Let $G$ be a finite group and $H$ a normal subgroup. By $D(H; G)$, we denote the crossed product of $C(H)$ and $C^* G$, which is only a subalgebra of the quantum double $D(G)$ of $G$. One can construct a $C^*$-subalgebra $\mathbb{H}$ of the field algebra $\mathbb{H}$ of $G$-spin models, such that $\mathbb{H}$ is a $D(H; G)$-module algebra. The concrete construction of $D(H; G)$-invariant subalgebra $\mathbb{H}$ of $\mathbb{H}$ is given. Moreover, the $C^*$-index of the conditional expectation $z_H$ from $\mathbb{H}$ onto $\mathbb{H}$ is calculated in terms of the quasi-basis for $z_H$.

**KEYWORDS**

$C^*$-index, conditional expectation, quantum double, quasi-basis

**MSC CLASSIFICATION**

46L05; 16S35

1 INTRODUCTION

Assume that $G$ is a finite group with a unit $e$. The $G$-valued spin configuration on the two-dimensional square lattice is the map $\sigma : \mathbb{Z}^2 \to G$ with Euclidean action functional:

$$S(\sigma) = \sum_{(x,y)} f(\sigma^{-1}_x \sigma_y),$$

in which the summation runs over the nearest neighbour pairs in $\mathbb{Z}^2$ and $f : G \to \mathbb{R}$ is a function of the positive type. This kind of classical statistical system or the corresponding quantum field theories are called $G$-spin models; see previous works.\(^1\)\(^-\)\(^3\) Such models provide the simplest examples of lattice field theories exhibiting quantum symmetry. In general, $G$-spin models with an Abelian group $G$ are known to have a symmetry group $G \times \hat{G}$, where $\hat{G}$ is the group of characters of $G$, namely the Pontryagin dual of $G$. If $G$ is non-Abelian, the Pontryagin dual loses its meaning, and the models have a symmetry of a quantum double $D(G)$.\(^4\),\(^5\) Here, $D(G)$ is defined as the crossed product of $C(G)$, the algebra of complex functions on $G$, and group algebra $CG$ with respect to the adjoint action of the latter on the former. Then $D(G)$ is a Hopf $^*$-algebra of finite dimension.\(^6\)\(^-\)\(^8\) Also as in the traditional quantum field theory, one can define a field algebra $F$ associated with this model.\(^9\) There is a natural action $\gamma$ of $D(G)$ on $F$ such that $F$ is a $D(G)$-module algebra with respect to the map $\gamma$. Namely, there is a bilinear map $\gamma : D(G) \times F \to F$ satisfying: $\forall a, b \in D(G), F_1, F_2, F \in F$,

\[
\begin{align*}
(ab)(F) &= a(b(F)), \\
\gamma(F_1 F_2) &= \sum_{(g)} a_{(1)}(F_1) a_{(2)}(F_2), \\
\gamma(F^*) &= (S(a^*)(F))^*.
\end{align*}
\]
Here and from now on, by \( a(F) \), we always denote \( \gamma(a \times F) \) in \( F \). Under the action of \( \gamma \) on \( F \), the observable algebra \( \mathcal{A}_{(G,G)} \) as the \( D(G) \)-invariant subalgebra of \( F \) is obtained. And there exists a duality between \( \mathcal{A}_G \) and \( D(G) \), i.e., there is a unique \( C^* \)-representation of \( D(G) \) such that \( D(G) \) and \( \mathcal{A}_{(G,G)} \) are commutants with each other.

In Xin and Jiang,\(^{10}\) we consider a more general situation. Let \( H \) be a normal subgroup of \( G \), then \( D(H; G) \) is defined as the crossed product of \( C(H) \) and \( C_H \) with respect to the adjoint action of the latter on the former. One can construct a \( C^* \)-subalgebra \( P_H \) of the field algebra \( F \) of \( G \)-spin models, called the field algebra of \( G \)-spin models determined by \( H \), such that \( P_H \) is a \( D(H; G) \)-module algebra even though \( D(H; G) \) is not a Hopf subalgebra of \( D(G) \). Then the observable algebra \( \mathcal{A}_{(H,G)} \), which is the set of fixed points of \( P_H \) under the action of \( D(H; G) \) is given. Also there exists a duality between \( D(H; G) \) and \( \mathcal{A}_{(H,G)} \).

