Rotor-routing on Galton-Watson trees

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December 18, 2014

Abstract

A rotor-router walk on a graph is a deterministic process, in which each vertex is endowed with a rotor that points to one of the neighbors. A particle located at some vertex first rotates the rotor in a prescribed order, and then it is routed to the neighbor the rotor is now pointing at. In the current work we make a step toward understanding the behavior of rotor-router walks on random trees. More precisely, we consider random i.i.d. initial configurations of rotors on Galton-Watson trees, and give a classification in recurrence and transience for transfinite rotor-router walks on these trees.

Keywords: Galton-Watson trees, rotor-router walk, recurrence, transience, return probability.

Mathematics Subject Classification: 60J80; 05C81; 05C05.

1 Introduction

A rotor-router walk on a graph is a deterministic process in which the exits from each vertex follow a prescribed periodic sequence. For an overview and other properties, see the expository paper [HLM+08]. Rotor-router walks capture in many aspects the expected behavior of simple random walks, but with significantly reduced fluctuations compared to a typical random walk trajectory; for more details see [CS06, FL11, HP10]. However, this similarity breaks down when one looks at recurrence or transience of the walks, where the rotor-router walk may behave differently than the corresponding random walk.

By an unpublished argument of Schramm the rotor-router walk is recurrent on any graph where the simple random walk is recurrent. The converse is not true; there exist graphs where the simple random walk is transient, but the rotor-router walk is still recurrent; see [LL09] for homogeneous trees and [FGLP14] for initial rotor configurations with all rotors aligned on \( \mathbb{Z}^d \). For random initial configuration of rotors on homogeneous trees, the issue of transience and recurrence was studied in [AH11], and by [HS12] on directed covers of graphs.

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In this note we investigate the recurrence and transience of rotor-router walks with random initial rotor configuration $\rho$ on Galton-Watson trees. In order to measure how transient this configuration is, we run $n$ rotor-walks starting from the root and we record whether each walk returns to the root or escapes to infinity. A rotor-router walk is called transient if the relative density of the number of escapes is positive. The main result Theorem 3.2 gives a criterion for recurrence and transience of the rotor-router walk and states that in the transient regime the relative density of escapes of the rotor-router walk equals almost surely the return probability of the simple random walk on Galton-Watson trees. As a consequence of the main result we get the following.

**Theorem 1.1.** Let $\rho$ be a uniformly distributed rotor configuration on a Galton-Watson tree $T$ with mean offspring number $m$. The rotor-router walk on $T$ is recurrent if and only if $m \leq 2$.

## 2 Preliminaries

### 2.1 Galton-Watson trees

Let $\{Z_h\}_{h \in \mathbb{N}_0}$ be a Galton-Watson process with offspring distribution $\xi$ given by $p_k = \mathbb{P}[\xi = k]$ for $k \in \mathbb{N}_0$. Throughout the paper we assume $p_0 = 0$; this assumption is made for presentational reasons. Informally, a Galton-Watson process is defined as follows: we start with one particle $Z_0 = 1$, which has $k$ children with probability $p_k$. Then each of these children also have children with the same offspring distribution, independently of each other and of their parent. This continues forever. Formally, let $\{\xi^h\}_{i,h \in \mathbb{N}}$ be i.i.d. random variables with the same distribution as $\xi$. Define $Z^h$ to be the size of the $h$-th generation, that is,

$$Z_{h+1} = \sum_{i=1}^{Z_h} \xi^h_i,$$

and $Z_0 = 1$. Our assumption $p_0 = 0$ means that each vertex has at least one child, and the process survives almost surely. Also, the mean offspring number $m = \mathbb{E}[\xi] \geq 1$. Here and thereafter we denote by $\mathbb{E}[\xi] = \sum_{k \geq 1} k\xi_k$ for a stochastic vector $\xi$. Observe hereby that a supercritical Galton-Watson process conditioned on survival is a Galton-Watson process with $p_0 = 0$ plus some finite bushes. The existence of finite bushes does not influence recurrence and transience properties.

