Non-oscillating solutions to uncoupled Ermakov systems and the semiclassical limit

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Abstract. The amplitude-phase formulation of the Schrödinger equation is investigated within the context of uncoupled Ermakov systems, whereby the amplitude function is given by the auxiliary nonlinear equation. The classical limit of the amplitude and phase functions is analyzed by setting up a semiclassical Ermakov system. In this limit, it is shown that classical quantities, such as the classical probability amplitude and the reduced action, are obtained only when the semiclassical amplitude and the accumulated phase are non-oscillating functions respectively of the space and energy variables. Conversely, among the infinitely many arbitrary exact quantum amplitude and phase functions corresponding to a given wavefunction, only the non-oscillating ones yield classical quantities in the limit $\hbar \to 0$.

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1. Introduction

Systems of the form
\[
\partial_t^2 u(t) + k^2(t)u(t) = \frac{1}{\rho u^2(t)} Y(\alpha(t)/u(t)) \tag{1}
\]
\[
\partial_t^2 \alpha(t) + k^2(t)\alpha(t) = \frac{1}{u\alpha^2(t)} Z(u(t)/\alpha(t)), \tag{2}
\]
where \(Y\) and \(Z\) are arbitrary functions of their arguments, are generically known as Ermakov systems. They are characterized by the existence of a first integral, the Ermakov (or Lewis-Ray-Reid) invariant linking the solutions of Eqs. (1) and (2), thereby giving rise to the so-called nonlinear superposition principle. Intensive studies of their properties, such as their linearization \cite{1} or their generalization to higher dimensions \cite{2, 3} have been undertaken to extend the remarkable results concerning uncoupled systems, ie \(Y(\zeta) = 0\) and \(Z(\zeta) = a^2\zeta\) where \(a\) is a constant, obtained by Ray and Reid \cite{6, 7}.

Although the paradigm in physical applications of uncoupled Ermakov systems has been the classical linear time-dependent harmonic oscillator (\(t\) being the time variable), where the full power of Hamiltonian structure \cite{4}, Lagrangian mechanics and Noether symmetries \cite{5} have been employed, it has also been remarked that uncoupled systems link the time-independent linear Schrödinger equation to a nonlinear 'auxiliary' equation in the following way:
\[
\hbar^2 \partial_x^2 u(x) + p^2(x)u(x) = 0 \tag{3}
\]
\[
\hbar^2 \partial_x^2 \alpha(x) + p^2(x)\alpha(x) = \frac{\hbar^2 a^2}{\alpha^3(x)}, \tag{4}
\]
where \(x\) refers to the space variable, and \(p(x, E)\) to its conjugate momentum in classical mechanics. \(E\) is the energy, assumed to be conserved, of the system. We have recently shown \cite{8} that the non-linear equation (4) corresponds to the equation for the amplitude function \(\alpha(x)\) in the amplitude-phase formulation of the Schrödinger equation, which arises by performing a so-called Milne transform on the wave-function; the phase function \(\phi(x)\) is obtained by integrating the relation \(\partial_x \phi = \alpha^{-2}\) (see Sec. 2).

We shall be concerned throughout this paper by the oscillatory properties of the solutions of the uncoupled Ermakov system formed by Eqs. (1) and (2) in the specific case of a potential energy function having a single minimum. Actually, since Eq. (3) defines a Sturm-Liouville problem, the oscillatory properties of the solutions are well-known \cite{9}, and we will restrict our analysis to the oscillations as a function of \(x\) and \(E\), of the amplitude and phase functions. Our first aim will be to show that, absolutely smooth, that is non-oscillating amplitude-phase functions can be constructed. Let us recall that the amplitude-phase formulation of the one-dimensional Schrödinger equation is frequently used in quantum scattering theories that explicitly include closed channels, such as quantum defect theory (see \cite{8} and references therein). In those situations, it is of prime importance for \(\alpha(x, E)\) to be a smooth function of both the space (usually radial)
coordinate $x$ and the energy $E$, since the scattering parameters (for example the phase-shifts) are defined in terms of amplitude-phase functions. However, by the principle of nonlinear superposition (Sec. 2.1), $\alpha$ may be expressed in terms of 2 independent solutions $u_1$ and $u_2$ of Eq. (3) – solutions which as known oscillate between the turning points of the potential. It follows that $\alpha(x)$ generally oscillates between the turning points. In practical implementations of amplitude-phase formalisms, numerical methods aiming at minimizing the amplitude of the oscillations have been devised. We have proposed in [8] such a method based on the invariant and the nonlinear superposition principle in the context of Ermakov systems. Here, the main point to be examined consists in the relationship between non-oscillating amplitude-phase functions and the functions obtained in the semiclassical ($\hbar \to 0$) limit. More specifically, we will prove that in this limit, the only non-oscillating solutions are the ones that yield classical quantities: in particular, the only semiclassical phase function that does not oscillate is the classical reduced action, and conversely the quantum continuation, for finite $\hbar$, of the classical reduced action is a non-oscillating function.

To this end, some properties of amplitude-phase functions, their behaviour as a function of $x$ and $E$, as well as the connection with Ermakov systems will be recalled in Sec. 2. In Sec. 3, we shall discuss semiclassical amplitude-phase functions by setting up a semiclassical Ermakov system; the classical probability amplitude and reduced action will appear as a particular solution of the semiclassical amplitude and phase functions. Those results will then be employed in a formal $\hbar$ expansion of the quantum amplitude-phase functions, to prove that provided the first order functions are non-oscillating, the solutions to each order in $\hbar$ will then not oscillate (Sec. 4). This will be followed in Sec. 5 by a discussion of the results, in particular in relation to quantization of classically integrable systems.

2. Amplitude-phase functions

Our concern here will be the 'auxiliary' amplitude and phase functions of the Schrödinger Eq. (3), with $p^2(x) = 2m(E - V(x))$, ie a particle of mass $m$ and energy $E$ trapped in a potential well $V(x)$ having a single minimum. $V(x)$ is defined on an interval $[s_1, s_2]$ (typically $s_{1,2} = \pm \infty$ or 0). Atomic units will be used throughout, except for the $\hbar$ factors which will be reestablished where appropriate.

