Supersymmetric quantum mechanics and the Korteweg-de Vries hierarchy

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Abstract

The connection between supersymmetric quantum mechanics and the Korteweg-de Vries (KdV) equation is discussed, with particular emphasis on the KdV conservation laws. It is shown that supersymmetric quantum mechanics aids in the derivation of the conservation laws, and gives some insight into the Miura transformation that converts the KdV equation into the modified KdV equation. The construction of the $\tau$-function by means of supersymmetric quantum mechanics is discussed.
I INTRODUCTION

It is well known that both the Korteweg-de Vries (KdV) equation [1] and supersymmetric quantum mechanics [2] have intimate connections to the inverse scattering problem [3-6]. In Ref. [4] it was shown that supersymmetric quantum mechanics can be used to construct reflectionless one-dimensional potentials with arbitrarily prescribed bound state energies. On the other hand, one can solve the KdV equation by means of the inverse scattering transform [1], wherein one associates a solution of KdV with a Schrödinger potential. Further links between supersymmetric quantum mechanics and the KdV equation have been found in connection with Bäcklund transformations [4], which can be used either to add a soliton to a solution of KdV, or to add a bound state to a one-dimensional potential. It has been shown that supersymmetric quantum mechanics allows one to obtain an expression for the \( \tau \)-function that comes up in solving the KdV equation [5], and to obtain one- and two-soliton solutions for an infinite family of equations related to KdV [6]. In addition, it has been noted [7] that the change of variable connecting the modified KdV equation with the usual KdV equation comes up quite naturally in factorizing the Schrödinger equation.

We have two parallel ways of adding a soliton to an already existing \( n \)-soliton solution. (1) We can perform a transformation within supersymmetric quantum mechanics [3-6]. (2) We can construct a \( \tau \)-function for the \( n \)-soliton solution which, when acted upon by a suitable vertex operator, yields a \( \tau \)-function for the \( n + 1 \)-soliton solution [8]. Our purpose in the present paper is to explore the relation between these two methods.

In Sec. II we review the construction of one-dimensional potentials using supersymmetric quantum mechanics. In Sec. III we derive conservation laws for deformations of a potential that leave the bound state energies unchanged, while Sec. IV introduces a hierarchy of equations of the KdV type as a means of generating such deformations. In Sec. V, we introduce the \( \tau \)-function and the vertex operator, and present a method for constructing the \( n \)-soliton \( \tau \)-function using supersymmetric quantum mechanics. We conclude in Sec. VI.

II SUPERSYMMETRIC QUANTUM MECHANICS

In this section we consider the construction of reflectionless one-dimensional potentials by means of supersymmetric quantum mechanics. We begin by writing the one-dimensional Schrödinger Hamiltonian

\[ H_+ = -\frac{d^2}{dx^2} + V_+(x) \quad (1) \]

in factorized form as

\[ H_+ = A^\dagger A, \quad (2) \]

1
where $A$ is given by

$$A = -\frac{d}{dx} + f(x).$$

In order that Eq. (2) hold, $f(x)$ must obey the Ricatti equation

$$f^2 + f' = V_+.$$  

(4)

The form of Eq. (2) suggests the introduction of a ‘partner’ Hamiltonian $H_-$ given by

$$H_- = -\frac{d^2}{dx^2} + V_-(x) = AA^\dagger.$$  

(5)

The potentials $V_\pm$ are given in terms of $f$ by

$$V_\pm = f^2 \pm f'.$$

It turns out that there are certain interesting relationships between the spectra of $H_+$ and $H_-$. For suppose that $\psi_+$ is an eigenfunction of $H_+$. We then have

$$A^\dagger A\psi_+ = E_+\psi_+.$$  

(6)

Multiplying on the left by $A$ gives

$$AA^\dagger[A\psi_+] = E_+[A\psi_+].$$  

(7)

From this it follows either that $\psi_\equiv A\psi_+$ is an eigenfunction of $H_-$ with eigenvalue $E_+$, or that $\psi_\equiv 0$. Now the latter of these two possibilities implies that

$$0 = \langle \psi_- | \psi_- \rangle = \langle A\psi_+ | A\psi_+ \rangle = \langle \psi_+ | A^\dagger A\psi_+ \rangle = E_+ \langle \psi_+ | \psi_+ \rangle,$$

(8)

so that $\psi_\equiv 0$ only when $E_+ = 0$. It follows that $H_+$ and $H_-$ have the same spectra, apart from the single $E = 0$ eigenvalue which is present in the spectrum of $H_+$, but absent from the spectrum of $H_-$. A similar argument shows that if $\psi_-$ is an eigenfunction of $H_-$, then $\psi_+ = A^\dagger \psi_-$ is an eigenfunction of $H_+$.

The factorization method can be used to construct reflectionless potentials possessing an arbitrary spectrum of bound states. To see this, suppose we have a Hamiltonian $H^{(1)}$ with potential $V^{(1)}$ having $n$ eigenvalues $E = E_1, E_2, \ldots, E_n$. We may construct from $V^{(1)}$ a potential $V^{(2)}$ having the $n$ eigenvalues $E_k$, plus one more eigenvalue $E_{n+1}$. To do this, first choose the zero of energy such that $V^{(1)} \to 0$ as $x \to \pm\infty$, and define $V_-$ by $V_- = V^{(1)} - E_{n+1}$. We factorize the corresponding Schrödinger equation as above: define $f(x)$ by

$$f^2 - f' = V_-,$$

(9)

and construct the partner potential

$$V_+ = f^2 + f'.$$

(10)

The potential $V_-$ has the $n$ eigenvalues $E_1 - E_{n+1}, E_2 - E_{n+1}, \ldots, E_n - E_{n+1}$, while $V_+$ has $n+1$ eigenvalues consisting of the $n$ eigenvalues of $V_-$, plus the additional
eigenvalue $E = 0$. Defining $V^{(2)} = V_+ + E_{n+1}$ yields the desired potential, possessing the $n + 1$ eigenvalues $E_1, E_2, \ldots, E_{n+1}$. In this way, one can start from the constant potential $V^{(1)} = 0$ and by iteration build up a potential possessing an arbitrary spectrum of eigenvalues.

