X-ray emission from relativistic electrons in a transverse high intensity optical lattice

I.A. Andriyash
E-mail: igor.andriyash@gmail.com

E. d’Humières

V.T. Tikhonchuk

Ph. Balcou
Univ. Bordeaux, CNRS, CEA, CELIA (Centre Lasers Intenses et Applications), UMR 5107, F-33400 Talence, France

Abstract. A new scheme of all-optical free electron laser (FEL), operating in the X-ray range and triggered by interaction of a moderately relativistic electron bunch with a transverse high intensity optical lattice is presented. The ponderomotive potential of the optical lattice forms guiding channels, where the electrons are trapped and oscillate transversely. The collective oscillations of the guided electron bunch scatter the laser beams and amplify the signal at the Stokes frequency, thereby yielding a coherent beam at the Doppler-shifted frequency in the XUV or X-ray domain. Analytic descriptions of a small signal amplification in such a system are presented, and the optimal resonance conditions are defined. The gain scaling laws are confirmed by numerical simulations using a simplified kinetic model in the co-propagating reference frame.

1. Introduction
We present here a new scheme of an all-optical free electron laser (FEL) operating in the X-ray range and triggered by interaction of a moderately relativistic electron bunch with a transverse high intensity optical lattice. The lattice is created by two co-polarized counter-propagating laser beams overlapping in the interaction region with the electrons. The ponderomotive potential of the optical lattice forms guiding channels where the electrons are trapped and oscillate transversely [1, 2]. The collective oscillations of the guided electron bunch [3] can amplify the scattered signal wave at the Stokes frequency, thereby yielding a coherent beam at the Doppler-shifted frequency in the XUV or X-ray domain. Such a scheme combines some properties of conventional XFELs [4] and the XFEL based on the stimulated Raman scattering [5] of electron plasma waves; however, the physics of amplification is significantly different due to the guiding effect of the optical lattice [6]. It is expected to be less demanding with respect to the electron beam quality.

In section 2 we describe the injection and the state of electron beam, trapped in the lattice potential. In the sections 3 and 4 we derive the gain scaling laws in linear fluid and kinetic...
descriptions, and analyze the typical amplification regimes. The theoretical estimates are confirmed by the comparison with numerical simulations in section 3 using a simplified kinetic model in the co-propagating reference frame. Numerical estimates in the conclusion show that this scheme holds the potential to achieve X-ray amplification over millimeter-scale lengths, while coupled to the Laser Wakefield Acceleration of electrons section 6.

2. Injection and the ground state of electron beam in the optical lattice

Let us consider a beam of relativistic electrons, which propagates along the z-axis into an optical lattice with a wave vector oriented along the x-axis (the direction of laser pulse propagation) and polarized along the y-axis. The lattice is formed by two counter-propagating plane waves:

\[ a_L = a_1 + a_2 = 2a_e e_y \sin k_0 x \cos \omega_0 t, \]

where the normalized vector potential \( a(z) \) characterizes the field amplitude in the relativistic units \( m_e \omega_e c/e \). Within the interaction region the lattice amplitude is considered as uniform along \( x \) and \( y \) axes, and does not depend on time. The profile of the lattice along the \( z \)-axis has a ramp \( a(0 < z < l_{ramp}) = a_0 z/l_{ramp} \) and is assumed to be constant \( a_0 \) for \( z > l_{ramp} \).

Entering the ramp, electrons become trapped in the ponderomotive potential, \( U_p \propto \sin^2 k_0 x \), of the lattice channels. Propagation along the ramp modifies the electron distribution in the \((x, p_x)\) plane and deforms the beam spatial profile. The maximum density is localized on the channel axes, where \( U_p = 0 \), and the electrons are less affected by injection. Without entering in the details, it is reasonable to assume that the maximal density of the trapped beams is the same as the average beam density before injection. The maximal transverse kinetic energy of trapped electrons is limited as, \( \epsilon_\bot < m_e c^2 (a_0/\gamma_b)^2 \), where \( \gamma_b = 1/\sqrt{1 - \beta_b^2} \) is the beam Lorentz factor.

