Nonexistence of positive supersolutions to a class of semilinear elliptic equations and systems in an exterior domain

Huyuan Chen\textsuperscript{1}, Rui Peng\textsuperscript{2,}* & Feng Zhou\textsuperscript{3}

\textsuperscript{1}Department of Mathematics, Jiangxi Normal University, Nanchang 330022, China;
\textsuperscript{2}School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou 221116, China;
\textsuperscript{3}Center for PDEs and School of Mathematical Sciences, East China Normal University, Shanghai 200241, China

Email: chenhuyuan@yeah.net, pengrui\_seu@163.com, fzhou@math.ecnu.edu.cn

Received March 27, 2018; accepted December 10, 2018; published online June 28, 2019

Abstract In this paper, we consider the following semilinear elliptic equation:
\[
\begin{cases}
-\Delta u = h(x, u) & \text{in } \Omega, \\
u > 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega$ is an exterior domain in $\mathbb{R}^N$ with $N \geq 3$, $h : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is a measurable function, and derive optimal nonexistence results of positive supersolutions. Our argument is based on a nonexistence result of positive supersolutions of a linear elliptic problem with Hardy potential. We also establish sharp nonexistence results of positive supersolutions to an elliptic system.

Keywords semilinear elliptic problem, supersolution, nonexistence

MSC(2010) 35J60, 35B53

Citation: Chen H Y, Peng R, Zhou F. Nonexistence of positive supersolutions to a class of semilinear elliptic equations and systems in an exterior domain. Sci China Math, 2020, 63: 1307–1322, https://doi.org/10.1007/s11425-018-9447-y

1 Introduction

In this paper, we are mainly concerned with the nonexistence of positive supersolutions to the following semilinear elliptic equation:
\[
\begin{cases}
-\Delta u = h(x, u) & \text{in } \Omega, \\
u > 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $\Omega$ is a punctured or an exterior domain in $\mathbb{R}^N$ with $N \geq 3$, i.e., $\Omega = \mathbb{R}^N \setminus \mathcal{O}$ with $\mathcal{O}$ a bounded, closed smooth subset of $\mathbb{R}^N$ and $h : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is a measurable function with $\mathbb{R}_+ = [0, +\infty)$. We assume, unless otherwise specified, that $N \geq 3$ throughout the paper. Without loss of generality, we also assume that $\mathcal{O} \subset \overline{B}_R(0)$, where $\overline{B}_R(0)$ represents the ball with the radius $R$, centered at the origin. A typical punctured domain is $\Omega = \mathbb{R}_+ \setminus \{0\}$, and a typical exterior domain is $\Omega = \mathbb{R}_+ \setminus \overline{B}_R(0)$ with $\mathcal{O} = \overline{B}_R(0)$.

*Corresponding author
A function $u \in C^2(\Omega)$ is said to be a positive supersolution of (1.1) if $u(x) > 0$ and $-\Delta u(x) \geq h(x, u)$ for all $x \in \Omega$.

The existence and nonexistence of solutions or supersolutions to Problem (1.1) have attracted great attention for many years; see [2–5, 11–13, 17, 19, 23, 25–27] and the references therein. In the special case that $h$ only depends on $u$, given $R_0 > 0$, Alarcón et al. [2] proved that Problem (1.1) with $\Omega = \mathbb{R}^N \setminus B_{R_0}(0)$ admits a positive solution if and only if $h$ satisfies

$$
\int_0^{\sigma_0} h(t)t^{-\frac{2N-2}{N-2}} dt < +\infty
$$

for some $\sigma_0 > 0$, by appealing to ODE (ordinary differential equation) techniques. Nevertheless, such an approach fails to apply to (1.1) if the nonlinear term $h$ is not radially symmetric.

In particular, when $h(x, u) = V(x)u^p$, (1.1) becomes

$$
\begin{cases}
-\Delta u = V(x)u^p & \text{in } \Omega, \\
u \geq 0 & \text{on } \partial \Omega.
\end{cases}
$$

(1.2)

The traditional method to establish nonexistence results for solutions or supersolutions to (1.2) is to make use of the fundamental solution and Hadamard property [4, 6, 7, 13]. It is worth mentioning that if $p = -2$, Problem (1.2) in a bounded domain is used to describe the MEMS (Mico-Electro-Mechanical-System) model [21, 24], and if $p = -1$, Problem (1.2) is related to the study of singular minimal hypersurfaces with symmetry\(^1\) [28].

When $V(x) = (1 + |x|)^\beta$ and $\Omega$ is a punctured or an exterior domain, Bidaut-Véron and Pohozaev [6, 7] showed that Problem (1.2) has no solution if

$$p \leq \frac{N + \beta}{N - 2} := p_{\beta}^* \quad \text{with } \beta \in (-2, 2).$$

On the other hand, when $\Omega = \mathbb{R}^N \setminus \{0\}$ and $V(x) = |x|^{a_0}(1 + |x|)^{\beta - a_0}$ with $a_0 \in (-N, +\infty)$ and $\beta \in (-\infty, a_0)$, Chen et al. [14] derived infinitely many positive solutions of (1.2) if

$$p \in \left(p_{\beta}^*, \frac{N + a_0}{N - 2}\right) \cap (0, +\infty).$$

Armstrong and Sirakov [4] and Chen and Felmer [13] dealt with the more general potential

$$V(x) \geq |x|^\beta (\ln |x|)^\tau \quad \text{for } |x| > e,$$

where $\beta > -2$ and $\tau \in \mathbb{R}$. Especially, [4, Theorem 3.1] and [13, Theorem 1.1] imply the following result:

- Let $\beta \in (-2, 2)$. Problem (1.2) with $\Omega = \mathbb{R}^N \setminus B_{e}(0)$ has no positive supersolution provided that either $p < p_{\beta}^*$, $\tau \in \mathbb{R}$ or $p = p_{\beta}^*$, $\tau \geq 0$.

In the current paper, we will provide a sharp improvement of the above result. Indeed, we can conclude the following:

- Let $\beta > -2$. Problem (1.2) with $\Omega = \mathbb{R}^N \setminus B_{e}(0)$ has no positive supersolution provided that either $1 \leq p < p_{\beta}^*$, $\tau \in \mathbb{R}$ or $p = p_{\beta}^*$, $\tau > -1$ (see Proposition 4.2).

- Let $\beta = -2$ and $p = p_{\beta}^* = 1$. Problem (1.2) with $\Omega = \mathbb{R}^N \setminus B_{e}(0)$ has no positive supersolution provided that $\liminf_{|x| \to +\infty} V(x|x|)^2 > \frac{(N-2)^2}{2}$ (see Theorem 2.1).

- Let $\beta \in \mathbb{R}$. Problem (1.2) with $\Omega = \mathbb{R}^N \setminus B_{e}(0)$ has a positive supersolution for properly large $\ell$ provided that either $p > p_{\beta}^*$, $\tau \in \mathbb{R}$ or $p = p_{\beta}^*$, $\tau < -1$ (see Proposition 4.3).

As a consequence, the above results show that both $p = p_{\beta}^*$ and $\tau = -1$ are the critical values for the existence of positive supersolutions to (1.2).

---

\(^1\) Simon L. Some examples of singular minimal hypersurfaces. Private communication, 2001
For the more general nonlinear problem (1.1), let us assume that
\[ h(x, u) \geq \tilde{h}(x, u) \geq 0 \quad \text{for all } x \in \mathbb{R}^N \setminus B_{\varepsilon}(0), \quad u \geq 0, \]
and \( \tilde{h} : \Omega \times \mathbb{R}^{+} \to \mathbb{R}^{+} \) is a function satisfying the following:

(H) \quad (a) for any \( x \in \mathbb{R}^N \setminus B_{\varepsilon}(0) \),
\[ \frac{\tilde{h}(x, s_1)}{s_1} \geq \frac{\tilde{h}(x, s_2)}{s_2} \quad \text{if } s_1 \geq s_2 > 0; \]
(b) for any \( t > 0 \),
\[ \liminf_{|x| \to +\infty} \tilde{h}(x, t|x|^{2-N})|x|^N > \frac{(N-2)^2}{4}; \]
if (b) fails, we assume that
(b1) there exists \( \sigma_1 \in (0, 1) \) such that for any \( t > 0 \),
\[ \liminf_{|x| \to +\infty} \tilde{h}(x, t|x|^{2-N})|x|^N (\ln |x|)^{\sigma_1} > 0; \]
and
(b2) there exists \( \sigma_2 > 0 \) such that for any \( t > 0 \),
\[ \liminf_{|x| \to +\infty} \frac{\tilde{h}(x, t|x|^{2-N}) (\ln |x|)^{\sigma_2}}{t|x|^{-N} (\ln |x|)^{\sigma_2}} > \frac{(N-2)^2}{4}. \]

Then we have the following theorem.

**Theorem 1.1.** \textit{Under the assumption (H), Problem (1.1) has no positive supersolution.}

We would like to mention that (H)(a) means that \( h \) is linear or superlinear, (H)(b) is related to the subcritical case while (H)(b1) and (H)(b2) deal with the critical case. As one will see below, Theorem 1.1 allows us to obtain some optimal nonexistence results of positive supersolutions to Problem (1.2).

