Bavrin’s Type Factorization of the Temljakov Operator for Holomorphic Functions in Circular Domains of $\mathbb{C}^n$

Renata Długoš1, 2 · Piotr Liczberski2 · Edyta Trybucka3

Received: 11 July 2017 / Accepted: 18 January 2018 / Published online: 20 March 2018
© The Author(s) 2018

Abstract The paper concerns investigations of holomorphic functions of several complex variables with a factorization of their Temljakov transform. Firstly, there were considered some inclusions between the families $C_G, M_G, N_G, R_G, V_G$ of such holomorphic functions on complete $n$-circular domain $G$ of $\mathbb{C}^n$ in some papers of Bavrin, Fukui, Higuchi, Michiwaki. A motivation of our investigations is a condensation of the mentioned inclusions by some new families of Bavrin’s type. Hence we consider some families $K^k_G, k \geq 2$, of holomorphic functions $f : G \to \mathbb{C}, f(0) = 1$, defined also by a factorization of $L_f$ onto factors from $C_G$ and $M_G$. We present some interesting properties and extremal problems on $K^k_G$.

Keywords Holomorphic functions on $n$-circular domains in $\mathbb{C}^n$ · Minkowski function · Estimates of homogeneous polynomials of Taylor series · Temljakov operator · Bavrin’s families of functions
Mathematics Subject Classification 32A30 · 30C45

1 Introduction

We say that a domain \( G \subset \mathbb{C}^n \), is complete \( n \)-circular if \( z^\lambda = (z_1^\lambda_1, \ldots, z_n^\lambda_n) \in G \) for each \( z = (z_1, \ldots, z_n) \in G \) and every \( \lambda = (\lambda_1, \ldots, \lambda_n) \in U^n \), where \( U \) is the unit disc \( \{ \zeta \in \mathbb{C} : |\zeta| < 1 \} \). From now by \( G \) will be denoted a bounded complete \( n \)-circular domain in \( \mathbb{C}^n \), \( n \geq 2 \).

By \( H_G \) let us denote the space of all holomorphic functions \( f : G \rightarrow \mathbb{C} \) and by \( H_G(1) \) the collection of all \( f \in H_G \), normalized by \( f(0) = 1 \).

Many authors (cf., eg., [1,2,5–7,11,18,19,23]) considered some Bavrin’s subfamilies \( X_G \) of the family \( H_G(1) \). In the definitions of these families the main role play the families \( C_G(\alpha) \), \( \alpha \in [0,1) \),

\[
C_G(\alpha) = \{ f \in H_G(1) : \text{Re}\ f(z) > \alpha, z \in G \}
\]

and the following invertible Temljakov [24] linear operator \( L : H_G \rightarrow H_G \)

\[
L f(z) = f(z) + Df(z)(z), z \in G,
\]

where \( Df(z) \) is the Fréchet derivative of \( f \) at the point \( z \). By a Bavrin’s family \( X_G \) we mean a collection of functions \( f \in H_G(1) \) whose the Temljakov transform \( L f \) has a functional factorization \( L f = p \cdot g \), where \( p \in C_G \equiv C_G(0) \) and \( g \) is from a fixed subfamily of \( H_G(1) \). Below, we recall the factorizations which define a few well known Bavrin’s families \( X_G \), like as

\[
\begin{align*}
V_G : L f &= p \cdot 1, p \in C_G, \\
M_G : L f &= p \cdot f, p \in C_G, \\
N_G : L f &= p \cdot L L f, p \in C_G, \\
R_G : L f &= p \cdot L \varphi, \varphi \in N_G, p \in C_G.
\end{align*}
\]

It is known that functions of these families were used to construct biholomorphic mappings in \( \mathbb{C}^n \) (cf., eg., [10,13,20]). Let us note that the above families have geometric interpretation, in particular the functions \( f \in M_G \) map biholomorphically some planar intersections \( S \) of \( G \) onto starlike domains in \( \mathbb{C} \) (see [1]). It is very important, because the starlikeness plays a central role in many different subjects of geometry and topology and in particular, in geometric function theory.

Let us recall also that Bavrin showed the inclusions \( N_G \subset R_G, V_G \subset R_G \) and pointed that the first of them can be complete to the following double inclusion \( N_G \subset M_G \subset R_G \). Thus, it is natural to ask whether is possible to do the same in the case of the second above inclusion. In the paper [12] the authors defined a family \( K_G^- \), which satisfies the inclusion \( V_G \subset K_G^- \subset R_G \). An adequate definition of \( K_G^- \) has the form: A function \( f \in H_G(1) \) belongs to \( K_G^- \) if its Temljakov transform \( L f \) has the factorization
\[ \mathcal{L} f(z) = p(z) \cdot h(z) \cdot h(-z), \quad z \in \mathcal{G}, \quad h \in \mathcal{M}_{\mathcal{G}} \left( \frac{1}{2} \right), \quad p \in \mathcal{C}_{\mathcal{G}}, \]

where the family \( \mathcal{M}_{\mathcal{G}}(\alpha), \alpha \in [0, 1) \), is defined similarly as \( \mathcal{M}_{\mathcal{G}} \), but in this case \( p \in \mathcal{C}_{\mathcal{G}}(\alpha) \).

In the present paper we consider Bavrin’s type families \( \mathcal{K}_{\mathcal{G}}^k, k \geq 2(\mathcal{K}_{\mathcal{G}}^2 = \mathcal{K}_{\mathcal{G}}^1) \) separating also the families \( \mathcal{V}_{\mathcal{G}}, \mathcal{R}_{\mathcal{G}} \), i.e., satisfying the inclusions \( \mathcal{V}_{\mathcal{G}} \subsetneq \mathcal{K}_{\mathcal{G}}^k \subsetneq \mathcal{R}_{\mathcal{G}}, k \geq 2 \).

The formal definition of such family has the following form.