2.1. Nonlinear superposition principle

We collect in this paragraph the main results concerning amplitude-phase functions that will be useful in what follows, omitting details (for more details and the relevant references, see [3]). Eq. (3) defines a Sturm-Liouville problem, typically a singular problem on the half line or real line when vanishing boundary conditions at $s_1$ and $s_2$ are implemented. However, our interest lies not in the specific eigenfunctions or eigenvalues of the Schrödinger equation, but in relating linearly independent solutions
of the linear equation to amplitude and phase functions. We will denote by \( u_1 \) and \( u_2 \) two independent solutions of Eq. (3) respectively regular at \( s_1 \) and \( s_2 \) and with Wronskian \( W = W[u_1, u_2] = (\partial_x u_1) u_2 - u_1 (\partial_x u_2) \).

A general solution \( u(x) \) of Eq. (3), which is readily written in terms of the independent solutions \( u_1 \) and \( u_2 \), may also be obtained as

\[
u(x) = b_1 \alpha(x) \sin [\phi(x) + b_2],
\]

where \( b_1 \) and \( b_2 \) are complex constants. A straightforward substitution in Eq. (3) leads to the two equations

\[
\partial_x^2 \alpha(x) + p^2(x) \alpha(x) = \alpha(x) [\partial_x \phi(x)]^2
\]

\[
\alpha^2(x) = \frac{a}{\partial_x \phi}
\]

where we can set \( a^2 = 1 \) without any loss of generality (since \( a^2 \) can be absorbed into \( \alpha \) by redefining \( \alpha \rightarrow \alpha/a^{1/2} \), thus recovering Eq. (4). For obvious reasons, \( \alpha \) and \( \phi \) are known respectively as the amplitude and phase functions. In terms of \( u_1 \) and \( u_2 \), it follows from standard results on Ermakov systems that the general solution for \( \alpha \) is given by

\[
\alpha(x) = \left[ \left( \frac{1}{2I} + 2Ic^2 \right) u_1^2(x) + \frac{2I}{W^2} u_2^2(x) - \frac{4Ic}{W} u_1(x) u_2(x) \right]^{1/2},
\]

and the equation for the phase is readily integrated to give

\[
\phi(x) = \arctan \left[ \left( \frac{1}{2I} + 2Ic^2 \right) W \frac{u_1(x)}{u_2(x)} - 2Ic \right] + \arctan 2Ic,
\]

where the integration constant is chosen so that \( \phi(s_1) = 0 \). Eq. (8) is an illustration of the nonlinear superposition principle [7]. \( I \) and \( c \) are two constants, independent of \( x \) (\( I \) is the Ermakov, or Lewis-Ray-Reid, invariant). Note that the value of the phase function at \( s_2 \), known as the accumulated phase, does not depend on the constants \( I \) and \( c \) for the eigenvalues \( E_0 \) of the Sturm-Liouville problem, since

\[
\phi(s_2, E = E_0) = \pi n,
\]

\[
\phi(s_2, E \neq E_0) = \arctan [2Ic(E)] + \frac{2n + 1}{2} \pi;
\]

here \( n \) is an integer giving the number of nodes of \( u_2 \). Note also that \( \alpha(x_1) \) is independent of \( c \) if \( x_1 \) is a zero of \( u_1 \).

2.2. Boundary conditions and energy dependence

The boundary conditions for \( \alpha \) and \( \phi \) are therefore incorporated through the parameters \( I(E) \) and \( c(E) \), which as indicated depend on the energy. Normalization of the eigenfunctions \( f \) of the Sturm-Liouville problem with vanishing boundary conditions at \( s_1 \) and \( s_2 \), yields

\[
\int_{s_1}^{s_2} f^2(x) dx = I \partial_E \phi(s_2, E = E_0).
\]
By choosing the eigenfunctions to be normalized per unit energy increment, $I$ becomes an energy-independent positive constant, and amplitude and phase functions depend on the single parameter $c(E)$.

We now introduce another solution $g(x)$ of the Schrödinger Eq. (3), defined in terms of the solutions regular at $s_1$ and $s_2$ by

$$g(x, c) = 2I \left( \frac{u_2(x)}{W} - cu_1(x) \right),$$

which fulfills $W[u_1, g] = 2I$ and gives $\alpha(x) = \left[ \frac{1}{2I} \left( u_1^2(x) + g^2(x) \right) \right]^{1/2}$. This simply means that $u_1$ and $g$ lag $\pi/2$ out of phase, and that with the conventions of Sec. 2.1 ($a = 1$ and $\phi(s_1) = 0$), we have

$$u_1(x) = \sqrt{2I} \alpha(x, c) \sin \phi(x, c),$$

$$g(x, c) = \sqrt{2I} \alpha(x, c) \cos \phi(x, c).$$

We have emphasized the $c$-dependence of the different functions (though self-consistency requires it, it may be checked explicitly that $u_1$ does not depend on $c$).

**2.3. Oscillatory properties**

### 2.3.1. Oscillations of the amplitude

Let $t_1(E)$ ($t_2(E)$) be the inner (outer) turning point. For values beyond the turning points (when $x < t_1$ or $x > t_2$), we recast Eq. (3) as

$$\frac{1}{2} \langle \phi; x \rangle = p^2(x) - \alpha^4(x),$$

where $\langle \phi; x \rangle \equiv \partial^2 \phi/\partial x^2 - \frac{2}{3} (\partial^2 \phi/\partial x \partial \phi)^2$ denotes the Schwartzian derivative. Since $\alpha$ is a positively defined quadratic form and $p^2(x) < 0$ beyond the turning points, the Schwartzian derivative of the phase is negative, and as it can be verified, if $\langle \phi; x \rangle < 0$ on an interval then $\partial_x \phi$ cannot have a positive local minimum on this interval, i.e. $\alpha$ cannot have a local maximum. By noting that $\alpha(x) \to +\infty$ when $x \to s_1$ and $x \to s_2$, we conclude that if $\alpha$ does not oscillate between the turning points, it will not oscillate on the whole interval $[s_1, s_2]$. Note that there is then a unique value $x_0$, with $t_1 < x_0 < t_2$, such that $\partial_x \alpha(x_0) = 0$.

