ARITHMETIC OVER TRIVIALLY VALUED FIELD AND ITS APPLICATIONS

by

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Abstract. — By some result on the study of arithmetic over trivially valued field, we find its applications to Arakelov geometry over adelic curves. We prove a partial result of the continuity of arithmetic $\chi$-volume along semiample divisors. Moreover, we give an upper bound estimate of arithmetic Hilbert-Samuel function.

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1. Introduction

Arakelov geometry is a theory to study varieties over $\mathcal{O}_K$ where $K$ is a number field. Since the closed points of Spec$\mathcal{O}_K$ only give rise to non-Archimedean places of $K$, the main idea of Arakelov Geometry is to "compactify" Spec$\mathcal{O}_K$ by adding Archimedean places, and the corresponding fibers are nothing but analytification of the generic fiber. In order to generalize the theory to the case over a function field or even more general cases, Chen and Moriwaki established the theory of Arakelov geometry over adelic curves\cite{4}. Here an adelic curve is a field equipped with a set of absolute values parametrised by a measure space. Such a structure can be easily constructed for any global fields
by using the theory of adÅlles. The first step is to give a theory of geometry of numbers. Inspired by the study of semistability of vector bundles over projective regular curves, the arithmetic slope theory was established, and the tensorial minimal slope property was firstly proved in [2]. It’s easy to find that some slope-related results can be roughly described by arithmetic over trivially valued field, especially the analogous Harder-Narasimhan filtration of a adelic vector bundle due to the obvious fact that any \( \mathbb{R} \)-filtration induces an ultrametric norm over trivially valued field.

This article is to dig deeper on this, and give mainly two applications. The first is the application to the study of \( \chi \)-volume function \( \hat{\text{vol}}_\chi(\cdot) \), which is an analogy of Euler characteristic. We can easily show some boundedness of \( \hat{\text{vol}}_\chi(\cdot) \) with the assumption on the finite generation of section ring. Therefore we obtain the following result on the continuity of \( \hat{\text{vol}}_\chi(\cdot) \).

**Theorem 1.1.** — Let \( \mathcal{D} = (D,g), \mathcal{E}_1 = (E_1,h_1), \ldots, \mathcal{E}_r = (E_r,h_r) \) be adelic \( \mathbb{Q} \)-Cartier divisors on \( X \) such that \( D \) and \( E_i \)'s are semiample.

\[
\lim_{\epsilon_1 + \cdots + \epsilon_r \to 0} \hat{\text{vol}}_\chi(D + \sum_{i=1}^r \epsilon_i E_i) = \hat{\text{vol}}_\chi(D)
\]

The second result worth to mention is about arithmetic Hilbert-Samuel function. In [6], H. Chen give an upper bound for arithmetic Hilbert-Samuel function of big arithmetic varieties with an assumption on the relationship between slopes and successive minima. Here in the case of \( \text{char } K = 0 \), we use the arithmetic over trivially valued to make a modification such that we can remove the assumption.

**Theorem 1.2** (Upper bound of arithmetic Hilbert-Samuel function)

Let \( (D,g) \) be an adelic divisor on \( X \) with \( D \) being big. For each \( n \in \mathbb{N} \), let \( \mathcal{E}_n \) denote the pair of \( E_n = H^0(X, \mathcal{O}_X(nD)) \) and norm family \( \xi_{ng} \). Then it holds that

\[
\hat{\text{deg}}_+(\mathcal{E}_n) \leq \hat{\text{vol}}(D,g) \frac{r^{d+1}}{(d+1)!} + O(n^d) + (C + 1/2) \cdot n \cdot \ln(n).
\]

Moreover, if \( D \) is ample, then

\[
\hat{\text{deg}}(\mathcal{E}_n) \leq \hat{\text{vol}}(D,g) \frac{r^{d+1}}{(d+1)!} + O(n^d) + (C + 1/2) \cdot n \cdot \ln(n).
\]

2. Normed vector spaces and graded linear series over trivially valued field

2.1. Arithmetic over trivially valued field. — Let \( E \) be a finitely dimensional vector space over \( K \). Then any \( \mathbb{R} \)-filtration \( \mathcal{F}^l \) on \( E \) induces an
ultrametric norm $\|\cdot\|_F$ over $(K, |\cdot|_0)$ which is given by

$$\|x\|_F := \exp(-\sup\{t \mid x \in F^t E\}).$$

Moreover, this is actually a bijection between the sets of ultrametric norms and $\mathbb{R}$-filtrations. Moreover, the minimal slope corresponding to the $\mathbb{R}$-filtration can be computed by $\hat{\mu}_{\min}(E, F^t) = -\ln(\max\{\|x\|_F \mid x \in E\})$. Therefore we give the following definition.

**Definition 2.1.** — Let $E = (E, \|\cdot\|)$ be an ultrametric normed vector space over $(K, |\cdot|_0)$. We define the minimal slope of $E$ by

$$\hat{\mu}_{\min}(E) = \begin{cases} -\ln(\max\{\|x\| \mid x \in E\}), & \text{if } E \neq 0, \\ +\infty, & \text{if } E = 0. \end{cases}$$

In particular, for any injective homomorphism $f : F \to E$, we denote by $\|\cdot\|_{\text{sub}(f)}$ the induced ultrametric norm on $F$, then

$$\hat{\mu}_{\min}(F, \|\cdot\|_{\text{sub}(f)}) \geq \hat{\mu}_{\min}(E).$$

**Proposition 2.2.** — Let $E = (E, \|\cdot\|)$ be an ultrametrically normed vector space over $(K, |\cdot|_0)$. Let

$$0 \to F \xrightarrow{f} E \to G \to 0$$

be an exact sequence of vector spaces over $K$. Then it holds that

$$\hat{\mu}_{\min}(E) = \min\{\hat{\mu}_{\min}(F, \|\cdot\|_{\text{sub}(f)}), \hat{\mu}_{\min}(G, \|\cdot\|_{\text{quot}})\}.$$

