Comparative variational studies of $0^+$ states in three-$\alpha$ models

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Abstract

Three variational approaches, the hyperspherical-harmonics, Gaussian-basis and Lagrange-mesh methods involving different coordinate systems, are compared in studies of $0^+$ bound-state energies in $3\alpha$ models. Calculations are performed with different versions of the shallow Ali-Bodmer potential (with and without Coulomb) and with the deep Buck-Friedrich-Wheatley potential. All three methods yield very accurate energies. Their advantages and drawbacks are evaluated. The implications of the disagreement between the obtained results and the experimental $^{12}\text{C}$ energies in $3\alpha$ models with Coulomb interaction are discussed.

1 Introduction

The analysis of the structure of halo nuclei such as $^6\text{He}$ and $^{11}\text{Li}$ has renewed the interest for accurate calculations of three-body systems. Various dynamical approaches to the three-body problem are known: the hyperspherical-harmonics method [1, 2], the Faddeev coordinate-space method [3, 4], the variational Gaussian-expansion method [5, 6, 7, 8], the Lagrange-mesh method [9, 10], ... Comparing the respective merits of such approaches is not easy because different groups usually make different choices for the conditions of calculation and even for the physical constants. Moreover the accuracy of the presented results is not always discussed.

Therefore, we think that it is timely to make a comparison of three of the available methods which are in our expertise field. Our aim is to provide results with established high accuracy to which other methods can be compared, under exactly the same conditions. To this end, we have selected a simple example: the $3\alpha$ system with realistic local $\alpha$-$\alpha$ potentials. This example contains all the difficulties of this type of problem: (i) Coulomb interaction between all particles, (ii) orbital-momentum dependence of the nuclear interaction, (iii) occurrence of forbidden states in the two-body interactions, (iv) full symmetrization of the three-boson system. Methods of calculation can not

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always accurately treat all these difficulties. On physical grounds, this system is also interesting for several reasons.

The three-α-boson system has a long history in nuclear physics (see Refs. [11, 12] for discussions of early results). Accurate calculations could be performed with purely local α-α realistic forces of the shallow type [13]. They clearly indicated that this model does not provide a realistic description of the $^{12}$C nucleus [14, 15]. Non-local two-body forces and/or three-body forces arising from the Pauli antisymmetrization play a non-negligible role. Strangely, a fair qualitative description of the bound spectrum is obtained when the Coulomb forces are ignored [14, 15]. Therefore three-body attractive forces can solve the problem, at least phenomenologically [12, 13].

Deep potentials involving forbidden states are supposed to simulate the effect of the Pauli principle [16, 17]. Performing accurate calculations with such forces turned out to be much more complicated because the two-body forbidden states must be eliminated from the three-body calculation. Otherwise, many unphysical states show up [18]. Attempts to solve that problem have led to contradictory results [19, 20, 18, 21]. Because of computer time limitations, whether the results obtained in Ref. [21] by one of us have reached convergence could not be established. It is thus timely to reexamine this question with several accurate methods and to establish firmly the $3\alpha$ bound states energies.

The $3\alpha$ system has also recently gained a lot of interest as a possible example of a new phenomenon: the possible Bose-Einstein condensation of α bosons in the light nuclei $^{12}$C, $^{16}$O, ... In Ref. [22], on the basis of microscopic calculations, it was proposed that the astrophysically significant $0^+_2$ state of the $^{12}$C nucleus at the excitation energy 7.65 MeV has a dominant relative s-wave structure as expected for a boson condensate. Moreover, a Generator Coordinate Method study [23] of the $^{12}$C and $^{16}$O nuclei structure predicts the existence of near-threshold $n\alpha$ states which would be the analog for finite systems of an $\alpha$ condensation in infinite matter. However, recent variational calculations based on point-like bosons in 3$\alpha$ and 4$\alpha$ models with a local $\alpha$-$\alpha$ force obtain rather small s-wave components and lead to the conclusion that an interpretation as a Bose-Einstein condensation is too far from reality for these nuclei [24]. In this physical issue, it is important to work with as exact as possible wave functions. The accuracy of the results must be ensured to validate any physical interpretation.

