Group-Case Commutative Association Schemes and Their Character Tables

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Abstract

Leading towards the classification of primitive commutative association schemes as the ultimate goal, Bannai and some of his school have been trying to

• identify the major sources of (primitive) commutative association schemes,
• collect known group-case primitive commutative association schemes, and
• compute their character tables

over the last twenty years. The construction of their character tables are important first step for a systematic study of such association schemes and towards the classification of those schemes. In this talk, we briefly survey the progress made in this direction of research, and list some open problems.

Dedicated to Eiichi Bannai on the occasion of his 60th birthday

1 Introduction

Let a finite group $G$ act on a finite set $X$ transitively. Then $G$ naturally acts on $X \times X$ by $(x, y)^g = (x^g, y^g)$. Let $R_0, R_1, \ldots, R_d$ be the orbits of $G$ on $X \times X$ with $R_0 = \{(x, x) : x \in X\}$. Then $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ is an association scheme, called a group-case (or Schurian) association scheme, and denoted by $\mathcal{X}(G, X)$.

Let $G$ be a finite group, and let $H$ be a subgroup of $G$. Let $X = H \backslash G$ be the set of the cosets of $H$ in $G$. Then $G$ acts transitively on $X$ under the action $(Hx)^g = H(xg)$. The group-case scheme $\mathcal{X}(G, H \backslash G)$ is commutative if and only if the permutation character of $G$ on $H \backslash G$ is multiplicity-free. Any group-case scheme $\mathcal{X}(G, X)$ can be viewed as $\mathcal{X}(G, H \backslash G)$ with the point stabilizer $H = G_x$ for an $x \in X$. The condition that the group-case association scheme is primitive is equivalent to that of the permutation group acts on the cosets by a maximal subgroup.

In early nineteen eighties, Bannai had a conviction that the works by many group theorists on the classification of maximal subgroups of finite simple groups (by using
the classification of finite simple groups) would eventually lead to the complete list of group-case primitive commutative association schemes. He seemed to believe that the calculations of parameters and character tables of known association schemes of such kind were to be feasible. He began to investigate the major sources of group-case primitive commutative association schemes, collect examples, and calculate their character tables.

The major sources of group-case (primitive) commutative association schemes of large class which Bannai (cf. [2]) has considered were as follows.

1. The actions of classical groups (or Chevalley groups) $G$ on appropriate subspaces $X$ of vector spaces over finite fields. (If the subspaces are isotropic, they are well understood because then the groups act on the cosets by parabolic subgroups. The cases of non-isotropic subspaces require further study. See examples in Table 1 below.)

2. The actions of classical groups (or Chevalley groups) $G$ on the cosets $X = H\backslash G$ of multiplicity-free (maximal) subgroups $H$. (A pair $(G, H)$ of a finite group $G$ and a subgroup $H$ whose permutation character $1^G_H$ is multiplicity-free, is often called a Gelfand pair. See examples in Tables 2 and 3 below.)

3. Finite (simple) groups, loops, and quasigroups $G$. (Commutative association schemes obtained from these algebraic structures by using their conjugacy classes as we will see in the sequel.)

4. The $n$-dimensional vector spaces $V$ over $GF(q)$ and subgroups $H$ of $GL(n, q)$; for example, the action of the semidirect product $G = V \rtimes H$ on $V$, in other words, $G$ acts on $X = H\backslash G$.

We note that the items in this list are not necessarily mutually exclusive nor cover all primitive commutative association schemes. Character tables of many commutative association schemes coming from the permutation groups in the above list have been investigated by many people including, Bannai-Song [8, 9, 10], Bannai-Shen-Song [6], Bannai-Kawanaka-Song [4], Bannai-Kwok-Song [5], Bannai-Shen-Song-Wei [7], Kwok [25, 26], Henderson [18, 19, 20, 21], Tanaka [29, 30], Bannai-Tanaka [12], Fujisaki [14, 15], and Bannai-Song-Yamada [11].

In Section 2, we briefly recall the definition of character tables of commutative association schemes and related basic facts. In Section 3, we discuss Paige’s simple Moufang loops and their character tables. In Section 4, we give a list of known examples of group-case association schemes whose character tables are either calculated or conjectured. Some open cases coming from Gelfand pairs are listed. In Section 5, we illustrate another example that is constructed in a different way from the previous ones. The character tables illustrated in this note are happened to be closely related to each other in an interesting way.

