PRICING OF HIGH-DIMENSIONAL OPTIONS

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Chapter 1

Preface

Pricing of high dimensional or spread options is one of the oldest and important problems in Mathematical Finance. Such options are important in equity, foreign exchange and commodity markets. Electricity spark spread options are traded over a wide range of markets for exchanging a specific fuel for electricity. A class of spreads which exchanges raw soybeans with a combination of soybean oil and soybean meal is popular in agricultural markets \[19\]. The spread is defined as the instrument \( S_t, t \geq 0 \) whose value at time \( t \) is given by the difference \( S_t = S_{1,t} - S_{2,t}, t \geq 0 \). Buying such a spread is buying \( S_{1,t} \) and selling \( S_{2,t} \). We should not limit ourselves to the case of the spread defined by \( S_t \), and instead we think of \( S_t \) as a price of traded financial instrument.

It is known that pricing of spread options requires models with jumps very different from geometric Brownian motion, and pricing of such options can be challenging \[52\]. We use Lévy processes to model returns.

Known methods for pricing spread options can be divided into two big groups: analytical approximations (approximating formulas) and numerical methods. We shall concentrate on analytical methods which are aimed to develop closed-form formulas to approximate the spread option price. There are two main approaches here: PDE’s and martingales. Experience shows that PDE’s methods are suitable if the dimension is low \[32, 99\]. We shall adapt martingale pricing approach. In this case the price \( V \) of the common spread option at time 0 is given by \( V = \exp (-rT) \mathbb{E}^Q [H] \), where \( H : \mathbb{R}^n \to [0, \infty) \) is the reward (pay-off) function, \( T > 0 \) is maturity time and the expectation is taken with respect to the equivalent martingale measure \( Q \) which corresponds to the chosen model (see Appendixes I and III for more details). In many cases of practical interest \( Q \) admits a density function \( p_Q^T \). Hence, in this case,

\[ V = \exp (-rT) \int_{\mathbb{R}^n} H p_Q^T d\mathbf{x}. \]  

(1.1)

It is important in applications to construct a pricing theory which includes a wide range of reward functions \( H \). In many practical cases the reward function \( H \) grows exponentially. For example, consider a frictionless market with no
arbitrage opportunities and with a constant riskless interest rate \( r > 0 \). Let \( S_t = \{ S_{j,t}, 1 \leq j \leq n, t \geq 0 \} \), be \( n \) asset prices which are modeled by an exponential Lévy processes \( S_{j,t} = S_{j,0} \exp(X_{j,t}) \). A European call option is defined by date \( T \), called the date of maturity, and a number \( K > 0 \), called the strike of exercise price, and it gives the right to its owner to acquire at time \( T \) one unit of the underlying instrument at the unit price \( K \). Assuming that this instrument can be resold for \( S_T \), this means that the owner of the option will receive the payout \( \max\{ S_T - K, 0 \} \) at maturity \( T \). Consider an option on the price spread \( S_{1,T} = S_{1,0} \exp(X_{1,T}) - \sum_{j=2}^{n} S_{j,0} \exp(X_{j,T}) - K, 0 \) at time \( T > 0 \). Clearly \( H(x_1, \cdots, x_n) \sim \exp(x_1), x_1 \to \infty \). Hence the characteristic function \( \Phi^{\mathbb{Q}}(x,T) \) of our model process, which is the Fourier transform of \( p^\mathbb{Q}_t(x) \) must admit an analytic extension into sufficiently wide strip to guarantee convergence of the pricing integral \( \mathbb{Q}_t \). Thus, we say that the model process is adapted to the payoff \( H \) if \( \mathbb{E}^{\mathbb{Q}}[H] < \infty \). This is a very restrictive condition on the model.

Let us discuss now customary used models in the one-dimensional situation. Consider a common frictionless market consisting of a riskless bond and stock which is modeled by an exponential Lévy process \( S_t = S_0 \exp(X_t) \) under a fixed equivalent martingale measure \( \mathbb{Q} \) with a given constant riskless rate \( r > 0 \). Observe that the idea of modeling the option price via a log-normal distribution is due to Samuelson [95]. Since in our model the stock does not pay dividends then the discounted stock price \( \exp(-rt)S_t \) must be a martingale under \( \mathbb{Q} \). Consider a contract (European call option) which gives to its owner the right but not the obligation to buy the underlying asset for the fixed strike price \( K \) at the specified expiry date \( T \). We need to evaluate its price \( V \). In this case the payoff has the form

\[
H(x_1, \cdots, x_n, T) = (S_T - K) \chi_{\{S_T > K\}}
\]

\[
= \max \left\{ S_{1,0} \exp(X_{1,T}) - \sum_{j=2}^{n} S_{j,0} \exp(X_{j,T}) - K, 0 \right\}
\]

at time \( T > 0 \). Clearly \( H(x_1, \cdots, x_n) \sim \exp(x_1), x_1 \to \infty \). Hence the characteristic function \( \Phi^{\mathbb{Q}}(x,T) \) of our model process, which is the Fourier transform of \( p^\mathbb{Q}_t(x) \) must admit an analytic extension into sufficiently wide strip to guarantee convergence of the pricing integral \( \mathbb{Q}_t \). Thus, we say that the model process is adapted to the payoff \( H \) if \( \mathbb{E}^{\mathbb{Q}}[H] < \infty \). This is a very restrictive condition on the model.

In the classical Black-Scholes model [11] the price of a stock follows the Geometric Brownian motion defined as \( S_t = S_0 \exp(X_t) \), where \( X_t, t \geq 0 \) is the Brownian motion with the probability density function

\[
p_{\Delta t}(x) = \left(2\pi \sigma^2 \Delta t\right)^{-1/2} \exp\left(-\frac{(x - \mu \Delta t)^2}{2\sigma^2 \Delta t}\right)
\]

for the increments \( X_{t+\Delta t} - X_t \) and parameters \( \mu \) and \( \sigma \) are known as drift and volatility respectively [14], p. 2. The dynamics for stock prices are given by

\[
ds_t = \mu S_t dt + \sigma S_t dW_t,
\]
where $W_t$ is a standard Brownian motion. This stochastic differential equation can be solved,

$$S_t = S_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right)$$

and the arbitrage free price $V$ at time $t = 0$ of a call option with maturity $T$ and strike price $K$ can be expressed as

$$V = S_0 \Phi (b_1) - K \exp (-rT) \Phi (b_2), \quad (1.3)$$

where

$$b_1 = \frac{\ln (S_0/K) + (r + \sigma^2/2) T}{\sigma T^{1/2}},$$

$$b_2 = \frac{\ln (S_0/K) + (r - \sigma^2/2) T}{\sigma T^{1/2}}$$

and $\Phi$ is the standard Normal cumulative distribution function \[11\]. In this model there exists a unique martingale measure $Q$ which is given by Girsanov theorem presented in Appendix III. See \[71\], \[53\] and \[36\] for more information.

As we can see, only the volatility parameter $\sigma$ appears in $(1.3)$ and the drift $\mu$ term vanishes. There are two common approaches to estimate $\sigma$. The first is based on empirical estimation from historical data. The stock price is observed at fixed time intervals (e.g. every day). Then we calculate the log-returns and estimate $\sigma$ by $s \sqrt{a/2}$, where $s$ is the standard deviation and $a$ is the number of trading days. The second approach is connected with the so-called implied volatility, which is the volatility of the underlying which, when substituted into $(1.3)$ gives a theoretical price equal to the market price. This equation can be solved numerically. If we calculate the implied volatility for different strikes $K$ and expiration times $T$ then we find that the volatility is not constant. The shape of the implied volatility versus $S_T / K$ for a fixed $T$ is called volatility smile.

This phenomenon is a consequence of the fact that the Normal distribution is a poor model for the log-returns \[22\]. Observe that if Black-Scholes’s formula $(1.3)$ were correct the implied volatility would be independent on $T$ and $K$ and equal to the historic volatility $s \sqrt{a/2}$, which is not true in reality. During the past decades the Black-Scholes model was increasingly criticized. Mandelbrot was the first who presented evidence against the log-normal distribution hypothesis \[82\]. Namely, he found that the empirical distribution is more concentrated in the tails and around the origin when compared with the Normal distribution. On the basis of these observations Mandelbrot proposed to consider a class of pure jump processes instead of the continuous Brownian motion.

It is well-known that the Black-Scholes theory becomes much more efficient if additional stochastic factors are introduced. Consequently, it is important to consider a wider family of Lévy processes. Stable Lévy processes have been used first in this context by Mandelbrot \[82\] and Fama \[37\]. From the 90’s Lévy processes became more popular (see e.g. \[83\], \[84\], \[13\], \[14\], \[59\], \[5\], \[7\], \[8\], \[21\] and references therein).
There are several different ways to construct high dimensional Lévy processes. A general method is based on a well-known Lévy-Khintchine formula \((4.1)\) which gives a representation of the characteristic exponent \(\psi\) of any Lévy process \(X_t\) on \(\mathbb{R}^n\). The characteristic function \(\Phi(x, t)\) of any Lévy process on \(\mathbb{R}^n\) can be formally defined as

\[
\Phi(x, t) = \mathbb{E}[\exp(i \langle x, X_t \rangle)] = \exp(-t \psi(x)),
\]

where \(\psi\) is the characteristic exponent of \(X_t\) which is uniquely determined. Then the density function \(p_t\) can be expressed as

\[
p_t(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(-i \langle x, x \rangle - t \psi(x)) \, dx.
\]

According to the Lévy-Khintchine formula, for any Lévy process \(X_t\) the characteristic exponent \(\psi\) admits the representation

\[
\psi(x) = \langle A, x \rangle - i \langle h, x \rangle - \int_{\mathbb{R}^n} (1 - \exp(i \langle \phi, x \rangle) - i \langle \phi, x \rangle \chi_D(x)) \, d\mu(x),
\]

where \(\chi_D\) is the characteristic function of the unit ball in \(\mathbb{R}^n\), \(h \in \mathbb{R}^n\), \(A\) is a symmetric nonnegative-definite matrix and \(d\mu(x)\) is a measure such that

\[
\int_{\mathbb{R}^n} \min\{1, \langle x, x \rangle\} \, d\mu(x) < \infty, \quad \mu(\{0\}) = 0.
\]

The triplet \((A, \mu, h)\) in \((1.4)\) is called the generating triplet (or the Lévy triplet). Selecting different Lévy densities \(\mu\) in the representation \((1.4)\) we get the set of characteristic exponents of Lévy processes (see e.g. [14], p. 200). However, this approach is connected with numerical computation of integrals over manifolds. For instance, a known class of high-dimensional Lévy models is based on so-called KoBoL family which is defined by

\[
\Pi(dx) = \rho^{-\nu-1} \exp(-\lambda(\phi) \rho) \, d\rho d\phi,
\]

where \(d\phi\) is a normalised rotation invariant measure on \(S^{n-1} \subset \mathbb{R}^n\) and \(\lambda\) is a continuous positive function on \(S^{n-1}\). It is possible to show that the associated characteristic exponent \(\psi\) has the form

\[
\psi(x) = -i \langle \mu, x \rangle + \Gamma(-\nu) \int_{S^{n-1}} \left(\lambda^\nu(\phi) - \lambda(\phi) - i \langle A \xi, \phi \rangle \right) d\phi,
\]

if \(\nu \in (0, 1) \cup (0, 2), \mu \in \mathbb{R}^n\) and \(A\) is a positive-definite matrix [14], p. 200. Note that similar models can be obtained if instead of \(S^{n-1} \subset \mathbb{R}^n\) we consider a homogeneous infinitely smooth \(m\)-dimensional (in the sense of the Lebesgue-Brower dimension) Riemannian manifold \(M^m \subset \mathbb{R}^n, m < n\). We shall not discuss here
this line of research. Observe that some very specific approaches of modeling the dependence structure of multivariate Lévy processes were discussed in [24].

We will adapt a general and practical approach which still allows to get explicit approximation formulas for pricing of spread options without involving of numerical methods. This allows application of analytic methods in our analysis. To model return processes we introduce a class of stochastic systems of the form

\[ U_t = X_t + BZ_t, \quad B = (b_{m,k}, 1 \leq m, k \leq n) \]

(1.5)

where \( X_t = (X_{1,t}, \ldots, X_{n,t}) \) and \( Z_t = (Z_{1,t}, \ldots, Z_{n,t}) \) have independent components defined by their characteristic exponents \( \psi^{(1)}_s, 1 \leq s \leq n \) and \( \psi^{(2)}_m, 1 \leq m \leq n \) respectively and \( U_t = (U_{1,t}, \ldots, U_{n,t}) \). The matrix \( B \) reflects the dependence between the processes \( U_{1,t}, \ldots, U_{n,t} \). As a linear combination of Lévy processes \( U_t \) is a Lévy process (see e.g. [96], p. 65) and return process is

\[ S_t = \{S_{j,t} = S_{j,0} \exp(U_{j,t}), 1 \leq j \leq n\}. \]

(1.6)

Empirical studies show that the stock prices are highly correlated (which is modeled by the matrix \( B \)) if the market is in crisis (see e.g. [http://www.economicsofcrisis.com/lit.html] for more information). We give an explicit form of the characteristic function \( \Phi(z, t) \) of \( U_t \),

\[ \Phi(z_1, \ldots, z_n, t) = \Phi(z, t) = \exp(-t\psi(z)), \]

where

\[ \psi(z) := \sum_{s=1}^n \psi^{(1)}_s(z_s) + \sum_{k=1}^n \psi^{(2)}_m \left( \sum_{s=1}^n b_{k,m} z_s \right). \]

(1.7)

We specify sufficient equivalent martingale measure conditions for our model (1.6). Under the equivalent martingale measure \( \mathbb{Q} \) all assets have the same expected rate of return which is a risk free rate \( r \). This means that under no-arbitrage conditions the risk preferences of investors acting on the market do not enter into valuation decisions. It is known [26] that the existence of equivalent martingale measure \( \mathbb{Q} \) is equivalent to the no-arbitrage condition. Remark that \( \mathbb{Q} \) is absolutely continuous with respect to \( \mathbb{P} \), the historic measure inferred from the observations of returns (see Appendix I for details). We show that under equivalent martingale measure condition \( \psi \) must satisfy the condition \( \psi^{\mathbb{Q}}(-ie_s) = -r, 1 \leq s \leq n \), where \( \{e_s, 1 \leq s \leq n\} \) is the standard basis in \( \mathbb{R}^n \). In general, \( \mathbb{Q} \) is not unique. Moreover, the class of equivalent martingale measures is sufficiently large to generate option prices from some dense set of an interval which depends on model parameters. One mathematically tractable choice is the so-called Esscher equivalent measure (see Appendix III for more information). We assume that \( \mathbb{Q} \) has been fixed and all expectations have been calculated with respect to this measure. Also, we shall not be concerned here with the problem of model calibration (see [18] for more information). One-dimensional characteristic exponents \( \psi^{(1)}_s \) and \( \psi^{(2)}_m \) in (1.7) are building blocks
of our model. Selecting different \( \psi_s^{(1)} \) and \( \psi_m^{(2)} \) and \( B = (b_{m,k}) \) in (1.4) we get a wide range of high-dimensional jump-diffusion models.

As a motivating example we consider a popular among practitioners class of models, so-called KoBoL family. Characteristic exponents \( \psi \) of such models have been considered in [13, 14, 15, 16] and can be obtained directly from the one-dimensional Lévy-Khintchine formula (1.4),

\[
\psi (\xi) = -i\mu \xi + c_\mu \Gamma (-\nu) \left( (-\lambda_-) \nu - (-\lambda_- - i\xi) \nu \right)
\]

(1.8)

where \( \nu \in (0, 1), \mu \in \mathbb{R}, c_+, c_- > 0, \lambda_- < 0 < \lambda_+ \) are one-dimensional model parameters. Observe that the parameters \( (\nu, \mu, c_+, c_-, \lambda_+, \lambda_-) \) determine the probability distribution. Larger values of \( \nu \) and \( c_+, c_- \) produce a larger peak of the probability distribution while the parameters \( c_+, c_- \) control asymmetry and \( \lambda_-, \lambda_+ \) determine the rate of exponential decay as \( |\xi| \to \infty \). For our applications is sufficient to notice that the function \( \psi (\xi) \) defined by (1.8) is analytic in the domain \( \mathbb{C} \setminus \{(-i\infty, i\lambda_-) \cup [i\lambda_+, +i\infty]\} \) and

\[
|\Phi (\xi, t)| = |\exp (-t\psi (\xi))| \approx \exp (-C |\xi|\nu),
\]

as \( |\xi| \to \infty \), \( \mathrm{Im}\xi \in (\lambda_-, \lambda_+), \nu \in (0, 1/2) \). Here \( C > 0 \) is an absolute constant since \( \Gamma (-\nu) < 0 \) and \( \cos (\nu \pi / 2) > 0 \) if \( \nu \in (0, 1) \) and \( c_+, c_- > 0 \). Hence applying Cauchy theorem in the strip \( \kappa_- \leq \mathrm{Im}\xi \leq \kappa_+ \), \( \lambda_- < \kappa_- < 0 < \kappa_+ < \lambda_+ \) we get

\[
p_t^Q (x) = M (x) N (x, t), \tag{1.9}
\]

where

\[
M (x) := \frac{1}{2\pi (\exp (\kappa_- x) + \exp (\kappa_+ x))}
\]

and

\[
N (x, t) := \int_\mathbb{R} \exp (-ix\xi) \left( \Phi^Q (\xi - i\kappa_-, t) + \Phi^Q (\xi - i\kappa_+, t) \right) d\xi
\]

is a bounded function on \( \mathbb{R} \). Observe if \( \int_\mathbb{R} M (x) H (x) dx < \infty \) then our model process is adapted to the reward function \( H \). In particular, if \( H \) is European call reward function (1.2) then \( H (x) \sim \exp (x) \), as \( x \to \infty \) and we should assume \( \lambda_+ > 1 \) to guarantee convergence of pricing integral (1.1). In general, if \( \Phi^Q (\xi, t), (\xi, t) \in \mathbb{R} \times \mathbb{R}_+ \) does not admit analytic extension with respect to \( \xi \) then we may apply stationary phase approximation to establish asymptotic for \( p_t^Q (x) \) as \( x \to \infty \) to select admissible reward functions. In the multidimensional settings we have a similar situation. Assume, for simplicity, that characteristic function \( \Phi^Q (z, t), (z, t) \in \mathbb{R}^n \times \mathbb{R}_+ \) admits analytic extension with respect to each variable \( z_k, 1 \leq k \leq n \) into the strips \( \mathrm{Im} z_k \in [-b_k, b_k], \) where \( \lim_{|z_k| \to \infty} |\Phi^Q (z_1, \cdots, z_n, t)| = 0, 1 \leq k \leq n \). Then we show that

\[
p_t^Q (x) = M (x) N (x, t),
\]
where
\[
M(x) = 2^{-2n} \pi^{-n} \left( \prod_{k=1}^{n} \cosh(b_k x_k) \right)^{-1} \tag{1.10}
\]
and \(N(x,t)\) is a bounded function,
\[
N(x,t) = \int_{\mathbb{R}^n} \exp(-i \langle x, z \rangle) \Phi^Q_n(z,t) \, dz,
\]
where the function \(\Phi^Q_n(z,t)\) is defined as,
\[
\Phi^Q_1(z,t) := \Phi^Q(z+ieb_1,t) + \Phi^Q(z-ieb_1,t),
\]
\[
\Phi^Q_k(z,t) := \Phi^Q_{k-1}(z+iebk,t) + \Phi^Q_{k-1}(z-iebk,t), \quad 2 \leq k \leq n.
\]

Our method of reconstruction of density functions \(p^Q_t\) is based on the Poisson summation formula justified by (1.10). Let \(P\) be a truncation parameter. Application of the Poisson summation allows us to construct a periodic extension
\[
\tilde{p}^Q_t(x) \approx \sum_{m \in \mathbb{Z}^n} \Phi^Q\left(-\frac{2\pi}{P}m, t\right) \exp\left(\frac{2\pi i}{P} \langle m, x \rangle\right) \tag{1.11}
\]
of \(p^Q_t\) of the same smoothness as \(p^Q_t\). Observe that the characteristic function \(\Phi^Q\) is known explicitly and \(\Phi^Q(x)\) decays exponentially fast as \(|x| \to \infty\) in many cases of practical interest. Hence the series (1.11) converges absolutely and represents an infinitely differentiable function on \(2^{-1}PQ_n\), where \(Q_n\) is the unit cube in \(\mathbb{R}^n\). For example, if the characteristic exponent \(\psi\) is defined by (1.7) then
\[
\left| p^Q_t(x) - \tilde{p}^Q_t(x) \right| \ll \exp\left(-2^{-1}Pb\right), \quad P \to \infty \quad \text{for any} \quad x \in 2^{-1}PQ_n, \quad \text{where} \quad b \quad \text{is a model parameter.}
\]
Next, we approximate \(\tilde{p}^Q_t\) by the Fourier projection with the spectrum in the domain \(\Omega_1/R \subset \mathbb{R}^n\) defined in the Theorem 25. Observe that \(\Omega_1/R\) has the shape of an exponential hyperbolic cross whose shape depends on model parameters. We show that \(\Omega_1/R\) contains \(m \asymp P^n (\ln R)^{\nu}, \quad R \to \infty\) points with integer components, where \(\nu\) is a model parameter and
\[
\left| p^Q_t(x) - \sum_{m \in \mathbb{Z}^n \cap \Omega_1/R} \Phi^Q\left(-\frac{2\pi}{P}m, t\right) \exp\left(\frac{2\pi i}{P} \langle m, x \rangle\right) \right|
\]
\[
\ll (mP^{-n})^{1-\nu} \exp\left(- (mP^{-n})^{\nu-1}\right), \quad m, P \to \infty
\]
for any \(x \in 2^{-1}PQ_n\).

We give a detailed treatment of the problem of comparison of numerical methods. To show that the method of approximation given by (1.11) is an optimal in the exponential scale we use \(m\)-widths instead of a commonly used tabulation approach. This allows us to compare a wide range of methods of
approximation and reconstruction (including nonlinear). All technical details are presented in Section 5.4 and Appendix IV.

Applying this approach for any concrete model process we can construct almost optimal method of recovery of $p_i^Q(x)$ (which reflects the course of dimensionality).

Final Chapter 6 deals with the problem of option pricing. We give a detailed proof of Hurd-Zhou theorem which is important in our applications. On the basis of this theorem and results developed in Chapter 5 we construct explicit approximation formulas for the price $V$ of a spread option given by (1.1) in the case when all one-dimensional Lévy processes $\psi^{(1)}$ and $\psi^{(2)}$, which define the characteristic exponent $\psi(v)$ in (1.7), are KoBoL processes. Theorem 36 indicates an exponential rate of convergence of approximation formulas presented. A similar analysis is applicable for general jump-diffusion models.

In the Appendices I-IV we collected all necessary results which we use in the text. In Appendix I we introduce $L_p$ spaces, present Fubini and Radon-Nikodym theorems which are important in our applications. In Appendix II we collect fundamental facts from Harmonic Analysis which are useful in Pricing Theory, such as Plancherel, Riesz and Riesz-Thorin theorems. Appendix III introduces martingales and presents two results on martingale conversion, the Doob-Meyer decomposition and Girsanov’s theorem which are important ingredients of the Theory of Pricing. Also, we present basic properties of the set of equivalent martingale measures. Appendix IV contains results on optimal approximation which are important in comparison of numerical algorithms.

The results obtained have been presented and discussed on Applied and Financial Mathematics seminars of the Department of Mathematics, University of Leicester 2010-2014, International Workshop-Radial Basis Functions, 2014, Birkbeck College, University of London, seminar of the Department of Economics, Mathematics and Statistics, Birkbeck College, University of London, 2014, Actuarial Teachers and Researchers Conference 2012, European Numerical Mathematics and Applications-2011, Leicester, Festivals of PhD students, University of Leicester, 2011 and 2012, British Mathematical Colloquium-2011, Leicester, and many other National and International meetings. I thank all the participants of these meetings for providing me with opportunities to talk on my research and to learn from their talks.

This book may be considered as a research report mostly based on results of the author and his colleagues. We review the basic material which is needed and give proofs of new results and of assertions not available in relevant books. In this sense we have tried to present a self-contained treatment, accessible to non-specialists. We hope that the book may now be reasonably free from error, in spite of the mass detail which it contains.
Chapter 2

Remarks on notation

\(\mathbb{N}, \mathbb{Z}, \mathbb{R}\) and \(\mathbb{C}\) are, respectively, the sets of all positive integers, all integers, all real numbers, and all complex numbers. \(\mathbb{Z}_+\) and \(\mathbb{R}_+\) are the collections of nonnegative elements of \(\mathbb{Z}\) and \(\mathbb{R}\), respectively. \(\mathbb{R}^n\) is the \(n\)-dimensional Euclidean space with the canonical basis \(e_1, \ldots, e_n\). Its elements \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\) are vectors with \(n\) real components. The inner product in \(\mathbb{R}^n\) is \(\langle x, y \rangle = \sum_{j=1}^n x_j y_j\); the norm is \(|x| = \left(\sum_{j=1}^n x_j^2\right)^{1/2}\).

\(\mathbb{C}^n\) is the \(n\)-dimensional complex space. Its elements \(z = (z_1, \ldots, z_n)\) are vectors with \(n\) complex components. Similarly we define \(\mathbb{N}^n\) and \(\mathbb{Z}^n\).

For a matrix \(A = (a_{j,k})\), \(A^T = (a_{k,j})\) means its transpose.

Let \(X\) be a vector space over reals. Let \(x_1, \ldots, x_m \in X\). By \(\text{lin} \{x_1, \ldots, x_m\}\) and \(\text{aff} \{x_1, \ldots, x_m\}\) we denote the linear span and affine combination of \(x_1, \ldots, x_m\) respectively. \(\text{lin} \{A, B\}\) means the linear span of \(A, B \subset X\).

For \(A, B \subset X\), \(z \in X\), and \(c \in \mathbb{R}\), \(A + z = \{x + z | x \in A\}\), \(A - z = \{x - z | x \in A\}\), \(cA = \{cx | x \in A\}\), \(-A = \{-x | x \in A\}\), \(A\setminus B = \{x | x \in A \& x \notin B\}\).

Minkovski’s sum and difference of \(A \subset X\) and \(B \subset X\) are defined as

\[
A + B = \{x + y | x \in A, y \in B\} \quad (2.1)
\]

and \(A - B = \{x - y | x \in A, y \in B\}\) respectively. For sets \(A\) and \(B\), \(A \times B\) denotes the Cartesian product.

Let \((X, \varnothing)\) be a metric space. The open ball \(B(x, r)\) of radius \(r > 0\) about \(x \in X\) is the set \(B(x, r) = \{y \in X | \varnothing (x, y) < r\}\). A subset \(U \subset X\) is called open if for every \(x \in U\) there exists an \(r > 0\) such that \(B(x, r) \subset U\). The complement \(X \setminus U\) of an open set \(U\) is called closed. The interior, the closure and the boundary of a set \(U \subset X\) are denoted by \(\text{int} U, \overline{U}\) and \(\partial U\) respectively.

