On a generalization of the iterative soft-thresholding algorithm for the case of non-separable penalty

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Abstract
An explicit algorithm for the minimization of an $\ell_1$-penalized least-squares functional, with non-separable $\ell_1$ term, is proposed. Each step in the iterative algorithm requires four matrix vector multiplications and a single simple projection on a convex set (or equivalently thresholding). Convergence is proven and a $1/N$ convergence rate is derived for the functional. In the special case where the matrix in the $\ell_1$ term is the identity (or orthogonal), the algorithm reduces to the traditional iterative soft-thresholding algorithm. In the special case where the matrix in the quadratic term is the identity (or orthogonal), the algorithm reduces to a gradient projection algorithm for the dual problem. By replacing the projection with a simple proximity operator, other convex non-separable penalties than those based on an $\ell_1$-norm can be handled as well.

(Some figures may appear in colour only in the online journal)

1. Introduction

Non-smooth minimization problems involving a sum of a quadratic data misfit term and a non-smooth penalty term have received a lot of attention in inverse problems and imaging in recent years. In this paper, we are interested in finding the minimizer $\hat{x}$ of the $\ell_1$-penalized least-squares functional $F$:

$$\hat{x} = \arg\min_x F(x) \quad \text{with} \quad F(x) = \frac{1}{2} \|Kx - y\|^2 + \lambda \|Ax\|_1, \quad (1)$$

by means of an iterative algorithm. Here, $\|u\|^2 = \sum u_i^2$ with $u_i \in \mathbb{R}$ and $\|w\|_1 = \sum |w_i|$ ($w_i$ may be an element of $\mathbb{R}$, $\mathbb{R}^2$, etc. and $|w_i|$ stands for the Euclidean length of $w_i$; other choices of $|w_i|$ are discussed in section 6). $K$ is a matrix mixing the variables in the quadratic data misfit term and $A$ is a linear operator mixing the variables in the penalty term. The quadratic term is convex and smooth, but the penalty term $\|Ax\|_1$ is convex and non-smooth. We work in a finite-dimensional setting.
For the case where the non-smooth penalty term in (1) is simple \((A = 1)\), many algorithms have appeared in recent years. One of the earliest (not necessarily the most efficient) is the iterative soft-thresholding algorithm [1] (see also section 3). As \(\ell_1\)-norm penalties promote sparsity, such algorithms are used in ‘compressed sensing’ [2] for finding a sparse solution (up to noise level) of a large-scale under-determined linear system. As problems in 2D and 3D imaging are large-scale problems, with many unknowns, such simple first-order iterative algorithms can still be useful.

The principal difference of this paper with respect to [1] is the presence of the matrix \(A\) in the penalty term. In image processing, the total variation penalty, which favors piecewise constant images, is popular for its ability to maintain sharp edges. The total variation penalty is defined by the \(\ell_1\)-norm of the gradient of the unknown \((A = \text{grad})\). It has mostly been studied for denoising \((K = 1)\) or for other special operators \(K\) (e.g. deconvolution).

Our aim here is to provide a simple iterative algorithm for the problem (1) with proven convergence (see theorem 1). We also desire an algorithm that is fully explicit: each step in the proposed iteration only uses four matrix–vector multiplications (one by \(K\), \(K^T\), \(A\) and \(A^T\)) and a simple projection on the \(\ell_\infty\) ball (or equivalently a single thresholding).

Although our main aim is to solve problem (1), we will formulate an algorithm and a convergence theorem for the more general problem:

\[
\hat{x} = \arg \min_x F(x) \quad \text{with} \quad F(x) = \frac{1}{2}\|Kx - y\|^2 + H(Ax),
\]

where \(H\) is a convex function (we assume that the solution to (2) exists). For problem (2) the projection operator mentioned before is replaced with the proximity operator \(\text{prox}_H\) of the convex conjugate \(H^*\) and soft-thresholding is replaced with the proximity operator \(\text{prox}_{H^*}\) of \(H\). It is not necessary to know the proximity operator of \(H(A \cdot)\).

The second goal of the paper is to bridge the gap between the well-known iterative soft-thresholding algorithm (used for the special case \(A = 1\)) and the general case \(A \neq 1\) in problem (1). The iterative soft-thresholding algorithm is well understood and has a \(1/N\) convergence rate for the decrease of the functional. It is also the basis of an accelerated algorithm with an improved \(1/N^2\) rate of decrease of the functional [3, 4]. The averages of the first \(N\) iterates of the proposed generalized soft-thresholding algorithm are proven to have a \(1/N\) rate on the functional.

Our results differ from several existing algorithms for solving (1) where each iteration step requires either the solution of another (non-trivial) minimization problem, the solution of a linear system, or a non-trivial projection on a convex set. Our proposed algorithm may therefore be of use in cases where the matrices involved \((K\) and \(A)\) have no special structure that makes such sub-problems easily solvable (i.e. not limited to deconvolution problems on regular grids, to orthogonal matrices, etc).

Iterative algorithms for the denoising case \((K = 1)\) can, amongst others, be found in [5, 6]. For general \(K\), an algorithm that uses a smoothing parameter is found in [7], an algorithm which needs a projection on a non-trivial convex set is in [8] and an algorithm which needs the solution of a non-trivial sub-problem is in [9–11]. These are results for \(A = \text{grad}\), but this is not essential in those algorithms.

Zhu and Chan [12] studied a primal-dual formulation and a so-called primal-dual hybrid gradient descent (PDHG) algorithm but concentrated on deconvolution. Connections with (more general) algorithms for variational inequalities were mentioned. This PDHG algorithm was placed in a general framework for primal-dual algorithms in [13] and many interconnections can be found there. The plethora of algorithms mentioned there still require either the solution of a linear system (which may easy in some special cases) or the minimization of a non-trivial sub-problem. Applications to image recovery of an
algorithm that is an instance of the so-called alternating direction method of multipliers, are tested in [14].

Recently, an explicit algorithm was proposed in [15, equation 5.11] with proven convergence. No rate on the functional was given. That explicit algorithm is different from the one presented here. It does not reduce to the iterative soft-thresholding algorithm when \( A = 1 \). Another explicit algorithm can also be derived using [16, equation 74] by the introduction of additional dual variables.

It remains a subject of study what speed increase can be gained (if any) from using an algorithm that solves a linear system at every iteration. The derivation of an \( O(1/N^2) \) algorithm, if at all possible for this problem, would be more interesting. Our analysis and proof is inspired by [17, 18] (who discuss a primal-dual algorithm for another problem) and by [16]. It is worth pointing out that no smoothing parameter is introduced in the non-smooth part of the functional. The proposed algorithm is not an iteratively reweighted least-squares algorithm.

