MOTION OF SLIGHTLY COMPRESSIBLE FLUIDS IN A BOUNDED DOMAIN. II

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Abstract. We study the problem of inviscid slightly compressible fluids in a bounded domain. We find a unique solution to the initial-boundary value problem and show that it is close to the analogous solution for an incompressible fluid. Furthermore we find that solutions to the compressible motion problem in Lagrangian coordinates depend differentiably on their initial data, an unexpected result for non-linear equations. The results were announced by the first author in [E4].

Contents

1. Introduction 1
2. Equations of Fluid Motion 2
3. Function Spaces, the Equation of State and Main Theorems 6
4. Estimates for \( f \) and \( \dot{f} \) 9
5. Proofs of Main Theorems 27
6. Differentiable Dependence on Initial Conditions 34
7. Differentiable Dependence on Initial Conditions for Quasi-linear Symmetric-Hyperbolic Equations. Proof of Proposition 6.1 37
8. Asymptotic Approximation to Compressible Fluid Motion 47
References 49

1. Introduction

This paper is a continuation of [E4] in which we prove the statements that were announced but not proven there. In that paper we studied the initial value problem for the motion of inviscid fluids in a bounded domain. We looked at both compressible and incompressible fluids and showed that the flow of the former is close to that of the latter if the compressibility is low. We announced but did not prove that for compressible fluids the fluid density has one more spatial derivative than the velocity. This fact is curious because as it is well known the fluid density and velocity together satisfy a quasilinear symmetric hyperbolic system and thus one would not expect either to be smoother than the other. Furthermore, as a consequence of the extra smoothness we find that the solution of the system, when expressed in Lagrangian coordinates is differentiably dependent on the initial data. This is unexpected because for quasilinear systems the dependence is usually continuous, but not differentiable. See for example [K2].
We note that in a previous paper [E2] similar results were proven for fluid motions which are periodic in space, but here we prove them in a domain with boundary. The boundary case is more difficult because it involves more complicated estimates (cf. section 4), but it is also more important because most fluid problems do involve boundaries. Furthermore, our proof is more direct than of [E2] in that it avoids operator theoretic methods.

Finally, (section 8) following a suggestion of Professor C. S. Morawetz, we explain how the construction used to show that slightly compressible motion is near to incompressible actually gives us a sequence of successive approximations to the compressible motion. The first term in the sequence is the incompressible motion, and the second is derived by studying the propagation of sound caused by the first motion.

We will use the same notation as in [E4] and for the reader’s convenience repeat some of the formulas and estimates written there. Thus throughout the paper, $u$, $\rho$, and $p$ will denote the velocity, density, and pressure of compressible motion. They are all functions of time $t$ and of position $x = (x^1, \ldots, x^n)$; that is, $u(t, x)$ is the velocity of that fluid particle which is at position $x$ at time $t$; $x$ is known as the Euler coordinate of the particle.

Given the velocity $u(t, x)$, one can find the flow of the fluid, $\zeta(t, x)$, as the solution of the ordinary differential equation

$$\partial_t \zeta(t, x) = u(t, \zeta(t, x)), \quad \zeta(0, x) = x.$$ 

Then $\zeta(t, x)$ is the position at time $t$ of the fluid particle which at time zero was at $x$. $x$ is known as the Lagrange coordinate of the particle; its Euler coordinate at time $t$ is $\zeta(t, x)$.

For incompressible fluid motion the velocity will be called $v = v(t, x)$, and will also be the velocity of the particle with Euler coordinate $x$. Its flow will be called $\eta(t, x)$ so $\eta$ will satisfy

$$\partial_t \eta(t, x) = v(t, \eta(t, x)), \quad \eta(0, x) = x.$$ 

The density of an incompressible fluid is always taken to be 1, and the pressure, if used, will also be called $p$. However, as we shall see, there is no notion of pressure for the idealized incompressible fluid.

Our analysis of fluid motion will be partly in Euler and partly in Lagrangian coordinates. Spatial derivatives and many $L^2$ estimates involving them will be done using Euler coordinates. Time derivatives will usually be written in Lagrangian coordinates or (what is essentially the same) with material derivatives in Euler coordinates. This seems to be the best way to study the situation because the $L^2$ estimates are simpler in Euler coordinates, but the Lagrange coordinates make it possible to write some of the equations as ordinary differential equations on a function space. In fact one result, the $C^1$ dependence on initial data, is true in Lagrange coordinates, but not in Euler.

2. Equations of Fluid Motion

We consider the motion of a fluid which fills a fixed bounded domain $\Omega \subset \mathbb{R}^n$, and we assume that $\Omega$ has a smooth boundary $\partial \Omega$, which is a compact $(n - 1)$-dimensional submanifold of $\mathbb{R}^n$.

We restrict our attention to barotropic fluids, (fluids whose pressure depends only on the density). Their equations of motion are:
\[ \begin{align*}
\frac{\partial u}{\partial t} + \nabla u &= -\frac{1}{\rho} \nabla p, \quad (2.1a) \\
\frac{\partial \rho}{\partial t} + \text{div}(\rho u) &= 0, \quad (2.1b)
\end{align*} \]

for compressible motion, and

\[ \begin{align*}
\frac{\partial v}{\partial t} + \nabla v &= -\nabla p, \quad (2.2a) \\
\text{div}(v) &= 0, \quad (2.2b)
\end{align*} \]

for incompressible motion. Here, \( \nabla \) means gradient, div means divergence and \( \nabla_w \) means derivative in direction \( w \), \( w \) a vector. Thus in coordinates, where all indices run from 1 to \( n \) and repeated indices are summed:

\[ (\nabla_v v)^j = v^i \frac{\partial v^i}{\partial x^j} \quad \text{and} \quad \text{div}(\rho u) = \frac{\partial (\rho u^i)}{\partial x^j} \]

The equations of motion are accompanied by the following boundary conditions which express the physical condition that the fluid not pass through the boundary. To write them we let \( \nu \) be the unit outward normal vector field on \( \partial \Omega \). Then, for compressible motion,

\[ \langle u, \nu \rangle = 0, \quad \text{which implies} \quad \langle \nabla_u u, \nu \rangle = -\frac{1}{\rho} \nabla_\nu p, \quad (2.3) \]

and for incompressible motion,

\[ \langle v, \nu \rangle = 0, \quad \text{which implies} \quad \langle \nabla_v v, \nu \rangle = -\nabla_\nu p. \quad (2.4) \]

Thus, equations (2.1a)-(2.1b) with boundary condition (2.3) shall be known as the compressible motion problem, while (2.2a)-(2.2b) with boundary condition (2.4) shall be the incompressible motion problem.

For the compressible case the pressure can be written as a function of the density so in addition to (2.1a)-(2.1b) we specify this function, writing

\[ p = p(\rho). \quad (2.5) \]

The core of this paper will be an analysis of the density for a compressible motion. We shall show that it satisfies a hyperbolic equation and use this equation to make estimates. As density is positive, it is more convenient to work with its logarithm, so we define the function

\[ f = \log \rho. \]

In addition to partial derivatives of \( f \) we shall use the material derivative which we denote by “\( \dot{\cdot} \)”. This derivative depends on the fluid velocity which we here label \( u = u(t, x) \). Specifically, for any function \( h \) depending on \( x \) and \( t \) we define the material derivative of \( h \) with respect to a fluid moving with velocity \( u \) to be

\[ \dot{h} = \frac{\partial h}{\partial t} + \nabla_u h. \]
Using (2.1b) we find:

\[ \dot{f} = \frac{1}{\rho} \dot{\rho} = - \text{div}(u). \]  

(2.6)

Taking another material derivative and using (2.1a) we find

\[ \ddot{f} = \text{div}(p'(\rho) \frac{1}{\rho} \nabla \rho) + u_j^i u_i^j \]  

(2.7)

where \( u_j^i \) means \( \frac{\partial u_j}{\partial x^i} \) and we sum over repeated indices. We shall rewrite (2.7) as a hyperbolic equation in \( f \).

In order to rewrite (2.7) we shall need additional notation. We note that for physical reasons \( p'(\rho) \) is non-negative (pressure cannot decrease as density increases). Thus we can define a function \( c \) by

\[ c = \sqrt{p'(\rho)}. \]  

(2.8)

This \( "c" \) is the sound speed of the fluid, and, like \( \rho \), it depends on \( x \) and \( t \).

Also, we introduce the operator \( \delta = - \text{div} \) which is the formal adjoint of \( \nabla \).

With this notation (2.7) can be rewritten as

\[ \ddot{f} = -\delta c^2 \nabla f + u_j^i u_i^j \]  

(2.9)

and can be regarded as a linear hyperbolic equation in \( f \) with inhomogeneous term \( u_j^i u_i^j \) (although the equation is not quite linear since \( c \) depends on \( f \)). It is sometimes referred to as a convected wave equation — convected because the usual time derivative is replaced by the material derivative.

The boundary condition (2.3) gives a boundary condition for (2.9) as well. From our definitions we find that

\[ \frac{1}{\rho} \nabla p = c^2 \nabla f, \]

so (2.3) implies

\[ \nabla \nu \cdot f = -\frac{1}{c^2} \langle \nabla u, \nu \rangle. \]  

(2.10)

This is an inhomogeneous Neumann boundary condition for the equation (2.9).

For incompressible fluid motion, the density is constant (in (2.2a) we have taken it to be equal to 1), and \( p \) does not correspond to any physical quantity. In fact for any given time \( t \), \( p(t) \) is determined by \( v(t) \) (up to a constant), although the determination is not local (in \( x \)).

This is easily shown as follows: (2.2b) implies that

\[ \delta \left( \frac{\partial v}{\partial t} \right) = 0, \]  

so\( \delta (\nabla \nu v) = -\delta \nabla p. \)

But since

\[ -\delta \nabla p = \Delta p \]

we get

\[ \Delta p = \delta (\nabla \nu v). \]  

(2.11)
Also (2.4) gives a Neumann boundary condition for $p$, so given $v$ and (2.4), (2.11) has a solution $p$ which is unique modulo an additive constant. $\nabla p$ is thus determined uniquely by $v$.

Given a vector field $w$ in $\Omega$, if we define an operator $Q$ by

$$Q(w) = \nabla g,$$

where $g$ solves

$$\begin{cases}
\Delta g = -\delta w & \text{in } \Omega, \\
\nabla_\nu g = \langle w, \nu \rangle & \text{on } \partial \Omega,
\end{cases}$$

then the system (2.2a)-(2.2b) with (2.4) is equivalent to

$$\frac{\partial v}{\partial t} + \nabla v \cdot v = Q(\nabla v v)$$  \hspace{1cm} (2.12)

so the artificial $p$ is eliminated.

The operator $Q$ is useful in the compressible motion as well. We let $P = I - Q$, where $I$ is the identity operator, and decompose the velocity as

$$u = P(u) + Q(u) = w + \nabla g.$$  

This is the well known decomposition of a vector field into its solonoidal (or divergence free) and gradient parts. We shall apply $P$ and $Q$ to (2.1a) to get equations for $w$ and $\nabla g$.

First note that $Q^2 = Q$, so $P^2 = P$. Also, for any function $g$, $Q(\nabla g) = \nabla g$, and $P(\nabla g) = 0$.

Furthermore since $p$ is a function of $\rho$

$$\frac{1}{\rho} \nabla p = \frac{1}{\rho} \rho'(\rho) \nabla \rho = \nabla \int_1^\rho \frac{1}{\lambda} \rho'(\lambda) d\lambda$$

so

$$P(\frac{\partial u}{\partial t} + \nabla_u u) = 0.$$  \hspace{1cm} (2.13)

Also

$$\nabla_\nu \nabla g \cdot \nabla g = \frac{1}{2} \nabla \langle \nabla g, \nabla g \rangle.$$  \hspace{1cm} (2.14)

From (2.13) and (2.14) we get

$$\frac{\partial w}{\partial t} + P(\nabla_u w) = -P(\nabla_w \nabla g).$$  \hspace{1cm} (2.15)

Applying $Q$ to (2.1a) we can get a similar equation for $\nabla g$. However, since $\delta \nabla g = \delta u$ and since by (2.6) we have

$$\dot{f} = \delta u,$$

we can investigate $\nabla g$ by studying $\dot{f}$.

We are concerned with the initial-boundary value problem for our fluid motion, so we prescribe $u_0(x) = u(0, x)$ and $\rho_0(x) = \rho(0, x)$, the initial velocity and density. As we showed in [E3], the system (2.1a)-(2.1b) and (2.5) with initial data $u_0$, $\rho_0$ and with boundary condition (2.3) has a unique solution on some time interval provided $u_0$ and $\rho_0$ satisfy the following conditions:
a) For each \( x \in \Omega \), \( |u_0(x)| < c(0, x) = \sqrt{p'(\rho_0(x))} \);

b) \( \rho_0 \) is close to a constant;

c) \( u_0, \rho_0 \) satisfy compatibility conditions on \( \partial \Omega \). That is,
\[
\langle u_0, \nu \rangle = 0,
\]
\[
\langle \nabla u_0 u_0 + \frac{1}{\rho_0} p'(\rho_0) \nabla \rho_0, \nu \rangle = 0,
\]

etc.

Since we want to compare the solutions of \( (2.1a)-(2.1b) \) to that of \( (2.2a)-(2.2b) \) when the initial data are close we will assume that \( v_0 \), the initial datum for \( (2.2a)-(2.2b) \) is near \( u_0 \).

Since \( v \) must satisfy \( (2.2b) \) we have \( \text{div}(v_0) = 0 \), and therefore \( \text{div}(u_0) \) must be small. Then from \( (2.7) \) we find that \( \dot{f}(0) \) is small.

Also since \( \rho \) is taken to be identically one in the incompressible fluid problem, we want \( \rho_0 \) to be nearly one as well. This means that \( f(0) = \log \rho_0 \) must also be small.

In section 4, we shall show that \( f(0) \) and \( \dot{f}(0) \) being small implies that \( f(t) \) and \( \dot{f}(t) \) are small as well.

3. Function Spaces, the Equation of State and Main Theorems

Our estimates on \( f \) and \( \dot{f} \) will be made by considering \( f(t), \dot{f}(t) : \Omega \to \mathbb{R} \). We shall estimate for each \( t \) the \( L^2 \) norms of the functions and of their derivatives. For this reason we shall need to introduce function spaces with \( L^2 \) norms and we do that here.

Given \( g \) and \( h \), \( C^k \) functions from \( \Omega \) to \( \mathbb{R} \), we define the inner product
\[
(g, h)_k = \int_{\Omega} \sum_{\ell=0}^{k} \langle \nabla^\ell g, \nabla^\ell h \rangle
\]
where \( \nabla^\ell g \) is the vector valued function consisting of all \( \ell \)th order partial derivatives of \( g \). This inner product induces a norm (which we call \( \| \cdot \|_k \)) on \( C^k(\Omega, \mathbb{R}) \), the set of all \( C^k \) functions on \( \Omega \). \( H^k(\Omega, \mathbb{R}) \) is defined to be the completion of \( C^k(\Omega, \mathbb{R}) \) in this norm. \( C^k(\Omega, \mathbb{R}^m) \) and \( H^k(\Omega, \mathbb{R}^m) \) will be analogous spaces for functions with values in \( \mathbb{R}^m \). Sometimes we write only \( H^k \). We shall also use the norm
\[
\| g \|_{C^k} = \sup_{x \in \Omega} \left( \sum_{\ell=0}^{k} |\nabla^\ell g(x)| \right)
\]
the usual \( C^k \) norm on \( C^k(\Omega, \mathbb{R}) \). It is well known that in this norm \( C^k(\Omega, \mathbb{R}) \) is complete.

By the Sobolev embedding theorem, \( H^s(\Omega, \mathbb{R}^m) \subset C^k(\Omega, \mathbb{R}^m) \) if \( s > k + \frac{n}{2} \) and for any \( w \in H^s(\Omega, \mathbb{R}^m) \):
\[
\| w \|_{C^k} \leq K \| w \|_s
\]
where \( K \) depends only on \( k, s \) and \( \Omega \). (see [T], chapter 4.) Also, if \( s > \frac{n}{2} \), \( h \in H^s \) and \( g \in H^{s_1} \), with \( s \geq s_1 \geq 0 \), then \( gh \in H^{s_1} \) and we have the product estimate
\[
\| gh \|_{s_1} \leq K \| g \|_{s_1} \| h \|_s
\]
(3.1)
where \( K \) depends on \( s_1, s \) and \( \Omega \). (see [T], chapter 13.) Equation (3.1) will be used throughout the paper to estimate the several products involved.
Since $\partial \Omega$ is a compact manifold we can also define $C^k(\partial \Omega, \mathbb{R})$, the space of $C^k$-functions on $\partial \Omega$, and completing this space with respect to an appropriate inner product we get the Hilbert space $H^k(\partial \Omega, \mathbb{R})$. Analogously one constructs $H^k(\partial \Omega, \mathbb{R}^n)$.

Using Fourier series we can construct $H^s(\Omega)$ and $H^s(\partial \Omega)$ for all non-negative real numbers $s$ (cf. [P]), and with this we get a restriction inequality as follows. Let $R : C^s(\Omega) \to C^s(\partial \Omega)$ be defined by restricting each $C^s$ function to its values on $\partial \Omega$. As is shown in [P], if $s > \frac{1}{2}$, $R$ extends to a bounded linear map $R : H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial \Omega)$. Denote by $\|\|_{\partial,s}$ the norm of $H^s(\partial \Omega)$. Then we get the inequality

$$\| Rh \|_{\partial,s-\frac{1}{2}} \leq K \| h \|_s, \quad s > \frac{1}{2},$$

where $K$ depends on $s$ and $\Omega$. (3.2) will be used throughout the paper to estimate the restrictions to $\partial \Omega$.

If $s > \frac{n}{2} + 1$ we define $D^s(\Omega) \equiv D^s$ to be the set of bijective maps from $\Omega$ to itself which together with their inverses are in $H^s(\Omega, \mathbb{R}^n)$. It is easy to prove the following:

$$D^s = \left\{ \zeta \in H^s(\Omega, \mathbb{R}^n) \mid \zeta : \Omega \to \Omega \text{ is bijective, and } J(\zeta) \text{ is nowhere zero} \right\}$$

where $J$ is the Jacobian. Also $D^s$ can be shown to be a subgroup of the group of $C^1$ diffeomorphisms of $\Omega$ (see [EM] or [E1]).

Our estimates for $f$ and $\dot{f}$ will depend on the fluid motion $\zeta(t)$, but also on the equation of state (2.5); i.e., on the relationship between $p$ and $\rho$. We are interested in slightly compressible fluids and their motion is governed by functions $p(\rho)$ where $p'(\rho) = \frac{dp}{d\rho}$ is large. Thus we shall consider a family $\{p_k(\rho)\}$ parameterized by $k \in \mathbb{R}_+$, and we shall assume that the parameterization is chosen such that:

$$p_k'(\rho)|_{\rho=1} = k.$$ 

We are concerned with the fluid motion when $k$ is large and in its limit as $k \to \infty$.

We shall need conditions on $p_k(\rho)$ for $\rho$ near 1 so we make the following assumption:

**Assumption 3.1.** There exist positive constants $a_0$ and $a_1$ such that

$$\| p^{(\ell)}_k(\rho) \|_{s+\ell} \leq k a_1 \quad \ell = 1, 2, \ldots, s + 2$$

provided that $\| \rho - 1 \|_{s+1} \leq \frac{a_0}{\sqrt{k}}$.

Assumption 3.1 involves the norm of a composite function; that is $\rho$ and $p^{(\ell)}(\rho)$ are considered as functions on $\Omega$, the latter being $x \mapsto p^{(\ell)}(\rho(t, x))$. The $\|\|_s$ norms of such functions are estimated in a straightforward manner using the derivatives of $p$ and $\| \rho \|_s$ (see for example Lemma A.2 of [BB]).

Assumption 3.1 is actually more restrictive than necessary since it bounds several derivatives by a multiple of $k$. It is possible to use a more general assumption, as was done in equations (5.3)-(5.4) of [E2], but this would make the argument more complicated. Hence we content ourselves to the less general case and leave extensions to the interested reader.

Before starting our theorems, we mention the compatibility conditions required for an $H^s$ compressible motion. As is stated in [E3], the equations (2.1a)-(2.1b) admit an $H^s$ solution $u,$
if the initial data \( u_0, \rho_0 \) satisfy compatibility conditions up to order \( s - 1 \). The zeroth order condition is simply
\[
\langle u_0, \nu \rangle = 0 \quad \text{on} \quad \partial \Omega. \tag{3.3}
\]
All time derivatives of \( u \) must also have zero normal component and using this and the equations of motion we get the first order compatibility condition:
\[
\langle \nabla u_0 \partial_t u + c_0^2 \nabla f_0, \nu \rangle = 0 \quad \text{on} \quad \partial \Omega, \tag{3.4}
\]
where \( c_0 = \sqrt{p'/(\rho_0)} \) is the sound speed of the fluid with density \( \rho_0 \) and \( f_0 = \log \rho_0 \).
Taking another derivative we get
\[
\langle \nabla u \partial_t u + \nabla \partial_t u + \partial_t(c^2)\nabla f + c^2 \nabla \partial_t f, \nu \rangle = 0 \quad \text{on} \quad \partial \Omega. \tag{3.5}
\]
To analyze (3.5) we use the so called second fundamental form of \( \partial \Omega \) which we call \( S_2 \). Given any two vector fields \( z_1 \) and \( z_2 \) on \( \partial \Omega \) we define
\[
S_2(z_1, z_2) = \langle \nabla z_1 z_2, \nu \rangle = -\langle z_1, \nabla z_2 \nu \rangle
\]
which is a function on \( \partial \Omega \). \( S_2 \) is easily seen to be symmetric in \( z_1 \) and \( z_2 \), and at each \( p \in \partial \Omega \) it depends only on \( z_1(p) \) and \( z_2(p) \) and not on their derivatives. Using it, (3.5) can be written
\[
2S_2(u, \partial_t u) + \partial_t(c^2)\nabla f + c^2 \nabla \partial_t f = 0 \quad \text{on} \quad \partial \Omega.
\]
Substituting for the time derivatives using (2.1a) and (2.1b) we get the second order compatibility condition
\[
-2S_2(u_0, \nabla u_0 u_0 + c_0^2 \nabla f_0) + p''(\rho_0)\rho_0(-\nabla u_0 f_0 + \delta_0 u_0)\nabla f_0 + c_0^2 \nabla (\nabla u_0 f_0 + \delta_0 u_0) = 0 \quad \text{on} \quad \partial \Omega. \tag{3.6}
\]
Higher order conditions are computed in the same way. One sees inductively that the condition of order \( \ell \) depends on \( \ell \) derivatives of \( f_0 \) or \( \rho_0 \) but only \( \ell - 1 \) derivatives of \( u_0 \) and \( \delta_0 u_0 \). That is, although there are \( \ell \text{th} \) order derivatives of \( u_0 \), they can all be expressed as \( (\ell - 1)\text{st} \) derivatives of \( u_0 \) and \( \delta_0 u_0 \). In (3.6), for example, we have only first order derivatives of \( u_0 \) and \( \delta_0 u_0 \). With these compatibility conditions we are ready to state our theorems.