However, as $u_1(x)$ and $u_2(x)$ oscillate for $t_1 < x < t_2$, $\alpha$ will generally oscillate, by virtue of the nonlinear superposition principle Eq. (8), between the turning points, the local wavelength being half that of $u_1$ or $u_2$. Nonetheless, at a specified energy, there may be infinitely many values of $c$ giving a non-oscillating amplitude function. We give in the following paragraphs a sufficient (but not necessary) condition for $c(E)$, that is the value of $c$ and its explicit energy-dependence, because this value has remarkable properties related to classical quantities in the semiclassical limit, as will be seen in Sec. 3.

† For example, taking the second derivative of $\alpha$, it is seen that any value of $c$ enclosed between $c_{\pm}(x)$, with $c_{\pm}(x) = u_2(x) [W u_1(x)]^{-1} \pm [2I u_1(x)]^{-1} [2I/p(x) - u_1^2(x)]^{1/2}$ and where $x$ spans the interval between the turning points, will do.
2.3.2. Inverted phase accumulation  Proceeding as in Sect. 2.1, we can define a phase function \( \bar{\phi}(x) \) such that \( \bar{\phi}(s_2) = 0 \), i.e., the phase starts accumulating at \( s_2 \) instead of \( s_1 \). As can be easily seen, this amounts to exchange the roles of \( u_1 \) and \( u_2 \); for instance, we now have \( u_2 = b_2\bar{\alpha}\sin \bar{\phi} \), where \( \partial_\phi \bar{\phi} = \bar{\alpha}(x)^{-2} \) and \( b_2 \) is a constant to be set below. \( \bar{\alpha} \) is given by

\[
\bar{\alpha}^2(x) = \left( \frac{1}{2I} + 2I\bar{c}^2 \right) u_2^2(x) + \frac{2I}{W^2} u_1^2(x) + \frac{4I\bar{c}}{W} u_1(x) u_2(x),
\]

with \( W[u_2, u_1] = -W \) and where we assumed \( I = \bar{I} \) for simplicity. Generally \( \alpha(x, c) \) and \( \bar{\alpha}(x, \bar{c}) \) are very different functions (for example the \( \bar{c} \) independent points of \( \bar{\alpha} \) are now located at the zeros of \( u_2 \)). Notwithstanding, it is apparent that \( \alpha(x, c) = \bar{\alpha}(x, \bar{c}) \) iff \( c^2 = \bar{c}^2 = W^{-2} - (2I)^{-2} \) and \( \bar{c} = -c \). We shall set

\[
c_o(E) = - \left[ |W(E)|^{-2} - |2I|^{-2} \right]^{1/2}.
\]

To keep the quadratic form real, this implies that \( W^2 < 4I^2 \), condition to be assumed in the rest of the paper (this is not a problem in practice because \( \alpha \) and \( \phi \) are left unchanged by the transformations \( u_1 \to \kappa u_1, W \to \kappa W, I \to \kappa^2 I, c \to c/\kappa^2 \), so the Wronskian can be conveniently rescaled).

Let us now suppose that \( c = -\bar{c} = \pm c_o \). We then have \( \alpha = \bar{\alpha} \). \( b_2 \) is found by evaluating \( W[u_1, u_2] \) at \( s_2 \), which yields, by choosing a proper sign convention \( b_2 = \sqrt{2I} \).

From Eqs. (13)-(15), it follows that

\[
\sin \bar{\phi}(\mp c_o) = W \left[ \cos \phi(\pm c_o)/2I \mp \left[ W^{-2} - (2I)^{-2} \right]^{1/2} \sin \phi(\pm c_o) \right],
\]

thereby obtaining the relation between \( \phi \) and \( \bar{\phi} \), which give the oscillations of \( u_1 \) and \( u_2 \) only if \( \alpha \) does not oscillate.

2.3.3. Auxiliary quadratic form on the unit circle  \( \alpha^2(x) \) is a positive definite quadratic form. Labelling \( M \) the matrix of the coefficients, we have \( \text{det } M = W^{-2} \) and \( \text{Tr } M = 1/2I + 2I\bar{c}^2 + 2I/W^2 \). \( \alpha^2(x) \) can be reduced to the canonical form

\[
\alpha^2(x, c) = \lambda_1(c)v_1^2(x, c) + \lambda_2(c)v_2^2(x, c),
\]

where \( \lambda_i(c) \) are the eigenvalues of \( M \) (\( \lambda_1 \geq \lambda_2 \)) and \( v_i(x, c) \) are the eigenvectors, normalized so that \( v_1^2(x, c) + v_2^2(x, c) = u_1^2(x) + u_2^2(x) \). We now introduce a quadratic form \( Q \) defined by

\[
Q(x, c) = \lambda_1(c)w_1^2(x, c) + \lambda_2(c)w_2^2(x, c),
\]

where \( w_i^2(x, c) = v_i^2(x, c) [u_1^2(x) + u_2^2(x)]^{-1} \). As indicated, \( Q \), as well as the \( \lambda_i \) and \( v_i \) are \( c \)-dependent. \( Q \) oscillates between its maximum and minimum values, which are by construction given respectively by \( \lambda_1 \) and \( \lambda_2 \).