In this paper, we continue to describe such models in terms of \( C^* \)-index. We devote Section 2 to algebraic generators for \( \mathcal{A}_{(H,G)} \) by means of discussing the local net structure to \( \mathcal{A}_{(H,G)} \). In Section 3, we construct a quasi-basis for the conditional expectation \( z_H : P_H \to \mathcal{A}_{(H,G)} \), and then obtain the corresponding \( C^* \)-index \( \text{Index}_{z_H} = |G||H| \), where \(|G|\) and \(|H|\) denote the order of the group \( G \) and \( H \), respectively.

Throughout this paper, all algebras are complex unital associative algebras. For more details on Hopf algebras one can refer to the books of Sweedler\(^{11}\) and Abe.\(^{12}\) We shall adopt its notation, such as \( S, \triangle, \varepsilon \) for the antipode, the comultiplication and the counit, respectively. Also we shall use the summation convention, which is standard in Hopf algebra theory:

\[
\begin{align*}
\triangle(a) &= \sum_{(a)} a_{(1)} \otimes a_{(2)}, \\
\triangle^{(2)}(a) &= \triangle \circ (\text{id} \otimes \triangle)(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes a_{(3)}, \\
\triangle^{(n)}(a) &= \triangle^{(n-1)} \circ (\text{id} \otimes \triangle)(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)} \otimes \ldots \otimes a_{(n+1)},
\end{align*}
\]

where the second one holds since \( (\text{id} \otimes \triangle) \circ \triangle = (\triangle \otimes \text{id}) \circ \triangle \), and so on.

### 2 THE STRUCTURE OF THE OBSERVABLE ALGEBRA IN \( P_H \)

Suppose that \( H \) is a normal subgroup of \( G \). In the previous paper,\(^{10}\) we defined a Hopf *-algebra \( D(H; G) \) and then constructed a \( C^* \)-subalgebra \( P_H \) in the field algebra \( F \) of \( G \)-spin models. Under the action \( \gamma \) of \( D(H; G) \) on it, \( P_H \) becomes a \( D(H; G) \)-module algebra and the observable algebra \( \mathcal{A}_{(H,G)} \) as the \( D(H; G) \)-invariant subalgebra of \( P_H \) is given. This section will discuss a local net structure on \( \mathcal{A}_{(H,G)} \), which can be achieved by finding algebraic generators for \( \mathcal{A}_{(H,G)} \) with local commutation relations. Let us begin with the following definition.

**Definition 2.1** (Xin and Jiang\(^{10}\)). \( D(H; G) \) is the crossed product of \( C(H) \) and group algebra \( C_H \), where \( C(H) \) denotes the set of complex functions on \( H \), with respect to the adjoint action of the latter on the former.

Using the linear basis elements \((h, g)\) of \( D(H; G) \), the multiplication can be written as

\[
(h, g)(t, s) = \delta_{hg,gt}(h, gs),
\]

where \( \delta_{hg} \) is the unit of \( D(H; G) \). Also, the structure maps are defined as

\[
\begin{align*}
(h, g)^* &= (g^{-1}hg, g^{-1}), & (\ast \text{-operation}) \\
\triangle(h, g) &= \sum_{t \in H} (t, g) \otimes (t^{-1}h, g), & (\text{coproduct}) \\
\varepsilon(h, g) &= \delta_{h, e}, & (\text{counit}) \\
S(h, g) &= (g^{-1}h, g^{-1}), & (\text{antipode})
\end{align*}
\]
for \((h, g) \in D(H; G)\). One can prove \(D(H; G)\) is a Hopf *-algebra with a unique element
\[
z_{i_0} = \frac{1}{|G|} \sum_{g \in G} (e, g),
\]
called a cointegral, satisfying
\[
avz_{i_0} = z_{i_0} a = \epsilon(a)z_{i_0}, \quad \forall a \in D(H; G),
\]
and \(\epsilon(z_{i_0}) = 1\). As a result, \(D(H; G)\) is a semisimple *-algebra of finite dimension. Consequently it can be a \(C^*\)-algebra in a natural way.\(^{10}\)

