Ascendancy of potentials over fields in electrodynamics

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Abstract

Multiple bases are presented for the conclusion that potentials are fundamental in electrodynamics, with electric and magnetic fields as quantities auxiliary to the scalar and vector potentials – opposite to the conventional ordering. One foundation for the concept of basic potentials and auxiliary fields consists of examples where two sets of gauge-related fields are such that one is physical and the other is erroneous, with the information for the proper choice supplied by the potentials. A major consequence is that a change of gauge is not a unitary transformation in quantum mechanics; a principle heretofore unchallenged. The primacy of potentials over fields leads to the concept of a hierarchy of physical quantities, where potentials and energies are primary, while fields and forces are secondary. Secondary quantities provide less information than do primary quantities. Some criteria by which strong laser fields are judged are based on secondary quantities, making it possible to arrive at inappropriate conclusions. This is exemplified by several field-related misconceptions as diverse as the behavior of charged particles in very low frequency propagating fields, and the fundamental problem of pair production at very high intensities. In each case, an approach based on potentials gives appropriate results, free of ambiguities. The examples encompass classical and quantum phenomena, in relativistic and nonrelativistic conditions. This is a major extension of the quantum-only Aharonov-Bohm effect, both in supporting the primacy of potentials over fields, and also in showing how field-based conceptions can lead to errors in basic applications.
I. INTRODUCTION

For most of the history of exploring electromagnetic phenomena, it had been believed that knowledge of the electric and magnetic fields in a physical problem is sufficient to define the problem. The scalar and vector potentials, whose spatial and temporal derivatives yield the fields, had been regarded as auxiliary quantities that are useful but not essential. This conclusion was apparently reinforced by the fact that the set of potentials to represent the fields is not unique. Subject to modest restrictions, there exist transformations (called gauge transformations) to other sets of potentials that produce the same fields.

This seemingly straightforward situation was upset by the Aharonov-Bohm effect [1, 2]. The simplest realization of this phenomenon is that an electron beam passing outside a solenoid containing a magnetic field will be deflected, even though there is no field outside the solenoid. There is, however, a potential outside the solenoid that suffices to explain the deflection. The effect remained controversial until it was verified experimentally [3]. This has the basic consequence that potentials are more fundamental than fields. The Aharonov-Bohm effect is founded on a single explicitly quantum-mechanical phenomenon, and commentary about its significance has been in terms of quantum mechanics [4, 5].

The concept explored here is different from that of the Aharonov-Bohm effect, and much more consequential. It is shown that a change of gauge can introduce a violation of basic symmetries, even when the usual constraints on allowable gauge transformations have been satisfied. Furthermore, these symmetry violations can occur in classical physics as well as in quantum mechanics with external electromagnetic fields. The consequences of these results are profound. There exist contrasting sets of potentials that yield exactly the same fields, but where one set is consonant with physical requirements but another is not. This proves directly that the selection of the proper set of potentials is the decisive matter, since the predicted fields are the same in both cases. A corollary is that gauge transformations are not unitary transformations. This contradicts the field-based assumption that gauge transformations must preserve the values of measurable quantities. The assumption of unitarity (often implicit) underlies some of the influential articles that have been published on the subject of gauge choice.

Examples employed here to demonstrate the primacy of potentials – a charged particle in interaction with a constant electric field, and a bound electron subjected to a plane-wave
field – represent basic physics problems, unlike the narrow specificity of the Aharonov-Bohm effect. An important feature revealed by these examples is that, although the physical consequences of static or quasistatic-electric (QSE) fields are quite similar to plane-wave effects at the low field intensities that exist in the usual atomic, molecular and optical (AMO) physics processes, at the high field intensities now achievable with laser fields, they can be profoundly different. These differences have yet to be fully appreciated in the AMO literature, leading to misconceptions that persist nearly forty years after the first laboratory observation \[^6\] of explicit intense-field effects.

A concept introduced here is that of a hierarchy of physical quantities. Since potentials are primary and fields are secondary, it follows that energies are primary and forces are secondary. This ranking resolves the long-standing mystery about why the Schrödinger equation cannot be written directly in terms of electric and magnetic fields, even though the fields were conventionally assumed to be basic physical quantities. All attempts to express the Schrödinger equation directly in terms of fields have resulted in nonlocality \[^7–12\]. This apparent anomaly is one of the enduring puzzles of quantum mechanics. A hierarchy of physical quantities also serves to clarify the current confused situation in the strong-laser community, where field-based intensity measures are employed that are inconsistent with energy-based criteria. One important example of this misdirection is the introduction of the concept of the “critical electric field” \[^13, 14\] into the discussion of strong laser effects, despite the fact that lasers produce transverse fields and the critical field has well-defined meaning only for longitudinal fields. The basic differences between transverse fields and longitudinal fields are also exhibited in the macroscopic world in terms of the properties of extremely-low-frequency radio waves.

The range of applicability of the concepts examined here is very large, since it encompasses classical electromagnetism, and also relativistic and nonrelativistic quantum mechanics in which the electromagnetic field is regarded as an external classical field.

The limitation to external classical fields is significant, since it places the present work outside the scope of a Yang-Mills theory \[^15\]. Quantum electrodynamics (QED) is a Yang-Mills theory, but standard QED does not incorporate strong-field theory. Strong-field theories contain an apparent intensity-dependent “mass shift”, discovered independently by Sengupta \[^16\] and by the present author \[^17, 18\]. This mass shift can be explained in terms of a demand for covariance in external fields \[^19\]. The mass shift is a fundamental phe-
nomenon in strong-field physics [20], but it does not exist in the context of the quantized fields of QED [17, 18, 21].

Section II below discusses two basic examples that exhibit pairs of potential choices that describe exactly the same fields, but where one set of potentials is physically acceptable and the other is not. One example is the simplest possible case: the classical interaction of a charged particle with a constant electric field. Of the two possibilities for gauge choice, one contradicts Noether’s Theorem [22]. There is no such problem with the alternative gauge. The next example is the interaction of a charged particle with a plane-wave field, such as the field of a laser. In this case, the key factor is that the symmetry principle in question – preservation of the propagation property of a plane-wave field – is not often mentioned, even in the context of very strong fields where this symmetry is crucial [19].

