MEAN HAUSDORFF DIMENSION OF SOME INFINITE DIMENSIONAL FRACTALS

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Abstract. Mean Hausdorff dimension is a dynamical version of Hausdorff dimension. It provides a way to dynamicalize geometric measure theory. We pick up the following three classical results of fractal geometry.

(1) The calculation of Hausdorff dimension of homogeneous sets in the circle.
(2) The coincidence of Hausdorff and Minkowski dimensions for self-similar sets.
(3) The calculation of Hausdorff dimension of Bedford–McMullen carpets.

We develop their analogues for mean Hausdorff dimension:

(1’) The calculation of mean Hausdorff dimension of homogeneous systems in the infinite dimensional torus.
(2’) The coincidence of mean Hausdorff dimension and metric mean dimension for self-similar systems.
(3’) The calculation of mean Hausdorff dimension of infinite dimensional carpets.

1. BACKGROUND

In the classical dimension theory, there are three famous notions of dimension: topological dimension, Hausdorff dimension and Minkowski dimension. Their basic relation is the following.

Topological dimension ≤ Hausdorff dimension ≤ Minkowski dimension.

At the end of the 20th century, Gromov [Gro99] found a way to dynamicalize topological dimension theory. He introduced a dynamical version of topological dimension called mean dimension. Given a dynamical system, its mean dimension measures how many parameters per iterate we need for describing its orbits. Mean dimension is also called mean topological dimension.

Lindenstrauss–Weiss [LW00] introduced a dynamical version of Minkowski dimension called metric mean dimension in order to better understand relations between mean dimension and topological entropy. The definition of metric mean dimension is a fusion of the definitions of topological entropy and Minkowski dimension. We will review it in §2.

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In order to connect mean dimension to ergodic theory, Lindenstrauss–Tsukamoto [LT19] introduced a dynamical version of Hausdorff dimension called *mean Hausdorff dimension*. See §2 for the definition. Mean Hausdorff dimension is better suited for measure theoretic studies than metric mean dimension, as (ordinary) Hausdorff dimension is more closely related to measure theory than Minkowski dimension.

The following is the basic relation between the above three dynamical dimensions.

\[
\text{mean dimension} \leq \text{mean Hausdorff dimension} \leq \text{metric mean dimension}.
\]

See Proposition 2.1 and Remark 2.2 in §2 for more precise statements.

We mainly study mean Hausdorff dimension and metric mean dimension in this paper. Mean Hausdorff dimension is more difficult to evaluate than metric mean dimension. So we often pay more attention to mean Hausdorff dimension.

We pick up the following three classical results of fractal geometry.

- **Homogeneous sets** ([Fur67]). Let \( \mathbb{R}/\mathbb{Z} \) be the circle with a metric
  \[
d(x, y) := \min_{n \in \mathbb{Z}} |x - y - n|.
\]
  Let \( a > 1 \) be a natural number greater than one. We consider the “\( \times a \) map” on the circle \( \mathbb{R}/\mathbb{Z} \):
  \[
  T_a : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \quad x \mapsto ax.
  \]
  Suppose \( X \subset \mathbb{R}/\mathbb{Z} \) is a closed subset satisfying \( T_a(X) \subset X \). Furstenberg [Fur67, Proposition III.1] proved that its Hausdorff dimension \( \dim_H(X, d) \) coincides with its Minkowski dimension \( \dim_M(X, d) \) and that they are given by
  \[
  \dim_H(X, d) = \dim_M(X, d) = \frac{h_{\text{top}}(X, T_a)}{\log a}.
  \]
  Here \( h_{\text{top}}(X, T_a) \) is the topological entropy of the dynamical system \( (X, T_a) \).

- **Self-similar sets** ([Fal89]). Let \( f_1, f_2, \ldots, f_k \) be contracting similarity transformations of the Euclidean space \( \mathbb{R}^n \). Let \( X \) be the attractor of the family \( \{f_1, f_2, \ldots, f_n\} \). Namely, \( X \) is the unique nonempty and compact subset of \( \mathbb{R}^n \) satisfying
  \[
  X = \bigcup_{i=1}^{k} f_i(X).
  \]
  Let \( d \) be the Euclidean metric on \( \mathbb{R}^n \). Falconer [Fal89, Example 2] proved that the Hausdorff dimension of \( (X, d) \) is equal to its Minkowski dimension:
  \[
  \dim_H(X, d) = \dim_M(X, d).
  \]

- **Bedford–McMullen carpets** ([Bed84, Mc84]). Let \( a \) and \( b \) be natural numbers with \( a \geq b \geq 2 \). Set \( A = \{0, 1, 2, \ldots, a-1\} \) and \( B = \{0, 1, 2, \ldots, b-1\} \). Let
$R \subset A \times B$ be a non-empty subset. We define a closed subset $X$ of the unit square $[0, 1]^2$ by

$$X = \left\{ \left( \sum_{n=1}^{\infty} \frac{x_n}{a^n}, \sum_{n=1}^{\infty} \frac{y_n}{b^n} \right) \in [0, 1]^2 : \forall n \geq 1 : (x_n, y_n) \in R \right\}. $$

This space was first introduced and studied by Bedford [Bed84, Chapter 4] and McMullen [Mc84], so it is called a Bedford–McMullen carpet. It is a famous example of fractal sets whose Hausdorff and Minkowski dimensions do not coincide.

Let $d$ be the Euclidean metric on the plane. For each $j \in B$ we denote by $t_j$ the number of $i \in A$ with $(i, j) \in R$. The Hausdorff dimension of $(X, d)$ is given by

$$\dim_H(X, d) = \log_b \left( \sum_{j=0}^{b-1} t_j \log_a b \right).$$

Let $r$ be the cardinality of $R$, and let $s$ be the number of $j \in B$ for which there exists $i \in A$ with $(i, j) \in R$. The Minkowski dimension is given by

$$\dim_M(X, d) = \log_b s + \log_a \left( \frac{r}{s} \right).$$

Except for some special cases\(^1\), the Hausdorff dimension $\dim_H(X, d)$ is strictly smaller than the Minkowski dimension $\dim_M(X, d)$.

The purpose of this paper is to develop analogues of these results for mean Hausdorff dimension and metric mean dimension. Our main results are Theorem 3.1 (an analogue of Furstenberg’s theorem), Theorem 4.3 (an analogue of Falconer’s theorem) and Theorem 5.3 (the mean Hausdorff dimension of “infinite dimensional carpets”) below.

This paper is a starting point for the study of “infinite dimensional fractals”. Our primary purpose is just to show how to formulate meaningful mathematical theorems about infinite dimensional fractals. Hopefully such study will become a fruitful research area in a future.

The plan of this paper is as follows: In §2 we explain the definitions of mean Hausdorff dimension and metric mean dimension. In §3 we study an analogue of Furstenberg’s theorem for mean Hausdorff dimension. In §4 we introduce a dynamical version of self-similar sets and study an analogue of Falconer’s theorem. In §5 we study the mean Hausdorff dimension of infinite dimensional carpets. In the Appendix we explain one more example of the calculations of mean Hausdorff dimension.

### 2. Basic definitions

The purpose of this section is to review the definitions of metric mean dimension ([LW00]) and mean Hausdorff dimension ([LT19]).

\(^1\)Namely, either $n = m$ or all nonzero $t_j$ are equal to each other.
First we prepare some basic quantities of compact metric spaces. Let \((X, d)\) be a compact metric space. For a positive number \(\varepsilon\), we define the \(\varepsilon\)-covering number \(#(X, d, \varepsilon)\) as the minimum cardinality \(n\) of open covers \(\{U_1, \ldots, U_n\}\) of \(X\) satisfying \(\text{Diam } U_i < \varepsilon\) for all \(1 \leq i \leq n\). We define the \(\varepsilon\)-scale Minkowski dimension of \(X\) by

\[
\dim_{\text{M}}(X, d, \varepsilon) = \frac{\log \#(X, d, \varepsilon)}{\log(1/\varepsilon)}.
\]

The upper and lower Minkowski dimensions of \((X, d)\) are defined by

\[
\overline{\dim}_{\text{M}}(X, d) = \limsup_{\varepsilon \to 0} \dim_{\text{M}}(X, d, \varepsilon), \quad \underline{\dim}_{\text{M}}(X, d) = \liminf_{\varepsilon \to 0} \dim_{\text{M}}(X, d, \varepsilon).
\]

When these two values coincide, it is called the Minkowski dimension of \((X, d)\) and denoted by \(\dim_{\text{M}}(X, d)\).

For \(s \geq 0\) and \(\varepsilon > 0\) we define \(H^s_{\varepsilon}(X, d)\) by

\[
\mathcal{H}^s_{\varepsilon}(X, d) = \inf \left\{ \sum_{i=1}^{\infty} (\text{Diam } E_i)^s \mid X = \bigcup_{i=1}^{\infty} E_i \text{ with Diam } E_i < \varepsilon \ (\forall i \geq 1) \right\}.
\]

The meaning of the term \((\text{Diam } E_i)^s\) becomes ambiguous when \(s = 0\) (with \(\text{Diam } E_i = 0\)) or \(E_i = \emptyset\). We use the convention that \(0^0 = 1\) and \((\text{Diam } \emptyset)^s = 0\) for all \(s \geq 0\). Note that this convention implies \(\mathcal{H}^0_{\varepsilon}(X, d) \geq 1\) (assuming \(X \neq \emptyset\)). For \(\varepsilon > 0\) we define the \(\varepsilon\)-scale Hausdorff dimension \(\dim_{\text{H}}(X, d, \varepsilon)\) as the supremum of \(s \geq 0\) satisfying \(\mathcal{H}^s_{\varepsilon}(X, d) \geq 1\). The Hausdorff dimension of \((X, d)\) is defined by

\[
\dim_{\text{H}}(X, d) = \lim_{\varepsilon \to 0} \dim_{\text{H}}(X, d, \varepsilon).
\]

For \(0 < \varepsilon < 1\) we have

\[
(2.1) \quad \dim_{\text{H}}(X, d, \varepsilon) \leq \dim_{\text{M}}(X, d, \varepsilon).
\]

Hence

\[
\dim_{\text{H}}(X, d) \leq \dim_{\text{M}}(X, d) \leq \overline{\dim}_{\text{M}}(X, d).
\]

Next we consider dynamical versions of Hausdorff and Minkowski dimensions. A pair \((X, T)\) is called a dynamical system if \(X\) is a compact metrizable space and \(T : X \to X\) is a continuous map\(^2\).

Let \((X, T)\) be a dynamical system with a metric \(d\) on \(X\). For each natural number \(N\) we define a metric \(d_N\) on \(X\) by

\[
d_N(x, y) = \max_{0 \leq n < N} d(T^n x, T^n y).
\]

\(^2\)In many literature of mean dimension theory (e.g. [LW00, Lin99, LT19]), one usually considers only invertible dynamical systems (namely, the case that \(T\) is a homeomorphism). But in this paper we do not assume that \(T\) is a homeomorphism. This is mainly because the “×a map” in Furstenberg’s theorem is non-invertible and we would like to study its variation in mean dimension theory.
We sometimes use the notation $d^T_N$ for $d_N$ in order to clarify the map $T$. The **topological entropy** of $(X, T)$ is defined by

$$h_{\text{top}}(X, T) = \lim_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{\log \#(X, d_N, \varepsilon)}{N} \right).$$

The topological entropy is a topological invariant of dynamical systems. Namely the value is independent of the choice of a metric $d$.

We define the **upper and lower metric mean dimensions** of $(X, T, d)$ by

$$\overline{\text{mdim}}_M(X, T, d) = \limsup_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{\dim_M(X, d_N, \varepsilon)}{N} \right),$$

$$\underline{\text{mdim}}_M(X, T, d) = \liminf_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{\dim_M(X, d_N, \varepsilon)}{N} \right).$$

These values depend on the choice of a metric $d$. When the upper and lower metric mean dimensions coincide, the common value is called the **metric mean dimension** of $(X, T, d)$ and denoted by $\text{mdim}_M(X, T, d)$.

We define the **upper and lower mean Hausdorff dimensions** of $(X, T, d)$ by

$$\overline{\text{mdim}}_H(X, T, d) = \lim_{\varepsilon \to 0} \left( \limsup_{N \to \infty} \frac{\dim_H(X, d_N, \varepsilon)}{N} \right),$$

$$\underline{\text{mdim}}_H(X, T, d) = \lim_{\varepsilon \to 0} \left( \liminf_{N \to \infty} \frac{\dim_H(X, d_N, \varepsilon)}{N} \right).$$

These also depend on the choice of $d$. When they coincide, the common value is called the **mean Hausdorff dimension** of $(X, T, d)$ and denoted by $\text{mdim}_H(X, T, d)$.

**Proposition 2.1.**

$$\underline{\text{mdim}}_H(X, T, d) \leq \text{mdim}_H(X, T, d) \leq \overline{\text{mdim}}_M(X, T, d) \leq \overline{\text{mdim}}_M(X, T, d).$$

**Proof.** Let $0 < \varepsilon < 1$. From (2.1)

$$\frac{\dim_H(X, d_N, \varepsilon)}{N} \leq \frac{\dim_M(X, d_N, \varepsilon)}{N}.$$

Hence we have $\overline{\text{mdim}}_H(X, T, d) \leq \text{mdim}_H(X, T, d)$. The rests are trivial. \qed

**Remark 2.2.** We denote by $\text{mdim}(X, T)$ the mean topological dimension of a dynamical system $(X, T)$. Then for any metric $d$ on $X$ we have [LT19, Proposition 3.2]

$$\underline{\text{mdim}}(X, T) \leq \text{mdim}_H(X, T, d).$$

We do not explain the definition of mean topological dimension here because we will not use it in the sequel.

Throughout the paper we denote the set of natural numbers by $\mathbb{N} = \{1, 2, 3, \ldots \}$. 
**Example 2.3.** Let $[0,1]^N = [0,1] \times [0,1] \times [0,1] \times \cdots$ be the infinite dimensional cube. We define the shift map $\sigma$ on it by

$$\sigma((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}.$$ 

We define a metric $d$ on $[0,1]^N$ by

$$d((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|.$$ 

Then

$$\text{mdim}([0,1]^N, \sigma) = \text{mdim}_H([0,1]^N, \sigma, d) = \text{mdim}_M([0,1]^N, \sigma, d) = 1.$$ 

**Example 2.4.** Let $K = \{0\} \cup \{\frac{1}{n} \mid n \geq 1\} = \{0, \frac{1}{2}, \frac{1}{3}, \ldots\}$. It is well-known that the Hausdorff dimension and Minkowski dimension of $K$ with respect to the Euclidean metric are zero and $\frac{1}{2}$ respectively. We consider $K^N = K \times K \times K \times \cdots$.

We define the shift map $\sigma : K^N \to K^N$ and a metric $d$ on $K^N$ as in Example 2.3. Then

$$\text{mdim}_H(K^N, \sigma, d) = 0, \quad \text{mdim}_M(K^N, \sigma, d) = \frac{1}{2}.$$ 

The proof of $\text{mdim}_M(K^N, \sigma, d) = \frac{1}{2}$ is straightforward. But it is not so easy to prove $\text{mdim}_H(K^N, \sigma, d) = 0$. We will explain it in the Appendix.

**3. Mean Hausdorff Dimension of Homogeneous Systems**

We develop an analogue of Furstenberg’s theorem [Fur67, Proposition III.1] in this section.

3.1. **Topological entropy of $\mathbb{N}^2$-actions.** We need to introduce topological entropy of $\mathbb{N}^2$-actions in order to explain the main result of this section. A triple $(X, S, T)$ is called a $\mathbb{N}^2$-action if $X$ is a compact metrizable space, and if $S : X \to X$ and $T : X \to X$ are continuous maps with $S \circ T = T \circ S$.

Let $(X, S, T)$ be a $\mathbb{N}^2$-action with a metric $d$ on $X$. For a subset $\Omega \subset \mathbb{N}^2$ we define a metric $d_{\Omega}^{S,T}$ on $X$ by

$$d_{\Omega}^{S,T}(x, y) = \sup_{(m,n) \in \Omega} d(S^m T^n x, S^m T^n y).$$ 

It is convenient to use this notation also for the case that $\Omega$ is a subset of $\mathbb{R}^2$: For a subset $\Omega \subset \mathbb{R}^2$ we set

$$d_{\Omega}^{S,T}(x, y) = \sup_{(m,n) \in \Omega \cap \mathbb{N}^2} d(S^m T^n x, S^m T^n y).$$ 

We define the **topological entropy of** $(X, S, T)$ by

$$h_{\text{top}}(X, S, T) = \lim_{\varepsilon \to 0} \left( \lim_{N \to \infty} \log \# \left( X, d_{[0,N]^2}^{S,T}, \varepsilon \right) \right).$$
It is easy to check that the limits exist. We also have
\[
    h_{\text{top}}(X, S, T) = \lim_{\varepsilon \to 0} \left( \lim_{M \to \infty} \lim_{N \to \infty} \frac{\log \#(X, d_{[0,M] \times [0,N]}^{S,T}, \varepsilon)}{MN} \right).
\]
Here
\[
    d_{[0,N]^2}^{S,T}(x, y) = \max_{0 \leq m < N} d(S^m T^n x, S^m T^n y),
\]
\[
    d_{[0,M] \times [0,N]}^{S,T}(x, y) = \max_{0 \leq m < M} \max_{0 \leq n < N} d(S^m T^n x, S^m T^n y).
\]

3.2. **Main result for homogeneous systems.** Let \( \mathbb{R}/\mathbb{Z} \) be the circle with a metric \( \rho \) defined by
\[
    \rho(x, y) = \min_{n \in \mathbb{Z}} |x - y - n|.
\]
We consider the “infinite dimensional torus”:
\[
    (\mathbb{R}/\mathbb{Z})^\mathbb{N} = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \times \cdots.
\]
We define a metric \( d \) on it by
\[
    d((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} 2^{-n} \rho(x_n, y_n).
\]
We define the shift map \( \sigma : (\mathbb{R}/\mathbb{Z})^\mathbb{N} \to (\mathbb{R}/\mathbb{Z})^\mathbb{N} \) by
\[
    \sigma((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}.
\]
Let \( a > 1 \) be a natural number greater than one. We define the “\( \times a \) map” \( T_a : (\mathbb{R}/\mathbb{Z})^\mathbb{N} \to (\mathbb{R}/\mathbb{Z})^\mathbb{N} \) by
\[
    T_a((x_n)_{n \in \mathbb{N}}) = (ax_n)_{n \in \mathbb{N}}.
\]
Notice that \( T_a \) and \( \sigma \) commute.

