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Abstract. We first obtain a locally uniform a priori bound of the dynamics of rational functions of degree > 1 on the Berkovich projective line over an algebraically closed and complete non-archimedean field of any characteristic, and an equidistribution result for moving targets towards the equilibrium (or canonical) measure, under the no potentially good reductions condition. This answers a question posed by Favre and Rivera-Letelier. We then establish a complex counterpart to the above a priori bound, on the dynamics of an endomorphism of degree > 1. As a special case, this yields a Diophantine-type estimate of the dynamics of f on its domaines singuliers (rotation domains).

1. Introduction

Let $K$ be an algebraically closed field of any characteristic that is complete with respect to a non-trivial and non-archimedean absolute value $| \cdot |$. The Berkovich projective line $P^1 = P^1(K)$ compactly augments $\mathbb{P}^1 = \mathbb{P}^1(K)$ (see [8]) and is canonically regarded as a tree in the sense of Jonsson [25, Definition 2.2], the weak topology of which coincides with the Gelfand topology of $P^1$. In particular, the classical projective line $\mathbb{P}^1$ is dense in $P^1$ and is in the set of all end points of $P^1$. The action on $\mathbb{P}^1$ of a rational function $f ∈ K(z)$ of degree $d > 1$ canonically extends to that on $P^1$, which is continuous, open, surjective, and discrete. The local degree function $\text{deg}(f) : P^1 \rightarrow \{1, \ldots, d\}$ also canonically extends to an upper semicontinuous function $P^1$ satisfying in particular $\sum_{S ∈ P^1} \text{deg}_S(f) = d$ for every $S ∈ P^1$, and induces the pullback action $f^*$ of $f$ on the space of all Radon measures on $P^1$. By the seminal Baker–Rumely [3], Chambert-Loir [13], and Favre–Rivera-Letelier [19], for every $f ∈ K(z)$ of degree $d > 1$, we have the equilibrium (or canonical) measure $μ_f$ of $f$ on $P^1$, which has no masses on polar subsets in $P^1$ and satisfies the $f$-balanced property $f^*μ_f = d · μ_f$ on $P^1$. Moreover, letting $δ_S$ be the Dirac measure on $P^1$ at each $S ∈ P^1$, the equidistribution

$$\lim_{n \to ∞} \frac{(f^n)^*δ_S}{d^n} = μ_f$$

holds for every $S ∈ P^1$ but the at most countable exceptional set $E(f) := \{a ∈ P^1 : \#(\bigcup_{n ∈ \mathbb{N} \cup \{0\}} f^{-n}(a)) < ∞\}$ of $f$ (if char $K = 0$, then even $\#E(f) ≤ 2$). In particular, $μ_f$ is mixing under $f$. 

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Our first aim is to contribute to a locally uniform quantitative study and an equidistribution problem on the dynamics of $f$ on $\mathbb{P}^1$. The equilibrium (or canonical) measure $\mu_f$ on $\mathbb{P}^1$ is a key tool.

1.1. An a priori bound of the dynamics of $f$. Recall (that the absolute value $|\cdot|$ is said to be non-trivial if $|K| \not\subset \{0, 1\}$ and) that the absolute value $|\cdot|$ is said to be non-archimedean if the strong triangle inequality

$$|z + w| \leq \max\{|z|, |w|\} \quad \text{for any } z, w \in K$$

holds. Then the chordal metric $[z, w]_{\mathbb{P}^1}$ on $\mathbb{P}^1$ (see (2.1) below, following the notation in Nevanlinna’s and Tsuji’s books [30, 33]) is written as

$$[z, w]_{\mathbb{P}^1} = \frac{|z - w|}{\max\{1, |z|\} \max\{1, |w|\}} \leq 1$$
on an affine chart $\cong K$ of $\mathbb{P}^1$.

Let $f \in K(z)$ be a rational function on $\mathbb{P}^1$ of degree $d > 1$. We say that $f$ has no potentially good reductions if $\deg(h \circ f \circ h^{-1}) < \deg f$ for every $h \in \text{PGL}(2, K)$, where $h \circ f \circ h^{-1} \in k(z)$ (of the degree $\in \{1, \ldots, \deg f\}$ is the reduction of the minimal expression of $h \circ f \circ h^{-1}$ modulo the maximal ideal $m_K$ of the ring $\mathcal{O}_K$ of $K$-integers (cf. Kawaguchi–Silverman [26, Definition 2]) and $k = \mathcal{O}_K/m_K$ is the residue field of $K$ (and we identify $\text{PGL}(2, K)$ with the projective transformations group on $\mathbb{P}^1$). This no potentially good reductions condition (on $f$) is equivalent to that the measure $\mu_f$ is supported by no singleton in $\mathbb{P}^1 \setminus \mathbb{P}^1$ (cf. [3 Corollary 9.27]).

Our first principal result is the following locally uniform a priori bound of the dynamics of $f$.

**Theorem 1.** Let $K$ be an algebraically closed field of any characteristic that is complete with respect to a non-trivial and non-archimedean absolute value. Then for every rational function $f \in K(z)$ on $\mathbb{P}^1$ of degree $d > 1$ having no potentially good reductions, every rational function $g \in K(z)$ on $\mathbb{P}^1$ of degree $> 0$, and every non-empty open subset $D$ in $\mathbb{P}^1$, we have

$$\lim_{n \to \infty} \sup_{w \in D} \frac{\log|f^n(w), g(w)|_{\mathbb{P}^1}}{d^n + \deg g} = 0. \quad (1.2)$$

1.2. Equidistribution towards $\mu_f$ for moving targets. For every $g \in K(z)$ of degree $> 0$ and every $n \in \mathbb{N}$, let $[f^n = g]$ be the effective divisor on $\mathbb{P}^1$ defined by the roots of the algebraic equation $f^n = g$ on $\mathbb{P}^1$, taking into account their multiplicities, which is regarded as a Radon measure on $\mathbb{P}^1$.

The following equidistribution theorem for moving targets under the no potentially good condition is an application of Theorem 1 and partly answers the question posed by Favre–Rivera-Letelier [19 après Théorème B].

**Theorem 2.** Let $K$ be an algebraically closed field of any characteristic that is complete with respect to a non-trivial and non-archimedean absolute value. Then for every $f \in K(z)$ of degree $d > 1$ having no potentially good reductions and every $g \in K(z)$ of degree $> 0$, we have

$$\lim_{n \to \infty} \frac{|f^n = g|}{d^n + \deg g} = \mu_f \quad \text{weakly on } \mathbb{P}^1. \quad (1.3)$$
In [19 Théorème B], they established (1.3) in the char $K = 0$ case (even without the no potentially good reductions assumption), and asked about the situation in the char $K > 0$ case. In Theorem 2 in the char $K > 0$ case, the no potentially good reductions assumption can be relaxed but cannot be omitted (as pointed out in [19 après Théorème B]).

