On the continuity of \( u(t, x) := Y^{t,x}_t \) from Feynman-Kac formula for a Neumann-Dirichlet problem

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Abstract

In this note we prove the continuity of the deterministic function \( u : [0, T] \times \bar{D} \to \mathbb{R} \), \( u(t, x) := Y^{t,x}_t \), where process \((Y^{t,x}_s)_{s \in [t, T]}\) is defined by the generalized multivalued backward stochastic differential equation:

\[
\begin{aligned}
    dY^{t,x}_s + F(s, X^{t,x}_s, Y^{t,x}_s)ds + G(s, X^{t,x}_s, Y^{t,x}_s)dA^{t,x}_s \\ + \partial \psi(Y^{t,x}_s)dA^{t,x}_s + Z^{t,x}_s dW_s, & \quad t \leq s \leq T, \\
    Y_T = \xi.
\end{aligned}
\]

The process \((X^{t,x}_s, A^{t,x}_s)_{s \geq t}\) is given by a stochastic differential equation with reflecting boundary conditions.

Keywords or phrases: Reflected diffusion processes; Continuity w.r.t. initial data

The aim of this paper is to correct the statement of [10, Proposition 13-inequality (40)] and the proof of [10, Corrolary 14-(c)]. In order to obtain the conclusion we should assume the additional conditions (11) and (12). Moreover, we will restrict to the case where coefficient \( f \) does not depend on \( Z \). We mention that our approach follows [2, Section 4].

This note is connected with many other papers; starting with [13] the viscosity solution of various type of parabolic partial differential equation with Neumann boundary condition, via probabilistic methods, represent the subject of: [3, 16, 4, 17, 5, 14, 15, 1].

We adopt all the notations and assumptions used in [10]. Let \( \mathcal{D} \) be a open connected bounded subset of \( \mathbb{R}^d \) of the form

\[ \mathcal{D} = \{ x \in \mathbb{R}^d : \ell(x) < 0 \}, \quad \text{Bd}(\mathcal{D}) = \{ x \in \mathbb{R}^d : \ell(x) = 0 \}, \]

where \( \ell \in C^3_b(\mathbb{R}^d), |\nabla \ell(x)| = 1 \), for all \( x \in \partial \mathcal{D} \).

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Functions

\[ b : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \]
\[ \sigma : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}, \]
\[ f : [0, \infty) \times \overline{\mathcal{D}} \times \mathbb{R} \rightarrow \mathbb{R}, \]
\[ g : [0, \infty) \times \text{Bd} (\mathcal{D}) \times \mathbb{R} \rightarrow \mathbb{R}, \]
\[ h : \overline{\mathcal{D}} \rightarrow \mathbb{R} \]

are continuous.

We assume that for all \( T > 0 \) there exist \( \alpha \in \mathbb{R} \) and \( L, \beta, \gamma \geq 0 \) (which can depend on \( T \)) such that \( \forall t \in [0, T] \), \( \forall x, \tilde{x} \in \mathbb{R}^d \):

\[
|b(t, x) - b(t, \tilde{x})| + ||\sigma(t, x) - \sigma(t, \tilde{x})|| \leq L \|x - \tilde{x}\|,
\]
and \( \forall t \in [0, T], \forall x \in \overline{\mathcal{D}}, u \in \text{Bd} (\mathcal{D}), y, \tilde{y} \in \mathbb{R} : \)

\[
(i) \quad (y - \tilde{y}) (f(t, x, y) - f(t, \tilde{x}, \tilde{y})) \leq \alpha \|y - \tilde{y}\|^2,
(ii) \quad |f(t, x, y)| \leq \gamma (1 + |y|),
(iii) \quad (y - \tilde{y}) (g(t, u, y) - g(t, u, \tilde{y})) \leq \alpha \|y - \tilde{y}\|^2,
(iv) \quad |g(t, u, y)| \leq \gamma (1 + |y|).
\]

With respect to the functions \( \varphi \) and \( \psi \) we assume

\[
(i) \quad \varphi, \psi : \mathbb{R} \rightarrow (-\infty, +\infty] \text{ are proper convex l.s.c. functions,}
(ii) \quad \varphi(y) \geq \varphi(0) = 0 \text{ and } \psi(y) \geq \psi(0) = 0, \forall y \in \mathbb{R},
\]

and there exists a positive constant \( M \) such that

\[
(i) \quad |\varphi(h(x))| \leq M, \quad \forall x \in \overline{\mathcal{D}},
(ii) \quad |\psi(h(x))| \leq M, \quad \forall x \in \text{Bd} (\mathcal{D}).
\]

We define

\[
\text{Dom} (\varphi) = \{ u \in \mathbb{R} : \varphi(u) < \infty \},
\]
\[ \partial \varphi (u) = \{ u^* \in \mathbb{R} : u^*(v - u) + \varphi(u) \leq \varphi(v), \forall v \in \mathbb{R} \}, \]
\[ \text{Dom}(\partial \varphi) = \{ u \in \mathbb{R} : \partial \varphi(u) \neq \emptyset \}, \]
\[ (u, u^*) \in \partial \varphi \iff u \in \text{Dom} \partial \varphi, \quad u^* \in \partial \varphi(u) \]

(for the function \( \psi \) we have the similar notation).

We introduce compatibility assumptions for all \( \epsilon > 0, t \geq 0, x \in \text{Bd} (\mathcal{D}), \tilde{x} \in \overline{\mathcal{D}} \) and \( y \in \mathbb{R} \):

\[
(i) \quad \nabla \varphi_{\epsilon} (y) g (t, x, y) \leq [\nabla \psi_{\epsilon} (y) g(t, x, y)]^+, \\
(ii) \quad \nabla \psi_{\epsilon} (y) f (t, \tilde{x}, y) \leq [\nabla \varphi_{\epsilon} (y) f(t, x, y)]^+,
\]

where \( a^+ = \max \{0, a\} \) and \( \nabla \varphi_{\epsilon} (y), \nabla \psi_{\epsilon} (y) \) are the unique solutions \( U \) and \( V \), respectively, of equations

\[
\partial \varphi (y - \epsilon U) \ni U \quad \text{and} \quad \partial \psi (y - \epsilon V) \ni V.
\]
Let now $T > 0$ be arbitrary and fixed. It follows from [8, Theorem 3.1] that for each $(t, x) \in [0, T] \times \mathcal{D}$ there exists a unique pair of continuous $\mathcal{F}^t_s$-p.m.s.p. $(X^t_{s,x}, A^t_{s,x})_{s \in [t, T]}$, with values in $\mathcal{D} \times \mathbb{R}_+$, solution of reflected stochastic differential equation:

$$
\begin{cases}
X^t_{s,x} = x + \int_t^s b(r, X^t_{r,x})dr + \int_t^s \sigma(r, X^t_{r,x})dW_r - \int_t^s \nabla \ell(X^t_{r,x})dA^t_{r,x}, \\
s \mapsto A^t_{s,x} \text{ is increasing,} \\
A^t_{s,x} = \int_t^s 1_{\{X^t_{r,x} \in \text{Bd}(\mathcal{D})\}}dA^t_{r,x}, \forall s \in [0, T],
\end{cases}
$$

(7)

where

$$
\mathcal{F}^t_s = \sigma \{ W_r - W_t : t \leq r \leq s \} \cup \mathcal{N}.
$$

We introduce a notation for the martingale part of the reflected diffusion process $X^t_{s,x}$:

$$
M^X_{s,x} := \int_t^s \sigma(X^t_{r,x})dW_r.
$$

Since $\mathcal{D}$ is a bounded set, then

$$
\sup_{s \geq 0} |X^t_{s,x}| \leq M
$$

(8)

and with calculus similar to that in [13] we have for all $\mu, T, p > 0$ there exists a positive constant $C$ such that $\forall t, t' \in [0, T], x, x' \in \mathcal{D}$:

$$
\mathbb{E} \sup_{s \in [0, T]} |X^t_{s,x} - X^t_{s,x'}|^p \leq C \left( |x - x'|^p + |t - t'|^\frac{p}{2} \right),
$$