We will not be interested only in the size $Z_h$ of the $h$-th generation, but as well in the underlying family trees. Let $T$ be the family tree of this Galton-Watson process, with vertex set $V(T) = \{x^h_i : h \in \mathbb{N}_0, 1 \leq i \leq Z_h\}$. If $x^h_{j+1}$ is a descendant of $x^h_i$ there is an edge in the family tree between these two vertices. For ease of notation we will always identify $T$ with its vertex set. For technical reasons we will add one additional vertex $o$ to the tree, which will act as the parent of the root vertex $r = x^1_0$. The vertex $o$ is called the sink of the tree. We always consider $T$ together with its natural planar embedding, that is for each generation $h$ we draw the vertices $x^h_i$ from right to left, for $i = 1, \ldots, Z_h$. In particular this means that each possible embedding of a given combinatorial tree has the same probability.
For each vertex $x \in \mathcal{T}$, denote by $d_x$ the number of children of $x$. Given the planar embedding described above, we denote by $x^{(k)}$, $k = 0, \ldots, d_x$, the neighbours of $x$ in $\mathcal{T}$ in counterclockwise order beginning at the parent $x^{(0)}$ of $x$.

2.2 Rotor-router walks

A rotor configuration $\rho$ is a map $\rho : \mathcal{T} \to \mathbb{N}_0$ such that $\rho(x) \in \{0, \ldots, d_x\}$ for all $x \in \mathcal{T}$. For a given rotor configuration $\rho_0$ and a starting vertex $x_0 \in \mathcal{T}$, a rotor-router walk is a sequence of pairs $\{(x_i, \rho_i)\}_{i \geq 0}$ such that for all $i \geq 1$ we have the transition rule

$$
\rho_{i+1}(x) = \begin{cases} 
\rho_i(x_i) + 1 \mod (d_{x_i} + 1), & \text{if } x = x_i \\
\rho_i(x), & \text{otherwise,}
\end{cases}
$$

and $x_{i+1} = x_i^{(\rho_{i+1}(x_i))}$. Informally this means that a particle performing a rotor-router walk, when reaching the vertex $x$ first increments the rotor at $x$ and then moves to the neighbour of $x$ the rotor is now pointing at.

We are interested in the recurrence or transience of transfinite rotor-router walks, as defined in [HP10], for random initial configuration of rotors. For a given rotor configuration, a rotor walk either visits each vertex infinitely often or visits each vertex only a finite number of times, (see e.g. [HP10, Lemma 6]). Thus in the second case, the rotor-router walk escapes to infinity, leaving behind a well defined limit rotor configuration. A transfinite rotor-router walk with initial rotor configuration $\rho$, is a sequence of rotor-router particles starting at the root $r$. Denote by $\rho_n$ the initial rotor configuration of the $n$-th particle, and let $\rho_1 = \rho$. For all $n \geq 1$, run the $n$-th rotor-router particle until it returns to $o$ for the first time. If this occurs after a finite number of steps, let $\rho_{n+1}$ be the rotor configuration left behind by this particle. In case the $n$-th particle never returns to $o$ we define $\rho_{n+1}$ to be the limit configuration created by the particle escaping to infinity. Let now

$$
e_k = \begin{cases} 
1, & \text{if the n-th particle escaped to infinity} \\
0, & \text{otherwise,}
\end{cases}
$$

and let $E_n(\mathcal{T}, \rho) = \sum_{k=1}^{n} e_k$ count the number of escapes of the first $n$ walks.

The next result, due to Schramm states that a rotor-router walk is no more transient than a random walk. A proof of this result can be found in [HP10, Theorem 10].

**Theorem 2.1.** For any locally finite graph $G$, any starting vertex, any cyclic order of neighbours and any initial rotor configuration $\rho$

$$
\limsup_{n \to \infty} \frac{E_n(G, \rho)}{n} \leq \gamma(G),
$$

where $\gamma(G)$ represents the probability that the simple random walk on the graph $G$ never returns to the starting point.

