KINEMATIC MORPHOLOGY OF LARGE-SCALE STRUCTURE: EVOLUTION FROM POTENTIAL TO ROTATIONAL FLOW

XIN WANG1, ALEX SZALAY1, MIGUEL A. ARAGÓN-CALVO1, MARK C. NEYRINCK1, and GREGORY L. EYINK1,2

1 Department of Physics & Astronomy, Johns Hopkins University, Baltimore, MD 21218, USA
2 Department of Applied Mathematics & Statistics, Johns Hopkins University, Baltimore, MD 21218, USA

Received 2013 September 20; accepted 2014 July 22; published 2014 September 5

ABSTRACT

As an alternative way to describe the cosmological velocity field, we discuss the evolution of rotational invariants constructed from the velocity gradient tensor. Compared with the traditional divergence–vorticity decomposition, these invariants, defined as coefficients of the characteristic equation of the velocity gradient tensor, enable a complete classification of all possible flow patterns in the dark-matter comoving frame, including both potential and vortical flows. We show that this tool, first introduced in turbulence two decades ago, is very useful for understanding the evolution of the cosmic web structure, and in classifying its morphology. Before shell crossing, different categories of potential flow are highly associated with the cosmic web structure because of the coherent evolution of density and velocity. This correspondence is even preserved at some level when vorticity is generated after shell crossing. The evolution from the potential to vortical flow can be traced continuously by these invariants. With the help of this tool, we show that the vorticity is generated in a particular way that is highly correlated with the large-scale structure. This includes a distinct spatial distribution and different types of alignment between the cosmic web and vorticity direction for various vortical flows. Incorporating shell crossing into closed dynamical systems is highly non-trivial, but we propose a possible statistical explanation for some of the phenomena relating to the internal structure of the three-dimensional invariant space.

Key words: cosmology: theory – large-scale structure of universe

Online-only material: color figures

1. INTRODUCTION

The peculiar velocity, \( \mathbf{u} \), together with the density distribution, \( \rho \), of dark matter, encode abundant information about the evolution of the large-scale structure of our Universe. Aside from its statistical manifestation via redshift-space distortions (Kaiser 1987; Davis & Peebles 1983; Hamilton 1992; Cole et al. 1994), the line-of-sight component of the peculiar flow of the individual galaxies can also be obtained using distance indicators, such as the Tully–Fisher relation (Tully & Fisher 1977) between galaxy luminosity and rotational velocity, and the transverse component from the proper motion of galaxies (Nusser et al. 2012) by highly accurate astrometric experiments like Gaia. Such a data set of large-scale, three-dimensional peculiar velocities would provide valuable information for our theory of the evolution of the dark matter velocity field.

An advantage of three-dimensional velocity data is the ability to study the vorticity contribution \( \mathbf{w} = \nabla \times \mathbf{u} \) (Peebles 1980; Pueblas & Scocicmarro 2009; Hahn et al. 2014), which in the standard cosmology model is negligible on a large scale and is mainly generated after the shell crossing at a later epoch, as any initial rotational component will decay rapidly due to the expansion of the universe. The divergence contribution \( \Theta = \nabla \cdot \mathbf{u} \), as the only remaining degree of freedom of, \( \mathbf{u} \), has long dominated most of the raw theoretical and observational investigations in this field. However, in the nonlinear regime the vectorial rotational component could contain valuable morphological information that would be lost if only the divergence is studied. To learn about the cosmic flow in more detail, in this paper we are interested in the gradient tensor of the velocity field, \( \mathbf{u} \),

\[
A_{ij}(\mathbf{x}, \tau) = \frac{\partial u_i}{\partial x_j}(\mathbf{x}, \tau),
\]

where \( \mathbf{x} \) is the Eulerian position and \( \tau \) the comoving time.

The tensor \( A_{ij} \) is closely related to cosmic web structure, because it characterizes the velocity variations around a mass element as it moves away from a void or toward halo/filamentary/wall structures. For irrotational flow, \( \mathbf{u} = -\nabla \psi \), where \( \psi \) is the velocity potential. At the linear order, \( \psi \) is proportional to the gravitational potential \( \Phi \); therefore, the tensor \( A_{ij} \approx \nabla \nabla \psi \) encodes information about the anisotropic gravitational field, which eventually leads to the formation of the cosmic web structure. Given the importance of the cosmic web in understanding large-scale structure formation (Bond et al. 1996), many techniques have been developed to classify it in numerical simulation (Pogosyan et al. 2009; Aragon-Calvo et al. 2010; Sousbie 2011; Sousbie et al. 2011; Bond et al. 2010; Forero-Romero et al. 2009; Hahn et al. 2007b; Sousbie et al. 2009; Stoica et al. 2005). Similar to the method using the tidal field, or equivalently the Hessian matrix of density, one usually considers the eigenvalues, \( \lambda_i \), of tensor \( A_{ij} \). Neglecting the subtleties in selecting the criteria, in general, entirely positive (negative) eigenvalues correspond to an entirely stretching (compressing) region, such as a void (halo). Thus a matrix with both positive and negative eigenvalues gives wall or filament structures, e.g., Hoffman et al. (2012). However, aside from its implicit coordinate system dependence, once the vorticity is generated, the anti-symmetric tensor, \( A_{ij} \), gives complex eigenvalues, \( \lambda_i \), and therefore complicates the discussion of flow morphology in such language.

In this paper we consider an alternative way of describing flow morphologies, as well as its applications.
coordinate dependence, one would prefer scalar fields that are rotational invariants built from the velocity gradient tensor $A_{ij}$. The eigenvalue problem of tensor $A_{ij}$ is given by
\[
\det[A_{ij} - \lambda \delta_{ij}] = \lambda^3 + s_1 \lambda^2 + s_2 \lambda + s_3 = 0, \tag{2}
\]
where $\delta_{ij}$ is the identity matrix. The coefficients $s_i$, $i = 1, 2, 3$, in Equation (2), are natural quantities to study. For various irrotational flows that have been discussed, the criteria in eigenvalue space are then mapped into different regions in the invariant space (Chong et al. 1990). One advantage of working in this parameter space is to avoid the complex domain after the emergence of vorticity, and $A_{ij}$ becoming anti-symmetric. This enables one to continuously explore the evolution from potential to vortical flow in a uniform framework.

Furthermore, compared with the rotation-dependent matrix $A_{ij}$, these invariants are also an appropriate device for the analytical study of the dynamical evolution of the cosmic flow. Starting from initial fluctuations with the negligible rotational degrees of freedom, one is able to trace the evolution of kinematic morphologies of a dark matter element as it travels away from an underdense region toward filaments or halos in the invariant space, eventually experiencing various types of vorticity in the multi-streaming regime. However, analytical study of the growth of vorticity is highly non-trivial (Pueblas & Scoccimarro 2009), because it mainly occurs in a non-linear, small scale that is beyond the reach of approaches like perturbation theory. In addition, efforts to incorporate the shell crossing would result in a hierarchy of fluid dynamical equations that must be closed. One approach is to model or measure the velocity dispersion and truncate the hierarchy (e.g., Pueblas & Scoccimarro 2009). In this paper, as suggested by our numerical measurement of vortical morphologies, the stochasticity of the process may provide some new insight.

Even for collapsed objects like halos, having the full information of tensor $A_{ij}$ is important. Theoretically, the angular momentum of the halo is largely explained by the tidal torque theory (Hoyle 1949; Peebles 1969; Doroshkevich 1970; White 1984; Wesson 1985), which explains the formation of halo spin from laminar flow by the misalignment between shear and inertia, and predicts a linear growth of angular momentum. However, this mechanism is only effective prior to turn-around in the spherical-collapse picture; after this, the collapse dramatically reduces the lever arm (Schafer 2009), and the alignment between the halo spin and vorticity direction found by Libeskind et al. (2013a) suggests a separate phase of the growth of halo angular momentum. The correlation picture between the halo spin and vorticity supports our finding from invariants because investigations using N-body simulations also indicate the existence of preferential orientation between the halo spin and filaments (Hahn et al. 2007b; Sousbie et al. 2009; Zhang et al. 2009; Laigle et al. 2013; Dubois et al. 2014; Aragon-Calvo & Yang 2014). Indeed, several authors found mass-dependent alignment of halo spins with filaments—low-mass halos tend to be aligned with filaments, and high-mass halos have spin perpendicular to the filaments (Aragon-Calvo et al. 2007; Hahn et al. 2007a; Paz et al. 2008; Sousbie et al. 2009; Codis et al. 2012; Aragon-Calvo & Yang 2014). Tempel & Libeskind (2014) found evidence for this alignment in SDSS galaxies.

The purpose of this paper is several-fold. After introducing the invariants of the velocity gradient tensor at the beginning of Section 2, we show its ability to classify flow morphologies for both potential and vortical flow, and discuss the connection to the cosmic web structure. Then we try to investigate the dynamical evolution of these invariants, first for irrotational flow in Section 3, and then in Section 4 we study the emergence of vorticity through shell crossing. For simplicity, after deriving the full dynamical equations before shell crossing, we study the Zel’dovich approximation (ZA), obtaining the exact numerical solution and the probability distribution of invariants. In Section 4 we present the result relating to the vorticity measured from N-body simulation, and then set up both the dynamical and statistical view for physical interpretation. Finally, we discuss and conclude in Section 5.

Throughout the paper, we make our numerical measurement with two different sets of N-body simulations. For the purpose of irrotational flow only, we use a simulation with the box size $100 \ h^{-1}$ Mpc and $512^3$ particles (hereafter Sing100). However the spatial resolution is not high enough to study vortical flow, which is generated at a very small scale. To achieve a clean signal with a high signal-to-noise ratio and also explore the relation to large-scale structure, we use the MIP (MUltipole In Parvo; many things in the same place) ensemble simulation (Aragon-Calvo 2012), which is a suite of $N$-body simulations constrained to have the same large-scale structure for initial Fourier modes with wavelengths over $4 \ h^{-1}$ Mpc, but independent small-scale realizations. The combining of many realizations enables one to visually inspect the correspondence between the rotational flow and cosmic web. The ensemble has 256 realizations$^4$ of a $32 \ h^{-1}$ Mpc box with $256^3$ particles each. This is equivalent in terms of effective volume and number of particles to a box of 193 $h^{-1}$ Mpc of side with $\sim 1540^3$ particles containing $\sim 5 \times 10^6$ haloes with a minimum mass of $3.25 \times 10^9 \ h^{-1}$ M$_\odot$. To numerically estimate the spatial gradient $A_{ij}$, we first construct the velocity field on a regular Eulerian grid with Delaunay tessellation (Bernardeau & van de Weygaert 1996; Pueblas & Scoccimarro 2009; Schaap & van de Weygaert 2000; pelupessy et al. 2003), and then take the gradient in real space. Although this method is known to have a better noise property in measuring velocity divergence and vorticity (Pueblas & Scoccimarro 2009), it still involves coarse graining and is less accurate than more sophisticated methods with an explicit phase space projection (Hahn et al. 2014).

2. INVARIANTS OF VELOCITY GRADIENT TENSOR AND LOCAL FLOW MORPHOLOGY

The invariants of the velocity gradient tensor have been studied in turbulence for more than two decades, since their introduction by Chong et al. (1990). They describe the fundamental and intrinsic properties of small-scale motions in turbulence (Meneveau 2011). In this section, we present the definition of the invariants of the velocity gradient tensor, discuss their ability to classify various potential and rotational flow morphologies, and then show the connection with cosmic web structure.

