A method for the determination of the coefficient of rolling friction using cycloidal pendulum

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Abstract. The paper presents a method for experimental finding of coefficient of rolling friction appropriate for biomedical applications based on the theory of cycloidal pendulum. When a mobile circle rolls over a fixed straight line, the points from the circle describe trajectories called normal cycloids. To materialize this model, it is sufficient that a small region from boundary surfaces of a moving rigid body is spherical. Assuming pure rolling motion, the equation of motion of the cycloidal pendulum is obtained - an ordinary nonlinear differential equation. The experimental device is composed by two interconnected balls rolling over the material to be studied. The inertial characteristics of the pendulum can be adjusted via weights placed on a rod. A laser spot oscillates together to the pendulum and provides the amplitude of oscillations. After finding the experimental parameters necessary in differential equation of motion, it can be integrated using the Runge-Kutta of fourth order method. The equation was integrated for several materials and found values of rolling friction coefficients. Two main conclusions are drawn: the coefficient of rolling friction influenced significantly the amplitude of oscillation but the effect upon the period of oscillation is practically imperceptible. A methodology is proposed for finding the rolling friction coefficient and the pure rolling condition is verified.

1. Introduction

The coefficient of rolling friction, characterizing the rolling resistance, [1-7] is a subject for numerous biomechanics studies regarding articular surfaces of bones. Complex motions resulting from rolling with sliding and spinning are met in human articulations (shoulder, knee, ankle, hip, elbow, jaw etc.) and the design of artificial joints develops considering among other phenomena, the rolling friction, [8]. Estimation of friction is an extremely difficult task considering the multitude of parameters influencing the phenomenon, [9-13]. For rolling friction characterization, there also are various methods, some of which (pendulum method) allow estimation of local tribological properties, [4], [14-18], and the others give a global estimation of rolling friction [7], [19].
2. Theoretical considerations

A mobile circle with radius $r$ is compelled to roll without sliding over a horizontal straight line $Ox$ as shown in figure 1. A straight line is attached to the circle, initial being in vertical position. The contact point between the mobile circle and the fixed straight line is denoted $C$. In the initial position, the straight line fixed to the circle is vertical and the contact point occupies the position $C_0$. The pure rolling condition imposes that the length of the arch $CC_0$ equals the length of the closed line segment $OC$.

![Figure 1. Rolling of a mobile circle over a fixed straight line.](image)

During the rolling motion of the mobile circle over the fixed straight line, the points of the straight line $\Delta$ describe arcs of ordinary cycloid when they belong to the mobile circle, arcs of prolate cycloids $C_b$ when the point is outside the circle and curtate cycloids $C_a$ when the point is positioned inside the circle [20-21].

![Figure 2. Cycloidal trajectories of points belonging to the mobile circle.](image)

It is considered that the circle $\Gamma$ and the straight line $\Delta$ are fixed to a rigid body having the mass centre on the mobile line $\Delta$ and $d$ is the distance between the centre of the circle and the centre of mass. The rigid body is characterised by the mass $M$ and the axial moment of inertia with respect to
an axis normal to the plane of motion passing through the centre of mass, \( J_z \). Figure 1. The rigid body is supported such as that the circle \( \Gamma \) contacts the fixed horizontal straight line \( D \), the rigid develops into a pendulum that oscillates. In the case when the radius of the mobile circle becomes zero, the pendulum turns into a physical pendulum. In reverse case, as mentioned, the points of the pendulum describe cycloidal trajectories and as a consequence the pendulum is a cycloidal one.