The following is the main result of this section.

**Theorem 3.1.** Let \( X \subset (\mathbb{R}/\mathbb{Z})^\mathbb{N} \) be a closed subset such that \( \sigma(X) \subset X \) and \( T_a(X) \subset X \). Then
\[
    \text{mdim}_H(X, \sigma, d) = \text{mdim}_M(X, \sigma, d) = \frac{h_{\text{top}}(X, \sigma, T_a)}{\log a}.
\]
Here \( h_{\text{top}}(X, \sigma, T_a) \) is the topological entropy of the \( \mathbb{N}^2 \)-action \( (X, \sigma, T_a) \).

A “symbolic dynamics version” of this theorem was presented in [ST21].

The proof of Theorem 3.1 consists of two parts: the proofs of the upper bound
\[
    \text{mdim}_M(X, T, d) \leq \frac{h_{\text{top}}(X, \sigma, T_a)}{\log a}
\]
and the lower bound
\[(3\cdot 2) \quad \mathsf{mdim}_H(X, T, d) \geq \frac{h_{\text{top}}(X, \sigma, T_a)}{\log a}.\]

The upper bound \((3\cdot 1)\) directly follows from the definitions. The proof of the lower bound \((3\cdot 2)\) is more involved. The next subsection is a preparation for it.

3.3. **Lipschitz map and mean Hausdorff dimension.** Dai–Zhou–Geng [DZG98, Theorem 2] proved that if \(T : X \to X\) is a Lipschitz map of a compact metric space \((X, d)\) with a Lipschitz constant \(L > 1\) then
\[(3\cdot 3) \quad \dim_H(X, d) \geq \frac{h_{\text{top}}(X, T)}{\log L}.\]

See also [Mis04, Corollary 2.2]. The purpose of this subsection is to prove a variation of this result for mean Hausdorff dimension.

First we prove a “finite accuracy version” of \((3\cdot 3)\).

**Lemma 3.2.** Let \((X, T)\) be a dynamical system with a metric \(d\) on \(X\). Suppose there is \(L > 1\) satisfying
\[d(Tx, Ty) \leq L \cdot d(x, y), \quad (x, y \in X).\]

Let \(t, \delta, \varepsilon\) be positive numbers satisfying
\[0 < t < 1, \quad 0 < \delta < 1, \quad \delta^{1-t} < \varepsilon.\]

Then
\[(3\cdot 4) \quad \lim_{N \to \infty} \frac{\log \# (X, d^{tN}_N, \varepsilon)}{N} \leq \frac{\log L}{t} \cdot \dim_H(X, d, \delta).\]

Notice that if we let \(\delta \to 0\) in the inequality \((3\cdot 4)\) then we get
\[\lim_{N \to \infty} \frac{\log \# (X, d^{tN}_N, \varepsilon)}{N} \leq \frac{\log L}{t} \cdot \dim_H(X, d).\]

Letting \(t \to 1\) and \(\varepsilon \to 0\), we get \((3\cdot 3)\). So \((3\cdot 3)\) follows from \((3\cdot 4)\).

**Proof of Lemma 3.2.** The following proof is motivated by the arguments of [Fur67, Proposition III.1] and [Bow73, Proposition 1].

Let \(s\) be a positive number satisfying \(\dim_H(X, d, \delta) < s\). There exists an open cover \(X = U_1 \cup \cdots \cup U_M\) such that \(\text{Diam}(U_m, d) < \delta\) for all \(1 \leq m \leq M\) and
\[\sum_{m=1}^M (\text{Diam}(U_m, d))^s < 1.\]

Choose positive numbers \(\delta_m\) \((1 \leq m \leq M)\) such that \(\text{Diam}(U_m, d) < \delta_m < \delta\) and
\[\sum_{m=1}^M \delta_m^s < 1. \]
Set
\[ N_m = \lceil \log L \delta_m^{-t} \rceil \quad (\geq 1). \]
Here \( \lceil x \rceil = \min \{ n \in \mathbb{Z} \mid n \geq x \} \) for real numbers \( x \). Then \( L^{N_m} \geq \delta_m^{-t} \) and hence
\[
(3.5) \quad \sum_{m=1}^{M} L^{-sN_m/t} \leq \sum_{m=1}^{M} \delta_m^s < 1.
\]
From the Lipschitz condition of \( T \)
\[
\text{Diam} \left( U_m, d_{N_m}^T \right) \leq L^{N_m} \cdot \text{Diam}(U_m, d)
\]
\[
\leq L^{\log L \delta_m^{-t}} \cdot \text{Diam}(U_m, d) \quad (\text{by } N_m = \lceil \log L \delta_m^{-t} \rceil \leq \log L \delta_m^{-t} + 1)
\]
\[
= \delta_m^{-t} \cdot \text{Diam}(U_m, d)
\]
\[
< \delta_m^{-1-t} < \delta^{1-t} < \varepsilon.
\]
Let \( N \) be a natural number. Let \( I_N \) be the set of sequences \((m_1, m_2, \ldots, m_k)\) of natural numbers \( m_i \) satisfying
\begin{itemize}
  \item \( 1 \leq m_i \leq M \) for all \( 1 \leq i \leq k \),
  \item \( N_{m_1} + N_{m_2} + \cdots + N_{m_{k-1}} < N \leq N_{m_1} + N_{m_2} + \cdots + N_{m_{k-1}} + N_{m_k} \).
\end{itemize}
Here \( k \) is not a fixed number. It also varies.
We define an open covering \( \mathcal{U} \) of \( X \) as the family of
\[
U_{m_1} \cap T^{-N_{m_1}} U_{m_2} \cap T^{-N_{m_1} - N_{m_2}} U_{m_3} \cap \cdots \cap T^{-N_{m_1} - N_{m_2} - \cdots - N_{m_{k-1}}} U_{m_k},
\]
where \((m_1, m_2, \ldots, m_k) \in I_N\). Every member \( U \in \mathcal{U} \) satisfies \( \text{Diam} \left( U, d_N^T \right) < \varepsilon \). Hence \( \#(X, d_N^T, \varepsilon) \leq |I_N| \). Here \(|I_N|\) denotes the cardinality of \( I_N \).
Set \( \bar{N} := \max(N_1, \ldots, N_M) \). We have
\[
|I_N| \cdot L^{-s(N + \bar{N})/t} \leq \sum_{(m_1, \ldots, m_k) \in I_N} L^{-s(N_{m_1} + \cdots + N_{m_k})/t}
\]
\[
\leq \sum_{k=1}^{\infty} \left( \sum_{m=1}^{\infty} L^{-sN_m/t} \right)^k < \infty \quad \text{(by } (3.5))\).
\]
Hence
\[
\log |I_N| \leq \frac{s(N + \bar{N})}{t} \cdot \log L + \text{const},
\]
where const is a positive constant independent of \( N \). Therefore
\[
\log \#(X, d_N^T, \varepsilon) \leq \frac{s(N + \bar{N})}{t} \cdot \log L + \text{const}.
\]
Dividing this by \( N \) and letting \( N \to \infty \), we get
\[
\lim_{N \to \infty} \frac{\log \#(X, d_N^T, \varepsilon)}{N} \leq \frac{s \log L}{t}.
\]
Since \( s \) is an arbitrary number larger than \( \dim_H(X, d, \delta) \), this shows the statement. \( \square \)
Theorem 3.3. Let $(X, S, T)$ be a $N^2$-action with a metric $d$ on $X$. Suppose there exists $L > 1$ such that

\[ d(Tx, Ty) \leq L d(x, y), \quad (x, y \in X). \]

Then

\[ \text{mdim}_H(X, S, d) \geq \frac{h_{\text{top}}(X, S, T)}{\log L}. \]

Proof. For any $M > 0$ we have $d_S^M(Tx, Ty) \leq L \cdot d_S^M(x, y)$. Let $0 < \delta < 1$, $0 < t < 1$ and $0 < \varepsilon < 1$ be positive numbers with $\delta^{1-t} < \varepsilon$.

From Lemma 3.2, for any $M > 0$

\[ \lim_{N \to \infty} \log \# \left( X, \frac{d_S^{S,T}_{\lfloor 0,M \rfloor \times [0,N)} \cdot \varepsilon}{N} \right) \leq \frac{\log L}{t} \cdot \dim_H \left( X, d_S^M, \delta \right). \]

Divide the both sides by $M$ and let $M \to \infty$. We get

\[ \lim_{N \to \infty} \frac{\log \# \left( X, \frac{d_S^{S,T}_{\lfloor 0,M \rfloor \times [0,N)} \cdot \varepsilon}{MN} \right)}{M} \leq \frac{\log L}{t} \cdot \liminf_{M \to \infty} \frac{\dim_H \left( X, d_S^M, \delta \right)}{M}. \]

Letting $\delta \to 0$, we have

\[ \lim_{N \to \infty} \frac{\log \# \left( X, \frac{d_S^{S,T}_{\lfloor 0,M \rfloor \times [0,N)} \cdot \varepsilon}{MN} \right)}{M} \leq \frac{\log L}{t} \cdot \text{mdim}_H(X, S, d). \]

We can let $t \to 1$ and $\varepsilon \to 0$, so this proves the statement. \qed

3.4. Proof of Theorem 3.1. Now we prove Theorem 3.1. Recall that $\rho$ is a metric on the circle $\mathbb{R}/\mathbb{Z}$ defined by

\[ \rho(x, y) = \min_{m \in \mathbb{Z}} |x - y - m|. \]

We write the statement of Theorem 3.1 again.

Theorem 3.4 (= Theorem 3.1). Let $d$ be a metric on $(\mathbb{R}/\mathbb{Z})^N$ defined by

\[ d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \rho(x_n, y_n). \]

Let $a > 1$ be a natural number and define $T_a$ on $(\mathbb{R}/\mathbb{Z})^N$ by the component-wise multiplication of $a$. Let $\sigma : (\mathbb{R}/\mathbb{Z})^N \to (\mathbb{R}/\mathbb{Z})^N$ be the shift. If $X \subset (\mathbb{R}/\mathbb{Z})^N$ is a closed subset satisfying $\sigma(X) \subset X$ and $T_a(X) \subset X$ then

\[ \text{mdim}_H(X, \sigma, d) = \text{mdim}_M(X, \sigma, d) = \frac{h_{\text{top}}(X, \sigma, T_a)}{\log a}. \]

Proof. Obviously $d(T_a(x), T_a(y)) \leq a d(x, y)$. So by Theorem 3.3

\[ \text{mdim}_H(X, \sigma, d) \geq \frac{h_{\text{top}}(X, \sigma, T_a)}{\log a}. \]
The remaining task is to show the upper bound
\[ \text{mdim}_M(X, \sigma, d) \leq \frac{h_{\text{top}}(X, \sigma, T_a)}{\log a}. \]

A key fact is that, for any natural number \( M \), if two points \( u, v \in \mathbb{R}/\mathbb{Z} \) satisfy
\[ \max_{0 \leq m < M} \rho(a^m u, a^m v) < \frac{1}{2a}, \]
then
\[ \rho(u, v) < \frac{1}{2a^M}. \]

From this, for any natural numbers \( L \) and \( M \), if two points \( x, y \in X \) satisfy
\[ d_{\sigma,T_a}[0,L] \times [0,M)(x, y) < \frac{1}{4a} \]
then
\[ d(x, y) < \frac{1}{2a^L} + 2^{-L}. \]

Indeed, \( d_{\sigma,T_a}[0,L] \times [0,M)(x, y) < \frac{1}{4a} \) implies that for all \( 1 \leq n \leq L \)
\[ \max_{0 \leq m < M} \rho(a^m x_n, a^m y_n) < \frac{1}{2a} \]
and hence
\[ \rho(x_n, y_n) < \frac{1}{2a^L} \quad (1 \leq n \leq L). \]

So
\[ d(x, y) \leq \sum_{n=1}^{L} 2^{-n} \rho(x_n, y_n) + \sum_{n=L+1}^{\infty} 2^{-n} \]
\[ < \frac{1}{2a^L} + 2^{-L}. \]

Let \( 0 < \varepsilon < 1 \) be arbitrary. We choose natural numbers \( L \) and \( M \) satisfying
\[ 2^{-L} < \frac{\varepsilon}{2}, \quad a^{-M} \leq \varepsilon < a^{-M+1}. \]

From the above consideration, for any natural number \( N \)
\[ d_{\sigma,T_a}[0,N+L] \times [0,M)(x, y) < \frac{1}{4a} \implies d_{\sigma}^N(x, y) < \frac{1}{2a^L} + 2^{-L} < \varepsilon. \]

So
\[ \#(X, d_{\sigma}^N, \varepsilon) \leq \#(X, d_{\sigma,T_a}[0,N+L] \times [0,M), \frac{1}{4a}). \]

Hence
\[ \lim_{N \to \infty} \frac{\log \#(X, d_{\sigma}^N, \varepsilon)}{N} \leq \lim_{N \to \infty} \frac{\log \#(X, d_{\sigma,T_a}[0,N+L] \times [0,M), \frac{1}{4a})}{N} \]
\[ = \lim_{N \to \infty} \frac{\log \#(X, d_{\sigma,T_a}[0,N] \times [0,M), \frac{1}{4a})}{N}. \]
From $\varepsilon < a^{-M+1}$, we have $(M - 1) \log a < \log(1/\varepsilon)$ and
\[
\frac{1}{\log(1/\varepsilon)} < \frac{1}{(M - 1) \log a} = \frac{M}{(M - 1) \log a} \cdot \frac{1}{M}.
\]
Therefore
\[
\lim_{N \to \infty} \frac{\dim_M (X, d_N^\varepsilon)}{N} = \lim_{N \to \infty} \frac{\log \#(X, d_N^\varepsilon, \varepsilon)}{N \log(1/\varepsilon)} \leq \frac{M}{(M - 1) \log a} \cdot \frac{\log \#(X, d_{(0,N)\times[0,M]}^{\sigma T_a}, \frac{1}{4a})}{NM}.
\]
$M$ goes to infinity as $\varepsilon$ goes to zero. So
\[
\limsup_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{\dim_M (X, d_N^\varepsilon)}{N} \right) \leq \frac{1}{\log a} \cdot \frac{\log \#(X, d_{(0,N)\times[0,M]}^{\sigma T_a}, \frac{1}{4a})}{NM} \leq \frac{1}{\log a} \cdot \htop(X, \sigma, T_a).
\]
This proves $\mdim_M(X, \sigma, d) \leq \frac{\htop(X, \sigma, T_a)}{\log a}$. \hfill $\Box$

4. Self-similarity and mean Hausdorff dimension

In this section we introduce a “self-similar system”, which is a dynamical version of a self-similar set. We show that mean Hausdorff dimension and metric mean dimension coincide for such systems.

4.1. Self-similar systems. Let
\[
\ell^\infty = \left\{ (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N} \mathbb{N} \mid \sup_{n \geq 1} |x_n| < \infty \right\}
\]
be the space of bounded sequences with the norm $\|x\|_\infty := \sup_{n \geq 1} |x_n|$ for $x = (x_n)_{n \in \mathbb{N}}$. We always assume that $\ell^\infty$ is endowed with the weak* topology as the dual space of $\ell^1$.

We define the shift map $\sigma : \ell^\infty \to \ell^\infty$ by
\[
\sigma ((x_n)_{n \in \mathbb{N}}) = (x_{n+1})_{n \in \mathbb{N}}.
\]
This is continuous with respect to the weak* topology. We will consider a certain self-similar set of $\ell^\infty$ invariant under the shift map $\sigma$.

Let $(\Omega, T)$ be a dynamical system. Suppose that for each $\omega \in \Omega$ we are given a point $a(\omega) = (a(\omega)_n)_{n \in \mathbb{N}} \in \ell^\infty$ so that the map
\[
\Omega \ni \omega \mapsto a(\omega) \in \ell^\infty
\]
is continuous (with respect to the weak* topology of $\ell^\infty$) and equivariant (i.e. $\sigma (a(\omega)) = a(T\omega)$). Since $\Omega$ is compact, we have
\[
\sup_{\omega \in \Omega} \|a(\omega)\|_\infty < \infty.
\]
Fix a real number $c$ with $0 < c < 1$. For each $\omega \in \Omega$ we define a contracting similarity transformation $S_\omega : \ell^\infty \to \ell^\infty$ by

$$S_\omega(x) = cx + a(\omega).$$

Then $\sigma(S_\omega(x)) = S_{T\omega}(\sigma(x))$.

**Proposition 4.1** (Definition of a self-similar system). There uniquely exists a non-empty compact subset $X$ of $\ell^\infty$ satisfying

$$X = \bigcup_{\omega \in \Omega} S_\omega(X).$$

This $X$ is $\sigma$-invariant (i.e. $\sigma(X) \subset X$). The dynamical system $(X, \sigma)$ is called a **self-similar system** defined by the family of contracting similarity transformations $\{S_\omega\}_{\omega \in \Omega}$.