2.1. Definition of Invariants

Among all nine elements in tensor $A_{ij}$, six degrees of freedom are rotationally invariant, and the other three encode coordinate information. As discussed in the introduction, to extract the morphological information we are particularly interested in the coefficients of the characteristic equation $\det[A - \lambda I] = 0$, i.e.,
\[
\lambda^3 + s_1 \lambda^2 + s_2 \lambda + s_3 = 0, \tag{3}
\]
$^4$ Although only 64 realizations are analyzed in this paper.
where the invariants $s_1, s_2, s_3$ are defined as (Chong et al. 1990)

\[ s_1 = -\text{tr}[A] = -\theta_i j, \]
\[ s_2 = \frac{1}{2} (s_i^2 - \text{tr}[A^2]) = \frac{1}{2} (s_i^2 - \theta_{ij} \theta_{ji} - \omega_{ij} \omega_{ji}) \]
\[ s_3 = -\det[A] = \frac{1}{3} (-s_1^3 + 3s_1s_2 - \text{tr}[A^3]) \]
\[ = \frac{1}{3} (-s_1^3 + 3s_1s_2 - \theta_{ij} \theta_{ji} \theta_{ki} - 3\omega_{ij} \omega_{ji} \theta_{ki}). \] (4)

Here we have decomposed $A_{ij}$ into a symmetric part $\theta_{ij}$ and an anti-symmetric part $\omega_{ij},$

\[ A_{ij} = A_{(ij)} + A_{(ji)} = \theta_{ij} + \omega_{ij}, \] (5)

where

\[ \theta_{ij} = \frac{1}{2} (A_{ij} + A_{ji}), \quad \omega_{ij} = \frac{1}{2} (A_{ij} - A_{ji}). \] (6)

The tensor $\theta_{ij}$ denotes the rate of the deformation of a fluid element and $\omega_{ij}$ shows the rate of rotation, which relates to the vorticity vector $\omega_i$ through $\omega_i = -\epsilon_{ijk} \omega_{jk},$ where $\epsilon_{ijk}$ is the totally anti-symmetric Levi–Civita symbol. To better understand the physics of these invariants $s_i,$ we could further decompose the rate of deformation into isotropic and anisotropic parts (i.e., the divergence $\theta$ and the shear tensor $\sigma_{ij})$

\[ \theta_{ij} = \sigma_{ij} + \frac{1}{3} \theta \delta_{ij}, \] (7)

where $\delta_{ij}$ is the Kronecker delta, and also define the magnitude of tensor $\sigma_{ij}$ and $\omega_{ij}$ as

\[ \sigma = \left( \frac{1}{2} \sigma_{ij} \sigma_{ij} \right)^{1/2}, \quad \omega = \left( \frac{1}{2} \omega_{ij} \omega_{ij} \right)^{1/2}. \] (8)

Using these definitions, the invariants just introduced can be expressed as

\[ s_1 = -\theta \]
\[ s_2 = \frac{1}{2} \theta^2 - \sigma^2 + \omega^2 \]
\[ s_3 = -\frac{1}{27} \theta^3 + \frac{1}{3} \theta (\sigma^2 - \omega^2) - \frac{1}{3} \sigma_{ij} \sigma_{jk} \sigma_{ki} - \frac{1}{4} \sigma_{ij} \omega_{ij} \omega_{ji}. \] (9)

In turbulence, many studies concentrate on the incompressible flow at $\theta = 0,$ invariants $s_2 = \omega^2 - \sigma^2$ and $s_3 = -\sigma_{ij} (\sigma_{jk} \sigma_{ki}/3 + \omega_{jk} \omega_{ki}),$ and then describe the balance between the shearing $\sigma_{ij}$ and vorticity $\omega_{ij}.$ It is then reasonable to discuss these two contributions separately, which means one could define the remaining scalar degree of freedom of $A_{ij}$ as, for example, the symmetric $s_2^{(s)}$ (or anti-symmetric $s_3^{(a)}$) contribution to $s_2$ and $s_3,$ where

\[ s_i = s_i^{(s)} + s_i^{(a)}, \quad i = (2, 3). \] (10)

In cosmology, however, $\theta$ is usually nonzero, so the antisymmetric contribution could be expressed as

\[ s_2^{(a)} = -\frac{1}{2} \omega_{ij} \omega_{ji} = \omega^2, \]
\[ s_3^{(a)} = s_1 \omega_{ij}^{(a)} - \omega_{ij} \omega_{ji} \theta_{ki} \]
\[ = -\frac{1}{3} \theta \omega^2 - \frac{1}{4} \sigma_{ij} \omega_{ij} \omega_{ji}. \] (11)

As will be seen later, before shell crossing $s_2^{(a)}$ and $s_3^{(a)}$ vanish the eigenvalues $\lambda_i$ of matrix $A_{ij}$ and therefore the local cosmic web is entirely determined by $s_1^{(s)}(i = 1, 2, 3).$ Once the vorticity is generated after the shell crossing, it will then couple to symmetric contribution, via $s_1^{(a)},$ and affect the morphological classification of the local flow. Therefore, the generation of the rotational flow is much more complicated, and not merely related to vorticity. Moreover, because $\omega_{ij}$ contributes to both $s_2$ and $s_3$ it is possible to generate laminar flow with a non-zero vorticity (Appendix B).

2.2. Classification of the Local Flow

Because they are the coefficients of the characteristic equation, the most direct application of $s_i$ is the classification of the solutions $\lambda_i.$ The trajectory around a given point $x_0$ can be approximated with a Taylor expansion,

\[ \frac{dx_i}{d\tau}(x) = \omega_{ij}(x) = \omega_{ij}(x_0) + A_{ij}(x_0) x_j + \cdots \] (12)

In the dark-matter comoving frame where $u_i(x_0) = 0,$ by ignoring higher-order terms one obtains the local trajectory of a fluid element via the differential equation

\[ \frac{dx_i}{d\tau} = A_{ij} x_j. \] (13)

Thus, to first order, $A$ characterizes the local trajectory of the fluid element. As will be shown later, by neglecting the rotational degree of freedom of the coordinate system various flow morphologies presented by solving Equation (13) in the canonical form of $A$ can be mapped into invariant space. The three-dimensional real parameter space can then be divided into various adjacent regions. Among them, the most important criteria is the surface dividing the real and complex roots of $\lambda_i,$ which is defined by the equation

\[ 27 s_3^2 + (4 s_1^3 - 18 s_1 s_2) s_3 + (4 s_2^3 - s_1^2 s_2^2) = 0. \] (14)

Using the discriminant $s_2 \leq s_1^2/3$ of the above quadratic equation, one is able to define the real solutions of $s_3,$ which we denote as $s_{3a}$ and $s_{3b},$ assuming $s_{3a} \leq s_{3b}.$ For real eigenvalues, the inequality $s_{3a} \leq s_3 \leq s_{3b}$ should hold, otherwise there will be complex eigenvalues.

In the following, we will review the work of Chong et al. (1990), and show how different flow morphologies could be classified according to the eigenvalues of $A_{ij}$ and its image in the invariant space. Here, we only discuss the most relevant categories in a cosmological context. A more complete classification, including all the degenerating special cases, can be found in the original paper. For convenience, in Appendix A, a list of the categories is discussed and some typos in Chong et al. (1990) are corrected.
sets $\lambda_i$ the visual impression (Hoffman et al. 2012), our method simply using either the Hessian matrix of gravitational potential regime, many similar algorithms have already been proposed, or the corresponding regions in the invariant space. In this Figure 1.

Illustration of various cosmic flow types in invariant space. We only show two-dimensional $s_1 - s_2$ plane with fixed $s_1$; the left panel shows negative $s_1$, and the right panel shows positive $s_1$.

The discussion of flow morphology is possible only in an appropriate coordinate system. The potential flow, $A_{ij}$ is symmetric, and therefore could always be diagonalized over the field of real numbers.

$$A = R^{-1} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} R, \quad \lambda_i \in \mathbb{R}, \quad (15)$$

where $R$ is a rotation matrix, and we assume $\lambda_1 \leq \lambda_2 \leq \lambda_3$. From Equation (15) and the definition of $s_i$, the invariants can simply be expressed as

$$s_1 = -\sum_i \lambda_i, \quad s_2 = \sum_{i \neq j} \lambda_i \lambda_j, \quad s_3 = -\prod_i \lambda_i. \quad (16)$$

For a general anti-symmetric matrix $A$ it is not easy to extract geometric information, although in many cases Equations (15) and (16) are still valid over complex field $\lambda_i \in \mathbb{C}$. Therefore, it is more convenient to rotate the matrix into a well-suited frame, as described in Section 2.2.2. In some situations, $A$ is not diagonalizable. However, these are very special cases and are unlikely to be important in the cosmological context.

2.2.1. Potential Flow

For the potential flow, our method categorizes different morphologies according to the sign of the eigenvalues of $A_{ij}$ or the corresponding regions in the invariant space. In this regime, many similar algorithms have already been proposed, using either the Hessian matrix of gravitational potential $\nabla^2 \Phi$ (Hahn et al. 2007b; Forero-Romero et al. 2009) or, similar to our method, the symmetric part of $A_{ij}$ (“V-web”; Hoffman et al. 2012). Instead of introducing a free parameter of the threshold eigenvalues $\lambda_{th}$, which is usually adjusted to match the visual impression (Hoffman et al. 2012), our method simply sets $\lambda_{th} = 0$. However, it is not difficult to explicitly incorporate this parameter. One just defines $\lambda' = \lambda - \lambda_{th}$ and rewrites the characteristic Equation (3) as a function of $\lambda'$,

$$(\lambda' + \lambda_{th})^3 + s_1(\lambda' + \lambda_{th})^2 + s_2(\lambda' + \lambda_{th}) + s_3 = 0, \quad (17)$$

and instead discusses the sign of variable $\lambda'$. Expanding Equation (17) is equivalent to defining a new set of invariants $s_i'$,

$$s_1' = s_1 + 3\lambda_{th}$$
$$s_2' = s_2 + 2\lambda_{th} s_1 + 3\lambda_{th}^2$$
$$s_3' = s_3 + \lambda_{th} s_1 + \lambda_{th}^2 s_1 + \lambda_{th}^3. \quad (18)$$

Then the classification conditions of the invariants that are introduced in the following remain the same.

Because the symmetric matrix $A_{ij}$ can be diagonalized into canonical form (15), three eigen-planes that contain solution trajectories will occur. These trajectories may look like nodes and saddles on the plane, depending on the sign of variable $\lambda'$. Thus the mass elements would all flow toward the position of interest $x_0$ in each eigen-plane, which is more likely to occur in the over-dense region around halos. Mapping from the eigenvalue space into the invariant space is straightforward; from Equation (16) we see $s_i > 0$ for all $i = (1, 2, 3)$. Combined with the real solution condition (Equation A3), this type corresponds to a region in invariant space,

$$s_1 > 0, \quad 0 < s_2 < s_2'/3, \quad \max(s_{3a}, 0) < s_3 < s_{3b}. \quad (19)$$

As illustrated in Figure 1, the streamlines in this region are stable nodes in all three eigen-planes, and therefore will be denoted as $SN/\bar{SN}/\bar{SN}$ in the following.

---

6 In this paper, we adopt the same terminology for describing the kinematic morphologies as in turbulence literature, e.g., Chong et al. (1990), where the nodes or saddles denote the configuration of the physical flow trajectory in the eigen-plane. These do not necessarily correspond to the nodes that correspond to halos or clusters in a density field classification.
On the other hand, when all eigenvalues \( \lambda_i > 0 \), three nodes still occur, but the flow heads outward from the origin, which would most likely happen in under-dense region, in the void. Similar to Equation (16), \( s_1, s_3 < 0 \), and \( s_2 > 0 \), and with real root conditions, it corresponds to the saddle point in invariant space

\[
s_1 < 0, \quad 0 < s_2 < s_2^2/3, \quad s_3a < s_3 < s_3 = \min(s_{3b}, 0). \tag{20}
\]

We denote this type as \( \mathcal{UN}/\mathcal{UN}/\mathcal{UN} \), where \( \mathcal{UN} \) stands for unstable node.

