A quantum dynamical framework for Brownian heat engines

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We present a self contained formalism modelled after the Brownian motion of a quantum harmonic oscillator for describing the performance of microscopic Brownian heat engines like Carnot, Stirling and Otto engines. Our theory, besides reproducing the standard thermodynamics results in the steady state enables permits us to study the role dissipation plays in determining the efficiency of Brownian heat engines under actual laboratory conditions. In particular, we analyse in detail the dynamics associated with decoupling a system in equilibrium with one bath and recoupling it to another bath and obtain exact analytical results which are shown to have significant ramifications on the efficiencies of engines involving such a step. We also develop a simple yet powerful technique for computing corrections to the steady state results arising from finite operation time and use it to arrive at the thermodynamic complementarity relations for various operating conditions and also to compute the efficiencies of the three engines cited above at maximum power. Some of the methods and techniques and exactly solvable models presented here are interesting in their own right and, in our opinion, would find useful applications in other contexts as well.

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I. INTRODUCTION

In recent years there has been an upsurge of interest in the interface between thermodynamics and quantum mechanics [1], macro and nano/micro [2], leading to a thorough reexamination of the basic concepts and principles of thermodynamics with ramifications in biological processes and soft condensed matter systems. New paradigms for notions of work, thermal machines etc. have emerged that provide deep insights into thermodynamics which in turn help enlarge its scope far beyond that envisaged originally and open up new possibilities [3]. These developments force one to formulate questions concerning efficiencies of various heat engines using appropriate microscopic considerations [4]. Indeed, one has even started considering quantum heat engines which in principle have efficiency larger than Carnot efficiency though such cases require non equilibrium steady states. Such steady states can be reached via the use of the coherent laser fields or via quantum interference effects [5]-[6]. In a very recent experiment [7] realized a microscopic Brownian heat engine [8]-[10]. The most important ingredient both in the work of Blickle and Bechinger [7] as well as in the proposals of Scully and collaborators [5] is the possibility that all the relevant parameters can be very well controlled experimentally and thus the heat engine cycle can be precisely realized and it becomes desirable to have exactly soluble models of microscopic heat engines.

In view of the way the experiments are carried out we need a fully dynamical model which should account for the way the system parameters like potentials or the external parameters like temperature are changed. Further the behavior of the engine should depend on various time scales foe example the time taken to reach equilibrium state. Such a time would depend on the scales of the damping in the system. Motivated by these requirements we develop in the present work an exactly soluble model of a microscopic Brownian heat engine. The model that we present is fully quantum mechanical. Our model enables us to examine many different possible experimental scenarios- (a) low temperature behavior where quantum effects are likely to dominate, (b) behavior under different relaxation conditions-for example the system could be underdamped or overdamped, (c) possibilities for the system to pass through nonequilibrium stages depending on the rate of change of the external parameters, (d) nonequilibrium conditions because the experimental time scales are smaller than the time it takes for the system to reach steady state. Our formulation is based on the Wigner function and quantum Langevin equations for an harmonic oscillator whose frequency is modulated in time. We also assume that the temperature of the environment is time dependent as well. These time dependences are needed to implement the heat engine cycle realistically. We calculate the time dependent Wigner function, all the fluctuation parameters and the entropy. These enable us to calculate thermodynamic quantities like work, heat and internal energy.

A brief outline of the work is as follows. In Section II for later reference we briefly recapitulate the expressions for the efficiencies, both classical as well as quantum, for the three engines based on standard thermodynamic considerations with usual assumptions regarding the speed with which the various steps are carried out. In Section III we present a
quantum thermodynamic framework based on Wigner phase space description for quantum systems which contains the classical framework as a limiting case. In Section IV we develop a general set up for computing various quantities of interest and give two models of frequency modulation where the relevant equations are amenable to exact analytical results. In Section V we consider the situation when the diffusion constant is varied linearly and in Section VI analyse its ramifications on the efficiencies of Brownian heat engines. In Section VII we develop a systematic scheme for computing finite time corrections to the efficiencies of classical and quantum Brownian motors and then use these results in Section VII to examine the role they play in determining the efficiency of the Stirling engine at maximum power. Section IX contains our concluding remarks and further outlook.

As noted above our working model for a heat engine is based on a quantum harmonic oscillator with a frequency $\omega$ interacting with a thermal bath at temperature $T$. Three typical engines that have been discussed extensively in the literature based on varying $\omega$ and $T$ appropriately are (a) The Stirling (b) Carnot and (c) Otto engines. Their cycles adapted to the harmonic oscillator model are schematically given below:

**Stirling Engine**

\[
\begin{align*}
\omega_1, T_c & \quad \text{Isothermal} & \omega_2, T_c \\
3 & \quad \overset{\tau_c}{\longrightarrow} 4 & \\
\text{Isochoric} & \quad \uparrow \quad \tau_h \quad \downarrow \quad \text{Isochoric} & \\
2 & \quad \overset{\tau_h}{\leftarrow} 1 & \\
\omega_1, T_h & \quad \text{Isothermal} & \omega_2, T_h
\end{align*}
\]

(1)

$\omega_2 > \omega_1, \; T_h > T_c$

**Carnot Engine**

\[
\begin{align*}
\omega_1, T_h & \quad \text{Isothermal} & \omega_2, T_h \\
1 & \quad \overset{\tau_h}{\longrightarrow} 2 & \\
\text{Isentropic} & \quad \uparrow \quad \tau_c \quad \downarrow \quad \text{Isentropic} & \\
4 & \quad \overset{\tau_c}{\leftarrow} 3 & \\
\omega_4, T_c & \quad \text{Isothermal} & \omega_3, T_c
\end{align*}
\]

(2)

$\omega_1 > \omega_2 > \omega_3 > \omega_4, \; T_h > T_c, \; \beta_c \omega_2 = \beta_c \omega_3, \; \beta_h \omega_1 = \beta_h \omega_4$

**Otto Engine**

\[
\begin{align*}
\omega_c, T_2 & \quad \text{Isentropic} & \omega_h, T_h \\
4 & \quad \overset{\tau_1}{\leftarrow} 3 & \\
\text{ Isochoric} & \quad \downarrow \quad \tau_2 \quad \uparrow \quad \text{ Isochoric} & \\
1 & \quad \overset{\tau_2}{\longrightarrow} 2 & \\
\omega_c, T_c & \quad \text{Isentropic} & \omega_h, T_1
\end{align*}
\]

(3)

$\omega_h > \omega_c > \omega_1, \; T_h > T_c, \; \beta_c \omega_c = \beta_1 \omega_h, \; \beta_h \omega_h = \beta_2 \omega_c$

Here, the $\tau$’s indicate the time taken to carry out the indicated step and $\beta$ stands for $1/K_B T$. The calculations that we give in subsequent sections can be applied to any of these engines.

