Separable unsteady nonparallel flow stability problems

Georgy I. Burde

Jacob Blaustein Institute for Desert Research,
Ben-Gurion University, Sede-Boker Campus, 84990, Israel

Alexander Zhalij

Institute of Mathematics of the Academy of Sciences of Ukraine,
Tereshchenkivska Street 3, 01601 Kyiv-4, Ukraine

Abstract

The so-called 'direct' approach to separation of variables in linear PDEs is applied to the hydrodynamic stability problem. Calculations are made for the complete linear stability equations in cylindrical coordinates. Several classes of the exact solutions of the Navier-Stokes equations describing spatially developing and unsteady flows, for which the linear stability problems can be rigorously reduced to eigenvalue problems of ordinary differential equations, are defined. Those exactly solvable nonparallel and unsteady flow stability problems can be used for testing approximate approaches and the methods based on direct numerical simulations of the (linearized) Navier-Stokes equations. The exact solutions of the viscous incompressible Navier-Stokes equations determined as the basic states, for which the linear stability problem is exactly separable, may be themselves of interest from theoretical and engineering points of view.

PACS numbers: 47.15.Fe, 47.20.Gv, 02.30.Jr

*Electronic address: georg@bgu.ac.il
†Electronic address: zhaliy@imath.kiev.ua
I. INTRODUCTION

The classical linear stability theory of viscous incompressible flows is concerned with the development in space and time of infinitesimal perturbations around a given basic flow. Then small disturbances are resolved into normal modes which, for a steady-state basic flow, depend on time exponentially with a complex exponent $\lambda$. For parallel shear basic flows, further separation of variables in the governing stability equations leads to a set of ordinary differential equations which, with taking recourse to Squire’s theorem and considering only 2-D disturbances, reduces to the Orr-Sommerfeld equation. When this equation is solved with proper boundary conditions, the problem of linear stability of parallel flows is reduced to a 2-point boundary (eigen) value problem.

For nonparallel basic flows, when the equations for disturbance flow are dependent not only on the normal coordinate, the corresponding operator does not separate unless certain terms are ignored. The approximation, that neglects the nonparallel terms and relates the stability characteristics to those of the equivalent parallel flow, has been extensively used for the boundary layer type flows to retain the great advantage of reducing the disturbance equations to ordinary differential equations (see, e.g., Reed and Saric, 1996). A number of weakly nonparallel theories, which seek to account for the affects of the flow divergence through equations that include higher-order terms than those in the Orr-Sommerfeld equation, have been developed (see reviews in Reed and Saric, 1996; Herbert 1997; Saric et al. 2003).

It is worth mentioning, in this context, the works on stability of conical flows by Shtern and Hussain (1998), Shtern and Drazin (2000) where an exact transformation, reducing the stability problem to an ordinary differential equation despite the non-parallelism of the basic flow, is found. However, the advantage of such a transformation in the analysis is limited to the particular class of steady perturbation modes. In Shtern and Hussain (2003), the approach of Shtern and Hussain (1998) has been applied to the time-oscillatory disturbances by using a far-field approximation in the equations for the disturbances.

Several methods have been designed for numerical solution of the nonparallel flow stability problems. In the context of boundary-layer type flows, the most successful effort to-date is the parabolic-stability-equation (PSE) approach, introduced and recently reviewed by Herbert (1997). However, the PSE based numerical studies are not able to accommodate
the upstream propagation of disturbances. The 'global' linear stability analysis (see review by Theofilis 2003) was developed for analyzing stability of the two-dimensional flows. The nonparallel stability effects have been also investigated on the basis of direct numerical simulations of the linearized Navier-Stokes equations (e.g., Davies and Carpentier 2003) or the complete Navier-Stokes equations (e.g., Fazel and Konzelmann 1990). Such numerical simulations are not equivalent to a stability analysis, and, in fact, have more in common with physical experiments than stability theory. The numerical methods suffer from the problem of boundary conditions on the 'open', inflow and outflow, boundaries which (especially outflow boundary conditions) can lead to spurious effects, even when carefully implemented.

If the basic flow is non-steady, this brings about great difficulties in theoretical studies of the instability since the method of normal modes in its traditional form, with the modes depending on time as $\exp(\lambda t)$, is not applicable at all. (Some success has been achieved in studying stability of the time-periodic basic states when Floquet theory can be applied - see Drazin and Reid 1981.) If an unsteady flow is non-parallel, it should further complicate matters. As a matter of fact, there are no examples of the linear stability problem for viscous incompressible flows developing both in space and time which is exactly solvable via separation of variables.

Recently, the so-called direct approach to separation of variables in linear PDEs has been developed by a proper formalizing the features of the notion of separation of variables (see, e.g., Zhdanov and Zhalij 1999a, 1999b). In this approach, a form of the 'ansatz' for a solution with separated variables in a new coordinate system as well as a form of reduced ODEs, that should be obtained as a result of the variable separation, are postulated from the beginning. The method has been successfully applied to several equations of mathematical physics (see, e.g., Zhalij, 1999; Zhdanov and Zhalij, 1999a, 1999b; Zhalij, 2002).

In the present paper, we apply this approach to the linear stability equations which govern the disturbance behavior in viscous incompressible fluid flows. The calculations are made for the linear stability equations written in cylindrical coordinates. The new coordinate systems and the forms of basic flows, which permit the postulated form of separation of variables in the equations for disturbances, are determined as the result of application of the method. Then the basic flows are specified by the requirement that they exactly satisfy the Navier-Stokes equations.

The paper is organized as follows. In Section II, we give a description of the method,
as applied to the problem of linear stability of a three-dimensional unsteady basic flow with respect to the three-dimensional unsteady perturbations, and present the results. An example of application of the method to the linear stability equations with a restriction to the two-dimensional perturbations is also presented. In Section III, we discuss the fluid dynamics interpretation of some basic flows, defined in Section II as the exact solutions of the Navier-Stokes equations possessing exactly solvable stability problems, and formulate the corresponding two-point boundary value problems of ordinary differential equations. Concluding remarks on the results obtained are furnished in Section IV.