A function \( f \in \mathcal{H}_{\mathcal{G}}(1) \) belongs to \( \mathcal{K}_{\mathcal{G}}^k \) if there exist a function \( p \in \mathcal{C}_{\mathcal{G}} \) and a function \( h \in \mathcal{M}_{\mathcal{G}}(k-1) \) such that the Temljakov transform \( \mathcal{L} f \) of \( f \) has the factorization

\[ \mathcal{L} f(z) = p(z) \cdot \prod_{l=0}^{k-1} h(\varepsilon^l z), \quad z \in \mathcal{G}, \quad (1.1) \]

where \( \varepsilon = \varepsilon_k = \exp \frac{2\pi i}{k} \) is a generator of the cyclic group of \( k \)th roots of unity.

Let us observe that \( \mathcal{K}_{\mathcal{G}}^k, k \geq 2 \) are nonempty families. Indeed, the function \( f = 1 \) belongs to \( \mathcal{K}_{\mathcal{G}}^k \), because it satisfies the factorization \((1.1)\) with \( p = 1 \in \mathcal{C}_{\mathcal{G}} \) and \( h = 1 \in \mathcal{M}_{\mathcal{G}}(k-1) \).

In the future, we will use a characterization of the family \( \mathcal{K}_{\mathcal{G}}^k \) by a notion of \((j, k)\)-symmetry, which is connected with a functional decomposition with respect to the above group.

Let us observe that bounded complete \( n \)-circular domains \( \mathcal{G} \) are \( k \)-symmetric sets for \( k = 2, 3, \ldots \), that is \( \varepsilon_\mathcal{G} = \mathcal{G} \). For \( j = 0, 1, \ldots, k-1 \) we define the collections \( \mathcal{F}_{j,k}(\mathcal{G}) \) of functions \((j, k)\)-symmetrical, i.e., all functions \( f: \mathcal{G} \to \mathbb{C} \) such that

\[ f(\varepsilon z) = \varepsilon^j f(z), \quad z \in \mathcal{G}. \]

If \( n = 1 \) and \( \mathcal{G} = \mathcal{U} \), then we write \( \mathcal{F}_{j,k}(\mathcal{U}) \).

The mentioned functional decomposition appears in the following result from [14].

**Theorem A** For every function \( f: \mathcal{G} \to \mathbb{C} \) there exists exactly one sequence of functions \( f_{j,k} \in \mathcal{F}_{j,k}(\mathcal{G}), j = 0, 1, \ldots, k-1 \), such that

\[ f = \sum_{j=0}^{k-1} f_{j,k}. \]

Moreover,

\[ f_{j,k}(z) = \frac{1}{k} \sum_{l=0}^{k-1} \varepsilon^{-jl} f\left( \varepsilon^l z \right), \quad z \in \mathcal{G}. \]

The functions \( f_{j,k}, \) which are uniquely determined by the above decomposition, will be called \((j, k)\)-symmetrical components of the function \( f \). Some interesting applications of the above partition may also be found in [15, 16] and [17].
2 Results

Now we can present a characterization of \( f \in \mathcal{K}^k_G \), simpler than (1.1).

**Theorem 1** A function \( f \in \mathcal{H}_G(1) \) belongs to the family \( \mathcal{K}^k_G \), \( k \geq 2 \) if and only if there exists a function \( g \in \mathcal{M}_G \cap \mathcal{F}_{0,k}(\mathcal{G}) \) and a function \( p \in \mathcal{C}_G \) such that

\[
\mathcal{L} f = p \cdot g.
\]  

(2.1)

**Proof** Let \( f \in \mathcal{K}^k_G \). Then there exists \( p \in \mathcal{C}_G \) and \( h \in \mathcal{M}_G(\frac{k-1}{k}) \) such that

\[
\mathcal{L} f(z) = p(z) \cdot g(z) \quad z \in \mathcal{G},
\]

where

\[
g(z) = \prod_{l=0}^{k-1} h(\epsilon^l z), \quad z \in \mathcal{G}.
\]

It is obvious that \( g \in \mathcal{F}_{0,k}(\mathcal{G}) \). We show that \( g \in \mathcal{M}_G \). To do it, using the differentiation product rule and the form of the operator \( \mathcal{L} \), we have at \( z \in \mathcal{G} \)

\[
\frac{\mathcal{L} g(z)}{g(z)} = 1 + \frac{Dg(z)(z)}{g(z)} = 1 + \sum_{l=0}^{k-1} \frac{Dh(\epsilon^l z)(\epsilon^l z)}{h(\epsilon^l z)} = 1 - k + \sum_{l=0}^{k-1} \frac{\mathcal{L} h(\epsilon^l z)}{h(\epsilon^l z)}.
\]

Hence and by the fact that \( h \in \mathcal{M}_G(\frac{k-1}{k}) \), we obtain that \( \text{Re} \frac{\mathcal{L} g(z)}{g(z)} > 1 - k + k \frac{k-1}{k} = 0 \).

Thus \( g \in \mathcal{M}_G \).

Now, let us suppose that \( f \) satisfies the equality (2.1), with a \( p \in \mathcal{C}_G \) and a \( g \in \mathcal{M}_G \cap \mathcal{F}_{0,k}(\mathcal{G}) \). Let us put \( h(z) = (g(z))^\frac{1}{k} \), \( z \in \mathcal{G} \), with the power function taking the value 1 at the point 1. Since \( g(z) \neq 0 \) (see [1]), the function \( h \) is holomorphic. It remains to show that \( h \in \mathcal{M}_G(\frac{k-1}{k}) \) and the equality (1.1) is fulfilled. To this end we compute step by step

\[
\text{Re} \frac{\mathcal{L} h(z)}{h(z)} = \text{Re} \frac{\frac{1}{k} \log(g(z))}{(g(z))^\frac{1}{k}} = 1 + \frac{1}{k} \text{Re} \frac{(g(z))^\frac{1}{k} - 1 Dg(z)(z)}{g(z)} \geq \frac{k - 1}{k}.
\]

The formula (1.1) follows from the definition of the function \( h \). Indeed,

\[
g(z) = (h(z))^k = \prod_{l=0}^{k-1} h(\epsilon^l z), \quad z \in \mathcal{G},
\]

because \( h \in \mathcal{F}_{0,k}(\mathcal{G}) \).

