Tighter Undecidability Bounds for Matrix Mortality, Zero-in-the-Corner Problems, and More

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Abstract

We study the decidability of three well-known problems related to integer matrix multiplication: Mortality ($M$), Zero in the Left-Upper Corner ($Z$), and Zero in the Right-Upper Corner ($R$).

Let $d$ and $k$ be positive integers. Define $M^d(k)$ as the following special case of the Mortality problem: given a set $\mathcal{X}$ of $d$-by-$d$ integer matrices such that the cardinality of $\mathcal{X}$ is not greater than $k$, decide whether the $d$-by-$d$ zero matrix belongs to $\mathcal{X}^+$, where $\mathcal{X}^+$ denotes the closure of $\mathcal{X}$ under the usual matrix multiplication. In the same way, define the $Z^d(k)$ problem as: given an instance $\mathcal{X}$ of $M^d(k)$ (the instances of $Z^d(k)$ are the same as those of $M^d(k)$), decide whether at least one matrix in $\mathcal{X}^+$ has a zero in the left-upper corner. Define $R^d(k)$ as the variant of $Z^d(k)$ where “left-upper corner” is replaced with “right-upper corner”. In the paper, we prove that $M^3(6)$, $M^5(4)$, $M^9(3)$, $M^{15}(2)$, $Z^3(5)$, $Z^5(3)$, $Z^9(2)$, $R^3(6)$, $R^4(5)$, and $R^{16}(3)$ are undecidable. The previous best comparable results were the undecidabilities of $M^3(7)$, $M^{13}(3)$, $M^{21}(2)$, $Z^7(7)$, $Z^{13}(2)$, $R^3(7)$, and $R^{10}(2)$.

1 Introduction

1.1 Notation and definition

Given two decision problems $P$ and $P'$, we say that $P$ reduces to $P'$ if there exists an oracle Turing machine [27] $T$ such that: if the oracle solves $P'$ then $T$ solves $P$. Two decision problems are called equivalent if they reduce to each other.

As usual, $\mathbb{N}$ denotes the semiring of non-negative integers and $\mathbb{Q}$ denotes the field of rational numbers. For every $n \in \mathbb{N}$, $[1,n]$ denotes the set of all $k \in \mathbb{N}$ such that $1 \leq k \leq n$.

1.1.1 Matrices

For every $m, n \in \mathbb{N} \setminus \{0\}$, $\mathbb{Q}^{m \times n}$ denotes the set of all $m$-by-$n$ matrices with entries in $\mathbb{Q}$, $I_n$ denotes the $n$-by-$n$ identity matrix, and $O_{m,n}$ denotes the $m$-by-$n$ zero matrix; subscripts
are sometimes dropped when there is no ambiguity. For every matrix \( X \), \( X^t \) denotes the transpose of \( X \).

Let \( d \in \mathbb{N} \setminus \{0\} \). For every \( \mathcal{X} \subseteq \mathbb{Q}^{d \times d} \), define \( \mathcal{X}^+ \) as the closure of \( \mathcal{X} \) under the usual matrix multiplication and define \( \mathcal{X}^* = \mathcal{X}^+ \cup \{I_d\} \). For every \( X \in \mathbb{Q}^{d \times d} \), \( X^* \) is understood as a shorthand for \( \{X^n : n \in \mathbb{N}\} \).

### 1.1.2 Semigroups

A semigroup is a set equipped with an associative operation. A monoid is a semigroup that has an identity element. For instance, \( \mathbb{Q}^{d \times d} \) is a monoid under the usual matrix multiplication. For every \( \mathcal{X} \subseteq \mathbb{Q}^{d \times d} \), \( \mathcal{X}^+ \) is the multiplicative subsemigroup of \( \mathbb{Q}^{d \times d} \) generated by \( \mathcal{X} \) and \( \mathcal{X}^* \) is the multiplicative submonoid of \( \mathbb{Q}^{d \times d} \) generated by \( \mathcal{X} \).

Let \( S \) and \( S' \) be multiplicative semigroups. A morphism from \( S \) to \( S' \) is a function \( \Phi : S \to S' \) such that \( \Phi(XY) = \Phi(X)\Phi(Y) \) for all \( X, Y \in S \). Throughout the paper, “morphism” always means “multiplicative semigroup morphism”.

### 1.2 Problems

Let \( d \in \mathbb{N} \setminus \{0\} \).

The Zero Reachability problem over \( \mathbb{Q}^{d \times d} \) [14, 12, 9, 25], denoted \( \mathcal{Z}^d \), is defined as: given \( L \in \mathbb{Q}^{1 \times d} \), \( C \in \mathbb{Q}^{d \times 1} \), and a finite \( \mathcal{X} \subseteq \mathbb{Q}^{d \times d} \), decide whether there exists \( Y \in \mathcal{X}^+ \) such that \( LYC = 0 \).

For every \( i, j \in [1, d] \), the following problem is denoted \( \mathcal{Z}_{i,j}^d \): given a finite \( \mathcal{X} \subseteq \mathbb{Q}^{d \times d} \), decide whether there exists \( Y \in \mathcal{X}^+ \) such that the \((i, j)\)th entry of \( Y \) equals 0. \( \mathcal{Z}_{1,1}^d \) is the Zero in the Left-Upper Corner problem over \( \mathbb{Q}^{d \times d} \) [6, 14, 12, 10]. Put \( \mathcal{R}^d = \mathcal{Z}_{1,d}^d \). \( \mathcal{R}^d \) is the Zero in the Right-Upper Corner problem over \( \mathbb{Q}^{d \times d} \) [19, 9, 7, 15, 1, 14, 12, 8, 3].

The Mortality problem over \( \mathbb{Q}^{d \times d} \) [6, 12, 10, 23, 7, 4, 18, 26, 21, 15, 2], denoted \( \mathcal{M}^d \), is defined as: given a finite \( \mathcal{X} \subseteq \mathbb{Q}^{d \times d} \), decide whether the \( d \)-by-\( d \) zero matrix belongs to \( \mathcal{X}^+ \).

Let \( k \in \mathbb{N} \). Define \( \mathcal{Z}^d(k) \) as the restriction of \( \mathcal{Z}^d \) to those instances \((L, C, \mathcal{X})\) for which the cardinality of \( \mathcal{X} \) is not greater than \( k \). For every \( i, j \in [1, d] \), define \( \mathcal{Z}_{i,j}^d(k) \) as the restriction of \( \mathcal{Z}_{i,j}^d \) to those subsets of \( \mathbb{Q}^{d \times d} \) that have cardinality \( k \) or less. Put \( \mathcal{R}^d(k) = \mathcal{Z}_{1,d}^d(k) \). Define \( \mathcal{M}^d(k) \) as the restriction of \( \mathcal{M}^d \) to those subsets of \( \mathbb{Q}^{d \times d} \) that have cardinality \( k \) or less. We convene that \( \mathcal{Z}^d(\infty) = \mathcal{Z}^d \), \( \mathcal{Z}_{i,j}^d(\infty) = \mathcal{Z}_{i,j}^d \), \( \mathcal{R}^d(\infty) = \mathcal{R}^d \), and \( \mathcal{M}^d(\infty) = \mathcal{M}^d \).

Note that restricting the previously defined problems to matrices with integer entries does not modify their decidabilities. Restricting them to matrices with non-negative integer entries makes them decidable [4, 9].

### 1.3 Organization of the paper

The paper is divided into five sections. Let \( d \in \mathbb{N} \setminus \{0\} \) and let \( k \in \mathbb{N} \cup \{\infty\} \). In Section 2, we prove the following four propositions:

**Proposition 1.** For every \( i, j \in [1, d] \) with \( i \neq j \), \( \mathcal{Z}_{i,j}^d(k) \) is equivalent to \( \mathcal{R}^d(k) \).
Table 1: Previous work.

| Problem     | Status | Reference(s) |
|-------------|--------|--------------|
| $Z^3(5)$    | D      | [13]         |
| $Z^3(5)$    | U      | [12]         |
| $Z^3(2)$    | U      | [14]         |
| $Z_{1,1}^3(7)$ | U    | [12]         |
| $Z_{1,1}^3(2)$ | U    | [14]         |
| $R^3(7)$    | U      | [8, 15, 20]  |
| $R^{10}(2)$ | U      | [14]         |
| $M^2(2)$    | D      | [6]          |
| $M^2$       | NP-hard | [2]          |
| $M^3(7)$    | U      | [12]         |
| $M^{13}(3)$ | U      | [14]         |
| $M^{21}(2)$ | U      | [12]         |

Proposition 2. For every $i \in [1, d]$, $Z_{i,i}^d(k)$ is equivalent to $Z^d(k)$.

Proposition 3. $Z^d(k)$ reduces to $R^{d+1}(k)$.