Our aim is to review the progress that has been made in the construction of character tables by paying attention to the construction methods employed. In doing this we would like to point out some connections between association schemes and classical groups and geometries. Experts in algebraic combinatorics, groups and geometry will hopefully find some useful information and sufficient pointers in this note.
2 The character tables

Let $\mathcal{X} = (X, \{R_i\}_{0 \leq i \leq d})$ be a commutative association scheme. Let $A_0, A_1, \ldots, A_d$ be the adjacency matrices and let $E_0, E_1, \ldots, E_d$ be the primitive idempotents of $\mathcal{X}$. The Bose-Mesner algebra $\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle = \langle E_0, E_1, \ldots, E_d \rangle$ of $\mathcal{X}$ over the field $\mathbb{C}$ of complex numbers, satisfies

$$A_j = \sum_{i=0}^{d} p_j(i) E_i.$$ 

Equivalently, with the character table $P = [p_j(i)]$ of $\mathcal{X}$, we have

$$[A_0 \ A_1 \ \cdots \ A_d] = [E_0 \ E_1 \ \cdots \ E_d] \begin{bmatrix} 1 & k_1 & k_2 & \cdots & k_d \\ 1 & p_1(1) & p_2(1) & \cdots & p_d(1) \\ 1 & p_1(2) & p_2(2) & \cdots & p_d(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & p_1(d) & p_2(d) & \cdots & p_d(d) \end{bmatrix}.$$ 

The character table $P$ of the association scheme satisfies the (i) row and (ii) column orthogonality relations [3, Theorem 3.5]. For $i, j \in \{0, 1, \ldots, d\}$,

$$\text{(i)} \quad \sum_{l=0}^{d} \frac{1}{k_l} p_l(i) \overline{p_l(j)} = \frac{|X|}{m_i} \delta_{ij}, \quad \text{(ii)} \quad \sum_{l=0}^{d} m_l p_l(l) \overline{p_l(l)} = |X| \delta_{ij},$$

where $m_l = \text{rank}(E_l) = \text{trace}(E_l) \ (0 \leq l \leq d)$, $\delta_{ij}$ is the Kronecker delta, and $\overline{a}$ denotes the complex conjugate of $a$. The numbers $m_l$ are the multiplicities of the scheme.

Let $G$ be a finite (simple) group, and let $C_0, C_1, \ldots, C_d$ be the conjugacy classes of $G$. Then by defining the associate classes $R_i$ by

$$(x, y) \in R_i \quad \text{iff} \quad yx^{-1} \in C_i,$$

we obtain a (primitive) commutative scheme $\mathcal{X}(G) = (G, \{R_i\}_{0 \leq i \leq d})$ which is often referred to as the group scheme. Given a finite (simple) quasigroup, we also obtain a (primitive) commutative association scheme by defining the associate relations as above. For $i = 0, 1, \ldots, d$, let $k_i$ and $f_i$ denote the sizes of conjugacy classes $C_i$ and the degrees of the irreducible characters of $G$, respectively. Then $f_i = \sqrt{\text{rank}(E_i)}$ and the character table $P$ of $\mathcal{X}(G)$ and the group character table $T$ has the following relation:

$$T = \begin{bmatrix} f_0 & 0 & \cdots & 0 \\ f_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & f_d \end{bmatrix} \cdot P \cdot \begin{bmatrix} 1/k_0 & 0 & \cdots & 0 \\ 1/k_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1/k_d \end{bmatrix}.$$ 

3 Character tables of Paige’s Moufang loops

A set $Q$ with one binary operation is a quasigroup if the equation $xy = z$ has a unique solution in $Q$ whenever two of $x, y, z \in Q$ are specified. A loop is a quasigroup with a neutral element 1 satisfying $1x = x = x1$ for every $x \in Q$. 


A Moufang loop is a loop in which any of the following (equivalent) Moufang identities holds: \((xy)xz = x(yxz),\ x(yzy) = ((xy)z)y,\ (xy)(zx) = x((yz)x),\) or \((xy)(zx) = (x(yz))x.\)

Paige (1956)\(^{28}\) introduced a class of finite simple Moufang loops which we are referring to as Paige’s simple Moufang loops. Liebeck (1987)\(^{27}\) proved that there are no other finite (non-associative) simple Moufang loops besides these Moufang loops. For every finite field \(\mathbb{F}_q\), there is exactly one simple Moufang loop \(\mathcal{M}^* = \mathcal{M}^*(q)\) of order \(q^3(q^2 - 1)/(q - 1, 2)\). Bannai and Song (1989)\(^9\) calculated the character tables of \(\mathcal{M}^*(q)\). We now recall the definition of \(\mathcal{M}^*(q)\).

