Moduli spaces for Lamé functions and Abelian differentials of the second kind

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Abstract

The topology of the moduli space for Lamé functions of degree $m$ is determined: this is a Riemann surface which consists of two connected components when $m \geq 2$; we find the Euler characteristics and genera of these components. As a corollary we prove a conjecture of R. Maier on degrees of Cohn’s polynomials. These results are obtained with the help of a geometric description of these Riemann surfaces, as quotients of the moduli spaces for certain singular flat triangles.

An application is given to the study of metrics of constant positive curvature with one conic singularity with the angle $2\pi(2m + 1)$ on a torus. We show that the degeneration locus of such metrics is a union of smooth analytic curves and we enumerate these curves.

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1 Introduction

1.1 Statement of main results

Consider an elliptic curve in the form of Weierstrass

\[ u^2 = 4x^3 - g_2x - g_3, \quad g_2^3 - 27g_3^2 \neq 0. \]  \hspace{1cm} (1)

*Lamé equation* (algebraic version) is the second order differential equation

\[ \left( \frac{u}{dx} \right)^2 - m(m+1)x - \lambda \right) w = 0, \]  \hspace{1cm} (2)

with respect to a function \( w(x) \), where \( m \geq 0 \) is an integer. The more familiar form of the Lamé equation is

\[ \frac{d^2W}{dz^2} - (m(m+1)\wp(z) + \lambda) W = 0, \]  \hspace{1cm} (3)

which is obtained from (2) by the change of the independent variable \( x = \wp(z) \), \( u = \wp'(z) \), \( W(z) = w(\wp(z)) \). Here \( \wp \) is the Weierstrass function of the lattice \( \Lambda \) with invariants \( g_2 = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-4} \), \( g_3 = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-6} \).

Changing the variables in (2) to \( x_1 = kx, \ w_1(x_1) = w(x_1/k) \), we obtain a new equation (2) with parameters

\[ (k \lambda, k^2g_2, k^3g_3), \quad k \in \mathbb{C}^*. \]  \hspace{1cm} (4)

Two equations obtained by such a change of the variables are called *equivalent*. The set of equivalence classes is the *moduli space for Lamé equations*, Lame\(_m\). The quotient of \( \mathbb{C}^3 \setminus \{0\} \) by the \( \mathbb{C}^* \) action (4) is the weighted projective space \( \mathbb{CP}(1,2,3) \), see for example [19], and Lame\(_m\) is obtained from it by removing the curve \( g_2^3 - 27g_3^2 = 0 \). It parametrizes projective structures on tori with one conic singularity with angle \( 2\pi(2m+1) \). For a general discussion of projective structures with conic singularities we refer to [26]. Lame\(_m\) has two singularities corresponding to two points with non-trivial stabilizers: \( (0,1,0) \) and \( (0,0,1) \), which are associated with square and hexagonal tori in [1].
The homogeneous function

\[ J = \frac{g_2^3}{g_2^3 - 27g_3^2} \]

is called the absolute invariant of the elliptic curve (1); it defines a map \( \pi_m : \text{Lame}_m \rightarrow \mathbb{C}J \) which is called the forgetful map.

A Lamé function is a non-trivial solution \( w \) of (2) with the property that \( w^2 \) is a polynomial in \( x \). It is easy to see that the degree of this polynomial must be \( m \). For given \( g_2, g_3 \), such solutions exist if and only if the accessory parameter \( \lambda \) satisfies an algebraic equation

\[ F_m(\lambda, g_2, g_3) = 0, \] (5)

where \( F_m \) is a polynomial in three variables, monic with respect to \( \lambda \), which is invariant under the \( \mathbb{C}^* \) action (4). For each \( \lambda \) satisfying (5), the Lamé function is unique up to a constant factor, \([35, 23.41]\).

Two Lamé functions are equivalent if \( w(x) = cw_1(kx) \), \( c, k \in \mathbb{C}^* \), and the set of equivalence classes is the moduli space for Lamé functions \( \text{L}_m \subset \text{Lame}_m \). We consider this space \( \text{L}_m \) as an abstract Riemann surface which is obtained by taking the quotient of the surface

\[ \{ (\lambda, g_2, g_3) \in \mathbb{C}^3 : F_m(\lambda, g_2, g_3) = 0, \ g_2^3 - 27g_3^2 \neq 0 \} \]

by the \( \mathbb{C}^* \) action (4). It is the normalization of the curve defined by equation (5) in \( \mathbb{C}P(1, 2, 3) \backslash \{ g_2^3 - 27g_3^2 = 0 \} \). Since \( F_m \) in (5) is a monic polynomial, the forgetful map \( \text{L}_m \rightarrow \mathbb{C}J \) is proper.

Equations (5) for \( m \leq 8 \) are explicitly written in Table 3 in \([27]\), see also Appendix to the present paper. Maier calls \( F_m \) the Lamé spectral polynomials.

Lamé equation and Lamé functions have long history going back to the work of Gabriel Lamé \([22, 23]\), and they have been intensively studied ever since, because of their importance for mathematical physics. Good reference for the classical work is \([35]\), and a modern survey is contained in the first three sections of \([27]\).

Most of the classical work on Lamé functions was concentrated on the real case with positive discriminant \( (g_2, g_3, \lambda \) are real and \( g_2^3 - 27g_3^2 > 0 \), and we are not aware of any systematic study of general properties of \( F_m \) and \( \text{L}_m \).

In this paper we determine the topology of the Riemann surfaces \( \text{L}_m \). To state our main result, we recall the notion of a 2-dimensional orbifold \([8, 33]\).
It is a compact Riemann surface $S$ with a function $n : S \to \mathbb{N}_{\geq 1} \cup \{\infty\}$, where $\mathbb{N}_{\geq 1}$ is the set of positive integers, such that $n(x) = 1$ for all points, except finitely many orbifold points of order $n(x) > 1$. Points with $n(x) = \infty$ are interpreted as punctures. For example, the moduli space for tori $\mathbf{C}_J$ has a natural orbifold structure with $S = \mathbf{C}$ and three orbifold points: $n(0) = 3$, $n(1) = 2$, and $n(\infty) = \infty$. $J = 0$ corresponds to the hexagonal torus, and $J = 1$ to the square torus.

The orbifold Euler characteristic is defined as

$$
\chi^O = \chi(S) - \sum_{x \in S} \left(1 - 1/n(x)\right),
$$

where $\chi(S) = 2 - 2g(S)$ is the ordinary Euler characteristic of the underlying compact surface, and $g$ is its genus. Here we follow [33]. Notice that in [17] a different definition is used: the Euler characteristic in [17] is smaller by a factor of 2. A ramified covering $f : S_1 \to S_2$ is called an orbifold map if $n(f(x))$ divides $n(x) \text{deg}_x f$ for all $x \in S_1$, and an orbifold covering if

$$
n(f(x)) = n(x) \text{deg}_x f \quad \text{for all} \quad x \in S_1. \quad (6)
$$

If $f$ is an orbifold covering, the Riemann-Hurwitz formula gives

$$
\chi^O(S_1) = \deg(f)\chi^O(S_2). \quad (7)
$$

We introduce the following functions of non-negative integers $m$.

$$
d_m^I := \begin{cases} m/2 + 1, & m \equiv 0 \pmod{2} \\
(m - 1)/2, & m \equiv 1 \pmod{2}, \end{cases} \quad (8)
$$

$$
d_m^{II} := 3\lceil m/2 \rceil. \quad (9)
$$

It is easy to see that $d_m^I$ and $d_m^{II}$ always have opposite parity: when $m \in \{0, 3\} \pmod{4}$, $d_m^I$ is odd and $d_m^{II}$ is even; when $m \in \{1, 2\} \pmod{4}$, $d_m^I$ is even and $d_m^{II}$ is odd.

$$
\epsilon_0 = \begin{cases} 0, & m \equiv 1 \pmod{3}, \\
1 & \text{otherwise}, \end{cases} \quad (10)
$$

$$
\epsilon_1 = \begin{cases} 0, & m \in \{1, 2\} \pmod{4}, \\
1 & \text{otherwise}. \end{cases} \quad (11)
$$

One can restate the definitions of $\epsilon_j$ as follows: $\epsilon_0 = 0$ if and only if $d_m^I$ is divisible by 3, and $\epsilon_1 = 0$ if and only if $d_m^I$ is even.

Our main result is the following.
Theorem 1.1. When \( m \geq 2 \), \( L_m \) is a Riemann surface consisting of two connected components which we call \( L^I_m \) and \( L^{II}_m \), while \( L_0 \) and \( L_1 \) are connected: \( L_0 = L^I_0 \) and \( L_1 = L^{II}_1 \).

The Riemann surface \( L_m \) has a natural orbifold structure with \( \epsilon_0 \) orbifold points of order 3 on \( L^I_m \), and one orbifold point of order 2 which belongs to \( L^I_m \) when \( \epsilon_1 = 1 \) and to \( L^{II}_m \) otherwise. The component \( L^I_m \) has \( d^I_m \) punctures, and the component \( L^{II}_m \) has \( 2d^{II}_m / 3 = 2 \lceil m/2 \rceil \) punctures.

The restrictions of the forgetful map to these components are orbifold maps and their degrees are \( d^I_m \) and \( d^{II}_m \). The orbifold Euler characteristics are

\[
\chi^O(L^I_m) = -(d^I_m)^2/6 \quad \text{and} \quad \chi^O(L^{II}_m) = -(d^{II}_m)^2/18. \tag{12}
\]

Remark. Ordinary Euler characteristics \( \chi \) are obtained from the \( \chi^O \) by adding the orbifold corrections which in our case are

\[
E^I = (4\epsilon_0 + 3\epsilon_1)/6 \quad \text{and} \quad E^{II} = (1 - \epsilon_1)/2. \tag{13}
\]

Euler characteristics can be expressed as functions of \( m \) rather than \( d \), see Appendix.

It is well known that for \( m \geq 2 \) each polynomial \( F_m \) factors into four factors in \( \mathbb{C}[\lambda, e_1, e_2, e_3] \), where \( e_i \) are related to \( g_2, g_3 \) by the equation

\[
4x^3 - g_2 x - g_3 = 4(x - e_1)(x - e_2)(x - e_3),
\]

see for example [34, Thm. 2] for an explicit statement of this. However there is no discussion of irreducibility of these four factors in the literature. Our theorem says that for \( m \geq 2 \), \( F_m \) has exactly two irreducible factors in \( \mathbb{C}[\lambda, g_2, g_3] \) and implies that the four factors of \( F_m \) in \( \mathbb{C}[\lambda, e_1, e_2, e_3] \) are irreducible.

Theorem 1.1 implies the formulas for the genera of \( L^K_m, K \in \{I, II\} \) in terms of \( m \) or \( d^K_m \) which are given in the Appendix.

We give several applications of Theorem 1.1. As a first application, we prove that the two irreducible components of the surface \( F_m(\lambda, g_2, g_3) = 0 \) in \([5]\) have no singularities in \( \mathbb{C}^3 \) except the lines \((0, t, 0)\) and \((0, 0, t)\).

To obtain a non-singular curve in \( \mathbb{CP}^2 \) parametrizing \( L_m \), we use Legendre’s family of elliptic curves

\[
v^2 = z(z - 1)(z - a), \quad a \in \mathbb{C}_a := \mathbb{C} \setminus \{0, 1\}. \tag{14}
\]
For the $J$-invariant of this curve we have
\[
J = \psi(a) = \frac{4}{27} \frac{(a^2 - a + 1)^3}{a^2(a - 1)^2}.
\] (15)

If $\overline{C}_J$ is considered as an orbifold with $n(0) = 3$, $n(1) = 2$, $n(\infty) = \infty$, and in $\overline{C}_a$ we set $n(a) = \infty$ for $a \in \{0, 1, \infty\}$, and $n(a) = 1$ otherwise, then $a \mapsto \psi(a)$ is an orbifold covering.

The form of the Lamé equation corresponding to (14) is
\[
Py'' + \frac{1}{2}P'y' - ((m(m + 1)z + B)y = 0, \quad P(z) = 4z(z - 1)(z - a),
\] (16)
where the accessory parameter $B$ is an affine function of $\lambda$ (see Section 8 for the details of this transformation). Lamé functions correspond to non-trivial solutions $y$ of (16) such that $y^2$ is a polynomial in $z$ of degree $m$. For such a solution to exist, a polynomial equation
\[
H_m(B, a) = 0
\] (17)
must be satisfied. The Riemann surface defined by this equation will be denoted by $H_m$. It is the normalization of the algebraic curve $H_m(B, a) = 0$ in $\mathbb{C}_a \times \mathbb{C}_B$. For $m \geq 2$, we will show that it consists of four irreducible components, $H^j_m$, $j \in \{0, 1, 2, 3\}$. These components are defined as follows: $H^0_m$ corresponds to polynomial solutions $y$ when $m$ is even and to solutions of the form $y = Q\sqrt{P}$ when $m$ is odd. The other three components $j \in \{1, 2, 3\}$ correspond to solutions
\[
Q(z)\sqrt{z}, \quad Q(z)\sqrt{z - 1}, \quad Q(z)\sqrt{z - a}, \quad \text{when } m \text{ is odd},
\]
and
\[
Q(z)\sqrt{(z - 1)(z - a)}, \quad Q(z)\sqrt{z(z - a)}, \quad Q(z)\sqrt{z(z - 1)},
\]
when $m$ is even, where $Q$ is a polynomial. The forgetful maps $\sigma^j_m : H^j_m \rightarrow \mathbb{C}_a$ are defined by $(B, a) \mapsto a$.

The polynomial $H_m$ is a product of four factors $H^j_m$, and we have ramified coverings $\Psi^0_m : H^0_m \rightarrow \mathbb{L}^j_m$, and $\Psi^j_m : H^j_m \rightarrow \mathbb{L}^{j'}_m$, $j \in \{1, 2, 3\}$ such that
\[
\pi^K_m \circ \Psi^j_m = \psi \circ \sigma^j_m,
\]
where $\pi^K_m : \mathbb{L}^K_m \rightarrow \mathbb{C}_j$, and $\sigma^j_m : H^j_m \rightarrow \mathbb{C}_a$ are the forgetful maps, and $\psi$ is the function (15). We will show that these maps $\Psi^j_m$, are orbifold
coverings with respect to the appropriate orbifold structures defined on some compactifications of $L_m^K$ and $H_m^j$.

This will permit us to compute the genera of components of $H_m$ via the Riemann–Hurwitz formula. Once the genera and degrees are known one can conclude that these curves are non-singular by the "genus–degree formula":

$$g \leq \frac{(d - 1)(d - 2)}{2},$$

(18)

where we have equality only for non-singular curves. We consider compactifications $H_m^j$ obtained from $H_m^j$ by filling in the punctures. Equivalently, $H_m^j$ is the normalization of the algebraic curve obtained as the projective closure of the zero set of equation $H_m^j = 0$.

**Theorem 1.2.** The maps $H_m^j \rightarrow \mathbb{CP}^2$ for $j \in \{0, \ldots, 3\}$ are non-singular embeddings, in particular $H_m^j$ are irreducible. The degrees of the ramified coverings $\Psi_m^j$ are 6 for $j = 0$ and 2 for $j \in \{1, 2, 3\}$.

So we have interesting sequences of non-singular planar curves $H_m^j$ defined over $\mathbb{Q}$ for which degrees and genera have been explicitly determined. Only a few such examples of high genus are known to the authors, see [9], [28].

One can deduce Theorem 1.1 from Theorem 1.2 if we know that $H_m^j$ are non-singular, we can find their genera from the equality in (18) and obtain all topological characteristics of $L_m$ using the orbifold coverings $\Psi_m^j$.

**Corollary 1.1.** All singularities of irreducible components of surfaces (5) are contained in the lines $(0, t, 0)$ and $(0, 0, t)$.

Computation for $m \leq 6$ shows that the only singularities of surfaces (5) are the zeros of the discriminant, but we do not prove this in this paper.

These results allow us to prove a conjecture of Maier about degrees of Cohn’s polynomials [27, Conj. 3.1(ii)]. We recall the definition. Let $F_m = F_m^I F_m^H$ be the irreducible factorization. Let $D_m^K$ be the discriminant of $F_m^K$ with respect to $\lambda$. Then $D_m^K(g_2, g_3)$ is quasi-homogeneous, that is the curves $D_m^K(g_2, g_3) = 0$ are invariant under the scaling transformations [4]. Therefore, the equations $D_m^K(g_2, g_3) = 0$ can be rewritten as $C_m^K(J) = 0$, and these $C_m^K$ are called Cohn’s polynomials.

**Corollary 1.2.** (Maier’s conjecture) $\deg C_m^I = \lfloor((d_m^I)^2 - d_m^I + 4)/6 \rfloor$ and $\deg C_m^H = d_m^I(d_m^H - 1)/2$, where $d_m^K = \deg_{\lambda} F_m^K$, as in (8), (9).
Our second application, and the original motivation of this work is the problem of describing degeneration of metrics of constant positive curvature with conic singularities which recently attracted substantial attention, [10, 11, 12, 17, 24, 25, 29, 30]. Let $S$ be a compact surface, and $\alpha_1, \ldots, \alpha_n$ positive numbers. Consider Riemannian metrics on $S$ with $n$ conic singularities with the angles $2\pi \alpha_j$. Each such metric defines a conformal structure on $S$ with $n$ marked points, so we have the forgetful map assigning this conformal structure to the metric. The goal is understanding the space of such metrics and the properties of the forgetful map.

In this paper we restrict ourselves to the case when $S$ is a torus with one singularity with the angle $2\pi \alpha$, where $\alpha = 2m + 1$ is an odd integer. Following [30] we denote the set of all such metrics by $\text{Sph}_{1,1}(\alpha)$. One can embed $\text{Sph}_{1,1}(\alpha)$ into $\text{Lame}_m$: the image of this embedding consists of those Lamé equations whose projective monodromy is unitarizable (that is conjugate to a subgroup of $\text{PSU}(2)$.)

We have the forgetful map $\text{Sph}_{1,1}(\alpha) \to C_J$ which assigns to each metric its conformal class. It is known that when $\alpha > 1$ is not an odd integer, this forgetful map is proper and surjective [5], [30], and $\text{Sph}_{1,1}(\alpha)$ is properly embedded in $\text{Lame}_m$.

This is not the case for odd integers $\alpha$, the fact discovered in [25] (see also [6] for a shorter proof of the main result of [25]). As the conformal class varies, a spherical metric can degenerate.

Let us define the set $\text{LW}_m \in \text{Lame}_m$ consisting of all Lamé equations whose projective monodromy consists of collinear translations (by the periods of the integral (20) below). In Section 10 we show that

**Theorem 1.3.**

$$\partial \text{Sph}_{1,1}(2m + 1) = \text{LW}_m,$$

where the boundary is with respect to $\text{Lame}_m$.

Then we describe the set $\text{LW}_m$.

**Theorem 1.4.** The set $\text{LW}_m$ consists of $m(m + 1)/2$ curves. These curves and their projections on $C_J$ are smooth real analytic curves (images of intervals under analytic functions with non-vanishing derivatives).

We propose to call projections of the curves $\text{LW}_m$ to $C_J$ Lin–Wang curves. They can be seen in the pictures in [25, 6] and our Figs. 12, 13 for $m = 1$; in [24, 12] and our Figs, 14, 15 for $m = 2$, and our Figs. 16–18 for $m = 3$. 8
Large part of the papers [10], [11] and [24] is dedicated to analytic study of these curves for small $m$; here we propose a different, geometric description of them.

1.2 Description of the method

Our main tool in this paper is a new geometric interpretation of Lamé functions and their moduli space $L_m$. Lamé functions correspond to what we call translation structures on tori with one conic singularity with the angle $2\pi(2m+1)$ and $m$ simple poles. (“Simple pole” refers to the developing map; its differential has double poles).

Let $(S, O)$ be an elliptic curve, that is $S$ is a torus with a marked point $O \in S$, and $m \geq 0$ an integer. Translation structures we are talking about can be identified with Abelian differentials of the second kind $g(z)dz$ on $S$ with single zero of multiplicity $2m$ at $O$, and $m$ double poles, subject to the condition that all residues vanish. Two translation structures are equivalent if the differentials differ by a non-zero constant factor.

We refer to a survey [36] of translation structures. Structures considered in this survey have no poles and correspond to Abelian differentials of the first kind. To explain the name “translation structure”, consider the Abelian integral

$$f(z) = \int_{z_0}^{z} g(\zeta)d\zeta.$$  \hspace{1cm} (20)

This is a multi-valued function on $S$ with a single critical point at $O$, and the monodromy of $f$ consists of translations by the periods of the integral coming from the fundamental group of $S$. This function $f$ is a developing map of a singular flat structure on $S$: it has one conic singularity with the angle $2\pi(2m+1)$ at $O$ and $m$ simple poles; the monodromy of this structure consists of translations, and the local monodromy at all points is trivial.

