Russell and the Neo-Logicists

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Abstract

For the Neo-Fregeans, logicism is first and foremost a means to meet Benacerraf’s challenge. The contention is that Hume’s Principle provides us with an attractive semantic and epistemological theory, which avoids both extreme Platonism and fictionalism. This answer does not extend, however, to earlier versions of logicism – the ones defended by the historical Frege and Russell, which do not use any abstraction principle. From the neo-logicist perspective, the old versions of logicism no longer constitute credible philosophies of mathematics.

In this paper, I suggest that the central position occupied today by the Benacerraf’s dilemma blinds us to the possibility of other forms of philosophical agenda, which the ancient logicians attempted to fulfill. Focusing on geometry and the theory of reals, I show that, beside the unification and reduction of all mathematics to logic, another issue was at stake in The Principles as in Principia: how to carve mathematics at its joint? Russell wanted to arbitrate between the various conceptions of mathematical architecture, and found a rational way of doing this. If both mathematics and logic have changed since Russell’s time, there is reason to believe that the architectonic issue is still alive today.

Key words: Logicism, Russell, neo-logicism, Benacerraf

1. Introduction

The reception of Russell’s two masterpieces, The Principles of Mathematics and Principia Mathematica, has been strange. Certain parts of these works are today universally celebrated as the main sources of modern logic, of analytic philosophy and of philosophical logic, while others have been completely forgotten and have had no influence on anything in XXth Century philosophy. Schematically, the logical beginnings of both books have attracted much attention, while the remote and mathematically more advanced sections have been completely neglected. Thus, parts I (on logic) and II (on the definition of integers) of The Principles have been widely commented, while parts IV (on order) and V (on continuity and infinity) have been less considered, and parts III (on quantity), VI (on space) and VII (on bodies) have

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fallen into oblivion. In the same way, the first three parts of *Principia Mathematica* (on logic and cardinal arithmetic) are well-known, while the last three parts (on relation arithmetic, series and quantity) have been forgotten. Russell confessed in [Russell, 1959, 86] that he knew of only six people who had read the later parts of [Russell and Whitehead, 1913]. And I am willing to bet that the copy of *Principia Mathematica* owned by the reader (if she/he owns one) is the same as mine, *Principia Mathematica* up to section 56!

Are there any good reasons (by good reasons, I mean philosophical reasons) to come back today to the neglected parts of *The Principles* and *Principia*? Russell has written many books in his long career – and some of them, it seems, deserve the fate, i.e. oblivion, they shared. Why shouldn’t this be the case for the remote sections of [Russell, 1903], and [Russell and Whitehead, 1913]?

That the history of philosophy can inform contemporary debate cannot be taken for granted. Richard Heck’s last book on *Grundgesetze* is interesting in this respect, since Heck explicitly maintains that the remote parts of Frege’s works (Frege’s theory of real numbers) can be bypassed. More precisely, Heck claims that, if Frege’s definition of natural numbers still has a philosophical importance, this is not the case of Frege’s analysis of real numbers, which is expounded in Part III of [Frege, 2013]. Heck extends this diagnosis to the Russellian version of logicism. According to him, it is not only the last part of *Grundgesetze* which has only a patrimonial importance; the whole of Russell’s logical construction falls in this unenviable category. I like Heck’s irreverence. I believe, like him, that historians have to explain why they think it worthwhile to spend their time commenting old books. I do believe however, that taking into account the whole of Russell’s logicist works has something to teach us today about the meaning and the limitation of our own conception of logicism.

In section 2, I explain the reason Heck gives to justify his neglect of Frege’s theory of quantity, and show how this argument applies to the whole of Russell’s approach. In section 3, I make a historical detour and give some background elements on the remote parts of *The Principles* and *Principia*, that will help us to understand better what Russell’s project was. I insist on the fact that Russell did not espouse the arithmetization of mathematics. In section 4, I come back to Heck’s diagnosis and explain in which respect Russell’s approach escapes Heck’s objection.

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1 [Heck, 2012, 13]: ‘Before we continue, I want to issue an apology. […] We are not going to consider *Grundgesetze* in its entirety. I shall have very little to say about Part III, which is concerned with the real numbers. […] This is not because the material is uninteresting […] so I owe an explanation of why I am not discussing it.’

2 Taking into account Frege’s theory of reals has also probably something to teach us about the the meaning of logicism. But I won’t argue for this in this article.
2. Heck’s diagnosis

As Heck explains it, Frege’s reals can be seen as linear transformations of an underlying, one-dimensional metric space. The important point in the construction is that the structure $\mathbb{R}$ is derived from an underlying structure, from which it inherits the properties. This is the source of a philosophical difficulty [Heck, 2012, 14]:

The obvious question is where one is supposed to get this underlying space, and the question becomes all the more pressing once one realizes that, if the group of its linear transformations is going to look like the reals, then the structure must already be isomorphic to the reals.

For instance, to have the right sort of continuity on the group of linear transformations (that is, on the reals), one has to postulate the right sort of continuity in the underlying space (in the quantitative domain). When the one-dimensional space is endowed with the right sort of properties, then the definition is formally correct. But Heck believes this brings no gain, since what one is trying to obtain via the logicist definition (that is, the familiar properties of the structure $\mathbb{R}$) should be introduced at the outset (by postulating the right constraints on the underlying space).

According to Heck, this feature stands in sharp contrast with what one finds in Frege’s definition of natural numbers. Of course, Frege’s historical construction (based on law V) is flawed. But neo-Fregeans explain that law V is used by Frege to derive what is called Hume’s Principle (hereafter HP), and that the derivation of the laws of arithmetic from HP is correct. [Wright, 1983] has proved Frege’s theorem: the axioms for Peano Arithmetic can be derived in second-order logic from HP.