3 Random initial rotor configuration

We construct random initial rotor configurations on Galton-Watson trees $\mathcal{T}$ introduced above. In order to do this, for each $k \geq 0$ we choose a probability distribution $Q_k$
supported on \( \{0, \ldots, k\} \). That is, we have the sequence of distributions \((Q_k)_{k \in \mathbb{N}_0} \), where

\[ Q_k = (q_{k,j})_{0 \leq j \leq k} \]

with \( q_{k,j} \geq 0 \) and \( \sum_{j=0}^{k} q_{k,j} = 1 \). Let \( Q \) be the infinite lower triangular matrix having \( Q_k \) as row vectors, i.e.:

\[
Q = \begin{pmatrix}
q_{00} & 0 & 0 & 0 & \ldots \\
q_{10} & q_{11} & 0 & 0 & \ldots \\
q_{20} & q_{21} & q_{22} & 0 & \ldots \\
q_{30} & q_{31} & q_{32} & q_{33} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

**Definition 3.1.** A random rotor configuration \( \rho \) on the Galton-Watson tree \( T \) is \( Q \)-distributed, if for each realization \( T(\omega) \), and for each \( v \in T(\omega) \) the rotor \( \rho(v) \) is a random variable with the following properties:

(a) \( \rho(v) \) is \( Q_{d_v} \) distributed, i.e., \( \mathbb{P}[\rho(v) = d_v - l \mid d_v = k] = q_{k,l} \),

(b) \( \rho(v) \) and \( \rho(u) \) are independent if \( u \neq v \), with \( u, v \in T(\omega) \).

We are now ready to state our main result.

**Theorem 3.2.** Let \( \rho \) be a random \( Q \)-distributed rotor configuration on a Galton-Watson tree \( T \) with offspring distribution \( \xi \), and let \( \nu = \xi \cdot Q \). Then we have almost surely:

(a) \( E_n(T, \rho) = 0 \) for all \( n \geq 1 \), if \( \mathbb{E}[\nu] \leq 1 \),

(b) \( \lim_{n \to \infty} \frac{E_n(T, \rho)}{n} = \gamma(T) \), if \( \mathbb{E}[\nu] > 1 \),

where \( \gamma(T) \) represents the probability that simple random walk started at the root of \( T \) never returns to \( o \).

Theorem 3.2 is a generalization of [AH11, Theorem 6] which holds for regular trees. See also [HS12, Theorem 3.5] for rotor-router walks on periodic trees.

Theorem 1.1 is now a corollary of Theorem 3.2 in the case of uniformly distributed rotors. More precisely, if \( Q_k \) is the uniform distribution on \( \{0, \ldots, k\} \) for all \( k \), we get a particularly simple recurrence condition involving only the mean offspring number \( m = \mathbb{E}[\xi] \).

**Proof of Theorem 1.1.** \( \mathbb{E}[\nu] = \sum_{l=0}^{\infty} \sum_{k=l}^{\infty} \frac{1}{k+1} p_k = \frac{1}{2} \sum_{k=0}^{\infty} k p_k = \frac{1}{2} \mathbb{E}[\xi] \). \( \square \)

**Remark 3.3.** Notice here the difference between simple random walk on Galton-Watson trees which is transient for mean offspring number \( m \in (1, 2] \), while the rotor-router walk with uniformly distributed initial rotors is recurrent.
3.1 Recurrent part

Proof of Theorem 3.2. Let $\rho$ be a random $Q$-distributed rotor configuration. Recall that for a vertex $x \in T$ we denote by $x^{(k)}$, $k = 1, \ldots, d_x$, the children of $x$. We call a child $x^{(k)}$ good if $\rho(x) < k$. This means that the rotor walk at $x$ will visit the good children before visiting the parent $x^{(0)}$. Since $\rho$ is $Q$-distributed,

$$P[x \text{ has } l \text{ good children} | d_x = k] = q_{k,l},$$

Therefore the distribution of the number of good children of a vertex $x$ in $T$ is given by

$$P[x \text{ has } l \text{ good children}] = \sum_{k=l}^{\infty} p_k q_{k,l},$$

which is the $l^{th}$ component of the vector $\nu = \xi \cdot Q$. Thus, for each vertex $x$ the set of descendants of $x$ which are connected to $x$ by a path consisting of only good children forms a Galton-Watson tree with offspring distribution $\nu$. By [AH11, Proposition 8] the rotor walk can only escape to infinity along paths that consist exclusively of good children. By assumption we have $E[\nu] \leq 1$, hence subtrees consisting of only good children die out almost surely. This implies that there are no escapes to infinity.