**Proposition 1.1.** The correspondence $w \mapsto \Omega = dx/(uw^2)$ is a bijection between the space of Lamé functions and the space of triples $(S, O, \Omega)$, where $S$ is a torus, $O \in S$ a point, and $\Omega$ is an Abelian differential on $S$, which switches sign under the conformal involution, and has a single zero of multiplicity $2m$ at the point $O$, and $m$ double poles with vanishing residues. This bijection defines a biholomorphic map between $L_m$ and the moduli space of Abelian differentials of considered type, up to scaling.
One can pull back the flat metric on $\mathbb{C}$ via $f$ and obtain a flat metric on the torus with one conic singularity at $O$ with the angle $2\pi(2m + 1)$ and $m$ simple “poles”. A pole of a flat metric is a point which has a punctured neighborhood isometric to $\{z \in \mathbb{C} : |z| > R\}$ with Euclidean metric, for some $R > 0$. We call our torus equipped with this metric a flat singular torus. Two flat singular tori are considered equivalent if there is an orientation-preserving diffeomorphism between them multiplying the metric by a non-zero constant.

To study flat singular tori we cut each of them into two congruent flat singular triangles. Congruent means “related by an orientation-preserving isometry”.

**Definition 1.1.** A flat singular triangle (FT) is a closed disk with three marked boundary points which are called corners, equipped with a flat metric with conic singularities at the corners and possibly simple poles inside the disk or on the open boundary arcs (sides) between adjacent corners, and such that the sides are geodesic. A side passing through a pole must be “unbroken” at this pole: in the chart $\{z \in \mathbb{C} : |z| > R\}$ it corresponds to two rays of the the same line.

Alternatively an FT can be described as a triple $(D, \{a_j\}, f)$, where $D$ is a closed disk in the complex plane, $(a_1, a_2, a_3)$ three distinct boundary points, and $f$ a locally univalent meromorphic function on $D \setminus \{a_j\}$ with conic singularities at $a_j$, which means

$$f(z) = f(a_j) + (z - a_j)^{\alpha_j}h_j(z),$$

where $\alpha_j > 0$, and $h_j$ is analytic near $a_j$, $h_j(a_j) \neq 0$, and such that the images of the sides $f([a_j, a_{j+1}]) \subset \ell_j \cup \{\infty\}$ where $\ell_j$ are three straight lines (not necessarily distinct) in the complex plane.

Two such triples $(D, \{a_j\}, f)$ and $(D', \{a'_j\}, g)$ are equivalent if there is a conformal homeomorphism $\phi : D \to D'$, $\phi(a_j) = a'_j$, and complex constants $c_1, c_2$ such that $f = c_1g \circ \phi + c_2$.

These two definitions of FT are equivalent: for a given triple, we pull back the Euclidean metric from $\mathbb{C}$ via $f$ and obtain a metric on $D$ satisfying the first definition. In the opposite direction, given such a metric we obtain $f$ as its developing map.

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1 We assume that the complex plane has the standard orientation, and conformal maps preserve it otherwise we call them anti-conformal.
It is easy to see that the sum of the interior angles $\pi \alpha_j$ of a flat singular triangle is an odd integer multiple of $\pi$. If one angle is an integer multiple of $\pi$, then all three of them are integer multiples of $\pi$.

An FT can be visualized by making a picture of its image in the plane under the developing map $f$. Such a picture consists of three lines (not necessarily distinct), three (pairwise distinct) points of their intersections $f(a_j)$, and a marking of the angles. See Figs. 1a, 1b, 1d, 1e, 2a-f. We mark the angles by little arcs near the images of the corners, and write $a_j$ instead of $f(a_j)$ in the pictures. Two such pictures define the same FT if they are related by a complex affine map.

An FT is called balanced if its angles $\pi \alpha_1, \pi \alpha_2, \pi \alpha_3$ satisfy the triangle inequalities
\[
\alpha_i \leq \alpha_j + \alpha_k,
\]
for all permutations of $(i,j,k)$. For example, a triangle with angle sum $\pi$ in Fig. 1a is balanced if and only if all its angles are $\leq \pi/2$. A triangle with angle sum $3\pi$ in Fig. 1b is balanced if and only if the largest angle is $\leq 3\pi/2$.

All flat singular triangles whose angles are integer multiples of $\pi$ are balanced. A balanced triangle is called marginal if we have equality in (21) for at least one permutation $(i,j,k)$. Otherwise it is called strictly balanced.

We abbreviate the expression “balanced flat singular triangle” as BFT.

Our main technical result is the following

**Theorem 1.5.** Every flat singular torus has a decomposition into two congruent BFT. When the triangles are strictly balanced, this decomposition is unique up to a cyclic permutation of the corner labels. If they are marginal, there are at most two such decompositions: a marginal triangle and its reflection define the same torus.

This is similar to Theorem 1.3 in [17] for spherical tori with one singularity. Theorem 1.5 gives a parametrization of our moduli space $L_m$ by a simpler moduli space $T_m$ for BFT’s with the sum of the angles $\pi(2m + 1)$. This last space admits a nice partition into open cells which is used to prove Theorem 1.1.

The plan of the paper is the following. In section 2 we recall basic facts about Lamé equations and Lamé functions, and explain the connection between Lamé functions and translation structures.
In section 3 we discuss BFT and define a map $\Phi = \Phi_m : T_m \to L_m$. Explicit local coordinates on $T_m$ are described in Section 4 and we show that $\Phi$ is complex analytic and proper. The proof of properness is based on the study of geodesics on flat singular tori.

In section 5 we prove the first part of Theorem 1.5, surjectivity of $\Phi$.

In section 6 we develop a classification of BFT and explicitly describe a partition of $T_m$ into open cells. We show that $T_m$ consists of two connected components, and that $\Phi$ is in fact 3-to-1 on the subset of strictly balanced triangles, and 6-to-1 on the subset of marginal triangles. Factoring $T_m$ by an appropriate equivalence relation we obtain a space $T^*_m$ and show that the induced map $\Phi^*_m : T^*_m \to L_m$ is injective, thus completing the proof of Theorem 1.5.

In section 7 we analyze the natural partition of $T_m$ and prove Theorem 1.1.

In section 8 we prove Theorem 1.2 and its two corollaries. This is based on Lemma 8.2 which is proved in Section 9.

The last two sections are devoted to spherical metrics: in Section 10 we discuss the monodromy of Lamé equations, and in Section 11 we produce equations and pictures of Lin–Wang curves, enumerate them, and show that they are smooth and real analytic.

Since we refer to some figures many times, in different places, all figures are collected at the end of the paper, after the reference list.

We thank Walter Bergweiler, Robert Maier and Vitaly Tarasov for useful discussions. Walter Bergweiler produced Figs. 12–18 and a Maple program generating polynomials $F_m$. Vitaly Tarasov proved Lemma 8.2. We also thank Eduardo Chavez Heredia for bringing [31] to our attention.

2 Lamé equations, Lamé functions and translation structures

We use the form (3) of the Lamé equation. Every solution $W$ of (3) is meromorphic in the $z$-plane. Indeed, by the existence theorem for linear ODE, the singularities of solutions belong to the lattice $\Lambda$, and plugging a power series for $W(z)$ at 0 shows that there are two linearly independent meromorphic solutions.
Proof of Proposition 1.1

We start with a Lamé function and assign to it a translation structure. Let \( W \) be a meromorphic solution of (3) whose square is even and \( \Lambda \)-periodic. Then
\[
W(z + \omega) = \pm W(z), \quad \omega \in \Lambda. \tag{22}
\]
Now
\[
W_1(z) := W(z) \int^z W^{-2}(\zeta) d\zeta \tag{23}
\]
is another solution of (2), linearly independent of \( W \), which can be seen by direct computation, so it must be also meromorphic, since every solution of (3) is meromorphic. It follows that all residues of \( W^{-2}(\zeta) d\zeta \) vanish. Now the ratio of two solutions \( f = W_1/W \) is an Abelian integral of the second kind, and as a ratio of two solutions of a second order linear equation, it also satisfies the Schwarz equation
\[
\frac{f'''}{f'} - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 = -2 (m(m+1)\wp + \lambda), \tag{24}
\]
thus all critical points of \( f \) are in \( \Lambda \) and \( f \) is \((2m+1)\)-to-1 at these points.

So every Lamé function defines a translation structure of the desired type. The differential \( W^{-2}(z) dz \) descends via the map
\[
z \mapsto (\wp(z), \wp'(z)) = (x, u),
\]
to the differential \( dx/(u^2 u) \) on the curve (1) as stated in Proposition 1.1.

Conversely, suppose that a translation structure of the described type is given and its developing map is
\[
f(z) = \int^z g(\zeta) d\zeta \tag{25}
\]
is given. Then \( g \) is an even elliptic function with \( m \) double poles per period parallelogram, vanishing residues, and zeros of order \( 2m \) at the points of \( \Lambda \). So \( g = W^{-2} \) for some meromorphic function \( W \) satisfying (22). Now we define \( W_1 \) by (23), and a direct computation shows that
\[
WW' - W'W_1 = 1.
\]
This means that \( W \) and \( W_1 \) are two linearly independent solutions of some equation \( W'' = PW \), where \( P \) is an elliptic function with periods \( \Lambda \). As the
only pole of $P$ can occur at a critical point of $f$, we conclude that poles of $P$ must be at the points of $\Lambda$, and a simple calculation with power series at $0$ shows that

$$P(z) = m(m + 1)z^{-2} + O(1), \quad z \to 0,$$

so $P(z) = m(m + 1)\varphi(z) + \lambda$ with some $\lambda \in \mathbb{C}$, and thus $W$ is a Lamé function.

We recall that two Lamé functions $W_1$ and $W_2$ are equivalent if $W_1(z) = cW_2(kz)$ for some $c$ and $k$ in $\mathbb{C}^*$. Translation structures are equivalent if their developing maps $f_1, f_2$ are related by post-composition with an affine map: $f_1(z) = af_2(kz) + b, a, k \in \mathbb{C}^*, b \in \mathbb{C}$.

We proved that equivalence classes of degree $m$ Lamé functions are in one-to-one correspondence with classes of translation structures on the torus with one conic singularity with the angle $2\pi(2m+1)$.

It is clear that this correspondence is continuous and holomorphic, therefore it is biholomorphic.

Let us recall how the polynomial $F_m$ is computed (another method is described in Section 6). It is convenient to use the algebraic form of the Lamé equation [2] which can be also written as

$$w'' + \frac{1}{2} \left( \sum_{j=1}^{3} \frac{1}{\zeta - e_j} \right) w' = \frac{m(m + 1)\zeta + \lambda}{4(\zeta - e_1)(\zeta - e_2)(\zeta - e_3)}w, \quad (26)$$

where $w = y \circ \varphi$. Since the only singularities in $\mathbb{C}$ of this equation are $e_j$, and the local exponents at these singularities are $\{0, 1/2\}$, each Lamé function can be written as

$$w(\zeta) = c \prod_{j=1}^{3} (\zeta - e_j)^{k_j/2} \prod_{j=1}^{n} (\zeta - \zeta_j), \quad (27)$$

where $k_j \in \{0, 1\}$ and

$$\sum_{j=1}^{3} k_j + 2n = m. \quad (28)$$

Plugging $\zeta = \zeta_k$ into (26) we obtain after simple calculations (see, for example, [35, 23.21]) the following system of equations for $\zeta_j$

$$2 \sum_{j,j \neq k} \frac{1}{\zeta_k - \zeta_j} + \sum_{j=1}^{3} \frac{k_j + 1/2}{\zeta_k - e_j} = 0, \quad 1 \leq k \leq n. \quad (29)$$
This system of equations determines Lamé functions. According to a theorem of Heine and Stieltjes [35, 23.46], system (29) has at most \( n + 1 \) solutions and exactly \( n + 1 \) for generic \( e_j \). Moreover, it has exactly \( n + 1 \) solutions when all \( e_j \) are real. System (29) is a very special case of the Bethe ansatz equations which frequently occur in mathematical physics and in the study of metrics with conic singularities.

Lamé functions are classified according to the values of \( k_j \) in (27): traditionally the number

\[
1 + \sum_{j=1}^{3} k_j \in \{1, 2, 3, 4\}
\]

is called the kind of a Lamé function [35].

Using (28) and Stieltjes theorem we conclude that the total number of Lamé functions for a given generic lattice is \( 2m + 1 \), and this is the degree of the Lamé spectral polynomial \( F_m \) with respect to \( \lambda \). The number of Lamé functions is exactly \( 2m + 1 = d^I_m + d^II_m \) for lattices with real \( e_j \), or equivalently with real \( g_j \) with \( g_3^2 - 27g_2 > 0 \).

It is easy to see that for even \( m \geq 2 \) there exist Lamé functions of the first and third kind; we denote the corresponding subsets by \( L^I_m \) and \( L^II_m \). Similarly, when \( m \geq 3 \) is odd, we define \( L^I_m \) as the set of Lamé functions of the fourth kind, and \( L^II_m \) as the set of Lamé functions of the second kind. This explains why \( L_m \) has at least two components when \( m \geq 2 \). The more difficult result, which is a part of Theorem 1.1, is that these components are in fact irreducible.

**Connection with spin structures.**

Connected components of moduli spaces of Riemann surfaces endowed with nonzero holomorphic differentials were classified by Kontsevich and Zorich [21]; the case of meromorphic differentials was treated by Boissy [7]. A consequence of Theorem 4.1 in [7] is that the moduli space of triples \((S, O, \Omega)\), where \((S, O)\) is an elliptic curve and \( \Omega \) is a meromorphic differential on \( S \) with a zero at \( O \) of order \( 2m \) and \( m \) poles \( q_1, \ldots, q_m \) of order 2 (and arbitrary residues), has exactly two connected components. Moreover, such components are distinguished by the spin invariant (already defined in [21 Sect. 2.2]). In our particular case, the spin invariant of \((S, O, \Omega)\) is odd if there exists a function on \( S \) with simple poles at \( q_1, \ldots, q_m \) and a zero of order \( m \) at \( O \), and it is even if such function does not exist.
Proposition 2.1. A Lamé function \( w \) is in \( L^I_m \) if the spin invariant of the corresponding translation surface is odd, and \( w \) is in \( L^{II}_m \) if the spin invariant is even.

Proof. Let \((S,O,\Omega)\) be the translation surface associated to the Lamé function \( w \). In particular, \( \Omega = \varphi/w^2 \), where \( \varphi \) is a nonzero holomorphic differential on \( S \), see Proposition 1.1. Then the spin invariant of \((S,O,\Omega)\) is odd if and only if \( w \) is a well-defined function on \( S \), which happens if and only if \( w = Q(x) \) or \( w = Q(x)u \) for a suitable \( Q \in \mathbb{C}[x] \). Hence, the spin invariant is odd if and only if \( w \) is of type \( I \). \( \square \)

As a consequence of [7, Thm. 4.1] we obtain

Corollary 2.1. The space \( L_m \) of Lamé functions is the disjoint union of the subset \( L^I_m \) of Lamé functions of type \( I \) and of the subset \( L^{II}_m \) of Lamé functions of type \( II \).

The techniques of [7] do not allow to study connected components of moduli of meromorphic differentials with vanishing residues. So it does not follow from [7] that \( L_m \) has exactly two connected components, and so that \( L^I_m \) and \( L^{II}_m \) are connected.

3 Balanced flat singular triangles

In this section we consider balanced flat singular triangles (BFT) with marked corners \( a_1, a_2, a_3 \), enumerated according to the positive orientation of the boundary (so that the region is on the left when we trace the boundary).

We denote the interior angles at these corners by \( \pi \alpha_1, \pi \alpha_2, \pi \alpha_3 \). As we already noticed, the sum of the angles is an odd multiple of \( \pi \), more precisely

\[
\alpha_1 + \alpha_2 + \alpha_3 = 4n + 2k + 1 = 2m + 1,
\]

where \( n \) is the number of interior poles, and \( k \) is the number of boundary poles. To prove (30) we recall the argument used in the proof of the Schwarz–Christoffel formula. Consider the developing map \( f \) defined in the upper half-plane with corners at \((a_1, a_2, a_3) = (0, 1, \infty)\). Since the monodromy of \( f \) is affine, \( f''/f' \) must extend to the complex plane as a rational function whose poles are symmetric with respect to the real line, and we have

\[
\frac{f''}{f'}(z) \sim \frac{a_j - 1}{z - a_j}, \quad z \to a_j, \quad j \in \{1, 2\}, \quad \frac{f''}{f'}(z) \sim -\frac{a_3 + 1}{z}, \quad z \to \infty,
\]

where

\[
f''(z) \sim \frac{a_j - 1}{z - a_j}, \quad z \to a_j, \quad j \in \{1, 2\}, \quad \frac{f''}{f'}(z) \sim -\frac{a_3 + 1}{z}, \quad z \to \infty,
\]

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and
\[ \frac{f''}{f'}(z) \sim -\frac{2}{z-z_j}, \quad z \to z_j \]
at the poles. So (30) follows by the Residue Theorem.

Fig. 1a shows a triangle with \( n = k = 0 \). In Fig. 1b \( n = 0, k = 1 \). Fig. 1d shows a triangle with \( n = 1, k = 0 \) (left) and two triangles with \( n = 0, k = 2 \) (right). Fig. 1e shows three triangles with \( n = 1, k = 0 \). Fig. 2 shows all types of triangles with sum of the angles \( 5\pi \) (\( m = 2 \)).

We also recall that either none of the \( \alpha_j \) or all of them are integers.

**Proposition 3.1.** An FT with non-integer \( \alpha_j \) is completely determined by the angles, and any positive angles \( \pi\alpha_j \) where \( \alpha_j \) are not integers and their sum is odd can occur.

For integer angles, the necessary and sufficient condition of existence of FT is that the sum of \( \alpha_j \) is odd and (21) is satisfied. For any such angles, there are three one-parametric families of FT’s.

A similar result for spherical triangles was proved in [15].

**Proof.** The first statement is essentially due to Klein [20]. If our triangle is modeled on the upper half-plane \( D = \mathbb{H} \) with vertices \( (a_1, a_2, a_3) = (0, 1, \infty) \) then the developing map \( f : \mathbb{H} \to \mathbb{C} \) satisfies the Schwarz equation with three singularities \( (0, 1, \infty) \) and \( \alpha_j \) determine this equation completely. When the \( \alpha_j \) are not integers, the monodromy representation corresponding to this equation is non-trivial, and there is only one choice, up to an affine transformation, of a solution with affine monodromy.

If all \( \alpha_j \) are integers, then the monodromy representation is trivial, and the developing map \( f \) extends from its domain \( D \) to the whole Riemann sphere and we obtain a rational function with three critical points \( a_1, a_2, a_3 \). The images of all three sides under the developing map \( f \) belong the same line \( \ell \). Preimage \( f^{-1}(\ell) \cap D \) is called the net of \( f \) (see [16]). The net is a cell decomposition of \( D \) with three vertices \( a_1, a_2, a_3 \). The 1-cells of this decomposition are disjoint chords of the disk \( D \) and three arcs of \( \partial D \). The number of chords in the interior of \( D \) which are adjacent to \( a_j \) is \( \alpha_j - 1 \geq 0 \). If \( m_j \) is the number of chords from \( a_i \) to \( a_k \), then \( \alpha_j - 1 = m_i + m_k \), and these three equations have unique non-negative solution \( (m_1, m_2, m_3) \) if and only if the integers \( \alpha_j \geq 1 \) satisfy (21) and their sum is odd. Thus the angles determine the net. Once the net is given, the developing map can be recovered from the values \( f(a_1), f(a_2), f(a_3) \) which in the considered case...
belong to a line. It is clear that one of the 2-faces of a net is a triangle while the others are digons. Suppose that the triangular face is mapped by \( f \) onto the upper half-plane. Then we have three possible orderings of \( f(a_j) \) on the real line. By scaling we may arrange that \( \min_j f(a_j) = 0 \) and \( \max_j f(a_j) = 1 \), then the intermediate point of the three \( f(a_j) \) serves as a parameter. So the set of triangles with given integer angles is parametrized by three intervals. □

Examples of nets of triangles are shown in Fig. 1c, where the stars mark the location of the poles of \( f \). These three nets correspond to triangles with the angles \((2\pi, 3\pi, 4\pi)\). In Fig. 1d, three triangles with the angles \((\pi, 2\pi, 2\pi)\) are shown together with their nets.

Let us define two types of triangles which we call primitive. A primitive triangle of type \( A \) has all angles in \((0, \pi)\) and their sum is \( \pi \) (Fig. 1a), and the primitive triangle of type \( B \) has one angle in \([\pi, 2\pi)\), two others in \((0, \pi]\), and the sum of the angles is \( 3\pi \) (Fig. 1b).

**Proposition 3.2.** Every BFT can be obtained from one of the two primitive triangles \( A \) or \( B \) by gluing half-planes to the sides.

