Generalized Play Hysteresis Operators in Limits of Fast-Slow Systems

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Abstract

Hysteresis operators appear in many applications such as elasto-plasticity and micromagnetics, and can be used for a wider class of systems, where rate-independent memory plays a role. A natural approximation for systems of evolution equations with hysteresis operators are fast-slow dynamical systems, which - in their used approximation form - do not involve any memory effects. Hence, viewing differential equations with hysteresis operators in the non-linearity as a limit of approximating fast-slow dynamics involves subtle limit procedures. In this paper, we give a proof of Netushil’s “observation” that broad classes of planar fast-slow systems with a two-dimensional critical manifold are expected to yield generalized play operators in the singular limit. We provide two proofs of this “observation” based upon the fast-slow systems paradigm of decomposition into subsystems. One proof strategy employs suitable convergence in function spaces, while the second approach considers a geometric strategy via local linearization and patching adapted originally from problems in stochastic analysis. We also provide an illustration of our results in the context of oscillations in forced planar non-autonomous fast-slow systems. The study of this example also strongly suggests that new canard-type mechanisms can occur for two-dimensional critical manifolds in planar systems.

Keywords: Fast-slow system, multiple time scale dynamics, hysteresis operator, generalized play, canard, Netushil’s observation.

1 Introduction

In this (non-technical) introduction, we are going to outline the main topic. We also present the main result and proof strategy used in this paper.

Depending upon the context, the term 'hysteresis' is used in technically different, yet strongly related, forms. Classical results in magnetic materials [SW48, Tor00] and mechanical systems [Lov13, MX91] refer to hysteresis as the description of memory effects, particularly...
with a focus on rate-independent memory \cite{MT04}; see Section 2.2 for the relevant mathematical definitions for this work and \cite{May03, BS96, KP89, Vis94, MR15} for further background literature. One observation for several classes of dynamical systems defined by hysteresis operators is that trajectories often form 'loops' in phase space as shown in Figure 1(a). A second common use of hysteresis is to describe a system exhibiting switching between two locally stable states upon parameter variation. A typical situation occurs when two fold bifurcations \cite{GH83} are connected in an S-shaped curve as shown in Figure 1(b). This hysteresis effect has been found and described in essentially all disciplines involving mathematical models ranging from neuroscience \cite{Fit55}, geoscience \cite{GR01}, engineering \cite{vdP26}, solid-state physics \cite{Str00}, ecology \cite{BHC03} to economics \cite{Cro93}, just to name a few; see also the references in \cite{Kue15}. A suitable mathematical formulation to approximate systems, in which this hysteresis effect appears, are fast-slow ordinary differential equations (ODEs), where parameters are viewed as slowly-driven variables. Also in this case, limit cycles are frequently observed, e.g., slowly moving through an S-shaped bifurcation diagram can lead to relaxation oscillations; see also Section 2.1 for more precise definitions and background on geometric singular perturbation theory (GSPT) for fast-slow systems \cite{Kue15}.

Figure 1: Sketch of two hysteresis models. (a) Trajectory for a hysteresis operator with memory; note that if we start at the lower left point, then the trajectory does not make sense as a trajectory of a planar ODE as it would violate local uniqueness. (b) Hysteresis behaviour given by a relaxation oscillation (solid curve) in a fast-slow system. The S-shaped critical manifold (dashed curve) can also be interpreted as one limiting part of the dynamics in the infinite-time scale separation limit, i.e., there are three branches of steady states, which meet at two fold bifurcations.

Although these two uses of hysteresis differ in their mathematical setup, it is evident from Figure 1 that one might expect at least some relation between the model classes for certain cases. The “observation” that certain hysteresis operators can be obtained as a singular limit from fast-slow dynamics is often attributed \cite{PS05, MOPS05} to Netushil \cite{Net68, Net70}. However, taking limits directly to relate fast-slow dynamics to systems with hysteresis operators is difficult as the limits are singular, in the sense that we have to bridge via the limiting process two different classes of equations. This difficulty is one possible motivation to look at the singular limit by exploiting gradient-type structures \cite{MR15}, if they are available, or going deeper into the theory of differential inclusions. Albeit yielding limits for certain subclasses of fast-slow systems, these approaches tend to make very strong assumptions on the structure of the system and, more importantly, the interpretation and proof strategies provided by GSPT, i.e., guided by the individual decomposition of trajectories, are lost. Hence, we are going to refer to this situation more precisely as Netushil’s conjecture, i.e., how one can rigorously unify GSPT and
the abstract hysteresis operator viewpoint, without making too stringent assumptions on the structure of the fast-slow system; see Sections 3 & 4 for the precise statements.

In this work, we give two rigorous proofs of Netushil’s conjecture for a very general class of planar fast-slow systems, which have two-dimensional critical manifolds; see also Section 3 for the technical statement of Netushil’s conjecture. We highlight that in our setting, we allow for a full coupling of fast and slow variables. We establish our results via two different techniques. The first approach is more functional-analytic, yet carefully exploits the fast-slow decomposition. The second approach is very geometric and transfers a proof-strategy initially developed for stochastic fast-slow systems to the deterministic hysteresis situation. More precisely, we show how to obtain the generalized play operator, which can be viewed as a model for perfect plasticity. In addition, to the overall GSPT approach, each of the two proofs also contains further technical advances, e.g., the use of suitable projectors to deal with small-scale oscillations or a careful matching argument near two-dimensional critical manifolds in planar systems. In summary, our results aim to provide a better bridge between two adjacent areas of mathematical modelling, and to provide a step in establishing Netushil’s conjecture, i.e., to link GSPT and hysteresis operators in even more generality; see Section 2.2 for background and comparison of our results to other approaches.

The paper is structured as follows. In Section 2 we explain the necessary background from fast-slow systems and hysteresis operators. We also provide a brief introduction to motivate our approach and compare it to other possible approaches not using GSPT and fast-slow decomposition. In Section 3 we state the first version of our main result and provide a more functional-analytic decomposition proof. In Section 4 we state a variant of the convergence result, which provides stronger convergence under stronger assumptions. The proof of this result follows a different strategy patching fast dynamics geometrically. In Section 5 we consider an application of our results to a prototypical periodically forced planar fast-slow system, which displays interesting oscillatory patterns. This example also illustrates the possible occurrence and the role of canard oscillatory trajectories.

Remark on notation: If not further specified, we always write $\| \cdot \| = \| \cdot \|_2$ for the Euclidean norm. The same remark applies to matrix norms and metrics on finite-dimensional spaces, they will all be understood in the Euclidean sense unless specified otherwise.

2 Background

In this section we provide some necessary background for the two main areas in this paper. We hope that this is going to make the results more accessible for experts in different fields. We restrict ourselves here to the case of planar systems to simplify the notation and present the main ideas. The techniques we present are expected to generalize to several classes of higher-dimensional systems.

2.1 Fast-Slow Systems

Let $(x, y) \in \mathbb{R}^2$ and consider the planar fast-slow system

\[ \varepsilon \frac{dx}{dt} = \varepsilon \dot{x} = f(x, y; \varepsilon), \]
\[ \frac{dy}{dt} = \dot{y} = g(x, y; \varepsilon), \]  

(1)
where $\varepsilon > 0$ is a small parameter indicating the time-scale separation between the fast variable $x$ and the slow variable $y$. For now we shall assume that $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are sufficiently smooth but for our main setting we shall considerably weaken this assumption and also allow the slow vector field to be non-autonomous. A classical goal in fast-slow systems is to understand the dynamics of (1) for sufficiently small $\varepsilon$ by showing persistence results from the singular limit $\varepsilon \to 0$. Setting $\varepsilon = 0$ in (1) yields the slow subsystem

$$
0 = f(x, y; 0),
\dot{y} = g(x, y; 0),
$$

which is a differential-algebraic equation on the critical set

$$
C_0 := \{ (x, y) \in \mathbb{R}^2 : f(x, y; 0) = 0 \}.
$$

Therefore, the slow subsystem (2) only covers the dynamics in a part of phase space. Another possibility is to consider the fast-slow system (1) on the fast time scale $s := \varepsilon t$ instead of the slow time scale $t$. Taking the singular limit on the fast time scale leads to the fast subsystem

$$
\frac{dx}{ds} = x' = f(x, y; 0),
\frac{dy}{ds} = y' = 0,
$$

which is a parametrized differential equation, where the slow variable $y$ acts as a parameter. The fast and slow subsystems are different types of differential equations, which illustrates the singular perturbation character of fast-slow systems. For (4) the critical set $C_0$ consists of the equilibrium points (or steady states), while it is a constraint for (2).

Several approaches to analyze fast-slow systems exist [Kue15]. Probably the most classical technique is to use asymptotic methods, such as matched asymptotic expansions [MR80, KC96, BO99]. In this approach, we aim to write the solution as

$$(x(s), y(s)) \sim \left(\sum_{k=1}^{K} x_k(s)a_k(\varepsilon), \sum_{k=1}^{K} y_k(s)b_k(\varepsilon)\right) + \mathcal{O}(b_{k+1}(\varepsilon), a_{k+1}(\varepsilon)), \quad \text{as } \varepsilon \to 0,$$

for some $K \in \mathbb{N}$, where $\{a_k(\varepsilon)\}_{k=1}^{\infty}, \{b_k(\varepsilon)\}_{k=1}^{\infty}$ are asymptotic sequences; the simplest example are polynomials $a_k(\varepsilon) = \varepsilon^k = b_k(\varepsilon)$. In asymptotic analysis, one usually has to match different series by combining solutions obtained of a slow time scale from (2) and via a fast time scale from (4). In particular, the crucial point is to exploit the decomposition algebraically.

More recently, the development of geometric singular perturbation theory (GSPT) has provided very significant additional geometric insight, how we may combine (2)–(4). In this work, we restrict to the case when the critical set $C_0$ is a manifold but for other cases see [KS01, Sch85, KS15]. Consider a compact submanifold $S_0 \subset C_0$. Then $S_0$ is called normally hyperbolic [Fen79, Jon95, Kap99] if $\partial_x f(p; 0) \neq 0$ for all $p \in S_0$; for a higher-dimensional fast variable, normal hyperbolicity is defined by requiring that the matrix $D_x f(p; 0)$ is a hyperbolic matrix. Fenichel’s Theorem [Fen79, Jon95, Wig94] guarantees that $S_0$ perturbs to a locally invariant slow manifold $S_\varepsilon$, which is $\mathcal{O}(\varepsilon)$-close to $S_0$. The dynamics on $S_\varepsilon$ is conjugate to the dynamics on $S_0$, which converts the singular perturbation problem to a regular perturbation problem near a normally hyperbolic critical manifold. In this view, asymptotic matching becomes geometric...
matching of singular limit trajectories from the fast and slow subsystems; e.g., in Figure 1(b),
we have to match two fast segments with two slow segments near the two fold points, where
the critical manifold loses normal hyperbolicity \cite{Kue15}.

In summary, the crucial point of this discussion for the current work is, that modern GSPT,
as well as more algebraic asymptotic matching, crucially exploit the views of decomposition,
scaled subsystems, and matching regions/points. There is a very detailed literature available on
many further important topics in fast-slow systems utilizing this strategy and we refer to the
literature review in \cite{Kue15} for a broader view of the area.

![Figure 2](image)

**Figure 2:** Sketch of two possible configurations of the critical manifold $C_0$ for planar fast-slow
systems. (a) Classical situation with a dimension one (and codimension one) critical manifold.
(b) Situation discussed in this paper, when $C_0$ has the same dimension as the ambient phase
space, i.e., dimension two and codimension zero; see also equation (5).

One crucial aspect of the critical manifold in planar systems is that it is generically of
dimension one (or even the empty set) if it is defined via the zeros of a generic smooth mapping
$f$. However, the disadvantage is that we cannot model two-dimensional constrained dynamics
in this context in planar fast-slow systems. For example, consider the fast variable vector field
\[
f(x, y; \varepsilon) := \begin{cases} 
-x + y & \text{if } y > x, \\
0 & \text{if } x - 1 < y < x, \\
-x + y + 1 & \text{if } y < x - 1.
\end{cases}
\]  

The critical manifold $C_0 = \{(x, y) \in \mathbb{R}^2 : x - 1 < y < x\}$ is a two-dimensional strip. The
classical normal hyperbolicity assumption does not hold for $C_0$ since the fast vector field is
identically zero inside the strip. In particular, one may ask, what happens in the singular limit
$\varepsilon \to 0$ to the flow of the fast-slow system (1) with fast vector field similar to (5). Netushil’s
observation \cite[Sec. 1.1.2]{PS05} is that hysteresis operators play a key role to capture the singular
limit in this situation.

One major obstacle to carry out GSPT directly using Fenichel’s Theorem and related results
is that one has to solve the problem in another limiting direction, which is not customary for
the GSPT approach. More precisely, the limit $\varepsilon = 0$ is unknown, whereas it is usually assumed
in GSPT that it is easier to analyze the fast and slow subsystems, i.e., usually the difficult
part is to control the case $0 < \varepsilon \ll 1$. In the context of Netushil’s observation, the geometry
is significantly worse as we only have $\partial C_0$ as the main geometric object available with just
one-sided estimates near $\partial C_0$ for the fast dynamics. Additionally, observe that we work in the
coupled setting of a fast-slow system rather than only with one equation with an input function
as in the classical statement for Netushil’s observation. Here we overcome these difficulties using
a suitable local decomposition of trajectories as well as projections to preserve the geometric viewpoint.

Other approaches to the problem use very different technical approaches, and we briefly review these approaches in the next section, which also contains some basic background on hysteresis operators.

2.2 Hysteresis Operators

Amongst others, hysteresis effects appear in physics in fields like ferromagnetism, ferroelectricity or plasticity [May03, BS96, KP89, Vis94, MR15]. They can also be observed in shape memory effects of certain materials, they are relevant for thermostats in engineering [Vis94], and are used for modelling of certain systems in mathematical biology [GST13, HJ80, Kop06, PKK12, CGT16]. Mathematically, these effects can be described by hysteresis operators such as the scalar play [BK15], the scalar stop [BR05] or the Prandtl-Ishlinskii operator [Kuh03], to only name a few.

We focus on scalar hysteresis operators in this paper. Given a time interval \([0, T]\), scalar hysteresis operators take an admissible time-dependent input function \(y : [0, T] \rightarrow \mathbb{R}\) together with an initial value \(x_0\) and return a time-dependent output function \(x = x(y, x_0)\), where we can also view \(x\) as a map \(x : [0, T] \rightarrow \mathbb{R}\) if \(x_0\) is fixed. All scalar rate-independent hysteresis operators have two properties in common [Vis94, BS96]:

Definition 2.1.