2. Mathematical tools

We assume that the function \( H \) in (2) and its convex conjugate \( H^* \) given by \( H^*(w) = \sup_x \langle w, x \rangle - H(x) \) are two proper, lower semi-continuous, convex functions on a finite-dimensional real vector space and with image in \( \mathbb{R} \cup \{+\infty\} \) [19]. For example in the case of problem (1), \( H(u) = \|u\|_1 \) and therefore \( H^*(w) = \begin{cases} 0 & \|w\|_\infty \leq \lambda \\ +\infty & \|w\|_\infty > \lambda \end{cases} \) (3) such that \( \lambda \|u\|_1 = H(u) = \max_w \langle w, u \rangle - H^*(w) = \sup_{\|w\|_\infty \leq \lambda} \langle w, u \rangle \).

The proximity operators [20] of the convex functions \( H \) and \( H^* \) are defined as

\[
\begin{align*}
\text{prox}_H(u) &= \arg\min_x H(x) + \frac{1}{2} \|x - u\|^2 \\
\text{prox}_{H^*}(u) &= \arg\min_w H^*(w) + \frac{1}{2} \|w - u\|^2.
\end{align*}
\]

Therefore, when \( H(u) = \lambda \|u\|_1 \) and \( H^* \) is given by expression (3), we find that \( \text{prox}_{H^*}(u) = P_\lambda(u) \), the projection on the \( \ell_\infty \) ball of radius \( \lambda \). It has a simple explicit expression

\[
P_\lambda(u) = \begin{cases} 
\frac{u}{|u|} & |u| \leq \lambda \\
\lambda & |u| > \lambda 
\end{cases}
\]

(applied componentwise). On the other hand, the proximity operator of \( H(u) = \lambda \|u\|_1 \) is the so-called soft-thresholding operator, \( \text{prox}_{H^*}(u) = S_\lambda(u) \). It has the explicit expression

\[
S_\lambda(u) = \begin{cases} 
\frac{u}{|u|} & |u| > \lambda \\
0 & |u| \leq \lambda 
\end{cases}
\]

(also applied componentwise). Clearly, soft-thresholding \( S_\lambda \) and projection \( P_\lambda \) are connected by

\[
P_\lambda(u) = u - S_\lambda(u).
\]

In the formulas for \( P_\lambda(u) \) and \( S_\lambda(u) \), \( u \) can be an element of \( \mathbb{R}, \mathbb{R}^2, \ldots \) depending on the context (in particular, \( (Ax)_i \in \mathbb{R}^2 \) when \( A = \text{grad} \) of a 2D image). We shall use the same notation \( S_\lambda, P_\lambda \) when applied componentwise to a list of elements of \( \mathbb{R}, \mathbb{R}^2, \ldots \).

Proximity operators are Lipschitz-continuous mappings [20]:

\[
\|\text{prox}_{H^*}(u) - \text{prox}_{H^*}(v)\| \leq \|u - v\| \quad \forall u, v.
\]
The subdifferential $\partial H(x) = \{y \mid H(y) \geq H(x) + \langle y, y - x \rangle \}$ of $H$ in $x$ can be characterized using the proximity operator of $H$. Indeed, from definition (4) it follows that $u^+ = \text{prox}_H(u^-)$ if and only if $0 \in \partial H(u^+) + u^+ - u^- + u^- - u^+ \in \partial H(u^-)$. In other words, setting $u = u^+ - u^-$, we have that $u \in \partial H(u^-)$ if and only if $u^+ = \text{prox}_H(u^- + u)$.

Finally, it can also be shown that the proximity operator of $H$ and its dual $H^*$ are related by the following identity [21]:

$$\text{prox}_H(u) + \text{prox}_{H^*}(w) = u,$$

as already verified in equation (7) for the special case $\text{prox}_H = S_\lambda$ and $\text{prox}_{H^*} = P_\lambda$.

We refer to [20] for a table of further examples and properties of proximity operators. The proximity operators $\text{prox}_{H^*} = P_\lambda$ and $\text{prox}_H = S_\lambda$, that will be used for problem (1), i.e. for $H(u) = \lambda \|u\|_1$, have explicit expressions that are easy to implement.

3. Variational equations and special cases

The variational equations of the minimization problem (2) are

$$K^T(Kx - y) + A^T w = 0,$$

where $w$ is an element of the subdifferential of $H$ at $Ax$. As mentioned before, this means that $Ax = \text{prox}_H(w + Ax)$ or equivalently, using (9), that $w = \text{prox}_{H^*}(w + Ax)$. The variational equations corresponding to the problem (2) are therefore

$$K^T(y - Kx) - A^T w = 0 \quad \text{and} \quad w = \text{prox}_{H^*}(w + Ax).$$

The goal of this paper is to write an iterative algorithm that converges to a solution of these equations. We assume that these equations have at least one solution $(\hat{x}, \hat{w})$.

By using that $H(Ax) = \sup_w(Ax, w) - H^*(w)$, the minimization problem (2) can also be written as a saddle-point problem:

$$\min_x \\max_w F(x, w),$$

(11)

where we have set

$$F(x, w) = \frac{1}{2} \|Kx - y\|^2 + \langle Ax, w \rangle - H^*(w).$$

(12)

A saddle point $(\hat{x}, \hat{w})$ of (11) is a point such that

$$F(\hat{x}, w) \leq F(\hat{x}, \hat{w}) \leq F(x, \hat{w})$$

(13)

for all $x$ and $w$. For completeness, we show in the following section that solutions $(\hat{x}, \hat{w})$ of equations (10) are saddle points of (11). We define the gap with respect to the saddle point $(\hat{x}, \hat{w})$ by

$$G(x, w) = F(x, \hat{w}) - F(\hat{x}, w).$$

(14)

It follows from (13) that this gap is non-negative for all $x$ and $w$.

In the special case $A = 1$, the problem (2) reduces to

$$\min_x \frac{1}{2} \|Kx - y\|^2 + H(x),$$

(15)

for which a forward–backward splitting algorithm

$$x^{n+1} = \text{prox}_H(x^n + K^T(y - Kx^n))$$

(16)

can be used. This algorithm converges for $\|K\| < \sqrt{2}$ [22]. More specifically, the minimization problem with $A = 1$ and $H(x) = \lambda \|x\|_1$, \n
$$\min_x \frac{1}{2} \|Kx - y\|^2 + \lambda \|x\|_1,$$

(17)
can be solved by the iterative soft-thresholding algorithm [1]
\[ x^{n+1} = S_{\lambda}(x^n + K^T (y - Kx^n)). \] (18)

Many other algorithms exist. One feature of this algorithm is that, as a consequence of the soft-thresholding, all the iterates \( x^n \) (not just the limit) have many exact zeros.

