R}), t \in [0, T)$ and $0 < \eta < N$, the function $f = I_{u > \xi}$ satisfies
\[
- \int_D \int_{-N}^N \Psi_\eta f^*(t) \varphi d\xi dx + \int_0^t \int_D \int_{-N}^N \Psi_\eta f a(\xi) \cdot \nabla \varphi d\xi dx ds
\]
\[
+ \int_D \int_{-N}^N \Psi_\eta f_0 \varphi d\xi dx
\]
\[ + \int_0^t \int_{\partial D} \int_{-N}^N \Psi_\gamma(-\mathbf{n}(\tilde{x})) \tilde{f} \varphi d\xi d\sigma(\tilde{x}) ds \]

\[ = - \sum_{k \geq 1} \int_0^t \int_{\partial D} \int_{-N}^N \Psi_\gamma g_k(x, \xi) \varphi d\nu_{x,s}(\xi) dxd\beta_k(s) \]

\[ - \frac{1}{2} \int_0^t \int_{\partial D} \int_{-N}^N \Psi_\gamma \partial_\xi \varphi G^2(x, \xi) d\nu_{x,s}(\xi) dxds \]

\[ + \int_{[0,t] \times D \times (-N,N)} \Psi_\gamma \partial_\xi \varphi dm + \frac{1}{2} \int_0^t \int_{\partial D} \int_{-N}^N \left( \psi_\gamma(N - \xi - \eta) - \psi_\gamma(N + \xi - \eta) \right) G^2(x, \xi) \varphi d\nu_{x,s}(\xi) dxds \]

\[ - \int_{[0,t] \times D \times (-N,N)} \left( \psi_\gamma(N - \xi - \eta) - \psi_\gamma(N + \xi - \eta) \right) \varphi dm \quad a.s. \quad (2.16) \]

and \( \tilde{f} = I_{u_\leq \xi} \) satisfies

\[ - \int_D \int_{-N}^N \Psi_\gamma \tilde{f}^+(t) \varphi d\xi dx + \int_0^t \int_D \int_{-N}^N \Psi_\gamma \tilde{f}(\xi) \cdot \nabla \varphi d\xi \\
\[ + \int_D \int_{-N}^N \Psi_\gamma \tilde{f}_0 \varphi d\xi dxds \\
\[ + \int_0^t \int_{\partial D} \int_{-N}^N \Psi_\gamma (-a(\xi) \cdot \mathbf{n}(\tilde{x})) \tilde{f} \varphi d\sigma(\tilde{x}) ds \]

\[ = \sum_{k \geq 1} \int_0^t \int_{\partial D} \int_{-N}^N \Psi_\gamma g_k(x, \xi) \varphi d\nu_{x,s}(\xi) dxd\beta_k(s) \]

\[ + \frac{1}{2} \int_0^t \int_{\partial D} \int_{-N}^N \Psi_\gamma \partial_\xi \varphi G^2(x, \xi) d\nu_{x,s}(\xi) dxds \]

\[- \int_{[0,t] \times D \times (-N,N)} \Psi_\gamma \partial_\xi \varphi dm \]

\[- \frac{1}{2} \int_0^t \int_{\partial D} \int_{-N}^N \left( \psi_\gamma(N - \xi - \eta) - \psi_\gamma(N + \xi - \eta) \right) G^2(x, \xi) \varphi d\nu_{x,s}(\xi) dxds \]

\[ + \int_{[0,t] \times D \times (-N,N)} \left( \psi_\gamma(N - \xi - \eta) - \psi_\gamma(N + \xi - \eta) \right) \varphi dm \quad a.s. \quad (2.17) \]

where \( \nu = -\partial_\xi f = \partial_\xi \tilde{f} = \delta_{u=\xi} \).

\[(ii) P-a.s., \text{ for a.e. } (t, x) \in \Sigma, \text{ we have} \]

\[ -a(\xi) \cdot \mathbf{n}(\tilde{x}) \tilde{f}(t, x, \xi) = M_N \tilde{f}_b(t, x, \xi) + \partial_\xi \tilde{m}^+_N(t, x, \xi), \quad (2.18) \]

\[ -a(\xi) \cdot \mathbf{n}(\tilde{x}) \tilde{f}(t, x, \xi) = M_N \tilde{f}_b(t, x, \xi) - \partial_\xi \tilde{m}^-_N(t, x, \xi) \quad (2.19) \]

for a.e. \( \xi \in (-N, N) \).

We remark that the weak star limit \( \tilde{f}^{(\lambda)} \) may depend on the chosen subsequence \( \{\gamma_n\}_{n \geq 1} \subset \{\gamma\}_{\gamma > 0} \). From now on, when considering \( \gamma \to 0 \), we always refer to a subsequence of \( \{\gamma_n\}_{n \geq 1} \) converging to 0.
At the end of this subsection, we mention that for almost all \( s \in (0, T) \), by making modification of the test functions \( \{\psi_n\} \) in Proposition 1 of [20] such that \( \psi_n \) approximates to \( I_{[s, t]} \) as \( n \to \infty \), and using (2.10), it follows that

**Lemma 2.2** For almost all \( 0 < s < t \leq T \) and \( \varphi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}) \), the function \( f = I_{s > \xi} \) associated to the kinetic solution \( u \) of (1.1)–(1.3) satisfies that

\[
\begin{align*}
- \int_D \int_{-N}^N \Psi_\eta f^*(t) \varphi d\xi dx + & \int_s^t \int_D \int_{-N}^N \Psi_\eta f a(\xi) \cdot \nabla \varphi d\xi dxdr \\
+ & \int_D \int_{-N}^N \Psi_\eta f^*_\varepsilon \varphi d\xi dx + \int_s^t \int_D \int_{-N}^N \Psi_\eta (-a(\xi) \cdot \mathbf{n}(\bar{x})) \tilde{f} \varphi d\xi d\sigma(\bar{x})dr \\
= & - \sum_{k \geq 1} \int_s^t \int_D \int_{-N}^N \Psi_\eta g_k(x, \xi) \varphi d\nu_{x,r}(\xi) dxd\beta_k(r) \\
- & \frac{1}{2} \int_s^t \int_D \int_{-N}^N \Psi_\eta \partial_\xi \varphi G^2(x, \xi) d\nu_{x,r}(\xi) dxdr \\
+ & \int_{[s,t] \times D \times (-N,N)} \Psi_\eta \partial_\xi \varphi dm + \frac{1}{2} \int_s^t \int_D \int_{-N}^N \left( \psi_\eta(N - \xi - \eta) - \psi_\eta(N + \xi - \eta) \right) G^2(x, \xi) \varphi d\nu_{x,r}(\xi) dxdr \\
- & \int_{[s,t] \times D \times (-N,N)} \left( \psi_\eta(N - \xi - \eta) - \psi_\eta(N + \xi - \eta) \right) \varphi dm \quad \text{a.s.} \quad (2.20)
\end{align*}
\]

2.3. **Renormalized kinetic solution.** As the invariant measures are living in \( L^1_x \), in this part, we need to extend the initial data \( \vartheta \) from \( L^q \) to \( L^1_x \). This generalization, the so-called \( L^1 \)-theory, has been done in several papers. The \( L^1 \)-theory for the periodic scalar conservation laws driven by stochastic forcing was developed in [12], which generalized the deterministic results established by Chen and Perthame [5]. Later, Noboriguchi [23] developed the \( L^1 \)-theory for periodic stochastic scalar conservation laws driven by multiplicative noise.

To state the \( L^1 \)-theory of (1.1)–(1.3), we shall extend the notion of solutions from kinetic solutions to renormalized kinetic solutions. Firstly, we need a weak version of kinetic measures.

**Definition 2.4** (Weak kinetic measure). A map \( m \) from \( \Omega \) to \( \mathcal{M}^*_0(D \times [0, T) \times \mathbb{R}) \) is said to be a weak kinetic measure if

1. \( m \) is weakly measurable,
2. \( m \) vanishes at infinity in average:

\[
\lim_{R \to +\infty} \frac{1}{R} \mathbb{E}\left[ m(D \times [0, T) \times \{\xi \in \mathbb{R}; R \leq |\xi| \leq 2R\}) \right] = 0,
\]
3. for every \( \phi \in C_b(D \times \mathbb{R}) \), the process

\[
(\omega, t) \to \int_{D \times [0,t] \times \mathbb{R}} \phi(x, \xi) dm(x, s, \xi) \quad \text{is predictable.}
\]
The following is a weak version of kinetic solution called renormalized kinetic solution.

**Definition 2.5** (Renormalized kinetic solution). Let \( \vartheta \in L^2_\omega L^1_\Omega \). A measurable function \( u : \Omega \times [0, T] \times D \to \mathbb{R} \) is called a renormalized kinetic solution to (1.1)–(1.3) with datum \( \vartheta \), if

1. \( u \in L^2_{\omega, t} L^1_\Omega \) and

\[
\mathbb{E} \sup_{0 \leq t \leq T} \| u(t) \|^2_{L^1_\Omega} < +\infty, \tag{2.21}
\]

2. there exists a weak kinetic measure \( m \) and if, for any \( N > 0 \), there exist non-negative functions \( m^\pm_N \in L^1(\Omega \times \mathbb{R} \times (-N, N)) \) such that \( \{m^\pm_N(t)\} \) are predictable,

\[
\lim_{\xi \uparrow N} m^+_N(t, x, \xi) = \lim_{\xi \downarrow -N} m^-_N(t, x, \xi) = 0,
\]

for all \( \varphi \in C^\infty_c([0, T) \times \mathbb{R} \times (-N, N)) \), \( f := I_{u \geq \xi} \) satisfies

\[
\int_0^T (f(t), \partial_t \varphi(t)) dt + (f_0, \varphi(0)) + \int_0^T (f(t), a(\xi) \cdot \nabla \varphi(t)) dt
\]

\[
+ M_N \int_{\Omega \times \mathbb{R}} f_b \varphi d\xi d\sigma(\bar{x}) dt
\]

\[
= - \sum_{k \geq 1} \int_0^T \int_D g_k(x, u(t, x)) \varphi(t, x, u(t, x)) dxd\beta_k(t)
\]

\[
- \frac{1}{2} \int_0^T \int_D \partial_\xi \varphi(t, x, u(t, x)) G^2(x, u(t, x)) dxdt
\]

\[
+ \int_{[0, T) \times D \times \mathbb{R}} \partial_\xi \varphi dm + \int_{\Omega \times \mathbb{R}} \partial_\xi \varphi dm^+_N d\xi d\sigma(\bar{x}) dt, \text{ a.s.,} \tag{2.22}
\]

and \( \bar{f} := 1 - f = I_{u \leq \xi} \) satisfies

\[
\int_0^T (\bar{f}(t), \partial_t \varphi(t)) dt + (\bar{f}_0, \varphi(0)) + \int_0^T (\bar{f}(t), a(\xi) \cdot \nabla \varphi(t)) dt
\]

\[
+ M_N \int_{\Omega \times \mathbb{R}} \bar{f}_b \varphi d\xi d\sigma(\bar{x}) dt
\]

\[
= \sum_{k \geq 1} \int_0^T \int_D g_k(x, u(t, x)) \varphi(t, x, u(t, x)) dxd\beta_k(t)
\]

\[
+ \frac{1}{2} \int_0^T \int_D \partial_\xi \varphi(t, x, u(t, x)) G^2(x, u(t, x)) dxdt
\]

\[
- \int_{[0, T) \times D \times \mathbb{R}} \partial_\xi \varphi dm - \int_{\Omega \times \mathbb{R}} \partial_\xi \varphi dm^-_N d\xi d\sigma(\bar{x}) dt, \text{ a.s..} \tag{2.23}
\]

3. **Statement of Main Results**

In this section, we state the main results whose proofs are given in Sects. 4, 5, and 6.
3.1. The contraction inequality in the weighted space. From Theorem 2.1, when the initial datum \( \vartheta \in L^q_0 T^q_x \) for all \( q \geq 1 \), (1.1)–(1.3) admits a unique kinetic solution \( u(t; \vartheta) \in L^q_0 T^q_x \) for all \( q \in [1, \infty) \) and for almost all \( t \in [0, T] \). Our first result reads as follows.

**Theorem 3.1** Let \( u(t; \vartheta) \) and \( u(t; \tilde{\vartheta}) \) are kinetic solutions of \( \mathcal{E}(A, \Phi, \vartheta) \) and \( \mathcal{E}(A, \Phi, \tilde{\vartheta}) \), respectively. Under Hypotheses (H1)–(H3),

\[
\text{ess sup}_{0 \leq t \leq T} \mathbb{E}\|u(t; \vartheta) - u(t; \tilde{\vartheta})\|_{L^1_{w;\vartheta}} \leq \mathbb{E}\|\vartheta - \tilde{\vartheta}\|_{L^1_{w;\vartheta}}. 
\]

(3.1)

3.2. The continuous extension in the weighted space. In this part, we state the various extensions of the kinetic solutions of (1.1)–(1.3).

**Theorem 3.2** Assume Hypotheses (H1) and (H3) hold. Let \( \vartheta \in L^\infty_0 T^\infty_x \). If \( u(t; \vartheta) \) is a kinetic solution to (1.1)–(1.3), then \( u \in C([0, T]; L^1_0 T^1_{w;\vartheta}) \).

With the help of (3.1) and Theorem 3.2, we get an extension of \( u \) with respect to the initial value.

**Proposition 3.3** Under Hypotheses (H1) and (H3), the mapping

\[
L^\infty_0 T^\infty_x \ni \vartheta \mapsto u(\cdot; \vartheta) \in C([0, T]; L^1_0 T^1_{w;\vartheta})
\]

extends uniquely to a continuous map \( v \) from \( L^2_0 T^1_{w;\vartheta} \) to \( C([0, T]; L^1_0 T^1_{w;\vartheta}) \). Furthermore, for all \( \vartheta, \tilde{\vartheta} \in L^2_0 T^1_{w;\vartheta} \),

\[
\sup_{t \in [0, T]} \mathbb{E}\|v(t; \vartheta) - v(t; \tilde{\vartheta})\|_{L^1_{w;\vartheta}} \leq \mathbb{E}\|\vartheta - \tilde{\vartheta}\|_{L^1_{w;\vartheta}}. 
\]

(3.2)

Combining [20] and using a similar method as in the proof of Proposition 3.2 in [23], we have

**Proposition 3.4** Assume Hypotheses (H1) and (H4) hold. Then the extension \( v(t; \vartheta) \) established by Proposition 3.3 is the unique renormalized kinetic solution to (1.1)–(1.3) on \([0, T]\) in the sense of Definition 2.5.

3.3. Ergodicity for renormalized kinetic solutions. Let \( B_b(L^1_{w;\vartheta}) \) be the space of bounded measurable functions from \( L^1_{w;\vartheta} \) to \( \mathbb{R} \) and \( C_b(L^1_{w;\vartheta}) \) the space of continuous bounded measurable functions from \( L^1_{w;\vartheta} \) to \( \mathbb{R} \). From Proposition 3.4, we know that for any \( \vartheta \in L^2_0 T^1_{w;\vartheta} \), the extension \( v(t; \vartheta) \) defined by Proposition 3.3 is the unique renormalized kinetic solution to (1.1)–(1.3) on \([0, T]\). Now, we can define the Markovian semigroup associated with the renormalized kinetic solution \( v(t; \vartheta) \) as follows

**Definition 3.1** For any \( t \geq 0 \), define \( P_t : B_b(L^1_{w;\vartheta}) \to B_b(L^1_{w;\vartheta}) \) by

\[
P_t F(\vartheta) := \mathbb{E}F(v(t; \vartheta)), \quad F \in B_b(L^1_{w;\vartheta}), \quad \vartheta \in L^1_{w;\vartheta}.
\]

**Proposition 3.5** (Feller) Assume Hypotheses (H1) and (H4) are in force. Then the family \( (P_t)_{t \geq 0} \) is a Feller semigroup, that is, \( P_t \) maps \( C_b(L^1_{w;\vartheta}) \) into \( C_b(L^1_{w;\vartheta}) \).
The following result not only reveals the existence and uniqueness of the invariant measures but also provides a mixing rate uniformly with respect to the initial condition.

**Theorem 3.6** Under Hypotheses (H1) and (H4), there exists a unique invariant measure \( \mu \in \mathcal{M}_1(L^1_{w,x}) \) for the semigroup \( P_t \). Furthermore, there exists \( C > 0 \), depending only on \( q_0 \), such that for all \( t > 0 \),

\[
\sup_{\vartheta \in L^1_{w,x}} \|F\|_{\text{Lip}(L^1_{w,x})} \leq 1 \quad \sup_{\vartheta \in L^1_{w,x}} \left| P_t F(\vartheta) - \int_{L^1_{w,x}} F(\xi) \mu(d\xi) \right| \leq C \|w\|_{L^q_x}^q t^{-\frac{1}{q_0}},
\]

where \( q^* = \frac{q_0 + 1}{q_0} \) and \( \text{Lip}(L^1_{w,x}) \) is the space of Lipschitz continuous functions from \( L^1_{w,x} \) to \( \mathbb{R} \).

Let \( \mathcal{D}_\varrho(v(t; \vartheta)) = \mathbb{P} \circ v(t; \vartheta)^{-1} \) be the law of \( v(t; \vartheta) \) under \( \mathbb{P} \). In view of the equivalence between \( \| \cdot \|_{L^1_{w,x}} \) and \( \| \cdot \|_{L^1_x} \), using Kantorovich-Rubinstein formula (see Theorem 5.10 in [26]), it follows immediately that

**Corollary 3.7** Under Hypotheses (H1) and (H4), there exists a unique invariant measure \( \mu \in \mathcal{M}_1(L^1_x) \) for the semigroup \( P_t \). Furthermore, there exists \( C > 0 \), depending only on \( q_0 \), such that for all \( t > 0 \),

\[
\sup_{\vartheta \in L^1_{w,x}} \mathcal{W}_1(\mathcal{D}_\varrho(v(t; \vartheta)), \mu) \leq C \|w\|_{L^q_x}^q t^{-\frac{1}{q_0}},
\]

where \( q^* = \frac{q_0 + 1}{q_0} \) and \( \mathcal{W}_1 \) is the \( L^1 \)-Wasserstein distance.

### 4. Proof of the Contraction Inequality in the Weighted Space

In this section, we will prove the contraction inequality \( (3.1) \) for kinetic solutions of \( \mathcal{E}(A, \Phi, \vartheta) \) and \( \mathcal{E}(A, \Phi, \tilde{\vartheta}) \). Firstly, we prove a technical proposition using the doubling variables method, which has been applied in several papers, e.g. [11,13].