**Theorem 3.1.** Let \( s > \frac{n}{2} + 1 \) and let \( u_{0k} \in H^s(\Omega, \mathbb{R}^n) \) and \( \rho_{0k} \in H^{s+1}(\Omega, \mathbb{R}) \) be families of functions parameterized by \( k \). Also let \( p_k \) be a family of smooth functions satisfying the Assumption 3.1. Assume that
1) For each \( k, u_{0k}, \rho_{0k} \) satisfy compatibility conditions up to order \( s \).
2) There exists a constant \( a_2 \) such that
\[
a) \| u_{0k} \|_{s} \leq a_2 \quad \text{and} \quad \| \delta u_{0k} \|_{s} \leq \frac{a_2}{\sqrt{k}}
\]
where \( f_{0k} = \log \rho_{0k} \).
Then there exists a number \( k_0 \) and a positive function \( T(k) \) such that if \( k > k_0 \) the system (2.1a)-(2.1b) with initial conditions \( \rho_{0k}, u_{0k} \) and with boundary condition (2.3) has a unique \( H^s \) solution \( u_k(t), \rho_k(t) \) defined on a time interval \([0, T(k))\). Furthermore \( p_k(t) \in H^{s+1} \) and \( \delta u_k(t) \in H^s \).
Remark 3.1. Most of this theorem was proven in [E3]. It is only the last statement, $\rho_k \in H^{s+1}$ and $\delta u_k \in H^s$ that is new, for in [E3] we only got $\rho_k \in H^s$. We shall see the significance of the additional differentiability in section 6.

Theorem 3.2. Assume that $u_{0k}$, $\rho_{0k}$ and $p_k$ are as above and satisfy all hypothesis of Theorem 3.1. Assume also that there exists a $v_0 \in H^s(\Omega, \mathbb{R}^n)$ such that $u_{0k} \to v_0$ in $H^s(\Omega, \mathbb{R}^n)$ as $k \to \infty$. Then there exists an interval $[0, T]$ and a unique curve $v : [0, T] \to H^s(\Omega, \mathbb{R}^n)$ satisfying (2.2a)-(2.2b) with initial condition $v_0$ and boundary condition (2.4). Furthermore if $u_k(t)$, $\rho_k(t)$ are the solutions of (2.1a)-(2.1b) from theorem 3.1, then if $T(k)$ is maximal, we find that $T(k) > T$ for large $k$ and $u_k(t) \to v(t)$ in $H^s(\Omega, \mathbb{R}^n)$ as $k \to \infty$. Also $\rho_k(t) \to 1$ in $H^{s+1}(\Omega, \mathbb{R})$.

A consequence of this theorem is that the solution to the slightly compressible motion (in Lagrangian coordinates) depends differentially on the initial data. To state this precisely requires additional constructions so we delay both the detailed statement and proof of this fact until sections 6 and 7.

Remark 3.2. All of Theorem 3.2 except the last statement about $\rho$ was proven in [E4]. The solution $v(t)$ was constructed in [EM] and has also been dealt with by other authors (cf. [K1], [K3] or [G] for example).

In the next section we will derive the basic estimates required for the proofs of these theorems, and the proofs themselves will follow in section 5.

4. Estimates for $f$ and $\dot{f}$

Now we shall consider $f$, the solution of (2.9), and find estimates for $\| f(t) \|_{s+1}$ and $\| \dot{f}(t) \|_s$. These estimates will be similar to the estimates of [E2] §12, but the method of estimation is simpler because it avoids operator theoretic techniques. Also we will get more information about the density function. For example our estimate of $\| f \|_{s+1}$ gives an estimate of the $H^{s+1}$-norm of log $\rho$ and hence of $\rho$. With this we will show that $\rho$ is actually smoother than the velocity $u$.

For simplicity of exposition we shall restrict ourselves to the case $s = 3$ and $n = 2$ or 3. The method can be used for arbitrary $s$ and $n$, (provided $s > \frac{n}{2} + 1$), but the expressions are more cumbersome. Also, the above case seems to be the one of primary importance.

We shall assume that we are given a family of functions $\{\rho_k\}$ which satisfies Assumption 3.1 and that for each $k$ we have $u_k$ and $\rho_k$ which satisfy (2.1a)-(2.1b) and the boundary condition (2.3) on some interval $[0, T]$. We further assume that $f_k = \log \rho_k \in H^4(\Omega)$, and that

$$\| u_k(0) \|_3 \leq a_2, \quad \| \delta u_k(0) \|_3 \leq \frac{a_2}{\sqrt{k}} \text{ and } \| f_k(0) \|_4 \leq \frac{a_2}{k} \quad (4.1)$$

as is assumed in theorem 3.1.

Also we assume that there exists a constant $a_3$ such that

Assumption 4.1.

$$\| u_k(t) \|_3 \leq a_3 \text{ for all } t \in [0, T], \quad (4.2)$$
and another constant and $a_4$ such that

**Assumption 4.2.**

\[
\begin{align*}
\|f_k(t)\|_4 & \leq \frac{a_4}{\sqrt{k}}, \\
\|\dot{f}_k(t)\|_3 & \leq a_4, \\
\|\ddot{f}_k(t)\|_2 & \leq a_4\sqrt{k}, \\
\|\dddot{f}_k(t)\|_1 & \leq a_4 k,
\end{align*}
\] (4.3)

for $t \in [0, T]$.

As we shall see, this latter assumption can eventually be discarded. With these assumptions we will derive estimates which hold for sufficiently large $k$. We will in fact find a constant $a_5$ such that for large $k$

\[
\begin{align*}
\|f_k(t)\|_4 & \leq \frac{a_5}{k}, \\
\|\dot{f}_k(t)\|_3 & \leq \frac{a_5}{\sqrt{k}}, \\
\|\ddot{f}_k(t)\|_2 & \leq a_5, \\
\|\dddot{f}_k(t)\|_1 & \leq a_5\sqrt{k},
\end{align*}
\] (4.4)

for $t \in [0, T]$.

Our method will be to derive a type of energy estimate for $f$, and its first material derivatives. Let

\[
L = -\delta c^2 \nabla,
\]

and

\[
F = u_j^i u_i^j,
\]

so that (2.9) be written as

\[
\dddot{f} = Lf + F.
\] (4.5)

Then let

\[
E(t) = \int_{\Omega} \left( |c\nabla \dddot{f}(t)|^2 + |L\dddot{f}(t)|^2 \right).
\] (4.6)

$E(t)$ will give us estimates for $\|\dddot{f}(t)\|_1$ and $\|\dddot{f}(t)\|_2$, and using these and (4.5) we can get estimates for $f$ and $\dot{f}$.

We now begin the estimate of $E(t)$. We shall use

\[
E(t) = E(0) + \int_0^t \frac{dE(s)}{ds} ds,
\] (4.7)

but in order to do so we must compute $\frac{dE}{dt}$. Differentiating $E$ we find

\[
\frac{1}{2} \frac{dE}{dt} = \int_{\Omega} \left( (\partial_t (c\nabla \dddot{f})), c\nabla \dddot{f} \right) + (\partial_t L\dddot{f})(L\dddot{f}),
\]
but
\[ \partial_t(c\nabla \ddot{f}) = (\partial_t c)\nabla \ddot{f} + c\nabla \partial_t \ddot{f} = (\partial_t c)\nabla \ddot{f} - c\nabla \nabla u \ddot{f} + c\nabla \dddot{f} \]
and similarly
\[ \partial_t L \dddot{f} = -\delta \partial_t (c^2)\nabla \dddot{f} - L \nabla u \ddot{f} + L \dddot{f}. \]
Therefore
\[ \frac{1}{2} \frac{dE}{dt} = \int_\Omega \left( \langle c \nabla \ddot{f}, c \nabla \dddot{f} \rangle + (L \dddot{f})(L \dddot{f}) \right) + R_1 \]
where
\[ R_1 = \int_\Omega \left( \langle (\partial_t c)\nabla \dddot{f} - c\nabla \nabla u \ddot{f}, c \nabla \dddot{f} \rangle - (\delta \partial_t c^2)\nabla \dddot{f} - (L \nabla_u \ddot{f}) L \dddot{f} \right). \quad (4.8) \]
Integrating by parts we find that
\[ \frac{1}{2} \frac{dE}{dt} = \int_\Omega \langle c \nabla \ddot{f}, c \nabla \dddot{f} \rangle - c \nabla L \dddot{f} \rangle + \int_{\partial \Omega} (c^2 \nabla \dddot{f}) L \dddot{f} + R_1 \]
\[ = I_1 + B_1 + R_1. \]
We proceed to evaluate \( I_1 \). First we take two material derivatives of equation (4.5). Letting \( L_1 \) be the operator defined by
\[ L_1 h = (Lh) - Lh \]
and computing directly we find
\[ (Lf) \dddot{f} = L \dddot{f} + L_1 \dddot{f} + (L_1 f) \]
so we get
\[ \dddot{f} = L \dddot{f} + L_1 \dddot{f} + (L_1 f) + \dddot{f}. \quad (4.10) \]
Therefore
\[ I_1 = \int_\Omega \langle c \nabla \ddot{f}, c \nabla (L_1 \dddot{f} + (L_1 f) + \dddot{f}) \rangle. \]
We now calculate \( L_1 \). Using (4.9) we find
\[ L_1 = -\delta \partial_t (c^2) \nabla + [\nabla_u, L] = -\delta (c^2) \nabla - [\nabla_u, \delta] c^2 \nabla - \delta c^2 [\nabla_u, \nabla] \quad (4.11) \]
where \([ , ]\) denotes the commutator.
Of course the commutators on the rightmost side of (4.11) are first order operators whose coefficients are first derivatives of \( u \). Therefore \( L_1 \) is a second order operator whose coefficients depend on (derivatives of) \( c \) and \( u \).
In order to make estimates with such operators we will need several formulas and also a convenient way to bound many different expressions. For the latter purpose we introduce a generic constant \( K \). It will have different values in different expressions but in each case it will depend only on \( \Omega \) and \( \{a_i|i = 1, \ldots, 4\} \).
To begin our formulas we note that from (2.1a) and the definition of \( c^2 \)
\[ \dot{u} = \partial_t u + \nabla_u u = -c^2 \nabla f. \]
Applying \( \partial_j \) to this we get
\[
(u^i_j) = -(c^2 f_i)_j - u^k_j u^i_k. \tag{4.12}
\]
For any function \( h \) a direct computation gives
\[
(\partial_i h) = \partial_i (\dot{h}) - u^i_t \partial_i h. \tag{4.13}
\]
Applying (4.13) to \( f \) we find
\[

\ddot{u}^i = -(c^2 \gamma) \partial_i f - c^2 \partial_i (\dot{f}) + c^2 u^j_i f_j,
\]
and
\[

\dddot{u}^i = -(c^2 \gamma) \partial_i f - 2(c^2 \gamma) (\partial_i (\dot{f}) - u^i_j f_j) - c^2 (\partial_i (\dot{f}) - 2u^i_\ell (\dot{f})_\ell) - ((c^2 f_i)_j + 2u^k_j u^i_k) f_j. \tag{4.14}
\]
Thus we have expressions for material derivatives of \( u \). We shall need formulas for material derivatives of \( c^2 \) as well. From the definition \( c^2 = p'(\rho) \) and the equation \( \dot{f} = \frac{\dot{p}}{\rho} \) we find
\[
(c^2 \gamma) = \frac{p''(\rho)}{\rho} = p''(\rho) \rho \dot{f}. \tag{4.15}
\]

Thus
\[
(c^2 \gamma) = \frac{p''(\rho)}{\rho} \dot{f} = p''(\rho) (\rho \dot{f} + \rho^2 \ddot{f}), \tag{4.16}
\]
and
\[
(c^2 \gamma) = \frac{p''''(\rho)}{\rho} (\rho \dot{f})^3 + 3p''(\rho) (\rho \ddot{f}) (\rho \dot{f} + \rho^2 \ddot{f}) + p''(\rho) (\rho \dddot{f} + \rho \dot{f} \ddot{f} + 2\rho^2 \dddot{f} + 2\rho^3 \dddot{f}^3). \tag{4.17}
\]

Since \( p'(1) = k, c^2 = k + \int_1^\rho p''(\lambda) d\lambda \). Also, since \( \| f \|_4 \leq \frac{4}{\sqrt{k}}, \rho = e^f, \) and \( |p''| \leq a_1 k \) we find
\[
\| c^2 - k \|_4 \leq K \sqrt{k}. \tag{4.18}
\]

Then using the formulas (4.15)-(4.17) we get the estimates
\[
\| (c^2 \gamma) \|_3 \leq Kk, \tag{4.19}
\]
\[
\| (c^2 \gamma) \|_2 \leq K(k(\sqrt{k} + \| \dot{f} \|_2) \leq Kk^\frac{3}{2}, \tag{4.20}
\]
\[
\| (c^2 \gamma) \|_1 \leq K(k(\sqrt{k} + \| \ddot{f} \|_1) \leq Kk^2. \tag{4.21}
\]

Using these estimates and the formulas for derivatives of \( u \) we get
\[
\| \dot{u} \|_3 \leq Kk \| f \|_4 \leq K \sqrt{k}, \tag{4.22}
\]
\[
\| \ddot{u} \|_2 \leq K(\sqrt{k} + k \| \dot{f} \|_3) \leq Kk, \tag{4.23}
\]
\[
\| \dddot{u} \|_1 \leq K(k + k \| \ddot{f} \|_2 + \| \dot{f} \|_3) \leq Kk^\frac{3}{2}. \tag{4.24}
\]

Using (4.11) with (3.1) and then with (4.2), (4.18), (4.19) we get
\[
\| L_1 \dot{f} \|_1 \leq K(\| u \|_3 \| c^2 \|_3 + \| (c^2 \gamma) \|_2 \| \dddot{f} \|_3 \leq Kk \| \dot{f} \|_3. \tag{4.25}
\]

This is the first step in the estimate of \( I_1 \). In order to estimate \( (L_1 f) \) we will need the following additional formula:
\[
(\nabla u, \partial_j h) = ((c^2 f_i)_j + u_j^k u^i_k) h_i - u^i_j h_i \dot{h} = ((c^2 f_i)_j + 2u_j^k u^i_k) h_i - u^i_j (\dot{h})_i \tag{4.26}
\]
which follows from (4.12) and (4.13). From (4.11) we have
\[
L_1 f = -\delta (c^2) \nabla f - [\nabla u, \delta] c^2 \nabla f - \delta c^2 [\nabla u, \nabla] f.
\]
Using the above formulas we get
\[
(\delta(c^2) \nabla f) = \delta(c^2) \nabla f + u^2_i((c^2) f_i) + \delta(c^2)(\nabla \dot{f} - (\nabla u^2) f_j),
\] (4.27)
and
\[
([\nabla_u, \delta] c^2 \nabla f) = ((c^2 f_i) + 2u^2_j k u^i_k) (c^2 f_j) - \dot{u}^i_j ((c^2 f_j) i).
\] (4.28)
But
\[
((c^2 f_j))_i = ((c^2) f_j + c^2((\dot{f})_j - u^2_j f_k))).
\] (4.29)
Therefore
\[
([\nabla_u, \delta] c^2 \nabla f) = ((c^2 f_i) + 2u^2_k u^i_k) (c^2 f_j) - \dot{u}^i_j ((c^2) f_j + c^2((\dot{f})_j - u^2_j f_k))).
\] (4.30)
Also
\[
(\delta c^2[\nabla_u, \nabla] f) = \delta(c^2)[\nabla_u, \nabla] f - (c^2(((c^2 f_i) + 2u^2_k u^i_k) f_i - \dot{u}^i_j ((\dot{f})_i) j - \dot{u}^i_j ((c^2 u^2_k f_k) j).
\] (4.31)
From (4.27), (4.30) and (4.31) we find
\[
\| (L_1 f) \|_1 \leq K \left( \| (c^2) \|_2 + \| (c^2) \|_2 \| u \|_3 + \| c^2 \|_3 \| f \| \right) \| f \|_4
\] (4.32)
\[
+ K(\| (c^2) \|_2 + \| c^2 \|_2 \| u \|_3) \| \dot{f} \|_3
\]
\[
\leq K(k^2 \| f \|_4 + k \| \dot{f} \|_3).
\]
To complete the estimate of \( I_1 \), we now estimate the norm of
\[
\dot{F} = (u^i_j u^i_j) = -((c^2 f_i) + u^2_k u^i_k) u^i_j + ((c^2 f_i) + u^2_k u^i_k) u^i_j).
\]
Note that
\[
((c^2 f_i))_j = ((c^2 f_i)_j - u^2_k (c^2 f_i) k),
\]
which by (4.13) equals
\[
((c^2) f_i + c^2((\dot{f})_i - u^2_k f_k))_j - u^2_k (c^2 f_i) k.
\] (4.33)
Using (4.33) and (4.12) we find that
\[
\| \dot{F} \|_1 \leq K \left( \| u \|_4 + \| u \|_2 \| c^2 \|_3 \| f \|_3 + \| u \|_3 \| (c^2) \|_2 \| f \|_3
\] (4.34)
\[
+ \| u \|_3 \| c^2 \|_3 \| \dot{f} \|_3
\]
\[
\leq K k (\| f \|_3 + \| \dot{f} \|_3).
\]
Combining (4.25), (4.32) and (4.34) we find that
\[
|I_1| \leq K k \| \dot{f} \|_1 (k \| \dot{f} \|_3 + k^2 \| f \|_4 + 1).
\] (4.35)
We now proceed to estimate $R_1$. By commuting $\nabla$ and $\nabla u$ and integrating by parts we find

$$\int_\Omega \langle c\nabla \nabla_u \ddot{f}, c\nabla \ddot{f} \rangle = \int_\Omega \langle c[\nabla, \nabla_u] \ddot{f}, c\nabla \ddot{f} \rangle - \int_\Omega \langle (\nabla_u c) \nabla \ddot{f}, c\nabla \ddot{f} \rangle + \frac{1}{2} \int_\Omega \delta(u) \langle c\nabla \ddot{f}, c\nabla \ddot{f} \rangle. \tag{4.36}$$

Similarly

$$\int_\Omega (L\nabla_u \ddot{f}) L\ddot{f} = \int_\Omega \left( - (\delta(\nabla_u(c^2))) \nabla \ddot{f})L\ddot{f} + \text{com} + \frac{1}{2} \delta(u)(L\ddot{f})^2 \right), \tag{4.37}$$

where “com” indicates second order terms in $\ddot{f}$ involving the commutators $[\nabla, \nabla_u]$ and $[\delta, \nabla_u]$.

Plugging (4.36) and (4.37) in the expression (4.8) for $R_1$ we get

$$R_1 = \int_\Omega \langle (\partial_t c) \nabla \ddot{f}, c\nabla \ddot{f} \rangle - \int_\Omega \langle c[\nabla, \nabla_u] \ddot{f}, c\nabla \ddot{f} \rangle + \int_\Omega \langle (\nabla_u c) \nabla \ddot{f}, c\nabla \ddot{f} \rangle - \frac{1}{2} \int_\Omega \delta(u) \langle c\nabla \ddot{f}, c\nabla \ddot{f} \rangle - \int_\Omega ( - (\delta(\nabla_u(c^2))) \nabla \ddot{f})L\ddot{f} - \int_\Omega \left( - (\delta(\nabla_u(c^2))) \nabla \ddot{f})L\ddot{f} \right) - \frac{1}{2} \int_\Omega \delta(u)(L\ddot{f})^2 - \text{com.}$$

The first and third integrals combined yield $\int_\Omega \langle c\nabla \ddot{f}, c\nabla \ddot{f} \rangle$. The fifth and sixth integrals can be rewritten as

$$- \int_\Omega \langle \delta(\partial_t c^2) \nabla \ddot{f}, L\ddot{f} \rangle - \int_\Omega \langle (\delta(\nabla_u(c^2))) \nabla \ddot{f})L\ddot{f} \rangle = - \int_\Omega \langle \delta(\partial_t c^2) \nabla \ddot{f}, L\ddot{f} \rangle + \int_\Omega \langle \delta(\nabla_u(c^2)) \nabla \ddot{f})L\ddot{f} \rangle + \int_\Omega \langle (\delta(\nabla_u(c^2))) \nabla \ddot{f})L\ddot{f} \rangle = - \int_\Omega \langle \delta(c^2) \nabla \ddot{f}, L\ddot{f} \rangle + 2 \int_\Omega \langle (\delta(\nabla_u(c^2))) \nabla \ddot{f})L\ddot{f} \rangle.$$

Therefore

$$R_1 = \int_\Omega \left( \langle c\nabla \ddot{f}, c\nabla \ddot{f} \rangle - (\delta(c^2) \nabla \ddot{f})L\ddot{f} - \frac{1}{2} \delta(u) \langle (c\nabla \ddot{f}, c\nabla \ddot{f}) \rangle + (L\ddot{f})^2 \right) - \frac{1}{2} \langle c[\nabla, \nabla_u] \ddot{f}, c\nabla \ddot{f} \rangle - \text{com} + 2 \langle (\delta(\nabla_u(c^2))) \nabla \ddot{f})L\ddot{f} \rangle.$$

Hence

$$|R_1| \leq K \left( \| c^2 \|_2 \| \ddot{f} \|_1^2 + \| (c^2') \|_2 \| c^2 \|_2 \| \ddot{f} \|_1 \right) + \frac{1}{2} \| \langle c[\nabla, \nabla_u] \ddot{f}, c\nabla \ddot{f} \rangle - \text{com} + 2 \langle (\delta(\nabla_u(c^2))) \nabla \ddot{f})L\ddot{f} \rangle \right). \tag{4.38}$$

To estimate $B_1$ we must use the boundary condition (2.3), and compute its material derivatives. In fact we will not get a good estimate for $B_1$ itself, but we will for $\int_0^T B_1$. Since we will use (4.7) to estimate $E$, this will be sufficient.
Applying (4.10) we find that

\[ B_1 = \int_{\partial \Omega} (c_\nu^2 \nabla \nu \tilde{f}) (\tilde{f}'' - L_1 \tilde{f} - (L_1 f) - \bar{F}) \]

\[ = \int_{\partial \Omega} (c_\nu^2 \nabla \nu \tilde{f}) \tilde{f}'' - \int_{\partial \Omega} (c_\nu^2 \nabla \nu \tilde{f}) (L_1 \tilde{f} - (L_1 f) - \bar{F}) = B_{11} + B_{1R}. \]

To estimate these terms we must compute \( c_\nu^2 \nabla \nu \tilde{f} \), and we do this by using the boundary condition for \( f \).