On the other hand, the problem
\[ \min_x \frac{1}{2} \|x - g\|^2 + \lambda \|Ax\| \] (19)
\((K = 1, y \rightarrow g\) in problem (1)) can be solved by a gradient projection algorithm:
\[ w^{n+1} = P_{\lambda}(w^n + A(x^n - A^T w^n)), \] (20)
where \( x^n = g - A^T w^n \), if \( \|A\| < 1 \) (as in [6, equation (11)] for \( A = \text{grad}\)). This is a special case of the gradient projection algorithm that can be used for minimization of a quadratic function over a convex set \( C: \min_{x \in C} \|g - A^T w\|^2 \). The quantities \( Ax^n \) are not sparse in every step; only in the limit will \( Ax^n \) be sparse.

4. Algorithm

Writing the variational equations (10) as fixed-point equations,
\[
\begin{aligned}
    x &= x + K^T (y - Kx) - A^T w \\
    w &= \text{prox}_{\lambda A^T}(w + Ax),
\end{aligned}
\] (21)
provides the usual ansatz for deriving iterative first-order algorithms for (2). Here, we choose to study the iteration
\[
\begin{aligned}
    x^{n+1} &= x^n + K^T (y - Kx^n) - A^T w^n \\
    w^{n+1} &= \text{prox}_{\lambda A^T}(w^n + A \bar{x}^{n+1}) \\
    \bar{x}^{n+1} &= x^n + K^T (y - Kx^n) - A^T w^{n+1},
\end{aligned}
\] (22)
the fixed point of which is a solution to the variational equations (10). Specifically, starting from \((x^n, w^n)\) one does a gradient descent step on \( F(x, w) \) in the \( x\)-variable to arrive at \((\tilde{x}^{n+1}, \bar{w}^{n+1})\), followed by a proximal ascent step in the \( w \) variable to compute \( w^{n+1} \). Finally one does a gradient descent step in \((\bar{x}^{n+1}, w^{n+1})\) to arrive at \((x^{n+1}, w^{n+1})\). This algorithm can therefore be interpreted as a ‘predict-correct’ algorithm for the saddle-point problem (11). On the other hand, algorithm (22) can equivalently be written in a ‘pseudo-implicit’ form as
\[
\begin{aligned}
    \tilde{x}^{n+1} &= x^{n+1} - A^T (w^n - w^{n+1}) \\
    \bar{w}^{n+1} &= \text{prox}_{\lambda A^T}(w^n + A \bar{x}^{n+1}) \\
    \bar{x}^{n+1} &= x^n + K^T (y - Kx^n) - A^T w^{n+1},
\end{aligned}
\] (23)
This form is useful for proving convergence.

Writing algorithm (22) as
\[
\begin{aligned}
    \tilde{x}^{n+1} &= x^n + K^T (y - Kx^n) \\
    \bar{w}^{n+1} &= \text{prox}_{\lambda A^T}(w^n + A \bar{g}^{n+1} - A^T w^n) \\
    \bar{x}^{n+1} &= g^{n+1} - A^T w^{n+1}
\end{aligned}
\] (24)
leads to the interpretation of a gradient descent step on the quadratic part of the functional, followed by a single step in a dual variable (compare with (20)) starting from the previous dual variable \( w^n \).

In the following section, we show that the proposed algorithm (22) converges to a solution of the fixed-point equations (21), i.e. to a saddle point of the min–max problem (11) and to a minimizer of the functional (2). Under some additional condition on \( H \), we also derive a convergence rate estimate for the functional \( \mathcal{F} \) in the average of the iterates.
For the special case when $AA^T = A^TA = 1$, the second line of algorithm (22) reduces to
\[ x^{n+1} = \text{prox}_{H^*}(A(x^n + K^T(y - Kx^n))) \]
which implies
\[ x^{n+1} = x^n + K^T(y - Kx^n) - A^T\text{prox}_{H^*}(A(x^n + K^T(y - Kx^n))). \]
Using $\text{prox}_{H^*}(u) = u - \text{prox}_{H^*}(u)$, one has
\[ x^{n+1} = A^T\text{prox}_{H^*}(A(x^n + K^T(y - Kx^n))). \]
This is the forward–backward splitting algorithm (16) for the variable $Ax$ and the operator $KAT$. In particular, for $H = \lambda \cdot \| \cdot \|_1$, algorithm (22) reduces to the iterative soft-thresholding algorithm (18) when $AA^T = A^TA = 1$. Similarly, when $K$ is orthogonal and $H(\cdot) = \lambda \cdot \| \cdot \|_1$, then algorithm (22) reduces to
\[
\begin{cases}
  w^{n+1} = P_{\delta A^*}(w^n + A(K^T y - A^T w^n)) \\
  x^{n+1} = K^T y - A^T w^{n+1}
\end{cases}
\]
which is the gradient projection algorithm (20) for the data $g = K^T y$.

5. Convergence

We will prove the convergence of algorithm (22).

**Lemma 1.** If $w^+ = \text{prox}_{H^*}(w^- + \Delta)$, then
\[ \| w - w^+ \|^2 \leq \| w - w^- \|^2 - \| w^+ - w^- \|^2 - 2\langle w - w^+, \Delta \rangle + 2H^*(w) - 2H^*(w^+) \] (25)
for all $w$.

**Proof.** If $w^+ = \text{prox}_{H^*}(w^- + \Delta) = \arg\min_{w'} H^*(w) + \frac{1}{2}\| w - (w^- + \Delta) \|^2$, then $w^- + \Delta - w^+ \in \partial H^*(w^+)$. We then have for all $w$ that $H^*(w) \geq H^*(w^+) + \langle \gamma, w - w^+ \rangle$ if $\gamma \in \partial H^*(w^+)$ and therefore
\[
\begin{align*}
  H^*(w) &\geq H^*(w^+) + \langle w^- + \Delta - w^+, w - w^+ \rangle \\
 &\geq H^*(w^+) + \langle \Delta, w - w^+ \rangle + \langle w^- - w^+, w - w^+ \rangle \\
 &\geq H^*(w^+) + \langle \Delta, w - w^+ \rangle + \frac{1}{2}\| w^- - w^+ \|^2 + \frac{1}{2}\| w - w^+ \|^2 - \frac{1}{2}\| w^- - w \|^2
\end{align*}
\]
which gives (25). $\square$

We will also use the following special case ($H^* = 0$ and $\delta s$ replaced by $\gamma s$). If $x^+ = x^- + \Delta$, then
\[ \| x - x^+ \|^2 = \| x - x^- \|^2 - \| x^- - x^+ \|^2 - 2\langle x - x^+, \Delta \rangle \] (26)
for all $x$.

For completeness we show that a solution of the variational equations is a saddle point of (11). This implies that the gap $G(x, w)$ with respect to the fixed point $(\hat{x}, \hat{w})$ is always non-negative.

