Symmetry-prohibited thermalization after a quantum quench

Peter Reimann

Fakultät für Physik, Universität Bielefeld, 33615 Bielefeld, Germany

(Dated: November 3, 2021)

The observable long-time behavior of an isolated many-body system after a quantum quench is considered, i.e., an eigenstate (or an equilibrium ensemble) of some pre-quench Hamiltonian $H$ serves as initial condition which then evolves in time according to some post-quench Hamiltonian $H_p$. Absence of thermalization is analytically demonstrated for a large class of quite common pre- and post-quench spin Hamiltonians. The main requirement is that the pre-quench Hamiltonian must exhibit a $Z_2$ (spin-flip) symmetry, which would be spontaneously broken in the thermodynamic limit, though we actually focus on finite (but large) systems. On the other hand, the post-quench Hamiltonian must violate the $Z_2$ symmetry, but for the rest may be non-integrable and may obey the eigenstate thermalization hypothesis for (sums of) few-body observables.

I. INTRODUCTION

The issue of thermalization in isolated many-body quantum systems has been investigated for almost a century [1,2], leading to fascinating experimental, numerical, and analytical progress in recent years [3-5]. The key question in this context is whether or not the long-time expectation values of experimentally relevant observables are sufficiently well approximated by the corresponding microcanonical expectation values, as predicted by textbook statistical mechanics.

While analytical results are still rather scarce, numerical evidence and heuristic arguments provide quite convincing support for the common expectation that the problem of thermalization is closely connected with the issue of integrability and with the eigenstate thermalization hypothesis (ETH) [3-8]. In fact, thermalization, non-integrability, and the ETH were initially presumed to be largely equivalent, but this simple picture has subsequently been challenged by various counter-examples, among others in Refs. [9-14], though some questions regarding their significance are still open [15-20]. Further prominent counter-examples are supposed to be system exhibiting many-body localization (MBL) [3,4], which in turn have been recently questioned, e.g., in Refs. [21,22].

As far as experimentally relevant observables are concerned, the common consensus is that focusing on few-body operators or sums thereof will be sufficient. As far as the generally out-of-equilibrium initial states are concerned, the most common examples are quantum quenches [3,4,21,22], where an eigenstate (or an equilibrium ensemble) of a so-called pre-quench Hamiltonian serves as initial condition, whose subsequent temporal evolution is governed by some different, so-called post-quench Hamiltonian.

Within this general framework, and notwithstanding the above mentioned counter-examples, the presently prevailing expectation is that thermalization after a quantum quench can be taken for granted if the post-quench system is non-integrable and obeys the ETH. The main objective of the present paper is to analytically show that this is not the case.

II. SETUP

For simplicity, we restrict ourselves to spin-models on a $d$-dimensional hypercubic lattice $\Lambda := \{1, \ldots, L\}^d$ with a large but finite number $|\Lambda| := L^d$ of sites (degrees of freedom) and periodic boundary conditions (some generalizations will be briefly mentioned later). Furthermore, we mainly consider “extensive” (translation invariant) Hamiltonians of the form

$$H = \sum_{i \in \Lambda} H_i ,$$

where the $H_i$ are translational copies of the same local (few-body and short-range) operator which only act non-trivially on lattice sites sufficiently close to $i$. Likewise, we often consider “intensive” observables of the form $A = |\Lambda|^{-1} \sum_{i \in \Lambda} A_i$.

A common example is the transverse-field Ising model (TFIM) with

$$H_i = -J (|i - j|) \sigma^x_i \sigma^x_j ,$$

where $\sigma^x_i$ and $\sigma^z_i$ are Pauli matrices at lattice site $i$, and $| i - j |$ is the natural distance between $i$ and $j$ on the (periodic) lattice. For instance, the interactions $J(|i - j|)$ may be unity if $i$ and $j$ are nearest neighbors, and zero otherwise. A particularly interesting observable is then the longitudinal magnetization

$$M = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \sigma^z_i .$$

III. ILLUSTRATION OF THE MAIN RESULT

Before working out the general theory, it may be instructive to illustrate the main message of the paper by means of some particular examples.

Therefore, let us focus on the following two-dimensional TFIM with periodic boundary conditions...
and nearest neighbor interactions,
\[ H = -\sum_{\langle ij \rangle} \sigma_i^x \sigma_j^x - g \sum_{i \in \Lambda} \sigma_i^z, \]
where the first sum is over all nearest-neighbor sites on the two-dimensional (periodically closed) square lattice \( \Lambda \). It is well-known that for \( g \)-values in the range \( 0 < g < g_c \approx 0.304 \) this model is non-integrable and exhibits a phase transition (spontaneous symmetry breaking) in the thermodynamic limit \( L \to \infty \). Put differently, in the canonical ensemble the order parameter (3) vanishes above some critical temperature \( T_c > 0 \) (which depends on \( g \)), and assumes a finite value below \( T_c \).

Returning to arbitrarily large but finite system sizes \( L \), let us now choose an eigenstate of such a pre-quench Hamiltonian (1) (with \( 0 < g < g_c \)) as initial condition, which then evolves in time according to a post-quench Hamiltonian (on the same two-dimensional lattice \( \Lambda \)) of the general form
\begin{equation}
H_p = H_s + \lambda \sum_{i \in \Lambda} \sigma_i^z, \tag{5}
\end{equation}
\begin{equation}
H_s := -\sum_{\langle ij \rangle} (J_x \sigma_i^x \sigma_j^x + J_y \sigma_i^y \sigma_j^y + J_z \sigma_i^z \sigma_j^z) - h \sum_{i \in \Lambda} \sigma_i^z, \tag{6}
\end{equation}
with largely arbitrary parameters \( \lambda, J_x, J_y, J_z, h \), except that \( \lambda \) must be non-zero. In particular, \( H_s \) may be (but need not be) equal to the pre-quench Hamiltonian (1), while the last term in (5) (with \( \lambda \neq 0 \)) is known to exclude any phase transition for the post-quench system. Moreover, many Hamiltonians of the general form (5), (6) are known or expected to be non-integrable and to obey the ETH [25, 26].

