Some linear Jacobi structures on vector bundles

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Abstract. We study Jacobi structures on the dual bundle $A^*$ to a vector bundle $A$ such that the Jacobi bracket of linear functions is again linear and the Jacobi bracket of a linear function and the constant function 1 is a basic function. We prove that a Lie algebroid structure on $A$ and a 1-cocycle $\phi \in \Gamma(A^*)$ induce a Jacobi structure on $A^*$ satisfying the above conditions. Moreover, we show that this correspondence is a bijection. Finally, we discuss some examples and applications.

Quelques structures de Jacobi linéaires sur des fibrés vectoriels

Résumé. On étudie des structures de Jacobi sur le fibré dual $A^*$ d’un fibré vectoriel $A$ tels que le crochet de Jacobi de fonctions linéaires est à nouveau linéaire et le crochet de Jacobi d’une fonction linéaire et la fonction constante 1 est une fonction basique. On démontre qu’une structure d’algébroïde de Lie sur $A$ et un 1-cocycle $\phi \in \Gamma(A^*)$ induisent une structure de Jacobi sur $A^*$ qui vérifie les conditions antérieures. On voit aussi que cette correspondance est une bijection. On montre finalement quelques exemples et applications.

Version française abrégée

Soit $M$ une variété différentiable et $\pi : A \to M$ un fibré vectoriel sur $M$.

Un cocycle pour une structure d’algébroïde de Lie sur $\pi : A \to M$ est une section $\phi$ du fibré dual $A^* \to M$ telle que $\phi[\mu, \eta] = \rho(\mu)(\phi(\eta)) - \rho(\eta)(\phi(\mu))$, pour tout $\mu, \eta \in \Gamma(A)$, où $[,]$ est le crochet de Lie sur l’espace $\Gamma(A)$ des sections de $\pi : A \to M$ et $\rho : A \to TM$ est l’application ancre (voir [13]). On dénote donc par $\tilde{A}$ l’ensemble des paires $(([,] , \rho), \phi)$, où $([,] , \rho)$ est une structure d’algébroïde de Lie sur $\pi : A \to M$ et $\phi \in \Gamma(A^*)$ un 1-cocycle. D’ailleurs, on dénote par $\mathcal{J}$ l’ensemble des structures de Jacobi $(\Lambda, E)$ sur $A^*$, lesquelles satisfont les deux conditions suivantes:

(C1) Le crochet de Jacobi de deux fonctions linéaires est linéaire.

(C2) Le crochet de Jacobi d’une fonction linéaire et la fonction constante 1 est une fonction basique.

On démontre donc, dans cette note, qu’il y a une correspondance bijective $\Psi : \tilde{A} \to \mathcal{J}$ entre les ensembles $\tilde{A}$ et $\mathcal{J}$. L’application $\Psi$ est définie par $\Psi(([[,] , \rho), \phi) = (\Lambda_{(A^*, \phi)}, E_{(A^*, \phi)})$ avec

$\Lambda_{(A^*, \phi)} = \Lambda_{A^*} + \Delta \wedge \phi^v$, \hspace{1cm} $E_{(A^*, \phi)} = -\phi^v$,

où $\Lambda_{A^*}$ est le bi-vecteur de Poisson sur $A^*$ induit par la structure d’algébroïde de Lie $([,], \rho)$ (voir [2, 3]), $\Delta$ est le champ de Liouville sur $A^*$ et $\phi^v$ est le relèvement vertical de $\phi$. Observons que les paires dans $\tilde{A}$ de la forme $(([[,] , \rho), 0)$ correspondent, à travers $\Psi$, aux structures de Poisson dans $\mathcal{J}$. Ainsi, comme conséquence, on déduit un résultat démontré dans [2, 3].

Les conditions (C1) et (C2) établies ci-dessus sont naturelles. En fait, on démontre que celles-ci sont vérifiées pour quelques structures de Jacobi, bien connues et importantes, définies sur l’espace
total de quelques fibrés vectoriels. En même temps, la correspondance Ψ nous permet d’obtenir de nouveaux et intéressants exemples de structures de Jacobi. On voit finalement, comme une autre application, qu’une structure d’algébroïde de Lie sur un fibré vectoriel \( A \rightarrow M \) et un 1-cocycle \( φ \in Γ(A^*) \) induisent une structure d’algébroïde de Lie sur le fibré vectoriel \( A \times \mathbb{R} \rightarrow M \times \mathbb{R} \).