**Lemma 2.** If $(\hat{x}, \hat{w})$ satisfies the fixed-point equations (21), then
\[ F(\hat{x}, w) \leq F(\hat{x}, \hat{w}) \leq F(x, \hat{w}) \] (27)
and hence
\[ G(x, w) \equiv F(x, \hat{w}) - F(\hat{x}, w) \geq 0 \] (28)
for all $x$ and $w$. 

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Proof. The first inequality $F(\hat{x}, w) \leq F(x, \hat{w})$ comes down to showing that $0 \leq \langle A\hat{x}, \hat{w} - w \rangle + H^*(w) - H^*(\hat{w})$ for all $w$. This follows immediately from choosing $w^+ = w_\gamma = \hat{w}$ and $\Delta = A\hat{x}$ in lemma 1.

The second inequality $F(x, \hat{w}) \leq F(x, \hat{v})$ can be written as

$$0 \leq \frac{1}{2} \|Kx - y\|^2 - \frac{1}{2} \|K\hat{x} - y\|^2 + \langle A(x - \hat{x}), \hat{w}\rangle \quad \forall x.$$ 

To show this, we choose $x^+ = x^- = \hat{x}$ and $\Delta = K^T(y - K\hat{x}) - A^T\hat{w}$ in expression (26) to find

$$0 = -2(x - \hat{x}, K^T(y - K\hat{x}) - A^T\hat{w})$$

$$= -2\|K(x - \hat{x}, y - K\hat{x}) + 2(x - \hat{x}, A^T\hat{w})\|$$

$$= -\|K(x - \hat{x})\|^2 - \|y - K\hat{x}\|^2 + \|Kx - y\|^2 + 2\langle x - \hat{x}, A^T\hat{w} \rangle$$

for all $x$, or

$$\|K(x - \hat{x})\|^2 = \|Kx - y\|^2 - \|K\hat{x} - y\|^2 + 2\langle A(x - \hat{x}), \hat{w}\rangle,$$

which is a slightly stronger result than needed.

The gap $G(x, w)$ equals

$$G(x, w) = \frac{1}{2} \|K(\hat{x} - x)\|^2 + \langle \hat{w} - w, A\hat{x} \rangle + H^*(w) - H^*(\hat{w})$$

(29)

as can be verified from its definition (and relation (26)). The sum of the last three terms on the right-hand side is non-negative, so

$$G(x, w) \geq \frac{1}{2} \|K(\hat{x} - x)\|^2.$$ 

The gap $G(x, w)$ is not a measure of closeness of $(x, w)$ to a saddle point $(\hat{x}, \hat{w})$ as $G(x, w) = 0$ does not imply that $(x, w)$ is a saddle point.

Lemma 3. If $(x^n, w^n)$ are given by iteration (22), then

$$\|x - x^{n+1}\|^2 + \|w - w^{n+1}\|^2 \leq \|x - x^n\|^2 + \|w - w^n\|^2$$

for all $x$ and $w$.

Proof. From lemma 1 and relation (26), we find

$$\|w - w^{n+1}\|^2 \leq \|w - w^n\|^2 - \|w^n - w^{n+1}\|^2 - 2\langle w - w^{n+1}, Ax^{n+1}\rangle + 2H^*(w) - 2H^*(w^{n+1})$$

$$\|x - x^{n+1}\|^2 = \|x - x^n\|^2 - \|x^n - x^{n+1}\|^2 - 2\langle x - x^{n+1}, K^T(y - Kx^n) - A^T w^{n+1}\rangle$$

which together yield

$$\|x - x^{n+1}\|^2 + \|w - w^{n+1}\|^2 \leq \|x - x^n\|^2 - \|x^n - x^{n+1}\|^2$$

$$+ \|w - w^n\|^2 - \|w^n - w^{n+1}\|^2$$

$$- 2\langle w - w^{n+1}, Ax^{n+1}\rangle - 2\langle x - x^{n+1}, K^T(y - Kx^n) - A^T w^{n+1}\rangle$$

$$+ 2H^*(w) - 2H^*(w^{n+1}).$$
As (22) implies \( x^{n+1} = x^n - A^T (w^n - w^{n+1}) \), this can be written as
\[
\|x - x^{n+1}\|^2 + \|w - w^{n+1}\|^2 \leq \|x - x^n\|^2 - \|x^n - x^{n+1}\|^2
\]
\[
+ \|w - w^n\|^2 - \|w^n - w^{n+1}\|^2
\]
\[
- 2\langle w - w^{n+1}, A(x^{n+1} - A^T (w^n - w^{n+1})) \rangle
\]
\[
- 2\langle x - x^{n+1}, K^T (y - K x^n) - A^T w^{n+1} \rangle
\]
\[
+ 2H^*(w) - 2H^*(w^{n+1}).
\]
The two \( \langle w^{n+1}, Ax^{n+1} \rangle \) terms cancel:
\[
\|x - x^{n+1}\|^2 + \|w - w^{n+1}\|^2 \leq \|x - x^n\|^2 - \|x^n - x^{n+1}\|^2
\]
\[
+ \|w - w^n\|^2 - \|w^n - w^{n+1}\|^2
\]
\[
2\langle A^T (w - w^{n+1}), A^T (w^n - w^{n+1}) \rangle
\]
\[
- 2\langle K(x - x^{n+1}), y - K x^n \rangle = \|Kx - y\|^2 - \|K x^{n+1} - y\|^2 - \|K(x - x^n)\|^2
\]
\[
- 2\langle Ax^{n+1}, w \rangle + 2\langle x, A^T w^{n+1} \rangle
\]
\[
+ 2H^*(w) - 2H^*(w^{n+1}).
\]
Now, by using the equalities
\[
2\langle A^T (w - w^{n+1}), A^T (w^n - w^{n+1}) \rangle = -\|A^T (w - w^n)\|^2 + \|A^T (w - w^{n+1})\|^2
\]
\[
- 2\langle K(x - x^{n+1}), y - K x^n \rangle = \|Kx - y\|^2 - \|K x^{n+1} - y\|^2 - \|K(x - x^n)\|^2
\]
\[
- 2\langle Ax^{n+1}, w \rangle + 2\langle x, A^T w^{n+1} \rangle = 2F(x, w^{n+1}) - 2F(x^{n+1}, w) - \|Kx - y\|^2
\]
\[
+ \|K x^{n+1} - y\|^2 + 2H^*(w^{n+1}) - 2H^*(w),
\]
the previous inequality reduces to
\[
\|x - x^{n+1}\|^2 + \|w - w^{n+1}\|^2 \leq \|x - x^n\|^2 - \|x^n - x^{n+1}\|^2
\]
\[
+ \|w - w^n\|^2 - \|w^n - w^{n+1}\|^2
\]
\[
- \|A^T (w - w^n)\|^2 + \|A^T (w^n - w^{n+1})\|^2
\]
\[
+ \|A^T (w - w^{n+1})\|^2
\]
\[
- \|K(x - x^n)\|^2 + \|K(x^n - x^{n+1})\|^2
\]
\[
+ 2F(x, w^{n+1}) - 2F(x^{n+1}, w),
\]
which is the desired result. \( \square \)