The proof is complete. \( \square \)
Now we consider an extremal problem for \( f \in \mathcal{K}_G^k \). More precisely, we look for some estimates for \( \mathcal{G} \)-balances of \( m \)-homogeneous polynomials \( Q_{f,m} \) of its unique power series expansion

\[
f(z) = 1 + \sum_{m=1}^{\infty} Q_{f,m}(z), \quad z \in \mathcal{G}.
\]

(2.2)

In our considerations the Minkowski function

\[
\mu_G(z) = \inf \{ t > 0 : \frac{1}{t} z \in \mathcal{G}, \quad z \in \mathbb{C}^n, \}
\]

will be very useful. This function gives a possibility to redefine the domain \( \mathcal{G} \) and its boundary \( \partial \mathcal{G} \) as follows:

\[
\mathcal{G} = \{ z \in \mathbb{C}^n : \mu_G(z) < 1 \}, \quad \partial \mathcal{G} = \{ z \in \mathbb{C}^n : \mu_G(z) = 1 \}.
\]

The notion of \( \mathcal{G} \)-balance of \( m \)-homogeneous polynomial \( Q_m : \mathbb{C}^n \to \mathbb{C}, \ m \in \mathbb{N} \cup \{0\} \), was defined in [3] as the quantity

\[
\mu_G(Q_m) = \sup_{w \in \mathbb{C}^n \setminus \{0\}} \frac{|Q_m(w)|}{(\mu_G(w))^m} = \sup_{v \in \partial \mathcal{G}} |Q_m(v)| = \sup_{u \in \mathcal{G}} |Q_m(U)|.
\]

The \( \mathcal{G} \)-balance \( \mu_G(Q_m) \) generalizes the norm \( \| Q_m \| \) of the polynomial \( Q_m \) and if \( \mathcal{G} \) is convex, then \( \mu_G(Q_m) \) reduces to \( \| Q_m \| \), because

\[
|Q_m(w)| \leq \mu_G(Q_m)(\mu_G(w))^m, \quad w \in \mathbb{C}^n
\]

and for bounded convex complete \( n \)-circular domains \( \mathcal{G} \) also \( \mu_G(w) = \| w \| \) (see, e.g., [21]).

We present the announced estimates of \( \mathcal{G} \)-balances \( \mu_G(Q_{f,m}) \) of \( m \)-homogeneous polynomials \( Q_{f,m} \) from the Taylor series of \( f \in \mathcal{M}_G^k \) in the following theorem.

**Theorem 2** If the expansion of the function \( f \in \mathcal{K}_G^k \), \( k \geq 2 \), into a series of \( m \)-homogenous polynomials \( Q_{f,m} \) has the form (2.2), then for the \( \mathcal{G} \)-balances \( \mu_G(Q_{f,m}) \) of polynomials \( Q_{f,m} \) the following sharp estimate hold:

\[
\mu_G(Q_{f,m}) \leq \begin{cases}
\frac{1}{m} \prod_{p=1}^{m-1} \left( 1 + \frac{2}{pk} \right) & \text{for } m = k, 2k, 3k, \ldots \\
\frac{2}{m+1} \prod_{p=1}^{\lfloor q \rfloor} \left( 1 + \frac{2}{pk} \right) & \text{for remaining } m \in \mathbb{N}
\end{cases}
\]

where \( \lfloor q \rfloor \) means the integral part of the number \( q \). We use a standard convention that the product \( \prod_{l=l_1}^{l_2} a_l \) is equal to 1 for \( l_2 < l_1 \).
Proof Let \( f \in \mathcal{K}_G^k \) be arbitrarily fixed. Then, by Theorem 1, the factorization (2.1) holds with a function \( p \in \mathcal{C}_G \) of the form
\[
p(z) = 1 + \sum_{\nu=1}^{\infty} Q_{p,\nu}(z), \quad z \in \mathcal{G}
\]
and a function \( g \in \mathcal{M}_G \cap \mathcal{F}_{0,k}(\mathcal{G}) \) of the form
\[
g(z) = 1 + \sum_{\nu=1}^{\infty} Q_{g,\nu}(z), \quad z \in \mathcal{G}.
\] (2.3)

From the above, by the series expansion of \( Lf \)
\[
L f(z) = 1 + \sum_{m=1}^{\infty} Q_{L f,m}(z) = 1 + \sum_{m=1}^{\infty} (m + 1) Q_{f,m}(z), \quad z \in \mathcal{G}
\]
and by the equalities \( Q_{f,0} = Q_{p,0} = Q_{g,0} = 1 \), we obtain the recursive formula for \( m \in \mathbb{N} \)
\[
(m + 1) Q_{f,m}(z) = \sum_{l=0}^{\lfloor \frac{m}{k} \rfloor} Q_{g,kl}(z) Q_{p,m-kl}(z), \quad z \in \mathcal{G}.
\]
Hence
\[
(m + 1) \left| Q_{f,m}(z) \right| \leq \sum_{l=0}^{\lfloor \frac{m}{k} \rfloor} \left| Q_{g,kl}(z) \right| \left| Q_{p,m-kl}(z) \right| , \quad z \in \mathcal{G}.
\] (2.4)

Since
\[
\left| Q_{p,\nu}(z) \right| \leq 2, \quad \nu \in \mathbb{N}, \quad z \in \mathcal{G},
\] (2.5)
(see [1]) we need some bounds for \( \left| Q_{g,k\mu}(z) \right| \). We show that for \( g \in \mathcal{M}_G \cap \mathcal{F}_{0,k}(\mathcal{G}) \) and \( \mu \in \mathbb{N} \) there hold the inequalities
\[
\left| Q_{g,k\mu}(z) \right| \leq \frac{2^{\mu-1}}{k\mu} \prod_{v=1}^{\mu} \left( 1 + \frac{2}{kv} \right), \quad z \in \mathcal{G}.
\] (2.6)