The three prototype engines above thus involve suitable combinations of the following three steps: (a) isothermal i.e. $\omega$ changing, $T$ fixed or (b) isochoric i.e. $\omega$ held fixed, $T$ changing or (c) isentropic i.e both $\omega$ and $T$ changing with $\omega/T$ fixed and one needs an appropriate formalism to compute the efficiencies under specific physical circumstances in which these steps are actually executed in an experiment. The present work has this as its major objective. Our principal results include (i) development of a self contained formalism for computing efficiencies of Brownian engines both in the classical as well as quantum contexts (ii) an exact analysis of the role of damping in the process of coupling the system to a bath at a higher temperature and its influence on the performance of the Stirling engine (iii) computation of the irreversible heat in isothermal processes and the derivation of complementarity relations (iv) a detailed analysis of the role of damping as well as finite time corrections on the efficiency of the Stirling engine at maximum power.
II. STEADY STATE EFFICIENCIES FROM THERMODYNAMICS

To set the notation and for later reference we assemble here the standard thermodynamic considerations that enable us to compute the efficiencies for the three engines listed above both in classical and as well as in the quantum contexts. These are:

1. the thermodynamic conservation law $\Delta U = \Delta Q - \Delta W$; where $\Delta U$: Change in the internal energy $U$; $\Delta Q$: Heat absorbed by the system; $\Delta W$: Work done by the system

2. $\Delta Q$ in an isentropic process $a \rightarrow b = 0$,

3. work done in an isochoric process $a \rightarrow b = 0$,

4. work done in an isothermal process $a \rightarrow b = -[F(b) - F(a)]$ where $F$ denotes the free energy of the system,

5. the expressions for $U$ and $F$ for the harmonic oscillator:

\[
\frac{1}{\beta}, \quad \beta \equiv \frac{1}{k_B T} \quad \text{(Classical)}
\]

\[
U = \frac{1}{\hbar \omega [n(\omega, T) + 1/2]}, \quad n(\omega, T) \equiv \frac{1}{(e^{\beta \hbar \omega} - 1)} \quad \text{(Quantum)},
\]

\[
F(\omega, T) = \frac{1}{\beta} \ln(\beta \hbar \omega) \quad \text{(Classical)}
\]

\[
F(\omega, T) = \frac{1}{\beta} \ln(2 \sinh(\beta \hbar \omega/2)) \quad \text{(Quantum)}
\]

6. the expression for the entropy of a classical harmonic oscillator

\[
S = k_B \left[ 1 + \ln \left( \frac{1}{\beta \hbar \omega} \right) \right].
\]

In an isothermal process one has

\[
\Delta Q = \Delta W = T \Delta S.
\]

In the quantum case, for the thermal states $\rho_{th}$,

\[
\rho_{th} = \frac{e^{-\beta \hat{H}}}{\text{Tr}[e^{-\beta \hat{H}}]},
\]

where $\hat{H}$ denotes the hamiltonian for a quantum harmonic oscillator, one has for the von Neumann entropy

\[
S = k_B [(n(\omega, T) + 1) \ln(n(\omega, T) + 1) - n(\omega, T) \ln n(\omega, T)].
\]

With this preparation we now proceed to compute the efficiencies of the three engines mentioned earlier both in the classical as well as quantum cases. These would then be compared with the results obtained from the microscopic theory developed later.

A. Stirling Engine

The efficiency $\eta_s$ of the Stirling engine is defined as

\[
\eta_s = \frac{\text{Work done by the system}}{\text{Heat flow into the system at } T_h}.
\]

Classical
In the classical case the work done by the engine is given by
\[
\Delta W_{1\to 2} + \Delta W_{3\to 4} = -[F(\omega_1, T_h) - F(\omega_2, T_h)] - [F(\omega_2, T_c) - F(\omega_1, T_c)]
\]
\[= K_B(T_h - T_c) \ln \left(\frac{\omega_2}{\omega_1}\right), \tag{10}\]
and the heat absorbed at \(T_h\) by
\[
\Delta W_{1\to 2} + \Delta U_{1\to 2} + \Delta U_{4\to 1} = -[F(\omega_1, T_h) - F(\omega_2, T_h)] + \frac{1}{2} \left(\frac{1}{\beta_h} - \frac{1}{\beta_c}\right) + \frac{1}{2} K_B(T_h - T_c).
\]
\[= K_B T_h \ln \left(\frac{\omega_2}{\omega_1}\right) + \frac{1}{2} K_B(T_h - T_c). \tag{11}\]
[ Note the factor of half in the second term on the RHS of the above equation. We will return to this later.]
Hence
\[\eta^c_s = \frac{\eta_c}{1 + \eta_c/(\ln(\omega_2/\omega_1))}, \quad \eta_c = 1 - \frac{T_c}{T_h}. \tag{12}\]

**Quantum**

Proceeding as before and using the expressions for \(U\) and \(F\) appropriate to the quantum case, we have for the work done
\[
\Delta W_{1\to 2} + \Delta W_{3\to 4} = -[F(\omega_1, T_h) - F(\omega_2, T_h)] - [F(\omega_2, T_c) - F(\omega_1, T_c)],
\]
\[= K_B T_h \ln \left(\frac{\sinh(\beta_h \omega_2/2)}{\sinh(\beta_h \omega_1/2)}\right) - K_B T_c \ln \left(\frac{\sinh(\beta_c \omega_2/2)}{\sinh(\beta_c \omega_1/2)}\right), \tag{13}\]
and for the heat absorbed at \(T_h\)
\[
\Delta W_{1\to 2} + \Delta U_{1\to 2} + \Delta U_{4\to 1} = -[F(\omega_1, T_h) - F(\omega_2, T_h)]
\]
\[+ [\hbar \omega_1(n(\omega_1, T_h) + 1/2) - \hbar \omega_2(n(\omega_2, T_h) + 1/2)]
\]
\[+ \frac{1}{2} \left([\hbar \omega_2(n(\omega_2, T_h) + 1/2)] - [\hbar \omega_2(n(\omega_2, T_c) + 1/2)]\right), \tag{14}\]
and hence
\[\eta^q_s = \frac{1 - Y/X}{1 + Z/X},
\]
\[X = \ln \left(\frac{\sinh(\beta_h \omega_2/2)}{\sinh(\beta_h \omega_1/2)}\right), \quad Y = \frac{\beta_h}{\beta_c} \ln \left(\frac{\sinh(\beta_c \omega_2/2)}{\sinh(\beta_c \omega_1/2)}\right), \tag{15}\]
\[Z = \frac{\beta_h}{2} \left[\hbar \omega_1 \coth(\beta_h \omega_1/2) - \frac{\hbar \omega_2}{2} \{\coth(\beta_h \omega_2/2) + \coth(\beta_c \omega_2/2)\}\right].\]
In the limit \(\beta \omega << 1\) \(\eta^q_s\) goes over to the classical efficiency \(\eta^c_s\) as expected.

**B. Carnot Engine**

For the Carnot engine the efficiency defined as before
\[\eta_c = \frac{\text{Work done by the System}}{\text{Heat flow into the system at } T_h},\]
turns out to be the same in both classical and quantum cases and is given by
\[\eta^c_s = \eta^q_s = \eta_c = 1 - \frac{T_c}{T_h}. \tag{16}\]
C. Otto Engine

Here again the efficiency defined as

\[ \eta_o = \frac{\text{Work done by the System}}{\text{Heat flow into the system during } 2 \rightarrow 3}, \]

turns out to be the same in both quantum and classical cases and is given by

\[ \eta^e_o = \eta^q_o = 1 - \frac{U(4) - U(1)}{U(3) - U(2)} = \left(1 - \frac{\omega_c}{\omega_h}\right). \tag{17} \]

The expressions for efficiencies for the three engines, realized here through a harmonic oscillator by appropriate changes of its frequency (or equivalently its ‘spring constant’) and the ambient temperature, hold for idealized operating conditions as stipulated in equilibrium thermodynamics. These, for instance, demand that the isothermal changes of frequency involved in the Stirling or the Carnot cycles be carried out quasistatically i.e. so slowly that at each instance the oscillator remains in the state of equilibrium at that temperature and frequency. Such conditions are hardly ever met in practice and particularly in the light of the experimental work reported in [7] there is an obvious need for developing a framework which brings into play aspects of approach to equilibrium, both in classical and quantum contexts, and is capable of furnishing a self-contained scheme for computing the efficiencies under realistic conditions. We develop such a scheme in the next section.

