A single-spin precessing gravitational wave in closed form

Andrew Lundgren\(^1\) and R. O’Shaughnessy\(^2\)

\(^1\)Albert-Einstein-Institut, Callinstr. 38, 30167 Hannover, Germany

\(^2\)Center for Gravitation and Cosmology, University of Wisconsin-Milwaukee, Milwaukee, WI 53211, USA

In coming years, gravitational wave detectors should find black hole-neutron star binaries, potentially coincident with astronomical phenomena like short GRBs. These binaries are expected to precess. Gravitational wave science requires a tractable model for precessing binaries, to disentangle precession physics from other phenomena like modified strong field gravity, tidal deformability, or Hubble flow; and to measure compact object masses, spins, and alignments. Moreover, current searches for gravitational waves from compact binaries use templates where the binary does not precess and are ill-suited for detection of generic precessing sources. In this paper we provide a closed-form representation of the single-spin precessing waveform in the frequency domain by reorganizing the signal as a sum over harmonics, each of which resembles a nonprecessing waveform. This form enables simple analytic calculations (e.g., a Fisher matrix) with easily-interpreted results. We have verified that for generic BH-NS binaries, our model agrees with the time-domain waveform to 2%. Straightforward extensions of the derivations outlined here (and provided in full online) allow higher accuracy and error estimates.

PACS numbers:

For the astrophysically most plausible strong gravitational wave sources – coalescing compact binaries of black holes and neutron stars with total mass \(M = m_1 + m_2 \leq 16M_\odot\) – ground based gravitational wave detector networks (notably LIGO \(^1\) and Virgo \(^2\,3\)) are principally sensitive to each binary’s nearly-adiabatic and quasi-circular inspiral \(^4\,12\). For nonprecessing binaries, the inspiral signal model has been explored in detail, both analytically and with numerical simulation. Fast, stationary-phase approximations to the signal exist which faithfully reproduce the signal \(^13\). By contrast, plausible astrophysical processes will produce merging black hole binaries with arbitrary spin orientations, fully populating the 15-dimensional model space for quasi-circular inspiral. Generic binaries orbits’ both shrink and rapidly precess \(^14\,16\). For many astrophysically interesting sources, including many binaries containing at least one black hole, significant spin precession is expected while the signal passes through LIGO and Virgo’s sensitive frequency band. Precession-induced modulations compete with effects from all other phenomena, including modifications to gravity or the nature of nuclear matter. Precession breaks severe degeneracies \(^17\) and allows high-precision measurements of binary parameters including high-precision mass, spin, and spin-orbit misalignment distributions, providing high-precision constraints on astrophysical processes like the central engines of short GRBs. Precessing sources have been substantially less-thoroughly modeled. At present, the current state-of-the-art SpinTaylorT4 \(^5\,6\) and SpinTaylorT2 models \(^13\) solve coupled time-domain ordinary differential equations to evolve the orbits and spins. Until this work, no fast, stationary-phase approximation existed which faithfully reproduces the signal from a generic single-spin binary. A frequency-domain signal provides an analytically tractable tool for theoretical analysis; for example, the Fisher matrix can be efficiently evaluated and its physical significance understood \(^17\). Also, these frequency-domain waveforms can be evaluated and explored with significantly reduced computational cost. Previous searches balanced accuracy and coverage against computational cost, usually adopting low-cost but easily-understood approximate waveforms (“detection templates”), though others proposed the use of accurate but expensive time-domain signals \(^5\,7\,8\,10\,18\). Present searches explicitly omit precession, hoping to be efficient at finding precessing binaries. Our revised signal model may change the balance, allowing direct searches for precessing sources.