The demand for the preservation of the propagation property imposes a strong limitation on possible gauge transformations. In the presence of a simultaneous scalar interaction, like a Coulomb binding potential, only the radiation gauge is possible [23]. This limitation to a unique gauge exists in both classical and quantum domains. An important aspect of this problem is that the widely-used dipole approximation in the description of laser-caused effects suppresses this symmetry, thus masking the errors that follow from ignoring this basic property. Both the constant-electric-field and the propagating-field examples admit of only one possible gauge. This lack of gauge-equivalent alternatives is extremely important, since both situations represent commonplace physical environments. This is in contrast to the specialized Aharonov-Bohm effect.

An immediate consequence of the demonstrated fact that some nominally valid gauge transformations can have unphysical consequences is that a gauge transformation is not a unitary transformation. This is discussed in Section III, where it is shown to be related to the construction of exact transition amplitudes.

The impossibility of writing the Schrödinger equation directly in terms of electric and magnetic fields is discussed in Section IV. This is further evidence of the basic nature of potentials, and it also supports the notion of a hierarchy of physical phenomena. Quantum mechanics can be constructed from classical mechanics when expressed in terms of system functions like the Lagrangian or Hamiltonian, whereas a Newtonian form of classical mechanics has such an extension only by extrapolation to a desired result. System functions are related to energies, whereas Newtonian physics involves forces, and forces are directly
connected with electric and magnetic fields, as shown by the Lorentz force expression.

The ambiguities inherent in the view that the $E^2 - B^2$ and $E \cdot B$ Lorentz invariants reliably characterize the electrodynamic environment is another topic examined in Section IV. (The Lorentz invariants, as are all electromagnetic quantities throughout this paper, are stated in Gaussian units.) This concept has an important failure when both invariants are zero, since it associates propagating plane-wave fields with the completely different constant crossed fields. The commonly-held assumption that constant crossed fields are a zero-frequency limit of plane-wave fields (see, for example, Refs. [24, 25]), is shown to be untenable.

Another topic in Section IV is the apparent dominance of the electric component of the Lorentz force expression at low frequencies, a field-related conception that draws attention away from the rising importance of the magnetic component of a propagating field as the frequency declines [26, 27]. Inappropriate emphasis on the electric field has caused conceptual errors even in relativistic phenomena, as discussed in Section IV in the context of vacuum pair production. A potentials-related approach obviates this electric-field-dominance hazard. The concept of the critical field is often mentioned in connection with strong-laser interactions [28, 29]. The critical field refers to that value of electric field at which spontaneous pair production from the vacuum becomes significant. It has been applied to laser fields in terms of the electric component of a plane-wave field. This is devoid of meaning for laser beams in vacuum because pure electric fields and plane-wave fields are disjoint concepts, as is evident from Section II. The conservation conditions applicable to critical-field considerations cannot be satisfied by a laser. Even were the electric component of a laser field equal to the critical field, pair production cannot occur because pair production from the vacuum by a laser pulse cannot occur unless there is a counter-propagating field to provide the necessary conservation of momentum [17, 18, 30, 31]. The fact that photons convey momentum is incompatible with the concept of a critical electric field for laser-induced processes.

Section IV concludes with the practical problem of communicating with submerged submarines. This has been done under circumstances that emphasize how different plane-wave fields are from QSE fields.

Section V explores the notion of a hierarchy of physical quantities. Potentials are directly related to energies, so they are identified as primary quantities. Fields are derived from potentials, so they are secondary. Forces are determined by fields and so forces are also secondary. The hierarchy concept is related to classical mechanics in that Newtonian physics
is couched in terms of forces, and so it is secondary to versions of classical mechanics based on energy-based system functions like the Lagrangian and Hamiltonian. Mechanics formulated with system functions infer Newtonian mechanics, but the converse is not true.

II. SYMMETRY VIOLATION

The two examples presented have an important qualitative difference. The first example – a constant electric field – is so elementary that the proper choice of potentials is obvious, and there is no motivation to explore the properties of the symmetry-violating alternative potentials. The next example is quite different in that the improper choice of potentials is very attractive to a laser-physics community that is accustomed to the dipole approximation. The requisite propagation property never appears within the dipole approximation, and its violation is thereby invisible.

A preliminary step is to introduce the units and conventions employed in this article, and to add some general remarks about terminology.

A. Units and conventions

Gaussian units are employed for all electromagnetic quantities. The expressions for the electric field $E$ and magnetic field $B$ in terms of the scalar potential $\phi$ and the 3-vector potential $A$ are

$$E = -\nabla \phi - \frac{1}{c} \partial_t A, \quad B = \nabla \times A.$$  (1)

A gauge transformation generated by the scalar function $\Lambda$ is

$$\tilde{\phi} = \phi + \frac{1}{c} \partial_t \Lambda, \quad \tilde{A} = A - \nabla \Lambda,$$  (2)

where $\Lambda$ must satisfy the homogeneous wave equation

$$\left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) \Lambda = \partial^\mu \partial_\mu \Lambda = 0.$$  (3)

Relativistic quantities are expressed with the time-favoring Minkowski metric, with the signature $(+ - - -)$, where the scalar product of two 4-vectors $a^\mu$ and $b^\mu$ is

$$a \cdot b = a^\mu b_\mu = a^0 b^0 - a \cdot b.$$  (4)
The 4-vector potential $A^\mu$ incorporates the scalar and 3-vector potentials as

$$A^\mu : (\phi, A).$$  \hspace{1cm} (5)

In 4-vector notation, the two gauge transformation expressions in Eq. (2) become the single expression

$$\tilde{A}^\mu = A^\mu + \partial^\mu \Lambda.$$  \hspace{1cm} (6)

Both the initial and gauge-transformed 4-vector potentials must satisfy the Lorenz condition

$$\partial^\mu A_\mu = 0, \quad \partial^\mu \tilde{A}_\mu = 0.$$  \hspace{1cm} (7)

The propagation 4-vector $k^\mu$ consists of the propagation 3-vector $k$ as the space part, and the amplitude $|k| = \omega/c$ as the time component:

$$k^\mu : (\omega/c, k).$$  \hspace{1cm} (8)

The 4-vector $k^\mu$ defines the light cone and, according to the rule (4), it is “self-orthogonal”:

$$k^\mu k_\mu = (\omega/c)^2 - k^2 = 0,$$  \hspace{1cm} (9)

which is an important possibility in this non-Euclidean space.

