Bayesian Heterogeneity Pursuit Regression Models for Spatially Dependent Data

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Abstract

Most existing spatial clustering literatures discussed the cluster algorithm for spatial responses. In this paper, we consider a Bayesian clustered regression for spatially dependent data in order to detect clusters in the covariate effects. Our proposed method is based on the Dirichlet process which provides a probabilistic framework for simultaneous inference of the number of clusters and the clustering configurations. A Markov chain Monte Carlo sampling algorithm is used to sample from the posterior distribution of the proposed model. In addition, Bayesian model diagnostic techniques are developed to assess the fitness of our proposed model, and check the accuracy of clustering results. Extensive simulation studies are conducted to evaluate the empirical performance of the proposed models. For illustration, our methodology is applied to a housing cost dataset of Georgia.

keywords: Clustered Coefficients Regression, Dirichlet process, MCMC, Spatial Random Effects

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1 Introduction

Spatial regression models have been widely applied in many different fields such as environmental science (Hu and Bradley, 2018), biological science (Zhang and Lawson, 2011), and econometrics (Brunsdon et al., 1996) to explore the relation between a response variable and a set of predictors over a region. One of the most important tasks for a spatial regression model is to capture the spatial dependent structure for the response variable. Cressie (1992) proposed a spatial regression model with Gaussian process. Diggle et al. (1998) extended Cressie’s model to generalized linear regression. In both models, spatial random effects are accounted for by the intercepts, and the regression coefficients are assumed to be constant over space. Brunsdon et al. (1996) proposed a geographically weighted regression (GWR) to capture spatially varying regression coefficients. The idea of GWR has been subsequently extended to the Cox model framework by Xue et al. (2018). From the Bayesian perspective, Gelfand et al. (2003) incorporated Gaussian process to regression coefficients to build a model with spatially varying coefficients. The literature mentioned above all assume that each location has its own set of regression parameters, which sometimes leads to overfitting. Cluster effects over the space of interest has not been taken into account.

Detection of heterogeneous covariate effects in many different fields, such as real estate applications, spatial econometrics, and environmental science are becoming of increasing research interest. For example, a country’s administrative divisions, such as regions, provinces, states, or territories, often have different economic statuses and development patterns. More advanced divisions and less developed divisions could be put into separate clusters and analyzed. Such clustering information is of great interest to regional economics researchers. One of the most popular methods for spatial cluster detection is the scan statistic based method (e.g., Kulldorff and Nagarwalla, 1995; Jung et al., 2007), where a scan statistic is constructed via a likelihood ratio statistic to test the potential clusters. Another important approach is to use the Bayesian framework to pursue spatial homogeneity (e.g., Carlin et al., 2014; Li et al., 2010). These two important approaches mainly focus on estimating cluster configurations of spatial response. Recently, methods for cluster detection of spatial regression coefficients have been proposed to detect the homogeneity of the covariates effects among subareas. Lee et al. (2017) proposed a hypothesis testing procedure to detect the spatial cluster under the GWR framework. Lee et al. (2019) extended their framework to a mixed-effects model, which used a two-stage approach based on the likelihood ratio test to detect clusters. The clustering configuration in both works is detected via a series of hypothesis testing. Estimation of slopes and intercepts are performed in two consecutive stages. Li and Sang (2019) incorporated spatial neighborhood information in a penalized approach to detect spatially clustered patterns in the regression coefficients. An integrated framework to estimate the number of clusters, the clustering configuration, and regression
coefficients simultaneously is more desired.

Bayesian inference provides a probabilistic framework for simultaneous inference of the number of clusters and the clustering configurations. In a fully Bayesian framework, complicated search algorithms in variable dimensional parameter space such as the reversible jump Markov chain Monte Carlo (MCMC) algorithm (Green, 1995), assigns a prior on the number of clusters, which is required to be updated at each iteration of an MCMC chain. Such algorithms are difficult to implement and automate, and are known to suffer from lack of scalability and mixing issues. Nonparametric Bayesian approaches, such as the Dirichlet process mixture model (DPMM; Ferguson, 1973), offer choices to allow uncertainty in the number of clusters.

In this paper, we propose a Bayesian spatial clustered linear regression model with DP prior, which considers spatially dependent structure and cluster the covariate effects simultaneously. In addition, implementation of our proposed methods based on nimble (de Valpine et al., 2017), a relatively new and powerful R package, is discussed. The model diagnostic technique, Logarithm of the Pseudo-Marginal Likelihood (LPML; Ibrahim et al., 2013), is introduced to assess the fitness of our proposed model. Our proposed Bayesian approach reveals interesting features of the state-level data of Georgia.

The remainder of the paper is organized as follows. In Section 2, we develop a spatial clustered linear regression model with DP prior. In Section 3, a MCMC sampling algorithm based on nimble and post MCMC inference are discussed. Extensive simulation studies are carried out in Section 4. For illustration, our proposed methodology is applied to Georgia health analytics data in Section 5. Finally, we conclude this paper with a brief discussion in Section 6.

2 Methodology

In this section, a Bayesian spatial clustered mixture model using Dirichlet processes mixture model is proposed for coefficient grouping in spatially dependent data. Based on the spatial regression model, the spatially-varying coefficients are assigned with a nonparametric Dirichlet process prior to achieve the goal of grouping.