What impresses Heck in the arithmetical case is that something seemingly as innocent as HP provides us with the full strength of arithmetic. Contrary to what happened in the definition of the reals, one does not have to postulate the existence of an underlying space isomorphic to $\mathbb{N}$ in order to get the natural numbers. In particular, one does not need to postulate an infinity axiom. This point is crucial in

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3 For more on Frege’s theory, see [Hale, 2000] and [Simons, 1987].
4 Hume’s Principle says that the number belonging to the concept $F$ is identical with the number belonging to the concept $G$ if, and only if, the concept $F$ is equinumerous with the concept $G$, or in a more formal setting: $Nx : Fx = N x : Gx \iff \forall x (Fx \equiv Gx)$. ‘$Nx : \ldots x$’ designates the abstraction operator, which, when applied to a sortal concept $Fx$ gives the number of objects having the property $F$; ‘$\equiv$’ designates the relation of equinumerosity.
5 Today Peano Arithmetic is the proper name for a first-order theory of arithmetic. Here, I follow the neo-Logicians and use Peano Arithmetic to designate a (version of) second-order formalization of arithmetic (close to the one set by Peano).
6 On neo-logicism and the relation of the neo-logicist interpretation to Frege, see [Heck, 2011].
7 By ‘infinity axiom’, I mean the ‘assumption which may be enunciated as follows: “If
Frege characterizes the numbers in terms of HP and then proves directly that there are infinitely many numbers, even if there are no non-numbers. Of course it is essential to how Frege works this trick that numbers are objects. Only then will HP endlessly disgorge new numbers: We get the infinity of the numbers-series by considering: $Nx : (x = 0)$, $Nx : (x = 0) \lor (x = 1)$, and so forth, which means that 0 and 1 must be within the domain of the variable ‘$x$’, which is an objectual variable.

The use of the already defined integers to abstract their successor is the trick which allows Frege to ‘endlessly disgorge new numbers’ and to avoid resorting to an infinity axiom.

Heck emphasizes that this feature sets apart neo-Fregean logicism from Russell’s construction. In 1903, Russell (falsely) thought he had proved that there is an infinity of entities. He doesn’t make the mistake in 1912, where the infinity axiom always appears as an antecedent clause in a conditional. When dealing with finite arithmetic, Russell avoids the use of this hypothesis by resorting to a costly ascent-in-type strategy. But as soon as he comes to real analysis, Russell has to introduce the conditional clause. Heck is then right when he says that Russell ‘has to invoke an axiom that says that there are infinitely many things that aren’t numbers to prove that there are infinitely many things that are’ [Heck, 2012, 14], and that this stands in sharp contrast with what Frege does in his *Grundgesetze*.

There is thus something special in the neo-Fregean deduction of natural numbers one does not find elsewhere. HP is, ontologically speaking, a thin principle, in the sense that it does not by itself posit an infinity of objects in an underlying space. Furthermore, HP is connected to the concept of abstraction which has played a key role in the theory of knowledge since Aristotle. These two features explain why Frege’s approach is, from a philosophical perspective, better [Heck, 2012, 15]:

To be sure, HP plays an important formal role in neo-logicism, but its ontological and epistemological roles are what make Frege’s Theorem worthy of all the philosophical attention it has received.

Frege’s theory of quantity is formally correct – as is Russell’s theory of inte-
gers. But both frameworks are ontologically and epistemologically worthless. The ontological gain is null, since one has first to endow a domain of objects with the properties (continuity or infinity) one wants to recover at the end. And there is no epistemological story which explains how one comes to know that the underlying space is continuous or infinite. From a philosophical perspective, HP, and the use Frege makes of it in the *Grundgesetze* has, then, no rival. So goes Heck’s argument.

Let me reformulate it. From a philosophical perspective, neo-logicism can be viewed as an attempt to settle the dilemma elaborated in [Benacerraf, 1973]. In this paper, Benacerraf sets two goals to any philosophy of mathematics: first, accounting for the truth of mathematical statements;\(^{11}\) second, accounting for the knowledge of those truths.\(^{12}\) Benacerraf maintains that each constraint could only be satisfied at the expense of the other. To account for the truth of a basic arithmetical statement, one must admit there are arithmetical objects – but as these objects are not perceptible, one is at pains to explain how one can know anything about them. On the other hand, any attempt to give a reasonable explanation of mathematical knowledge threatens the idea that mathematical statements say something about independent mathematical facts. For Heck and the neo-logicians, HP explains how one can refer to numbers and acquire arithmetic knowledge, without relying on any unreasonable intellectual intuition. In [Hale and Wright, 2002], Hale and Wright present neo-logicism as an ‘intellectual’ response to the Benacerraf’s dilemma.\(^{13}\) According to them, HP provides us with a plausible solution to Benacerraf dilemma, which avoids both extreme Platonism and fictionalism.

Russell’s approach does not provide us with that type of goods. In *The Principles*, Russell is a full-fledged Platonist and endorses a very unreasonable epistemology based on acquaintance with abstract objects. At the time of *Principia* however, Russell’s position is much more refined, and it might compete with neo-Fregean approaches to solve Benacerraf’s dilemma. Indeed, Russell does not hold there that numbers and classes are *bona fide* objects, but are what he calls ‘incomplete symbols’. This basis could be elaborated to provide a nominalist answer to Benacerraf’s

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\(^{11}\) [Benacerraf, 1973, 666]: ‘Any theory of mathematical truth [should] be in conformity with a general theory of truth · · · which certifies that the property of sentences that the account calls ‘truth’ is indeed truth.’

\(^{12}\) [Benacerraf, 1973, 667]: ‘A satisfactory account of mathematical truth · · · must fit into an over-all account of knowledge in a way that makes it intelligible how we have the mathematical knowledge that we have.’

