Resonance and fractal geometry

Henk Broer

Johann Bernoulli Institute for Mathematics and Computer Science
Rijksuniversiteit Groningen
Summary

i. resonance

ii. two oscillators: torus- en circle dynamics

iii. driven oscillators, examples:
   - Hopf-Neĭmark-Sacker bifurcation
   - parametric resonance
   - Hopf saddle-node bifurcation for maps
What is resonance?

**Resonance:** interaction of oscillating subsystems, where

**rational** ratio of frequencies and with **compatible** motion

voorbeelden:

1 : 1 resonance: Huygens’s clocks, Moon and Earth, Charon and Pluto

1 : 2 resonance: Botafumeiro (Santiago de Compostela)

2 : 3 resonance: Mercury
Christiaan Huygens (1629-1695)

and title page *Horologium Oscillatorium* 1673
cycloids and synchronisation
Huygens’s clocks synchronous

Chr. Huygens, Œuvres Complètes de Christiaan Huygens, publiées par la Société Hollandaise des Sciences 16, Martinus Nijhoff, The Hague 1929, Vol. 5, 241-262; Vol. 17, 156-189
Tidal resonance

Moon ‘caught’ by Earth in 1 : 1 resonance
Pluto and Charon caught each other:
als the ultimate fate of the Earth-Moon system . . .
Mercury ‘caught’ in 3 : 2 resonance

A. Correia and J. Laskar, Mercury’s capture into the 3/2 spin-orbit resonance as a result of its chaotic dynamics, *Nature* 429 (2004) 848-850
Botafumeiro

Santiago de Compostela

incense container brought into 1:2 resonance by pulley:
period exactly equals twice that of the forcing
Mathematical programme

**modelling** in terms of dynamical systems depending on parameters

**emergence** of several kinds of dynamics: periodic, quasi-periodic and chaotic

**bifurcations** (fase transitions) in between

**applications** from climate change to (biological) cell systems
Torus en circle dynamics
simplest model: torus dynamics

Poincaré map: \( \varphi \mapsto \varphi + 2\pi\alpha + \varepsilon f(\varphi) \)
\( \sim \) dynamics of the circle (by iteration)

resonant \( \equiv \) periodic
Example: Arnol’d family

\[ \varphi \mapsto \varphi + 2\pi \alpha + \varepsilon \sin \varphi \]

resonance tongues in \((\alpha, \varepsilon)\)-vlak
catalogue of circle and torus dynamics
Explanation torus dynamics

within tongues: periodicity ⇔ resonance ⇔ phase-locking ∼ ‘synchronisation’

within ‘main tongue’: 1:1 resonance ∼ ‘entrainment’

outside tongues: quasi-periodicity
each orbit densely fills torus / circle

H.W. Broer and F. Takens, *Dynamical Systems and Chaos*, Epsilon-Uitgaven 64, 2009; revised edition Appl. Math. Sc., Springer-Verlag, 2010 (to appear)

H.W. Broer, K. Efstathiou and E. Subramanian, Robustness of unstable attractors in arbitrarily sized pulse-coupled systems with delay, *Nonlinearity* 21(1) (2008), 13-49
Devil’s staircase

rotation number $\varrho \equiv$ mean rotation

$\varrho$ as a function of $\alpha$, continuous, non-decreasing and constant on plateaux for rational values of $\varrho$

resonance $\Leftrightarrow$ $\varrho$ rational

H.W. Broer, C. Simó and J.C. Tatjer, Towards global models near homoclinic tangencies of dissipative diffeomorphisms, *Nonlinearity* 11 (1998), 667-770
Geometry in parameter space

non-resonance \sim fractal geometry

\textbf{Cantor} set, topologically small
  (nowhere dense)

\textbf{positive} Lebesgue measure

\textbf{fractal} set with ‘Droste’ effect

\textbf{motion} \textbf{quasi-periodic}
  with irrational rotation number;
  ‘strongly’ irrational \Rightarrow
  differentiable conjugation with rigid rotation

