Local Theory of \( t \)-bonded Sets

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Abstract

The local theory for regular and multi-regular systems was developed in the assumption that these systems are Delone sets, or \((r, R)\)-systems. The requirement for a set to be a \((r, R)\)-system particularly implies that any two points in a Delone set can be connected by a sequence of points from the set with sequel inter-point distances bounded by \(2R\). In the terminology we adopted in this paper, it means that a Delone set is a \(2R\)-bonded set. Meanwhile, there are crystals, e.g. zeolites, whose atomic structure is multi-regular microporous point set. In these structures there are cavities that are relatively large compared to the "length"of bonds between atomes. In other words, the parameter \(R\) in this Delone set significantly exceeds a natural link parameter. For a better description of such "microporous"structures it is worthwhile to take into consideration a parameter that represents atomic bonds within the matter. In the paper we generalize some results of the local theory to the sets that we called \( t \)-bonded sets even without making an assumption that a set is a Delone set.

1 Introduction

In this section we present basic definitions related to the mathematical concept of crystal in the light of the local theory with the overarching goal to extend the theory’s results by re-introducing the concept of \( t \)-bonded sets [Definition 2.1] and considering the class of sets that includes the Delone sets as a subclass.

The concept of \( t \)-bonded sets was briefly introduced by one of the authors in [1] under the name of \( d \)-connected sets (in Russian), though it has not received due consideration. In light of the developments in the local theory for crystals that occurred since 1976 and demands in chemistry and crystallography, we believe the local theory for \( t \)-bonded sets deserves to be developed.

The above mentioned definitions single the family of crystals out of the family of more general sets, which fulfils the requirements for point sets to be uniformly not very close to each other (see the \( r \)-condition below), and relatively dense (the \( R \)-condition below). Sets with these conditions were introduced and studied by B. Delone who called them \((r, R)\)-systems ([2],[3]).

Definition 1.1 (Delone Set). Let \( \mathbb{R}^d \) be an Euclidean space and \( r \) and \( R \) some positive numbers. A set \( X \subset \mathbb{R}^d \), is called a Delone set with parameters \( r \) and \( R \) (or \((r, R)\)-system) if

(i) \((r\text{-condition})\): any open ball \( B(r) \) of radius \( r \) has at most one point from \( X \), and

(ii) \((R\text{-condition})\): any closed ball \( B(R) \) of radius \( R \) has at least one point from \( X \).
Remark 1.1. The definition of a Delone set requires the existence of numbers \( r \) and \( R \) with specified properties. However, for the sake of shortening the theorems’ statements and proofs we included these two parameters into the definition of a Delone set as a characteristic of the set in the assumption that they exist. Even more, we chose \( r \) as the supremum of all numbers such that set \( X \) satisfies \( r \)-condition, and \( R \) as the infimum of the set of all numbers that satisfy \( R \)-condition.

Definition 1.2 (Regular System). A Delone set \( X \) is called a regular system if for any two points \( x \) and \( y \) from \( X \) there is a symmetry \( s \) of \( X \) such that \( s(x) = y \), i.e. if the symmetry group \( \text{Sym}(X) \) acts transitively on \( X \).

Remark 1.2. It follows immediately from definition 1.2 that a regular point set \( X \) is an orbit \( G \cdot x \), where \( x \) is a point from \( X \), and \( G \) is, generally speaking, a subgroup of \( \text{Sym}(X) \). We remind that \( G \)-orbit of \( x \) is the set \( G \cdot x = g(x) | g \in G \).

Definition 1.3 (Discrete Group). A group \( G \subset \text{Iso}(d) \) is called a discrete subgroup, if the orbit \( G \cdot x \) of any point \( x \in \mathbb{R}^d \) is a discrete subset of \( \mathbb{R}^d \).

Definition 1.4 (Fundamental Domain). Let \( G \) be a discrete subgroup of \( \text{Iso}(d) \). We call the closed domain \( F(G) \) in \( \mathbb{R}^d \) a fundamental domain of group \( G \) if:
(i) for any point \( x \) from \( \mathbb{R}^d \), the intersection of \( F(G) \) and the orbit \( G \cdot x \) is not empty;
(ii) for any point \( x \) from \( \mathbb{R}^d \), the interior of \( F(G) \) contains at most one point from \( G \cdot x \).

Remark 1.3. For a discrete group \( G \) a fundamental domain does exist. It suffices to take an orbit \( G \cdot x \) of a non-fixed point \( x \) with respect to \( G \) and construct the Voronoi tessellation for the \( G \cdot x \). The Voronoi domain is a fundamental domain of the group. A fundamental domain can be chosen in a non-unique way, sometimes it can be unbounded.

Definition 1.5. [Crystallographic Group]. Let \( \text{Iso}(d) \) be the complete group of all isometries of Euclidean \( d \)-space \( \mathbb{R}^d \). A subgroup \( G \) of the group \( \text{Iso}(d) \) is called crystallographic if any orbit \( G \cdot x \) is a discrete set, and the fundamental domain of \( G \) is compact.

Important results for crystallographic groups were obtained in [4],[5].

Statement 1.1. A Delone set \( X \) is a regular system if and only if there is a crystallographic group \( G \) such that \( X \) is a \( G \)-orbit of some point \( x \).

E.S. Fedorov defined crystal as a finite union of regular point sets [6].

Definition 1.6. [Crystal]. We say that a subset \( X \) of \( \mathbb{R}^d \) is a crystal if \( X \) is the \( G \)-orbit of a finite set \( X_0 = \{x_1, ..., x_k\} \), i.e. \( X = \bigcup_{i=1}^{k} G \cdot x_i \).

Thus a crystal can be regarded as a union of orbits of several points with respect to the same crystallographic group \( G \).

The main goal of the local theory for crystals is to develop a sound mathematical theory and methodology that would serve as a model of crystalline structure and formation from the pairwise identity of local arrangements around each atom. However, before 1970s, there were neither formal statements that used mathematical language and concepts, nor rigorously proven results in this regard until B. Delone and R. Galulin formulated the problem, and Delone’s students N. Dolbilin, M. Stogrin, and others (see for instance, [7]-[10]) developed a mathematically sound local theory of crystals.
We would like to point out that Delone sets have the following property that plays a significant role in most proofs of the local theory of crystals.

**Statement 1.2.** For a Delone set $X$ and for any two points $x$ and $x' \in X$ there is a finite sequence of points from $X$ $x = x_0, x_1, \ldots, x_m = x'$ such that $|x_{i-1}x_i| \leq 2R$, $i \in [1, m]$.

We call such sequence a $2R$-chain and denote it as $[x, \ldots, x']$. We call each closed interval $[x_{i-1}, x_i]$ a link of the $2R$-chain.

Following the terminology of Definition 2.1, that the next section starts with, we can say that any Delone set is a $2R$-bonded set.

For Delone sets we can even make a stronger statement than statement 1.2. In fact, the following statement that is proved in the next section holds true. For any Delone $(r, R)$-set $X$ there is such $\varepsilon = \varepsilon(r, R) > 0$ that for any two points $x$ and $x' \in X$ there is $(2R - \varepsilon)$-chain $[x, \ldots, x']$ [Statement 2.1].

For some Delone sets the value of $\varepsilon$ is very small, and therefore the length of links is bounded from above by an upper bound close to $2R$. However, there are many crystalline structures, e.g. for zeolites, such that they are presented as $(r, R)$-sets, and any two points of the structure could be connected by a chain with the links which are significantly smaller than $2R$, i.e. the parameter $t$ [Definition 2.1] is significantly smaller than $2R$.

We can also note that in the proofs of theorems in the local theory that use $2R$-linkage of Delone sets, the length of links in a chain that connects the given two points of $X \subset \mathbb{R}^d$ is not essential, but what is essential is that any two points of $X$ could be connected by a chain with the links’ length not greater than the fixed number $t$, ($t$ depends on the set $X$). The size of a local region that determines global properties of the set $X$ also depends on the value of $t$. The lesser the value of $t$, the smaller region could be considered. In this respect, the value of $t = 2R$ in many structures seems to be unnecessary too large, though for some $X$ the value of $t$ can be very close to $2R$.

On the other hand, in the assumption that $X$ is a Delone $(r, R)$-set, if the parameter $t$ is significantly smaller than $2R$, local conditions expressed in of $R$ happen to be not very efficient. We believe that local conditions that determine global properties of a set could be found in terms of $t$, the parameter that shows the lengths of chains’ links that any two points of the set could be connected with.

All these observations inspired us to develop the local theory for the $t$-bonded sets. In this theory we do not assume (unless it is stated as a premise), that the set $X$ under consideration is a Delone set, and therefore these sets do not possess some properties that were used in developing the local theory for Delone sets.

## 2 Definitions and Main Results

As we have already mentioned, in this paper, unless stated differently, we consider subsets (that we denote $X, Y, Z, \ldots$ etc.) of $d$-space $\mathbb{R}^d$ that are uniformly discrete point sets, i.e. $X$ is a set such that there exists $r > 0$ such that any ball $B(r) \subset \mathbb{R}^d$ contains at most one point from $X$ (condition (i) in definition 1.1 of a Delone set). Thus $X$ fulfills just the $r$-condition and, generally speaking, it is not a Delone set. However, in this paper we reserve letter $r$ for the parameter in the $r$-condition and letter $R$ for the parameter for the $R$-condition respectively in the definition of a Delone $(r, R)$-set.
Like for a Delone set we’ll choose \( r \) as the supremum of all numbers such that the set \( X \) satisfies the r-condition, and \( R \) as the infimum of the set of all numbers that satisfy the R-condition.