As an example of this procedure we construct a potential possessing a single bound state with energy $E = -\kappa^2$. We begin from $V^{(1)} = 0$, so that $V_- = \kappa^2$, and $f$ obeys

$$f^2 - f' = \kappa^2.$$  \hspace{1cm} (11)

This may be linearized by the substitution $f = -w'/w$, yielding

$$f = -\kappa \tanh \kappa (x - x_0),$$  \hspace{1cm} (12)

so that $V_+$ is given by

$$V_+ = \kappa^2 \left[ 1 - 2 \text{sech}^2 \kappa (x - x_0) \right],$$  \hspace{1cm} (13)

and the desired potential, possessing a single bound state, is given by

$$V^{(2)} = -2\kappa^2 \text{sech}^2 \kappa (x - x_0).$$  \hspace{1cm} (14)

To see that $V^{(2)}$ is reflectionless, consider a plane wave solution for the potential $V_-$, $\psi_-(x) = e^{ikx}$. The corresponding solution for the potential $V_+$ is given by $\psi_+ = A^\dagger \psi_-$. From Eq. (12), it follows that $\psi_+$ behaves asymptotically like $(ik \mp \kappa) e^{ikx}$ as $x \to \pm \infty$. Since no term proportional to $e^{-ikx}$ arises, the potential is reflectionless. Defining the reflection and transmission coefficients $R$ and $T$ by

$$\psi(x) \to \begin{cases} e^{ikx} + R(k)e^{-ikx}, & x \to -\infty; \\ T(k)e^{ikx}, & x \to +\infty, \end{cases}$$  \hspace{1cm} (15)

we find that $R = 0$, while

$$T(k) = \frac{ik - \kappa}{ik + \kappa}.$$  \hspace{1cm} (16)

By repeating the above procedure, it is possible to construct potentials possessing arbitrarily many bound states. It is shown in the Appendix that such potentials are reflectionless, and that furthermore they have transmission coefficients given by

$$T(k) = \prod_{i=1}^{N} \frac{ik - \kappa_i}{ik + \kappa_i},$$  \hspace{1cm} (17)

where the $N$ bound states have energies $E_i = -\kappa_i^2$.  

3
III CONSERVATION LAWS FOR ISOSPECTRAL DEFORMATIONS OF A POTENTIAL

Consider again the one-dimensional Schrödinger equation, with a reflectionless potential $V(x) = -u(x)$:

$$\psi_{xx} + [k^2 + u(x)]\psi = 0. \quad (18)$$

We wish to derive certain quantities which will be conserved under any deformation of the potential which leaves the spectrum of bound states unchanged. These conserved quantities may be obtained via an asymptotic expansion of the transmission coefficient for large values of $k$. We write the wavefunction as

$$\psi(x) = \exp\left[ikx + \int_{-\infty}^{x} \phi(x') \, dx'\right]. \quad (19)$$

and expand $\phi$ as

$$\phi(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{(2ik)^n}. \quad (20)$$

Substitution of (19) into (18) yields the recursion relation

$$f_{n+1} = -f_{n,x} - \sum_{k=1}^{n-1} f_k f_{n-k}, \quad (21)$$

for $n \geq 1$, while $f_1 = -u(x)$. The first few terms in the series are

$$f_1 = -u, \quad f_2 = u_x, \quad f_3 = -u_{xx} - u^2, \quad f_4 = (u_{xx} + 2u^2)_x, \quad \text{and} \quad f_5 = -(u_{xx} + 3u^2)_{xx} + u_x^2 - 2u^3.$$

Taking the limit $x \to \pm\infty$ in Eq. (19) and comparing with Eq. (15) shows that the transmission coefficient given in Eq. (17) may be written in two ways:

$$\log T(k) = \sum_{n=1}^{\infty} \frac{1}{(2ik)^n} \int_{-\infty}^{\infty} f_n(x) \, dx = \sum_{i=1}^{N} \left[ \log(ik - \kappa_i) - \log(ik + \kappa_i) \right]. \quad (22)$$

Expanding the quantity on the far right in a power series in $1/k$ and equating coefficients of $1/k^n$ gives

$$\int_{-\infty}^{\infty} f_n(x) \, dx = \begin{cases} 0 & \text{if } n \text{ is even;} \\ -\frac{2^{n+1}}{n} \sum_{i=1}^{N} \kappa_i^n & \text{if } n \text{ is odd.} \end{cases} \quad (23)$$

The integrals vanish for even $n$ because the integrands in this case are total derivatives of a function having the same values at $x = \pm\infty$, as we see for $n = 2$ and $n = 4$ in the list above. It follows from Eq. (23) that the integrals of the functions $f_n$ are constants of the motion for any spectrum-preserving deformation of the potential $u(x)$: since the eigenvalues $\kappa_i^2$ are constant for such a deformation, the integrals must...
also be constant. These integrals are precisely the KdV Hamiltonians, up to an overall multiplicative factor. The choice of this multiplicative factor is simply a matter of convenience, so we choose to define the Hamiltonians as

\[ H_{2n+1}[u] = -\frac{1}{2^{2n+1}} \int_{-\infty}^{\infty} f_{2n+3}(x) \, dx. \] (24)