To prove Theorem 1.1, it turns out that a nonexistence result of positive supersolutions of the linear Hardy elliptic problem (2.1) (see Section 2) is vital in our analysis. Such a nonexistence result can be established by Agmon-Allegretto-Piepenbrink theory [1]. In this paper, we shall provide a different proof which seems simpler.

Another focus of our paper is on the following semilinear elliptic system:

\[
\begin{cases}
-\Delta u = h_1(x, u, v) & \text{in } \Omega, \\
-\Delta v = h_2(x, u, v) & \text{in } \Omega, \\
u, v \geq 0 & \text{on } \partial \Omega.
\end{cases}
\]

(1.3)

It is known that Liouville-type theorems for System (1.3) have been established (see [16, 22, 31–33]) primarily on the whole space \( \Omega = \mathbb{R}^N \). In particular, when \( h_1(x, u, v) = v^p \), \( h_2(x, u, v) = u^q \), the nonexistence of positive solutions to (1.3) in \( \Omega = \mathbb{R}^N \) has been investigated by [16,32,33] in the subcritical case that
\[ \frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}. \]

It seems that there is little research work devoted to the nonexistence/existence of solutions supersolutions to (1.3) when \( \Omega \) is an exterior domain. We say a function pair \( (u, v) \in C^2(\Omega) \times C^2(\Omega) \) is a positive supersolution of (1.3) if \( u(x), v(x) > 0, -\Delta u(x) \geq h_1(x, u, v) \) and \(-\Delta v \geq h_2(x, u, v) \) for all \( x \in \Omega \).

For System (1.3), two functions \( \tilde{h}_1 \) and \( \tilde{h}_2 \) are involved to control functions \( h_1 \) and \( h_2 \) respectively.

(SH) \quad Suppose that the non-negative function \( \tilde{h}_i \) (\( i = 1, 2 \)) defined on \( (\mathbb{R}^N \setminus B_{\varepsilon}(0)) \times [0, +\infty) \times [0, +\infty) \) satisfies that
(a) the maps \( t \rightarrow h_1(x, s, t) \) and \( s \rightarrow h_2(x, s, t) \) are nondecreasing and for any \( x \in \mathbb{R}^N \setminus B_{e}(0), \)
\[
\frac{\tilde{h}_1(x, s_1, t_1)}{s_1} \geq \frac{\tilde{h}_1(x, s_2, t_2)}{s_2} \quad \text{and} \quad \frac{\tilde{h}_2(x, s_1, t_1)}{t_1} \geq \frac{\tilde{h}_2(x, s_2, t_2)}{t_2}
\]
if \( s_1 \geq s_2 > 0, \ t_1 \geq t_2 > 0; \)
(b) for any \( t > 0, \)
\[
\liminf_{|x| \to +\infty} \tilde{h}_1(x, t|x|^{2-N}, t|x|^{2-N})|x|^{N} > \frac{(N-2)^2}{4},
\]
or
\[
\liminf_{|x| \to +\infty} \tilde{h}_2(x, t|x|^{2-N}, t|x|^{2-N})|x|^{N} > \frac{(N-2)^2}{4};
\]
if (b) fails, we assume that

(b1) there exist positive constants \( \sigma_3 \) and \( \sigma_4 \) with either \( \sigma_3 < 1 \) or \( \sigma_4 < 1 \), such that for any \( t > 0, \)
\[
\liminf_{|x| \to +\infty} \tilde{h}_1(x, t|x|^{2-N}, t|x|^{2-N})|x|^{N} (\ln |x|)^{\sigma_3+2} > 0, \quad i = 1, 2;
\]
and

(b2) if \( \sigma_3 < 1, \sigma_4 \geq 1 \) there exists \( \sigma_5 > 0 \) such that for any \( t > 0, \)
\[
\liminf_{|x| \to +\infty} \frac{\tilde{h}_1(x, t|x|^{2-N}(\ln |x|)^{\sigma_3}, t|x|^{2-N})}{t|x|^{-N}(\ln |x|)^{\sigma_5}} > \frac{(N-2)^2}{4};
\]
if \( \sigma_3 \geq 1, \sigma_4 < 1 \), there exists \( \sigma_6 > 0 \) such that for any \( t > 0, \)
\[
\liminf_{|x| \to +\infty} \frac{\tilde{h}_2(x, t|x|^{2-N}(\ln |x|)^{\sigma_3}, t|x|^{2-N}(\ln |x|)^{\sigma_6})}{t|x|^{-N}(\ln |x|)^{\sigma_6}} > \frac{(N-2)^2}{4};
\]
(b3) if \( \sigma_3, \sigma_4 < 1 \), there exist \( \sigma_5, \sigma_6 > 0 \) such that for any \( t > 0, \)
\[
\liminf_{|x| \to +\infty} \frac{\tilde{h}_1(x, t|x|^{2-N}(\ln |x|)^{\sigma_3}, t|x|^{2-N}(\ln |x|)^{\sigma_6})}{t|x|^{-N}(\ln |x|)^{\sigma_5}} > \frac{(N-2)^2}{4}
\]
or
\[
\liminf_{|x| \to +\infty} \frac{\tilde{h}_2(x, t|x|^{2-N}(\ln |x|)^{\sigma_3}, t|x|^{2-N}(\ln |x|)^{\sigma_6})}{t|x|^{-N}(\ln |x|)^{\sigma_6}} > \frac{(N-2)^2}{4}.
\]
Then we can state the following theorem.

\textbf{Theorem 1.2.} \textit{Assume that for all} \((x, u, v) \in (\mathbb{R}^N \setminus B_{e}(0)) \times [0, +\infty) \times [0, +\infty), \textit{the functions} h_1 \textit{and} h_2 \textit{satisfy}
\[
h_1(x, u, v) \geq \tilde{h}_1(x, u, v), \quad h_2(x, u, v) \geq \tilde{h}_2(x, u, v),
\]
with \( \tilde{h}_1 \) and \( \tilde{h}_2 \) fulfilling (SH). \textit{Then System (1.3) has no positive supersolution.}

Assumption (SH)(b) is related to the subcritical case, (SH)(b2) says that one of the nonlinearities is critical, and (SH)(b3) represents that both nonlinear terms are critical. In particular, when the nonlinearities \( h_1 \) and \( h_2 \) take the forms \( h_1(x, u, v) = |x|^{\beta_1}(\ln |x|)^{p_1} u^{p_2} v^{q_1} \) and \( h_2(x, u, v) = |x|^{\beta_2}(\ln |x|)^{p_2} u^{p_2} v^{q_2}, \)
we are able to clarify the nonexistence and existence of positive supersolutions in terms of the parameters \( p_1, p_2, q_1, q_2, \tau_1, \) and \( \tau_2; \) see Propositions 4.5 and 4.6 below for the precise details.

The rest of the paper is organized as follows. In Section 2, we show the nonexistence of supersolutions of a linear Hardy problem. In Section 3, we prove our main results Theorems 1.1 and 1.2. In Section 4, we apply Theorems 1.1 and 1.2 to two concrete examples to obtain sharp nonexistence results.
2 Nonexistence of positive supersolutions of a linear Hardy problem

In this section, we shall investigate the linear elliptic problem with Hardy potential:

\[
\begin{aligned}
-\Delta u &= V(x)u \quad \text{in } \Omega, \\
\quad u &> 0 \quad \text{on } \partial\Omega.
\end{aligned}
\]  

(2.1)

The nonexistence result of positive supersolutions to (2.1) reads as follows.

**Theorem 2.1.** Assume that \(\Omega\) is a punctured or an exterior domain, and \(V\) is a non-negative function satisfying

\[
\liminf_{|x| \to +\infty} V(x)|x|^2 > \frac{(N-2)^2}{4}.
\]  

(2.2)

Then Problem (2.1) has no positive supersolution.

We remark that \((N-2)^2/4\) is the best constant in the Hardy-Sobolev inequality

\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \geq \frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx.
\]

Related Hardy problems have been studied extensively; one may refer to, for example, [8–10, 15, 20]. In particular, Chen et al. [15] considered the Hardy problem \(-\Delta u = \frac{\mu}{|x|^p} u + f(x)\) in \(\mathcal{D} \setminus \{0\}\), subject to the zero Dirichlet boundary condition, where \(f \geq 0\) and \(\mathcal{D}\) is a bounded domain containing the origin, and proved that this problem has no positive solution once \(\mu > \frac{(N-2)^2}{4}\). To this end, they used new distributional identities to classify the isolated singular solutions of \(-\Delta u = \frac{\mu}{|x|^p} u + f\) in \(\mathcal{D} \setminus \{0\}\) and found fundamental solutions of \(-\Delta u = \frac{\mu}{|x|^p} u\) in \(\mathbb{R}^N\).

It is worth noting that [18, 29, 30] indicate that the nonexistence of positive supersolutions to (2.1) can be obtained by using Agmon-Allegretto-Piepenbrink theory [1]. We will provide a different and elementary proof. Our strategy is to employ the Kelvin transform to transfer the unbounded domain \(\Omega\) into a bounded one containing the origin.

For the linear elliptic equation involving the general homogeneous potential in the punctured domain \(\mathbb{R}^N \setminus \{0\}\):

\[
-\Delta u = \mu|x|^{-\alpha}u \quad \text{in } \mathbb{R}^N \setminus \{0\},
\]  

(2.3)

we obtain the following theorem.