1 Jacobi manifolds and Lie algebroids

Let \( M \) be a differentiable manifold of dimension \( n \). We will denote by \( C^∞(M, \mathbb{R}) \) the algebra of \( C^∞ \) real-valued functions on \( M \), by \( Ω^1(M) \) the space of 1-forms, by \( X(M) \) the Lie algebra of vector fields and by \( [\, , \, ] \) the Lie bracket of vector fields.

A Jacobi structure on \( M \) is a pair \((\Lambda, E)\), where \( \Lambda \) is a 2-vector and \( E \) is a vector field on \( M \) satisfying the following properties:

\[
([\Lambda, \Lambda])_{SN} = 2E \wedge \Lambda, \quad [E, \Lambda]_{SN} = 0. \tag{1}
\]

Here \([\, , \, ]_{SN}\) denotes the Schouten-Nijenhuis bracket \((I, IV)\). The manifold \( M \) endowed with a Jacobi structure is called a Jacobi manifold. A bracket of functions (the Jacobi bracket) is defined on \( M \) by \( \{f, g\} = \Lambda(d\bar{f} \wedge dg) + f\bar{E}(g) - g\bar{E}(f) \), for all \( f, g \in C^∞(M, \mathbb{R}) \). Note that

\[
\{\bar{f}, \bar{g}\} = \bar{g}\{\bar{f}, \bar{h}\} + \bar{h}\{\bar{f}, \bar{g}\} - \bar{g}\bar{h}\{\bar{f}, 1\}. \tag{2}
\]

In fact, the space \( C^∞(M, \mathbb{R}) \) endowed with the Jacobi bracket is a local Lie algebra in the sense of Kirillov (see \([8]\)). Conversely, a structure of local Lie algebra on \( C^∞(M, \mathbb{R}) \) defines a Jacobi structure on \( M \) (see \([3]\)). If the vector field \( E \) identically vanishes then \((M, \Lambda)\) is a Poisson manifold. Jacobi and Poisson manifolds were introduced by Lichnerowicz (\([10, 11]\)) (see also \([3, 4, 12]\)).

A Lie algebroid structure on a differentiable vector bundle \( π : A \rightarrow M \) is a pair that consists of a Lie algebra structure \([\, , \, ]\) on the space \( Γ(A) \) of the global cross sections of \( π : A \rightarrow M \) and a homomorphism of vector bundles \( ρ : A \rightarrow TM \), the anchor map, such that if we also denote by \( ρ : Γ(A) \rightarrow X(M) \) the homomorphism of \( C^∞(M, \mathbb{R}) \)-modules induced by the anchor map then: (i) \( ρ : (Γ(A), [\, , \, ]) \rightarrow (X(M), [\, , \, ]) \) is a Lie algebra homomorphism and (ii) for all \( f, g \in C^∞(M, \mathbb{R}) \) and for all \( μ, η \in Γ(A) \), one has \([μ, f] = f[μ, η] + (ρ(μ)(f))η \) (see \([13]\)).

If \((A, [\, , \, ], ρ)\) is a Lie algebroid over \( M \), one can introduce the Lie algebroid cohomology complex with trivial coefficients (for the explicit definition of this complex we remit to \([13]\)). The space of 1-cochains is \( Γ(A^*) \), where \( A^* \) is the dual bundle to \( A \), and a 1-cochain \( φ \in Γ(A^*) \) is a 1-cocycle if and only if

\[
\phi[μ, η] = ρ(μ)(φ(η)) - φ(ρ(η)(μ)), \quad \text{for all} \quad μ, η \in Γ(A). \tag{3}
\]