Proposition 4. $Z^d(k)$ reduces to $M^d(k + 1)$.

Note that the equivalence of $Z^d(k)$ and $Z_{1,1}^d(k)$, which follows from Proposition 2, was previously overlooked. In Section 3, we prove that $Z^3(5)$ and $R^3(6)$ are undecidable. In Section 4, we prove that $Z^5(3)$, $Z^9(2)$, and $M^{15}(2)$, are undecidable. In Section 5, we put forward some remaining open questions.

1.4 Contribution

The undecidabilities of $Z^3(5)$, $Z^5(3)$, and $Z^9(2)$ imply those of $R^4(5)$, $R^6(3)$, and $R^{10}(2)$ by Proposition 3 and those of $M^3(6)$, $M^5(4)$, and $M^9(3)$ by Proposition 4. Hence, the following problems are proven undecidable in the present paper: $Z^3(5)$, $Z^5(3)$, $Z^9(2)$, $R^3(6)$, $R^4(5)$, $R^6(3)$, $R^{10}(2)$, $M^3(6)$, $M^5(4)$, $M^9(3)$, and $M^{15}(2)$; the undecidabilities of $Z^3(5)$, $Z^9(2)$, and $R^{10}(2)$ were previously known [12, 14]. Previous results about our problems are summarized in Table 1. Our contribution is depicted in Tables 2, 3, 4, and 5. The contents of the five tables are to be understood as follows: D stands for “decidable”, U and U stand for “undecidable”, ? stands for “unknown”, and U denotes our contribution.

2 General results

In this section, we prove some basic properties of our problems. Unsurprisingly, we shall see that they are closely related to each other. Let $d \in \mathbb{N} \setminus \{0\}$ and let $k \in \mathbb{N} \cup \{\infty\}$. 


Table 2: Current knowledge about the undecidability of $3^d(k)$.

| $d$ | 2 | 3 | 4 | 5 | 6 | ... |
|-----|---|---|---|---|---|-----|
| 2   | ? | ? | ? | ? | ? | ... |
| 3   | ? | ? | ? | U | U | ... |
| 4   | ? | ? | ? | U | U | ... |
| 5   | ? | U | U | U | U | ... |
| 6   | ? | U | U | U | U | ... |
| 7   | ? | U | U | U | U | ... |
| 8   | ? | U | U | U | U | ... |
| 9   | U | U | U | U | U | ... |
| 10  | U | U | U | U | U | ... |

Table 3: Current knowledge about the undecidability of $3^{d}_{1,1}(k)$.

| $d$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... |
|-----|---|---|---|---|---|---|---|-----|
| 2   | ? | ? | ? | ? | ? | ? | ? | ... |
| 3   | ? | ? | ? | U | U | U | U | ... |
| 4   | ? | ? | ? | U | U | U | U | ... |
| 5   | ? | U | U | U | U | U | U | ... |
| 6   | ? | U | U | U | U | U | U | ... |
| 7   | ? | U | U | U | U | U | U | ... |
| 8   | ? | U | U | U | U | U | U | ... |
| 9   | U | U | U | U | U | U | U | ... |
| 10  | U | U | U | U | U | U | U | ... |
| 11  | U | U | U | U | U | U | U | ... |
| 12  | U | U | U | U | U | U | U | ... |
| 13  | U | U | U | U | U | U | U | ... |
| 14  | U | U | U | U | U | U | U | ... |

| ... | ... | ... | ... | ... | ... | ... | ... | ...

...
Set

\[ E_i = \begin{pmatrix} O_{i-1,1} \\ 1 \\ O_{d-i,1} \end{pmatrix} \]

for every \( i \in [1,d] \): the \( d \)-tuple \( (E_i)_{i \in [1,d]} \) is the canonical basis of the linear space \( \mathbb{Q}^{d \times 1} \).

Remark that, for any \( L \in \mathbb{Q}^{1 \times d} \), any \( C \in \mathbb{Q}^{d \times 1} \), any \( Y \in \mathbb{Q}^{d \times d} \), and any \( i, j \in [1,d] \), \( LE_j \) equals the \( j \)th entry of \( L \), \( E_i^tC \) equals the \( i \)th entry of \( C \), \( E_i^tY \) equals the \( i \)th row of \( Y \), \( YE_j \) equals the \( j \)th column of \( Y \), and \( E_i^tYE_j \) equals the \((i,j)\)th entry of \( Y \).

**Lemma 1.** Let \( D \subseteq [1,d] \) and let \( \pi: D \to [1,d] \) be injective. There exists \( P \in \mathbb{Q}^{d \times d} \) such that \( P \) is non-singular and \( PE_j = E_{\pi(j)} \) for every \( j \in D \).

*Proof.* Let us first consider the case where \( D = [1,d] \), i.e., where \( \pi \) is a permutation of \([1,d]\). Let \( P \) be the permutation matrix associated with \( \pi \):

\[ P = \sum_{i=1}^{d} E_{\pi(i)}E_i^t. \]

It is easy to see that \( P \) satisfies the desired properties; in particular, note that \( P^t = P^{-1} \) is the permutation matrix associated with \( \pi^{-1} \).

Let us now deal with the general case. Remark that there exists a permutation \( \bar{\pi} \) of \([1,d]\) such that \( \bar{\pi}(j) = \pi(j) \) for every \( j \in D \). Hence, the general case reduces to the case where \( D = [1,d] \). \( \square \)

**Lemma 2.** For every \( i \in [1,d] \), \( \mathcal{R}^d_{i,i}(k) \) is equivalent to \( \mathcal{R}^d_{1,1}(k) \).
| $d$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … |
|-----|---|---|---|---|---|---|---|---|
| 2   | D | ? | ? | ? | ? | ? | ? | … |
| 3   | ? | ? | ? | ? | U | U | U | … |
| 4   | ? | ? | ? | ? | U | U | U | … |
| 5   | ? | ? | U | U | U | U | U | … |
| 6   | ? | ? | U | U | U | U | U | … |
| 7   | ? | ? | U | U | U | U | U | … |
| 8   | ? | ? | U | U | U | U | U | … |
| 9   | ? | U | U | U | U | U | U | … |
| 10  | ? | U | U | U | U | U | U | … |
| 11  | ? | U | U | U | U | U | U | … |
| 12  | ? | U | U | U | U | U | U | … |
| 13  | ? | U | U | U | U | U | U | … |
| 14  | ? | U | U | U | U | U | U | … |
| 15  | U | U | U | U | U | U | U | … |
| 16  | U | U | U | U | U | U | U | … |
| 17  | U | U | U | U | U | U | U | … |
| 18  | U | U | U | U | U | U | U | … |
| 19  | U | U | U | U | U | U | U | … |
| 20  | U | U | U | U | U | U | U | … |
| 21  | U | U | U | U | U | U | U | … |
| 22  | U | U | U | U | U | U | U | … |
| :   | : | : | : | : | : | : | : | … |

Table 5: Current knowledge about the decidability of $\mathcal{M}^d(k)$. 
Proof. Let \( i, j \in [1, d] \) be fixed. Applying Lemma 1 with \( D = \{ i, j \} \), we see that there exists \( P \in \mathbb{Q}^{d \times d} \) such that \( P \) is non-singular, \( PE_i = E_j \), and \( PE_j = E_i \). Let \( \Phi : \mathbb{Q}^{d \times d} \rightarrow \mathbb{Q}^{d \times d} \) be the morphism defined by: \( \Phi(X) = PXP^{-1} \) for every \( X \in \mathbb{Q}^{d \times d} \). For every \( Y \in \mathbb{Q}^{d \times d} \), the \((i, i)\)th entry of \( Y \) equals the \((j, j)\)th entry of \( \Phi(Y) \). Hence, \( \Phi \) induces a reduction from \( 3^d_{i, i}(k) \) to \( 3^d_{j, j}(k) \). \qed

**Proposition 1.** For every \( i, j \in [1, d] \) with \( i \neq j \), \( 3^d_{i, j}(k) \) is equivalent to \( 9^d(k) \).

**Proof.** Let \( i_1, j_1, i_2, j_2 \in [1, d] \) be such that \( i_1 \neq j_1 \) and \( i_2 \neq j_2 \). Applying Lemma 1 with \( D = \{ i_1, j_1 \} \), we see that there exists \( P \in \mathbb{Q}^{d \times d} \) such that \( P \) is non-singular, \( PE_{i_1} = E_{i_2} \), and \( PE_{j_1} = E_{j_2} \). Let \( \Phi : \mathbb{Q}^{d \times d} \rightarrow \mathbb{Q}^{d \times d} \) be the morphism defined by: \( \Phi(X) = PXP^{-1} \) for every \( X \in \mathbb{Q}^{d \times d} \). For every \( Y \in \mathbb{Q}^{d \times d} \), the \((i_1, j_1)\)th entry of \( Y \) equals the \((i_2, j_2)\)th entry of \( \Phi(Y) \). It follows that \( \Phi \) induces a reduction from \( 3^d_{i_1, j_1}(k) \) to \( 3^d_{i_2, j_2}(k) \). \qed

**Lemma 3.** For every \( i, j \in [1, d] \), \( 3^d_{i, j}(k) \) reduces to \( 3^d(k) \).

**Proof.** For every finite \( \mathcal{X} \subseteq \mathbb{Q}^{d \times d} \), \( \mathcal{X} \) is a yes-instance of \( 3^d_{i, j} \) if, and only if, \((E_i^T, E_j, \mathcal{X})\) is a yes-instance of \( 3^d \).

