An explicit description of (1, 1) L-space knots, and non-left-orderable surgeries

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Abstract
Greene, Lewallen and Vafaee characterized (1, 1) L-space knots in $S^3$ and lens space in the notation of coherent reduced (1, 1)-diagrams. We analyze these diagrams, and deduce an explicit description of these knots. With the new description, we prove that any L-space obtained by Dehn surgery on a (1, 1)-knot in $S^3$ has non-left-orderable fundamental group.

1 Introduction
An L-space is a rational homology 3-sphere with minimal Heegaard Floer homology, that is, $\dim \hat{HF} = |H_1(Y)|$. A nice topological property of L-spaces is that [18], they do not admit co-orientable taut foliations, and its converse statement is only partially verified. Another conjectural property of L-spaces is the non-left-orderability of fundamental groups [2], that is, there does not exist a total order $\leq$ on the fundamental group such that $g \leq h$ implies $fg \leq fh$. Although we have multiple computational tools, the Heegaard Floer data is not easy to utilize. Therefore, a better characterization of L-spaces would be helpful.

One way to construct L-spaces is via Dehn surgeries. A knot $K$ is called an L-space knot, if it admits an L-space surgery. It is a positive (resp. negative) L-space knot if it admits a positive (resp. negative) L-space surgery. The Dehn surgery along a nontrivial positive L-space knot $K$ in $S^3$ with slope $\frac{p}{q}$ yields an L-space if and only if $\frac{p}{q} \geq 2g(K) − 1$ [19]. Similar results also hold for knots in other L-spaces [21].

For a closed orientable 3-manifold $Y$, we say that a knot $K$ in $Y$ is a $(g, b)$-knot, if there exists a Heegaard splitting $Y = U_0 \cup U_1$ of genus $g$, such that each of $K \cap U_0$ and $K \cap U_1$ consists of $b$ trivial arcs. The family of $(1, 1)$-knots (also called 1-bridge torus knot in the literature) in the 3-sphere and lens spaces is widely studied. The knot Floer invariants arises diagrammatically [8, 10, 20] if we can find a $(1, 1)$-decomposition.

A $(1, 1)$-diagram for a $(1, 1)$-knot $K$ in the three-sphere or lens space $Y$ a doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$, which consists of two simple closed curves $\alpha$ and $\beta$ on the torus $\Sigma$ and two basepoints $w$ and $z$ in $\Sigma − \alpha − \beta$. The diagram $(\Sigma, \alpha, \beta, w, z)$ is called reduced if each bigon contains a basepoint. In this case, the diagram can be specified [20] by four parameters $p, q, r, s$. Via successive isotopies to removing empty bigons, every $(1, 1)$-knot has a reduced $(1, 1)$-diagram. In [9], Greene, Lewallen and Vafaee established the following criterion to determine whether a reduced $(1, 1)$-diagram represents an L-space knot.
Previous Work. [9, Theorem 1.2] A reduced \((1,1)\)-diagram represents an L-space knot if and only if it is coherent, that is, there exist orientations on \(\alpha\) and \(\beta\) that induce coherent orientations on the boundary of every embedded bigon \((D, \partial D) \subseteq (\Sigma, \alpha \cup \beta)\). It represents a positive or negative L-space knot according to the sign of \(\alpha \cdot \beta\) with coherent orientation.

Building on their work, we describe the family of \((1,1)\) L-space knots explicitly as follows.

**Theorem 1.** Let \(Y = U_0 \cup_\Sigma U_1\) be a genus one Heegaard splitting of a three-sphere or lens space, with standard geometry. A knot in \(Y\) is a \((1,1)\) L-space knot if and only if it is isotopic to a union of three arcs \(\rho \cup \tau_0 \cup \tau_1\), such that

(a) \(\rho\) is a geodesic of \(\Sigma\);
(b) \(\tau_0\) is properly embedded in some meridional disk of \(U_0\);
(c) \(\tau_1\) is properly embedded in some meridional disk of \(U_1\).

