CONNECTIVE VERSIONS OF $\text{TMF}(3)$

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Abstract. We study three connective versions of the spectrum for topological modular forms of level 3. All three were described briefly by Mahowald and Rezk in [8], but we add much detail to their discussion. Letting $\text{tmf}(3)$ denote our connective model which is a ring spectrum, we compute $\text{tmf}(3)_*(\mathbb{R}P^\infty)$.

1. Introduction

In [8], the second author and Rezk discuss the periodic spectrum $\text{TMF}(\Gamma_0(3))$, abbreviated here as $\text{TMF}(3)$, associated to topological modular forms of level 3. In Section 7 of [8], they discuss briefly three connective models of $\text{TMF}(3)$. The main purpose of this paper is to clarify and fill in details for these connective models.

The first model is $X \wedge \text{tmf}$, where $X$ is a certain 10-cell complex; it was first introduced by the second author and Gorbounov in their study of $\text{MO}[8]$. It is probably the best of our three models because it is a ring spectrum. In Section 2, we define it and compute its homotopy groups. Our description and method of computation differ somewhat from that of [6].

In [8], another connective model for $\text{TMF}(3)$ is discussed, which is $Z \wedge \text{tmf}$, where $Z$ is a certain 8-cell complex. Although $Z \wedge \text{tmf}$ is not a ring spectrum, its importance is primarily that the dimensions of the cells of $Z$ allow one to easily construct a map $Z \to \text{TMF}(3)$ thanks to certain homotopy groups of $\text{TMF}(3)$ being 0. The other models are then related to $\text{TMF}(3)$ via the $Z$-model. In Section 3, we provide some additional details to the sketch given in [8].

In Section 4, we consider a third model which was also introduced in [8]. This one is closely related to consideration of a splitting of $\text{tmf} \wedge \text{tmf}$. There is a Brown-Gitler-type splitting of the $A$-module $H^*(\text{tmf} \wedge \text{tmf})$, and we show that it is not realized by

Date: May 20, 2010.

Key words and phrases. Topological modular forms, Adams spectral sequence.

2000 Mathematics Subject Classification: 55P42.
a spectrum splitting. Again we add some clarity and detail to the description in [8] of this model and its homotopy groups.

All three of our models are equivalent after inversion of $v_2$, but as connective models they are different. The homotopy groups of the second and third models are subsets of those of the first, obtained by omitting certain initial portions. One nice feature of our approach is to relate the Ext calculations for the second and third models directly to that of the first, even though the constructions of the spectra are very different.

In Section 5 we compute $\pi_*(P_1 \wedge X \wedge \text{tmf})$, where $P_1 = RP^\infty$. If we think of $X \wedge \text{tmf}$ as our best model of $\text{tmf}(3)$, then this is $\text{tmf}(3)_*(P_1)$. Our original goal in undertaking this study was to use TMF(3) in obstruction theory, and this computation would be a first step toward doing that.

2. THE MODEL OF GORBOUNOV AND MAHOWALD

In their study of $\pi_*(MO[8])$ in [6], the second author and Gorbounov introduced a new spectrum, which turns out to be the best model for a connective version of TMF(3). Certain aspects of the construction in [6] were unclear to the first author, and so we have prepared this alternative account. In Theorem 2.1 we define the spectrum, and in Theorems 2.11 and 2.12 we determine its homotopy groups. In Section 3 we will establish its relationship with TMF(3).

**Theorem 2.1.** (a) There is a 9-cell CW complex $Y$ with one cell of each dimension $0, 2, 3, 4, 6, 7, 8, 9,$ and $10$, in which the following Steenrod operations are nonzero on the bottom class $g$:

$$
\begin{align*}
\text{Sq}^2, \text{Sq}^3, \text{Sq}^4, \text{Sq}^4 \text{Sq}^2, \text{Sq}^5 \text{Sq}^2, \text{Sq}^6 \text{Sq}^2 = \text{Sq}^8, \text{Sq}^6 \text{Sq}^3, \text{Sq}^7 \text{Sq}^3.
\end{align*}
$$

This together with $\text{Sq}^6 g = 0$ completely describes $H^*(Y)$ as an $A$-module.

(b) There is a map $\Sigma^3 Y \xrightarrow{\alpha} S^0$ extending $2\nu$.

(c) Let $X$ denote the mapping cone of $\alpha$. There is a map $X \xrightarrow{f} \text{bo}$ which is the identity on the bottom cell.

(d) Let $\tilde{f}$ denote the composite

$$
X \wedge \text{tmf} \xrightarrow{f \wedge 1} \text{bo} \wedge \text{tmf} \xrightarrow{\mu} \text{bo},
$$
and let $C$ denote the mapping cone of $\tilde{f}$. There is an isomorphism of $A$-modules

$$H^*(C) \approx \Sigma^4 A/A(Sq^4, Sq^5 Sq^1).$$

(e) $X \wedge \text{tmf}$ is a ring spectrum.

Remark 2.3. This $X \wedge \text{tmf}$ will be our preferred model for the connective $\text{tmf}(3)$, because it is a ring spectrum. The spectrum $\Sigma^{16} X \wedge \text{tmf}$ is apparently a subspace of $\text{MO}(8)/\text{tmf}$, but this will not enter into our argument. This was the motivation for the initial discussion of $X \wedge \text{tmf}$ in [6].

Throughout the paper, $A_n$ denotes the subalgebra of the mod 2 Steenrod algebra $A$ generated by $Sq^i$ for $i \leq 2^n$. Also $\eta$ and $\nu$ denote the (class of the) Hopf maps in the 1- and 3-stems. All cohomology groups have coefficients in $\mathbb{Z}_2 = \mathbb{Z}/2$. Our spectra are localized at 2.

Proof. (a.) Let $X_3 = S^0 \cup_\eta e^2 \cup_2 e^3$ and $X_7 = S^0 \cup_\nu e^4 \cup_\eta e^6 \cup_2 e^7$. Let $Q$ denote the quotient of $X_3 \wedge X_7$ by its 4-skeleton. The Steenrod algebra structure, or equivalently the cell structure, of $Q$ is depicted in Diagram 2.4. Here a symbol $(i, j)$ is the product class or cell of an $i$-cell of $X_3$ and a $j$-cell of $X_7$. We indicate both $Sq^1$ and $Sq^2$ by straight lines, and $Sq^4$ by a curved line.

Diagram 2.4. Cell structure of quotient of $X_3 \wedge X_7$

```
(2, 4) ——— (3, 4) ——— (3, 6)
|                |                |
(0, 6) ——— (0, 7) ——— (2, 7)
```

There is a map $g$ from this $Q$ to $S^6 \cup_2 e^7 \cup_\eta e^9$ which sends the cells $(2, 4)$, $(3, 4)$, and $(3, 6)$ by degree 1, and the cells $(0, 6)$, $(0, 7)$, and $(2, 7)$ by degree $-1$. The fiber of the composite

$$X_3 \wedge X_7 \xrightarrow{\text{coll}} Q \xrightarrow{g} S^6 \cup_2 e^7 \cup_\eta e^9$$

is the desired complex $Y$. The Steenrod operations in $Y$ can be determined from the Cartan formula together with the fact that $Sq^2$ and $Sq^3$ are nonzero in $X_3$, and $Sq^4$, ...
Sq^6, and Sq^7 are nonzero in \( X_7 \). For example, \( Sq^4 Sq^2 \) on the bottom class is \((2,4)\), while \( Sq^6 \) is \((2,4) + (0,6)\), which is \( g^*(x_6) \) and hence is 0 in the fiber.

(b.) Let \( DY \) denote the \( S \)-dual of \( Y \), with cells of dimensions the negative of those of \( Y \). Thus the top cell of \( DY \) has dimension 0. Note that \( Sq^8 = 0 \) in \( H^*(DY) \), since it is dual to \( \chi Sq^8 \), which is 0 in \( H^*(Y) \). Let \((DY)^{(-1)}\) denote the \((-1)\)-skeleton of \( DY \). We use the ASS to show that \( 2\nu \) is in the image of \( \pi_3(DY) \xrightarrow{c} \pi_3(S^0) \), where \( c \) collapses \((DY)^{(-1)}\). We use Bruner’s software \([2]\) to compute \( Ext_{A^1}^r(H^*(DY)) \) for \( 2 \leq t - s \leq 4 \) as in Diagram 2.5. Here and throughout, we omit writing \( \mathbb{Z}_2 \) as the second argument of our Ext groups.

**Diagram 2.5. Ext groups for \( 2 \leq t - s \leq 4 \)**

\[
Ext_A(H^*((DY)^{(-1)})) \rightarrow Ext_A(H^*DY) \rightarrow Ext_A(H^*S^0) \xrightarrow{\delta}
\]

The desired class \( 2\nu \) is indicated with \( A \) in the diagram, and is the image of the circled class. The class \( \nu \), indicated by \( B \), maps to \( B' \) in the exact sequence.

(c.) Let \( DX \) denote the \( S \)-dual of \( X \), with 10 cells, in dimensions \(-14 \) up to 0. Then \([\Sigma^i X, bo] \approx \pi_i(DX \wedge bo)\), and this can be computed by the ASS with \( E_2 = Ext_{A_1}(H^*(DX)) \). The \( A_1 \)-structure of \( H^*(DX) \) is easily seen, and the \( Ext_{A_1} \)-calculation easily made, giving the result in Diagram 2.6 in dimension \(< 4 \). There are clearly no possible differentials, and our desired map is detected in filtration 0 by the circled element.
Diagram 2.6. $\text{Ext}_{A_1}(H^*DX)$ in $t - s < 4$

\[
\begin{array}{c}
\begin{array}{c}
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(3) 5
\end{array}
\end{array}
\]

(d.) There is a commutative diagram in which horizontal and vertical sequences are fiber sequences.

\[
\begin{array}{c}
\begin{array}{c}
S^0 \wedge \text{tmf} \longrightarrow \text{tmf} \\
\downarrow \quad \downarrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
X \wedge \text{tmf} \longrightarrow b_0 \longrightarrow C \\
\downarrow \quad \downarrow \quad \downarrow
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
(X/S^0) \wedge \text{tmf} \longrightarrow b_0/\text{tmf} \longrightarrow C \longrightarrow \Sigma(X/S^0) \wedge \text{tmf}
\end{array}
\end{array}
\]

The restriction of $\tilde{f}$ to the 4-skeleton is $S^0 \cup_{2v} e^4 \to S^0 \cup_v e^4$ of degree 1 on the bottom cell. Thus $\tilde{f}$ has degree 2 on its bottom 4-cell. The $A$-module $H^*(bo/\text{tmf})$ is isomorphic to $A \otimes_{A_2} A_2/\!\!/A_1$, and the $A_2$-module $A_2/\!\!/A_1$ has basis

\[
\{ g_4, \text{Sq}^2 g_4, \text{Sq}^3 g_4, \text{Sq}^4 \text{Sq}^2 g_4, \text{Sq}^4 \text{Sq}^3 g_4, \text{Sq}^6 \text{Sq}^3 g_4, \text{Sq}^4 \text{Sq}^6 \text{Sq}^3 g_4 \}.
\]

Thus $(\tilde{f})^* = 0$, and, since $X/S^0 = \Sigma^4 Y$, there is a short exact sequence of $A$-modules

\[
0 \to H^*(\Sigma^5 Y \wedge \text{tmf}) \to H^*(C) \to A \otimes_{A_2} A_2/\!\!/A_1 \to 0,
\]

and $\text{Sq}^1 g_4 \neq 0$ in $H^* C$. We conclude that $H^*(C)$ is an extended cyclic $A_2$-module on a 4-dimensional generator, with nonzero operations being those in (2.7) and $\text{Sq}^1$ and the operations listed in (2.2) applied to $\text{Sq}^1$. One easily checks that this $A_2$-module equals $A_2/(\text{Sq}^4, \text{Sq}^5 \text{Sq}^1)$, and so the $A$-module $H^*(C)$ is as claimed.
(e.) We will prove there is a map $m' : X \wedge X \to bo$ extending the inclusion of the bottom cell and that when followed by the map $bo \to C$ of part (d), the composite is trivial. Thus by the definition of $C$, $m'$ factors through a map $m : X \wedge X \to X \wedge \text{tmf}$ extending the inclusion of the bottom cell. Smashing this twice with \text{tmf} and following by two multiplications of \text{tmf} yield the desired product on $X \wedge \text{tmf}$.