(9)

$$(t, x) \mapsto A^t_{s,x} : [0, T] \times \mathcal{D} \rightarrow C([0, T], \mathbb{R}_+) \text{ is continuous } \mathbb{P}\text{-a.s.}
$$

and

$$
\mathbb{E}[e^{\mu A^t_{s,x}}] < \infty.
$$

(10)

In addition to paper [10] we impose two supplementary assumptions:

for all $(r, x) \in [0, T] \times \mathcal{D}$ the matrix $\sigma(r, x)$ is invertible

(11)

and $\forall t, \tilde{t} \in [0, T], x \in \text{Bd}(\mathcal{D}), y, \tilde{y} \in \mathbb{R}$

$$
|g(t, x, y) - g(\tilde{t}, \tilde{x}, \tilde{y})| \leq \beta \left( |t - \tilde{t}| + |x - \tilde{x}| + |y - \tilde{y}| \right).
$$

(12)

Under assumptions (1)-(6), it follows from [10, Theorem 9] (with $k = 1$ and $\tau$ replaced by $T$) that for each $(t, x) \in [0, T] \times \mathcal{D}$ there exists a unique 4-tuple $(Y^t_{x, x}, Z^t_{x, x}, U^t_{x, x}, V^t_{x, x})$ of p.m.s.p. such that $Y^t_{x}$ has continuous trajectories,

$$
\mathbb{E} \sup_{s \in [0, T]} e^{\lambda s + \mu A^t_{s,x}} |Y^t_{s,x}|^2 + \mathbb{E} \int_0^T e^{\lambda s + \mu A^t_{s,x}} |Y^t_{s,x}|^2 (ds + dA^t_{s,x}) < \infty,
$$

$$
\mathbb{E} \int_0^T e^{\lambda s + \mu A^t_{s,x}} |Z^t_{s,x}|^2 ds < \infty,
$$

$$
\mathbb{E} \int_0^T e^{\lambda s + \mu A^t_{s,x}} |U^t_{s,x}|^2 ds < \infty, \quad \mathbb{E} \int_0^T e^{\lambda s + \mu A^t_{s,x}} |V^t_{s,x}|^2 dA^t_{s,x} < \infty,
$$

3
and

\[(Y_{s}^{t,x}, U_{s}^{t,x}) \in \partial \varphi, \ P (d\omega) \otimes dt, \quad (Y_{s}^{t,x}, V_{s}^{t,x}) \in \partial \psi, \ P (d\omega) \otimes A (\omega, dt), \ \text{a.e. on } \Omega \times [t, T], \quad (13)\]

and satisfy the following BSDE:

\[
Y_{s}^{t,x} + \int_{s}^{T} U_{r}^{t,x} dr + \int_{s}^{T} V_{r}^{t,x} dA_{r}^{t,x} = h(X_{T}^{t,x}) + \int_{s}^{T} 1_{[t,T]} (r) f (r, X_{r}^{t,x}, Y_{r}^{t,x}) dr \\
+ \int_{s}^{T} 1_{[t,T]} (r) g (r, X_{r}^{t,x}, Y_{r}^{t,x}) dA_{r}^{t,x} - (M_{T}^{t,x} - M_{s}^{t,x}), \ \text{for all } s \in [0, T] \ \text{a.s.} \quad (14)\]

where

\[M_{s}^{t,x} := \int_{0}^{s} \hat{Z}_{r}^{t,x} dM_{r}^{X_{t,x}}.\]

Above we have extended the solution to interval \([0, t)\) by defining

\[Y_{s}^{t,x} = Y_{t}^{t,x}, \ \hat{Z}_{s}^{t,x} = 0, \ U_{s}^{t,x} = 0, \ V_{s}^{t,x} = 0, \ M_{s}^{X_{t,x}} = 0, \ \forall s \in [0, t).\]

We mention that, if we denote

\[K_{s}^{1,t,x} := \int_{0}^{s} U_{r}^{t,x} dr \quad \text{and} \quad K_{s}^{2,t,x} := \int_{0}^{s} V_{r}^{t,x} dA_{r}^{t,x},\]

then conditions (13) are equivalent with

\[
\int_{s_{1}}^{s_{2}} \langle u - Y_{r}^{t,x}, dK_{r}^{1,t,x} \rangle + \int_{s_{1}}^{s_{2}} \varphi (Y_{r}^{t,x}) dr \leq \int_{s_{1}}^{s_{2}} \varphi (u) dr, \\
\forall u \in \mathbb{R}, \ \forall 0 \leq t \leq s_{1} \leq s_{2} \quad (15)\]

and

\[
\int_{s_{1}}^{s_{2}} \langle v - Y_{r}^{t,x}, dK_{r}^{2,t,x} \rangle + \int_{s_{1}}^{s_{2}} \psi (Y_{r}^{t,x}) dA_{r}^{t,x} \leq \int_{s_{1}}^{s_{2}} \psi (v) dA_{r}^{t,x}, \\
\forall v \in \mathbb{R}, \ \forall 0 \leq t \leq s_{1} \leq s_{2}. \quad (16)\]

Notations

\[dK_{s}^{1,t,x} \in \partial \varphi(Y_{s}^{t,x}) d\omega, \ \mathbb{P}\text{-a.e.} \quad \text{and} \quad dK_{s}^{2,t,x} \in \partial \psi(Y_{s}^{t,x}) dA_{s}^{t,x}, \ \mathbb{P}\text{-a.e.} \quad (17)\]

mean that inequalities (15) and (16) are respectively satisfied.

We remark in addition that functions \(f, g\) depends by \(\omega\) only via function \(X_{t,x}\).

Below is the correct statement of [10, Proposition 13]. The only difference is due to the presence, in inequality (19), of the integral with respect to the measure generated by the total variation \(\mathcal{V} A_{t}^{t,x} - A_{t}^{t,x} \mathcal{V}\). The proof uses the same techniques and inequalities as declared in [10, Proposition 13].
Proposition 1 Under assumptions (1)-(6), we have

$$
\mathbb{E} \sup_{s \in [0, T]} e^{\lambda s + \mu A_s} |Y^t, x|^2 \leq C(T)
$$

(18)

and

$$
\mathbb{E} \sup_{s \in [0, T]} e^{\lambda s + \mu A_s} |Y^t, x - Y^t, x'|^2 \leq \mathbb{E} \left[ e^{\lambda T + \mu A_T} |h(X^T_T) - h(X^T_T')|^2 
+ \int_0^T e^{\lambda r + \mu A_r} \left( \mathbb{1}_{[t, T]}(r) f(r, X^r_T, Y^r_T, Z^r_T) - \mathbb{1}_{[t', T]}(r) f(r, X^r_T, Y^r_T, Z^r_T) \right)^2 dr 
+ \int_0^T e^{\lambda r + \mu A_r} \left( \mathbb{1}_{[t, T]}(r) g(r, X^r_T, Y^r_T) - \mathbb{1}_{[t', T]}(r) g(r, X^r_T, Y^r_T) \right)^2 dA^r_T 
+ \int_0^T e^{\lambda r + \mu A_r} \mathbb{1}_{[t, T]}(r) \left| g(r, X^r_T, Y^r_T) \right|^2 dA^{t,x}_r \right].
$$

(19)

We define

$$
u(t, x) = Y^t_t, \quad (t, x) \in [0, T] \times \mathcal{D},$$

(20)

which is a determinist quantity since $Y^t_t$ is $\mathcal{F}^t_t \equiv \mathcal{N}$-measurable.

From the Markov property we have

$$
u(s, X^r_s) = Y^r_s.
$$

(21)

The proof of the first two points in [10, Corollary 14] is the same. Concerning the point (c), we highlight that now the continuity of application $(t, x) \mapsto \nu(t, x)$ does not follows anymore directly from inequality (19) (as in [10, Corollary 14-(c)]). Our note involves new arguments. Since the function $\nu$ is defined through $Y^t_t$, the problem of continuity of $\nu$ is reduced to the continuity of the stochastic process with respect to the initial data $(t, x)$. We will give first a generalization of [2, Proposition 15] to our backward stochastic equation; more precisely, we will show that $(Y^t, x)_{s \in [0, T]}$ is tight in a suitable topological space and we will use the techniques presented in [2, Section 4] and [3, Section 3]. This approach forces us to restrict to the case where coefficient $f$ does not depend on $Z$ (for a more detailed explanation see the comments from [11, Section 6, page 535]).