The aim of the present paper is to compare the results and merits of three different approaches of the $3\alpha$ bound-state problem for two types of effective $\alpha$-$\alpha$ interactions. The compared methods are the three-dimensional Lagrange-mesh method (LMM) [9, 10], the hyperspherical-harmonics method (HHM) on a Lagrange mesh [25], and the variational method on a Gaussian basis (VGM) [7, 21]. The LMM is under some conditions very accurate and can help calibrating the other two. The HHM is representative of variational techniques in hyperspherical coordinates. The VGM and its extension the stochastic variational method are versatile tools which can also treat larger numbers of particles.

The selected effective $\alpha$-$\alpha$ interactions are the shallow AB potential of Ali and Bodmer with a strong repulsive core [13] and the deep BFW potential of Buck, Friedrich and Wheatley [17]. For elastic $\alpha$-$\alpha$ scattering, these potentials are essentially equivalent [26]. However, not only are their form factors very different but their use for a three-body system lead to quite different difficulties. Since the effects of the Pauli principle affecting the twelve nucleons can not be treated in an exact way in a $3\alpha$ model,
one can try to simulate them with a microscopically founded potential. The deep BFW potential possesses unphysical bound states in the lowest s and d waves of relative motion to simulate the Pauli-forbidden states. However these forbidden states can not be kept in the three-body calculation because it becomes very difficult to find the physical states among the many eigenvalues of the Hamiltonian [18]. The forbidden states can be approximately eliminated from the solutions of the three-body Schrödinger equation with the method of orthogonalising pseudopotentials [27]. However, this 3α model only takes into account two-body Pauli effects due to nucleon exchanges between two α particles. Using the full three-body Pauli projector is a very complicated problem [21] which will not be considered here.

The 3α model is described in section 2, with emphasis on the potential choice. In section 3, the three methods of resolution of the three-body Schrödinger equation are summarized. In section 4, the results and the respective merits and limitations of the methods are compared. Some comments are made in section 5. Concluding remarks are presented in section 6.

2 Model and potentials

The Hamiltonian of three identical bosons with mass \( m_\alpha \) has the simple form

\[
H = -\frac{\hbar^2}{2m_\alpha}(\nabla_1^2 + \nabla_2^2 + \nabla_3^2) + V(r_{23}) + V(r_{31}) + V(r_{12}),
\]

where \( r_{ij} = r_j - r_i \) is the relative coordinate between particles \( j \) and \( i \). For deep two-body potentials with forbidden states in some partial waves, we use the method of orthogonalising pseudopotentials [27, 21]. Each two-body potential \( V(r_{ij}) \) of the Hamiltonian is replaced by the corresponding pseudopotential

\[
\tilde{V}(r_{ij}) = V(r_{ij}) + \Lambda \sum_f \hat{\Gamma}^{(f)}_{ij},
\]

where the sum runs over the forbidden states. The constant \( \Lambda \) must be taken large enough to push the forbidden states to high energies. The projector on a two-body forbidden state \( f \) is given by

\[
\hat{\Gamma}^{(f)}_{ij} = \sum_{m_f} |\varphi_{f_{m_f}}\rangle \langle \varphi_{f_{m_f}}|,
\]

where \( \varphi_{f_{m_f}}(r_{ij}) \) is the wave function of a forbidden state between \( \alpha \) particles \( i \) and \( j \). As shown below, this wave function must be accurate. Because of the symmetry properties of identical bosons, only even partial waves of the \( \alpha-\alpha \) potentials contribute.

We use \( \hbar^2/m_\alpha = 10.4465 \) MeV fm\(^2\) and \( e^2 = 1.44 \) MeV fm in all calculations. The Coulomb interaction is taken as

\[
V_C(r) = 4e^2 \text{erf} (\beta r)/r.
\]

The first model of the \( \alpha-\alpha \) interaction used in this work is the AB potential d of Ref. [13],

\[
V_{AB}(r) = V_1 \exp(-\eta_1 r^2) + V_2 \exp(-\eta_2 r^2) + V_C(r).
\]
The $s$-wave parameters are $V_1 = 500$ MeV, $\eta_1 = 0.49$ fm$^{-2}$ and $V_2 = -130$ MeV, $\eta_2 = 0.225625$ fm$^{-2}$ for potential d. In the simplified $s$-wave version (ABd$_0$), these parameters are used in all partial waves and we choose $\beta = 0.75$ fm$^{-1}$ like in the BFW potential below. In the $l$-dependent potential (ABd), the parameters are as in the $s$ wave, except $V_1 = 320$ MeV for $l = 2$, $V_1 = 0$ for $l \geq 4$ and $\beta = \sqrt{3}/(2 \times 1.44)$ in the Coulomb potential for all $l$ [13]. The second model makes use of the BFW potential of Ref. [17],