1.3. Complex counterpart. From now on, pick $k \in \mathbb{N}$. In [1.3, 1.4 and 1.5] we would work over $\mathbb{C}$, so denote $\mathbb{CP}^k$ by $\mathbb{P}^k$ for simplicity.

Let $f$ be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d > 1$ (so of topological degree $d^k$). By the seminal Fornæss–Sibony [20] (see also the survey [13]), there is a weak limit $\mu_f := \lim_{n \to \infty} (f^{*} \omega_{FS})^{\wedge k}/d^{kn}$ on $\mathbb{P}^k$, where $\omega_{FS}$ is the Fubini-Study Kähler form on $\mathbb{P}^k$ normalized as $\omega_{FS}^{\wedge k}(\mathbb{P}^k) = 1$. Let us equip $\mathbb{P}^k$ with the chordal metric $[x,y]_{\mathbb{P}^k}(\leq 1)$ (see (5.1) below), which is equivalent to the spherical metric on $\mathbb{P}^k$ induced by $\omega_{FS}$. The probability measure $\mu_f$, which is called the equilibrium measure of $f$, has no masses on pluripolar subsets in $\mathbb{P}^k$, satisfies the $f$-balanced property $f^{*} \mu_f = d^k \cdot \mu_f$ on $\mathbb{P}^k$ and is supported by $J(f)$, where $J(f)$ is the (first) Julia set of $f$, i.e., the set of all points in $\mathbb{P}^k$ at each of which the family $\{f^n : n \in \mathbb{N}\}$ is not normal. Moreover, letting $\delta_x$ be the Dirac measure on $\mathbb{P}^k$ at each $x \in \mathbb{P}^k$, for every $x \in \mathbb{P}^k$ but an at most pluripolar subset $\mathcal{E}_f$ in $\mathbb{P}^k$,

$$\lim_{n \to \infty} (f^n)^{*} \delta_x/d^{kn} = \mu_f \quad \text{weakly on } \mathbb{P}^k,$$

so $\mu_f$ is mixing under $f$ (by the further investigation [11, 15, 17] on (1.4), the pluripolar $\mathcal{E}_f$ is in fact algebraic in $\mathbb{P}^k$).

Our second aim is to contribute to a locally uniform quantitative study of the dynamics of $f$, aiming at obtaining a Diophantine-type estimate on domaines singuliers (rotation domains) of the dynamics of $f$. The equilibrium measure $\mu_f$ on $\mathbb{P}^k$ is a key tool.

1.4. The mass of $\mu_f$ on the boundary of a cyclic Fatou component.

The (first) Fatou set $F(f)$ of $f$ is by definition $\mathbb{P}^k \setminus J(f)$, and each component of $F(f)$ is called a Fatou component of $f$. Both $J(f)$ and $F(f)$ are totally invariant under $f$, and $f$ maps each Fatou component properly to a Fatou component, and the preimage of a Fatou component under $f$ is the union of (at most $d^k$) Fatou components. A Fatou component $W$ of $f$ is said to be cyclic if $W$ is invariant under $f^p$ for some $p \in \mathbb{N}$ in that $f^p(W) = W$, and then we call the minimal such $p$ the exact period of $W$ under $f$.

For cyclic Fatou components $W$ of $f$, since $\mu_f$ is ergodic under $f$, we have $\mu_f(\partial W) \in \{0,1\}$. The following answers the question on when $\mu_f(\partial W) = 1$.

**Theorem 3.** Let $f$ be a holomorphic endomorphism of $\mathbb{P}^k$ of degree $> 1$.

Then for every cyclic Fatou component $W$ of $f$ having the exact period $p \in \mathbb{N}$, we have $\mu_f(\partial W) \in \{0,1\}$, and $W$ is totally invariant under $f^p$ (in that $(f^p)^{-1}(W) = W$) if and only if $\mu_f(\partial W) = 1$. Moreover, if $\mu_f(\partial W) = 0$, then for every component $U$ of $\bigcup_{n \in \mathbb{N} \cup \{0\}} f^{-n}(W)$, we have $\mu_f(\partial U) = 0$.

Following Fatou [18 Sec. 28], we call a cyclic Fatou component $W$ of $f$ having the exact period, say $p \in \mathbb{N}$, a domaine singulier (a singular domain, or a rotation domain) if $f^p : W \to W$ is biholomorphic (i.e., if le morphisme
holomorphic $f^p : W \to W$ est singulière). Theorem 3 yields $\mu_f(\partial W) = 0$ for every domaine singulier $W$ of $f$. In the dimension $k = 1$ case (then $\text{supp}\mu_f = J(f)$ and a domaine singulier of $f$ is either a Siegel disk or an Herman ring of $f$), this is equivalent to

\[
\partial W \subset J(f) \quad \text{for every domaine singulier } W \text{ of } f.
\]

This might be of independent interest. In the dimension $k = 1$ case, Theorem 3 (with Sullivan’s no wandering domain theorem, Fatou’s finiteness of the number of cyclic Fatou components, and Riemann-Hurwitz’s formula) also implies that if there are no cyclic Fatou components of $f^2$ totally invariant under $f^2$, then $J(f)$ is strictly larger than the union of the boundaries of all Fatou components of $f$. This corresponds to Abikoff [1] for Kleinian groups.

1.5. An a priori bound of the dynamics of $f$ on $\mathbb{P}^k$. The following a priori bound of $f$ is an application of Theorem 3.

**Theorem 4** (an a priori bound). Let $f$ be a holomorphic endomorphism of $\mathbb{P}^k$ of algebraic degree $d > 1$. Then for every holomorphic endomorphism $g$ of $\mathbb{P}^k$ of degree $> 0$ and every non-empty open subset $D$ in $\mathbb{P}^k$,

\[
\lim_{n \to \infty} \sup_{y \in D} \log\left[\frac{\left[f^n(y), g(y)\right]_{\mathbb{P}^k}}{d^n + \deg g}\right] = 0.
\]

The argument in the proof is similar to those in Buff–Gauthier [12] and Gauthier [22], using a domination principle (Bedford–Taylor [6]; see also Bedford–Smillie [31, Page 77]) from pluripotential theory. Theorem 4 improves [31, Theorem 1 when $K = \mathbb{C}$ and $\deg g > 0$, where “$= 0$” in (1.6) was “$> -\infty$” (but possibly $\deg g = 0$). In Theorem 4 the assumption that $\deg g > 0$ can be relaxed but cannot be omitted.

