Distribution of particles which produces a desired radiation pattern

A.G. Ramm
Mathematics Department, Kansas State University, Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu, fax 785-532-0546, tel. 785-532-0580

Abstract

A method is given for calculation of a distribution of small particles, embedded in a medium, so that the resulting medium would have a desired radiation pattern for the plane wave scattering by this medium.

1 Introduction

Let $D_0 \subset \mathbb{R}^3$ be a bounded domain, $k_0$ be the wave number in $D_0$ and $k < k_0$ be the wave number in $D'_0 = \mathbb{R}^3 \setminus D_0$. We assume that $D_0$ is a homogeneous medium, so that $k_0$ is a constant. This assumption can be weakened: we may assume that $k_0 = k_0(x)$ is a known function. Let $D_m$ be a particle, $d_m$ be its diameter, $1 \leq m \leq M$, $M$ is the number of small particles, $a = \max_{1 \leq m \leq M} \frac{d_m}{2}$ is an estimate for the radius of a small particle. We assume that $k_0 a \ll 1$, and then $k a \ll 1$, i.e., particles are small, that $d \gg a$, where $d = \min_{j \neq m} \text{dist}(D_m, D_j)$ and that the particles are acoustically soft, i.e., $u = 0$ on $S_m$, the boundary of $D_m$. Denote $U_{m=1}^M D_m := U, \ R^3 \setminus U = V, \ \partial U$ is the boundary of $U, \ C_m$ is the electrical capacitance of a perfect conductor with the shape of $D_m$.

The Inverse Problem is:

Can one distribute many small particles in $D_0$ so that a plane wave $u_0 := e^{i k_0 x}$, (where $\alpha$ is a given unit vector, $\alpha \in S^2, \ S^2$ is a unit sphere), scattered by $D_0$, would produce a desired radiation pattern (scattering amplitude) $A(\alpha', \alpha)$?

The acoustic pressure $u$ solves the problem (1)-(3):

$$(\nabla^2 + k^2 - q(x))u = 0 \text{ in } V, \quad q(x) = \begin{cases} k^2 - k_0^2 & \text{in } D_0 \\ 0 & \text{in } D'_0, \end{cases} \quad (1)$$

PACS 03.40.Kf MSC 35J05, 35J10, 70F10, 81U40, 35R30
key words: acoustic scattering, many-body problem, nanotechnology, inverse problems
\[ u = 0 \text{ on } \partial U, \]  
\[ u = e^{ik\alpha \cdot x} + A(\alpha', \alpha) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \alpha' = \frac{x}{r}, \]  
and the coefficient \( A(\alpha', \alpha) \) is called the scattering amplitude or radiation pattern.

Let \( G = G(x, y) \) solve the problem
\[ \left[ \nabla^2 + k^2 - q(x) \right] G = -\delta(x-y) \text{ in } \mathbb{R}^3, \quad r \left( \frac{\partial G}{\partial r} - ikG \right) = o(1) \quad r \to \infty. \]  
This \( G \) exists, is unique, and solves the integral equation
\[ G(x, y) = g(x, y) - (k^2 - k_0^2) \int_{D_0} g(x, z) G(z, y) \, dz, \quad g = g(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}. \]  
For brevity, we drop the \( k \)-dependence in \( g, G \) and other functions below. One may consider \( G(x, y) \) known since \( D_0 \) and \( k_0 \) are known.

Let us look for the solution to (1)–(3) of the form
\[ u = U_0(x, \alpha) + \sum_{m=1}^{M} \int_{S_m} G(x, t) \sigma_m(t) \, dt, \]  
where \( \sigma_m(s) \) are unknown functions and \( U_0(x, \alpha) \) is the scattering solution corresponding to the potential \( q(x) \) in the absence of small bodies, i.e., in the whole space. For arbitrary \( \sigma_m \) the right-hand side of (6) solves equation (1) (because \( G \) solves (4)), satisfies the radiation condition (3), and
\[ A(\alpha', \alpha) = A_q(\alpha', \alpha) + \frac{1}{4\pi} \sum_{m=1}^{M} \int_{S_m} U_0(s, -\alpha') \sigma_m(s) \, ds, \]  
where the scattering solution \( U_0(s, \alpha) \) can be defined by the formula ([1, p.232]):
\[ G(x, s) = \frac{e^{ikr}}{4\pi r} U_0(s, \alpha) + o\left(\frac{1}{r}\right), \quad r := |x| \to \infty, \quad \alpha = -\frac{x}{r}. \]  
Formula (8) was proved in [3, p.46], where it was shown that \( U_0(x, \alpha) \) solves the problem
\[ \left[ \nabla^2 + k^2 - q(x) \right] U_0(x, \alpha) = 0 \text{ in } \mathbb{R}^3, \quad U_0 = e^{ik\alpha \cdot x} + A_q(\alpha', \alpha) \frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \]  
where \( r := |x| \to \infty \) along the direction \( \alpha' \). The function \( U_0 \) can also be considered known, and \( G \) can be considered as Green’s function for the Schrödinger equation (4).
The right-hand side of (6) solves problem (1)–(3) if and only if \( \sigma_m \) are such that boundary condition (2) is satisfied:

\[
-U_0(s, \alpha) - \sum_{m \neq j} \int_{S_m} G(s, t) \sigma_m(t) \, dt = \int_{S_j} G(s, t) \sigma_j(t) \, dt, \quad s \in S_j, \quad 1 \leq j \leq M.
\]

(10)

So far the smallness of the particles was not used. If \( ka \ll 1 \), then equation (10) can be simplified:

\[
-U_0(s_j, \alpha) - \sum_{m \neq j} G(s_j, t_m) Q_m = \int_{S_j} g_0(s_j, t) \sigma_j \, dt, \quad 1 \leq j \leq M, \quad Q_m := \int_{S_m} \sigma_m \, dt,
\]

(11)

where \( s_j \in S_j \) is any point in \( S_j \), and \( g_0 = \frac{1}{4\pi|x-y|} \). In equation (11) we have used two approximations. The first one is

\[
G(s, t) \approx g(s, t), \quad s, t \in S_j.
\]

(12)

This approximation is justified by (5) when \( k|x-y| \ll 1 \), because the term \( g(x, y) \) is the main term on the right-hand side of (5) as \( x \to y \), and \( |s_j - t| \ll 1 \). The second one is \( g \approx g_0 \) if \( k|s-t| \ll 1 \), and the error of this approximation is \( O(ka) \), so this approximation is also justified because \( ka \ll 1 \).

Equation (11) is equation for the charge distribution \( \sigma_j \) on the surface \( S_j \) of a perfect conductor charged to the potential \( -U_0(s_j, \alpha) - \sum_{m \neq j} G(s_j, t_m) Q_m \). Therefore the total charge \( Q_j := \int_{S_j} \sigma_j \, dt \) on the surface \( S_j \) can be calculated by the formula

\[
Q_j = C_j \left( -U_0(s_j, \alpha) - \sum_{m \neq j} G(s_j, t_m) Q_m \right), \quad 1 \leq j \leq M,
\]

(13)

where \( C_j \) is the electrical capacitance of the perfect conductor \( D_j \). In [1, p.385] formulas for calculation of \( C_j \) with arbitrary desired accuracy are given. Equation (13) is a linear algebraic system for finding unknown \( Q_j, 1 \leq j \leq M \).

Consider the limiting case \( M \to \infty \) of the distribution of particles in \( D_0 \). Define \( C(y) \) as follows:

\[
\int_D C(y) \, dy = \lim_{M \to \infty} \sum_{D_m \subset D} C_m,
\]

(14)

where \( D \subset D_0 \) is an arbitrary subdomain of \( D_0 \). The above definition means that \( C(y) \) is the limiting density of the capacitances of the small particles in \( D \). Formula (13) shows that the self-consistent field \( u_e \) can be defined as

\[
u_e(x, \alpha) = \begin{cases} U_0(x, \alpha) + \sum_{m=1}^M G(x, t_m) Q_m, & \min_{1 \leq m \leq M} |x - t_m| \gg a, \\ U_0(x, \alpha) + \sum_{m \neq j} G(x, t_m) Q_m, & |x - t_j| \sim a. \end{cases}
\]