The concept of transversality refers to the property of plane-wave fields expressed in a relativistic context as covariant transversality

$$k^\mu A_\mu = 0,$$  \hspace{1cm} (10)

in terms of the 4-potential $A^\mu$. In many textbooks on classical electromagnetic phenomena, transversality is defined as geometrical transversality

$$k \cdot E = 0 \text{ and } k \cdot B = 0,$$  \hspace{1cm} (11)

in terms of the electric and magnetic fields. It can be shown that covariant transversality infers geometrical transversality.

B. Terminology

Despite the conclusion in this paper that potentials are more basic than fields, it is not possible to avoid the use of the term “field” in a generic sense. For example, one important
conclusion reached herein is that vector and scalar potentials provide more information than do electric and magnetic fields in the description of the effects of laser fields. In the preceding sentence, the term “laser field” is used generically to identify the radiation created by a laser, despite the particular result that potentials are the better approach in the description of that radiation. A similar problem arises when it is concluded that the dipole approximation amounts to the replacement of the “transverse field” of a laser by the more elementary “longitudinal field”. In each of the phrases demarcated by quotation marks, the word “field” is used in a generic sense to identify an electromagnetic phenomenon.

C. Constant electric field

The problem of a particle of mass $m$ and charge $q$ immersed in a constant electric field of magnitude $E_0$ is inherently one-dimensional. For present purposes, nothing is gained by going to three spatial dimensions. The problem is clearly one in which energy is conserved. By Noether’s Theorem [22], the Lagrangian must be independent of time $t$, so that the connection between the electric field and potentials given in Eq. (1) must depend only on the scalar potential $\phi$. Equation (1) can then be integrated to give the potentials

$$\phi = -xE_0, \quad A = 0,$$

since an additive constant of integration has no physical meaning. The potentials descriptive of this problem are unique, and given by Eq. (12).

The Lagrangian function is the difference of the kinetic energy $T$ and the potential energy $U$:

$$L = T - U$$

$$= \frac{1}{2} m \dot{x}^2 + qxE_0.$$  \hspace{1cm} (13)

$$= \frac{1}{2} m \ddot{x}^2 + qxE_0.$$  \hspace{1cm} (14)

The Lagrangian equation of motion is

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} - qE_0 = 0,$$

which is just the elementary Newtonian equation

$$m\ddot{x} = qE_0.$$  \hspace{1cm} (15)

$$m\ddot{x} = qE_0.$$  \hspace{1cm} (16)
The simplest initial conditions for this problem – initial position and velocity set to zero – lead to the solution

\[ x = \frac{qE_0}{2m} t^2. \quad (17) \]

From Eqs. (13) and (14), it follows that

\[ T = \frac{1}{2m} (qE_0t)^2, \quad U = -\frac{1}{2m} (qE_0t)^2, \quad T + U = 0. \quad (18) \]

The anticipated conservation of energy holds true.

Despite the uniqueness of the potentials of Eq. (12), there exists an apparently proper gauge transformation generated by the function

\[ \Lambda = c t x E_0. \quad (19) \]

The gauge-transformed potentials are

\[ \tilde{\phi} = 0, \quad \tilde{A} = -c t E_0, \quad (20) \]

and the Lagrangian function is

\[ \tilde{L} = \frac{1}{2} m \tilde{x}^2 - q t E_0 \tilde{x}. \quad (21) \]

The kinetic energy is unaltered (\( \tilde{T} = T \)), but the new potential energy is

\[ \tilde{U} = q t E_0 \tilde{x}, \quad (22) \]

which is explicitly time-dependent. The new equation of motion is

\[ \frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial \tilde{x}} \right) - \frac{\partial \tilde{L}}{\partial \tilde{x}} = m \ddot{x} - q E_0 = 0, \quad (23) \]

which is identical to that found in the original gauge, so that the solution is the same as Eq. (17). However, the altered gauge has introduced a fundamental change. The gauge-transformed potential energy is evaluated as

\[ \tilde{U} = \frac{1}{m} (qE_0t)^2, \quad (24) \]

so that

\[ \tilde{T} + \tilde{U} = \frac{3}{2m} (qE_0t)^2. \quad (25) \]
The total energy is not conserved, as was presaged by the explicit time dependence of the gauge-transformed Lagrangian \([21]\).

How did this happen? One constraint placed on gauge transformations (see, for example, the classic text by Jackson \([33]\)) is that the generating function must be a scalar function that satisfies the homogeneous wave equation, as in Eq. \([3]\). This is satisfied by the function \([19]\). The only other condition is the Lorenz condition \([7]\), which is satisfied by the potentials before and after transformation. However, there is no condition that guarantees preservation of symmetries inherent in the physical problem. It is not enough to employ appropriate fields; it is necessary to employ the appropriate potentials to ensure that all aspects of the physical problem are rendered properly.

This writer is unaware of any instance where inappropriate potentials have been accepted and employed in this exceedingly simple problem. The same cannot be said for the next example.

**D. Plane-wave field**

Laser fields are of central importance in contemporary physics, and laser fields are plane-wave fields. A plane-wave field is the only electromagnetic phenomenon that has the ability to propagate indefinitely in vacuum without the continued presence of sources. In the typical laboratory experiments with lasers, the practical consequence of this ability to propagate without need for sources means that all fields that arrive at a target can only be a superposition of plane-wave fields. Any contamination introduced by optical elements like mirrors or gratings can persist for only a few wavelengths away from such elements. On the scale of a typical laboratory optical table, this is negligible.