On the set
\[
\left\{ \begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix} : a, b \in \mathbb{F}_q, \ \alpha, \beta \in \mathbb{F}_q^3 \right\}
\]
where \(\mathbb{F}_q^3 := \{(x, y, z) : x, y, z \in \mathbb{F}_q\}\), using the dot product \(\alpha \cdot \beta\) and vector product \(\alpha \times \beta\) in \(\mathbb{F}_q^3\), define the Zorn’s multiplication
\[
\begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix} \begin{bmatrix} c & \gamma \\ \delta & \delta \end{bmatrix} := \begin{bmatrix} ac + \alpha \cdot \delta & a\gamma + db\alpha - \beta \times \delta \\ c\beta + b\delta + \alpha \times \gamma & \beta \cdot \gamma + bd \end{bmatrix}.
\]

Given \(M = \begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix}\), we define its determinant by
\[
\det(M) := ab - \alpha \cdot \beta.
\]

Then both sets \(\mathcal{L} := \{M : \det(M) \neq 0\}\) and \(\mathcal{M} := \{M : \det(M) = 1\}\) are (non-associative) Moufang loops. The center of \(\mathcal{L}\) is
\[
Z(\mathcal{L}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in \mathbb{F}_q^* \right\}.
\]

We see that \(|Z(\mathcal{M})| = 1\) if the characteristic of \(\mathbb{F}_q\) is 2, otherwise, \(|Z(\mathcal{M})| = 2\). The quotient loop \(\mathcal{M}^* := \mathcal{M}/Z(\mathcal{M})\) is referred to as the Paige’s simple Moufang loop.

The conjugacy classes and character tables of \(\mathcal{M}^*(q)\) were calculated in \(^9\). Here we recall the case with \(q = 2^r\).

**Theorem 1.**\(^9\) Table 5) The character table of \(\mathcal{X}(\mathcal{M}^*), q = 2^r\) is given by

\[
\begin{array}{cccc}
1 & (q^6 - 1) & q^6 - q^3 & \cdots & q^6 - q^3 \\
1 & q^2 - 1 & -q^3 + q^2 & \cdots & q^3 + q^2 \\
1 & -q^3 - 1 & q^2 [a_{kl}] & 0 & \cdots & q^3 + q^2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & q^2 - 1 & q^2 [b_{mn}] & \end{array}
\]

where
\[
a_{kl} = -q(\sigma^{kl} + \sigma^{-kl}),\ 1 \leq k, l \leq q/2;\ \sigma = \exp(2\pi i/(q + 1))
\]
\[
b_{mn} = q(\rho^{mn} + \rho^{-mn}),\ 1 \leq m, n \leq (q - 2)/2;\ \rho = \exp(2\pi i/(q - 1)).
\]
This character table resembles the following table of $\mathcal{X}(\text{PSL}(2, q)), q = 2^r$ which is derived from the group character table of $\text{PSL}(2, 2^r)$ found in [13].

**Proposition 1.** [13, §38] The character table of $\mathcal{X}(\text{PSL}(2, q)), q = 2^r$ is given by

$$
\begin{array}{c|cccc|cccc}
1 & (q^2 - 1) & q^2 - q & \cdots & q^2 - q & q^2 + q & \cdots & q^2 + q \\
1 & 0 & -q + 1 & \cdots & -q + 1 & q + 1 & \cdots & q + 1 \\
1 & -q - 1 & \vdots & \vdots & [a_{kl}] & 0 \\
1 & q - 1 & \vdots & \vdots & 0 & [b_{mn}] \\
1 & q - 1 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
$$

where $a_{kl}$ and $b_{mn}$ as in the above theorem.

So, it is evident how the character table of $\mathcal{X}(\mathcal{M}^*)$ can be expressed in terms of $\mathcal{X}(\text{PSL}(2, q))$, and vice versa. It is also shown that the corresponding result for odd $q$ is very similar to the above. Namely, if we replace all $q^3$ by $q$ and $q^2$ by 1 in the character table of $\mathcal{X}(\mathcal{M}(q)^*)$ for $q = p^r$ with $p$ an odd prime, then the resulting table is the character table of the fusion scheme $\mathcal{X}(\text{PSL}(2, q))$ obtained from that of $\mathcal{X}(\text{PSL}(2, q))$ by combining two conjugacy classes of order $p$ of $\text{PSL}(2, q)$, which are conjugate in $\text{PGL}(2, q)$, into a single class. (See [9, Theorem 2.3.3].)