II. APPLICATION OF THE DIRECT METHOD TO SEPARATION OF VARIABLES IN THE STABILITY EQUATIONS

A. Procedure

We formulate the linear stability problem based on the Navier-Stokes equations written in cylindrical coordinates \((r, \varphi, z)\). As usual in stability analysis, we split the velocity and pressure fields \((\hat{v}_r, \hat{v}_\varphi, \hat{v}_z, \hat{p})\) into two problems: the basic flow problem \((V_r, V_\varphi, V_z, P)\) and a perturbation one \((v_r, v_\varphi, v_z, p)\),

\[
\hat{v}_r = V_r + v_r, \quad \hat{v}_\varphi = V_\varphi + v_\varphi, \quad \hat{v}_z = V_z + v_z, \quad \hat{p} = P + p
\]  

(1)

Introducing (1) into the Navier-Stokes equation written in terms of the variables \((\hat{v}_r, \hat{v}_\varphi, \hat{v}_z, \hat{p})\) and neglecting all terms that involve the square of the perturbation amplitude while imposing the requirement that the basic flow variables \((V_r, V_\varphi, V_z, P)\) themselves satisfy the Navier-Stokes equations, one arrives at the following set of linear stability equations in cylindrical coordinates:

\[
\frac{\partial v_r}{\partial t} + V_r \frac{\partial v_r}{\partial r} + v_r \frac{\partial V_r}{\partial r} + V_\varphi \frac{\partial v_r}{\partial \varphi} + v_\varphi \frac{\partial V_r}{\partial \varphi} + V_z \frac{\partial v_r}{\partial z} + v_z \frac{\partial V_r}{\partial z} - \frac{2 V_\varphi v_r}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \varphi^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{2 \partial v_\varphi}{r^2} \right),
\]

\[
\frac{\partial v_\varphi}{\partial t} + V_r \frac{\partial v_\varphi}{\partial r} + v_r \frac{\partial V_\varphi}{\partial r} + V_\varphi \frac{\partial v_\varphi}{\partial \varphi} + v_\varphi \frac{\partial V_\varphi}{\partial \varphi} + V_z \frac{\partial v_\varphi}{\partial z} + v_z \frac{\partial V_\varphi}{\partial z} + \frac{V_r v_\varphi}{r} + \frac{v_r V_\varphi}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \varphi} + \nu \left( \frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_\varphi}{\partial \varphi^2} + \frac{\partial^2 v_\varphi}{\partial z^2} + \frac{1}{r} \frac{\partial v_\varphi}{\partial r} + \frac{2 \partial v_r}{r^2} - \frac{v_\varphi}{r^2} \right),
\]  

(2)
\[
\frac{\partial v_r}{\partial t} + V_r \frac{\partial v_r}{\partial r} + v_r \frac{\partial V_r}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial \rho} + \nu \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \varphi^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} \right),
\]
\[
\frac{\partial v_\varphi}{\partial r} + \frac{1}{r} \frac{\partial v_\varphi}{\partial \varphi} + \frac{v_z}{r} = 0,
\]
\[
\frac{\partial v_z}{\partial t} + V_z \frac{\partial v_z}{\partial z} + v_z \frac{\partial V_z}{\partial z} = -1 \frac{\partial p}{\rho} + \nu \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \varphi^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} \right).
\]

Let us introduce a new coordinate system \( t, \xi = \xi(t, r), \gamma = \gamma(t, \varphi), \eta = \eta(t, z) \).

We will say that the system \([2]\) is \textit{separable in the non-stationary cylindrical coordinate system} \( \xi, \gamma, \eta \) if the separation ansatz,

\[
v_r = T(t) \exp(a_\eta + m_\gamma + sS(t)) f(\xi),
\]
\[
v_\varphi = T(t) \exp(a_\eta + m_\gamma + sS(t)) g(\xi),
\]
\[
v_z = T(t) \exp(a_\eta + m_\gamma + sS(t)) h(\xi),
\]
\[
p = T_1(t) \exp(a_\eta + m_\gamma + sS(t)) \pi(\xi)
\]

reduces the system of PDEs \([2]\) to a system of three second-order and one first order ordinary differential equations for four functions \( f(\xi), g(\xi), h(\xi), \pi(\xi) \) of the following form

\[
h''(\xi) = U_{11} g'(\xi) + U_{12} h'(\xi) + U_{13} \pi'(\xi) + U_{14} f(\xi) + U_{15} g(\xi) + U_{16} h(\xi) + U_{17} \pi(\xi),
\]
\[
f''(\xi) = U_{21} g'(\xi) + U_{22} h'(\xi) + U_{23} \pi'(\xi) + U_{24} f(\xi) + U_{25} g(\xi) + U_{26} h(\xi) + U_{27} \pi(\xi),
\]
\[
g''(\xi) = U_{31} g'(\xi) + U_{32} h'(\xi) + U_{33} \pi'(\xi) + U_{34} f(\xi) + U_{35} g(\xi) + U_{36} h(\xi) + U_{37} \pi(\xi),
\]
\[
f'(\xi) = U_{41} f(\xi) + U_{42} g(\xi) + U_{43} h(\xi) + U_{44} \pi(\xi).
\]

Here \( U_{ij} \) are second order polynomials with respect to spectral parameters \( a, s, m \) with coefficients, which are some smooth functions on \( \xi \).

Note, that equations \([3]-[4]\) form the input data of the method. We can change these conditions and thereby modify the definition of separation of variables. For instance, we can change the order of the reduced equations \([4]\) or the number of essential parameters \( a, s, m \). So, our claim of obtaining the \textit{complete description} of basic flows and non-stationary coordinate systems providing separation of variables in \([2]\) makes sense only within the framework of the given definition. If one uses a more general definition, it might be possible to construct new coordinate systems and basic flows providing separability of the system \([2]\).