**Theorem 2.2.** Problem (2.3) has no positive supersolution provided that one of the following conditions holds:

(i) \(\alpha \neq 2, \mu > 0\).

(ii) \(\alpha = 2, \mu > \frac{(N-2)^2}{4}\).

Theorem 2.2 is optimal in a certain sense, and it also reveals essential differences between Problem (1.1) with a punctured domain and Problem (1.1) with an exterior domain; see the following remark.

**Remark 2.3.** Concerning Theorem 2.2, we would like to make some comments as follows.

(i) When \(\alpha < 2\) and \(\mu > 0\), one can easily see from the proof of Theorem 2.2 that the linear problem

\[
\begin{aligned}
-\Delta u &= \mu|x|^{-\alpha}u \quad \text{in } \mathbb{R}^N \setminus B_\ell(0), \\
\quad u &> 0 \quad \text{on } \partial B_\ell(0)
\end{aligned}
\]  

(2.4)

has no positive supersolution for any \(\ell > 0\).

(ii) When \(\alpha > 2\) and \(\mu > 0\), Problem (2.4) has a positive supersolution for properly large \(\ell\). This can be seen from Proposition 4.3(i) below (by taking \(p = 1, \beta < -2, \tau = 0\) there). Such a result is in sharp contrast with the above (i) and Theorem 2.2(ii).

(iii) When \(\alpha = 2\) and \(\mu \leq (N-2)^2/4\), Problem (2.3) (and so Problem (2.4)) has a positive supersolution by considering the fundamental solutions of Hardy operators; one can refer to, for example, [15]. Hence, for \(\alpha = 2, \mu = (N-2)^2/4\) is the critical value of existence of positive supersolutions of Problems (2.3) and (2.4).
2.1 Proof of Theorem 2.1

In the case that $\beta = -2$ and so the critical exponent $p_0^* = 1$, Problem (2.1) is related to the Hardy-Leray potentials. The following result plays an essential role in the proof of Theorem 2.1.

**Lemma 2.4** (See [15, Proposition 5.2]). Assume that $\mu > (N - 2)^2/4$, $\mathcal{D}$ is a bounded smooth domain containing the origin and $f \in L^\infty_{\text{loc}}(\mathcal{D} \setminus \{0\})$ is a non-negative function. Then the Hardy problem

$$
\begin{cases}
-\Delta u = \frac{\mu}{|x|^2} u + f & \text{in } \mathcal{D} \setminus \{0\}, \\
\mu u = 0 & \text{on } \partial \mathcal{D}
\end{cases}
$$

(2.5)

has no positive solution.

**Proof of Theorem 2.1.** We argue indirectly and suppose that $u$ is a positive supersolution of (2.1). The main idea below is to reflect $\Omega$ to a bounded punctured domain through the Kelvin transform and then to obtain a contradiction by Lemma 2.4.

Without loss of generality, we may assume that $\Omega$ is a connected exterior domain satisfying $0 \notin \overline{\Omega}$. Denote

$$
\Omega^\sharp = \left\{ x \in \mathbb{R}^N : \frac{x}{|x|^2} \in \Omega \right\}
$$

and $v(x) = u\left(\frac{x}{|x|^2}\right)$ for $x \in \Omega^\sharp$.

Clearly, $\Omega^\sharp$ is a bounded punctured domain. By a direct computation, for $x \in \Omega^\sharp$, we have

$$
\nabla v(x) = \nabla u\left(\frac{x}{|x|^2}\right) + \frac{1}{|x|^2}\nabla u\left(\frac{x}{|x|^2}\right) \cdot x
$$

and

$$
\Delta v(x) = \frac{1}{|x|^4} \Delta u\left(\frac{x}{|x|^2}\right) + \frac{2(2 - N)}{|x|^4} \left(\nabla u\left(\frac{x}{|x|^2}\right) \cdot x\right).
$$

Let

$$
u^\sharp(x) = |x|^{2-N} v(x), \quad V^\sharp(x) = |x|^{-4} V\left(\frac{x}{|x|^2}\right).
$$

Then for $x \in \Omega^\sharp$, using the fact that $\Delta(|x|^{-2-N}) = 0$, we observe that

$$
-\Delta u^\sharp(x) = -\Delta v(x)|x|^{2-N} - 2 v(x) \cdot (\nabla |x|^{2-N})
\begin{align*}
&= \left[ \frac{1}{|x|^4} \Delta u\left(\frac{x}{|x|^2}\right) + \frac{2(2 - N)}{|x|^4} \left(\nabla u\left(\frac{x}{|x|^2}\right) \cdot x\right) \right]|x|^{2-N} \\
&\quad - \frac{2(2 - N)}{|x|^2} \nabla u\left(\frac{x}{|x|^2}\right) \cdot \frac{1}{|x|^2} - 2 \left(\nabla u\left(\frac{x}{|x|^2}\right) \cdot x\right) \frac{x}{|x|^2} |x|^{2-N} \\
&= |x|^{-2-N} - \Delta u\left(\frac{x}{|x|^2}\right) \\
&\geq V^\sharp(x) u^\sharp(x).
\end{align*}
$$

Set $V^*(x) = \frac{-\Delta u^\sharp(x)}{u^\sharp(x)}$. Thus, we notice that $V^*$ is continuous in $\Omega^\sharp$ and $V^* \geq V^\sharp$ in $\Omega^\sharp$. Therefore,

$$
V^*(x) \geq |x|^{-4} V\left(\frac{x}{|x|^2}\right), \quad \forall x \in \Omega^\sharp.
$$

Because of (2.2), it follows that

$$
\liminf_{|x| \to 0^+} V^*(x)|x|^2 > \frac{(N - 2)^2}{4},
$$

which in turn implies that there exist $\mu_1 > (N - 2)^2/4$ and $r_1 > 0$ such that

$$
V^*(x) \geq \mu_1 |x|^{-2}, \quad \forall x \in B_{r_1}(0) \setminus \{0\}.
$$
Denote \( u_0 = u^d - \varphi_0 \), where \( \varphi_0 \) is the unique positive solution of
\[
\begin{align*}
-\Delta \varphi_0 &= 0 \quad \text{in} \ B_{r_1}(0), \\
\varphi_0 &= u^d \quad \text{on} \ \partial B_{r_1}(0).
\end{align*}
\]
Then \( u_0 \) satisfies \( u_0 = 0 \) on \( \partial B_{r_1}(0) \) and
\[
-\Delta u_0(x) - \frac{\mu_2}{|x|^{2}} u_0 = V^*(x)u^d(x) - \frac{\mu_1}{|x|^2} u^d(x) + \frac{\mu_1}{|x|^2} \varphi_0(x) =: f^*(x) \geq 0, \quad \forall \ x \in B_{r_1}(0) \setminus \{0\}.
\]
As a consequence, \( u_0 \) is a positive solution of (2.5) with \( D = B_{r_1}(0), \mu = \mu_1 \) and \( f = f^* \). This contradicts Lemma 2.4. Thus, (2.1) admits no positive supersolution, and the proof is completed.

2.2 Proof of Theorem 2.2

Proof of Theorem 2.2. On the contrary, suppose that the problem
\[
-\Delta u = \mu |x|^{-\alpha} u \quad \text{in} \ \mathbb{R}^N \setminus \{0\}
\]
has a positive supersolution \( u_0 \), i.e.,
\[
-\Delta u_0 \geq \mu |x|^{-\alpha} u_0 \quad \text{in} \ \mathbb{R}^N \setminus \{0\} \quad \text{pointwise.} \tag{2.6}
\]
When \( \alpha = 2 \) and \( \mu > \frac{(N-2)^2}{4} \), a contradiction can be seen directly from Theorem 2.1. When \( \alpha < 2 \), (2.6) can be written as
\[
-\Delta u_0 \geq \mu |x|^{2-\alpha} |x|^{-2} u_0 \geq (N-2)^2 |x|^{-2} u_0 \quad \text{in} \ \mathbb{R}^N \setminus B_{r_\mu}(0),
\]
where
\[
r_\mu = \left[ \frac{(N-2)^2}{\mu} \right]^{\frac{1}{2-\alpha}} > 0.
\]
This contradicts Theorem 2.1.

When \( \alpha > 2 \), it follows from (2.6) that there exists \( \bar{f} \geq 0 \) such that
\[
-\Delta u_0 = \mu |x|^{2-\alpha} |x|^{-2} u_0 + \bar{f} \geq (N-2)^2 |x|^{-2} u_0 \quad \text{in} \ B_{r_\mu}(0) \setminus \{0\}.
\]
Denote by \( \varphi_0 \) the unique positive solution of
\[
\begin{align*}
-\Delta \varphi_0 &= 0 \quad \text{in} \ B_{r_\mu}(0), \\
\varphi_0 &= u_0 \quad \text{on} \ \partial B_{r_\mu}(0).
\end{align*}
\]
Then \( \varphi_0 \in C^2(B_{r_\mu}(0)) \cap C(\overline{B_{r_\mu}(0)}) \), and \( v_0 := u_0 - \varphi_0 \) is bounded from below and is a solution of
\[
\begin{align*}
-\Delta v_0 &= \bar{f} \quad \text{in} \ B_{r_\mu}(0) \setminus \{0\}, \\
v_0 &= 0 \quad \text{on} \ \partial B_{r_\mu}(0),
\end{align*}
\]
where \( \bar{f} = \mu |x|^{-\alpha} u_0 + \bar{f} > 0 \).