$$V_{\text{BFW}}(r) = V_0 \exp(-\eta_0 r^2) + V_C(r),$$

(6)

with $V_0 = -122.6225$ MeV, $\eta_0 = 0.22$ fm$^{-2}$ and $\beta = 0.75$ fm$^{-1}$. Both potentials describe fairly well the experimental phase shifts of the $\alpha$-$\alpha$ scattering for $l = 0, 2, 4$ up to about 20 MeV. The BFW potential has three nonphysical bound states forbidden by the Pauli principle at the energies $E(0^+_1) = -72.625691755$ MeV, $E(0^+_2) = -25.618638588$ MeV and $E(2^+) = -22.000501732$ MeV. The $0^+_1$ state corresponds to the forbidden shell configuration $s^8$, while the $0^+_2$ and $2^+$ states correspond to $s^6p^2$.

3 Methods of resolution of the three-body Schrödinger equation

Basically, the three methods that we consider are all variational calculations of the three-body bound-state energies, but with different coordinate systems on one hand and different types of basis on the other hand. Two of the methods contain an additional Gauss-quadrature approximation. The three methods are based on a minimization of the expectation value of the Hamiltonian over linear or nonlinear variational parameters in a trial wave function. The comparison of their results is thus a significant test.

The LMM [9, 10] makes use of three-dimensional Lagrange basis functions, i.e. infinitely differentiable functions which vanish at all points of an associated mesh, with the exception of one. Because of the Gauss quadrature associated with the mesh, the potential matrix in the Lagrange basis is diagonal and its diagonal elements are the potential values at mesh points. Remarkably the Lagrange-mesh method appears to be as accurate as the corresponding variational calculation [28, 25]. However, it does not apply to $l$-dependent potentials.

The Lagrange basis functions are defined in the system of perimetric coordinates

$$x = r_{12} - r_{32} + r_{13},$$

$$y = r_{12} + r_{32} - r_{13},$$

$$z = -r_{12} + r_{32} + r_{13},$$

(7)

where $r_{ij} = |r_i - r_j|$ is the distance between two particles. The basis functions read

$$\Psi^{0^+}(x,y,z) = \sum_{ijk} C_{ijk} S f_i(x/h)f_j(y/h)f_k(z/h)$$

(8)

where $S$ is the symmetrization projector and $h$ is a scaling parameter. The one-dimensional Lagrange functions read

$$f_i(x/h) = (-1)^i (hu_i)^{1/2} \frac{L_N(x/h)}{x - hu_i} e^{-x/2h}$$

(9)
where $L_N(u)$ is the Laguerre polynomial of degree $N$ and $u_i$ is one of its zeros, i.e. $L_N(u_i) = 0$. The interest of this approach is that the mesh equations resulting from the Gauss approximation are rather simple and only involve potential values at the $(hu_i, hu_j, hu_k)$ mesh points (see Ref. [9] for details).

In the following the calculations are performed with up to $N = 38$ mesh points for each of the three dimensions. The scaling factor $h$ is around 0.5 fm for $0_1^+$ and 1.2 fm for $0_2^+$ (the results are independent of the precise value of $h$). After symmetrization, the size of the largest matrix is $N(N+1)(N+2)/6 = 9880$ but computing times are short because the filling rate is about 10%.

In the HHM [1, 2, 25], the variational wave function is expanded over hyperspherical harmonics as

$$
\Psi^{0+}(\rho, \Omega_5) = \sum_{lKl} C_{lKl} S Y^{ll}_{K0}(\Omega_5) \hat{f}_i(\rho/h),
$$

(10)

where $Y^{ll}_{KL}$ are hyperspherical harmonics depending on five angular variables noted as $\Omega_5$. The hyperradial wave functions depend on the rotationally and permutationally invariant hyperradius $\rho$ and are expanded on regularized Lagrange-Laguerre basis functions

$$
\hat{f}_i(\rho/h) = (\rho/hu_i)^{3/2} f_i(\rho/h)
$$

(11)

where $f_i$ is given by Eq. (9). Here also a Gauss approximation eliminates the need for calculations of potential matrix elements and significantly reduces the computing times (see Ref. [25] for details and tests). One of the main advantages of this method is its validity for both bound-state and scattering problems.