We construct the dual of $m'$, an element of $\pi_0(DX \wedge DX \wedge bo)$. The $E_2$-term of the ASS converging to $\pi_*(DX \wedge DX \wedge bo)$ is $\text{Ext}_{A_1}(H^*(DX \wedge DX))$. The $A_1$-structure of $H^*(DX)$ is easily seen and $\text{Ext}_{A_1}$ of tensor products of the summands is easily computed, as, for example, in [3]. We obtain that in the vicinity of $t - s = 0$, the chart has a copy of $bo_*$ beginning in position $(0,0)$ and 15 additional copies of $bo_*$ beginning in positions $(0,s)$ for $3 \leq s \leq 12$. The groups in $t - s = -1$, i.e. corresponding to $\pi_{-1}$, are all 0. Thus there are no possible differentials from $t - s = 0$ in the ASS, and we deduce the existence of our map $S^0 \to DX \wedge DX \wedge bo$, whose dual is $m'$.

Next we compute the ASS for $\pi_*(DX \wedge DX \wedge C)$. Let $Y$ be as in part (a). Then $DX = \Sigma^{-4}DY \cup \nu \epsilon^0$, and so $H^*(DX) \approx H^*(\Sigma^{-4}DY) \oplus H^*(S^0)$ as $A$-modules. Thus the ASS converging to $\pi_*(DX \wedge DX \wedge C)$ has

$$E_2 \approx \text{Ext}_A(H^*(\Sigma^{-4}DY \wedge DY) \otimes A/(\text{Sq}^4, \text{Sq}^{5,1})) \oplus \text{Ext}_A(H^*(DY) \otimes A/(\text{Sq}^4, \text{Sq}^{5,1}))$$

$$\oplus \text{Ext}_A(H^*(DY) \otimes A/(\text{Sq}^4, \text{Sq}^{5,1})) \oplus \text{Ext}_A(\Sigma^4A/(\text{Sq}^4, \text{Sq}^{5,1})).$$

Note that the bottom class of $DY$ is in grading $-10$. We can use Bruner’s software to see that each of these Ext groups is 0 in $t - s = 0$. For example,

$$\text{Ext}_A(H^*(\Sigma^{-4}DY \wedge DY) \otimes A/(\text{Sq}^4, \text{Sq}^{5,1}))$$

has 15 $\mathbb{Z}$-towers in the $(-3)$-stem, beginning in filtrations 2 through 8. It is 0 in stems $-2$, $-1$, and 0, and then in the 1-stem has 21 $\mathbb{Z}$-towers, on each of which $\eta$ and $\eta^2$ are nonzero.

Thus $\pi_0(DX \wedge DX \wedge C) = 0$ and hence $[X \wedge X, C] = 0$. Therefore the map $X \wedge X \xrightarrow{m'} bo \to C$ is trivial, implying the result by the argument of the first paragraph of the proof. $\square$

The main step toward describing $\pi_*(X \wedge \text{tmf})$ is, because of [2.11(d)], the Ext calculation in Theorem 2.9. This calculation was first made in [6], but our approach
will be somewhat different. Our approach will be useful in performing other related Ext calculations. The description is in terms of $bo_\ast$ and $bsp_\ast$, which are depicted in Diagram 2.8.

**Diagram 2.8.** $bo_\ast$ and $bsp_\ast$

We will denote by $a_{x,y}$ an element of $\text{Ext}^{x,y}(A/A, Z_2)$. This corresponds to the usual $(x, y)$ components in an ASS. There are standard elements $h_1, h_2,$ and $v_2$ of $(x, y)$-grading $(1, 1), (3, 1),$ and $(24, 4)$, respectively. Here and throughout, $R[a]/(b_1, \ldots, b_r)$ denotes a free module over a polynomial algebra $R[a]$ with basis $\{b_1, \ldots, b_r\}$.

**Theorem 2.9.** As a bigraded abelian group, $\text{Ext}_A^\ast(A/A(Sq^4, Sq^5 Sq^1), Z_2)$ is isomorphic to

$$\mathbb{Z}_2[v_2^8](a_{0,0}, h_2a_{0,0}, a_{14,2}, h_1a_{14,2}, h_2a_{14,2}, a_{31,5}, h_2a_{31,5}, a_{39,7})$$

$$\oplus \ker(bo_\ast[v_2^4](a_{5,1}, a_{21,3}) \rightarrow \mathbb{Z}_2[v_2^8](a_{21,3}))$$

$$\oplus \, bsp_\ast[v_2^4](a_{9,2}, a_{17,4}).$$

**Proof.** By the Change-of-Rings Theorem, it is equivalent to compute

$$\text{Ext}_A^\ast(A_2/A_2(Sq^4, Sq^5 Sq^1), Z_2).$$

One can verify that there is an exact sequence of $A_2$-modules:

$$0 \leftarrow A_2/(Sq^4, Sq^5, 1) \leftarrow A_2 \leftarrow \Sigma^4 A_2 \oplus \Sigma^6 A_2 // A_1 \leftarrow \Sigma^{11} A_2/(Sq^1, Sq^5) \oplus \Sigma^{16} A_2$$

$$\leftarrow \Sigma^{18} A_2/(Sq^3) \oplus \Sigma^{20} A_2 \leftarrow \Sigma^{25} A_2 \oplus \Sigma^{26} A_2 // \Sigma^{25} A_2 \oplus \Sigma^{26} A_2$$

$$\leftarrow \Sigma^{34} A_2//A_1 \oplus \Sigma^{36} A_2 // (Sq^3) \leftarrow \Sigma^{40} A_2$$

$$\leftarrow \Sigma^{46} A_2 // (Sq^3) \oplus \Sigma^{52} A_2 // A_1 \leftarrow \Sigma^{56} A_2/(Sq^1, Sq^5, 1) \leftarrow 0$$
with
\[
\begin{align*}
d_1(I_4) &= Sq^4 \\
d_1(I_6) &= Sq^5 Sq^1 \\
d_2(I_{11}) &= Sq^7 I_4 \\
d_2(I_{16}) &= (Sq^{6.6} + Sq^{7.5}) I_4 + Sq^{4.6} I_6 \\
d_3(I_{18}) &= Sq^2 I_{16} + Sq^7 I_{11} \\
d_3(I_{20}) &= Sq^4 I_{16} + Sq^{6.3} I_{11} \\
d_4(I_{25}) &= Sq^7 I_{18} + Sq^5 I_{20} \\
d_4(I_{26}) &= Sq^{7.1} I_{18} + Sq^6 I_{20} \\
d_5(I_{34}) &= Sq^{2.7} I_{25} \\
d_5(I_{36}) &= (Sq^{5.6} + Sq^{6.5}) I_{25} + Sq^{4.6} I_{26} \\
d_6(I_{40}) &= Sq^4 I_{36} + Sq^6 I_{34} \\
d_7(I_{46}) &= Sq^6 I_{40} \\
d_7(I_{52}) &= Sq^{7.5} I_{40} \\
d_8(I_{56}) &= Sq^4 I_{52} + (Sq^{4.6} + Sq^{6.3.1}) I_{46}.
\end{align*}
\]

For \(0 \leq i \leq 7\), let \(C_i\) denote the \(A_2\)-module which is the domain of \(d_i\). Because the domain of \(d_8\) is \(\Sigma^{56}\) of the beginning module, the exact sequence could be continued periodically with the \(\Sigma^{56} A_2/(Sq^4, Sq^{5.1})\) removed, and \(C_{i+8} \approx \Sigma^{56} C_i\). There is a spectral sequence building \(\text{Ext}(A_2/(Sq^4, Sq^{5.1}))\) from \(\bigoplus_{i \geq 0} \phi^i \text{Ext}(\Sigma^{-i} C_i)\), where \(\phi^i\) increases filtration by \(i\). Of the modules that appear in \(C_i\), \(\text{Ext}(A_2)\) is just \(\mathbb{Z}_2\) in \((0,0)\), \(\text{Ext}(A_2/A_1)\) is \(b_0\), \(\text{Ext}(A_2/(Sq^1, Sq^5))\) is \(bsp\), \(\text{Ext}(A_2/(Sq^3))\) is \(\text{Ext}(A_2) \oplus \phi \text{Ext}(\Sigma^2 bsp)\), and \(\text{Ext}((A_2 \oplus \Sigma^1 A_2)/(Sq^1 I_0, Sq^3 I_0 + Sq^2 I_1))\) is \(\phi^{-1}(\ker(b_0 \to \mathbb{Z}_2))\).

When these are put together, one obtains exactly the claim of the theorem. There can be no differentials because differentials are \(h_i\)-natural. The differentials would go from position \((x,y)\) of \(\phi^i \text{Ext}(\Sigma^{-i} C_i)\) to position \((x-1,y+1)\) of \(\phi^{i+r} \text{Ext}(\Sigma^{-(i+r)} C_{i+r})\).

In Diagram 2.10 we depict this chart for \(x \leq 48\), to show the impossibility of differentials in both this SS converging to Ext, and in an ASS to be considered later. Note that the \(\mathbb{Z}_2\) in the 48-stem is \(v_2^8\) times the initial \(\mathbb{Z}_2\).
Diagram 2.10. \( \text{Ext}_A(A/(\text{Sq}^4, \text{Sq}^{5.1})) \) through degree 48

The following result is an easy consequence of Theorems 2.1 and 2.9.
Theorem 2.11. There is an isomorphism of graded abelian groups

\[ \pi_*(X \wedge \text{tmf}) \cong \text{bo}_*[v_2^4] \langle v_1v_2 \rangle \oplus \ker\left(\text{bo}_*[v_2^4] \to \mathbb{Z}_2[v_2^8]\langle v_2^4 \rangle\right) \]

\[ \oplus \text{bsp}_*[v_2^4](2v_2^2, 2v_1v_2^3) \]

\[ \oplus \mathbb{Z}_2[v_2^8]\langle \nu, \nu^2, x, \eta x, \nu x, x^2, \eta x^2, y \rangle, \]

where the (homotopy group, Adams filtration) of elements is (2, 1) for \( v_1 \), (6, 1) for \( v_2 \), (17, 3) for \( x \), and (42, 8) for \( y \).