**Theorem 1.** Let \( \|K\| < \sqrt{2} \) and \( \|A\| < 1 \). If equations (10) have a solution and the sequence \( (x^n, w^n) \) is defined by the iteration
\[
\begin{align*}
x^{n+1} &= x^n + K^T (y - K x^n) - A^T w^n \\
w^{n+1} &= \text{prox}_{H_U} (w^n + A x^{n+1})
\end{align*}
\]
then
\[
\begin{align*}
(1) \text{the sequence } (x^n, w^n) \text{ converges to a solution } (x^*, w^*) \text{ of the variational equations (10)}
\end{align*}
\]
\[
\begin{align*}
\text{thereby providing a minimizer of (2) and a saddle point of (11)};
\end{align*}
\]
\[
\begin{align*}
(2) \text{the average of the first } N \text{ iterates } (\bar{x}^N, \bar{w}^N) = \frac{1}{N}(x^n, w^n) \text{ converges to the saddle point } (x^*, w^*) \text{ and there exists a constant } C_1 > 0 \text{ independent of } N \text{ such that}
\end{align*}
\]
\[
\begin{align*}
F(\bar{x}^N, \bar{w}^N) - F(x, \tilde{w}^N) \leq \frac{\|x - x^0\|^2 + \|w - w^0\|^2 + C_1}{2N}
\end{align*}
\]
for all \( x, w \) (with \( C_1 = 0 \) if \( \| K \| \leq 1 \), in particular,
\[
0 \leq G(x^N, \hat{w}^N) \leq \frac{\| x^N - x^\dagger \|^2 + \| w^N - w^\dagger \|^2 + C_1}{2N}.
\]
(33)

For \( 1 < \| K \| < \sqrt{2} \), the constant \( C_1 \) depends on the starting point \( (x^0, w^0) \). An explicit expression is given in equation (39);

(3) if the penalty function \( H \) has full domain (i.e. \( H \) does not take the value \(+\infty\)), there exists a constant \( C_2 \) independent of \( N \) such that
\[
0 \leq \mathcal{F}(\hat{x}^N) - \mathcal{F}(x^\dagger) \leq \frac{C_2}{N} \quad \forall N.
\]
(34)

**Proof.**

(i) Let \((\hat{x}, \hat{w})\) be a saddle point of (11). From lemma 3 we find
\[
\| \hat{x} - x^{n+1} \|^2 + \| \hat{w} - w^{n+1} \|^2 \leq \| \hat{x} - x^n \|^2 - \| K(\hat{x} - x^n) \|^2
\]
\[
+ \| \hat{w} - w^n \|^2 - \| A^T(\hat{w} - w^n) \|^2,
\]
\[
- \| x^n - x^{n+1} \|^2 + \| K(x^n - x^{n+1}) \|^2
\]
\[
- \| w^n - w^{n+1} \|^2 + \| A^T(w^n - w^{n+1}) \|^2
\]
\[
+ \| A^T(\hat{w} - w^{n+1}) \|^2 - \| K(\hat{x} - x^{n+1}) \|^2,
\]
where we have used relation (30) to set \( 2F(\hat{x}, w^{n+1}) - 2F(x^{n+1}, \hat{w}) = -2G(x^{n+1}, w^{n+1}) \leq \| K(\hat{x} - x^{n+1}) \|^2 \). Using the inequality
\[
-\| K(\hat{x} - x^n) \|^2 - \| K(\hat{x} - x^{n+1}) \|^2 = -\frac{1}{2} \| K(\hat{x} - x^n) + K(\hat{x} - x^{n+1}) \|^2
\]
\[
- \frac{1}{2} \| K(\hat{x} - x^n) - K(\hat{x} - x^{n+1}) \|^2
\]
\[
\leq -\frac{1}{2} \| K(x^n - x^{n+1}) \|^2,
\]
we find
\[
\| \hat{x} - x^{n+1} \|^2 + \| \hat{w} - w^{n+1} \|^2 \leq \| \hat{x} - x^n \|^2 + \| \hat{w} - w^n \|^2 - \| A^T(\hat{w} - w^n) \|^2
\]
\[
- \| x^n - x^{n+1} \|^2 + \frac{1}{2} \| K(x^n - x^{n+1}) \|^2
\]
\[
- \| w^n - w^{n+1} \|^2 + \| A^T(w^n - w^{n+1}) \|^2
\]
\[
+ \| A^T(\hat{w} - w^{n+1}) \|^2.
\]

As we assume that \( \| K \| < \sqrt{2} \) and \( \| A \| < 1 \), we can introduce regular square matrices \( L \) and \( B \) by \( L = \frac{1}{2} K^T K \) and \( B = I - L A^T \) and deduce
\[
\| \hat{x} - x^{n+1} \|^2 + \| B(\hat{w} - w^{n+1}) \|^2 \leq \| \hat{x} - x^n \|^2 + \| B(\hat{w} - w^n) \|^2
\]
\[
- \| L(x^n - x^{n+1}) \|^2 - \| B(w^n - w^{n+1}) \|^2.
\]

Summing from \( N \) to \( M \), one also finds
\[
\sum_{n=N}^{M} (\| L(x^n - x^{n+1}) \|^2 + \| B(w^n - w^{n+1}) \|^2) \leq \| \hat{x} - x^N \|^2 + \| B(\hat{w} - w^N) \|^2
\]
\[
- \sum_{n=N}^{M} (\| L(x^n - x^{n+1}) \|^2 + \| B(w^n - w^{n+1}) \|^2).
\]
(35)

As \( B \) is invertible, it follows that the sequence \( (x^n, w^n) \) is bounded. Hence, there is a convergent subsequence \( (x^{n_j}, w^{n_j}) \xrightarrow{j \to \infty} (x^\dagger, w^\dagger) \) (the same subsequence for \( x^n \) and \( w^n \)).
It also follows from inequality (35) that
\[
\sum_{n=N}^{M} (\| L(x^n - x^{n+1}) \|^2 + \| B(w^n - w^{n+1}) \|^2) \leq \| \hat{x} - x^N \|^2 + \| B(\hat{w} - w^N) \|^2.
\]
(36)
Hence \( \|L(x^n - x^{n+1})\|^2 \) and \( \|B(w^n - w^{n+1})\|^2 \) tend to zero for large \( n \), which implies that \( \|x^n - x^{n+1}\| \) and \( \|w^n - w^{n+1}\| \) tend to zero. It follows that the subsequence \((x^{n_j}, w^{n_j})\) also converges to \((x^+, w^+)\) and, by continuity of \( \text{prox}_{\gamma H_\cdot} \), \((x^+, w^+)\) satisfies the fixed-point equations (21). We can therefore choose \((\hat{x}, \hat{w}) = (x^+, w^+)\) in relation (35) to find \( \|x^1 - x^{M+1}\|^2 + \|B(w^1 - w^{M+1})\|^2 \leq \|x^1 - x^0\|^2 + \|B(w^1 - w^0)\|^2 \) (37) for all \( M \geq N \). As there is a convergent subsequence of \((x^n, w^n)\), the right-hand side of this expression can be made arbitrarily small for large enough \( N \) (\( N = n_j \), for some \( j \)). Hence the left-hand side will be arbitrarily small for all \( M \) larger than this \( N \). This proves convergence of the whole sequence \((x^n, w^n)\) to \((x^+, w^+)\).