The main result of the present paper is the analytical prediction that in any such post-quench system there exists a large number of pre-quench eigenstates which do not exhibit thermalization after an initial quantum quench as described above.

### IV. PRE-QUENCH SYSTEMS

On top of the assumptions around Eq. (1), the pre-quench Hamiltonian \( H \) is required to exhibit a so-called spin-flip (or \( Z_2 \)) symmetry.

Let us first exemplify the general idea for the TFIM in (1-3). One readily verifies that the Pauli matrices \( \sigma^a_i \) are, for any \( a \in \{x, y, z\} \) and \( i \in \Lambda \), unitary operators, satisfying \( \sigma^a_i \sigma^a_j = -\sigma^a_j \sigma^a_i \). It follows that also \( U := \prod_{i \in \Lambda} \sigma_i^z \) is a unitary operator, satisfying \( \sigma_i^z U \sigma_i^z = -U \sigma_i^z \) and \( U \sigma_i^z U = \sigma_i^z U \) for any \( i \in \Lambda \). With (1-3) this implies
\begin{equation}
U H = H U, \tag{7}
\end{equation}
\begin{equation}
U M = -M U. \tag{8}
\end{equation}

The extension to more general models in terms of spin-operators \( S_i^a \) is straightforward, well-known, and therefore only briefly sketched: Each summand contributing to \( H \) now must consist of a product of factors of the form \( S_i^a \) with \( a \in \{x, y, z\} \) and \( i \in \Lambda \), so that the number of factors with the property \( a \in \{y, z\} \) is even. For instance, in the TFIM from (1), (2) these products are of the form \( S_i^y S_j^y \) with \( S_i^y = \hbar a_i^y/2 \). Analogously, \( M \) may for the moment be any operator with the following property: Each summand contributing to \( M \) must consist of a product of factors \( S_i^a \) so that the number of factors with the property \( a \in \{y, z\} \) is odd, as exemplified by (4).

Denoting \( S^z := \sum_{i \in \Lambda} \sigma_i^z \) and \( U := \exp(i \pi S^z/h) \) implies again that \( U \) is unitary, and by exploiting the Hadamard lemma of the Baker-Campbell-Hausdorff formula one recovers again the key symmetries (7) and (8).

Since \( H \) and \( U \) commute (see (7)), there exists a common set of eigenvectors \( |n\rangle \), and since \( U \) is unitary, all its eigenvalues are of unit modulus. It follows that \( \langle n|U^1 M U|n\rangle = \langle n|M|n\rangle \), while (8) implies \( \langle n|U^1 M U|n\rangle = -\langle n|M|n\rangle \), hence
\begin{equation}
\langle n|M|n\rangle = 0 \tag{9}
\end{equation}
for all \( |n\rangle \). Likewise, \( \langle n|H M|n\rangle \) can be identified with \( \langle n|U^1 H M U|n\rangle \), implying with (7) and (8) that (9) implies
\begin{equation}
\langle n|H M|n\rangle = 0 \tag{10}
\end{equation}
for all pre-quench eigenstates \( |n\rangle \).

Denoting by \( E_n \) the corresponding eigenvalues of \( H \) and by \( \beta_\delta := Z^{-1} e^{-\beta H} \) with \( Z := \text{Tr}\{e^{-\beta H}\} \) the canonical ensemble, the expectation value \( \text{Tr}\{\rho_\beta M\} \) can be rewritten as \( \sum_n p_n \langle n|M|n\rangle \) with \( p_n := Z^{-1} e^{-\beta E_n} \). Eq. (9) thus implies \( \text{Tr}\{\rho_\beta M\} = 0 \), hence the variance of \( M \) in the canonical ensemble takes the form
\begin{equation}
\sigma_{\beta M,L}^2 := \text{Tr}\{\rho_\beta M^2\} = \sum_n p_n \langle n|M^2|n\rangle. \tag{11}
\end{equation}

It is known (see below) or intuitively expected that for many of the above specified spin-lattice models there exists some suitable \( M \), for which the variance in (11) converges in the thermodynamic limit to a non-vanishing value \( \sigma_{\beta L}^2 := \lim_{L \to \infty} \sigma_{\beta M,L}^2 \) below some critical temperature \( T^{-1}_c > 0 \). In this context, \( M \) is then commonly referred to as order parameter, the variance in (11) as its thermal fluctuations, and their finite value for large \( L \) as long-range order, announcing a phase transition via spontaneous symmetry breaking (SSB) in the thermodynamic limit. Here, we adopt these standard notions without recapitulating the well-known underlying physics, since we are actually only interested in the fact that \( \sigma_{\beta M,L}^2 \approx \sigma_{\beta M}^2 > 0 \) for sufficiently large (but finite) \( L \) if \( \beta > \beta_c \).

Indeed, such a behavior of \( \sigma_{\beta M,L}^2 \) has been rigorously derived for a considerable variety of short-range spin-lattice models in \( d \geq 2 \) dimensions, quite often with an order parameter \( M \) which is identical or similar to the magnetization in (3), see e.g. Refs. [24, 36] and further references therein. Incidentally, these rigorous results also cover various generalizations of the model class.
specified around (1), including natural instead of periodic boundary conditions, other lattice geometries, one-dimensional models with long-range interactions, and lattice-gas instead of spin models.