**Proposition 4.1** Assume Hypotheses (H1)–(H3) are in force. Let \( u(t; \vartheta) \) and \( u(t; \tilde{\vartheta}) \) be kinetic solutions of \( \mathcal{E}(A, \Phi, \vartheta) \), \( \mathcal{E}(A, \Phi, \tilde{\vartheta}) \), respectively. Then, for any \( 0 \leq t < T \), \( \gamma, \delta > 0 \), \( N \in \mathbb{N} \), and for any element \( \lambda \) of the partition of unity \( \{\lambda_i\}_{i=0,1,\ldots,M} \) on \( \overline{D} \), the functions \( f_1(t) := f_1(t, x, \xi) = I_{u(t,x; \vartheta)}(\xi) \) and \( f_2(t) := f_2(t, y, \zeta) = I_{u(t,y; \tilde{\vartheta})}(\zeta) \) with data \( (f_{1,0}, f_{2,0}), i = 1, 2 \), satisfy

\[
\mathbb{E} \int_{D^*_1} \int_{D^*_2} \int_{J_{-N}^*} \int_{J_{-N}^*} (f_{1,0}^\pm(t) f_{2,0}^\pm(t) + f_{1,0}^\pm(t) f_{2,0}^\pm(t)) \rho_\gamma^\delta(\xi - \zeta) \lambda(x) w(x) d\xi d\zeta d\xi_d d\xi_d dy dx \\
\leq \mathbb{E} \int_{D^*_1} \int_{D^*_2} \int_{J_{-N}^*} \int_{J_{-N}^*} (f_{1,0}^\pm(t) f_{2,0}^\pm(t) + f_{1,0}^\pm(t) f_{2,0}^\pm(t)) \rho_\gamma^\delta(\xi - \zeta) \lambda(x) w(x) d\xi d\zeta d\xi_d d\xi_d dy dx \\
+ \sum_{i=1}^5 J_i + I_N,
\]

(4.1)
where

\[
J_1 = \mathbb{E} \int_0^t \int_{D_x^N} \int_{D_y} \int_{-N}^N \int_{-N}^N (f_1^{(i)}(s) \tilde{f}_2(s) + f_1^{(i)}(s) f_2(s)) (-a(\xi)) \\
\cdot n(\bar{\xi})) \rho_\gamma(y - x) \psi_{\delta}(\xi - \zeta) \lambda(x) w(x) d\bar{\xi} d\zeta d\bar{y} d\sigma(\bar{\xi}) ds,
\]

\[
J_2 = \mathbb{E} \int_0^t \int_{D_x^N} \int_{D_y} \int_{-N}^N \int_{-N}^N (f_1(s) \tilde{f}_2(s) + f_1(s) f_2(s)) (a(\xi) - a(\zeta)) \\
\cdot \nabla_x \rho_\gamma(y - x) \lambda(x) w(x) \psi_{\delta}(\xi - \zeta) d\xi d\zeta dy ds,
\]

\[
J_3 = \mathbb{E} \int_0^t \int_{D_x^N} \int_{D_y} \int_{-N}^N \int_{-N}^N \lambda(x) w(x) \rho_\gamma(y - x) \psi_{\delta}(\xi - \zeta) d\xi d\zeta dy ds,
\]

\[
J_4 = -\sum_{j=1}^d \mathbb{E} \int_0^t \int_{D_x^N} \int_{D_y} \int_{-N}^N \int_{-N}^N (f_1(s) \tilde{f}_2(s) + f_1(s) f_2(s)) a_j(\xi) \rho_\gamma(y - x) \lambda(x) \psi_{\delta}(\xi - \zeta) d\xi d\zeta dy ds,
\]

\[
J_5 = \mathbb{E} \int_0^t \int_{D_x^N} \int_{D_y} \int_{-N}^N \int_{-N}^N \lambda(x) w(x) \rho_\gamma(y - x) \psi_{\delta}(\xi - \zeta) \\
\sum_{k \geq 1} |g_k(x, \xi) - g_k(y, \zeta)|^2 \nu_{x,x}^1 \otimes \nu_{y,y}^2(\xi, \zeta) dy ds,
\]

with \( I_N \) being defined by (4.6) satisfying that

\[
\limsup_{N \to \infty} I_N = 0.
\]

Moreover, for \( i = 1, 2 \), \( \tilde{f}_i^{(i)} \) are defined in Proposition 2.3, \( n(\bar{\xi}) \) is given by (2.7), \( f_{1,0} = I_{\theta > \xi} \), \( f_{2,0} = I_{\theta > \xi} \), \( f_{1,b} = I_{0 > \xi} \), \( i = 1, 2 \), \( \nu_{x,x}^1(\xi) = \delta_{u(s,x; \theta) = \xi} \) and \( \nu_{y,y}^2(\zeta) = \delta_{u(s,y; \theta') = \zeta} \).

**Proof** Denote by \( m_1 \) and \( m_2 \) the two kinetic measures associated to \( \mathcal{E}(A, \Phi, \theta) \) and \( \mathcal{E}(A, \Phi, \bar{\theta}) \), respectively. Let \( \varphi_1 \in C^\infty_c(\mathbb{R}^d_x \times \mathbb{R}_\xi) \) and \( \varphi_2 \in C^\infty_c(\mathbb{R}^d_y \times \mathbb{R}_\zeta) \). Set \( \alpha(x, \xi, y, \zeta) = \varphi_1(x, \xi) \varphi_2(y, \zeta) \).

Employing the same method as in [11] and [20], using (2.16)–(2.17), we obtain

\[
\mathbb{E} \int_{D_x^N} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) f_1^+(t) \tilde{f}_2^+(t) \alpha^\lambda w(x) d\xi d\zeta dy dx \\
= \mathbb{E} \int_{D_x^N} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) f_1(s) \tilde{f}_2(s) \alpha^\lambda w(x) d\xi d\zeta dy ds + \sum_{i=1}^{12} I_i,
\]

where

\[
I_1 = \mathbb{E} \int_0^t \int_{D_x^N} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) f_1(s) \tilde{f}_2(s) (a(\xi) \\
\cdot \nabla_x + a(\zeta) \cdot \nabla_y)(\alpha^\lambda w(x)) d\xi d\zeta dy ds,
\]
\[ I_2 = \frac{1}{2} \mathbb{E} \int_0^t \int_{D_2} \int_{D_3} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) \tilde{f}_2(s) \partial_\xi \alpha^\lambda w(x) G^2(x, \xi) d\nu_{x, s}^1(\xi) d\zeta dy ds, \]

\[ I_3 = \mathbb{E} \int_0^t \int_{D_2} \int_{D_3} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) \tilde{f}_1(\lambda)(s, x, \xi) \tilde{f}_2(s)(-a(\xi)) \cdot n(x) \alpha^\lambda w(x) d\xi d\zeta dy ds \]

\[ I_4 = -\mathbb{E} \int_{(0, t] \times D_2 \times (-N, N)} \int_{D_3} \int_{-N}^N \Psi_\eta(\xi, \zeta) \tilde{f}_2(s) \partial_\xi \alpha^\lambda w(x) d\zeta dy dm_1(s, x, \xi), \]

\[ I_5 = -\frac{1}{2} \mathbb{E} \int_0^t \int_{D_2} \int_{D_3} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi) \tilde{f}_2(s) \]

\[ \left( \psi_\eta(-\eta + N - \xi) - \psi_\eta(-\eta + N + \xi) \right) \alpha^\lambda w(x) G^2(x, \xi) d\nu_{x, s}^1(\xi) d\zeta dy ds, \]

\[ I_6 = \mathbb{E} \int_{(0, t] \times D_2 \times (-N, N)} \int_{D_3} \int_{-N}^N \Psi_\eta(\xi) \tilde{f}_2(s) \]

\[ \left( \psi_\eta(-\eta + N - \xi) - \psi_\eta(-\eta + N + \xi) \right) \alpha^\lambda w(x) d\zeta dy dm_1(s, x, \xi), \]

and

\[ I_7 = -\frac{1}{2} \mathbb{E} \int_0^t \int_{D_2} \int_{D_3} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) f_1(s) \partial_\xi \alpha^\lambda w(x) G^2(y, \zeta) d\xi d\nu_{y, s}^2(\zeta) dy ds, \]

\[ I_8 = \mathbb{E} \int_0^t \int_{D_2} \int_{D_3} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) f_1(s) \tilde{f}_2(s, y, \zeta)(-a(\xi)) \cdot n(y) \alpha^\lambda w(x) d\xi d\zeta d\sigma(y) dy ds, \]

\[ I_9 = +\mathbb{E} \int_{(0, t] \times D_2 \times (-N, N)} \int_{D_3} \int_{-N}^N \Psi_\eta(\xi, \zeta) f_1^-(s) \partial_\xi \alpha^\lambda w(x) d\zeta dy dm_2(s, y, \zeta), \]

\[ I_{10} = \frac{1}{2} \mathbb{E} \int_0^t \int_{D_2} \int_{D_3} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi) f_1(s) \left( \psi_\eta(-\eta + N - \xi) - \psi_\eta(-\eta + N + \xi) \right) \]

\[ \alpha^\lambda w(x) G^2(y, \zeta) d\zeta d\nu_{y, s}^2(\zeta) dy ds, \]

\[ I_{11} = -\mathbb{E} \int_{(0, t] \times D_2 \times (-N, N)} \int_{D_3} \int_{-N}^N \Psi_\eta(\xi) f_1^-(s) \]

\[ \left( \psi_\eta(-\eta + N - \zeta) - \psi_\eta(-\eta + N + \zeta) \right) \alpha^\lambda w(x) d\zeta dy dm_2(s, y, \zeta), \]

\[ I_{12} = -\sum_{k \geq 1} \mathbb{E} \int_0^t \int_{D_2} \int_{D_3} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) g_{k, 1}(x, \xi) g_{k, 2}(y, \zeta) \alpha^\lambda w(x) d\nu_{x, s}^1 \]

\[ \otimes \nu_{y, s}^2(\xi, \zeta) dy ds, \]

where \( \Psi_\eta(\xi, \zeta) = \Psi_\eta(\xi) \Psi_\eta(\zeta) \) and \( \alpha^\lambda = \alpha(x, \xi, y, \zeta) \lambda(x) \). By a density argument, (4.3) remains true for any test function \( \alpha \in C_c^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi \times \mathbb{R}_y^d \times \mathbb{R}_\zeta) \). Thanks to (2.9) and (2.10), the assumption that \( \alpha \) is compactly supported can be relaxed. By a
truncation argument, we can take \( \alpha(x, \xi, y, \zeta) = \rho_\gamma^\lambda(y - x) \psi_\delta(\xi - \zeta) \) in (4.3). In this case, \( \alpha^\lambda = \lambda(x) \rho_\gamma^\lambda(y - x) \psi_\delta(\xi - \zeta) \). Note that \( \rho_\gamma^\lambda(y - x) = 0 \) on \( D_x^\lambda \times \partial D_y \), it implies that \( I_8 = 0 \). Also we have

\[
(\partial_\xi + \partial_\zeta)\alpha^\lambda = 0, \quad \nabla_x \rho_\gamma^\lambda(y - x) = -\nabla_y \rho_\gamma^\lambda(y - x). \tag{4.4}
\]

Since \( \partial_x w(x) = -1 \), by (4.4), we have

\[
I_1 = \mathbb{E} \int_0^t \int_{D_x^\lambda} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) f_1(s) \tilde{f}_2(s) (a(\xi) - a(\zeta))
\]

\[
\cdot \left( \nabla_x \rho_\gamma^\lambda(y - x) \right) \lambda(x) w(x) \psi_\delta(\xi - \zeta) d\xi d\zeta dydxds
\]

\[
+ \mathbb{E} \int_0^t \int_{D_x^\lambda} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) f_1(s) \tilde{f}_2(s) a(\xi)
\]

\[
\cdot \left( \nabla_\lambda \lambda(x) \right) \rho_\gamma^\lambda(y - x) w(x) \psi_\delta(\xi - \zeta) d\xi d\zeta dydxds
\]

\[
- \sum_{j=1}^d \mathbb{E} \int_0^t \int_{D_x^\lambda} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) f_1(s) \tilde{f}_2(s) a_j(\xi)
\]

\[
\rho_\gamma^\lambda(y - x) \lambda(x) \psi_\delta(\xi - \zeta) d\xi d\zeta dydxds.
\]

Utilizing (4.4) again, by integration by parts, we deduce that

\[
I_2 = \frac{1}{2} \mathbb{E} \int_0^t \int_{D_x^\lambda} \int_{D_y} \int_{-N}^N \int_{-N}^N \partial_\xi \left[ \Psi_\eta(\xi, \zeta) \tilde{f}_2(s) \right] \alpha^\lambda w(x) G^2(x, \xi) dv_{x,s}^1 (\xi) d\zeta dydxds
\]

\[
= \frac{1}{2} \mathbb{E} \int_0^t \int_{D_x^\lambda} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) \alpha^\lambda w(x) G^2(x, \xi) dv_{x,s}^1 \otimes v_{y,s}^2 (\xi, \zeta) dydxds
\]

\[
- \frac{1}{2} \mathbb{E} \int_0^t \int_{D_x^\lambda} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi) (\psi_\eta(N - \eta - \zeta)
\]

\[- \psi_\eta(N - \eta + \zeta)) \tilde{f}_2(s) \alpha^\lambda w(x) G^2(x, \xi) dv_{x,s}^1 (\xi) d\zeta dydxds
\]

\[= I_{2,1} + I_{2,2}.
\]

Similarly, it follows that

\[
I_7 = -\frac{1}{2} \mathbb{E} \int_0^t \int_{D_x^\lambda} \int_{D_y} \int_{-N}^N \int_{-N}^N \partial_\xi \left[ \Psi_\eta(\xi, \zeta) f_1(s) \right] \alpha^\lambda w(x) G^2(y, \xi) dv_{x,s}^2 (\xi) dydxds
\]

\[
= \frac{1}{2} \mathbb{E} \int_0^t \int_{D_x^\lambda} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) \alpha^\lambda w(x) G^2(y, \xi) dv_{x,s}^1 \otimes v_{y,s}^2 (\xi, \zeta) dydxds
\]

\[
+ \frac{1}{2} \mathbb{E} \int_0^t \int_{D_x^\lambda} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi) (\psi_\eta(N - \xi - \eta) - \psi_\eta(N + \xi - \eta))
\]

\[f_1(s) \alpha^\lambda w(x) G^2(y, \xi) dv_{y,s}^2 (\xi) dydxds
\]

\[= I_{7,1} + I_{7,2}.
\]
By integration by parts formula, we have
\[ I_4 = -E \int_{[0,t]} \int_{D_x^x \times (-N,N)} \int_{D_y} \int_{-N}^{N} \partial_{\xi} \left[ \Psi_\eta(\xi, \xi) \tilde{f}_2^+(s) \right] \alpha^\lambda w(x) d\xi dy dm_1(s, x, \xi) \]
\[ = E \int_{[0,t]} \int_{D_x^x \times (-N,N)} \int_{D_y} \int_{-N}^{N} \Psi_\eta(\xi)(\Psi_\eta(N - \xi - \eta) - \Psi_\eta(N + \xi - \eta)) \]
\[ - \Psi_\eta(N + \xi - \eta)) \tilde{f}_2^+(s) \alpha^\lambda w(x) d\xi dy dm_1(s, x, \xi) \]
\[ - E \int_{[0,t]} \int_{D_x^x \times (-N,N)} \int_{D_y} \int_{-N}^{N} \Psi_\eta(\xi, \xi) \alpha^\lambda w(x) d\nu_{2,s}^+ \Psi_\eta(N - \xi - \eta) - \Psi_\eta(N + \xi - \eta) \Psi_\eta(N - \xi - \eta)) \]
\[ \tilde{f}_2^+(s) \alpha^\lambda w(x) d\xi dy dm_1 =: I_{4,1}. \]

Similarly, we can bound \( I_9 \) as follows
\[ I_9 \leq - E \int_{[0,t]} \int_{D_x^x \times (-N,N)} \int_{D_y} \int_{-N}^{N} \Psi_\eta(\xi)(\Psi_\eta(N - \xi - \eta) - \Psi_\eta(N + \xi - \eta)) \]
\[ f_1^-(s) \alpha^\lambda w(x) d\xi dx dm_2 =: I_{9,1}. \]

By a similar argument as in the proof of Proposition 2 in [20] and using Lemma 2.1, we obtain
\[ |I_5| \leq C E \int_0^T \int_{D_x^x} \int_{D_y} \int_{-N}^{N} \int_{-N}^{N} \left( \Psi_\eta(-\eta + N - \xi) + \Psi_\eta(-\eta + N + \xi) \right) \]
\[ \times \alpha^\lambda \rho_\eta^\lambda(y - x) \Psi_\eta(\xi - \xi) w(x)(1 + |\xi|^2) d\nu_{s,x}^1(\xi) d\xi dy dx ds \]
\[ \leq C \int_{\mathbb{R}} \left( \Psi_\eta(-\eta + N - \xi) + \Psi_\eta(-\eta + N + \xi) \right) (1 + |\xi|^2) E \int_0^T \int_D d\nu_{s,x}^1(\xi) d\xi dx ds \]
\[ \leq C \int_{\mathbb{R}} \left( \Psi_\eta(-\eta + N - \xi) + \Psi_\eta(-\eta + N + \xi) \right) (1 + |\xi|^2) d\mu_{\nu_1}(\xi) \]
\[ \rightarrow C(1 + N^2)(\mu_{\nu_1}'(N) + \mu_{\nu_1}'(-N)), \]
as \( \eta \rightarrow 0^+ \), where \( \mu_{\nu_1}' \) is defined by (2.14). Similarly,
\[ |I_6| \leq E \int_{[0,t]} \int_{D_x^x \times (-N,N)} \int_{D_y} \int_{-N}^{N} \Psi_\eta(\xi, \xi) \tilde{f}_2^+(s) \left( \Psi_\eta(-\eta + N - \xi) + \Psi_\eta(-\eta + N + \xi) \right) \]
\[ \alpha^\lambda w(x) d\xi dy dm_1(s, x, \xi) \]
\[ \leq C E \int_{[0,t]} \int_{D_x^x \times (-N,N)} \int_{D_y} \int_{-N}^{N} \left( \Psi_\eta(-\eta + N - \xi) \right) \]
\[ + \Psi_\eta(-\eta + N + \xi)) \rho_\eta^\lambda(y - x) \Psi_\eta(\xi - \xi) d\xi dy dm_1(s, x, \xi) \]
\[ \leq C \int_{\mathbb{R}} \left( \Psi_\eta(-\eta + N - \xi) + \Psi_\eta(-\eta + N + \xi) \right) E \int_0^T \int_D d\mu_1(s, x, \xi) \]
\[ \rightarrow C(\mu_{m_1}'(N) + \mu_{m_1}'(-N)), \] (4.5)
as $\eta \to 0^+$. In addition, the terms $I_{10}, I_{11}, I_{2,2}, I_{7,2}, I_{4,1}, I_{9,1}$ containing $\psi_\eta$ can be estimated from above as $\eta \to 0^+$ by

$$I_N := C(\mu'_{m_1}(\pm N) + \mu'_{m_2}(\pm N) + (1 + N^2)(\mu'_{v_1}(\pm N) + \mu'_{v_2}(\pm N))).$$

(4.6)

Due to Lemma 2.1, we see that $\limsup_{N \to \infty} I_N = 0$. Moreover,

$$I_{12} + I_{2,1} + I_{7,1}$$

$$= \frac{1}{2} \mathbb{E} \int_0^t \int_{D_x} \int_{D_y} \int_{-N} f_1(t) \tilde{f}_2(t) \alpha^x w(x) d\xi d\xi dx$$

$$\leq \mathbb{E} \int_0^t \int_{D_x} \int_{D_y} \int_{-N} f_{1,0} \tilde{f}_{2,0} \alpha^x w(x) d\xi d\xi dx$$

$$+ \mathbb{E} \int_0^t \int_{D_x} \int_{D_y} \int_{-N} f_{1,0} \tilde{f}_{2,0} (\lambda(x) - \xi - \psi_\xi) d\xi d\xi dx$$

$$+ \mathbb{E} \int_0^t \int_{D_x} \int_{D_y} \int_{-N} f_1(s) \tilde{f}_2(s) \alpha(x) d\xi d\xi dx$$

Combining all the previous estimates and letting $\eta \downarrow 0$ in (4.3), we get

$$\mathbb{E} \int_0^t \int_{D_x} \int_{D_y} \int_{-N} \int_{-N} f_1(t) \tilde{f}_2(t) \alpha^x w(x) d\xi d\xi dx$$

$$\leq \mathbb{E} \int_0^t \int_{D_x} \int_{D_y} \int_{-N} f_{1,0} \tilde{f}_{2,0} \alpha^x w(x) d\xi d\xi dx$$

$$+ \mathbb{E} \int_0^t \int_{D_x} \int_{D_y} \int_{-N} f_{1,0} \tilde{f}_{2,0} (\lambda(x) - \xi - \psi_\xi) d\xi d\xi dx$$

$$+ \mathbb{E} \int_0^t \int_{D_x} \int_{D_y} \int_{-N} f_1(s) \tilde{f}_2(s) \alpha(x) d\xi d\xi dx$$

$$\cdot \nabla_x \rho^{\Lambda}_\psi(y - x) \lambda(x) \psi_\xi d\xi d\xi dx$$

$$+ \mathbb{E} \int_0^t \int_{D_x} \int_{D_y} \int_{-N} f_1(s) \tilde{f}_2(s) \alpha(x) d\xi d\xi dx$$

$$\cdot \nabla_x \lambda(x) \rho^{\Lambda}_\psi(y - x) \psi_\xi d\xi d\xi dx$$

$$- \sum_{j=1}^d \mathbb{E} \int_0^t \int_{D_x} \int_{D_y} \int_{-N} \int_{-N} f_1(s) \tilde{f}_2(s) \alpha_j(x) \rho^{\Lambda}_\psi(y - x) d\xi d\xi dx$$

$$- \frac{1}{2} \mathbb{E} \int_0^t \int_{D_x} \int_{D_y} \int_{-N} \int_{-N} \lambda(x) \psi_\xi d\xi d\xi dx$$

Adding them together, we get the desired result (4.1) for $f_1^+$. To obtain the result for $f_1^-$, we simply take $t_n \uparrow t$, write (4.1) for $f_1^+(t_n)$ and let $n \to \infty$. \hfill \Box