The principal steps of the procedure of variable separation in the system \([2]\) are as follows.
1. We insert the ansatz (3) into equation (2) and express the derivatives \(f''(\xi), g''(\xi), h''(\xi), f'\(, h')\), in terms of functions \(g'(\xi), h'(\xi), \pi'(\xi), f(\xi), g(\xi), h(\xi), \pi(\xi)\) using equations (4).

2. We regard \(g'(\xi), h'(\xi), \pi'(\xi), f(\xi), g(\xi), h(\xi), \pi(\xi)\) as the new independent variables. As the functions \(\xi(t, r), \gamma(t, \varphi), \eta(t, z), T(t), T_1(t), S(t)\), basic flows \(V_r, V_\varphi, V_z\) and coefficients of the polynomials \(U_{ij}\) (which are some smooth functions on \(\xi\)) are independent on these variables, we can demand that the obtained equality is transformed into identity under arbitrary \(g'(\xi), h'(\xi), \pi'(\xi), f(\xi), g(\xi), h(\xi), \pi(\xi)\). In other words, we should split the equality with respect to these variables. After splitting we get an overdetermined system of nonlinear partial differential equations for unknown functions \(\xi(t, r), \gamma(t, \varphi), \eta(t, z), T(t), T_1(t), S(t)\), basic flows \(V_r, V_\varphi, V_z\) and coefficients of the polynomials \(U_{ij}\). This have been done with the aid of 
\textit{Mathematica} package.

3. After solving the above system we get an exhaustive description of coordinate systems providing separability of equations (2) according to our definition.

Thus, the problem of variable separation in equation (2) reduces to integrating the overdetermined system of PDEs for unknown functions \(\xi(t, r), \gamma(t, \varphi), \eta(t, z), T(t), T_1(t), S(t)\), basic flows \(V_r, V_\varphi, V_z\) and coefficients of the polynomials \(U_{ij}\). This have been done with the aid of 
\textit{Mathematica} package.

B. Results

We will consider the stability problems with separated variables for the basic flows specified by the requirement that they exactly satisfy the Navier-Stokes equations.

1. Three-dimensional perturbations

The forms of the perturbations \(v_r, v_\varphi, v_z\) and \(p\) are:

\[
v_r = T(t) \exp \left( a\eta + m\varphi + s \int T(t)^2 dt \right) f(\xi),
\]

\[
v_\varphi = T(t) \exp \left( a\eta + m\varphi + s \int T(t)^2 dt \right) g(\xi),
\]
\[ v_z = T(t) \exp \left( a\eta + m\varphi + s \int T(t)^2 dt \right) h(\xi), \]
\[ p = \rho T(t)^2 \exp \left( a\eta + m\varphi + s \int T(t)^2 dt \right) \pi(\xi). \tag{5} \]

where
\[ \xi = T(t)r, \quad \eta = T(t)z + c(t). \tag{6} \]

Two classes of basic flows satisfying the Navier-Stokes equations are found as the result of the analysis. Velocity fields for both classes are defined by
\[ V_z = A(\xi)T(t) - zT'(t) - \beta(t), \quad \beta(t) = \frac{c'(t)}{T(t)} \]
\[ V_r = B(\xi)T(t) - rT'(t) \]
\[ V_\varphi = C(\xi)T(t), \tag{7} \]

where the functions \( T(t) \) and \( B(\xi) \) are specified in different ways for the two classes.

**Class I:**
\[ T(t) = \frac{1}{\sqrt{t}}, \quad B(\xi) = -\frac{3\xi}{4} + \frac{k}{\xi}, \tag{8} \]

where the functions \( A(\xi) \) and \( C(\xi) \) satisfy the equations
\[ (4k + 3\xi^2 - 4\nu)A'(\xi) + \xi(-4k + 3\xi^2 + 4\nu)A''(\xi) + 4\xi^2\nu A''(\xi) = 0, \]
\[-4\nu k_0 \xi + (-4k + 3\xi^2 - 4\nu)C(\xi) + \xi(-4k + 3\xi^2 + 4\nu)C'(\xi) + 4\nu \xi^2 C''(\xi) = 0. \tag{9} \]

and the pressure distribution is given by
\[
\frac{P}{\rho} = \frac{\nu k_0 \varphi}{t} + \frac{\varphi^2}{8t^2} + x \left[ \beta'(t) + \frac{\beta(t)}{2t} + t^{-3/2} \left( \nu A''(\xi) - \frac{4k - 3\xi^2 - 4\nu}{4\xi} A'(\xi) \right) \right]
+ \frac{1}{t} \int \frac{16k^2 - 5\xi^2 + 16\xi^2 C^2(\xi)}{16\xi^3} d\xi + p_0(t) \tag{10} \]

The ODEs (9) can be explicitly solved in terms of the incomplete gamma functions.

**Class II:**
\[ T(t) = 1, \quad B(\xi) = \frac{k}{\xi} \]

where \( A(\xi) \) and \( C(\xi) \) satisfy the equations
\[ (k - \nu)A'(\xi) + \xi(\nu - k)A''(\xi) + \xi^2 \nu A'''(\xi) = 0, \]
\[ \nu k_0 \xi + (k + \nu)C(\xi) + \xi(k - \nu)C'(\xi) - \xi^2 \nu C''(\xi) = 0. \tag{11} \]
and the corresponding pressure distribution is

$$\frac{P}{\rho} = \nu k_{0} \varphi + x \left( \beta'(t) + \nu A''(\xi) + \frac{\nu - k}{\xi} A'(\xi) \right) + \int \frac{k^2 + \xi^2 C^2(\xi)}{\xi^3} d\xi + p_0(t)$$

The ODEs (11) can be explicitly solved in elementary functions.

