Some properties of the symbol algebras

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ABSTRACT. In this paper, we obtain some properties of the symbol algebras, starting from their connections with the quaternion and cyclic algebras over a field \( \mathbb{K} \), where \( \mathbb{K} \) is an algebraic number field, \( p \) is a prime in \( \mathbb{K} \) and \( \mathbb{K}_p \) is the completion of \( \mathbb{K} \) with respect to \( p \)-adic valuation, in the case when \( \mathbb{K} = \mathbb{Q} (\varepsilon) \), \( \varepsilon^3 = 1 \), \( \varepsilon \neq 1 \).

1. Introduction

Symbol algebras have many applications in number theory (class field theory), as can be seen in [4], [6], [7]. Since they are a natural generalization of the quaternion algebras, in this paper we find some interesting examples of split quaternion algebras and non division symbol algebras and we give a necessary and sufficient condition for a \( K \)-cyclic central simple algebra \( A = \left( \alpha, \beta \right) \) to be a division algebra.

First, we recall some definitions in the theory of associative algebras.

Let \( A \neq 0 \) be an algebra over the field \( \mathbb{K} \). If the equations \( ax = b, ya = b, \forall a, b \in A, a \neq 0 \), have unique solutions, then the algebra \( A \) is called a division algebra. If \( A \) is a finite-dimensional algebra, then \( A \) is a division algebra if and only if \( A \) is without zero divisors \( x \neq 0, y \neq 0 \Rightarrow xy \neq 0 \). (see [9])

Let \( K \) be a field with char\( K \neq 2 \). Let \( \mathbb{H}_K (\alpha, \beta) \) be a quaternion algebra with basis \( \{1, e_1, e_2, e_3\} \) and the multiplication given by

\[
\begin{array}{c|cccc}
\cdot & 1 & e_1 & e_2 & e_3 \\
1 & 1 & e_1 & e_2 & e_3 \\
e_1 & e_1 & \alpha & e_3 & \beta e_2 \\
e_2 & e_2 & -e_3 & \beta & -\beta e_1 \\
e_3 & e_3 & -\alpha e_2 & \beta e_1 & -\alpha \beta \\
\end{array}
\]

Each element \( x \in \mathbb{H}_K (\alpha, \beta) \) has the form \( x = x_0 \cdot 1 + x_1 e_1 + x_2 e_2 + x_3 e_3 \), with \( x_i \in \mathbb{K}, i = 0, 1, 2, 3 \). For \( a \in \mathbb{H}_K (\alpha, \beta) \), \( a = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 \), the element \( \bar{a} = a_0 - a_1 e_1 - a_2 e_2 - a_3 e_3 \) is called the conjugate of the element \( a \). Let \( a \in \mathbb{H}_K (\alpha, \beta) \). We have that \( t (a) \cdot 1 = a + \overline{a} \in \mathbb{K} \), \( n (a) \cdot 1 = a \overline{a} \in \mathbb{K} \) and these are called

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The positive integer \( D \) is called an algebr\( a \) over \( K \). The division algebra \( D \) is a field, also known as a power norm residue algebra. A quaternion generalized algebra and symbol algebras are central simple \( K \)-algebras. The quaternion algebra \( H \) is split if and only if \( \alpha \) and \( \beta \) are units in \( K \). Otherwise, the algebra \( H \) is a split algebra.

An important invariant for a quaternion algebra \( H \) is the associated conic, denoted \( C \). The associated conic is the projective plane curve defined by the homogeneous equation \( \alpha x^2 + \beta y^2 = z^2 \).

Let \( K \) be an algebraic number field. By a prime of \( K \) we mean a class of equivalent valuations of \( K \). Recall that the finite primes of \( K \) are in one-to-one correspondence with the prime ideals of the ring of integers of \( K \), and the infinite primes are in correspondence with the embedding of \( K \) into the field of complex numbers \( \mathbb{C} \). If \( v \) is a prime of \( K \), we denote with \( K_v \) the completion of \( K \) with respect to the \( v \)-adic valuation.

**Proposition 1.1.** [4, pag. 7] The quaternion algebra \( H \) is split if and only if the conic \( C \) has a rational points over \( K \) (i.e. if there are \( x_0, y_0, z_0 \in K \) such that \( \alpha x_0^2 + \beta y_0^2 = z_0^2 \)).

