A Review of the Instability of Hot Electroweak Theory and its Bounds on $m_h$ and $m_t$*

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The electroweak vacuum need not be absolutely stable. For certain top and Higgs masses in the Minimal Standard Model, it is instead metastable with a lifetime exceeding the present age of the Universe. The decay of our vacuum may be nucleated at low temperature by quantum tunneling or at high temperature in the early Universe by thermal excitation. I briefly review the constraints on top and Higgs masses from requiring that the electroweak vacuum be sufficiently stable to have survived to the present day.

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1. Introduction

Today I discuss a question of life and death importance—though, admittedly, only to our descendants a billion times removed. Is the electroweak vacuum absolutely stable? Or will it instead someday decay away, billions of years hence, and wipe out the Universe as we know it? In pursuing this question, we shall be led to focus on a related question that should have concerned our ancestors a billion times removed: was the electroweak vacuum stable in the hot Early Universe? Investigating these concerns of the far-flung past and far-flung future will eventually address a question of relevance today: what are the best bounds on the top and Higgs masses that can be derived from requiring that the electroweak vacuum be sufficiently stable that it could have persisted to the present day? This discussion will all be in the context of the Minimal Standard Model, and I shall rashly assume that the standard single-Higgs-doublet model of electroweak symmetry breaking happens to be a good effective theory of nature below some large scale $\Lambda \gg 1$ TeV.

Much of this discussion is a synopsis of Ref. [1], which is work that I did with Stamatis Vokos. For a good review of much of the earlier work on the subject, see Ref. [2].

To understand that the electroweak vacuum can be unstable, consider the one-loop running of the Higgs self-coupling $\lambda$. It is of the form:

$$\beta_\lambda \equiv \frac{d\lambda}{d(\ln M)} \sim a\lambda^2 + bg^2 - cg_y^4$$  \hspace{1cm} (1)

where $a$, $b$ and $c$ are positive constants. The three terms are due to the contributions of scalar, vector, and fermion loops respectively, where $g$ and $g_y$ are the gauge and Yukawa couplings. The minus sign in front of the fermion loop contribution comes from Fermi statistics. Now suppose that the top mass is large compared to the W and Higgs masses. Then $g_y^2$ is large compared to $g^2$ and $\lambda$ and the last term dominates:

$$\beta_\lambda \sim -cg_y^4.$$  \hspace{1cm} (2)

The running of $\lambda$ is shown qualitatively in fig. 1. At large scales $M$, $\lambda$ runs negative. This means that the Higgs sector becomes unstable at very large mass scales or very short distances.

Now consider the classical Higgs potential shown in fig. 2a:

$$V_{cl}(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4.$$  \hspace{1cm} (3)
When radiative corrections are included, one finds that the dominant effect is to replace coupling constants by running coupling constants evaluated at the scale of $\phi$ itself:

$$V_{\text{eff}}(\phi) \approx -\frac{1}{2} \mu^2 (\bar{\phi})^2 + \frac{1}{4} \lambda(\bar{\phi})\bar{\phi}^4.$$  (4)

(More specifically, this is the result of summing leading-logarithms to all orders for the effective potential.) $\bar{\phi}$ is $\phi$ rescaled by its anomalous dimension. But I shall not keep track of the distinction between $\phi$ and $\bar{\phi}$; see Ref. [1] for details.

For large top mass, fig. 1 and eq. (4) imply that effective potential has the qualitative form of fig. 2b. Our vacuum is only meta-stable, and the potential is unstable at large $\phi$. Because the coupling $\lambda$ runs logarithmically, the scale $\phi_B$ at which the potential becomes unstable in fig. 2b will in general be exponentially large compared to the scale $\phi_A$ of the electroweak vacuum.

Fig. 3 shows for which values of top and Higgs masses our vacuum is unstable. I treat the Minimal Standard Model as a good effective theory up to some scale $\Lambda$. The determination of vacuum stability then depends on whether or not the instability B of fig. 2b manifests, if it manifests at all, for $\phi < \Lambda$. If so, then our vacuum is definitely unstable. If not, then it is stable within the assumed range of validity of the Minimal Standard Model, and any instability could only arise from the unspecified, new physics above $\Lambda$.

