NATURAL DIFFERENTIAL OPERATIONS ON
MANIFOLDS: AN ALGEBRAIC APPROACH

P. I. KATSYLO AND D. A. TIMASHEV

ABSTRACT. We consider natural algebraic differential operations acting on geometric quantities over smooth manifolds. We introduce a method of study and classification of such operations, called IT-reduction. It reduces the study of natural operations to the study of polynomial maps between (vector) spaces of jets which are equivariant with respect to certain algebraic groups. Using the IT-reduction, we obtain short and conceptual proofs of some known results on the classification of certain natural operations (the Schouten theorem, etc) together with new results including the non-existence of a universal deformation quantization on Poisson manifolds.

INTRODUCTION

In differential geometry, there are many nontrivial formulæ and theorems based on local calculations. Typical examples are: the Bianchi identities, the Gilkey theorem [Gi1], the Weitzenböck formula [Ber Ch. 1, I]. Such theorems and formulæ of purely local nature lie in the basis of differential geometry. Moreover, it often happens that a discovery of some local formula allows to solve an important problem. For example, the Gilkey theorem led to a new proof of the index theorem [ABP, Gi2]. Thus it is tempting to look for a universal approach to local problems of differential geometry.

One of possible approaches, called formal geometry, was suggested by I. M. Gelfand and D. A. Kazhdan in 1971 [GK]. Approximately at the same time, E. B. Vinberg observed that using simple arguments from the representation theory of algebraic groups and invariant theory makes local calculations in differential geometry much easier and more intelligible. (This was one of initial motivations for the study of invariant theory at the Moscow school of invariant theory leaded by Vinberg.) This approach reduces local problems of differential geometry to problems in the invariant theory of finite-dimensional representations.

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of linear algebraic groups. We call it the invariant-theoretic reduction (IT-reduction).

A simplified version of the IT-reduction method is exposed in [Ka]. In this paper we introduce this method in maximal generality which is necessary for applications. The idea of local study of geometric quantities and natural differential operations between them by considering jet spaces and the actions of coordinate transformations on them was exploited by many researchers, see e.g. the monograph [KMS]. We develop an instrumental approach concentrating on effective representation- and invariant-theoretic methods for solving concrete local problems of differential geometry. For instance, in [KMS] the concept of natural operations is developed in very general context and much effort is put on proving that certain natural operations are of finite order, using Peetre-type theorems, while we impose the finite order assumption from the very beginning and consider mainly differential operations given by algebraic formulae believing that this case is most interesting and essential in applications. We illustrate the IT-reduction method by solving several local problems. Some of them were previously solved by other authors using ad hoc methods, and other results are new.

Now we briefly describe the content of the paper.

In §1 we introduce basic notions of geometric objects and quantities, natural bundles and differential operations. Essentially, geometric quantities (e.g. tensor fields) are sections of fibre bundles associated with coframe bundles and natural differential operations act on geometric quantities by universal differential formulae that are invariant under coordinate transformations. Morally, such operations, like exterior differential or the curvature of a Riemannian metric, should have intrinsic “physical meaning” since they do not depend on a chosen frame of reference. In this section, we explain the IT-reduction method. Also, we recall some basic facts from the representation theory of classical groups, which are used in computations.

Then we concentrate on polynomial natural differential operations acting on tensor fields. Starting with some simple reductions and general finiteness results in §2 we derive in §3 the classification of natural linear differential operations, which goes back to Schouten. In §4 we use the IT-reduction to extend this result to manifolds with additional symplectic structure, cf. [Ru]. This requires an extension of the notion of a natural differential operation to manifolds equipped with an additional structure, see §4. Finally, we prove in §5 that there exists no universal formula for deformation quantization on Poisson manifolds which is invariant under coordinate transformations.

**Convention.** In formulæ of tensor calculus, we systematically use the Einstein summation rule, i.e., assume by default the summation in each
pair of coinciding upper and lower indices which run from 1 up to the
dimension of a manifold.

1. Preliminaries

We work over smooth real or complex manifolds. However our con-
siderations will be purely algebraic and the ground field \( K \) will not play
any essential rôle. So we consider smooth \( K \)-manifolds making no dis-
tinction between the cases \( K = \mathbb{R} \) (differential geometry) and \( K = \mathbb{C} \)
(complex analytic geometry).

1.1. Geometric quantities. The concept of a geometric quantity goes
back to Riemann, see [AVL, Ch. 6, §1]. Loosely speaking, a geometric
quantity is a scalar value or a tuple of scalars associated with each
point of a manifold in a way depending on chosen local coordinates
which transforms under a coordinate change in a regular way depend-
ing only on initial values and partial derivatives, up to a certain order,
of new coordinates with respect to old ones. Natural examples are ten-
sor fields (order 1) and connections (order 2). A rigorous definitio n of
a geometric quantity can be given in several ways.

The classical analytic definition sounds as follows. Let \( M \) be a man-
ifold of dimension \( n \). A geometric quantity is a function \( f_\alpha = f_\alpha(x_\alpha) \) of
local coordinates \( x_\alpha = (x_1^\alpha, \ldots, x_n^\alpha) \) on \( M \) which takes values in a space
(vector space or, more generally, manifold) \( F \) and transforms under a
coordinate change \( x_\alpha \to x_\beta \) in the following way:

\[
(1) \quad f_\beta = \Phi \left( \left\{ \frac{\partial^{l} x_\beta^i}{\partial (x_\alpha^l)^{i_1} \cdots \partial (x_\alpha^l)^{i_n}} \right\}_{1 \leq i \leq k, 1 \leq i \leq n, l_1 + \cdots + l_n = l}, f_\alpha \right),
\]

where \( \Phi \) is a differentiable map. In other words, \( f_\beta \) depends only on \( f_\alpha \)
and on the \( k \)-jet \( J^k g_\beta_\alpha \) of the coordinate transformation \( x_\beta = g_\beta_\alpha(x_\alpha) \).

In order to reformulate this definition in modern terms, consider th e
group \( GL_n^{(k)} \) of \( k \)-jets of local diffeomorphisms \( \mathbb{K}^n \to \mathbb{K}^n \) at 0. Elements
of \( GL_n^{(k)} \) are represented in the form:

\[
x \mapsto g(x) = g_1(x) + g_2(x, x) + \cdots + g_k(x, \ldots, x),
\]

\[
x = (x^1, \ldots, x^n), \quad g_l \in \mathfrak{S}^l(\mathbb{K}^n)^* \otimes \mathbb{K}^n \quad (l = 1, \ldots, k), \quad \det g_1 \neq 0.
\]

\( GL_n^{(k)} \) is a linear algebraic group isomorphic to the automorphism group
of the truncated polynomial algebra

\[
J_n^{(k)} = \mathbb{K}[x^1, \ldots, x^n]/(x^1, \ldots, x^n)^{k+1}.
\]

Its unipotent radical \( NGL_n^{(k)} \) is defined by the equation \( g_1(x) = x \) and
the subgroup \( GL_n \) of linear transformations is a Levi subgroup.

The Lie algebra \( \mathfrak{g}l_n^{(k)} \) of \( GL_n^{(k)} \) is identified with the space of polynomial
vector fields of degree \( \leq k \) vanishing at 0, which are represented
by polynomial maps $x \mapsto g(x)$ as above, but without the restriction $\det g_1 \neq 0$. The Lie algebra $\mathfrak{gl}_n^{(k)}$ of $\text{NGL}_n^{(k)}$ is distinguished by $g_1 = 0$.

It is easy to see that the map $\Phi : \text{GL}_n^{(k)} \times F \to F$ in (1) defines an action of $\text{GL}_n^{(k)}$ on $F$. In most applications, $F$ is a vector or affine space, or an open subset in such a space, and the action $\text{GL}_n^{(k)} : F$ is a rational linear or affine representation.

This consideration leads to a geometric reformulation of the above analytic definition of geometric quantities.

Let $\text{Fr}^k(M)$ denote the coframe bundle of order $k$ on $M$ [AVL, Ch. 6, 1.2]. The fibre of $\text{Fr}^k(M) \to M$ over $z \in M$ consists of coframes of order $k$ at $z$, i.e., $k$-jets of coordinate systems $x = (x^1, \ldots, x^n)$ in a neighborhood of $z$ with $x(z) = 0$. $\text{Fr}^k(M)$ is a principal bundle with respect to the natural action of $\text{GL}_n^{(k)}$. Instead of coframes, one may consider frames of order $k$, which are $k$-jets of inverse coordinate maps sending a neighborhood of $0 \in \mathbb{K}^n$ onto a neighborhood of $z \in M$. Frames of order 1 are determined by fixing a basis of $T_zM$ (= the image of the standard basis of $T_0\mathbb{K}^n = \mathbb{K}^n$), i.e., a usual frame on $M$. The bundles of frames and coframes of order $k$ are canonically isomorphic.

