Mass and Axial current renormalization in the Schrödinger functional scheme for the RG-improved gauge and the stout smeared $O(a)$-improved Wilson quark actions

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We present the quark mass and axial current renormalization factors for the RG-improved Iwasaki gauge action and three flavors of the stout smeared $O(a)$-improved Wilson quark action. We employ $\alpha = 0.1$ and $n_{\text{step}} = 6$ for the stout link smearing parameters and all links in the quark action are replaced with the smeared links. Using the Schrödinger functional scheme we evaluate the renormalization factors at $\beta = 1.82$ where large scale simulations are being carried out.
1. Introduction

We are accumulating the configurations near the physical masses of up-down (degenerated) and strange quarks \( N_f = 2 + 1 \) on a 96\(^4\) lattice with the lattice cutoff \( a^{-1} \sim 2.3 \) [GeV] under the project of HPCI (High Performance Computing Infrastructure) Strategic Programs for Innovative Research (SPIRE) Field 5, “The origin of matter and the universe” [1]. The properties of these ensembles have to be determined via the detailed analysis on the physical quantities such as the light hadron spectrum, quark masses, and hadronic observables, extracted from the ensembles.

In this poster, we report on the mass and axial current renormalization factors determined with the Schrödinger functional (SF) scheme for the RG-improved Iwasaki gluon action and three flavors of the stout smeared \( O(a) \)-improved Wilson quark action. We employ \( \alpha = 0.1 \) and \( n_{\text{step}} = 6 \) for the stout link smearing [2] parameters, and all link variables contained in the \( O(a) \)-improved Wilson quark action are replaced with the stout smeared ones. The \( O(a) \)-improvement parameter \( c_{\text{SW}} \) has been determined nonperturbatively in Ref. [3] using the SF method. We determine the renormalization factors at \( \beta = 1.82 \) (\( a^{-1} \sim 2.3 \) [GeV]) where the simulations on the 96\(^4\) lattice with this lattice action are being carried out.

The renormalization constants for the axial and pseudo-scalar operators (and vector operator as a byproduct) with the SF scheme are determined using the standard method described in Refs. [4, 5, 6, 7, 8]. The temporal and spatial lattice sizes are finite at \( T = aN_T \) and \( L = aN_S \), and a Dirichlet boundary condition in the temporal direction is imposed to define the SF scheme. The boundary gauge fields at \( n_4 = 0 \) and \( N_T \) are kept fixed by the Dirichlet boundary condition and the same boundary condition is applied on the smeared gauge field during the stout smearing steps. The up, down and strange quark masses are degenerate and tuned to vanish in determining the renormalization constants. The HMC (two-flavor part) and RHMC (single-flavor part) algorithms with the SF boundary condition are employed to generate the gauge configuration. The action parameters are set to be \( \beta = 1.82 \) and \( c_{\text{SW}} = 1.11 \) [3], and the boundary parameters are to be \( c_{P}^R = 1 \) \( (c_{I}^R = 3/2) \) for the plaquette (rectangular) term for the gauge action and \( \bar{c}_t = 1 \) for the quark action.

2. Axial, vector, and pseudo-scalar operator renormalization constants

The operators to be renormalized are defined by

\[
A_4^a(x) = \overline{q}(x)\gamma_4 \gamma_5 T^a q(x), \quad V_4^a(x) = \overline{q}(x)\gamma_4 T^a q(x), \quad P^a(x) = \overline{q}(x)\gamma_5 T^a q(x). \tag{2.1}
\]