Next, we show \( v_0 > 0 \) in \( B_{r_\mu}(0) \setminus \{0\} \). Indeed, consider
\[
\begin{align*}
-\Delta z &= \mu |x|^{-\alpha} u_0 \chi_{B_{r_\mu}(0) \setminus B_{2r_\mu}(0)} \quad \text{in} \ B_{r_\mu}(0), \\
z &= 0 \quad \text{on} \ \partial B_{r_\mu}(0),
\end{align*}
\]
which admits a unique positive bounded solution, denoted by \( z_0 \). For any small \( \epsilon > 0 \), set
\[
\psi_\epsilon(x) = \epsilon (\mu |x|^{-\alpha} - |x|^{2-N}) + z_0(x).
\]
Then there exists $r_\varepsilon \in (0, r_\mu)$ such that $\lim_{\varepsilon \to 0^+} r_\varepsilon = 0$ and $\psi_\varepsilon \leq v_0$ on $\overline{B}_{r_\varepsilon}(0)$. By the classical comparison principle, we have $v_0 \geq \psi_\varepsilon$ in $B_{r_\varepsilon}(0) \setminus B_{r_\mu}(0)$. Sending $\varepsilon \to 0$, we can conclude $v_0 \geq z_0 > 0$ in $B_{r_\mu}(0) \setminus \{0\}$.

Furthermore, we see that $v_0 = 0$ on $\partial B_{r_\mu}(0)$ and

$$-\Delta v_0 = \mu |x|^{-\alpha} u_0 + \tilde{f} \geq (N-2)^2 \frac{1}{|x|^2} u_0 + F \quad \text{in } B_{r_\mu}(0) \setminus \{0\},$$

where

$$F(x) = \mu |x|^{-\alpha} \varphi_0 + \tilde{f} + (\mu |x|^{-\alpha} - (N-2)^2 \frac{1}{|x|^2}) u_0 \geq 0 \quad \text{for } 0 < |x| < r_\mu.$$  

Thus, we obtain a contradiction to Lemma 2.4 and the proof is completed. \qed

**Remark 2.5.** With a slight modification of the proof for the case $\alpha > 2$ in Theorem 2.2, we can observe the following result: for $\mu > (N-2)^2/4$ with the dimension $N \geq 2$, and $f \in L^\infty_{\text{loc}}(\overline{D} \setminus \{0\})$ a non-negative function with $D$ a bounded smooth domain containing the origin, then the Hardy problem (2.5) has no nontrivial non-negative supersolution.

## 3 Nonexistence of positive supersolutions of nonlinear problems

### 3.1 Proof of Theorem 1.1

Our proof is based on Theorem 2.1 and the following comparison principle.

**Lemma 3.1.** Assume that $\Omega$ is a smooth exterior domain, $f_1, f_2$ are continuous in $\Omega$, $g_1, g_2$ are continuous on $\partial \Omega$, and $f_1 \geq f_2$ in $\Omega$ and $g_1 \geq g_2$ on $\partial \Omega$. Let $u_1$ and $u_2$ satisfy $-\Delta u_1 \geq f_1$ in $\Omega$, $u_1 \geq g_1$ on $\partial \Omega$, and $-\Delta u_2 \leq f_2$ in $\Omega$, $u_2 \leq g_2$ on $\partial \Omega$. If $\liminf_{|x| \to +\infty} u_1(x) \geq \limsup_{|x| \to +\infty} u_2(x)$, then we have $u_1 \geq u_2$ on $\overline{\Omega}$.

**Proof.** Such a comparison principle may be folklore. We provide a simple proof here for the sake of completeness. Let $w = u_2 - u_1$, and then $w$ satisfies

$$-\Delta w \leq 0 \text{ in } \Omega, \quad w \leq 0 \text{ on } \partial \Omega \quad \text{and} \quad \limsup_{|x| \to +\infty} w(x) \leq 0.$$

Thus, given $\varepsilon > 0$, there exists $r_\varepsilon > 0$ converging to infinity as $\varepsilon \to 0$ such that

$$w \leq \varepsilon (r_\varepsilon^{2-N} + 1) \quad \text{on } \partial B_{r_\varepsilon}(0).$$

As a result, we have

$$w(x) \leq \varepsilon < \varepsilon (|x|^{2-N} + 1) \quad \text{on } \partial (\Omega \cap B_{r_\varepsilon}(0)).$$

Note that $\Delta(|x|^{2-N}) = 0$. It then follows from the classical comparison principle in any bounded smooth domain that

$$w(x) \leq \varepsilon (|x|^{2-N} + 1) \quad \text{in } \Omega \cap B_{r_\varepsilon}(0).$$

According to the arbitrariness of $\varepsilon > 0$, we can conclude that $w \leq 0$ on $\overline{\Omega}$. \qed
\[-\Delta w_0(x) = - \left[ w_\theta''(|x|) + w_\theta'(|x|) \cdot \frac{N-1}{|x|} \right] \\
= \sigma (N-2)|x|^{-N}(\ln |x|)^{\sigma-1} - \sigma (\sigma-1)|x|^{-N}(\ln |x|)^\sigma - 2 \\
\leq \frac{3}{2} \sigma (N-2)|x|^{-N}(\ln |x|)^{\sigma-1}.
\] (3.1)

Now, with the aid of the function \(w_0\), we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** We use a contradiction argument and suppose that \((1.1)\) has a positive super-solution \(u\). Since \(h(x, u) \geq \tilde{h}(x, u)\) for \((x, u) \in (\mathbb{R}^N \setminus B_\varepsilon(0)) \times [0, +\infty)\), we have that \(u\) fulfills

\[-\Delta u \geq \tilde{h}(x, u) \quad \text{in } \mathbb{R}^N \setminus B_\varepsilon(0).
\] (3.2)

By the positivity of \(u\), one can find a constant \(c_0 > 0\) such that

\[u(x) \geq c_0 e^{2-N}, \quad \forall x \in \partial B_\varepsilon(0).
\]

Hence, Lemma 3.1 gives

\[u(x) \geq c_0 |x|^{2-N}, \quad \forall |x| \geq \varepsilon.
\] (3.3)

By our assumption \((H)(a)\), one can see that

\[\frac{\tilde{h}(x, u(x))}{u(x)} \geq \frac{\tilde{h}(x,c_0 |x|^{2-N})}{c_0 |x|^{2-N}}, \quad \forall |x| \geq \varepsilon,
\]

and if \((H)(b)\) holds, then

\[\liminf_{|x| \to +\infty} \frac{\tilde{h}(x,c_0 |x|^{2-N})}{c_0 |x|^{2-N}} |x|^2 > \frac{(N-2)^2}{4}.
\] (3.4)

Take \(V(x) = \frac{\tilde{h}(x,u(x))}{u(x)}\), which satisfies \(\liminf_{|x| \to +\infty} V(x)|x|^2 > (N-2)^2/4\). It is easily observed that \(u\) is a positive supersolution of

\[-\Delta u = V(x)u \quad \text{in } \mathbb{R}^N \setminus B_\varepsilon(0).
\]

This contradicts Theorem 2.1. Thus, (3.2) has no positive supersolution provided \((H)\) holds.

If \((H)(b)\) fails, in the sequel we shall establish the nonexistence result in Theorem 1.1 using the assumptions \((H)(b1)\) and \((H)(b2)\). Under \((H)(b1)\), by (3.3), there exists \(\varrho_0 \geq \varepsilon\) such that

\[\tilde{h}(x,u(x)) \geq \tilde{h}(x,c_0 |x|^{2-N}) \frac{u(x)}{c_0 |x|^{2-N}} \geq \tilde{h}(x,c_0 |x|^{2-N})
\geq m_0 |x|^{-N}(\ln |x|)^{-\sigma_1}, \quad \forall |x| \geq \varrho_0,
\] (3.5)

where

\[m_0 = \min \left\{ \left. \frac{\tilde{h}(x,c_0 |x|^{2-N})}{c_0 |x|^{2-N}} |x|^N(\ln |x|)^{-\sigma_1} \right| \right\} > 0.
\]

In light of (3.1) (by taking \(\sigma = 1 - \sigma_1 > 0\) there), one can find \(t_1 \in (0, \frac{m_0}{3(N-2)^2\sigma_1})\) and \(\varrho_1 \geq \varrho_0\) such that

\[u(x) \geq t_1 w_0(x), \quad |x| = \varrho_1
\]

and

\[-\Delta u(x) \geq m_0 |x|^{-N}(\ln |x|)^{-\sigma_1} \geq -\Delta(t_1 w_0(x)), \quad \forall |x| > \varrho_1.
\]

