The stabilizer for $n$-qubit symmetric states

Xian Shi

Institute of Mathematics, Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing 100190, China
University of Chinese Academy of Sciences, Beijing 100049, China
UTS-AMSS Joint Research Laboratory for Quantum
Computation and Quantum Information Processing,
Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing 100190, China

Abstract

The stabilizer group for an $n$-qubit state $|\phi\rangle$ is the set of all invertible local operators (ILO) $g = g_1 \otimes g_2 \otimes \cdots \otimes g_n$, $g_i \in \mathcal{G}(2, \mathbb{C})$ such that $|\phi\rangle = g|\phi\rangle$. Recently, [Gour et al. 2017 J. Math. Phys. (N.Y.) 58 092204] presented that almost all $n$-qubit state $|\psi\rangle$ own a trivial stabilizer group when $n \geq 5$. In this article, we consider the case when the stabilizer group of an $n$-qubit symmetric pure state $|\psi\rangle$ is trivial. First we show that the stabilizer group for an $n$-qubit symmetric pure state $|\phi\rangle$ is nontrivial when $n \leq 4$. Then we present a class of $n$-qubit symmetric states $|\phi\rangle$ with trivial stabilizer group when $n \geq 5$. At last, we prove that when $m \geq 5$, almost all $n$-qubit symmetric pure states own a trivial stabilizer group, due to the results in [Gour et al. 2017 J. Math. Phys. (N.Y.) 58 092204], we have that almost all $n$-qubit symmetric pure states are isolated.

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*Electronic address: shixian01@gmail.com
I. INTRODUCTION

Quantum entanglement \([2]\) is a valuable resource for a variety of tasks that cannot be finished by classical resource. Among the most popular tasks are quantum teleportation \([3]\) and quantum superdense coding \([4]\). Due to the importance of quantum entanglement, the classification of quantum entanglement states is a big issue for the quantum information theory.

Entanglement theory is a resource theory with its free transformation is local operations and classical communication (LOCC). As LOCC is hard to deal with mathematically, and with the number of the parties of the quantum systems grows, the classification of all entanglement states under the LOCC restriction becomes very hard. A conventional way is to consider other operations, such as stochastic LOCC (SLOCC), local unitary operations (LU), separable operations (SEP).

Two \(n\)-partite states \(|\phi\rangle\) and \(|\psi\rangle\) are SLOCC equivalent \([5]\) if and only if there exists \(n\) invertible local operations (ILO) \(A_i, i = 1, 2, \cdots, n\) such that

\[|\phi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n |\psi\rangle.\]  

Then the classification for multi-qubit pure states under SLOCC attracts much attention \([6-11]\). However, there are uncountable number of SLOCC inequivalent classes in \(n\)-qubit systems when \(n \geq 4\), so it is a formidable task to classify multipartite states under SLOCC.

SEP is simple to describe mathematically and contains LOCC strictly, as there exists pure state transformations belonging to SEP, but cannot be achieved by LOCC. The authors in \([12]\) presented that the existence of transformations under separable operations between two pure states depends largely on the stabilizer of the state. Recently, Gour et al. showed that almost all of the stabilizer group for 5 or more qubits pure states contains only the identity \([1]\). And the authors in \([13]\) generalized this result to \(n\)-qudit systems when \(n > 3, d > 2\).

Symmetric states belong to the space that is spanned by the pure states invariant under particle exchange, and there are some results done on the classifications under SLOCC limited to symmetric states \([14-18]\). The authors in \([15]\) proved if \(|\psi\rangle\) and \(|\phi\rangle\) are \(n\)-qubit symmetric pure states, and there exists \(n\) invertible operations \(A_i, i = 1, 2, \cdots, n\) such that

\[|\psi\rangle = A_1 \otimes A_2 \otimes \cdots \otimes A_n |\phi\rangle,\]  

then there exists an invertible matrix \(A\) such that

\[|\psi\rangle = A^{\otimes n} |\phi\rangle.\]
Moreover, P. Migdal et al. [16] generalized the results from qubit systems to qudit systems.

In the article, we consider the problem on the stabilizer groups for \( n \)-qubit symmetric states. This article is organized as follows. In section II, we present preliminary knowledge on \( n \)-qubit symmetric pure states. In section III, we present our main results. First, we present the stabilizer group for an \( n \)-qubit symmetric state is nontrivial when \( n \leq 4 \), then we present a class of \( n \)-qubit symmetric states whose stabilizer group is trivial when \( n \geq 5 \), at last, when \( m = 2 \), the pure state \( |\psi\rangle \) owns a nontrivial stabilizer group, when \( m = 4 \), there exists only one case when \( |\psi\rangle \) owns a nontrivial stabilizer group, when \( m \geq 5 \), the stabilizer group of almost all \( n \)-qubit symmetric pure state is trivial. In section IV, we will end with a summary.

II. PRELIMINARY KNOWLEDGE

In this section, we will first recall the definition of symmetric states, and then we present the Majorana representation for an \( n \)-qubit symmetric pure state.

A pure state \( |\psi\rangle \in \mathcal{H}_2 \) can be represented by a point on a Bloch sphere geometrically as

\[
|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle,
\]

here two parameters \( \theta \in [0, \pi], \phi \in [0, 2\pi] \).