Note that, if \(\tau_0\) or \(\tau_1\) is of length zero, then by definition, the knot is a 1-bridge braid in \(Y\). The study of 1-bridge braids originates from the classification of knots in a solid torus with nontrivial solid torus surgeries [1, 6, 7], where it is shown that every such knot is a torus knot or a 1-bridge braid. If we put solid torus in the standard position in \(S^3\), these knots has nontrivial lens space surgeries. And as its name suggests, any lens space is an L-space. It is also proved that any 1-bridge braid in the three-sphere or lens space is an L-space knot [9], and the L-spaces obtained by surgeries along 1-bridge braids in \(S^3\) has non-left-orderable fundamental groups [17]. In line with these researches, we deduce similar properties of \((1,1)\) L-space knots in \(S^3\).

**Theorem 2.** A nontrivial positive \((1,1)\) L-space knot in \(S^3\) can be represented as the closure of the braid

\[
(\sigma_{\omega} \sigma_{\omega-1} \cdots \sigma_{\omega-b_0+1}) (\sigma_{\omega} \sigma_{\omega-1} \cdots \sigma_1)^{b_1} (\sigma_{\omega-1} \sigma_{\omega-2} \cdots \sigma_1)^{t-b_1}
\]

in the braid group \(B_{\omega+1}\) on \(\omega + 1\) strands, where \(1 \leq b_0 \leq \omega\) and \(1 \leq b_1 \leq t\).

An example is shown in Figure 1 below.

![Figure 1: The braid when \((\omega, t, b_0, b_1) = (6, 7, 4, 3)\).](image)

In [17], the author introduced the property (D) as follows.

**Definition 3.** For a nontrivial knot \(K\) in \(S^3\) with \(\mu\) and \(\lambda\) representing a meridian and a longitude in the knot group, we say \(K\) has property (D) if

1. for any homomorphism \(\rho\) from \(\pi_1(S^3 - K)\) to \(\text{Homeo}_+(\mathbb{R})\), if \(s \in \mathbb{R}\) is a common fixed point of \(\rho(\mu)\) and \(\rho(\lambda)\), then \(s\) is a fixed point of every element in \(\pi_1(S^3 - K)\);
2. \(\mu\) is in the root-closed, conjugacy-closed submonoid generated by \(\mu^{2^{g(K)-1}}\lambda\) and \(\mu^{-1}\).
The author proved that [17, Theorem 1.3] nontrivial knots which are closures of positive 1-bridge braids have property (D). And by [17, Theorem 4.1], it implies the non-left-orderability of the fundamental groups of the L-spaces obtained by Dehn surgeries on closures of 1-bridge braids. In this paper, we prove the following result in a similar way. Thanks to the additional symmetry, our proof is simplified compared to the proof of [17, Theorem 1.3].

**Theorem 4.** Nontrivial positive \((1,1)\) L-space knots in \(S^3\) have property (D).

Therefore, by [17, Theorem 4.1], we have the following conclusion.

**Theorem 5.** The fundamental group of an L-space obtained by Dehn surgery on a \((1,1)\)-knot in \(S^3\) is not left orderable.

Because a \((1,1)\)-decomposition eases the computation of knot Floer homology, many examples of L-space knots which were studied in the literature are \((1,1)\)-knots. Theorem 5 serves as the generalization of relevant non-left-orderability results [3, 4, 11, 12, 13, 14, 15, 16, 17, 22, 23].

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2 Description of \((1,1)\) L-space knots

This section is dedicated to prove Theorem 4.

Let \((\Sigma, \alpha, \beta, w, z)\) denote a reduced \((1,1)\)-diagram representing an L-space knot in the three-sphere or lens space \(Y\). Since the \(\alpha\) curve is simple, it represents a primitive element in \(H_1(\Sigma)\). We can straighten out the \(\alpha\) curve via a self-homeomorphism of \(\Sigma\). Assume that \(\alpha\) curve is horizontal, with an orientation from left to right. The \(\beta\) curve is cut into strands of two bands and two rainbows by the \(\alpha\) curve. By [9, Theorem 1.2], we can choose an orientation of \(\beta\) which induces an orientation from left to right on every rainbow strand, which is the opposite of the coherent orientation. Assume that \(w\) is on the left side of \(\alpha\), and \(z\) is on the right side of \(\alpha\).

2.1 The first step

We define the positive curves on the torus \(\Sigma\) as follows.

**Definition 6.** An oriented curve \(\gamma\) on \(\Sigma\) is called positive, if at each inner intersection point of \(\gamma\) and \(\alpha \cup \beta\), the \(\gamma\) curve goes from the right side of the \(\alpha\) or \(\beta\) curve to the left side transversally.