Notice that here $X$ is compact with respect to the weak* topology, not the norm topology, and that $X$ becomes bounded and closed in the $\ell^\infty$-norm by the uniform boundedness principle (the Banach–Steinhaus theorem).

**Proof.** We define $X \subset \ell^\infty$ by

$$X = \left\{ \sum_{k=0}^\infty c^k a(\omega_k) \bigg| \omega_k \in \Omega \,(k \geq 0) \right\}.$$ 

Notice that the sum $\sum_{k=0}^\infty c^k a(\omega_k)$ absolutely converges because $\sup_{\omega \in \Omega} \|a(\omega)\|_\infty < \infty$.

$X$ is the image of a continuous map

$$\Omega \times \Omega \times \cdots \to \ell^\infty, \quad (\omega_k)_{k \geq 0} \mapsto \sum_{k=0}^\infty c^k a(\omega_k),$$

where $\Omega \times \Omega \times \cdots$ is endowed with the product topology. Since $\Omega$ is compact, $X$ is also compact. We have

$$S_\omega \left( \sum_{k=0}^\infty c^k a(\omega_k) \right) = a(\omega) + \sum_{k=0}^\infty c^{k+1} a(\omega_k).$$

From this, it is easy to see that $X$ satisfies

$$X = \bigcup_{\omega \in \Omega} S_\omega(X).$$

We also have

$$\sigma \left( \sum_{k=0}^\infty c^k a(\omega_k) \right) = \sum_{k=0}^\infty c^k a(\omega_k) = \sum_{k=0}^\infty c^k a(T\omega_k).$$

Therefore $\sigma(X) \subset X$.

Next we study the uniqueness of $X$. Suppose a non-empty compact subset $Y \subset \ell^\infty$ satisfies

$$Y = \bigcup_{\omega \in \Omega} S_\omega(Y).$$
Recall that the compactness (with respect to the weak* topology) implies that $Y$ is bounded and closed in the $\ell^\infty$-norm by the uniform boundedness principle.

Suppose $Y \not\subset X$. We set
\[
\delta := \sup_{y \in Y} \left( \inf_{x \in X} \| x - y \|_\infty \right) > 0.
\]
Since $Y = \bigcup_{\omega \in \Omega} S_\omega(Y)$, we have
\[
\delta = \sup_{y \in Y, \omega \in \Omega} \left( \inf_{x \in X} \| x - S_\omega y \|_\infty \right).
\]
However
\[
\inf_{x \in X} \| x - S_\omega y \|_\infty \leq \inf_{x \in X} \| S_\omega x - S_\omega y \|_\infty \quad \text{(by $S_\omega(X) \subset X$)}
\]
\[
= c \cdot \inf_{x \in X} \| x - y \|_\infty \leq c \cdot \delta.
\]
Hence $\delta \leq c \cdot \delta < \delta$. This is a contradiction. Hence $Y \subset X$. By switching the roles of $X$ and $Y$, we also have $X \subset Y$. So $X = Y$. This shows the uniqueness of $X$. \hfill \square

**Remark 4.2.** We recall that the map $S_\omega : \ell^\infty \to \ell^\infty$ has the form
\[
S_\omega(u) = cu + a(\omega), \quad (c \text{ is a fixed constant}).
\]
The point is that the linear part of $S_\omega$ is a scalar multiplication. Probably many readers feel that this form is too restricted. This severe restriction comes from the fact that the $\ell^\infty$-geometry does not admit many similarity transformations. There is no “rotation” for $\ell^\infty$ (except for permutations of coordinates). We expect that it is more interesting to study “self-affine sets” of $\ell^\infty$, rather than “self-similar sets”, because the space $\ell^\infty$ seems to admit many interesting affine transformations. We hope to come back to this study in a future paper.

Let $X \subset \ell^\infty$ be the self-similar system defined by $\{S_\omega\}_{\omega \in \Omega}$. We define a metric $d$ on $X$ by
\[
d\left((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}\right) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|.
\]
We are interested in the mean Hausdorff dimension and metric mean dimension of $(X, \sigma, d)$. The following is the main result of this section.

**Theorem 4.3.** Under the above setting, the self-similar system $X$ satisfies
\[
\mdim_H(X, \sigma, d) = \mdim_M(X, \sigma, d) \leq \frac{\htop(\Omega, T)}{\log(1/c)}.
\]
Here $\htop(\Omega, T)$ is the topological entropy of $(\Omega, T)$.

Therefore, the mean Hausdorff dimension and metric mean dimension coincide for self-similar systems.
Remark 4.4. Let \(a_1, \ldots, a_m\) be vectors in \(\mathbb{R}^N\), and define contracting similarity transformations \(f_i : \mathbb{R}^N \to \mathbb{R}^N\) \((1 \leq i \leq m)\) by
\[
f_i(u) = cu + a_i.
\]
Let \(K \subset \mathbb{R}^N\) be an attractor of \(\{f_1, \ldots, f_m\}\). Then the similarity dimension of \(K\) is given by
\[
\frac{\log m}{\log(1/c)}.
\]
It is well-known that the Hausdorff and Minkowski dimensions of \(K\) (with respect to the Euclidean metric) are bounded by the similarity dimension. The term \(\frac{h_{\text{top}}(\Omega, \sigma)}{\log(1/c)}\) in Theorem 4.3 is an analogue of the similarity dimension in our setting.

We prove Theorem 4.3 in the next subsection. Before going into the proof, we study a simple example.

Example 4.5 (\(\beta\)-expansions). Let \(a > 1\) be an integer greater than one. Let \(\{0, 1, 2, \ldots, a-1\}^N\) be the full-shift on the alphabet \(0, 1, 2, \ldots, a-1\). We naturally consider that this is a subset of \(\ell^\infty\):
\[
\{0, 1, 2, \ldots, a-1\}^N \subset \ell^\infty.
\]
Let \(\Omega \subset \{0, 1, 2, \ldots, a-1\}^N\) be a subshift (a closed subset invariant under the shift map \(\sigma\)). We fix a real number \(\beta\) with \(\beta \geq a\). For each \(\omega \in \Omega\) we define \(S_\omega : \ell^\infty \to \ell^\infty\) by
\[
S_\omega(x) = \frac{x + \omega}{\beta}.
\]
Let \(X \subset \ell^\infty\) be a self-similar system defined by these transformations. This is given by
\[
X = \left\{ \sum_{n=1}^{\infty} \frac{\omega_n}{\beta^n} \mid \omega_n \in \Omega \right\}.
\]
By Theorem 4.3 we have
\[
\text{mdim}_H(X, \sigma, d) = \text{mdim}_M(X, \sigma, d) \leq \frac{h_{\text{top}}(\Omega, \sigma)}{\log \beta}.
\]
Actually the equality holds here as we will see below. We need the next claim.

Claim 4.6. Let \(0 \leq u_k, v_k \leq a-1\) be integers \((1 \leq k \leq n)\). If \((u_1, \ldots, u_n) \neq (v_1, \ldots, v_n)\) then
\[
\left| \sum_{k=1}^{n} \frac{u_k}{\beta^k} - \sum_{k=1}^{n} \frac{v_k}{\beta^k} \right| \geq \frac{1}{\beta^n}.
\]

\(3\) Notice that, when \(\beta = a\), the system \((X, \sigma)\) provides an example for Theorem 3.1 by projecting it to the infinite dimensional torus \((\mathbb{R}/\mathbb{Z})^N\). We also note that, when \(\beta > a\), our setting is simpler than general \(\beta\)-expansions because we restrict “digits” to \(\{0, 1, 2, \ldots, a-1\}\) and \(a\) is forbidden. For example, when \(\beta = \frac{1+\sqrt{5}}{2} = 1.618\ldots\), a complication of \(\beta\)-expansions may occur from \(\frac{1}{\beta} = \frac{1}{\beta^2} + \frac{1}{\beta^3}\). We sidestep this complication simply by forbidding the digit 1 to appear. We do not dig deeper into this problem in this paper.
Proof. Take an integer \( m \in [1, n] \) with \((u_1, \ldots, v_{m-1}) = (v_1, \ldots, v_{m-1})\) and \( u_m \neq v_m \). We have
\[
\left| \sum_{k=1}^{n} \frac{u_k}{\beta^k} \sum_{k=1}^{n} \frac{v_k}{\beta^k} \right| = \left| \frac{u_m - v_m}{\beta^m} + \sum_{k=m+1}^{n} \frac{u_k - v_k}{\beta^k} \right| \\
\geq \left| \frac{u_m - v_m}{\beta^m} \right| - \sum_{k=m+1}^{n} \left| \frac{u_k - v_k}{\beta^k} \right| \\
\geq \frac{1}{\beta^m} - \sum_{k=m+1}^{n} \frac{a-1}{\beta^k} \\
= \frac{1}{\beta^m} - (a-1) \cdot \frac{\beta^{n-m} - 1}{\beta^n (\beta - 1)} \\
\geq \frac{1}{\beta^m} - \frac{\beta^{n-m} - 1}{\beta^n} \quad \text{(by } a \leq \beta) \\
= \frac{1}{\beta^n}.
\]
\(\square\)

Let \( N \) be a natural number and let \( \pi_N : \{0, 1, 2, \ldots, a-1\}^N \rightarrow \{0, 1, 2, \ldots, a-1\}^N \) be the projection to the first \( N \) coordinates. Fix \( \xi \in \Omega \). For \( \omega_1, \ldots, \omega_n, \omega'_1, \ldots, \omega'_n \in \Omega \), if \((\pi_N(\omega_1), \ldots, \pi_N(\omega_n)) \neq (\pi_N(\omega'_1), \ldots, \pi_N(\omega'_n))\) then by Claim 4.6
\[
d_N \left( \sum_{k=1}^{n} \frac{\omega_k}{\beta^k} + \sum_{k=n+1}^{\infty} \frac{\xi}{\beta^k}, \sum_{k=1}^{n} \frac{\omega'_k}{\beta^k} + \sum_{k=n+1}^{\infty} \frac{\xi}{\beta^k} \right) \geq \frac{1}{\beta^n}.
\]
Therefore
\[
\# \left( X, d_N, \frac{1}{\beta^n} \right) \geq |\pi_N(\Omega)|^n.
\]
Then
\[
\dim_M \left( X, d_N, \frac{1}{\beta^n} \right) \geq \frac{\log |\pi_N(\Omega)|}{N \log \beta}.
\]
Letting \( N \rightarrow \infty \), the right-hand side becomes \( h_{\text{top}}(\Omega, \sigma)/ \log \beta \):
\[
\lim_{N \rightarrow \infty} \frac{\dim_M \left( X, d_N, \frac{1}{\beta^n} \right)}{N} \geq \frac{h_{\text{top}}(\Omega, \sigma)}{\log \beta}.
\]
Letting \( n \rightarrow \infty \), we conclude
\[
\text{mdim}_M(X, \sigma, d) \geq \frac{h_{\text{top}}(\Omega, \sigma)}{\log \beta}.
\]
Thus we have
\[
\text{mdim}_H(X, \sigma, d) = \text{mdim}_M(X, \sigma, d) = \frac{h_{\text{top}}(\Omega, \sigma)}{\log \beta}.
\]
4.2. **Proof of Theorem 4.3.** The following is a key lemma for the proof of Theorem 4.3. This is a finite accuracy version of a theorem of Falconer [Fal89, Theorem 4]. The proof closely follows Falconer’s original argument.

**Lemma 4.7.** For any real numbers $0 < \varepsilon, a, \tau < 1$ there exists $\delta_0 = \delta_0(\varepsilon, a, \tau) > 0$ such that the following statement holds true. Let $(X, d)$ be a compact metric space such that for every closed ball $B \subset X$ of radius $\varepsilon$ there exists a map $\varphi : X \to B$ satisfying

$$d(\varphi(x), \varphi(y)) \geq a\varepsilon d(x, y) \quad (x, y \in X).$$

Then

$$\dim_H(X, d, \delta_0) \geq \tau \cdot \log \left( \frac{\#(X, d, 9\varepsilon)}{\log \left( \frac{1}{a\varepsilon} \right)} \right).$$

**Proof.** Set $N = \#(X, d, 9\varepsilon)$. There exists points $x_1, \ldots, x_N \in X$ such that $d(x_i, x_j) > 3\varepsilon$ for $i \neq j$. Let $B_i$ be the closed ball of radius $\varepsilon$ centered at $x_i$. Then

$$d(B_i, B_j) := \inf_{x \in B_i, y \in B_j} d(x, y) > \varepsilon \quad (> a\varepsilon).$$

We can take a map $\varphi_i : X \to B_i$ satisfying $d(\varphi_i(x), \varphi_i(y)) \geq a\varepsilon d(x, y)$.

For $1 \leq i_1, i_2, \ldots, i_n \leq N$ we set

$$B_{i_1 i_2 \ldots i_n} = \varphi_{i_n} \circ \varphi_{i_{n-1}} \circ \cdots \circ \varphi_{i_2}(B_{i_1}).$$

We have

$$B_{i_1 i_2 \ldots i_n} \subset B_{i_2 i_3 \ldots i_n} \subset \cdots \subset B_{i_{n-1} i_n} \subset B_{i_n}.$$

See Figure 1.

![Figure 1](image_url)

**Claim 4.8.** For $(i_1, i_2, \ldots, i_n) \neq (j_1, j_2, \ldots, j_n),

$$d(B_{i_1 i_2 \ldots i_n}, B_{j_1 j_2 \ldots j_n}) > (a\varepsilon)^n.$$
Proof. If $i_n \neq j_n$ then $B_{i_1 \ldots i_n} \subset B_{i_n}$ and $B_{j_1 \ldots j_n} \subset B_{j_n}$ imply
\[
d(B_{i_1 \ldots i_n}, B_{j_1 \ldots j_n}) \geq d(B_{i_n}, B_{j_n}) > a\varepsilon \geq (a\varepsilon)^n.
\]
If $i_n = j_n$ and $i_{n-1} \neq j_{n-1}$ then $B_{i_1 \ldots i_n} \subset \varphi_{i_n}(B_{i_{n-1}})$ and $B_{j_1 \ldots j_n} \subset \varphi_{i_n}(B_{j_{n-1}})$ imply
\[
d(B_{i_1 \ldots i_n}, B_{j_1 \ldots j_n}) \geq d(\varphi_{i_n}(B_{i_{n-1}}), \varphi_{i_n}(B_{j_{n-1}}))
\geq a\varepsilon d(B_{i_{n-1}}, B_{j_{n-1}})
\geq (a\varepsilon)^2 \geq (a\varepsilon)^n.
\]
If $(i_n, i_{n-1}) = (j_n, j_{n-1})$ and $i_{n-2} \neq j_{n-2}$ then $B_{i_1 \ldots i_n} \subset \varphi_{i_n} \circ \varphi_{i_n-1}(B_{i_{n-2}})$ and $B_{j_1 \ldots j_n} \subset \varphi_{i_n} \circ \varphi_{i_n-1}(B_{j_{n-2}})$ imply
\[
d(B_{i_1 \ldots i_n}, B_{j_1 \ldots j_n}) \geq d(\varphi_{i_n} \circ \varphi_{i_n-1}(B_{i_{n-2}}), \varphi_{i_n} \circ \varphi_{i_n-1}(B_{j_{n-2}}))
\geq a\varepsilon d(\varphi_{i_n-1}(B_{i_{n-2}}), \varphi_{i_n-1}(B_{j_{n-2}}))
\geq (a\varepsilon)^2 d(B_{i_{n-2}}, B_{j_{n-2}})
\geq (a\varepsilon)^3 \geq (a\varepsilon)^n.
\]
We can proceed similarly and prove the claim. \qed

For each $n \geq 1$ we take a Borel probability measure $\mu_n$ on $X$ such that for every $1 \leq i_1, \ldots, i_n \leq N$ we have
\[
\mu_n(B_{i_1 \ldots i_n}) = \frac{1}{N^n}.
\]
Notice that this implies
\[
\mu_n\left(\bigcup_{1 \leq i_1, \ldots, i_n \leq N} B_{i_1 \ldots i_n}\right) = 1.
\]
Moreover, for any $1 \leq m \leq n$ and $1 \leq i_1, \ldots, i_m \leq N$ we also have
\[
\mu_n(B_{i_1 \ldots i_m}) = \frac{1}{N^m}.
\]
We fix $0 < \delta_0 < 1$ satisfying
\[
(4.1) \quad (1 - \tau) \cdot \frac{\log \delta_0}{\log(a\varepsilon)} \geq 1.
\]
Set
\[
s = \frac{\tau \log N}{\log \left(\frac{1}{a\varepsilon}\right)}.
\]
Claim 4.9. Let $E \subset X$ be a Borel subset satisfying $0 < \text{Diam } E < \delta_0$ then
\[
\mu_n(E) \leq (\text{Diam } E)^s
\]
for all sufficiently large $n$. 
Proof. Set \( r = \text{Diam} E \). Take a natural number \( m \) with \((a\varepsilon)^{m+1} \leq r < (a\varepsilon)^m\). Then \( E \) intersects with at most one set in \( \{ B_{i_1i_2...i_m} \} \) by Corollary 4.8. For any integer \( n \geq m \) we have

\[
\mu_n(E) \leq \frac{1}{N^m}.
\]

From \((a\varepsilon)^{m+1} \leq r\), we have \((m + 1) \log(a\varepsilon) \leq \log r\) and hence

\[
m \geq -1 + \frac{\log r}{\log(a\varepsilon)}.
\]

From (4.1)

\[
m \geq -(1 - \tau) \cdot \frac{\log \delta_0}{\log(a\varepsilon)} + \frac{\log r}{\log(a\varepsilon)} > -(1 - \tau) \cdot \frac{\log r}{\log(a\varepsilon)} + \frac{\log r}{\log(a\varepsilon)} \quad \text{(by } r < \delta_0\text{)}
\]

\[
= \tau \cdot \frac{\log r}{\log(a\varepsilon)}.
\]

Then

\[
\frac{1}{N^m} = \exp(-m \log N)
\]

\[
< \exp \left(-\tau \cdot \frac{\log r}{\log(a\varepsilon)} \cdot \log N \right)
\]

\[
= \exp(s \log r) \quad \text{(by } s = \frac{\tau \log N}{\log \left(\frac{1}{a\varepsilon}\right)}\text{)}
\]

\[
= r^s.
\]

Thus we have \( \mu_n(E) < r^s \) for any \( n \geq m \).