In order to describe the motion of the cycloidal pendulum, a fixed system of reference is considered, with the vertical axis passing through the initial contact point \( C \) and the horizontal axis coincident to the straight line \( D \). The versors of the system are \( i \) and \( j \). A second mobile coordinate system is considered, with the origin \( Q \) in the centre of the circle and the \( Qy' \) axis coincident to the straight line \( \Delta \) and the \( Qx' \) axis normal to the line \( \Delta \). The versors of the mobile system are \( i' \) and \( j' \). It is obvious that the versor \( k' \) of \( Oz' \) axis from mobile system has unmovable direction and corresponds to the direction of \( k \) versor of \( Oz \) axis from fixed system. The following forces and moments act upon the pendulum:

- The gravitational force of the body \( G \)
- The normal reaction \( N \), perpendicular to the straight line \( \Delta \)
- The tangential (frictional) reaction \( T \) tangent to the straight line \( \Delta \)
- The rolling friction torque, \( M_r \), normal to the plane of motion

The theorem of motion of the centre of mass is applied for expressing the movement of the pendulum, with the concrete form:

\[
M_{rG} = G + N + T \tag{1}
\]

and the moment of momentum theorem written with respect to the centre of mass has the form:

\[
J_E = \overline{GC} \times (N + T) \tag{2}
\]

Equation (1) projected on the axes of the fixed system provide two scalar equations. The equation (2) has non-zero projection only with respect to the versor \( k' \). But since \( k' \equiv k \) it results that the equation of moments can be projected on \( Oz \) axis with the mention that the moment of inertia \( J_z \) should be calculated with respect to a system with the origin in the centre of mass \( G \) and having the axes oriented by the versors \( i' \), \( j' \) \( k' \). To these equations, the relations characterising the friction must be added; the frictional reaction:

\[
T \leq \mu N \tag{3}
\]

and the rolling friction moment:

\[
M_r \leq sN \tag{4}
\]

In the above relations 3 and 4, \( \mu \) represents the dynamic coefficient of sliding friction and \( s \) is the rolling friction coefficient. The equal sign in relation 3 should be considered when relative sliding occurs in \( C \) point, situation when the friction force can be determined, and a new unknown is not introduced and the pendulum has two degrees of freedom. In opposite case, the inequality sign should be kept, but the friction force is in this case undetermined and the pendulum has one degree of freedom. So, the lack of sliding between the points in contact in point \( C \) indicates the presence of pure rolling between the circle \( \Gamma \) and the immobile straight line \( D \). The equality or inequality sign must be maintained in relation 4 according to the value of relative angular velocity from pair \( C \): zero or non-zero.

The consequence of the contact between the circle \( \Gamma \) and the straight line \( D \) is:

\[
y_Q = r \tag{5}
\]
thus the position of the pendulum is completely determined by the two parameters $x_G$ and $\phi$ angle. In
the case when between the $C$ points there is sliding, the unknowns of the problem are the two
kinematics parameters $x_G$ and rotation angle $\phi$, together to $N$, $T$ and $M_r$ these unknowns being
determined by the three projection equations and the equations:

$$ T = \mu N $$

(6)

$$ M_x = sN $$

(7)

If pure rolling exists between the mobile circle and the straight line $D$, mathematically expressed
by the relation:

$$ x_Q = r \phi, $$

(8)

the number of degrees of freedom reduces with one unit. Solving the pendulum dynamics problem in
this case is made adding the equation 8 to the projection equations of equations 1 and 2 and by
imposing that at every instant the following inequality is necessary to be verified:

$$ T < \mu N $$

(9)

For the position vector of the centre of mass is written the relation:

$$ \overline{OG} = \overline{OC} + \overline{CQ} + \overline{OG} $$

(10)

Explicitly:

$$ \overline{OG} = x_G i + y_G j = -r \phi i + r j + d \sin \phi i - d \cos \phi j = $$

$$ = (-r \phi + d \sin \phi) j + (r - d \cos \phi) j $$

(11)

From the projections of vector equation 11 there are obtained :

$$ \begin{cases}
    x_G = -r \phi + d \sin \phi \\
    y_G = r - d \cos \phi
\end{cases} $$

(12)

The concrete form of equations 1 and 2 are:

$$ M(\ddot{x}_G i + \ddot{y}_G j) = -Mg j + Nj + Ti $$

(13)

respectively:

$$ J \dot{\omega} k' = (\overline{CO} + \overline{OG}) \times (N + T) - sN \text{sign} \phi \ k' $$

(14)