A Jacobi manifold \((M, \Lambda, E)\) has an associated Lie algebroid \((T^*M \times \mathbb{R}, [, ], #, (\Lambda, E))\), where \( T^*M \) is the cotangent bundle of \((M, E)\), \( #, (\Lambda, E) \) are defined by

\[
[(α, \bar{f}), (β, \bar{g})](\Lambda, E) = \left( \mathcal{L}_{#, (\Lambda)}β - \mathcal{L}_{#, (\Lambda)}α - d(Λ(α, β)) + \bar{f}\mathcal{L}_E β - \bar{g}\mathcal{L}_E α - i_E(α \wedge β), \right. \\
\left. \Lambda(β, α) + #, (\Lambda)(\bar{g}) - #, (\Lambda)(\bar{f}) + \bar{f}\mathcal{L}_E(\bar{g}) - \bar{g}\mathcal{L}_E(\bar{f}), \right)
\]

\[
#(\Lambda, E)(α, \bar{f}) = #, (\Lambda)(α) + \bar{f}E, \tag{4}
\]

for \((α, \bar{f}), (β, \bar{g}) \in Ω^1(M) \times C^∞(M, \mathbb{R})\), \( \mathcal{L} \) being the Lie derivative operator and \( #, (\Lambda) : Ω^1(M) \rightarrow X(M) \) the mapping given by \( β(#, (\Lambda)(α)) = Λ(α, β) \) (see \([9]\)).

In the particular case when \((M, \Lambda, E)\) is a Poisson manifold we recover, by projection, the Lie algebroid \((T^*M, [, ], #, (\Lambda, E))\), where \([, ]\) is the bracket of 1-forms defined by (see \([3, 2, 14]\)):

\[
[\, , \, ]_\Lambda : Ω^1(M) \times Ω^1(M) \rightarrow Ω^1(M), \quad [α, β]_\Lambda = \mathcal{L}_{#, (\Lambda)}β - \mathcal{L}_{#, (\Lambda)}α - d(Λ(α, β)).
\]
2 Some linear Jacobi structures on vector bundles

Let \( \pi : A \to M \) be a vector bundle and \( A^* \) the dual bundle to \( A \). Suppose that \( \pi^* : A^* \to M \) is the canonical projection. If \( \mu \in \Gamma(A) \) and \( f \in C^\infty(M, \mathbb{R}) \) then \( \mu \) determines a linear function on \( A^* \) which we will denote by \( \tilde{\mu} \) and \( f = \tilde{f} \circ \pi^* \) is a \( C^\infty \) real-valued function on \( A^* \) which is basic.

Now, assume that \( (A, [,], \rho) \) is a Lie algebroid over \( M \). Then \( A^* \) admits a Poisson structure \( \Lambda_{A^*} \), such that the Poisson bracket of linear functions is again linear (see [2, 3]). The local expression of \( \Lambda_{A^*} \) is given as follows. Let \( U \) be an open coordinate neighbourhood of \( M \) with coordinates \( (x^1, \ldots, x^n) \) and \( \{\epsilon_i\}_{i=1, \ldots, n} \) a local basis of sections of \( \pi : A \to M \) in \( U \). Then, \( (\pi^*)^{-1}(U) \) is an open coordinate neighbourhood of \( A^* \) with coordinates \( (x^i, \mu_j) \) such that \( \mu_j = \tilde{\epsilon}_j \), for all \( j \). In these coordinates the structure functions and the components of the anchor map are

\[
[e_i, e_j] = \sum_{k=1}^n c_{ij}^k e_k, \quad \rho(e_i) = \rho_i^j \frac{\partial}{\partial x^j}, \quad i, j \in \{1, \ldots, n\},
\]

with \( c_{ij}^k, \rho_i^j \in C^\infty(U, \mathbb{R}) \), and the Poisson structure \( \Lambda_{A^*} \) is given by

\[
\Lambda_{A^*} = \sum_{i<j} \sum_k c_{ij}^k \mu_k \frac{\partial}{\partial \mu_i} \wedge \frac{\partial}{\partial \mu_j} + \sum_{i,l} \rho_i^j \frac{\partial}{\partial \mu_i} \wedge \frac{\partial}{\partial x^l}.
\]

Next, we will show an extension of the above results for the Jacobi case.