Our first step is to construct a positive curve \(\gamma_0\) connecting \(w\) to \(z\).

Let \(S\) be the set of endpoints of all positive curves originating from \(w\). If \(z \in S\), our first step is completed. Otherwise, we assume \(z \not\in S\). Let \(S_w\) be the connected component of \(S - \alpha\) containing \(w\). Since \(w \in S_w\), each point on a rainbow strand around \(w\) is an interior point of \(S_w\). Since \(z \not\in S\), each point on a rainbow strand around \(z\) is in the exterior of \(S_w\). Therefore, the shape of \(S_w\) is a rectangle. Let the vertices of \(S_w\) be \(P_1, P_2, P_3, P_4\) counterclockwise, with \(P_1P_2, P_1P_3\) being parts of the \(\alpha\) curve, \(P_2P_3, P_3P_4\) being parts of
the $\beta$ curve, and the basepoint $w$ being close to the edge $P_1P_2$. From the definition of $S_w$, we can derive the orientation of $P_4P_1$ and $P_2P_3$ as shown in Figure 2.

![Figure 2: The rectangle $S_w$.](image)

If there exists another embedded open rectangle $P_4P_3P_5P_6$ on the left of $P_4P_3$ which does not contain any basepoints, then we replace $P_1P_2P_3P_4$ by the immersed rectangle $P_1P_2P_3P_6$ and try the same extension again. Because the $\beta$ curve is connected, the edge $P_2P_3$ extends to the right strand of the basepoint $z$ in finite steps. Hence, we assume that the sequence of extensions ends at an immersed rectangle $P_1P_2Q_2Q_1$, as shown in Figure 3.

![Figure 3: The immersed rectangle $P_1P_2Q_2Q_1$.](image)

There are two possibilities for not able to extend the immersed rectangle: one of the edge $P_1Q_1$ and $P_2Q_2$ extends to a rainbow strand, or the embedded rectangle on the left of $Q_1Q_2$ contains at least one basepoint. If the embedded rectangle on the left of $Q_1Q_2$ contains the basepoint $z$. Then by the definition of $S$, we have $z \in S$. Otherwise, the edge $Q_1Q_2$ intersects with the edge $P_1P_2$ on the torus $\Sigma$.

By the definition of $S_w$, the strands $P_4P_1$ and $P_2P_3$ are on the boundary of $S$, so we have $Q_1Q_2 \subseteq P_1P_2$. If $P_1 = Q_1$ or $P_2 = Q_2$, then the edge $Q_1P_1$ or the edge $P_2Q_2$ covers the $\beta$ curve. In that case, $Q_1P_1$ or $P_2Q_2$ contains the left and right strand of the basepoint $z$, which is a contradiction. Therefore, the edge $Q_1Q_2$ lies in the interior of the edge $P_1P_2$.

Suppose that there are $q$ rainbow strands in the middle, $r_1 \geq 1$ band strands on the left (including $Q_1P_1$) and $r_2 \geq 1$ band strands on the right (including $P_2Q_2$) in the immersed rectangle $P_1P_2Q_2Q_1$. Suppose that the $i$-th intersection point on $Q_1Q_2$ is the $(i + k)$-th intersection point on $P_1P_2$ for $1 \leq i \leq r_1 + r_2$. Then we have $1 \leq k \leq 2q - 1$.

For $1 \leq i \leq 2q + r_1 + r_2$, let $\varepsilon_i = 1$ if the $\beta$ curve goes from the right side of the $\alpha$ curve to the left side at the $i$-th intersection point on $P_1P_2$. Otherwise, let $\varepsilon_i = -1$. Then
we have
\[ \varepsilon_i = \begin{cases} 
\varepsilon_{i+k} & \text{if } 1 \leq i \leq r_1; \\
1 & \text{if } r_1 + 1 \leq i \leq r_1 + q; \\
-1 & \text{if } r_1 + q + 1 \leq i \leq r_1 + 2q; \\
\varepsilon_{i-2q+k} & \text{if } 2q + r_1 + 1 \leq i \leq 2q + r_1 + r_2.
\end{cases} \]

If \( 1 \leq k \leq q \), then we have \( \varepsilon_1 = 1 \) by induction. If \( q + 1 \leq k \leq 2q - 1 \), then we have \( \varepsilon_{2q+r_1+r_2} = -1 \) by induction. Either case leads to a contradiction.

