Hamiltonian Analysis of $1 + 1$ dimensional Massive Gravity

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ABSTRACT: We perform the Hamiltonian analysis of $1 + 1$ dimensional non-linear massive gravity studied in arXiv:107.3820. We find the constraint structure of given theory and perform the counting of the physical degrees of freedom.

KEYWORDS: Massive Gravity.
1. Introduction and Summary

One of the most challenging problems of modern theory of gravity is to decide whether it is possible to find consistent formulation of massive gravity. The first attempt for construction of this theory is dated to the year 1939 when Fierz and Pauli formulated its version of linear massive gravity. The central problem of theories of massive gravity is that they suffer from the problem of the ghost instability, for very nice review, see [2]. Since the general relativity is completely constrained system there are four constraint equations along the four general coordinate transformations that enable to eliminate four of the six propagating modes of the metric, where the propagating mode corresponds to a pair of conjugate variables. As a result the number of physical degrees of freedom is equal to two which corresponds to the massless graviton degrees of freedom. On the other hand in case of the massive gravity the diffeomorphism invariance is lost and hence the theory contains six propagating degrees of freedom which only five correspond to the physical polarizations of the massive graviton while the additional mode is ghost.

De Rham and Gobadadze argued recently in [5] that it is possible to find such a formulation of the massive gravity which is ghost free in the decoupling limit. Then it was shown in [6] that this action that was written in the perturbative form can be resumed into fully non-linear actions. It was claimed there that this is the first successful construction of potentially ghost free non-linear actions of massive gravity.

However it is still open problem whether this theory contains ghost or not, see for example [12]. On the other hand S.F. Hassan and R.A. Rosen argued recently in series of papers [7, 8, 9, 10] on the non-perturbative level that it is possible to perform such a redefinition of the shift function so that the resulting theory still contains the Hamiltonian constraint. Then it was argued that the presence of this constraint allows to eliminate the scalar mode and hence the resulting theory is the ghost free massive gravity. This result was however questioned in [11] where it was shown that the Hamiltonian constraint is the second class constraint which however implies that all non physical modes cannot be eliminated.

\[\text{For review, see } [3].\]
On the other hand it was argued recently in [13] that there exists formulation of the ghost free massive gravity using its Stückelberg formulation. This version of massive gravity certainly deserves very careful Hamiltonian analysis which is however very difficult due to the complexity of given action. On the other hand simpler toy model of 1 + 1 dimensional massive gravity was presented in [13] which could be explicitly analyzed. In fact, some preliminary analysis of this model was given in [13] but we feel that it deserves more careful treatment. Explicitly we mean that it is necessary to carefully identify the collections of the primary constraints where we have to take into account the fact that the 1 + 1 gravity is non dynamical. This is an important difference from the analysis given in [13] where this fact was not considered. Then we proceed by the standard way and we find collection of the secondary constraints. We analyze whether these constraints are preserved during the time evolution of the system and we find that no new additional constraints are generated. This result is again different from the conclusion presented in [13]. Now using this known structure of constrains we determine the number of physical degrees of freedom. Remarkably we find that there are no physical degrees of freedom left exactly as in the standard case of scalar theory coupled to two dimensional gravity. In other words we confirm the results obtained in [13] even if we derive them in different way.

We would like to stress that 1 + 1 dimensional massive gravity is rather special due to the fact that 1 + 1 dimensional gravity is non-dynamical. For that reason we mean that the result that all non physical degrees of freedom are eliminated should be taken with care. We expect that the situation will be different in case of higher dimensional massive gravities due to the fact that now the gravity becomes dynamical. It would be certainly nice to see whether the massive gravity in Stückelberg picture is ghost free or not especially in the context of the recent results found in [11]. We hope to return to the problem of the Hamiltonian formulation of four dimensional the massive gravity in Stückelberg picture in future.

The structure of this note is as follows. In the next section we perform the Hamiltonian analysis of the toy model of the 1 + 1 dimensional massive gravity introduced in [13]. In Appendix (A) we present the Hamiltonian analysis of the model that consists from two scalar fields minimally coupled to 1 + 1 dimensional gravity in order to see the difference with the Hamiltonian analysis of 1 + 1 dimensional non-linear massive gravity studied in the main body of the paper.