We will denote by \( \Delta \) the Liouville vector field of \( A^* \) and by \( \phi^* \in \mathcal{X}(A^*) \) the vertical lift of \( \phi \in \Gamma(A^*) \). Note that if \( (x^i, \mu_j) \) are fibred coordinates on \( A^* \) as above and \( \phi = \sum_{i=1}^n \phi_i \epsilon^i \), with \( \phi_i \in C^\infty(U, \mathbb{R}) \) and \( \{\epsilon^i\} \) the dual basis of \( \{\epsilon_i\} \), then

\[
\Delta = \sum_{i=1}^n \mu_i \frac{\partial}{\partial \mu_i}, \quad \phi^* = \sum_{i=1}^n \phi_i \frac{\partial}{\partial \mu_i}.
\]

Thus, using \( [\hbar, \hbar] = 0 \), \( [\cdot, \hbar] \), \( (\cdot, \cdot) \) and \( (\cdot, \cdot, \cdot) \), we deduce

**Theorem 1** Let \( (A, [,], \rho) \) be a Lie algebroid over \( M \) and \( \phi \in \Gamma(A^*) \) a 1-cocycle. Then, there is a unique Jacobi structure \( (\Lambda_{(A^*, \phi)}, E_{(A^*, \phi)}) \) on \( A^* \) with Jacobi bracket \( \{\cdot, \cdot\}_{(A^*, \phi)} \) satisfying

\[
\{\tilde{\mu}, \tilde{\eta}\}_{(A^*, \phi)} = [\tilde{\mu}, \tilde{\eta}], \quad \{\tilde{\mu}, \tilde{f} \circ \pi^*\}_{(A^*, \phi)} = (\rho(\mu)(\tilde{f}) + \phi(\mu)\tilde{f}) \circ \pi^*, \quad \{\tilde{f} \circ \pi^*, \tilde{g} \circ \pi^*\}_{(A^*, \phi)} = 0,
\]

for \( \mu, \eta \in \Gamma(A) \) and \( \tilde{f}, \tilde{g} \in C^\infty(M, \mathbb{R}) \). The Jacobi structure is given by

\[
\Lambda_{(A^*, \phi)} = \Lambda_{A^*} + \Delta \wedge \phi^*, \quad E_{(A^*, \phi)} = -\phi^*.
\]

Now, we will prove a converse of Theorem 1.

**Theorem 2** Let \( \pi : A \to M \) be a vector bundle over \( M \) and let \( (\Lambda, E) \) be a Jacobi structure on the dual bundle \( A^* \) satisfying:

(C1) The Jacobi bracket of linear functions is again linear.

(C2) The Jacobi bracket of a linear function and the constant function 1 is a basic function.

Then, there is a Lie algebroid structure on \( \pi : A \to M \) and a 1-cocycle \( \phi \in \Gamma(A^*) \) such that \( \Lambda = \Lambda_{(A^*, \phi)} \) and \( E = E_{(A^*, \phi)} \).
Proof: Denote by \( \{,\} \) the Jacobi bracket on \( A^* \) induced by the Jacobi structure \( (\Lambda, E) \) and suppose that \( \mu, \eta \in \Gamma(A) \) and that \( f, g \in C^\infty(M, \mathbb{R}) \). If \( \pi^*: A^* \to M \) is the canonical projection, the function \( \{(f \circ \pi^*)\mu, 1\} = \{f\mu, 1\} \) is basic. Thus, from (2) and (C2), we have that
\[
\{f \circ \pi^*, 1\} = 0. \tag{8}
\]
On the other hand, the function \( \{\tilde{\mu}, (f \circ \pi^*)\tilde{\eta}\} = \{\tilde{\mu}, f\tilde{\eta}\} \) is linear. Therefore, from (3), (C1) and (C2), we obtain that the function \( \{\tilde{\mu}, f \circ \pi^*\} \) is basic. Consequently, the Jacobi bracket of a linear function and a basic function is a basic function. In particular, \( \{f \circ \pi^*, (g \circ \pi^*)\tilde{\mu}\} = \{f \circ \pi^*, g\tilde{\mu}\} \) is basic. This implies that (see (3) and (8))
\[
\{f \circ \pi^*, g \circ \pi^*\} = 0. \tag{9}
\]
Now, we define the section \( [\mu, \eta] \) of the vector bundle \( \pi: A \to M \) and the \( C^\infty \) real-valued functions on \( M \), \( \phi(\mu) \) and \( \rho(\mu)(f) \), which are characterized by the following relations
\[
[\mu, \eta] = \{\tilde{\mu}, \tilde{\eta}\}, \quad \phi(\mu) \circ \pi^* = \{\tilde{\mu}, 1\}, \quad \rho(\mu)(f \circ \pi^*) = \{\tilde{\mu}, f \circ \pi^*\} - (f \circ \pi^*)(\{\tilde{\mu}, 1\}). \tag{10}
\]
From (3), (4), (6) and (10), we deduce that \( \phi \) can be considered as a \( C^\infty(M, \mathbb{R}) \)-linear map \( \phi: \Gamma(A) \to C^\infty(M, \mathbb{R}) \) (that is, \( \phi \in \Gamma(A^*) \)) and that \( \rho \) can be considered as a \( C^\infty(M, \mathbb{R}) \)-linear map \( \rho: \Gamma(A) \to \mathfrak{X}(M) \). Moreover, using (2), (4), (10) and the fact that \( \{,\} \) is the Jacobi bracket of a Jacobi structure (see Section 4), it follows that the triple \( (A, [\ [, ] ], \rho) \) is a Lie algebroid over \( M \) and that \( \phi \in \Gamma(A^*) \) is a \( 1 \)-cocycle. Finally, by (2), (10) and Theorem 3, we conclude that \( (\Lambda, E) = (\Lambda_{(A^*, \phi)}, E_{(A^*, \phi)}) \).