**Proof.** — When $E = 0$, the proposition is trivial. We thus assume that $E \neq 0$. By definition, we can easily see that

$$\hat{\mu}_{\min}(E) \leq \min\{\hat{\mu}_{\min}(F, \|\cdot\|_{\text{sub}(f)}), \hat{\mu}_{\min}(G, \|\cdot\|_{\text{quot}})\}.$$

Let $x \in E$ be an element with $\|x\| = \exp(-\hat{\mu}_{\min}(E))$. If $x$ is contained in the image of $F$, then $\hat{\mu}_{\min}(F, \|\cdot\|_{\text{sub}(f)}) = \hat{\mu}_{\min}(E)$, we are done. Otherwise, $\tilde{x} \neq 0 \in G$, it holds that

$$\|\tilde{x}\|_{\text{quot}} = \inf_{y \in F} \|x + f(y)\| = \inf_{y \in F} \max\{\|x\|, \|f(y)\|\} = \|x\|$$

which implies that $\hat{\mu}_{\min}(G, \|\cdot\|_{\text{quot}}) = \hat{\mu}_{\min}(E)$. □

**Remark 2.3.** — The above proposition can be also obtained by using the inequality in \[4\] Proposition 4.3.32 since an ultrametric normed vector space can be viewed as an adelic vector bundle over the trivially valued field $K$. 


Definition 2.4. — Let $\mathcal{E}_i = (E_i, \| \cdot \|_i)_{i=1}^n$ be a collection of ultrametric normed vector spaces over $(K, |\cdot|_0)$. We define the a norm $\| \cdot \|$ on $\bigoplus_{i=1}^n E_i$ by

$$\|(a_1, \cdots, a_n)\| := \max_{i=1}^n \|a_i\|_i$$

for $(a_1, \cdots, a_n) \in \bigoplus_{i=1}^n E_i$, which is called the direct sum of norms $\{\| \cdot \|_i\}_{i=1}^n$, and the ultrametric normed vector space $(\bigoplus_{i=1}^n E_i, \| \cdot \|)$ can be denoted by $\bigoplus_{i=1}^n \mathcal{E}_i$.

Remark 2.5. — It’s easy to see that

$$\hat{\mu}_{\min} \left( \bigoplus_{i=1}^n \mathcal{E}_i \right) = \min_{i=1}^n \{\hat{\mu}_{\min}(\mathcal{E}_i)\}.$$

Moreover, the successive slopes of $\bigoplus_{i=1}^n \mathcal{E}_i$ is just the sorted sequence of the union of successive slopes of $\mathcal{E}_i$.

Definition 2.6. — Let $\mathcal{E} = (E, \| \cdot \|_E)$ and $\mathcal{F} = (F, \| \cdot \|_F)$ be ultrametric normed vector spaces over $(K, |\cdot|_0)$. The tensor product $\mathcal{E} \otimes \mathcal{F}$ is defined by equipping $E \otimes F$ with the ultrametric norm

$$\|x\|_{E \otimes F} := \min \left\{ \max_i \{\|s_i\|_E \cdot \|t_i\|_F\} \mid x = \sum_i s_i \otimes t_i \right\}$$

2.2. Filtered graded linear series. —

Definition 2.7. — Let $E_\bullet = \{E_n\}_{n \in \mathbb{N}}$ be a collection of vector subspaces of $K(X)$ over $K$. We say $E_\bullet$ is a graded linear series if $\bigoplus_{n \in \mathbb{N}} E_n^m$ is a graded sub-$K$-algebra of $K(X)[Y]$. If $\bigoplus_{n \in \mathbb{N}} E_n^m$ is finitely generated $K$-algebra, we say $E_\bullet$ is of finite type. We say $E_\bullet$ is of subfinite type if $E_\bullet$ is contained in a graded linear series. Moreover, if we equip each $E_n$ with an $\mathbb{R}$-filtration $\mathcal{F}_n^t E_n$, then we call $E_\bullet$ a filtered graded linear series.

Denote by $\delta : \mathbb{N} \to \mathbb{R}$ the function maps $n$ to $C \ln \dim_K(E_n)$. We say $\{\mathcal{F}_n^t E_n\}_{n \in \mathbb{N}, t \in \mathbb{R}}$ satisfies $\delta$-superadditivity if

$$\mathcal{F}_n^{t_1} E_n \mathcal{F}_m^{t_2} E_m \subset \mathcal{F}_{n+m}^{t_1 + t_2 - \delta(n) - \delta(m)}$$

holds for any $n, m \in \mathbb{N}$ and $t_1, t_2 \in \mathbb{R}$. Moreover, we say $\{\mathcal{F}_n^t E_n\}_{n \in \mathbb{N}, t \in \mathbb{R}}$ is strong $\delta$-superadditive if

$$\mathcal{F}_n^{t_1} E_{n_1} \mathcal{F}_{n_2}^{t_2} E_{n_2} \cdots \mathcal{F}_n^{t_r} E_{n_r} \subset \mathcal{F}_{n_1+n_2+\cdots+n_r}^{t_1 + t_2 + \cdots + t_r - \delta(n_1) - \delta(n_2) - \cdots - \delta(n_r)} E_{n_1+n_2+\cdots+n_r}$$

holds for any $r \geq 2$, $i = 1, \ldots, r$, $n_i \in \mathbb{N}$ and $t_i \in \mathbb{R}$.
Since each $\mathbb{R}$-filtration on $E_n$ corresponds to an ultrametric norm $\|\cdot\|_n$ on $E_n$. The collection $\mathcal{E}_\bullet = (E_n, \|\cdot\|_n)$ is called an ultrametrically normed graded linear series. The strong $\delta$-superadditivity gives the following inequality:

$$\prod_{i=1}^{r} \|s_i\|_{n_i} \leq \left\| \prod_{i=1}^{r} s_i \right\|_{n_1 + n_2 + \ldots + n_r} \prod_{i=1}^{r} \dim_K(E_{n_i})^C$$

holds for any $r \geq 2$, $i = 1, \ldots, r$, $n_i \geq 0$ and $s_i \in E_{n_i}$.

