Scalar Curvature, Injectivity Radius and Immersions with Small Second Fundamental Forms

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To 80th birthday of B.L.

Abstract
We prove in special cases the following.

• Bounds on the injectivity radii of "topologically complicated" Riemannian $n$-manifolds $X$, where the scalar curvatures of $X$ are bounded from below, $\text{Sc}(X) \geq \sigma > 0$.

• Lower bounds on focal radii of smooth immersions from $k$-manifolds, e.g., homeomorphic to the $k$-torus, to certain Riemannian manifolds of dimensions $n = k + m$, e.g., to the cylinders $S^{n-1} \times \mathbb{R}^1$.

• Topological lower bounds on the mean curvatures of domains in Riemannian manifolds, e.g., in the Euclidean $n$-space $\mathbb{R}^n$.

At the present moment, our results are limited by the spin condition and the $n \leq 8$ restriction.

Contents

1 Definitions, Conjectures and Illustrative Examples 2
1.1 Corollaries, Questions, Generalizations 8

2 $T^n$-Stabilized Curvatures $\text{Sc}^n$, $\text{Sc}^\partial$, $\text{Sc}_{\partial\partial}$ and Multispreads $\tilde{\Box}^n(h)$ on Homology 12

3 Maps to Spheres and to Sphere Bundles 16

4 Injectivity Radii and Maps to Spheres 23

5 Focal Radii and Normal Curvatures 29

6 Mean Convex Domains in $\mathbb{R}^n$ 31

7 A Few Words on Foliations 37

8 References 40
1 Definitions, Conjectures and Illustrative Examples

Terminology and Notation. Given a Riemannian manifold $X = (X, g)$, possibly non-complete and/or with a boundary let

$$\text{inrad}_x(X) = \text{dist}(x, \partial_{\text{metr}} X), \ x \in X,$$

be the radius $R$ of the maximal open ball $B = B_x(R) \subset X$, which doesn’t intersect the boundary $\partial X \subset X$ as well the "infinity" of $X$, i.e. the boundary of $X$ in the metric completion of $X$, where our $\partial_{\text{metr}} X$ stands for the topological boundary of the interior of $X$ in the metric completion of $X$.

For instance $\text{inrad}_x(X) = \text{dist}(x, \partial X)$ for (metrically) complete manifolds with boundaries.

The conjugacy radius

$$\text{conj.rad}_x(X) \leq \text{inrad}_x(X),$$

is the maximal $r$, such that the map $\exp_x$ is a local diffeomorphisms on the open $r$-ball $B_T x(r) \subset T_x(X)$, where, in fact, "$\exp_x$" is defined on the ball $B_T x(R = \text{inrad}_x(X)) \subset T_x(X)$.

The injectivity radius

$$\text{inj.rad}_x(X) \leq \text{conj.rad}_x(X)$$

is the maximal $r$, such that the map $\exp_x : T_x(X) \to X$ is a diffeomorphism from the open $r$-ball $B_T x(r) \subset T_x(X)$ to its image in $X$, where, in fact, "$\exp_x$" is defined the ball $B_T x(R = \text{inrad}_x(X))$.

Observe, that the injectivity radius of the universal covering $\tilde{X}$ of $X$ mediates between the two above radii,

$$\text{conj.rad}_x(X) \geq \text{inj.rad}_{\tilde{x}}(\tilde{X}) \geq \text{inj.rad}_x(X),$$

where $\text{inj.rad}_{\tilde{x}}(\tilde{X})$ stands for $\text{inj.rad}_{\tilde{x}}(\tilde{X})$, where $\tilde{x} \in \tilde{X}$ is a lift of $x$ to $\tilde{X}$ and where the latter radius is independent of the lift.

Remark. The ratio $\text{conj.rad}(X)/\text{inj.rad}(\tilde{X})$ can be arbitrarily large. For instance, one can construct Riemannian metrics $g_i$, $i = 1, 2, ..., $ on all compact 3-manifolds $X$, such that $\text{diameter}(X, g_i) \leq 1$ and $\text{sect.curv}(g_i) \leq 1/i$, which makes $\text{conj.rad}_x(X, g_i) \geq \sqrt{i}$ for all $x \in X$.

(Probably, similar metrics exist on all manifolds of dimensions $n \geq 3$.)

The focal radius of a smooth immersion\footnote{"Immersions" are locally (smooth) embeddings, or, equivalently, smooth maps with non-vanishing differentials (on the non-zero tangent vectors).} map $f : Z \to X$, denoted

$$\text{foc.rad}_x(Z) = \text{foc.rad}_x(Z \xrightarrow{f} X),$$

is the maximal $r$, for which the differential of the normal exponential map

$$\exp_x^f = \exp_{f(x)}^x : B_{T_{f(x)}(Z)} \to X,$$

is an immersion.
The scalar curvature of a Riemannian manifold \(X = (X, g)\), denoted
\[
Sc = Sc(X, x) = Sc(X, g) = Sc(g) = Sc_g(x),
\]
is a function on \(X\), which is equal to the sum of the values of the sectional curvatures at the \(n(n - 1)\) ordered bivectors of an orthonormal frame in \(X\),
\[
Sc(X, x) = \sum_{i,j} \text{sect.curv}_x(\tau_{ij}), \ i \neq j = 1, ..., n,
\]
where this sum doesn’t depend on the choice of the frame by the Pythagorean theorem and where a characteristic property of the scalar curvature is additivity under Cartesian-Riemannian products:
\[
Sc(X_1 \times X_2, g_1 + g_2) = Sc(X_1, g_1) + Sc(X_2, g_2).
\]

**Examples.** (a) The scalar curvature of the \(n\)-sphere of radius \(R\) is
\[
Sc(S^n(R)) = n(n - 1)\text{sect.curv}(S^n(R)) = \frac{n(n - 1)}{R^2}.
\]
(b) The scalar curvature of products of spheres by Euclidean spaces, \(X = S^m(R) \times \mathbb{R}^k\) satisfy
\[
Sc(X) = \frac{m(m - 1)}{R^2},
\]
where, observe, the sectional curvatures of these products are pinched between 0 and \(\frac{Sc(X)}{m(m-1)}\),
\[
0 \leq \text{sect.curv}(X) \leq \frac{Sc(X)}{m(m-1)} = \frac{1}{R^2},
\]
and their conjugacy and injectivity radii are
\[
\text{conj.rad}(X) = \text{inj.rad}(X) = \text{inj.rad}(S^m(R)) = R\pi.
\]
(c) If \(X \subset \mathbb{R}^{n+1}\) is a smooth cooriented hypersurface with principal curvatures \(\lambda_i = \lambda_i(x), \ i = 1, ..., n = \dim(X)\), then, by the Gauss formula, the scalar curvature of the Riemannian metric \(g\) on \(X\) induced from the Euclidean/Riemannian one is
\[
Sc_g(x) = \sum_{i,j} \lambda_i \lambda_j = \left(\sum_i \lambda_i\right)^2 - \sum_i \lambda_i^2.
\]

**Leon Green’s Conjugate Radius Inequality** (theorem 5.1 in [G1963]).
The minimal conjugacy radius of a complete Riemannian manifold \(X\) with finite volume and with the Ricci curvature bounded from below, \(\text{Ricci}(X) \geq R > -\infty\) of dimension \(n\) is bounded by the mean scalar curvature of \(X\) as follows:
\[
\inf_{x \in X} (\text{conj.rad}_x(X))^2 \cdot \text{vol}(X)^{-1} \int_X Sc(X, x) dx \leq \pi^2 \cdot n(n-1),
\]

[2] In the original paper [G1963], \(X\) is assumed compact, but if \(X\) is non-compact and geodesically complete, then \(\text{Vol}(X) < \infty\) and \(\inf \text{Ricci}(X) > -\infty\) suffice for the proof.
where the equality holds only for (necessarily compact) manifolds with constant sectional curvatures \( > 0 \) and where, observe,

\[
\text{vol}(x)^{-1} \int_X \text{Sc}(X, x)dy \geq \inf_{x \in X} \text{Sc}(X, x).
\]

Consequently,

\[
[\text{Sc} | \text{conj}]_n \inf \text{Sc}(X) \cdot (\text{conj.rad}(X))^2 \leq n(n + 1)\pi^2
\]

for all compact Riemannian \( n \)-manifolds \( X \).

Below is a conjectural topological sharpening of \([\text{Sc} | \text{conj}]_n\].

**1.A. \([\text{Sc} | \text{conj}]_{n-k}\)-Conjecture.** Let \( X \) be a complete Riemannian \( n \)-manifold, such that there exists a homology class \( h \in H_k(X) \), such that no non-zero multiple of \( h \) is representable by a continuous map from a compact oriented \( k \)-manifold with positive scalar curvature.

where this is expressed in writing by the inequality

\[
\text{Sc}(X) \not\succ H_k, 0.
\]

(Informally speaking, the "Q-homological scalar curvature" of \( X \) is not "fully positive in dimension \( k \)."

Then

\[
[\text{Sc} | \text{conj}]_{n-k} \inf \text{Sc}(X) \cdot (\text{conj.rad}(X))^2 \leq \pi^2(m - 1), m = n - k,
\]

Moreover, if \( X \) is connected non-compact complete, then

\[
\inf \text{Sc}(X) \cdot (\inf \text{conj.rad}(X))^2 \leq \pi^2(m - 1)(m - 2).
\]

**Two Obvious (conjectural) Corollaries.** Since \( \text{inj.rad} \leq \text{conj.rad} \) and

\[
\text{conj.rad}(X) \geq \frac{\pi}{\sqrt{\max(0, \text{sup sect.curv}(X))}}
\]

where \( \text{sup sect.curv}(X) \) is the supremum of the sectional curvatures over all tangent planes in \( X \), the \([\text{Sc} | \text{conj}]_m\)-inequality \( \inf \text{Sc}(X) \cdot (\text{conj.rad}(X))^2 \leq \pi^2m(m - 1) \)

implies that

\[
[\text{Sc} \text{Inj}]_m \inf \text{Sc}(X) \cdot (\text{inj.rad}(\tilde{X}))^2 \leq \pi^2m(m - 1)
\]

where \( \tilde{X} \) is the universal covering of \( X \), and that

\[
[\text{Sc/sect}]_m \inf \text{Sc}(X) \leq \text{sup sect.curv}(X) \leq m(m - 1),
\]

\footnote{
Products \( Y^m \times T^k \) are basic examples of such \( X \), see [SY1979], [GL1980]

If \( X \) has subexponential volume growth and \( \text{Ricci}(X) > \infty \), then the inequality \( \inf \text{Sc}(X) \cdot (\text{conj.rad}(X))^2 \leq \pi^2n(n - 1) \) follows by Green’s integration argument.

But no bound \( \inf \text{Sc}(X) \cdot (\text{inj.rad}(X))^2 \leq C = C_n \) (not even, for \( C = \infty \)) is known for general non-compact complete manifolds, except for \( n \leq 5 \), where quantifying the uniform contractibility argument from [CL2020] (compare with [CL 2020], 3.10.3 in [Gr 2021] and 1.1.A in the next section) shows that \( \inf \text{Sc}(X) \cdot (\text{inj.rad}(X))^2 \leq 10^{100} \) for complete Riemannian manifolds \( X \) of dimensions \( n \leq 5 \).
}
Although, both inequalities remain conjectural, we shall show in section 4 that some manifolds \( X \) must satisfy at least one of the two.

Below is an instance of this.

**1.B. Two Examples.** Let \( X \) be a Riemannian manifold of dimension \( n = m + k \), let \( Z \) be a closed connected orientable manifold \( Z \), which admits a metric with non-positive sectional curvature and let

\[
\phi : X \to Z,
\]

be a continuous map.

(For instance, \( X = Y \times \mathbb{T}^k \), where \( \mathbb{T}^k \) is the \( k \)-torus and \( \phi : (y, t) \mapsto t \).)

Let the scalar curvature of \( X \) be bounded from below by

\[
Sc(X) \geq \sigma > 0.
\]

Let the sectional curvature of \( X \) be bounded from above by

\[
\text{sect.curv}(X) \leq \kappa, \kappa > 0.
\]

**1.B(i). Compact Case.** Let \( X \) be compact without boundary, let \( \dim(Z) = k \) and let the \( k \)-dimensional induced homology homomorphism

\[
\phi_* : H_k(X) \to H_k(Z) = \mathbb{Z}
\]

doesn’t vanish.

Let

\[
m(m - 1)\kappa \leq \sigma.
\]

If

\begin{itemize}
  \item \text{spin} either \( m \leq 3 \) or the universal covering \( \tilde{X} \) of \( X \) is spin,
  \item \text{spin} \( n = \dim(X) \leq 8 \),
\end{itemize}

then the universal covering of \( X \) satisfies

\[
inj.rad(\tilde{X}) \leq \pi \sqrt{\frac{1}{\kappa}}.
\]

Remark. Albeit highly constrained, this inequality is non-vacuous. For instance, when applied to

\[
X_\lambda = (S^m \times Z, g_{S^m} + \lambda^2 g_Z), \ \lambda \to \infty,
\]

it shows that compact Riemannian manifolds \( Z = (Z, g_Z) \) with \( Sc(g_Z) > 0 \) of dimension \( k = 8 - m \) admit no metrics with \( \text{sect.curv} \leq 0 \). (Of course, this is known for all \( k \).)

**1.B(ii). Non-Compact Case.** Let \( \dim(Z) = k - 1 \), let \( \phi : X \to Z \) be a fibration (e.g. a topological bundle of balls), let \( \psi : \tilde{Z} \to X \) be a "lift" of the universal covering of \( Z \) to \( X \), i.e. \( \phi \circ \psi = \pi \). (For instance, if the fibration \( \phi : X \to Z \) admit a section \( Z \to X \), then the universal covering of the image of

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5This is a rather annoying (possibly redundant for complete \( X \)) condition, where, necessarily \( \kappa \geq \frac{1}{8(n-1)} \).

6This means that the restrictions of the tangent bundle \( T(\tilde{X}) \) to surfaces in \( \tilde{X} \) are trivial, for which vanishing of the second homotopy group \( \pi_2(X) \) is sufficient.
this section may be taken for \( \hat{\psi} \).) Let the scalar and the sectional curvatures of \( X \) be bounded as earlier by

\[
Sc(X) \geq \sigma \quad \text{and} \quad sect.curv(X) \leq \kappa.
\]

Let \( \hat{R} \) be the distance from the image of \( \hat{\psi} \) to the metric boundary of \( X \),

\[
\hat{R} = dist_{\text{metr}}(\hat{\psi}(\hat{Z}), \partial X),
\]

(where, by definition, \( \hat{R} = \infty \) for geodesically complete \( X \)), let

\[
\hat{U} = U_{\frac{\hat{R}}{\hat{R}}}(\hat{\psi}(\hat{Z})) \subset X
\]

be the \( \frac{5}{6} \hat{R} \)-neighbourhood of \( \hat{\psi}(\hat{Z}) \subset X \), and let

\[
\hat{\sigma} = \sigma - \frac{36(n-1)\pi^2}{nR^2} \geq m(m-1)\kappa.
\]

If

- \( \bullet_{\text{spin}} \) either \( m \leq 3 \) or the universal covering of \( X \) is spin,
- \( \bullet_{\#} \) \( n = \dim(X) \leq 8 \),

then the injectivity radius of the universal covering \( \hat{X} \) of \( X \) is bounded in \( U \) as follows:

\[
\inf_{x \in \hat{U}} inj.rad_x(\hat{X}) \leq \pi \sqrt{\frac{n(m-1)}{\hat{\sigma}}}
\]

1.C. Corollary: Focal Radius Bound. Let \( X \) be a complete Riemannian \( n \)-manifold such that

\[
Sc(X) \geq \sigma > 0, \quad sect.curv(X) \leq \kappa \quad \text{and} \quad inj.rad(X) \geq r \geq \frac{\pi}{\sqrt{\kappa}}.
\]

Let \( Z \) be a compact \((n-m)\)-dimensional manifold, which admits a metric with \( sect.curv \leq 0 \) and let \( Z \hookrightarrow X \) be a smooth immersion.

If \( m \geq 0 \) and

- \( \bullet_{\text{spin}} \) either \( m \leq 4 \) or the universal covering of \( X \) is spin,
- \( \bullet_{\#} \) \( n = \dim(X) \leq 8 \),

Then

\[
\text{foc.rad}(Z) \leq 6\pi \sqrt{\frac{n-1}{n}} \sqrt{\frac{1}{\sigma - (m-1)(m-2)\kappa}}
\]

Proof. Let \( T^i(Z) \to Z \) be the normal bundle of \( Z \to X \) and let \( BT^i(R) \subset T^i(Z) \) be the open \( R \)-ball subbundle.

Observe that the exponential map \( \exp^i : BT^i(R) \to X \) is defined for \( R \leq \text{foc.rad}(Z) \) and is an immersion for these \( R \).

Then 1.B(ii) applies to \( X^i = BT^i(\text{foc.rad}(Z)) \) with the metric induced by \( \exp^i \) from \( X \) and the proof follows.

Mean Curvature Inequalities. All of the above generalizes to Riemannian manifolds with mean convex boundaries, where the simplest instance of this is as follows.

1.D. Narrow Tunnel Theorem. Let \( X \subset \mathbb{R}^n, \ n \geq 3 \), be a smooth (not necessarily connected and possibly infinite) domain with

\[
\text{mean.curv}(\partial X) \geq n-2 = \text{mean.curv}(S^{n-1}(R_+)) \quad \text{for} \ R_+ = 1 + \frac{1}{n-2}.
\]
If \( n \leq 8 \), then the diameters of all connected components of the subset
\[
X_{-(1+\varepsilon)} = X \setminus U_{1+\varepsilon}(\partial X) = \{ x \in X \mid \text{dist}(x, \partial X) > 1+\varepsilon \} \subset X, \ \varepsilon > 0,
\]
are bounded by \( \text{const}_{n,\varepsilon} < \infty \). (This confirms conjecture 10 in [Gr2019] for \( n_1 = 1 \).)

**Remark/Conjecture.** Probably, the connected components of the subset \( X_{-1} \subset X \) are bounded.

**Plan of the Paper and Organisation of the Proofs.** In the next (sub)section, we discuss of our results in the general context of the metric scalar curvature geometry.

In section 2 we introduce a (not quite) numerical invariant on homology of a Riemannian manifold \( X \), denoted \( \text{Sc}^2_{\overline{\partial}}(h), \ h \in H_m(X) \).