III. EFFICIENCIES BEYOND THE STEADY STATE: A DYNAMICAL MODEL

To go beyond the standard thermodynamic assumptions regarding the rate at which which various steps in a heat engine are carried out so that one can evaluate the performance of an engine under actual laboratory conditions we need a framework which treats the system modelling the engine as an open system and permitting proper inclusion of dissipative effects and the possibility of varying the system potential and the ambient temperature. In the present context, such a framework is provided by the dynamics of a quantum Brownian oscillator of frequency \( \omega \) in contact with a heat bath at temperature \( T \).is described by the master equation \[13\]

\[ \frac{\partial}{\partial t} \rho = -\frac{i}{\hbar} \left[ \hat{p}^2/2m + \frac{1}{2} m \omega^2 q^2, \rho \right] - \frac{2 \kappa m \omega}{\hbar} (n(\omega, T) + 1/2) (\{\hat{q}, \{\hat{q}, \rho\}\}) - \frac{i \kappa}{\hbar} (\{\hat{q}, \{\hat{p}, \rho\}\}), \tag{18} \]

where \( \hat{q} \) and \( \hat{p} \) denote the position and momentum operators obeying the commutation relations \( [\hat{q}, \hat{p}] = i \hbar \).

For reasons given later it proves expedient to transcribe the quantum dynamics described by the master equation using the Wigner phase space description of quantum systems \[14,15\] which associates with a density operator \( \rho \) a phase space function \( W(q, p) \) of classical variables \( q, p \) as follows:

\[ \hat{\rho} \mapsto W_\rho(q, p) = \text{Tr} \left\{ \hat{\rho} \, \hat{W}(q, p) \right\} ; \]

\[ \hat{W}(q, p) = \frac{1}{(2\pi \hbar)} \int dq' \, \left| q + \frac{1}{2} q' \right\rangle \left\langle q - \frac{1}{2} q' \right| e^{i \, p q'/\hbar}, \tag{19} \]

We note here that we use the Wigner phase space description in preference to other phase space descriptors for two reasons: (a) it is the only one that that maps the quantum mechanical average of a product of two operators to the phase space average of the corresponding Wigner functions (b) its moments \( < q^m p^n > \) correspond to quantum averages of of the symmetrised operator \( (\hat{q}^m \hat{p}^n)_S \). For example \( < q^2 p > \) corresponds to the expectation value of the operator \( (\hat{q}^2 \hat{p} + \hat{p} \hat{q} + \hat{p}^2)/3 \).

Use of the Wigner description turns the master equation into a Fokker-Planck equation for \( W(q, p) \) \[15\]

\[ \frac{\partial}{\partial t} W(q, p, t) = \left[ -\frac{\partial}{\partial q} \left( \frac{p}{m} \right) + \frac{\partial}{\partial p} \left( 2 \kappa p + \left( \frac{\partial V(q, a)}{\partial q} \right) \right) + D \frac{\partial^2}{\partial p^2} \right] W(q, p, t). \]
where
\[ V(q, a) = \frac{1}{2} a q^2, \quad a \equiv m \omega^2, \] (20)

and
\[ D = 2 \hbar \omega \kappa (n(\omega, T) + \frac{1}{2}), \quad n(\omega, T) = (e^{\beta \hbar \omega} - 1)^{-1}. \] (21)

In the following the parameter \( a \), the 'spring constant', will be taken to be controlled externally.

The Langevin equations equivalent to the above FPE read:
\[ \dot{q} = \frac{p}{m}, \] (22)
\[ \dot{p} = -2 \kappa p - \frac{\partial}{\partial q} V(q, a) + f(t), \] (23)
\[ \langle f(t)f(t') \rangle = 2D \delta(t-t'). \] (24)

The Langevin equations (22) – (24) lend themselves to a nice thermodynamics interpretation [8]: Rewriting (23) as
\[ -(-2 \kappa p + f(t)) + \dot{p} + \frac{\partial}{\partial q} V(q, a) = 0, \] (25)

and multiplying it by \( dq \) and using
\[ dV = \frac{\partial V(q, a)}{\partial q} dq + \frac{\partial V(q, a)}{\partial a} da, \] (26)

one obtains
\[ -(-2 \kappa p + f(t)) dq + d(p^2/2m + V(q, a)) - \frac{\partial V(q, a)}{\partial a} da. \] (27)

The three terms in the above equation may now be identified in an intuitively plausible manner as:
\[ dQ = (-2 \kappa p + f(t)) dq, \quad dU = d(p^2/2m + V(q, a)), \quad dW = -\frac{\partial V(q, a)}{\partial a} da, \] (28)

leading to the energy balance equation:
\[ -dQ + dU + dW = 0, \] (29)

with \( dQ \) (-\( dQ \)) understood as the heat flow into of (out) the system and \( dW \) (-\( dW \)) as the work done by (on) the system. The stochastic averages of these quantities denote by \( dQ, dU \) and \( dW \) respectively relate directly to the corresponding thermodynamic quantities and capture the thermodynamic conservation laws. This self-contained approach is clearly more microscopic than thermodynamics as it provides a framework for computing not only the averages of these quantities but their probability distributions as well.

We note here that while it is certainly possible to transcribe the master equation dynamics directly into equivalent quantum Langevin equations for the operators \( \hat{q} \) and \( \hat{p} \) but owing to their noncommutativity the crucial step (27) needed to obtain a clear thermodynamic interpretation of the such Langevin equations would now involve terms like \( \hat{p} dq \hat{q} \) and would therefore be fraught with ordering ambiguities.

The scheme described above for computing \( dQ, dU \) and \( dW \) together with the expression for von Neumann entropy
\[ S = K_B \left[ (\sigma + 1) \ln(\sigma + 1) - \sigma \ln \sigma \right], \quad \sigma = \sqrt{\text{Det}[\mathcal{V}]} - \frac{1}{2}, \] (30)

for Gaussian states [16] (which is what we would exclusively deal with) i.e. states \( \rho \) for which the Wigner distribution is a Gaussian:
\[ W(q, p) = \frac{1}{\sqrt{(2\pi)^2 \text{Det}[\mathcal{V}]}} \exp \left[ -\frac{x^T \mathcal{V} x}{2 \text{Det}[\mathcal{V}]} \right], \quad x \equiv \begin{pmatrix} q \\ p \end{pmatrix}, \] (31)
provide all that we need for the considerations below. Here \( V \) stands for the variance matrix,

\[
V = \begin{pmatrix}
<q^2 > & <qp > \\
<qp > & <p^2 >
\end{pmatrix},
\]

and \(< \cdot >\) denote averages with respect to the Wigner distribution. The uncertainty relations require that \( \sigma \) be positive. Note that the set of Gaussian states contains the set of harmonic oscillator thermal states \( \rho_{th} \) as a special case.

Before proceeding further it is instructive to check that the Wigner description above together with the thermodynamic interpretation implied by (28) in the steady state does indeed reproduce the results given earlier for the efficiencies of the three engines using standard thermodynamic considerations. Thus, for instance, calculation of the efficiency of the Stirling engine involves computing \( \Delta W_{1\to 2} \Delta U_{1\to 2} \), \( \Delta U_{4\to 1} \) which in the present framework are given by

\[
\Delta W_{1\to 2} = \int_{1}^{2} dW = -\int_{\omega_2}^{\omega_1} m\omega <q^2 >_{T=T_e} d\omega,
\]

\[
\Delta W_{3\to 4} = \int_{3}^{4} dW = -\int_{\omega_4}^{\omega_3} m\omega <q^2 >_{T=T_h} d\omega,
\]

\[
\Delta U_{1\to 2} = \int_{1}^{2} dU = \left(\frac{<p^2 >}{2m} + \frac{1}{2}m\omega^2 <q^2 >\right)_1 - \left(\frac{<p^2 >}{2m} + \frac{1}{2}m\omega^2 <q^2 >\right)_2,
\]

\[
\Delta U_{4\to 1} = \int_{4}^{1} dU = \left(\frac{<p^2 >}{2m} - \frac{<p^2 >}{2m}\right).
\]

Further, from the FPE or the Langevin eqns it follows that in the steady state

\[
<p^2 > = D/2\kappa, \quad m\omega^2 <q^2 > = D/2m\kappa.
\]