**Remark 2.1.**

1. If \(H\) is a subgroup of \(G\), not a normal subgroup. One can prove there is not the adjoint action of \(\mathbb{C}H\) on \(C(G)\), and then \(D(H; G)\) can not be defined.
2. Different from the case of \(D(G; H)\), which is the crossed product of \(C(G)\) and \(\mathbb{C}H\) with respect to the adjoint action of the latter on the former,\(^9,13\) \(D(H; G)\) is not a Hopf subalgebra of \(D(G)\), even though it is a subalgebra of \(D(G)\). As to nonbalanced quantum double, one can refer to previous works.\(^{14,15}\)
3. Also, the relation \(S^2 = \text{id}\) holds in \(D(H; G)\), which implies that \(\forall a \in D(H; G),\)
\[
\sum_{(a)} S(a_{(2)})a_{(1)} = \sum_{(a)} a_{(2)}S(a_{(1)}) = \epsilon(a)1_{DXH;G}.
\]

As in the traditional case, one can define the local quantum field algebra associated with the models.

**Definition 2.2** (Xin and Jiang\(^9\)). \(F_{\text{loc}}\) is an associative algebra with a unit \(I\) generated by \(\{\delta_g(x), \rho_h(l) : g \in G, h \in H; x \in \mathbb{Z}, l \in \mathbb{Z} + \frac{1}{2}\}\) subject to
\[
\sum_{g \in G} \delta_g(x) = I = \rho_s(l),
\]
\[
\delta_{g_1}(x)\delta_{g_2}(x) = \delta_{g_1g_2}(x),
\]
\[
\rho_{h_1}(l)\rho_{h_2}(l) = \rho_{h_1h_2}(l),
\]
\[
\delta_{g_1}(x)\delta_{g_2}(x') = \delta_{g_1}(x')\delta_{g_2}(x),
\]
\[
\rho_{h_1}(l)\delta_{g_1}(x) = \begin{cases} 
\delta_{\rho_h(l,g_1}(x), & l < x, \\
\delta_{g_1}(x), & l > x,
\end{cases}
\]
\[
\rho_{h_1}(l)\rho_{h_2}(l') = \rho_{h_1h_2}(l')\rho_{h_1^{-1}h_2^{-1}}(l), & l > l'.
\]

for \(x, x' \in \mathbb{Z}; l, l' \in \mathbb{Z} + \frac{1}{2}\) and \(h_1, h_2 \in H, g_1, g_2 \in G\). In particular, if \(H = G\), by \(F_{\text{loc}}\) we denote \(F_{\text{loc}}\).

The *-operation is defined on the generators as \(\delta_{g}^*(x) = \delta_{g^{-1}}(x), \rho_h^*(l) = \rho_{h^{-1}}(l)\) and is extended antilinearly and antimultiplicatively to \(F_{\text{loc}}\). In this way, \(F_{\text{loc}}\) becomes a unital *-algebra.

For any finite subset \(\Lambda \subseteq 1/2\mathbb{Z}\), let \(F_{\mu} (\Lambda)\) be the subalgebra of \(F_{\text{loc}}\) generated by
\[
\left\{ \delta_g(x), \rho_h(l) : g \in G, h \in H, x \in \Lambda \cap \mathbb{Z}, l \in \Lambda \cap (\mathbb{Z} + 1/2) \right\}.
\]

In particular, we consider an increasing sequence of intervals \(\Lambda_n \equiv \Lambda_{l_n,x_n}\), for \(l_n \in \mathbb{Z} + 1/2, x_n \in \mathbb{Z}, n \in \mathbb{N}\), where
\[
\Lambda_{n+1} = \begin{cases} 
\Lambda_{l_{n+1},x_{n+1}}, & n \in 2\mathbb{Z} - 1, \\
\Lambda_{l_{n-1},x_{n-1}}, & n \in 2\mathbb{Z},
\end{cases}
\]