For this purpose let us observe that for each \( z \in \mathcal{G} \), the function
\[
G(\zeta) = \zeta g(\zeta z), \quad \zeta \in \mathcal{U}
\]
begins to the family \( S^* \cap \mathcal{F}_{1,k}(\mathcal{U}) \) of \((1,k)\)-symmetric univalent starlike mappings (in the unit disc \( \mathcal{U} \)) and its Taylor series has the form
\[
G(\zeta) = \zeta + \sum_{\mu=1}^{\infty} b_{k\mu+1} \zeta^{k\mu+1} = 1 + \sum_{\mu=1}^{\infty} Q_{g,k\mu}(z) \zeta^{k\mu+1}, \quad \zeta \in \mathcal{U}.
\]
Thus, in view of the estimates [25] of the coefficients of functions from \( S^* \cap \mathcal{F}_{1,k}(U) \) we get the announced bounds (2.6).

In two next parts of the proof we use also the fact [4] that for every \( k, s \in \mathbb{N} \setminus \{1\} \) there holds the identity:

\[
1 + \frac{2}{k} + \sum_{l=2}^{s} \frac{2}{lk} \prod_{v=1}^{l-1} \left( 1 + \frac{2}{vk} \right) = \prod_{v=1}^{s \wedge v} \left( 1 + \frac{2}{vk} \right). \tag{2.7}
\]

Now, we will estimate the quantities \( |Q_{f,m}(z)|, z \in \mathcal{G} \), using all the conditions (2.4)–(2.7).

First let us assume that \( m = ks \), where \( s \in \mathbb{N} \). Since \( Q_{p,m-kl}(z) = 1 \) for \( l = s \), we get from (2.4) that

\[
(m + 1) Q_{f,m}(z) \leq Q_{g,ks}(z) + 2 \sum_{l=0}^{s-1} Q_{g,kl}(z), z \in \mathcal{G}.
\]

Thus for \( z \in \mathcal{G} \), in view of (2.6) and (2.7),

\[
(m + 1) |Q_{f,m}(z)| \leq \frac{2}{sk} \prod_{v=1}^{s \wedge v} \left( 1 + \frac{2}{vk} \right) + 2 \left[ 1 + \frac{2}{k} + \sum_{l=2}^{s-1} \frac{2}{lk} \prod_{v=1}^{l-1} \left( 1 + \frac{2}{vk} \right) \right]
\]

\[
= \frac{-2}{sk} \prod_{v=1}^{s \wedge v} \left( 1 + \frac{2}{vk} \right) + 2 \left[ 1 + \frac{2}{k} + \sum_{l=2}^{s} \frac{2}{lk} \prod_{v=1}^{l-1} \left( 1 + \frac{2}{vk} \right) \right]
\]

\[
\leq \frac{-2}{sk} \prod_{v=1}^{s \wedge v} \left( 1 + \frac{2}{vk} \right) + 2 \prod_{v=1}^{s \wedge v} \left( 1 + \frac{2}{vk} \right)
\]

\[
= 2 \left( \frac{sk + 1}{sk} \right) \prod_{v=1}^{s \wedge v} \left( 1 + \frac{2}{vk} \right).
\]

Hence, for \( m = k, 2k, 3k, \ldots \)

\[
|Q_{f,m}(z)| \leq \frac{2}{m} \prod_{v=1}^{m \wedge v} \left( 1 + \frac{2}{vk} \right), z \in \mathcal{G}.
\]

Now let us consider the case \( m = ks + r \), where \( s \in \mathbb{N} \cup \{0\} \) and \( r \in \{1, 2, \ldots, k-1\} \). In this case we apply in (2.4) the inequality \( |Q_{p,m-kl}(z)| \leq 2, l = 0, \ldots, s = \left\lfloor \frac{m}{k} \right\rfloor \), which follows from estimates (2.5), because \( m - kl > 0 \). Thus, in view of (2.6) and (2.7) we get step by step
\[(m + 1)|Q_{f,m}(z)| \leq 2 \sum_{l=0}^{\left\lfloor \frac{m}{k} \right\rfloor} |Q_{g,kl}(z)| \leq 2 \left[ 1 + \frac{2}{k} + \sum_{l=2}^{\left\lfloor \frac{m}{k} \right\rfloor} \frac{2}{lk} \prod_{v=1}^{l-1} \left( 1 + \frac{2}{vk} \right) \right] \]

\[
\leq 2 \prod_{v=1}^{\left\lfloor \frac{m}{k} \right\rfloor} \left( 1 + \frac{2}{vk} \right) .
\]

Summing up the results of both cases we get

\[
|Q_{f,m}(z)| \leq \begin{cases} 
\frac{2}{m} \prod_{v=1}^{m-1} \left( 1 + \frac{2}{vk} \right) & \text{for } m = k, 2k, 3k, \ldots \\
\frac{2}{m+1} \prod_{v=1}^{\left\lfloor \frac{m}{k} \right\rfloor} \left( 1 + \frac{2}{vk} \right) & \text{for remaining } m \in \mathbb{N}
\end{cases}, \quad z \in \mathcal{G}
\]

and consequently

\[
\sup_{z \in \mathcal{G}} |Q_{f,m}(z)| \leq \begin{cases} 
\frac{2}{m} \prod_{v=1}^{m-1} \left( 1 + \frac{2}{vk} \right) & \text{for } m = k, 2k, 3k, \ldots \\
\frac{2}{m+1} \prod_{v=1}^{\left\lfloor \frac{m}{k} \right\rfloor} \left( 1 + \frac{2}{vk} \right) & \text{for remaining } m \in \mathbb{N}
\end{cases}.
\]

These inequalities and the definition of \( \mathcal{G} \)-balances \( \mu_{\mathcal{G}}(Q_{f,m}) \) of \( m \)-homogeneous polynomials imply the estimates from the statement of the theorem.

Now, we will show the sharpness of the above estimates.