**Geometric definition.** Suppose that $F$ is a manifold equipped with a differentiable action of $\text{GL}_n^{(k)}$. The associated fibre bundle

$$\mathcal{F} = \text{Fr}^k(M) \times^{\text{GL}_n^{(k)}} F := (\text{Fr}^k(M) \times F)/\text{GL}_n^{(k)}$$

is said to be the space of geometric objects of type $F$ on $M$. If the action $\text{GL}_n^{(k)} : F$ is not reduced to the action of the quotient group $\text{GL}_n^{(k-1)}$, then we say that geometric objects of type $F$ have order $k$.

Spaces of geometric objects are also called natural bundles. A geometric quantity of type $F$ is a section of the natural bundle $\mathcal{F} \to M = \text{Fr}^k(M)/\text{GL}_n^{(k)}$. The set of geometric quantities is denoted by $\Gamma(\mathcal{F}) = \Gamma(M, \mathcal{F})$.

**Example 1.** Let $F = (\mathbb{K}^n)^{\otimes p} \otimes (\mathbb{K}^n^*)^{\otimes q}$, with the natural linear action of $\text{GL}_n = \text{GL}_n^{(1)}$. Then $\mathcal{F} = \mathcal{T}^{p,q}$ is the tensor bundle of type $(p, q)$.

Generally, every natural vector bundle $\mathcal{F}$ of order 1, with an additional requirement that the representation $\text{GL}_n : F$ be rational, embeds as a subbundle into a product of $\mathcal{T}^{p,q}$'s. For this reason, we call such $\mathcal{F}$ tensor bundles.

**Example 2.** [AVL, Ch. 6, §4]). The $l$-jets of local sections of $\mathcal{F} \to M$ form the $l$-jet bundle $\mathcal{F}^{(l)}$ of order $k + l$. Every section $f : M \to \mathcal{F}$ defines a section $J^l f : M \to \mathcal{F}^{(l)}$ whose value at $z \in M$ is the $l$-jet $J^l_z f$ of $f$ at $z$. The action of $\text{GL}_n^{(k+l)}$ on the typical fibre $F^{(l)}$ of $\mathcal{F}^{(l)}$ is derived from the transition rule for local coordinate presentations of a section of $\mathcal{F}$. It is given by the formula

$$J^{k+l}_0 g \cdot J^l_0 f = J^l_0 \left( J^{k}_{g^{-1}(x)} g \cdot f \left( g^{-1}(x) \right) \right),$$

(2)
for every local diffeomorphism $g$ of $\mathbb{K}^n$ with $g(0) = 0$ and local section $f$ of $\mathbb{K}^n \times F \to \mathbb{K}^n$ defined in a neighborhood of 0.

If $F$ is a vector space, then $F^{(l)} = J_{g}^{(l)} \otimes F$. (Caution: the $GL_{n}^{(k+l)}$-action on $F^{(l)}$ is not given by the tensor product of linear representations.)

For the sequel, we need a more explicit description of how jets of diffeomorphisms act on jets of tensor fields.

**Lemma 1.** Let $R : GL_{n} \to GL(F)$ be a rational representation and $\rho : gl_{n} \to gl(F)$ the corresponding Lie algebra representation. The natural actions $GL_{n}^{(k+1)} : F^{(k)}$ and $gl_{n}^{(k+1)} : F^{(k)}$ are given by the formulae:

$$(g \cdot f)(x) = R \left( \sum_{l=1}^{k+1} l \cdot g_{l}(g^{-1}(x), \ldots, g^{-1}(x), \cdot) \right) f(g^{-1}(x)),$$

$$(\xi \cdot f)(x) = \sum_{l=1}^{k+1} l \cdot \rho(\xi_{l}(x, \ldots, x, \cdot)) f(x) - \sum_{l=1}^{k} l \cdot f_{l}(x, \ldots, x, \xi(x)),$$

$$\forall g \in GL_{n}^{(k+1)}, \ \xi \in gl_{n}^{(k+1)}, \ f \in F.$$  

Here $g_{l}, \xi_{l} \in S^{l}\mathbb{K}^{n*} \otimes \mathbb{K}^{n}$ and $f_{l} \in S^{l}\mathbb{K}^{n*} \otimes F$ are the homogeneous components of $g, \xi, f$, and the r.h.s. are interpreted as follows. The arguments of $R, \rho$ are polynomial functions in $x$ taking values in $GL_{n}$, resp. in $gl_{n}$, at $x$ from a neighborhood of 0. The r.h.s. are expanded in Taylor series in $x$ and truncated from the order $k + 1$.

**Corollary.**

$$(3) \quad (\xi \cdot f)_{k} = \sum_{l=1}^{k+1} l\gamma(\xi_{l} \otimes f_{k+1-l}) - (k+1-l)\sigma(\xi_{l} \otimes f_{k+1-l}),$$

where $\gamma, \sigma : S^{l}\mathbb{K}^{n*} \otimes \mathbb{K}^{n} \otimes S^{k+1-l}\mathbb{K}^{n*} \otimes F \to S^{k}\mathbb{K}^{n*} \otimes F$ are linear maps defined as follows. To compute $\gamma$, consider $S^{l}\mathbb{K}^{n*} \otimes \mathbb{K}^{n}$ as a subspace of $S^{l-1}\mathbb{K}^{n*} \otimes gl_{n}$, apply $gl_{n}$ to $F$ via $\rho$, and conclude by the multiplication $S^{l-1}\mathbb{K}^{n*} \otimes S^{k+1-l}\mathbb{K}^{n*} \to S^{k}\mathbb{K}^{n*}$. The map $\sigma$ is the contraction of $\mathbb{K}^{n}$ with $S^{k+1-l}\mathbb{K}^{n*}$ followed by the multiplication $S^{l}\mathbb{K}^{n*} \otimes S^{k-1}\mathbb{K}^{n*} \to S^{k}\mathbb{K}^{n*}$.

The proof is a straightforward calculation based on (2).

Note that the space of geometric objects of given type is naturally defined over any $n$-dimensional manifold. This observation leads to a reformulation of the definition of geometric objects in the categorical language $	ext{[PT]}$.

Let $\text{Man}_{n}$ denote the category of $n$-dimensional manifolds where the morphisms are open embeddings. Let $\text{Fib}_{n}$ denote the category of fibre bundles over $n$-dimensional manifolds, the morphisms being differentiable maps of bundles covering the morphisms of their bases in $\text{Man}_{n}$. 
Categorical definition. A type of geometric objects is a functor $\mathcal{F} : \mathcal{M} \rightrightarrows \mathcal{Fib}$ such that $\mathcal{F}(M)$ is a bundle over $M$ for any $n$-dimensional manifold $M$ and $\mathcal{F}(M')$ is the restriction of $\mathcal{F}(M)$ for any open submanifold $M' \subset M$, the morphism $\mathcal{F}(M') \to \mathcal{F}(M)$ induced by $M' \hookrightarrow M$ being the inclusion.

It is easy to see that all $\mathcal{F} = \mathcal{F}(M)$ have one and the same typical fibre $F$. Every local diffeomorphism $g_{\beta\alpha} : U_{\alpha} \to U_{\beta}$ between two sufficiently small neighborhoods of $0 \in \mathbb{K}^n$ with $g_{\beta\alpha}(0) = 0$ induces a diffeomorphism $\mathcal{F}(g_{\beta\alpha}) : \mathcal{F}(U_{\alpha}) \simeq U_{\alpha} \times F \to \mathcal{F}(U_{\beta}) \simeq U_{\beta} \times F$. These $\mathcal{F}(g_{\beta\alpha})$ are compatible with shrinking of $U_{\alpha}, U_{\beta}$ and thus induce transformations of $F$ ( = the fibre at 0) depending only on the germs of $g_{\beta\alpha}$ at 0. Palais and Chuu-Lian Terng [PT] proved that in fact these transformations of $F$ depend only on $J^k g_{\beta\alpha}$ for sufficiently large $k$. This yields an action $\text{GL}_n^k : F$ and isomorphism $\mathcal{F}(M) \simeq \text{Fr}_n^k(M) \times \text{GL}_n^k F$.

1.2. Differential operations. Differential operations act on geometric quantities. Given two natural bundles $\mathcal{V}, \mathcal{W} \to M$, a differential operation $D$ from $\mathcal{V}$ to $\mathcal{W}$ transforms (local) sections of $\mathcal{V}$ into those of $\mathcal{W}$ according to a formula of the following kind, in local coordinates:

$$(4) \quad (Dv)^p = \delta^p \left( \left\{ x^i, v^q, \partial_1^i \cdots \partial_n^q v^q \right\}_{1 \leq i \leq n, 1 \leq q \leq \dim \mathcal{V}, 1 \leq l_1 + \cdots + l_n \leq k} \right), \quad \forall v \in \Gamma(\mathcal{V}),$$

where $\delta^p$ are differentiable functions ($p = 1, \ldots, \dim W$), $V, W$ are typical fibres of $\mathcal{V}, \mathcal{W}$, and $\partial_j = \partial/\partial x^j$ ($j = 1, \ldots, n$).

