Adaptive drivers in a model of urban traffic

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Abstract. – We introduce a simple lattice model of traffic flow in a city where drivers optimize their route-selection in time in order to avoid traffic jams, and study its phase structure as a function of the density of vehicles and of the drivers’ behavioral parameters via numerical simulations and mean-field analytical arguments. We identify a phase transition between a low- and a high-density regime. In the latter, inductive drivers may surprisingly behave worse than randomly selecting drivers.

After the seminal works [1–3], models of vehicular traffic have enjoyed a continuously increasing interest among physicists (see e.g. [4–6]), and substantial progress has been achieved in understanding the origin of many empirically observed features. In several cases, traffic models have also revealed connections to important out-of-equilibrium systems in statistical mechanics. The NaSch cellular automaton [2,7], for instance, is a close relative of the totally asymmetric simple exclusion process (TASEP) [8], and of the KPZ-class of surface growth models [9,10]. Urban traffic has also been extensively studied. Examples are BML models [1,5], where vehicles are bound to travel on a lattice, representing the network of streets, with time-evolution governed by TASEP-like rules. Typically, increasing the density of vehicles, a sharp transition occurs from an unjammed regime with finite average velocity to a jammed state where cars are blocked in a single large cluster spanning the entire lattice.

In these cellular-automaton type of models, the dynamics of drivers does not pursue an explicit goal, like minimizing traveling times. Here we introduce a different class of models. We postulate that each driver has a finite set of feasible routes for going between two points in a city and that, on each day, he/she tries to choose the least crowded one using a simple learning process [11]. Our principal aim is to investigate whether – and in which traffic conditions – inductive drivers behave more efficiently than drivers who choose their routes at random. A similar question arises in the Minority Game (MG) [12], and this affinity constitutes the starting point of our analysis. More precisely, upon increasing the density of vehicles we find a transition from a phase where drivers are unevenly distributed over streets to one where streets are equally loaded. In this latter phase, depending on their learning rates, inductive drivers can behave worse than randomly selecting drivers because of crowd effects.

We consider, as a model of a city, a square lattice with $L \times L$ nodes and $P = 2L^2$ edges (assuming periodic boundary conditions). Edges represent the different streets. On each
day, each one of \( N \) drivers must travel from a node \( A \) (say, his/her workplace) to a node \( B \) (say, home) \( (A \) and \( B \) are different for different drivers) following one of \( S \) possible routes. Routes are sets of consecutive edges on the lattice (see Fig. 1). Indexing edges by \( \mu \), it is natural to identify a route with a vector \( q_{ig} = \{q_{ig}^\mu\}_{\mu=1}^P \), where \( i = 1, \ldots, N \) labels drivers and \( g = 1, \ldots, S \) labels the feasible paths of each driver, while \( q_{ig}^\mu \) is the number of times driver \( i \) passes via street \( \mu \) in route \( g \). For each \( i \), \( S \) randomly generated routes of equal length \( \ell \) (length = number of edges) joining two randomly chosen nodes \( A \) and \( B \) are given\(^{(1)}\). Denoting by \( \tilde{g}_i(n) \) the route selected by driver \( i \) on day \( n \), we assume that

\[
\text{Prob}\{\tilde{g}_i(n) = g\} = C e^{\Gamma U_{ig}(n)}
\]

with \( C \) a normalization constant and \( \Gamma \geq 0 \) the ‘learning rate’ of drivers. The functions \( U_{ig} \) are the scores of the different feasible routes of driver \( i \), and are updated according to

\[
U_{ig}(n+1) = U_{ig}(n) - \frac{1}{P} \sum_{\mu=1}^{P} q_{ig}^\mu Q_{\mu}(n) + \frac{1}{2}(1 - \delta_{g,\tilde{g}_i(n)})\zeta_{ig}(n),
\]

where \( Q_{\mu}(n) = \sum_{i=1}^{N} q_{i,\tilde{g}_i(n)}^\mu \) is the number of drivers passing through \( \mu \) on day \( n \). We assume this to be known to every driver, e.g. via the GPS technology or the traffic bulletin, to a degree specified by the last term on the r.h.s. of (2), which accounts for the fact that the driver’s knowledge of the \( Q_{\mu}(n) \) of routes he/she has not visited on day \( n \) (i.e. streets such that \( q_{i,\tilde{g}_i(n)}^\mu = 0 \)) is subject to the noise \( \zeta_{ig}(n) \), with mean value and correlations given by

\[
\langle \zeta_{ig}(n) \rangle = \eta \quad \langle \zeta_{ig}(n)\zeta_{jh}(m) \rangle = \Delta \delta_{ij}\delta_{gh}\delta_{nm}
\]

If \( \eta = 0 \), the driver has unbiased information about routes he/she does not follow; whereas if \( \eta > 0 \) (resp. \( \eta < 0 \)) he/she overestimates (resp. underestimates) the performance of such routes. We shall first consider the simpler case \( \eta = \Delta = 0 \), and later discuss the effects introduced by \( \eta \neq 0 \) and \( \Delta > 0 \).

