Barrow Holographic Dark Energy Model - A new Perspective

Nandhida Krishnan.P* and Titus K Mathew†

Department of Physics, Cochin University of Science and Technology,
Kochi, Kerala 682022, India,
* nandhidakrishnan@cusat.ac.in
†titus@cusat.ac.in

Abstract

Recently Barrow holographic dark energy has been proposed to describe the late acceleration of the universe, originated from the entropy correction due to the quantum gravitational effect on the geometry of the black hole surface. In contrast to the existing models in the literature, here we considered the Barrow holographic dark energy as a dynamical vacuum, with Granda-Oliveros length as IR cut-off. We have analytically solved for the Hubble parameter and studied the evolution of cosmological parameters with the best estimated model parameters extracted using the combined dataset of supernovae type Ia pantheon (SN Ia) and observational Hubble data (OHD). We have found that, in the absence of interaction between the dark sectors, the model evolves like ΛCDM model with effective mass density parameters. The analysis on thermodynamics of the present model, confirms the validity of generalised second law of thermodynamics through out the evolution and it attain a maximum entropy state, the end de Sitter epoch. Additionally, We performed dynamical system analysis and it reveals that the end de Sitter phase is a stable one.

Keywords: Barrow entropy, holographic dark energy model, Granda-Oliveros cut-off, Generalized second law of thermodynamics.

1 Introduction

Contemplation of the data from type 1a supernovae (SN Ia) for different redshift leads to one of the most pivotal discovery in the twentieth century that the current universe is not just expanding but follows an accelerating expansion [1, 2, 3, 4, 5, 6, 7, 8]. This implies that the universe is dominated by an exotic component with negative pressure termed as dark energy, which is responsible for the current accelerated expansion. The best-fitting model for the present scenario of the accelerating universe is the ΛCDM model, referred to as the standard model of cosmology, explaining successfully the current cosmological observational data. This model incorporates cosmological constant, Λ as the dark energy component. Even though the standard model satisfying extremely well with the current observational data [9], two major flaws encountered must be addressed, the cosmological constant problem and the coincidence problem. The first one points out the existing inconsistency in the magnitude of the observed value of the cosmological constant and the theoretically predicted value. The latter seeking the reason for the coincidence of the present value of dark energy density with that of matter, since both have evolved with different expansion history. To burring these problems, one can either consider the dark energy density as varying or modify the gravitational theories. The second approach yields modified gravity theories, such as f(R). Lovelock, etc. The first one, equally competing, rely on the construction of various dynamical dark energy models. Among these, holographic dark energy models are become recently popular. However previously proposed holographic dark energy models was strongly disfavoured with observational data [10]. Later acceptable holographic dark energy models were proposed and its theoretical studies were done, also the parameters are constraints with the observational data [11]. In the case of highly gravitating systems like black hole, the entropy is proportional to the area of the horizon rather the volume enclosed [12, 13, 14]. That is the black hole entropy scales non extensively, [15, 16, 17, 18]. However, entropy defined in an effective quantum field theory, behaves extensively as $S \sim L^3 \Lambda^3$ where $L$ is the length with UV cut-off $\Lambda$. In order resolve the breakdown of
quantum field theory to describe a black hole, a longer distance IR and short distance UV cut-off relation was proposed by Cohen et al. [19, 20, 21] as,

$$L^3 \Lambda^4 \leq L M_p^2.$$  

(1)

Near the saturation the entropy thus satisfies [19],

$$S_{\text{max}} \approx S_{BH}^{3/4}$$  

(2)

Where $$M_p$$, plank mass, $$S_{BH}$$ is the black hole entropy with radius $$L$$ as IR cut-off and UV cut-off $$\Lambda$$. The holographic principle for the universe upholds the idea that the energy density that resides inside the horizon is described in terms of reduced plank mass $$M_p$$ and characteristic length scale $$L$$. Dark energy density is constructed through dimensional analysis as [22],

$$\rho_\Lambda = 3 c^2 M_p^2 L^{-2}$$  

(3)

c^2 is the model parameter and $$L$$ is the cosmological length scale. It permits various approaches to choose the length scale in which the most straightforward choice is Hubble horizon $$L = 1/H$$. Even though this assumption yields the present value of energy density [22], it failed badly in explaining the current accelerated expansion, and also disfavoured the value of the equation of state (EoS) of the dark energy, $$\omega_{DE} \leq -1/3$$ instead predicts a value around, $$\omega_{DE} = 0$$ [23]. Alternatively, particle horizon and event horizon becomes the reasonable choices for the length scale. On consideration of particle horizon, the model results in the equation of state parameter $$\omega_{DE} > -1/3$$, thus fails to explain accelerated expansion [24]. However, with the future event horizon as the length scale, the resulting model predicts the recent acceleration of the universe, but it faces the causality problems [11], that dark energy at present seems to depend on the future evolution of the scale factor, which in turn violate causality. Attention has then turn to define the holographic dark energy with curvature scalar as the IR cut-off and later a hybrid cut-off scale known as the Granda-Oliveros scale has been used to holographic dark energy models. After these models, holographic dark energy density becomes one of the well-satisfying selections for dark energy densities. It was found that holographic dark energy explains the background cosmological evolutions and also conceded with the observational data [25, 26, 11, 27, 28, 29, 30, 31, 32]. In reference [33, 34], it was shown that The holographic dark energy models with future event horizon as IR cut-off posses big rip singularity in the late evolution of the universe. On considering phenomenologically the possible interaction of the dark energy with matter, interacting holographic dark energy models are being constructed to study the background evolution and the thermodynamic behaviour [35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. The cosmological implications of holographic dark energy are extended to different gravity theories as Gauss-bonnet gravity [47] also applying different entropy forms implies numerous models, Renyi holographic dark energy [48], Tsallis holographic dark energy [49].

Barrow holographic dark energy model is an innovative one among the current approaches in explaining the recent acceleration of the universe. By considering quantum gravitational effects, Barrow have expressed that, the black hole surface has got a fractal structure. This could have implications on the form of horizon entropy. Following this perspective, Barrow have proposed that the black hole entropy would satisfy a more general relation [50],

$$S_h = \left( \frac{A}{A_0} \right)^{1+\Delta/2}$$  

(4)

Where $$A$$ and $$A_0$$ are the area of the black hole horizon and plank area respectively. The exponent $$\Delta$$ quantifies the amount of quantum-gravitational deformation effects. In literature the Barrow exponent, takes the range $$0 \leq \Delta \leq 1$$. The upper limit of $$\Delta$$ is corresponding to the most intricate and fractal structure of the black hole horizon, while $$\Delta = 0$$ corresponds to the the simplest horizon structure, which reduces the entropy into the standard Bekenstein-Hawking entropy [51, 58, 17]. The above deformation in entropy is differs from the logarithmic corrections to the black hole entropy [52, 53]. Even though the above expression bears some resemblance to Tsallis’s non-extensive entropy, but the underlying principles are completely different [54, 55]. In reference [56], the author formulates a form of dark energy called, Barrow holographic dark energy by considering the holographic principle in the cosmological scenario with Barrow entropy for the horizon. The
determining the nature of dark energy. For $\Delta$ cut-off, and non-relativistic matter. They have shown that, the Barrow exponent $\Delta$ has a significant role in the thermal history of flat universe having Barrow holographic dark energy with future event horizon as IR different length scale as IR cut-off, in either interacting or non-interacting scenario. In [56], authors analysed dark energy density allows one to construct various models of Barrow holographic dark energy by taking the effect of barrow exponent $\Delta$ on the EoS parameter. For successfully explains the thermal history of the universe with the matter and dark energy eras [67].