These on using \( D = 2m\kappa\omega(n(\omega, T) + 1/2) \) then give

\[
\Delta W_{1\to 2} = -\int_{\omega_2}^{\omega_1} [n(\omega, T_h + 1/2)]d\omega
\]

\[
= K_B T_h \ln \left( \frac{\sinh(\beta_h\omega_2/2)}{\sinh(\beta_h\omega_1/2)} \right) = F(1) - F(2),
\]

\[
\Delta W_{3\to 4} = -\int_{\omega_4}^{\omega_3} [n(\omega, T_c) + 1/2]d\omega
\]

\[
= -K_B T_c \ln \left( \frac{\sinh(\beta_c\omega_2/2)}{\sinh(\beta_c\omega_1/2)} \right) = [F(3) - F(4)],
\]

\[
\Delta U_{1\to 2} = [\omega_1(n(\omega_1, T_h) + 1/2) - [\omega_2(n(\omega_2, T_h) + 1/2)],
\]

\[
\Delta U_{4\to 1} = \frac{1}{2} \left[ [\omega_2(n(\omega_2, T_c) + 1/2)] - [\omega_2(n(\omega_2, T_c) + 1/2)] \right],
\]

which are the same expressions as before and therefore one recovers the expression for efficiency given in Section II. (Note here that in computing \( \Delta U_{4\to 1} \) we considered only the contribution from \(<p^2 >\) and not from \(<q^2 >\), a question that will be examined in greater detail later.)

**IV. QUANTUM DYNAMICS UNDER TIME DEPENDENT CHANGES OF TEMPERATURE AND POTENTIAL**

We have seen in the previous section that the Langevin equations equivalent to the Fokker-Planck equation obeyed by the Wigner distribution lend themselves to a direct and transparent thermodynamic interpretation and that this interpretation in the steady state limit reproduces the standard thermodynamic results. To prepare ground for going beyond the steady state limit we now analyse the structure of the solutions of the Langevin equations at hand allowing for arbitrary time dependence in the potential and the diffusion coefficients and apply this framework to arrive at the exact solutions of the Langevin equations for three fairly realistic models.
The Langevin equations which in the present case are linear stochastic equations with additive noise may be solved to yield:

\[
\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = M(t) \begin{pmatrix} q(0) \\ p(0) \end{pmatrix} + \int_0^t dt' M(t')M(t')^{-1} \begin{pmatrix} 0 \\ \sqrt{2D(t')}f(t') \end{pmatrix},
\]

(42)

where

\[
M(t) = \begin{pmatrix} u(t) & v(t) \\ m\dot{u}(t) & m\dot{v}(t) \end{pmatrix},
\]

(43)
solves the homogeneous equations

\[
\frac{d}{dt} \begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} 1/m & 0 \\ -m\omega^2 - 2\kappa & 0 \end{pmatrix} \begin{pmatrix} q(t) \\ p(t) \end{pmatrix}.
\]

(44)

From (22) for the variance matrix

\[
\mathcal{V}(t) \equiv \begin{pmatrix} <q^2(t)> & <q(t)p(t)> \\ <q(t)p(t)> & <p^2(t)> \end{pmatrix},
\]

(45)
one has

\[
\mathcal{V}(t) = M(t)[\mathcal{V}(0) + \int_0^t dt' M^{-1}(t') \begin{pmatrix} 0 & 0 \\ 0 & 2D(t') \end{pmatrix} M^{-1}(t') M^T(t)].
\]

(46)

It is therefore clear that finding explicit solutions for the variances in situations where both \(\omega\) and \(T\) depend on time depends on our ability to solve for \(M(t)\). We list below three physically meaningful cases where this is indeed possible.

**Case I: \(\omega\) independent of time**

For this familiar case the functions \(U(t)\) and \(V(t)\) which determine the matrix \(M(t)\) are explicitly given by

\[
u(t) = \frac{(\lambda_+ e^{-\lambda_+ t} - \lambda_- e^{-\lambda_- t})}{(\lambda_+ - \lambda_-)}, \quad v(t) = \frac{(e^{-\lambda_+ t} - e^{-\lambda_- t})}{m(\lambda_+ - \lambda_-)}; \quad \lambda_\pm = \kappa \pm \sqrt{\kappa^2 - \omega^2}.
\]

(47)

Further, owing to time translation available in this case, we have \(M^{-1}(t) = M(-t)\), \(M(t)M(t') = M(t + t')\), and (45) simplifies to

\[
\mathcal{V}(t) = M(t)\mathcal{V}(0)M^T(t) + \int_0^t dt' M(t') \begin{pmatrix} 0 & 0 \\ 0 & 2D(t-t') \end{pmatrix} M^T(t').
\]

(48)

**Case II: \(\omega^2(t) = \omega_0^2 (1 + \alpha t^2)\), \(0 \leq t \leq T\)**

In this case investigated in [17], the functions \(u(t)\) and \(v(t)\) in the range \(0 \leq t \leq T\) are given by

\[
u(t) = \left[\frac{f_+(t)\dot{f}_-(0) - f_-(t)\dot{f}_+(0)}{f_+(0)\dot{f}_-(0) - f_-(0)\dot{f}_+(0)}\right], \quad v(t) = m \left[\frac{f_+(t)\dot{f}_-(0) - f_-(t)\dot{f}_+(0)}{\dot{f}_+(0)\dot{f}_-(0) - f_-(0)\dot{f}_+(0)}\right],
\]

(49)

\[
f_\pm(t) = e^{-\kappa t}(t + a)^{1/2}J_{\pm 1/3} \left(2\frac{3}{1/2}(t + a)^{3/2}\right), \quad a = \left(1 - \frac{\kappa^2}{\omega_0^2}\right) \frac{T}{\mu}, \quad b = \frac{\omega_0^2 \mu}{T}.
\]

(50)

**Case III: \(\omega^2(t) = \omega_0^2 e^{\frac{\mu t}{2}}\), \(0 \leq t \leq T\)**

In this case the functions \(f_+(t)\) and \(f_-(t)\) in [50] are again given in terms of Bessel functions as

\[
f_\pm(t) = e^{-\kappa t}J_{\pm \alpha} \left(a e^{\frac{2\kappa t}{\mu}}\right), \quad a = \frac{2T\omega_0}{\mu} \quad \alpha = \frac{2T\kappa}{\mu}.
\]

(51)

Having dealt with some exactly solvable cases where the frequency is changed in a specific way but the temperature may be varied arbitrarily, we now illustrate how the formalism developed above lends itself to useful exact or approximate calculations leading to finite time corrections.
V. LINEAR VARIATION OF THE DIFFUSION CONSTANT

We first consider the case in which the harmonic oscillator with frequency \( \omega \) is in equilibrium with a bath at temperature \( T_0 \) characterized by a diffusion constant \( D_0 \). With \( \omega \) held fixed, the diffusion coefficient is changed linearly from its initial value \( D_0 \) appropriate to temperature \( T_0 \) to its final value \( D_1 \) appropriate to temperature \( T_1 \) in a time \( \tau \) and then kept at that value thereafter.

\[
\omega : \text{held fixed} \\
D(t) = \begin{cases} 
D_0 + (D_1 - D_0) \frac{t}{\tau}, & 0 \leq t \leq \tau \\
D_1, & t > \tau.
\end{cases}
\] (52)

This situation pertains to the isochoric step in the Brownian engines and is relevant for discussions on aspects of decoupling the system from a heat bath at one temperature and recoupling it to another heat bath at a different temperature.