Altogether, (1) and (2) imply that drivers prefer less crowded routes. If one assumes that the transit time through street \( \mu \) is proportional to \( Q_{\mu} \), then the above dynamics describes a driver trying to learn which of his feasible routes is faster. For \( \Gamma \to \infty \), drivers always choose

\(^{(1)}\)In our simulations, loops are admitted, but all results remain valid for more realistic loop-less routes.

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**Fig. 1** – Two routes of length \( \ell = 16 \) for going from \( A \) to \( B \) in a 11 \( \times \) 11 city.
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Fig. 2 – Stationary values of $\sigma^2/N$ and $H/N$ as a function of $c = N/P$ for randomly selecting drivers ($\Gamma = 0$, open markers) and adaptive drivers ($\Gamma = +\infty$, filled markers). In these simulations: $S = 2$, $P = 200$, $\ell = 50$ and averages are taken over at least 50 disorder samples for each point. The initial conditions of (2) were taken to be $U_{ig}(0) = 0$ for all $i$ and $g$. The solid line in the graphs is the analytic estimate of $H$ and $\sigma^2$ (for $\Gamma = 0^+$) for the model with uncorrelated disorder.

the route with the highest score, whereas for $\Gamma < \infty$ they randomize the choice. In particular, for $\Gamma = 0$ they select one of their feasible routes at random with equal probability.

As in the MG [13], we expect the collective performance in the stationary state of (2) to depend only on the density of vehicles $c = N/P = N/(2L^2)$. We focus on the asymptotic (long-time) values of the observables [14]

$$H = \langle (Q^\mu)^2 - \langle Q^\mu \rangle^2 \rangle$$

$$\sigma^2 = \langle (Q^\mu)^2 - \langle Q^\mu \rangle^2 \rangle$$

where $\langle \cdot \rangle$ stands for a time average over the stationary state of (2) and $x^\mu = (1/P) \sum_{\mu=1}^P x^\mu$. $H$ describes the distribution of drivers over the street network in the stationary state. If $H = 0$, the distribution is uniform ($\langle Q^\mu \rangle = \langle Q^\mu \rangle$ for all $\mu$) and it is not possible to find convenient routes. If $H > 0$, instead, the distribution is not uniform and fast pathways do exist. The optimal road usage is achieved when $H = 0$ and traffic fluctuations (namely $\sigma^2 - H$) are minimized. Notice that if transit times are assumed to be proportional to the street loads $Q^\mu$, then $\sigma^2$ measures the total traveling time of drivers. Notice also that, since all routes have the same total length $\ell$ (i.e. $\sum_{\mu} q^\mu_{ig} = \ell$ for all $i$ and $g$), $\langle Q^\mu \rangle = c\ell$ is just a constant.

In Fig. 2 we report the results obtained from computer experiments with $\eta = \Delta = 0$ as a function of $c$ for drivers with $\Gamma = \infty$ and $\Gamma = 0$. The latter lead to a stationary state where a uniform distribution of vehicles is never achieved, as for them $H > 0$ for all $c$. The former, instead, behave in a similar way only for small $c$. As $c$ is increased, the traffic load becomes more and more uniform ($H$ decreases) and fluctuations ($\sigma^2$) decrease, indicating that inductive drivers manage to behave better than random ones. At a critical point $c_c \approx 3$ the distribution becomes uniform (i.e. $H = 0$) and vehicles fill the available streets uniformly. Now drivers can’t find a convenient way and are forced to change route very frequently. As a consequence, global fluctuations increase dramatically. Notice that above the critical point traffic fluctuations are significantly smaller for randomly selecting drivers than for optimizers.
This behavior is very similar to that of the MG. It is easy to check, along the lines of [15], that in the stationary state drivers choose routes with frequencies \( f_{ig} = \langle \text{Prob}\{\hat{g}_i(n) = g\} \rangle \) that minimize the ‘Hamiltonian’