Computing as above we find

\[ \nabla \nu \tilde{f} = (\nabla \nu f)'' + (\nabla_{\nu,[u]} f)'' + (\nabla_{\nu,[u]} \dot{f}) + \nabla_{\nu,[u]} \ddot{f}. \]

(4.39)

But by (2.10)

\[ (\nabla \nu f)'' = -\frac{1}{c_\nu^2} \langle \nu, \nabla u \rangle''. \]

(4.40)

Since \( \langle \nu, u \rangle = 0 \) on \( \partial \Omega \),

\[-\langle \nu, \nabla u \rangle = S_2(u, u), \]

where \( S_2 \) is the second fundamental form of \( \partial \Omega \). We shall sometimes write simply \( S_2(u) \).

Taking a material derivative we find

\[ (S_2(u))'' = 2S_2(\ddot{u}, u) + S_3(u), \]

(4.41)

where \( S_3 \) is a symmetric tri-linear form.

Continuing we find

\[ (S_2(u))''' = 2S_2(\dddot{u}, u) + 2S_2(\ddot{u}, \dot{u}) + 5S_3(\dddot{u}, u, u) + S_4(u) \]

where \( S_4 \) is a symmetric quadri-linear form.

Finally

\[ (S_2(u))'''' = 2S_2(\ddddot{u}, u) + S_R(u, \dddot{u}, \ddot{u}) \]

where \( S_R \) is a polynomial in \( u, \dot{u}, \dddot{u} \) of degree five, and each term of \( S_R \) contains at most three dot’s. That is, in a given term of \( S_R \), \( \dddot{u} \) and \( \ddot{u} \) appear with power \( i \) and \( j \) where \( i + 2j \leq 3 \).

From (4.40) we get a formula for \( (\nabla \nu f)'' \) involving \( \frac{1}{c_\nu^2} \) and \( S_2(u) \) and the first three material derivatives of each of them. Letting \( q = \frac{1}{c_\nu^2} \) we write this as

\[ (\nabla \nu f)''' = 2qS_2(\dddot{u}, u) + \dot{q} S_2(u) + f_Rq \]

(4.42)

where \( f_{Rq} \) is a polynomial in \( u, \dddot{u}, \ddot{u}, q, \dot{q} \), and \( \dot{q} \) each of whose terms contains at most three dot’s.

In order to estimate this, we must compute material derivatives of \( q \), and we do this using (4.18)-(4.21). Direct computation yields

\[ \dot{q} = -q^2(c_\nu^2) = -q^2 \rho''(\rho) \dot{f}, \]

(4.43)

\[ \ddot{q} = -q^2(c_\nu^2) - (q^2)(c_\nu^2) = -q^2(c_\nu^2) + 2q^3((c_\nu^2))^2, \]

(4.44)

\[ \dddot{q} = -q^2(c_\nu^2) + 6q^3(c_\nu^2)(c_\nu^2) - 6q^4((c_\nu^2))^3. \]

(4.45)
From (4.18) we find that \( \| \frac{\partial^2}{\partial t^2} - 1 \|_4 \leq \frac{K}{\sqrt{k}} \) and from this it follows that \( \| q \|_4 \leq \frac{K}{\sqrt{k}} \). Then from (4.19)-(4.21) and (4.43)-(4.45) we get
\[
\| \ddot{q} \|_3 \leq \frac{K}{k},
\]
(4.46)
\[
\| \dot{q} \|_2 \leq \frac{K}{\sqrt{k}},
\]
(4.47)
\[
\| \ddot{q} \|_1 \leq K.
\]
(4.48)
From these formulas and the estimates (4.22)-(4.24) for \( u \) we get
\[
\| (\nabla_u f) \|_{\partial \Omega} \leq K (1 + \| \dot{f} \|_2 + \sqrt{k} \| f \|_4).
\]
(4.49)
Now we estimate the other terms of (4.39). Clearly
\[
\| \nabla_{[\nu,\eta]} \ddot{f} \|_{\partial \Omega} \leq K \| \ddot{f} \|_2
\]
and using (4.26) we get
\[
\| (\nabla_{[\nu,\eta]} \dot{f}) \|_{\partial \Omega} \leq K (1 + k \| f \|_4 + \| \dot{f} \|_2).
\]
(4.50)
Finally we estimate
\[
(\nabla_{[\nu,\eta]} f)^* = (\nabla_{[\nu,\eta]} f + \nabla_{[\nu,\eta]} \dot{f} - [u, \nu]^i u_i^j f_j).
\]
(4.51)
By taking a material derivative of (4.12), using (4.19) and (4.22)-(4.23) we find
\[
\| \nabla_{[\nu,\eta]} \ddot{f} \|_{\partial \Omega} \leq K k \| f \|_4.
\]
The other terms of (4.51) are easily estimated and we get
\[
\| (\nabla_{[\nu,\eta]} \ddot{f}) \|_{\partial \Omega} \leq K (k \| f \|_4 + \| \dot{f} \|_3 + \| \ddot{f} \|_2).
\]
(4.52)
Combining (4.49)-(4.50) and (4.52) we get
\[
\| \nabla_{\nu} \dddot{f} \|_{\partial \Omega} \leq K (1 + \| \dot{f} \|_2 + \| \ddot{f} \|_3 \| f \|_2).
\]
(4.53)
Therefore
\[
\| \mathcal{A} \dddot{f} \|_{\partial \Omega} \leq K (k \| \dddot{f} \|_2 + k \| \dot{f} \|_3 + k^2 \| f \|_4),
\]
so from (4.25), (4.32) and (4.34) we get
\[
|B_{11}| \leq K (k \| \dot{f} \|_3 + k^\frac{3}{2} \| f \|_4)(k \| \dddot{f} \|_2 + k \| \dot{f} \|_3 + k^2 \| f \|_4). \]
(4.54)
It remains to estimate \( \int_0^t B_{11} \), and to do this we integrate by parts with respect to the material derivative, obtaining
\[
\int_0^t B_{11} = \int_{\partial \Omega} (\mathcal{A} \dddot{f}) \nabla_{\nu} \dddot{f} \bigg|_0^t - \int_0^t \int_{\partial \Omega} (\mathcal{A} \dddot{f}) \dddot{f} + \int_0^t \int_{\partial \Omega} \delta_\nu(u)(\mathcal{A} \dddot{f}) \dddot{f},
\]
(4.55)
where \( \delta_\nu \) is the formal adjoint of the gradient on the manifold \( \partial \Omega \). The last term of this expression is clearly bounded by
\[
K \int_0^t \| u \|_3 \| \mathcal{A} \dddot{f} \|_{\partial \Omega} \| \dddot{f} \|_1.
\]
(4.56)
To compute the next to last term, we take the material derivative of \( c^2 \nabla \nu \dddot{f} \). Using (4.39) we get
\[
(c^2 \nabla \nu \dddot{f}) = (c^2) \nabla \nu \dddot{f} + c^2((\nabla \nu f)^{\ldots} + (\nabla_{[\nu,u]} f)^{\ldots} + (\nabla_{[\nu,u]} \dddot{f}) + (\nabla_{[\nu,u]} \dddot{f})).
\]
From (4.53) we find
\[
\| (c^2) \nabla \nu \dddot{f} \|_{\partial, \frac{1}{2}} \leq K (k + k \| \dddot{f} \|_2 + k \| \dddot{f} \|_3 + k^2 \| f \|_4)
\]
so since \( \| \dddot{f} \|_3 \) is assumed to be bounded
\[
\left| \int_{\partial \Omega} (c^2) (\nabla \nu \dddot{f}) \dddot{f} \right| \leq K \| \dddot{f} \|_1 (k + k \| \dddot{f} \|_2 + k^2 \| f \|_4) \quad (4.57)
\]
From (4.42) we find that
\[
(\nabla \nu f)^{\ldots} = 2q(S_2(\dddot{u}, \dddot{u}) + S_2(\dddot{u}, \dddot{u}) + S_3(u, \dddot{u}, u) + q\dot{S}_2(\dddot{u}, u) + \dddot{q}(S_2(u)) + (f_{R\theta})^\prime \quad (f_{R\theta})\]
involves at most third material derivatives of \( q \) and \( u \) and so do all the other terms except \( 2q \dot{S}_2(\dddot{u}, u) \) and \( \dddot{q} S_2(u) \). Collecting them as \( f_{R\theta} \) we get
\[
(\nabla \nu f)^{\ldots} = 2q \dot{S}_2(\dddot{u}, u) + \dddot{q} S_2(u) + f_{R\theta}. \quad (4.58)
\]
However, estimating as before we find
\[
\| f_{R\theta} \|_{\partial, \frac{1}{4}} \leq K (\sqrt{k} + k \| f \|_4 + k \| \dddot{f} \|_3 + \| \dddot{f} \|_2) \quad (4.59)
\]
so we need only worry about the first two terms on the right side of (4.58). Taking a material derivative of (4.14) we find that
\[
\dddot{u} = u_R - c^2 \nabla \dddot{f}
\]
where \( u_R \) is a polynomial in (up to) third material derivatives of \( c^2 \) and in \( u_j^I, \nabla f, \nabla \dddot{f}, \) and \( \nabla \dddot{f} \). Thus we get
\[
\| u_R \|_1 \leq K (k \| \dddot{f} \|_2 + k^3 \| \dddot{f} \|_3 + k^2 \| f \|_4). \quad (4.60)
\]
Similarly, differentiating (4.45) we get
\[
\dddot{q} = -q^2 (c^2)^{\ldots} + q_R,
\]
where \( q_R \) is a polynomial in \( q \) and up to third material derivatives of \( c^2 \). Using the formulas (4.18)-(4.21) we get
\[
\| q_R \|_1 \leq K (\sqrt{k} + \frac{1}{k} \| \dddot{f} \|_1) \quad (4.61)
\]
But differentiating (4.17), we find
\[
(c^2)^{\ldots} = p''(\rho) \rho \dddot{f} + C_R,
\]
where \( C_R \) is a polynomial in \( \rho \) and material derivatives of \( f \) whose coefficients are up to fifth derivatives of the function \( p(\rho) \).
Estimating as before we get
\[
\| C_R \|_1 \leq K (k + k^3 \| \dddot{f} \|_2 + k \| \dddot{f} \|_1)
\]
so

\[ \| q^2 C_R \|_1 \leq K \left( \frac{1}{k} + \frac{1}{k} \| \dddot{f} \|_1 + \frac{1}{\sqrt{k}} \| \dddot{f} \|_2 \right). \]  

(4.62)

Using (4.58) and the above, we can write

\[ (\nabla \nu f)^{\nu} = 2qS_2(c^2\nabla \dddot{f}, u) - q^2 p''(\rho) \dddot{f} S_2(u) + f_{R\nu} \]  

where

\[ f_{R\nu} = 2qS_2(u_R, u) - q_\nu S_2(u) - q^2 C_R S_2(u) + f_{R,\nu}. \]

Using (4.59)-(4.61) and (4.62) we get

\[ \| f_{R\nu} \|_{\partial \Omega} \leq K(\sqrt{k} + \frac{1}{k} \| \dddot{f} \|_1 + \| \dddot{f} \|_2 + k \| \dddot{f} \|_3 + k \| f \|_4). \]  

(4.64)

We now use (4.63) to estimate

\[ \int_0^t \int_{\partial \Omega} c^2 (\nabla \nu f)^{\nu} \dddot{f}. \]

On \( \partial \Omega \) we can write

\[ \nabla \dddot{f} = \nabla_{\partial f} \dddot{f} + \nabla \nu \dddot{f} \]  

(4.65)

where \( \nabla_{\partial f} \dddot{f} \) is the gradient of \( \dddot{f}_{|\partial \Omega} \). Then

\[ \int_0^t \int_{\partial \Omega} (\nabla \nu f)^{\nu} \dddot{f} = \int_0^t \int_{\partial \Omega} \left( S_2(c^2 \nabla_{\partial f}(\dddot{f})^2, u) + 2S_2(c^2 \nabla \nu \dddot{f}, u) \dddot{f} \right) \]

(4.66)

\[ - \frac{1}{2} q_{p''}(\rho) S_2(u)(\dddot{f})^2 + c^2 (f_{R\nu}) \dddot{f} \).

From (4.53) and (4.64) we see that the second and forth terms of (4.66) are bounded by

\[ K \int_0^t (k^{\frac{3}{2}} + \| \dddot{f} \|_1 + k \| \dddot{f} \|_2 + k^2 \| \dddot{f} \|_3 + k^2 \| f \|_4) \| \dddot{f} \|_1. \]  

(4.67)

Since \( \nabla_{\partial f} \) involves derivative only along \( \partial \Omega \) we can integrate by parts to get

\[ \left| \int_0^t \int_{\partial \Omega} S_2(c^2 \nabla_{\partial f}(\dddot{f})^2, u) \right| \leq K \int_0^t k \| \dddot{f} \|_1^2. \]  

(4.68)

Also integrating by parts with respect to material derivative we find that

\[ \left| \int_0^t \int_{\partial \Omega} q_{p''}(\rho) S_2(u)(\dddot{f})^2 \right| \leq \left| \int_0^t \int_{\partial \Omega} q_{p''}(\rho) S_2(u)(\dddot{f})^2 \right| \]

(4.69)

\[ + K \int_0^t \| \dddot{f} \|_1^2 (1 + \| \dddot{f} \|_3 + k \| f \|_4). \]

But

\[ \int_{\partial \Omega} q_{p''}(\rho) S_2(u)(\dddot{f})^2 \leq K \| \dddot{f} \|_1^2 \]
so combining (4.67), (4.68), (4.69) we get
\[
\left| \int_0^t \int_{\partial \Omega} c^2 (\nabla_{\nu} f) \cdot \ddot{f} \right| \leq K (\| \ddot{f} (0) \|_1^2 + \| \ddot{f} (t) \|_1^2) + K \int_0^t \| \ddot{f} \|_1 \left( k^2 + \| \ddot{f} \|_1 \right) + k \| \ddot{f} \|_2 + k^2 \| \dot{f} \|_3 + k^2 \| f \|_4).
\]
(4.70)

To complete our estimate of \( \int_0^t \int_{\partial \Omega} (c^2 \nabla_{\nu} f) \cdot \ddot{f} \) we must estimate
\[
\int_0^t \int_{\partial \Omega} c^2 \left\{ (\nabla_{[\nu,u]} f)^{\nu} + (\nabla_{[\nu,u]} \dot{f})^{\nu} + (\nabla_{[\nu,u]} \ddot{f})^{\nu} \right\} \ddot{f}.
\]
Computing the material derivative we find that the term in braces equals
\[3 \nabla_{[\nu,u]} \ddot{f} + \nabla_{[\nu,u]} f + f_{R_3}\]
where \( f_{R_3} \) is a polynomial in \( c^2 \) and its first two material derivatives as well as \( u_j, \nabla f, \nabla \dot{f}, \)
and \( \nabla \ddot{f} \). Thus we get
\[\| f_{R_3} \|_1 \leq K (\sqrt{k} \| \dot{f} \|_2 + k \| \ddot{f} \|_3 + k \| f \|_4),\]
so
\[
\left| \int_0^t \int_{\partial \Omega} c^2 f_{R_3} \ddot{f} \right| \leq K \int_0^t (\sqrt{k} \| \dot{f} \|_2 + k \| \ddot{f} \|_3 + k \| f \|_4) \| \ddot{f} \|_1.
\]
(4.71)

We decompose \( \nabla_{[\nu,u]} \ddot{f} \) in to tangential and normal derivatives of \( \ddot{f} \) (as we did in (4.65)), getting
\[\nabla_{[\nu,u]} \ddot{f} = (\nabla_{[\nu,u]} \ddot{f})_{\theta} + (\nabla_{[\nu,u]} \ddot{f})_{\nu}.
\]
Then integration by parts gives
\[
\left| \int_{\partial \Omega} c^2 \left( \nabla_{[\nu,u]} \ddot{f} \right)_{\theta} \ddot{f} \right| \leq K k \| \ddot{f} \|_1^2.
\]
(4.72)

Also \( (\nabla_{[\nu,u]} \ddot{f})_{\nu} = \langle [\nu, u], \nu \rangle \nabla_{\nu} \ddot{f} \) so from (4.53) we get
\[
\left| \int_{\partial \Omega} c^2 \left( \nabla_{[\nu,u]} \ddot{f} \right)_{\nu} \ddot{f} \right| \leq K (k + k \| \dot{f} \|_2 + k \| \ddot{f} \|_3 + k^2 \| f \|_4) \| \ddot{f} \|_1.
\]
(4.73)

We now deal with the remaining term \( \nabla_{[\nu,u]} f \). Computing, we find that
\[\langle [\nu, u], \nu \rangle = -c^2 \nabla_{[\nu,u]} \ddot{f} + u_{R_2},\]
where \( u_{R_2} \) obeys the inequality
\[\| u_{R_2} \|_{\partial \Omega} \leq K (k^2 \| f \|_4 + k \| \dot{f} \|_3).
\]
(4.74)

Using the tangential and normal decomposition on \( \partial \Omega \) we get
\[c^2 \nabla_{[\nu,u]} \ddot{f} = -c^2 \nabla_{\theta} \nabla_{\nu} \ddot{f} - c^2 \nabla_{\nu} \nabla_{\nu} \ddot{f}.
\]

But
\[c^2 \nabla_{\nu} \nabla_{\nu} \ddot{f} = L \ddot{f} + c^2 D_{\tau} \nabla_{[\nu,u]} \ddot{f} - L_{\dot{\theta}} \ddot{f} + L_{R} \ddot{f}.
\]
where $D_r$ is a first order operator involving only tangential derivatives, $L_\delta = -\delta_\delta c^2 \nabla_\delta \left( \nabla_\delta \right.$ being the gradient of the manifold $\partial \Omega$ and $\delta_\delta$ its formal adjoint), and $L_R$ is a first order operator whose coefficients involve $c^2$ and its first derivatives.