We call an \( n \)-partite pure state \( |\psi\rangle \) symmetric state if it is invariant under permuting the particles. That is, for any permutation operator \( P_\pi, P_\pi |\psi\rangle = |\psi\rangle \). Generally, there are two main characterizations for an \( n \)-qubit symmetric pure state \( |\psi\rangle \), Majorana representation [19] and Dicke representation. The Majorana representation for an \( n \)-qubit symmetric pure
state is that there exists single particles $|\phi_i\rangle$, $i = 1, 2, \cdots, n$, such that

$$|\psi\rangle = \frac{e^{i\theta}}{\sqrt{K}} \sum_{\sigma \in \text{perm}} |\phi_{\sigma(1)}\rangle |\phi_{\sigma(2)}\rangle \cdots |\phi_{\sigma(n)}\rangle,$$

(3)

where the sum runs over all distinct permutations $\sigma$, $K$ is a normalization prefactor and the $|\phi_i\rangle$ are single qubit states $|\phi_i\rangle = \cos \frac{\theta_i}{2} |0\rangle + e^{i\phi_i} \sin \frac{\theta_i}{2} |1\rangle$, $i = 1, 2, \cdots, n$, we would denote $|\phi_i\rangle$ as $|\phi_i\rangle = a_i |0\rangle + b_i |1\rangle$ below. An n-qubit symmetric pure state $|\psi\rangle$ can also be characterized as the sum of the Dicke states $|D(n, k)\rangle$, that is,

$$|\psi\rangle = \sum x_k |D(n, k)\rangle.$$

(4)

Here the Dicke states $|D(n, k)\rangle$ are defined as

$$|D(n, k)\rangle = \frac{1}{\sqrt{C^k_n}} \sum |0\cdots 01\cdots 1\rangle,$$

(5)

where the sum runs over all the permutations of the qubits. Up to a global phase factor, the parameters $a_i/b_i$ in a pure state $|\phi_i\rangle = a_i |0\rangle + b_i |1\rangle$ are the roots of the polynomial $P(z) = \sum (-1)^k x_k \sqrt{C^k_n} z^k$, here $C^k_n$ denotes the binomial coefficient of $n$ and $k$.

Next we introduce an isometric linear map

$$\mathcal{M} : \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$$

(6)

$$|b\rangle \langle a| \rightarrow |b\rangle |a\rangle$$

(7)

here we denote that $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is the set of all linear maps from the Hilbert space $\mathcal{H}_1$ to the Hilbert space $\mathcal{H}_2$. This map is useful for considering the stabilizer group for a 2-qubit symmetric state. Now we introduce some properties of this map:

(1). Assume $A \in \mathcal{L}(\mathcal{H}_1), B \in \mathcal{L}(\mathcal{H}_1)$, we have that

$$A \otimes B \mathcal{M}(X) = \mathcal{M}(AXB^T).$$

(8)

(2). Assume $|\psi\rangle \in \mathcal{H}_1, |\phi\rangle \in \mathcal{H}_2$, then

$$\mathcal{M}(|\phi\rangle \langle \psi|) = |\phi\rangle |\overline{\psi}\rangle.$$

(9)

Then we recall the definition and some important properties of Möbius transformation, which is useful for the last part of this article. Möbius transformation is defined on the extended complex plane onto itself [20], it can be represented as

$$f(z) = \frac{az + b}{cz + d},$$

(10)
with \(a, b, c, d \in \mathbb{C}, ad - bc \neq 0\). From the above equality, we see that when \(c \neq 0\), this function \(f : \mathbb{C} - \{-d/c\} \to \mathbb{C} - \{a/c\}, f(-d/c) = \infty, f(\infty) = a/c\), when \(c = 0\), this function \(f : \mathbb{C} \to \mathbb{C}, f(\infty) = \infty\). The Möbius transformation owns the following properties:

1. Möbius transformation map circles to circles.
2. Möbius transformation are conformal.
3. If two points are symmetric with respect to a circle, then their images under a Möbius transformation are symmetric with respect to the image circle. This is called the ”Symmetry Principle.”
4. With the exception of the identity mapping, a Möbius transformation has at most two fixed points.
5. There exists a unique Möbius transformation sending any three points to any other three points.
6. The unique Möbius transformation \(z \to M(z)\) sending three points \(q, r, s\) to any other three points \(q', r', s'\) is given by

\[
\frac{(M(z) - q')(r' - s')}{(M(z) - s')(r' - q')} = \frac{(z - q)(r - s)}{(z - s)(r - q)}.
\]

7. The Möbius transformation forms a group, Möbius transformation is isotropic to the projective linear group \(PSL(2, \mathbb{C}) \cong SL(2, \mathbb{C})/\{I, -I\}\).

As we know, the stereographic projection is a mapping that projects a sphere onto a plane. This projection is defined on the whole sphere except a point, and this map is smooth and bijective. It is conformal, i.e. it preserves the angels at where curves meet. By transforming the majorana points of a pure state \(|\psi\rangle\) to an extended complex plane, we may get the following proposition [21].

**Lemma 1.** Assume \(|\psi_1\rangle, |\psi_2\rangle\) are two pure symmetric states, if \(|\psi_1\rangle\) and \(|\psi_2\rangle\) are SLOCC-equivalent iff there exists a Möbius transformation (10) between their Majorana points.

At last, we recall two parameters defined in [14], diversity degree and degeneracy configuration of an \(n\)-qubit symmetric pure state. Both two parameters can be used to identify the SLOCC entanglement classes of all \(n\)-qubit symmetric pure state. Assume \(|\psi\rangle\) is an \(n\)-qubit symmetric pure state, \(|\psi\rangle = \frac{1}{\sqrt{R}} \sum_{\sigma \in \text{perm}} |\phi_{\sigma(1)}\rangle |\phi_{\sigma(2)}\rangle \cdots |\phi_{\sigma(n)}\rangle\), up to a global phase factor, two states \(|\phi_i\rangle\) and \(|\phi_j\rangle\) are identical if and only if \(a_i b_j - a_j b_i = 0\), and we define their number the degeneracy number. Then we define the degeneracy configuration \(\{n_i\}\) of a symmetric
state $|\psi\rangle$ as the list of its degeneracy numbers $n_i$ ordered in decreasing order. We denote the number of the elements in the set $\{n_i\}$ as the diversity degree $m$ of the symmetric state, it stands for the number of distinct $|\phi_i\rangle$ in the Eq.(3). For example, a 3-qubit GHZ state $|GHZ\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$, we have $|\phi_1\rangle = \frac{|0\rangle + w|1\rangle}{2}$, $|\phi_2\rangle = \frac{|0\rangle + w^2|1\rangle}{2}$, $|\phi_3\rangle = \frac{|0\rangle + |1\rangle}{2}$, $w$ is roots of $P(z) = 1 - z^3$, the degeneracy number of $|GHZ\rangle$ is 3, the degeneracy configuration of $|GHZ\rangle$ is $\{1, 1, 1\}$.