(15)
Therefore, defining \( u_e \) we neglect the influence of any fixed single small particle on the field. This is justified when \( M \to \infty \). Using (13) and (15) one gets
\[
\begin{align*}
  u_e(x, \alpha) &= U_0(x, \alpha) - \sum_m G(x, t_m) C_m u_e(t_m, \alpha),
\end{align*}
\]
and in the limit \( M \to \infty \) one obtains the equation:
\[
\begin{align*}
  u_e(x, \alpha) &= U_0(x, \alpha) - \int_{D_0} G(x,y) C(y) u_e(y, \alpha) \, dy,
\end{align*}
\]
where \( C(y) \) is defined in (14). Equation (16) is equivalent to the Schrödinger scattering problem:
\[
\begin{align*}
  [\nabla^2 + k^2 - q(x) - C(x)] u_e &= 0 \text{ in } \mathbb{R}^3,
\end{align*}
\]
where \( q(x) \) is known, the function \( u_e \) has the following asymptotics
\[
\begin{align*}
  u_e &= e^{i k \alpha \cdot x} + A(\alpha', \alpha) \frac{e^{i k r}}{r} + o \left( \frac{1}{r} \right), \quad r := |x| \to \infty, \quad \alpha' = \frac{x}{r},
\end{align*}
\]
and \( A(\alpha', \alpha) \) is the scattering amplitude at a fixed \( k > 0 \).

Therefore the Inverse Problem, stated above, is reduced to inverse scattering problem of finding the potential \( q(x) + C(x) \) from the knowledge of the corresponding fixed-energy scattering amplitude \( A(\alpha', \alpha) \).

This problem was solved by the author (see [1, Chapter 5] and references therein). In Section 2 we outline the author’s algorithm for solving this inverse scattering problem.

If the potential \( q(x) + C(x) \) is found and \( q(x) \) is known, then \( C(x) \) is found and one knows the density of the particle distribution in \( D \) which produces the desired radiation pattern \( A(\alpha', \alpha) \). Assuming that the particles are identical, one has \( C_m = C \), where \( C \) is the electrical capacitance of one particle, and \( C(x) = N(x) C \), where \( N(x) \) is the density of particles, that is, the number of particles per unit volume around point \( x \). See [2] for the theory of wave scattering by small bodies.

## 2 Solution to inverse scattering problem

We follow [1, p.264] and take \( k = 1 \) without loss of generality. Given \( A(\alpha', \alpha) \) one finds \( A_\ell(\alpha) := \int_{S^2} A(\alpha', \alpha) Y_{\ell m}(\alpha') \, d\alpha' \), where \( Y_{\ell m} := Y_{\ell m} \) are the normalized spherical harmonics, so that
\[
\begin{align*}
  A(\alpha', \alpha) &= \sum_{\ell=0}^{\infty} A_\ell(\alpha) Y_{\ell m}(\alpha'),
  \quad \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \ell =: \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell}.
\end{align*}
\]

If the data \( A(\alpha', \alpha) \) are exact, then ([1, p.262])
\[
\begin{align*}
  \max_{\alpha \in S^2} |A_\ell(\alpha)| &\leq O \left( \sqrt{\frac{b_0}{\ell}} \left( \frac{b_0 e}{2\ell} \right)^{\ell+1} \right),
  \quad |Y_{\ell m}(\theta')| \leq \frac{1}{\sqrt{4\pi}} \frac{e^{\kappa r}}{|j_\ell(r)|} \forall r > 0, \quad \theta \in \mathcal{M},
\end{align*}
\]
where \( b_0 > 0 \) is the radius of the smallest ball containing the domain \( D_0, \mathcal{M} := \{ \theta : \theta \in \mathbb{C}^3, \theta \cdot \theta = 1 \}, \theta \cdot \omega := \sum_{j=1}^3 \theta_j \omega_j \), \( \kappa = |Im \theta| \), \( j_\ell(r) \) is the spherical Bessel function. Fix an arbitrary \( \xi \in \mathbb{R}^3 \). One can find (nonuniquely and explicitly) \( \theta', \theta \in \mathcal{M} \), such that \( \theta' - \theta = \xi, \theta \to \infty \). For example, if \( \xi = te_3, t = |\xi| > 0 \), (which can be assumed without loss of generality), then \( \theta' = \frac{t}{2} e_3 + z_1 e_1 + z_2 e_2, \theta = -\frac{t}{2} e_3 + z_1 e_1 + z_2 e_2 \), and the condition