Let \((\Omega, \mathcal{F}, \nu)\) be a measure space. If \(\mathcal{F}\) is the Lebesgue \(\sigma\)-algebra \(\mathcal{L}\) then we write \((\Omega, \mathcal{L}, \nu)\). \(\text{Vol}_n(B)\) is the Lebesgue measure of a set \(B \subset \mathbb{R}^n\). \(\chi_B\) is the indicator function of a set \(B\), that is, \(\chi_B(x) = 1\) for \(x \in B\) and \(0\) for \(x \notin B\).
The abbreviation a.s. denotes almost surely, that is, with probability 1. The abbreviation a.e. denotes almost everywhere, or almost surely, with respect to the Lebesgue measure. Similarly, $\nu$-a.e. denotes almost everywhere, or almost every, with respect to a measure $\nu$.

The symbol $\delta_a$ represents the probability measure concentrated at $a \in \mathbb{R}^n$. If $a = 0$ then we shall write $\delta_a = \delta$. The expression $\nu_1 \ast \nu_2$ represents the convolution of finite measures $\nu_1$ and $\nu_2$; $\nu^{(m)}$ is the $m$-fold convolution of $\nu$.

When $m = 0$, $\nu^{(m)}$ is understood to be $\delta_0$. $C(\mathbb{R}^n)$ be the space of continuous functions on $\mathbb{R}^n$ and $L_p(\mathbb{R}^n)$ be the usual space of $p$-integrable functions $f : \mathbb{R}^n \to \mathbb{R}$ (or $f : \mathbb{R}^n \to \mathbb{C}$) equipped with the norm

$$
\|f\|_p = \|f\|_{L_p(\mathbb{R}^n)} := \begin{cases} 
\left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}, & 1 \leq p < \infty, \\
es^\ast \sup_{x \in \mathbb{R}^n} |f(x)|, & p = \infty.
\end{cases}
$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be an integrable function, $f \in L_1(\mathbb{R}^n)$. Define the Fourier transform

$$
\mathbf{F}(f)(y) = \int_{\mathbb{R}^n} \exp(-i \langle x, y \rangle) f(x) \, dx
$$

and its formal inverse as

$$
\mathbf{F}^{-1}(f)(y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(i \langle x, y \rangle) f(x) \, dx.
$$

$\mathbb{P}(A)$ is the probability of an event $A$. $\mathbb{E}[X]$ is the expectation of a random variable $X$.

$I$ is the identity matrix. $A^T$ and $A^*$ are, respectively, the transpose and the conjugate of a matrix $A$.

Let $X$ be a Banach space and $f$ be a function, $f \in X$. The notation $\|f(\cdot, \alpha)\|_X$ means that we are taking the norm of $f(\cdot, \alpha)$ with respect to the argument denoted by $(\cdot)$. Let $X$ and $Y$ be Banach spaces. The norm $\|A\| := \sup \{ \|Ax\|_Y : \|x\|_X \leq 1 \}$ of a linear operator $A : X \to Y$ is denoted by $\|A\|_{X \to Y}$ and the space of bounded linear operators $A$ is denoted by $L(X,Y)$. Let $X$, $Y$ and $Z$ be Banach spaces $A \in L(X,Y)$ and $B \in L(Y,Z)$ then the composition of $A$ and $B$ is denoted by $B \circ A : X \to Z$.

The expression $f(x) \sim g(x)$ means that $\lim_{x \to \infty} f(x)/g(x) = 1$. We shall write $f(x) \lesssim g(x)$ if $\lim_{x \to \infty} f(x)/g(x) \leq 1$ and $A \approx B_n$, $n \in \mathbb{N}$ if $B_n$ is a sequence of formal approximants to $A$ without regard to any type of convergence.

Different positive universal constants are mostly denoted by the letter $C$. We did not carefully distinguish between the different constants, neither did we try to get good estimates for them. The same letter will be used to denote different universal constants. For the easy of notation we put $a_m \gg b_m$ for two sequences, if $a_m \geq Cb_m$ and $a_m \asymp b_m$ if $C_1b_m \leq a_m \leq C_2b_m$ for all $m \in \mathbb{N}$ and some constants $C$, $C_1$, and $C_2$. Through the text $|a|$ means integer part of $a \in \mathbb{R}$.
Chapter 3

General definitions

3.1 Market and derivative instruments

A *market* is a system of institutions, procedures, social relations and infrastructures where parties engage in exchange. *Market participants* consist of all the *buyers* and *sellers of a good* who influence its *price*. A market allows any tradable item to be evaluated and priced. In general, the structure of a well-functioning market can be approximated as following:

1. Many small buyers and sellers.
2. Buyers and sellers have equal access to information.
3. Products are comparable.

An *investor* is someone who puts money into something with the expectation of a financial return. *Assets* are economic resources, i.e. value of ownership which has a positive economic value and that can be converted into cash. *Finance* is the study of how investors allocate their assets over time under conditions of certainty and uncertainty. A *derivative instrument* is a contract between two parties that specifies conditions under which payments are to be made between the parties. We say that a financial contract is a *derivative security* (or a *contingent claim*) if its value at expiration date $T$ is determined exactly by the market price of the underlying cash instrument at time 0. An *option* (in finance) is a derivative instrument that specifies a contract between two parties for a future transaction on an *asset* (commonly a *stock*, a *bond*, a currency or a futures contract) at a reference price (the *strike*). A *stock* represents the original capital invested in the business by its founders. A *bond* is a negotiable certificate that acknowledges the indebtedness of the bond issuer to the holder. A *forward contract* is an obligation to buy (or sell) an underlying asset at a fixed price (*forward price*) on a known date $T$. A *European call option* on a security $S_t$ is the right to buy the security at a fixed strike price $K$ at the
expiration date $T$. The call option can be purchased for a price $C_t$ (called the premium) at time $t < T$. A *European put option* gives the owner the right to sell an asset at a specified price at expiration $T$. Instead, *American options* can be exercised at any time $0 < t \leq T$. Before the option is first written at time $t$, its value $C_t$ is unknown. That is why it is important to get some estimates of what this price will be if the option is written. Hence, the problem is to get a good approximation for $C_t$ as a function of the underlying assets price and the relevant market parameters. The *bid-ask spread* is the difference between the bid and ask price.

To simplify our model we assume that our market is such that:

1. There are no commissions and fees (the price of an asset in trade is much bigger than commissions and fees).

2. The bid-ask spreads on $S_t$ and $C_t$ are zero (the market is in equilibrium).

With these assumptions we have the following two possibilities. If $S_T \leq K$ (the option is out-of-money) then the option will have no value. Hence, $C_T = 0$. Otherwise, if $S_T > K$ (the option is in-the money) then (since by our assumption, there are no commissions and bid-ask spreads) the net profit will be $C_T = S_T - K > 0$. Joining these possibilities we get $C_T = \max \{S_T - K, 0\} = (S_T - K)^+,$ where

$$(a)^+ := \begin{cases} a, & a > 0, \\ 0, & a \leq 0. \end{cases}$$

A state of nature is said to be *insurable state* when there exists a portfolio which has a non-zero return in that state. For a market where every state is insurable, a price vector can be uniquely determined. Hence, a *complete market* can be defined as a market in which all the contingent claims are attainable.

A complete market can be defined with respect to the concept of a viable financial market. If any strategy which is implemented at the initial time with a zero cost has a zero terminal payoff then we have the absence of *riskless arbitrage opportunities*. A *viable financial market* is defined as a market where there is no profitable riskless arbitrage opportunities. Note that there is an important relationship between arbitrage and the martingale property of securities prices. It means that the best estimation of the future price is derived from the latest information, i.e only the most recent information matters. A financial market is viable iff there is a probability $Q$ which is equivalent to a historical probability $P$, under which the discounted asset prices have the martingale property. We say that a viable market is complete iff there is such a probability $Q$.

### 3.2 The time value of money

The *time value of money* is one of the central concepts in finance theory which states that a unit of currency today is worth more than the same unit of currency
tomorrow due to its potential earning capacity. In other words, £1 paid now is worth more than £1 paid in a year because by depositing £1 in the bank today, one gets more than a pound in a year. Present value (or present discounted value) is a future value of an asset that has been discounted to reflect its value today. Similarly, future value is the value of an asset in the future which is equivalent to a specified sum at present. For a fixed time period \([T_1, T_2]\), interest is the additional gain between the beginning \(T_1\) and the end \(T_2\) of the time period. Present value \(P\) of a future sum \(F\) can be obtained using continuous compound interest rate \(r\) as:

\[
P = F \exp \left( -\int_{T_1}^{T_2} r(t) dt \right).
\]

### 3.3 Arbitrage theorem

All known methods of pricing derivatives employ the notion of arbitrage. An arbitrage can be defined as a way to make guaranteed profit from nothing by selling an asset at time \(T_1\) and then settling accounts at \(T_2\). An existence of arbitrage provides an investment opportunity with infinite rate of return. Hence, investors would try to use arbitrage to make money without putting up anything at time \(T_1\). Consequently, to eliminate this possibility we need to introduce so-called Efficient Market Hypothesis which are essentially are:

1. All known information is reflected on prices of all securities.
2. The current prices are the best estimates of the values of securities.
3. The prices will instantaneously adjust according to any new information.
4. An investor cannot outperform the market price using all known information.

To give an analytic definition of an arbitrage consider a simple model with two time points \(T_1\) and \(T_2\), \(T_1 < T_2\) and zero interest. Let \(a\) be the value of \(S(T)\) at \(T_2\) with probability \(p\) and \(b\) be the value of \(S(T)\) at \(T_2\) with probability \(1 - p\), \(a < b\). By this way we specify \(\mathbb{P}\) on \((\Omega, \mathcal{F})\), where \(\Omega = \{a, b\}\) and \(\mathcal{F} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}\). Hence, we get a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Consider a portfolio \((N, MS(T))\) consisting of \(N\) units of money and \(M\) units of stocks. The value \(V(T)\) of this portfolio at \(T_1\) is \(V(T_1) = N + MS(T_1)\) and at \(T_2\) is \(V(T_2) = N + MS(T_2)\). We say that there exists an arbitrage opportunity if there exists a portfolio \((N, MS(T))\) such that \(V(T_1) = 0\), \(V(T_2) \geq 0\) and \(\mathbb{P}(V(T_2) > 0) > 0\). It is possible to show that there exist no arbitrage opportunities iff \(a < S(T_1) < b\) [35].

**Theorem 1** (Fundamental Theorem of Asset Pricing) There exist no arbitrage opportunities iff there exist a probability measure \(\mathbb{Q}\) equivalent to the
original probability measure $\mathbb{P}$ such that the stock price process $(S(T_1), S(T_2))$ satisfies $\mathbb{E}^Q [S(T_2) | S(T_1)] = S(T_1)$.

A probability measure $Q$ is called an equivalent martingale measure. Observe that Theorem 1 explicitly relates a fundamental notion of arbitrage to a far advanced theory of martingales. In the case of multi-period model we have a similar result [104].

**Theorem 2** There are no arbitrage opportunities in the multi-period model iff for every $t$, the one-period model $(S_t, S_{t+1})$, with respect to the filtration $(\mathcal{F}_t, \mathcal{F}_{t+1})$, admits no arbitrage opportunities.

See Appendix III for more information. Consider the case of continuous-time settings (see, e.g. [104]).

**Definition 3** A probability measure $Q$ on a measure space $(\Omega, \mathcal{F})$ is called equivalent martingale measure if it is equivalent to $\mathbb{P}$ and $S_t$ is a martingale with respect to $Q$. The collection of all equivalent martingale measures on the measure space $(\Omega, \mathcal{F})$ is denoted by $\mathcal{M}_S(\Omega, \mathcal{F})$.

The change of measure spaces $(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\Omega, \mathcal{F}, Q)$ is based on the Radon-Nikodim theorem (see Appendix I, Theorem 41).

**Theorem 4** There are no arbitrage opportunities iff there exists an equivalent martingale measure.

The proof of this statement is based on the Hahn-Banach theorem for locally convex topological vector spaces and Banach-Alaoglu theorem which we shall not discuss here.

If a probability measure $\mathbb{P}$ is estimated using historical return data for the underlying stock, the measure is referred to as the market measure (or the physical measure, or historical measure). Asset prices are modeled by stochastic processes $(S_t)_{t \geq 0}$ whose evolutions are determined by a fixed probability measure. In the theory of arbitrage pricing there exists a risk neutral probability measure under which asset prices are arbitrage free. The absence of arbitrage is equivalent to the existence of a risk neutral equivalent martingale measure $Q$ for $(S_t)_{t \geq 0}$ making the underlying process become a martingale. Under the equivalent martingale measure all assets have the same expected rate of return which is the risk free rate. It means that under no-arbitrage conditions the risk preferences of investors acting on the market do not enter into valuation decisions [27]. For a general overview on financial derivatives from a mathematical and an economic point of view we refer to [50, 17, 35, 36].
Chapter 4

Lévy processes and characteristic exponents

4.1 Introduction

In this section we present important for our applications properties of Lévy processes on $\mathbb{R}^n$. We introduce a class of stochastic systems to model return processes which will be studied in the later chapters and develop sufficient equivalent martingale measure conditions for such kind of models. The building blocks of our model are one-dimensional processes. To make our results more specific, we assume that all one-dimensional components are KoBoL processes with different parameters. For such kind of processes we give a complete proof for the representation of characteristic exponent for a particular choice of parameters. We introduce the notion of $(\lambda_-, \lambda_+)$-analyticity which is a useful tool in study density functions. It is shown that any KoBoL process of order $\nu \in (0, 1/2)$ is $(0, \lambda_+)$-analytic. It allows us to consider a general class of contour deformations in representations of density functions. Observe that a very specific class of contour deformations has been considered in [15].

4.2 Basic results

We start with basic definitions and results. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a triplet of a set $\Omega$, an admissible family $\mathcal{F}$ of subsets, $\mathcal{F} \subset \{\emptyset, 2^\Omega\}$ and a mapping $\mathbb{P} : \mathcal{F} \rightarrow [0, 1]$ such that

1. $\Omega \in \mathcal{F}$ and $\emptyset \in \mathcal{F}$.
2. If $A_n \in \mathcal{F}$ for any $n \in \mathbb{N}$, then
\[
\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}, \quad \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}.
\]

3. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.

4. $0 \leq \mathbb{P}(A) \leq 1$, $\mathbb{P}(\Omega) = 1$, and $\mathbb{P}(\emptyset) = 0$.

5. If $A_n \in \mathcal{F}$ for any $n \in \mathbb{N}$ and $A_n \cap A_m = \emptyset$, $\forall n, m \in \mathbb{N}$, $n \neq m$, then
\[
\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).
\]

A family $\mathcal{F} \subset \{\emptyset, 2^\Omega\}$ satisfying 1, 2 and 3 is called a $\sigma$-algebra and a mapping $\mathbb{P}$ with the properties 4 and 5 is called a probability measure. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{B}(\mathbb{R}^n)$ be the collection of all Borel sets on $\mathbb{R}^n$ which is the $\sigma$-algebra generated by all open sets in $\mathbb{R}^n$, i.e. the smallest $\sigma$-algebra that contains all open sets in $\mathbb{R}^n$. A real valued function is called measurable (Borel measurable) if it is $\mathcal{B}(\mathbb{R}^n)$ measurable. A mapping $X : \Omega \rightarrow \mathbb{R}^n$ is an $\mathbb{R}^n$-valued random variable if it is $\mathcal{F}$-measurable, i.e. for any $B \in \mathcal{B}(\mathbb{R}^n)$ we have $\{\omega | X(\omega) \in B\} \in \mathcal{F}$. A stochastic process $X = \{X_t\}_{t \in \mathbb{R}^+}$ is a one-parametric family of random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The trajectory of the process $X$ is a map
\[
\mathbb{R}^+ \rightarrow \mathbb{R}^n
\]
\[
t \mapsto X_t(\omega),
\]
where $\omega \in \Omega$ and $X_t = (X_{1,t}, \ldots, X_{n,t})$. For a fixed $0 \leq t_0 < t_1 < \cdots < t_m$, $m \in \mathbb{N}$ and Borel measurable sets $B_k \subset \mathbb{R}^n$, $0 \leq k \leq m$ consider the map
\[
\mathcal{B}(\mathbb{R}^{mn}) \rightarrow \mathbb{R}^+
\]
\[
\prod_{1 \leq k \leq m} B_k \mapsto \mathbb{P}(X_{t_1} \in B_1, \ldots, X_{t_m} \in B_m),
\]
which defines a probability measure on $\mathcal{B}(\mathbb{R}^{mn})$. The system of finite-dimensional distributions of $X$ is the family of all such measures over all choices $0 \leq t_0 < t_1 < \cdots < t_m$, $m \in \mathbb{N}$. Two stochastic processes $X$ and $Y$ are identical in law, written as $X \overset{d}{=} Y$ (or $X = Y$ mod (law)) if the systems of their finite-dimensional distributions are identical.

Consider the $\sigma$-algebra $\mathcal{F}$ generated by the cylinder sets, known as Kolmogorov's $\sigma$-algebra.

**Theorem 5** (Kolmogorov's extension theorem) Suppose that for any $0 \leq t_1 \leq \cdots \leq t_m$ and $m \in \mathbb{N}$ a distribution $\nu_{t_1, \ldots, t_m}$ is given. If for any $B_1, \ldots, B_m \in \mathcal{B}(\mathbb{R}^n)$ we have
\[
\nu_{t_1, \ldots, t_m} \left( \prod_{s=1}^{m} B_s \right) = \nu_{t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_m} \left( \prod_{1 \leq s \leq m, s \neq k} B_s \right),
\]
\[
B_k = \mathbb{R}^n
\]
then there exists a unique probability measure \( P \) on \( F \) that has \( \{ \nu_{t_1}, \ldots, t_m \} \) as its system of finite-dimensional distributions.

Different proofs of this statement can be found in [58] and [16].

\( X = \{ X_t \}_{t \in \mathbb{R}_+} \) is called a \( \text{Lévy process} \) (process with stationary independent increments) if

1. The random variables \( X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_m} - X_{t_{m-1}}, \) for any \( 0 \leq t_0 < t_1 < \cdots < t_m \) and \( m \in \mathbb{N} \) are independent (independent increment property).
2. \( X_0 = 0 \) a.s.
3. The distribution of \( X_{t+\tau} - X_t \) is independent of \( \tau \) (temporal homogeneity or stationary increments property).
4. It is stochastically continuous, i.e.

\[
\lim_{\tau \to t} P[|X_\tau - X_t| > \epsilon] = 0
\]

for any \( \epsilon > 0 \) and \( t \geq 0 \).
5. There is \( \Omega_0 \in F \) with \( P(\Omega_0) = 1 \) such that, for any \( \omega \in \Omega_0 \), \( X_t(\omega) \) is right-continuous on \( [0, \infty) \) and has left limits on \( (0, \infty) \).

A process satisfying 1-4 is called a \( \text{Lévy process in law} \). An additive process is a stochastic process which satisfies 1,2,4,5 and an additive process in law satisfies 1,2,4.

Let \( x, y \in \mathbb{R}^n \), \( x = (x_1, \ldots, x_n) \), \( y = (y_1, \ldots, y_n) \), \( \langle x, y \rangle \) be the usual scalar product in \( \mathbb{R}^n \), i.e.

\[
\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k \in \mathbb{R}
\]

and \( |x| := \langle x, y \rangle^{1/2} \). Let \( C(\mathbb{R}^n) \) be the space of continuous functions on \( \mathbb{R}^n \) and \( L_p(\mathbb{R}^n) \) be the usual space of \( p \)-integrable functions \( f : \mathbb{R}^n \to \mathbb{R} \) (or \( f : \mathbb{R}^n \to \mathbb{C} \)) equipped with the norm

\[
\|f\|_p = \|f\|_{L_p(\mathbb{R}^n)} := \left\{ \begin{array}{ll}
\left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}, & 1 \leq p < \infty, \\
\text{ess sup}_{x \in \mathbb{R}^n} |f(x)|, & p = \infty.
\end{array} \right.
\]

For a finite measure \( \nu \) on \( \mathbb{R}^n \) (i.e. if \( \nu(\mathbb{R}^n) < \infty \)) we define its Fourier transform as

\[
\mathbf{F}(\nu)(y) = \int_{\mathbb{R}^n} \exp(-i \langle x, y \rangle) \, \nu(dx)
\]

and its formal inverse

\[
\nu(dx) = \mathbf{F}^{-1} \circ \mathbf{F}(\nu)(dy) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp(i \langle x, y \rangle) \mathbf{F}(\nu)(y) \, dy.
\]
The convolution $v = v_1 * v_2$ of two measures $v_1$ and $v_2$ on $\mathbb{R}^n$ is defined as

$$v(B) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \chi_B(x + y) v_1(dx) v_2(dy) < \infty,$$

where

$$\chi_B(x) := \begin{cases} 1, & x \in B, \\ 0, & x \notin B \end{cases}$$

is the characteristic function of a Borel (Lebesgue) measurable set $B \subset \mathbb{R}^n$. A probability measure $v$ is called infinitely divisible if for any $m \in \mathbb{N}$ there is a probability measure $v^{(m)}$ such that

$$v = v^{(m)} * \cdots * v^{(m)}.$$

It is known that if $v$ is infinitely divisible then there exists a unique continuous function $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$ such that $\phi(0) = 0$ and $\exp(\phi(y)) = F(v)(y)$ (see, e.g. [96], p. 37).

The characteristic function $\Phi(x,t)$ of the distribution of $X_t$ of any Lévy process can be formally defined as

$$\Phi(x,t) := \mathbb{E}[\exp(i\langle x, X_t \rangle)] = \exp(-t\psi(x)) = \left(2\pi\right)^{-n/2} \int_{\mathbb{R}^n} \exp(-i\langle x, y \rangle - t\psi(y)) d\nu_t,$$

where $p_t(x)$ is the density function of $X_t$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$ and the function $\psi(x)$ is uniquely determined. This function is called the characteristic exponent. Vice versa, a Lévy process $X = \{X_t\}_{t \in \mathbb{R}_+}$ is determined uniquely by its characteristic exponent $\psi(x)$. In particular, $p_t$ can be expressed as

$$p_t(\cdot) = \left(2\pi\right)^{-n/2} \int_{\mathbb{R}^n} \exp(-i\langle \cdot, x \rangle - t\psi(x)) d\nu_t,$$

We say that a matrix $A$ is nonnegative-definite (or positive-semidefinite) if $x^*Ax \geq 0$ for all $x \in \mathbb{C}^n$ (or for all $x \in \mathbb{R}^n$ for the real matrix), where $x^*$ is the conjugate transpose. A matrix $A$ is nonnegative-definite if it arises as the Gram matrix of some set of vectors $v_1, \cdots, v_n$, i.e. $A = (v_{i,j}) = (v_j, v_i)$.

The following classical result, plays a key role in our analysis.

**Theorem 6** (Lévy-Khintchine formula) Let $X = \{X_t\}_{t \in \mathbb{R}_+}$ be a Lévy process on $\mathbb{R}^n$. Then its characteristic exponent admits the representation

$$\psi(y) = -\frac{1}{2} \langle Ay, y \rangle - i \langle h, y \rangle - \int_{\mathbb{R}^n} (1 - \exp(i\langle y, x \rangle) - i\langle y, x \rangle \chi_D(x)) \Pi(dx),$$

(4.1)
where \( \chi_D(x) \) is the characteristic function of \( D := \{ x \in \mathbb{R}^n, |x| \leq 1 \} \), \( A \) is a symmetric nonnegative-definite \( n \times n \) matrix, \( h \in \mathbb{R}^n \) and \( \Pi(dx) \) is a measure on \( \mathbb{R}^n \) such that

\[
\int_{\mathbb{R}} \min\{1, |x|^2\} \Pi(dx) < \infty, \quad \Pi(\{0\}) = 0.
\]

(4.2)

The density of \( \Pi \) is known as the Lévy density and \( A \) is the covariance matrix. In particular, if \( A = 0 \) (or \( A = (a_{j,k})_{1 \leq j,k \leq n}, a_{j,k} = 0 \)) then the Lévy process is a pure non-Gaussian process and if \( \Pi = 0 \) the process is Gaussian.

**Definition 7** We say that the Lévy process has bounded variation if its sample paths have bounded variation on every compact time interval.

A Lévy process has bounded variation iff \( A = 0 \) and

\[
\int_{\mathbb{R}^n} \min\{|x|, 1\} \Pi(dx) < \infty, \quad \Pi(\{0\}) = 0
\]

(see e.g. [10], p. 15).

The systematic exposition of the theory of Lévy processes can be found in [43, 44, 45, 96, 1, 87].

### 4.3 A class of stochastic systems

In this section we introduce a class of stochastic systems to model multidimensional return processes. Let \( X_{1,t}, \cdots, X_{n,t} \) and \( Z_{1,t}, \cdots, Z_{n,t} \) be independent random variables, with the density functions \( p^{(1)}_1(x_1), \cdots, p^{(1)}_n(x_n) \) and \( p^{(2)}_1(x_1), \cdots, p^{(2)}_n(x_n) \) and characteristic exponents \( \psi^{(1)}_s \) and \( \psi^{(2)}_m, 1 \leq s, m \leq n \) respectively. Let \( X_t = (X_{1,t}, \cdots, X_{n,t})^T, Z_t = (Z_{1,t}, \cdots, Z_{n,t})^T \) and \( B = (b_{j,k}) \) be a real matrix of size \( n \times n \). Consider random vector \( U_t = (U_{1,t}, \cdots, U_{n,t})^T \),

\[
U_t = X_t + BZ_t.
\]

(4.3)

A matrix \( B \) reflects dependence between the processes \( U_{1,t}, \cdots, U_{n,t} \) in our model. Assume for simplicity that \( \mathbb{E}[X_{s,t}] = 0 \) and \( \mathbb{E}[Z_{s,t}] = 0, 1 \leq s \leq n \), \( \text{var}(X_{s,t}) = \text{var}(Z_{s,t}) = v_t \) and \( b_{s,k} = 1, 1 \leq s, k \leq n \). It is easy to check that for any \( s \) and \( l, 1 \leq s \neq l \leq n \) the correlation coefficient

\[
\rho(U_{s,t}, U_{l,t}) := \frac{\mathbb{E}[U_{s,t}U_{l,t}]}{\left( \mathbb{E}[U_{s,t}^2] \mathbb{E}[U_{l,t}^2] \right)^{1/2}}
\]

between \( U_{s,t} \) and \( U_{l,t} \), where

\[
U_{s,t} = X_{s,t} + \sum_{k=1}^{n} b_{s,k} Z_{k,t}, U_{l,t} = X_{l,t} + \sum_{k=1}^{n} b_{l,k} Z_{k,t}
\]
is $\rho(U_{s,t}, U_{l,t}) = n(n + 1)^{-1}$. This reflects our empirical experience: if the market is in crisis then the prices of stocks are highly correlated (see http://www.economicsofcrisis.com/lit.html for more information).