**Definition 2.8.** — We define the volume of a graded linear series $E_\bullet$ of Kodaira-dimension $d$ by

$$\text{vol}(E_\bullet) := \limsup_{n \to +\infty} \frac{\dim_K(E_n)}{n^d}.$$  

If $E_\bullet$ is a ultrametrically normed graded linear series satisfying $\delta$-superadditivity, then we define its arithmetic volume and arithmetic $\chi$-volume by

$$\hat{\text{vol}}(E_\bullet) := \limsup_{n \to +\infty} \frac{\sum \max(\hat{\mu}_i(E_n), 0)}{n^{d+1}/(d+1)!},$$

$$\hat{\text{vol}}_\chi(E_\bullet) := \limsup_{n \to +\infty} \frac{\sum \hat{\mu}_i(E_n)}{n^{d+1}/(d+1)!}.$$  

Its asymptotic maximal slope, lower asymptotic minimal slope, and lower asymptotic minimal slope is defined respectively by

$$\hat{\mu}_{\text{asy}}^\max(E_\bullet) := \limsup_{n \to +\infty} \frac{\hat{\mu}_{\max}(E_n)}{n},$$

$$\hat{\mu}_{\text{asy}}^\inf(E_\bullet) := \liminf_{n \to +\infty} \frac{\hat{\mu}_{\min}(E_n)}{n},$$

$$\hat{\mu}_{\text{asy}}^\inf(E_\bullet) := \limsup_{n \to +\infty} \frac{\hat{\mu}_{\min}(E_n)}{n}.$$

### 3. Reminders on Arakelov geometry over adelic curves

#### 3.1. Adelic curves. —

**Definition 3.1.** — Let $K$ be a field and $M_K$ be all its places. An adelic curve is a 3-tuple $S = (K, (\Omega, \mathcal{A}, \nu), \phi)$ where $(\Omega, \mathcal{A}, \nu)$ is a measure space consisting of the space $\Omega$, the $\sigma$-algebra $\mathcal{A}$ and the measure $\nu$, and $\phi$ is a function ($\omega \in \Omega$) $\mapsto |\cdot|_\omega \in M_K$ such that

$$\omega \mapsto \ln|s|_\omega$$

is $\nu$-integrable for any non-zero element $s$ of $K$. Moreover, if the integral of above function is always zero, then we say $S$ is proper. For each $\omega \in \Omega$, we denote by $K_\omega$ the completion field of $K$ with respect to $|\cdot|_\omega$. 

Some typical examples are number fields, projective curves, polarised varieties. We refer to [4] Section 3.2 for detailed constructions. From now on, we assume $S$ to be proper unless it's specified.

3.2. Adelic vector bundles. — Let $E$ be a vector space over $K$ of dimension $n$. Let $\xi = \{\|\cdot\|_\omega\}$ be a norm family where each $\|\cdot\|_\omega$ is a norm on $E_{K_\omega} := E \otimes_K K_\omega$. We can easily define restriction $\xi_F$ of $\xi$ to a subspace $F$, quotient norm family $\xi_E \rightarrow G$ induced by a surjective homomorphism $E \rightarrow G$, the dual norm family $\xi^\vee$ on $E^\vee$, exterior power norm family and tensor product norm family. Please check [4] Section 1.1 and 4.1 for details.

Definition 3.2 (Adelic vector bundles). — We say a norm family $\xi$ is upper dominated if

$$\forall s \in E^*, \int_{\Omega} \ln \|s\|_\omega \nu(d\omega) < +\infty.$$ 

Moreover, we say $\xi$ is dominated if its dual norm $\xi^\vee$ on $E^\vee$ is also upper dominated. We say $\xi$ is measurable if for any $s \in E^*$, the function $s \mapsto \|s\|_\omega$ is $\mathcal{A}$-measurable. If $\xi$ is both dominated and measurable, we say the pair $\mathcal{E} = (E, \xi)$ is an adelic vector bundle. Note that the property of being an adelic vector bundle is well-preserved after taking restriction to subspace, quotient, exterior power and tensor product.

Definition 3.3 (Arakelov degrees and slopes). — Let $\mathcal{E} = (E, \xi)$ be an adelic vector bundle. If $E \neq 0$, then we define its Arakelov degree as

$$\hat{\deg}(\mathcal{E}) := -\int_{\omega} \ln \|s\|_{\text{det} \xi, \omega} \nu(d\omega)$$ 

where $s \in \text{det} E \setminus \{0\}$ and $\text{det} \xi$ is the determinant norm family on $\text{det} E$ (the highest exterior power). Note that this definition is independent with the choice of $s$ since the adelic curve $S$ is proper. If $E = 0$, then by convention we define that $\hat{\deg}(\mathcal{E}) = 0$. Moreover, we define the positive degree as

$$\hat{\deg}_+(\mathcal{E}) := \sup_{F \subset E} \hat{\deg}(F, \xi_F),$$ 

i.e. the supremum of all its adelic vector sub-bundles’ Arakelov degrees. If $E \neq 0$, its slope $\hat{\mu}(\mathcal{E})$ is defined to be the quotient $\hat{\deg}(\mathcal{E}) / \dim_K(E)$. The
maximal slope and minimal slope is defined respectively as
\[
\hat{\mu}_{\text{max}}(\mathcal{E}) := \begin{cases} 
\sup_{0 \neq F \subset \mathcal{E}} \hat{\mu}(F, \xi_F), & \text{if } E \neq 0 \\
-\infty, & \text{if } E = 0
\end{cases}
\]
\[
\hat{\mu}_{\text{min}}(\mathcal{E}) := \begin{cases} 
\inf_{E \not\subseteq G \neq 0} \hat{\mu}(G, \xi_{E \not\subseteq G}), & \text{if } E \neq 0 \\
+\infty, & \text{if } E = 0
\end{cases}
\]