with \(x_1 = 0, l_1 = -1/2\). Szlachányi and Vecsényesi\(^3\) have shown that \(F(\Lambda_n), n \in \mathbb{N}\) are full matrix algebras; they can be identified with \(M_{|G|}\). Moreover, under the induced norm, \(F(\Lambda_n)\) are finite dimensional \(C^*\)-algebras. Hence \(F_{\mu} (\Lambda_n), n \in \mathbb{N}\) are subalgebras of full matrix algebras, and then they are finite dimensional \(C^*\)-algebras. The natural embeddings \(\iota_n : F_{\mu} (\Lambda_n) \to F_{\mu} (\Lambda_{n+1})\), that identify the \(\delta\) and \(\rho\) generators, are norm preserving. Using the \(C^*\)-inductive limit,\(^{16}\) a \(C^*\)-algebra \(F_{\mu}\) can be given by
Then the map $\gamma$ is called a conditional expectation. If $H$ in the field algebra $\Gamma$ is a subalgebra of $I$ where $I$ is the unit of $\Lambda$.

Proof. We know that $\Lambda$ is a $\gamma$-subalgebra of $\Lambda$, since $\Lambda$ is a projection of norm one. The conditional expectation $z_n : \Lambda \to \Lambda$ will be addressed in the next section.

In this section, we will give the concrete construction of $\Lambda$. In order to do this, for $g \in G, x \in \mathbb{Z}$, and $l \in \mathbb{Z} + \frac{1}{2}$, set

$$
\psi_g(x) = \sum_{h \in G} \phi_{h^{-1}}(x - \frac{1}{2}) \phi_{h^{-1}}(x + \frac{1}{2})
$$

and

$$
\psi_g(l) = \sum_{h \in G} \delta_{h}(l - \frac{1}{2}) \phi_{h^{-1}}(l + \frac{1}{2}).
$$

Lemma 2.2. Let $\Lambda^{\frac{1}{2}} \subseteq \mathbb{Z}$ be a finite interval for $n, m \in \mathbb{Z}$. The $D(H; G)$-invariant subalgebra of $\Lambda^{\frac{1}{2}}$ is generated by

$$
\{ \psi_g(x), \psi_h(l) : g \in G, h \in H, x \in \Lambda^{\frac{1}{2}} \cap \mathbb{Z}, l \in \Lambda^{\frac{1}{2}} \cap (\mathbb{Z} + \frac{1}{2}) \}.
$$

That is,

$$
z_n \left( \Lambda^{\frac{1}{2}} \right) = \left\{ \psi_g(x), \psi_h(l) : g \in G, h \in H, x \in \Lambda^{\frac{1}{2}} \cap \mathbb{Z}, l \in \Lambda^{\frac{1}{2}} \cap (\mathbb{Z} + \frac{1}{2}) \right\}.
$$

Proof. We know that $\Lambda^{\frac{1}{2}}$ is a $\gamma$-subalgebra of $\Lambda$, generated by

$$
\{ \delta_g(x), \phi_h(l) : g \in G, h \in H, x \in \Lambda^{\frac{1}{2}} \cap \mathbb{Z}, l \in \Lambda^{\frac{1}{2}} \cap (\mathbb{Z} + \frac{1}{2}) \}.
$$
Notice that for $h_i \in H, i = 1, 2, \ldots, m$ with $h_1h_2 \ldots h_m = e$,

$$z_{ii} \left( \delta_{g_1}(1)\delta_{g_2}(2) \ldots \delta_{g_m}(m)\phi_{h_1} \left( \frac{1}{2} \right) \phi_{h_2} \left( \frac{1}{2} \right) \ldots \phi_{h_m} \left( m - \frac{1}{2} \right) \right)$$

$$= \frac{1}{|G|} \sum_{f \in G} \left( f, e \right) \left( \delta_{g_1}(1)\delta_{g_2}(2) \ldots \delta_{g_m}(m)\phi_{h_1} \left( \frac{1}{2} \right) \phi_{h_2} \left( \frac{1}{2} \right) \ldots \phi_{h_m} \left( m - \frac{1}{2} \right) \right)$$

$$= \frac{1}{|G|} \sum_{f \in G} \sum_{g \in G, f \in H} \left( t_1, f \right) \delta_{g_1}(1)(t_1t_2^{-1}, f)\delta_{g_2}(2) \ldots \left( t_{m-1}t_m^{-1}, f \right)\delta_{g_m}(m)$$