If the matrix \( A_{ij} \) becomes indefinite (i.e., with both positive and negative eigenvalues), then it possess saddle points in two eigenvector planes. As shown in Figure 1, in the eigeneplane with a saddle point the flow approaches inward from one direction and departs toward the other without passing through the origin. Especially when the smallest two eigenvalues \( \lambda_1, \lambda_2 \) are negative, while \( \lambda_3 \) positive, the two saddle points reside in the planes spanned by \( v_1 - v_3 \) and \( v_2 - v_3 \), where \( v_1 \) is the eigenvector corresponding to \( \lambda_1 \), and the node in plane \( v_1 - v_2 \) is stable (i.e., the mass elements flow inward). Therefore, this category is called \( \mathcal{SN}/\mathcal{SN}/\mathcal{SN} \), where the last two \( \mathcal{SN} \) denote the saddle point. From the illustration of the trajectory in Figure 1, this resembles the situation when the mass flows along the filamentary structures. In invariant space, the sum of \( \lambda_1 \) could be either positive or negative, depending on the relation between \( |\lambda_1 + \lambda_2| \) and \( |\lambda_3| \). If \( s_1 \geq 0 \) (i.e., \( \lambda_1 + \lambda_2 \geq \lambda_3 \)), the flow will change faster in the node plane \( v_1 - v_2 \), and cause the net compression of the fluid element. If \( s_1 < 0 \), it is the opposite happens. In summary, we have

\[
s_1 \geq 0, \quad s_2 < s_2^2/4, \quad s_3a < s_3 < 0, \quad \text{or } s_1 < 0, \quad s_2 < 0, \quad s_3a < s_3 < 0. \tag{21}
\]

When only the smallest eigenvalue \( \lambda_1 \) is negative, and \( \lambda_2, \lambda_3 \) positive, the two saddle points reside in the planes spanned by \( v_1 - v_2 \) and \( v_1 - v_3 \), and the node in plane \( v_2 - v_3 \) is unstable because \( \lambda_{2,3} > 0 \). This corresponds to the case when matter flows toward a wall structure. Again we have two conditions in the invariant space

\[
s_1 \leq 0, \quad s_2 < s_2^2/4, \quad 0 < s_3 < s_{3b}, \quad \text{or } s_1 > 0, \quad s_2 < 0, \quad 0 < s_3 < s_{3b}. \tag{22}
\]

Similarly, \( s_1 \leq 0 \) corresponds to a faster velocity change in the node plane \( v_2 - v_3 \), and a net expansion of the fluid element. If \( s_1 > 0 \), the fluid element contracts.

2.2.2. Vortical Flow

After shell crossing, the anti-symmetric part \( \omega_{ij} \) of the matrix \( A_{ij} \) will be generated. In terms of real numbers, the canonical form of \( A_{ij} \) can be expressed as

\[
A = R^{-1} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} R,
\]

where \( a, b, c \) are all real, and the complex eigenvalue \( \lambda_{1,2} = a \pm ib, \lambda_3 = c \). Then the invariants simply read as

\[
s_1 = -(2a + c), \quad s_2 = a^2 + b^2 + 2ac, \quad s_3 = -c(a^2 + b^2). \tag{24}
\]

In this case, one plane exists that corresponds to eigenvalues \( \lambda_{1,2} \), and contains the solution trajectory, which can be expressed as

\[
r = r_0 e^{ma} \tag{25}
\]

in polar coordinates \((r, \alpha)\). Here the factor \( m = a/b \) denotes the rate of spiraling, and \( r_0 \) is a constant depending on the initial condition. Along the direction of \( \lambda_3 \), the behavior of the trajectory is controlled by the value of \( c \).

Depending on the value of \( m \) and \( c \), and neglecting degenerate cases, four major categories of rotational flow remain. For negative \( m \), the trajectory spirals inward to the origin of the plane, hereafter the stable focal flow. If \( c > 0 \), mass elements flow away from the plane, a situation denoted \( SFS \) (i.e., stable focal stretching). If \( c < 0 \), the mass elements flow toward the plane, hereafter stable focal compressing, and abbreviated as \( SFC \). For positive \( m \), the trajectory spirals outward from the origin, and the flow is called unstable focal flow. Depending on the value of \( c \), the latter two flow types are called unstable focal stretching, hereafter \( UFS \) for \( m > 0, c > 0 \), and unstable focal compressing, hereafter \( UFC \) for \( m > 0, c < 0 \). The detailed criteria for vortical regions are listed in Appendix A.

2.3. Cosmic Web and Kinematic Morphologies

To show the relationship between large-scale structure morphologies and the kinematic classification, we highlight different categories of the flow patterns of \( \text{Sing}100 \) simulation at redshift \( z = 0 \), and compare those with the density field in Figure 2. Because the process of cosmic web formation is largely associated with potential flow, and to avoid disturbance from the rotational degree of freedom, we smooth the velocity field using a Gaussian filter with a smoothing length varying from \( R = 0.2 \, \text{Mpc} \, h^{-1} \) to 1.5 Mpc \, h^{-1}. Notice that different categories are labeled by their kinematic names, such as \( \mathcal{SN}/\mathcal{SN}/\mathcal{SN} \) instead of the more familiar halo, because by setting \( \chi_{\text{th}} = 0 \), our morphology classification does not necessarily reproduce the best visual structures compared with V-web (Hoffman et al. 2012). Since our method includes both potential and rotational morphologies, cosmic-web structures, especially filaments, defined with irrotational flow could only be identified with a relatively large smoothing length. However, with an appropriate smoothing (e.g., \( R = 0.8 \, \text{Mpc} \, h^{-1} \) as shown in the lower left panel), one does see a good visual correspondence between the filamentary cosmic web in the density (first panel) and the corresponding kinematic categories, \( \mathcal{SN}/\mathcal{SN}/\mathcal{SN} \) (halo-like) and \( \mathcal{SN}/\mathcal{SN}/\mathcal{SN} \) (filament-like). This mainly reflects the connection between the velocity potential \( \psi \) and gravitational potential \( \Phi \) on large scales. As smoothing is performed more aggressively, the features at small scales vanish, as expected. However, unlike the wall/sheet structure identified in Hoffman et al. (2012), our wall-like \( \mathcal{UN}/\mathcal{SN}/\mathcal{SN} \) type fills almost half the simulation volume. More studies are desired, but we believe that this is a combined effect of the larger Eulerian volume filling factor for underdense regions and the relative larger volume in the invariant space, as could be seen in Figure 1. Figure 2 also shows that the large-scale structure identified kinematically is reliable with various degrees of smoothing, although the rotational categories, shown in yellow, are reduced dramatically as the smoothing length increases.

To better understand the rotational regions in Figure 2, we use the MIP N-body ensemble simulations and show the spatial distribution of four vortical flow categories in Figure 3. One benefit of using the MIP simulation is that, since vorticity is only generated at small scales after shell crossing, there will not be many pixels classified as rotational in a single realization of a moderate resolution simulation. Moreover, because the
classification criteria is mutually exclusive, each pixel can only be categorized as one of them. On the other hand, with the full suite of MIP simulation, one can explore the ensemble probability of generating one particular morphology given the large-scale environment. For this purpose, we make a complete classification of all realizations, and then highlight the region where more than \( n_{\text{ct}} \) out of 64 are categorized as a certain type. As a result, each pixel of the simulation could be multi-valued. In Figure 3 the first row illustrates the vortical classification of an arbitrary realization, while the middle and bottom row are the combined classification with \( n_{\text{ct}} = 5 \) and 2, respectively. The underlying density distribution of the last two rows is produced by stacking all 64 realizations, therefore removing the small-scale structure as compared with the first. Similar to the irrotational morphologies, the most distinctive feature of the figure is that various flow categories display different spatial distributions. For example, \( SFC \) resides around knots, while and \( SFS \) and \( UFC \) are located along filaments; this is especially clear for lower \( n_{\text{ct}} \). Furthermore, as will be discussed later, the direction of the vorticity of different categories also varies among different categories.

Aside from the spatial distribution, we also plot in Figure 4 the probability density function (PDF) of matter density for various irrotational flow categories in the upper panels and vortical types in the lower. While the PDF is displayed on the left, we also present curves rescaled by the volume fraction of each morphology type on the right. The left panels show that the locations of PDF maxima of each category in \((1 + \delta)\)-axis, \( \delta_{\text{max}} \), exhibit a sequence consistent with their cosmic web morphologies. That is, the PDF of type \( SN/SN/SN \) peaks at a higher density value \( \delta \) than that of type \( SN/S/S \); we identify these as halo and filament, respectively; following these are wall/sheet and void types. On the other hand, as shown in the lower panel, \( \delta_{\text{max}} \) of vortical types are all greater than that of the entire field (long-dashed line), which is reasonable because they are only generated after shell crossing in denser regions. Although the differences of peak positions among all vortical categories are narrower than for potential types, they are still consistent with the spatial distribution from Figure 3: the type \( SFS \) around knots are the highest density, followed by \( SFS \) and \( UFC \), which trace filaments. The right panels further show the relative volume abundance of each category. As the distributions are volume-weighted in Eulerian space, higher density regions are expected to be less abundant. This explains the lower amplitude of category \( SN/SN/SN \) and \( SFC \) in the figure. Also, as consistent with the visual impression of Figure 2, more regions are classified as \( UN/S/S \) instead of \( UN/UN/UN \).

To further examine our invariant-based cosmic web classification, we also investigate the isotropy of each category,
Figure 3. Four different categories of vortical flow in the MIP ensemble, a suite of simulations with the same large-scale structure, but independent small-scale realizations. Every column presents the same morphologies labeled in the upper right corner of each panel. The top row makes use of only one realization, showing that very small fractions of the region are classified as rotational categories. This is reasonable given the resolution of our simulation. To more clearly display the spatial distribution of various rotational categories, we use the whole ensemble simulation in the middle and bottom rows, where a much larger area highlights the region where more than $n_{ct}$ out of 64 realizations are labeled as a particular rotational category. We adopt $n_{ct} = 5$ for the middle row and 2 for the bottom. The density fields for these two are also stacked over all realizations, which is why small-scale structures are smeared out.

(A color version of this figure is available in the online journal.)

because, to some level, halos and voids should be more isotropic than filaments and walls. Similarly, Libeskind et al. (2013b) measured the fractional anisotropy for its shear based V-web classification,

$$f_a = 1 - \frac{1}{\sqrt{3}} \frac{\sum_i \lambda_i^2}{\sum_i \lambda_i},$$  

(26)

where $\lambda_i$ is the eigenvalue of the symmetric tensor $\theta_{ij}$. It takes values between zero and unity, with $f_a = 0$ for totally isotropic expansion/contraction and $f_a = 1$ for anisotropic motion. Due to their selected eigenvalue threshold $\lambda_{th} = 0.44$, Libeskind et al. (2013b) found that voids in their classification have the highest anisotropy, whereas filaments and walls have a broad distribution of $f_a$. However, in our classification, as shown in the upper panel of Figure 5, filaments and walls exhibit the highest anisotropy, whereas halos and voids are relatively isotropic. Contrary to Libeskind et al. (2013b), the voids here show the lowest anisotropy. This suggests that, although a fine-tuned $\lambda_{th}$ might produce a better visual web structure, some of its results may be harder to interpret physically.