III. MAIN RESULTS

First we present the stabilizer group for a two-qubit symmetric pure state $|\psi\rangle \in \mathcal{H}_2 \otimes \mathcal{H}_2$ is nontrivial.

**Theorem 1.** Assume that $|\psi\rangle$ is a 2-qubit symmetric pure state, $|\psi\rangle = \sum_{i=0}^{2} c_i |D(2, i)\rangle$, then the stabilizer group for the state $|\psi\rangle$ is nontrivial.

**Proof.** Assume $A \in \mathcal{L}(\mathcal{H}_1)$ and $B \in \mathcal{L}(\mathcal{H}_2)$ and $A \otimes B |\psi\rangle = |\psi\rangle$, then we have $\mathcal{M}^{-1}(|\psi\rangle) = \mathcal{M}^{-1}(A \otimes B |\psi\rangle)$, according to the equality (8), we have that

$$AX = X(B^T)^{-1}.$$  \hfill (11)

Here we assume that $X = \begin{pmatrix} c_0 & c_1 \\ c_1 & c_2 \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B^T = \begin{pmatrix} h & -f \\ -g & e \end{pmatrix}$ and $\det(B) = 1$, then we can obtain the following equation set:

$$\begin{cases} 
  c_0(c - a) = c_1(b - g) \\
  c_0f - c_2b = c_1(a - h) \\
  c_0c - c_2f = c_1(d - e) \\
  c_1(f - c) = c_2(d - h) 
\end{cases}$$  \hfill (12)

As there are four equations and eight variables, then we know that the rank of the solution vectors is more than 1, that is, the stabilizer group for the state $|\psi\rangle$ contains more than the identity. \hfill \square

Now we present a lemma to show an $n$-qubit symmetric pure state owns a nontrivial stabilizer group when $n = 3, 4$. 

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Lemma 2. Assume $|\psi_1\rangle$ and $|\psi_2\rangle$ are symmetric pure states, and $|\psi_1\rangle$ is SLOCC equivalent to $|\psi_2\rangle$, if there exists a nontrivial ILO $g$ such that $g^\otimes n|\psi_1\rangle = |\psi_1\rangle$, then the stabilizer group for $|\psi_2\rangle$ is nontrivial.

Proof. As $|\psi_1\rangle$ and $|\psi_2\rangle$ are symmetric states, then there exists an ILO $h$ such that $|\psi_2\rangle = h^\otimes n|\psi_1\rangle$. $|\psi_2\rangle = h^\otimes n g^\otimes n(h^{-1})^\otimes n|\psi_2\rangle$, that is, the stabilizer group for $|\psi_2\rangle$ is nontrivial. □

In [14], the authors presents a three-qubit symmetric pure state is SLOCC equivalent to $|W\rangle$ or $|GHZ\rangle$. As we know, when we choose $g =$ \begin{pmatrix} \exp(i\pi/4) & 0 \\ 0 & \exp(-i\pi/2) \end{pmatrix}, $g^\otimes 3|W\rangle = |W\rangle, \sigma_z^\otimes 3|GHZ\rangle = |GHZ\rangle$. From Lemma 2, we see that the stabilizer group for all three-qubit pure symmetric states is nontrivial. And a four-qubit symmetric state is SLOCC equivalent to one of the elements in $S = \{|D(4,0)\rangle, |D(4,1)\rangle, |D(4,2)\rangle, |D(4,2)\rangle + |D(4,0)\rangle + \frac{\mu}{\sqrt{1+\mu^2}} |D(4,2)\rangle\}$, then we present a nontrivial stabilizer for the elements in the set $S$, for the state $|D(4,0)\rangle$, we have $\sigma_z^\otimes 4|D(4,0)\rangle = |D(4,0)\rangle$, for the state $|D(4,1)\rangle$, we choose an ILO $g =$ \begin{pmatrix} \exp(\frac{\pi i}{6}) & 0 \\ 0 & \exp(\frac{3\pi i}{2}) \end{pmatrix}, for the state $|D(4,2)\rangle$, we can choose an ILO $g = \sigma_z$, and for the last element in the set $S$, we can also choose an ILO $g = \sigma_z$, then due to the lemma 2, we have the stabilizer group for all four-qubit symmetric pure state is nontrivial. Note that for four-qubit pure states, the authors in [18] also proposed the similar results.

Here we denote

\[ G \equiv SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C}) \otimes \cdots \otimes SL(2, \mathbb{C}) \]
\[ K \equiv SU(2) \otimes SU(2) \otimes \cdots \otimes SU(2) \]
\[ \tilde{G} \equiv GL(2, \mathbb{C}) \otimes GL(2, \mathbb{C}) \otimes \cdots \otimes GL(2, \mathbb{C}) \]
\[ \tilde{K} \equiv U(2) \otimes U(2) \otimes \cdots \otimes U(2) \] (13)