(ii) As \( (x^n, w^n) \xrightarrow{n \to \infty} (x^+, w^+) \), the Cesàro averages \((\bar{x}^N, \bar{w}^N) = \sum_{n=1}^{N}(x^n, w^n)/N\) also converge to \((x^+, w^+)\). It follows from Lemma 3 that

\[
2 \left( F(x^{n+1}, w) - F(x, w^{n+1}) \right) \leq \|x - x^n\|^2 + \|B(w - w^n)\|^2
- \|x - x^{n+1}\|^2 - \|B(w - w^{n+1})\|^2
- \|x^n - x^{n+1}\|^2 + \|K(x^n - x^{n+1})\|^2.
\]

Then, using convexity, one finds

\[
F(\bar{x}^N, w) - F(x, \bar{w}^N) \leq \frac{1}{N} \sum_{n=0}^{N-1} \left( F(x^{n+1}, w) - F(x, w^{n+1}) \right)
\]

(38)

\[
\leq \frac{1}{2N} \sum_{n=0}^{N-1} \left( \|x - x^n\|^2 + \|B(w - w^n)\|^2
- \|x - x^{n+1}\|^2 - \|B(w - w^{n+1})\|^2
- \|x^n - x^{n+1}\|^2 + \|K(x^n - x^{n+1})\|^2 \right)
\]

\[
= \frac{1}{2N} \left( \|x - x^0\|^2 + \|B(w - w^0)\|^2
- \|x - x^N\|^2 - \|B(w - w^N)\|^2
- \sum_{n=0}^{N-1} (\|x^n - x^{n+1}\|^2 - \|K(x^n - x^{n+1})\|^2) \right)
\]

\[
\leq \frac{1}{2N} \left( \|x - x^0\|^2 + \|B(w - w^0)\|^2
- \sum_{n=0}^{N-1} (\|x^n - x^{n+1}\|^2 - \|K(x^n - x^{n+1})\|^2) \right).
\]

If \( \|K\| \leq 1 \), then the summation on the right-hand side can be dropped outright. If \( 1 < \|K\| < \sqrt{2} \), it can be bounded as follows:

\[
\sum_{n=0}^{N-1} (\|x^n - x^{n+1}\|^2 + \|K(x^n - x^{n+1})\|^2) \leq (\|K\|^2 - 1) \sum_{n=0}^{N-1} \|x^n - x^{n+1}\|^2
\]

\[
\leq (\|K\|^2 - 1) \sum_{n=0}^{\infty} \|x^n - x^{n+1}\|^2
\]

\[
\leq \frac{\|K\|^2 - 1}{1 - \|K\|^2/2} (\|x^0 - x^+\|^2 + \|B(w^0 - w^+)\|^2),
\]
where the last line follows from $(1 - \|K\|^2/2) \sum_{n=0}^{\infty} \|x^n - x^{n+1}\|^2 \leq \sum_{n=0}^{\infty} \|L(x^n - x^{n+1})\|^2$ and inequality (36). Therefore, a constant $C_1$ independent of $N$,

$$
C_1 = \begin{cases} 
0 & \|K\| \leq 1 \\
\|K\|^2 - 1 & 1 - \|K\|^2/2 
\end{cases} \left(\|x^n - x^0\|^2 + \|w^n - w^0\|^2\right) \quad 1 < \|K\| < \sqrt{2},
$$

(39)
can be introduced such that

$$
F(\tilde{x}^N, w) - F(x, \tilde{w}^N) \leq \frac{\|x - x^0\|^2 + \|w - w^0\|^2 + C_1}{2N},
$$

(where we have used $\|B\| < 1$).

Relation (33) follows from choosing $(x, w) = (x^1, w^1)$ in equation (32).

(iii) For the functional $F$, we find

$$0 \leq F(\tilde{x}^N) - F(x^1) = F(\tilde{x}^N) - F(x^1, w^1) \leq \sup_{w} \|F(\tilde{x}^N, w) - F(x^1, \tilde{w}^N)\|,$$

In the last expression, $\sup_{w} F(\tilde{x}^N, w)$ is reached in a $w^{*, N} \in \partial H(A\tilde{x}^N)$. We already know that the $\tilde{x}^N$ converge and therefore that the $A\tilde{x}^N$ are bounded (independent of $N$). As we assume that $H$ has full domain, it follows that $H$ is locally Lipschitz and these $w^{*, N}$ are bounded as well (by an appropriate local Lipschitz constant of $H$, say $R$). We can therefore write

$$0 \leq F(\tilde{x}^N) - F(x^1) \leq \max_{\|w\| \leq R} \|F(\tilde{x}^N, w) - F(x^1, \tilde{w}^N)\| \leq \max_{\|w\| \leq R} \frac{\|x^1 - x^0\|^2 + \|w - w^0\|^2 + C_1}{2N} = C_2/N,$$

which proves relation (34).

\[\square\]

6. Discussion

In case the conditions $\|K\| < \sqrt{2}$ and/or $\|A\| < 1$ are not satisfied, it is possible to rescale the matrices, the data $y$ and the variable $w$ to write a convergent algorithm. In the special case of functional (1), it suffices to rewrite the problem equivalently as

$$\min x \frac{1}{2} \|\sqrt{\tau}K x - (\sqrt{\tau} y)\|^2 + \frac{\tau \lambda}{\sqrt{\sigma}} \|\sqrt{\sigma} A x\|_1,$$

and use algorithm (22) with $\text{prox}_{\mu\tau} = P_{\frac{\tau}{\sqrt{\sigma}}\sqrt{\sigma}}$, for the matrices $\sqrt{\tau}K$, $\sqrt{\sigma}A$ and the data $\sqrt{\tau}y$. Renaming $w \leftarrow \sqrt{\sigma}w/\tau$ and using the scaling property $P_{\frac{\tau}{\sqrt{\sigma}}\sqrt{\sigma}}(u) = \tau/\sqrt{\sigma} P_{\sqrt{\sigma}}(\sqrt{\sigma}u/\tau)$, one finds the following iteration:

$$
\begin{align*}
\tilde{x}^{n+1} &= x^n + \tau K^T(y - Kx^n) - \tau A^T w^n \\
w^{n+1} &= P_{\frac{\tau}{\sqrt{\sigma}}\sqrt{\sigma}}(u^n + \sigma/\tau A^T \tilde{x}^{n+1}) \\
x^{n+1} &= \tilde{x}^{n+1} + \tau K^T(y - Kx^n) - \tau A^T w^{n+1}
\end{align*}
$$