The equations with separated variables can be written for both classes in the forms

$$f(\xi)(\xi^2 s + \nu - m^2 \nu - a^2 \xi^2 \nu + a \xi^2 A(\xi) + m \xi C(\xi) + \xi^2 B'(\xi)) +$$

$$2(m \nu - \xi C(\xi)) g(\xi) + \xi((\nu + \xi B(\xi)) f'(\xi) + \xi(\nu'(\xi) - \nu f''(\xi))) = 0,$$

$$(\xi^2 s + \nu - m^2 \nu - a^2 \xi^2 \nu + a \xi^2 A(\xi) + \xi B(\xi) + m \xi C(\xi)) g(\xi) +$$

$$f(\xi)(-2m \nu + \xi C(\xi) + \xi^2 C'(\xi)) + \xi(m \pi(\xi) + (-\nu + \xi B(\xi)) g'(\xi) - \xi \nu g''(\xi)) = 0,$$

$$(\xi^2 s - m^2 \nu - a^2 \xi^2 \nu + a \xi^2 A(\xi) + m \xi C(\xi)) h(\xi) +$$

$$\xi(a \xi \pi(\xi) + \xi f(\xi) A'(\xi) - \nu h(\xi) + \xi B(\xi) h'(\xi) - \xi \nu h''(\xi)) = 0,$$

$$f(\xi) + m g(\xi) + \xi(a h(\xi) + f'(\xi)) = 0.$$

2. Two-dimensional perturbations

The stability properties of a given flow may be tested by considering perturbations of specific structures. For example, the problem may be restricted to the two-dimensional perturbations even though an analog of the Squire theorem cannot be proved (see, e.g., Griffond and Casalis 2001, Joslin 1996), or the perturbation flow field may be taken to have the same general form as the basic state (Duck and Dry 2001). Although the stability analysis restricted to perturbations of specific forms is not complete, it enables one to show that the flow is susceptible to a special kind of instability. To demonstrate that a specification of the disturbance field may lead to new possibilities we consider the results of application of the direct method to the linear stability equations with a restriction to the two-dimensional perturbations with \(v_z = 0\) and \(v_r\) and \(v_\phi\) not dependent on \(z\).

The separability analysis leads to the perturbations of the form

$$v_r = T(t) \exp \left( m \varphi + s \int T(t)^2 dt \right) f(\xi),$$

$$v_\phi = T(t) \exp \left( m \varphi + s \int T(t)^2 dt \right) g(\xi),$$

$$v_z = 0,$$
\begin{align*}
p &= \rho T(t)^2 \exp \left( m\varphi + s \int T(t)^2 dt \right) \pi(\xi), \\
\xi &= T(t)r
\end{align*}

which is a particular case of (5) for \( a = 0 \). However, for the perturbations of the form (12), the corresponding basic flows are not restricted to those listed in Section IIB1. In addition, the following basic flows are permitted

\begin{align*}
V_z &= -kz + \beta(t), \\
V_r &= kr/2 + q/r, \\
V_\varphi &= \nu B(\xi) T(t).
\end{align*}

\[
\frac{P}{\rho} = -\frac{1}{2} k^2 x^2 + x (k\beta(t) - \beta'(t)) - \frac{4q^2 + k^2 r^4}{8r^2} + T^2(t) \left( \nu^2 \int \frac{B^2(\xi)}{\xi} d\xi - \frac{1}{2} \nu k_0 \varphi \right) + p_0(t)
\]

where the functions \( T(t) \) and \( B(\xi) \) satisfy the equations

\[
T''(t) - \frac{1}{2} \left( QT^3(t) - kT(t) \right) = 0
\]

\[
k_0 \xi - (2q + 2\nu + Q\xi^2) B(\xi) - \xi (2q - 2\nu + Q\xi^2) B'(\xi) + 2\nu \xi^2 B''(\xi) = 0
\]

which leads to the following cases

\[
T(t) = \frac{1}{\sqrt{e^{kt} + 1}} \left( \frac{Q}{k} = 1 \right), \quad T(t) = \frac{1}{\sqrt{e^{kt} - 1}} \left( \frac{Q}{k} = -1 \right),
\]

\[
T(t) = 1 \left( \frac{Q}{k} = 1 \right), \quad T(t) = e^{-kt/2} \quad (Q = 0).
\]

**III. SPECIFIC FLOW STABILITY PROBLEMS**

In this section, we will discuss the fluid dynamics interpretation of some basic flows, defined above as the \textit{exact solutions of the Navier-Stokes equations} for which the corresponding \textit{stability problems are exactly separable}, and will formulate the corresponding two-point boundary value problems.

We will consider particular cases of the class of the exact solutions of the Navier-Stokes equations in cylindrical coordinates identified in Section IIB1 as \textit{Class I}. It is possible to enrich the solution defined by the formulas (6) - (10) using the invariance of the solution
with respect to a shift of time variable. Making change of variables \( t = t' - 1/b \), where \( b \) is a constant, and omitting primes in what follows, we will have the solution of the Navier-Stokes equations in the form

\[
V_z = -\frac{bz}{2(1-bt)} + \frac{F(\zeta)}{\sqrt{1-bt}} - \beta(t), \quad V_r = \frac{1}{\sqrt{1-bt}} \left( \frac{b\zeta}{4} + \frac{k}{\zeta} \right), \quad V_\phi = \frac{M(\zeta)}{\sqrt{1-bt}},
\]

\[
\zeta = \frac{r}{\sqrt{1-bt}}, \quad \beta(t) = c'(t)\sqrt{1-bt}
\]

where \( b \) can be both positive and negative.