The study is performed in the reference system moving with the average velocity electron beam \( \beta_b c \). In such a system the electron beam parameters and the laser field are modified due to the Lorentz transformation. The wave vectors of both laser pulses obtain a component along the electron beam, \( k_{0\parallel} = \gamma_b \beta_b k_0^l \), while the transverse projections remain invariant \( k_{0\bot} = k_0^l \).

The superscript "L" hereafter indicates the parameters in the laboratory reference system. The electron velocity spread in the beam frame is assumed to be non-relativistic, it can be estimated as \( \delta \beta_\bot = \epsilon^L_\bot / (\delta \sigma_\bot^l) \), \( \delta \beta_\parallel = \delta \gamma_\parallel^l / (\gamma_b \delta \beta_b) \), where \( \epsilon^L_\bot \) is a normalized emittance of the beam, \( \sigma_\bot^l \) its radius and \( \delta \gamma_\parallel^l \) characterizes its energy spread. The laser wavelength \( \lambda_0 \) and the electron density are decreased by the factor of \( \gamma_b \), and assuming a time independent beam density one may calculate the ratio of the electron beam density to the critical density for the laser field as follows:

\[ \frac{n_e}{n_c} = \frac{\lambda_0^2}{\pi \sigma_\bot^l \gamma_b^3} \frac{J_L}{J_A}, \]

where \( J \) is the electron current and \( J_A = m_e c^3/e \simeq 17 \) kA is the Alfvén current.

Considering a perturbative approach to the lattice–beam interaction, we assume the electron distribution function (EDF) to be a sum of the EDFs of the ground state \( f_0 \) and a small perturbation \( f \), while the vector potential includes the laser field \( a_L \) and a small signal wave \( a_s \). Thus, the ponderomotive potential \( U_p = (m_e c^2/4)\nabla|\tilde{\alpha}|^2 \) includes the zeroth-order and first-order perturbation terms,

\[ U_p^{(0)} = m_e (c/2)^2 |\tilde{a}_L|^2, \quad (1a) \]

\[ U_p^{(1)} = m_e (c/2)^2 (\tilde{a}_s \tilde{a}_L^* + \text{c.c.}). \quad (1b) \]

We describe the interaction process using the Vlasov equation averaged over the laser period \( \tau_0 = \lambda_0/c \), and the enveloped wave equation for the electromagnetic vector potential. In the
zeroth order there is no signal wave, and the system reads:

$$\left( \partial_t + v_x \partial_x + v_z \partial_z - \partial_L U_p^{(0)} \partial_v \right) f_0 = 0 ,$$  \hspace{1cm} (2a)

$$\left( \partial_t^2 - c^2 \nabla^2 \right) \bar{a}_L = - \bar{a}_L \frac{\omega_p^2}{n_0} \int \text{d}v f_0 ,$$  \hspace{1cm} (2b)

where $\omega_p = \sqrt{4\pi e^2 n_0 / m_e}$ is the electron plasma frequency on the channel axis, where the electron density reaches its maximum value, $n_0 = n_e (x = 0)$. The enveloped field of the optical lattice reads $\bar{a}_L = 2 a_0 \sin(k_{0 \perp} x) e^{i k_{0 \perp} z}$. For the practically interesting cases, the relative electron density in the proper frame is very low (below $10^{-5}$ for the available sources). Thus, we may neglect the dispersion term in the right hand side of eq. (2a) and consider $\omega_0 = k_0 \epsilon_0$.

Considering a single trapped beam, we may approximate the sinusoidal channel potential with a parabolic one, $U_p^{(0)} = \Omega^2 x^2 / 2$, where $\Omega = \sqrt{2 a_0 k_{0 \perp} \epsilon_0}$ is a frequency of small-amplitude oscillations. The electron trajectories in such a potential are the characteristics of eq. (2a), and they read:

$$X(x_0, v_{x0}, t) = x_0 \cos \Omega t + \left( v_{x0} / \Omega \right) \sin \Omega t ,$$
$$V_x(x_0, v_{x0}, t) = v_{x0} \cos \Omega t - x_0 \Omega \sin \Omega t ,$$
$$Z(z_0, v_{z0}, t) = z_0 + v_{z0} t , \quad V_z(z_0, v_{z0}, t) = v_{z0}$$

where $x_0, v_{x0}, z_0$ and $v_{z0}$ are coordinates of the particle at $t = 0$. Its easy to see, that along these trajectories the equation, $df_0 / dt = 0$, is satisfied, so the ground state EDF is constant with respect to the motion integrals $v_x, v_z^2 + \Omega^2 x^2$.