**Definition 2.1.** [\( t \)-bonded Set]. A set \( X \subset \mathbb{R}^d \) is said to be a t-bonded set in \( \mathbb{R}^d \), or just t-bonded set in \( \mathbb{R}^d \), where \( t \) is some positive number if:

(1) \( \text{aff } X = \mathbb{R}^d \), where \( \text{aff } X \) stands for affine hull of \( X \).

(2) For any two points \( x \) and \( x' \in X \) there is a finite sequence of points from \( X \)

\[ x = x_0, x_1, \ldots, x_m = x' \text{ such that } |x_{i-1}x_i| \leq t, \quad i \in [1, m] \]

We will call the sequence a \( t \)-chain and denote it as \([x, \ldots, x']\). Each closed interval \([x_{i-1}, x_i]\) will be called a link of the \( t \)-chain.

**Statement 2.1.** For any Delone \((r, R)\)-set \( X \) there is such \( \delta = \delta(r, R) > 0 \) that for any two points \( x \) and \( x' \in X \) there is \( t \)-chain \([x, \ldots, x']\) with all links that are no longer than \( 2R - \delta \), i.e. any Delone set is a \( t \)-bonded set, where the parameter \( t \) can be chosen less than \( 2R \).

**Proof.** First of all, we’ll show that due to the \( R \)-property of a Delone set \( X \) for any two points \( x \) and \( x' \in X \) there is a \( 2R \)-chain \([x, \ldots, x']\). Let \( B_z(R) \) denote a ball with radius \( R \) and the center \( z \) such that \( z \in [xx'] \) and \(|xz| = R\). The ball \( B_z(R) \) contains, generally speaking, several points \( y, y_1, \ldots, y_m \) from \( X \) different from \( x \). Each of them fulfills \( 0 < |xy| \leq 2R \) and \(|yx'| < |xx'|\). If the point \( x' \) is among these points \( y, y_1, \ldots \), the required chain is already complete. If among points \( y, y_1, \ldots \) there is no \( x' \), each of them can be chosen as the first point \( x_1 \) of the \( 2R \)-chain \([x, x_1, \ldots, x']\) being constructed. Let us take a ball \( B_{z_2}(R) \) with the radius \( R \) and the center \( z_2 \in [x_1, x'] \). In \( B_{z_2}(R) \) there are again finitely many points such that each of them can be chosen as the second point of \( 2R \)-chain \([x, x_1, x_2, \ldots, x']\) being constructed. Applying this argument again and again we come to the \( 2R \)-chain \([x, x_1, x_2, \ldots, x']\).

Proving the existence of a \((2R - \delta)\)-chain will require additional arguments. First, it is easy to show that if on a circle with a radius no greater than \( R \) there are at least three points \( y_1, y_2, y_3, \ldots \) such that for all the points \(|y_iy_j| \geq 2r\), then any two of these points can be connected by a \( t \)-chain of the points \( y_1, y_2, y_3, \ldots \), where

\[ t \leq \sqrt{4R^2 - r^2} = 2R(1 - \sqrt{1 - (r/2R)^2}). \]

Now we construct a \( t \)-chain \([x, \ldots, x']\) which connects \( x \) to \( x' \in X \). We start with a ‘small’ ball \( B_z(\rho) \), where the center \( z \in [xx'] \) and \(|xz| = \rho\), i.e. \( x \) is on the boundary of the ball. If \( \rho < r \) the ball \( B_z(\rho) \) contains only one point from \( X \). It is the point \( x \). Now we keep shifting the ball’s center \( z \) from the point \( x \) along the segment \([xx']\). At the same time we keep increasing the radius \( \rho \) of the ball so that the boundary of the inflated ball passes through the point \( x \). Due to the \( R \) and \( r \)-conditions of a Delone set at some point in time the inflated ball \( B_z(\rho) \) ‘catches up’ a new point \( x_1 \in X \). It is obvious that the radius of the ball does not exceed \( R \).

We can assume that there are no other points from \( X \) on the boundary \( \partial B_z(\rho) \). Therefore we can continue to inflate the ball and at the same time keep points \( x \) and \( x_1 \) on the ball’s boundary, until the ball is constructed with at least three points from \( X \) on its boundary. Since there are no points from \( X \) inside the ball \( B_z(\rho) \), its radius does not exceed \( R \). Furthermore, any three points on a sphere are necessarily non-collinear. They have to be on a circle which is a plane section of the boundary sphere of the ball. In other words, we have at least three points \( x, x_1 \) and \( x_1 \in X \) on a circle of the radius not greater than \( R \). Due to the above mentioned remark
on three points on a circle in $X$ there is a $t$-chain that connects $x$ to $x_1$. This chain 
$x, . . . , x_1$ can be chosen as the starting fragment of the $[x, . . . , x']$ to be constructed. We emphasize that this fragment is already not monotonic. Indeed, though $|x'x_1| < |x'x|$, other intermediate points of the fragment can be further from $x'$ than points $x$ and $x_1$.

If the point $x_1$ differs from $x'$, we can apply the same argument and construct a new fragment $[x_1, . . . , x_2]$ of the $t$-chain, where $|x_2x' | < |x_1,x'|$. Continuing the process we will construct the $t$-chain $[x . . . , x']$.

It is clear that $t \geq 2r$. For some Delone sets the value of $t$ can be chosen significantly less than $2R$. For instance, if $X = \mathbb{Z}^d$ is the cubic lattice, then $t$ can be chosen as the edge length of the cube and $2R = t\sqrt{d}$.

In the local theory the concept of cluster plays a significant role and there could be different approaches to this concept. In this paper we consider a version of the cluster which was mainly used in the local theory for Delone sets. We note that because the concept of cluster we adopt in this paper is the same for both Delone sets and $t$-sets, the concepts of clusters’ equivalence and cluster’s group of symmetries $S_x(\rho)$ are also the same.

**Definition 2.2.** [Cluster]. Let $\rho > 0$, a $\rho$-cluster $C_x(\rho)$ centered at point $x \in X$ be defined as a set of all points $x' \in X$ such that $|xx'| \leq \rho$, i.e. $C_x(\rho) = X \cap B_x(\rho)$

Local conditions for a set to be regular are normally expressed in terms of $R$. However, in case of $t$-bonded sets, since we do not require the $R$-condition, there are some interesting properties of a set that could not be expressed in terms of $R$. Some properties of a set fail to be true if we try to replace $R$ with other parameters that seem to play a similar role. For example, it is natural that parameter $t$ for a $t$-bonded set has a role similar to that of the parameter $2R$ in a Delone set. However, there is an important difference between $2R$-clusters in case of Delone sets and $t$-clusters in case of $t$-bonded sets. As is known, in a Delone set any $2R$-cluster has its affine hull of full dimension $d$. In a $t$-bonded set $X \subset \mathbb{R}^d$ which is not a Delone set, an analogous statement that the dimension of the affine hull of the $t$-cluster (rank of a $t$-cluster) is also equal to $d$ fails to be true. Moreover, there are Delone sets $X$, and even regular systems, in which parameter $t$ is significantly smaller than $2R$ and the affine hulls of all clusters are two-dimensional, though Delone sets are full-dimensional. We’ll start section 3 with an example of a $t$-bonded set that is a regular system, therefore a Delone set, though the affine hull of a $t$-cluster in not equal to $d$. In the same section we will introduce some conditions that guarantee that a given cluster has rank $d$, i.e. the dimension of the affine hull of the cluster equals $d$ [Theorem 3.1].

The same conditions will guarantee that the rank of a cluster would not increase with the growth of its radius. This fact will play an important role for the proofs in the local theory for $t$-bonded sets.

**Statement 2.2.** Given $t$-bonded set $X$, $\rho > 0$, let $X \setminus C_x(\rho) \neq \emptyset$. Then there is a point $x' \in X$ such that $x' \in C_x(\rho + t) \setminus C_x(\rho)$, and it is linked to the center $x$ by a $t$-chain contained in $C_x(\rho + t)$.

It should be noted that in the spherical layer $B_x(\rho + t) \setminus B_x(\rho)$, generally speaking, there could also be points $x'' \in X$, that are linked to $x$ just by a ‘long’ $t$-chain starting at the center $x$ of the cluster $C_x(\rho + t)$, leaving it, and then coming back to the cluster $C_x(\rho + t)$ to get eventually connected to $x''$. 

5
The concept of cluster was used to develop the local theory for crystals that by definition are Delone sets. Here we would like to mention that in case of $t$-bonded sets, two more, different from the traditional, concepts of cluster might play an interesting role - combinatorial clusters and "mixed"clusters. However, development of the local theory of crystals that uses a different concept of clusters requires a separate discussion that goes well beyond the goals of this paper.