2. 1 + 1 dimensional Massive Gravity in Stückelberg Picture

Four dimensional non-linear massive gravity action that was introduced in [3] consists from two parts. The first one is the standard Einstein-Hilbert action while the second one is a specific form of the Lagrangian for the scalar fields $\phi^A$ where $A = 0, 1, 2, 3$, for more detailed treatment, see [3]. While it is very difficult to find the Hamiltonian formulation of the four dimensional non-linear massive gravity with Stückelberg fields it is much easier task to perform the Hamiltonian analysis of the toy model of non-linear massive gravity action that was introduced in [13]. Before we proceed to the explicit analysis of this theory we introduce following notation where $\gamma_{\alpha\beta}$ is a two-dimensional metric and
\(\sigma^\alpha, \alpha, \beta = 0, 1, \sigma^0 = \tau, \sigma^1 = \sigma\) are corresponding coordinates. Then in order to formulate the Hamiltonian formalism it is convenient to use ADM formalism \cite{14} for the 1 + 1 dimensional metric

\[
\gamma_{\alpha\beta} = \left( -n_\tau^2 + \frac{1}{\omega} n_\sigma^2 n_\sigma \right), \quad (2.1)
\]

where \(n_\tau\) is the lapse, \(n_\sigma\) is the shift and \(\omega\) is spatial part of metric. Then it is easy to see that

\[
\det \gamma = -n_\tau^2 \omega, \quad \gamma^{\alpha\beta} = \left( -\frac{n_\sigma^2}{n_\tau^2} \frac{1}{\omega} \frac{n_\sigma}{n_\tau^2} \right), \quad (2.2)
\]

where we defined

\[
n_\sigma^\sigma \equiv \frac{n_\sigma}{\omega}. \quad (2.3)
\]

The non-linear massive gravity action arises from the coupling of the 1 + 1 dimensional gravity to two scalar fields \(\phi^a, a = 0, 1\) where the scalar field Lagrangian density has the form

\[
\mathcal{L}_m = 2n_\tau \sqrt{\omega} \left[ -1 + \sqrt{(D^- \phi^-)(D^+ \phi^+)} = 2n_\tau \sqrt{\omega} \left[ -1 + \frac{1}{2\lambda} (D^- \phi^-)(D^+ \phi^+) + \frac{\lambda}{2} \right], \quad (2.4)
\]

where

\[
\phi^\pm = \phi^0 \pm \phi^1, \quad D^\pm = \frac{1}{\sqrt{\omega}} \partial_\tau \pm \frac{1}{n_\tau} [\partial_\tau - n^\sigma \partial_\sigma]. \quad (2.5)
\]

In order to see the equivalence of these two forms of the Lagrangian densities given above note that the equation of motion for \(\lambda\) gives

\[
\lambda^2 = (D^+ \phi^+)(D^- \phi^-). \quad (2.6)
\]

Inserting this result into the expression on the second line in (2.4) we recover the form of the Lagrangian given on the first line.

Now we are ready to proceed to the Hamiltonian analysis of the action \(S = \int d\tau d\sigma \mathcal{L}_m\), where \(\mathcal{L}_m\) is given in (2.4). As the first step we introduce the momenta \(\pi^\tau, \pi^\sigma, \pi^\omega, p_\pm\) conjugate to \(n_\tau, n^\sigma, \omega\) and \(\phi^\pm\) that have non-zero Poisson brackets

\[
\{ n_\tau(\sigma), \pi^\tau(\sigma') \} = \delta(\sigma - \sigma'), \quad \{ n^\sigma(\sigma), \pi^\sigma(\sigma') \} = \delta(\sigma - \sigma'), \quad \{ \omega(\sigma), \pi^\omega(\sigma') \} = \delta(\sigma - \sigma'), \quad \{ \lambda(\sigma), \pi^\lambda(\sigma') \} = \delta(\sigma - \sigma'), \quad \{ \phi^+(\sigma), p_+(\sigma') \} = \delta(\sigma - \sigma'), \quad \{ \phi^-(\sigma), p_-(\sigma') \} = \delta(\sigma - \sigma'). \quad (2.7)
\]

\(^2\text{For review of ADM formalism, see \cite{15}.}\)
Now due to the fact that the $1+1$ dimensional gravity is non-dynamical we find following primary constraints

$$\pi^\tau = \frac{\delta S}{\delta \partial_\tau n^\tau} \approx 0, \quad \pi_\sigma = \frac{\delta S}{\delta \partial_\tau n^\sigma} \approx 0, \quad \pi^\omega = \frac{\delta S}{\delta \partial_\tau \omega} \approx 0. \quad (2.8)$$