For the linear functional \( I = (\mu_{\mathcal{G}}(J))^{-1} J \), with

\[
J(z) = \sum_{l=1}^{n} z_l, \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^n,
\]

let us denote by \( \mathcal{Z} \) an analytic set \( \mathcal{G} \cap I^{-1}(0) \) and let \( I^m(z) = (Iz)^m, z \in \mathcal{G}, m \in \mathbb{N} \cup \{0\} \). The equalities in our estimates are achieved for the following function \( f \in k_G^{k}, k \geq 2 \),

\[
f(z) = \begin{cases} 
\frac{\sum_{l=0}^{k-1} l^{-1}(z)}{(1-I^k(z))^2} - \frac{1}{I(z)} - \sum_{l=3}^{k-1} \frac{l-2}{l} l^{l-1}(z) H(\frac{z}{k}, \frac{l}{k}, \frac{l+k}{k}, I^k(z)) & \text{for } z \in \mathcal{G} \setminus \mathcal{Z}, \\
1 & \text{for } z \in \mathcal{Z}
\end{cases},
\]

where \( H(a, b, c, \xi) : \mathcal{U} \to \mathbb{C} \) is a hypergeometric function

\[
H(a, b, c, \xi) = \sum_{v=0}^{\infty} \frac{(a)_v (b)_v \xi^v}{(c)_v v!}, \quad \xi \in \mathcal{U},
\]

\( (a)_v = a(a+1)(a+2)\ldots(a+v-1) \) and \( (a)_0 = 1 \).
defined by Pochhammer symbols \((a)_v, (b)_v, (c)_v:\)

\[
(a)_v = \begin{cases} 
   a(a + 1) \ldots (a + v - 1), & v \in \mathbb{N} \\
   1, & v = 0
\end{cases},
\]

and the branch of the power function \(x^{\frac{2}{k}}\) takes the value 1 at the point \(x = 1\). In the case \(k = 2, 3\) we use a standard convention that the sum

\[
\sum_{l=3}^{k-1} \frac{l - 2}{l} I^{l-1}(z) H \left( \frac{2}{k}, \frac{l}{k}, \frac{l + k}{k}, I^k(z) \right), \ z \in \mathcal{G}
\]

is equal to zero, if the superscript of the sum is smaller than the subscript.

In the paper [4], it was proven that the above function gives the equalities in the bounds from the statement of the theorem. It remains to show that \(f \in \mathcal{K}^k_{\mathcal{G}}\) for \(k \geq 2\). To do it, let us observe that as shown in [4]

\[
\mathcal{L} f(z) = \frac{1 + I(z)}{1 - I(z)} \frac{1}{\left(1 - I^k(z)\right)^{\frac{2}{k}}}, \ z \in \mathcal{G}.
\]

This implies, in view of Theorem 1, the relation \(f \in \mathcal{K}^k_{\mathcal{G}}\), because the functions

\[
p(z) = \frac{1 + I(z)}{1 - I(z)}, g(z) = \frac{1}{\left(1 - I^k(z)\right)^{\frac{2}{k}}}, \ z \in \mathcal{G}
\]

belong to \(\mathcal{C}_{\mathcal{G}}\) and to \(\mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})\), respectively.

We use the estimates of \(\mathcal{G}\)-balances \(\mu_{\mathcal{G}}(Q_{f,m})\) of polynomials \(Q_{f,m}\) to solve the mentioned separation problem for the families \(\mathcal{V}_{\mathcal{G}}, \mathcal{K}^k_{\mathcal{G}}, \mathcal{R}_{\mathcal{G}}\). We prove the following theorem:

**Theorem 3** For every \(k \geq 2\) there holds the double inclusion

\[
\mathcal{V}_{\mathcal{G}} \subset \mathcal{K}^k_{\mathcal{G}} \subset \mathcal{R}_{\mathcal{G}}.
\]

**Proof** We start with the inclusion \(\mathcal{V}_{\mathcal{G}} \subset \mathcal{K}^k_{\mathcal{G}}\). To do it, let us assume that \(f \in \mathcal{V}_{\mathcal{G}}\), then \(\mathcal{L} f \in \mathcal{C}_{\mathcal{G}}\). Putting \(p = \mathcal{L} f\) and \(h = 1\), we obtain the factorization (1.1) with \(p \in \mathcal{C}_{\mathcal{G}}\) and \(g = 1 \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})\). Hence \(f \in \mathcal{K}^k_{\mathcal{G}}\). It remains to show the relation \(\mathcal{V}_{\mathcal{G}} \neq \mathcal{K}^k_{\mathcal{G}}\). To do it, let us observe that for \(f \in \mathcal{V}_{\mathcal{G}}\) there hold the sharp estimates \(\mu_{\mathcal{G}}(Q_{f,m}) \leq \frac{1}{m+1}, m \in \mathbb{N}\) (cf., eg., [1]), while for \(f \in \mathcal{K}^k_{\mathcal{G}}\) the sharp estimates \(\mu_{\mathcal{G}}(Q_{f,m}) \leq B(m)\) (Theorem 2.), with the obvious bound \(B(m) > \frac{2}{m+1}, m \in \mathbb{N} \setminus \{1\}\). Hence, the extremal function \(f \in \mathcal{K}^k_{\mathcal{G}}\) does not belong to \(\mathcal{V}_{\mathcal{G}}\).

Now we prove that \(\mathcal{K}^k_{\mathcal{G}} \subset \mathcal{R}_{\mathcal{G}}\). To this end, let us suppose that \(f \in \mathcal{K}^k_{\mathcal{G}}\). Then there exist functions \(p \in \mathcal{C}_{\mathcal{G}}, g \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G})\) such that \(\mathcal{L} f = p \cdot g\). Denoting \(\varphi = \mathcal{L}^{-1} g\), we have that \(\varphi \in \mathcal{N}_{\mathcal{G}}\) (by the Aleksander type theorem [1]) and \(\mathcal{L} f = p \mathcal{L} \varphi\). Thus
f \in \mathcal{R}_G$. It remains to show the relation $\mathcal{K}^k_G \neq \mathcal{R}_G$. For this purpose, let us observe that in the above estimates $\mu_G(Q_{f,m}) \leq B(m), m \in \mathbb{N}$, we have $B(m) \leq 1, m \in \mathbb{N}$ (see below), while for $f \in \mathcal{R}_G$ there hold the sharp estimates $\mu_G(Q_{f,m}) \leq m + 1$(see for instance [1]). Therefore, the extremal function $f \in \mathcal{R}_G$ does not belong to $\mathcal{K}^k_G$.