We represent \( h \) in some cases by stable \( \mu \)-bubbles \( X_{\mu} \subset X \) with a properly tailored function \( \mu(x) \) and thus bound \( \text{Sc}^2_{\overline{\partial}}(h) \) in terms of \( \text{Sc}(X) \) in manifolds \( X \), where the universal covering \( \tilde{X} \) is spin.

(These \( Y_{\mu} \) are constructed as in section 5 of [Gr2021] by using the inductive dimension descent scheme originated in [SY1979] and combined with symmetrization-by-\( T^{n-m} \)-warping from [GL1983], also see [SY2017], following an idea from [F-CS1980]. A presence of possible stable singularities in \( Y_{\mu} \) entails here the possibly unnecessary constraint \( \text{dim}(X) \geq 8 \).

In section 3 we introduce another invariant, denoted \( \text{Rad}(h/S^N) \), which concerns area decreasing maps \( X \to S^N \), and show that
\[
\text{Rad}(h/S^N) \leq \sqrt{\frac{m(m-1)}{\text{Sc}^2_{\overline{\partial}}(h)}}.
\]

(This is done essentially, but not quite, as in [GL1983] by relying on the relative index theorem for families of twisted Dirac operators over spin manifold\(^7\) combined with the "twisted" Lichnerowitz formula, which is sharpened in the present case by Llarull’s inequality\(^8\) compare with section 3.4.3 in [Gr2021]).

In section 4, we show that the inequalities \( \text{sect.\,curv}(X) \leq \kappa \) and \( \text{inj.\,rad}(X) \geq \pi/\sqrt{\kappa} \) imply that \( \text{Rad}(h/S^N) \geq 1/\sqrt{\kappa} \) for \( N \geq 2 \text{dim}(X) \).

We combine this with the above and obtain the inequality
\[
\kappa \geq \text{Sc}^2_{\overline{\partial}}(h)/m(m-1)
\]
for manifolds \( X \) with the universal coverings spin, which is a generalized quantitative version of the following non-existence theorem (see 13.8 in [GL1983]).

\begin{itemize}
\item \textbf{Non-torsion} homology classes\(^9\) in complete manifolds with non-positive sectional curvatures are not representable by continuous maps from oriented spin manifolds \( X \) with positive scalar curvatures\(^10\).
\end{itemize}

\(^7\)Embarrassingly, I see no direct elementary proof of this even for \( X_{-(1+\varepsilon)} \), where \( \varepsilon > 0 \).

\(^8\)Compare with sections 5.3, 5.5 in [Gr2021].

\(^9\)This theorem generalizes to manifolds, where the universal coverings are spin, see §§9.3, 9.4 in [Gr1996] and section 10 in [Gr2019].

\(^10\)Mario Llarull was Blaine Lawson’s student who, by Blaine’s suggestion, algebraically evaluated the bottom of the spectrum of the the curvature operator on the spinors on \( S^n \), (formula 4.6 in [Ll1998]) and thus obtained the optimal bound for the area dilations of maps from manifolds with \( \text{Sc} \geq \sigma \) to spheres.

\(^11\)I am not certain on what is known in this respect about \( l \)-torsion, say for \( l = 2 \).

\(^12\)The origin of the argument used in [GL1983] can be traced to the proof of the Novikov conjecture for spaces with \( \text{sect.\,curv} \leq 0 \) in [Mi1974].
Finally, for \( n \leq 8 \), we combine the inequality \( \kappa \geq Sc^{33}(h)/m(m-1) \) with a lower bound on \( Sc^{33}(h) \) obtained with \( \mu \)-bubbles \( Y^m_{BM} \) in section 2, which is essential to gain the correct \( m(m-1)/n(n-1) \) factor in our inequalities.

Thus we arrive at our main theorem 4.C for distance decreasing maps from manifolds with lower bounds on their \( T^n \)-stabilized scalar curvatures to manifolds with "large" metric balls.

In section 5 we say more on focal radii and discuss lower bounds on curvatures of immersions.

In section 6 we generalize 4.C to manifolds with mean convex boundaries, for which we adapt the results from [GS2000] and [Lo2021], which we slightly improve along the lines of [Li1998] and [Li2010].

In section 7 we prove a version of 1.B for Riemannian foliations (defined in 1.1.B below) on 4-manifolds.

Everywhere, we indicate limitations of our results and arguments, (most pronounced for Riemannian foliations with \( Sc > 0 \)) and try to formulate conceivable generalizations and improvements.

### 1.1 Corollaries, Questions, Generalizations

(a) Most known inequalities obtained by combining (Dirac) index theoretic and \( (\mu\text{-bubble}) \) geometric measure theoretic arguments have been generalized and proved with the index theorems for Callias type operators with no use of the geometric measure theory.\(^\text{13}\)

However, for all I know, there are no such proofs of 1.B, 1.C, 1.D at the present moment and the constrain \( n \leq 8 \) remains intact.

(b) The \( \Box^{33}(n = 2, N) \)-inequality in 2.B of the next section and the (weakened) \( T^n \)-stable 2d Bonnet-Myers diameter inequality for surfaces \( Y \) with \( Sc^2 \geq 2 \) imply the following.

\[ \left[ \text{[\cdots]} \right] \text{Pairs of Riemannian metrics } g \text{ and } g \leq g \text{ on compact } n \text{-manifolds } X \text{ homeomorphic to } S^2 \times Z, \text{ where } Z \text{ supports Riemannian metrics } sect \cdot curv \leq 0, \text{ satisfy, for } n \leq 8, \text{ the following:} \]

\[ \inf Sc(X, g) \cdot inj.\text{rad}(\tilde{X}, \tilde{g})^2 \leq 2\pi^2 \frac{2(n-1)}{n}, \]

where \( \tilde{X} \) is the universal covering of \( X \).\(^\text{14}\)

Furthermore, if \( \inf Sc(g)/\sup sect \cdot curv(g) \geq n(n-1) \), then Zhu’s area inequality \(^\text{15}\) implies 1.B(i) for \( m = 2 \), that is:

\[ \inf Sc(X, g) \cdot inj.\text{rad}(\tilde{X}, \tilde{g})^2 \leq 2\pi^2. \]

\(^{13}\)The extra \( T^n-m \)-warping factor associated with \( Y^m_{BM} \) is handled by twisting the relevant (already twisted) Dirac operator with a family of flat bundles over \( T^N \) or with a single almost flat bundle as in [GL1980] and in [Gr1996]. (See [Gr2021] for a pedestrian presentation of these topics).

\(^{14}\)See [Ze2019] [Ze2020] [Ce2020] [CZ2021] [WXY2021].

\(^{15}\)In fact, \( \inf Sc(X, g) \cdot inj.\text{rad}(\tilde{X}, \tilde{g})^2 < 2\pi^2 \frac{2(n-1)}{n} \), since, by the proof of [BMd], the inequality \( Sc^2(Y) \geq 2 \) implies that \( Y \) contains a pair of points with \( \text{dist}(y_1, y_2) < 2\pi^2 \frac{2(n-1)}{n} \).

Probably, it is not hard to show that \( \inf Sc(X, g) \cdot inj.\text{rad}(\tilde{X}, \tilde{g})^2 \lesssim 2\pi^2 \frac{2(n-1)}{n} - 0.01 \), but the corresponding sharp inequality \( \inf Sc(X, g) \cdot inj.\text{rad}(\tilde{X}, \tilde{g})^2 \leq 2\pi^2 \) remain problematic.\(^\text{16}\)See [Zho2019], [Zhu2020] and 2.8 in [Gr2021].
Here, the Dirac operator is not used and the spin issue doesn’t arise.

Besides, granted a mild generalization of theorem 4.6 from [SY2017], the condition $n \leq 8$ can be removed while 1.B in the spin case may follow for all $m$ and $n$ by adopting the techniques/ideas of J. Lohkamp [Loh2018]. (One has no idea what to do with spin for $m \geq 4$.)

(d) Contractibility $\lambda$-Radius. Given a function $\lambda(r) \geq r$, $r \geq 0$, define $\text{contr}_\lambda \text{rad}(X)$ for a metric space $X$ as the supremum of the radii of the balls $B_x(r) \subset X$, which are contractible in the concentric ball $B_x(\lambda(r)) \supset B_x(r)$ for all $x \in X$.

Observe that $\text{contr}_{1\lambda} \text{rad}(X) \geq \text{inj.rad}(X)$ and that if $X$ is a contractible manifold with a cocompact isometry group, e.g. $X$ is the universal covering of a compact aspherical manifold, then $\text{contr}_\lambda \text{rad}(X) = \infty$ for some function $\lambda(r)$.

Similarly to $[\bullet \cdot]_{m=2}$ one shows that if $n \leq 8$, then the metrics $g \geq \tilde{g}$ from $[\bullet \cdot]_{m=2}$ satisfy

$$\left[\circ \lambda \bullet\right]_{m=2} \text{contr}_\lambda \text{rad}(\tilde{X}, \tilde{g}) \leq \lambda \left(2 \pi \sqrt{\frac{n-1}{\inf \text{Sc}(X, g)}}\right),$$

for all function $\lambda(r)$. (This generalizes $[\bullet \cdot]_{m=2}$ for $\lambda(r) = r$.)

Next let $m = 3$ and recall (see 3.10.1 in [Gr2021]) that the filling radii of 3-manifolds are bounded in terms of their $\mathbb{T}^3$-stabilized scalar curvature $\text{Sc}^\ast(X) \geq \inf \text{Sc}(X)$ (defined in he next section) as follows

$$\text{fillrad}(X) \leq \frac{6 \pi}{\sqrt{\text{Sc}^\ast(X)}}$$

If $n = \text{dim}(X) \leq 8$, then the $\square^{33}(n, m = 3, N)$-inequality implies (by an easy argument, see (see [Gr1983]) that

$$\left[\circ \lambda \bullet\right]_{m=3} \text{contr}_\lambda \text{rad}(\tilde{X}, \tilde{g}) \leq \lambda \left(\frac{6 \pi}{\sqrt{\text{Sc}^\ast(X, g)}}\right),$$

for all $\lambda = \lambda(r)$ and

$$\lambda(r) = 2\lambda(2\lambda(2\lambda(r)))$$

(Desingularization of $\mu$-bubles needed for $n = \text{dim}(X) \geq 8$ seems here more difficult than that for $m = 2$.)

(e) Stable Quasi-conjugate $\lambda$-Radii. There are several candidates for a (bi)Lipschitz stable version of conjugate radius, where our definition below is, probably, provisional.

This "radius" of a Riemannian manifold $X = (X, g)$,

$$\text{conj.rad}(X) \geq \text{conjugate}(X),$$

is the supremum of the numbers $r$, such that there exist the following:

- (a) a smooth Riemannian manifold $\Theta$ of dimension $2n = 2\text{dim}(X)$ (instead of the space of tangent $r$-balls, $\mathbb{T}^n(X) = \{B_{0}(x)(r) \subset T_x(X)\}_{x \in X}$),
- (b) smooth maps

$$\Pi : \Theta \rightarrow X, \; \Upsilon : \Theta \rightarrow X, \; O : X \rightarrow \Theta,$$

17I can’t claim this as a theorem, since I haven’t mastered the argument in [SY2017].
(c) a homotopy \( I_t : \Theta \to \Theta, 0 \leq t < \infty \), with the following properties.

- The map \( \Pi \) is a smooth submersion, the fibers of which are denoted \( \Theta_x = \Pi^{-1}(x) \subset \Theta \).
- The maps \( \Upsilon_x = \Upsilon_{|\Theta_x} : \Theta_x \to X \) are locally diffeomorphic on all \( \Theta_x, x \in X \).
- The map \( O : X \to \Theta \) (which corresponds to the zero section \( \Theta : X \to T(X) \)) satisfies: \( O(x) \in \Theta_x = \Pi^{-1}(x) \subset \Theta \) for all \( x \in X \).
- The homotopy starts from the identity map, \( I_0 = id \), it preserves all \( \Theta_x \), i.e. \( \Theta_x \overset{\simeq}{\to} \Theta_x \) for all \( t \) and all \( x \in X \) and it fixes \( O(X) \), i.e. \( O \circ I_t(x) = x \) for all \( x \in X \).

Moreover, the maps \( \Theta \overset{I_t}{\to} \Theta \) are homeomorphisms\(^{18}\) for sufficiently large \( t \), and they converge to \( O \) for \( t \to \infty \), that is

\[ I_t(\theta) \to O(x) \text{ for } \theta \in \Theta_x \text{ and } t \to \infty. \]

Now let us turn to metric properties of \( O \) and \( I_t \) with respect to the induced metrics \( g_x = \Upsilon_x^*(g) \) on \( \Theta_x \).

- The distances from \( O(x) \in \Theta_x \) to the boundary \( \partial \Theta_x \) satisfy

\[ \text{dist}(x, \partial \Theta_x) > r, \]

- For all \( t \geq 0 \), all \( \rho > 0 \) and all \( x \in X \), the map \( I_t : \Theta_x \to \Theta_x \) sends \( \rho \)-balls around \( O(x) \in \Theta_x \) to the concentric \( \lambda(r) \)-balls.

1.1.A. Observation. The arguments in [CL2020] and [Gr2020], as expanded in section 3.10.3 of [Gr 2021], show that for all functions \( \lambda = \lambda(r) \geq r \), there exists a positive number \( R = R_\lambda < \infty \), such that all compact Riemannian manifolds \( X \) of dimensions \( n \leq 5 \) with \( \text{Sc}^c(X) \geq 0 \) satisfy the following \( \lambda \)-version of Green's [Sc] [conjr]_n:

\[ \text{conj}_{\lambda \text{rad}}(X) \leq \frac{R_\lambda}{\sqrt{\text{Sc}^c(X)}} \]

Notice that this does not yield the corresponding inequality for the contractibility radius of (the universal covering of) \( X \), since our definition doesn’t make \( \text{conj}_{\lambda \text{rad}}(X) \geq \text{contr}_{\lambda \text{rad}}(\tilde{X}) \); However, the inequality

\[ \text{contr}_{\lambda \text{rad}}(\tilde{X}) \leq \frac{R_\lambda}{\sqrt{\text{Sc}^c(X)}} \]

holds for \( \text{dim}(X) \leq 5 \) with the same \( R_\lambda \) and for the same reason as \( \text{conj}_{\lambda \text{rad}} \) does.

QUESTIONS. All of the above, including the \([\text{Sc}]_{n-k} \text{conj} \) conjecture, tells us precious little about bounds on "radii" \( \text{rad}^c(X) \) of a Riemannian manifold \( X = (X, g) \) with \( \text{Sc}(g) > 1 \), where such a bound must depend on the topology and/or metric geometry of \( X \) and where \( \text{rad}^c \) may stand for \( \text{inj}, \text{rad}, \text{conj} \), \( \text{rad} \), \( \text{inj}, \text{rad}(\tilde{X}) \), the contractibility radius \( \text{contr}_{\lambda \text{rad}}(X) \) or \( \tilde{X} \) for some \( \lambda(r) \) or for \( \text{conj}_{\lambda \text{rad}} \).

\(-\)

\(^{18}\)This bizarre condition is need to justify 1.2.A below.

\(^{19}\)Here we assume that \( X \) is complete with no boundary and this inequality means that the \( r \)-ball in \( \Theta_x \) around \( O(x) \) is compact.
Here are a few specific questions.

Let \((X, g_0)\) be a compact Riemannian \(n\)-manifold with \(\text{Sc}(g_0) > 0\).

When can \(g_0\) be \([\text{Sc} > 0]\)-homotopic\(^{20}\) to a metric \(g_1\) with \(\text{Sc}(g_1) \geq n(n-1) = \text{Sc}(S^n)\) and \(\text{inj.rad}(X, g_1) \geq \rho\) for a given \(\rho > 0\)?

(I) For instance, – this seems unlikely but not impossible – does such \([\text{Sc} > 0]\)-homotopy \(g_t\) exist for all \(\rho < \pi\) and all simply connected manifolds \(X\)?

(II) Or, does the inequality \(\text{Sc}(g) \geq n(n-1)\) for a metric \(g\) on \(X\) imply that \(\text{conj.rad}(X, g_1) \leq \pi - \varepsilon_n\) for some universal \(\varepsilon_n > 0\), e.g. for \(\varepsilon_n = 1/10^n\), unless \(g\) is \([\text{Sc} > 0]\)-homotopic to (is closed to?) a metric with constant sectional curvature on \(X\)?

(III) What are Scalrad\(^{-}\)-extremal manifolds \((X, g_0)\), i.e. such that
\[
\inf \text{Sc}(g_0) \cdot \text{rad}^*(X, g_0)^2 \geq \inf \text{Sc}(g) \cdot \text{rad}^*(X, g)^2
\]
for all Riemannian metrics \(g\) on \(X\)?

For instance, are compact irreducible symmetric spaces Scalrad\(^{-}\)-extremal, say for \(\text{rad}^* = \text{inj.rad}\)?

(IV) How much does a bound on the diameter of \(X\) contribute to the inequality \(\text{rad}^*(X) \geq \rho\) for manifolds with \(\text{Sc}(X) \geq 1\)?

For instance, what can be said about the topological complexity of an \(X\) which admits a metric \(g\), such that
\[
\text{Sc}(g) \geq 1, \text{inj.rad}(X, g) \geq 0.01, \text{diam}(X, g) \leq 1?
\]

Do these inequalities limit some complexity of the \([\text{Sc} > 0]\)-homotopy class of \(g\)?

Remark. If we drop the inequality \(\text{Sc}(g_1) \geq 1\), the remaining two, which amounts to \(\frac{\text{diam}}{\text{inj.rad}} \leq 100\), don’t much restrict the topology of \(X\).

For instance all surfaces \(X\) admit Riemannian metrics with \(\frac{\text{diam}}{\text{inj.rad}} \leq 4\) obtained with ramified coverings \(X \rightarrow \mathbb{T}^2\) and/or \(X \rightarrow S^2\) (see [Ba1996]).

Similarly, by ramifying \(X_n \rightarrow S^n\) one can produce topologically complicated manifolds with small ratios \(\frac{\text{diam}}{\text{inj.rad}}\).

In fact, using Alexander’s braid-knot theorem + Hilden’s 3-fold branched covering theorem (applied to \(S^3\) preliminary very strongly ramified over two linked Hopf circles) one can show that

all compact 3-manifolds admit Riemannian metrics with \(\frac{\text{diam}}{\text{inj.rad}} \leq 100\).

I am not certain if this is true for all \(n\)-manifolds.