Proof. We use the exact sequence

\[ \to \pi_*(\Sigma^{-1}C) \to \pi_*(X \wedge \text{tmf}) \to \pi_*(\text{bo}) \to \]

from 2.1. The bottom class of \( H^*(C) \) causes \( \text{Sq}^4 \iota_0 = 0 \) in \( H^*(X \wedge \text{tmf}) \), and so a chart for a spectral sequence converging to \( \pi_*(X \wedge \text{tmf}) \) can be formed from \( \text{bo}_* \) of Diagram 2.8 together with Diagram 2.10 shifted by (3, 1) units. By \( h_i \)-naturality there are no differentials or extensions, and so the chart depicts \( \pi_*(X \wedge \text{tmf}) \). The names \( v_1v_2, 2v_2^2, \) and \( 2v_1v_3^2 \) which we give to certain generators are, at least at this point, meant to only describe stem and filtration. \( \square \)

Our next result simplifies the \( \text{bo}_*-\text{bsp}_* \)-part of this description and also incorporates as much as we can say about the ring structure from our approach. Our limitation is that our approach can only give the ring structure of \( \pi_*(X \wedge \text{tmf}) \) up to elements of higher filtration in the Adams-type spectral sequence we have been using. Note that we say “Adams-type” because we have elevated the filtrations of the part from \( C \) by 1 compared to an actual ASS. The reason that we can’t say any better than “up to higher filtration” is, first of all the usual limitation of an ASS, and secondly that our multiplication of \( X \wedge \text{tmf} \) is only defined up to maps of higher filtration. It seems that such deviations would change the product structure in \( \pi_*(X \wedge \text{tmf}) \). For example, the product of classes that we call \( 2v_2^2 \) and \( 2v_1^4v_2^4 \) (so-called because of their image in \( \text{BP}_* \); note that these classes are generators—the elements without the factor 2 are not present in \( \pi_*(X \wedge \text{tmf}) \)) would naturally be \( 4v_1^4v_2^2 \), an element which would be divisible by 4 in \( \pi_*(X \wedge \text{tmf}) \). However, we cannot assert that this product of generators is divisible by 4; it might equal, for example, \( 4v_1^4v_2^2 + v_1^{16} \).
Theorem 2.12. There is an isomorphism of graded abelian groups 
\[ \pi_*(X \wedge \text{tmf}) \cong K \oplus \mathbb{Z}_2[v_2^8](\nu, \nu^2, x, \eta x, \nu x, x^2, \nu x^2, v_1 v_2 x^2), \]
where 
\[ K = \ker(R \to \mathbb{Z}_2[v_2^8](v_2^4)) \]
with \( R \) the subring of \( \mathbb{Z}[v_1, v_2, \eta]/(2\eta, \eta^3) \) generated by \( 2v_1^2, v_1^4, v_1 v_2, 2v_2^2, \) and \( v_2^4 \). The isomorphism is, up to elements of higher filtration, an isomorphism of rings, with the additional relations \( v_1^4 x = \eta v_1^3 v_2^2, v_1 v_2 x = \eta v_2^4, x^3 = \nu v_2^8, \) and \( x^7 = 0 \).

Stems of elements are as in Theorem 2.11. Note that \( x^7 = 0 \), not just up to elements of higher filtration, as it lies in a zero group.

Proof. It is not difficult to check that this description is consistent as an Adams-filtered graded abelian group with the description in Theorem 2.11. We must establish various product formulas.

First we show that \( x^2 \) is nonzero, corresponding to \( a_{31,5} \) in 2.9. The multiplication of \( X \wedge \text{tmf} \) restricts to a filtration-1 map \( \Sigma^{-1} C \wedge \Sigma^{-1} C \to \Sigma^{-1} C \). Note that \( H^*(\Sigma^{-1} C) \approx \Sigma^3 A/(\text{Sq}^4, \text{Sq}^5, 1) \), and so the multiplication can be considered as

\[ \text{Ext}_{A_2}^{s,t}(A_2/(\text{Sq}^4, \text{Sq}^5, 1)) \otimes \text{Ext}_{A_2}^{s,t}(A_2/(\text{Sq}^4, \text{Sq}^5, 1)) \to \text{Ext}_{A_2}^{s+1,t+4}(A_2/(\text{Sq}^4, \text{Sq}^5, 1)) \]

(2.13)

with \( \iota_0 \otimes \iota_0 \mapsto h_2 \iota_0 \). With \( x \in \text{Ext}_{A_2}^{2,16}(A_2/(\text{Sq}^4, \text{Sq}^5, 1)) \), the image of \( x \otimes x \) is in \( \text{Ext}_{A_2}^{5,36}(A_2/(\text{Sq}^4, \text{Sq}^5, 1)) \). We wish to show it is nonzero.

Note that \( x \iota_0 \otimes \alpha \iota_0 \mapsto \alpha h_2 x \iota_0 \), so we want the Yoneda product of \( h_2 x \) with \( x \). Using the minimal “resolution” in the proof of 2.9, we consider the following diagram:

\[
\begin{array}{ccc}
C_3 & \leftarrow & C_4 & \leftarrow & C_5 \\
\downarrow h_2 x & & \downarrow f_4 & & \downarrow f_5 \\
\Sigma^{20} C_0 & \leftarrow & \Sigma^{20} C_1 & \leftarrow & \Sigma^{20} C_2 \\
\downarrow x & & \downarrow \Sigma^{36} \mathbb{Z}_2, & & \end{array}
\]
The relevant parts are
\[
\begin{array}{cccccc}
\Sigma^{20} A_2 & \xleftarrow{\text{Sq}^4 \text{Sq}^6} & (\Sigma^{25} A_2 \oplus \Sigma^{26} A_2) / R & \xleftarrow{\text{Sq}^4 + \text{Sq}^6} & \Sigma^{36} A_2 \\
1 & \downarrow f_4 & \downarrow f_5 & \downarrow & \downarrow & \downarrow \Sigma^{36} Z_2 \\
\Sigma^{20} A_2 & \xleftarrow{\text{Sq}^4 \text{Sq}^5,1} & \Sigma^{24} A_2 \oplus \Sigma^{26} A_2 / / A_1 & \xleftarrow{\text{Sq}^6 + \text{Sq}^7,5,\text{Sq}^4} & \Sigma^{36} A_2
\end{array}
\]

We find that \( f_4(\iota_{25}) = \text{Sq}^1 \iota_{24} \) and \( f_4(\iota_{26}) = \text{Sq}^2 \iota_{24} + \iota_{26} \), and then that \( f_5 \) is the identity.

A similar argument works to show \( x^3 = \nu v_2^8 \). Relations for \( x^4 \), \( x^5 \), and \( x^6 \) can be deduced from the stated relations.

The elements \( \eta x \), \( x^2 \), and \( y \) generate the three occurrences of \( A_2 / (\text{Sq}^3) \) in the resolution in the proof of 2.9. The bsp's on \( a_{17,4,4} \), \( v_2^3 a_{9,2} \), and \( v_2^3 a_{17,4} \) in Theorem 2.11 are obtained from \( \text{Ext}_{A_2} \) of these three \( A_2 / (\text{Sq}^3) \)'s by omitting the initial \( Z_2 \). This implies that \( v_1^4 x = \eta^2 v_1^3 v_2^3 \), \( v_1^4 x^2 = \eta^2 v_1^2 v_2^6 \), and \( v_1^4 y = \eta^2 v_1^3 v_2^7 \). One of our relations is obtained by dividing the first of these by \( \eta \), while the latter two imply that \( y = v_1 v_2 x^2 \).

The elements which we call \( v_1^i v_2^j \cdot (2v_2^2)^e \) with \( 1 \leq e \leq 3 \) in \( E_2^{*,*}(H^*X) \) are in the image of the ring map from \( \text{Ext}_{A_2}(Z_2) \), and so products among them are as we claim because of the ring structure of \( \text{Ext}_{A_2}(Z_2) \). That the products of \( (v_1 v_2)^i \) with \( 2v_2^2 \) are as claimed can be proved by a Yoneda product argument with the element \( 2v_2^2 \) of \( \text{Ext}_{A_2}^{3,15}(Z_2) \). To verify this using a minimal resolution of \( A / (\text{Sq}^4, \text{Sq}^5,1) \), one should expand the efficient resolution used in the proof of 2.9 to use only \( A_2 \) and \( A_2 / (\text{Sq}^1) \) (and not the more efficient \( A_2 / A_1 \) and \( A / (\text{Sq}^3) \)). This produces some additional \( \text{Sq}^4 \text{Sq}^6 \) terms in the resolution. The following not-quite-commutative diagram of not-quite-exact sequences shows the most relevant terms in the morphism from a portion of the resolution built on \( (v_1 v_2)^i \) to the most relevant terms of the resolution of \( Z_2 \), and can be used to establish that the Yoneda product of the element that we call \( (v_1 v_2)^i \) followed by the element that we call \( 2v_2^2 \) equals the element that we call...
suggests, we consider the morphism of minimal resolutions inducing (2.13). Let

\[ A_2/(\text{Sq}^1) \xleftarrow{\text{Sq}^{4,6}} A_2 \xleftarrow{\text{Sq}^2} A_2 \xleftarrow{\text{Sq}^3} A_2 \]

For example, when \( i = 2 \), the top row of this diagram corresponds to elements in (13, 3), (22, 4), (23, 5), and (25, 6) in Diagram 2.10.

To see that the elements that we call \( v_1^iv_2^j \) multiply by one another as the notation suggests, we consider the morphism of minimal resolutions inducing (2.13). Let

\[ C : C_0 \leftarrow C_1 \leftarrow \cdots \quad \text{(resp. } D : D_0 \leftarrow D_1 \leftarrow \cdots) \]

be a minimal \( A_2 \)-resolution of \( \Sigma^3 A_2/(\text{Sq}^4, \text{Sq}^{5,1}) \) (resp. \( \Sigma^2 \ker(A_2 \rightarrow A_2/(\text{Sq}^4, \text{Sq}^{5,1})) \). Then (2.13) is induced by a morphism \( \psi : C \rightarrow D \). The class which we call \( v_1^iv_2^j \) is dual to a generator \( \alpha_i \in (C_{2i-1})_{10i-1} \) and to a generator \( \beta_i \in (D_{2i-2})_{10i-2} \).

First we show that the square of our \( v_1v_2 \) class equals our class called \( v_1^iv_2^j \). The relevant parts are that \( C_1 \xrightarrow{d_1} C_0 \) has \( C_0 = \Sigma^3 A_2 \), \( C_1 = \Sigma^7 A_2 \oplus \Sigma^9 A_2/A_0 \), with \( d_1(\iota_7) = \text{Sq}^4 \iota_3 \) and \( d_1(\iota_9) = \text{Sq}^5_1 \iota_3 \), while the relevant part of \( D \) is

\[ \Sigma^{18} A_2 \xrightarrow{d_2} \Sigma^{13} A_2/A_0 \xrightarrow{d_1} \Sigma^6 A_2 \]

with \( d_2(\iota_{18}) = \text{Sq}^5 \iota_{13} \) and \( d_1(\iota_{13}) = \text{Sq}^7 \iota_6 \). In the commutative diagram of exact sequences

\[
\begin{array}{ccc}
D_2 & \xrightarrow{d_2} & D_1 \\
\downarrow f_2 & & \downarrow f_1 \\
C_1 \otimes C_1 & \xrightarrow{d_1} & C_0 \otimes C_0 \\
\end{array}
\]

we must have

\[
\begin{align*}
\nonumber f_0(\iota_6) &= \iota_3 \otimes \iota_3 \\
\nonumber f_1(\iota_{13}) &= (1 + T)(\iota_3 \otimes \text{Sq}^3 \iota_7 + \text{Sq}^1 \iota_3 \otimes (\text{Sq}^2 \iota_7 + \iota_9) + \text{Sq}^2 \iota_3 \otimes \text{Sq}^1 \iota_7 + \text{Sq}^3 \iota_3 \otimes \iota_7) \\
f_2(\iota_{18}) &= \iota_9 \otimes \iota_9,
\end{align*}
\]

implying the result. Note that the important term here was the \( \iota_9 \), which occurred because of the difference between \( \text{Sq}^6 \) and \( \text{Sq}^2 \text{Sq}^4 \).
Now we show that the class which we call \( v_1^2 v_2^3 \) times the class which we call \( v_1^1 v_2^2 \) equals the class that we call \( v_1^{i+2} v_2^{i+2} \). This, with the result of the preceding paragraph, implies that all powers of \( v_1 v_2 \) are as claimed.