Then by Lemma 3.1, it follows that

\[u(x) \geq t_1 |x|^{2-N}(\ln |x|)^{\theta_1}, \quad \forall |x| \geq \varrho_1
\] (3.6)

with \(\theta_1 = 1 - \sigma_1\).
As a next step, we are going to improve the decay of $u$ at infinity by an induction argument. To this end, let $\{\theta_j\}_j$ be the sequence generated by
\[
\theta_{j+1} = \theta_j + \theta_1 = (j+1)\theta_1, \quad j = 1, 2, 3, \ldots
\] (3.7)
In view of (H)(a), we may assume that $\sigma_2 \geq 1$, where $\sigma_2 > 0$ appears from the assumption (H)(b2). Furthermore, by observing that $\lim_{j \to +\infty} \theta_j = +\infty$, one can assert that there exists $j_0 \in \mathbb{N}$ such that
\[
\theta_{j_0} \geq \sigma_2 \quad \text{and} \quad \theta_{j_0} < \sigma_2.
\] (3.8)

We now claim that for any integer $j$, there exist $\varrho_j \geq \varrho_0$ and $\epsilon_j > 0$ such that
\[
u(x) \geq \epsilon_j |x|^{2-N}(\ln |x|)^{\theta_j}, \quad \forall |x| \geq \varrho_j.
\] (3.9)

Indeed, when $j = 1$, (3.9) has been proved above. Assume that (3.9) holds for some $j$. We will show that (3.9) holds true for $j+1$. Notice that there exists $r_j \geq \varrho_j$ such that
\[
\epsilon_j t_j |x|^{2-N}(\ln |x|)^{\theta_j} \geq c_0 |x|^{2-N}, \quad \forall |x| \geq r_j.
\]

By (H)(a), (H)(b1), (3.5) and (3.9), we obtain
\[
\tilde{h}(x, u(x)) \leq \hat{h}(x, t_j |x|^{2-N}(\ln |x|)^{\theta_j}) \frac{u(x)}{t_j |x|^{2-N}(\ln |x|)^{\theta_j}} \geq \hat{h}(x, t_j |x|^{2-N}(\ln |x|)^{\theta_j}) \geq \hat{h}(x, c_0 |x|^{2-N}) \frac{t_j |x|^{2-N}(\ln |x|)^{\theta_j}}{c_0 |x|^{2-N}} \geq m_0 \frac{t_j |x|^{2-N}(\ln |x|)^{\theta_j-\sigma_1}}{c_0 |x|^{2-N}}, \quad \forall |x| \geq r_j.
\]

Taking $\sigma = 1+\theta_j-\sigma_1 > 0$ in (3.1), and then using Lemma 3.1 (by comparing $u$ with $|x|^{2-N}(\ln |x|)^{\theta_j+\theta_1}$), we can conclude that $u(x) \geq t_{j+1} |x|^{2-N}(\ln |x|)^{\theta_j+\theta_1}, \forall |x| \geq \varrho_{j+1}$, for some $t_{j+1} > 0$ and $\varrho_{j+1} \geq r_j$. This verifies the previous claim (3.9).

Therefore, (3.8) and (3.9) imply that $u(x) \geq t|x|^{2-N}(\ln |x|)^{\sigma_2}, \forall |x| \geq \varrho_{j_0}$, and in turn by (H)(a),
\[
\frac{\tilde{h}(x, u(x))}{u(x)} \geq \frac{\hat{h}(x, t|x|^{2-N}(\ln |x|)^{\sigma_2})}{t|x|^{2-N}(\ln |x|)^{\sigma_2}} \geq \frac{1}{4} + \epsilon_0, \quad \forall x \in \mathbb{R}^N \setminus B_{\varrho_{j_0}}(0).
\] (3.10)

By (H)(b2), there exist $\varrho > \varrho_{j_0}$ and $\epsilon_0 > 0$ such that
\[
\frac{\tilde{h}(x, u(x))}{u(x)} \geq \frac{(N-2)^2}{4} + \epsilon_0, \quad \forall x \in \mathbb{R}^N \setminus B_{\varrho}(0).
\]

Let $V(x) = \frac{\tilde{h}(x, u(x))}{u(x)}$. Then
\[
V(x)|x|^2 > \frac{(N-2)^2}{4} + \epsilon_0, \quad \forall x \in \mathbb{R}^N \setminus B_{\varrho}(0).
\]

Clearly, $u$ is a positive supersolution of $-\Delta u = V(x)u$ in $\mathbb{R}^N \setminus B_\varrho(0)$, which is contrary to Theorem 2.1. Thus, (1.1) admits no positive supersolution. Then the proof is completed. \hfill \square

3.2 Proof of Theorem 1.2

Proof of Theorem 1.2. Suppose that (1.3) has a positive supersolution $(u, v)$. Clearly, $(u, v)$ fulfills
\[
-\Delta u \geq \tilde{h}_1(x, u, v) \geq 0 \quad \text{and} \quad -\Delta v \geq \tilde{h}_2(x, u, v) \geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_\varrho(0).
\] (3.11)
Since one can find a small $c_0 > 0$ such that
\[ u(x) \geq c_0 e^{2-N}, \quad v(x) \geq c_0 e^{2-N}, \quad \forall x \in \partial B_e(0), \]
an analysis similar to that of obtaining (3.3) yields that
\[ u(x) \geq c_0 |x|^{2-N}, \quad v(x) \geq c_0 |x|^{2-N}, \quad \forall |x| \geq e. \]  
(3.12)

Thus, it follows from (SH)(a) that
\[ \frac{\tilde{h}_1(x, u(x), v(x))}{u(x)} \geq \frac{\tilde{h}_1(x, u(x), c_0 |x|^{2-N})}{u(x)} \geq \frac{\tilde{h}_1(x, c_0 |x|^{2-N}, c_0 |x|^{2-N})}{c_0 |x|^{2-N}}, \quad \forall |x| \geq e \]
and
\[ \frac{\tilde{h}_2(x, u(x), v(x))}{v(x)} \geq \frac{\tilde{h}_2(x, c_0 |x|^{2-N}, v(x))}{v(x)} \geq \frac{\tilde{h}_2(x, c_0 |x|^{2-N}, c_0 |x|^{2-N})}{c_0 |x|^{2-N}}, \quad \forall |x| \geq e. \]

If (SH)(b) holds, we then have
\[ \liminf_{|x| \to +\infty} \frac{\tilde{h}_1(x, c_0 |x|^{2-N}, c_0 |x|^{2-N})}{c_0 |x|^{2-N}} |x|^2 > \frac{(N-2)^2}{4} \]  
(3.13)

or
\[ \liminf_{|x| \to +\infty} \frac{\tilde{h}_2(x, c_0 |x|^{2-N}, c_0 |x|^{2-N})}{c_0 |x|^{2-N}} |x|^2 > \frac{(N-2)^2}{4}. \]  
(3.14)

By taking $V(x) = \frac{\tilde{h}_1(x, u(x), v(x))}{u(x)}$ if (3.13) holds (or $V(x) = \frac{\tilde{h}_1(x, u(x), v(x))}{v(x)}$ if (3.14) holds), we see that $u$ (or $v$) is a positive supersolution of $-\Delta u = V(x)u$ in $\mathbb{R}^N \setminus B_e(0)$ with $\liminf_{|x| \to +\infty} V(x)|x|^2 > \frac{(N-2)^2}{4}$. This is impossible due to Theorem 2.1. Therefore, (1.3) has no positive supersolution.

If (SH)(b) fails, we continue to prove the nonexistence result in Theorem 1.2 under the assumptions (SH)(b1)–(SH)(b3). There are three cases to distinguish as follows.

**Case 1.** $0 < \sigma_3 < 1$, $\sigma_4 \geq 1$. By (3.12) and the assumption (SH)(a), we have
\[ \tilde{h}_1(x, u(x), v(x)) \geq \tilde{h}_1(x, u(x), c_0 |x|^{2-N}) =: \tilde{h}(x, u), \quad \forall |x| > e. \]

Then $u$ verifies that
\[ -\Delta u \geq \tilde{h}(x, u) \quad \text{in} \quad \mathbb{R}^N \setminus B_e(0). \]  
(3.15)

By virtue of the assumptions (SH)(a) and (SH)(b1), there exists $\rho_0 \geq e$ such that
\[ \tilde{h}(x, u) \geq \tilde{h}_1(x, c_0 |x|^{2-N}, c_0 |x|^{2-N}) \geq \frac{1}{2} m_1 |x|^{-N} (\ln |x|)^{-\sigma_3}, \quad \forall |x| \geq \rho_0, \]  
(3.16)

where
\[ m_1 = \min \left\{ 1, \liminf_{|x| \to +\infty} \tilde{h}_1(x, c_0 |x|^{2-N}, c_0 |x|^{2-N}) |x|^N (\ln |x|)^{\sigma_3} \right\}. \]

This implies that (H)(b1) holds. Moreover, (SH)(b2) indicates that $\tilde{h}$ satisfies (H)(b2). Thus, an application of Theorem 1.1 to Problem (3.15) leads to a contradiction. Hence, (1.3) has no positive supersolution in Case 1.