Finally, from (2.4) we find the momenta $p_\pm, \pi_\lambda$ conjugate to $\phi^\pm$ and $\lambda$

$$p_\lambda \approx 0, \quad p^+ = \frac{\delta L}{\delta \partial_\tau \phi^+} = \sqrt{\omega} \frac{1}{\lambda} D_\phi^+, \quad p^- = \frac{\delta L}{\delta \partial_\tau \phi^-} = -\sqrt{\omega} \frac{1}{\lambda} D_\phi^- . \quad (2.9)$$

Using these results we easily derive the Hamiltonian density

$$\mathcal{H} = n^\tau \mathcal{H}_\tau + n^\sigma \mathcal{H}_\sigma - 2 + \Gamma_\tau \pi^\tau + \Gamma_\sigma \pi_\sigma + \Gamma_\omega \pi^\omega + \Gamma_\lambda \pi_\lambda , \quad (2.10)$$

where

$$\mathcal{H}_\tau = \frac{1}{\sqrt{\omega}} (p_- \partial_\sigma \phi^- - p_+ \partial_\sigma \phi^+) + 2 \sqrt{\omega} - \lambda \left[ \frac{1}{\sqrt{\omega}} p^+ p^- + \frac{\sqrt{\omega}}{2} \right] ,$$

$$\mathcal{H}_\sigma = -2 \omega \nabla_\sigma \pi_\omega + p_+ \partial_\sigma \phi^+ + p_- \partial_\sigma \phi^- .$$

(2.11)

and where $\Gamma_\tau, \Gamma_\sigma, \Gamma_\omega, \Gamma_\lambda$ are Lagrange multipliers corresponding to the primary constraints

$$\pi^\tau \approx 0, \quad \pi_\sigma \approx 0, \quad \pi^\omega \approx 0, \quad \pi_\lambda \approx 0. \quad (2.12)$$

Note that we used the freedom in the form of the Lagrange multiplier when we added the primary constraint $\pi^\omega$ into the definition of the Hamiltonian. Explicitly, we added following expression to the Hamiltonian $\Gamma_\omega \pi^\omega = -2 n^\sigma \omega \nabla_\sigma \pi_\omega + \Gamma_\omega \pi^\omega$, where $\nabla_\sigma$ is one dimensional spatial covariant derivative. The reason for such a form of the Lagrange multiplier will be clear from the analysis of the time evolution of the secondary constraints. More precisely, the requirement of the preservation of the primary constraints imply secondary ones

$$\partial_\tau \pi^\tau = \{ \pi^\tau, \mathcal{H} \} = -\mathcal{H}_\tau \approx 0 ,$$

$$\partial_\tau \pi_\sigma = \{ \pi_\sigma, \mathcal{H} \} = -\mathcal{H}_\sigma \approx 0 ,$$

$$\partial_\tau \pi_\lambda = \{ \pi_\lambda, \mathcal{H} \} = -\frac{1}{\sqrt{\omega}} p^+ p^- - \frac{\sqrt{\omega}}{2} = -G_\lambda \approx 0 ,$$

$$\partial_\tau \pi^\omega = \{ \pi^\omega, \mathcal{H} \} \approx \frac{n^\tau \lambda}{2 \omega} \equiv 0 .$$

(2.13)

Since we presume that two dimensional metric is non-singular we have that $\omega \neq 0, n_\tau \neq 0$ and consequently the last equation implies the secondary constraint $\lambda \approx 0$. As a result we have following collection of secondary constraints $\mathcal{H}_\tau \approx 0, \mathcal{H}_\sigma \approx 0, G_\lambda \approx 0, \lambda \approx 0$, where we defined an independent constraint $\tilde{\mathcal{H}}_\tau$ as

$$\tilde{\mathcal{H}}_\tau = \frac{1}{\sqrt{\omega}} (p_- \partial_\sigma \phi^- - p_+ \partial_\sigma \phi^+) + 2 \sqrt{\omega} . \quad (2.14)$$
As the next step we introduce the smeared form of the constraints \( \tilde{\mathcal{H}}_\tau, \mathcal{H}_\sigma \)

\[
\mathbf{T}_\sigma(N^\sigma) = \int d\sigma N^\sigma \mathcal{H}_\sigma, \quad \mathbf{T}_\tau(N) = \int d\sigma N \tilde{\mathcal{H}}_\tau .
\]  