Next we will use the method proposed in [1] to give a class of symmetric states with its stabilizer group containing only the identity. First we introduce the definition of $SL$-invariant polynomials. A polynomial $f : \mathcal{H} \to \mathbb{C}$ is $SL$-invariant if $f(g|\psi\rangle) = f(|\psi\rangle), \forall g \in SL(2, \mathbb{C}) \otimes SL(2, \mathbb{C}) \cdots \otimes SL(2, \mathbb{C})$, here we denote $\mathcal{H} = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_2$ and $SL(2, \mathbb{C}) = \{ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} | a_{ij} \in \mathbb{C}, i, j = 1, 2, \det(A) = 1 \}$. $f_2$ is a $SL$-invariant
polynomial with degree 2, which is defined as \( f_2(|\psi\rangle) = (|\psi\rangle, |\psi\rangle) = \langle \psi^* | \sigma_y^{\otimes n} | \psi \rangle \), here \( \sigma_y \) is the Pauli operator with its matrix representation \( \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \). Due to the property of \( \sigma_y \), we have that when \( n \) is odd, \( f_2(\cdot) = 0 \). Another polynomial, \( f_4(|\psi\rangle) \), is a polynomial with degree 4, it is defined as \( f_4(|\psi\rangle) = \det \begin{pmatrix} (|\psi_0\rangle, |\psi_0\rangle) & (|\psi_0\rangle, |\psi_1\rangle) \\ (|\psi_1\rangle, |\psi_0\rangle) & (|\psi_1\rangle, |\psi_1\rangle) \end{pmatrix} \), here we assume that \( |\psi\rangle = |0\rangle |\psi_0\rangle + |1\rangle |\psi_1\rangle \). Below we denote the stabilizer group for a pure state \( |\psi\rangle \) as \( \tilde{G}_\psi = \{ g_1 \otimes g_2 \otimes \cdots \otimes g_n \in \tilde{G} | g_1 \otimes g_2 \otimes \cdots \otimes g_n |\psi\rangle = |\psi\rangle \} \).

**Example 1.** Assume \( |\psi\rangle \) is an \( n \)-partite symmetric pure state \( (n \geq 5) \), \( |\psi\rangle = x_k |D(n, k)\rangle + x_l |D(n, l)\rangle + x_n |D(n, n)\rangle \), \( k + l = n + 1 \), \( k, l \neq \frac{n}{2} \), \( \gcd(k, l) = 1 \), here we denote that \( \gcd(k, l) \) is the greatest common divisor of \( k \) and \( l \), Then \( \tilde{G}_\psi = \{ I \} \).

**Proof.** First we denote \( g_2 \otimes g_3 \otimes \cdots \otimes g_n \) as \( h \) and let \( g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \), then

\[
g_1 \otimes h |\psi\rangle
= |0\rangle h[a_1(x_k |D(n - 1, k)\rangle + x_l |D(n - 1, l)\rangle) + b_1(x_k |D(n, k - 1)\rangle + x_l |D(n - 1, l - 1)\rangle) \\
+ x_n |D(n - 1, n - 1)\rangle] + |1\rangle h[c_1(x_k |D(n - 1, k)\rangle + x_l |D(n - 1, l)\rangle) \\
+ d_1(x_k |D(n - 1, k - 1)\rangle + x_l |D(n - 1, l - 1)\rangle) + x_n |D(n - 1, n - 1)\rangle]
= |0\rangle(x_k |D(n - 1, k)\rangle + x_l |D(n - 1, l)\rangle) \\
+ |1\rangle(x_k |D(n - 1, k - 1)\rangle + x_l |D(n - 1, l - 1)\rangle) + x_n |D(n - 1, n - 1)\rangle)
\] (14)

Apply \( \langle 0 | \otimes I^{(n-1)} \) to the left hand side (LHS) and the right hand side (RHS) of the above equality, and we denote that \( |\zeta_1\rangle = x_k |D(n - 1, k)\rangle + x_l |D(n - 1, l)\rangle \), \( |\zeta_2\rangle = x_k |D(n - 1, k - 1)\rangle + x_l |D(n - 1, l - 1)\rangle + x_n |D(n - 1, n - 1)\rangle \), then

\[
h[a_1|\zeta_1\rangle + b_1|\zeta_2\rangle] = x_k |D(n - 1, k)\rangle + x_l |D(n - 1, l)\rangle.
\] (15)

Assume \( n \) is odd, then by using \( f_2(\cdot) \) to the equality (15), we have \( 2b_1^2 x_k x_l = 0, b_1 = 0 \). As \( |\psi\rangle \) is a symmetric state, we may assume \( g_i = \begin{pmatrix} a_i & 0 \\ c_i & d_i \end{pmatrix} \), \( i = 2, \ldots, n \), due to \( g_1 \otimes g_2 \otimes \cdots \otimes g_n |\psi\rangle = |\psi\rangle \) and through simple computation, we have that \( c_i = 0 \). Applying \( g_i = \begin{pmatrix} a_i & 0 \\ 0 & d_i \end{pmatrix} \) to the equality (14), we have \( d_i = \lambda a_i, \prod d_i = 1, \lambda^k = \lambda^l = 1 \), as \( \gcd(k, l) = 1 \), we have
First we prove

**Proof.**

Example 2. Assume

Then we present a class of symmetric critical states $|ψ⟩$ with $G_ψ = \{I\}$. First we present the definition of critical states and a meaningful characterization of critical states. The set of critical states is defined as:

$$\text{Crit}(H_n) = \{ψ|⟨ψ|X|ψ⟩ = 0, X ∈ Lie(G)\}. \quad (16)$$

Here $Lie(G)$ is the Lie algebra of $G$. The critical set is valuable, as many important states in quantum information theory, such as the Bell states, GHZ states, cluster states and graph states, belong to the set of critical states. Then we present a fundamental characterization of critical states.

**Lemma 3.** (The Kempf-Ness theorem)

1. $ψ ∈ \text{Crit}(H_n)$ if and only if $||gψ|| ≥ ||ψ||$ for all $g ∈ G$, $||⋅||$ denotes the Euclidean norm of $|ψ⟩ ∈ H_n$.
2. If $ψ ∈ \text{Crit}(H_n)$, then $||gψ|| ≥ ||ψ||$ with equality if and only if $gψ ∈ Kψ$. Moreover, if $g$ is positive semidefinite then the equality condition holds if and only if $gψ = ψ$.
3. If $ψ ∈ H_n$, then $Gψ$ is closed in $H_n$ if and only if $Gψ ∩ \text{Crit}(H_n) ≠ \emptyset$.