In section 4 we study the structure of cluster’s group symmetries that plays a pivotal role in the local theory of $t$-sets (and crystals). At the end of the section the $t$-extension theorem is proved, that gives sufficient conditions to extend $\rho_0$-cluster isometry to the $(\rho_0 + t)$-cluster isometry.

**Definition 2.4.** [Cluster Equivalency]. Given $t$-bonded set $X$ in $\mathbb{R}^d$, $\rho > 0$, two points $x \in X$ and $x' \in X$, we say that the $\rho$-cluster $C_x(\rho)$ is equivalent to the $\rho$-cluster $C_{x'}(\rho)$, if there is a space isometry $g$ of $\mathbb{R}^d$, such that $g(x) = x'$ and $g(C_x(\rho)) = C_{x'}(\rho)$.

In section 5 and 6 we prove two theorems [Theorem 5.1 and Theorem 6.1] for $t$-sets that are similar to the Criterion for Regular (Delone) Systems (see, e.g. [7], [11], [12], [13]) and Criterion for Crystals (see, e.g. [13], [14], [15]). Though the statements of the theorems are almost identical for both Delone sets and $t$-sets the main challenge of the proofs is related to the rank of the clusters, which as we have already mentioned is $d$ for 2R-clusters in Delone sets, however, in case of $t$-sets it may not be equal to $d$ for $\rho$-clusters when $\rho$ is equal to $t$. The cluster’s rank naturally affects the structure of the group $S_x(\rho)$ of the cluster’s symmetries. The statements of both theorems, as well as their proofs depend on the concept of cluster counting function that we define below.

It is clear that with a given $\rho > 0$, the relation of clusters to be equivalent as defined above, is an equivalence relation on a set of all $\rho$-clusters in $X$. Therefore, the set of all $\rho$-clusters in $X$ could be presented as a disjoint union of equivalence classes.

**Definition 2.5.** [Cluster Counting Function] For a given $t$-bonded set $X$ in $\mathbb{R}^d$ and $\rho > 0$, the cluster counting function $N(\rho)$ is defined as the cardinal number of the set of equivalence classes of $\rho$-clusters in $X$ provided the cardinal number of equivalence clusters is finite.

**Definition 2.6.** [Set of Finite Type]. Set $X$ is said to be of finite type if the cardinal number $N(\rho)$ is finite for any $\rho > 0$.

**Statement 2.3.** For a set $X$ of finite type the cluster counting function $N(\rho)$ is defined and finite for any $\rho \geq 0$; it is a positive, piecewise constant, integer valued, monotonically non-decreasing, and continuous from the left function.

Statement 2.3 is true for both Delone sets and $t$-sets. In section 7 we discuss $t$-sets of finite type, and we show how different Delone sets and $t$-sets are, as far as the property "to be a set of finite type"is concerned.

In section 7 we also included the proof of an anecdotal fact [Statement 7.1] that in case of Delone sets there exists a local condition of a Delone set $X$ to be a set of finite type. If for a Delone set $X$ the counting function $N(\rho)$ is finite for $\rho = 2R$, then it is finite for all $\rho > 0$. In case of $t$-bonded sets the situation is quite different. We will prove the following theorem.

**Theorem 7.1.** Given $t > 0$, for an arbitrarily large $k > 0$ there is a $t$-bonded set $X \subset \mathbb{R}^2$ such that $N(kt) < \infty$, but for any $\varepsilon > 0$ $N(kt + \varepsilon) = \infty$.

An example of such $t$-bonded set will be presented in the proof of theorem 7.1.
3 The Rank of a Cluster

In this section we will discuss the rank of a cluster, i.e. the dimension of the affine hull of a cluster. As we have already stated in the previous section, given a point set \( X \subseteq \mathbb{R}^d \), there is an important difference between \( 2R \)-clusters in case of Delone sets and \( t \)-clusters in case of \( t \)-bonded sets. As is well-known for Delone sets, a \( 2R \)-cluster for any point \( x \in X \) has the rank equal to \( d \). Though \( 2R \) for Delone sets and \( t \) for \( t \)-bonded sets play similar roles in local theories for Delone sets and \( t \)-bonded sets respectively, as is shown below, an analogous to the previous statement for \( t \)-bonded sets fails to be true.

**Statement 3.1.** There are Delone sets \( X \) of rank 3 and even regular systems in which the parameter \( t \) is significantly smaller that \( 2R \), and all \( t \)-clusters have rank 2, although Delone sets have rank 3.

**Example.** Let \( \Lambda \subset \mathbb{R}^3 \) be a lattice of rank 3 constructed on the orthonormal basis, and \( X := \Lambda \setminus (2\Lambda + (1,1,1)) \). Then \( X \) is a Delone set with parameters \( r = 1/2 \) and \( R = 1 \). Since all \( 2R \)-clusters in \( X \) are centrally symmetrical and pairwise equivalent, then \( X \) is a regular system ([15], [16],[17]). On the other hand, \( X \) is a \( t \)-bonded set where \( t = 1 < 2 = 2R \). Each \( t \)-cluster \( C_x(1) \) is a cross of rank 2. However, the set \( X \) is a regular system of rank 3.

Though the situation with clusters’ ranks in case of Delone sets and \( t \)-bonded sets is different, the following statements (lemma 3.1, theorems 3.1 and 3.2) on ranks of \( \rho \)-clusters hold true. To shorten the notation, we’ll use \( d_x(\rho) := \dim(\text{aff}C_x(\rho)) \) for the rank of the cluster \( C_x(\rho) \).

In all discussions below \( \Pi^n \) stands for the \( n \)-dimensional plane that is the affine hull of a \( t \)-bonded set \( X \), \( n \leq d \).

**Lemma 3.1.** Let \( X \subset \mathbb{R}^d \) be a \( t \)-bonded set, \( \rho \) a positive real number, and \( x, x' \) two points from \( X \) such that \( |xx'| \leq t \), and the following conditions hold true,

\[
d_x(\rho) = d_x(\rho + t) \quad \text{and} \quad d_{x'}(\rho) = d_{x'}(\rho + t) \tag{3.1}
\]

Then \( \text{aff}C_{x'}(\rho) = \text{aff}C_{x'}(\rho + t) = \text{aff}C_x(\rho + t) = \text{aff}C_x(\rho + t) \).

**Proof.** Since \( |xx'| \leq t \), it follows that \( C_x(\rho) \subset C_{x'}(\rho + t) \) and \( C_{x'}(\rho) \subset C_x(\rho + t) \). Hence, \( \text{aff}C_x(\rho) \subset \text{aff}C_{x'}(\rho + t) \) and \( \text{aff}C_{x'}(\rho) \subset \text{aff}C_x(\rho + t) \). From the premises of the lemma \( d_x(\rho) = d_x(\rho + t) \) and \( d_{x'}(\rho) = d_{x'}(\rho + t) \), it follows that \( \text{aff}C_x(\rho + t) = \text{aff}C_{x'}(\rho + t) \) and \( \text{aff}C_{x'}(\rho) = \text{aff}C_x(\rho + t) \). Therefore, \( \text{aff}C_x(\rho) = \text{aff}C_{x'}(\rho + t) = \text{aff}C_x(\rho + t) = \text{aff}C_{x'}(\rho + t) \). \( \square \)

**Theorem 3.1.** Let \( X \subset \mathbb{R}^d \) be a \( t \)-bonded set, and there is some \( \rho > 0 \) such that for any point \( x \) from \( X \), the following condition holds true,

\[
d_x(\rho) = d_x(\rho + t) = n(x) \tag{3.2}
\]

Then \( n(x) \equiv d \) and \( \forall x \in X \text{ aff}C_x(\rho) = \text{aff}(X) = \mathbb{R}^d \).

**Proof.** Let us take a point \( x_0 \in X \) and denote \( \text{aff}C_{x_0}(\rho) := \Pi^n \) where \( n = d_{x_0}(\rho) \). We’ll show that any point \( x \in X \) belongs to \( \Pi^n \), and therefore \( \Pi^n = \mathbb{R}^d \).

Let \([x_0, x_1...x_i...x_n = x]\) be a \( t \)-chain that connects \( x_0 \) and \( x \). By the \( t \)-bonded set definition for any points \( x \) and \( x' \) there exists a \( t \)-chain that connects \( x \) and \( x' \). Since
for any $i$, $0 \leq i \leq n - 1$ the distance $|x_i x_{i+1}| \leq t$, if follows from lemma3.1, that for any $i$, $0 \leq i \leq n - 1$, $\text{aff} C_{x_i}(\rho) = \text{aff} C_{x_{i+1}}(\rho)$. Therefore, $\text{aff} C_{x_0}(\rho) = \text{aff} C_{x}(\rho)$. We proved that any point $x \in X$, belongs to $\text{aff} C_{x_0}(\rho)$. Meanwhile, $\text{aff} C_{x_0}(\rho) \subset \text{aff} X$, and therefore $\text{aff} C_{x_i}(\rho) = \text{aff}(X) = \mathbb{R}^d$ for any $x_0 \in X$. \(\square\)

**Theorem 3.2.** Let $X \subset \mathbb{R}^d$ be a t-bonded set, such that for every given $\rho \leq t \cdot (d - 1)$ the ranks of all $\rho$-clusters are equal ($d_\rho(x) = d(\rho)$, $\forall x \in X$). Then, for any $\rho' \geq d \cdot t$ and any $x \in X$, the rank $d_\rho(x') = d$.