(40)

for problem (1). Now step-size parameters $\sigma, \tau > 0$ should satisfy $\tau < 2/\|K^T K\|$ and $\sigma < 1/\|A A^T\|$. In this case, the bound (34) is valid as $\lambda \|\cdot\|_1$ has full domain.
For the general case (2), the scaled version of the algorithm can be derived in a similar fashion. It takes the form

$$\begin{align*}
F^{n+1} &= x^n + \tau K^T (y - Kx^n) - \tau A^T w^n \\
\bar{w}^{n+1} &= \text{prox}_{\mathcal{H}^n} (w^n + \sigma / \tau x^{n+1}) \\
x^{n+1} &= x^n + \tau K^T (y - Kx^n) - \tau A^T w^{n+1}
\end{align*}$$

(41)

with $\tau < 2/\|K^T K\|$ and $\sigma < 1/\|A A^T\|$. Here, we have used that $(\tau H(\cdot / \sqrt{\sigma}))^* = \tau H^*(\sqrt{\sigma} / \tau \cdot)$ and $\text{prox}_f(\alpha u) = \alpha \text{prox}_{\alpha^2 f / \alpha}(u)$ for $\alpha > 0$.

It can be verified numerically that the functional $\mathcal{F}(x^n)$ does not necessarily decrease monotonically as a function of $n$ (this can be shown to hold in the special case $A = 1$ and $H(u) = \lambda \|u\|_1$, see [3]). The gap function $G(x^n, w^n)$ does not decrease monotonically as a function of $n$ either. The error between $(x^n, w^n)$ and $(x^1, w^1)$ decreases monotonically as a function of $n$ in the norm $(\|x\|^2 + \|B w\|^2)^{1/2}$. This is a consequence of relation (37).

The condition $\|A\| < 1$ used in the proof of convergence excludes the case $A = 1$. Nevertheless, the proof of convergence in theorem 1 can be slightly adapted to cover the case $A = 1$ as well.

The strength of algorithm (22) lies in the fact that only $\text{prox}_{\mathcal{H}}$, is needed and not $\text{prox}_{H(A \cdot)}$, $\text{prox}_{H(A \cdot)}$, may have a simple expression, whereas the proximity operator of $H(A \cdot)$ may not. In particular, for $H = \lambda \|\cdot\|_1$, one has expressions (5) and (6) for $\text{prox}_{\mathcal{H}}$, and $\text{prox}_{H}$. One can also find closed-form expressions for $\text{prox}_{\mathcal{H}}$, when $|\cdot|$ (used in the expression $\|Ax\|_i = \sum_i |(Ax)_i|$) refers to the 1- or $\infty$-norms instead of the 2-norm.

We believe that the proposed algorithm (40), its connection with the traditional iterative soft-thresholding algorithm and its proof of convergence are new. The combination of a gradient step with the dual algorithm (20) has been proposed several times already [10, 11]; as such that would not be an explicit algorithm as it requires infinitely many dual iterations in each outer iteration. Here, we have shown convergence in the case when just one dual step is made in each iteration. The series of algorithms discussed in [13, 15] mostly make use of a non-explicit step in the iteration, or of the solution of a linear system at every iteration. These existing algorithms are often special cases of more general methods. The explicit algorithm in [15] is also different. In [16, equation 74], the authors propose another explicit method using additional dual variables.

In [23, 24], Korpelevich introduced an extragradient algorithm for the solution of a general saddle-point problem. It relies on updating two copies of primal and dual variables (say $(x^n, w^n)$ and $(\tilde{x}^n, \tilde{w}^n)$), and combining the one with the gradient in the other point. No distinction is made there between primal and dual variables, as we do here.

In [25], a proximal-point algorithm is introduced that reduces to the Korpelevich algorithm in a special case. Moreover, it was shown that the iterates of that algorithm converge ‘ergodically’ with rate $1/N$ on the primal objective function (i.e. the Césaro means decrease in the functional with rate $1/N$). It is, however, assumed there that both primal and dual variables are bounded (saddle-point problem on a compact domain). This is a major difference with our result, where we only require that the dual variable $w$ be bounded in order to derive a $1/N$ bound on the functional (point (iii) of theorem 1); the primal variable $x$ is unbounded.

We did not try to extend the convergence proof to an infinite-dimensional setting, as was done in [1] for the iterative soft-thresholding algorithm. The most useful example of problem (1) is perhaps the case where $A = \text{grad}$ (total variation penalty), but this operator is unbounded in the infinite-dimensional case.

Algorithm (40) for problem (1) can also be used for signal recovery under analysis style sparsity requirements [26]: finding $x$ with many $(A x)_i$ equal to zero for a given frame operator $A$. Another application of algorithm (40) is solving a linear inverse problem while imposing group sparsity (possibly with overlapping groups) [27]. In this case, matrix $A$ is chosen in
Min=−1
Max=1
−1.5 −0.75 0.0 0.75 1.5
Min=−1.0171
Max=1.0128
−1.5 −0.75 0.0 0.75 1.5
Figure 1. Synthetic seismic tomography experiment using total variation penalty (see section 7): (a) input model with both sharp and smooth edges between zones of constant model value; (b) reconstruction from 8490 noisy data with 1000 iterations of algorithm (40) and $A = \text{grad}$; (c) evolution of the distance to the limit model and of the functional to the limit value (here $\hat{x}$ is obtained from 100 000 iterations of the same algorithm; $\hat{x}$ is not equal to $x^\text{in}$); results for two choices of the step length parameter $\tau$ are given; (d) sum of the rows of the matrix $K$ to indicate the illumination of the sphere by the rays in the data set.

such a way that $\|Ax\|_1 = \sum_{e \in \text{groups}} |(x_e)_{e \in \text{group}}|$, i.e. $A$ has a single 1 on each row (and all other elements are zero). Columns may have more than a single nonzero entry (this would correspond to overlapping groups). In this expression, $|\cdot|$ is again the Euclidean norm of a vector.