Eq. (11) implies that \( \langle n | M^2 | n \rangle \geq \sigma^2_{M,\beta}/2 \) for at least one eigensate \( |n\rangle \), given \( L \) is sufficiently large and \( \beta > \beta_c \). Focusing on \( \beta \to \infty \), this pertains, in particular, to the ground state of \( \hat{H} \) (or at least to one of them in case of degeneracy). More generally, it seems reasonable to expect that there actually exist many eigensates \( |n\rangle \) for which \( \langle n | M^2 | n \rangle \) exceeds some suitable threshold, for instance \( \sigma^2_{M,\beta}/2 \). The straightforward but somewhat tedious verification of this expectation is worked out in the Appendix, showing that the number of these eigensates is actually exponentially large in \( L \). Numerically, this issue is at (or even beyond) current feasibility limits \(^{27,37,38}\). In particular, the sophisticated numerical explorations in Ref. \(^{38}\) suggest that our above expectations may actually apply even to all eigensates \( |n\rangle \) with sufficiently low energies, e.g., \( E_n < \text{Tr} \{ \rho_{\beta} \hat{H} \} \) for some \( \beta > \beta_c \). Analytically, the same conclusion can also be recovered under the assumption that the diagonal matrix elements \( \langle n | M^2 | n \rangle \) obey the ETH \(^{38}\).

V. POST-QUENCH SYSTEMS

As announced in the introduction, we focus on the most common quantum quench scenario, where the initial condition \( |\psi(0)\rangle \) is given by an eigensate \( |n\rangle \) of the pre-quench Hamiltonian \( \hat{H} \), while the actual dynamics \( |\psi(t)\rangle = e^{-i\hat{H}_p t/\hbar} |n\rangle \) is governed by a different, post-quench Hamiltonian \( \hat{H}_p \). \(^{3,4}\) In particular, we are not interested in the thermalization of the pre-quench system. Moreover, we restrict ourselves to initial states \( |n\rangle \) for which \( \langle n | M^2 | n \rangle \) exceeds some \( L \)-independent threshold value \( \mu \), for instance \( \mu = \sigma^2_{M,\beta}/2 \) (see above). Finally, instead of a pre-quench eigensate we will also consider initial conditions in the form of a pre-quench equilibrium ensemble.

As exemplified by \(^{5}\) and \(^{6}\), the post-quench Hamiltonian is required to be of the form

\[
\hat{H}_p = \hat{H}_s + \lambda \hat{V} , \tag{12}
\]

where \( \hat{H}_s \) is of the same general structure as the Hamiltonians discussed around Eqs. (11) and (17), and thus obeys again symmetry relations (hence the index “s”) analogous to (7) and (10), i.e. \(^{28}\),

\[
\langle n | \hat{H}_s \hat{M} | n \rangle = 0 . \tag{13}
\]

Furthermore, \( \lambda \) must be non-zero and \( L \)-independent, while \( \hat{V} \) is required to be of the form

\[
\hat{V} = |\Lambda| \hat{M} , \tag{14}
\]

where \( \hat{M} \) is the order parameter of the pre-quench system \( \hat{H} \). Hence, \( \hat{V} \) is usually an extensive quantity, as for instance in \(^{3}\) and \(^{4}\).

In general, \( \hat{H}_s \) may but need not agree with the pre-quench Hamiltonian \( \hat{H} \). In particular, and as exemplified by \(^{5}\) and \(^{6}\), \( \hat{H}_s \) may be a Heisenberg-type model \(^{38}\), possibly with some anisotropy (e.g. XXZ- or XY-models) and/or next-nearest-neighbor interactions etc. Moreover, \( \hat{H}_s \) may for instance contain – similarly as in \(^{3} \) – a contribution proportional to \( \sum_{i \in \Lambda} S_i^z \). In fact, since only the property \(^{19}\) will actually be needed below, the post-quench system may even exhibit some disorder, possibly giving rise to MBL.

Importantly, the perturbation \( \hat{V} \) in (12) breaks the symmetry of the unperturbed \( \hat{H}_s \). Hence, the post-quench system \( \hat{H}_p \) generically does not exhibit long-range order, and no SSB and phase transitions will occur in the thermodynamic limit.

Furthermore, for many of the above specified examples, the Hamiltonian \( \hat{H}_p \) is commonly expected to be non-integrable and to obey the ETH \(^{2,3,25,26,37,38}\), though rigorous proofs are usually not available, and even the precise meaning of “integrability” is still not entirely clear \(^{2,4}\). Since these issues are not at the focus of our present paper, we tacitly take for granted those commonly expected properties.

VI. NON-THERMALIZATION

In a first step, the essential arguments are worked out in case the initial condition is given by an eigensate \( |n\rangle \) of the pre-quench Hamiltonian. Then, the modifications for initial conditions in the form of a canonical or microcanonical pre-quench equilibrium ensemble are addressed. Finally, additional details and physical arguments are provided, for simplicity focusing again on pre-quench eigensates.