**Proof.** Let us consider two cases $d(t) = 1$ and $d(t) \geq 2$.

Case $d(t) = 1$. $C_x(t) \subset L_0$. We’ll show that $X \subset L$, i.e. any $x' \in X$ is on the $L_0$. In fact, let us connect $x$ to $x'$ by a $t$-chain ($x = x_0, \ldots, x_k = x'$). Let $L_i$ be a line of cluster $C_x(t)$ for $i \in [0, k]$. Since for any $i \in [1, k]$ $C_{x_{i-1}}(t) \cap C_{x_i}(t) \supset \{x_{i-1}, x_i\}$, it implies that the lines $L_{i-1}$ and $L_i$ of neighboring clusters are passing through both these points: $x_{i-1}, x_i \in L_{i-1}$ and $x_{i-1}, x_i \in L_i$, $i \in [1, k]$. Hence, $L_{i-1} = L_i$. Since the last identity is true for all $i \in [1, k]$, we conclude that all lines $L_i$ coincide with the line $L_0$. Therefore, $L_0 = L_1 = \cdots = L_k \equiv x'$. We showed that any point $x' \in L_0$.

We proved that $d(t) = 1$, implies $d(\rho') = 1$ for all $\rho' \geq d \cdot t$. This concludes the proof of the case $d(t) = 1$.

Let us assume now that $d(t) \geq 2$, and consider the function $d(\rho)$ defined at $[1, d \cdot t]$ and its values at points $kt$: $d(t) = 2 \leq d(2t) \leq d(3t) \leq \cdots \leq d(d \cdot t)$. If all these inequalities are strong, then $d(d \cdot t) = d + 1$ which is impossible since $X \subset \mathbb{R}^d$. Therefore, there is $k$, $1 \leq k \leq d - 1$ such that $d(k \cdot t) = d((k + 1) \cdot t)$. Hence, by Theorem 3.1, $d(k \cdot t) = d((k + 1) \cdot t) = d(\rho') = d$ for all $\rho' \geq d \cdot t$. \(\square\)

**Remark.** We proved that under the conditions of Theorem 3.2 stabilization of the rank of any cluster definitely occurs when $\rho \geq d \cdot t$. However, for some sets it might occur even if $\rho \leq d \cdot t$.

4 Symmetry of Clusters

In this paragraph we again assume that $X$ is a $t$-set in $\mathbb{R}^d$ which by definition implies $\text{aff} X = \mathbb{R}^d$. Let us denote by $O(x, d)$ a group of all isometries of space $\mathbb{R}^d$ which leave $x \in \mathbb{R}^d$ fixed.

**Definition 4.1.** [The Symmetry of a Cluster.] Assume $x \in X$, then an isometry $\tau \in O(x, d)$ is called a symmetry of the cluster $C_x(\rho)$ if $\tau(C_x(\rho)) = C_x(\rho)$.

We want to emphasize that since $\tau \in O(x, d)$, any symmetry $\tau$ of a cluster leaves its center $x$ fixed. We denote by $S_x(\rho)$ a group of all symmetries $\tau$ of the cluster $C_x(\rho)$.

Now let $\text{aff} C_x(\rho) = \Pi_x^n$ where $\Pi_x^n$ is an $n$-dimensional affine plane, $x \in \Pi_x^n$, and $n \leq d$. We denote by $\mathcal{S}_x(\rho)$ a group of all isometries from $O(x, n)$ that leave invariant the plane $\Pi_x^n$ and the cluster $C_x(\rho)$.

If $n = d$, then $\mathcal{S}_x(\rho) = S_x(\rho)$. Let $n < d$, then we denote the affine hull of $C_x(\rho)$ by $\Pi_x^n$, and the complementary orthogonal $(d - n)$-plane passing through $x$ by $Q_x^{d-n}$. Let $s \in S_x(\rho)$ be a symmetry of the $\rho$-cluster $C_x(\rho)$. It is clear that any such symmetry is an orthogonal transformation of the $d$-space which is a product of the transformation $s \in \Pi_x^n$ and of an arbitrary transformation $g \in O(x, d - n)$ of the complementary plane. The following lemma summarizes some facts on the cluster group.
Lemma 4.1. The following statements hold true:

(1) If \( \text{aff}C_x(\rho) = \Pi^0_x \) and \( n < d \), then \( S_x(\rho) = \overline{S}_x(\rho) \oplus O(x, d - n) \), where \( \overline{S}_x(\rho) \subset O(x, n), \) and \( O(x, d - n) \) is the full group of isometries of the plane \( \mathbb{R}^d \) complementary to \( \Pi_x^n \) and passing through the point \( x \).

(2) The group \( \overline{S}_x(\rho) \) is a finite subgroup of \( O(x, n) \). Particularly, if \( \text{aff}C_x(\rho) = \mathbb{R}^d \), then group \( S_x(\rho) = \overline{S}_x(\rho) \) is a finite subgroup of \( O(x, d) \);

(3) The group \( S_x(\rho) \) is finite if and only if \( \text{aff}C_x(\rho) = \mathbb{R}^d \) or \( \text{aff}C_x(\rho) = \mathbb{R}^{d-1} \).

Proof. (1) Any symmetry \( \tau \) from \( S_x(\rho) \) can be represented as a product of two isometries \( f \cdot g \) where \( f \in \overline{S}_x(\rho) \) and \( g \in O(x, d - n) \) is an arbitrary isometry that operates in the \((d - n)\)-plane \( \Pi_x^{d-n} \), complementing to the plane \( \Pi_x^n \), and leaves the center \( x \) fixed. At the same time, the product of any symmetry from the group \( \overline{S}_x(\rho) \) and any symmetry \( g \in O(x, d - n) \), is a symmetry from \( S_x(\rho) \). Therefore \( S_x(\rho) = \overline{S}_x(\rho) \oplus O(x, d - n) \).

(2) Let us prove first that if \( \text{aff}C_x(\rho) = \Pi^0_x \) and \( n < d \), then \( \overline{S}_x(\rho) \) is a finite subgroup of \( O(x, n) \). According to (1), \( S_x(\rho) = \overline{S}_x(\rho) \oplus O(x, d - n) \) where \( \overline{S}_x(\rho) \) is a group of all symmetries \( f \in O(x, n) \) that operates on \( C_x(\rho) \) as a subset of \( \Pi_x^n \). Since \( C_x(\rho) \) is a finite set and any point in \( \mathbb{R}^n \) is an affine combination of points from \( C_x(\rho) \), we conclude that any \( \tau \) from \( \overline{S}_x(\rho) \) is completely determined by its values on the points \( x \) from the finite set \( C_x(\rho) \). Therefore, \( \overline{S}_x(\rho) \) is a finite subgroup of \( O(x, n) \). If \( n = d \) then \( \overline{S}_x(\rho) = S_x(\rho) \), hence, \( S_x(\rho) = \overline{S}_x(\rho) \) is a finite subgroup of \( O(x, d) \).

(3) It follows immediately from the second part of the lemma that the condition \( \text{aff}C_x(\rho) = \mathbb{R}^d \) is sufficient for the group \( S_x(\rho) \) to be finite. It follows from (1) that \( n < d \) implies then \( S_x(\rho) = \overline{S}_x(\rho) \oplus O(x, d - n) \). If \( d - n > 1 \), then \( O(x, d - n) \) is an infinite group, and therefore \( S_x(\rho) \) is infinite. If \( d - n = 1 \), then \( O(x, d - n) \) is a finite group and \( S_x(\rho) = \overline{S}_x(\rho) \oplus O(x, d - n) \) is a finite group as a product of two finite groups. Hence, the condition \( \text{aff}C_x(\rho) = \mathbb{R}^d \) or \( \text{aff}C_x(\rho) = \mathbb{R}^{d-1} \) is also necessary for the group \( S_x(\rho) \) to be finite.

Lemma 4.2. Assume \( \overline{S}_x(\rho_0) \) and \( \overline{S}_x(\rho_0 + t) \) are finite groups as defined at the beginning of the paragraph for clusters \( C_x(\rho_0) \) and \( C_x(\rho_0 + t) \) respectively. The following statements hold true.

(1) If \( S_x(\rho_0 + t) = S_x(\rho_0) \), then \( \text{aff}C_x(\rho_0) = \text{aff}C_x(\rho_0 + t) = \Pi^0_x \), \( n \leq d \).

(2) The equality \( S_x(\rho_0) = S_x(\rho_0 + t) \) is equivalent to \( \overline{S}_x(\rho_0) = \overline{S}_x(\rho_0 + t) \).