Thus we find

$$
\int_{\partial \Omega} c^2 (\nabla_{[\nu,u]} f) \tilde{\ddot{f}} = \int_{\partial \Omega} \left( c^2 \nabla_{uR_2} f - c^4 (\nabla_\delta \nabla_\nu \tilde{f}) (\nabla_\delta f) - c^2 (L \tilde{f} + c^2 D_r \nabla_\nu \tilde{f} - L_\delta \tilde{f} + L_R \tilde{f}) \nabla_\nu f \right) \tilde{\ddot{f}}.
$$

Using (4.74) we find that

$$
\left| \int_{\partial \Omega} (c^2 \nabla_{uR_2} f - (c^2 L_R \tilde{f}) \nabla_\nu f) \tilde{\ddot{f}} \right| \leq K \parallel \tilde{f} \parallel_1 \left( k^\frac{3}{2} \parallel f \parallel_4 + k \parallel \dot{f} \parallel_3 + k \parallel \ddot{f} \parallel_2 \right).
$$

Since

$$
\nabla_\nu \tilde{f} = (\nabla_\nu f) + (\nabla_{[\nu,u]} f) + \nabla_{[\nu,u]} \dot{f}
$$

and

$$
(\nabla_\nu f) = -(q S_2(u))^\gamma
$$

we also get the estimate

$$
\left| \int_{\partial \Omega} c^4 ((\nabla_\delta \nabla_\nu \tilde{f}) \nabla_\delta f + (D_r \nabla_\nu \tilde{f}) \nabla_\nu f) \tilde{\ddot{f}} \right| \leq K \parallel \tilde{f} \parallel_1 \left( k^\frac{3}{2} + k^2 \parallel f \parallel_4 + k^\frac{3}{2} \parallel \dot{f} \parallel_3 \right)
$$

$$
\leq K k^\frac{3}{2} \parallel \tilde{f} \parallel_1.
$$

It remains to estimate the terms

$$
\int_0^t \int_{\partial \Omega} c^2 (L_\delta \tilde{f}) \tilde{\ddot{f}} \quad \text{and} \quad \int_0^t \int_{\partial \Omega} c^2 (L \tilde{f}) \tilde{\ddot{f}}.
$$

The first of these equals

$$
- \int_0^t \int_{\partial \Omega} c^2 \langle \nabla_\delta \tilde{f}, \nabla_\delta c^2 \tilde{f} \rangle,
$$

which equals

$$
- \frac{1}{2} \int_0^t \int_{\partial \Omega} c^4 (|\nabla_\delta \tilde{f}|^2) + \int_0^t R_2,
$$

where

$$
|R_2| \leq K k^2 \parallel \tilde{f} \parallel_2^2.
$$

Using (4.10) we find that the second term equals

$$
\int_0^t \int_{\partial \Omega} c^2 (\tilde{\ddot{f}} - L_1 \dot{f} - (L_1 f) - \tilde{F}) \tilde{f},
$$

which equals

$$
\frac{1}{2} \int_0^t \int_{\partial \Omega} c^2 (\tilde{f})^2 + \int_0^t R_3,
$$
and from (4.25), (4.32) and (4.34) we find that

\[ |R_3| \leq K \parallel \ddot{f} \parallel_1 (k \parallel \dot{f} \parallel_3 + k^{\frac{3}{2}} \parallel f \parallel_4). \]

Therefore

\[ \int_0^t \int_{\partial \Omega} c^2(\nabla_{[\nu,u]} f) \ddot{f} = -\frac{1}{2} \int_0^t \left( c^4 |\nabla \tilde{f}|^2 + c^2 |\tilde{f}|^2 \right) \bigg|_0^t + \int_0^t R_4 \quad (4.75) \]

where

\[ |R_4| \leq K(k^{\frac{3}{2}} \parallel \ddot{f} \parallel_1 + k \parallel \dot{f} \parallel_1^2 + k^2 \parallel f \parallel_2^2). \]

We let \( P(t) \) stand for the term \( \frac{1}{2} \int_{\partial \Omega} \left( c^4 |\nabla \tilde{f}|^2 + c^2 |\tilde{f}|^2 \right) \) so that (4.75) becomes

\[ \int_0^t \int_{\partial \Omega} c^2(\nabla_{[\nu,u]} f) \ddot{f} = P(0) - P(t) + \int_0^t R_4. \quad (4.76) \]

Combining (4.71), (4.72), (4.73) and (4.76), and using our assumed bounds on \( \parallel f \parallel_4, \parallel \dot{f} \parallel_3, \) and \( \parallel \ddot{f} \parallel_2 \) we find

\[ \int_0^t \int_{\partial \Omega} c^2 \left\{ (\nabla_{[\nu,u]} f)^{-} + (\nabla_{[\nu,u]} \tilde{f})^{-} + (\nabla_{[\nu,u]} \dot{f}) \right\} \ddot{f} = P(0) - P(t) + \int_0^t R_5 \quad (4.77) \]

where

\[ |R_5| \leq K(k \parallel \ddot{f} \parallel_1^2 + k^{\frac{3}{2}} \parallel \dot{f} \parallel_1 + k^2 \parallel f \parallel_2^2). \]

Then combining (4.57), (4.70) and (4.77) we find

\[ P(t) + \int_0^t \int_{\partial \Omega} (c^2 \nabla_{[\nu,u]} f) \ddot{f} \leq P(0) + K(\parallel \ddot{f}(0) \parallel_1^2 + \parallel \dot{f}(t) \parallel_1^2) \]

\[ + K \int_0^t (k^{\frac{3}{2}} \parallel \ddot{f} \parallel_1 + k \parallel \dot{f} \parallel_1^2 + k^2 \parallel f \parallel_2^2 + k^3 \parallel \ddot{f} \parallel_1 \parallel f \parallel_3). \]

Using (4.55), and (4.56) and the fact that

\[ k^2 \parallel \ddot{f} \parallel_1 \parallel f \parallel_3 \leq \frac{1}{2}(k \parallel \ddot{f} \parallel_1^2 + k^3 \parallel f \parallel_3^2) \]

we get

\[ P(t) + \int_0^t B_{11} \leq P(0) + K(Q(t) + Q(0)) \]

\[ + K \int_0^t (k^{\frac{3}{2}} \parallel \ddot{f} \parallel_1 + k \parallel \dot{f} \parallel_1^2 + k^2 \parallel f \parallel_2^2 + k^3 \parallel \ddot{f} \parallel_3^2), \quad (4.78) \]

where \( Q \) is a function of time defined by

\[ Q = \parallel \ddot{f} \parallel_1^2 + \parallel \dot{f} \parallel_1 (k + k^2 \parallel \ddot{f} \parallel_2 + k \parallel \dot{f} \parallel_3 + k^2 \parallel f \parallel_4). \]

Clearly this \( Q \) obeys the inequality

\[ Q \leq K(k^{\frac{3}{2}} \parallel \ddot{f} \parallel_1^2 + k^{\frac{3}{2}} + k^{\frac{3}{2}} \parallel \ddot{f} \parallel_2^2 + k^{\frac{3}{2}} \parallel \ddot{f} \parallel_3^2 + k^{\frac{3}{2}} \parallel f \parallel_4^2). \quad (4.79) \]
Finally using (4.35), (4.38), (4.54) and (4.78) we get
\[ E(t) + P(t) \leq E(0) + P(0) + K(Q(t) + Q(0)) + K \int_0^t (k \| \ddot{f} \|_2^2 + k^2 \| \dot{f} \|_2^2 + k^3 \| \dot{f} \|_3^2 + k^4 \| f \|_4^2 + k^2). \] (4.80)

This is our estimate for the growth of $E(t)$.

We now proceed to show that $E$ gives a bound for the norms of $f$ and its material derivatives. First we note that since $c^2$ is in $H^4$, $L : H^s \to H^{s-2}$ ($2 \leq s \leq 5$) is a bounded operator with null space and co-null space equal to the constant functions. Because of this we will need the decomposition
\[ H^s = H^s_0 \oplus \text{Con} \] (4.81)
given by
\[ g = g_1 + g_2 \]
where $g_2$ is the constant function
\[ g_2 = \frac{\int_\Omega g}{\int_\Omega 1} \]
Of course, $H^s_0$ is the space of $H^s$ functions whose integral is zero and Con stands for the constants. We shall use this decomposition on $f$ and its material derivatives. To simplify calculations we shall assume that $\int_\Omega 1 = 1$.

First we note that
\[ \int_\Omega \dot{f} = \int_\Omega \delta(u) = 0 \] (4.82)
so
\[ (\dot{f})_2 = 0 \text{ and } \dot{f} = (\dot{f})_1. \] (4.83)

Note also that
\[ f_2(t) = f_2(0) + \int_0^t \partial_t f_2 = f_2(0) + \int_0^t \int_\Omega \partial_t f = f_2(0) + \int_0^t \int_\Omega \dot{f} - \int_0^t \int_\Omega \nabla u f. \] (4.84)
Therefore
\[ |f_2(t)| \leq |f_2(0)| + \int_0^t \int_\Omega |\nabla u f| \leq |f_2(0)| + K \int_0^t \| f \|_0. \]
But for any function $g$, and any $s \geq 0$
\[ \| g \|_s \leq K(\| g_1 \|_s + \| g_2 \|_s). \]
Hence
\[ |f_2(t)| \leq |f_2(0)| + K \int_0^t (\| f_2 \|_0 + \| f_1 \|_0). \] (4.85)
Let
\[ \| g \|_{s_{\text{max}}} = \max\{\| g(\tau) \|_s \mid 0 \leq \tau \leq t\}. \]
Then iterating inequality (4.85) we get
\[ |f_2(t)| \leq e^{Kt} |f_2(0)| + \| f_1 \|_{0, \text{max}} (e^{Kt} - 1). \]  
(4.86)

From this and (4.83) we find
\[ \| f(t) \|_s \leq K (|f_2(0)| + \| f_1 \|_{s, \text{max}}). \]  
(4.87)

We now proceed to get similar inequalities for the higher material derivatives of \( f \).
\[
|(\dddot{f})_2| = \left| \int_\Omega (\partial_t \dot{\dot{f}} + \nabla u \dot{\dot{f}}) \right| \leq K \| \dot{\dot{f}} \|_0 .
\]

Therefore
\[ \| \dddot{f} \|_s \leq K (|\dddot{f}_1|_s + \| \dddot{f} \|_0). \]  
(4.88)

Also
\[
(\dddot{f})_2 = \int_\Omega (\partial_t^2 \dddot{f} + \partial_t \nabla u \dddot{f} + \nabla u \nabla u \dddot{f}).
\]

But
\[
\int_\Omega \partial_t^2 \dddot{f} = 0 \quad \text{and} \quad \left| \int_\Omega \nabla u \nabla u \dddot{f} \right| \leq K \| \dddot{f} \|_3 .
\]

Furthermore
\[
2 \int_\Omega \partial_t \nabla u \dddot{f} = 2 \int_\Omega \delta(u) \partial_t \dot{f} = 2 \int_\Omega \dot{\dddot{f}} = 2 \int_\Omega (\dddot{f} - \nabla u \dddot{f}).
\]

And since \( \| \dddot{f} \|_3 \) is assumed to be bounded
\[
\left| \int_\Omega \dddot{f} (\dddot{f} - \nabla u \dddot{f}) \right| \leq K (\| \dddot{f} \|_2 + \| \dddot{f} \|_3) .
\]

It follows that
\[
|((\dddot{f})_2| \leq K (\| \dddot{f} \|_2 + \| \dddot{f} \|_3),
\]
so
\[
\| \dddot{f} \|_1 \leq K (\| \dddot{f} \|_2 + \| \dddot{f} \|_3 + \| \dddot{f}_1 \|_1) .
\]  
(4.89)

Combining (4.83), (4.87), (4.88) and (4.89) we find that the norms of the first components of \( f \) and its material derivatives (in the decomposition (4.81)) bound the norms of \( f \) and its derivatives. Specifically we get, after multiplication by various powers of \( k \),
\[
k^2 \| f \|_4 + k^{3/2} \| \dot{f} \|_3 + k \| \dddot{f} \|_2 + k^{3/2} \| \dddot{f} \|_1 \leq K (k^2 \| f_1 \|_{4, \text{max}} + k^{3/2} \| (\dddot{f})_1 \|_3 + k \| (\dddot{f})_1 \|_2 + k^{3/2} \| (\dddot{f})_1 \|_1 + k^2 |f_2(0)|).
\]

From this inequality we know that it is sufficient to estimate the \( H_{0, s}^s \) components. This is equivalent to working with the functions modulo additive constants. Therefore, for the rest of this section we shall work modulo constants; that is, we will simply equate \( \| g \|_s \) and \( \| g_1 \|_s \).

To get bounds for \( f \) and its derivatives we will need some results about elliptic boundary value problems. First we consider the Neumann problem for the Laplacian. We let \( N : \)
$H^s(\Omega) \rightarrow H^{s-\frac{3}{2}}(\partial\Omega)$ (s > $\frac{3}{2}$) be the operator defined by: $Ng = R(\nabla_\nu g)$, where $R : H^{s-1}(\Omega) \rightarrow H^{s-\frac{3}{2}}(\partial\Omega)$ is simply the restriction of a function to $\partial\Omega$.

It is well known that $N$ is continuous and surjective and admits a continuous right inverse $G_\Delta : H^{s-\frac{3}{2}}(\partial\Omega) \rightarrow H^s(\Omega)$ where $G_\Delta$ solves the Neumann problem, i.e., $G_\Delta h = g$ is the solution of

\[
\begin{cases}
\Delta g = 0, \\
R\nabla_\nu g = h.
\end{cases}
\]

Also it is well known that $\Delta : H^s_\nu(\Omega) \rightarrow H^{s-2}(\Omega)$ (s ≥ 2) is bijective (modulo additive constants) where $H^s_\nu(\Omega) = \{ g \in H^s(\Omega) \mid R\nabla_\nu g = 0 \}$. We denote its inverse by $\Delta^{-1} : H^{s-2} \rightarrow H^s_\nu$.

From this we conclude that for any $g \in H^s(\Omega)$, s ≥ 2, we get

\[
g = \Delta^{-1}\Delta g + G_\Delta Ng
\]

and it follows that

\[
\| g \|_s \leq K (\| \Delta g \|_{s-2} + \| Ng \|_{\partial\Omega, s-\frac{3}{2}}). \tag{4.90}
\]

Now we shall derive an inequality like (4.90) using $L$ instead of $\Delta$. Since $\| c^2 \|_4 \leq Kk$ we know that $L : H^s \rightarrow H^{s-2}$ has operator norm bounded by $Kk$ for 2 ≤ s ≤ 5. Also from (4.18) we find that $\frac{1}{K}L - \Delta = \delta(\frac{\alpha^2}{k} - 1)\nabla : H^s \rightarrow H^{s-2}$ has operator norm bounded by $\frac{K}{\sqrt{k}}$. Therefore $\frac{1}{K}L\Delta^{-1} - I : H^{s-2} \rightarrow H^{s-2}$ has norm bounded by $\frac{K}{\sqrt{k}}$, where $I$ means the identity operator. But since one can invert operators near $I$ by a Neumann series we find that for any $\lambda > 1$, if $\sqrt{k} > \lambda K$, then $\frac{1}{K}L\Delta^{-1}$ has an inverse whose norm is less than $\frac{1}{\lambda^2}$. Also $L : H^s_\nu \rightarrow H^{s-2}$ equals $k(\frac{1}{K}L\Delta^{-1})\Delta$, so for large $k$, $L : H^s_\nu \rightarrow H^{s-2}$ is bijective and $L^{-1} = \frac{1}{K}\Delta^{-1}(\frac{1}{K}L\Delta^{-1})^{-1}$ has norm bounded by $\frac{K}{k}$.

Furthermore we can define $G_L : H^{s-\frac{3}{2}}(\partial\Omega) \rightarrow H^s(\Omega)$ (2 ≤ s ≤ 5) which plays a role analogous to $G_\Delta$. We let $G_L = G_\Delta - L^{-1}LG_\Delta$. Then one sees easily that $G_L h$ is a solution to the boundary value problem:

\[
\begin{cases}
Lg = 0, \\
R\nabla_\nu g = h.
\end{cases}
\]

Also $G_L : H^{s-\frac{3}{2}}(\partial\Omega) \rightarrow H^s(\Omega)$ clearly has a norm bounded independent of $k$, i.e.,

\[
\| G_L \| \leq \| G_\Delta \| + \| LL^{-1}G_\Delta \| \leq \| G_\Delta \| (1 + K\frac{1}{k}).
\]

As with the Laplacian, we get the equation

\[
g = L^{-1}Lg + G_LNg \tag{4.91}
\]
and from this we get the inequality

$$\| g \|_{s} \leq K \left( \frac{1}{k} \| Lg \|_{s-2} + \| Ng \|_{\partial, s-\frac{3}{2}} \right). \tag{4.92}$$

This inequality will be our main tool used to estimate norms of $f$ and its derivatives. We begin with the norm of $f$ itself. From our basic equations we find that

$$L f = \ddot{f} - F \quad \text{and} \quad N f = qS_2(u).$$

Therefore

$$\| f \|_{4} \leq K \frac{k}{k} (\| \dot{f} \|_{2} + 1). \tag{4.93}$$

Also differentiating these equations we get

$$L \dot{f} = \ddot{f} - (L_1 \ddot{f} + \dot{F}) \quad \text{and} \quad N \dot{f} = (qS_2)(u) + \nabla_{[\nu,u]} f.$$

But $\dot{F} = (u^i_j)u^i_t + u^j_i(u^i_t)$ so from (4.12) we get

$$\| \dot{F} \|_{2} \leq K(k \| f \|_{4} + 1) \leq K \sqrt{k}$$

and from the formula (4.11) for $L_1$ we find

$$\| L_1 f \|_{1} \leq K k \| f \|_{4} \leq K \sqrt{k}.$$

From (4.41) and (4.46) we find

$$\| (qS_2(u)) \hat{\gamma} \|_{\partial, \frac{3}{2}} \leq K \left( \frac{1}{k} + \| f \|_{4} \right) \leq \frac{K}{\sqrt{k}}$$

and of course

$$\| \nabla_{[\nu,u]} f \|_{\partial, \frac{3}{2}} \leq K \| f \|_{4} \leq \frac{K}{\sqrt{k}}.$$

Combining these four inequalities and using (4.92), we get

$$\| \dot{f} \|_{3} \leq K \left( \frac{1}{k} \| \ddot{f} \|_{1} + \frac{1}{\sqrt{k}} \right). \tag{4.94}$$

From the definition of $E$, we know $\| L \ddot{f} \|_{0} \leq \sqrt{E}$, so to bound $\ddot{f}$ we need only look at boundary data. But

$$N \ddot{f} = (qS_2(u))^\gamma + (\nabla_{[\nu,u]} f)^\gamma + \nabla_{[u,u]} \ddot{f}.$$

Since $\| \dot{f} \|_{3}$ is assumed bounded and $\| f \|_{4} \leq \frac{K}{\sqrt{k}}$, we get $\| N \ddot{f} \|_{\partial, \frac{3}{2}} \leq K$. Therefore

$$\| \ddot{f} \|_{2} \leq K \left( \frac{1}{k} \sqrt{E} + 1 \right). \tag{4.95}$$

Finally we have the obvious inequality

$$\| \ddot{f} \|_{1} \leq K \sqrt{\frac{E}{k}}, \tag{4.96}$$
so we have bounds for $f$ and its first three material derivatives. Combining (4.93) and (4.95) we find

$$\| f \|_4 \leq K \left( \frac{1}{k} + \frac{1}{k^2} \sqrt{E} \right)$$

(4.97)

and combining (4.94) and (4.96) we get

$$\| \dot{f} \|_3 \leq K \left( \frac{1}{\sqrt{k}} + \frac{1}{k^2 \sqrt{E}} \right).$$

(4.98)

Thus combining (4.95)-(4.98) we get all the bounds in terms of $E$ itself.

But since the assumptions on $E$ and its material derivatives, (4.2) and (4.3d) hold, then for large $k$, $E_1(t) \leq K K_1 k^2 e^{K_1 t}$. That is Assumption 4.2 is no longer necessary. The inequalities (4.1) and (4.2) imply that for large $k$, (4.100) holds. But since $E_1$ bounds $f$ and its material derivatives, (4.100) implies

$$\| f \|_4 \leq \frac{K}{k}, \quad \| \dot{f} \|_3 \leq \frac{K}{\sqrt{k}}, \quad \| \ddot{f} \|_2 \leq K \quad \text{and} \quad \| \dddot{f} \|_1 \leq K \sqrt{k}.$$
This $K$ is the desired $a_5$ of (4.4a)-(4.4d).

Thus we have shown that the assumptions (4.1) and (4.2) imply that there is a constant $a_5$ depending only on $\Omega$, $T$, $a_2$ and $a_3$ such that (4.4a)-(4.4d) hold for large $k$.

**Remark 4.1.** The argument given to prove (4.4a)-(4.4d) is similar to that used for Proposition 12.13 of [E2] and for Lemma 4.6 of [E4].

5. Proofs of Main Theorems

We begin this section with the proof of Theorem 3.1. It will be a combination of the results of [E3] with the estimates of section 4. As in section 4, we shall restrict ourselves to the case $s = 3$. The proof for general $s$ is essentially the same.

We first consider for fixed $k$ the initial data $u_{0k}$, $\rho_{0k}$ and assume that $k$ is large enough so that $\rho_{0k}$ is near 1, and so that for each $x \in \Omega$, $|u_{0k}(x)|^2 < p'(\rho_{0k}(x))$. This can be done because of assumption 2a) of Theorem 3.1 which implies that the $C^0$-norm of $u_{0k}$ is bounded independent of $k$ and because of 2c) of that theorem which says that

$$\| \log \rho_{0k} \|_4 \leq \frac{a_2}{k}.$$ 

Then from [E3] we know that there exists an interval $[0, T]$ and $H^3$ functions $u$, and $\rho$ defined on $[0, t] \times \Omega$ which satisfy the compressible motion problem with initial data $u_{0k}$, $\rho_{0k}$. Thus we need only show that $\rho(t) \in H^4$ and $\delta u(t) \in H^3$.

In order to show this we must first approximate $u_{0k}$, $\rho_{0k}$ by $H^4$ functions which also satisfy the compatibility conditions. That is we want a sequence of $H^4$ functions $(u_n, \rho_n)$ such that

1) $u_n \to u_{0k}$ in $H^3$ and $\delta u_n \to \delta u_{0k}$ in $H^3$ as $n \to \infty$.

2) $\rho_n \to \rho_{0k}$ in $H^4$ as $n \to \infty$.

3) For each $n$, the pair $(u_n, \rho_n)$ satisfies the compatibility conditions up to order 3.

We now proceed to construct such a sequence.

First we decompose $u_{0k}$ into its divergence free and gradient parts:

$$u_{0k} = w_{0k} + \nabla g_{0k}.$$ 

We let $\tilde{w}_n$ be a sequence of $H^4$ divergence free vector fields which converge to $w_{0k}$ in $H^3(\Omega, \mathbb{R}^n)$. Such a sequence is easily found; one simply approximates $w_{0k}$ by a sequence of $H^4$ vector fields and lets the $\{\tilde{w}_n\}$ be $P$ applied to this sequence. Now let $\tilde{u}_n = \tilde{w}_n + \nabla g_{0k}$ and $\tilde{\rho}_n = \rho_{0k}$. Then $\delta \tilde{u}_n = -\Delta g_{0k} = \delta u_{0k}$, and $\Delta g_{0k} \in H^3$ implies $\nabla g_{0k} \in H^4$, so clearly the sequence $(\tilde{u}_n, \tilde{\rho}_n)$ satisfies the conditions 1) and 2). We now show that it can be modified so that it will satisfy 3) as well.

Of course $\tilde{u}_n$ satisfies the zeroth compatibility condition because

$$\langle \tilde{u}_n, \nu \rangle = \langle \nabla g_{0k}, \nu \rangle = 0 \text{ on } \partial \Omega$$
We let \( \varphi_i = \varphi_i(u, \rho) \) be the function defining the \( i \)th compatibility condition; that is we define \( \varphi_i \) so that the \( i \)th condition for initial data \((u, \rho)\) is
\[
\varphi_i(u, \rho) = 0
\]
Then from section 3 we know that (letting \( f = \log \rho \))
\[
\varphi_1(u, \rho) = \langle \nabla_u u + c^2(\rho)\nabla f, \nu \rangle
\]  \hspace{1cm} (5.1)
To get formulas for \( \varphi_i, i > 1 \) it is convenient to use the quadratic form
\[
S_2(v_1, v_2) = -\langle v_1, \nabla v_2 \nu \rangle
\]
which we defined on section 3.