An immediate consequence of Theorem 4 for $g = \text{Id}_{\mathbb{P}^k}$ is that for every domaine singulier $W$ of $f$ having the exact period $p \in \mathbb{N}$ and every open subset $D$ in $\mathbb{P}^k$ contained in $W$,

\[
\lim_{n \to \infty} \frac{\log \sup_{y \in D} |f^{pn}(y), y|_{\mathbb{P}^k}}{d^{pn} + 1} = 0.
\]

This Diophantine-type estimate (1.7) of $f^p$ on a domaine singulier $W$ has been known in [31, Theorem 3] under the additional assumption that $W$ is of maximal type in that, setting $q := \min\{j \in p\mathbb{N} : f^j|W \in G_0\}$, where $G_0$ is the component of the closed subgroup generated by $f^p|W$ in the biholomorphic automorphisms group $\text{Aut}(W)$ containing $\text{Id}_W$, there exists a Lie groups isomorphism $G_0 \to \mathbb{T}^k$ that maps $f^0|W$ to $(e^{2\pi i \alpha_1}, \ldots, e^{2\pi i \alpha_k})$ for some $\alpha_1, \ldots, \alpha_k \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$; in general, $G_0$ at least contains a torus $\mathbb{T}^j$ for some $j \in \{1, \ldots, k\}$ (Ueda [37]). This is the reason why a domaine singulier is also called a rotation domain. However, it has been unclear whether $\mathbb{T}^j$ is geometric, except for the maximal case $j = k$ (Barrett–Bedford–Dadok [4], see also Mihaiescu [29] in the case $k = 2$ and $j = 1$). Under the above maximal type assumption, also in terms of the notation there, the estimate (1.7) has been illustrated as

\[
\lim_{n \to \infty} \frac{\log \max_{j \in \{1, \ldots, k\}} |e^{nh^{2\pi i \alpha_j}} - 1|}{d^{pn} + 1} = 0
\]

(cf. Cremer [14, p157] in the case of $k = 1$).
1.6. Organization of the article. In Sections 2 and 3 we show Theorems 1 and 2 respectively. In Section 4 we show Theorem 3. In Section 5 we show Theorem 4 and conclude with a few remarks including a Nevanlinna-theoretic reformulation (1.6′ below) of Theorem 4.

2. Proof of Theorem 1

Let $K$ be an algebraically closed field of any characteristic that is complete with respect to a non-trivial and non-archimedean absolute value $| \cdot |$.

For the potential theory on $P^1 = P^1(K)$ including the fully general study of harmonic analysis on $P^1$, i.e., subharmonic functions on open subsets in $P^1$, see Baker–Rumely and Thuillier, and also the study of the class of “$\delta_{\text{can}}$”-subharmonic functions” $g : P^1 \to \mathbb{R}_{\leq 0} \cup \{-\infty\}$ such that $\Delta g + \delta_{\text{can}}$ are probability Radon measures on $P^1$, see Favre–Rivera-Letelier and Thuillier. Here $\mathcal{S}_{\text{can}} \in P^1 \setminus P^1$ ($P^1 = P^1(K)$) is the Gauss (or canonical) point in $P^1$ and $\Delta = \Delta_0$ is the Laplacian on $P^1$, and in subsection 2.2 the opposite sign convention on $\Delta$ is adopted. In the following, $\delta_{\text{can}}$ plays a role similar to that of the (spherical area element induced by) $\omega_F$ on $\mathbb{CP}^1$.

Let $f \in K(z)$ of degree $d > 1$. Recall that for every $\mathcal{S} \in P^1$, $f^* \delta_{\mathcal{S}} = \sum_{\nu \in f^{-1}(\mathcal{S})} (\deg_{\mathcal{S}}(f)) \delta_{\nu}$ on $P^1$, and that for every Radon measure $\nu$ on $P^1$, $f^* \nu = \int_{P^1} (f^* \delta_{\mathcal{S}}) d\nu(S)$ on $P^1$.

A lift of $f$ is a pair $F = (F_0, F_1) \in (K[z_0, z_1] \cdot)^2$ of homogeneous polynomials of degree $d$ in the indeterminants $z_0, z_1$, which is unique up to multiplication in $K^*$, such that $\pi \circ F = f \circ \pi$ on $K^2 \setminus \{0\}$ (and that $F^{-1}(0) = \{0\}$). Here the 0 of the $K$-linear space $K^2$ is the origin $(0, 0)$ of $K^2$, and we let $\pi : K^2 \setminus \{0\} \to \mathbb{P}^1$ be the canonical projection.

Let $\|(z_0, z_1)\| = \max\{|z_0|, |z_1|\}$ be the maximal norm on $K^2$. Noting that $K^2 \cong K^2$ and the chordal metric $[z, w] \in \mathbb{P}^1$ is defined as

\[(z, w)[P^1] := |Z \wedge W|/(||Z|| \cdot ||W||) \leq 1, \quad z, w \in \mathbb{P}^1,\]

where $Z \in \pi^{-1}(z)$ and $W \in \pi^{-1}(w)$. The function $-\log \max\{|1, | \cdot |\} = \log[1, \infty]$ on each affine chart $\cong K$ of $P^1$ extends continuously to $P^1 \setminus \{\infty\}$ (writing as $P^1 = K \cup \{\infty\}$) and moreover, extends to a function $P^1 \to \mathbb{R}_{\leq 0} \cup \{-\infty\}$ such that this extension (we still write it as $-\log \max\{|1, | \cdot |\}$ for simplicity) satisfies

\[\Delta(-\log \max\{|1, | \cdot |\}) = \delta_{\infty} - \delta_{\text{can}} \quad \text{on } P^1.\]

Fix a lift $F$ of $f$. Then for every $n \in \mathbb{N}$, $F^n$ is a lift of $f^n$ and $\deg(f^n) = d^n$, and the function

\[T_{F^n} := \log ||F^n|| - d^n \cdot \log || \cdot ||\]

on $K^2 \setminus \{0\}$ descends to $\mathbb{P}^1$ and in turn extends continuously to $P^1$, satisfying

\[\Delta T_{F^n} = (f^n)^* \delta_{\text{can}} - d^n \cdot \delta_{\text{can}} \quad \text{on } P^1\]

(see, e.g., [22 Definition 2.8]). Moreover, there is the uniform limit

\[g_F := \lim_{n \to \infty} \frac{T_{F^n}}{d^n} \quad \text{on } P^1,\]
which is continuous on $\mathbb{P}^1$ and in fact satisfies
\[
\Delta g_F = \mu_f - \delta_{\text{can}} \quad \text{on } \mathbb{P}^1
\]
(see [31 §10], [19 §6.1]).