Let consider the Skorohod space $\mathcal{D} = \mathcal{D} \left( [0, T], \mathbb{R} \right)$ of càdlàg functions $\nu : [0, T] \rightarrow \mathbb{R}$ (i.e. right continuous and with left-hand limits) endowed with $\mathcal{S}$-topology (introduced by Jakubowski in [6]). Space $C \left( [0, T], \mathbb{R}^d \right)$ of continuous functions is equipped with the supremum norm topology.

We work with $\mathcal{S}$-topology because we need the continuity of application $\mathcal{D} \ni y \mapsto \int_0^s g(r, y(r)) dA_r$, where $h$ is a continuous function and $A$ is a continuous non-decreasing function. This property is not true in Meyer-Zheng topology (unless the measure induced by $A$ is absolutely continuous with respect to the Lebesgue measure).

For the convenience of the reader we summarize the introduction of $\mathcal{S}$-topology (see [6]). First, let $\mathcal{V}^+ \subset \mathcal{D}$ denote the space of non-negative and non-decreasing functions $\mathcal{D} \ni \nu : [0, T] \rightarrow \mathbb{R}$ and therefore $\mathcal{V} := \mathcal{V}^+ - \mathcal{V}^+$ is the space of bounded variation functions. $\mathcal{S}$-topology is a sequential topology defined by:
Definition 2 Sequence \((x_n)_{n \in \mathbb{N}} \subset \mathbb{D}\) converges to \(x_0 \in \mathbb{D}\) (denoted by \(x_n \longrightarrow x_0\)) if for every \(\varepsilon > 0\), there exists \((v_{n,\varepsilon})_{n \in \mathbb{N}} \subset \mathbb{D}\) such that:

(a) elements \(v_{n,\varepsilon}\) are \(\varepsilon\)-uniformly close to \(x_n\), i.e.

\[
\sup_{[0,T]} |x_n(t) - v_{n,\varepsilon}(t)| \leq \varepsilon, \ \forall n \in \mathbb{N};
\]

(b) \(v_{n,\varepsilon}\) is weakly-* convergence to \(v_{0,\varepsilon}\) (denoted by \(\longrightarrow\)), i.e for every continuous function \(f : [0,T] \rightarrow \mathbb{R}\),

\[
\int_0^T f(t) \, dv_{n,\varepsilon}(t) \longrightarrow \int_0^T f(t) \, dv_{0,\varepsilon}(t), \text{ as } n \rightarrow \infty.
\]

Remark 3 ([6, Remark 2.4]) From this definition we deduce the pointwise convergence outside a countable set \(Q_\varepsilon \subset [0,T]\), i.e.

\[
v_{n,\varepsilon}(t) \longrightarrow v_{0,\varepsilon}(t), \ \forall t \in [0,T] \setminus Q_\varepsilon.
\]

In addition

\[x_n(t) \longrightarrow x_0(t), \ \forall t \in [0,T] \setminus \bigcup_{i=1}^{\infty} Q_{1/i}.
\]

A very important ingredient is the following result (a generalization of the Helly-Bray theorem) due to Boufoussi, Van Casteren [3]. We precise that, with a slight change of the proof, we can consider weaker assumptions in order to obtain the same convergence result:

Lemma 4 ([3, Lemma 3.3]) Let \(\Phi_n : [0,T] \times \mathbb{R} \rightarrow \mathbb{R}\) with \(1 \leq n \leq \infty\), be uniformly locally Lipschitz functions, such that

\[
\lim_{n \rightarrow \infty} \Phi_n(s,y) = \Phi_\infty(s,y), \ \forall (s,y) \in [0,T] \times \mathbb{R},
\]

and \((H^n)_{1 \leq n \leq \infty}\) be a sequence of elements of \(\mathcal{V}_+^c\) (the subspace of continuous functions \(V \in \mathcal{V}_+\) vanishing at 0) such that

\[
\lim_{n \rightarrow \infty} \|H^n(\cdot) - H_\infty(\cdot)\|_\infty = 0.
\]

Let sequence \((Y^n)_{1 \leq n \leq \infty} \subset \mathbb{D}\) such that

\[Y^n \longrightarrow Y_\infty \text{ as } n \rightarrow \infty.
\]

Then there exists a countable subset \(Q \subset [0,T]\) such that for all \(t \in [0,T] \setminus Q\) :

\[
\lim_{n \rightarrow \infty} \int_0^t \Phi_n(s,Y^n(s)) \, dH^n(s) = \int_0^t \Phi_\infty(s,Y_\infty(s)) \, dH_\infty(s).
\]

The main result of this note is the following:

Proposition 5 Under assumptions (1)-(6) and (11)-(12), function

\[(t, x) \mapsto u(t, x) = Y^{t,x}_t : [0,T] \times \mathcal{D} \rightarrow \mathbb{R}\]

is continuous.
Remark 6 Using the continuity of $u$ it was proved in [10] that this function is the unique viscosity solution of the following parabolic variational inequality with a mixed nonlinear multivalued Neumann-Dirichlet boundary condition:

$$
\begin{cases}
\frac{\partial u(t, x)}{\partial t} - \mathcal{L}_t u(t, x) + \partial \varphi(u(t, x)) \geq f(t, x, u(t, x)), & t > 0, \ x \in \mathcal{D}, \\
\frac{\partial u(t, x)}{\partial n} + \partial \psi(u(t, x)) \geq g(t, x, u(t, x)), & t > 0, \ x \in \partial \mathcal{D}, \\
u(0, x) = h(x), & x \in \mathcal{D}.
\end{cases}
$$

Proof of Proposition 5. Let $(t_n, x_n) \to (t, x)$, as $n \to \infty$. To prove that $u(t_n, x_n) \to u(t, x)$ is equivalent with proving that any subsequence has a further subsequence which converges to $u(t, x)$. Let $(t_{n_k}, x_{n_k})$ be an arbitrary subsequence still denoted in the sequel by $(t_n, x_n)$.

Using the definitions

$$f_n(r, x, y) := \mathbb{1}_{[t_n, T]}(r) f(r, x, y) \quad \text{and} \quad g_n(r, x, y) := g(r \vee t_n, x, y)$$

it is clearly that the processes

$$X^n := X^{t_n, x_n}, \ A^n := A^{t_n, x_n} \quad \text{and} \quad Y^n := Y^{t_n, x_n}, \ \dot{Z}^n := \dot{Z}^{t_n, x_n}, \ U^n := U^{t_n, x_n}, \ V^n := V^{t_n, x_n}$$

satisfy equation (7)

$$\begin{cases}
X^n_s = x + \int_{t_n}^{s \wedge t} b(r, X^n_r)dr + \int_{t_n}^{s \wedge t} \sigma(r, X^n_r)dW_r - \int_{t_n}^{s \wedge t} \nabla \ell(X^n_r)dA^n_r, \\
A^n_s = \int_{t_n}^{s \wedge t} \mathbb{1}_{\{X^n_r \in \partial \mathcal{D}\}}dA^n_r, \forall s \in [0, T]
\end{cases} \quad (22)$$

and backward equation

$$Y^n_s + \int_s^T U^n_r dr + \int_s^T V^n_r dA^n_r = h(X^n_T) + \int_s^T f_n(r, X^n_r, Y^n_r)dr + \int_s^T g_n(r, X^n_r, Y^n_r)dr$$

$$+ \int_s^T \dot{Z}^n_r dM^n_r, \ s \in [0, T] \quad (23)$$

such that (13) is satisfied.

The first part of the proof is adapted from [2, Proposition 15] and [3, Theorem 3.1].

