Phase-ordering kinetics: ageing and local scale-invariance

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Abstract. Dynamical scaling in ageing systems, notably in phase-ordering kinetics, is well-established. New evidence in favour of Galilei-invariance in phase-ordering kinetics is described.

DYNAMICAL SCALING IN AGEING SYSTEMS

The study of the long-time dynamics of statistical systems far from equilibrium has been a topic of intensive study. In many instances, the relaxation times towards equilibrium can become extremely long such that the system stays for all intents and purposes out of equilibrium. A paradigmatic example is provided by glassy systems which might be considered as an extremely viscous liquid. In principle, the presence of very long relaxation time-scales might suggest that quantitative properties of glassy dynamics should depend on a huge variety of microscopic ‘details’ and furthermore, as the behaviour of a glass may depend on its previous (thermal, mechanic,. . . ) history, those system age. However, as first pointed out by Struik in 1978 [1], the ageing dynamics of many physically very different glass-forming systems can be described in terms of universal master curves. The challenge is to try to understand the origin of this dynamical scale-invariance and to compute the form of these master curves from the essential characteristics of the system.

Here we shall consider an analogous situation in supposedly more simple systems without intrinsic disorder or frustration. From some initial state (typically one chooses a fully disordered initial state) the system is rapidly ‘quenched’ either onto its critical point or else into a region of the phase diagram with at least two stable stationary states. The dynamics and an eventual ageing behaviour (that is, a breaking of time-translation invariance) is then observed. One distinguishes physical ageing, where the underlying microscopic processes are reversible, from chemico-biological ageing, where irreversible microscopic processes may occur. In terms of models, a simple example for physical ageing is provided by the phase-ordering kinetics of a simple Ising ferromagnet with purely relaxational dynamics quenched to below its critical temperature \( T_c > 0 \). On the other hand, the ageing behaviour of the contact process (particles of a single species \( A \) move diffusively on a lattice and react according to \( A + A \rightarrow \emptyset \) and \( A \rightarrow 2A \)) provides a paradigmatic case of ageing with underlying irreversible processes.

Physically, these two kinds of ageing phenomena are quite distinct. In relaxing ferromagnets, see e.g. [2, 3, 4] for reviews, there is for \( T < T_c \) a non-vanishing surface tension between the ordered domains which leads to the formation and growth of or-
dered clusters of linear size \(L = L(t) \sim t^{1/z}\), where \(z\) is the dynamical exponent. For purely relaxational dynamics, it can be shown that \(z = 2\) for \(T < T_c\) whereas for \(T = T_c\), the non-trivial value of \(z\) equals the one found for equilibrium critical dynamics. If \(\phi(t, \vec{r})\) denotes the time- and space-dependent order parameter, it is convenient to characterize the ageing behaviour in terms of the two-time correlation and linear response functions

\[
C(t, s; \vec{r}) = \left\langle \phi(t, \vec{r}) \phi(s, \vec{0}) \right\rangle, \quad R(t, s; \vec{r}) = \left. \frac{\delta \left\langle \phi(t, \vec{r}) \right\rangle}{\delta h(s, \vec{0})} \right|_{h=0}
\]  

(1)

where \(h\) is the magnetic field conjugate to \(\phi\). The autocorrelation and linear autoresponse functions are given by \(C(t, s) = C(t, s; \vec{0})\) and \(R(t, s) = R(t, s; \vec{0})\). Here and later space-translation invariance will be assumed. In phase-ordering, dynamical scaling is found in the regime where

\[
t \gg \tau_{\text{micro}}, \quad s \gg \tau_{\text{micro}} \quad \text{and} \quad t-s \gg \tau_{\text{micro}}
\]

(2)

where \(\tau_{\text{micro}}\) is some microscopic reference time-scale. In the ageing regime \((2)\) the only relevant lengths scales are describes in terms of \(L(t)\) and one expects

\[
C(t, s) = s^{-b} f_C(t/s), \quad R(t, s) = s^{-1-a} f_R(t/s)
\]