$$= \frac{1}{|G|} \sum_{g \in G} \left( \delta_{fg_1}(1)\delta_{fg_2}(2) \ldots \delta_{fg_m}(m)\phi_{fh_1} \left( \frac{1}{2} \right) \phi_{fh_2} \left( \frac{1}{2} \right) \ldots \phi_{fh_m} \left( m - \frac{1}{2} \right) \right)$$

which together with the following equation

$$w_{i1} \left( \frac{1}{2} \right) w_{i2} \left( \frac{1}{2} \right) \ldots w_{im-1} \left( m - \frac{1}{2} \right) v_{j1} \left( 1 \right) v_{j2} \left( 2 \right) \ldots v_{jm-1} \left( m - 1 \right)$$

$$= \sum_{s \in E \cap H} \sum_{t \in G} \delta_{g_1}(1)\delta_{g_2}(2) \ldots \delta_{g_m}(m)\phi_{h_1} \left( \frac{1}{2} \right) \phi_{h_2} \left( \frac{1}{2} \right) \ldots \phi_{h_m} \left( m - \frac{1}{2} \right)$$

$$= \frac{1}{|G|} \sum_{g \in G} \left( \delta_{g_1}(1)\delta_{g_2}(2) \ldots \delta_{g_m}(m)\phi_{h_1} \left( \frac{1}{2} \right) \phi_{h_2} \left( \frac{1}{2} \right) \ldots \phi_{h_m} \left( m - \frac{1}{2} \right) \right)$$

yields that

$$z_{ii} \left( F_{ii} \left( \Lambda_{1, m} \right) \right)$$

is a $C^*$-subalgebra of $A(H, G)$, generated by

$$\left\{ \omega_g(x), \nu_h(l) : g \in G, h \in H, x \in \Lambda_{1, m-\frac{1}{2}} \cap \mathbb{Z}, l \in \Lambda_{1, m-\frac{1}{2}} \cap \left( \mathbb{Z} + \frac{1}{2} \right) \right\}.$$

It follows from induction that $z_{ii} \left( F_{ii} \left( \Lambda_{n, \frac{1}{2}, m} \right) \right)$ is generated by

$$\left\{ \omega_g(x), \nu_h(l) : g \in G, h \in H, x \in \Lambda_{n, m-\frac{1}{2}} \cap \mathbb{Z}, l \in \Lambda_{n, m-\frac{1}{2}} \cap \left( \mathbb{Z} + \frac{1}{2} \right) \right\}.$$

The proof is completed.

**Remark 2.2.** For $\Lambda \subseteq \frac{1}{2} \mathbb{Z}$, let

$$A_n(\Lambda) = \left\{ \nu_h(x), w_g(l) : h \in H, g \in G, x \in \Lambda \cap \mathbb{Z}, l \in \Lambda \cap \left( \mathbb{Z} + \frac{1}{2} \right) \right\}.$$

Lemma 2.2 together with Lemma 2.1 implies that $z_{ii} : F_{ii} \left( \Lambda_{n, \frac{1}{2}, m} \right) \rightarrow A_n(\Lambda_{n, m-\frac{1}{2}})$ is a conditional expectation.
Theorem 2.1. The observable algebra $A_{(H,G)}$ is the $C^*$-algebra given by the $C^*$-inductive limit

$$A_{(H,G)} = \bigcup_{\Lambda} A_{(\Lambda)}(\Lambda).$$

Proof. If $A \in A_{(H,G)}$ and $\epsilon > 0$, then from Lemma 2.2 and the continuity of the projection $z_H$, we know $A = z_H(A)$ and there is $B \in F_{(\Lambda)}(\Lambda_{n,\frac{1}{m}})$ with $\|A - B\| < \epsilon$, which implies that

$$\|A - z_H(B)\| = \|z_H(A - B)\| \leq \|A - B\| < \epsilon,$$

and $z_H(B) \in A_{(\Lambda)}(\Lambda_{n,m-\frac{1}{m}})$. \qed

3 | $C^*$-INDEX

This section aims to calculate the $C^*$-index of conditional expectation $z_H : F_{(\Lambda)} \to A_{(H,G)}$, where $H$ is a normal subgroup of $G$ with $[G:H] = k$ and $t_1 = e, t_2, \ldots, t_k$ is a left coset representation of $H$ in $G$, namely $G = \bigcup_{i=1}^{k} t_i H$ and $i \neq j$ induces that $t_i H \cap t_j H = \emptyset$, where $e$ is the unit of $G$.