In other words, a differential operation of order $\leq k$ is a map $D : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{W})$ induced by a morphism $\mathcal{V}^{(k)} \to \mathcal{W}$ (denoted by the same letter) so that $Dv(z) = D(J^k_z v), \forall v \in \Gamma(\mathcal{V}), z \in M$ [AVL, Ch. 6, 4.6]. (The order is exactly $k$ if $D$ does not factor through the canonical projection $\mathcal{V}^{(k)} \to \mathcal{V}^{(k-1)}$.)

Remark. $D$ produces a series of differential operations $\mathcal{V}^{(k+l)} \to \mathcal{W}^{(l)}$ (denoted by the same letter) in an obvious way.

Definition 1. Suppose that $V, W$ are open invariant subsets in vector or affine spaces equipped with rational representations of some $\text{GL}_n^l$. A differential operation $D : \mathcal{V}^{(k)} \to \mathcal{W}$ is algebraic if the maps of fibres $\delta_x : \mathcal{V}^{(k)} \to \mathcal{W}$ are algebraic morphisms of bounded degree. In other words, $\delta^p$ in (4) are rational functions in $v^q, \partial_1^i \cdots \partial_n^q v^q$ whose coefficients are differentiable functions in $x$, denominators depend only on $v^q$, and the degrees of numerators are bounded on $M$.

Remark. One may extend Definition 1 replacing typical fibres $V, W$ by more general algebraic varieties with algebraic $\text{GL}_n^l$-actions. However our formulation is sufficient for many applications.

We focus our attention at algebraic differential operations as the most customary case.
Algebraic differential operations may be regarded as geometric objects, too. In the simplest case where $V, W$ are vector spaces, all $\delta$ are polynomial maps of degree $\leq d$. The set $\text{Mor}_d(V^{(k)}, W)$ of polynomial maps $V^{(k)} \rightarrow W$ of degree $\leq d$ is a vector space with the natural action of $\text{GL}_n^{(k+l)}$ by conjugation. It is easy to see that $D$ is nothing but a section of $\text{Fr}^{k+l}(M) \times^{\text{GL}_n^{(k+l)}} \text{Mor}_d(V^{(k)}, W)$. The general case is handled in the same way if one restricts from above the degrees of denominators, too.

Example 3. Let $V, W$ be vector bundles. Linear differential operators of order $\leq k$ from $V$ to $W$ are geometric quantities taking values in $\mathcal{D}^k(V, W) = \text{Fr}^{k+l}(M) \times^{\text{GL}_n^{(k+l)}} (V^{(k)} \otimes W)$.

Example 4. Let $\mathcal{T} = \mathcal{T}^{1,0}$ be the tangent bundle of $M$. Consider the subbundle $\mathcal{C} \subset \mathcal{D}^1(\mathcal{T}, \mathcal{T} \otimes \mathcal{T}^*)$ consisting of homomorphisms $\nabla$ splitting the natural exact sequence:

$$0 \rightarrow \mathcal{T} \otimes \mathcal{T}^* \xrightarrow{\nabla} \mathcal{T}^{(1)} \rightarrow \mathcal{T} \rightarrow 0.$$  

Sections $\nabla \in \Gamma(\mathcal{C})$ act on vector fields as covariant derivations: in local coordinates, given a vector field $\xi(x) = \xi_0 + \xi_j x^j + \cdots$ in a neighborhood of $z \in M$, we have

$$\nabla \xi(z) = \nabla(\xi_0 \partial_0) + \nabla(\xi_j x^j \partial_j) = \Gamma^k_{ij} \xi_0 \partial_k \otimes dx^j + \xi_j^i \partial_i \otimes dx^j.$$  

Hence geometric quantities with values in $\mathcal{C}$ are linear connections on $M$. They are affine geometric objects of order 2.

1.3. Natural operations. Natural differential operations on geometric quantities are distinguished by the property that their coordinate expression is one and the same for any choice of local coordinates. This property may be reformulated as follows.

Definition 2. A differential operation $D : \mathcal{V}^{(k)} \rightarrow \mathcal{W}$ is called natural if the respective map $\delta : \mathcal{V}^{(k)} \rightarrow \mathcal{W}$ of fibres at $z \in M$ does not depend on $z$ and is $\text{GL}_n^{(k+l)}$-equivariant (assuming that $\mathcal{V}, \mathcal{W}$ have order $\leq l$).

Since a natural differential operation $D$ is uniquely determined by the map of typical fibres $\delta$, it follows that $D$ is naturally defined on geometric quantities of given type over any $n$-dimensional manifold. This observation leads to a functorial point of view on natural operations.

For each type of geometric objects $\mathfrak{F}$, consider the respective functor of geometric quantities $M \rightsquigarrow \Gamma(M, \mathcal{F}) = \Gamma(\mathcal{F})$, which associates with an $n$-manifold $M$ the space of sections of $\mathcal{F} = \mathfrak{F}(M) \rightarrow M$. It becomes a contravariant functor from $\text{Man}_n$ to topological spaces, if we equip $\Gamma(\mathcal{F})$ with the topology of locally uniform convergence of sections and all their partial derivatives.
A natural operation $D : V^{(k)} \to W$ induces a natural transformation of functors: there is a commutative square

\[
\begin{array}{ccc}
\Gamma(M, V) & \xrightarrow{D} & \Gamma(M, W) \\
\downarrow & & \downarrow \\
\Gamma(M', V) & \xrightarrow{D} & \Gamma(M', W),
\end{array}
\]

for every open embedding $M' \hookrightarrow M$. Conversely, under certain conditions a natural transformation of geometric quantities is given by a natural differential operation, by Peetre-type theorems [KMS, Ch. 5].

Algebraic natural operations are given by everywhere defined rational maps $\delta : V^{(k)} \to W$ that are equivariant with respect to the action of an appropriate algebraic group $\text{GL}_n^{(k+l)}$. Thus the study of such operations is a purely algebraic problem belonging to the representation theory of algebraic groups and invariant theory. For this reason, we call our approach to algebraic differential operations the \textit{IT-reduction}. (IT stands for “invariant-theoretic”.)

In this paper, we consider natural algebraic differential operations on tensor bundles. By the above discussion, they are in a bijective correspondence with polynomial (or, more generally, rational) maps $\delta : J_n^{(k)} \otimes V \to W$ that are $\text{GL}_n$-equivariant and $\text{NGL}_n^{(k+l)}$-invariant, where $V, W$ are rational representations of $\text{GL}_n$ and $\text{GL}_n^{(k+l)}$ acts on $J_n^{(k)} \otimes V = V^{(k)}$ in the natural way (cf. Lemma 1).

\textbf{Example 5.} A classical example of a natural (algebraic) differential operation is the exterior differential $d : \Gamma(\Omega^m) \to \Gamma(\Omega^{m+1})$, where $\Omega^m = \bigwedge^m \mathcal{T}^*$ is the bundle of exterior $m$-forms. The respective map of fibres

\[
\delta : J_n^{(1)} \otimes \bigwedge^m \mathbb{K}^n^* \to \mathbb{K}^n \otimes \bigwedge^m \mathbb{K}^n^* \to \bigwedge^{m+1} \mathbb{K}^n^*
\]

is the canonical projection (ignoring the constant term) followed by the alternation:

\[
\delta(y_0 \otimes y_1 \wedge \cdots \wedge y_m) = d(y_0 \ dy_1 \wedge \cdots \wedge dy_m) = y_0 \wedge y_1 \wedge \cdots \wedge y_m,
\]

$\forall y_0, \ldots, y_m \in \mathbb{K}^n^*$.

\textbf{Example 6.} Lie derivative on a tensor bundle $V$ may be regarded as a natural bilinear operation of order 1 from $\mathcal{T} \times V$ to $V$. The respective map of fibres is

\[
\delta : (\mathbb{K}^n \oplus (\mathbb{K}^n^* \otimes \mathbb{K}^n)) \times (V \oplus (\mathbb{K}^n^* \otimes V)) \to V,
\]

$$(\xi_0 + \xi_1, v_0 + v_1) \mapsto \rho(\xi_1)v_0 - \sigma(\xi_0 \otimes v_1),$$

where $\rho$ is the tensor representation of $\mathfrak{gl}_n \simeq \mathbb{K}^n^* \otimes \mathbb{K}^n$ in the typical fibre $V$, and $\sigma$ denotes the contraction of $\mathbb{K}^n$ with $\mathbb{K}^n^*$.

Other examples are curvatures of Riemannian metrics, etc.
1.4. Classical groups: representations and invariants. We recall some basic facts about rational representations of classical linear groups and classical invariant theory, which we use in the sequel. Our basic references will be [FH], [VP].