A specific form for \( f_n(x) \) may be written for the single-soliton solution by noting that for a soliton with \( x_0 = 0 \),

\[ \psi = A \, e^{ikx} \sim \left( \frac{ik - \kappa \tanh \kappa x}{ik + \kappa} \right) e^{ikx} \]

\[ = \exp(ikx + \log[ik - \kappa \tanh \kappa x] - \log[ik + \kappa]) \equiv \exp[ikx + \int_{x_0}^{x} \phi(x')dx'] \]

Expanding

\[ \phi(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{(2ik)^n} = (d/dx)(\log[ik - \kappa \tanh \kappa x] - \log[ik + \kappa]) \] (25)

gives the result

\[ f_n = -\frac{(2\kappa)^n}{n} \frac{d}{dx} \tanh^n \kappa x. \] (26)

### IV  EVOLUTION EQUATIONS

We can use the results of the previous two sections to derive a set of spectrum-preserving evolution equations for the potential \( u(x) \). We know from the last section that any such evolution equation must conserve the transmission coefficient \( T(k) \), and, consequently, must also conserve each of the Hamiltonians \( H_{2n+1} \). A natural choice for an evolution equation conserving a specific Hamiltonian \( H_{2k+1} \) is a member of the Korteweg-de Vries hierarchy, which may be written in the form

\[ u_t = \frac{\partial}{\partial x} \frac{\delta H_{2k+1}}{\delta u}. \] (27)

We have introduced the variational derivative \( \delta/\delta u \), defined by

\[ \delta H[u] = \int_{-\infty}^{\infty} \frac{\delta H}{\delta u} \delta u \, dx, \] (28)

so that if \( H[u] = \int_{-\infty}^{\infty} h(u, u_x, \ldots) \, dx \),

\[ \frac{\delta H}{\delta u} = \sum_{k=0}^{\infty} \left( -\frac{d}{dx} \right)^k \frac{\partial h}{\partial u^{(k)}}, \] (29)
where $u^{(k)}$ is the $k^{th}$ derivative of $u(x,t)$ with respect to $x$. Eq. (27) automatically conserves the Hamiltonian $H_{2k+1}$ since

$$\frac{dH_{2k+1}}{dt} = \int_{-\infty}^{\infty} \frac{dH_{2k+1}}{du} u_t \, dx = \int_{-\infty}^{\infty} \frac{1}{2} \frac{d}{dx} \left( \frac{\delta H_{2k+1}}{\delta u} \right)^2 \, dx = 0. \quad (30)$$

The proof that Eq. (27) also conserves all of the Hamiltonians is somewhat more subtle.

To show that Eq. (27) conserves all of the Hamiltonians, we first derive a new expression for the transmission coefficient $T(k)$. We again employ a factorization method to write the Schrödinger equation in the form

$$\left(-\frac{d^2}{dx^2} - [k^2 + u]\right) \psi = A A^\dagger \psi = 0, \quad (31)$$

where $A = -\frac{d}{dx} + f$, and

$$f_x - f^2 - k^2 = u. \quad (32)$$

Since $AA^\dagger \psi = 0$ implies $A^\dagger \psi = 0$, we find that

$$f = -\frac{d}{dx} \log \psi. \quad (33)$$

Since $\psi \to T(k) \exp(ikx)$ as $x \to \infty$, we conclude that the transmission coefficient may be written as

$$\log T(k) = -\int_{-\infty}^{\infty} (ik + f) \, dx. \quad (34)$$

It follows that if the evolution equation for $f$ corresponding to Eq. (27) has the form $f_t = \partial J/\partial x$, then $T(k)$, and all of the Hamiltonians, will be conserved. It turns out that this is in fact the case. To see this, we make use of the following observations: First, we note that the variational derivatives of the Hamiltonians obey the recursion relation [1]

$$\frac{\partial}{\partial x} \frac{\delta H_{2k+1}}{\delta u} = \frac{1}{4} \left( \frac{\partial^3}{\partial x^3} + 2u_x + 4u \frac{\partial}{\partial x} \right) \frac{\delta H_{2k-1}}{\delta u} \equiv \frac{1}{4} M \frac{\delta H_{2k-1}}{\delta u}. \quad (35)$$

Next, we observe that the linear operator $M$ may be written as

$$M = -\frac{1}{4} \left( \frac{\partial}{\partial x} - 2f \right) \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} - 2f \right) - 4k^2 \frac{\partial}{\partial x}. \quad (36)$$

Finally, note that the variational derivatives $\delta H/\delta u$ and $\delta H/\delta f$ are related in a simple way: we have

$$\delta H = \int_{-\infty}^{\infty} \frac{\delta H}{\delta u} u \, dx = \int_{-\infty}^{\infty} \frac{\delta H}{\delta u} \left( \frac{\partial}{\partial x} - 2f \right) f \, dx = \int_{-\infty}^{\infty} \frac{\delta H}{\delta f} f \, dx, \quad (37)$$
so that
\[
\frac{\delta H}{\delta f} = \left( -\frac{\partial}{\partial x} - 2f \right) \frac{\delta H}{\delta u}.
\] (38)

We can now derive an evolution equation for \( f(x,t) \). Beginning from Eq. (27), we find

\[
u_t = \left( \frac{\partial}{\partial x} - 2f \right) f_t = \frac{\partial}{\partial x} \frac{\delta H_{2k+1}}{\delta u}
\]
\[= -\frac{1}{4} \left( \frac{\partial}{\partial x} - 2f \right) \frac{\partial}{\partial x} \left( -\frac{\partial}{\partial x} - 2f \right) \frac{\delta H_{2k-1}}{\delta u} - k^2 \frac{\partial}{\partial x} \frac{\delta H_{2k-1}}{\delta u}
\]
\[= \left( \frac{\partial}{\partial x} - 2f \right) \frac{\partial}{\partial x} \left( \frac{1}{4} \frac{\delta H_{2k-1}}{\delta f} + \frac{k^2}{4} \frac{\delta H_{2k-1}}{\delta f} \right) + \frac{k^4}{4} \frac{\partial}{\partial x} \frac{\delta H_{2k-3}}{\delta u}.
\] (39)