The corresponding pressure distribution is

\[
\frac{P}{\rho} = \frac{b^2 x^2}{8(1-bt)^2} + x \left[ \beta'(t) - \frac{b\beta(t)}{2(1-bt)} + (1-bt)^{-3/2} \left( \nu F''(\zeta) - \frac{(4k-4\nu+3b\zeta^2)F'(\zeta)}{4\zeta} \right) \right] + \frac{1}{1-bt} \int \frac{16k^2 - 5b^2\zeta^4 + 16\zeta^2 M^2(\zeta)}{16\zeta^3} d\zeta - \frac{k_0\nu \varphi}{1-bt} + p_0(t)
\]

The functions \( F(\zeta) \) and \( M(\zeta) \) satisfy the equations

\[
(4k-3b\zeta^2-4\nu)F'(\zeta) + \zeta(-4k-3b\zeta^2+4\nu)F''(\zeta) + 4\zeta^2\nu F'''(\zeta) = 0,
\]

\[
4k_0\nu \zeta + (-4k-3b\zeta^2-4\nu)M'(\zeta) + \zeta(-4k-3b\zeta^2+4\nu)M'(\zeta) + 4\zeta^2\nu M''(\zeta) = 0.
\]

Equations (16) and (17) can be solved in quadratures

\[
F(\zeta) = c_1 + c_2 \Gamma \left( \frac{k}{2\nu}, -Z(\zeta) \right) + c_3 \int e^{Z(\zeta)} \zeta^{\frac{k}{2\nu}-1} \Gamma \left( 1 - \frac{k}{2\nu}, Z(\zeta) \right) d\zeta,
\]

\[
Z(\zeta) = \frac{3bk^2}{8\nu},
\]

\[
M(\zeta) = \frac{1}{\zeta} \left[ c_4 + c_5 \Gamma \left( 1 + \frac{k}{2\nu}, -Z(\zeta) \right) \right] + \frac{k_0}{2} \left( \frac{3b}{8\nu} \right)^{\frac{k}{2\nu}} \int e^{Z(\zeta)} \zeta^{\frac{k}{2\nu}-1} \Gamma \left( -\frac{k}{2\nu}, Z(\zeta) \right) d\zeta,
\]

where \( \Gamma(A, Z) \) is the incomplete Gamma function and \( c_1, \ldots, c_5 \) are arbitrary constants.

Note also the expression for \( F'(\zeta) \)

\[
F'(\zeta) = e^{Z(\zeta)} \zeta^{\frac{k}{2\nu}-1} \left[ -2c_2 \left( \frac{3b}{8\nu} \right)^{\frac{k}{2\nu}} + c_3 \Gamma \left( 1 - \frac{k}{2\nu}, Z(\zeta) \right) \right].
\]

The correspondingly specified perturbations \( \varphi \) take the forms

\[
v_r = (1-bt)^s \exp (a\eta + m\varphi) f(\zeta),
\]

\[
v_\varphi = (1-bt)^s \exp (a\eta + m\varphi) g(\zeta),
\]

10
\[ v_z = (1 - bt)^s \exp (a\eta + m\varphi) h(\zeta), \]
\[ \frac{p}{\rho} = (1 - bt)^{s-1/2} \exp (a\eta + m\varphi) \pi(\zeta). \]  

(21)

where

\[ \eta = \frac{z}{\sqrt{1 - bt}} + c(t) \]  

(22)

The equations for the perturbation amplitudes are

\[-4k + b\zeta^2 - 4b\zeta^2 s + 4\nu - 4m^2\nu - 4a^2\zeta^2\nu + 4a\zeta^2 F(\zeta) + 4m\zeta M(\zeta) \]  
\[ f(\zeta) \]
\[ + \zeta (4k - 4\nu + 3b\zeta^2) f'(\zeta) - 4\nu\zeta^2 f''(\zeta) \]
\[ + 8 (m\nu - \zeta M(\zeta)) g(\zeta) + 4\zeta^2 \pi'(\zeta) = 0, \]
\[ (4k + b\zeta^2 - 4b\zeta^2 s + 4\nu - 4m^2\nu - 4a^2\zeta^2\nu + 4a\zeta^2 F(\zeta) + 4m\zeta M(\zeta)) g(\zeta) \]
\[ + \zeta (4k - 4\nu + 3b\zeta^2) g'(\zeta) - 4\nu\zeta^2 g''(\zeta) \]
\[ + (-8m\nu + 4\zeta M(\zeta) + 4\zeta^2 M'(\zeta)) f(\zeta) + 4m\pi(\zeta) = 0, \]
\[ -2 (b\zeta^2 + 2b\zeta^2 s + 2m^2\nu + 2a^2\zeta^2\nu - 2a\zeta^2 F(\zeta) - 2m\zeta M(\zeta)) h(\zeta) \]
\[ + \zeta (4k - 4\nu + 3b\zeta^2) h'(\zeta) - 4\nu\zeta^2 h''(\zeta) + 4\zeta^2 F'(\zeta) f(\zeta) + 4a\zeta^2 \pi(\zeta) = 0, \]
\[ f(\zeta) + mg(\zeta) + \zeta (ah(\zeta) + f'(\zeta)) = 0. \]  

(23)

This system can be reduced to a system of two third-order equations for two functions \( f(\zeta) \) and \( g(\zeta) \), for example.

The above formulas (14)-(23) remain valid if we introduce the nondimensional variables, with the time scale \( 1/|b| \) and the correspondingly defined velocity scale. In the dimensionless equations (we will retain the same notation for the nondimensional variables), the parameter \( b \) takes one of the two values: \( b = 1 \) or \( b = -1 \), and \( \nu \) is replaced by \( 1/\text{Re} \) where \( \text{Re} \) is the Reynolds number.

Note that the solution (14) - (15) for \( b = 1 \) undergoes the finite-time 'breakdown' (see, e.g., Duck and Dry 2001, Hall et al. 1992) and the 'normal mode' forms (21)-(22) are naturally adjusted to the description of the disturbed flow as the breakdown time \( t = 1 \) is approached.

Next we will consider several two-point boundary value problems corresponding different specifications of the basic flow (14)-(19).

(i) **Axially symmetrical stagnation-point type flows.** These are the simplest basic flows, that are obtained from (14)-(15) by setting

\[ F(\zeta) = 0, \ M(\zeta) = 0, \ c(t) = 0, \ k = 0. \]  

(24)
In the case of $b = 1$, the solution describes impingement of two axially opposite stagnation point flows with velocities growing with time, and, in the case of $b = -1$, the solution describes the flow where fluid flowing radially from infinity approaches the axis and spreads along it, with the flow velocity decreasing with time as $(1 + t)^{-1}$ (Fig. 1). In both cases, the boundary conditions to equations (23) are set at the axis $\zeta = 0$ and at $\zeta = \infty$, as follows

$$f(0) = 0, \ g(0) = 0, \ h'(0) = 0; \quad f(\infty) = 0, \ g(\infty) = 0, \ h(\infty) = 0 \quad (25)$$

(ii) **Flow outside an expanding cylinder.** This case corresponds to $b = -1$. The radius of the cylinder changes with time as $R = \sqrt{1 + t}$. (In the dimensional variables, marked with stars, it is $R^* = R_0^* \sqrt{1 - b^* t^*}$ and the value $R_0^*$ is used as a length scale for the nondimensional variables while the time scale is $1/|b^*|$). The surface of the cylinder stretches in the longitudinal direction according to the law $U = K z$ where $K = 1/2 (1 + t)^{-1}$ and $U$ is an axial velocity at $r = R$.