The following is the so-called “super $L^1_{w;x}$ – contraction principle” mentioned in the introduction.
Theorem 4.2 The kinetic solutions \( u(t; \vartheta), u(t; \tilde{\vartheta}) \) of \( \mathcal{E}(A, \Phi, \vartheta) \) and \( \mathcal{E}(A, \Phi, \tilde{\vartheta}) \) satisfy

\[
\mathbb{E}\|u(t; \vartheta) - u(t; \tilde{\vartheta})\|_{L_{w; \infty}} \leq \mathbb{E}\|\vartheta - \tilde{\vartheta}\|_{L_{w; \infty}}
- \sum_{j=1}^{d} \mathbb{E} \int_{0}^{t} \int_{D} |A_{j}(u(s; \vartheta)) - A_{j}(u(s; \tilde{\vartheta}))| dx ds.
\]

(4.7)

Proof For any \( t \geq 0, N \in D \) and any element \( \lambda \) of the partition of unity \( \{\lambda_{i}\} \) on \( D \), define the error term

\[
\mathcal{E}_{t}^{N, \lambda}(y, \delta) = \mathbb{E} \int_{D_{x}^{\lambda}} \int_{D_{y}^{\lambda}} \int_{-N}^{N} \int_{-N}^{N} (f_{1}^{\pm}(t) f_{2}^{\pm}(t) + f_{1}^{\pm}(t) f_{2}^{\pm}(t))
\]

\[
\rho_{y}^{\lambda}(y - x) \psi_{y}(\xi - \zeta) \lambda(x) w(x) d\xi d\zeta dy dx
- \mathbb{E} \int_{D_{x}^{\lambda}} \int_{D_{y}^{\lambda}} \int_{-N}^{N} (f_{1}^{\pm}(t, x, \xi) f_{2}^{\pm}(t, y, \xi)) \rho_{y}^{\lambda}(y - x) \lambda(x) w(x) d\xi dy dx
\]

\[
= H_{1}^{N, \lambda}(t) + H_{2}^{N, \lambda}(t),
\]

(4.8)

where

\[
H_{1}^{N, \lambda}(t) = \mathbb{E} \int_{D_{x}^{\lambda}} \int_{D_{y}^{\lambda}} \int_{-N}^{N} \int_{-N}^{N} (f_{1}^{\pm}(t) f_{2}^{\pm}(t) + f_{1}^{\pm}(t) f_{2}^{\pm}(t))
\]

\[
\rho_{y}^{\lambda}(y - x) \psi_{y}(\xi - \zeta) \lambda(x) w(x) d\xi d\zeta dy dx
- \mathbb{E} \int_{D_{x}^{\lambda}} \int_{D_{y}^{\lambda}} \int_{-N}^{N} (f_{1}^{\pm}(t, x, \xi) f_{2}^{\pm}(t, y, \xi)) \rho_{y}^{\lambda}(y - x) \lambda(x) w(x) d\xi dy dx,
\]

\[
H_{2}^{N, \lambda}(t) = \mathbb{E} \int_{D_{x}^{\lambda}} \int_{D_{y}^{\lambda}} \int_{-N}^{N} \int_{-N}^{N} (f_{1}^{\pm}(t, x, \xi) f_{2}^{\pm}(t, y, \xi)) \rho_{y}^{\lambda}(y - x) \lambda(x) w(x) d\xi dy dx
- \mathbb{E} \int_{D_{x}^{\lambda}} \int_{D_{y}^{\lambda}} \int_{-N}^{N} (f_{1}^{\pm}(t, x, \xi) f_{2}^{\pm}(t, x, \xi)) \lambda(x) w(x) d\xi dx.
\]

We start with the estimate of \( H_{1}^{N, \lambda}(t) \). Note that

\[
\int_{D_{x}^{\lambda}} \int_{D_{y}^{\lambda}} \int_{-N}^{N} \rho_{y}^{\lambda}(y - x) f_{1}^{\pm}(t, x, \xi) f_{2}^{\pm}(t, y, \xi) \lambda(x) w(x) d\xi dy dx
= \int_{D_{x}^{\lambda}} \int_{D_{y}^{\lambda}} \int_{-N}^{N} \int_{-N}^{N} \rho_{y}^{\lambda}(y - x) \psi_{y}(\xi - \zeta) f_{1}^{\pm}(t, x, \xi) f_{2}^{\pm}(t, y, \xi) \lambda(x) w(x) d\xi d\zeta dy dx
\]

\[
+ \int_{D_{x}^{\lambda}} \int_{D_{y}^{\lambda}} \int_{-N}^{N} \rho_{y}^{\lambda}(y - x) f_{1}^{\pm}(t, x, \xi) f_{2}^{\pm}(t, y, \xi) \lambda(x) w(x)
\]
\[
\left(1 - \int_{-N}^{N} \psi_{\delta}(\xi - \zeta) d\xi\right) d\xi dy dx,
\]
which implies
\[
\left| \int_{D_x^\pm} D_y \int_{-N}^{N} \rho_{\gamma}^\pm (y - x) f_1^\pm(t, x, \xi) \tilde{f}_2^\pm(t, y, x, \xi) \lambda(x) w(x) d\xi dy dx \right|
\]
\[
- \int_{D_x^\pm} D_y \int_{-N}^{N} f_1^\pm(t, x, \xi) \tilde{f}_2^\pm(t, y, x, \xi) \rho_{\gamma}^\pm (y - x) \psi_{\delta}(\xi - \zeta) \lambda(x) w(x) d\xi dy dx
\]
\[
\leq \int_{D_x^\pm} D_y \rho_{\gamma}^\pm (y - x) \lambda(x) w(x) \int_{-N}^{N} I_{u^\pm(t, x; \bar{\theta}) > \xi} \int_{-N}^{N} \psi_{\delta}(\xi - \zeta)
\]
\[
(I_{u^\pm(t, y; \bar{\theta}) \leq \xi} - I_{u^\pm(t, y; \bar{\theta}) \leq \xi}) d\xi dy dx
\]
\[
+ C \int_{-N}^{N} \left(1 - \int_{-N}^{N} \psi_{\delta}(\xi - \zeta) d\xi\right) d\xi =: K_{1}^{N, \lambda}(\gamma, \delta) + \Upsilon^{N}(\delta).
\]
Applying the dominated convergence theorem, we get \(\lim_{\delta \to 0} \Upsilon^{N}(\delta) = 0\). Moreover, by the fact that \(\int_{0}^{\delta} \psi_{\delta}(\xi) d\xi' = \int_{-\delta}^{0} \psi_{\delta}(\xi') d\xi' = \frac{1}{2}\), we deduce that
\[
K_{1}^{N, \lambda}(\gamma, \delta) \leq \int_{D_x^\pm} D_y \int_{-N}^{N} \rho_{\gamma}^\pm (y - x) \lambda(x) w(x) \int_{-N}^{N} I_{u^\pm(t, x; \bar{\theta}) > \xi} \int_{(\xi - \zeta) \vee (-N)}^{\xi} \psi_{\delta}
\]
\[
(\xi - \zeta) I_{\xi < u^\pm(t, y; \bar{\theta}) < \xi} d\xi dy dx
\]
\[
+ \int_{D_x^\pm} D_y \int_{-N}^{N} \rho_{\gamma}^\pm (y - x) \lambda(x) w(x) \int_{-N}^{N} I_{u^\pm(t, x; \bar{\theta}) > \xi} \int_{(\xi + \delta) \wedge N}^{\xi} \psi_{\delta}
\]
\[
(\xi - \zeta) I_{\xi < u^\pm(t, y; \bar{\theta}) < \xi} d\xi dy dx
\]
\[
\leq \int_{D_x^\pm} D_y \int_{-N}^{N} \rho_{\gamma}^\pm (y - x) \lambda(x) w(x) \int_{u^\pm(t, y; \bar{\theta})}^{N \wedge u^\pm(t, x; \bar{\theta}) \wedge (u^\pm(t, y; \bar{\theta}) + \delta)} \int_{(\xi - \zeta) \vee (-N)}^{\xi} \psi_{\delta}
\]
\[
(\xi - \zeta) d\xi dy dx
\]
\[
+ \int_{D_x^\pm} D_y \int_{-N}^{N} \rho_{\gamma}^\pm (y - x) \lambda(x) w(x) \int_{-N \vee (u^\pm(t, y; \bar{\theta}) - \delta)}^{u^\pm(t, x; \bar{\theta}) \wedge (u^\pm(t, y; \bar{\theta}) + \delta)} \int_{(\xi + \delta) \wedge N}^{\xi} \psi_{\delta}
\]
\[
(\xi - \zeta) d\xi dy dx
\]
\[
\leq \delta \int_{D_x^\pm} D_y \int_{-N}^{N} \rho_{\gamma}^\pm (y - x) \lambda(x) w(x) d\xi dy dx \leq C \delta \int_{D_x^\pm} \lambda(x) dx, \text{ a.s.}
\]
Hence, we get
\[
\left| \int_{D_x^\pm} D_y \int_{-N}^{N} \rho_{\gamma}^\pm (y - x) f_1^\pm(t, x, \xi) \tilde{f}_2^\pm(t, y, x, \xi) \lambda(x) w(x) d\xi dy dx \right|
\]
\[
- \int_{D_x^\pm} D_y \int_{-N}^{N} f_1^\pm(t, x, \xi) \tilde{f}_2^\pm(t, y, x, \xi) \rho_{\gamma}^\pm (y - x) \psi_{\delta}(\xi - \zeta) \lambda(x) w(x) d\xi dy dx
\]
Similarly, it follows that

\[
\begin{align*}
\left| \int_{D_{\delta}^+) \int_{D_{\gamma}} \int_{-N}^N \rho_{\gamma}^\delta(y-x) f_1^\pm(t, x, \xi) f_2^\pm(t, y, \xi) \lambda(x) w(x) d\xi dy dx \\
- \int_{D_{\delta}^+) \int_{D_{\gamma}} \int_{-N}^N f_1^\pm(t, x, \xi) f_2^\pm(t, y, \xi) \lambda(x) w(x) d\xi dy dx \right| & \\
\leq C \delta \int_{D_{\delta}^+} \lambda(x) dx + \mathcal{Y}^N(\delta), \quad a.s.. 
\end{align*}
\] (4.9)

Based on (4.9) and (4.10), by using the dominated convergence theorem, we obtain

\[
\sum_{i=0}^M |H_{1,i}^N(\lambda_t)| \leq C \delta \sum_{i=0}^M \int_{D_{\delta}^+} \lambda_i(x) dx + 2M \mathcal{Y}^N(\delta)
= C \delta \int_D \sum_{i=0}^M \lambda_i(x) dx + 2M \mathcal{Y}^N(\delta)
\leq C \delta + 2M \mathcal{Y}^N(\delta) \to 0 \quad \text{as } \delta \to 0. \quad (4.11)
\]

Moreover, by utilizing \( \rho_{\gamma}^\delta(y-x) = 0 \) on \( D_{\delta}^+ \times D^c \), it follows that

\[
\begin{align*}
\left| \int_{D_{\delta}^+} \int_{D_{\gamma}} \int_{-N}^N \rho_{\gamma}^\delta(y-x) \lambda(x) w(x) f_1^\pm(t, x, \xi) f_2^\pm(t, y, \xi) d\xi dy dx \\
- \int_{D_{\delta}^+} \int_{D_{\gamma}} \int_{-N}^N f_1^\pm(t, x, \xi) f_2^\pm(t, y, \xi) \lambda(x) w(x) d\xi dy dx \right| & \\
= \left| \int_{D_{\delta}^+} \int_D \int_{-N}^N \rho_{\gamma}^\delta(y-x) \lambda(x) w(x) f_1^\pm(t, x, \xi) (f_2^\pm(t, y, \xi) - f_2^\pm(t, x, \xi)) d\xi dy dx \right| \\
& \leq \sup_{[z_l \in (-\gamma, \gamma), z_d \in (\gamma L_{\lambda,y} L_{\lambda,2\gamma})]} \left| \int_{D_{\delta}^+} \lambda(x) w(x) \int_\mathbb{R} |f_1^\pm(t, x, \xi) f_2^\pm(t, x + z, \xi) \right| \\
& - f_2^\pm(t, x, \xi) d\xi dx \\
\leq C \sup_{[z_l \in (-\gamma, \gamma), z_d \in (\gamma L_{\lambda,y} L_{\lambda,2\gamma})]} \left| \int_{D_{\delta}^+} \lambda(x) \int_\mathbb{R} |f_2^\pm(t, x + z, \xi) + I_{0>\xi} - I_{0>\xi} + f_2^\pm(t, x, \xi) d\xi dx \right| \\
\leq C \sup_{[z_l \in (-\gamma, \gamma), z_d \in (\gamma L_{\lambda,y} L_{\lambda,2\gamma})]} \int_{D_{\delta}^+} \lambda(x) \left| \int_\mathbb{R} \Delta f_2^\pm(t, x + z, \xi) - \Delta f_2^\pm(t, x, \xi) d\xi dx, \right|
\end{align*}
\]
where we have used the boundedness of $w$. As a result, the integrability of $\Lambda f^\pm_2$ on $D \times \mathbb{R}$ implies that
\[
\lim_{\gamma \to 0} \left| \int_{D \times \mathbb{R}} f^\pm_1(t, x, \xi) \tilde{f}^\pm_2(t, y, \xi) \rho^\lambda(\gamma (y - x)) \lambda(x) w(x) d\xi d\gamma (y - x) \right| = 0, \quad a.s.\
\]
Thus, by the dominated convergence theorem, we obtain
\[
\lim_{\gamma \to 0} \sum_{i=0}^{N} \mathbb{E} \left| \int_{D \times \mathbb{R}} f^\pm_1(t, x, \xi) \tilde{f}^\pm_2(t, y, \xi) \rho_{\gamma, i}^\lambda (y - x) \lambda_i(x) w(x) d\xi d\gamma (y - x) \right| = 0.\
\]
Symmetrically, we have the same estimation for $\tilde{f}^\pm_2(t, x, \xi) f^\pm_1(t, y, \xi)$. Thus, we conclude that
\[
\lim_{\gamma \to 0} \sum_{i=0}^{N} \mathbb{E} \left| H_{2,N}^i (t) \right| = 0. \quad (4.12)\]
Combining (4.8), (4.11) with (4.12), for any $t \in [0, T]$, we have
\[
\lim_{\gamma, \delta \to 0} \sum_{i=0}^{N} \mathbb{E} \left| \tilde{\xi}_{t,N}^i (\gamma, \delta) \right| = 0. \quad (4.13)\]
Using the dominated convergence theorem, it follows from (4.13) that
\[
\lim_{\gamma, \delta \to 0} \sum_{i=0}^{N} \int_0^T |\tilde{\xi}_{t,N}^i (\gamma, \delta)| dt = 0. \quad (4.14)\]
For any $t \in [0, T]$ and $N > 0$, define the error term
\[
r^N_{t, \lambda}(\gamma, \delta) := \mathbb{E} \left[ \int_0^t \int_{\partial D^\lambda_2} \int_{\mathbb{R}} \int_{-N}^{N} (f_1^{(\lambda)}(s) \tilde{f}_2(s) + \tilde{f}_1^{(\lambda)}(s) f_2(\tilde{f}^{(\lambda)}(s) + \tilde{f}_1^{(\lambda)}(s) \tilde{f}_2^{(\lambda)}(s) (-a(\xi) \cdot n) \lambda(x) w(x) d\xi d\gamma (y - x) ds 
\times \psi_3(\xi - \zeta) \lambda(x) w(x) d\xi d\gamma (y - x) ds
\right. \\
- \mathbb{E} \left[ \int_0^t \int_{\partial D^\lambda_2} \int_{-N}^{N} (f_1^{(\lambda)}(s) \tilde{f}_2(s) + \tilde{f}_1^{(\lambda)}(s) f_2(\tilde{f}^{(\lambda)}(s) + \tilde{f}_1^{(\lambda)}(s) \tilde{f}_2^{(\lambda)}(s) (-a(\xi) \cdot n) \lambda(x) w(x) d\xi d\gamma (y - x) ds 
\int_{-N}^{N} (f_1^{(\lambda)}(s) \tilde{f}_2(s)) (-a(\xi) \cdot n) \lambda(x) w(x) d\xi d\gamma (y - x) ds 
\right. \\
\left. + \tilde{f}_1^{(\lambda)}(s) f_2(s) (-a(\xi) \cdot n) \rho_{\gamma, i}^\lambda (y - x) \end{array} \right. \\
\left. \times \psi_3(\xi - \zeta) \lambda(x) w(x) d\xi d\gamma (y - x) ds 
\right. \\
\left. + \tilde{f}_1^{(\lambda)}(s) \tilde{f}_2^{(\lambda)}(s) (-a(\xi) \cdot n) \lambda(x) w(x) d\xi d\gamma (y - x) ds 
\right]. \quad (4.15)\]
According to Proposition 2.3, there exists a subsequence, still denoted by $\{\gamma\}_{\gamma > 0}$, such that $\tilde{f}_2 * \rho_{\gamma}^\lambda \to \tilde{f}_2^{(\lambda)}$ and $f_2 * \rho_{\gamma}^\lambda \to f_2^{(\lambda)}$ in the weak star topology in $L^\infty (\Omega \times \Sigma^\lambda \times \mathbb{R})$,
as $\gamma \to 0$. Employing a similar method as in the estimation of $H^{1,\lambda}_1$, we deduce that for each $N > 0$, $t > 0$,

$$\lim_{\gamma, \delta \to 0} \sum_{i=0}^{M} r_{i, \lambda}^{N, \gamma, \delta}(\gamma, \delta) = 0. \quad (4.16)$$

Applying the dominated convergence theorem again, it follows that

$$\lim_{\gamma, \delta \to 0} \sum_{i=0}^{M} \int_0^T r_{i, \lambda}^{N, \gamma, \delta}(\gamma, \delta) dt = 0. \quad (4.17)$$

Based on the above estimates, and by (4.1), we reach

$$\mathbb{E} \int_{D^N} \int_{-N}^{N} (f_1^{\pm}(t, x, \xi) \tilde{f}_2^{\pm}(t, x, \xi) + f_1^{\pm}(t, x, \xi) \tilde{f}_2^{\pm}(t, x, \xi)) \lambda(x) w(x) d\xi dx$$

$$\leq \mathbb{E} \int_{D^N} \int_{-N}^{N} (f_{1,0}^{\tilde{f}_2}(\xi) + f_{1,0}^{\tilde{f}_2}(\xi)) \lambda(x) w(x) d\xi dx$$

$$+ \mathbb{E} \int_0^t \int_{D^N} \int_{-N}^{N} (\tilde{f}_1^{\lambda}(s) \tilde{f}_2^{\lambda}(s) + \tilde{f}_1^{\lambda}(s) \tilde{f}_2^{\lambda}(s)) (-a(\xi) \cdot \mathbf{n}) \lambda(x) w(x) d\xi d\sigma(\tilde{x}) ds$$

$$+ \sum_{i=2}^{5} J_i + r_{i, \lambda}^{N, \lambda}(\gamma, \delta) + \mathcal{E}_{\gamma, \delta}^{N, \lambda}(\gamma, \delta) + \mathcal{E}_{0}^{N, \lambda}(\gamma, \delta) + I_N, \quad (4.18)$$

where $J_i, i = 2, \ldots, 5$ were defined in the statement of Proposition 4.1, $I_N$ was given by (4.6).