In [60], the author extracted the modified Friedmann equations for a (3+1) dimensional FRW universe with Barrow holographic dark energy, and equations reduces to the conventional Friedmann equation on neglecting the Barrow exponent. The author commented on the significance of the Barrow exponent in determining the nature of the dark energy. It has been argued that, although at intermediate times the dark energy equation of state shows departure form the ΛCDM behaviour, but at asymptotically large times, the model predicts a de Sitter epoch independent of the Barrow exponent. Sheykhi also extracted a modified Friedmann equation with Barrow entropy [61]. Additionally, the author explored generalized second law, and have shown that the generalized second law of thermodynamics is valid for a universe with fractal boundary. The interacting BHDE model with $H$ as IR cut-off has been studied in [62]. These authors also indicated the possibility of conditional violation of GSL. Authors in [63] studied the Barrow holographic dark energy for a non-flat FLRW universe with an apparent horizon as IR cut-off and have shown that the dark energy have a quintessence behaviour. However for flat universe, the dark energy exhibits quintessence and Chaplygin gas behaviour [64]. In reference [65], the authors carried out the dynamical analysis and the state-finder diagnostic on interacting Barrow holographic dark energy with different IR cut-offs. Recently, the Thermodynamics of the BHDE model were studied with a specific cut-off, called Nijori-Odintsov cut-off, which is special combination of event horizon scale [66]. In this paper authors incorporated the bulk viscosity in BHDE sector and studied the evolution of the cosmological parameters and have shown the model satisfies the generalised second law. Very recently, the BHDE framework has been studied in non-flat universe. It successfully explains the thermal history of the universe with the matter and dark energy eras [67].

On detailed investigation, the author examined the effect of barrow exponent $\Delta$ on the EoS parameter. For $\Delta = 0$, EoS parameter lies entirely in the quintessence region, while for $\Delta > 0.03$ they claimed that model crosses phantom divide. So far, all the works in the present literature were done by considering the dark energy equation of state as varying with the expansion of the Universe. In the present work, we propose an Interacting BHDE model, in which the Barrow holographic dark energy treated as running vacuum energy. Since UV cut-off used in defining the BHDE is in fact related to the vacuum, it is more logical to consider the BHDE as a decaying vacuum energy with EoS, $-1$. We organize the paper as follows. The first section introduces an interacting BHDE model with GO IR cut-off, and the corresponding Hubble parameter of the model is evaluated. The second section presents the extraction of model parameters using $\chi^2$ square statistics with corresponding error-bars. Third section consists of the evolution of the various cosmological parameters. In the section, the fourth, the present model is distinguished from the standard ΛCDM model by analysing the evolution of the model in $r$-$s$ plane. In section five, We extended our work to study the thermodynamics of the model and analysed the status of the GSL in this model. Later in section six, the phase space analysis is performed to confirm the future stability of the model. We conclude our work in
the last section. Throughout this paper, $G$ is the gravitational constant, $M_p$ is plank mass, $L_p$ is the plank length. The dot over a parameter shows the derivative with cosmic time $t$. Subscript zero over any parameter indicates the present value of that quantity.

2 Interacting Barrow Holographic dark energy

Consider a spatially flat, homogenous, and isotropic universe with BHDE and pressure-less dark matter as cosmic components. The dark energy density defined by equation (3). We use the Granda-Oliveros IR cut-off for the holographic dark energy model, which appeared as a function of both the square and time derivative of the Hubble parameter as $[68]$

$$L^{-2} = (\alpha_1 H^2 + \beta_1 \dot{H})$$

with $\alpha_1$ and $\beta_1$ are dimensionless parameters, $H$ is the Hubble parameter ($H = \dot{a}/a$) and $\dot{H}$ is the time derivative of Hubble parameter. Then using equation(5) the dark energy density becomes, (taken the natural units $\hbar = c = 1, 8\pi G = 1$)

$$\rho_\Lambda = 3(\alpha H^2 + \beta \dot{H})^{\frac{\Delta}{2\alpha}}$$

Where $\alpha$ and $\beta$ are constant parameters with dimension $[L]^2\Delta$. The evolution equation of spatially flat($k = 0$) Friedmann universe with dark matter density $\rho_m$ and dark energy density $\rho_\Lambda$,

$$3H^2 = \rho_m + \rho_\Lambda$$

We consider that, the matter and dark energy sectors are interacting with each other and this interaction is characterised through an interacting term $Q$, which characterized by the amount of energy density transfer with two possibilities, $Q < 0$, energy transfer is from dark matter to BHDE or energy is being transferred from BHDE to dark matter for a non-negative value of $Q$ $[69]$. For such an interacting scenario, the conservation equation becomes,

$$\dot{\rho}_\Lambda + 3H(\rho_\Lambda + p_\Lambda) = -Q$$
$$\dot{\rho}_m + 3H(\rho_m + p_m) = Q$$

Where $Q$ takes many phenomenological forms, either linear or non-linear function of Hubble parameter and either matter density or dark energy density, or combination of both $\rho_m$ and $\rho_\Lambda$ $[43, 70, 71, 72, 25, 73, 74]$. For the proposed model we accounting an interaction among the dark sectors as a linear combination of Hubble parameter and matter density takes the form, $Q = 3bH\rho_m$ $[75]$ with $b$ as the coupling constant. Since the interaction is considerably small and in order to satisfying Le Chatelier’s brun principle in cosmological frame discussed in $[76]$ constraint the coupling constant as $b << 1$. For FLRW universe comprises of an interacting BHDE with equation of state $\omega_{DE} = -1$ ie, $p_\Lambda = -\rho_\Lambda$ and pressure-less matter, $p_m = 0$, the conservation equation takes the differential form as,

$$\frac{d\Omega_\Lambda}{dx} = -3b\Omega_m$$
$$\frac{d\Omega_m}{dx} = -3(1 - b)\Omega_m.$$ (10)

Here the differentiation involve with a new variable $x$ instead of cosmic time $t$, where $x = \ln a$, and $a$ is the scale factor. Solution of the above equation is $\Omega_m = \Omega_{m0}e^{-3(1-b)x}$, with $\Omega_{m0} = \frac{\rho_m}{3H^2}$ the present matter density parameter. By combining the equations (8) and (10) gives a second order differential equation, for the evolution of the Hubble parameter,

$$\frac{d^2 H^2}{dx^2} + 3\frac{dH^2}{dx} + 9b\Omega_{m0}e^{-3(1-b)x} = 0$$

Solving this we get the evolution of the Hubble parameter as,

$$H^2 = \frac{\Omega_{m0}}{1 - b}e^{-3(1-b)x} - \frac{\Gamma_1}{3} e^{-3x} + \Gamma_2$$

(12)
\( \Gamma_1 \) and \( \Gamma_2 \) are constants depends on the model. To evaluate these constant we use the conditions,

\[
\begin{align*}
\dot{H}^2 \big|_{x=0} &= 1, \\
\frac{d\dot{H}^2}{dx} \bigg|_{x=0} &= \frac{2}{\beta} \left( (H_0^2 \Omega_0) \frac{\dot{x}^2}{2} - \alpha \right)
\end{align*}
\]  

(13)

where the second relation is obtained from the equation of dark energy density and \( \Omega_0 = 1 - \Omega_{m0} \). Applying these conditions to the solution (12), we get the constants as,

\[
\Gamma_1 = 3\Omega_{m0} + \frac{2}{\beta} \left[ (1 - \Omega_{m0}) H_0^2 \frac{\dot{x}^2}{2} - \alpha \right]
\]

(14)

\[
\Gamma_2 = 1 - \frac{b\Omega_{m0}}{1 - b} + \frac{2}{3\beta} \left[ (1 - \Omega_{m0}) H_0^2 \frac{\dot{x}^2}{2} - \alpha \right]
\]

(15)

In the limit \( x \to -\infty \), \( \dot{H}^2 \) in equation (12) approaches to \( \dot{H}^2 \to \left( \frac{\Omega_{m0}}{1 - b} a^3 - \frac{\Gamma_1}{3} \right) a^{-3} \) shows a decelerated expansion (Since \( b << 1 \), the term inside the bracket can be taken as almost constant). For the future limit as \( x \to +\infty \), \( \dot{H}^2 \to \Gamma_2 \) analogues to a future de Sitter phase. So the model predicts a transition to a late accelerated epoch during the expansion history of the universe. For non-interacting, i.e. with \( b = 0 \), equation (12) became,

\[
\dot{H}^2 = \Omega_{m0} e^{-3x} + \Omega_{\Lambda 0}
\]

(16)

With effective mass parameter [78] for non-relativistic matter \( \Omega_{m0} \) and dark energy \( \Omega_{\Lambda 0} \) takes the form,

\[
\bar{\Omega}_{m0} = \frac{2\alpha}{3\beta} \left[ 1 - \frac{(1 - \Omega_{m0}) H_0^2 \frac{\dot{x}^2}{2} }{\alpha} \right], \quad \bar{\Omega}_{\Lambda 0} = 1 - \bar{\Omega}_{m0}
\]

(17)

Hence in the non-interacting case, the present model reduces to a \( \Lambda \)CDM like model with effective dark matter density parameter \( \bar{\Omega}_{m0} \) and corresponding dark energy density parameter. The reason for this behaviour is that, when \( b = 0 \) the conservations laws in equation (9) reduces to separate conservations of matter and dark energy, as a result the dark energy density will effectively becomes a constant.