In this paper, we show that the gravitational wave signal for a single-spin binary \(^29\) system can be well approximated with a simple analytic form in the frequency domain. Our decomposition expresses the signal as a sum of five terms, each a weak modulation (sideband) of a leading-order (“carrier”) function. Our decomposition relies on simple precession \(^14\) of the angular momenta and breaks down precisely when that approximation does. We have implemented our waveform as SpinTaylorF2SingleSpin in the open-source lalsimulation package and compared it to the time-domain SpinTaylorT2 approximant \(^13\). The noise-weighted inner product of two waveforms \(h_1(t)\) and \(h_2(t)\) can be expressed in terms of their fourier transforms \(\hat{h}(f)\) as \(^19\)

\[
(h_1|h_2) = 4\text{Re} \int_{f_1}^{f_2} df \frac{\hat{h}_1(f)\hat{h}_2^*(f)}{S_n(f)},
\]

and the overlap is \((h_1|h_2) / \sqrt{(h_1|h_1)(h_2|h_2)}\). We generate both signals with the exact same mass, spin, and orientation parameters, then maximize the overlap over only the time and phase. Figure \(^11\) shows the overlap distribution for our fiducial scenario: \(4 \times 10^4\) randomly
chosen 10 + 1.4M\odot BH-NS binaries, each with a random source orientation, random BH spin direction, and random BH spin magnitude \( \chi \in [0.5, 1] \). For \( S_n(f) \), we use the Advanced LIGO high-power zero-detune noise spectrum\[^{[20]}\]\[^{[21]}\] with \( f_l = 15\text{Hz} \) and terminate the signal at \( f_h = 6^{-3/2}/M\pi \), the last stable orbit of a BH with the binary’s total mass. The two models agree to better than 2% in most cases, with mismatches of up to 10% occurring only for strong spin-orbit misalignment \( L \cdot \dot{S} \lesssim -0.5 \) and unfavorable viewing orientations. Under these extreme conditions, the spins undergo transitional precession\[^{[14]}\] in the detectors’ sensitive band, breaking the simple precession approximation used here.

In our notation, the components of the binary have masses \( m_1 \) and \( m_2 \), with \( m_1 \) the larger mass; we also use \( M = m_1 + m_2 \), \( \eta = m_1m_2/M^2 \). Only the larger mass has spin, with a dimensionless spin parameter \( \chi_1 \) and spin \( \mathbf{S}_1 = m_1^2 \chi_1 \). [Here and henceforth we adopt \( G = c = 1 \).] The orbital angular velocity is denoted \( \omega \) and is related to the gravitational wave frequency by \( \omega = \pi f \) and to the orbital velocity by \( v = (M\omega)^{1/3} \). We omit trivial dependence on the source sky location. To define the orientation of the binary with respect to the solar system barycenter, we use the polar angles \( \theta, \psi \) of the total angular momentum \( \mathbf{J} = \mathbf{L} + \mathbf{S} \) relative to the line of sight, where \( \psi \) is the angle of \( \mathbf{J} \) projected into the plane of the sky and \( \theta \) is the angle between \( \mathbf{J} \) and the line of sight; see\[^{[22]}\], \[^{[23]}\], and references therein. The \( \mathbf{J} \) direction is nearly fixed during the inspiral; see\[^{[14]}\].

The precessing waveform can be decomposed into a non-precessing waveform acting on by a time-dependent rotation\[^{[24]}\]\[^{[27]}\]. The rotation accounts for the precession of the orbital angular momentum of the binary. The waveform in the source frame is

\[
\tilde{h}(t) = \sum_{\ell,m} \tilde{h}_{\ell,m}(t)e^{-i\ell \phi}e^{-im \phi(t)} \tag{2}
\]

where \( \Phi \) is the orbital phase and where here and henceforth \( \hat{\ } \) denotes the frame co-rotating with the precession of \( \mathbf{L} \). In this frame, the source amplitudes nearly obey \( \tilde{h}_{\ell,m} = (-1)^{\ell}\tilde{h}^{*}_{\ell,m} \), to the extent spin-dependent higher harmonics can be neglected; expressions for \( \tilde{h}^{*}_{\ell,m} \) exist in the literature\[^{[4]}\].

The source frame and inertial frame are related by a time-dependent rotation, specified by three Euler angles \( (\alpha, \beta, \zeta) \). [For brevity, we omit \( t \) here and henceforth; time dependence is understood for all quantities except \( \theta, \phi \) and \( \psi \).] The source frame is aligned with the instantaneous angular momentum \( \mathbf{L} \); the inertial frame is aligned with the total angular momentum \( \mathbf{J} \) of the binary. In addition to the two Euler angles \( (\alpha, \beta) \) that express \( \mathbf{L} \) relative to \( \mathbf{J} \), we apply a third Euler angle \( \zeta \equiv -\int \cos(\beta)(d\alpha/dt)dt \) to minimize superfluous coordinate changes associated with the orbital plane\[^{[24]}\]. Keeping in mind the definition of \( \psi \), the waveform in an inertial frame can be expressed as a weighted sum of the co-rotating-frame amplitudes \( \tilde{h}^{\ell,m} \):