For the case at hand, with \( V(0) \) chosen to be the variance matrix corresponding to the oscillator being at equilibrium with the bath at temperature appropriate to \( D_0 \)

\[
V(0) = \begin{pmatrix}
\frac{D_0}{2\kappa m} & 0 \\
0 & \frac{D_0}{2\kappa m^2\omega^2}
\end{pmatrix},
\] (53)

we have from (52)

\[
< q^2(t) > = \frac{D_0}{2\kappa} \left( \frac{u^2(t)}{m^2\omega^2} + v^2(t) \right) + 2 \int_0^t dt' v^2(t-t')D(t'), 
\] (54)

\[
< q(t)p(t) > = 2 \int_0^t dt' v(t-t')\dot{v}(t-t')D(t'),
\] (55)

\[
< p^2(t) > = \frac{mD_0}{2\kappa} \left( \frac{u(t)\dot{u}(t)}{m^2\omega^2} + v(t)\dot{v}(t) \right) + 2m^2 \int_0^t dt' \dot{v}^2(t-t')D(t').
\] (56)

Using the relations

\[
v^2 = -\frac{1}{2\kappa} \frac{1}{2\kappa} \frac{d}{dt} \left( \frac{u^2}{m^2\omega^2} + v^2 \right),
\] (57)

\[
v\dot{v} = -\frac{1}{2\kappa} \frac{d}{dt} \left( \frac{u\dot{u}}{m^2\omega^2} + v\dot{v} \right),
\] (58)

\[
\dot{v}^2 = -\frac{1}{2\kappa} \frac{1}{2\kappa} \frac{d}{dt} \left( \frac{u^2}{m^2\omega^2} + \dot{v}^2 \right),
\] (59)

which follow from

\[
\dot{v} = -2\kappa v + \frac{u}{m}; \quad \dot{u} = -m\omega^2 v,
\] (60)

we obtain for \( t > \tau \)

\[
< q^2(t) > = \alpha(t) < q^2 >_0 + (1 - \alpha(t)) < q^2 >_1 
\] (61)

\[
< p^2(t) > = \beta(t) < p^2 >_0 + (1 - \beta(t)) < p^2 >_1 .
\] (62)

where

\[
\alpha(t) = \frac{1}{\tau} \int_{t-\tau}^t dt' [m^2\omega^2 v^2(t') + u^2(t')],
\] (63)

\[
\beta(t) = \frac{1}{\tau} \int_{t-\tau}^t dt' [m^2\dot{v}^2(t') + \frac{\dot{u}^2(t')}{\omega^2}].
\] (64)
In the limit $t \to \infty$ limit both $\alpha(t)$ and $\beta(t)$ go to zero and hence $< q^2 >$ and $< p^2 >$ assume their respective equilibrium values. The parameters $\alpha(t)$ and $\beta(t)$ thus interpolate between the initial and the final equilibrium values of $< q^2 >$ and $< p^2 >$ and quantify the approach to equilibrium. In the following we consider the case when $t = \tau$ i.e. the situation that obtains immediately after the bath has reached the state characterized by the final value of the diffusion coefficient. Evidently as far as the system is concerned we are dealing here with a non equilibrium state as the system has not yet had time to equilibrate with the 'final' bath.

Putting $t = \tau$ in (63) and (64) and denoting $\alpha(\tau)$ and $\beta(\tau)$ simply as $\alpha$ and $\beta$ we obtain on substituting for $u$ and $v$ from (47) and carrying out the relevant integrals

$$\alpha = \frac{1}{(x-y)^2} \left[ (x+y) \left( \frac{x}{2y} (1-e^{-2y}) + \frac{y}{x} (1-e^{-2x}) \right) - \frac{4xy}{x+y} (1-e^{-(x+y)}) \right],$$

(65)

$$\beta = \frac{1}{(x-y)^2} \left[ (x+y) \left( 1 - \frac{e^{-2y} + e^{-2x}}{2} \right) - \frac{4xy}{x+y} (1-e^{-(x+y)}) \right],$$

(66)

where $x = [\kappa + \sqrt{\kappa^2 - \omega^2}] \tau$, $y = [\kappa - \sqrt{\kappa^2 - \omega^2}] \tau$. We now examine the behaviour of $\alpha$ and $\beta$ in the overdamped and weak dissipation regimes respectively.

**Overdamped Case**

In the overdamped regime i.e. $\kappa >> \omega$, $x \approx 2\kappa \tau >> 1$, $y \approx \frac{\kappa^2 \tau}{2\omega} << 1$ one finds that

$$\alpha \approx \frac{1 - e^{-2y}}{2y} \to 1 \text{ as } y \to 0,$$

(67)

$$\beta \approx \frac{1 - e^{-2x}}{2x} \to 0 \text{ for } x >> 1.$$

(68)

**Underdamped Case**

On the other hand, in the weak dissipation regime $\kappa << \omega$, $x \approx \kappa + i\omega$, $y \approx \kappa - i\omega$ and we have in the limit $\kappa \tau \to 0$

$$\alpha \approx \left[ \frac{(1-e^{-2\kappa \tau})}{2\kappa \tau} + \kappa \tau \left( \frac{\sin \omega \tau}{\omega \tau} \right)^2 \right],$$

(69)

$$\approx 1 - \kappa \tau \left( 1 - \left( \frac{\sin \omega \tau}{\omega \tau} \right)^2 \right),$$

(70)

$$\beta \approx \left[ \frac{(1-e^{-2\kappa \tau})}{2\kappa \tau} - \kappa \tau \left( \frac{\sin \omega \tau}{\omega \tau} \right)^2 \right],$$

(71)

$$\approx 1 - \kappa \tau \left( 1 + \left( \frac{\sin \omega \tau}{\omega \tau} \right)^2 \right).$$

(72)

Note that $\alpha > \beta$ in both the cases. In fact this is always true – it can easily be shown that with $x, y$ defined as before

$$\alpha - \beta = \frac{x+y}{x-y} \int_0^1 dt \left[ e^{-xt} - e^{-yt} \right],$$

(73)

and hence $\alpha > \beta$ by virtue of the fact that the integrand is always positive.

**VI. EFFECT OF TIME SCALES ON EFFICIENCIES OF BROWNIAN MOTORS**

We recall that in the calculation of the efficiency of the Stirling engine from standard thermodynamic considerations presented in Section II, we had drawn attention to the factor of $1/2$ in the expression for $\Delta U_{4 \to 1}$. Likewise in the computation of the Stirling engine using the quantum stochastic thermodynamics in the steady state we had noted that in computing $\Delta U_{4 \to 1}$ only $< p^2 > /2m$ contributes to $\Delta U_{4 \to 1}$ and not $m \omega^2 < q^2 > /2$. This seemingly ad
hoc prescription can now be understood at a deeper level in the light of the analysis in Section V leading to eqns. (61)-(64). It is clear from the discussion therein that, in general, the expression for $\Delta U_{4 \rightarrow 1}$ should be taken to be

$$\Delta U_{4 \rightarrow 1} = (1 - \beta) \left( \frac{< p^2 >_1}{2m} - \frac{< p^2 >_4}{2m} \right) + (1 - \alpha) m \omega^2 \left( \frac{< q^2 >_1}{2} - \frac{< q^2 >_4}{2} \right),$$

(74)

where $\alpha$ and $\beta$ depend on various time scales involved. Indeed in the overdamped regime $\alpha \rightarrow 1$ and $\beta \rightarrow 0$ and one recovers the earlier results. The mystery behind the factor of $1/2$ in (11) and that behind retaining the contribution from $< p^2 > /2m$ in (36) alone is thus resolved. On the other hand, in the weak dissipation regime where both $\alpha$ and $\beta$ are close to 1 the situation is very different and this has significant consequences for the the relative magnitude of classical and quantum efficiencies under same operating conditions as discussed later. Further, since for a harmonic oscillator $< p^2 > /2m = m \omega^2 < q^2 > /2m$ we can rewrite the above equation as

$$\Delta U_{4 \rightarrow 1} = 2 \mu \left( \frac{< p^2 >_1}{2m} - \frac{< p^2 >_4}{2m} \right); \quad \mu = 1 - \alpha + \beta.$$  

(75)

Using this expression in the calculation of the classical and quantum efficiencies for the Stirling engine given earlier respectively we obtain

$$\eta_{cl}^{\phi} = \frac{\eta_c}{1 + \eta_c \mu \ln \left( \frac{\omega_2}{\omega_1} \right)},$$

(76)

and

$$\eta_{q}^{\phi} = \frac{1 - Y/X}{1 + Z/X},$$

$$X = \ln \left( \frac{\sinh(\beta_c \hbar \omega_2/2)}{\sinh(\beta_c \hbar \omega_1/2)} \right), \quad Y = \frac{\beta_h}{\beta_c} \ln \left( \frac{\sinh(\beta_c \hbar \omega_2/2)}{\sinh(\beta_c \hbar \omega_1/2)} \right),$$

$$Z = \frac{\beta_h}{2} \left[ \hbar \omega_1 \coth(\beta_h \omega_1/2) - \hbar \omega_2 \{(1 - \mu) \coth(\beta_h \omega_2/2) + \mu \coth(\beta_c \omega_2/2)\} \right].$$

(77)

The appearance of the parameter $\mu$ appearing here may be viewed as a phenomenological way of incorporating non equilibrium effects arising from decoupling of the system from one bath and recoupling it to another.