2.1 Spatial Regression Model

The basic geostatistical model (Gelfand and Schliep, 2016) for spatially dependent responses over a spatial domain is denoted by

\[ Y(s) = X(s)\beta + w(s) + \epsilon(s), \]  

(1)
where \( \mathbf{s} = (s_1, \cdots, s_n)^\top \) denotes the \( n \)-dimensional location vector of the observations, \( \mathbf{X}(\mathbf{s}) \) is the location-specific covariate matrix, \( w(\mathbf{s}) \) is usually a stationary Gaussian process with variance \( \sigma^2 \), while \( \epsilon(\mathbf{s}) \) adds the nugget effect (\( \tau^2 \) in Carlin et al., 2014), which is usually white noise. The above spatial regression model can also be rewritten as

\[
Y(s) \mid \mathbf{\beta}, w(s), \sigma_y^2 \sim N(\mathbf{X}(s)\mathbf{\beta} + w(s), \sigma_y^2),
\]

\[
\mathbf{W} \sim \text{MVN}(0, \Sigma_W),
\]

where \( \sigma_y^2 \) is the variance of the response, \( \mathbf{W} = (w(s_1), \cdots, w(s_n))^\top \) denotes the spatial structure with a covariance matrix \( \Sigma_W \), and MVN denotes the multivariate normal distribution. Conventionally, the covariance matrix is given as \( \Sigma_W = \sigma_w^2 \mathbf{H} \), where \( \mathbf{H} \) is a matrix constructed using the great circle distance matrix, denoted as GCD, among different locations, and \( \sigma_w^2 \) is a scalar. There are three common weighting schemes for capturing \( \mathbf{H} \), including:

- the unity scheme: \( \mathbf{H} = \text{diag}(1) \)
- the exponential scheme: \( \mathbf{H} = \exp(-\text{GCD}/\phi) \) \hfill (2)
- the Gaussian scheme: \( \mathbf{H} = \exp(-(\text{GCD}/\phi)^2) \),

where \( \phi \) is a tuning parameter that control the spatial correlation.

Another spatial regression model is the spatially varying coefficients model (Gelfand et al., 2003):

\[
Y(s) = \mathbf{X}^\top(s)\mathbf{\tilde{\beta}}(s) + \epsilon(s),
\]

where \( \mathbf{X}(s) \) is a \( p \times 1 \) covariate matrix at location \( s \), and \( \mathbf{\tilde{\beta}}(s) \) is assumed to follow a \( p \)-variate spatial process model. If we have observations \( (Y(s_i), \mathbf{X}(s_i)) \) for \( i = 1, \cdots, n \), they can be written into

\[
\mathbf{Y} = \mathbf{X}^\top \mathbf{\tilde{\beta}} + \mathbf{\epsilon},
\]

where \( \mathbf{Y} = (Y(s_1), \ldots, Y(s_n))^\top \), \( \mathbf{X}^\top \) is an \( n \times (np) \) block diagonal matrix which has the row vector \( \mathbf{X}^\top(s_i) \) as its \( i \)-th diagonal entry, \( \mathbf{\tilde{\beta}} = (\mathbf{\tilde{\beta}}(s_1)^\top, \ldots, \mathbf{\tilde{\beta}}(s_n)^\top)^\top \), and \( \mathbf{\epsilon} \sim N(0, \sigma^2 \mathbf{I}) \).

Gelfand et al. (2003) proposed the following hierarchical model:

\[
\mathbf{Y} \mid \mathbf{\tilde{\beta}}, \sigma^2 \sim N_n(\mathbf{X}^\top \mathbf{\tilde{\beta}}, \sigma^2 \mathbf{I})
\]

\[
\mathbf{\tilde{\beta}} \mid \mathbf{\mu_\beta}, \mathbf{T} \sim N(1_{n\times1} \otimes \mathbf{\mu_\beta}, \mathbf{H}(\phi) \otimes \mathbf{T})
\]

where \( \mathbf{\mu_\beta} \) is a \( p \times 1 \) vector, \( \mathbf{H}(\phi) \) is a \( n \times n \) matrix measuring spatial correlations between the \( n \) observed locations, \( \mathbf{T} \) is a \( p \times p \) covariance matrix associated with an observation vector at any spatial location, and \( \otimes \) denotes the Kronecker product.
In model (1), $\beta$ is constant over space, which means the covariate effects remain the same over all locations; in model (4), the correlation of the spatially varying coefficients is solely dependent on distance between pairs of locations. For many spatial economics data, however, some regions will share similar covariate effects regardless of their geographical distance. Taking China as an example, Beijing tends to have similar economic development pattern with Shanghai or Jiangsu (Ma et al., 2019b). The models in (1) and (4), however, do not take into account such inherent similarities in spatially dependent data.