**Proof.** This is essentially due to Klein [20], but we sketch a proof. Consider the case of non-integer angles. Let \( \alpha_i = \{\alpha_i\} + [\alpha_i] \), be the decomposition into fractional and integer parts.

a) If \( \sum [\alpha_i] \) is even, then \( \sum \{\alpha_i\} \) is odd. Since the last sum belongs to \((0, 3)\), we must have

\[
\sum \{\alpha_i\} = 1. \tag{31}
\]

Since our triangle is balanced, and we have (31), we obtain for all permutations \((i, j, k)\) of \((1, 2, 3)\):

\[
[\alpha_i] = \alpha_i - \{\alpha_i\} \leq \alpha_j + \alpha_k - \{\alpha_i\} = [\alpha_j] + [\alpha_k] + \{\alpha_j\} + \{\alpha_k\} - \{\alpha_i\} \leq [\alpha_j] + [\alpha_k] + 1.
\]

So \( [\alpha_i] \leq [\alpha_j] + [\alpha_k] \). It follows that the following quantities are non-negative integers:

\[
x_i = ([\alpha_j] + [\alpha_k] - [\alpha_i]) / 2,
\]

and we have \( [\alpha_i] = x_j + x_k \), for all permutations \((i, j, k)\) of \((1, 2, 3)\). So we can take a triangle with the angles \( \{\alpha_j\} \) of the type \( A \) and glue \( x_j \) half-planes
to the side opposite to $a_j$, for each $j$. The resulting triangle has the same angles as our original triangle, so it is the same triangle by Proposition 3.1.

b) If $\sum \lfloor \alpha_i \rfloor$ is odd, then we decrease one of the integer parts (for example, the largest one) by 1, and increase the corresponding fractional part by 1. So we set for some $i$

$$p_i = \lfloor \alpha_i \rfloor - 1, \quad \alpha'_i = \{\alpha_i\} + 1,$$

leaving other numbers unchanged ($\alpha'_j = \{\alpha_j\}$, $p_j = \lfloor \alpha_j \rfloor$, and similarly for $k$). Now we have

$$\alpha'_i \in (1, 2), \quad \alpha'_j \in (0, 1), \quad \alpha'_k \in (0, 1),$$

and since $\sum \{\alpha_{\ell}\}$ is even and less than 3 in this case, it must be 2, and thus

$$\sum \alpha'_\ell = 3.$$ (33)

Now, since our triangle is balanced, and using (33) and (32) we obtain

$$p_i = \alpha_i - \alpha'_i \leq \alpha_j + \alpha_k - \alpha'_i = p_j + p_k + \alpha'_j + \alpha'_k - \alpha'_i \leq p_j + p_k + 1,$$

and since the sum of $p_{\ell}$ is even, we conclude that $p_i \leq p_j + p_k$. So we can define $x_i, x_j, x_k$ as in part a) and conclude that our triangle is obtained from a triangle with the angles $\alpha'_\ell$ of the type B, by gluing $x_{\ell}$ half-planes to the side opposite to $a_{\ell}$.

Triangles with integer angles can be considered using their nets introduced in the proof of Proposition 3.1. A net is a chord diagram in a disk with three vertices on the circle. Evidently each net has one triangular face, and the rest of the faces are digons. Triangular face corresponds to the primitive triangle and digons are half-planes. Since the angle sum is odd, only case b) can occur, and the primitive triangle is of type B with the angles $(\pi, \pi, \pi)$. Thus the primitive triangle in this case is just a half-plane with three marked boundary points.

\[\square\]

Corollary 3.1. Each side of a BFT contains at most one pole, and the developing map sends each side $(a_i, a_{i+1})$ injectively either to the interval $(f(a_i), f(a_{i+1}))$, or to the complement of this interval on the line in $\mathbb{C} \cup \{\infty\}$ containing it.
Remarks. 1. By analyzing the proof of Proposition 3.2 one can obtain the following criterion: the side opposite to $a_i$ contains a pole if and only if
\[
\left\lfloor \left( \left\lfloor \alpha_j \right\rfloor + \left\lfloor \alpha_k \right\rfloor - \left\lfloor \alpha_i \right\rfloor - 1 \right) / 2 \right\rfloor \text{ is odd.}
\]

2. It follows from this Corollary and from (30) that for each $m \geq 2$ we have exactly two possibilities for the number of sides with a pole. When $m$ is even, we have either none or 2 sides with poles. If $m$ is odd, we have either one or three sides with poles. This shows that the set $T_m$ of all BFT’s with the angle sum $\pi(2m+1)$, $m \geq 2$ must have at least four connected components. We show in Section 6 that there are exactly four.

3. The decomposition into a primitive triangle and half-planes stated in Proposition 3.2 is canonical when $\sum \left\lfloor \alpha_j \right\rfloor$ is even, but not canonical when it is odd. In the latter case, we can obtain from one to three such different decompositions, depending on the number of positive $\left\lfloor \alpha_j \right\rfloor$.

4. The primitive triangle $T'$ obtained from a balanced triangle $T$ may be unbalanced. In this case, there is always at least one half-plane in $T$ glued to the side of $T'$ which is opposite to the largest angle of $T'$. Indeed, let $\alpha'_j$ be the angles of $T'$, and $\alpha_1$ is the largest one. Then the angles of $T$ are $\alpha_i = \alpha'_i + x_j + x_k$, where $(i, j, k)$ is a permutation of $(1, 2, 3)$, and $x_j$ are the numbers of half-planes glued to the sides opposite to $\alpha'_j$, $j = 1, 2, 3$. Since $T$ is balanced,
\[
\alpha'_1 + x_2 + x_3 = \alpha_1 \leq \alpha_2 + \alpha_3 = \alpha'_2 + \alpha'_3 + x_2 + x_3 + 2x_1,
\]
so $\alpha'_1 \leq \alpha'_2 + \alpha'_3 + 2x_1$. Thus if $T'$ is unbalanced, we have $x_1 > 0$.

Construction of the map $\Phi$.

Let $T_m$ be the set of all balanced triangles with the sum of the angles $\pi(2m+1)$. For every $T \in T_m$ we define a singular flat torus $\Phi(T)$ in the following way. We take two copies of $T$ and identify each pair of corresponding sides by the orientation-reversing isometry.

Thus all corners of both copies are glued into one point, and the sides are glued into three simple loops on the torus based at this point and otherwise disjoint.

Notice that the resulting torus has an orientation-preserving isometric involution which interchanges the two triangles. The four fixed points of this involution are: the conic singularity (corresponding to the vertices of the
triangle) and the “midpoints of the sides” which are points of order 2 on the elliptic curve. If a side is unbounded, its midpoint is a pole.

With such gluing we obtain a flat singular torus with one singularity with the angle $2\pi(2m + 1)$. There are $m$ simple poles on the torus coming from the poles of the metric on the triangle. An interior pole of $T$ gives two poles on $\Phi(T)$, while a pole on a side of $T$ gives one pole on $\Phi(T)$.

Let $T$ be a BFT with corners $a_1, a_2, a_3$ and developing map $f$. Let $b_j = (f(a_i) + f(a_k))/2$, where $(i, j, k)$ is a permutation of $(1, 2, 3)$. We define affine maps $s_j$ to be rotations by $\pi$ about $b_j$. Consider the group generated by $s_1, s_2, s_3$. It contains a subgroup $G$ of index 2 consisting of translations; elements of $G$ are products of even numbers of $s_j$. The following proposition is evident:

**Proposition 3.3.** The monodromy group of the developing map of the flat singular torus $\Phi(T)$ is $G$. □

In particular, the monodromy consists of collinear translations if and only if the angles of the triangle are integer multiples of $\pi$.

Next we address the question when two different BFT’s can give the same (isometric up to a constant factor) tori. This can happen in at least two ways:

1. The triangles are obtained by a cyclic permutation of the corners.
2. Some pairs of marginal triangles define the same torus.

More precisely, we have

**Proposition 3.4.** For a marginal triangle $T$ with the angles $(\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_1 = \alpha_2 + \alpha_3$, and the triangle $T^*$ with the angles $(\alpha_1, \alpha_3, \alpha_2)$ the tori $\Phi(T)$ and $\Phi(T^*)$ are congruent.

$T$ and $T^*$ are related by a reflection, an orientation-reversing isometry.

Proof of Proposition 3.4. First we notice that a marginal triangle cannot have integer angles (since the angle sum is odd), so it is completely determined by the angles. We use Klein’s decomposition described in the proof of Proposition 3.2. Our triangle is obtained from a primitive triangle by gluing half-planes to the sides.

We claim that for a marginal triangle, no half-planes are glued to the side opposite to the larger angle. Indeed, let $\alpha'_j$ be the angles of the primitive
triangle, and \( \alpha_j = \alpha'_j + p_j = \alpha'_j + x_i + x_k \), where \( p_j, x_i, x_k \) are non-negative integers. Then \( \alpha_1 = \alpha_2 + \alpha_3 \) implies

\[
\alpha'_1 = \alpha'_2 + \alpha'_3 + 2x_1,
\]

and we obtain that \( x_1 = 0 \) since \( \alpha_1 < 2 \), \( \alpha'_2 > 0 \), \( \alpha'_3 > 0 \). This proves the claim.

We also conclude from (34) with \( x_1 = 0 \) that the primitive triangle \( T' \) with the angles \( \alpha'_1, \alpha'_2, \alpha'_3 \) corresponding to our triangle \( T \) is also marginal, with the larger angle \( \alpha_1 \). If \( T' \) is of the type A, then \( \alpha'_1 = 1/2 \). If \( T' \) is of the type B, then \( \alpha'_1 = 3\pi/2 \). Gluing together two copies of \( T' \) along the side \((a_3, a_1)\) we obtain either a rectangle in the plane, or a complement of a rectangle on the Riemann sphere. It is clear that rectangles obtained from \( T \) and \( T^* \) are congruent, and the corresponding tori \( \Phi(T) \) and \( \Phi(T^*) \) are also congruent, see Fig. 3. □

We give two examples illustrating Theorem 1.5 in the simplest cases.

**Example 1.** Consider a triangle \( T \) shown in Fig. 1a, whose angle sum is \( \pi \). It is balanced iff all angles are at most \( \pi/2 \). Gluing two congruent copies of such a triangle along a common side, we obtain a parallelogram in the plane. Identifying the opposite sides of this parallelogram by translations we obtain a flat non-singular torus \( \Phi(T) \) (\( m = 0 \)).

Now consider a flat non-singular torus \( L \) with a marked point \( O \). Let \( f : \mathbb{C} \to L, \, f(0) = O \) be the universal cover. Then there is a lattice \( \Lambda \subset \mathbb{C} \) such that \( L = \mathbb{C}/\Lambda \). A fundamental region \( D \) of this lattice can be taken in the form of a parallelogram which can be normalized so that the shorter side is \([0, 1]\). Let \([0, \tau]\) be the longer side. It is well known that \( \tau \) can be always chosen in the fundamental region of the modular group

\[
G = \{ \tau : |\tau| \geq 1, |\text{Re} \, \tau| \leq 1/2 \}.
\]

To achieve this one can normalize so that the shortest non-zero element of \( \Lambda \) is 1, then the shortest non-real element of \( \Lambda \) is \( \tau \). For \( \tau \in G \), each diagonal of \( D \) breaks \( D \) into a pair of congruent triangles, such that at least one pair consists of balanced triangles. Both pairs consists of balanced triangles if and only if \( D \) is a rectangle, in which case the triangles of different pairs are marginal and are reflections of each other. It is easy to see that at least one diagonal breaks \( D \) into balanced triangles if and only if \( \tau \in G \). This proves
Theorem 1.5 for $m = 0$. The proof in the general case is a generalization of this argument. □

Example 2. Consider a triangle $T$ in Fig. 1b. The angle sum is $3\pi$ ($m = 1$). Rotating $T$ by $\pi$ about the point $(a_1 + a_3)/2$ we obtain a congruent triangle $T'$. Gluing $T$ and $T'$ along the side which contains pole, we obtain the “exterior parallelogram” $Q$. Identifying the opposite sides of $Q$ by translations, we obtain the flat singular torus $\Phi(T)$ with the angle $6\pi$ at the singularity. It will be proved later that every flat singular torus with the angle $6\pi$ can be obtained from an exterior parallelogram by identifying the opposite sides by translations. We claim that every exterior parallelogram $Q$ can be obtained by gluing two balanced triangles of the type B. Indeed, consider the diagonals of the parallelogram $C \setminus Q$, and extend them to $Q$. Each extension of a diagonal breaks $Q$ into two triangles of the type B, and it is easy to see that only for one diagonal these triangles are balanced, unless our $Q$ is an exterior rectangle as in Fig. 3. When $Q$ is an exterior rectangle, our triangles are marginal, and they are related by reflection as in Proposition 3.4. □

Now we define an equivalence relation $\sim$ on $T_m$, in accordance with statements 1, 2 before Proposition 3.4. Two BFT’s $T$ and $T'$ are equivalent if either there is a congruence between them cyclically permuting the corners, or if they are marginal and related by a reflection as described Proposition 3.4. Then our map $\Phi$ is well defined on the equivalence classes, and we prove in Section 5 that the induced map

$$\Phi^* : T^*_m \rightarrow L_m, \quad T^*_m = T_m / \sim$$

is bijective. We denote by $\Phi^*$ the map defined by this formula on the disjoint union $T^*$ of $T^*_m$, $m \geq 0$, and mapping it to the disjoint union $L$ of $L_m$.

4 Complex analytic coordinates on $T_m$

We introduce a complex analytic structure on the set of BFT. We define functions on the set of BFT:

$$\phi_{i,j,k} = \frac{f(a_i) - f(a_j)}{f(a_k) - f(a_j)}.$$ 

These functions are locally injective and we use them as complex coordinates on $T^*_m$. The correspondence maps between charts are linear-fractional.
So they define a complex analytic structure on $T^*_m$ (and even a projective structure).

To show that the map $\Phi_m^*$ is complex analytic, we recall Proposition 3.3 which implies that $\phi_{i,j,k}$ are ratios of periods of the differential in Proposition 1.1. It is clear that periods and their ratios are analytic on $L_m$ (which can be identified with a space of proportionality classes of differentials). We refer to a much more general statement of this kind in [3, Corollary 2.3].

So $\phi_{i,j,k}$ are local complex analytic coordinates on $T^*_m$, and we have

**Proposition 4.1.** The map $\Phi_m^* : T^*_m \to L_m$ is complex analytic. □

Next we prove

**Proposition 4.2.** The maps $\Phi_m^*$ are proper.

**Proof.** We are going to show that a degenerating sequence of triangles gives a degenerating sequence of tori. First we clarify the notions of degenerating sequences.

For every flat singular torus $L$ with one conic singularity at $O$, we define the set $G$ of all simple geodesics loops based at $O$. Some of these loops may pass through a pole and thus have infinite length. The systole $s_1$ is the minimal length of all elements of $G$. The second systole $s_2 \geq s_1$ is the minimal length of all elements of $G$ whose homotopy class is not a multiple of the class of some element of $G$ of length $s_1$. Since we identify tori with proportional metrics, only the ratio $s_2/s_1$ is defined as a function on $L_m$. It is clear that $s_2/s_1$ is bounded on every compact subset of $L_m$.

If $s_2/s_1 \to \infty$ for a sequence of tori in $L_m$, then this sequence diverges in $L_m$, and we say that tori of this sequence degenerate.

Proposition 3.2 defines for every balanced $T$ a primitive triangle $T'$ which is of type A or B (Fig. 1a,b). When the sum of $\lfloor \alpha_j \rfloor$ is even, then $T'$ is of type A and it is defined uniquely. When this sum is odd, we may have up to three choices for $T'$; they are of type B (see Remark 3 after Corollary 3.1). We pick one of them, as we did in the proof of Proposition 3.2.

This primitive triangle $T'$ may be unbalanced. If this is the case, then there is at least one half-plane in $T$ attached to the side of $T'$ opposite to the largest angle of $T'$ (Remark 4 after Corollary 3.1). Then we denote by $T''$ the union (more precisely the result of gluing) of $T'$ with this half-plane. When $T'$ is balanced we set $T'' = T'$. We call $T''$ the balanced extension of $T'$.

There are 4 types of $T''$: balanced of type A or B and balanced extensions of unbalanced A and B. We call these last two types $A''$ and $B''$. 
The numbers $|f(a_i) - f(a_j)|$ are all the same for $T, T'$ and $T''$. Consider the ratios
\[ \frac{f(a_i) - f(a_j)}{f(a_k) - f(a_j)} \]

It is easy to see that if a sequence of triangles in $T_m$ leaves every compact in $T_m$ then one of these ratios tends to infinity. We call triangles of such a sequence degenerate, and notice that $T, T'$ and $T''$ degenerate simultaneously. (Unbalanced triangle of type $A$ can degenerate in a different way: when a vertex tends to an interior point of the opposite side. But its balanced extension does not degenerate in this case.)

We claim that $s_1, s_2$ for $\Phi(T'')$ are the same as for $\Phi(T)$. Indeed, attaching $n$ half-planes to a side of a triangle $T''$ results in attaching a digon $D$ (with poles) with angles $2\pi n$ at at its two corners to the torus $\Phi(T)$. Every curve in such a digon with endpoints at $\partial D$ is at least as long as the segment between its endpoints. On the other hand, there are four types of $T''$ of which $A$ and $B''$ have all sides bounded, while the complementary segment of the unbounded side of $A''$ or $B$ is at least $s_2$ for $T''$.

It remains to consider the tori $\Phi(T'')$, and to show that when $T''$ degenerates then $s_2/s_1 \to \infty$ for these tori. There are 4 cases to consider:

a) $T'$ is balanced, then $T'' = T'$. Then $T'$ is of type $A$ or $B$, and balanced, and this is essentially the cases of Examples 1, 2 in the end of Section 3.

b) $T'$ is unbalanced, $T''$ is the balanced extension of $T'$ of the type $A$ or $B$. One has to find the first and second systole of such tori $\Phi(T'')$.

We denote $A''$ and $B''$ the classes of triangles consisting of balanced extensions of unbalanced triangles of type $A$ and $B$.

We are interested in the length spectrum of the set of geodesic loops in $\Phi(T_1)$ where $T_1$ is a balanced triangle of one of the types $A, B, A'', B''$. To find it we describe all geodesic loops of finite length in $G$ for each type.

**Lemma 4.1.** For $T_1$ of type $A$, all elements of $G$ have finite length, and they are in bijective correspondence with indivisible elements of the lattice generated by the sides of $T_1$. So $s_1$ and $s_2$ are the smallest and the second smallest lengths of sides of $T_1$.

\[ ^2 \text{An element of } \mathbb{Z}^2 \text{ is called indivisible if it is not an integer multiple of any other element.} \]
For $T_1$ of type $B$, there are two elements in $G$ of finite length. They correspond to the two sides of $T_1$ of finite length.

For $T_1$ of type $A''$, let $a_3$ be the corner opposite to the side of infinite length, and denote $a = f(a_2) - f(a_1)$, $b = f(a_3) - f(a_1)$ Then the length spectrum is

$$\{ |na + b| : n \in \mathbb{Z} \}$$

and $\{ s_1, s_2 \} = \{|b|, |a - b|\}$.

For $T_1$ of type $B''$, there are exactly three elements of $G$ of finite length. They correspond to the sides of $T_1$.

Proof. We look at all geodesics starting at a vertex of $T_1$ with all possible slopes.

If $T_1$ is of type $A$, then the torus $\Phi(T_1)$ is a flat non-singular torus from Example 1 in Section 3. Elements of $G$ are in bijective correspondence with segments whose endpoints are elements of the lattice and which contain no lattice points in their interiors.

For $T_1$ of type $B$, the torus $\Phi(T_1)$ can be represented as an exterior parallelogram $Q$ whose opposite sides are identified by translations. Any geodesic starting from a vertex which is not a side of $Q$, visits the pole and thus has infinite length. The set $G$ contains two elements of finite length (corresponding to two pairs of parallel sides of $Q$).

Let $T_1$ be the balanced extension of an unbalanced triangle $T_2$ with angle sum $\pi$, and $f$ the developing map of $T_1$. We normalize so that $f(T_2) = (0, 1, b)$, so that $a = 1$, and $|b| \leq 1$. The torus $\Phi(T_1)$ is partitioned into two triangles congruent to $T_2$ and two half-planes. Suppose that a geodesic starting from a vertex $v$ is not a side of $T_2$. Then it visits the regions of our partition, one after another. Once the geodesic enters a half-plane it must stay there until it hits a pole, so the length of such geodesic is infinite. A geodesic in $G$ of finite length must cross the two finite sides of $T_1$ alternatively, and its length is given by (35). See Fig. 4, which shows the images under the developing map: $f(T_1)$ is dark, the image of one congruent copy of $f(T_1)$ is grey, and images other congruent copies of $T_2$ are white triangles. Images of several geodesics issued from one corner are shown: the dotted lines are images of geodesics of infinite length which contain poles, and dashed lines are images of some geodesics in $G$ of finite length. Since $|b| \leq 1$, the smallest and the second smallest elements of (35) are $|b|$ and $|1 - b|$.

Now consider a torus $\Phi(T_1)$ where $T_1$ is of type $B''$, that is $T_1$ is a balanced extension of some unbalanced $T_2$ of type $B$. Then $T_1$ is an exterior
triangle (exterior of a bounded triangle with angle sum $\pi$). The torus $\Phi(T_1)$ is obtained by gluing two copies of exterior triangles $T_1$. Every geodesic on this torus which does not correspond to a side of $B''$ passes through a pole. □

We conclude from Lemma 4.1:

For a balanced triangle $T$, the first and second systoles of $\Phi(T)$ are two of the three numbers $|f(a_i) - f(a_j)|$.