3.2 The frontier process

To prove the transient part of Theorem 3.2, we will use the frontier process introduced in [HSH14]. For sake of completeness we state the definition of the process here.

Fix an infinite tree $T$ with root $r$ and without leaves and a rotor configuration $\rho$ on $T$. As before we attach an additional vertex $o$ to the root $r$ of the tree. Consider the following process which generates a sequence $F_\rho(n)$ of subsets of vertices of the tree $T$. $F_\rho(n)$ is constructed by a rotor-router process consisting of $n$ rotor-router walks starting at the root $r$, such that each vertex of $F_\rho(n)$ contains exactly one particle. In the first step put a particle at the root $r$ and set $F_\rho(1) = \{r\}$. Inductively given $F_\rho(n)$ and the rotor configuration that was created by the previous step, we construct the next set $F_\rho(n+1)$ using the following rotor-router procedure. Perform rotor-router walk with a particle starting at the root $r$, until one of the following stopping conditions occurs:

(a) The particle reaches $o$. Then set $F_\rho(n+1) = F_\rho(n)$.

(b) The particle reaches a vertex $x$, which has never been visited before. Then set $F_\rho(n+1) = F_\rho(n) \cup \{x\}$.

(c) The particle reaches an element $y \in F_\rho(n)$. We delete $y$ from $F_\rho(n)$, i.e., set $F'(n) = F_\rho(n) \setminus \{y\}$. Now there are two particles at $y$, both of which are restarted until stopping condition (a), (b) or (c) for the set $F'(n)$ applies to them. Note that since we are on a tree at least one particle will stop at a child of $y$ after one step, due to halting condition (d).

We will call the set $F_\rho(n)$ the frontier of $n$ particles. Basic properties of this process can be found in [HSH14]. Remark that the frontier process $F_\rho(n)$ depends on the underlying tree $T$. 

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Following [HSH14], let us introduce

\[ M(n) = \max_{\rho} \max \{ |x| : x \in F_\rho(n) \}, \]

where \( |x| \) is the distance of \( x \) to the root \( r \). \( M(n) \) represents the maximal height of the frontier \( F_\rho(n) \). In order to get an upper bound for \( M(n) \), we will use the anchored expansion constant.

Let \( G \) be an infinite graph with vertex set \( V = V(G) \) and edge set \( E = E(G) \). For \( S \subset V(G) \) we denote by \( |S| \) the cardinality of \( S \) and by \( \partial S \) the vertex boundary of \( S \), that is, the set of vertices in \( V(G) \setminus S \) that have one neighbor in \( S \). We say that the set \( S \) is connected if the induced subgraph on \( S \) is connected. Fix the root \( r \in V(G) \). The anchored expansion constant of \( G \) is

\[ \iota'_E(G) = \liminf_{n \to \infty} \left\{ \frac{\left| \partial S \right|}{|S|} : r \in S \subset V(G), \ S \text{ is connected}, \ n \leq |S| < \infty \right\}. \]

For the remainder of this section, we assume that the tree \( T \) has positive anchored expansion constant \( \iota'_E(T) > 0 \).

The next result is a generalization of [HSH14] Lemma 1.5.

**Lemma 3.4.** Assume \( \iota'_E(T) > 0 \). Then there exists \( c = c(T) < 1 \) such that \( M(n) < cn \), for \( n \) large enough.

**Proof.** Consider the frontier process \( F_\rho(n) \) on \( T \). Let \( x \) be an element of \( F_\rho(n) \) with maximal distance \( M = |x| \) to the root \( r \). Denote by \( p = (r = x_0, x_1, \ldots, x_M = x) \) the shortest path between \( r \) and \( x \). Since \( F_\rho(1) = \{r\} \) and by the iterative construction of \( F_\rho(n) \), there exist \( 1 = n_0 < n_1 < \cdots < n_M = n \), such that \( x_i \in F_\rho(n_i) \) for all \( i \in \{0, \ldots, M\} \).