Here we use hyperspherical harmonics up to $K_{\text{max}} = 30$ and $N = 30$ hyperradial mesh points with a scaling factor $h = 0.3$ fm. Because of symmetrization, the basis size is $N(K_{\text{max}} + 2)(K_{\text{max}} + 6)/16 = 2160$.

The VGM [5, 6, 7, 21] is a high-accuracy method for the study of the structure of quantum-mechanical few-body systems. Its combination with stochastic methods [8] makes it applicable to few-body problems (up to six clusters at present [29]) in atomic, nuclear and quark physics. The variational wave function reads

$$
\Psi^{0+}(x, y) = \sum_{lkl} C_{lkl} S[Y^l(\hat{x}) \otimes Y^l(\hat{y})]^{00} x^l y^l \exp(-\alpha_i x^2 - \beta_j y^2)
$$

(12)

where $x$ and $y$ are the Jacobi coordinates of any of the three possible sets, and the $\alpha_i$ and $\beta_j$ are non-linear parameters (see Ref. [21] for details and for the choice of non-linear parameters).

Matrix elements of the Hamiltonian are calculated analytically for $l$-independent potentials with simple form factors such as ABd0 (see [7, 8]). A numerical evaluation is necessary for the moments of potentials depending on $l$ or with a complicated form factor. However, in most cases, one can use recurrence formulas [21] which allow to tabulate these moments. The high flexibility of a many-particle Gaussian basis makes it possible to describe several-particle configurations that are formed in the ground and excited states of multicluster systems [7]. For $l$-dependent potentials, extensions of the code are required with respect to Ref. [21]. If the potential term depends on a specific Jacobi coordinate $x$, it is necessary to perform a change of variable in the basis functions appearing in the matrix element and depending on another set of Jacobi coordinates in order to express them in the corresponding $(x, y)$ set. After an analytic
integration over $y$ and over the angular components of $x$, the remaining integration over $x$ is performed numerically. A drawback of the VGM is the non-orthogonality of the basis which may restrict the basis size. In the VGM calculations below, the number of Gaussians is 680, except otherwise indicated.

4 Results

The three numerical methods are used to determine energy values for the ground and lowest excited bound states of the $3\alpha$ system with $J^\pi = 0^+$. The numerical results obtained with the ABd$_0$ potential are presented in Table 1. As in Ref. [24], let us first discuss calculations without the Coulomb term. The three methods yield close numbers which prove a high accuracy of all of them. The LMM is faster and more accurate. Its accuracy can be estimated by varying the scaling factor $h$ and the number $N$ of mesh points. From these tests, all displayed digits should be correct. The LMM accuracy is much better for the ground state than for the excited state. The error of the VGM is about $2 \times 10^{-6}$ MeV. With 372 Gaussians, the results are $-5.12205$ and $-1.3523$ MeV while with 280 Gaussians they are $-5.1215$ and $-1.341$ MeV. The HHM accuracy is $10^{-3}$ MeV for the ground state but is less good for the excited state. To improve this accuracy, higher values of $K_{\text{max}}$ should be used.

With the Coulomb term in ABd$_0$, only one weakly bound ground state is obtained (see Table 1). The convergence of all methods is slower in that case. The accuracy is better than $10^{-8}$ for the LMM, $10^{-5}$ for the VGM and $10^{-3}$ for the HHM, in MeV. The $3\alpha$ energy is far from the experimental $^{12}\text{C}$ energy $-7.275$ MeV [30] because off-shell effects are not well reproduced by the potential. Other Coulomb terms lead to similar results. With the Coulomb parameter $\beta$ from the AB paper, one finds $-0.61733886$. For the point Coulomb potential $-4e^2/r$, the energy is $-0.57238115$. With the value $\hbar^2/m_\alpha = 10.36675$ of Ref. [24], we obtain $-5.18093389$ and $-0.62106627$ MeV without and with point Coulomb, respectively, in excellent agreement with the values $-5.18$ and $-0.62$ obtained by these authors.

Results with the full ABd potential are also presented in Table 1. The LMM does not apply to $l$-dependent potentials. We think that the accuracies of VGM and HHM are comparable to the ABd$_0$ case. They agree within $10^{-3}$ MeV. The binding energy is increased by about 1.3 MeV (without Coulomb) or 1 MeV (with Coulomb) with respect to the ground-state energy of the ABd$_0$ potential. In spite of this increase, the obtained energy remains not very realistic for $^{12}\text{C}$.