A. Pre-quench eigenstates

Eqs. (9), (12), and (14) imply

\[
\langle n | \hat{H}_p | n \rangle = \langle n | \hat{H}_s | n \rangle . \tag{15}
\]

Furthermore, we can conclude from Eqs. (12)–(14) that

\[
\langle n | \hat{H}_s^2 | n \rangle = \langle n | \hat{H}_s^2 | n \rangle + \lambda^2 \langle n | \hat{V}^2 | n \rangle . \tag{16}
\]

Combining (15) and (10) yields

\[
\sigma^2_{p,n} = \sigma^2_{s,n} + \lambda^2 \langle n | \hat{V}^2 | n \rangle , \tag{17}
\]

\[
\sigma^2_{p,n} := \langle n | \hat{H}_s^2 | n \rangle - \langle n | \hat{H}_s | n \rangle^2 , \tag{18}
\]

\[
\sigma^2_{s,n} := \langle n | \hat{H}_s^2 | n \rangle - \langle n | \hat{H}_s | n \rangle^2 . \tag{19}
\]

Observing (14) and \( \sigma^2_{s,n} \geq 0 \), we thus obtain

\[
\sigma^2_{p,n} \geq |\Lambda|^2 \lambda^2 \langle n | \hat{M}^2 | n \rangle . \tag{20}
\]

As detailed above (12), \( \langle n | \hat{M}^2 | n \rangle \) is lower bounded by an \( L \)-independent constant on the order of \( \sigma^2_{M,\beta} \), yielding

\[
\sigma_{p,n} \geq c |\Lambda| , \tag{21}
\]
where $c$ is on the order of $|\lambda|\sigma_{\beta}^2$ and independent of the system size $|\Lambda|$

A well-established prerequisite for thermalization is that the system’s energy distribution must be sufficiently narrow, i.e., the energy spread must be small (subextensive) in comparison with the typical (extensive) system energies themselves [3, 4, 23, 40–44]. On the other hand, Eqs. (18) and (21) tell us that the energy spread (standard deviation) $\sigma_{p,n}$ is (at least) extensive in the system size $|\Lambda|$. As a consequence, the system cannot exhibit thermalization.

Before expounding in more detail this very condensed line of reasoning, we next address some modifications of the considered initial states.

### B. Canonical and microcanonical pre-quench ensembles

Instead of a pre-quench eigenstate, let us now turn to initial conditions in the form of a canonical ensemble. As detailed above Eq. (11), we are thus dealing with a mixed initial state (density operator) of the form

$$\rho(0) = \rho_\beta = \sum_{n=1}^{N} p_n |n\rangle \langle n|$$  \hspace{1cm} (22)

with $p_n := Z^{-1} e^{-\beta E_n}$. Similarly as in (15), the post-quench energy expectation value

$$\langle H_p \rangle := \text{Tr} \{ \rho_\beta H_p \}$$  \hspace{1cm} (23)

thus satisfies the relations

$$\langle H_p \rangle = \sum_{n=1}^{N} p_n \langle n|H_p|n\rangle = \sum_{n=1}^{N} p_n \langle n|H_s|n\rangle = \langle H_s \rangle .$$  \hspace{1cm} (24)

Likewise, the corresponding second moment now takes, similarly in (19), the form

$$\langle H_p^2 \rangle = \langle H_s^2 \rangle + \lambda^2 \langle V^2 \rangle .$$  \hspace{1cm} (25)

Concerning the post-quench energy variance

$$\sigma_p^2 := \langle H_p^2 \rangle - \langle H_p \rangle^2$$  \hspace{1cm} (26)

we then can conclude, similarly as in (17)–(20) that

$$\sigma_p^2 \geq |\Lambda| \lambda^2 \langle M^2 \rangle \hspace{1cm} (27)$$

and with (11) that

$$\sigma_p^2 \geq |\Lambda| \lambda^2 \sigma_{M,\beta,L}^2 .$$  \hspace{1cm} (28)

If the temperature $\beta^{-1}$ of the canonical ensemble in (22) is smaller than the critical temperature $\beta_c^{-1}$ (see below (11)), thermalization can thus again be ruled out analogously as below (21).

Closely related to the discussion at the end of Sec. [11] we remark that the usually expected equivalence of ensembles has not been rigorously proven until now for systems which exhibit long-range order [15–17]. Yet it seems reasonable to expect that, at least qualitatively, the mere existence of long-range order would also be recovered in the microcanonical ensemble in cases where it provably occurs in the canonical ensemble [20–36]. If so, one can readily modify the above line of reasoning to show non-thermalization also in cases where the initial condition $\rho(0)$ is given by a microcanonical ensemble with a comitant temperature which is smaller than $\beta_c^{-1}$.

### C. More detailed discussion

A more detailed version of the arguments at the end of Sec. [VI A] is as follows: Denoting the eigenvalues and eigenvectors of $H_p$ by $E_n$ and $|\psi_n\rangle$, a projective measurement of the observable $H_p$ yields the outcome $E_n$ with probability $p_n := |\langle n|\psi\rangle|^2$ for any given (normalized) system state $|\psi\rangle$. On the average over many repetitions of the measurement, the mean value is thus $\sum_n p_n E_n = \langle \psi|H_p|\psi\rangle$ and the second moment $\sum_n p_n E_n^2 = \langle \psi|H_p^2|\psi\rangle$. Hence, $\sigma_{p,n}^2$ in (18) is the variance when repeatedly measuring the (post-quench) system energy $H_p$ in the initial state $|\psi(0)\rangle = |n\rangle$ of the post-quench dynamics. But since the energy is a conserved quantity, $\sigma_{p,n}^2$ is at the same time the energy variance for any later system state $|\psi(t)\rangle$.

Accordingly, the energy spread (standard deviation) is given by $\sigma_{p,n}$ for all times $t$ and thus also in the long-time limit, and scales according to (21) at least linearly with the system size $|\Lambda|$.