### 3 Extraction of Model Parameters

We adopted the chi-square statistics method to obtain the best fit model parameters [79]. For any observable physical quantity \( \xi \) there is an observational value \( \xi_{\text{obs}} \), and a corresponding theoretically predicted value, \( \xi_{\text{theo}}(P) \), which depends on the model parameter say \( P \). In the present model the parameters are \( P = \alpha, \beta, b, \Delta, H_0 \) and \( \Omega_{m0} \). Then the \( \chi^2 \) function defined by [22],

\[
\chi^2_{\xi}(P) = \sum_i \frac{\left[ \xi_{\text{theo}}(P) - \xi_{\text{obs}} \right]^2}{\sigma^2_{\xi}}
\]

(18)

where \( \sigma_{\xi} \) is the standard deviation in the observation of the physical property. The most probable values of the model parameters can be extracted by statistically minimising the \( \chi^2 \) function. We have used the observational Hubble dataset containing 36 data points (OHD36) in the redshift range \( 0.07 \leq z \leq 2.36 \) [80, 81] and observational data from Type Ia Supernovae (SN Ia) Pantheon dataset with 1048 apparent magnitude versus redshift measurements in the redshift range \( 0.01 \leq z \leq 2.3 \) [82]. The Hubble dataset is a complete collection of currently available, reliable H(z) data. In these 36 H(z) measurements, 31 data are determined using the cosmic chronometric technique, three measurements are from the radial BAO signal in the galaxy distribution, and the last two are measured from the BAO signal in the Lyman forest distribution alone or cross correlated with QSOs [83]. The expression for \( \chi^2_{\text{OHD36}} \) for Hubble dataset is,

\[
\chi^2_{\text{OHD36}}(\alpha, \beta, b, \Delta, H_0, \Omega_{m0}) = \sum_{i=1}^{36} \left[ \frac{H(\alpha, \beta, b, \Delta, H_0, \Omega_{m0}, z_i) - H_i}{\sigma_i} \right]^2
\]

(19)
Table 1: Observational constraints on model parameters using SN Ia+OHD36 dataset

|   | $b$   | $H_0$       | $\Omega m_0$ | $\alpha$   | $\beta$   | $\Delta$ |
|---|-------|-------------|---------------|-------------|-----------|-----------|
|   | 0.025±0.022 | 69.460±0.270 | 0.292±0.008   | 1.065±0.004 | 0.214±0.010 | 0.075±0.001 |

where $H(\alpha, \beta, b, \Delta, H_0, \Omega_{m0}, z(i))$ and $H_i$ are the theoretical and observed Hubble parameter respectively and $\sigma_i$ is the corresponding standard deviation. The $\chi^2$ function corresponding to the Supernovae data can be expressed as,

$$
\chi^2_{SN Ia}(\alpha, \beta, b, \Delta, H_0, \Omega_{m0}) = \sum_{i=1}^{1048} \frac{(m(\alpha, \beta, b, \Delta, H_0, \Omega_{m0}, z_i) - m_i)^2}{\sigma_i^2}
$$

(20)

where $m(\alpha, \beta, b, \Delta, H_0, \Omega_{m0}, z_i)$ is the theoretical value of the apparent magnitude of the Supernovae at a given redshift and $m_i$ is the corresponding observed magnitude. The $\sigma_i$ in the denominator of the above equation is now the standard deviation in the observation of the supernovae. The theoretical magnitude is evaluated by determining the distance modulus,

$$
dL(\alpha, \beta, b, \Delta, H_0, \Omega_{m0}, z_i) = c(1 + z_i) \int_0^{z_i} \frac{dz}{H(\alpha, \beta, b, \Delta, H_0, \Omega_{m0}, z_i)}.
$$

(21)

Using this the magnitude can be determined using the standard relation,

$$
m(\alpha, \beta, b, \Delta, H_0, \Omega_{m0}, z_i) = M + 5\log_{10} \left( \frac{dL(\alpha, \beta, b, \Delta, H_0, \Omega_{m0}, z_i)}{Mpc} \right) + 25,
$$

(22)

where $M$ is absolute magnitude of the supernovae. The most probable values of the model free parameters is then estimated corresponds to the minimal $\chi^2_{total}$ value of the combined dataset of Supernovae type Ia and observational Hubble data (SN Ia + OHD36) as, $\chi^2_{total} = \chi^2_{SN Ia} + \chi^2_{OHD36}$ with $\chi^2_{min} = 1053.73$ and $\chi^2_{d.o.f} = 98$. The absolute magnitude $M$ is extracted as $-19.37$. The 1$\sigma$(68.3%), 2$\sigma$(95.4%) and 3$\sigma$(99.73%) two dimensional confidence contours for the parameters, $\alpha, \beta, b, \Delta, H_0, \Omega_{m0}$ of IBHDE model is as given in Figures 2, 1, 3, 4, 5. The best estimated values of various model parameters for 1$\sigma$ error bar are given in Table 1. The Present value of Hubble parameter is extracted as 69.46±0.27 which shows a slight variation from the observational value (Plank 2018 observation 67.4±0.5 km s$^{-1}$ Mpc$^{-1}$ [83], and WMAP sky survey value 71.9±2.6 km s$^{-1}$ Mpc$^{-1}$ [31].
Using the best fit model parameters, we compared the predicted distance moduli of various supernovae with the corresponding observational data shown in Figure 6. We have also compared the predicted Hubble parameter values with the observed one in Figure 7. Both figures show good agreement between the prediction and observation.
4 Behaviour of Cosmological parameters

The equation 12 shows the evolution of the Hubble parameter with respect to the cosmic time. The asymptotic behaviour of the Hubble parameter reveals that the evolution ultimately leads to a de Sitter epoch. Figure 8 shows the evolution of Hubble parameter with scale factor, $a$ where we used the best estimated values of the model parameters.

\[ \Omega_m = \Omega_{m0} e^{-3(1-b)x}, \quad \Omega_\Lambda = \frac{1}{H_0^2} \left( \alpha H^2 + \frac{\beta dH^2}{dx} \right)^{2-\Delta} \] (23)

The behavior of these density parameters, corresponding to dark matter and dark energy, are given in Figures 9 and 10. The co-evolution of these mass density parameters are evident in logarithmic scale as in Figure 11. It clearly shows the dominance of matter over the dark energy density prior to the transition and also the later dominance of dark energy. This adequately solves the coincidence problem.
Figure 9: Matter density parameter vs scale factor with present value, $\Omega_{m0} = 0.292$.  

Figure 10: Dark energy density parameter vs scale factor with present value, $\Omega_{\Lambda0} = 0.707$.  

Figure 11: Evolution of density parameter corresponds to non-relativistic matter and interacting barrow holographic dark energy in logarithmic scale. Red line corresponding to IBHDE density and blue for matter density.