In terms of dimensionless quantities $a, b, c$ as

$$\beta_c \hbar \omega_1 = a, \quad \frac{\omega_2}{\omega_1} = b, \quad \frac{\beta_h}{\beta_c} = c$$

(78)

the expression above for the efficiencies $\eta_{q}^{\phi}$ and $\eta_{cl}^{\phi}$ in the classical and quantum read

$$\eta_{cl}(b, c) = \frac{1 - c}{1 + \mu \frac{(1 - c)}{\ln b}},$$

(79)

$$\eta_{q}(a, b, c) = \frac{\ln \left( \frac{\sinh(abc/2)}{\sinh(ac/2)} \right) - c \ln \left( \frac{\sinh(ab/2)}{\sinh(a/2)} \right)}{\ln \left( \frac{\sinh(ab/2)}{\sinh(ac/2)} \right) + \frac{ac}{2} \coth(ac/2) - \frac{abc}{2} \{ (1 - \mu) \coth(ab/2) - \mu \coth(ab/2) \}}.$$

(80)

In the experiments of Blickle and Bechinger [7]

$$a = 9.50065 \times 10^{-7}, \quad b = 2.04922, \quad c = 0.845272.$$  

(81)

With $b$ fixed at this values we plot below the ratio $R = \eta_{q}^{\phi}/\eta_{cl}^{\phi}$ as a function of $a, c$ for two representative values of $\mu$. 


Fig. 1 The ratio $R = \frac{\eta^2}{\eta^1}$ plotted as a function of $a = \beta_c \hbar \omega_1$ and $c = \beta_h / \beta_c$ for $\omega_2 / \omega_1 = 2.04$ and (a) $\mu = 0.01$ (b) $\mu = 0.5$

VII. FINITE TIME CORRECTIONS: COMPLEMENTARITY RELATIONS

We next consider the situation when the system starts out at equilibrium with a bath at temperature $T$, and the frequency is changed from its initial value $\omega_0$ to its final value $\omega_1$ in a finite time either isothermally ($T$ held fixed) or isentropically ($\omega/T$ held fixed) and focus on computing finite time corrections to the standard thermodynamic results. Referring to (1)-(3) we recall that while Stirling and Carnot engines involve the former operation, Carnot and Otto engines involve the latter. The scheme for computing finite time corrections developed below is similar in spirit to the adiabatic approximation in quantum mechanics and is a variant of the method formulated in [18] in the context of the Fokker-Planck equation adapted to the equations for the moments themselves.

The equations for the second moments that follow from the Langevin or the Fokker-Planck equation may be written as

$$\frac{d}{dt} X(t) = A(t) X(t) + Y(t), \quad \text{ (82)}$$

where

$$X(t) = \begin{pmatrix} <q^2> \\ <qp> \\ <p^2> \end{pmatrix}, \quad A(t) = \begin{pmatrix} 0 & -\frac{m}{\hbar} \omega(t) & 0 \\ -m \omega(t) & -2\kappa & \frac{1}{\hbar} \\ 0 & -2m \omega^2 & -4\kappa \end{pmatrix}, \quad Y(t) = \begin{pmatrix} 0 \\ 0 \\ 2D(t) \end{pmatrix}. \quad \text{ (83)}$$

At this stage, as indicated, we allow the frequency and the diffusion coefficients to be independent functions of $t$. Later however, we would specialise to situations appropriate to isothermal or isentropic variation of the frequency.

Putting $t = s\tau$ and expanding $X(t)$ as

$$X(t) = X^{(0)}(s) + \frac{1}{\tau} X^{(1)}(s) + \cdots, \quad \text{ (84)}$$

we obtain

$$A(s) X^{(0)}(s) + Y(s) = 0 \Rightarrow X^{(0)}(s) = -A^{-1}(s) Y(s), \quad \text{ (85)}$$

$$X^{(1)}(s) = A^{-1}(s) \frac{d}{ds} X^{(0)}(s). \quad \text{ (86)}$$

The first of these equations can be taken to describe the situation where the system is in the steady state corresponding to the instantaneous values of $\omega$ and $D$ and the second as describing deviations from this steady state. These equations then give

$$<q^2>^{(0)} = \frac{D(s)}{2m^2\omega^2(s)\kappa}; \quad \text{ (87)}$$

$$<q(s)p(s)>^{(0)} = 0; \quad <p^2>^{(0)} = \frac{D(s)}{2\kappa}.$$
and

\[ < q^2(s) >^{(1)} = -\left[ \frac{8\kappa^2 + 2\omega^2(s)}{8\kappa\omega^2(s)} \frac{d}{ds} < q^2(s) >^{(0)} \right] + \frac{1}{m\omega^2(s)} \frac{d}{ds} < q(s)p(s) >^{(0)} + \frac{1}{4\kappa m^2\omega^2(s)} \frac{d}{ds} < p^2(s) >^{(0)}, \]  

(88)

\[ < q(s)p(s) >^{(1)} = \frac{m}{2} \frac{d}{ds} < q^2(s) >^{(0)} \]  

(89)

\[ < p^2(s) >^{(1)} = -\left[ \frac{m^2\omega^2(s)}{4\kappa} \frac{d}{ds} < q^2(s) >^{(0)} + \frac{d}{4\kappa} \frac{d}{ds} < p^2(s) >^{(0)} \right]. \]  

(90)

These equations together with (87) give finite time corrections to the variances. As the diffusion coefficient is a function of both \( \omega \) and \( T \) we now specialize to the situations where

(a) \( \omega \) is time dependent, \( T \) is held fixed (Isothermal Case)

(b) \( \omega, T \) both are time dependent but \( \omega/T \) is held fixed (Isentropic case). With this in mind we may rewrite expression for \( < q^2(s) >^{(1)} \) which we would need shortly as

\[ < q^2(s) >^{(1)} = \frac{\hbar}{4\kappa m\omega^2} \left[ \left( \frac{4\kappa^2}{\omega^2} \right) \left( n(\omega, T) + \frac{1}{2} \right) \frac{d\omega}{ds} - \left( \frac{4\kappa^2}{\omega^2} + 2 \right) \omega \frac{d}{ds} \left( n(\omega, T) + \frac{1}{2} \right) \right]. \]  

(91)

In the isothermal case both the terms on the RHS contribute. On the other hand in the isentropic case only the first term contributes as during this process \( \omega/T \) and hence \( n(\omega, T) \) are held constant. In the following we confine ourselves to the isothermal case and give the results for two physically interesting limiting cases corresponding to the overdamped and weak dissipation dissipation regimes.