\[
\tilde{h}(t) - i\tilde{h}_x = e^{-2i\psi} \sum_{\ell,m} D_{\ell,m}^{(f)}(\alpha, \beta, \zeta)\tilde{h}^{\ell,m}(t) - 2Y_{\ell,0}(\theta, \phi) \exp(-im \Phi) \tag{3}
\]

where \( D_{\ell,m}^{(f)} \) is the usual Wigner rotation matrix representation of SU(2)\[^{[24]}\]\[^{[29]}\]. Because \( h_+ - i h_\times \) has spin weight \(-2\), this expression is proportional to \( \exp(-2i\psi) \).

For the special case of leading-order quadrupole emission \( (\tilde{h}^{lm}_0 = 0 \text{ unless } l = |m| = 2) \), the real part of the above expression can be rewritten as\[^{[22]}\]

\[
h_+ = 2M\eta/\bar{D}v^2 \text{Re} \left[ z(\alpha - \phi, \theta, \psi, \beta)e^{2i(\Phi - \zeta)} \right] \tag{4}
\]

where \( z(\alpha) \) is a complex variable that captures the modulation due to precession. Rather than decompose \( z \) into amplitude-phase modulations\[^{[14]}\] or use spherical harmonics\[^{[5]}\], we expand \( z = \sum \tilde{z}_m e^{im\alpha} \) where

\[
\tilde{z}_m = -2Y_{2,m}(\beta, \phi) \left[ e^{-2i\psi - 2Y_{2,m}(\theta, 0)} + e^{2i\psi - 2Y_{2-m}(\theta, 0)} \right] . \tag{5}
\]

The \( \tilde{z}_m \) are normalized so \( \tilde{z}_m = \delta_{m-2} \) when \( \beta = \theta = \psi = 0 \). In this sum, each term is proportional to \( e^{im\phi} \) and is therefore modulated at a harmonic of the precession frequency; multiplying the leading-order \( \exp(-2i\Phi) \), each term therefore produces a sideband, offset from the carrier frequency. While this expression applies in general,
we substantially increase its value by adopting coordinates with a hierarchy of timescales, so \( \Phi \) changes on an orbital timescale; \( \alpha \) on a radiation reaction timescale; and \( \beta \) and \( v \) on the inspiral timescale. This separation of timescales occurs naturally in single-spin BH-NS binaries. To an excellent approximation, a single-spin binary undergoes simple precession \[14\], where the total angular momentum direction is fixed; on short (precession) timescales the orbital angular momentum precesses around \( \hat{J} \) at a uniform rate; and on longer (inspiral) timescales the precession cone opening angle gradually increases. Adopting coordinates aligned with \( J \), the polar angle \( \beta \) is identified as the precession cone opening angle \[22\] and therefore changes slowly.

In the co-rotating frame, orbital evolution can be calculated using the instantaneous binary’s binding energy \( E(v) \) and the gravitational wave flux \( F(v) \) \[6\]:

\[
\frac{dt}{dv} = -\frac{dE}{dJ} \cdot F(v). 
\]

Both \( E(v) \) and \( F(v) \) are known as post-Newtonian expansions in the velocity \( v \) \[11\] \[28\]. These are currently known to order \( v^5 \) past the leading order for non-spinning terms and \( v^6 \) in terms involving spin (note that terms at order \( v^5 \) and beyond may also have terms containing powers of \( \log v \)). We expand \( dt/dv \) as a power series in \( v \), to the same order as we know the flux and energy, to find \( t(v) \) as a closed-form series in \( v \). We use the additional relation for the orbital phase \( \frac{d\Phi}{dv} = \frac{dt}{dv} \cdot \hat{M} \) to obtain \( \Phi(v) \) in closed form.