For every $g \in K(z)$ of degree $> 0$ and every $n \in \mathbb{N}$, the function $z \mapsto [f^n(z), g(z)]_{\mathbb{P}^1}$ on $\mathbb{P}^1$ extends continuously to a function
\[
S \mapsto [f^n, g]_{\text{can}}(S) : \mathbb{P}^1 \to [0, 1],
\]
which does not necessarily coincide with the evaluation $S \mapsto [S', S'']_{\text{can}}$, at $(S', S'') = (f^n(S), g(S)) \in (\mathbb{P}^1)^2$, on $\mathbb{P}^1$ but satisfies
\[
\Delta \log[f^n, g]_{\text{can}} = [f^n = g] - (f^n)^*\delta_{\text{can}} - g^*\delta_{\text{can}} \quad \text{on } \mathbb{P}^1
\]
([32 Proposition 2.9 and Remark 2.10]), recalling that $[f^n = g]$ is the effective divisor on $\mathbb{P}^1$ defined by the roots of the algebraic equation $f^n = g$ on $\mathbb{P}^1$, taking into account their multiplicities, and is regarded as a Radon measure on $\mathbb{P}^1$.

For the non-archimedean dynamics from the Fatou-Julia strategy, we refer to [33 4]. The Berkovich Julia set $J(f)$ of $f$ is $\text{defined by supp } \mu_f$, and the Berkovich Fatou set $F(f)$ of $f$ is by $\mathbb{P}^1 \setminus J(f)$. Then (the classical Fatou set) $F(f) \cap \mathbb{P}^1$ coincides with the region of equicontinuity of the family $\{f^n : (\mathbb{P}^1, [z, w]_{\mathbb{P}^1}) \to (\mathbb{P}^1, [z, w]_{\mathbb{P}^1}) : n \in \mathbb{N}\}$. A Berkovich Fatou component $W$ of $f$ is a component of $F(f)$. Both $J(f)$ and $F(f)$ are totally invariant under $f$, $f$ maps a Berkovich Fatou component properly to a Berkovich Fatou component, and the preimage of a Berkovich Fatou component under $f$ is the union of (at most $d$) Berkovich Fatou components. A Berkovich Fatou component $W$ is said to be cyclic if $f^p(W) = W$ for some $p \in \mathbb{N}$, and then the minimal such $p \in \mathbb{N}$ is called the exact period of $W$ under $f$.

A cyclic Berkovich Fatou component $W$ of $f$ having the exact period $p$ is called a Berkovich $\text{domaine singulier}$ of $f$ if $f^p : W \to W$ is injective.

The former half in the following is a non-archimedean counterpart of ([15]), and follows from an observation of the Gelfand topology of $\mathbb{P}^1$.

**Lemma 2.1.** If $f$ has no potentially good reductions, then for any Berkovich $\text{domaine singulier}$ $W$ of $f$, we have $\partial W \subseteq J(f)$. Moreover, for every component $U$ of $\bigcup_{n \in \mathbb{N} \cup \{0\}} f^{-n}(W)$, we have $\mu_f(\partial U) = 0$.

**Proof.** Let us see the former half. Let $W$ be a $\text{domaine singulier}$ of $f$ having the exact period $p \in \mathbb{N}$, and suppose that $\partial W = J(f)$. By deg $f > 1$ and the injectivity of $f^p : W \to W$, there is a component $U$ of $f^{-1}(W) \setminus W$, and then $\partial U \setminus W \subset J(f) \setminus \partial W = \emptyset$. Then $\partial W$ is not only a singleton but also in $\mathbb{P}^1 \setminus \mathbb{P}^1$, so that $\mu_f$ is supported by a singleton in $\mathbb{P}^1 \setminus \mathbb{P}^1$, and we are done.

Let us see the latter half. Suppose that $f$ has no potentially good reductions, and fix a $\text{domaine singulier}$ $W$ of $f$. By the former half and the ergodicity of $\mu_f$ under $f$, we have $\mu_f(\partial W) = 0$ and, in turn, for every $n \in \mathbb{N}$ and every component $U$ of $f^{-n}(W)$, by $f^* \mu_f = d \cdot \mu_f$ on $\mathbb{P}^1$, we compute as $0 \leq d^n \cdot \mu_f(\partial U) = ((f^n)^* \mu_f)(\partial U) = \int_{\mathbb{P}^1} ((f^n)^* \delta_v)(\partial U) d\mu_f(v) = \int_{\partial U} (f^n)^* \delta_v(\partial U) d\mu_f(v) \leq d^n \cdot \mu_f(\partial W) = 0$, so that $\mu_f(\partial U) = 0$ (since $d > 1 > 0$).

Suppose now that there are $g \in K(z)$ of degree $> 0$ and a non-empty open subset $D$ in $\mathbb{P}^1$ such that ([12]) does not hold, or equivalently, replacing
D with some component of the minimal open subset in \( \mathbb{P}^1 \) containing the original \( D \) as the dense subset, there is a sequence \( (n_j) \) in \( \mathbb{N} \) tending to \( \infty \) as \( j \to \infty \) such that

\[
\lim_{j \to \infty} \frac{\sup_{S \in D} \log|f^{n_j}|}{d^{n_j} + \deg g} < 0.
\]

Then \( D \subset F(f) \) (since \( F(f) \cap \mathbb{P}^1 \) is the region of equicontinuity of \( \{f^n : n \in \mathbb{N}\} \), and let \( U \) be the Berkovich Fatou component of \( f \) containing \( D \). Then since \( \deg g > 0 \), we have \( \lim_{j \to \infty} f^{n_j+1-n_j} = \text{Id}_{g(U) \cap \mathbb{P}^1} \) locally uniformly on \( g(U) \cap \mathbb{P}^1 \), and there exists \( N \in \mathbb{N} \) such that \( V := f^{n_N}(U) \cup g(U) \) is a cyclic Berkovich Fatou component of \( f \). By Rivera-Letelier’s counterpart of Fatou’s classification of cyclic (Berkovich) Fatou components ([19, Proposition 2.16] and its esquisse de démonstration, see also [4, Remark 7.10]), this \( V \) is a Berkovich domaine singular of \( f \).