### 2.2 The second step

We have constructed a positive curve \( \gamma_0 \) connecting \( w \) to \( z \). By eliminating self-loops, we assume that \( \gamma_0 \) is simple and intersects each connected component of \( \Sigma - \alpha - \beta \) at most once. Our second step is to construct a positive simple closed curve \( \gamma \) passing through \( w \) and \( z \).

Let \( T_1, T_2, \ldots, T_p \) be all intersection points between the \( \alpha \) curve and the \( \beta \) curve, ordered along the orientation of \( \alpha \). Via a self-homeomorphism of \( \Sigma \), we assume the following condition: for \( 1 \leq i \leq p \), if the \( \alpha \)-segment \( T_iT_{i+1} \) does not intersect with the \( \gamma_0 \) curve, then it has unit length; otherwise, it has length \( 2q + 1 \), where \( q \) is the number of strands in each rainbow.

For a downward-oriented band strand \( e_1 \) and an upward-oriented band strand \( e_2 \) on the \( \beta \) curve, there exists an embedded open rectangle \( R \) with two edges being \( e_1 \) and \( e_2 \) and the other two edges \( e_3, e_4 \) on the \( \alpha \) curve. We can further assume that the rectangle is on the left of \( e_1, e_2, e_3 \) and on the right of \( e_4 \).

Let \( l_i \) denote the length of \( e_i \) for \( i = 3, 4 \), then
\[ l_i = |e_i \cap \beta| + 2q |e_i \cap \gamma_0| - 1. \]

The difference \( |e_4 \cap \beta| - |e_3 \cap \beta| \) depends on whether each basepoint lies in \( R \), that is,
\[ |e_4 \cap \beta| - |e_3 \cap \beta| = 2q |\{z\} \cap R| - 2q |\{w\} \cap R|. \]

The difference \( |e_4 \cap \gamma_0| - |e_3 \cap \gamma_0| \) depends on how \( \gamma_0 \) intersects \( R \). Since \( \gamma_0 \) is positive, if it intersects \( e_1, e_2 \) or \( e_3 \) at a point, then it enters \( R \) there; if it intersects \( e_4 \) at a point, then it exits \( R \) there. Hence we have
\[ |e_4 \cap \gamma_0| - |e_3 \cap \gamma_0| = |e_1 \cap \gamma_0| + |e_2 \cap \gamma_0| + |\{w\} \cap R| - |\{z\} \cap R|. \]

By combining these equations, we get \( l_3 \leq l_4 \).

Therefore, there exists a linear foliation \( F \) of the torus \( \Sigma \), such that, up to isotopy, each strand of the \( \beta \) curve is contained in a leaf of \( F \) or transverse to \( F \) in a fixed direction.

Via another isotopy, we can assume that the entire \( \beta \) curve is either contained in a leaf of \( F \) or transverse to \( F \). In either case, we can assume that the slope of \( F \) is irrational under a perturbation of the foliation, so the leaves of \( F \) are dense. We extend the curve in both directions from the basepoint \( w \) along a leaf of \( F \) until the endpoints reach the connected component of \( \Sigma - \alpha - \beta \) containing the basepoint \( z \). After closing the curve by connecting two endpoints within the connected component, we get the positive simple closed curve \( \gamma \) passing through \( w \) and \( z \).
2.3 The third and the last steps

Our third step is to complete the proof of the “only if” part of Theorem 1. Via a self-homeomorphism of $\Sigma$, we abandon the horizontality of the $\alpha$ curve, and assume that the $\gamma$ curve is horizontal instead. Since $\gamma$ is positive, we can either assume $\alpha$ is a geodesic or assume $\beta$ is a geodesic, but not simultaneously. In fact, there exists an isotopy $f : (\alpha \cup \beta \cup \gamma) \times [0, 1] \to \Sigma$, such that $f(z, t), f(\gamma, t)$ are independent of $t$, and $f(\alpha, 0), f(\beta, 1), f(\gamma, 0)$ are geodesics. Let $\rho$ be the curve in $\Sigma \times [0, 1]$ defined by $\rho(t) = (f(w, t), t)$. Let $\tau_0$ (resp. $\tau_1$) be a geodesic in $(\Sigma, 0)$ (resp. $(\Sigma, 1)$) which does not intersect with $(f(\alpha, 0), 0)$ (resp. $(f(\beta, 1), 1)$). After attaching the solid tori $U_0$ and $U_1$ to the boundary components of $\Sigma \times [0, 1]$, such that $(f(\alpha, 0), 0)$ (resp. $(f(\beta, 1), 1)$) is a meridional disk of $U_0$ (resp. $U_1$), we recover the knot $\rho \cup \tau_0 \cup \tau_1$ from the $(1, 1)$-diagram.