A partition is a weakly decreasing sequence of non-negative integers \( \lambda = (\lambda_1, \ldots, \lambda_n), \lambda_1 \geq \ldots \geq \lambda_n \geq 0. \) The length of \( \lambda \) is the number of \( \lambda_i \neq 0, \) and \( |\lambda| = \lambda_1 + \cdots + \lambda_n \) is the number partitioned by \( \lambda. \) Fragments of the form \( d, \ldots, d \) (\( s \) times) in \( \lambda \) are often written as \( ds. \) Omitting the non-negativity condition \( \lambda_n \geq 0 \) yields the definition of a virtual partition.

Let \( S^\lambda \) denote the Schur functor corresponding to a partition \( \lambda. \) To any vector space \( V, \) it relates a subspace \( S^\lambda V \subset V^{\otimes |\lambda|} \) constructed as follows. We may assume that the tensor factors are indexed by the boxes of the Young diagram corresponding to \( \lambda. \) Then \( S^\lambda V \) is obtained from \( V^{\otimes |\lambda|} \) by applying first the symmetrization in each row of the Young diagram, denoted \( \text{Sym}_\lambda, \) and then the alternation in each column \( \text{Alt}_\lambda. \)

\( S^\lambda V \) is an irreducible polynomial \( \text{GL}(V) \)-module spanned by

\[
\text{Alt}_\lambda \left( v_1^{\otimes \lambda_1} \otimes \cdots \otimes v_n^{\otimes \lambda_n} \right), \quad v_1, \ldots, v_n \in V.
\]

Every rational \( \text{GL}(V) \)-module decomposes into a direct sum of irreducible submodules isomorphic to \( S^\lambda V \otimes \det d, \) \( d \in \mathbb{Z}. \)

In our considerations it will be convenient to realize the irreducible rational \( \text{GL}_n \)-modules as \( S^\lambda \mathbb{K}^{n*} \otimes \det d. \) Such a module is determined, up to isomorphism, by a virtual partition \( \overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_n), \overline{\lambda}_i = \lambda_i - d. \) It contains unique, up to proportionality, eigenvectors with respect to the mutually opposite lower- and upper-triangular Borel subgroups \( B^-, B^+ \subset \text{GL}_n, \) namely

\[
v^-_\lambda = \text{Alt}_\lambda \left( (x_1^{\otimes \lambda_1} \otimes \cdots \otimes x_n^{\otimes \lambda_n}) \right),
\]

\[
v^+_\lambda = \text{Alt}_\lambda \left( (x_n^{\otimes \lambda_1} \otimes \cdots \otimes x_1^{\otimes \lambda_n}) \right),
\]

called lowest, resp. highest, weight vectors. Note that \( v^\pm_\lambda \) generates \( S^\lambda \mathbb{K}^{n*} \otimes \det d \) as a \( B^\mp \)-module.

Similarly, the irreducible representations of \( \text{SL}_n \) are realized in \( S^\lambda \mathbb{K}^{n*}, \) \( \lambda_n = 0. \) The irreducible representations of \( \text{Sp}_n \) (\( n \) even) are parameterized by partitions \( \lambda \) of length \( l \leq n/2 \) and realized in the subspaces \( S^{(\lambda)} \mathbb{K}^{n*} \subset S^\lambda \mathbb{K}^{n*} \) spanned by

\[
\text{Alt}_\lambda \left( y_1^{\otimes \lambda_1} \otimes \cdots \otimes y_l^{\otimes \lambda_l} \right),
\]

where \( y_1, \ldots, y_l \) span an isotropic subspace in \( \mathbb{K}^{n*}. \) \( B^\pm \cap \text{Sp}_n \) are mutually opposite Borel subgroups in \( \text{Sp}_n, \) with highest/lowest vectors \( v^\pm_\lambda \in S^{(\lambda)} \mathbb{K}^{n*}, \) provided that the symplectic form has secondary-diagonal matrix.

There are effective formulae for decomposing certain tensor products.
Pieri formulæ.

\[ S^\lambda K_n^* \otimes S^k K_n^* \simeq \bigoplus_{\lambda', |\lambda'|=|\lambda|+k, \lambda'_i \geq \lambda_i, \lambda'_i \geq \lambda_{i+1}} S^{\lambda'} K_n^* \] as GL_n-modules,

\[ S^{(\lambda)} K_n^* \otimes S^k K_n^* \simeq \bigoplus_{\lambda', \mu, |\mu|=|\lambda|-p, |\lambda'|=|\lambda|+k-p, p \leq k, \lambda_i, \lambda'_i \geq \mu_i \geq \lambda_{i+1}, \lambda'_{i+1}} S^{(\lambda')} K_n^* \] as Sp_n-modules.

In more visual terms, the Young diagrams of various \( \lambda' \) are obtained from that of \( \lambda \) by first removing \( p \) boxes from the right of some rows (\( p = 0 \) for GL_n) and then adding \( k-p \) boxes on the right of some rows in such a way that the horizontal positions of removed or added boxes do not overlap with those in other rows and with lower rows of \( \lambda \).

The algebraic study of natural differential operations on tensor bundles involves polynomial maps between various tensor spaces which are equivariant with respect to classical groups. These maps can be described with the aid of classical invariant theory.

**Theorem 1.** Let \( G \) be one of the classical linear groups GL_n, SL_n, O_n, SO_n, Sp_n and \( V_1, \ldots, V_s, W \) be tensor spaces over \( K^n \). Every \( G \)-equivariant polynomial map \( V_1 \times \cdots \times V_s \rightarrow W \) is obtained by composition and linear combination from the following basic tensor operations:

1. tensor product of elements of various \( V_i \) (maybe occurring repetitively);
2. tensor product with basic \( G \)-invariant tensors, which are: the identity operator (= the Kronecker delta), the co- and contravariant skew-symmetric \( n \)-tensors \( \text{det} \) and \( \text{det}^* \) (for \( G = \text{SL}_n, \text{SO}_n \)), the co- and contravariant metric tensors \( \omega \) and \( \omega^* \) (for \( G = \text{O}_n, \text{SO}_n, \text{Sp}_n \));
3. (partial) contraction;
4. permutation of indices.

**Proof.** The problem reduces to a description of polynomial functions on \( V_1 \times \cdots \times V_s \times W^* \) that are linear with respect to \( W^* \). The assertion in this case stems from the symbolic method of classical invariant theory [VP, 9.5]: it suffices to involve basic operations (1), (2), (3) with basic tensors \( \text{det}, \text{det}^*, \omega, \omega^* \) (depending on \( G \)). Contractions involving only indices of \( W^* \) correspond to tensor products with identity operators. Contracting indices of \( W^* \) with those of \( V_1, \ldots, V_s \), and basic tensors in various orders corresponds to (1). \( \square \)

### 2. Finiteness theorems

2.1. **Reduction to multilinear case.** We start the study of natural differential operations on tensor fields by some easy reductions.
Let $V, W$ be tensor bundles with typical fibres $V, W$. An algebraic natural differential operation $D : \Gamma(V) \to \Gamma(W)$ of order $\leq k$ is given by a $GL_{n(k+1)}$-equivariant polynomial map $\delta : V^{(k)} \to W$. Since $GL_{n(k+1)}$ acts on the vector spaces $V^{(k)}, W$ linearly, the homogeneous components of $\delta$ are equivariant maps, too. Thus it suffices to study homogeneous operations, i.e., those $D$ corresponding to homogeneous $\delta$.

Assume that $D$ is homogeneous of degree $\deg D := \deg \delta = d$. The polarization of $\delta$ yields a multilinear equivariant map $\delta : V^{(k)} \times \cdots \times V^{(k)} \to W$. (We denote it by the same letter, because the initial homogeneous map is the restriction of the multilinear map to the diagonal.) By decomposing the $GL_n$-modules $V, W$ into irreducibles and by multilinearity, we reduce $\delta$ to finitely many equivariant linear maps of the form

$$\delta : V_1^{(k)} \otimes \cdots \otimes V_d^{(k)} \to W,$$

where $V_i, W$ are now assumed to be irreducible $GL_n$-modules. In this case we say that the associated tensor bundles $V_i, W$ are indecomposable. Thus we have reduced (to a certain extent) the study of arbitrary algebraic natural differential operations between tensor bundles to the case of multilinear natural operations on indecomposable tensor bundles.

Consider the map (5). Since

$$V_1^{(k)} \otimes \cdots \otimes V_d^{(k)} = \bigoplus_{l_1,\ldots,l_d \leq k} S^{l_1}K^n \otimes \cdots \otimes S^{l_d}K^n \otimes V_1 \otimes \cdots \otimes V_d,$$

$\delta$ decomposes into a sum of $GL_n$-equivariant maps

$$\delta_{l_1,\ldots,l_d} : S^{l_1}K^n \otimes \cdots \otimes S^{l_d}K^n \otimes V_1 \otimes \cdots \otimes V_d \to W.$$

For any irreducible $GL_n$-module $U = S^\lambda K^n \otimes \det p$ corresponding to a virtual partition $\lambda = \lambda - (p^n)$, put $|U| = |\lambda| = |\lambda| - np$.