Then (3.6) tells us that
\[
\varphi_2(u, \rho) = -2S_2(u, \nabla_u u + c^2(\nabla f) + p''(\rho)(-\nabla_u f + \delta u)\nabla_f + c^2\nabla(\nabla_u f + \delta u).
\]  \hspace{1cm} (5.2)
To compute \( \varphi_3 \) we take a time derivative of (3.5) to get
\[
\varphi_3(u, \rho) = -2S_2(\partial_t u, \partial_t u) - 2S_2(\partial_t u, \partial_t u) + \partial_t^2(\nabla u f + 2\partial_t(c^2)\nabla_u \partial_t f + c^2\nabla_u \partial_t^2 f. \]
\]  \hspace{1cm} (5.3)
From our usual computations we get
\[
\partial_t u = -\nabla_u u + c^2 \nabla f,
\]
\[
\partial_t f = -\nabla_u f + \delta u,
\]
\[
\partial_t(c^2) = p''(\rho)\rho\partial_t f,
\]
\[
\partial_t^2 u = \nabla_{\partial_t u} u + \nabla_u \partial_t u - \partial_t(c^2)\nabla f - c^2\nabla \partial_t f,
\]
\[
\partial_t^2 f = -\nabla_{\partial_t u} f - \nabla_u \partial_t f + \delta \partial_t u = -\nabla_{\partial_t u} f - \nabla_u \partial_t f - \delta \nabla_u u - \delta c^2 \nabla f,
\]
and substituting these expressions for the time derivatives in (5.3) gives an expression for \( \varphi_3(u, \rho) \). Other \( \varphi \)'s are computed in the same way, each one involving one more derivative of (3.5).

Now let \( u \in H^3 \) (with \( \langle u, \nu \rangle = 0 \)), \( \rho \in H^4 \) and assume \( \delta u \in H^3 \). Then since \( \varphi_i \) involves \( i \) derivatives of \( \rho \) but only \( i - 1 \) derivatives of \( u \) and \( \delta u \), we find
\[
\varphi_i(u, \rho) \in H^{3^{1/2} - i}(\partial \Omega, \mathbb{R})
\]
Letting \( u = w + \nabla g \) be the usual decomposition we can think of each \( \varphi_i \) as a function of \( w, g, \) and \( f \). In this way we get
\[
\varphi_i : P(H^3(\Omega, \mathbb{R}^n)) \times \nabla(H^5(\Omega, \mathbb{R})) \times H^4(\Omega, \mathbb{R}) \to H^{3^{1/2} - i}(\partial \Omega, \mathbb{R}) \hspace{1cm} (i = 1, 2, 3)
\]
where \( H^5(\Omega, \mathbb{R}) = \{g \in H^5 \mid \nabla_g g = 0 \text{ on } \Omega\} \). We shall call the domain of this map \( X^{3,4,4} \). We use \( H^5(\Omega, \mathbb{R}) \) to insure that \( \langle u, \nu \rangle = 0 \) on \( \partial \Omega \) and that \( \delta u = -\Delta g \) is in \( H^3 \). Then if we let \( \widetilde{\Phi} = (\varphi_1, \varphi_2, \varphi_3) \) we get a smooth map
\[
\widetilde{\Phi} : X^{3,4,4} \to H^{3^{1/2}}(\partial \Omega, \mathbb{R}) \times H^{1^{1/2}}(\partial \Omega, \mathbb{R}) \times H^{1/2}(\partial \Omega, \mathbb{R}) \hspace{1cm} (5.4)
\]
We let \( Y \) denote the range of this map. Of course \( \widetilde{\Phi}(u_{ok}, \rho_{ok}) = 0 \) so \( \lim_{n \to \infty} \widetilde{\Phi}(u_n, \rho_n) = 0 \).
Now we show the existence of \( (u_n, \rho_n) \) in three steps. First we let
\[
X^{4,4,4} = \{(w, \nabla g, f) \in X^{3,4,4} \mid w \in H^4 \}
\]
and let \( \Phi \) be the restriction of \( \hat{\Phi} \) to \( X^{4,4,4} \). Second we show that the derivative of the map

\( \Phi : X^{4,4,4} \to Y \) at each point \( (\tilde{u}_n, \tilde{\rho}_n) \) is a surjection. Third we use an implicit function theorem type of argument to show that for large \( n \) there exists \((u_n, \rho_n)\) near \((\tilde{u}_n, \tilde{\rho}_n)\) such that \( \Phi(u_n, \rho_n) = 0 \). This sequence \( \{(u_n, \rho_n)\} \) will satisfy the required conditions 1), 2) and 3).

We proceed with the second step, computing the derivative of \( \Phi \). From (5.1) we see that

\[
D_{(\tilde{u}, \tilde{\rho})} \varphi_1(w, \nabla g, f) = -2S_2(\tilde{u}, w + \nabla g) + p''(\tilde{\rho}) f \nabla_\nu \tilde{f} + c^2(\tilde{\rho}) \nabla_\nu f
\]

where \( D_{(\tilde{u}, \tilde{\rho})} \varphi_1(w, \nabla g, f) \) means the derivative of \( \varphi_1 \) at \((\tilde{u}, \tilde{\rho})\) in direction \((w, \nabla g, f)\).

Assuming that \( \| \tilde{f} \|_4 \leq K \) we find that

\[
D_{(\tilde{u}, \tilde{\rho})} \varphi_1(w, \nabla g, f) = A_{11}(w, \nabla g, f) + c^2(\tilde{\rho}) \nabla_\nu f
\]

(5.5)

where \( A_{11} : X^{4,4,4} \to H^{3/2}_{\nu}(\partial \Omega, \mathbb{R}) \) is a bounded linear operator depending on \((\tilde{u}, \tilde{\rho})\) but with operator norm bounded by some \( K \) independent of \( k \).

Similarly from (5.2) we find

\[
D_{(\tilde{u}, \tilde{\rho})} \varphi_2(w, \nabla g, f) = A_{21}(w, \nabla g) + kA_{22}f + c^2(\tilde{\rho}) \nabla_\nu g,
\]

(5.6)

and from (5.3)

\[
D_{(\tilde{u}, \tilde{\rho})} \varphi_3(w, \nabla g, f) = A_{31}(w, \nabla g) + kA_{32}(\nabla g) + kA_{33}f - c^2(\tilde{\rho}) \nabla_\nu Lf
\]

(5.7)

where \( L = -\delta c^2(\tilde{\rho}) \nabla \), and \( A_{21}, A_{22} : X^{4,4,4} \to H^{11/2}_{\nu}(\partial \Omega, \mathbb{R}) \) are bounded operators with bounds independent of \( k \), as are \( A_{31}, A_{32}, A_{33} : X^{4,4,4} \to H^{3/2}_{\nu}(\partial \Omega, \mathbb{R}) \). Furthermore since the \( \varphi_i \)'s involve no more than two derivatives of \( \tilde{u} \) and \( \delta \tilde{u} \), the bounds on the operators \( A_{ij} \) are uniform for all \((\tilde{u}, \tilde{\rho})\) which are near \((u_{0k}, \rho_{0k})\) in \( X^{3,4,4} \). Formulas (5.5)-(5.7) give an expression for the linear operator \( D_{(\tilde{u}, \tilde{\rho})} \Phi : X^{4,4,4} \to Y \). From the above we know that it is uniformly bounded for all \((\tilde{u}, \tilde{\rho})\) in an \( X^{3,4,4} \)-neighborhood of \((u_{0k}, \rho_{0k})\).

Computing in the same way we get a formula for \( D^2_{(\tilde{u}, \tilde{\rho})} \Phi \), the second derivative of \( \Phi \) at \((\tilde{u}, \tilde{\rho})\) which is a bilinear map from \( X^{4,4,4} \times X^{4,4,4} \) to \( Y \). Using the same reasoning as above, we find that \( D^2_{(\tilde{u}, \tilde{\rho})} \Phi \) is bounded as a bilinear operator and again we see that the bound is uniform for \((\tilde{u}, \tilde{\rho})\) near \((u_{0k}, \rho_{0k})\) in \( X^{3,4,4} \).

Now we shall use (5.5)-(5.7) to find a right inverse to \( D_{(\tilde{u}, \tilde{\rho})} \Phi \). This amounts to finding a solution \((w, \nabla g, \nabla f)\) to the linear equations

\[
A_{11}(w, \nabla g, f) + c^2\nabla_\nu f = h_1
\]

(5.8)

\[
A_{21}(w, \nabla g) + kA_{22}f + c^2\nabla_\nu g = h_2
\]

(5.9)

\[
A_{31}(w, \nabla g) + kA_{32}(\nabla g) + kA_{33}f - c^2\nabla_\nu Lf = h_3
\]

(5.10)

where \( h_i \in H^{3/2}_{\nu - i}(\partial \Omega, \mathbb{R}) \), \( i = 1, 2, 3 \).

When \( k \) is large this system can be solved by a Neumann series technique. To begin, we let \( w = 0 \) so \( u = \nabla g \). Then we let

\[
f_1 = G_L(\frac{1}{c^2}h_1) - L^{-1}G_L(\frac{1}{c^2}h_3)
\]

\[
g_1 = \Delta^{-1}G(\frac{1}{c^2}h_2 - \frac{k}{c^2}A_{22}f_1)
\]
where \( G, G_L, \Delta^{-1} \) and \( L^{-1} \) are as in section 4.

Then
\[
\frac{1}{c^2}A_{11}(0, \nabla g_1, f_1) + \nabla_y f_1 = \frac{1}{c^2}h_1 + \frac{1}{c^2}A_{11}(0, \nabla g_1, f_1),
\]
\[
\frac{1}{c^2}A_{21}(0, \nabla g_1) + \frac{k}{c^2}A_{22}f_1 + \nabla_y \Delta g_1 = \frac{1}{c^2}h_2 + \frac{1}{c^2}A_{21}(0, \nabla g_1),
\]
\[
\frac{1}{c^2}A_{31}(0, \nabla g_1) + \frac{k}{c^2}A_{32}(\nabla g_1) + \frac{k}{c^2}A_{33}f_1 - \nabla_y Lf_1 = \frac{1}{c^2}(h_3 + A_{31}(0, \nabla g_1) + kA_{32}(\nabla g_1) + kA_{33}f_1)
\]
\[= \frac{1}{c^2}(h_3 + R_1),\]
where \( R_1 \) is defined by the last equality. Therefore let
\[
f_2 = f_1 + G_L\frac{1}{c^2}A_{11}(0, \nabla g_1, f_1) + L^{-1}G_L\frac{1}{c^2}(R_1)
\]
and let
\[
g_2 = g_1 + \Delta^{-1}G(\frac{1}{c^2}A_{21}(0, \nabla g_1) - \frac{k}{c^2}A_{22}(f_2 - f_1)).
\]
continuing we get a sequence \( \{g_\ell, f_\ell\} \) and it is clear that
\[
\| g_\ell - g_{\ell-1} \|_5 + \| f_\ell - f_{\ell-1} \|_4 \leq \frac{K}{\ell^4}.
\]
Therefore for \( k \) large we get the limits \( g_\ell \rightarrow g \) and \( f_\ell \rightarrow f \) and \( (0, \nabla g, f) \) satisfies the system (5.8)-(5.10). Furthermore it is clear that
\[
\| f \|_4 + \| \nabla g \|_4 \leq K(\| h_1 \|_{\partial, 21/2} + \| h_2 \|_{\partial, 11/2} + \| h_3 \|_{\partial, 3/2})
\]
where \( K \) depends only on the norms of the operators \( A_{ij} \). Therefore we have found a right inverse to \( D(\tilde{u}, \tilde{\rho})\Phi \) which is uniformly bounded for \( (\tilde{u}, \tilde{\rho}) \) near \( (u_{k0}, \rho_{k0}) \).

Now we proceed to our third step, the implicit function theorem argument, which will show the existence of the sequence \( (u_n, \rho_n) \). Our proof follows from the proof of the implicit function theorem. Let \( A_n : Y \rightarrow X^{4,4,4} \) be the right inverse to \( D(\tilde{u}_n, \tilde{\rho}_n)\Phi \) which we constructed above, and let
\[
\psi_n : Y \rightarrow Y
\]
be defined by
\[
\psi_n(y) = \Phi(A_n(y) + (\tilde{u}_n, \tilde{\rho}_n)) - \Phi(\tilde{u}_n, \tilde{\rho}_n).
\]
Clearly \( \psi_n \) takes zero to zero and its derivative at zero is the identity map from \( Y \) to \( Y \). Also, since \( D^2 \Phi \) and \( A_n \) are bounded so is \( D^2 \psi_n \). Assuming \( D^2 \psi_n : Y \times Y \rightarrow Y \) has bound \( K \), a standard estimate tells us that for \( \epsilon < \frac{1}{4K} \), \( I - \psi_n : B_{2\epsilon}(0) \rightarrow B_{\epsilon}(0) \) is a contraction with Lipshitz constant \( \frac{1}{4} \). \( B_{\epsilon}(0) \) is the ball about zero of radius \( \epsilon \) in \( Y \).

Now given \( z \in B_{\epsilon}(0) \), let \( \theta : Y \rightarrow Y \) be defined by
\[
\theta(y) = y - \psi_n(y) + z = (I - \psi_n)(y) + z.
\]
Then \( \theta : B_{2\epsilon}(0) \rightarrow B_{2\epsilon}(0) \) is also a contraction and therefore has a unique fixed point, call it \( y_n \).
Given any positive \( \epsilon \) which is less than \( \frac{1}{10} \), we can find \( n \) so that \( \| \Phi(\tilde{u}_n, \tilde{\rho}_n) \|_Y < \epsilon \). Then let \( z = \Phi(\tilde{u}_n, \tilde{\rho}_n) \), so that \( y_n \) satisfies:

\[
y_n = y_n - \psi_n(y_n) + \Phi(\tilde{u}_n, \tilde{\rho}_n) = y_n - \Phi(A_n(y_n) + (\tilde{u}_n, \tilde{\rho}_n)).
\]

Thus \( \Phi(A_n(y_n) + (\tilde{u}_n, \tilde{\rho}_n)) = 0 \) so \( (u_n, \rho_n) = A_n(y_n) + (\tilde{u}_n, \tilde{\rho}_n) \) satisfies the compatibility conditions. Since \( \| y_n \| < 2\epsilon \), \( \| (u_n, \rho_n) - (\tilde{u}_n, \tilde{\rho}_n) \|_{X^{3.4.4}} < 2\epsilon \). Therefore \((u_n, \rho_n)\) converges to \((u_0, \rho_0)\) in \( X^{3.4.4} \), and hence is the desired sequence.

Now that we have the sequence \((u_n, \rho_n)\), Theorem 3.1 follows from the estimates of section 4. From [E3] we know that for each pair \((u_n, \rho_n)\) we get curves \((u_n(t), \rho_n(t))\) in \( H^4 \) defined on some interval \([0, T_n]\); we take \( T_n \) to be as large as possible.

However since \((u_n, \rho_n) \rightarrow (u_0, \rho_0)\) in \( H^3 \) we know that if \( n \) is large, \( \{(u_n(t), \rho_n(t))\} \) and \( (u(t), \rho(t)) \) all exist as \( H^3 \) functions on \([0, T]\), and \((u_n(t), \rho_n(t))\) converges to \((u(t), \rho(t))\) in \( H^3 \). Therefore there exist constants \( a_2 \) and \( a_3 \) as required in section 4 so that (4.1) and (4.2) hold for the functions \( u_n(t) \) and \( f_n(t) = \log \rho_n(t) \), and since \((u_n, \rho_n) \rightarrow (u, \rho)\) in \( H^3 \) one can choose one set of constants for all \( n \). Hence we find a constant \( a_5 \) such that (4.4a)-(4.4d) hold for all \( f_n \) also. But it is well known (see [K2] for example) that if \( h_n \rightarrow h \) in \( H^s \) and \( \{h_n\} \) is a bounded sequence in \( H^{s+1} \), then in fact \( h \in H^{s+1} \). Therefore \( f_n \rightarrow f = \log \rho \) in \( H^3 \) implies \( f \in H^4 \), and \( \dot{f}_n \rightarrow \dot{f} \) implies \( \dot{f} \in H^3 \). Hence we have \( \rho \in H^4 \) and \( \dot{f} = \delta u \in H^3 \). This completes the proof of theorem 3.1.

To prove Theorem 3.2 we first state an equivalent theorem and prove the latter using the estimates of section 4 together with the proof of theorem 5.5 of [E2].

In order to state the equivalent theorem we will use the map \( \zeta(t) : \Omega \rightarrow \Omega \) to describe the compressible fluid position at time \( t \), and \( \eta(t) : \Omega \rightarrow \Omega \) to describe the incompressible position. We think of both \( \zeta \) and \( \eta \) as curves in \( \mathcal{D}^s \), with \( \zeta(0) = \eta(0) = \text{id} \), the identity diffeomorphism. If we use \( "\cdot\)" to denote time derivatives of \( \eta \) and \( \zeta \) we have (as in section 1)

\[
\dot{\eta}(t)(x) = v(t, \eta(t)(x)) = v(t)(\eta(t)(x)) \tag{5.11}
\]

so

\[
v(t)(x) = \dot{\eta}(t) \circ (\eta(t))^{-1}(x)
\]

and

\[
\dot{\zeta}(t)(x) = u(t, \zeta(t)(x)) = u(t)(\zeta(t)(x)) \tag{5.12}
\]

so

\[
u(t)(x) = \dot{\zeta}(t) \circ (\zeta(t))^{-1}(x) \tag{5.13}
\]

where \((\eta(t))^{-1}\) and \((\zeta(t))^{-1}\) are the inverses of the diffeomorphisms \( \eta(t), \zeta(t) : \Omega \rightarrow \Omega \) and \( \circ \) means composition. Therefore

\[
\ddot{\eta}(t)(x) = \partial_t v(t, \eta(t)(x)) + \nabla v(t, \eta(t)(x)) = \dot{v}(t, \eta(t)(x)) = \dot{v}(t)(\eta(t)(x))
\]

and similarly

\[
\ddot{\zeta}(t)(x) = \partial_t u(t, \zeta(t)(x)) + \nabla u(t, \zeta(t)(x)) = \dot{u}(t, \zeta(t)(x)) = \dot{u}(t)(\zeta(t)(x))
\]

Hence (2.12) is equivalent to the equation

\[
\ddot{\eta}(t)(x) = (Q(\nabla v))(\eta(t)(x)) \tag{5.14}
\]
and (2.1a) is equivalent to
\[
\ddot{\zeta}(t)(x) = -\left(\frac{1}{\rho} \nabla p\right)(\zeta(t)(x)) = -(c^2 \nabla f)(\zeta(t)(x))
\] (5.15)

Since (5.15) involves \( \rho \) or \( f \) we must find a relation between one of them and \( \zeta \). If we denote by \( J(\zeta(t)) \) the Jacobian determinant of \( \zeta(t) \), a direct computation gives
\[
\frac{\partial}{\partial t}(J(\zeta(t)))(t, x) = J(\zeta(t))(t, x)(\text{div}(u)(t, \zeta(t)(x))).
\]

Thus we find that if we let
\[
h(t, x) = -\log(J(\zeta(t)))((\zeta(t)^{-1})(x))
\]
then \( h \) satisfies
\[
\dot{h} = -\delta u.
\]

Therefore since \( f \) satisfies (2.6) with initial condition \( f(0, x) = f_{0k}(x) \) and since \( h(0, x) = \log(J(\text{id})) = 0 \) we must have
\[
f(t, x) = f_{0k}(x) + h(t, x)
\] (5.16)

Combining (5.15) and (5.16) we find that \( (u, f) \) satisfy (2.1a) and (2.6) with initial condition \( f(0, x) = f_{0k}(x) \) if and only if \( \zeta \) satisfies
\[
\ddot{\zeta}(t)(x) = -(c^2 \nabla (f_{0k} + h))(t, \zeta(t)(x))
\] (5.17)

where \( h \) is defined in terms of \( \zeta \) as above and \( c^2 \) depends on \( \rho \) or \( f \) which in turn depends on \( h \). Also we note that if \( \zeta \) and \( \eta \) are curves in \( \mathcal{D}^* \), and \( u \) and \( v \) are defined in terms of \( \zeta \) and \( \eta \) using (5.11)-(5.12), then we find that \( u \) and \( v \) must satisfy the boundary conditions
\[
\langle u, \nu \rangle = \langle v, \nu \rangle = 0 \text{ on } \partial \Omega.
\]

Now we state the equivalent theorem

**Theorem 5.1.** Assume that \( u_{0k}, \rho_{0k}, p_k, \) and \( v_0 \) are as in Theorem 3.2. Then there exist an interval \( [0, T] \) and a unique smooth curve \( \eta : [0, T] \to \mathcal{D}^* \) satisfying (5.14) such that \( \eta(0) = \text{id}, \) the identity diffeomorphism, and \( \dot{\eta}(0) = v_0 \). Also for each \( k \), there exists a unique \( C^2 \) curve \( \zeta_k(t) \) in \( \mathcal{D}^* \) defined on an interval \( [0, T(k)] \) satisfying (5.17), and such that \( \zeta_k(0) = \text{id}, \) \( \dot{\zeta}_k(0) = u_{0k} \).