Due to this lemma, Gour and Wallach in [22] showed that $ψ ∈ H_n$ is critical if and only if all the local density matrices of $|ψ⟩$ are proportional to the identity. And by the lemma in [1], for a class of pure states $|ψ⟩ ∈ \text{Crit}(H_n)$, if we could show the set $K_ψ = \{U_1 ⊗ U_2 ⊗ \cdots ⊗ U_n ∈ K|U_1 ⊗ U_2 ⊗ \cdots ⊗ U_n|ψ⟩ = |ψ⟩\}$ contains only the identity, then $G_ψ = \{I\}$.

**Example 2.** Assume $|ψ⟩ = a_0|D(n, 0)⟩ + a_{n−1}|D(n, n−1)⟩$ or $|ψ⟩ = a_1|D(n, 1)⟩ + a_n|D(n, n)⟩$ is an $n$-qubit symmetric critical pure state with $n > 4$, then $G_ψ = \{I\}$.

**Proof.** First we prove $K_ψ = \{I\}$. When $|ψ⟩ = a_1|D(n, 1)⟩ + a_n|D(n, n)⟩$, assume $U^{⊗n} ∈ K$ satisfies $U^{⊗n}|ψ⟩ = |ψ⟩$, then

$$(U ⊗ U)ρ_{1,2}(U ⊗ U)^+ = ρ_{1,2}, \quad (17)$$

the equality (17) can be changed as

$$(U ⊗ U)[\frac{(n−2)|a_1|^2}{n}|00⟩⟨00| + \frac{|a_1|^2}{n}(|01⟩⟨01| + |10⟩⟨10|) + |a_n|^2|11⟩⟨11|](U ⊗ U)^+$$

$$= \frac{|a_1|^2(n−2)}{n}|00⟩⟨00| + \frac{|a_1|^2}{n}(|01⟩⟨01| + |10⟩⟨10|) + |a_n|^2|11⟩⟨11| \quad (18)$$

$\lambda = 1$, that is $G_ψ = \{I\}$.

When $n$ is even, we use $f_k(\cdot)$ to the equality (15), we have $2b_1^2x_k^2x_l^2 = 0, b_1 = 0$. From the same method as when $n$ is odd, we have that $G_ψ = \{I\}\Box$.

Then we present a class of symmetric critical states $|ψ⟩$ with $G_ψ = \{I\}$. First we present the definition of critical states and a meaningful characterization of critical states. The set of critical states is defined as:

$$\text{Crit}(H_n) = \{ψ|⟨ψ|X|ψ⟩ = 0, X ∈ Lie(G)\}. \quad (16)$$

Here $Lie(G)$ is the Lie algebra of $G$. The critical set is valuable, as many important states in quantum information theory, such as the Bell states, GHZ states, cluster states and graph states, belong to the set of critical states. Then we present a fundamental characterization of critical states.

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2. If $ψ ∈ \text{Crit}(H_n)$, then $||gψ|| ≥ ||ψ||$ with equality if and only if $gψ ∈ Kψ$. Moreover, if $g$ is positive semidefinite then the equality condition holds if and only if $gψ = ψ$.
3. If $ψ ∈ H_n$, then $Gψ$ is closed in $H_n$ if and only if $Gψ ∩ \text{Crit}(H_n) ≠ \emptyset$.

Due to this lemma, Gour and Wallach in [22] showed that $ψ ∈ H_n$ is critical if and only if all the local density matrices of $|ψ⟩$ are proportional to the identity. And by the lemma in [1], for a class of pure states $|ψ⟩ ∈ \text{Crit}(H_n)$, if we could show the set $K_ψ = \{U_1 ⊗ U_2 ⊗ \cdots ⊗ U_n ∈ K|U_1 ⊗ U_2 ⊗ \cdots ⊗ U_n|ψ⟩ = |ψ⟩\}$ contains only the identity, then $G_ψ = \{I\}$.

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**Proof.** First we prove $K_ψ = \{I\}$. When $|ψ⟩ = a_1|D(n, 1)⟩ + a_n|D(n, n)⟩$, assume $U^{⊗n} ∈ K$ satisfies $U^{⊗n}|ψ⟩ = |ψ⟩$, then

$$(U ⊗ U)ρ_{1,2}(U ⊗ U)^+ = ρ_{1,2}, \quad (17)$$

the equality (17) can be changed as

$$(U ⊗ U)[\frac{(n−2)|a_1|^2}{n}|00⟩⟨00| + \frac{|a_1|^2}{n}(|01⟩⟨01| + |10⟩⟨10|) + |a_n|^2|11⟩⟨11|](U ⊗ U)^+$$

$$= \frac{|a_1|^2(n−2)}{n}|00⟩⟨00| + \frac{|a_1|^2}{n}(|01⟩⟨01| + |10⟩⟨10|) + |a_n|^2|11⟩⟨11| \quad (18)$$

$\lambda = 1$, that is $G_ψ = \{I\}$.

When $n$ is even, we use $f_k(\cdot)$ to the equality (15), we have $2b_1^2x_k^2x_l^2 = 0, b_1 = 0$. From the same method as when $n$ is odd, we have that $G_ψ = \{I\}\Box$.
As the formula on the RHS of the equality above can be seen as a Schmidt decomposition, and \( U \otimes U \) cannot increase the Schmidt rank, then \( U|1\rangle = u_1|1\rangle \), as \( U \) is a unitary operator, then \( U|0\rangle = u_0|0\rangle \). At last, due to \( U^{\otimes n}|\psi\rangle = |\psi\rangle \), then we have \( u_0^{n-1}u_1 = 1, u_1^n = 1 \), i.e. \( u_0 = u_1 \). The other case is similar.

Here we present another proof on \( G_{L_n} = \{I\} \), \( |L_n\rangle \) is defined in the equality (8) of the article [1]. Note that the examples above tells us \( G_\psi = \{I\} \), however, \( \tilde{G}_\psi \supset \{I\} \). It seems that this result is simple, However, this method is very useful to present nontrivial examples of states in n-qudit systems with nontrivial stabilizer groups [13].