A natural generalization of the quaternion algebra is the symbol algebra, also known as a power norm residue algebra. J. Milnor, in his book *Introduction to Algebraic K-Theory*, calls it the symbol algebra because of its connection with the \( K \)-theory and with the Steinberg symbols. (see [8])

A symbol algebra is a unitary associative algebra over a field \( K \) with \( \zeta \in K, \zeta^n = 1, \zeta \) a primitive root, generated by the elements \( x, y \) which satisfy the relations \( x^2 = \alpha, y^2 = \beta \) and \( yx = \zeta xy \). This algebra is denoted \( \left( \frac{\alpha}{\beta} \right) \).

Obviously, for \( n = 2 \) we obtain the algebra \( H \) of \( \left( \frac{\alpha}{\beta} \right) \).

The quaternion generalized algebras and symbol algebras are central simple algebras.

**Proposition 1.2.** [8, pag. 237] If \( K \) is an algebraic number field and \( A \) is a central simple \( K \)-algebra, then the dimension of \( A \) over \( K \) is a square.

**Definition 1.3.** Let \( A \) be a central simple algebra of finite dimension \( n \) over \( K \). The positive integer \( d = \sqrt{n} \) is called the degree of the algebra \( A \).

**Theorem (Weddeburn).** [8, pag. 50] Let \( A \) be a central simple algebra over the field \( K \). There are \( n \in \mathbb{N}^* \) and a division algebra \( D, K \subseteq D, \text{ such that } A \simeq M_n(D) \). The division algebra \( D \) is unique up to an isomorphism.

**Definition 1.4.** With the notation of the above Theorem, the degree of the algebra \( D \) over \( K \) (as an algebra) is called the index of the algebra \( A \).

For some \( h \in \mathbb{N}^* \), the tensor product over the field \( K \) \( A \otimes \ldots \otimes A \) \((h-\) times) is isomorphic to a full matrix algebra over \( K \).

**Definition 1.5.** The smallest such an \( h \) is called the exponent of the algebra \( A \).

**Theorem 1.6.** [1] The algebra \( A \) is a division algebra if and only if its index and its degree are the same.
Theorem 1.7. (Brauer-Hasse-Noether).[8] Every central simple algebra over an algebraic number field is cyclic and its index is equal to its exponent. We shall use in the third section some results from the theory of algebraic number fields and we recall these here.

Theorem 1.8. ([1]) Let \( K \subseteq E \) be a cyclic extension of commutative fields of degree \( d \). The cyclic \( K \)–algebra \( A = \left( \frac{\mathbb{Z}}{\mathbb{Z}_\alpha} \right) \) has the exponent \( d \) if and only if \( \alpha \not\in N_{L/K}(L^*) \), for each minimal subfield \( L \) of \( E \) over \( K \).

Theorem 1.9. ([4]) Let \( K \) be a field such that \( \zeta \in K, \zeta^n = 1, \zeta \) is a primitive root, and let \( \alpha, \beta \in K^* \). Then the following statements are equivalent:

i) The cyclic algebra \( A = \left( \frac{\mathbb{Z}}{\mathbb{Z}_\alpha} \right) \) is split.

ii) The element \( \beta \) is a norm from the extension \( K \subseteq K(\sqrt[n]{\alpha}) \).

Theorem 1.10. ([1;2;6]) Let \( K \) be an algebraic number field, \( v \) be a prime of \( K \) and \( K \subseteq L \) a Galois extension. Let \( w \) be a prime of \( L \) lying above \( v \) such that \( K_v \subseteq L_w \) is an unramified extension of \( K_v \) of (residual) degree \( f \). Let \( b = \pi_w^m \cdot u_w \in K_v^* \), where \( \pi_w \) denote a prime element in \( K_v \) and \( u_w \), a unit in the ring of integers \( \mathcal{O}_v \), \( m \in \mathbb{Z} \). Then \( b \in N_{L_w/K_v} (L_w^*) \) if and only if \( f \mid m \). In particular, every unit of \( \mathcal{O}_v \) is the norm of a unit in \( L_w \).

Theorem 1.11. ([2;7]) Let \( K \) be an algebraic number field, \( e \) be an admissible modulus of \( K \), \( v \) be a prime of \( K \), \( v \) divides \( e \). Let \( K \subseteq L \) be a Galois extension. Let \( w \) be any prime of \( L \) lying above \( v \). Then an element \( \alpha \in N_{L_w/K_v} (L_w^*) \) if and only if the Artin symbol \( \left( \frac{L_w/K_v}{\alpha} \right) \) is the identity in the Galois group \( \text{Gal}(L_w/K_v) \), where \( (\alpha) \) denotes the ideal generated by \( \alpha \) in the ring of integers \( \mathcal{O}_v \).