Suppose we are given an open cover \( X = U_1 \cup \cdots \cup U_K \) with \( 0 < \text{Diam} U_k < \delta_0 \) for all \( 1 \leq k \leq K \). Then by Claim 4.9

\[
\mu_n(U_k) < (\text{Diam} U_k)^s
\]

for all \( k \) and any sufficiently large \( n \). Therefore

\[
\sum_{k=1}^K (\text{Diam} U_k)^s > \sum_{k=1}^K \mu_n(U_k) \geq \mu_n(X) = 1.
\]

This implies \( \mathcal{H}^s_{\delta_0}(X, d) \geq 1 \). Thus

\[
\dim_H(X, d, \delta_0) \geq s = \tau \cdot \frac{\log \#(X, d, 9\varepsilon)}{\log \left(\frac{1}{a\varepsilon}\right)}.
\]

This proves the statement of the lemma.
We return to the setting of §4.1. First we recall various definitions. Let \((\Omega, T)\) be a dynamical system. Suppose that we are given a continuous equivariant map
\[ \Omega \ni \omega \mapsto a(\omega) \in \ell^\infty. \]

Fix \(0 < c < 1\) and define \(S_\omega : \ell^\infty \to \ell^\infty\) for each \(\omega \in \Omega\) by
\[ S_\omega(x) = cx + a(\omega). \]

Let \(X \subset \ell^\infty\) be the self-similar system defined by \(\{S_\omega\}_{\omega \in \Omega}\). It is a shift-invariant, non-empty, compact subset of \(\ell^\infty\) (with respect to the weak* topology) satisfying
\[ X = \bigcup_{\omega \in \Omega} S_\omega(X). \]

We define a metric \(d\) on \(X\) by
\[ d((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \sum_{n=1}^\infty 2^{-n} |x_n - y_n|. \]

We are interested in the mean Hausdorff dimension and metric mean dimension of \((X, \sigma, d)\), where \(\sigma : X \to X\) is the shift map.

Since \(X\) is compact, we can find \(A \geq 1\) such that
\[ \text{Diam}(X, d) < A. \]

For \(N \geq 1\) we define a metric \(d_N\) on \(X\) by
\[ d_N(x, y) = \max_{0 \leq n < N} d(\sigma^n(x), \sigma^n(y)). \]

**Claim 4.10.** Let \(0 < \varepsilon < 1\) and let \(N\) be a natural number. For any point \(p \in X\) there exists a map \(\varphi : X \to B_\varepsilon(p, d_N)\) satisfying
\[ d_N(\varphi(x), \varphi(y)) \geq \frac{c\varepsilon}{A} d_N(x, y). \]

Here \(B_\varepsilon(p, d_N)\) is the closed \(\varepsilon\)-ball centered at \(p\) with respect to the metric \(d_N\).

**Proof.** For any \(\omega \in \Omega\)
\[ d(S_\omega(x), S_\omega(y)) = d(cx + a(\omega), cy + a(\omega)) = \sum_{n=1}^\infty 2^{-n} |cx_n - cy_n| = cd(x, y). \]

Similarly we have
\[ d_N(S_\omega(x), S_\omega(y)) = cd_N(x, y). \]

Take a natural number \(k\) with \(c^k A \leq \varepsilon < c^{k-1} A\). From \(X = \bigcup_{\omega \in \Omega} S_\omega(X)\) we can find a sequence \(\omega_1, \ldots, \omega_k \in \Omega\) satisfying
\[ p \in S_{\omega_1} \circ \cdots \circ S_{\omega_k}(X). \]

Set \(\varphi := S_{\omega_1} \circ \cdots \circ S_{\omega_k} : X \to X\). For any \(x, y \in X\), by (4.2)
\[ d_N(\varphi(x), \varphi(y)) = c^k d_N(x, y) \leq c^k A \leq \varepsilon. \]
Therefore $\varphi(X) \subset B_\varepsilon(p, d_N)$. From $\varepsilon < c^{k-1} A$ we have $c^k > \frac{\varepsilon}{A}$ and hence
\[ d_N (\varphi(x), \varphi(y)) = c^k d_N(x, y) \geq \frac{c\varepsilon}{A} d_N(x, y). \]

Now we start the proof of Theorem 4.3. We rewrite the statement:

**Theorem 4.11** (= Theorem 4.3). For the above self-similar system $(X, \sigma)$ we have
\[ \text{mdim}_H(X, \sigma, d) = \text{mdim}_M(X, \sigma, d) \leq \frac{h_{\text{top}}(\Omega, T)}{\log(1/c)}. \]

**Proof.** We first prove $\text{mdim}_H(X, \sigma, d) = \text{mdim}_M(X, \sigma, d)$. It is enough to show\[ \text{mdim}_H(X, \sigma, d) \geq \text{mdim}_M(X, \sigma, d). \]
Let $0 < \varepsilon, \tau < 1$. By Lemma 4.7 and Claim 4.10, there exists $\delta_0 = \delta_0(\varepsilon, c/A, \tau) > 0$ such that for any natural number $N$
\[ \dim_H(X, d_N, \delta_0) \geq \tau \cdot \frac{\log \#(X, d_N, 9\varepsilon)}{\log(\frac{\delta_0}{\varepsilon})}. \]
Divide this by $N$ and let $N \to \infty$:
\[ \text{mdim}_H(X, \sigma, d) \geq \liminf_{N \to \infty} \frac{\dim_H(X, d_N, \delta_0)}{N} \geq \lim_{N \to \infty} \tau \cdot \frac{\log \#(X, d_N, 9\varepsilon)}{N \log(\frac{\delta_0}{\varepsilon})}. \]
Letting $\varepsilon \to 0$, we get
\[ \text{mdim}_H(X, \sigma, d) \geq \tau \cdot \text{mdim}_M(X, \sigma, d). \]
Letting $\tau \to 1$, we conclude:
\[ \text{mdim}_H(X, \sigma, d) \geq \text{mdim}_M(X, \sigma, d). \]
Next we prove
\[ \text{mdim}_M(X, \sigma, d) \leq \frac{h_{\text{top}}(\Omega, T)}{\log(1/c)}. \]
Let $\rho$ be a metric on $\Omega$. For $N \geq 1$ we define a metric $\rho_N$ on $\Omega$ by
\[ \rho_N(\omega, \omega') = \max_{0 \leq n < N} \rho(T^n \omega, T^n \omega'). \]
For $\omega, \omega' \in \Omega$ and $x, y \in X$
\[ d(S_\omega(x), S_{\omega'}(y)) = d(cx + a(\omega), cy + a(\omega')) \]
\[ = \sum_{n=1}^{\infty} 2^{-n} |cx_n + a(\omega)_n - cy_n - a(\omega')_n| \]
\[ \leq cd(x, y) + d(a(\omega), a(\omega')). \]
Similarly, for any natural number $N$
\[ d_N(S_\omega(x), S_{\omega'}(y)) \leq cd_N(x, y) + d_N(a(\omega), a(\omega')). \]
By repeatedly applying this inequality, for \(\omega_1, \ldots, \omega_n, \omega'_1, \ldots, \omega'_n \in \Omega\) and \(x, y \in X\) (4.3)

\[
d_N \left( S_{\omega_1} \circ \cdots \circ S_{\omega_n}(x), S_{\omega'_1} \circ \cdots \circ S_{\omega'_n}(y) \right) \leq c^n d_N(x, y) + \sum_{i=1}^{n} c^{i-1} d_N(a(\omega_i), a(\omega'_i))
\]

\[
\leq c^n d_N(x, y) + \frac{\max_{1 \leq i \leq n} d_N(a(\omega_i), a(\omega'_i))}{1 - c}
\]

\[
< c^n A + \frac{\max_{1 \leq i \leq n} d_N(a(\omega_i), a(\omega'_i))}{1 - c}.
\]

In the last inequality we have used \(\text{Diam}(X, d_N) = \text{Diam}(X, d) < A\).

Let 0 < \(\varepsilon\) < 1. We take \(\delta > 0\) such that

\[
\rho(\omega, \omega') < \delta \implies d(a(\omega), a(\omega')) < (1 - c)\frac{\varepsilon}{6}.
\]

Then for any natural number \(N\) we have

\[
\rho_N(\omega, \omega') < \delta \implies d_N(a(\omega), a(\omega')) < (1 - c)\frac{\varepsilon}{6}.
\]

We take a natural number \(n\) satisfying

\[
A c^n < \frac{\varepsilon}{6} \leq A c^{n-1}.
\]

Let \(\Omega_{N, \delta} \subset \Omega\) be a \(\delta\)-spanning set with respect to \(\rho_N\) with \(|\Omega_{N, \delta}| = \#(\Omega, \rho_N, \delta)\). Fix \(p \in X\). For any \(\omega_1, \ldots, \omega_n \in \Omega\) we can find \(\omega'_1, \ldots, \omega'_n \in \Omega_{N, \delta}\) satisfying \(\rho_N(\omega_i, \omega'_i) < \delta\) (and then \(d_N(a(\omega_i), a(\omega'_i)) < (1 - c)\varepsilon/6\)). Then for any \(x \in X\), by (4.3)

\[
d_N \left( S_{\omega_1} \circ \cdots \circ S_{\omega_n}(x), S_{\omega'_1} \circ \cdots \circ S_{\omega'_n}(p) \right) < \frac{\varepsilon}{3}.
\]

Since we know that

\[
X = \bigcup_{\omega_1, \ldots, \omega_n \in \Omega} S_{\omega_1} \circ \cdots \circ S_{\omega_n}(X),
\]

this implies that the set

\[
\{ S_{\omega'_1} \circ \cdots \circ S_{\omega'_n}(p) \mid \omega'_1, \ldots, \omega'_n \in \Omega_{N, \delta} \}
\]

is a \(\varepsilon/3\)-spanning set of \(X\) with respect to \(d_N\). Therefore

\[
\#(X, d_N, \varepsilon) \leq |\Omega_{N, \delta}|^n = \#(\Omega, \rho_N, \delta)^n.
\]

Since \(\varepsilon/6 \leq A c^{n-1}\), we have

\[
n \log (1/c) \leq \log \left( \frac{6A}{c^2} \right).
\]

Hence

\[
\frac{\log \#(X, d_N, \varepsilon)}{N \log \left( \frac{6A}{c^2} \right)} \leq \frac{\log \#(\Omega, \rho_N, \delta)}{N \log (1/c)}.
\]

Letting \(N \to \infty\)

\[
\lim_{N \to \infty} \frac{\log \#(X, d_N, \varepsilon)}{N \log \left( \frac{6A}{c^2} \right)} \leq \lim_{N \to \infty} \frac{\log \#(\Omega, \rho_N, \delta)}{N \log (1/c)} \leq \frac{h_{\text{top}}(\Omega, T)}{\log (1/c)}.
\]
Letting $\varepsilon \to 0$, we conclude

$$\text{mdim}_M(X, \sigma, d) \leq \frac{h_{\text{top}}(\Omega, T)}{\log(1/c)}.$$ 

\[\square\]

5. Infinite dimensional carpets

In this section we study a mean dimension version of Bedford–McMullen carpets. This provides a natural example for which mean Hausdorff dimension and metric mean dimension do not coincide.

5.1. Weighted topological entropy. Here we review the theory of weighted topological entropy. We will need this notion for formulating the main result of this section.

The weighted topological entropy was originally introduced by Feng–Huang [FH16]. The presentation here follows the approach of [Tsu21].

Let $(X, T)$ and $(Y, S)$ be dynamical systems. A map $\pi : X \to Y$ is called a factor map if $\pi$ is a continuous surjection satisfying $\pi \circ T = S \circ \pi$. We often use the notation $\pi : (X, T) \to (Y, S)$ for clarifying the underlying dynamics.

For a real number $0 \leq w \leq 1$ and a factor map $\pi : (X, T) \to (Y, S)$ we will define a weighted topological entropy $h^w_{\text{top}}(\pi, X, T)$. Let $d$ and $d'$ be metrics on $X$ and $Y$ respectively. For a natural number $N$ we define new metrics $d_N$ and $d'_N$ on $X$ and $Y$ by

$$d_N(x_1, x_2) = \max_{0 \leq n < N} d(T^nx_1, T^nx_2), \quad d'_N(y_1, y_2) = \max_{0 \leq n < N} d'(S^ny_1, S^ny_2).$$

For $\varepsilon > 0$ and a subset $U \subset X$ we define

$$\#(U, d_N, \varepsilon) = \min \left\{ n \geq 1 \mid \begin{array}{l}
\exists \text{ open subsets } U_1, \ldots, U_n \text{ of } X \text{ with } \\
U \subset U_1 \cup \cdots \cup U_n \text{ and } \text{Diam}(U_k, d_N) < \varepsilon \end{array} \right\}.$$

When $U$ is empty, we set $\#(U, d_N, \varepsilon) = 0$.

Recall $0 \leq w \leq 1$. We set

$$\#^w(\pi, X, d_N, d'_N, \varepsilon)$$

$$= \min \left\{ \sum_{k=1}^n \left( \#(\pi^{-1}(V_k), d_N, \varepsilon) \right)^w \mid \begin{array}{l}
Y = V_1 \cup \cdots \cup V_n \text{ is an open cover with } \\
\text{Diam}(V_k, d'_N) < \varepsilon \text{ for all } 1 \leq k \leq n \end{array} \right\}.$$ 

This quantity is sub-multiplicative in $N$ and monotone in $\varepsilon$. We define the $w$-weighted topological entropy of the factor map $\pi : (X, T) \to (Y, S)$ by

$$h^w_{\text{top}}(\pi, X, T) = \lim_{\varepsilon \to 0} \left( \lim_{N \to \infty} \frac{\log \#^w(\pi, X, d_N, d'_N, \varepsilon)}{N} \right).$$

The value of $h^w_{\text{top}}(\pi, X, T)$ is independent of the choices of metrics $d$ and $d'$. So it provides a topological invariant.
The weighted topological entropy satisfies the following variational principle ([FH16, Theorem 1.4, Corollary 1.5] and [Tsu21, Theorem 1.3]):

\[(5.1) \quad h^w_{\text{top}}(\pi, X, T) = \sup_{\mu \in \mathcal{M}^T(X)} \{wh_\mu(X, T) + (1-w)h_{\pi_*\mu}(Y, S)\}.\]

Here \(\mathcal{M}^T(X)\) is the set of \(T\)-invariant Borel probability measures on \(X\), and \(h_\mu(X, T)\) and \(h_{\pi_*\mu}(Y, S)\) are the Kolmogorov–Sinai entropy of the measure preserving systems \((X, T, \mu)\) and \((Y, S, \pi_*\mu)\) respectively.

In the next subsection we need to use the weighted topological entropy for a factor map between symbolic dynamical systems. In the case of symbolic dynamics, the above formulation of the weighted topological entropy is essentially the same with the one given in [BF09, Theorem 1.1] and [BF12, Theorem 3.1].

Let \(A\) and \(B\) be finite sets, and let \(((A \times B)^N, \sigma)\) and \((B^N, \sigma)\) be the full-shifts on the alphabets \(A \times B\) and \(B\) respectively. Let \(\pi : (A \times B)^N \to B^N\) be the natural projection. Let \(\Omega \subset (A \times B)^N\) be a subshift (shift-invariant closed subset). Set \(\Omega' := \pi(\Omega) \subset B^N\).

For a natural number \(N\), we define \(\Omega|_N \subset (A \times B)^N\) and \(\Omega'|_N \subset B^N\) as the images of the projections of \(\Omega\) and \(\Omega'\) to the first \(N\) coordinates, respectively. We denote by \(\pi_N : \Omega|_N \to \Omega'|_N\) the natural projection map.

**Lemma 5.1.** In the above setting, for \(0 \leq w \leq 1\), the weighted topological entropy of the factor map \(\pi : (\Omega, \sigma) \to (\Omega', \sigma)\) is given by

\[h^w_{\text{top}}(\pi, \Omega, \sigma) = \lim_{N \to \infty} \frac{1}{N} \log \sum_{v \in \Omega'|_N} |\pi^{-1}_N(v)|^w.\]

Here \(|\pi^{-1}_N(v)|\) is the cardinality of \(\pi^{-1}_N(v) \subset \Omega|_N\).

Notice that \(\sum_{v \in \Omega'|_N} |\pi^{-1}_N(v)|^w\) is sub-multiplicative in \(N\). So the above limit exists.