The lower panel of Figure 5 also displays the same quantities for vortical flows. Clearly, the filament-tracing types SFS and UFC are highly anisotropic. This suggests that even after generating rotational degrees of freedom, the fluid elements in the filamentary environment generally follow a similar shearing movement. Conversely, other categories show relatively flat distributions.

3. GRAVITATIONAL EVOLUTION BEFORE SHELL CROSSING

In the next two sections, we theoretically investigate the evolution of the velocity gradient tensor before and after shell crossing. Before shell crossing, it is possible to neglect the stress tensor in the Euler equation and consider the so-called dust model (Peebles 1980; Bernardeau et al. 2002),

$$\frac{\partial \mathbf{u}}{\partial \tau} + \mathcal{H}(\tau) \mathbf{u}(\mathbf{x}, \tau) + \mathbf{u}(\mathbf{x}, \tau) \cdot \nabla \mathbf{u}(\mathbf{x}, \tau) = -\nabla \Phi(\mathbf{x}, \tau),$$  

(27)

where $\mathcal{H} = d \ln a/d\tau$, and $\Phi$ is the Newtonian potential, which satisfies the Poisson equation

$$\nabla^2 \Phi(\mathbf{x}, \tau) = \frac{3}{2} \Omega_m(\tau) \mathcal{H}^2(\tau) \delta(\mathbf{x}, \tau).$$  

(28)

One can then close the system with the matter continuity equation,

$$\frac{\partial \delta}{\partial \tau} + \nabla \cdot [(1 + \delta) \mathbf{u}] = 0.$$  

(29)

Equation (27)–(29) are basic equations of large-scale structure in Newtonian cosmology. For potential flow, it is sufficient to take the velocity potential $\phi(\mathbf{x})$, or equivalently, the divergence $-s_1(\mathbf{x}) = \nabla^2 \phi(\mathbf{x})$, as the only dynamical degree of freedom of $u_i(\mathbf{x})$, as other quantities including $u_1(\mathbf{x})$ and $A_{ij}(\mathbf{x})$ could be recovered by explicit spatial derivatives. For our purpose,
however, the flow morphology classification at given position \( \mathbf{x} \) requires more information from tensor \( A_{ij}(\mathbf{x}) \) than a single scalar. Moreover, instead of examining the tensorial field \( A_{ij}(\mathbf{x}) \), in the following, we follow a fluid element and investigate its Lagrangian morphological evolution. To proceed, one takes the gradient of the Euler equation (27) and obtains the Lagrangian evolution equation of the tensor \( A_{ij} \),

\[
\frac{dA_{ij}}{d\tau} + \mathcal{H}(\tau)A_{ij} + A_{ik}A_{kj} = \sigma_{ij},
\]

where we have written the Lagrangian total derivative \( d/d\tau = \partial/\partial\tau + \mathbf{u} \cdot \nabla \), and defined \( \sigma_{ij}(\mathbf{x}, \tau) = -\nabla /\nabla \Phi(\mathbf{x}, \tau) \). Here, the Lagrangian frame enables us to trace the change of the flow morphology of a mass element, and as will be shown later, in some cases it simplifies the dynamical equations. Equation (30) holds for each element of the gradient tensor. To marginalize over these degrees of freedom, one is interested in the evolution equation of the invariants \( s_i \). Multiplying both sides of Equation (30) by \( \mathbf{I}, \mathbf{A}, \) and \( \mathbf{A}^2 \), respectively, and then taking the trace, gives the dynamical equation of \( s_1, s_2, \) and \( s_3 \),

\[
\begin{align*}
\frac{d}{d\tau} s_1 + \mathcal{H}(\tau)s_1 - s_1^2 + 2s_2 &= -\sigma s_1 \\
\frac{d}{d\tau} s_2 + 2\mathcal{H}(\tau)s_2 - s_1 s_2 + 3s_3 &= -s_1 \sigma - \sigma A s_1 \\
\frac{d}{d\tau} s_3 + 3\mathcal{H}(\tau)s_3 - s_1 s_3 &= -s_2 \sigma - s_1 \sigma A - \sigma A^2 s_1 
\end{align*}
\]

(31)

with source terms defined as \( \sigma = \sigma_{ij} = -\nabla^2 \Phi, \sigma_A = \sigma_{ij}A_{ij}, \) and \( \sigma_{A^2} = \sigma_{ij}A_{ij}A_{ij} \). Here we used the following identities to simplify the expression,

\[
\begin{align*}
\text{tr}[\mathbf{A}^2] &= A_{ij}A_{ji} = s_1^2 - 2s_2 \\
\text{tr}[\mathbf{A}^3] &= A_{ij}A_{jk}A_{kl} = -s_1^3 + 3s_1 s_2 - 3s_3 \\
\text{tr}[\mathbf{A}^4] &= A_{ij}A_{jk}A_{kl}A_{li} = s_1^4 - 4s_1^2 s_2 + 4s_1 s_3 + 2s_2^2. \quad (32)
\end{align*}
\]
The final equation is derived with the help of the Cayley–Hamilton theorem. Equation (31) depends on the full tidal tensor and its coupling with $A_{ij}$, and hence is non-trivial to solve. The scalar $\sigma$ is simply proportional to the density, and can be obtained via the continuity equation, $\sigma = -(3/2)\Omega_m H_0$. To close Equation (31), one must supplement the evolution of tidal tensor, either using new equation(s) or a known solution from another technique. On the other hand, as will be shown shortly, Equation (31) is dramatically simplified in the ZA.

3.1. Dynamical Evolution of Invariants in the Zel’dovich Approximation

In Lagrangian dynamics, the mass element moves in the gravitational potential along the trajectory (Zel’dovich 1970; Bernardeau et al. 2002)

$$\mathbf{x}(\mathbf{q}, \tau) = \mathbf{q} + \Psi(\mathbf{q}, \tau),$$

(33)

from initial Lagrangian position $\mathbf{q}$. To first order (i.e., the ZA; Zel’dovich 1970), the displacement field $\Psi(\mathbf{q}, \tau)$ is given by

$$\nabla \cdot \Psi(\mathbf{q}, \tau) = -D(\tau)\delta(\mathbf{q}),$$

(34)

where $D(\tau)$ the linear growth factor of the density perturbation. In Eulerian space this is equivalent to replacing the Poisson equation with (Munshi 1994; Hui & Bertschinger 1996; Bernardeau et al. 2002)

$$u_i(x, \tau) = -\frac{2f(\tau)}{3\Omega_m(\tau)H(\tau)}\nabla_i\Phi(x, \tau),$$

(35)

which then closes the system using the Euler equation (27). Here, $\Omega_m(\tau)$ is the matter density fraction at epoch $\tau$, and $f = d\ln D/d\ln a$ is the growth rate. In this approximation, $\sigma, \sigma_A$ and $\sigma_{Ai}$ equal

$$\sigma = -\mathcal{H}(\tau)\mathcal{L}(\tau)s_1,$$

$$\sigma_A = \mathcal{H}(\tau)\mathcal{L}(\tau)(s_1^2 - 2s_2),$$

$$\sigma_{Ai} = \mathcal{H}(\tau)\mathcal{L}(\tau)(-s_1^3 + 3s_1s_2 - 3s_3).$$

(36)

where we have defined $\mathcal{L}(\tau) = 3\Omega_m(\tau)/2f(\tau)$.

Substituting Equation (36) back into Equation (31), obtains the full dynamical equation of the invariant $s_i$ in the ZA. This can be further simplified by defining the rescaled velocity $\tilde{u} = u/D^{(v)}$, where $D^{(v)}(\tau) = dD/d\tau = \mathcal{H}fD$, and changing the time variable $\tau$ in Equation (27) into the linear growth rate $D$. Then, Euler equation reads (Shandarin & Zel’dovich 1989)

$$\frac{d\tilde{u}}{d\tau} = \left(\frac{\partial}{\partial D} + \tilde{u} \cdot \nabla\right)\tilde{u} = 0,$$

(37)

where the prime denotes the Lagrangian derivative $d/d\tau$, and we have also used the differential equation of the linear growth rate (Peebles 1980; Bernardeau et al. 2002)

$$\frac{d^2D(\tau)}{d\tau^2} + \mathcal{H}(\tau)\frac{dD(\tau)}{d\tau} = \frac{3}{2}\Omega_m(\tau)H^2(\tau)D(\tau).$$

(38)

For the velocity gradient tensor, we similarly define the rescaled quantity $\tilde{A}_{ij} = A_{ij}/D^{(v)}$, and

$$\frac{d\tilde{A}_{ij}}{d\tau} + \tilde{A}_{ik}\tilde{A}_{kj} = 0.$$  

(39)

From the definition of various categories in Appendix A, all boundaries classifying different flow categories are invariant if the velocity is scaled with any positive number. Therefore using Equation (4), we define the rescaled invariants $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3$ as

$$\tilde{s}_i(\tau) = \frac{s_i(\tau)}{[D^{(v)}]^{3}}, \quad i \in \{1, 2, 3\}.$$  

(40)

One can then derive a set of solvable ordinary differential equations for the reduced invariants:

$$\tilde{s}_1' - \tilde{s}_2^2 + 2\tilde{s}_2 = 0,$$

$$\tilde{s}_2' - \tilde{s}_1\tilde{s}_2 + 3\tilde{s}_3 = 0,$$

$$\tilde{s}_3' - \tilde{s}_1\tilde{s}_3 = 0.$$  

(41)

In Figure 6, we plot the numerical solution of the evolution of invariants for various sets of initial conditions, where different colors indicate distinct initial flow morphologies. The upper panel illustrates the time evolution in the projected $s_3 - s_2$ plane, starting from the vicinity around the zero point, because $s_1$ roughly grows as $(D^{(v)})^{3/2}$ to the lowest order. The thin dashed boundaries are drawn at a certain $s_1$, which are marked with a solid circle on top of each trajectory. Thus even though some of the trajectories seem to visually pass through these boundary lines, it doesn’t mean that the morphologies have really changed, simply because the boundaries also evolve with time. In the lower part, we also plot the time evolution of each invariant using the same color scheme.
3.2. Initial Condition and Evolution of Probability Distribution

The fact that we obtain a set of ordinary differential equations (Equation (41)) in the ZA reflects the local assumption of the dynamical evolution, which means that the flow morphologies, as well as other properties of the mass elements, decouple from the nearby environment and are entirely determined by the initial conditions. It is straightforward but still physically relevant to study the evolution of the distribution of the invariants \( s_i \) away from the initial conditions in the ZA.