Definition 3.1 (Watatani\textsuperscript{17}). Let $\Gamma$ be a conditional expectation from a unital $C^*$-algebra $B$ onto its unital $C^*$-subalgebra $A$. A finite family $\{u_1, u_1^*, (u_2, u_2^*), \ldots, (u_n, u_n^*)\} \subseteq B \times B$ is called a quasi-basis for $\Gamma$ if for all $a \in B,$

$$\sum_{i=1}^{n} u_i \Gamma(u_i^* a) = a = \sum_{i=1}^{n} \Gamma(a u_i) u_i^*.$$

Furthermore, if there exists a quasi-basis for $\Gamma$, we call $\Gamma$ of finite-index type. In this case, we define the index of $\Gamma$ by

$$\text{Index } \Gamma = \sum_{i=1}^{n} u_i u_i^*.$$

Remark 3.1.

(1) If $\Gamma$ is a conditional expectation of finite-index type, then Index $\Gamma$ is in the center of $A$ and does not depend on the choice of quasi-basis.\textsuperscript{17}

(2) Let $N \subseteq M$ be factors of type $\text{II}_1$ and $\Gamma : M \to N$ the canonical conditional expectation determined by the unique normalized trace on $M$, then Index $\Gamma$ is exactly Jones index $[M, N]$ based on the coupling constant.\textsuperscript{18} More generally, let $M$ be a ($\sigma$-finite) factor with a subfactor $N$ and $\Gamma$ a normal conditional expectation from $M$ onto $N$, then $\Gamma$ is of finite-index if and only if Index $\Gamma$ is finite in the sense of Kosaki,\textsuperscript{19} and the values of Index $\Gamma$ are equal.

In order to get the $C^*$-index of $z_H : F_{(\Lambda)} \to A_{(H,G)}$, it is crucial to find its quasi basis.

Theorem 3.1. For fixed $k \in \mathbb{Z}, x \in G, y \in H$, set

$$u_{x,y} = \sqrt{|G|} \delta_x(k) \rho_y(k + \frac{1}{2}).$$

Then $\{(u_{x,y}, u_{x,y}^*) : x \in G, y \in H\}$ is a quasi-basis for $z_H : F_{(\Lambda)} \to A_{(H,G)}$.

Proof. Without loss of generality, one can consider the case $k = 1$.

Firstly, one can show that $\{(u_{x,y}, u_{x,y}^*) : x \in G, y \in H\}$ is a quasi-basis for $z_H : F_{(\Lambda)}(\Lambda_{1,m}) \to A_{(\Lambda_{1,m-\frac{1}{m}})}$, for any $m \in \mathbb{Z}$ and $m > 1$. 

Note that

\[
\sum_{x \in G} \sum_{y \in H} u_{x,y} z_{ul} \left[ u_{x,y}^* \delta_{g_1} (1) \delta_{g_2} (2) \ldots \delta_{g_m} (m) \rho_{h_1} \left( \frac{1}{2} \right) \rho_{h_2} \left( \frac{3}{2} \right) \ldots \rho_{h_n} \left( m - \frac{1}{2} \right) \right] = |G| \sum_{x \in G} \sum_{y \in H} u_{x,y} z_{ul} \left[ \delta_{g_1} (1) \delta_{g_2} (2) \ldots \delta_{g_m} (m) \rho_{h_1} \left( \frac{1}{2} \right) \rho_{h_2} \left( \frac{3}{2} \right) \ldots \rho_{h_n} \left( m - \frac{1}{2} \right) \right]
\]

\[
= |G| \sum_{x \in G} \sum_{y \in H} u_{x,y} z_{ul} \left[ \delta_{g_1} (1) \delta_{g_2} (2) \delta_{g_3} (3) \ldots \delta_{g_m} (m) \rho_{h_1} \left( \frac{1}{2} \right) \rho_{h_2} \left( \frac{3}{2} \right) \ldots \rho_{h_n} \left( m - \frac{1}{2} \right) \right]
\]