\(^{13}\) [Hale and Wright, 2002, 103–104]: ‘Two broad approaches seem possible: intuitional and intellectual. It may be proposed, first that an epistemology of math should reckon with a special faculty – traditionally ‘intuition’ – which enables an awareness of systems of abstract · · · broadly as ordinary sense perception makes us aware of ordinary concrete objects and their properties. Or it may be proposed that access to the objects of pure math is afforded by our general abilities of reason and understanding.’ The two intellectual approaches are neo-logicism and Shapiro’s *ante rem* structuralism.
dilemma. This is the promising line taken by Kevin Klement recently. Since my focus is on [Russell, 1903] as well as [Russell and Whitehead, 1913], I won’t follow this path here. I will thus stick to the official ‘unreasonable’ epistemological position that Russell took in 1903, and that he still held around 1910–13 with regard to logical constants and logical forms. In the rest of the paper, I will therefore follow Heck’s claim according to which Frege’s use of HP gives us reasons to hope for a resolution to Benacerraf’s challenge in a way that neither Russell’s logical constructions nor Frege’s definition of the reals do. But if this is the case, then a new question arises: If Russell’s thought is not driven by the kind of concerns that Heck calls ‘philosophical’, then by what kind of concerns was he driven? If Russell does not give any interesting answer to Benacerraf’s dilemma, how could we redefine the goals of the philosophical inquiry in order to enable us to say that, by reading *The Principles* and *Principia*, we have found what we were looking for?

Before confronting the question, let me make two remarks on the neo-Fregean view:

1. In neo-logicism, arithmetic plays a central role. This is of course partly due to HP, but this is also due to the works of the arithmetizers of the end of XIXth Century, which showed that the gap between discrete and continuous quantity is not irreducible. Real analysis and geometry, and then the whole of mathematics, can be seen as the study of complicated structures based on integers. Once one succeeds in explaining the nature of arithmetical knowledge, then one can extend the explanation to the other kinds (geometrical, etc.) of mathematical knowledge. Arithmetical knowledge is not merely a part of mathematics, it is the foundation of the whole building. Of course, I do not want here to challenge the fact that one can represent mathematics as an extension of arithmetic. What I want to point out is merely the fact that, in neo-logicism, the whole of mathematical knowledge is viewed as a philosophically non-problematic extension of arithmetical knowledge. Any question about the way to organize mathematics, to systematize what is, as a matter of fact, a multifarious domain made of several different parts, is not considered an important point. I am aware to slightly oversimplify the situation: neo-logicists do not focus their attention only on integers; attempts to derive set-theory, or the theory of real numbers, exist. But these attempts play a marginal role in the ongoing debate, in the sense that these extensions are not supported by every neo-logicist— in the sense also that the disputes about the nature of abstraction principles is most of the

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14 See his [Klement, 2013] and his forthcoming paper [Klement, forth].
15 [Shapiro, 2003] is a discussion of the prospects of basing set-theory on abstraction principle. [Hale, 2000] is an attempt to found Frege’s theory of real numbers on abstraction.
16 See for instance [Wright, 2000] and below.
time centered on HP and the arithmetical case.

2. The second, related, point is that, in neo-logicism, the main task is to construct a deductive line which goes from logic (extended by HP) to mathematics (arithmetic). That is, the neo-logicist claim is first and foremost an existence claim: is there a way to derive the mathematical content from the thin logical basis? Frege’s theorem shows there is such a path. The failure of the historical Frege can be interpreted as the discovery that the expected deduction did not, in fact, exist. Of course, I do not want to contest here that logicists, of all sorts, claim that there is a deductive path going from logic to mathematics. What I want to stress however is that this focus on existence relegates the question of unicity into the background. If more than one path proves to exist, how to choose the one to take? This is not an issue that is discussed by the neo-logicists. Again, I slightly oversimplify here: there have been debates about the various ways of deriving the sequence of the natural numbers from abstraction principles. But there is no doubt that the problem of arbitrating between different legitimate abstractionist constructions is incidental compared to the issue related to the legitimacy of using abstraction principles.

To summarize, in neo-logicism, the key philosophical issue is: How HP can meet Benacerraf’s challenge? This implies that arithmetical knowledge and that HP definition of integers occupy the center of the stage. Compared to this issue, the two other questions (how to structure mathematical knowledge, and how to choose a definition in a context where there are many possible ones) have no importance. In section 4, I will attempt to show that these two side issues are precisely those which govern Russell’s thought in the remote parts of *The Principles* and of *Principia*. But before doing this, I need to explain what is going on in these little known and unfamiliar texts.

3. Russell’s foundationalism put upside-down: logicism and arithmetization

The idea that Russell, in *The Principles* as in *Principia*, supported the arithmetization program, is widely shared, apparently for good reasons. Russell himself is the first to have presented his logicism as an extension of the reductivist approach of the arithmetizers. Indeed, at the beginning of [Russell, 1919, 5–6], one finds:

All traditional pure mathematics, including analytical geometry, may be regarded as consisting wholly of propositions about the natural numbers. That

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17 See for instance Boolos’ proposal of deriving the sequence of natural numbers by modifying Frege’s axiom V in [Boolos, 1989] and the analysis of Shapiro in [Shapiro, 1999].
is to say, the terms which occur can be defined by means of the natural numbers, and the propositions can be deduced from the properties of the natural numbers – with the addition, in each case, of the ideas and propositions of pure logic. That all traditional pure mathematics can be derived from the natural numbers is a fairly recent discovery . . . For the present, we shall take for granted the arithmetization of mathematics, though this was a feat of the very greatest importance. Having reduced all traditional pure mathematics to the theory of the natural numbers, the next step in logical analysis was to reduce this theory itself to the smallest set of premises and undefined terms from which it could be derived.