Remark 1 That condition (C1) does not necessarily imply condition (C2) is illustrated by the following simple example. Let \( M \) be a single point and \( A^* = \mathbb{R}^2 \) endowed with the Jacobi structure \( (\Lambda, E) \), where \( \Lambda = xy \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \) and \( E = x^2 \frac{\partial}{\partial x} \). It is easy to prove that the Jacobi bracket satisfies (C1) but not (C2).

Let \( M \) be a differentiable manifold and \( \pi: A \to M \) a vector bundle. Denote by \( A \) and \( \mathcal{J} \) the following sets. \( A \) is the set of the pairs \( ([\ [, ] ], \rho) \), where \( ([\ [, ] ], \rho) \) is a Lie algebroid structure on \( \pi: A \to M \) and \( \phi \in \Gamma(A^*) \) is a \( 1 \)-cocycle. \( \mathcal{J} \) is the set of the Jacobi structures \( (\Lambda, E) \) on \( A^* \) which satisfy the conditions (C1) and (C2) (see Theorem 2).

Then, using Theorems 2 and 3, we obtain

**Theorem 3** The mapping \( \Psi: A \to \mathcal{J} \) between the sets \( A \) and \( \mathcal{J} \) given by
\[
\Psi(([\ [, ] ], \rho)) = (\Lambda_{(A^*, \phi)}, E_{(A^*, \phi)})
\]
is a bijection.

Note that \( \Psi(A) = \mathcal{P} \), where \( \mathcal{P} \) is the subset of the Jacobi structures of \( \mathcal{J} \) which are Poisson and \( A \) is the subset of \( A \) of the pairs of the form \( ([\ [, ] ], 0) \), that is, \( A \) is the set of the Lie algebroid structures on \( \pi: A \to M \). Therefore, from Theorem 3, we deduce a well known result (see 2.3): the mapping \( \Psi \) induces a bijection between the sets \( A \) and \( \mathcal{P} \).

### 3 Examples and applications

In this section we will present some examples and applications of the results obtained in Section 2.
1.- Let $(\mathfrak{g},[ , ]\rangle$ be a real Lie algebra of dimension $n$. Then, $\mathfrak{g}$ is a Lie algebroid over a point. The resultant Poisson structure $\Lambda_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is the well known Lie-Poisson structure (see [3]). Thus, if $\phi \in \mathfrak{g}^*$ is a 1-cocycle then, using Theorem 1, we deduce that the pair $(\Lambda_{(\mathfrak{g}^*,\phi)},E_{(\mathfrak{g}^*,\phi)})$ is a Jacobi structure on $\mathfrak{g}^*$, where

$$
\Lambda_{(\mathfrak{g}^*,\phi)} = \Lambda_{\mathfrak{g}^*} + R \wedge C_\phi, \quad E_{(\mathfrak{g}^*,\phi)} = -C_\phi,
$$

$R$ is the radial vector field on $\mathfrak{g}^*$ and $C_\phi$ is the constant vector field on $\mathfrak{g}^*$ induced by $\phi \in \mathfrak{g}^*$. From (1), (3) and (4), it follows that

$$
\eta = \text{canonical projection over the first factor and}
$$

Thus, if $\Lambda$ is the radial vector field on $\mathfrak{g}$ and $C_\phi$ is the constant vector field on $\mathfrak{g}$ induced by $\phi \in \mathfrak{g}$.