1.1.B. Riemannian Foliations. Let
\[
\mathcal{X} = (\mathcal{X}, g) = (Q, \mathcal{X}, g)
\]
be a smooth Riemannian \(n\)-foliation, that is a smooth manifold \(Q\) foliated into smooth \(n\)-dimensional manifolds \(X\), and \(g\) is a smooth positive definite quadratic form on the (sub)bundle \(T(X) \subset T(Q)\) tangent to the leaves.

Let \(X_q \subset Q, q \in Q\) denotes the leaf passing trough the point \(q\) and let \(\text{Sc}(X, q), \text{conj.rad}(X), \text{etc.}\) be the corresponding invariants of \(X_q\) at \(q\) defined in section 1, where the corresponding \(\inf\)-invariant refer to \(\inf_{q \in Q}\).

Observe that Green’s integration argument applies to Riemannian foliations with transversal measures on compact manifolds\(^{22}\) and that \([\text{Sc}(\text{conj})]_{n-k}\)

\(^{20}\text{This means a homotopy } g_t \text{ of Riemannian metrics with } \text{Sc}(g_t) > 0, \text{ for all } t \in [0,1].\)

\(^{21}\text{This may be also true for some reducible spaces such as } S^2 \times S^2, \text{ for instance.}\)

\(^{22}\text{Here one needs only transversal continuity of } \mathcal{X} \text{ and } g.\)
Conjecture 1.A makes sense for foliations.

2 $T^\infty$-Stabilized Curvatures $Sc^\infty$, $Sc^3\infty$, $Sc^{33}\infty$ and Multispreads $\overset{\perp}{\overset{\ominus}{\Delta}}(h)$ on Homology

$T^\infty$-Extensions. A "warped" $\mathbb{T}^N$-extension, $N = 0, 1, \ldots$, where $\mathbb{T}^N$ is the $N$-torus, of a Riemannian manifold $Y = (Y, g)$, where $Y$ may have a boundary, is a Riemannian manifold denoted $Y^*_N = Y \times \mathbb{T}^N$, that is the product $Y \times \mathbb{T}^N$ with a Riemannian metric $g^*_N$ of the following form:

$$g^*_N = dy^2 + \sum_{i=1}^{N} \varphi_i^2 dt_i^2$$

for some smooth positive functions $\varphi_i(y) \geq 0$ on $Y$, which are strictly positive ($>0$) in the interior $Y_N \setminus \partial Y$ of $Y$.

Clearly, this $g^*_N$ is invariant under the obvious action of $\mathbb{T}^N$ on $Y^*_N$ and $\varphi_i(y)$ are equal to the $g^*$-lengths of the $\mathbb{T}^1$-orbits of the points $g^* = (y, \sigma) \in Y^*$ under this action.

$T^\infty$-Stabilized Scalar Curvature(s). Let us agree that the inequality

$$Sc^3\infty_N(Y, y) > \sigma(y)$$

for a given function $\sigma(y)$ on $Y$ signifies that

there exists a $\mathbb{T}^N$-extension $Y^*_N$ of $Y$ the scalar curvature of which satisfies

$$Sc(Y^*_N, y) > \sigma(y),$$

where this scalar curvature is a function $Y = Y^*_N/\mathbb{T}^N$, since the metric $g^*_N$ is $\mathbb{T}^N$-invariant.

Clearly

$$Sc(Y) \leq Sc^3\infty(Y) \leq \ldots \leq Sc^{3\infty}(Y) \leq \ldots .$$

We agree that $Sc^\infty(Y)$ stands for $Sc^{3\infty}(Y) = \sup_N Sc^{3\infty}_N(Y)$, i.e. $Sc^{3\infty}$ with an arbitrarily large $N$ and observe that $Sc^{3\infty}(Y)$ (and even $Sc^{3\infty}_N(Y)$) of manifolds $Y$ with boundaries (see below) can be significantly greater than $Sc(X)$. Yet, most known geometric properties of manifolds with $Sc > \sigma$ are also enjoyed by those with $Sc^{3\infty} > \sigma$.

To have a single non-ambiguous number, let

$$Sc^\infty(Y)$$

be the supremum of the numbers $\sigma$, such that $Sc^{3\infty}(Y) > \sigma$.

Examples. It is not hard to show the following (see [Gr2023]).

\[\text{In fact, } Sc\left( g + \sum_{i=1}^N \varphi_i^2 dt_i^2 \right) = Sc(y) - 2\Delta \Phi - ||\nabla \Phi||^2 - \sum_i ||\nabla \log \varphi_i||^2 \text{ for } \Phi(\sigma) = \log(\varphi_1(y) \cdot \ldots \cdot \varphi_N(y)), \text{ see [GL1983], [SY2017], [Zhu2019], or compute yourself.}\]

\[\text{The proofs of the bounds on the 2-waists of 3-manifolds } Y \text{ with } Sc(Y) \geq \sigma > 0 \text{ (see [MN2011], section 3.10 in [Gr2021] and LM2021]) do not apply, at least not immediately, to } Y \text{ with } Sc^\infty(Y) \geq \sigma.\]

\[\text{Also the } T^\infty\text{-stabilization of the known geometric 4d-inequalities obtained with the Seiberg-Witten equations (see [LB2021]) remains problematic.}\]
(a) **Rectangular solids** satisfy:

\[
Sc^a \left( \bigtimes_{i=1}^n [-a_i, b_i] \right) = 4 \lambda_1 \left( \bigtimes_{i=1}^n [-a_i, b_i] \right) = \sum_{i=1}^n \frac{4 \pi^2}{(b_i - a_i)^2}.
\]

where this \( \lambda_1 \) is the first eigenvalue of the Laplace operator with the Dirichlet boundary condition in \( \bigtimes_{i=1}^n [-a_i, b_i] \).

(b) **Unit hemispheres** satisfy

\[
Sc^a \left( S_n^a \right) = n(n + 3) = Sc(S_n^a) + 4 \lambda_1(S_n^a).
\]

(c) **Unit balls** satisfy

\[
Sc^a \left( B_n^a \right) = 4 \lambda_1(B_n^a) > n^2 + 5n + n^{4/3}.
\]

**Definition of \( Sc^{23} \).** Given a homology class in a Riemannian manifold \( X \),

\[ h \in H_n(X), \]

write

\[ Sc_N^{23}(h) = Sc_N^{23}(h)_{\text{dist}} > \psi \]

for a function \( \psi = \psi(x) \) on \( X \), if there exists a compact oriented (a priori disconnected) Riemannian \( k \)-manifold \( Y \) and a 1-Lipschitz (i.e. distance non-increasing) map

\[ f : Y \to X, \text{ such that } f_\ast[Y] = h, \]

where \([Y] \in H_k(Y)\) is the fundamental class of \( Y \), and such that the \( \mathbb{T}^N \)-stabilized scalar curvature of \( Y \) is bounded from below by \( \psi \), i.e.

\[ Sc_N^{23}(Y) > \psi \circ f \]

for the composed function \( \psi \circ f(y) = \psi(f(y)) \).

\[ [Sc^{23}(X)]. \] Here \( Sc^{23}(X) = Sc^{23}[X] \) for the fundamental homology class

\[ [X] \in H_{\text{dim}(X)}(X) \]

of an oriented manifold \( X \).

Observe that

\[ Sc^{23}(X) \geq Sc^a(X), \]

where this inequality is strict, for instance, for products \( X = X_1 \times X_2 \), where \( X_1 \) has constant scalar curvature \( \sigma_1 > 0 \) and \( Sc(X_2) < 0 \).

On the other hand the equality is expected for many manifolds with positive (sectional, Ricci?) curvatures, but at the present moment the proof is available only for spheres \( S^3 \), \( S^4 \) and closely related manifolds of dimensions 3 and 4. Yet higher dimensional equalities of this kind e.g. for compact symmetric spaces with non-zero Euler characteristics, are available (see GS2000 and section 8.1) for the spinor versions of \( Sc^{23} \) defined below.

**Open manifolds and Homology with Infinite Support.** The above definition of \( Sc^{23} \) naturally generalizes to homology classes with *infinite supports* in non-compact manifolds, denoted

\[ h \in H_n(X, \partial_\infty X), \]

where \( \partial_\infty X \) stands for the complement of an unspecified "arbitrarily large" compact subset in \( X \).
In other words, \( H_m(X, \partial_\infty X) \) is the projective limit of \( H_m(X, X_i) \) over all \( X_i \subset X \) with compact complements.

Here the manifolds \( Y \) may be non-compact and the maps \( f : Y \to X \) are required to be proper, that is \( f \) sends \( \partial_\infty Y \to \partial_\infty X \), which means that the \( f \)-pullbacks of compact subsets in \( X \) are compact.

\[ \exists^* \text{-Spin.} \] One defines two spin versions of \( S^{33*}_\text{sp}(h) \):
\[ S^{33*}_\text{spin}(h), \] where the manifolds \( Y \) are spin, \( \tilde{h} \), where the universal coverings \( \tilde{Y} \) of \( Y \) are spin.

Clearly,
\[ S^{33*}_\text{spin}(h) \leq S^{33*}_\text{sp}(h) \leq S^{33*}(h) . \]

**Homological Pullbacks.** Let \( f : V \to W \), where \( \dim(W) - \dim(V) = k \), be a proper map between orientable manifolds, and recall that the cohomology homomorphism \( f^* : H^i(W) \to H^i(V) \) composed with the Poincare duality isomorphisms defines the "pullback homomorphism" \( H_m(W) \to H_{m+k}(V) \) for \( m = \dim(V) - l \), which, for generic smooth maps, is implemented by taking \( f \)-pullbacks of \( m \)-cycles in \( W \).

Also, such a homomorphism is defined for topological fibrations and topological submersions \( V \to W \) with manifold fibers. \( ^{25} \)

**Multi-Spreads.** (Compare with section 7.1 in [Gr2021].) Let \( V \) be an \( n \)-dimensional Riemannian manifold (possibly non-compact) with a boundary, and let \( h \in H_m(V, \partial_\infty V) \) be a homology class (with infinite, i.e. non-compact, support if \( \partial_\infty \) is non-empty, i.e. \( V \) is non-compact).

The \( \square^+ \)-spread \( \square(h) \) is the supremum of the numbers \( d \geq 0 \), for which there exists

a continuous boundary-to-boundary map from \( V \) to the \( k \)-cube for \( k = n - m \), \( \bar{\psi} = (\psi_1, \ldots, \psi_i, \ldots, \psi_k) : V \to \square = [-1,1]^k \), \( \psi_i : V \to [-1,1] \), such that the following two conditions are satisfied.

1. The class \( h \) is equal to the \( \psi \)-pullback of the generator of \( h_0 \in H_0([-1,1]^k) \), that is, if \( \bar{\psi} \) is smooth, then \( h \) is represented by the \( \psi \)-pullback of a generic point \( p \) in the interior of the cube.

2. The distances between the pullbacks of the opposite faces in the cube \([−1,1]^k\),
\[ d_i = \text{dist}_V(\psi_i^{-1}(-1), \psi_i^{-1}(1)), i = 1, \ldots, k, \]
are bounded from below by the following inequality
\[ \left( \frac{1}{k} \sum_{i=1}^{k} \frac{1}{d_i^2} \right)^{1/2} \geq d, \]
that is
\[ \frac{1}{k} \sum_{i=1}^{k} \frac{1}{d_i^2} \leq \frac{1}{d^2}, \]
(e.g. if \( d_i \geq d \) for all \( i = 1, \ldots, k \).
(Equivalently, one could require the functions \( \psi_i : V \to [-1,1] \) to be \( \frac{1}{d_i} \)-Lipschitz.)

\( ^{25} \) Conceivably, some nonzero multiples of all homology classes \( h \) satisfy \( S^{33*}_{\text{sp}}(i \cdot h) \leq S^{33*}_{\text{spin}}(i \cdot h) \).

\( ^{26} \) This is explained in purely geometric terms in [Gr 2014].
Next define $\tilde{d}^i(h) \geq \Box^i(h)$ of a homology class $h \in H_n(X,\partial_\infty X)$ for an $n$-dimensional Riemannian manifold $X$ (possibly non-compact) with a boundary, as the supremum of the numbers $d \geq 0$, such that there exist

(i) a Riemannian manifold $\tilde{V}$ of dimension $n = \dim(X)$
(ii) a homology class $\tilde{h} \in H_n(\tilde{V};\partial_\infty \tilde{V})$ with $\Box^i(\tilde{h}) \geq d$.
(iii) a proper locally isometric map $\varphi : \tilde{V} \to X$, such that the induced homology homomorphism $\varphi_* : H_n(\tilde{V};\partial_\infty \tilde{V}) \to H_n(X;\partial_\infty X)$ sends $\tilde{h}$ to $h$, in writing: $\varphi_*(\tilde{h}) = h$.

2.A. *Enlargeable Manifolds and sect.curv $\leq 0$. A Riemannian manifold $Z$ is called enlargeable if there exist zero dimensional homology classes $h_i \in H_0(Z)$ with $\tilde{d}^i(h_i) \geq i$, $i = 1, 2, \ldots$.

For instance, complete manifolds with non-positive sectional curvatures are enlargeable, since their universal coverings contain arbitrarily large compact domains $\tilde{V}$, i.e., with arbitrarily large $1$-multispreads of their zero-dimensional homology.

It follows that if $X = Y \times Z$, where $Y$ is an oriented $m$-manifold and where $Z$ is a compact manifold, which admits a metric with non-positive sectional curvature, e.g., it is the $(n-m)$-torus, then the $\tilde{d}^i$-spread (like the $\Box^i$-spread) of the fundamental homology class $[Y] \in H_n(X,\partial_\infty X)$ of $Y = Y \times z_0 \subset X$, is infinite.

2.B. $\Box^{3\times}(n,m,N)$-Inequality. Let $X = (X,g)$ be an $n$-dimensional orientable Riemannian manifold with a boundary and let $h \in H_n(X,\partial_\infty X)$.

If $n = \dim(X) \leq 8$, then

$$S_c^{\Box^{3\times}}_{N+k}(h) \geq S_c^{\Box^X}(X) - \frac{n + m - 1}{n + N} \cdot \frac{4k\pi^2}{\tilde{d}^i(h)^2}, \quad k = n - m,$$

that signifies the implication

$$S_c^{\Box^X}(X) > \psi \implies S_c^{\Box^{3\times}}_{N+k}(h) > \psi$$

for all continuous functions $\psi = \psi(x)$ on $X$.

Furthermore, if $X$ is spin, then

$$S_c^{\Box^{3\times}}_{N+k,sp}(h) \geq S_c^{\Box^X}(X) - \frac{n + m - 1}{n + N} \cdot \frac{4k\pi^2}{\tilde{d}^i(h)^2},$$

and if the universal covering $\tilde{X}$ of $X$ is spin, then

$$S_c^{\Box^{3\times}}_{N+k,sp}(h) \geq S_c^{\Box^X}(X) - \frac{n + m - 1}{n + N} \cdot \frac{4k\pi^2}{\tilde{d}^i(h)^2}.$$

Proof. If $k = 1$ this follows from the equivariant separation theorem applied to $X \times \mathbb{T}^N$ with a metric $g^*$ and the general case follows by induction on $k$.

(See the following section for immediate corollaries of this and for open questions.)

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27This map does not have to send $\partial \tilde{V} \to \partial X$.
28See 5.4 in [Gr2021] also [Gr2021′] and compare with [Ri2020] and [GZ2021], where similar arguments are used.
It is commonly believed that a presence of singularities in minimal hypersurfaces and/or in stable $\mu$-bubbles for $n \geq 8$ is non-consequential for scalar curvature, which is confirmed by Natan Smale’s generic regularity for $n = 8$ [Sm2003].

In the present case, this belief turns into the following.

2.C. $\Box^{33}(n > 8)$-Conjecture. The inequalities $[Sc^{33}]$, $[Sc^{33}_p]$ and $[Sc^{33}_p]$ hold true for all $m$, $n$, and $N$.

(It is conceivable that the inequality $[Sc^{33}_p]$ holds for non-zero multiples $i \cdot h \in H_m(X)$ in non-spin manifolds $X$.)

2.D. $\exists \&\Sigma$-Stabilization of Riemannian Foliations. Given a Riemannian foliation $(Q, \mathcal{Y}, g)$ with $m$-dimensional leaves $Y$ (see 1.1.B), define its $\mathbb{T}^n$-extension

$$(Q^n = Q \times \mathbb{T}^N, \mathcal{Y}_n, g^n),$$

by foliating $Q^n$ into $Y^n = Y \times \mathbb{T}^N$ and taking $g^n = g + \sum_i \varphi_i^2 dt_i^2$ for smooth positive functions $\varphi_i(q) \geq 0$ on $Q$.

Accordingly, we introduce $Sc^{3\&}(\mathcal{Y})$, where the sentence

$*$Sc$^{3\&}(\mathcal{Y})$ has property $P^n$

means that there exist functions $\varphi_i$, such that the corresponding $\mathcal{Y}^n$ has property $P$.

Conceivably, one may also define $Sc^{3\&}(h)$, $h \in H_M(R)$, for Riemannian manifolds $R$ by considering oriented $M$-dimensional manifolds $Q$ foliated by $m$-dimensional Riemannian $Y$ and maps $f : Q \to R$, which are distance decreasing on all leaves $Y$ with respect to the metrics $Sc^{3\&} \cdot g$ on $Y$.

Then one may formulate a counterpart of $\Box^{33}$-inequality 2.B for manifolds $R$ foliated into $n$-dimensional leaves for $\dim(R) - n = M - m$.

But there is little evidence for such a generalization of 2.B, where a more realistic conjecture should allow foliations with singular leaves $Y$. (See section 7.)

3 Maps to Spheres and to Sphere Bundles

Let $X = (X, g)$ and $S = (S, g_S)$ be Riemannian manifolds, let $\mathcal{P} \subset C(X, S)$ be a subset in the space of continuous maps $X \to S$, let $\Upsilon : X \times \mathcal{P} \to S$ be the evaluation map $\Upsilon : (x, a) \mapsto a(x)$. Let $B$ be an aspherical topological space and $\beta : X \times \mathcal{P} \to B$ a continuous map, e.g. $\beta$ is the composition of the coordinate projection $X \times \mathcal{P} \to X$ a map $\beta : X \to B$ and the classifying map map $\beta : X \to B = B(\Gamma)$, where $\Gamma$ is the fundamental group $\pi_1(X)$.

What we do in this section is motivated by the following.