**Remark 3.4.** — If the field \( K \) is of characteristic zero, then there exists a constant \( C > 0 \) such that for any two adelic vector bundles \( \mathcal{E} \) and \( \mathcal{F} \), it holds that
\[
\hat{\mu}_{\text{min}}(\mathcal{E} \otimes \mathcal{F}) \geq \hat{\mu}_{\text{min}}(\mathcal{E}) + \hat{\mu}_{\text{min}}(\mathcal{F}) - C \ln(\dim_K(\mathcal{E}) \dim_K(\mathcal{F}))
\]
which is firstly proved in [2] and reformulated in [4, Chapter 5], called minimal slope property of level \( \geq C \). All the constant \( C \) showed up in this article refers to the constant in this sense.

**Definition 3.5.** — Let \( \mathcal{E} = (E, \xi) \) be an adelic vector bundle of dimension \( n \). The Harder-Narasimhan \( \mathcal{R} \)-filtration is given by
\[
\mathcal{F}^t_{\text{hn}}(\mathcal{E}) = \sum_{0 \neq F \subset E, \hat{\mu}_{\text{min}}(F, \xi_F) \geq t} F.
\]
We denote by \( \hat{\mu}_{\text{min}}(\mathcal{E}) = \hat{\mu}_n \leq \hat{\mu}_{n-1} \leq \cdots \leq \hat{\mu}_1 = \hat{\mu}_{\text{max}}(\mathcal{E}) \) the jumping points of the \( \mathcal{R} \)-filtration, then it holds that
\[
\sum \hat{\mu}_i \leq \deg(\mathcal{E}) \leq \sum \hat{\mu}_i + \frac{1}{2} n \ln n,
\]
\[
\sum \max(\hat{\mu}_i, 0) \leq \deg_+(\mathcal{E}) \leq \sum \max(\hat{\mu}_i, 0) + \frac{1}{2} n \ln n.
\]

**3.3. Adelic Cartier divisors.** — Let \( X \) be a geometrically irreducible normal projective variety. Let \( H \) be an very ample line bundle whose global sections \( E := H^0(X, H) \) is equipped with a dominated norm family \( \xi = \{\|\|_\omega\}_{\omega \in \Omega} \) i.e. \( \mathcal{E} = (E, \xi) \) is an adelic vector bundle. For each place \( \omega \in \Omega \), we denote by \( X^\text{an}_\omega \) the analytification of \( X \times_{\text{Spec} K} \text{Spec} K_\omega \) in the sense of Berkovich(see [1]). As in [4, 2.2.3], the norm \( \|\|_\omega \) induces a Fubini-Study metric \( \varphi_{FS, \mathcal{E}, \omega} = \{\|\cdot\|_\omega(x)\} \), which is continuous in the sense that for any open subset \( U \subset X \) and a section \( s \in H^0(X, U) \), the function
\[
x \in U^\text{an}_\omega \mapsto |s|_\omega(x)
\]
is continuous with respect to Berkovich topology, where \( U^\text{an}_\omega \) is the analytification of \( U \times_{\text{Spec} K} \text{Spec} K_\omega \).
For arbitrary line bundle $L$, and two continuous metric families $\varphi = \{\varphi_\omega\}$, $\varphi' = \{\varphi'_\omega\}$, we define the distance function

$$\text{dist}(\varphi, \varphi') : (\omega \in \Omega) \mapsto \sup_{x \in X_{\omega}^{an}} |1|_{\varphi_\omega - \varphi'_\omega}(x)$$

where $|1|_{\varphi_\omega - \varphi'_\omega}(x)$ is a continuous function on $X$ since $\varphi_\omega - \varphi'_\omega$ is a continuous metric of $O_X$.

We say the pair $(H, \varphi)$ of a very ample line bundle and a continuous metric family is an adelic very ample line bundle if $\varphi$ is measurable (see [4, 6.1.4]), and there exists an dominated norm family $\xi$ on the global sections $E = H^0(X, H)$ such that the distance function $\text{dist}(\varphi, \varphi_{FS})$ is $\nu$-dominated.

As the very ample line bundles generates the Picard group $\text{Pic}(X)$, we define the arithmetic Picard group $\hat{\text{Pic}}(X)$ be the abelian group generated by all adelic very ample line bundles. An element in $\hat{\text{Pic}}(X)$ is called an adelic line bundle.

For a Cartier divisor $D$ on $X$, we define the $D_\omega$-Green function $g_\omega$ to be an element of the set

$$C^0_{gen}(X^{an}_\omega) := \{f \text{ is a continuous function on } U \mid \emptyset \neq U \subset X^{an}_\omega\} / \sim$$

where $f \sim g$ if they are identical on some non-empty open subset $V \subset X$ such that for any local equation $f_D$ of $D$ on $U$, $\ln |f_D| + g_\omega$ is continuous on $U^{an}_\omega$. It's easy to see that each $D_\omega$-Green function induce a continuous metric on the corresponding line bundle (see [4, Section 2.5] for details). Moreover, we say a pair $(D, g = \{g_\omega\})$ of Cartier divisor $D$ and Green function family $g$ is an adelic Cartier divisor if the it corresponds to an adelic line bundle. We denote by $\text{Div}_R(X)$ the group of all adelic Cartier divisor. Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$, then we can define the set of adelic $\mathbb{K}$-Cartier divisors as

$$\text{Div}_\mathbb{K}(X) = \text{Div}(X) \otimes \mathbb{K} / \sim$$

where "$\sim$" is the equivalence relationship generated by $\sum (0, g_i) \otimes k_i \sim (0, \sum g_i k_i)$, where $g_i$'s are continuous function families and $k_i \in \mathbb{K}$.