Furthermore if \( T(k) \) is maximal, then \( T(k) > T \) for large \( k \), and as \( k \to \infty, \zeta_k(t) \to \eta(t) \) as a \( C^1 \) curve in \( \mathcal{D}^* \). In addition
\[
J(\zeta_k(t)) \circ \zeta_k(t)^{-1} \to 1 \text{ in } H^{s+1}
\]

It is clear that Theorem 5.1 implies 3.2; one simply let \( u_k(t) = \dot{\zeta}_k(t) \circ (\zeta(t))^{-1}, \) \( v(t) = \eta(t) \circ (\eta(t))^{-1} \). Then \( \zeta_k \to \eta \) in \( C^1 \) imply that \( u_k \to v \) in \( C^0 \). Also since
\[
\rho_k(t) = \frac{\rho_{0k}}{J(\zeta_k(t)) \circ (\zeta_k(t)^{-1})}
\]
we find \( \rho_k \to 1 \) in \( H^{s+1} \).
We now proceed with the proof of Theorem 5.1. It is essentially the same as the proof of Theorem 5.5 of [E2] so we shall go through it rather briefly expecting the reader to refer to

\[ E2 \]

for details.

First we define\( Z : \mathcal{D}^3 \times H^3(\Omega, \mathbb{R}^n) \to H^3(\Omega, \mathbb{R}^n) \) by

\[
Z(\xi, \alpha) = (Q(\nabla_{\alpha\xi^{-1}} P(\alpha \circ \xi^{-1}))) \circ \xi
\]

as in \( E2 \) section 13. Then since \( v = \dot{\eta} \circ \eta^{-1} \) and \( P(v) = v \) we find that (5.14) can be written

\[
\ddot{\eta} = Z(\eta, \dot{\eta}).
\]

Solving this with initial condition \( v_0 \) is of course the same as solving

\[
\dot{\eta}(t) = v_0 + \int_0^t Z(\eta(s), \dot{\eta}(s))ds.
\]

(5.18)

We will now find an equation similar to (5.18) for \( \zeta = \zeta_k \). Note that

\[
\dot{\zeta} = u \circ \zeta = w \circ \zeta + \nabla g \circ \zeta
\]

where \( u = w + \nabla g \) is the decomposition of \( u \) into divergence free and gradient parts.

But from (2.15) we find that

\[
\frac{\partial w}{\partial t} + \nabla_u w = Q(\nabla_u w) - P(\nabla_w \nabla g)
\]

and therefore

\[
(w \circ \zeta) = (Q(\nabla_u w) - P(\nabla_w \nabla g)) \circ \zeta.
\]

Define \( R : \mathcal{D}^3 \times H^3(\Omega, \mathbb{R}^n) \to H^3(\Omega, \mathbb{R}^n) \) by

\[
R(\xi, \alpha) = (P(\nabla_{\alpha\xi^{-1}} Q(\alpha \circ \xi^{-1}))) \circ \xi
\]

as in \( E2 \) section 13. Then

\[
\dot{\zeta} = Z(\zeta, \dot{\zeta}) - R(\zeta, \dot{\zeta})
\]

so

\[
\dot{\zeta} = P(u_{0k}) + \int_0^t (Z(\zeta(s), \dot{\zeta}(s)) - R(\zeta(s), \dot{\zeta}(s)))ds + (\nabla g(t)) \circ \zeta(t).
\]

(5.19)

We shall show that this equation is close to (5.18), and from this the theorem will follow.

We let \( T_1(k) = \min\{T, T(k)\} \) and assume that \( (\zeta_k, \dot{\zeta}_k) : [0, T_1(k)) \to \mathcal{D}^3 \times H^3(\Omega, \mathbb{R}^n) \) is within \( \epsilon \) of the curve \( (\eta, \dot{\eta}) : [0, T_1(k)) \to \mathcal{D}^3 \times H^3(\Omega, \mathbb{R}^n) \). This gives a constant \( a_3 \) such that (4.2) holds. The hypothesis of Theorem 3.1 implies that (4.1) holds so if \( k \) is large, the estimates of section 4 are valid and we find \( a_5 \) such that

\[
\| f_k(t) \|_4 \leq \frac{a_5}{k} \quad \text{and} \quad \| \tilde{f}_k(t) \|_3 \leq \frac{a_5}{\sqrt{k}}
\]

But \( \nabla g = \nabla \Delta^{-1} \tilde{f} \) and \( R(\zeta, \dot{\zeta}) = P(\nabla_w \nabla g) \circ \zeta \) so we get a constant \( a_6 \) depending on \( \epsilon \), such that

\[
\| \nabla g \circ \zeta \|_3 \leq \frac{a_6}{\sqrt{k}}
\]
and

\[ \| R(\zeta, \dot{\zeta}) \|_3 \leq \frac{a_6}{\sqrt{k}} \]

Also as \( k \to \infty \), \( u_{ok} \to v_0 \), so \( P(u_{ok}) \to v_0 \) also. Thus we find

\[ \| \dot{\eta}(t) - \dot{\zeta}_k(t) \|_3 \leq \| P(u_{ok}) - v_0 \|_3 + (1 + t) \frac{a_6}{\sqrt{k}} \]

\[ + \int_0^t \| Z(\eta(s), \dot{\eta}(s)) - Z(\zeta_k(s), \dot{\zeta}_k(s)) \|_3 \, ds. \]  \tag{5.20}

Iterating this inequality we find that for large \( k \) we can find \( \epsilon_2 < \epsilon \) such that if \( t \in [0, T_1(k)] \)

\[ \| (\zeta_k(t), \dot{\zeta}_k(t)) - (\eta(t), \dot{\eta}(t)) \|_3 \leq \epsilon_2. \]

Furthermore, since \( \| f \|_4 \leq \frac{a_6}{k} \) we can pick \( a_6 \) so that \( \| \rho_k - 1 \|_4 \leq \frac{a_6}{k} \) also. Therefore we find that since \( T(k) \) is maximal \( T(k) > T_1 \) so \( T = T_1(k) \). Also as \( k \to \infty \), \( \epsilon_2 \) can be taken arbitrarily small. Hence \( \zeta_k \to \eta \) as a \( C^1 \) curve in \( \mathcal{D}^3 \) and \( \rho_k \to 1 \) in \( H^4 \), so \( J(\zeta_k) \circ \zeta_k^{-1} \to 1 \) in \( H^4 \) also. \( \zeta \) is a \( C^2 \) curve in \( \mathcal{D}^3 \) because \( f \in H^4 \) and by (5.15), \( \dot{\zeta} = -(c^2 \nabla f) \circ \zeta \in H^3(\Omega, \mathbb{R}^n) \).

This concludes the proof.

**Remark 5.1.** It is most curious that \( \rho(t) \) or \( f(t) \) are in \( H^4 \) while \( u(t) \) or \( \zeta(t) \) are only in \( H^3 \).

In the first place \( \rho = J(\zeta) \circ \zeta^{-1} \) involves first derivatives of \( \zeta \) so one would expect \( \rho \) to be less rather than more differentiable. Secondly, \( (u, \rho) \) satisfies (2.1a)-(2.1b) which is a quasi-linear symmetric hyperbolic system. The usual methods of solution of such systems would produce equally differentiable \( u \) and \( \rho \).

## 6. Differentiable Dependence on Initial Conditions

In this section we shall show that the solution of the compressible fluid problem depends differentiably on the initial conditions \( u_{ok}, \rho_{ok} \). This solution, \( u_k(t), \rho_k(t) \) is given by Theorem 3.1.

As we explained in section 5, the motion described by \( u_k(t), \rho_k(t) \) (which we sometimes call \( u(t), \rho(t) \)) can be equivalently described by \( \zeta(t) : \Omega \to \Omega \) where \( \zeta(t) \) is a differentiable curve in \( \mathcal{D}^s \), and

\[ \dot{\zeta}(t) = u(t)(\zeta(t)(x)) \]  \tag{6.1}

We shall show that for fixed \( t \), \( \zeta(t) \) and \( \dot{\zeta}(t) \) depend differentiably on \( u_{ok}, \rho_{ok} \). First we note that \( u_k(t), \rho_k(t) \) are given by theorem 3.1 only when \( u_{ok} \in H^s, \delta u_{ok} \in H^s, \) and \( \rho_{ok} \in H^{s+1} \), and when \( u_{ok} \) satisfies some inequalities and the compatibility conditions.

Thus (taking \( s = 3 \)), we let

\[ \mathcal{I} = \{ (u, \rho) \in X^{3,4,4} \mid (u, \rho) \text{ satisfy the inequalities of Theorem 3.1, and } \Phi(u, \rho) = 0 \} \]

where \( \Phi : X^{3,4,4} \to Y \) is defined in (5.4). The set \( \mathcal{I} \) is then the set of possible initial conditions for Theorem 3.1.

Given \( t \) we define \( \psi_t(u, \rho) = (\zeta(t), \dot{\zeta}(t)) \) where \( \zeta(t) \) is the compressible fluid motion with initial data \( (u, \rho) \). Then \( \psi_t : U_t \to \mathcal{D}^3 \times H^3(\Omega, \mathbb{R}^n) \) where \( U_t \) is the set of initial conditions for which the fluid motion is defined for at least time \( t \). From Theorem 3.1 it follows that \( U_t \) is open in \( \mathcal{I} \).
The goal of this section is to show the following:

**Theorem 6.1.** $U_t$ is a submanifold of $X^{3,4,4}$ and $\psi_t : U_t \to \mathcal{D}^3 \times H^3(\Omega, \mathbb{R}^n)$ is $C^1$.

**Proof.** Since $U_t$ is open in $\mathcal{I}$, to show that $U_t$ is a submanifold it suffices to check that $\mathcal{I}$ is a submanifold. This we proceed to do.

First let

$$\mathcal{I} = \{(u, \rho) \in X^{3,4,4} \mid (u, \rho) \text{ satisfy the inequalities of Theorem } 3.1 \}$$

Clearly $\mathcal{I}$ is open in $X^{3,4,4}$. Then let $\tilde{\Phi} : \mathcal{I} \to Y$ be defined as in (5.4). As we showed in section 5, $\tilde{\Phi}$ is a smooth map and its derivative at any point $(u, \rho)$ is a surjective map from $X^{3,4,4}$ to $Y$. Hence $\mathcal{I} = \{(u, \rho) \in \mathcal{I} \mid \tilde{\Phi}(u, \rho) = 0\}$ is a submanifold of $\mathcal{I}$ or $X^{3,4,4}$.

Now we must show that $\psi_t : U_t \to \mathcal{D}^3 \times H^3(\Omega, \mathbb{R}^n)$ is differentiable, and to do so we will use equation (5.19). Note that $\nabla g$ of that equation is equal to $\nabla \Delta^{-1} \dot{f}$ where $f$ is the solution of (2.9). Of course $f$ and therefore $\nabla g$ depend on the initial data $(u, \rho)$ so we define for each $t$

$$\Xi_t : U_t \to H^3(\Omega, \mathbb{R}^n),$$

$$\Xi_t(u, \rho) = \nabla g.$$

We shall need the following proposition which will be proven in section 7.

**Proposition 6.1.** $\Xi_t : U_t \to H^3(\Omega, \mathbb{R}^n)$ is a $C^1$ map.

**Remark 6.2.** Since $\dot{f}$ is in $H^3$, $\nabla g$ is actually in $H^4$. However, the map $(u, \rho) \mapsto \nabla g$ is continuous in $H^4$, but probably not differentiable.

Given Proposition 6.1, our proof that $\psi_t$ is differentiable will be a modification of the proof that the solution of an ordinary differential equation depends differentiably on initial conditions (see for example [L]). In this case the equation will be (5.19).

We shall need the following.

**Lemma 6.3.** Let $w \in H^4(\Omega, \mathbb{R}^n)$. Then the map $\zeta \mapsto w \circ \zeta$ is a $C^1$ map from $\mathcal{D}^3$ to $H^3(\Omega, \mathbb{R}^n)$.

**Proof.** This is just a calculus lemma, see [E1], [BB], or [EM].

To show that $\psi_t$ is differentiable, it suffices to show that $(u, \rho) \mapsto \dot{\zeta}$ is differentiable, because

$$\zeta(t) = \text{id} + \int_0^t \dot{\zeta}(s) ds. \quad (6.2)$$

To show that $\dot{\zeta}$ depends differentiably on $(u, \rho)$ we rewrite (5.19), using (6.2) to replace $\zeta$ with $\dot{\zeta}$.

First we rewrite the terms of (5.19), letting

$$A(u, \rho) = Pu,$$

$$B(\zeta, \dot{\zeta}) = Z(\zeta, \dot{\zeta}) - R(\zeta, \dot{\zeta})$$

$$C(u, \rho, \zeta, t) = (\nabla g(t)) \circ \zeta(t) = \Xi_t(u, \rho) \circ \zeta(t)$$
Then (5.19) becomes
\[ \dot{\zeta}(t) = A(u, \rho) + \int_0^t B\left(\zeta(s), \dot{\zeta}(s)\right) ds + C(u, \rho, \zeta, t) \]
or
\[ \dot{\zeta}(t) = A(u, \rho) + \int_0^t B\left(\text{id} + \int_0^s \dot{\zeta}(s') ds', \dot{\zeta}(s)\right) ds + C(u, \rho, \zeta, t). \] (6.3)

Now we use an argument from [L], to get the required differentiability. Let
\[ X = C([0, t], H^3(\Omega, \mathbb{R}^n)) \]
be the Banach space of continuous curves from [0, t] to \( H^3(\Omega, \mathbb{R}^n) \) with norm
\[ \|z\| = \sup_{0 \leq s \leq t} \{\|z(s)\|_3\}. \]

Let \( T : U_t \times X \to X \) be defined by
\[ T(u, \rho, z)(s) = A(u, \rho) + \int_0^s B\left(\text{id} + \int_0^\ell z(t') dt', z(\ell)\right) d\ell \]
\[ + C(u, \rho, \text{id} + \int_0^s z(\ell) d\ell, s) - z(s). \] (6.4)

Checking carefully, we see that \( z(t) \) is a solution of (6.3) if and only if \( T(u, \rho, z) = 0 \). Thus given \((u, \rho)\) we find that
\[ T(u, \rho, \dot{\zeta}) = 0 \]

We now use the implicit function theorem to show that \( \dot{\zeta} \) is a \( C^1 \) function of \((u, \rho)\). First note that the maps \( A, B, \) and \( C \) are all \( C^1 \): \( A : X^{3,4,4} \to H^3(\Omega, \mathbb{R}^n) \) is a continuous linear map; \( B : \mathcal{D} \times H^3(\Omega, \mathbb{R}^n) \to H^3(\Omega, \mathbb{R}^n) \) is \( C^\infty \) since \( Z \) and \( R \) are \( C^\infty \) as is shown in [E2]; \( C : U_t \times \mathcal{D} \times [0, t] \to H^3(\Omega, \mathbb{R}^n) \) is \( C^1 \) by Proposition 6.1 and Lemma 6.3. Therefore \( T \) is also \( C^1 \).

Now we compute the partial derivative of \( T \) with respect to its last (i.e, \( z \)) variable. Using (6.4) we find that the \( z \)-partial derivative of \( T \) in direction \( \xi \) is:
\[ D_z T(u, \rho, z)(\xi)(s) = \int_0^s DB\left(\text{id} + \int_0^\ell z(t') dt', z(\ell)\right) \left(\int_0^\ell \xi(t') dt', \xi(\ell)\right) d\ell \]
\[ + DC\left(u, \rho, \text{id} + \int_0^s z(\ell) d\ell, s\right) + 0, 0, 0, \int_0^s \xi(\ell) d\ell) - \xi(s). \]
(6.5)

Therefore we can write
\[ D_z T(u, \rho, z)\xi = \epsilon(u, \rho, z)\xi - \xi \]
where \( \epsilon(u, \rho, z) : X \to X \) is a linear map defined by (6.5).

Since \( \epsilon(u, \rho, z) : X \to X \) involves \( \int_0^t \xi \), we can pick \( t > 0 \) small enough that the operator norm of \( \epsilon(u, \rho, z) \) is less than 1. In that case \( D_z T(u, \rho, z) : X \to X \) is an invertible map.

Now fix a fluid motion \( \zeta \) with initial data \((u, \rho)\), so \( T(u, \rho, \dot{\zeta}) = 0 \). Then if \( D_z T(u, \rho, \dot{\zeta}) \) is invertible, the implicit function theorem tells us that there is a neighborhood \( V_t \) of \((\bar{u}, \bar{\rho}) \) in \( U_t \) and a \( C^1 \) map \( \Lambda : V_t \to X \) such that for any \((\bar{u}, \bar{\rho}) \in V_t, T(\bar{u}, \bar{\rho}, \Lambda(\bar{u}, \bar{\rho})) = 0 \). Thus \( \dot{\zeta} = \Lambda(\bar{u}, \bar{\rho}) \)
gives a fluid motion with initial data \((\tilde{u}, \tilde{\rho})\). But if \(t\) is small, \(D_z T(u, \rho, \dot{\zeta})\) is invertible, so we get the \(C^1\) map \(\psi_t\). Therefore since \(\psi_t(\tilde{u}, \tilde{\rho}) = (\tilde{u}, \tilde{\rho})(t)\), \(\psi_t : U_t \to H^3(\Omega, \mathbb{R}^n)\) is \(C^1\) as well.

To show that \(\psi_t\) is \(C^1\) for large \(t\), we note that for \(0 < t_1 < t\)
\[
\dot{\zeta}(t) = P(\dot{\zeta}(t_1) \circ \zeta^{-1}(t_1)) \circ \zeta(t_1) + \int_{t_1}^t B(\zeta(t_1)) + \int_{t_1}^s \dot{\zeta}(s')ds', \dot{\zeta}(s))ds
\]
\[
+ C(u, \rho, \zeta(t_1) + \int_{t_1}^t \dot{\zeta}(s)ds, t) \tag{6.6}
\]

Using (6.6) instead of (6.3), and defining \(T\) analogously, we find that \(\dot{\zeta}(t)\) is a \(C^1\) function of \((u, \rho, \zeta(t_1), \dot{\zeta}(t_1))\). Iterating this argument we get that \(\dot{\zeta}(t)\) is a \(C^1\) function of \((u, \rho, \dot{\zeta}(0)) = (u, \rho, u)\). It follows that \(\psi_t\) is \(C^1\) for all \(t\), so Theorem 6.1 is proven. \(\square\)

**Remark 6.4.** The map \((u_0, \rho_0) \mapsto u(t)\) defined from \(U_t\) to \(H^3(\Omega, \mathbb{R}^n)\) cannot be shown to be \(C^1\) by this method. The use of Lagrange coordinates is essential. Indeed \((u_0, \rho_0) \mapsto u(t)\) is probably not \(C^1\) or even Hölder continuous. See the counter example of [K2] and also section 7 of this paper.

7. **Differentiable Dependence on Initial Conditions for Quasi-linear Symmetric-Hyperbolic Equations. Proof of Proposition 6.1**

The specific goal of this section is the proof of Proposition 6.1, which says that the gradient part of a compressible fluid velocity depends differentiably on the initial conditions. However, since the proof is basically the same for a general quasi-linear symmetric hyperbolic system, we will discuss the general case as well.

Consider the initial value problem
\[
\begin{cases}
\partial_t u + \sum_{i=1}^n a_i(t, x, u)\partial_i u = f(t, x, u) \\
u(0, x) = u_0(x)
\end{cases} \tag{7.1}
\]
where \(0 \leq t \leq T\), \(x \in \mathbb{R}^n\), \(f\) and \(u\) take values in \(\mathbb{R}^m\) and \(a_i(t, x, u)\) is a symmetric \(m \times m\) matrix. To avoid extra technicalities, we will assume that \(\{a_i\}\) and \(f\) are smooth.

It is well known (see [K2] for example) that for small \(t\), this problem has a unique solution. We shall discuss this solution using the results and function spaces of [K2]. However, in order to avoid imposing conditions at \(|x| \to \infty\), we shall assume that \(a_i, f, u_0\) and \(u\) are all periodic in \(x\), or equivalently \(x \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n\).

**Theorem 7.1.** If \(s > \frac{n}{2} + 1\), then for each \(u_0 \in H^s(\mathbb{T}^n)\) there exists a \(T > 0\) and a unique solution \(u(t, x)\) on \([0, T] \times \mathbb{T}^n\)

**Proof.** See [K2]. \(\square\)

Letting \(u(t, x) = u(t)(x)\), we consider \(u\) as a curve of functions on \(\mathbb{T}^n\). This curve is continuous in \(H^s(\mathbb{T}^n)\), that is, it is an element of
\[
CH^s([0, T]) = C^0([0, T], H^s(\mathbb{T}^n))
\]
Also there exists a neighborhood $U$ of $u_0$ in $H^s(\mathbb{T}^n)$, such that for each $v_0$ in $U$, there exists a solution $v$ of (7.1) with $v \in CH^s([0, T])$ and $v(0) = v_0$. Furthermore $v$ depends continuously on $v_0$.

Counter examples in [K2] show that $u(t) \in H^s(\mathbb{T}^n)$ do not depend differentiably (or even Hölder continuously) on $u_0$ however. We shall investigate this phenomenon and show that the dependence of $u$ on $u_0$ is $C^1$ if we reduce $s$ by one in the range. That is, the map from $u_0 \in H^s$ to $u \in CH^{s-1}([0, T])$ is $C^1$. Then using this idea, we will prove Proposition 6.1.

First we consider a $C^1$-curve of initial data $u^\lambda_0$ and let $u^\lambda(t)$ be a solution of (7.1) with initial data $u^\lambda_0$. When $\lambda = 0$ we suppress the superscript so that $u^0_0 = u_0$ and $u^0(t) = u(t)$. Let us assume that all derivatives exist and let

$$z(t) = \partial_\lambda(u^\lambda(t))|_{\lambda=0}.$$  

Then differentiating (7.1) we find:

$$\partial_t z + \sum_{i=1}^n a_i(t, x, u)\partial_i z + \sum_{i=1}^n \partial_u(a_i(t, x, u))z\partial_i u = \partial_u f(t, x, u)z$$

(7.2)

and we rewrite this as

$$\partial_t z + A(u)\nabla z + B(u, \nabla u)z = F(u)z.$$  

(7.3)

This equation is linear symmetric-hyperbolic and such equations are also analyzed in [K2]. The solution of (7.2) with $H^s$ initial data is shown to be a continuous curve in $H^{s-1}$. It is not shown to be in $H^s$ because the operator $B(u, \nabla u)$ is multiplication by $H^{s-1}$-functions (since it involves $x$-derivatives of $u$). Thus $B(u, \nabla u) : H^{s-1} \rightarrow H^{s-1}$ but $B(u, \nabla u)(H^s) \not\subset H^s$.