The class which we call \( 2v_1^i v_2^i \) is dual to a generator \( \gamma_{i+1} \in (C_{2i+2})_{10i+14} \) and to a generator \( \delta_{i+1} \in (D_{2i+1})_{10i+13} \). In the resolutions, \( d(\alpha_{i+1}) \equiv Sq^5 \gamma_i \) and \( d(\beta_{i+1}) \equiv Sq^5 \delta_i \) mod other terms, where \( \alpha_{i+1} \) and \( \beta_{i+1} \) are dual to \( v_1^{i+1} v_2^{i+1} \) as above. Because the product of our \( 2v_2^2 \) class and our \( v_1^1 v_2^1 \) class equals our \( 2v_1^i v_2^i \) class, as was shown earlier, we conclude that in \( D \xrightarrow{\psi} C \otimes C \), \( \psi(\delta_{i+1}) = \gamma_1 \otimes \alpha_i \) plus other terms. Thus, modulo other terms, we have

\[
d(\psi(\beta_{i+2})) = \psi(d(\beta_{i+2})) \equiv Sq^5 \gamma_1 \otimes \alpha_i
\]

and

\[
d(\alpha_2 \otimes \alpha_i) \equiv Sq^5 \gamma_1 \otimes \alpha_i,
\]

from which we conclude \( \psi(\beta_{i+2}) = \alpha_2 \otimes \alpha_i \), which is equivalent to our claim.

Now that we know that the classes which we have named by monomials in \( v_1 \) and \( v_2 \) multiply consistently with these names, we can deduce the final relation \( v_1 v_2 x = \eta v_2^4 \) from \( v_1^4 x = \eta v_1^3 v_2^3 \) by multiplying the latter by \( v_1 v_2 \) and then dividing by \( v_1^4 \). \( \square \)

### 3. An 8-cell model related to TMF(3)

In [8 §7], another connective model for TMF(3) is discussed, which is \( Z \wedge \tmf \), where \( Z \) is a certain 8-cell complex. Although \( Z \wedge \tmf \) is not a ring spectrum, it is still true that \( v_2^{-1} Z \wedge \tmf \simeq \text{TMF}(3) \). The importance of this model is primarily that the dimensions of the cells of \( Z \) allow one to construct a map \( Z \to \text{TMF}(3) \) thanks to certain homotopy groups of \( \text{TMF}(3) \) being 0. The other models are then related to \( \text{TMF}(3) \) via the \( Z \)-model. In this section, we provide some additional details to the sketch given in [8].

Let \( X_7 = S^0 \cup e \cup e^6 \cup e \cup e^7 \) be as in the proof of Theorem 2.1.1a, and let \( X_{421} = \Sigma^7 DX_7 = S^0 \cup e \cup e^3 \cup \eta \cup e^7 \).

**Lemma 3.1.** The map \( S^6 \xrightarrow{\nu^2} S^0 \hookrightarrow X_{421} \) extends to a map \( \Sigma^6 X_7 \to X_{421} \).

The proof in [8], using the ASS of \( X_{421} \) through dimension 13, is perfectly clear.
**Definition 3.2.** Let $Z$ denote the mapping cone of the map $\Sigma^{23}X_7 \to \Sigma^{17}X_{421}$ obtained from Lemma 3.1.

**Proposition 3.3.** There is an element $x \in \pi_{17}^{\ast}(\text{TMF}(3))$ of order 2 which is not divisible by $\eta$, and a map $Z \to \text{TMF}(3)$ which extends this map $x$.

**Proof.** This is where we need input from the theory of topological modular forms. In [8], a 48-periodic ring spectrum TMF(3) (called there TMF($\Gamma_0(3)$)) is defined and its homotopy groups calculated, using a spectral sequence defined using results about elliptic curves. Their result ([8, 4.1]), localized at 2, is a $v_2^8$-inverted version of our Theorem 2.12, but with their ring structure being precise, not just up to elements of higher filtration. We emphasize that our Theorem 2.12 and [8, 4.1] are totally independent calculations. Our 2.12 uses only homotopy theory (and the existence of a ring spectrum tmf with $H^\ast(\text{tmf}) \approx A//A_2$), while [8, 4.1] uses the Weierstrass curve.

A schematic of $\pi_{\ast}(\text{TMF}(3))$ from the ASS viewpoint is given in Diagram 3.4. Each collection of four closely-spaced towers represents infinitely many such towers in the same stem. If the lowest of these begins in filtration $s$, then there are such towers in filtration $s + 2i$ for all $i \geq 0$, with a slight exception in dimension 24. The names of the bottom generators are $1$, $2v_1^2$, $v_1v_2$, $2v_2^2$, $v_1^2v_2^2$, $2v_1v_3^2$, $2v_1^2v_4^2$, $v_1v_5^2$, $2v_6^2$, $v_1^2v_6^2$, and $2v_7v_2^2$. The name of the generator in filtration $s + 2i$ is $v_1^{3i}v_2^{-i}$ times that of the bottom generator, except that in dimension 24, we have $2v_2^4$ and $v_1^{3i}v_4^{-i}$ for all $i > 0$. The eight $\mathbb{Z}_2$’s along the bottom, indicated by a solid dot, occur only once, in the indicated filtration. Because of period 48, this is the whole picture.
Diagram 3.4. Schematic of $\pi_*(\text{TMF}(3))$

The extension of $x$ over $\mathbb{Z}$ occurs because $2x = 0$ and $\pi_i(\text{TMF}(3)) = 0$ for $i = 19$, 23, 27, 29, and 30, showing that the obstructions to extending over the remaining cells are all 0.

The spectrum $\mathbb{Z} \wedge \text{tmf}$ will be one of our connective models of $\text{TMF}(3)$. The following result gives its homotopy groups, which are closely related to those of $X \wedge \text{tmf}$.

**Theorem 3.5.** There is an isomorphism of graded abelian groups

$$\pi_*(\mathbb{Z} \wedge \text{tmf}) \approx \tilde{K} \oplus \mathbb{Z}_2[v_2^8](x, \eta x, \nu x, x^2, \nu x^2, v_1 v_2 x^2, v_2^8 \nu, v_2^8 \nu^2),$$

where

$$\tilde{K} = \ker(\tilde{R} \to \mathbb{Z}_2[v_2^8](v_2^4))$$

with $\tilde{R}$ the subgroup of the ring $R$ of $2.12$ spanned by all elements divisible by $v_2^3$.

In dimension $\leq 51$, $\pi_*(\mathbb{Z} \wedge \text{tmf})$ may be seen in Diagram 2.10 by removing the first two $\mathbb{Z}_2$’s, and the $b_0$’s starting in the 5-stem, and the $b_{sp}$’s starting in the 9-stem, and increasing stems of all elements by 3. Thus the first element would be the $\mathbb{Z}_2$ class $x$, which appears in 2.10 in position (14, 2), and is in the 17-stem for $\mathbb{Z}$. For the ASS-type chart that we will describe in our proof, filtrations should be decreased by 2, so that $x$ appears in filtration 0.
Proof. Let $M_7 = H^*(X_7)$ be the $A$-module (or $A_2$-module) whose only nonzero groups are $\mathbb{Z}_2$ in dimensions 0, 4, 6, and 7 with $Sq^7 \neq 0$, and let $M_{421}$ be the $A$-module or $A_2$-module whose only nonzero groups are $\mathbb{Z}_2$ in dimensions 0, 1, 3, and 7 with $Sq^1 Sq^2 Sq^4 \neq 0$. There is an exact sequence
\[
\cdots \to \text{Ext}^{s-2,t-1}_{A_2}(\Sigma^{24} M_7) \to \text{Ext}^{s,t}_{A_2}(\Sigma^{17} M_{421}) \to E_2^{s,t}(Z \wedge \text{tmf}) \to \text{Ext}^{s-1,t-1}_{A_2}(\Sigma^{24} M_7) \to \cdots
\]
with $d(\iota_{24}) = h^2_{24} t_{17}$. Here $E_2(Z \wedge \text{tmf})$ is the $E_2$-term of a spectral sequence converging to $\pi_*(Z \wedge \text{tmf})$. We could compute $E_2(Z \wedge \text{tmf})$ by first computing $\text{Ext}_{A_2}(M_7)$ and $\text{Ext}_{A_2}(M_{421})$ (and these have been computed in [4] and [6]), but we prefer the following method which relates it directly to $E_2(X \wedge \text{tmf})$.

Let $P = \ker(d_1)$ in the resolution in the proof of Theorem 2.9. One easily verifies that there is an exact sequence of $A_2$-modules
\[
0 \to \Sigma^{24} M_7 \overset{i}{\to} \Sigma^{11} A_2/(Sq^1, Sq^5) \overset{d_2}{\to} P \overset{q}{\to} \Sigma^{16} M_{421} \to 0
\]
with $d_2(\iota_{11}) = Sq^7 I_4$, $q(Sq^6+7.5 I_4 + Sq^4.6 I_6) = \text{gen}_{16}$, and $i(\iota_{24}) = Sq^6.7+4.6^3 \iota_{11}$.

Note that $\text{Ext}_{A_2}(P)$ consists of a shifted version of Diagram 2.10 minus the first two $\mathbb{Z}_2$’s and the first $bo_*$’s. It is shifted so that the (now) initial tower, which did begin in (9, 2), now begins in (11, 0). Note also that
\[
\text{Ext}_{A_2}(P) \overset{d_2}{\to} \text{Ext}_{A_2}(\Sigma^{11} A_2/(Sq^1, Sq^5))
\]
is surjective, because of the $bsp_*$ in 2.10 beginning in (9, 2).

Let $K = \text{im}(d_2) = \ker(q)$. There is a commutative diagram of exact sequences, with $\text{Ext} = \text{Ext}_{A_2}$ and all $\text{Ext}$ groups having the same second superscript $t$. 
in which $d_2^*$ is surjective. By a diagram chase, this implies exactness of

$$\rightarrow \Ext^{s-2}(\Sigma^{24}M_7) \xrightarrow{\delta} \Ext^{s}(\Sigma^{16}M_{421}) \rightarrow \Ext^{s}(P) \xrightarrow{j} \Ext^{s}(K) \rightarrow \Ext^{s-1}(\Sigma^{24}M_7) \rightarrow .$$

(3.8)

This $\delta$ must send $\iota_{24}$ to $h_{24}^{2}$ since $\Ext^{2,24}(P) = 0$. Thus it must agree totally with $d$ of (3.6), and so the exact sequences (3.6) and (3.8) are identical. Therefore, $E_2(Z \wedge \text{tmf}) \approx \ker(d_2^*)$, and this is the chart obtained from 2.10 extended indefinitely, by removing the first two dots, the initial $bo_*$, and the $bsp_*$ starting in $(9,2)$, and regrading so that the $\mathbb{Z}_2$ in $(14,2)$ in 2.10 is now in $(17,0)$.