V.I. Arnol’d, \textit{Geometrical Methods in the Theory of Ordinary Differential Equations},
Springer-Verlag 1983

J. Oxtoby, \textit{Measure and Category}, Springer-Verlag 1971
Conclusions Huygens’s clocks

torus / circle model weakly coupled oscillators as before

almost identical oscillators \( \Rightarrow \rho \approx 0 \) modulo 1
\( \Rightarrow \) parameters \((\alpha, \varepsilon)\) in ‘main tongue’

M. Bennett, M.F. Schatz, H. Rockwood and K. Wiesenfeld, Huygens’s clocks, 
*Proc. R. Soc. Lond. A* 458 (2002), 563-579
Hopf-Neimark-Sacker I

**more general:** map \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

fixed point \( f(0) = 0 \)

**eigenvalues derivative** \( e^{2\pi(\alpha \pm i\beta)} \)

with \( \alpha \approx 0 \) and \( \beta \approx p/q \)

**example:** \( f \) is Poincaré map of driven oscillator

\[
\ddot{x} + ax + c\dot{x} = \varepsilon q(x, \dot{x}, t)
\]

with \( q(x, \dot{x}, t + 2\pi) \equiv q(x, \dot{x}, t) \)

**locally:** geometry with universal singularities
(saddle node / fold and other bifurcations)

**globaal:** quasi-periodicity and fractal geometry
Hopf-Neimark-Sacker II

non-degenerate case $q \geq 5$

F. Takens, Forced oscillations and bifurcations. In: Applications of Global Analysis I, *Comm. of the Math. Inst. Rijksuniversiteit Utrecht* (1974). In: H.W. Broer, B. Krauskopf and G. Vegter (eds.), *Global Analysis of Dynamical Systems*, IoP Publishing (2001), 1-62
Hopf-Neimark-Sacker III

mildly-degenerate case $q \geq 7$
Conclusions ‘Arnol’d’ tongues

universal: fold, cusp, swallowtail and Whitney umbrella

H.W. Broer, M. Golubitsky and G. Vegter, The geometry of resonance tongues: A Singularity Theory approach, *Nonlinearity* 16 (2003), 1511-1538

H.W. Broer, S.J. Holtman and G. Vegter, Recognition of the bifurcation type of resonance in mildly degenerate Hopf-Neĭmark-Sacker families, *Nonlinearity* 21 (2008), 2463-2482

H.W. Broer, S.J. Holtman, G. Vegter and R. Vitolo, Geometry and dynamics of mildly degenerate Hopf-Neĭmarck-Sacker families near resonance, *Nonlinearity* 22 (2009), 2161-2200

H.W. Broer, S.J. Holtman and G. Vegter, Recognition of resonance type in periodically forced oscillators. *Physica-D* (2010) (to appear)
Parametric resonance

driven oscillator

\[ \ddot{x} + (a + \varepsilon q(t)) \sin x = 0 \text{ (swing)} \]

with \( q(t + 2\pi) \equiv q(t) \)

for example:

- \( q(t) = \cos t \)
- \( q(t) = \cos t + \frac{3}{2} \cos(2t) \)
- \( q(t) = \text{signum}(\cos t) \)

loss of stability in discrete tongues

emanating from \((a, \varepsilon) = (\frac{1}{4}k^2, 0), k = 0, 1, 2, \ldots\)

subharmonic bifurcations

H.W. Broer and G. Vegter, Bifurcational aspects of parametric resonance, *Dynamics Reported, New Series* 1 (1992), 1-51
Resonance tongues swing

H.W. Broer and M. Levi, Geometrical aspects of stability theory for Hill’s equations, *Archive Rat. Mech. An.* 131 (1995), 225-240

H.W. Broer and C. Simó, Resonance tongues in Hill’s equations: a geometric approach, *Journal of Differential Equations* 166 (2000), 290-327
Botafumeiro revisited