\[
= |G| \sum_{x \in G} \sum_{y \in H} u_{x,y} z_{ul} \left[ \delta_{g_1} (1) \delta_{g_2} (2) \delta_{g_3} (3) \ldots \delta_{g_m} (m) \rho_{h_1} \left( \frac{1}{2} \right) \rho_{h_2} \left( \frac{3}{2} \right) \ldots \rho_{h_n} \left( m - \frac{1}{2} \right) \right]
\]

\[
= |G| \sum_{x \in G} \sum_{y \in H} \delta_{g_1} (1) \delta_{g_2} (2) \delta_{g_3} (3) \ldots \delta_{g_m} (m) \rho_{h_1} \left( \frac{1}{2} \right) \rho_{h_2} \left( \frac{3}{2} \right) \ldots \rho_{h_n} \left( m - \frac{1}{2} \right)
\]

\[
\times \rho_{h_1} \left( \frac{3}{2} \right) \ldots \rho_{h_n} \left( m - \frac{1}{2} \right)
\]

\[
= |G| \sum_{x \in G} \sum_{y \in H} \delta_{g_1} (1) \delta_{g_2} (2) \delta_{g_3} (3) \ldots \delta_{g_m} (m) \rho_{h_1} \left( \frac{1}{2} \right) \rho_{h_2} \left( \frac{3}{2} \right) \ldots \rho_{h_n} \left( m - \frac{1}{2} \right)
\]

which yields that for any \( a \in F_{\#} (\Lambda_{\frac{1}{2}}, m) \),

\[
\sum_{x \in G} \sum_{y \in H} u_{x,y} z_{ul} (u_{x,y}^* a) = a.
\]

Similarly, one can verify

\[
\sum_{x \in G} \sum_{y \in H} z_{ul} (au_{x,y}) u_{x,y}^* = a, \forall a \in F_{\#} (\Lambda_{\frac{1}{2}}, m).
\]

By induction, we can show that \( \{(u_{x,y}, u_{x,y}^*) : x \in G, y \in H\} \) is a quasi-basis for \( z_{ul} : F_{\#} (\Lambda_{\frac{1}{2}}, m) \to A_{\#} (\Lambda_{\frac{1}{2}}, m) \), for any \( n, m \in \mathbb{Z} \) and \( n < m \).

Since \( z_{ul} \) is a projection of norm one, \( z_{ul} \) can therefore be extended to the map of \( \bigcup_{n < m} F_{\#} (\Lambda_{\frac{1}{2}}, m) \) onto \( \bigcup_{n < m} A_{\#} (\Lambda_{\frac{1}{2}}, m) \) by continuity, and then \( \{(u_{x,y}, u_{x,y}^*) : x \in G, y \in H\} \) is a quasi-basis for \( z_{ul} : F_{\#} (\Lambda_{\frac{1}{2}}, m) \to A_{\#} (\Lambda_{\frac{1}{2}}, m) \).

Finally, the uniqueness of the C*-inductive limit\(^{16}\) implies that \( F_{\#} = \bigcup_{n < m} F_{\#} (\Lambda_{\frac{1}{2}}, m) \) and \( A_{\#} (H,G) = \bigcup_{n < m} A_{\#} (\Lambda_{\frac{1}{2}}, m) \).

As a result, \( \{(u_{x,y}, u_{x,y}^*) : x \in G, y \in H\} \) is a quasi-basis for \( z_{ul} : F_{\#} \to A_{\#} (H,G) \).

From Theorem 3.1, we know \( z_{ul} \) is a conditional expectation of finite-index type, which can guarantee that \( z_{ul} \) is nondegenerate.
Remark 3.2.

(1) For \( k, l \in \mathbb{Z} \) with \( k \leq l \), \( x \in G, y \in H \), set

\[
w_{x,y} = \sqrt{|G|} \delta_x(k) \rho_y(l + \frac{1}{2}).
\]

One can verify that \( \{(w_{x,y}, w_{x,y}^*) : x \in G, y \in H\} \) is a quasi-basis for \( z_u : F_{n} \to A_{(H,G)} \).