We have not only the uniform bound \( \sup_{j \in \mathbb{N}} \sup_{s \in D} \log|f^{n_j}| \leq 0 \) from above but also, since \( V \neq \mathbb{P}^1 \), the bound \( \lim_{j \to \infty} \sup_{j \in \mathbb{N}} \sup_{s \in D} (\log|f^{n_j}|)/d^{n_j} \geq 0 > -\infty \) from below. Hence by (2.2), (2.3), (2.5), and a version of Hartogs’s lemma (cf. [19, Proposition 2.18], [3, Proposition 8.57]), taking a subsequence if necessary, there exists a function \( \phi : \mathbb{P}^1 \to \mathbb{R}_{\leq 0} \) such that the (pointwise, which corresponds to \( L^1_{\text{loc}} \) in the \( \mathbb{C} \) case) convergence

\[
\phi = \lim_{j \to \infty} \frac{\log|f^{n_j}|}{d^{n_j} + \deg g}
\]

holds and that \( \Delta(\phi + gF) + \delta_{S_{\text{can}}} \) is a probability Radon measure on \( \mathbb{P}^1 \) (cf. [19, §3.4]). Since

\[
\Delta(\phi + gF + \log \max\{1, |\cdot|\}) + \delta_{\infty} = \Delta(\phi + gF) + \delta_{S_{\text{can}}} \geq 0 \quad \text{on} \quad \mathbb{P}^1
\]

for each affine chart \( \cong K \) (and writing as \( \mathbb{P}^1 = K \cup \{\infty\} \), the function \( \phi \) is upper semicontinuous on \( \mathbb{P}^1 \), and the restriction \( \phi|U \) is subharmonic (by letting in addition \( \infty \in f^{-1}(U) \setminus U \), so that \( U \in \mathbb{P}^1 \setminus \{\infty\} \)). By (2.3) and (2.6), the open subset \( \{\phi < 0\} \) in \( \mathbb{P}^1 \) contains \( D \setminus \mathbb{P}^1 \neq \emptyset \), and in turn by the maximum principle applied to \( \phi|U \) (and \( \phi \leq 0 \)), we must have \( U \subset \{\phi < 0\} \).

On the other hand, by (2.1), the upper semicontinuity of \( \phi \), a version of Hartogs lemma already recalled in the above, and (1.1), we have \( \phi \equiv 0 \) on \( J(f) \) (so on \( \partial U \)). Define the function

\[
\psi := \begin{cases} \phi & \text{on} \ U \\ 0 & \text{on} \ \mathbb{P}^1 \setminus U \end{cases} \quad : \mathbb{P}^1 \to \mathbb{R}_{\leq 0} \cup \{-\infty\}.
\]

Pick such an affine chart \( \cong K \) of \( \mathbb{P}^1 \) that \( \infty \in f^{-1}(U) \setminus U \), writing as \( \mathbb{P}^1 = K \cup \{\infty\} \), so that \( U \in \mathbb{P}^1 \setminus \{\infty\} \). Then the function \( \psi + gF + \log \max\{1, |\cdot|\} \) is not only upper semicontinuous on \( \mathbb{P}^1 \setminus \{\infty\} \) (since \( \phi \leq 0 \)) but also subharmonic on \( U \) and \( \mathbb{P}^1 \setminus (\overline{U} \cup \{\infty\}) \), and is indeed subharmonic (or equivalently, domination subharmonic [3, §8.2, §7.3]) on \( \mathbb{P}^1 \setminus \{\infty\} \) (since so is \( \phi + gF + \log \max\{1, |\cdot|\} \), and \( \phi \leq 0 \)). In particular, we have the probability Radon measure

\[
\Delta(\psi + gF) + \delta_{S_{\text{can}}} = \Delta(\psi + gF + \log \max\{1, |\cdot|\} ) + \delta_{\infty} \quad \text{on} \quad \mathbb{P}^1.
\]
Now suppose to the contrary that $f$ has no potentially good reductions. We claim that $\Delta(\psi + g_F) + \delta_{S_{\text{can}}} = \mu_f$ on $\mathbb{P}^1$; for, under the no potentially good reductions assumption, by Lemma 2.1 we have $\mu_f(\partial U) = 0$. The definition of $\psi$ with (2.4) yields $\Delta(\psi + g_F) + \delta_{S_{\text{can}}} = \Delta g_F + \delta_{S_{\text{can}}} = \mu_f$ on $\mathbb{P}^1 \setminus \overline{U}$ and, moreover, by $\text{supp} \mu_f = \{J(f)\}$ and the vanishing $\mu_f(\partial U) = 0$, we compute as

\[
(\Delta(\psi + g_F) + \delta_{S_{\text{can}}})(\overline{U}) = 1 - (\Delta(\psi + g_F) + \delta_{S_{\text{can}}})(\mathbb{P}^1 \setminus \overline{U}) \\
= 1 - \mu_f(\mathbb{P}^1 \setminus \overline{U}) = \mu_f(\overline{U}) = \mu_f(U) + \mu_f(\partial U) = 0.
\]

Hence the claim holds. Once this claim is at our disposal, also by (2.4), we must have $\phi \equiv 0$ on $U \setminus \mathbb{P}^1 \neq \emptyset$. This contradicts $U \setminus \mathbb{P}^1 \subset \{\phi < 0\} \setminus \mathbb{P}^1$.

3. Proof of Theorem 2

Let $K$ be an algebraically closed field of any characteristic that is complete with respect to a non-trivial and non-archimedean absolute value $| \cdot |$. Let $f \in K(z)$ of degree $d > 1$ and $g \in K(z)$ of degree $> 0$, and fix a lift of $f$. We continue the discussion and to use the notation in Section 2. By (2.2), (2.3), (2.4), and (2.5), the equidistribution (1.3) follows from the (pointwise, which corresponds to $L^1_{\text{loc}}$ in the $\mathbb{C}$ case) convergence

\[
\lim_{n \to \infty} \frac{\log [f^n, g]_{\text{can}}}{d^n + \deg a} = 0 \quad \text{on } \mathbb{P}^1 \setminus \mathbb{P}^1.
\]