We first notice that the velocity gradient tensor is closely related to the tensor \( \xi_{ij} = \partial \Psi_i / \partial q_j (\tau) \),

\[
\tilde{A}_{ij} = \frac{A_{ij}(\tau)}{d \ln D / d \tau(\tau)} = \frac{\partial \Psi_i(q, \tau)}{\partial x_j} = \frac{\partial \Psi_i(q, \tau)}{\partial q_k} (J^{-1})_{kj},
\]

(42)

For convenience, we define another reduced quantity \( \tilde{A} \), where \( d \ln D / d \tau = D^{(0)} / D = \tilde{A} \), and \( J_{ij} \) is the Jacobian matrix from Lagrangian \( q \) to Eulerian space \( x \), where \( J_{ij} = \partial x_i / \partial q_j = \delta_{ij} + \xi_{ij} (\tau) \). We also assume that the Jacobian is invertible, which is true before shell crossing. Concentrating on the initial conditions, components of \( \partial \Psi_i (q, \tau) / \partial q_j \) are small compared to unity, so \( J_{ij} \sim I_{ij} \) and \( \tilde{A}_{ij} \sim \tilde{\xi}_{ij} \). In the following, we write all relevant quantities in this limit with superscript \((\tilde{\cdot})\), such as \( \tilde{A}^{(\tilde{\cdot})}_{ij} \) and \( \tilde{s}^{(\tilde{\cdot})}_i \). Writing the diagonal representation of \( \tilde{A}^{(\tilde{\cdot})}_{ij} \) as

\[
\tilde{A}_{ij}^{(\tilde{\cdot})} = \text{diag} (\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3),
\]

(43)

with the help of the distribution function of ordered eigenvalues \( \mathcal{P}(\tilde{\lambda}_i) \) for a Gaussian field (Doroshkevich 1970), one could derive the distribution function of invariants \( \tilde{s}_{i}^{(\tilde{\cdot})} \) (Bardeen et al. 1986; Pogosyan 2009, 2010)

\[
\mathcal{P}(\{\tilde{s}_{i}^{(\tilde{\cdot})}\}) = \frac{15^3}{8 \pi \sqrt{5} \sigma^6} e^{-\frac{1}{2} \frac{\sigma^2}{\sqrt{5} \sigma^6} \theta(\tilde{s}_{i}^{(\tilde{\cdot})})^2 - \tilde{\sigma}_{i}^{(\tilde{\cdot})}} \times \text{Real}(\{\tilde{s}_{i}^{(\tilde{\cdot})}\}).
\]

(44)

Here, the variance \( \sigma \) of the density fluctuation is

\[
\sigma^2 = \frac{1}{2 \pi^2} \int dk k^2 \mathcal{P}(k) W^2(k R),
\]

(45)

and \( W(k R) \) is a window function with smoothing length \( R \). The function \( \text{Real}(\{\tilde{s}_{i}^{(\tilde{\cdot})}\}) = \Theta(\tilde{s}_{1}^{(\tilde{\cdot})} - 3 \tilde{s}_{2}^{(\tilde{\cdot})}) \text{Rec}(\tilde{s}_{3}^{(\tilde{\cdot})}, \tilde{s}_{1}^{(\tilde{\cdot})}, \tilde{s}_{3}^{(\tilde{\cdot})}) \) defines the region in \( \tilde{s}_i \) space with real eigenvalue solutions, where \( \Theta(x) \) is Heaviside step function, and \( \text{Rec}(x, x_1, x_2) \) is a rectangular function that equals one if \( x_1 < x < x_2 \), and zero otherwise.

As tensor \( \tilde{\xi}_{ij} (\tau) \) grows large enough that the effect of the Eulerian gradient becomes non-negligible, and the matrix \( \tilde{A}_{ij} \) becomes non-Gaussian even in the ZA. Using the definition Equation (42), we again consider the diagonalized matrix

\[
\tilde{A}_{ij} = \text{diag}(\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3).
\]

(46)

In general, the eigenvector system defined by \( \tilde{A}_{ij} \) and \( \tilde{\xi}_{ij} \) will differ. However, since the addition and the inverse of a non-singular matrix does not change the eigenvectors, we have

\[
\tilde{\eta}_i = \frac{\lambda_i}{1 + \lambda_i}, \quad 1 + \lambda_i \neq 0.
\]

(47)

This allows us to estimate a Lagrangian-space (i.e., mass-weighted) PDF of the invariants of \( \tilde{A}_{ij} \)

\[
\mathcal{P}(\{\tilde{s}_{i}\}) = \mathcal{P}(\{\tilde{\lambda}_i\}) \left| \frac{\partial \tilde{\eta}_i}{\partial \tilde{\eta}_k} \frac{\partial \tilde{\eta}_k}{\partial \tilde{s}_j} \right| = \frac{15^3}{8 \pi \sqrt{5} \sigma^6} \text{Real}(\{\tilde{s}_{i}\}) \times \exp \left[ -\frac{3(\tilde{s}_{1} + 2 \tilde{s}_{2} + 3 \tilde{s}_{3})^2}{\sigma^2 T^2} + \frac{15(3 \tilde{s}_{3} + \tilde{s}_{2})}{2 \sigma^2 T} \right].
\]

(48)
where we have defined the notation

\[ T = 1 + \sum \tilde{s}_i = 1 + \tilde{s}_1 + \tilde{s}_2 + \tilde{s}_3. \]  

(49)

Real(\(\tilde{s}_i\)) is the same function defined previously.

In Figure 7, we show the two-dimensional probability distribution of \(\tilde{s}_3 - \tilde{s}_2\) at two different \(\tilde{s}_1\)s. For the dispersion, we use \(\sigma = 0.1\) to represent an initial epoch in the first row, and \(\sigma = 1\) for a later time in the second. Similar to previous figures, the dashed lines illustrate the boundaries between classifications. Since initial conditions with the same \(\tilde{s}_1(\tau_{\text{init}})\) but different \(\tilde{s}_2(\tau_{\text{init}})\) and \(\tilde{s}_3(\tau_{\text{init}})\) do not necessarily evolve to the same \(\tilde{s}_1(\tau_{\text{final}})\), a comparison of the two-dimensional probability distribution at given \(\tilde{s}_1\) between two epochs is by no means rigorous as they do not consist of the same set of samples. Therefore, the values of \(\tilde{s}_1\) for plotting the figure are chosen more or less arbitrarily. Nevertheless, the asymmetry still increases with \(\sigma\). This remains valid and is even more evident for \(\tilde{s}_1 = 0\), because initially \(\tilde{s}_{2,3}\) are symmetrically distributed. However, the skewed distribution in the lower panels does not necessarily mean that the probability of a Lagrangian fluid element belonging to a particular morphology changes with time significantly in the ZA, because \(\tilde{s}_1\) and the morphology boundaries evolve together, and the sample points contributing to the last panel at \(\tilde{s}_1 = 0\) come from various initial \(\tilde{s}_1(\tau_{\text{init}})\).

3.3. Nonlinear Evolution beyond Zel’dovich Approximation

Even without the shell crossing, the gravitational nonlinearity of the system (Equations (28)–(30)) makes the analytical study of the evolution of velocity gradient invariants very complicated. One possibility is to investigate the Lagrangian evolution of all the relevant quantities, at least numerically, similar to Equation (41) in ZA, but including density \(\delta\), velocity gradient \(A_{ij}\), and tidal tensor \(\epsilon_{ij} = \varpi_{ij} - \varpi \delta_{ij}/3\), the last of which does not exist in Newtonian cosmology due to the nonlocality of the theory. In the following, we only concentrate on the numerical measurement from the simulation.

In Figure 8, we show the two-dimensional probability distribution of the invariants from the \(N\)-body simulations Sing100. Instead of measuring the invariants themselves, we normalize the velocity gradient tensor

\[ \hat{A}_{ij} = \frac{A_{ij}}{\sqrt{A_{nn} A_{nn}}} \]  

(50)

first, given that any rescaling of \(A_{ij}\) with a positive constant would not change the boundaries between kinematic classes. After this normalization, the invariants were confined within the cubic region \(\tilde{s}_1 \in [-\sqrt{3}, \sqrt{3}], \tilde{s}_2 \in [-1/2, 1]\) and \(\tilde{s}_3 \in [-\sqrt{3}/9, \sqrt{3}/9]\). From top to bottom, left to right, we plot the distribution at \(\tilde{s}_1 = (-1.6, -1.2, -0.8, -0.4, 0, 0.4, 0.8, 1.2, 1.6)\). Before further...
Two-dimensional probability of $\hat{s}_3$ and $\hat{s}_2$ for nine different bins of $\hat{s}_1$, as measured from simulations at $z = 0$. Note that the hat quantities $\hat{s}_i$ denote the rescaling of the tensor $A_{ij}$ Equation (50) so that invariants are confined within given rectangular region. From left to right, $\hat{s}_1$ goes from $-1.6$, $-1.2$, $-0.8$ in the first row, to $-0.4$, $0$, $0.4$ in the middle, and then $0.8$, $1.2$, $1.6$ in the bottom row. At larger $|\hat{s}_1|$, mass elements are more likely to spread into vortical regions, although part of the spreading is due to our finite bin size of $\hat{s}_1$.

(A color version of this figure is available in the online journal.)

proceeding, one should realize that a direct comparison between Figures 7 and 8 could be misleading because here we are measuring the volume-weighted probability distribution in Eulerian space, whereas Figure 7 is mass-weighted. A consistent comparison of various theoretical models, as well as simulations in both Eulerian and Lagrangian space, are desired but will not be the main topic of this paper.

Note that the whole data set is divided into nine thick $\hat{s}_1$ bins, while the dashed boundaries in each panel correspond to the median value of the $\hat{s}_1$ in each bin. In panels with a medium $\hat{s}_1$, where the shape of the distribution is sharp and clear, one would expect that most of the offset is due to this finite bin size effect. This occurs from the third to the sixth panel in the figure at least, because the sample points at these $\hat{s}_1$ bins are much more than the others. The fraction of data in all nine panels are $\{0.003, 0.009, 0.38, 0.31, 0.15, 0.06, 0.041, 0.029, 0.018\}$, respectively. For the first two panels at $\hat{s}_1 = -1.6$ and $-1.2$, however, theoretical arguments may suggest the same because they correspond to the outflowing void region where gravitational nonlinearity is less significant. In the bottom panels with a large positive $\hat{s}_1$, in which regions the fluid elements collapse fast enough, the PDFs are much flatter and there is much spreading, especially in the last two panels. This could be attributed to the nonlinear evolution and the effect of shell crossing.

4. SHELL CROSSING AND EMERGENCE OF VORTICITY

In the multi-streaming regime, vorticity is often generated. Because velocity and density perturbations are strongly coupled, it is reasonable to expect that the rotational degrees of freedom correlate with large-scale structures, as in the potential flow. Indeed, from the $N$-body simulation, we observe this association not only via the distinct spatial distribution of various vortical flows, but also from the alignment between the vorticity and the cosmic web. The theory, as will be seen later in this section, is complicated and highly non-trivial quantitatively. One obstacle is to incorporate the shell crossing in a closed, dynamical fluid system. To do so, we start from the fundamental Vlasov–Poisson system, and then derive evolution equations of invariants, including the phase space information of multi-streaming. However, instead of solving the dynamical system, we instead try to propose a statistical description that relates to the internal structure of the invariant space.
Figure 9. PDF of the cosine of the angle between the velocity and vorticity as measured in all 64 realizations of the MIP simulation. The alignment is strongest for the categories SFS and UFC, which trace filamentary structures. (A color version of this figure is available in the online journal.)

4.1. The Spatial and Orientation Distribution of Vorticity

The distinct spatial distributions revealed in both potential and rotational flows suggest that the emergence of vorticity could in principle relate to the cosmic web. Unlike in irrotational flow, the spatial distribution of all the categories shown in Figure 3 are visually correlated with filaments, walls, and knots, where shell crossing takes place. This is especially clear for the categories SFS and UFC with smaller \( n_c \) (i.e., the last row in the figure), when almost all visible filaments have been painted by those labels. Together with flow trajectories illustrated in Figure 1, the spatial distribution of some categories also seem to be consistent with physical intuition. For example, within knots the spatial contraction and the energy transfer from direct infall to orbiting leads to the inward winding and compression along the radial direction (i.e., the SFC category). The type of vorticity generated along the filaments funneling matter to knots should involve inward spiraling and stretching. However, this agreement is not as large as expected. On the one hand, this reveals the complexity of the problem; on the other hand, it might suggest a statistical/stochastic view of the vorticity generation.