\[
\mathcal{H}_q(f_{ig}) = \langle Q^\mu \rangle^2 + \frac{\eta}{2} \sum \limits_{i,g} f_{ig}(1 - f_{ig}/2), \quad \langle Q^\mu \rangle = \sum \limits_{i,g} f_{ig} q_{ig}^\mu
\]

subject to \( \sum g f_{ig} = 1 \). In fact, taking the average of \( \hat{f}_{ig} \) in the stationary state, one sees that the stationarity conditions for \( f_{ig} \) are exactly the conditions for the minima of \( \mathcal{H}_q \). Note that, up to a constant, \( \mathcal{H}_q = H \). The stationary frequencies \( f_{ig} \) turn out to be independent of \( \Gamma \) for \( \Gamma > 0 \) (see [15] for details) and of \( \Delta \). In turn, because of \( \hat{f}_{ig} \), \( H \) has the same property.

A direct application of the standard replica-based minimization tools to compute \( \text{min} H \) is made difficult by the presence of spatial correlations\(^{(2)}\) in the disorder variables \( q_{ig}^\mu \). In order to test the effects of spatial correlations, we considered a ‘naive’ version of (2), in which \( \langle Q^\mu \rangle \) is made difficult by the presence of spatial correlations\(^{(2)}\). To begin with, we restrict ourselves to the case \( S = 2 \) (stochastic) differential version of (2). Following [15], we shall now derive the continuous-time method (we do not report this analysis here). It turns out (see Fig. 2) that the shape of the characteristic times are independent of \( \Gamma \) for \( \Gamma \) large while keeping \( \Delta \) small. A large \( \Delta \) will allow us to use the central limit, while with a small \( \Delta \) we can resort to a continuous-time approximation. At odds with the on-line MG, where characteristic times are \( \mathcal{O}(N) \) and this procedure is well defined as long as \( \Gamma \ll N \) [15], in the present ‘batch’ version this analysis is in principle correct only for \( \Gamma \ll 1 \). However, it provides useful insights whose validity for \( \Gamma \gg 1 \) can be checked against numerical simulations.

The knowledge of \( f_{ig} \), however, does not allow us in principle to compute \( \sigma^2 \), which requires a fully dynamical approach. Following [15], we shall now derive the continuous-time (stochastic) differential version of (2). To begin with, we restrict ourselves to the case \( S = 2 \) (\( g = 1, 2 \)) and introduce the variables \( y_i(n) = (U_{1i}(n) - U_{12}(n))/2 \) and \( s_i(n) = 3 - 2 \hat{g}_i(n) \), such that \( s_i(n) = \pm 1 \) is an Ising spin. The dynamics can then be re-cast in the form

\[
y_i(n+1) = y_i(n) - \frac{1}{\Gamma} \sum \limits_{\mu=1}^{P} \xi_{i}^\mu Q^\mu(n) + \frac{1}{8} [(1 - s_i(n)) \zeta_{1i}(n) - (1 + s_i(n)) \zeta_{12}(n)]
\]

where \( \xi_{i}^\mu = (q_{1i}^\mu - q_{12}^\mu)/2 \). Simulations show that relaxation and correlation times are proportional to \( 1/\Gamma \). Following [15], we introduce the re-scaled time \( t = \Gamma n \) and study the dynamics in a time interval \( \Delta t \) corresponding to \( \Delta n = \Delta t/\Gamma \) time steps. When \( \Gamma \ll 1 \), we can take \( \Delta n \) large while keeping \( \Delta t \) small. A large \( \Delta n \) will allow us to use the central limit, while with a small \( \Delta t \) we can resort to a continuous-time approximation. At odds with the on-line MG, where characteristic times are \( \mathcal{O}(N) \) and this procedure is well defined as long as \( \Gamma \ll N \) [15], in the present ‘batch’ version this analysis is in principle correct only for \( \Gamma \ll 1 \). However, it provides useful insights whose validity for \( \Gamma \gg 1 \) can be checked against numerical simulations.