Proof. (1) Let us prove the first statement by contradiction. Assume that \( \text{aff}C_x(\rho_0) = \Pi^{n_1}_x \subset \text{aff}C_x(\rho_0 + t) = \Pi^{n_2}_x \), \( n_1 < n_2 \). The group \( \overline{S}_x(\rho_0 + t) \) of symmetries operates in \( \Pi^{n_2}_x \) where \( n_2 > n_1 \). It is also a finite group, and the restriction of any isometry \( f' \in \overline{S}_x(\rho_0 + t) \) onto plane \( \Pi^{n_1}_x \) is an isometry from \( \overline{S}_x(\rho_0) \). Without loosing generality we can assume that \( n_2 \) is equal to \( d \), i.e. \( S_x(\rho_0 + t) = \overline{S}_x(\rho_0 + t) \). It follows from lemma 4.1 (4) that \( S_x(\rho_0 + t) \) is a finite group, and \( S_x(\rho_0) \) is an infinite group. Hence, \( S_x(\rho_0 + t) \) is not equal (proper subgroup) of \( S_x(\rho_0) \). That is a contradiction with the premise of the lemma \( S_x(\rho_0 + t) = S_x(\rho_0) \).

(2) Assume that \( S_x(\rho_0) = S_x(\rho_0 + t) \). It follows from the first part of the lemma that both of these groups operate in the same plane \( \text{aff}C_x(\rho_0) = \Pi^n_x = \text{aff}C_x(\rho_0 + t) = \Pi^n_x \). From the lemma 4.1 it follows \( S_x(\rho_0) = \overline{S}_x(\rho_0) \oplus O(x, d - n) \) and \( S_x(\rho_0 + t) = \overline{S}_x(\rho_0 + t) \oplus O(x, d - n) \). Since \( S_x(\rho_0) = S_x(\rho_0 + t) \), we conclude that \( \overline{S}_x(\rho_0) = \overline{S}_x(\rho_0 + t) \).
Assume now that $S_x(\rho_0) = S_x(\rho_0 + t)$. First of all, it means that they operate in the same planes. Since by lemma 4.1, $S_x(\rho) = S_x(\rho_0) \bigoplus O(x, d - n)$ and $S_x(\rho_0 + t) = S_x(\rho_0 + t) \bigoplus O(x, d - n)$, we conclude that $S_x(\rho_0) = S_x(\rho_0 + t)$. □

Let us remind that according to Definition 2.4 given a $t$-bonded set $X$ in $\mathbb{R}^d$ and $\rho > 0$, the $\rho$-cluster $C_x(\rho)$ is equivalent to the $\rho$-cluster $C_{x'}(\rho)$, if there is a space isometry $g$ of $\mathbb{R}^d$, such that $g(x) = x'$ and $g(C_x(\rho)) = C_{x'}(\rho)$.

**Statement 4.1.** Given $t$-set $X \subset \mathbb{R}^d$ and $\rho_0 > 0$, if clusters $C_x(\rho_0)$ and $C_{x'}(\rho_0)$ are equivalent, then groups $S_x(\rho_0)$ and $S_{x'}(\rho_0)$ are conjugate.

**Proof.** Since $C_x(\rho)$ and $C_{x'}(\rho)$, i.e. there is an isometry $g$ of $\mathbb{R}^d$, such that $g(x) = x'$ and $g(C_x(\rho)) = C_{x'}(\rho)$. Let $s \in S_x(\rho)$, then $g \circ s \circ g^{-1}$ maps $C_{x'}(\rho_0)$ onto $C_{x'}(\rho_0)$, i.e. $g \circ s \circ g^{-1} \in S_{x'}(\rho_0)$. We proved that $S_{x'}(\rho_0) = gS_x(\rho_0)g^{-1}$.

**Statement 4.2.** Let $X$ be a $t$-bonded set in $\mathbb{R}^d$, and there is a point $x \in X$ and $\rho_0 > 0$ such that $S_x(\rho_0) = S_x(\rho_0 + t)$. If the cluster $C_x(\rho_0 + t)$ is equivalent to a centered at some point $x'$ cluster $C_{x'}(\rho_0 + t)$, then $S_x(\rho_0) = S_{x'}(\rho_0 + t)$.

**Proof.** Since $C_x(\rho_0 + t)$ is equivalent to $C_{x'}(\rho_0 + t)$, there is an isometry $g$ of $\mathbb{R}^d$ such that $g(x) = x'$ and $g(C_x(\rho_0 + t)) = C_{x'}(\rho_0 + t)$. It follows from the previous statement and the assumption $S_x(\rho_0) = S_x(\rho_0 + t)$, that $S_{x'}(\rho_0 + t) = g \circ S_x(\rho_0 + t) \circ g^{-1} = g \circ S_x(\rho_0) \circ g^{-1} = S_{x'}(\rho_0)$.

Now we'll prove a technical theorem which plays an important role in proving local theorems.

**Theorem 4.1** (t-extension Theorem). Let in the $t$-bonded set $X$ for two points $x$ and $x' \in X$ and some $\rho_0 > 0$, clusters $C_x(\rho_0 + t)$ and $C_{x'}(\rho_0 + t)$ are equivalent, and the groups $S_x(\rho_0)$ and $S_{x'}(\rho_0 + t)$ coincide:

$$S_x(\rho_0) = S_{x'}(\rho_0 + t). \quad (4.1)$$

Then any isometry $g \in Iso(\mathbb{R}^d)$ such that $g(x) = x'$ and that maps $C_x(\rho_0)$ onto $C_{x'}(\rho_0)$ (i.e. $g(C_x(\rho_0)) = C_{x'}(\rho_0)$) also maps $(\rho_0 + t)$-cluster $C_x(\rho_0 + t)$ onto $C_{x'}(\rho_0 + t)$ (i.e $g(C_x(\rho_0 + t)) = C_{x'}(\rho_0 + t)$).

**Proof.** By the assumption of the theorem, clusters $C_x(\rho_0 + t)$ and $C_{x'}(\rho_0 + t)$ are equivalent. Therefore there is an isometry $g \in Iso(\mathbb{R}^d)$ such that $g(x) = x'$ and $g(C_x(\rho_0 + t)) = C_{x'}(\rho_0 + t)$.

Let $f$ be an arbitrary isometry that maps $\rho_0$-cluster $C_x(\rho_0)$ onto $\rho_0$-cluster $C_{x'}(\rho_0)$. Let us take the composition $f^{-1} \circ g$. Then we have

$$(f^{-1} \circ g)(C_x(\rho_0)) = f^{-1}(g(C_x(\rho_0))) = f^{-1}(C_{x'}(\rho_0)) = C_{x'}(\rho_0). \quad (4.2)$$

From (4.2) it follows that $f^{-1} \circ g \in S_x(\rho_0)$. Hence, by condition (4.1) of Theorem 4.1 $f^{-1} \circ g \in S_x(\rho_0) = S_x(\rho_0 + t)$. Let us put $f^{-1} \circ g := s$, $s \in S_x(\rho_0 + t)$, Thus, $f = g \circ s^{-1}$. Since $g$ maps $C_x(\rho_0 + t)$ onto $C_{x'}(\rho_0 + t)$ and $s^{-1}$ maps $C_x(\rho_0 + t)$ onto $C_{x}x(\rho_0 + t)$, we conclude that $f$ maps $C_x(\rho_0 + t)$ onto $C_{x'}(\rho_0 + t)$. □

### 5 Criterion for Regular t-bonded Systems

Let us remind that by Definition 2.5 a cluster counting function $N_t(\rho)$ is equal to the cardinal number of equivalence classes of clusters with radius $\rho$ provided the cardinal number is finite.
Definition 5.1 (Regular t-bonded System). A t-bonded set $X$ is called a regular t-system if for any two points $x$ and $y$ from $X$ there is a symmetry $g$ of $X$ such that $g(x) = y$, i.e. if the symmetry group $\text{Sym}(X)$ acts transitively on $X$.

Theorem 5.1. Given t-set $X$ in $\mathbb{R}^d$, assume that there is $\rho_0$ such that the following two conditions hold:

1. $N(\rho_0 + t) = 1$;
2. for some point $x_0 \in X$ $S_{x_0}(\rho_0) = S_{x_0}(\rho_0 + t)$.

Then:

1. Group $G \subset \text{Iso}(\mathbb{R}^d)$ of all symmetries of $X$ operates on $X$ transitively.
2. For any point $x \in X$ $\text{aff}(C_x(\rho_0)) = \text{aff}(C_x(\rho_0 + t)) = \text{aff}X = \mathbb{R}^d$.

Proof. First of all, note that because of Statement 4.2 and condition (1) of the theorem (any two $(\rho_0 + t)$-clusters are equivalent), condition (2) of the theorem holds true not only for the point $x_0$, but for any point $x$ in the set $X$ $(S_x(\rho_0) = S_x(\rho_0 + t))$.

Let us prove that the subgroup $G \subset \text{Iso}(\mathbb{R}^d)$ of all symmetries of $X$ operates on $X$ transitively.

By condition (1) of the theorem for any two points $x$ and $x'$ from $X$, there exists $g \in \text{Iso}(\mathbb{R}^d)$ such that $g$ maps $C_x(\rho_0 + t)$ onto $C_{x'}(\rho_0 + t)$ and $g(x) = g(x')$. We’ll prove that $g$ maps $X$ onto $X$.

Let us take an arbitrary point $z \in X$ and connect $x$ to $z$ by a $t$-chain $x = x_0, x_1, \ldots, x_n, z$. We will show that $g$-images of all points of the chain starting with $x_1$ and ending with $x_n = z$ belong to $X$.