The deceleration parameter, $q$ which explains nature of the expansion of the universe, that is whether it is decelerating or accelerating. If $q < 0$ universe is in the accelerated phase, and if $q > 0$ it could be in the decelerated phase. Deceleration parameter $q$ defined as,

$$q = -1 - \frac{\dot{H}}{H^2}$$  

(24)

For the present Interacting BHDE, $q$ takes the form

$$q = -1 + \frac{3\Omega_{m0} a^{-3(1-b)} - \Gamma_1 a^{-3}}{2 \left( \Omega_{m0} a^{-3(1-b)} - \frac{\Gamma_1}{3} a^{-3} + \Gamma_2 \right)}.$$  

(25)

As $a \rightarrow -\infty$ the second term on the r.h.s. of the above equation dominates hence $q > 0$ corresponding to the prior deceleration era, while the second terms vanishes as $a \rightarrow \infty$, hence $q \rightarrow -1$ corresponding to the end de Sitter epoch. As $a = 1$, the deceleration parameter for current epoch results,

$$q_0 = -1 + \frac{3\Omega_{m0} - \Gamma_1}{2 \left( \Omega_{m0} - \frac{\Gamma_1}{3} + \Gamma_2 \right)}.$$  

(26)

The evolutionary behaviour of the deceleration parameter $q$ as a function of redshift $z$ is plotted for the best fit model parameters obtained from OHD36+SN Ia dataset in Figure[12]. The value of deceleration parameter for the current epoch($z = 0$) is estimated as $-0.54$, shows close agreement with the
observational results $q_0 = -0.63 \pm 0.12$ [85], $0.644 \pm 0.22$ [86]. The transition redshift characterising the onset of the late acceleration is found to be around $z_T = 0.71$ and is comparable with the observational constraint using SN Ia+BAO/CMB dataset $0.72 \pm 0.05$ [87] and slightly higher compared to constraints using SN Ia+CMB data, $z_T=0.57\pm0.07$, [85], SN Ia+CMB+LSS joint analysis $z_T = 0.61$ [88]. However it should be noted that, the extracted values of $q_0$ and $z_T$ in the present model are comparable to the corresponding values obtained for the standard ΛCDM model ($z_t=0.74$ and $q_0=-0.59$) [87].

![Figure 12: Behaviour of q parameter vs z with transition redshift $z_T = 0.71$.](image)

![Figure 13: Evaluation of curvature scalar with scale factor for the best estimate parameters](image)

The curvature scalar $R$ is a useful parameter to asses about the early epoch of universe and is defined in terms of Hubble parameter by [89],

$$R = -6(\dot{H} + 2H^2)$$  \hspace{1cm} (27)

The evolution of curvature scalar parameter is as shown in Figure 13. It is evident that, as $a \to 0$, curvature scalar approaches to infinity, which indicates the existence of the Big-bang.

The age of the universe can be obtained by the relation,

$$t_0 - t_B = \int_0^1 \frac{1}{aH(a)} da$$  \hspace{1cm} (28)

Where $t_0$ is the present time, $t_B = 0$ is the big bang time ; hence the universe has the age of the order of $t_0$. We obtained the age by suitably evaluating the above integral by substituting the derived Hubble parameter of the present model. We fund that the age of universe, corresponding to the best estimated model parameters as 14.026 Gyr. The age obtained is very much compatible with that obtained from the CMB anisotropic data, 14 ± 0.5 Gyr [90] and from the study based on the 42 high redshift supernovae deduced the age of the flat universe, 14.9±1.4 Gyr [4]. Also not too much different from, 13.7±0.2 Gyr using WMAP data analysis [5], 13.69±0.13 Gyr from WMAP and 13.72 ±0.12 Gyr from WMAP+BAO+SN [84]. It may be noted that, the extracted age in the present model is near to the upper age limit set by the The age mentioned above are extracted from different observational datasets, and analytical techniques lead to constraint observation of oldest globular clusters, around 14.56 Gyr, however the lower set by this observation is around 12.07 Gyr [91].

5 Statefinder Diagnostics

We have analysed the model using the statefinder diagnostics first prosed by Sahni et al. [92] and is effective in contrasting a given dark energy model with the standard ΛCDM or with any other given model, in a model-independent way. Here we use this tool to compare the status of the present model with the standard model. The statefinder parameters $r, s$ are formulated from the scale factor $a(t)$ and its derivatives with respect to cosmic time up to the third order. Following the evolutionary trajectory of the model in $(r,s)$ plane, it can classified as quintessence, if $r < 1, s > 0$ and if $r > 1$ and $s < 0$, then it corresponds to chaplygin gas model [93, 94, 92]. For ΛCDM model, the parameter values are $\{r,s\} = \{1,0\}$. The statefinder pair $\{r,s\}$
takes the form \[ r = \frac{\dot{a}}{aH^3} \] (29)

\[ s = \frac{r - 1}{3(q - \frac{1}{2})} \] (30)

In terms of $\ddot{H}$ as,

\[ r = 1 + \frac{1}{2H^2} \frac{d^2H^2}{dx^2} + \frac{3}{2H^2} \frac{dH^2}{dx} \] (31)

\[ s = \frac{r - 1}{2H^2} \frac{dH^2}{dx} + \frac{9}{2} \] (32)

After substituting the respective terms from equation (11) and (12), the parameters becomes,

After substituting the respective terms from equation (11) and (12), the parameters becomes,

\[ r = 1 - \frac{9b\Omega_m a^{-3(1-b)} + 2\Gamma_1 a^{-3}}{2 \left( \frac{\Omega_m \Omega_0}{1-b} a^{-3(1-b)} - \frac{\Gamma_1 a^{-3}}{3} + \Gamma_2 \right)} \] (33)

\[ s = \frac{3\Omega_m a^{-3(1-b)}}{\frac{b}{1-b} \Omega_m a^{-3(1-b)} + \Gamma_2} \] (34)

At the asymptotic limit $a \to \infty$ the statefinder parameters takes the values, $r \to 1$ and $s \to 0$, shows that the present model will tends to the $\Lambda$CDM in the far future of the evolution. The trajectory of the model in the $r - s$ plane with the best-fit parameters is depicted in Figure [14]. The current value of $r$ and $s$ are determined from the two-dimensional parametric plane as $\{r_0, s_0\} = \{0.97, 0.01\}$. This, however, reveals that the model is arguably different form the standard $\Lambda$CDM. The trajectory shows that, the condition $r < 1, s > 0$ is satisfied through the evolution and hence the proposed model has a quintessence behavior.