The next statement gives us an explicit form of the characteristic function of the return process $U_t$.

**Theorem 8** Let $U_t = X_t + BZ_t$, $B = (b_{m,k})$. Then in our notation the characteristic function $\Phi(v,t)$ of $U_t$ has the form

$$
\Phi(v,t) = (2\pi)^n \left( \prod_{s=1}^{n} F^{-1} \left( p_{(1)}^{(s,t)} \right) \right)(v) F^{-1} \left( \prod_{m=1}^{n} p_{(2)}^{(m,t)} \right) (B^T v),
$$

where

$$
\psi(v) = \sum_{s=1}^{n} \psi_{(1)}^{(s)}(v_s) + \sum_{m=1}^{n} \psi_{(2)}^{(m)} \left( \sum_{k=1}^{n} b_{k,m} v_k \right).
$$

**Proof.** Consider the transformation $\mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ defined as

$$
U_t = X_t + BZ_t,
$$

$$
Z_t = Z_t.
$$

The inverse is given by

$$
X_t = U_t - BZ_t,
$$

$$
Z_t = Z_t,
$$

or

$$
\begin{pmatrix}
X_t \\
Z_t
\end{pmatrix} =
\begin{pmatrix}
I & -B \\
0 & I
\end{pmatrix}
\begin{pmatrix}
U_t \\
Z_t
\end{pmatrix}
$$

and the Jacobian $J$ of this transformation is

$$
J = \det \left( \begin{pmatrix}
I & -B \\
0 & I
\end{pmatrix} \right) = 1,
$$

where $I = I_{n \times n}$ is an identity. The density function

$$
\tilde{p}_t(u,z) = \tilde{p}_t(u_1, \cdots, u_n, z_1, \cdots z_n)
$$

is given by

$$
\tilde{p}_t(u,z) = \prod_{s=1}^{n} p_{(1)}^{(s,t)} \left( u_s - \sum_{m=1}^{n} b_{s,m} z_m \right) \prod_{l=1}^{n} p_{(2)}^{(l,t)}(z_l).
$$

This means that the density function $p_t(u)$ of $U_t$ is

$$
p_t(u) = \int_{\mathbb{R}^n} \tilde{p}_t(u,z) dz
$$

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and the characteristic function has the form

\[
\Phi(v,t) := \mathbb{E} [\exp (i \langle U_t, v \rangle)] := \exp (-t \psi(v))
\]

\[
= \int_{\mathbb{R}^n} \exp (i \langle u, v \rangle) p_t(u) \, du
\]

\[
= \int_{\mathbb{R}^n} \exp (i \langle u, v \rangle) \left( \int_{\mathbb{R}^n} \tilde{p}_s(u,z) \, dz \right) \, du
\]

\[
= \int_{\mathbb{R}^n} \exp (i \langle u, v \rangle) \left( \int_{\mathbb{R}^n} \prod_{s=1}^n p_{s,t}(u_s - \sum_{m=1}^n b_{s,m} z_m) \prod_{m=1}^n p_{m,t}^{(2)}(z_m) \, dz \right) \, du
\]

\[
= \int_{\mathbb{R}^n} \prod_{s=1}^n \int_{\mathbb{R}} p_{s,t}^{(1)}(u_s - \sum_{m=1}^n b_{s,m} z_m) \exp (i u_s v_s) \, du_s \prod_{m=1}^n p_{m,t}^{(2)}(z_m) \, dz \tag{4.4}
\]

In the last line we applied Fubini theorem (see Appendix I, Theorem 39 Fubini’s theorem). Let \( \xi_s = u_s - \sum_{m=1}^n b_{s,m} z_m, 1 \leq s \leq n. \) Then

\[
\int_{\mathbb{R}} p_{s,t}^{(1)}(u_s - \sum_{m=1}^n b_{s,m} z_m) \exp (i u_s v_s) \, du_s
\]

\[
= \int_{\mathbb{R}} p_{s,t}^{(1)}(\xi_s) \exp \left( i \left( \xi_s + \sum_{m=1}^n b_{s,m} z_m \right) v_s \right) \, d\xi_s
\]

\[
= \exp \left( iv_s \sum_{m=1}^n b_{s,m} z_m \right) \int_{\mathbb{R}} p_{s,t}^{(1)}(\xi_s) \exp (i \xi_s v_s) \, d\xi_s
\]

\[
= \exp \left( iv_s \sum_{m=1}^n b_{s,m} z_m \right) 2\pi F^{-1} \left( p_{s,t}^{(1)}(v_s) \right) \tag{4.5}
\]

Comparing (4.4) and (4.5) we get

\[
\Phi(v,t) = \int_{\mathbb{R}^n} \prod_{s=1}^n \exp \left( iv_s \sum_{m=1}^n b_{s,m} z_m \right) 2\pi F^{-1} \left( p_{s,t}^{(1)}(v_s) \right) \prod_{m=1}^n z_{m,t}(z_m) \, dz
\]

\[
= \prod_{s=1}^n 2\pi F^{-1} \left( p_{s,t}^{(1)}(v_s) \right) \int_{\mathbb{R}^n} \prod_{s=1}^n \exp \left( iv_s \sum_{m=1}^n b_{s,m} z_m \right) \prod_{m=1}^n p_{m,t}^{(2)}(z_m) \, dz
\]

\[
= \prod_{s=1}^n 2\pi F^{-1} \left( p_{s,t}^{(1)}(v_s) \right) \int_{\mathbb{R}^n} \exp \left( i \sum_{s=1}^n v_s \sum_{m=1}^n b_{s,m} z_m \right) \prod_{m=1}^n p_{m,t}^{(2)}(z_m) \, dz
\]

\[
= \prod_{s=1}^n 2\pi F^{-1} \left( p_{s,t}^{(1)}(v_s) \right) \int_{\mathbb{R}^n} \exp (i \langle v, Bz \rangle) \prod_{m=1}^n p_{m,t}^{(2)}(z_m) \, dz
\]
\[
\prod_{s=1}^{n} 2\pi \mathcal{F}^{-1} \left( p^{(1)}_{s,t}(v_s) \right) \int_{\mathbb{R}^n} \exp \left( i \langle B^T v, z \rangle \right) \left( \prod_{m=1}^{n} p^{(2)}_{m,t}(z_m) \right) \, dz
\]

\[
= \prod_{s=1}^{n} 2\pi \mathcal{F}^{-1} \left( p^{(1)}_{s,t}(v_s) \right) \mathcal{F}^{-1} \left( \prod_{m=1}^{n} 2\pi p^{(2)}_{m,t}(B^T v) \right),
\]

where \( A^T = (a_{k,j}) \) is the transpose of \( A \). Hence

\[
\Phi(v, t) = \prod_{s=1}^{n} \exp \left( -t \psi^{(1)}_{s}(v_s) \right) \prod_{m=1}^{n} \exp \left( -t \psi^{(2)}_{m} \left( \sum_{k=1}^{n} b_{k,m} v_k \right) \right)
\]

\[
= \exp \left( -t \left( \sum_{s=1}^{n} \psi^{(1)}_{s}(v_s) + \sum_{m=1}^{n} \psi^{(2)}_{m} \left( \sum_{k=1}^{n} b_{k,m} v_k \right) \right) \right).
\]

\( \square \)

### 4.4 Sufficient equivalent martingale measure conditions for basket options

In this section we specify the equivalent martingale measure condition for our model. Under the equivalent martingale measure all assets have the same expected rate of return which is a risk free rate. This means that under no-arbitrage conditions the risk preferences of investors acting on the market do not enter into valuation decisions. Recall that in general, \( Q \) is not unique. It was shown in [33] that the class of equivalent martingale measures is so rich that every price in some interval \([a, b]\) can be obtained by a particular martingale measure. We assume that \( Q \) has been fixed and all expectations will be computed with respect to this measure (see Appendix III for more information).

Consider a frictionless market consisting of a riskless bond \( B \) and stock \( S \). In this market \( S \) is modeled by an exponential Lévy process \( S_t = S_0 \exp(X_t) \) under a chosen equivalent martingale measure \( Q \). Assume that the riskless rate \( r \) is constant.

**Theorem 9** Let \( D \) be the domain of \( \psi^Q(\xi) \) and \( \mathbb{R} \cup \{-i\} \subset D \), then in our notations \( \psi^Q(-i) = -r \).

**Proof.** The discounted price process which is given by

\[
\tilde{S}_t = \exp(-rt)S_t = \exp(-rt)S_0 \exp(X_t)
\]

must be a martingale under a chosen equivalent martingale measure \( Q \), i.e. for any \( 0 \leq l < t \leq T \) the martingale condition must hold,

\[
\tilde{S}_t = \mathbb{E}^Q \left[ \tilde{S}_l | \mathcal{F}_l \right]
\]

(see Appendix III for more information). Without loss of generality we may assume \( l = 0 \). Then for any \( t \in (0, T] \) we have

\[
\tilde{S}_0 = S_0 \exp(-r0) = S_0 = \mathbb{E}^Q [S_0 \exp(-rt) \exp(X_t) | \mathcal{F}_0]
\]

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\[ E^Q [S_0 \exp(-rt) \cdot \exp(X_t)] = S_0 E^Q [\exp(-rt) \cdot \exp(X_t)]. \]

Since \( S_0 > 0 \) then
\[ E^Q [\exp(-rt) \cdot \exp(X_t)] = 1, \]
or
\[ \exp(rt) = E^Q [\exp(X_t)]. \tag{4.6} \]

Since \( \psi(-i) \subset D \) then by the definition of the characteristic exponent
\[ \exp(-t\psi(-i)) = E^Q [\exp(i(-i)X_t)] = E^Q [\exp(X_t)]. \]

Hence, since \( t > 0 \), then from (4.6) it follows that \( r = -\psi(-i). \) \( \square \)

A commonly used condition on \( \psi^Q(\xi) \) is that it admits the analytic continuation into the strip \( \{z| -1 \leq \Im z \leq 0\} \) (see, e.g. [72] p. 83).

We specify now the equivalent martingale measure condition for the system (4.3).

**Theorem 10** Let the stock prices be modeled by
\[ S_{s,t} = S_{s,0} \exp(U_{s,t}), \quad 1 \leq s \leq n, \]
and the domain \( D \subset \mathbb{R}^n + i\mathbb{R}^n \) of the characteristic exponent \( \psi^Q \) contains \( \mathbb{R}^n \cup (\cup_{k=1}^n \{-ie_k\}) \) where \( \{e_k, 1 \leq k \leq n\} \) is the standard basis in \( \mathbb{R}^n \). Then
\[ \psi^Q(-ie_s) = -r, \quad 1 \leq s \leq n. \tag{4.7} \]

**Proof.** Observe that for any \( 1 \leq s \leq n \) the discount price process \( S_{s,t} \) must be a martingale under a chosen equivalent martingale measure \( Q \). Let \( \psi^Q_s(x_s) \) be the characteristic exponent of \( U_{s,t} \). Then
\[ \exp(-t\psi^Q_s(x_s)) = E^Q [\exp(ix_sU_{s,t})] \]
\[ = E^Q [\exp((ix_sU_{s,t},e_s))] = \exp(-t\psi^Q_s(x_s,e_s)). \]
Thus by Theorem 9 we get \( r = -\psi^Q_s(-i) \), which gives a system of \( n \) equations
\[ \psi^Q(-ie_s) = -r, \quad 1 \leq s \leq n. \]
\( \square \)

Observe that in general riskless interest rate may depend on \( s \). In this case we get the system \( \psi^Q(-ie_s) = -r_s, 1 \leq s \leq n. \)
4.5 KoBoL family

In this section we study characteristic exponents of so-called KoBoL family. The idea is based on a simple observation. From the Lévy-Khintchine formula (4.1) it follows that it is possible to find $\psi(\xi)$ explicitly if we can compute explicitly the inverse Fourier transform of $\Pi(dx)$. Therefore, it was suggested by the authors of [14] to consider the following form of $\Pi(dx)$,

$$\Pi(dx) = |x|^\alpha \exp \left(-\beta |x|\right) dx,$$

where $\alpha$ and $\beta$ are fixed parameters. Let $\lambda_- < 0 < \lambda_+$.

$$\Pi^+(\nu, \lambda_+, dx) = (\max \{x, 0\})^{-\nu - 1} \exp(-\lambda_+ x) dx$$

and

$$\Pi^-(\nu, \lambda_-, dx) = (\max \{-x, 0\})^{-\nu - 1} \exp(-\lambda_- x) dx,$$

where $\nu < 2$.

**Definition 11** A Lévy process is called a KoBoL process of order $\nu < 2$ if it is purely non-Gaussian with the Lévy measure of the form

$$\Pi(dx) = c_+ \Pi^+(\nu, \lambda_+, dx) + c_- \Pi^-(\nu, \lambda_-, dx),$$

where $c_+ > 0$, $c_- > 0$, $\lambda_- < 0 < \lambda_+$.

We call $\nu$ the order of the process, $\lambda_+$ and $\lambda_-$ the steepness parameters and $c_+$ and $c_-$ the intensity parameters of the process. The parameter $\lambda_-$ ($\lambda_+$ respectively) determines the rate of the exponential decay of the right (left respectively) tail of the density function. It is easy to see that the condition (4.2) is satisfied, i.e.

$$\int_{\mathbb{R}} \min \left\{1, x^2 \right\} \left(c_+ \Pi^+(\nu, \lambda_+, dx) + c_- \Pi^-(\nu, \lambda_-, dx)\right) < \infty.$$ 

Moreover, if $\nu < 1$ then

$$\int_{\mathbb{R}} \min \left\{1, |x| \right\} \left(c_+ \Pi^+(\nu, \lambda_+, dx) + c_- \Pi^-(\nu, \lambda_-, dx)\right) < \infty,$$

i.e. a KoBoL process is of finite variation iff $\nu < 1$.

**Lemma 12** ([14], p. 70) If $\nu \in (0, 1) \cup (1, 2)$ then

$$\psi(\xi) = -i\mu \xi + c_- \Gamma(-\nu) \left((-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu\right) + c_+ \Gamma(-\nu) \left(\lambda_+^\nu - (\lambda_+ + i\xi)^\nu\right).$$

(4.8)

If $\nu = 0$, then

$$\psi(\xi) = -i\mu \xi + c_- \left[\ln(-\lambda_- - i\xi) - \ln(-\lambda_-)\right]$$

and
If $\nu = 1$, then

$$
\psi (\xi) = -i\mu \xi + c_- \left[ (-\lambda_-) \ln (-\lambda_-) - (-\lambda_- - i\xi) \ln (-\lambda_- - i\xi) \right]
+ c_+ \left[ \lambda_+ \ln \lambda_+ - (\lambda_+ + i\xi) \ln (\lambda_+ + i\xi) \right],
$$

where $\mu \in \mathbb{R}$, $c_\pm > 0$, and $\lambda_- < 0 < \lambda_+$. The proof of Lemma 12 presented in [14] is incomplete. The next statement gives a complete proof of the representation (4.8) which is important in our applications.

**Theorem 13** Let $\nu \in (0, 1)$ then in our notation

$$
\psi (\xi) = -i\mu \xi + c_- \Gamma (-\nu) \left[ (-\lambda_-)\nu - (-\lambda_- - i\xi)\nu \right]
+ c_+ \Gamma (-\nu) \left( \lambda_+\nu - (\lambda_+ + i\xi)\nu \right),
$$

(4.9)

where $\mu$ is a real parameter.

**Proof.** It is sufficient to prove the statement just for the $\Pi^+ (\nu, \lambda, dx)$, i.e. to find

$$
-\psi^+ (\xi) := \int_\mathbb{R} \left( \exp (ix\xi) - 1 - ix\xi \chi_{[-1,1]} (x) \right) \Pi^+ (dx)
$$

$$
= \int_\mathbb{R} \left( \exp (ix\xi) - 1 - ix\xi \chi_{[-1,1]} (x) \right) \max \{x, 0\}^{-\nu-1} \exp (-\lambda x) dx
$$

$$
= \int_0^\infty \left( \exp (ix\xi) - 1 - ix\xi \chi_{[-1,1]} (x) \right) x^{-\nu-1} \exp (-\lambda x) dx
$$

$$
= \int_0^\infty \left( \exp (ix\xi) - 1 \right) x^{-\nu-1} \exp (-\lambda x) dx
$$

$$
i\xi \int_0^1 x^{-\nu} \exp (-\lambda x) dx
$$

$$
:= I_1 (\xi, \nu, \lambda) - i\xi D (\nu, \lambda),
$$

where $D (\nu, \lambda) := \int_0^1 x^{-\nu} \exp (-\lambda x) dx$ and

$$
I_1 (\xi, \nu, \lambda) = -\frac{1}{\nu} \int_0^\infty \left( \exp (- (\lambda - i\xi) x) - \exp (-\lambda x) \right) dx^{-\nu}
$$

$$
= -\frac{1}{\nu} \left( \left. \exp (- (\lambda - i\xi) x) - \exp (-\lambda x) \right|_0^\infty x^{-\nu} \right)
$$

$$
- \left( \frac{1}{\nu} \right) \int_0^\infty (- (\lambda - i\xi) \exp (- (\lambda - i\xi) x) + \lambda \exp (-\lambda x)) x^{-\nu} dx
$$

$$
= -\frac{\lambda - i\xi}{\nu} \int_0^\infty \exp (- (\lambda - i\xi) x) x^{-\nu} dx - \lambda^\nu \Gamma (-\nu) := I_2 - \lambda^\nu \Gamma (-\nu).
$$
Making change of variable \( z = (\lambda - i\xi) x \) in \( I_2 \) we get

\[
I_2 = -\left(\frac{\lambda - i\xi}{\nu}\right)^\nu \int_\gamma \exp(-z) z^{-\nu} dz,
\]

where \( \gamma \) is the ray \( \{ z = (\lambda - i\xi) x, \lambda > 0, \xi \in \mathbb{R} \} \), \( \lambda \) and \( \xi \) are fixed parameters and \( x \geq 0 \). Assume that \( \xi \geq 0 \). The case \( \xi \leq 0 \) can be treated similarly.

Consider the contour \( \eta := \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \), where

\[
\begin{align*}
\gamma_1 &:= \{ z = \rho \exp(i\theta) \mid 0 \leq \theta \leq \arg(\lambda - i\xi), \lambda > 0, \xi \in \mathbb{R} \}, \\
\gamma_2 &:= \{ z \mid \rho \leq z \leq R, z \in \mathbb{R} \}, \\
\gamma_3 &:= \{ z = R \exp(i\theta) \mid 0 \leq \theta \leq \arg(\lambda - i\xi), \lambda > 0, \xi \in \mathbb{R} \}, \\
\gamma_4 &:= \{ z = (\lambda - i\xi) x, \rho \leq |z| \leq R \}.
\end{align*}
\]

The function \( \exp(-z) z^{-\nu} \) is analytic in the domain bounded by \( \eta \), hence from the Cauchy theorem it follows that

\[
\oint_\eta \exp(-z) z^{-\nu} dz = 0
\]

and since \( \xi \geq 0 \) then for some \( \delta > 0 \) we get \( -\pi/2 + \delta \leq \arg(\lambda - i\xi) \leq 0 \). Hence

\[
\begin{align*}
\lim_{R \to \infty} \left| \int_{\gamma_3} \exp(-z) z^{-\nu} dz \right| &= \lim_{R \to \infty} \left| \int_0^{\arg(\lambda - i\xi)} \exp(-R \exp(i\theta)) R^{-\nu} \exp(-i\nu \theta) R \exp(i\theta) d\theta \right| \\
&\leq \frac{\pi}{2} \lim_{R \to \infty} \exp(-R \cos \delta) R^{1-\nu} = 0.
\end{align*}
\]

Observe that

\[
\begin{align*}
\lim_{\rho \to 0} \left| \int_{\gamma_1} \exp(-z) z^{-\nu} dz \right| &= \lim_{\rho \to 0} \left| \int_0^{2\pi} \exp(-\rho \exp(i\theta)) \rho^{-\nu} \exp(-i\nu \theta) \rho \exp(i\theta) d\theta \right| \\
&\leq 2\pi \lim_{\rho \to 0} \rho^{-\nu+1} = 0.
\end{align*}
\]

Hence

\[
\int_\gamma \exp(-z) z^{-\nu} dz = \int_{\mathbb{R}^+} \exp(-z) z^{-\nu} dz = \Gamma(-\nu + 1) = -\nu \Gamma(-\nu).
\]

Consequently,

\[
I_2 = -\left(\frac{\lambda - i\xi}{\nu}\right)^\nu \int_\gamma \exp(-y) y^{-\nu} dy = \Gamma(-\nu) (\lambda - i\xi)^\nu
\]

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\[\psi^+ (\xi) = \Gamma (-\nu) \left( \lambda^\nu - ((\lambda - i\xi)^\nu) \right) + i \xi D (\nu, \lambda).\]

Finally, the term \(i \xi D (\nu, \lambda)\) can be considered as a part of \(i \mu \xi\), where \(\mu \in \mathbb{R}\) is a free parameter. □

Observe that the parameters \((\nu, c_+, c_-, \lambda_+, \lambda_-)\) determine the probability density. For larger \(\nu\) and \(c_\pm\) we get a larger peak of the probability distribution. The parameters \(c_+\) and \(c_-\) control asymmetry of the probability distribution while \(\lambda_-\) and \(\lambda_+\) determine the rate of exponential decay as \(\xi \to \pm \infty\).

Consider the asymptotic behavior of KoBoL exponent \(\psi (\xi)\) in the strip \(\Im \xi \in (\lambda_-, \lambda_+)\) as \(|\xi| \to \infty\). In what follows we shall adapt the standard notations, \(z^\nu = \exp (\nu \ln z)\), where \(\nu, z \in \mathbb{C}\) such that \(z \not\in (-\infty, 0]\) and \(\ln z\) denotes the branch of \(\ln z\) defined on \(\mathbb{C} \setminus (-\infty, 0]\) and such that that \(\ln(1) = 0\).

**Lemma 14** Let \(c_+ = c_- = c > 0\), \(\xi = \rho \exp (i\phi)\) and \(\Im \xi \in (\lambda_-, \lambda_+)\). Then

\[
\text{Re} \psi (\xi) \sim -\rho \nu 2c \Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right)
\]

if \(\text{Re} \xi \to \infty\) and

\[
\text{Re} \psi (\xi) \sim -\rho \nu 2c \Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \cos (\nu \pi)
\]

if \(\text{Re} \xi \to -\infty\).

**Proof.** Clearly

\[
(-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu \sim -\rho \nu \exp \left( i \left( -\frac{\pi}{2} + \phi \right) \nu \right)
\]

and

\[
\lambda_+^\nu - (\lambda_+ + i\xi)^\nu \sim -\rho \nu \exp \left( i \left( \frac{\pi}{2} + \phi \right) \nu \right)
\]

as \(\rho \to \infty\). Since \(c_+ = c_- = c\) then

\[
\psi (\xi) = -i \mu \xi + c_+ \Gamma (-\nu) ((-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu) + c_+ \Gamma (-\nu) (\lambda_+^\nu - (\lambda_+ + i\xi)^\nu)
\]

\[
\sim -i \mu \rho \exp (i\phi) - 2c \Gamma (-\nu) \exp (i\nu\phi) \cos \left( \frac{\pi \nu}{2} \right) \rho \nu.
\]

Hence

\[
\text{Re} \psi (\xi) \sim \rho \mu \sin \phi + \rho \nu 2c \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \cos (\nu \phi).
\] (4.10)

To complete the proof we remark that \(\phi \to 0\) if \(\text{Re} \xi \to \infty\) and \(\phi \to \pi\) if \(\text{Re} \xi \to -\infty\) in the strip \(\Im \xi \in (\lambda_-, \lambda_+)\). □

**Corollary 15** Let \(\nu \in (0, 1/2), \xi = \rho \exp (i\phi)\) and \(\Im \xi \in (\lambda_-, \lambda_+)\). Then the respective characteristic function \(\Phi (\xi, t) = \exp (-t \psi (\xi))\) can be estimated as

\[
|\Phi (\xi, t)| = |\exp (-t \psi (\xi))|.
\]
\[
\exp (-t \text{Re} \psi (\xi)) \lesssim \exp \left( 2t \rho' c \Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right)
\]
if \( \text{Re} \xi \to \infty \) and
\[
|\Phi (\xi, t)| \lesssim \exp \left( 2t \rho' c \Gamma (-\nu) \cos \left( \nu \pi \right) \cos \left( \frac{\pi \nu}{2} \right) \right)
\]
if \( \text{Re} \xi \to -\infty \). In particular, if \( \nu \in (0, 1/2) \) then
\[
\text{tc} \Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \cos \left( \nu \pi \right) < 0
\]
and
\[
|\Phi (\xi, t)| \ll \exp (-Ct |\xi|^\nu), \ |\xi| \to \infty, \ \text{Im} \xi \in (\lambda_-, \lambda_+).
\]

**Example 16** At this point we present two more important examples of characteristic exponents \( \psi (\xi) \) which are of practical interest in empirical studies of financial markets. Remark that Madan and collaborators [80], [81] were first who applied Variance Gamma processes in studies of financial markets. The respective characteristic exponent has the form
\[
\psi (\xi) = -i\mu \xi + c_+ [\ln(-\lambda_- - i\xi) - \ln(-\lambda_-)] + c_- [\ln(\lambda_+ + i\xi) - \ln(\lambda_+)],
\]
where \( \lambda_- < 0 < \lambda_+, \ c > 0 \) and \( \mu \in \mathbb{R} \). A Variance Gamma process with these parameters is also a Lévy process of exponential type \( (\lambda_-, \lambda_+) \).