### 2.2 Spatial Regression with Dirichlet Process Mixture Prior

Within the Bayesian framework, coefficient grouping can be accomplished using a Dirichlet process mixture model (DPMM) by nonparametrically linking the spatial response variable to covariates through cluster membership (Molitor et al., 2010). Formally, a probability measure $G$ following a Dirichlet process (DP; Ferguson, 1973) with a concentration parameter $\alpha$ and a base distribution $G_0$ is denoted by $G \sim DP(\alpha, G_0)$ if

$$ (G(A_1), \cdots, G(A_r)) \sim \text{Dirichlet}(\alpha G_0(A_1), \cdots, \alpha G_0(A_r)), $$

where $(A_1, \cdots, A_r)$ are finite measurable partitions of the space $\Omega$. Several different formulations can be used for determining the DP. In this work, we use the stick-breaking construction proposed by Sethuraman (1991) for DP realization, which is given as

$$
\theta_c \sim G_0, \quad G = \sum_{c=1}^{\infty} \pi_c \delta_{\theta_c}(\cdot),
$$

$$
\pi_1 = V_1, \quad \pi_c = V_c \prod_{\ell < c} (1 - V_\ell),
$$

$$
V_c \sim \text{Beta}(1, \alpha),
$$

where $\theta_c$ is the $c$-th matrix consisting of the possible values of the parameters of $G_0$, $\delta_{\theta_c}(\cdot)$ denotes a discrete probability measure concentrated at $\theta_c$, and $\pi_c$ is the random probability weight between 0 and 1.

For a DPMM, the observed data $y_i$ for $i = 1, \ldots, n$ follow an infinite mixture distribution, where a vector of latent allocation variables $\mathcal{Z}$ is introduced to enable explicit characterization of the clustering. Let $\mathcal{Z}_{n,k} = \{(z_1, \ldots, z_n) : z_i \in \{1, \ldots, k\}, 1 \leq i \leq n\}$ denote all possible clusterings of $n$ observations into $k$ clusters, where $z_i = c \in \{1, \ldots, k\}$ denotes the
cluster assignment of the $i$th observation. The DPMM can be written as

$$y_i \mid Z, \theta \sim f(y_i \mid \theta_z),$$

$$\theta_c \sim G_0,$$

$$P(z_i = c \mid \pi) = \pi_c,$$

$$\pi_1 = V_1, \quad \pi_c = V_c \prod_{\ell < c} (1 - V_\ell), \quad V_c \sim \text{Beta}(1, \alpha),$$

where $\pi = (\pi_1, \ldots, \pi_c, \ldots)$.

Adapting the DPMM to the spatial regression setting, we focus on the clustering of spatially-varying coefficients $\beta(s) = (\beta^T(s_1), \ldots, \beta^T(s_n))^T$, where $\beta(s_i)$ is the $p$-dimensional coefficient vector for location $s_i$. In our setting, we assume that the $p \times n$ coefficient matrix can be clustered into a $p \times k$ matrix, i.e., $\beta(s) = \beta_z \in \{\beta_1, \ldots, \beta_K\}$, then the model can be written as

$$y(s_i) \mid \beta_z, w(s_i), \sigma_y^2 \sim N(X(s_i)\beta(s_z) + w(s_i), \sigma_y^2),$$

$$W \sim \text{MVN}(0, \Sigma_W),$$

$$\beta_{z_i} \sim \text{MVN}(\mu, \Sigma),$$

$$P(z_i = c \mid \pi) = \pi_c,$$

$$\pi_1 = V_1, \quad \pi_c = V_c \prod_{\ell < c} (1 - V_\ell), \quad V_c \sim \text{Beta}(1, \alpha).$$

### 3 Bayesian Inference

MCMC is used to draw samples from the posterior distributions of the model parameters.

In this section we present the sampling scheme, the posterior inference of cluster belongings, and measures to evaluate the estimation performance and clustering accuracy.

#### 3.1 The MCMC Sampling Schemes

We present the main R function written using the nimble package (de Valpine et al., 2017). The model is wrapped in a nimbleCode() function. For ease of exposition, we break it into separate snippets.

Define $S$ as the number of locations. The following code represent Equation (7). At each location, $y[i]$ has a normal distribution with $\text{mu}\_y[i]$ and precision $\text{tau}\_y$, which is equivalent to $1/\sigma_y^2$. A Gamma(1,1) prior is given to $\text{tau}\_y$. The coefficient vector for location $i$, $b[i, 1:6]$, equals the coefficient vector estimated for the cluster it belongs to, represented by $\text{latent}[i]$, which follows a multinomial distribution with probability vector...
zlatent[1:M], where M denotes the number of potential clusters.

SLMMCode <- nimbleCode(
  for (i in 1:S) {
    y[i] ~ dnorm(mu_y[i], tau = tau_y)
    mu_y[i] <- b[i, 1] * x1[i] + b[i, 2] * x2[i] +
    b[i, 3] * x3[i] + b[i, 4] * x4[i] + b[i, 5] * x5[i] +
    b[i, 6] * x6[i] + W[i]
    b[i, 1:6] <- bm[latent[i], 1:6]
    latent[i] ~ dcat(zlatent[1:M])
  }
)

The following code represent Equation (8). H represents the matrix Σ_W, where phi is a tuning parameter that controls spatial correlation. The random effects at locations 1 to S follow a multivariate normal distribution with mu_w[1:S] and precision matrix, which equals the product of σ_w^2, tau_w, and the inverse of H. The prior distribution of the bandwidth phi is specified to be a uniform distribution from 0 to a certain upper limit, denoted by D. The prior of tau_w is set to Gamma(1,1).