For the special case of BH-NS binaries, we can analytically solve the spin-precession equations that determine \( \hat{L} \) and hence \( \alpha, \beta \), and \( \zeta \) as functions of the velocity. These three angles can be substituted into the rotation operator to transform the waveform from the precessing source frame into the inertial frame. The leading order (“Newtonian”) expression for the orbital angular momentum is \( \hat{L} = \frac{m_1 m_2}{m_2} \hat{\ell}_N \), and for the total angular momentum is \( \hat{J} = \hat{L} + \hat{S}_1 \). The magnitude of the spin is conserved. In terms of the dimensionless spin, it is \( \hat{S}_1 = \frac{m_2^2}{\hat{\chi}} \hat{S} \). The angle between \( \hat{L} \) and \( \hat{S}_1 \) does not evolve in the single-spin case, giving the conserved quantity \( \kappa = \hat{L}_N \cdot \hat{S}_1 \). We can additionally define two ratios

\[
\gamma \equiv \frac{|\hat{S}_1|}{|\hat{\ell}_N|} = \frac{m_1 \chi}{m_2} v, \\
\Gamma_J \equiv |\hat{J}|/|\hat{\ell}_N| = \sqrt{1 + 2\kappa \gamma + \gamma^2}. 
\]

The global behavior of \( \Gamma_J \) is not well-fit by a single low-order polynomial, as it approaches 1 at \( v \to 0 \) and is proportional to \( v \) in the limit of large \( v \). As a result, expressions involving \( \Gamma_J \) are generally not well-fit by a standard PN expansion.

In terms of these quantities, the opening angle \( \beta \) of the cone swept out by \( \hat{L} \) (the “precession cone”) is \( \cos \beta \equiv \hat{L}_N \cdot \hat{J} = \frac{1 + \kappa \gamma}{\Gamma_J} \). The time evolution of \( \hat{L}_N \) is given by \( \frac{d\hat{L}_N}{dt} = \Omega_p \hat{J} \times \hat{L}_N \), which causes \( \hat{L}_N \) to rotate around \( \hat{J} \) with an angular rate of

\[
\Omega_p = \eta \left( 2 + \frac{3m_2}{2m_1} \right) v^5 \Gamma_J. 
\]

The definitions of \( \alpha \) and \( \zeta \) are then

\[
\dot{\alpha} = \Omega_p, \\
\dot{\zeta} = \alpha \cos \beta = \Omega_p \cos \beta. 
\]

Substituting the definitions above leads to

\[
\alpha(v) = \eta \left( 2 + \frac{3m_2}{2m_1} \right) \int v^5 \Gamma_J \left( \frac{dt}{dv} \right) dv 
\]

\[
\zeta(v) = \eta \left( 2 + \frac{3m_2}{2m_1} \right) \int v^5 \left( 1 + \kappa \gamma \right) \left( \frac{dt}{dv} \right) dv. 
\]

We use the TaylorT2 \[1\] expression for \( dt/dv \) as a power series in \( v \) to express \( \alpha \) and \( \zeta \) as integrals in \( v \) rather than in \( t \). Despite the non-polynomial behavior in \( \Gamma_J \), both integrals can be evaluated term-by-term to produce closed-form expressions for \( \alpha(v) \) and \( \zeta(v) \) at least to 3 PN order, where terms of the form \( v^n \log v \) appear. These closed-form expressions are provided as supplementary online material, and in the lalsimulation code. Since \( \alpha, \zeta \) and the orbital phase influence each terms’ phase in a similar way but \( \alpha, \zeta \) depend on integrals over \( dt/dv \) multiplied by at least \( v^3 \), at least two fewer terms in a post-Newtonian series expansion for \( dt/dv \) are needed to reproduce the precessional dynamics of \( \alpha, \zeta \) at the accuracy needed.

For nonprecessing signals, a commonly-used approximation to a Fourier transform is provided by the stationary-phase approximation, which for a real-valued signal \( ReAexp(i\Phi(t)) \) has the form

\[
A(v(f))/2\sqrt{d^2\Phi/dt^2} exp(\psi(f)), 
\]