6 Thermodynamics of IBHDE Model

In this section we will analyse the thermal evolution of the model, concentrating on the evolution of the entropy. The universe evolving as an ordinary macroscopic system, hence advances towards an equilibrium state with maximum entropy. The aforementioned statement is valid only if the system satisfies the
appropriate constraints given by [95, 96, 97],
\[ \dot{S} \geq 0 \quad \text{for always} ; \quad \ddot{s} < 0 \quad \text{for atleast later time of evolution}. \]
where the over-dot represents the derivative with respect to cosmic time. Regarding entropy generation, the generalized second law state that sum of the entropy of the matter enclosed by the horizon, \( S_m \) and that of horizon, \( S_h \) itself, should always a non-decreasing function of cosmic time [98, 99, 100, 51, 95, 101]. That is,
\[ \dot{S}_h + \dot{S}_m \geq 0, \quad (35) \]
Using conservation law and integrability condition,
\[ dS = d\left( \frac{\rho + P}{T} + \text{constat} \right) \quad (36) \]
Thus the entropy becomes,
\[ S = \frac{(\rho + P)V}{T}, \quad (37) \]
where \( V \) is the volume of the horizon and \( T \) is the temperature. The entropy of the non-relativistic matter within the apparent horizon of volume \( V = 4\pi c^3/3H^3 \) is,
\[ S_m = \frac{8\pi^2 c^3 H_0^3 \Omega_m a^{-3(1-b)}}{H^4}, \quad (38) \]
where we have used, the Hawking temperature relation, \( T = H/2\pi \). For horizon entropy we have used the Barrow entropy relation such that,
\[ S_h = \left( \frac{\pi c^5}{\hbar G H^2} \right)^{1+\Delta/2} k_B \]
(39)
Here \( c \) is the speed of light in vacuum, \( \hbar = h/2\pi \) with the plank’s constant, and \( G \) is the gravitational constant. The rate of change of the total entropy with the respect to the variable, \( x = lna \) is given as,
\[ S_{tot}' = S_m' + S_h' \quad (40) \]
\[ S_{tot}' = 8\pi^2 c^3 H_0^3 \Omega_m \left( \frac{-3(1-b)}{H^4} - 2 \frac{dH^2}{dx} \right) - \left( \frac{4\pi c^5}{\hbar G} \right)^{1+\Delta/2} \frac{2 + \Delta}{2H^{4+\Delta}} \frac{dH^2}{dx} \quad (41) \]
The evolution of \( S_{tot}' \) with scale factor is plotted in Figure 15 and is always positive. This shows that total entropy of the horizon plus that of matter within the horizon is non-decreasing through out the evolution of the universe. So the generalized second law of thermodynamics(GSL) is valid for the universe which comprises Barrow holographic dark energy and pressure-less matter. This is a point to be noted in contrast with some recent works, which claim a conditional violation of GSL for non-zero Barrow exponent, \( \Delta \). In reference [58] it is demonstrated that the GSL may be conditionally violated depending on the evolution of the universe. For example, if the Hubble parameter assume \( \Lambda \)CDM like behaviour, then the generalised second will be satisfied for any reasonable value of \( \Delta \). Otherwise the law need not satisfied in general [58].
Such a conditional violation of the generalised second law is noted in interacting Barrow holographic dark energy model in reference [62]. The basic difference of the above two works with the present model is that, the Barrow holographic dark energy has been treated as one with varying equation of state in that works. In contrary, we have treated the dark energy as a running vacuum with constant equation of state, \(-1\). However, there some other Barrow holographic dark energy models, which advocates the validity of GSL. It has been shown that generalized second law is valid for interacting BHDE with Hubble horizon and event horizon as IR cut-off in DGP(Dvali-Gabadadze-porrati) braneworld cosmology [102]. Also GSL found to be valid in Barrow holographic dark energy with NO(Nojiri-Odintsov) cut-off [103]. Now we will check whether the present model predicts an evolution of the universe to a state of maximum entropy. We have seen that the first condition for the maximisation of entropy is satisfied. To check the second condition we have obtained the second derivative of the total entropy as,
\[ S''_{\text{tot}} = \frac{24\pi^2 c^3 H^3_0 \Omega_m}{H^4} \left[ 3(1 - b)^2 + \frac{4(1 - b)}{H^2} \frac{dH^2}{dx} - \frac{2\Omega_m}{3H^2} \frac{d^2 H^2}{dx^2} + \frac{2\Omega_m}{H^4} \left( \frac{dH^2}{dx} \right)^2 \right] + \left( \frac{\pi c^5}{\bar{h}G} \right) \frac{1 + \Delta}{2} + \Delta \frac{2}{2H^4} \left[ 4 + \Delta \frac{1}{2H^2} \right] - \frac{1}{H^4} \frac{d^2 H^2}{dx^2} \right] \] (42)

Since the relation (42) seems to be slightly tricky to analyse, Figure 16 is plotted for the second derivative of the total entropy for best-estimate model parameters. It is obtained that \( S'' < 0 \) at least at the later time of evolution, i.e., as \( a \to \infty \) and it approach zero form below, thus satisfies the entropy maximization condition. Thus the present IBHDE model shows the thermal behavior analogues to the \( \Lambda \)CDM universe. At this juncture, it is to be noted that, as mentioned in the previous section, the present model is reduced to a \( \Lambda \)CDM like model for the interaction parameter, \( b = 0 \), with an effective mass density parameter and dark energy density parameter. For estimated model parameters these density parameters become, \( \Omega_{m0} = 0.285 \) and \( \Omega_{\Lambda 0} = 0.715 \). In this limit, the Generalized second law will surely be satisfied throughout the evolution[58]. This shows that, the validity of the generalised second law in the present model, doesn’t have any dependence on the evolution form of the Hubble parameter.

### Figure 15: Evolution of \( S'_\text{tot} \) vs scale factor \( a \).

The values of model parameters is considered from the combined dataset of SN Ia+OHD36.

### Figure 16: \( S''_\text{tot} \) vs scale factor \( a \).

The values of model parameters is considered from the combined dataset of SN Ia+OHD36.

## 7 Phase-Space Analysis

Phase space analysis aims to study the asymptotic behavior of the present model. The method has been widely used to obtain the asymptotic evolution of the universe in various cosmological model [104, 105, 106]. The method involves the formulation of a set of autonomous differential equations of appropriately chosen dimensionless phase space variables \( z_i \) as,

\[ z'_i = f_i(z_j) \quad i, j = 1, 2, 3, \ldots \] (43)

where the prime represents the derivative with respect to a suitably chosen variable. The equilibrium solutions called the critical points \( (z_i = z^*_i) \), are obtained by considering, \( z'_i = 0 \) for all \( i \) [107]. The stability of the model corresponding to these solutions can then be obtained by analysing the sign of the eigenvalues of Jacobian matrix of the critical points(or equilibrium points). If all the eigenvalues are negative, then the critical point is a global attractor, and the neighbouring trajectories will always converge to that point, thus it is a stable state. The obtained critical point is said to be unstable for all the eigenvalues are positive, for which the neighbouring trajectories seems to be diverging from the critical point. If one of the eigenvalues is negative and other is positive, then the critical point is said to be saddle, the neighbouring trajectories either converge or diverge depending upon the initial conditions [108].

To analyse asymptotic behavior of the present model, we chose the dimensionless phase space variable \( u \) and \( v \) defined as,

\[ u = \frac{\rho_m}{3H^2} \quad v = \frac{\rho_{\Lambda}}{3H^2} \] (44)
Table 2: Critical points and Eigenvalues

| Critical points | Eigenvalues       | Nature   |
|-----------------|-------------------|----------|
| (0.926, 0)      | (0.926, 0.091)    | unstable |
| (0, 1)          | (−2.926, −0.463)  | stable   |

Figure 17: Phase space trajectories of the model in u-v plane with best estimated model parameters from combined dataset SN Ia+OHD36. The point (u,v) = (0,1) is a global attractor.

By considering Friedmann equation (8) and conservation relations (9)-(10), the autonomous differential equations corresponding to these variables are,

\[
u' = f(u, v) = 3(b - 1)u + u^2 - 2uv + 2u
\]

\[
v' = g(u, v) = \frac{(2 - \Delta)u \alpha(2v - u - 2) - \beta(\frac{3}{2}bu + \frac{3}{2}u + 2v - 2)}{\alpha - \beta(\frac{u}{2} - v + 1) - (2v^2 - vu - 2v)}
\]

Here the differentiation is with respect to the variable \(x = \ln a\). The following critical points are obtained by equating \(u' = 0\) and \(v' = 0\),

\[(v^*, v^*) = (0, 1), (1 - 3b, 0)\]

In order to study the nature of these critical points, consider a small perturbation to the phase variables, say \(u = u^* + \delta u\) and \(v = v^* + \delta v\),

\[
\delta u' \quad \delta v'
\]

The 2 x 2 matrix in the above equation is the general form of Jacobian matrix. Diagonalizing the above matrix gives the corresponding eigenvalues. We have obtained the matrix, and derived the eigenvalues as shown in Table 2 along with the inference regarding stability. The phase space plot is illustrated in figure 17.

The eigenvalues corresponding to the critical point (0.926, 0.091), are both positive, hence it is unstable. The figure reveals that trajectories around the this critical point are diverging. This point is in fact corresponds to prior matter dominated epoch, which is decelerating. On the other hand the eigenvalues corresponds to the critical point (0, 1) are both negative and hence stable. This critical points is corresponding to the dominance of Barrow dark energy, hence corresponds to the end de Sitter epoch. The phase plot shows that the trajectories are converging to this critical point, hence it can be a global attractor. So the dynamical system analysis of the IBHDE model holds the result that the model approaches an end de Sitter stable state in the far future. A similar dynamical system analysis were performed for a varying equation.
of state, Barrow holographic dark energy model with Hubble horizon as IR cut-off for different interacting terms and have shown that the end de sitter phase is stable [65].