Unless (1.3) holds, by an argument similar to that in the previous section involving a version of Hartogs’s lemma (cf. [19] Proposition 2.18], [3] Proposition 8.57], there exist a sequence $(n_j)$ in $\mathbb{N}$ tending to $\infty$ as $j \to \infty$ and a function $\phi : \mathbb{P}^1 \to \mathbb{R}_{\geq 0} \cup \{-\infty\}$ such that the (pointwise, which corresponds to $L^1_{\text{loc}}$ in the $\mathbb{C}$ case) convergence

\[
\phi := \lim_{j \to \infty} \frac{\log [f^{n_j}, g]_{\text{can}}}{d^{n_j} + \deg g} \quad \text{on } \mathbb{P}^1 \setminus \mathbb{P}^1
\]

holds and that $\Delta(\phi + g_F) + \delta_{S_{\text{can}}}$ is a probability Radon measure on $\mathbb{P}^1$, that $\phi$ is upper semicontinuous on $\mathbb{P}^1$, and moreover that $\{\phi < 0\} \neq \emptyset$. Then fixing a non-empty open subset $D \subset \{\phi < 0\}$, we have $\sup_D \phi < 0$, and in turn by a version of Hartogs lemma already recalled, we have

\[
\limsup_{j \to \infty} \sup_{S \in D} \frac{\log [f^{n_j}, g]_{\text{can}}(S)}{d^{n_j} + \deg g} \leq \sup_D \phi < 0.
\]

This is impossible if $f$ has no potentially good reductions, by Theorem 1.

Remark 3.1. The difference between the proof of Theorem 2 and Favre–Rivera-Letelier’s one of [19] Théorème B (in the char $K = 0$ case but even without the no potentially good reductions assumption) is that in the char $K > 0$ case, no (geometric) structure theorems on a (subset of a) domaine singulier (appearing as $V$ in the proof of Theorem 1) have been known, like in the complex dimension $> 1$ case. In [19] §3.4. Démonstration du Théorème B), a structure theorem on quasiperiodicity domains (appearing as $g(U)$ in the proof of Theorem 1) was substantial.
4. Proof of Theorem \[3\]

Let \( f \) be a holomorphic endomorphism of \( \mathbb{P}^k = \mathbb{C} \mathbb{P}^k \) of algebraic degree \( d > 1 \). The critical set \( C(f) \) of \( f \) is the set of all points \( p \in \mathbb{P}^k \) at each of which \( f \) is not locally biholomorphic. Then not only \( C(f) \) but also \( f(C(f)) \) are proper algebraic subsets, so pluripolar, in \( \mathbb{P}^k \). Recall that for every \( x \in \mathbb{P}^k \), \( f^* \delta_x = \sum_{y \in f^{-1}(x)} (\deg f_y) \delta_y \) on \( \mathbb{P}^k \), where for each \( y \in \mathbb{P}^k \), \( \deg f_y \in \{1, \ldots, d^k \} \) is the local degree of \( f \) at \( y \), and that for every Radon measure \( \nu \) on \( \mathbb{P}^k \),

\[
(f^* \nu)(\partial U) \leq (\deg f : U \to V) \cdot \nu(\partial V).
\]

**Proof.** For every \( v \in (\partial V) \setminus f(C(f)) \), there is an open neighborhood \( D \) of \( v \) in \( \mathbb{P}^k \) such that each component of \( f^{-1}(D) \) is mapped by \( f \) biholomorphically onto \( D \). Then fixing a point \( v' \in D \cap V \), we have

\[
(f^* \delta_v)(\partial U) \leq \# \{ \text{components of } f^{-1}(D) \text{ intersecting } \partial U \} \\
\leq (f^* \delta_{v'}) f^{-1}(D) \cap U \leq \deg f : U \to V \\
= (\deg f : U \to V) \cdot \delta_v(\partial V \setminus f(C(f))).
\]

Hence for every positive Radon measure \( \nu \) on \( \mathbb{P}^1 \) having no mass on \( f(C(f)) \), we compute as

\[
(f^* \nu)(\partial U) = \int_{\mathbb{P}^k} (f^* \delta_v)(\partial U) \nu(v) = \int_{(\partial V) \setminus f(C(f))} (f^* \delta_{v'})(\partial U) \nu(v) \\
\leq (\deg f : U \to V) \cdot \nu(\partial V \setminus f(C(f))) = (\deg f : U \to V) \cdot \nu(\partial V),
\]

which completes the proof. \( \square \)

Pick a cyclic Fatou component \( W \) of \( f \) having the exact period \( p \in \mathbb{N} \). Since \( \mu_f \) has no masses on pluripolar subsets in \( \mathbb{P}^k \), by the balanced property of \( \mu_f \) under \( f \) and Lemma \[1.1\] we have

\[
d^{kp} : \mu_f(\partial W) = ((f^p)^* \mu_f)(\partial W) \leq (\deg(f^p : W \to W)) \cdot \mu_f(\partial W).
\]

Hence if \( \mu_f(\partial W) > 0 \), then \( \deg(f^p : W \to W) = d^{kp} \), i.e., \( (f^p)^{-1}(W) = W \); and then \( (f^p)^{-1}(\partial W) \subset \partial W \) and \( \partial W \not\subset \mathcal{E}_f \) (since \( \mu_f(\mathcal{E}_f) = 0 \)), so by \[1.3\], \( \mu_f(= \mu_{f^p}) \) is supported by \( \partial W \), i.e., \( \mu_f(\partial W) = 1 \). Conversely, if \( (f^p)^{-1}(W) = W \), then we have \( \partial W \not\subset \mathcal{E}_f \) since \( \mathbb{P}^k \setminus \mathcal{E}_f \) is connected, \( J(f) := \text{supp } \mu_f \), and \( \mu_f(\mathcal{E}_f) = 0 \), and then we have \( \mu_f(\partial W) = 1 \) as already seen. Hence the former half of Theorem \[3\] holds.

The latter half can be seen by a computation similar to that in the proof of the latter half of Lemma \[2.1\]. \( \square \)
5. Proof of Theorem 4

Let $f$ be a holomorphic endomorphism of $\mathbb{P}^k = \mathbb{C}P^k$ of algebraic degree $d > 1$.

A lift of $f$ is a $(k+1)$-tuple $F = (F_0, F_1, \ldots, F_k) \in (\mathbb{C}[z_0, z_1, \ldots, z_k])^{k+1}$ of homogeneous polynomials of degree $d$ in the $k+1$ indeterminates $z_0, \ldots, z_{k+1}$, which is unique up to multiplication in $\mathbb{C}^*$, such that $\pi \circ F = f \circ \pi$ on $\mathbb{C}^{k+1} \setminus \{0\}$ and if in addition $F$ is normal for some sequence $(d_n)$ of homogeneous polynomials of degree $C$ which is unique up to multiplication in $\mathbb{C}^*$, then $\pi \circ F = f \circ \pi$ on $\mathbb{C}^{k+1} \setminus \{0\}$, and if in addition $F$ is normal for some sequence $(d_n)$ of homogeneous polynomials of degree $C$ which is unique up to multiplication in $\mathbb{C}^*$, then $\pi \circ F = f \circ \pi$ on $\mathbb{C}^{k+1} \setminus \{0\}$. By Ueda [36, Theorem 2.2], the family $(5.2)$ is normal in $\mathbb{C}^{k+1} \setminus \{0\}$, and we let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ be the canonical projection.