Russell starts from the fact that the arithmetizers (Dedekind, Cantor, Weierstrass) have grounded all mathematics on arithmetic. From this perspective, geometry can be reduced to the theory of the reals, that can be, in its turn, reduced to arithmetic:

| Geometry | Real Analysis | Arithmetic |

Russell’s logicism would go one step further: all mathematics, arithmetic included, would be grounded on logic. Thus, the picture of mathematics Russell articulated would resemble this one:

| Geometry | Real Analysis | Arithmetic | Logic |

This would explain why the definition of integers are central: once arithmetic is deduced from logic, one can use the works of Cantor, Dedekind and Weierstrass, to deduce the rest of mathematics.

This picture, where the mathematical disciplines are piled on top of each others, and where all of them are based on logic, is the standard one. One even sometimes equates logicism with this schema. Note however that the vertical structure is not forced on us. One could very well be a logicist, and refuse to give arithmetic such an
importance. Indeed, that all mathematics is deducible from logic is perfectly compatible with a situation in which the various mathematical branches are not all reducible to arithmetic. Thus the following schema, where geometry, and real analysis, are directly deduced from logic, without being first reduced to arithmetic, encapsulates a logicist view of mathematics:

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| Arithmetic | Real Analysis | Geometry |
|-------------|---------------|----------|
|             |               | Logic    |
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Table 2  Logicism without arithmetization

My goal in this section is to show that this picture is actually the one that Russell endorsed. In other words, I want to demonstrate that, contrary to what he suggested in [Russell, 1919], the logicist Russell did not endorse the arithmetization program. That the arithmetization of mathematics was looked at with a critical eye is first indicated in a letter from Whitehead to Russell dated from 1909, September the 14th:

The importance of quantity grows upon further considerations — *The modern arithmeticisation [sic] of mathematics is an entire mistake*. ...To consider [arithmetical entities] as the sole entities involves in fact complicated ideas by involving all sorts of irrelevancies — In short the old fashioned algebras which talked of ‘quantities’ were right, if they had only known what ‘quantities’ were — which they did not.

Whitehead can hardly be clearer: arithmetization is an entire mistake. But the passage is not from Russell, and it is extracted from an unpublished private letter. Shouldn’t we cautiously stick to the standard picture and disregard this unorthodox opinion?

As a matter of fact, what one finds in the advanced parts of [Russell, 1903], and [Russell and Whitehead, 1913] complies with what Whitehead said in his letter. It is thus the well-known descriptions coming from [Russell, 1919], not Whitehead’s claim, that misrepresent the content of *Principia. Introduction to Mathematical Philosophy* was meant to be a popular book, where Russell oversimplified his position. Note however that even there, Russell alluded at times to alternatives to the arithmetization program. For instance, after having defined a rational in an ‘arithmetic way’ (as an equivalence class in \( \mathbb{N} \times \mathbb{N} \)), Russell explains [Russell, 1919, 64]:

We need fractions ...most obviously for purposes of measurement. My friend and collaborator A. N. Whitehead has developed a theory of fractions specially
adapted for their application to measurement, which is set forth in Principia Mathematica.

In his [Russell, 1919], Russell only hinted at these non-arithmetical constructions, and referred the reader to Principia for a full treatment of the question.\(^{18}\) Let me follow his advice and enter into the terra incognita. I will focus on the important examples of the theory of reals in [Russell and Whitehead, 1913] and the theory of space in [Russell, 1903]. In both cases, we will see that Russell’s approach opposed ‘the modern arithmetization’. I will however, for reasons of space, be brief and won’t delve deeply in developments that are complicated in their detail.\(^{19}\)

3.1. Real numbers in Principia

How do Russell and Whitehead define rational and real numbers in Principia? Standard techniques were used at the time (Cauchy sequences or Dedekind’s cuts, for instance) to define the reals in terms of rationals – which in turn were defined in terms of integers. As a matter of fact, Russell and Whitehead heavily relied on Dedekind’s cut method in Part V of their work. In section *275, after having given a formal definition of the series of order type \(\eta\) (i.e., the series isomorphic to \(\mathbb{Q}\)) and of the series of order type \(\theta\) (i.e., the series isomorphic to \(\mathbb{R}\)), the authors proved that the series of Dedekindian cuts of a \(\eta\)-series is a \(\theta\)-series (*275.21). This theorem is a reformulation of Dedekind’s construction of \(\mathbb{R}\) as the set of cuts in \(\mathbb{Q}\).

Does this use of Dedekind’s technique make Russell and Whitehead partisans of the arithmetization program? Not at all. Principia V is devoted to the general theory of series. No mention of the notion of number is made at this stage. It is only in Principia VI that rational and real numbers are defined. Like in Frege, real numbers are connected in Principia VI to measurement of quantities. But whereas Frege derived the definition of a real number from a very specific underlying structure, Russell and Whitehead distinguish two kinds of numbers, the ‘pure’ and the ‘applied’ numbers.\(^{20}\) Applied numbers resemble Frege’s numbers in the sense that they can be said to measure quantities, and also in the sense that they are transformations (Russell and Whitehead use the term ‘relations’) between the elements of an underlying space. Pure numbers do not correspond, however, to anything in Frege. They are relations as well; but the domain over which they act is very general and is not constrained by some postulates.

To clarify the distinction between pure and applied numbers, and to expound the insight that governs Russell and Whitehead’s development, let me focus on the

\(^{18}\) This is the case as well in [Russell, 1919, 114], where limit of function is defined for function with non numerical arguments.

\(^{19}\) For more on this, see [Gandon, 2012].