8 Conclusion

In the present work, we have analysed a cosmological model with Barrow holographic dark energy interacting with matter, to explain the current accelerated expansion of the universe. For obtaining the BHDE density we have used Granda-Oliveros(GO) length scale as IR cut-off. Unlike the studies existing in the current literature, we consider BHDE as dynamical vacuum with constant equation of state, $\omega = -1$. By choosing the interaction term as $Q = 3bH\rho_m$, we have solved for the Hubble parameter and found that, it is capable of explaining the transition from prior decelerated epoch to the late accelerated stage. For zero interaction parameter, $b = 0$ the model reduces to a $\Lambda$CDM model, with effective mass densities $\bar{\Omega}_m$ and $\bar{\Omega}_\Lambda$. We extracted best-fit values of the model parameters at 1σ confidence interval, by contrasting the model with the cosmological observational dataset SN Ia+OHD36. We have studied the evolution of the cosmological parameters such as Hubble parameter, density parameter, deceleration parameter, and curvature scalar. The present value of the Hubble parameter is estimated as $H_0 = 69.458 \text{ km s}^{-1} \text{ Mpc}^{-1}$, and age of the universe as around 14.026 Gyr. These are in concordance with the observational results. The behavior of the density parameter solves the existing coincidence problem. The transition from prior decelerated phase to an accelerating phase is found to occur at transition redshift $z_T = 0.71$, and the present value of deceleration parameter $q_0 = -0.54$ and are moderately consistent with the observational results. On analysing the thermal evolution of the model, we have shown that it obeys the GSL, in contrast to some of the recent speculations that, the models with Barrow holographic dark energy can violate the GSL conditionally. This is mainly due to the consideration of the BHDE as a dynamical vacuum. On extending the thermodynamics study we found that the model satisfies the entropy maximization, such that the universe evolve in to a stable equilibrium state with maximum entropy. Moreover, On ignoring the interaction between the dark sectors, the model guarantees the validity of GSL due to its $\Lambda$CDM like behaviour with effective mass parameter $\bar{\Omega}_m = 0.285$ and $\bar{\Omega}_\Lambda = 0.715$. Finally, we extended our work to study the dynamical system behavior of the IBHDE model with GO cut-off, which the model predicts a stable asymptotic stable de Sitter epoch followed by prior decelerated epoch.

To summarise, treating Barrow holographic dark energy as a dynamical vacuum, has some advantageous over the models in which it is treated as a dark energy of varying equation of state. The first and foremost thing is that, it respects that GSL, such that there is no chance at all for any conditional violation of the entropy increase as predicted by the conventional varying equation of state models. Secondly when there is no interaction between the dark energy and dark matter, the model mimics the standard $\Lambda$CDM model but with effective mass density parameters, which is not a new thing as there are several models which predicts effective mass density parameters against the conventional density parameter values. The overall performance of the model in explaining the background evolution of the late accelerated epoch of the universe is found to be reasonably good.

References

[1] P. A. R. Ade et al., “Planck 2015 results. XIII. Cosmological parameters,” Astron. Astrophys., vol. 594, p. A13, 2016.

[2] J. A. Frieman, M. S. Turner, and D. Huterer, “Dark energy and the accelerating universe,” Annual Review of Astronomy and Astrophysics, vol. 46, p. 385–432, Sep 2008.

[3] A. G. Riess et al., “Observational evidence from supernovae for an accelerating universe and a cosmological constant,” Astron. J., vol. 116, pp. 1009–1038, 1998.

[4] S. Perlmutter et al., “Measurements of $\Omega$ and $\Lambda$ from 42 high redshift supernovae,” Astrophys. J., vol. 517, pp. 565–586, 1999.

[5] D. N. Spergel et al., “First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Determination of cosmological parameters,” Astrophys. J. Suppl., vol. 148, pp. 175–194, 2003.
D. N. Spergel et al., “Wilkinson Microwave Anisotropy Probe (WMAP) three year results: implications for cosmology,” *Astrophys. J. Suppl.*, vol. 170, p. 377, 2007.

M. Tegmark et al., “Cosmological parameters from SDSS and WMAP,” *Phys. Rev. D*, vol. 69, p. 103501, 2004.

D. J. Eisenstein et al., “Detection of the Baryon Acoustic Peak in the Large-Scale Correlation Function of SDSS Luminous Red Galaxies,” *Astrophys. J.*, vol. 633, pp. 560–574, 2005.

N. Aghanim et al., “Planck 2018 results. VI. Cosmological parameters,” *Astron. Astrophys.*, vol. 641, p. A6, 2020. [Erratum: *Astron.Astrophys.* 652, C4 (2021)].

S. D. H. Hsu, “Entropy bounds and dark energy,” *Phys. Lett. B*, vol. 594, pp. 13–16, 2004.

M. Li, “A model of holographic dark energy,” *Physics Letters B*, vol. 603, no. 1, pp. 1–5, 2004.

L. Susskind, “The world as a hologram,” *Journal of Mathematical Physics*, vol. 36, p. 6377–6396, Nov 1995.

R. Bousso, “The Holographic principle,” *Rev. Mod. Phys.*, vol. 74, pp. 825–874, 2002.

W. Fischler and L. Susskind, “Holography and cosmology,” *arXiv preprint hep-th/9806039*, 1998.

G. ’t Hooft, “Dimensional reduction in quantum gravity,” *Conf. Proc. C*, vol. 930308, pp. 284–296, 1993.

G. Hooft, “The holographic principle,” *Basics and Highlights in Fundamental Physics*, Apr 2001.

J. D. Bekenstein, “Entropy bounds and black hole remnants,” *Phys. Rev. D*, vol. 49, pp. 1912–1921, Feb 1994.

J. D. Bekenstein, “Black holes and entropy,” *Phys. Rev. D*, vol. 7, pp. 2333–2346, Apr 1973.

A. G. Cohen, D. B. Kaplan, and A. E. Nelson, “Effective field theory, black holes, and the cosmological constant,” *Phys. Rev. Lett.*, vol. 82, pp. 4971–4974, Jun 1999.

Y. S. Myung, “Holographic principle and dark energy [rapid communication],” *Physics Letters B*, vol. 610, pp. 18–22, Mar. 2005.

C. Rong-Gen, H. Bin, and Z. Yi, “Holography, uv/ir relation, causal entropy bound, and dark energy,” *Communications in Theoretical Physics*, vol. 51, p. 954–960, May 2009.

S. Wang, Y. Wang, and M. Li, “Holographic dark energy,” *Physics Reports*, vol. 696, pp. 1–57, 2017. Holographic Dark Energy.

S. D. Hsu, “Entropy bounds and dark energy,” *Physics Letters B*, vol. 594, no. 1, pp. 13–16, 2004.

Y. Gong, “Extended holographic dark energy,” *Physical Review D*, vol. 70, Sep 2004.

X. Zhang and F.-Q. Wu, “Constraints on holographic dark energy from type ia supernova observations,” *Phys. Rev. D*, vol. 72, p. 043524, Aug 2005.

Q.-G. Huang and Y.-G. Gong, “Supernova constraints on a holographic dark energy model,” *JCAP*, vol. 08, p. 006, 2004.

R. Horvat, “Holography and variable cosmological constant,” *Phys. Rev. D*, vol. 70, p. 087301, 2004.

Q.-G. Huang and M. Li, “The Holographic dark energy in a non-flat universe,” *JCAP*, vol. 08, p. 013, 2004.

C. Feng, B. Wang, Y. Gong, and R.-K. Su, “Testing the viability of the interacting holographic dark energy model by using combined observational constraints,” *JCAP*, vol. 09, p. 005, 2007.
[30] M. Li, X.-D. Li, S. Wang, and X. Zhang, “Holographic dark energy models: A comparison from the latest observational data,” JCAP, vol. 06, p. 036, 2009.

[31] X. Zhang and F.-Q. Wu, “Constraints on holographic dark energy from the latest supernovae, galaxy clustering, and cosmic microwave background anisotropy observations,” Phys. Rev. D, vol. 76, p. 023502, Jul 2007.