We want to find a lower bound for \( n_{i+2} - n_i \), that is, for the number of steps needed to replace \( x_i \) by \( x_{i+2} \) in the frontier. At time \( n_i \), the vertex \( x_i \) is added to the frontier. The next time after \( n_i \) that a particle visits \( x_i \) halting condition \( \Box \) occurs, thus the rotor at \( x_i \) is incremented two times. As long as not all children of \( x_i \) are part of the frontier, every particle can visit \( x_i \) at most once, since it either stops immediately at a child of \( x_i \) on stopping condition \( \Box \) or is returned to the ancestor of \( x_i \). This means that at subsequent visits the rotor at \( x_i \) is incremented exactly once. In order for \( x_{i+2} \) to be added to the frontier, the rotor at \( x_i \) has to point at direction \( x_{i+1} \) twice. Thus replacing \( x_i \) with \( x_{i+2} \) in the frontier, needs at least \( d_{x_i} + 1 \) particles which visit \( x_i \). Hence, \( n_{i+2} - n_i \geq 1 + d_{x_i} \), for \( i = 0, \ldots, M - 2 \). Denote by \( \tilde{p} = (x_0, x_1, \ldots, x_{M-3}, x_{M-2}) \). By assumption, we have \( \iota'_E(T) > 0 \). Thus, for \( M \) big enough there exists a constant \( \kappa > 0 \), such that \( |\partial \tilde{p}| \geq \kappa |\tilde{p}| \).

Since \( \tilde{p} \) is a path of a tree we get

\[ |\partial \tilde{p}| + |\tilde{p}| = 1 + \sum_{i=0}^{M-2} d_{x_i}. \]

We have therefore

\[ \sum_{i=0}^{M-2} n_{i+2} - n_i \geq \sum_{i=0}^{M-2} (1 + d_{x_i}) = (M - 1) + |\tilde{p}| + |\partial \tilde{p}| - 1 \geq (M - 1)(\kappa + 2) - 1. \]
On the other hand
\[
\sum_{i=0}^{M-2} n_{i+2} - n_i = n_M + n_{M-1} - n_1 - n_0 < 2n. \tag{4}
\]
Combining (3) and (4) gives \( M \leq \frac{2}{\kappa+2} n + \frac{1}{\kappa+2} + 1 \), which proves the claim.

We aim now at getting a lower bound for the size of \( F_\rho(n) \). Following [HSH14], we first estimate the number of particles stopped at \( o \) during the formation of the frontier \( F_\rho(n) \).

This is accomplished using Theorem 1 from [HP10]. For a tree \( T \), define
\[
\ell(n) = \{ x \in T : |x| = M(n) \text{ and the path from } r \text{ to } x \text{ contains no vertex of } F_\rho(n) \}, \tag{5}
\]
where \( M(n) \) is defined in (1). By construction, the set \( F_\rho(n) \) may have “holes”. We can fill these holes by adding additional vertices \( \ell(n) \) on the maximal level \( M(n) \). All these additional vertices were not touched by a rotor particle during the formation of \( F_\rho(n) \).

Fix \( n \) and a rotor configuration \( \rho \), and let \( S = F_\rho(n) \cup \ell(n) \) (6) be the sink determined by the frontier process \( F_\rho(n) \). Denote by \( T^S \) the finite tree which is obtained by truncating \( T \) at \( S \).

Let \( (X_t) \) be the simple random walk on \( T \). Let \( \tau_o = \inf \{ t \geq 0 : X_t = o \} \) and \( \tau_S = \inf \{ t \geq 0 : X_t \in S \} \) be the first hitting time of \( o \) and \( S \) respectively. Consider now the hitting probability
\[
h(x) = h^S_0(x) = \Pr_x[\tau_o < \tau_S], \quad x \in T^S, \tag{7}
\]
that is, the probability to hit \( o \) before \( S \), when the random walk starts in \( x \). We have \( h(o) = 1 \) and \( h(x) = 0 \), for all \( x \in S \). For \( x \in T \setminus T^S \), we set \( h(x) = 0 \).