(2) Let \( k, l \in \mathbb{Z} \) with \( k > l \), \( x \in G, y \in H \), put

\[
v_{x,y} = \sqrt{|G|} \delta_x(k) \rho_y(l + \frac{1}{2}),
\]

then \( \{(v_{x,y}, v_{x,y}^*) : x \in G, y \in H\} \) is not a quasi-basis for \( z_u : F_{n} \to A_{(H,G)} \). In fact, one can show that \( \{(v_{x,y}, v_{x,y}^*) : x \in G, y \in H\} \) is not a quasi-basis for \( z_u : F_{n}(\Lambda_{\frac{1}{2},\frac{3}{2}}) \to A_{n}(\Lambda_{\frac{1}{2},\frac{3}{2}}) \), where \( v_{x,y} = \sqrt{|G|} \delta_x(1) \rho_y \left( \frac{1}{2} \right) \).

Notice that

\[
\sum_{x \in G, y \in H} v_{x,y}^* \varphi_{x,y} \left[ \delta_x(1) \delta_y(2) \phi_{x,y} \left( \frac{1}{2} \right) \phi_{y,z} \left( \frac{1}{2} \right) \right] = |G| \sum_{x \in G, y \in H} v_{x,y} \varphi_{x,y} \left[ \delta_x(1) \delta_y(1) \phi_{x,y} \left( \frac{1}{2} \right) \phi_{y,z} \left( \frac{1}{2} \right) \right] = |G| \sum_{x \in G, y \in H} v_{x,y} \varphi_{x,y} \left[ \delta_x(1) \delta_y(1) \phi_{x,y} \left( \frac{1}{2} \right) \phi_{y,z} \left( \frac{1}{2} \right) \right] = |G| \sum_{x \in G, y \in H} v_{x,y} \varphi_{x,y} \left[ \delta_x(1) \phi_{x,y} \left( \frac{1}{2} \right) \phi_{y,z} \left( \frac{1}{2} \right) \right] = |G| \sum_{x \in G, y \in H} v_{x,y} \varphi_{x,y} \left[ \delta_x(1) \phi_{x,y} \left( \frac{1}{2} \right) \phi_{y,z} \left( \frac{1}{2} \right) \right] = |G| \sum_{x \in G, y \in H} v_{x,y} \varphi_{x,y} \left[ \delta_x(1) \phi_{x,y} \left( \frac{1}{2} \right) \phi_{y,z} \left( \frac{1}{2} \right) \right]
\]

which can imply that for some \( a \in F_{n} \left( \Lambda_{\frac{1}{2},\frac{3}{2}} \right) \), we have

\[
\sum_{x \in G, y \in H} v_{x,y} \varphi_{x,y} (v_{x,y}^* a) \neq a.
\]

Now it is time to arrive at the main result of the paper.

**Theorem 3.2.** The \( \text{C}^* \)-index of \( z_u : F_{n} \to A_{(H,G)} \) is \( |G||H|I \).

**Proof.** Since \( \text{Index}_{z_u} \) does not depend on the choice of quasi-basis, then

\[
\text{Index}_{z_u} = \sum_{x \in G, y \in H} u_{x,y} \varphi_{x,y} = |G| \sum_{x \in G, y \in H} \delta_x(1) \rho_y \left( \frac{3}{2} \right) \delta_x(1) \rho_y \left( \frac{3}{2} \right) = |G| \sum_{x \in G, y \in H} \delta_x(1) \rho_y \left( \frac{3}{2} \right) \rho_y \left( \frac{3}{2} \right) = |G| \sum_{x \in G, y \in H} \delta_x(1) \rho_y \left( \frac{3}{2} \right) = |G||H|I.
\]

** Remark 3.3.** In particular, if \( H = G \), then \( z_u = \frac{1}{|G|} \sum_{g \in G} (e, g) : F \to A_{(G,G)} \) is a conditional expectation of finite-index type, and \( \text{Index}_{z_u} = |G|^2 I \).
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CONFLICT OF INTEREST
This work does not have any conflicts of interest.

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