where \( \psi(f) = 2\pi f(t(v) - \Phi(v)) \); see, e.g., Eq. (2.18) in \[10\]. Using this approximation, the Fourier transform of our modulated waveform [Eq. \[16\]] can be efficiently computed term-by-term, in general using the phase \( \Phi_m = 2(\Phi - \zeta) + ma \). At the level of approximation used here, the relation \( t(f) \) and hence \( \Psi(f) \equiv 2\pi ft(f) - 2\Phi(t(f)) \) is independent of \( \alpha \); the factor \( \exp(i(-2\zeta + ma)) \) can be viewed as a slowly-varying term, like the prefactor \( A \). Following custom in gravitational wave data analysis, we approximate the stationary phase amplitude factor as \( 1/\sqrt{d^2\Phi/dt^2} \approx (d^2\Phi/dt^2)^{-1/2} \) by its leading order term, proportional to \( f^{-\gamma/6} \). We therefore find

\[
\bar{a}_+(f) \approx \frac{2M\eta}{D \sqrt{D^2\Phi/dt^2}} v^2 \sum_m z_m e^{i(\Psi - 2\zeta)} + im\alpha 
\]

\[
\approx \frac{2\pi M_c^2}{D} \sqrt{\frac{5}{96\pi}(\pi \mathcal{M} f)^{-7/6}} \sum_m z_m e^{i(\Psi - 2\zeta)} + im\alpha 
\]

where the expressions for \( \alpha(v(f)) \) and \( \zeta(v(f)) \) follow by explicit substitution. The complete expression for \( \Psi(f) \) can be found in Eq. (3.18) of \[16\] or Appendix B3 of \[28\].

Our fast, faithful frequency-domain waveform provides an efficient, analytically-tractable representation of gravitational waves from generic precessing BH-NS binaries. Our method generalizes naturally to include higher harmonics. As noted above, this method allows for fast,
accurate analytic Fisher matrices, to estimate the performance of parameter estimation. More broadly, overlaps between generic single-spin sources can both be calculated and understood analytically. As a concrete example, Brown et al. \cite{22} evaluated the overlap between a generic precessing signal and a nonprecessing search template. In our representation, each sideband has a unique time-frequency trajectory, offset from the “chirp” of the orbital frequency versus time. Hence, the best fit between the nonprecessing model’s time-frequency path lies on one of the sidebands; a nonprecessing search misses all power, except that associated with the optimal sideband; and the best-fitting match will be proportional to $|z_m|$, reproducing their general result. Finally, our model suggests several strategies for a viable search for precessing single-spin signals. First and foremost, our model allows us to identify precisely which masses, spins, and viewing orientations would benefit from a multi-modal search. Second, the functional form of $z_m$ ensures that usually two or fewer $m$ will contribute significantly to the signal. Each of the harmonics is essentially a nonprecessing waveform. A promising search strategy would be to perform a nonprecessing search, then recombine the power from two (or more) suitable spots. Third, our expression simplifies the form of overlaps between the $l, m$ modes of precessing signals. Combined with a massive reduction in computational cost, our model may allow effective searches for generic precessing systems, using physical templates and maximizing over source orientation \cite{5}.

Figure \ref{fig:figure1} suggests our approximation is effective. A subsequent publication will expand on the material provided here and online, providing a detailed analysis and error estimates, identifying limits of our approximation and areas for improvement (e.g., better amplitudes via expanding $d^2 \Phi /dt^2$ beyond leading order), exploring its utility for data analysis and parameter estimation, and discussing the accuracy of its assumptions (e.g., our model for $\alpha(t)$ and $\alpha(f)$; simple precession; et cetera).

**Acknowledgements** ROS is supported by NSF award PHY-0970074. We thank Will Farr and Thomas Dent for helpful discussions and the referees for carefully reading and helpful feedback.