We have made repeated use of Eq. (37) to lower the index \( k \), and Eq. (38) to express \( \delta/\delta u \) in terms of \( \delta/\delta f \). We may continue lowering \( k \) in this manner until the “remainder” term (i.e. the term which is not multiplied by \( (\partial/\partial x - 2f) \) on the right hand side of Eq. (39)) has the form \( \partial/\partial x(\delta H_{-1}/\delta u) \). At this point, the remainder vanishes and the sequence terminates. Hence the evolution equation for \( f \) has the form

\[
\left( \frac{\partial}{\partial x} - 2f \right) \left( f_t - \frac{1}{4} \frac{\partial}{\partial x} \sum_{l=0}^{k} (-1)^{l+1} k^2 l \frac{\delta H_{2(k-l)-1}}{\delta f} \right) = 0.
\] (40)

An argument due to Miura, Kruskal and Gardner [9] can now be used to show that Eq. (40) implies

\[
f_t = \frac{\partial}{\partial x} \left( \frac{1}{4} \sum_{l=0}^{k} (-1)^{l+1} k^2 l \frac{\delta H_{2(k-l)-1}}{\delta f} \right).
\] (41)

The argument runs as follows: We know from Sec. III that \( f \), which is the logarithmic derivative of the wavefunction, may be expanded in an asymptotic series of the form

\[
f = ik + \sum_{n=1}^{\infty} \frac{a_n(x,t)}{(2ik)^n}.
\] (42)

Consequently, the quantity

\[
F \equiv f_t - \frac{1}{4} \frac{\partial}{\partial x} \sum_{l=0}^{k} (-1)^{l+1} k^2 l \frac{\delta H_{2(k-l)-1}}{\delta f}
\] (43)

may also be written as a series in the form

\[
F = \sum_{n=-\infty}^{N} b_n(x,t)(2ik)^n.
\] (44)

Substituting these expansions into Eq. (40) and equating the coefficients of each power of \( k \) to zero, we find that the coefficient of \( k^{N+1} \) is simply \( b_N \), so that \( b_N = 0 \). But
this in turn implies that \( b_{N-1} = b_{N-2} = b_{N-3} = \ldots = 0 \). As a result, we conclude that \( F \equiv 0 \), and Eq. (11) follows.

The evolution equation (11) is of the form

\[
f_t = \frac{\partial}{\partial x} J(x).
\]

From Eq. (13) it then follows that the transmission coefficient is a constant of the motion. Since the coefficients of the asymptotic expansion (22) of \( \log T(k) \) are just constant multiples of the KdV Hamiltonians, it follows that each evolution equation of the form (27) conserves all of the Hamiltonians. Consequently, each such equation also preserves the eigenvalue spectrum of the potential \( u(x) \).

The above formalism permits a simple derivation of the time dependence of the one soliton solution of KdV under the evolution equations (27). The fact that each of these evolution equations conserves all of the Hamiltonians implies that the various evolutions commute with one another [1]. As a result, the function \( u \) may be thought of as depending on an infinite number of time variables, with the dependence on the various times fixed by

\[
u_{t_{2k+1}} = \frac{\partial}{\partial x} \frac{\delta H_{2k+1}}{\delta u}.
\]

We illustrate this by considering the one soliton solution of the KdV hierarchy. We make the ansatz \( u(x,t_1,t_3,\ldots) = 2\kappa^2 \text{sech}^2(\kappa x + \sum_{l=0}^{\infty} \alpha_{2l+1} t_{2l+1}) \), and substitute into the evolution equations (11) to determine the constants \( \alpha_i \). We find that Eq. (11) with \( k = 0 \) gives, after some manipulation,

\[
2\kappa \alpha_1 \frac{\partial}{\partial x} \text{sech}^2(\kappa x + \sum_{l=0}^{\infty} \alpha_{2l+1} t_{2l+1}) = 2\kappa^2 \frac{\partial}{\partial x} \text{sech}^2(\kappa x + \sum_{l=0}^{\infty} \alpha_{2l+1} t_{2l+1}),
\]

so that \( \alpha_1 = \kappa \). Using the recursion relation (33), it can be shown that

\[
\frac{\partial}{\partial x} \frac{\delta H_{2k+1}}{\delta u} = \kappa^2 \frac{\partial}{\partial x} \frac{\delta H_{2k-1}}{\delta u}.
\]

From this all of the constants \( \alpha_i \) can be determined from \( \alpha_1 \). The result is \( \alpha_{2l+1} = \kappa^{2l+1} \), so that the full time dependence of the one-soliton solution is given by

\[
u(x,t_1,t_3,\ldots) = 2\kappa^2 \text{sech}^2(\kappa x + \kappa t_1 + \kappa^3 t_3 + \kappa^5 t_5 + \ldots).
\]

An equivalent result has been given in [6].

V THE \( \tau \)-FUNCTION AND THE VERTEX OPERATOR

In this section we discuss the construction of reflectionless potentials and multisoliton solutions of KdV using the \( \tau \)-function. A generic potential can be expressed in terms of the \( \tau \)-function by

\[
V(x) = -2 \frac{d^2}{dx^2} \log \tau(x).
\]
The $\tau$-function can be constructed in either of two ways: First, we can use supersymmetric quantum mechanics to build up the $\tau$-function from a sort of “vacuum state” using the techniques discussed above. Alternatively, we can use the vertex operator [1,8] to construct the desired $\tau$-function.