Since $|x_1x| \leq t$, it follows that $C_{x_1}(\rho_0) \subset C_x(\rho_0 + t)$ and $g(C_{x_1}(\rho_0)) = C_{y_1}(\rho_0)$ where $y_1 = g(x_1) \in C_{y_1}(\rho_0 + t) \subset X$. By the Theorem 4.1 $g(C_{x_1}(\rho_0 + t)) = C_y(\rho_0 + t)$

Hence we proved that $g(C_{x_1}(\rho_0 + t)) = C_{y_1}(\rho_0 + t)$ and $g(x_1) = y_1 \in X$. Since the distance $|x_i, x_{i-1}| \leq t$ for all $i$ such that $1 \leq i \leq n - 1$, applying the same argument to points $x_i$ and $x_{i+1}$ as we applied to $x_0$ and $x_1$, we prove that for all non negative integers $i \leq n - 1$, $g(x_{i+1}) = y_{i+1} \in X$ and $g(C_{x_{i+1}}(\rho_0 + t)) = C_{y_{i+1}}(\rho_0 + t)$.

Hence, $g(z) = g(x_n) = y_n \in X$, and $g(X) \subset X$.

To show that $g$ is a surjection we note that the inverse isometry $g^{-1}$ maps $x'$ onto $x$ and $C_{x'}(\rho)$ onto $C_x(\rho_0)$. Applying the same argument to $g^{-1}$ as we applied to $g$ we show that $g^{-1}$ maps $X$ into $X$. Therefore, for any $y \in X$, $g^{-1}y \in X$. Hence, $g$ is a surjection.

As we already mentioned, $S_x(\rho_0) = S_x(\rho_0 + t)$ for any point $x \in X$. Therefore, by lemma 4.2 (part 1) $\text{aff}C_x(\rho) = \Pi_x^a = \text{aff}C_x(\rho_0 + t) = \Pi_x^a$, i.e for every $x \in X$ the following condition holds

$$d_x(\rho) = d_x(\rho + t) = n(x).$$

Then, by Theorem 4.1 $\forall x \in X n(x) \equiv d$, therefore $\text{aff}C_x(\rho_0) = \text{aff}C_x(\rho_0 + t) = \text{aff}X = \mathbb{R}^d$.

6 Multi-regular t-bonded Systems: Criterion

Definition 6.1. A t-bonded set $X \subset \mathbb{R}^d$ is a multi-regular t-bonded system if there is a finite set $X_0 = \{x_1, ..., x_m\}$ such that

$$X = \cup_{i=1}^k \text{Sym}(X) \cdot x_i$$
This definition is analogous to that of a crystal [Definition 1.6]. However, the situation is quite different in some respects. For instance, in case of a crystal we deal with Delone sets which are always infinite sets. Therefore the requirement to represent a Delone set as a disjoint union of a finite number of regular sets determines the selection of the subclass of Delone sets, called crystals. In case of $t$ sets any finite set $X$ is a $t$-bonded set for some value $t$ and can be thought as a multi-regular system: $X = \bigcup_{x_i \in X_0} \text{Sym}(X) \cdot x_i$, where $X_0 = X$ and $\text{Sym}(X)$ is a trivial group. Nevertheless, the following question makes sense in any case (finite or infinite) for $t$-bonded sets.

Let us call a $t$-bonded set an $m$-regular $t$-bonded system if the number of classes in $X/\text{Sym}(X) = m$. Are there conditions which guarantee that a $t$-bonded set $X$ is an $m$-regular system? The following criterion answers the question.

**Theorem 6.1** (Local Criterion for $m$-regular $t$-systems). A $t$-bonded set $X \subset \mathbb{R}^d$ is an $m$-regular $t$-system if and only if there is some $\rho_0 > 0$ such that two conditions hold:

1) $N(\rho_0) = N(\rho_0 + t) = m$;
2) $S_x(\rho_0) = S_x(\rho_0 + t)$, $\forall x \in X$.

**Proof.** We preceede the formal proof of the theorem with several remarks and lemma 6.1, which from our point of view, is not only technical, but also has its own value. The idea of the proof is similar to that of an analogous criterion for a crystal (see, e.g. [13], [14], [15]). Though in case of the $t$-set on order to prove this criterion we do not need to prove that the group $\text{Sym}(X)$ is a crystallographic group.

**Remark 1.** The local criterion for regular systems [Theorem 5.1] is a particular case of Theorem 6.1. Indeed, the condition $N(\rho_0) = 1$ implies $N(\rho_0) = N(\rho_0 + t) = 1$.

**Remark 2.** Condition 1) of Theorem 6.1 means that with the increasing radius $\rho$, the number of cluster classes on segment $[\rho_0, \rho_0 + t]$ does not increase, i.e. remains unchanged: $N(\rho) = N(\rho + t)$. In addition, due to the condition 2), the cluster group $S_x(\rho_0)$, $\forall x \in X$, does not get smaller under the $t$-extension of $\rho_0$-cluster: $S_x(\rho_0) = S_x(\rho_0 + t)$.

The stabilization of these two parameters (the number of cluster classes and the order of cluster groups) on segment $[\rho_0, \rho_0 + t]$ implies their stabilization on the half-line $[\rho_0, \infty)$.

**Remark 3.** The set $X$ can be represented as a disjoint union $X = \bigcup_{i=1}^m X_i$, of not empty subsets $X_i$, where $X_i$ is a set of all points of $X$ that are centers of equivalent $\rho_0$-clusters; i.e. $x$ and $x'$ belong to the same $X_i$ if and only if there is an isometry $g \in \text{Iso}(d)$ that maps $C_x(\rho_0)$ onto $C_{x'}(\rho_0)$. With a given $\rho_0$, we will call two points that belong to the same $X_i$ $\rho_0$-equivalent points. It is clear that in a general situation (without any requirements like condition 1) in the theorem’s statement, the representation of $X$ as a disjoint union of subsets $X_i$ is finer for $(\rho_0 + t)$-equivalent classes than for $(\rho_0)$-equivalent classes. However, condition 1) of the theorem means that these two representations of $X$ as unions of $\rho_0$-equivalent and $(\rho_0 + t)$-equivalent subsets are the same, i.e $x$ and $x'$ are $\rho_0$-equivalent if and only if $x$ and $x'$ are $(\rho_0 + t)$-equivalent. We should also note that without any conditions if $x$ and $x'$ are $(\rho_0 + t)$-equivalent, then these points are $\rho_0$-equivalent. It means that in the previous statement written as "if and only if" statement, condition 1) of the theorem actually guarantees that $\rho_0$-equivalence of two points implies $(\rho_0 + t)$-equivalence.
Remark 4. Without losing generality, the condition 2) of the theorem could be required not for all points in $X$, but rather for a finite number of points $X_0 = \{x_1, \ldots, x_n\}$ such that $x_i \in X_i$. By statement 4.2, since all points in each $X_i$ are $(\rho_0 + t)$-equivalent, $S_x(\rho_0) = S_x(\rho_0 + t)$ for some $x_i \in X_i$ implies that $S_x(\rho_0) = S_x(\rho_0 + t), \forall x \in X_i$.

Remark 5. It follows from the $t$-extension theorem [Theorem 4.1] that if $x$ and $x'$ belong to $X_i$, then any isometry $g \in Iso(\mathbb{R}^d)$ that maps $C_x(\rho_0)$ onto $C_{x'}$ (i.e. $g(C_x(\rho_0)) = C_{x'}(\rho_0))$ and $x$ onto $x'$ ($g(x) = x'$) also maps $(\rho_0 + t)$-cluster $C_x(\rho_0 + t)$ onto $C_{x'}(\rho_0 + t)$ (i.e $C_x(\rho_0 + t) = C_{x'}(\rho_0 + t)$).

Lemma 6.1. Let a $t$-bonded set $X$ fulfill conditions 1) and 2) of the theorem and $X_i$ a subset of $X$ of all $\rho_0$-equivalent points from $X$, $i \in [1, m]$. If $G_i$ is a group generated by all isometries $f$ such that $f(x) = x$ and $f(C_x(\rho_0)) = C_{x'}(\rho_0)$, $\forall x, x' \in X_i$ then:

1) $G_i$ operates transitively on every set $X_j$, $\forall j \in [1, m]$.
2) The group $G_i$ does not depend on $i$, $G_i = \text{Sym}(X)$.

Proof. Since for any $i$, $X_i$ is not empty, for any two points $x, x' \in X_i$ there is an isometry $g$ that maps $C_x(\rho_0)$ onto $C_{x'}(\rho_0)$ and $x$ onto $x'$. Therefore for any $i$, $G_i$ is not empty. Because of the way we defined $X_i$, at least one isometry exists in $G_i$ though it could be more than one.

To prove that for any point $z \in X$, $g(z) \in X$ we can apply the same method that was used to prove Theorem 5.1, though due to the fact that unlike the conditions of Theorem 5.1, not all points in the set $X$ are $(\rho_0 + t)$-equivalent, and therefore we must be sure that the $t$-extension theorem (remark 5) is applicable to the situation under consideration.

Let us take an arbitrary point $z \in X$ and connect $x$ to $z$ by a $t$-chain $x = x_0, x_1, \ldots, x_n = z$. We will show that $g$-images of all points in the chain starting with $x_1$ and ending with $x_n = z$ belong to $X$.