Energies of the $3\alpha$ ground state calculated with the BFW potential including the Coulomb term are presented in Table 2 for the HHM and VGM. Because of the projection on forbidden states, the LMM can not be applied. To show the convergence of the orthogonalising-pseudopotentials method, we present numerical results for several values of the projection parameter $\Lambda$ ($10^5$, $10^6$, $10^7$, $10^8$ and $10^9$ MeV). It was shown in Ref. [21] that lower $\Lambda$ values can not fully eliminate the effect of the forbidden states. The VGM calculations are here performed with 280 Gaussians. Including more Gaussians leads to a numerical instability for large $\Lambda$ values. However, this rather small Gaussian basis already yields a good estimation for the ground-state energy as shown by the close agreement between the two variational approaches for each $\Lambda$ value. They agree within one or two percent. A fair convergence with respect to $\Lambda$ is obtained for
values higher than $10^7$ MeV. Extrapolation with respect to $K_{\text{max}}$ in the HHM yields $E = -0.30$ for $\Lambda = 10^7$, $-0.28$ for $10^8$ and $-0.27$ for $10^9$ (in MeV). The best result of Ref. [21] ($-0.283$ for $N = 7$ and $\Lambda = 10^8$) is not far from these values. The apparent lack of convergence observed in that work is due to a comparison with calculations with an insufficient accuracy on the forbidden states.

The $3\alpha$ binding energy is 0.3 MeV smaller with the BFW potential than with the ABd$_0$ potential and 1.25 MeV smaller than with the ABd potential. The convergence of our results allows us to draw a conclusion that was not accessible with the basis sizes in Ref. [21]: a deep local potential accurately reproducing the $\alpha$-$\alpha$ phase shifts does not provide a better description of the $3\alpha$ system than shallow potentials. Independently of the choice of local potential, the $3\alpha$ model is not valid for the shell-model-like states of $^{12}\text{C}$.

5 A few comments on the $3\alpha$ model

First we must emphasize the quality of the results obtained by Visschers and Van Wageningen more than 30 years ago [14]. For ABd without Coulomb, they obtain $-6.37$ MeV for the ground-state energy, only 0.05 MeV about our result. This result is excellent taking into account the computer limitations of that time. With Coulomb, their result is less good as expected for the convergence of a smaller binding energy.

Reliable results now exist for a deep potential. They confirm that the $3\alpha$ model with local forces is not able to reproduce the $^{12}\text{C}$ ground-state energy. The deep-potential energy is even above the shallow-potential one. A similar effect is observed for the three-nucleon system [31] but not for the $^6\text{He}$ halo nucleus [25].

An attractive three-body force can be used to cure this problem [12, 3]. The effective three-body force employed in Ref. [3] depends however on the hyperradius $\rho$ and is specific for the HHM (see also Ref. [2]). This type of form factor has little physical meaning.

The origin of the weakness of the $3\alpha$ model should be understandable microscopically. Microscopic $3\alpha$-cluster models [32, 33] provide results much closer to experiment. With an effective nucleon-nucleon interaction, they are able to reproduce fairly well the $\alpha$-$\alpha$ scattering and the $^{12}\text{C}$ properties, simultaneously. In particular, the $^{12}\text{C}$ ground state has a compact shell-model structure which can be reproduced by overlapping microscopic $\alpha$ clusters. On the contrary, the $3\alpha$-boson picture with local, deep or shallow, $\alpha$-$\alpha$ forces is far from reality. States with a compact shell-model structure are not satisfactorily described by the model. Moreover, the high sensitivity of the ground-state binding energy of the $3\alpha$ system to the description of the two-body forbidden states [21] shows the importance of a correct treatment of the Pauli principle in cluster-model calculations.