Introducing the new variable \( \tilde{g}_i(t) = \Gamma g_i(n) \) for \( t = n \Gamma \) and iterating \( \hat{f}_{ig} \) from time \( n \) to time \( n + \Delta n \) we obtain

\[
\tilde{g}_i(t + \Delta t) - \tilde{g}_i(t) = \sum \limits_{n=\Delta t/\Gamma}^{(t+\Delta t)/\Gamma} \left\{ -\xi_{i}^\mu Q^\mu + \frac{1}{8} [(1 - s_i(n)) \zeta_{1i}(n) - (1 + s_i(n)) \zeta_{12}(n)] \right\}
\]

If \( \Delta n \) is large enough, by the central limit theorem, we can approximate the sum using the first two moments, whereas if \( \Delta t \) is small enough, the change in \( \tilde{g}_i(t) \) is small, which means that \( \tilde{g}_i(t) \) can be considered constant. The averages over the fast spin variables can then be

\(^{(2)}\)If a route goes through street \( \mu \) \( (q_{ig}^\mu > 0) \) then nearby streets are more likely to belong to the same path than streets which are far away.
performed with the “instantaneous” distribution

\[ \text{Prob}\{s_i(n) = \pm 1\} \sim [1 + e^{\mp\hat{y}_i(t)}]^{-1} \]  

which depends on the “slow” variables \( \hat{y}_i(t) \). After simple calculations we arrive, in the limit \( \Delta t \to 0 \), at the continuous-time Langevin equation

\[ \partial_t \hat{y}_i(t) = -\sum_{j=1}^{N} J_{ij} \tanh \hat{y}_j - \frac{\eta}{4} \tanh \hat{y}_i + \epsilon_i(t) \]  

\[ \langle \epsilon_i(t) \epsilon_j(t') \rangle = \Gamma \sum_{k=1}^{N} J_{ik} J_{jk}[1 - (\tanh \hat{y}_k)^2] \delta(t - t') + \frac{\Gamma \Delta}{16} \delta_{i,j} \delta(t - t') \]  

where \( J_{ij} = \xi_i \xi_j^\mu \). The case \( \Gamma = 0 \) turns out to differ from the case \( \Gamma = 0^+ \) because the limits \( t \to \infty \) and \( \Gamma \to 0 \) don’t commute.

Let us first consider the case \( \eta = 0 \). For \( \Gamma = 0^+ \), when this equation is exact, the dynamics becomes deterministic and the values of \( \hat{y}_i \) (and hence of \( \sigma^2 \)) can be obtained by minimizing the function \( H_{\eta} \) given in (5). We were unable to compute the stationary state of (9) for \( \Gamma > 0 \) because, as in the case of the on-line MG for \( H > 0 \), the Fokker-Planck equation associated to it has no simple factorized solutions (see [15]). Hence \( \sigma^2 \) depends on \( \Gamma \) and on \( \Delta \) for all values of \( c \). Such a dependence is weak for \( c < c_c \) but it becomes very strong in the high density phase. In particular, \( \sigma^2 \) increases with \( \Gamma \), as in the MG.

When \( \Delta = 0 \), the stationary state depends on the initial conditions \( \hat{y}_i(0) \) for \( c > c_c \): the larger the initial spread, the smaller the value of \( \sigma^2 \) (see [15] for the analogous phenomenon in the MG). For \( \Delta > 0 \), the dependence on initial conditions disappears and is replaced by a non-trivial dynamical behavior (see Fig. 3). In the high density phase, where drivers would behave worse than random with \( \Delta = 0 \), global efficiency can improve beyond the random
threshold if $\Delta > 0$. Taking averages after a fixed equilibration time $t_{eq}$, we find that $\sigma^2$ reaches, for $\Delta \approx 40$, a minimum that is well below the value of $\sigma^2$ for $\Gamma = 0^+$ with the same homogeneous initial conditions $y_i(0) = 0$. For $\Delta \to \infty$ we recover the behavior of random drivers. However, when we increase $t_{eq}$, the curve shifts to the left, showing that the system is not in a steady state. Rescaling $\Delta$ by $t_{eq}^{-1/4}$, the decreasing part of the plot collapses, while the rest of the curve flattens. This suggests that the equilibrium value of $\sigma^2$ drops suddenly as soon as $\Delta > 0$. In order to understand this behavior, it is useful to remind that large values of $\sigma^2$ are due to the off-diagonal nature of the noise covariance (10), which is responsible for a dynamic feedback effect, as shown in [16]. The presence of the diagonal noise term $\Delta$ destroys the correlated fluctuations created by the first term of (10), and removes the dependence on initial conditions thereby allowing a slow diffusive dynamics toward the states of maximal spread in the $y_i$ variables (i.e. with minimal $\sigma^2$) which are consistent with the stationary state condition $H = 0$. In loose words: noise-corrupted information can avoid crowd effects when the vehicle density is very high.$^3$