An instance \((L, C, \mathcal{X})\) of \( 3^d \) is called non-degenerated if \( LC \neq 0 \).

**Lemma 4 ([9]).** \( 3^d(k) \) reduces to its restriction to non-degenerated instances.

**Proof.** Let \((L, C, \mathcal{X})\) be an instance of \( 3^d(k) \).

First, assume that \( LXC = 0 \) for some \( X \in \mathcal{X} \). Then, \((L, C, \mathcal{X})\) is a yes-instance of \( 3^d \).

Second, assume that \( LXC \neq 0 \) for every \( X \in \mathcal{X} \). Then, \((L, XC, \mathcal{X})\) is a non-degenerated instance of \( 3^d(k) \) for every \( X \in \mathcal{X} \). Moreover, \((L, C, \mathcal{X})\) is a yes-instance of \( 3^d \) if, and only if, there exists \( X \in \mathcal{X} \) such that \((L, XC, \mathcal{X})\) is a yes-instance of \( 3^d \). \qed

**Lemma 5.** Let \( L \in \mathbb{Q}^{1 \times d} \) and let \( C \in \mathbb{Q}^{d \times 1} \) be such that \( LC \neq 0 \). There exists \( P \in \mathbb{Q}^{d \times d} \) such that \( P \) is non-singular, \( LP = LCE_1 \), and \( C = PE_1 \).

**Proof.** First, consider the case where both the leftmost entry of \( L \) and the uppermost entry of \( C \) equal 1. Then, there exist \( L' \in \mathbb{Q}^{1 \times (d-1)} \) and \( C' \in \mathbb{Q}^{(d-1) \times 1} \) such that

\[
L = \begin{pmatrix} 1 & L' \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 \\ C' \end{pmatrix}.
\]

Put

\[
U = \begin{pmatrix} 1 & O \\ C' & -I \end{pmatrix}, \quad V = \begin{pmatrix} 1 & -(LC)^{-1}L' \\ O & -I \end{pmatrix}, \quad \text{and} \quad P = UV.
\]

It is clear that \( U, V, \) and \( P \) are non-singular with \( U^{-1} = U, V^{-1} = V \) and \( P^{-1} = VU \).

Moreover, we have \( LU = (LC - L') \) and \( VE_1 = E_1 \), so \( LP = LCE_1 \) and \( C = PE_1 \).
Let us now deal with the general case. For each $i \in [1, d]$, put $\lambda_i = LE_i$ and $\gamma_i = E_i^t C$. Since
\[ \sum_{i=1}^{d} \lambda_i \gamma_i = LC \neq 0, \]
there exists $j \in [1, d]$ such that $\lambda_j \gamma_j \neq 0$. Applying Lemma 1 with $D = \{1\}$, we see that there exists $T \in \mathbb{Q}^{d \times d}$ such that $T$ is non-singular and $T E_1 = E_j$. Let $\tilde{L} = \lambda_j^{-1} LT$ and $\tilde{C} = \gamma_j^{-1} T^{-1} C$. By construction, we have $\tilde{L} \tilde{C} = \lambda_j^{-1} \gamma_j^{-1} LC \neq 0$ and both the leftmost entry of $\tilde{L}$ and the uppermost entry of $\tilde{C}$ equal 1. Therefore, there exists $\tilde{P} \in \mathbb{Q}^{d \times d}$ such that $\tilde{P}$ is non-singular, $\tilde{L} P = L C E_1^t$, and $C = P E_1$. Put $P = \gamma_j T \tilde{P}$. It is easy to see that $P$ satisfies the desired properties.

Let $R$ be a division ring, let $L, L' \in R^{1 \times d}$, and let $C, C' \in R^{d \times 1}$ be such that none of $C, L, C'$, and $L'$ is a zero matrix. We claim that $LC = L'C'$ if, and only if, there exists $P \in R^{d \times d}$ such that $P$ is multiplicatively invertible in $R^{d \times d}$, $LP = L'$, and $C = PC'$. Our claim nicely generalizes Lemma 5; its proof is left to the reader. It follows from our claim that the restriction of $3^d(k)$ to degenerated instances is equivalent to $\mathfrak{A}^d(k)$; the verification is left to the reader.

**Proposition 2.** For every $i \in [1, d]$, $3^d_{1,i}(k)$ is equivalent to $3^d(k)$.

**Proof.** By Lemmas 2 and 3, it suffices to show that $3^d(k)$ reduces to $3^d_{1,1}(k)$. Moreover, by Lemma 4, we only need to reduce non-degenerated instances of $3^d(k)$.

Let $(L, C, \mathcal{X})$ be a non-degenerated instance of $3^d(k)$. By Lemma 5, there exists $P \in \mathbb{Q}^{d \times d}$ such that $P$ is non-singular, $LP = L C E_1^t$, and $C = P E_1$. Put $\mathcal{X}' = \{ P^{-1} X P : X \in \mathcal{X} \}$. Since the cardinality of $\mathcal{X}'$ equals that of $\mathcal{X}$, $\mathcal{X}'$ is an instance of $3^d_{1,1}(k)$. Moreover, $P$ is computable from $L$ and $C$ (a more efficient method than brute-force enumeration can be derived from a simple examination of the proof of Lemma 5), so $\mathcal{X}'$ is computable from $(L, C, \mathcal{X})$. Finally, remark that for every $Y \in \mathbb{Q}^{d \times d}$, the $(1, 1)$th entry of $P^{-1} Y P$ equals $(LC)^{-1} L Y C$. Therefore, $(L, C, \mathcal{X})$ is a yes-instance of $3^d$ if, and only if, $\mathcal{X}'$ is a yes-instance of $3^d_{1,1}$.

Lemma 3 ensures that $\mathfrak{A}^d(k)$ reduces to $3^d(k)$; whether $3^d(k)$ reduces to $\mathfrak{A}^d(k)$ is an open question. However, it holds true that:

**Proposition 3.** $3^d(k)$ reduces to $\mathfrak{A}^{d+1}(k)$.

**Proof.** By Proposition 2, it suffices to prove that $3^d_{1,1}(k)$ reduces to $\mathfrak{A}^{d+1}(k)$.

Let $\Phi : \mathbb{Q}^{d \times d} \to \mathbb{Q}^{(d+1) \times (d+1)}$ be the morphism defined by:
\[
\Phi(X) = \begin{pmatrix} X &XE_1 \\ O & 0 \end{pmatrix}
\]
for every $X \in \mathbb{Q}^{d \times d}$. For every $Y \in \mathbb{Q}^{d \times d}$, the $(1, 1)$th entry of $Y$ equals the $(1, d+1)$th entry of $\Phi(Y)$. Hence, $\Phi$ induces a reduction from $3^d_{1,1}(k)$ to $\mathfrak{A}^{d+1}(k)$.

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Proposition 3 improves on the following result, which is implicitly used in at least two papers:

**Proposition 5** ([14, 9]). \( \mathcal{Z}^d(k) \) reduces to \( \mathcal{R}^{d+2}(k) \).

**Proof.** For every \( L \in \mathbb{Q}^{1 \times d} \) and every \( C \in \mathbb{Q}^{d \times 1} \), let \( \Phi_{L,C}: \mathbb{Q}^{d \times d} \to \mathbb{Q}^{(d+2) \times (d+2)} \) be the morphism defined by:

\[
\Phi_{L,C}(X) = \begin{pmatrix}
0 & LX & LXC \\
O & X & XC \\
0 & O & 0
\end{pmatrix}
\]

for every \( X \in \mathbb{Q}^{d \times d} \). For every instance \((L, C, \mathcal{X})\) of \( \mathcal{Z}^d \), \((L, C, \mathcal{X})\) is a yes-instance of \( \mathcal{Z}^d \) if, and only if, \( \Phi_{L,C}(\mathcal{X}) \) is a yes-instance of \( \mathcal{R}^{d+2}(k) \).

**Proposition 6.**

- \( \mathcal{R}^d(k) \) reduces to \( \mathcal{R}^{d+1}(k) \).
- \( \mathcal{Z}^d(k) \) reduces to \( \mathcal{Z}^{d+1}(k) \).
- \( \mathcal{M}^d(k) \) reduces to \( \mathcal{M}^{d+1}(k) \).