Plane-wave fields propagate at the speed of light in vacuum; they are fundamentally relativistic. The 1905 principle of Einstein is basic: the speed of light is the same in all inertial frames of reference \([34]\). The mathematical statement of this principle is that any description of a plane-wave field can depend on the spacetime coordinate \(x^\mu\) only as a scalar product with the propagation 4-vector \(k^\mu\). The consequence of this projection of the spacetime 4-vector onto the light cone is that any change of gauge must be such as to be confined to the light cone. That is, with the definition

\[
\varphi \equiv k^\mu x_\mu, \tag{26}
\]
the field 4-vector must be such that

$$A^\mu_{pw} = A^\mu_{pw}(\varphi),$$

(27)

where the subscript $pw$ stand for plane-wave. When the gauge transformation of Eq. (6) is applied, the gauge-altered 4-vector potential is confined by the condition (27) to the form

$$\tilde{A}^\mu = A^\mu + k^\mu \Lambda',$$

(28)

where the gauge-change generating function can itself depend on $x^\mu$ only in the form of $\varphi$, and

$$\Lambda' = \frac{d}{d\varphi} \Lambda(\varphi).$$

(29)

As is evident from Eq. (9), transversality is maintained by the gauge transformation (28).

A further limitation arises if an electron is subjected to a scalar binding potential in addition to the vector potential associated with the laser field. A relativistic Hamiltonian function for a charged particle in a plane-wave field contains a term of the form

$$\left( i\hbar \partial^\mu - \frac{q}{c} A^\mu \right) \left( i\hbar \partial_\mu - \frac{q}{c} A_\mu \right).$$

(30)

This occurs in the classical case, in the Klein-Gordon equation of quantum mechanics, and in the second-order Dirac equation of quantum mechanics [36, 37]. The expansion of the expression in Eq. (30) contains the squared time part

$$\left( i\hbar \partial_t - \frac{q}{c} A^0 \right)^2.$$

(31)

If $A^0$ contains contributions from both a scalar potential and the time part of the plane-wave 4-vector potential, then executing the square in Eq. (31) would give a term containing the product of these two scalar potentials that is not physical; it does not occur in the reduction of relativistic equations of motion to their nonrelativistic counterparts [23]. This applies specifically to applications in AMO physics. That is, it must be true that [23, 38]

$$A^0_{pw} = \phi_{pw} = 0.$$

(32)

This means that gauge freedom vanishes. Only the radiation gauge (also known as Coulomb gauge) is possible. This is the gauge in which scalar binding influences are described by scalar potentials $\phi$ and laser fields are described by 3-vector potentials $A$. 

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Consider the gauge transformation generated by the function

\[ \Lambda = -A^\mu (\varphi) x_\mu, \]  

(33)

This leads to the transformed gauge

\[ \tilde{A}^\mu = -k^\mu x^\nu \left( \frac{d}{d \varphi} A_\nu \right), \]  

(34)

which was introduced in Ref. [38] in an attempt to base the Keldysh approximation [39] on plane-wave fields rather than on quasistatic electric fields. The transformed 4-potential can also be written as

\[ \tilde{A}^\mu = -\frac{k^\mu}{\omega/c} \cdot \mathbf{E} (\varphi), \]  

(35)

thus suggesting a relativistic generalization of the nonrelativistic length gauge used by Keldysh and widely employed within the AMO community. The problem with the \( \tilde{A}^\mu \) of Eq. (34) or (35) is that it violates the symmetry (27) required of a propagating field. Nevertheless, this \( \tilde{A}^\mu \) satisfies the Lorenz condition (7) and the transversality condition (10); and the generating function of Eq. (33) satisfies the homogeneous wave equation of Eq. (3) [38]. That is, all the usual requirements for a gauge transformation are met even though the transformed 4-vector potential \( \tilde{A}^\mu \) of Eq. (34) or (35) violates the symmetry required of a propagating field like a laser field.

This violation of a basic requirement for a laser field has unphysical and hence unacceptable consequences. The most obvious is that the covariant statement of the all-important [19] ponderomotive energy \( \tilde{U}_p \) produces a null result since

\[ \tilde{U}_p \sim \tilde{A}^\mu \tilde{A}_\mu = 0 \]  

(36)

as a consequence of the self-orthogonality of the propagation 4-vector \( k^\mu \). The resemblance of Eq. (35) to the length-gauge representation of a quasistatic electric field suggests a tunneling model for the relativistic case [40, 41], which is inappropriate for strong laser fields. Tunneling can occur only through interference between scalar potentials, and a strong laser field is inherently vector, not scalar.

The basic defect of the potentials (34) or (35) is violation of the Einstein condition of the constancy of the speed of light in all Lorentz frames, despite the validity of the gauge transformation leading to those potentials. The importance of the physical situation in which this occurs is robust evidence of the significance of the proper choice of potentials,
since the electric and magnetic fields attained from the unacceptable potentials (34) or (35) are exactly the same as those that follow from potentials that satisfy properly the condition (27).

III. GAUGE TRANSFORMATIONS AND UNITARITY

Unitary transformations in quantum physics preserve the values of physical observables. It was shown above that not all gauge transformations produce physically acceptable results. Therefore, gauge transformations are not unitary transformations. This conclusion is supported by the basic structure of transition amplitudes.