(3)

with the asymptotics \(f_{C,R}(y) \sim y^{-\lambda_{C,R}/z}\) for \(y\) large. We stress the importance of the third condition in eq. \((2)\) for the validity of the scaling forms \((3)\). Throughout, the scaling limit \(t, s \to \infty\) with \(y = t/s > 1\) fixed will be implied. While the autocorrelation (autoresponse) exponents \(\lambda_{C,R}\) are new, independent exponents, the exponents \(a, b\) can be explicitly given. At criticality \(T = T_c\), one has \(a = b = (d - 2 + \eta)/z = 2\beta/\nu_z\), where \(\beta, \nu, \eta\) are standard equilibrium critical exponents. For \(T < T_c\), one has \(b = 0\) always. In simple scalar systems with short-ranged equilibrium correlations, such as the Ising or Potts models (with \(d > 1\)), one has \(a = 1/z = 1/2\). In the case of long-ranged equilibrium correlations, \(a\) may be different, i.e. \(a = (d - 2)/z\) in the \(d\)-dimensional spherical model.

On the other hand, in irreversible ageing systems such as the critical voter-model or the critical contact-process, there is no surface tension and the dynamics proceeds through cluster dissolution \([5]\). Still, in the ageing regime \((2)\) one observes again the same formal scaling behaviour eq. \((3)\). However, the exponents \(a\) and \(b\) need no longer be the same even if one considers the ageing at a phase-transition in the non-equilibrium steady-state. For example, there is recent numerical evidence from the critical contact process in both \(1D\) and \(2D\) that \(b = 2\beta/\nu_{\perp} z\) which naturally generalizes the result of critical systems with detailed balance, but \(a = b - 1\) \([6, 7]\).

**GALILEI-INVARINACE IN PHASE-ORDERING KINETICS**

Having reviewed current knowledge on the dynamical scaling of ageing systems, notably on the values of the ageing exponents \(a\) and \(b\), we now consider the scaling functions \(f_{C,R}(y)\) themselves. In particular, we ask whether there exist any generic, model-independent argument which might inform us about the form of the functions \(f_{C,R}(y)\).
In this context, the following general statements about the dynamical symmetries of phase-ordering kinetics can be made.

1. Time-translation invariance is broken.
2. There is dynamical scaling \[2, 3\], i.e. formally the order parameter satisfies the covariance condition

\[ \phi(t, \vec{r}) = \alpha^{x_\phi} \phi(\alpha^2 t, \alpha \vec{r}) \]  

where \( \alpha \) is a constant rescaling factor and \( x_\phi \) an exponent.
3. There is new evidence for Galilei-invariance which we now discuss.

Conventionally, one starts from a coarse-grained order parameter which is assumed to satisfy a Langevin equation, for example the one for model A dynamics \[2\]

\[ 2M \frac{\partial \phi}{\partial t} = \Delta \phi - \frac{dV(\phi)}{d\phi} + \eta \]  

where \( \Gamma = (2M)^{-1} \) is a kinetic coefficient, \( \Delta \) is the spatial Laplacian, \( V(\phi) \) is a typical double-well potential (e.g. \( V(\phi) = (\phi^2 - 1)^2 \)) and \( \eta \) is the thermal gaussian noise with covariance \( \langle \eta(t) \eta(t') \rangle = 2T \delta(t-t') \). In addition, one assumes a gaussian uncorrelated initial state with covariance \( \langle \phi(0, \vec{r}) \phi(0, \vec{0}) \rangle = a_0 \delta(\vec{r}) \). The associated field-theoretic action in the Martin-Siggia-Rose (MSR) formalism is

\[ S[\phi, \tilde{\phi}] = S_0[\phi, \tilde{\phi}] + S_b[\tilde{\phi}] \]  

where \( \tilde{\phi} \) is the response field and

\[ S_0[\phi, \tilde{\phi}] = \int d\vec{r} d\vec{r'} \left[ \tilde{\phi}(2M \frac{\partial}{\partial t} - \Delta) \phi + \phi \frac{\delta V(\phi)}{\delta \phi} \right] \]

\[ S_b[\tilde{\phi}] = -T \int d\vec{r} d\vec{r'} \tilde{\phi}^2(t, \vec{r}) - \frac{a_0}{2} \int d\vec{r} \tilde{\phi}^2(0, \vec{r}) \]  