**Corollary 3.9.** There is a map $Z \wedge \text{tmf} \rightarrow X \wedge \text{tmf}$ such that the induced map $v_2^{-1}Z \wedge \text{tmf} \rightarrow v_2^{-1}X \wedge \text{tmf}$ is an equivalence.

**Proof.** There is a map $Z \rightarrow X \wedge \text{tmf}$ extending $x$ for the same reason as in the proof of Proposition 3.3, namely 0 obstructions. Smashing with tmf and following by the multiplication of tmf yields the desired map. The proof of 3.5 identified $\pi_*(Z \wedge \text{tmf})$ with the kernel of (3.7), which is contained in $\pi_*(X \wedge \text{tmf})$. Thus $\pi_*(Z \wedge \text{tmf})$ injects into all of $\pi_*(X \wedge \text{tmf})$ except $\nu$, $\nu^2$, and the multiples of $v_i^2v_j^i$ for $i \leq 2$. These latter classes are, for $i = 0$ the $bo_*$ which is $\coker(\pi_*(\Sigma^{-1}C) \rightarrow \pi_*(X \wedge \text{tmf}))$, for $i = 1$ the initial $bo_*$ in 2.10, and for $i = 2$ the $bsp_*$ which appears in 2.10 to begin in $(9,2)$. Since $v_8^8$ times these classes are in the image from $\pi_*(Z \wedge \text{tmf})$, we deduce the claim that it is an equivalence after $v_2^8$ is inverted. □
Theorem 3.10. The map $Z \to \text{TMF}(3)$ of Proposition 3.3 induces an equivalence $v_2^{-1}Z \wedge \text{tmf} \to \text{TMF}(3)$.

Proof. We need a fact from topological modular forms that there is a map $\text{tmf} \wedge \text{TMF}(3) \to \text{TMF}(3)$ making $\text{TMF}(3)$ a tmf-module. Using this and the map $Z \to \text{TMF}(3)$, we obtain a map $Z \wedge \text{tmf} \to \text{TMF}(3)$.

We will show it sends $\pi_*(Z \wedge \text{tmf})$ to elements of $\pi_*(\text{TMF}(3))$ with the same names (as those of Theorem 3.5). In the proof of Proposition 3.3, we discussed how [8, 4.1] can be interpreted to give $\pi_*(\text{TMF}(3))$ as a $v_2$-inverted version of our Theorem 2.12. Then the same argument as was used in the proof of 3.9 gives the asserted equivalence.

The class $x$ maps across by construction. We must deduce from this, by various types of naturality, that all other classes map across. Our map is one of tmf$_*$-modules. The relation $v_1^ix = \eta v_1^3v_2^3$ is present in both $\pi_*(Z \wedge \text{tmf})$ and $\pi_*(\text{TMF}(3))$ (by Theorem 2.12 and [8, 4.1], resp.), and hence $\eta v_1^3v_2^3$ maps across, and then so also does $v_1^3v_2^3$. Since $16v_2^2$ is in tmf$_*$, we deduce that all $v_1^3v_2^3$ with $i \equiv 3 \mod 4$ and $j$ odd map across.

By the Toda bracket formula $2v_1^5v_2^3 = \langle \eta v_1^3v_2^3, \eta, 2 \rangle$, which is valid in both $Z \wedge \text{tmf}$ and TMF(3), we now have that all $v_1^3v_2^3$ with $i$ and $j$ odd map across.

In [8, 4.1], it is noted that $\pi_{20}(S^0) \to \pi_{20}(\text{TMF}(3))$ sends $\kappa$ to $\nu x$. One can show, for example using Yoneda products, that $\kappa$ acting on $x \in \pi_{17}(Z \wedge \text{tmf})$ yields the class that we call $\nu x^2$. Thus $\nu x^2$ maps across, and hence so does $x^2$. There is a bracket formula $2v_2^6 = \langle x^2, \eta, 2 \rangle$ in both spectra, and so $v_2^6$ maps across. Arguing as before, we deduce that all $v_1^iv_2^j$ with $i$ and $j$ even map across. Knowing that $v_2^6$ maps across implies the same for $\nu v_2^8$ and $\nu^2v_2^8$. We have now accounted for all of $\pi_*(Z \wedge \text{tmf})$. ☐

The following corollary is immediate from 3.9 and 3.10.

Corollary 3.11. There is an equivalence $v_2^{-1}X \wedge \text{tmf} \to \text{TMF}(3)$.

Thus both $X \wedge \text{tmf}$ and $Z \wedge \text{tmf}$ can serve as connective models of TMF(3). We prefer $X \wedge \text{tmf}$ because it is a ring spectrum and gives a better approximation to $\pi_*(\text{TMF}(3))$ prior to inverting $v_2$, but $Z \wedge \text{tmf}$ was useful because it was so easy to get a map from it into TMF(3).
4. A model related to $\text{tmf} \wedge \text{tmf}$

In this section we study a third model of $\text{tmf}(3)$ introduced in [8]. This one is closely related to $\text{tmf} \wedge \text{tmf}$, and we provide a proof that a plausible splitting of $\text{tmf} \wedge \text{tmf}$ does not occur. We clarify some aspects of the construction in [8] and compute the homotopy groups.

Let $A^* = \mathbb{Z}_2[\zeta_1, \zeta_2, \ldots]$ denote the dual of the mod 2 Steenrod algebra. Here $\zeta_i = \chi(\xi_i)$, the conjugates of the usual generators. Assign a weight $\text{wt}$ on $A^*$ by $\text{wt}(\zeta_i) = 2^{i-1}$ and $\text{wt}(ab) = \text{wt}(a) + \text{wt}(b)$. It is well-known and easily verified that

$$(A//A_2)^* = \mathbb{Z}_2[\zeta_8, \zeta_4^2, \zeta_8^3, \zeta_4, \zeta_5, \ldots]$$

and there is a splitting as $A_2$-modules

$$(A//A_2)^* \approx \bigoplus_{n \geq 0} M_n,$$

where $M_n$ is spanned by all monomials in $(A//A_2)^*$ of weight $8n$. The $A$-action is given by $\zeta_i(\chi \text{Sq}) = \zeta_i + \zeta_{i-1}^2$. Note that $H_*(\text{tmf}) \approx (A//A_2)^*$.

Similarly $H_*(\text{bo}) = (A//A_1)^*$ is isomorphic to a polynomial algebra on $\zeta_1^4$, $\zeta_2^2$, and $\zeta_i$ for $i \geq 3$. There are bo-Brown-Gitler spectra $\text{bo}_n$ satisfying that $H_*(\text{bo}_n)$ is the span of all monomials in $H_*(\text{bo})$ with weight $\leq 4n$. For $n \geq 0$ one easily verifies that there is an isomorphism of $A_2$-modules

$$\bigoplus \phi_n : \bigoplus_{n \geq 0} H_*(\Sigma^8n\text{bo}_n) \to H_*(\text{tmf})$$

defined by $\phi_n(\sigma^{8n_1}\zeta_1^{i_1}\zeta_2^{i_2} \cdots) = \zeta_1^{8n_i} - \sum 2^{i_j} \zeta_1^j \zeta_2^{i_2} \cdots$. The image of $\phi_n$ is $M_n$, the span of monomials of weight $8n$. One might ask if this isomorphism is induced by an equivalence of the spectra $\text{tmf} \wedge \text{tmf}$ and $\bigvee \Sigma^8n\text{bo}_n \wedge \text{tmf}$. An analogous equivalence $\text{bo} \wedge \text{bo} \simeq \bigvee \Sigma^8n\overline{\text{B}}_n \wedge \text{bo}$ was proved in [7]. In that case $\overline{\text{B}}_n$ was an integral Brown-Gitler spectrum.

We answer this question and prepare for a new TMF(3) model by proving the following result.
Theorem 4.1. The spectra $\text{tmf} \wedge \text{tmf}$ and $\bigvee_{n \geq 0} \Sigma^{8n} \text{bo}_n \wedge \text{tmf}$ are not homotopy equivalent. Indeed, in the ASS converging to $\pi_*(\text{tmf} \wedge \text{tmf})$, which has

$$E_2 \approx \bigoplus_{n \geq 0} \text{Ext}_{A_2}(H^*(\Sigma^{8n} \text{bo}_n)),$$

there is a class $g \in \text{Ext}_{A_2}^{0,24}(H^*(\Sigma^{16} \text{bo}_2))$ and an element $w \in \text{Ext}_{A_2}^{3,26}(H^*(\Sigma^{8} \text{bo}_1))$ such that $d_3(g) = w$.

Proof. Let $\overline{\text{tmf}}$ denote the cofiber of $S^0 \to \text{tmf}$. Since $\text{tmf}$ is a ring spectrum, there is a splitting

$$\text{tmf} \wedge \text{tmf} \simeq (S^0 \wedge \text{tmf}) \vee (\overline{\text{tmf}} \wedge \text{tmf}).$$

We will use the cofibration

$$\overline{\text{tmf}} \wedge S^0 \to \overline{\text{tmf}} \wedge \text{tmf} \to \overline{\text{tmf}} \wedge \overline{\text{tmf}} \quad (4.2)$$

and a differential in the ASS of $\overline{\text{tmf}}$ to deduce the claimed differential in the ASS of $\overline{\text{tmf}} \wedge \text{tmf}$.

In Diagram 4.3, we depict $\text{Ext}_{A_2}^{s,t}(H^*(\Sigma^{8} \text{bo}_1 \vee \Sigma^{16} \text{bo}_2))$ for $s < 8$, $t-s < 40$. Elements suggested by solid dots come from the first summand, and those with open circles (or connected to open circles by lines) come from the second summand.

Diagram 4.3. $\text{Ext}_{A_2}^{s,t}(H^*(\Sigma^{8} \text{bo}_1 \vee \Sigma^{16} \text{bo}_2))$ in a range

The cofibration which defines $\overline{\text{tmf}}$ induces an exact sequence

$$\to \text{Ext}_{A}^{s,t}(H^*(\text{tmf})) \to \text{Ext}_{A}^{s,t}(H^*(\overline{\text{tmf}})) \to \text{Ext}_{A}^{s+1,t}(H^*(S^0)) \to \text{Ext}_{A}^{s+1,t}(H^*(\text{tmf})) \to .$$
There is a lower vanishing line in $\text{Ext}_A(H^*(\text{tmf})) \approx \text{Ext}_A(\mathbb{Z}_2)$ (e.g. [4, 2.6]) which implies that $\text{Ext}_A^{s,t}(H^*(\text{tmf})) \approx \text{Ext}_A^{s+1,t}(H^*(S^0))$ if $s \leq 6$ and $t - s \geq 31$. In [1], it was shown that in the ASS of $S^0$ there are nonzero elements $e_1 \in \text{Ext}_A^{4,42}(H^*(S^0))$ and $h_1t \in \text{Ext}_A^{7,44}(H^*(S^0))$ satisfying $d_3(e_1) = h_1t$. These elements are in the range of our asserted isomorphism, and so there must be corresponding elements $\overline{e_1} \in \text{Ext}_A^{3,42}(H^*(\text{tmf}))$ and $\overline{h_1t} \in \text{Ext}_A^{6,44}(H^*(\text{tmf}))$ related by a $d_3$-differential.