Transition amplitudes without resort to perturbation theory are best expressed by S matrices. These are of two (equivalent) types. The direct-time or post amplitude is

\[
(S - 1)_{fi} = -\frac{i}{\hbar} \int_{-\infty}^{\infty} dt \left( \Phi_f, H_I \Psi_i \right),
\]  

and the time-reversed or prior amplitude is

\[
(S - 1)_{fi} = -\frac{i}{\hbar} \int_{-\infty}^{\infty} dt \left( \Psi_f, H_I \Phi_i \right).
\]  

The indices \( f \) and \( i \) label the final and initial states. The \( \Phi \) states are non-interacting states and the \( \Psi \) states are fully interacting states satisfying, respectively, the Schrödinger equations

\[
i\hbar \partial_t \Phi = H_0 \Phi,
\]

\[
i\hbar \partial_t \Psi = (H_0 + H_I) \Psi,
\]

where \( H_I \) is the interaction Hamiltonian.

In a gauge transformation, the matrix elements within the time integrations in Eqs. (37) and (38) transform as

\[
(\Phi_f, H_I \Psi_i) \rightarrow \left( \Phi_f, \tilde{H}_I \tilde{\Psi}_i \right),
\]

\[
(\Psi_f, H_I \Phi_i) \rightarrow \left( \tilde{\Psi}_f, \tilde{H}_I \Phi_i \right).
\]

Because the noninteracting states are unaltered in a gauge transformation, there is no necessary equivalence between the two sides of the expressions in Eqs. (41) and (42).
Those authors that endorse the favored status of the length gauge\textsuperscript{[42–46]} “solve” this problem by attaching a unitary operator to all states, including non-interacting states: $\tilde{\Phi} = U\Phi$, $\tilde{\Psi} = U\Psi$. All $U$ and $U^{-1}$ operators exactly cancel in the matrix element, and the transition amplitude is unchanged. This is what leads to the property “gauge-invariant formalism” sometimes ascribed to the length gauge. However, this procedure amounts to an identity or to a change of quantum picture, but not to a gauge transformation.

IV. FUNDAMENTAL CONTRASTS IN THE APPLICABILITY OF FIELDS AND POTENTIALS

The first example to be presented is the very basic one of the impossibility of expressing the Schrödinger equation directly in terms of electric and magnetic fields, which should be possible if fields are truly more fundamental than potentials. Other direct examples of difficulties posed by the assumption of the fundamental importance of fields are shown, many of them long employed unnoticed within the strong-field community.

A. Schrödinger equation

The Schrödinger equation

$$i\hbar \partial_t \Psi (t) = H \Psi (t), \quad (43)$$

when viewed as a statement in a Hilbert space (that is, without selecting a representation such as the configuration representation or the momentum representation) states that the effect of rotating a state vector $\Psi (t)$ by the operator $H$ within the Hilbert space produces the same effect as differentiating the vector with respect to time (multiplied by $i\hbar$). Time $t$ is an external parameter upon which the state vectors depend, which accounts for why Eq. (43) specifies $t$ as a label independent of the Hilbert space. A unitary transformation preserves this equivalence. This can be stated as

$$i\hbar \partial_t - \tilde{H} = U \left( i\hbar \partial_t - H \right) U^{-1}. \quad (44)$$

Since Eq. (44) can be written as

$$\tilde{H} = UHU^{-1} + U \left( i\hbar \partial_t U^{-1} \right), \quad (45)$$
this shows explicitly that the Hamiltonian operator does not transform unitarily if there is any time dependence in $U$.

An important gauge transformation is that introduced by Göppert-Mayer [47], widely employed in the AMO community. This transformation is given by

$$U_{GM} = \exp \left( \frac{ie}{\hbar c} \mathbf{r} \cdot \mathbf{A}(t) \right),$$

which depends explicitly on time when $\mathbf{A}(t)$ describes a laser field within the dipole approximation, meaning that Eq. (45) is consequential.

The fact that, in general,

$$\tilde{H} \neq UHU^{-1},$$

is the explanation for the curious result to be found in many papers (for example, Refs. [42–46]) that the $\mathbf{r} \cdot \mathbf{E}$ potential is a preferred potential. If any other potential is employed in solving the Schrödinger equation, then the claim is made that a transformation factor must be employed even on a non-interacting state. There is a logical contradiction inherent in the requirement that a non-interacting state must incorporate a factor that depends on an interaction, but the list of published papers that accept this premise is much longer than the salient examples cited here. The underlying problem is the assumption that a gauge transformation transforms the Hamiltonian unitarily. That problem exists in all of the references just cited, although it is usually submerged in complicated manipulations. It is especially clear in Ref. [46], where it is specified that all operators $O$ transform under a gauge transformation according to the unitary-transformation rule

$$\tilde{O} = UOU^{-1}, \quad U^{-1} = U^\dagger.$$  

That specification is applied to $H$, in violation of the condition (45), and to the interaction Hamiltonian $H_I$, with no explanation for how it is possible to gauge-transform from the length-gauge interaction to any other gauge in view of the absence of operators in the scalar potential $\mathbf{r} \cdot \mathbf{E}$. In the scheme proposed in Refs. [42–46], if the problem is initially formulated in the context of the $\mathbf{r} \cdot \mathbf{E}$ potential, it is never possible to transform to any other gauge. This explains the use of the phrase “gauge-invariant formulation” with respect to $\mathbf{r} \cdot \mathbf{E}$ to be found in some published works.
B. Locality and nonlocality

Fields are derived from potentials by the calculus process of differentiation, as exhibited in Eq. (1). Differentiation is carried out at a point in spacetime. It is local. If potentials are to be expressed from fields, that requires integration, consisting of information from a range of spacetime values; it is nonlocal. The fact that the Schrödinger equation requires the local information from potentials, and cannot be described by fields without inferring nonlocality in spacetime, is direct evidence that potentials are more fundamental than fields.