**Construction A.** The character table of $\mathcal{X}(\mathcal{M}(q)^*)$ was constructed as follows.

1. We calculated the parameters $p_{ij}^h$ of $\mathcal{X}(\mathcal{M}(q)^*)$ and expressed them in terms of parameters of $\mathcal{X}(\text{PSL}(2, q))$ as in [9, Lemma 2.2.2].

2. Using the relationship $\sum_{h=0}^d p_{ij}^h p_h(r) = p_i(r) p_j(r)$ between the parameters and the characters for $\mathcal{X}(\text{PSL}(2, q))$, and the relationship between the two sets of parameters obtained in (1), we found the relations between the parameters and entries of the character table of $\mathcal{X}(\mathcal{M}(q)^*)$.

3. Finally, we examined if the table satisfied the orthogonality conditions of rows and columns to be a character table.

Of course, this method only works when two schemes are intimately related so that the relations in (1) are simple enough to figure out the relations in (2). The construction of the character tables of Paige’s Moufang loops has led us to be able to construct those for many other association schemes via this method. Despite the ‘intimacy’ requirement, it has been used effectively in many cases where one scheme ‘controls’ many others. See, for example [6, 7, 29].

**4 Group-case association schemes**

All association schemes coming from the major sources listed in Introduction are essentially coming from either a finite simple groups or quasigroups, Gelfand pairs, the primitive permutation groups and the suitable subgroups that can be the point stabilizers of the actions of groups. All group schemes $\mathcal{X}(G)$ can be viewed as groups $G \times G$. 
acting on $G$ by $x \mapsto g^{-1}xh$. For given a quasigroup $Q$ and $x \in Q$, Suppose $G$ is the multiplicative group $Gr(Q)$ of $Q$ generated by all permutations $L(x)$ and $R(x)$ of $Q$ defined by

$$L(x): y \mapsto xy; \quad R(x): y \mapsto yx.$$ 

Then the association scheme defined by the orbits of $Gr(Q)$ on $Q \times Q$ as the associate relations, is isomorphic to the quasigroup association scheme $\mathcal{X}(Q)$ defined by its conjugacy classes.

The class of group schemes and quasigroup schemes contain all finite simple groups and quasigroups. All finite simple groups have been classified (cf. [16]), but quasigroups seems to require a lot more work. Many maximal subgroups of finite simple groups have been discovered by the work of many group theorists (cf., [22, 23]). So there are a lot of group-case primitive commutative association schemes to be studied. The character tables of some of these association schemes have been calculated. In doing so, the following facts play an important role.

**Construction B.** Let $G$ be a group and $H$ be a subgroup of $G$. Let $c_0, c_1, \ldots, c_d$ be class representatives of the conjugacy classes $C_0, C_1, \ldots, C_d$. Let $\{Hg_jH : 0 \leq j \leq d\}$ be the set of all double cosets of $H$. Suppose

$$1^G_H = \rho_0 + \rho_1 + \cdots + \rho_d$$

is the decomposition of $1^G_H$ into irreducible characters $\rho_0, \rho_1, \ldots, \rho_d$ of $G$. Then by [3, Corollary 11.7], the entries $p_j(i)$ of the character table of $\mathcal{X}(G, H\setminus G)$ are given by

$$p_j(i) = \frac{1}{|H|} \sum_{c \in Hg_jH} \rho_i(c) = \frac{1}{|H|} \sum_k |Hg_jH \cap C_k| \cdot \rho_i(c_k).$$

So, in order to calculate $p_j(i)$ for $\mathcal{X}(G, H\setminus G)$, we need to know

- the conjugacy classes and group characters of $G$,
- the set of double cosets of $H$ in $G$,
- the size of the intersection of each conjugacy class and each double coset, and
- the decomposition of $1^G_H$ into irreducible characters.