Start now \( n \) rotor particles at the root \( r \), and stop them when they either reach \( o \) or \( S \). By the Abelian property of rotor-router walks (see [AH12, Lemma 24]) and by the construction of the frontier process \( F_\rho(n) \) we will have exactly one rotor particle at each vertex of \( F_\rho(n) \), no particles at \( \ell(n) \), and the rest of the particles are at \( o \). In order to estimate the proportion of rotor particles stopped at \( o \) we use Theorem 1 from [HP10], which we state here adapted to our case.

**Theorem 3.5** (Theorem 1, [HP10]). Consider the sink \( S \) as above, and let \( (X_t) \) be the simple random walk on \( T \). Let \( E \) be the edge set of \( T \) and suppose that the quantity
\[
K = 1 + \sum_{(x,y) \in E} |h(x) - h(y)| \tag{8}
\]
is finite. If we start \( n \) rotor particles at the root \( r \), then
\[
|h(r) - \frac{n_o}{n}| \leq \frac{K}{n}, \tag{9}
\]
where \( n_o \) represents the number of particles stopped at \( o \).

**Lemma 3.6.** Let \( K \) be the constant defined in (8). Then \( K = 1 + (M(n) + 1)(1 - h(r)) \).
In this section we will prove the transient part of Theorem 3.2. Let

\[ l(T) = \liminf_{n \to \infty} \frac{E_n(T, \rho)}{n} \quad \text{and} \quad l_j(T) = \liminf_{n \to \infty} \frac{E_n(T_j, \rho_j)}{n}, \quad j = 1, \ldots, d_r, \]

be a tree with root \( r \), and rotor configuration \( \rho \). Denote by \( T_1, \ldots, T_{d_r} \) the principal branches of \( T \), and by \( \rho_j \) the restriction of \( \rho \) to \( T_j \). Write

\[ l(T) = \liminf_{n \to \infty} \frac{E_n(T, \rho)}{n} \quad \text{and} \quad l_j(T) = \liminf_{n \to \infty} \frac{E_n(T_j, \rho_j)}{n}, \quad j = 1, \ldots, d_r, \]
3.3 Transient part

From [HP10] Lemma 25, we have that for any tree \( T \)
\[
    l(T) \geq 1 - \frac{1}{1 + \sum_{j=1}^{d_r} l_j(T)}.
\]  
(10)

If \( T \) is a Galton-Watson tree we have that all \( l_j = l_j(T) \), \( j = 1, \ldots, d_r \) are i.i.d. Furthermore \( l = l(T) \) has the same distribution as the \( l_j \), but is not independent of the \( l_j \). Since (10) is valid for any tree,
\[
    l \geq 1 - \frac{1}{1 + \sum_{j=1}^{d_r} l_j}
\]  
(11)
holds stochastically. We first show that under the conditions of Theorem 3.2(b) the random variable \( l \) is almost surely bounded away from zero.

**Proposition 3.9.** Let \( \rho \) be a random \( Q \)-distributed rotor configuration on a Galton-Watson tree \( T \) with offspring distribution \( \xi \), and let \( \nu = \xi \cdot Q \). Suppose \( \mathbb{E}[\nu] > 1 \). Then there exists \( \delta > 0 \), such that
\[
    \mathbb{P}[l \geq \delta] = 1.
\]

**Proof.** Consider \( n \) rotor walks particles and build the frontier process \( F_\rho(n) \) on \( T \). The Galton-Watson process with offspring distribution \( \nu \), with \( \mathbb{E}[\nu] > 1 \) survives with positive probability \( p \). Therefore with positive probability there exists a live path starting at the root \( r \) of \( T \). Existence of a live path implies that the first particle escapes: \( \mathbb{P}[E_1(T, \rho) = 1] = p > 0 \). Denote by \( X \) the set of vertices in \( F_\rho(n) \), for which there is a live path starting at \( x \). Then \( \#X = \sum_{x \in F_\rho(n)} Y_x \), where the \( Y_x \)'s are i.i.d. Bernoulli\((p)\) random variables. By Corollary 3.7 and Theorem 3.8, using Wald’s inequality we get \( \mathbb{E}[\#X] = \mathbb{E}[\#F_\rho(n)]p > \kappa pn \) with \( \kappa = \mathbb{E}[\hat{\kappa}] \). We shall first prove that
\[
    E_n(T, \rho) \geq \#X.
\]  
(12)