**Lemma 2.** Suppose $\delta_{l_1,\ldots,l_d} \neq 0$; then $l_1 + \cdots + l_d = |W| - \sum |V_i|$. In other words, every natural multilinear differential operation on indecomposable tensor bundles is homogeneous with respect to the total order of derivation.

**Proof.** Consider the subgroup of homotheties $K^\times \subset GL_n$. For any irreducible $GL_n$-module $U$ we have: $t \cdot u = t^{-|U|}u, \forall t \in K^\times, u \in U$. Now the lemma stems from the $K^\times$-equivariance of $\delta_{l_1,\ldots,l_d}$. $\square$

### 2.2. Finiteness

Now we prove two general finiteness results. The first one is easy.

**Theorem 2.** Given two tensor bundles $V, W$, the differential order of an algebraic natural operation $D$ from $V$ to $W$ is $O(\deg D)$. 
Corollary (cf. [K1 §4]). Natural algebraic differential operations of degree \( \leq d \) between two given tensor bundles form a finite-dimensional space.

**Proof.** The polarization and decomposition of \( V, W \) into indecomposables reduces the problem to the case where \( D : \Gamma(V_1) \times \cdots \times \Gamma(V_d) \to \Gamma(W) \) is a multilinear operation on indecomposable tensor bundles. Now by Lemma 2, the order of \( D \) is

\[
\max\{\sum_{\ell} l_i \mid l_i \in \mathbb{Z}_{\geq 0}, \delta_{l_1, \ldots, l_d} \neq 0\} \leq |W| - \sum |V_i| \leq |W| + d \cdot \max\{-|V_i|\} = O(d).
\]

The second result is much stronger.

**Theorem 3.** Given a tensor bundle \( V \) and two positive integers \( k, d \), there exist finitely many indecomposable tensor bundles \( W_i \) and natural operations \( D_i : V^{(k)} \to W_i \) of degree \( \leq d \) in partial derivatives such that every natural operation \( D : V^{(k)} \to W \) of degree \( \leq d \) in partial derivatives is represented as

\[
DV = \sum_{i,p} \Phi_{ip}(D_i v \otimes v \otimes p), \quad \forall v \in \Gamma(V),
\]

where \( \Phi_{ip} : W_i \otimes S^p V \to W \) are natural linear maps (i.e., compositions of contractions, permutations of indices, tensor product with the identity operator, and linear combinations, by Theorem 1).

**Proof.** Passing to jets, we may reformulate the assertion as follows: there exist finitely many irreducible \( GL_n \)-modules \( W_i \) and \( GL_{(k+1)n} \)-equivariant polynomial maps \( \delta_i : V^{(k)} = \bigoplus_{l=0}^k S^l \mathbb{K}^n \otimes V \to W_i \) of degree \( \leq d \) in the coordinates of

\[
V^{(k)}_+ = \bigoplus_{l=1}^k S^l \mathbb{K}^n \otimes V
\]

such that every \( GL_{(k+1)n} \)-equivariant polynomial map \( \delta : V^{(k)} \to W \) of degree \( \leq d \) in \( V^{(k)}_+ \) is of the form

\[
\delta(v) = \sum_{i,p} \Phi_{ip}(\delta_i(v) \otimes v_0 \otimes p), \quad \forall v = v_0 + v_+ \in V^{(k)} = V \oplus V^{(k)}_+,
\]

where \( \Phi_{ip} : W_i \otimes S^p V \to W \) are \( GL_n \)-equivariant linear maps. The \( GL_{(k+1)n} \)-equivariance condition means that the maps are \( GL_n \)-equivariant and \( NGL_{(k+1)n} \)-invariant.

All polynomial functions on \( V^{(k)} \) of degree \( \leq d \) in \( V^{(k)}_+ \) form a free \( S^*V^* \)-module \( S^*V^* \otimes \text{Mor}_d(V^{(k)}_+, \mathbb{K}) \) of finite rank. The \( NGL_{(k+1)n} \)-invariant functions form a \( GL_n \)-stable submodule \( M \). Clearly, operations of
order \( \leq k \) and degree \( \leq d \) in partial derivatives are identified with \( \text{GL}_n \)-fixed elements of \( \mathcal{M} \otimes W \) or \( \text{GL}_n \)-equivariant linear maps \( W^* \to \mathcal{M} \).

As a submodule of a Noetherian module, \( \mathcal{M} \) is finitely generated. Choose finitely many \( \delta_i : W_i^* \to \mathcal{M} \) whose images generate \( \mathcal{M} \) as an \( S^* V^* \)-module. By complete reducibility of \( \text{GL}_n \)-modules, every \( \delta : W^* \to \mathcal{M} \) lifts to a \( \text{GL}_n \)-equivariant linear map \( W^* \to \bigoplus S^* V^* \otimes W_i^* \) along the module epimorphism \( \bigoplus S^* V^* \otimes W_i^* \to \mathcal{M} \). The component mappings \( W^* \to S^p V^* \otimes W_i^* \) are nothing but \( \Phi_{ip}^* \).

\[ \square \]

3. Linear natural operations

In this section we deduce the known classification of natural linear differential operations on tensor bundles in a short and conceptual way using our algebraic approach (the IT-reduction). Apparently, the exterior differential is essentially the unique such operation.

\textbf{Theorem 4.} Every natural linear differential operation of order \( > 0 \) on tensor bundles is obtained from the exterior differential by composition with tensor operations (contraction, permutation of indices, tensor product with the identity operator) and linear combination.

\textbf{Remark.} This theorem is usually referred to as the Schouten theorem, although Schouten just formulated it in 1951 without proof. It was proved for differential forms by Palais (1959), for arbitrary covariant tensors by Leicher (1973), and in full generality by Rudakov (1974), Chuu-Lian Terng (1976), and Kirillov (1977), see [Ki]. For operations of order 1, a proof based on the IT-reduction was first obtained by Smirnov [Sm].

\textbf{Proof.} Let \( \mathcal{V}, \mathcal{W} \) be two tensor bundles with typical fibres \( V, W \). A linear differential operation \( D : \mathcal{V}^{(k)} \to \mathcal{W} \) is determined by a \( \text{GL}_n \)-equivariant linear map \( \delta : V^{(k)} \to W \) which vanishes on \( \mathfrak{ng}^{(k+1)} \cdot V^{(k)} \).

Without loss of generality we may assume that \( \mathcal{V}, \mathcal{W} \) are indecomposable. (Injections and projections onto indecomposable summands are given by tensor operations.)

We prove that \( S^k \mathbb{K}^{n*} \otimes V \subseteq \ker \delta \) unless \( k = 1 \), \( V \simeq \bigwedge^m \mathbb{K}^{n*}, m < n \).

It suffices to show that \( \xi \cdot v \) span \( S^k \mathbb{K}^{n*} \otimes V \) for some \( \xi \in \mathfrak{ng}^{(k+1)} \), \( v \in V^{(k)} \). Suppose that \( V = S^\lambda \mathbb{K}^{n*} \otimes \det^d \) corresponds to a virtual partition \( \lambda = \lambda - (d^n) \).

If \( k > 1 \), then \( S^k \mathbb{K}^{n*} \otimes V \) is spanned by \( \xi \cdot v, \xi = \xi_k \in S^k \mathbb{K}^{n*} \otimes \mathbb{K}^n, v = v_1 \in \mathbb{K}^{n*} \otimes V \). Indeed, take

\[ \xi = (x^i)^k \otimes e_i, \quad v = x_i \otimes v^-_\lambda, \]

where \( e_1, \ldots, e_n \) are the standard basic vectors of \( \mathbb{K}^n \). (Here and below in the proof, we do \textit{not} sum over \( i \).) By (3) we have

\[ \xi \cdot v = - (k\lambda_i + 1)(x^i)^k \otimes v^-_\lambda. \]
For \( i = n \), \( \xi \cdot v \) generates \( S^k K^{n*} \otimes V \) as a \( GL_n \)-module, because it is the product of a highest and a lowest weight vectors.

If \( k = 1 \), then we are left with \( \xi \in S^2 K^{n*} \otimes K^n, v \in V \). Take
\[
\xi = (x^i)^2 \otimes e_i, \quad v = v_\lambda \implies \xi \cdot v = -2\lambda_i x^i \otimes v_\lambda.
\]
If \( \lambda_n \neq 0 \), then we conclude as above. Otherwise suppose \( \lambda_1 > 1 \). Put
\[
\xi' = x^i x^j \otimes e_i \implies \begin{cases}
\xi' \cdot v = -\lambda_i x^j \otimes v - x^i \otimes v', \\
\xi \cdot v' = -2(\lambda_i - 1)x^j \otimes v',
\end{cases}
\]
where
\[
v' = \sum_{p=1}^{\lambda_i} \mathrm{Alt}_\lambda ( (x^1)^{\otimes \lambda_1} \cdots (x^i)^{\otimes p-1} \otimes x^j \otimes (x^i)^{\otimes \lambda_i-p} \cdots (x^n)^{\otimes \lambda_n} ).
\]
For \( i = 1, j = n \), we obtain \( x^n \otimes v_\lambda \in \ker \delta \) and conclude as above.