(Vol) The output function \(x(t)\) at time \(t \in [0, T]\) may depend not only on the value of the input function \(y(t)\) at time \(t\), but on the whole history of \(y\) in the interval \([0, t]\). This non-locality in time is often referred to as memory effect, causality or Volterra property: for all \(y_1, y_2\) in the domain of the operator, for all initial values \(x_0\), and any \(t \in [0, T]\) it follows that if \(y_1 = y_2\) in \([0, t]\), then \([x(y_1, x_0)](t) = [x(y_2, x_0)](t)\); cf. [Vis94, Chapter III].

(RI) The output function \(x\) is invariant under time transformations. This means that for any monotone increasing and continuous function \(\phi : [0, T] \rightarrow [0, T]\) with \(\phi(0) = 0\) and \(\phi(T) = T\) and for all admissible input functions \(y\) it holds

\[x(y \circ \phi, x_0)(t) = x(y, x_0)(\phi(t)), \quad \forall t \in [0, T].\]

In [Vis94, Chapter III], the function \(\phi\) is also assumed to be bijective, i.e., the definitions differ in the literature. For our purpose one may consider either definition of admissible time transformations. Invariance under time transformations is also called rate-independence in the literature [MR15, Definition 1.2.1].

Furthermore, we recall that hysteresis operators may not be described by planar differential equations. To see this, consider an ODE of the form

\[
\frac{d}{dt}x_i(t) = f(x_i(t), y_i(t)), \quad x_i(0) = x_{0,i}, \quad i \in \{1, 2\}
\]

for two different input functions \(y_1, y_2\) and initial values \(x_{0,1}, x_{0,2}\). If for a given time \(t \in [0, T]\) it holds \(x_1(t) = x_2(t)\) and if \(y_1|_{[0,t]} = y_2|_{[0,t]}\), then clearly \(x_1|_{[0,T]} = x_2|_{[0,T]}\) no matter if \(y_1|_{[0,t]} \neq y_2|_{[0,t]}\).
Figure 3: Typical finite-time trajectory of $y$ and $x(y, x_0)$ in the $(y, x)$-plane. The initial and final points are marked with black dots. Note that we allow quite general curves $F_+, F_-$ as long as the graphs are increasing and Lipschitz.

$y_2|_{[0,t]}$ or even $x_{0,1} \neq x_{0,2}$. Therefore, the Volterra property cannot be captured in general. The solutions $x_1, x_2$ of the ODEs on the remaining time interval $(t, T]$ depend only on the current values $x_1(t)$ and $x_2(t)$ at time $t$ and on the behaviour of the input functions $y_1$ and $y_2$ in the interval $(t, T]$; see also Figure 1(a). Differential equations like

$$\frac{dx(t)}{dt} = f(x(t), y(t)), \quad x(0) = x_0,$$

are also not rate-independent. To see this, consider a time transformation $\phi$. Then for the solution operator $x = x(y, x_0)$ of the differential equation it holds

$$x(y \circ \phi, x_0)(t) = x_0 + \int_0^t f(x(s), y(\phi(s))) \, ds,$$

whereas we find that

$$[x(y, x_0) \circ \phi](t) = x_0 + \int_0^{\phi(t)} f(x(s), y(s)) \, ds$$

for $t \in [0, T]$. In general those two functions do not coincide. In summary, the appearance of differential equations with hysteresis as the singular limit of systems of differential equations brings along completely new features in the evolution dynamics.

The class of hysteresis operators represents merely a part of the more general class of rate-independent processes. There are various (often equivalent) ways to represent such processes. In this paper, we will be concerned with so-called scalar generalized play operators which we introduce in the following \cite[Chapter III.2]{Vis94}. Let $F_-, F_+: \mathbb{R} \rightarrow \mathbb{R}$ be two increasing functions with $F_- \leq F_+$. If $F_-$ and $F_+$ are Lipschitz continuous, then one way to represent the solution $x = x(y, x_0) \in W^{1,p}(0, T)$ of the generalized play operator, which corresponds to the functions $F_-, F_+$ with input $y \in W^{1,p}(0, T)$, $1 \leq p \leq \infty$, and $x_0 \in \mathbb{R}$, is the solution of the following variational inequality:

$$\dot{x}(t)(x(t) - \xi) \leq 0 \quad \forall \xi \in [F_-(y(t)), F_+(y(t))], \text{ a.e. in } (0, T),$$

$$x(t) \in [F_-(y(t)), F_+(y(t))],$$

$$x(0) = \min \{ \max \{F_-(y(0)), x_0\}, F_+(y(0)) \}.$$
We do not list all the properties of a generalized play operator here, but refer to the literature for certain features once we need them. The typical behaviour of the solution $x(y, x_0)$ is depicted in Figure 3. The appearance of rate-independent systems as the (singular) limit of regularized problems has been analyzed via several approaches. We only refer to several results, which are related to our work, but emphasize that many other research directions in the area of hysteresis operators are currently being pursued.

The first equation in (1) given by

$$\varepsilon \dot{x} = f(x, y; \varepsilon)$$

(7)

can be viewed as one possible regularization of (6). Equivalent to (6), $x(y, x_0)$ can be represented by a differential inclusion, see e.g. [Vis94] or [BS96]. Many rate-independent processes are equally described by an energetic formulation via a (local/global) stability condition and an energy balance condition [MR15, Chapter 2 or Chapter 3]. This leads to the notion of local/global energetic solutions. There are several other concepts used to describe rate-independent processes. We refer to [Mie11] or [MRS12] for overviews.

In the context of energetic solutions, and the representation of the corresponding solutions by rate-independent differential inclusions, one often considers the so-called vanishing-viscosity limit, see [MRS12]. The latter is achieved by regularizing the rate-independent differential inclusion, frequently by a term modelling viscosity. The solutions of $\varepsilon$-dependent regularized problems are then proved to converge to an energetic solution of the initial problem. One goal of the approach is to reveal properties of the energetic solution such as the behaviour at discontinuity points, see [MRS12].

Consider (7) with a given function $y$. Special choices of $f$, $F_+$ and $F_-$ lead to a convex and/or coercive energy functional for the regularized, as well as for the limit problem. In this case, uniform-in-$\varepsilon$-estimates of the norm of $x_\varepsilon$ by the norm of $y$ or $\dot{y}$ in the appropriate spaces can be derived [MR15, Chapter 1.7 or Chapter 3.8] or [MRS12]. Moreover, either (I) an equi-continuity estimate for $\{x_\varepsilon\}_\varepsilon$, (II) a uniform-in-$\varepsilon$ bound of (some) norm of $\dot{x}_\varepsilon$, or (III) a bound of the total variation of $\{x_\varepsilon\}_\varepsilon$ by $y$ and $\dot{y}$ can be derived. For (I), a suitable application of the Arzelà-Ascoli Theorem yields the desired convergence of $x_\varepsilon$ to a singular limit $x(y, x_0)$. For (II) or (III), a weak compactness argument together with Helly’s selection theorem can be used to prove convergence. For special choices of $f$, $F_+$ and $F_-$, it should be possible to use them also for the coupled system (1). Our situation may include non-convex and non-coercive energy functionals as well as systems without any energy structure.

Furthermore, in our setting, in the formulation as variational inequalities, the generality of $f$, $F_+$ and $F_-$ together with the coupling in the fast-slow system (1) complicates the limit procedure, since uniform-in-$\varepsilon$-estimates for $\dot{x}_\varepsilon$, even in $L^1(0,T)$, can no longer be derived. We bypass this problem by projecting $x_\varepsilon$ in $y-x$-phase space vertically onto the set $C_0$ of roots of $f$, i.e., we employ the geometric one-sided limit available for the critical manifold as discussed above. For the projection $p_\varepsilon$ we can even show $W^{1,\infty}(0,T)$-bounds, which are uniform in $\varepsilon$. That is, the question of convergence of $x_\varepsilon$ and $y_\varepsilon$ to $x(y, x_0)$ and $y$ is shifted to the problem of showing that $x_\varepsilon - p_\varepsilon$ converges to zero in the limit $\varepsilon \rightarrow 0$ in an appropriate space, and that the limit $x(y, x_0) = \lim_{\varepsilon \rightarrow 0} p_\varepsilon$ follows the hysteresis law (6), where $y = \lim_{\varepsilon \rightarrow 0} y_\varepsilon$ solves the fast equation in (1) with $x = x(y, x_0)$. The latter can be proved in our setting.
Another result about hysteresis as the singular limit in ODEs, which can be compared to our problem is derived in [Kre05] assuming the Liénard case

\[ f(x, y; \varepsilon) = -F(x) + y. \]

The source function \( y \) is a priori fixed, i.e., independent of \( \varepsilon \) and independent of fast-slow coupling. Furthermore, the set \( C_0 \) of this problem has co-dimension 1, different from our setting, where major difficulties appear as \( C_0 \) has co-dimension 0. Another main difference in [Kre05] is that the singular limit \( x_\varepsilon \) with \( y \) independent of \( \varepsilon \) is the output of a hysteresis operator of switch type with input \( y \). Since the solutions \( x_\varepsilon \) are continuous, while the limit \( x \) is in general discontinuous, uniform convergence in classical spaces cannot be expected in [Kre05]. However, uniform convergence results have been obtained for the concept of \( r \)-convergence in the space of regulated functions involving uniform bounds. We completely bypass the problem of showing uniform-in-\( \varepsilon \) bounds for the oscillations or for the derivative of the solutions \( x_\varepsilon \) in [1] by introducing suitable projection functions \( p_\varepsilon \) below.

In fact, our approach substantially differs from previous approaches already by its fundamental principles and the direct relation to the fast-slow GSPT viewpoint. It also provides several new technical tools, and it does not require specialized spaces, ODE structure assumptions, or the existence of an energy. In particular, this setting is designed to link GSPT and fast-slow systems a lot more directly to hysteresis limits than has been possible so far.

3 The Main Result I - Functional Approach

The main system we are going to study is a planar fast-slow system with the added generalization that the slow vector field may also depend upon time

\[ \varepsilon \dot{x} = f(x, y), \]
\[ \dot{y} = g(x, y, t), \]

augmented with the initial condition \((x(0), y(0)) = (x_0, y_0)\). Our argument also allows for the generalization \( g(x, y, t) = g(x, y, t; \varepsilon) \) and \( f(x, y) = f(x, y; \varepsilon) \) as long as \( f, g \) are bounded for all \( \varepsilon \in [0, \varepsilon_0] \) for some sufficiently small \( \varepsilon_0 > 0 \). We shall not make this additional generalization explicit and just omit the \( \varepsilon \)-dependence in \( f, g \). We denote the solutions of (8)-(9) by \((x_\varepsilon(t), y_\varepsilon(t))\) to emphasize the dependence upon \( \varepsilon \). We are interested in the limit as \( \varepsilon \to 0 \) on the finite time interval \( J_T := (0, T) \). The situation outlined already in Sections 2.1-2.2 is made precise by the following main assumptions on \( f, g \):

(A1) \( f, g \) are Lipschitz continuous with Lipschitz constants \( L_f \) and \( L_g \) respectively; we can also allow only local Lipschitz continuity in our results but refrain from doing so as it just complicates the notation.

(A2) The function \( f \) satisfies

\[ f(x, y) < 0 \quad \text{if } x > F_+(y), \]
\[ f(x, y) = 0 \quad \text{if } x \in [F_-(y), F_+(y)], \]
\[ f(x, y) > 0 \quad \text{else.} \]

for two functions \( F_+ > F_- \).
Figure 4: Sign behaviour of $f$ and projection to $p_\varepsilon$ in the $(y, x)$-plane. Note that the critical manifold $\mathcal{C}_0 = \{f = 0\}$ is two-dimensional and bounded by the two curves $F_\pm$; see also assumption (A2). The projection along the vertical fast coordinate to $\mathcal{C}_0$ is denoted by $p_\varepsilon$ and only acts on the fast variable.

(A3) $F_+, F_-$ are Lipschitz continuous with Lipschitz constants $L_+$ and $L_-$. We define $L_\pm := \max\{L_+, L_-\}$. $F_+, F_-$ are monotone increasing functions.

(A4) $g$ satisfies the growth assumption

$$|g(x, y, t)| \leq M(t)(1 + G(x) + |y|) \quad (11)$$

with $M > 0$ continuous and $G$ bounded.

Assumption (A1) essentially assures the existence of unique (local) solutions of the system (8)-(9). (A2) forces the solutions $(x_\varepsilon, y_\varepsilon)$ to converge to the critical manifold $\mathcal{C}_0$ as $\varepsilon \to 0$, in particular, the signs of $f$ are given so that $\mathcal{C}_0$ can be viewed as attracting with respect to any fast trajectory movement. Assumption (A3) is going to yield the required Lipschitz continuity of the corresponding limiting generalized play operator and enables us to carry over Lipschitz bounds of $y_\varepsilon$, which are independent of $\varepsilon$, to $F_+(y_\varepsilon)$ and $F_-(y_\varepsilon)$. Amongst others, (A2) and (A3) together are eventually going to allow us to show bounds of $x_\varepsilon$ in $C(J_T) \subset C^0(J_T)$, which are independent of $\varepsilon$. In particular, it provides a growth bound of the slow variable, which could just become unbounded, i.e., other growth bounds are expected to work as well.

Since the derivative of $x_\varepsilon$ cannot be bounded independently of $\varepsilon$, we introduce a projection function for which such a bound can be derived. As shown in Figure 4, let $p_\varepsilon = p(x_\varepsilon, y_\varepsilon)$ be defined by

$$p(x_\varepsilon, y_\varepsilon) := \min\{\max\{x_\varepsilon, F_-(y_\varepsilon)\}, F_+(y_\varepsilon)\}.$$ 

The family of projections $p_\varepsilon$ is continuous because $F_-, F_+, x_\varepsilon, y_\varepsilon$, max and min are all continuous. The main theorem we are going to prove in several steps below is the following:

**Theorem 3.1.** Suppose the assumptions (A1)-(A4) hold and let $q \in (1, +\infty)$. Denote by $(\overline{\tau}, \overline{\eta})$
the unique solution of
\begin{align}
\dot{y}(t) &= g(x(t), y(t), t) &\text{a.e. in } J_T, \\
y(0) &= y_0, \\
\dot{x}(t)(x(t) - \xi) &\leq 0 &\forall \xi \in [F_-(y(t)), F_+(y(t))], \text{ a.e. in } J_T, \\
x(0) &= \min\{\max\{F_-(y_0), x_0\}, F_+(y_0)\}, \\
x(t) &\in [F_-(y(t)), F_+(y(t))] &\text{in } J_T.
\end{align}

Then we have \( \bar{x} \in W^{1,\infty}(J_T) \) and \( \bar{y} \in C^1(J_T) \cap C(J_T) \). Moreover, the main convergence result is

\[
x_\varepsilon \to \bar{x} \text{ in } L^q(J_T) \text{ and } y_\varepsilon \to \bar{y} \text{ in } W^{1,q}(J_T)
\]
as \( \varepsilon \to 0 \), i.e., the singular limit \( \bar{x} \) is the solution of a generalized play operator for the curves \( F_+ \) and \( F_- \) with input \( \bar{y} \).