Now we consider the exact sequences in $\text{Ext}_A(\mathcal{F})$ and $\pi_*(\mathcal{F})$ induced by (4.2). Using Bruner’s software, we see that $\text{Ext}_A^{s,t}(H^*(\text{tmf} \wedge \text{tmf})) = 0$ if $t - s = 39$ and $s > 3$. Thus neither of the elements $\overline{e_1}$ or $\overline{h_1t}$ can be in the image from $\text{Ext}_A(H^*(\text{tmf} \wedge \text{tmf}))$, the second since there is nothing to hit it, and the first since a class which hits it would have to support a differential, but there is nothing for it to hit. Thus the elements $\overline{e_1}$ and $\overline{h_1t}$ related by the $d_3$ in the ASS of $\text{tmf}$ map nontrivially to $\text{Ext}_A(H^*(\text{tmf} \wedge \text{tmf}))$. One easily checks that $\text{Ext}_A(H^*(\Sigma^{24}bo_3) \oplus H^*(\Sigma^{32}bo_4))$ is 0 in these bigradings. Thus the elements $\overline{e_1}$ and $\overline{h_1t}$ must map nontrivially to classes in $\text{Ext}_A(H^*(\Sigma^8bo_1) \oplus H^*(\Sigma^{16}bo_2))$ involved in a $d_3$-differential. These must be the two classes at the extreme right end of Diagram 4.3, one in filtration 6 from $\Sigma^8bo_1$ and the other in filtration 3 from $\Sigma^{16}bo_2$.

This already implies the first conclusion of the theorem, that $\text{tmf} \wedge \text{tmf}$ does not split as $\bigvee_{n \geq 0} \Sigma^{8n}bo_1 \wedge \text{tmf}$. We would like to infer from this differential the claimed nontrivial $d_3$ on the class $g$ in position $(24,0)$. Clearly the $h_2$-action and the nonzero $d_3$ from (39,3) implies that $d_3$ is nonzero on the class in $(33,1)$. Let $X_7 = S^0 \cup_{e^4} e^6 \cup_{e^7} e^7$ as before. If $d_3(g) = 0$, then the homotopy class $g$ would extend to a map $\Sigma^{24}X_7 \rightarrow \text{tmf} \wedge \text{tmf}$, since Diagram 4.3 shows that there are no obstructions to the extension. Smashing with tmf and following by the multiplication of tmf would yield a map $\Sigma^{24}X_7 \wedge \text{tmf} \rightarrow \text{tmf} \wedge \text{tmf}$ extending $g$. Since $X_7 = bo_1$, the ASS of $\Sigma^8X_7 \wedge \text{tmf}$ is just the black elements in Diagram 4.3. The 16-suspension of the element in $(17,1)$ in that diagram does not support a differential in $\Sigma^{24}X_7 \wedge \text{tmf}$ but would map to the class in $(33,1)$ in $\text{tmf} \wedge \text{tmf}$ which we showed does support a differential. This contradicts the assumption that $d_3(g) = 0$.

\[\square\]
Now we begin working toward the construction of our third connective model of TMF(3).

**Proposition 4.4.** There is a subcomplex $W_1$ of $\text{tmf}$ such that there is a cofibration

$$\Sigma^8bo_1 \to W_1 \to \Sigma^{16}bo_2$$

which has a short exact sequence in mod-2 cohomology.

**Proof.** We use the description of $H_*(\text{tmf})$ given in the second paragraph of this section. All elements of weight $\leq 16$ are in dimension $\leq 31$, and the first few elements of weight greater than 16 are $\zeta_1^{24}$, $\zeta_1^{16}\zeta_2^4$, $\zeta_1^{16}\zeta_2^2$, and $\zeta_1^{16}\zeta_4$. The $A$-module structure of $H^*(\text{tmf}(31)/\text{tmf}(23))$ is

$$\langle Sq^0, Sq^2, Sq^3, Sq^4, Sq^5, Sq^6, Sq^7 \rangle \zeta_8^2 \oplus \langle Sq^0, Sq^4, Sq^6, Sq^7 \rangle \zeta_2^{24}, \ (4.5)$$

with the first (resp. second) summand dual to monomials of weight 16 (resp. 24). Here the $\widehat{\ (\ )}$ represents duality. Bruner’s software shows that there is a map

$$\text{tmf}(31)/\text{tmf}(23) \to \Sigma^{24}X_7$$

which induces the identity homomorphism from the second summand of (4.5) and 0 from the first. This is done by computing $\text{Ext}_A$ of the tensor product of the dual of the module in (4.5) with $M_7$, and seeing that there are no possible differentials from the obvious filtration-0 class. The desired complex $W_1$ is the fiber of the composite

$$\text{tmf}(31) \to \text{tmf}(31)/\text{tmf}(23) \to \Sigma^{24}X_7,$$

where the second map is the one just noted. □

The $E_2$-term of the ASS for $W_1 \wedge \text{tmf}$ in dimension less than 40 is given in Diagram 4.3 and, as established in Theorem 4.1 there are $d_3$-differentials on the classes in positions $(24,0)$, $(33,1)$, $(36,2)$, and $(39,3)$. Let $f: S^{32} \to W_1 \wedge \text{tmf}$ be a nontrivial map of Adams filtration 1, which exists by Diagram 4.3. Smash with $\text{tmf}$ and follow by the multiplication of $\text{tmf}$, obtaining a map $S^{32} \wedge \text{tmf} \to W_1 \wedge \text{tmf}$.

**Definition 4.6.** Define $W$ to be the cofiber of this map $S^{32} \wedge \text{tmf} \to W_1 \wedge \text{tmf}$.

This $W$ will be our third connective model of TMF(3). Note that, unlike the first two, it is not obtained as the smash product of a finite complex with $\text{tmf}$, since the above map $f$ does not factor through $W_1$ itself.
Similarly, let $S^{16} \rightarrow bo_2 \wedge tmf$ correspond to essentially the same class, as the open circles in Diagram 4.3 depict the ASS of $\Sigma^{16}bo_2$. Extend this to a map $S^{16} \wedge tmf \rightarrow bo_2 \wedge tmf$, and let $\tilde{bo}_2$ denote the cofiber of this. There is a cofiber sequence

$$\Sigma^8bo_1 \wedge tmf \rightarrow W \rightarrow \Sigma^{16}\tilde{bo}_2.$$  \hfill (4.7)

The short exact sequence of $A$-modules

$$0 \rightarrow \Sigma^{17}A//A_2 \rightarrow H^*(\tilde{bo}_2) \rightarrow A \otimes_{A_2} H^*(bo_2) \rightarrow 0$$

induces an exact sequence in $\text{Ext}_A$ which implies that $\text{Ext}_A(H^*(\tilde{bo}_2))$ begins as the 16-desuspension of the open circles in Diagram 4.3 with the portion connected to the element in $(32, 1)$ removed. It contains no unpictured elements in filtration 0 or 1. Therefore, $H^*(\tilde{bo}_2) = A \otimes_{A_2} B$, where $B$ sits in a short exact sequence of $A_2$-modules

$$0 \rightarrow \Sigma^{17}\mathbb{Z}_2 \rightarrow B \rightarrow H^*(bo_2) \rightarrow 0,$$  \hfill (4.8)

with the new class in $B$ equal to $Sq^4Sq^6Sq^7\iota_0$, or equivalently $Sq^4Sq^2Sq^3\iota_8$. It also equals $Sq^2$ of the top class of $H^*(bo_2)$. The $A_2$-module $B$ cannot be given the structure of $A$-module, as the Adem relation $Sq^2Sq^{15} = Sq^1Sq^{16}+Sq^{16}Sq^1$ would be violated.

Our next result gives a direct relationship among $\text{Ext}_{A_2}(A_2/(Sq^4, Sq^5))$, which was depicted through degree 48 in Diagram 2.10 and is very closely related to the homotopy groups described in Theorem 2.12 and $\text{Ext}_{A_2}(B)$ and $\text{Ext}_{A_2}(H^*(X_7))$, which two together are related to the ASS of $W$. After stating and proving this result, we will use it to determine $\pi_*(W)$ and see that $v_2^{-1}W$ is another model for $\text{TMF}(3)$.

We begin by noting that $\text{Ext}_{A_2}(A_2/(Sq^4, Sq^5)) \approx \text{Ext}_{A_2}(A_2/(Sq^4, Sq^5))$.

**Theorem 4.9.** Let $\widetilde{\text{Ext}}_{A_2}(A_2/(Sq^4, Sq^5))$ denote $\text{Ext}_{A_2}(A_2/(Sq^4, Sq^5))$ without the $\mathbb{Z}_2$ in $\text{Ext}^{0,4}$ or the tower beginning in $\text{Ext}^{1,11}$. There is an exact sequence

$$\text{Ext}^{s+2,t}_{A_2}(\Sigma^6M_7) \rightarrow \widetilde{\text{Ext}}^{s+2,t}_{A_2}(A_2/(Sq^4, Sq^5)) \rightarrow \text{Ext}^{s,t}_{A_2}(\Sigma^4B) \rightarrow \text{Ext}^{s,t+3}_{A_2}(\Sigma^6M_7) \rightarrow .$$

**Proof.** One can verify that there is an exact sequence of $A_2$-modules

$$0 \rightarrow K \rightarrow i: \Sigma^4A_2 \rightarrow A_2(Sq^4, Sq^5) \rightarrow \phi: \Sigma^6M_7 \rightarrow 0,$$

where $\Sigma^6M_7$ is generated by $\phi(Sq^{5})$, and $i(K)$ is the submodule of $\Sigma^4A_2$ generated by $Sq^7\iota_{4}$, and that there is a short exact sequence of $A_2$-modules

$$0 \rightarrow \Sigma^{16}B \rightarrow \Sigma^{11}A_2//A_0 \rightarrow K \rightarrow 0.$$
with $B$ as above, and the $A_2$-generators of $\Sigma^{16}B$ mapping to $Sq^5 \iota_{11}$ and $Sq^{4,3} \iota_{11}$.

Let $R = \text{coker}(i) = \ker(\phi)$. Except for the classes omitted in forming $\widetilde{\text{Ext}}$, we have isomorphisms

$$\text{Ext}_{A_2}^s(\Sigma^{16}B) \xrightarrow{\cong} \text{Ext}_{A_2}^{s+1}(K) \xrightarrow{\cong} \text{Ext}_{A_2}^{s+2}(R)$$

and an exact sequence

$$\to \text{Ext}_{A_2}^{s+2}(\Sigma^6M_7) \to \text{Ext}_{A_2}^{s+2}(A_2(Sq^4, Sq^5, 1)) \to \text{Ext}_{A_2}^{s+2}(R) \to \text{Ext}_{A_2}^{s+3}(\Sigma^6M_7),$$

from which the result follows.

Similarly to Theorem 3.5, we can now deduce the following result without using complete information about $\text{Ext}_{A_2}(B)$.