This procedure has been employed when W. Kwok (1991) [25] calculated the character table of $\mathcal{X}(O_3(q), O^+_2(q)\setminus O_3(q))$, the scheme obtained from the action of the general orthogonal group $O_3(q)$ acting on the sets of hyperplanes (for odd $q$; see [30] for even $q$). This character table controls the character tables of association schemes coming from the orthogonal groups on the sets of hyperplanes in the corresponding orthogonal geometries. We remark that the character tables of the association schemes coming from the action of $O^+_{2m}(q)$ on the set of non-isotropic points are obtained by modifying the character tables of the group $PSL(2, q)$ in exactly the same way as that of $\mathcal{X}(M^*)$ is obtained from that of $\mathcal{X}(PSL(2, q))$. The reason for this may be explained as follows. Let $V$ be a $2m$-dimensional vector space over $GF(q)$. Let $X$ be the set
of non-isotropic points corresponding to a non-singular quadratic form of Witt index \( m \). Then \(|X| = q^{m-1}(q^m - 1)\). The group \( G = O^{\pm}_{2m}(q) \) acts transitively on \( X \) if \( q \) is even, and it acts transitively on each half of \( X \) if \( q \) is odd. Any of these transitive permutation groups gives a symmetric association scheme of class \( q \) if \( q \) is even, and class \( (q+1)/2 \) if \( q \) is odd. It is shown that when \( m = 4 \) this permutation group \( O^+_8(q) \) on \( X \) (or the half of \( X \)) is isomorphic to the permutation group \( Gr(\mathcal{M}^*(q)) \) on \( \mathcal{M}^*(q) \).

The following table summarizes the results from [6, 7, 10, 25, 29]. In every case, the character tables of association schemes corresponding to permutation groups \( G \) on corresponding geometries are controlled by the character table of a ‘canonical’ one. This happens to almost all cases (cf. [5]).

**Table 1. Examples of schemes from source group 1.**

| Groups \( O^{\pm}_{2m}(q) \), \( q \) even | Geometries | Controlled by |
| --- | --- | --- |
| \( O^+_{2m}(q) \), \( q \) odd | nonisotropic points (or lines) | \( PGL(2, q) \) |
| \( O^+_{2m}(q) \), \( q \) odd | each half of nonisotropic points | \( PSL(2, q) \) |
| \( O^+_{2m+1}(q) \), \( q \) even | \( \pm \)-type hyperplanes | \( PGL(2, q)/D(2q \pm 1) \) |
| \( O^+_{2m+1}(q) \), \( q \) odd | \( \pm \)-type nonisotropic points | \( PGL(2, q)/D(2q \pm 1) \) |
| \( U_m(q) \) | nonisotropic points | \( PGL(2, q)/Z_{q+1} \) |
| \( Sp_{2m}(q) \) | nonisotropic lines | \( PGL(2, q)/Z_{q-1} \) |
| \( PTL(n, q) \) | non-incident point-hyperplane pairs | \( PGL(2, q)/Z_{q-1} \) |

The following table includes some examples of known Gelfand pairs for which corresponding association schemes have been investigated. However, the character tables of the associated commutative association schemes are not yet known for (4).

**Table 2. Examples of Gelfand pairs from the source group 2.**

| Labels | Groups \( G \) | Subgroups \( H \) | References |
| --- | --- | --- | --- |
| (1) | \( GL(2n, q) \) | \( Sp(2n, q) \) | Klyachko [24]; Bannai-Kawanaka-Song [1] |
| (2) | \( GL(n, q^2) \) | \( GL(n, q) \) | Gow [17]; Henderson [18, 19] |
| (3) | \( GL(n, q^2) \) | \( GU(n, q) \) | Gow [17]; Henderson [18, 19] |
| (4) | \( GL(2n, q) \) | \( GL(n, q^2) \) | Inglis-Liebeck-Saxl [23]; Terras [31]; Bannai-Tanaka [12]; Henderson [20] |
| (5) | \( GU(2n, q^2) \) | \( Sp(2n, q) \) | Inglis [22]; Henderson [20, 21] |
| (6) | \( Sp(4, q) \) | \( Sz(q) \) | Inglis [22]; Bagchi-Sastry [1]; Bannai-Song [8] |

There are many instances where the knowledge of all ingredients in Construction 13 does not automatically determine the character tables. Still there are many other examples of known Gelfand pairs, and character tables of their corresponding association schemes need to be determined. For example, among the association schemes \( \mathcal{X}(G, H \setminus G) \) coming from the Gelfand pairs \( G \) and \( H \) that appeared in the Arjeh
Table 3. Examples of Gelfand pairs whose character tables need to be determined.