It follows that when a balanced triangle $T$ degenerates, then the torus $\Phi(T)$ also degenerates, therefore the map $\Phi$ is proper, and thus $\Phi^*$ is proper as well. This completes the proof of Proposition 4.2. □

5 Surjectivity of $\Phi^*$

We recall that $T_m$ is the set of BFT with the angle sum $\pi(2m + 1)$, and $T_m^* = T_m/\sim$ is the quotient by the following equivalence relation:

(i) we identify triangles obtained from each other by a cyclic permutation of the three vertices, and

(ii) we identify pairs of triangles described in Proposition 3.4.

In this section we prove that the map $\Phi^*_m : T_m^* \to L_m$ is surjective establishing the first part of Theorem 1.5. Injectivity will be proved in Section 6.

Proof of Surjectivity of $\Phi^*$.

The plan of the proof is the following: for a given flat singular torus $L$ we find two special geodesic loops whose complement is a quadrilateral $Q$. Then we construct cell decompositions $C_2, C_4$ of $Q$, and reassembling certain cells of $C_4$ we obtain a decomposition of $L$ into two congruent balanced triangles.

Let $L$ be a torus with the singular point $O$. Consider the germ at $O$ of the developing map $f : L \to \mathbb{C}, f(O) = 0$. Let $g : \mathbb{C} \to L$ be a universal covering with $g(0) = O$. Then the composition

$$F = f \circ g$$

has a meromorphic continuation to the whole plane. This meromorphic function satisfies

$$F(z + \omega) = F(z) + \eta, \quad \omega \in \Lambda.$$  \hspace{1cm} (36)

\footnote{Many authors call this $F$ a developing map.}
Here $\omega \mapsto \eta(\omega)$ is a group homomorphism $\Lambda \to \mathbb{C}$, and there are two possibilities:

$a)$ *Generic case.* The image of $\Lambda$ is another lattice $\Lambda' \subset \mathbb{C}$, of rank 2, and $F : \Lambda \to \Lambda'$ is an isomorphism, or

$b)$ *Degenerate case.* The image of $\Lambda$ belongs to a line through the origin.

The pull back the flat metric via $F$ has the length element

$$ds = |F'(z)||dz|.$$ 

This metric has conic singularities at the critical points of $F$ which are the points of $\Lambda$, and some poles.

Let $\gamma_1$ be a shortest curve among all curves from 0 to some point $\omega \in \Lambda \setminus \{0\}$. We denote its endpoint other than 0 by $\omega_1$. It is clear that $\gamma_1$ is a simple curve; $F(\gamma_1)$ is a segment $[0, \eta_1]$, where $\eta_1 \in \Lambda' \setminus \{0\}$, and the map $F : \gamma_1 \to [0, \eta_1]$ is a homeomorphism.

Let $\gamma_2$ be a shortest of all curves from 0 to some point $\omega_2 \in \Lambda \setminus \{\mathbb{Z}\omega_1\}$. The following lemma implies that $\gamma_1$ and $\gamma_2$ are disjoint except their common endpoint at 0.

**Lemma 5.1.** The curves $g(\gamma_1)$ and $g(\gamma_2)$ in $L$ intersect only at $O$.

*Proof.* Suppose that this is not so, and let $p \neq O$ be a point of intersection. Since $p$ is not a conic singularity, and our curves are geodesic, they must make a non-zero angle at $p$. It follows that the ratio of the periods of the differential $df$ over $g(\gamma_1)$ and $g(\gamma_2)$ is not real.

Now we construct a loop $\Gamma$ in $L$ based at $O$ which is shorter than $g(\gamma_2)$ and whose homology class is not a multiple of $g(\gamma_1)$. The point $p$ breaks $g(\gamma_1)$ into two arcs, and we denote the shorter of these arcs by $I_1$. Similarly $I_2$ is the shorter of the two arcs into which $p$ breaks $g(\gamma_2)$. Let $\Gamma$ be the concatenation of $I_1$ and $I_2$. From our observations on the periods of $df$ we conclude that

$$\int_\Gamma df \neq 0,$$

therefore $\Gamma$ is non-trivial. Here we used that $df$ has no residues at the poles. Moreover, this integral cannot be a real multiple of the integral over $g(\gamma_1)$, so $\Gamma$ is not a multiple of $g(\gamma_1)$. Finally, the length of $\Gamma$ is at most the length of $g(\gamma_2)$, but $\Gamma$ can be shortened since it has a non-zero angle at $p$, so we obtain a contradiction. $\Box$
The loops \(g(\gamma_1)\) and \(g(\gamma_2)\) cut the torus into a quadrilateral. Preimage of this quadrilateral under \(g\) is a quadrilateral in the plane bounded by \(\gamma_1, \gamma_2\) and their shifts \(\gamma_1 + \omega_2\) and \(\gamma_2 + \omega_1\). From Lemma 5.1 we conclude that all four curves are pairwise disjoint except their endpoints.

Thus we obtain a Jordan quadrilateral that will be called \(Q\) (the boundary is included).

Since the curves \(g(\gamma_1)\) and \(g(\gamma_2)\) have intersection index \(\pm 1\), they generate the fundamental group of the torus, and it follows that \(Q\) does not contain other lattice points except \(0, \omega_1, \omega_2\) and \(\omega_1 + \omega_2\). The image \(F(\partial Q)\) consists of 4 straight segments which form a parallelogram in the plane in the non-degenerate case. In the degenerate case these 4 segments belong to the same line. Next we study

*Topology of the map* \(F : Q \to \overline{\mathbb{C}}\).

The following argument is purely topological, so we consider an arbitrary Jordan quadrilateral \(Q\) in the plane (a closed disk with four distinct marked boundary points \(a_1, \ldots, a_4\), which we call corners, enumerated according to the standard orientation). The boundary arcs \((a_i, a_{i+1})\), where \(i\) is a residue modulo 4, are called the sides.

Let \(F : Q \to \overline{\mathbb{C}}\) be a continuous function which is a local homeomorphism on the complement of the corners, and topologically holomorphic\(^4\) at the corners.

About the boundary behavior we make one of the two assumptions:

a) Generic case: \(F(\partial Q)\) is a Jordan curve \(\gamma\) and \(F : \partial Q \to \gamma\) is a homeomorphism, or

b) Degenerate case: the restrictions of \(F\) to the sides are homeomorphisms onto the image of each side, and these images are segments of the same straight line \(\ell\) in \(\mathbb{C}\). The images of opposite sides have equal length.

We want to obtain a topological description of possible partitions of \(Q\) by \(F^{-1}(\gamma)\) in case a) and by \(F^{-1}(\ell)\) in case b).

First we address the generic case a). Consider the cell decomposition \(C_1\) of the Riemann sphere which has two 2-cells: the interior \(I\) and the exterior \(E\) of \(\gamma\) (Fig. 5d). The 0-cells are \(F(a_j)\) and 1-cells are the four arcs into which \(F(a_j)\) divide \(\gamma\). We assign the labels to 0- and 1-cells by the following rules: \(F(a_j)\) has label \(j\); the arc \((F(a_j), F(a_{j+1}))\) has label \(j\).

\(^4\)Topologically equivalent to \(z \mapsto z^{n_i}, 1 \leq i \leq 4\).
Now consider the preimage $C_2 = F^{-1}(C_1)$ in $Q$. Our assumptions about $F$ imply that $C_2$ is a finite cell decomposition of $Q$. It is called the net of $F$. Closures of the cells of $C_2$ are mapped onto the closures of the cells of $C_1$ homeomorphically, and we label cells of $C_2$ by their images. Since $F$ is a local homeomorphism on $Q \setminus \{a_j\}$, the 1-skeleton of $C_2$ consists of simple curves which can meet only at the corners. We call the intersections of these curves with the interior of $Q$ arcs and define the length of an arc as the number of 1-cells that it contains. An example of the cell decomposition $C_2$ is shown in Fig. 5a, where the black dots are 0-cells.

The faces of $C_2$ are quadrilaterals, and we classify them as follows:

A face is called lateral if its boundary consists of one arc of length 1 and one arc of length 3, both arcs having as endpoints two adjacent corners of $Q$.

A face is called diagonal if its boundary consists of two arcs of length 2, both having as endpoints two opposite corners of $Q$.

A face is called triangular if its boundary consists of two arcs of length 1 and one arc of length 2, arcs of length 1 connecting pairs of adjacent corners, while the arc of length 2 connects opposite corners.

A face is called quadrilateral if its boundary consists of 4 arcs of length 1, each connecting a pair of adjacent corners of $Q$.

Fig. 5a contains 8 lateral, 1 diagonal and 2 triangular faces.

Let us show that this classification exhausts all possibilities for the faces of $C_2$. A face of $C_2$ cannot have all 4 boundary vertices in the interior of $Q$, since then there would be an adjacent face which is not simply connected. Neither a face of $C_2$ can have two vertices at the same corner, because the restriction of $f$ on the boundary of a face is a homeomorphism onto $\gamma$. A face of $C_2$ cannot have only one vertex at a corner, because if this were the case, an adjacent 2-cell will have all its 4 boundary edges the same as the original face, which is impossible.

**Lemma 5.2.** Under the assumption a) there are the following possibilities:

(i) The net contains one quadrilateral face and some (possibly none) lateral faces.

(ii) The net contains at least one diagonal face, two triangular faces and several (possibly none) lateral faces. All diagonal faces share the same opposite corners on their boundaries.

This lemma and its proof are illustrated in Fig. 5. In Fig. 5b case (i) is illustrated ($C_2$ is shown with bold lines). Fig. 5a is an example of case (ii).
**Proof.** Notice that lateral faces come in pairs, so the number of lateral faces sharing two given corners $a_i, a_{i+1}$ on their boundaries must be even. So the innermost arc in $Q$, connecting $(a_i, a_{i+1})$, has length 1. Removing all lateral faces, we obtain a smaller quadrilateral $Q'$, and a cell decomposition $C_3$ of it which has no lateral faces. The restriction of $f$ to $Q'$ satisfies the same conditions as $f$ on $Q$: the boundary $\partial Q$ is mapped onto $\gamma$ homeomorphically.

If $C_3$ consists of a single face, we are in case (i). If $C_3$ contains a diagonal face, suppose it has $a_1$ and $a_3$ on the boundary. Then all diagonal faces must have $a_1$ and $a_3$ on their boundaries. Removing all of them, we obtain two triangular faces, so we are in case (ii).

If $C_2$ contains no diagonal faces, then there are no triangular faces. Indeed, suppose that the cell decomposition of $Q'$ consists of just two triangles. The 1-cells on the boundary of each triangle have 4 distinct labels, and two of these 1-cells are in the common boundary of these two triangles. But the 1-cells on the boundary of $Q'$ also have 4 distinct labels, and this is evidently impossible. Thus if there are no diagonal faces in $Q'$, then there are also no triangular faces, and we are in case (i). □

**Transformation of the cell decomposition $C_2$ into another cell decomposition $C_4$ of $Q$.**

The edges of $C_4$ are defined as follows. First, they are arcs of length 1 of $C_2$. Then we discard all arcs of length at least 2, and add new edges by the following rules:

Suppose that $C_2$ has a diagonal face $G$. We recall that cells of $C_2$ are labeled by their images in $C_1$. If $G$ is a cell of $C_2$ is labeled $I$, we draw the **diagonal**: the $F$-preimage in $G$ of that diagonal of the parallelogram $I$ which has two corners of $Q$ as its extremities. If $G$ is labeled $E$, we use one of the two **exterior diagonals** of the parallelogram $I$. An exterior diagonal is the complement to a diagonal in the line which contains this diagonal. In Fig. 5b the added diagonal is red (dotted), and the discarded arcs are grey.

If $C_2$ has no diagonal faces, then it has one quadrilateral face. If this quadrilateral face is labeled $I$, we break this quadrilateral face by the preimage of the **shorter** diagonal of the parallelogram $I$. If the quadrilateral face of $C_2$ is labeled $E$ we break this quadrilateral face by the preimage of the exterior diagonal of $I$ which connects the two vertices of this parallelogram with the larger exterior corner (Fig. 5g). Partition of a quadrilateral face is shown if Fig. 5c.

By these rules, we obtain a cell decomposition $C_4$ of $Q$ which has no
interior vertices. This decomposition contains two congruent triangles and a number of digons. Each digon is mapped to the sphere with a cut along a segment. We break it into two digons by the $F$-preimage of the complement of this segment in the line that contains it. These lines are shown as red/dotted lines in Fig. 5b,c. After these cuts are made, the number of digons in every “bunch” becomes even. Adding half of them to the adjacent side of the triangle we obtain a decomposition of our torus into two triangles.

The final decomposition of $Q$ into two primitive triangles and digons isometric to half-planes is shown in red/dashed and bold black lines in Figs. 5b,c in two cases: 4b) when $C_2$ has a diagonal face, and 4c) when it does not.

Now we show that these two triangles are balanced. We refer to the decomposition of a singular triangle described in Proposition 3.2.

Gluing any numbers of half-planes to the sides of a balanced triangle results in a balanced triangle. Primitive triangles are balanced in the following cases. Primitive triangle of the type A is balanced if all angles are less than $\pi/2$. If the greater angle is $> \pi/2$ and at least one half-plane is glued to the opposite side, the resulting triangle is balanced. If the cell decomposition $C_2$ contains a diagonal face, this implies that at least one half-plane was glued opposite the largest angle of the triangular face. If the triangular face is of the type A, then its longest side is the diagonal, so the largest angle is opposite to it. If this triangle is of type B, then it is balanced (a triangle of this type is always balanced).

If there was no diagonal in $C_2$, then we obtained triangular faces of $C_4$ by drawing either the smaller diagonal in a parallelogram, or the exterior diagonal in its exterior which has endpoints at the bigger exterior angles. In both cases the triangle is balanced.

So we obtained a partition of $Q$ into two balanced triangles. We can re-assemble it by moving digons adjacent to a side of $Q$ to the opposite side to make the two balanced triangles congruent. This completes the proof in the non-degenerate case.

Now we consider the degenerate case. $F : Q \to \overline{C}$ maps the sides of $Q$ into a line $\ell$, and we assume without loss of generality that $\ell = \mathbb{R} \cup \{\infty\}$. The images of sides occupy some segment $(a, b) \in \mathbb{R}$, where $a < b$. It is evident that $a$ and $b$ are the images of two opposite corners of $Q$. Without loss of generality, these corners are $a_1$ and $a_3$. The preimage $F^{-1}(\mathbb{R})$ defines a cell decomposition $C_5$ of $Q$. It is exactly of the same type as nets studied in [16]: they consist of simple curves with endpoints at the corners and disjoint
interiors, and each curve is mapped homeomorphically onto its image.

**Lemma 5.3.** Under these assumptions $C_5$ contains a curve from $a_1$ to $a_3$. So the faces of $C_5$ are two triangles and several (possibly none) digons.

**Proof.** Any component of $F^{-1}(\{a, b\} \cup \{\infty\})$ must be a curve from $a_1$ to $a_3$ in the interior of $Q$. This proves the lemma.

Each digon of $C_5$ is mapped by $F$ to $\mathcal{C}$ with a cut (bounded or containing $\infty$). We partition digons into two halves by complements of these cuts to the $\mathbb{R} \cup \{\infty\}$. Then we split these half-planes in each “bunch” into two equal parts and add them to the corresponding sides of triangular faces. This defines a decomposition of our torus into two triangles. That these triangles are balanced is proved in the same way as in the non-degenerate case.

This completes the proof of surjectivity of $\Phi^*$. □

6 The spaces $A_m$, $T_m$ and $T_m^*$

6.1 Connected components

To visualize Proposition 3.1, we introduce the space of angles $A_m$. In the intersection of the plane

$$P = \{\alpha \in \mathbb{R}^2 : \alpha_1 + \alpha_2 + \alpha_3 = 2m + 1\}, \quad (37)$$

with the open first octant in $\mathbb{R}^3$ (Fig. 9) we consider the triangle $\Delta_m$ defined by the inequalities

$$0 < \alpha_j \leq \alpha_i + \alpha_k \quad \text{for all permutations} \quad (i, j, k);$$

it is shaded in Fig. 9. The vertices of $\Delta_m$ are

$$(m + 1/2, m + 1/2, 0), \quad (m + 1/2, 0, m + 1/2), \quad (0, m + 1/2, m + 1/2).$$

Notice that the vertices do not belong to $\Delta_m$ but the sides do belong, so $\Delta_m$ is neither open nor closed.

To obtain $A_m$ we remove from $\Delta_m$ all lines where some $\alpha_j$ is an integer, and add all points where all three $\alpha_j$ are integers. The intersections of lines $\alpha_j = k$ with $\Delta_m$ will be called segments. A segment is called even or odd
depending on the parity of \( k \). There are three families of parallel segments, each containing \( m \) segments. Spaces of angles for \( m = 1, \ldots, 5 \) are shown in Figs. 6, 7.

The set \( A_m \) has a natural partition into open topological disks (faces) open intervals (edges) and points (vertices): the faces are components of the interior of \( A_m \) (they are triangles or quadrilaterals), the vertices are the points where all \( \alpha_j \) are integers, and the edges are open intervals in \( A_m \cap \partial \Delta_m \).

The set of vertices of \( A_m \) will be denoted by \( V_m \).

We have a natural projection

\[
\varphi : T_m \to A_m
\]

which to every balanced triangle with angles \((\pi \alpha_1, \pi \alpha_2, \pi \alpha_3)\) puts into correspondence the point \((\alpha_1, \alpha_2, \alpha_3) \in A_m \subset \mathbb{R}^3\). It follows from Proposition 3.1 that this correspondence maps the part of \( T_m \) where \( \alpha_j \) are not integers bijectively onto \( A_m \setminus V_m \). Triangles with integer angles are mapped to \( V_m \) and the preimage of each point in \( V_m \) consists of three intervals.

This induces a partition of \( T_m \): the faces of \( T_m \) are \( \varphi \)-preimages of the faces of \( A_m \), the edges are of two types: interior edges which are mapped by \( \varphi \) to the vertices of \( A_m \) and boundary edges which are mapped bijectively onto intervals of \( A_m \cap \partial \Delta_m \). There are no vertices in this partition of \( T_m \).

The faces of \( T_m \) are adjacent when their images in \( A_m \) share a boundary vertex and their angles at this vertex are vertical\(^5\). Notice that the map \( \phi \) switches the orientation when one passes through any interior edge of \( T_m \) from a face to an adjacent face. This can be seen from the explicit formula for the angles in terms of the conformal coordinate \( z = \phi_{i,j,k} \) introduced in Section 4 in the chart where \( f(a_1) = 0, f(a_2) = 1, f(a_3) = z = x + iy \) we have

\[
\alpha_1 = p + \arctan(y/x), \quad \alpha_2 = q + \arctan(y/(1-x)),
\]

where \( p, q \) are integers. Assuming that \( x \in (0,1) \) we compute the Jacobian and see that it switches sign simultaneously with \( y \).

**Remark.** Gluing of two 2-cells along their common boundary 1-cell corresponding to a vertex of \( A_m \) reverses the natural orientation of these 2-cells induced from the \((\alpha_1, \alpha_2)\)-plane. Nevertheless, it is easy to check that the surface \( T_m \) is orientable. To do this one paints the 2-cells of \( T_m \) into two

\(^5\)Opposite angles among the four angles made by crossing of two lines.
colors, so that each two 2-cells with a common vertex have different colors. It is clear that such a coloring is possible, see Fig. 8.

To study the surface $T_m$ we introduce the graph $\Gamma_m$, which will be called the nerve. Examples of these graphs are shown in figures 6, 7. Their vertices correspond to 2-cells of $T_m$ (or faces of $A_m$) and two vertices of $\Gamma_m$ are connected by an edge if the corresponding two 2-cells of $T_m$ share an edge or, which is the same, if the corresponding 2-cells of $A_m$ share a vertex and their angles at this vertex are vertical. Then we find

**Proposition 6.1.** When $m \geq 2$, the graph $\Gamma_m$ has 4 connected components. Exactly one of them, $\Gamma'_m$ is invariant under the order 3 rotation about the center of $A_m$. Three others are permuted by this rotation. $\Gamma_1$ has only three components, permuted by the rotation. $\Gamma_0$ consists of one vertex only. □

In Fig. 6, $\Gamma'_m$ is blue/dotted, while in Fig. 7 one of the three components permuted by the order 3 rotation is red, any of these three components is called $\Gamma''_m$ (they are isomorphic graphs embedded in the plane).

We give first a geometric sketch which makes our proposition evident, and then a more formal proof.

**Sketch of a proof of Proposition 6.1.** Let us consider the plane $P$ which is defined in (37). Intersections of $P$ with the planes $\{\alpha_j = \text{integer}\}$ break $P$ into triangles. Connecting the centers of pairs of triangles which share a vertex and whose angles at this vertex are vertical, we obtain four honeycomb structures $X_j$ with disjoint vertices. See Fig. 8 which shows two of these honeycombs. Choosing one vertex of one honeycomb as a center, we see that this honeycomb is invariant under rotation by 120° about this vertex, while the other three are permuted.