Since we have the conclusion of the existence Theorem 9 from [10] and $\lambda, \mu \geq 0$ we easily see that:

$$\sup_{n \in \mathbb{N}} \mathbb{E}\left( \sup_{s \in [0, T]} |Y^n_s|^2 \right) + \mathbb{E}\left( \int_0^T |Y^n_r|^2 dA^n_r \right) + \mathbb{E}\left( \int_0^T |Z^n_r|^2 dr \right) < \infty.$$ 

The $S$-tightness we will be obtained using the sufficient condition given, e.g., in [7, Appendix A]; for this we shall prove the uniform boundedness for the quantities of type

$$CV_T(L) + \mathbb{E}\left( \sup_{s \in [0, T]} |L_s| \right),$$
where the conditional variation $CV_T$ is defined by

$$CV_T(L) := \sup_{\pi} \sum_{i=0}^{m-1} \mathbb{E}\left[\left|\mathbb{E}^{F_i} [L_{t_{i+1}} - L_{t_i}]\right|\right]$$

(24)

with the supremum taken on partitions $\pi : t = t_0 < t_1 < \cdots < t_m = T$.

Let

$$M_n := \int_0^s Z_r^n dM_r^X$$

and

$$K_{s,n} := \int_0^s U_r^n dr, \quad K_{s,n}^2 := \int_0^s V_r^n dA_r^n, \forall s \in [0,T].$$

Equation (23) becomes

$$Y_n^s + (K_{s,n}^1 - K_{s,n}^1) + (K_{s,n}^2 - K_{s,n}^2) = h(X_n^s) + \int_s^T f_n(r, X_r^n, Y_r^n) dr$$

$$+ \int_s^T g_n(r, X_r^n, Y_r^n) dA_r^n - (M_n^s - M_n^s), \quad s \in [0,T]$$

(25)

with

$$dK_{s,n}^1 \in \partial \varphi (Y_n) ds, \quad \mathbb{P}\text{-a.e.} \quad \text{and} \quad dK_{s,n}^2 \in \partial \psi (Y_n) dA_n, \quad \mathbb{P}\text{-a.e., } \forall s \in [t_n, T].$$

(26)

It is easy to make the computations (as in [2, Proposition 15] or [3, Theorem 3.1]) and to deduce that

$$\sup_{n \in \mathbb{N}} \left[ CV_T (Y_n^s) + \mathbb{E}\left( \sup_{s \in [0,T]} |Y_n^s| \right) + \mathbb{E}\left( \sup_{s \in [0,T]} |M_n^s| \right) \right] < \infty$$

and

$$\sup_{n \in \mathbb{N}} \left[ CV_T (K_{s,n}^1) + \mathbb{E}\left( \sup_{s \in [0,T]} |K_{s,n}^1| \right) + CV_T (K_{s,n}^2) + \mathbb{E}\left( \sup_{s \in [0,T]} |K_{s,n}^2| \right) \right] < \infty.$$

As example,

$$CV_T (K_{s,n}^2) + \mathbb{E}\left( \sup_{s \in [0,T]} |K_{s,n}^2| \right) \leq \mathbb{E} \int_0^T |V_r^n| dA_r^n$$

$$\leq \left( \mathbb{E}(A_T^n) \right)^{1/2} \left[ \mathbb{E} \int_0^T |V_r^n|^2 dA_r^n \right]^{1/2} \leq C.$$

Now the criterion presented in [7, Appendix A] ensures tightness with respect to the $S$-topology of the sequence $(Y_n^s, M_n^s, K_{s,n}^1, K_{s,n}^2)$ and therefore

$$\Gamma^n := (X^n, W, A^n, Y^n, M^n, K_{s,n}^1, K_{s,n}^2)$$

is tight in $C^2 ([0,T], \mathbb{R}^d) \times C([0,T], \mathbb{R}) \times \mathbb{D}^4$.

From [6, Theorem 3.4 & Definition 3.3], it follows that there exists a subsequence (still denoted by $n$) and the following processes, defined on the same probability space $( [0,1], B_{[0,1]}, \lambda)$,

$$\bar{\Gamma}^n := (\bar{X}^n, \bar{W}^n, \bar{A}^n, \bar{Y}^n, \bar{M}^n, \bar{K}_{s,n}^1, \bar{K}_{s,n}^2) \in C^2 ([0,T], \mathbb{R}^d) \times C([0,T], \mathbb{R}) \times \mathbb{D}^4$$
and 
\[ \Gamma := (\bar{X}, \bar{W}, \bar{A}, \bar{Y}, \bar{M}, \bar{K}^1, \bar{K}^2) \in C^2([0, T], \mathbb{R}^d) \times C([0, T], \mathbb{R}) \times \mathbb{D}^4 \]

such that 
\[ \Gamma^n \sim \Gamma^n \]

and 
\[ \forall \omega \in [0, 1], \bar{\Gamma}^n (\omega) \xrightarrow{U^3 \times \mathbb{S}^4} \bar{\Gamma} (\omega), \text{ as } n \to \infty. \]

As a consequence we obtain 
\[ \forall \omega \in [0, 1], (\bar{X}^n (\omega), \bar{W}^n (\omega), \bar{A}^n (\omega)) \to (\bar{X} (\omega), \bar{W} (\omega), \bar{A} (\omega)), \]

in \( C^2([0, T], \mathbb{R}^d) \times C([0, T], \mathbb{R}) \), as \( n \to \infty. \)

From the S-convergence we see that we have in addition the pointwise convergence of the sequence \((Y^n, M^n, K^{1,n}, K^{2,n})_n\), outside a countable set \(Q_1 \subset [0, T]\).

We are now able to pass to the limit in (22): since 
\[ (X^n, W, A^n) \sim (\bar{X}^n, \bar{W}^n, \bar{A}^n) \]

we deduce, using classical arguments, that \((\bar{X}^n, \bar{W}^n, \bar{A}^n)\) satisfy equation (22), the limit process \((\bar{X}, \bar{W}, \bar{A})\) satisfy equation (7) and \(W^n, \bar{W}\) are Brownian motion with respect to the filtration \(\mathcal{F}^{\bar{X}, \bar{W}}\). Hence we have 
\[ (\bar{X}, \bar{A}) = (\bar{X}_{t,x}^r, \bar{A}_{t,x}^r), \]

where \((\bar{X}_{t,x}^r, \bar{A}_{t,x}^r)_{r \in [0,T]}\) is the solution of equation (7) considered on the probability space \(([0,1], \mathcal{B}[0,1], \lambda)\).

Concerning equation (25), we state first the following technical result (which proof is left to the reader):

**Lemma 7** Let 
\[ (X^n, Y^n, M^n, K^{1,n}, K^{2,n}) \sim (\bar{X}^n, \bar{Y}^n, \bar{M}^n, \bar{K}^1, \bar{K}^2, n) \]

and two continuous functions \(G : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}\) and \(\phi : C([0, T], \mathbb{R}^d) \times \mathbb{D}^4 \to \mathbb{R}\). If 
\[ \phi(X^n, Y^n, M^n, K^{1,n}, K^{2,n}) = \int_{s_1}^{s_2} G(r, X^n_r, Y^n) \, dA^n_r \]

then 
\[ \phi(\bar{X}^n, \bar{Y}^n, \bar{M}^n, \bar{K}^{1,n}, \bar{K}^{2,n}) = \int_{s_1}^{s_2} G(r, \bar{X}^n_r, \bar{Y}^n) \, d\bar{A}^n_r. \]

Hence we deduce that 
\[ Y^n_s + (\bar{K}^{1,n}_T - \bar{K}^{1,n}_s) + (\bar{K}^{2,n}_T - \bar{K}^{2,n}_s) = h(\bar{X}^n_T) + \int_s^T f_n(r, \bar{X}^n_r, \bar{Y}^n) \, dr \]

\[ + \int_s^T g_n(r, \bar{X}^n_r, \bar{Y}^n) \, d\bar{A}^n_r - (M^n_T - M^n_s), \quad s \in [0, T]. \]  

(27)

In order to prove that (26) holds true for \((\bar{Y}^n, \bar{K}^{1,n}, \bar{K}^{2,n})\), the next Lemma can be proved:
Lemma 8 Let

\[ (A^n, Y^n, K^{2,n}) \sim (\bar{A}^n, \bar{Y}^n, \bar{K}^{2,n}) \]

and a proper convex l.s.c. function \( \psi : \mathbb{R} \to (-\infty, +\infty] \) such that \( \psi(y) \geq \psi(0) = 0, \forall y \in \mathbb{R} \).