Then autocorrelators and autoresponses can be found as follows

\[ C(t, s) = \langle \phi(t) \phi(s) \rangle , \ R(t, s) = \langle \phi(t) \tilde{\phi}(s) \rangle \]  

We shall consider as an extension of dynamical scaling the transformations of the so-called Schrödinger group, defined by \[8\]

\[ t \rightarrow \frac{\alpha t + \beta}{\gamma t + \delta} , \ \vec{r} \rightarrow \mathcal{R} \vec{r} + \vec{v} t + \vec{a} \]  

where \( \alpha \delta - \beta \gamma = 1 \) and \( \mathcal{R} \) is a \( d \)-dimensional rotation. Besides translations in time and space, rotations and scale transformations with \( z = 2 \), the Schrödinger group also contains the so-called ‘special’ transformations parametrized by \( \gamma \). The Schrödinger group is the maximal dynamical symmetry group of the free (and also of several non-linear) Schrödinger/diffusion equations and acts projectively (i.e. up to a phase factor) on the wave function \[8, 9\].
The relationship of the Schrödinger group to phase-ordering kinetics can now be formulated in terms of the following three theorems.

We consider an arbitrary space-time infinitesimal coordinate transformation \( \delta r_\mu = \varepsilon_\mu \) with \( \mu = 0, 1, \ldots, d \). Let \( \eta \) stand for the phase picked up by the wave function \( \phi \) under such a transformation. We call a MSR-theory local if the MSR-action \( (6) \) transforms as

\[
\delta S = \int \! dt \! d\vec{r} \left( T_{\mu\nu} \partial_\mu \varepsilon_\nu + J_\mu \partial_\mu \eta \right) + \int_{(t=0)} \! d\vec{r} \left( U_\nu \varepsilon_\nu + V \eta \right)
\]

Here \( T_{\mu\nu} \) is the energy-momentum tensor, \( J_\mu \) a conserved current and the second integral is only over the initial line \( t = 0 \).

**Theorem 1:** \([10]\) For a local MSR-theory, one has

\[
\text{phase-shift invariance} \\
\text{space-translation invariance} \\
\text{scale-invariance with } z = 2 \\
\text{Galilei-invariance}
\]

\[ \implies \text{special Schrödinger invariance} \quad (11) \]

This result is completely analogous to a well-known result in conformal field-theory. We point out that the requirement of time-translation invariance is not required in order to derive the invariance under special Schrödinger transformations for local theories.

To prove this, it suffices to write down the various infinitesimal transformations explicitly. From invariance under phase shifts it follows \( V = 0 \), space-translation invariance implies \( U_1 = \ldots = U_d = 0 \), from scale-invariance it follows \( 2T_{00} + T_{11} + \ldots + T_{dd} = 0 \) and finally Galilei-invariance implies \( T_{0i} + 2 \mathcal{M} J_i = 0 \) for \( i = 1, \ldots, d \). Consequently \( \delta S = 0 \) under special Schrödinger transformations, as asserted. q.e.d.

One may write down the tensor \( T_{\mu\nu} \) and the current \( J_\mu \) explicitly, for free fields this has been done in \([10]\).