To complete the proof, we show that $B(m) \leq 1, m \in \mathbb{N}$. To do it, we consider two cases, according to the partition $m = ks + r, r \in \{0, 1, \ldots, k - 1\}$, from the proof of Theorem 2.

1. Let us suppose that $r = 0$. Then, if $s = \frac{m}{k} = 1$, we see that the superscript $s - 1$ of the first product in Theorem 2 is smaller than its subscript 1. Hence, we replace the referred product by 1 and consequently, we get $\mu_G(Q_{f,m}) \leq \frac{2}{m} \leq 1$, because $m = k \geq 2$. Next, if $s \geq 2$, then from Theorem 2, by the inequality $1 + \frac{2}{vk} \leq \frac{v + 1}{v}, v \in \mathbb{N}, k \in \mathbb{N} \setminus \{1\}$, we obtain

$$\mu_G(Q_{f,m}) \leq \frac{2}{m} \prod_{v=1}^{s-1} \frac{v + 1}{v} \leq \frac{2}{m} \frac{s}{k} = \frac{2}{k} \leq 1.$$ 

2. Let us suppose that $r \in \{1, \ldots, k - 1\}$. Then, if $s = \left\lfloor \frac{m}{k} \right\rfloor = 0$, we see that the superscript of the second product in Theorem 2 is smaller than its subscript. Hence we replace the referred product by 1 and consequently, we get $\mu_G(Q_{f,m}) \leq \frac{2}{m+1} \leq 1$, because $m \leq k - 1$. Next, if $s = \left\lfloor \frac{m}{k} \right\rfloor \geq 1$, then similarly as in step 1, we obtain

$$\mu_G(Q_{f,m}) \leq \frac{2}{m+1} \prod_{v=1}^{s} \frac{v + 1}{v} \leq \frac{2}{m+1} (s + 1) \leq \frac{2(s+1)}{ks+2} \leq \frac{2(s+1)}{2s+2} \leq 1.$$ 

\[ \square \]

Now, we give a growth theorem for $f \in \mathcal{K}^k_G$ and its Temljakov transform $\mathcal{L} f$.

**Theorem 4** For functions $f \in \mathcal{K}^k_G$ there follow the following sharp estimates

$$\frac{1 - r}{1 + r} \frac{1}{(1 + r^k)^2} \leq |\mathcal{L} f(z)| \leq \frac{1 + r}{1 - r} \frac{1}{(1 - r^k)^2}, r = \mu_G(z) \in [0, 1),$$

(2.9)

$$\frac{1}{r} \int_0^r \frac{1 - \varrho}{1 + \varrho} \frac{1}{(1 + \varrho^k)^2} d\varrho \leq |f(z)| \leq \frac{1}{r} \int_0^r \frac{1 + \varrho}{1 - \varrho} \frac{1}{(1 - \varrho^k)^2} d\varrho, \quad r = \mu_G(z) \in [0, 1).$$

(2.10)

**Proof** First, let us observe that the above estimates are true for $z = 0$ (in (2.10) the values at $r = 0$, of the left and right hand sides, mean the limit if $r \to 0^+$). Thus, in the sequel we will assume that $z \in \mathcal{G} \setminus \{0\}$. We start with the estimates (2.9). Since $f \in \mathcal{K}^k_G$, there exist a function $p \in \mathcal{C}_G$ and a function $g \in \mathcal{M}_G \cap \mathcal{F}_{0,k}(\mathcal{G})$ such that the factorization (2.1) holds. Therefore, we show for such functions $g$ the following inequalities...
\[
\frac{1}{(1 + r^k)^\frac{1}{k}} \leq |g(z)| \leq \frac{1}{(1 - r^k)^\frac{1}{k}}, \quad r = \mu_G(z) \in (0, 1).
\]

To this aim, let us fix arbitrarily a point \( z \in G \) such that \( \mu_G(z) = r \in (0, 1) \) and let us consider the function

\[
G(\xi) = \xi g(\xi - \frac{z}{\mu_G(z)}), \quad \xi \in \mathcal{U}.
\]

Then \( G \) is \((1, k)\)-symmetric, holomorphic, normalized and satisfies the condition

\[
\text{Re} \frac{\xi G'(\xi)}{G(\xi)} = \text{Re} \frac{\mathcal{L} g(\xi - \frac{z}{\mu_G(z)})}{g(\xi - \frac{z}{\mu_G(z)})} > 0, \quad \xi \in \mathcal{U}.
\]

Hence \( G \in S^* \cap \mathcal{F}_{1,k}(\mathcal{U}) \) and by [9, Thm. 2.2.13]

\[
\frac{|\xi|}{(1 + |\xi|^k)^\frac{1}{k}} \leq |G(\xi)| \leq \frac{|\xi|}{(1 - |\xi|^k)^\frac{1}{k}}, \quad \xi \in \mathcal{U}.
\]

Putting \( \xi = \mu_G(z) \) in the above we obtain, by the definition of the function \( G \), the announced inequality.