C. Ambiguity in the electromagnetic field tensors

The basic field tensor of electrodynamics is defined as

\[ F^\mu\nu = \partial^\mu A^\nu - \partial^\nu A^\mu. \] (49)

It is important to note that this expression is in terms of the derivatives of potentials rather than the potentials themselves. Thus it is not surprising that the Lorentz invariant found from the inner product of \( F^\mu\nu \) with itself yields an expression in terms of fields:

\[ F^\mu\nu F^\mu\nu = 2 (B^2 - E^2). \] (50)

A dual tensor can be defined as

\[ G^\mu\nu = \frac{1}{2} \epsilon^{\mu\nu\rho\lambda} F_{\rho\lambda}, \] (51)

where \( \epsilon^{\mu\nu\rho\lambda} \) is the completely asymmetric fourth-rank tensor. (The conventions of Jackson [33] are being employed.) The inner product of the basic and dual tensors gives a second Lorentz invariant:

\[ G^\mu\nu F^\mu\nu = -4B \cdot E, \] (52)

also in terms of fields. The two Lorentz invariants

\[ E^2 - B^2, \quad E \cdot B \] (53)

are said to characterize the electrodynamic environment.

An important special case is that of transverse, propagating fields, where

\[ E^2 - B^2 = 0, \quad E \cdot B = 0. \] (54)
The properties (54) lead to radiation fields as sometimes being called “null fields”. (The terms radiation field, propagating field, transverse field, plane-wave field, are here used interchangeably.) Radiation fields propagate at the speed of light in vacuum, and they have the unique character that, after initial formation, they propagate indefinitely in vacuum without the presence of sources.

However, the invariants (54) are not unique to radiation fields; they apply also to “constant crossed fields”. That is, it is always possible to generate static electric and magnetic fields of equal magnitude that are perpendicular to each other, and will thus possess zero values for both of the Lorentz invariants of the electromagnetic field. Constant crossed fields do not propagate, and they cannot exist without the presence of sources. They are unrelated to radiation fields despite sharing the same values of the Lorentz invariants. Most importantly, constant crossed fields cannot be considered as the zero-frequency limit of radiation fields, as they are sometimes described [24, 25]. All radiation fields propagate at the speed of light for all frequencies, no matter how low. There is no possible zero-frequency static limit [49].

There is no ambiguity when radiation fields and constant crossed fields are expressed in terms of their potentials. Radiation-field potentials possess the periodicity inherent in trigonometric dependence on the $\varphi$ of Eq. (26). This is unrelated to the $\phi = -r \cdot E_0$ potential of (12) for a constant electric field $E_0$, and $A = - (r \times B_0) / 2$ for a constant magnetic field $B_0$, both of which require source terms for their existence.

D. Lorentz force

The force exerted on a particle of charge $q$ moving with velocity $v$ in a field with electric and magnetic components $E$ and $B$ is given by the Lorentz force expression

$$\mathbf{F} = q \left( \mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \right).$$

(55)

In a plane-wave field, the electric and magnetic fields are of equal magnitude: $|E| = |B|$. Thus, under conditions where $|v| / c \ll 1$, the magnetic component of the force is minor as compared to the electric component. The implication is that, as the field frequency declines, the motion-related magnetic component reduces to an adiabatic limit. This concept of adiabaticity justifies the complete neglect of the magnetic field that is a key element of the
The dipole approximation, applied within the AMO community in the form:

$$\mathbf{E} = \mathbf{E}(t), \quad \mathbf{B} = 0.$$  \hfill (56)

The adiabaticity line of reasoning suggests a so-called adiabatic limit where the field frequency declines to zero, and the plane wave field behaves as a constant crossed field that satisfies the plane-wave condition \[^{54}\].

The entire line of reasoning that involves the concepts of adiabaticity, adiabatic limit, and a zero-frequency limit for plane-wave fields is field-based and erroneous \[^{48,49}\]. When the problem is treated in terms of potentials, it becomes clear that $v/c$ approaches unity for very strong fields even when the frequency is very low, and the magnetic force becomes equivalent to the electric force for very strong fields.

### E. Critical field

The “critical field” in electrodynamics is related to the spontaneous breakdown of the vacuum into electron-positron pairs. The critical field is defined as the electric field strength at which the $\pm mc^2$ limits for the rest energies of electron and positron in a particle-hole picture are “tilted” by the electric field to allow tunneling between positive and negative energy states when the spatial limits of the tunnel are separated by an electron Compton wavelength. This type of pair production is called Sauter-Schwinger pair production \[^{13,14}\]. It is fundamentally different from Breit-Wheeler pair production \[^{17,18,50}\] to be discussed below.

The reason for the fundamental difference is that Sauter-Schwinger pair production is a phenomenon due to electric fields and Breit-Wheeler is due to plane-wave fields. The distinction between pure electric fields and plane-wave fields could hardly be more clear, since both types of fields have unique gauge choices that are contrasting.

The critical field is often mentioned as a goal of strong-field laser facilities, which is basically a non-sequitur. The critical field applies only to electric fields and lasers produce plane-wave fields. Even if a laser field were sufficiently intense that its electric component had the magnitude of the QSE critical field, pair production from the vacuum cannot occur because the photons of the laser field convey momentum. A counter-propagating plane-wave field is necessary to satisfy momentum conservation as well as energy conservation in the
production of pairs from the vacuum. This is then the two-fields Breit-Wheeler process, which is unrelated to the single-field Sauter-Schwinger process.

F. Pair production from the vacuum

For many years the stated ultimate goal of large-laser programs was to achieve a laser intensity such that the electric component of the laser is equal to the critical field discussed in the preceding subsection. This magnitude of electric field corresponds to an intensity of about $4.6 \times 10^{29} \text{W/cm}^2$.

The problem is that the Schwinger limit applies only to electric fields. It does not apply to laser fields [51], as explained above. The only way to produce momentum balance with a laser field while still producing pairs from the vacuum is to have the laser beam collide with oppositely-directed photons, as proposed in Ref. [30]. This was predicted to be done on a practical basis at a linear accelerator facility such as that at SLAC (Stanford Linear Accelerator Center), with a laser intensity of only slightly greater than $10^{18} \text{W/cm}^2$. An important note is that the prediction of Ref. [30] was for the use of the then-important ruby laser at a different pulse length and a different energy of the energetic electron beam used to produce the countervailing photon field. The predicted threshold of 25 photons from the laser field in Ref. [30] is altered to 5 photons for the parameters of the experiment that was actually done at SLAC in 1997 [31]. The theoretical prediction of an effective threshold intensity of about $10^{18} \text{W/cm}^2$ is maintained because of the essential independence of the laser frequency that was remarked in Ref. [30].