Noting that $a(\xi) \cdot \mathbf{n} \tilde{f}_2^{\lambda}(s) = a(\xi) \cdot \mathbf{n} \tilde{f}_2$ a.e. on $[0, T] \times \partial D^N \times (-N, N)$, $\sum_{i=0}^{M} \mathcal{E}_i \lambda(x) f_1^{\lambda}(s) = \tilde{f}_1$ and $\sum_{i=0}^{M} \mathcal{E}_i \lambda(x) f_1^{\lambda}(s) = \tilde{f}_1$, it follows that

$$\sum_{i=0}^{M} \int_0^t \int_{\partial D^N} \int_{-N}^{N} (\tilde{f}_1^{\lambda}(s) \tilde{f}_2^{\lambda}(s) + \tilde{f}_1^{\lambda}(s) \tilde{f}_2^{\lambda}(s)) (-a(\xi) \cdot \mathbf{n}) \lambda(x) w(x) d\xi d\sigma(\tilde{x}) ds$$

$$= \sum_{i=0}^{M} \int_0^t \int_{\partial D^N} \int_{-N}^{N} (f_1^{\lambda}(s) \tilde{f}_2^{\lambda}(s) + \tilde{f}_1^{\lambda}(s) \tilde{f}_2^{\lambda}(s)) (-a(\xi) \cdot \mathbf{n}) \lambda(x) w(x) d\xi d\sigma(\tilde{x}) ds$$

$$= \int_0^t \int_{\partial D} \int_{-N}^{N} \sum_{i=0}^{M} \lambda(x) (f_1^{\lambda}(s) \tilde{f}_2^{\lambda}(s) + \tilde{f}_1^{\lambda}(s) \tilde{f}_2^{\lambda}(s)) (-a(\xi) \cdot \mathbf{n}) w(x) d\xi d\sigma(\tilde{x}) ds$$

$$= \int_0^t \int_{\partial D} \int_{-N}^{N} (\tilde{f}_1^{\lambda}(s) \tilde{f}_2^{\lambda}(s) + \tilde{f}_1^{\lambda}(s) \tilde{f}_2^{\lambda}(s)) (-a(\xi) \cdot \mathbf{n}) w(x) d\xi d\sigma(\tilde{x}) ds$$

$$\leq |a(\xi)| \mathbb{E} \int_0^t \int_{\partial D} \int_{-N}^{N} (f_1^{\lambda}(s) \tilde{f}_2^{\lambda}(s) + \tilde{f}_1^{\lambda}(s) \tilde{f}_2^{\lambda}(s)) (-a(\xi) \cdot \mathbf{n}) w(x) d\xi d\sigma(\tilde{x}) ds. \quad (4.19)$$

where we have used (3.11) in [20].

Moreover, due to (3.10) in [20], for any element $\lambda$ of the partition of unity $\{\lambda_i\}$, it gives

$$|J_2| \leq C \delta \gamma^{-1}, \quad |J_5| \leq C (\gamma^2 \delta^{-1} + \delta). \quad (4.20)$$
Thus, summing (4.18) over $i = 0, \ldots, M$, and using $\sum_{i=0}^{M} \lambda_i = 1$, we get

$$\mathbb{E} \int_D \int_{-N}^{N} (f_{1}^\pm(t) \bar{f}_{2}^\pm(t) + \bar{f}_{1}^\pm(t) f_{2}^\pm(t))w(x)d\xi dx$$

$$\leq \mathbb{E} \int_D \int_{-N}^{N} (f_{1,0} \bar{f}_{2,0} + \bar{f}_{1,0} f_{2,0})w(x)d\xi dx + |a(0)| \mathbb{E} \int_0^t \int_D \int_{\mathbb{R}} (f_{1,b} \bar{f}_{2,b} + \bar{f}_{1,b} f_{2,b})w(x)d\xi d\sigma(\bar{x})ds$$

$$+ CM \delta \gamma^{-1} + CM (\gamma^2 \delta^{-1} + \delta) + \sum_{i=0}^{M} \left( J_3 + J_4 + r_i^{N,\lambda_i}(\gamma, \delta) + \mathcal{E}_t^{N,\lambda_i}(\gamma, \delta) + I_N \right).$$

(4.21)

On the other hand, by (3.9) in [20], we have

$$\lim_{\gamma, \delta \to 0} \sum_{i=0}^{M} J_3 = \mathbb{E} \int_0^t \int_D \int_{-N}^{N} (f_{1}(s) \bar{f}_{2}(s) + \bar{f}_{1}(s) f_{2}(s))a(\xi)$$

$$\cdot \nabla_x \left( \sum_{i=0}^{M} \lambda_i(x) \right) d\xi dxds = 0. \quad (4.22)$$

Applying similar method to (4.13), by utilizing (2.10) and $\max_{x \in [-N, N]} |a(\xi)| = M_N$, we obtain

$$\lim_{\gamma, \delta \to 0} \sum_{i=0}^{M} J_4 = -\sum_{j=1}^{d} \mathbb{E} \int_0^t \int_D \int_{-N}^{N} (f_{1}(s) \bar{f}_{2}(s) + \bar{f}_{1}(s) f_{2}(s))a_j(\xi)d\xi dxds. \quad (4.23)$$

Now, taking $\delta = \gamma^{\frac{4}{3}}$ and letting $\gamma \to 0$, we deduce from (4.13), (4.16), (4.22) and (4.23) that

$$\mathbb{E} \int_D \int_{-N}^{N} (f_{1}^\pm(t) \bar{f}_{2}^\pm(t) + \bar{f}_{1}^\pm(t) f_{2}^\pm(t))w(x)d\xi dx$$

$$\leq \mathbb{E} \int_D \int_{-N}^{N} (f_{1,0} \bar{f}_{2,0} + \bar{f}_{1,0} f_{2,0})w(x)d\xi dx$$

$$- \sum_{j=1}^{d} \mathbb{E} \int_0^t \int_D \int_{-N}^{N} (f_{1}(s) \bar{f}_{2}(s) + \bar{f}_{1}(s) f_{2}(s))a_j(\xi)d\xi dxds$$

$$+ |a(0)| \mathbb{E} \int_0^t \int_D \int_{\mathbb{R}} (f_{1,b} \bar{f}_{2,b} + \bar{f}_{1,b} f_{2,b})w(x)d\xi d\sigma(\bar{x})ds + \sum_{i=0}^{M} I_N. \quad (4.24)$$

Recall that $f_1(t, x, \xi) = I_{u(t,x;\bar{\theta}) > \xi}$, $f_2(t, x, \xi) = I_{u(t,x;\bar{\theta}) > \xi}$. Denote by $u_1 = u_1(s) = u(s; \bar{\theta})$, $u_2 = u_2(s) = u(s, \bar{\theta})$. Next, we prove that for any $1 \leq j \leq d$,

$$\lim_{N \to \infty} \mathbb{E} \int_0^t \int_D \int_{-N}^{N} (f_{1}(s) \bar{f}_{2}(s) + \bar{f}_{1}(s) f_{2}(s))a_j(\xi)d\xi dxds$$
\[
\begin{align*}
&= \lim_{N \to \infty} \mathbb{E} \int_0^t \int_D \int_{-N}^N (I_{u_1 > \xi} \tilde{I}_{u_2 > \xi} + \tilde{I}_{u_1 > \xi} I_{u_2 > \xi}) a_j(\xi) d\xi dx ds \\
&= \mathbb{E} \int_0^t \int_D |A_j(u_1) - A_j(u_2)| dx ds. \quad (4.25)
\end{align*}
\]

As \( A_j \) is increasing, we have
\[
\begin{align*}
\int_{-N}^N I_{u_1 > \xi} \tilde{I}_{u_2 > \xi} a_j(\xi) d\xi &= (A_j(u_1 \land N) - A_j(u_2 \lor (-N)))^+, \\
\int_{-N}^N \tilde{I}_{u_1 > \xi} I_{u_2 > \xi} a_j(\xi) d\xi &= (A_j(u_1 \lor (-N)) - A_j(u_2 \land N))^-.
\end{align*}
\]

On the other hand, it follows that
\[
|\langle A_j(u_1 \land N) - A_j(u_2 \lor (-N)) \rangle^+ | \leq C |u_1|^q_{q_0+1} + |u_2|^q_{q_0+1},
\]
\[
|\langle A_j(u_1 \lor (-N)) - A_j(u_2 \land N) \rangle^- | \leq C |u_1|^q_{q_0+1} + |u_2|^q_{q_0+1}.
\]

Now, (4.25) follows from the dominated convergence theorem.

Letting \( N \to \infty \) in (4.24), using (4.2) and (4.25), and the dominated convergence theorem, we obtain
\[
\begin{align*}
\mathbb{E} \int_D \int_{\mathbb{R}} (f_1^\pm(t) \tilde{f}_2^\pm(t) + f_1^\pm(t) f_2^\pm(t)) w(x) d\xi dx \\
\leq \mathbb{E} \int_D \int_{\mathbb{R}} (f_{1,0} \tilde{f}_{2,0} + \tilde{f}_{1,0} f_{2,0}) w(x) d\xi dx \\
- \sum_{j=1}^d \mathbb{E} \int_0^t \int_D |A_j(u(s; \vartheta)) - A_j(u(s; \tilde{\vartheta}))| dx ds \\
+ |a(0)| \mathbb{E} \int_0^t \int_{\partial D} \int_{\mathbb{R}} (f_{1,b} \tilde{f}_{2,b} + \tilde{f}_{1,b} f_{2,b}) w(x) d\xi ds d\sigma(\tilde{x}). \quad (4.26)
\end{align*}
\]

Since \( f_1(t, x, \xi) = I_{u(t,x; \vartheta) > \xi} \), \( f_2(t, x, \xi) = I_{u(t,x; \tilde{\vartheta}) > \xi} \), \( f_{i,0} = I_{\vartheta > \xi} \), \( f_{2,0} = I_{\tilde{\vartheta} > \xi} \), using
\[
\int_{\mathbb{R}} I_{u > \xi} \tilde{I}_{v > \xi} d\xi = (u - v)^+, \quad \int_{\mathbb{R}} I_{u > \xi} I_{v > \xi} d\xi = (u - v)^-,
\]
we deduce from (4.26) that
\[
\begin{align*}
\mathbb{E} \int_D |u(t; \vartheta) - u(t; \tilde{\vartheta})| w(x) dx \\
- \sum_{j=1}^d \mathbb{E} \int_0^t \int_D |A_j(u(s; \vartheta)) - A_j(u(s; \tilde{\vartheta}))| dx ds.
\end{align*}
\]

\( \square \)

**Proof of Theorem 3.1** As a consequence of Theorem 4.2, we have
\[
\mathbb{E} \|u(t; \vartheta) - u(t; \tilde{\vartheta})\|_{L^1_{w; \xi}} \leq \mathbb{E} \|\vartheta - \tilde{\vartheta}\|_{L^1_{w; \xi}}.
\]

\( \square \)
To close this section, we mention that with the help of Lemma 2.2, along the same arguments as in the proof of Proposition 4.1 and Theorem 4.2, we can prove the following result.

**Lemma 4.1** Let \( u(t; \vartheta), u(t; \tilde{\vartheta}) \) be kinetic solutions of \( \mathcal{E}(A, \Phi, \vartheta) \) and \( \mathcal{E}(A, \Phi, \tilde{\vartheta}) \) on \([0, T]\), respectively. Under Hypotheses (H1)–(H3), for almost every \( 0 \leq s < t \leq T \), we have

\[
\mathbb{E}\|u(t; \vartheta) - u(t; \tilde{\vartheta})\|_{L_{w; x}^1} \leq \mathbb{E}\|u(s; \vartheta) - u(s; \tilde{\vartheta})\|_{L_{w; x}^1} - \sum_{j=1}^{d} \mathbb{E}\int_{s}^{t} \int_{D} |A_j(u(r; \vartheta)) - A_j(u(r; \tilde{\vartheta}))| dxdr.
\]

### 5. Proof of Continuity Extension in the Weighted Space

In this section, we will prove that when the initial data \( \vartheta \in L_{w}^{-\infty} L_{x}^{-\infty} \), the kinetic solution admits a continuous extension in the time variable. To this end, for \( n \geq 1 \), we consider the following approximating equation

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{du^n}{dt} - \frac{1}{n} \Delta u^n dt + \text{div}(A(u^n)) dt = \Phi(u^n) dW(t) \quad \text{in } D \times (0, T),
\end{array} \right.
\end{aligned}
\]

where the initial value \( \vartheta \in L_{0}^{-\infty}(\Omega \times D) \) and \( \mathcal{F}_{0} \otimes \mathcal{B}(D) \) –measurable random variable. According to [16], under Hypotheses (H1) and (H3), for any \( n \geq 1 \), (5.1) admits a unique \( L_{x}^{-q} \) –valued continuous solution \( u^n \in L_{x}^{-2}(\Omega; C([0, T]; H) \cap L_{x}^{-2}(\Omega; H^{1}(D))) \) for large enough \( n \). Moreover, similar to (4.8) in [20], for every \( p \geq 2 \), there exists a constant \( C \) independent of \( n \) such that

\[
\mathbb{E}\sup_{t \in [0, T]} \|u^n(t)\|_{L_{x}^{-p}}^p + \frac{1}{n} \mathbb{E}\int_{0}^{T} \|\nabla u^n\|_{L_{x}^{-2}}^2 dt \leq C(1 + \mathbb{E}\|\vartheta\|_{L_{w}^{-p}}^p).
\]

Similar to Proposition 23 in [11], \( f = I_{w^n_{x} \leq \tilde{\vartheta}} \) satisfies that there exists a kinetic measure \( m^n \) such that for any \( \varphi \in C_{c}^{\infty}((0, T) \times \bar{D} \times \mathbb{R}) \),

\[
\begin{aligned}
&- \int_{0}^{T} \int_{D} \int_{\mathbb{R}} f(\vartheta_t + a \cdot \nabla) \varphi d\xi dxdt + \frac{1}{n} \int_{0}^{T} \int_{D} \int_{\mathbb{R}} \nabla x f \cdot \nabla x \varphi d\xi dxdt \\
&- \int_{D} \int_{\mathbb{R}} f_{0}\varphi(0) d\xi dx \\
&= \sum_{k \geq 1} \int_{0}^{T} \int_{D} \int_{\mathbb{R}} g_{k}(x, \xi) \varphi(x, t, \xi) d\nu_{x,t}(\xi) dx d\beta_{k}(t) \\
&+ \frac{1}{2} \int_{0}^{T} \int_{D} \int_{\mathbb{R}} \partial_{\xi} \varphi(x, t, \xi) G^{2}(x, \xi) d\nu_{x,t}(\xi) dxdt - (m^n + q^n)(\partial_{\xi} \varphi), \text{a.s.}
\end{aligned}
\]

where \( f_{0} = I_{\vartheta_{x} \geq \tilde{\vartheta}}, \nu = -\partial_{\xi} f = \delta_{w_{x} = \tilde{\vartheta}} \) and \( q^n : \Omega \rightarrow M_{0}^{\#}(D \times [0, T] \times \mathbb{R}) \) is defined by \( q^n = \frac{1}{n} |\nabla u^n|^{2} \delta_{w_{x} = \tilde{\vartheta}} \).

Further, note that when \( \Phi \equiv 0 \), (5.1) turns to be the equation (1.1) in [19] with \( \beta(\xi) = \frac{1}{n} \xi \). As stated in [19], we do not know whether the space kinetic traces exist or
not for (5.1), hence the weak star cluster points will be employed to take place of those. Applying techniques used in the proof of Proposition 2.1 (b) in [19] (see P689-P693) to (5.3) with \( W_{\rho, \lambda}(x) = \int_{0}^{x \cdot \theta_{\rho, \lambda}(\tau)} d\tau \) replaced by \( \int_{0}^{x \cdot \theta_{\rho, \lambda}(\tau)} \psi_{\tau}(r - \tau(\tilde{L}_{\lambda} + 1)) dr \), where \( \psi \) is a standard modifier, we get an identity, which is similar to (2.9) in [19]. Concretely, for any \( \phi \in C_{C}^{\infty}([0, T) \times \mathbb{R}^{d} \times \mathbb{R}) \),

\[
- \int_{0}^{T} \int_{D} \int_{\mathbb{R}} f(\partial_{t} + a \cdot \nabla) \phi^{\lambda} d\xi dxdt + \frac{1}{n} \int_{0}^{T} \int_{D} \int_{\mathbb{R}} \nabla_{x} f \cdot \nabla_{x} \phi^{\lambda} d\xi dxdt
\]

satisfies such that for any \( \phi \), not for (5.1), hence the weak star cluster points will be employed to take place of those.

Concretely, for any \( \phi \), not for (5.1), hence the weak star cluster points will be employed to take place of those.

Moreover, (5.4) can be strengthened to be weak only with respect to \( C \) and referring to [10, 19], we can derive the following kinetic formulation satisfied by \( f = \left. E_{\rho}^{n} \right|_{x = \xi} \). Firstly, for any \( \phi \in C_{C}^{\infty}([0, T) \times \mathbb{R}^{d} \times \mathbb{R}) \), a chain rule formula holds true:

\[
\partial_{x_{d}} f^{+}(t, \bar{\lambda}, h_{\lambda}(\bar{\xi}) + r, \xi) \geq 0, \quad a.s.
\]

(5.5)

Proceeding as Proposition 1 (ii) in [20] (applying \( \phi^{\lambda} = \Psi_{\eta}(\xi) \psi^{\lambda}(x, t, \xi) \) to (5.4) with \( \varphi \in C_{C}^{\infty}([0, T) \times \mathbb{R}^{d} \times \mathbb{R}) \) and referring to [10, 19], we can derive the following kinetic formulation satisfied by \( f = \left. E_{\rho}^{n} \right|_{x = \xi} \). Firstly, for any \( \phi \in C_{C}^{\infty}([0, T) \times \mathbb{R}^{d} \times \mathbb{R}) \), a chain rule formula holds true:

\[
\partial_{x_{d}} f^{+}(t, \bar{\lambda}, h_{\lambda}(\bar{\xi}) + r, \xi) \geq 0, \quad a.s.
\]

Moreover, (5.4) can be strengthened to be weak only with respect to \( x \) and \( \xi \) by using the same method as Proposition 1 (ii) in [20]. That is, there exists a kinetic measure \( m^{n} \) such that for any \( \varphi \in C_{C}^{\infty}([0, T) \times \mathbb{R}^{d} \times \mathbb{R}) \), any \( t \in [0, T) \) and for any \( 0 < \eta < N \), \( f = \left. E_{\rho}^{n} \right|_{x = \xi} \) satisfies

\[
- \int_{D} \int_{-N}^{N} \Psi_{\eta}(\xi) f^{+}(t) \varphi d\xi dx + \int_{D} \int_{-N}^{N} \Psi_{\eta}(\xi) a(\xi) \cdot \nabla \varphi d\xi dx ds
\]
\[-\frac{1}{n} \int_0^t \int_D \int_{-N}^N \Psi_\eta(\xi) \nabla_x f^+ \cdot \nabla_x \varphi d\xi \, dx \, ds + \int_D \int_{-N}^N \Psi_\eta(\xi) f_0^+ \varphi d\xi \, dx\]

\[+ \frac{1}{n} \sum_{i=0}^M \int_0^t \int_{\Pi \lambda_i} \int_{-N}^N \Psi_\eta(\xi) (\nabla_x h_{\lambda_i}(\bar{x}) \cdot \nabla_x f_0^+ - \bar{\varphi}) \, d\xi \, d\bar{x} \, ds\]

\[+ \int_0^t \int_{\partial D} \int_{-N}^N \Psi_\eta(\xi) (-a(\xi) \cdot \mathbf{n}(\bar{x})) \bar{f} \varphi d\xi \, d\sigma(\bar{x}) \, ds\]

\[-\frac{1}{n} \sum_{i=0}^M \lim_{\tau \to +0} \int_0^t \int_{\Pi \lambda_i} \int_{-N}^N \Psi_\eta(\xi) \]

\[
\left[ \int_0^\infty \psi_\tau (r - \tau(\bar{L}_{\lambda_i} + 1)) (\partial_{x_d} f^+ \varphi^{\lambda_i} (\bar{x}, h_{\lambda_i}(\bar{x}) + r, \xi) ) \, dr \right] d\xi \, d\bar{x} \, ds
\]

and \(\bar{f} := 1 - f\) satisfies

\[-\int_D \int_{-N}^N \Psi_\eta(\xi) \bar{f}^+(t) \varphi d\xi \, dx + \int_0^t \int_D \int_{-N}^N \Psi_\eta(\xi) \bar{f}^+ a(\xi) \]

\[
\cdot \nabla \varphi d\xi \, dx \, ds - \frac{1}{n} \int_0^t \int_D \int_{-N}^N \Psi_\eta(\xi) \nabla_x \bar{f}^+ \cdot \nabla_x \varphi d\xi \, dx \, ds
\]

\[+ \int_D \int_{-N}^N \Psi_\eta(\xi) \bar{f}_0^+ \varphi d\xi \, dx\]

\[+ \frac{1}{n} \sum_{i=0}^M \int_0^t \int_{\Pi \lambda_i} \int_{-N}^N \Psi_\eta(\xi) (\nabla_x h_{\lambda_i}(\bar{x}) \cdot \nabla_x \bar{f}_0^+ \varphi^{\lambda_i} \, d\xi \, d\bar{x} \, ds\]

\[+ \int_0^t \int_{\partial D} \int_{-N}^N \Psi_\eta(\xi) (-a(\xi) \cdot \mathbf{n}(\bar{x})) \bar{f} \varphi d\xi \, d\sigma(\bar{x}) \, ds\]

\[-\frac{1}{n} \sum_{i=0}^M \lim_{\tau \to +0} \int_0^t \int_{\Pi \lambda_i} \int_{-N}^N \Psi_\eta(\xi) \]

\[
\left[ \int_0^\infty \psi_\tau (r - \tau(\bar{L}_{\lambda_i} + 1)) (\partial_{x_d} f^+ \varphi^{\lambda_i} (\bar{x}, h_{\lambda_i}(\bar{x}) + r, \xi) ) \, dr \right] d\xi \, d\bar{x} \, ds
\]
\[
\begin{align*}
\sum_{k \geq 1} & \int_0^t \int_D \int_{-N}^N \Psi_\eta(\xi) g_k(x, \xi) \varphi(x, \xi) d\nu_{x,x}(\xi) dx d\beta_k(s) \\
& + \frac{1}{2} \int_0^t \int_D \int_{-N}^N \Psi_\eta(\xi) \partial_\xi \varphi(x, \xi) G^2(x, \xi) d\nu_{x,x}(\xi) dx ds \\
& - \int_{[0,t] \times D \times (-N,N)} \Psi_\eta(\xi) \partial_\xi \varphi d(m^n + q^n) \\
& - \frac{1}{2} \int_0^t \int_D \int_{-N}^N (\psi_\eta(N - \xi - \eta) - \psi_\eta(N - \eta)) G^2 \varphi d\nu_{x,x}(\xi) dx ds \\
& + \int_{[0,t] \times D \times (-N,N)} (\psi_\eta(N - \xi - \eta) - \psi_\eta(N - \eta)) \varphi d(m^n + q^n), \text{ a.s.,}
\end{align*}
\]

where \( f_0 = I_{\theta > \xi}, f_b = I_{0 > \xi} \), \( \tilde{f} \) is defined by (2.15) with \( f = I_{u^n > \xi} \) and \( q^n = \frac{1}{n} |\nabla u^n|^2 \delta_{u^n = \xi} \).