4. Applications on $\chi$-volume function over an adelic curve

In this section, we let $S = (K, (\Omega, A, \nu), \phi)$ be a proper adelic curve where $K$ is of characteristic 0. Let $X$ be a geometrically irreducible smooth $K$-variety, and $(D, g)$ be a adelic $\mathbb{R}$-Cartier divisor on $X$. Then each $E_n := H^0(X, O_X([nD]))$ admits with an adelic vector bundle structure by the norm family $\xi_{ng}$. Then we can equip $E_n$ with the Harder-Narasimhan filtration $F^i_n E_n$ induced by $\xi_{ng}$. Then this filtered linear series satisfies the $\delta$-superadditivity. Notice that there is a correspondence between $\mathbb{R}$-filtrations and ultrametric norms on $E_n$ over trivially valued $K$. We denote by $\|\cdot\|_n$ the norm induced by $F^i_n E_n$. The volume, $\chi$-volume, asymptotic maximal slope, lower asymptotic
minimal slope, upper asymptotic minimal slope of \((D, g)\) is defined to be the same with the definition \(\hat{\mu}_{\text{min}}(E)\) of the corresponding ultrametrically normed graded linear series \(E_n = \{(E_n, \|\cdot\|_{n})\}\).

We recall some previous concerning volume and \(\chi\)-volume here.

1. \(\hat{\text{vol}}(\cdot)\) is continuous on any finitely dimensional subspace of \(\text{Div}_{\mathbb{R}}(X)\).
2. If \(\hat{\mu}_{\text{min}}^\text{sup}(D, g) > -\infty\), then there exists a integrable function \(\psi\) on \(\Omega\) such that
   \[
   \hat{\deg}(E_n, \xi_{n(g+\psi)}) = \hat{\deg}_{\sigma}(E_n, \xi_{n(g+\psi)})
   \]
   for every \(n \in \mathbb{N}_+\). In particular, \(\hat{\text{vol}}(D, g + \psi) = \hat{\text{vol}}_{\chi}(D, g + \psi)\).

4.1. Boundedness of asymptotic minimal slope. —

**Proposition 4.1.** — Let \(E_n = \{(E_n, \|\cdot\|_{n})\}_{n \in \mathbb{N}}\) be an ultrametrically normed graded linear series of finite type satisfying \(\delta\)-superadditivity. Then it holds that
\[
\hat{\mu}_{\text{min}}^\text{inf}(E) := \liminf_{n \to +\infty} \frac{\hat{\mu}_{\text{min}}(E_n)}{n} > -\infty.
\]

**Proof.** — We fix a set of generators, and assume that they are of degree at most \(N\). Then for any \(n \in \mathbb{N}_+\), let
\[
\Psi_n := \{(\lambda_1, \lambda_2, \ldots, \lambda_r) \mid 1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r \leq \lambda, \sum_i \lambda_i = n, r = 1, \ldots, n\}.
\]
Then the map
\[
\bigoplus_{(\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Psi_n} (\otimes_{i=1}^r (E_{\lambda_i}, \|\cdot\|_{\lambda_i}, \dim_K(E_n)^C) \rightarrow E_n
\]
is surjective and of operator norm \(\leq 1\). Therefore we have
\[
\hat{\mu}_{\text{min}}(E_n) \geq \min_{(\lambda_1, \lambda_2, \ldots, \lambda_r) \in \Psi_n} \{\sum_{i=1}^r (\hat{\mu}_{\text{min}}(E_{\lambda_i}) - \delta(\lambda_i))\}.
\]
Let \(L = \min \left\{\frac{\hat{\mu}_{\text{min}}(E_l) - \delta(l)}{l} \right\}_{1 \leq l \leq N}\). Then \(\frac{\hat{\mu}_{\text{min}}(E_n)}{n} \geq L\) for every \(n \in \mathbb{N}_+\).

**Corollary 4.2.** — If the section ring of \(D\) is finitely generated, then for any Green function family \(g\) on \(D\), it always holds that
\[
\hat{\mu}_{\text{min}}^\text{inf}(D, g) > -\infty.
\]
In particular, the above inequality holds if \(D\) is semiample.
4.2. Continuity of $\chi$-volume. —

**Proposition 4.3.** — Let $(L_1, \varphi_1), (L_2, \varphi_2), \ldots, (L_r, \varphi_r)$ be adelic line bundles on $X$ such that all $L_i$’s are semiample. For any $a = (a_1, a_2, \ldots, a_r) \in \mathbb{N}^r$, denote that

$$a \cdot L := \sum_{i=1}^r a_i L_i$$

and

$$a \cdot \varphi := \sum_{i=1}^r a_i \varphi_i.$$ 

Then there exists constants $S$ and $T$ such that

$$\hat{\mu}_{\text{min}}(H^0(X, a \cdot L), \xi_{a \cdot \varphi}) \geq S \cdot |a| + T$$

where $|a| := a_1 + a_2 + \cdots + a_r$.