3.A. Aspherical Parametric Mapping Conjecture. Let $X$ be a complete $n$-manifold, let $S$ be the unit $K$-sphere $S^K$, $K \geq 1$, let $[S^K]^* \in H^K(S^K) = \mathbb{Z}$ be the fundamental cohomology class and let $\alpha \in H^K(B)$.

If all $f \in \mathcal{P}$ are smooth maps $f : X \to S^K$, which are constant at infinity and which are strictly area decreasing on all smooth surfaces in $X$ with respect to the Riemannian metrics $Sc(g) \cdot g$ on $X$ and $m(m - 1)g_{S^K}$ on $S^K$ for $m = n - k$, then the cup product

$$\Upsilon^*[S^K]^* \sim \beta^*(\alpha) \in H^{K+k}(X \times \mathcal{P}, \partial_{\infty} X \times \mathcal{P})$$

16
vanishes on

\[ H_n(X) \otimes H_{K-m}(P) \subset H_{K+k}(X \times P). \]

(Recall that \( m = n - k \) and that \( \partial_\infty X \) stands for the complement of a large unspecified compact subset in \( X \).

Remarks (i\(_A\)) Theorem 3.E below confirms this conjecture, where \( \alpha \) is the fundamental cohomology class of an enlargable \( k \)-manifold, provided that \( \text{dim}(X) = n \leq 8 \) and the universal covering of \( X \) is spin.

(ii\(_A\)) This conjecture makes sense for singular spaces \( X \), where one can define scalar curvature, e.g. for Alexandrov spaces with lower curvature bounds.

(iii\(_A\)) An ultimate form of this conjecture must apply to the space/category of germs of \( n \)-submanifolds (or singular spaces) in \( S^K \) with induced metrics having \( Sc \geq \sigma > 0 \) (e.g. \( \sigma = m(m-1) \)).

The following results (corresponding to \( m = n \)) gives an idea of what may be expected in this regard.

3.B. Su-Wang-Zhang Foliated Area Contraction Theorem. Let \( \mathcal{X} = (\mathcal{X}, G) \) be a complete oriented Riemannian \( K \)-manifold smoothly foliated into \( m \)-manifolds \( \tilde{X} \subset \mathcal{X} \) and let \( f : \mathcal{X} \to S^K \) be a smooth map locally constant at infinity.

If the leaf-wise scalar curvature of \( \mathcal{X} \), denoted \( Sc^\ast(X) \) is bounded from below by \( Sc^\ast(X) > m(m-1) \) and if the map \( f \) is area decreasing on all smooth surfaces contained in the leaves, then, provided either the manifold \( \mathcal{X} \) is spin or the tangent bundle to the leaves is spin, the map \( f \) has zero degree,

\[ \deg(f) = 0. \]

Remarks. (i\(_B\)) The inequality \( Sc^\ast(X) > m(m-1) \) is required in [SWZ2021] only on the support of the differential of \( f \), while everywhere else \( Sc^\ast \) must be non-negative.

(ii\(_B\)) The conclusion \( \deg(f) = 0 \) holds (unless I misunderstand something in [SWZ2021]) if the map \( f \) is leaf-wise area decreasing for the metrics \( Sc^\ast \cdot G \) in \( \mathcal{X} \) and \( m(m-1)g_{Sk} \) in \( S^K \).

(iii\(_B\)) Probably, (I haven’t properly checked this) the spin condition can be relaxed to the spin of the lifts of the tangent bundles (of the manifold \( \mathcal{X} \) and of the leaves) to the universal covering of \( \mathcal{X} \).

Preparation for 3.C. Let \( P \) and \( Q \) be smooth manifolds possibly with boundaries and \( \Psi : Q \to P \) be a smooth submersion\(^{29}\) the fibers of which, denoted

\[ \tilde{X}_p = \Psi^{-1}(p) \subset Q, \quad p \in P, \]

are thought of as a family of manifolds parametrized by \( P \).

Let \( \mathcal{X} \) be the foliations of \( Q \) into (the connected components of) the fibers \( \tilde{X}_p \) and let \( g \) be a Riemannian metric on \( \mathcal{X} \).

(Recall the fibers \( \tilde{X}_p \).)

Let \( S^\ast \to P \) be a \( K \)-dimensional sphere bundle, let \( \delta : P \to S^\ast \) be a section of this bundle and let \( P_\ast \) be the image of the opposite section,

\[ P_\ast = -\delta(P) \subset S^\ast. \]

\(^{29}\)The differential \( d\Psi : T(Q) \to T(P) \) has everywhere rank \( = \text{dim}(P) \)

17
Let $\theta^* \in H^K(S^*, P_\bullet)$ be the Thom cohomology class of the $K$-ball bundle $S^* \times P_\bullet \to P$, that is the Poincaré dual to

$$\delta_\bullet [P] \in H_d(S^*, \partial S^*),$$

where $d = \dim(P)$ and $[P] \in H_d(P, \partial P)$ is the fundamental class of $P$.

Let

$$H_{ij}(Q, Q \setminus Q_0) = H_{ij}(\Psi(Q, Q \setminus Q_0), Q_0 \subset Q,$

be the group of relative homology classes of possibly infinite relative $i$-cycles $C \subset Q$, such that the images $\Psi(C) \subset P$ are precompact and such that $\dim(\Psi(C)) \leq j$ and $\dim(\Psi(\partial C)) \leq j - 1$.

Let

$$\mathcal{F} : Q \to S^*$$

be a smooth fiber bundle map\(^{30}\) such that

(i) the pullback $\Psi^{-1}(\partial P) \subset Q$ is sent by $\mathcal{F}$ to $P_\bullet \subset S^*$,

(ii) the complements of "uniformly controllably large" compact subsets $R_p$ in the fibers $X_p \subset Q$ are also sent to $P_\bullet$, i.e. there is a closed subset, $R \subset Q$ such that the restriction $\Psi|_R : R \to P$ is a proper map and $\mathcal{F}(Q \setminus R) \subset P_\bullet$.

Let

$$\mathcal{F}^*(\theta^*) \in H^K(Q, \mathcal{F}^{-1}(P_\bullet))$$

be the (relative) cohomology class induced by $\mathcal{F}$ from $\theta^* \in H^K(S^*, P_\bullet)$.

**Spherical Metric $\gamma$.** Let $\gamma = \{\gamma_p\}$, be a smooth family of Riemannian metrics in the $K$-spheres $S^*_p \subset S^*$ with constant sectional curvatures $\kappa_p$.

**3.C. Preliminary Area Contraction Inequality.** Let the above map $\mathcal{F}$ be fiberwise area decreasing with respect to the metrics $Sc^{\delta_{x^*}}(X) \cdot g$ in $\Omega_1^m$ and $n(n - 1)\kappa_\gamma = \{\kappa_p \cdot n(n - 1)\gamma_p\}$ in $S^*$, i.e. the $\mathcal{F}$-images of all smooth surfaces $E \times X_p, p \in P$, satisfy

$$\text{area}_{\gamma_p}(\mathcal{F}(E)) < \int_E Sc^{\delta_{x^*}}(\tilde{X}_p)(x_p) d_{g_p} x_p.$$

Let the fibers $\tilde{X}_p$ be manifolds without boundaries and the metrics $g_p$, on $\tilde{X}_p$ be complete.

If the sphere bundle $S^* \to P$ and the tangent bundle $T(X) \to Q$ are spin, then the cohomology class $\mathcal{F}^*(\theta^*)$ vanishes on $H_{K(K-n)}(Q, \Psi^{-1}(\partial P))$.

**About the Proof.** If the bundle is trivial, then 3.C for area decreasing maps $(X, Sc(X) \cdot g) \to Q)$ (instead of [area $<_{\delta_{x^*}}$]) follows from 3.B which need o be augmented with (ii$_B$); this, as explained in ?? below, yields 3.C for $Sc^{\delta_{x^*}}$ as well.

In fact, since the leaves $\tilde{X}_p$ are closed, the proof of 3.C is much simpler than that of 3.B in [SWZ2021] (where non-trivial holonomy creates the major problem) and it goes along the usual lines, roughly, as follows.

---

\(^{30}\)p-Fibers of $Q$ go to $p$-fibers of $S^*$ for all $p \in P$,

\(^{31}\)Recall that $Sc(g^\gamma)$ actually is a function on $Q$; see 2.D.
Let $S^p_\ast P \to P$ be the Clifford bundle of spinors in the fibers $S^p_\ast = S^K$, let $T^* \to T^N$ be an almost flat bundle with non zero top Chern class and all other classes zero and let $D_p^n$, $p \in P$, be the Dirac operators on the fibers $X^p = X_p \times T^N$ with coefficients in $S^p(S^p_\ast) \otimes T^* \to P$.

$$D_p^n : C^\infty(S^p_\ast(X^p) \otimes S^p_\ast(F(p)) \otimes T^* \to P).$$

Then, by Llarull’s trace evaluation (4.6 in [Ll1998]) of the curvature term in the twisted Lichnerowicz formula (see [GL1980], [GL1983] and 4.16 and 8.17 in [LM1989]), the inequality $\{area < \epsilon\}$ implies that the operators $D_p^n$ are positive. (Zhang’s Dirac deformation argument [Zha2020] allows $inf_x Sc(X, x) = 0$ at infinity, but this is unneeded for our applications.)

Hence, the $K(P)$-theoretic relative $\tau$-index (see §13 in [GL1983] and references therein) is zero. Then, if $K$, $N$ and $dim(X_p)$ are even, the relative version of the Atiyah-Singer index theorem for families [GL1983] shows that

$$F^\tau(\theta^\tau)(h) = 0 \text{ for all } h \in H_{K/K-n}(Q, \Psi^{-1}(\partial P)).$$

Finely, a spherical suspension construction, (see [Li1998] and sections 3.4 and 6.3.4 in [Gr2021] reduces the case of odd dimensions to that of the above even ones. QED.

Remarks. (iC) The above mentioned spherical suspension construction also allows a reduction of the theorem to the case, where sphere bundle $S^\ast \to P$ is trivial, as follows (compare with he proof of theorem 13.8 in [GL1983]).

Let $S^\ast \supset S^\ast r$ be an extension of $S^\ast \to P$ to a trivial sphere bundle, $S^\ast = \ast = \ast \ast$ and $dim(P)$ and let $\mathbb{R}^{K^\ast+1} = S^1 \supset S^\ast_\ast = p \times S^\ast$ be the Euclidean spaces containing the fibers of this trivial sphere bundle.

Let $R^\ast_p$, $p \in P$, be the $(K^\ast - K^\ast)$-dimensional family of unit vectors $\nu_r \in T^{-\delta(p)}(\mathbb{R}^{K^\ast+1})$ normal to $S^\ast_\ast$ and pointing inward the unit ball in $\mathbb{R}^{K^\ast+1}$ bounded by $S^\ast_\ast$.

Let $S^p_\ast \subset S^p_\ast r$, $r \in R^\ast_p$, be $K$-dimensional subspheres $S^p_\ast \subset S^p_\ast$ obtained by intersecting $S^p_\ast$ with affine subspaces of dimension $K^\ast + 1$ in $\mathbb{R}^{K^\ast+1}$, spanned by tangent subspaces $T_{\delta(p)}(S^p_\ast) \subset T_{\delta(p)}(\mathbb{R}^{K^\ast+1}) \subset S^\ast$ and vectors $\nu_r$, where we allow vectors $\nu_r$ tangent to $S^p_\ast \subset S^p_\ast$, where the corresponding $K$-spheres $S^p_\ast$ collapse to the point $-\delta(p) \in S^p_\ast$.

Let $R^\ast \to P$ be the fibration with the fibers $R^\ast_p$ and let $Q^\ast \to Q$ be the induced fibration over $Q$.

The map $F : Q \to S^\ast$ naturally suspends to a map $F^\ast : Q^\ast \to S^{\ast \ast}$, where $S^{\ast \ast} \to R^\ast$ is the (trivial!) sphere bundle induced from $S^\ast \to P$ by the map $R^\ast \to P$, and the inequality 3.A for $F$ reduces (this is an exercise) to that for $F^\ast$.

(iiC) A similar suspension argument allows us to replace $\{area < \epsilon\}$ by the corresponding inequality with the plane $Sc$ for the foliation of the Riemannian product $Q \times \mathbb{R}^N$ (regarded as the $\mathbb{Z}^N$-covering of $Q \times T^N$ in the definition of $Sc^\ast$) by the leaves $X_p \times \mathbb{R}^N$ endowed with the metrics $g^\ast_p$ (lifted from $Q \times T^N$ to $Q \times \mathbb{R}^N$).

This $Q \times \mathbb{R}^N$ is mapped to the suspension $S^\ast \wedge S^N$ via the one point compactification maps $\mathbb{R}^N \to S^N$ composed with the $\epsilon$-scaling map

$$Q \times \mathbb{R}^N \xrightarrow{\varepsilon} Q \times \mathbb{R}^N \text{ for } (q, r) \mapsto (q, \varepsilon r).$$

19
(Recall that the spherical suspension corresponds to adding the trivial vector bundle of rank \( N \) to the \((K + 1)\) -vector bundle associated with \( S^* \).)

What is essential is that
- the map \( \varepsilon \) preserves the splitting of the tangent bundle \( T(Q \times \mathbb{R}^N) = T(Q) \times \mathbb{R}^N \);
- the map \( \varepsilon \) is \( g^* \)-isometric on the tangent spaces to \( Q \times r \);
- the map \( \varepsilon \) is \( \varepsilon \)-contracting on all \( q \times \mathbb{R}^N \) with respect to the metric \( g^* \).

Therefore, Llarull’s trace evaluation (4.6 in [Ll1998]) of the curvature term in the twisted Lichnerowitz formula applied to a small positive \( \varepsilon \) yields positivity of leaf-wise Dirac operator twisted with \( S^*_{\mathcal{F}(p)}(S_p^* \wedge S_N^*) \) and the proof follows.

Notice that this argument automatically takes care of odd dimensions and it needs no additional twist with almost flat bundles \( \mathcal{T}^* \). However, when it comes to rigidity theorems, such a twist, or rather twisting with families of flat bundles, makes he proofs easier.

(iiiC) The spin condition on \( T(X) \) can be relaxed to spin of the lift of \( T(X) \) to the universal covering of \( X \) (as in §§9, 9, 1 in [Gr1996] and section 10 in [Gr2019] and footnote 78 in section 2.7 in [Gr2021]) but one has no idea what to do in the fully non-spin situation.

(ivC) The recent development in the Dirac index theory for manifolds with boundaries (see [CZ2021], [WXY2021] and references therein) shows – I haven’t truly checked this – that completeness of the fibers \( X_p \) can be replaced by a specific (finite!) lower bound on the distances from the supports of the differentials of the maps \( \mathcal{F}_p : X_p \to S_p^* \) to \( \partial X_p \) (and /or to \( \partial \omega X_p \)).

Let us now turn to the case \( m \leq n \) and let us define the \( \tilde{\mathcal{D}}^i \)-multispread of an \( h \in H_{j/k}(Q, Q \setminus \Psi^{-1}(\partial P)) \) by exhausting \( P \) by compact domains \( P_1, \ldots, P_i, \ldots, P \),

\[
P_1 \subset \ldots \subset P_i \subset \ldots \subset P,
\]

observing that \( h \) can be represented by infinite \( j \)-cycles \( C_i \subset P_i \) for all sufficiently large \( i \) letting \( h(i) = [C_i] \in H_j(\Psi^{-1}(P_i^\circ)) \) for the interiors \( P_i^\circ \subset P_i \), and setting

\[
\tilde{\mathcal{D}}^i(h) = \limsup_{i \to \infty} \tilde{\mathcal{D}}^i(h(i)).
\]

### 3.D. Area Contraction Conjecture

Let us allow the fibers \( \tilde{X}_p = \Psi^{-1}(p) \subset Q \) of the submersion \( \Psi : Q \to P \) to have boundaries and allow \( m \leq n = \text{dim}(\tilde{X}_p) \).

Let

\[
h_K \in H_{K(K) - m}(Q, \Psi^{-1}(\partial P)),
\]

and

\[
\sigma_N^* = \sigma_N^*(q) = Sc(g^*_p(q)) = \frac{n + N - 1}{n + N} \cdot \frac{4(n - m)\pi^2}{\tilde{\mathcal{D}}^i(h_K)^2}.
\]

Let the map \( \mathcal{F} \) be fiberwise area decreasing with respect to the metrics \( \sigma_N^* \cdot g \) in \( Q \) and \( \gamma_m \) in \( S^* \).

If the fibers \( \tilde{X}_p \) are metrically complete, then

\[
\mathcal{F}^*(\theta^*) (h_K) = 0.
\]

(This conjecture points toward our ultimate goal, but it may need an adjustment to avoid a possible counterexample in its present formulation, e.g. obtained with a parametric thin surgery.)
We prove below a special case of his conjecture by applying 3.B to families of $m$-dimensional submanifolds $Y_p \subset \tilde{X}_p$ for $m < n = \dim(X_p)$, where the scalar curvatures of $Y_p$ are bounded from below according to the $\Delta^2_\gamma(n,m)$-inequality 2.B.

**Preparation for Theorem 3.E.**

Let $P$ and $Q$ be smooth manifolds, possibly with boundaries, and let $\Psi : Q \to P$ be a true fibration, with an oriented, possibly disconnected, $n$-dimensional Riemannian fiber $\tilde{X} = (\tilde{X}, g)$, where the structure group $\Pi$ of the fibration is countable and its action on $\tilde{X}$ discrete, free, orientation preserving and where, since $Q$ is connected, the quotient space $\overline{X} = \tilde{X}/\Pi$ is connected.

Here, the fundamental group $\pi_1(P)$ acts on $\tilde{X}$ via a surjective homomorphism $\pi_1(P) \to \Pi$ and

$$Q = (\tilde{P} \times \tilde{X})/\pi_1(P)$$

for the Galois action of $\pi_1(P)$ on the universal covering $\tilde{P} \to P$ and the diagonal action of $\pi_1(P)$ on the product $\tilde{X} \times \tilde{P}$.

Thus, there is a natural covering map, say

$$\Phi : Q \to P \times \overline{X} = (\tilde{P} \times \tilde{X})/\pi_1(P) \times \pi_1(P).$$

Given homology classes $h_m \in H_m(\overline{X}, \partial_{\infty} \overline{X})$, where $m \leq n = \dim(\overline{X}) = \dim(\tilde{X})$, and $h_{K-m} \in H_{K-m}(P, \partial P)$, let

$$h_K \in H_{K|K-m}(Q, \Psi^{-1}(\partial P))$$

be the pullback of the product $h_{K-m} \otimes h_m \in H_{K}(P \times \overline{X}, \partial P \times \overline{X} \cup P \times \partial_{\infty} \overline{X})$ to $Q$ under the map $\Phi : Q \to P \times \overline{X}$.