We say a function $H : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{R} \cup \{-\infty\}$ satisfies the log-homogeneity (of order 1). By Ueda [36, Theorem 2.2], the family $(5.2)$ is normal in $\mathbb{C}^{k+1} \setminus \{0\}$, and we let $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ be the canonical projection.

Let $|| \cdot ||$ be the Euclidean norm on $\mathbb{C}^{k+1}$. Then the function log $|| \cdot ||$ on $\mathbb{C}^{k+1} \setminus \{0\}$ is continuous and plurisubharmonic and satisfies the log-homogeneity (of order 1). The complex Laplacian $dd^\ast$ is normalized as usual, so in particular that $\pi^* \omega_{\mathbb{C}^*} = dd^\ast \log || \cdot ||$ on $\mathbb{C}^{k+1} \setminus \{0\}$. Set $\ell(k) := \binom{k+1}{2} \in \mathbb{N}$, and if in addition $H$ is plurisubharmonic on $\mathbb{C}^{k+1} \setminus \{0\}$, also say $H$ is a $\omega_{\mathbb{C}^*}$-plurisubharmonic function on $\mathbb{P}^k$.

Let $|| \cdot ||$ be the Euclidean norm on $\mathbb{C}^{k+1}$. Then the function log $|| \cdot ||$ on $\mathbb{C}^{k+1} \setminus \{0\}$ is continuous and plurisubharmonic and satisfies the log-homogeneity (of order 1). The complex Laplacian $dd^\ast$ is normalized as usual, so in particular that $\pi^* \omega_{\mathbb{C}^*} = dd^\ast \log || \cdot ||$ on $\mathbb{C}^{k+1} \setminus \{0\}$. Set $\ell(k) := \binom{k+1}{2} \in \mathbb{N}$, and if in addition $H$ is plurisubharmonic on $\mathbb{C}^{k+1} \setminus \{0\}$, also say $H$ is a $\omega_{\mathbb{C}^*}$-plurisubharmonic function on $\mathbb{P}^k$.

The chordal metric on $\mathbb{P}^k$ is

$$ (5.1) \quad [x, y]_{\pi k} := ||Z \wedge W||/(||Z|| \cdot ||W||) \leq 1, \quad x, y \in \mathbb{P}^k, $$

where $Z \in \pi^{-1}(x)$ and $W \in \pi^{-1}(y)$, and is equivalent to the spherical metric on $\mathbb{P}^k$ induced by $\omega_{\mathbb{C}^*}$.

Fix a lift $F$ of $f$. Then there exists the uniform limit

$$ (5.2) \quad G^F := \lim_{n \to \infty} \frac{1}{d^n} \log ||F^n|| : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{R} $$

which is continuous (so locally bounded) and plurisubharmonic and satisfies the log-homogeneity (of order 1). By Ueda [36, Theorem 2.2], the region of plurisubharmonicity of $G^F$ in $\mathbb{C}^{k+1} \setminus \{0\}$ coincides not only with $\pi^{-1}(F(f))$ but also with $\pi^{-1}(\hat{F}(f))$, where $\hat{F}(f)$ is the set of all points in $\mathbb{P}^k$ at each of which the family $\{f^n_j : (\mathbb{P}^k, [x, y]_{\mathbb{P}^k}) \to (\mathbb{P}^k, [x, y]_{\mathbb{P}^k}) : j \in \mathbb{N}\}$ is normal for some sequence $(n_j)$ in $\mathbb{N}$ tending to $\infty$ as $j \to \infty$. Let us start the proof of Theorem 4. Suppose to the contrary that there are a holomorphic endomorphism $g$ of $\mathbb{P}^k$ of degree $> 0$ and a domain $D$ in $\mathbb{P}^k$ such that $\mathbb{P}^k$ does not hold, and then there is a sequence $(n_j)$ in $\mathbb{N}$ tending to $\infty$ as $j \to \infty$ such that

$$ (5.3) \quad \lim_{j \to \infty} \sup_{g \in D} \log |f^{n_j}(y) - g(y)|_{\mathbb{P}^k} < 0. $$

Then $D \subset F(f)$, and let $U$ be the Fatou component of $f$ containing $D$. Then since $\deg g > 0$, we have $\lim_{j \to \infty} f^{n_j+1-n_j} = \text{Id}_{\mathbb{P}^k}$ locally uniformly on $g(D)$, and then there exists $N \in \mathbb{N} \cup \{0\}$ such that $V := f^N(U) = g(U)$ is a cyclic Fatou component of $f$ having the exact period, say, $p \in \mathbb{N}$ and satisfies $\deg(g^p : V \to V) = 1 < d^p$. Hence by Theorem 4 we have $\mu_f(\partial U) = 0$.

Now fix a lift $G$ of $g$. By $(5.1)$ and $(5.2)$, the family $\{\log |F^{n_j} \wedge G||/d^{n_j} : j \in \mathbb{N}\}$ is locally uniformly bounded from above on $\mathbb{C}^{k+1} \setminus \{0\}$. By $(5.1)$, $(5.2)$, and $J(f) \neq \emptyset$, we also have the bound $\lim \sup_{j \to \infty} \sup_{G \in \mathbb{C}^{k+1} \setminus \{0\}} (\log |F^{n_j} \wedge$
$G)/d^n\psi \geq 0 > -\infty$ from below. Hence by a version of Hartogs lemma for a sequence of plurisubharmonic functions (see [23, Theorem 4.1.9(a)] or [2, Theorem 1.1.1]), taking a subsequence if necessary, the plurisubharmonic function
\[
\phi := \lim_{j \to \infty} \frac{\log \| F^{n_j} \wedge G \|}{d^n \psi + \deg g} (\leq G^F) \quad \text{in } L^1_{\text{loc}}(\mathbb{C}^{k+1}, m_k)
\]
exists, where $m_k$ is the Lebesgue measure on $\mathbb{C}^{k+1} \cong \mathbb{R}^{2(k+1)}$. The function $\phi$ also satisfies the log-homogeneity (of order 1). By the log-homogeneity of both $G^F$ and $\phi$, the function $\phi - G^F$ on $\mathbb{C}^{k+1} \setminus \{0\}$ descends to a function $\mathbb{F}^k \to \mathbb{R}_{\leq 0} \cup \{-\infty\}$, which is upper semicontinuous on $\mathbb{F}^k$ and is plurisubharmonic on $U$. Then by (5.2) and (5.3), the open subset $\{\phi - G^F < 0\}$ in $\mathbb{F}^k$ contains $D$, and in turn by the maximum principle applied to the plurisubharmonic function $(\phi - G^F)/U$ (and $\phi - G^F \leq 0$), we must have
\[
U \subset \{\phi - G^F < 0\}.
\]