To build up the $\tau$-function using supersymmetric quantum mechanics, we consider a sequence of potentials $V_1, V_2, \ldots V_n$, where $V_m$ has $m$ bound states, the highest $m-1$ of which are shared with $V_{m-1}$. From the appendix, these potentials may be written in terms of a set of functions $f_m$ as

$$V_m(x) = f_m^2 + f'_m - \kappa_m^2$$  \hspace{1cm} (51)

or

$$V_m(x) = f_{m+1}^2 - f'_{m+1} - \kappa_{m+1}^2.$$  \hspace{1cm} (52)

Using this, $V_n$ can be expressed as

$$V_n = \sum_{m=1}^{n} (V_m - V_{m-1}) = 2 \frac{d}{dx} \sum_{m=1}^{n} f_m,$$  \hspace{1cm} (53)

where $V_0 \equiv 0$. Introducing functions $w_m$ defined by

$$f_m = -w'_m/w_m,$$  \hspace{1cm} (54)

the potential $V_n$ becomes

$$V_n = -2 \frac{d^2}{dx^2} \log \prod_{m=1}^{n} w_m,$$  \hspace{1cm} (55)

so that the $\tau$-function corresponding to $V_n$ is given by

$$\tau_n = \prod_{m=1}^{n} w_m.$$  \hspace{1cm} (56)

We can use the operators $A_m = -d/dx + f_m$ to construct the functions $w_m$ from simple linear combinations of exponentials. First observe that the sequence of Hamiltonians $H_m = -d^2/dx^2 + V_m$ may be written as

$$H_m = A_m^\dagger A_m - \kappa_m^2 = A_{m+1}^\dagger A_{m+1} - \kappa_{m+1}^2.$$  \hspace{1cm} (57)

Furthermore, the functions $w_m$ obey

$$[A_m A_m^\dagger - \kappa_m^2] w_m = -\kappa_m^2 w_m.$$  \hspace{1cm} (58)

Now suppose $w_m^0$ is chosen to satisfy

$$- \frac{d^2}{dx^2} w_m^0 = [A_1 A_1^\dagger - \kappa_1^2] w_m^0 = -\kappa_m^2 w_m^0.$$  \hspace{1cm} (59)

Multiplying on the left by $A_1^\dagger$ and using Eq. (57), we find that $A_1^\dagger w_m^0$ obeys

$$[A_2 A_2^\dagger - \kappa_2^2] A_1^\dagger w_m^0 = -\kappa_m^2 A_1^\dagger w_m^0.$$  \hspace{1cm} (60)
Repeating this process we find that the function \( w_m \) is given by

\[
w_m = \prod_{k=1}^{n-1} A_k^1 w_m^0,
\]

so that the \( \tau \)-function can be written as

\[
\tau_n = \prod_{m=1}^{n} \left( \prod_{k=1}^{n-1} A_k^1 \right) w_m^0.
\]

Since the functions \( w_m^0 \) are solutions of Eq. (59), they have the form

\[
w_m^0 = a_m^+ e^{\kappa m x} + a_m^- e^{-\kappa m x},
\]

with the constants \( a_m^\pm \) chosen so as to ensure that \( w_m \) has no nodes. For the case of potentials symmetric about the origin, we choose \( a_m^+ = 1 \), \( a_m^- = (-1)^{m+1} \). The first few \( \tau \)-functions are

\[
\tau_0(x) = 1,
\]

\[
\tau_1(x) = a_1^+ e^{\kappa_1 x} + a_1^- e^{-\kappa_1 x},
\]

\[
\tau_2(x) = a_1^+ a_2^+ (\kappa_2 - \kappa_1) e^{\kappa_1 x} e^{\kappa_2 x} + a_1^+ a_2^- (-\kappa_2 - \kappa_1) e^{\kappa_1 x} e^{-\kappa_2 x} + a_1^- a_2^+ (\kappa_2 + \kappa_1) e^{-\kappa_1 x} e^{\kappa_2 x} + a_1^- a_2^- (-\kappa_2 + \kappa_1) e^{-\kappa_1 x} e^{-\kappa_2 x},
\]

and

\[
\tau_3(x) = a_1^+ a_2^+ a_3^+ (\kappa_3 - \kappa_2) (\kappa_3 - \kappa_1) (\kappa_2 - \kappa_1) e^{\kappa_1 x + \kappa_2 x + \kappa_3 x} + a_1^+ a_2^- a_3^- (-\kappa_3 - \kappa_2) (-\kappa_3 - \kappa_1) (\kappa_2 - \kappa_1) e^{\kappa_1 x + \kappa_2 x - \kappa_3 x} + a_1^- a_2^+ a_3^+ (\kappa_3 + \kappa_2) (\kappa_3 + \kappa_1) (-\kappa_2 - \kappa_1) e^{\kappa_1 x - \kappa_2 x + \kappa_3 x} + a_1^- a_2^- a_3^- (-\kappa_3 + \kappa_2) (-\kappa_3 + \kappa_1) (-\kappa_2 - \kappa_1) e^{\kappa_1 x - \kappa_2 x - \kappa_3 x} + a_1^+ a_2^+ a_3^- (\kappa_3 - \kappa_2) (\kappa_3 + \kappa_1) (\kappa_2 + \kappa_1) e^{-\kappa_1 x + \kappa_2 x + \kappa_3 x} + a_1^- a_2^- a_3^+ (-\kappa_3 - \kappa_2) (-\kappa_3 + \kappa_1) (\kappa_2 + \kappa_1) e^{-\kappa_1 x + \kappa_2 x - \kappa_3 x} + a_1^- a_2^+ a_3^- (\kappa_3 + \kappa_2) (\kappa_3 + \kappa_1) (-\kappa_2 + \kappa_1) e^{-\kappa_1 x - \kappa_2 x + \kappa_3 x} + a_1^- a_2^- a_3^- (-\kappa_3 + \kappa_2) (-\kappa_3 + \kappa_1) (-\kappa_2 + \kappa_1) e^{-\kappa_1 x - \kappa_2 x - \kappa_3 x}.
\]