**Proof.** — Let $E = \bigoplus_{i=1}^r L_i$, $\mathbb{P}(E) := \text{Proj}_X(\text{Sym}(E))$ the projective bundle on $X$ associated with $E$. Let $H = \mathcal{O}_{\mathbb{P}(E)}(1)$. Then

$$H^0(\mathbb{P}(E), mH) = \bigoplus_{a_1 + \cdots + a_r = m} H^0(X, \sum_{i=1}^r a_i L_i)$$

Note that since $L_i$’s are semiample, so is $H$ on $\mathbb{P}(E)$. For each $a \in \mathbb{N}^r$, we denote by $\|\cdot\|_{(a)}$ the norm on $H^0(X, a \cdot L)$ over trivially valued $K$ induced by the Harder-Narasimhan filtration of $\xi_{a \cdot \varphi}$. If we write $a = a^{(1)} + a^{(2)} + \cdots + a^{(l)}$ where $a^{(j)} \in \mathbb{N}^r$, then it holds that for any $u = u_1 \cdots u_l \in H^0(X, a \cdot L)$ where $u_j \in H^0(X, a^{(j)} \cdot L)$,

$$\|u\|_{(a)} \leq \prod_{i=1}^l \|u_j\|_{(a^{(j)})} \dim_K(H^0(X, a^{(j)} \cdot L))^C$$

Let $\|\cdot\|_m$ be the direct sum of $\{|\cdot|_{(a)} | a \in \mathbb{N}^r, |a| = m\}$. Then it suffices to prove that the normed graded algebra

$$\bigoplus_{m \geq 0} (H^0(\mathbb{P}(E), mH), \|\cdot\|_m)$$

is strong $\delta$-supperadditive. Let $s_i = \sum_{|a|=m_i} u^{(i)}_{a} \in H^0(\mathbb{P}(E), m_i H)$ where $i = 1, 2, \ldots, l$, $m_i \in \mathbb{N}$, $u^{(i)}_{a} \in H^0(X, a \cdot L)$. It holds that

$$\|s_i\|_{m_i} = \max_{|a|=m_i} \{\|u^{(i)}_{a}\|_{(a)}\}.$$
Moreover,

\[ \|s_1 \cdots s_l\|_m = \max_{\|a\|=m} \left\{ \left\| \sum_{a^{(1)}+\cdots+a^{(l)}=a} \prod_{i=1}^l u_a^{(i)} \right\|_{\alpha(a)} \right\} \]

\[ \leq \max_{\|a^{(i)}|=m_i} \left\| \prod_{i=1}^l u_a^{(i)} \right\|_{\alpha(a)} \left( \dim_K(H^0(X, a^{(i)} \cdot L))^C \right) \]

\[ \leq \prod_{i=1}^l \max_{\|a\|=m_i} \{ \|u_a^{(i)}\|_{\alpha(a)} \} \dim_K(H^0(X, a \cdot L))^C \]

\[ \leq \prod_{i=1}^l \|s_i\|_m \dim_K(H^0(\mathbb{P}(E), m_iH))^C. \]

Therefore there exists constants \( S \) and \( T \) such that

\[ \hat{\mu}_{\min}(H^0(X, a \cdot L), \xi_{a \cdot L}) \geq \hat{\mu}_{\min}(H^0(\mathbb{P}(E), |a|H), \|\|_{|a|}) \geq S \cdot |a| + T. \]

\[ \square \]

**Theorem 4.4.** — Let \( \overline{D} = (D, g), \overline{E}_1 = (E_1, h_1), \ldots, \overline{E}_r = (E_r, h_r) \) be adelic \( \mathbb{Q} \)-Cartier divisors on \( X \) such that \( D \) and \( E_i \)'s are semiaffine.

(3) \[ \lim_{\epsilon_i+\cdots+\epsilon_r \to 0} \overline{\text{vol}}_X(\overline{D} + \sum_{i=1}^r \epsilon_i E_i) = \overline{\text{vol}}_X(\overline{D}) \]

**Proof.** — Due to the homogeneity, we may assume that all \( D \) and \( E_i \) are integral semiaffine Cartier divisors. Then there exists constants \( S \) and \( T \) depending on \( \overline{D} \) and \( \overline{E}_1, \ldots, \overline{E}_r \), such that

\[ \hat{\mu}_{\min}(H^0(X, n_0D + \sum_{i=1}^r n_iE_i), \xi_{n_0g + \sum_{i=1}^r n_ih_i}) \geq T + S \sum_{i=0}^r n_i \]

where \( n_i \in \mathbb{N} \). Assume that \( \epsilon_i = p_i/q_i \) where \( p_i \) and \( q_i \) are coprime positive integers and \( q_i \geq 1 \). Let \( q = \prod_{i=1}^r q_i \), then it holds that

\[ \hat{\mu}_{\min}(H^0(X, mq(D + \sum_{i=1}^r \epsilon_i E_i)), \xi_{nq(g + \sum_{i=1}^r r_ih_i)}) \geq T + Smq(1 + \sum_{i=1}^r \epsilon_i) \]
for every \( m \in \mathbb{N} \). Therefore
\[
\hat{\rho}_{\min}^{\sup}(D + \sum_{i=1}^{r} \epsilon_iE_i) \geq S(1 + \sum_{i=1}^{r} \epsilon_i).
\]

Take \( \nu \)-integrable functions \( \phi \) such that \( \int_{\Omega} \phi \nu(d\omega) = S \). Denote that \( |\epsilon| = \sum_{i=1}^{r} \epsilon_i \), it holds that
\[
(4) \quad \hat{\text{vol}}(D + \sum_{i=1}^{r} \epsilon_iE_i + (0, 1 + |\epsilon|\phi)) = \hat{\text{vol}}(D + \sum_{i=1}^{r} \epsilon_iE_i + (0, 1 + |\epsilon|\phi)).
\]
Due to the continuity of \( \hat{\text{vol}}(\cdot) \), (3) can be easily derived from (4).

5. Applications on arithmetic Hilbert-Samuel function

5.1. Asymptotically modified norm. —

Let \( \mathcal{E}_n = \{ E_n = (E_n, \| \cdot \|_n) \}_{n \in \mathbb{N}} \) be an ultrametrically normed graded linear series of finite type satisfying \( \delta \)-superadditivity. Let \( r_n = \dim_K(E_n) \).