(One caveat about the experiment is that it was reported as a high-order perturbative result, whereas it is readily shown to be at an intensity beyond the radius of convergence of perturbation theory [52]. A second problem is that it was described as “light-by-light scattering”, which is a different process altogether. Feynman diagrams of these processes have electron and positron as emergent particles in a pair production process, while light-by-light scattering has emergent photons.)

The striking difference between the $4.6 \times 10^{29} \text{W/cm}^2$ required to attain a laser electric component equal to the critical field and the actual $1.3 \times 10^{18} \text{W/cm}^2$ required for the SLAC experiment is evidence of a misplaced focus on the electric field required for vacuum pair production. The required intensity of the laser field depends on the properties of the counter-
propagating field, but it is never as large as the Sauter-Schwinger critical field.

**G. Low frequency limit of a plane-wave field**

It is conventional to view low-frequency laser-induced phenomena from the standpoint of the Lorentz force expression of Eq. (55). In a plane-wave field, the electric and magnetic fields are of equal magnitude: \(|E| = |B|\). Thus, under conditions where \(|v|/c \ll 1\), the magnetic component of the force is minor as compared to the electric component. The implication is that, as the field frequency declines, the motion-related magnetic component reduces to an adiabatic limit. This concept of adiabaticity appears to justify the complete neglect of the magnetic field that is a key element of the dipole approximation, applied within the AMO community in the form given in Eq. (56). Adiabaticity suggests a so-called adiabatic limit, where the field frequency declines to zero, and the plane wave field behaves as a constant crossed field that satisfies the plane-wave condition (54).

The entire line of reasoning that involves the concepts of adiabaticity, adiabatic limit, and a zero-frequency limit for plane-wave fields is field-based and erroneous [48, 49]. When the problem is treated in terms of potentials, it becomes clear that \(v/c\) approaches unity for very strong fields. The magnetic force becomes equivalent to the electric force for very strong fields. The magnetic force becomes equivalent to the electric force for very strong fields.

Analysis from a potentials standpoint is in stark contrast to a fields-based approach.

The ponderomotive potential energy of a charged particle in a plane-wave field is a fundamental property of the particle [19], and it becomes divergent as the field frequency approaches zero. The immediate consequence is that the limit \(\omega \to 0\) causes the dipole approximation to fail [48, 49], and corresponds to an extremely relativistic environment [19]. This contradicts maximally the field-based conclusions.

The ponderomotive energy \(U_p\) is given by

\[
U_p = \frac{q^2}{2mc^2} \left\langle |A_\mu A_\mu| \right\rangle ,
\]

(57)

where the angle brackets denote a cycle average, and the absolute value needs to be indicated because the 4-vector potential \(A_\mu\) is a spacelike 4-vector and its square is thus negative with the metric being employed. The ponderomotive energy is based on potentials. When
expressed in terms of field intensity $I$, $U_p$ behaves as

$$U_p \sim I/\omega^2,$$  \hspace{1cm} (58)

which explains the relativistic property of the charged particle as the frequency approaches zero.

**H. Extremely-low-frequency radio waves**

A central issue in this article is that transverse fields and longitudinal fields are fundamentally different, even when they seem to have some properties in common. An effective example is the matter of communicating with submerged submarines with extremely low frequency (ELF) radio waves. The U. S. Navy operated such a system [53], designed to communicate with submarines submerged at depths of the order of 100 m over an operational range of about half of the Earth's surface. The point of using extremely low frequencies is the large skin depth in a conducting medium (seawater) that can be achieved. What is most remarkable about the system is that the frequency of 76 Hz that was used has a wavelength of about $4 \times 10^6$ m, which is approximately 0.62 times the radius of the Earth. The system could convey an intelligible signal over about half of the Earth's surface. Considering the length of the submarine (about 100 m) to be the size of the receiving antenna, this means that the wavelength to antenna-length ratio is about $4 \times 10^4$. That is, the received radio wave is constant over the entire length of the receiving antenna to a very high degree of accuracy. A nearly-constant electric field cannot be detected at a distance of half of our planet away from its source; a nearly-constant electric field cannot penetrate 100 m through a conducting medium; and so a nearly-constant electric field cannot convey an intelligible signal in the presence of all of these extremely effective barriers. A longitudinal field is fundamentally different from a transverse field.

The proposed “local constant-field approximation” (LCFA) for relativistic laser effects [25, 34] is based on the presumed similarity of low-frequency laser fields to constant crossed fields. This was shown earlier in this manuscript to be a meaningless association. The application involved in the communication with submerged submarines is a practical demonstration that the LCFA is not a valid concept.
V. HIERARCHY OF PHYSICAL EFFECTS

In basic calculus, if a function $f(x)$ is known, then so is $df/dx$; differentiation is local. If the derivative is all that is known, the process of integration to find $f(x)$ requires the knowledge of $df/dx$ over a range of values; integration is nonlocal. In electromagnetism, potentials are the analog of $f(x)$ and fields correspond to $df/dx$. The example of the Schrödinger equation shows that knowledge of potentials is primary and fields are secondary. This ordering can be extended to other physical quantities. For example, the Lorentz force expression in Eq. (55) relates forces directly to fields, meaning that forces are secondary. A further implication is that Newtonian mechanics, expressed in terms of forces, is secondary to Lagrangian and Hamiltonian mechanics that are expressed in terms of energies; that is, in terms of potentials. It is no accident that mechanics textbooks show that formalisms based on system functions like the Lagrangian or Hamiltonian infer the Newtonian formalism, but to go in the reverse direction requires an extrapolation of concepts.