We now consider the rôle of Galilei-invariance more closely. It is well-known that a system coupled to a heat bath with a uniform temperature \( T > 0 \) cannot be Galilei-invariant. This can be easily seen from the MSR-action \( (6) \), since the noise terms \( S_\beta[\hat{\phi}] \) are not invariant under a phase-shift \( \phi \rightarrow e^\eta \phi, \hat{\phi} \rightarrow e^{-\eta} \hat{\phi} \). At most, the deterministic part \( S_0 \) of the action might be Galilei-invariant. If that is the case, one has the following Bargman superselection rule

\[
\langle \phi \cdots \phi \check{\phi} \cdots \check{\phi} \rangle_0 \sim \delta_{n,m} \quad (12)
\]

The index 0 refers to the average taken with respect to the deterministic part \( S_0 \) only.

**Theorem 2:** \([11]\) If the deterministic part \( S_0 \) of the MSR-action \( (6) \) is Galilei-invariant such that \((12)\) holds true, then

\[
R(t,s) = R_0^{(2)}(t,s) \\
C(t,s) = \frac{a_0}{2} \int \! d\vec{r} \, R_0^{(3)}(t,s,0;\vec{r}) + T \int \! du \! d\vec{r} \, R_0^{(3)}(t,s,u;\vec{r}) \quad (14)
\]

where the two-point function \( R_0^{(2)}(t,s) = \langle \phi(t)\check{\phi}(s) \rangle_0 \) and the three-point function \( R_0^{(3)}(t,s,u;\vec{r}) = \langle \phi(t,\vec{r})\phi(s,\vec{r})\check{\phi}^2(u,\vec{r}+\vec{y}) \rangle_0 \) are fixed by the deterministic part \( S_0 \).
To see this, one merely has to include the noise term $S_b[\tilde{\phi}]$ into the average, viz. $R = \langle \phi \tilde{\phi} \rangle = \langle \phi \tilde{\phi} e^{-S_b[\tilde{\phi}]} \rangle_0 = \langle \phi \tilde{\phi} \rangle_0$ because of the Bargman superselection rule \[12\]. $C(t,s)$ is found similarly.

Consequently, given the Galilei-invariance of the deterministic part, one can find the contributions of both the thermal and the initial noise. Remarkably, the response function is noise-independent, whereas the correlator vanishes in the absence of noise.

Finally, we have to address the question whether the deterministic part of the Langevin equation (5) can be Galilei-invariant. At first sight, the answer seems to be negative, since a well-known mathematical fact \[9\] states that the non-linear Schrödinger equation

\[
(2mi\partial_t - \Delta) \phi = F(t, \vec{r}, \phi, \phi^*)
\]

is Schrödinger-invariant only for the special potential $F = c(\phi \phi^*)^{2/d} \phi$, where $c$ is a constant. Furthermore, the solutions $\phi$ are necessarily complex. These difficulties can be circumvented by considering the mass $m$ as a new dynamical variable \[12, 10\]. We introduce a new wave function $\psi$ via

\[
\phi(t, r) = \int_{-\infty}^{\infty} d\zeta e^{-im\zeta} \psi(\zeta, t, r)
\]

and look for the symmetries of the new non-linear ‘Schrödinger’-equation

\[
(2\partial_{\zeta} \partial_t - \partial_r^2) \psi = gF(\zeta, t, r, \psi, \psi^*)
\]

where we wrote the coupling constant $g$ of the non-linear term explicitly. Since from dimensional counting, $g$ is in general dimensionful, it should transform under the action of the Schrödinger group. We have systematically constructed all such representations of the Schrödinger Lie algebra and also of its subalgebra when time-translations are left out \[13\]. We then find all invariant non-linear equations of the type (17). The full analysis will be presented elsewhere, here we merely quote one special result.

**Theorem 3:** \[13\] If we write the solutions of (17) $\psi = \psi_g(\zeta, t, r) = g^{(1-2x)/4y} \psi(\zeta, t, r)$, where $x$ is the scaling dimension of $\psi$ and $y$ the scaling dimension of $g$, then the equation

\[
(2\partial_{\zeta} \partial_t - \partial_r^2) \Psi = g^{-5/4y} f \left( g^{1/4y} \psi \right)
\]

is Schrödinger-invariant, where $f$ is an arbitrary function.