**Proof of Proposition 6.1.**

Let $\Delta'_m$ be the intersection of the plane

$$\alpha_1 + \alpha_2 + \alpha_3 = 2m + 1$$

with the closed first octant $\alpha_j \geq 0$, $1 \leq j \leq 3$. The segments

$$\{(\alpha_1, \alpha_2, \alpha_3) \in \Delta'_m : \alpha_j = k\}, \quad 1 \leq j \leq 3, \quad 0 \leq k \leq 2m$$

divide $\Delta'_m$ into open triangles which we call faces of $\Delta'_m$. There are three families of these segments, depending on the value of $j$ which were discussed
in the beginning of this section. We recall that a segment with even/odd \( k \) is called \textit{even/odd}. Since \( \alpha_1 + \alpha_2 + \alpha_3 \) is odd, among three segments intersecting at an integer vertex, either one or all of them are odd. This implies that a face has either one or three sides on even segments.

We classify faces of \( \Delta'_m \) into four types:

- type \( I \), if all three sides of the triangular face belong to even segments,
- type \( II_j \), \( j \in \{1, 2, 3\} \) if one side belongs to an even segment of family \( j \), while two other sides belong to odd segments of the other two families.

Two faces sharing an integer vertex are called \textit{vertical} if their angles at this vertex are vertical (opposite).

We claim that vertical faces are of the same type. Indeed if the two segments bounding the vertical angles are both even, then each of our two faces must have all three sides on even segments, thus both faces are of type \( I \). If exactly one of the segments is even, and belongs to family \( j \), then both faces are of the type \( II_j \). If both segments are odd, then the sides of our faces opposite to the considered vertex are even and parallel, so they are in the same family and our two faces are in the same family. This proves the claim.

Now we claim that faces of the same type cannot have a common side. If two faces have a common side on an even segment then one of them is of type \( I \) and another is of type \( II \). If the common side is on an odd segment then both faces are of type \( II \) and their sides on even segments are not parallel. Thus they belong to different types. This proves the claim.

Our next claim is that the closure of the union of faces of the same type is connected. Consider faces of type \( I \). The closure of their union consists of the faces themselves and all even segments. It is clear that the union of all even segments is connected.

The proof for other types is similar: the closure of the union of faces of type \( II_j \) consists of the faces themselves, the even segments of family \( j \), and odd segments of two other families. The union of these segments is connected.

If we restrict now to \( A_m \) and consider the union of those faces of a single type which intersect \( A_m \) and their vertices, this union is still connected. Indeed if two faces of one family share a vertex and both intersect \( A_m \), then this common vertex belongs to \( A_m \).

This proves Proposition 6.1. \( \square \)
Now consider the map $\varphi : T_m \to A_m$. Component $L'_m$ of $T_m$ consists of preimages of faces of type $I$ of $A_m$ and common edges of pairs of these preimages that project to the vertices of faces of type $I$ of $A_m$. Similarly for $i \in \{1, 2, 3\}$, components $II_i$ of $T_m$, consist of preimages of faces of type $II_i$ and common edges of pairs of these preimages which project to the vertices of faces of type $II_i$ of $A_m$. So we obtain

**Corollary 6.1.** When $m \geq 2$, $T_m$ consists of four connected components $I, II_1, II_2, II_3$. Cyclic permutation of vertices preserves component $I$ and permutes components $II_i$. As a consequence, $T^*_m$ has two components, $I$ and $II$. These components are distinguished by the number of sides with poles:

- When $m$ is even, $L'_m$ consists of BFT with no poles on the sides, and $L''_m$ consists of BFT with two poles on the sides.

- When $m$ is odd, $L'_m$ consists of BFT with 3 poles on the sides, and $L''_m$ consists of BFT with one pole on the side.

**Example.** Figure 2 shows all types of BFT for $m = 2$ (angle sum $5\pi$). Triangles of types a), b), c) belong to $L'_m$. Suppose for example that triangle a) is deformed so that the top vertex moves towards the opposite (horizontal) side. Eventually we obtain a triangle b) with the angles $2\pi, 2\pi, \pi$. If the middle vertex of b) continues moving downwards, we obtain triangle c). Its developing map is 2-to-1 onto the darkly shaded region and 1-to-1 onto the lightly shaded region. There are three types of such triangles c) if the vertices are labeled, but only one type with unlabeled vertices.

Triangles of types d), e), f) belong to $L''_m$. It is easy to visualize how they are deformed to each other.

Triangles a), b), c) have one pole inside, while triangles d), e), f) have 2 poles, one on each of the two unbounded sides, and the third side is free of poles. Fig. 2 should be compared with Figs. 6, 7, $m = 2$: the set $A_2$ shows the location of all these triangles in the parameter space.

### 6.2 Proof of injectivity of $\Phi^*$

We established in Section 4 that $\Phi^*_m : T^*_m \to L_m$ is a proper holomorphic map between punctured Riemann surfaces. So to prove injectivity it is sufficient to show that every component of $T^*_m$ contains an interval $I$ such that for every $T \in I$, $T$ is the unique $\Phi^*$-preimage of $\Phi^*(T)$.

Assume that $m \geq 1$. Then every component contains a triangle with integer angles. Let $T$ be a triangle with integer angles. According to Propo-
sition 3.3 the monodromy group of $\Phi^*(T)$ is a subgroup of a line, so for the torus $\Phi^*(T)$ alternative b) (degenerate case) holds in the proof of surjectivity. It follows that any $\Phi^*$-preimage of $\Phi^*(T)$ also has all integer angles (see Proposition 3.3).

Recall that a triangle with integer angles is obtained from a triangle in Fig. 1b in which $a_1, a_2, a_3$ belong to the same line, by gluing half-planes to the sides (see Proposition 3.2). Let us normalize so that $a_3 = 0$, $a_1 = 1$, and we choose our interval $I$ so that $a_2 := a \in (0, 1/2)$. Then the following properties of $\Phi^*(T)$ are evident:

The shortest non-trivial loop $\gamma_1$ based at the singularity has length $a_2$. We define the orientation of this loop by orienting $(0, a)$ from $a$ to 0 on the boundary of the reduced triangle $T'$. We recall that the reduced triangle $T'$ is the upper half-plane with corners at 0, $a$, $1$.

So the parameter $a$ is uniquely defined by $\Phi^*(T)$, as the shortest length of a loop based at $O$ on $\Phi^*(T)$. Now we define $\gamma_2$ as the loop corresponding to $(a, 1)$ in $\partial T'$, oriented from $a$ to 1. This loop $\gamma_2$ is characterized as the shortest loop whose class does not belong to $Z\gamma_1$.

Each side of $T'$ defines a homotopy class of loops based at $O$. Two of them are $\gamma_1$ and $\gamma_2$. Suppose that $m_1$ half-planes were glued to the side $(0, a)$, and $m_2$ half-planes were glued to the side $(a, 1)$. Then the torus $\Phi^*(T)$ contains $m_1 + 1$ disjoint (except the base point) geodesic loops in the class $[\gamma_1]$, and $m_2 + 1$ disjoint geodesic loops in the class $[\gamma_2]$.

This implies that the angles $\pi \alpha_i$ of $T$ are defined by the properties of the torus $\Phi^*(T)$, namely $\alpha_i = 1 + m_j + m_k$. This proves injectivity of the map $\Phi^*$ and completes the proof of Theorem 1.5.

7 Euler characteristics of components of $L_m$

and completion of the proof of Theorem 1.1

Theorem 1.5 reduces the study of topology of $L_m$ to the study of topology of $T_m$.

It is convenient to use the nerves $\Gamma'_m$ and $\Gamma''_m$ introduced in Section 6.

First we recall that $\Gamma_m/Z_3$ consists of 2 components. One of them comes from the component $\Gamma'_m$ which is invariant with respect to the $Z_3$ action. This is our component $L'_m$. Component $L''_m$ comes from the three components of $\Gamma''_m$ which are permuted by the $Z_3$ action. See Figs. 6, 7.
Computation of the Euler characteristic for component $L^I_m$.

Let $\Gamma'_m$ be the component of $\Gamma_m$ which is invariant with respect to the $\mathbb{Z}_3$ action. The numbers $\epsilon_0, \epsilon_1$ are defined in (10) and (11). We add to them

$$
\epsilon_2 = \begin{cases} 
0, & \text{if } m \equiv 1 \pmod{2} \\
1, & \text{if } m \equiv 0 \pmod{2} 
\end{cases}
$$

and interpret these numbers in terms of $A_m$:

$\epsilon_0 = 1$ if the center of $A_m$ belongs to a 2-cell. Equivalent condition is that a vertex of $\Gamma'_m$ is fixed by the $\mathbb{Z}_3$ action.

$\epsilon_1 = 1$ if there is a vertex of $\Gamma'_m$ representing a face which has the middle of the side of $\Delta_m$ on the boundary.

$\epsilon_2 = 1$ if there is a vertex of $\Gamma'_m$ representing a face which has a corner of $\Delta_m$ on the boundary.

We introduce further notation:

$V_1, V_2, V_3$ are the numbers of vertices of $\Gamma'_m$ of degrees 1, 2, 3, and $V$ is the total number of vertices.

$E$ is the number of edges of $\Gamma'_m$.

Taking into account all identifications on $\Gamma'_m$, we obtain the following formula for the Euler characteristic:

$$
\chi(L^I_m) = V_3/3 + V_1/6 + V_2/6 - E/3 + 2\epsilon_0/3 + (\epsilon_1/2 - \epsilon_2)/2.
$$

(39)

To explain this formula, we compute the contributions to the (ordinary) Euler characteristic of $L^I_m$.

The group $\mathbb{Z}_3$ fixes the center of $A_m$ which belongs to $\Gamma'_m$ if and only if $\epsilon_0 = 1$, and acts freely on the set of remaining 3-valent vertices, so the number of 2-cells in $L^I_m$ corresponding to 3-valent vertices is $V_3/3 + 2\epsilon_0/3$.

Also, the group $\mathbb{Z}_3$ acts freely on the set of edges, and thus there are $E/3$ corresponding 1-cells in $L^I_m$.

For a 1-valent or 2-valent vertex $v$ of $\Gamma'_m$, its class under the equivalence relation consists of

a) 6 elements, if $v$ is neither a corner of $\Delta_m$ nor a midpoint of a side of $\Delta_m$;

b) 3 elements, if $v$ is a corner of $\Delta_m$;
c) 3 elements, if \( v \) is the midpoint of a side of \( \Delta_m \).

If we have \( V^a \) vertices of type a), such vertices correspond to \( V^a/6 \) 2-cells in \( L_m^I \). The number \( V^b \) of vertices of type b) can be 0 or 3: the latter case occurs if and only if \( \epsilon_2 = 1 \), in which case such vertices correspond to a punctured disk in \( L_m^I \). In both cases, vertices of type b) contribute \( V_b/6 - \epsilon_2/2 = 0 \) to the Euler characteristic. Similarly, the number \( V^c \) of vertices of type c) can be 0 or 3: the latter case occurs if and only if \( \epsilon_1 = 1 \), in which case such vertices correspond to a disk in \( L_m^I \). In both cases, vertices of type c) contribute \( V_c/6 + \epsilon_1/2 = 1 \) to the Euler characteristic. Adding these contributions, we obtain (39).

It is easy to see that for odd \( m \)

\[
E = 3(m^2 - 1)/8, \quad V_1 = 3(m - 1)/2, \quad V_2 = 0, \quad V = (m^2 + 4m - 5)/4, \quad V_3 \]

and \( V_3 \) can be computed by the formula \( V_3 = V - V_1 - V_2 \). This gives the formula for \( \chi(L_m^I) \) when \( m \) is odd. When \( m \) is even, we have

\[
E = 3(m^2 + 2m)/8, \quad V_1 = 3, \quad V_2 = 3(m - 1)/2, \quad V = (m/2 + 1)^2, \quad V_3 \]

and again \( V_3 = V - V_1 - V_2 \). This gives the formula for \( \chi(L_m^I) \) when \( m \) is even. The resulting formulas for \( \chi \) in terms of \( m \) are written in the Appendix. Expressing \( m \) in terms of \( d_m^I \) in (8) and subtracting the orbifold correction we obtain \( \chi^O(L_m^I) = -(d_m^I)^2/6 \).

**Computation of the Euler characteristic of component** \( L_m^{II} \)

Let \( \Gamma_m'' \) be one of the three components of \( \Gamma_m \) which are permuted by the \( Z_3 \) action. We use the following notation

\( E \) is the number of edges of \( \Gamma_m'' \)

\( V_1, V_2, V_3 \) and \( V \) are the numbers of vertices of \( \Gamma_m'' \) of orders 1, 2, 3 and the total number of vertices.

\( \epsilon_1 \) and \( \epsilon_2 \) have the same meaning as before. A computation analogous to that for \( L_m^I \) gives

\[
\chi(L_m^{II}) = V_3 + V_1/2 + V_2/2 - E + (\epsilon_2 - \epsilon_1)/2, \]

as vertices of type b) (respectively, of type c)) belong to \( L_m^{II} \) if and only if \( \epsilon_2 = 0 \) (respectively \( \epsilon_1 = 0 \)). When \( m \) is odd,

\[
E = (3m^2+4m+1)/8, \quad V_1 = (m+3)/2, \quad V_2 = m-1, \quad V = (m^2+4m+3)/4. \]

40
When $m$ is even,

$$E = (3m^2 + 2m)/8, \quad V_1 = m, \quad V_2 = m/2, \quad V = (m/2 + 1)^2 - 1.$$ 

This gives the formulas for $\chi(L_{m}^{II})$ in terms of $m$ and $\epsilon_1$ (written in the Appendix). Expressing $m$ in terms of $d_{m}^{II}$ from (9) and subtracting the orbifold correction we obtain $\chi^{O}(L_{m}^{II}) = -(d_{m}^{II})^2/18$.

### Computation of the number of punctures.

Consider a small simple loop around a puncture of $T_{m}^{*}$. This loop projects to a contour in $A_m$ which goes close to the lines $\alpha_j = k$, switching the side at each integer point. For component $L_{m}^{I}$, the contour goes near lines with the same even $k$, and $j = 1, 2, 3$ and closes. See Fig. 10. So there is a $1-1$ correspondence between these contours and triples of segments (one in each family) with even $k$. So there are $\lfloor m/2 \rfloor$ of such loops. In addition, when $m$ is even there is a puncture corresponding to the vertices of $\Delta_m$. Thus the total number of punctures on $L_{m}^{I}$ is $m/2 + 1$ when $m$ is even and $(m - 1)/2$ when $m$ is odd. In other words, the number of punctures on Component $L_{m}^{I}$ equals

$$h_{m}^{I} := d_{m}^{I}, \quad (40)$$

where $d_{m}^{I}$ was defined in (8). For component $L_{m}^{II}$, the computation is similar, see Fig. 11. Each contour goes either near an even segment, in which case it closes after describing three segments, one of each family. If a contour accompanies an odd segment, it ends on the other side of the odd segment after describing three segments. So the total number of contours is $m$ when $m$ is even and $m + 1$ when $m$ is odd (the extra puncture for odd $m$ coming from the corners of $\Delta_m$), in other words

$$h_{m}^{II} := 2 \lfloor m/2 \rfloor = 2d_{m}^{II}/3. \quad (41)$$

Component $L_{m}^{I}$
We include two tables for $1 \leq m \leq 13$. Notation, besides that already introduced is: $g$ for the genus, $h$ for the number of punctures, $d$ for the degree of the forgetful map as in (8), (9). The formulas for the degrees follow from [35, sections 23.21-23.24].

**Remarks.** There is an alternative method of counting the punctures, based on the description on compactifications of the spaces of Abelian differentials in [3, 4]. In recent preprints [13, 14] a general method of computation

| $m$ | $\epsilon_0$ | $\epsilon_1$ | $\epsilon_2$ | $V_1$ | $V_2$ | $V_3$ | $E$ | $V$ | $\chi$ | $h$ | $g$ | $d$ |
|-----|---------------|---------------|---------------|-------|-------|-------|-----|-----|-------|-----|-----|-----|
| 2   | 1             | 0             | 1             | 3     | 0     | 1     | 3   | 4   | 0     | 2   | 0   | 2   |
| 3   | 1             | 1             | 0             | 3     | 0     | 1     | 3   | 4   | 1     | 1   | 0   | 1   |
| 4   | 0             | 1             | 1             | 3     | 3     | 3     | 9   | 9   | -1    | 3   | 0   | 3   |
| 5   | 1             | 0             | 0             | 6     | 0     | 4     | 9   | 10  | 0     | 2   | 0   | 2   |
| 6   | 1             | 0             | 1             | 3     | 6     | 7     | 18  | 16  | 18    | -2  | 4   | 0   | 4   |
| 7   | 0             | 1             | 0             | 9     | 0     | 9     | 18  | 18  | 18    | -1  | 3   | 0   | 3   |
| 8   | 1             | 1             | 1             | 3     | 9     | 13    | 30  | 25  | -3    | 5   | 0   | 5   |
| 9   | 1             | 0             | 0             | 12    | 0     | 16    | 30  | 28  | -2    | 4   | 0   | 4   |
| 10  | 0             | 0             | 1             | 3     | 12    | 21    | 45  | 36  | -6    | 6   | 1   | 6   |
| 11  | 1             | 1             | 0             | 15    | 0     | 25    | 45  | 40  | -3    | 5   | 0   | 5   |
| 12  | 1             | 1             | 1             | 3     | 15    | 63    | 49  | -7  | 7     | 1   | 7   |
| 13  | 0             | 0             | 0             | 18    | 0     | 36    | 63  | 54  | -6    | 6   | 1   | 6   |

| Component $L^{II}_m$ |
|----------------------|
| $m$ | $\epsilon_1$ | $\epsilon_2$ | $V_1$ | $V_2$ | $V_3$ | $E$ | $V$ | $\chi$ | $h$ | $g$ | $d/3$ |
|-----|---------------|---------------|-------|-------|-------|-----|-----|-------|-----|-----|-------|
| 1   | 0             | 0             | 2     | 0     | 0     | 1   | 2   | 0     | 2   | 0   | 1     |
| 2   | 0             | 1             | 2     | 1     | 0     | 2   | 3   | 0     | 2   | 0   | 1     |
| 3   | 1             | 0             | 3     | 2     | 1     | 5   | 6   | -2    | 4   | 0   | 3     |
| 4   | 1             | 1             | 4     | 2     | 2     | 7   | 8   | -2    | 4   | 0   | 2     |
| 5   | 0             | 0             | 4     | 4     | 4     | 12  | 12  | -4    | 6   | 0   | 3     |
| 6   | 0             | 1             | 6     | 3     | 2     | 15  | 15  | -4    | 6   | 0   | 3     |
| 7   | 1             | 0             | 5     | 6     | 9     | 22  | 20  | -4    | 8   | 1   | 4     |
| 8   | 1             | 1             | 8     | 4     | 12    | 26  | 24  | -4    | 8   | 1   | 4     |
| 9   | 0             | 0             | 6     | 4     | 16    | 35  | 30  | -12   | 10  | 2   | 5     |
| 10  | 0             | 1             | 10    | 5     | 20    | 40  | 35  | -12   | 10  | 2   | 5     |
| 11  | 1             | 0             | 7     | 10    | 25    | 51  | 42  | -18   | 12  | 4   | 6     |
| 12  | 1             | 1             | 12    | 6     | 30    | 57  | 48  | -18   | 12  | 4   | 6     |
of Euler’s characteristics for spaces of Abelian differentials with prescribed multiplicities of zeros and poles is developed. However, our results do not follow from the results stated in these preprints, mainly because of the additional condition that residues vanish.

*Orbifold points.*

By definition, an orbifold point in $L_m$ is a point which corresponds to a flat singular torus with a non-trivial automorphism. An automorphism here means an orientation-preserving isometry. The trivial automorphism is the involution which exists on every flat singular torus. There are two types of tori with non-trivial automorphisms: hexagonal ones with an automorphism of order 3, and square ones, with non-trivial automorphism of order 4. In the representation of tori as $\Phi^*(T)$, hexagonal tori correspond to triangles whose all angles are equal, while square tori correspond to marginal triangles whose two smaller angles are equal. So in the space of angles $A_m$, the hexagonal torus arises from the center of $\Delta_m$ when this center belongs to $A_m$, and the square torus corresponds to the middles of the sides of $\Delta_m$. In Figs. 6, 7, these points are denoted by little circles in the center of the picture, and little black triangles in the middles of the sides.

In the next section we will use the following

**Proposition 7.1.** In the Lamé equation (2) or (3) corresponding to a hexagonal or square torus (in the metric sense), the accessory parameter $\lambda$ is equal to 0 (see the text after (4)).

*Proof.* Since a metric automorphism is also a conformal automorphism, it corresponds to an automorphism of the Lamé equation, that is to a fixed point of transformation (4). For both fixed points we have $\lambda = 0$. □

8 **Theorem 1.2 and Maier’s conjecture**

To prove Theorem 1.2 and its corollaries we first state the exact relation between $L_m$ and $H_m$.