At last, I would like to apply Möbius transformation to show when the diversity number \( m \) of an \( n \)-qubit symmetric pure state \( |\psi\rangle \) is 5 or 6, the stabilizer group of \( |\psi\rangle \) is trivial, when \( m \geq 7 \), under a conjecture we make, the stabilizer group of \( |\psi\rangle \) is trivial.

Assume a pure symmetric state \( |\psi\rangle \) can be represented in terms of Majorana representation:

\[
|\psi\rangle = \frac{e^{i\alpha}}{\sqrt{K}} \sum_{\sigma \in \text{perm}} |\phi_{\sigma(1)}\rangle |\phi_{\sigma(2)}\rangle \cdots |\phi_{\sigma(n)}\rangle, \tag{19}
\]

where the sum takes over all the permutations and \( K \) is the normalization for the state \( |\psi\rangle \). Due to the main results proposed by Mathonet et al. [15], we see that if \( |\psi\rangle = g_1 \otimes g_2 \otimes \cdots \otimes g_n |\psi\rangle \), then there exists an ILO \( g \) such that \( |\psi\rangle = g^{\otimes n} |\psi\rangle \), i.e. if we could prove \( \{g|g^{\otimes n} |\psi\rangle = |\psi\rangle \} = \{I\} \), then the stabilizer group of \( |\psi\rangle \) is trivial. From the Eq.(19),

\[
g^{\otimes n} |\psi\rangle = |\psi\rangle, \tag{20}
\]

\[
\sum_{\sigma \in \text{perm}} \bigotimes_{i} g |\phi_{\sigma(i)}\rangle = \sum_{\sigma \in \text{perm}} \bigotimes_{i} |\phi_{\sigma(i)}\rangle. \tag{21}
\]

Due to the uniqueness of the Majorana representation for a symmetric state and according to the equality (21), we see that there is a permutation \( \sigma \) such that

\[
g |\phi_{\sigma(i)}\rangle = \lambda_i |\phi_{\sigma(i)}\rangle \tag{22}
\]

with \( \Pi_i \lambda_i = 1 \).

Assume the diversity number of a symmetric state \( |\psi\rangle \) is \( m \), the divergence configuration for the state \( |\psi\rangle \) is \( \{k_1, k_2, \cdots, k_m\} \) with \( k_1 \geq k_2 \cdots \geq k_m \) and \( \sum_i k_i = n \). Then from the Lemma 1 and the property (4) of Möbius transformation, we have:
Corollary 1. Assume $|\psi\rangle$ is an $n$-qubit symmetric state, its degeneracy configuration is $\{k_1, k_2, \cdots, k_m\}, m \geq 3$, if there exists $k_i, k_j$ and $k_l$ such that these three values are unequal to the residual elements in the set, then the stabilizer group for the state $|\psi\rangle$ contains only the identity.

At last, we talk about the stabilizer group for an $n$-qubit symmetric pure state $|\psi\rangle$ with the increase of the diversity number $m$ of $|\psi\rangle$, when $m = 2$ the stabilizer group for $|\psi\rangle$ is nontrivial, when $m \geq 5$, under a conjecture we make, the stabilizer group for $|\psi\rangle$ is trivial.

$m=1$ (separable states): When a symmetric state $|\psi\rangle$ is separable, then it can be represented as

$$|\psi\rangle = |\epsilon\epsilon\cdots\epsilon\rangle,$$

(23)

it is easy to see when an ILO $g$ satisfies that $|\epsilon\rangle$ is an eignvector of $g$, $g^\otimes n$ is the stabilizer for the state $|\psi\rangle$.

$m=2$: In this case, the state $|\psi\rangle$ can be represented as

$$|\psi\rangle = \frac{e^{i\theta}}{\sqrt{K}} \sum_{k_1} |\epsilon_1\epsilon_2\cdots\epsilon_{n-k_1}\rangle,$$

(24)

where the sum takes over all the permutation of $k_1 |\epsilon_1\rangle s$ and $n - k_1 |\epsilon_2\rangle s$ in this case, when $k_1 \neq n - k_1$, first we denote a local operator $h = -[[|\epsilon_1\rangle, |\epsilon_2\rangle]]^{-1}$, this means if $|\epsilon_1\rangle = a_1|0\rangle + b_1|1\rangle, |\epsilon_2\rangle = a_2|0\rangle + b_2|1\rangle$, $h = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$, then $g$ can be written as

$g = h^{-1} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} h$. And when $k_1 = n - k_1$, the ILO $g$ can also be written as

$g = h^{-1} \begin{pmatrix} 0 & \lambda_2 \\ \lambda_1 & 0 \end{pmatrix} h, h = [[|\epsilon_1\rangle, |\epsilon_2\rangle]]^{-1}$.

When $m \geq 3$, due to Lemma 3, by searching a nontrivial Möbius transformation between the Majorana points of the pure state $|\psi\rangle$, we can see whether a pure state $|\psi\rangle$ owns a nontrivial stabilizer group.

When $m = 3$, assume the degeneracy configuration of a pure state $|\psi\rangle$ is $\{k_1, k_2, k_3\}$, due to the property (5), we see that except when $k_1 \neq k_2 \neq k_3$, $|\psi\rangle$ owns a nontrivial stabilizer group.