[32] M. Li, X.-D. Li, Y.-Z. Ma, X. Zhang, and Z. Zhang, “Planck Constraints on Holographic Dark Energy,” JCAP, vol. 09, p. 021, 2013.

[33] M. Cruz and S. Lepe, “Modeling holographic dark energy with particle and future horizons,” Nuclear Physics, vol. 956, p. 115017, 2020.

[34] T. Bandyopadhyay, “Modified holographic dark energy and phantom behaviour of randall-sundrum brane,” International Journal of Theoretical Physics, vol. 50, no. 10, pp. 3284–3288, 2011.

[35] B. Wang, C.-Y. Lin, D. Pavón, and E. Abdalla, “Thermodynamical description of the interaction between holographic dark energy and dark matter,” Physics Letters B, vol. 662, p. 1–6, Apr 2008.

[36] M. Setare and E. C. Vagenas, “Thermodynamical interpretation of the interacting holographic dark energy model in a non-flat universe,” Physics Letters B, vol. 666, no. 2, pp. 111–115, 2008.

[37] B. Wang, Y.-g. Gong, and E. Abdalla, “Transition of the dark energy equation of state in an interacting holographic dark energy model,” Phys. Lett. B, vol. 624, pp. 141–146, 2005.

[38] B. Wang, C.-Y. Lin, and E. Abdalla, “Constraints on the interacting holographic dark energy model,” Phys. Lett. B, vol. 637, pp. 357–361, 2006.

[39] M. R. Setare, “Interacting holographic dark energy model and generalized second law of thermodynamics in non-flat universe,” JCAP, vol. 01, p. 023, 2007.

[40] A. Sheykhi, “Thermodynamics of interacting holographic dark energy with apparent horizon as an IR cutoff,” Class. Quant. Grav., vol. 27, p. 025007, 2010.

[41] A. Sheykhi, “Interacting holographic dark energy in Brans-Dicke theory,” Phys. Lett. B, vol. 681, pp. 205–209, 2009.

[42] P. Praseetha and T. K. Mathew, “Entropy of holographic dark energy and the generalized second law,” Class. Quant. Grav., vol. 31, p. 185012, 2014.

[43] P. Praseetha and T. K. Mathew, “Evolution of holographic dark energy with interaction term \( Q \propto H \rho_{de} \) and generalized second law,” Pramana, vol. 86, no. 3, pp. 701–712, 2016.

[44] D. Pavon and W. Zimdahl, “Holographic dark energy and cosmic coincidence,” Phys. Lett. B, vol. 628, pp. 206–210, 2005.

[45] A. Oliveros and M. A. Acero, “New holographic dark energy model with non-linear interaction,” Astrophys. Space Sci., vol. 357, no. 1, p. 12, 2015.

[46] L. Feng and X. Zhang, “Revisit of the interacting holographic dark energy model after Planck 2015,” JCAP, vol. 08, p. 072, 2016.

[47] M. Setare and E. Saridakis, “Correspondence between holographic and gauss–bonnet dark energy models,” Physics Letters B, vol. 670, p. 1–4, Dec 2008.

[48] H. Moradpour, S. A. Moosavi, I. P. Lobo, J. P. Morais Graça, A. Jawad, and I. G. Salako, “Thermodynamic approach to holographic dark energy and the Rényi entropy,” Eur. Phys. J. C, vol. 78, no. 10, p. 829, 2018.

[49] M. Tavayef, A. Sheykhi, K. Bamba, and H. Moradpour, “Tsallis Holographic Dark Energy,” Phys. Lett. B, vol. 781, pp. 195–200, 2018.
[50] J. D. Barrow, “The area of a rough black hole,” *Physics Letters B*, vol. 808, p. 135643, 2020.

[51] J. D. Bekenstein, “Black holes and the second law,” *Lett. Nuovo Cim.*, vol. 4, pp. 737–740, 1972.

[52] S. Carlip, “Logarithmic corrections to black hole entropy, from the cardy formula,” *Classical and Quantum Gravity*, vol. 17, p. 4175–4186, Sep 2000.

[53] R. K. Kaul and P. Majumdar, “Logarithmic correction to the bekenstein-hawking entropy,” *Phys. Rev. Lett.*, vol. 84, pp. 5255–5257, Jun 2000.

[54] C. Tsallis and L. J. L. Cirto, “Black hole thermodynamical entropy,” *The European Physical Journal C*, vol. 73, Jul 2013.

[55] G. Wilk and Z. Włodarczyk, “Interpretation of the nonextensivity parameter $q$ in some applications of tsallis statistics and lévy distributions,” *Phys. Rev. Lett.*, vol. 84, pp. 2770–2773, Mar 2000.

[56] E. N. Saridakis, “Barrow holographic dark energy,” *Phys. Rev. D*, vol. 102, p. 123525, Dec 2020.

[57] F. K. Anagnostopoulos, S. Basilakos, and E. N. Saridakis, “Observational constraints on Barrow holographic dark energy,” *Eur. Phys. J. C*, vol. 80, no. 9, p. 826, 2020.

[58] E. N. Saridakis and S. Basilakos, “The generalized second law of thermodynamics with barrow entropy,” *The European Physical Journal C*, vol. 81, Jul 2021.

[59] M. P. Dabrowski and V. Salzano, “Geometrical observational bounds on a fractal horizon holographic dark energy,” *Phys. Rev. D*, vol. 102, no. 6, p. 064047, 2020.

[60] E. N. Saridakis, “Modified cosmology through spacetime thermodynamics and barrow horizon entropy,” *Journal of Cosmology and Astroparticle Physics*, vol. 2020, pp. 031–031, jul 2020.

[61] A. Sheykhi, “Barrow Entropy Corrections to Friedmann Equations,” *Phys. Rev. D*, vol. 103, no. 12, p. 123503, 2021.

[62] A. A. Mamon, A. Paliathanasis, and S. Saha, “Dynamics of an Interacting Barrow Holographic Dark Energy Model and its Thermodynamic Implications,” *Eur. Phys. J. Plus*, vol. 136, no. 1, p. 134, 2021.

[63] A. Dixit, V. K. Bharadwaj, and A. Pradhan, “Barrow HDE model for Statefinder diagnostic in non-flat FRW universe,” 3 2021.

[64] A. Pradhan, A. Dixit, and V. K. Bhardwaj, “Barrow hde model for statefinder diagnostic in flrw universe,” *International Journal of Modern Physics A*, vol. 36, p. 2150030, Feb 2021.

[65] Q. Huang, H. Huang, B. Xu, F. Tu, and J. Chen, “Dynamical analysis and statefinder of Barrow holographic dark energy,” *Eur. Phys. J. C*, vol. 81, no. 8, p. 686, 2021.

[66] G. Chakraborty, S. Chattopadhyay, E. Güdekli, and I. Radinschi, “Thermodynamics of Barrow Holographic Dark Energy with Specific Cut-Off,” *Symmetry*, vol. 13, no. 4, p. 562, 2021.

[67] P. Adhikary, S. Das, S. Basilakos, and E. N. Saridakis, “Barrow Holographic Dark Energy in non-flat Universe,” 4 2021.

[68] L. Granda and A. Oliveros, “Infrared cut-off proposal for the holographic density,” *Physics Letters B*, vol. 669, p. 275–277, Nov 2008.

[69] S. H. Pereira and J. F. Jesus, “Can dark matter decay in dark energy?,” *Physical Review D*, vol. 79, Feb 2009.

[70] A. Pasqua, S. Chattopadhyay, and R. Myrzakulov, “Power-law entropy-corrected holographic dark energy in hořava-lifshitz cosmology with granda-oliveros cut-off,” 2016.

[71] F. Arévalo, A. P. Bacalhau, and W. Zimdahl, “Cosmological dynamics with nonlinear interactions,” *Classical and Quantum Gravity*, vol. 29, p. 235001, Oct 2012.
[72] Y. L. Bolotin, A. Kostenko, O. A. Lemets, and D. A. Yerokhin, “Cosmological Evolution With Interaction Between Dark Energy And Dark Matter,” Int. J. Mod. Phys. D, vol. 24, no. 03, p. 1530007, 2014.