A *marked* elliptic curve is an elliptic curve on which the three points of (exact) order 2 are labeled. Legendre’s family (14) parametrizes marked elliptic curves: the labels are 0, 1, $a$. The permutation group $S_3$ acts on the space of marked elliptic curves by permuting the labels. Explicitly, the orbit
of \( a \) under this action is
\[
a, \quad 1 - a, \quad 1/a, \quad 1 - 1/a, \quad 1/(1 - a), \quad a/(a - 1). \tag{42}
\]
This action lifts to the moduli space \( \mathbb{C} \times \mathbb{C}_a \) of Lamé equations in the form of Legendre: the generators \( a \mapsto 1 - a \) and \( a \mapsto 1/a \) lift to
\[
(B, a) \mapsto (-B - m(m + 1), 1 - a), \quad \text{and}
\]
\[
(B, a) \mapsto (B/a, 1/a).
\]
To obtain these two transformations, one changes the independent variable in (16)
\[
z \mapsto 1 - z \quad \text{and} \quad z \mapsto z/a,
\]
respectively. Taking the quotient of the space \( \mathbb{C} \times \mathbb{C}_a \) of equations (16) by this \( S_3 \) action we obtain an orbifold covering \( \Psi_m \) of degree 6 from the moduli space of equations (16) to the moduli space \( \text{Lame}_m \), such that the following diagram is commutative:
\[
\begin{array}{ccc}
H^j_m & \xrightarrow{\Psi^j_m} & L^K_m \\
\downarrow \sigma_m & & \downarrow \pi_m \\
\mathbb{C}_a & \xrightarrow{\psi} & \mathbb{C}_J
\end{array}
\tag{43}
\]
Here \( \Psi^j_m \) are restrictions of \( \Psi_m \) on \( H^j_m \), and \( K = I \) for \( j = 0 \), \( K = II \) for \( j \in \{1, 2, 3\} \). (We have not proved yet that \( H^j_m \) are irreducible; this will be done only in the end of this section).

The explicit expression of \( \psi \) is in (15). To obtain an explicit expression of \( \Psi_m \) we change the independent variable \( z \) in the equation (16) to \( z - (1 + a)/3 \). Then we easily obtain \( \Psi_m = (R_1, R_2, R_3) \) modulo scaling (4), where
\[
\begin{align*}
\lambda &= R_1(B, a) := B + m(m + 1)(a + 1)/3, \tag{44} \\
g_2 &= R_2(B, a) := 4(a^2 - a + 1)/3, \\
g_3 &= R_3(B, a) := 8(a^3 - 3a^2/2 - 3a/2 + 1)/27.
\end{align*}
\]
We define compact Riemann surfaces \( \overline{L}_m^K, \overline{H}^j_m, \overline{C}_J, \) and \( \overline{C}_a \) by filling the punctures. Later we will endow them with orbifold structures. The forgetful maps \( \pi_m, \sigma_m \) and maps \( \psi, \Psi \) extend uniquely to these compactifications.

**Definition 8.1.** A point \( x \in \overline{L}_m \) is called special if \( \pi_m(x) \in \{0, 1, \infty\} \). A point \( x \in \overline{H}_m \) is called special if
\[
\sigma_m(x) \in \{0, 1, \infty, 2, 1/2, -1, (1 \pm i\sqrt{3})/2\}.
\]
We will later show (in the proof of Corollary 1.1 in this section) that
Ψ_m^j : H_m^j \to L_m^j are orbifold coverings, so the maps Ψ_m^j : H_m^j \to L_m^j, as
maps between Riemann surfaces, can be ramified only at special points.

Next we study ramification properties of forgetful maps at the special
points. For this we need two lemmas, the first one is classical, see for example
[18, Ch. II, §1, Thm 1]:

Lemma 8.1. Let \( A = (a_{i,j}) \) be an \( n \times n \) matrix with
\( a_{i,i+1} > 0, \ 1 \leq i \leq n-1, \) and
\( a_{i,i-1} > 0, \ 2 \leq i \leq n, \) the rest of the entries are zeros. Then all
roots of the characteristic polynomial are real and simple. The characteristic
polynomial is either even or odd, in other words it has the form \( \lambda^k P(\lambda^2), \)
where \( k \in \{0, 1\} \), and \( P \) is a real polynomial.

The second lemma was communicated to us by V. Tarasov; it is inspired
by [31, Prop. 3]:

Lemma 8.2. Let \( A = (a_{i,j}) \) be an \( n \times n \) matrix with
\( a_{i,i+1} > 0, \ 1 \leq i \leq n-1, \) and \( a_{i,i-2} > 0, \ 3 \leq i \leq n, \) the rest of the entries are zeros. Then all
roots of the characteristic polynomial, except possibly 0, are simple and their
arguments are of the form \( 2\pi k/3, \ k \in \{0, 1, 2\} \). In fact this characteristic
polynomial has the form \( \lambda^k P(\lambda^3), \) where \( k \in \{0, 1, 2\} \) and \( P \) is a real polynomial.

A proof of Lemma 8.2 will be given in the next section.

The following proposition lists ramification of forgetful maps over special
points. We use the word “ramification” in the sense of maps between
Riemann surfaces, not orbifolds.

Proposition 8.1. 1. Ramification of \( \pi_m^K \) over special points is the following:
Over \( J = 0 \) there are \( \lfloor d_m^K/3 \rfloor \) triple points, and one additional point \( x \) when
\( d \) is not divisible by 3. This additional point \( x \) is the orbifold point of order
3, and \( \pi_m^K \) has \( x \) as a double point when \( d \equiv 2 \mod 3 \), and a simple point
when \( d \equiv 1 \mod 3 \).
Over \( J = 1 \) there are \( \lfloor d_m^K/2 \rfloor \) double points, and one simple point when \( d_m^K \)
is odd. This simple point is the orbifold point of order 2.
Over \( J = \infty \) there are \( d_m^{II}/3 \) double points when \( K = II \). The rest \( d_m^{II}/3 \)
points are simple. For \( K = I \) all points over \( \infty \) are simple.

2. Ramification of \( \sigma_m^j \) over special points is the following: Over each \( a = 1/2 \pm i\sqrt{3}/2, \) there is one double point when \( d_m^j \equiv 2 \mod 3 \). There is no other ramification over special points.
Proof. For component $L^I_m$ with even $m$ and $J = 0$, we consider polynomial solutions $Q$ of equation (2) with $g_2 = 0$, $g_3 = 1$, that is
\[(4x^3 - 1)Q'' + 6x^2Q' - m(m + 1)xQ = \lambda Q.\]
The matrix of the linear operator in the left-hand side in the basis of monomials has the form as in Lemma [8.2]. Therefore the characteristic polynomial of this matrix has the form $\lambda^k P(\lambda^3)$. This has a root of multiplicity 2 at 0 when $d \equiv 2 \pmod{2}$. Other roots come in triples, each triple lies on the same orbit under the $C^*$ action [4], so we have $\lfloor dK/3 \rfloor$ triple points.

Similar considerations apply to other special points.

For component $L^I_m$ with odd $m$ and $J = 0$, we consider solutions of (2) of the form $\sqrt{4x^3 - g_2x - g_3}Q(x)$, where $Q$ is a polynomial. The equation for $Q$ becomes
\[(4x^3 - 1)Q'' + 18x^2Q' + (12 - m(m + 1))xQ = \lambda Q,\]
and this leads to a matrix of the same form described in Lemma [8.2] so the same argument as in the case of even $m$ applies.

For component $L^I_m$ with even $m$ and $J = 1$, we set $g_2 = 1, g_3 = 0$, and obtain
\[(4x^3 - x)Q'' + (6x^2 - 1/2)Q' - m(m + 1)xQ = \lambda Q\]
which leads to a matrix described in Lemma [8.1]. The characteristic polynomial is of the form $\lambda^k P(\lambda^2)$, $k \in \{0, 1\}$ which has one simple root $\lambda = 0$ when $k = 1$ and other roots come in pairs which are on the same orbit under the $C^*$ action.

For component $L^I_m$ with odd $m$ and $J = 1$ we obtain the equation
\[(4x^3 - x)Q'' + (18x^2 - 3/2)Q' + (12 - m(m + 1))xQ = \lambda Q\]
which again leads to a matrix described in Lemma [8.1]. The conclusion is similar.

For component $L^{II}_m$ we use the Legendre’s form of Lamé equation (16). When $m$ is odd, and $J = 1$, we plug the solution of the form $\sqrt{z}Q(z)$ and obtain
\[
4z(z - 1)(z - a)Q'' + (10z^2 + 8z(1 + a) + 6a)Q' - ((m^2 - m - 2)z + 1 + a + B)Q = 0.
\]
When $m$ is even, and $j = 1$, we plug the solution of the form $\sqrt{z(z - 1)} Q(z)$ and obtain

$$4z(z - 1)(z - a)Q'' + (14z^2 - (12a + 8)z + 6a)Q' - \left((m^2 + m - 6)z + B + 4a + 1\right) Q = 0.$$ 

Both these equations lead to Jacobi matrices as in Lemma 8.1.

To study ramification at the punctures, we use again Legendre’s form. Take, for example, $a = 0$. The matrix of the operator in the left-hand side of (3) is triangular, with distinct eigenvalues. So $\sigma_m$ is unramified at a point $x$ with $\sigma_m(x) = 0$. Now we have $\deg_0 \psi = 2$, so by (43)

$$\deg_{\psi_m(x)}(\pi_m) \cdot \deg_x \Psi_m = 2,$$

thus each multiple is either 1 or 2. But we know the total number of points in $L^K_m$ over $J = \infty$ (punctures) and this implies the statement of Proposition 8.1 for $J = \infty$.

That $\pi_m$ is an orbifold map follows from the identification of the orbifold points in $L^K_m$ in Proposition 7.1. □

The difference between $\pi_m$ and $\sigma_m$ is that there is no $\mathbb{C}^*$ action in the second case.

Proposition 8.1 together with relation (43) and known ramification of $\psi$ allows us to define the orbifold structure on the compactified spaces, so that the $\psi$ and $\Psi_m$ extend to orbifold coverings of these compactifications.

For what follows we define compactifications of our orbifolds:

$$\overline{C}_J = C(0(3), 1(2), \infty(2)), \quad \overline{C}_a = C.$$ 

Then $a \mapsto J = \psi(a)$ which is defined in (15) is an orbifold covering. Then we define $\overline{L}^K_m$ by adding the punctures $x$, $\pi_m^K(x) = \infty$, and defining $n(x) = 1$ if $\deg_x(\pi_m^K) = 2$ and $n(x) = 2$ when $\deg_x(\pi_m^K) = 1$. Finally we define $\overline{C}_a$ as the Riemann sphere with $n(a) = 1$ for all $a$, and define $\overline{H}^j_m$ as $\overline{H}^j_m$ with filled punctures. The orbifold structure on $\overline{H}^j_m$ is trivial: $n(x) \equiv 1$. With these definitions Theorem 1.1 gives:

$$\chi^O(\overline{L}^K_m) = \chi^O(\overline{L}^K_m) + d^K_m/2.$$ 

(45)
Proposition 8.2. The following diagram is commutative:

\[
\begin{array}{ccc}
\mathcal{H}^j_m & \xrightarrow{\Psi^j_m} & \mathcal{L}^K_m \\
\downarrow \sigma_m & & \downarrow \pi_m \\
\overline{C}_a & \xrightarrow{\psi} & \overline{C}_j
\end{array}
\]

(46)

Here all four spaces are orbifolds, with orbifold functions just defined, the horizontal arrows are orbifold coverings, and vertical arrows are maps of orbifolds. We have
\[
\deg \Psi^0_m = 6 \quad \text{and} \quad \deg \Psi^j_m = 2, \quad j \in \{1, 2, 3\}.
\]

Furthermore,
\[
\begin{align*}
\chi(\mathcal{H}^0_m) &= \chi^O(\mathcal{H}^0_m) = 6 \chi^O(\mathcal{L}^I_m), \\
\chi(\mathcal{H}^j_m) &= \chi^O(\mathcal{H}^j_m) = 2 \chi^O(\mathcal{L}^{II}_m).
\end{align*}
\]

(47) \hspace{1cm} (48)

Proof. Since in the diagram (43) the horizontal arrows are orbifold coverings and vertical arrows are orbifold maps, it remains to check the points over \( J = \infty \) and over \( a \in \{0, 1, \infty\} \). That \( \psi : \overline{C}_a \to \overline{C}_j \) is an orbifold covering is well known and follows from the explicit formula (15).

Let \( x \in \mathcal{H}^j_m, \sigma_m(x) \in \{0, 1, \infty\} \). By Proposition 8.1, \( \deg_x(\sigma_m) = 1 \) and we know that \( \deg_{\sigma(x)}(\psi) = 2 \). Therefore,
\[
\deg_x(\Psi^j_m) \cdot \deg_{\psi^j_m(x)}(\pi_m) = 2,
\]

thus \( \deg_x \Psi^j_m \) is either 1 or 2, and the definition of \( n(\Psi^j_m(x)) \) ensures that \( \Psi^j_m \) is an orbifold covering.

That the vertical arrows are orbifold maps follows from Proposition 8.1. Formulas (47), (48) follow from (7).

Now we are ready to prove Theorem 1.2.

Proposition 8.3. The Riemann surfaces \( \overline{\mathcal{H}}^j_m \) are connected, and their images in \( \mathbb{C}P^2 \) are non-singular.

Proof. Using (7), (45) and (47), we obtain
\[
2 - \chi(\overline{\mathcal{H}}^0_m) = 2 - 6 \chi^O(\mathcal{L}^I_m) = 2 - 6 \chi^O(\mathcal{L}^I_m) - 3d^I_m
\]
\[
= 2 + (d^I_m)^2 - 3d^I_m = (d^I_m - 1)(d^I_m - 2).
\]

48
Similarly, using (7), (45) and (48), we obtain

\[ 2 - \chi(H^j_m) = 2 - 2\chi^O(L^j_m) = 2 - 2\chi^O(L^j_m) - d^{II}_m \]
\[ = 2 + (d^{II}_m)^2/9 - d^{II}_m = (d^{II}_m/3 - 1)(d^{II}_m/3 - 2), \quad j \in \{1, 2, 3\}. \]

Therefore, in any case we have

\[ 2 - \chi(H^j_m) = (\deg H^j_m - 1)(\deg H^j_m - 2). \quad (49) \]

Suppose that for some \( j \) and \( m \), \( H^j_m \) has \( N \) irreducible components of degrees \( d_k \) genera \( g_k \) and degrees \( d_k \) for \( 1 \leq k \leq N \). Then

\[ \deg H^j_m = \sum_{k=1}^{N} d_k, \quad \chi(H^j_m) = \sum_{k=1}^{N} \chi_k, \quad (50) \]
\[ \chi_k = 2 - 2g_k, \quad \text{and} \quad 2g_k \leq (d_k - 1)(d_k - 2); \quad (51) \]

the last inequality follows from (18). Substituting the expressions \( \deg H^j_m \) and \( \chi(H^j_m) \) from (50) to (49) and using (51) we obtain after simple manipulation

\[ \left( \sum_{k=1}^{N} d_k \right)^2 \leq \sum_{k=1}^{N} d_k^2; \]

since all \( d_k \geq 1 \), this is possible only when \( N = 1 \). Thus \( H^j_m \) is irreducible. Then from (49) we obtain its genus,

\[ g(H^j_m) = (2 - \chi(H^j_m))/2 = (\deg H^j_m - 1)(\deg H^j_m - 2)/2, \]

so it is non-singular since it satisfies (18) with equality. \( \square \)

This proposition completes the proof of Theorem 1.2.

**Proof of Corollary 1.1**

Consider the map \( R : H_m \to \{ F_m(\lambda, g_2, g_3) = 0 \} \) defined in (44). We will show that it is transversal to the orbits of the \( C^* \) action (4) at non-special points.

A trajectory of restriction of this action onto the \( (g_2, g_3) \) plane has the form \( (g_2, g_3) = (t^2, ct^3), t \in C^* \), so the tangent vector is \( (2t, 3ct^2) \) which
is parallel to \((2/g_3, 3/g_2) = (2/R_3, 3/R_2)\). If the vectors \((2/R_3, 3/R_2)\) and \((R'_2, R'_3)\) are collinear, we must have

\[ S := R_2R_3(3R'_2/R_2 - 2R'_3/R_3) = 0. \]

But an explicit computation shows that

\[ S = -16a(a - 1)/3, \]

which can be zero only at the special points.

Therefore the maps \(\Psi^j_m\) are ramified only at the special points. Diagram (46) is clear from the definition. □

Proof of Corollary 1.2.

We compute the ramification of the forgetful map \(\pi\) and then make the correction for special points.

The usual (not orbifold) Euler characteristic of the compactification of \(L^I_m\) is \(\chi(L^I_m) = \chi(L^I_m) + h\), where \(h\) equals the number of punctures. So from Theorem 1.1 for \(L^I_m\) and Riemann–Hurwitz formula the total ramification of \(\pi\) is

\[
2d - \chi(L^I_m) = 2d - \chi(L^I_m) - h = d - \chi(L^I_m)
\]

\[
= d + d^2/6 - (4e_0 + 3e_1)/6 = [(d^2 - d + 4)/6] + 2[d/3] + [d/2],
\]

where \(d = \deg_\lambda F^I_m\). The first summand is the degree of the Cohn polynomial, and the other two reflect the additional ramification over the special points 0 and 1 (Proposition 8.1). Indeed, since the only singularities of the surface \(F^I_m(\lambda, g_2, g_3) = 0\) lie over \(g_2 = 0\) and \(g_3 = 0\), the zeros of the Cohn polynomial at all points \(J \in C \setminus \{0, 1\}\) come from ramification points of \(\pi\). For \(J = 0\), our curve has the form \(\lambda^k P(\lambda^3)\), where \(P\) has only simple zeros, so only \(\lambda = 0\) is a multiple zero when \(k = 2\). Other ramification points of \(\pi\) over \(J = 0\) do not contribute to zeros of Cohn’s polynomial.

Similarly, \(J = 1\) is not a zero of Cohn’s polynomial.

For \(L^H_m\) the total ramification is

\[
d + d^2/18 - (1 - \epsilon_1)/2 = (d/3)(d/3 - 1)/2 + 2d/3 + [d/2].
\]

where \(d = \deg_\lambda F^H_m\). Again, the first summand corresponds to the degree of the Cohn polynomial while the other two reflect additional ramification over the special points (Proposition 8.1). □
Next we briefly describe an alternative approach to our main results. The following remarks are not necessary for understanding the rest of the paper.

**Remarks on parametrization of $H_m$ by a space of triangles**

In the beginning of the previous section we mentioned that $H_m$ represents the space of marked singular tori. In view of the above interpretation of $H_m$, we can construct a natural lift of the isomorphism $\Phi^* : T^*_m \rightarrow L_m$ to an isomorphism $\hat{\Phi}^* : \hat{T}^*_m \rightarrow H_m$ that makes the following diagram commutative:

$$
\begin{array}{ccc}
\hat{T}^*_m & \stackrel{\Sigma}{\longrightarrow} & T^*_m \\
\downarrow\hat{\Phi}^* & & \downarrow\Phi^* \\
H_m & \stackrel{\Psi}{\longrightarrow} & L_m
\end{array}
$$

Here $\hat{T}^*_m$ is a suitable space of flat singular triangles. The point is that one could fully describe the topology of the open Riemann surface $\hat{T}^*_m$ and so of $H_m$ in a direct way, and from that deduce the topology of $L_m$ by taking the quotient by a natural $S_3$-action described below.

Recall that in a balanced flat singular triangle $(D, \{a_i\}, f) \in T_m$, the cyclic order of the corners $(a_1, a_2, a_3)$ on $\partial D$ matches the orientation induced by the developing map $f$, and that we coordinatize $T_m$ using the functions $\phi_{i,j,k}$. Denote by $-T_m$ the space of balanced flat singular triangles $(D, \{a_i\}, f)$ for which the cyclic order $(a_1, a_3, a_2)$ matches the orientation induced by $f$, and coordinatize $-T_m$ using the complex conjugates $\bar{\phi}_{i,j,k}$. Moreover, let $\hat{T}_m$ be the disjoint union of $T_m$ and $-T_m$. The permutation group $S_3$ acts on $\hat{T}_m$ by relabeling the corners.

By identifying each marginal triangle $(D, \{a_i\}, f) \in \hat{T}_m$ with its conjugate $(D, \{a_i\}, f)$, we obtain a space $\hat{T}'_m$. It is immediate that the charts defined above for $\hat{T}_m$ induce a complex structure on $\hat{T}'_m$, and that the $S_3$-action descends to $\hat{T}'_m$. The quotient space $\hat{T}'_m/S_3$ is naturally identified with $T^*_m$ and such identification induces the map $\Sigma$.