At last, the “if” part of Theorem 1 can be proved in a way similar to the 1-bridge braid case. In [9, Section 3], a coherent reduced $(1, 1)$-diagram was constructed for each 1-bridge braid in $S^3$ and lens space to utilize [9, Theorem 1.2], as shown in Figure 4. We make a tiny change in the construction here: the basepoint $z$ is no longer restricted to be the starting point of the $\gamma'$, but can be any point in $\Sigma - \alpha - \beta$. The topological meaning of the diagram is as explained in the previous paragraph: if we move the basepoint $w$ along a geodesic $\gamma'$, we can untwist the $\beta$ curve at the expense of twisting the $\alpha$ curve. With this change, a $(1, 1)$-diagram as the middle one in Figure 4 can represent the knot described in Theorem 1. This $(1, 1)$-diagram is coherent in the sense that certain orientations on the $\alpha$ and $\beta$ curves induce coherent orientations on the boundary of every embedded bigon $(D, \partial D) \subseteq (\Sigma, \alpha \cup \beta)$. Each isotopy to remove an empty bigon preserves the coherence, so we get a reduced $(1, 1)$-diagram in finite steps. By [9, Theorem 1.2], we proved the “if” part of Theorem 1.

![Figure 4: The construction of a coherent diagram of the 1-bridge braid $K(-2, 3, 7)$ in $S^3$, modified from [9, Figure 3].](image)

3 Non-left-orderable surgeries

3.1 A positive braid representation

In this subsection, we prove Theorem 2 and derive the genus formula.

Let $K$ be a nontrivial positive $(1, 1)$ L-space knot in $S^3$. Let $S^3 = U_0 \cup_S U_1$ be a genus one Heegaard splitting with standard geometry. By Theorem 1, $K$ is isotopic to $\rho \cup \tau_0 \cup \tau_1$, we get the knot $\rho \cup \tau_0 \cup \tau_1$ from the $(1, 1)$-diagram.
where $\rho$ is a geodesic of $\Sigma$, and $\tau_0$ (resp. $\tau_1$) is properly embedded in some meridional disk of $U_0$ (resp. $U_1$).

An orientation on the geodesic $\rho$ induces orientations on the cores of $U_0$ and $U_1$. If the cores of $U_0$ and $U_1$ are negatively linked, then the construction in Subsection 2.3 yields a negative coherent reduced $(1, 1)$-diagram. By [9, Theorem 1.2], $K$ is a negative $(1, 1)$ L-space knot, which contradicts the assumption that $K$ is a nontrivial positive $(1, 1)$ L-space knot in $S^3$. Thus, the cores of $U_0$ and $U_1$ with induced orientations are positively linked. So $\rho$ can be realized as a part of a positive braid. After appending the arcs $\tau_0$ and $\tau_1$, we get a positive braid as shown in Figure 4.

Let $K$ be the closure of the positive braid represented by

$$(\sigma_\omega \sigma_{\omega-1} \cdots \sigma_{\omega-b_0+1}) (\sigma_\omega \sigma_{\omega-1} \cdots \sigma_1)^{b_1} (\sigma_{\omega-1} \sigma_{\omega-2} \cdots \sigma_1)^{t-b_1}.$$ 

If $b_0 = b_1 = 0$, then $K$ has an unknot component, which is not allowed. If $b_0 = 0$, we can decrease $t$ and $b_1$ by one and set $b_0$ to $\omega$. If $b_1 = 0$, we can decrease $\omega$ and $b_0$ by one and set $b_1$ to $t$. For the representation with minimal $t + \omega$, we have $1 \leq b_0 \leq \omega$ and $1 \leq b_1 \leq t$. Therefore, Theorem 2 holds.