Here we assume that each $z_i$ is not $\infty$, as if there exists $k$ such that $z_k = \infty$, we can
always make $z_k$ be not $\infty$ by Möbius transformation. When $m = 4$, assume the degeneracy configuration of the pure state $|\psi\rangle$ is $\{k_1, k_2, k_3, k_4\}$, when all the four number are different from each other or there are only two of them are equal, the stabilizer group for $|\psi\rangle$ is trivial. First we show when $k_1 = k_2, k_3 = k_4$, the stabilizer group is nontrivial, i.e. there exists a nontrivial Möbius transformation $f$ that can permute $z_i, i = 1, 2, 3, 4$. Let $f(z_1) = z_2, f(z_2) = z_1, f(z_3) = z_4, f(z_4) = z_3$, from the property (6) of the Möbius transformation, we see $f$ exists. When three of them are equal, we can always assume $k_1 = k_2 = k_3 \neq k_4$, then the Möbius transformation $f$ should satisfy one of the following cases,

(1). $f(z_1) = z_2, f(z_2) = z_3, f(z_3) = z_1, f(z_4) = z_4$
(2). $f(z_1) = z_2, f(z_2) = z_1, f(z_3) = z_3, f(z_4) = z_4$
(3). $f(z_1) = z_1, f(z_2) = z_3, f(z_3) = z_2, f(z_4) = z_4$
(4). $f(z_1) = z_3, f(z_3) = z_1, f(z_2) = z_2, f(z_4) = z_4$

Here we only consider the first two cases, the other two is similar to the second one. In the case 1, from the property (6) of the Möbius transformation $f$, we have

$$
\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \frac{(z_2 - z_3)(z_1 - z_4)}{(z_2 - z_4)(z_1 - z_3)} = \frac{(z_3 - z_1)(z_2 - z_4)}{(z_3 - z_4)(z_2 - z_1)},
$$

(25)

from the above equalities, we have

$$
(z_1 - z_4)^2(z_2 - z_3)^2 + (z_1 - z_2)(z_3 - z_4)(z_2 - z_4)(z_1 - z_3) = 0, \quad (26)
$$
$$
(z_1 - z_2)^2(z_3 - z_4)^2 + (z_1 - z_4)(z_2 - z_3)(z_3 - z_1)(z_2 - z_4) = 0, \quad (27)
$$

then (26) − (27), we have

$$
(z_1 - z_4)(z_2 - z_3)(z_2 - z_4)(z_1 - z_3) = 0, \quad (28)
$$

i.e. $z_1 = z_4$, $z_2 = z_3$, $z_4 = z_2$ or $z_3 = z_1$, however, these four conditions is invalid, that is, the case 1 is invalid. For the case 2, we may also from the property (6) of the Möbius transformation $f$, we have

$$
\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{(z_2 - z_1)(z_3 - z_4)}{(z_2 - z_4)(z_3 - z_1)},
$$

(29)

$$
\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_3)(z_2 - z_4)},
$$

(30)
from the above equalities, we have

\[(z_1 - z_3)(z_2 - z_4) - (z_2 - z_3)(z_1 - z_4) = 0, \tag{31}\]
\[(z_1 - z_2)(z_3 - z_4) = 0, \tag{32}\]

this is invalid. Then we see the above four conditions are invalid. When \(k_1 = k_2 \neq k_3 \neq k_4\), similar to the above analysis, the stabilizer group for the pure states is trivial. Then we see that only when \(k_1 = k_2, k_3 = k_4\) the stabilizer group is nontrivial.

Here we note that when \(|\psi\rangle\) is a four qubit symmetric pure state, \(|\psi\rangle\) always owns a non-trivial stabilizer group. As the diversity configuration can only be \(\{1, 1, 1, 1\}, \{2, 1, 1\}, \{3, 1\}\) or \(\{4\}\), from the above analysis, we see that the stabilizer group is nontrivial.

When \(m = 5\), assume the degeneracy configuration of the pure state \(|\psi\rangle\) is \(\{k_1, k_2, k_3, k_4, k_5\}\). First we analyze the case when \(k_1 = k_2 = k_3 = k_4 = k_5\), if the stabilizer group is nontrivial, then there exists a nontrivial Möbius transformation \(f\) that can permute \(z_i, i = 1, 2, 3, 4, 5\), and we need to consider the following cases,

1. \(f(z_1) = z_2, f(z_2) = z_3, f(z_3) = z_4, f(z_4) = z_5, f(z_5) = z_1,\)
2. \(f(z_1) = z_2, f(z_2) = z_3, f(z_3) = z_4, f(z_4) = z_1, f(z_5) = z_5,\)
3. \(f(z_1) = z_2, f(z_2) = z_3, f(z_3) = z_1, f(z_4) = z_4, f(z_5) = z_5,\)
4. \(f(z_1) = z_2, f(z_2) = z_3, f(z_3) = z_1, f(z_4) = z_5, f(z_5) = z_4,\)
5. \(f(z_1) = z_2, f(z_2) = z_1, f(z_3) = z_4, f(z_4) = z_3, f(z_5) = z_5.\)

Similar to the analysis when \(m = 4\), we have the above case (3) is invalid. For the case (4), we can consider the Möbius transformation \(g = f \circ f\), as \(g(z_1) = z_3, g(z_2) = z_1, g(z_3) = z_2, g(z_4) = z_4, g(z_5) = z_5\), so the case (4) is invalid. For the case (5), we can get

\[(z_1 - z_4)(z_2 - z_5)(z_3 - z_5) + (z_1 - z_5)(z_2 - z_3)(z_4 - z_5) = 0, \tag{33}\]
\[(z_1 - z_5)(z_2 - z_4)(z_3 - z_5) + (z_1 - z_3)(z_2 - z_5)(z_4 - z_5) = 0, \tag{34}\]
\[(z_1 - z_5)(z_2 - z_3)(z_4 - z_5) = 0. \tag{35}\]

Then the case (5) is invalid. For the case (2), similar to the analysis when \(m = 4\), we see this is invalid. However, for the case (1), it may be valid.