We are left with the case \( k = 1, V = \bigwedge^{m+1} K^{n*} \oplus S^{(2,1^{m-1})} K^{n*} \).

All \( \xi \cdot v \) are killed by \( \mathrm{Alt}_{(1^{m+1})} \), hence span the 2-nd direct summand.

The unique natural operation, given by the projection \( \mathrm{Alt}_{(1^{m+1})} \) onto the 1-st summand, is the exterior differential, cf. Example \ref{example-7}.

Remark. It would be interesting to reproduce by this method the classification of natural bilinear differential operators obtained by Grorman \cite{Gr}. However the computations here will be more involved.

4. Natural operations on symplectic manifolds

4.1. Geometric structures. Manifolds are often equipped with additional geometric structures, and it is important to study differential operations that are “natural” with respect to these structures. There are several possible ways to formalize these concepts. Here we adopt the following one.

Definition 3. A type of geometric structures is a functor on \( \mathcal{M}an_n \) associating with each \( n \)-manifold \( M \) a subsheaf \( A = A(M) \) in the sheaf of sections of a natural bundle \( F \to M \) so that, for any open embedding \( M' \hookrightarrow M \), \( A(M') \) is the pullback of \( A(M) \). A geometric structure (of given type) is a section \( \alpha \in \Gamma(A) \).

Remark. As a rule, \( A \) consists of sections satisfying a certain natural differential equation.

Example 7. A Riemannian structure is given by a section of a natural bundle \( (S^2 T^*)^+ \) of positive quadratic forms on tangent spaces (\( \mathbb{K} = \mathbb{R} \)).

Example 8. A symplectic structure is a section of
\[
A = \{ \omega \in \Gamma((\Omega^2)^{\text{reg}}) \mid d\omega = 0 \},
\]
where \( (\Omega^2)^{\text{reg}} \) is a natural bundle of non-degenerate 2-forms.
Example 9. A Poisson structure is a section of
\[
\mathcal{A} = \{ \beta \in \Gamma(\Lambda^2 \mathcal{T}) \mid [\beta, \beta] = 0 \},
\]
where \([\beta, \beta] = \beta^{ij} \partial_i \beta^{kl} \partial_j \wedge \partial_k \wedge \partial_l\) is the Schouten bracket.

Definition 4. A natural differential operation on manifolds with a geometric structure of type \(\mathcal{A}\) is a natural operation (in the sense of [L,M])
\[
D : \Gamma(\mathcal{V}) \times \Gamma(\mathcal{A}) \to \Gamma(\mathcal{W}),
\]
where \(\mathcal{V}, \mathcal{W}\) are natural bundles.

In other words, \(Dv\) depends on \(v\) and \(\alpha\) (as a parameter), and on their partial derivatives (up to a certain order) in a way independent of chosen local coordinates, where \(v\) and \(\alpha\) are (local) sections of \(\mathcal{V}\) and \(\mathcal{A}\), respectively.

Example 10. The covariant differential on tensor fields is a natural operation on Riemannian manifolds. The Poisson bracket of functions is a natural operation on symplectic or Poisson manifolds.

Alike ordinary natural differential operations, those on manifolds with an additional structure are uniquely determined by their action on jets:
\[
\delta : V^{(k)} \times A^{(l)} \to W,
\]
where \(A^{(l)} \subseteq F^{(l)}\) is the space of jets (at some point) of local sections from \(\mathcal{A}\), and \(\delta\) is \(\text{GL}_n^{(r)}\)-equivariant for an appropriate \(r\).

In the presence of an additional geometric structure, the group of coordinate transformations, which must preserve the coordinate expression of a natural operation, often can be reduced by the following standard invariant-theoretic trick.

Definition 5. Suppose a Lie (or algebraic) group \(G\) acts on a manifold (algebraic variety) \(X\) and \(H \subseteq G\) is a Lie (algebraic) subgroup. An \(H\)-stable submanifold (subvariety) \(Y \subseteq X\) is said to be a \((G, H)\)-section if the natural map \(G \times^H Y \to X\) is an isomorphism.

The significance of this notion is that \(G\)-invariant maps of \(X\) are in a bijective correspondence (by restriction to \(Y\)) with \(H\)-invariant maps of \(Y\).

In particular, for some important geometric structures it happens that \(A^{(l)}\) admits a \((\text{GL}_n^{(r)}, G^{(r)})\)-section \(A_0^{(l)}\) for some subgroup \(G^{(r)} \subseteq \text{GL}_n^{(r)}\). Therefore natural differential operations are in a bijective correspondence with \(G^{(r)}\)-invariant maps
\[
\delta : V^{(k)} \times A_0^{(l)} \to W.
\]

Example 11 ([Ka, §3]). The space of \(k\)-jets of Riemannian metrics
\[
A^{(k)} = (S^2 \mathbb{R}^{n^*})^+ \bigoplus \bigoplus_{l=1}^{k} S^l \mathbb{R}^{n^*} \otimes S^2 \mathbb{R}^{n^*}
\]
admits a \((\text{GL}_n^{(k+1)}, \text{GL}_n)\)-section

\[ A_0^{(k)} = (S^2 \mathbb{R}^{n*})^+ \oplus \bigoplus_{l=2}^k \text{Ker Sym}_{l+1}, \]

where \(\text{Sym}_{l+1} : S^l \mathbb{R}^{n*} \otimes S^l \mathbb{R}^{n*} \rightarrow S^{l+1} \mathbb{R}^{n*} \otimes \mathbb{R}^{n*}\) is the symmetrization in the first \(l+1\) indices. In fact, \(\text{Ker Sym}_{l+1} \simeq S^{l(2)} \mathbb{R}^{n*}\). Moving a metric to \(A_0^{(k)}\) by a coordinate transformation corresponds to writing the metric in geodesic coordinates with center at given point.

4.2. **Symplectic structure.** Jets of symplectic structures form a homogeneous space, i.e., one may take just one point for a section, by the Darboux theorem. It is instructive to give a purely algebraic proof of this fact in the spirit of this paper.

**Formal Poincaré Lemma.** The De Rham complex

\[ \ldots \xrightarrow{d} S^l \mathbb{K}^{n*} \otimes \bigwedge^m \mathbb{K}^{n*} \xrightarrow{d} S^{l-1} \mathbb{K}^{n*} \otimes \bigwedge^{m+1} \mathbb{K}^{n*} \xrightarrow{d} \ldots \]

is exact.

**Proof.** By the Pieri formulæ, we have

\[ S^l \mathbb{K}^{n*} \otimes \bigwedge^m \mathbb{K}^{n*} \simeq S^{(l+1,m-1)} \mathbb{K}^{n*} \oplus S^{(l,m)} \mathbb{K}^{n*}. \]

(In fact, \(S^{(l+1,m-1)} \mathbb{K}^{n*} \subset S^l \mathbb{K}^{n*} \otimes \bigwedge^m \mathbb{K}^{n*}\) and \(S^{(l,m)} \mathbb{K}^{n*}\) embeds as \(\text{Ker Sym}_{l+1}\).) The map \(d : S^l \mathbb{K}^{n*} \otimes \bigwedge^m \mathbb{K}^{n*} \rightarrow S^{l-1} \mathbb{K}^{n*} \otimes \bigwedge^{m+1} \mathbb{K}^{n*}\) is nothing but the alternation in the last \(m+1\) indices. It maps \(S^{(l,m)} \mathbb{K}^{n*}\) isomorphically into \(S^{l-1} \mathbb{K}^{n*} \otimes \bigwedge^{m+1} \mathbb{K}^{n*}\), while its kernel \(S^{(l+1,m-1)} \mathbb{K}^{n*}\) is exactly the image of \(S^{l+1} \mathbb{K}^{n*} \otimes \bigwedge^m \mathbb{K}^{n*}\). \(\square\)

**Formal Darboux Theorem.** Every jet \(\omega(x) = \omega_0 + \omega_1(x) + \cdots + \omega_k(x, \ldots, x) \in J_n^{(k)} \otimes \bigwedge^2 \mathbb{K}^{n*}\) such that \(\omega_0\) is non-degenerate and \(d\omega = 0\) is \(\text{GL}_n^{(k+1)}\)-equivalent to \(\omega_0\).