We now set \( c = \pm c_o \). Let \( x_1 \) (\( x_2 \)) label the points where \( u_1 \) (\( u_2 \)) vanishes; we then have \( Q(x_{1,2}, \pm c_o) = 2I/W^2 \), so that between 2 zeros of \( u_1 \) and \( u_2 \), \( Q(x, \pm c_o) \) has at least one extremum on the unit circle (for those points, the equality \( u_1^2(x) = u_2^2(x) \) is fulfilled). Note that \( \alpha(x, c = \pm c_o) \) goes through both the \( c \)-independent points of \( \alpha(x, c) \).
and $\tilde{a}(x, \tilde{c})$, so if $\alpha$ oscillates, then there is an extremum of $\alpha$ between $x_1$ and $x_2$, and the sign of $\partial_x \alpha$ alternates between the consecutive zeros of $u_1$ and $u_2$. This is illustrated on Fig. 1 for the specific case of the harmonic oscillator (to be discussed in details in Sec. 5.4); the zeros of $u_1$ and $u_2$ are respectively shown as triangles and rectangles. Note also that the maxima of $Q(c_o)$ correspond to the minima of $Q(-c_o)$, since $Q(-c_o) = \bar{Q}(c_o)$. Combining Eqs. (20) and (21) and taking the derivative $\partial_x \alpha(x_{1,2})$ as a function of $\partial_x Q$, $Q$, $u_1$ and $u_2$, it can be seen indeed that for $c = -c_o$ the sign of $\partial_x \alpha$ at 2 consecutive zeros, $x_1$ and $x_2$, alternates. However, for $c = c_o$, the sign of $\partial_x \alpha$ between 2 consecutive arbitrary zeros of $u_1$ and $u_2$ does not change, and thus $\alpha(c_o)$ does not oscillate.

2.3.4. Oscillations of the accumulated phase  We have explained in [8] why obtaining non-oscillatory functions is important when amplitude-phase methods are employed in scattering theory. The goal there is to extend energy-normalization, which for the eigenfunctions of the Sturm-Liouville problem is given by Eq. (12), to functions $f(x)$ which converge at $s_2$ but diverge at $s_1$ (the phase-shifted or scattered wavefunctions). By combining the continuity equation for the probability density and L’Hôpital’s rule, improper energy normalization is defined by

$$\int_{s}^{s_2} f^2(x, E) dx = I\partial_E c \left( \frac{1}{2I} + 2Ic^2(E) \right)^{-1}, \quad (22)$$

where $r$ is a cut-off radius (and as above, $I$ is assumed to be energy-independent). This normalization is of course arbitrary, since it is governed by $c(E)$, but it conditions the energy-dependence of the different scattering parameters. In particular, the accumulated phase, which is unambiguously defined (Eq. (10)) for the eigenfunctions of the Sturm-Liouville problem, crucially depends (Eq. (11)) on the normalization when $E$ is not an eigenvalue.

More precisely, let us assume the eigenvalues of the Sturm-Liouville problem to be given by $E_0 = \xi(n)$, where $n$ is the number of zeros of the corresponding eigenfunction (thus, of $u_2$), and $\xi(n)$ is an a-priori arbitrary, but monotonous function admitting a differentiable inverse, $n(E) = \xi^{-1}(E)$. The functional relation between $E_0$ and integer values of $n$ is thereby extended to any energy $E$ lying between two eigenvalues, ie $E = \xi(n)$, $n$ real. The energy-dependence for $c(E)$ can now be chosen so as to extend the normalization of the eigenfunctions to non-integer values of $n$ by equating Eq. (22) to $I\pi \partial_E \xi^{-1}(E)$ (cf Eqs. (11) and (12)), yielding

$$c(E) = -\frac{1}{2I} \cot \pi \xi^{-1}(E). \quad (23)$$

Substituting in Eq. (11) gives the following expression for the accumulated phase:

$$\phi(s_2, E \neq E_0) = \pi \xi^{-1}(E) \equiv \pi n(E). \quad (24)$$

§ We noted, however, that given the behaviour of the amplitude function at $s_1$ and $s_2$, there is necessarily a point between $t_1$ and $t_2$ where $\partial_x \alpha$ vanishes. The argument sketched here relies on the signs of the basis functions $u_1$ and $u_2$ and their derivatives when $V(x)$ is monotonous on a full cycle of oscillation of the basis functions, and excludes the neighbourhood around the bottom of the potential, where $\partial_x \alpha$ changes sign (but $\partial_x \alpha$ does not vanish exactly at the bottom of the potential, as would be the case in the WKB approximation).
Thus the accumulated phase does not oscillate as a function of the energy (it is a simple straight line as a function of \( n \)) and the value of \( c \) given in Eq. (23) is the only value compatible with energy normalization leading to a non-oscillating accumulated phase function. We shall mention in Sec. 5.3 below the relation the specific form (24) has with the canonical action variable in classical mechanics. Note finally, that provided the basis functions \( u_1 \) and \( u_2 \) are redefined so that their Wronskian is proportional to \( 2I \sin \pi n(E) \), Eq. (23) becomes a particular form of the more general Eq. (18): with such a choice, the amplitude \( \alpha \) is a non-oscillating function of \( x \) and the accumulated phase \( \phi(s_2) \) is a non-oscillating function of \( E \).

3. Semi-classical Ermakov system

3.1. Asymptotic solutions to the linear equation

The approximate solutions to the one dimensional Schrödinger equation when \( \hbar \to 0 \) are well-known from the asymptotic theory of ordinary linear differential equations \[10\]. It follows from Sec. 2.3.1 that it is sufficient to consider the solutions between the turning points (ie for real \( p(x) \)). Real solutions are of the form

\[
\tilde{u}(x) = \frac{a_1}{\sqrt{p(x)}} \sin \left[ \pm \int p(x')dx' + a_2 \right]
\]

where \( a_1 \) and \( a_2 \) are constants. Tilded ('\( \tilde{\} \)) quantities will henceforth denote asymptotic (semiclassical) functions when these are to be distinguished from the corresponding exact quantum solutions. It is well-known from Hamilton-Jacobi theory that

\[
S(x, E) = \pm \int p(x', E)dx' + a_2,
\]

where \( S(x, E) \) is known as the Hamilton-Jacobi characteristic function or reduced action: the characteristics in phase-space are made up of the points \( (x, \partial_x S) \).