This interpretation can be further verified by examining the alignment between the direction of the vorticity, the velocity, and the cosmic web structure. In Figure 9, we present the distribution of the angle between vorticity and velocity for various categories. Interestingly, two of these mainly trace the filamentary structures, showing the most significant correlation signal between the direction of the two vectors, but in an opposite way. In SFS, the velocity tends to be aligned with vorticity, whereas the velocity in type UFC tends to be perpendicular. In general, although not as obviously as in previous examples, the mass elements seem to flow along the vorticity direction if it is stretching, and perpendicularly when compressing along the direction of vorticity.

Figure 10 shows the alignment of the vorticity with cosmic-web structure. We display the angle distribution between the vorticity and the eigenvectors of the density Hessian matrix \( \nabla \nabla \rho \), for different vortical categories. Here \( e_i \) denotes the \( i \)th eigenvector, with eigenvalue \( \lambda_i \), where \( \lambda_i \) are sorted according to the absolute value, \( |\lambda_1| \leq |\lambda_2| \leq |\lambda_3| \). Therefore, \( e_1 \) corresponds to the direction with the slowest changing rate of density distribution, and \( e_3 \) corresponds to the fastest. In filaments, \( e_1 \) is aligned with the filament’s axis. We smoothed the density field using a Gaussian filter at smoothing length \( R = 0.5 \) Mpc \( h^{-1} \) and 1 Mpc \( h^{-1} \). Similar to the situation in the velocity-vorticity alignment, the correlation is strongest for categories SFS and UFC, which trace the filaments. Particularly, type SFS is more aligned with the direction of the smallest eigenvalues \( |\lambda_1| \). On the other hand, the distinctive alignment of UFC is also visible, where the vorticity is perpendicular to \( |\lambda_1| \), and parallel to \( |\lambda_2| \) or \( |\lambda_3| \). We also notice the smoothing variations that were encountered by Codis et al. (2012) and Aragon-Calvo & Yang (2014).

Figures 9 and 10 reinforce our previous physical interpretations. For instance, they show that the vorticity direction of the SFS type aligns with velocity and the filament directions. Given the schematic flow trajectory from Figure 1, this is consistent with the physical picture of the accretion along the filament, with an additional rotation around the main path (Pichon et al. 2011; Codis et al. 2012; Laigle et al. 2013), which is expected to be generated during the formation of the filamentary structure (Zel’dovich 1970; Bond et al. 1996; Pichon et al. 2011). For the category UFC the major component of the
The Astrophysical Journal, 793:58 (18pp), 2014 September 20

Wang et al.

velocity is still along the filament because the vorticity direction is perpendicular to both filament and velocity. This suggests that vorticity of this type is related to the halo spin formed by major mergers along the filaments (Codis et al. 2012). Therefore as demonstrated previously, shell crossing and the generation of vorticity are physically rich processes. In the invariant space, this means that certain types of trajectories exist that are transitioning from potential flow to a particular rotational flow region.

4.2. Emergence of Vorticity: Dynamical View

Before discussing our statistical view of the vorticity generation, we first examine the dynamical system. To proceed, one must restore the extra source contribution of the velocity dispersion in the Euler equation (27), because the term  in Equation (30) is symmetric, and therefore the initial irrotational flow will not be able to generate vorticity. The standard approach is to start from the more fundamental Vlasov equation of the one-particle, phase-space density (Peebles 1980; Bernardeau et al. 2002),

$$\frac{df}{dt} + \frac{p}{am} \cdot \nabla f - am \Phi \frac{df}{dp} = 0, \quad (51)$$

and then takes moments of the velocity, given the fluid quantities defined as

$$\rho = \int d^3p f(x, p, \tau), \quad u_i = \int d^3p f(x, p, \tau) \frac{p_i}{am},$$

$$\zeta_{ij} = \int d^3p f(x, p, \tau) \left( \frac{p_i}{am} - u_i \right) \left( \frac{p_j}{am} - u_j \right), \quad (52)$$

Here $x$ and $p = amu$ are the comoving positions and momenta of the particles, with $m$ the mass of particle and $u$ the peculiar velocity. The velocity dispersion, $\zeta_{ij}$, was ignored in the previous dust model. Eventually the Euler equation is recovered with an extra term,

$$\frac{d\mathbf{u}}{dt} + \mathcal{H}(\tau)u(x, \tau) + u(x, \tau) \cdot \nabla u(x, \tau) = -\nabla \Phi(x, \tau) - \frac{1}{\rho} \nabla_j (\rho \zeta_{ij}). \quad (53)$$

The gradient tensor $A_{ij}$ reads as

$$\frac{dA_{ij}}{dt} + \mathcal{H}(\tau) A_{ij} + A_{ik} A_{kj} = \sigma_{ij} + \zeta_{ij}, \quad (54)$$

and we define the dispersion tensor $\zeta_{ij}$ as

$$\zeta_{ij} = -\nabla_j \left[ \frac{1}{\rho} \nabla_k (\rho \zeta_{ik}) \right]. \quad (55)$$

Replacing $\sigma_{ij}$ and its derivatives with $\sigma_{ij} + \zeta_{ij}$ in Equation (31), one obtains the dynamical equations of invariants with extra source terms,

$$\frac{d}{d\tau} s_1 + \frac{3}{2} \mathcal{H}(\tau) s_1 - s_1^2 + 2s_2 = -\sigma - \zeta,$$

$$\frac{d}{d\tau} s_2 + 2\mathcal{H}(\tau) s_2 - s_1 s_2 + 3s_3 = -s_1(\sigma + \zeta) - \sigma_A - \zeta_A,$$

$$\frac{d}{d\tau} s_3 + 3\mathcal{H}(\tau) s_3 - s_1 s_3 = -s_2(\sigma + \zeta) - s_1(\sigma_A + \zeta_A) - \sigma_{A^2} - \zeta_{A^2}. \quad (56)$$

However, the system has to be closed (i.e., the infinite hierarchy series needs to be truncated; Pueblas & Scoccimarro 2009). A handful of suggestions have been proposed (Pueblas & Scoccimarro 2009; Domínguez 2000; Buchert & Domínguez 2005). In general, one desires to know the full phase space information from Equation (51).

4.3. Emergence of Vorticity: Statistical View

Instead of explicitly solving the dynamical system discussed previously, it is also possible to establish a statistical description for the generation of vorticity in the invariant space. In the multi-streaming region, one defines the bulk velocity, which projected from the phase space as the density-weighted average among all streams

$$u(x) = \frac{1}{\rho(x)} \sum_s \rho(q_s) u(q_s), \quad (57)$$

where $\rho(q_s)$ and $u(q_s)$ are density and velocity of each stream, and $\rho(x) = \sum_s \rho(q_s)$ is the Eulerian density. By taking the gradient of Equation (57), two separate contributions appear to the averaged gradient tensor $A_{ij}$: the density average of $A_{ij}$ among all streams

$$\frac{1}{\rho(x)} \left[ \sum_s u_s(q_s) \frac{\partial \rho(q_s)}{\partial x_j} - u_s(x) \frac{\partial \rho(x)}{\partial x_j} \right]. \quad (58)$$

and the coupling between the density gradient and velocity, which arises due to the projection of a multi-valued field from the phase space (Pichon & Bernardeau 1999; Hahn et al. 2014).

$$\frac{1}{\rho(x)} \left[ \sum_s u_s(q_s) \frac{\partial \rho(q_s)}{\partial x_j} - u_s(x) \frac{\partial \rho(x)}{\partial x_j} \right]. \quad (59)$$

In the cold dark matter scenario, matter is assumed to reside on a three-dimensional thin sheet in the six-dimensional phase space. The generalized Kelvin’s circulation theorem ensures that the average vorticity remains zero within any circle in the phase space (Lynden-Bell 1967), and is also zero for all closed loops in the three-dimensional position space that are non-intersecting when projected from six dimensions. However, in the multi-streaming region after the shell crossing, a simple circle in the phase space could be projected as interacting “8”-like loops, with nonzero vorticity for each in the three-dimensional position space. However, depending on the scale of interest, at any epoch, one is free to choose loops that have not been deformed large enough to generate any net effect except near the vicinity of singular points, which might become non-negligible in the deeply nonlinear regime, for example in virialized halos.

For simplicity, we assume that the velocity of each stream is still potential after shell crossing, so the first contribution (Equation (58)) of $A_{ij}$ is symmetric. Then, the rotation arises from the coupling between the density gradient and the velocity of the streams (Equation (59)). Since various morphological structures are distinct in the density gradient, as well as their coupling with the velocity field, this contribution depends on the environment. This could be responsible for the different spatial and orientation distribution of various vortical categories.

4.3.1. Shell Crossing in the Invariants Space

Given the expression of $A_{ij}$ in Equations (58) and (59), it is possible to set up a statistical description of the vorticity
emergence in the invariant space. At the arbitrary Eulerian position \( x \), we assume there are many streams, labeled from 1 to \( \alpha \), flowing potentially, and we denote the invariants as

\[
1-a \frac{\partial}{\partial x} s_i^{(i)} = \{ s_i^{(1)} , s_i^{(2)} , \ldots , s_i^{(\alpha)} \}, \quad i = (1, 2, 3), \tag{60}
\]

where \( s_i^{(i)} \) are potential invariants for \( m \)th stream, and have the probability distribution

\[
P(1-a s_i^{(i)}). \tag{61}
\]

As seen in previous sections, this distribution characterizes the morphological information of the cosmic web. After a short, yet finite amount of time \( \Delta t \), the winding of the dark matter sheet in the phase space becomes large enough to form a multi-streaming region. These \( \alpha \) streams encounter and couple with each other to generate final invariants \( s_i \) through Equations (58) and (59), with distribution functions \( P(s_i) \). This shell crossing process can then be described by the conditional probability distribution of final \( s_i \) given initial invariants \( s_i^{(1)} , \ldots , s_i^{(\alpha)} \)

\[
P(s_i^{(1)} \ldots a s_i^{(\alpha)}). \tag{62}
\]

From Equation (59), these distributions also encode information about cosmic-web morphologies, because velocity and density gradients couple differently depending on the environment. Here we explicitly write down the time lapse \( \Delta t \) to simplify the description, as in reality, multi-value regions emerge gradually, from three-flow regions to more complicated cases. In this situation, shell crossings occur more than \( m \) times during \( \Delta t \), what happens is essentially a random walk in invariant space, where the final distribution \( P_{\Delta t}(s_i^{(1)} \ldots a s_i^{(\alpha)}) \) is a series multiplication of separate probability functions

\[
P_{\Delta t}(s_i^{(1)} \ldots a s_i^{(\alpha)}) = \prod_{m} P_{s_i^{(m)}(1-a s_i^{(m)})}. \tag{63}
\]

Here we assume that only a subset \( \tau_{m} (1 \ldots \alpha) \) was involved in the process at each step.