We finally come to the case $\eta \neq 0$. If $\eta > 0$ drivers overestimate the profitability of routes they do not take. This changes the phase transition into a smooth crossover, but the qualitative picture remains the same. For $\eta < 0$ instead the situation changes radically. This effect has been studied in detail in the MG [14, 17, 18]. Based on these results, we argue that while the stationary state is unique for $c < c_c(\eta)$, many stationary states exist in the high density phase, each of which may be selected by the dynamics, depending on initial conditions. This feature, which corresponds to the occurrence of replica-symmetry breaking in the minimization of (5), means dynamically that the system acquires long term memory, as found in [18] for the ‘na¨ıve’ model with uncorrelated disorder and $\Delta = 0$. As $\eta \to 0^-$ we find $c_c(\eta) \to c_c$, while for $\eta \to -2$ we find $c_c(\eta) \to 0$. $\eta < 0$ implies that drivers overestimate the traffic on routes they do not take. This means that they know that if they had actually taken one of those route, the traffic there would have been slowed down. At the value $\eta = -2$, drivers correctly account for this fact and indeed the dynamics converges to an optimal solution, i.e. to a Nash equilibrium in game theoretic terminology.$^4$ The stationary state is characterized by no traffic fluctuations ($\sigma^2 = H$) because each driver selects one route and sticks to it. The value of $\sigma^2$ for $\eta = -2$ is shown in Fig. 4 as a function of $c$.

In summary, we have introduced a highly simplified model for the behavior of drivers in a city, with the aim of analyzing the emergent collective behavior. The model is formally related to the ‘batch’ MG [19], but displays features that are substantially different from all previously studied variations. In absence of information noise, inductive drivers turn out to behave better than random drivers for low car densities, while at high densities the opposite occurs. However, when $\Delta > 0$ and/or $\eta < 0$ inductive drivers can be guided to a more efficient state, though the dynamics becomes substantially more complex. Clearly, within its skinny definition, we are unable to obtain a level of description as detailed as previous approaches to urban traffic [5, 20]. However this model highlights the relevance of the inductive behavior of drivers as opposed to zero-intelligence random drivers, and hence the impact that new technologies of information broadcasting can have on urban traffic. Furthermore, the model is amenable to be made more realistic (e.g. by considering routes with different $\ell$). This approach to urban traffic can draw from the rich literature on the MG and its variations, whose collective behavior has been clarified in detail, as well as contribute to it.

$^3$We stress, however, that the noise $\zeta_{ig}$ has to be uncorrelated across agents.

$^4$A Nash equilibrium here means that each driver takes the best route available to him, given the choices of all other drivers.
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REFERENCES

[1] O. Biham, A. Middleton and D. Levine, Phys. Rev. A, 46 (1992) R6124.
[2] K. Nagel and M. Schreckenberg, J. Phys. I (France), 2 (1992) 2221.
[3] B.S. Kerner and P. Kornhäuser, Phys. Rev. E, 48 (1993) R2335.
[4] D. Helbing, Verkehrsdynamik (Springer, Berlin) 1997 (in German).
[5] D. Chowdhury, L. Santen and A. Schadschneider, Phys. Rep., 329 (2000) 199.
[6] D. Helbing, Rev. Mod. Phys., 73 (2001) 1067.
[7] A. Schadschneider and M. Schreckenberg, J. Phys. A, 26 (1993) L697.
[8] B. Derrida, Phys. Rep., 301 (1998) 65.
[9] M. Kardar, G. Parisi and Y.-C. Zhang, Phys. Rev. Lett., 56 (1986) 889.
[10] T. Halpin-Healy, Phys. Rep., 254 (1995) 215.
[11] D. Helbing, Dynamic decision behavior and optimal guidance through information services: Models and experiments. Preprint (2002, to appear).
[12] D. Challet and Y.-C. Zhang, Physica A, 246 (1997) 407. See also [http://www.unifr.ch/econophysics/minority/](http://www.unifr.ch/econophysics/minority/) for an extensive and commented list of references.
[13] R. Savit, R. Manuca and R. Riolo, Phys. Rev. Lett., 82 (1999) 2203.
[14] D. Challet, M. Marsili and R. Zecchina, Phys. Rev. Lett., 84 (2000) 1824.
[15] M. Marsili and D. Challet, Phys. Rev. E, 64 (2001) 056138
[16] D. Challet and M. Marsili, e-print cond-mat/0210549
[17] A. De Martino and M. Marsili, J. Phys. A, 34 (2002) 2525
[18] J.A.F. Heimel and A. De Martino, J. Phys. A, 34 (2001) L539
[19] J.A.F. Heimel and A.C.C. Coolen, Phys. Rev. E, 63 (2001) 056121
[20] D. Chowdhury and A. Schadschneider, Phys. Rev. E, 59 (1999) R1311