The inductive generalization to the \( n \)-bound state \( \tau \)-function is evidently (cf. [10])

\[
\tau_n(x) = \prod_{m=1}^{n} \left( \left[ \sum_{s_m=\pm} a_m^{s_m} e^{s_m \kappa_m x} \right] \prod_{l=1}^{m-1} (s_m \kappa_m - s_l \kappa_l) \right).
\]

An alternative method of adding a soliton to a \( \tau \)-function is provided by the vertex operator, which is a linear operator that, when applied to a \( \tau \)-function, adds a soliton. In Eqs. (53-57) we have displayed the first few \( \tau \)-functions generated by supersymmetric quantum mechanics. The given expressions are somewhat deceptive, however, since they over-count the number of degrees of freedom that one has when
adding a bound state to a potential. From the given expressions, it would seem that one is free to independently choose all three of the constants $a_m^1$, $a_m$, and $\kappa_m$. Contrary to this, one in fact has only two degrees of freedom: since the multiplication of a $\tau$-function by an overall constant leaves the potential unchanged, the true independent variables are the ratio $a_m^1/a_m$ and the constant $\kappa_m$. The former of these two is in fact a measure of the “position” of the soliton that one adds when adding a bound state. With these facts in mind, we re-write the above $\tau$-functions in an alternative form:

$$\tau_1(x) = 1 + e^{2\kappa_1(x-x_1)}, \quad (69)$$

$$\tau_2(x) = 1 + e^{2\kappa_1(x-x_1)} + e^{2\kappa_2(x-x_2)} + \left(\frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1}\right)^2 e^{2\kappa_1(x-x_1)+2\kappa_2(x-x_2)}, \quad (70)$$

and,

$$\tau_3(x) = 1 + e^{2\kappa_1(x-x_1)} + e^{2\kappa_2(x-x_2)} + e^{2\kappa_3(x-x_3)} + \left(\frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1}\right)^2 e^{2\kappa_1(x-x_1)+2\kappa_2(x-x_2)} + \left(\frac{\kappa_3 - \kappa_1}{\kappa_3 + \kappa_1}\right)^2 e^{2\kappa_2(x-x_2)+2\kappa_3(x-x_3)} + \left(\frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1}\right)^2 \left(\frac{\kappa_3 - \kappa_1}{\kappa_3 + \kappa_1}\right)^2 \left(\frac{\kappa_3 - \kappa_2}{\kappa_3 + \kappa_2}\right)^2 e^{2\kappa_1(x-x_1)+2\kappa_2(x-x_2)+2\kappa_3(x-x_3)}. \quad (71)$$

In deriving these expressions from those given above, we have made use of the fact that multiplication of a $\tau$-function by an overall constant or by an exponential whose argument is linear in $x$ leaves the potential unchanged. Using these expressions, we can construct a linear operator that converts an $n$-soliton $\tau$-function into an $n+1$-soliton $\tau$-function. We denote this operator by $A(\kappa, \bar{x})$ to make its dependence on the relevant variables explicit, and construct it so as to satisfy

$$A(\kappa, \bar{x})\tau_n = \tau_{n+1}. \quad (72)$$

Certain properties of $A(\kappa, \bar{x})$ can be deduced by considering its action on $\tau_0 \equiv 1$: we require that

$$A(\kappa_1, x_1)\tau_0 = A(\kappa_1, x_1)1 = 1 + e^{2\kappa_1(x-x_1)}. \quad (73)$$

This equation is automatically satisfied by $A$ of the form $A(\kappa, \bar{x}) = 1 + B(\kappa, \bar{x})$, with $B$ constructed such that

$$B(\kappa_1, x_1)1 = e^{2\kappa_1(x-x_1)}. \quad (74)$$

Further constraints are obtained by considering the action of $A(\kappa, \bar{x})$ on $\tau_1$. Using (74), and applying $A$ to $\tau_1$, we find

$$A(\kappa_2, x_2)\tau_1 = 1 + e^{2\kappa_1(x-x_1)} + e^{2\kappa_2(x-x_2)} + B(\kappa_2, x_2)e^{2\kappa_1(x-x_1)}. \quad (75)$$

$B$ must therefore obey

$$B(\kappa_2, x_2)e^{2\kappa_1(x-x_1)} = \left(\frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1}\right)^2 e^{2\kappa_1(x-x_1)+2\kappa_2(x-x_2)}. \quad (76)$$
Here we encounter the difficulty that such a constraint on $B$ leads to inconsistencies when we apply it to $\tau$-functions of higher order. A way around this difficulty [11] is suggested by re-expressing the $\kappa$-dependent coefficient in $\tau_2$ in the following form:

$$
\left(\frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1}\right)^2 = e^{2(\log(1-\kappa_1/\kappa_2)-\log(1+\kappa_1/\kappa_2))} = e^{-4\sum_{k=0}^{\infty} \frac{1}{\kappa_2+\kappa_1} (\frac{\kappa_2}{\kappa_1})^{2k+1}}. \quad (77)
$$

We can generate precisely such a factor through the action of $B$ by introducing a set of auxiliary variables $t_3, t_5, t_7, \ldots$ in the $\tau$-function according to the following prescription: replace the constant of integration $x_1$ in $\tau_1$ with the “time-dependent” expression $x_1(\kappa, t_3, t_5, \ldots) = \bar{x}_1 - \sum_{k=0}^{\infty} \kappa_1^{2k} t_{2k+1}$, with $\bar{x}_1$ constant. These auxiliary variables can then be set to zero in the final expression for the potential. The $\kappa$-dependent constant can be generated by making use of the identity [based on Eq. (77)]

$$
e^{-2\left(\frac{1}{\kappa_2} \frac{\partial}{\partial x} + \frac{1}{\kappa_2} \frac{\partial}{\partial t_3} + \frac{1}{5 \kappa_2} \frac{\partial}{\partial t_5} + \ldots\right)} e^{2\kappa_1(x-x_1(\kappa, t_1, \ldots))} = \frac{\left(\kappa_2 - \kappa_1\right)^2}{\left(\kappa_2 + \kappa_1\right)} e^{2\kappa_1(x-x_1(\kappa, t_1, \ldots))}. \quad (78)
$$