\(^{20}\) [Russell and Whitehead, 1913, 407].
rational number. The pure rational number \( m/n \), with \( m \) and \( n \) two positive integers \((n \neq 0)\), is defined in *303.01 [Russell and Whitehead, 1913, 260].

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m/n =_{\text{def}} \{(R, S) \mid R^n \cap S^m \neq \emptyset\}
\]

The definition makes use of the notion of relational product: if \( A \) and \( B \) are two binary relations (of the same type), then it is possible to form the relation \( A \mid B \), which is such that \( A \mid B(x, y) \) if, and only if, there is a \( z \) such that \( A(x, z) \) and \( B(z, y) \). The relational product is thus the extension to binary relations of the familiar notion of composition of functions. Now, \( R^n \) is the relational product of \( R \), \( n \) times with itself. According to *303.01, two binary relations \( R \) and \( S \) (of the same type) have the ratio \( m/n \) if, and only if, there are at least two objects \( x \) and \( y \) such that \( R^n(x, y) \) and \( S^m(x, y) \).

To understand the idea, let us imagine that \( R \) and \( S \) are two oriented vectors acting on the Euclidean line; then \( R \) and \( S \) have the ratio \( m/n \) if, and only if, \( n \) steps of size \( R \) from a point \( a \) leads to exactly the same point \( b \) as \( m \) steps of size \( S \).

Here the vectors \( R \) and \( S \) have the ratio 2/3 because three steps of size \( R \) from \( a \) leads to exactly the same point \( b \) as two steps of size \( S \). The definition *303.01 recovers and generalizes the insight behind the Euclidean definition of a ratio. Having introduced addition, multiplication and order on pure rational numbers (see *304, *305, *306), Russell and Whitehead succeed in proving (with the help of the axiom of infinity) that the set of ratios generates a dense Archimedean ordered field. This means that *Principia* pure ratios have exactly the same properties as our usual rational numbers.

The difference with Frege comes from the fact that the relational structure to which *Principia* pure ratios are applied is not as specific as Frege’s underlying metric space. Typological restrictions aside, no constraints are put on the domain which the relations \( R \) and \( S \) belong to. This is a feature which is occulted by the example I have just given of vectors acting over the Euclidean lines. Indeed, contrary to what the example suggests, pure rational numbers do not measure anything: two relations \( R \) and \( S \) can have distinct ratios, and one and the same ratio can hold between \( R \) and two different relations. To be considered as a measure of relations, supplementary

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21 I slightly simplify the definition and modernize the notation.
constraints should be set on the domain of relations to which the rational numbers are applied. This is the reason Russell and Whitehead, after having presented the properties of pure ratios in section A of Principia VI, concentrate their examination, in section B, on vector families, that is, on more specific sets of relations. The general theory of measurement (or the theory of applied numbers), which combines the results of section A on pure numbers with that of section B on vector family, is presented in section C. Whereas Frege begins right from the start with some very special quantitative domains (the analog of Principia vector families), Russell and Whitehead first define a very general notion of number (endowed with all the usual arithmetical properties) that applies to any relational structure; second, they particularize the domain of relations in order to guarantee that numbers, thus defined, measure (in a sense that is specified at the beginning of section C) the vectors to which they are applied.22

Russell and Whitehead’s construction of rational and real numbers is intricate and complex,23 but enough has been said, I think, to show that Whitehead did not ramble, when he held, in his letter to Russell, that ‘the modern arithmetization of mathematics is an entire mistake’, and that ‘the old fashioned algebras which talked of “quantities” were right, if they had only known what “quantities” were.’ In Principia, rational and real numbers are not seen as set-theoretical constructions based on integers; they are viewed as relations of relations, and they are connected, in a more flexible way than in Frege’s, to measurement of quantity.

Let me add something. The connection made in Principia VI between numbers and quantities takes its roots in geometry, and more particularly in the issue related to the introduction of coordinates into a geometrical space. The problem of the articulation between synthetic (Euclidean) and analytic (Cartesian) approaches of geometry occupied a central place in the mathematics of the XIXth Century. One way of dealing with this issue was to introduce the coordinate grid as the result of a geometrical construction – or rather as the result of the iteration of one and the same geometrical construction. M"obius designed such a method, which was generalized and systematized, after him, by von Staudt. In the volume II of their classical Projective Geometry, Veblen and Young based the classification of the various projective spaces on the analysis of the properties of the M"obius net (i.e., the coordinates grid generated by M"obius construction). Russell and Whitehead’s theory of measurement in Principia VI is explicitly an attempt to extend M"obius’ technique to a very general kind of (non-necessarily projective) structure, called measurable vector family.

22 In this respect, Russell and Whitehead’s definition of rational numbers escape the objection Heck addressed to Frege’s definition of real numbers: the authors of Principia do not set particular constraints on the quantitative domain to recover the familiar ordinal properties of the rational numbers.

23 For more on this fascinating doctrine, I refer to my [Gandon, 2012, chap. 5].
Indeed, in the key sections *352 to *355, devoted to measurement by real numbers, Russell and Whitehead define the minimal conditions which would allow them to give a sense to the notion of a Möbius net. Instead of using Möbius’ technique as a tool for classifying the various species of projective spaces, the authors of Principia seek to describe the abstract and general scaffolding surrounding the use of such a technique. Despite the difference in their projects, there is thus a genuine proximity between [Russell and Whitehead, 1913] and [Veblen and Young, 1918]: both works find a common source of inspiration in the pure synthetic approach focusing on the geometrical construction of coordinate. And this strengthens the point I want to make: far from being rooted in the arithmetization of analysis, Russell and Whitehead’s doctrine of rational and real numbers finds its source in the tradition of pure synthetic geometry.