Vacuum decay is initiated by localized, random fluctuations—either quantum or thermal—of the scalar field $\phi$. Imagine a localized region of space of radius $R$ that probes the instability of the potential of fig. 2b: the scalar field is of order $\phi_C$ inside $R$ and equals the meta-stable vacuum $\phi_A$ outside. This “bubble” of unstable phase wants to grow because the inside has lower potential energy than the outside, which means that the total potential energy of the bubble has a negative contribution which grows as $-(\text{volume})$. On the other hand, the bubble wants to shrink because of surface tension; there is a positive contribution to the total potential energy [arising from the $(\nabla \phi)^2$ term] which grows with $R$. The potential energy of a bubble is shown qualitatively as a function of radius $R$ in fig. 4. (This picture artificially assumes the order of magnitude of $\phi$ inside the bubble is held fixed. See Ref. [5] for a qualitative picture of the more general case.) Bubbles smaller than the critical bubble size $R_c$ will collapse; larger bubbles will grow, eventually sucking the entire universe into the unstable phase. (In most cases, the $\phi$ field will continue rolling down the unstable potential until it reaches the cut-off $\phi \sim \Lambda$, at which point its fate depends on what new physics exists beyond the Minimal Standard Model.)

2. Instability at Zero Temperature

At zero energy, the only way our vacuum can decay is by quantum tunneling beneath the barrier in fig. 4. The rate for quantum tunneling through a barrier is exponentially small, and the exponent is given by the uncertainty principle. If $\Delta E$ is the amount one
must temporarily violate energy conservation while tunneling beneath the barrier, and $\Delta t$ is the amount of time one spends doing it, then very roughly speaking
\[
\text{amplitude} \sim e^{-(\Delta E)(\Delta t)} \sim e^{-S_E},
\]
where $S_E$ is the Euclidean action of the tunneling process. In practice, one solves for the classical Euclidean “bounce” solution that represents vacuum decay in the field theory problem and evaluates its Euclidean action to get the amplitude $\exp(-S_E)$ for vacuum decay. In the case at hand, this was done numerically many years ago by Flores and Sher.\[6\]

The observation that we exist does not require that our vacuum be absolutely stable, but instead only requires that the lifetime of our vacuum exceed the present age of the Universe. Intriguingly, there is a significant slice of parameter space for which our vacuum is ultimately unstable but has a lifetime exceeding 10 billion years. The dashed and dot-dash lines of Fig. 3 delineate this slice for the case $\Lambda = 10^{19}$ GeV. This computation was first performed by Flores and Sher.

There is a simple toy model for the tunneling problem\[5\] which can be solved analytically and which gives an excellent approximation (to within 1% or so) to the full numerical results. The effective potential (4) only becomes unstable at exponentially large values of $\phi$. So to good approximation I may drop the quadratic term in favor of the quartic one. Since $\lambda$ only runs logarithmically with $\phi$, let me approximate it in the unstable region by a negative constant $-\kappa$. The toy model is then defined by the potential
\[
V(\phi) = -\frac{1}{4}\kappa \phi^4.
\]
(6)
The meta-stable vacuum of this model is at $\phi = 0$. At first glance this may seem absurd because $V(\phi)$ above has no barrier against vacuum decay. But remember that the potential energy is $(\nabla \phi)^2 + V(\phi)$ rather than just $V(\phi)$. As a result of the gradient term, $\phi = 0$ is meta-stable against any local perturbations.

To estimate the vacuum decay amplitude in this toy model, rescale $\phi$ to factor the coupling constant out in front of the action:
\[
S_E \to \frac{1}{\kappa} \int d^4 x \left[ (\partial \tilde{\phi})^2 - \frac{1}{4} \tilde{\phi}^4 \right].
\]
(7)
So any non-trivial classical Euclidean solution has $S_E \sim 1/\kappa$. This model is in fact well-known, and the relevant solution is the Fubini instanton (or Lipaton):
\[
S_E = \frac{8\pi^2}{3\kappa}, \quad \phi(r) = \sqrt{\frac{2}{\kappa}} \frac{2R}{\sqrt{\kappa^2 + R^2}}.
\]
(8)
where the size scale $R$ of the solution is arbitrary because of the scale invariance of (7). The rate of false vacuum decay in this model then has exponential dependence
\[
\text{rate} \sim e^{-8\pi^2/3\kappa}.
\]
(9)
To return to the Minimal Standard Model, we only need to realize that the scale invariance of the toy model is slightly broken by the running of the coupling constants. Tunneling will take place at the scale which maximizes the rate \( \lambda(\phi) \), and so
\[
rate \sim \max_{\lambda(\phi)<0} \left[ e^{-8\pi^2/3|\lambda(\phi)|} \right].
\] (10)
This gives excellent agreement with numerical results.