Theorem 1.12. ([6]) Let \( \zeta \) be a primitive root of the unity of \( l \)–order, where \( l \) is a prime natural number and let \( A \) be the ring of integers of the Kummer field \( Q(\zeta, \sqrt[l]{\alpha}) \). A prime ideal \( P \) in the ring \( \mathbb{Z}[\zeta] \) is in \( A \) in one of the situations:

i) It is equal with the \( l \)–power of a prime ideal from \( A \), if the \( l \)–power character \( \left( \frac{\zeta}{P} \right)_l = 0; \)

ii) It is a prime ideal in \( A \), if \( \left( \frac{\zeta}{P} \right)_l = 1 \) if and only if \( \lambda_1 \) is a root of order \( l \) of unity, different from 1.

iii) It decomposes in \( l \) different prime ideals from \( A \), if \( \left( \frac{\zeta}{P} \right)_l = 1 \).

Theorem 1.13. ([5;6]) Let \( l \) be a natural number, \( l \geq 3 \) and \( \zeta \) be a primitive root of the unity of \( l \)–order. If \( p \) is a prime natural number, \( l \) is not divisible with \( p \) and \( f \) is the smallest positive integer such that \( p^f \equiv 1 \mod{l} \), then we have

\[ p\mathbb{Z}[\zeta] = P_1 P_2 ... P_r, \]

where \( r = \frac{\phi(l)}{f}, \phi \) is the Euler’s function and \( P_j, j = 1, ..., r \) are different prime ideals in the ring \( \mathbb{Z}[\zeta] \).
In the following, we consider the symbol algebra for \( n = 3 \) and \( K = \mathbb{Q}(\varepsilon) \) or \( \mathbb{Q}_p(\varepsilon) \), where \( \varepsilon \) is a primitive cubic root of unity and \( p \) a prime number.

2. Some example of quaternion and symbol algebras

**Proposition 2.1.** For \( \alpha = -1, \beta = p, p = 4k + 3, a \) prime number, \( K = \mathbb{Q} \), the algebra \( \mathbb{H}_Q(-1, p) \) is a division algebra.

Proof. Let \( x \in \mathbb{H}_Q(-1, p), x = x_0 + x_1e_2 + x_2e_2 + x_3e_3, x_i \in \mathbb{Q}, i = 0, 1, 2, 3 \) such that \( n(x) = 0 \). It results \( x_0^2 + x_1^2 - px_2^2 - px_3^2 = 0 \), then \( p \mid (x_0^2 + x_1^2) \). Since \( p = 4k + 3 \) is a prime and \( p \mid (x_0^2 + x_1^2) \), and the powers of \( p \) in the factorization of \( x_0^2 + x_1^2 \) and \( x_2^2 + x_3^2 \) are even. We obtain a contradiction, therefore \( x = 0. \) □

**Theorem (Gauss).** If \( p \equiv 1 \mod 3 \), then there are integers \( a, b \) such that \( 4p = a^2 + 27b^2 \).

**Proposition 2.2.** If \( K = \mathbb{Q}(\sqrt{3}) \), then the quaternion algebra \( \mathbb{H}_K(-1, p) \), where \( p \equiv 1 \mod 3 \) is a split algebra.

Proof. Indeed, \( \mathbb{H}_K(-1, p) \) is a split algebra if and only if the associated conic \( x^2 + py^2 = z^2 \) has \( \mathbb{Q}(\sqrt{3}) \) - rational points. Using the Gauss’s theorem, there are \( a, b \in \mathbb{Z} \) such that \( 4p = a^2 + 27b^2 \). Then for \( y_0 = 1, z_0 = \frac{a}{2}, x_0 = \frac{3\sqrt{3b}}{2} \), the point \( \left( \frac{3\sqrt{3b}}{2}, 1, \frac{a}{2} \right) \) is a \( \mathbb{Q}(\sqrt{3}) \) - rational point for the associated conic, and we use Proposition 1.1. □

From the Wedderburn theorem, we know that a finite dimensional simple algebra \( A \) over a field \( K \) is isomorphic to a matrix algebra \( M_n(D) \), for \( D \) a division algebra. Let \( K = \mathbb{Q}(\varepsilon) \) where \( \varepsilon \) is a cubic root of unity and let \( d = [D : K] \) be the index of the algebra \( A \). The algebra \( A = \left( \frac{\alpha, \beta}{K, \varepsilon} \right) \) is a central simple algebra of degree \( 3 \), hence \( d \mid 3 \).