So-called Normal Inverse Gaussian processes were introduced and studied by Barndorff-Nielsen [3]-[8]. The respective characteristic exponent is
\[
\psi (\xi) = -i\mu \xi + \delta \left[ (\alpha^2 - (\beta + i\xi)^2)^{\nu/2} - (\alpha^2 - \beta^2)^{\nu/2} \right].
\]

### 4.6 Representations of density functions of KoBoL processes

**Definition 17** For a fixed \( R > 0 \) consider two piecewise smooth curves
\[
\lambda_+ (x) := x + i (\alpha_+ + a_+ (x)) : [-R, R] \to \{ z | \text{Im} z > 0 \},
\]
and
\[
\lambda_- (x) := x + i (\alpha_- + a_- (x)) : [-R, R] \to \{ z | \text{Im} z < 0 \},
\]
where \( \alpha_+ > 0, \ a_+ (x) \geq 0 \) is an even function increasing on \([0, R]\) and decreasing on \([-R, 0]\). Similarly, \( \alpha_- < 0, \ a_- (x) \leq 0 \) is an even function increasing on \([-R, 0]\) and decreasing on \([0, R]\). Consider six contours,
\[
\gamma_1 (R) := \{ z | z = |\lambda_+ (R)| \exp (i\phi), \ \phi \in [\arg (R + i (\alpha_+ + a_+ (R))), 0] \},
\]
for any \( t > 0 \), \( \gamma_2 (R) := [\lambda_+ (R), \lambda_- (R)] \),
\( \gamma_3 (R) := \{ z | z = [\lambda_- (R) \exp (i \phi), \ \phi \in [0, \arg (R + i (\alpha_- + a_- (R)))]) \} \),
\( \gamma_4 (R) := \{ z | z = [\lambda_- (-R) \exp (i \phi), \ \phi \in [\arg (-R + i (\alpha_- + a_- (-R))), -\pi]) \} \),
\( \gamma_5 (R) := [\lambda_- (-R), \lambda_+ (-R)] \),
\( \gamma_6 (R) := \{ z | z = [\lambda_+ (-R) \exp (i \phi), \ \phi \in [\arg (-\pi, -R + i (\alpha_+ + a_+ (-R)))]) \} \).

We say that a Lévy process \( X = \{ X_t, t > 0 \} \) is \((\lambda_-, \lambda_+)-analytic\) if for any \( R > 0 \) its characteristic exponent \( \psi (z) \) admits analytic extension into the domain \( \Omega_R \) bonded by

\[
\lambda_- (\cdot) \cup \lambda_+ (\cdot) \cup \bigcup_{k=1}^6 \gamma_k (R)
\]

and

\[
\lim_{R \to \infty} \int_{\gamma_k (R)} \exp (iyz - t \psi (z)) \, dz = 0, \ 1 \leq k \leq 6, \ t, y > 0
\]

for any \( t > 0, y > 0 \).

Recall that in the case of European call option the reward function has the form \( H (y) = \max \{ S_0 \exp (y) - K, 0 \} \). Hence we need just to consider the case \( 0 < \ln (K/S_0) < y \). Observe that usually \( K > S_0 \) because of potential earning capacity of the stock \( S_t \) during the time interval \( (0, T) \).

A useful representation of density functions is given by the following statement.

**Theorem 18** Let \( X = \{ X_t, t > 0 \} \) be a \((\lambda_-, \lambda_+)-analytic\) process. Then

\[
p_t (y) = \frac{1}{2\pi (\exp (\alpha_+ y) + \exp (\alpha_- y))} \times \int_{-\infty}^{\infty} \exp (iy (x + ia_+ (x))) N (x) + \exp (iy (x + ia_- (x))) M (x) \, dx,
\]

where

\[
N (x) := \exp (-t \psi (x + i (\alpha_+ + a_+ (x)))) (1 + ia_+ (x))
\]

and

\[
M (x) := \exp (-t \psi (x + i (\alpha_- + a_- (x)))) (1 + ia_- (x)).
\]

**Proof.** Let \( \gamma_7 (R) := \{ z | z = x + i \lambda_+ (x), \ x \in [-R, R]\} \). Since the process \( X = \{ X_t, t \geq 0 \} \) is \((\lambda_-, \lambda_+)-analytic\) then using Cauchy theorem we get

\[
\int_{\gamma_1 \cup [\lambda_+ (R), -|\lambda_- (-R)|] \cup \gamma_7 (R) \cup \gamma_5 (R)} \exp (iyz - t \psi (z)) \, dz = 0.
\]
Applying \((\lambda_-, \lambda_+)-\)analyticity and letting \(R \to \infty\) we get

\[
p_t(y) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp(iy\xi - t\psi(\xi)) \, d\xi
\]

\[
= \frac{1}{2\pi} \lim_{R \to \infty} \int_{[-R,R]} \exp(iy\xi - t\psi(\xi)) \, d\xi
\]

\[
= \frac{\exp(-\alpha_+ y)}{2\pi}
\]

\[
\times \int_{\mathbb{R}} \exp(iyx - ya_+(x) - t\psi(x + i\alpha_+ + ia_+(x))) (1 + ia_+(x)) \, dx.
\]

Similarly,

\[
p_t(y) = \frac{\exp(-\alpha_- y)}{2\pi}
\]

\[
\times \int_{\mathbb{R}} \exp(iyx - ya_- (x) - t\psi(x + i\alpha_- + ia_- (x))) (1 + ia_- (x)) \, dx.
\]

The proof follows from the last two representations of \(p_t\). □

In particular, if \(X = \{X_t, t > 0\}\) is a \((0, \lambda_+)-\)analytic process then the respective density function \(p_t(y)\) can be represented as

\[
p_t(y) = \frac{\exp(-\alpha_+ y)}{2\pi}
\]

\[
\times \int_{\mathbb{R}} \exp(iyx - ya_+(x) - t\psi(x + i\alpha_+ + ia_+(x))) (1 + ia_+(x)) \, dx.
\]

The following statement gives a wide range of examples of \((0, \lambda_+)-\)analytic processes.

**Theorem 19** Any KoBoL process with parameters \(\mu \geq 0, c_+ = c_- = c > 0\) and \(\nu \in (0, 1/2)\) is \((0, \lambda_+)-\)analytic Lévy process.

**Proof.** Clearly, characteristic exponent \(\psi(\xi)\) given by (4.9) is analytic in the domain

\[\mathbb{C} \setminus \{i\lambda_+, i\infty\} \cup \{i\lambda_-, -i\infty\}\]

Hence it is sufficient to show that

\[
\lim_{R \to \infty} I_+(R, y, t) = \lim_{R \to \infty} I_-(R, y, t) = 0,
\]

where

\[
I_+(R, y, t) := \left| \int_{\xi \in \Gamma^+_R} \exp(iy\xi - t\psi(\xi)) \, d\xi \right|
\]

(4.11)

\[
I_-(R, y, t) := \left| \int_{\xi \in \Gamma^-_R} \exp(iy\xi - t\psi(\xi)) \, d\xi \right|
\]

(4.12)
\[ \Gamma^+_R = \{ \xi | \xi = |\lambda_+ (R)| \exp (i\phi), \phi \in [0, \arg (R + i\alpha_+ (R))] \} \]

and

\[ \Gamma^-_R = \{ \xi | \xi = |\lambda_- (-R)| \exp (i\phi), \phi \in [\pi, \arg (-R + i\alpha_- (-R))] \}. \]

Let us present estimates for the integral (4.11) first. Clearly,

\[
I^+_R (R, y, t) \leq R \left| \int_0^{\arg(R + i\alpha_+ + i\alpha_+ (R))} \exp (iyR \exp (i\phi)) - t\psi (R \exp (i\phi)) Ri \exp (i\phi) \, d\phi \right| \leq R \int_0^{\arg(R + i\alpha_+ + i\alpha_+ (R))} \chi (y, R, \nu, \phi) \, d\phi, \tag{4.13}
\]

where

\[ \chi (y, R, \nu, \phi, t) := \exp (\text{Re} (iyR \exp (i\phi) - t\psi (R \exp (i\phi)))) \].

Since \( y \geq 0, \mu \geq 0, c > 0, t > 0 \) and \(-\Gamma (-\nu) \cos (\pi \nu/2) > 0 \) if \( \nu \in (0, 1/2) \) then applying (4.10) we get

\[ \chi (y, R, \nu, \phi, t) \leq C \exp \left( -yR \sin \phi - tR\mu \sin \phi - 2ctR^{2\nu} \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \cos (\nu \phi) \right) \]

\[ \leq C \exp \left( -2ctR^{2\nu} \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \cos (\nu \phi) \right) \tag{4.14} \]

Comparing and (4.13) and (4.14) we get

\[ I^+_R (R, y, t) \leq CR \int_0^{\pi/2} \exp \left( -2ctR^{2\nu} \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \left( 1 - \frac{2\nu}{\pi} \phi \right) \right) \, d\phi, \]

where we used the fact that \( \cos (\nu \phi) \geq 1 - 2\phi \nu / \pi \) if \( \phi \in [0, \pi/2] \). This means that

\[ I^+_R (R, y, t) \leq CR \exp \left( -2ctR^{2\nu} \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \right) \times \int_0^{\pi/2} \exp \left( 2ctR^{2\nu} \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \frac{2\nu}{\pi} \phi \right) \, d\phi \]

\[ = \frac{\pi \Gamma^{1-\nu}}{4ct\nu \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right)} \exp \left( -2ctR^{2\nu} \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \right) \]
\[
\times \left( \exp \left( 2ctR^{\nu} \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \nu \right) - 1 \right) \leq \frac{\pi CR^{1-\nu}}{4ct\nu \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right)} \exp \left( -2ctR^{\nu} \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \right) (1 - \nu) .
\]

From the last line we see that

\[ I_+ (R, y, t) \ll R^{1-\nu} \exp (-CtR^\nu), \quad R \to \infty \]

for any \( y > 0 \). Now we get estimates of the integral \([1,12]\). In this case \( \phi \in \left[ \arg (-R + ia_+ (-R)) , \pi \right] \subset [\pi/2, \pi] \). Hence \( -\Gamma (-\nu) \cos (\pi \nu/2) \cos (\nu \phi) > 0 \) since \( \nu \in (0,1/2) \). Recall that \( y > 0, \mu > 0 \) and \( t > 0 \). Applying the same line of arguments as above we get

\[
I_- (R, y, t) \leq CR \exp \left( -2ctR^{\nu} \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \right) \\
\times \int_{\pi/2}^{\pi} \exp \left( 2ctR^{\nu} \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \frac{2\nu}{\pi} \phi \right) d\phi \\
= \frac{\pi CR^{1-\nu}}{4ct\nu \left( -\Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right)} \exp \left( 2ctR^{\nu} \Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \\
\times \left( \exp \left( -4\nu ctR^{\nu} \Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \right) - \exp \left( -2\nu ctR^{\nu} \Gamma (-\nu) \cos \left( \frac{\pi \nu}{2} \right) \right) \\
\ll R^{1-\nu} \exp (-Ct (1 - 2\nu) R^\nu), \quad R \to \infty.
\]

□
Chapter 5

Recovery of density functions in jump-diffusion models

5.1 Introduction

Recall that the pricing formula has the form $V = \exp(-rT) \mathbb{E}^Q[H]$, where $Q$ is a fixed equivalent martingale measure. Since the reward function $H$ has usually a simple structure the main problem is to approximate the respective risk-neutral density function $p^Q_T$, where $T > 0$ is a maturity time. Hence it is important to construct a simple, saturation free and adapted to the course of dimensionality method of approximation of density functions which are important in the theory of spread options. Our method is based on the Poisson summation formula and approximation of density functions by harmonics in the respective exponential hyperbolic cross. The advantage of this approach is that the application of the Poisson summation formula gives a periodic extension of the density function of the same smoothness as the original function \[67, 68\] instead of known approaches discussed in \[38, 39\]. Also, this approach allows us to get approximation formulas without application of numerical methods.

Approximation of smooth functions by subspaces of entire functions of exponential type and sk-splines was considered in \[65, 66, 69, 77, 71, 72, 73, 74, 76\]. These methods are saturation free on a wide range of sets of smooth functions (including analytic and entire functions) and give almost optimal rate of convergence in the sense of respective $m$-widths. However, application of these methods requires use of numerical methods. We shall not discuss this line of research here.
5.2 Representations of density functions

Assume for simplicity that all characteristic exponents $\psi^{(1),Q}_s, 1 \leq s \leq n$ and $\psi^{(2),Q}_m, 1 \leq m \leq n$ in Theorem 8 correspond to a KoBoL process and hence are analytically extendable into the strips $\text{Im} z_s \in [\kappa_{s,-}, \kappa_{s,+}]$ and $\text{Im} z_m \in [\kappa_{m,-}, \kappa_{m,+}]$ respectively, where $\lambda_{s,-} < \kappa_{s,-} < 0 < \kappa_{s,+} < \lambda_{s,+}$ and $\lambda_{m,-} < \kappa_{m,-} < 0 < \kappa_{m,+} < \lambda_{m,+}$, $1 \leq s, m \leq n$. Let $b_{k,m} \geq 0, 1 \leq k, m \leq n$. It is easy to check that the function $z = (z_1, \cdots, z_n) \rightarrow \psi(z)$, 

$$
\psi(z) = \sum_{s=1}^{n} \psi^{(1),Q}_s(z_s) + \sum_{m=1}^{n} \psi^{(2),Q}_m \left( \sum_{k=1}^{n} b_{k,m} z_k \right)
$$

defined in Theorem 8 is analytically extendable into the domain 

$$
T_n := \left( \bigcup_{s=1}^{n} \{ \text{Im} z_s \in [\kappa_{s,-}, \kappa_{s,+}] \} \right) \cap \left( \bigcup_{s=1}^{n} \left\{ \text{Im} z_s \in \left[ \kappa_{s,-} \left( \sum_{k=1}^{n} b_{k,m} \right)^{-1}, \kappa_{s,+} \left( \sum_{k=1}^{n} b_{k,m} \right)^{-1} \right] \right\} \right), \quad (5.1)
$$

or $b_{-,s} \leq \text{Im} z_s \leq b_{+,s}$, where 

$$
b_{-,s} := \max \left\{ \kappa_{s,-}, \kappa_{s,-} \left( \sum_{k=1}^{n} b_{k,m} \right)^{-1} \right\}, \\
b_{+,s} := \min \left\{ \kappa_{s,+}, \kappa_{s,+} \left( \sum_{k=1}^{n} b_{k,m} \right)^{-1} \right\},
$$

$1 \leq s \leq n$ and $b_{k,m} \geq 0, 1 \leq k, m \leq n$. In this case $\Phi^Q(z,t) = \Phi^Q(z_1, \cdots, z_n, t)$ admits an analytic extension into the same domain $T_n \subset \mathbb{C}^n$. Let $b_{+} := (b_{+,1}, \cdots, b_{+,n})$ and $b_{-} := (b_{-,1}, \cdots, b_{-,n})$.

Theorem 20 Let $\psi^{(1),Q}_s, 1 \leq s \leq n$ and $\psi^{(2),Q}_m, 1 \leq m \leq n$ be defined by (5.8), i.e. 

$$
\psi^{(1),Q}_s(\xi_s) = -i\mu_s \xi_s + c_s \Gamma (-\nu_s) \left( (-\lambda_{s,-})^{\nu_s} - (-\lambda_{s,-} - i\xi_s)^{\nu_s} \right) \\
+ c_s \Gamma (-\nu_s) \left( \lambda^{\nu_s}_{+,s} - (\lambda_{+,s} + i\xi_s)^{\nu_s} \right), \ \nu_s \in (0,1/2)
$$

and 

$$
\psi^{(2),Q}_m(\xi_m) = -i\mu_m \xi_m + c_m \Gamma (-\nu_m) \left( (-\lambda_{-,m})^{\nu_m} - (-\lambda_{-,m} - i\xi_m)^{\nu_m} \right) \\
+ c_m \Gamma (-\nu_m) \left( \lambda^{\nu_m}_{+,m} - (\lambda_{+,m} + i\xi_m)^{\nu_m} \right), \ \nu_m \in (0,1/2),
$$

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where \( c_s, c_m > 0, \nu_s, \nu_m \in (0, 1/2) \). Then the respective density function \( p_T^Q (\cdot) \) can be represented as

\[
p_T^Q (\cdot) = \frac{1}{(2\pi)^n \left( \exp(\langle \cdot, b_+ \rangle) + \exp(\langle \cdot, b_- \rangle) \right)} \times \int_{\mathbb{R}^n} \exp (-i \langle \cdot, z \rangle) \left( \Phi^Q (z - ib_+, T) + \Phi^Q (z - ib_-, T) \right) dz.
\]

In particular, if \(-b_- = b_+ := b\) then

\[
p_T^Q (\cdot) = \frac{1}{2 (2\pi)^n} \left( \cosh (\langle \cdot, b \rangle) \right)^{-1} \times \int_{\mathbb{R}^n} \exp (-i \langle \cdot, z \rangle) \left( \Phi^Q (z + ib, T) + \Phi^Q (z - ib, T) \right) dz. \quad (5.2)
\]

Let

\[
\Phi^Q_1 (z, T) := \Phi^Q (z + i\epsilon_1 b_1, T) + \Phi^Q (z - i\epsilon_1 b_1, T),
\]

\[
\Phi^Q_k (z, T) := \Phi^Q_{k-1} (z + i\epsilon_k b_k, T) + \Phi^Q_{k-1} (z - i\epsilon_k b_k, T), \quad 2 \leq k \leq n.
\]

Then

\[
p_T^Q (x) = \frac{1}{2^{2n} \pi^n} \left( \prod_{s=1}^n \cosh (b_s x_s) \right)^{-1} \int_{\mathbb{R}^n} \exp (-i \langle x, z \rangle) \Phi^Q_n (z, T) dz. \quad (5.3)
\]

**Proof.** We shall prove just (5.2) since (5.3) follows in a similar manner. In our notation density function can be represented as

\[
p_T^Q (\cdot) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp (-i \langle \cdot, z \rangle) \Phi^Q (z, T) dz = (2\pi)^{-n} \Phi \left( \Phi^Q (z, T) \right) (\cdot).
\]

Recall that \( \psi^{(1),Q}_s (\xi_s), 1 \leq s \leq n \) admits an analytic extension into the strip \( \text{Im} \xi_s \in [\kappa_{s,-}, \kappa_{s,+}] \), where \( \lambda_{s,-} < \kappa_{s,-} < 0 < \kappa_{s,+} < \lambda_{s,+}, 1 \leq s \leq n \) and \( \psi^{(2),Q}_m (\xi_m), 1 \leq m \leq n \) admits an analytic extension into the strip \( \text{Im} \xi_m \in [\kappa_{m,-}, \kappa_{m,+}] \), where \( \lambda_{m,-} < \kappa_{m,-} < 0 < \kappa_{m,+} < \lambda_{m,+}, 1 \leq m \leq n \). From Corollary 15 it follows that

\[
|\Phi (z, T)| = |\exp (-T \psi (z))|
\]

\[
= \left| \exp \left( -T \left( \sum_{s=1}^n \psi^{(1)}_s (z_s) + \sum_{m=1}^n \psi^{(2)}_m \left( \sum_{k=1}^n b_{k,m} z_k \right) \right) \right) \right|
\]

\[
\ll \left| \exp \left( -T \sum_{s=1}^n \psi^{(1)}_s (z_s) \right) \right|
\]

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\[
\exp \left( -CT \sum_{s=1}^{n} |z_s|^\nu_s \right),
\]
where $|z_k| \to \infty$, $z_k \in T_n$, $1 \leq k \leq n$, where the domain $T_n$ is defined by (5.1). Hence, applying Cauchy theorem (see, e.g. [91]) $n$ times in the domain $T_n$, which is justified by (5.4), we get

\[
p_Q^\Theta (\cdot) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp \left( -i \langle \cdot, z \rangle \right) \Phi_Q^\Theta (z, T) \, dz
\]

or

\[
p_Q^\Theta (\cdot) \exp \left( - \langle \cdot, b_+ \rangle \right) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp \left( -i \langle \cdot, z \rangle \right) \Phi_Q^\Theta (z + i b_+, T) \, dz.
\]

Similarly,

\[
p_Q^\Theta (\cdot) \exp \left( - \langle \cdot, b_- \rangle \right) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp \left( -i \langle \cdot, z \rangle \right) \Phi_Q^\Theta (z + i b_-, T) \, dz.
\]

Comparing (5.5) and (5.6) we get the proof. □

### 5.3 Approximation of density functions by Poisson summation

We will need the following result which is known as the Poisson summation formula.

**Theorem 21** ([98] p. 252) Suppose that for some $A > 0$ and $\delta > 0$ we have

\[
\max \{ f(x), Ff(x) \} \leq A (1 + |x|)^{-n-\delta}.
\]

Then

\[
\sum_{m \in \mathbb{Z}^n} f(x + Pm) = \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} F(f) \left( \frac{2\pi}{P} \right)^n \exp \left( \frac{2\pi i}{P} \langle m, x \rangle \right)
\]

for any $P > 0$. The series converges absolutely.

Assume that $\nu_s, \nu_m \in (0, 1/2)$, $1 \leq s, m \leq n$ as before. Put

\[
\tilde{M} := \frac{1}{2^{2n} \pi^n} \left\| \int_{\mathbb{R}^n} \exp \left( -i \langle \cdot, v \rangle \right) \Phi_n^\Theta (v, T) \, dv \right\|_{L_\infty(\mathbb{R}^n)}.
\]

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Observe that \( \tilde{M} < \infty \) because of the estimate (5.4). Fix \( T > 0, \epsilon > 0 \) and select such \( P \in \mathbb{N} \) that

\[
\tilde{M} \prod_{s=1}^{n} \sum_{m_k \in \mathbb{Z}, m \neq 0} \left( \cosh \left( b_s \frac{2m_k - 1}{2} P \right) \right)^{-1} \leq \epsilon,
\]

(5.7)

where \( m = (m_1, \ldots, m_n) \) and \( m \neq 0 \) means \( m \neq (0, \ldots, 0) \). Clearly

\[
\epsilon \asymp \exp \left( - \frac{P}{2} \min \{ b_s \mid 1 \leq s \leq n \} \right), \quad P \to \infty.
\]

(5.8)

**Theorem 22** Let \( Q_n := \{ x \mid x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ |x_k| \leq 1, 1 \leq k \leq n \} \) be the unit cube in \( \mathbb{R}^n \) and \( -b_- = b_+ := b \). Then in our notation

\[
E_1(P) := \left\| p^Q_T(x) - \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q_{-x,T}(m,T) \exp \left( \frac{2\pi i}{P} \langle m, x \rangle \right) \right\|_{L_{\infty}(\mathbb{R}^n)} \lesssim \exp \left( - \frac{P}{2} \min \{ b_s \mid 1 \leq s \leq n \} \right) P^{n/q}, \quad P \to \infty,
\]

where \( 1 \leq p \leq \infty \).

Proof. Using Theorem 20 we get

\[
\Phi^Q_{-x,T}(m,T) = (2\pi)^n F^{-1} \left( \Phi^Q_T \right) (-x) = (2\pi)^n \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp (i \langle x, y \rangle) \Phi^Q_T (-y) dy \right)
\]

Applying (5.9) we can check that the conditions of Theorem 21 are satisfied. Hence using condition (5.4) we get

\[
\left\| p^Q_T(x) - \sum_{m \in \mathbb{Z}^n} p^Q_T(x + Pm) \right\|_{L_{\infty}(\mathbb{R}^n)} \leq \tilde{M} \prod_{s=1}^{n} \sum_{m_k \in \mathbb{Z}, m \neq 0} \left( \cosh \left( b_s \frac{2m_k - 1}{2} P \right) \right)^{-1} \leq \epsilon.
\]
\[
\int_{\mathbb{R}^n} \exp(i \langle -x, y \rangle) p_T^Q(y) \, dy = \mathbf{F}\left(p_T^Q\right)(x).
\]

Consequently,
\[
\left\| p_T^Q(x) - \sum_{m \in \mathbb{Z}^n} p_T^Q(x + Pm) \right\|_{L_\infty(\frac{\mathbb{T}}{P}Q_n)} = \left\| p_T^Q(x) - \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \mathbf{F}\left(p_T^Q\right)\left(\frac{2\pi}{P}m\right) \exp\left(\frac{2\pi i}{P} \langle m, x \rangle\right) \right\|_{L_\infty(\frac{\mathbb{T}}{P}Q_n)}
\]
\[
= \left\| p_T^Q(x) - \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q\left(-\frac{2\pi}{P}m, T\right) \exp\left(\frac{2\pi i}{P} \langle m, x \rangle\right) \right\|_{L_\infty(\frac{\mathbb{T}}{P}Q_n)} \leq \epsilon 
\]
and
\[
\left\| p_T^Q(x) - \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q\left(-\frac{2\pi}{P}m, T\right) \exp\left(\frac{2\pi i}{P} \langle m, x \rangle\right) \right\|_{L_1(\frac{\mathbb{T}}{P}Q_n)} \leq \epsilon P^n.
\]

Finally, applying Riesz-Thorin interpolation theorem (see Appendix II, Theorem 43 and 5.3) we obtain
\[
\left\| p_T^Q(x) - \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q\left(-\frac{2\pi}{P}m, T\right) \exp\left(\frac{2\pi i}{P} \langle m, x \rangle\right) \right\|_{L_q(\frac{\mathbb{T}}{P}Q_n)} \leq \epsilon P^{n/q} \approx \exp\left(-\frac{P}{2} \min\{b_s | 1 \leq s \leq n\}\right) P^{n/q}, \quad P \to \infty.
\]

\[\square\]

Observe that according to (5.4) the function \(|\Phi^Q\left(-\frac{2\pi}{P}m, T\right)|\) exponentially decays as \(|m| \to \infty\). Hence the series
\[
\tilde{p}_T^Q(x) := \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q\left(-\frac{2\pi}{P}m, T\right) \exp\left(\frac{2\pi i}{P} \langle m, x \rangle\right)
\]
converges absolutely and represents an infinitely differentiable and \(\frac{P}{T}Q_n\)-periodic function which will be denoted again by \(\tilde{p}_T^Q(x)\).

Example 23 Let
\[
p(x, y) = (2\pi)^{-1} \exp\left(-\frac{x^2 + y^2}{2}\right).
\]
be a Gaussian density then its Fourier transform is $\exp\left(-\frac{1}{2}\left(x^2 + y^2\right)\right)$. For a fixed $m$ and $P$ consider the approximant

$$g(p, m, P, x, y) := \frac{1}{P^2} \sum_{|k| \leq m} \sum_{|s| \leq m} F(p) \left( -\frac{2\pi k}{P}, -\frac{2\pi s}{P} \right) \exp\left( ik \frac{2\pi}{P} + isy \frac{2\pi}{P} \right).$$

The respective error of approximation is

$$\varepsilon(p, P, m) := \max \{|p(x, y) - g(p, m, P, x, y)|, x, y \in [-P/2, P/2]\}.$$ 

In particular, let $n = 2$, $P = 6$ and $m = 3$ then $\varepsilon(p, P, m) = 1.747 \times 10^{-3}$.

### 5.4 Comparison of methods of approximation

Theorem 22 shows an exponential rate of convergence of $\tilde{p}_Q^n(x)$ which is given by an infinite trigonometric series to $p_Q^n(x)$ as $P \to \infty$. In this section we discuss the problem of optimal recovery of density functions in the sense of $m$-widths and $m$-cowidths. This allows us to compare a wide range of numerical methods and construct a quasi optimal truncation of the series $\tilde{p}_Q^n(x)$.