This is almost what we need. The correct form of $B$ is evidently

$$
B(\kappa, \bar{x}) = e^{2\kappa_1(x+\sum_{k=0}^{\infty} \kappa_1^{2k} t_{2k+1} - \bar{x})} e^{-2\left(\frac{1}{\kappa_2} \frac{\partial}{\partial x} + \frac{1}{\kappa_2} \frac{\partial}{\partial t_3} + \frac{1}{5 \kappa_2} \frac{\partial}{\partial t_5} + \ldots\right)}
= e^{2\left(\sum_{k=0}^{\infty} \kappa_1^{2k+1} t_{2k+1} - \kappa_2 \bar{x}\right)} e^{-2\left(\sum_{k=0}^{\infty} \frac{1}{(2k+1) \kappa_2} \frac{\partial}{\partial t_{2k+1}}\right)}. \quad (79)
$$

The latter form of $B$ has been simplified by making the identification $x \equiv t_1$. We see that the form of the $\kappa$-dependent factors, or “phase shift functions” appearing in $\tau_2$ enable us to determine the form of the vertex operator relatively easily. This fact has been noted elsewhere [11], in connection with the Kadomtsev-Petviashivili equation. The form of the operator $A$ is then

$$
A(\kappa, \bar{x}) = 1 + e^{2\left(\sum_{k=0}^{\infty} \kappa_1^{2k+1} t_{2k+1} - \kappa_2 \bar{x}\right)} e^{-2\left(\sum_{k=0}^{\infty} \frac{1}{(2k+1) \kappa_2} \frac{\partial}{\partial t_{2k+1}}\right)}. \quad (80)
$$

This form of the vertex operator has been cited elsewhere [1,12], and does in fact convert an $n$-bound state $\tau$-function into one possessing $n + 1$ bound states. This is most readily seen by direct computation using the identity

$$
B(\kappa_n, \bar{x}_n) \cdots B(\kappa_1, \bar{x}_1) = \prod_{k=1}^{n} \left\{ \prod_{l=1}^{k-1} \left(\frac{\kappa_k - \kappa_l}{\kappa_k + \kappa_l}\right)^2 e^{2\left(\sum_{n=0}^{\infty} \kappa_1^{2n+1} t_{2n+1} - \kappa_2 \bar{x}_l\right)} \right\} \quad (81)
$$

and comparing, after suitable multiplication by an overall constant and an overall exponential factor, with the form of the $\tau$-function given in Eq. (52). The $n$-soliton $\tau$-function can be generated by repeated action of the vertex operator on $\tau_0 \equiv 1$:

$$
\tau_n = A(\kappa_n, x_n) A(\kappa_{n-1}, x_{n-1}) \cdots A(\kappa_1, x_1) 1. \quad (82)
$$

A particularly compact form of the vertex operator (52) can given if we choose $\beta = \exp(-2\kappa \bar{x})$ and make use of the fact that $B(\kappa, \bar{x})^2 = 0$. We may write $A$ as

$$
A(\kappa, \beta) = e^{\beta B(\kappa, \beta)}. \quad (83)
$$
In this form, it is apparent that the operator \( B(\kappa, 0) \) is a generator of an infinitesimal symmetry that maps \( \tau \)-functions onto new \( \tau \)-functions. It has been shown [12] that these symmetries are generated by a certain class of infinite-dimensional Lie algebras.

Another frequently used form of the vertex operator is

\[
X(\kappa) = \exp \sum_{0}^{\infty} \left( \kappa^{2k+1} t_{2k+1} \right) \exp \sum_{0}^{\infty} \left( -\frac{1}{(2k+1)\kappa^{2k+1}} \frac{\partial}{\partial t_{2k+1}} \right).
\]  

This form of the vertex operator can also be used to add a soliton to a \( \tau \)-function: we have

\[
\tau_n = \left( b_n^+ X(\kappa_n) + b_n^- X(-\kappa_n) \right) \tau_{n-1}.
\]  

In this case, it is the ratio \( b_n^+/b_n^- \) that determines the position of the new soliton. The equivalence of this construction to that given above can be verified by making use of the identity

\[
X(\kappa_n)X(\kappa_{n-1}) \ldots X(\kappa_1)1 = \prod_{m=1}^{n} \exp \sum_{0}^{\infty} \left( \kappa_m^{2k+1} t_{2k+1} \right) \prod_{l=1}^{m-1} \frac{\kappa_m - \kappa_l}{\kappa_m + \kappa_l}^{1/2},
\]

to construct the \( n \)-soliton \( \tau \)-function, and comparing the result, modulo irrelevant factors, to that given above.

In the above discussion we have treated the variables \( t_{2k+1} \) as auxiliary variables which are to be set to zero after they have been used to construct a given potential. Comparison of Eq. (49) with the vertex operator (80) shows that the auxiliary variables \( t_{2k+1} \) introduced in this section are precisely the KdV time variables employed in Sec. IV. Consequently, the above construction can also be used to construct multi-soliton solutions of the KdV hierarchy. Defining \( u(x, t_3, t_5, \ldots) \) by

\[
u(x, t_3, t_5, \ldots) = 2 \frac{\partial^2}{\partial x^2} \log \tau(x, t_3, \ldots),
\]

with \( \tau \) constructed by repeated application of the vertex operator on 1, we see that \( u \) obeys

\[
u_{t_{2k+1}} = \frac{\partial}{\partial x} \frac{\delta}{\delta u} H_{2k+1}.
\]

As the time variables run over all possible values, \( u \) maps out all of the isospectral deformations of the potential \( V(x) = -u(x)|_{t_3, t_5, \ldots \text{fixed}} \).