3.2. Ermakov system

By direct substitution of a general asymptotic solution into the Schrödinger equation, and by labeling \( \tilde{\alpha} \) and \( \tilde{\phi} \) the semiclassical amplitude and phase functions, we obtain a semiclassical Ermakov system

\[
\hbar^2 \frac{\partial^2 \tilde{u}}{\partial x^2} \tilde{u} + \left[ p^2(x) + \frac{\hbar^2}{2} \langle S; x \rangle \right] \tilde{u} = 0
\]

\[
\hbar^2 \frac{\partial^2 \tilde{\alpha}}{\partial \tilde{\phi}^2} + \left[ p^2(x) + \frac{\hbar^2}{2} \langle S; x \rangle \right] = \hbar^2 \left( \partial_x \tilde{\phi} \right)^2,
\]

where again the bracket \( \langle ; x \rangle \) denotes a Schwartzian derivative and we have, as for the usual amplitude-phase functions \( \tilde{\alpha}^2 = \tilde{\alpha}/\partial_x \tilde{\phi} \) and thus \( \partial_x^2 \tilde{\alpha}/\tilde{\alpha} = -\langle \tilde{\phi}; x \rangle /2 \). Eqs. (27)-(28) are the semiclassical version of the quantum system given by Eqs. (3)-(4). Eq. (27) is the modified Schrödinger equation exactly obeyed by the semiclassical wavefunctions, and Eq. (28) is the nonlinear equation fulfilled by the semiclassical amplitude function.
The passage from the exact (quantum) Ermakov system to the semiclassical one simply consists in a redefinition of the potential energy function, and is identical if \( \langle S; x \rangle \) vanishes. Although generally \( \langle S; x \rangle \) is nonzero (except for the free particle), \( \langle S; x \rangle \) does tend to zero or to a finite value in the limit of high quantum numbers (eg the harmonic oscillator for the former, the centrifugal Coulomb potential for the latter). Only if this value is negligible compared to the other terms in the energy function does the high quantum numbers condition fit with the semiclassical limit.

### 3.3. General solutions

\( \tilde{\alpha} \) and \( \tilde{\phi} \) are given in terms of two independent functions \( \tilde{u}_1 \) and \( \tilde{u}_2 \) of Eq. (27) by the same relations, Eqs. (8) and (9), as in the exact (quantum) case, with now tilded quantities. It is convenient, however, to set the tilded constants \( \tilde{a}, \tilde{I} \) and \( \tilde{W} \) equal to their quantum counterpart \( a, I \) and \( W \). This is done by first noting that \( \mathcal{W}[\alpha \sin \phi, \alpha \cos \phi] = a \), which we then set equal to \( \mathcal{W}[\tilde{\alpha} \sin \tilde{\phi}, \tilde{\alpha} \cos \tilde{\phi}] \), so \( \tilde{a} = a = 1 \). To preserve the Wronskians (cf Eqs. (14)–(15)), the semiclassical function \( \tilde{u}_1 \), of the form given by Eq. (25), is thus set as

\[
\tilde{u}_1(x) = \sqrt{\frac{2I}{p(x)}} \sin S(x),
\]

where we have implicitly included \( a_2 \) in the reduced action so that \( \tilde{u}_1 \) is the asymptotic approximation to \( u_1 \) in the neighbourhood of an arbitrary \( x \) lying between the turning points. An independent solution \( \tilde{u}_2 \) with Wronskian \( \mathcal{W}[\tilde{u}_1, \tilde{u}_2] = W \) is then obtained under the form \( \cos(S(x) + b)/\sqrt{p(x)} \) as

\[
\tilde{u}_2(x) = \sqrt{\frac{2I}{p(x)}} \kappa \cos \left[ S(x) + \arccos \frac{W}{2I\kappa} \right],
\]

where we have introduced the scaling factor \( \kappa \) to keep all quantities real. In what follows, we shall set \( \kappa = 1 \), which is tantamount to rescaling \( u_2 \) (a similar rescaling was performed in the quantum case, see below Eq. (18)).

### 3.4. Non-oscillating solutions

Substituting Eqs. (29)–(30) in the expression for \( \tilde{\alpha} \) readily yields

\[
\tilde{\alpha}^2 = \frac{1}{p(x)} \left[ 4I^2 W^{-2} \sin^2 S(x) + (\cos S(x) - 2Ic \sin S(x)) \times \left\{ \cos S(x) - 2I \left( c + W^{-1} \left[ 4 - W^2 / I^2 \right]^{1/2} \right) \sin S(x) \right\} \right],
\]

which is a highly oscillatory function for an arbitrary value of the parameter \( c \). However, it may be noted by inspection that for \( c = -[W^{-2} - (2I)^{-2}]^{1/2} \), the oscillating terms are cancelled out. Remark that this is the same expression that was labelled \( c_o \) in the quantum case (Eq. (15)). Reestablishing \( \hbar \), the amplitude now reads \( \tilde{\alpha}^2(c_o) = \hbar/p(x) \), which given our assumptions is a non-oscillating function of \( x \). Identical substitutions
may be done for $\tilde{\phi}$, from which it follows that the semiclassical phase function is highly oscillatory for an arbitrary value of $c$ except if $c = c_o$, and in that case,

$$\tilde{\phi}(x, c_o) = S(x)/\hbar.$$ (32)

In short, non-oscillating functions are obtained for a unique value of the parameter $c$, for which the semiclassical quantities match their classical counterpart ($\sqrt{\hbar/p(x)}$ and $S(x)/\hbar$ are respectively the classical probability amplitude and phase functions).

Writing Eq. (28) as

$$\frac{\hbar^2}{2} \left[ \langle \tilde{\phi}; x \rangle - \langle S; x \rangle \right] = p^2(x) - \frac{\hbar^2}{2},$$ (33)

this means that each side of the nonlinear equation of the semiclassical Ermakov system vanishes independently. Note also that $p^2(x) + \hbar^2 \langle S; x \rangle /2 = \hbar^2 \langle \tan \tilde{\phi}(c); x \rangle /2$ (this is established by using the Möbius invariance of the Schwartzian derivative and establishing a $c$-dependent linear transformation relating $\tan \tilde{\phi}(x, c)$ to $\tan S(x)$).

4. $\hbar$ expansions

The link between the solutions of the quantum and semiclassical Ermakov systems is done by employing a formal $\hbar$ expansion of the amplitude function. As in the previous section, we assume $I$ and $W$ to be identical in both the quantum and the semiclassical case. It is then straightforward to show that to each order in $\hbar$ (as well as to infinite order), non-oscillatory functions are obtained.