**Theorem 4.10.** There is an isomorphism of graded abelian groups

$$\pi_*(W) \approx K' \oplus \mathbb{Z}_2[v_2^8](x, \eta x, \nu x, x^2, \nu x^2, v_1v_2x^2, v_2^8 \nu, v_2^8 \nu^2),$$

where

$$K' = \ker(R' \to \mathbb{Z}_2[v_2^8](v_2^4))$$

with $R'$ the subgroup of the ring $R$ of $2.12$ spanned by all elements divisible by $v_2$ but not including the cyclic group generated by $2v_2^2$.

**Proof.** The map $\Sigma^{15} \widetilde{bo_2} \to \Sigma^{8}bo_1 \wedge \text{tmf}$ whose cofiber is $W$ has Adams filtration 3 since $H^i(\Sigma^{15} \widetilde{bo_2}) = 0$ for $i < 15$ and for $i = 17, 18, \text{ and } 20$, the values of $i$ for which $\pi_*(\Sigma^{8}bo_1 \wedge \text{tmf})$ has nonzero classes in filtration less than 3. We obtain a homomorphism

$$\text{Ext}_{A_2}^{s,t}(\Sigma^{15}B) \to \text{Ext}_{A_2}^{s+3,t+3}(\Sigma^8M_7).$$

We claim that this is the same homomorphism as the one at the end of the exact sequence in Theorem 4.9.

Both of them are nontrivial on the class in $\text{Ext}_{A_2}^{0,24}(\Sigma^{16}B))$, the first by Theorem 4.1 and the second since Diagram 2.10 is 0 in position $(21,3)$. Let $C$ (resp. $D$) be a minimal $A_2$-resolution of $\Sigma^8M_7$ (resp. $\Sigma^{15}B$). There is a morphism $C_3 \xrightarrow{\phi} \Sigma^{15}B$ which lifts to a morphism $C_3 \to D_0$ and then to $C_{s+3} \to D_s$ for all $s$. Since $B_5 = 0$, $\phi$ must be 0 on the generators in 8, 12, and 20, and it must send the generator in 23 nontrivially to get the correct Ext morphism. This completely determines the entire
Ext morphism. The same is true of the Ext morphism at the end of the sequence of 
4.9. Thus the two Ext morphisms are equal.

We obtain that $E^{s,t}_{2}(W) \approx \widetilde{\text{Ext}}_{A_{2}}(A_{2}(\text{Sq}^{4}, \text{Sq}^{5,1}))$. We have already seen that
there are no possible differentials in an ASS with $E_{2} \approx \widetilde{\text{Ext}}_{A_{2}}(A_{2}(\text{Sq}^{4}, \text{Sq}^{5,1}))$. Thus
$\pi_{*}(W)$ is like the groups described in Theorem 2.12 without the initial $bo_{*}, \nu, \nu^{2}$, or
the $2v_{2}^{2}$-tower.

Similarly to Corollary 3.11, we obtain the following result, giving a third connective
model for TMF(3). The significance of this one is its close relationship to tmf.

**Corollary 4.11.** There is an equivalence $v_{2}^{-1}W \to \text{TMF}(3)$.

**Proof.** Similarly to 3.9, we construct a map $Z \to W$, then use the tmf-module struc-
ture of $W$ to extend to a map $Z \wedge \text{tmf} \to W$. This becomes an equivalence after
inverting $v_{2}$. Then we use Theorem 3.10. \qed

5. **tmf(3)-homology of real projective space**

In this section, we compute $\pi_{*}(X \wedge \text{tmf} \wedge P_{1})$, where $X$ is as in Theorem 2.1 and
$P_{1} = RP^\infty$. Because $X \wedge \text{tmf}$ is probably the best connective model for TMF(3), this
could be considered as tmf(3)$_{*}(P_{1})$. More work will be required to deduce results for
$P_{n}$ or $P_{1}^{m}$ from this, but this should provide a model. One possible application of this
calculation would be to obstruction theory, which was an initial motivation for this
project.

It is convenient to state and prove the result for $\Sigma P_{1}$. Some of the tmf$_{*}$-module
structure is included in the result. We now state the main theorem of this section.
Although it is not exactly an ASS, we describe the groups in an ASS-like way, with
bigrading $(i, s)$ for an element of $\pi_{i}(X \wedge \text{tmf} \wedge \Sigma P_{1})$ of filtration $s$. Many elements
are expressed as $a^{e_{1}}v_{2}^{e_{2}}$ of bigrading $(2e_{1} + 6e_{2}, e_{2})$. Thus $a$ (resp. $v_{2}$) is thought of as
having bigrading $(2, 0)$ (resp. $(6, 1)$), although $a$ and $v_{2}$ themselves are not actually
elements of $\pi_{*}(X \wedge \text{tmf} \wedge \Sigma P_{1})$. Certain powers of $v_{2}$ can be thought of as being part
of the tmf$_{*}$-module structure. Note that the elements $a^{e_{1}}v_{2}^{e_{2}}$ are not really products,
since $X \wedge \text{tmf} \wedge \Sigma P_{1}$ is not a ring spectrum. The element $a$ roughly corresponds to
$v_{1}/2$. 

Theorem 5.1. For each pair \((e_1, e_2)\) such that \(e_1 > 0\), \(e_2 \geq 0\), and \(e_1 \equiv e_2 \pmod{2}\), \(\pi_*(X \wedge \text{tmf} \wedge \Sigma P_1)\) has a summand \(\mathbb{Z}/2^{e_1}\) generated by

\[
\begin{cases}
a^{e_1}v_2^{e_2} & \text{if } e_1 \equiv e_2 \pmod{4} \\
2a^{e_1}v_2^{e_2} & \text{if } e_1 \equiv e_2 + 2 \pmod{4},
\end{cases}
\quad (5.2)
\]

with the following two variations:

- if \(e_1 = 2\) and \(e_2 \equiv 0 \pmod{8}\), it is \(\mathbb{Z}/8\) generated by \(a^2v_2^{e_2}\);
- if \(e_1 = 1\) and \(e_2 \equiv 1\) or \(3 \pmod{8}\), it is \(\mathbb{Z}/4\) generated by \(av_2^{e_2}\).

If \(e_1 \geq 5\) and \(e_1 \equiv e_2 \pmod{4}\), or if \((e_1, (e_2 \mod 8)) = (4, 0)\) or \((3, 3)\), then \(\eta a^{e_1}v_2^{e_2} \neq 0\). If \((e_1, (e_2 \mod 8)) = (1, 1), (4, 4), (2, 6),\) or \((3, 7)\), then \(\eta a^{e_1}v_2^{e_2} \neq 0\).

If \(e_1 \geq 3\) and \(e_1 \equiv e_2 + 2 \pmod{4}\), or \(e_1 = 2\) and \(e_2 \equiv 0 \pmod{8}\), then there exists \(b_{e_1, e_2}\) of bigrading \((e_1 + e_2 - 2, 2e_1 + 6e_2 - 2)\) and order 2 satisfying \(\eta^2 b_{e_1, e_2} = 2^{e_1}a^{e_1}v_2^{e_2}\). If \((e_1, (e_2 \mod 8)) = (1, 3)\) or \((2, 4)\), there exists \(b'_{e_1, e_2}\) of bigrading \((e_1 + e_2 - 1, 2e_1 + 6e_2 - 1)\) and order 2 satisfying \(\eta b'_{e_1, e_2} = 2^{e_1}a^{e_1}v_2^{e_2}\).

In addition, there are the following \(\mathbb{Z}_2\) classes \(x_{i, s}\) of bigrading \((i, s)\):\footnote{Note that the subscripts of \(x\) refer to bigrading, while the subscripts of \(b\) and \(b'\) do not.}

- \(x_{8i+2, 1}\) for \(i \geq 1\).
  All the rest are acted on freely by \(v_2^8\).
- \(x_{5, 1} = \nu b_{2, 0}, x_{7, 1} = \nu a^2\);
- \(x_{6, 1}\) satisfying \(\nu x_{6, 1} = \eta a v_2\);
- \(x_{21, 3}\) and \(\nu x_{21, 3}\);
- \(x_{22, 4} = \nu b'_{1, 3}, x_{23, 4} = \nu a v_2^3\);
- \(x_{36, 6}\) and \(\nu x_{36, 6}, x_{37, 6}\) and \(\nu x_{37, 6}\);
- \(x_{38, 6}\) satisfying \(\nu x_{38, 6} = \eta a^2 v_2^6\).

In Diagrams 5.3 and 5.4 we depict the groups of Theorem 5.1. All elements except those in position \((8i + 2, 1)\) for \(i \geq 1\) in Diagram 5.3 are acted on freely by \(v_2^8\).
Diagram 5.3. $\pi_*(X \wedge \text{tmf} \wedge \Sigma P_1)$ in $* < 32$
Diagram 5.4. \( \pi_*(X \wedge \text{tmf} \wedge \Sigma P_1) \), \( 32 \leq * < 48 \)

The remainder of this section is devoted to the proof of Theorem 5.1. By Theorem 2.1, there is an exact sequence

\[
\begin{align*}
&bo_*(P_1) \xrightarrow{g_*} \pi_*(C \wedge P_1) \xrightarrow{\delta_*} \pi_*(X \wedge \text{tmf} \wedge \Sigma P_1) \xrightarrow{\tilde{f}_*} bo_*(\Sigma P_1).
\end{align*}
\]
As is well-known, $b_0(P_1)$ can be computed from $\text{Ext}_{A_1}(H^*(P_1))$, and from $\text{2.1\,(d)}$, $\pi_*(C \wedge P_1)$ can be computed from
\[
\text{Ext}_{A_2}(\Sigma^4A_2/(Sq^4,Sq^{5,1}) \otimes H^*(P_1)).
\] (5.6)

We can use Bruner’s software to compute (5.6) through a large range of dimensions, enough to see patterns. In order to prove that these patterns continue, $v_8^2$-periodicity, which follows from the resolution in the proof of $\text{2.9}$, is very helpful, but we still need to prove what happens in filtration less than 8 and dimension greater than 48. Most of our analysis will go into computing (5.6), but we begin by analyzing (5.5).

It is convenient to use (5.5) to form a chart for $\pi_*(X \wedge \text{tmf} \wedge \Sigma P_1)$ from $\phi \text{Ext}_{A_2}(\Sigma^4A_2/(Sq^4,Sq^{5,1}) \otimes H^*(P_1)) \oplus \text{Ext}_{A_1}(H^*(\Sigma P_1))$.

Recall that $\phi$ increases filtration by 1. The behavior for $10 \leq i \leq 18$ is typical, and is depicted in Diagram 5.7 in which black dots are from $\text{Ext}_{A_1}(H^*(\Sigma P_1))$ and $\circ$’s are from $\phi \text{Ext}_{A_2}(\Sigma^4A_2/(Sq^4,Sq^{5,1}) \otimes H^*(P_1))$.