We have obtained in this way a formulation of Schrödinger-invariance which allows for real-valued functions and furthermore should be flexible enough to include the double-well potentials which enter into the Langevin equation (5).

**TESTS OF GALILEI-INVARIANCE**

The results quoted above in the theorems 1-3 lead to quantitative predictions which have been successfully tested in simulations. We first consider the response function, which
according to theorem 2 can be found from the assumption of its covariant transformation under the Schrödinger group. This leads to \([14, 11]\)

\[
R(t, s; \vec{r}) = R(t, s) \exp \left( -\frac{\mathcal{M}}{2} \frac{\vec{r}^2}{t-s} \right), \quad R(t, s) = s^{-1-a} f_R(t/s)
\]

\[
f_R(y) = f_0 y^{1+a'} - \lambda_R/s (y-1)^{-1-a'}
\]

(19)

where \(\mathcal{M}\) and \(f_0\) are non-universal constants. In most cases, one has \(a = a'\), but exceptions are known to occur, e.g. in the 1D Glauber-Ising model. The gaussian space-dependence of \(R(t, s; \vec{r})\) is characteristic of Galilei-invariance. Since response functions are very much affected by noise and hence difficult to measure directly, a convenient way of testing (19) is to study the spatio-temporally integrated response

\[
\int_0^s du \int_0^{\sqrt{15s}} dr r^{d-1} R(t, u; \vec{r}) = r_0 s^{d/2-a} \rho^{(2)}(t/s, \mu) + \cdots
\]

(20)

where \(\mu\) is a control parameter, \(r_0\) a constant related to \(f_0\) and the function \(\rho^{(2)}\) can be found explicitly from (19) [14].

In figure 1 we compare the prediction derived from (19) with simulational data [14] in the 2D kinetic Ising model, quenched into its ordered phase and with a purely relaxational heat-bath dynamics. We used the exponents \(z = 2\), \(a = a' = 1/z = 1/2\) and \(\lambda_R = 1.26\). From the plots, one sees a nice collapse of the data obtained for several values of the waiting time \(s\). Finally, the full curve agrees perfectly with the data. Similar results also hold true in the 3D Ising model [14] and also for the 2D three-state Potts model [15]. Furthermore, the prediction (19) can be reproduced in the exactly solvable \(d\)-dimensional spherical and 1D Glauber-Ising models.

We remark that \(R(t, s)\) is well reproduced in the critical 1D contact-process \([6]\).
As a second example, we consider the calculation of the autocorrelator \( C(t,s) \). From theorem 2, this requires the calculation of a noiseless three-point response function. Since Schrödinger-invariance fixes the three-point function only up to an undetermined scaling function, a further argument is needed in order to determine \( C(t,s) \) completely. For models which are described by an underlying free-field theory, it turns out that a simple heuristic idea based on the absence of singularities in \( C(t,s) \) leads to the following simple expression for \( T = 0 \) in the scaling limit:

\[
C(t,s) = f_C(t/s), \quad f_C(y) \approx C_0 \left((y+1)^2/y\right)^{-\lambda c/2}
\]  

(21)

This agrees indeed with the exact solutions of the spherical model, the spin-wave approximation of the XY model, the critical voter model and the free random walk. It can also be checked that the terms coming from thermal noise are irrelevant, as expected from renormalization group arguments.

Finally, for models not described by a free-field theory one can invoke an extension of Schrödinger invariance to a new form of conformal invariance. Then \( C(t,s) \) can be written in terms of hypergeometric functions and this prediction again agrees nicely with simulational data in both the 2D Ising and three-states Potts models.

Summarizing, we have presented evidence, both conceptual and simulational, which strongly suggest that dynamical scaling in phase-ordering kinetics can be extended to the larger Schrödinger group (without time-translations) of local scale-transformations. In particular, it appears that equations of the form used traditionally, allow for a simple symmetry characterization. However, the derivation of equations of the type from a physical argument remains to be understood.

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