For any \( \vartheta_1, \vartheta_2 \in L_{\infty}^{\infty}, \) denote by \( u^n(t; \vartheta_1), u^n(t; \vartheta_2) \) the solutions of (5.1) with boundary values \( u_{1,0}^n = u_{2,0}^n = 0 \) on \( \Sigma \), respectively. Similar to the proof of Proposition 4.1 and using (4.20), we derive the following comparison theorem associated to \( u^n(t; \vartheta_1) \) and \( u^n(t; \vartheta_2) \).

**Lemma 5.1** Assume Hypotheses (H1) and (H3) hold. For any \( t \in (0, T), \gamma, \delta > 0, \) \( N \in \mathbb{D}, \) and any element \( \lambda \) of the partition of unity \( \{\lambda_i\}_{i=0,1,\ldots,M} \) on \( \overline{D} \), the functions \( f_1(t) := f_1(t, x, \xi) = I_{u^n(t; \vartheta_1) > \xi} \) and \( f_2(t) := f_2(t, y, \xi) = I_{u^n(t; \vartheta_2) > \xi} \) with data \( f_{1,0} = I_{\vartheta_1 > \xi}, f_{2,0} = I_{\vartheta_2 > \xi}, f_{1,b} = I_{0 > \xi}, (m^n_1, q^n) \), \( i = 1, 2, \) satisfy

\[
\begin{align*}
& E \int_{D_\delta} \int_{D_\delta} \int_{-N}^N \int_{-N}^N (f_1^+(t) f_2^+(t) + f_1^+(t) f_2^+(t)) \rho_Y(y - x) \\
& \psi_\delta(\xi - \zeta) \lambda(\lambda) w(x) d\xi d\zeta d\sigma d\lambda \\
& \leq E \int_{D_\delta} \int_{D_\delta} \int_{-N}^N (f_1(0) f_2(0) + \tilde{f}_1(0) f_2(0)) \lambda(\lambda) w(x) d\xi d\zeta d\sigma d\lambda \\
& + E \int_0^t \int_{\partial D_\delta} \int_{-N}^N (f_1^+(s) f_2^+(s) + \tilde{f}_1^+(s) f_2^+(s)) (-a(\xi) \cdot n) \lambda(\lambda) w(x) d\xi d\sigma d\lambda d\sigma d\lambda ds \\
& - \frac{1}{n} \int_0^t \int_{\partial D_\delta} \int_{-N}^N (f_{1,b}^+ f_{2,b}^+ + \tilde{f}_{1,b}^+ f_{2,b}^+) \frac{\lambda(\lambda) w(x)}{\lambda(\lambda) w(x)} d\xi d\sigma d\lambda d\sigma d\lambda ds \\
& \sum_{j=1}^{d-1} \int_0^t \int_{\partial D_\delta} \int_{-N}^N (f_{1,b}^+ f_{2,b}^+ + \tilde{f}_{1,b}^+ f_{2,b}^+) \alpha(\lambda) \lambda \beta \delta x_j h(\lambda, \alpha) d\xi d\sigma d\lambda d\sigma d\lambda ds \\
& - \frac{1}{n} \int_0^t \int_{\partial D_\delta} \int_{-N}^N (f_{1,b}^+ f_{2,b}^+ + \tilde{f}_{1,b}^+ f_{2,b}^+) \alpha(\lambda) \lambda \beta \delta x_j h(\lambda, \alpha) d\xi d\sigma d\lambda d\sigma d\lambda ds \\
& + \tilde{r}_t^{N,\lambda}(\gamma, \delta) + \tilde{r}_t^{N,\lambda}(\gamma, \delta) + C \delta^{-1} + C (\gamma^2 \delta^{-1} + \delta) + \tilde{L}^{N,\lambda} + \tilde{L}^{N,\lambda} + I_N, \tag{5.7}
\end{align*}
\]

where

\[
I_N = C(\mu_{q_1}^{\prime}(\pm N) + \mu_{m_1}^{\prime}(\pm N) + \mu_{q_2}^{\prime}(\pm N) + \mu_{m_2}^{\prime}(\pm N) + (1 + N^2) \mu_{v_1}^{\prime}(\pm N) + (\pm N) + (1 + N^2) \mu_{v_2}^{\prime}(\pm N)).
\]
Here, \( r^N_i(\gamma, \delta) \) is defined by (4.15), \( \tilde{L}^N_1 \), \( \tilde{L}^N_2 \) are \( J_3 \), \( J_4 \) in Proposition 4.1 with \( f_1 = I_u^a \cdot x \) and \( f_2 = I_u^a \cdot \xi \). Moreover, \( \tilde{r}^N_i(\gamma, \delta) \), \( \tilde{r}^N(\gamma, \delta) \) and \( \tilde{r}^N_i(\gamma, \delta) \) are defined by the following (5.11), (5.13) and (5.15). \( v_1 = \delta_u(\gamma; \vartheta_1) = \delta \), \( v_2 = \delta_u(\gamma; \vartheta_2) = \xi \) and \( \mu_{\vartheta_i}, i = 1, 2 \) are defined by the same way as \( m^N_\vartheta \) in (2.14) satisfying Lemma 2.1, which implies that \( \lim_{N \to \infty} I_N = 0 \).

**Proof** According to Proposition 4.1 and (4.18), it suffices to handle the additional terms generated by \( n^{-1} \Delta_x u^n(t, x; \vartheta_1) \) and \( n^{-1} \Delta_y u^n(t, y; \vartheta_2) \). Denote by \( J^\ast_{i, n} \) and \( J^\ast_{2, n} \) the terms related to \( f_1^+ f_2^- \) and \( f_1^+ f_2^+ \), respectively.

\[
J^\ast_{1, n} = -\frac{1}{n} \mathbb{E} \left[ \int_0^t \int_{D_N^+} \int_{D_N} \int_{-N}^{N} \psi_\eta(\xi, \zeta) f_2^+ \varphi_2 \nabla_x f_1^+ \cdot \nabla_x (\varphi_1 \lambda w) d\xi d\zeta dx dy ds \right]
-
\frac{1}{n} \mathbb{E} \left[ \int_0^t \int_{D_N^+} \int_{D_N} \int_{-N}^{N} \psi_\eta(\xi, \zeta) f_1^+ \varphi_1 \lambda w \nabla_y f_2^- \cdot \nabla_y \varphi_2 d\xi d\zeta dx dy ds \right]
+
\frac{1}{n} \mathbb{E} \left[ \int_0^t \int_{D_N^+} \int_{D_N} \int_{-N}^{N} \psi_\eta(\xi, \zeta) f_2^+ (\nabla_\xi h_\lambda(\tilde{x}) \cdot \nabla_\xi f_1^+) \alpha^\phi w d\xi d\zeta dx dy ds \right]
+
\frac{1}{n} \mathbb{E} \left[ \int_0^t \int_{D_N^+} \int_{D_N} \int_{-N}^{N} \psi_\eta(\xi, \zeta) f_1^+ \varphi_1 \lambda w(x) (\nabla_\xi h_\lambda(\tilde{y})) \cdot \nabla_\xi \tilde{f}_2^+ \varphi_2^+ d\xi d\zeta dx dy ds \right]
-
\frac{1}{n} \mathbb{E} \lim_{r \to +0} \left[ \int_0^t \int_{D_N^+} \int_{D_N} \int_{-N}^{N} \psi_\eta(\xi, \zeta) f_2^+ \varphi_2 \right]
\left[ \int_0^\infty \psi_\tau (r - \tau (L_\lambda + 1)) (\partial_{x_d} f_1^+) \varphi_1^+ w dr \right] d\xi d\zeta dx dy ds
-
\frac{1}{n} \mathbb{E} \lim_{r \to +0} \left[ \int_0^t \int_{D_N^+} \int_{D_N} \int_{-N}^{N} \psi_\eta(\xi, \zeta) f_1^+ \varphi_1^+ w(x) \right]
\left[ \int_0^\infty \psi_\tau (r - \tau (L_\lambda + 1)) (\partial_{y_d} \tilde{f}_2^+) \varphi_2^+ w dr \right] d\xi d\zeta dx dy ds
-
\mathbb{E} \left[ \int_{[0, t] \times D_N^+ \times (-N, N)} \int_{D_N} \int_{-N}^{N} \psi_\eta(\xi, \zeta) f_2^+ \partial_\xi \alpha \lambda w(x) dq_1^N(s, x, \xi) d\xi d\gamma \right]
+
\mathbb{E} \left[ \int_{[0, t] \times D_N \times (-N, N)} \int_{D_N} \int_{-N}^{N} \psi_\eta(\xi, \zeta) f_1^+ \lambda w(x) \partial_\xi \alpha dq_2^N(s, y, \xi) d\xi d\gamma \right]
+
\mathbb{E} \left[ \int_{[0, t] \times D_N^+ \times (-N, N)} \int_{D_N} \int_{-N}^{N} \psi_\eta(\xi, \zeta) \tilde{f}_2^+ (\psi_\eta(N - \xi - \eta)) \right]
-
\psi_\eta(\xi + N - \eta) \alpha \lambda w(x) dq_1^N(s, x, \xi) d\gamma d\zeta
-
\mathbb{E} \left[ \int_{[0, t] \times D_N \times (-N, N)} \int_{D_N} \int_{-N}^{N} \psi_\eta(\xi, \zeta) f_1^+ (\psi_\eta(N - \xi - \eta)) \right]
-
\psi_\eta(\xi + N - \eta) \alpha \lambda w(x) dq_2^N(s, y, \xi) d\gamma d\zeta

= \sum_{i=1}^{10} I_i.

(5.8)
Similar to (4.5), we have
\[
|I_9| + |I_{10}| \leq C \int_{\mathbb{R}} (\psi_\eta(-\eta + N - \xi) + \psi_\eta(-\eta + N + \xi)) d\mu_{q_1}(\xi)
\]
\[
+ C \int_{\mathbb{R}} (\psi_\eta(-\eta + N - \xi) + \psi_\eta(-\eta + N + \xi)) d\mu_{q_2}(\xi)
\]
\[
\rightarrow C(\mu_{q_1}'(\pm N) + \mu_{q_2}'(\pm N)),
\]
(5.9)
as \eta \to +0, where \(\mu_{q_i}'\), \(i = 1, 2\) are defined by the same way as \(m_{i}''\) in (2.14) satisfying Lemma 2.1.

From now on, keeping in mind that \(\alpha(x, \xi, y, \zeta) = \varphi_1(x, \xi)\varphi_2(y, \zeta) = \rho_{\gamma}^{y}(y - x)\psi_{\delta}(\xi - \zeta)\) and \(\nabla_x \alpha = -\nabla_y \alpha\). Since \(\rho_{\gamma}^{y}(y - x) = 0\) on \(D^y_\xi \times \partial D_y\), we get \(I_4 = 0\) and \(I_5 = 0\). Moreover, by utilizing (5.5), it follows that \(I_5 \leq 0\). By using the divergence theorem and \(\partial_x \psi(\lambda) = -1\), we have

\[
I_1 = -\frac{1}{n} \mathbb{E} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Psi_\eta(\xi, \zeta) f_2^+ \nabla_x f_1^+ \cdot (\nabla_x \alpha) \lambda w d\xi d\zeta dxdyds
\]
\[
- \frac{1}{n} \mathbb{E} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Psi_\eta(\xi, \zeta) f_2^+ \alpha \nabla_x f_1^+ \cdot \nabla_x (\lambda w) d\xi d\zeta dxdyds
\]
\[
= -\frac{1}{n} \mathbb{E} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Psi_\eta(\xi, \zeta) (\nabla_y f_2^+) \cdot (\nabla_x f_1^+ \alpha \lambda w) d\xi d\zeta dxdyds
\]
\[
- \frac{1}{n} \mathbb{E} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Psi_\eta(\xi, \zeta) f_1^+ \alpha \Delta_x (\lambda w) d\xi d\zeta dxdyds,
\]
where we used the fact that for any \(\phi \in C^\infty_{0}(\mathbb{R})\), \(\int_{\mathbb{R}} f_1^+(x, t, \xi) \phi(\xi) d\xi\) has the trace \(\int_{\mathbb{R}} f_1^+(x, t, \xi) \phi(\xi) d\xi\) on \(\Sigma^\lambda\). Similarly, we get

\[
I_2 = \frac{1}{n} \mathbb{E} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Psi_\eta(\xi, \zeta) f_1^+ \tilde{\alpha}(\zeta) \tilde{\alpha}(\xi) \nabla_y f_2^+ \cdot (\nabla_y \tilde{f_2}) d\xi d\zeta dxdyds
\]
\[
- \frac{1}{n} \mathbb{E} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Psi_\eta(\xi, \zeta) (\nabla_x f_1^+) \cdot (\nabla_y \tilde{f_2}) \alpha \lambda w d\xi d\zeta dxdyds
\]
\[
- \frac{1}{n} \mathbb{E} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Psi_\eta(\xi, \zeta) f_1^+ \alpha \nabla_x (\lambda w) \cdot (\nabla_y \tilde{f_2}) d\xi d\zeta dxdyds.
\]
Then, it follows that
\[
I_1 + I_2 = -\frac{2}{n} \mathbb{E} \int_0^t \int_{D_2} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta)(\nabla_y \tilde{f}_2^+) \cdot (\nabla_x f^+)\alpha w d\xi d\zeta dx dy ds
- \frac{1}{n} \mathbb{E} \int_0^t \int_{D_2} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) f^+_{1,b} \tilde{f}_2^+ \alpha n(\tilde{x}) \cdot \nabla_x (\lambda w) d\xi d\zeta d\sigma(\tilde{x}) dy ds
+ \frac{1}{n} \mathbb{E} \int_0^t \int_{D_2} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) f^+_{1,b} \tilde{\alpha}_x (\lambda w) \nabla_y \tilde{f}_2^+ \cdot \nabla_y \tilde{f}_2^+ \alpha d\xi d\zeta d\sigma(\tilde{x}) dy ds
+ \frac{1}{n} \mathbb{E} \int_0^t \int_{D_2} \int_{D_y} \int_{-N}^N \int_{-N}^N \Psi_\eta(\xi, \zeta) f^+_{1,b} \tilde{\alpha}_x (\lambda w) d\xi d\zeta d\sigma(\tilde{x}) dy ds
=: K_1 + K_2 + K_3 + K_4.
\]

Based on (5.6), using the same method as the proof of Theorem 3.3 in [10] (the estimates of \(J_2\)), we get
\[
K_1 = 2n^{-1} E \int_0^t \int_{D_2} \int_{D_y} \int_{-N}^N \int_{-N}^N f^+_{1,b} \tilde{\alpha} w \tilde{\alpha}_x \tilde{f}_2^+ \cdot \nabla_x h_\lambda(\tilde{x}) \tilde{\sigma} d\xi d\zeta d\tilde{x} d\tilde{y} ds
- \frac{1}{n} \mathbb{E} \int_0^t \int_{D_2} \int_{D_y} \int_{-N}^N \int_{-N}^N f^+_{1,b} \tilde{\alpha} w \tilde{\alpha}_x \tilde{f}_2^+ \alpha d\xi d\zeta d\tilde{x} d\tilde{y} ds
= : K_{3,1} + K_{3,2}.
\]

Taking into account the fact that for \(1 \leq i \leq d - 1\),
\[
\tilde{\alpha}_y \tilde{\sigma} = -\tilde{\alpha}_x \tilde{\sigma} - \sum_{k=1}^{d-1} \psi_\gamma(y_k - x_k) \partial_{x_k} h_\lambda(\tilde{x}) \psi_\gamma'(y_d - h_\lambda(\tilde{x})) - \gamma(\tilde{L}_\lambda + 1) \psi_\delta(\xi - \zeta),
\]

by integration by parts, we deduce that
\[
K_{3,1} = -\frac{1}{n} \mathbb{E} \int_0^t \int_{D_2} \int_{D_y} \int_{-N}^N \int_{-N}^N \tilde{\alpha}_x \tilde{w} \tilde{f}_2^+ \nabla_x h_\lambda(\tilde{x}) \cdot \nabla_x f^+_{1,b} d\xi d\zeta d\tilde{x} d\tilde{y} ds
- \frac{1}{n} \mathbb{E} \int_0^t \int_{D_2} \int_{D_y} \int_{-N}^N \int_{-N}^N f^+_{1,b} \tilde{f}_2^+ \tilde{\alpha}_x \tilde{w} \nabla_x h_\lambda(\tilde{x}) \cdot \nabla_x \tilde{f}_2^+ d\xi d\zeta d\tilde{x} d\tilde{y} ds
+ \frac{1}{n} \sum_{j=1}^{d-1} \mathbb{E} \int_0^t \int_{D_2} \int_{D_y} \int_{-N}^N \int_{-N}^N f^+_{1,b} \tilde{f}_2^+ \tilde{\alpha}_x \tilde{\alpha}_j h_\lambda(\tilde{x}) d\xi d\zeta d\tilde{x} d\tilde{y} ds
- \frac{1}{n} \mathbb{E} \int_0^t \int_{D_2} \int_{D_y} \int_{-N}^N \int_{-N}^N f^+_{1,b} \tilde{f}_2^+ \tilde{\alpha}_x \tilde{w} \tilde{\alpha}_x \tilde{w} \nabla_x h_\lambda(\tilde{x}) d\xi d\zeta d\tilde{x} d\tilde{y} ds.
\]
\[
- \frac{1}{n} \mathbb{E} \int_0^t \int_{\partial D_x^\epsilon} \int_{D_x^\epsilon} \int_{-N}^N \int_{-N}^N f_{1,b}^+ \partial_y \tilde{f}_2^+ |\nabla \tilde{h}_x(\bar{x})|^2 \bar{\alpha} \bar{\lambda} w d\xi d\zeta d\tilde{x} dy ds.
\]

Combining all the above estimates, we have

\[
\lim_{\eta \to 0} J_{1}^{\eta} \leq - \frac{1}{n} \mathbb{E} \int_0^t \int_{\partial D_x^\epsilon} \int_{D_x^\epsilon} \int_{-N}^N \int_{-N}^N f_{1,b}^+ \tilde{f}_2^+ \bar{\omega} n(\bar{x}) \cdot \nabla \bar{\chi}(\bar{\lambda}w) d\xi d\zeta d\sigma(\bar{x}) dy ds
\]

\[
+ \frac{1}{n} \sum_{j=1}^{d-1} \mathbb{E} \int_0^t \int_{\partial D_x^\epsilon} \int_{D_x^\epsilon} \int_{-N}^N \int_{-N}^N f_{1,b}^+ \tilde{f}_2^+ \bar{\omega} \frac{\nabla h_x(\bar{x})}{\nabla \bar{\chi}(\bar{\lambda}w)} d\xi d\zeta d\tilde{x} dy ds
\]

\[
- \frac{1}{n} \mathbb{E} \int_0^t \int_{\partial D_x^\epsilon} \int_{D_x^\epsilon} \int_{-N}^N \nabla h_x(\bar{x}) \left( \frac{\alpha^2 w}{\Pi_1^2 \lambda} \right) d\xi d\zeta d\tilde{x} dy ds.
\]

Due to the symmetry of \( f_{1,b}^+ \tilde{f}_2^+ \) and \( \tilde{f}_1^+ f_2^+ \), we have a similar estimation for \( \lim_{\eta \to 0} J_{2}^{\eta} \).