For \( \alpha = -1, \beta = 1 \), the algebra \( A \) is generated, for example, by the elements

\[
X = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & -\varepsilon^2 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},
\]

where \( X^3 = -I_3, Y^3 = I_3 \) and \( XY = \varepsilon YX \). (see[3]) We obtain that \( A \simeq M_3(\mathbb{Q}(\varepsilon)) \). Therefore \( d = 1 \) and the algebra \( A \) is not a division algebra.

We obtain the following proposition

**Proposition 2.3.** The algebras \( A = \left( \frac{\alpha, \beta}{\mathbb{Q}(\varepsilon), \varepsilon} \right) \), for \( \alpha, \beta \in \{-1, 1\} \) are not division algebras.

Proof. The algebra \( A \) has dimension 9, hence degree 3, with basis \( B = \{1, x, y, x^2, y^2, xy, x^2y, xy^2, x^2y^2\} \), \( x^3 = a, y^3 = b \). With the correpondence \( x \rightarrow X, y \rightarrow Y \), we have that \( A \simeq M_3(\mathbb{Q}(\varepsilon)) \), the index \( d = 1 \neq 3 \), where 3 is the algebra’s degree, then \( A \) is not a division algebra. (We used Proposition 1.6.) □
If the central simple algebra $A$ is a division algebra, since it has the degree three, it results that it is a cyclic algebra. It results that there is an element $x \in A - K$, $\alpha \in K$ such that $x^3 = \alpha \in K$. From the Noether-Skolem theorem, it results that there is an element $y \in A - K$ such that $yxy^{-1} = \tau x$. We have $y^3x = xy^3$ and $y^3y = yy^3$, then $y^3$ commutes with the generators $x, y$, therefore $y^3 \in K = C(A)$, the centralizer of the algebra $A$. Hence, there is $\beta \in K$ such that $y^3 = \beta$, and $A = \left(\frac{\alpha, \beta}{K_v, \tau}\right) \simeq M_n(D)$, with $[D : K] = 3$.

3. The algebra $A = \left(\frac{\alpha, \beta}{K_v, \tau}\right)$

We consider the case of the algebra $A = \left(\frac{\alpha, \beta}{K_v, \tau}\right)$ where $\tau$ is a primitive cubic root of unity. We give a necessary and sufficient condition for a $K_v$-cyclic central simple algebra $A = \left(\frac{\alpha, \beta}{K_v, \tau}\right)$ to be a division algebra and finally we find when $\beta$ is a norm for the field $K_v(\sqrt[3]{\alpha})$, where $K_v$ is the completion of the field $K$ with respect the $v$-adic valuation.

Let $K$ be an algebraic number field and $v$ be a prime (finite or infinite) of $K$ such that $\tau \in K_v$, where $\tau$ is a primitive cubic root. We consider the $K_v$-central simple algebra $A = \left(\frac{\alpha, \beta}{K_v, \tau}\right), \alpha, \beta \in K_v^*$. 

**Proposition 3.1.** With the above notation, if $L = K(\sqrt[3]{\alpha})$, the following statements are equivalent:

i) The algebra $A = \left(\frac{\alpha, \beta}{K_v, \tau}\right)$ is a division algebra.

ii) $\beta \notin N_{L_w/K_v}(L_w^*), \text{ for each } w \text{ a prime of } L \text{ lying above } v$.

**Proof.** We consider the cyclic extension of fields $K_v \subseteq L_w$ and we apply the Theorems 1.6, 1.7, 1.8. We obtain that the $K_v$-cyclic central simple algebra $A = \left(\frac{\alpha, \beta}{K_v, \tau}\right)$ is a division algebra if and only if $\beta \notin N_{L_w/K_v}(L_w^*)$. □

From the above proposition and the Theorem 1.9, result that a $K_v$-cyclic central simple algebra $A = \left(\frac{\alpha, \beta}{K_v, \tau}\right)$ is either split or a division algebra.
In the following, we will study the central simple algebra $A = \left( \frac{\alpha, p^3l}{K_p, \varepsilon} \right)$, where $p$ is a prime natural number, $p > 3$, $l \in \mathbb{N}^*$, $\varepsilon$ is a primitive cubic root of unity, $K = \mathbb{Q} (\varepsilon)$.

**Proposition 3.2.** Let $p$ be a prime natural number, $p \equiv 2 \pmod{3}$ and let be given the $K_p-$ algebra where $l \in \mathbb{N}^*, \alpha \in K, K = \mathbb{Q} (\varepsilon).$ Let $P$ be a prime ideal of the ring of integers of the field $L = K (\sqrt[3]{\alpha})$, lying above $p$. Then $p^3l$ is a norm from $L^*$ and the local Artin symbol $\left( \frac{L_P / K_p}{(p^3l)} \right)$ is the identity.