(2.15)

Note that \( \mathbf{T}_\sigma(N^\sigma) \) has these non-zero Poisson brackets with canonical variables

\[
\{ \mathbf{T}_\sigma(N^\sigma), \omega \} = -\partial_\sigma \omega N^\sigma - 2\omega \partial_\sigma N^\sigma ,
\]

\[
\{ \mathbf{T}_\sigma(N^\sigma), p_\pm \} = -\partial_\sigma (N^\sigma p_\pm) ,
\]

\[
\{ \mathbf{T}_\sigma(N^\sigma), \phi^\pm \} = -N^\sigma \partial_\sigma \phi^\pm .
\]  

(2.16)

Then it is easy to determine following Poisson brackets

\[
\{ \mathbf{T}_\sigma(N^\sigma), \mathbf{T}_\tau(M) \} = \mathbf{T}_\tau(N^\sigma \partial_\sigma M) ,
\]

\[
\{ \mathbf{T}_\sigma(N^\sigma), \mathbf{T}_\sigma(M^\sigma) \} = \mathbf{T}_\sigma(\partial_\sigma M^\sigma - M^\sigma \partial_\sigma N^\sigma)
\]

\[
\{ \mathbf{T}_\tau(N), \mathbf{T}_\tau(M) \} = \mathbf{T}_\sigma \left( \frac{1}{\omega} (N \partial_\sigma M - M \partial_\sigma N) \right) .
\]  

(2.17)

In other words \( \mathbf{T}_\sigma(N^\sigma), \mathbf{T}_\tau(N) \) form the closed algebra of constraints. Finally we list non-zero Poisson brackets between the constraint \( G_\lambda \) and remaining constraints

\[
\{ \mathbf{T}_\sigma(N^\sigma), G_\lambda \} = -\partial_\sigma G_\lambda N^\sigma - \partial_\sigma N^\sigma G_\lambda ,
\]

\[
\{ \mathbf{T}_\tau(N), G_\lambda \} = \frac{N}{\omega} (\partial_\sigma p^- p^+ - \partial_\sigma p^+ p^-) ,
\]

\[
\{ \pi^\omega, G_\lambda \} = \frac{1}{2\omega} \left( \frac{p^+ p^-}{\sqrt{\omega}} - \sqrt{\omega} \right) \approx \frac{1}{2\omega} G_\lambda - \frac{1}{\sqrt{\omega}} ,
\]

\[
\{ \pi^\omega, \tilde{\mathcal{H}}_\tau \} = \frac{1}{2\omega^{3/2}} \left( \partial_\sigma \phi^- - \partial_\sigma \phi^+ \right) - \frac{1}{2\sqrt{\omega}} = \frac{1}{2\omega} \tilde{\mathcal{H}}_\tau - \frac{1}{\sqrt{\omega}} .
\]  

(2.18)

Now we are ready to analyze the time evolution of primary and secondary constraints. Note that the total Hamiltonian takes the form

\[
H_T = \mathbf{T}_\tau(n_\tau) + \mathbf{T}_\sigma(n^\sigma) + \int d\sigma (\Gamma_\tau \pi^\tau + \Gamma^\sigma \pi_\sigma + \Gamma_\omega \pi^\omega + \Gamma_\lambda \pi^\lambda + \Sigma^\lambda G_\lambda + \Sigma^\omega \lambda ) .
\]  

(2.19)

Clearly the time evolution of the primary constraints \( \pi^\tau, \pi_\sigma, \pi^\omega \) is preserved during the time evolution of the system. The time evolution of the constraint \( \pi^\lambda \) takes the form

\[
\partial_\tau \pi^\lambda = \{ \pi^\lambda, H_T \} \approx -\Sigma^\omega = 0
\]  

(2.20)

and hence determines the value of the Lagrange multiplier \( \Sigma^\omega = 0 \). In the same way the requirement of the preservation of the constraint \( \lambda \approx 0 \) during the time evolution of the system determines the value of the constraint \( \Gamma_\lambda = 0 \). In other words \( \pi^\lambda \approx 0, \lambda \approx 0 \) are the second class constraints.
Now we proceed to the analysis of the requirement of the preservation of the constraints \( T_\tau(N_\tau), T_\sigma(N_\sigma) \) and \( G_\lambda \) during the time evolution of the system. In case of \( T_\tau(N_\tau) \) we find

\[
\partial_\tau T_\tau(N_\tau) = \{ T_\tau(N_\tau), H_T \} \approx \int d\sigma \left[ \frac{N_\tau \Sigma^\lambda}{\omega} (\partial_\sigma p^- p^+ - \partial_\sigma p^+ p^-) + N_\tau \frac{1}{\sqrt{\omega}} \Gamma_\omega \right] = 0 .
\]