Let $\{\varphi_k(x), k \in \mathbb{N}\}$ be a set of continuous orthonormal and uniformly bounded functions on a measure space $(\Omega, \mathcal{F}, \nu)$. Let

$$L := \sup_{k \in \mathbb{N}} \|\varphi_k\|_\infty < \infty.$$ 

For any $f \in L_1 := L_1(\Omega, \mathcal{F}, \nu)$ we construct a formal Fourier series

$$s[f] = \sum_{k=1}^{\infty} c_k(f) \varphi_k, \quad c_k(f) := \int_\Omega f \varphi_k dx.$$ 

Consider the set of functions

$$\Lambda := \{f | |c_k(f)| \leq \lambda_k, \ k \in \mathbb{N}\},$$

where $\lambda_k > 0, k \in \mathbb{N}$. It is easy to check that $\Lambda$ is a convex and symmetric set. Also, $\Lambda$ is compact in $L_1(\Omega, \mathcal{F}, \nu)$ (see Appendix I for definitions) if

$$\sum_{k=1}^{\infty} \lambda_k < \infty.$$ 

Let $\gamma_m$ be one of the widths $d_m(\Lambda, L_q(\Omega, \mathcal{F}, \nu)), a_m(\Lambda, L_q(\Omega, \mathcal{F}, \nu)), a^m(\Lambda, L_q(\Omega, \mathcal{F}, \nu)), \lambda^m(\Lambda, L_q(\Omega, \mathcal{F}, \nu))$ (see Appendix IV for definitions).
Theorem 24  Let $\lambda_k, k \in \mathbb{N}$ be a nonincreasing sequence of positive numbers, $\sum_{k=1}^{\infty} \lambda_k < \infty$ then in our notation

$$\kappa_m \geq \eta L^{1-1/q} \left( \int_{\Omega} dv \right)^{1/q} \lambda_{m+1}, \ q \geq 1,$$

where $\eta = 1$ if $\kappa_m$ is $d_m$ or $a_m$ and $\eta = 2^{-1}$ if $\kappa_m$ is $a^m$ or $\lambda^m$.

Proof. Fix $L_{m+1} := \text{lin} \{ \varphi_k, 1 \leq k \leq m + 1 \}$ and consider the set

$$Q_{m+1} := \left\{ t_{m+1} := \sum_{k=1}^{m+1} c_k \varphi_k, \ |c_k| \leq 1 \right\}$$

which is the unit ball in $L_{m+1}$. The respective norm in $L_{m+1}$ is denoted by $\| t_{m+1} \|_{Q_{m+1}}$. Since $\lambda_k, k \in \mathbb{N}$ is a nonincreasing sequence of positive numbers then

$$\lambda_{m+1} Q_{m+1} \subset \Lambda. \quad (5.10)$$

Applying Riesz’s theorem (see Appendix II, Theorem 44 we get

$$\| t_{m+1} \|_{L_1} \geq L^{-1} \max \{|c_k|, 1 \leq k \leq m + 1\}.$$ 

This means that

$$\| t_{m+1} \|_{L_1} \geq L^{-1} \| t_{m+1} \|_{Q_{m+1}} = \| t_{m+1} \|_{LQ_{m+1}},$$

for any $t_{m+1} \subset L_{m+1}$, or

$$B_1 \cap L_{m+1} \subset LQ_{m+1},$$

where $B_1 := \{ f \ | \| f \|_{L_1} \leq 1 \}$. Hence, applying (5.10) we get

$$L^{-1} \lambda_{m+1} B_1 \cap L_{m+1} \subset \lambda_{m+1} Q_{m+1} \subset \Lambda.$$

From the last line and the definition of Bernstein’s $m$-width (see Appendix IV)

$$b_m (\Lambda, L_1 (\Omega, F, \nu)) \geq L^{-1} \lambda_{m+1}. \quad (5.11)$$

From Jensen’s inequality (see Appendix I, Theorem 38 it follows that

$$\left( \int_{\Omega} dv \right)^{1-1/q} \| f \|_{L_1} \leq \| f \|_{L_q}$$
for any $f \in L_q$, $q \geq 1$. Hence by the definition of Bernstein’s $n$-widths and (5.11) we get

$$b_m (\Lambda, L_q (\Omega, F, \nu)) \geq L^{-1} \left( \int_{\Omega} d\nu \right)^{-1+1/q} \lambda_{m+1}, \ q \geq 1. \quad (5.12)$$

Applying Corollary 62 (see Appendix IV) we get

$$d_m (\Lambda, L_q (\Omega, F, \nu)) \geq L^{-1} \left( \int_{\Omega} d\nu \right)^{-1+1/q} \lambda_{m+1}, \ q \geq 1.$$  

The same lower bound for $a_m (\Lambda, L_q (\Omega, F, \nu))$ follows from Theorem 66, Appendix IV.

Let us obtain lower bounds for the respective cowidths. From the Theorem 6.3 it follows that for any compact and symmetric set $A$ in a Banach space $X$

$$b_m (A, X) \leq 2a_m (A, X),$$

It is known that [102] p. 222,

$$a_m (A, X) \leq u_m (A, X)$$

and [101] p. 190,

$$u_m (A, X) \leq a^m (A, X).$$

Hence

$$b_m (A, X) \leq 2a^m (A, X). \quad (5.13)$$

Finally, comparing (5.11)-(5.13) we get

$$a^m (\Lambda, L_q (\Omega, F, \nu)) \geq 2^{-1} L^{-1} \left( \int_{\Omega} d\nu \right)^{-1+1/q} \lambda_{m+1}, \ q \geq 1.$$  

A similar result can be obtained for $\lambda^m (\Lambda, L_q (\Omega, F, \nu))$. □

Now we apply Theorem 24 to estimate from below the rate of convergence. First we need the following result.

**Theorem 25** Let

$$\Omega (\Phi^Q, T, \varrho, P) := \left\{ z \in \mathbb{R}^n, \left| \Phi^Q \left( \frac{2\pi}{P} z, T \right) \right| \geq \varrho \right\}.$$  

Then

$$\text{Card} (\Omega (\Phi^Q, T, \varrho, P) \cap \mathbb{Z}^n) \ll P^n \left( \ln \varrho^{-1} \right)^{\sum_{s=1}^{n} \nu_s},$$

for any $\varrho > 0$ and fixed $T > 0$ and $\nu_s \in (0, 1/2)$, $1 \leq s \leq n$, as $P \to \infty$. Let

$$\Omega'_{\varrho} (\Phi^Q, T, \varrho, P)$$

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\[ z = (z_1, \ldots, z_n) \in \mathbb{R}^n, \quad \exp \left( -CT \sum_{s=1}^{n} \left| \frac{2\pi}{P} z_s \right|^{\nu_s} \right) \geq \theta \]

then \( \Omega (\Phi^Q, T, \varrho, P) \subset \Omega'_\varrho (\Phi^Q, T, \varrho, P) \) and

\[
\text{Card} \left( \Omega'_\varrho (\Phi^Q, T, \varrho, P) \cap \mathbb{Z}^n \right) \geq P^n \left( \ln \frac{1}{\varrho} \right)^{\sum_{s=1}^{n} \nu_s^{-1}}, \quad (5.14)
\]

as \( P \to \infty \).

Proof. From (5.4) it follows that

\[
\Omega (\Phi^Q, T, \varrho, P) \subset \Omega'_\varrho (\Phi^Q, T, \varrho, P) \]

\[
= \left\{ z \in \mathbb{R}^n, \sum_{s=1}^{n} \left| \frac{CT}{\ln \frac{1}{\varrho} - 1} \frac{2\pi}{P} z_s \right|^{\nu_s} \leq 1 \right\}.
\]

Since the boundary of \( \Omega'_\varrho (\Phi^Q, T, \varrho) \) is piecewise smooth then

\[
\text{Card} \left( \Omega'_\varrho (\Phi^Q, T, \varrho, P) \cap \mathbb{Z}^n \right) \sim \text{Vol}_n \left( \Omega'_\varrho (\Phi^Q, T, \varrho, P) \right),
\]

as \( P \to \infty \). Hence

\[
\text{Vol}_n \left( \Omega'_\varrho (\Phi^Q, T, \varrho, P) \right) = \int_{\Omega'_\varrho (\Phi^Q, T, \varrho, P)} d\mathbf{z}
\]

\[
= \prod_{s=1}^{n} \left( \frac{\ln \frac{1}{\varrho} - 1}{CT} \right)^{\nu_s^{-1}} \left( \frac{P}{2\pi} \right)^n \text{Vol}_n \left( B (\nu_1, \ldots, \nu_n) \right),
\]

\[
= (CT)^{-\sum_{s=1}^{n} \nu_s^{-1}} \left( \frac{P}{2\pi} \right)^n \text{Vol}_n \left( B (\nu_1, \ldots, \nu_n) \right) \left( \ln \frac{1}{\varrho} \right)^{\sum_{s=1}^{n} \nu_s^{-1}},
\]

where

\[
B (\nu_1, \ldots, \nu_n) := \left\{ z = (z_1, \ldots, z_n) \in \mathbb{R}^n, \sum_{s=1}^{n} |z_s|^{\nu_s} \leq 1 \right\}
\]

and \( \nu_1 > 0, \ldots, \nu_n > 0 \). It is known [105] that

\[
\text{Vol}_n B (\nu_1, \ldots, \nu_n) = 2^n \prod_{s=1}^{n} \frac{\Gamma (1 + \nu_s)}{\Gamma (1 + \sum_{s=1}^{n} \nu_s)}.
\]

Hence

\[
\text{Card} \left( \Omega (\Phi^Q, T, \varrho, P) \cap \mathbb{Z}^n \right) \ll \text{Card} \left( \Omega'_\varrho (\Phi^Q, T, \varrho, P) \cap \mathbb{Z}^n \right)
\]

\[
\approx P^n \left( \ln \frac{1}{\varrho} \right)^{\sum_{s=1}^{n} \nu_s^{-1}}, \quad (5.15)
\]

as \( P \to \infty \). \( \square \)
Consider the measure space \( (\mathbb{P}^n, \mathcal{L}, dx) \), where \( \mathbb{P}^n = \mathbb{R}^n / P\mathbb{Z}^n \) is the \( n \)-dimensional torus, \( \mathcal{L} \) is the Lebesgue \( \sigma \)-algebra and \( dx \) is the Lebesgue measure on \( \mathbb{P}^n \). Define the function class

\[
\Lambda := \left\{ f(x) = \sum_{\mathbf{m} \in \mathbb{Z}^n} c_{\mathbf{m}} \varphi_{\mathbf{m}}(x) \right\}, \quad \mathbf{m} = (m_1, \ldots, m_n),
\]

where \( |c_{\mathbf{m}}| \leq \lambda_{\mathbf{m}} \) and

\[
\varphi_{\mathbf{m}}(x) := P^{-n/2} \exp \left( \frac{2\pi i}{P} \langle \mathbf{m}, x \rangle \right), \quad \mathbf{m} \in \mathbb{Z}^n.
\]

Observe that the system \( \{ \varphi_{\mathbf{m}}(x), \mathbf{m} \in \mathbb{Z}^n \} \) is orthonormal and

\[
L = \sup_{\mathbf{m} \in \mathbb{N}} \| \varphi_{\mathbf{m}} \|_\infty = P^{-n/2}.
\]

Let

\[
\lambda_{\mathbf{m}} = \exp \left( -CT \sum_{s=1}^{n} \left| \frac{2\pi}{P} m_s \right|^{\nu_s} \right).
\]

Recall that

\[
\left| \Phi^Q \left( -\frac{2\pi}{P} \mathbf{m}, T \right) \right| \leq \exp \left( -CT \sum_{s=1}^{n} \left| \frac{2\pi}{P} m_s \right|^{\nu_s} \right).
\]

Hence

\[
\hat{p}^Q_T(x) = \frac{1}{P^n} \sum_{\mathbf{m} \in \mathbb{Z}^n} \Phi^Q \left( -\frac{2\pi}{P} \mathbf{m}, T \right) \exp \left( \frac{2\pi i}{P} \langle \mathbf{m}, x \rangle \right) \in \Lambda.
\]

**Theorem 26**  In our notation

\[
\kappa_m(\Lambda, L_q(\mathbb{P}^n, \mathcal{L}, dx)) \gg P^{-1/2+1/q} \exp \left( - \left( P^{-n} m \right) \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1} \right),
\]

as \( m \to \infty \).

**Proof.** Let \( dx \) be a Lebesgue measure on \( \Omega = 2^{-1} P Q_n \), where

\[
Q_n := \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \max |x_k| \leq 1, 1 \leq k \leq n \}
\]

is the unit cube in \( \mathbb{R}^n \). From Theorem 24 we get

\[
\kappa_m(\Lambda, L_q(\mathbb{P}^n, \mathcal{L}, dx)) \gg L^{-1} \left( \int_{2^{-1} P Q_n} dx \right)^{-1+1/q} \lambda_{m+1}
\]

\[
\gg P^{n/2} P^{n(-1+1/q)} \lambda_{m+1} = P^{-1/2+1/q} \lambda_{m+1}.
\]

(5.16)
Let \( m > P^n (\ln \varrho^{-1}) \sum_{s=1}^{n} \nu_s^{-1} \) then, by Theorem 25,
\[
\varrho \asymp \exp \left( - \left( P^{-n} m \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1} \right) \right) \tag{5.17}
\]
and applying (5.14) we get \( \lambda_{m+1} \asymp \varrho \). Hence, using (5.17) and (5.16) we obtain
\[
\kappa_m (\Lambda, L_q (P^T, L, dx)) > P^{-1/2+1/q'} \exp \left( - \left( P^{-n} m \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1} \right) \right),
\]
as \( m \to \infty \). □

### 5.5 Approximation of density functions by \( m \)-term exponential sums

The next statements deal with approximation of functions using \( m \)-term exponential sums with spectrum in the domain \( \Omega'_{1/R} \).

**Theorem 27** Let \( 2 \leq q \leq \infty \), \( 1/q + 1/q' = 1 \), \( \nu_s \in (0, 1/2) \), \( 1 \leq s \leq n \),
\[
m := P^n (\ln R) \sum_{s=1}^{n} \nu_s^{-1} .
\]
Then in our notation
\[
E (m, P) := \left\| \hat{p}^P_T (x) - \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n \cap \Omega'_{1/R}} \Phi^Q \left( \frac{-2\pi i}{P} m, T \right) \exp \left( \frac{2\pi i}{P} (m, x) \right) \right\|_{L_q (\mathbb{R}^n)}
\]
\[
\ll (m P^{-n}) \left( 1 - \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1} \right)^{q'/q} \exp \left( - (m P^{-n}) \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1} \right),
\]
as \( m, P \to \infty \).

**Proof.** Recall that the system of functions
\[
\varphi_m (x) := P^{-n/2} \exp \left( \frac{2\pi i}{P} (m, x) \right), \quad m \in \mathbb{Z}^n, \quad x \in \frac{P}{2} Q_n
\]
is uniformly bounded, \( |\varphi_m (x)| \leq P^{-n/2}, \forall m \in \mathbb{Z}^n \) and orthonormal in \( L_2 (\frac{P}{2} Q_n) \).

Let \( \rho \to \infty \). Then, by (5.15)
\[
\text{Vol}_n (\Omega'_{1/\rho}) \asymp P^n (\ln \rho) \sum_{s=1}^{n} \nu_s^{-1} := V (\rho).
\]

Applying Riesz theorem (see Appendix I, Theorem 44 we get
\[
E (m, P) = \left\| \frac{1}{P^n} \sum_{m \in (\mathbb{R}^n \cap \Omega'_{1/R}) \cap \Omega^n} \Phi^Q \left( \frac{-2\pi i}{P} m, T \right) \exp \left( \frac{2\pi i}{P} (m, x) \right) \right\|_{L_q (\mathbb{R}^n)}
\]

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\[
\frac{1}{P^{n/2}} \left\| \sum_{m \in \left( \mathbb{R}^n \setminus \Omega_{1/R} \right) \cap \mathbb{Z}^n} \Phi^Q \left( -\frac{2\pi}{P} m, T \right) \varphi_m (x) \right\|_{L_q \left( \frac{Q_n}{4} \right)} \lesssim P^{-n/2} P^{-\left( n/2 \right)} \left( \int_{\mathbb{R}^n} \rho^{-q'} dV (\rho) \right)^{1/q'} := P^{-n/q'} (I (R))^{1/q'} ,
\]

where

\[
I (R) = \int_{\mathbb{R}^n} \rho^{-q'} dV (\rho)
\]

\[
= P^n \left( \sum_{s=1}^n \nu_s^{-1} \right) \int_{\mathbb{R}^n} \rho^{-q'-1} \left( \ln \rho \right)^{\sum_{s=1}^n \nu_s^{-1} - 1} d\rho. \tag{5.18}
\]

Observe that \( \nu_s \in (0, 1/2) \). Hence \( \sum_{s=1}^n \nu_s^{-1} - 1 > 0 \). Let \( \alpha > 1 \) and \( \beta > 0 \).

Then

\[
\int_{\mathbb{R}^n} x^{-\alpha} (\ln x)^\beta dx
\]

\[
= \frac{1}{-\alpha + 1} x^{-\alpha+1} (\ln x)^\beta \bigg|_{-\alpha+1} - \int_{\mathbb{R}^n} \frac{1}{-\alpha + 1} x^{-\alpha+1} (\ln x)^\beta - 1 x^{-1} dx
\]

\[
= \frac{1}{\alpha - 1} R^{-\alpha+1} (\ln R)^\beta + \frac{\beta}{\alpha - 1} \int_{\mathbb{R}^n} x^{-\alpha} (\ln x)^{\beta-1} dx
\]

\[
= \frac{1}{\alpha - 1} R^{-\alpha+1} (\ln R)^\beta, \quad R \to \infty, \tag{5.19}
\]

since

\[
\lim_{R \to \infty} \frac{\int_{\mathbb{R}^n} x^{-\alpha} (\ln x)^{\beta-1} dx}{\int_{\mathbb{R}^n} x^{-\alpha} (\ln x)^\beta dx} = 0.
\]

Comparing (5.18) and (5.19) we get

\[
I (R) \ll P^n R^{-q'} (\ln R)^{\sum_{s=1}^n \nu_s^{-1} - 1}, \quad R \to \infty.
\]

Hence

\[
E (m, P) \ll R^{-1} (\ln R)^{\left( \sum_{s=1}^n \nu_s^{-1} - 1 \right)/q'}, \quad R \to \infty.
\]

This means that using

\[
m = P^n (\ln R)^{\sum_{s=1}^n \nu_s^{-1}}
\]

harmonics from \( \Omega_{1/R} \cap \mathbb{Z}^n \) we get the error of approximation

\[
E (m, P) \ll \left( m P^{-n} \right)^{\left( 1 - \left( \sum_{s=1}^n \nu_s^{-1} \right)^{-1} \right)/q'} \exp \left( - \left( m P^{-n} \right)^{\left( \sum_{s=1}^n \nu_s^{-1} \right)^{-1}} \right),
\]

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as \( m, P \to \infty \). □

Comparing Theorem 26 and Theorem 27 we see that the domain of truncation \( \Omega_{1/R} \) is optimal in exponential scale in the sense of \( n \)-cowidth.

Applying Theorem 22 and Theorem 27 we get the following statement.

**Corollary 28**

Let \( 2 \leq q \leq \infty \) and \( b := \min \{ b_s \mid 1 \leq s \leq n \} \) Then in our notation

\[
E_1(P) + E(m, P) = \left\| p_1^Q(x) - \frac{1}{P^{m}} \sum_{m \in \Omega_{1/R} \cap \mathbb{Z}^n} \Phi^Q \left( - \frac{2\pi}{P} m, T \right) \exp \left( \frac{2\pi i}{P} \langle m, x \rangle \right) \right\|_{L_q(Q_R)}
\]

\[\ll \exp (-Pb) P^{n/q} + (mP^{-n}) \left( \frac{1 - (\sum_{s=1}^{n} \nu_s^{-1})^{-1}}{q} \right) \exp \left( - (mP^{-n}) \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1} \right),\]

as \( m, P \to \infty \).

Let for simplicity \( q = \infty \). Let \( P \) in Corollary 28 be such that

\[ Pb = (mP^{-n}) \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1}, \]

or

\[ P = \left( b^{-1} m \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1} \right)^{1 + n \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1}}, \tag{5.20} \]

then

\[ E_1(P) + E(m, P) \ll \exp \left( -am^k \right) m^h, \ m \to \infty, \]

where

\[ a := b^{-1} \left( 1 + n \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1} \right)^{-1}, \]

\[ k := \frac{\left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1}}{1 + n \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1}}, \]

and

\[ h := \frac{1 - \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1}}{1 + n \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1}}. \]

This means that using \( m \) harmonics with spectrum in \( \Omega_{1/R} \), where \( R = \exp \left( (P^{-n} m) \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1} \right) \)

and \( P \) is defined by (5.20) we get the error of convergence \( \exp \left( -am^k \right) m^h \) as \( m \to \infty \).
Chapter 6

Option pricing

6.1 Introduction

Pricing of high-dimensional options is a deep problem of Financial Mathematics. The main aim of this chapter is to develop new simple and practical methods of pricing of basket options. As a motivating example consider a frictionless market with no arbitrage opportunities with a constant riskless interest rate $r > 0$. Let $S_{j,t}, 1 \leq j \leq n, t \geq 0,$ be $n$ asset price processes. The common spread option with maturity $T > 0$ and strike $K \geq 0$ is the contract that pays $H = \left(S_{1,T} - \sum_{j=2}^{n} S_{j,T} - K\right)_+$ at time $T$. There is a wide range of such options traded across different sectors of the financial markets. Assuming the existence of a risk-neutral equivalent martingale measure $Q$ we get the following pricing formula for the value $V$ of the spread option at time 0,

$$V = \exp \left(-rT\right) \mathbb{E}^Q \left[H\right],$$

where $H$ is a reward function and the expectation is taken with respect to the equivalent martingale measure.

There is an extensive literature on spread options and their applications. In particular, if $K = 0$ a spread option is the same as an option to exchange one asset for another. An explicit solution in this case has been obtained by Margrabe [85]. Margrabe model assumes that $S_{t,1}$ and $S_{t,2}$ follow a geometric Brownian motion whose volatilities $\sigma_1$ and $\sigma_2$ do not need to be constant, but the volatility $\sigma$ of $S_{t,1}/S_{t,2}$ is a constant, $\sigma = \left(\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho\right)$, where $\rho$ is the correlation coefficient of the Brownian motions $S_{1,t}$ and $S_{2,t}$. Margrabe formula states that

$$V = \exp \left(-q_1 T\right) S_{0,1} \Phi\left(d_1\right) - \exp \left(-q_2 T\right) S_{0,2} \Phi\left(d_2\right),$$

where $\Phi$ denotes the cumulative distribution for a standard Normal distribution,

$$d_1 = \frac{1}{\sigma T^{1/2}} \left(\ln \left(\frac{S_{0,1}}{S_{0,2}}\right) + \left(q_1 - q_2 + \frac{\sigma}{2}\right) T\right),$$

$$d_2 = \frac{1}{\sigma T^{1/2}} \left(\ln \left(\frac{S_{0,1}}{S_{0,2}}\right) + \left(q_1 - q_2 - \frac{\sigma}{2}\right) T\right).$$
\[ d_2 = d_1 - \sigma T^{1/2} \] and \( q_1, q_2 \) are the constant continuous dividend yields.

Unfortunately, in the case where \( K > 0 \) and \( S_{t,1}, S_{t,2} \) are geometric Brownian motions, no explicit pricing formula is known. In this case various approximation methods have been developed. There are three main approaches: Monte Carlo techniques which are most convenient for high-dimensional situations because the convergence is independent of the dimension, fast Fourier transform methods studied in [20] and PDEs. Observe that PDE based methods are suitable if the dimension of the PDE is low (see, e.g. [90, 32, 99, 106] for more information). The usual PDE’s approach is based on numerical approximation resulting in a large system of ordinary differential equations which can then be solved numerically.

Approximation formulas usually allow quick calculations. In particular, a popular among practitioners Kirk formula [57] gives a good approximation to the spread call (see also Carmona-Durrleman procedure [19, 78]). Various applications of the fast Fourier transform have been considered in [28, 79]. Different approaches of pricing basket options using geometric Brownian motion have been discussed in [9, 77, 56, 86, 88].

It is well-known that the Merton-Black-Scholes theory becomes much more efficient if additional stochastic factors are introduced. Consequently, it is important to consider a wider family of Lévy processes. Stable Lévy processes have been used first in this context by Mandelbrot [82] and Fama [37]. From the 90th Lévy processes became very popular (see, e.g. [83, 84, 13, 14, 29] and references therein). We present here a general pricing formula which is applicable for a wide range of jump-diffusion models [67, 68].

6.2 Hurd-Zhou theorem

In this section we prove a technical result (see [52, 51]) which is important in our applications. Let \( \Gamma (z) \) be the gamma function, \( \Gamma (\xi) := \int_0^\infty x^{\xi - 1} \exp (-x) \, dx, \, \xi \in \mathbb{C} \setminus \{ -\mathbb{N} \cup \{0\} \} \).

The proof is based on several lemmas.