Let us consider a model EDF of non-relativistic particles homogeneously distributed along $z$ axis with no $z$ velocity component. In the plane $(x, v_x)$ they occupy the Kapchinsky-Vladimirsky (K-V) ellipse $v_x^2 + \Omega^2 x^2 = \epsilon_\perp$, (for more on the K-V distribution see [7]):

$$f_0 = n_0 / (2 \sqrt{\epsilon_\perp}) \eta (\epsilon_\perp - v_x^2 - \Omega^2 x^2) ,$$

where the maximal kinetic energy of the electrons transverse motion is limited by the ponderomotive potential amplitude, $\epsilon_\perp = 2 a_0^2 \epsilon_0$. Using the definition eq. (4), we may calculate the distributions of electron density and pressure:

$$n_e = n_0 \sqrt{1 - \xi^2} ,$$  \hspace{1cm} (5a)

$$P_{xx} = n_0^2 v_x^2 / 3 n_0^2 , \quad P_{zz} = n_e v_\perp^2 ,$$  \hspace{1cm} (5b)

where $\xi = x / L$ is a transverse coordinate normalized to the “effective” beam half-width $L = \sqrt{\epsilon_\perp / \Omega}$. The system eq. (5a) represents an equation of state, that may be used subsequently to describe the perturbation dynamics in a fluid approach.

### 3. Fluid description of the stimulated scattering of electrons in the optical lattice

We first study the collective electron dynamics in a fluid approach, i.e. electron motion follows the hydrodynamic equations, interacting with the electromagnetic field via the first order of ponderomotive potential eq. (1b). In this simplified model the electrostatic potential and the beam longitudinal temperature are neglected and the transverse electron temperature does not exceed the trapping limit. The interaction is described with the following set of equations:

$$\partial_t n_e + \nabla \cdot (n_e \mathbf{u}) = 0 ,$$

$$\partial_t u_x + \nabla_x P_{xx} / m_e n_e + \nabla_z U_p = 0 ,$$

$$\partial_t u_z + \nabla_z P_{zz} / m_e n_e + \nabla_x U_p = 0 ,$$

$$\left( (\partial_t - i \omega_0)^2 - c^2 \nabla^2 \right) \bar{a} = - (4 \pi e^2 / m_e) n_e \bar{a} ,$$

\hspace{1cm} (6d)
where \( \mathbf{u} \) is an electron fluid velocity. For the signal wave propagating in the positive direction of \( z \) axis with frequency and wavenumber close to those of the pump \((\omega_s, k_s) \approx (\omega_0, k_0)\), we may consider the electromagnetic field in the enveloped form \( \bar{a}_s \), defined as

\[
\alpha_s = \text{Re}[\bar{a}_s(t, z) e^{-i(\omega_0 t - k_0 z)}].
\]

Equation (6) may be further simplified by assuming the signal wave to be a plane wave, \( \partial^2_\perp a_s = 0 \).

We describe the collective electron modes as a perturbation of the beam position \( \Delta_x \), defined as \( \xi = (x + \Delta_x)/L \). Without a signal wave, \( \partial_t U_p = 0 \), and from equations (6) it follows that:

\[
\partial^2_\perp \Delta_x = -\Omega^2 \Delta_x.
\]

This equation describes the collective beam mode with a frequency \( \omega_b = \Omega \). For a small amplitude signal, considering eq. (6) in the Fourier domain, we obtain the dispersion equation:

\[
(\omega^2 - \Omega^2) \left[ (\omega + \omega_0)^2 - (k - k_0 \parallel)^2 c^2 - \omega_p^2 \right] = 4\alpha \omega_0^4,
\]

where the coefficient

\[
\alpha = K (a_0 \omega_p kc / 2\omega_0^2)^2,
\]

defines the coupling between the beam and signal modes, and the factor \( K = (k_0 \perp L)^2 \xi^2 \sqrt{1 - \xi^2} \), accounts for the filling factor \( k_0 \perp L \) and the coupling localization within the beam, \( |\xi| < 1 \). For realistic beam parameters the filling factor is \( k_0 \perp L \approx 1 \), and we may estimate \( K \) by averaging the coupling localization over \( \xi \). In the case of a plane signal wave, the localization shape is \( \propto \xi^2 \sqrt{1 - \xi^2} \), and we evaluate the averaged factor as \( K \approx 0.2 \). If the scattered wave is diffracted, its profile must be considered, and the value of \( K \) will be estimated numerically.