On the other hand the preservation of the constraint \( G_\lambda \) implies

\[
\partial_\tau G_\lambda = \{ G_\lambda, H_T \} \approx -\frac{\tau}{\omega} (\partial_\sigma p^- p^+ - \partial_\sigma p^+ p^-) - \Gamma_\omega \frac{1}{\sqrt{\omega}} = 0 .
\]

Finally the requirement of the preservation of the constraint \( \pi^\omega \approx 0 \) gives

\[
\partial_\tau \pi^\omega = \{ \pi^\omega, H_T \} \approx -\int d\sigma \frac{1}{\sqrt{\omega}} (n_\tau + \Sigma^\lambda) = 0
\]

which implies that \( \Sigma^\lambda = -n_\tau \). Then we however see that the equations (2.21) and (2.22) are not independent. Then one of them determines the Lagrange multiplier \( \Gamma_\omega \) as

\[
\Gamma_\omega = -\frac{n_\tau}{\sqrt{\omega}} (\partial_\sigma p^- p^+ - \partial_\sigma p^+ p^-)
\]

while \( n_\tau \) is still free parameter. In other words \( G_\lambda \) and \( \pi^\omega \) are the second class constraints. The constraint \( \pi^\omega = 0 \) vanishes strongly while the constraint \( G_\lambda = 0 \) implies

\[
\omega = -p^+ p^- .
\]

Finally we should replace the Poisson brackets with corresponding Dirac brackets. In case of the constraints \( \tilde{H}_\tau \) we find

\[
\left\{ \tilde{H}_\tau(\sigma), \tilde{H}_\tau(\sigma') \right\}_D = \left\{ \tilde{H}_\tau(\sigma), \tilde{H}_\tau(\sigma') \right\} - \\
- \int d\sigma_1 d\sigma_2 \left\{ \tilde{H}_\tau(\sigma), \pi^\omega(\sigma_1) \right\} \Delta^{-1}(\sigma_1, \sigma_2) \left\{ G_\lambda(\sigma_2), \tilde{H}_\tau(\sigma') \right\} + \\
+ \int d\sigma_1 d\sigma_2 \left\{ \tilde{H}_\tau(\sigma), G_\lambda(\sigma_1) \right\} \Delta^{-1}(\sigma_1, \sigma_2) \left\{ \pi^\omega(\sigma_2), \tilde{H}_\tau(\sigma') \right\} = \\
= \left\{ \tilde{H}_\tau(\sigma), \tilde{H}_\tau(\sigma') \right\} ,
\]

where we defined

\[
\left\{ \pi^\omega(\sigma), G(\sigma') \right\} \approx -\frac{1}{\sqrt{\omega(\sigma)}} \delta(\sigma - \sigma') , \quad \Delta^{-1} = -\sqrt{\omega(\sigma)} \delta(\sigma - \sigma')
\]

and we used the fact that all Poisson brackets are proportional to the delta functions so that the contributions from the Poisson brackets between \( \tilde{H}_\tau \) and the second class constraints
vanish. In case of the constraints $T_\sigma$ we find that the Dirac brackets are the same as the corresponding Poisson brackets due to the fact that the Poisson brackets between $T_\sigma$ and the second class constraints vanish on the constraint surface. Finally, the Dirac brackets between $\phi^\pm$ and $p^\pm$ coincide with Poisson brackets due to the fact that the Poisson brackets between $\phi^\pm$ and $\pi^\pm$ vanish.