**Proof.** It is immediate to check

\[h^w_{\text{top}}(\pi, \Omega, \sigma) \leq \lim_{N \to \infty} \frac{1}{N} \log \sum_{v \in \Omega'|_N} |\pi^{-1}_N(v)|^w.\]

Here we prove the reverse inequality. We define metrics \(d\) and \(d'\) on \((A \times B)^N\) and \(B^N\) by

\[d((x, y), (x', y')) = 2^{-\min\{n|(x_n, y_n)\neq(x'_n, y'_n)\}}, \quad d'(y, y') = 2^{-\min\{|y_n|y_n\neq y'_n\}}.\]

Let \(N\) be a natural number. For subsets \(U \subset (A \times B)^N\) and \(V \subset B^N\), we denote by \(U|_N \subset (A \times B)^N\) and \(V|_N \subset B^N\) the projections to the first \(N\)-coordinates. If \(\text{Diam}(U, d_N) < 1\) then \(U|_N\) is a singleton or empty. Similarly, if \(\text{Diam}(V, d'_N) < 1\) then so is \(V|_N\).

Let \(0 < \varepsilon < 1\). Let \(\Omega' = \Omega_1 \cup \cdots \cup \Omega_n\) be an open cover with \(\text{Diam}(\Omega_k, d'_N) < \varepsilon\) and

\[\#^w(\pi, \Omega, d_N, d'_N, \varepsilon) = \sum_{k=1}^n \left(\#(\pi^{-1}(\Omega_k), d_N, \varepsilon)\right)^w.\]
We also assume that any $V_k$ is not empty. (Hence $V_k|_N$ is a singleton.) Set $t_k = \#(\pi^{-1}(V_k), d_N, \varepsilon)$.

For each $V_k$ we take an open cover $\pi^{-1}(V_k) = U_{k1} \cup U_{k2} \cup \cdots \cup U_{kt_k}$ with $\text{Diam} (U_{kl}, d_N) < \varepsilon$ for all $1 \leq l \leq t_k$. Any $U_{kl}$ is not empty. So $U_{kl}|_N$ is a singleton. We have

$$
\#^w (\pi, \Omega, d_N, d'_N, \varepsilon) = \sum_{k=1}^n t_k^w
$$

$$
= \sum_{v \in \Omega'} \left( \sum_{k: V_k|_N = \{v\}} t_k^w \right).
$$

Here, in the sum $\sum_{k: V_k|_N = \{v\}}$ in the second line, $k$ runs over all index $k \in [1, n]$ satisfying $V_k|_N = \{v\}$. Since we assume $0 \leq w \leq 1$,

$$
\sum_{k: V_k|_N = \{v\}} t_k^w \geq \left( \sum_{k: V_k|_N = \{v\}} t_k \right)^w \quad \text{(by } x^w + y^w \geq (x+y)^w \text{ for } x, y \geq 0).\]

For $v \in \Omega'|_N$

$$
\pi_N^{-1}(v) = \bigcup_{k: V_k|_N = \{v\}} (\pi^{-1}(V_k))|_N = \bigcup_{k: V_k|_N = \{v\}} \bigcup_{l=1}^{t_k} (U_{kl}|_N).
$$

Since every $U_{kl}|_N$ is a singleton,

$$
|\pi_N^{-1}(v)| \leq \sum_{k: V_k|_N = \{v\}} t_k.
$$

Therefore

$$
\sum_{v \in \Omega'} \left| \pi_N^{-1}(v) \right|^w \leq \sum_{v \in \Omega'} \left( \sum_{k: V_k|_N = \{v\}} t_k \right)^w
$$

$$
\leq \sum_{v \in \Omega'} \left( \sum_{k: V_k|_N = \{v\}} t_k^w \right)
$$

$$
= \#^w (\pi, \Omega, d_N, d'_N, \varepsilon).
$$

Thus

$$
\lim_{N \to \infty} \frac{1}{N} \log \sum_{v \in \Omega'} \left| \pi_N^{-1}(v) \right|^w \leq \lim_{N \to \infty} \frac{1}{N} \log \#^w (\pi, \Omega, d_N, d'_N, \varepsilon) \leq h_{\text{top}}^w (\pi, \Omega, \sigma).
$$

\[\square\]

**Example 5.2.** Let $R \subset A \times B$ be a nonempty subset. Set $\Omega = R^N$. This is a subshift of $(A \times B)^N$. For each $v \in B$ we denote by $t(v)$ the number of $u \in A$ with $(u,v) \in R$. Then for each $v = (v_1, \ldots, v_N) \in \Omega'|_N$ we have

$$
|\pi_N^{-1}(v)| = t(v_1) \cdots t(v_N).
$$
By Lemma 5.1
\[
h^w_{\top}(\pi, \Omega, \sigma) = \lim_{N \to \infty} \frac{1}{N} \log \sum_{v \in \Omega^N} |\pi_N^{-1}(v)|^w
\]
\[
= \lim_{N \to \infty} \frac{1}{N} \log \left( \sum_{v \in B} t(v)^w \right)^N
\]
\[
= \log \sum_{v \in B} t(v)^w.
\]

Readers can find in [KP96] many wonderful calculations of \(h^w_{\top}(\pi, \Omega, \sigma)\) in the case that \(\Omega\) is a subshift of finite type or a sofic subshift\(^4\).

5.2. Main result for infinite dimensional carpets. Let \(a \geq b \geq 2\) be two natural numbers and set
\[
A = \{0, 1, 2, \ldots, a - 1\}, \quad B = \{0, 1, 2, \ldots, b - 1\}.
\]

Let \((A \times B)^N\) be the one-sided full-shift on the alphabet \(A \times B\) with the shift map \(\sigma : (A \times B)^N \to (A \times B)^N\). Let \(\pi : (A \times B)^N \to B^N\) be the natural projection. We also denote by \(\sigma : B^N \to B^N\) the shift map on \(B^N\).

Let \([0, 1]^N = [0, 1] \times [0, 1] \times [0, 1] \times \ldots\) be the infinite dimensional cube, and consider the product \([0, 1]^N \times [0, 1]^N\) with a metric \(d\) defined by
\[
d((x, y), (x', y')) = \sum_{n=1}^{\infty} 2^{-n} \max(|x_n - x'_n|, |y_n - y'_n|),
\]
where \(x = (x_n)_{n \in N}, y = (y_n)_{n \in N}\) and \(x' = (x'_n)_{n \in N}, y' = (y'_n)_{n \in N}\) are points in \([0, 1]^N\). We define a shift map on \([0, 1]^N \times [0, 1]^N\) (also denoted by \(\sigma : [0, 1]^N \times [0, 1]^N \to [0, 1]^N \times [0, 1]^N\)) by
\[
\sigma ((x_n)_{n \in N}, (y_n)_{n \in N}) = ((x_{n+1})_{n \in N}, (y_{n+1})_{n \in N}).
\]

Let \(\Omega \subset (A \times B)^N\) be a subshift, namely a closed subset satisfying \(\sigma(\Omega) \subset \Omega\). We assume \(\Omega \neq \emptyset\). We define a carpet system \(X_\Omega \subset [0, 1]^N \times [0, 1]^N\) by
\[
X_\Omega = \left\{ \left( \sum_{m=1}^{\infty} \frac{x_m}{a^m}, \sum_{m=1}^{\infty} \frac{y_m}{b^m} \right) \in [0, 1]^N \times [0, 1]^N \right\} \times \Omega \text{ for all } m \geq 1.
\]

Here \(x_m \in A^N \subset \ell^\infty\) and we consider the summation \(\sum_{m=1}^{\infty} \frac{a^m}{a^m}\) in \(\ell^\infty\). Then \(\sum_{m=1}^{\infty} \frac{x_m}{a^m}, \sum_{m=1}^{\infty} \frac{y_m}{b^m} \in [0, 1]^N\). Similarly for the term \(\sum_{m=1}^{\infty} \frac{y_m}{b^m}\).

\((X_\Omega, \sigma)\) is a subsystem of \(([0, 1]^N \times [0, 1]^N, \sigma)\). We are interested in its mean Hausdorff dimension and metric mean dimension with respect to the metric (5.2). Recall that \(\pi : (A \times B)^N \to B^N\) is the natural projection. We consider its restriction to \(\Omega\) and also denote it by \(\pi : \Omega \to \pi(\Omega)\). Set \(\Omega' = \pi(\Omega)\). This is a subshift of \(B^N\).

\(^4\)Kenyon–Peres [KP96] did not use the terminologies of weighted topological entropy, but we can interpret their results in terms of \(h^w_{\top}(\pi, \Omega, \sigma)\).
Theorem 5.3. In the above setting, the mean Hausdorff dimension and metric mean dimension of \((X_\Omega, \sigma, d)\) are given by

\[
\text{mdim}_H(X_\Omega, \sigma, d) = \frac{h_{\log a}^b (\pi, \Omega, \sigma)}{\log b},
\]

\[
\text{mdim}_M(X_\Omega, \sigma, d) = \frac{h_{\log a} (\Omega, \sigma)}{\log a} + \left( \frac{1}{\log b} - \frac{1}{\log a} \right) h_{\log a}^{\log b}(\Omega', \sigma).
\]

Here \(h_{\log a}^b (\pi, \Omega, \sigma)\) is the weighted topological entropy of the factor map \(\pi : (\Omega, \sigma) \to (\Omega', \sigma)\) with the weight \(\log a \log b = \log b / \log a\).

By the variational principle (5.1) for weighted topological entropy, the above formula of the mean Hausdorff dimension can be also expressed as

\[
\text{mdim}_H(X_\Omega, \sigma, d) = \frac{1}{\log b} \sup_{\mu \in M^\bullet(\Omega)} \left\{ (\log a \log b) h_{\log a} (\mu, \Omega, \sigma) + (1 - \log a \log b) h_{\pi, \mu} (\Omega', \sigma) \right\},
\]

On the other hand, by the standard variational principle for topological entropy

\[
\text{mdim}_M(X_\Omega, \sigma, d) = \sup_{\mu \in M^\bullet(\Omega)} \left\{ \frac{h_{\mu} (\Omega, \sigma)}{\log a} + \left( \frac{1}{\log b} - \frac{1}{\log a} \right) h_{\pi, \mu} (\Omega', \sigma) \right\}.
\]

So the difference lies in whether we take supremum simultaneously or separately for the two terms \(h_{\mu} (\Omega, \sigma)\) and \(h_{\nu} (\Omega', \sigma)\).

Example 5.4. Let \(a = 3\) and \(b = 2\). Then \(A = \{0, 1, 2\}\) and \(B = \{0, 1\}\). Set

\[R = \{(0, 0), (1, 1), (2, 0)\} \subset A \times B.\]

Define \(\Omega = R^\mathbb{N}\). Then \(\Omega' = B^\mathbb{N}\). By Example 5.2

\[h_{\log 3}^2 (\pi, \Omega, \sigma) = \log (1 + 2^{\log 3}.2)\]

It is also easy to see

\[h_{\log a} (\Omega, \sigma) = \log 3, \quad h_{\log a} (\Omega', \sigma) = \log 2.\]

Therefore by Theorem 5.3

\[
\text{mdim}_H(X_\Omega, \sigma, d) = \log 2 \left(1 + 2^{\log 3}.2\right) = 1.3496838201 \ldots,
\]

\[
\text{mdim}_M(X_\Omega, \sigma, d) = 1 + \left( \frac{1}{\log 2} - \frac{1}{\log 3} \right) \log 2
\]

\[= 2 - \log 3.2 = 1.3690702464 \ldots.\]

As we already mentioned at the end of §5.1, readers can find many interesting calculations of \(h_{\log a}^b (\pi, \Omega, \sigma)\) for sofic subshifts \(\Omega\) in [KP96].
5.3. Calculation of metric mean dimension. Here we prove the formula (5.4) of the metric mean dimension of the carpet system $(X_\Omega, \sigma^d)$. We use the notations of Theorem 5.3. Set
\[ w = \log_a b = \frac{\log b}{\log a}. \]
Since \( a \geq b \geq 2 \), we have \( 0 < w \leq 1 \). For a natural number \( N \) we denote by \( \Omega|_N \) and \( \Omega'|_N \) the images of \( \Omega \) and \( \Omega' \) under the projections
\[ (A \times B)^N \to A^N \times B^N, \quad ((u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}}) \mapsto ((u_1, \ldots, u_N), (v_1, \ldots, v_N)), \]
\[ B^N \to B^N, \quad (v_n)_{n \in \mathbb{N}} \mapsto (v_1, \ldots, v_N). \]
We also define \( X_{\Omega|_N} \) as the image of \( X_\Omega \) under the projection
\[ [0, 1]^N \times [0, 1]^N \to [0, 1]^N \times [0, 1]^N, \quad ((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) \mapsto ((x_1, \ldots, x_N), (y_1, \ldots, y_N)). \]
We have
\[ X_{\Omega|_N} = \left\{ \left( \sum_{m=1}^{\infty} \frac{x_m}{a^m}, \sum_{m=1}^{\infty} \frac{y_m}{b^m} \right) \in [0, 1]^N \times [0, 1]^N \mid (x_m, y_m) \in \Omega|_N \right\}. \]
Let \((x, y) \in (\Omega|_N)^N\) where \( x = (x_m)_{m \in \mathbb{N}} \) and \( y = (y_m)_{m \in \mathbb{N}} \) with \( x_m \in A^N, \ y_m \in B^N \) and \( (x_m, y_m) \in \Omega|_N \). For natural numbers \( N \) and \( M \), we define a subset \( Q_{N,M}(x, y) \subset X_{\Omega|_N} \) by
\[ Q_{N,M}(x, y) = \left\{ \left( \sum_{m=1}^{\infty} \frac{x'_m}{a^m}, \sum_{m=1}^{\infty} \frac{y'_m}{b^m} \right) \mid \begin{array}{l} (x'_m, y'_m) \in \Omega|_N \text{ for all } m \geq 1 \text{ with } \\
| x'_m = x_m (1 \leq m \leq \lfloor wM \rfloor) \text{ and } \\
y'_m = y_m (1 \leq m \leq M) \end{array} \right\}. \]
Here \( \lfloor wM \rfloor \) is the largest integer not greater than \( wM \). The set \( Q_{N,M}(x, y) \) depends only on the coordinates \( x_1, \ldots, x_{\lfloor wM \rfloor}; y_1, \ldots, y_M \). So we also denote it by
\[ Q_{N,M}(x_1, \ldots, x_{\lfloor wM \rfloor}; y_1, \ldots, y_M) \quad (= Q_{N,M}(x, y)). \]
From \( w = \log_a b \), we have \( a^{-wM} = b^{-M} \) and
\[ b^{-M} \leq a^{-\lfloor wM \rfloor} < ab^{-M}. \]
We have
\[ \text{Diam} (Q_{N,M}(x, y), \| \cdot \|_\infty) \leq \max (a^{-\lfloor wM \rfloor}, b^{-M}) = a^{-\lfloor wM \rfloor} < ab^{-M}. \]
Here \( \| \cdot \|_\infty \) is the \( \ell^\infty \)-distance (i.e. the distance defined by the \( \ell^\infty \)-norm) on \( X_{\Omega|_N} \subset \mathbb{R}^{2N} \).

**Lemma 5.5.** Let \( \varepsilon \) be a positive number. For any \( N \geq 1 \) we have
\[ \# (X_{\Omega|_N}, \| \cdot \|_\infty, \varepsilon) \leq \# (X_\Omega, d_N, \varepsilon). \]
Let \( L \) be a natural number satisfying \( \sum_{n \leq L} 2^{-n} < \varepsilon/2 \). Then
\[ \# (X_{\Omega}, d_N, \varepsilon) \leq \# (X_{\Omega|_{N+L}}, \| \cdot \|_\infty, \frac{\varepsilon}{2}). \]
**Proof.** The first inequality follows from
\[ \|x|_N - y|_N\|_\infty \leq d_N(x, y) \quad (x, y \in X_N), \]
where $x|_N$ denotes the projection of $x$ to $X_N|_N$. The second inequality follows from
\[ d_N(x, y) < \|x|_{N+L} - y|_{N+L}\|_\infty + \frac{\varepsilon}{2}. \]

□

**Lemma 5.6.** For any natural numbers $N$ and $M$ we have
\[ \# \left( X_{N,M}, \|\cdot\|_\infty, ab^{-M} \right) \leq |\Omega|_N^{[wM]} \cdot |\Omega'|_N^{[wM] - [wM]}, \]
\[ \# \left( X_{N,M}, \|\cdot\|_\infty, b^{-M} \right) \geq |\Omega|_N^{[wM]} \cdot |\Omega'|_N^{[wM] - [wM]}. \]

Here $|\Omega|_N$ and $|\Omega'|_N$ denote the cardinalities of $\Omega|_N$ and $\Omega'|_N$ respectively.