If

\[
\int_{s_1}^{s_2} \psi(Y^n_r) dA^n_r \leq \int_{s_1}^{s_2} (Y^n_r - v) dK^{2,n}_r + \int_{s_1}^{s_2} \psi(v) dA^n_r, \forall v \in \mathbb{R}, \forall 0 \leq t \leq s_1 \leq s_2.
\]

then

\[
\int_{s_1}^{s_2} \psi(\bar{Y}^n_r) d\bar{A}^n_r \leq \int_{s_1}^{s_2} (\bar{Y}^n_r - v) d\bar{K}^{2,n}_r + \int_{s_1}^{s_2} \psi(v) d\bar{A}^n_r, \forall v \in \mathbb{R}, \forall 0 \leq t \leq s_1 \leq s_2.
\]

Hence we deduce that

\[ d\bar{K}^{1,n}_s \in \partial \varphi(\bar{Y}^n_s) ds, \mathbb{P}\text{-a.e. and } d\bar{K}^{2,n}_s \in \partial \psi(\bar{Y}^n_s) d\bar{A}^n_s, \mathbb{P}\text{-a.e., } \forall s \in [t_n, T]. \]  

\[ (28) \]

Now we can pass to the limit in \((27)\).

First, applying \([6, \text{Corollary 2.11}]\), we see that

\[
\lim_{n \to \infty} \int_s^T f_n(r, \bar{X}_r^n, \bar{Y}_r^n) dr = \int_s^T f(r, \bar{X}_r^{t,x}, \bar{Y}_r) dr,
\]

since

\[
\int_s^T f_n(r, \bar{X}_r^n, \bar{Y}_r^n) dr = \int_s^T \mathbb{1}_{[t_n, T]}(r) f(r, \bar{X}_r^n, \bar{Y}_r^n) dr
\]

\[
= \int_s^T \mathbb{1}_{[t, T]}(r) f(r, \bar{X}_r^n, \bar{Y}_r^n) dr + \int_s^T f(r, \bar{X}_r^n, \bar{Y}_r^n)(\mathbb{1}_{[t_n, T]}(r) - \mathbb{1}_{[t, T]}(r)) dr.
\]

For the Riemann-Stieltjes integral we will apply Lemma 4. Hence, there exists a countable set \( Q_2 \subset [0, T] \) such that, for any \( s \in [0, T] \setminus Q_2 \),

\[
\lim_{n \to \infty} \int_0^s g_n(r, \bar{X}_r^n, \bar{Y}_r^n) d\bar{A}^n_r = \int_0^s g(r, \bar{X}_r^{t,x}, \bar{Y}_r) d\bar{A}_r^{t,x}.
\]

It follows that

\[
\bar{Y}_s + (\bar{K}_s^1 - \bar{K}_s^2) + (\bar{K}_s^2 - \bar{K}_s^2) = h(\bar{X}_T) + \int_s^T \mathbb{1}_{[t, T]}(r) f(r, \bar{X}_r^{t,x}, \bar{Y}_r) dr
\]

\[
+ \int_s^T g(r, \bar{X}_r^{t,x}, \bar{Y}_r) d\bar{A}_r^{t,x} - (\bar{M}_T - \bar{M}_s), s \in [0, T] \setminus (Q_1 \cup Q_2).
\]

\[ (29) \]

Since the processes \( \bar{Y}, \bar{M}, \bar{K}^1 \) and \( \bar{K}^2 \) are càdlàg the above equality take place for any \( s \in [0, T] \).

From the above equation, it is immediately that \( \bar{M} \) is \( \mathcal{F}_s^X, W, A, Y, M, K^1, K^2 \)-adapted and it can be show (see e.g. the proof \([3, \text{Theorem 3.1 (step 3)}]\)) that both \( M^X \) and \( M \) are martingale.
with respect to the same filtration \( \mathcal{F}_s^X, \mathcal{W}, \hat{A}, \mathcal{Y}, \mathcal{M}, \mathcal{K}_1, \mathcal{K}_2 \) (and this is the reason to work not with the filtration generated by the Brownian motion).

On our new probability space \([0, 1], \mathcal{B}_{[0,1]}, \lambda\) we consider the solution \((\bar{Y}^{t,x}, \bar{Z}^{t,x}, \bar{U}^{t,x}, \bar{V}^{t,x})\) of the BSDE (14):

\[
\begin{align*}
\bar{Y}^{t,x}_s + (\bar{K}^1_{T} - \bar{K}^1_s) + (\bar{K}^2_{T} - \bar{K}^2_s) = h(X^{t,x}_T) + \int_s^T \mathbb{1}_{[t,T]}(r) f(r, \bar{X}^{t,x}_r, \bar{Y}^{t,x}_r) dr \\
+ \int_s^T g(r, \bar{X}^{t,x}_r, \bar{Y}^{t,x}_r) d\bar{A}^{t,x}_r - (\bar{M}^{t,x}_T - \bar{M}^{t,x}_s), \quad s \in [0, T],
\end{align*}
\]

with

\[
d\bar{K}^1_s \in \partial \varphi(\bar{Y}^{t,x}_s) ds, \quad \mathbb{P}\mbox{-a.e.} \quad \text{and} \quad d\bar{K}^2_s \in \partial \psi(\bar{Y}^{t,x}_s) d\bar{A}^{t,x}_s, \quad \mathbb{P}\mbox{-a.e.},
\]

where

\[
\begin{align*}
\bar{K}^1_{t,x} := \int_0^t \bar{U}^{t,x}_s ds, \quad \bar{K}^2_{t,x} := \int_0^t \bar{V}^{t,x}_s d\bar{A}^{t,x}_s, \quad \bar{M}^{t,x}_s := \int_0^s \bar{Z}^{t,x}_r d\bar{M}^{t,x}_r.
\end{align*}
\]

From Itô’s formula applied to (29) and (30) and since \( \bar{M} \) and \( \bar{M}^{t,x} \) are martingale with respect to the same filtration, we obtain

\[
\begin{align*}
&\left| \bar{Y}_s - \bar{Y}^{t,x}_s \right|^2 + \left( [\bar{M} - \bar{M}^{t,x}]_T - [\bar{M} - \bar{M}^{t,x}]_s \right) \\
&\quad + 2 \int_s^T (\bar{Y}_r - \bar{Y}^{t,x}_r)(d\bar{K}^1_r - d\bar{K}^1_{t,x}) + 2 \int_s^T (\bar{Y}_r - \bar{Y}^{t,x}_r)(d\bar{K}^2_r - d\bar{K}^2_{t,x}) \\
&\quad = 2 \int_s^T (\bar{Y}_r - \bar{Y}^{t,x}_r)(f(r, \bar{X}^{t,x}_r, \bar{Y}_r) - f(r, \bar{X}^{t,x}_r, \bar{Y}^{t,x}_r)) dr \\
&\quad + 2 \int_s^T (\bar{Y}_r - \bar{Y}^{t,x}_r)(g(r, \bar{X}^{t,x}_r, \bar{Y}_r) - g(r, \bar{X}^{t,x}_r, \bar{Y}^{t,x}_r)) d\bar{A}^{t,x}_r \\
&\quad - 2 \int_s^T (\bar{Y}_r - \bar{Y}^{t,x}_r) d(\bar{M}_r - \bar{M}^{t,x}_r), \quad s \in [t, T],
\end{align*}
\]

where \([\bar{M} - \bar{M}^{t,x}] \) is the quadratic variation process of \( \bar{M} - \bar{M}^{t,x} \).