As said in the introduction, a necessary condition for thermalization is that the long-time behavior of all relevant (measurable) observables must be well-approximated by the corresponding microcanonical expectation values. One such relevant observable is the system energy $H_p$ (or its “intensive” counterpart in Eq. (51) below). Furthermore, for our system states $|\psi(t)\rangle$ with initial condition $|n\rangle$, the pertinent microcanonical ensemble $\rho_{\text{mc}}$ must satisfy the usual condition

$$E := \text{Tr} \{ \rho_{\text{mc}} H_p \} = \langle n|H_p|n\rangle ,$$  \hspace{1cm} (29)

i.e., the “microcanonical energy window” must be chosen so that the energy of the actual system under consideration is correctly reproduced. Note that in spite of the fact that $|n\rangle$ may be the ground state of the pre-quench Hamiltonian $H$, the corresponding post-quench energy [29] is in general not close to the ground state of the post-quench Hamiltonian $H_p$.

Though the post-quench system is in general not assumed to be at thermal equilibrium, we may nevertheless ask how it would behave at thermal equilibrium. Then, the post-quench system would, essentially by definition, comply with the textbook microcanonical formalism. In
particular, the microcanonical energy variance
\[ \sigma_{\text{mc}}^2 := \text{Tr}(\rho_{\text{mc}}H_p^2) - E^2 \]
(30)
is then known to be – also depending on the actual choice of the microcanonical energy window \[40\] – at most on the order of \([k_B T(E)]^2\), where \(k_B\) is Boltzmann’s constant and \(T(E)\) the microcanonical temperature corresponding to the system energy \(E\) from \(29\). Under any reasonable “upscaling procedure”, the temperature \(T(E)\) is furthermore expected to approach some finite value in the thermodynamic limit \(|\Lambda| \to \infty\) (or at least to grow slower than \(|\Lambda|\)). Hence, the energy spread of the actual system is, according to \(21\), incompatible with the corresponding spread \(\sigma_{\text{mc}}\) at thermal equilibrium \[48\].

Again from a different viewpoint, let us consider the “intensive” observable (energy density)
\[ A := H_p/|\Lambda| , \]
(31)
see also below \[1\]. Given \(A\) is a relevant (experimentally measurable) observable, the same must apply to \(A^2\). (One simply has to square the outcome of each (projective) measurement of \(A\) \[19\].) Exploiting \(18\), \(21\), and \(29\)–\(31\), one readily verifies that
\[ \langle n|A^2|n\rangle - \text{Tr}(\rho_{\text{mc}}A^2) = \frac{\sigma_{p,n}^2 - \sigma_{\text{mc}}^2}{|\Lambda|^2} \geq c^2 - \frac{\sigma_{\text{mc}}^2}{|\Lambda|^2} . \]
(32)
As before, the right hand side generically approaches a positive value for sufficiently large systems, hence the difference on the left hand side is not negligible compared to the the two expectation values themselves. In conclusion, the system does not exhibit thermalization.

As far as the non-thermalization of other observables than the energy density in \[61\] is concerned, obtaining rigorous statements turns out to be rather difficult, while non-rigorous arguments are straightforward and still quite convincing: For simplicity, let us focus on observables \(A\) which satisfy the ETH (non-thermalization in the absence of ETH is quite common anyway). Then, the long-time average of \(\langle \psi(t)|A|\psi(t)\rangle\) is well-known \[3, 38, 40, 41\] to be closely approximated by \(\sum_m \tilde{p}_m A(\tilde{E}_m)\), where \(p_m := |\langle \tilde{m}|n\rangle|^2\) are the populations of the post-quench energy levels \(|\tilde{m}\rangle\) by the initial state \(\psi(0) = |n\rangle\), \(\tilde{E}_m\) are the corresponding post-quench eigenvalues, and \(A(x)\) is a smooth function of its argument \(x\). Likewise, the microcanonical expectation value \(\text{Tr}(\rho_{\text{mc}}A)\) is closely approximated by \(A(E)\), where \(E\) is given by \(29\). A necessary condition for thermalization is that the long-time average must be close to the microcanonical expectation value, i.e., the approximation
\[ \sum_m \tilde{p}_m A(\tilde{E}_m) = A(E) \]
(33)
must be fulfilled very well. However, since the energy distribution of the initial state \(|n\rangle\) is not narrow according to \(18\) and \(21\), the deviations of \(A(\tilde{E}_m)\) from \(A(E)\) are in general not negligible on the left hand side of \(83\), and hence the approximation will not be fulfilled sufficiently well.

In hindsight it may be worthwhile to recall that if a system thermalizes it is understood that every relevant (measurable) quantity must exhibit thermalization. In turn, in order to show non-thermalization it is sufficient that at least one measurable quantity exhibits non-thermalization. As argued above, the energy spread is such a quantity, hence our demonstration of non-thermalization is completed. Nevertheless, it is interesting to consider the behavior of other quantities, as was done in the previous paragraph. Though we were not able to show their non-thermalization with similar rigor as for the energy spread, this does not undermine our already completed (rigorous) demonstration of non-thermalization.

VII. CONCLUSIONS AND COMPARISON WITH RELATED WORKS

A large class of quite common spin-lattice models have been identified which do not exhibit thermalization after a quantum quench. Among them are many examples which are generally considered to be non-integrable and to obey the ETH. The salient point is that the pre-quench system must exhibit long-range order in the canonical ensemble (due to an underlying spin-flip symmetry), hence an analogous property is inherited by exponentially many pre-quench eigenstates (including the ground state), and (presumably) also by the corresponding microcanonical ensemble. Choosing one of these eigenstates or equilibrium ensembles as initial condition, and properly incorporating the concomitant order parameter into the post-quench Hamiltonian, then gives rise to a non-narrow post-quench energy distribution, which is known to be incompatible with the assumption that the system exhibits thermalization in the long run \[3, 4, 23, 40, 41\].