4.3.2. Boundary Crossing and the Internal Structure of Invariants Space

To further explore the validity of this description, we explore the probability distribution \( P(s_i^{(1)} \ldots a s_i^{(\alpha)}) \) numerically, and show the result in Table 1. At an arbitrary location where shell crossing occurs, we assume that a random number of streams overlap. From Equations (58) and (59), both the symmetric and anti-symmetric contributions are environmentally dependent, so we generate samples of \( A_{ij}^{(i)} \) for various kinematical categories from the ZA, and identify these irrerotational kinematical types as morphology structures. That is, we consider \( SN/SN/SN \) as halo, \( SN/S/S/S \) as filament, and \( UN/UN/UN \) as wall structures. Then, based on their morphological classes, we generate the velocity and density gradient for all flow elements, which roughly resemble the physical process near each morphological structure. For example, the velocity vectors and the density gradients are both sampled spherically for halos. The velocities are assumed to flow close to a line for the filaments, and the density gradients around the radial direction of a cylinder. Near the wall, both velocities and density gradients are chosen close to a given direction. With \( u_i, \nabla, \rho, \) and \( A_{ij}^{(i)} \), for each stream, we are then able to construct the full velocity gradient tensor \( A_{ij} \) via Equations (58) and (59).

| Table 1 | Kinematic Morphologies Generated by the Toy Model of Stream Crossing |
|---------|---------------------------------------------------------------|
| \( SN/\bar{SN}/SN \) (Halo) | \( SN/S/S \) (Filament) | \( UN/UN/UN \) (Wall) | \( UN/\bar{UN}/\bar{UN} \) (Void) | Total (after) |
| \( SN/\bar{SN}/SN \) | 0.3(10.1) | 0.5(1.6) | 0.3(0.5) | \ldots | 1.1 |
| \( SN/S/S \) | 1.1(34.3) | 10.5(37.0) | 11.3(22.5) | \ldots | 22.9 |
| \( UN/\bar{UN}/\bar{UN} \) | 0.2(6.0) | 4.3(15.1) | 25.6(51.3) | \ldots | 30.3 |
| \( SN/S/S \) | 0.0(0.6) | 0.1(0.2) | 3.6(7.1) | 18.2(100) | 21.8 |
| \( SFS \) | 0.4(13.2) | 5.3(18.5) | 2.6(5.2) | \ldots | 8.3 |
| \( UF \) | 0.0(0.4) | 0.6(2.1) | 2.5(5.0) | \ldots | 3.1 |
| \( SF \) | 0.8(26.5) | 2.3(8.0) | 0.6(1.1) | \ldots | 3.7 |
| \( UF \) | 0.3(9.5) | 4.9(17.4) | 3.6(7.1) | \ldots | 8.8 |

Notes. Each column shows the percentage of flow morphologies that are generated from a particular type displayed at the top, the numbers in parentheses give the normalized fraction within every column (i.e., the probability of generating other categories from that morphology). The last row (column) is the marginal sum of each column (row). Therefore, the last row gives the total fraction of each morphology before the crossing, and the last column gives total percentage after the crossing. Compared with the spatial distributions in Figure 3, the pre-crossing fractions (i.e., the last row) are volume-weighted in the Eulerian space at redshift \( z = 2 \). All the numbers are displayed in percentage (%), and have been rounded for display purpose. The dashes in the void column mean our model does not assume any stream crossing in the void region, and therefore are set to zero.

We also list the fractions of all kinematical types generated by our numerical shell crossing model in Table 1. The total number of samples we generated is roughly on the order of 256\(^3\). Each column corresponds to the particular morphological structure that we started from, which from left to right are halo \( SN/\bar{SN}/SN \), filament \( SN/S/S \), wall \( UN/\bar{UN}/\bar{UN} \), and void \( UN/\bar{UN}/\bar{UN} \), respectively. In our probably over-simplified toy model, stream crossing happens for streams with the same kinematical class. After that, however, all categories can be generated, including both potential and vortical classes, as shown in the different rows. In each column of the table, we list the percentage of flow morphologies generated from the particular type displayed at the top, the numbers in parentheses give the normalized probability within every column. The last row gives the total volume-weighted fraction of each morphology before the crossing, and the last column gives the total percentage after the crossing.

From the table, physically implausible situations, like voids generated in highly multi-streaming filaments and halos, are produced only in a very tiny amount. Although a substantial fraction of potential flows are produced, we are more interested in the vortical ones. From the table, there is a higher probability that vorticity is generated with class \( SF \) around halos (26.5%) compared with 13.2% of \( SFS \) and 9.5% of \( UF \); type \( SF \) (18.5%) and \( UF \) (17.4%) near filaments; and a similar fraction of almost all vortical types around walls. Given the simplicity of our model, the exact value in Table 1 should not be taken too seriously. However, from the fifth row of the table, among 8.3% of vortical category \( SFS \) are generated from stream crossing, and more than 60% are formed near filaments \( SN/S/S \). This is also true for category \( UF \) (4.9% compared with 8.8% in total). From Figure 3, this is what is shown from the filamentary spatial distribution of these two categories. For type \( SF \), however, due to the small fraction of type \( SN/\bar{SN}/SN \) initially, the dominant contribution comes from filaments. This might also be physically plausible because the spatial distribution of this type tends to extend a little from the central halos to the filamentary
structures. Finally, both our model and Figure 3 suggest that type \( UFC \) is more likely to be generated in walls (2.5% compared with 3.1% in total).

Even without any more detailed information about the conditional distribution of the shell crossing \( P(S_i \mid a_i^{(s)}) \), it is not hard to understand the result. Figure 1 shows that the region of irrotational flow that follows the filament structures (‘\( S N/S/S \)’) is adjacent to the \( SFS \) vortical region in the invariant space, which is also associated with filaments. Similarly, the triangle-like region of the halos \( SN/SN/SN \) resides just inside the \( SFC \) regime. This is also consistent with the formation of cosmic web structures. Consider the mass elements flowing around the filaments potentially. In the canonical form of the tensor \( A_{ij} \), there are three real eigenvalues \( \lambda_i \). At some point shell crossing occurs, and from Zel’dovich’s structure-formation theory, the filaments are formed from the second collapse, after walls. The trajectory position in space will produce a spiral in the plane perpendicular to the filament. In the idealized case, where \( \lambda_1 \sim \lambda_2 < 0 \) for any of two eigenvalues initially, and only a small amount \( \epsilon \) has been changed to the real part of \( \lambda_1 \), one obtains the canonical form

\[
A = \begin{pmatrix}
\lambda_1 + \epsilon & -b \\
b & \lambda_1 + \epsilon \\
0 & 0
\end{pmatrix}.
\]

From Equation (24), in this case, \( s_2 \) will be enlarged and \( s_3 \) will be smaller, giving a shift in the upper left direction in the invariant space. So, just around filaments, category \( SFS \) will be generated, and the spatial association with the large-scale structure is preserved after shell crossing. Furthermore, this physical picture also suggests that the vorticity direction is aligned with filaments, which is consistent with our measurement in the simulation for this category. The situation around halos is even simpler, as from Figure 1, category \( S N/SN/SN \) is almost entirely surrounded by the \( SFC \) type in the invariant space. This, therefore, gives a much higher probability to generate this type of rotational flow around the halos.

5. CONCLUSION AND DISCUSSIONS

In this paper, we concentrated on the velocity gradient tensor \( A_{ij} \) and its rotational invariants defined as the coefficients of the characteristic equation of \( A_{ij} \). They enable the classification of all possible flow patterns, including both potential and rotational flows. For the irrotational flow, our invariant-based categorization is equivalent to the cosmic web finder V-web (Hoffman et al. 2012), with the threshold eigenvalue \( A_{ab} = 0 \). The PDFs of the density and fractional anisotropy of various flows are consistent with their morphology types, although the visual impression of the web structure may not be as accurate as in other techniques (Pogosyan et al. 2009; Aragon-Calvo et al. 2010; Sousbie 2011; Sousbie et al. 2011; Bond et al. 2010; Forero-Romero et al. 2009; Hahn et al. 2007b; Sousbie et al. 2009; Stoica et al. 2005; Hoffman et al. 2012). To improve it, one could introduce a non-zero, fine-tuned threshold \( A_{ab} \) as done in the V-web finder. In our method, this can be achieved by a simple invariant transformation (Equation (18)). However, from the fractional anisotropy distribution, the physical interpretation might become difficult. Because all invariants are one-to-one mapped to eigenvalues, they could also be applied to other dynamical classification methods (e.g., invariants defined for deformation tensor or the Hessian matrix of gravitational potential; Hahn et al. 2007b; Forero-Romero et al. 2009).

As a region develops vorticity, the invariants change continuously, which is a clear benefit of working with them. By combining an ensemble of N-body simulations with the same large-scale modes, we show that various rotational flows also trace the cosmic web structure differently. This reveals the dynamical nature of these variables and provides an alternative view on the emergence of vorticity. Therefore, an understanding of their dynamical evolution would be valuable. We first stepped back and concentrated on the irrotational flow—which is important on its own as an indicator of the cosmic web—and started from the simplest ZA, where the accurate solution of time evolution and the analytical formula of the PDF are both available. We also wrote down a set of dynamical equations for the invariants from the Euler equation. However, further investigation is difficult because of the gravitational nonlinearity. One possibility is to study the Lagrangian evolution of all relevant quantities, including the density \( \delta \), velocity gradient \( A_{ij} \), and tidal tensor \( \epsilon_{ij} = \sigma_{ij} - \sigma \delta_{ij}/3 \). Due to the non-locality of Newtonian cosmology, however, the evolution equation of \( \epsilon_{ij} \) is missing and will only be feasible under certain approximations. We follow this approach in a separate paper (X. Wang et al., in preparation).

Given the complications of the dynamical modeling of the nonlinear gravitational evolution, as well as the shell crossing, the key insight came after realizing the stochastic nature of the multi-streaming. The distinctive spatial distribution of different rotational flow morphologies is very likely to be a consequence of some diffusion process in the invariant space. Indeed, the internal structure of the invariant space (Figure 1) shows the adjacency between the potential morphology \( S N/SN/SN \) and rotational type \( SFC \), which both spatially follow halos, and the contiguity between type \( S N/S/S \) and \( SFS \), which appear around filaments. Without a sophisticated model, we numerically investigated the PDFs of invariants generated from a toy model of stream crossing. The result, Table 1, qualitatively supports our conjecture.

However, many details of this stochastic process are still missing. For example, it is unclear whether a simple diffusion would be able to explain the physics associated with halo formation. Unlike the vorticity, the angular momenta of halos can be formed from laminar flow by the misalignment between the shear and inertia of a given region that encloses the material of a protogalaxy (i.e., from the tidal torque theory; Hoyle 1949; Peebles 1969; Doroshkevich 1970; White 1984; Wesson 1985). Compared with the fast growth ~ \( D^5/2 \) (Pueblas & Scoccimarro 2009) of vorticity after its emergence, the halo spin grows slower because of the very early stages of structure formation. However, as suggested by Libeskind et al. (2013a), the strong alignment between the direction of vorticity and the halo angular momentum suggests that there is another phase of the coevolution between them after the tidal torque becomes ineffective, and after the turn-around when the lever arm is dramatically reduced.

This coevolution phase seems to be reasonable as shown by our findings of two different alignments between the vorticity and filaments (category \( SFS \) and \( UFC \), respectively). Recent studies also suggest two states of preferential orientation between halo spins and filaments (Hahn et al. 2007b; Sousbie et al. 2009; Zhang et al. 2009; Codis et al. 2012; Aragon-Calvo & Yang 2014). Moreover, it has been confirmed that this orientation is mass-dependent (Codis et al. 2012; i.e., low-mass halos tend to be aligned with filaments, and high-mass halos have spin perpendicular to the filaments; Aragon-Calvo et al. 2007; Hahn et al. 2007a; Paz et al. 2008; Codis et al. 2012).
Codis et al. (2012) proposed an interpretation that more massive halos acquire their spins to be perpendicular to the filaments via a merger process along the filament. If this is the case, the trajectories of a mass element in the invariant space might be much more complicated. Further investigations of how such a process would manifest itself in the invariant space would be interesting.