**Proof.** Remark that \( \mathcal{R}^d(k) \) reduces to \( \mathcal{Z}^d(k) \) by Lemma 3 and that \( \mathcal{Z}^d(k) \) reduces to \( \mathcal{R}^{d+1}(k) \) by Proposition 3. Therefore, the first part of the proposition holds true.

The second part of the proposition can be proven in the same way: \( \mathcal{Z}^d(k) \) reduces to \( \mathcal{R}^{d+1}(k) \) by Proposition 3 and \( \mathcal{R}^{d+1}(k) \) reduces to \( \mathcal{Z}^{d+1}(k) \) by Lemma 3.

Let \( \Phi: \mathbb{Q}^{d \times d} \to \mathbb{Q}^{(d+1) \times (d+1)} \) be the morphism defined by:

\[
\Phi(X) = \begin{pmatrix}
x & O \\
O & 0
\end{pmatrix}
\]

for every \( X \in \mathbb{Q}^{d \times d} \). For every \( Y \in \mathbb{Q}^{d \times d} \), \( Y \) equals the \( d \)-by-\( d \) zero matrix if, and only if, \( \Phi(Y) \) equals \((d + 1)\)-by-\((d + 1)\) zero matrix. Hence, \( \Phi \) induces a reduction from \( \mathcal{M}^d(k) \) to \( \mathcal{M}^{d+1}(k) \), and thus the third part of the proposition holds true.

**Lemma 6.** Let \( L \in \mathbb{Q}^{1 \times d} \), let \( C \in \mathbb{Q}^{d \times 1} \), and let \( \mathcal{X} \subseteq \mathbb{Q}^{d \times d} \). The following two assertions are equivalent:

1. There exists \( Y \in \mathcal{X}^* \) such that \( LYC = 0 \).
2. The \( d \)-by-\( d \) zero matrix belongs to \((\mathcal{X} \cup \{CL\})^+\).

**Proof.** Note that \( LX^*C \subseteq \mathbb{Q} \). Since \((LYC)CL \in (\mathcal{X} \cup \{CL\})^+\) for every \( Y \in \mathcal{X}^* \), the first considered assertion implies the second one. Since

\[
L(\mathcal{X} \cup \{CL\})^*C = (LX^*)^+, 
\]

the second considered assertion implies \( 0 \in (LX^*)^+ \). Besides, \( 0 \in (LX^*)^+ \) is equivalent to \( 0 \in LX^*C \) because \( \mathbb{Q} \) has the zero-product property. Therefore, the considered assertions are equivalent.
Proposition 4. $Z^d(k)$ reduces to $M^d(k+1)$.

Proof. By Lemma 4, we only need to reduce non-degenerated instances of $Z^d(k)$.

Let $(L, C, \mathcal{X})$ be a non-degenerated instance of $Z^d(k)$. Clearly, $\mathcal{X} \cup \{CL\}$ is an instance of $M^d(k+1)$ and $\mathcal{X} \cup \{CL\}$ is computable from $(L, C, \mathcal{X})$. To conclude the proof of the proposition, we only need to check that the following three assertions are equivalent:

1. $(L, C, \mathcal{X})$ is a yes-instance of $Z^d$.
2. There exists $Y \in \mathcal{X}^*$ such that $LYC = 0$.
3. $\mathcal{X} \cup \{CL\}$ is a yes-instance of $M^d$.

The first two considered assertions are equivalent because $LC \neq 0$; the last two considered assertions are equivalent by Lemma 6. \hfill \Box

Proposition 7 ([6]). $Z^2(k)$ is equivalent to $M^2(k+1)$.

Proof. By Proposition 4, it suffices to show that $M^2(k+1)$ reduces to $Z^2(k)$. The proof is based on Lemma 6 and the following property of 2-by-2 matrices: for every $X \in \mathbb{Q}^{2 \times 2}$, either $X$ is non-singular or $X$ can be written as an outer product.

Let $\mathcal{X}$ be an instance of $M^2(k+1)$.

First, assume that all matrices in $\mathcal{X}$ are non-singular. Then, $\mathcal{X}$ is a no-instance of $M^2$ because all matrices in $\mathcal{X}^+$ are non-singular.

Second, assume that some matrix in $\mathcal{X}$ can be written as an outer product. Then, there exist $L \in \mathbb{Q}^{1 \times 2}$ and $C \in \mathbb{Q}^{2 \times 1}$ such that $CL \in \mathcal{X}$. Clearly, $(L, C, \mathcal{X} \setminus \{CL\})$ is an instance of $Z^2(k)$ and $(L, C, \mathcal{X} \setminus \{CL\})$ is computable from $\mathcal{X}$. To conclude the proof of the proposition, we only need to check that the following three assertions are equivalent:

1. $\mathcal{X}$ is a yes-instance of $M^2$.
2. There exists $Y \in (\mathcal{X} \setminus \{CL\})^*$ such that $LYC = 0$.
3. $LC = 0$ or $(L, C, \mathcal{X} \setminus \{CL\})$ is a yes-instance of $Z^2$.

The first two considered assertions are equivalent by Lemma 6; the last two considered assertions are clearly equivalent. \hfill \Box

3 Three-by-three matrices

In this section, we prove that $Z^3(5)$ and $M^3(6)$ are undecidable by reduction from the generalized Post correspondence problem. Let $k \in \mathbb{N} \cup \{\infty\}$.
3.1 The (generalized) Post correspondence problem

Precise definitions of the Post Correspondence Problem (PCP) [24] and its best-known generalization are presented in this section.

An alphabet is a finite set of symbols. The canonical alphabet is the binary alphabet \{1, 2\}. A word is a finite sequence of symbols. Word concatenation is denoted multiplicatively. For every word \(w\), \(|w|\) denotes the length of \(w\). The word of length 0 is called the empty word and denoted \(\varepsilon\). Let \(A\) be an alphabet. The set of all words over \(A\) is denoted \(A^*\). Note that \(A^*\) is a monoid under concatenation. Set \(A^+ = A^* \setminus \{\varepsilon\}\).

Two slightly different definitions of the Generalized Post Correspondence Problem (GPCP) can be found in the literature. Let \(e \in \{\star, +\}\). Define GPCP\(_e\) as the following problem: given an alphabet \(A\), two morphisms \(f, g: A^* \to \{1, 2\}^*\), and \(x, x', y, y' \in \{1, 2\}^*\), decide whether there exists \(w \in A^e\) such that \(xf(w)x' = yg(w)y'\); it is understood that the instance \((A, f, g, x, x', y, y')\) is encoded by the quintuple \(((f(a), g(a)) : a \in A\}, x, x', y, y')\).

Define GPCP\(_e\)(\(k\)) as the restriction of GPCP\(_e\) to those instances \((A, f, g, x, x', y, y')\) for which the cardinality of \(A\) is not greater than \(k\). The subscript \(e\) is sometimes dropped when there is no ambiguity.

**Proposition 8.** GPCP\(_\star\)(\(k\)) and GPCP\(_+\)(\(k\)) are equivalent.

**Proof.** Let \(I = (A, f, g, x, x', y, y')\) be an instance of GPCP\(_\star\)(\(k\)).

First, \(I\) is a yes-instance of GPCP\(_\star\) if, and only if, at least one of the following two holds true: \(xx' = yy'\) or \(I\) is a yes-instance of GPCP\(_+\). Therefore, GPCP\(_\star\)(\(k\)) reduces to GPCP\(_+\)(\(k\)). Second, \(I\) is a yes-instance of GPCP\(_+\) if, and only if, there exists \(a \in A\) such that

\[(A, f, g, xf(a), x', yg(a), y')\]

is a yes-instance of GPCP\(_\star\). Therefore, GPCP\(_+\)(\(k\)) reduces to GPCP\(_\star\)(\(k\)). \(\square\)

Define PCP\(_k\) as the restriction of GPCP\(_+\)(\(k\)) to those instances \((A, f, g, x, x', y, y')\) that satisfy \(xx'y' = \varepsilon\). PCP\(_\infty\) is the PCP. The fundamental property of PCP is its undecidability [24, 17, 27, 19]. The undecidabilities of many decision problems are proven by reductions from PCP [17, 19]. As far as we know, undecidability in 3-by-3 matrices has always been proven by reductions from PCP or GPCP. Note that the restriction of PCP\(_k+2\) to *Claus instances* [12] is equivalent to GPCP\(_k\) [12, 8, 15].

