The process matrix framework for a single-party system

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The process matrix framework [O. Oreshkov, F. Costa, and C. Brukner, Nature Communications 3, 1092 (2012)] can describe general physical theory where locally operations are described by completely-positive maps but globally no fixed causal structure is assumed. In this framework, two parties who perform measurements on each single-qubit system can violate a “causal inequality”, which is not violated if the global fixed causal structure exists. Since the standard quantum physics assumes a fixed global causal structure, the process matrix framework can describe more general physical theory than the standard quantum physics. In this paper, we show that for a single-party system the process matrix framework is reduced to the standard quantum physics, and therefore no exotic effect beyond the standard quantum physics can be observed. This result is analogous to the well known fact in the Bell inequality violation: a single-party system can be described by a local hidden variable theory, whereas more than two parties can violate the Bell inequality.

I. INTRODUCTION

Exploring more general physical theory beyond the standard quantum physics has great practical importance as well as pure academic interest. It can give some (and hopefully full) explanations why quantum physics is as it is while quantum physics is not the most general no-signaling theory [1–5]. It also provides plenty of new insights for researches in other fields, such as statistical physics, field theory, and, interestingly, even computer science. For example, it is known that certain exotic effects, such as the closed-time-like curve [6, 7], non-linear time evolutions [8], and postselections [9], enable super strong computing (such as PP). It was also pointed out that the fact that universal quantum computing with postselections (postBQP) is very strong (i.e., PP) [9] can be used to show the hardness of classical efficient simulations of some superficially innocent non-universal quantum computing models, such as depth-four circuits [10], commuting gates [11], non-interacting bosons [12], and the one-clean qubit model (DQC1) [13, 14].

There are several theoretical formalisms to study general physics beyond the standard quantum physics [13–18]. Oreshkov, Costa, and Brukner [18] recently proposed a new framework, so called the process matrix (PM) framework, to study a physical theory where locally operations are described by CP maps but globally no fixed causal structure is assumed (see also Refs. [13, 20]). Since the standard quantum physics assumes the global fixed causal structure, the PM framework can describe more general physical theory beyond the standard quantum physics. In fact, it was shown in Ref. [18] that for two parties who perform measurements on each single-qubit system the PM framework can violate a “causal inequality”, which is not violated in the theory where the global fixed causal structure exists. (See the next section.)

Does the PM framework always exhibit some exotic effects beyond the standard quantum physics? Or exotic effects are exceptional for some special circumstances, such as certain specific system dimension or number of parties, etc.?

The purpose of the present paper is to study the question. In this paper, we show that for a single-party system we cannot see any exotic effect beyond the standard quantum physics: the PM framework is reduced to the standard quantum physics. It is interesting to point out that this result has an analogy in the Bell inequality violation: a single-party system can be described by a local hidden variable theory, whereas more than two parties can violate the Bell inequality (or its multipartite generalizations).

II. TWO PARTIES EXPERIMENT

It was shown in Ref. [18] that the PM framework for two parties exhibits the exotic effect, namely, the violation of the causal inequality. Let us consider the following game (Fig. 1). Alice and Bob are in the different laboratories. Operations in the inside of each laboratory are described by CP maps, but in the outside of the laboratories, no fixed causal structure is assumed. Alice is given a random bit $a \in \{0, 1\}$ and has to output $x \in \{0, 1\}$. Bob is given two random bits $b, b' \in \{0, 1\}$ and has to output $y \in \{0, 1\}$. A quantum state enters into Alice's laboratory, and she performs a measurement on it. She sends the post-measurement state out to the laboratory. Also in Bob's laboratory, he performs a measurement on an entering quantum state, and sends the post-measurement state out to his laboratory.

If a fixed global causal structure exists in the outside of the laboratories, the inequality \[ p_{OCB} \equiv \frac{1}{2} \left[ p(x = b | b' = 0) + p(y = a | b' = 1) \right] \leq \frac{3}{4} \]
is satisfied [18]. This bound is called the causal inequality. (Analogous to the Bell inequality, which is satisfied if the system is described by a local hidden variable theory.)
where $d_1$ is the dimension of $H_1$, $T$ is the matrix transposition, and

$$|ME\rangle \equiv \sum_{j=1}^{d_1} |j\rangle \otimes |j\rangle \in H_1 \otimes H_1$$

is the (non-normalized) maximally-entangled state.

For example, if the CP operation is to project onto $|\psi\rangle$ and change the post-measurement state into $|\eta\rangle$, the corresponding CJ operator is

$$\sum_{i,j=1}^{d_1} |i\rangle \langle i| \otimes |\psi\rangle \langle \psi| |\eta\rangle = |\psi\rangle \otimes |\eta\rangle$$

where $|0\rangle \equiv |x\rangle$ [27].