4.1. Series expansions

The formal asymptotic solution to the Schrödinger equation for small values of the parameter $\hbar$ is usually done by transforming it to the Riccati form and then obtaining a recurrence relation between complex function of order $j$ and the functions of lower order [10]. Here we proceed slightly differently, because we want the relations between the amplitude and the phase to be verified to each order. We look for a generic solution of Eq. (3) under the form $u(x) = a(x) \exp if(x)/\hbar$ where $a(x)$ and $f(x)$ are real functions admitting the series expansions

$$a(x) = \sum_{j=0}^{\infty} a_j(x)\hbar^j, \quad f(x) = \sum_{j=0}^{\infty} f_j(x)\hbar^j.$$ (34)

Substitution into the Schrödinger equation gives $a_j(x) = 0$, $f_j(x) = 0$ for odd $j$ and the following recurrence relations for even $j$, $j \geq 2$:

$$\partial_x f_j(x) = \frac{1}{2a_0\partial_x f_0} \left[ \partial_x^2 a_{j-2} - \sum_{m=2}^{j-2} a_m \sum_{n=0}^{j-m} \partial_x f_n \partial_x f_{j-m-n} \right],$$ (35)

$$a_j(x) = \frac{b_j}{\sqrt{\partial_x f_0}} - \sum_{n=0}^{j-2} \int \left( 2\partial_y a_n \partial_y f_{j-n} + a_n \partial_y^2 f_{j-n} \right) / \sqrt{\partial_y f_0 dy}.$$ (36)
Non-oscillating solutions to uncoupled Ermakov systems and the semiclassical limit

with \( f_0(x) = \pm S(x) \), \( a_0(x) = \pm \partial_x f_0^{-1/2} \), and where the constants \( b_j \) appearing in the solution to the homogeneous equations for \( a_j(x) \) are all set to 0 for \( j \geq 2 \). The Wronskian relations are then preserved to each order in \( \hbar \), that is

\[
\mathcal{W}[a(x) \sin f(x), a(x) \cos f(x)] = a_0^2(x) \partial_x f_0(x) = \partial_x S(x)/p(x) = 1. \tag{37}
\]

Between the turning points, the \( \pm \) branches are combined to yield real oscillatory functions. The formal expansion for \( u_1 \) (assuming again the adequate integration constant to be included in \( f_0 \)) is then given by

\[
u_1(x) = \sqrt{2I} \sum_{j=0}^{\infty} a_j(x) \hbar^j \sin \left( \sum_{i=0}^{\infty} f_i(x) \hbar^i \right). \tag{38}\]

The expansion for the function lagging \( \pi/2 \) out of phase is trivially obtained by using the cos function; from Eqs. (13) to (15) the formal expansion for \( u_2 \) is then found as

\[
u_2(x) = \sum_{j=0}^{\infty} a_j(x) \hbar^j \left( W \cos \left( \sum_{i=0}^{\infty} f_i(x) \hbar^i \right) - I \left[ 4 - W^2/I^2 \right]^{1/2} \sin \left( \sum_{i=0}^{\infty} f_i(x) \hbar^i \right) \right). \tag{39}\]

To first order these functions coincide by construction with the semiclassical wavefunctions \( \tilde{u}_1 \) and \( \tilde{u}_2 \).

4.2. Amplitude and phase expansions

The \( \hbar \) expansions for the amplitude and phase functions are obtained by combining the nonlinear superposition principle [Eqs. (8)-(9)] with the formal series expansions for \( u_1 \) and \( u_2 \). The expansions may be done to finite or infinite order. In the first case, the functions are Taylor expanded around \( \hbar = 0 \) after separating the classical terms \( a_0 \) and \( f_0 \). The infinite order case is analogous to the first order case treated in Sec. 3.4. For example substituting the series expansions in Eq. (9) gives the expression (mod \( \pi \)) of the phase which can be simplified as

\[
\phi(x, c) = \arccot \left( \cot \left( \frac{S(x)}{\hbar} + \sum_{i=1}^{\infty} f_i(x) \hbar^{i-1} \right) - \left[ 2Ic(E) + IW^{-1} \left[ 4 - W^2/I^2 \right]^{1/2} \right] \right). \tag{40}\]

The amplitude function may be obtained by deriving this last equation, keeping in mind the Wronskian relations (37), yielding an expression involving the sines and cosines of the expression between parentheses in Eq. (40). In both cases, the highly oscillatory terms are cancelled by setting \( c = c_o \) (in Eq. (40) for example, the term between square brackets then vanishes).

5. Discussion

5.1. General Remarks

We have thus seen that same value of \( c \) gives rise to non-oscillating functions both in the semiclassical and quantum cases. This is not surprising, if the similarities between Eqs. (8), (13) and (19) on the one hand, together with the nonlinear superposition principle in both the quantum and semiclassical Ermakov systems on the other hand
are considered: the same functional relation gives the amplitude-phase functions in both cases. We note however that the simple arguments we have given above, although physically appealing because of the direct connection to standard classical quantities, are liable to a more rigorous treatment. For example the series expansion obtained for the amplitude function does not necessarily converge, and even its asymptotic properties as a function of the parameter $c$ deserve a more thorough investigation. Actually, Lewis had made the same remarks when studying the adiabatic invariant series of the time-dependent classical harmonic oscillator in powers of an adiabatic parameter $\varepsilon$ [11], which is defined by the same uncoupled Ermakov system as the one studied in this paper. Lewis’s work was the first (albeit implicit) application to Ermakov systems of Kruskal’s asymptotic theory of Hamiltonian systems extending the study of the adiabatic invariants beyond the first order in $\varepsilon$ [12]. The transposition of these theories to the present problem is not straightforward because we lack here the Hamiltonian formalism on which these theories are based (eg, the integral invariants that appear in Kruskal’s theory would have here a rather obscure interpretation). Nonetheless, the present results on the oscillatory properties can be directly transposed to any uncoupled Ermakov system depending on an ‘adiabatic’ parameter $\varepsilon$ (which in the present context corresponds of course to the Planck constant).