2.- Let $(TM,[ , ],\text{Id})$ be the trivial Lie algebroid. In this case, the Poisson structure $\Lambda_{T^*M}$ on $T^*M$ is the canonical symplectic structure. Therefore, if $\phi$ is a closed 1-form on $M$, then the pair

$$
\Lambda_{(T^*M,\phi)} = \Lambda_{T^*M} + \Delta \wedge \phi^v, \quad E_{(T^*M,\phi)} = -\phi^v,
$$

is a Jacobi structure on $T^*M$. Furthermore, we can prove that the map $\#_{\Lambda_{(T^*M,\phi)}} : \Omega^1(T^*M) \to \mathfrak{X}(T^*M)$ is an isomorphism and consequently, using the results of [5, 8] (see also [4]), it follows that $(\Lambda_{(T^*M,\phi)},E_{(T^*M,\phi)})$ is a locally conformal symplectic structure.

3.- Let $(M,\Lambda)$ be a Poisson manifold and $(T^*M,[ , ]_\Lambda,\#_\Lambda)$ the associated cotangent Lie algebroid (see Section 3). The induced Poisson structure on $TM$ is the complete lift $\Lambda^c$ to $TM$ of $\Lambda$ (see [3]). Thus, if $X \in \mathfrak{X}(M) = \Gamma(TM)$ is a 1-cocycle, that is, $X$ is a Poisson infinitesimal automorphism ($\mathcal{L}_X \Lambda = 0$), we deduce that

$$
\Lambda_{(TM,X)} = \Lambda^c + \Delta \wedge X^v, \quad E_{(TM,X)} = -X^v,
$$

is a Jacobi structure on $TM$.

4.- The triple $(TM \times \mathbb{R},[ , ],\pi)$ is a Lie algebroid over $M$, where $\pi : TM \times \mathbb{R} \to TM$ is the canonical projection over the first factor and $[ , ]$ is the bracket given by

$$
[(X,\dot{f}),(Y,\dot{g})] = ([X,Y],X(\dot{g}) - Y(\dot{f})), \quad \text{for } (X,\dot{f}),(Y,\dot{g}) \in \mathfrak{X}(M) \times C^\infty(M,\mathbb{R}). \tag{11}
$$

In this case, the Poisson structure $\Lambda_{T^*M \times \mathbb{R}}$ on $T^*M \times \mathbb{R}$ is just the canonical cosymplectic structure of $T^*M \times \mathbb{R}$, that is, $\Lambda_{T^*M \times \mathbb{R}} = \Lambda_{T^*M}$. Now, it is easy to prove that $\phi = (0,-1) \in \Omega^1(M) \times C^\infty(M,\mathbb{R}) = \Gamma(T^*M \times \mathbb{R})$ is a 1-cocycle (see [3] and (11)). Moreover, using Theorem 4, we have that the Jacobi structure $(\Lambda_{(T^*M \times \mathbb{R},\phi)},E_{(T^*M \times \mathbb{R},\phi)})$ on $T^*M \times \mathbb{R}$ is the one defined by the canonical contact 1-form $\eta_M$. We recall that $\eta_M$ is the 1-form on $T^*M \times \mathbb{R}$ given by $\eta_M = dt + \lambda_M$, $\lambda_M$ being the Liouville 1-form of $T^*M$ (see [2]).