Thus, we get

\[
\lim_{\eta \to 0} (J_{1}^{\eta} + J_{2}^{\eta})
\]
\[ + C(\mu_{q_1}^\prime(\pm N) + \mu_{q_2}^\prime(\pm N)) \]
\[ =: \sum_{i=1}^{7} L_i + C(\mu_{q_1}^\prime(\pm N) + \mu_{q_2}^\prime(\pm N)). \]  \tag{5.10}

Recall that \( \alpha(x, y, \xi, \zeta) = \rho_{\gamma}^\lambda(y - x)\psi_{\delta}(\xi - \zeta) \). Define
\[
\tilde{r}_t^{N, \lambda}(\gamma, \delta) := L_1 + L_3 + L_4
\]
\[
+ \frac{1}{n} \mathbb{E} \int_0^t \int_{\partial D_2^+} \int_{D_2^+} \int_{-N}^N (f_{1, b}^+, f_{2, b}^+) \n(\bar{x}) \cdot \nabla x(\lambda w) d\xi d\sigma(\bar{x}) d\sigma
\]
\[
- \frac{1}{n} \sum_{j=1}^{d-1} \mathbb{E} \int_0^t \int_{\Pi^j} \int_{-N}^N (f_{1, b}^+, f_{2, b}^+) \alpha_{\lambda j} \partial_{x_j} h(\bar{x}) d\xi d\bar{x} ds
\]
\[
+ \frac{1}{n} \mathbb{E} \int_0^t \int_{\Pi^j} \int_{-N}^N (f_{1, b}^+, f_{2, b}^+) \alpha_{\lambda j} \bar{w} \Delta_{\xi} h(\bar{x}) d\xi d\bar{x} ds. \tag{5.11}
\]

Taking into account (5.2), we know that \( u^n \) has trace on the boundary in the classical sense, that is, \( u^n = 0 \) on \( \Sigma \). Hence, it follows that
\[
\lim_{\gamma, \delta \to 0} \sum_{i=0}^M |\tilde{r}_t^{N, \lambda_i}(\gamma, \delta)| = 0, \quad \lim_{\gamma, \delta \to 0} \sum_{i=0}^M \int_0^T |\tilde{r}_t^{N, \lambda_i}(\gamma, \delta)| dt = 0, \]  \tag{5.12}

for each \( N > 0 \). Let
\[
\tilde{r}_t^{N, \lambda}(\gamma, \delta) := L_5 + L_6. \tag{5.13}
\]

By changing variables method and utilizing \( \partial_{yd} f_2^+ = \delta(\xi - u^n_1(y, s))\partial_{yd} u^n_2(y, s) \) in \( D'(D_y \times \mathbb{R}) \), we have
\[
L_6 = \frac{1}{n} \mathbb{E} \int_0^t \int_{\partial D_2^+} \int_{D_2^+} \int_{-N}^N \rho_{\gamma}^\lambda(y - x)\bar{w} \psi_{\delta}(\xi - u^n_1(\bar{y}, y_d)) \partial_{yd} u^n_2(\bar{y}, y_d) d\xi d\bar{x} dy dx ds
\]
\[
- \frac{1}{n} \mathbb{E} \int_0^t \int_{\partial D_2^+} \int_{D_2^+} \int_{-N}^N \rho_{\gamma}^\lambda(y - x)\bar{w} \psi_{\delta}(\xi + u^n_1(\bar{y}, y_d)) \partial_{yd} u^n_2(\bar{y}, y_d) d\xi d\bar{x} dy dx ds,
\]

Since for any \( \delta > 0, \psi_{\delta}(\xi \pm u^n_1(\bar{y}, y_d)) \partial_{yd} u^n_2(\bar{y}, y_d) \in L^2(D_y) \) and \( u^n_2(\bar{x}, h(\bar{x})) = u^n_{2, b} = 0 \), we deduce that
\[
\lim_{\gamma \to 0} L_6 = \frac{1}{n} \mathbb{E} \int_0^t \int_{\partial D_2^+} \int_{D_2^+} \int_{-N}^N \mathbb{E} \bar{w} \psi_{\delta}(\xi - u^n_1(\bar{x}, h(\bar{x}))) \partial_{yd} u^n_2(\bar{x}, x_d) | x_d = h(\bar{x}) d\xi d\bar{x} dx ds
\]
\[
- \frac{1}{n} \mathbb{E} \int_0^t \int_{\partial D_2^+} \int_{D_2^+} \int_{-N}^N \bar{w} \psi_{\delta}(\xi + u^n_1(\bar{x}, h(\bar{x}))) \partial_{yd} u^n_2(\bar{x}, x_d) | x_d = h(\bar{x}) d\xi d\bar{x} dx ds
\]
\[
= \frac{1}{n} \mathbb{E} \int_0^t \int_{\partial D_2^+} \int_{D_2^+} \int_{-N}^N \bar{w} \psi_{\delta}(\xi - u^n_{2, b}) \partial_{yd} u^n_2(\bar{x}, h(\bar{x})) d\xi d\bar{x} dx ds
\]
\[
- \frac{1}{n} \mathbb{E} \int_0^t \int_{\partial D_2^+} \int_{D_2^+} \int_{-N}^N \bar{w} \psi_{\delta}(\xi + u^n_{2, b}) \partial_{yd} u^n_2(\bar{x}, h(\bar{x})) d\xi d\bar{x} dx ds.
\]
By the same method as above, we get \( \lim_{\gamma \to 0} L5 = 0 \), for any \( \delta > 0 \) and \( N > 0 \). Thus, it follows that
\[
\lim_{\gamma \to 0} \sum_{i=0}^{M} |\tilde{r}_i^{N,\lambda_i}(\gamma, \delta)| = 0, \quad \lim_{\gamma \to 0} \sum_{i=0}^{M} \int_0^T |\tilde{r}_i^{N,\lambda_i}(\gamma, \delta)| dt = 0, \quad (5.14)
\]
for each \( \delta > 0 \) and \( N > 0 \). Define
\[
\tilde{r}_i^{N,\lambda}(\gamma, \delta) := L2 + L7. \quad (5.15)
\]
Using \( \sum_{i=0}^{M} \lambda_i = 1 \) and by the definition of \( w \), we have
\[
\sum_{i=0}^{M} |\tilde{r}_i^{N,\lambda_i}(\gamma, \delta)| = 0, \quad \sum_{i=0}^{M} \int_0^T |\tilde{r}_i^{N,\lambda_i}(\gamma, \delta)| dt = 0, \quad (5.16)
\]
for each \( \gamma, \delta > 0 \) and \( N > 0 \).

Finally, by using (5.10) and with the aid of (4.18)–(4.20), we get the desired result by taking \( \tilde{L}_1^{N,\lambda} := J3 \) and \( \tilde{L}_2^{N,\lambda} := J4 \), where \( J3 \) and \( J4 \) are in Proposition 4.1 with \( f_1 = I_{u^n > \xi} \) and \( f_2 = I_{u^n > \zeta} \), respectively.

The following result states that the solution \( u^n \) of (5.1) converges to the solution \( u \) of (1.1)–(1.3) in \( L^1(\Omega \times [0, T]; L^1_{w; x}) \).

**Proposition 5.1** Assume Hypotheses (H1) and (H3) are in force and let \( \vartheta \in L^\infty_\omega L^\infty_x \), then
\[
\lim_{n \to \infty} \mathbb{E} \int_0^T \| u(t; \vartheta) - u^n(t; \vartheta) \|_{L^1_{w; x}} dt = 0. \quad (5.17)
\]

**Proof** Let \( f_1(t) := f_1(t, x; \xi) = I_{u^n(t,x;\vartheta)>\xi} \) and \( f_2(t) := f_2(t, y; \zeta) = I_{u^n(t,y;\vartheta)>\zeta} \) with data \( f_{1,0} = I_{\vartheta > \xi}, f_{2,0} = I_{\vartheta > \zeta}, f_{1,b} = I_{\vartheta > \xi}, f_{2,b} = I_{\vartheta > \zeta} \). The corresponding kinetic measures are denoted by \( m \) and \( (m^n, q^n) \). Compared with Lemma 5.1, we only need to deal with additional terms generated by the term \( n^{-1} \Delta_y u^n(t, y; \vartheta) \). Let \( K^*_1 \) be the terms related to \( f_1^+ f_2^+ \) and \( K^*_2 \) be the terms related to \( f_1^+ f_2^- \). We have
\[
K^*_1 = -\frac{1}{n} \mathbb{E} \int_0^t \int_{D^*_x} \int_{D_y} \int_{\Omega} \psi_{\eta}(\xi, \zeta) f_1(\nabla_x \tilde{f}_2) \cdot \nabla_x \alpha \lambda w d\xi d\zeta dxdyds
+ \frac{1}{n} \mathbb{E} \int_0^t \int_{\Pi_y} \int_{D^*_x} \int_{\Omega} \psi_{\eta}(\xi, \zeta) f_1 \varphi \lambda w(x)(\nabla_x h_\xi(\tilde{y}) \cdot \nabla_x \tilde{f}_2,\tilde{b}) \frac{\psi_{\eta}}{\nabla_x \tilde{f}_2,\tilde{b}} d\xi d\zeta dxdyds
- \frac{1}{n} \mathbb{E} \lim_{\tau \to 0} \int_0^t \int_{\Pi_y} \int_{D^*_x} \int_{\Omega} \psi_{\eta}(\xi, \zeta) f_1^+ \varphi \lambda w(x)
\int_0^{\psi_T(r - \tau(\tilde{L}_\lambda + 1))} (\partial_x \tilde{f}_2^+,\varphi_2) dr d\xi dxdyds
+ \mathbb{E} \int_{[0,t] \times D_y \times (-N,N)} \int_{D_x} \int_{\Omega} \psi_{\eta}(\xi, \zeta) f_1^- \lambda(x) w(x) \partial_x \alpha dq^n(s,y,\zeta) dx \]
\[ - \mathbb{E} \int_{[0,1] \times D_y \times (-N, N)} \int_{D^N_x} \int_{-N}^N \Psi_\eta(\xi) f_1^-(\psi_\eta(N - \xi - \eta) \\
- \psi_\eta(\xi + N - \eta)) \alpha \lambda w(x) dq^M(s, y, \xi, \eta) d\xi dx \\
=: \sum_{i=1}^5 J^{\ast}_i, \]

where \( \alpha(x, y, \xi, \zeta) = \varphi_1(x, \xi) \varphi_2(y, \zeta) = \rho_{\gamma}^2(y-x) \psi_\theta(\xi-\zeta) \) and \( q^M = \frac{1}{N} |\nabla_x u^M|^2 \delta_{u^M = \zeta} \).

Since \( \rho_{\gamma}^2(y-x) = 0 \) on \( D^N_y \times \partial D_y \), we get \( J^*_2 = 0 \). Similar to the estimations of \( I_9 \) and \( I_{11} \) in Proposition 4.1, we have \( J^*_4 + J^*_5 \leq C \mu_{q^M}(\pm N) \) with \( \lim_{N \to \infty} \mu_{q^M}(\pm N) = 0 \).

Define
\[
\Upsilon(\xi, \zeta) = \int_{-\infty}^\xi \int_{-\infty}^\zeta \Psi_\eta(\xi', \zeta') \psi_\delta(\xi' - \zeta') d\xi' d\zeta',
\]

it gives that \( \Upsilon(\xi, \zeta) \leq C (|\xi| + |\zeta| + \delta) \). Then, by integration by parts and the boundedness of \( w \), we get
\[
J^*_1 = -\frac{1}{n} \mathbb{E} \int_0^t \int_{D^N_x} \int_{-N}^N f_1 \int_{D^N_y} \Delta_{\gamma} \rho_{\gamma}^2(y-x) \partial_\xi \partial_\eta \Upsilon(\xi, \zeta) \lambda(x) w(x) d\xi d\zeta dy ds
\]
\[
\leq C \frac{1}{n} \mathbb{E} \int_0^t \int_{D^N_x} \int_{-N}^N |\Delta_{\gamma} \rho_{\gamma}^2(y-x)| \lambda(x) w(x) \int_{\mathbb{R}^2} (|\xi| + |\zeta| + \delta) d\nu_{\delta, \gamma}(\xi, \zeta) dy ds
\]
\[
\leq C n^{-1} \gamma^{-2} + C n^{-1} \gamma^{-2} \delta \int_0^t \int_{D^N_x} \lambda(x) dx ds.
\]

For \( J^*_2 \), we have
\[
J^*_2 = -\frac{1}{n} \mathbb{E} \int_0^t \int_{D^N_x} \int_{-N}^N \Psi_\eta(\xi, \zeta) f_1 \int_{D^N_y} \Delta_{\gamma} h_{\lambda}(\bar{y}) \varphi_1 \varphi_2 \lambda(x) w(x) d\xi d\zeta dx ds \leq C n^{-1} L,
\]

where \( L \) is defined by (2.8).

Utilizing similar method as above, we deduce that \( K^*_2 \) has the same bound as \( K^*_1 \). Hence, we get
\[
K^*_1 + K^*_2 \leq C n^{-1} \gamma^{-2} + C n^{-1} \gamma^{-2} \delta \int_0^t \int_{D^N_x} \lambda(x) dx ds + C n^{-1} L + C \mu_{q^M}(\pm N).
\]

Due to Proposition 4.1, we obtain that for any \( 0 \leq t \leq T \),
\[
\mathbb{E} \int_{D^N_x} \int_{-N}^N \int_{-N}^N (f_1^+(t) \hat{f}_2^+(t) + f_1^+(t) \bar{f}_2^+(t)) \rho_{\gamma}^2(y-x) \psi_\gamma(\xi - \zeta) \lambda(x) w(x) d\xi d\zeta dy dx
\]
\[
\leq \mathbb{E} \int_{D^N_x} \int_{-N}^N \int_{-N}^N (f_1^0 \hat{f}_2^0 + \bar{f}_1^0 \bar{f}_2^0) \rho_{\gamma}^2(y-x) \psi_\gamma(\xi - \zeta) \lambda(x) w(x) d\xi d\zeta dy dx
\]
\[
+ \sum_{i=1}^5 J_i + I_N + C n^{-1} \gamma^{-2} + C n^{-1} \gamma^{-2} \delta \int_0^t \int_{D^N_x} \lambda(x) dx ds + C n^{-1} L,
\]

(5.18)
where \( \tilde{J}_i \) are the corresponding terms to \( J_i, i = 1, \ldots, 5 \) in Proposition 4.1 with \( f_1 = I_{u(t; \vartheta) > \xi} \) and \( f_2 = I_{u_n(t; \vartheta) > \xi} \). Moreover,

\[
I_N = C(\mu'_m(\pm N) + \mu'_{m_n}(\pm N) + \mu'_{q_n}(\pm N) + (1 + N^2)\mu'_{v,1}(\pm N) + (1 + N^2)\mu'_{v,2}(\pm N))
\]

satisfying \( \limsup_{N \to \infty} I_N = 0 \).

For (5.18), by integrating \( t \) from 0 to \( T \) and summing \( \lambda_i \) over \( i = 0, \ldots, M \), by (4.15), (4.19)–(4.23) in Theorem 4.2 and using \( \bar{J}_4 \leq 0 \), we get

\[
\mathbb{E} \int_0^T \int_D \int_{-N}^N (f_1^\pm(t) \tilde{f}_2^\pm(t) + \tilde{f}_1^\pm(t) f_2^\pm(t)) w(x) d\xi dx dt \\
\leq T \mathbb{E} \int_0^T \int_D \int_{-N}^N (f_1^0 \tilde{f}_2^0 + \tilde{f}_1^0 f_2^0) w(x) d\xi dx + |a(0)|
\]

\[
\mathbb{E} \int_0^T \int_0^T \int_D \int_{-N}^N (f_1^0 \tilde{f}_2^0 + \tilde{f}_1^0 f_2^0) w(x) d\xi dx ds dt
\]

\[
+ CMT \delta \gamma^{-1} + CTM (\gamma^2 \delta^{-1} + \delta) + TMI_N + CMTn^{-\gamma-2}
\]

\[
+ CT^2 n^{-1} \gamma^{-2} + Cn^{-1} LMT + \sum_{i=0}^M \mathcal{E}_N^{i, \lambda_i}(\gamma, \delta) T
\]

\[
+ \sum_{i=0}^M \left( \int_0^T r_t^{N, \lambda_i}(\gamma, \delta) dt + \int_0^T \mathcal{E}_t^{N, \lambda_i}(\gamma, \delta) dt \right),
\]

where error terms \( \mathcal{E}_N^{i, \lambda}(\gamma, \delta) \) and \( r_t^{N, \lambda}(\gamma, \delta) \) are defined by (4.8) and (4.15), respectively.

By the dominated convergence theorem, we have

\[
\lim_{N \to \infty} \mathbb{E} \int_0^T \int_D \int_{-N}^N (f_1^0 \tilde{f}_2^0 + \tilde{f}_1^0 f_2^0) w(x) d\xi dx = \mathbb{E} \int_0^T \int_D \int_{-N}^N (f_1^0 \tilde{f}_2^0 + \tilde{f}_1^0 f_2^0) w(x) d\xi dx.
\]

(5.19)

Moreover, employing a similar method as in the proof of (4.25) with \( f_1 = I_{u(t; \vartheta) > \xi} \) and \( f_2 = I_{u_n(t; \vartheta) > \xi} \), by (5.2), we get

\[
\lim_{N \to \infty} \mathbb{E} \int_0^T \int_D \int_{-N}^N (f_1^\pm(t) \tilde{f}_2^\pm(t) + \tilde{f}_1^\pm(t) f_2^\pm(t)) w(x) d\xi dx dt
\]

\[
= \mathbb{E} \int_0^T \int_D \int_0^T (f_1^\pm(t) \tilde{f}_2^\pm(t) + \tilde{f}_1^\pm(t) f_2^\pm(t)) w(x) d\xi dx dt \text{ uniformly on } n.
\]

(5.20)

Taking into account (5.19), (5.20) and \( \limsup_{N \to \infty} I_N = 0 \), we know that for any \( t > 0 \), there exists a big enough constant \( N_0 \) independent of \( \gamma, \delta, n \) such that

\[
\mathbb{E} \int_0^T \int_D \int_{-N_0}^N (f_1^\pm(t) \tilde{f}_2^\pm(t) + \tilde{f}_1^\pm(t) f_2^\pm(t)) w(x) d\xi dx dt \\
\leq \mathbb{E} \int_0^T \int_D \int_{-N_0}^N (f_1^\pm(t) \tilde{f}_2^\pm(t) + \tilde{f}_1^\pm(t) f_2^\pm(t)) w(x) d\xi dx dt + t
\]
\[
\leq T \mathbb{E} \int_D \int_{\mathbb{R}} \left( f_{1,0} \tilde{f}_{2,0} + \bar{f}_{1,0} f_{2,0} \right) w(x) d\xi dx + (T + 1)t \\
+ \lvert a(0) \rvert \mathbb{E} \int_0^T \int_0^t \int_{\partial D} \int_{\mathbb{R}} \left( f_{1,b} \tilde{f}_{2,b} + \bar{f}_{1,b} f_{2,b} \right) w(x) d\xi d\sigma(\bar{x}) ds dt \\
+ CTM(\gamma^2 \delta^{-1} + \delta) + CMTn^{-1} \gamma^{-2} + CT^2 n^{-1} \gamma^{-2} \delta \\
+ Cn^{-1} LMT + \sum_{i=0}^M \mathcal{E}_0^{N_0,\lambda_i}(\gamma, \delta) T \\
+ \sum_{i=0}^M \left( \int_0^T r_i^{N_0,\lambda_i}(\gamma, \delta) dt + \int_0^T \mathcal{E}_t^{N_0,\lambda_i}(\gamma, \delta) dt \right). \]

Taking \( \delta = \gamma^{\frac{4}{5}}, \gamma = n^{-\frac{1}{3}} \) and letting \( n \to \infty \) (in this case, \( \gamma, \delta \to 0 \)), we deduce from (4.13)–(4.14) and (4.17) that

\[
\lim_{n \to \infty} \mathbb{E} \int_0^T \int_D \int_{\mathbb{R}} \left( f_{1,0} \tilde{f}_{2,0} + \bar{f}_{1,0} f_{2,0} \right) w(x) d\xi dx + \lvert a(0) \rvert \\
\leq T \mathbb{E} \int_D \int_{\mathbb{R}} \left( f_{1,0} \tilde{f}_{2,0} + \bar{f}_{1,0} f_{2,0} \right) w(x) d\xi dx + \lvert a(0) \rvert \\
\mathbb{E} \int_0^T \int_0^t \int_{\partial D} \int_{\mathbb{R}} \left( f_{1,b} \tilde{f}_{2,b} + \bar{f}_{1,b} f_{2,b} \right) w(x) d\xi d\sigma(\bar{x}) ds dt,
\]

where we have used the arbitrary of \( \iota \).