If the cylinder surface is impermeable, then using the boundary condition $V_r = V$ at $r = R$, where $V = dR/dt = 1/2 (1 + t)^{-1/2}$ is the radial velocity of the cylinder surface, yields $k = 3/4$. In the case of porous cylinder, we have $k = 3/4 - V_0$, where $V_0$ is a constant defining a magnitude of the suction ($V_0 > 0$) or injection ($V_0 < 0$) velocity as $V_s = V_0 (1 + t)^{-1/2}$.

Since the domain is infinite in radial direction, it should be set $c_3 = 0$ in the expression (20) for $F'(\zeta)$ not to have an unbounded behavior for $F(\zeta)$ at infinity. (The incomplete Gamma function $\Gamma(A, Z) \sim Z^{A-1} e^{-Z}$ as $Z \to \infty$ which results in $F'(\zeta) \sim \zeta^{-1}$ as $\zeta \to \infty$ and produces the logarithmic behavior for $F(\zeta)$ at infinity - this can be confirmed by considering a behavior of $F(\zeta)$ itself for specific values of $k = 2n\nu$, with $n$ being a positive integer, when closed-form solutions of equation (16) for $F(\zeta)$ can be found.) Then the flow at infinity

FIG. 1: Unsteady axially symmetric stagnation point flows: $b = 1$ (left), $b = -1$ (right)
represents a combination of a stagnation point flow and a uniform stream, and the two constants in the expression for \( F(\zeta) \) are determined from the boundary condition \( V_z = U \) at \( r = R \) \( (F(1) = 0) \) and the condition for the uniform part of the flow velocity at infinity be of the form \( U_\infty/\sqrt{1 + t} \) where \( U_\infty \) is a constant.

For not swirling flows (a swirl can be also added with the swirl velocity defined by (14) and (19)) we have the following to be introduced into the equations for perturbations (21)-(23):

\[
F(\zeta) = U_\infty \left[ 1 - \frac{\Gamma \left( \frac{k \text{Re}}{2}, \frac{3 \text{Re} z^2}{8} \right)}{\Gamma \left( \frac{k \text{Re}}{2}, \frac{3 \text{Re} z^2}{8} \right)} \right], \quad M(\zeta) = 0, \quad c(t) = 0, \quad k = \frac{3}{4} - V_0. \tag{26}
\]

where \( \text{Re} = R_0^2 |b^*|/\nu^* \) is the Reynolds number (the corresponding flow structure is illustrated by Fig. 2). Note that the solution (26) is expressed in elementary functions for \( \text{Re} = 2n/k \), where \( n \) is an integer, with the use of the specific values of the incomplete Gamma function (Abramowitz and Stegun 1965):

\[
\Gamma(n, Z) = 1 - \left( 1 + Z + \frac{Z^2}{2!} + \ldots + \frac{Z^{n-1}}{(n-1)!} \right) e^{-Z}
\]

In the equations for perturbations (21)-(23), it should be also set \( b = -1 \) and \( \nu = 1/\text{Re} \).

The boundary conditions to equations (23) are set at \( \zeta = 1 \) and at \( \zeta = \infty \), as follows

\[
f(1) = 0, \quad g(1) = 0, \quad h(1) = 0; \quad f(\infty) = 0, \quad g(\infty) = 0, \quad h(\infty) = 0 \tag{27}
\]

(iii) **Flow inside an expanding porous cylinder.** In this case, like as in the previous one, \( b = -1 \), the radius of the cylinder changes with time as \( R = \sqrt{1 + t} \) and the surface stretches according to the law \( U = Kz \) where \( K = \frac{1}{2}(1 + t)^{-1} \). The difference is in that the fluid is now inside the cylinder and there is an injection of fluid through the porous pipe surface, which may be either normal to the surface or oblique, with the blowing velocity varying with time as \( V_b = V_0 (1 + t)^{-1/2} \), where \( V_0 \) is a constant.

The conditions \( V_r = 0 \) at \( r = 0 \) requires \( k = 0 \) and using the condition \( \partial V_z / \partial r = 0 \) at \( r = 0 \) in (20) yields

\[
F'(\zeta) = c_1 \frac{1 - e^{-\frac{3c^2R}{8}}}{\zeta},
\]

where \( \text{Re} \) is the Reynolds number. Then two arbitrary constants in the expression for \( F(\zeta) \) are determined from the condition at the cylinder surface \( F(1) = (3/4) \tanh \theta \), where the
FIG. 2: Flow outside an expanding impermeable cylinder for Re = 100 and $U = -0.5$ at different time moments: a) $t = 1$; b) $t = 5$; c) $t = 10$. 
angle $\theta$ defines the direction of blowing (with respect to the inward radial direction), and from the condition at the axis $F(0) = U_0$, where $U_0$ is a constant defining the axial flow velocity. Restricting ourselves to not swirling flows and normal blowing ($\theta = 0$), we have the following to be introduced into the equations for perturbations (21)-(23):

$$F(\zeta) = U_0 \frac{\text{Ei} \left( -\frac{3\zeta^2\text{Re}}{8} \right) - \text{Ei} \left( -\frac{3\text{Re}}{8} \right) - \ln \zeta^2}{\gamma + \Gamma \left( 0, \frac{3\text{Re}}{8} \right) + \ln \left( \frac{3\text{Re}}{8} \right)}, \quad M(\zeta) = 0, \quad c(t) = 0, \quad k = 0. \quad (28)$$

where $\text{Ei}(Z)$ is the exponential integral function and $\gamma$ is Euler’s constant (the corresponding flow is shown in Fig. 3). Note that, despite the presence of the logarithmic term in the nominator, the expression (28) for $F(\zeta)$ is finite at $\zeta = 0$ since the expansion of $\text{Ei}(Z)$ for small $Z$ includes the term $\ln Z$.