Let $g$ be a Riemannian metric in $\overline{X}$ for which $(\overline{X}, g)$ is metrically complete (bounded subsets are precompact), and let $g_p$ be the Riemannian metrics in the fibers $\tilde{X}_p$ induced by the covering maps $\tilde{X}_p = \tilde{X} \to \overline{X}$.

Let $\varphi_i(x)$, $i = 1, \ldots, N$, be smooth positive functions on $\overline{X}$ and

$$g^* = g + \sum_{i=1}^{N} \varphi(x)^2 dt_i^2,$$

be the metric on $\overline{X} \times \mathbb{T}^N$.

Recall the homology class $h_m \in H_m(\overline{X}, \partial_{\infty} \overline{X})$ and let

$$\gamma^*_m = \gamma^*_m(x) = \text{Sc}(g^*) - \frac{n + N - 1}{n + N} \cdot \frac{4(n-m)^2}{\overline{\Delta}_p(h_m)^2}.$$

**3.E. Area Contraction Theorem.** Let the above map $\mathcal{F} : Q \to S^*$ be fiberwise area decreasing with respect to the metrics $\gamma^*_m \cdot g_p$ in the fibers $\tilde{X}_p = \tilde{X}$ of the fibration $\Psi : Q \to P$ and with the above normalized spherical metrics $\gamma_m$ in the fibers of $S^* \to P$.

If

- $\text{spin}$ either the manifold $\tilde{X}$ be spin or $m \leq 3$,
- $\text{compt}$ $\tilde{X} = (\tilde{X}, g)$ be metrically complete,
- $\text{cs}$ $n = \dim(\overline{X}) \leq 8$.

Then

$$\mathcal{F}^*(\theta^*)(h_K) = 0.$$

21
In fact, as we have already stated, this is a direct corollary of 2.B+3.B.

3.F. Representative Example of 3.D. Let $Y$ and $P$ be smooth compact oriented manifolds and let $X$ be a Riemannian manifold homeomorphic to $Y \times Z$, where $Z$ is a compact enlargeable $k$-manifold, e.g. $Z = \mathbb{T}^k$.

Let $\text{Sc}(X) \geq m(m-1)$ for $m = \dim(Y)$.

Let $\dim(P) = K$, let $S^{m+K} \subset \mathbb{R}^{m+K+1}$ be the unit sphere and let

$$f : X \times P \rightarrow S^{m+K}$$

be a smooth map, which is area decreasing on all $X = X_p = X \times p \subset X \times P, p \in P$.

If the universal covering of $Y$ is spin and if $\dim(X) \leq 8$, then

$$f_*([Y \times P]) = 0 \in H_{m+K}(S^{m+K}) = \mathbb{Z},$$

where $[Y \times P] \in H_{m+K}(X \times P)$ denotes the homology class of the submanifold $Y \times P = Y \times 0 \times P \subset X = Y \times \mathbb{T}^k \times P, 0 \in \mathbb{T}^k$. 

Remarks. (i) $\text{Grant validity of } \square^{32}(n > 8)$-conjecture 3.C (irrelevance of singularities) the proof of 3.D applies to all $n = m + k$.

But this proof breaks down for all dimensions if we allow mutually non-isometric fibers $X_p = (X, g_p)$, (even for 3.E), since the minimal hypersurfaces and stable $\mu$-bubbles are not continuous in $g_p$. \[32\]

This begs for a \textit{purely Dirac theoretic} proof of 3.D in the spirit of [CZ2021] and [WXY2021] that could yield more general inequalities, e.g. formulated in terms of the $K$-area. \[33\]

corrected on May 12 2023

3.G. $\text{Rad}_{\text{area}}(h_*/S^K)$-Invariant. Much (but not all) of the above can be expressed in the following terms.

Given a Riemannian manifold $X$ and $h_m \in H_m(X, \partial_{\infty}X)$, let $\text{Rad}_{\text{area}}(h_m/S^K)$ be the supremum of the numbers $R > 0$ with the following property.

There exists a cellular topological space $P$, a bundle $S^* \rightarrow P$ of $K$-spheres of radii $R$ with a distinguished section $P_0 \hookrightarrow S^*$ and a continuous map $F : X \times P \rightarrow S^*$, such that

- the map $F$ sends $\partial_{\infty}X \rightarrow P_0 \subset S^*$;
- the $p$-fiber maps $F_p = F_{|X \times p} : X \rightarrow S^*_p = S^K$ of $S$, are smooth and $C^1$-continuous in $p \in P$; logycl $\text{area}$ the maps $F_p$ are area decreasing for all $p \in P$;
- there exists a rational homology class $h_{K-m} \in H_{K-m}(P)$ such that the $F$-pullback of the Thom class $\theta^* \in H^K(S^*, P_0)$ doesn’t vanish on $h_m \otimes h_{K-m} \in H_K(X \times P, \partial_X \times P)$,

$$F^*(\theta^*)(h_m \otimes h_{K-m}) \neq 0.$$

The proof of 3.C shows that if the universal covering of $X$ is spin, then

$$\text{Sc}^{33^{\text{a}}}(h_m) \geq \sigma > 0 \implies \text{Rad}_{\text{area}}(h_m/S^K) \leq \sqrt{\frac{m(m-1)}{\sigma}}.$$

This, together with the $\square^{33}(n, m, N)$-inequality 2.B,

$$\text{Sc}^{33^{\text{a}}}(h_m) \geq \text{Sc}^{33^{\text{a}}}(X) = \frac{n + N - 1}{n + N} \cdot \frac{4(n - m)\pi^2}{\sigma^2}.$$

\[32\] We go around this discontinuity for $m = 3$ and $K = 1$ in the 4d-example 7.C.

\[33\] The only feasible approach to such inequalities lies in the Dirac theoretic realm but the scalar curvature geometry of minimal hypersurface and $\mu$-bubbles depending on parameters is interesting in its own right; we say a few words about it in section 7.
proved for \( n = \text{dim}(X) \leq 8 \) yields the corresponding lower bound on \( \text{Radarea}(h_{m}/S^K) \) in terms of \( \text{Sc}(X) \) and the \( \Box^1 \)-spread of \( h_{m} \).

**Remark on \( \text{Rad}_{\text{vol}}(h_{m}/S^K) \).** The above definition obviously generalizes from \( \text{area} = \text{vol}_2 \) to all \( \text{vol}_n \), where, for instance, Almgren’s max-min argument (details need checking) yields the inequality

\[
\text{Rad}_{\text{vol}}([X]_*/S^K) \leq \sqrt[n]{\text{vol}(X)/\text{vol}(S^n)}, \quad n = \text{dim}(X),
\]

and the convex partition argument yields the corresponding \( \varepsilon \)-waist inequality under the \( \mathbb{Z}_2 \)-version of \( \bullet_{\text{top}} \). (See [Gu2014] and references therein.)

### 4 Injectivity Radii and Maps to Spheres

**Notation and Definitions.** Let \( P \) be a Riemannian manifold, \( \hat{P} \to P \) be the universal covering over \( P \) regarded as a principal fibration with the fiber equal to the fundamental group \( \pi(P) \).

Let \( \Psi = \Psi_P : \hat{P}^\Delta \to P \)

be the fibration over \( P \) which has \( \hat{P} \) as a fiber, and which is associated with the universal covering \( \hat{P} \) over \( P \) for the deck (Galois) action of the fundamental group \( \pi(P) \) in this fiber.

Thus, the space \( \hat{P}^\Delta \) is equal to \( (\hat{P} \times \hat{P})/\pi_1(P) \) for the diagonal \( \pi_1(P) \)-action, where the two \( \hat{P} \)-factors of \( (\hat{P} \times \hat{P}) \) play here different roles.

Let \( \delta : P \to \hat{P}^\Delta \) be the section of this fibration defined by the diagonal embedding \( \hat{P} \to (\hat{P} \times \hat{P})/\pi_1(P) \).

Let \( r = r(p) \geq 0 \) be a continuous function on \( P \), let \( BT(r) = BT(P,r) \subset T(P) \) be the "subbundle" of \( r(P) \)-balls in the tangent spaces \( T_p(P) \).

Let \( \hat{B}^\Delta(r) \subset \hat{P}^\Delta \) be the "subbundle" of the \( r \)-balls \( \hat{B}_\delta(p)(r(p)) \) in the fibers \( \hat{P}_p^\Delta \subset \hat{P}^\Delta \), where these balls degenerate to points for \( r(p) = 0 \).

Let \( r(p) \leq \text{inj.rad}(\hat{P}_p)(\hat{P}_p) \)\(^{34} \) and let \( r(p) \) be strictly positive inside \( P \), i.e. \( r(p) > 0 \) for \( p \in P \setminus \partial P \), and observe that these "subbundles" are true ball bundles over \( P \setminus \partial P \).

In fact, the inverse exponential maps in the fibers \( \hat{B}_\delta(p) = \hat{P} \) isomorphically send \( \hat{B}_\delta^\Delta(r) \) to the bundle \( BT(r) \),

\[
\exp^{-1} : \hat{B}_\delta^\Delta(r) \to BT(r) = BT(P,r),
\]

where the tangent \( r \)-balls \( BT_p(r(p)) \) are identified with the tangent balls to the fibers \( \hat{P}_p^\Delta \) at \( \delta(p) \in \hat{P}_p^\Delta \).

Let \( S^* = S^*(r) \to P \) be the \( K \)-sphere "bundle", \( K = \text{dim}(P) \) that is obtained by shrinking the complements of the open balls \( BT_x(r(x)) \subset T_P(P) \) to points and let \( P_\bullet \subset S^* \) be the image of the section \( \delta_\bullet : P \to S^* \) which sends \( p \in P \) to these points.

---

\(^{34}\)This fails to be a true bundle at the points where \( r(p) = 0 \).  
\(^{35}\)If \( P \) is geodesically incomplete, i.e. the exponential map at an \( p \in P \) is defined only on certain open tangent ball \( BT_p(R_p) \subset T_P(P) \), then, by definition, the inequality \( \text{inj.rad}_p \geq r \) signifies that \( r \leq R_p \), i.e. the exponential map \( \exp_p : BT_p(r) \to P \) is defined and it is diffeomorphism on its image.
If $p \in \partial P$ and $r(p) = 0$, we agree that the "fiber" $S_p^* \subset S^*$ consists of a single point $\bullet_p = \delta_s(p)$.

Let
$$\tilde{B} : \tilde{P}^\Delta \to S^*(r)$$
be the composition of $\tilde{\exp}^{-1} : \tilde{B}^\Delta_0(r) \to BT(r) \subset T(P)$ with the (tautological) quotient map $T(X) \to S^*(r)$.

Let $E : P^\Delta \to P$ be the map, where each fiber $\tilde{P}_p^\Delta = \tilde{P} \subset \tilde{P}^\Delta$ goes to $P$ by the covering map $E_p = \tilde{P} \to P$, which sends $\delta(p) \mapsto p$.

Let $X$ be a smooth $m$-dimensional manifold and let $f : X \to P$ be a proper map.

Let $\tilde{Q}_X$ be the product of $\tilde{P}^\Delta \to P$ and $X \to P$ over $P$, i.e.
$$\tilde{Q}_X = \{(\tilde{p}^\Delta, x) \in (\tilde{P}^\Delta, X)_{E(\tilde{p}^\Delta) \circ f(x)} \}.$$

(If $X \subset P$, then $\tilde{Q}_X = E^{-1}(X).$

Let the map
$$\tilde{f}^\Delta : \tilde{Q}_X \to \tilde{P}^\Delta,$$
be defined by the projection $(\tilde{p}^\Delta, x) \mapsto \tilde{p}^\Delta \in \tilde{P}^\Delta$, thus $E \circ \tilde{f}^\Delta(\tilde{p}^\Delta, x) = f(x)$, and let
$$\Psi_X = \Psi_p \circ \tilde{f}^\Delta : \tilde{Q}_X \to P.$$
Observe that this $\Psi_X$ is a fibration over $P$, where the fiber $\tilde{Q}_X,p$ is equal to the covering of $X$ induced by the map $f : X \to P$ from the covering $E_p : \tilde{P}_p^\Delta \to P$, and that $\tilde{Q}_X$ covers the product of $P$ and $X$, where this covering (map), say
$$\Psi^X_X : \tilde{Q}_X \to P \times X,$$
is induced from the covering $\tilde{P}^\Delta = (\tilde{P} \times \tilde{P})/\pi_1(P) \to P \times P$ by the map $P \times X \to P \times P$ for $(p, x) \mapsto (p, f(x))$.

Let
$$\tilde{F}_{X,r} = \tilde{B} \circ \tilde{f}^\Delta : \tilde{Q}_X \to S^*(r).$$

4.A. Nonvanishing Lemma.\textsuperscript{150} Let $P$ and $X$ be Riemannian manifolds, possibly with boundaries and $f : X \to P$ be a proper map with the image in the interior of $P$, i.e. $f(X) \subset P \setminus \partial P.$ and let $h_m \in H_m(X, \partial_\infty X)$.

Let $r(x) \leq \text{inj.rad}_x(P)$ be a continuous function on $P$, which is strictly positive inside $P$ and which is bounded by distance to $P_0$ outside a compact subset $P_1 \subset P$

$$r(x) \leq \text{dist}(x, P_0) \text{ for } x \in P \setminus P_1,$$

where necessarily $P_1 \supset P_0$.

If the $\mathbb{Q}$-tensorisation of the image of the class $h_m$ in $P$ doesn’t vanish,

$$0 \neq f_*(h_m) \otimes \mathbb{Q} \in H_m(P, \partial_\infty P; \mathbb{Q}),$$

then the pullback of the Thom class $\theta^* \in H^k(S^*, P_\bullet)$ under the map $\tilde{F}_{X,r}$,

$$\tilde{F}_{X,r}^*(\theta^*) \in H^k(\tilde{Q}_X, \tilde{F}_{X,r}^1(P_\bullet)),$$

\textsuperscript{150}Compare with lemma 13.5 in [GL1983].
doesn’t vanish on \( H_{K/K-m}(\bar{Q}_X, \Psi_X^{-1}(\partial P)) \).

Proof. Let \( h_{K-m} \in H_{K-m}(P, \partial P) \) for \( K = \text{dim}(P) \) be a homology class, such that the intersection number \([h_m \cap h_{K-m}]_{\text{int}} \in \mathbb{Z}\) doesn’t vanish (we assume \( P \) is connected oriented) and let

\[
\check{h}_K \in H_{K(J-K-m)}(\check{Q}_X, (\check{Q}_X, \Psi_X^{-1}(\partial P \times X \cup P \times \partial \infty X)) \subset H_K(\check{Q}_X, \Psi_X^{-1}(\partial P \times X \cup P \times \partial \infty X))
\]

(for the above \( \check{Q}_X \subset \check{P} \times X \) and \( \Psi_X : \check{Q}_X \to P \)) be the pullback of the product

\[
h_{K-m} \otimes h_m \in H_K(P \times X, \partial P \times X \cup P \times \partial \infty X)
\]

under the natural (covering) map

\[
\Psi : \check{Q}_X \to P \times X.
\]

The class \( \check{F}^*_\nu_r(\theta^*) \) doesn’t change if we replace \( r \) by a smaller continuous positive function \( r'(x) \leq r(x) \); thus our geometrical lemma reduces to a purely topological one, where \( r \) is taken arbitrarily small.

To simplify further, properly embed \( P \subset \mathbb{R}^{2K} \) and replace \( P \) by a small regular neighbourhood \( P^* \subset \mathbb{R}^{2K} \).

By the Thom isomorphism, this reduces the lemma to the case where the \( K \)-sphere bundle bundle \( S^* \) is trivial

\[
S^* = P \times S^K \quad \text{(with } 2K \text{ instead of } K).\]

Here the Thom class \( \theta^* \) is induced from the fundamental class of the sphere \( S^K \) by the map \( S^* \to S^K \) and non-vanishing of \( \check{F}^*_\nu_r(\theta^*) \) follows by the obvious degrees comparison argument as in [GL1983]. QED.

4.B. Let us give geometric conditions that make the class \( \check{F}^*_\nu_r(\theta^*) \), hence homology image class \( f_*(X) \in H_m(P) \), vanish.

Definition of \( \text{Rad}_p^*(P) \). The radius

\[
\text{Rad}^*_p(P, r) = \text{Rad}^*_p(P, r)_{\text{area}} \quad \text{for } r \leq \text{inj.rad}_p(P)
\]

is the supremum of the numbers \( R > 0 \), such that the open ball \( BT_p(r) \subset T_p(P) = \mathbb{R}^n \) for \( n = \text{dim}(P) \) and \( r = \text{inj.rad}_p(P) \) admits a smooth \( O(n-1) \)-equivariant map to the \( n \)-sphere of radius \( R \), say

\[
\alpha_{x,r} : BT_p(r) \to S^n(R),
\]

where \( 0 \in BT_p(r) \) goes to the south pole and the boundary sphere \( S^{n-1}_p(r) = \partial BT_p(r) \) to the north pole, and such that the composition of this \( \alpha \) with the inverse exponential map that sends the open ball \( B_p(r) \subset P \) to the \( n \)-sphere,

\[
\alpha_{x,r} \circ (\exp_p)^{-1} : B_p(r) \to S^n(R),
\]

is area decreasing.

Example. Let \( a(r, \kappa) \) denote the area of the disk of radius \( r \) in the complete simply connected surface with constant curvature \( \kappa \), e.g.

\[
a(r, 1) = 2\pi(1 - \cos r), \quad a(\pi, 0) = r^2, \quad a(r, -1) = 2\pi(\cosh r - 1),
\]

\[
\text{for } a \in R.
\]

Example. Let \( a(r, \kappa) \) denote the area of the disk of radius \( r \) in the complete simply connected surface with constant curvature \( \kappa \), e.g.

\[
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\]

\[
\text{for } a \in R.
\]
and let
\[ R^*(r, \kappa) = \sqrt{\frac{a(r, \kappa)}{4\pi}} \]
be the radius of the 2-sphere with area \( a(r, \kappa) \).

Then, by the textbook comparison theorem, the inequalities
\[ \text{sect.curv}(P) \leq \kappa \text{ and inj.rad}_p(P) \geq r, \text{ where } \kappa \leq \frac{1}{r}, \]
imply that
\[ \text{rad}\_p^*(P) \geq R^*(r, \kappa). \]

for all complete Riemannian \( n \)-manifold \( P \).