In summary, the reduced phase space is spanned by the variables $\phi^\pm, p^\pm$ and $n_\tau, n^\sigma$ together with $\pi^\tau, \pi^\sigma$. The dynamics is governed by the Hamiltonian

$$H_{\text{red}} = \int d\sigma (n_\tau \tilde{H}_{\tau}^{\text{red}} + n^\sigma \mathcal{H}_\sigma + \Gamma_\tau \pi^\tau + \Gamma_\sigma \pi^\sigma),$$

(2.28)

where

$$\tilde{H}_{\tau}^{\text{red}} = \frac{1}{\sqrt{-p_+ p_-}} (p_- \partial_\sigma \phi^- - p_+ \partial_\sigma \phi^+) + 2\sqrt{-p_+ p_-}.$$  

(2.29)

Note that $\tilde{H}_{\tau}^{\text{red}}, \mathcal{H}_\sigma, \pi^\tau, \pi^\sigma$ are the first class constraints and that the symplectic structure of given theory is canonical. Finally we should stress that there are no physical degrees of freedom left due to the fact that four first class constraints listed above eliminate all physical degrees of freedom $\phi^\pm, \pi^\pm, n_\tau, \pi^\tau, n^\sigma, \pi^\sigma$. Explicitly, in order to fix the constraints $\mathcal{H}_\tau, \mathcal{H}_\sigma$ we consider following gauge fixing functions

$$G^+ \equiv \phi^+ - k\sigma - \tau \approx 0, \quad G^- \equiv \phi^- - l\sigma - \tau \approx 0,$$

(2.30)

where $k, l$ are integers. The diffeomorphism constraint gives

$$p_+ k + p_- l = 0, \quad p_+ = -\frac{l}{k} p_-,$$

(2.31)

while the Hamiltonian constraint $\tilde{H}_\tau = 0$ implies

$$p_- = -k, \quad p_+ = -l.$$  

(2.32)

Finally note that the requirement of the preservation of the gauge fixing functions $G^\pm \approx 0$ during the time evolution of the system determines the value of the variables $n_\tau, n^\sigma$ which in turns could serve as the gauge fixing conditions for the primary constraints $\pi^\tau \approx 0, \pi^\sigma \approx 0$. Then however the requirement of the preservation of these gauge fixing functions determine the value of the Lagrange multipliers $\Gamma_\tau, \Gamma_\sigma$. In other words in the process of the gauge fixing we completely determined all gauge symmetry parameters and also all canonical variables.

A. Hamiltonian Analysis of 1 + 1 dimensional Scalar Field Theory

In this appendix we briefly review the Hamiltonian analysis of 1 + 1 dimensional scalar theory minimally coupled to gravity. In other words we consider the Lagrangian density

$$\mathcal{L}_{\text{scal}} = -\frac{1}{2} \sqrt{-\gamma \gamma^{\alpha\beta} \partial_\alpha \phi^A \partial_\beta \phi^B \eta_{AB}},$$

(A.1)
where $A, B = 1, 2$. Using $1 + 1$ formalism we find the Lagrangian density in the form

$$\mathcal{L} = \frac{1}{2} \sqrt{\omega n_\tau} \left( \nabla_n \phi^A \nabla_n \phi^B - \frac{1}{\omega} \partial_\sigma \phi^A \partial_\sigma \phi^B \right) \eta_{AB}. \quad (A.2)$$

Then it is easy to find corresponding Hamiltonian density in the form

$$\mathcal{H} = \frac{n_\tau}{\sqrt{\omega}} \mathcal{H}_\tau + n^\sigma \mathcal{H}_\sigma + \Gamma_\tau \pi^\tau + \Gamma^\sigma \pi_\sigma + \Gamma_\omega \pi_\omega, \quad (A.3)$$

where

$$\mathcal{H}_\tau = \frac{1}{2} \pi_A \eta^{AB} \pi_B + \frac{1}{2} \partial_\sigma \phi^A \eta_{AB} \partial_\sigma \phi^B,$$

$$\mathcal{H}_\sigma = p_A \partial_\sigma \phi^A. \quad (A.4)$$

Observe that now $\omega$ appears in the combination with $n_\tau$. As a result the requirement of the preservation of the primary constraint $\pi^\omega$ does not generate any additional constraint. More precisely, the requirement of the preservation of the constraints $\pi^\tau \approx 0, \pi^\sigma \approx 0$ implies the secondary constraints

$$\mathcal{H}_\tau \approx 0, \quad \mathcal{H}_\sigma \approx 0. \quad (A.5)$$

Then the standard analysis shows that these constraints are the first class constraints. The gauge fixing of these constraints eliminate 2 scalar fields and corresponding conjugate momenta.

We see that the main difference with respect to the analysis presented in the main body of the paper is that in case of $1 + 1$ scalar field theory Lagrangian density (A.1) the requirement of the preservation of the primary constraint $\pi^\omega$ does not generate new additional constraint and hence corresponding structure of constraints is much simpler.

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