Amplification of the Stokes mode is described by the complex frequency in eq. (7) with a real part \( \omega \approx -\Omega \) and imaginary part \( \Gamma \) corresponding to an exponential growth rate. Considering the resonant wave vector, \( k_s = k - k_0 \parallel \approx (\omega_0 - \Omega)/c \), and assuming \( \Omega \ll \omega_0 \), we estimate the imaginary part as:

\[
\Gamma/\omega_0 = \omega_0 \sqrt{\alpha / \Omega (\omega_0 - \Omega)}.
\]

4. Kinetics of the beam and field perturbations

The amplification growth rate may be also estimated in a linear kinetic model of the perturbation dynamics. To first order, the enveloped electromagnetic equation reads:

\[
(\partial_t + c \partial_z) \bar{a}_s - i \frac{c}{k_0} \partial^2_\perp \bar{a}_s = -i a_0 \sin(k_0 \perp x) e^{-ikz} \frac{\omega_p^2}{n_0 \omega_0} \int d\mathbf{r} f, \tag{9}
\]

where \( k = k_0 + k_0 \parallel \). As in section 3, we assume the plane signal wave, so \( \partial^2_\perp a_s = 0 \), and reduce the order of eq. (9). The electron dynamics is described by the Vlasov equation:

\[
\left( \partial_t + v_x \partial_x + v_z \partial_z - \partial_x U_p^{(0)} \partial_{v_x} + \partial_z U_p^{(1)} \partial_{v_z} \right) f = \left( \partial_x U_p^{(1)} \partial_{v_x} + \partial_z U_p^{(1)} \partial_{v_z} \right) f_0, \tag{10}
\]

where two terms in the right hand side represent the sources of EDF perturbations along \( x \) and \( z \) axes respectively. These perturbations are related to the components of the ponderomotive force \( \partial_x U_p^{(1)} \) to \( \partial_z U_p^{(1)} \), and their ratio can be estimated as \( k/k_0 \parallel \approx \gamma_b \gg 1 \), so that the \( x \) component may neglected. Integrating eq. (10) over the time along the electron trajectories eq. (3) and over the electron velocity, we calculate the electron density perturbation:

\[
\frac{\delta n}{n_0} = \sum_{\pm} (N_\pm e^{ikz} + c.c.) , \tag{11}
\]
where,

\[ N_\pm = \frac{1}{4} a_0^2 N_s e^{i k c^2 / 4 k_{0\perp} L} \xi \sqrt{1 - \xi^2} \int_{t_0}^t dt' \bar{a}_s(t') e^{i \alpha \omega (t - t')} . \]

Here the components \( N_+ e^{ikz} \), \( N_- e^{ikz} \), \( N_+ e^{-ikz} \), \( N_- e^{-ikz} \) may be resonantly coupled to the up- or down-shifted waves, propagating either along the positive or negative direction of \( z \) axis. For the signal propagating along the \( z \) axis we may neglect the short wavelength terms \( \propto e^{-2kz} \), and describe the coupled of electromagnetic (9) and beam (11) modes as follows:

\[ (\partial_t + c \partial_z) \bar{a}_s = -i a_0 k_{0\perp} x \frac{\omega_p^2}{\omega_0} \sum_{\pm} N_\pm . \tag{12} \]

We consider the harmonic solutions of eq. (12), \( N_\pm, \bar{a}_s \propto e^{-i \omega t + i \Delta k z} \), where \( \omega = \omega_0 - \omega_s \) and \( \Delta kc = \omega_0 - k_s c \). The values of \( \omega \) and \( \Delta k \) define the frequency and the wave number detuning of the signal field with respect to the laser wave. They are related by the dispersion equation:

\[ \omega - \Delta kc = -\alpha \omega_0^3 \sum_{\pm} (\omega \pm \Omega)^{-2} , \tag{13} \]