It should be also set $b = -1$ and $\nu = 1/\text{Re}$ in equations (21)-(23). The boundary conditions for the perturbation amplitudes are set at the axis $\zeta = 0$ and at the cylinder surface $\zeta = 1$, as follows

$$f(0) = 0, \quad g(0) = 0, \quad h'(0) = 0; \quad f(1) = 0, \quad g(1) = 0, \quad h(1) = 0 \quad (29)$$

(iv) Flow inside a contracting porous cylinder. In this case $b = 1$ and the radius of the cylindrical tube changes with time as $R = \sqrt{1 - t}$ ($t < 1$). The surface of the tube shrinks according to the law $U = -Kz$ where $K = \frac{1}{2}(1-t)^{-1}$ and there is a suction of fluid through the permeable cylinder surface, which may be either normal to the surface ($F(1) = 0$) or oblique ($F(1) = (3/4) \tanh \theta$ with the angle $\theta$ defining the direction of suction with respect to the outward radial direction). The suction velocity varies with time as $V_b = V_0(1-t)^{-1/2}$ where $V_0 = [9/16 + F^2(1)]^{1/2}$. For not swirling flows and normal suction ($\theta = 0$), we have the following to be introduced into the equations for perturbations (21)-(23):

$$F(\zeta) = U_0 \frac{\text{Ei} \left( \frac{3\zeta^2\text{Re}}{8} \right) - \text{Ei} \left( \frac{3\text{Re}}{8} \right) - \ln \zeta^2}{\gamma - \text{Ei} \left( \frac{3\text{Re}}{8} \right) + \ln \left( \frac{3\text{Re}}{8} \right)}, \quad M(\zeta) = 0, \quad c(t) = 0, \quad k = 0. \quad (30)$$

where $U_0$ is a constant defining the axial flow velocity (the corresponding basic flow is shown in Fig. 4).

It should be set $b = 1$ and $\nu = 1/\text{Re}$ in equations (21)-(23), and the boundary conditions are set at the axis $\zeta = 0$ and at the cylinder surface $\zeta = 1$, as follows

$$f(0) = 0, \quad g(0) = 0, \quad h'(0) = 0; \quad f(1) = 0, \quad g(1) = 0, \quad h(1) = 0 \quad (31)$$

15
FIG. 3: Flow inside an expanding porous cylinder for Re = 100 and $U = 5$ at different time moments: a) $t = 0$; b) $t = 1$; c) $t = 5$.

(v) Flow in the gap between concentric cylinders. Here, like as in the previous cases, different boundary conditions can be considered.

IV. CONCLUDING REMARKS

Several classes of the exact solutions of the Navier-Stokes equations describing spatially developing and unsteady flows, for which the linear stability problems can be rigorously reduced to eigenvalue problems of ordinary differential equations, have been defined. Those
FIG. 4: Flow inside a contracting permeable cylinder for $\text{Re} = 100$ and $U = 6$ at different time moments: a) $t = 0$; b) $t = 0.5$; c) $t = 0.75$.

effectively solvable nonparallel and unsteady flow stability problems can provide a necessary foundation for a number of approximate approaches used in the stability analysis so far. The results can be also used for testing the methods based on direct numerical simulations of the (linearized) Navier-Stokes equations. Note that the basic flows considered in the paper belong to the category of the so-called 'open' flows (see, e.g., Huerre and Monkewitz 1990), for which the numerical instability simulations can be quite challenging because of the problem of boundary conditions on the inflow and outflow boundaries.

It is worth remarking that the general forms of the basic flows, which have been obtained from the only requirement of separability of the corresponding stability problem, are reacher
than those remaining after specification to the exact solutions of the Navier-Stokes equations. Thus, using the approach accepted in many stability studies, where the form of the basic flow is chosen quite freely to approximate the physical situation of interest, we could considerably enrich the list of relevant flows. However, our purpose was to provide examples of a completely rigorous analysis that reduced the stability problem to an eigenvalue problem of ordinary differential equations.

In addition, note that we have not yet exhausted the "direct approach" to separation of variables in the hydrodynamic stability equations. Changing the input data of the method given by equations (3)-(4) may lead to new results. We considered here only the most natural generalization of the normal modes of the steady-state parallel flow analysis, which allows periodicity of perturbations in two new variables, and the order of the reduced equations was taken the same as that obtained in the parallel flow stability problem. If one uses other input data, it might be possible to construct new coordinate systems and basic flows providing separability of the system (2).

We will also remark on the practical importance of specific basic flows that have been defined in the course of our analysis (some of them are discussed in the previous section). Those flows, mainly, are either ones over the stretching surfaces or the flows within porous channels possessing moving walls. The description of the flow near a stretching surface has many important applications in manufacturing processes in industry. A literature on the subject (see, e.g., the book by Pop and Ingham 2001) shows considerable research activities in this area. Solutions for physical situations, close to those considered in our Section III, can be found, for example, in Burde (1995a, 1995b), Youssef (1997), Nahapatra and Gupta (2003), Nazar et al. (2004).

Laminar, incompressible and time-dependent flows that develop within a channel possessing permeable, moving walls have received considerable attention in the past due to their relevance in a number of engineering applications. Instances of direct application of such flows include the modeling of sweat cooling or heating, isotope separation, filtration, paper manufacturing, irrigation, and the grain regression during solid propellant combustion. From a different perspective, the sequences of expansions and contractions completed by channel walls enable a researcher to mimic more realistically peristaltic motion caused by pulsating walls and involving fluid absorption and filtration processes. For the cases, similar to those considered in the present paper, which pertain to a pipe that exhibits either inject...
tion or suction across porous boundaries while undergoing uniform expansion or contraction see, e.g., Uchida and Aoki (1977), Goto and Uchida (1990), Majdalani and Zhou (2003), Dauenhauer and Majdalani (2003).