On the other hand, there hold for \( p \in \mathcal{C}_G \) the following estimates [1]

\[
\frac{1 - r}{1 + r} \leq |p(z)| \leq \frac{1 + r}{1 - r}, \quad r = \mu_G(z) \in (0, 1),
\]

Using the estimates of \(|p(z)| \) and \(|g(z)| \) we get the estimates (2.9). The sharpness of the upper bounds (2.9) confirms the function given by (2.8). Indeed, for \( r \in (0, 1) \) and function \( f \in \mathcal{K}^k_G \) given by (2.8), we get

\[
\mathcal{L} f(z) = \frac{1 + r}{1 - r} \frac{1}{(1 - r^k)^\frac{1}{k}}
\]

at points \( z \in G \), \( \mu_G(z) = r \in (0, 1) \) such that \( I(z) = r \) (this condition is fulfilled by the points \( z = rz^* \), where \( z^* \in \partial G \) and \( I(z^*) = 1 \)).

The sharpness of the lower bounds (2.9) can be proven in a similar way.

Now, we prove the estimates (2.10). To obtain the upper bound (2.10), we use the proved above upper bound (2.9) and the fact that the Temljakov operator \( \mathcal{L} \) is invertible and

\[
\mathcal{L}^{-1} u(z) = \int_0^1 u(tz) dt, \quad u \in \mathcal{H}_G, \quad z \in G.
\]
Indeed, we have for \( f \in K^k_G \) and \( z \in G \), \( \mu_G(z) = r \in (0, 1) \),

\[
|f(z)| = |L^{-1}L f(z)| = \left| \int_0^1 L(tz) dt \right| \leq \int_0^1 \frac{1 + rt}{(1 - rt)(1 - (rt)^k)^k} dt
\]

\[
= \frac{1}{r} \int_0^r \frac{1 + \varrho}{(1 - \varrho)(1 - \varrho^k)^k} d\varrho.
\]

To prove the lower bound (2.10) let us consider the function

\[
F(\xi) = \xi f\left(\xi \frac{z}{\mu_G(z)}\right), \xi \in \mathcal{U},
\]

with arbitrarily fixed \( f \in K^k_G \) and \( z \in G \), \( \mu_G(z) = r \in (0, 1) \). Since

\[
F'(\xi) = L f\left(\xi \frac{z}{\mu_G(z)}\right), \xi \in \mathcal{U},
\]

we get, by Theorem 1, that there exist functions \( g \in \mathcal{M}_G \cap \mathcal{F}_{0,k}(G) \) and \( p \in \mathcal{C}_G \) such that the factorization (2.1) is true. Thus

\[
F'(\xi) = P(\xi) \cdot G(\xi), \xi \in \mathcal{U},
\]

where for \( \xi \in \mathcal{U} \)

\[
G(\xi) = \xi g\left(\xi \frac{z}{\mu_G(z)}\right), P(\xi) = p\left(\xi \frac{z}{\mu_G(z)}\right).
\]

Moreover, \( G \in \mathcal{S}^* \cap \mathcal{F}_{1,k}(\mathcal{U}) \) (see the proof of the estimates (2.9)) and \( P : \mathcal{U} \to \mathbb{C}, P(0) = 1 \), is a holomorphic function with a positive real part. Therefore, \( F \) belongs to a subclass \( \mathcal{K}^{(k)} \) (considered in [22] and for \( k = 2 \) in [8]) of the class of close-to-convex functions. Hence, \( F \) is univalent in the disc \( \mathcal{U} \).

On the other hand, by the lower bound (2.9), we have that

\[
|F'(\xi)| \geq \frac{1 - |\xi|}{1 + |\xi|} \frac{1}{(1 + |\xi|^k)^k},
\]

because \( r = \mu_G\left(\xi \frac{z}{\mu_G(z)}\right) = |\xi| \). Now we show that

\[
|F(\xi)| \geq \int_0^r \frac{1 - \varrho}{1 + \varrho} \frac{1}{(1 + \varrho^k)^k} d\varrho, |\xi| = r \in (0, 1).
\]
To this aim, it is sufficient to show that it holds for the nearest point $F(\zeta_0)$ from zero ($|\zeta_0| = r \in (0, 1)$), otherwise, we have $|F(\zeta)| \geq |F(\zeta_0)|$, $|\zeta| = r$. Since $F$ is univalent in the disc $U$, the original image of the line segment $0, F(\zeta_0)$ is a piece of arc $F^{-1}(0, F(\zeta_0))$ in the disc $rU$. Thus

$$|F(\zeta_0)| = \int_{0, F(\zeta_0)} |dw| = \int_{F^{-1}(0, F(\zeta_0))} |F'(\zeta)| \, d\zeta \geq \int_0^r \frac{1 - \varrho}{1 + \varrho} \frac{1}{(1 + \varrho^k)^{\frac{3}{2}}} \, d\varrho, \quad r \in (0, 1).$$

Thus, by the definition of $F$, we get

$$\left| \zeta f \left( \zeta \frac{z}{\mu_G(z)} \right) \right| \geq \int_0^r \frac{1 - \varrho}{1 + \varrho} \frac{1}{(1 + \varrho^k)^{\frac{3}{2}}} \, d\varrho, \quad |\zeta| = r \in (0, 1).$$

Hence, putting $\zeta = \mu_G(z) = r \in (0, 1)$, we have the lower bound (2.10).

Finally, let us note that we obtain the equalities in the inequalities (2.10) for the function (2.8) in adequate points $z \in G$. \hfill \qed

We close the paper with a sufficient condition guaranteeing that a function $f \in \mathcal{H}_G(1)$ belongs to $K^k_G$. We formulate it in the term of $G$-balances of $m$-honogeous polynomials in developments of functions from $\mathcal{H}_G(1)$.

**Theorem 5** Let $f \in \mathcal{H}_G(1)$ has the form (2.2). If there exists a function $g \in \mathcal{M}_G \cap \mathcal{F}_{0,k}(G)$ of the form (2.3) such that

$$\sum_{m=1}^{\infty} (m + 1) \mu_G(Q_{f,m}) + \sum_{m=1}^{\infty} \mu_G(Q_{g, mk}) \leq 1,$$

then $f \in K^k_G$.