Two terms in the right hand side of eq. (13) represent the Stokes (+) and anti-Stokes (-) modes. It can be shown, that there are no unstable solutions in the anti-Stokes domain. Considering the instability near the resonance \( \omega \simeq -\Omega \), one may neglect the anti-Stokes term, and find:

\[ (\omega - \Delta kc) (\omega + \Omega)^2 = \alpha \omega_0^3 . \tag{14} \]

where factors in the left hand side characterize the electromagnetic and electron beam modes and the right hand side defines the coupling. Assuming only the temporal signal amplification (that is, a periodic system), the detuning term has to be real \( \Delta kc \in \mathbb{R} \).

The maximal growth rate corresponds to detuning \( \Delta kc = -\Omega \), where the dispersion equation reads: \( (\omega + \Omega)^3 = \alpha \omega_0^3 \). The imaginary part of the frequency is evaluated as

\[ \Gamma/\omega_0 = (\sqrt{3}/2) \alpha^{1/3} , \tag{15} \]

which corresponds to the signal frequency, \( \omega_s = \omega_0 (1 - \alpha^{1/3}/2) - \Omega \). For the case, \( \alpha^{1/3} \ll \Omega/\omega_0 \), it is close to the Stokes frequency \( \omega_s \simeq \omega_0 - \Omega \). Note, that a the scaling eq. (15) is similar to the SRS in a strong coupling regime, while eq. (15) corresponds to the weak coupling [5]. For the non-detuned wave, the instability develops in a weak coupling regime, and eq. (13) gives the following growth rate:

\[ \Gamma/\omega_0 = \sqrt{\alpha \omega_0 / \Omega} . \tag{16} \]

Our linear models are based on the perturbation theory in the approximation of a small amplitude signal and electron density perturbations, and a constant electron ground state temperature. In numerical simulations, we observe that electrons are rapidly heated along the propagation direction and trapped in the longitudinal potential due to the interference of the signal and pump waves. Trapping continues till all resonant (interacting with the wave) electrons are trapped. The density perturbations reach the maximum value \( |N_{\pm}| \simeq 1 \), which results in the signal saturation at the level that follows from eq. (12):

\[ |a_s/a_0| \simeq \omega_p^2 / (\omega_0 \Omega) . \tag{17} \]

These theoretical models of the beam trapping and the signal amplification are compared with the numerical simulations presented in the next section. They account for the dynamics of electrons in the ground state and the signal wave saturation.
5. Numerical modeling of the signal amplification in the optical lattice

For the numerical simulations we use the two-dimensional code EWOK, where the electron motion is averaged over the laser period $\tau_0$ and is described with help of the particle-in-cell (PIC) method. The electromagnetic field is calculated from the single-component, reduced-order Maxwell equation eq. (9), and its transverse distribution is presented by the Fourier harmonics.

Here we study a temporal amplification of the signal wave in a periodic system. The vertical size of the simulation domain presents the width of one potential channel $\gamma_b \lambda_0 / 2$. The length in the horizontal dimension is limited by the condition that the signal and the beam modes are periodic. It is chosen as an integer number of two wavelengths: the longitudinal potential $a_s a_{L*} \sim e^{i(k_0 + k_s)z}$ and the enveloped signal wave $\bar{a}_s \sim e^{i(k_s - k_0)z}$. This choice of the length of the simulation domain may limit the detuning of the signal wave and result in its artificial diffraction. The electrostatic force violates the periodicity and is neglected in what follows.

![Figure 1](image1.png)

**Figure 1.** (a) Amplitude of the scattered field in the center of right boundary, $(x, z) = (0, 70.2) \lambda_0$. (b) Spatial-spectral distribution of the amplified wave along the right boundary.

Let us consider the interaction of an electron beam having the Lorentz factor $\gamma_b = 10$ with the optical lattice, created by two laser waves with the amplitudes $a_0 = 0.1$. The dimensions of the simulation domain are $5 \lambda_0 \times 70.2 \lambda_0$, and the injection duration is $t_{\text{inj}} = 20 \lambda_0 / c$. The transverse momentum spread is close to the trapping conditions $\beta_\perp = 0.1$, while the longitudinal momentum spread and the electron density vary in the calculations.