1. **Overdamped regime** (\( \kappa >> \omega \))

\[ < q^2(s) >^{(1)} = \frac{\kappa}{m\omega^2\beta} \left[ \left( \frac{\beta h\omega}{2} \right) \coth \left( \frac{\beta h\omega}{2} \right) + \left( \frac{\beta h\omega}{2} \right)^2 \cosech^2 \left( \frac{\beta h\omega}{2} \right) \right] \left( \frac{d\omega}{ds} \right). \]  

(92)

2. **Weak dissipation** (\( \kappa << \omega \))

\[ < q^2(s) >^{(1)} = \frac{1}{2\kappa m\omega^3\beta} \left[ \left( \frac{\beta h\omega}{2} \right) \cosech^2 \left( \frac{\beta h\omega}{2} \right) \right]^2 \left( \frac{d\omega}{ds} \right). \]  

(93)

We now compute expressions for the irreversible heat \( Q_{irr} \):

\[ Q_{irr} = F(i) - F(f) - \Delta W_{i \rightarrow f}, \]  

(94)

in an isothermal process at temperature \( T \) from \( i \rightarrow f \) arising from finite time corrections. Recalling that

\[ \Delta W_{i \rightarrow f} = - \int_{\omega_i}^{\omega_f} m\omega < q^2 > d\omega \]

\[ = - \int_{\omega_i}^{\omega_f} m\omega \left( < q^2 >^{(0)} + \frac{1}{\tau} < q^2 >^{(1)} \right) d\omega, \]  

(95)

and that

\[ - \int_{\omega_i}^{\omega_f} m\omega < q^2 >^{(0)} d\omega = F(i) - F(f), \]  

(96)

we have

\[ Q_{irr} = \frac{1}{\tau} \int_{\omega_i}^{\omega_f} m\omega < q^2 >^{(1)} d\omega \equiv T \Sigma \]  

(97)
From the way the quantity $\Sigma$ is defined above it is clear that it would, in general, depend on both $T$ as well as on the manner in which $\omega$ is varied from its initial value $\omega_i$ to its final value $\omega_f$ in the time $\tau$. We now turn to the question as to what would be the minimum value of $Q_{\text{irr}}$ and hence that of $\Sigma$ in the weak dissipation and overdamped regimes.

Using (92) and (93) and parametrizing $\omega(s)$ such that $\omega(0) = \omega_i$, $\omega(1) = \omega_f$ we have

\[
Q_{\text{irr}} = \frac{2\kappa}{\beta T} \int_0^1 ds g(\omega(s)) \left( \frac{d\omega}{ds} \right)^2 \quad (\kappa >> \omega)
\]

and

\[
Q_{\text{irr}} = \frac{1}{2\kappa \beta T} \int_0^1 ds g(\omega(s)) \left( \frac{d\omega}{ds} \right)^2 \quad (\kappa << \omega).
\]

where

\[
g(\omega) = \frac{1}{2\omega^4} \left[ \left( \frac{\beta \hbar \omega}{2} \right) \coth \left( \frac{\beta \hbar \omega}{2} \right) + \left( \frac{\beta \hbar \omega}{2} \right)^2 \cosech^2 \left( \frac{\beta \hbar \omega}{2} \right) \right] \quad (\kappa >> \omega)
\]

and

\[
g(\omega) = \frac{1}{\omega^2} \left[ \left( \frac{\beta \hbar \omega}{2} \right) \cosech \left( \frac{\beta \hbar \omega}{2} \right) \right]^2 \quad (\kappa << \omega)
\]

The expression $Q_{\text{irr}}$ involve the functional

\[
I[\omega] = \int_0^1 ds g(\omega(s)) \left( \frac{d\omega}{ds} \right)^2.
\]

If we define $\Omega = f(\omega)$ and its inverse as $\omega = h(\Omega)$ and choose $f(\omega)$ to satisfy

\[
\frac{df}{d\omega} = \sqrt{g(\omega)},
\]

then we find that curve $\omega(s)$; $0 \leq s \leq 1$ which minimizes $I[\omega]$ is given by

\[
\omega(s) = h(f(\omega(0))(1-s) + f((\omega(1))s)),
\]

and its minimum value of by

\[
I_{\text{min}}[\omega] = (f(\omega(1)) - f(\omega(0)))^2.
\]

In the classical limit ($\beta \hbar \omega << 1$) in the strong damping regime ($\kappa >> \omega$) we have

\[
g(\omega) = \frac{1}{\omega^2} \Rightarrow f(\omega) = -\frac{1}{\omega}; h(\Omega) = -\frac{1}{\Omega},
\]

and hence

\[
Q_{\text{irr}}^{\text{min}} = \frac{2\kappa K_B T}{\tau} \left[ \frac{1}{\omega(1)} - \frac{1}{\omega(0)} \right]^2,
\]

in agreement with the results of Sekimoto and Sasa [18] (modulo an erroneous factor of 1/4 in the value of $Q_{\text{irr}}^{\text{min}}$ as quoted). This minimum value is realised along the curve

\[
\omega(s) = \left[ \frac{s}{\omega(0)} + \frac{1-s}{\omega(1)} \right]^{-1} \quad ; 0 \leq s \leq 1.
\]

Thus in the classical limit, in the overdamped regime, we obtain the following inequality for the product of the irreversible heat and the time taken to execute the step:

\[
\tau \times Q_{\text{irr}} \geq 2\kappa K_B T \left[ \frac{1}{\omega(1)} - \frac{1}{\omega(0)} \right]^2.
\]

Such a relation is referred to in the literature as a thermodynamic complementarity relation, an analogue, both in spirit and form, of the energy-time uncertainty relations in quantum mechanics. It should however be noted that the RHS of the above complementarity relation is independent of $\hbar$. 
Again in the overdamped regime, but now in the low-temperature limit i.e. $\beta \hbar \omega \to 0$, we find that $g(\omega) \approx \beta \hbar/4\omega^3$ and the complementarity relation becomes

$$\tau \times Q_{\text{irr}} \geq \frac{\hbar}{2\kappa} \left( \frac{1}{\sqrt{\omega(1)}} - \frac{1}{\sqrt{\omega(0)}} \right)^2,$$

and one finds that $\hbar$ now does appear on the RHS as one would expect in the limit of low temperature where quantum effects become significant.

Turning to the weak dissipation case, one finds that expression for $g(\omega)$ is such that the relevant integral in (101) can be given in a closed form leading to the following complementarity relation:

$$\tau \times Q_{\text{irr}} \geq \frac{K_B T}{2\kappa} \left[ \log \left( \frac{\tanh(\beta \hbar \omega(1)/4)}{\tanh(\beta \hbar \omega(0)/4)} \right) \right]^2,$$

valid for all values of $T$. In particular, in the classical limit it becomes:

$$\tau \times Q_{\text{irr}} \geq \frac{K_B T}{2\kappa} \left[ \log \left( \frac{\omega(0)}{\omega(1)} \right) \right]^2,$$

and the curve $\omega(s), 0 \leq s \leq 1$ which minimizes $Q_{\text{irr}}$ now turns out to be

$$\omega(s) = \omega(0)^{(1-s)/2} \omega(1)^{s/2}.$$

We emphasise here that though we have presented explicit expressions for $Q_{\text{irr}}$ for the weak and strong damping regimes, the results in (87) and (88) enable us to derive expressions for $Q_{\text{irr}}$ both for classical as well as quantum case without any specific assumptions on relative magnitudes of $\kappa$ and $\omega$. Further, in the classical limit we find that for an isothermal process from $i \to f$ carried out in a finite time $\tau$, $Q_{\text{irr}}$ has the structure

$$Q_{\text{irr}} = T \Sigma \frac{\tau}{\tau},$$

where $\Sigma$ is independent of $T$. On using the fact that for an isothermal process $F(i) - F(f) = T \Delta S_{i \to f}$, we may rewrite (114) as

$$\Delta W_{i \to f} = T \left( \Delta S_{i \to f} - \Sigma_{i \to f} \right).$$

This provides a convenient and physically useful way of parametrizing deviations from quasi-staticity in that in the limit $\tau_{i \to f} \to \infty$ one recovers the familiar results of equilibrium thermodynamics.