From this we might suspect that $\partial_\lambda(u^\lambda)$ exists as a curve in $CH^{s-1}$, but not generally in $CH^s$. We shall show that this is in fact what happens.

To begin with, we present a simple example which will help us both with the symmetric hyperbolic system and with the proof of proposition 6.1 (compare to the counter example of [K2]). Let $n = m = 1$, and let $u^\lambda_0$ be a $C^1$ curve in $H^s(\mathbb{T}, \mathbb{R})$ ($s > \frac{3}{2}$) parameterized by $\lambda$. Then for each $\lambda$, let $u^\lambda$ be the solution of

$$\partial_t u^\lambda + u^\lambda \partial_x u^\lambda = 0, \quad u^\lambda(0) = u^\lambda_0.$$  

(7.4)

(this is a one dimensional compressible fluid motion with $p(\rho) = 0$). Such a solution is easily found: Let $\zeta^\lambda(t)$ be a curve in $\mathscr{D}^s$ defined by $\zeta^\lambda(t) = x + tu^\lambda_0(x)$, so $\dot{\zeta}^\lambda(t) = u^\lambda_0$. Then let $u^\lambda(t) = \zeta^\lambda(t) \circ (\zeta^\lambda(t))^{-1} = u^\lambda_0 \circ (\zeta^\lambda(t))^{-1}$ (compare with (5.12) ). Clearly:

$$u^\lambda(t) \circ \zeta^\lambda(t) = u^\lambda_0$$

(7.5)

and differentiating (7.5) with respect to $t$, we find that $u^\lambda$ satisfies (7.4). Since $\zeta^\lambda(t)$ is a continuous curve in $\mathscr{D}^s$ (at least for $t$ near zero), $(\zeta^\lambda(t))^{-1}$ is also continuous in $\mathscr{D}^s$. Hence $u^\lambda(t)$ is continuous in $H^s(\mathbb{T}, \mathbb{R})$. Thus if $T$ is small enough so that $\zeta^\lambda(t) \in \mathscr{D}^s$ for all $t$ between zero and $T$, then $u^\lambda \in CH^s([0, T])$. Also it is clear from the construction that $u^\lambda$ depends continuously on $\lambda$ and that we can in fact find numbers $T$ and $\lambda$ such that the map $\lambda \mapsto u^\lambda$ is continuous from $(-\lambda, \lambda)$ to $CH^s([0, T])$. 

Now we consider differentiability with respect to $\lambda$. Let $z_0 = \partial_\lambda(u_0^\lambda)|_{\lambda=0}$ and $z = \partial_\lambda(u^\lambda)|_{\lambda=0}$. Then $\partial_\lambda(\zeta^\lambda(t))|_{\lambda=0} = tz$, so differentiating (7.5) with respect to $\lambda$ and letting $\lambda = 0$, we get

$$z(t) \circ \zeta(t) + (\partial_x u(t)) \circ \zeta(t) t z_0 = z_0$$

(7.6)

where we omit the “$\lambda$” when $\lambda = 0$. Also applying $\partial_\lambda$ to (7.5) we get:

$$((\partial_x u^\lambda(t)) \circ \zeta^\lambda(t))\partial_x(\zeta^\lambda(t)) = \partial_x u_0^\lambda.$$ 

But

$$\partial_x(\zeta^\lambda(t)) = 1 + t\partial_x u_0^\lambda$$

so

$$\partial_x(u^\lambda(t)) \circ \zeta^\lambda(t) = \frac{\partial_x u_0^\lambda}{1 + t\partial_x u_0^\lambda}.$$ 

(7.7)

Combining (7.6) and (7.7) we find:

$$z(t) \circ \zeta(t) = z_0 \left(1 - \frac{t\partial_x u_0^\lambda}{1 + t\partial_x u_0^\lambda}\right) = z_0 \left(\frac{1}{1 + t\partial_x u_0^\lambda}\right).$$

(7.8)

From (7.8) we see that $z$ is an element of $CH^{s-1}$ which depends continuously on $u_0$, and from this it follows that $u \in CH^{s-1}$ is a $C^1$-function of $u_0$.

Differential dependence on $u_0$ can also be shown by another method which will prove useful later. Differentiating (7.4) with respect to $\lambda$ we find that $z$, if it exists, must satisfy:

$$\partial_\lambda z + u\partial_\lambda z + (\partial_x u)z = 0.$$ 

(7.9)

This equation is of the same type as (7.3), so it can be solved for $z \in CH^{s-1}$ by the methods of [K2]. Also the difference quotient $z^\lambda = \frac{1}{\lambda}(u^\lambda - u)$ satisfies

$$\partial_\lambda z^\lambda + u\partial_\lambda z^\lambda + (\partial_x u^\lambda)z^\lambda = 0, \quad z^\lambda(0) = z_0^\lambda = \frac{1}{\lambda}(u_0^\lambda - u_0).$$ 

(7.10)

so $y = z^\lambda - z$ satisfies

$$\partial_\lambda y + u\partial_\lambda y + (\partial_x u)y = \lambda z^\lambda \partial_\lambda z^\lambda, \quad y(0) = y_0 = z_0^\lambda - z_0.$$ 

(7.11)

The estimates of [K2] show that $y \to 0$ in $CH^{s-1}$ as $\lambda \to 0$, and from this it follows that $u \in CH^{s-1}$ depends differentiably on $u_0$.

We will use this method to show differentiability of the solution of (7.1), and also in the proof of Proposition 6.1.

One can also solve (7.9)-(7.11) more directly as follows: (7.9) is equivalent to

$$(z(t) \circ \zeta(t))' + (\partial_x u(t) \circ \zeta(t))(z(t) \circ \zeta(t)) = 0$$

so

$$z(t) \circ \zeta(t) = \exp(-\int_0^t (\partial_x u(s)) \circ \zeta(s)ds)z_0.$$ 

(7.12)

Similarly from (7.10) we get

$$z^\lambda \circ \zeta(t) = \exp(-\int_0^t (\partial_x u^\lambda(s)) \circ \zeta(s)ds)z^\lambda_0.$$ 

(7.13)
Also, (7.11) is equivalent to:

\[(y(t) \circ \zeta(t))' + ((\partial_x u(t)) \circ \zeta(t))(y(t) \circ \zeta(t)) = \lambda(z^\lambda \partial_z z^\lambda) \circ \zeta(t).\]  

(7.14)

Therefore

\[y(t) \circ \zeta(t) = \exp(-\int_0^t (\partial_x u)(s) \circ \zeta(s) ds)y_0 + \int_0^t \exp(-\int_s^t (\partial_x u)(\tau) \circ \zeta(\tau)d\tau)(z^\lambda(s)\lambda \partial_z z^\lambda(s)) \circ \zeta(s) ds\]

As \(\lambda \to 0\), \(u^\lambda \to u\) in \(CH^s\). Hence \(\partial_z u^\lambda - \partial_z u = \lambda \partial_z z^\lambda \to 0\) in \(CH^{s-1}\). Also from (7.13) we find that \(z^\lambda\) is bounded in \(CH^{s-1}\) uniformly in \(\lambda\). Therefore the last term of (7.15) goes to zero in \(CH^{s-1}\) as \(\lambda \to 0\). But \(z_0^\lambda \to z_0\) by definition, so \(y_0 \to 0\). Hence by (7.15) \(y(t) \circ \zeta(t) \to 0\) in \(CH^{s-1}\) as \(\lambda \to 0\). An argument of this type will be used in the proof of Proposition 6.1 also.

We now proceed to prove differentiability for the symmetric hyperbolic case.

**Proposition 7.1.** Let \(u_0 \in U \subset H^s(\mathbb{T}^n)\) and \(u \in CH^s([0,T])\) be as in Theorem 7.1. Also let \(v\) be the solution of (7.1) for \(v_0 \in U\) as in that theorem. Define \(\Phi : U \to CH^{s-1}([0,T])\) by \(\Phi(v_0) = v\). Then \(\Phi\) is \(C^1\). In fact, \(D\Phi(u_0)z_0\) is the solution of (7.3) with initial data \(z_0\).

To prove this proposition we will need the following Lemma which is a variant of the \(\Omega\)-Lemma (see e.g. [E1] or [P]).

**Lemma 7.2.** Let \(h = h(t,x,u)\) be a smooth function from \([0,T] \times \mathbb{R}^n \times \mathbb{R}^m\) to some \(\mathbb{R}^k\) and for \(s' > \frac{n}{2}\) let \(\Omega_h : CH^{s'} \to CH^{s'}\) be defined by \(\Omega_h(u)(t)(x) = h(t,x,u(t)(x))\). Then \(\Omega_h\) is a smooth map. Also the derivative of \(\Omega_h\) at \(u\) in the direction \(z\) obeys the formula:

\[(D(\Omega_h)(u)z)(t)(x) = \partial_\lambda h(t,x,u(t)(x))(z(t)(x))\]

or more succinctly

\[D(\Omega_h)(u)z = \partial_\lambda h(u)z.\]  

(7.16)

**Proof.** The proof is the same as the proof of the \(\Omega\)-Lemma in [E1] or [P], so we shall omit most of it. We shall however compute the first derivative of \(\Omega_h\).

Let \(u^\lambda = u + \lambda z \in CH^{s'}\). Then for each \(t\) and \(x\)

\[\Omega_h(u^\lambda)(t)(x) - \Omega_h(u)(t)(x) = \int_0^1 \partial_\lambda h(t,x,u^\lambda(t,x))\lambda z(t,x)d\tau.\]

Therefore

\[\Omega_h(u^\lambda)(t)(x) - \Omega_h(u)(t)(x) = \int_0^1 \Omega_{\partial_\lambda h}(u^\lambda)\lambda z d\tau = \lambda \left( \int_0^1 \Omega_{\partial_\lambda h}(u^\lambda) d\tau \right) z.\]

Hence

\[\frac{1}{\lambda}(\Omega_h(u^\lambda) - \Omega_h(u)) - \partial_\lambda h(u)z = \left( \int_0^1 (\Omega_{\partial_\lambda h}(u^\lambda) - \Omega_{\partial_\lambda h}(u)) d\tau \right) z.\]  

(7.17)

The right hand side of (7.17) goes to zero in \(CH^{s'}\) as \(\lambda \to 0\), and (7.16) follows. \(\Box\)
Proof of Proposition 7.1: Let \( u \) be the solution of (7.1) with initial data \( u_0 \) and let \( u^\lambda \) be the solution with initial data \( u_0^\lambda = u_0 + \lambda z_0 \). Also let \( z \) be the solution of (7.3) with initial data \( z_0 \) and let \( z^\lambda \) equal \( \frac{1}{\lambda} (u^\lambda - u) \).

We shall show that \( \lim_{\lambda \to 0} z^\lambda = z \) where the limit is in \( CH^{s-1} \). Since \( u \) and \( u^\lambda \) satisfy (7.1), \( z^\lambda \) satisfies:

\[
\partial_t z^\lambda + \sum_{i=1}^n a_i(t, x, u) \partial_i z^\lambda + \sum_{i=1}^n \frac{1}{\lambda} (a_i(u^\lambda) - a_i(u)) \partial_i u^\lambda = \frac{1}{\lambda} (f(u^\lambda) - f(u)) \quad (7.18)
\]

We want to show that the solution of (7.18) is near \( z \) and to do so we rewrite (7.18) as:

\[
\partial_t z^\lambda + A(u) \nabla z^\lambda + B(u, \nabla u) z^\lambda = F(u) z^\lambda + E \quad (7.19)
\]

where \( A, B, \) and \( F \) are as in (7.3) and

\[
E = \sum_{i=1}^n \left( \partial_a a_i(u) z^\lambda \partial_i u - \frac{1}{\lambda} (a_i(u^\lambda) - a_i(u)) \partial_i u^\lambda \right) + \frac{1}{\lambda} (f(u^\lambda) - f(u)) - \partial_a f(u) z^\lambda \quad (7.20)
\]

Subtracting (7.3) from (7.19) we find

\[
\partial_t (z^\lambda - z) + A(u) \nabla (z^\lambda - z) + B(u, \nabla u) (z^\lambda - z) = F(u) (z^\lambda - z) + E \quad (7.21)
\]

and \( z^\lambda(0) - z(0) = 0 \). From standard energy estimates (or from Theorem I of [K2]) we find that the solution of (7.21) obeys:

\[
\| z^\lambda(t) - z(t) \|_{s-1} \leq e^{Kt} \int_0^t \| E(\tau) \|_{s-1} \, d\tau \quad (7.22)
\]

where \( K \) depends only on \( \max_{0 \leq t \leq T} \{ \| u(t) \|_s \} \), the norm of \( u \) in \( CH^s \).

Thus to show that \( z^\lambda(t) \to z(t) \) in \( H^{s-1} \) it suffices to estimate \( \| E(t) \|_{s-1} \). To do this it will be convenient to use the notation \( O(\lambda) \). We say that a function \( h(\lambda, t) \) is \( O(\lambda) \) if \( \lim_{\lambda \to 0} h(\lambda, t) = 0 \) uniformly in \( t \in [0, T] \).

From (7.20) we find:

\[
\| E \|_{s-1} \leq \sum_{i=1}^n \left\| \frac{1}{\lambda} (a_i(u^\lambda) - a_i(u)) \partial_i a^i(u) z^\lambda \right\|_{s-1} \| \partial_i u \|_{s-1}
\]

\[
+ \sum_{i=1}^n \left\| a_i(u^\lambda) - a_i(u) \right\|_{s-1} \| \partial_i u^\lambda - \partial_i u \|_{s-1} + \left\| \frac{1}{\lambda} (f(u^\lambda) - f(u)) - \partial_a f(u) z^\lambda \right\|_{s-1}
\]

\[
= I + II + III
\]

We note that \( I \) is \( O(\lambda) \) by Lemma 7.2.

We proceed to estimate \( II \). \( \| \partial_i u^\lambda - \partial_i u \|_{s-1} \) is \( O(\lambda) \), and \( \frac{1}{\lambda} \| a_i(u^\lambda) - a_i(u) \|_{s-1} \) is \( O(\lambda) \) + \( \| \partial_a a^i(u) z^\lambda \|_{s-1} \). Therefore

\[
II \leq O(\lambda) (1 + \| z^\lambda \|_{s-1})
\]

But then since \( \| z \|_{s-1} \) is independent of \( \lambda \) we get:

\[
II \leq O(\lambda) (1 + \| z^\lambda - z \|_{s-1})
\]
From Lemma 7.2, we find that $III$ is $O(\lambda)$ also and thus we get
\[ \| E \|_{s-1} \leq O(\lambda)(1 + \| z^\lambda - z \|_{s-1}) \] (7.23)
Hence from (7.22) we get:
\[ \| z^\lambda(t) - z(t) \|_{s-1} \leq O(\lambda)e^{Kt} \int_0^t \| z^\lambda(r) - z(r) \|_{s-1} dr + O(\lambda) \]
from this it follows, iterating the inequality, that \( \| z^\lambda - z \|_{s-1} \) is $O(\lambda)$. Thus the derivative of $\Phi$ at $u_0$ in direction $z_0$ is $z$, the solution of (7.3). But this $z$ is clearly a continuous linear function of $z_0$ and from [K2] we know that $z \in CH^{s-1}$ is a continuous function of $u \in CH^s$. Therefore $\Phi$ is $C^1$ and the proposition if proved.

Proposition 6.1 does not follow as a special case of Proposition 7.1 because $\Omega$ is a domain with boundary and we must consider initial-boundary value problems. But we will see that much of the argument is the same.

Proposition 6.1 says that $\Xi_t$ is a $C^1$ map, where $\Xi_t(u_0, \rho_0) = \nabla g(t)$, but since $\nabla g(t) = \nabla \Delta^{-1} f$, is suffices to show that $(u_0, \rho_0) \mapsto f$ is a $C^1$ map from $U_t$ to $H^2(\Omega, \mathbb{R})$. We shall prove a slightly different proposition that clearly implies this.

Given $(u_0, \rho_0) \in U_t$ pick an interval $[0, T]$ such that $t \in [0, T]$ and the fluid motion $u(t), \rho(t)$ with initial data $(u_0, \rho_0)$ is defined on $[0, T]$. As before we let $f_0 = \log \rho_0$ and $f(t) = \log \rho(t)$. Let
\[
Z = \left\{ x \in C^0([0, T], H^2(\Omega, \mathbb{R}^n)) \mid \dot{x} \in \bigcap_{k=0}^n C^k([0, T], H^{2-k}(\Omega, \mathbb{R}^n)) \right\}
\]
\[
X_0^k = \bigcap_{j=0}^k C^j([0, T], H^{2-j}(\Omega, \mathbb{R}))
\]
\[
X_{1/2, 0}^{k+1/2} = \bigcap_{j=0}^k C^j([0, T], H^{k+1/2-j}(\partial \Omega, \mathbb{R}))
\]
and let $\| \cdot \|_Z$, $\| \cdot \|_{X_0^k}$ and $\| \cdot \|_{X_{1/2, 0}^{k+1/2}}$ be the appropriate norms for these spaces (see below).

Let $\Xi(\tilde{u}_0, \tilde{f}_0) = (\tilde{u}, \tilde{f})$, where $\tilde{u}, \tilde{f}$ defines the fluid motion with initial data $\tilde{u}_0$, $\tilde{f}_0$. Then $\Xi$ is defined from a neighborhood $V$ of $(u_0, f_0)$ in $U_t$ to $Z \times X_0^3$.

**Proposition 7.3.** $\Xi : V \to Z \times X_0^3$ is a $C^1$ map.

**Proof.** First we note that $(u, f)$ satisfies the system
\[
\begin{align*}
\dot{u} &= -c^2(f)\nabla f \quad (7.24a) \\
\dot{f} &= -\varepsilon c^2(f)\nabla f + u\cdot w
\end{align*}
\] (7.24b)
with boundary condition
\[ c^2\nabla \nu f = -S_2(u, u) \text{ on } \partial \Omega, \] (7.25)
and initial condition

\[ f(0) = f_0, \quad \dot{f}(0) = \delta u_0, \quad u(0) = u_0. \]  

(7.26)

Proceeding as in the proof of Proposition 7.1, we let \((u^\lambda, f_0^\lambda)\) be a \(C^1\) curve of initial conditions in \(V\) and let \((u^\lambda, f^\lambda)\) be the corresponding solutions of (7.24a)-(7.24b), (7.25), (7.26). If all derivatives exist, \((z_0, h_0) = \partial_{\lambda}(u_0^\lambda, f_0^\lambda)|_{\lambda=0}\) and \((z, h) = \partial_{\lambda}(u^\lambda, f^\lambda)|_{\lambda=0}\), then \((z, h)\) must satisfy

\[
\begin{cases}
\dot{z} + \nabla z u = - (c^2)'(f) h \nabla f - c^2 \nabla h \\
\dot{h} + \nabla \dot{f} + (\nabla z f) = - \delta c^2 \nabla h - (c^2)'(f) h \nabla f + 2 z_i^j u^j_i
\end{cases}
\]  

(7.27a)

with initial-boundary conditions

\[
\begin{align*}
(c^2)' h \nabla \nu f + c^2 \nabla \nu h &= - 2 S_2(u, z) \text{ on } \partial \Omega, \\
z(0) &= z_0, \quad h(0) = h_0, \quad \text{and } \dot{h}(0) + \nabla z(0) f_0 = \delta z_0.
\end{align*}
\]  

(7.28)

(7.29)

Here the extra terms on the left of (7.27a)-(7.29) come from the fact that \(\cdot\) depends on \(\lambda\). For example

\[
\partial_{\lambda}(\dot{u}^\lambda) = \partial_{\lambda}(\partial_t u^\lambda + \nabla_{u^\lambda} u^\lambda) = \partial_t \partial_{\lambda} u^\lambda + \nabla_{u^\lambda} \partial_{\lambda} u^\lambda + \nabla_{\partial_{\lambda} u^\lambda} u^\lambda,
\]

so

\[
\partial_{\lambda}(\dot{u}^\lambda)|_{\lambda=0} = \dot{z} + \nabla z u.
\]

Also \(c\) is a function of \(f\) (see (2.8)) and \((c^2)'(f)\) means \(\partial_f(c^2(f))\). Below we sometimes write \(c^2\) for \(c^2(f)\).