**Proof.** Recall that $\text{Diam} \left( Q_{N,M}(x, y), \|\cdot\|_\infty \right) < ab^{-M}$. The first inequality follows from
\[ X_{N,M} = \bigcup \left\{ Q_{N,M}(x_1, \ldots, x_{[wM]}, y_1, \ldots, y_M) \mid \begin{array}{l}
(x_m, y_m) \in \Omega|_N \text{ for } 1 \leq m \leq [wM] \\
y_m \in \Omega'|_N \text{ for } [wM] + 1 \leq m \leq M
\end{array} \right\}. \]

Next we consider the second inequality. We fix a point $(\xi, \eta) \in \Omega|_N$. For each $v \in \Omega'|_N$ we pick up $s(v) \in A^N$ satisfying $(s(v), v) \in \Omega|_N$. For $(x_m, y_m) \in \Omega|_N$ $(1 \leq m \leq [wM])$ and $y_m \in \Omega'|_N$ $( [wM] + 1 \leq m \leq M)$, we set
\[ p(x_1, \ldots, x_{[wM]}, y_1, \ldots, y_M) = \left( \sum_{m=1}^{[wM]} \frac{x_m}{a^m} + \sum_{m=[wM]+1}^{M} \frac{s(y_m)}{a^m} + \sum_{m=M+1}^{\infty} \frac{\xi}{a^m} \sum_{m=1}^{M} \frac{y_m}{b^m} + \sum_{m=M+1}^{\infty} \frac{\eta}{b^m} \right). \]

This is a point in $X_{N,M}$. For $(x_1, \ldots, x_{[wM]}, y_1, \ldots, y_M) \neq (x'_1, \ldots, x'_{[wM]}, y'_1, \ldots, y'_M)$, we have
\[ \|p(x_1, \ldots, x_{[wM]}, y_1, \ldots, y_M) - p(x'_1, \ldots, x'_{[wM]}, y'_1, \ldots, y'_M)\|_\infty \geq \min \left( a^{-[wM]}, b^{-M} \right) = b^{-M}. \]

Hence the set
\[ \left\{ p(x_1, \ldots, x_{[wM]}, y_1, \ldots, y_M) \mid \begin{array}{l}
(x_m, y_m) \in \Omega|_N \text{ for } 1 \leq m \leq [wM] \\
y_m \in \Omega'|_N \text{ for } [wM] + 1 \leq m \leq M
\end{array} \right\} \]
is $b^{-M}$-separated with respect to the $l^\infty$-distance. The second inequality follows from this. □
From Lemmas 5.5 and 5.6 we can calculate the metric mean dimension of \((X,\sigma,d)\):

\[
\text{mdim}_M (X,\sigma,d) = \lim_{M \to \infty} \left\{ \lim_{N \to \infty} \frac{\log \left( |\Omega|_N^{\left\lfloor \frac{wM}{M-\lfloor wM \rfloor} \right\rfloor} \cdot |\Omega'|_N^{M-\lfloor wM \rfloor} \right)}{NM \log b} \right\}
\]

\[
= \lim_{M \to \infty} \left\{ \frac{1}{M \log b} \lim_{N \to \infty} \left( \frac{\lfloor wM \rfloor \log |\Omega|_N + (M-\lfloor wM \rfloor) \log |\Omega'|_N}{N} \right) \right\}
\]

\[
= \lim_{M \to \infty} \frac{\lfloor wM \rfloor h_{\text{top}}(\Omega,\sigma) + (M-\lfloor wM \rfloor) h_{\text{top}}(\Omega',\sigma)}{M \log b}
\]

\[
= \frac{w}{\log b} h_{\text{top}}(\Omega,\sigma) + \frac{1-w}{\log b} h_{\text{top}}(\Omega',\sigma)
\]

\[
= \frac{h_{\text{top}}(\Omega,\sigma)}{\log a} + \left( \frac{1}{\log b} - \frac{1}{\log a} \right) h_{\text{top}}(\Omega',\sigma), \quad (w = \log_a b = \frac{\log b}{\log a}).
\]

This proves the formula (5.4) in Theorem 5.3.

5.4. Preparations. The calculation in the last subsection was rather straightforward. It is more difficult to calculate the mean Hausdorff dimension. This subsection is a preparation for it.

First we prepare a general result on metric geometry. This result will be also used in the Appendix.

**Lemma 5.7.** Let \(c,\varepsilon, s\) be positive numbers. Let \((X,d)\) be a compact metric space with a Borel probability measure \(\mu\). Suppose:

- \(6\varepsilon^c < 1\).
- For any \(x \in X\) there exists a Borel subset \(A \subset X\) satisfying \(x \in A\) and

\[
0 < \text{Diam} A < \frac{\varepsilon}{6}, \quad \mu(A) \geq (\text{Diam} A)^s.
\]

Then \(\dim_H(X,d,\varepsilon) \leq (1+c)s\).

**Proof.** For any \(x \in X\) there exists \(A_x \subset X\) with \(x \in A_x\) satisfying \(0 < \text{Diam} A_x < \varepsilon/6\) and \(\mu(A_x) \geq (\text{Diam} A_x)^s\). The set \(A_x\) is contained in \(B(x,\text{Diam} A_x)\) (the closed ball of radius \(\text{Diam} A_x\) centered at \(x\)). By the compactness, there are \(x_1,\ldots,x_n \in X\) satisfying (let \(r_k := \text{Diam} A_{x_k}\))

\[
X = \bigcup_{k=1}^n B(x_k, r_k), \quad \mu(B(x_k, r_k)) \geq \mu(A_{x_k}) \geq r_k^s.
\]

By the finite Vitali covering lemma [EW11, Lemma 2.27], we can pick \(x_{k_1},\ldots,x_{k_m}\) such that

- \(B(x_{k_1}, r_{k_1}), \ldots, B(x_{k_m}, r_{k_m})\) are mutually disjoint.
- \(X = \bigcup_{j=1}^m B(x_{k_j}, 3r_{k_j})\).
Then \( \text{Diam} B(x_{k_j}, 3r_{k_j}) \leq 6r_{k_j} < \varepsilon \) and

\[
\sum_{j=1}^{m} (6r_{k_j})^{(1+c)s} = \sum_{j=1}^{m} (6r_{k_j})^{c s} \cdot (6r_{k_j})^{s} < \sum_{j=1}^{m} \varepsilon^{c s} \cdot (6r_{k_j})^{s} = \sum_{j=1}^{m} (6\varepsilon^{c})^{s} \cdot r_{k_j}^{s} < \sum_{j=1}^{m} r_{k_j}^{s} \quad \text{(by } 6\varepsilon^{c} < 1) \]

\[
\leq \sum_{j=1}^{m} \mu (B(x_{k_j}, r_{k_j})) \quad \text{(by } \mu (B(x_k, r_k)) \geq r_{k}) \]

\[
\leq 1 \quad \text{(since } B(x_{k_1}, r_{k_1}), \ldots, B(x_{k_m}, r_{k_m}) \text{ are mutually disjoint).}
\]

This shows \( \operatorname{dim}_H(X, d, \varepsilon) \leq (1 + c)s \).

\[ \square \]

We also need some basic results of probability theory. For a random variable \( \xi \) we denote its mean and variance by \( E(\xi) \) and \( V(\xi) := E[(\xi - E(\xi))^2] \) respectively. Recall the Kolmogorov maximal inequality (see, e.g. [Dur10, Theorem 2.5.2], [Fel68, Chapter IX, §7]):

**Theorem 5.8** (Kolmogorov maximal inequality). Let \( \xi_1, \ldots, \xi_n \) be independent random variables with \( E(\xi_k) = 0 \) and \( V(\xi_k) < \infty \) for all \( 1 \leq k \leq n \). Set \( S_k = \xi_1 + \cdots + \xi_k \). Then for any positive number \( h \)

\[
P \left( \max_{1 \leq k \leq n} |S_k| \geq h \right) \leq \frac{V(S_n)}{h^2}.
\]

The next lemma shows simple consequences of this theorem.

**Lemma 5.9.**

1. Let \( \xi_1, \xi_2, \xi_3, \ldots \) be independent and identically distributed (i.i.d.) random variables with \( \xi_1 \in L^2 \). Then for any positive number \( \delta \) and natural number \( m \)

\[
P \left( \sup_{n \geq m} \left| \frac{\xi_1 + \cdots + \xi_n}{n} - E(\xi_1) \right| > \delta \right) \leq \frac{4V(\xi_1)}{\delta^2 n}.
\]

2. For any positive numbers \( C \) and \( \delta \) there exists a natural number \( m_0 = m_0(C, \delta) \) for which the following statement holds true: If \( \xi_1, \xi_2, \xi_3, \ldots \) be i.i.d. random variables with \( \xi_1 \in L^2 \) and \( V(\xi_1) \leq C \), then

\[
P \left( \sup_{n \geq m_0} \left| \frac{\xi_1 + \cdots + \xi_n}{n} - E(\xi_1) \right| \leq \delta \right) \geq \frac{3}{4}.
\]
(3) For any $0 < w \leq 1$ and any positive numbers $C$ and $\delta$, there exists a natural number $m_1 = m_1(w, C, \delta)$ for which the following statement holds true: If $\xi_1, \xi_2, \xi_3, \ldots$ be i.i.d. random variables with $\xi_1 \in L^2$, $|E\xi_1| \leq C$ and $\forall \xi_1 \leq C$ then
\[
P \left( \sup_{n \geq m_1} \left| \frac{\xi_1 + \cdots + \xi_n}{\sqrt{n}} - \frac{\xi_1 + \cdots + \xi_{\lfloor wn \rfloor}}{\sqrt{\lfloor wn \rfloor}} \right| \leq \delta \right) \geq \frac{1}{2}.
\]

Proof. (1) Let $\sigma^2 = \mathbb{V}(\xi_1)$. Set $\eta_n = \xi_n - E\xi_n$ and $S_n = \eta_1 + \cdots + \eta_n$. We have $\mathbb{V}(S_n) = n\sigma^2$. By the Kolmogorov maximal inequality, for any $h > 0$ and natural number $n$
\[
P \left( \max_{1 \leq k \leq \frac{\delta n}{2}} |S_k| \geq \delta \right) \leq \frac{2\sigma^2}{\delta^2 n}.
\]

Letting $h = \delta n$, we get
\[
P \left( \max_{n \leq k \leq 2n} |S_k| \geq \delta n \right) \leq \frac{2\sigma^2}{\delta^2 n}.
\]

We have
\[
\left\{ \max_{n \leq k \leq 2n} \frac{|S_k|}{k} \geq \delta \right\} \subseteq \left\{ \max_{n \leq k \leq 2n} |S_k| \geq \delta n \right\}.
\]

Hence
\[
P \left( \max_{n \leq k \leq 2n} \left| \frac{S_k}{k} \right| \geq \delta \right) \leq \frac{2\sigma^2}{\delta^2 n}.
\]

We use this inequality for $n = m, 2m, 2^2m, 2^3m, \ldots$ Then for each $i = 0, 1, 2, \ldots$
\[
P \left( \max_{2^im \leq k \leq 2^{i+1}m} \left| \frac{S_k}{k} \right| \geq \delta \right) \leq \frac{2\sigma^2}{2^i\delta^2m}.
\]

We have
\[
\left\{ \sup_{k \geq m} \left| \frac{S_k}{k} \right| > \delta \right\} = \bigcup_{i=0}^{\infty} \left\{ \max_{2^im \leq k \leq 2^{i+1}m} \left| \frac{S_k}{k} \right| > \delta \right\}.
\]

Therefore
\[
P \left( \sup_{k \geq m} \left| \frac{S_k}{k} \right| > \delta \right) \leq \sum_{i=0}^{\infty} \mathbb{P} \left( \max_{2^im \leq k \leq 2^{i+1}m} \left| \frac{S_k}{k} \right| \geq \delta \right) \leq \frac{4\sigma^2}{\delta^2 m}.
\]

(2) Take a natural number $m_0$ with $\frac{4C}{\delta^2 m_0} \leq \frac{1}{4}$. By (1)
\[
P \left( \sup_{n \geq m_0} \left| \frac{\xi_1 + \cdots + \xi_n}{n} - E\xi_1 \right| > \delta \right) \leq \frac{1}{4}.
\]

Hence
\[
P \left( \sup_{n \geq m_0} \left| \frac{\xi_1 + \cdots + \xi_n}{n} - E\xi_1 \right| \leq \delta \right) \geq \frac{3}{4}.
\]

(3) Let $m_0 = m_0(C, \delta/3)$ be the natural number given in (2). We take a natural number $m_1$ satisfying
\[
m_1 \geq \frac{m_0}{w}, \quad m_1 \geq \frac{3C}{w \delta}.
\]

By (2)
\[
P \left( \sup_{n \geq m_0} \left| \frac{\xi_1 + \cdots + \xi_n}{n} - E\xi_1 \right| \leq \frac{\delta}{3} \right) \geq \frac{3}{4}.
\]
For \( n \geq m_1 \) we have \([nw] \geq m_0\). Hence

\[
\mathbb{P} \left( \sup_{n \geq m_1} \left| \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} - \mathbb{E}\xi_1 \right| \leq \frac{\delta}{3} \right) \geq \frac{3}{4}.
\]

We have

\[
\left| \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} - \mathbb{E}\xi_1 \right| \leq \left| \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} - \mathbb{E}\xi_1 \right| + \left( \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} - 1 \right) \mathbb{E}\xi_1
\]

\[
\leq \left| \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} - \mathbb{E}\xi_1 \right| + \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} \mathbb{E}\xi_1
\]

\[
\leq \left| \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} - \mathbb{E}\xi_1 \right| + \frac{C}{wn}.
\]

For \( n \geq m_1 \) we have \( \frac{C}{wn} \leq \frac{C}{w_{m_1}} \leq \frac{\delta}{3} \). Hence

\[
\sup_{n \geq m_1} \left| \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} - \mathbb{E}\xi_1 \right| \leq \sup_{n \geq m_1} \left| \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} - \mathbb{E}\xi_1 \right| + \frac{\delta}{3}.
\]

Therefore

\[
\mathbb{P} \left( \sup_{n \geq m_1} \left| \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} - \mathbb{E}\xi_1 \right| \leq \frac{2\delta}{3} \right) \geq \frac{3}{4}.
\]

We have

\[
\left\{ \sup_{n \geq m_1} \left| \frac{\xi_1 + \cdots + \xi_n}{n} - \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} \right| \leq \delta \right\}
\]

\[
\sup \left\{ \sup_{n \geq m_1} \left| \frac{\xi_1 + \cdots + \xi_n}{n} - \mathbb{E}\xi_1 \right| \leq \frac{\delta}{3} \right\} \cap \left\{ \sup_{n \geq m_1} \left| \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} - \mathbb{E}\xi_1 \right| \leq \frac{2\delta}{3} \right\}.
\]

Therefore

\[
\mathbb{P} \left( \sup_{n \geq m_1} \left| \frac{\xi_1 + \cdots + \xi_n}{n} - \frac{\xi_1 + \cdots + \xi_{[wn]}}{wn} \right| \leq \delta \right) \geq \frac{1}{2}.
\]

\small
\[\square\]

We also need the following elementary lemma.

**Lemma 5.10.** Let \( 0 < w \leq 1 \) and \( C \geq 0 \). Let \( \{a_n\}_{n=0}^{\infty} \) be a sequence of real numbers with \( 0 \leq a_n \leq C \). For any \( \delta > 0 \) and any natural number \( m \) there exists \( n \) with

\[
m \leq n \leq w^{-[C/\delta]} \left( m + \left[ \frac{C}{\delta} \right] \right) + 1
\]

satisfying

\[
a_n - a_{[wn]} \geq -\delta.
\]

The number \( w^{-[C/\delta]} \left( m + \left[ \frac{C}{\delta} \right] \right) + 1 \) looks complicated. We will not need its precise form in the sequel. The point is that it depends only on \( w, C, \delta, m \).
Proof. Set \( k = \lceil C/\delta \rceil \). Suppose that for all integers \( n \in [m, w^{-k}(m + k) + 1] \) we have \( a_n - a_{\lfloor wn \rfloor} < -\delta \).

We define a non-increasing sequence \( n_0 \geq n_1 \geq \cdots \geq n_k \) by

\[
n_0 = \lfloor w^{-k}(k + m) + 1 \rfloor, \quad n_{j+1} = \lfloor wn_j \rfloor.
\]

We have \( n_{j+1} \geq wn_j - 1 \) and hence

\[
n_{j+1} + \frac{1}{1-w} \geq w \left( n_j + \frac{1}{1-w} \right).
\]

Then

\[
n_k \geq w^k \left( n_0 + \frac{1}{1-w} \right) - \frac{1}{1-w} = w^k n_0 - \frac{1 - w^k}{1-w} = w^k n_0 - (1 + w + w^2 + \cdots + w^{k-1}) \geq w^k n_0 - k \geq w^k (w^{-k}(m + k)) - k \quad (\text{by } n_0 = [w^{-k}(k + m) + 1])
\]

\[= m.
\]

So all \( n_0, n_1, \ldots, n_k \) belong to \( [m, w^{-k}(m + k) + 1] \). Therefore

\[a_{n_j} - a_{n_{j+1}} < -\delta \quad \text{for all } 0 \leq j \leq k - 1.
\]

Since we assumed \( a_{n_k} \leq C \),

\[
a_{n_0} < a_{n_k} - \delta k \leq C - C \quad (\text{by } k = \lceil C/\delta \rceil)
\]

\[= 0.
\]

Namely \( a_{n_0} < 0 \). This contradicts with the assumption \( a_n \geq 0 \). \( \square \)

5.5. Calculation of mean Hausdorff dimension. We calculate the mean Hausdorff dimension of the carpet system \((X_\Omega, \sigma, d)\) in this subsection. We use the notations introduced in §5.2 and §5.3. Our argument is motivated by [Mc84] and [BP17, Section 4.2].

Recall \( w = \log_a b \). Let \( N \) be a natural number. For each \( v \in \Omega'|_N \) we define \( t_N(v) \) as the number of \( u \in A^N \) with \((u, v) \in \Omega|_N \). Set

\[
Z_N = \sum_{v \in \Omega'|_N} t_N(v)^w.
\]

By Lemma 5.1 we have

\[h^w_{\text{top}}(\pi, \Omega, \sigma) = \lim_{N \to \infty} \frac{\log Z_N}{N}.
\]
For \((u, v) \in \Omega_N\) we set
\[
f_N(u, v) = \frac{1}{Z_N} t_N(v)^{w-1}.
\]

We have
\[
\sum_{(u, v) \in \Omega_N} f_N(u, v) = 1.
\]

So we can consider \(f_N(u, v)\) as a probability measure on \(\Omega_N\). Let \(\mu_N = (f_N)^{\otimes N}\) be the product of infinite copies of the measure \(f_N\). This is a probability measure defined on \((\Omega_N)^N\).