**Preparation for Theorem 4.C.**

Let, as earlier, \( P \) and \( X = (X, g) \) be complete Riemannian manifolds, possibly with boundaries, and let \( f : X \to P \setminus \partial P \)
be a smooth proper map. Let \( U = U_f \subset P \) be the set of points \( p \in P \), which are closer to the the image of \( f \), than to the boundary of \( P \),
\[ U = \{ p \in P \mid \text{dist}(p, f(X)) \leq \text{dist}(p, \partial P) \} \]
and let
\[ \tilde{R}^* = \tilde{R}^*_U = \inf_{p \in U} \text{Rad}_{\tilde{p}}(\tilde{P}, r = \text{inj.rad}_p(\tilde{P})). \]

for the points \( \tilde{p} \) over \( p \in P \) in the universal covering \( \tilde{P} \) of \( P \).

Now we are able to state the **main result** of the present paper.

**4.C. Injectivity Radius Theorem.** Let the map \( f : X \to P \setminus \partial P \)
be area decreasing with respect to the metric \( \text{Sc}^\alpha_N(X) \cdot g \) in \( X \) and the original metric in \( P \) (used in the definition of \( \tilde{R}^*_p \) and let
\[ h \in H_m(X, \partial_{\infty} X). \]

If the \( \mathbb{Q} \)-tensorisation of the \( f \)-image of \( h \) in the rational homology group \( H_k(P, \partial_{\infty}P; \mathbb{Q}) \) doesn’t vanish,
\[ f_* (h) \otimes \mathbb{Q} \neq 0, \]
then, in the following \( \bullet_1,\bullet_2,\bullet_3 \) cases, the \( \Box^1 \)-spread of \( h \) is related to the above \( \tilde{R}^* \) by the inequality
\[ [\Box^1 \& \tilde{R}^*] \]
\[ \frac{n + N - 1}{n + N} \frac{4(n-m)\pi^2}{(\Box^1(h))^2} \leq \text{Sc}_N^\alpha(X) + \frac{m(m-1)}{(\tilde{R}^*)^2}. \]

\( \bullet_1 \quad \text{The universal covering of } X \text{ is spin and } m = n, \)
\( \bullet_2 \quad \text{the universal covering of } X \text{ is spin and } n \leq 8, \)
\( \bullet_3 \quad n \leq 8 \text{ and } m \leq 3. \)

**Proof.** The above example 4.B shows that he map \( \tilde{F}_{X,r} = \tilde{B}_o f^\Delta : \tilde{Q}_X \to S^*(r) \)
is area decreasing and the proof follows by using the \( \Box^{12}(n,m,N) \)-inequality 2.B
(this is where \(n \leq 8\) comes from) and confronting non-vanishing lemma 4.A with
the area contraction theorem 3.C (where the spin condition resides).

**Corollaries, Remarks, Conjectures.**

4.D. \(\mathbf{m} = \mathbf{n}\)-Example. The inequality \([\tilde{\alpha} \mu, \hat{\alpha} \mu]\) for \(m = n\) translates as follows.

\(\star_{m=n}\) Let \(X = (X, g)\) be a complete oriented Riemannian \(n\)-manifold, \(n \geq 2\)
with spin universal covering \(\tilde{X}\) and let \(P\) be a complete Riemannian manifold
of dimension \(K \geq n\), such that that \(\text{sec.curv}(P) \leq \kappa \geq 0\) and the injectivity radius of
the universal covering of \(P\) satisfies
\[
inj.rad(\tilde{P}) \geq \frac{\pi}{\sqrt{\kappa}}
\]

Let \(f : X \rightarrow P\) be a smooth quasi-proper map\(^{35}\) (e.g. \(f\) is proper or it is locally constant
at infinity).

Let \(Sc^{3n}(X, x) \geq \sigma(x) > 0\) for a positive continuous function \(\sigma\) on \(X\), (e.g.
\(Sc(X) > \sigma(X)\)).

If \(f\) is area decreasing with respect to the metric \(\sigma \cdot g\), then the rational fundamental homology class
of \(X\) is sent by \(f\) to zero in \(H_n(P, \partial_\infty P; \mathbb{Q})\),
\[
f_*[X]_\mathbb{Q} = 0.
\]

(If \(\kappa = 0\), this generalizes theorem 13.8 from [GL1983] and if \(P\) is the unit \(n\)-sphere, this is
the \(\alpha\)-stabilized Llarull-Listing-Zhang theorem\(^{39}\)).

4.E. Generalization and Proof of 1.B. Let \(X = (X, g)\) be a Riemannian manifold of dimension \(n = m + k\),
such that \(\tilde{Sc}^{3n}(X) \geq \sigma = \sigma(x) > 0\) (e.g. \(\tilde{Sc}(X) \geq \sigma\)), let \(Z\) be a compact connected enlargeable manifold.

Let
\[
\phi : X \rightarrow Z
\]
be a smooth map, let be a covering of \(X\) induced by \(\phi\) from the universal covering
\(\tilde{Z} \rightarrow \tilde{Z}\) and let \(\hat{\phi} : \tilde{X} \rightarrow \tilde{Z}\) be the lift of \(\phi\) to \(\tilde{X}\).

Let \(Y = \hat{\phi}^{-1}(\tilde{z}) \subset \tilde{X}\) be the \(\hat{\phi}\)-pullback of a generic point \(\tilde{z} \in \tilde{Z}\),

Let \(P\) be a Riemannian manifold with \(\text{sect.curv}(P) \leq \kappa \geq 0\) and let \(f : \tilde{X} \rightarrow P\)
be a smooth map, which is strictly area decreasing with respect to the metric
\[
\frac{\kappa}{m(m-1)} \cdot \sigma \cdot g \text{ in } X.
\]

Let
\[
\bullet_{\text{spin}} ^{\text{Conjecturally}} \text{ either } m \leq 3 \text{ or the universal covering of } X \text{ is spin,}
\]

\(^{35}\)Conjecturally the inequality \(\text{inj.rad}(\tilde{P}) \geq \pi/\sqrt{\kappa}\) can be relaxed to \(\text{conj.rad}(\tilde{P}) \geq \pi/\sqrt{\kappa}\)
and, possibly, the inequality \(\text{sect.curv}(P) \leq \kappa\) can be dropped at the same time.

\(^{39}\)A map \(f\) is quasi-proper if it extends to a continuous map between the compactified spaces, from \(X^{\text{ends}} \rightarrow X\) to \(P^{\text{ends}} \rightarrow P\).

\(^{39}\)This argument, which relies on the trace inequality 4.6 in [Li1998], automatically delivers,
as it was observed in [Li2010], a sharper result, where the area decreasing condition on \(f\) is
replaced by the corresponding bound on trace norm of \(\kappa^2 df\) as it is explained in detail in
section 3.4.1 in [Gr2021].

Similarly, \(m < n\), one may use the supremum of the \(\kappa^2\)-trace norms over the restrictions
of the differential \(df\) to all \(m\)-dimensional subspaces in the tangent bundle \(T(X)\).
\[ n = \text{dim}(X) \leq 8. \]

**4.E(i). Compact Case.** Let \( X \) be compact without boundary, let \( \text{dim}(Z) = k \) and let the \( f \)-image of the fundamental class \( h_m = [Y] \in H_m(\hat{X}) \) doesn’t vanish in the rational homology \( H_m(P; \mathbb{Q}) \),

\[ \hat{f}_*(h_m) \otimes \mathbb{Q} \neq 0. \]

Then the universal covering of \( P \) satisfies

\[ \text{inj.rad}(\tilde{P}) \leq \frac{\pi}{\sqrt{\kappa}}. \]

**4.E(ii). Non-compact Case.** Let \( \text{dim}(Z) = k - 1 \), let \( \hat{Z} \subset \hat{X} \) be a closed subset, and let \( h_m \in H_m(\hat{X} \setminus \hat{Z} \cap Y) \) be a homology class, such that the \( f \)-image of \( h_m \) doesn’t vanish in the rational homology \( H_m(P; \mathbb{Q}) \).

Let \( \hat{R} = \text{dist}_{\text{metr}}(\hat{Z}, \partial \hat{X}) \) (where, by definition, \( \hat{R} = \infty \) for geodesically complete \( X \)), let \( \hat{U} = U_r(\hat{Z}) \subset X \) be the closed (but non-compact) \( r \)-neighbourhood of \( \hat{Z} \subset X \), and let

\[ \hat{\sigma} = \sigma - \frac{36(n-1)^2}{n\hat{R}^2} \geq m(m-1)\kappa. \]

Then the injectivity radius of the universal covering \( \tilde{P} \) at the pullbacks of the points \( f(x) \in P \) to \( \tilde{P} \) satisfies

\[ \inf_{x \in U} \text{inj.rad}_{f(x)}(\tilde{P}) \leq \frac{\pi}{\sqrt{\kappa}}. \]

**Proof.** These propositions are obvious corollaries of 4.C, while the examples 2B(i) and 2B(ii) for a manifold \( X \) follow from 4.E(i) and 4.E(ii) applied to the covering map \( f : \hat{X} \to P = X \).

**4.F. About Rigidity.** The \( \mu \)-bubbles inequalities present in our argument, as well as area inequalities for maps to spheres are accompanied by the corresponding rigidity phenomena in the extremal cases (see [LL1998], [Li2010], [Br2011], [Zh2019], [GZ2021] and section 5.7 in [Gr2021]). Then bringing these rigidity properties together shows that the inequalities

\[ \text{sect.curv}(g) \leq \frac{\sigma}{m(m-1)} \]

and

\[ \text{inj.rad}(X, \tilde{g}) \geq \pi \sqrt{\frac{m(m-1)}{\sigma}} \]

together with non-vanishing of \([Y] \otimes \mathbb{Q} \in H_m(X, \partial_{\infty}X; \mathbb{Q})\) imply that the \( g = \tilde{g} \) and the universal covering \((\hat{X}, \hat{g}) \) is isometric to \( S^m(R) \times \mathbb{R}^k \) for some \( R > 0 \).

**4.E. \( \mathcal{R}^*_{\text{area}} \)-Invariant.** The proof of the injectivity radius theorem 4.C remains valid if \( R^* \) is replaced by \( \mathcal{R}^*_{\text{area}}(P) \geq R^* \), which quantifies the \( \lambda^2 \)-enlargeability from [GL1983] and which is defined as follows.

Recall the map \( \tilde{B} : \tilde{P}^\Delta \to S^*(r) \) and let us agree that the inequality

\[ \mathcal{R}^*_{\text{area}}(P) \geq r = r(p) \]
signify that there exists a smooth map
\[ \tilde{B} : \tilde{P}^\Delta \to S^\bullet(r) \]
with the following two properties.
- $\bullet_{hom}$ the map $\tilde{B}$ sends $p$-fibers to $p$-fibers
\[ \tilde{P}_p \xrightarrow{\tilde{B}} S^\bullet_p(r(p)), \quad p \in P, \]
such that $\partial \tilde{P}_p \cup \partial_{\infty} P_p$ is sent to the point $\bullet_p \in S^\bullet_p(r(p))$ for all $p \in P$, and such that $\tilde{B}$ is homotopic to $\bar{B}$ by such fiber preserving continuous maps $\bar{P}^\Delta \to S^\bullet(r)$
- $\bullet_{area}$ The map $\tilde{B}$ is fiber-wise smooth, it is $C^1$-continuous in $p \in \bar{P}$ and area decreasing on all fibers $\bar{P}_p$.

Example. It is shown in (different terms) in [Gl1983] that locally homogeneous spaces $P$ with contractible universal coverings have $\tilde{R^\bullet}_{length}(P) = \infty$\footnote{I am not certain if this is truly necessary for the validity of 4.C.} i.e. $\tilde{R^\bullet}_{length}(P) \geq r$ for all $r$.

It follows, for instance, that the Riemannian products of these $P$ with unit spheres satisfy $\tilde{R^\bullet}_{area}(P \times S^l) \geq \pi - \varepsilon$ for all $\varepsilon > 0$.

5 Focal Radii and Normal Curvatures

Let $X = (X, g)$ be a complete Riemannian $n$-manifold, possibly with a boundary, e.g. $X = B^n(1) \times \mathbb{R}^n$.

Let $Z$ be a $(k + l)$-dimensional manifold, e.g. homeomorphic to $T^k \times S^l$ or to $\mathbb{R}^k \times S^l$ and let $f : Z \to X$ be a smooth immersion.

Let (compare with the proof of 1.C)
\[ BT^i(R) \subset T^i(Z) \]
be the $R$-ball subbundle in the normal bundle of $Z$ for $R < foc.rad(Z)$, remove arbitrarily small open $\varepsilon$-neighbourhood of the zero section $Z \to T^i(R)$ from this ball bundle, let
\[ P = BT^i(R) \times U_{\varepsilon}(Z) \]
and endow this $P$ with the same Riemannian metric as $X$, that is induced by the normal exponential map $\exp^i : BT^i(R) \to X$ from the Riemannian metric $g$ in $X$.

Let $S(r/2) \subset T = BT^i(R)$ be the $(R/2)$-sphere subbundle, and let
\[ X^\bullet = X^\bullet_{r, R-r} \subset P, \quad r < R/2, \]
be the bundle of annuli pinched between of the sphere subbundles
\[ S(r), S(R-r) \subset P. \]

Let $inj.rad_f(z)(X) \geq \frac{R}{2} - r, \quad z \in Z$, and observe that
\[ \text{dist}_P(S(r), S(R-r)) = R - 2r \]
in this case.

As in the proof of 1.C, we derive a bound on $R = foc.rad(Z) - \varepsilon, \varepsilon \to 0$, hence on $foc.rad(Z)$\footnote{"Length" instead of "area" means that the corresponding maps $B : \bar{P} \xrightarrow{\tilde{B}} S^\bullet_p(r(p))$ are length decreasing.} by applying 4.C to the imbedding $X^\bullet \to P$ as follows.

\footnote{Since we prefer the closed ball bundle for $P$ we can’t use $R = foc.rad(Z)$.}
Let $$h_l \in H_l(Z, \partial_{\infty}Z; \mathbb{Q})$$ be a non-zero rational homology class, which has infinite □$$^1$$-spread with respect to the induced metric in $$Z$$.

\[ □^1(h_l) = \infty. \]

Let $$m = l + (n - k - l - 1) = n - k - 1$$ and let

$$h_m \in H_m(S(R/2), \partial_{\infty}S(R/2) = H_m(X^\perp, \partial_{\infty}X^\perp) = H_m(P, \partial_{\infty}P)$$

be image of $$h_l \in H_l(\tilde{Z}, \partial_{\infty}\tilde{Z}; \mathbb{Q})$$ under the rational Gysin’s homomorphism

$$H_l(Z; \partial_{\infty}Z; \mathbb{Q}) \rightarrow H_m(P; \partial_{\infty}P; \mathbb{Q})$$

for the $$(n - k - l - 1)$$-sphere bundle $$S(R/2) \rightarrow Z$$.

Observe that if the Euler class of the normal bundle $$T^\perp$$ under a smooth proper map $$\rho : S(R/2) \rightarrow Z$$ vanishes on some $$\tilde{h}_m \in H_m(\tilde{Z}, \partial_{\infty}\tilde{Z}; \mathbb{Q})$$ then the corresponding $$\tilde{h}_m \in H_m(\tilde{X}^\perp, \partial_{\infty}\tilde{X}^\perp) = H_m(S(R/2), \partial_{\infty}S(R/2)$$ satisfies

$$\square^1(\tilde{h}_m) \geq \text{dist}_{\rho}(S(r), S(R - r)) = R - 2r$$

for all coverings $$\tilde{Z}$$ of $$Z$$ and that $$U \subset P$$ defined in 4.C as

$$U = \{ p \in P \mid \text{dist}(p, X) \leq \text{dist}(p, \partial P) \}$$

is equal to the $$(\rho, R - \rho)$$ annulus bundle for $$\rho = r/2$$.

5.A. Conclusion. If the universal covering of $$T^\perp(Z)$$ is spin and $$n = \text{dim}(X) = n \leq 8$$, then theorem 4.C applied with this $$U$$, with

$$r = \frac{1}{3}R, \rho = \frac{1}{6}R$$

and with

$$\tilde{R}^\ast \inf_{p \in U} \text{Rad}_p^\ast(\tilde{\partial}_p r = \text{inj.rad}_{\tilde{p}}(\tilde{\partial}_p))$$

The Gysin homomorphism is defined for orientable sphere bundles; if the bundle $$S(R/2) \rightarrow Z$$ is non-orientable, we pass to an orientable double covering of it.
for the points \( \hat{p} \) over \( p \in P \) in the universal covering \( \hat{P} \) of \( P \) shows that

\[
\frac{n + N - 1}{n + N} \frac{4(n - m)\pi^2}{9 \text{foc.rad}(Z)^2} \geq \text{Sc}_N(X) + \frac{m(m - 1)}{(R^*)^2}.
\]

**Discussion.** Focal radius is closely related to the **maximal normal curvature** of an immersions \( f : Z \to X \), that is the supremum of the \( X \)-curvatures of the geodesics lines in \( Z \), i.e. of the curvatures measured in the Riemannian geometry of \( X \) of geodesics in \( Z \) for the induced Riemannian metric in \( Z \).

However, the inequality \( [\Box^1 & \text{foc.rad}] \) yields a significant lower bound on the curvatures of immersions to "simple" manifolds \( X \), e.g. to the unit ball \( B^n \subset \mathbb{R}^n \) (where \( \text{foc.rad}(Z) = \min(\text{dist}(Z, \partial B^n), 1/\text{curv}(Z)) \)) only for large \( n/m \gg 8 \), where this inequality remains conjectural.

In contrast, Anon Petrunin recently proved the following.

5.B. **Asymptotically Optimal \( \sqrt{3} \)-Inequality.** If a compact \( k \)-manifold \( Z \) admits no metric with \( \text{Sc} > 0 \), e.g. \( Z \) is enlargeable, then immersions \( Z \to B^n \) have \( \text{curv}(Z) \geq \sqrt{3k/(k + 2)} \) for all \( k \) and \( n \).

(Optimality of this result is proved in [Gr2022], where we construct immersions \( Z^k \to B^{8k^2} \) with \( \text{curv}(Z) = \sqrt{3k/(k + 2)} + \varepsilon \) and where we also present upper and lower bounds on the curvatures of immersions \( Z^k \to X^n \) for \( n - k \).)