The remainder of our proof will be like the proof of proposition 7.1 except that (7.3) will be replaced by (7.27a)-(7.29). We let \((z^\lambda, h^\lambda) = \frac{1}{\lambda}(u^\lambda - u, f^\lambda - f)\). Then \((z^\lambda, h^\lambda)\) satisfies

\[
\begin{cases}
\dot{z}^\lambda + \nabla z^\lambda u^\lambda = - c^2 \nabla h^\lambda - \frac{1}{\lambda} (c^2(f^\lambda) - c^2(f)) \nabla f^\lambda \\
\dot{h}^\lambda + \nabla \dot{z}^\lambda f^\lambda + (\nabla z^\lambda f^\lambda) + \lambda \nabla z^\lambda \nabla z^\lambda f = - \delta c^2 \nabla h^\lambda \\
- \frac{1}{\lambda} \delta (c^2(f^\lambda) - c^2(f)) \nabla f^\lambda + z_i^j(u_i^j + u_i^j)
\end{cases}
\]  

(7.30a)

with boundary condition

\[
\frac{1}{\lambda} (c^2(f^\lambda) - c^2(f)) \nabla \nu f^\lambda + c^2 \nabla \nu h^\lambda = - S_2(u^\lambda + u, z^\lambda) \text{ on } \partial \Omega.
\]  

(7.31)

Assuming that \((z, h)\) satisfies (7.27a)-(7.29) we find that \((z^\lambda - z, h^\lambda - h)\) satisfies the following:

\[
(z^\lambda - z) + \nabla z^\lambda - u = -(c^2)'(f^\lambda - h) \nabla f - c^2 \nabla (h^\lambda - h) + E_1,
\]  

(7.32)

where

\[
E_1 = - \nabla u^\lambda u^\lambda - (\frac{1}{\lambda} (c^2(f^\lambda) - c^2(f)) - (c^2)'(f^\lambda)) \nabla f - (c^2(f^\lambda) - c^2(f)) \nabla h^\lambda,
\]

(7.33)

and

\[
(h^\lambda - h)^i + \nabla z^\lambda - \dot{f} + (\nabla z^\lambda - f)^i = - \delta c^2 \nabla (h^\lambda - h)
\]

\[
- \delta (c^2)'(h^\lambda - h) \nabla f + 2 u_j^i (z^\lambda - z)^j_i + E_2,
\]  

(7.34)
For a fixed $t$

\[ E_2 = -\lambda (\nabla_{z^\lambda} h^\lambda + (\nabla_{z^\lambda} h^\lambda) + \nabla_{z^\lambda} \nabla_{z^\lambda} f) - \delta \left( \frac{1}{\lambda} (c^2 (f^\lambda) - c^2 (f)) - (c^2)'(h^\lambda) \nabla f \right) \]

\[ - \delta (c^2 (f^\lambda) - c^2 (f)) \nabla h^\lambda + (u^\lambda - u)^i \zeta_i^\lambda, \]

where

\[ c^2 \nabla_\nu (h^\lambda - h) + (c^2)'(h^\lambda - h) \nabla_\nu f = 2 S_2 (u, z^\lambda - z) + E_\theta, \]  

(7.36)

with boundary condition

\[ E_\theta = -S_2 (u^\lambda - u, z^\lambda) - \left( \frac{1}{\lambda} (c^2 (f^\lambda) - c^2 (f)) - (c^2)'(h^\lambda) \nabla_\nu f - (c^2 (f^\lambda) - c^2 (f)) \nabla_\nu h^\lambda. \]  

(7.37)

Notice that (7.32)-(7.37) is the same system as (7.27a)-(7.29) except for the inhomogeneous terms $E_1$, $E_2$, $E_\theta$.

We shall show that both these systems have solutions in $Z \times X^3_0$. Also since $(u^\lambda_0, f^\lambda_0)$ is a $C^1$ curve in $V$, $(z^\lambda_0 - z_0, h^\lambda_0 - h_0)$ is $O(\lambda)$ in $X^{3,4,4}$. We shall also get bounds for $E_1$, $E_2$ and $E_\theta$ in terms of $O(\lambda)$. From this it will follow that $(z^\lambda - z, h^\lambda - h)$ is $O(\lambda)$ in $Z \times X^3_0$, so $\Xi$ is differentiable at $(u_0, f_0)$ and $D\Xi(u_0, f_0)(z_0, h_0) = (z, h)$. One routinely sees that $(z, h)$ depends continuously on $(u, f)$, so $\Xi$ is in fact $C^1$, which is what we want to prove.

To solve (7.32)-(7.36), we shall rewrite it as:

\[
\begin{cases}
\dot{x} + \nabla_x u = B_1 r + A_1 r + E_1, \\
\dot{r} + \nabla_x \dot{f} + (\nabla_x f) = L r + B_2 r + A_2 x + E_2, \\
c^2 \nabla_\nu r + B_\theta \nu = S \nu + E_\theta \text{ on } \partial \Omega
\end{cases}
\]  

(7.38)

where $x$ replaces $z^\lambda - z$, $r$ replaces $h$, $L$ is as in (4.5), and the other terms are defined in the obvious way.

We first consider the case $E_1 = E_2 = E_\theta = 0$ which is a homogeneous linear system in $(x, r)$. Also let $(x_0, r_0) \in X^{3,4,4}$ be the initial data which we assume satisfy the compatibility conditions.

Given $r \in X^3_0$, we can consider (7.38a) as an inhomogeneous linear equation in $x$. We will solve it following the idea of our example (see (7.9)-(7.15)). We first assume $r = 0$, so we have:

\[ \dot{x} + \nabla_x u = 0 \]  

(7.39)

But this is equivalent to

\[ \partial_t (x(t) \circ \zeta(t)) = - (\nabla_{x(t)} u(t)) \circ \zeta(t). \]  

(7.40)

For a fixed $t$, the operator $B_3(t) : H^2(\Omega, \mathbb{R}^n) \to H^2(\Omega, \mathbb{R}^n)$, defined by

\[ B_3(t) y = - (\nabla_{y \circ \zeta(t)^{-1}} u \circ \zeta(t)^{-1}) \circ \zeta(t), \]

is bounded since $u, \zeta \in H^3$ and since $u(t)$ and $\zeta(t)$ are continuous in $t$, so is $B_3(t)$ in the operator norm. Thus (7.40), which is

\[ \partial_t y = - B_3 y \]

is simply a linear ordinary differential equation in $H^2(\Omega, \mathbb{R}^2)$. Hence we can solve it for $y(t)$ and then we define $x(t) = y(t) \circ (\zeta(t))^{-1}$. It is clear that this $x(t)$ satisfies (7.39) and that
it is in \( Z \). We let \( U(t, s) : H^2(\Omega, \mathbb{R}^n) \to H^2(\Omega, \mathbb{R}^n) \) be the operator which gives the solution of (7.39). That is, for any \( t, s \in [0, T] \), \( x(t) = U(t, s)x_0 \) is the solution of (7.39) with initial condition \( x(s) = x_0 \). By standard means we can show that \( \| U(t, s) \| \leq e^{K|t-s|} \) where \( K \) depends only on \( \| u \|_3 \).

Now we can solve (7.38a) for general \( r \in X^3_0 \). We have the usual (Duhamel’s) formula

\[
x(t) = U(t, 0)x_0 + \int_0^t U(t, s)(A_1r(s) + B_1r(s))ds
\]

Since \( r(s) \in H^3 \), \( A_1r(s) \in H^2 \), and \( B_1r(s) \in H^3 \), so \( x(t) \in H^2 \). Also since \( x \) satisfies (7.38a) and \( r \in X^3_0 \) it follows that \( x \in Z \). Furthermore we get:

\[
\| x \|_Z \leq K(\| x_0 \|_2 + \| r \|_{X^3_0})
\]

(7.41)

Give \( x \in Z \), we can also solve (7.38b)-(7.38c) for \( r \in X^3_0 \). In fact (7.38b) is a second order hyperbolic equation for \( r \) with inhomogeneous term \(-\nabla_x \tilde{f} - (\nabla_x f) + A_2x \) and with Neumann type boundary condition (7.38c). Such equations are considered in [M] where good estimates for their solutions are given. In particular we consider

\[
\begin{cases}
\ddot{r} = Lr + B_2r + A_2x + F, \\
c^2\nabla_\nu^2r + B_\nu r = G \text{ on } \partial \Omega,
\end{cases}
\]

(7.42)

and assume that compatibility conditions hold up to order \( k - 1 \) we get:

\[
\| r(t) \|_{X^3_0} \leq K(\| r(0) \|_{X^3_0} + \int_0^t (\| F(s) \|_{X^{k-1}_0} + \| G(s) \|_{X^{k-1, \frac{1}{2}}_0})ds
\]

(7.43)

where

\[
\| r(t) \|_{X^3_0} = \sum_{j=0}^k \| \partial_t^jr(t) \|_{k-j}
\]

\[
\| G(t) \|_{X^{k-\frac{1}{2}}_0} = \sum_{j=0}^{k-1} \| \partial_t^jG(t) \|_{\partial,k-j-\frac{1}{2}}
\]

In the cases of (7.38b)-(7.38c) the inhomogeneous term \( A_2x(t) \) is in \( H^1 \), but not in \( H^2 \), and the other terms are at least as smooth. Thus if we let

\[
\| x(t) \|_Z = \| x(t) \|_2 + \| \dot{x}(t) \|_2 + \| \partial_t \dot{x} \|_1 + \| \partial_t^2 \dot{x} \|_0
\]

(7.44)

we get

\[
\| r(t) \|_{X^3_0} \leq K(\| r(0) \|_{X^3_0} + \int_0^t (\| x(s) \|_Z + \| Sx(s) \|_{X^{1/2}_{\frac{1}{2},0}})ds.
\]

(7.45)

Furthermore, since restriction of functions to \( \partial \Omega \) gives a continuous map from \( H^s(\Omega) \) to \( H^{s-\frac{1}{2}}(\partial \Omega) \) \( (s > \frac{1}{2}) \), the term \( \| Sx(s) \|_{X^{1/2}_{\frac{1}{2},0}} \) is bounded by \( \| x(s) \|_Z \). Therefore we can omit it in (7.45).
Similarly from
\begin{equation}
\dot{r} + (\nabla_x \dot{f}) + (\nabla_x f) = L \dot{r} + L_1 r + B_2 \dot{r} + B_{21} r + (A_2 x),
\end{equation}
and
\begin{equation}
c^2 \nabla \nu \dot{r} + c^2 \nabla_{[u,\nu]} r + (c^2) \nabla \nu r + B_0 \dot{r} + B_{01} r = (S x) \text{ on } \partial \Omega,
\end{equation}
where \( L_1 \) is defined in (4.9), \( B_{21} r = (B_2 r) - B_2 \dot{r} \) and \( B_{01} r = (B_0 r) - B_0 \dot{r} \). (7.46) is an equation of form (7.42) for \( \dot{r} \) with inhomogeneous terms
\begin{equation}
F = - (\nabla_x \dot{f}) - (\nabla_x f) + L_1 r + B_{21} r + (A_2 x)
\end{equation}
and
\begin{equation}
G = - c^2 \nabla_{[u,\nu]} r + (c^2) \nabla \nu r - B_{01} r + (S x).
\end{equation}
Since \( (A_2 x) \|_{X_0^3} \leq K \| x \|_Z \) and other terms are sufficiently smooth, from (7.43) we get
\begin{equation}
\| \dot{r}(t) \|_{X_0^3} \leq K(\| \dot{r}(0) \|_{X_0^3} + \int_0^t \| x(s) \|_Z + \| r(s) \|_{X_0^3}) ds.
\end{equation}
Combining (7.47) and (7.44) with the equations (7.46) and (7.38b) we obtain
\begin{equation}
\| r \|_{X_0^3} \leq K(\| r(0) \|_{X_0^3} + \int_0^t \| x(s) \|_Z ds).
\end{equation}
The inequalities (7.41) and (7.48) show that if \( T \) is small enough we can solve (7.38a)-(7.38c) by an iteration. Given initial data \((x_0, r_0)\) we let \( x_1 \) be a curve in \( Z \) starting at \( x_0 \). Then let \( r_1 \) be the solution of (7.38b)-(7.38c) with initial data \( r_1(0) = r_0, \dot{r}_1(0) = \delta x_0 - \nabla x_0 f_0 \) and with \( x_1 \) in place of \( x \). Let
\begin{equation}
x_2(t) = U(t, 0)x_0 + \int_0^t (U(t, s)(A_1 r_1(s) + B_1 r_1(s))) ds
\end{equation}
and let \( r_2 \) be the solution of (7.38b)-(7.38c) with the same initial data but with \( x_2 \) in place of \( x \). Continue inductively to get sequences \( \{x_n\} \subset Z \) and \( \{r_n\} \subset X_0^3 \). Clearly
\begin{equation}
(x_n - x_{n-1}) + \nabla_{x_n-x_{n-1}} u = A_1(x_{n-1} - x_{n-2}) + B_1(x_{n-1} - x_{n-2})
\end{equation}
and \( x_n(0) - x_{n-1}(0) = 0 \). Hence by (7.41) we get
\begin{equation}
\| x_n - x_{n-1} \|_Z \leq K \| x_{n-1} - x_{n-2} \|_{X_0^3}.
\end{equation}
Similarly from (7.48) we find
\begin{equation}
\| r_n - r_{n-1} \|_{X_0^3} \leq K \int_0^t \| x_n(s) - x_{n-1}(s) \|_Z ds.
\end{equation}
From these two inequalities it follows that if \( K^2 T < 1 \), \( \{(x_n, r_n)\} \) is a Cauchy sequence in \( Z \times X_0^3 \). Its limit, which we call \((z, h)\), is clearly a solution of (7.38a)-(7.38c) or equivalently of (7.27a)-(7.29). Since the system is linear we can piece together short time solutions to get a solution on any interval. Thus we can drop the restriction \( K^2 T < 1 \), and get a solution on any interval \([0, T]\) on which \( u \) and \( f \) are defined.
Having solved the homogeneous system (7.38a)-(7.38c) we can solve the inhomogeneous system by the usual application of Duhamel’s formula. Then using the inequalities (7.41) and (7.48) with inhomogeneous terms included we find that the solution of (7.38a)-(7.38c) obeys
\[ \| x \|_Z \leq K(\| x(0) \|_2 + \| r \|_{X_0^3} + \| E_1 \|_{X_0^3}) \]
and
\[ \| r \|_{X_0^3} \leq K(\| r(0) \|_{X_0^3} + \int_0^T (\| x(t) \|_Z + \| E_2(t) \|_{X_0^3} + \| E_\theta(t) \|_{X_0^3}) dt) \]
Iterating these inequalities we find a new K such that
\[ \| x \|_Z + \| r \|_{X_0^3} \leq K(\| x(0) \|_2 + \| r(0) \|_{X_0^3} + \| E_1 \|_{X_0^3} + \| E_2 \|_{X_0^3} + \| E_\theta \|_{X_0^3}) \] (7.49)

Now we are ready to show that \((z^\lambda - z, h^\lambda - h)\) is \(O(\lambda)\) in \(Z \times X_0^3\). We simply consider the system (7.32)-(7.37) with initial conditions \(z_0^\lambda - z_0, h_0^\lambda - h_0\) and \(\delta(z_0^\lambda - z_0) - \nabla z_0^\lambda = f_0\). These initial conditions are \(O(\lambda)\) in \(H^3 \times H^4 \times H^3\) because \((u_0^\lambda, f_0^\lambda)\) is a \(C^1\)-curve in \(X^{3,4,4}\). Hence we need only show that the last three terms of (7.49) are \(O(\lambda)\). This one does in the same way as in the estimate (7.23) of \(\| E \|_{s-1}\). We omit the details. \(\square\)

8. Asymptotic Approximation to Compressible Fluid Motion

In this section we show how the equation (5.19) can be used to find approximate solutions to the compressible fluid motion problem. We will construct a sequence \(\{\zeta_n\}\) of curves in \(H^3\) which will approach the actual fluid motion \(\zeta(t)\). The methods used to estimate the \(\{\zeta_n\}\) are some of the methods of sections 4 through 7. Hence our presentation will be brief.

First we note that the estimate (5.20) shows that incompressible motion is itself an approximation of order \(\frac{1}{\sqrt{k}}\). We let \((u_0, \rho_0)\) be as in Theorem 3.1, let \(v_0 = Pu_0\) and let \(\eta \) and \(\zeta\) be incompressible and compressible fluid motions as in Theorem 5.1. Then by (5.20)
\[ \| \dot{\eta}(t) - \dot{\zeta}(t) \|_3 \leq K\left(\frac{1+t}{\sqrt{k}}\right) + \int_0^t Z(\eta(\tau), \dot{\eta}(\tau)) - Z(\zeta(\tau), \dot{\zeta}(\tau)) d\tau. \]
Iterating this inequality we get
\[ \| (\eta(t), \dot{\eta}(t)) - (\zeta(t), \dot{\zeta}(t)) \|_3 \leq K\frac{t}{\sqrt{k}}. \] (8.1)

To get better approximations we will construct a sequence of curves \(\zeta_n(t)\) such that
\[ \| (\zeta_n(t), \dot{\zeta}_n(t)) - (\zeta_{n-1}(t), \dot{\zeta}_{n-1}(t)) \|_3 \leq Kk^{-\frac{3}{4}} \] (8.2)
and
\[ \| (\zeta(t), \dot{\zeta}(t)) - (\zeta_n(t), \dot{\zeta}_n(t)) \|_3 \leq Kk^{-\frac{n+1}{4}}. \] (8.3)

To begin we simply let \(\zeta_0(t) = \eta(t)\) be the incompressible motion. Then we define the \(\zeta_n\) inductively as follows:
Let \( f_0(t) = \log \rho_{0k}(t) \). Then let \( f_n \) be the solution of
\[
\begin{align*}
\dot{f}_n &= -\delta c^2(f_n) \nabla f_n + (u_{n-1})^i_j (u_{n-1})^j_i, \\
c^2(f_n) \nabla \nu \dot{f}_n &= -S_2(u_{n-1}, u_{n-1}) \text{ on } \partial \Omega, \\
f_n(0) &= \log \rho_{0k}, \quad \dot{f}(0) = \delta \rho_{0k}.
\end{align*}
\] (8.4)
\[
\begin{align*}
\dot{f}_n &= -\delta c^2(f_n) \nabla f_n + (u_{n-1})^i_j (u_{n-1})^j_i, \\
c^2(f_n) \nabla \nu \dot{f}_n &= -S_2(u_{n-1}, u_{n-1}) \text{ on } \partial \Omega, \\
f_n(0) &= \log \rho_{0k}, \quad \dot{f}(0) = \delta \rho_{0k}.
\end{align*}
\] (8.5)

where \( u_{n-1} = \zeta_{n-1} \circ (\zeta_{n-1})^{-1} \) as in (5.12)-(5.13) and \( \cdot \) means \( \partial_t + \nabla u_{n-1} \). Let \( \nabla g_n(t) = \nabla \Delta^{-1} \dot{f}_n \) and let \( \zeta_n \) be the solution of
\[
\begin{align*}
\dot{\zeta}_n(t) &= Pu_{0k} + \int_0^t \dot{Z}(\zeta_n(s), \zeta_n(s)) ds + \nabla g_n(t) \circ \zeta_n(t), \\
\zeta_n(0) &= \text{id},
\end{align*}
\] (8.6)

where \( \dot{Z} \) is defined by: \( \dot{Z}(\xi, \alpha) = Z(\xi, \alpha) - R(\xi, \alpha) \) (cf. section 5).

Since \( \nabla g_n(t) \in H^4 \), the map \( \xi \mapsto \nabla g_n(\cdot) \circ \xi \) is \( C^1 \) as a map from \( \mathcal{D}^3 \) to \( H^3(\Omega, \mathbb{R}^n) \). Also \( \dot{Z} \) is a smooth map from \( \mathcal{D}^3 \times H^3 \to H^3 \). Therefore from standard estimates involving the Lipshitz constants of \( \dot{Z} \) and \( \Omega_{\nabla g(t)} \) we find
\[
\| (\zeta_n(t), \dot{\zeta}_n(t)) - (\zeta_{n-1}(t), \dot{\zeta}_{n-1}(t)) \|_3 \leq K \| \nabla g_n(t) - \nabla g_{n-1}(t) \|_3
\] (8.7)

However \( \| \nabla g_n(t) - \nabla g_{n-1}(t) \|_3 \leq K \| \dot{f}_n - \dot{f}_{n-1} \|_2 \) and using estimates like those of sections 4 and 7 we find that
\[
\| \dot{f}_n(t) - \dot{f}_{n-1}(t) \|_2 \leq \frac{K}{\sqrt{k}} \| u_{n-1} - u_{n-2} \|_3.
\] (8.8)

Combining (8.7) and (8.8) we get
\[
\| (\zeta_n, \dot{\zeta}_n) - (\zeta_{n-1}, \dot{\zeta}_{n-1}) \|_3 \leq \frac{K}{\sqrt{k}} \| (\zeta_{n-1}, \dot{\zeta}_{n-1}) - (\zeta_{n-2}, \dot{\zeta}_{n-2}) \|_3 \quad (n \geq 2).
\] (8.9)

Also \( \zeta_0 = \eta \) so
\[
\dot{\zeta}_0(t) = Pu_{0k} + \int_0^t \dot{Z}(\zeta_0(\tau), \dot{\zeta}_0(\tau)) d\tau
\]

and
\[
\dot{\zeta}_1(t) = Pu_{0k} + \int_0^t \dot{Z}(\zeta_1(\tau), \dot{\zeta}_1(\tau)) d\tau + \nabla g_1(t) \circ \zeta_1(t).
\]

But since \( f_1 \) satisfies (8.4)-(8.5) we know \( \| \dot{f}_1 \| \leq \frac{K}{\sqrt{k}} \) so \( \| \nabla g_1(t) \|_3 \leq \frac{K}{\sqrt{k}} \). Hence it follows as before that
\[
\| (\zeta_1, \dot{\zeta}_1) - (\zeta_0, \dot{\zeta}_0) \|_3 \leq \frac{K}{\sqrt{k}}
\] (8.10)

(8.9) and (8.10) together imply (8.2).

Now we prove (8.3). If \( n = 0 \) (8.3) is simply (8.1). For \( n > 0 \), we note that \( \zeta \) satisfies (8.6) with \( g \) replacing \( g_n \). Thus we get as before
\[
\| (\zeta, \dot{\zeta}) - (\zeta_n, \dot{\zeta}_n) \|_3 \leq K \| \nabla g - \nabla g_n \|_3 \leq K \| \dot{f} - \dot{f}_n \|_2.
\] (8.11)
Also
\[ \| \dot{f} - \dot{f}_n \|_2 \leq \frac{K}{\sqrt{k}} \| u - u_{n-1} \|_3 \leq \frac{K}{\sqrt{k}} \| (\zeta, \dot{\zeta}) - (\zeta_{n-1}, \dot{\zeta}_{n-1}) \|_3 . \] (8.12)

(8.3) follows from (8.11) and (8.12) by induction.

**Remark 8.1.** Equation (2.9) which is (8.4) without the subscripts “n” and “n − 1” is the equation of sound on a fluid moving with velocity \( u \). Acoustical engineers sometimes find an approximate solution to this equation by solving (8.4) with \( n = 1 \) (cf. [H]).

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