Theorem 3.1 shows that the singular limit of the non-autonomous planar fast-slow system is \( [8] \) with \( [9] \) replaced by a hysteresis operator. An example of a trajectory of a generalized scalar play operator is shown in Figure 3. Note that the time-dependence of the slow vector field can indeed generate highly nontrivial dynamics inside \( C_0 \). Of course, Theorem 3.1 only provides a partial solution of Netushil’s conjecture as we have not characterized all classes of hysteresis operators, which arise as singular fast-slow limits. We need to derive several important auxiliary results before we can prove Theorem 3.1.

**Lemma 3.2.** Fix \( \varepsilon > 0 \) and assume \( [A1], [A4] \) hold. Then we can bound \( y_\varepsilon \in C(J_T) \cap C^1(J_T) \) and \( p_\varepsilon \) in \( W^{1,\infty}(J_T) \) independent of \( \varepsilon \). More precisely, there exists a constant \( K_1 > 0 \) such that

\[
\|y_\varepsilon\|_{C(J_T)} + \|y_\varepsilon\|_{C^1(J_T)} + \|p_\varepsilon\|_{W^{1,\infty}(J_T)} \leq K_1.
\]

More precisely, there exists a constant \( K_2 > 0 \) such that

\[
\|x_\varepsilon - p_\varepsilon\|_{C(J_T)} \leq K_2.
\]

The constants \( K_1, K_2 > 0 \) are independent of \( \varepsilon \).

**Proof.** For this proof, let \( c > 0 \) denote a generic constant, which does not depend on \( \varepsilon \). First, we are going to show that \( y_\varepsilon \) is bounded in \( C(J_T) \cap C^1(J_T) \). In a second step, we prove that \( p_\varepsilon \) is bounded in \( W^{1,\infty}(J_T) \). By assumption \( [A4] \) we obtain an estimate of the form

\[
|y_\varepsilon(t)| \leq |y_0| + \int_0^t M(s)(1 + G(x_\varepsilon(s)) + |y_\varepsilon(s)|) \, ds \leq c_0 + c_1 \int_0^t |y_\varepsilon(s)| \, ds
\]

for some constants \( c_0, c_1 > 0 \); note that the constants in \( [19] \) may grow in time as \( M \) is time-dependent but the constants remain finite for every fixed final time \( T > 0 \). Applying Gronwall’s Lemma to \( [19] \) implies \( \|y_\varepsilon\|_{C(J_T)} \leq c \). Equation \( [9] \) and assumption \( [A4] \) yield \( \|y_\varepsilon\|_{C^1(J_T)} \leq c \). For further use we note that this implies that \( y_\varepsilon \) is Lipschitz continuous on \( J_T \) with a Lipschitz constant independent of \( \varepsilon \). Regarding \( p_\varepsilon \), it follows from the definition of the projection that

\[
p_\varepsilon(t) \in [F_-(y_\varepsilon(t)), F_+(y_\varepsilon(t))].
\]
Since $F_-$ and $F_+$ are Lipschitz continuous by (A3) we obtain from (20) that $\|p_\varepsilon\|_{C(J_T)} \leq c$. It remains to show that the norm of $p_\varepsilon$ in $W^{1,\infty}(J_T)$ can be bounded independently of $\varepsilon$. Again from the definition of $p_\varepsilon$ it follows for a.e. $t \in J_T$ that
\[
p_\varepsilon(t) \in (F_-(y_\varepsilon(t)), F_+(y_\varepsilon(t))) \quad \text{if and only if} \quad \dot{p}_\varepsilon(t) = 0 = \dot{x}_\varepsilon(t),\]
and otherwise, $p_\varepsilon(t) \in \{F_-(y_\varepsilon(t)), F_+(y_\varepsilon(t))\}$. Assumption (A3) together with the bounds for $y_\varepsilon$, which we have shown already, yield that $F_-(y_\varepsilon(\cdot))$ and $F_+(y_\varepsilon(\cdot))$ are Lipschitz continuous with Lipschitz constant independent of $\varepsilon$. Let $t_1 < t_2$ be given and suppose $p_\varepsilon(t_1) = F_+(y_\varepsilon(t_1))$. If $p_\varepsilon(t) = F_+(y_\varepsilon(t))$ in the time interval $[t_1, t_2]$ then
\[
|p_\varepsilon(t_1) - p_\varepsilon(t_2)| = |F_+(y_\varepsilon(t_1)) - F_+(y_\varepsilon(t_2))| \leq c|t_1 - t_2|
\]
by Lipschitz continuity of $F_+(y_\varepsilon(\cdot))$. Otherwise, there exist times $t^{(1)} \in [t_1, t_2)$ and $t^{(2)} \in (t^{(1)}, t_2]$ such that
\[
x_\varepsilon(t) \geq F_+(y_\varepsilon(t)) \quad \forall t \in [t_1, t^{(1)}] \quad \text{and} \quad x_\varepsilon(t) = F_+(y_\varepsilon(t^{(1)})) \quad \forall t \in [t^{(1)}, t^{(2)}].
\]
Note that in the case when $t_1 = t^{(1)}$ we imply the empty set if we write $[t_1, t^{(1)}]$. In this setting we find
\[
|p_\varepsilon(t_1) - p_\varepsilon(t^{(2)})| = |F_+(y_\varepsilon(t_1)) - F_+(y_\varepsilon(t^{(1)}))| \leq c|t_1 - t^{(1)}| \leq c|t_1 - t^{(2)}|.
\]
We first choose $t^{(1)} \in [t_1, t_2)$ and then $t^{(2)} \in (t^{(1)}, t_2]$ both maximal. If $t^{(2)} \neq t_2$ and $F_+(y_\varepsilon(t^{(1)})) = F_-(y_\varepsilon(t^{(2)}))$ then we apply the same reasoning with $F_-$ on the interval $[t^{(2)}, t_2]$. We obtain a partition $t^{(0)} = t_1 \leq t^{(1)} \leq \cdots \leq t^{(k)} = t_2$ of $[t_1, t_2]$ such that
\[
|p_\varepsilon(t_1) - p_\varepsilon(t_2)| \leq \sum_{i=1}^{k} |p_\varepsilon(t^{(i-1)}) - p_\varepsilon(t^{(i)})| \leq c \sum_{i=1}^{k} |t^{(i-1)} - t^{(i)}| = c|t_1 - t_2|.
\]
This implies that $p_\varepsilon$ is Lipschitz continuous independent of $\varepsilon$. By a standard embedding theorem [Eva02, Chapter 5.8.2.b, Theorem 4] it follows that
\[
\|p_\varepsilon\|_{W^{1,\infty}(J_T)} \leq c.
\]
This concludes the proof of the first bound (17) in the result. It remains to show that $\|x_\varepsilon - p_\varepsilon\|_{C(J_T)}$ is bounded. We calculate
\[
|x_\varepsilon(t) - p_\varepsilon(t)| = |x_\varepsilon(0) - p_\varepsilon(0)| + \int_0^t (\dot{x}_\varepsilon(s) - \dot{p}_\varepsilon(s)) \frac{x_\varepsilon(s) - p_\varepsilon(s)}{|x_\varepsilon(s) - p_\varepsilon(s)|} \, ds
\]
\[
= |x(0) - p(0)| + \frac{1}{\varepsilon} \int_0^t f(x_\varepsilon(s), y_\varepsilon(s)) \frac{x_\varepsilon(s) - p_\varepsilon(s)}{|x_\varepsilon(s) - p_\varepsilon(s)|} \, ds - \int_0^t \dot{p}_\varepsilon(s) \frac{x_\varepsilon(s) - p_\varepsilon(s)}{|x_\varepsilon(s) - p_\varepsilon(s)|} \, ds.
\]
By the stability assumption (A2) and the definition of $p_\varepsilon$ it holds
\[
f(x_\varepsilon(s), y_\varepsilon(s)) \frac{x_\varepsilon(s) - p_\varepsilon(s)}{|x_\varepsilon(s) - p_\varepsilon(s)|} \leq 0
\]
(21).
for all \( s \in [0, t] \), cf. Figure 4. Hence, it follows that
\[
|x_\varepsilon(t) - p_\varepsilon(t)| - \frac{1}{\varepsilon} \int_0^t f(x_\varepsilon(s), y_\varepsilon(s)) \frac{x_\varepsilon(s) - p_\varepsilon(s)}{|x_\varepsilon(s) - p_\varepsilon(s)|} \, ds \leq |x(0) - p(0)| + \int_0^t |\dot{p}_\varepsilon(s)| \, ds \tag{22}
\]
and the left side is greater or equal than zero. We have already shown that the right side in (22) is bounded independently of \( \varepsilon \). Consequently, the estimate (22) implies
\[
\|x_\varepsilon - p_\varepsilon\|_{C(J_T)} \leq c,
\]
which finishes the proof.

Since the bounds in Lemma 3.2 are independent of \( \varepsilon \), we can proceed by a compactness argument and prove convergence results for appropriate subsequences.

**Lemma 3.3.** Fix any \( q \in (1, +\infty) \) and assume (A1)-(A4), then the following results hold:

(C1) \( \lim_{\varepsilon \to 0} \|x_\varepsilon - p_\varepsilon\|_{L^q(J_T)} = 0 \).

(C2) Given any sequence \( \{\varepsilon_j\}_{j=1}^\infty \) such that \( \varepsilon_j \to 0 \) as \( j \to +\infty \), there exists a subsequence \( \{\varepsilon_{j_k}\}_{k=1}^\infty \) and functions \( \overline{x}, \overline{y} \in W^{1,q}(J_T) \) such that
\[
p_{\varepsilon_{j_k}} \to \overline{x} \quad \text{and} \quad y_{\varepsilon_{j_k}} \to \overline{y} \tag{23}
\]
as \( \varepsilon_k \to 0 \) weakly in \( W^{1,q}(J_T) \) and strongly in \( C(J_T) \).

(C3) In \( L^q(J_T) \) as \( \varepsilon_k \to 0 \), we have
\[
x_{\varepsilon_{j_k}} \to \overline{x}. \tag{24}
\]

**Proof.** In order to prove (C1) we compute for arbitrary \( \varepsilon > 0 \) and \( t \in J_T \)
\[
(x_\varepsilon(t) - p_\varepsilon(t))^2 = (x(0) - p(0))^2 + 2 \int_0^t (\dot{x}_\varepsilon(s) - \dot{p}_\varepsilon(s))(x_\varepsilon(s) - p_\varepsilon(s)) \, ds
\]
\[
= (x(0) - p(0))^2 + 2 \int_0^t f(x_\varepsilon(s), y_\varepsilon(s))(x_\varepsilon(s) - p_\varepsilon(s)) \, ds
\]
\[
- 2 \int_0^t \dot{p}_\varepsilon(s) (x_\varepsilon(s) - p_\varepsilon(s)) \, ds.
\]
This calculation together with Lemma 3.2 and (21) yields
\[
0 \leq (x_\varepsilon(t) - p_\varepsilon(t))^2 - 2 \int_0^t f(x_\varepsilon(s), y_\varepsilon(s))(x_\varepsilon(s) - p_\varepsilon(s)) \, ds \leq c, \tag{25}
\]
where \( c > 0 \) is, as before, a generic constant independent of \( \varepsilon \). The bounds in (25) and the sign condition (21) imply
\[
f(x_\varepsilon(s), y_\varepsilon(s))(x_\varepsilon(s) - p_\varepsilon(s)) \to 0
\]
for a.e. \( s \in J_T \) as \( \varepsilon \to 0 \). By definition of \( p_\varepsilon \) and assumption (A2) we conclude that \( x_\varepsilon(s) - p_\varepsilon(s) \) tends to zero for a.e. \( s \in J_T \) as \( \varepsilon \to 0 \). This result, together with (18) and the Lebesgue dominated convergence theorem, implies
\[
\lim_{\varepsilon \to 0} \|x_\varepsilon - p_\varepsilon\|_{L^q(J_T)} = 0,
\]
for any \( q \in (1, +\infty) \), which concludes the proof of \((C1)\). It remains to show \((C2)\)-(C3) concerning subsequences for a given sequence \( \{\varepsilon_j\}_{j=1}^{\infty} \) with \( \lim_{j \to \infty} \varepsilon_j = 0 \). By Lemma 3.2.2 the functions \( y_\varepsilon \) and \( p_\varepsilon \) are bounded in \( W^{1,\infty}(J_T) \) independently of \( \varepsilon \) and hence are in \( W^{1,q}(J_T) \) for any \( q \in (1, +\infty) \). Using the Banach-Alaoglu Theorem \cite{Rud91}, it follows that there is a subsequence \( \{\varepsilon_{j_k}\}_{k=1}^{\infty} \) of \( \{\varepsilon_j\}_{j=1}^{\infty} \) and that there exist functions \( \overline{\varphi}, \overline{\eta} \in W^{1,q}(J_T) \) such that

\[
p_{\varepsilon_{j_k}} \to \overline{\varphi} \quad \text{and} \quad y_{\varepsilon_{j_k}} \to \overline{\eta}
\]

as \( \varepsilon_{j_k} \to 0 \) weakly in \( W^{1,q}(J_T) \). Because \( W^{1,q}(J_T) \) is compactly embedded in \( C(J_T) \) for \( 1 < q \leq \infty \) \cite{AF03} Theorem 6.3], this convergence is strong in \( C(J_T) \). Since \( x_\varepsilon - p_\varepsilon \to 0 \) in \( L^q(J_T) \) we finally conclude \( x_{\varepsilon_{j_k}} \to \overline{\varphi} \) in \( L^q(J_T) \) as \( \varepsilon_{j_k} \to 0 \).

Having proven that subsequences of \( x_\varepsilon, y_\varepsilon \) and \( p_\varepsilon \) actually converge in a certain sense, we would like to understand the behaviour of the limit functions. We can also improve the type of convergence of the subsequence \( \{y_{\varepsilon_k}\}_{k=1}^{\infty} \).

**Lemma 3.4.** Consider the same assumptions and the notation of Lemma 3.3. The functions \( \overline{\varphi} \) and \( \overline{\eta} \) solve the system

\[
\begin{align*}
\dot{y}(t) &= g(x(t), y(t), t) \quad \text{a.e. in } J_T, \quad (27) \\
y(0) &= y_0, \quad (28) \\
\dot{x}(t)(x(t) - \xi) &\leq 0 \quad \forall \xi \in [F_-(y(t)), F_+(y(t))], \ \text{a.e. in } J_T, \quad (29) \\
x(0) &= \min\{\max\{F_-(y_0), F_+(y_0)\}\}, \quad (30) \\
x(t) &\in [F_-(y(t)), F_+(y(t))] \quad \text{in } \overline{J_T}. \quad (31)
\end{align*}
\]

Furthermore, \( y_{\varepsilon_k} \to \overline{\eta} \) in \( W^{1,q}(0,T) \) as \( k \to \infty \), i.e., we have strong convergence in \( W^{1,q}(0,T) \) of the slow dynamics for a subsequence.