Next we could partition the degeneracy configuration of \(|\psi\rangle\) into \(d\) parts according to whether the degeneracy number of the majorana point \(|\phi_i\rangle\) is equal, when \(d \geq 2\), we show that the stabilizer group of a symmetric pure state \(|\psi\rangle\) is trivial. Assume the stabilizer group of the state \(|\psi\rangle\) is nontrivial, then we have a nontrivial Möbius transformation which
permutes the majorana points, that is, we could assume \( f(z_1) = z_2, f(z_2) = z_3, \cdots, f(z_i) = z_1, f(z_{i+1}) = z_{i+2}, f(z_{i+2}) = z_{i+3}, \cdots, f(z_{m}) = z_{i+1}, \) if \( i_1, i_2 - i_1 \geq 3, (i_1, i_2 - i_1) = 1, \) here we denote that \( (i_1, i_2 - i_1) \) is the greatest common divisor of \( i_1 \) and \( i_2 - i_1, \) then we could have \( f^{i_1}(z_j) = z_j, j = 1, 2, \cdots, i_1, \) that is, \( f^{i_1} = 1, \) however, \( f(z_j) \neq z_j, j \geq i_1 + 1. \) Similarly, we could prove the case when \( i_1, i_2 - i_1 \geq 3, (i_1, i_2 - i_1) = r \neq 1. \) When \( i_1 = i_2 - i_1 = i_3 - i_2 = 2, \) there exists \( f \) such that \( f(z_1) = z_2, f(z_2) = z_1, \cdots, f(z_5) = z_6, f(z_6) = z_5. \) Similar to the analysis above, we have

\[
\frac{(z_1 - z_2)(z_3 - z_5)}{(z_1 - z_2)(z_3 - z_2)} = \frac{(z_2 - z_1)(z_4 - z_6)}{(z_2 - z_6)(z_4 - z_1)},
\]

\[
(z_3 - z_5)(z_2 - z_6)(z_1 - z_4) = (z_1 - z_5)(z_3 - z_2)(z_4 - z_6),
\]

\[
(z_3 - z_5)(z_2 - z_6)(z_1 - z_4)(z_1 - z_2) = (z_2 - z_1)(z_1 - z_5)(z_3 - z_2)(z_4 - z_6),
\]

\[
\frac{(z_1 - z_4)(z_2 - z_6)}{(z_1 - z_2)(z_2 - z_4)} = \frac{(z_2 - z_3)(z_1 - z_5)}{(z_2 - z_5)(z_1 - z_3)},
\]

\[
(z_1 - z_4)(z_2 - z_6)(z_2 - z_5)(z_1 - z_3) = (z_1 - z_6)(z_2 - z_4)(z_2 - z_3)(z_1 - z_5),
\]

that is,

\[
(z_2 - z_5)(z_3 - z_1)(z_4 - z_6) = (z_3 - z_5)(z_1 - z_6)(z_2 - z_4),
\]

\[
(36) - (39),
\]

\[
(z_4 - z_6)(z_1 - z_2)(z_3 - z_5) = 0,
\]

as these six points are not identical, this is invalid. Then we show the following theorem,

**Theorem 2.** Assume a pure state \( |\psi\rangle \) is symmetric with its degeneracy configuration \( \{k_1, k_2, \cdots, k_m\}, \) then we could partition the degeneracy configuration into \( d \) parts according to whether the degeneracy number is equal. When \( d \geq 2, \) we have the stabilizer group of the state \( |\psi\rangle \) is trivial. When \( d = 1, \) if there doesnot exist a Möbius transformation \( f \) such that \( f(z_1) = z_2, f(z_2) = z_3, \cdots, f(z_{m-1}) = f(z_m), f(z_m) = z_1, \) then we have the stabilizer group of the state \( |\psi\rangle \) is trivial.

From the above theorem, we have that when \( m \geq 5, \) the symmetric pure state \( |\psi\rangle \) with trivial stabilizer group is of full measure among the subspace of symmetric states.

This fact is interesting, as it implies much in the entanglement theory. As when an \( n \)-qubit symmetric pure state is entangled, it is fully entangled. Due to the results in [1], we have the following theorem.
Theorem 3. Assume $|\psi\rangle$ is an n-qubit symmetric pure state with $\tilde{G}_\psi = \{I\}$, $|\phi\rangle$ is an n-qubit symmetric entangled state, then $|\psi\rangle$ can be converted deterministically to $|\phi\rangle$ by LOCC or SEP operations if and only if there is $u \in \tilde{K}$ such that $|\psi\rangle = u|\phi\rangle$.

From the above theorem, we have if two states satisfy the conditions in the above theorem, if they are not LU equivalent, they cannot be transformed into other by LOCC, even SEP. And from the statement below the theorem 2, we have that when $m \geq 5$, almost all symmetric pure states $|\psi\rangle$ are isolated. This fact may represent the complexity of the structure of the multipartite entanglement.

From the above results, we see that when two symmetric pure states satisfy conditions in the above theorems, a state can be transformed into the other by local transformations only with probability. At last, we present the maximal probability with which $|\psi\rangle \rightarrow |\phi\rangle$ can be converted by LOCC by using the theorem 7 in [1].

Theorem 4. Assume $|\psi\rangle$ is a symmetric pure state with $\tilde{G}_{|\psi\rangle} = \{I\}$, $|\phi\rangle = g^\otimes n|\psi\rangle$ is a symmetric pure state, then the maximal probability with which $|\psi\rangle$ can be converted to $|\phi\rangle$ by LOCC or SEP is given by

$$p_{\text{max}}(|\psi\rangle \rightarrow |\phi\rangle) = \frac{1}{\lambda_{\text{max}}(g_1^\otimes n g_2^\otimes n)},$$

(41)

here we denote that $\lambda_{\text{max}}(X)$ is the maximal eigenvalue of the matrix $X$.

IV. CONCLUSION

In this article, we consider the stabilizer group for a symmetric state $|\psi\rangle$. First we present that the stabilizer group for an n-qubit symmetric state $|\psi\rangle$ contains more than the identity when $n = 2, 3, 4$, then similar to the method presented in [1], we give a class of states whose stabilizer group contains only the identity, we also propose a class of states $|\psi\rangle$ with $G_\psi = \{I\}$, $\tilde{G}_\psi \neq \{I\}$, at last, we present that when $m \geq 5$, almost all n-qubit symmetric pure states owns a trivial stabilizer group, then due to the results in [1], we have that almost all n-qubit symmetric pure states cannot be transformed among symmetric pure entangled states by nontrivial LOCC transformations deterministically, and we also present the optimal probability of the local transformations among two SLOCC-equivalent symmetric states.
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