The general setup and reasoning in the present paper are somewhat similar to those in Ref. \[51\], but there are also significant differences. Most importantly, the main focus of Ref. \[51\] is on the implications of long-range order for so-called prethermalization phenomena, not on the possible absence of thermalization in the “true” long-time limit. Moreover, the present paper addresses systems which exhibit long-range order at non-vanishing temperatures, while Ref. \[51\] mainly considers systems where long-range order is present only at zero temperature (corresponding to a quantum phase transition in the thermodynamic limit). Finally, some of the arguments adopted in Ref. \[51\] are less rigorous than in the present paper.

Note that the requirement of a narrow energy distribution in the context of thermalization is well-established. However, it would not be right to say \[51\] that, as a consequence, the requirement must always be fulfilled, nor that the requirement already implies as a more or less ob-
vious consequence the main result of our paper, namely the existence of relevant examples with a non-narrow energy distribution.

Overall, our so-far understanding of thermalization is predominantly based on numerical evidence and non-rigorous arguments \[8, 34\]. The present main result belongs to the still rather scarce analytical statements in this context. The key ingredients for its derivation were symmetry considerations and previously established proofs of canonical long-range order for instance in Refs. \[29, 30\].

Incidentally, a general argument that the energy distribution must be narrow after a quantum quench has been provided in the Supplemental Information of Ref. \[42\], however explicitly assuming “the absence of long-range correlations”. A similar argument in Appendix A of Ref. \[25\] assumes a closely-related, so-called cluster decomposition property. Apparently, both assumptions are not fulfilled by our examples. In this sense, the present paper is complementary to Refs. \[25, 42\] and numerous subsequent works which built on them.

**Acknowledgments**

Invaluable discussions with Lennart Dabelow, Jürgen Schnack, Fabian Essler, and Michael Kastner are gratefully acknowledged. This work was supported by the Deutsche Forschungsgemeinschaft (DFG) within the Research Unit FOR 2692 under Grant No. 355031190, and by the International Centre for Theoretical Sciences (ICTS) during a visit for the program - Thermalization, Many-body localization and Hydrodynamics (Code: ICTS/hydrodynamics2019/11).

**Appendix A: Eigenstates with non-negligible variance of $M$**

Taking for granted that the order parameter $M$ is an intensive observable, as exemplified by \[43\] (see also below \[11\]), there exists some finite upper bound of the operator norm $\|M^2\|$ for all $L$. For instance, \[43\] readily implies that $\|M^2\|$ is upper bounded by $\Lambda^{-2} \sum_{ij \in \Lambda} \|\sigma_i^z \sigma_j^z\|$ and with $\|\sigma_i^z \sigma_j^z\| \leq \|\sigma_i^z\| \|\sigma_j^z\| = 1$ that $\|M^2\| \leq 1$ for all $L$. Without loss of generality we denote by $a$ the smallest such bound of $\|M^2\|$, implying

$$\langle n|M^2|n\rangle \leq a$$  \[(A1)\]

for all $n$ and all $L$.

Since $\sigma_{M,\beta,L}^2$ converges to $\sigma_{M,\beta}^2$ for large $L$ and any given $\beta > \beta_c$ (see below \[11\]), we know that

$$\sigma_{M,\beta,L}^2 \geq (3/4) \sigma_{M,\beta}^2$$  \[(A2)\]

for all sufficiently large $L$.

Finally, for any given $L$ and $\beta > \beta_c$, we denote by $I_{\beta,L}$ the set of all indices $n$ with the property that

$$\langle n|M^2|n\rangle \geq \sigma_{M,\beta,L}^2/2$$  \[(A3)\]

and by $N_{\beta,L}$ the number of elements contained in the set $I_{\beta,L}$. In other words, $N_{\beta,L}$ represents the number of eigenstates $|n\rangle$ with the property \[A3\].

From now on we tacitly restrict ourselves to sufficiently large $L$, so that both \[A1\] and \[A2\] are fulfilled, and to temperatures $\beta^{-1}$ below the critical temperature $\beta_c^{-1}$, so that $\sigma_{M,\beta}^2 > 0$ (see below \[11\]).

Without loss of generality we furthermore assume that the eigenvalues $E_n$ are ordered by magnitude and that the indices $n$ run from 1 to some upper limit $N$ (which is exponentially large in the system size $|\Lambda| = L^d$ for any given spin-model, but which in principle may also be infinite). Accordingly, \[11\] implies

$$\sigma_{M,\beta,L}^2 = \sum_{n=1}^N p_n \langle n|M^2|n\rangle = K_1 + K_2 ,$$  \[(A4)\]

$$K_1 := \sum_{n \in I_{\beta,L}} p_n \langle n|M^2|n\rangle ,$$  \[(A5)\]

$$K_2 := \sum_{n \notin I_{\beta,L}} p_n \langle n|M^2|n\rangle ,$$  \[(A6)\]

$$p_n := \frac{e^{-\beta E_n}}{\sum_{m=1}^N e^{-\beta E_m}} .$$  \[(A7)\]

With \[A1\] it follows that $K_1$ in \[A5\] is upper bounded by $\sum_{n \in I_{\beta,L}} p_n a$, and since $E_1 \leq E_n$ and thus $p_n \leq p_1$ for all $n$ that

$$K_1 \leq \sum_{n \in I_{\beta,L}} p_1 a = p_1 a N_{\beta,L} .$$  \[(A8)\]