The success of 12-nucleon descriptions of the $3\alpha$ system has inspired a model which is apparently very similar to ours but which is based on a quite different philosophy. In this model local $\alpha$-$\alpha$ interactions are obtained by folding an effective nucleon-nucleon interaction and the forbidden states are those of the microscopic $\alpha$-$\alpha$ norm kernel. These forbidden states are simple oscillator states which are used in the pseudopotential [3] in place of the exact bound states of the potential. This procedure leads to an overbinding of the $3\alpha$ system [31], in opposition with our results.
Using oscillator forbidden states with the BFW potential would be inconsistent. In the simple $3\alpha$ model that we have discussed, no information about the underlying structure of the $\alpha$ particle is available. The choice of the oscillator parameter in the forbidden states derived from a microscopic 12-body calculation would be quite arbitrary. We have tested this variant by replacing in the pseudopotential the exact forbidden wave functions of the BFW potential by oscillator wave functions. According to the choice of oscillator parameter, various ground-state energies can be obtained. For each choice, the energy is much lower than the energy $-0.26$ MeV given in Table 2 since the bound states of the BFW potential are only partly eliminated. In principle, with arbitrary forbidden-state wave functions, the pseudopotential technique can provide any result between the energy $-240.65$ MeV derived without elimination of forbidden states and the energy $-0.26$ MeV corresponding to full elimination. The more-or-less arbitrary choice of the wave functions of the forbidden state to be eliminated becomes a parameter of the model with which the experimental $^{12}\text{C}$ binding energy could be reproduced or which can lead to overbinding as well as underbinding.

The preceding discussion concerns local $\alpha-\alpha$ potentials. The role of non-locality in the interaction is not yet clearly evaluated. An attempt to elucidate this problem within the $3\alpha$ model has been performed in Ref. [35] where a non-local interaction derived from a resonating-group microscopic calculation fitting the $\alpha-\alpha$ scattering provides a better-bound $^{12}\text{C}$ ground state. However, the calculation and the elimination of forbidden states are directly performed with the $\alpha-\alpha$ $T$-matrix and the corresponding potential obtained after this elimination (which is presumably also non-local) is not available.

6 Conclusion

To summarize, the energies of the $0^+$ ground and lowest excited states in the $3\alpha$ model were evaluated in the framework of three variational approaches based on different coordinate systems. The LMM is faster and more accurate but is restricted to potentials which do not depend on the orbital momentum. The VGM is flexible and accurate but is more difficult to apply with $l$-dependent potentials. The non-orthogonality of the basis may cause problems. The slowness of convergence with respect to $K$ makes the HHM slightly less accurate than the other two. However, it can easily be adapted to $l$-dependent potentials and be extended to positive energies. The elimination of forbidden states with pseudopotentials shows convergence with respect to the parameter $\Lambda$ in contradiction with the conclusion of Ref. [21]. In order to reach convergence, the wave functions of the forbidden states must be very accurate.

On the physical side, when used in a realistic way with the Coulomb interaction, both AB and BFW $\alpha-\alpha$ potentials yield poor results for the $3\alpha$ system. The $3\alpha$ binding energies are too small compared with the $^{12}\text{C}$ experimental value. The $^{12}\text{C}$ ground-state structure can be described very well with microscopic $3\alpha$ models [32, 33] but it cannot be described with three-$\alpha$-boson models using realistic local interactions. The role of nonlocality in the two-body interaction still needs to be understood. It has recently started to be explored in Ref. [35].
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Table 1: $0^+$ energies (in MeV) of the $3\alpha$ system calculated with the $s$-wave ($\text{ABd}_0$) or $l$-dependent ($\text{ABd}$) Ali-Bodmer potentials of d type.

| state   | LMM     | HHM     | VGM     |
|---------|---------|---------|---------|
|         | $\text{ABd}_0$ without Coulomb |         |         |
| $0^+_1$ | $-5.122093595$ | $-5.1219$ | $-5.1220913$ |
| $0^+_2$ | $-1.3606$ | $-1.20$ | $-1.3566$ |
|         | $\text{ABd}_0$ with Coulomb ($\beta = 0.75$) |         |         |
| $0^+_1$ | $-0.58427008$ | $-0.5836$ | $-0.584266$ |
|         | $\text{ABd}$ without Coulomb |         |         |
| $0^+_1$ | $-6.423$ | $-6.42285$ |         |
| $0^+_2$ | $-1.92$ | $-1.934$ |         |
|         | $\text{ABd}$ with Coulomb |         |         |
| $0^+_1$ | $-1.523$ | $-1.523$ |         |
Table 2: $3\alpha$ ground-state energy calculated with the BFW potential for different values of the projection parameter $\Lambda$ (in MeV).

| $\Lambda$ | HHM   | VGM   |
|-----------|-------|-------|
| $10^5$    | $-0.638$ | $-0.644$ |
| $10^6$    | $-0.416$ | $-0.426$ |
| $10^7$    | $-0.284$ | $-0.288$ |
| $10^8$    | $-0.261$ | $-0.263$ |
| $10^9$    | $-0.259$ | $-0.261$ |