**Proof.** We first show that \( \overline{\eta} \) solves \((27)-(28)\) with \( x = \overline{\varphi} \) and improve the convergence of \( \{y_{\varepsilon_k}\}_{k=1}^{\infty} \). Lemma 3.3, assumptions \((A1)\) and \((A4)\) and the Lebesgue dominated convergence theorem yield that

\[
g(x_{\varepsilon_k}, y_{\varepsilon_k}, t) \to g(\overline{\varphi}, \overline{\eta}, t)
\]

in \( L^q(J_T) \) as \( \varepsilon_k \to 0 \). By Lemma 3.3, \( y_{\varepsilon_k} \) converges to \( \overline{\eta} \) uniformly in \( J_T \). For \( t \in \overline{J_T} \) we obtain by using \( y_{\varepsilon_k} = y_0 + \int_0^t g(x_{\varepsilon_k}, y_{\varepsilon_k}, s) \, ds \) that

\[
\left| \overline{\eta}(t) - y_0 - \int_0^t g(\overline{\varphi}, \overline{\eta}, s) \, ds \right| \leq |\overline{\eta}(t) - y_{\varepsilon_k}(t)| + \int_0^t |g(\overline{\varphi}, \overline{\eta}, s) - g(x_{\varepsilon_k}, y_{\varepsilon_k}, s)| \, ds \to 0 \quad (33)
\]

as \( \varepsilon_k \to 0 \) by using Cauchy-Schwarz and \( t < +\infty \) for the last term to get convergence in \( L^1 \). This shows that \( \overline{\eta} \) solves \((27)-(28)\) with \( x = \overline{\varphi} \). Together with \((32)\) we may conclude that \( y_{\varepsilon_k} \to \overline{\eta} \) in \( W^{1,q}(0,T) \).

Next, we are going to show that \( \overline{\varphi} \) solves \((29)-(31)\) with \( y = \overline{\eta} \). First, we will deal with \((30)-(31)\). By definition of \( p_\varepsilon \) and with Lemma 3.3 we have

\[
\overline{\varphi}(0) = \lim_{k \to \infty} p_{\varepsilon_k}(0) = \min\{\max\{x_0, F_-(y_0)\}, F_+(y_0)\}
\]

as well as

\[
\overline{\varphi}(t) \in [F_-(\overline{\eta}(t)), F_+(\overline{\eta}(t))] \quad \forall \ t \in \overline{J_T},
\]

\[
\overline{\eta} \in W^{1,q}(0,T)
\]

\[
\overline{\eta} \in C(J_T)
\]

\[
\overline{\eta} \in L^q(J_T)
\]
which proves (30)–(31). Hence, it remains to show the variational inequality (29), which we accomplish in two steps. First, we deal with initial data in the interior of the critical manifold $\mathcal{C}_0$ and in a second step we are going to consider dynamics on the boundary. Fix $t_0 \in \overline{T}$ and suppose we start in the interior

$$\overline{\pi}(t_0) \in (F_-(\overline{y}(t_0)), F_+(\overline{y}(t_0))).$$

Continuity of $\pi$, $F_-(\overline{y}(\cdot))$ and $F_+(\overline{y}(\cdot))$ implies that there is some interval $J \subset J_T$ with $t_0 \in J$ such that $\overline{\pi}(t) \in (F_-(\overline{y}(t)), F_+(\overline{y}(t)))$ for all $t \in J$. We define the distance to the boundary as

$$\delta_J := \min_{t \in J}\{\overline{\pi}(t) - F_-(\overline{y}(t)), F_+(\overline{y}(t)) - \overline{\pi}(t)\}.$$

By Lemma 3.3 and assumption [A3] we can find some $\varepsilon(0) > 0$ such that for all $\varepsilon_k < \varepsilon(0)$ and all $t \in J$ the following estimate holds

$$|F_-(y_{\varepsilon_k}(t)) - F_-(\overline{y}(t))| + |F_+(y_{\varepsilon_k}(t)) - F_+(\overline{y}(t))| + |p_{\varepsilon_k}(t) - \overline{\pi}(t)| < \frac{\delta_J}{4}.$$ (36)

This implies $p_{\varepsilon_k}(t) \in (F_-(y_{\varepsilon_k}(t)), F_+(y_{\varepsilon_k}(t)))$ as well as

$$\min_{t \in J}\{p_{\varepsilon_k}(t) - F_-(y_{\varepsilon_k}(t)), F_+(y_{\varepsilon_k}(t)) - p_{\varepsilon_k}(t)\} \geq \frac{\delta_J}{2}.$$

for all $\varepsilon_k < \varepsilon(0)$ and $t \in J$. By definition of the projection $p_{\varepsilon_k}$ we immediately find $p_{\varepsilon_k}(t) = x_{\varepsilon_k}(t)$ and $\dot{p}_{\varepsilon_k}(t) = 0$ for a.e. $t \in J$ so that $p_{\varepsilon_k}(t) = p_{\varepsilon_k}(t_0)$ for $t \in J$ and the variational inequality is just satisfied with zero almost everywhere in $J$. As the second step, suppose we start on the boundary of the critical manifold which occurs e.g. if $\overline{\pi}(t_0) = F_+(\overline{y}(t_0))$. Then there is some interval $J$ with $t_0 \in J$ such that $\overline{\pi}(t) > F_-(\overline{y}(t))$ for all $t \in J$; note that we slightly overload the notation here and again use $J$. A similar argument leads to (36) and the conclusion that for all for all $\varepsilon_k < \varepsilon(0)$ and $t \in J$ we now have

$$p_{\varepsilon_k}(t) > F_-(y_{\varepsilon_k}(t)) + \frac{\delta_J}{2}.$$

By definition of $p_{\varepsilon_k}$ it follows $x_{\varepsilon_k}(t) \geq F_-(y_{\varepsilon_k}(t))$ and $\dot{x}_{\varepsilon_k}(t) \leq 0$ for a.e. $t \in J$. In particular, we have a negative sign for $\dot{x}_{\varepsilon_k}(t)$, which we want to transfer to the limit and show that

$$\overline{\pi} \leq 0, \quad \text{a.e. in } J.$$

(37)

To prove (37), assume in contradiction that

$$\overline{\pi}(t_2) > \overline{\pi}(t_1) \quad \text{for some } t_1, t_2 \in J, \ t_1 < t_2.$$ (38)

Then there exists $\varepsilon(1) > 0$ such that $p_{\varepsilon_k}(t_2) > p_{\varepsilon_k}(t_1)$ if $\varepsilon_k < \varepsilon(1)$. Using the Lipschitz continuity of the $p_{\varepsilon_k}$ from Lemma 3.2 we can find $t^{(1)} \in (t_1, t_2)$ and $\delta_1 > 0$ such that $p_{\varepsilon_k}(t) < p_{\varepsilon_k}(t_2) - \delta_1$ for all $t \in [t_1, t^{(1)}]$ and all $\varepsilon_k < \varepsilon(1)$. By Lemma 3.2 $x_{\varepsilon_k} - p_{\varepsilon_k}$ converges to zero a.e. in $J_T$. Hence there must be some $t^{(2)} \in [t_1, t^{(1)}]$ and some $\varepsilon(2) < \varepsilon(1)$ such that $x_{\varepsilon_k}(t^{(2)}) < p_{\varepsilon_k}(t_2) - \delta_2$ for some $\delta_2 < \delta_1$ and all $\varepsilon_k < \varepsilon(2)$. But then because $\dot{x}_{\varepsilon_k} \leq 0$ a.e. in $J$ it follows

$$x_{\varepsilon_k}(t) < p_{\varepsilon_k}(t_2) - \delta_2$$
for all $\varepsilon_k < \varepsilon^{(2)}$ and all $t \in [t^{(2)}, t_2]$. Again because $x_{\varepsilon_k} - p_{\varepsilon_k}$ converges to zero a.e. in $J_T$ this yields

$$
\overline{\tau}(t) = \lim_{k \to \infty} (p_{\varepsilon_k}(t) - x_{\varepsilon_k}(t) + x_{\varepsilon_k}(t)) < \lim_{k \to \infty} (p_{\varepsilon_k}(t) - x_{\varepsilon_k}(t) + p_{\varepsilon_k}(t_2) - \delta_2) = \overline{\tau}(t_2) - \delta_2
$$

for all $\varepsilon_k < \varepsilon^{(2)}$ and a.e. $t \in [t^{(2)}, t_2]$. By continuity of $\overline{\tau}$ this estimate holds for all $t \in [t^{(2)}, t_2]$ which gives the contradiction

$$
\overline{\tau}(t_2) < \overline{\tau}(t_2) - \delta_2.
$$

Hence, (37) indeed holds also in the limit. From (35) and the previous results we may now conclude that $\overline{\tau} < 0$ in a subset of $J$, which has positive measure, is only possible if $\overline{\tau} = F_+ (\overline{y})$ a.e. in this set. Note that this precisely gives one case of the variational inequality.

Similarly, we can deal with the case $\overline{\tau} (t_0) = F_- (\overline{y}(t_0))$ to show that $\overline{\tau} \geq 0$ and that $\overline{\tau} > 0$ in a set of positive measure is only possible if $\overline{\tau} = F_- (\overline{y})$ a.e. in this set. Hence, it follows, we have for a.e. $t \in J_T$

$$
\overline{\tau}(t)(\overline{\tau}(t) - \xi) \leq 0 \quad \forall \xi \in [F_- (\overline{y}(t)), F_+ (\overline{y}(t))].
$$

This proves that $\overline{\tau}$ solves (29)-(31) with $y = \overline{y}$. \hfill \square

We observe that the proof of Lemma 3.4 essentially relied on the convergence of the fast projection as $\varepsilon \to 0$. Furthermore, the argument treats the critical manifold $C_0$ in three different phases according whether we are on the boundaries defined by $F_+$, $F_-$ or in the interior. Since many other hysteresis operators can also be defined by variational inequalities, we actually may hope that our strategy can be carried over to other singular limits not covered by standard normally hyperbolic Fenichel theory or other fast-slow systems methods. Finally, we can summarize the previous results and prove the main result.

**Proof.** (of Theorem 3.1) Most of the statements already follow from combining Lemma 3.2, Lemma 3.3 and Lemma 3.4. However, we still have to prove that $(\overline{\tau}, \overline{y})$ are uniquely determined by (12)-(16).

First, we make a few observations. The limit $\overline{\tau}$ is a generalized play operator for the Lipschitz continuous curves $F_+$ and $F_-$ with input $y = \overline{y}$. By [Vis94, III.2., Theorem 2.2], this generalized play is a Lipschitz continuous hysteresis operator from $C(J_T) \times \mathbb{R}$ to $C(J_T)$, where the second input variable is the initial condition $x_0$. The map $g$ is Lipschitz continuous by (A1).

To prove uniqueness, we argue by contradiction. Suppose that there are two pairs of solutions $(x_1, y_1)$ and $(x_2, y_2)$ of (12)-(16). Since $y_1$ and $y_2$ are continuous, for $t$ close enough to 0, we have

$$
|y_1(t) - y_2(t)| \leq \int_0^t |g(x_1(s), y_1(s), s) - g(x_2(s), y_2(s), s)| \, ds
$$

$$
\leq L_g \int_0^t |x_1(s) - x_2(s)| + |y_1(s) - y_2(s)| \, ds
$$

$$
\leq c \int_0^t \sup_{0 \leq \tau \leq s} |y_1(\tau) - y_2(\tau)| + |y_1(s) - y_2(s)| \, ds.
$$

We used Lipschitz continuity of the hysteresis operator for the last estimate. Therefore,

$$
\sup_{0 \leq \tau \leq t} |y_1(\tau) - y_2(\tau)| \leq c \int_0^t \sup_{0 \leq \tau \leq s} |y_1(\tau) - y_2(\tau)| \, ds,
$$
and by Gronwall’s Lemma it follows \( \sup_{0 \leq \tau \leq t} |y_1(\tau) - y_2(\tau)| = 0 \). This argument can be repeated for another small time interval so that finally \( y_1 = y_2 \). Consequently, \( x_1 = x_2 \) as well. This shows uniqueness of \( \overline{x} \) and \( \overline{y} \).

Concerning the regularity of \( \overline{x} \) and \( \overline{y} \) note that \( \overline{y} \in C^1(J_T) \) because \( g(\overline{x}(\cdot), \overline{y}(\cdot), \cdot) \) is continuous. Since \( \overline{y} \) is also continuous in \( J_T \) it follows \( \overline{y} \in W^{1,\infty}(J_T) \). We then obtain \( \overline{x} \in W^{1,\infty}(J_T) \) by \cite[III.2.,Theorem 2.3]{VVis94}.

The last step is the convergence result, which is relatively simple using the intermediate results. Indeed, by Lemma 3.3 and Lemma 3.4, every sequence \( \{\varepsilon_j\}_{j=1}^\infty \) with \( \varepsilon_j \to 0 \) has a subsequence \( \{\varepsilon_k =: \varepsilon_k\}_{k=1}^\infty \) such that
\[
y_{\varepsilon_k} \to \overline{y} \quad \text{in } W^{1,q}(J_T) \quad \text{and} \quad x_{\varepsilon_k} \to \overline{x} \quad \text{in } L^q(J_T).
\]
Because \( \overline{x} \) and \( \overline{y} \) are the unique solution of (12)-(16) we conclude that this convergence holds for the whole sequence \( \{(x_{\varepsilon_j}, y_{\varepsilon_j})\}_{j=1}^\infty \).

Note that the strategy of our proof presented in this section depends crucially on the fast variable convergence, which is dealt with via weak convergence first. Our second approach replaces this step of the argument using a more geometric strategy based upon local linearization, which actually relies on additional assumptions on the differentiability of the vector field; hence, it complements the approach presented in this section.

## 4 The Main Result II - Linearization Approach

As before, we consider the fast-slow non-autonomous planar system \((8)-(9)\) on a finite time interval \( J_T = (0,T) \). However, we strengthen the assumption \((A1)\) to the following:

\begin{itemize}
  \item \((A1')\) \( g \in C^2(\mathbb{R}^3) \) and \( f \in C^2(\mathbb{R}^2) \).
\end{itemize}

Essentially we are going to get a stronger convergence result if we assume more differentiability. The result we are going to prove in this section is a variant/extension of Theorem 3.1.