**Proof.** We shall successively kill the non-constant terms of \(\omega\) by transformations \(x \mapsto g(x) = x + g_0(x, x) + \cdots + g_{k+1}(x, \ldots, x)\) from \(\text{NGL}_n^{(k+1)}\). Arguing by induction on \(k\), we may assume that \(\omega_l = 0, 0 < l < k\). Take \(g \in \text{NGL}_n^{(k+1)}\) such that \(g_l = 0, 2 \leq l \leq k\). Using Lemma I and its corollary and adopting the notation therein, we have

\[ (g \cdot \omega)(x) = \omega(x) + (k+1)\rho(g_{k+1}(x, \ldots, x), \omega_0) \]

\[ = \omega(x) + (k+1)\gamma(g_{k+1} \otimes \omega_0) = \omega(x) + d(g_{k+1} \ast \omega_0), \]

where \((\cdot) \ast \omega_0\) is the lowering of the upper index by contraction with \(\omega_0\) in its 2-nd index. Thus \(\omega_k\) may be shifted by an arbitrary vector in the image of

\[ S^{k+1} \mathbb{K}^{n*} \otimes \mathbb{K}^n \xrightarrow{S^{k+1} \mathbb{K}^{n*} \otimes \omega_0} S^{k+1} \mathbb{K}^{n*} \otimes \mathbb{K}^{n*} \xrightarrow{d} S^k \mathbb{K}^{n*} \otimes \bigwedge^2 \mathbb{K}^{n*}. \]

By the formal Poincaré lemma, the image consists of all (jets of) closed forms. Hence \(\omega_k\) may be shifted to 0. \(\square\)
Acting by $GL_n$ we may obtain that $\omega_0$ be the standard symplectic form on $\mathbb{K}^n$ such that

$$\omega_0(e_i, e_j) = \begin{cases} 1, & i + j = n + 1, \ i < j, \\ -1, & i + j = n + 1, \ i > j, \\ 0, & \text{otherwise}. \end{cases}$$

We identify $\mathbb{K}^n$ with $\mathbb{K}^{n*}$ by lowering the indices. The symplectic form $\omega_0^*$ on $\mathbb{K}^{n*}$ induced from $\omega_0$ is given by $\omega_0^*(x^i, x^j) = \omega_0(e_i, e_j)$.

The stabilizer of $\omega_0$ in $GL_n^{(k)}$ is the group $Sp_n^{(k)}$ of $k$-jets of symplectomorphisms $\mathbb{K}^n \to \mathbb{K}^n$ at $0$. It has Levi decomposition $Sp_n^{(k)} = Sp_n \times NSp_n^{(k)}$, with the unipotent radical $NSp_n^{(k)} = \text{NGL}_n^{(k)} \cap Sp_n^{(k)}$. Let us describe the Lie algebra $sp_n^{(k)}$ of $Sp_n^{(k)}$.

**Lemma 3.** $\xi \in sp_n^{(k)} \iff \xi_l \ast \omega_0 \in S^{l+1}\mathbb{K}^{n*}, \ \forall l = 1, \ldots, k$ (in particular, $\xi_1 \in sp_n$)

**Proof.** By the corollary of Lemma [1]

$$\xi \cdot \omega_0 = d(\xi \ast \omega_0) = \sum_{l=1}^{k} d(\xi_l \ast \omega_0) = 0 \iff d(\xi_l \ast \omega_0) = 0, \ \forall l = 1, \ldots, k.$$ 

We have $\xi_l \ast \omega_0 \in S^l\mathbb{K}^{n*} \otimes \mathbb{K}^{n*} \simeq S^{l+1}\mathbb{K}^{n*} \oplus S^{(l,1)}\mathbb{K}^{n*}$, and the 1-st summand is exactly $\text{Ker} \ d$. \qed

By the above reasoning, natural differential operations on symplectic manifolds are given by $Sp_n^{(r)}$-equivariant maps $\delta : V^{(k)} \to W$.

**Remark.** Another approach to natural differential operations on symplectic manifolds is to extend the notion of geometric objects by considering fibre bundles $F = \text{SFr}_k^{(k)} F$ associated with the symplectic coframe bundle $\text{SFr}_k^{(k)} F$ of order $k$. The latter consists of $k$-jets of symplectic coordinate systems transforming the symplectic form $\omega$ on $M$ into the standard symplectic form $\omega_0$ on $\mathbb{K}^n$. Note that $F$ is acted on only by $Sp_n^{(k)}$, not $GL_n^{(k)}$, but if the action extends to $GL_n^{(k)}$, then $F \simeq \text{SFr}_k^{(k)} F \times GL_n^{(k)}$ is a natural bundle in the sense of [1].

Now natural differential operations $D : \Gamma(V) \to \Gamma(W)$ are defined as those having one and the same expression in all *symplectic* coordinate systems. This is equivalent to the map of jets $\delta : V^{(k)} \to W$ be $Sp_n^{(r)}$-equivariant.

**4.3. Linear operations.** Now we describe linear natural differential operations on symplectic manifolds in a way similar to Section 3. The classification of such operations, straightforward via our approach, was first obtained by Rudakov [Ru] in a rather indirect way from the study of irreducible representations of certain infinite-dimensional Lie algebras.
On symplectic $n$-manifolds, the natural isomorphism $\Omega^m \simeq \Omega^{m*}$ allows to define the differential operation $d^* : \Gamma(\Omega^m) \to \Gamma(\Omega^{m-1})$ contragredient to $d : \Gamma(\Omega^{m-1}) \to \Gamma(\Omega^m)$. Actually $d^*$ is the composition of $d : \Gamma(\Omega^{n-m}) \to \Gamma(\Omega^{n-m+1})$ with the natural isomorphisms $\Omega^m \simeq \Omega^{m*} \sim \Omega^{n-m}$ and $\Omega^{n-m+1} \sim (\Omega^{m-1})^* \simeq \Omega^{m-1}$ given by tensor operations.

**Theorem 5.** Every natural linear differential operation of order $> 0$ on tensor bundles over symplectic manifolds is obtained from the exterior differential $d$ and the symplectic Laplacian $dd^*$ by composition with tensor operations (contraction, permutation of indices, tensor product with the symplectic form $\omega$ or the dual bivector $\omega^*$) and linear combination.

**Proof.** Similarly to the proof of Theorem 4 it suffices to classify $\text{Sp}_n$-equivariant linear maps $\delta : V^{(k)} \to W$ which vanish on $\text{nsp}_n^{(k+1)} \cdot V^{(k)}$, where $V, W$ are irreducible $\text{Sp}_n$-modules. Assuming $V = S^{(\lambda)} \mathbb{K}^{n*}$, we prove that $S^k \mathbb{K}^{n*} \otimes V \subseteq \text{nsp}_n^{(k+1)} \cdot V^{(k)} \subseteq \text{Ker} \delta$ unless $\lambda = (1^m)$, $k \leq 2$.

Recall that, by Lemma 3 we may identify $\bigoplus_{l=2}^{k+1} S^{l+1} \mathbb{K}^{n*}$ with $\text{nsp}_n^{(k+1)}$ by raising an index by contraction with $\omega_0^*$ in its 1-st index.

If $k > 1$, $\lambda_1 \neq 1$, then $S^k \mathbb{K}^{n*} \otimes V$ is spanned by $S^{k+1} \mathbb{K}^{n*} \cdot (\mathbb{K}^{n*} \otimes V)$. Indeed, take
\[
\xi = (x^n)^{k+1}, \quad v = x^1 \otimes v^-_\lambda,
\]
\[
\xi' = (k+1)x^1(x^n)^k, \quad v' = x^n \otimes v^-_\lambda;
\]
then by (3) we have
\[
\xi \cdot v = (x^n)^k \otimes v^-_\lambda + k x^1 (x^n)^{k-1} \otimes \bar{v},
\]
\[
\xi' \cdot v' = (k\lambda_1 - 1)(x^n)^k \otimes v^-_\lambda + k(k-1)x^1 (x^n)^{k-1} \otimes \bar{v}, \quad \text{where}
\]
\[
\bar{v} = \sum_{p=1}^{\lambda_1} \text{Alt}_{\lambda_1} \left( (x^1)^{\otimes p-1} \otimes x^n \otimes (x^1)^{\otimes \lambda_1-p} \otimes (x^2)^{\otimes \lambda_2} \cdots \otimes (x^{n/2})^{\otimes \lambda_{n/2}} \right).
\]

We obtain $k(\lambda_1 - 1)x^n \otimes v^-_\lambda \in S^{k+1} \mathbb{K}^{n*} \cdot (\mathbb{K}^{n*} \otimes V)$. Since $\lambda_1 \neq 1$, this vector generates $S^k \mathbb{K}^{n*} \otimes V$ as an $\text{Sp}_n$-module, because it is the product of a highest and a lowest weight vectors.

In the case $k = 1$ or $\lambda_1 = 1$, take
\[
\xi = 3x^1(x^n)^2, \quad v = (x^n)^{k-1} \otimes v^-_\lambda,
\]
\[
\xi' = 3(x^1)^2 x^n, \quad v' = (x^n)^{k-1} \otimes \bar{v};
\]
then
\[
\xi \cdot v = (2\lambda_1 + 1 - k)(x^n)^k \otimes v^-_\lambda + 2x^1 (x^n)^{k-1} \otimes \bar{v},
\]
\[
\xi' \cdot v' = -2\lambda_1(x^n)^k \otimes v^-_\lambda + 2(\lambda_1 - 1 - k)x^1 (x^n)^{k-1} \otimes \bar{v}.
\]
We derive that $(\lambda_1 + 1 - k)(2\lambda_1 - 1 - k)(x^n)^k \otimes v_\lambda \in S^3 K^{n*}(S^{k-1} K^{n*} \otimes V)$. As above, this vector is nonzero and generates $S^3 K^{n*} \otimes V$ as an $Sp_n$-module unless $k \leq 2$, $\lambda_1 \leq 1$.