A minimal genus Seifert surface is obtained [5] by applying Seifert’s algorithm to a positive diagram, so the genus of $K$ is

$$g(K) = \frac{1}{2}(\#\text{crossings} - \#\text{strands} + 1)$$
$$= \frac{1}{2}(b_0 + b_1 \omega + (t - b_1)(\omega - 1) - (\omega + 1) + 1)$$
$$= \frac{1}{2}(t \omega - t - \omega + b_0 + b_1).$$

3.2 The knot group

In this subsection, we investigate the knot group $\pi_1(S^3 - K)$. As a $(1, 1)$-knot, the knot group has a 2-generator presentation. However, to keep the symmetry, we specify four elements $x_0, y_0, x_1, y_1$ in the knot group instead.

Let $D_0$ (resp. $D_1$) be the meridional disk of $U_0$ (resp. $U_1$) containing $\tau_0$ (resp. $\tau_1$). Then $D_0$ (resp. $D_1$) is divided by $\tau_0$ (resp. $\tau_1$) into two disks $D_{x,0}$ and $D_{y,0}$ (resp. $D_{x,1}$ and $D_{y,1}$). Let the points $P, Q, R$ on $\Sigma$ be $\rho \cap \tau_0, \tau_0 \cap \tau_1, \tau_1 \cap \rho$, respectively. Orient the knot $K$ so that $P, Q, R$ appears in order. Orient the cores of $U_0, U_1$ and the disks $D_0, D_1, D_{x,0}, D_{y,0}, D_{x,1}, D_{y,1}$ accordingly. Let $Q'$ be a point near $Q$ in $\Sigma - \rho - D_0 - D_1$, so that $Q'$ is on the negative side of $D_0$ and on the positive side of $D_1$. Let $Q''$ be a point in $\rho$ on the boundary of the connected component of $\Sigma - \rho - D_0 - D_1$ containing $Q'$, as shown in Figure 4.

The fundamental group of $U_0 - \tau_0$ (resp. $U_1 - \tau_1$) is freely generated by two elements $x_0, y_0$ (resp. $x_1, y_1$), where $x_0$ (resp. $y_0, x_1, y_1$) is represented by a loop based at $Q'$ intersecting $D_{x,0}$ (resp. $D_{y,0}, D_{x,1}, D_{y,1}$) once positively and not intersecting other disks. Then $\pi_1(S^3 - K)$ based at $Q'$ is generated by $x_0, y_0, x_1, y_1$.

Without loss of generality, we assume $x_0$ (resp. $x_1$) has larger norm than $y_0$ (resp. $y_1$) in $H_1(S^3 - K)$. Then

$$\mu = x_0y_0^{-1} = y_1^{-1}x_1$$

represents a meridian of $K$ around $Q$. 

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The boundary of $D_1$ (resp. $D_0$) intersects $\rho$ in $t$ (resp. $w$) points, not counting $P$, $Q$ and $R$. Starting from $Q$ along positive direction, let the points be $R_1, R_1-1, \ldots, R_1$ (resp. $P_\omega, P_{\omega-1}, \ldots, P_1$) in order. For each $i$ with $1 \leq i \leq t$ (resp. $1 \leq i \leq \omega$), let $g_i$ (resp. $h_i^{-1}$) represent the loop based at $Q'$ in $U_0 - \tau_0$ (resp. $U_1 - \tau_1$) which first travels to $R_i$ (resp. $P_i$) without intersecting $D_0$ (resp. $D_1$), then follows $\rho$ in positive (resp. negative) direction to $P$ (resp. $R$) but not past it, lastly travels back to $Q'$ without intersecting $D_0$ (resp. $D_1$). Then each $g_i$ (resp. $h_i$) represented by a word in $x_0$ and $y_0$ (resp. $x_1$ and $y_1$), and we have

$$
y_0 = (g_1 \mu g_1^{-1}) (g_2 \mu g_2^{-1}) \cdots (g_t \mu g_t^{-1}),
$$

$$
y_1 = (h_1^{-1} \mu h_1) (h_2^{-1} \mu h_2) \cdots (h_\omega^{-1} \mu h_\omega).
$$