**Proposition 5.1.** — For any \( n \in \mathbb{N} \) and \( s \in E_n \), we define
\[
\| s \|^\prime_n := \liminf_{m \to \infty} \| s^m \|^\frac{1}{m}.
\]
Then the following four properties holds:

1. \( \| s \|^\prime_n = \lim_{m \to \infty} \| s^m \|^\frac{1}{m} \).
2. \( \| s \|^\prime_n \) is an ultrametric norm on \( E_n \) over trivially valued \( K \).
3. \( \| s \|^\prime_n \| t \|^\prime_m \geq \| st \|^\prime_{n+m} \) for any \( s \in E_n \) and \( t \in E_m \).
4. \( \| s \|^\prime_n \leq r_n^C \).

**Proof.** — (1)Let \( a_m := -\ln \| s^m \|_{nm} \). Since \( \mathcal{F}_n \) is \( \delta \)-superadditive, we have
\[
\mathcal{F}_{nm}F_{nm}F_{nm}^mE_{nm'} \subset \mathcal{F}^{a_m + a_{m'} - \delta(nm) - \delta(nm')}E_{n(n+m')}
\]
which implies that \( a_{m+m'} \geq a_m + a_{m'} - \delta(nm) - \delta(nm') \). So the sequence \( \{ a_m \} \) converges in \( \mathbb{R} \).

(2) For any \( s, t \in E_n \), \( \| (s + t)^m \|_{nm} \leq \max_{i=0,\ldots,m} \{ \| s^i t^{m-i} \|_{nm} \} \). Since the filtration is \( \delta \)-superadditive, we can deduce that
\[
\| s^i t^{m-i} \|_{nm} \leq \| s^i \|_{ni} \| t^{m-i} \|_{n(m-i)} \exp(\delta(ni) + \delta(n(m-i))).
\]
Let \( A = \max\{ \| s \|^\prime_n, \| t \|^\prime_n \} \). Then for any \( \epsilon > 0 \), there is an integer \( N \) such that
\[
\| s \|^\prime_n \exp(\delta(ni)) \leq A + \epsilon \quad \text{and} \quad \| t \|^\prime_n \exp(\delta(ni')) \leq A + \epsilon \quad \text{for every} \quad l > N.
\]
Let $B = \max_{i=0, \ldots, N} \{ \max(\|s_i\|_{n_i}, \|t_i\|_{n_i}) \exp(\delta(n_i)) \}$. When $m > 2N$, either $i$ or $m - i$ is greater than $N$, thus

$$
\|(s + t)^m\|_{\frac{1}{nm}} \leq \max_{i=0, \ldots, N} \{(A + \epsilon)^{\frac{m-i}{mn}} B^\frac{1}{m}\}.
$$

The right hand side of the inequality has the limit $A + \epsilon$ which implies that there exists $N' \in \mathbb{N}_+$ such that $\|(s + t)^m\|_{\frac{1}{nm}} \leq A + 2\epsilon$ for every $m > N'$. Therefore $\|s + t\|_n \leq A$.

(3) Due to the $\delta$-superadditivity, we have

$$
-\ln \|st\|_{l(n+m)} \geq -\ln \|s\|_{nl} + -\ln \|t\|_{ml} - \delta(nl) - \delta(ml)
$$

for every $l \in \mathbb{N}_+$. Let $l \to +\infty$, we obtain (3).

(4) This is a direct result from an estimate of the limit.

By (2) of 5.1 we can give an $\mathbb{R}$-filtration of $E_n$ induced by $\|\cdot\|'_n$. Let $\overline{E}_n := (E_n, \|\cdot\|'_n)$. It’s easy to see that $\hat{\mu}_l(\overline{E}_n) + \delta(n) \geq \hat{\mu}_l(\overline{E}_n)$ because of (4) of Proposition 5.1.

**Proposition 5.2.** — It holds that

(5) $\hat{\mu}^{asy}_{\max}(\overline{E}_n) = \hat{\mu}^{asy}_{\max}(\overline{E}_n)$.

**Proof.** — It’s obvious that $\hat{\mu}^{asy}_{\max}(\overline{E}_n) \leq \hat{\mu}^{asy}_{\max}(\overline{E}_n)$. Conversely, for any $t < \hat{\mu}_{\max}(\overline{E}_n)$, there exists an increasing sequence of integers $\{n_k \in \mathbb{N}\}_{k \in \mathbb{N}}$ and a sequence of sections $\{s_k \in E_{n_k}\}_{k \in \mathbb{N}}$, such that for all $k \in \mathbb{N}$

$$
-\ln \|s_k\|_{n_k} > t.
$$

By the definition of $\|\cdot\|'_n$, for each $k \in \mathbb{N}$, $-\ln \|s_k^m\|_{\frac{1}{mn}} > t$ for every $m \gg 0$. Therefore we can obtain another increasing sequence of integers $\{t_k := m_kn_k | m_k \in \mathbb{N}_+\}$ and $t_k := s_k^{m_k}$ such that

$$
-\ln \|t_k\|_{t_k} > t
$$

which implies that $\hat{\mu}^{asy}_{\max}(\overline{E}_n) \geq t$. Hence the equation (5) is obtained.

For each $t \in \mathbb{R}$, we define $E^t_n := \{s \in E_n | -\ln \|s\|'_n \geq nt\}$ and $E^t_{\star} := \{E^t_n\}_{n \in \mathbb{N}}$. Then $E^t_{\star}$ is a graded linear series of subfinite type due to property (3) of Proposition 5.1.
5.2. Estimate on graded series of subfinite type. — This part is a result of [5] after some modifications. Let $X$ be a projective $K$-scheme of dimension $d$ admitting the following collection of morphisms $\{p_i : X_i \to C_i\}_{i=1,\ldots,n}$ given by following construction:

1. If $d = 1$, then $C_1 := X$, $X_1 \to C_1$ is the normalization of $X$.
2. If $d > 1$, then $C_1$ is a projective regular curve over $\text{Spec} K$, $X_1 = X$ and $p_1$ is a projective and flat $k$-morphism.
3. For any $i \in \{2, \ldots, d-1\}$, $C_i$ is a projective regular curve over $K(C_{i-1})$, $X_i$ is the the generic fiber of $p_{i-1}$ and $p_i$ is a projective flat morphism of $K(C_{i-1})$-schemes.
4. $X_d$ is the normalization of the generic fiber of $p_{d-1}$.