5.- Let $(M,\Lambda,\mathbf{E})$ be a Jacobi manifold and $(T^*M \times \mathbb{R},[ , ]_{(\Lambda,\mathbf{E})},\#_{(\Lambda,\mathbf{E})})$ the associated Lie algebroid (see Section 4). From (3), (5) and (6), it follows that $\phi = (-\mathbf{E},0) \in \mathfrak{X}(M) \times C^\infty(M,\mathbb{R}) = \Gamma(T^*M \times \mathbb{R})$ is a 1-cocycle. On the other hand, a long computation, using (3), (5), (6) and Theorem 4, shows that

$$
\Lambda_{(TM \times \mathbb{R},\phi)} = \Lambda^c + \frac{\partial}{\partial t} \wedge E^c - t\left(\Lambda^v + \frac{\partial}{\partial t} \wedge E^v\right), \quad E_{(TM \times \mathbb{R},\phi)} = E^v,
$$

where $\Lambda^c$ (resp. $\Lambda^v$) is the complete (resp. vertical) lift to $TM$ of $\Lambda$ and $E^c$ (resp. $E^v$) is the complete (resp. vertical) lift to $TM$ of $E$. We remark that in [6] the authors characterize the conformal infinitesimal automorphisms of $(M,\Lambda,\mathbf{E})$ as Legendre-Lagrangian submanifolds of the Jacobi manifold $(TM \times \mathbb{R},\Lambda_{(TM \times \mathbb{R},\phi)},E_{(TM \times \mathbb{R},\phi)})$.

6.- Let $(\mathfrak{A},[ , ],\rho)$ be a Lie algebroid over $M$ and $\phi \in \Gamma(A^*)$ a 1-cocycle. Denote by $\hat{\Lambda}_{A^* \times \mathbb{R}}$ the Poissonization of the Jacobi structure $(\Lambda_{(A^*,\phi)},E_{(A^*,\phi)})$, that is, $\hat{\Lambda}_{A^* \times \mathbb{R}}$ is the Poisson structure
on $\hat{A}^* = A^* \times \mathbb{R}$ given by (see \cite{1,11})

$$\hat{A}_{A^* \times \mathbb{R}} = e^{-t} \left( \Lambda_{(A^*, \phi)} + \frac{\partial}{\partial t} \wedge E_{(A^*, \phi)} \right).$$  \hfill (12)

$\hat{A}^*$ is the total space of a vector bundle over $M \times \mathbb{R}$ and, from (12), we obtain that the Poisson bracket of two linear functions on $\hat{A}^*$ is again linear. This implies that the dual vector bundle $\hat{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ admits a Lie algebroid structure $\langle \cdot, \cdot \rangle$. Note that the space $\Gamma(\hat{A})$ can be identified with the set of time-dependent sections of $A \to M$. Under this identification, we deduce that (see \cite{10} and \cite{12})

$$\left[ \hat{\mu}, \hat{\eta} \right] = e^{-t} \left( \left[ \hat{\mu}, \hat{\eta} \right] + \phi(\hat{\mu}) \left( \frac{d\hat{\eta}}{dt} - \hat{\eta} \right) - \phi(\hat{\eta}) \left( \frac{d\hat{\mu}}{dt} - \hat{\mu} \right) \right), \quad \hat{\rho}(\hat{\mu}) = e^{-t} \left( \rho(\hat{\mu}) + \phi(\hat{\mu}) \frac{\partial}{\partial t} \right),$$

for all $\hat{\mu}, \hat{\eta} \in \Gamma(\hat{A})$, where $\frac{d\hat{\mu}}{dt}$ (resp. $\frac{d\hat{\eta}}{dt}$) is the derivative of $\hat{\mu}$ (resp. $\hat{\eta}$) with respect to the time. Note that if $t \in \mathbb{R}$ then the sections $\hat{\mu}$ and $\hat{\eta}$ induce, in a natural way, two sections $\hat{\mu}_t$ and $\hat{\eta}_t$ of $A \to M$ and that $\left[ \hat{\mu}, \hat{\eta} \right]$ is the time-dependent section of $A \to M$ given by $\left[ \hat{\mu}, \hat{\eta} \right](x,t) = \left[ \hat{\mu}_t, \hat{\eta}_t \right](x)$, for all $(x,t) \in M \times \mathbb{R}$.

The construction of the Lie algebroid $(\hat{A}, \langle \cdot, \cdot \rangle, \hat{\rho})$ from the Lie algebroid $(A, \langle \cdot, \cdot \rangle, \rho)$ and the cocycle $\phi$ plays an important role in $\hat{A}^*$.

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