Since $b_0, b_1 \neq 0$, the point $Q''$ is on the arc $R_1 P_\omega \subset \rho$ which is a part of boundary of the connected component of $\Sigma - \rho - D_0 - D_1$ containing $Q'$. The longitude $\lambda$ of $K$ starting from $Q$ is determined by

$$
\mu^{b_0} \lambda = h_\omega g_1.
$$

The integer $k_0$ can be found by counting the crossings between $K$ and a loop represented by $h_\omega g_1$ on a planar diagram. The loop represented by $h_\omega g_1$ differs from the blackboard framing of $K$ as shown in Figure 1 by $2t$ additional positive crossings, so we have

$$
k_0 = \# \text{crossings} + t
$$

$$
= b_0 + b_1 \omega + (t - b_1)(\omega - 1) + t
$$

$$
= t \omega + b_0 + b_1.
$$

Therefore we have

$$
\mu^{t \omega + b_0 + b_1} \lambda = h_\omega g_1,
$$

and

$$
\mu^{2g(K) - 1} \lambda = h_\omega g_1 \mu^{-t - \omega - 1}.
$$

Furthermore, the word representing $g_1$ starts with an $x_0$, and the word representing $h_\omega$ ends with an $x_1$.

### 3.3 The property (D)

In this subsection, we prove that $K$ has property (D).

The first part of the property (D) is the following.

**Lemma 7.** For any homomorphism $\rho$ from $\pi_1(S^3 - K)$ to $\text{Homeo}^+(R)$, if $s \in R$ is a common fixed point of $\rho(\mu)$ and $\rho(\lambda)$, then $s$ is a fixed point of every element in $\pi_1(S^3 - K)$.

**Proof.** Since $s$ is a common fixed point of $\rho(\mu)$ and $\rho(\lambda)$, it is a common fixed point of $\rho(x_0 y_0^{-1})$, $\rho(y_1^{-1} x_1)$ and $\rho(h_\omega g_1)$. Without loss of generality, we assume $\rho(x_0)s \geq s$, then we have $\rho(y_0)s \geq s$. We also have $\rho(x_1)s \geq s$ (resp. $\rho(x_1)s \leq s$) if and only if $\rho(y_1)s \geq s$ (resp. $\rho(y_1)s \leq s$).

Starting from the base point $Q'$, we construct a geodesic $\gamma$ on $\Sigma - \rho$ parallel to $\rho$. Because $K$ is nontrivial, the arc $\rho$ is not parallel to $\partial D_0$ or $\partial D_1$. Extend $\gamma$ until it crosses each disk $D_0, D_1$ at least once and reaches the connected component of $\Sigma - \rho - D_0 - D_1$ containing $Q'$ again. Then we close up the curve to obtain the knot group element $g_0$, which can be represented by a nontrivial word in $x_0$ and $y_0$, and also by a nontrivial word.
Remark. By symmetry, we have \( \rho(y_0)s \geq s \). By the second condition, we have \( \rho(x_0)s \geq s \). Because \( h_\omega g_1 \) is represented by a word in \( x_0, y_0, x_1, y_1 \) with at least one \( x_0 \) and one \( x_1 \), we have \( \rho(x_0)s = \rho(y_0)s = \rho(x_1)s = \rho(y_1)s = s \). Therefore \( s \) is a fixed point of every element in \( \pi_1(S^3 - K) \).

\[ \square \]

Lemma 8. The element \( \mu \) is in the root-closed, conjugacy-closed submonoid generated by \( \mu^{2g(K) - 1} \lambda \) and \( \mu^{-1} \).

Proof. As in [17 Section 3], we define the preorder \( \leq_k \) generated by \( \mu \) and \( (\mu^{2g(K) - 1} \lambda)^{-1} \) on \( \pi_1(S^3 - K) \). Since \( \mu = x_0 y_0^{-1} = y_1^{-1} x_1 \), we have \( x_0 \geq_k y_0 \) and \( x_1 \geq_k y_1 \). Since \( \mu^{2g(K) - 1} \lambda = h_\omega g_1 \mu^{-t} \omega \), we have \( h_\omega g_1 \leq_k \mu t^\omega + 1 \).