**Theorem 4.1.** Suppose the assumptions \((A1'), (A2), (A4)\) hold. Let \((\overline{x}, \overline{y})\) be the unique solution of
\[
\dot{y}(t) = g(x(t), y(t), t) \quad \text{a.e. in } J_T, \\
y(0) = y_0, \\
\dot{x}(t)(x(t) - \xi) \leq 0 \quad \forall \xi \in [F_-(y(t)), F_+(y(t))], \text{ a.e. in } J_T, \\
x(0) = \min\{\max\{F_-(y_0), x_0\}, F_+(y_0)\}, \\
x(t) \in [F_-(y(t)), F_+(y(t))] \quad \text{in } J_T.
\]

Then \( \overline{y} \in C(J_T) \cap C^1(J_T) \) and \( \overline{y} \in W^{1,\infty}(J_T) \). For arbitrary \( \eta_1 > 0 \), there exists an \( \varepsilon_{\eta_1} > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_{\eta_1}) \) there exists a time \( t(\varepsilon) \in J_T \) with
\[
\|y_\varepsilon - \overline{y}\|_{C(J_T)} + \|x_\varepsilon - \overline{x}\|_{C[J(t(\varepsilon)), T]} < \eta_1.
\]

If \( x_0 \in [F_-(y_0), F_+(y_0)] \), then \( t(\varepsilon) = 0 \). Otherwise, \( t(\varepsilon) \leq \varepsilon C(\eta_1) \) for some \( C(\eta_1) > 0 \). This implies the following:
\((N1)\) If \(x_0 \in [F_- (y_0), F_+ (y_0)]\), then \(y_\varepsilon \to \overline{y}\) in \(C(\overline{J}_T) \cap C^1(J_T)\) and \(x_\varepsilon \to \overline{x}\) in \(C(\overline{J}_T)\).

\((N2)\) Otherwise, for arbitrary \(\eta_2 > 0\), \(y_\varepsilon \to \overline{y}\) in \(C(\overline{J}_T) \cap C^1(\eta_2, T)\) and \(x_\varepsilon \to \overline{x}\) in \(C[\eta_2, T]\).

\((N3)\) For any \(q \in (1, +\infty)\), \(y_\varepsilon \to \overline{y}\) in \(W^{1,q}(J_T)\) and \(x_\varepsilon \to \overline{x}\) in \(L^q(J_T)\).

In particular, note that the conclusions of convergence to a hysteresis operator in the singular limit are now obtained in stronger norms in \((N2)\) but the convergence result of Theorem 3.1 stated in \((N3)\) obviously still holds. We do not expect any stronger convergence in \((N2)\) for the fast variable, even if the differentiability of \(f, g\) is increased. This is reminiscent of the classical results from Fenichel Theory \([\text{Fen79, Jon95, Kue15}]\) as fast subsystem trajectories generically develop non-differentiable points when connecting to a critical manifold. To prove Theorem 4.1 we need some additional notation. First, note that by Lemma 3.2 trajectories remain bounded.

**Definition 4.2.** Let \(M\) be a compact rectangle such that \((x_\varepsilon(t), y_\varepsilon(t))\) is contained in \(M\) for all \(\varepsilon > 0\) and all \(t \in \overline{J}_T\).

- We introduce \(M_0 := M \cap \{(x, y) : f(x, y) = 0\}\), \(M_+ := M \cap \{(x, y) : f(x, y) < 0\}\) and \(M_- := M \cap \{(x, y) : f(x, y) > 0\}\).

- We define the constants \(C_f, C_g, C_{D_f}, C_{D_g}, C_{D^2_f}, C_{D^2_g} \geq 0\) by the upper bounds of the norms of \(f, g, Dg, Df, D^2 f \text{ and } D^2 g\) on \(M\). Moreover, we set \(C_M := \max\{|w| : w \in M\}\), \(C_{D_2} := \max\{C_{D^2_f}, C_{D^2_g}\}\) and \(L := \max\{L_f, L_g\}\); cf. assumptions \([A1]\) and \([A1']\).

We remark that the notation of \(M_+\) and \(M_-\) corresponds to \(F_+\) and \(F_-\) above and the sign subscripts are chosen so that \([F_-, F_+]\) is an interval.

**Definition 4.3.** For \(w_0 = (x_{w_0}, y_{w_0}) \in M\) and \(\varepsilon > 0\) we write \(x_{\varepsilon, w_0}\) and \(y_{\varepsilon, w_0}\) for the solution of \((8)-(9)\) with initial value \((x(0), y(0)) = w_0\) and on the time interval \(J_T\). For \((\tau_0, \tau_1) \subset J_T\) we denote by \(x_{\varepsilon, w_0, (\tau_0, \tau_1)}\) and \(y_{\varepsilon, w_0, (\tau_0, \tau_1)}\) the solution of \((8)-(9)\) with initial value \((x(\tau_0), y(\tau_0)) = w_0\) and on the time interval \((\tau_0, \tau_1)\).

The additional subscript \(w_0\) will be necessary in the proof as we piece together several local results comparing linear and nonlinear terms. This approach is similar to a strategy of patching sample paths used in the context of stochastic fast-slow systems \([\text{BG06, BGK15}]\); see also Figure 5. With \(w \in M, t_0 \in \overline{J}_T\) and \(\varepsilon > 0\) consider the following linearized system of evolution equations:

\[
\varepsilon \ddot{x}(t) = f(w) + [Df(w)] \left( \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - w \right) \quad \text{for } t > t_0,
\]

\[
x(t_0) = x_w,
\]

\[
\dot{y}(t) = g(w, t_0) + [\partial_{(x,y)} g(w, t_0)] \left( \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - w \right) \\
+ [\partial_t g(w, t_0)](t - t_0) \quad \text{for } t > t_0,
\]

\[
y(t_0) = y_w.
\]
We group the terms containing \((x(t), y(t))\) and the remaining terms together, write

\[
F_1(w) + [F_2(w)]\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} := f(w) + [Df(w)]\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - w,
\]

\[
G_1(w, t_0, t) + [G_2(w, t_0)]\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} := g(w, t_0) + [\partial_{x,y}g(w, t_0)]\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - w
\]

\[
+ [\partial_tg(w, t_0)](t - t_0),
\]

and denote by \(C_{F_1}, C_{F_2}, C_{G_1}, C_{G_2} > 0\) upper bounds of the maximum norms of the corresponding functions \(F_j, G_j\) with \(j \in \{1, 2\}\) in \(M\) and \(\overline{J_T}\). We also set

\[
C_{F,G} := \max\{C_{F_1}, C_{F_2}, C_{G_1}, C_{G_2}\}.
\]

The next step is to define the local approximations of the solution and patch them together on a given finite-time interval.

**Definition 4.4.** For \(v = (x_v, y_v) \in M\), for given \(\varepsilon, \theta > 0\) and \((\tau_0, \tau_1) \subset J_T\), we define \(x^{\text{lin}}_{\varepsilon, \theta, v, (\tau_0, \tau_1)}\) and \(y^{\text{lin}}_{\varepsilon, \theta, v, (\tau_0, \tau_1)}\) in the following way:

(S1) Let \(v^{(0)} := v, \tau^{(0)} := \tau_0\) and define \(x^{\text{lin}}_{\varepsilon, \theta, v, (\tau_0, \tau_1)}\) and \(y^{\text{lin}}_{\varepsilon, \theta, v, (\tau_0, \tau_1)}\) first on the interval \([\tau^{(0)}, \tau^{(1)}]\) as the solution of (46)-(49) with initial value \(w = v^{(0)}\) and initial time \(\tau^{(0)}\). Let \(\tau^{(1)}\) be the first time in \((\tau_0, \tau_1)\) with

\[
|v_x^{(0)} - x^{\text{lin}}_{\varepsilon, \theta, v, (\tau_0, \tau_1)}(\tau^1)| = \frac{\theta}{2} \quad \text{or} \quad \tau^{(1)} = \tau_1.
\]

We set \(v^{(1)} := (x^{\text{lin}}_{\varepsilon, \theta, v, (\tau_0, \tau_1)}(\tau^1), y^{\text{lin}}_{\varepsilon, \theta, v, (\tau_0, \tau_1)}(\tau^1))\).
(S2) For given \(v^{(i)} \in M\) and \(\tau^{(i)} \in (\tau_0, \tau_1), \ i \geq 1\), we define \(x^{\text{lin}}_{\varepsilon, v, (\tau_0, \tau_1)}\) and \(y^{\text{lin}}_{\varepsilon, v, (\tau_0, \tau_1)}\) inductively on the interval \([\tau^{(i)}, \tau^{(i+1)}]\) by the solution of (16)-(49) with initial value \(w = v^{(i)}\) and initial time \(\tau^{(i)}\). \(\tau^{(i+1)}\) is the first time in \((\tau^{(i)}, \tau_1]\) with

\[
|v^{(i)}_x - x^{\text{lin}}_{\varepsilon, v, (\tau_0, \tau_1)}(\tau^{(i+1)})| = \frac{\theta}{2} \quad \text{or} \quad \tau^{(i+1)} = \tau_1.
\]

We set \(v^{(i+1)} := (x^{\text{lin}}_{\varepsilon, v, (\tau_0, \tau_1)}(\tau^{(i+1)}), y^{\text{lin}}_{\varepsilon, v, (\tau_0, \tau_1)}(\tau^{(i+1)})\). If \(\tau^{(i+1)} < \tau_1\) we repeat Step (S2).

(S3) After finitely many steps this defines \(x^{\text{lin}}_{\varepsilon, v, (\tau_0, \tau_1)}\) and \(y^{\text{lin}}_{\varepsilon, v, (\tau_0, \tau_1)}\) on the interval \([\tau_0, \tau_1]\).

Although Definition 4.4 may look complicated, it is actually just a piecewise definition of the linearized solution using hitting times in (50)-(51).

**Definition 4.5.** Let \(\delta > 0\) be given. Then by assumption \(\text{(A2)}\) we can define \(f_+(\delta) < 0\) by

\[
f_+(\delta) := \frac{1}{2} \max \{f(x, y) : (x, y) \in M_+, \dist((x, y), M_0) \geq \delta\}.\]

Similarly we define \(f_-(\delta) > 0\) by

\[
f_-(\delta) := \frac{1}{2} \min \{f(x, y) : (x, y) \in M_-, \dist((x, y), M_0) \geq \delta\}.\]

Furthermore, it will be helpful to introduce the following notation

\[
\min \{|f_+(\delta)|, f_-(\delta)| =: f_m(\delta).\]

The next lemma is the main step to estimate the deviation of the solution obtained from the linearized process by patching to the true solution. To simplify the statement and the proof, we use for the next result the notation \(x_{\varepsilon, w_0} := x_{w_0}, y_{\varepsilon, w_0} := y_{w_0}, x^{\text{lin}} := x^{\text{lin}}_{\varepsilon, v, (\tau_0, \tau_1)}\) and \(y^{\text{lin}} := y^{\text{lin}}_{\varepsilon, v, (\tau_0, \tau_1)}\).

**Lemma 4.6.** Let \(\delta > 0\) be given such that \(M \setminus (M_0 + B(0, \delta)) \neq \emptyset\) and let \(v = (x_v, y_v) \in M \setminus (M_0 + B(0, \delta))\). Furthermore, consider any \(\tau_0 \in [0, T)\) and define

\[
\varepsilon_\delta := \min \left\{\frac{f_m(\delta)}{2(1 + L_+)C_{F,G}} \left[\sqrt{2} \left(C_M + \frac{2C_{F,G}}{f_m(\delta)}\right) e^{2C_{F,G}/f_m(\delta)} + 1\right]\right\}.\]

Let \(\varepsilon \in (0, \varepsilon_\delta)\) and \(\theta \in \left(0, \min \left\{\frac{f_m(\delta)}{C_{D^1}}, 1\right\}\right)\) be arbitrary but fixed. Denote by \(\tau_1 > \tau_0\) the first hitting time such that \((x^{\text{lin}}, y^{\text{lin}}) \in M_0 + B(0, \delta)\) or \(\tau_1 = T\). We have the local time estimate

\[
|\tau^{(j+1)} - \tau^{(j)}| \leq \frac{\varepsilon \theta}{f_m(\delta)}, \quad \text{for all} \quad j \in \{0, \cdots, K - 1\},
\]

and the global number \(K\) of time intervals \((\tau^{(j)}, \tau^{(j+1)})\) in Definition 4.4 satisfies

\[
K \leq \left\lfloor \frac{2(1 + L_+)}{\theta (1 - \frac{\varepsilon}{\varepsilon_\delta})} \dist(v, M_0 + B(0, \delta))\right\rfloor \leq \frac{2(1 + L_+)}{\theta (1 - \frac{\varepsilon}{\varepsilon_\delta})} \dist(v, M_0 + B(0, \delta)) + 1.
\]
In particular, this implies the global time estimate

$$|\tau_1 - \tau_0| \leq \varepsilon \left( \frac{2 \left( 1 + L_\pm \right) \text{dist}(v, M_0 + B(0, \delta))}{f_m(\delta) \left( 1 - \frac{\varepsilon}{\delta} \right)} + \frac{1}{C_{DF}} \right).$$

(55)

Moreover, for \(w_0 \in M\) and arbitrary \(t \in [\tau_0, \tau_1]\) there holds

$$|x_{w_0}(t) - x^{\text{lin}}(t)| + |y_{w_0}(t) - y^{\text{lin}}(t)| \leq \left( |x_{w_0}(\tau_0) - x_v| + |y_{w_0}(\tau_0) - y_v| + \theta^2 K_1 \right) K_2,$$

where the constants \(K_1, K_2\) depend upon the given data as follows

$$K_1 = \left( \frac{2 \left( 1 + L_\pm \right) \text{dist}(v, M_0 + B(0, \delta))}{f_m(\delta) \left( 1 - \frac{\varepsilon}{\delta} \right)} + \frac{1}{C_{DF}} \right) \left( 2 C_{D2} + \frac{\varepsilon^3}{f_m(\delta)^2} \right),$$

$$K_2 = \exp \left[ 2L \left( \frac{2 \left( 1 + L_\pm \right) \text{dist}(v, M_0 + B(0, \delta))}{f_m(\delta) \left( 1 - \frac{\varepsilon}{\delta} \right)} + \frac{1}{C_{DF}} \right) \right].$$

Proof. For \(j \in \{0, \ldots, K - 1\}\) let \((\tau^{(j)}, \tau^{(j+1)})\) be an interval as in Definition [4.4]. On this interval, we have

$$\begin{pmatrix} x^{\text{lin}}(t) \\ y^{\text{lin}}(t) \end{pmatrix} = \exp \left[ (t - \tau^{(j)}) \begin{pmatrix} \frac{1}{\varepsilon} F_2(v^{(j)}) \\ G_2(v^{(j)}, \tau^{(j)}) \end{pmatrix} \right] v^{(j)}$$

$$+ \int_{\tau^{(j)}}^t \exp \left[ (t - s) \begin{pmatrix} \frac{1}{\varepsilon} F_2(v^{(j)}) \\ G_2(v^{(j)}, \tau^{(j)}) \end{pmatrix} \right] \begin{pmatrix} \frac{1}{\varepsilon} F_1(v^{(j)}, s) \\ G_1(v^{(j)}) \end{pmatrix} ds. \tag{57}$$