As it was observed that the nonperturbative \( O(a) \)-mixing correction to the axial current was consistent with zero [3], we assume a negligible \( O(a) \)-mixing and employ the unimproved current operators in determining the renormalization factors. The correlation functions used for setting the
renormalization conditions are
\[ f_{XY}(t,s) = \frac{2}{N_f^2(N_f^2 - 1)} \sum_{\xi \beta} f^{abc} f^{cde} (O^{\prime \prime} X^a(\bar{x},t) Y^b(\bar{y},s) O^c), \tag{2.2} \]
\[ f_X(t) = -\frac{1}{N_f^2 - 1} \sum_{\xi} (X^a(\bar{x},t) O^a), \tag{2.3} \]
\[ f_1 = -\frac{1}{N_f^2 - 1} (O^{\prime \prime} O^a), \tag{2.4} \]
\[ f_V(t) = \frac{1}{N_f(N_f^2 - 1)} \sum_{\xi} i f^{abc} (O^{\prime \prime} V^b(\bar{x},t) O^c), \tag{2.5} \]
where \( f^{abc} \) is the structure constant of \( SU(N_f) \). The operator \( X \) (\( Y \)) can be \( A_\delta \) or \( P \). \( O^a \) and \( O^{\prime \prime} \) are boundary operators defined by
\[ O^a = \frac{1}{(L/a)^3} \sum_{\bar{y},\bar{z}} \bar{\zeta}(\bar{y}) \gamma T^a \zeta(\bar{z}), \quad O^{\prime \prime} = \frac{1}{(L/a)^3} \sum_{\bar{y},\bar{z}} \bar{\zeta}(\bar{y}) \gamma T^a \zeta'(\bar{z}), \tag{2.6} \]
where \( \zeta \) and \( \zeta' \) are boundary quark fields located at \( n_4 = 0 \) and \( n_4 = N_T \) respectively.

Using the correlation functions \((2.2)-(2.5)\), the renormalization factors are determined with
\[ Z_A = \sqrt{Z_A(2T/3)} \bigg|_{m_q \to 0}, \quad \tilde{Z}_A(t) = \frac{f_1}{n_A} \left[ f_{AA}(t, T/3) - 2m_{PCAC} f_{PA}(t, T/3) \right]^{-1}, \tag{2.7} \]
\[ Z_V = \tilde{Z}_V(T/2) \bigg|_{m_q \to 0}, \quad \tilde{Z}_V(t) = \frac{f_1}{n_V f_V(t)}, \tag{2.8} \]
\[ Z_P = \tilde{Z}_P(T/2) \bigg|_{m_q \to 0}, \quad \tilde{Z}_P(t) = \frac{\sqrt{3} f_1}{n_P f_P(t)}, \tag{2.9} \]
where \( n_A \), \( n_V \), and \( n_P \) are normalization constants evaluated at the tree-level so as to be \( Z_A = 1 \), \( Z_V = 1 \), and \( Z_P = 1 \). To take the mass-less limit we employ an averaged PCAC mass for \( Z_A \) and \( Z_V \);
\[ a m_q = a m_{PCAC}^{PCAC} = \frac{1}{3} \sum_{t=T/2-a}^{T/2+a} \frac{a f_A(t + a) - a f_A(t - a)}{4 f_P(t)}, \tag{2.10} \]
while a non-averaged mass for \( Z_P \);
\[ a m_q = a m_{PCAC}^{PCAC} = \frac{a f_A(T/2 + a) - a f_A(T/2 - a)}{4 f_P(T/2)}. \tag{2.11} \]

The simulation parameters are shown in Table [1]. The phase angle \( \theta \) is the parameter of the generalized periodic boundary condition in each spatial direction for the quark field. The boundary gauge fields are fixed to the identity matrix. The data are measured at every trajectory and the statistical errors are estimated with the jackknife method after blocking data with the size of 100 trajectories. The simulations (A1S) and (A1L) are dedicated for \( Z_A \) (and \( Z_V \) as a byproduct), while (P4a) and (P4b) are for \( Z_P \). The hopping parameter \( \kappa = 0.126110 \) corresponds to the critical value \( \kappa_c \) determined in Ref. [3]. Almost vanishing masses are realized in the (A1S) and (A1L) runs.

The time dependence of \( \tilde{Z}_A(t) \) and \( \tilde{Z}_V(t) \) from (A1L) is shown in Figure [1]. \( \tilde{Z}_V(t) \) is almost time-independent, and a short plateau around \( t = 17 - 22 \) \((\sim 2T/3)\) is observed for \( \tilde{Z}_A(t) \) when
disconnected diagrams are included properly. A similar behavior is observed in (A1S). The renormalization factors extracted with the definitions (2.7)-(2.9) are tabulated in Table 2. We assign the discrepancy of \( Z_A \) and \( Z_V \) between two runs, (A1S) and (A1L), to the systematic error. We observe that \( Z_A \simeq Z_V \) and \( Z_A \simeq 1 \). This could indicate a better chiral property of the stout smeared quark action we employed.