On the other hand, since all $n \notin I_{\beta,L}$ satisfy $\langle n|M^2|n\rangle \leq \sigma_{M,\beta,L}^2$ according to \[A3\], we can conclude that

$$K_2 \leq \sum_{n \notin I_{\beta,L}} p_n b_\beta \leq \sum_{n=1}^N p_n b_\beta = b_\beta = \sigma_{M,\beta,L}^2/2 .$$  \[(A9)\]

With \[A1\], \[A8\], \[A9\] it follows that

$$p_1 a N_{\beta,L} \geq K_1 = \sigma_{M,\beta,L}^2 - K_2 \geq \sigma_{M,\beta,L}^2 - \sigma_{M,\beta,L}^2/2$$  \[(A10)\]

and with \[A2\] and \[A7\] that

$$N_{\beta,L} \geq Z \frac{\sigma_{M,\beta,L}^2}{4 a} ,$$  \[(A11)\]

$$Z := \sum_{m=1}^N e^{-\beta (E_m - E_1)} .$$  \[(A12)\]

Disregarding extremely small temperatures $\beta^{-1}$, the number of energies $E_n$ with the property that $\beta (E_n - E_1) \approx 1$ may be expected to grow exponentially with the
system size $L$, hence $Z$ will be exponentially large in $L$ as well. Alternatively, $Z$ in (A12) may be viewed as a canonical partition function with the natural convention that energies are measured relatively to the ground state energy $E_0$. Taking for granted generic thermodynamic properties, it follows that $Z = e^{-\beta F}$, where $F$ is the free energy, and where $\beta F$ is an extensive, negative quantity, implying once again that $Z$ is exponentially large in the system size $L$.

On the other hand, the last factor on the right hand side of (A11) is independent of $L$. Moreover, this quantity is dimensionless, positive, and generically not expected to be extremely small compared to unity (except if the temperature $\beta^{-1}$ is very close to the critical temperature $\beta_c^{-1}$).

Altogether, the number of eigenstates $|n\rangle$, which exhibit a non-negligible variance of $M$ in the sense of (A3), is given by the left hand side of (A11) and is found to grow exponentially with $L$ for any given temperature $\beta^{-1}$ below (and not too close to) $\beta_c^{-1}$.

[1] J. von Neumann, Beweis des Ergodensatzes und des H-Theorems in der neuen Mechanik, Z. Phys. 57, 30 (1929); English translation: R. Tumulka, Proof of the Ergodic Theorem and the H-Theorem in Quantum Mechanics, Eur. Phys. J. H 35, 201 (2010).

[2] S. Goldstein and R. Tumulka, Long-Time Behavior of Macroscopic Quantum Systems: Commentary Accompanying the English Translation of John von Neumann’s 1929 Article on the Quantum Ergodic Theorem, Eur. Phys. J. H 35, 173 (2010).

[3] L. D’Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, From Quantum Chaos and Eigenstate Thermalization to Statistical Mechanics and Thermodynamics, Adv. Phys. 65, 239 (2016).

[4] C. Gogolin and J. Eisert, Equilibration, thermalization, and the emergence of statistical mechanics in closed quantum systems, Rep. Prog. Phys. 79, 056001 (2016).

[5] T. Mori, T. N. Ikeda, E. Kaminishi, and M. Ueda, Thermalization and prethermalization in isolated quantum systems: a theoretical overview, J. Phys. B 51, 112001 (2018).

[6] R. Nandkishore and D. A. Huse, Many-body localization and thermalization in quantum statistical mechanics, Annu. Rev. Condens. Matter Phys. 6, 15 (2015).

[7] M. Ueda, Quantum equilibration, thermalization and prethermalization in ultracold atoms, Nat. Rev. Phys. 2, 669 (2020).

[8] T. Langen, T. Gasenzer, and J. Schmiedmayer, Prethermalization and universal dynamics in near-integrable quantum systems, J. Stat. Mech. 064009 (2016).

[9] M. C. Banuls, J. I. Cirac, and M. B. Hastings, Strong and weak thermalization of infinite nonintegrable quantum systems, Phys. Rev. Lett. 106, 050405 (2011).

[10] N. Shiraishi and T. Mori, Systematic construction of counterexamples to the eigenstate thermalization hypothesis, Phys. Rev. Lett. 119, 030601 (2017).

[11] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papic, Weak ergodicity breaking from quantum many-body scars, Nat. Phys. 14, 745 (2018).

[12] C.-J. Lin and O. I. Motrunich, Exact quantum many-body scar states in the Rydberg-blockaded atom chain, Phys. Rev. Lett. 122, 173401 (2019).

[13] A. A. James, R. M. Konik, and N. J. Robinson, Nonthermal states arising from confinement in one and two dimensions, Phys. Rev. Lett. 122, 130603 (2019).

[14] N. J. Robinson, A. A. James, and R. M. Konik, Signatures of rare states and thermalization in a theory with confinement, Phys. Rev. B 99, 195108 (2019).

[15] H. Kim, M. C. Banuls, J. I. Cirac, M. B. Hastings, and D. A. Huse, Slowest local operators in quantum spin chains, Phys. Rev. E 92, 012128 (2015).

[16] C.-J. Lin and O. I. Motrunich, Quasiparticle explanation of weak-thermalization regime under quench in a nonintegrable quantum spin chain, Phys. Rev. A 95, 023621 (2017).

[17] T. Farrelly, F. G. S. L. Brandano, and M. Cramer, Thermalization and return to equilibrium on finite quantum lattice systems, Phys. Rev. Lett. 118, 140601 (2017).

[18] R. Mondaini, K. Mallayya, L. F. Santos, and M. Rigol, Comment on “Systematic construction of counterexamples to the eigenstate thermalization hypothesis”, Phys. Rev. Lett. 121, 038901 (2018).