Define \(k_G\) as the smallest \(k \in \mathbb{N}\) such that GPCP\(_k\) is undecidable; define \(k_P\) as the smallest \(k \in \mathbb{N}\) such that PCP\(_k\) is undecidable. The exact values of \(k_P\) and \(k_G\) are still unknown. However, it is known that \(k_P \leq k_G + 2\) [15], \(2 < k_G, [11]\), \(k_P \leq 7\) [20], and \(k_G \leq 5\) [12]:

\[3 \leq k_G \leq k_P \leq k_G + 2 \leq 7\]

The decidabilities of GPCP(3), GPCP(4), PCP(3), PCP(4), PCP(5), and PCP(6) are open.
| Year | Undecidable problem | Reference |
|------|---------------------|-----------|
| 1970 | $M^3(2k_P + 2)$    | [23]      |
| 1974 | $3_{3,2}^3(k_P)$    | [19]      |
| 1980 | PCP(10)             | [8]       |
| 1981 | $R^3(k_P)$          | [8] (see also [15] and Theorem 2) |
| 1996 | GPCP(7)             | [16] (see also [15]) |
| 1997 | $M^3(2k_P + 1)$     | [15]      |
| 1999 | $M^3(k_P + 2)$      | [5] (see also [6]) |
| 2001 | $3_{3,1,1}^3(2k_P)$ | [10]      |
| 2005 | PCP(7)              | [20]      |
| 2007 | GPCP(5)             | [12]      |
|      | $3^3(k_G)$          | [12] (see also Theorem 1) |
|      | $3_{3,1,1}^3(2k_G + 2)$ | [12] |
|      | $M^3(k_G + 2)$      | [12]      |

Table 6: Undecidability in 3-by-3 matrices and the (generalized) Post correspondence problem.

### 3.2 Undecidability bounds

In this section, we prove that $3^3(k_G)$, $R^3(k_P)$, and $R^3(k_G + 1)$ are undecidable; the undecidabilities of $3^3(k_G)$ and $R^3(k_P)$ were already known [8, 12]. However, it is still unknown whether $k_P \leq k_G + 1$. Besides, the undecidability of $3^3(k_G)$ implies that of $R^4(k_G)$ by Proposition 3 and that of $M^3(k_G + 1)$ by Proposition 4. Previous related undecidability results are listed in Table 6. As $k_G \leq 5$ [12], $3^3(5)$, $R^3(6)$, $R^4(5)$, and $M^3(6)$ are undecidable.

**Lemma 7** ([19, 15, 1, 8]). There exists a morphism $\Psi: \{1,2\}^* \times \{1,2\}^* \rightarrow \mathbb{Q}^{3 \times 3}$ such that for all $u, v \in \{1,2\}^*$, the $(1,3)$th entry of $\Psi(u, v)$ equals 0 if, and only if, $u = v$.

**Proof.** Let $\sigma: \{0,1,2\}^* \rightarrow \mathbb{N}$ be the function defined by: for each $w \in \{0,1,2\}^+$, $w$ is a base-3 representation of the integer $\sigma(w)$ (we convene that $\varepsilon$ is a representation of 0). Hence, $\sigma$ satisfies $\sigma(0) = 0$, $\sigma(1) = 1$, $\sigma(2) = 2$, and

$$\sigma(ww') = 3^{|w'|} \sigma(w) + \sigma(w')$$

for all $w, w' \in \{0,1,2\}^*$. Set

$$\Psi(u, v) = \begin{pmatrix} 1 & \sigma(v) & \sigma(u) - \sigma(v) \\ 0 & 3^{|v|} & 3^{|u|} - 3^{|v|} \\ 0 & 0 & 3^{|u|} \end{pmatrix}$$

(1)

for every $u, v \in \{1,2\}^*$. Straightforward computations yield

$$\Psi(ww', vv') = \Psi(u, v)\Psi(u', v')$$

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for all \( u, v, u', v' \in \{1, 2\}^* \), so \( \Psi \) is a morphism. Now, remark that \( \sigma \) is not injective because 
\( \sigma(0w) = \sigma(w) \) for every \( w \in \{0, 1, 2\}^* \). However, \( \sigma \) is injective on the set of those words in 
\( \{0, 1, 2\}^* \) that do not begin with 0. In particular, \( \sigma \) is injective on \( \{1, 2\}^* \). Since the \( (1, 3) \)th entry of \( \Psi(u, v) \) equals \( \sigma(u) - \sigma(v) \) for all \( u, v \in \{1, 2\}^* \), \( \Psi \) satisfies the desired property.

Let \( S \) be a multiplicative semigroup, let \( A \) and \( B \) be alphabets, and let \( \Psi : A^* \times B^* \rightarrow S \) be a morphism. If the operation of \( S \) is computable then \( \Psi \) is computable.

**Theorem 1** ([12]). \( 3^3(k_G) \) is undecidable.

*Proof.* Let us show that GPCP\(_+\)(\( k \)) reduces to \( 3^3(k) \) for any \( k \). Set \( E_1 = (1 \ 0 \ 0) \) and \( E_3 = (0 \ 0 \ 1) \); such a notation is consistent with Section 2. Let \( \Psi \) be as in Lemma 7.

Let \( I = (A, f, g, x, x', y, y') \) be an instance of GPCP(\( k \)). Put

\[
L = E_1^t \Psi(x, y), \\
C = \Psi(x', y') E_3, \\
X(w) = \Psi(f(w), g(w))
\]

for every \( w \in A^* \), and

\[
\mathcal{X} = \{ X(a) : a \in A \}.
\]

Since \( \Psi \) is computable, \( (L, C, \mathcal{X}) \) is computable from \( I \). Moreover, the cardinality of \( \mathcal{X} \) is not greater than that of \( A \), so \( (L, C, \mathcal{X}) \) is an instance of \( 3^3(k) \). To conclude the proof of the theorem, we only need to check that the following three assertions are equivalent:

1. \( I \) is a yes-instance of GPCP\(_+\). 
2. There exists \( w \in A^+ \) such that \( LX(w)C = 0 \). 
3. \( (L, C, \mathcal{X}) \) is a yes-instance of \( 3^3 \). 

For every \( w \in A^* \), the \( (1, 3) \)th entry of \( \Psi(xf(w)x', yg(w)y') \) equals \( LX(w)C \), and thus

\[
LX(w)C = 0 \iff xf(w)x' = yg(w)y'.
\] (2)

Therefore, the first two considered assertions are equivalent. Now, remark that \( X(ww') = X(w)X(w') \) for all \( w, w' \in A^* \). It follows that

\[
\mathcal{X}^+ = \{ X(w) : w \in A^+ \}.
\] (3)

Therefore, the last two considered assertions are equivalent. \( \square \)

**Theorem 2** ([19, 15, 8]). \( 3^3(k_P) \) is undecidable.

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Proof. Let us show that PCP(\(k\)) reduces to \(R^3(k)\) for any \(k\). Let the notation be as in the proof of Theorem 1. Without loss of generality, we assume \(\Psi(\varepsilon, \varepsilon) = I_3\). Hence, if \(xx'y' = \varepsilon\) then \(L = E_1^4\) and \(C = E_3\). The following three assertions are thus equivalent in the case where \(I\) is an instance of PCP:

1. \(I\) is a yes-instance of PCP.
2. There exists \(w \in A^+\) such that \(E_1^4X(w)E_3 = 0\).
3. \(X\) is a yes-instance of \(R^3\).

The last important result of Section 3 is:

**Theorem 3.** \(R^3(kG + 1)\) is undecidable.

Our proof of Theorem 3 requires the introduction of additional material, including the proofs of two lemmas. An instance \((A, f, g, x, x', y, y')\) of GPCP is called *Claus-like* if it satisfies the following three conditions for any \(w \in A^*\):

\[
xf(w) \neq yg(w),
\]

\[
f(w)x' \neq g(w)y',
\]

and

\[
f(w) = g(w) \iff w = \varepsilon.
\]

Let \(\lambda\) and \(\rho\) be the morphisms from \(\{1, 2\}^*\) to itself defined by: \(\lambda(a) = 12a\) and \(\rho(a) = a12\) for each \(a \in \{1, 2\}\). The useful properties of \(\lambda\) and \(\rho\) are summarized in the following lemma:

**Lemma 8.** The following properties hold true for any \(u, v \in \{1, 2\}^*\):

\[
\lambda(u)12 = 12\rho(v) \iff u = v, \tag{4}
\]

\[
\lambda(u) \neq 12\rho(v), \tag{5}
\]

\[
\lambda(u)12 \neq \rho(v), \tag{6}
\]

and

\[
\lambda(u) = \rho(v) \iff uv = \varepsilon. \tag{7}
\]

**Proof.** The proof of Equation (4) is left to the reader. The length of \(\lambda(u)\) is a multiple of 3 whereas the length of \(12\rho(v)\) is congruent to 2 modulo 3. Therefore, Equation (5) holds true. Equation (6) is proven in the same way as Equation (5). It remains to prove Equation (7). If \(uv = \varepsilon\) then \(\lambda(u) = \varepsilon = \rho(v)\). If \(u = \varepsilon\) and \(v \neq \varepsilon\) then \(\lambda(u) = \varepsilon \neq \rho(v)\). If \(u \neq \varepsilon\) and \(v = \varepsilon\) then \(\lambda(u) \neq \varepsilon = \rho(v)\). Let us now deal with the last case: \(u \neq \varepsilon\) and \(v \neq \varepsilon\). The lengths of \(\lambda(u)\) and \(\rho(v)\) are then larger than or equal to 3. Furthermore, the second letter of \(\lambda(u)\) equals 1 whereas the second letter of \(\rho(v)\) equals 2. It follows \(\lambda(u) \neq \rho(v)\). \(\square\)
Lemma 9. For each $e \in \{ \star, + \}$, GPCP$_e(k)$ reduces to its restriction to Claus-like instances.