In order to see a violation of the causal inequality, authors of Ref. [18] proposed the following PM:

$$W_{OCB} \equiv \frac{1}{4} \left[ I \otimes Z \otimes Z \otimes I + Z \otimes I \otimes X \otimes Z \right].$$

If Alice’s CJ operator is

$$[x] \otimes [a],$$

and Bob’s CJ operator is

$$b'[y] \otimes [\eta] + (b' + 1)(-1)^y \otimes [b + y],$$

where $|\eta\rangle$ is any state, then,

$$p_{OCB} = \frac{2 + \sqrt{2}}{4} > \frac{3}{4}.$$

In this way, the PM framework for two laboratories can exhibit the exotic effect beyond the standard quantum physics.

### III. Single-Party System

Now let us show that the PM framework is reduced to the standard quantum physics for a single-party system. We assume that Alice is in her laboratory (Fig. 2). A state $\rho$ enters into the laboratory, and she measures it. Then she sends the post-measurement state $\sigma$ out to the laboratory. Her operation in the inside of the laboratory is described by a CP map, but there is no fixed causal structure in the outside of her laboratory.

In the PM framework, this experiment is described in the following way. Alice’s measurement on the entering state $\rho$ is described by a CP map

$$\mathcal{E}_j : L(H_1) \rightarrow L(H_2)$$

if the measurement result is $j$, where $H_1$ and $H_2$ are the input and output Hilbert spaces, respectively. The
is the PM. It is easy to show that where \( d \) is the CJ operator corresponding to \( E \) up to the constant factor (Appendix). In other words, the PM enters into the laboratory, and sends the post-measurement state \( \sigma \) out to the laboratory. Her operation in the inside of the laboratory is described by a CP map, but there is no fixed causal structure in the outside of the laboratory.

probability \( P(E_j) \) that Alice’s measurement result is \( j \) is given by

\[
P(E_j) = \text{Tr}(WM_{E_j}),
\]

where

\[
M_{E_j} = [(I \otimes E_j)|ME\rangle\langle ME|^T = \sum_{i,j=1}^{d_1} |i\rangle\langle j| \otimes E_j(|j\rangle\langle i|) \in L(H_1 \otimes H_2)
\]
is the CJ operator corresponding to \( E_j \), and the operator

\[
W \in L(H_1 \otimes H_2),
\]
is the PM. It is easy to show that

\[
\text{Tr}(W) = d_2,
\]

where \( d_2 \) is the dimension of \( H_2 \) (for a proof, see Appendix). In other words, the PM \( W \) is a density matrix up to the constant factor \( d_2 \).

For example, if Alice’s CJ operator is \( \overline{\psi} \otimes \overline{\eta} \) the probability for the event is given by

\[
\text{Tr} \left[ W \left( \overline{\psi} \otimes \overline{\eta} \right) \right].
\]

One might think that if \( W \) could be a maximally-entangled state between \( H_1 \) and \( H_2 \), then the closed timeline curve can be implemented, since Alice can “postselect” the \( H_2 \) part of \( W \) into \( |\eta\rangle \), which affects the “past state”, i.e., the \( H_1 \) part of \( W \).

However, such a “strange” effect does not happen. We now show that in the single-party setup the PM framework is reduced to the standard quantum physics. (For simplicity, we here give a proof for the single-qubit case. A generalization to the multi-qubit case is given in Appendix.) Let us decompose \( W \) in the Pauli basis:

\[
W = \sum_{\alpha \in \{1,x,y,z\}} \sum_{\beta \in \{1,x,y,z\}} w_{\alpha\beta} \sigma_\alpha \otimes \sigma_\beta,
\]

where \( \{w_{\alpha\beta}\} \) are complex numbers, and

\[
\begin{align*}
\sigma_1 &= I = |0\rangle\langle 0| + |1\rangle\langle 1|, \\
\sigma_x &= X = |0\rangle\langle 1| + |1\rangle\langle 0|, \\
\sigma_y &= Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|, \\
\sigma_z &= Z = |0\rangle\langle 0| - |1\rangle\langle 1|,
\end{align*}
\]

are Pauli operators.