**Definition 5.3.** — Let $L$ be a big line bundle on $X$. Let $E_{\bullet}$ be a graded subalgebra of the section ring $R(X, L)$ of $L$. We say that $E_{\bullet}$ contains an ample divisor if

1. There exists an $m \gg 0$ and a decomposition $mL = A + F$ where $A$ is ample and $F$ is effective.
2. For every $k \gg 0$, we have the inclusion $H^0(X, kA_m) \subset E_{km} \subset H^0(X, kmL)$

**Lemma 5.4.** — Let $(L, \varphi)$ be an adelic Cartier divisor on $X$ with $L$ being big. Let $E_{\bullet}$ be the corresponding ultrametrically normed graded linear series. Then for any $t \leq \hat{\mu}_{\text{asy}}(L, \varphi)$, the graded linear series $E_{\bullet}$ given by asymptotically modified norms as described in previous subsection contains an ample divisor.

Proof. — See [3] Lemma 1.6.

**Theorem 5.5.** — Let $L$ be a big line bundle on $X$. There exists a $f(n) \simeq O(n^{d-1})$ such that for any graded subalgebra $F_{\bullet}$ of the section ring $R(X, L)$ containing an ample divisor, it holds that

$$\dim_K(F_n) \leq \text{vol}(F_{\bullet})n^d + f(n).$$

Proof. — See [5] Theorem 5.1.

**Theorem 5.6 (Upper bound of arithmetic Hilbert-Samuel function)**

Let $(D, g)$ be an adelic divisor on $X$ with $D$ being big. For each $n \in \mathbb{N}$, let $E_n$ denote the pair of $E_n = H^0(X, O_X(nD))$ and norm family $\xi_{ng}$. Then it holds that

$$\deg_+(E_n) \leq \hat{\text{vol}}(D, g) \frac{n^{d+1}}{(d+1)!} + O(n^d) + (C + 1/2) \cdot r_n \ln(r_n).$$
Proof. — Let \( \| \cdot \|_n \) be the ultrametric norm induced by the Harder-Narasimhan filtration on \( E_n \). We apply the asymptotic modification on \( \{ (E_n, \| \cdot \|_n) \} \)

\[
\hat{\deg}_+(E_n) \leq \sum_{i=1}^{r_n} \max(\hat{\mu}_i(E_n), 0) + 1/2 \cdot r_n \ln(r_n)
\]

\[
\leq \sum_{i=1}^{r_n} \max(\hat{\mu}'_i(E_n), 0) + (C + 1/2) \cdot r_n \ln(r_n)
\]

\[
= n \int_0^{+\infty} \dim_K(E^*_n) dt + C \cdot r_n \ln(r_n) + \delta(E_n)
\]

\[
\leq n \int_0^{p_{max}(D,g)} \vol(E^*_n) n^d dt + O(n^d) + (C + 1/2) \cdot r_n \ln(r_n)
\]

\[
= \vol(D, g) \frac{n^{d+1}}{(d+1)!} + O(n^d) + (C + 1/2) \cdot r_n \ln(r_n)
\]

The last equation is obtained by the virtue of [4, Theorem 6.3.16].

Corollary 5.7. — We keep the hypothesis in Theorem 5.6. If \( D \) is semiample, then

\[
\hat{\deg}(E_n) \leq \hat{\vol}(D, g) \frac{n^{d+1}}{(d+1)!} + O(n^d) + (C + 1/2) \cdot r_n \ln(r_n)
\]

Proof. — Since there exists an integrable function \( \psi \) on \( \Omega \) such that \( \forall n \in \mathbb{N}_+, \hat{\deg}(E_n, \xi + M_n \psi) = \hat{\deg}(E_n, \xi + M_n \psi) \)

and

\[
\hat{\vol}(D, g + \psi) = \hat{\vol}(D, g + \psi),
\]

applying previous proposition, we obtain that

\[
\hat{\deg}(E_n) + n \cdot r_n A
\]

\[
\leq \hat{\vol}(D, g) \frac{n^{d+1}}{(d+1)!} + \vol(D) \frac{n^{d+1}}{d!} A + O(n^d) + (C + 1/2) \cdot r_n \ln(r_n)
\]

where \( A = \int_\Omega \psi(\omega) d\omega \). Then the formula (6) is obtained by Riemann-Roch type inequality of Koll\&Acute and Matsusaka [7].

References

[1] V. G. Berkovich – Spectral theory and analytic geometry over non-archimedean fields, no. 33, American Mathematical Soc., 2012.

[2] J.-B. Bost & H. Chen – “Concerning the semistability of tensor products in arakelov geometry”, Journal de Math\&Eacutematiques Pures et Appliqu\&Eacute\'es 99 (2013), no. 4, p. 436 – 488.
[3] S. Boucksom & H. Chen – “Okounkov bodies of filtered linear series”, *Compositio Mathematica* **147** (2011), no. 4, p. 1205–1229.

[4] H. Chen & A. Moriwaki – *Arakelov geometry over adelic curves*, Lecture Notes in Mathematics, vol. 2258, 2019.

[5] H. Chen – “Majorations explicites des fonctions de hilbert-.samuel géométrique et arithmétique”, *Mathematische Zeitschrift* **279**(1-2) (2014), p. 99 – 137.

[6] _______ , “Hodge index inequality in geometry and arithmetic: a probabilistic approach”, *Journal de l’École polytechnique - Mathématiques* **3** (2016), p. 231 – 262.

[7] J. Kollár & T. Matsusaka – “Riemann-roch type inequalities”, *American Journal of Mathematics* **105** (1983), no. 1, p. 229–252.