We are left with the case $\lambda = (1^m)$, $k \leq 2$. For $k = 1$, we have

$$K^{n*} \otimes V \simeq S^{(1m-1)} K^{n*} \oplus S^{(1m+1)} K^{n*} \oplus S^{(21m-1)} K^{n*}.$$  

The projections onto the first two summands correspond to $d^*$ and $d$, respectively, while the 3-rd summand is exactly $S^3 K^{n*} \cdot V$.

For $k = 2$, we have

$$S^2 K^{n*} \otimes V \simeq S^{(1m)} K^{n*} \oplus S^{(21m-2)} K^{n*} \oplus S^{(21m)} K^{n*} \oplus S^{(31m-1)} K^{n*}.$$  

It is easy to verify that the last 3 summands are in $S^3 K^{n*} \cdot (K^{n*} \otimes V)$. Indeed, for $i < n$ we have

$$(x^1)^i \cdot (x^i \otimes (x^n \wedge x^2 \wedge \cdots \wedge x^m)) = -2x^1 x^i \otimes (x^1 \wedge \cdots \wedge x^m) =: w.$$  

For $i = 1$, $w$ is a lowest weight vector of $S^{(31m-1)} K^{n*}$. For $i = m + 1$, the invariant projector $Alt_{(21m)}$ maps $w$ to a lowest weight vector $(-1)^{m+1} x^1 \otimes x^1 \wedge \cdots \wedge x^{m+1} \in S^{(21m)} K^{n*}$. For $i = n + 1 - m$, another invariant projector, namely the contraction with $\omega^n_1$ in, say, the first and last indices, maps $w$ to a lowest weight vector $x^1 \otimes x^1 \wedge \cdots \wedge x^{m-1} \in S^{(21m-2)} K^{n*}$. Since these irreducible modules occur in $S^2 K^{n*} \otimes V$ exactly once, they are in $S^3 K^{n*} \cdot (K^{n*} \otimes V)$ by the Schur lemma. The projection onto the remaining 1-st summand corresponds to $dd^*$. □

Remark. Natural bilinear differential operators on symplectic manifolds were partially classified by Grozman [Gr2]. It would be interesting to complete the classification using our methods.

5. DEFORMATION QUANTIZATION

In this section, we address the problem of existence of a natural deformation quantization on Poisson manifolds. Let us recall the notion of deformation quantization.

Given a Poisson manifold $M$ with a Poisson bivector $\beta \in \Gamma(M, \wedge^2 T)$ (cf. Example 9), the sheaf $\mathcal{O}$ of differentiable functions on $M$ comes equipped with the Poisson bracket $\{f, g\} = \beta(df, dg) = \beta^{ij} \partial_i f \partial_j g$. A deformation quantization is an associative product $\ast$ on the sheaf $\mathcal{O}[[\varepsilon]]$ of formal power series with coefficients in $\mathcal{O}$ that is $\mathbb{K}[[\varepsilon]]$-linear with respect to infinite formal sums and is defined on $\mathcal{O}$ by a formula

$$f \ast g = fg + \varepsilon \{f, g\} + \cdots + \varepsilon^m \beta_m(f, g) + \cdots,$$

where $\beta_m$ ($m = 1, 2, \ldots$) are bilinear differential operators. The $\ast$-product may be considered as a non-commutative deformation of the usual commutative product of functions, $\varepsilon$ being the parameter of deformation, so that the 1-st order term of $f \ast g - g \ast f$ is $2\{f, g\}$. Deformation quantization is one of possible approaches to mathematical foundations of quantum mechanics [BFFLS].
Several constructions of a deformation quantization for a given Poisson structure are known: by Moyal, by De Wilde–Lecomte [WL], by Fedosov [Fe], [EW], by Kontsevich [Ko], etc. But all of them involve some additional geometric structure on a Poisson manifold: affine structure, linear connection, etc. A natural question arises: does there exist a canonical deformation quantization given by one and the same universal formula for all Poisson manifolds? In terms of this paper, this may be reformulated as follows: does there exist a $*$-product (6) such that its terms $\beta_m$ are natural differential operations on Poisson manifolds in the sense of Definition 4? We answer this question negatively.

**Theorem 6.** There exists no natural deformation quantization on Poisson manifolds.

**Proof.** First we prove the theorem for symplectic manifolds. In the symplectic case, $\beta = \omega^*$ is the bivector dual to the symplectic form $\omega$. The operations $\beta_m$ are determined by the respective bilinear maps of jets

$$\beta_m : J^{(k)}_n \otimes J^{(l)}_n \rightarrow \mathbb{K},$$

which must be $\text{Sp}_n^{(r)}$-equivariant, $r \geq \max\{k, l\}$.

Since $S^k\mathbb{K}^n\ast$ are pairwise distinct self-dual irreducible $\text{Sp}_n$-modules, a non-zero $\text{Sp}_n$-invariant linear map $S^k\mathbb{K}^n\ast \otimes S^l\mathbb{K}^n\ast \rightarrow \mathbb{K}$ exists (and is then given by the full contraction with $(\omega_0^\ast)^\otimes k$, up to proportionality) iff $k = l$.

However, if $k > 1$, then the unique trivial irreducible factor $\mathbb{K} \hookrightarrow S^k\mathbb{K}^n\ast \otimes S^k\mathbb{K}^n\ast$ is in $\text{Ker} \beta_m$. Indeed, consider the action $\text{nsp}_n^{(r)} : J^{(k)}_n \otimes J^{(l)}_n$. In particular, we have

$$S^3\mathbb{K}^n\ast \cdot (S^k\mathbb{K}^n\ast \otimes S^k\mathbb{K}^n\ast) \subseteq (S^k\mathbb{K}^n\ast \otimes S^k\mathbb{K}^n\ast) \oplus (S^{k-1}\mathbb{K}^n\ast \otimes S^{k+1}\mathbb{K}^n\ast).$$

Put $\xi = 3(x^1)^2 x^n, v = (x^1)^{k-1} \otimes (x^n)^k$; then

$$\xi \cdot v = (k - 1)(x^1)^k \otimes (x^n)^k - 2k(x^1)^{k-1} \otimes x^1(x^n)^k$$

has a non-zero projection to $\mathbb{K}$, whence $\mathbb{K} \subset S^3\mathbb{K}^n\ast \cdot (S^k\mathbb{K}^n\ast \otimes S^k\mathbb{K}^n\ast)$. Since $\beta_m$ is $\text{NSp}_n^{(r)}$-invariant, $\text{Ker} \beta_m \supseteq \text{nsp}_n^{(r)} \cdot (J^{(k)}_n \otimes J^{(l)}_n)$, whence the claim.

It follows that $\beta_m$ is a linear combination of the usual multiplication $\mathbb{K} \otimes \mathbb{K} \rightarrow \mathbb{K}$ and the Poisson bracket $\mathbb{K}^n\ast \otimes \mathbb{K}^n\ast \rightarrow \mathbb{K}$. However, it is easy to see that these two operations cannot be combined into an associative $*$-product: in fact, the associativity will be violated already at order 2 (i.e., in the coefficient at $\varepsilon^2$).

Now it is obvious that a natural quantization cannot exist on Poisson manifolds of even dimension, because otherwise it would restrict to a natural quantization of symplectic manifolds as a particular case.

On a Poisson manifold $M$ of odd dimension, the Poisson structure $\beta$ is always degenerate. However, an open subset of $M$ where $\text{rk} \beta = \max$ has a foliation with symplectic leaves. If we trivialize this foliation in a
neighborhood of a point and consider functions depending only on the coordinates along the leaves and coordinate transformations which do not involve the remaining coordinates, then a natural quantization on $M$ would restrict to a natural quantization of the symplectic leaves, a contradiction.

\[ \square \]

**Remark.** Theorem 6 is a deformation quantization analogue of the Van Hove theorem in the theory of geometric quantization [Hu 5.2.2], cf. [XT].

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**Independent University of Moscow, Bolshoi Vlasievskii 11, 119002, Moscow, Russia**

*E-mail address*: p57taras@yandex.ru

**Moscow State University, Faculty of Mechanics and Mathematics, Leninskie gory, 119992, Moscow, Russia**

*E-mail address*: timashev@mech.math.msu.su

*URL*: [http://mech.math.msu.su/dePARTMENT/Algebra/stAFF/timashev](http://mech.math.msu.su/dePARTMENT/Algebra/stAFF/timashev)