Since $|x_1| \leq t$, it follows that $C_{y_1}(\rho_0) \subset C_{x}(|\rho_0 + t|$ and $g(C_{y_1}(\rho_0)) = C_{y_1}(\rho_0)$ where $y_1 = g(x_1) \in C_{y_1}(\rho_0) \subset X$. Since $g(C_{x_1}(\rho_0)) = C_{y_1}(\rho_0)$ and $y_1 = g(x_1)$, it follows that $x_1$ and $y_1$ belong to the same set $X$. Therefore it follows from Theorem 4.1 (see also remark 5) that $g(C_{y_1}(\rho_0 + t)) = C_{y_1}(\rho_0 + t)$.

Hence, we proved that $g(C_{x}(\rho_0 + t)) = C_{y_1}(\rho_0 + t)$ and $g(x_i) = y_i \in X_j \subset X$.

Since for any positive integer $i \leq m$ the distance $|x_i|_{x_{i-1}} \leq t$, applying the same argument to the points $x_i$ and $x_{i+1}$ as we applied to $x_0$ and $x_1$, we prove that for any positive integer $i \leq m$, $g(C_{x_{i+1}}(\rho_0 + t)) = C_{y_{i+1}}(\rho_0 + t)$ and $g(x_{i+1}) = y_{i+1} \in X_j \subset X$ for some $j \leq m$.

Hence, $g(z) = g(x_n) = y_n \in X$, and $g(X) \subseteq X$.

To show that $g$ is a surjection, we notice that the inverse isometry $g^{-1}$ maps $x'$ onto $x$ and $C_{x'}(\rho)$ onto $C_x(\rho)$. Applying the same argument to $g^{-1}$ as we applied to $g$ we show that $g^{-1}$ maps $X$ onto $X$. Therefore, for any $y \in X g^{-1}y \in X$. Hence, $g$ is a surjection. Therefore $G_i$ is a subgroup of the group $G := \text{Sym}(X)$ (i.e. $G_i \subseteq G$).

Let us take now any $f \in \text{Sym}(X)$, and any point $x \in X_i$. Since $f$ maps $X$ onto $X$. It is clear that $f$ establishes $(\rho_0 + t)$-equivalency of points $x$ and $f(x)$, therefore $f \in G_i$. Hence, we proved that for any positive integer $i \leq m$, $G_i = \text{Sym}(X)$ \[ \Box \]

To complete the proof of the theorem we need to make two observations. First, by the definition of the set $X_i$ and group $G_i$, the group $G_i$ acts transitively on $X_i$,
therefore, \( X_i = G \cdot x_i \). Second, since for any positive integer \( i \leq m \), \( G_i = Sym(X) \), it follows that \( X_i = Sym(X) \cdot x_i \). If we denote the set that consists of one point from each \( X_i \) by \( X_0 \) we obtain

\[
X = \bigcup_{x_i \in X_0} Sym(X) \cdot x_i.
\]

This concludes the proof of the theorem. \( \square \)

7 On t-bonded Sets of Finite Type and Infinite type

**Definition 7.1.** A set \( X \) is said to be of the finite type if for each \( \rho > 0 \) the number \( N(\rho) \) of classes of \( \rho \)-clusters is finite.

It is easy to see that for any uniformly discrete set \( X \) the function \( N(\rho) \) is always defined and equal to 1 for all \( \rho < r \). It is not hard to prove the following:

**Statement 7.1.** If \( X \) is a Delone set with \( N(2R) < \infty \), then for all \( \rho > 0 \) the cluster counting function \( N(\rho) < \infty \), i.e. \( X \) is a set of the finite type.

**Proof.** The key reason for this fact is as follows. Given the Delone set \( X \), we can construct the Delone tiling corresponding to the Delone set. Let us take a ball \( B_x(\rho) \) centered at point \( x \in X \). Then the Delone tiles which overlap with the ball form a pavement of the ball. The \( \rho \)-cluster \( C_x(\rho) \) is obviously a subset of all those vertices of the pavement of the ball \( B_x(\rho) \) which are located in the ball.

Now we take a point \( z \in \mathbb{R}^d \) and consider the family \( P \) of all possible face-to-face pavements \( P \) of the ball \( B_z(\rho) \) by tiles with the following conditions:

a) a tile of pavement \( P \) is congruent to a tile from the Delone tiling for the set \( X \);

b) the center \( z \) of the ball \( B_z(\rho) \) is a vertex of the pavement \( P \).

We emphasize that we do not assume that any pavement is congruent to a fragment of the Delone tiling for \( X \). On the other hand, it is obvious that for any \( x \in X \) a pavement of the ball \( B_z(\rho) \) which is a fragment of the Delone tiling for \( X \) is congruent to some pavement \( P \in P \). Moreover, the cluster \( C_z(\rho) \subset X \) is congruent to a set of those vertices of the \( P \) which belong to the ball \( B_z(\rho) \).

We note that if in the family \( P \) there are just finitely many non-congruent pavements, then in \( X \) for a given \( \rho > 0 \) there are also finitely many non-equivalent \( \rho \)-clusters. Now we show that the family \( P \) is finite.

In fact, the condition \( N(2R) < \infty \) implies that in the Delone tiling for the set \( X \) there are just finitely many pairwise non-congruent Delone tiles. It is known that the Delone tiling is a face-to-face tiling. Assume two tiles \( Q \) and \( Q' \) have a congruent hyperface. It is easy to see that the polytope \( Q \) can be put to the polytope \( Q' \) by a common hyperface only in a finite number of ways. Due to the two conditions (finiteness of classes of Delone tiles for \( X \) and finiteness of non-congruent pairs \((Q, Q')\) of tiles adjacent on a common hyperface), only finitely many face-to-face pavements \( P \) of the ball \( B_z(\rho) \) with the above-mentioned properties a) and b) can be constructed. It follows from this that \( N(\rho) < \infty \). \( \square \)

Now we return to a more general case when \( X \) is a t-bonded set. The following theorem shows that the case of t-bonded sets differs from the case of Delone sets.
**Theorem 7.1.** Given $t > 0$, for an arbitrarily large $k > 0$ there is a $t$-bonded set $X \subset \mathbb{R}^2$ such that $N(kt) < \infty$, but for any $\varepsilon > 0$ $N(kt + \varepsilon) = \infty$. Thus the above mentioned $t$-bonded set $X$ is not a set of the finite type.

**Proof.** We will construct an example of such set $X$.

Let us take two positive numbers $a$ and $b$ such that $a/b$ is irrational and $t < a, b < 2t$, where $t > 0$ is a given parameter for $X$. Assume that $L_1 : v = 0$ and $L_2 : v = kt$ are two horizontal lines in the plane $(u, v)$.

Along each of these lines we construct a "horizontal"broken line whose vertices will be a subset of the set $X$.

The set $X_1$ of vertices of the first broken line is determined by the formulas:

$$x_i = (ia/2, 0),$$

if $i$ is even

$$x_i = (ia/2, -\theta_1)$$

if $i$ is odd, $\theta_1 > 0$, $i \in \mathbb{Z}$.

The altitude $\theta_1$ in an equilateral triangle $\triangle x_{i-1}x_i x_{i+1}$ is chosen so that side-lengths $|x_0x_1|, |x_1x_2|, \ldots$ of the broken line are equal to $t$.

The set $X_2$ of vertices of the second horizontal broken line along the line $L_2$ is determined by the formulas:

$$y_i = (ib/2, kt),$$

if $i$ is even

$$y_i = (ib/2, kt + \theta_2)$$

if $i$ is odd, $\theta_2 > 0$, $i \in \mathbb{Z}$.

The altitude $\theta_2$ in the equilateral triangle $\triangle y_{i-1}y_i y_{i+1}$ is chosen so that sides $|y_0y_1|, |y_1y_2|$, ... of the broken line are equal to $t$.

Note that all points of $X_1$ with even indices are on the line $L_1$, and all points of $X_1$ with odd indices are below this line.

Similarly all points of $X_2$ with even indices are on the line $L_2$, and all points of $X_2$ with odd indices are above this line.

The distance between $(0, 0)$ and $(0, kt)$ is equal to $kt$. At the same time, the distance between any other pairs of points from $X_1$, and $X_2$ (but the pair $(0, 0)$ and $(0, kt)$) is greater than $kt$. It is clear that for the pair $x_i \in X_1$ and $y_j \in X_2$ the distance is greater than $kt$ provided either $i$ or $j$ is odd. Assume that $i$ and $j$ are both even. If $|x_iy_j| = kt$, then the interval that connects $x_i$ and $y_j$ is parallel to the interval that connects $(0, 0)$ and $(0, kt)$, which in turn implies that $ia/2 = jb/2$, and $b/a$ is rational. However, by choice of $b$ and $a$ the ratio $b/a$ is irrational.

To complete the construction of the set $X$ we add to the sets $X_1$ and $X_2$ the third set $X_3$ which is described below.

Let $X_3$ be a set of vertices of a broken line along the interval that connects $(0, 0)$ to $(0, kt)$. Below we explain how this broken line is constructed.