[73] J. Sadeghi, M. Setare, A. Amani, and S. Noorbakhsh, “Bouncing universe and reconstructing vector field,” Physics Letters B, vol. 685, p. 229–234, Mar 2010.

[74] H. M. Sadjadi, “Particle versus future event horizon in interacting holographic dark energy model,” JCAP, vol. 02, p. 026, 2007.

[75] E. Sadri, M. Khurshudyan, and D.-f. Zeng, “Scrutinizing various phenomenological interactions in the context of holographic ricci dark energy models,” The European Physical Journal C, vol. 80, May 2020.

[76] D. Pavón and B. Wang, “Le châtelier–braun principle in cosmological physics,” General Relativity and Gravitation, vol. 41, p. 1–5, Jun 2008.

[77] P. George, V. M. Shareef, and T. K. Mathew, “Interacting holographic Ricci dark energy as running vacuum,” Int. J. Mod. Phys. D, vol. 28, no. 04, p. 1950060, 2018.

[78] A. A. Starobinsky, “How to determine an effective potential for a variable cosmological term,” Journal of Experimental and Theoretical Physics Letters, vol. 68, p. 757–763, Nov 1998.

[79] J. Solà Peracaula, J. de Cruz Pérez, and A. Gomez-Valent, “Possible signals of vacuum dynamics in the Universe,” Mon. Not. Roy. Astron. Soc., vol. 478, no. 4, pp. 4357–4373, 2018.

[80] H. Yu, B. Ratra, and F.-Y. Wang, “Hubble Parameter and Baryon Acoustic Oscillation Measurement Constraints on the Hubble Constant, the Deviation from the Spatially Flat ΛCDM Model, the Deceleration-Acceleration Transition Redshift, and Spatial Curvature,” The Astrophysical Journal, vol. 856, p. 3, Mar. 2018.

[81] H. Amirhashchi and S. Amirhashchi, “Constraining bianchi type i universe with type ia supernova and h(z) data,” Physics of the Dark Universe, vol. 29, p. 100557, 2020.

[82] D. M. Scolnic, D. O. Jones, A. Rest, Y. C. Pan, R. Chornock, R. J. Foley, M. E. Huber, R. Kessler, G. Narayan, A. G. Riess, S. Rodney, E. Berger, D. J. Brout, P. J. Challis, M. Drout, D. Finkbeiner, R. Lunnan, R. P. Kirshner, N. E. Sanders, E. Schlafly, S. Smartt, C. W. Stubbs, J. Tonry, W. M. Wood-Vasey, M. Foley, J. Hand, E. Johnson, W. S. Burgett, K. C. Chambers, P. W. Draper, K. W. Hodapp, N. Kaiser, R. P. Kudritzki, E. A. Magnier, N. Metcalfe, F. Bresolin, E. Gall, R. Kotak, M. McCrum, and K. W. Smith, “The Complete Light-curve Sample of Spectroscopically Confirmed SNe Ia from Pan-STARRS1 and Cosmological Constraints from the Combined Pantheon Sample,” apj, vol. 859, p. 101, June 2018.

[83] Planck Collaboration, N. Aghanim, Y. Akrami, M. Ashdown, and j. . A. k. . c. y. . m. . s. v. . . e. . A. p. . A. d. . a. . a. e. . p. . a. a. . h. a. . P. Aumont, et al title = "Planck 2018 results. VI. Cosmological parameters"

[84] E. Komatsu, J. Dunkley, and M. R. al, “FIVE-YEAR WILKINSON MICROWAVE ANISOTROPY PROBE OBSERVATIONS: COSMOLOGICAL INTERPRETATION,” The Astrophysical Journal Supplement Series, vol. 180, pp. 330–376, feb 2009.

[85] U. Alam, V. Sahni, and A. A. Starobinsky, “The Case for dynamical dark energy revisited,” JCAP, vol. 06, p. 008, 2004.

[86] Z. Li, P. Wu, and H. Yu, “Examining the cosmic acceleration with the latest union2 supernova data,” Physics Letters B, vol. 695, no. 1, pp. 1–8, 2011.

[87] A. A. Mamon, K. Bamba, and S. Das, “Constraints on reconstructed dark energy model from sn ia and bao/cmb observations,” The European Physical Journal C, vol. 77, Jan 2017.
[88] A. A. Mamon, “Constraints on a generalized deceleration parameter from cosmic chronometers,” Modern Physics Letters A, vol. 33, p. 1850056, Apr 2018.

[89] E. W. Kolb and M. S. Turner, The Early Universe, vol. 69. 1990.

[90] L. Knox, N. Christensen, and C. Skordis, “The age of the universe and the cosmological constant determined from cosmic microwave background anisotropy measurements,” The Astrophysical Journal, vol. 563, p. L95–L98, Dec 2001.

[91] B. Chaboyer, P. Demarque, P. J. Kernan, and L. M. Krauss, “A lower limit on the age of the universe,” vol. 271, no. 5251, pp. 957–961, 1996.

[92] V. Sahni, T. D. Saini, A. A. Starobinsky, and U. Alam, “Statefinder—a new geometrical diagnostic of dark energy,” Journal of Experimental and Theoretical Physics Letters, vol. 77, p. 201–206, Mar 2003.

[93] Y.-B. Wu, S. Li, M.-H. Fu, and J. He, “A modified Chaplygin gas model with interaction,” Gen. Rel. Grav., vol. 39, pp. 653–662, 2007.

[94] U. Alam, V. Sahni, T. Deep Saini, and A. A. Starobinsky, “Exploring the expanding universe and dark energy using the statefinder diagnostic,” Monthly Notices of the Royal Astronomical Society, vol. 344, p. 1057–1074, Oct 2003.

[95] D. Pavón and N. Radicella, “Does the entropy of the universe tend to a maximum?,” General Relativity and Gravitation, vol. 45, p. 63–68, Sep 2012.

[96] P. B. Krishna and T. K. Mathew, “Holographic equipartition and the maximization of entropy,” Phys. Rev. D, vol. 96, no. 6, p. 063513, 2017.

[97] K. P. B and T. K. Mathew, “Emergence of cosmic space and the maximization of horizon entropy,” 2021.

[98] W. G. Unruh and R. M. Wald, “Acceleration radiation and the generalized second law of thermodynamics,” Phys. Rev. D, vol. 25, pp. 942–958, Feb 1982.

[99] J. D. Bekenstein, “Generalized second law of thermodynamics in black hole physics,” Phys. Rev. D, vol. 9, pp. 3292–3300, 1974.

[100] J. M. Bardeen, B. Carter, and S. W. Hawking, “The four laws of black hole mechanics,” Communications in Mathematical Physics, vol. 31, no. 2, pp. 161 – 170, 1973.

[101] G. W. Gibbons and S. W. Hawking, “Cosmological event horizons, thermodynamics, and particle creation,” Phys. Rev. D, vol. 15, pp. 2738–2751, May 1977.

[102] S. Rani and N. Azhar, “Braneworld inspires cosmological implications of barrow holographic dark energy,” Universe, vol. 7, p. 268, 07 2021.

[103] G. Chakraborty, S. Chattopadhyay, E. Güdekli, and I. Radinschi, “Thermodynamics of barrow holographic dark energy with specific cut-off,” Symmetry, vol. 13, no. 4, 2021.

[104] A. A. Coley, Dynamical systems and cosmology. Dordrecht, Netherlands: Kluwer, 2003.

[105] N. Mazumder, R. Biswas, and S. Chakraborty, “Interacting holographic dark energy at the ricci scale and dynamical system,” 2011.

[106] R. Biswas, N. Mazumder, and S. Chakraborty, “Interacting holographic dark energy model as a dynamical system and the coincidence problem,” 2011.

[107] N. Roy and N. Banerjee, “Phase space analysis of a holographic dark energy model,” 12 2014.

[108] R. D. Gregory, Frontmatter, pp. i–iv. Cambridge University Press, 2006.