**VIII. EFFICIENCY OF THE STIRLING ENGINE AT MAXIMUM POWER**

In this section we would use the results of the previous section to analyse the efficiency of the Stirling engine at maximum power very much in the spirit of the earlier works in the context the Carnot cycle. We would closely follow the works of Schmiedl et al [9] and of Esposito et al [11] who analysed the question of the efficiency of the Carnot cycle at maximum power in the limit of low dissipation from fairly general considerations. In particular, in [11] it was shown that the Carnot efficiency at maximum power $\eta^*_{\text{c}}$ is bounded below by $\eta_c/2$ and above by $\eta_c/(2 - \eta_c)$ and that while the Curzon-Ahlborn efficiency [19] is reached in the limit of 'symmetric' dissipation, the upper bound is realized in a completely asymmetric limit and coincides with the universal upper bound derived in [20] from somewhat different considerations.

Consider the situation when the isothermal steps $1 \to 2$ and $3 \to 4$ are carried out in finite times $\tau_h$ and $\tau_c$ respectively as indicated in (14). Power generated during the Stirling cycle is then

$$P = \frac{\Delta W_{1 \to 2} + \Delta W_{3 \to 4}}{\tau_c + \tau_h},$$

Also, as we have seen in Section VI, that the expression for the efficiency for the Stirling engine can be written as

$$\eta^{\text{cl}}_h = \frac{\Delta W_{1 \to 2} + \Delta W_{3 \to 4}}{\mu K_B (T_h - T_c) + \Delta W_{1 \to 2}}.$$
where $\mu \approx 0$ in the weak coupling regime and equal to 1/2 in the overdamped regime.

Using (113) and putting $\Sigma_{1\rightarrow 2} \equiv \Sigma_n$, $\Sigma_{3\rightarrow 4} \equiv \Sigma_c$, $\Delta S_{1\rightarrow 2} = -\Delta S_{3\rightarrow 4} = \Delta S$ [114] and (115) become

$$P = \frac{(T_h - T_c)\Delta S - T_h \Sigma_h/\tau_h - T_c \Sigma_c/\tau_c}{\tau_c + \tau_h} \quad (116)$$

$$\eta_s^{ci} = \frac{(T_h - T_c)\Delta S - T_h \Sigma_h/\tau_h - T_c \Sigma_c/\tau_c}{\mu K_B(T_h - T_c) + T_h \Delta S - T_h \Sigma_h/\tau_h}. \quad (117)$$

Maximizing $P$ with respect to $\tau_h$ and $\tau_c$ one finds that $P$ attains its maximum value for

$$\tau_h = \tau_h^* = \frac{T_h \Sigma_h}{(T_h - T_c)\Delta S} \left(1 + \sqrt{\frac{T_c \Sigma_c}{T_h \Sigma_h}}\right), \quad (118)$$

$$\tau_c = \tau_c^* = \frac{T_c \Sigma_c}{(T_h - T_c)\Delta S} \left(1 + \sqrt{\frac{T_h \Sigma_h}{T_c \Sigma_c}}\right). \quad (119)$$

Substituting these values for $\tau_h$ and $\tau_c$ in (117) one finds that the efficiency for the Stirling engine at maximum power is given by

$$\eta_s^{ci} = \frac{\eta_c \left(1 + \sqrt{\frac{\Sigma_c}{\Sigma_h}}\right)}{\left(1 + \frac{T_c \Sigma_c}{T_h \Sigma_h}\right) + \frac{T_h}{T_c} \left(1 - \frac{\Sigma_c}{\Sigma_h}\right) + \frac{2\mu \eta_c}{\log(\eta_c^*)}}. \quad (120)$$

We now consider two cases:

**Case A $\mu = 0$**

In the extreme weak dissipation regime i.e. $\mu = 0$, one recovers results similar to those in [9,11] in the context of the Carnot cycle:

1. In the symmetric case i.e $\Sigma_c/\Sigma_h = 1$, $\eta_s^{ci}$ equals the Curzon-Ahlborn efficiency $\eta_{CA} = 1 - \sqrt{T_c/T_h}$:

   $$\frac{\Sigma_c}{\Sigma_h} = 1 : \quad \eta_s^{ci} = \eta_{CA}. \quad (121)$$

2. $\eta_s^{ci}$ is bounded by $\eta_c/2$ and $\eta_c/(2 - \eta_c)$

   $$\frac{\eta_c}{2} \leq \eta_s^{ci} \leq \frac{\eta_c}{2 - \eta_c}. \quad (122)$$

   The upper and the lower bounds respectively correspond to $\Sigma_c/\Sigma_h \rightarrow 0$ and $\Sigma_c/\Sigma_h \rightarrow \infty$

**Case B $\mu \neq 0$**

For small but non zero $\mu < \frac{1}{16} \log(\omega_2/\omega_1)$ these results get modified to those given below

1. In the symmetric case i.e $\Sigma_c/\Sigma_h = 1$, $\eta_s^{ci}$ is less than the Curzon-Ahlborn efficiency [10] $\eta_{CA} = 1 - \sqrt{T_c/T_h}$:

   $$\frac{\Sigma_c}{\Sigma_h} = 1 : \quad \eta_s^{ci} = \frac{\eta_{CA}}{1 + \left(\frac{\mu}{\log(\omega_2/\omega_1)}\right) \left(\frac{2\eta_{CA}}{2 - \eta_{CA}}\right)} < \eta_{CA}. \quad (123)$$

2. $\eta_s^{ci}$ is bounded by $\eta_c/2$ and $\eta_c/(2 - \eta_c)$

   $$\frac{\eta_c}{2} \leq \eta_s^{ci} \leq \frac{\eta_c}{2 - \eta_c}. \quad (124)$$

   As before, the upper and the lower bounds respectively correspond to $\Sigma_c/\Sigma_h \rightarrow 0$ and $\Sigma_c/\Sigma_h \rightarrow \infty$
On the other hand if \( \mu > \frac{1}{2} \log(\omega_2/\omega_1) \), one finds that

\[
\eta_{cl}^{\text{cl}} \leq \eta_c/2.
\]  

(125)

In the figures below we display the bounds on \( \eta_{cl}^{\text{cl}} \) for \( \mu = 0.001, 0.1, 0.2, 0.4 \) with \( \omega_2/\omega_1 \) taken to be 2.05 where we also give the plots for \( \eta_c, \eta_{CA} \) and \( \eta_c/2 \) for comparison.

**IX. CONCLUSIONS**

In this work we have developed a microscopic framework for computing efficiencies of quantum/classical brownian motors realized by a harmonic oscillator. Two exactly solvable models for frequency modulation are presented. In the context of the Stirling Engine we have analytically treated the question of coupling the system at equilibrium with a bath at low temperature to a bath at higher temperature and the role that various time scales play in this process and have shown that these have strong influence on its efficiency. Further we have developed a procedure for computing finite time corrections to the quantities of interest needed for calculating the efficiency of the the three engines considered - Stirling, Carnot and Otto, and have derived the thermodynamic complementarity relations in the overdamped and underdamped situations both in the high as well as low temperature limits. In the spirit of the
works reported in [9] and [11] on the Carnot engine, we have analysed in detail the efficiency of the Stirling engine at maximum power and have investigated the role of dissipation parameters thereon. Though in this work we have exclusively considered interactions between the system and a thermal bath, the formalism can easily be extended to situations where the thermal bath is replaced by a squeezed thermal bath bringing with it new parameters and thereby ushering in new possibilities that have no classical analogues.

In the present work we have modelled the three heat engines after the quantum harmonic oscillator. It is of interest to carry out a similar analysis for finite state quantum systems. Indeed the entire perspective on heat pumps, refrigerators and heat engines developed in [3] is based on the simplest of quantum systems – a qubit. Though in that context a convenient Fokker-Planck framework would no longer be available we expect that the methodology developed here applied directly to the master equation would be useful there as well. We hope to return to this and related questions in the near future.

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