Poincaré map  swing in $1 : 2$ resonance
period doubling
quasi-periodicity
chaos . . .
From gaps to tongues

universal geometry from gaps to tongues with \( \varepsilon \) extra parameter

collapse theory of gaps

quasi-periodical analogue

\[ q(t) = Q(\omega_1 t, \omega_2 t, \ldots, \omega_n t) \text{ with } Q : \mathbb{T}^n \rightarrow \mathbb{R} : \]

global fractal geometry and ‘Droste’ effect

for example: \( n = 2 \) with \( \omega_1 = 1 \) and \( \omega_2 = \frac{1}{2} (\sqrt{5} - 1) \)

rotatie number \( \varrho \) as before

H.W. Broer, H. Hanßmann, Á. Jorba, J. Villanueva and F.O.O. Wagener, Normal-internal resonances in quasi-periodically forces oscillators: a conservative approach. *Nonlinearity* **16** (2003), 1751-1791
Devil’s staircase revisited

rotation number $\varrho$ as a function of $\alpha$

tongues $\rightarrow$ gaps spectrum Schrödinger operator $\sim\rightarrow$ Cantor spectrum and devils staircases . . .

J. Moser and J. Pöschel, An extension of a result by Dinaburg and Sinai on quasi-periodic potentials, *Comment. Math. Helvetici* **59** (1984), 39-85

H.W. Broer, J. Puig and C. Simó, Resonance tongues and instability pockets in the quasi-periodic Hill-Schrödinger equation, *Commun. Math. Phys.* **241** (2003), 467-503
Hopf saddle-node I

map \( f : \mathbb{R}^3 \to \mathbb{R}^3 \), fixed point \( f(0) = 0 \)

eigenvalues derivative \( \approx 1 \) en \( e^{2\pi(\alpha \pm i\beta)} \)

with \( \alpha \approx 0 \) and \( \beta \approx p/q \)

math more experimental

inspired by climate models . . .

H.W. Broer, C. Simó and R. Vitolo, Bifurcations and strange attractors in the Lorenz-84 climate model with seasonal forcing, *Nonlinearity* 15(4) (2002), 1205-1267

A.E. Sterk, R. Vitolo, H.W. Broer, C. Simó and H.A. Dijkstra, New nonlinear mechanisms of midlatitude atmospheric low-frequency variability. *Physica D: Nonlinear Phenomena* 239 (2010), 701-718

H.W. Broer, H.A. Dijkstra, C. Simó, A.E. Sterk and R. Vitolo, The dynamics of a low-order model for the Atlantic Multidecadal Oscillation, *DCDS-B* (2010) (to appear)
H.W. Broer, C. Simó and R. Vitolo, The Hopf-Saddle-Node bifurcation for fixed points of 3D-diffeomorphisms, analysis of a resonance ‘bubble’, *Physica D* **237** (2008), 1773-1799

H.W. Broer, C. Simó and R. Vitolo, The Hopf-Saddle-Node bifurcation for fixed points of 3D-diffeomorphisms, the Arnol’d resonance web, *Bull. Belgian Math. Soc. Simon Stevin* **15** (2008), 769-787
Hopf saddle-node III

corresponding dynamics: quasi-periodicity and chaos
Conclusions

coeexistence periodicity (including resonance), quasi-periodicity and chaos in product state- and parameter space

bifurcations (phase transitions): singularities

non-resonances: ⇝ Kolmogorov-Arnol’d-Moser theory

fractal geometry and ‘Droste’ effect

modelling at a larger scale

D. Ruelle and F. Takens, On the nature of turbulence, Comm. Math. Phys. 20 (1971), 167-192; 23 (1971), 343-344

H.W. Broer, KAM theory: the legacy of Kolmogorov’s 1954 paper, Bull. AMS (New Series) 41(4) (2004), 507-521

H.W. Broer, B. Hasselblatt and F. Takens (eds.): Handbook of Dynamical Systems, Volume 3 North-Holland, 2010 (to appear)