**VI CONCLUSIONS**

We have explored two methods for adding a soliton to a multi-soliton solution of the Korteweg-de Vries equation (and related higher-order equations). The method more familiar from the standpoint of soliton theory [1] employs a function (the \( \tau \)-function) whose logarithm, when differentiated twice, gives the solution (up to a factor). A
vertex operator, a function of infinitely many times, acts upon this $\tau$-function to add a soliton.

The alternative method for adding a soliton relies upon supersymmetric quantum mechanics. In order to add a soliton, one must solve a (first-order) Riccati equation, thereby adding one integration constant for each soliton. The physical interpretation of this integration constant is the position of the new soliton relative to all the others. This position is best visualized at large separations of the individual solitons, which occurs at asymptotic values of the times $t_{2k+1}$ governing the evolution according to the Hamiltonians $H_{2k+1}$. Under such circumstances, the multi-soliton solution closely resembles a set of individual one-soliton “humps,” each lump propagating with a speed governed by its size.

We have shown that the two methods for adding solitons are equivalent. Nonetheless, the method of demonstration still seems somewhat roundabout. There are many features of single-soliton solutions which suggest that they may be useful in constructing the vertex operator (80), but we have so far been unable to find a more direct route to this result.

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VIII APPENDIX: DERIVATION OF THE TRANSMISSION COEFFICIENT USING SUPERSYMMETRIC QUANTUM MECHANICS

We wish to show that potentials constructed using the method of Sec. II are reflectionless and have transmission coefficients $T(k)$ given by Eq. (17):

$$T(k) = \prod_{i=1}^{N} \frac{ik - \kappa_i}{ik + \kappa_i}.$$  \hfill (89)

These results may be proven by induction on $N$.

For, suppose that we have constructed a series of potentials $V_n, n = 0, 1, 2, \ldots, N$, such that $V_n$ has bound states at $E = -\kappa_1^2, -\kappa_2^2, \ldots, -\kappa_N^2$. Using the methods of Sec. II, we may construct $V_{n+1}$ from $V_n$ as follows: We write

$$V_{-}^{(n+1)} = V_n + \kappa_{n+1}^2 = f_{n+1}^2 - f'_{n+1},$$  \hfill (90)

so that the partner potential $V_{+}^{(n+1)}$ is

$$V_{+}^{(n+1)} = V_{n+1} + \kappa_{n+1}^2 = f_{n+1}^2 + f'_{n+1}.$$  \hfill (91)
We therefore find that each potential $V_n$ has two equivalent representations:

$$V_n = f_{n+1}^2 - f_{n+1}' - \kappa_{n+1}^2 = f_n^2 + f_n' - \kappa_n^2.$$  \hspace{1cm} (92)

The eigenfunctions of $V_n$ can be related to those of $V_{n+1}$ using the operators $A$ and $A^\dagger$. Comparing with Sec. II, we find that if $\psi_n$ is an eigenfunction of $V_n$, then the corresponding eigenfunction $\psi_{n+1}$ for the potential $V_{n+1}$ is given by

$$\psi_{n+1} = A_{n+1}^\dagger \psi_n = \left( \frac{d}{dx} + f_{n+1} \right) \psi_n.$$  \hspace{1cm} (93)

We now make the following induction hypotheses: we suppose for some $n$ that (i) $f_n \to \mp \kappa_n$ as $x \to \pm \infty$, and that (ii) the transmission coefficient for $V_n$ is given by Eq. (89) with $N = n$. We know from Sec. II that both hypotheses hold for $n = 1$. It follows that $V_n \to 0$ as $x \to \pm \infty$, and that $f_{n+1}$ obeys

$$f_{n+1}^2 - f_{n+1}' = V_n + \kappa_{n+1}^2.$$  \hspace{1cm} (94)

As $x \to \pm \infty$, this reduces to $f_{n+1}^2 - f_{n+1}' = \kappa_{n+1}^2$, which has the solution $f_{n+1} = -\kappa_{n+1} \tanh \kappa_{n+1} (x - x_0)$. Consequently $f_{n+1}$ has the asymptotic forms

$$f_{n+1} \to \begin{cases} -\kappa_{n+1} \tanh \kappa_{n+1} (x - x_+) & \to -\kappa_{n+1}, \quad x \to +\infty; \\ -\kappa_{n+1} \tanh \kappa_{n+1} (x - x_-) & \to \kappa_{n+1}, \quad x \to -\infty. \end{cases}$$  \hspace{1cm} (95)

This proves that our first induction hypothesis holds for all $n$. To see that the second holds, observe that a plane wave solution in the potential $V_n$ has the asymptotic form

$$\psi_n(x) \to \begin{cases} e^{ikx}, \quad x \to -\infty; \\ \Pi_{i=1}^n \frac{ik - \kappa_i}{ik + \kappa_i} e^{ikx}, \quad x \to +\infty, \end{cases}$$  \hspace{1cm} (96)

Applying the operator $A_{n+1}^\dagger$ to $\psi_n$ gives the corresponding solution for the potential $V_{n+1}$. After dividing through by $ik + \kappa_{n+1}$, we find that this solution has the form

$$\psi_{n+1}(x) \to \begin{cases} e^{ikx}, \quad x \to -\infty; \\ \Pi_{i=1}^{n+1} \frac{ik - \kappa_i}{ik + \kappa_i} e^{ikx}, \quad x \to +\infty, \end{cases}$$  \hspace{1cm} (97)

which proves the second of our induction hypotheses, as well as Eq. (89).
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