Consider the matrix

$$\begin{pmatrix} [F_2(v^{(j)})]^{(1)} \\ [G_2(v^{(j)}, \tau^{(j)})]^{(1)} \end{pmatrix} = \begin{pmatrix} F_2(v^{(j)}) \\ G_2(v^{(j)}, \tau^{(j)}) \end{pmatrix}.$$

Note that the entries of the matrix are bounded by \(C_{F,G}\). By definition of the matrix exponential it follows that for all \(s, t \in [\tau^{(j)}, \tau^{(j+1)}]\)

$$(0, 1) \left[ \exp \left[ (t - s) \begin{pmatrix} \frac{1}{\varepsilon} F_2(v^{(j)}) \\ G_2(v^{(j)}, \tau^{(j)}) \end{pmatrix} \right] \right] = (m_1(t, s), 1 + m_2(t, s)),$$

where for \(l \in \{1, 2\}\) a direct calculation yields

$$|m_l(t, s)| \leq \sum_{k=1}^{\infty} \frac{(t - s)C_{F,G})^k}{k!} \left( 1 + \frac{1}{\varepsilon} \right)^{k-1} = C_{F,G}(t - s) \sum_{k=0}^{\infty} \left( \frac{(t - s)C_{F,G} (1 + \varepsilon)}{\varepsilon} \right)^k \frac{1}{(k + 1)!}$$

$$\leq C_{F,G}(t - s) \exp \left( \frac{2C_{F,G}(t - s)}{\varepsilon} \right) \leq C_{F,G}(\tau^{(j+1)} - \tau^{(j)}) \exp \left( \frac{2C_{F,G}(\tau^{(j+1)} - \tau^{(j)})}{\varepsilon} \right).$$

Consequently, we also deduce that

$$\left( |m_1(t, s)| + |m_2(t, s)| \right)^{1/2} \leq \sqrt{2} C_{F,G}(\tau^{(j+1)} - \tau^{(j)}) \exp \left( \frac{2C_{F,G}(\tau^{(j+1)} - \tau^{(j)})}{\varepsilon} \right).$$
We multiply with $(0, 1)$ from the left in (57) for $j \in \{0, \cdots, K-1\}$ and take the absolute value to obtain

$$|y_{\text{lin}}(t) - y_{\nu(j)}| = \left| (m_1(t, \tau(j)), m_2(t, \tau(i)))v^{(j)} + \int_{\tau(i)}^t (m_1(t,s), 1 + m_2(t,s)) \left( \frac{1}{\varepsilon} F_1(v^{(j)}), \frac{1}{\varepsilon} G_1(v^{(j)}, s) \right) ds \right|$$

$$\leq \sqrt{2} C_{F,G}(\tau(j+1) - \tau(j)) e^{2C_{F,G}/f_{m}(\delta)} \left( C_M + (\tau(j+1) - \tau(j)) C_{F,G} \left( 1 + \frac{1}{\varepsilon} \right) \right)$$

$$+ (\tau(j+1) - \tau(j)) C_{F,G}.$$

Since $\theta \in (0, 1]$, by the definition of $\varepsilon_\delta$, using $\varepsilon \in (0, \varepsilon_\delta)$ and if (53) would hold, i.e., $|\tau(j+1) - \tau(j)| \leq \frac{\theta \varepsilon}{f_{m}(\delta)}$, then it follows that

$$|y_{\text{lin}}(t) - y_{\nu(j)}| \leq \frac{\theta \varepsilon C_{F,G}}{f_{m}(\delta)} \left( \sqrt{2} e^{2C_{F,G}/f_{m}(\delta)} \left( C_M + \frac{2C_{F,G}}{f_{m}(\delta)} \right) + 1 \right)$$

$$\leq \frac{\theta \varepsilon}{2(1 + L_\pm ) \varepsilon_\delta} \leq \frac{\theta}{2}.$$

Next, we are going to prove (53). We already know that if $|\tau(j+1) - \tau(j)| \leq \frac{\theta \varepsilon}{f_{m}(\delta)}$ then $(x_{\text{lin}}(t), y_{\text{lin}}(t)) \in B(v^{(j)}, \theta)$ for all $t \in [\tau(j), \tau(j+1)]$. Recall that $\tau_j$ was defined such that $(x_{\text{lin}}(t), y_{\text{lin}}(t))$ lies outside of a $\delta$-neighbourhood of the critical manifold, i.e., in $M \setminus (M_0 + B(0, \delta))$. Assume that $v \in M_+$. Then we can obtain a local upper bound on the fast vector field of the following form

$$f(v^{(j)}) + [Df(v^{(j)})] \left( \begin{array}{c} x_{\text{lin}}(t) \\ y_{\text{lin}}(t) \end{array} \right) - v^{(j)} < 2 f_+(\delta) + C_{Df} \theta < f_+(\delta)$$

(58)

for all $t \in [\tau(j), \tau(j+1)]$. This actually implies a helpful bound at the end of the small time interval, namely

$$x_{\text{lin}}(\tau(j+1)) = v^{(j)} + \int_{\tau(j)}^{\tau(j+1)} \frac{1}{\varepsilon} \left( F_1(v^{(j)}) + F_2(v^{(j)}) \left[ \left( (x_{\text{lin}}(s)) \right) - v^{(j)} \right] \right) ds$$

$$\leq v^{(j)} + \frac{(\tau(j+1) - \tau(j))}{\varepsilon} f_+(\delta) \leq v^{(j)} - \frac{(\tau(j+1) - \tau(j))}{\varepsilon} f_{m}(\delta).$$

(59)

The corresponding result for $v \in M_-$ follows analogously. By definition of the patched linear solution, the result (53) follows. We have collected enough estimates to derive the upper bound (54) for $K$. Suppose first that $v \in M_+$. Then for all $j \in \{0, \cdots, K-1\}$ there holds $x_{\nu(j+1)} \leq x_{\nu(j)}$ and

$$|y_{\nu(j)} - y_{\nu(j+1)}| \leq \frac{\theta \varepsilon}{2(1 + L_\pm ) \varepsilon_\delta}.$$

By assumption (A3), we have that $F_+$ is monotone increasing with Lipschitz constant $L_+$. Therefore for all $j \in \{0, \cdots, K-1\}$:

$$\text{dist}(v^{(j)}, M_0 + B(0, \delta)) \leq (1 + L_\pm ) \text{dist}(v, M_0 + B(0, \delta)) + \sum_{k=0}^{j-1} \left( L_+ |y_{\nu(k+1)} - y_{\nu(k)}| - \frac{\theta}{2} \right)$$

$$\leq (1 + L_\pm ) \text{dist}(v, M_0 + B(0, \delta)) + j \frac{\theta}{2} \left( \frac{\varepsilon}{\varepsilon_\delta} - 1 \right).$$

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Hence, \( v^{(K)} \in M_0 + B(0, \delta) \) if \( K \) is large enough, so that
\[
(1 + L_{\pm}) \text{dist}(v, M_0 + B(0, \delta)) + K \frac{\theta}{2} \left( \frac{\varepsilon}{\varepsilon_\delta} - 1 \right) \leq 0,
\]
which can be attained for some
\[
K \leq \left[ \frac{2(1 + L_{\pm}) \text{dist}(v, M_0 + B(0, \delta))}{\theta \left( 1 - \frac{\varepsilon}{\varepsilon_\delta} \right)} \right] \leq \frac{2(1 + L_{\pm}) \text{dist}(v, M_0 + B(0, \delta))}{\theta \left( 1 - \frac{\varepsilon}{\varepsilon_\delta} \right)} + 1
\]
and (54) follows as analogous estimates for \( v \in M_- \) lead to the same bounds for \( K \); note that in this case we have \( x_{v(j+1)} \geq x_{v(j)} \) for \( j \in \{0, \ldots, K - 1\} \). The upper bound for \( K \) implies almost immediately the total time estimate (55). For example, consider the direct calculation in the case of \( v \in M_+ \)
\[
|\tau_1 - \tau_0| \leq K \frac{\varepsilon\theta}{f_m(\delta)} \leq \left( \frac{2(1 + L_{\pm}) \text{dist}(v, M_0 + B(0, \delta))}{\theta \left( 1 - \frac{\varepsilon}{\varepsilon_\delta} \right)} + 1 \right) \frac{\varepsilon\theta}{f_m(\delta)} \leq \varepsilon \left( \frac{2(1 + L_{\pm}) \text{dist}(v, M_0 + B(0, \delta))}{f_m(\delta) \left( 1 - \frac{\varepsilon}{\varepsilon_\delta} \right) + 1} \right) \frac{1}{C_f(Df)}.
\]
With the bounds on time intervals, one may now inductively show the worst-case upper bound (56) by a second-order approximation of \( f \) and \( g \) in combination with Gronwall’s Lemma and \( \varepsilon < \varepsilon_\delta \leq 1 \).

We are also going to need a preliminary estimate for the full nonlinear solution near the boundary of the critical manifold.

**Lemma 4.7.** Let \( \delta, \varepsilon > 0 \) and \( v = (x_v, y_v) \in M_0 + B(0, \delta) \) be given. Consider any \( \tau_0 \subset [0, T) \) and denote by \( \tau_1 \) the first time after \( \tau_0 \) such that \( (x_{\varepsilon,v, (\tau_0, \tau_1)}, y_{\varepsilon,v, (\tau_0, \tau_1)}) = (x, y) \in \partial(M_0 + B(0, 2\delta)) \) or \( \tau_1 = T \). Then either
\[
\tau_1 = T \quad \text{or} \quad |\tau_1 - \tau_0| > \frac{\delta}{C_g}. \tag{60}
\]
Furthermore, for arbitrary \( w_0 \in M_+ \), \( (x_{w_0}, y_{w_0}) := (x_{\varepsilon, w_0}, y_{\varepsilon, w_0}) \) and \( t \in (\tau_0, \tau_1) \) there holds
\[
|x_{w_0}(t) - x(t)| + |y_{w_0}(t) - y(t)| \leq \left( |x_{w_0}(\tau_0) - x_v| + |y_{w_0}(\tau_0) - y_v| \right) e^{(\tau_1 - \tau_0)L(1 + 1/\varepsilon)}.
\]

**Proof.** We consider the case \( v \in M_+ \). Then because \( f(x, y) < 0 \) for all \( (x, y) \in M_+ \) and because \( F_+ \) is monotone increasing, the closest point in \( \partial(M_0 + B(0, 2\delta)) \), which can be reached from \( v \) is given by
\[
(x_v, y_v) := \{(x_v, y_v - \alpha) : \alpha > 0\} \cap \partial(M_0 + B(0, 2\delta)),
\]
and \( y_v - y > y_v - y_\alpha \) for all points \( (x, y) \in (M_0 + B(0, 2\delta)) \) with \( x \leq x_v \). Since \( v \in (M_0 + B(0, \delta)) \) we have \( y_v - y_\alpha > \delta \). Hence, either \( \tau_1 = T \) or \( |y(\tau_1) - y_v| > \delta \). This yields
\[
\delta < |y(\tau_1) - y_v| = \left| \int_{\tau_0}^{\tau_1} \dot{y}(s) \, ds \right| = \left| \int_{\tau_0}^{\tau_1} g(x(s), y(s), s) \, ds \right| \leq C_g(\tau_1 - \tau_0).
\]
Therefore, we find \( |\tau_1 - \tau_0| > \frac{\delta}{C_g} \). Analogous estimates for \( v \in M_- \) prove (60). The last statement in the result follows by a direct Gronwall Lemma argument. \( \square \)
It is helpful to describe the solution near the critical manifold by the full nonlinear dynamics as it reduces to the slow dynamics in the singular limit, while still using the patched linearized solution for the fast dynamics. This motivates the following definition:

**Definition 4.8.** Let $w_0 = (x_0, y_0) \in M$ and $\delta > 0$ be given. Adopt the assumptions and the notation from Lemma 4.4. Let $\varepsilon \in (0, \frac{\varepsilon}{2})$ be arbitrary. Furthermore, choose $\theta_\varepsilon \in \left(0, \min \left\{ \frac{f_{lin}(\delta)}{C_D}, 1 \right\} \right)$ such that $\theta_\varepsilon$ is of order $o \left( e^{-\frac{4\delta}{\varepsilon^2}} \right)$. We define $\tilde{x}_\varepsilon$ and $\tilde{y}_\varepsilon$ inductively as follows:

(T1) Let $t_0 := 0$ be the initial time. If $\text{dist}(w_0, M_0) \leq \delta$, define $t_1 > t_0$ to be the first time such that $(x_\varepsilon, w_0, (t_0, t_1), y_\varepsilon, w_0, (t_0, t_1)) \in \partial(M_0 + B(0, 2\delta))$ or $t_1 = T$. In this case we set

$$(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) := (x_\varepsilon, w_0, (t_0, t_1), y_\varepsilon, w_0, (t_0, t_1)) \quad \text{on} \quad [t_0, t_1].$$

If $\text{dist}(w_0, M_0) > \delta$ we define $t_1 > t_0$ by the first time such that the linearized solution satisfies $(x_\varepsilon^{lin}, w_0, (t_0, t_1), y_\varepsilon^{lin}, w_0, (t_0, t_1)) \in (M_0 + B(0, \delta))$ or $t_1 = T$. We then set

$$(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) := (x_\varepsilon^{lin}, w_0, (t_0, t_1), y_\varepsilon^{lin}, w_0, (t_0, t_1)) \quad \text{on} \quad [t_0, t_1].$$

In both cases we define $w_1 := (\tilde{x}_\varepsilon(t_1), \tilde{y}_\varepsilon(t_1)).$

(T2) Let $(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon)$ be defined on the interval $J_{t_i}$ and let $w_0, \ldots, w_i$ be chosen. If $t_i = T$ we are done. Otherwise, $t_i < T$. If $w_i \in \partial(M_0 + B(0, \delta))$, define $t_{i+1} > t_i$ by the first time such that $(x_\varepsilon, w_i, (t_i, t_{i+1}), y_\varepsilon, w_i, (t_i, t_{i+1})) \in \partial(M_0 + B(0, 2\delta))$ or $t_{i+1} = T$. In this case we set

$$(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) := (x_\varepsilon, w_i, (t_i, t_{i+1}), y_\varepsilon, w_i, (t_i, t_{i+1})) \quad \text{on} \quad [t_i, t_{i+1}].$$