3.2. Geometry in The Principles

Before the XIXth Century, mathematics was defined as the science of quantity. The genus quantity was itself subdivided in two species: the continuous quantities, whose properties were investigated in geometry, and the discrete quantities, whose properties were studied by arithmetic. The arithmetization program put an end to this division, by showing that geometrical quantities can be defined in terms of arithmetical quantities. Did Russell in The Principles endorse this perspective? Did he regard space as a numerical structure?

One finds at least three characterizations of geometrical spaces in [Russell, 1903]. One first finds an arithmetical definition of space. In §474, a numerical model $\mathbb{R}^n$ of a Euclidean $n$-dimensional space is constructed: ‘from the existence of $\theta$ [the real line seen as a set-construction based on $\mathbb{N}$], by the definition of complex numbers . . . , we prove the existence of the class of Euclidean spaces of any number of dimensions’. Then, using standard techniques, Russell constructs a numerical model of projective space, and the various numerical models of metrical spaces with constant curvature. This way of defining space is in line with the arithmetizing view according to which geometry is an extension of arithmetic.

But this is not the only definition of space one finds in The Principles – nor the most developed one. In Part V of the book, devoted to geometry, Russell discusses two other definitions, not arithmetical in character. In chapter 46, space is defined as a purely ordinal structure. Here Russell elaborates on the construction presented in Pasch’s Vorlesungen über neure Geometrie (1882). A non-metrical structure (called descriptive space) is first defined by an axiom system, which includes two non-logical terms: the point and the betweenness relation. Once the so-called descriptive theory

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24 For a presentation of the different stages of their construction, I refer the reader to [Gandon, 2012, 151–153].
is developed, congruence axioms are introduced to recover the full strength of metrical geometry. In *The Principles*, Russell shows how the three indefinables (point, betweenness, congruence) can be explained away, by substituting variables of the appropriate types to these non-logical constants. This second definition of space is logical (in so far as it contains no non-logical terms), but it is not arithmetical: space is not identified to a numerical manifold.

The last definition of space is expounded in chapter 45. It is grounded on Pieri’s works and based on the idea that projective space (and then any kind of space) is basically an incidence structure. It is this last characterization which plays the most important role in [Russell, 1903]. Let me present it in more detail.

The insight that real projective geometry can be conceived as the theory of incidence relation between points, lines and planes was first articulated by the German geometer Georg von Staudt. But von Staudt, in proving what will be called the fundamental theorem of real projective geometry (which says that a projective transformation on a line is fixed once the projective images of three points are fixed), made a mistake, generalizing to the real line what held for the rational points only. Klein, who first pointed out this deficiency, thought that the gap in Staudt’s proof could be filled by introducing ordinal hypotheses. For Klein then, projective geometry dealt with incidence, but also with order. In 1898, Pieri shows that Klein’s diagnosis was too hasty. In [Pieri, 1898], he completes von Staudt’s construction without introducing extraneous ordinal considerations. More precisely, Pieri succeeds in defining a separation relation (i.e., a projective order) on the projective line by setting constraints on the way lines intersect in the plane. In this way, Pieri restores von Staudt’s conception, according to which projective space is an incidence structure. This beautiful extension of von Staudt’s view is at the basis of Russell’s definition of geometry as ‘the study of series of two or more dimensions’ [Russell, 1903, §352]. In *The Principles*, Russell expounds Pieri’s formal theory, and he shows how, by replacing the non-logical constants (points, lines, planes and their incidence relations) by variables of the appropriate types, one can extract a purely logical definition of projective space as an incidence structure.

The connection between incidence and space has been emphasized by Whitehead, who, in the introduction of [Whitehead, 1906, p. 4-5], explains:

Geometry, in the widest sense in which it is used by modern mathematicians, is a department of what in a certain sense may be called the general science of clas-

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25 For more on this, see [Pasch, 1882].
26 See [Russell, 1903, 429–430].
27 Russell resumes Klein’s view according to which metrical geometry can be derived from within a projective setting.
28 I refer to [Gandon, 2012, chapter 2] for more on this complicated story.
sification. This general science may be defined thus: given any class of entities \( K \), the subclasses of \( K \) form a new class of classes, the science of classification is the study of sets of classes selected from this new class so as to possess certain assigned properties.

For example, in the traditional Aristotelian branch of classification by species and genera, the selected set from the class of subclasses of \( K \) are (1) to be mutually exclusive, and (2) to exhaust \( K \) ... The importance of this process of classification is obvious, and is sufficiently emphasized by writers on Logic. ... Geometry is the science of cross-classification. The fundamental class \( K \), is the class of points; the selected set of subclass of \( K \) is the class of (straight) lines. This set of subclasses is to be such that any two points lie on one and only one line, and that any line possesses at least three points. These properties of straight lines represent the properties which are common to all branches of the science which usage terms Geometrical, when the modern Geometries with finite numbers of points are taken account of.

Cross-classification is Whitehead’s term for incidence (incidence axioms are called cross-classification axioms by him). The abstract and general character of the definition, and the contrast made between logical and geometrical classification, help to understand Russell’s idea: a space is nothing else than a classification in which the elements can belong to more than one subsets. The final reference to geometries over finite fields, which began to emerge in the works of Veblen and his collaborators, shows that Whitehead was aware of the scope of this definition. In finite space, there is no continuity, no infinity, no order relation – a finite space is a space just because it is an incidence structure.

Let me summarize what I have said so far. There are at least three logical definitions of geometrical space in [Russell, 1903]. The first one defines a space as a numerical manifold (points are \( n \)-uplets of real numbers, \( n > 1 \)). In the two last ones, a space is regarded as a model of an axiomatic theory (in which no non-logical constants occur). In the second one, projective space is viewed as an ordinal structure, while in the third one, projective space is considered as an incidence structure. Which characterization did Russell endorse at the end?