Using the assumptions on \( f \) and \( g \), the below auxiliary result (inequality 33) and generalized Gronwall’s lemma from [9, Lemma 12] we deduce identification of the limit:

\[
\bar{Y} = \bar{Y}^{t,x} \quad \text{and} \quad \bar{M} = \bar{M}^{t,x}.
\]

**Lemma 9** (i) The limit processes \( \bar{K}^1 \) and \( \bar{K}^2 \) satisfy

\[
(d\bar{K}^1_s + d\bar{K}^2_s) \in \partial \varphi(\bar{Y}_r) dr + \partial \psi(\bar{Y}_r) d\bar{A}^{t,x}_r, \quad \text{for} \ r \in [t, T],
\]

in the sense that for any \( v \in \mathbb{R} \) and any \( 0 \leq t \leq s_1 \leq s_2 \),

\[
\begin{align*}
&\int_{s_1}^{s_2} (v - \bar{Y}_r) d(\bar{K}^1_r + \bar{K}^1_{t,x}) + \int_{s_1}^{s_2} \varphi(\bar{Y}_r) dr + \int_{s_1}^{s_2} \psi(\bar{Y}_r) d\bar{A}^{t,x}_r \\
&\quad \leq \int_{s_1}^{s_2} \varphi(v) dr + \int_{s_1}^{s_2} \psi(v) d\bar{A}^{t,x}_r.
\end{align*}
\]
(ii) The following inequality holds true:

\[
\int_s^T \left[ (\bar{Y}_r - \bar{Y}^{t,x}_r)(d\bar{K}_r^1 - d\bar{K}^{t,x}_r) + (\bar{Y}_r - \bar{Y}^{t,x}_r)(d\bar{K}_r^2 - d\bar{K}^{t,x}_r) \right] \geq 0. \tag{33}
\]

Finally, from equality

\[
\bar{Y}^n_{t_n} = -\bar{K}^{1,n}_T - \bar{K}^{2,n}_T + h(\bar{X}^n_{t_n}) + \int_{t_n}^T f(r, \bar{X}^n_r, \bar{Y}^n_r)dr + \int_{t_n}^T g(r, \bar{X}^n_r, \bar{Y}^n_r)d\bar{A}^n_r - \bar{M}^n_T
\]

and the pointwise convergence outside a countable set \(Q \subset [0, T)\) we deduce that

\[
\bar{Y}^n_{t_n} \to \bar{Y}^{t,x}_t = -\bar{K}^{1,t,x}_T - \bar{K}^{2,t,x}_T + h(\bar{X}^{t,x}_T) + \int_t^T f(r, \bar{X}^{t,x}_r, \bar{Y}^{t,x}_r)dr + \int_t^T g(r, \bar{X}^{t,x}_r, \bar{Y}^{t,x}_r)d\bar{A}^{t,x}_r - \bar{M}^{t,x}_T.
\]

Hence, on a subsequence,

\[
u(t_n, x_n) \to u(t, x),
\]

since \(u\) is deterministic function.

The last part of the proof consists in showing Lemma 9.

**Proof of Lemma 9**

(i) We know (see, e.g., [12, Proposition 6.24]) that if \(\varphi\) is a l.s.c. function such that \(\varphi(x) \geq 0, \forall x \in \mathbb{R}\), then there exists a sequence of locally Lipschitz functions \(\varphi_n : \mathbb{R} \to \mathbb{R}\) such that

\[
0 \leq \varphi_1(x) \leq \cdots \leq \varphi_j(x) \leq \cdots \leq \varphi(x)
\]

and

\[
\lim_{j \to \infty} \varphi_j(x) = \varphi(x)
\]

(the same conclusion holds true for \(\psi\)).

From the above, using conditions (28) and Itô’s formula for (14), we deduce that, for any \(j \in \mathbb{N}\) and any \(s_1, s_2\) such that \(0 \leq t \leq s_1 \leq s_2\),

\[
\mathbb{E} \left[ \int_{s_1}^{s_2} \varphi_j(\bar{Y}^n_r)dr + \int_{s_1}^{s_2} \psi_j(\bar{Y}^n_r)d\bar{A}^n_r \right]
\]

\[
\leq \mathbb{E} \left[ \int_{s_1}^{s_2} \varphi(\bar{Y}^n_r)dr + \int_{s_1}^{s_2} \psi(\bar{Y}^n_r)d\bar{A}^n_r \right]
\]

\[
\leq \mathbb{E} \left[ \int_{s_1}^{s_2} (\bar{Y}^n_r - v)d\bar{K}^{1,n}_r + \int_{s_1}^{s_2} \varphi(v)dr + \int_{s_1}^{s_2} (\bar{Y}^n_r - v)d\bar{K}^{2,n}_r + \int_{s_1}^{s_2} \psi(v)d\bar{A}^n_r \right]
\]

\[
= \mathbb{E} \left[ -\int_{s_1}^{s_2} v(d\bar{K}^{1,n}_r + d\bar{K}^{2,n}_r) + \int_{s_1}^{s_2} \varphi(v)dr + \int_{s_1}^{s_2} \psi(v)d\bar{A}^n_r \right] + \mathbb{E} \left[ |\bar{Y}^n_{s_2}|^2 - |\bar{Y}^n_{s_1}|^2 \right]
\]

\[
+ \mathbb{E} \left[ \int_{s_1}^{s_2} \bar{Y}^n_r f(r, \bar{X}^n_r, \bar{Y}^n_r)dr + \int_{s_1}^{s_2} \bar{Y}^n_r g(r, \bar{X}^n_r, \bar{Y}^n_r)d\bar{A}^n_r - ([M^n]_{s_2} - [M^n]_{s_1}) \right]
\]

Now,

\[
\liminf_{n \to \infty} \mathbb{E} \left[ \int_{s_1}^{s_2} \varphi_j(\bar{Y}^n_r)dr + \int_{s_1}^{s_2} \psi_j(\bar{Y}^n_r)d\bar{A}^n_r + ([M^n]_{s_2} - [M^n]_{s_1}) \right]
\]

\[
\geq \liminf_{n \to \infty} \mathbb{E} \int_{s_1}^{s_2} \varphi_j(\bar{Y}^n_r)dr + \liminf_{n \to \infty} \mathbb{E} \int_{s_1}^{s_2} \psi_j(\bar{Y}^n_r)d\bar{A}^n_r + \liminf_{n \to \infty} \mathbb{E} ([M^n]_{s_2} - [M^n]_{s_1}).
\]
By Fatou’s lemma we have

\[ \liminf_{n \to \infty} \mathbb{E} \int_{s_1}^{s_2} \varphi_j (Y^n_r) \, dr \geq \mathbb{E} \int_{s_1}^{s_2} \varphi_j (\bar{Y}_r) \, dr. \]

From Lemma 4 we see that for any \( s_1, s_2 \in [0, T] \setminus Q_1 \)

\[ \lim_{n \to \infty} \int_{s_1}^{s_2} \psi_j (Y^n_r) \, dA^n_r = \int_{s_1}^{s_2} \psi_j (\bar{Y}_r) \, d\bar{A}^{1,x}_r, \quad \text{P-a.s.} \]

hence, using Fatou’s lemma,

\[ \liminf_{n \to \infty} \mathbb{E} \int_{s_1}^{s_2} \psi_j (Y^n_r) \, d\bar{A}^{1,x}_r \geq \liminf_{n \to \infty} \mathbb{E} \int_{s_1}^{s_2} \psi_j (Y^n_r) \, dA^n_r = \mathbb{E} \int_{s_1}^{s_2} \psi_j (\bar{Y}_r) \, d\bar{A}^{1,x}_r. \]

Finally,

\[
\liminf_{n \to \infty} \mathbb{E} (|\bar{M}_n| - |\bar{M}_n|) = \liminf_{n \to \infty} \mathbb{E} (|\bar{M}_n|^2 - |\bar{M}_n|^2) = \liminf_{n \to \infty} \mathbb{E} (|\bar{M}_n|^2 - |\bar{M}_n|^2) \\
\geq \mathbb{E} (|\bar{M}_n|^2 - |\bar{M}_n|^2) = \mathbb{E} (|\bar{M}_n|^2 - |\bar{M}_n|^2) = \mathbb{E} (|\bar{M}_n|^2 - |\bar{M}_n|^2).
\]

On the other hand, as a consequence of S-convergence of sequence \((\bar{K}^{1,n}, \bar{K}^{2,n})_n\), we deduce that there exists a countable set \(Q_2 \subset [0, T)\) such that

\[ \lim_{n \to \infty} \left[ (\bar{K}^{1,n}_{s_2} - \bar{K}^{1,n}_{s_1}) + (\bar{K}^{2,n}_{s_2} - \bar{K}^{2,n}_{s_1}) \right] = (\bar{K}^{1}_{s_2} - \bar{K}^{1}_{s_1}) + (\bar{K}^{2}_{s_2} - \bar{K}^{2}_{s_1}), \quad \forall s_1, s_2 \in [0, T] \setminus Q_2. \]