If $w_i \in \partial(M_0 + B(0, 2\delta))$, define $t_{i+1} > t_i$ by the first time such that the linearized solution satisfies $(x_\varepsilon^{lin}, w_i, (t_i, t_{i+1}), y_\varepsilon^{lin}, w_i, (t_i, t_{i+1})) \in \partial(M_0 + B(0, \delta))$ or $t_{i+1} = T$. In this case we set

$$(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) := (x_\varepsilon^{lin}, w_i, (t_i, t_{i+1}), y_\varepsilon^{lin}, w_i, (t_i, t_{i+1})) \quad \text{on} \quad [t_i, t_{i+1}].$$

In both cases we define $w_{i+1} := (\tilde{x}_\varepsilon(t_{i+1}), \tilde{y}_\varepsilon(t_{i+1})).$

As before, we need a projection operator for the solution. Define $\tilde{p}_\varepsilon = p(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon)$ by

$$\tilde{p}_\varepsilon := p(\tilde{x}_\varepsilon, \tilde{y}_\varepsilon) = \min \{ \max \{ \tilde{x}_\varepsilon, F_-(\tilde{y}_\varepsilon) \}, F_+(\tilde{y}_\varepsilon) \}. \quad (61)$$

**Lemma 4.9.** Consider the same assumptions and the notation from Definition 4.8. Since $w_0 \in M$ is fixed we denote $x_\varepsilon, w_0$ by $x_\varepsilon$ and $y_\varepsilon, w_0$ by $y_\varepsilon$. Then there exists a constant $C(\delta) > 0$ such that

$$|x_\varepsilon(t) - \tilde{x}_\varepsilon(t)| + |y_\varepsilon(t) - \tilde{y}_\varepsilon(t)| \leq \theta_\varepsilon^2 e^{T_{lin}} C(\delta), \quad \forall t \in J_T. \quad (62)$$

Since $\theta_\varepsilon$ is of order $o \left( e^{-\frac{4\delta}{\varepsilon^2}} \right)$, this implies that

$$\|(x_\varepsilon, y_\varepsilon) - (\tilde{x}_\varepsilon, \tilde{y}_\varepsilon)\|_{C(J_T; \mathbb{R}^2)} \leq \theta_\varepsilon^2 e^{T_{lin}} C(\delta) \to 0$$

as $\varepsilon \to 0$. Furthermore, if $w_0 = (x_{w_0}, y_{w_0}) \in M_0 + B(0, \delta)$ then

$$\|\tilde{p}_\varepsilon - \tilde{x}_\varepsilon\|_{C(J_T)} \leq 2\delta(1 + L_{\pm}). \quad (63)$$

If $w_0 = (x_{w_0}, y_{w_0}) \in M \backslash \left( M_0 + B(0, \delta) \right)$ then there exists a constant $C_0(\delta) > 0$ and a time $t_1 \in J_T$ with $t_1 \leq \varepsilon C_0(\delta)$ such that

$$\|\tilde{p}_\varepsilon - \tilde{x}_\varepsilon\|_{C(t_1, T)} \leq 2(1 + L_{\pm})\delta. \quad (64)$$

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Proof. During the proof we denote $\bar{x} := \bar{x}_\varepsilon$, $\bar{y} := \bar{y}_\varepsilon$, $x := x_\varepsilon$, and $y := y_\varepsilon$. By Lemma 4.7, each subinterval of $J_T$ in which $(\bar{x}, \bar{y})$ behaves according to (8)-(9) can be estimated from below by $\delta/C_\varepsilon$. Hence, the total number $K$ of subintervals in Definition 4.8 is bounded from above by $(TC_\varepsilon)/\delta$. Let $K = K_1 + K_\varepsilon$, where $K_1$ is the number of time intervals in which $(\bar{x}, \bar{y})$ is defined via (46)-(49), and where $K_\varepsilon$ denotes the number of time intervals in which $(\bar{x}, \bar{y})$ is given by (8)-(9). We first assume $w_0 = (x_{w_0}, y_{w_0}) \in M \backslash (M_0 + B(0, \delta))$. In this case, $K_1 \in \{K_\varepsilon, K_\varepsilon + 1\}$. Without loss of generality, we assume $K_1 = K_\varepsilon + 1 = \frac{K_\varepsilon + 1}{2}$. By Lemma 4.3 and because $\varepsilon \leq \frac{\varepsilon_\delta}{2}$, the first time interval $(t_0, t_1)$ in Definition 4.8 is bounded from above by

$$|t_1 - t_0| \leq \varepsilon \left( \frac{4(1 + L_\varepsilon) \text{dist}(w_0, M_0 + B(0, \delta))}{f_m(\delta)} + \frac{1}{C_{DF}} \right) =: \varepsilon C_0(\delta).$$

We introduce the notation

$$C_1(\delta) := \max \left\{ C_0(\delta), \left( \frac{4(1 + L_\varepsilon) \delta}{f_m(\delta)} + \frac{1}{C_{DF}} \right) \right\}, \quad C_2(\delta) := C_1(\delta) \left( 2C_{D^2} + \frac{\varepsilon^3}{2} \right), \quad C_3(\delta) := 2LC_1(\delta).$$

Lemma 4.6 implies that for $i \in \{0, \ldots, \frac{K_\varepsilon - 1}{2}\}$ and $t \in [t_{2i}, t_{2i+1}]$ we may estimate the difference between the full and approximate solutions by

$$|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)| \leq \left[ |x(t_{2i}) - x_{w_{2i}}| + |y(t_{2i}) - y_{w_{2i}}| + \theta^2C_2(\delta) \right] e^{C_3(\delta)}.$$

Lemma 4.7 proves that for $i \in \{1, \ldots, \frac{K_\varepsilon - 1}{2}\}$ and $t \in [t_{2i-1}, t_{2i}]$ we obtain

$$|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)| \leq \left[ |x(t_{2i-1}) - x_{w_{2i-1}}| + |y(t_{2i-1}) - y_{w_{2i-1}}| \right] e^{(t_{2i} - t_{2i-1})L(1 + \frac{1}{\varepsilon})}.$$

This together with $\sum_{i=1}^{\frac{K_\varepsilon - 1}{2}} (t_{2i} - t_{2i-1}) \leq T$ implies that we can estimate for any $t \in \overline{J_T}$:

$$|x(t) - \bar{x}(t)| + |y(t) - \bar{y}(t)| \leq \left[ |x(t_0) - x_{w_0}| + |y(t_0) - y_{w_0}| + \frac{K - 1}{2} \theta^2C_2(\delta) \right] \exp \left( \frac{K - 1}{2}C_3(\delta) + TL \left( 1 + \frac{1}{\varepsilon} \right) \right) = \frac{K - 1}{2} \exp \left( \frac{K - 1}{2}C_3(\delta) + TL \left( 1 + \frac{1}{\varepsilon} \right) \right) =: \theta^2 e^{T_k} C_4(\delta).$$

Analogous estimates apply for $w_0 = (x_{w_0}, y_{w_0}) \in (M_0 + B(0, \delta))$. This proves (62). Now the results (63)-(64) follow directly from the definition of the mapping $\tilde{p}_\varepsilon$ in (61).

Finally we can prove the main result. Some elements of the proof of Theorem 3.1 will be kept. However, we can improve the convergence norm and also simplify the argument that in the singular limit, solutions satisfy the variational inequality for the generalized play operator.

Proof. (of Theorem 4.1) As in the proof of Theorem 3.1 we shall argue with sequences and converging subsequences $\{\varepsilon_k\}$. There exist functions $\bar{x}, \bar{y} \in W^{1,q}(J_T)$ such that

$$p_{\varepsilon_k} \to \bar{x} \quad \text{and} \quad y_{\varepsilon_k} \to \bar{y} \quad (65)$$

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as $\varepsilon_k \to 0$ weakly in $W^{1,q}(J_T)$ and strongly in $C(J_T)$ respectively. Furthermore, one shows as previously that $x_{\varepsilon_k} \to \overline{x}$ in $L^q(J_T)$, that the convergence of $y_{\varepsilon_k}$ is strong and that $\overline{y}$ solves (39)-(10) with $x = \overline{x}$. The strategy is now to improve the convergence norm and to simplify the argument that $\overline{x}$ solves (11)-(13).

Let $\eta > 0$ be arbitrary. Recall that by assumption (A3) we must have $F_- < F_+$ and both functions are monotone increasing. Using these facts and the definition (61) of $\bar{p}_\varepsilon$ yields

$$|\bar{p}_\varepsilon(t) - p_\varepsilon(t)| \leq \max\{|F_- (y_\varepsilon(t)) - F_- (\bar{y}_\varepsilon(t))|, |F_+ (y_\varepsilon(t)) - F_+ (\bar{y}_\varepsilon(t))|, |x_\varepsilon(t) - \bar{x}_\varepsilon(t)|\}$$

for all $t \in J_T$. Consider the assumptions and the notation from Definition 4.8 as well as the notation from Lemma 4.9. If $w_0 = (x_{w_0}, y_{w_0}) = (x_0, y_0) \in M \setminus M_0$ then we set

$$\delta_\eta := \min \left\{ \text{dist}(w_0, M_0), \frac{\eta}{6(1 + L_\pm)} \right\}$$

(66)
in Definition 4.8 so that $w_0 \in M \setminus (M_0 + \overline{B(0, \delta_\eta)})$, and define $\varepsilon_{\delta_\eta}$ as in Definition 4.8. In this case, we further let

$$t_{\varepsilon} := t_1 \leq \varepsilon C_0(\delta_\eta)$$

for $\varepsilon \in (0, \frac{\varepsilon_{\delta_\eta}}{2})$ with $t_1$ and $C_0(\delta_\eta)$ from Lemma 4.9. If $w_0 = (x_{w_0}, y_{w_0}) \in M_0$ then let

$$\delta_\eta := \frac{\eta}{6(1 + L_\pm)}$$

(67)
in Definition 4.8 and again consider $\varepsilon_{\delta_\eta}$ from Definition 4.8. Then we can define $t_{\varepsilon} = 0$ for $\varepsilon \in (0, \frac{\varepsilon_{\delta_\eta}}{2})$. Now we can find $\varepsilon_\eta \in (0, \frac{\varepsilon_{\delta_\eta}}{2})$ such that for all $\varepsilon \in (0, \varepsilon_\eta)$ the following key bound is satisfied

$$\max \{2, L_- + L_+\} \theta_\varepsilon^2 e^{3L_\pm} C(\delta_\eta) < \frac{\eta}{3}.$$  

(68)

Lemma 4.9 can then be used to estimate for all $\varepsilon \in (0, \varepsilon_\eta)$ the fast variable

$$\max_{t \in J_T : t \geq t_{\varepsilon}} |x_\varepsilon(t) - p_\varepsilon(t)| \leq \max_{t \in J_T : t \geq t_{\varepsilon}} |x_\varepsilon(t) - \bar{x}_\varepsilon(t)| + |\bar{x}_\varepsilon(t) - \bar{p}_\varepsilon(t)| + |\bar{p}_\varepsilon(t) - p_\varepsilon(t)|$$

$$\leq 2\delta_\eta (1 + L_\pm) + \max \{2, L_\pm\} \theta_\varepsilon^2 e^{3L_\pm} C(\delta_\eta) < \frac{2}{3} \eta.$$  

(69)

Given a subsequence $\varepsilon_k \to 0$, we choose $k_\eta > 0$ such that $\varepsilon_k \in (0, \varepsilon_\eta)$ and

$$\max_{t \in J_T} |\eta_{\varepsilon_k}(t) - \overline{y}(t)| + |p_{\varepsilon_k}(t) - \overline{x}(t)| < \frac{\eta}{3}.$$  

(70)

for all $k \geq k_\eta$. From the last two maximum norm bounds (69)-(70) it then follows that we have

$$\|x_{\varepsilon_k} - \overline{x}\|_{C([t_{\varepsilon_k}, T])} + \|y_{\varepsilon_k} - \overline{y}\|_{C(J_T)} \leq \|y_{\varepsilon_k} - \overline{y}\|_{C(J_T)} + \max_{t \in [t_{\varepsilon_k}, T]} |x_{\varepsilon_k}(t) - p_{\varepsilon_k}(t)| + |p_{\varepsilon_k}(t) - \overline{x}(t)|$$

$$< \eta.$$  

(71)

The bound (71) is the crucial step. It is going to provide that the variational inequality is solved and it is going to show the convergence in the $C^0$-norm. We start by proving the former, i.e., by showing that $\overline{x}$ solves (11)-(13) with $y = \overline{y}$. The proof of the properties

$$\overline{x}(0) = \min \{\max \{x_0, F_-(y_0)\}, F_+(y_0)\}$$
and
\[ \pi(t) \in [F_-(\overline{y}(t)), F_+(\overline{y}(t))] \quad \forall \ t \in J_T \]
remain the same as in the proof of Theorem 3.1. Let \( t_0 \in J_T \) be given. Suppose that \( \pi(t_0) = F_+(\overline{y}(t_0)) \). Assume first that \( g(x, y, t) < 0 \) in a neighbourhood \( U \times I \subset M \times J_T \) of \( (\pi(t_0), \overline{y}(t_0), t_0) \) and \( (\pi(t), \overline{y}(t)) \in U \) for all \( t \in I \). Using [Rud91, Theorem 4.15 and Theorem 4.16] and continuity of \( g \), after eventually making \( U \) smaller, we may assume that
\[ -C_g < g(x, y, t) < -c < 0 \quad \text{for some constant } c \text{ and all } (x, y, t) \in U \times I. \] (72)

Now we are going to apply the crucial bound (71). Define a parameter \( \eta > 0 \) (which will be fixed later) with
\[ \eta < \min_{t \in I} \text{dist}((\pi(t), \overline{y}(t)), \partial U). \]
In dependence of this parameter, let \( \delta_\eta, \varepsilon_\eta, k_\eta \) be defined as in (66), (67), (68) and (70). Moreover, we choose \( \varepsilon_\eta \in (0, \varepsilon_\eta) \) such that
\[ t_\varepsilon \leq \varepsilon C_0(\delta_\eta) < \min\{t : t \in I\} \quad \text{for all } \varepsilon \in (0, \varepsilon_\eta). \]

Finally we define \( k_I \geq k_\eta \) such that \( \varepsilon_k \in (0, \varepsilon_1) \) for all \( k \geq k_I \). These choices then lead us to the estimate
\[ \|y_{\varepsilon_k} - \overline{y}\|_{C(I_T)} + \|x_{\varepsilon_k} - \pi\|_{C[0, T]} < \eta \]
for all \( k \geq k_I \). Therefore, we obtain \( (x_{\varepsilon_k}(t), y_{\varepsilon_k}(t), t) \in U \times I \) for all \( k \geq k_I \) and \( t \in I \). Because \( (\pi(t_0), \overline{y}(t_0)) \in \partial M_0 \) by assumption, (73) yields \( \text{dist}((x_{\varepsilon_k}(t_0), y_{\varepsilon_k}(t_0)), \partial M_0) < \eta \) for all \( k \geq k_I \).