5.2. Scattering basis functions

Previously to the work of Fano et al. [13], the use of amplitude-phase methods in scattering theory was limited to the high kinetic-energy limit, and the derivatives of the phase $\phi$ above first order neglected, thereby effectively restricting the treatment from the start to the standard WKB approximation (see eg, Ch. 4.3 of [14]). The more recent application of these methods to define a pair of basis functions for phase-shifted wavefunctions in a potential (as outlined above), relies on numerical treatments to minimize the oscillations; these treatments are preferred even in the cases for which an analytic pair of basis functions is known, such as the Whittaker functions for the centrifugal Coulomb problem or the parabolic cylinder functions for the harmonic oscillator. It is interesting to note that the approach suggested in Sec. 2.3 yields, for an arbitrary potential with a single minimum, the same relations that are known to be valid (and non-oscillating) for the analytic functions in the mentioned special cases (eg in the Coulomb case where the effective quantum number $\nu$ is defined by $\nu = \xi^{-1}(E) + l$, the accumulated phase obtained with the Whittaker functions is $\phi(\infty) = \pi(\nu - l)$, and improper normalization follows the normalization of the eigenfunctions by normalizing to $\nu^3/2$, independently of $\nu$ being real or an integer [15]).

¿From a formal standpoint, defining a specific basis of functions is equivalent to defining the Green’s function of the scattering process in the asymptotic field. This is the Green’s function that appears in the Lippmann-Schwinger equations and through which the collision operators are defined. This is why the collision operators depend on the parameter $c$. Though at first sight this may appear as an unexpected feature, it must
be remembered that the explicit inclusion of closed channels leads to a modification of the usual Green’s functions through a term depending on the accumulated phase \[16\].

5.3. Semiclassical limit and classical quantities

Standard semiclassical physics is usually not concerned by the quantum to classical limiting procedure, because the strategy there is to start from classical quantities at the outset and then proceed to quantization. However, we have seen in Sec. 3 that the classical reduced action is only one of the many phase functions that are obtained in the semiclassical limit (namely the non-oscillating one). This type of problem is frequent in 'classicalization' procedures: a particular, often arbitrary choice has to be made to recover classical quantities (action, Liouville equation, etc.) A recent example is given in Ref. \[17\], where the passage from Hilbert-space to classical-phase space operators involves particular choices for the parameters in order to recover the classical dynamics. In other works, this requirement takes the form of an additional ad-hoc condition usually termed as the 'correspondence principle' (eg in Ref. \[18\] where a quantum version of the Hamilton-Jacobi equation is given an additional boundary condition \( p_{\text{quantum}}(x,E) \to p_{\text{classical}}(x,E) \) when \( h \to 0 \) and \( E \) is fixed)\[1\]. Though much less general, our treatment is more transparent in that the continuation of the classical reduced action in the quantum domain is readily identified: it is the non-oscillating phase function (and the continuation of the classical probability amplitude is the non-oscillating amplitude function).

Other aspects of the classical-quantum correspondence for a classically integrable and separable system deserve to be mentioned. The accumulated phase (Eqs. (10)-11) is seen to be directly related to the line integral around a closed loop of \( \alpha^{-2} \):

\[
\oint \left[ \partial_x \phi(x,c) \right] dx = 2\phi(s_2,c). \tag{41}
\]

Contrary to EBK (torus) quantization, there are no caustics when dealing with exact quantization, and the quantization condition reads \( \oint \partial_x \phi(x,c) dx = 2\pi n \), ie, \( n \) is an integer and the line integral does not depend on the particular value of \( c \). Here we understand by "exact quantization" the quantization of the exact quantum phase, and not the exact WKB quantization of non-solvable potentials, as employed by Voros \[19\]. Note however that when \( c = c_o \), the (unquantized) integral \( (11) \) reads, according to Eq. (24):

\[
\oint \left[ \partial_x \phi(x,c_o) \right] dx = 2\pi n(E). \tag{42}
\]

Not only does \( c = c_o \) preserve for non-integer real numbers the functional relation valid for exact quantization, but it gives a parameterization of the quantum equivalent of the canonical action variable.

\[1\] Other authors crudely suppress the \( h \)--dependent terms in selected equations where this suppression leads to classical relations (for example this would be done in Eq. (4) by giving the amplitude squared the dimensions of a classical quantity, getting thereby rid of \( \partial_x^2 \alpha \), but would not be done in Eq. (3), which does not support an obvious classical interpretation). This procedure has often been criticized because the functions and the parameters appearing in the equations depend on \( h \).
In a similar vein, the period of motion \( T \) is given in Hamilton-Jacobi theory by taking the energy derivative of the reduced action along the closed loop. The transposition to the phase function in the quantum case would imply taking the energy derivative of Eq. (41), which by Eqs. (11) and (22) is proportional to the normalization. Again, the normalization depends on \( c(E) \) and to take \( \partial_E \phi(c) \) as the time parameterization doesn’t appear to make much sense unless \( c = c_0 \), since any other value would lead to an oscillating function, which would further not collapse to the classical period in the \( \hbar \to 0 \) limit.

5.4. Example

We illustrate the properties mentioned above on the harmonic oscillator, a paradigm both in the Hamilton-Jacobi formulation of classical mechanics [20] and in the semiclassical theory of bound states [21]. The reduced action \( S(x, E) \) of Eq. (26) is readily obtained, from which it follows that
\[
\partial_x S = \left( 2mE - m^2 \omega^2 x^2 \right)^{1/2}
\]
\[
J = \oint [\partial_x S] dx = 2\pi E/\omega,
\]
where \( m \) and \( \omega \) are the mass and frequency of the oscillator; we slightly depart from usual conventions and define \( J \) to be the canonical action variable. The period is recovered as \( T = \partial_E J \). The standard WKB solutions between the caustics, Eqs. (29)-(30), are obtained from these classical quantities. Semiclassical quantization must take into account the singularities at the turning points, from which it follows that \( E_0 = \hbar \omega \left( n + 1/2 \right) \), with \( n \) being an integer.