Since \( \text{Tr}(W) = 2 \), we obtain \( w_{11} = \frac{1}{2} \). For \( \alpha \in \{x,y,z\} \) and \( \beta \in \{x,y,z\} \), let \( |\alpha(s)\rangle \) be the eigenvector of \( \sigma_\alpha \) corresponding to the eigenvalue \( (-1)^s \) (\( s \in \{0,1\} \)), and \( |\beta(m)\rangle \) be the eigenvector of \( \sigma_\beta \) corresponding to the eigenvalue \( (-1)^m \) (\( m \in \{0,1\} \)).

\[
1 = \sum_{s=0}^{1} \text{Tr} \left[ W \left( |\alpha(s)\rangle \otimes |\beta(m)\rangle \right) \right] = \sum_{s=0}^{1} \left( w_{1,1} + (-1)^m w_{1,\beta} + (-1)^s w_{\alpha,1} + (-1)^{s+m} w_{\alpha,\beta} \right)
\]

\[
= 1 + \sum_{s=0}^{1} \left( (-1)^m w_{1,\beta} + (-1)^{s+m} w_{\alpha,\beta} \right),
\]

which leads to

\[
\sum_{s=0}^{1} \left( (-1)^m w_{1,\beta} + (-1)^{s+m} w_{\alpha,\beta} \right) = 0.
\]

If we take \( m = s \), then

\[
0 = \sum_{s=0}^{1} \left( (-1)^s w_{1,\beta} + (-1)^{2s} w_{\alpha,\beta} \right) = 2w_{\alpha,\beta}.
\]

If we take \( m = 0 \), then

\[
0 = \sum_{s=0}^{1} \left( (-1)^0 w_{1,\beta} + (-1)^s w_{\alpha,\beta} \right) = 2w_{1,\beta}.
\]

In summary, we have shown that for the single-qubit system the PM has the form of

\[
W = \left( \frac{1}{2}I + w_{x1}X + w_{y1}Y + w_{z1}Z \right) \otimes I \equiv W_1 \otimes I.
\]

Since \( W \geq 0 \), we obtain \( W_1 \geq 0 \). This fact and \( \text{Tr}(W_1) = 1 \) means that \( W_1 \) is a density operator. In this case, the PM framework recovers the standard quantum physics.
since for any Kraus operator $E_k$,

$$\text{Tr}\left[W\left(\sum_{i,j} |i\rangle\langle j| \otimes \sum_k E_k |j\rangle \langle i| E_k^\dagger\right)\right]$$

$$= \text{Tr}\left[(W_i \otimes J)\left(\sum_{i,j} |i\rangle\langle j| \otimes \sum_k E_k |j\rangle \langle i| E_k^\dagger\right)\right]$$

$$= \sum_{i,j,k} \langle j|W_i |i\rangle \langle i| E_k^\dagger E_k |j\rangle$$

$$= \text{Tr}\left(\sum_k E_k W_1 E_k^\dagger\right),$$

which is the probability rule in the standard quantum physics.

### IV. DISCUSSION

In this paper, we have shown that the process matrix framework for a single-party system is reduced to the standard quantum physics. This result has the analogy in the Bell inequality violation. It will be a future research subject to clarify whether the analogy is only a superficial one or some deep connections are underlying. For example, the causal inequality has the same bound as that of the Bell inequality. Furthermore, an upper-bound of the causal inequality was derived [19], which is an analog to the Tsirelson bound of the Bell inequality.

One consequence of our result for experiments would be the following: imagine that an experimental project team is trying to test quantum physics. In order to capture any exotic effect beyond the standard quantum physics, like a closed time like curve, they have constructed a large accelerator by spending huge amount of budget. However, due to the shortage of money, they have managed to get only a single laboratory. Can they see any exotic effect beyond the standard quantum physics? Our result shows that unless the “CP map description rule” is locally wrong they cannot see any exotic effect with a single laboratory. In other words, they have to find new sponsor to construct another laboratory.

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**Note added.**— After uploading (the first version of) this paper on arXiv, the author was informed that the result was already shown by Oreshkov, Costa, and Brukner.

### Appendix

We first show $\text{Tr}(W) = d_2$. Let us consider the Kraus operator

$$E_{j,k} = \frac{1}{\sqrt{d_2}} |j\rangle\langle k|$$

for $j = 1, 2, ..., d_2$ and $k = 1, 2, ..., d_1$, which satisfies

$$\sum_{j=1}^{d_2} \sum_{k=1}^{d_1} E_{j,k}^\dagger E_{j,k} = \frac{1}{d_2} \sum_{j=1}^{d_2} \sum_{k=1}^{d_1} |j\rangle\langle j| \langle k|$$

$$= I_{d_1}.$$

Then, we obtain

$$1 = \sum_{j=1}^{d_2} \sum_{k=1}^{d_1} \text{Tr}(WM_{E_{j,k}})$$

$$= \frac{1}{d_2} \sum_{j=1}^{d_2} \sum_{k=1}^{d_1} \text{Tr}\left[W\left(K \otimes L\right)\right]$$

$$= \frac{1}{d_2} \text{Tr}(W),$$

which means $\text{Tr}(W) = d_2$.