In equation 14, the last term from the right side represents the vector of rolling friction moment and
it must have opposite sign with respect to angular velocity.

$$ \omega = \dot{\phi} k' $$

(15)

Replacing relation 12 in equation 13 and completing the computations, the following equations of
projections are obtained:

$$ M \left[ M \ d \ddot{\phi} \cos \phi - r \ddot{\phi} - Md \ \phi^2 \sin \phi \right] = T $$

(16)

$$ M \left[ d \phi^2 \cos \phi + d \dot{\phi} \sin \phi \right] = N - Mg $$

(17)

The system made of equations 16 and 17 is solved with respect to $N$ and $T$.
The moment of momentum theorem has the next equation of projection:

\[ J \ddot{\phi} + (d \sin \varphi + s \text{sgn} \dot{\phi}) N + (d \cos \varphi - r) T = 0 \]  
(19)

Replacing the relation 18 into 19, the differential equation of motion of the pendulum is obtained:

\[
\dot{\phi} = -\left( \frac{r}{d} \sin \varphi + \frac{s}{d} \cos \varphi \text{sgn} \dot{\phi} \right) \varphi^2 + \left( \sin \varphi + \frac{s}{d} \text{sgn} \dot{\phi} \right) g \\
1 + \left( \frac{r}{d} \right)^2 - 2 \frac{r}{d} \cos \varphi + \frac{s}{d} \sin \varphi \text{sgn} \dot{\phi} + \frac{J z}{Md^2} d
\]
(20)

The equation 20 is an ordinary nonlinear differential equation.

3. Experimental set-up and principle of the method
The cycloidal pendulum used in experimental tests is presented in figure 3. It consists in a cylindrical aluminium rod 1 to which is attached a cylindrical body 2, having on frontal faces two identical conical depressions, coaxial, where two bearing balls of diameter 41.25mm are fastened. The two balls are contacting two horizontal metallic rulers. By using two balls, two contact points were obtained with the rulers and thus, these two points completely define the oscillation axis of the pendulum. Using a single ball, this is not achievable. To modify the position of the centre of mass and the moment of inertia of the pendulum, on the rod 1 are placed a series of weights 4, that can glide along the rod and then are fixed by a screw. A laser stick of small dimensions is attached to the inferior end of the rod for the study of pendulum motion.
The image of the laser spot is visualised on a scale positioned at $L$ distance under the oscillation axis of the pendulum. When the pendulum is moved from the position of equilibrium and then released, it approaches to vertical position due to the moment of gravitational force to which opposes the rolling friction torque. In order to obtain accurate results it is desirable that the two moments are comparable and this requirement is fulfilled using the weights $4$. When the values of the torques are comparable, the position of the pendulum is no longer vertical. To identify the vertical position of equilibrium, a wire with weight is brought in the vicinity of the rod to materialize the vertical direction, figure 4. This fact is necessary for making the zero indication from the ruler to coincide with the equilibrium position of the laser spot. As it can be observed from equation 20, in order to characterise the motion of the pendulum, it is necessary knowing the central moment of inertia with respect to an axis parallel to the oscillation axis that passes through the centre of mass, $J_z$ and the distance between the centre of mass to the oscillation axis, $d$. To this purpose, the method proposed in [22] is applied. The principle of the method consist in finding the experimental periods of oscillation $T_1$ and $T_2$ with respect to two axes parallel to the axis of pendulum, having stipulated the distance $\ell$ between them. To this end, a cylindrical part of small dimensions is attached to the inferior end of the pendulum, figure 5, set with two small ball bearings 2 on a shaft. Reverting the pendulum and finding the oscillation periods with respect to the axis of the ball bearings, for two given positions of the cylindrical part, the two parameters $J_z$ and $d$ can be found. To validate the results concerning the position of the centre of mass and the moment of inertia, the actual pendulum, figure 5, was modelled using a CAD software, figure 6, that allows finding all inertial characteristics.