Moreover, to a triangle $T$ in $\hat{T}_m$ we can associate the torus $\Phi(T)$ with the marking $t(T) = (t_1, t_2, t_3)$, where $t_1, t_2, t_3$ are the midpoints of $[a_1, a_2]$, $[a_1, a_3]$, $[a_2, a_3]$ respectively. Thus we can define $\hat{\Phi}^*(T) = (\Phi(T), t(T))$. Since the diagram is manifestly commutative, the map $\hat{\Phi}^* : \hat{T}_m \rightarrow H_m$ is an
isomorphism of Riemann surfaces (with trivial orbifold structure on both surfaces), and \( \Sigma \) is an orbifold cover.

Connected components of \( H_m \) can be studied by analyzing \( \hat{T}_m^* \), instead of exploring the orbifold cover \( \Psi_m \) as we did in the previous section. Similarly to what we did with \( T_m \), we can construct a nerve graph \( \hat{\Gamma}_m \) analogous to \( \Gamma_m \). It consists of the disjoint union of two isomorphic components: \( \Gamma_m \) associated to \( T_m \) and \( -\Gamma_m \) associated to \( -T_m \). Note that, if \( v \) is a lateral vertex of \( \Gamma_m \), namely a vertex corresponding to a face of \( A_m \) adjacent to a boundary edge in \( \partial \Delta_m \), then the preimage of \( v \) and the preimage of \( -v \) belong to the same connected component of \( \hat{T}_m^* \). Since each component of \( \Gamma_m \) has a lateral vertex, it follows that every component of \( \Gamma_m \) exactly corresponds to a component in \( \hat{T}_m^* \), which thus has four connected components.

Similarly to what was done in Section 7 one can also compute the genera of \( H_j^m \) using the parametrization \( \hat{\Phi} : \hat{T}_m^* \to H_m \) and obtain an alternative proof of our results about \( H_m \) in this section. Once the number of components, their genera and non-singularity are established for \( H_m \), one can obtain Theorem 1.1 via (46).

9 Proof of Lemma 8.2

In this section, we give a proof of Lemma 8.2 which is due to V. Tarasov. It is inspired by an argument from [31, Proposition 3] which was brought to our attention by Eduardo Chavez Heredia from the University of Bristol.

The proof is a generalization of the classical arguments, going back to Ch. Sturm, which are used in the proof of Lemma 8.1.

Let \( D = \{d_{ij}\} \) be an \( n \times n \) matrix with entries

\[
\begin{align*}
d_{i,i} &= s, & i &= 1, \ldots, n, \\
d_{i,i+1} &= a_i, & i &= 1, \ldots, n - 1, \\
d_{i,i-2} &= b_i, & i &= 3, \ldots, n.
\end{align*}
\]

and consider the principal minors

\[
D_k = \det(d_{ij})_{i,j=1,\ldots,k}
\]

They satisfy the recurrences

\[
D_{k+3} = s D_{k+2} + c_k D_k,
\]
where
\[ c_k = a_{k+1}a_{k+2}b_{k+3} > 0, \]
with the initial conditions
\[ D_0 = 1, \quad D_1 = s, \quad D_2 = s^2. \]
Then
\[ D_{3j}(s) = P_j(s^3), \quad D_{3j+1}(s) = sQ_j(s^3), \quad D_{3j+2}(s) = s^2R_j(s^3) \]
where the polynomials \( P_j, Q_j, R_j \) satisfy the recurrences
\begin{align*}
P_{j+1} &= sR_j + A_jP_j, \quad (52) \\
Q_{j+1} &= P_{j+1} + B_jQ_j, \quad (53) \\
R_{j+1} &= Q_{j+1} + C_jR_j, \quad (54)
\end{align*}
where
\[ A_j = c_{3j}, \quad B_j = c_{3j+1}, \quad C_j = c_{3j+2} \]
and the initial conditions
\[ P_0 = Q_0 = R_0 = 1. \quad (55) \]

Lemma 9.2 follows from

**Proposition 9.1.** Let us define polynomials \( P_j(s), Q_j(s), R_j(s) \) by recurrences (52), (53), (54) where all \( A_j, B_j, C_j \) are strictly positive, and initial conditions (55). Then all polynomials \( P_j, Q_j \) and \( R_j, j = 1, \ldots \) are monic, have degree \( j \) and positive coefficients, and all their roots are negative and simple. Moreover, if \( p_{j1} > \ldots > p_{jj}, q_{j1} > \ldots > q_{jj}, r_{j1} > \ldots > r_{jj} \), are respective roots of the polynomials \( P_j, Q_j, R_j \), then \( p_{jk} > q_{jk} > r_{jk} \) for all \( k = 1, \ldots, j \), and \( r_{jk} > p_{j,k+1} \) for all \( k = 1, \ldots, j-1 \).

**Proof.** We prove this by induction. It is evident that our polynomials are monic and have positive coefficients. Therefore their real roots are negative. To find the number of real roots and to show that they are interlacent we look at the signs of our polynomials at the roots of other polynomials using (52)–(54). The base of induction is given by \( j = 1 \) and is clear. The induction procedure is as follows.
By (52) and the induction assumption,
\[ P_{j+1}(0) P_{j+1}(p_{j1}) = p_{j1} A_j P_j(0) R_j(p_{j1}) < 0, \]
\[ P_{j+1}(t_j k) P_{j+1}(p_{j,k+1}) = p_{j,k+1} A_j P_j(t_j k) R_j(p_{j,k+1}) < 0, \]
for all \( k = 1, \ldots, j - 1, \) and
\[ (-1)^j P_{j+1}(t_j) = (-1)^j A_j P_j(t_j) > 0. \]
This implies that \( P_{j+1} \) has roots \( p_{j+1,k}, \ k = 1, \ldots, j + 1, \) located as follows:
\[ 0 > p_{j+1,1} > p_{j1}, \quad r_{j,k-1} > p_{j+1,k} > p_{jk}, \quad k = 2, \ldots, j, \quad r_{jj} > p_{j+1,j+1}. \] (56)
Thus all roots of \( P_{j+1} \) are negative and simple.

By the induction assumption and (56),
\[ p_{j+1,1} > q_{j1}, \quad k = 2, \ldots, j, \quad q_{jj} > p_{j+1,j+1}. \] (57)
Then by (53) and (57),
\[ Q_{j+1}(p_{j+1,k}) Q_{j+1}(q_{jk}) = B_j Q_j(p_{j+1,k}) \quad P_{j+1}(q_{jk}) < 0, \]
for all \( k = 1, \ldots, j, \) and
\[ (-1)^j Q_{j+1}(p_{j+1,j+1}) = (-1)^j B_j Q_j(p_{j+1,j+1}) > 0. \]
This implies that \( Q_{j+1} \) has roots \( q_{j+1,k}, \ k = 1, \ldots, j + 1, \) located as follows:
\[ p_{j+1,k} > q_{j+1,k} > q_{jk}, \quad k = 1, \ldots, j, \quad p_{j+1,j+1} > q_{j+1,j+1}. \] (58)
Thus all roots of \( Q_{j+1} \) are negative and simple.

By the induction assumption, (58), and (56),
\[ q_{j+1,1} > r_{j1}, \quad r_{j,k-1} > q_{j+1,k} > r_{jk}, \quad k = 2, \ldots, j, \quad r_{jj} > q_{j+1,j+1}. \] (59)
Then by (54) and (59),
\[ R_{j+1}(q_{j+1,k}) R_{j+1}(r_{jk}) = C_j R_j(q_{j+1,k}) Q_{j+1}(r_{jk}) < 0, \]
for all \( k = 1, \ldots, j, \) and
\[ (-1)^j R_{j+1}(q_{j+1,j+1}) = (-1)^j C_j R_j(q_{j+1,j+1}) > 0. \]
This implies that \( R_{j+1} \) has roots \( r_{j+1,k}, \ k = 1, \ldots, j + 1, \) located as follows:
\[ q_{j+1,k} > r_{j+1,k} > r_{jk}, \quad k = 1, \ldots, j, \quad q_{j+1,j+1} > r_{j+1,j+1}. \] (60)
Thus all roots of \( R_{j+1} \) are negative and simple.

The inequalities \( p_{j+1,k} > q_{j+1,k} > r_{j+1,k} \) for all \( k = 1, \ldots, j + 1, \) follow from (58) and (60), and the inequalities \( r_{j+1,k} > p_{j+1,k+1} \) for all \( k = 1, \ldots, j, \) follow from (60) and (56). This completes the proof. \( \square \)
10 Projective monodromy of the Lamé equation

In this section we prove Theorem 1.3.

Lamé equation with integer \( m \) has trivial local monodromy about the origin. Since the fundamental group of the torus is \( \mathbb{Z}^2 \), the projective monodromy is represented by a pair of commuting elements \( PSL(2, \mathbb{C}) \).

So we investigate the set of pairs of commuting elements of \( PSL(2, \mathbb{C}) \) modulo conjugation. Every such pair \((A, B)\) can be conjugated to one of the following forms:

\[
(z \mapsto \mu_1 z, \quad z \mapsto \mu_2 z), \quad (\mu_1, \mu_2) \in (\mathbb{C}^*)^2, \quad (61)
\]

\[
(z \mapsto z + a_1, \quad z \mapsto z + a_2), \quad (a_1, a_2) \in \mathbb{C}^2, \quad (62)
\]

or

\[
(z \mapsto -z, \quad z \mapsto 1/z). \quad (63)
\]

It is proved in [10, Theorem 2.2] that the third possibility (63) cannot happen for the projective monodromy of Lamé equations with integer \( m \). Notice that \( PSL(2, \mathbb{C}) \) representations (61) and (62) can be lifted to \( SL(2, \mathbb{C}) \). The pair \((A, B) = (id, id)\) is also excluded.

Conjugacy classes of pairs of the form (61) are parametrized by \((\mathbb{C}^*)^2\). Two pairs of the form (62) are conjugate iff \((a_1 : a_2) = (a'_1 : a'_2)\), so they are parametrized by the projective line \( \mathbb{CP}^1 \) and one point \((0, 0)\).

Suppose that a sequence of pairs \((A, B)\) of type (61) converges in \( PSL(2, \mathbb{C}) \) to a pair of type (62). To figure out how \((a_1, a_2)\) in (62) are related to \((\mu_1, \mu_2)\) we consider commuting pairs of linear-fractional transformations \((\phi_1, \phi_2)\),

\[
\phi_j(z) = \frac{(1 + f_j)z + (a_j + p_j)}{q_jz + (1 + g_j)}, \quad j \in \{1, 2\},
\]

where \(f_j, g_j, p_j, q_j\) are small numbers, and \(a_j\) are constants not simultaneously equal to 0.

The condition that matrices representing \(\phi_j\) have determinant 1 gives

\[
f_j + g_j \equiv a_jq_j, \quad (64)
\]

where \(\equiv\) means that we neglected the terms of order 2 and higher. The condition that \(\phi_1\) and \(\phi_2\) commute implies by comparing the diagonal elements of the product matrices

\[
a_1q_2 \equiv a_2q_1. \quad (65)
\]
Now it follows from (64) and (65) that
\[(f_1 + g_1) : (f_2 + g_2) \to (a_1^2 : a_2^2).\] (66)

Similar equation can be obtained when the eigenvalue of one or both limit matrices is \(-1\). In other words, if \(A\) and \(B\) are \(SL(2, \mathbb{C})\) matrices representing \(\phi_1\) and \(\phi_2\) tending to parabolic \(\phi_1^*\) and \(\phi_2^*\), then we have
\[
\lim \frac{\text{tr}^2 A - 4}{\text{tr}^2 B - 4} = (a_1^2 : a_2^2),
\] (67)

where \((a_1 : a_2)\) is the “ratio invariant” of the pair of commuting parabolic transformations.

So we obtain that the space of projective monodromy representations for Lamé equations is the blow up of \((\mathbb{C}^*)^2\) at the point \((1, 1)\). Monodromies of unitarizable Lamé equations form the real torus
\[
\{(\mu_1, \mu_2) \in (\mathbb{C}^*)^2 : |\mu_1| = |\mu_2| = 1\} \setminus (1, 1),
\]
and the boundary of this real torus in the blow up is the real projective line. Since \(A\) and \(B\) are elliptic, the left hand side of (67) is positive, so the ratio \((a_1 : a_2)\) is real.

Thus we obtain

**Proposition 10.1.** Abelian integrals arising as limits of developing maps of spherical metrics have real period ratios.

So we have
\[\partial \text{Sph}_{1,1}(2m + 1) \subset \text{LW}_m.\]

Let us prove the opposite inclusion. We recall that \(\text{Lame}_m\) is biholomorphic to \(\mathbb{CP}(1, 2, 3)\), and each point of \(\text{Lame}_m\) corresponds to a Schwarz equation (24) whose solution is a developing map of a translation structure on a torus with one conic singularity with angle \(2\pi(2m + 1)\). Let us fix \(L \in \text{LW}_m\). We want to prove that there exists a sequence \(L_n \in \text{Lame}_m, L_n \to L\) such that \(L_n\) have unitarizable monodromy.

Since \(L \in \text{LW}_m\), is obtained as \(\Phi(T)\) from some BFT, we can normalize \(T\) so that the images of corners under the developing map are
\[(-\delta, k\delta, \delta),\]
where \( k \in (-1, 1) \) depends on the similarity class of \( T \) and thus is fixed, while \( \delta > 0 \) is arbitrary. We glue a congruent copy \(-T\) to \( T\) by the side with endpoints \( \{-\delta, \delta\} \). Notice that the gluing map \( z \mapsto -z \) is the orientation-reversing isometry of this side, with respect to both flat and spherical metric. The result of this gluing is a “parallelogram” \( Q \), which can be represented by \((\Delta, \{a_j\}, f)\), where \( \Delta \) in the unit disk, \( a_1, a_2, a_3, a_4 \) are boundary points (corners), and \( f \) is a meromorphic function, a developing map of \( Q \). This developing map is a local homeomorphism at all points of \( \Delta \setminus \{a_j\} \) while at the corners it behaves like an integer power. The torus \( \Phi(T) \) is obtained by gluing the opposite sides of \( Q \), so we have

\[
f(p(z)) = t_i \circ f(z), \quad z \in \partial \Delta, \quad i = 1, \ldots, 4,
\]

where \( p(z) \) is an involution of \( \partial \Delta \) defining the gluing of opposite sides, \( t_j \) are translations, and \( i \) depends on the side to which \( z \) belongs. Subscript \( i \) has to be interpreted similarly in all following formulas (69), (70), (71). Each of these translations \( t_i \) is a product of two Euclidean rotations by \( \pi \) about the “middles” of the sides (see Proposition 3.3).

Notice that \( T \) can be also considered as a spherical triangle, since its sides belong to the real line, which is both spherical and Euclidean geodesic. So we can produce from \( T \) a spherical torus with one conic singularity with the same angle as in \( \Phi(T) \), by identifying the opposite sides of \( Q \) by spherical isometries \( s_i, 1 \leq i \leq 4 \). Similarly to \( t_i \), each \( s_i \) is a product of two spherical rotations by \( \pi \) about the “spherical middles” of the sides.

Notice the following: let \( a, b \) be two points in \([-\delta, \delta]\). There exists a unique Euclidean rotation \( e \) by angle \( \pi \), and a unique spherical rotation \( s \) by angle \( \pi \) which interchange \( a \) and \( b \). Moreover, when \( \delta \) is small, then \( e \) and \( s \) are close to each other in the sense that \( s = e \circ \phi \), where \( \phi \) is a linear fractional homeomorphism of \( R \) which is \( \epsilon \)-bi-Lipschitz with respect to the spherical metric, where \( \epsilon = \epsilon(\delta) \to 1 \) as \( \delta \to 0 \). We call such homeomorphisms \( \phi \) close to the identity. Using this statement, we obtain that

\[
s_i = t_i \circ \phi_i, \quad (69)
\]

where \( \phi_i \) are close to the identity. Now we define homeomorphisms \( \psi_i \) of the sides of \( \Delta \) and the involution \( q \) of the unit circle \( \partial \Delta \) by

\[
f \circ \psi_i = \phi_i^{-1} \circ f, \quad q := \psi_i^{-1} \circ p, \quad (70)
\]
and the function \( g : \partial \Delta \to \mathbb{R} \),

\[
g = f \circ \psi_i. \tag{71}
\]

It is easy to see that \( \psi_i \) are close to the identity: the only points where \( f \) is not bi-Lipschitz are the corners, where \( f \) behaves like \( z^n \), and conjugating a diffeomorphism which is \( C^1 \)-close to the identity by \( z^m \) result in a diffeomorphism which is \( C^1 \) close to the identity. Now we use the theorem from [32, Thms. 5.3, 5.17] which says that every \( \epsilon \)-bi-Lipschitz homeomorphism of the unit circle with \( \epsilon \) sufficiently close to 1 has an \( \epsilon_1 \)-bi-Lipschitz extension to a homeomorphism of the unit disk. So we extend \( \psi \) and define \( g \) in \( \Delta \) by (71). This extended \( g \) satisfies

\[
g(q(z)) = s_i \circ g(z), \quad z \in \partial \Delta,
\]

which is analogous to (68), and follows from (68), (69), (70) and (71). So \( g \) can be considered as a developing map of a spherical metric on a torus obtained by identifying the opposite sides by the involution \( q \). The map \( g \) is not conformal with respect to the standard conformal structure on \( \Delta \) but conformal in a new structure defined by \( \psi \). This new structure is close to the standard one in the sense of the theory of quasiconformal mappings [1]. Indeed, a bi-Lipschitz map with constant close to 1 is quasiconformal with dilatation close to 1. Let \( L_\delta \) be the spherical torus obtained by this construction, and let \( F \) and \( G_\delta \) be the lifts the developing maps of the tori \( \Phi(T) \) and \( L_\delta \) to the universal cover. Then \( F \) is a meromorphic function, and \( G_\delta \) a quasiregular one, conformal with respect to the non-standard conformal structure in \( \mathbb{C} \). We normalize \( F \) and \( G_\delta \) so that the preimages of the conic singularities form lattices \( \Lambda_F \) and \( \Lambda_{G_\delta} \), with primitive periods 1 and \( \omega_F, \omega_{G_\delta} \).

To compare \( F \) and \( G_\delta \) we find a homeomorphism \( \eta_\delta : \mathbb{C} \to \mathbb{C} \) such that \( F_\delta := G_\delta \circ \eta_\delta \) is a meromorphic function, and \( \eta_\delta \) is normalized by \( \eta_\delta(0) = 0, \eta_\delta(1) = 1 \). This homeomorphism is obtained as a normalized solution of the Beltrami equation [1, Chap. V], and since the quasiconformal dilatation of \( \eta_\delta \) is small, \( \eta_\delta \to \text{id} \), as \( \delta \to 0 \). We conclude that the lattices of poles \( \Lambda_{F_\delta} \) and \( \Lambda_F \) are close to each other uniformly on compacts, and \( F_\delta \to F \) as \( \delta \to 0 \) uniformly (with respect to the spherical metric) on compact subsets of the plane. This means that when \( \delta \to 0 \), the lattice invariants \( g_j(\Lambda_{F_\delta}) \to g_j(\Lambda_F) \), \( j = 2, 3 \), and the Schwarzian of \( F_\delta \) converges to the Schwarzian of \( F \). Since the Schwarzians of \( F \) and \( F_\delta \) are both of the form

\[
m(m + 1) \varphi(z, \Lambda) + \lambda,
\]

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where \( \lambda = \lambda(F) \) or \( \lambda = \lambda(F_\delta) \), we conclude that \( \lambda(F_\delta) \to \lambda(F) \) as \( \delta \to 0 \), and this proves that \( L_\delta \to L \) in \( \text{Lame}_m \). This completes the proof of Theorem 1.3.

11 Lin–Wang curves

Proposition 10.1 gives the following characterization of Lin–Wang curves: \( J \in \pi_m(\text{LW}_m) \) if some Lamé equation with invariant \( J \) has translational monodromy which belongs to a straight line. In particular, there is a flat singular torus corresponding to \( (J, \lambda) \), and this torus is of the form \( \Phi^*(T) \) where \( T \) is a BFT with integer angles.

**Proof of Theorem 1.4**

The set \( \text{LW}_m \) is the image of the set of triangles with angles integer multiples of \( \pi \) under the map \( \Phi_m \). Triangles with angles integer multiples of \( \pi \) form straight intervals in the local coordinates \( \phi_{i,j,k} \) introduced in Section 4, and the map \( \Phi_m \) is biholomorphic.

We conclude that the set \( \text{LW}_m \) consists of analytic (non-singular) curves. There are three such curves corresponding to each integer point in \( \text{A}_m \). So all together we have \( m(m + 1)/2 \) Lin–Wang curves.

Proposition 10.1 shows how to find equations of Lin–Wang curves. They are curves in \( \text{L}_m \), the moduli space of Lamé functions, where the ratio of periods of the integral (25) is real. According to [17], to each triple of integers satisfying the triangle inequalities corresponds a component of the space \( \text{Sph}_{1,1}(2m + 1) \). This component is parametrized by an open triangle, and the sides of this triangle correspond to three interior edges of \( T_m \) which are mapped by \( \phi \) in (38) to integer points of \( \text{A}_m \) by the map \( \phi \) in (38). These three edges parametrize Lin–Wang curves by the map \( \pi_m \circ \Phi_m^* \).