\[
\frac{t^2}{4} + z_1^2 + z_2^2 = 1, \quad z_1, z_2 \in \mathbb{C},
\]

implies \( \theta, \theta' \in \mathcal{M} \). One may find many \( z_1 z_2 \in \mathbb{C} \), such that \( (\text{ii}) \) holds and \( |z_1| \to \infty \). For example, take \( z_1 = re^{i\varphi}, z_2 = re^{-i\varphi} \), then \( r^2 \sin(2\varphi) - r^2 \sin(2\varphi) = 0, r^2[\cos(2\varphi + \cos(2\varphi)] = 1 - \frac{t^2}{4}, \) so \( r^2 \cos(2\varphi) = \frac{1}{2} - \frac{t^2}{8} \). One can take \( r \geq \left| \frac{1}{2} - \frac{t^2}{8} \right|^{1/2} \) and find \( \varphi \) such that \( \cos(2\varphi) = \left( \frac{1}{2} - \frac{t^2}{8} \right) \frac{1}{r} \).

In what follows we always assume

\[
\theta' - \theta = \xi, \quad \theta', \theta \in \mathcal{M}, \quad |\theta| \to \infty.
\]

Because of \( (\text{ii}) \), the series

\[
A(\theta', \alpha) = \sum_{\ell=0}^{\infty} A_{\ell}(\alpha) Y_{\ell}(\theta'), \quad \theta \in \mathcal{M},
\]

converges absolutely and uniformly on compact subsets of \( S^2 \times \mathcal{M} \).

Fix positive numbers \( b_0 < b_1 < b_2 \) such that \( D_0 \subset B_{b_0} := \{ x : |x| \leq b_0 \} \). Note that the scattering solution \( u(x, \alpha) \) for the scattering potential \( q(x) + C(x) \) can be written explicitly in the region \( |x| > b_0 \):

\[
u(\alpha) \in L^2(S^2). \quad \text{Consider the problem:}
\]

\[
F(\nu) = \min,
\]

where

\[
F(\nu) := \int_{b_1 \leq |x| \leq b_2} \left| e^{-i\theta \cdot x} \int_{S^2} u(x, \alpha) \nu(\alpha) d\alpha - 1 \right|^2 \ dx.
\]

The function \( u(x, \alpha) \) in \( (\text{ii}) \) is defined in \( (\text{iii}) \), and the minimization in \( (\text{iv}) \) is with respect to all \( \nu \in L^2(S^2) \). One can prove \( (\text{v} \ p.265 \ |) \) that

\[
\inf F(\nu) := d(\theta) \leq \frac{\text{const}}{|\theta|}, \quad \theta \in \mathcal{M}, \quad |\theta| \gg 1,
\]

\[5\]
Let $\nu(\alpha, \theta)$ be an arbitrary approximate solution to (25) in the following sense:

$$\mathcal{F}(\nu(\alpha, \theta)) \leq 2d(\theta).$$  \hfill (28)

For this $\nu$ define

$$\hat{C} := -4\pi \int_{S^2} A(\theta', \alpha) \nu(\alpha, \theta) d\alpha,$$  \hfill (29)

where $A(\theta', \alpha)$ is defined in \[22\].

Let $\tilde{C}(\xi) = \int_{D_0} C(x)e^{-i\xi \cdot x} dx$, where $C(x) \in L^2(D_0)$ vanishes in $D'_0$. Let \[21\] hold.

The following theorem is proved by the author in [1, p.266].

**Theorem.** Under the above assumptions one has

$$|\hat{C} - \tilde{C}(\xi)| = O \left( \frac{1}{|\theta|} \right).$$  \hfill (30)

**Conclusion:** An algorithm is given for embedding many small particles in a domain $D_0$ in such a way that the plane wave, scattered by such domain, would have a desired radiation pattern $A(\alpha', \alpha)$.

The algorithm consists of solving the inverse scattering problem, namely, finding the potential $q(x) + C(x)$, vanishing outside $D_0$, from the fixed-energy ($k^2 = \text{const} > 0$) scattering amplitude $A(\alpha', \alpha)$. If this potential is found, then $C(x)$ is found, and the small acoustically soft identical particles should be distributed in $D_0$ with the density $N(x) = C^{-1}C(x)$, where $C > \tau$ is the electrical capacitance of a perfect conductor which has the shape of the particle.

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