**Lemma 29** Let

\[ H (x_1, x_2) := (\exp (x_1) - \exp (x_2) - 1)_+ . \]

Then for any real numbers \( \epsilon = (\epsilon_1, \epsilon_2), \, \epsilon_2 > 0, \, \epsilon_1 + \epsilon_2 < -1, \)

\[ H (x_1, x_2) = (2\pi)^{-2} \int_{\mathbb{R}^2 + i\epsilon} \exp (i \langle u, x \rangle) g (u) \, du \]

\[ = (2\pi)^{-2} \int_{-\infty + i\epsilon_1}^{\infty + i\epsilon_1} \int_{-\infty + i\epsilon_2}^{\infty + i\epsilon_2} \exp (i (x_1 u_1 + x_2 u_2)) g (u_1, u_2) \, du_1 du_2, \]

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where
\[ g(u_1, u_2) = \frac{\Gamma(i(u_1 + u_2) - 1)\Gamma(-iu_2)}{\Gamma(iu_1 + 1)}. \]

**Proof.** Let \( \epsilon_2 > 0 \) and \( \epsilon_1 + \epsilon_2 < -1 \) then using definition of \( H(x) \) it is possible to show that

\[ \exp(\langle x, \epsilon \rangle) H(x) = \exp(x_1\epsilon_1 + x_2\epsilon_2) H(x_1, x_2) \]
\[ = \exp(x_1\epsilon_1 + x_2\epsilon_2) (\exp(x_1) - \exp(x_2) - 1)_+ \in L_2(\mathbb{R}^2). \]

Hence, by the Plancherel theorem (see Appendix II, Theorem 42) there is such function \( r(u) \in L_2(\mathbb{R}^2) \) that

\[ \exp(\langle x, \epsilon \rangle) H(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} \exp(i \langle x, u \rangle) r(u) du. \]

Consequently,
\[ H(x) = (2\pi)^{-2} \int_{\mathbb{R}^2} \exp(i \langle x, u \rangle - \langle x, \epsilon \rangle) r(u) du \]
\[ = (2\pi)^{-2} \int_{\mathbb{R}^2} \exp(i \langle x, u + i\epsilon \rangle) r(u) du \]
\[ = (2\pi)^{-2} \int_{\mathbb{R}^2+i\epsilon} \exp(i \langle x, u \rangle) r(u - i\epsilon) du. \]

and

\[ r(u) = \int_{\mathbb{R}^2} \exp(-i \langle x, u \rangle) \exp(\langle x, \epsilon \rangle) H(x) dx \]
\[ = \int_{\mathbb{R}^2} \exp(-i \langle x, u + i\epsilon \rangle) H(x) dx. \]

Let \( r(u - i\epsilon) := g(u) \) then
\[ g(u) = \int_{\mathbb{R}^2} \exp(-i \langle x, u \rangle) H(x) dx \]
\[ = \int_{\mathbb{R}^2} \exp(-i(x_1u_1 + x_2u_2)) (\exp(x_1) - \exp(x_2) - 1)_+ dx_1dx_2. \]

Clearly, \( \exp(x_1) - \exp(x_2) - 1)_+ \geq 0 \) if \( x_1 \geq 0 \) and \( \exp(x_1) - \exp(x_2) - 1 \geq 0. \)

Hence
\[ g(u_1, u_2) \]
\[ = \int_0^\infty \exp(-iu_1x_1) \]
\begin{align*}
\times & \left(\int_{-\infty}^{\ln(\exp(x_1)-1)} \exp(-iu_2x_2) \left(\left(\exp(x_1)-1\right) - \exp(x_2)\right) dx_2\right) dx_1 \\
&= \int_0^\infty \exp(-iu_1x_1)(\exp(x_1)-1)^{1-iu_2} \left((-iu_2)^{-1} - (1 - iu_2)^{-1}\right) dx_1.
\end{align*}

Making change of variable \(z = \exp(-x_1)\) we get

\begin{align*}
g(u_1, u_2) &= \frac{1}{(-iu_2)(1-iu_2)} \int_0^1 z^{iu_1-1} \left(\frac{1-z}{z}\right)^{1-iu_2} dz \\
&= \frac{1}{(-iu_2)(1-iu_2)} \left(\int_0^1 z^{i(u_1+u_2)-1} \left(1-z\right)^{(2-iu_2)-1} dz \right) \\
&= \frac{1}{(-iu_2)(1-iu_2)} B(i(u_1+u_2)-1, (2-iu_2)),
\end{align*}

where

\[ B(a, b) := \int_0^1 z^{a-1} (1-z)^{b-1} dz = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} \]

is the Beta function which is defined for \(\text{Re}a > 0, \text{Re}b > 0\). Hence,

\begin{align*}
g(u_1, u_2) &= \frac{\Gamma(i(u_1+u_2)-1) \Gamma(-iu_2 + 2)}{(-iu_2)(1-iu_2) \Gamma(iu_1 + 1)} \\
&= \frac{\Gamma(i(u_1+u_2)-1) \Gamma(-iu_2)}{\Gamma(iu_1 + 1)},
\end{align*}

since \(\Gamma(-iu_2 + 2) = (1 - iu_2) \Gamma(-iu_2 + 1) = (-iu_2)(1 - iu_2) \Gamma(-iu_2 + 1)\). □

**Lemma 30** Let \(z \in \mathbb{R}, x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) and \(u = (u_1, \ldots, u_n) \in \mathbb{C}^n\), \(\text{Im}u_k > 0, 1 \leq k \leq n\). Then

\[ \int_{\mathbb{R}^n} \delta \left( \exp(z) - \sum_{k=1}^n \exp(x_k) \right) \exp(z - i \langle u, x \rangle) \, dx \]

\[ = \prod_{k=1}^n \frac{\Gamma(-i u_k)}{\Gamma(-i \sum_{k=1}^n u_k)} \exp \left(-iz \sum_{k=1}^n u_k \right), \]

where \(\delta(\cdot)\) denotes the delta function.

**Proof.** Making change of variables \(\rho = \exp(z)\) and \(\sigma_k = \exp(x_k)\) we get

\[ I_n := \int_{\mathbb{R}^n} \delta \left( \exp(z) - \sum_{k=1}^n \exp(x_k) \right) \exp(z - i \langle u, x \rangle) \, dx \]

\[ = \rho \int_{\rho Q^n} \delta \left( \rho - \sum_{k=1}^n \sigma_k \right) \prod_{k=1}^n \sigma_k^{-iu_k} - 1 \prod_{k=1}^n d\sigma_k, \]

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where

\[ Q^n := \{ x = (x_1, \ldots, x_n) \mid 0 < x_k \leq 1, 1 \leq k \leq n \} \]

since

\[ \int_{\mathbb{R}^n \setminus Q^n} \delta \left( \rho - \sum_{k=1}^{n} \sigma_k \right) \prod_{k=1}^{n} \sigma_k^{-i\mu_k - 1} \prod_{k=1}^{n} d\sigma_k = 0. \]

We proceed by induction. It is easy to check that \( I_1 = \rho^{i\mu_1} \), or Lemma 30 is true for \( n = 1 \). If Lemma 12 is true for \( m = n \), then for \( m = n + 1 \) we get

\[ I_{n+1} = \rho \int_{\rho Q^{n+1}} \delta \left( \rho - \sigma_{n+1} \right) \prod_{k=1}^{n} \sigma_k^{-i\mu_k - 1} \prod_{k=1}^{n} \sigma_k^{-i\mu_k - 1} d\sigma_{n+1} \prod_{k=1}^{n} d\sigma_k, \]

We can rewrite \( I_{n+1} \) as

\[ I_{n+1} = \rho \int_{\rho Q^{n+1}} \delta \left( \rho - \sigma_{n+1} \right) J_n (\rho, \sigma_{n+1}) d\sigma_{n+1}, \]

where

\[ J_n (\rho, \sigma_{n+1}) := (\rho - \sigma_{n+1}) \int_{\rho Q^n} \prod_{k=1}^{n} \sigma_k^{-i\mu_k - 1} \delta \left( \rho - \sigma_{n+1} \right) \prod_{k=1}^{n} d\sigma_k. \]

By the induction hypothesis

\[ J_n (\rho, \sigma_{n+1}) = \prod_{k=1}^{n} \frac{\Gamma (-i\mu_k)}{\Gamma (-i \sum_{k=1}^{n} u_k)} \exp \left( -i \sum_{k=1}^{n} u_k \ln (\rho - \sigma_{n+1}) \right) \]

\[ = \prod_{k=1}^{n} \frac{\Gamma (-i\mu_k)}{\Gamma (-i \sum_{k=1}^{n} u_k)} (\rho - \sigma_{n+1})^{-i \sum_{k=1}^{n} u_k}. \]

Hence

\[ I_{n+1} = \prod_{k=1}^{n} \frac{\Gamma (-i\mu_k)}{\Gamma (-i \sum_{k=1}^{n} u_k)} \rho \int_{\rho Q^{n+1}} \prod_{k=1}^{n} \sigma_k^{-i\mu_k - 1} \left( \rho - \sigma_{n+1} \right)^{-i \sum_{k=1}^{n} u_k} d\sigma_{n+1} \]

\[ = \prod_{k=1}^{n} \frac{\Gamma (-i\mu_k)}{\Gamma (-i \sum_{k=1}^{n} u_k)} \rho^{-i \sum_{k=1}^{n} u_k} \int_{0}^{\rho} \sigma_{n+1}^{-i \sum_{k=1}^{n} u_k - 1} \left( 1 - \frac{\sigma_{n+1}}{\rho} \right)^{-1 - i \sum_{k=1}^{n} u_k} d\sigma_{n+1}. \]

Making change of variables \( \xi := \sigma_{n+1}/\rho \), we obtain

\[ I_{n+1} = \prod_{k=1}^{n} \frac{\Gamma (-i\mu_k)}{\Gamma (-i \sum_{k=1}^{n} u_k)} \rho^{-i \sum_{k=1}^{n} u_k} \int_{0}^{1} (\rho \xi)^{-i \sum_{k=1}^{n} u_k - 1} (1 - \xi)^{-1 - i \sum_{k=1}^{n} u_k} \rho d\xi \]

\[ = \prod_{k=1}^{n} \frac{\Gamma (-i\mu_k)}{\Gamma (-i \sum_{k=1}^{n} u_k)} \rho^{-i \sum_{k=1}^{n} u_k} \int_{0}^{1} \xi^{-i \sum_{k=1}^{n} u_k - 1} (1 - \xi)^{-1 - i \sum_{k=1}^{n} u_k} d\xi \]
\[ \prod_{k=1}^{n} \frac{\Gamma(-iu_k)}{\Gamma(-i \sum_{k=1}^{n} u_k)} \exp(-iz \sum_{k=1}^{n} u_k). \]

**Theorem 31** (Hurd-Zhou) Let \( n \geq 2 \). For any real numbers \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) with \( \epsilon_m > 0 \) for \( 2 \leq m \leq n \) and \( \epsilon_1 < -1 - \sum_{m=2}^{n} \epsilon_m \),

\[
\left( \exp(x_1) - \sum_{m=2}^{n} \exp(x_m) - 1 \right) = (2\pi)^{-n} \int_{\mathbb{R}^{n+ie}} \exp(i \langle u, x \rangle) g(u) \, du,
\]

where \( x = (x_1, \ldots, x_n) \) and, for \( u = (u_1, \ldots, u_n) \in \mathbb{C}^n \),

\[
g(u) = \frac{\Gamma(i \sum_{m=1}^{n} u_m - 1) \prod_{m=2}^{n} \Gamma(-iu_m)}{\Gamma(iu_1 + 1)}. \tag{6.1}
\]

**Proof.** We need to show (6.1). Observe that

\[
\int_{\mathbb{R}} \delta \left( \exp(z) - \sum_{k=2}^{n} \exp(x_k) \right) \exp(z) \, dz = 1.
\]

Hence

\[
g(u)
\]

\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}} \delta \left( \exp(z) - \sum_{k=2}^{n} \exp(x_k) \right) \left( \exp(x_1) - \sum_{k=2}^{n} \exp(x_k) - 1 \right) \exp(z - i \langle u, x \rangle) \, dz \, dx
\]

\[
= \int_{\mathbb{R}^2} \left( \exp(x_1) - \exp(z) - 1 \right) \exp(z - i \langle u, x \rangle) \, dz \, dx
\]

\[
\times \left( \int_{\mathbb{R}^{n-1}} \delta \left( \exp(z) - \sum_{k=2}^{n} \exp(x_k) \right) \exp(z - i \langle u, x \rangle) \, dx_2 \cdots dx_n \right) \, dx_1 \, dz.
\]

Applying Lemma 30 and Lemma 29 we get

\[
g(u) = \frac{\prod_{k=2}^{n} \Gamma(-iu_k)}{\Gamma(-i \sum_{k=2}^{n} u_k)}
\]

\[= \prod_{k=1}^{n} \frac{\Gamma(-iu_k)}{\Gamma(-i \sum_{k=1}^{n} u_k)} \exp(-iz \sum_{k=1}^{n} u_k). \]
\[
\times \int_{\mathbb{R}^2} \exp \left( -iu_1 x_1 - iz \sum_{k=2}^{n} \exp (u_k) \right) (\exp (x_1) - \exp (z) - 1)_+ dx_1 dz
\]

\[
= \frac{\Gamma \left( i \sum_{k=1}^{n} u_k - 1 \right) \prod_{k=2}^{n} \Gamma (-iu_k)}{\Gamma (iu_1 + 1)}.
\]

\[\square\]

### 6.3 Approximation formulas

In applications it is important to construct a pricing theory which includes a wide range of reward functions \( H \). For instance, the reward function for a spread option which is given by

\[
H = H (x) = H (x_1, \cdots, x_n)
\]

\[
= \left( S_{0,1} \exp (x_1) - \sum_{j=2}^{n} S_{0,j} \exp (x_j) - K \right)_+
\]

admits an exponential growth with respect to \( x_1 \) as \( x_1 \to \infty \). Hence we need to introduce the following definition.

**Definition 32** We say that the model process \( S_t = \{S_{j,t}, \ 1 \leq j \leq n\} \) is adapted to the payoff \( H \) if \( \mathbb{E}^{Q}[H] < \infty \).

Clearly, if \( \mathbb{E}^{Q}[H] = \infty \) then the option can not be priced. Recall that the operator of expectation is taken with respect to the density function \( p^{Q}_T \) which satisfies the equivalent martingale measure condition (4.7).

The next statement reduces the reward function to a canonical form.

**Lemma 33** In our notation

\[
V = K \exp (-rT) \int_{\mathbb{R}^n} \left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right)_+ p^{Q}_T (y - \mathbf{d}) \, dy,
\]

where

\[
\mathbf{d} := (d_1, \cdots, d_n), \quad d_j = \ln \left( \frac{S_{0,j}}{K} \right), \quad 1 \leq j \leq n.
\]

**Proof.** Recall that \( V = \exp (-rT) \mathbb{E}^{Q}[H] \). In our case

\[
H = \left( S_{1,T} - \sum_{j=2}^{n} S_{j,T} - K \right)_+.
\]
where
\[ S_{j,T} = S_{j,0} \exp (U_{j,T}) , \quad 1 \leq j \leq n . \]

This means that
\[
V = \exp (-rT) \int_{R^n} \left( S_{0,1} \exp (x_1) - \sum_{j=2}^{n} S_{0,j} \exp (x_j) - K \right) p_T^0(x) \, dx ,
\]
\[
= K \exp (-rT) \times \int_{R^n} \left( \exp \left( x_1 + \ln \left( \frac{S_{0,1}}{K} \right) \right) - \sum_{j=2}^{n} \exp \left( x_j + \ln \left( \frac{S_{0,j}}{K} \right) \right) - 1 \right) p_T^0(x) \, dx ,
\]
where \( S_{0,j} , \ 1 \leq j \leq n \) are the respective spot prices. Making the change of variables
\[ y_j = x_j + \ln \left( \frac{S_{0,j}}{K} \right) , \quad 1 \leq j \leq n , \]
we get
\[
V = K \exp (-rT) \int_{R^n} \left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right) p_T^0(y-d) \, dy ,
\]
where
\[
\mathbf{d} := (d_1, \cdots , d_n) , \quad d_j = \ln \left( \frac{S_{0,j}}{K} \right) , \quad 1 \leq j \leq n .
\]

□

**Theorem 34** In our notation, for any \( \mathbf{m} = (m_1, \cdots , m_n) \in \mathbb{Z}^n \) and \( \epsilon = (\epsilon_1, \cdots , \epsilon_n) \) with \( \epsilon_m > 0 \) for \( 2 \leq m \leq n \) and \( \epsilon_1 < -1 - \sum_{m=2}^{n} \epsilon_m \), we have
\[
\int_{R^n} \exp \left( \left\langle \frac{2\pi i}{\mathbf{F}} \mathbf{m} + \epsilon , \mathbf{x} \right\rangle \right) H(\mathbf{x}) \, d\mathbf{x}
= \Gamma \left( -\frac{2\pi i}{\mathbf{F}} \sum_{s=1}^{n} m_s - \sum_{s=1}^{n} \epsilon_s - 1 \right) \prod_{s=2}^{n} \Gamma \left( \frac{2\pi i}{\mathbf{F}} m_s + \epsilon_s \right)
\]
\[
\Gamma \left( -\frac{2\pi i}{\mathbf{F}} m_1 - \epsilon_1 + 1 \right).
\]

**Proof.** Observe that
\[
H(\mathbf{x}) = (2\pi)^{-n} \int_{R^n+i\epsilon} \exp (i \langle \mathbf{u}, \mathbf{x} \rangle) g(\mathbf{u}) \, d\mathbf{u}
= (2\pi)^{-n} \int_{R^n} \exp (i \langle \mathbf{z}+i\epsilon, \mathbf{x} \rangle) g(\mathbf{z}+i\epsilon) \, d\mathbf{z}
\]
Then the formal approximat

\[ \tilde{g}(\epsilon, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp \left( \epsilon (z, x) \right) g(z + i\epsilon) \, dz, \]

where the function \( g \) is defined by (6.1). Hence

\[ H(x) \exp(\langle \epsilon, x \rangle) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(\epsilon (z, x)) \, g(z + i\epsilon) \, dz. \]

Since \( H(x) \exp(\langle \epsilon, x \rangle) \in L_2(\mathbb{R}^n) \) then, applying Plancherel theorem (see Appendix II, Theorem 42) and Theorem 31, we get

\[ \mathcal{F}(H(x) \exp(\langle \epsilon, x \rangle))(u) = \int_{\mathbb{R}^n} \exp(-i \langle u, x \rangle) \, H(x) \exp(\langle \epsilon, x \rangle) \, dx = g(u + i\epsilon) \]

This means that

\[ \int_{\mathbb{R}^n} \exp \left( \frac{2\pi i}{P} \langle m, x \rangle \right) H(x) \exp(\langle \epsilon, x \rangle) \, dx = g \left( -\frac{2\pi i}{P} m + i\epsilon \right) \]

\[ = \frac{\Gamma \left( \frac{1}{2} \sum_{s=1}^{n} m_s - \sum_{s=1}^{n} \epsilon_s - 1 \right) \prod_{s=2}^{n} \Gamma \left( \frac{2\pi}{P} m_s + \epsilon_s \right)}{\Gamma \left( -\frac{2\pi}{P} m_1 - \epsilon_1 + 1 \right)}. \]

\[ \square \]

The next statement gives a general approximation formula for spread options which is important in various applications. Observe that it does not show the rate of convergence. This problem will be discussed later. At this stage we just explain how to construct the approximation formula.

Theorem 35 Let

\[ \mathbf{d} := (d_1, \cdots, d_n), \quad d_j = \ln \left( \frac{S_{0,j}}{K} \right), \quad 1 \leq j \leq n \]

and \( \epsilon = (\epsilon_1, \cdots, \epsilon_n), \quad 0 < \epsilon_j \leq b_{+j}, \quad 2 \leq j \leq n, \quad b_{-1} \leq \epsilon_1 < -1 - \sum_{m=2}^{n} \epsilon_m. \)

Then the formal approximat \( \tilde{V} \) for \( V \) can be written as

\[ \tilde{V} = \frac{K \exp (-rT - \langle \mathbf{d}, \epsilon \rangle)}{P^n} \sum_{m \in \Omega_{1/R}} \Phi_Q \left( -\frac{2\pi}{P} m + i\epsilon, T \right) \exp \left( -\frac{2\pi i}{P} \langle \mathbf{m}, \mathbf{d} \rangle \right) \]

\[ \times \frac{\Gamma \left( -\frac{2\pi}{P} \sum_{s=1}^{n} m_s - \sum_{s=1}^{n} \epsilon_s - 1 \right) \prod_{s=2}^{n} \Gamma \left( \frac{2\pi}{P} m_s + \epsilon_s \right)}{\Gamma \left( -\frac{2\pi}{P} m_1 - \epsilon_1 + 1 \right)}, \quad R \to \infty, \quad P \to \infty, \]

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where
\[
\Omega_{1/R}' = \left\{ x \in \mathbb{R}^n, \sum_{s=1}^{n} \left| \frac{CT}{\ln R} \nu_s^{-1} \frac{2\pi}{P} x_s \right| \leq 1 \right\}.
\]

**Proof.** Applying Lemma 33 we get
\[
V = \exp (-rT) \mathbb{E}^Q [H]
\]
\[
= \exp (-rT) \int_{\mathbb{R}^n} \left( S_{0,1} \exp (x_1) - \sum_{j=2}^{n} S_{0,j} \exp (x_j) - K \right) p_{T}^Q (x) \, dx,
\]
\[
= K \exp (-rT) \int_{\mathbb{R}^n} \left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right) p_{T}^Q (y - d) \, dy,
\]
where
\[
d := (d_1, \cdots, d_n), \quad d_j = \ln \left( \frac{S_{0,j}}{K} \right), \quad 1 \leq j \leq n.
\]

For a given \( \epsilon = (\epsilon_1, \cdots, \epsilon_n) \), \( b_{-s} \leq \epsilon_s \leq b_{+s}, \quad 1 \leq s \leq n \) we can apply Cauchy theorem \( n \) times in the domain \( T_n \) defined by (5.1), which is justified by (5.4). Hence
\[
p_{T}^Q (y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp (-i \langle y, x \rangle) \Phi^Q (x, T) \, dx
\]
\[
= (2\pi)^{-n} \int_{\mathbb{R}^n+i\epsilon} \exp (-i \langle y, x \rangle) \Phi^Q (x, T) \, dx
\]
\[
= (2\pi)^{-n} \int_{\mathbb{R}^n} \exp (-i \langle y, x + i\epsilon \rangle) \Phi^Q (x + i\epsilon, T) \, dx
\]
\[
= \exp (\langle y, \epsilon \rangle) (2\pi)^{-n} \int_{\mathbb{R}^n} \exp (-i \langle y, x \rangle) \Phi^Q (x + i\epsilon, T) \, dx.
\]

Let \( y \in \mathbb{F}Q_n \). Recall that \( Q_n = \{ x \mid x = (x_1, \cdots, x_n) \in \mathbb{R}^n, \quad |x_k| \leq 1 \} \). Then from Corollary 5.5 we get
\[
p_{T}^Q (y) \approx \exp (\langle y, \epsilon \rangle) \left( \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q \left( -\frac{2\pi}{P} m + i\epsilon, T \right) \exp \left( \frac{2\pi i}{P} \langle m, y \rangle \right) \right)
\]
and
\[
p_{T}^Q (y - d) \approx \exp (\langle y - d, \epsilon \rangle)
\]
\[
\times \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q \left( -\frac{2\pi}{P} m + i\epsilon, T \right) \exp \left( \frac{2\pi i}{P} \langle m, y - d \rangle \right)
\]

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\[
\approx \exp((y - d, \epsilon)) \times \frac{1}{P^n} \sum_{m \in \Omega_{1/R}^n} \left( \Phi^Q \left( -\frac{2\pi}{P} m + i\epsilon, T \right) \exp \left( -\frac{2\pi i}{P} \langle m, d \rangle \right) \right) \exp \left( \frac{2\pi i}{P} \langle m, y \rangle \right).
\]

Since \( \epsilon_j > 0, 2 \leq j \leq n, \epsilon_1 < -1 - \sum_{j=2}^n \epsilon_j \) then we can apply Theorem 34 to obtain

\[
V = K \exp(-rT) \int_{\mathbb{R}^n} \left( \exp(y_1) - \sum_{j=2}^n \exp(y_j) - 1 \right) \frac{p_1^Q(y - d)}{iT} dy + \]

\[
\approx \frac{K \exp(-rT)}{P^n} \sum_{m \in \Omega_{1/R}^n} \left( \Phi^Q \left( -\frac{2\pi}{P} m + i\epsilon, T \right) \exp \left( -\frac{2\pi i}{P} \langle m, d \rangle \right) \right) \exp \left( \frac{2\pi i}{P} \langle m, y \rangle \right) dy
\]

\[
\times \int_{\mathbb{R}^n} \left( \exp(y_1) - \sum_{j=2}^n \exp(y_j) - 1 \right) \exp(\langle y - d, \epsilon \rangle) \exp \left( -\frac{2\pi i}{P} \langle m, y \rangle \right) dy
\]

\[
= \frac{K \exp(-rT - \langle d, \epsilon \rangle)}{P^n} \sum_{m \in \Omega_{1/R}^n} \left( \Phi^Q \left( -\frac{2\pi}{P} m + i\epsilon, T \right) \exp \left( -\frac{2\pi i}{P} \langle m, d \rangle \right) \right)
\]

\[
\times \int_{\mathbb{R}^n} \left( \exp(y_1) - \sum_{j=2}^n \exp(y_j) - 1 \right) \exp(\langle y, \epsilon \rangle) \exp \left( \frac{2\pi i}{P} \langle m, y \rangle \right) dy
\]

\[
= \frac{K \exp(-rT - \langle d, \epsilon \rangle)}{P^n} \sum_{m \in \Omega_{1/R}^n} \left( \Phi^Q \left( -\frac{2\pi}{P} m + i\epsilon, T \right) \exp \left( -\frac{2\pi i}{P} \langle m, d \rangle \right) \right)
\]

\[
\times \frac{\Gamma \left( -\frac{2\pi i}{P} \sum_{s=1}^n m_s - \sum_{s=1}^n \epsilon_s - 1 \right) \Pi_{s=2}^n \Gamma \left( \frac{2\pi i}{P} m_s + \epsilon_s \right)}{\Gamma \left( -\frac{2\pi i}{P} m_1 - \epsilon_1 + 1 \right)}
\]

\[
= \tilde{V}.
\]

\[ \square \]

**Theorem 36** Let in our notations \( T > 0, 0 < \epsilon_j \leq b_{+j}, 2 \leq j \leq n, \)

\( b_{-1} \leq \epsilon_1 < -1 - \sum_{m=2}^n \epsilon_m, d := (d_1, \cdots, d_n), d_j = \ln \left( \frac{s_{11}^{d_j}}{R} \right) \), \( 1 \leq j \leq n, \)

\( b = \min \{ -b_{-s}, b_{+s}, 1 \leq s \leq n \}, \)

\( M(P, R) := \left\| (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(i \langle -d, x \rangle) \Phi^Q (-x + i\epsilon, T) dx \right\| \]

\[
\left| \frac{1}{P^n} \sum_{m \in \Omega_{1/R}^n} \left( \Phi^Q \left( -\frac{2\pi}{P} m + i\epsilon, T \right) \exp \left( -\frac{2\pi i}{P} \langle m, d \rangle \right) \right) \exp \left( \frac{2\pi i}{P} \langle m, \cdot \rangle \right) \right|_{L^\infty(\mathbb{R}^n)}.
\]

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and \( \tilde{V} \) be the approximant for \( V \) from Theorem 35, then

\[
\delta := |V - \tilde{V}|
\]

\[
\ll \frac{K \exp(-rT) \Gamma \left( -\sum_{s=1}^{n} \epsilon_s \right)}{\Gamma(1-\epsilon_1)} \prod_{s=2}^{n} \Gamma(\epsilon_s) \\
\times \left( \exp(-Pb) + (mP^{-n})^{1 - \left( \sum_{s=1}^{n} \nu_s \right)^{-1}} \exp \left( - (mP^{-n})^{\left( \sum_{s=1}^{n} \nu_s \right)^{-1}} \right) \right)
\]

\[
+ K \exp(-rT) M(P,R) \times \left\| \left( \exp(y_1) - \sum_{j=2}^{n} \exp(y_j) - 1 \right) \exp(\langle y, \epsilon \rangle) \right\|_{L_1(\mathbb{R}^n; \langle \frac{\mu}{\|d\|} \rangle Q_n)}
\]