But

\[ \sup_{n \in \mathbb{N}} \mathbb{E} |\bar{K}^{1,n}|^p \leq C \sup_{n \in \mathbb{N}} \mathbb{E} \int_0^s |U^n_r|^2 \, dr < \infty \]

and, for any \( 1 < p < 2 \),

\[
\sup_{n \in \mathbb{N}} \mathbb{E} |\bar{K}^{2,n}|^p \leq \sup_{n \in \mathbb{N}} \mathbb{E} \left[ (A^n_t)^{p/2} \left( \int_0^s |V^n_r|^2 \, dA^n_r \right)^{p/2} \right] \\
\leq \sup_{n \in \mathbb{N}} \mathbb{E} \left[ (A^n_t)^{p/2} \left( \int_0^s |V^n_r|^2 \, dA^n_r \right)^{p/2} \right] < \infty.
\]

Hence, by a convergence criterion (see, e.g. [12, Corollary 1.21]) we see that

\[ \lim_{n \to \infty} \mathbb{E} \int_{s_1}^{s_2} (d\bar{K}^{1,n}_{r} + d\bar{K}^{2,n}_{r}) = \mathbb{E} \int_{s_1}^{s_2} (d\bar{K}^{1}_{r} + d\bar{K}^{2}_{r}), \quad \forall s_1, s_2 \in [0, T] \setminus Q_2. \]

Obviously,

\[ \lim_{n \to \infty} \mathbb{E} \left[ \int_{s_1}^{s_2} \psi (v) \, d\bar{A}^{1,x}_r \right] = \mathbb{E} \left[ \int_{s_1}^{s_2} \psi (v) \, d\bar{A}^{1,x}_r \right], \quad \forall s_1, s_2 \in [0, T]. \]

and, as above,

\[ \lim_{n \to \infty} \mathbb{E} |\bar{Y}^{n}_{s_2}|^2 - |\bar{Y}^{n}_{s_1}|^2 = \mathbb{E} |\bar{Y}^{n}_{s_2}|^2 - |\bar{Y}^{n}_{s_1}|^2, \quad \forall s_1, s_2 \in [0, T] \setminus Q_2. \]
By Lebesgue’s theorem we deduce the convergence
\[ \lim_{n \to \infty} E \int_{s_1}^{s_2} \bar{Y}^n f(r, \bar{X}^n_r, \bar{Y}^n_r) dr = E \int_{s_1}^{s_2} \bar{Y}_r f(\bar{r}, \bar{X}_r^t, \bar{Y}_r) dr \text{, } \forall s_1, s_2 \in [0, T] \setminus Q_2 \]
and, for the last term, we can use Lemma 4, since \((x, y) \mapsto y \cdot g(r, x, y)\) is locally Lipschitz (with the constant independent of \(n\)). Therefore, using again the convergence criterion as above,
\[ \lim_{n \to \infty} E \int_{s_1}^{s_2} \bar{Y}^n g(r, \bar{X}^n_r, \bar{Y}^n_r)d\bar{A}^n = E \int_{s_1}^{s_2} \bar{Y}_r g(\bar{r}, \bar{X}_r^t, \bar{Y}_r)d\bar{A}_r^t, \text{ } \forall s_1, s_2 \in [0, T] \setminus Q_1 \]
In order to finish the proof of the Lemma we conclude, passing to the \(\lim \inf\) in (34), that
\[
E \left[ \int_{s_1}^{s_2} \varphi_j (\bar{Y}_r) dr + \int_{s_1}^{s_2} \psi_j (\bar{Y}_r) d\bar{A}^t_r + ([\bar{M}]_{s_2} - [\bar{M}]_{s_1}) \right]
\]
\[
= E \left[ - \int_{s_1}^{s_2} v(\bar{d}K^t_r + \bar{d}K^2_r) + \int_{s_1}^{s_2} \varphi (v) dr + \int_{s_1}^{s_2} \psi (v) d\bar{A}^t_r \right] + E [Y_{s_2}^2 - |y_{s_1}|^2]
\]
\[
+ E \left[ \int_{s_1}^{s_2} \bar{Y}_r f(\bar{r}, \bar{X}_r^t, \bar{Y}_r) dr + \int_{s_1}^{s_2} \bar{Y}_r g(\bar{r}, \bar{X}_r^t, \bar{Y}_r)d\bar{A}_r^t \right]
\]
which represents the conclusion.

\(\text{(ii) Let}
\]
\[ \bar{Q}_s (\omega) := s + \bar{A}^t_s (\omega), \text{ } s \in [0, T] \]
\(\text{(which is strictly increasing) and } \{\bar{\alpha}_s : s \geq 0\} \text{ be the real positive progressively measurable stochastic process (given by Radon-Nikodym’s representation theorem) such that } \bar{\alpha} \in [0, 1] \text{ and}
\]
\[ ds = \bar{\alpha}_s d\bar{Q}_s \text{ and } d\bar{A}^t_s = (1 - \bar{\alpha}_s) d\bar{Q}_s . \]

We define now
\[ \Psi (\omega, s, y) = [\bar{\alpha}_s (\omega) \varphi (y) + (1 - \bar{\alpha}_s (\omega)) \psi (y)] \]
which implies that \(\partial_y \Psi\) is maximal monotone operator and
\[ \partial_y \Psi (\omega, s, y) = [\bar{\alpha}_s (\omega) \partial \varphi (y) + (1 - \bar{\alpha}_s (\omega)) \partial \psi (y)]. \]