From [HP10] Lemmas 18,19, it suffices to prove (12) for \( T^H = \{ x \in T : |x| \leq H \} \), with \( H > M(n) \), i.e.,
\[
    E_n(T^H, S^H, \rho^H) \geq \#X.
\]  
(13)

Here, \( E_n(T^H, S^H, \rho^H) \) represents the number of particles that stop at \( S^H = \{ x \in T : |x| = H \} \) when we start \( n \) rotor-router walks at the root \( r \) of \( T \) and rotor configuration \( \rho^H \) (the restriction of \( \rho \) on \( T^H \)). In \( T^S \), truncated at the frontier \( S \), start \( n \) particles at \( r \), and stop them when they either reach \( S \) or return to \( o \). The vertices at distance greater than \( M(n) \) were not reached, and the rotors there are unchanged. Now for every vertex \( x \) in \( X \) restart one particle. Since there is a live path a \( x \) the particle will reach the level \( H \) without leaving the cone of \( x \), at which point the particle is stopped again. Hence if we restart all particles which are located in \( F_\rho(n) \) at least \( \#X \) of them will reach level \( H \) before returning to the root. Because of the Abelian property of rotor-router walks, (13) follows, therefore also (12). Using the standard Chernoff bound, there exists \( \delta \in (0,1) \) such that
\[
    \mathbb{P}[E_n(T, \rho) < \delta n] \leq \mathbb{P}[\#X < \delta n] \leq \mathbb{P}[\#X < \frac{\delta}{\kappa p} \mathbb{E}[\#X]]
\]
\[
    \leq \exp \left\{ -\frac{(1 - \frac{\delta}{\kappa p})^2}{2} \mathbb{E}[\#X] \right\} \leq \exp \left\{ -\frac{(1 - \frac{\delta}{\kappa p})^2}{2} \kappa pn \right\}.
\]
We can then choose $c > 0$ such that $\mathbb{P}[E_n(T, \rho) < \delta n] \leq e^{-cn}$, for $n$ big enough. Applying Borel-Cantelli Lemma yields the claim.

Recall that $\gamma(T)$ is the probability that simple random walk started at the root of $T$ will never visit $o$. Then $\gamma(T)$ in probability satisfies the recursion

$$\gamma(T) = 1 - \frac{1}{1 + \sum_{j=1}^\infty \gamma(T_j)}, \quad (14)$$

see [LPP97, Equation (4.1)].

Next, we want to prove that the random variable $l$ stochastically dominates the random variable $\gamma(T)$. This requires some additional work. Denote by $F_\gamma = F_\gamma(T)$ and $F_l = F_l(T)$ the cumulative distribution function (c.d.f.) of $\gamma$ and $l$ respectively. The recursive structure of Galton-Watson trees with offspring distribution $\xi$ and $\mathbb{P}[\xi = k] = p_k$ gives that $F_\gamma$ satisfies

$$F_\gamma(s) = \begin{cases} \sum_{k=0}^\infty p_k F_\gamma^k \left( \frac{1}{1 - s} \right), & \text{if } s \in (0, 1) \\ 0, & \text{if } s \leq 0 \\ 1, & \text{if } s \geq 1. \end{cases} \quad (15)$$

Moreover, for the c.d.f. $F_l$ we have using (11)

$$F_l(s) \leq \sum_{k=0}^\infty p_k F_l^k \left( \frac{s}{1 - s} \right), \quad \text{if } s \in (0, 1). \quad (16)$$

We shall use [LPP97, Theorem 4.1], which we state here.