Proof. Let $\mathcal{I} = (A, f, g, x, x', y, y')$ be an instance of GPCP$(k)$.

First, let
\[
\tilde{A} = \{ a \in A : f(a)g(a) \neq \varepsilon \},
\]
let $\tilde{f}$ be the restriction of $f$ to $\tilde{A}^*$, let $\tilde{g}$ be the restriction of $g$ to $\tilde{A}^*$, and let
\[
\tilde{\mathcal{I}} = (\tilde{A}, \tilde{f}, \tilde{g}, x, x', y, y').
\]

It is clear that $\tilde{\mathcal{I}}$ is an instance GPCP$(k)$ and that $\tilde{\mathcal{I}}$ is computable from $\mathcal{I}$. Moreover, if $A \neq \tilde{A}$ then $\mathcal{I}$ is a yes-instance of GPCP$_e$ if, and only if, at least one the following two holds true: $xx' = yy'$ or $\tilde{\mathcal{I}}$ is a yes-instance of GPCP$_e$. Replacing $\mathcal{I}$ with $\tilde{\mathcal{I}}$ if needed, we may assume that $A = \tilde{A}$, or equivalently, that
\[
f(w)g(w) = \varepsilon \iff w = \varepsilon
\]
for every $w \in A^*$.

Now, put
\[
\tilde{x} = \lambda(x), \quad \tilde{f} = \lambda \circ f, \quad \tilde{x}' = \lambda(x')12,
\]
\[
\tilde{y} = 12\rho(y),\quad \tilde{g} = \rho \circ g, \quad \tilde{y}' = \rho(y'),
\]
and
\[
\tilde{\mathcal{I}} = (A, \tilde{f}, \tilde{g}, \tilde{x}, \tilde{x}', \tilde{y}, \tilde{y}').
\]

It is clear that $\tilde{\mathcal{I}}$ is an instance of GPCP$(k)$ and that $\tilde{\mathcal{I}}$ is computable from $\mathcal{I}$. Moreover, let $w \in A^*$. By letting $u = xf(w)x'$ and $v = yg(w)y'$ in Equation (4), we get
\[
\tilde{x} \tilde{f}(w) \tilde{x}' = \tilde{y} \tilde{g}(w) \tilde{y}' \iff xf(w)x' = yg(w)y'.
\]

Therefore, $\mathcal{I}$ is a yes-instance of GPCP$_e$ if, and only if, $\tilde{\mathcal{I}}$ is a yes-instance of GPCP$_e$. It remains to prove that $\tilde{\mathcal{I}}$ is a Claus-like instance of GPCP. By letting $u = xf(w)$ and $v = yg(w)$ in Equation (5), we get
\[
\tilde{x} \tilde{f}(w) \neq \tilde{y} \tilde{g}(w).
\]

By letting $u = f(w)x'$ and $v = g(w)y'$ in Equation (6), we get
\[
\tilde{f}(w) \tilde{x}' \neq \tilde{g}(w) \tilde{y}.
\]

By letting $u = f(w)$ and $v = g(w)$ in Equation (7), we get
\[
\tilde{f}(w) = \tilde{g}(w) \iff f(w)g(w) = \varepsilon.
\]

Finally, combining the latter equivalence with Equation (8) yields
\[
\tilde{f}(w) = \tilde{g}(w) \iff w = \varepsilon.
\]
Proof of Theorem 3. Let us show that \( \text{GPCP}(k) \) reduces to \( \mathfrak{R}^3(k+1) \) for any \( k \). By Lemma 9, we only need to reduce Claus-like instances of \( \text{GPCP}_*(k) \). Let the notation be as in the proof of Theorem 1. Without loss of generality, we assume \( \Psi(\varepsilon, \varepsilon) = I_3 \). Combining the latter assumption and Equation (3), we get
\[
\mathcal{X}^* = \{ X(w) : w \in A^* \}. \tag{9}
\]
To prove the theorem, it suffices to check that the following four assertions are equivalent in the case where \( \mathcal{I} \) is a Claus-like instance of \( \text{GPCP} \):

1. \( \mathcal{I} \) is a yes-instance of \( \text{GPCP}_* \).
2. There exists \( w \in A^* \) such that \( LX(w)C = 0 \).
3. \( 0 \in LA^*C \).
4. \( \mathcal{X} \cup \{ CL \} \) is a yes-instance of \( \mathfrak{R}^3 \).

The first two considered assertions are equivalent because Equation (2) holds for every \( w \in A^* \). The second and the third considered assertions are equivalent by Equation (9). Let us now show that the last two considered assertions are equivalent. Let \( w \in A^* \). Clearly,

- the \((1,3)\)th entry of \( \Psi(xf(w), yg(w)) \) equals \( LX(w)E_3 \),
- the \((1,3)\)th entry of \( \Psi(f(w)x', g(w)y') \) equals \( E^t_1X(w)C \), and
- the \((1,3)\)th entry of \( \Psi(f(w), g(w)) \) equals \( E^t_1X(w)E_3 \).

As \( \mathcal{I} \) is a Claus-like instance of \( \text{GPCP} \), it follows that both \( LX(w)E_3 \) and \( E^t_1X(w)C \) are non-zero and that \( E^t_1X(w)E_3 = 0 \) is equivalent to \( w = \varepsilon \). Combining the latter facts with Equations (9) and (3), we obtain that \( 0 \) is not in \( L\mathcal{X}^*E_3 \), \( E^t_1\mathcal{X}^*C \), or \( E^t_1\mathcal{X}^*E_3 \). Besides, remark that
\[
E^t_1(\mathcal{X} \cup \{ CL \})^+ E_3 = (E^t_1\mathcal{X}^*E_3) \cup (E^t_1\mathcal{X}^*C) (L\mathcal{X}^*C)^* (L\mathcal{X}^*E_3) .
\]
Hence, we have
\[
0 \in E^t_1(\mathcal{X} \cup \{ CL \})^+ E_3 \iff 0 \in L\mathcal{X}^*C ,
\]
as desired.

4 Trading dimension for matrices

In this section, we prove that \( \mathfrak{M}^{15}(2), \mathfrak{Z}^5(3), \) and \( \mathfrak{Z}^9(2) \) are undecidable. Let \( d, h, k \in \mathbb{N} \setminus \{0\} \).

Theorem 4. \( \mathfrak{M}^d(hk + 1) \) reduces to \( \mathfrak{M}^{kd}(h + 1) \).
Proof. Let $X$ be an instance of $M^d(hk + 1)$. Write $X$ in the form

$$X = \{U\} \cup \{X_{i,j} : (i, j) \in [1, h] \times [1, k]\}.$$

Put

$$V = \begin{pmatrix} O & U \\ I_{kd - d} & O \end{pmatrix}, \quad \Gamma = \begin{pmatrix} I_d \\ O_{kd - d, d} \end{pmatrix}, \quad Y_i = (X_{i,1} \ X_{i,2} \ X_{i,3} \ \cdots \ X_{i,k})$$

for every $i \in [1, h]$, and

$$Y = \{V\} \cup \{\Gamma Y_i : i \in [1, h]\}.$$

Clearly, $Y$ is an instance of $M^{kd}(h + 1)$ and $Y$ is computable from $X$.

**Lemma 10.** For every $X_1, X_2, X_3, \ldots, X_k \in \mathbb{Q}^{d \times d}$, equality

$$(X_1 \ X_2 \ X_3 \ \cdots \ X_k) V^* \Gamma = \{X_1, X_2, X_3, \ldots, X_k\} U^*$$

holds true.