![Figure 5. Actual pendulum.](image1)

![Figure 6. CAD model of pendulum.](image2)

![Figure 7. Laser spot moving along the ruler.](image3)

With the parameters occurring in differential equation 20 now known, it was integrated using the method Runge–Kutta of fourth order, [23]. The characteristic parameters of the used pendulum are:

- ball radius $r = 20.63 \cdot 10^{-3} m$;
- distance from the centre of mass to the axis of oscillation $d = 0.0178 m$;
- mass of pendulum $M = 1.350 kg$;
- moment of inertia with respect to a central axis, parallel to the oscillation axis
  \[ J_G = 0.0275 \text{ kg} \cdot \text{m}^2; \]
- distance from the oscillation axis to the ruler on which the spot moves \( L = 0.906 \text{ m} \).

With the law of motion of the pendulum found, the motion of the spot on the ruler, shown in figure 7, is given by the law:

\[ x(\varphi) = (r + L)\tan \varphi - r\varphi \quad (21) \]

Using the above data, the equation 20 was integrated for six values of coefficient of rolling friction: \((10, 30, 50, 100, 150, 200) \mu \text{m}\) and the results are presented in figure 8.

Two main conclusions can be drawn from the experimental study, figure 8, concerning the coefficient of rolling friction:
- has a strong influence upon oscillation amplitude;
- the effect upon the period of oscillation is practically unnoticeable.

Considering the above aspects, the following procedure is proposed for finding the coefficient of rolling friction:
- the pendulum is set into oscillation motion
- the motion of the spot on the ruler is video-captured
- the movie is transferred to a computer and split into frames for finding the maximum values of spot amplitudes and the instants of reaching these amplitudes. The amplitude is precisely found since at maximum elongation the velocity is zero. During the motion, there are errors in finding the elongation.
- with known amplitudes of the motion of the spot, the values corresponding to angular amplitudes of the pendulum are found using the equation 21.

The experimental data are represented on a graph. On the same graph there are plotted the results of integration of the equation of motion for different values of the coefficient of rolling friction \( s \), as carried on for figure 8. Using the network of curves \( s = \text{const} \) it is found a range over which the experimental value of the coefficient of rolling friction should be searched for. Refining the network in the chosen domain, coefficient of rolling friction is found. For the established value of coefficient of rolling friction it must be verified that the condition of pure rolling 9 is satisfied at any instant. After integration of the equation of motion 20 using the Runge-Kutta method, three vectors having the same number of elements, are obtained, \( t, \varphi \) and \( \omega \) representing the time, rotation angle and angular velocity of the pendulum. Validation of the pure rolling condition assumes verifying if the condition 9
is satisfied. This matter supposes knowing the dependence of time for normal reaction $N$ and tangential reaction $T$. As it can be observed, for these calculi the variation of angular acceleration of the pendulum $\varepsilon = \ddot{\varphi}$ must also be known and it can be obtained by applying the relation:

$$\varepsilon_k = \frac{\varphi_k - \varphi_{k-1}}{t_k - t_{k-1}}$$

The drawback of the above relation is that it generates a vector whose size is smaller with one unit than the size of the vectors $t$, $\varphi$, $\omega$ and thus the relations 18 become inoperable. Then, to obtain the vector of angular acceleration with the same size as angular velocity and elongation, the following relation is applied:

$$\varepsilon_k = \left[ \left( \sin \varphi_k + \frac{s}{d} \sin \omega_k \cos \varphi_k \right) d\omega_k^2 + \left( \sin \varphi_k + \frac{s}{d} \sin \omega_k \right) g \right] \frac{Md}{\left[ 1 + \frac{r^2}{d^2} - 2 \frac{r}{d} \cos \varphi_k + \frac{s}{d} \sin \varphi_k \sin \omega_k \right] Md^2 + J_G}$$

(23)

With known vectors elongation, angular velocity and angular acceleration, there can be found the tangential force $T$ and normal force $N$ and finally, the rolling friction condition can be verified.

The differential equation 20 was integrated for properly chosen data and the results were employed in the constructions of the plots from figures 9-11 for phase space, hysteresis loop and angular acceleration of the pendulum. To be remarked that in the points of maximum amplitude, the angular acceleration of the pendulum presents finite jumps. The pure rolling condition was verified in the supporting points of the pendulum and the results are presented in figures 12 and 13. The equation of motion was integrated for the same date, changing only the launching amplitude of the pendulum $\varphi_0$.