for (j in 1:S) {
  for (k in 1:S) {
    H[j, k] <- exp(-Dist[j, k] / phi)
  }
}

W[1:S] ~ dmnorm(mu_w[1:S], prec = prec_W[1:S, 1:S])
prec_W[1:S, 1:S] <- tau_w * inverse(H[1:S, 1:S])

phi ~ dunif(0, D)
tau_w ~ dgamma(1, 1)

mu_w[1:S] <- rep(0, S)

The distribution of β for each location s_i is defined next. They each come from a multivariate normal distribution with mean mu_bmm and covariance matrix var_bmm, which is a diagonal matrix with all diagonal entries being 1/tau_bmm. The inverse variance term, tau_bmm, is again
given a Gamma(1,1) prior, and the entries in the mean vector are all given independent standard normal priors.

```r
for (k in 1:M) {
    bm[k, 1:6] ~ dmnorm(mu_bm[1:6], cov = var_bm[1:6, 1:6])
}
var_bm[1:6, 1:6] <- 1/tau_bm * diag(rep(1, 6))
tau_bm ~ dgamma(1, 1)
for (j in 1:6) {
    mu_bm[j] ~ dnorm(0, 1)
}
```

Finally for the model, the stick breaking process corresponding to Equations (10) and (11) is depicted.

```r
zlatent[1:M] <- stick_breaking(vlatent[1:(M - 1)])
for (j in 1:(M - 1)) {
    vlatent[j] ~ dbeta(1, alpha)
}
alpha ~ dgamma(1, 1)
)
```

With the full model defined, we next declare the data list, which is made up of the response, the covariates, and the matrix of distances. The constants in the model also need to be supplied, including the number of locations $S$, the number of starting clusters $M$, and the upper endpoint $D$ for the uniform distribution of bandwidth. In addition, the initial values are specified. Code to compile the model, supply the initial values, and invoke the MCMC process is included in the supplementary package.

```r
SLMMdata <- list(y = y, x1 = X[,1], x2 = X[,2], x3 = X[,3],
    x4 = X[,4], x5 = X[,5], x6 = X[,6],
    Dist = distmatrix)
SLMMConsts <- list(S = 159, M = 50, D = 100)
```
SLMMInits <- list(tau_y = 1, 
        latent = rep(1, SLMMConsts$S), alpha = 2, 
        tau_bm = 1, 
        mu_bm = rnorm(6), phi = 1, tau_w = 1, 
        vlatent = rbeta(SLMMConsts$M - 1, 1, 1))

3.2 Inference of MCMC results

The estimated parameters, together with the cluster assignments \( z \), are determined for each replicate from the best post burn-in iteration selected using the Dahl’s method (Dahl, 2006). Dahl (2006) proposed a least-squares model-based clustering for estimating the clustering of observations using draws from a posterior clustering distribution. In this method, membership matrices for each iteration, \( B^{(1)}, \ldots, B^{(M)} \), are calculated. The matrix \( B \) is defined as:

\[
B = (B(i, j))_{i,j \in \{1:n\}} = (z_i = z_j)_{n \times n},
\]

with \( B(i, j) = \{0, 1\} \) for all \( i, j = 1, \ldots, n \). Having \( B(i, j) = 1 \) means observations \( i \) and \( j \) are in the same cluster in a certain iteration. Then we calculate the least squares distance to Euclidean mean for each MCMC iteration and choose the best of these iterations. The procedure can be described as follows:

- Calculate the Euclidean mean for all membership matrices \( \overline{B} = \frac{1}{M} \sum_{i=1}^{M} B^{(t)} \).
- Find the iteration that has the least squared distance to \( \overline{B} \) as:

\[
C_{LS} = \arg\min_{c \in \{1:M\}} \sum_{i=1}^{n} \sum_{j=1}^{n} (B(i, j)^{(c)} - \overline{B}(i, j))^2.
\]

An advantage of the least-squares clustering is the fact that information from all clusterings are utilized via the usage of the empirical pairwise probability matrix \( \overline{B} \). It is also intuitively appealing, as the average clustering is selected instead of formed via an external, ad hoc clustering algorithm.

3.3 Model Assessment

In the spatial regression model, the Gaussian process spatial structure \( W \) can be constructed via several different weighting schemes including the aforementioned unity, expo-
nential, and Gaussian schemes in (2). In order to determine which weighting scheme is the most suitable for the data, a commonly used model comparison criterion, the Logarithm of the Pseudo-Marginal Likelihood (LPML; Ibrahim et al., 2013), is applied. The LPML can be obtained through the Conditional Predictive Ordinate (CPO) values. Let $Y_{(-i)}^* = Y_j : j = 1, \cdots, i - 1, i + 1, n$ denote the observations with the $i$th subject response deleted. The CPO for the $i$th subject is defined as:

$$\text{CPO}_i = \int f(y(s) | \beta(s), w(s), \sigma_y^2) \pi(w(s), \beta(s), \sigma_y^2 | D_{(-i)}) d(w(s), \beta(s), \sigma_y^2),$$