6. **Mean Convex Domains in \( \mathbb{R}^n \)**

Let \( X \) be an infinite tunnel in the 3d space, that is a closed subset \( X \subset \mathbb{R}^3 \) diffeomorphic to the cylinder \( B^2 \times \mathbb{R} \).

6.A. **Example: Fat Mouse in a Narrow Tunnel.** If the mean curvature of the boundary of \( X \) is everywhere \( \geq 1 = \text{mean.curv}(S^2(2)) \), then a "mouse", which contains an \((1 + \varepsilon)\) ball \( B^3(1 + \varepsilon) \) inside its body won't be able to crawl through this tunnel:

*If a connected subset \( Z \subset X \) infinitely stretches out to the both ends of \( X \), i.e. it is not contained in \( B^2 \times S \) for a proper subset \( S \subset \mathbb{R} \), then the 1-neighbourhood \( U_1(Z) \subset \mathbb{R}^3 \) of \( Z \) intersects the boundary of \( X \).*

**Remarks.** (a) The model \( X \) is the infinite round cylinder \( B^2(1) \times \mathbb{R} \subset \mathbb{R}^3 \), where \( \text{mean.curv}(\partial X) = 1 \), where the optimal \( Z \) is the central line \( 0 \times \mathbb{R} \subset X \) with \( U_1(Z) = X \).

(b) Instructive examples of tunnels with \( \text{mean.curv}(\partial X) \geq 1 \) are made by interconnecting infinite chains of disjoint balls \( B^2_\varepsilon(2 - \varepsilon) \subset \mathbb{R}^3 \) of radii \( 2 - \varepsilon \), \( \varepsilon > 0 \), by narrow tubes, such that \( X \) contains all these balls, the mean curvature of \( X \) remains \( \geq 2 - 2\varepsilon \) close to the balls and is arbitrarily large everywhere else.\(^4\)

**Proof of 6.A.** This follows from the following proposition together with the Gehring-Bombieri-Simon linking/filling theorem (see section 8.1 in [Gr1983] and references therein.)

---

\(^4\) Similarly to the thin codimension 2 surgery of manifolds with \( \text{Sc} > \sigma \) (1.3 in [Gr2021]), hypersurfaces with \( \text{mean.curv} > \mu \) admit codimension 1 surgery, (24 in [Gr2017]).

Namely, let \( V \) be a domain with a smooth boundary in a Riemannian \( n \)-manifold \( U \) and \( Y \subset U \) be a smooth submanifold of dimension \( \text{dim}(Y) \leq n - 2 \) with a boundary, such that \( V \cap Y = \partial Y \cap \partial V \), where the intersection between \( Y \) and \( \partial V \) is transversal.

Let \( \mu(u) \) be a continuous function on \( U \), such that \( \mu(u) > \text{mean.curv}(\partial \bar{V}) \) at all boundary points \( u \in \partial V \). Then the union \( V \cup Y \) admits an arbitrarily small regular neighbourhood \( V_\varepsilon \supset V \cup Y \) with smooth boundary, such that \( \text{mean.curv}\partial(V_\varepsilon, u) > \mu(u) \) for all \( u \in \partial V_\varepsilon \).
6.B. Lemma. Let $X \subset \mathbb{R}^3$ be a smooth closed connected, possibly infinite, domain with smooth non-simply connected boundary. If $\text{mean.curv}(\partial X) \geq 1$, then the boundary of $X$ contains non-contractible closed curves $\Theta \subset \partial X$ of lengths $\leq 2\pi + \epsilon$ for all $\epsilon > 0$.

In fact, since $S\sigma (X) \geq 0$, theorem 1.1 in [GZ2021] applied to large compact parts of $X$, shows that the manifold $X$ contains simply connected surfaces $\Sigma_\epsilon$, for all $\epsilon > 0$ with $\partial \Sigma = \Theta \subset \partial X$, which represent non-trivial homology classes in $H_2(X, \partial X) = \mathbb{Z}$ and such that

$$\int_\Theta \text{mean.curv}(\partial X, \theta)d\theta \leq 2\pi + \epsilon.$$ 

Since $\text{mean.curv}(\partial X) \geq 1$, the lemma follows.

Remarks. (a) The proof in [GZ2021] elaborates on the following observation, which applies to all Riemannian 3-manifolds $X$ (compare with [SY1979] and 5.4 in [Gr2014]).

Let $\Sigma \subset X$ be a connected stable minimal surface with boundary $\Theta \subset \partial X$ and observe that the stability condition implies that the intrinsic curvature of $\partial \Sigma \subset \Sigma$ is bounded from below by the mean curvature of $\partial X$ at all points $\theta \in \partial \Sigma \subset X$. Then, as in [SY1979], the stability inequality to the unit normal field on $\Sigma$ one shows that

$$\int_\Sigma S\sigma (X, \sigma)d\sigma + \int_\Theta \text{mean.curv}(\partial X, \theta)d\theta \leq 2\pi,$$

which, for $S\sigma (X) \geq 0$ and $\text{mean.curv}(\partial X) \geq 1$, yields the inequality $\text{length}(\Theta) \leq 2\pi$.

(b) Somewhat paradoxically, locally length minimizing ("length-stable") geodesics $\Theta \subset \partial X$ may have arbitrarily large lengths and the minimal surfaces $\Sigma_\epsilon \subset X$ with boundaries $\Theta$ can't be deformed to locally minimizing ones with free boundaries by area decreasing homotopies.

In fact, the minimization process of areas of surfaces bounded by long $\Theta_{\text{min}} \subset \partial X$ divides $\Theta_{\text{min}}$ into several shorter curves in $\partial X$.

(c) If $X$ is a compact non-simply connected domain in $\mathbb{R}^3$ with $\text{mean.curv}(\partial X) \geq 1$, then, by the rigidity theorem 1.3 in [GZ2021], the shortest non-contractible curve $\Theta_{\text{min}} \subset \partial X$ has length $2\pi$.

Probably, if $\text{diam}(X) \leq 1$, then $\text{length}(\Theta_{\text{min}}) \leq 2\pi - 0.01$. (Round tori seem good candidates for extremal $X$.)

Let us generalize 6.A. to Riemannian manifolds $X$, $S\sigma (X) \geq \sigma$ of higher dimensions $n$, where the boundary of our $X$ is divided into two parts, $\partial X = \partial_{\Sigma} \cup \partial_{\text{side}}$, where these are smooth domains in $\partial X$, which meet along their common boundary, $\partial \partial_{\Sigma} = \partial \partial_{\text{side}} \subset \partial X$.\footnote{Our $\partial_{\Sigma}$ corresponds to $\partial_{\Sigma}^{\text{eff}}$ in [GZ2021].}

Example. If $X$ is the product of a smooth manifold $B$ by the $k$-cube $\Box^k = [-1,1]^k$ then $\partial_{\text{side}} X = \partial \times \Box^k$ and $\partial_{\Sigma} = B \times \partial \Box^k$.

(For the above cylindrical $X = B^2 \times \mathbb{R}$, the side boundary is $\partial B^2 \times \mathbb{R}$ and $\partial_{\Sigma}$ corresponds to the ideal top and bottom of the cylinder.)

Let $\Psi : X \rightarrow \Box^k = [-1,1]^k$ be a continuous map which sends $\partial_{\Sigma} X \rightarrow \partial \Box^k$ and let, as in section 2, $d_i, i = 1, \ldots, k$, be the distances between the pullbacks of the opposite faces of the cube $\Box^k$ and

$$d_{\Box} = d_{\Box}(\Psi) = \left(\frac{1}{k} \sum_{i=1}^k \frac{1}{d_i^2}\right)^{-\frac{1}{2}}.$$
Let $Z \subset X$ be a closed subset and $h_k \in H_k(Z, Z \cap \partial X)$ be a homology class which doesn’t vanish under the homology homomorphism induced by $\Psi$,

$$0 = \Psi_*(h_k) \in H_k(\mathbb{Z}, \partial \mathbb{Z}) = \mathbb{Z}.$$ \hfill (6.C)

6.C. (n, k)-Mouse Conjecture\&Theorem. Let $\sect.curv(X) \leq \kappa$, $\kappa \geq 0$, let

$$\mean.curv(\partial X, X) \geq \mu.$$ \hfill (6.C)

and let

$$[Sc_0(n, k)] \quad Sc(X) \geq \frac{n-1}{n} \cdot \frac{4k\pi^2}{d_0^2}.$$ \hfill (6.C)

Then

$$\inj.rad_{Z}(X) = \inf \inj.rad_{x, x'}(X) \leq r_k,$$

where $r_k$ is equal to the radius of the ball $B$ in the sphere $S^{n-k}(1/\sqrt{\kappa})$, such that $\mean.curv(\partial B) = \mu$ and where we prove this if the universal covering $X$ is spin and $n \leq 8$.

6.D. Corollary/Example. Let $X \subset \mathbb{R}^n$ be a closed domain homeomorphic to $B^2 \times Z_0 \times \mathbb{R}$, where $Z_0$ is a compact $(k-1)$-dimensional manifold, such that $\mathbb{G}^1(Z_0) = \infty$, e.g. $Z_0$ is the $(k-1)$-torus $\mathbb{T}^{k-1}$.

Let $f : Z_0 \times \mathbb{R} \to X$ be a proper map properly homotopic to the imbedding $Z_0 \times \mathbb{R} = 0 \times Z_0 \times \mathbb{R} \hookrightarrow X$. If $\mean.curv(\partial X) \geq n - k - 1$, then, provided $n \leq 8$,

$$\dist(f(Z_0 \times \mathbb{R}), \partial X) \leq 1.$$ \hfill (6.D)

In fact, 6.C applies to compact domains $X_1 \subset \ldots \subset X_i \subset \ldots \subset X$ which exhaust $X$ and where the Euclidean metrics are slightly perturbed to make $Sc(X_i) > \sigma_i$, for $0 \leq \sigma_i \to 0$.

Proof of 6.C. Our argument in the proof of the injectivity radius $\Omega_+^{3\Phi}(n, m)$-theorem 4.C shows that 6.C reduces to a generalization of the area contraction theorem 3. C to families of maps from manifolds with mean convex boundaries to balls in the spheres $S^N$.

This generalization reduces to two other propositions 6.E and 6.F below, where 6.E generalizes 2.C to manifolds with mean convex boundaries and which is resolved for $n \leq 8$, while 6.F generalizes codimension zero parametric area contraction inequality 3.A, where an intended proof needs an extension of Llarull’s (algebraic) inequality and where this inequality for maps from surfaces follows from [GZ2021]. (This is sufficient for the proof of 6.C for $n - k = 2$.)

Preparations for 6.E.

Given a Riemannian manifold $(Y, g = g_Y)$ with a distinguished domain $\partial_* \subset \partial Y$, e.g. $\partial_* = \partial Y$, let us incorporate the mean curvature of $\partial$ in the definition of $Sc^{3\Phi}$ from section 2 as follows.

Let $Sc(Y, \partial_*)$ denote the pair $(Sc(Y), \mean.curv_\partial(\partial_*))$, where $Sc(Y)$ is a function on $Y$ and $\mean.curv_\partial(\partial_*)$ is a function on $\partial_*$ and if

$$Y^* = Y \times \mathbb{T}^N, g^* = g_Y + \sum_{i=1}^{N} \varphi_i^2 dt_i^2,$$

One may also use $g^* = g_{Euc} + \phi^2 dt^2$ on $X \times \mathbb{T}^1$, for $\phi(x) = 1 + \varepsilon \cdot \dist(x, \partial X)$.

6.C for $n - k = 2$ follows from 6.E below, which itself, as in similar cases mentioned earlier, follows for all $n$ from a mild generalization of theorem 4.6 from [SY2017].

33
then $\text{Sc}(Y^* \& \partial^*)$ is the pair $(\text{Sc}(g^*), \text{mean.curv}_g(\partial_* \times \mathbb{T}^N))$.

Accordingly, we introduce the "inequality"

$$\text{Sc}^3(Y \& \partial_*) > (\sigma, \mu)$$

for functions $\sigma$ on $Y$ and $\mu$ on $\partial_*$ as the existence of $g^*$, such that $\text{Sc}(g^*) > \sigma$ and $\text{mean.curv}_g(\partial_* \times \mathbb{T}^N) \geq \mu$, where this is used below for $\partial_* = \partial Y$.

Then, we define $\text{Sc}^{3 \partial_*}$ on homology classes of Riemannian manifolds $X = (X, g = g_X^*)$ relative to a given $\partial_* \subset \partial X$, where we represent such classes $h_m \in H_m(X, \partial_*)$ by 1-Lipschitz maps from $m$-manifolds $Y$ to $X$, such that $\partial(Y) = \partial_*$, and where, similarly to section 2, we use notation:

$$\text{Sc}^{3 \partial_*}(h_m) > (\psi, \nu)$$

as the existence of $Y$ and $f$, such that

$$\text{Sc}^3(Y \& \partial Y) > (\sigma \circ f, \mu \circ f).$$

Next, let the boundary of $X$ be decomposed as above, $\partial X = \partial_c \cup \partial_{side}$, define $\tilde{\partial}^i(h_m)$ for $h_m \in H_m(X, \partial_{side} \cup \partial_{\infty}X)$ with maps $\Psi$ from $X$ to the cube $[-1,1]^k$, $k = n - m$, such that $\partial_{\infty}(X)$ goes to the boundary of the cube and the $\Psi$-pullback of a generic point is homologous to $h_m$.

If $n \leq 8$, then the proof of 2.C extended to $\mu$-bubbles with boundaries in $\partial_{side} X$ (see section in [Gr2021] and [GZ2021]) yields the following proposition, which remains conjectural for $n \geq 8$.

**6.E. $\Box^3(n, m)$-Theorem.** Let $X$ be a Riemannian $n$-manifold with decomposed boundary $\partial X = \partial_c \cup \partial_{side}$, let $X^* = (X \times \mathbb{T}^N, g^*)$ be a $\mathbb{T}^n$-extension of $X$ (for $g^* = g_X^* + \sum_{i=1}^k \phi^2_i dt_i^2$), denote $\nu^* = \text{mean.curv}_g^* (\partial_{side} \times \mathbb{T}^N)$ and let $h \in H_m(X, \partial_{side} \cup \partial_{\infty}X)$, $m = n - k$.

If $n \leq 8$, then

$$[\text{Sc}^{3 \partial_*}] \quad \text{Sc}^{3 \partial_*}(h) \geq \left( \text{Sc}(X^*) - \frac{N + n - 1}{N + n} \frac{4k\pi^2}{\tilde{\partial}^i(h)^2}, \nu^* \right),$$

$$[\text{Sc}_{\partial, sp}^{3 \partial_*}] \quad \text{Sc}_{\partial, sp}^{3 \partial_*}(h) \geq \left( \text{Sc}(X^*) - \frac{N + n - 1}{N + n} \frac{4k\pi^2}{\tilde{\partial}^i(h)^2}, \nu^* \right), \text{ if } X \text{ is spin},$$

$$[\text{Sc}_{\partial, sp}] \quad \text{Sc}_{\partial, sp}(h) \geq \left( \text{Sc}(X^*) - \frac{N + n - 1}{N + n} \frac{4k\pi^2}{\tilde{\partial}^i(h)^2}, \nu^* \right), \text{ if } \tilde{X} \text{ is spin},$$

**Preparations for 6.F.**

Let $Q = (Q, G)$ be an orientable $n$-dimensional Riemannian manifold with a boundary and let $\mathcal{X}$ be a smooth orientable $m$-dimensional foliation on $Q$ where all leaves are transversal to the boundary $\partial X$.

Let $\varphi_i(q), i = 1, \ldots, N$, be smooth positive functions on $Q$, let

$$G^* = G + \sum_{i=1}^N \varphi_i^2 dt_i^2$$

34
and let \( Q^* = Q \times \mathbb{T}^N = (Q \times \mathbb{T}^N, G^*) \).

Let \( \sigma^*(q) = \text{Sc}(X_q^*) \), \( q \in Q \), where \( X_q^* = X_q \times \mathbb{T}^N \subset Q^* \) for the leaf \( X_q \subset Q \) passing through \( q \in Q \) and where \( X_q^* \) is endowed with the Riemannian metric \( g_q^* \) induced from \( G^* \) on \( Q^* \to X_q^* \).

Let \( \mu^*(q) = \text{mean.curv}_{g_q^*}(\partial X_q^*) \), \( q \in \partial Q \).

Let \( S^n(\sqrt{1/e}) \) denote the complete simply connected \( n \)-space with constant curvature \( e \); spherical for \( e > 0 \), Euclidean for \( e = 0 \) and hyperbolic for \( e < 0 \).

Let \( \mathcal{B} = \mathcal{B}(e, r) \subset S^n(\sqrt{1/e}) \) be the \( r \)-ball in this "sphere" and let \( \mu = \mu(e, r) \) be the mean curvature of the boundary \( \partial \mathcal{B} \).

Let \( F : Q \to \mathcal{B} \) be a smooth map, which sends \( \partial Q \to \partial \mathcal{B} \) and which is locally constant at infinity, i.e. the complement of a compact subset in \( Q \) goes to a finite subset in \( \mathcal{B} \).

6.F. Conjecture&Theorem: Maps from Foliations to Balls. Let the manifold \( Q = (Q, G) \) be metrically complete and let \( \sigma^* > 0 \) and \( \mu^* > 0 \).

If the differential \( dF(q) \) restricted to the leaf \( X_q \) and its exterior square \( \wedge^2 dF(q) \) satisfy the following inequalities:

- \( \bullet \mu \ \| dF \|_{\partial X_q} \leq \frac{\mu}{\mu(q)} \) for all \( q \in \partial Q \),
- \( \bullet \ |\wedge^2 dF|_{X_q} \| \leq \frac{m(m-1)e}{\sigma^*(q)} \) for all \( q \in Q \);

then the map \( F \) has zero degree, where we prove this if

(i) \( \mathcal{X} \) is a fibration

(ii) the scalar curvature of \( X^* \) is uniformly positive, \( \sigma^* > 0 > \epsilon \),

(iii) the tangent bundle of the lift of the foliation \( \mathcal{X} \) to the universal covering \( \tilde{Q} \) of \( Q \) is spin.

About the Proof. If \( m = 2 \) and \( N = 0 \), this follows from the Gauss bonnet theorem and if \( N \geq 1 \) a similar proof seems plausible (but I didn’t check this).