[19] N. Shiraishi and T. Mori, Shiraishi and Mori reply, Phys. Rev. Lett. 121, 038902 (2018).

[20] C. J. Turner, A. A. Michailidis, D. A. Abanin, M. Serbyn, and Z. Papic, Quantum scarred eigenstates in a Rydberg atom chain: entanglement, breakdown of thermalization, and stability to perturbations, Phys. Rev. B 98, 155134 (2018).

[21] W. De Roeck and F. Huveneer, Stability and instability towards delocalization in many-body localization systems, Phys. Rev. B 95, 155129 (2017).

[22] J. Suntajs, J. Bonca, T. Prosen, and L. Vidmar, Quantum chaos challenges many-body localization, Phys. Rev. E 102, 062144 (2020).

[23] M. Kiefer-Emmanouilidis, R. Unanyan, M. Fleischhauer, and J. Sirker, Slow delocalization of particles in many-body localized phases, Phys. Rev. B 103, 024203 (2021).

[24] D. Sels and A. Polkovnikov, Thermalization through linked conducting clusters in spin chains with dilute defects, arXiv:2105.09348.

[25] F. H. L. Essler and M. Fagotti, Quench dynamics and relaxation in isolated integrable quantum spin chains, J. Stat. Mech. 064002 (2016).

[26] L. Vidmar and M. Rigol, Generalized Gibbs ensemble in integrable lattice models, J. Stat. Mech. 064007 (2016).

[27] K. R. Fratus and M. Srednicki, Eigenstate thermalization and the emergence of statistical mechanics in closed quantum spin systems with isotropic and nonisotropic interactions, J. Stat. Mech. 064002 (2016).

[28] F. J. Dyson, Existence of a phase-transition in a one-dimensional Ising ferromagnet, Commun. Math. Phys. 12, 91 (1969).

[29] F. J. Dyson, E. H. Lieb, and B. Simon, Phase transitions in quantum spin systems with isotropic and nonisotropic interactions, J. Stat. Phys. 18, 335 (1978).
[31] J. Fröhlich and E. H. Lieb, Phase transitions in anisotropic lattice spin systems, Commun. Math. Phys. 60, 233 (1978).

[32] J. Fröhlich, R. Israel, E. H. Lieb, and B. Simon, Phase transitions and reflection positivity I. General theory and long range lattice models, Commun. Math. Phys. 62, 1 (1978).

[33] J. Fröhlich, R. Israel, E. H. Lieb, and B. Simon, Phase transitions and reflection positivity II. Lattice systems with short-range and Coulomb interactions, J. Stat. Phys. 22, 297 (1978)

[34] T. Kennedy, Long range order in the anisotropic quantum ferromagnetic Heisenberg model, Commun. Math. Phys. 100, 447 (1985)

[35] N. Datta, R. Fernandez, and J. Fröhlich, Low-temperature phase diagrams of quantum lattice models I. Stability for quantum perturbations of classical systems with finitely-many ground states, J. Stat. Phys. 84, 455 (1996).

[36] C. Borgs, R. Kotecky, and D. Ueltschi, Low temperature phase diagrams for quantum perturbations of classical spin systems, Commun. Math. Phys. 181, 409 (1996).

[37] R. Mondaini, K. R. Fratus, M. Srednicki, and M. Rigol, Eigenstate thermalization in the two-dimensional transverse field Ising model, Phys. Rev. E 93, 032104 (2016).

[38] K. R. Fratus and M. Srednicki, Eigenstate thermalization and spontaneous symmetry breaking in the one-dimensional transverse-field Ising model with power-law interactions, arXiv:1611.03992

[39] For one-dimensional models, the pre-quench Hamiltonian $H$ must exhibit long-range (e.g. algebraically decaying) interactions, as mentioned below Eq. (11) and exemplified in Ref. [38].

[40] M. Srednicki, Thermal fluctuations in quantized chaotic systems, J. Phys. A 29, L75 (1996).

[41] M. Srednicki, The approach to thermal equilibrium in quantized chaotic systems, J. Phys. A: Math. Gen 32, 1163 (1999).

[42] M. Rigol, V. Dunjko, and M. Olshanii, Thermalization and its mechanism for generic isolated quantum systems, Nature (London) 452, 854 (2008).

[43] P. Reimann, Foundation of statistical mechanics under experimentally realistic conditions, Phys. Rev. Lett. 101, 190403 (2008).

[44] G. Biroli, C. Kollath, and A. M. Läuchli, Effect of Rare Fluctuations on the Thermalization of Isolated Quantum Systems, Phys. Rev. Lett. 105, 250401 (2010).

[45] H. Touchette, Equivalence and nonequivalence of ensembles: thermodynamic, macrostate, and measure levels, J. Stat. Phys. 159, 987 (2015).

[46] H. Tasaki, On the local equivalence between the canonical and the microcanonical ensembles for quantum spin systems, J. Stat. Phys. 172, 905 (2018).

[47] T. Kuwahara and K. Saito, Gaussian concentration bound and ensemble equivalence in generic quantum many-body systems including long-range interactions, Ann. Phys. 421, 168278 (2020).

[48] When employing a canonical instead of a microcanonical ensemble, the energy spread (standard deviation) is known to scale as $|\Lambda|^{1/2}$, which is again incompatible with [24].

[49] The situation is remarkably similar to the proposition in Refs. [38, 40, 41] that if an observable $A$ obeys the ETH, the same applies to $A^2$.

[50] Viewpoint of an anonymous test reader.

[51] V. Alba and M. Fagotti, Prethermalization at Low Temperature: The Scent of Long-Range Order, Phys. Rev. Lett. 119, 010601 (2017).