(14)

where

$$\pi(w(s), \beta(s), \sigma_y^2 | Y_{(-i)}^*) = \frac{\prod_{j \neq i} f(y(s_j) | \beta(s), w(s), \sigma_y^2) \pi(w(s), \beta(s), \sigma_y^2 | Y_{(-i)}^*)}{c(Y_{(-i)}^*)},$$

and $c(Y_{(-i)}^*)$ is the normalizing constant. Within the Bayesian framework, a Monte Carlo estimate of the CPO can be obtained as:

$$\hat{\text{CPO}}_{1-i}^{-1} = \frac{1}{T} \sum_{t=1}^{T} \frac{1}{f(y(s_i) | w_t(s), \beta_t(s), \sigma_{yt}^2)},$$

(15)

where $T$ is the total number of Monte Carlo iterations. An estimate of the LPML can subsequently be calculated as:

$$\hat{\text{LPML}} = \sum_{i=1}^{N} \log(\hat{\text{CPO}}_i).$$

(16)

A model with a larger LPML value is more preferred.

3.4 Convergence Diagnostics

We use the Rand index (RI; Rand, 1971) to measure the accuracy of clustering. The RI is defined as

$$\text{RI} = (a + b)/(a + b + c + d) = (a + b) / \left(\begin{array}{c} n \\ 2 \end{array}\right),$$

where $C_1 = \{X_1, \ldots, X_r\}$ and $C_2 = \{Y_1, \ldots, Y_s\}$ are two partitions of $\{1, 2, \ldots, n\}$, and $a, b, c$ and $d$ respectively denote the number of pairs of elements of $\{1, 2, \ldots, n\}$ that are (a) in a same set in $C_1$ and a same set in $C_2$, (b) in different sets in $C_1$ and different sets in $C_2$, (c) in a same set in $C_1$ but in different sets in $C_2$, and (d) in different sets in $C_1$ and a same set in $C_2$. The RI ranges from 0 to 1 with a higher value indicating better agreement between
the two partitions. In particular, \( RI = 1 \) indicates that \( C_1 \) and \( C_2 \) are identical in terms of modulo labeling of the nodes.

# 4 Simulation

In this section, we conduct simulation studies to assess the performance of the proposed methods under scenarios where there is no clustered covariate effect, and when there is indeed clustered covariate effect.

## 4.1 Simulation Without Clustered Covariate Effects

The spatial adjacency structure of counties in Georgia is used. As a starting point, to mimic the real dataset in Section 5, one observation is generated for each of the 159 counties. The covariates \( X_1, \ldots, X_6 \) are generated i.i.d. from \( N(0, 1) \). The spatial random effects \( W \) are simulated based on the matrix of great circle distance (GCD) between county centroids. The great circle distances are obtained using the function \( \text{distCosine()} \), and the centroids are calculated based on county polygons using the function \( \text{centroid()} \), both provided by the R package \texttt{geosphere} (Hijmans, 2017). The GCD matrix is subsequently normalized to have a maximum value of 10 for ease in computation, and the response \( Y \) is generated as

\[
Y = X\beta + W + \epsilon,
\]

where \( X = (X_1, \ldots, X_6) \), \( W \sim \text{MVN}(0, \exp(-\text{GCD}/4)) \), and \( \epsilon \sim N(0, 1) \). Different values of \( \beta \) are used: \((1, 0, 1, 0, 0, 5, 2)^\top\), \((2, 0, 1, 0, 4, 2)^\top\), and \((9, 0, -4, 0, 2, 5)^\top\), corresponding to scenarios where the signal is weak, moderate, and strong. For each of the three \( \beta \)'s, the average parameter estimate denoted by \( \bar{\beta}_{t,m} (t = 1, \ldots, 159; m = 1, \ldots, 6) \) in 100 simulations is calculated as

\[
\bar{\beta}_{t,m} = \frac{1}{100} \sum_{r=1}^{100} \hat{\beta}_{t,m,r},
\]

where \( \hat{\beta}_{t,m,r} \) denotes the posterior estimate for the \( m \)th coefficient of county \( t \) in the \( r \)th replicate. The performance of these posterior estimates are evaluated by the mean absolute bias (MAB), the mean standard deviation (MSD), the mean of mean squared error (MMSE) and mean coverage rate (MCR) of the 95% highest posterior density (HPD) intervals in the
following ways:

\[
MAB = \frac{1}{159} \sum_{\ell=1}^{159} \frac{1}{100} \sum_{r=1}^{100} \left| \hat{\beta}_{\ell,m,r} - \beta_{\ell,m} \right|, \tag{18}
\]

\[
MSD = \frac{1}{159} \sum_{\ell=1}^{159} \sqrt{\frac{1}{99} \sum_{r=1}^{100} \left( \hat{\beta}_{\ell,m,r} - \bar{\beta}_{\ell,m} \right)^2}, \tag{19}
\]

\[
MMSE = \frac{1}{159} \sum_{\ell=1}^{159} \frac{1}{100} \sum_{r=1}^{100} \left( \hat{\beta}_{\ell,m,r} - \beta_{\ell,m} \right)^2, \tag{20}
\]

\[
MCR = \frac{1}{159} \sum_{\ell=1}^{159} \frac{1}{100} \sum_{r=1}^{100} \left( \hat{\beta}_{\ell,m,r} \in 95\% \text{ HPD interval} \right), \tag{21}
\]

where \(1(\cdot)\) denotes the indicator function. Evaluation of clustering performance using the RI is facilitated by the R package fossil (Vavrek, 2011).