**Case 2.** $\sigma_3 \geq 1$, $0 < \sigma_4 < 1$. The proof is similar to Case 1.

**Case 3.** $0 < \sigma_3, \sigma_4 < 1$. Due to (SH)(b1), we can deduce
\[ \tilde{h}_1(x, u(x), v(x)) \geq \frac{1}{2} m_i |x|^{-N} (\ln |x|)^{-\sigma_3}, \quad \forall |x| \geq \rho_0, \]
for some $\rho_0 > e$ and
\[ m_i = \min \left\{ 1, \liminf_{|x| \to +\infty} \tilde{h}_1(x, c_0 |x|^{2-N}, c_0 |x|^{2-N}) |x|^N (\ln |x|)^{\sigma_3} \right\}. \]
Proceeding similarly as proving (3.6), one can assert that for some \( t_0 > 0 \) and \( \rho_1 > \rho_0 \),
\[
    u(x) \geq t_0 |x|^{2-N} (\ln |x|)^{1-\sigma_3} \quad \text{and} \quad v(x) \geq t_0 |x|^{2-N} (\ln |x|)^{1-\sigma_4} \quad \text{in} \quad \mathbb{R}^N \setminus B_{\rho_1}(0).
\]
(3.17)

Then, reasoning as in the part of the claim (3.9), we have
\[
    u(x) \geq t_j |x|^{2-N} (\ln |x|)^{j(1-\sigma_3)} \quad \text{and} \quad v(x) \geq t_j |x|^{2-N} (\ln |x|)^{j(1-\sigma_4)}, \quad \forall |x| \geq \rho_j
\]
for a sequence \( \{(t_j, \rho_j)\}_{j=1}^\infty \). Therefore, there exists a large integer \( j^* \) such that
\[
    j^*(1-\sigma_3) > \sigma_5 \quad \text{and} \quad j^*(1-\sigma_4) > \sigma_6.
\]

Thus, we obtain \( u(x) \geq t_{j^*} |x|^{2-N} (\ln |x|)^{\sigma_5}, v(x) \geq t_{j^*} |x|^{2-N} (\ln |x|)^{\sigma_6}, \forall |x| \geq \rho_{j^*} \). This then yields
\[
    \frac{\hat{h}_1(x, u(x), v(x))}{u(x)} \geq \frac{\hat{h}_1(x, t_{j^*} |x|^{2-N} (\ln |x|)^{\sigma_5}, t_{j^*} |x|^{2-N} (\ln |x|)^{\sigma_6})}{t_{j^*} |x|^{2-N} (\ln |x|)^{\sigma_5}}, \quad \forall |x| \geq \rho_{j^*},
\]
and
\[
    \frac{\hat{h}_2(x, u(x), v(x))}{u(x)} \geq \frac{\hat{h}_2(x, t_{j^*} |x|^{2-N} (\ln |x|)^{\sigma_5}, t_{j^*} |x|^{2-N} (\ln |x|)^{\sigma_6})}{t_{j^*} |x|^{2-N} (\ln |x|)^{\sigma_5}}, \quad \forall |x| \geq \rho_{j^*}.
\]

By the assumption (SH)(b3), there exist \( q > \rho_{j^*} \) and \( \epsilon_0 > 0 \) such that
\[
    \frac{\hat{h}_1(x, u(x), v(x))}{u(x)} \quad \text{or} \quad \frac{\hat{h}_2(x, u(x), v(x))}{v(x)} > \left( \frac{(N-2)^2}{4} + \epsilon_0 \right) \frac{1}{|x|^2}, \quad \forall x \in \mathbb{R}^N \setminus B_\rho(0).
\]
Taking \( V(x) = \frac{h_1(x, u(x), v(x))}{u(x)} \) (or \( V(x) = \frac{h_2(x, u(x), v(x))}{v(x)} \)), we have that \( u \) is a positive supersolution of
\[
    -\Delta u = V(x)u \quad \text{in} \quad \mathbb{R}^N \setminus B_\rho(0),
\]
contradicting Theorem 2.1. As a consequence, (1.3) has no positive supersolution in Case 3. The proof is now completed.
\[\square\]

4 Application: Two examples

In this section, we shall use two typical examples to illustrate the optimality of the nonexistence results obtained by this paper.

Example 4.1. \( h(x, u) = |x|^{\beta} (\ln |x|)^{\tau} u^p. \)

When \( \beta > -2 \), then \( p_3^* = \frac{N+\beta}{N-2} > 1 \). We have the following proposition.

Proposition 4.2. Assume that \( h(x, u) = |x|^{\beta} (\ln |x|)^{\tau} u^p, x \in \mathbb{R}^N \setminus B_0(0) \) with \( \beta > -2 \). Then the following assertions hold.

(i) Problem (1.1) has no positive supersolution provided that either \( 1 \leq p < p_3^*, \tau \in \mathbb{R} \) or \( p = p_3^*, \tau > -1 \).

(ii) Problem (1.1) has no positive bounded supersolution provided that \( p \in (-\infty, 1), \tau \in \mathbb{R}. \)

Proof. We shall apply Theorem 1.1 to obtain the desired results. It suffices to check the condition (H).

We first verify (i). When \( p = 1 \), the nonexistence follows from Theorem 2.1 directly. Clearly, the condition (H)(a) is fulfilled once \( p > 1 \). We also note that
\[
    h(x, |x|^{2-N}) |x|^N = |x|^\beta + N - (N-2)p (\ln |x|)\tau, \quad \forall |x| > e.
\]
If \( p < p_3^* \), then \( \beta + N - (N-2)p > 0 \). So for any \( t > 0 \) and \( \tau \in \mathbb{R}, \) then \( \lim_{|x| \rightarrow +\infty} h(x, t |x|^{2-N}) |x|^N = +\infty. \)

Thus, the assumption (H)(b) is satisfied.

If \( p = p_3^* \) and \( \tau \in (-1, 0) \), by taking \( \sigma_1 = -\tau > 0 \), we have
\[
    h(x, t |x|^{2-N}) |x|^N (\ln |x|)^{\sigma_1} = t^{\beta_3} (\ln |x|)^{\tau+\sigma_1} = t^{\beta_3}, \quad \forall |x| > e.
\]
If \( p = p_\beta^* \) and \( \tau \geq 0 \), by taking \( \sigma_1 = \frac{1}{2} > 0 \), we have
\[
\hat{h}(x, t|x|^{2-N})|x|^N(\ln |x|)^{\sigma_1} = t^{p_\beta^*}(\ln |x|)^{\frac{\tau + \frac{1}{2}}{2}} \geq t^{p_\beta^*}, \quad \forall |x| > e.
\]

Furthermore, let us choose \( \sigma_2 > 0 \) such that \( \tau + \sigma_2(p_\beta^* - 1) > 0 \). Then
\[
\frac{h(x, t|x|^{2-N}(\ln |x|)^{\sigma_2})}{t|x|^{-N}(\ln |x|)^{\sigma_2}} = t^{p_\beta^*-1}(\ln |x|)^{\tau + \sigma_2(p_\beta^*-1)}, \quad \forall |x| > e.
\]

Therefore, \( h \) satisfies the assumptions (H)(b1) and (H)(b2). Thus, the assertion (i) is proved.

We next verify the assertion (ii). Arguing indirectly, suppose that (1.1) has a positive bounded supersolution \( u \). Denote \( \hat{h}(x, u) = M^{p-1}|x|^{\beta}(\ln |x|)^{\tau}u, \) if \( p \in (-\infty, 1], \tau \in \mathbb{R}, \) where \( M = 1 + \sup_{x \in \Omega} u(x) > 0 \).

Then, it is easily checked that
\[
h(x, u) = |x|^{\beta}(\ln |x|)^{\tau}u^p \geq M^{p-1}|x|^{\beta}(\ln |x|)^{\tau}u = \hat{h}(x, u), \quad \forall |x| > e, \quad u \in (0, M],
\]
and \( \hat{h} \) fulfills the assumptions (H)(a) and (H)(b) for any \( \tau \in \mathbb{R} \). Consequently, \( u \) is a positive supersolution of
\[
-\Delta u = \hat{h}(x, u) \quad \text{in} \quad \mathbb{R}^N \setminus B_e(0).
\]
However, (4.1) has no positive supersolution by Theorem 1.1. Such a contradiction implies that (1.1) has no positive bounded supersolution. The assertion (ii) follows.

**Proposition 4.3.** Assume that \( h(x, u) = |x|^{\beta}(\ln |x|)^{\tau}u^p, \forall x \in \mathbb{R}^N \setminus B_t(0), \) where \( \beta \in \mathbb{R} \). Then for some large \( \ell > e \), Problem (1.1) with \( \Omega = \mathbb{R}^N \setminus B_t(0) \) has a positive bounded supersolution if one of the following conditions is satisfied:

(i) \( p > p_\beta^* \) and \( \tau \in \mathbb{R} \).

(ii) \( p = p_\beta^* \) and \( \tau < -1 \).