For the electron density $n_e / n_c = 0.8 \times 10^{-5}$, the signal amplitude at the center of the right boundary of simulation box, demonstrates the exponential growth with the rate $\Gamma / \omega_0 = 1.1 \times 10^{-3}$ as it is shown in fig. [1]. For these conditions $\alpha^{1/3} \ll \Omega / \omega_0$, the instability develops in the weak coupling regime, and the growth rate value is in good agreement with eqs. (8) and (16). The strong coupling scaling eq. (15) predicts a few times higher growth rate $\Gamma / \omega_0 = 3.7 \times 10^{-3}$.

The field spectrum at the right boundary $z / \lambda_0 = 70.2, x / \lambda_0 \in [-2.5, 2.5]$ presented in fig. [1b] gives the frequency of the amplified wave and its transverse profile. The amplified wave has a narrow spectrum around the resonant frequency $\omega_{\text{res}} = 0.985 \omega_0 \simeq \omega_0 - \Omega$, which corresponds to the Stokes mode. The cut of this picture at the resonant frequency $\omega_s = \omega_{\text{res}}$ along the $x$ coordinate gives the transverse distribution of the amplitude of the amplified wave $a_s$. It is shown with the white dashed curve in fig. [1b]. The frequency shift increases by 8% at the edges of the amplification zone.

The amplification of the signal wave implies the growth of the electron beam modes. The density of electrons that are trapped in this mode, $n_e \propto \xi \sqrt{1 - \xi^2} \exp(i(k_0 + k_s)z)$, is presented in fig. [2]. Trapping of all electrons corresponds to the instability saturation at the level given by...
eq. (17). For the parameters presented in fig. 1a, this field is estimated as $a_{sat} = 5.3 \cdot 10^{-5}$. It is in a good agreement with the field measured in the simulations: $\max[a_s] = 5.8 \cdot 10^{-5}$.

6. Conclusions
We presented two successive models of the stimulated Raman scattering of a relativistic electron beam in a high intensity optical lattice. A fluid and a kinetic perturbation theory were developed that describe the interaction of electrons trapped in the ponderomotive potential of the lattice with the electromagnetic field. The collective oscillations of electrons in the lattice potential enable a resonant transfer of the laser beam energy to the signal wave. The longitudinal momentum of the signal wave is transferred to electrons. The rate of the signal growth is obtained analytically and compared to the results of PIC simulations. A qualitatively good level of agreement is obtained.

The proposed interaction scheme may be used for amplification of XUV or X-ray coherent radiation. Let us consider for example a 4 kA electron beam produced in a laser wake field accelerator with $\gamma = 100$ and interacting with an optical lattice with an amplitude $a_0 = 0.5$. The maximal gain length for that case is estimated to $300 - 400 \mu m$. In the saturated regime, a microjoule pulse at a wavelength of 0.4 Å can be generated, of few femtosecond duration. Even though these theoretical estimates may be reduced in real experimental conditions, this XFEL-SRS scheme will enable a high gain of X-rays, along with an efficient stabilization of the electron beam. This all-optical Raman X-ray laser may thus result in ultra-compact, fully coherent source up to the hard X-ray range, with innumerable applications to science, technology and medicine.

References
[1] Fedorov M V, Oganesyan K B and Prokhorov A M 1988 Appl. Phys. Lett. 53 353–354
[2] Balcou Ph 2010 Eur. Phys. J. D 59(3) 525–537
[3] Andriyash I A, Ph Balcou and Tikhonchuk V T 2011 Eur. Phys. J. D 65(3) 533–540
[4] Freund H P and Antonsen T M 1996 Principles of Free-Electron Lasers (Chapman & Hall)
[5] Sprangle P and Drobot A T 1979 J. Appl. Phys. 50 2652–2661
[6] Andriyash I A, d’Humières E, Tikhonchuk V T and Balcou Ph 2012 Phys. Rev. Lett. 109(24) 244802
[7] Reiser M 2008 Theory and Design of Charged Particle Beams 2nd ed (Wiley-VCH)