3. Instability at High Temperature

At the high temperatures of the Early Universe, the situation is completely different. Return to the simple picture of the bubble energy in fig. 4 and consider temperatures large compared to the energy barrier \( E_c \) for vacuum decay. At such temperatures, energy states with \( E > E_c \) will be thermally excited, and the vacuum can then decay classically, without any exponential suppression associated with quantum tunneling. For more modest temperatures, the probability of having enough energy to cross the barrier classically is given by a simple Maxwell-Boltzmann factor:
\[
rate \sim e^{-\beta E_c},
\] (11)
where \( \beta \) is the inverse temperature.

The critical energy \( E_c \) is found by looking for a static, unstable solution to the classical field equations of motion that corresponds to resting precariously atop the barrier in fig. 4. (In various contexts, such solutions are called “sphalerons” or “O(3) bounces.”) Once the solution is found, \( E_c \) is identified as its classical energy. I shall discuss such solutions in the Minimal Standard Model in a moment, but first I want to preview the results. The solid line in fig. 5 shows the bound that may be derived\(^\text{13}\) by requiring that our vacuum was stable enough to persist through the hot temperatures of the early Universe. Note that it is stronger than the bound we derived earlier but not so strong as to require that our vacuum be absolutely stable. This is the strongest bound that has been derived from considerations of vacuum stability. The dotted line shows an earlier bound by Anderson\(^\text{7}\) that was based on considering temperatures only up to the critical temperature of symmetry restoration in electroweak theory; our stronger bounds come from investigating temperatures all the way up to the assumed scale \( \Lambda \) of new physics.

At first glance, these results may not make much sense. If one looks again at fig. 4, it appears that an unstable vacuum could never survive the early Universe since, at some point in time, the temperature must have been greater than the barrier energy \( E_c \). In field theory, however, it happens that \( E_c \) is itself effectively a function of \( T \), and so the matter is more subtle. At high temperature, the Higgs receives a thermal contribution to its mass of order \( g^2 T^2 \) that is analogous to the Debye or plasmon masses of the photon in a hot plasma:
\[
V_{\text{eff}}(\phi, T) \approx V_{\text{eff}}(\phi, 0) + \frac{1}{2} g^2 T^2 \phi^2,
\] (12)
where we are adopting $g^2$ as a short-hand for a particular linear combination of squared coupling constants. In the Minimal Standard Model,

$$g^2 \to \frac{1}{12} \left( \frac{3}{4} g^2_1 + \frac{9}{4} g^2_2 + 2 g^2_y + 6 \lambda \right).$$

This $g^2 T^2 \phi^2$ term is the standard effect responsible for symmetry restoration at high-temperature since it replaces $-\mu^2$ in the classical potential (3) by $-\mu^2 + g^2 T^2$, which turns positive for large $T$.

The thermal mass of the Higgs has an important impact in a problem similar to the one considered in this talk: the Linde-Weinberg bound. For small $m_h$ and $m_t$ (rather than large $m_t$), the effective potential is qualitatively of the form fig. 3a at zero temperature. Our vacuum is again unstable but this time is unstable against decay to $\phi \to 0$ rather than $\phi \to \infty$. Once again, there is a range of parameter space for which the lifetime of our vacuum exceeds 10 billion years. However, the thermal mass term in (12) favors smaller $\phi$ over larger $\phi$. As one increases the temperature, the energy barriers between the vacua get smaller and eventually disappear, so that the effective potential at high temperatures looks like fig. 3b and has only the vacuum at $\phi = 0$. So, in the early Universe, the average value of $\phi$ will be zero. By the time the Universe cools down enough for the non-zero meta-stable vacua to appear, it is too late! There is no way that the entire Universe will jump into the false vacuum state. As a result, the Higgs and top masses that give rise to fig. 3a may all be ruled out.

In our case, however, the situation is reversed. The barriers between A and C in fig. 2b will grow with temperature. (At the same time, the potential will turn upward at the origin so that the false vacuum A moves to $\phi = 0$.) Thus, increasing the temperature involves a trade-off for vacuum decay: (1) there is more energy available to thermally excite transitions across the barrier, but (2) the barrier is higher. If the universe started out in the false vacuum (which depends on the nature of the new physics above $\Lambda$), then it is not clear whether or not it would have decayed. To resolve which effect wins, one must examine the critical energy $E_c$ for vacuum decay in more detail.