**Proof.** Let \( \tilde{V} \) be the approximant for \( V \). Since 0 < \( \epsilon_j \leq b_{-j} \), 2 ≤ \( j \) ≤ \( n \), \( b_{-1} \leq \epsilon_1 < -1 - \sum_{m=2}^{n} \epsilon_m \) then we get

\[
\delta = |V - \tilde{V}|
\]

\[
= K \exp(-rT) \left\| \int_{\mathbb{R}^n} \left( \left( \exp(y_1) - \sum_{j=2}^{n} \exp(y_j) - 1 \right) \exp(\langle y, \epsilon \rangle) \right) \exp(i\langle y - d, x \rangle) \Phi^{\mathbb{Q}}(-x + i\epsilon, T) \, dx \right\|
\]

\[
- \frac{1}{P^n} \sum_{m \in \Omega_{1/R}} \left( \Phi^{\mathbb{Q}} \left( \frac{-2\pi x}{P} \right) \exp \left( -\frac{2\pi i}{P} \langle m, d \rangle \right) \right) \exp \left( \frac{2\pi i}{P} \langle m, y \rangle \right) \, dy
\]

\[
:= K \exp(-rT) \int_{\mathbb{R}^n} \left( \exp(y_1) - \sum_{j=2}^{n} \exp(y_j) - 1 \right) \exp(\langle y, \epsilon \rangle) \mu(y) \, dy.
\]

From the Corollary 5.5 it follows that for chosen \( P > 0 \) and \( m > 0 \) we have

\[
|\mu(y)| = \left| (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(i\langle y - d, x \rangle) \Phi^{\mathbb{Q}}(-x + i\epsilon, T) \, dx \right|
\]

\[
- \frac{1}{P^n} \sum_{m \in \Omega_{1/R}} \left( \Phi^{\mathbb{Q}} \left( \frac{-2\pi x}{P} \right) \exp \left( -\frac{2\pi i}{P} \langle m, d \rangle \right) \right) \exp \left( \frac{2\pi i}{P} \langle m, y \rangle \right) \, dy
\]

\[
\ll \exp(-Pb) + (mP^{-n})^{1 - \left( \sum_{s=1}^{n} \nu_s \right)^{-1}} \exp \left( - (mP^{-n})^{\left( \sum_{s=1}^{n} \nu_s \right)^{-1}} \right)
\]
for any $y \in \mathbb{R}^n Q_n - d$. Let us put $m = 0$ in the Theorem 34. Then

$$L_\epsilon := \left\| \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right\|_{L_1(\mathbb{R}^n)}^{\exp (\langle y, \epsilon \rangle)} + \Gamma (1 - \epsilon_1)$$

$$= \Gamma (-\sum_{s=1}^{n} \epsilon_s - 1) \prod_{j=2}^{n} \Gamma (\epsilon_s)$$

for a chosen $\epsilon = (\epsilon_1, \cdots, \epsilon_n)$, $0 < \epsilon_j \leq b_{+j}$, $2 \leq j \leq n$, $b_{+1} \leq \epsilon_1 < -1 - \sum_{m=2}^{n} \epsilon_m$. Observe that $\mathbb{P}_\epsilon Q_n - d \geq \left\{ \mathbb{P}_\epsilon - \|d\|_\infty \right\} Q_n$, where $\|d\|_\infty := \max \{|d_k|, 1 \leq k \leq n\}$. Therefore

$$\int_{(\mathbb{P}_\epsilon - \|d\|_\infty)Q_n} (\exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1) dy \exp (\langle y, \epsilon \rangle) \mu (y) dy$$

$$\leq L_\epsilon \left( \exp (-Pb) + (mP^{-n})^{1 - (\sum_{s=1}^{n} \epsilon_s - 1)^{-1}} \exp \left( - (mP^{-n})^{(\sum_{s=1}^{n} \epsilon_s - 1)^{-1}} \right) \right).$$

Finally, we have

$$\int_{\mathbb{R}^n \setminus (\mathbb{P}_\epsilon - \|d\|_\infty)Q_n} (\exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1) dy \exp (\langle y, \epsilon \rangle) \mu (y) dy$$

$$\leq M (P, R) \left\| \left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right) \exp (\langle y, \epsilon \rangle) \right\|_{L_1(\mathbb{R}^n \setminus (\mathbb{P}_\epsilon - \|d\|_\infty)Q_n)}.$$ 

\[\square\]

Assume, for simplicity, $n = 2$. Let, as before, $\epsilon_1 < -1 - \epsilon_2$ and $\epsilon_2 > 0$. Since $(\exp (y_1) - \exp (y_2) - 1)_+ \geq 0$ if $\exp (y_1) - \exp (y_2) - 1 \geq 0$ and $x_1 \geq 0$ then

$$\left\| (\exp (y_1) - \exp (y_2) - 1)_+ \exp (\langle y, \epsilon \rangle) \right\|_{L_1(\mathbb{R}^2 \setminus (\mathbb{P}_\epsilon - \|d\|_\infty)Q_2)}$$

$$= \int_{L_1(\mathbb{R}^2 \setminus (\mathbb{P}_\epsilon - \|d\|_\infty)Q_2)} (\exp (y_1) - \exp (y_2) - 1)_+ \exp (\langle y, \epsilon \rangle) dy$$

$$:= I_1 + I_2,$$

where

$$I_1 = \int_{-\mathbb{P}_\epsilon + \max \{d_1, d_1\}}^\infty \exp (\epsilon_1 y_1) \left( \int_{-\mathbb{P}_\epsilon + \max \{d_1, d_1\}}^{\ln (\exp (x_1) - 1)} (\exp (y_1) - 1 - \exp (y_2)) \exp (\epsilon_2 y_2) dy_2 \right) dy_1$$

and

$$I_2 = \int_{0}^{-\mathbb{P}_\epsilon + \max \{d_1, d_1\}} \exp (\epsilon_1 y_1) \left( \int_{-\mathbb{P}_\epsilon + \max \{d_1, d_1\}}^{-\ln (\exp (x_1) - 1)} (\exp (y_1) - 1 - \exp (y_2)) \exp (\epsilon_2 y_2) dy_2 \right) dy_1.$$

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It is possible to show that

\[ I_1 \ll \exp \left( -\frac{\epsilon_2 P}{2} \right) \]

and

\[ I_2 \ll \exp \left( \frac{(\epsilon_1 + \epsilon_2 + 1) P}{2} \right) \]

as \( P \to \infty \). Hence

\[
\| (\exp (y_1) - \exp (y_2) - 1)_+ \exp ((y, \epsilon)) \|_{L_1(\mathbb{R}^2 \setminus (\mathbb{R}^2 - \|d\|_\infty)Q_2)} \ll \exp \left( \frac{P}{2} \max \{ -\epsilon_2, \epsilon_1 + \epsilon_2 + 1 \} \right), \ P \to \infty
\]

where \( \epsilon_1 + \epsilon_2 + 1 < 0, \ \epsilon_2 > 0 \).
Appendix I: Measure and integral

$L_p$ spaces

For $0 < p < \infty$, $l_p$ is the space consisting of all sequences $c = \{c_k, \ k \in \mathbb{Z}^n\}$ satisfying

$$\sum_{k \in \mathbb{Z}^n} |c_k|^p < \infty.$$ 

If $p \geq 1$, then

$$\|c\|_p := \left( \sum_{k \in \mathbb{Z}^n} |c_k|^p \right)^{1/p}$$

defines a norm on $l_p$. If $p = \infty$, then the norm on $l_\infty$ is defined by

$$\|c\|_\infty := \sup_{k \in \mathbb{Z}^n} |c_k|.$$ 

If $1 \leq p \leq \infty$, then $l_p$ is a complete normed space with respect to the norm $\|c\|_p$, and therefore is a Banach space.

Let $1 \leq p \leq \infty$ and $(\Omega, \mathcal{F}, \nu)$ be a measure space of functions $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_{p,v} = \|f\|_p := \begin{cases} \left( \int_{\Omega} |f|^p \, d\nu \right)^{1/p}, & 1 \leq p < \infty, \\ \text{ess sup} |f|, & p = \infty \end{cases} < \infty.$$ 

In this case we say that $f \in L_p = L_p(\Omega, \mathcal{F}, \nu)$. For any $1 \leq p \leq \infty$, $L_p$ is a Banach space.

Let $\mathcal{L}$ be the Lebesgue $\sigma$-algebra and $dy$ be the Lebesgue measure.

**Theorem 37** (Young's inequality [98], [49]) Let $f \in L_p(\mathbb{R}^n, \mathcal{L}, dy)$, $g \in L_q(\mathbb{R}^n, \mathcal{L}, dy)$ and $1/p + 1/q = 1/r + 1$. Then

$$\|h\|_r \leq \|f\|_p \|g\|_q.$$
**Theorem 38** (Jensen inequality) Let \((\Omega, \mathcal{F}, \nu)\) be a probability space, i.e. \(\nu(\Omega) = 1\). Let \(f : \Omega \to \mathbb{R}\) be a \(\nu\)-integrable function and \(g : \mathbb{R} \to \mathbb{R}\) be a convex function. Then
\[
\int_{\Omega} g \circ f d\nu \geq g \left( \int_{\Omega} f d\nu \right).
\]

**Fubini and Tonelli theorems**

Fubini theorem allows us to compute a double integral using iterated integrals. As a consequence it gives us sufficient conditions to change the order of integration. It is one of the central tools of the probability theory.

**Theorem 39** (Fubini theorem) Suppose \((A, \mathcal{F}_1, \nu_1)\) and \((B, \mathcal{F}_2, \nu_2)\) are complete measure spaces. Assume that \(f(x, y)\) is \(\nu_1 \times \nu_2\) measurable on \(A \times B\). If
\[
\int_{A \times B} |f(x, y)| d\nu_1 d\nu_2 < \infty,
\]
where the integral is taken with respect to a product measure \(\nu_1 \times \nu_2\) on \(A \times B\), then
\[
\int_{A \times B} f(x, y) d\nu_1 d\nu_2 = \int_A \left( \int_B f(x, y) d\nu_2 \right) d\nu_1
\]
and
\[
= \int_B \left( \int_A f(x, y) d\nu_1 \right) d\nu_2.
\]

If \(\int_{A \times B} |f(x, y)| d\nu_1 d\nu_2 = \infty\), then the two iterated integrals from the right may have different values.

The measure \(\nu\) is called \(\sigma\)-finite if \(\Omega\) is the countable union of measurable sets with finite measure.

Another important theorem for much of probability theory is the following statement.

**Theorem 40** (Tonelli theorem) Let \((A, \mathcal{F}_1, \nu_1)\) and \((B, \mathcal{F}_2, \nu_2)\) are two \(\sigma\)-finite measure spaces and \(f(x, y)\) be a \(\nu_1 \times \nu_2\) measurable function such that \(f(x, y) \geq 0, \forall (x, y) \in A \times B\) then
\[
\int_{A \times B} f(x, y) d\nu_1 d\nu_2 = \int_A \left( \int_B f(x, y) d\nu_2 \right) d\nu_1
\]
and
\[
= \int_B \left( \int_A f(x, y) d\nu_1 \right) d\nu_2.
\]

Any probability space has a \(\sigma\)-finite measure. In this situation Tonelli theorem simply says that if \(f(x, y) \geq 0, \forall (x, y) \in A \times B\) then we can change the order of integration without a hard condition \(\int_{A \times B} |f(x, y)| d\nu_1 d\nu_2 < \infty\) of Fubini theorem.
Radon-Nikodym theorem

Let $(\Omega, \mathcal{F}, \nu)$ be a measure space. Assume that $\nu_1$ and $\nu_2$ are two measures on a measurable set $(\Omega, \mathcal{F})$ and $\nu_2(A) = 0 \Rightarrow \nu_1(A) = 0$ then we say that $\nu_1$ is absolutely continuous with respect to $\nu_2$ (or dominated by $\nu_2$). In this case we shall write $\nu_1 \ll \nu_2$. If $\nu_1 \ll \nu_2$ and $\nu_2 \ll \nu_1$, the measures $\nu_1$ and $\nu_2$ are said to be equivalent, $\nu_1 \asymp \nu_2$.

**Theorem 41** (Radon-Nikodym) Let $\nu_1$ and $\nu_2$ are two $\sigma$–finite measures on a measure space $(\Omega, \mathcal{F})$ and $\nu_1 \ll \nu_2$, then there exists a $\nu_2$–measurable function $f$ with the range $R(f) \subset [0, \infty)$, denoted by $f = d\nu_1/d\nu_2$, such that for any $\nu_2$–measurable set $A$ we have

$$
\nu_1(A) = \int_A f \cdot d\nu_2.
$$

See, e.g. [97] for more information.
Appendix II: Harmonic Analysis

Plancherel theorem

To justify an inversion formula we will need Plancherel theorem (see e.g. [30]). Let in our notation $L_2(\mathbb{R}^n) := L_p(\mathbb{R}^n, \mathcal{L}, dy)$.

**Theorem 42** (Plancherel) The Fourier transform is a linear continuous operator from $L_2(\mathbb{R}^n)$ onto $L_2(\mathbb{R}^n)$. The inverse Fourier transform, $\mathcal{F}^{-1}$, can be obtained by letting

$$(\mathcal{F}^{-1}g)(x) = \frac{1}{(2\pi)^n} (\mathcal{F}g)(-x)$$

for any $g \in L_2(\mathbb{R}^n)$.

Riesz-Thorin and Riesz theorems

The Riesz-Thorin interpolation theorem is an important tool in Harmonic Analysis and Probability. This theorem bounds norms of linear operators acting between $L_p = L_p(\Omega, \mathcal{F}, \nu)$ spaces.

**Theorem 43** (Riesz-Thorin, [107]) Let $(\Omega_1, \mathcal{F}_1, \nu_1)$ and $(\Omega_2, \mathcal{F}_2, \nu_2)$ be $\sigma$-finite measure spaces. Suppose $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, and let $A$ be a bounded linear operator $A \in L(L_{p_0}, L_{q_0}) \cap L(L_{p_1}, L_{q_1})$. Then

$$\|A|_{L_{p_0} \to L_{q_0}} \| \leq \|A|_{L_{p_0} \to L_{q_0}} \|^{1-\theta} \cdot \|A|_{L_{p_1} \to L_{q_1}} \|^\theta, \forall \theta \in [0, 1],$$

where

$$\frac{1}{p_0} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_0} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

**Theorem 44** (F. Riesz, [107], v. 2, p. 123) Let $(\Omega, \mathcal{F}, \nu)$ be a measure space and $\omega_k(x)$, $k \in \mathbb{Z}^n$ be any orthonormal and uniformly bounded system over $\Omega$, i.e.

$$\int_{\Omega} \omega_k(x) \overline{\omega}_m(x) \, dv = \delta_{k,m} := \begin{cases} 1, & k = m, \\ 0, & k \neq m \end{cases}$$
and
\[
\sup_{x \in \Omega} |\omega_k(x)| \leq L, \quad \forall k \in \mathbb{Z}^n,
\]

Let \(1 \leq p \leq 2\).

1. If \(f \in L_p(\Omega, F, \nu)\), then the Fourier coefficients
\[
c_k := \int_{\Omega} f(x) \omega_k(x) \, d\nu
\]
satisfy the inequality
\[
\|c\|_{p'} \leq \frac{L^{2/p-1}}{p'} \|f\|_p,
\]
where \(c = \{c_k, \; k \in \mathbb{Z}^n\}\), \(1/p + 1/p' = 1\) and
\[
\|c\|_q := \left( \sum_{k \in \mathbb{Z}^n} |c_k|^q \right)^{1/q}, \quad 1 \leq q \leq \infty.
\]

2. Given any sequence \(c := \{c_k, \; k \in \mathbb{Z}^n\}\) with \(\|c\|_p\) finite, there is an \(f \in L_{p'}(\Omega, F, \nu)\) satisfying
\[
c_k := \int_{\Omega} f(x) \omega_k(x) \, d\nu
\]
for all \(k \in \mathbb{N}^n\) and
\[
\|f\|_{p'} \leq \frac{L^{2/p-1}}{p'} \|c\|_p.
\]

See [107, 46, 2] for more information.
Appendix III: Martingales

Martingale methods and pricing

Observe that every forecast is an average of possible future values. All possible values that the random variable can assume in an unfolding future are weighted by the probabilities associated with these values. Hence we need to compute expected values of random variables $S_t$ based on the information revealed at time $\tau \leq T$. The theory of martingales is commonly used for these purposes. Martingales (semi-martingales) is an important class of random sequences with various applications in derivative pricing. We will need some basic definitions.

Definition 45 A binary relation $\preceq$ on a set $A$ is a collection of ordered pairs $(a, b)$ of elements of $A$. In other words, it is a subset of the Cartesian product $A^2 = A \times A$.

Definition 46 We say that a binary relation $\preceq$ is antisymmetric if $a \preceq b$ and $b \preceq a$ then $a = b$, transitive if $a \preceq b$ and $b \preceq c$ then $a \preceq c$ and total if $a \preceq b$ or $b \preceq a$.

Definition 47 A total order is a binary relation (denoted by $\preceq$) on some set $A$ which is transitive, antisymmetric, and total. A set $T$ paired with a total order $\preceq$ is called a totally ordered set (or a chain).

In general, the information used by decision makers will increase as time $t$ passes. It is natural to assume that the decision maker never forgets past data. Hence, the following definition.

Definition 48 A family of $\sigma$-algebras $\{\mathcal{F}_t | t \in T\}$, $\mathcal{F}_t \subset \mathcal{F}$, $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ if $t_1 \preceq t_2$ on a given probability space $(\Omega, \mathcal{F}, P)$ is called a current of $\sigma$-algebras (a current of experiments or filtration).

The set $\mathcal{F}_t$ can be interpreted as the class of all observed events in the experiments carried out up to the moment $t$ inclusively. Fix an arbitrary totally ordered set $T$. Let $\{\mathcal{F}_t | t \in T\}$ be a current of $\sigma$-algebras.

Definition 49 A family of random variables $\{\xi(t), \mathcal{F}_t, t \in T\}$ in which the random variables $\xi(t)$ are $\mathcal{F}_t$ measurable for each $t \in T$ is called a martingale if

$$E[|\xi(t)|] < \infty,$$

$$E[\xi(t) | \mathcal{F}_s] = \xi(s), \ P - a.s., \ s \preceq t, \ s, t \in T.$$
A family \( \{ \xi(t), F_t, t \in T \} \) is called a submartingale, if
\[
\mathbb{E}[\xi(t)|F_s] \geq \xi(s), \ P - a.s., \ s \leq t, \ s, t \in T,
\]
and supermartingale if
\[
\mathbb{E}[\xi(t)|F_s] \leq \xi(s), \ P - a.s., \ s \leq t, \ s, t \in T.
\]

Super and submartingales are called semimartingales.

Remark that the property \( \mathbb{E}[\xi(t)|F_s] = \xi(s), \ P - a.s., \ s \leq t, \ s, t \in T \) means that the best forecast of unobserved future values is the last observation on \( \xi(s) \). All expectations here are assumed to be taken with respect to the probability measure \( P \). Observe that a martingale is always defined with respect to some current of \( \sigma \)-algebras \( \{ F_t | t \in T \} \) and probability measure \( P \). Unfortunately, most financial assets are not martingales. For instance, the price of a bond is expected to increase over time. Also, the stock prices are expected to increase on average over time. It means that
\[
B_t < \mathbb{E}[B_s|F_t], \ t < s < T,
\]
where \( B_t \) is the price of a bond maturing at time \( t \) and \( T \) is a time horizon. Clearly, it contradicts the condition \( B_t = \mathbb{E}[B_s|F_t], \ t < s \). Similarly, a stock \( S_t \) will have a positive expected return. Hence, it does not behave as a martingale. The same observation is true for the price of European-type options. Although the majority of financial assets are not martingales, it is still possible to convert them into martingales.

The majority of known methods of pricing derivatives employ the notion of arbitrage which reflects market equilibrium. It means that if an arbitrage portfolio exists, there exist an opportunity of "free lunches". In a real financial market any arbitrage opportunity will be eliminated by the activity of brokers who will try to make money using that opportunity and marked naturally will enter into the state of equilibrium.

Our later discussion shows that no matter what the "true" (or historic) probabilities are, if there are no arbitrage opportunities, one can represent the fair market value of a financial instrument using probability measures constructed under the equilibrium assumption. See [94, 55, 31, 50, 24] for more details. There are two conventional ways to proceed. The first approach is based on Doob-Meyer’s theorem (see, e.g. [15], p. 25, [94], p. 141).

**Theorem 50** (Doob-Meyer decomposition) If \( \xi(t), t \geq 0 \) is a right-continuous submartingale with respect to \( F_t \), then \( \xi(t) \) admits the decomposition
\[
\xi(t) = M_t + A_t,
\]
where \( M_t \) is a right-continuous martingale with respect to probability \( P \) and \( A_t \) is an increasing process measurable with respect to \( F_t \).

The second approach is based on the idea of changing probability measure to make \( \exp(-rt) S_t \) a martingale. This commonly used in derivative pricing method is based on Girsanov’s theorem and is based on a proper change of
the underlying probability distribution $\mathbb{P}$. More precisely, if $\exp(-rt) S_t$ is a submartingale, i.e.

$$\mathbb{E}^\mathbb{P}[\exp(-rs) S_{t+s}|\mathcal{F}_t] > S_t, \quad \forall s > 0,$$

where $\mathbb{E}^\mathbb{P}[\exp(-rs) S_{t+s}|\mathcal{F}_t]$ is the conditional expectation calculated using a probability distribution $\mathbb{P}$ then applying Girsanov’s theorem we can find a probability distribution $\mathbb{Q}$ (on the same measure space), such that

$$\mathbb{E}^\mathbb{Q}[\exp(-rs) S_{t+s}|\mathcal{F}_t] = S_t, \quad \forall s > 0.$$

Hence, $\exp(-rs) S_t$ becomes a martingale. Such probability distributions $\mathbb{Q}$ are called equivalent martingale measures.

**Definition 51** A standard Brownian motion is a random process $X = \{X_t | t \in \mathbb{R}_+\}$ with state space that satisfies the following properties:

1. $X_0 = 0$ (with probability 1).
2. $X$ has stationary increments. That is $\forall s, t \in [0, \infty), \ s < t$, the distribution of $X_t - X_s$ is the same as the distribution $X_{t-s}$.
3. $X$ has independent increments, or $\forall t_1, \cdots, t_n \in [0, \infty)$ with $t_1 < \cdots < t_n$, the random variables $X_{t_1}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent.
4. $X_t$ is normally distributed, $\forall t \in [0, \infty) \Rightarrow X_t \sim N(0, t)$.
5. With probability 1, $t \mapsto X_t$ is continuous on $[0, \infty)$.

**Theorem 52** (Girsanov) Consider the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$. Assume that $(\Theta_t)_{0 \leq t \leq T}$ is an adapted to the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ process such that $\int_0^T \Theta^2_t ds < \infty$ and the process $(L_t)_{0 \leq t \leq T}$ is a martingale:

$$L_t := \exp\left(-\int_0^t \Theta_s dW_s - \int_0^t \Theta^2_s ds\right),$$

where $dW_t$ is a standard Brownian motion. Then under the probability $\mathbb{P}^{(L)}$ with the density $L_T$ with respect to $\mathbb{P}$, the process $(W^*_t)_{0 \leq t \leq T}$

$$W^*_t := W_t + \int_0^t \Theta_s ds$$

is a standard Brownian motion.

**The class of equivalent martingale measures**

A market model is called complete if the set $EMM$ of all equivalent martingale measures is a singleton. The necessary and sufficient conditions for the absence of arbitrage and for the completeness are given in [23].
Theorem 53 Let $W_t$ denote the standard Brownian motion and $N_t$ a standard Poisson process. Suppose $S_t$ is neither increasing nor decreasing. Let $F_t := \sigma (S_u, u \leq t)$ be the natural filtration of $S_t$. Then model $S_t = S_0 \exp (Z_t)$ is complete in the following cases only:

1. $Z_t = \alpha W_t + \beta t, (\alpha, \beta) \in \mathbb{R}^2 \setminus \{\alpha = 0, \beta \neq 0\}$:

2. $Z_t = \alpha W_t + \beta t, (\alpha, \beta) \in \mathbb{R}^2, \gamma > 0$ and $\alpha \beta < 0$.

We see that the Black-Scholes model is complete in contrast to the hyperbolic or KoBoL models. It was shown in [33] that the set $\mathcal{EMM}$ is so rich that every price in some interval $(a, b)$ can be obtained by a particular martingale measure $Q$. Let $r > 0$ be the constant rate, $\mu$ the drift and $\Pi$ be the Lévy measure of $Z_t$ under $P$. Let $\mathcal{EMM}'$ be the subset of all $Q \in \mathcal{EMM}$ under which $Z_t$ is again Lévy process. If the system

$$
\begin{align*}
\int_{\mathbb{R}} \left( y^{1/2} (x) - 1 \right)^2 \Pi (dx) + \int_{|x| > 1} (\exp (x) - 1) y(x) \Pi (dx) < \infty \\
\mu - r + \int_{\mathbb{R}} ((\exp (x) - 1) y(x) - \chi_D (x)) \Pi (dx) = 0
\end{align*}
$$

has a solution $y : \mathbb{R} \to (0, \infty)$, then $\mathcal{EMM} \supset \mathcal{EMM}' \neq \emptyset$.

Theorem 54 (Eberlein-Jacod [33]) Consider the range sets

$I_e := \left\{ \exp (-rT) \mathbb{E}_Q [H] \mid Q \in \mathcal{EMM} \right\}$,

$I'_e := \left\{ \exp (-rT) \mathbb{E}_Q [H] \mid Q \in \mathcal{EMM}' \right\}$.

If the Lévy measure $\Pi$ of $Z_t$ under $P$ satisfies

1. $\Pi ((-\infty, a]) > 0, \forall a \in \mathbb{R}$;

2. $\Pi$ has no atom and satisfies

$$
\int_{[-1,0]} |x| \Pi (dx) = \int_{[-1,0]} |x| \Pi (dx) = \infty
$$

then $\mathcal{EMM}$ is not empty, $I_e$ is the full interval

$$
(\exp (-rT) H (S_0 \exp (rT)), S_0),
$$

where $H$ is the pay-off function and $I'_e$ is dense in this interval.

In order to calculate option prices we need to choose an equivalent martingale measure in $\mathcal{EMM}'$. There are two common approaches, the Esscher transform and the so-called minimal Entropy measure.

Let $Z_t$ be a Lévy process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$. The Esscher transform is any change of $P$ to an equivalent measure $Q$ with a density process $\frac{dQ}{dP} |_{\mathcal{F}_t}$ (see Appendix I, Theorem 41 for the definition) of the form

$$
X_t = \frac{\exp (\theta Z_t)}{M(\theta)}, \quad \theta \in \mathbb{R},
$$

where $M(\theta)$ is the moment generating function of $Z_t$.