Proof. The idea of the proof is simply to compute

$$(X_1 \ X_2 \ X_3 \ \cdots \ X_k) V^n \Gamma$$

for every $n \in \mathbb{N}$. Let us extend the $k$-tuple $(X_j)_{j \in [1, k]}$ into an infinite sequence $(X_j)_{j \in \mathbb{N} \setminus \{0\}}$ of elements of $\mathbb{Q}^{d \times d}$ by means of the recurrence formula:

$$X_{j+k} = X_j U$$

for every $j \in \mathbb{N} \setminus \{0\}$. Let $n \in \mathbb{N}$ and let $j \in [1, k]$. Straightforward inductions on $n$ yield

$$(X_1 \ X_2 \ X_3 \ \cdots \ X_k) V^n = (X_{n+1} \ X_{n+2} \ X_{n+3} \ \cdots \ X_{n+k})$$

and

$$X_{j+kn} = X_j U^n.$$

It follows

$$(X_1 \ X_2 \ X_3 \ \cdots \ X_k) V^{kn+j-1} \Gamma = X_{j+kn} = X_j U^n,$$

which proves the lemma.

Put

$$X' = \{X_{i,j} : (i, j) \in [1, h] \times [1, k]\}$$

and

$$Y' = \{Y_i : i \in [1, h]\}.$$

**Lemma 11.** Equality $Y' Y^* \Gamma = X' X^*$ holds true.
Proof. Lemma 10 ensures

\[ Y_i V^* \Gamma = \{X_{i,1}, X_{i,2}, X_{i,3}, \ldots, X_{i,k}\} U^* \]

for every \( i \in [1, h] \), and thus we have

\[ Y^* V^* \Gamma = X^* U^*. \quad (10) \]

Besides, equalities \( \mathcal{X} = \{U\} \cup \mathcal{X}' \) and \( \mathcal{Y} = \{V\} \cup \Gamma \mathcal{Y}' \) yield

\[ (\mathcal{X}' U^*)^+ = \mathcal{X}' \mathcal{X}^* \]

and

\[ (Y^* V^* \Gamma)^+ = Y^* Y^* \Gamma, \]

respectively. Combining the last two equalities with Equation (10), we obtain

\[ Y^* Y^* \Gamma = (Y^* V^* \Gamma)^+ = (\mathcal{X}' U^*)^+ = \mathcal{X}' \mathcal{X}^*, \]

as desired. \( \square \)

Let us now complete the proof of the theorem. Combining Lemma 11 with inclusions \( \mathcal{X}' \subseteq \mathcal{X} \) and \( \Gamma \mathcal{Y}' \subseteq \mathcal{Y} \), we get

\[ Y^* Y^* \Gamma \subseteq \mathcal{X}^+ \]

and

\[ \Gamma \mathcal{X}' \mathcal{X}^* \mathcal{Y}' \subseteq \mathcal{Y}^+. \]

It follows from the former inclusion that \( O_{k_d,k_d} \in \mathcal{Y}^+ \) implies \( O_{d,d} \in \mathcal{X}^+ \); the converse follows from the latter inclusion. Hence, \( \mathcal{X} \) is a yes-instance of \( \mathcal{M}^d \) if, and only if, \( \mathcal{Y} \) is a yes-instance of \( \mathcal{M}^{kd} \). \( \square \)

Since \( \mathcal{M}^3(k_G + 1) \) is undecidable (see Section 3), it follows from Theorem 4 that \( \mathcal{M}^{3k_G}(2) \) is undecidable. As \( k_G \leq 5 \) [12], \( \mathcal{M}^{15}(2) \) is undecidable.

**Theorem 5.** \( \mathcal{Z}^d(hk + 1) \) reduces to \( \mathcal{Z}^{kd}(h + 1) \).

Proof. By Lemma 4, we only need to reduce non-degenerated instances of \( \mathcal{Z}^d(hk + 1) \).

Let \( (L, C, \mathcal{X}) \) be a non-degenerated instance of \( \mathcal{Z}^d(hk + 1) \). Let \( \Gamma, U, V, \mathcal{Y}, \mathcal{X}', \) and \( \mathcal{Y}' \) be as in the proof of Theorem 4; additionally, let \( \Lambda \in Q^{d \times kd} \) be given by:

\[ \Lambda = (I_d \ I_d \ I_d \ \cdots \ I_d). \]

Put \( \mathcal{I} = (L \Lambda, \Gamma C, \mathcal{Y}) \). Clearly, \( \mathcal{I} \) is an instance of \( \mathcal{Z}^{kd}(h + 1) \) and \( \mathcal{I} \) is computable from \( (L, C, \mathcal{X}) \). To complete the proof of the theorem, it suffices to check that \( (L, C, \mathcal{X}) \) is a yes-instance of \( \mathcal{Z}^d \) if, and only if, \( \mathcal{I} \) is a yes-instance of \( \mathcal{Z}^{kd} \).

Equalities \( \mathcal{X} = \{U\} \cup \mathcal{X}' \) and \( \mathcal{Y} = \{V\} \cup \Gamma \mathcal{Y}' \) yield

\[ \mathcal{X}^* = U^* \cup U^* \mathcal{X}' \mathcal{X}^* \]
\[ Y^* = V^* \cup V^* \Gamma Y Y^*, \]
respectively. Moreover, Lemma 10 ensures
\[ \Lambda V^* \Gamma = U^*. \]
Hence, using also Lemma 11, we get
\[
\Lambda Y^* \Gamma = \Lambda (V^* \cup V^* \Gamma Y Y^*) \Gamma \\
= (\Lambda V^* \Gamma) \cup (\Lambda V^* \Gamma)(Y Y^*) \Gamma \\
= U^* \cup U^* \mathcal{X}' \mathcal{X}^* \\
= \mathcal{X}^*,
\]
and then
\[ L \Lambda Y^* \Gamma C = L \mathcal{X}^* C. \]
Since \( L \Lambda \Gamma C = LC \neq 0 \), we obtain
\[ 0 \in L \Lambda Y^* \Gamma C \iff 0 \in L \mathcal{X}^* C, \]
as desired. \( \square \)

Lemma 12. Let \( \mathcal{L} \) be a non-zero linear subspace of \( \mathbb{Q}^{1 \times d} \). Let \( \ell \) denote the dimension of \( \mathcal{L} \). The restriction of \( \mathcal{Z}^d(k) \) to those instances \((L, C, \mathcal{X})\) for which \( L \mathcal{X}^* \subseteq \mathcal{L} \) reduces to \( \mathcal{Z}^\ell(k) \).

Proof. First, let us check that there exist \( P \in \mathbb{Q}^{\ell \times d} \) and \( P' \in \mathbb{Q}^{d \times \ell} \) such that \( LP'P = L \) for every \( L \in \mathcal{L} \). Let \( P \in \mathbb{Q}^{\ell \times d} \) be such that the rows of \( P \) form a basis of \( \mathcal{L} \). Since the row rank of \( P \) is full, there exists \( P' \in \mathbb{Q}^{d \times \ell} \) such that \( PP' = I_\ell \) [28]. Hence, we have
\[ \mathcal{L} = \{ KP : K \in \mathbb{Q}^{1 \times \ell} \} \]
and \( KPP'P = KP \) for every \( K \in \mathbb{Q}^{1 \times \ell} \). Therefore, \( P'P \) satisfies the desired property.

We are now ready to prove that the considered restriction of \( \mathcal{Z}^d(k) \) reduces to \( \mathcal{Z}^\ell(k) \).

Let \((L, C, \mathcal{X})\) be an instance of \( \mathcal{Z}^d(k) \) such that \( L \mathcal{X}^* \subseteq \mathcal{L} \). Put
\[ \mathcal{I} = (LP', PC, P \mathcal{X} P'). \]
Clearly, \( \mathcal{I} \) is an instance of \( \mathcal{Z}^\ell(k) \) and \( \mathcal{I} \) is computable from \((L, C, \mathcal{X})\). Moreover, let \( n \in \mathbb{N} \). Since \( L \mathcal{X}^n P'P = L \mathcal{X}^n \), a straightforward induction on \( n \) yields
\[ LP' (P \mathcal{X} P')^n = L \mathcal{X}^n P', \]
and thus
\[ LP' (P \mathcal{X} P')^n PC = L \mathcal{X}^n C. \]
Therefore, \((L, C, \mathcal{X})\) is a yes-instance of \( \mathcal{Z}^d \) if, and only if, \( \mathcal{I} \) is a yes-instance of \( \mathcal{Z}^\ell \). \( \square \)
Define $\mathcal{M}_d$ as the set of those $X \in \mathbb{Q}^{d \times d}$ that satisfy the following two equivalent conditions:

1. The leftmost column of $X$ equals \( \begin{pmatrix} 1 \\ O_{d-1,1} \end{pmatrix} \).

2. For every $K \in \mathbb{Q}^{1 \times d}$, the leftmost entry of $KX$ equals the leftmost entry of $K$.

Define $\tilde{\mathcal{F}}^d(k)$ as the restriction of $\mathcal{F}^d(k)$ to those instances $(L, C, \mathcal{X})$ for which $\mathcal{X} \subseteq \mathcal{M}_d$.