In each replicate, the MCMC chain length is set to be 50,000, with thinning 10 and the first 2,000 samples are discarded as burn-in, therefore we have 3,000 samples for posterior inference. The parameter \(D\) for the uniform prior of bandwidth, i.e. \(\phi\) in Equation (2), is set to 100 such that the prior for bandwidth is also noninformative. In Table 1 the average parameter estimates \(\hat{\beta}_{\ell,m}\) are reported together with the four performance measures in Equations (18),(19), (20) and (21) are reported for the three settings. Under all three settings, the parameter estimates are highly close to the true underlying values, and have very small MAB, MSD and MMSE, while maintaining the MCR at close to 95% level. The RI’s are all very close to or equal to 1, indicating that the clustering results are highly consistent and credible. It is worth noticing that, even when the signal is relatively weak, the clustering approach is quite precise.

4.2 Simulation with Clustered Covariate Effects

We consider an underlying setting where there exist clustered covariate effects. First we consider a setting where the clustered covariate effect is independent of spatial locations, i.e. where cluster belonging are set randomly. The 159 counties are randomly assigned to three clusters, visualized in Figure 1(a). There are, respectively, 51, 49, and 59 counties in the three clusters. Different parameter vectors are used for data generation in different clusters (see Table 2) to assess the estimation and clustering performance under different strengths of signals. The spatial random effect \(W\) is generated using the same setting as in Section 4.1. The performance measures are presented in Table 3. In another scenario, a setting where the clustered covariate effect depends on spatial locations. Consider a partition of Georgia counties into three large regions, visualized in Figure 1(b). The same parameter vectors in
Table 1: Average parameter estimates and performance of parameter estimates and clustering results when without clustered covariate effect.

| Setting | $\hat{\beta}$ | MAB | MSD | MMSE | MCR | RI |
|---------|---------------|-----|-----|------|-----|----|
| Setting 1 | $\beta_1$ 1.005 | 0.110 | 0.085 | 0.007 | 0.910 | 0.997 |
|          | $\beta_2$ 0.002 | 0.087 | 0.069 | 0.005 | 0.980 |    |
|          | $\beta_3$ 0.996 | 0.104 | 0.076 | 0.006 | 0.950 |    |
|          | $\beta_4$ -0.003 | 0.090 | 0.071 | 0.005 | 0.970 |    |
|          | $\beta_5$ 0.508 | 0.106 | 0.080 | 0.006 | 0.960 |    |
|          | $\beta_6$ 1.978 | 0.109 | 0.081 | 0.007 | 0.970 |    |
| Setting 2 | $\beta_1$ 1.971 | 0.218 | 0.117 | 0.014 | 0.930 | 0.999 |
|          | $\beta_2$ -0.001 | 0.090 | 0.072 | 0.005 | 0.980 |    |
|          | $\beta_3$ 0.987 | 0.129 | 0.085 | 0.007 | 0.960 |    |
|          | $\beta_4$ 0.002 | 0.098 | 0.076 | 0.006 | 0.950 |    |
|          | $\beta_5$ 3.934 | 0.362 | 0.145 | 0.021 | 0.910 |    |
|          | $\beta_6$ 1.955 | 0.194 | 0.105 | 0.011 | 0.960 |    |
| Setting 3 | $\beta_1$ 9.006 | 0.108 | 0.081 | 0.007 | 0.930 | 1.000 |
|          | $\beta_2$ -0.001 | 0.088 | 0.069 | 0.005 | 0.970 |    |
|          | $\beta_3$ -3.996 | 0.089 | 0.069 | 0.005 | 0.960 |    |
|          | $\beta_4$ -0.002 | 0.091 | 0.072 | 0.005 | 0.960 |    |
|          | $\beta_5$ 2.007 | 0.106 | 0.081 | 0.007 | 0.950 |    |
|          | $\beta_6$ 4.990 | 0.090 | 0.071 | 0.005 | 0.960 |    |

Figure 1: Visualization of (a) random cluster assignment and (b) regional cluster assignment for Georgia counties used for simulation studies.
Table 2: True parameter vectors used in data generation for three clusters.

| Setting 1 | Cluster 1          | Cluster 2          | Cluster 3          |
|-----------|--------------------|--------------------|--------------------|
| Setting 1 | (1, 0, 1, 0, 0.5, 2) | (1, 0.7, 0.3, 2, 0, 3) | (2, 1, 0.8, 1, 0, 1) |
| Setting 2 | (2, 0, 1, 0, 4, 2)  | (1, 0, 3, 2, 0, 3)  | (4, 1, 0, 3, 0, 1)  |
| Setting 3 | (9, 0, -4, 0, 2, 5) | (1, 7, 3, 6, 0, -1) | (2, 0, 6, 1, 7, 0)  |

Table 2 are used for the three clusters under three settings. Corresponding performance measures are reported in Table 4.