On the other hand, by (5.1), the upper semicontinuity of $\phi - G^F$, and a version of Hartogs lemma for a sequence of plurisubharmonic functions (see [23, Theorem 4.1.9(b)]), we also have $\phi - G^F \equiv 0$ on $J(f)$, so that $\phi = G^F$ on $\pi^{-1}(J(f))$. Let us define the locally bounded function
\[
\psi := \begin{cases} 
\max\{\phi, G^F - 1\} & \text{on } \pi^{-1}(U) \\
G^F & \text{on } (\mathbb{C}^{k+1} \setminus \{0\}) \setminus \pi^{-1}(U)
\end{cases}
\]
on $\mathbb{C}^{k+1} \setminus \{0\}$, which is still plurisubharmonic on $\mathbb{C}^{k+1} \setminus \{0\}$; for, it is upper semicontinuous on $\mathbb{C}^{k+1} \setminus \{0\}$ (since so is $G^F$) and plurisubharmonic on $(\mathbb{C}^{k+1} \setminus \{0\}) \setminus \pi^{-1}(\partial U)$, and satisfies the mean value inequality at each point in $\pi^{-1}(\partial U)$ on each complex line passing through it (since so does $\phi$, and $\phi \leq G^F$). The function $\psi$ also satisfies the log-homogeneity (of order 1). By the log-homogeneity of both $G^F$ and $\psi$, the function $\psi - G^F$ also descends to a function on $\mathbb{F}^k$.

Following the manner in [9, §2.1] for $\omega_{FS}$-plurisubharmonic functions on $\mathbb{F}^k$, let us also denote by $d^c\psi$ (resp. $d^c G^F$) the current on $\mathbb{F}^k$ whose pull-back under $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{F}^k$ coincides (the genuine) $d^c\psi$ (resp. the genuine $d^c G^F$) on $\mathbb{C}^{k+1} \setminus \{0\}$. Then the $k$-th Bedford–Taylor wedge products $(d^c\psi)^{\wedge k}$ and $(d^c G^F)^{\wedge k}$ on $\mathbb{F}^k$ are probability measures on $\mathbb{F}^k$.

From the definition of $G^F$, the latter probability measure $(d^c G^F)^{\wedge k}$ on $\mathbb{F}^k$ is nothing but the equilibrium measure $\mu_f$ of $f$. We claim that the former $(d^c\psi)^{\wedge k}$ also coincides with $\mu_f$: for, the definition of $\psi$ yields $(d^c\psi)^{\wedge k} = (d^c G^F)^{\wedge k} = \mu_f$ on $\mathbb{F}^k \setminus \overline{U}$ and, moreover, by supp $\mu_f \subset J(f)$ and the vanishing $\mu_f(\partial U) = 0$, we compute as $(d^c\psi)^{\wedge k}(\overline{U}) = 1 - (d^c\psi)^{\wedge k}(\mathbb{F}^k \setminus \overline{U}) = 1 - \mu_f(\mathbb{F}^k \setminus \overline{U}) = \mu_f(U) + \mu_f(\partial U) = 0$. Hence the claim holds.

Once this claim is at our disposal, we have $\psi - G^F \geq 0$ (indeed $= 0$) on $(d^c\psi)^{\wedge k}$-almost everywhere $\mathbb{F}^k$, and then by a classical domination principle ([10, Corollary 2.5]; for a summary on the properties of plurisubharmonic weights on big line bundles over complex compact manifolds, which applies to $\omega_{FS}$-plurisubharmonic functions on $\mathbb{F}^k$, see [9, §2]), we have $\psi \geq G^F$ on $\mathbb{C}^{k} \setminus \{0\}$, so in particular $G^F \leq \psi = \phi$ on $\pi^{-1}(U)$. This contradicts $U \subset \{\phi - G^F < 0\}$. \[\square\]
Remark 5.1. From (1.6), an argument similar to that in the proof of Theorem 2 shows that \( \lim_{n \to \infty} (\log \| F_n \land G \|)/(d^n + \deg g) = G^F \) in \( L^1_{loc}(\mathbb{C}^{k+1}, m_k) \), or equivalently, that the vanishing of the Valiron deficiency
\[
(1.6) \quad \limsup_{n \to \infty} \frac{1}{d^n + \deg g} \int_{\mathbb{P}^k} \log \frac{1}{\| f^n(y), g(y) \|^{k+1}} d\omega_{FS}(y) = 0
\]
of the sequence \( (f^n) \) with respect to \( g \); conversely, (1.6) implies (1.6).

Remark 5.2. In the \( k = 1 \) case, this (1.6) also gives the following.

Theorem 5.3. For every \( f \in \mathbb{C}(z) \) of degree \( d > 1 \) and every \( g \in \mathbb{C}(z) \) of degree \( > 0 \), we have
\[
(5.4) \quad \lim_{n \to \infty} \frac{[f^n = g]}{d^n + \deg g} = \mu_f \text{ weakly on } \mathbb{P}^1.
\]
Here \( [f^n = g] \) is the effective divisor on \( \mathbb{P}^1 \) defined by the zeros of the algebraic equation \( f^n = g \) on \( \mathbb{P}^1 \) and regarded as the Radon measure on \( \mathbb{P}^1 \).

This equidistribution theorem for moving targets on \( \mathbb{P}^1 \) was shown in Lyubich’s seminal [28, Theorem 3] by a (purely) dynamical argument on \( \mathbb{P}^1 \). Here we mainly worked on \( \mathbb{C}^2 \setminus \{0\} \). We refer to a potential theoretic proof of (5.4) for \( g = \text{Id}_{\mathbb{P}^1} \) in Fornæss–Sibony [21, the proof of Theorem 6.1], which used the analytic irrational rotationization of Siegel disks or Herman rings. Our proof of (5.4) is similar to theirs but succeeds in bypassing this structure theorem on complex 1-dimensional domaines singuliers (appearing as \( V \) in the proof of Theorem 4), using pluripotential theory.

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