Let \(N\) and \(M\) be natural numbers. Let \((x, y) \in (\Omega_N)^N\) where \(x = (x_m)_{m \in \mathbb{N}}\) and \(y = (y_m)_{m \in \mathbb{N}}\) with \(x_m \in A^N\), \(y_m \in B^N\) and \((x_m, y_m) \in \Omega_N\). We set
\[
P_{N,M}(x, y) = \left\{ (x', y') \in (\Omega_N)^N \mid \begin{cases} x'_m = x_m & (1 \leq m \leq \lfloor wM \rfloor) \\ y'_m = y_m & (1 \leq m \leq M) \end{cases} \right\}.
\]

The set \(Q_{N,M}(x, y)\) introduced in (5.5) is the image of \(P_{N,M}(x, y)\) under the map
\[
(\Omega_N)^N \to X_{\Omega_N}, \quad (x', y') \mapsto \left( \sum_{m=1}^{\infty} \frac{x'_m}{q^m}, \sum_{m=1}^{\infty} \frac{y'_m}{p^m} \right).
\]

The key argument below is a calculation of the measure \(\mu_N(P_{N,M}(x, y))\). Since the value of \(f_N(u, v) = \frac{1}{Z_N} t_N(v)^{w-1}\) depends only on \(v\), we have
\[
\mu_N(P_{N,M}(x, y)) = \prod_{m=1}^{M} f_N(x_m, y_m) \cdot \prod_{m=\lfloor wM \rfloor + 1}^{M} t_N(y_m).
\]

Taking a logarithm,
\[
\log \mu_N(P_{N,M}(x, y)) = \sum_{m=1}^{M} \log f_N(x_m, y_m) + \sum_{m=\lfloor wM \rfloor + 1}^{M} \log t_N(y_m)
\]
\[
= -M \log Z_N + (w - 1) \sum_{m=1}^{M} \log t_N(y_m) + \sum_{m=\lfloor wM \rfloor + 1}^{M} \log t_N(y_m)
\]
\[
= -M \log Z_N + w \sum_{m=1}^{M} \log t_N(y_m) - \sum_{m=1}^{\lfloor wM \rfloor} \log t_N(y_m)
\]

Set
\[
S_{N,M}(x, y) = \sum_{m=1}^{M} \frac{\log t_N(y_m)}{N}.
\]

We also set \(S_{N,0}(x, y) = 0\). We have
\[
(5.8) \quad \frac{1}{NM} \log \mu_N(P_{N,M}(x, y)) = -\frac{\log Z_N}{N} + w \left( \frac{S_{N,M}(x, y)}{M} - \frac{S_{N,\lfloor wM \rfloor}(x, y)}{wM} \right).
\]
Lemma 5.11. For any positive number $\delta$ and any natural number $K$ there exists a natural number $L = L(\delta, K) \geq K$ for which the following statement holds true: For any $N \geq 1$ and any $(x, y) \in (\Omega|_N)^N$ there exists a natural number $M \in [K, L]$ satisfying
\[ \frac{1}{NM} \log \mu_N (P_{N,M}(x, y)) \geq -\frac{\log Z_N}{N} - \delta. \]

Proof. From (5.8)
\[ \frac{1}{NM} \log \mu_N (P_{N,M}(x, y)) \geq -\frac{\log Z_N}{N} + w \left( \frac{S_{N,M}(x, y)}{M} - \frac{S_{N,[wM]}(x, y)}{[wM]} \right). \]
We apply Lemma 5.10 to the sequence
\[ a_M := \frac{S_{N,M}(x, y)}{M}. \]
We have $0 \leq a_M \leq \log a$. Let
\[ L = \left[ \left\lfloor \frac{\log a}{\log \frac{1}{w} + \delta} \right\rfloor + 1 \right]. \]
Then, by Lemma 5.10, there exists $M \in [K, L]$ satisfying
\[ a_M - a_{\lfloor wM \rfloor} \geq -\delta. \]

Lemma 5.12. For any $\delta > 0$ there exists a natural number $m_2 = m_2(\delta) > 0$ such that for any natural number $N$ there exists a Borel subset $R(\delta, N) \subset (\Omega|_N)^N$ satisfying the following two conditions.
\begin{itemize}
  \item $\mu_N (R(\delta, N)) \geq \frac{1}{2},$
  \item For any $M \geq m_2$ and $(x, y) \in R(\delta, N)$ we have
  \[ \left| \frac{1}{NM} \log \mu_N (P_{N,M}(x, y)) - \frac{\log Z_N}{N} \right| \leq \delta. \]
\end{itemize}

The point of the statement is that $m_2(\delta)$ is independent of $N$.

Proof. We define $\xi_m : (\Omega|_N)^N \to \mathbb{R}$ for $m \geq 1$ by
\[ \xi_m(x, y) = \frac{\log t_N(y_m)}{N}. \]
Notice that this depends not only on $m$ but also $N$. But we suppress the dependence on $N$ in our notation for simplicity. The random variables $\xi_1, \xi_2, \xi_3, \ldots$ are independent and identically distributed with respect to the measure $\mu_N = (f_N)^{\otimes N}$. We have $0 \leq \xi_m \leq \log a$. Hence its mean and variance (with respect to $\mu_N$) are bounded by
\[ 0 \leq \mathbb{E}(\xi_m) \leq \log a, \quad \mathbb{V}(\xi_m) \leq (\log a)^2. \]
We apply Lemma 5.9 (3) to \( \{\xi_m\} \). Then we can find \( m_2(\delta) > 0 \) such that the set

\[
R(\delta, N) := \left\{ (x, y) \in (\Omega|_N)^N \mid \sup_{M \geq m_2(\delta)} \left| \frac{\xi_1 + \cdots + \xi_M}{M} - \frac{\xi_1 + \cdots + \xi_{wM}}{wM} \right| \leq \delta \right\}
\]

satisfies \( \mu_N(R(\delta, N)) \geq \frac{1}{2} \).

We have

\[
S_{N,M}(x, y) = \sum_{m=1}^M \xi_m.
\]

From (5.8)

\[
\frac{1}{NM} \log \mu_N(P_{N,M}(x, y)) + \log Z_N = w \left( \frac{\xi_1 + \cdots + \xi_M}{M} - \frac{\xi_1 + \cdots + \xi_{wM}}{wM} \right).
\]

Hence for any \( (x, y) \in R(\delta, N) \)

\[
\sup_{M \geq m_2} \left| \frac{1}{NM} \log \mu_N(P_{N,M}(x, y)) + \log Z_N \right| \leq \delta.
\]

Recall that for \( (x, y) = (x_m, y_m) \in (\Omega|_N)^N \) we have denoted

\[
Q_{N,M}(x, y) = \left\{ \left( \sum_{m=1}^\infty \frac{x_m'}{a^m}, \sum_{m=1}^\infty \frac{y_m'}{b^m} \right) \in X_{\Omega|_N} \mid \left( x_m', y_m' \right) \in \Omega|_N \text{ for all } m \geq 1 \text{ with } x_m' = x_m \text{ and } y_m' = y_m \right\}.
\]

This is the image of the set \( P_{N,M}(x, y) \) under the map \( (\Omega|_N)^N \to X_{\Omega|_N} \). It is straightforward to check that for \( (x, y) = (x_m, y_m) \in (\Omega|_N)^N \) and \( (x', y') = (x_m', y_m') \) the following three conditions are equivalent to each other:

- \( P_{N,M}(x, y) = P_{N,M}(x', y') \).
- \( Q_{N,M}(x, y) = Q_{N,M}(x', y') \).
- \( (x_1, \ldots, x_{wM}, y_1, \ldots, y_M) = (x_1', \ldots, x_{wM}', y_1', \ldots, y_M') \).

For subsets \( E \) and \( F \) of \( X_{\Omega|_N} \) we set

\[
\mathrm{dist}_\infty(E, F) = \inf_{x \in E, y \in F} \|x - y\|_\infty.
\]

Here \( \|x - y\|_\infty \) is the \( \ell^\infty \)-distance on \( X_{\Omega|_N} \subset \mathbb{R}^{2N} \).

**Lemma 5.13.** Let \( N \) and \( M \) be natural numbers. Let \( (x^{(1)}, y^{(1)}), \ldots, (x^{(k)}, y^{(k)}) \in (\Omega|_N)^N \) with \( k \geq 4N + 1 \). Suppose that

\[
P_{N,M}(x^{(i)}, y^{(i)}) \neq P_{N,M}(x^{(j)}, y^{(j)}) \quad \text{for } i \neq j.
\]

Then there are \( i \) and \( j \) for which

\[
\mathrm{dist}_\infty(Q_{N,M}(x^{(i)}, y^{(i)}), Q_{N,M}(x^{(j)}, y^{(j)})) \geq b^{-M}.
\]
Proof. The assumption $P_{N,M}(x^{(i)}, y^{(i)}) \neq P_{N,M}(x^{(j)}, y^{(j)})$ implies
\[
\left( x_1^{(i)}, \ldots, x_{\lceil w M \rceil}^{(i)}, y_1^{(i)}, \ldots, y_{\lceil w M \rceil}^{(i)} \right) \neq \left( x_1^{(j)}, \ldots, x_{\lceil w M \rceil}^{(j)}, y_1^{(j)}, \ldots, y_{\lceil w M \rceil}^{(j)} \right).
\]
Since $k \geq 4^N + 1$, there are $i$ and $j$ for which we have either
\[
\left\| \sum_{m=1}^{\lceil w M \rceil} x_m^{(i)} a^m - \sum_{m=1}^{\lceil w M \rceil} x_m^{(j)} a^m \right\|_\infty \geq 2a^{-\lceil w M \rceil}
\] (5.9)
or
\[
\left\| \sum_{m=1}^M y_m^{(i)} b^m - \sum_{m=1}^M y_m^{(j)} b^m \right\|_\infty \geq 2b^{-M}.
\] (5.10)
We have
\[
Q_{N,M}(x, y) \subset \left( \sum_{m=1}^{\lceil w M \rceil} x_m a^m \right) + [0, a^{-\lceil w M \rceil}]^N \times [0, b^{-M}]^N.
\]
Therefore (5.9) implies
\[
dist_\infty (Q_{N,M}(x^{(i)}, y^{(i)}), Q_{N,M}(x^{(j)}, y^{(j)})) \geq a^{-\lceil w M \rceil} \geq b^{-M}.
\]
The condition (5.10) implies
\[
dist_\infty (Q_{N,M}(x^{(i)}, y^{(i)}), Q_{N,M}(x^{(j)}, y^{(j)})) \geq b^{-M}.
\]
So we get the statement in both of the cases. \hfill \square

**Proposition 5.14.** For any $\delta > 0$ there exists $\varepsilon > 0$ such that for any $N \geq 1$
\[
\dim_H(X_{\Omega}|N, \| \cdot \|_\infty, \varepsilon) \geq \log_b Z_N - \delta N.
\]

**Proof.** If $\Omega|N$ is a single point, then $Z_N = 1$ and the statement holds trivially. So we assume that $\Omega|N$ contains at least two points.

Since $Z_N$ is sub-multiplicative in $N$, there exists a positive constant $s$ satisfying $\log_b Z_N \leq sN$ for all $N \geq 1$. Let
\[
m_2 = m_2 \left( \frac{\delta \log b}{2} \right)
\]
be the natural number introduced in Lemma 5.12. Let $N$ be a natural number. By Lemma 5.12 there exists a Borel subset $R = R \left( \frac{\delta \log b}{2}, N \right) \subset (\Omega|N)^N$ satisfying $\mu_N(R) \geq 1/2$ and for any $M \geq m_2$ and $(x, y) \in (\Omega|N)^N$
\[
\left| \frac{1}{NM} \log \mu_N(P_{N,M}(x, y)) + \frac{\log Z_N}{N} \right| \leq \frac{\delta \log b}{2}.
\]
Take $\varepsilon > 0$ satisfying
\[
\varepsilon < b^{-m_2-1}, \quad 4b^s \varepsilon^{\delta/2} < \frac{1}{2}.
\] (5.11)
Suppose we are given a covering $X_\Omega|_N = \bigcup_{k=1}^\infty E_k$ with $0 < \text{Diam} (E_k, \|\cdot\|_\infty) < \varepsilon$ for all $k \geq 1$. We set $D(E_k) := \text{Diam} (E_k, \|\cdot\|_\infty)$ for simplicity of the notation. We will show that

$$\sum_{k=1}^\infty D(E_k)^{\log Z_N - \delta N} > 1.$$  

This proves $\dim_H (X_\Omega|_N, \|\cdot\|_\infty, \varepsilon) \geq \log Z_N - \delta N$. 

For each $E_k$ we take a natural number $M_k \geq m_2$ satisfying 

$$b^{-M_k-1} \leq D(E_k) < b^{-M_k}.$$  

We define 

$$C_k = \{P_{N,M_k}(x,y) \mid (x,y) \in R \text{ with } Q_{N,M_k}(x,y) \cap E_k \neq \emptyset\}.$$  

For $P_{N,M_k}(x,y)$ and $P_{N,M_k}(x',y')$ in $C_k$, we have 

$$\text{dist}_\infty (Q_{N,M_k}(x,y), Q_{N,M_k}(x',y')) \leq D(E_k) < \varepsilon < b^{-M_k}.$$  

By Lemma 5.13 we have $|C_k| \leq 4^N$. 

We have 

$$R \subset \bigcup_{k=1}^\infty \bigcup_{P \in C_k} P.$$  

Hence

$$(5\cdot12) \quad \frac{1}{2} \leq \mu_N(R) \leq \sum_{k=1}^\infty \sum_{P \in C_k} \mu_N(P).$$  

Since $M_k \geq m_2$, every $P = P_{N,M_k}(x,y) \in C_k$ with $(x,y) \in R$ satisfies 

$$\frac{1}{NM_k} \log \mu_N(P) \leq -\frac{\log Z_N}{N} + \frac{\delta \log b}{2}.$$  

Hence for $P \in C_k$

$$\mu_N(P) \leq \exp \left\{ M_k \left( -\log Z_N + \frac{N \delta \log b}{2} \right) \right\}$$

$$= \exp \left\{ M_k \log b \left( -\log b Z_N + \frac{\delta N}{2} \right) \right\}$$

$$= b^{-M_k \left( \log_b Z_N - \frac{\delta N}{2} \right)}.$$  

From $b^{-M_k-1} \leq D(E_k)$ we have $b^{-M_k} \leq b^D(E_k)$. So 

$$\mu_N(P) \leq (b \cdot D(E_k))^{\log_b Z_N - \frac{\delta N}{2}}, \quad (P \in C_k).$$
Therefore by (5·12)

\[
\frac{1}{2} \leq \sum_{k=1}^{\infty} \sum_{P \in C_k} (b \cdot D(E_k))^{\log_b Z_N - \frac{\delta N}{2}} 
\leq \sum_{k=1}^{\infty} 4^N (b \cdot D(E_k))^{\log_b Z_N - \frac{\delta N}{2}} \quad \text{(by } |C_k| \leq 4^N) 
\leq \sum_{k=1}^{\infty} 4^N b^{\log_b Z_N} D(E_k)^{\log_b Z_N - \frac{\delta N}{2}} 
\leq \sum_{k=1}^{\infty} 4^N b^{sN} D(E_k)^{\log_b Z_N - \frac{\delta N}{2}} \quad \text{(by } \log_b Z_N \leq sN) 
= \sum_{k=1}^{\infty} 4^N b^{sN} D(E_k)^{\frac{\delta N}{2}} \cdot D(E_k)^{\log_b Z_N - \delta N} 
\leq \sum_{k=1}^{\infty} 4^N b^{sN} e^{\frac{\delta N}{2}} \cdot D(E_k)^{\log_b Z_N - \delta N} \quad \text{(by } D(E_k) < \varepsilon) 
\leq \sum_{k=1}^{\infty} \frac{1}{2^N} \cdot D(E_k)^{\log_b Z_N - \delta N} \quad \text{(by } 4b^{s}e^{\delta/2} < \frac{1}{2} \text{ in (5·11)}).
\]

Thus

\[1 < \sum_{k=1}^{\infty} D(E_k)^{\log_b Z_N - \delta N}.
\]

\[\square\]

**Corollary 5.15.**

\[\text{mdim}_H (X_{\Omega}, d_N) \geq \frac{h^w_{\text{top}} (\pi, \Omega, \sigma)}{\log b}.
\]

**Proof.** For any natural number $N$, the natural projection $(X_{\Omega}, d_N) \to (X_{\Omega}|_N, \|\cdot\|_\infty)$ is one-Lipschitz. So for any $\varepsilon > 0$

\[\dim_H (X_{\Omega}, d_N, \varepsilon) \geq \dim_H (X_{\Omega}|_N, \|\cdot\|_\infty, \varepsilon).
\]

By Proposition 5.14, for any $\delta > 0$ there exists $\varepsilon > 0$ so that for any $N \geq 1$

\[\dim_H (X_{\Omega}, d_N, \varepsilon) \geq \dim_H (X_{\Omega}|_N, \|\cdot\|_\infty, \varepsilon) \geq \log_b Z_N - \delta N.
\]

Hence

\[
\liminf_{N \to \infty} \frac{\dim_H (X_{\Omega}, d_N, \varepsilon)}{N} \geq \lim_{N \to \infty} \frac{\log_b Z_N}{N} - \delta = \frac{1}{\log b} \lim_{N \to \infty} \frac{\log Z_N}{N} - \delta = \frac{h^w_{\text{top}} (\pi, \Omega, \sigma)}{\log b} - \delta,
\]

\[\text{by } h^w_{\text{top}} (\pi, \Omega, \sigma) = \lim_{N \to \infty} \frac{\log Z_N}{N}.
\]

Letting $\varepsilon \to 0$ and $\delta \to 0$, we get the statement. \[\square\]
Proposition 5.16. For any \( \varepsilon > 0 \) there exists a natural number \( N_0 = N_0(\varepsilon) \) such that for any \( N \geq N_0 \)
\[
\dim_H (X_\Omega, d_{N-N_0}, \varepsilon) \leq (1 + \varepsilon) (\log_b Z_N + \varepsilon N).
\]

Proof. We will use Lemma 5.7. As in the proof of Proposition 5.14 we take a positive number \( s \) satisfying \( \log_b Z_N \leq sN \) for all \( N \geq 1 \).