Recall that \( -C_g < g(x, y, t) < -c < 0 \) for all \( (x, y, t) \in U \times I \) by (72). This fact applied in the fast-slow system (8)-(9) implies that for all \( k \geq k_I \), \( y_{\varepsilon_k} \) is monotone decreasing in \( I \) with \( -C_g < \dot{y}_{\varepsilon_k} < -c \). We also already know from (69) that
\[ \max_{t \in J_T, t \geq t_\varepsilon} |x_{\varepsilon}(t) - p_\varepsilon(t)| < \frac{2\eta}{3} \quad \text{for } \varepsilon \in (0, \varepsilon_\eta) \]
and since \( (p_\varepsilon, y_\varepsilon) \in M_0 \) for all \( \varepsilon > 0 \) by definition and since \( t_\varepsilon < \max\{t \in I\} \), this yields
\[ \max_{t \in I} \text{dist}((x_{\varepsilon_k}(t), y_{\varepsilon_k}(t)), M_0) < \frac{2\eta}{3} \quad \text{for all } k \geq k_I. \]

This fact can be combined with the observation that there is no fast flow inside the critical manifold, i.e., \( \dot{x}_{\varepsilon_k}(t) = 0 \) if \( (x_{\varepsilon_k}(t), y_{\varepsilon_k}(t)) \in M_0 \) and with the fact that \( \text{dist}((x_{\varepsilon_k}(t_0), y_{\varepsilon_k}(t_0)), \partial M_0) < \eta \) and \( -C_f < \dot{y}_{\varepsilon_k} < -c \) in \( I \) for all \( k \geq k_I \). In particular, we may now conclude that
\[ \max_{t \in I \cap [t_0, T]} \text{dist}((x_{\varepsilon_k}(t), y_{\varepsilon_k}(t)), \partial M_0) < \eta \quad \text{for all } k \geq k_I. \]

Since we are still free in our choice of \( \eta \) (which then determines \( k_I \)), the last estimate, together with the crucial bound (73), proves that \( (\pi(t), \overline{y}(t)) \in \partial M_0 \) and therefore \( \pi(t) = F_+(\overline{y}(t)) \) for all \( t \in I \cap [t_0, T] \). Moreover, it follows that \( \overline{y} \) is monotone decreasing in \( I \). Since \( F_+ \) is monotone increasing, this implies that \( \pi \) is monotone increasing in \( I \cap [t_0, T] \), so that \( \pi(t) < 0 \) for a.e. \( t \in I \cap [t_0, T] \).

The case when \( g(x, y, t) > 0 \) in a neighbourhood of \( (\pi(t_0), \overline{y}(t_0), t_0) \) leads to a contradiction as we would have moved already inside \( M_0 \) earlier in this case. If \( t_0 \in J_T \) is a time such that
$g(x, y, t) \geq 0$ in $(U \times I) \cap (M \times [t_0, T])$ for some neighbourhood $U \times I$ of $(\overline{\pi}(t_0), \overline{\gamma}(t_0), t_0)$ then \( \overline{\gamma} \) is monotone increasing and \( \overline{\pi} \) is constant in $I \cap [t_0, T]$ with \( \overline{\pi}(t) = F_+(\overline{\gamma}(t_0)) \). The cases when \( \overline{\pi}(t_0) = F_-(\overline{\pi}(t_0)) \) or $$(x(t_0), y(t_0)) \in \text{int}(M_0)$$ are treated in a similar manner. Therefore, we have shown that $x$ solves (41)-(43) with $y = \overline{\gamma}$.

Uniqueness of $x$ and $y$ and convergence of the whole sequence follow just as in the proof of Theorem 3.1. More precisely, we may repeat the steps with any subsequence converging to zero and (71) then proves (45). From (45) we deduce the convergence result of $x_\varepsilon$. This yields the convergence result also for $y_\varepsilon$.

5 An Application to Forced Oscillations

So far, we have studied the singular limit convergence guided by Netushil’s conjecture of fast-slow systems coupled with hysteresis operators. Our results in Sections 3-4 were fully rigorous. However, it will be of interest to see, how we can practically analyze fast-slow systems (8)-(9). In this section we provide a numerical and formal analysis of an important subclass of (8)-(9).

First, note that (8)-(9) can be re-written in non-autonomous form as

$$
\begin{align*}
\frac{dx}{dt} &= f(x, y), \\
\frac{dy}{dt} &= g(x, y, t), \\
\frac{dt}{d\tau} &= \omega,
\end{align*}
$$

(74)

where we introduce an additional parameter $\omega$. From a dynamical systems point of view, fast-slow problems of the form (74) provide many highly investigated classical examples. For example, if we consider $f$ as the classical cubic non-linearity and assume that $g$ is periodic in $t$, say for concreteness,

$$
\begin{align*}
f_{\text{vdP}}(x, y) &= y - \frac{x^3}{3} + x, \\
g(x, y, t) &= g(x, y, t + 2\pi),
\end{align*}
$$

(75)

then (74) has a very prominent representative given by the forced van der Pol oscillator [vdP34, Kue15, Guc03] usually written as

$$
\begin{align*}
\frac{dx}{dt} &= f_{\text{vdP}}(x, y), \\
\frac{dy}{dt} &= a \sin(2\pi \theta) - x, \\
\frac{d\theta}{dt} &= \omega.
\end{align*}
$$

(76)

where $(x, y, \theta) \in \mathbb{R}^2 \times S^1$ with $S^1 = [0, 1]/(0 \sim 1)$ is the circle, $(y, \theta)$ are the slow variables and $a, \omega$ are the main amplitude and phase parameters used in bifurcation studies of (76); see [GHW03, BEG +03, GNV84, SW04]. The forced van der Pol equation is also one of the very few ODE models, where it has been rigorously proven that chaotic oscillations may occur [Hai09]. The model has also a strong link to one-dimensional or almost one-dimensional return maps and chaotic dynamics [GWY06]. The forced van der Pol equation is still being studied very actively [BDG +16].

In our setting of generalized play operators and Netushil’s conjecture, we have considered a different class of fast variable vector fields specified by assumptions [(A1)-(A3)]. This naturally
raises the question, what actually happens dynamically, if we replace the fast-vector field in the van der Pol oscillator with one satisfying (A1)-(A3). Our main example we propose to study is

\[
f(x, y) = \begin{cases} 
  y - x - 1 & \text{if } x < y - 1, \\
  y - x + 1 & \text{if } x > y + 1, \\
  0 & \text{else},
\end{cases}
\]

(77)

for the fast variable. Piecewise linear approximations are very classical in fast-slow systems, e.g., they have been studied in the van der Pol context many times [Lev49] but are still of high current interest [DFH+13, DGP+16]. For the slow variable, we propose a linear term and a sinusoidal forcing

\[
g(x, y, t) = a \sin(2\pi t) + bx + cy
\]

(78)

for parameters \(a, b, c \in \mathbb{R}\). The strategy to start with the lowest order Taylor expansion is well-known in fast-slow systems for the slow variables [Guc08, SW01] and so is starting with the lowest harmonic in many other contexts [Kur12]. One checks that (A1)-(A4) hold with \(F^+(y) = y + 1\) and \(F^-(y) = y - 1\) except for the boundedness of \(G(x)\); however, we shall observe that \(x\) is going to stay bounded for certain parameter choices to be investigated below so we can just cut off \(G(x) = bx\) smoothly outside a compact set.

Figure 6: Direct numerical integration of the fast-slow ODEs (74) with nonlinearities given by (77)-(78); the parameters are chosen as \(a = 1, b = -1, c = \frac{1}{5}, \omega = 4, \) and \(\varepsilon = 0.01\). The initial condition was chosen outside and \(O(1)\)-separated from \(C_0\). (a) Phase portrait showing a typical trajectory (black curve) projected into the \((y, x)\)-plane, i.e., not explicitly showing the non-autonomous periodic \(\tau\)-direction. \(C_0\) lies between the two curves (gray) defined by \(F^+\) and \(C_0\) has the same dimension as the ambient phase. (b) Time series of the trajectory with the \(y\)-coordinate (solid curve) and the \(x\)-coordinate (dashed curve). One clearly observes small scale behaviour in the region, where the fast and slow variables interact.

As a first step, we would like to check, whether we can find any interesting dynamics by selecting the basic nonlinearities (77)-(78). Figure 6 shows the results of numerical integration. We have selected an initial condition far separated of the critical manifold

\[
C_0 = \{(x, y, \tau) \in \mathbb{R}^2 \times S^1 : y - 1 \leq x \leq y + 1\}.
\]

(79)
The initial condition gets attracted very fast towards $C_0$ as shown in Figure 3(a) as expected already from the theoretical results based upon assumption (A2). The dynamics near the boundary

$$\partial C_0 = \{ x = F_-(y) \} \cup \{ x = F_+(y) \} =: C_- \cup C_+$$

is a lot more delicate. We observe in Figure 6 the case of many small amplitude oscillations (SAOs) near both parts of the boundary. Furthermore, there are relatively slow jumps between $C_-$ and $C_+$ in comparison to the long very slow drift time near each boundary piece. Essentially, we observe oscillations, which look similar to classical relaxation oscillations [MR80, Gra87, Kue15], just with high-frequency fast SAOs overlayed near the slowest scale pieces and the jumps in the relaxation cycle still occur on the slow time scale. Figure 6 does not provide an indication, whether the oscillations are actually periodic or potentially even chaotic.

The SAOs near $F_\pm$ are easy to explain formally. Suppose we use the standard slow subsystem reduction just along the lines $C_{\pm}$, then we obtain

$$\frac{dy}{dt} = a \sin(2\pi \omega t) + bx + cy = a \sin(2\pi \omega t) + (b + c)y \pm b, \quad y(0) = y_0. \quad (81)$$

Figure 7: Direct numerical integration of the fast-slow ODEs (74) with nonlinearities given by (77)-(78); the parameters are chosen as $a = 1, b = -1, c = \frac{1}{5}, \omega = 4, \varepsilon = 0.01$ and final time $T = 5 \cdot 10^4$. The initial condition was chosen outside and $O(1)$-separated from $C_0$. (a) Phase portrait showing a typical trajectory (black curve) projected into the $(y, x)$-plane. We have now also marked the two “average” equilibrium points $(Y_\pm, F_\pm(Y_\pm))$ as dots (black). (b1) Zoom near $(Y_-, F_-(Y_-)) = (1.25, 0.25)$. (b2) Zoom near a typical region with dynamics well-approximated by the formal slow subsystem (81) near the branch $C_-$. 

The ODE (81) can actually be solved explicitly. We denote the solutions corresponding to the respective signs in front of the constant term $\pm b$ by $y_\pm = y_\pm(t)$. We have

$$y_\pm(t) = Y_\pm + Y e^{(b+c)t} + Y_h(t) \quad (82)$$
where the individual ("constant, exponential prefactor, and harmonic") terms are given by

\[ Y_\pm = \mp \frac{b}{b+c}, \tag{83} \]
\[ Y_e = \pm \frac{b}{b+c} + \frac{2\pi a\omega}{(b+c)^2 + 4\pi^2\omega^2} + y_0, \tag{84} \]
\[ Y_h(t) = -\frac{a(b+c)\sin(2\pi\omega t) + 2\pi a\omega \cos(2\pi\omega t)}{(b+c)^2 + 4\pi^2\omega^2}. \tag{85} \]

Figure 8: Bifurcation diagram of the fast-slow ODEs (74) with nonlinearities given by (77)-(78); the parameters are chosen as \(a = 1, b = -1, \omega = 4,\) and \(\varepsilon = 0.01.\) (a) Main bifurcation diagram varying the parameter \(c\) and showing the maximum and minimum amplitudes \(A\) of the variable \(y\) for the global attractor. (b1) Time series for \(c = 0.1.\) (b2) Time series for \(c = -0.1.\) The dashed curves are \(x(t)\) and the solid curve with SAOs are \(y(t).\)

The formal slow subsystem (81) remains bounded for all \(y_0 \in \mathbb{R}\) if and only if \(b + c \leq 0.\) We shall not investigate the borderline case \(b = -c\) here and just assume \(b + c < 0\) from now on. Then \(Y_e \exp[(b+c)t] \to 0\) as \(t \to +\infty\) so the dynamics of \(y_\pm(t)\) is a harmonic oscillation around the points \(Y_\pm,\) i.e., we have

\[ \lim_{t \to +\infty} \frac{1}{t} \int_0^t y_\pm(s) \, ds = Y_\pm \]

so we may view \(Y_\pm\) as averaged equilibrium points. We now have to re-visit the numerical results from Figure 6 which are presented in phase space in a different way in Figure 7 where we clearly see that the slow subsystem approximation correctly describes the behaviour near the branches \(C_\pm,\) i.e., we move upwards via the terms \(Y_\mp e^{(b+c)t}Y_e\) on the right branch \(C_\mp\) towards \((y, x) = (Y_\mp, F_\mp(Y_\mp))\) and there are oscillations induced by the term \(Y_h(t).\) Similarly we move downwards on the left branch \(C_\pm\) with several oscillations induced by the time-dependent terms.

The next natural question is, how the global periodic large oscillations are generated under parameter variation. Figure 8 shows a basic bifurcation diagram fixing all parameters except \(c.\)
We observe a very rapid growth of the amplitude of the oscillations as \( c \) passes through \( c = 0 \). In particular, the transition could be viewed as being similar to a canard-type explosion \([\text{DD95, Eck83, DR96, Kue15}]\) as the growth of the amplitude occurs near a part of \( C_0 \), which is not normally hyperbolic and not attracting, i.e., inside \( \text{int}(C_0) \). Note carefully that if \( x \approx 0 \), then the slow equation for \( y \) has only a small \( x \)-dependence, so \( c \) can actually control growth or decay in this region. Indeed, in this case the singular limit generalized play operator from Theorem 3.1 precisely shows an equilibrium point at \( x = 0 \) for the \( y \)-dynamics if \( c < 0 \).

The last step we would like to check is to illustrate numerically the convergence of the fast-slow system to the system coupled with a generalized play operator depending upon \( \varepsilon \). Figure 9 shows how we converge from an oscillation with quite large excursions outside of \( C_0 \) to the singular limit generalized play operator, which is entirely constrained to \( C_0 \) after the projection of the initial condition. We observe that on the initial transient approach towards the oscillatory solution, there are significant differences in the patterns of the SAOs for different values of \( \varepsilon \). Furthermore, the patterns seem to stabilize a bit more as \( \varepsilon \to 0 \) with more oscillations near the averaged equilibrium points discussed above. This suggests that an asymptotic description of the precise patterns could be possible locally but we leave this as an aspect for future work. Similarly, one could consider a more detailed parameter study, which should also be considered in another context focusing more on several classes of models. Here we only wanted to illustrate the proof of Netushil’s conjecture in our setting of coupled fast-slow systems and show that the associated systems can have interesting nontrivial dynamics.

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