The renormalization factor for the pseudo-scalar operator depends on the renormalization scale. The renormalization scale corresponds to the physical box size \( L \) and the scale is implicitly defined by the value of the renormalized coupling \( g^2(L) \) in the SF scheme. To convert \( Z_{P}^{SF}(1/L) \) to the mass renormalization factor \( Z_{m}^{\overline{MS}}(\mu) \) at a reference scale \( \mu \) in the \( \overline{MS} \) scheme, we need the RG evolution of \( Z_P \) together with the running of the renormalized coupling constant \( g^2(\mu) \) in both the SF and \( \overline{MS} \) schemes. The details of the mass renormalization factor are described in the next section.

\[ Z_{V}(t) \, \beta=1.82, \, L/a=12, \, T/a=30, \, \theta=1/2 \]

\[ Z_{A}(t) \, \beta=1.82, \, L/a=12, \, T/a=30, \, \theta=1/2 \]

\[ \bar{Z}_{V}(t) \, \beta=1.82, \, L/a=12, \, T/a=30, \, \theta=1/2 \]

\[ \bar{Z}_{A}(t) \, \beta=1.82, \, L/a=12, \, T/a=30, \, \theta=1/2 \]
3. Scale setting and mass renormalization

The renormalization factor of the pseudo-scalar operator \( Z_P \) is evaluated at \( T = L = 4a \) and \( \beta = 1.82 \). The simulation parameters and results are shown in Tables [1] and [2]. We have two simulations, (P4a) and (P4b). The scale \( L = 4a \) is chosen so that the RG evolution by the step scaling function of the coupling is available from the scale \( L_{\text{max}} = 4a \). The renormalized coupling in the SF scheme at \( L = 4a \) and \( \beta = 1.82 \) is shown in Table [3] where the boundary condition defining the SF scheme coupling is imposed on the gauge field. The PCAC mass in Table [3] is defined by Eq. (2.11). The step scaling evolution for the SF scheme coupling from \( g_{\text{SF}}^2(1/L_{\text{max}}) \sim 3.7 - 3.8 \) is available in the continuum limit [1]. The PCAC masses at \( \kappa_c = 0.126110 \) is slightly off the vanishing point as seen in (P4a) and (G4a) runs. The discrepancy between \( \kappa = 0.12512 \) and \( \kappa_c \) are assigned to the systematic error of \( Z_m \).

Combining the RG evolutions for \( Z_P \) and the coupling in both schemes, we can extract the mass renormalization constant \( Z_m^{\overline{\text{MS}}}(g_0, \mu) \) in the \( \overline{\text{MS}} \) scheme [8, 9, 10, 11] by

\[
Z_m^{\overline{\text{MS}}}(g_0, \mu) = \left( \frac{m^{\overline{\text{MS}}}(\mu)}{m^{\text{RGI}}} \right) \left( \frac{M^{\text{RGI}}}{m^{\overline{\text{MS}}}(1/L_{\text{max}})} \right) \left( \frac{Z_A^{\text{SF}}(g_0, a/L)}{Z_{\text{SF}}(g_0, a/L_{\text{max}})} \right),
\]

where \( L_{\text{max}} = 4a \) will be applied. The mass ratios between the renormalization group invariant (RGI) mass \( M^{\text{RGI}} \) and the renormalized masses \( m^{\overline{\text{MS}}} \) (or \( m^{\overline{\text{MS}}} \)) are defined by

\[
\left( \frac{M^{\text{RGI}}}{m^{\overline{\text{MS}}}(\mu)} \right) = (2b_0 \bar{g}^2(\mu))^{-d_0/(2b_0)} \exp \left[ - \int_0^{\bar{g}(\mu)} \frac{d\bar{g}}{\bar{\beta}(\bar{g})} \left( \frac{\tau(\bar{g})}{\bar{\beta}(\bar{g})} - \frac{d_0}{b_0 \bar{g}} \right) \right]_{\overline{\text{MS}}},
\]