Quantum mechanically, the eigenvalues are given by \( \xi(n) = \hbar \omega \left( n + 1/2 \right) \) when \( n \) is an integer that counts the zeros of the eigenfunction, but this relation can be extended for any real value of \( n \), as discussed in Sec. 2.3.4. We then write \( \xi^{-1} \) as \( n(E) = E/\hbar \omega - 1/2 \). The derivative of the exact (quantum) phase is given by Eqs. (7)-(8) as
\[
\partial_x \phi(x, c) = \frac{m\omega}{\hbar} \left[ \left( \frac{1}{2I} + 2Ic^2 \right) u_1^2(x) + \frac{2I}{W^2} u_2^2(x) - \frac{4Ic}{W} u_1(x) u_2(x) \right]^{-2}.
\]
Fig. 2 compares the classical conjugate momentum as given by Eq. (43) with two exact phases \( \hbar \partial_x \phi(x, c) \) obtained from Eq. (45) by using numerical solutions of the Schrödinger equation \( u_1 \) and \( u_2 \) with respective vanishing boundary conditions at \( x = -\infty \) and \( x = +\infty \). One of the curves is for an arbitrary value of \( c \), the other corresponds to \( c = c_0 \). It may be seen that even for a moderate excitation \( (n \approx 12) \), the non-oscillating solution with \( c = c_0 \) can barely be distinguished from the classical momentum at the same energy, except near the turning points (the reason is that \( \langle S; x \rangle \) is negligible, hence the exact quantum phase, solution of Eq. (3) tends to the semiclassical quantum phase \( \tilde{\phi} \) of Eq. (28), which is simply \( S(x) \) when \( c = c_0 \).

The analog of the canonical action \( J \) appears as the line integral (41), which generally depends on \( c \) except when \( n \) is an integer in which case \( \oint [\hbar \partial_x \phi(x, c)] dx = \)
Non-oscillating solutions to uncoupled Ermakov systems and the semiclassical limit

Thus for the quantized energies, the quantal line integral differs from $J$ by the Maslov index. However, if $c = c_o$, we have for any $E$ [Eq. (42)]

$$\oint [\bar{h} \partial_x \phi(x, c_o)] dx = 2\pi E/\omega - \bar{h}/2,$$

which is the classical result with an action correction coming from the Maslov index. Note that taking the energy derivative of Eq. (41) crucially depends, for any energy $E$ (including the eigenvalues) on the energy-dependence $c(E)$. Only the energy dependence given by the relation (23) above, which yields Eq. (46) in this case, renders the usual relation for the period, and more generally follows the classical time parameterization for conservative systems in Hamilton-Jacobi theory. Thus, the parameter $c$, which appears free within quantum mechanics and has no classical counterpart, must be constrained if the usual classical relations for the oscillator are to be extended to quantum amplitude and phase functions.

The usual interpretation of the invariant within Ermakov systems hinges on the use of an original Hamiltonian or Lagrangian, from which the Ermakov equations are derived. The invariant is then associated, by means of Noether’s theorem, with the conserved quantity of an auxiliary motion [5, 6]. Such an interpretation is of course not available here, where the Ermakov equations are used in a quantum-mechanical context. The invariant $I$ is employed in this context only to define the normalization, through Eq. (12). However, for the choice $c = c_o$, a further interpretative step may be taken, since then the term on the right hand-side in Eqs. (12) or (22) gives

$$\frac{\hbar}{2m} I \partial_E \oint [\bar{h} \partial_x \phi(x, c_o)] dx.$$

(47)

For the harmonic oscillator, we have by Eq. (46) and by adopting unity normalization

$$I = \frac{m\omega}{\hbar\pi}.$$

(48)

For other systems, $I$ usually depends on $n(E)$ except if the wavefunctions are energy-normalized. Elementary manipulations yield the more general form

$$I = 2\pi \left[ \int \lambda(x, E) dx \right]^{-1},$$

(49)

where $\lambda(x, E)$ is the local de Broglie wavelength and the range of integration is restricted to the classical domain between the turning points.

6. Conclusion

Previous interest in amplitude-phase methods led us to investigate in this work the oscillatory properties of the nonlinear equation of uncoupled Ermakov systems. It was shown that non-oscillating amplitude-phase functions in the space and energy variables have a particular feature in the semiclassical limit: they yield classical quantities. We have seen that although standard quantum-mechanical quantities, such as the wavefunctions or the eigenvalues, are insensitive to the value of the parameter $c$ and its energy dependence, there is a unique value of $c$ which appears as connecting quantum
Non-oscillating solutions to uncoupled Ermakov systems and the semiclassical limit

amplitude and phase functions to their classical counterpart. Only in this case can 'quantum characteristics' \((x, \partial_x \phi)\) having a sense, and collapsing to \((x, \partial_x S)\) when \(\hbar \to 0\) be defined. We insist again that from the point of view of quantum mechanics, even in the semiclassical limit, this need not be the case: any amplitude and phase functions obeying Eq. (33) will yield correct semiclassical wavefunctions. A full study of these aspects on specific physical systems will be given elsewhere. Further links with current work on Ermakov systems may lead to a better appreciation, as well as to an extension, of the formalism.

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Figure captions

**Figure 1.** $Q(c_o)$ (broken line, left scale), $\alpha(c_o)$ (non-oscillating black line) and $\alpha(-c_o)$ (oscillating grey line) [right scale] are plotted for a harmonic oscillator (atomic units and $\omega = 1$, $n(E) = 4.4$, see Sec. 5.4 for the definitions). All amplitude functions $\alpha(c)$ for any $c$ go through the points $x_1$ (triangles), and any function $\bar{\alpha}(\bar{c})$ goes through the points $x_2$ (rectangles). Only the 2 functions $\alpha(\pm c_o)$ go through both the points $x_1$ and $x_2$. Since $Q$ oscillates and $Q(x = x_{1,2}, \pm c_o) = 2I/W^2$ is constant, $\partial_x Q$ has opposite signs at $x_1$ and $x_2$, as may be seen on the figure. This is also the case for $\alpha(-c_o)$ (grey line) which therefore oscillates, but not for $\alpha(c_o)$.

**Figure 2.** The positive branch of the classical canonical momentum $p(x)$ for a harmonic oscillator ($\omega = 1$, atomic units) is plotted (grey line) vs the "quantal momentum" $\partial_x \phi(x, c)$ i) for any arbitrary value of $c$ (dashed line) and ii) for the non-oscillating value $c = c_o$ (black line).