Note that \( f_1 = I_{u(t; \vartheta) > \xi} \) and \( f_2 = I_{u^n(t; \vartheta) > \xi} \) with the corresponding data \( f_{1,0} = I_{\vartheta > \xi}, f_{2,0} = I_{\vartheta > \xi} \) and \( f_1^b = I_{0 > \xi}, f_2^b = I_{0 > \xi} \). Applying (4.27), we get

\[
\lim_{n \to \infty} \mathbb{E} \int_0^T \int_D \left| u(t; \vartheta) - u^n(t; \vartheta) \right| w(x) dx dt = 0.
\]

Now, we are in a position to give the proof of Theorem 3.2.

**Proof of Theorem 3.2** For \( \vartheta \in L_0^\infty L_\infty^\infty \), under Hypotheses (H1) and (H3), we deduce from Proposition 5.1 that there exists a subset \( \mathcal{I} \subset [0, T] \) with \( |\mathcal{I}| = T \) and a (non-relabelled) subsequence such that

\[
\lim_{n \to \infty} \mathbb{E} \| u(t; \vartheta) - u^n(t; \vartheta) \|_{L_{w,x}^\infty} = 0, \quad \text{forevery} \, t \in \mathcal{I}. \tag{5.21}
\]

In the following, we will prove that for any \( \tau > 0 \), there exists \( \varsigma > 0 \) such that

\[
\mathbb{E} \| u^n(t; \vartheta) - u^n(t'; \vartheta) \|_{L_{w,x}^\infty} < \tau, \tag{5.22}
\]

for every \( n \geq 1 \) and \( t > t' \in \mathcal{I} \) with \( |t - t'| < \varsigma \). For simplicity, we write \( u^n(t, x; \vartheta) = u^n(t, x) \) and \( u^n(t', x; \vartheta) = u^n(t', x) \).

Noting that \( \rho_{\gamma}(y - x) = 0 \) on \( D^{\lambda}_x \times D^c \), we have

\[
\mathbb{E} \int_D \left| u^n(t, x) - u^n(t', x) \right| w(x) dx
\]
Letting $f_1(t') := f_1(t', x, \xi) = I_{u^n(t', x) > \xi}$, $f_2(t') := f_2(t', y, \xi) = I_{u^n(t', y) > \xi}$ with $f_{1,0} = I_{\theta > \xi}$, $f_{2,0} = I_{\theta > \xi}$, it follows that

$$K^n_2(\gamma) = \sum_{i=0}^{M} \mathbb{E} \int_{D_i^x} \int_{D_y} \int_{\mathbb{R}} (f_1(t', x, \xi) \tilde{f}_2(t', y, \xi) + \tilde{f}_1(t', x, \xi) f_2(t', y, \xi)) \lambda_i(x) w(x) \rho^{\gamma_i}(y - x) d\xi dy dx.$$

Define

$$\tilde{K}^{n,N}_2(\gamma) := \sum_{i=0}^{M} \mathbb{E} \int_{D_i^x} \int_{D_y} \int_{-N}^{N} (f_1(t', x, \xi) \tilde{f}_2(t', y, \xi) + \tilde{f}_1(t', x, \xi) f_2(t', y, \xi)) \lambda_i(x) w(x) \rho^{\gamma_i}(y - x) d\xi dy dx,$$

then, employing a similar method as in the proof of (4.25) with $f_1(t')$, $f_2(t')$, and by utilizing (5.2), it follows that

$$K^n_2(\gamma) = \lim_{N \to \infty} \tilde{K}^{n,N}_2(\gamma) \text{ uniformly on } n. \quad (5.23)$$

Applying (4.11), we get for any $N$,

$$\tilde{K}^{n,N}_2(\gamma) \leq C \delta + 2M \Upsilon^{N}(\delta) + \sum_{i=0}^{M} \mathbb{E} \int_{D_i^x} \int_{D_y} \int_{-N}^{N} \int_{-N}^{N} (f_1(t') \tilde{f}_2(t') + \tilde{f}_1(t') f_2(t')) \lambda_i(x) w(x) \rho^{\gamma_i}(y - x) \psi_\delta(\xi - \zeta) d\xi d\zeta dy dx$$

$$=: C \delta + 2M \Upsilon^{N}(\delta) + J^{n,N}(\gamma, \delta), \quad (5.24)$$

where $\lim_{\delta \to 0} \Upsilon^{N}(\delta) = 0$.

Applying Lemma 5.1 with $\vartheta_1 = \vartheta_2 = \vartheta$, by $\tilde{L}^{N,\lambda}_2 \leq 0$, it gives

$$J^{n,N}(\gamma, \delta)$$
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\[ \int_{-N}^{N} (f_{1,0} \tilde{f}_{2,0} + \tilde{f}_{1,0} f_{2,0}) \lambda_i(x) w(x) d\xi dx + \mathcal{E}_0^{N,\lambda_i}(\gamma, \delta) \]

\[ + \mathbb{E} \int_{0}^{t'} \int_{D_x}^{t-N} (f^+_1 b \tilde{f}^+_2 b + \tilde{f}^+_1 b f^+_2 b) \mathbf{n}(\tilde{x}) \cdot \nabla_x (\lambda_i w) d\xi d\sigma(\tilde{x}) ds \]

\[ - a(\xi) \cdot \mathbf{n} \lambda_i(x) w(x) d\xi d\sigma(\tilde{x}) ds \]

Further, in view of (4.19), we reach

\[ J^{n,N}(\gamma, \delta) \leq \int_{D} \int_{-N}^{N} (f_{1,0} \tilde{f}_{2,0} + \tilde{f}_{1,0} f_{2,0}) w(x) d\xi dx + \sum_{i=0}^{M} \mathcal{E}_0^{N,\lambda_i}(\gamma, \delta) + \Lambda_i^{n,N}(\gamma, \delta) \]

\[ + \sum_{i=0}^{M} r_{i}^{N,\lambda_i}(\gamma, \delta) + \sum_{i=0}^{M} \tilde{r}_{i}^{N,\lambda_i}(\gamma, \delta) + \sum_{i=0}^{M} \tilde{r}_{i}^{N,\lambda_i}(\gamma, \delta) + \sum_{i=0}^{M} \tilde{r}_{i}^{N,\lambda_i}(\gamma, \delta) \]

\[ + CM\delta^{r-1} + CM(\gamma^{2}\delta^{r-1} + \delta) + \sum_{i=0}^{M} (\tilde{L}_{i}^{N,\lambda_i} + I_N), \quad (5.25) \]

where

\[ \Lambda_i^{n,N}(\gamma, \delta) = |a(0)| \mathbb{E} \int_{0}^{t'} \int_{D_D}^{t-N} (f_{1,0} \tilde{f}_{2,0} + \tilde{f}_{1,0} f_{2,0}) w(x) d\xi ds \]

Since \( \int_{D} (f^+_1 b \tilde{f}^+_2 b + \tilde{f}^+_1 b f^+_2 b) d\xi = 0 \), by the dominated convergence theorem, we have

\[ \lim_{N \to \infty} |\Lambda_i^{n,N}(\gamma, \delta)| = 0, \quad (5.26) \]
for each \( n \) and \( \gamma, \delta > 0 \).

Taking into account that \( \lim_{N \to \infty} \sum_{i=0}^{M} I_N = 0 \), (5.23), (5.26) and

\[
\lim_{N \to \infty} \mathbb{E} \int_{-N}^{N} \int_{-N}^{N} (f_{1,0} \bar{f}_{2,0} + \bar{f}_{1,0} f_{2,0}) w(x) d\xi dx = \mathbb{E} \int_{D} \int_{-N}^{N} (f_{1,0} \bar{f}_{2,0} + \bar{f}_{1,0} f_{2,0}) w(x) d\xi dx = 0,
\]

for any \( \epsilon > 0 \), there exists a big enough constant \( N_0 \) independent of \( \gamma, \delta, n \) such that

\[
|K^{n}_{2}(\gamma) - \tilde{K}^{n,N_0}_{2}(\gamma)| + |\Lambda^{n,N_0}_{t}(\gamma, \delta)|
\]

\[
+ \mathbb{E} \int_{D} \int_{-N_0}^{N_0} (f_{1,0} \bar{f}_{2,0} + \bar{f}_{1,0} f_{2,0}) w(x) d\xi dx \sum_{i=0}^{M} I_N_0 < \epsilon.
\]

Then, in view of (5.24) and (5.25), one obtains

\[
K^{n}_{2}(\gamma) \leq \epsilon + \tilde{K}^{n,N_0}_{2}(\gamma) - |\Lambda^{n,N_0}_{t}(\gamma, \delta)|
\]

\[
- \mathbb{E} \int_{D} \int_{-N_0}^{N_0} (f_{1,0} \bar{f}_{2,0} + \bar{f}_{1,0} f_{2,0}) w(x) d\xi dx - \sum_{i=0}^{M} I_N_0
\]

\[
\leq \epsilon + C \delta + 2M \gamma N_0(\delta) + J^{n,N_0}(\gamma, \delta) - |\Lambda^{n,N_0}_{t}(\gamma, \delta)|
\]

\[
- \mathbb{E} \int_{D} \int_{-N_0}^{N_0} (f_{1,0} \bar{f}_{2,0} + \bar{f}_{1,0} f_{2,0}) w(x) d\xi dx - \sum_{i=0}^{M} I_N_0
\]

\[
\leq \epsilon + C \delta + 2M \gamma N_0(\delta) + \sum_{i=0}^{M} \xi^{N_0,\lambda_i}(\gamma, \delta) + \sum_{i=0}^{M} \tilde{r}^{N_0,\lambda_i}(\gamma, \delta) + \sum_{i=0}^{M} \tilde{r}^{N_0,\lambda_i}(\gamma, \delta)
\]

\[
+ \sum_{i=0}^{M} \bar{r}^{N_0,\lambda_i}(\gamma, \delta) + \sum_{i=0}^{M} \bar{r}^{N_0,\lambda_i}(\gamma, \delta) + CM \delta \gamma^{-1}
\]

\[
+ CM(\gamma^2 \delta^{-1} + \delta) + \sum_{i=0}^{M} \tilde{L}^{N_0,\lambda_i}.
\]  

From (4.13), (4.16), (5.12), (5.14), (5.16) and (4.22), we have

\[
\lim_{\gamma, \delta \to 0} \sum_{i=0}^{M} \xi^{N_0,\lambda_i}(\gamma, \delta) = 0, \quad \lim_{\gamma, \delta \to 0} \sum_{i=0}^{M} \tilde{r}^{N_0,\lambda_i}(\gamma, \delta) = 0, \quad \lim_{\gamma, \delta \to 0} \sum_{i=0}^{M} \tilde{r}^{N_0,\lambda_i}(\gamma, \delta) = 0,
\]

\[
\lim_{\gamma, \delta \to 0} \sum_{i=0}^{M} \bar{r}^{N_0,\lambda_i}(\gamma, \delta) = 0, \quad \lim_{\gamma, \delta \to 0} \sum_{i=0}^{M} \tilde{r}^{N_0,\lambda_i}(\gamma, \delta) = 0,
\]

\[
\lim_{\gamma, \delta \to 0} \sum_{i=0}^{M} \tilde{L}^{N_0,\lambda_i} = \lim_{\gamma, \delta \to 0} \sum_{i=0}^{M} J_3 = 0.
\]
Taking $\delta = \gamma^{\frac{4}{5}}$ independent of $n$, and letting $\gamma \to 0$ in (5.27), we get

$$\lim_{\gamma, \delta \to 0} K^n_2(\gamma) \leq \iota.$$  

Since $\iota$ is arbitrary, we deduce that

$$\lim_{\gamma \to 0} K^n_2(\gamma) = 0 \quad \text{uniformly on } n. \quad (5.28)$$

Now, we focus on the estimates of $K^n_1(\gamma)$. Let $\eta_\varepsilon$ be a symmetric approximation of $| \cdot |$ given by

$$\eta_\varepsilon(0) = \eta_\varepsilon'(0) = 0, \quad \eta_\varepsilon''(r) = \varepsilon^{-1} \tilde{\eta}(\varepsilon^{-1}|r|)$$

for some non-negative function $\tilde{\eta} \in C^\infty(\mathbb{R})$ which is bounded by 2, supported in $(0, 1)$ and integrates to 1. Moreover, the following properties of $\eta_\varepsilon$ hold:

$$|\eta_\varepsilon(r) - |r|| \lesssim \varepsilon, \quad \text{supp } \eta_\varepsilon'' \subset [-\varepsilon, \varepsilon], \quad |\eta_\varepsilon''(r)| \leq 2\varepsilon^{-1}, \quad \eta_\varepsilon'(r) \in [0, \frac{1}{2}]. \quad (5.29)$$

This implies that

$$K^n_1(\gamma) \lesssim \varepsilon \|w\|_{L^1_\gamma} + \sum_{i=0}^M \mathbb{E} \int_{D_y} \int_{D_x^i} \eta_\varepsilon(u^n(t', s), x)$$

$$u^n(t', y)) \lambda_i(x) w(x) \rho^{x_i}_{\gamma}(y - x) dy dx =: \varepsilon \|w\|_{L^1_\gamma} + \tilde{K}^n_1(\gamma).$$

Under Hypotheses (H1) and (H3), the generalized Itô formula from Proposition A.1 in [10] yields

$$\tilde{K}^n_1(\gamma) = \sum_{i=0}^M \mathbb{E} \int_{D_y} \int_{D_x^i} \eta_\varepsilon(u^n(t', s), x)$$

$$- u^n(t', y)) \lambda_i(x) w(x) \rho^{x_i}_{\gamma}(y - x) dy ds dy$$

$$- \frac{1}{n} \sum_{i=0}^M \mathbb{E} \int_{D_y} \int_{D_x^i} \eta_\varepsilon'(u^n(s, x))$$

$$- u^n(t', y)) \lambda_i(x) w(x) \rho^{x_i}_{\gamma}(y - x) dy ds dy$$

$$- \frac{1}{n} \sum_{i=0}^M \mathbb{E} \int_{D_y} \int_{D_x^i} \eta_\varepsilon'(u^n(s, x))$$

$$- u^n(t', y)) \lambda_i(x) w(x) \rho^{x_i}_{\gamma}(y - x) dy ds dy$$

$$- \sum_{i=0}^M \mathbb{E} \int_{D_y} \int_{D_x^i} \eta_\varepsilon'(u^n(s, x))$$

$$\cdot \Delta u^n(s, x) \lambda_i(x) w(x) \rho^{x_i}_{\gamma}(y - x) dy ds dy$$

$$+ \frac{1}{2} \sum_{i=0}^M \mathbb{E} \int_{D_y} \int_{D_x^i} \eta_\varepsilon''(u^n(s, x) - u^n(t', y))$$

$$G^2(x, u^n(s, x)) \lambda_i(x) w(x) \rho^{x_i}_{\gamma}(y - x) dy ds dy$$

$$=: \sum_{i=1}^5 I_i.$$
Clearly, \( I_2 \leq 0 \). By (5.29), we have

\[
I_1 \lesssim \varepsilon \|w\|_{L^1_x} + \sum_{i=0}^{M} \mathbb{E} \int_{D_x} \int_{D_y} \|u^n(t', x) - u^n(t', y)\|_{\lambda_i(x)w(x)\rho_{\nu_i}^{\lambda_i} (y - x)dydx} \leq \varepsilon \|w\|_{L^1_x} + K^n_2(\gamma).
\]

Using (5.2), we deduce that

\[
I_3 \leq \frac{1}{n} \|w\|_{W^{1,\infty}_x} (t - t')^{\frac{1}{2}} \mathbb{E} \left[ \int_{t'}^t \|\nabla u^n\|_{L^2_x}^2 ds \right]^{\frac{1}{2}} \leq \gamma^{-1} \|w\|_{W^{1,\infty}_x} (t - t')^{\frac{1}{2}} (1 + \mathbb{E} \|\vartheta\|_{L^2_x}^2).
\]

Letting

\[
H(\xi) = \int_0^\xi \eta_g'(\zeta - u^n(t', y))a(\zeta)d\zeta,
\]

by integration by parts formula and (5.2), we have

\[
I_4 = -\sum_{i=0}^{M} \mathbb{E} \int_{t'}^t \int_{D_x} \int_{D_y} \langle \nabla \cdot H(u^n(s, x)) \lambda_i(x)w(x)\rho_{\nu_i}^{\lambda_i} (y - x)dydxds = \sum_{i=0}^{M} \mathbb{E} \int_{t'}^t \int_{D_x} \int_{D_y} H(u^n(s, x)) \cdot \nabla (\lambda_i(x)w(x)\rho_{\nu_i}^{\lambda_i} (y - x))dydxds \lesssim \gamma^{-1} \varepsilon^{-1} \|w\|_{W^{1,\infty}_x} \mathbb{E} \int_{t'}^t \int_{D_x} (1 + \|u^n(s, x)\|_{L^{q_0+1}_x})dxds \lesssim \gamma^{-1} \varepsilon^{-1} \|w\|_{W^{1,\infty}_x} (t - t') \left[ 1 + \mathbb{E} \|\vartheta\|_{L^{q_0+1}_x}^2 \right].
\]

By Hypothesis (H3) and (5.2), it follows that

\[
I_5 \lesssim \varepsilon^{-1} \gamma^{-1} \|w\|_{L^\infty_x} (t - t') \mathbb{E} \sup_{t \in [0, T]} (1 + \|u^n(t)\|_{L^2_x}^2) \lesssim \varepsilon^{-1} \gamma^{-1} (t - t') (1 + \mathbb{E} \|\vartheta\|_{L^2_x}^2).
\]

Combining all the above estimates, we get

\[
K^n_1(\gamma) \leq 2\varepsilon \|w\|_{L^\infty} + K^n_2(\gamma) + \gamma^{-1} \|w\|_{W^{1,\infty}_x} (t - t')^{\frac{1}{2}} (1 + \mathbb{E} \|\vartheta\|_{L^2_x}^2) + \gamma^{-1} \varepsilon^{-1} \|w\|_{W^{1,\infty}_x} (t - t') \left[ 1 + \mathbb{E} \|\vartheta\|_{L^{q_0+1}_x}^2 \right] + \varepsilon^{-1} \gamma^{-1} (t - t') \mathbb{E} (1 + \|\vartheta\|_{L^2_x}^2).
\]

Thus, we conclude that

\[
\mathbb{E} \int_D |u^n(t, x) - u^n(t', x)|w(x)dx \leq 2\varepsilon \|w\|_{L^\infty} + 2K^n_2(\gamma) + \gamma^{-1} \|w\|_{W^{1,\infty}_x} (t - t')^{\frac{1}{2}} (1 + \mathbb{E} \|\vartheta\|_{L^2_x}^2) + \gamma^{-1} \varepsilon^{-1} \|w\|_{W^{1,\infty}_x} (t - t') \left[ 1 + \mathbb{E} \|\vartheta\|_{L^{q_0+1}_x}^2 \right] + \varepsilon^{-1} \gamma^{-1} (t - t') \mathbb{E} (1 + \|\vartheta\|_{L^2_x}^2).
\]
\[ + \gamma^{-1} \varepsilon^{-1} \|w\|_{W^1, \infty}(t - t') \left[ 1 + E\|\tilde{\vartheta}\|^{q_0 + 1}_{L^{q_0 + 1}} \right] + \varepsilon^{-1} \gamma^{-1}(t - t') E(1 + \|\vartheta\|^2_{L^2_x}). \]