**Proof.** Recall that \( w_0(x) = |x|^{2-N}(\ln |x|)^{\sigma} \) with \( \sigma > 0 \). If \( |x| > \max\{1, \exp\left(\frac{2(\ell - 1)}{N - 2}\right)\} \), it immediately follows from (3.1) that
\[
-\Delta w_0(x) = \sigma(N - 2)|x|^{-N}(\ln |x|)^{\sigma-1} - t\sigma(\sigma - 1)|x|^{-N}(\ln |x|)^{\sigma-2} \geq \frac{1}{2}\sigma(N - 2)|x|^{-N}(\ln |x|)^{\sigma-1},
\]
and
\[
h(x, w_0) = |x|^{\beta}(\ln |x|)^{\tau}(|x|^{2-N}(\ln |x|)^{\sigma})^p = |x|^{\beta + (2-N)p}(\ln |x|)^{\tau + \sigma p}.
\]

When \( p > p_\beta^* \), then \( \beta + (2-N)p < -N \). One can easily see that there exists a large constant \( \ell > e \) such that \( -\Delta w_0(x) \geq h(x, w_0(x)), \forall |x| > \ell \). Hence, \( w_0 \) is a desired supersolution.

When \( p = p_\beta^* \), we have \( \beta + (2-N)p = -N \). Similar to the above, for some large \( \ell > e \), Problem (1.1) has a positive supersolution \( w_0 \) if \( \sigma - 1 > \tau + p_\beta^* \), i.e.,
\[
\sigma(p_\beta^* - 1) < -\tau - 1,
\]
(4.2)

where \( -\tau - 1 > 0 \) by our assumption \( \tau < -1 \). So when \( \beta > -2 \), (4.2) holds if we take \( \sigma > \frac{-\tau - 1}{p_\beta^* - 1} \). When \( \beta \leq -2 \), then \( p_\beta^* - 1 \leq 0 \) and (4.2) is satisfied if we just take \( \sigma = 1 \).

In each case, the supersolution \( w_0 \) is bounded in \( \mathbb{R}^N \setminus B_t(0) \). \( \square \)

Our second example is the following one:

**Example 4.4.** \( h_1(x, u, v) = |x|^{\beta_1}(\ln |x|)^{\tau_1}u^{p_1}v^{q_1}, \) \( h_2(x, u, v) = |x|^{\beta_2}(\ln |x|)^{\tau_2}u^{p_2}v^{q_2} \).

**Proposition 4.5.** Assume that
\[
h_1(x, u, v) = |x|^{\beta_1}(\ln |x|)^{\tau_1}u^{p_1}v^{q_1}, \quad h_2(x, u, v) = |x|^{\beta_2}(\ln |x|)^{\tau_2}u^{p_2}v^{q_2}, \quad \forall x \in \mathbb{R}^N \setminus B_e(0),
\]
where \( p_1, q_1 \geq 1, p_2, q_2 \geq 0, \beta_1, \beta_2 > -2 \) and \( \tau_1, \tau_2 \in \mathbb{R} \). **Problem (1.3) has no positive supersolution if one of the following conditions holds:**

(i) \( p_1 + q_1 < p_{\beta_1}^*, \tau_1, \tau_2 \in \mathbb{R} \).
(ii) \( p_2 + q_2 < p_2^*, \tau_1, \tau_2 \in \mathbb{R} \).

(iii) \( p_1 + q_1 = p_2^*, p_2 + q_2 = p_2^*, p_1 > 1, \tau_1 > -1, \tau_2 \in \mathbb{R} \).

(iv) \( p_1 + q_1 = p_2^*, p_2 + q_2 = p_2^*, q_2 > 1, \tau_2 > -1, \tau_1 \in \mathbb{R} \).

(v) \( p_1 + q_1 = p_2^*, p_2 + q_2 = p_2^*, \tau_1 > -1, \tau_2 > -1 \).

Proof. In order to apply Theorem 1.2, we only need to check that the nonlinearities \( h_1, h_2 \) satisfy (SH).

First of all, when \( p_1, q_2 \geq 1, h_1 \) and \( h_2 \) satisfy (SH)(a). We further note that

\[
\begin{align*}
    h_1(x, s|x|^{-2N}, s|x|^{-2N})|x|^N &= s^{p_1+q_1}|x|^{\beta_1+-(N-2)(p_1+q_1)}(\ln |x|)^{\tau_1}, \quad \forall |x| > e, \\
    h_2(x, s|x|^{-2N}, s|x|^{-2N})|x|^N &= s^{p_2+q_2}|x|^{\beta_2+-(N-2)(p_2+q_2)}(\ln |x|)^{\tau_2}, \quad \forall |x| > e.
\end{align*}
\]

Condition (i) implies that \( \beta_1 + N - (N - 2)(p_1 + q_1) > 0 \) and Condition (ii) implies that \( \beta_2 + N - (N - 2)(p_2 + q_2) > 0 \). Hence, in each of these cases, (SH)(b) holds, so (1.3) has no positive supersolution. Thus, the proposition holds in the cases (i) and (ii).

When \( p_1 + q_1 = p_2^*, p_2 + q_2 = p_2^* \), we see that

\[
\begin{align*}
    h_1(x, t|x|^{-2N}, t|x|^{-2N})|x|^N &= t^{p_1+q_1}(\ln |x|)^{\tau_1}, \quad \forall |x| > e, \\
    h_2(x, t|x|^{-2N}, t|x|^{-2N})|x|^N &= t^{p_2+q_2}(\ln |x|)^{\tau_2}, \quad \forall |x| > e.
\end{align*}
\]

Let \( \sigma_3 = -\tau_1 \) if \( -1 < \tau_1 < 0 \) and \( \sigma_4 = -\tau_2 \) if \( -1 < \tau_2 < 0 \). Otherwise, we let \( \sigma_3 = \sigma_4 = \frac{1}{2} \). Then for \( |x| > e \), we find that

\[
\begin{align*}
    h_1(x, t|x|^{-2N}, t|x|^{-2N})|x|^N(\ln |x|)^{\sigma_3} &\geq t^{p_1+q_1}, \\
    h_2(x, t|x|^{-2N}, t|x|^{-2N})|x|^N(\ln |x|)^{\sigma_4} &\geq t^{p_2+q_2}.
\end{align*}
\]

Furthermore, if \( p_1 > 1 \), for any \( t > 0 \) we obtain that

\[
\frac{h_1(x, t|x|^{-2N}(\ln |x|)^{\sigma_3}, t|x|^{-2N})}{t|x|^{-N}(\ln |x|)^{\sigma_3}} = t^{p_1+q_1-1}(\ln |x|)^{\tau_1+\sigma_3(p_1-1)} \rightarrow +\infty, \quad \text{as } |x| \rightarrow +\infty
\]

by choosing \( \sigma_3 > 0 \) so that \( \tau_1 + \sigma_3(p_1-1) > 0 \).

If \( q_2 > 1 \), for any \( t > 0 \) we derive that

\[
\frac{h_2(x, t|x|^{-2N}(\ln |x|)^{\sigma_6}, t|x|^{-2N}(\ln |x|)^{\sigma_6})}{t|x|^{-N}(\ln |x|)^{\sigma_6}} = s^{p_2+q_2-1}(\ln |x|)^{\tau_2+\sigma_6(q_2-1)} \rightarrow +\infty, \quad \text{as } |x| \rightarrow +\infty
\]

by choosing \( \sigma_6 > 0 \) so that \( \tau_2 + \sigma_6(q_2-1) > 0 \).

If \( p_1 = q_2 = 1 \), then for any \( t > 0 \), it holds that

\[
\begin{align*}
    h_1(x, t|x|^{-2N}(\ln |x|)^{\sigma_3}, t|x|^{-2N}(\ln |x|)^{\sigma_3}) &= t^{q_1}(\ln |x|)^{\tau_1+\sigma_3q_1}, \quad \forall |x| > e, \\
    h_2(x, t|x|^{-2N}(\ln |x|)^{\sigma_6}, t|x|^{-2N}(\ln |x|)^{\sigma_6}) &= t^{q_1}(\ln |x|)^{\tau_2+\sigma_6q_2}, \quad \forall |x| > e.
\end{align*}
\]

Since \( p_2, q_1 \geq 0 \) and \( p_2 + q_1 > 0 \), we may choose either \( \sigma_3 > 0 \) or \( \sigma_6 > 0 \) such that

\( \tau_2 + \sigma_3p_2 > 0 \) or \( \tau_1 + \sigma_6q_1 > 0 \).

According to the above analysis, it is easily seen that (SH)(b1)–(SH)(b3) are fulfilled when one of the cases (iii), (iv) and (v) holds. Thus, Theorem 1.2 can be applied to assert that (1.3) has no positive supersolution in each of such cases. The proof is completed.