His final and considered view is that, despite the formal correctness of the two first approaches, space should be defined as an incidence structure.\(^{29}\) It is the Von Staudt-Pieri line which Russell finally chose. The definition of space one finds in The Principles does not then square with the portrait of Russell as a partisan of the

\(^{29}\) This is made clear in chapter 44, the first chapter of Part V, where Russell defines geometry as the general ‘study of series of two or more dimensions’ [Russell, 1903, §352]; see also [Russell, 1903, 421], where Russell credits Pieri to have completed von Staudt’s works.
arithmetization program. Russell did acknowledge, in 1903, that geometry can be, in this way, ‘reduced’ to arithmetic. But this is not the path Russell chose to take. On the contrary, his final view is that the pure synthetic geometers, who opposed the Cartesian view, were right.

What can be learned from this short presentation of two key theories developed in the remote parts of *The Principles* and *Principia?* The picture one gets is not the standard one, according to which logicism is a continuation of arithmetization. Of course, Russell and Whitehead stuck to the view that all mathematics is reducible to logic. But this claim does not imply that Russell and Whitehead held that all mathematics should be first reduced to arithmetic. A global reductionism (logicism) is compatible with various local anti-reductionisms (the refusals of reducing geometry and real analysis to arithmetic). The form of Russell’s logicism resembles what one finds in table 2 above.

4. Logicisms – new and old

In section 2, we agree with Heck that, viewed from a Benacerrafian perspective, Russell’s logicism did not stand comparison with neo-Fregean logicism. Thus, the question surfaces again: if Russell was not driven by a Benacerrafian-like issue, then by what kind of concerns was he driven?

In neo-logicism, the mathematical target is regarded as a well-defined and homogeneous one: arithmetic. The main problem is indeed to guarantee that at least one road from logic to arithmetic exists. The situation is different in Russell’s works. As we have just seen, mathematics for Russell, is not a homogeneous body of knowledge. Mathematics has distinct parts (at least three: arithmetic, real analysis and geometry), and the central problem is precisely to articulate the various mathematical subdisciplines. In Russell, one does not start with a pre-definite division of the mathematical field. On the contrary, one of the most difficult philosophical tasks in the advanced parts of *The Principles* and *Principia* is precisely to discuss and determine how mathematics should be organized – to solve what could be called

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30 Once again, following Klement, one could (and should) resist this diagnosis. But as we focus here on [Russell, 1903] as well as [Russell and Whitehead, 1913], this path is not, for us, an available option.

31 I am not claiming that Russell held that mathematics can be reduced to those three parts. Arithmetic, real analysis, geometry are here just typical examples of distinct mathematical disciplines. I am also not interested in determining how Russell would reduce other disciplines, like algebra, complex analysis, topology, etc. I am merely concerned by the philosophical consequences of the acknowledgment that mathematics is made of different parts.

32 That the real issue is not to guarantee the existence of one road from logic to math-
the ‘architectonic issue’. This question directly leads to the unicity problem. Indeed, choosing a definition among several possible ones is favoring one structuration of the mathematical field among other possible ones. In other words, the two questions raised at the end of section 2 (How to structure mathematical knowledge? How to choose a definition in a context where there are many possible ones?) are the two sides of the same Russellian coin. It is by meeting the latter (the unicity issue) that one solves the former (the architectonic issue).

It might be thought that the architectonic issue, while being recognized as an important question, is not a genuine philosophical problem. Basing the analysis on Benacerraf’s challenge, one might consider that problems in philosophy of mathematics are about the place that mathematical knowledge, taken as a whole, occupies in human knowledge in general – that philosophy of mathematics should not be concerned with the internal structure of mathematical knowledge. Furthermore, one could insist on the fact that mathematicians do speak about the structure of mathematical knowledge: they prove unifying conjectures, representation theorems, etc. This would provide philosophers with another argument showing that the architectonic issue should be discarded as a purely mathematical affair. This line of reasoning is present in Heck: the issue raised by Frege’s theory of reals is first and foremost mathematical, and not genuinely philosophical. How to answer this argumentative line?

Mathematicians do speak about the organization of their science, it is true – but they never stop disagreeing on the value of the results they manage to collect. In this matter, what is complicated is, indeed, not so much obtaining results, but assessing their relevance in view of the architectonic issue. Questions about the organization of mathematics are not like questions of knowing whether or not such and such conjecture is proved. They are more like questions concerning the best proof of such and such a theorem, or the best way to define such and such a notion. There is no consensus within the mathematical community about how to settle this kind of interrogation. Nor is there agreement among mathematicians about the best way mathematics should be structured. The lack of consensus concerning this issue leads mathematicians (and philosophers of mathematics alike) to endorse two opposed attitudes: dogmatism and skepticism. The dogmatist sticks to one of the available options, ignoring the others. The skeptic, on the contrary, acknowledges the plurality of approaches, but professes tolerance and renounces any opinion about the

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33 This attitude could be the one of a dogmatic arithmetizer, as pictured in Whitehead’s letter to Russell, who does not want to hear about the use of numbers in geometry or in the measurement of quantities.
This grim scenario is forced on us, however, only in so far as one regards the architectonic issue as a pure mathematical question, one that can be solved by proofs and theorems. Russell thought differently. He acknowledged, with the skeptics, that one cannot prove that one architecture is better than the other. But he nevertheless believed that facts and arguments on various sides of the architectonic dispute can be cautiously apprehended, and rationally weighed to reach a fair and considered judgment. What is striking in the remote parts of *The Principles* and *Principia* is that logic and the theory of relations is not used merely as a means to derive existing mathematics – it is used as a framework for representing and confronting the various organizations of mathematics that were developed by mathematicians. The logical reconstruction of the different perspectives provided Russell with a common ground upon which the mathematical and philosophical debate about the pros and the cons of the various approaches can proceed. At the end, Russell endorsed a certain view of the mathematical architecture. But, his choice was based on the discussion of the alternatives, and was never presented as the outcome of a proof. The skeptic and the dogmatic have in common the belief that, outside proofs, no rational arguments are available. Russell and Whitehead thought, on the contrary, that the architectonic issue could (and should) be the topic of a rational discussion, which must not take, however, the shape of a proof — in brief, they thought that the architectonic issue falls within the philosophical scope. What allowed them to go beyond dogmatism and skepticism was the idea that the architectonic issue is a philosophical problem.