In general, for any infinitely divisible distribution $\nu (dx)$ on $\mathbb{R}$ with a finite moment generating function on some interval $(c, d), c < 0 < d$ the Esscher
transform $v_\theta (dx)$ is infinitely divisible for any $\theta \in (c, d)$ with Lévy generating triplet $(a_\theta, \Pi_\theta, h_\theta)$ given by

$$a_\theta = a,$$

$$\Pi_\theta (dx) = \exp (\theta x) \Pi (dx),$$

$$h_\theta = h + \theta a + \int_{\mathbb{R}} (\exp (\theta x) - 1) \chi_D (x) \Pi (dx)$$

(see [93]). In this case the characteristic exponent should satisfy (see [14], p. 20),

$$\psi^Q (\xi) = \psi^P (\xi - i\theta) - \psi^P (-i\theta)$$

for some $\theta \in \mathbb{R}$. Comparing this with the Theorem 9 we get

$$r + \psi^P (-i(\theta + 1)) - \psi^P (-i\theta) = 0.$$

The method of minimal entropy is presented in [89, 40, 92]. See [18, 34, 41, 42, 48] for more information.
Appendix IV: Comparison of numerical methods

$m$-Widths

$m$-Widths were introduced by Kolmogorov [58] in 1936 to compare and classify a wide range of numerical methods. Let $X$ be a Banach space with the norm $\| \cdot \|$. Kolmogorov’s $n$-width $d_n(A, X)$ of a symmetric set $A$ in $X$ is defined as

$$d_m(A, X) = \inf_{L_m \subset X} \sup_{A \subset X} \inf_{y \in L_m} \| x - y \|,$$

where the last inf is taken over all subspaces $L_m \subset X$ of dimension $n$. The problem of calculating the $m$-widths usually splits into two parts: estimating the quantity

$$E(L_m, A, X) = \sup_{A \subset X} \inf_{y \in L_m} \| x - y \|,$$

where $L_m$ is a fixed subspace, which gives us a necessary upper bound, and obtaining a lower estimate of the width $d_m(A, X)$. The difficulty in finding lower bound for $m$-width is that all $m$-dimensional subspaces $L_m \subset X$ have to be considered. In 1960 Tikhomirov [100] proved a theorem on the diameter of a ball (see Theorem 61) where he first applied an interesting topological method, namely the theorem of Borsuk-Ulam, on the basis of which he proposed a method of obtaining lower estimates of widths. We present here a simple proof of Theorem 6.3 which is important in our applications.

Let us remind some definitions. Let $X$ be a Banach space with the unit ball $B$ and $A$ be a compact, centrally symmetric subset of $X$. Let $L_{m+1}$ be an $(m+1)$-dimensional subspace in $X$. Bernstein’s $m$-width is defined as

$$b_m(A, X) = \sup \left\{ L_{m+1} \subset X \mid \sup \{ \epsilon > 0 \mid \epsilon B \cap L_{m+1} \subset A \} \right\}.$$

The Alexandrov’s $m$-width is the value

$$a_m(A, X) = \inf_{\Sigma_m \subset X} \inf_{\sigma : A \to \Sigma_m} \sup \{ \| x - \sigma(x) \| : x \in A \}.$$
where the infimum is taken over all $m$-dimensional complexes $\Sigma_m$, lying in $X$ and all continuous mappings $\sigma: A \to \Sigma_m$. The Urysohn’s width $u_m (A,X)$ is the infimum of those $\epsilon > 0$ for which there exists a covering of $A$ by open sets (in the sense of topology induced by the norm $\|\cdot\|$ in $X$) of diameter $< \epsilon$ in $X$ and multiplicity $m+1$ (i.e. such that each point is covered by $\leq m+1$ sets and some point is covered by exactly $m+1$ sets). Observe that the width $u_m (A,X)$ was introduced by Urysohn [103] and inspired by the Lebesgue-Brouwer definition of dimension.

In problems of optimal recovery arise quantities which are known as cowidths. Let $(X,\vartheta)$ be a given metric (Banach) space, $Y$ a certain set (coding set), $A \subset X$, $\Theta$ a family of mappings $\theta: A \to Y$, then the respective cowidth can be defined as

$$c_{\Theta} (A,X) = \inf \sup_{\theta \in \Theta} \sup_{y \in \theta(A)} \mathrm{diam} \left\{ \theta^{-1} (y) \cap A \right\},$$

where

$$\theta^{-1} (y) = \{ x | x \in X, \theta (x) = \theta (y) \}.$$

In particular, let $Y$ be $\mathbb{R}^m$ and $\Theta: A \to \mathbb{R}^m$ be a linear application, $\Theta = L (A, \mathbb{R}^m)$, then we get a linear cowidth $\lambda^m (A,X)$. It is easy to check that $\lambda^m = 2d^m$, where $d^m$ is the Gelfand’s $m$-width defined by

$$d^m (A,X) = \inf \{ L_{-m} \subset X | \sup \{ \| x \| | x \in A \cap L_{-m} \} \},$$

where inf is taken over all subspaces $L_{-m} \subset X$ of codimension $m$. Letting $Y$ be the set of all $m$-dimensional complexes in $X$ and $\Theta = C (A,Y)$ be the set of all continuous mappings $\theta: A \to Y$, then we get Alexandrov’s cowidths $a^m (A,X)$.

### Functional and operator of best approximation

Here we present some known facts about functional and operator of best approximation. Let $X$ be a Banach space with the norm $\| \cdot \|_X = \| \cdot \|$. The deviation of $x \in X$ from the non-empty subset $M \subset X$, i.e.

$$E(x) = E(x,M) = E(x,M,X) := \inf_{y \in M} \| x - y \|_X \quad (6.2)$$

is known as the best approximation of $x$ from the set $M$. For a fixed set $M \subset X$ the equation (6.2) defines a functional on $X$, $E: X \to \mathbb{R}_+$ which is called the best approximation functional.

**Proposition 55** Let $M \subset X$ be a linear manifold, then the functional $E(\cdot, M)$ is uniformly continuous, subadditive:

$$E(x_1 + x_2) \leq E(x_1) + E(x_2), \quad \forall x_1, x_2 \in X,$$

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positively homogeneous:

\[ E(ax) = |a|E(x), \quad \forall a \in \mathbb{R} \]

and convex:

\[ E(ax_1 + (1-a)x_2) \leq aE(x_1) + (1-a)E(x_2), \]

\[ \forall a \in [0,1], \quad \forall x_1, x_2 \in X. \]

**Proof.** Let \( x_1 \in X \) and \( x_2 \in X \), then for any \( y \in M \)

\[ E(x_1) \leq \|x_1 - y\| \leq \|x_1 - x_2\| + \|x_2 - y\|. \]

Taking the infimum on \( y \in M \) we find

\[ E(x_1) \leq \|x_1 - x_2\| + E(x_2), \]

or

\[ E(x_1) - E(x_2) \leq \|x_1 - x_2\|. \]

Interchanging \( x_1 \) and \( x_2 \) we get \( E(x_2) - E(x_1) \leq \|x_1 - x_2\| \) or

\[ |E(x_1) - E(x_2)| \leq \|x_1 - x_2\| \]

which implies the uniform continuity of \( E \). To show that \( E \) is subadditive we remark that for any \( y_1 \in X \) and \( y_2 \in X \) we have

\[ E(x_1 + x_2) \leq \|x_1 + x_2 - y_1 - y_2\| \]

\[ \leq \|x_1 - y_1\| + \|x_2 - y_2\|. \]

Taking inf from the right with respect to \( y_1 \) and \( y_2 \) we get \( E(x_1 + x_2) \leq E(x_1) + E(x_2) \) which means that \( E \) is subadditive.

For any \( x \in X \) and \( a \in \mathbb{R} \setminus \{0\} \) we have

\[ E(ax) = \inf_{y \in M} \|ax - y\| = |a| \inf_{y \in M} \|x - y/a\| \]

\[ = |a| \inf_{y \in M} \|x - y\| = |a|E(x), \]

this proves that \( E \) is positively homogeneous. Finally, since \( E \) is subadditive and positively homogeneous then it is convex. \( \square \)

If the inf in (6.2) is attained for \( y_0 \in M \), i.e. \( E(x) = \|x - y_0\| \), then \( y_0 \) is called an element of best approximation for \( x \) in \( M \). The set \( M \subset X \) is called an **existence set** if for every \( x \in X \) there is an element of best approximation in \( X \).

**Proposition 56** Every closed locally compact subset \( M \subset X \) is an existence set. In particular, every finite dimensional subspace of \( X \) is an existence set.
Proof. Assume that \( x \in X \setminus M \) and \( E(x) = c > 0 \), otherwise the existence is obvious. By the definition of inf for every \( n \in \mathbb{N} \) there is such \( y_n \in M \) that \( \|x - y_n\| < E(x) + 1/n \) and the sequence \( \{y_n\} \) is bounded since

\[
\|y_n\| = \|x - x + y_n\| \leq \|x\| + E(x) + 1/n
\]

\[
= \|x\| + c + 1/n.
\]

Using local compactness of \( M \) we may find such a subsequence \( \{y_{n_m}\} \) that \( y_{n_m} \to y_0 \) as \( m \to \infty \). Remark that \( y_0 \in M \) because \( M \) is closed. It is clear that

\[
E(x) \leq \|x - y_{n_m}\| < E(x) + 1/m_n, \quad n \in \mathbb{N}
\]

and if we let \( m \to \infty \), we get \( \|x - y_0\| = E(x) \), which means that \( y_0 \) is an element of best approximation. \( \Box \)

The norm on \( X \) is called strictly convex if for any \( x \in X \) and \( y \in X \), \( \|x\| = \|y\| = 1 \) we have that \( \|ax + (1-a)y\| < 1 \) for any \( a \in (0,1) \). This means that the unit sphere in \( X \), \( \|x\| = 1 \) does not contain any segment.

**Proposition 57**  Let \( M \) be a convex subset of a strictly normalized space \( X \). If for some \( x \in X \) there is an element of best approximation in \( M \) then this element is unique.

**Proof.** Assume that there are two elements \( y_1 \in M \) and \( y_2 \in M \), \( y_1 \neq y_2 \) of best approximation for \( x \in X \),

\[
E(x) = \|x - y_1\| = \|x - y_2\|.
\]

Since \( M \) is convex then for any \( a \in [0,1] \) the element \( y_a = ay_1 + (1-a)y_2 \) is in \( M \) and

\[
E(x) \leq \|x - y_a\| = \|a(x - y_1) + (1-a)(x - y_2)\|
\]

\[
\leq a\|x - y_1\| + (1-a)\|x - y_2\|
\]

\[
= aE(x) + (1-a)E(x) = E(x).
\]

This means that the sphere \( \{z \mid z \in X; \|x - z\| = E(x)\} \) contains the segment \( y_a = ay_1 + (1-a)y_2 \), \( a \in [0,1] \) which is a contradiction with the strict convexity. \( \Box \)

The set \( M \in X \) with the property that for every \( x \in X \) there exists a unique element of best approximation is called a **Chebyshev set**.

Let \( M \) be a Chebyshev set then the operator of the best approximation (metric projection) \( P(x) \) is defined by the following equality

\[
E(x, M) = \|x - P(x)\|, \quad P(x) \in M.
\]

**Proposition 58**  If \( M \) is a locally compact Chebyshev set in \( X \), then
operator $P$ is continuous. If $M$ is a Chebyshev subspace, then $P$ is homogeneous and, in particular, odd $P(-x) = -P(x)$.

**Proof.** Let $x_0$ be a fixed point in $X$ and $x_m \to x_0$, Observe that

$$
\|P(x_m) - x_0\| \leq \|P(x_m) - x_m\| + \|x_m - x_0\|
$$

and the sequence $\{E(x_m, M)\}$ converge by the Proposition 55. Consequently, the sequence $\{P(x_m)\}$ is bounded. Assume that $P(x_m) \not\to P(x_0)$. Using local compactness of $M$ we find a subsequence $P(x_{m_n})$ such that $\lim_{n \to \infty} P(x_{m_n}) = z \neq P(x_0)$. Since $M$ is a Chebyshev set and therefore closed we have $z \in M$. Taking a limit when $n \to \infty$ in

$$
\|x_{m_n} - P(x_{m_n})\| = E(x_{m_n}, M) \leq \|x_{m_n} - P(x_0)\|
$$

we get $\|x_{m_n} - z\| \leq \|x_{m_n} - P(x_0)\|$, which means that $z$ is an element of best approximation for $x_0$ in $M$. This contradicts the assumption that $M$ is a Chebyshev set. Hence $P(x_m) \to P(x_0)$.

In the case when $M$ is a Chebyshev subspace for any $a \in \mathbb{R}$ we get

$$
\|ax - aP(x)\| = |a|\|x - P(x)\|
$$

or $P(ax) = aP(x)$. □

Let $M = M_m$ be an $m$-dimensional Chebyshev subspace of the normed space $X$ and $\{x_1, \cdots, x_m\}$ be a basis in $M_m$. The operator of best approximation can be represented as

$$
P(x) = \sum_{k=1}^{m} \alpha_k(x)x_k. \quad (6.3)
$$

From the Proposition 58 we get

**Proposition 59** The functionals $\alpha_k(x) : X \to M_m, 1 \leq k \leq m$ are homogeneous and continuous.

**Proof.** By the Proposition 58, $P(ax) = aP(x)$, which means that

$$
\sum_{k=1}^{m} a\alpha_k(x)x_k = \sum_{k=1}^{m} a\alpha_k(x)x_k.
$$

The representation (6.3) is unique, hence for any $a \in \mathbb{R}$ and $x \in X$ we have $a\alpha_k(x) = \alpha_k(ax), 1 \leq k \leq m$. Finally, remark that the convergence in a finite dimensional space $M_m$ (dim $M_m = m$) is equivalent to componentwise convergence and the operator $P : X \to M_m$ is continuous. This implies the continuity of the functionals $\alpha_k, 1 \leq k \leq m$. □
Borsuk-Ulam theorem

The next statement is an important result and is extensively used in the calculation of lower bounds for $n$-widths [12].

**Theorem 60** (Borsuk-Ulam) Let $X$ and $Y$ be finite-dimensional Banach space over $\mathbb{R}$ or $\mathbb{C}$ with $\dim Y < \dim X$ and let $S = S(X) = \{x \in X : \|x\| = 1\}$ be the unit sphere in $X$. If $f : S \to Y$ is a continuous map, then there is a point $x \in S$ such that $f(-x) = f(x)$. In particular, if $f$ is an odd function, then there is a point $x \in S$ such that $f(x) = 0$.

Theorem 60 was suspected by Ulam and proven by Borsuk and can be reformulated as following. Let $\Omega$ be a bounded, open, symmetric neighborhood of $0$ in $\mathbb{R}^m$, and $F$ a continuous odd map of the boundary $\partial \Omega$ into $\mathbb{R}^{m-1}$. Then there exists an $x^* \in \partial \Omega$ such that $F(x^*) = 0$.

**Theorem 61** Let $X_{n+1}$ be any $n+1$ dimensional subspace of a real normed linear space $X$, and let $B(X_{n+1})$ denote the unit ball of $X_{n+1}$. Then

$$d_k(B(X_{m+1}), X) = 1, \quad k = 0, 1, 2, \ldots, m.$$ 

**Proof.** It is clear that

$$d_m(B(X_{m+1}), X) \leq d_{m-1}(B(X_{m+1}), X)$$

$$\leq \cdots \leq d_0(B(X_{m+1}), X) = 1,$$

so it is sufficient to show that $d_m(B(X_{n+1}), X) \geq 1$. We show that for any given $m$-dimensional subspace $L_m \in X$ there exists $x \in \partial B(X_{m+1})$ with zero as a best approximation from $L_m$. Let $\{x_1, \cdots, x_{m+1}\}$ and $\{z_1, \cdots, z_m\}$ be bases for $X_{m+1}$ and $L_m$ respectively, then for any $x \in X_{m+1}$ and $z \in L_m$ we have representations

$$x = \sum_{s=1}^{m+1} a_s x_s, \quad z = \sum_{s=1}^{m} b_s z_s.$$ 

It is sufficient to take $X = \text{lin}\{X_{m+1}, L_m\}$ in the proof. Remark that $\dim X = l \leq 2m + 1$. Let $\{y_1, \cdots, y_l\}$ be a basis for $X = \text{lin}\{X_{m+1}, L_m\}$, so any $x \in X$ may be written in the form

$$x = \sum_{s=1}^{l} c_s y_s.$$ 

If the norm on $X$ is not strictly convex then it may be replaced by the norm

$$\|x\|_\epsilon = \|x\| + \epsilon \left( \sum_{s=1}^{l} |c_s|^2 \right)^{1/2}$$ (6.4)
which is strictly convex. Because \( \dim X < 2m + 1 \) we can take the limit \( \epsilon \to 0 \) while maintaining the validity of the theorem. This means that we can assume that the norm on \( X \) is strictly convex which implies the uniqueness and continuity of the best approximation operator and also implies its oddness. The domain

\[ \Omega = \left\{ (a_1, \ldots, a_{n+1}) : \, x = \sum_{s=1}^{m+1} a_s x_s, \, \| x \| < 1 \right\} \]

is a bounded, open, symmetric neighborhood of \( 0 \) in \( \mathbb{R}^{m+1} \). For any \( a \in \partial \Omega \) let \( F(a) = (b_1, \ldots, b_m) \in \mathbb{R}^m \) denote the vector of coefficients of the best approximation to

\[ x = \sum_{s=1}^{m+1} a_s x_s \in \partial B(X_{m+1}) \]

from \( L_m \). By the Proposition 59 the map \( F(\cdot) : \partial \Omega \to \mathbb{R}^m \) is an odd, continuous map of \( \partial \Omega \) into \( \mathbb{R}^m \). Hence, by the Borsuk-Ulam theorem there exist an

\[ x^* = \sum_{s=1}^{m+1} a^*_s x_s, \, \| x^* \| = 1, \]

for which the zero element is the best approximation from \( L_m \). □

From the Theorem 61 and the definition of Bernstein’s \( n \)-widths we get

**Corollary 62** Let \( A \) be a compact symmetric set in a Banach space \( X \) then,

\[ d_m (A, X) \geq b_m (A, X), \, m = 0, 1, \cdots. \]

**Brouwer theorem**

**Theorem 63** (Brouwer) For any continuous function \( F \) mapping a compact convex set \( B \) into itself there is a point \( x \in B \) such that \( F(x) = x \).

**Definition 64** Let \((X, \vartheta)\) be a metric space and \( F : (X, \vartheta) \to (X, \vartheta) \) be a continuous map such that \( \vartheta (x, F(x)) \leq \epsilon \) for any \( x \in X \). In this case we say that \( F \) is an \( \epsilon \)-shift.

**Corollary 65** Let \( X \) be a Banach space with the unit ball \( B \), \( \dim X < \infty \). Let \( F \) be an \( \epsilon \)-shift of \( B \) and \( \epsilon \in (0, 1) \). Then there is \( \delta > 0 \) such that \( \delta B \subset F(B) \).

**Proof.** Pick such \( \delta \) that \( \epsilon + \delta < 1 \). If for any \( x_0 \in \delta B \) we have \( x_0 \in F(B) \) then \( \delta B \subset F(B) \) and the statement is proved.

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Hence, assume that there is such $x_0 \in \delta B$ that $x_0 \notin F(B)$. Consider a continuous map $\xi = Y(x)$, $Y : B \rightarrow \partial B$ which is defined as following. Let $l(x)$, $x \in B$ be a ray from $F(x)$ which passes through $x_0 \in \delta B$ and $\xi := l(x) \cap \partial B$.

Assume that $x \in \text{int} B$. Then $x \neq \xi \in \partial B$. Let $x \in \partial B$. By assumption $x_0 \notin F(B)$ and $x_0 \in \delta B$ where $\delta < 1$. This means that $x_0 \neq F(x)$ for any $x \in B$ and $x_0 \notin \partial B$. By construction, $x_0$ is a convex combination of $F(x)$ and $\xi$. Hence $x_0 = (1 - \alpha) F(x) + \alpha \xi$, for some $\alpha \in (0, 1)$. Consequently,

$$\|\xi - F(x)\| > \|\xi - x_0\| \geq 1 - \delta$$

and, therefore

$$\|x - \xi\| = \|x - F(x) + F(x) - \xi\| > 1 - \delta - \epsilon = 1 - (\delta + \epsilon) > 0$$

since $\delta + \epsilon < 1$. This means that $x \neq \xi$. Hence the map $Y$ has not fixed points. This contradicts Brouwer’s theorem since the unit ball $B$ in $X$, dim $X < \infty$ is compact and convex. □

Let $(A, \vartheta)$ be a metric compact space and $\{U_1, \ldots, U_m\}$ be an open covering of $A$, i.e. $A \subset \bigcup_{s=1}^m U_s$. Let $Z$ be a linear metric space and $\{z_1, \ldots, z_m\}$ be a set of distinct points in $Z$. Let

$$F : A \rightarrow Z$$

$$F(x) = \sum_{s=1}^m \lambda_s(x) z_s,$$

where

$$\lambda_s(x) := \frac{d_s(x)}{\sum_{k=1}^m d_k(x)}$$

and

$$d_k(x) := \min \{\vartheta(x, y) \mid y \in A \setminus U_k\}.$$ 

Clearly $\lambda_s(x) \geq 0$ since $d_k(x) \geq 0$. Observe that the functions $\lambda_s(x)$, $1 \leq s \leq m$ are continuous,

$$\sum_{s=1}^m \lambda_s(x) = 1$$

and $\lambda_s(x) = 0$ if $x \notin \bigcup_s$. The set $F(A)$ is called the nerve of an open covering $\{U_1, \ldots, U_m\}$ generated by the set $\{z_1, \ldots, z_m\}$ and is denoted by $\mathcal{N}(z_1, \ldots, z_m)$.

**Proposition 66** Let $A$ be a compact in a Banach space $X$. Then for any $\epsilon > 0$ there exists $m = m(\epsilon)$, a linear manifold $M_m$, dim $M_m = m$ and an $\epsilon$-shift $F : A \rightarrow M_m$.

**Proof.** Since $A$ is a compact then by the Hausdorff theorem for any $\epsilon > 0$ there is a finite $\epsilon$-net $\{x_1, \ldots, x_m\}$, $m = m(\epsilon)$ in $A$, i.e. $A$ can be covered by the union of sets $\epsilon B + x_s$, $1 \leq s \leq m$, or $A = \bigcup_{s=1}^m (\epsilon B + x_s)$, where $B$
is the unit open ball in \( X \). Clearly \( \text{aff}\{x_1, \ldots, x_m\} \) is a linear manifold \( M_m \), 
\( m := \dim M_m \leq n \). The map \( F: A \to N(x_1, \ldots, x_m) \) is a required \( \epsilon \)-shift. □

**Theorem 66** (Tikhomirov [101], p. 221) Let \( X \) be a Banach space and \( A \subseteq X \) be a convex symmetric compact. Then
\[
b_m(A, X) \leq 2a_m(A, X).
\]

**Proof.** From the proof of Theorem 61 and (6.4) we may assume that \( X \) is a finite dimensional Banach space with infinitely smooth and strictly convex unit ball \( B \). Fix an \((m + 1)\)-dimensional subspace \( L_{m+1} \) in \( X \). Observe that \( L_{m+1} \) is a Chebyshev subspace. Hence the operator of metric projection \( P_{L_{m+1}}: X \to L_{m+1} \) is well defined. Assume that
\[
2a_m(A, X) < b_m(A, X) - 4\epsilon.
\]

(6.5)

Let \( K_m \) be an \( m \)-dimensional complex and \( F: A \to K_m \) be a continuous map such that
\[
\sup \{ x \in A \| x - Fx \| \} \leq a_m(A, X) + \epsilon.
\]

(6.6)

Let \( z_1, \ldots, z_s \) be the vertices of \( K_m \) and \( \zeta_1 := P_{L_{m+1}}z_1, \ldots, \zeta_s := P_{L_{m+1}}z_s \) are the elements of the best approximation of \( z_1, \ldots, z_s \) in \( L_{m+1} \). Since \( K_m \) is a simplicial complex then any \( x \in K_m \) can be represented in the form
\[
x = \sum_{j=1}^{s} \alpha_{s_j} z_{s_j}.
\]

Define the maps \( P: K_m \to L_{m+1}, \)
\[
P : z_j := \sum_{j=1}^{s} \alpha_{s_j} \zeta_{s_j} = \sum_{j=1}^{s} \alpha_{s_j} P_{L_{m+1}}z_{s_j} = \sum_{j=1}^{s} P_{L_{m+1}} \alpha_{s_j} z_{s_j}
\]

and \( \Psi := P \circ F \). Observe that \( P \) is a simplicial map. Hence \( \dim P(K_m) \leq \dim K_m = m \). The diameter of simplexes which constitute \( K_m \) can be assumed as small as we pleased. Hence, for any \( \epsilon > 0 \) we, by the definition of \( P_{L_{m+1}} \) and \( P \), may assume that
\[
\max \{ y \in K_m \| (P_{L_{m+1}} - P)y \| \leq \epsilon \}.
\]

(6.7)

Therefore, for any
\[
x \in (b_m(A, X) - \epsilon)B \cap L_{m+1} \subseteq A
\]

we get
\[
\| x - \Psi x \| = \| x - P \circ F x \|
\]

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\[ \|x - Fx + Fx - P \circ Fx + PL_{m+1} \circ Fx - PL_{m+1} \circ Fx\| \]
\[ \leq \|x - Fx\| + \|Fx - PL_{m+1} \circ Fx\| + \|PL_{m+1} \circ Fx - P \circ Fx\| \]
\[ := J_1 + J_2 + J_3. \]

By assumption (6.6),
\[ J_1 \leq a_m (A,X) + \epsilon \]
and
\[ J_2 = \|Fx - PL_{m+1} \circ Fx\| = \inf \{\xi \in L_{m+1} \|Fx - \xi\} \]
\[ \leq \|Fx - x\| \leq a_m (A,X) + \epsilon. \]

From (6.7) it follows that
\[ J_3 = \|PL_{m+1} \circ Fx - P \circ Fx\| \leq \epsilon. \]

Comparing these estimates we get
\[ \|x - \Psi x\| \leq (a_m (A,X) + \epsilon) + (a_m (A,X) + \epsilon) + \epsilon \]
\[ \leq 2a_m (A,X) + 3\epsilon \]
\[ < (b_m (A,X) - 4\epsilon) + 3\epsilon = b_m (A,X) - \epsilon, \]
where we used (6.5). Hence a continuous map \( \Psi \) of the ball \( (b_m (A,X) - \epsilon) B \cap L_{m+1} \) is an \( (b_m (A,X) - \epsilon) \)-shift. From Corollary 6.3 we get
\[ \dim (\Psi (b_m (A,X) - \epsilon) B \cap L_{m+1}) \geq m + 1. \]

But
\[ \dim (\Psi (b_m (A,X) - \epsilon) B \cap L_{m+1}) \leq \dim K_m = m. \]

Contradiction proofs that (6.5) is impossible. Consequently, \( b_m (A,X) \leq 2a_m (A,X). \)
\[ \square \]
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