For each signal strength and each of the two settings, we randomly selected four replicates from the total of 100 replicates and visualize the results in the Online Supplement. It is no surprise that under both settings, the accuracy of clustering increases with the strengthening of signals. It can be seen from Supplemental Figure 1 and 4 that with weak true signals, the proposed approach suffers from over-clustering, which is a known property of Dirichlet process mixtures that the posterior does not concentrate at the true number of clusters (see, e.g., Miller and Harrison, 2013). This over-clustering behavior, however, diminishes as the signals’ strength increase. When the signals are strong, the RI reaches near 0.85, indicating that 85% of the time, two counties that belong to the same cluster are correctly put into the same cluster. Together with increase in RI is decrease in MCR, which is an inevitable result of incorporating more counties in each cluster. For each county, taking other counties that do not belong to this county’s true cluster introduces bias in estimation.

5 Real Data Analysis

We consider analyzing influential factors for monthly housing cost in Georgia using the proposed methods. The dataset is available at www.healthanalytics.gatech.edu, with 159 observations corresponding to the 159 counties in Georgia. For each county, the dependent variable median monthly housing cost for all occupied housing units is observed. The independent variables considered here include: the percentage of adults aged 18 to 64 who are unemployed ($X_1$), the average total real and personal property taxes collected per person ($X_2$), the median home market value ($X_3$, in thousand dollars), the percentage of White race population ($X_4$), the median age ($X_5$), and size of a county’s population ($X_6$, in thousands). Figure 2 provides a visualization of the 6 covariates on the Georgia map. In the computation, the covariates are centered and scaled to have mean 0 and unit standard deviation. Also, following the common practice in economics to account for long-tailed distributions (see, e.g., Wooldridge, 2015), we take the logarithm of monthly housing cost before fitting the model. For model fitting, the covariates are centered and scaled to have mean 0 and
Table 3: Performance of parameter estimates and clustering results under the scenario where cluster belongings are set randomly.

| Setting  | \( \beta_1 \) | MAB | MSD | MMSE | MCR | RI |
|----------|----------------|-----|-----|------|-----|----|
| Setting 1 | \( \beta_1 \)  | 0.186 | 0.421 | 0.242 | 0.935 | 0.621 |
|          | \( \beta_2 \)  | 0.173 | 0.401 | 0.180 | 0.967 |
|          | \( \beta_3 \)  | 0.150 | 0.293 | 0.093 | 0.985 |
|          | \( \beta_4 \)  | 0.206 | 0.721 | 0.672 | 0.924 |
|          | \( \beta_5 \)  | 0.168 | 0.241 | 0.063 | 0.977 |
|          | \( \beta_6 \)  | 0.227 | 0.747 | 0.705 | 0.916 |
| Setting 2 | \( \beta_1 \)  | 0.967 | 0.812 | 0.735 | 0.757 | 0.690 |
|          | \( \beta_2 \)  | 0.443 | 0.378 | 0.150 | 0.834 |
|          | \( \beta_3 \)  | 0.958 | 0.806 | 0.743 | 0.753 |
|          | \( \beta_4 \)  | 0.961 | 0.826 | 0.698 | 0.762 |
|          | \( \beta_5 \)  | 1.390 | 1.071 | 1.339 | 0.786 |
|          | \( \beta_6 \)  | 0.670 | 0.608 | 0.408 | 0.766 |
| Setting 3 | \( \beta_1 \)  | 1.941 | 0.763 | 0.636 | 0.816 | 0.852 |
|          | \( \beta_2 \)  | 1.870 | 0.728 | 0.602 | 0.867 |
|          | \( \beta_3 \)  | 2.310 | 0.933 | 0.940 | 0.828 |
|          | \( \beta_4 \)  | 1.491 | 0.636 | 0.445 | 0.814 |
|          | \( \beta_5 \)  | 1.700 | 0.738 | 0.606 | 0.845 |
|          | \( \beta_6 \)  | 1.442 | 0.600 | 0.388 | 0.828 |
Table 4: Performance of parameter estimates and clustering results under the scenario where cluster belongings are set depending on the county centroid locations.

| Setting | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ |
|---------|---------|---------|---------|---------|---------|---------|
| Setting 1 | 0.201   | 0.154   | 0.165   | 0.216   | 0.182   | 0.210   |
|          | 0.413   | 0.426   | 0.289   | 0.720   | 0.256   | 0.683   |
|          | 0.237   | 0.195   | 0.091   | 0.657   | 0.068   | 0.619   |
|          | 0.966   | 0.977   | 0.991   | 0.937   | 0.988   | 0.904   |
|          | 0.597   |         |         |         |         |         |
| Setting 2 | 0.885   | 0.425   | 0.789   | 0.969   | 1.437   | 0.585   |
|          | 0.806   | 0.353   | 0.783   | 0.856   | 1.106   | 0.592   |
|          | 0.782   | 0.138   | 0.832   | 0.758   | 1.280   | 0.432   |
|          | 0.787   |         |         |         |         |         |
|          | 0.691   |         |         |         |         |         |
| Setting 3 | 1.981   | 1.711   | 2.389   | 1.363   | 1.617   | 1.471   |
|          | 0.764   | 0.706   | 0.951   | 0.609   | 0.724   | 0.603   |
|          | 0.625   | 0.710   | 0.983   | 0.486   | 0.634   | 0.389   |
|          | 0.831   | 0.855   | 0.838   | 0.852   | 0.830   | 0.838   |
|          | 0.855   |         |         |         |         |         |
standard deviation 1. The response variable is also centered and scaled, and therefore all models to follow are fitted without the intercept term.