Due to (5.28), for any \( \tau > 0 \), there exists a small positive constant \( \gamma_0 \) independent of \( n \) such that \( K^n_2(\gamma_0) < \frac{\tau}{2} \). For such \( \tau \), we can choose small positive constants \( \varepsilon_0 \) and \( \zeta \) independent of \( n \) such that for any \( t, t' \in I \) with \( |t - t'| < \zeta \) such that

\[ 2\varepsilon_0 \|w\|_{L^\infty} + \gamma_0^{-1} \|w\|_{W^{1, \infty}} \zeta \frac{1}{2} (1 + E\|\vartheta\|^2_{L^2_x}) + \gamma_0^{-1} \varepsilon_0^{-1} \|w\|_{W^{1, \infty}} \zeta \left[ 1 + E\|\tilde{\vartheta}\|^{q_0 + 1}_{L^{q_0 + 1}} \right] + \varepsilon_0^{-1} \gamma_0^{-1} \zeta E(1 + \|\vartheta\|^2_{L^2_x}) < \frac{\tau}{2}. \]

Thus, for any \( \tau > 0 \), there exists \( \zeta > 0 \) independent of \( n \) such that for every \( t, t' \in I \) with \( |t - t'| < \zeta \),

\[ E\|u^0(t; \vartheta) - u^0(t'; \vartheta)\|_{L^1_{w; x}} < \tau, \]

which is the desired result (5.22). Taking \( n \to \infty \) on (5.22), by (5.21), we get

\[ E\|u(t; \vartheta) - u(t'; \vartheta)\|_{L^1_{w; x}} \leq \tau. \]

As a result, \( u : I \to L^1_{w; x} L^1_{w; x} \) is uniformly continuous, hence it has a unique continuous extension on \([0, T] \). \( \square \)

**Proof of Proposition 3.3** For any \( \vartheta \in L^2_\omega L^1_{w; x} \), there exists a sequence \( \{\vartheta_n\}_{n \geq 1} \subset L^\infty_\omega L^\infty_x \) such that \( \vartheta_n \to \vartheta \) in \( L^2_\omega L^1_{w; x} \) (for example, let \( \vartheta_n = \max\{-n, \min\{\vartheta, n\}\} \)). From Theorem 3.2, we have \( u(\cdot; \vartheta_n) \in C([0, T]; L^1_\omega L^1_{w; x}) \) for each \( n \geq 1 \). Furthermore, we deduce from Theorem 3.1 that for any \( m > n \geq 1 \),

\[ \sup_{t \in [0, T]} E\|u(t; \vartheta_m) - u(t; \vartheta_n)\|_{L^1_{w; x}} \leq E\|\vartheta_m - \vartheta_n\|_{L^1_{w; x}}. \quad (5.30) \]

Thus, \( \{u(\cdot; \vartheta_n)\}_{n \geq 1} \) is a Cauchy sequence in \( C([0, T]; L^1_\omega L^1_{w; x}) \), which yields that the limit \( v(\cdot; \vartheta) := \lim_{n \to \infty} u(\cdot; \vartheta_n) \) exists in \( C([0, T]; L^1_\omega L^1_{w; x}) \). Moreover, using Theorem 3.1, we see that the limit \( v(\cdot; \vartheta) \) is independent of the choices of \( \{\vartheta_n\}_{n \geq 1} \). Clearly, for \( \vartheta \in L^\infty_\omega L^\infty_x \), we have \( v(\cdot; \tilde{\vartheta}) = u(\cdot; \vartheta) \) in \( C([0, T]; L^1_\omega L^1_{w; x}) \). Thus, \( v \) is the unique continuous extension of \( u \) on \( C([0, T]; L^1_\omega L^1_{w; x}) \). Finally, for \( \vartheta, \tilde{\vartheta} \in L^2_\omega L^1_{w; x} \), choosing appropriate approximating sequences \( \vartheta_n, \tilde{\vartheta}_n \) and passing to the limits in (5.30), we deduce that

\[ \sup_{t \in [0, T]} E\|v(t; \vartheta) - v(t; \tilde{\vartheta})\|_{L^1_{w; x}} \leq E\|\vartheta - \tilde{\vartheta}\|_{L^1_{w; x}}, \]

which is the desired result (3.2). \( \square \)
6. Ergodicity

In this section, we will prove the main result Theorem 3.6. First, we obtain a polynomial decay for the difference of kinetic solutions of (1.1)–(1.3) with different initial conditions.

**Proposition 6.1** Assume Hypotheses (H1) and (H3) are in force. Let \( u(t; \vartheta) \) and \( u(t; \tilde{\vartheta}) \) be the kinetic solutions of (1.1)–(1.3) with initial data \( \vartheta, \tilde{\vartheta} \in L^\infty_\omega L^\infty_x \). Then for all \( t \geq 0 \),

\[
\sup_{\vartheta, \tilde{\vartheta} \in L^\infty_\omega L^\infty_x} \mathbb{E} \| u(t; \vartheta) - u(t; \tilde{\vartheta}) \|_{L^{1}_{w;x}} \leq C_{q_0} \| w \|_{L^{q_0}_{x}}^{q^{*}} t^{-\frac{1}{q_0}},
\]

where the constant \( C_{q_0} \) depends only on \( q_0 \) and \( q^{*} = \frac{q_0+1}{q_0} \).

**Proof** By (4.28), Theorem 3.2 and using Hypothesis (H1), we deduce that for every \( 0 \leq s < t \leq T \),

\[
\mathbb{E} \| u(t; \vartheta) - u(t; \tilde{\vartheta}) \|_{L^{1}_{w;x}} \leq \mathbb{E} \| u(s; \vartheta) - u(s; \tilde{\vartheta}) \|_{L^{1}_{w;x}}
- C_{q_0} \mathbb{E} \int_{s}^{t} \| u(r; \vartheta) - u(r; \tilde{\vartheta}) \|_{L^{q_0+1}_{x}}^{q_0+1} dr,
\]

where the constant \( C_{q_0} \) depends only on \( q_0 \). Further, by Hölder inequality, it follows that

\[
\mathbb{E} \| u(r; \vartheta) - u(r; \tilde{\vartheta}) \|_{L^{1}_{w;x}} \leq \left( \mathbb{E} \| u(r; \vartheta) - u(r; \tilde{\vartheta}) \|_{L^{q_0+1}_{x}}^{q_0+1} \right)^{\frac{1}{q_0+1}} \| w \|_{L^{q^{*}}_{x}},
\]

where \( q^{*} = \frac{q_0+1}{q_0} \). Thus, we deduce that for any \( 0 \leq s < t \leq T \),

\[
\mathbb{E} \| u(t; \vartheta) - u(t; \tilde{\vartheta}) \|_{L^{1}_{w;x}} \leq \left( \mathbb{E} \| u(s; \vartheta) - u(s; \tilde{\vartheta}) \|_{L^{1}_{w;x}} \right)^{\frac{1}{q_0+1}} \int_{s}^{t} \mathbb{E} \| u(r; \vartheta) - u(r; \tilde{\vartheta}) \|_{L^{1}_{w;x}}^{q_0+1} dr.
\]

(6.2)

Set \( f(t) := \mathbb{E} \| u(t; \vartheta) - u(t; \tilde{\vartheta}) \|_{L^{1}_{w;x}} \). Due to Theorem 3.2, \( f(t) \) is continuous on \([0, T]\). (6.2) yields

\[
f(t) - f(s) \leq -C_{q_0} \| w \|_{L^{q^{*}}_{x}}^{-(q_0+1)} \int_{s}^{t} f^{q_0+1}(r) dr,
\]

for all \( 0 \leq s < t \leq T \). Applying Lemma B.2 in [9] to (6.3) and proceeding as the proof of Theorem 3.8 in [9], we obtain

\[
\mathbb{E} \| u(t; \vartheta) - u(t; \tilde{\vartheta}) \|_{L^{1}_{w;x}} \leq C_{q_0} \| w \|_{L^{q^{*}}_{x}}^{q^{*}} t^{-\frac{1}{q_0}}.
\]

\( \square \)

As a consequence of (3.2), we deduce the following result.
\textbf{Corollary 6.2} For the unique continuous extension \( v \) of \( u \) given by Proposition 3.3, for any \( t > 0 \),

\[
\sup_{\vartheta, \tilde{\vartheta} \in L_{w,x}^q} \mathbb{E} \| v(t; \vartheta) - v(t; \tilde{\vartheta}) \|_{L_{W,x}^1} \leq C_{q_0} \| w \|_{L_{w,x}^{q_0}} \cdot t^{-\frac{1}{q_0}}, \tag{6.4}
\]

where the constant \( C_{q_0} \) depends only on \( q_0 \).

For technical reasons, we extend the time horizon to \(-\infty\). We need the following notations. For \( \vartheta \in L_{x_1}^q \) for all \( q \geq 1 \) and arbitrary \( s > -\infty \), we denote by \( u_s(t; \vartheta) := u_s(t, x; \vartheta) \) the kinetic solution of

\[
\begin{aligned}
\partial_t u_s(t, x; \vartheta) &= \text{div} A(u_s(t, x; \vartheta)) + \sum_{k \geq 1} \sigma^k(x, u_s(t, x; \vartheta)) \beta_k(t), \\
u_s(s, \cdot; \vartheta) &= \vartheta, \\
u_s(s, \cdot; \vartheta) &= 0 \quad \text{on } \Omega \times \Sigma,
\end{aligned}
\tag{6.5}
\]

for \( t \geq s \), where we have extended \( \beta_k(t) \) for \( t < 0 \) by gluing at \( t = 0 \) an independent Brownian motion evolving backwards in time. According to this new notation, \( u_0(\cdot; \vartheta) = u(\cdot; \vartheta) \). The global well-posedness of (6.5) for the case \( s \neq 0 \) can be obtained analogously as the case \( s = 0 \). In addition, as stated in Theorem 3.2, under Hypotheses (H1) and (H4), the mapping

\[
L_{x_1}^\omega L_{x_1}^\infty \ni \vartheta \mapsto u_s(\cdot; \vartheta) \in C([s, \infty); L_{x_1}^1 L_{w,x}^1),
\]

has its unique continuous extension \( v_s(\cdot; \vartheta) \) from \( L_{x_1}^2 L_{w,x}^1 \) to \( C([s, \infty); L_{x_1}^1 L_{w,x}^1) \), which is the unique renormalized kinetic solution to (6.5).

For the renormalized kinetic solution \( v_s(\cdot; \vartheta) \) to (6.5), it admits the following relation.

\textbf{Proposition 6.3} Assume Hypotheses (H1) and (H4) are in force. For every \( \vartheta \in L_{x_1}^1 \) and \(-\infty < s_1 \leq s_2 \leq t \leq T \), it holds true that \( v_{s_1}(t; \vartheta) = v_{s_2}(t; \vartheta) \) in \( L_{x_1}^1 L_{w,x}^1 \).

\textbf{Proof} For simplicity, we prove this relation for \((s_1, s_2) = (0, s)\) for some \( s > 0 \). Let \( \vartheta \in L_{x_1}^1 \) and \( v(t; \vartheta) = v(t, x; \vartheta) := v_0(t, x; \vartheta) \) be the renormalized kinetic solution of (6.5) satisfying (2.22)–(2.23) on \([0, T]\). One can prove that \( v(s; \vartheta) \in L_{x_1}^2 L_{w,x}^1 \). In fact, by (2.21), we have

\[
\text{ess sup}_{0 \leq t \leq T} \mathbb{E} \| v(t; \vartheta) \|_{L_{x_1}^1}^2 \leq C, \tag{6.6}
\]

then, taking into account the fact that \( v(\cdot; \vartheta) \in C([0, T]; L_{x_1}^1 L_{w,x}^1) \), there exists a sequence \( s_n \to s \) such that \( v(s_n, x; \vartheta) \to v(s, x; \vartheta) \) for almost every \((\omega, x)\) and \( \sup_{n \geq 1} \mathbb{E} \| v(s_n, \vartheta) \|_{L_{x_1}^1}^2 \leq C \). As a result of Fatou’s lemma, we get

\[
\mathbb{E} \| v(s; \vartheta) \|_{L_{x_1}^1}^2 \leq \liminf_{n \to \infty} \mathbb{E} \| v(s_n, \vartheta) \|_{L_{x_1}^1}^2 \leq \sup_{n \geq 1} \mathbb{E} \| v(s_n, \vartheta) \|_{L_{x_1}^1}^2 \leq C,
\]

which implies \( v(s; \vartheta) \in L_{x_1}^2 L_{w,x}^1 \).

On the other hand, let \( \phi \in C_c^\infty([s, T) \times \overline{D} \times (-N, N)) \) which can be extended continuously on \([0, s]\) (taking the constant value \( \phi(s) \)) and choose a smooth sequence
\{\chi_\kappa\}_{\kappa > 0} \subset C^\infty_c((s, T)) \) approximating to \( I_{[s, T]} \) as \( \kappa \to \infty \). Substituting \( \varphi \) in (2.22)–(2.23) with \( \chi_\kappa(t) \phi \) and letting \( \kappa \to \infty \) (passing to a suitable subsequence), with the aid of Proposition 3.3 and the fact that \( \nu(s; \vartheta) \in L^2_\omega L^1_x \), it follows that \( f = I_{\nu>\xi} \) satisfies
\[
\int_s^T \langle f(t), \partial_t \phi \rangle dt + \langle f_s, \phi(s) \rangle + \int_s^T \langle f(t), a(\xi) \cdot \nabla \phi \rangle dt
+ MN \int_{\partial D \times [s, T] \times \mathbb{R}} f_\beta \phi(t) d\xi d\sigma(\tilde{x}) dt
= -\sum_{k \geq 1} \int_s^T \int_D g_k(x, u(t, x)) \phi(t, x, u(t, x)) dx d\beta_k(t)
- \frac{1}{2} \int_s^T \int_D \partial_x \phi(t, x, u(t, x)) G^2(x, u(t, x)) dx dt
+ \int_{[s, T] \times D \times \mathbb{R}} \partial_x \phi(t, x, u(t, x)) d\sigma(t, x)
+ \int_{\partial D \times [s, T] \times \mathbb{R}} \partial_x \phi(t, x, u(t, x)) d\sigma(t, x), \ a.s.,
\]
and \( \tilde{f} := 1 - f = I_{\nu \leq \xi} \) fulfills a similar equation as above. As a result, \( \nu(t; \vartheta) \) is a renormalized kinetic solution on \([s, T]\). Finally, by uniqueness of renormalized kinetic solutions and the continuity in \( L^1_\omega L^1_{w,x} \), we conclude that \( \nu(t; \vartheta) = \nu_s(t; \nu(s; \vartheta)) \) in \( L^1_\omega L^1_{w,x} \) for every \( t \in [s, T] \). For the case of \( s < 0 \), we also can obtain this relation by using similar method as above.

Proposition 3.5 says that the mappings \( P_t, t \geq 0 \) define a Feller semigroup.

**Proof of Proposition 3.5** After the preparations in Proposition 6.3, the proof now follows from standard arguments, see, e.g. Theorem 9.14 (or Theorem 9.8) in [8]. We omit the details.

Combining Propositions 6.3 and 3.5, we have identities \( P_{s,r} P_{r,t} = P_{s,t} \) for any \( s < r < t \), as well as \( P_{s,t} = P_{s+r,t+r} \) for every \( r \in \mathbb{R} \).

Now, we are in a position to prove the polynomial mixing of \( P_t \).

**Proof of Theorem 3.6** Let \( \vartheta \in L^1_x \), denote by \( \eta_s(\vartheta) = \nu_s(0; \vartheta) \) for arbitrary \( s < 0 \). By Proposition 6.3, for \( s_1 \leq s_2 \leq -1 \), it follows that \( \eta_{s_1}(\vartheta) = \nu_{s_2}(0; \nu_{s_1}(s_2; \vartheta)) \) in \( L^1_\omega L^1_{w,x} \). Hence,
\[
\eta_{s_2}(\vartheta) - \eta_{s_1}(\vartheta) = \nu_{s_2}(0; \vartheta) - \nu_{s_2}(0; \nu_{s_1}(s_2; \vartheta))
\]
in \( L^1_\omega L^1_{w,x} \). By Corollary 6.2, we have
\[
\mathbb{E} \| \eta_{s_2}(\vartheta) - \eta_{s_1}(\vartheta) \|_{L^1_{w,x}} \leq C q_0 \| u \|_{L^q_{w,x}} |s_2 - s_1|^{-\frac{1}{q_0'}} ,
\]
which implies that \( \eta_s(\vartheta) \) is a Cauchy sequence in \( L^1_\omega L^1_{w,x} \). Hence, there exists a random variable \( X(\vartheta) \in L^1_\omega L^1_{w,x} \) such that \( \eta_s(\vartheta) \to X(\vartheta) \) in \( L^1_\omega L^1_{w,x} \), as \( s \to -\infty \).
We claim that \( X(\vartheta) \) is independent of the initial data \( \vartheta \). Indeed, for any \( \vartheta, \tilde{\vartheta} \in L^1_{\omega} \), by Corollary 6.2, we have

\[
\mathbb{E}\|\eta_s(\vartheta) - \eta_s(\tilde{\vartheta})\|_{L^1_{\omega};x} \leq C_{q_0} \|w\|_{L^q_x}^*|s|^{-\frac{1}{40}}.
\]

Then letting \( s \to -\infty \), we have \( X(\vartheta) = X(\tilde{\vartheta}) \) in \( L^1_{\omega} L^1_{\omega;x} \).

Let \( X = X(0) \) and define \( \mu = \mathbb{P} \circ X^{-1} \in M_1(1_{\omega} L^1_{\omega;x}) \). Next, we verify that \( \mu \) is an invariant measure of \( P_t \). Denote by \( P_{s,t} \) the semigroup associated to (6.5) at time \( t \), then \( P_t = P_{0,t} \) for any \( t \geq 0 \). Keeping in mind that \( \eta_{-s}(0) \to X(0) \) in \( L^1_{\omega} L^1_{\omega;x} \), when \( s \to \infty \), it follows that

\[
\int_{L^1_{\omega;x}} P_{0,t} F(\xi) \mu(d\xi) = \mathbb{E} P_{0,t} F(X(0)) = \lim_{s \to \infty} \mathbb{E} P_{0,t} F(\eta_{-s}(0))
\]

for every \( F \in C_b(L^1_{\omega;x}) \), where we have used the Feller property of \( P_{0,t} \), \( P_{s,s} P_{r,t} = P_{s,t} \) for any \( s < r < t \), as well as \( P_{s,t} = P_{s+r,t+r} \) for every \( r \in \mathbb{R} \). (6.7) shows that \( P_{t*} \mu = \mu \) as measures on \( L^1_{\omega;x} \), hence, \( \mu \) is an invariant measure of \( P_t \) on \( L^1_{\omega;x} \).

From Corollary 6.2, for any \( F \in Lip(L^1_{\omega;x}) \) and \( \vartheta, \tilde{\vartheta} \in L^1_{\omega;x} \),

\[
|P_t F(\vartheta) - P_t F(\tilde{\vartheta})| \leq \|F\|_{Lip(L^1_{\omega;x})} \mathbb{E}\|v(t; \vartheta) - v(t; \tilde{\vartheta})\|_{L^1_{\omega;x}} \leq C_{q_0} \|F\|_{Lip(L^1_{\omega;x})} \|w\|_{L^q_x}^*|t|^{-\frac{1}{40}},
\]

which implies that any two invariant measures \( \mu \) and \( \tilde{\mu} \) on \( L^1_{\omega;x} \) coincide.

Finally, by utilizing (6.8), it follows that for all \( t > 0 \),

\[
\left| P_t F(\vartheta) - \int_{L^1_{\omega;x}} F(\xi) \mu(d\xi) \right| = |P_t F(\vartheta) - \int_{L^1_{\omega;x}} P_t F(\xi) \mu(d\xi)|
\]

\[
\leq \mathbb{E}|P_t F(\vartheta) - P_t F(X(0))| \leq C_{q_0} \|F\|_{Lip(L^1_{\omega;x})} \|w\|_{L^q_x}^*|t|^{-\frac{1}{40}}.
\]

Taking the supremum over \( \|F\|_{Lip(L^1_{\omega;x})} \leq 1 \) and \( \vartheta \in L^1_{\omega;x} \), we get the desired result.

\[\square\]

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