Thus each integer point in \( \text{A}_m \) corresponds to a component of the moduli space of \( \text{Sph}_{1,1}(2m + 1) \) of spherical tori. The boundary of this component consists of one or three Lin–Wang curves: when the integer point is the center of \( \text{A}_m \), which happens when \( m \equiv 1 \) (mod 3), there is one curve, otherwise there are three of them. If there is one curve, it belongs to \( \text{L}^I_m \), as it happens for \( m = 1 \) and \( m = 4 \).

For integer points other than the center of \( \text{A}_m \) we have three curves which can be all on component \( \text{L}^I_m \), or one of them can be on \( \text{L}^I_m \) and two on \( \text{L}^I_m \). Figs. 6, 7 and similar figures for other \( m \) permits to determine this for every
integer point on $A_m$.

We give explicit formulas for Lin–Wang curves for $1 \leq m \leq 3$. We recall that the developing map is given by (20). Since working with elliptic functions is easier than with Abelian integrals (this was the primary reason for introducing elliptic functions), we pass to the universal covering.

In what follows, $g$ is the integrand in (20). It is an even elliptic function with a single zero of multiplicity $2m$ at the origin, and double poles with vanishing residues. The general form of such function is

$$g(z) = c_0 + \sum_{j=1}^{m} c_j \wp(z - a_j).$$

By Abel’s theorem, $2(a_1 + \ldots + a_m) \equiv 0$, and we want to choose $c_j$ so that $g$ and its first $2m - 1$ derivatives vanish at the origin. So

$$c_0 = -\sum_{j=1}^{m} c_j \wp(a_j),$$

and for the rest of $c_j$ we have a system of equations

$$\sum_{j=1}^{m} c_j \wp^{(k)}(-a_j) = 0, \quad 1 \leq k \leq 2m - 1,$$

which has a non-trivial solution if the matrix of this system has rank at most $m - 1$.

Once $g$ is found, we are interested in the ratio of periods of the integral (20). Lin–Wang curves make the locus of points where this ratio is real. Below we give the results of computation for $1 \leq m \leq 3$. We use the standard notation of the theory of elliptic functions [2]: $\wp(\omega_j) = e_j$, $1 \leq j \leq 3$, so $\omega_j$ are half-periods, $\zeta$ is the Weierstrass zeta function, $\zeta' = -\wp$, satisfying

$$\zeta(z + 2\omega_j) = \zeta(z) + 2\eta_j.$$

Case $m = 1$. $g(z) = \wp(z - \omega_j) - e_j$, $j \in \{1, 2, 3\}$. In this case, $L_1 = L_1^{H}$. The equation of Lin–Wang curves is

$$\text{Im} \frac{\eta_1 + \omega_1 e_j}{\eta_2 + \omega_2 e_j} = 0, \quad j \in \{1, 2, 3\}.$$
This defines three curves in the fundamental region of the modular group in the \( \tau \)-half-plane, Fig. 12, which correspond to one curve in the \( J \)-plane (Fig. 13).

Case \( m = 2 \). For \( L_2^I \), we obtain

\[
g(z) = \varphi(z + a) + \varphi(z - a) - 2\varphi(a), \quad \text{where} \quad \varphi(a) = \sqrt{g_2/12}.
\]

The equation of the Lin–Wang curve is

\[
\text{Im} \frac{\eta_1 + \omega_1 \sqrt{g_2/12}}{\eta_2 + \omega_2 \sqrt{g_2/12}} = 0.
\]

For \( L_2^H \), we have

\[
g(z) = \varphi''(\omega_j)(\varphi(z + \omega_k) - e_k) - \varphi''(\omega_k)(\varphi(z + \omega_j) - e_j),
\]

and the equation of Lin–Wang curve is

\[
\text{Im} \frac{\omega_1(e_j \varphi''(\omega_k) - e_k \varphi''(\omega_j)) + \eta_1(\varphi''(\omega_k) - \varphi''(\omega_j))}{\omega_2(e_j \varphi''(\omega_k) - e_k \varphi''(\omega_j)) + \eta_2(\varphi''(\omega_k) - \varphi''(\omega_j))} = 0.
\]

These curves in the \( \tau \)-half-plane are shown in Fig. 14 and their images in the \( J \)-plane are in Fig. 15. These are the three curves bounding a single triangle which is the moduli space \( \text{Sph}_{1,1}(5) \). One of these curves, (which has a loop in Fig. 12) belongs to \( L_2^I \), other two belong to \( L_2^H \). Shading in Fig. 15 is the hypothetical projection of a component of \( \text{Sph}_{1,1}(5) \) by the forgetful map. We do not know whether the restriction of the forgetful map on \( \text{Sph}_{1,1}(2m + 1) \) is open. So we don’t know that the boundary of this projection is contained in Lin–Wang curves.

Case \( m = 3 \). For \( L_3^I \) we obtain:

\[
g(z) = c_0 + c_1\varphi(z + \omega_1) + c_2\varphi(z + \omega_2) + c_3\varphi(z + \omega_3),
\]

where

\[
c_0 = -\sum_{j=1}^{3} c_j e_j,
\]

\[
c_k = (6e_{k+2}^2 - g_2/2)(6e_{k+1}^2 - g_2/2)(e_{k+2} - e_{k+1}), \quad k \in \{1, 2, 3\},
\]

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where the subscripts are understood as residues mod 3, but we use 3 instead of 0 to prevent the confusion with previous formula. Setting $B = c_1 + c_2 + c_3$, the equation of the Lin-Wang curve is

$$\text{Im} \frac{\omega_1 c_0 - \eta_1 B}{\omega_2 c_0 - \eta_2 B} = 0.$$  

This curve in the $\tau$-plane is shown in Fig. 16.

For component $L_{III}$ we introduce the notation for $k \in \{1, 2, 3\}$:

$$P_k^\pm = -e_k/5 \pm \sqrt{3(5g_2/4 - 3e_k^2)},$$

$$c_{1,k}^\pm = -2 \frac{6(P_k^\pm)^2 - g_2/2}{6e_k^2 - g_2/2},$$

$$c_{0,k}^\pm = -c_{1,k}^\pm e_k - 2P_k^\pm.$$

Then

$$g(z) = c_{0,k}^\pm + c_{1,k}^\pm \wp(z + \omega_k) + \wp(z + a_k^\pm) + \wp(z - a_k^\pm),$$

where $\pm a_k^\pm$ are solutions of the equation $\wp(z) = P_k^\pm$. Then we have six Lin–Wang curves

$$\text{Im} \frac{c_{0,k}^\pm \omega_1 - (c_{1,k}^\pm + 2)\eta_1}{c_{0,k}^\pm \omega_2 - (c_{1,k}^\pm + 2)\eta_2} = 0.$$  

These curves in the $\tau$-half-plane are shown in Fig. 17, and their image in the $J$-plane in Fig. 18, where the curve from $L_{III}$ is also included (it is the one which looks like a vertical line in the middle). Fig. 18 shows in the right-hand side the detail which looks like a tripod in the left-hand side. Figure 17 contains 16 curves which give 5 images in Fig. 18. Three of these 5 curves in Fig. 18 constitute the full boundary of one triangle of the moduli space for spherical tori, and the remaining three curves in Fig. 18, including that one curve which comes from Fig. 16 constitute the boundary of the second triangle in the moduli space for spherical tori.
Appendix. List of formulas

Polynomials $F^I_m$:

| $m$ | Formula                                      |
|-----|----------------------------------------------|
| 2   | $\lambda^2 - 3g_2$                         |
| 3   | $\lambda$                                   |
| 4   | $\lambda^3 - 52g_2\lambda + 560g_3$        |
| 5   | $\lambda^2 - 27g_2$                        |
| 6   | $\lambda^4 - 294g_2\lambda^2 + 7776g_3\lambda + 3465g_2^2$ |
| 7   | $\lambda^3 - 196g_2\lambda + 2288g_3$      |
| 8   | $\lambda^5 - 1044g_2\lambda^3 + 48816g_3\lambda^2 + 112320g_2^2\lambda - 4665600g_2g_3$ |
| 9   | $\lambda^4 - 774g_2\lambda^2 + 21600g_3\lambda + 41769g_2^2$ |

Polynomials $F^{II}_m$:

| $m$ | Formula                                      |
|-----|----------------------------------------------|
| 1   | $4\lambda^3 - g_2\lambda - g_3$             |
| 2   | $4\lambda^2 - 9g_2\lambda + 27g_3$          |
| 3   | $16\lambda^6 - 504g_2\lambda^4 + 2376g_3\lambda^3 + 4185g_2^2\lambda^2 - 36450g_2g_3\lambda - 3375g_3^2 + 91125g_3^2$ |
| 4   | $16\lambda^6 - 1016g_2\lambda^4 + 8200g_3\lambda^3 + 10297g_2^2\lambda^2 - 41650g_2g_3\lambda - 27783g_2^3 - 42875g_3^2$ |
Polynomials $H^0_m$:

2  $B^2 + 4(a + 1)B + 12a$
3  $B + 4(a + 1)$
4  $B^3 + 20(a + 1)B^2 + (64a^2 + 336a + 64)B + 640(a^2 + a)$
5  $B^2 + 20(a + 1)B + 64(a^2 + 1)$
6  $B^4 + 56(a + 1)B^3 + (784a^2 + 2744a + 784)B^2$

   $+ (2304a^3 + 29472a^2 + 29472a + 2304)B$

   $+ 48384a^3 + 152208a^2 + 48384a$

Polynomials $H^1_m$:

1  $B + a + 1$
2  $B + 4a + 1$
3  $B^2 + 10(a + 1)B + 9a^2 + 78a + 9$
4  $B^2 + (20a + 10)B + 64a^2 + 136a + 9$
5  $B^3 + 35(a + 1)B^2 + (259a^2 + 1046a + 259)B$

   $+ 225a^3 + 5235(a^2 + a) + 225$

6  $B^3 + (56a + 35)B^2 + (784a^2 + 1568a + 259)B$

   $+ 2304a^3 + 13008a^2 + 7464a + 225$

Degrees of forgetful maps:

\[
\begin{align*}
  d^I_m &:= \begin{cases} 
m/2 + 1, & m \equiv 0 \pmod{2} \\
  (m - 1)/2, & m \equiv 1 \pmod{2}, \end{cases} \\
  d^{II}_m &:= 3 \lfloor m/2 \rfloor.
\end{align*}
\]
Euler characteristics:

\[ \chi(L^I_m) = \begin{cases} 
-\frac{(m+2)^2}{24} + \frac{4\epsilon_0 + 3\epsilon_1}{6}, & m \equiv 0 \pmod{2}, \\
-\frac{(m-1)^2}{24} + \frac{4\epsilon_0 + 3\epsilon_1}{6}, & m \equiv 1 \pmod{2}, \\
-\frac{m^2}{8} + \frac{1-\epsilon_1}{2}, & m \equiv 0 \pmod{2}, \\
-\frac{(m+1)^2}{8} + \frac{1-\epsilon_1}{2}, & m \equiv 1 \pmod{2},
\end{cases} \]

\[ \chi(L^II_m) = \begin{cases} 
-\frac{m^2}{8} + \frac{1-\epsilon_1}{2}, & m \equiv 0 \pmod{2}, \\
-\frac{(m+1)^2}{8} + \frac{1-\epsilon_1}{2}, & m \equiv 1 \pmod{2}.
\end{cases} \]

where

\[ \epsilon_0 = \begin{cases} 
0, & \text{if } m \equiv 1 \pmod{3}, \\
1, & \text{otherwise},
\end{cases} \]

\[ \epsilon_1 = \begin{cases} 
0, & \text{if } m \in \{1, 2\} \pmod{4}, \\
1, & \text{otherwise}.
\end{cases} \]

Numbers of punctures: \( h^I_m = d^I_m, \ h^II_m = 2d^{II}_m/3. \)

Genera in terms of \( d^K_m \):

\[ g(L^I_m) = 1 + \frac{(d^I_m)^2}{12} - \frac{d^I_m^2}{2} - \frac{4\epsilon_0 + 3\epsilon_1}{12}, \quad (74) \]

\[ g(L^II_m) = 1 + \frac{(d^{II}_m)^2}{36} - \frac{d^{II}_m^2}{3} - \frac{1-\epsilon_1}{4}. \quad (75) \]

Genera in terms of \( m \):

\[ g(L^I_m) = \frac{m^2 - 8m + 28}{48} - \frac{4\epsilon_0 + 3\epsilon_1}{12}, \quad m \equiv 0 \pmod{2}, \quad (76) \]

\[ g(L^I_m) = \frac{m^2 - 14m + 61}{48} - \frac{4\epsilon_0 + 3\epsilon_1}{12}, \quad m \equiv 1 \pmod{2}, \quad (77) \]

\[ g(L^{II}_m) = \frac{m^2 - 8m + 16}{16} - \frac{1-\epsilon_1}{4}, \quad m \equiv 0 \pmod{2}, \quad (78) \]

\[ g(L^{II}_m) = \frac{m^2 - 6m + 9}{16} - \frac{1-\epsilon_1}{4}, \quad m \equiv 1 \pmod{2}. \quad (79) \]
Degrees of Cohn’s polynomials

\[ \deg C_m^I = \left\lfloor \frac{(d_m^I)^2 - d_m^I + 4}{6} \right\rfloor, \]

\[ \deg C_m^{II} = \frac{d_m^{II}(d_m^{II} - 1)}{2}. \]

References

[1] L. Ahlfors, Lectures on quasiconformal mappings, second edition, AMS, Providence RI, 2002.

[2] N. I. Akhiezer, Elements of the theory of elliptic functions, AMS, Providence, RI, 1990.

[3] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky and M. Möller, Strata of \( k \)-differentials. Alg. Geom. 6 (2019) 196–233.

[4] M. Bainbridge, D. Chen, Q. Gendron, S. Grushevsky and M. Möller, Compactification of strata of Abelian differentials, Duke Math. J., 167, 12 (2018) 2348–2416.

[5] D. Bartolucci and G. Tarantello, Liouville type equations with singular data and their applications to periodic multivortices for the electroweak theory, Comm. Math. Phys. 229 (2002), no. 1, 3—47.

[6] W. Bergweiler and A. Eremenko, Green functions and antiholomorphic dynamics on tori, Proc. AMS 144 N 7 (2016) 2911–2922.

[7] C. Boissy, Connected components of the strata of the moduli space of meromorphic differentials, Comment. Math. Helv. 90 (2015) 255–286.

[8] A. Bonifant and J. Milnor, Group actions, divisors, and plane curves, Bull. AMS 57, 2 (2020) 171–267.

[9] E. Brieskorn and H. Knörrer, Plane algebraic curves, Birkhäuser/Springer, Basel, 1986.

[10] C.-L. Chai, C.-S. Lin, C.-L. Wang, Mean field equations, hyperelliptic curves and modular forms: I, Camb. J. Math. 3 (2015), no. 1-2, 127–274.
[11] C.-L. Chai, C.-S. Lin and C.-L. Wang, Mean field equations, hyperelliptic curves and modular forms: II, J. Éc. polytech. Math. 4 (2017), 557—593.

[12] Z. Chen and C.-S. Lin, Critical points of the classical Eisenstein series of weight two, J. Differential Geom. 113 (2019), no. 2, 189—226.

[13] M. Costantini, M. Möller, J. Zachhuber, The Chern classes and the Euler characteristic of the moduli spaces of Abelian differentials, arXiv:2006.12803.

[14] M. Costantini, M. Möller, J. Zachhuber, diffstrata – a Sage package for calculations in the tautological ring of the moduli space of Abelian differentials, arXiv:2006.12815.

[15] A. Eremenko, Metrics of positive curvature with conic singularities on the sphere, Proc. Amer. Math. Soc. 132 N 11 (2004) 3349–3355.

[16] A. Eremenko and A. Gabrielov, Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry, Ann of Math. 155 (2002), 105–129.

[17] A. Eremenko, G. Mondello and D. Panov, Moduli of spherical tori with one conical point, arXiv:2008.02772.

[18] R. Gantmakher and M. Krein, Oscillation matrices and kernels and small vibrations of mechanical systems, AMS Chelsea, Providence, RI, 2002.

[19] T. Hosgood, An introduction to varieties in weighted projective space, arXiv:1604.02441.

[20] F. Klein, Vorlesungen über die hypergeometrische Funktion, Springer-Verlag, Göttingen, 1981 (reprint of the 1933 edition).

[21] M. Kontsevich and A. Zorich, Connected components of the moduli spaces of Abelian differentials with prescribed singularities, Invent. math. 153 (2003) 631–678.

[22] G. Lamé, Mémoire sur les axes des surfaces isothermes du second degré considéré comme des fonctions de la température, J. de Math. pures et appl. IV (1839) 100–125.

67
[23] G. Lamé, Mémoire sur l’équilibre des températures dans un ellipsoïde à
trois axes inégaux, J. de Math. pures et appl. IV (1839) 126–163.

[24] C.-S. Lin, Green function, mean field equation and Painlevé VI equation,
in the book: Current developments in mathematics, 2015, Intl. Press, 2017.

[25] C.-S. Lin and C.-L. Wang, Elliptic functions, Green functions and the
mean field equations on tori, Ann. of Math. (2) 172 (2010), no. 2, 911–
954.

[26] F. Loray and D. Marin, Projective structures and projective bundles
over compact Riemann surfaces, Astérisque No. 323 (2009), 223—252.

[27] R. Maier, Lamé polynomials, hyperelliptic reductions and Lamé band
structure, Phil. Trans. R. Soc. A 366 (2008) 1115–1153.

[28] MathOverflow, Examples of plane algebraic curves,
https://mathoverflow.net/questions/352957.

[29] G. Mondello and D. Panov, Spherical metrics with conical singularities
on 2-sphere: angle constraints, IMRN 16 (2016) 4937–4995.

[30] G. Mondello and D. Panov, Spherical surfaces with conical points: sys-
tole inequality and moduli spaces with many connected components,
GAFA 29 4 (2019) 1110–1193.

[31] B. Shapiro and M. Tater, On spectral asymptotics of quasi-exactly solv-
able quartic and Yablonskii–Vorob’ev polynomials, arXiv:1412.3026.

[32] P. Tukia and J. Väisälä, Extension of embeddings close to isometries or
similarities, Ann. Acad. Sci. Fenn., Ser. A. I. Math., 9 (1984) 153–175.

[33] W. Thurston, On the combinatorics of iterated rational maps, preprint,
Princeton Univ., 1985.

[34] A. Turbiner, Lamé equation, sl(2) algebra and isospectral deformations,
J. Phys. A 22 (1989) L1–L3.

[35] E. T. Whittaker and G. N. Watson, A course of modern analysis, Cam-
bridge UP, 1927.
[36] A. Zorich, Flat surfaces, in: Frontiers in Number Theory, Physics, and Geometry. I, Springer-Verlag, New York, 2006, pp. 437-583.

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Figure 1: a),b) - primitive triangles; c) - examples of nets with the angles \((3\pi, 2\pi, 4\pi)\); d) - triangles with the angles \((2\pi, \pi, 2\pi)\) and their nets; e) - deformation of triangle with the angle sum \(5\pi\).
Figure 2: All types of BFT for $m = 2$.

Figure 3: To the proof of Proposition 3.4.
Figure 4: To the proof of Lemma 4.1 for a triangle of type $A''$. 
Figure 5: To the proof of Lemma 5.2
Figure 6: Spaces of angles $A_m$ for $m \leq 5$. The nerve of component $L'_m$ is blue/dotted.
Figure 7: Spaces of angles $A_m$ for $m \leq 5$. The nerves of components $II_1, II_2, II_3$ are shown.
Figure 8: Two components in the space of flat singular triangles.
Figure 9: Triangle $\Delta_3$ (shaded) in the intersection of the plane $\alpha_1+\alpha_2+\alpha_3 = 7$ with the first octant. Faces of type $I$ have darker shading.
Figure 10: Counting punctures for $L_m^I$. 
Figure 11: Counting punctures for $L_m^{II}$. Dotted lines represent gluings.
Figure 12: Lin–Wang curves for $m = 1$, $\tau$-half-plane. Shaded area corresponds to $\text{Sph}_{1,1}(3)$. 
Figure 13: Lin–Wang curve $m = 1$, $J$-plane. Projection of $\text{Sph}_{1,1}(3)$ is shaded.
Figure 14: Lin–Wang curves for $m = 2$, $\tau$-half-plane.
Figure 15: $m = 2$, $J$-plane. The curve with a loop is in $L^I_2$, the other two curves in $L^II_2$. Shaded area is the hypothetical projection of $\text{Sph}_{1,1}(5)$; it is not known whether the restriction of the forgetful map on $\text{Sph}_{1,1}(5)$ is open.
Figure 16: $m = 3$, $\tau$-half-plane, Lin–Wang curves from $\mathbf{L}_3^t$. 
Figure 17: $m = 3$, $\tau$-half-plane, Lin–Wang curves from $L^I_3$. 

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Figure 18: $m = 3$, $J$-plane, Lin–Wang curves from both components. Magnification of detail on the right, this detail is on component $L^I_3$. 