Thus, the exact solutions of the viscous incompressible Navier-Stokes equations determined in this paper as the basic states, for which the linear stability problem is exactly separable, may be themselves of interest from both theoretical and engineering points of view.

Acknowledgements. This research was supported by the Israel Science Foundation (grant No. 117/03).

[1] Abramowitz, M., and Stegun, I.A., *Handbook of mathematical functions*. (Dover Publications, Inc., New York, 1965).
[2] Burde, G.I., ”The construction of special explicit solutions of the boundary-layer equations. Unsteady flows,” Quart. J. Mech. Appl. Math., 48, 611-633 (1995a).
[3] Burde, G.I., ”Nonsteady stagnation-point flows over permeable surfaces: explicit solutions of the Navier-Stokes equations,” J. Fluid Engrg. Trans. ASME, 117, 189-191 (1995b).
[4] Dauenhauer, E.C., and Majdalani, J., ”Exact self-similarity solution of the Navier-Stokes equations for a porous channel with orthogonally moving walls,” Phys. Fluids, 15, 1485-1494 (2003).
[5] Davies, C., and Carpentier, P.W., ”Global behaviour corresponding to the absolute instability of the rotating-disc boundary layer,” J.Fluid. Mech., 486, 287-329 (2003).
[6] Drazin, P.G., and Reid, W.H., *Hydrodynamic stability*. (Cambridge University Press, 1995).
[7] Duck, P, and Dry, S.L., ”On a class of unsteady, non-parallel, three-dimensional disturbances to boundary-layer flows,” J. Fluid Mech., 441, 31-65 (2001).
[8] Fazel, H., and Konzelmann, U., ”Non-parallel stability of a flat plate boundary layer using the complete Navier-Stokes equations,” J. Fluid Mech., 221, 311-347 (1990).
[9] Goto, M. and Uchida, S., ”Unsteady flows in a semi-infinite expanding pipe with injection through the wall,” Trans. Japan. Soc. Aeronaut. Space Sci., 33, 14-27 (1990).
[10] Griffond, J., and Casalis, G., ”On the nonparallel stability of the injection induced two-dimensional Taylor flow,” Phys. Fluids, 13, 1635-1644 (2001).
[11] Hall, P, Balakumar, P. and Papageorgiu, D., "On a class of unsteady three-dimensional Navier-Stokes solutions relevant to rotating disk flows: threshold amplitudes and finite-time singularities.,” J. Fluid Mech., 238, 297-323 (1992).

[12] Herbert, Th., "Parabolized stability equations," Annu. Rev. Fluid. Mech., 29, 245-283 (1997).

[13] Huerre, P. and Monkewitz, P.A. "Local and global instabilities in spatially developing flows," Ann. Rev. Fluid. Mech., 22, 473-537 (1990).

[14] Joslin, R.D., "Simulation of nonlinear instabilities in an attachment-line boundary layer,” Fluid Dyn. Res., 18, 81-97 (1996).

[15] Majdalani, J., and Zhou, C., "Moderate-to-large injection and suction driven channel flows with expanding or contracting walls,” ZAMM (Z. Angew. Math. Mech.), 83, 181-196 (2003).

[16] Mahapatra, T.R., and Gupta, A.S., "Stagnation-point flow towards a stretching surface,” Can. J. Chem. Eng, 81, 258-263 (2003).

[17] Nazar R., Amin, N., Filip, D., and Pop, L., "Unsteady boundary layer flow in the region of the stagnation point on a stretching sheet,” Int. J. Eng. Sc., 42, 1241-1253 (2004).

[18] Pop I., and Ingham, D.B., Convective Heat Transfer: Mathematical and Computational Modelling of Viscous Fluids and Porous Medis, (Pergamon, Oxford, 2001).

[19] Reed, H.L., and Saric, W.S., "Linear stability theory applied to boundary layers,” Annu. Rev. Fluid Mech., 28, 389-428 (1996).

[20] Saric, W.S., Reid, H.L. and White, E.B., "Stability and transition of three-dimensional boundary layers,” Ann. Rev. Fluid. Mech., 35, 413-440 (2003).

[21] Shtern, V., and Drazin, P.G., "Instability of a free swirling jet driven by a half-line vortex,” Proc. R. Soc. Lond. A, 456, 1139-1161 (2000).

[22] Shtern, V., and Hussain, F., "Instabilities of conical flows causing steady bifurcations,” J. Fluid Mech., 366, 33-85 (1998).

[23] Shtern, V., and Hussain, F., "Effect of deceleration on jet instability,” J.Fluid. Mech., 480, 283-309 (2003).

[24] Theofilis, V., "Advances in global linear stability analysis of nonparallel and three-dimensional flows,” Progress in Aerospace Sciences, 39, 249-315 (2003).

[25] Uchida S., and Aoki H., "Unsteady flows in a semi-infinite contracting or expanding pipe,” J. Fluid Mech., 82, 371-387 (1977).

[26] Youssef, F.A., "On the heat transfer from an expanding cylinder in cross-flow,” Appl. Thermal
Engng., 17, 235-248 (1997).

[27] Zhalij, A., "On separable Fokker-Planck equations with a constant diagonal diffusion matrix," J. Phys. A: Math. Gen., 32, 7393-7404 (1999). [math-ph/9904034]

[28] Zhalij, A., "On separable Pauli equations," J. Math. Phys., 43, 1365-1389 (2002). [math-ph/0203001]

[29] Zhdanov, R., and Zhalij, A., "On separable Schrödinger equations," J. Math. Phys., 40, 6319-6338 (1999a). [math-ph/9911018]

[30] Zhdanov, R., and Zhalij, A., "Separation of variables in the Kramers equation," J. Phys. A: Math. Gen., 32, 3851-3863 (1999b). [math-ph/9906004]