But if \( m \geq 3 \), except maybe for \( m = 3 \) and \( N = 0 \), the only available approach to the proof is via Dirac operators where either \( T(Q) \) or \( T(\mathcal{X}) \) lifted to the universal covering of \( Q \) must be spin; in fact if \( X \) has no boundary, then 6.F is proved [SWZ2021], provided either \( T(Q) \) or \( T(\mathcal{X}) \) is spin.

For an individual manifold \( X = Q \) with a boundary, where the universal covering \( \tilde{X} \) is spin, 6.F follows from Goette-Semmler’s theorem [GS2000] applied to suitably smoothed double \( \Phi X = X \cup_{\partial X} X \) (see section 4.3 in [Gr2021]) mapped to a double of the receiving ball, and where a more satisfactory proof, which relies on a relative index theorem and which has an advantage of yielding rigidity of convex domains in symmetric spaces, is due to John Lott [Lo2021].

To extend either of these arguments to foliations one needs a version of inequality 4.6 from [LL1998], which allows replacement of the sup-norms in the inequalities \( \bullet \mu \) and \( \bullet \epsilon \) by the normalized trace norms

\[ \|dF\partial X\| \sim \|dF\partial X\|_{\text{trace}}(m-1) \] and \( \|\wedge d F\| \sim \|\wedge d F\|_{\text{trace}} m(m-1) \)

(see section 3.4 in [Gr2021], where this is explained for individual manifolds) and to prove 6.F, one needs only the following.

6.G. Local Comparison Lemma. Let a point \( s_0 \in S^n \) be taken for a pole in the sphere a let us call an equatorial subsphere \( S^m \subset S^n \) radial if it is made by geodesic segments radiating from the pole \( s_0 \in S^n \), which is equivalent to inclusion \( S^m \ni s_0 \).

\[ \text{This notation is different from that in section 2.1.B.1.} \]
Let $X_n$ be a Riemannian manifold, let $x_a \in X_n$ and let $f : X_n \to S^n$ be a smooth map, such that the differential $df : T(X_n) \to T(S^n)$ at a point $x_a \in X_n$ sends the tangent space $T_{x_a}(X_n)$ to the tangent space of a radial equatorial (sub)sphere $S^m \subset S^n$.

$$ df(T_{x_a}(X_n)) \subset T_{f(x_a)}(S^n). $$

Let $f' : X_n \to S^m$ be a smooth map, differential of which at $x_a$, is equal to that of $f$, i.e. $f'(x_a) = f(x_a) \in S^m \subset S^n$ and the linear map $d_{x_a}f' : T_{x_a}(X_n) \to T_{f(x_a)}(S^m)$ is equal to $d_{x_a}f$.

Let $g_o = \phi_1(r)^2dr^2 + \phi_2(r)^2ds^2$, $r(s) = \text{dist}(s, s_o)$, be a radial Riemannian metric on $S^n$, which is equivalent in the present case to invariance of $g_o$ under the orthogonal subgroup $O(n-1) \subset O(n)$, which fixes the pole $s_o \subset S^n$.

Let $D$ be the Dirac operator on $X$ with the coefficients in the pullback $f^*(S^+(S^m))$ for the Clifford (spinor) bundle $S^+$ over $S^n$.

$$ D : C^\infty((S^+(X_n) \otimes f^*(S^+(S^n))) S^+ $$

and

$$ D' : C^\infty((S^+(X_n) \otimes (f')^*(S^+(S^m))) S^+ $$

be Dirac operator with the coefficients in the $f'$-pullback $(f')^*(S^+(S^m))$.

Let

$$ B = \nabla\nabla^* : C^\infty((S^+(X_n) \otimes (f')^*(S^+(S^m))) S^+ $$

and

$$ B' = \nabla'(\nabla')^* : C^\infty((S^+(X_n) \otimes (f')^*(S^+(S^m))) S^+ $$

be the corresponding (positive) Bochner Laplacians and let us represent the (zero order) difference operators $D - B$ and $D' - B'$ by endomorphisms of the corresponding vector bundles, while keeping the notation

$$ D - B : S^+(X_n) \otimes (f)^*(S^+(S^m)) S^+ $$

and

$$ D' - B' : S^+(X_n) \otimes ((f')^*(S^+(S^m)) S^+ $$

Then the lowest eigenvalue of the (linear selfadjoint operator) action of $D - B$ on the fiber

$$ [S^+(X_n) \otimes (f^*(S^+(S^n)) S^+ ]_{x_a} = S^+(X_n)_{x_a} \otimes S^+(f(x_a))(S^n) $$

is equal to lowest eigenvalue of the operator $D' - B'$ on

$$ [S^+(X_n) \otimes ((f')^*(S^+(S^m)) S^+ ]_{x_a} = S^+(X_n)_{x_a} \otimes S^+(f(x_a))(S^m) $$

Proof. Since the $g_o$-Riemannian (Levi-Cevita) connection on the normal bundle $T^+(S^m) = T(S^n)|_{S^m} \otimes T(S^m)$ is (obviously) parallel, the $(S^n, g_o)$-spin bundle restricted to $S^m$ decomposes into a sum of $2^{n-m}$ copies of the $(S^m, g_o)$-spin bundle.

Then $S^+(X_n)_{x_a} \otimes S^+(f(x_a))(S^m)$ decomposes into the corresponding sum of copies of $S^+(X_n)_{x_a} \otimes S^+(f(x_a))(S^m)$, for $f(x_a) = f'(x_a)$ and $d_{x_a}f = d_{x_a}f'$ and, by
theorems II.4.16 and II.8.17 in [LM 1989] (compare with section 4 in [Li1998]),
the operator $D - B$ on $S^r(X_q)_{x_q} \otimes S^r(f(x_q))$ decomposes accordingly.

Thus, the two operators have the same eigenvalues but with different multiplicities.

6.H. Corollary. Let $X_q$ be a Riemannian manifold, let $x_q \in X_q$, let $(S^n, g_0)$
be the $n$-sphere with a radial metric with respect to a given pole in $S^n$ and let $f : X_q \to S^n$ be as smooth map.

If $\text{sect}.\text{curv}(g_0, x_q) \geq 0$ and if

$$|| \wedge^2 d_x f || < \frac{m(m-1)g(x_q)}{Sc(X_q, x_q)}$$

then, in the cases $\bullet_{\text{cnst}}$ and $\bullet_\circ$ below, the operator

$$(D - B)_x : [S^r(X_q) \otimes (S^r(S^n))]_{x_q}$$

is positive, i.e. the lowest eigenvalue of $(D - B)_x$ is $>0$.

- $\bullet_{\text{cnst}}$ The sectional curvature of the metric $g_0$ at the point $f(x_q) \in S^n$ is constant on the set of 2-planes in $T_f(x_q)(S^n)$.
- $\bullet \circ$ The image $d_x f(T_{x_q}) \subset T_{f(x_q)}(S^n)$ is contained in the tangent subbundle of a radial sphere $S^m \subset S^n$.

Proof. The case $\bullet_{\text{cnst}}$ follows from the trace inequality 4.6 from [LI1998],
(see proposition 2 in [Li2010]).

The above lemma reduces the general case of $\bullet_\circ$, where $n \geq m$, to $n = m$,
where inequality 1.11 in [GS2000], applies. (Also see proposition 1 in [Li2010].)

Remark. In both cases $\bullet_{\text{cnst}}$ and $\bullet_\circ$, the $f$-pullback of the curvature operator of $g_0$ to $X_q$ is diagonalizable at $x_q$ and 6.G follows by the argument from [LI1998].

Conclusion of the Proof of 6.F. The doubling&smoothing argument in section 3.5 of [Gr2021] reduces 6.F to the corresponding property of maps from foliations on manifolds without boundaries to spheres with radial metrics, where, in the spin case, the argument in the proof of the mapping theorem 1.2 from [SWZ2021] applies.

However, we need an extension of [SWZ2021]-arguments to a $\mathbb{T}^n$-stabilized scalar curvature $Sc^*(\mathcal{X}) = Sc(\mathcal{X} \times \mathbb{T}^N)$, instead of the plain $Sc(\mathcal{X})$ and also to where the universal covering $\tilde{\mathcal{X}}$, rather than $\mathcal{X}$ is spin.

For this reason, we claim 6.F only for fibration with uniformly positive $Sc^*$, but this is sufficient for the proof of 6.C for $n \leq 8$.

7 A Few Words on Foliations

Let $Q$ be a compact smooth orientable manifold and $\mathcal{X}$ a smooth $n$-dimensional Riemannian foliation $\mathcal{F}Q$ with leaves $X = X_q \subset Q$, the scalar curvatures of which are bounded from below by $Sc(X_q) > 0$, which is also written as $Sc(\mathcal{X}) > \sigma$.

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49 See [GL1980], [Mi2002] [BMN2011], [Gr,2014] [BH2021] for various versions of this argument.

50 “Riemannian” refers to a smooth positive quadratic form on the tangent bundle $T(\mathcal{X}) \to Q$.
Let $h \in H_K(Q)$ be a homology class and let $A : S^K \to S^K$ be a smooth leaf-wise area decreasing map, which doesn’t vanish at $h$ under the homology homomorphism induced by $A$,

$$A_*(h) \neq 0 \in H_K(S^K) = \mathbb{Z}.$$  

7.A. Problem. Given $\sigma - m(m-1)$ for $m = n-k$ and $k = \dim(Q) - K$, find topological and/or geometric conditions on $h$ that would imply an inequality

$$\sigma \leq \sigma_-.$$  

Remark. One could derive some (non-sharp) inequalities of this kind, from the corresponding (mainly conjectural) ones for individual manifolds with $Sc^\alpha \geq \sigma$, if one could proof the following.

7.B. Conjecture. The inequality $Sc(\mathcal{X}) \geq \sigma$, and even $Sc^{3\alpha}(\mathcal{X}) \geq \sigma$, imply that $Sc^{3\alpha}(Q) \geq \sigma$.

(This is proven in [Gr2021] for codimension 1 foliations $\mathcal{X}$ on manifolds $Q$ with $\dim(Q) \leq 7$.)

7.C. 4d-Example. Let $Q$ be a compact orientable 4-manifold and $\mathcal{X}$ an orientable 3-dimensional Riemannian foliations with $Sc(\mathcal{X}) > 2$ and such that

• $\infty$ the leaves $X \subset Q$ of $\mathcal{X}$ have at most finitely many ends.

• $0$ there exists a leaf $X_0 \subset Q$ of $\mathcal{X}$, and a homology class $h \in H_2(X_0)$ such that the image of $h$ in $H_2(Q)$, the image of which in $H_2(Q)$ under the homology inclusion homomorphism $H_2(X_0) \to H_2(Q)$ is non-torsion.

Then there exists a non-torsion homology class $h_3 \in H_3(Q)$, such that all smooth leaf wise area decreasing maps $f : Q \to S^3$ send $h_3$ to zero.

$$f_*(h_3) = 0 \in H_3(S^3) = \mathbb{Z}.$$  

Moreover, this conclusion holds if the inequality $Sc(\mathcal{X}) > 2$ is relaxed to $Sc^\alpha(\mathcal{X}) > 2$.

Proof. Start by recalling the following.

7.D. Lemma. Let $X$ be a complete orientable Riemannian 3-manifold with $Sc(X) > \sigma + 2\varepsilon > 0$, possibly, with a uniformly mean convex boundary (i.e. mean.curv($\partial X$) $\geq \mu > 0$), where $\sigma, \varepsilon > 0$, and let $Y_0 \subset X$ be a compact connected oriented surface with area$(Y_0) \geq \frac{8\pi}{\sigma}$.

Then there exists a one parameter deformation $Y_t$ of $Y_0$, $0 \leq t \leq 1$, where all $Y_t$ are piecewise smooth 2-cycles, which continuously depend on $t$ with respect to the flat topology and such that

(i) area$(Y_t) \leq area(Y_0)$ for all $t$;

(ii) $Y_t \subset X$ is a smooth surface with area$(Y_t) \leq \frac{8\pi}{\sigma}$.

In fact, $Y_0$ admits an area decreasing deformation to a stable minimal surface $Y_{min}$, where the the second variation formula and the Gauss-Bonnet theorem, imply that

$$\int_{Y_{min}} Sc(X, y) dy \leq \int_{Y_{min}} Sc(Y_{min}, y) dy \leq 8\pi.$$  

Furthermore, the same argument applies to $Sc^\alpha(X) > \sigma + 2\varepsilon > 0$ instead of $Sc(X) > \sigma + 2\varepsilon > 0$ with a use of the Zhu area inequality (see [Zh1999] and section 2.8 in [Gr2021]).

51 It is unclear if this condition is needed.

52 The second variation formula was brought to the needed form in [SY1979], see section 2.5 in [Gr2021] for details.
Sketch of the Proof of 7.C. Let $\nu$ be a nonvanishing vector field on $Q$ nowhere tangent to the leaves of $X$.

Let $Y_0 \subset X_0$ be a compact smooth connected surface, with non-torsion fundamental homology class $[Y] \in H_2(Q)$, where, by the Lemma, one may be assume that $\text{area}(Y_0) \leq 4\pi - 2\varepsilon$.

Continuously move $Y_0$ in the direction of $\nu$ by small distance $\delta > 0$ locally away from $X_0$ to another leave $X_1 \subset Q$ locally "downstream" of $X_0$ relative to $\nu$ while keeping $\text{area} \leq 4\pi - \varepsilon$ all along. Then move the resulting surface within $X_1$ to get $Y_1 \subset X_1$ with $\text{area}(Y_1) \leq 2 - 2\varepsilon$.

Keep doing this and simultaneously regularize the surfaces, thus obtaining a sequence of surfaces $Y_2, ..., Y_l, ...$ in different leaves, where the diameters of $Y_i$ with respect to the induced Riemannian metrics and the curvatures of $Y_i$ in the leaves are uniformly bounded.

It follows that there are arbitrarily large $i$ and $j$, such that $Y_i$ and $Y_{i+j}$ come arbitrarily close together in $Q$.

Join the two by a narrow cylinder, and obtain a 3-cycle $Y_{ij}$ sliced by surfaces (some of which may be singular) of areas $\leq 4\pi - \varepsilon$.

Now, the condition $\bullet_{\text{finite}}$ and the area bounds on (almost minimal) surfaces ($\mu - \text{bubbles}$) for 3-manifolds $X$ with $Sc^* > \sigma > 0$, shows that the leaves $X$ can be exhausted by compact domains

$$B(1) \subset ... \subset B(2) \subset ... \subset X$$

with

$$\text{area}(\partial B) \leq \text{const}.$$  

Hence, these $B_k$ can be completed to 3-cycles $C_k \supset B_i$ with

$$\text{vol}_3(C_k \setminus B_k) \leq \text{const}'.$$

It is also clear that some leaf $X$ transversally intersect $C$ over a very large (of order of $j$) number $l$ of our surfaces $Y_{i_1}, ..., Y_{i_l}$, which implies that the homology class of intersection $Y_{i_j} \cup C_k$ for large $j$ and $k$ represent a non-zero multiple of $h \in H_3(Q)$. Hence, the class $[Y_{ij}] \in H_3(Q)$ is non-torsion as well.

Finally, since $Y_{ij}$ is sliced by surfaces with areas $< \pi$, area decreasing maps $Y_{ij} \to S^3$ have degrees zero by Almgren’s waist inequality (See [Gr2003], [Gu2014]) and the proof is achieved by taking $h_0 = [Y_{ij}]$ with the above (large) $i$ and $j$.

Sub-Example. Let $X$ be a usual geodesic foliation on the 2-torus $T^2$ and $X_0$ be the corresponding foliation on $Q = T^2 \times S^2$ with the leaves $X = X \times S^2$, where these $X \subset T^2$ may be circles or lines.

Observe that the product metric $G_0 = dt^2 + ds^2$ in this foliation has $Sc = 2$, and that all non-zero $h \in H_3(Q) = \mathbb{Z} \oplus \mathbb{Z}$ allow area non-increasing maps $f = f_h : Q \to S^3$ with $f_*(h) \neq 0$.

Let $X \subset T^2$ be a closed geodesic transversal to $X$ and let $Y = X \times S^2 \subset Q$ and let

$$G_\varepsilon = dt^2 + (1 + \varepsilon(t))ds^2,$$

where $\varepsilon(t), t \in T^2$, is a smooth function which is negative away from a small $\delta$-neighbourhood $U_\delta(Y) \subset T^2$ and is small positive in this neighbourhood.

Apparently, (I haven’t check this carefully) there are functions $\varepsilon(t)$ of this kind arbitrarily $C^\infty$-close to zero, such that $Sc(X \times G_\varepsilon) > 2$ and where all leafwise $G_\varepsilon$-area decreasing maps $f : Q \to S^3$ send the class $[Y] \in H_3(Q)$ to zero.
Yet all $h' \in H_3(Q)$, which are not multiples of $h$, do allow leaf-wise $G_x$-area decreasing maps $f': Q \to S^3$ with $f'_*(h') \neq 0$.

7.C. Question. Is there a meaningful version of lemma 7.D for higher dimensions and codimensions?

One knows, for instance, that if a compact Riemannian manifold $X$ with $\text{Sc}(X) \geq 2$ is homeomorphic to $S^2 \times T^k$, and if $k \leq 6$, then the homology class of the 2-sphere $S^2 = S^2 \times \{t\} \in X$ is representable by a smooth surface with area $\leq 4\pi$ (see [Zh2019]).

But if $n = \dim(X) \geq 5$ (I am uncertain about $n = 4$) a manifold $X$ with $\text{Sc}(X) \geq 2$ may contain locally minimal 2-spheres homologous to $S^2$ of arbitrary large areas.

What is more promising in this regard is the geometry of the functional

$$Y \mapsto \frac{1}{\text{Sc}(Y)}$$

on the space $\mathcal{Y} = \mathcal{Y}(X)$, of closed cooriented smooth (singular?) hypersurfaces $Y \subset X$ endowed with induced Riemannian metrics.

Hopefully, the lower bound on $\text{Sc}^*(X)$, does have non-trivial influence on the topology of the sublevels $\mathcal{Y}_a \subset \mathcal{Y}$, of this (or a similar) functional, where $\mathcal{Y}_a$ is the space of hypersurfaces $Y \subset X$ with $\text{Sc}^*(Y) \geq a^{-1}$, and where we are concerned with connectedness of $\mathcal{Y}_a$ for $a = \text{Sc}^*(X)$, as well as with the images of the inclusion homology homomorphisms $H_*(\mathcal{Y}_a) \to H_*(\mathcal{Y}_{a+b})$ where $a$ and $b/a$ tend to infinity.

8 References

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54Defining an adequate topology in $\mathcal{Y}$ is quite an issue here.
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