This ordering, or hierarchy, of physical quantities has consequences in problems that go beyond simple electric-only or magnetic-only fields. In laser-related problems where both electric and magnetic fields are involved, it is possible to arrive at invalid inferences if only secondary influences are regarded as controlling. An example is the common practice of viewing electric fields as being the dominant quantities in long-wavelength circumstances where the dipole approximation appears to be valid. This disguises the fact that extremely low frequencies can lead to relativistic behavior, where electric fields supply inadequate information [48, 49], and inappropriate concepts such as adiabaticity exist.

A cogent example is the critical field. A widely used “strong-field QED parameter” is simply the ratio of the electric field to the critical field. This has meaning only for static-electric or for QSE fields. For strong-field laser applications, relevant intensity parameters are all ratios of energies. See, for example, the section entitled “Measures of intensity” in Ref. 55. (The early, but still useful review article by Eberly 56 is also instructive in this regard.) In particular, the “free-electron intensity parameter” $z_f = 2U_p/mc^2$ is known to measure the coupling between an electron and a nonperturbatively intense plane-wave field [17, 19, 57], replacing the fine-structure constant of perturbative electrodynamics.

A special insight arises when the $z_f$ parameter is related to the fine-structure constant $\alpha$.
where $\rho$ is the number of photons per unit volume, and $2\lambda C^2$ is approximately the volume of a cylinder of radius equal to the electron Compton wavelength $\lambda_C$ and a length given by the wavelength $\lambda$ of the laser field. That is, it appears that all of the photons within the cylinder participate in the coupling between the electron and the laser field. The electron Compton wavelength is the usual interaction distance for a free electron, but the wavelength can be a macroscopic quantity. The “strong-field physics” that arises from the application of the Volkov solution to problems involving the interaction of very intense radiation fields with matter [16–18], thus appears to be the bridge connecting quantum electrodynamics with the classical electrodynamics of Maxwell.

Other examples of the hazards of excessive dependence on secondary quantities have been mentioned above in the context of the assumed dominance of the electric component of the Lorentz force at low frequencies, and the inference from the Lorentz invariants that constant crossed fields are related to propagating fields.

**VI. FAILURE OF PERTURBATION THEORY**

Perturbation theory has been the cornerstone of QED since the Nobel-prize winning work of Feynman, Schwinger, and Tomonaga. The radius of convergence of perturbation theory was examined in depth a long time ago [17]. The motivation for the study was the demonstration by Dyson [58] that, despite its remarkable quantitative success, standard covariant QED has a zero radius of convergence for a perturbation expansion. The question about whether a strong-field theory based on the Volkov solution of the Dirac equation [59] could be convergent was the motivation for Ref. [17]. The answer was affirmative, but with intensity-dependent singularities in the complex coupling-constant plane that limited the convergence. (This explains the $z_f$ terminology, since $z_f$ was allowed to be complex, and the quantity $z$ is often used for complex numbers.) Physically, convergence fails whenever the field intensity is high enough to cause a channel closing. That is, if a process requires a certain threshold energy to proceed in a field-free process as measured by some photon number $n_0$, that threshold energy will be increased due to the need for a free electron to possess the ponderomotive energy $U_p$ in the field. When $U_p$ is large enough to cause $n_0$ to
index up to $n_0 + 1$, that is called a channel-closing, and it marks a sufficient condition for the failure of perturbation theory. The upper limit on perturbation theory is therefore set by

$$U_p < \hbar \omega, \quad \text{or} \quad I < 4\omega^3,$$

where the last expression is in atomic units, using the equivalence $U_p = I/4\omega^2$ in atomic units, and $I$ is the laser intensity.

Although the original convergence investigation was done for free electrons, the same limit was found for atomic ionization [60].

The relevance of using an index based on the ratio of the electric field to the critical field as an ad hoc limit on perturbation theory for laser effects is strongly questioned here. Such a basic matter as the applicability of perturbation methods to the treatment of the effects of radiation fields is governed by primary quantities like the energies $U_p$ and $mc^2$, and not by secondary quantities like the magnitude of the electric field.

VII. OVERVIEW

The Aharonov-Bohm effect introduced a major change in electrodynamics because it showed that potentials are indeed more fundamental than fields. However, although the effect relates to a quantum phenomenon, it has had little effect on the way in which quantum mechanics is employed. The ascendancy of potentials over fields as described above is much more consequential, especially for strong-field phenomena. A simple summarizing statement is that electromagnetic scalar and vector potentials convey more physical information than the electric and magnetic fields derivable from them.

Of special importance is the fact that two cases have been identified where there can be only one acceptable set of potentials, and these two cases are of very wide practical scope. One case is the constant electric field, which is exactly or approximately applicable to a wide variety of AMO and condensed-matter applications. The unique acceptable potential for static electric fields is the length-gauge, or $r \cdot E$ potential, and this is already widely employed for constant or slowly varying electric-field applications.

The other case with a unique allowable gauge is the propagating-field case, which is of profound importance in laser-based experiments. Such experiments constitute an increasingly large proportion of AMO activities. An essential reminder is that only propagating fields
can persist in the absence of sources, so that virtually all laser-based experiments occur in a superposition of propagating-field components. The radiation gauge (or Coulomb gauge) is the only gauge employable without risking the creation of hidden errors due to improper gauge choice. It is unfortunate that, in strong-field laser applications, the widespread use of the dipole approximation introduces precisely that hazard of hidden errors that affects both practical calculations and qualitative insights into the behavior of physical systems subjected to laser fields.

It has been shown that gauge transformations are not, in general, unitary. This has never previously been reported, and it can lead to further errors in addition to the important example explored in Section III above about the putative universality of the length gauge.

The concept of primary and secondary physical quantities has been introduced, with over-dependence on a secondary quantity like the electric field having the capability of leading to important misconceptions.

The criterion of a critical field for longitudinal fields has no relevance to the transverse field of laser-induced processes.

The appearance of mixed quantum and classical quantities in ascertaining the limits on perturbative methods in the application of strong-field theories identifies strong-field physics as the bridge connecting quantum electrodynamics and the classical electromagnetism of Maxwell.

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