We firstly apply the model assessment criteria, LPML, for selecting the best weighting scheme for the data. The LPML values for the unity weighting scheme, the exponential weighting scheme and the Gaussian weighting scheme are shown in Table 5. From Table 5 we can see that the model with exponential weighting scheme has the largest LPML value among the three candidate schemes, and is therefore preferred. To verify that there is indeed spatially varying covariate effects, we also fitted the spatially-varying coefficients model without clustering (4) to the dataset. The model is compared against a vanilla Bayesian regression where covariate effects are assumed constant over the entire state. From Table 6, the LPML values of the first two models that allow for spatially varying coefficients are larger than the vanilla regression model, and the differences are not minor. This indicates that there indeed exist spatially varying covariate effect, and more flexible models are preferred. Comparing the LPML and $p_D$ for the first two models, it can be seen that the proposed model reduces $p_D$ and provides better fit to the data. Combining the conclusions from Tables 5 and 6, the proposed model with exponential weighting scheme for $W(s)$ is fitted on the dataset.

A total of 3 clusters are identified. The cluster belongings of the 159 counties are visualized in Figure 3, and their corresponding parameter estimates are presented in Table 7. It can be seen that for cluster 1, which includes most of the counties (124 out of 159), higher
Table 5: LPML values for different weighting schemes in the proposed model.

| Scheme              | Unity   | Exponential | Gaussian |
|---------------------|---------|-------------|----------|
| LPML                | -165.784| -165.620    | -217.878 |

Table 6: LPML and $p_D$ values for different models.

| Model                               | LPML    | Spatial-varying coefficients model | Vanilla regression |
|-------------------------------------|---------|-----------------------------------|--------------------|
| The proposed model                  | -165.620| -174.171                          | -203.013           |
| $p_D$                               | 75.17   | 105.92                            | 11.44              |

unemployment rates, higher median home market value, and larger population sizes are significant indicators of high housing costs. For cluster 2 (26 out of 159), median home market value is also positively correlated with the monthly housing cost, while higher median age indicates lower housing cost. For cluster 3 (9 out of 159) median home market value turns out to be the only decisive factor and has significant increasing effect for housing cost.

6 Discussion

In this paper, we have proposed a Bayesian clustered coefficients linear regression model with spatial random effects to capture heterogeneity of regression coefficients. Multiple weighting schemes in modeling the spatial random effects have been proposed, and the corresponding Bayesian model selection criterion have been discussed. Compared to a vanilla regression model with no spatial random effect, allowing the covariate effects to be spatially varying provides better fit to the data, and more profound insight into heterogeneity in development at different locations. In addition, compared to observations made in Ma et al. (2019a), where each location is allowed to have its own set of parameter estimates, the clustering approach reduces the effective number of parameters without sacrificing the model goodness-of-fit. The usage of the method is illustrated both in simulation studies and an application to analysis of impacting factors for housing cost in Georgia.

A few topics beyond the scope of this paper are worth further investigation. In this paper, we only considered the full model that includes all covariates. Appropriate approaches for variable selection under a clustered regression context is worth investigating. The DPMM is used to get clustering information of regression coefficients. Miller and Harrison (2013) shows that the posterior on the number of clusters is not consistent based on the DPMM. Such pattern have been observed in both our simulation studies, where there are some small clusters which only contain a few counties. Proposing a consistent prior for clustered regression coefficients is an important future work. In addition, extending our approach in
Figure 3: Clusters produced by the proposed approach.

Table 7: Parameter estimates and their 95% HPD intervals for the three clusters identified.

| Cluster | 1      | 2      | 3      |
|---------|--------|--------|--------|
| $\beta_1$ | 0.154  | -0.088 | -0.349 |
|         | (0.020, 0.293) | (-0.424, 0.394) | (-1.533, 0.811) |
| $\beta_2$ | -0.091 | 0.043  | -0.047 |
|         | (-0.217, 0.040) | (-0.404, 0.433) | (-1.19, 1.08) |
| $\beta_3$ | 1.841  | 1.985  | 1.540  |
|         | (1.615, 2.072) | (1.463, 2.580) | (0.317, 2.778) |
| $\beta_4$ | -0.078 | -0.146 | 0.151  |
|         | (-0.295, 0.102) | (-0.648, 0.313) | (-0.910, 1.509) |
| $\beta_5$ | -0.004 | -0.998 | -0.994 |
|         | (-0.205, 0.192) | (-1.478, -0.578) | (-1.899, 0.186) |
| $\beta_6$ | 0.327  | 0.611  | 0.119  |
|         | (0.069, 0.698) | (-0.592, 1.421) | (-0.789, 1.435) |
non-gaussian model is an interesting topic. Considering spatial dependent structure for the regression coefficients is devoted to future research.

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