**Diagram 5.7.** Forming $\pi_*(X \wedge \text{tmf} \wedge \Sigma P_1)$, $10 \leq * < 18$

The content in this chart is the $d_1$-differential from $(12,0)$ and the $\eta$-extension from $(16,0)$. These are generalized and proved in Theorem 5.8

**Theorem 5.8.** In (5.5),

- $b_{0s_i+3}(P_1) \xrightarrow{g} \pi_{s_i+3}(C \wedge P_1)$ is nontrivial.
- There is an element $\gamma_{s_i} \in \pi_{s_i}(X \wedge \text{tmf} \wedge \Sigma P_1)$ such that $\tilde{f}_s(\gamma_{s_i})$ has Adams filtration 0, and $\eta \gamma_{s_i} = \delta_*(y_{s_i}) \neq 0$ with $y_{s_i}$ of Adams filtration 0 in $\pi_{s_i+1}(C \wedge P_1)$. 


**Proof.** We will see in 5.14 that Ext$^0_{A_{2}}(H^*(C\wedge P_1)) \approx \mathbb{Z}_2$ with nonzero class $\iota_4 \otimes x_{8i-1}$.

The morphism $g^*\mathord{\upharpoonright}_{\Sigma^4A}$ is induced by

$$
\Sigma^4A/(\text{Sq}^4, \text{Sq}^{5.1}) \otimes H^*P_1 \rightarrow A/\Lambda_1 \otimes H^*P_1 \approx A \otimes \Lambda_1 H^*P_1
$$

$$
\iota_4 \otimes x_{8i-1} \mapsto \text{Sq}^4 \otimes x_{8i-1} \leftrightarrow \text{Sq}^4(1 \otimes x_{8i-1}) + 1 \otimes x_{8i+3},
$$

which proves the first statement. The $\eta$-extension follows similarly from

$$
\iota_4 \otimes x_{8i-3} \mapsto \text{Sq}^4 \otimes x_{8i-3} \leftrightarrow \text{Sq}^4(1 \otimes x_{8i-3}) + \text{Sq}^2(1 \otimes x_{8i-1}).
$$

To know that the class $\gamma_{8i}$ is nonzero in $\pi_{8i}(-)$, we use 5.14 to see that, unless $i \equiv 5 \mod 8$, the only possible target of a differential from $\gamma_{8i}$ is ruled out by $h_2$-naturality.

If $i \equiv 5 \mod 8$, the differential, if nonzero in the ASS of $\Sigma P_1$, would have to also be nonzero in the ASS of the cofiber $R$ of the Kahn-Priddy map $\lambda: P_1 \rightarrow S^0$, but it is ruled out there by $h_2$-naturality. \hfill \Box

Let $L = A_2/(\text{Sq}^4, \text{Sq}^{5.1})$. A good way to obtain Ext$^*_{A_2}(L \otimes H^*(P_1))$ begins by computing Ext$^*_{A_2}(L \otimes Q)$, where $Q$ is the $A_2$-module which has as its only nonzero classes $x_i$ for $i \geq 1$ and $i \in \{-9, -5, -3, -2, -1\}$ with $\text{Sq}^j x_i = \binom{i}{j} x_{i+j}$. Then $Q$ is an extension of copies of $\Sigma^{8i-1}A_2/\Lambda_1$ for $i \geq -1$. See [4, p.299]. Thus there is a spectral sequence converging to Ext$^*_{A_2}(L \otimes Q)$ with

$$
E_2^{s,*} = \bigoplus_{i \geq -1} \text{Ext}^{s,*}_{\Lambda_1}(\Sigma^{8i-1}L).
$$

One easily computes Ext$^*_{\Lambda_1}(L)$ to be as in Diagram 5.10 from which it is immediate that the spectral sequence collapses and

$$
\text{Ext}^*_{A_2}(L \otimes Q) \cong \bigoplus_{i \geq -1} \text{Ext}^*_{\Lambda_1}(\Sigma^{8i-1}L). \tag{5.9}
$$

We obtain that, in grading $8i + 5$, Ext$^*_{A_2}(L \otimes Q)$ has a $bo_*$ beginning in filtration $s$ for all nonnegative $s \leq 4i + 1$ except $s = 4i$. This will explain the low-filtration form of Diagrams 5.3 and 5.4.
Diagram 5.10. $\text{Ext}_{A_1}(L)$

There is a short exact sequence of $A_2$-modules

$$0 \to H^* P_1 \to Q \to \Sigma^{-9} M_7 \oplus \Sigma^{-1} \mathbb{Z}_2 \to 0,$$

and also after tensoring with $L$. Thus there is an exact sequence

$$\text{Ext}^s_{A_2}(L \otimes \Sigma^{-9} M_7) \oplus \text{Ext}^s_{A_2}(\Sigma^{-1} L) \to \text{Ext}^s_{A_2}(L \otimes Q) \quad (5.11)$$

$$\to \text{Ext}^s_{A_2}(L \otimes H^* P_1) \to \text{Ext}^{s+1}_{A_2}(L \otimes \Sigma^{-9} M_7) \oplus \text{Ext}^{s+1}_{A_2}(\Sigma^{-1} L).$$

In Theorem 2.9 and Diagram 2.10, we computed and displayed $\text{Ext}_{A_2}(L)$. A nice computation of $\text{Ext}_{A_2}(L \otimes M_7)$ can be obtained by tensoring the exact sequence at the beginning of the proof of 2.9 with $M_7$. This yields a spectral sequence computing $\text{Ext}_{A_2}(L \otimes M_7)$ from things such as $\text{Ext}_{A_2}(M_7 \otimes A_2)$, which is just four $\mathbb{Z}_2$’s, and $\text{Ext}_{A_2}(M_7 \otimes A_2/[A_1]) \approx \text{Ext}_{A_1}(M_7)$, which is $b_{0*} \oplus \Sigma^{4} bsp_{*}$. The resulting spectral sequence has only a very few possible differentials, which are most easily settled using Brumer’s software, although they can be settled without it. Both $\text{Ext}_{A_2}(L)$ and $\text{Ext}_{A_2}(L \otimes M_7)$ have lower vanishing lines. From these and the exact sequence, we obtain that

$$\text{Ext}^{s,t}_{A_2}(L \otimes Q) \to \text{Ext}^{s,t}_{A_2}(L \otimes H^* P_1)$$

is an isomorphism if $s \leq 8$ and $t - s \geq 53$.

Thus a Brumer calculation of $\text{Ext}^{s,t}_{A_2}(L \otimes H^* P_1)$ for $t - s \leq 53$, which is easily done and is consistent with Theorem 5.14 together with the complete description of $\text{Ext}_{A_2}(L \otimes Q)$ in (5.9) and (5.10) and $v_2^8$-periodicity, gives a complete determination of the groups $\text{Ext}^{s,t}_{A_2}(L \otimes H^* P_1)$. Note that the Brumer software is not absolutely
necessary for this. First of all, it is just a finite calculation, and secondly there are rather simple patterns for the boundary homomorphism in (5.11), which could be determined directly.

There is one more thing required in order to determine the chart for \( \text{Ext}_{A_2}^{s,t}(L \otimes H^*P_1) \), and the resulting \( \pi_s(C \wedge P_1) \). In dimensions greater than 53 and congruent to 0 mod 4, we know from the determination of \( \text{Ext}_{A_2}^{s,t}(L \otimes Q) \) that in filtration \( \leq 8 \) \( \text{Ext}_{A_2}^{s,t}(L \otimes H^*P_1) \) has \( h_0 \)-towers beginning in each filtration (\( >0 \) in dimension 0 mod 8), and we know from the Bruner calculation and periodicity that in high filtration it has towers which end in every second filtration coming down from a certain maximum filtration. But how do we know the way these match up? We must show that, as suggested in Diagrams 5.3 and 5.4, the lowest bottoms match up with the highest tops.

One way to do this is to use the spectral sequence which builds \( \text{Ext}_{A_2}^{s,t}(L \otimes H^*P_1) \) from

\[
\bigoplus_{s \geq 0} \phi^s \text{Ext}_{A_2}^{s,s}((\Sigma^{-s}C_s \otimes H^*P_1)),
\]

(5.12)

where \( C_s \) are the \( A_2 \)-modules in the resolution of \( L \) at the beginning of the proof of 2.9. The \( s \)-summand provides a bunch of \( \mathbb{Z}_2 \)'s at height \( s \) in the resulting chart (coming from \( \phi^s \text{Ext}^0(-) \)) together with the portion of Diagram 5.13 consisting of towers beginning at height \( s \). Note that there are no such towers when \( s = 0 \).
Diagram 5.13. *Portion of spectral sequence building* $\text{Ext}_{A_2}(L \otimes H^*P_1)$

The desired form for the bottoms of the towers, as obtained from the complete description of $\text{Ext}_{A_2}(L \otimes Q)$ in (5.9) and (5.10) differs slightly from this, in that in dimensions congruent to 4 mod 8 most of the towers should begin one filtration lower. This can only be accounted for by an extension from a $\mathbb{Z}_2$ from the next smaller $s$-value.

For example, in dimension 28, Diagram 5.13 shows towers beginning at height 1, 2, 3, and 4, coming from summands $s = 1, 2, 3,$ and 4 in (5.12) with tops at height 12, 10, 8, and 6, respectively. These correspond to $\pi_{32}(C \wedge P_1)$, which, according to Theorem 5.14, corresponds to $\pi_{32}(X \wedge \text{tmf} \wedge \Sigma P_1)$ in Diagram 5.4 with its largest tower removed and filtrations decreased by 1; hence, towers beginning at height 0, 1, 2, and 3 ending at height 12, 10, 8, and 6. Then, for example, the tower in $\pi_{32}(C \wedge P_1)$ (corresponding to $\text{Ext}^{s,s+28}_{A_2}(L \otimes H^*P_1)$ going from filtration 0 to 12 can only come, in the spectral sequence of (5.12), from the $s = 1$ tower with an extension from a $\mathbb{Z}_2$ from $s = 0$.

The main thing that was obtained from using $Q$ which was not easily obtained from (5.12) is the $\eta^2$-hooks on the bottom of towers. In (5.12) these come about from the filtration-0 $\mathbb{Z}_2$’s in the various $s$-summands in a complicated way, but they are clear
in Diagram 5.10. The above remarks imply the following result, the computation of (5.6), since there are no possible differentials in the ASS.

**Theorem 5.14.** The ASS converging to \( \pi_*(C \wedge P_1) \) has \( E_2^{s,t} = \text{Ext}^{s,t}_{A_2}(\Sigma^4 L \otimes H^*(P_1)) \) and collapses. The description of \( \pi_*(C \wedge P_1) \) can be obtained from that of \( \pi_*(X \wedge \text{tmf} \wedge \Sigma P_1) \) in Theorem 5.1 by making the following changes:

- Remove summands in (5.2) for which \( e_2 = 0 \), (but do not remove \( \eta a^{e_1} \) and \( \eta^2 a^{e_1} \) when \( e_1 \equiv 0 \mod 4 \));
- Remove \( b_{e_1,0} \) and \( \eta b_{e_1,0} \) with \( e_1 \equiv 2 \mod 4 \);
- Add elements \( c_{8i+3,0} \) of order 2 for \( i \geq 1 \);
- Decrease filtrations by 1.

The proof of Theorem 5.1 is now immediate from the exact sequence (5.5), Theorem 5.14, and Theorem 5.8, which describes the only possible differentials and extensions in (5.5).

**Remark 5.15.** The way that we have chosen to describe these things is reversed from the way they are derived. We first compute the groups in 5.14 and then use them to determine the groups in 5.11. However, we are mostly interested in 5.1, and so we felt that it should be stated up front. It seemed like overkill to state the whole thing again for \( \pi_*(C \wedge P_1) \), since it is so similar.

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