| $\mathcal{SU}_n(q)$ | $\mathcal{SU}_3(q) \times \mathcal{SU}_3(q)$ | 20: $1^G_H$ decomposed |
|-------------------|---------------------------------|---------------------|
| $\mathcal{PSU}_4(q)$ | $\mathcal{SU}_2(q) \times \mathcal{SU}_2(q)$ | 20: $1^G_H$ decomposed |
| $\mathcal{PSU}_4(q)$ | $\mathcal{O}^-_4(q)$, $q$: odd | 20: $1^G_H$ decomposed |
| $\mathcal{P}\Omega^+_2(q)$ | stabilizer of an $\Omega^+_2(q)$ space | 14 [15]: double cosets described |
| $\mathcal{P}\Omega^-_2(q)$ | stabilizer of an $\Omega^-_2(q)$ space | 14 [15]: double cosets described |
| $\mathcal{P}\Omega_n(q)$ | stabilizer of an $\Omega^+_2(q)$ space, $nq$: odd | ? |

Using the classification of finite simple groups, multiplicity-free maximal subgroups of almost simple groups are getting well understood. This will eventually lead to the complete list of association schemes of this type which we are looking for.

5 Character tables of $\mathcal{X}(G_2(q), O_6^+(q))$

This is an example that we determine the character table by using fission relations together with orthogonality conditions of the character table.

Let $q$ be odd, and let $G = G_2(q)$ be the Chevalley group of type $G_2$. Let $\Omega_1$ and $\Omega_2$ denote the sets of hyperplanes of type $O^+_6(q)$ and $O^-_6(q)$ in the 7-dimensional orthogonal space over $\mathbb{F}_q$. $G$ acts transitively on $\Omega_1$ and $\Omega_2$. Let $H_1$ and $H_2$ be the one-point stabilizers of $G$ on $\Omega_1$ and $\Omega_2$, respectively. Then $H_1 \simeq SL_3(q).2$ and $H_2 \simeq SU_3(q).2$. The corresponding ranks of the permutation group are $\frac{1}{2}(q+5)$ and $\frac{1}{2}(q+3)$, respectively. It is shown that $\mathcal{X}(G_2(q), \Omega_1) \simeq \mathcal{X}(O_7(q), \Omega_2)$. The character table of $\mathcal{X}(O_7(q), \Omega_2)$ has been constructed in [6]. However, the character table of $\mathcal{X}(G_2(q), H_1\backslash G_2(q))$ is not isomorphic to $\mathcal{X}(O_7(q), \Omega_1)$ but to its fission table. We note that $|G| = q^{6}(q^2-1)(q^6-1)$, and $|\Omega_1| = |G:H_1| = \frac{1}{2}q^3(q^3+1)$ with rank($G, \Omega_1$) = $\frac{1}{2}(q+5) = 1 + \text{rank}(O_7(q), \Omega_1)$.

The character table of $\mathcal{X}(G_2(q), SL_3(q).2 \backslash G_2(q))$ and that of $\mathcal{X}(O_7(q), \Omega_1)$ are, respectively, given as follows

$$
\begin{bmatrix}
1 & 2(q^3 - 1) & (q^2 - 1)(q^3 - 1) & q^2(q^3 - 1) & \ldots & q^2(q^3 - 1) & \frac{1}{2}q^2(q^3 - 1) \\
1 & -q^2 + q - 2 & q^3 - q^2 - q + 1 & -2q^2 & \ldots & -2q^2 & -q^2 \\
1 & 2q^2 - 2q - 2 & q^3 - 4q^2 + 2q + 1 & -2q^2 & \ldots & -2q^2 & -q^2 \\
1 & -2 & -q^2 + 1 & \vdots & \vdots & \vdots & (q^2 \chi_{ij})_{1 \leq i \leq \frac{1}{2}(q-1)}^{1 \leq j \leq \frac{1}{2}(q-1)} \\
1 & -2 & -q^2 + 1 & \vdots & \vdots & \vdots & \vdots 
\end{bmatrix}
$$
where $\chi_{ij} \in \mathbb{Q}(\theta) \cup \mathbb{Q}(\rho)$, $\theta$ and $\rho$ are the $(q + 1)$th- and $(q - 1)$th- root of unity, are the entries of the character table of $\mathcal{X}(O_3(q), \Omega_1)$ described by Kwok [25] and can be calculated from the result in §6.

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