In what follows, we will establish the existence of positive supersolutions to System (1.3). For the sake of convenience, denote

\[
\begin{align*}
    \sigma_1 &= \frac{(\tau_2 + 1)q_1 - (\tau_1 + 1)(q_2 - 1)}{(p_1 - 1)(q_2 - 1) - p_2q_1} \quad \text{and} \quad \sigma_2 = \frac{(\tau_1 + 1)p_2 - (\tau_2 + 1)(p_1 - 1)}{(p_1 - 1)(q_2 - 1) - p_2q_1}.
\end{align*}
\]
Proposition 4.6. Assume that
\[ h_1(x, u, v) = |x|^{\beta_1 (\ln |x|)^{\tau_1}} u^{p_1} v^{q_1}, \quad h_2(x, u, v) = |x|^{\beta_2 (\ln |x|)^{\tau_2}} u^{p_2} v^{q_2}. \]

Problem (1.3) has a positive supersolution if \( \beta_1, \beta_2 \in \mathbb{R} \), and
\[ p_1 + q_1 \geq p_1^{\beta_1}, \quad p_2 + q_2 \geq p_2^{\beta_2}, \quad \sigma_1 > 0, \quad \sigma_2 > 0. \]

Proof. Set \( w_i(x) = |x|^{2-N(\ln |x|)^{\tau_i}} \) with \( \tau_i > 0 \), \( i = 1, 2 \). If \( |x| > \max\{1, e^{2(\max(\beta_1, \beta_2)-1)/N-1}\} \), it follows from (3.1) that
\[ -\Delta(t w_i(x)) = t\sigma_i(N-2)|x|^{-N(\ln |x|)^{\sigma_i,-1}} - t\sigma_i(\sigma_i-1)|x|^{-N(\ln |x|)^{\sigma_i,-2}} \]
\[ \geq \frac{1}{2} t\sigma_i(N-2)|x|^{-N(\ln |x|)^{\sigma_i,-1}}, \quad i = 1, 2 \]
and since \( p_1 + q_1 \geq p_1^{\beta_1}, \) we have that \( h_2(x, tw_1, tw_2) \geq p_2 + q_2 |x|^{-N(\ln |x|)^{\sigma_2}}. \)

Similarly, by \( p_2 + q_2 \geq p_2^{\beta_2}, \) we have that \( h_2(x, tw_1, tw_2) \geq t^{p_1+q_1} |x|^{(2-N)(\beta_1+\beta_2)} \). In view of the definitions of \( \sigma_1 \) and \( \sigma_2, \)
\[ \tau_1 + \sigma_1 p_1 + \sigma_2 q_1 = \sigma_1 - 1, \quad \tau_2 + \sigma_1 p_2 + \sigma_2 q_2 = \sigma_2 - 1. \quad (4.4) \]

By choosing \( t = \min\{[1/2]^{\beta_1(N-2)}|\tau_1+\sigma_1|, [1/2]^{\beta_2(N-2)}|\tau_2+\sigma_2|\} \), we then see that \( (tw_1, tw_2) \) is a supersolution of (1.3).

Remark 4.7. Concerning the condition \( \sigma_1, \sigma_2 > 0 \) imposed in Proposition 4.3, we want to make some comments as follows.

(i) When \( p_1 = q_2 = 1 \), we have \( \sigma_1 = -\tau_1 + 1 > 0 \) and \( \sigma_2 = -\tau_2 + 1 > 0 \) if \( \tau_1, \tau_2 < -1, p_2, q_1 > 0 \).

(ii) When \( q_2 = 1 \) and \( \tau_1 = -1 \), we have \( \sigma_1 = -\tau_2 + 1 > 0 \) and \( \sigma_2 = (\tau_2 + 1)(p_2^{-1}-1) > 0 \) if \( \tau_2 < -1, p_1 < 1, p_2, q_1 > 0 \).

Acknowledgements The first author was supported by National Natural Science Foundation of China (Grant Nos. 11726614 and 11661045), and Jiangxi Provincial Natural Science Foundation (Grant No. 20161ACB20007). The second author was supported by National Natural Science Foundation of China (Grant Nos. 11671175 and 11571200), the Priority Academic Program Development of Jiangsu Higher Education Institutions, Top-notch Academic Programs Project of Jiangsu Higher Education Institutions (Grant No. PPZY2015A013) and Qing Lan Project of Jiangsu Province. The third author was supported by National Natural Science Foundation of China (Grant Nos. 11726613, 11271133 and 11431005) and Science and Technology Commission of Shanghai Municipality (STCSM) (Grant No. 13dz2260400). The authors thank Professor Yehuda Pinchover for bringing to our attention some existing work on the linear Hardy potential problem (2.1) such as [18,29,30] and the two referees for their careful reading and helpful suggestions, which helped to improve the exposition of the paper.

References
1 Agmon S. On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds. In: Methods of Functional Analysis and Theory of Elliptic Equations. Naples: Liguori, 1983, 19–52
2 Alarcón S, García-Melián J, Quaas A. Optimal Liouville theorems for supersolutions of elliptic equations with the Laplacian. Ann Sc Norm Super Pisa Cl Sci (5), 2016, 16: 129–158
3 Armstrong S, Sirakov B. Nonexistence of positive supersolutions of elliptic equations via the maximum principle. Comm Partial Differential Equations, 2011, 36: 2011–2047
4 Armstrong S, Sirakov B. Sharp Liouville results for fully nonlinear equations with power-growth nonlinearities. Ann Sc Norm Super Pisa Cl Sci (5), 2011, 10: 711–728
5 Bae S, Kwon O. Nonexistence of positive solutions of nonlinear elliptic systems with potentials vanishing at infinity. Nonlinear Anal, 2012, 75: 4025–4032
6 Bidaut-Véron M. Local behaviour of solutions of a class of nonlinear elliptic systems. Adv Differential Equations, 2000, 5: 147–192
7 Bidaut-Véron M, Pohozaev S. Nonexistence results and estimates for some nonlinear elliptic problems. J Anal Math, 2001, 84: 1–49
8 Brezis H, Dupaigne L, Tesei A. On a semilinear elliptic equation with inverse-square potential. Selecta Math, 2005, 11: 1–7
9 Brezis H, Vázquez L. Blow-up solutions of some nonlinear elliptic problems. Rev Mat Univ Complut Madrid, 1997, 10: 443–469
10 Chaudhuri N, Cîrstea F. On trichotomy of positive singular solutions associated with the Hardy-Sobolev operator. C R Math Acad Sci Paris, 2009, 347: 153–158
11 Chen H. Liouville theorem for the fractional Lane-Emden equation in an unbounded domain. J Math Pures Appl (9), 2018, 111: 21–46
12 Chen H, Alhomadan S, Hajaiej H, et al. Fundamental solutions for Schrödinger operators with general inverse square potentials. Appl Anal, 2018, 97: 787–810
13 Chen H, Felmer P. On Liouville type theorems for fully nonlinear elliptic equations with gradient term. J Differential Equations, 2013, 255: 2167–2195
14 Chen H, Felmer P, Yang J. Weak solutions of semilinear elliptic equation involving Dirac mass. Ann Inst H Poincaré Anal Non Linéaire, 2018, 35: 729–750
15 Chen H, Quaas A, Zhou F. On nonhomogeneous elliptic equations with the Hardy-Leray potentials. ArXiv:1705.08047, 2017
16 De Figueiredo D, Felmer P. A Liouville-type theorem for elliptic systems. Ann Sc Norm Super Pisa Cl Sci (5), 1994, 21: 387–397
17 Deng Y, Li Y, Yang F. A note on the positive solutions of an inhomogeneous elliptic equation on $\mathbb{R}^N$. J Differential Equations, 2009, 246: 670–680
18 Devyver B, Fraas M, Pinchover Y. Optimal Hardy weight for second-order elliptic operator: An answer to a problem of Agmon. J Funct Anal, 2014, 266: 4422–4489
19 Du Y, Guo Z. Positive solutions of an elliptic equation with negative exponent: Stability and critical power. J Differential Equations, 2009, 246: 2387–2414
20 Dupaigne L. A nonlinear elliptic PDE with the inverse square potential. J Anal Math, 2002, 86: 359–398
21 Esposito P, Ghoussoub N, Guo Y. Compactness along the branch of semistable and unstable solutions for an elliptic problem with a singular nonlinearity. Comm Pure Appl Math, 2007, 60: 1731–1768
22 Fazly M, Ghoussoub N. On the Hénon-Lane-Emden conjecture. Discrete Contin Dyn Syst, 2013, 34: 2513–2533
23 García-Huidobro M, Yarur C. Classification of positive singular solutions for a class of semilinear elliptic systems. Adv Differential Equations, 1997, 2: 383–402
24 Guo Y. Global solutions of singular parabolic equations arising from electrostatic MEMS. J Differential Equations, 2008, 245: 809–844
25 Li Y. Asymptotic behavior of positive solutions of equation $-\Delta u + K(x)u^p = 0$ in $\mathbb{R}^N$. J Differential Equations, 1992, 95: 304–330
26 Li Y. On the positive solutions of the Matukuma equation. Duke Math J, 1993, 70: 575–590
27 Liu Y, Li Y, Deng Y. Separation property of solutions for a semilinear elliptic equation. J Differential Equations, 2000, 163: 381–406
28 Meadows A. Stable and singular solutions of the equation $-\Delta u = \frac{1}{u}$. Indiana Univ Math J, 2004, 53: 1419–1430
29 Pinchover Y. A Liouville-type theorem for Schrödinger operators. Comm Math Phys, 2007, 272: 75–84
30 Pinchover Y. Topics in the theory of positive solutions of second-order elliptic and parabolic partial differential equations. In: Proceedings of Symposia in Pure Mathematics, vol. 76. Providence: Amer Math Soc, 2007, 329–355
31 Quaas A, Sirakov B. Existence and nonexistence results for fully nonlinear elliptic systems. Indiana Univ Math J, 2009, 58: 751–788
32 Serrin J, Zou H. Non-existence of positive solutions of semilinear elliptic systems. Discourses Math Appl, 1994, 3: 55–68
33 Souto M, Souplet P. The proof of the Lane-Emden conjecture in four space dimensions. Adv Math, 2009, 221: 1409–1427