Let \( \tilde{g_0} = 1, \tilde{g_1}, \ldots, \tilde{g_\nu} = g_1 \) be all suffixes of \( g_1 \), and \( \tilde{h_0}, \tilde{h_1}, \ldots, \tilde{h_\omega} = h_\omega \) be all prefixes of \( h_\omega \), ordered by length. Suppose that \( \tilde{g_i} \) appears \( m_i \) times in \( g_1, g_2, \ldots, g_t \) for each \( 0 \leq i < t' \), and \( \tilde{h_i} \) appears \( n_i \) times in \( h_1, h_2, \ldots, h_\omega \) for each \( 0 \leq i < \omega' \). Then we have

\[
y_0 = (g_1 \mu g_1^{-1}) (g_2 \mu g_2^{-1}) \cdots (g_t \mu g_t^{-1}) \geq_k (g_1 \mu g_1^{-1}) (\tilde{g_i} \mu \tilde{g_i}^{-1})^{m_i},
\]

and

\[
y_1 = (h_1^{-1} \mu h_1) (h_2^{-1} \mu h_2) \cdots (h_\omega^{-1} \mu h_\omega) \geq_k (h_1^{-1} \mu h_1)^{n_i} (h_\omega^{-1} \mu h_\omega).
\]

For each \( 0 \leq i < t' \), we have either \( \tilde{g_{i+1}} = y_0 \tilde{g_i} \) or \( \tilde{g_{i+1}} = x_0 \tilde{g_i} = \mu y_0 \tilde{g_i} \). And for \( i = t' - 1 \), it is necessarily the latter case. So we have

\[ \tilde{g_{i+1}} \geq_k y_0 \tilde{g_i} \geq_k (g_1 \mu g_1^{-1}) \tilde{g_i} \mu^{m_i}. \]

By induction, we have

\[ \tilde{g_1} = \tilde{g_\nu} = \mu y_0 \tilde{g_\nu - 1} \geq_k \mu g_1 \mu^{t' - 1} \mu^{\sum_{i=0}^{t'-1} m_i}. \]

By symmetry, we have

\[ h_\omega \geq_k \mu^{\sum_{i=0}^{t'-1} m_i} h_\omega^{-1} \mu^{\omega'} \mu. \]

Here \( t' \) (resp. \( \omega' \)) is the number of intersection points between \( Q''P \subset \rho \) and \( D_0 \) (resp. \( RQ'' \subset \rho \) and \( D_1 \)), not counting \( P \) and \( R \). And \( \sum_{i=0}^{t'-1} m_i \) (resp. \( \sum_{i=0}^{\omega'-1} n_i \)) is the number of intersection points between \( Q''P \subset \rho \) and \( D_1 \) (resp. \( RQ'' \subset \rho \) and \( D_0 \)). So we have

\[ \sum_{i=0}^{t'-1} m_i = \omega - \omega'. \]
and

$$\sum_{i=0}^{\omega'-1} n_i = t - t'. $$

Then we have

$$h_\omega \geq k \mu^{t-t'} h_\omega^{-1} \mu^{\omega'} h_\omega \mu$$

$$\geq k \mu^{t-t'} h_\omega^{-1} \mu^{\omega'} h_\omega.$$ 

So $h_\omega \geq k \mu^{t-t'+\omega'}$. Because $h_\omega g_1 \leq k \mu^{t+\omega+1}$, we have $g_1 \leq k \mu^{t+\omega'-\omega+1}$. Then we have

$$g_1 \geq k \mu g_1 \mu' g_1^{-1} \mu^{\omega'-\omega'}$$

$$\geq k \mu g_1 \mu^{-1}.$$ 

Since $g_1 \geq k \mu g_1 \mu^{-1}$, we get $g_1 \mu' g_1^{-1} \geq k \mu'$. So we have

$$g_1 \geq k \mu g_1 \mu' g_1^{-1} \mu^{\omega'-\omega'}$$

$$\geq k \mu^{\omega+t'-\omega'+1}. $$

By symmetry, we have $h_\omega \geq k \mu^{t-t'+\omega'+1}$. By $h_\omega g_1 \leq k \mu^{t+\omega+1}$, we have $\mu \leq k 1$. In other words, the meridian $\mu$ is in the root-closed, conjugacy-closed submonoid generated by $\mu^{2g(K)}-1 \lambda$ and $\mu^{-1}$.

Combining Lemma 7 and Lemma 8 we proved Theorem 4. By [17 Theorem 4.1], we proved Theorem 5.

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