**Proof.** Since $p \equiv 2 \pmod{3}$, from Theorem 1.13., we obtain that $p$ is prime in the ring $\mathbb{Z}[\varepsilon]$. It results that cubic residual symbol $\left( \frac{\alpha}{p^3l, \mathbb{Z}[\varepsilon]} \right)_3 = 1$, from Theorem 1.12, we have that $p$ is totally split in $A$, where $A$ is the ring of integers of the field $L = K (\sqrt[3]{\alpha})$. We denote with $g$ the number of decomposition of the ideal $pA$ in the extension $K \subset L$. It results $g = 3$ and knowing that $efg = [L : K] = 3$, then $f = e = 1$. But $[L_P : K_p] = ef$, therefore $L_P = K_p$, for each $P \in \text{Spec} (A), P \mid pA$. In this case, we obtain that $p$ is the norm of itself in the trivial extension of $K_p$ and the Artin symbol $\left( \frac{L_P / K_p}{(p^3l)} \right)$ is the identity. □

**Proposition 3.3.** Let $p$ be a prime natural number, $p \equiv 1 \pmod{3}$ and let $K_{p_1}-$ algebra $A = \left( \frac{\alpha, p^3l}{K_{p_1}, \varepsilon} \right)$, where $l \in \mathbb{N}^*, \alpha \in K, K = \mathbb{Q} (\varepsilon)$ and $p_1$ is a prime element in $\mathbb{Z}[\varepsilon]$, $p_1 \mid p$. Let $P$ be a prime ideal in the ring of integers of the field $L = K (\sqrt[3]{\alpha})$, lying above $p_1$. Then $p^3l \in N_{L_P/K_{p_1}} (L_P)$ and the local Artin symbol $\left( \frac{L_P / K_{p_1}}{(p^3l)} \right)$ is the identity in the Galois group $Gal (L_P/K_{p_1}).$

**Proof.** From Theorem 1.13 and that $\mathbb{Z}[\varepsilon]$ is a principal ring, we have that the ideal $p\mathbb{Z}[\varepsilon] = p_1\mathbb{Z}[\varepsilon] \cdot p_2\mathbb{Z}[\varepsilon]$, where $p_1, p_2$ are prime distinct elements in $\mathbb{Z}[\varepsilon].$
We study the $K_{p_1}$-algebra $A = \left( \frac{\alpha, p^3_l}{K_{p_1}, \varepsilon} \right)$.

Case 1. If the cubic residual symbol $\left( \frac{\alpha}{p_1 \mathbb{Z}[\varepsilon]} \right)_3$ is a root of unity different from 1, from Theorem 1.12, we obtain that the ideal $p_1 \mathcal{A} \in \text{Spec}(\mathcal{A})$, where $\mathcal{A}$ is the ring of integers of the Kummer field $K(\sqrt[3]{\alpha})$. So that $e = 1, g = 1$ and since $efg = [K(\sqrt[3]{\alpha}) : K] = 3$, it results that $f = 3$, who obviously divides $3l$. From Theorem 1.10, we obtain that $p^3l \in N_{L_P/K_{p_1}}(L^*_P)$. Using Theorem 1.11 and Proposition 3.1, we have that the local Artin symbol $\left( \frac{L_P / K_{p_1}}{(p^3l)} \right)$ is the identity in the Galois group $Gal(L_P/K_{p_1})$ and the algebra $A = \left( \frac{\alpha, p^3_l}{K_{p_1}, \varepsilon} \right)$ is not a division $K_{p_1}$ algebra.

Case 2. If the cubic residual symbol $\left( \frac{\alpha}{p_1 \mathbb{Z}[\varepsilon]} \right)_3 = 1$, from Theorem 1.12, we obtain that $p_1 \mathcal{A} = P_1 P_2 P_3$, $P_i \in \text{Spec}(\mathcal{A}), i = 1, 3$, therefore $g = 3$. But $efg = [K(\sqrt[3]{\alpha}) : K] = 3$, therefore $e = f = 1$. Since $[L_P : K_{p_1}] = ef$, we obtain that $L_P = K_{p_1}$ for each $P \in \text{Spec}(\mathcal{A}), p \mid p_1 \mathcal{A}$. In this case, we have that $p_1$ is a norm of itself in the trivial extension of $K_{p_1}$ and the local Artin symbol $\left( \frac{L_P / K_{p_1}}{(p^3l)} \right)$ is the identity. □

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