Relations (31) and (32) become respectively
\[ d \left( \bar{K}^1_{r, t} + \bar{K}^2_{r, t} \right) \in \partial_y \Psi (r, \bar{Y}^t_r) d\bar{Q}_r \quad \text{and} \quad d \left( \bar{K}^1_r + \bar{K}^2_r \right) \in \partial_y \Psi (r, \bar{Y}_r) d\bar{Q}_r, \text{ for } r \in [t, T], \]
i.e., \(\forall v \in \mathbb{R}, \forall s_1 \leq s_2,\)
\[ (j) \int_{s_1}^{s_2} (v - \bar{Y}^t_r) d \left( \bar{K}^1_{r, t} + \bar{K}^2_{r, t} \right) + \int_{s_1}^{s_2} \Psi (r, \bar{Y}^t_r) d\bar{Q}_r \leq \int_{s_1}^{s_2} \Psi (r, v) d\bar{Q}_r \quad \text{and} \]
\[ (jj) \int_{s_1}^{s_2} (v - \bar{Y}_r) d \left( \bar{K}^1_r + \bar{K}^2_r \right) + \int_{s_1}^{s_2} \Psi (\bar{Y}_r) d\bar{Q}_r \leq \int_{s_1}^{s_2} \Psi (v) d\bar{Q}_r . \]
(35)
The next step is to prove that inequalities (35) hold true for any function \( v \in \mathbb{D} \). We will prove this assertion only for inequality \((jj)\) (in a similar way can be proved \((j)\)). Let \( v \in \mathbb{D} \) and \( \epsilon, s_1, s_2 \) be arbitrary chosen such that \( 0 < \epsilon < s_1 < s_2 \leq T \). If \( r, u \in [s_1, s_2] \), let \( r_\epsilon \) and \( u_\epsilon \) be given by
\[
Q_{r_\epsilon} = Q_r + Q_\epsilon \quad \text{and} \quad Q_{u_\epsilon} = Q_u - Q_\epsilon.
\]
From (35-\(jj\)) we deduce, by taking \( s_1 = u_\epsilon \) and \( s_2 = u_\epsilon \), that
\[
\int_{u_\epsilon}^{u} (v(\epsilon) - \bar{Y}_r) \, d\left(\bar{K}_1 + \bar{K}_2^2\right) + \int_{u_\epsilon}^{u} \Psi(\epsilon, \bar{Y}_r) \, d\bar{Q}_r \leq \int_{u_\epsilon}^{u} \Psi(v(\epsilon), \bar{Q}_r) \, d\bar{Q}_r
\]
and therefore
\[
\int_{s_1}^{s_2} \left( \frac{1}{Q_\epsilon} \int_{u_\epsilon}^{u} (v(\epsilon) - \bar{Y}_r) \, d\left(\bar{K}_1 + \bar{K}_2^2\right) \right) \, d\bar{Q}_u
\]
\[
+ \int_{s_1}^{s_2} \left( \frac{1}{Q_\epsilon} \int_{u_\epsilon}^{u} \Psi(\epsilon, \bar{Y}_r) \, d\bar{Q}_r \right) \, d\bar{Q}_u \leq \int_{s_1}^{s_2} \left( \frac{1}{Q_\epsilon} \int_{u_\epsilon}^{u} \Psi(v(\epsilon), \bar{Q}_r) \, d\bar{Q}_r \right) \, d\bar{Q}_u.
\]
Since
\[
\frac{1}{Q_\epsilon} \int_{u_\epsilon}^{u} \Psi(\epsilon, \bar{Y}_r) \, d\bar{Q}_r = \frac{1}{Q_\epsilon} \int_{Q_u}^{Q_u(Q^{-1}, \bar{Y}_{Q^{-1}})} \Psi(Q^{-1}, \bar{Y}_{Q^{-1}}) \, ds = \frac{1}{Q_\epsilon} \int_{Q_u - Q_\epsilon}^{Q_u} \Psi(Q^{-1}, \bar{Y}_{Q^{-1}}) \, ds
\]
\[
\rightarrow \Psi(u, \bar{Y}_u), \, \text{as} \, \epsilon \to 0, \, \text{a.e.} \, u \in [s_1, s_2],
\]
we deduce by Fatou’s lemma that
\[
\int_{s_1}^{s_2} \left( \frac{1}{Q_\epsilon} \int_{u_\epsilon}^{u} \Psi(\epsilon, \bar{Y}_r) \, d\bar{Q}_r \right) \, d\bar{Q}_u \rightarrow \int_{s_1}^{s_2} \Psi(u, \bar{Y}_u) \, d\bar{Q}_u, \, \text{as} \, \epsilon \to 0.
\]
Using Lebesgue’s theorem,
\[
\int_{s_1}^{s_2} \left( \frac{1}{Q_\epsilon} \int_{u_\epsilon}^{u} (v(\epsilon) - \bar{Y}_r) \, d\left(\bar{K}_1 + \bar{K}_2^2\right) \right) \, d\bar{Q}_u
\]
\[
= \int_{-\infty}^{+\infty} \left( \frac{1}{Q_\epsilon} \int_{-\infty}^{+\infty} \bar{1}_{[s_1,s_2]}(u) \, d\bar{Q}_u \right) \, d\bar{Q}_u
\]
\[
= \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} \frac{1}{Q_\epsilon} \bar{1}_{[s_1,s_2]}(u) \, \bar{1}_{[u,u]}(r) (v(\epsilon) - \bar{Y}_r) \, d\left(\bar{K}_1 + \bar{K}_2^2\right) \right) \, d\bar{Q}_u
\]
\[
= \int_{-\infty}^{+\infty} \left( \frac{1}{Q_\epsilon} \int_{-\infty}^{+\infty} \bar{1}_{[s_1,s_2]}(u) \, \bar{1}_{[r,r_\epsilon]}(u) (v(\epsilon) - \bar{Y}_r) \, d\left(\bar{K}_1 + \bar{K}_2^2\right) \right) \, d\bar{Q}_u
\]
\[
= \int_{-\infty}^{+\infty} \left( \frac{1}{Q_\epsilon} \int_{r}^{r_\epsilon} \bar{1}_{[s_1,s_2]}(u) (v(\epsilon) - \bar{Y}_r) \, d\left(\bar{K}_1 + \bar{K}_2^2\right) \right) \, d\bar{Q}_u
\]
\[
\rightarrow \int_{-\infty}^{+\infty} \bar{1}_{[s_1,s_2]}(r) (v(\epsilon) - \bar{Y}_r) \, d\left(\bar{K}_1 + \bar{K}_2^2\right), \, \text{as} \, \epsilon \to 0,
\]
since
\[
\frac{1}{Q_\epsilon} \int_{r}^{r_\epsilon} \bar{1}_{[s_1,s_2]}(u) (v(\epsilon) - \bar{Y}_r) \, d\bar{Q}_u = \frac{1}{Q_\epsilon} \int_{Q_r}^{Q_{r_\epsilon}} \bar{1}_{[s_1,s_2]}(Q^{-1}) (v(Q^{-1}) - \bar{Y}_{Q^{-1}}) \, ds
\]
\[
\rightarrow \bar{1}_{[s_1,s_2]}(r) (v(\epsilon) - \bar{Y}_r), \, \text{as} \, \epsilon \to 0.
\]
On the other hand, if we assume that
\[ \varphi (v (u)) + \psi (v (u)) \leq M, \ \forall u \in [s_1, s_2], \]  
we deduce using Lebesgue’s theorem that
\[ \int_{s_1}^{s_2} \frac{1}{Q_r} \int_{u}^{u} \Psi (r, v (u)) dQ_r \rightarrow \int_{s_1}^{s_2} \Psi (u, v (u)) dQ_u, \text{ as } \epsilon \to 0, \text{ a.e.} \]
Hence under assumption (36) we obtain
\[ \int_{s_1}^{s_2} (v (r) - \bar{Y}_r) d (K_r^1 + K_r^2) + \int_{s_1}^{s_2} \Psi (r, \bar{Y}_r) d\bar{Q}_r \leq \int_{s_1}^{s_2} \Psi (r, v (r)) d\bar{Q}_r. \]
Let’s take now
\[ v_n (u) = \mathbb{1}_{[0,n)} (\varphi (v (u)) + \psi (v (u))) \cdot v (u) + \mathbb{1}_{[n,\infty)} (\varphi (v (u)) + \psi (v (u))) \cdot 0 \]
which satisfy restriction (36) since \( v (u) \in \text{Dom} (\partial \varphi) \cap \text{Dom} (\partial \psi), \varphi, \psi \) are convex, \( \varphi (0) = \psi (0) = 0 \) and
\[ \varphi (v_n (u)) + \psi (v_n (u)) \leq \mathbb{1}_{[0,n)} (\varphi (v (u)) + \psi (v (u))) \cdot [\varphi (v (u)) + \psi (v (u))] \leq 2n. \]
Therefore, using the previous step, it follows that
\[ \int_{s_1}^{s_2} (v_n (r) - \bar{Y}_r) d (K_r^1 + K_r^2) + \int_{s_1}^{s_2} \Psi (r, \bar{Y}_r) d\bar{Q}_r \leq \int_{s_1}^{s_2} \Psi (r, v_n (r)) d\bar{Q}_r \]
\[ \leq \int_{s_1}^{s_2} \mathbb{1}_{[0,n)} (\varphi (v (u)) + \psi (v (u))) \Psi (r, v (r)) d\bar{Q}_r \]
Passing to the limit and using Lebesgue theorem (for the first integral) and Beppo-Levi theorem (for the last integral) it follows that, for any \( v, v' \in \mathbb{D} \) and any \( 0 \leq t \leq s_1 \leq s_2 \)
\[ \int_{s_1}^{s_2} (v (r) - \bar{Y}_r) d (K_r^1 + K_r^2) + \int_{s_1}^{s_2} \Psi (r, \bar{Y}_r) d\bar{Q}_r \leq \int_{s_1}^{s_2} \Psi (r, v (r)) d\bar{Q}_r \]  
(37)
and (in a similar manner)
\[ \int_{s_1}^{s_2} (v' (r) - \bar{Y}_r) d (K_r^1 + K_r^2) + \int_{s_1}^{s_2} \Psi (r, \bar{Y}_r) d\bar{Q}_r \leq \int_{s_1}^{s_2} \Psi (r, v' (r)) d\bar{Q}_r. \]  
(38)
Taking \( v = \bar{Y}_r \) in (37) and \( v' = \bar{Y} \) in (38) we deduce
\[ \int_{s_1}^{s_2} (\bar{Y}_r - \bar{Y}_r) (dK_r^1 + dK_r^2 - dK_r^{1,t,x} - dK_r^{2,t,x}) \geq 0 \]
and the proof of (ii) is complete. \[ \blacksquare \]
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