It can be noticed (figure 12), that for small launching amplitudes, $\varphi_0 = 3^\circ$, the ratio $|T|/N$ is inferior to the coefficient of sliding friction $\mu = 0.2 \pm 0.03$ while for greater values of launching amplitude, like $\varphi_0 = 60^\circ$ the ratio exceeds the values of sliding coefficient and confirms that sliding replaces rolling at the ends of the race.

Figure 9. Phase space.
Figure 10. Hysteresis loop.

Figure 11. Angular acceleration of the pendulum.

Figure 12. The variation of $T/N$ ratio for small launching amplitudes ($\varphi_0 = 3^\circ$).
4. Experimental results

Two materials were experimentally tested, rubber and aluminium and the results are presented subsequently. The principle for finding the rolling friction coefficient was described above. For the tested material, the motion of the laser spot along the ruler was video captured and after that, using a computer, the movie was split into frames and the instants and values of maximum amplitude were identified. These experimental amplitudes are represented alongside the theoretical curves of variation of angular elongation of the pendulum; the coefficient of rolling friction was changed until the coincidence between experimental and theoretical data was obtained. The curves obtained for rubber and aluminium are presented in figures 14 and 15 respectively. In order to compare the motion of the pendulum for the two materials, the data from figures 14 and 15 were plotted on the same graph, figure 16.

Figure 13. The variation of $T/N$ ratio for large launching amplitudes ($\phi_0 = 45^\circ$).

Figure 14. Theoretical signal and experimental data for rubber.
Figure 15. Theoretical signal and experimental data for aluminium.

Figure 16. Comparison between experimental data and theoretical signal for aluminium and rubber.

5. Conclusions

The proposed method presents advantages like the employment of small material probes and applicability for biological samples or very expensive materials. The characteristic parameters of the pendulum are adjustable and this permits the study of a broad variety of materials, from soft to stiff ones. The paper presents the method and experimental device used for finding the coefficient of rolling friction from non-conforming contacts. The principle of the method is based on finding the damping from the bearings of a pendulum, the damping value being directly related to the coefficient of rolling friction. The pendulum has a special design, since the axis of oscillation actually consists in
two sphere-plane contacts and the materials to be tested are the materials of the sphere and plane. The fact that from the pair of bodies that materialize the support the plane is kept immobile and the ball is mobile, it results that the points of the pendulum describe a cycloidal family and thus, the name of cycloidal pendulum.

For the actual pendulum, first there were determined the inertial characteristics: mass, position of centre of mass, moment of inertia with respect to the axis of oscillation and then the equation of motion of the pendulum was deduced. Other papers from literature consider the domain of small angular amplitudes with the purpose to obtain a linear differential equation of motion. In the actual work, the equation describing the motion of the pendulum is a nonlinear one and it was integrated from the general case using a numerical procedure.

A non-contact method was selected for characterization of the motion of the pendulum for a minimum effect upon the dynamics of the actual system. Practically a small laser generator was fixed on the pendulum body and therefore the spot moves along a scale. The motion of the spot was video captured and the movie split into frames for identification of maximum amplitudes and corresponding instants. The fact that the data from experimental tests are recorded and can be further processed is an advantage of the method. The experimental data were plotted. On the same graph there were traced the curves of theoretical variation of angular elongation, established for a set of values of coefficient of rolling friction. The curve that best approximates the experimental results was chosen. The method presents the advantage that the region where the contact point moves has small dimensions and thus permits the strictly local estimation of coefficient of rolling friction. A matter to be improved refers to the manner of finding the decrease of amplitude that requires a relatively long period of time. Employment of a sensor offering numerical results for the spot motion would substantially reduce the working time and present more expedite methodology.

The main contribution consist in finding and integration of the equation of motion and developing the method for identification of coefficient of rolling friction by comparing experimental data to the theoretical results.

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