So let us return to the toy model of the previous section and consider its high-temperature limit:

$$V(\phi) \to \frac{1}{2} g^2 T^2 \phi^2 - \frac{1}{4} \kappa \phi^4.$$  

Now consider the static energy $\int \frac{1}{2} (\nabla \phi)^2 + V$ and scale out both the coupling $\kappa$ and the dimension $gT$:

$$E \to \frac{gT}{\kappa} \int d^3 \bar{x} \left[ \frac{1}{2} (\nabla \bar{\phi})^2 + \frac{1}{2} \bar{\phi}^2 - \frac{1}{4} \bar{\phi}^4 \right].$$

The energy of any non-trivial static solution will therefore be of order $gT/\kappa$. Finding the extremum of the integral numerically, one finds

$$E_c = 6.015 \pi \cdot gT/\kappa.$$
and so the vacuum decay rate is roughly
\[ \text{rate} \sim e^{-\beta E_c(T)} \sim e^{-6.0 \pi g/\kappa}. \] \[ (13) \text{equation 17} \]

This result is independent of temperature.

The effect of the returning to the real problem in the Minimal Standard Model is to again break scale invariance. The rate is maximal at the temperature which maximizes \(-\lambda(T)\):
\[ \max_T [\text{rate}] \sim \max_{\lambda(T) > 0} \left[ e^{-6.0 \pi g(T)/|\lambda(T)|} \right]. \] \[ (13) \text{equation 18} \]

Note that, for sufficiently small coupling constants, this should always beat the rate from zero-temperature tunneling because the exponents are of order \(\beta E \sim g/|\lambda|\) in the thermal case and \(S_E \sim 1/|\lambda|\) in the zero-temperature tunneling case.

A discussion of the dimensional prefactors in (18) may be found in Ref. [1].

**(13)equation19section 4. Conclusion**

The result of applying (18) to the Minimal Standard Model is shown in fig. 7 for a variety of choices of scale \(\Lambda\) up to which the Minimal Standard Model is posited to be a good effective theory. What is all of this good for? Suppose that sometime in the next decade we happen to find the top and a candidate minimal Higgs at the point marked X. We shall then know that there must be new physics below 100 TeV. Alternatively, a sexier possibility is that we may someday discover that the Minimal Standard Model is in fact a good effective theory up to 10 TeV or so and find the top and Higgs in a region of fig. 3 that is unstable for \(\Lambda = 10^4\) GeV. We shall then know that the Universe as we know it is doomed and that our vacuum will decay someday billions of years hence.

The reader who is interested in much more speculative bounds, based on the controversial idea that weak interactions may necessarily become strong at high energy, should examine Refs. [3] and [4].

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**Figure Captions**

**Figure 11.** The running of $\lambda$ with scale $M$ for large top mass. The end-point on the left is fixed by the physical Higgs mass.

**Figure 22.** The effective potential (a) classically and (b) including radiative corrections from a heavy top.

**Figure 33.** Vacuum stability as a function of Higgs and top mass. Each line corresponds to a choice of the scale $\Lambda$ of physics beyond the Minimal Standard Model, and the solid line corresponds to choosing the Plank Scale. Considering only $\phi < \Lambda$, our vacuum is absolutely stable above, and only meta-stable below, the corresponding line.

**Figure 44.** The qualitative relationship between the potential energy and the radius of a bubble of unstable phase.

**Figure 55.** The dashed line is the $\Lambda = 10^{19}$ GeV line from fig. 3 and divides vacuum stability from meta-stability. If only zero-temperature tunneling is considered, the lifetime of the vacuum exceeds 10 billion years to the left of the dot-dash line. The solid line shows the improved bound obtained by requiring the vacuum to be sufficiently stable in the hot early Universe. The dotted line is the related but less stringent bound from ref. [7].

**Figure 66.** The form of the effective potential relevant to the Linde-Weinberg bound (a) at zero temperature, and (b) at high temperature.

**Figure 77.** The bound from vacuum decay in the early Universe shown for several values of the cut-off $\Lambda$. 

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The graph shows a decrease in $\lambda(M)$ as $\log M$ increases.
\[ \Lambda = 10^{19} \text{GeV} \]
(a) \[ \mathcal{V}(\phi) \]

(b) \[ \mathcal{V}(\phi) \]