We next show the generalization of the single-qubit result to multi-qubit result. Let $S_{s_1,...,s_n}$ be a certain subset of $\{0, 1\}^n$, which can depend on $(s_1, ..., s_n) \in \{0, 1\}^n$.

Consider certain $w_1, ..., w_{n}$, where $\eta_j \in \{1, \alpha, \beta\}$, and $\eta_k \in \{1, \alpha, \beta\}$, and $\eta_r \in \{1, \alpha, \beta\}$, and

$$S_{s_1,...,s_n} = \left\{(m_1, ..., m_n) | m_j \oplus m_k \oplus ... \oplus m_r = 0\right\},$$

not 1. If we take

$$S_{s_1,...,s_n} = \left\{(m_1, ..., m_n) | m_j \oplus m_k \oplus ... \oplus m_r = 0\right\},$$
then

\[ 1 = 2^n w_1, \ldots, 1, \ldots, 1 + 2^n w_1, \ldots, 1, \eta_1, \ldots, \eta_n, \]

which means that \( w_1, \ldots, 1, \eta_1, \ldots, \eta_n = 0 \).

Next consider certain \( w_{\xi_1}, \ldots, \xi_n, \eta_1, \ldots, \eta_n \), where \( \xi_t, \xi_u, \ldots, \xi_v \) and \( \eta_j, \eta_k, \ldots, \eta_r \) are not 1. Then, if we take

\[
S_{s_1, s_2, \ldots, s_n} = \{(m_1, \ldots, m_n) | s_t \oplus s_u \oplus \ldots \oplus s_v = m_j \oplus m_k \oplus \ldots \oplus m_r\},
\]

which means that \( w_{\xi_1}, \ldots, \xi_n, \eta_1, \ldots, \eta_n = 0 \).

[1] S. Popescu and D. Rohrlich, Found. Phys. 24, 379 (1994).
[2] M. Pawlowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Zukowski, Nature 461, 1101 (2009).
[3] D. Gross, M. Mueller, R. Colbeck, and O. C. O. Dahlsten, Phys. Rev. Lett. 104, 080402 (2010).
[4] H. Barnum, S. Beigi, S. Boixo, M. B. Elliott, and S. Wehner, Phys. Rev. Lett. 104, 140401 (2010).
[5] G. de la Torre, L. Masanes, A. J. Short, and M. P. Mueller, Phys. Rev. Lett. 109, 090403 (2012).
[6] D. Bacon, Phys. Rev. A 70, 032309 (2004).
[7] S. Aaronson and J. Watrous, Proc. R. Soc. A 465, 631 (2009).
[8] D. S. Abrams and S. Lloyd, Phys. Rev. Lett. 81, 3992 (1998).
[9] S. Aaronson, Proc. R. Soc. A 461, 3437 (2005).
[10] B. Terhal and D. DiVincenzo, Quant. Inf. Comput. 4, 134 (2004).
[11] M. J. Bremner, R. Jozsa, and D. J. Shepherd, Proc. R. Soc. A 467, 2126 (2011).
[12] S. Aaronson and A. Arkhipov, Theory of Computing 9, 143 (2013).
[13] E. Knill and R. Laflamme, Phys. Rev. Lett. 81, 5672 (1998).
[14] T. Morimae, K. Fujii, and J. F. Fitzsimons, Phys. Rev. Lett. 112, 130502 (2014).
[15] G. Chiribella, G. M. D’Ariano, P. Perinotti, and B. Vallorin, Phys. Rev. A 88, 022318 (2013).
[16] M. S. Leifer and R. W. Spekkens, Phys. Rev. A 88, 052130 (2013).
[17] J. Fitzsimons, J. Jones, and V. Vedral, arXiv:1302.2731.
[18] O. Oreshkov, F. Costa, and C. Brukner, Nat. Comm. 3, 1092 (2012).
[19] C. Brukner, arXiv:1404.0721.
[20] C. Brukner, Nat. Phys. 10, 259 (2014).
[21] A. Baumeler, A. Feix, and S. Wolf, arXiv:1403.7333.
[22] A. Baumeler and S. Wolf, arXiv:1312.5916.
[23] T. Morimae, Phys. Rev. A 90, 010101(R) (2014).
[24] L. Hardy, arXiv:0509120.
[25] L. Hardy, J. Phys. A 40, 3081 (2007).
[26] L. Hardy, arXiv:0701019.
[27] We call this notation Vedran’s notation.