Let $c > 0$ be such that $t \leq c < 2t$ and $c$ divides $kt$: $\frac{kt}{c} = n \in \mathbb{Z}$. Construct the following broken line with vertices $z_0, z_1, z_2n$ and links all equal to $t$.

$$z_i = (0, ci/2)$$

if $i$ is even, $z_0 = x_0$ and $z_{2n} = y_0$;

$$z_i = (\theta_3, ci/2)$$

if $i$ is odd.

Here the altitude $\theta_3 > 0$ in the equilateral triangle $z_{i-1}z_iz_{i+1}$ is chosen so that the lateral sides $z_{i-1}z_i$ and $z_iz_{i+1}$ are both equal to $t$.

With the construction of $X_3$ we completed constructing the $t$-bonded set $X$ which is defined as $X = X_1 \cup X_2 \cup X_3$.
The role of $X_3$ is to make the entire set $X$ $t$-connected. We emphasize that by construction of the set $X$, any $t$-cluster in the set $X$ has rank 2, i.e. the affine hull with dimension 2.

It is not hard to prove that $N(kt) < \infty$. In fact, all clusters $C_{x_i}(kt)$, $x_i \in X_1$ are equivalent if $|i| > \frac{2(kt+\theta_3)}{a}$.

Analogously, for points $y_j \in X_2$ all clusters $C_{y_j}(kt)$ are equivalent if $|j| > \frac{2(kt+\theta_3)}{b}$.

Besides these two classes of $kt$-clusters in $X$, there is a finite amount of $kt$-clusters centered at points $x_i$ and $y_j$ with $|i| \leq \frac{2(kt+\theta_3)}{a}$ and $|j| \leq \frac{2(kt+\theta_3)}{b}$ respectively, and there is a finite amount of $kt$-clusters centered at points of $X_3$. Therefore $N(kt) < \infty$.

Throughout the rest of the proof we will consider clusters centered at points $x_i \in X_1$ and $y_j \in X_2$ only with even indices $i$ and $j$. To be consistent with all the notations let us redesignate the points $x_i$ with even indices $i$ with new natural indices $p$, where $p = i/2$, $x_i = x_p$. A similar change is done for the points $y_j$ where $j$ is even, $j \mapsto q$, where $q = j/2$. Then $x_p = (pa, 0)$ and $y_q = (qb, kt)$, where $p$ and $q$ are positive integer numbers.

Let us take $\varepsilon > 0$ so small that the ball $B_{x_p} := B_{x_p}(kt + \varepsilon)$ and $L_2$ intersect over a chord with the the length $2\delta(\varepsilon)$ where $\delta < b/4$. Since $\delta < b/4$, $|B_{x_p}(kt + \varepsilon) \cap X_2| \leq 1$ for any $p \in \mathbb{N}$.

Let us consider a set $Q$ of all pairs $(p, q) \in \mathbb{N}^2$ such that $B_{x_p} \cap L_2 \ni y_q$, i.e the point $y_q$ belongs to the cluster $C_{x_p}(kt + \varepsilon)$-cluster centered in the point $x_p$. This is equivalent to the inequality $|pa - qb| < \delta$, or to the inequality

$$|\alpha - \frac{p}{q}| < \frac{\delta}{q},$$

where $\alpha = \frac{b}{a}$ is irrational. (7.1)

By the Dirichlet theorem, the inequality

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}$$

has infinitely many solutions in positive integers $(p, q)$.

Moreover, let $\alpha = [a_0; a_1, \ldots, a_n, \ldots]$ be the continued fraction of $\alpha$ and $\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$ the $n$-th convergent. The sequence of the convergents determines the following two sequences of positive integer numbers

$$q_1 < q_2 < q_3 < \ldots$$

and

$$p_1 < p_2 < p_3 < \ldots$$

(7.3)
Due to (7.4) for all large enough $n$ such that $q_{n+1} > 1/\delta$ we have

$$|\alpha - p_n| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}$$

(7.4)

Due to (7.4) for all large enough $n$ such that $q_{n+1} > 1/\delta$ we have

$$|\alpha - p_n| < \delta,$$

(7.5)

Now for all these values of $n$ we take the following pairs of points $(x_{pn}, y_{qn})$ where $x_{pn} = (p_n a, 0) \in X_1$ and $y_{qn} = (q_n b, kt) \in X_2$. Due to (7.5) the following inequality holds

$$|q_n b - p_n a| < q_n \frac{\delta}{q_n} \leq \delta.$$  

(7.6)

By choice of $\delta = \delta(\varepsilon)$ it follows from (7.6) that $y_{q_n}$ belongs to the cluster $C_{x_{pn}}(kt + \varepsilon)$. Therefore, $(p_n, q_n) \in Q$ for all $n > N_0(\delta)$.

Let $(p, q)$ and $(p', q')$ be two pairs of indices and $x_p = (pa, 0)$, $y_q = (qb, kt)$ and $x_{p'} = (p'a, 0)$, $y_{q'} = (q'b, kt)$ pairs of corresponding points. If the pairs $(p, q)$ and $(p'q')$ both belong to $Q$, then $y_q \in C_{x_p}(kt + \varepsilon)$ and $y_{q'} \in C_{x_{p'}}(kt + \varepsilon)$. We show that for the clusters $C_{x_p}(kt + \varepsilon)$ and $C_{x_{p'}}(kt + \varepsilon)$ there are at most two potentially possible isometries of the plane that establish equivalence of these clusters.

Since $kt + \varepsilon \geq 2t + \varepsilon > a$, points $x_{p-1} = ((p-1)a, 0)$ and $x_{p+1} = ((p+1)a, 0)$ also belong to the cluster $C_{x_{p}}(kt + \varepsilon)$. We can even say that these two points are the only two points that belong to the cluster at the distance $a$ from the center of the cluster $C_{x_{p}}(kt + \varepsilon)$, if $x_p \neq x_0$. The same observation is true regarding the cluster $C_{x_{p'}}(kt + \varepsilon)$, centered in the point $x_{p'}$. It means that any isometry of the plane that establishes equivalence of the clusters $C_{x_{p}}(kt + \varepsilon)$ and $C_{x_{p'}}(kt + \varepsilon)$, should map the center $x_p$ onto the $x_{p'}$, and points $x_{p-1}$ and $x_{p+1}$ onto points $x_{p'-1}$ and $x_{p'+1}$ (though we don’t claim that the order of these two points should be preserved). There could be only two isometries of this type. The first one is a parallel shift when $x_{p-1}$ maps onto $x_{p'-1}$, and $x_{p+1}$ maps onto $x_{p'+1}$ (the isometry preserves the order of the points $x_{p-1}$ and $x_{p+1}$). The second one, when $x_{p-1}$ maps onto $x_{p'+1}$, and $x_{p+1}$ maps onto $x_{p'-1}$, is a parallel shift followed by the reflection around the line $u = ap'$.

Let us prove now that $C_{x_{p}}(kt + \varepsilon)$ and $C_{x_{p'}}(kt + \varepsilon)$ are not equivalent if $(p, q)$ and $(p'q') \in Q$. Note, that $y_q \in C_{x_{p}}(kt + \varepsilon)$ and $y_{q'} \in C_{x_{p'}}(kt + \varepsilon)$. We will prove that neither a parallel shift, nor a parallel shift followed by reflection around line $u = p'a$, can establish equivalence of the clusters $C_{x_{p}}(kt + \varepsilon)$ and $C_{x_{p'}}(kt + \varepsilon)$.

Assume that for some real number $l$ (positive or negative) $bq = ap - l$. In case of parallel shift $bq' = ap' - l$, and $bq - bq' = ap - ap'$, hence, $qb - q'b = pa - p'a$, i.e. $b/a = (p - p')/(q - q')$. If the isometry is the shift and the reflection, as described above, then $bq' = ap + l$, hence $qb + q'b = pa + p'a$, or $b/a = (p + p')/(q + q')$. In either case $b/a$ is a rational number that contradicts the choice of $a$ and $b$.

Therefore, $(kt + \varepsilon)$-clusters $C_{x_{p}}(kt + \varepsilon)$ and $C_{x_{p'}}(kt + \varepsilon)$, are not equivalent for any different $p$ and $p'$ from the infinite sequence $p_1 < p_2 < p_3 < \ldots$. Hence, the set $X$ is a $t$-bonded set of infinite type. □
8 Summary

In the paper we developed the local theory of regular and multi-regular $t$-bonded sets. The significance of this theory is that the terms of $t$-bonded sets seems to be more appropriate for describing the chemical bonds existing between atoms in real structures. In many respects this theory follows in the tracks of the local theory of regular Delone systems. However, the $t$-bonded sets essentially extend the limits of the family of Delone sets, and therefore it is no surprise that in spite of the similarity of the theories, there are essentially new features in the behavior of the $t$-bonded sets that are not the Delone sets.

From our point of view the studies of $t$-bonded sets should be continued in two directions. First, to get the upper bound for the radius $\rho_0$ of a cluster such that the condition $N(\rho_0) = 1$ implies regularity of a $t$-bonded set in the 3D space. Second, regarding potential application of the theory, it makes sense to extend the above mentioned theory of regular sets for clusters defined by other metrics.

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