**Proof** Since $g$, as a function from $\mathcal{M}_G$ omits zero [1], we will prove that

$$\text{Re} \left( \frac{L f(z)}{g(z)} \right) > 0, \quad z \in G.$$
\[
|\mathcal{L} f(z) - g(z)| - |\mathcal{L} f(z) + g(z)|
= \left| \sum_{m=1}^{\infty} (m + 1) Q_{f,m}(z) - \sum_{m=1}^{\infty} Q_{g,mk}(z) \right| - 2 + \sum_{m=1}^{\infty} (m + 1) Q_{f,m}(z)
+ \sum_{m=1}^{\infty} Q_{g,mk}(z)
\leq 2 \left[ \sum_{m=1}^{\infty} (m + 1) \left| Q_{g,mk}(z) \right| + \sum_{m=1}^{\infty} \left| Q_{g,mk}(z) \right| - 1 \right]
\leq 2 \left[ \sum_{m=1}^{\infty} (m + 1) \mu_{G}(Q_{f,m}) + \sum_{m=1}^{\infty} \mu_{G}(Q_{g,mk}) - 1 \right] \leq 0.
\]

Thus
\[
\left| \frac{\mathcal{L} f(z)}{g(z)} - 1 \right| \leq \left| \frac{\mathcal{L} f(z)}{g(z)} + 1 \right|, \ z \in \mathcal{G}
\]
and hence
\[
\text{Re} \frac{\mathcal{L} f(z)}{g(z)} \geq 0, \ z \in \mathcal{G}.
\]

This gives the mentioned inequality by a maximum principle for pluriharmonic functions of several complex variables. Putting \( p(z) = \frac{\mathcal{L} f(z)}{g(z)} \), \( z \in \mathcal{G} \), we obtain that the transform \( \mathcal{L} f \) has the factorization (1) with \( g \in \mathcal{M}_{\mathcal{G}} \cap \mathcal{F}_{0,k}(\mathcal{G}) \) and \( p \in \mathcal{C}_{\mathcal{G}} \). Consequently, \( f \in K^{k}_{\mathcal{G}} \).

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Bavrin, I.I.: A class of regular bounded functions in the case of several complex variables and extreme problems in that class. Moskov Obl. Ped. Inst. Moskov (1976) (in Russian)
2. Długosz, R.: Embedding theorems for holomorphic functions of several complex variables. J. Appl. Anal. 19, 153–165 (2013)
3. Długosz, R., Leś, E.: Embedding theorems and extreme problems for holomorphic functions on circular domains of \( \mathbb{C}^{n} \). Complex Var. Elliptic Equ. 59, 883–899 (2014)
4. Długosz, R., Liczberski, P.: An application of hypergeometric functions to a construction in several complex variables. J. Anal. Math. (2017) accepted for publication
5. Dobrowolska, K., Liczberski, P.: On some differential inequalities for holomorphic functions of many variables. Demonstr. Math. 14, 383–398 (1981)
6. Dziubinski, I., Sitarski, R.: On classes of holomorphic functions of many variables starlike and convex on some hypersurfaces. Demonstr. Math. 13, 619–632 (1980)
7. Fukui, S.: On the estimates of coefficients of analytic functions. Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 10, 216–218 (1969)
8. Gao, C., Zhou, S.: On a class of analytic functions related to the starlike functions. Kyungpook Math. J. 45, 123–130 (2005)
9. Graham, I., Kohr, G.: Geometric Function Theory in One and Higher Dimensions. Marcel Dekker Inc, New York (2003)
10. Hamada, H., Honda, T., Kohr, G.: Parabolic starlike mappings in several complex variables. Manuscr. Math. 123, 301–324 (2007)
11. Higuchi, T.: On coefficients of holomorphic functions of several complex variables. Sci. Rep. Tokyo Kyoiku Daigaku 8, 251–258 (1965)
12. Leś-Bomba, E., Liczberski, P.: On some family of holomorphic functions of several complex variables. Sci. Bull. Chelm Sect. Math. Comput. Sci. 2, 7–16 (2007)
13. Liczberski, P.: On the subordination of holomorphic mappings in $\mathbb{C}^n$. Demonstr. Math. 2, 293–301 (1986)
14. Liczberski, P., Połubiński, J.: On $(j, k)$-symmetrical functions. Math. Bohem. 120, 13–28 (1995)
15. Liczberski, P., Połubiński, J.: Functions $(j, k)$-symmetrical and functional equations with iterates of the unknown function. Publ. Math. Debr. 60, 291–305 (2002)
16. Liczberski, P., Połubiński, J.: Symmetrical series expansion of complex valued functions. N. Z. J. Math. 27, 245–253 (1998)
17. Liczberski, P., Połubiński, J.: A uniqueness theorem of Cartan–Gutzmer type for holomorphic mappings in $\mathbb{C}^n$. Ann. Pol. Math. 79, 121–127 (2002)
18. Marchlewksa, A.: On a generalization of close-to-convexity for complex holomorphic functions in $\mathbb{C}^n$. Demonstr. Math. 4, 847–856 (2005)
19. Michiwaki, Y.: Note on some coefficients in a starlike functions of two complex variables. Res. Rep. Nagaoka Tech. Coll. 1, 151–153 (1963)
20. Pfaltzgraff, J.A., Suffridge, T.J.: An extension theorem and linear invariant families generated by starlike maps. Ann. Univ. Mariae Curie Sklodowska Sect. A 53, 193–207 (1999)
21. Rudin, W.: Functional Analysis. McGraw-Hill Inc, New York (1991)
22. Seker, B.: On certain new subclass of close-to-convex functions. Appl. Math. Comput. 218, 1041–1045 (2011)
23. Stankiewicz, J.: Functions of two complex variables regular in halfspace. Folia Sci. Univ. Techn. Rzeszów. Math. 19, 107–116 (1996)
24. Temljakov, A.: Integral representation of functions of two complex variables. Izv. Ross. Akad. Nauk. Ser. Mat. 21, 89–92 (1957)
25. Waadeland, H.: Über k-fach symmetrische sternfö rmige schlichte Abbildungen des Einheitskreises. Math. Scand. 3, 150–154 (1955)