\[
\left( \frac{M^{\text{RGI}}}{m^{\overline{\text{MS}}}(1/L_{\text{max}})} \right) = \prod_{j=1}^n \sigma_p(u_j) \left( 2b_0 \bar{g}^2(1/L_n) \right)^{-d_0/(2b_0)} \exp \left[ - \int_0^{\bar{g}(1/L_n)} \frac{d\bar{g}}{\bar{\beta}(\bar{g})} \left( \frac{\tau(\bar{g})}{\bar{\beta}(\bar{g})} - \frac{d_0}{b_0 \bar{g}} \right) \right]_{\text{SF}}.
\]

\( \bar{g}^2(\mu) \) is the renormalized coupling constant in the \( \overline{\text{MS}} \) scheme for Eq. (3.2), and that in the SF scheme for Eq. (3.3). \( \sigma_p(u) \) is the step scaling function for \( Z_P \) in the SF scheme. The argument \( u \) is the renormalized coupling defined by \( u_j = g_{\text{SF}}^2(2^j/L_{\text{max}}) \) which is evolved from \( u_0 = g_{\text{SF}}^2(1/L_{\text{max}}) \) using the step scaling function \( \sigma(u) \) for the coupling via \( u_{j+1} = \sigma(u_j) \).

In order to evaluate the mass renormalization constant \( Z_m^{\overline{\text{MS}}}(g_0, \mu) \), we employ \( \sigma_p(u) \) from Ref. [8] and \( \sigma(u) \) from Ref. [9]. The number of steps \( n \) is chosen to be 5 from which we can evaluate the exponent of Eq. (3.3) perturbatively. The two-loop mass anomalous dimension \( \tau(g) \) [12] and the three-loop beta function \( \beta(g) \) [13] are used in the SF scheme, while the four-loop estimates for the mass anomalous dimension and beta function are employed in the \( \overline{\text{MS}} \) scheme in evaluating the mass renormalization constant. In order to evaluate the mass renormalization constant at the reference scale \( \mu = 2 \) [GeV], we need the scale of \( a^{-1} \) in physical unit at \( \beta = 1.82 \).

4. Results

The axial and vector current renormalization factors at \( \beta = 1.82 \) are evaluated as

\[
Z_A = 0.9650(68)(95),
\]

\[
Z_V = 0.95153(76)(1487),
\]
where the central values are from (A1L). The first error is the statistical one and the second is the systematic one which is evaluated from the discrepancy between the two runs.

The mass renormalization constant $Z_{m}^{\text{MS}}(g_{0}, \mu)$ at $\mu = 2$ [GeV] and $\beta = 1.82$ is evaluated as

$$Z_{m}^{\text{MS}}(g_{0}, \mu = 2\text{[GeV]}) = 0.9950(111)(89),$$

where the central value is extracted by combining the factors $Z_A$ from (A1L), $Z_P$ from (P4a), and $u_0$ from (G4a). For the scale $a^{-1}$, we use the preliminary value $a^{-1} = 2.332(18)$ [GeV] from Ref. [14]. The first error is the statistical error and the second is the systematic one estimated from the discrepancy to the mass renormalization constant evaluated using (P4b) and (G4b).

**5. Summary**

In this poster, we presented the determination of the renormalization constants for the axial and vector currents and the quark mass in the Schrödinger functional scheme. The values in Eqs. (4.1)-(4.3) were obtained for the RG-improved Iwasaki gluon and three flavors of the stout smeared quarks at $\beta = 1.82$. By applying these renormalization factors to the results from the simulation on the $96^4$ lattice, we can obtain the quark masses and the decay constants precisely [14].

**Acknowledgments**

The numerical computations were performed using a PC-cluster of HPCI Strategic Program Field 5 and the HA-PACS GPU cluster system at the Center for Computational Sciences, University of Tsukuba. This work was supported in part by the Grant-in-Aid for Scientific Research (Nos. 15K05068, 25800138) from the Japan Society for the Promotion of Science (JSPS), and by MEXT and JICFuS as a priority issue (Elucidation of the fundamental laws and evolution of the universe) to be tackled by using Post “K” Computer.

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