APPENDIX A

LIST OF CLASSIFICATION

Most of the results in this appendix are originally from Chong et al. (1990). We quote the results here for the purpose of completeness. Considering the solution to the characteristic equation

$$\det(A - \lambda I) = \lambda^3 + s_1 \lambda^2 + s_2 \lambda + s_3 = 0, \quad (A1)$$

one can look at the coefficient space \( \{s_1, s_2, s_3\} \). The surface \( S_1 \) that divides the real and complex solutions is given by

$$27s_3^2 + 4s_2^3 - 18s_1 s_2 s_3 + (4s_2^3 - s_1 s_3^2) = 0 \quad (A2)$$

Denoting the solution of the above equation as \( s_{3a} \) and \( s_{3b} \), assuming \( s_{3a} \leq s_{3b} \), the regions with real eigenvalues at fixed \( s_1 \) correspond to

$$s_2 < s_1^2/3, \quad s_3 < s_3 \leq s_{3b}. \quad (A3)$$

When either equal sign of the second inequality holds, two of eigenvalues will be equal. If the first equality also holds, one gets three real and equal eigenvalues.

For the purpose of cosmology, we are particularly interested in the following potential flow categories:

| Stable Nodes (SN/SN/SN) | \( s_1 > 0 \), \( 0 < s_2 < s_1^2/3 \), \( \max(s_{3a}, 0) < s_3 < s_{3b} \) |
|-------------------------|--------------------------------------------------|
| Stable Node/Saddle (SN/S/S) | \( s_1 \geq 0 \), \( s_2 < s_1^2/4 \), \( s_{3a} < s_3 < s_{3b} \) |
| or \( s_1 < 0 \), \( s_2 < 0 \), \( s_{3a} < s_3 < 0 \) |
| Unstable Node/Saddle (UN/S/S) | \( s_1 \leq 0 \), \( s_2 < s_1^2/4 \), \( 0 < s_3 < s_{3b} \) |
| or \( s_1 > 0 \), \( s_2 < 0 \), \( 0 < s_3 < s_{3b} \) |
| Unstable Nodes (UN/UN/UN) | \( s_1 < 0 \), \( 0 < s_2 < s_1^2/3 \), \( s_{3a} < s_3 < \min(s_{3b}, 0) \), |

where we list in the order of halo, filament, wall, and void. For vortical flow, we have

| Stable Focal Stretching (SFS) | \( s_1 \geq 0 \), \( s_2 > s_1^2/4 \), \( s_3 < 0 \), |
| or \( s_1 > 0 \), \( s_2 < s_1^2/4 \), \( s_3 < s_{3a} \), |
| or \( s_1 < 0 \), \( s_2 > 0 \), \( s_3 < s_{1s2} \), |
| or \( s_1 < 0 \), \( s_2 < 0 \), \( s_3 < s_{sa} \), |
| Stable Focal Compressing (SFC) | \( s_1 > 0 \), \( s_2 > s_1^2/3 \), \( 0 < s_3 < s_1s_2 \), |
| or \( s_1 > 0 \), \( s_1^2/4 < s_2 < s_1^2/3 \), \( 0 < s_3 < s_{1sa} \), |
| or \( s_1 > 0 \), \( 0 < s_2 \leq s_1^2/3 \), \( s_{3b} < s_3 < s_1s_2 \) |
| Unstable Focal Stretching (UFS) | \( s_1 < 0 \), \( s_2 > s_1^2/3 \), \( s_1s_2 < s_3 < 0 \), |
| or \( s_1 < 0 \), \( s_1^2/4 < s_2 < s_1^2/3 \), \( s_{3b} < s_3 < 0 \), |
| or \( s_1 < 0 \), \( 0 < s_2 \leq s_1^2/3 \), \( s_1s_2 < s_3 < s_{3a} \) |
| Unstable Focal Compressing (UFC) | \( s_1 \leq 0 \), \( s_2 > s_1^2/4 \), \( s_3 > 0 \), |
| or \( s_1 \leq 0 \), \( s_2 < s_1^2/4 \), \( s_3 > s_{3b} \), |
| or \( s_1 > 0 \), \( s_2 > 0 \), \( s_3 > s_{1s2} \), |
| or \( s_1 > 0 \), \( s_2 < 0 \), \( s_3 > s_{sa} \), |

(A5)

For a complete classification, please see the Appendix in Chong et al. (1990).
APPENDIX B

VORTICITY AND VORTICAL FLOW

Furthermore, we discuss the distinction between vorticity and vortical flow, as defined via local trajectories in our paper. As a simple example, consider the velocity field

$$\mathbf{u}(\mathbf{x}) = x_1 u_0,$$  \hspace{1cm} (B1)

where $u_0$ is a constant vector and $x_1 = \mathbf{x} \cdot \mathbf{e}_1$ along direction $\mathbf{e}_1$. The flow is entirely aligned with $u_0$, but the amplitude varies. One can show that the gradient of $\mathbf{u}$ is anti-symmetric, (i.e., the vorticity is nonzero). However, the invariants of this expression are consistent with non-vortical flow, $s_2 = s_3 = 0$. Or more precisely, it belongs to the degenerate kinematical categories, which are not a main topic in this paper. A more physical example is near a caustic after the shell crossing. Among all streams at a given position, there are two categories, ordinary flow $c_1$ and singular flow $c_2$, where

$$J\left[\mathbf{x}(\mathbf{q}_2)\right] = \left|\frac{\partial \mathbf{x}(\mathbf{q}_2)}{\partial \mathbf{q}_2}\right| = 0,$$  \hspace{1cm} (B2)

Consider an Eulerian position $\mathbf{x}$ near the caustic edge $\mathbf{x}_0$, with a corresponding Lagrangian position $\mathbf{q}$ and $\mathbf{q}_0$. Following Pichon & Bernardeau (1999), there exists a direction orthogonal to the caustic edge, denoted as $\perp$ direction, where the Eulerian coordinate

$$(\mathbf{x} - \mathbf{x}_0)_{\perp} \approx -\eta (\mathbf{q} - \mathbf{q}_0)^2.$$  \hspace{1cm} (B3)

The Jacobian for the two singular flows is $J(\mathbf{x}) \approx 2\sqrt{\eta(\mathbf{x} - \mathbf{x}_0)_{\perp}}$. Then the Eulerian velocity $\mathbf{u}(\mathbf{x})$ could be expressed as

$$\mathbf{u}(\mathbf{x}) \approx \frac{2\mathbf{u}(\mathbf{q}_0)}{J(\mathbf{x})} \left(\frac{\rho(\mathbf{q}_0)}{\rho(\mathbf{q})}\right),$$  \hspace{1cm} (B4)

where $\mathbf{q}_0$ is ordinary flow. As shown in Pichon & Bernardeau (1999), the vorticity of Equation (B4) is nonzero. However, the invariants of this expression are consistent with non-vortical flow; $s_2 = s_3 = 0$.

REFERENCES

Aragon-Calvo, M. A. 2012, arXiv:1210.7871
Aragon-Calvo, M. A., Platen, E., Pranav, R., van de Weygaert, R., & Szalay, A. S. 2010, ApJ, 723, 364
Aragon-Calvo, M. A., van de Weygaert, R., Jones, B. J. T., & van der Hulst, J. M. 2007, ApJ, 655, L5
Aragon-Calvo, M. A., & Yang, L. F. 2007, ApJ, 655, L5
Bardeen, J. M., Bond, J. R., Kaiser, N., & Szalay, A. S. 1986, ApJ, 304, 15
Bernardeau, F., Colombi, S., Gaztañaga, E., & Scoccimarro, R. 2002, Phys. Rev., 367, 1
Bernardeau, F., & van de Weygaert, R. 1996, MNRAS, 279, 693
Bond, J. R., Kofman, L., & Pogosyan, D. 1996, Natur, 380, 603
Bond, N. A., Strauss, M. A., & Cen, R. 2010, MNRAS, 409, 156
Buchert, T., & Domínguez, A. 2005, A&A, 438, 443
Chung, M. S., Perry, A. E., & Cantwell, B. J. 1990, PhFl, 2, 765
Codis, S., Pichon, C., Devriendt, J., et al. 2012, MNRAS, 427, 3320
Cole, S., Fisher, K. B., & Weinberg, D. H. 1994, MNRAS, 267, 785
Davis, M., & Peebles, P. J. E. 1985, ApJ, 267, 465
Domínguez, A. 2000, PhRvD, 62, 103501
Dubois, Y., Pichon, C., Welker, C., et al. 2011, MNRAS, in press (arXiv:1102.1165)
Forero-Romero, J. E., Hoffman, Y., Gottlöber, S., Klypin, A., & Yepes, G. 2009, MNRAS, 396, 1815
Hahn, O., Angulo, R. E., & Abel, T. 2014, MNRAS, submitted (arXiv:1402.1165)
Hahn, O., Carollo, C. M., Porciani, C., & Dekel, A. 2007a, MNRAS, 381, 41
Hahn, O., Porciani, C., Carollo, C. M., & Dekel, A. 2007b, MNRAS, 375, 489
Hahn, O., Metuki, O., Yepes, G., et al. 2012, MNRAS, 425, 2049
Hoyle, F. 1949, Problems of Cosmical Aerodynamics, ed. J. M. Burgers & H. C. van der Hulst (Dayton, OH: Central Air Documents Office), 195
Hui, L., & Bertschinger, E. 1996, ApJ, 471, 1
Kaiser, N. 1987, MNRAS, 227, 1
Laigle, C., Pichon, C., Combe, S., et al. 2013, MNRAS, submitted (arXiv:1310.3801)
Libeskind, N. I., Hoffman, Y., Forero-Romero, J., et al. 2013b, MNRAS, 428, 2499
Libeskind, N. I., Hoffman, Y., Steinmetz, M., et al. 2013a, ApJ, 766, L15
Lynden-Bell, D. 1967, MNRAS, 136, 101
Meneveau, C. 2011, AnRFM, 43, 219
Munshi, D., & Starobinski, A. A. 1994, ApJ, 428, 433
Nusser, A., Branchini, E., & Davis, M. 2012, ApJ, 755, 58
Paz, D. J., Stassun, G., & Padilla, N. D. 2008, MNRAS, 389, 1127
Peebles, P. J. E. 1969, ApJ, 155, 393
Peebles, P. J. E. (ed.) 1980, in The Large-scale Structure of the Universe (Princeton, NJ: Princeton Univ. Press)
Pellupessy, F. I., Schaap, W. E., & van de Weygaert, R. 2003, A&A, 403, 389
Pichon, C., & Bernardeau, F. 1999, A&A, 343, 663
Pichon, C., Pogosyan, D., Kimm, T., et al. 2011, MNRAS, 418, 2493
Pogosyan, D., Gay, C., & Pichon, C. 2009, PhRvD, 80, 081301
Pogosyan, D., Gay, C., & Pichon, C. 2010, PhRvD, 81, 9901
Pogosyan, D., Pichon, C., Gay, C., et al. 2009, MNRAS, 396, 635
Pueblas, S., & Scoccimarro, R. 2009, PhRvD, 80, 3504
Schelp, W. E., & van de Weygaert, R. 2000, A&A, 363, L29
Sousbie, T., Pichon, C., & Kawahara, H. 2011, MNRAS, 418, 3504
Sousbie, T., Pichon, C., & Kawahara, H. 2011, MNRAS, 414, 384
Stoica, R. S., Martínez, V., Mateu, J., & Saar, E. 2005, A&A, 434, 423
Tully, R. B., & Fisher, J. R. 1977, A&A, 54, 661
Tresse, E., & Libeskind, N. I. 2014, ApJ, 775, 142
Wess, J. M. 1985, A&A, 151, 105
White, S. D. M. 1984, ApJ, 286, 38
Zel’dovich, Ya. B. 1970, A&A, 5, 84