**Theorem 3.10 (Theorem 4.1, [LPP97]).** The functional equation (15) has exactly two solutions, $F_\gamma$ and the Heavyside function $1_{[0, \infty)}$. Define the operator on c.d.f.'s

$$\mathcal{K} : F \mapsto \sum_{k=0}^\infty p_k F^k \left( \frac{s}{1 - s} \right), \quad \text{with } s \in (0, 1).$$

For any initial c.d.f. $F$ with $F(0) = 0$ and $F(1) = 1$ other than the Heavyside function, we have weak convergence under iteration to $F_\gamma$:

$$\lim_{n \to \infty} \mathcal{K}^n(F) = F_\gamma.$$ 

**Remark 3.11.** In particular, for c.d.f.'s $F_1 \leq F_2$, for all $n \geq 1$ it holds

$$\mathcal{K}^n(F_1) \leq \mathcal{K}^n(F_2).$$

**Lemma 3.12.** Let $F_l$ be the c.d.f. of the random variable $l$. For all $n \geq 1$, we have that $F_l \leq \mathcal{K}^n(F_l)$.

**Proof.** For $n = 1$, we have $F_l \leq \mathcal{K}(F_l)$ which holds by (10) and by the definition of the operator $\mathcal{K}$ in Theorem 3.10. For each $n > 1$, using Remark 3.11 for c.d.f.'s $F_l \leq \mathcal{K}(F_l)$ we get $\mathcal{K}^n(F_l) \leq \mathcal{K}^{n+1}(F_l)$. The claim follows then easily by induction.

**Theorem 3.13.** The random variable $l$ dominates $\gamma$ stochastically.
Proof. By Proposition 3.9, \( F_l \) is not equal to the Heavyside function \( 1_{(0, \infty)} \). Hence, applying Theorem 3.10 to \( F_l \) gives

\[
\lim_{n \to \infty} K^n(F_l) = F_\gamma.
\]

By Lemma 3.12

\[
F_l \leq \lim_{n \to \infty} K^n(F_l).
\]

Hence \( F_l \leq F_\gamma \), which is equivalent to the stochastic dominance of \( l \) over \( \gamma \).

We are finally able to prove the transient part of Theorem 3.2.

Proof of Theorem 3.2 (b). From Theorem 3.13, we have \( \gamma \leq l \), stochastically. Theorem 2.1 implies that

\[
l = \liminf_{n \to \infty} \frac{E_n(T, \rho)}{n} \leq \limsup_{n \to \infty} \frac{E_n(T, \rho)}{n} \leq \gamma, \text{ almost surely.}
\]

Putting both parts together we get the stochastic inequalities

\[
\gamma \leq \liminf_{n \to \infty} \frac{E_n(T, \rho)}{n} \leq \limsup_{n \to \infty} \frac{E_n(T, \rho)}{n} \leq \gamma,
\]

which implies that the limit \( L = \lim_{n \to \infty} \frac{E_n(T, \rho)}{n} \) exists, and that \( L = \gamma(T) \) in distribution.

It is now an easy exercise to show that \( L = \gamma \) almost surely. Since \( L = \gamma \) in distribution, we have \( \mathbb{E}[\gamma - L] = 0 \). From Theorem 3.14, we know that \( \gamma - L \geq 0 \) almost surely. The expectation of the nonnegative random variable \( \gamma - L \) can be zero only when \( \gamma - L = 0 \) almost surely. Therefore

\[
\lim_{n \to \infty} \frac{E_n(T, \rho)}{n} = \gamma(T),
\]

with probability one, which proves the transient part.

Acknowledgements. The authors would like to thank Walter Hochfellner for performing simulations of rotor-router walks on Galton-Watson trees. The research of Wilfried Huss was supported by the Austrian Science Fund (FWF): P24028-N18 and by the Erwin-Schrödinger scholarship (FWF): J3628-N26. The research of Ecaterina Sava-Huss was supported by the Erwin-Schrödinger scholarship (FWF): J3575-N26. Moreover, all three authors were supported by the exchange program Amadeus between Graz and Marseille, project number FR 11/2014.

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