**Theorem 6.** $\tilde{\mathcal{F}}^3(k_G)$ is undecidable.

*Proof.* Let us show that $\text{GPCP}_+(k)$ reduces to $\tilde{\mathcal{F}}^3(k)$ for any $k$. Let the notation be as in the proof of Theorem 1. By Equation (1), the range of $\Psi$ is a subset of $\mathcal{M}_3$. It follows that $X(w) \in \mathcal{M}_3$ for every $w \in A^*$, and thus $(L, C, \mathcal{X})$ is an instance $\tilde{\mathcal{F}}^3(k)$.

**Lemma 13.** $\tilde{\mathcal{F}}^d(k)$ reduces to its restriction to non-degenerated instances.

*Proof.* The proof is the same as that of Lemma 4.

**Theorem 7.** $\tilde{\mathcal{F}}^d(hk + 1)$ reduces to $\mathcal{F}^{1+k(d-1)}(h+1)$.

*Proof.* The proof relies on Lemma 12. For each $s \in \mathbb{Q}$, define $\mathcal{K}(s)$ as the set of those $K \in \mathbb{Q}^{1 \times kd}$ such that, for every $j \in [0, k-1]$, the $(jd + 1)$th entry of $K$ equals $s$. Let $K \in \mathbb{Q}^{1 \times kd}$ and let $K_1, K_2, K_3, \ldots, K_k \in \mathbb{Q}^{1 \times d}$ be such that

$$K = \begin{pmatrix} K_1 & K_2 & K_3 & \cdots & K_k \end{pmatrix}.$$

For every $s \in \mathbb{Q}$, $K$ belongs to $\mathcal{K}(s)$ if, and only if, the leftmost entry of $K_j$ equals $s$ for every $j \in [1, k]$. Put $\mathcal{L} = \bigcup_{s \in \mathbb{Q}} \mathcal{K}(s)$. Clearly, $\mathcal{L}$ is a linear subspace of $\mathbb{Q}^{1 \times kd}$ and the dimension of $\mathcal{L}$ equals $1 + k(d - 1)$. By Lemmas 12 and 13, it suffices to show that the restriction of $\tilde{\mathcal{F}}^d(hk + 1)$ to non-degenerated instances reduces to the restriction of $\mathcal{F}^{kd}(h + 1)$ to those instances $(L, C, \mathcal{X})$ for which $L\mathcal{X}^* \subseteq \mathcal{L}$.

Let $(L, C, \mathcal{X})$ be a non-degenerated instance of $\tilde{\mathcal{F}}^d(hk + 1)$. Let the notation be as in the proofs of Theorems 4 and 5. Let $s$ denote the leftmost entry of $L$. It is clear that

$$L\Lambda = \begin{pmatrix} L & L & L & \cdots & L \end{pmatrix} \in \mathcal{K}(s).$$

Moreover, if $K \in \mathcal{K}(s)$ then straightforward computations yield

$$KV = \begin{pmatrix} K_2 & K_3 & \cdots & K_k & K_1U \end{pmatrix} \in \mathcal{K}(s)$$

and

$$KTY_i = K_iY_i = \begin{pmatrix} K_1X_{i,1} & K_1X_{i,2} & K_1X_{i,3} & \cdots & K_1X_{i,k} \end{pmatrix} \in \mathcal{K}(s)$$

for $i \in [1, h]$. Hence, we have $L\Lambda \in \mathcal{K}(s)$ and $K\mathcal{Y} \in \mathcal{K}(s)$. It follows $L\Lambda Y^* \subseteq \mathcal{K}(s) \subseteq \mathcal{L}$, and thus $\mathcal{J}$ is an instance of the suitable restriction of $\mathcal{F}^{kd}$.

We claim that $\tilde{\mathcal{F}}^d(hk + 1)$ reduces to $\mathcal{F}^{1+k(d-1)}(h+1)$; the verification is left to the reader. As $k_G \leq 5$ [12], $\mathcal{F}^3(5)$ is undecidable by Theorem 6. It then follows from Theorem 7 that $\mathcal{F}^5(3)$ and $\mathcal{F}^9(2)$ are undecidable. Combining Theorems 6 and 7, we obtain that $\mathcal{F}^{2k_G-1}(2)$ is undecidable. Therefore, $\mathcal{R}^{2k_G}(2)$ and $\mathcal{M}^{2k_G-1}(3)$ are undecidable by Propositions 3 and 4.
5 Open questions

The cases where $d = 2$ and where $k = 1$ yield challenging open questions.

5.1 Two-by-two matrices

The undecidability of $M^3$ was first proven in 1970 [23]. It was later proven that $Z^2(1)$ and $M^2(2)$ are decidable [29, 13, 6] and that $M^2$ is NP-hard [2]. However, the decidabilities of $M^2(k+1)$, $Z^2(k)$, and $R^2(k)$ remain open for $2 \leq k \leq \infty$. The decidability of $M^2$ has been repeatedly reported as open since 1977 [26].

By Proposition 7, $Z^2(2)$ and $M^2(3)$ are equivalent. By Lemma 3, $R^2(2)$ reduces to $Z^2(2)$ and $M^2(3)$. Therefore, if there exist $d, k \in \mathbb{N}$ such that $d \geq 2$, $k \geq 2$, $(d, k) \neq (2, 2)$, and $M^d(k)$ is decidable then $M^3(2)$ or $R^2(2)$ is decidable. The decidabilities of the latter two problems remain open.

5.2 Linear recurrences

The decidability of $M^d(1)$ is easy to see: for every $X \in \mathbb{Q}^{d \times d}$, $\{X\}$ is a yes-instance of $M^d(1)$ if, and only if, $X^d$ equals the $d$-by-$d$ zero matrix. Moreover, it is known that $Z^d(1)$ is decidable [13], the proof being highly non-trivial. However, the decidabilities of $Z^d(1)$ and $R^d(1)$ remain open for $d \geq 6$. Let us briefly discuss the question.

Given a sequence $(u_n)_{n \in \mathbb{N}}$ of elements of $\mathbb{Q}$, we say that $(u_n)_{n \in \mathbb{N}}$ satisfies a linear recurrence relation (LRR) of order $d$ if the following three equivalent conditions [25] are met:

1. There exist $L \in \mathbb{Q}^{1 \times d}$, $C \in \mathbb{Q}^{d \times 1}$, and $X \in \mathbb{Q}^{d \times d}$ such that $u_n = LX^nC$ for every $n \in \mathbb{N}$.

2. There exist $a_0, a_1, \ldots, a_{d-1} \in \mathbb{Q}$ such that

$$u_{n+d} = \sum_{i=0}^{d-1} a_i u_{n+i}$$

for every $n \in \mathbb{N}$.

3. There exists two polynomials $f(x)$ and $g(x)$ over $\mathbb{Q}$ such that $g(0) \neq 0$, the degree of $f(x)$ is smaller than $d$, the degree of $g(x)$ is not greater than $d$, and the generating function of $(u_n)_{n \in \mathbb{N}}$ satisfies:

$$\sum_{n=0}^{\infty} u_n x^n = \frac{f(x)}{g(x)}.$$

Let $(u_n)_{n \in \mathbb{N}}$ be a sequence of elements of $\mathbb{Q}$ that satisfies an LRR of order $d$.

- If $u_0 \neq 0$ then there exist $X \in \mathbb{Q}^{d \times d}$ such that $u_n u_0^{-1}$ equals the $(1, 1)$th entry of $X^n$ for every $n \in \mathbb{N}$.
• If \( u_0 = 0 \) then there exists \( X \in \mathbb{Q}^{d \times d} \) such that \( u_n \) equals the \((1, d)\)th entry of \( X^n \) for every \( n \in \mathbb{N} \).

The following two problems are equivalent to \( \mathcal{Z}_d(1) \):

1. Given \( X \in \mathbb{Q}^{d \times d} \), decide whether there exists \( n \in \mathbb{N} \) such that the \((1, 1)\)th entry of \( X^n \) equals 0.

2. Given a sequence \((u_n)_{n \in \mathbb{N}}\) of elements of \( \mathbb{Q} \) that satisfies an LRR of order \( d \), decide whether there exists \( n \in \mathbb{N} \) such that \( u_n = 0 \).

The following two problems are equivalent to \( \mathcal{R}_d(1) \):

1. Given \( X \in \mathbb{Q}^{d \times d} \), decide whether there exists \( n \in \mathbb{N} \setminus \{0\} \) such that the \((1, d)\)th entry of \( X^n \) equals 0.

2. Given a sequence \((u_n)_{n \in \mathbb{N}}\) of elements of \( \mathbb{Q} \) that satisfies an LRR of order \( d \) and \( u_0 = 0 \), decide whether there exists \( n \in \mathbb{N} \setminus \{0\} \) such that \( u_n = 0 \).

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