7. Numerical example of total variation minimization

Algorithm (40) is applied to a stylized problem in seismic tomography. We try to reconstruct a simple synthetic 2D input model $x^\text{in}$ defined on the sphere (see figure 1(a)) from 8490 data. The input model has a number of zones of constant value with either a sharp edge or a smooth edge in between. The model space has dimension 98 304. The data $y$ are found from 8490 seismic surface rays that criss-cross the globe (these correspond to actual earthquakes and seismic stations [28]) and that make up the rows of a matrix $K$ (see figure 1(d)). More precisely, synthetic data $y$ are constructed through the formula $y = Kx^\text{in} + \epsilon$, where $\epsilon$ is Gaussian noise of magnitude $\|\epsilon\| = 0.1 \times \|Kx^\text{in}\|$ (i.e. 10% noise). The goal now is to reconstruct $x^\text{in}$ as well as possible from $y$ by imposing a total variation penalty in cost function (1). In other words, we will look for the minimizer of function (1), where $K$ and $y$ are given and where we choose $A = \text{grad}$. 

13
Algorithm (40), with $\tau = 0.99/\|K\|^2$ and $\sigma = 0.99/\|A\|^2$ and 1000 iterations, was used to produce a reconstruction: $x^{out} = x^{1000}$ (in about 20 s of computer time). The penalty parameter $\lambda$ was chosen to fit the data to the level of the noise: $\|Kx^{out} - y\| \approx \|\epsilon\|$. 

The original model and its reconstruction are shown in figure 1, panels (a) and (b). The total variation penalty results in a piecewise constant output model. Sharp edges (e.g. near North America) are reasonably well resolved given the small amount of data available. In panel (c), the distance of iterate $x^n$ to a reference minimizer $\hat{x}$ is shown in the solid line for the step length choice $\tau = 0.99/\|K\|^2$. The residual error (with respect to $\hat{x}$) after 1000 iterations is 10%, and the functional attains about three correct decimal compared to the ‘true’ minimal value. We repeated the experiment with the choice $\tau = 1.99/\|K\|^2$ and we show the results in panel (c) in dashed lines. In this case, the step length choice $\tau = 1.99/\|K\|^2$ performs slightly better than the choice $\tau = 0.99/\|K\|^2$. Our experiments show that this may depend on the value of the penalty parameter $\lambda$ and that it may not hold for all iterates $x^n$.

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References

[1] Daubechies I, Defrise M and De Mol C 2004 An iterative thresholding algorithm for linear inverse problems with a sparsity constraint Commun. Pure Appl. Math. 57 1413–57
[2] Donoho D L 2006 Compressed sensing IEEE Trans. Inf Theory 52 1289–306
[3] Beck A and Teboulle M 2009 A fast iterative shrinkage-threshold algorithm for linear inverse problems SIAM J. Imaging Sci. 2 183–202
[4] Nesterov Yu E 1983 A method for solving a convex programming problem with convergence rate $O(1/k^2)$ Sov. Math.—Dokl. 27 372–6
[5] Chambolle Antonin 2004 An algorithm for total variation minimization and applications J. Math. Imaging Vis. 20 89–97
[6] Chambolle A 2005 Total variation minimization and a class of binary MRF models Energy Minimization Methods in Computer Vision and Pattern Recognition (Lecture Notes in Computer Science vol 3757) (Berlin: Springer) pp 136–52
[7] Chan T F, Golub G H and Mulet P 1999 A nonlinear primal-dual method for total variation-based image restoration SIAM J. Sci. Comput. 20 1964–77
[8] Bect J, Blanc-Féraud L, Aubert G and Chambolle A 2004 A $l^1$-unified variational framework for image restoration ECCV: Proc. Eur Conf on Computer Vision (Prague, Czech Republic) (Lecture Notes in Computer Science vol 3024) ed T Pajdla and J Matas (Berlin: Springer) pp 1–13
[9] Daubechies I, Teschke G and Vese L 2007 Iteratively solving linear inverse problems under general convex constraints Inverse Probl. Imaging 1 29–46
[10] Beck A and Teboulle M 2009 Fast gradient-based algorithms for constrained total variation image denoising and deblurring problems IEEE Trans. Image Process. 18 2419–34
[11] Bresies K 2009 A forward-backward splitting algorithm for the minimization of non-smooth convex functionals in Banach space Inverse Probl. 25 045005
[12] Zhu M and Chan T 2008 An efficient primal-dual hybrid gradient algorithm for total variation image restoration. UCLA Computational and Applied Mathematics Report 08–34
[13] Esser E, Zhang X and Chan T F 2010 A general framework for a class of first order primal-dual algorithms for convex optimization in imaging science SIAM J. Imaging Sci. 3 1015–46
[14] Afonso M V, Bioucas-Dias J M and Figueiredo M A T 2010 Fast image recovery using variable splitting and constrained optimization IEEE Trans. Image Process. 19 2345–56
[15] Zhang X, Burger M and Osher S 2011 A unified primal-dual algorithm framework based on Bregman iteration J. Sci. Comput. 46 20–46
[16] Chambolle A and Pock T 2011 A first-order primal-dual algorithm for convex problems with applications to imaging J. Math. Imaging Vis. 40 120–45
[17] Popov L D 1980 A modification of the Arrow–Hurwicz method for search of saddle points Math. Notes 28 845–8
[18] Pock T, Cremers D, Bischof H and Chambolle A 2009 An algorithm for minimizing the Mumford–Shah functional 2009 IEEE 12th Int. Conf. on Computer Vision 1133–40
[19] Rockafellar R T 1970 Convex Analysis (Princeton, NJ: Princeton University Press)
[20] Combettes P L and Pesquet J-C 2011 Proximal splitting methods in signal processing Fixed-Point Algorithms for Inverse Problems in Science and Engineering ed H H Bauschke, R S Burachik, P L Combettes, V Elser, D R Luke and H Wolkowicz (Berlin: Springer) pp 185–212
[21] Moreau J J 1965 Proximité et dualité dans un espace hilbertien Bull. Soc. Math. France 93 273–99
[22] Combettes P L and Wajs V R 2005 Signal recovery by proximal forward–backward splitting Multiscale Model. Simul. 4 1168–200
[23] Korpelevich G M 1976 The extragradient method for finding saddle points and other problems Ekon.-Mat. Metody 12 747–56 (In Russian) (Engl. Transl. in [24])
[24] Korpelevich G M 1977 The extragradient method for finding saddle points and other problems Matiekon 13 35–49 (Engl. Transl.)
[25] Nemirovski A 2005 Prox-method with rate of convergence O(1/t) for variational inequalities with Lipschitz continuous monotone operators and smooth convex–concave saddle point problems SIAM J. Optim. 15 229–51
[26] Nama S, Davies M E, Elad M and Gribonval R 2011 The cosparse analysis model and algorithms Technical Report (arXiv:1106.4987v1)
[27] Yuan M and Lin Y 2006 Model selection and estimation in regression with grouped variables J. R. Stat. Soc. B 68 49–67
[28] Trampert J and Woodhouse J H 2001 Assessment of global phase velocity models Geophys. J. Int. 144 165–74