Take the theory of space as an illustration. There are three ways of defining the notion of a projective space – as a numerical manifold of a certain kind, as an ordinal structure of a certain kind, as an incidence structure. All these definitions are formally irreproachable; all of them lead to a reduction of the basic geometrical notions to logical constants. Now, this definitional plurality mirrors the various ways mathematicians, at the time, conceived of the relation between projective geometry and the rest of mathematics. Some (like Klein, Dini, Darboux) regarded geometry in a ‘Cartesian’ way, as a development of real or complex analysis; some (like Pasch, and Hilbert perhaps) thought that geometry was a science of order; some at last (von Staudt, Pieri) considered geometry as a theory of incidence. Each of these views, that are represented by distinct logical constructions within Russell’s system, was supported by prominent mathematicians, and so, no consensus prevailed at the time on this issue within the mathematical community. Russell chose to support the incidence approach. His reasons were roughly that projective space is the fundamental geometrical notion, and that von Staudt’s view is the only one to account for

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34 As an example of a ‘tolerant’ attitude, one can refer to Vailati’s pragmatical approach. See [Arrighi et al., 2010].
35 See [Russell, 1903, chaps. 44, 48].
what makes projective geometry so special.\textsuperscript{36} These arguments are, of course, not compelling: one can consider that projective space is a very specific concept, which will soon be superseded by topological structures;\textsuperscript{37} one can argue that the incidence theory does not succeed in capturing the ‘essence’ of projective space – that, for instance, algebraic approaches are more promising.\textsuperscript{38} In other words, there is no logical or mathematical proof, in \textit{The Principles}, that projective space is an incidence structure. But so what? Russell never claimed to prove that his view of geometry is the only possible one. On the contrary, he left room for alternative views. The issue at stake was not for him mathematical, and it is precisely on this point that Russell differed from both the skeptic and the dogmatic. Seen as an answer to a philosophical problem, Russell’s position, if not compelling, was nevertheless rational, since his argumentation proceeds from a careful comparison of the merits and shortcomings of the other possible hypotheses.

The idea that the architectonic issue is a purely mathematical affair, and not a genuine philosophical question, cannot be sustained. As a matter of fact, the idea that one could draw, once and for all, a sharp separation line between philosophical and mathematical problem is doomed to fail: problems concerning the internal organization of mathematics are typically pertaining both at the same time to mathematics and to philosophy. As I said above, mathematicians have something to say about the architectonic issue, and one cannot hope to make progress on this matter without taking into account their results. But this is not enough: mathematicians coming from different traditions often talk past each other, and room for discussion, within the mathematical community, is usually wanting. There is a place here for philosophy; in the remote part of \textit{The Principles} and \textit{Principia}, Russell used his logical framework as a tool for giving voice to, and arbitrating between, the various ways of carving the mathematical landscape.

\section*{5. Conclusion}

Let me summarize the discussion. Benaceraff’s challenge occupies a central place on the agenda of the philosophy of mathematics today.\textsuperscript{39} Neo-logicism is often presented as an attractive response to this dilemma; and, in this respect, Heck is right to emphasize the gap between Frege’s construction of integers and the rest of the old

\textsuperscript{36} See [Russell, 1903, 421], [Whitehead, 1906].
\textsuperscript{37} This was Poincaré’s view; see [Poincare, 1902].
\textsuperscript{38} The standard definition today of a projective space is algebraic in character: given any vector space $V$ over a field $F$, the associated projective space $P(V)$ is the structure $V - \{0\} / \sim$, where $\sim$ is the equivalence relation $u \sim v$ such that $u = \lambda v$, for $u, v \in V - \{0\}$, and $\lambda \in F$.
\textsuperscript{39} On this, see the preface of [Mancosu, 2008].
logicist construction. If one considers the philosophy of mathematics as an answer to Benacerraf’s dilemma, then it seems that the advanced parts (at least) of Russell’s books have only a patrimonial interest. I have here turned things around, and attempted to extract from Russell’s developments, not new answers to already known issues, but new problems that are no longer addressed today, or at least not enough.

Carefully reading the mathematical parts of *The Principles* and *Principia* shows that one should not regard logicism as an extension of arithmetization. Russell did not start with a pre-definite view of how existing mathematics is structured. On the contrary, the architectonic issue was an integral part of the logicist task, and it is this issue (which shape should one give to the mathematical material?) that is made optional when one puts Benacerraf’s dilemma at the center of the philosophical stage. Has this problem become obsolete?

Since Russell’s time, mathematics and logic have widely changed in all sorts of ways. The conceptual tools and the logical framework devised in *Principia* seem no longer relevant today. But the architectonic issue is still a current problem – even bigger today, it seems, than at Russell’s time. Indeed, mathematics has become much more heterogeneous than it was, the mathematical subdisciplines have exploded in number, emergence of new techniques (like computer-assisted proofs) have increased the fragmentation of mathematical knowledge. And there is even less consensus among mathematicians about the architecture of mathematics than there was at the beginning of the XXth Century. It seems then that Russell’s aspiration to provide mathematicians with the means to discuss and compare the various ways they could organize their material remains a valid philosophical ambition.

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