\begin{thebibliography}{99}
\bibitem{1} Abbott et al. (The LIGO Scientific Collaboration), (gr-qc/0308043) (2003), URL http://xxx.lanl.gov/abs/gr-qc/0308043
\bibitem{2} T. Accadia, F. Acernese, F. Antonucci, P. Astone, G. Ballardin, et al., Class. Quant. Grav. 28, 114002 (2011).
\bibitem{3} The Virgo Collaboration, *Advanced Virgo Baseline Design* (2009), [VIR-0027A-09].
\bibitem{4} K. G. Arun, A. Buonanno, G. Faye, and E. Ochsner, Phys. Rev. D 79, 104023 (2009), 0810.5336.
\bibitem{5} Y. Pan, A. Buonanno, Y. Chen, and M. Vallisneri, Phys. Rev. D 69, 104017 (2004).
\bibitem{6} A. Buonanno, B. R. Iyer, E. Ochsner, Y. Pan, and B. S. Sathyaprakash, Phys. Rev. D 80, 084043 (2009), 0907.0700.
\bibitem{7} A. Buonanno, Y. Chen, and M. Vallisneri, Phys. Rev. D 67, 104025 (2003).
\bibitem{8} A. Buonanno, Y. Chen, Y. Pan, and M. Vallisneri, Phys. Rev. D 70, 104003 (2004).
\bibitem{9} T. Damour, A. Gopakumar, and B. R. Iyer, Phys. Rev. D 70, 064028 (2004).
\bibitem{10} A. Buonanno, Y. Chen, Y. Pan, H. Tagoshi, and M. Vallisneri, Phys. Rev. D 72, 084027 (2005).
\bibitem{11} C. Königsdörffer and A. Gopakumar, Phys. Rev. D 73, 124012 (2006), arXiv:gr-qc/0603056.
\bibitem{12} M. Hannam, S. Husa, B. Brügmann, and A. Gopakumar, Phys. Rev. D 78, 104007 (2008).
\bibitem{13} D. Brown, I. Harry, D. Keppel, A. Lundgren, A. Nitz, and E. Ochsner, in preparation (2013).
\bibitem{14} T. A. Apostolatos, C. Cutler, G. J. Sussman, and K. S. Thorne, Phys. Rev. D 49, 6274 (1994).
\bibitem{15} J. D. Schnittman, Phys. Rev. D 70, 124020 (2004).
\bibitem{16} D. Gerosa, M. Kesden, E. Berti, R. O’Shaughnessy, and U. Sperhake, Submitted to PRD (arXiv:1302.4442) (2013), 1302.4442.
\bibitem{17} E. Poisson and C. M. Will, Phys. Rev. D 52, 848 (1995).
\bibitem{18} A. Buonanno, Y. Chen, and M. Vallisneri, Phys. Rev. D 67, 024016 (2003).
\bibitem{19} C. Cutler and E. Flanagan, Phys. Rev. D 49, 2658 (1994).
\bibitem{20} LIGO Scientific Collaboration (2009), URL https://dcc.ligo.org/cgi-bin/DocDB/ShowDocument?docid=2974
\bibitem{21} J. Aasi et al. (The LIGO Scientific Collaboration and the Virgo Collaboration), (arXiv:1304.0670) (2013), URL http://xxx.lanl.gov/abs/arXiv:1304.0670
\bibitem{22} D. A. Brown, A. Lundgren, and R. O’Shaughnessy, Phys. Rev. D 86, 064020 (2012), 1203.6060, URL http://arxiv.org/abs/1203.6060
\bibitem{23} H. Cho, E. Ochsner, R. O’Shaughnessy, C. Kim, and C. Lee, Phys. Rev. D 87, 024003 (2013), 1209.4494, URL http://xxx.lanl.gov/abs/arXiv:1209.4494
\bibitem{24} M. Boyle, R. Owen, and H. P. Pfeiffer, Phys. Rev. D 84, 124011 (2011), 1110.2965.
\bibitem{25} R. O’Shaughnessy, B. Vaishnav, J. Healy, Z. Meeks, and D. Shoemaker, Phys. Rev. D 84, 124002 (2011), 1109.5224, URL http://link.aps.org/doi/10.1103/PhysRevD.84.124002
\bibitem{26} P. Schmidt, M. Hannam, S. Husa, and P. Ajith, Phys. Rev. D 84, 024046 (2011), 1012.2879, URL http://xxx.lanl.gov/abs/arXiv:1012.2879
\bibitem{27} P. Schmidt, M. Hannam, and S. Husa, (arXiv:1207.3088) (2012), URL http://xxx.lanl.gov/abs/arXiv:1207.3088
\bibitem{28} A. H. Nitz, A. Lundgren, D. A. Brown, E. Ochsner, D. Keppel, and I. W. Harry, ArXiv e-prints (2013), 1307.1757.
\bibitem{29} We expect our scheme approximates the evolution of two-spin systems with moderate mass ratio, since the second spin’s influence on the orbit can be neglected.
\end{thebibliography}