Let \( N \) be a natural number. We define a probability measure \( \nu_N \) on \( X_\Omega|_N \) as the push-forward of \( \mu_N \) under the map
\[
(\Omega|_N)^N \to X_\Omega|_N, \quad (x_m, y_m)_{m \in \mathbb{N}} \mapsto \left( \sum_{m=1}^{\infty} x_m a^m, \sum_{m=1}^{\infty} y_m b^m \right).
\]

We set \( X_\Omega(N) = X_\Omega|_N \times [0, 1]^2 \times [0, 1]^2 \times \cdots \subset ([0, 1]^2)^N \), \( \) recall \( X_\Omega|_N \subset ([0, 1]^2)^N \).

We have \( X_\Omega \subset X(N) \). Recall that the metric \( d \) on \( ([0, 1]^2)^N \) is defined by
\[
d((x_n, y_n)_{n \in \mathbb{N}}, (x'_n, y'_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} 2^{-n} \max(|x_n - x'_n|, |y_n - y'_n|).
\]

We denote the Lebesgue measure on the square \([0, 1]^2\) by \( \text{Leb} \). We define
\[
\nu'_N := \nu_N \otimes \text{Leb} \otimes \text{Leb} \otimes \text{Leb} \otimes \cdots.
\]

This is a probability measure on \( X(N) \). For natural numbers \( N, M \) and \( (x, y) \in (\Omega|_N)^N \) we set
\[
Q'_{N,M}(x, y) := Q_{N,M}(x, y) \times [0, 1]^2 \times [0, 1]^2 \times \cdots \subset X(N).
\]

We have
\[
\nu'_N (Q'_{N,M}(x, y)) = \nu_N (Q_{N,M}(x, y)) \geq \mu_N (P_{N,M}(x, y)).
\]

Given \( \varepsilon > 0 \), we take \( \delta > 0 \) with \( 6\delta^\varepsilon < 1 \). We take a natural number \( K \) satisfying
\[
(5.13) \quad ab^{-K} < \frac{\delta}{12}, \quad (2a)^{s^+\varepsilon} \leq b^{K}. \tag{5.13}
\]

Let \( L = L \left( \frac{\varepsilon \log b}{2}, K \right) \) be the natural number introduced in Lemma 5.11. Then by Lemma 5.11 for any \( N \geq 1 \) and any \( (x, y) \in (\Omega|_N)^N \) there exists \( M \in [K, L] \) satisfying
\[
\frac{1}{NM} \log \mu_N (P_{N,M}(x, y)) \geq -\log Z_N - \frac{\varepsilon \log b}{2}.
\]
Then

\[
\nu'_N \left( Q'_{N,M}(x, y) \right) \geq \mu_N \left( P_{N,M}(x, y) \right) \\
\geq \exp \left\{ NM \left( -\frac{\log Z_N}{N} - \frac{\varepsilon \log b}{2} \right) \right\} \\
= \exp \left\{ -M \log b \left( \log Z_N + \frac{\varepsilon N}{2} \right) \right\} \\
= (b^{-M})^{\log_b Z_N + \frac{\varepsilon N}{2}}.
\]

**Claim 5.17.** For any \( N \geq 1 \) and any \((x, y) \in (\Omega_N)^N\) there exists \( M \in [K, L] \) satisfying

\[
\nu'_N \left( Q'_{N,M}(x, y) \right) \geq (2ab^{-M})^{\log_b Z_N + \varepsilon N}.
\]

**Proof.** By the above argument, there exists \( M \in [K, L] \) satisfying

\[
\nu'_N \left( Q'_{N,M}(x, y) \right) \geq (b^{-M})^{\log_b Z_N + \frac{\varepsilon N}{2}}.
\]

So it is enough to prove

\[
(b^{-M})^{\log_b Z_N + \frac{\varepsilon N}{2}} \geq (2ab^{-M})^{\log_b Z_N + \varepsilon N}.
\]

This is equivalent to

\[
(5.14) \quad b^{\frac{\varepsilon M}{2}} \geq (2a)^{\frac{\log_b Z_N + \varepsilon N}{N}}.
\]

From \( \log Z_N \leq s \) and the choice of \( K \) in (5.13),

\[
(2a)^{\frac{\log_b Z_N + \varepsilon N}{N}} \leq (2a)^{s + \varepsilon} \leq b^{\frac{sK}{2}}.
\]

Since \( M \geq K \), we have the above (5.14). \( \Box \)

We take a natural number \( N_0 \) satisfying

\[
\sum_{n > N_0} 2^{-n} < ab^{-L}.
\]

For \( u = (u_n)_{n \in \mathbb{N}} \) with \( u_n \in [0, 1]^2 \), we denote \( u|_N = (u_1, \ldots, u_N) \in ([0, 1]^2)^N \). Then for \( u, v \in ([0, 1]^2)^N \) and \( N > N_0 \)

\[
(5.15) \quad d_{N-N_0}(u, v) < \|u|_N - v|_N\|_{\infty} + ab^{-L}.
\]

**Claim 5.18.** For any \((x, y) \in (\Omega|_N)^N\) and any natural numbers \( N, M \) with \( N > N_0 \) and \( M \leq L \), we have

\[
0 < \text{Diam} \left( Q'_{N,M}(x, y) \right) < 2ab^{-M}.
\]

Notice that \( 2ab^{-M} < \delta/6 \) because we have assumed \( ab^{-K} < \delta/12 \) in (5.13).
Proof. Obviously \( Q'_{N,M}(x,y) = Q_{N,M}(x,y) \times [0,1]^2 \times [0,1]^2 \times \cdots \) is not a single point. So its diameter is positive. By the above (5·15)

\[
\text{Diam} \left( Q'_{N,M}(x,y), d_{N-N_0} \right) < \text{Diam} \left( Q_{N,M}(x,y), \| \cdot \|_{\infty} \right) + ab^{-L}.
\]

As we saw in (5·6) in §5.3, we have \( \text{Diam} \left( Q_{N,M}(x,y), \| \cdot \|_{\infty} \right) \leq ab^{-M} \). So

\[
\text{Diam} \left( Q'_{N,M}(x,y), d_{N-N_0} \right) < ab^{-M} + ab^{-L} \leq 2ab^{-M}, \quad (\text{by } M \leq L).
\]

\( \square \)

By Claims 5.17 and 5.18 for any natural number \( N > N_0 \) and any \((x,y) \in (\Omega|_N)^N\) there exists a natural number \( M \in [K, L] \) such that

\[
0 < \text{Diam} \left( Q'_{N,M}(x,y), d_{N-N_0} \right) < \frac{\delta}{6},
\]

\[
\nu'_N \left( Q'_{N,M}(x,y) \right) > \left( \text{Diam} \left( Q'_{N,M}(x,y), d_{N-N_0} \right) \right)^{\log_b Z_N + \varepsilon N}.
\]

Recall that we have assumed \( 6\delta < 1 \). Applying Lemma 5.7 to \((X(N), d_{N-N_0})\), we get

\[
\dim_H \left( X(N), d_{N-N_0}, \delta \right) \leq \left(1 + \varepsilon \right) \left( \log_b Z_N + \varepsilon N \right), \quad (N > N_0).
\]

Since \( 0 < \delta < \varepsilon \) and \( X_\Omega \subset X(N) \), we have

\[
\dim_H \left( X_\Omega, d_{N-N_0}, \varepsilon \right) \leq \left(1 + \varepsilon \right) \left( \log_b Z_N + \varepsilon N \right), \quad (N > N_0).
\]

\( \square \)

Corollary 5.19.

\[
\overline{\text{mdim}}_H (X_\Omega, \sigma, d) \leq \frac{h^w_{\text{top}} (\pi, \Omega, \sigma)}{\log b}.
\]

Proof. Recall

\[
\frac{h^w_{\text{top}} (\pi, \Omega, \sigma)}{\log b} = \lim_{N \to \infty} \frac{\log_b Z_N}{N}.
\]

Let \( \varepsilon \) be any positive number and let \( N_0 = N_0(\varepsilon) \) be the natural number given by Proposition 5.16. Then for any \( N > N_0 \)

\[
\frac{\dim_H (X_\Omega, d_{N-N_0}, \varepsilon)}{N} \leq \left(1 + \varepsilon \right) \left( \frac{\log_b Z_N}{N} + \varepsilon \right).
\]

Letting \( N \to \infty \), we get

\[
\limsup_{N \to \infty} \frac{\dim_H (X_\Omega, d_{N}, \varepsilon)}{N} \leq \left(1 + \varepsilon \right) \left( \lim_{N \to \infty} \frac{\log_b Z_N}{N} + \varepsilon \right)
\]

\[
= \left(1 + \varepsilon \right) \left( \frac{h^w_{\text{top}} (\pi, \Omega, \sigma)}{\log b} + \varepsilon \right).
\]

Letting \( \varepsilon \to 0 \), we get the statement of the corollary. \( \square \)
By Corollaries 5.15 and 5.19, we conclude
\[ \text{mdim}_H (X_\Omega, \sigma, d) = \frac{h_{\text{top}} (\pi, \Omega, \sigma)}{\log b} , \quad (w = \log_a b). \]

This completes the proof of Theorem 5.3.

**Appendix A. Mean Hausdorff Dimension of \( \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \}^\mathbb{N} \)**

The purpose of this Appendix is to prove the result stated in Example 2.4. Let
\[ K = \left\{ \frac{1}{n} \mid n \geq 1 \right\} \cup \{0\} = \left\{ 0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\}. \]

Let \( K^\mathbb{N} \) be the one-sided full shift on the alphabet \( K \) with the shift map \( \sigma : K^\mathbb{N} \to K^\mathbb{N} \). We define a metric \( d \) on it by
\[ d ((x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} 2^{-n} |x_n - y_n|. \]

In Example 2.4 we claimed that
\[ \text{mdim}_H (K^\mathbb{N}, \sigma, d) = 0, \quad \text{mdim}_M (K^\mathbb{N}, \sigma, d) = \frac{1}{2}. \]

We omit the proof of \( \text{mdim}_M (K^\mathbb{N}, \sigma, d) = \frac{1}{2} \) because it is rather straightforward. In this Appendix we explain the detailed proof of \( \text{mdim}_H (K^\mathbb{N}, \sigma, d) = 0 \). The proof is based on Lemma 5.7 in §5.4. We restate its special version (letting \( c = 1 \) on Lemma 5.7) here for the convenience of readers.

**Lemma A.1** (⊂ Lemma 5.7). Let \( \varepsilon \) and \( s \) be positive numbers with \( \varepsilon < 1/6 \). Let \((X, d)\) be a compact metric space with a Borel probability measure \( \mu \). Suppose that for any \( x \in X \) there exists a Borel subset \( A \subset X \) with \( x \in A \) satisfying
\[ 0 < \text{Diam} A < \frac{\varepsilon}{6}, \quad \mu(A) \geq (\text{Diam} A)^s. \]

Then \( \text{dim}_H(X, d, \varepsilon) \leq 2s \).

Set \( X = K^\mathbb{N} \). We define a probability measure \( \nu \) on \( K \) by
\[ \nu (\{u\}) = au^2 \quad (u \neq 0), \quad \nu (\{0\}) = \frac{1}{2}, \]
where \( a \) is a positive number satisfying
\[ a \left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) = \frac{1}{2}. \]
Indeed we have \( a = \frac{3}{\pi^2} \), but we do not need its precise value. We only need to use \( a < 1 \). We define \( \mu = \nu^\otimes \mathbb{N} \). This is a Borel probability measure on \( X \).
Let $\varepsilon$ be a positive number with $\varepsilon < \frac{1}{6}$, and let $m$ be a natural number. We take a positive number $\delta = \delta(\varepsilon, m)$ satisfying
\[(A-1) \quad \delta < \min \left( \frac{\varepsilon}{12}, \frac{a^m}{8}, \left( \frac{1}{2} \right)^{\frac{m}{3}+1} \right).\]

We take a natural number $L$ satisfying
\[\sum_{n>L} 2^{-n} < \delta^{m^m}.
\]

**Claim A.2.** For any $x \in X$ and any natural number $N$ there exists a Borel subset $A \subset X$ with $x \in A$ satisfying
\[0 < \text{Diam} (A, d_N) < \frac{\varepsilon}{6}, \quad \mu(A) \geq \left( \text{Diam}(A, d_N) \right)^{\frac{6}{m} \left( N+L \right)}.
\]

We assume this claim for the moment. Then by Lemma A.1, for any natural number $N$
\[\dim_H(X, d_N, \varepsilon) \leq \frac{12}{m} \left( N + L \right).
\]

Since $L$ is independent of $N$, we have
\[\limsup_{N \to \infty} \frac{\dim_H(X, d_N, \varepsilon)}{N} \leq \frac{12}{m}.
\]

Letting $\varepsilon \to 0$ and $m \to \infty$, we conclude
\[\overline{m\dim_H}(X, \sigma, d) = 0.
\]

So the rest of the problem is to prove Claim A.2.

**Proof of Claim A.2.** Take $x = (x_n)_{n \in \mathbb{N}}$ with $x_n \in K$. For this $x$ we introduce a partition
\[[1, N + L] \cap \mathbb{N} = I_0 \cup I_1 \cup I_2 \cup \cdots \cup I_m \cup I_{m+1} \quad (\text{disjoint union})
\]
by
\[I_0 = \{ n \mid x_n \leq \delta \}, \quad I_k = \left\{ n \mid \delta^{m} < x_n \leq \delta^{m+1} \right\} \quad (1 \leq k \leq m),
\]
\[I_{m+1} = \{ n \mid x_n \leq \delta^{m^m} \}.
\]

There exists $m_0 \in \{0, 1, 2, \ldots, m\}$ satisfying
\[(A-2) \quad |I_{m_0}| \leq \frac{N + L}{m+1} < \frac{N + L}{m}.
\]

Set
\[r = \delta^{m_{m_0}}.
\]

We have $r \leq \delta < \varepsilon/12$. We define $A \subset X$ by
\[A = \left\{ (y_n)_{n \in \mathbb{N}} \in K^{\mathbb{N}} \mid x_n - r \leq y_n \leq x_n \text{ for all } 1 \leq n \leq N + L \right\}.
\]
This is not a single point. So its diameter is positive. We have \( x \in A \) and
\[
\text{Diam}(A, d_N) \leq r + \sum_{n > L} 2^{-n} < r + \delta^{m^m} \leq 2r < \frac{\varepsilon}{6}.
\]

We need to estimate
\[
\mu(A) = \prod_{n=1}^{N+L} \nu([x_n - r, x_n]) .
\]

Let \( 1 \leq n \leq N + L \).

Case 1. If \( n \in I_k \) with \( 0 \leq k < m_0 \) then \( x_n > \delta^{m^k} \) and hence
\[
\nu([x_n - r, x_n]) \geq \nu(\{x_n\}) = ax_n^2
\]
\[
> a\delta^{2m^k}
\]
\[
\geq a\delta^{2m^{m_0} - 1} = a\left(\delta^{m^{m_0}}\right)^{\frac{1}{m}} = ar^\frac{2}{m}
\]
\[
>(2r)^\frac{2}{m} \quad \text{by} \quad r \leq \delta < \frac{a}{8} \quad \text{in (A.1)}
\]
\[
\geq (\text{Diam}(A, d_N))^{\frac{2}{m}}.
\]

Case 2. If \( n \in I_{m_0} \) then \( x_n > \delta^{m^{m_0}} = r \) and hence
\[
\nu([x_n - r, x_n]) \geq \nu(\{x_n\}) = ax_n^2
\]
\[
> ar^2
\]
\[
>(2r)^3 \quad \text{by} \quad r \leq \delta < \frac{a}{8} < \frac{a}{8}
\]
\[
\geq (\text{Diam}(A, d_N))^3 .
\]

Since we know that \(|I_{m_0}| \leq (N + L)/m \) by (A.2), we have
\[
\prod_{n \in I_{m_0}} \nu([x_n - r, x_n]) \geq (\text{Diam}(A, d_N))^\frac{3}{m}(N+L) .
\]

Case 3. If \( n \in I_k \) with \( k > m_0 \) then \( x_n \leq \delta^{m^{k-1}} \leq \delta^{m^{m_0}} = r \). This implies \( 0 \in [x_n - r, x_n] \).

Hence
\[
\nu([x_n - r, x_n]) \geq \nu(\{0\}) = \frac{1}{2}
\]
\[
>(2r)^\frac{1}{m} \quad \text{by} \quad r \leq \delta < \left(\frac{1}{2}\right)^\frac{m+1}{m}
\]
\[
\geq (\text{Diam}(A, d_N))^{\frac{1}{m}} .
\]

Summarizing the above, we get
\[
\mu(A) \geq (\text{Diam}(A, d_N))^\frac{3}{m}(N+L) \cdot (\text{Diam}(A, d_N))^\frac{m}{m}(N+L) = (\text{Diam}(A, d_N))^\frac{3}{m}(N+L) .
\]
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