One Loop Vacuum Polarization in a Locally de Sitter Background

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ABSTRACT

We compute the one loop vacuum polarization from massless, minimally coupled scalar QED in a locally de Sitter background. Gauge invariance is maintained through the use of dimensional regularization, whereas conformal invariance is explicitly broken by the scalar kinetic term as well as through the conformal anomaly. A fully renormalized result is obtained. The one loop corrections to the linearized, effective field equations do not vanish when evaluated on-shell. In fact the on-shell one loop correction depends quadratically on the inflationary scale factor, similar to a photon mass. The contribution from the conformal anomaly is insignificant by comparison.

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1 Introduction

One of many exciting and potentially testable consequences of inflationary cosmology is a mechanism for generating the primordial cosmic magnetic fields which may have served as the seeds for the currently observed galactic field of about $10^{-6}$ Gauss. The idea [1, 2] is that the inflationary production of light, minimally coupled, charged scalars — such as the Higgs — resulted in the photon acquiring a plasma mass of about $m_\gamma \sim eH$, where $H \sim 10^{12}$ GeV is the inflationary Hubble parameter. Of course this would suppress the creation of photons during inflation, but it would vastly amplify the zero point energy of the super-horizon modes,

$$\frac{1}{2} \hbar \omega \longrightarrow \frac{1}{2} \sqrt{k^2 e^{-2Ht} + m_\gamma^2},$$

where $k = 2\pi/\lambda$ is the co-moving wave number. After the end of inflation the charged plasma dissipates — either by annihilation or through being redshifted into insignificance. If this happens quickly enough the enormous zero point energies are shed as coherent ensembles of very long wave length photons which would manifest as magnetic fields on super-horizon scales.

To be more quantitative let us model the spacetime geometry during inflation as locally de Sitter. We can express the invariant element conveniently either in co-moving or conformal coordinates,

$$ds^2 = -dt^2 + e^{2Ht} d\vec{x} \cdot d\vec{x} = a^2 \left[ -d\eta^2 + d\vec{x} \cdot d\vec{x} \right].$$

The conformal factor and the transformation which relate the two coordinate systems are,

$$a(\eta) = \frac{1}{H \eta} = e^{Ht}.$$  \hspace{1cm} (3)

Gravity is a non-dynamical background. The dynamical variables are the vector potential $A_\mu(x)$ and a complex scalar $\phi(x)$. Their Lagrangian is,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} - (\partial_\mu - ieA_\mu)\phi^*(\partial_\nu + ieA_\nu)\phi g^{\mu\nu} \sqrt{-g}.$$ \hspace{1cm} (4)

One way of understanding the mass generation mechanism is by appealing to the result of Vilenkin and Ford for the coincidence limit of a free scalar in Bunch-Davies vacuum [3],

$$\left\langle \Omega | \phi^*(x) \phi(x) | \Omega \right\rangle_{\text{free}} = \left( \frac{H}{2\pi} \right)^2 \left\{ \text{UV} + Ht \right\}.$$ \hspace{1cm} (5)
Here “UV” stands for an ultraviolet divergent constant. If we infer an approximate action for the photons by replacing the various $\phi^*\phi$ terms by the finite part of their vacuum expectation values, the result seems to be a time-dependent photon mass,

$$m_\gamma^2 = \frac{e^2 H^2}{2\pi^2} H t = \frac{e^2 H^2}{2\pi^2} \ln(a). \quad (6)$$

Although very suggestive, the analysis of the preceding paragraph is not really consistent quantum field theory. The kinematic properties of a particle are encoded in its 2-point 1PI (one-particle-irreducible) function. The 2-photon 1PI function is known as “the vacuum polarization” and the one loop contributions to it are depicted in Figures 1-3. Making the replacement,

$$-e^2 A_\mu A_\nu \phi^*\phi g^{\mu\nu} \sqrt{-g} \rightarrow -e^2 A_\mu A_\nu \left(\frac{H}{2\pi}\right)^2 \left\{ \text{UV} + H t \right\} g^{\mu\nu} \sqrt{-g}. \quad (7)$$

corresponds to including only the diagram of Fig. 1. The other two graphs are the same order ($e^2$) in perturbation theory and there seems to be no good reason for ignoring them. The diagram of Fig. 2 is required to make the vacuum polarization gauge invariant. And the graph of Fig. 3 is needed to absorb the ultraviolet divergence.

The purpose of this paper is to compute all three diagrams in a consistent regularization and to demonstrate that they induce corrections to the photon wave function very like those of a photon mass. In Section 2 we review the familiar results from flat space. This serves as a useful introduction to using dimensional regularization in position space and establishes a crucial correspondence limit for checking the accuracy of our subsequent work. Section 3 gives the scalar propagator in $D$-dimensional de Sitter space. In Section 4 we first reduce the vacuum polarization to manifestly transverse form, then we renormalize it. In Section 5 we take the result on-shell to demonstrate that the tree order wave functions receive one loop corrections like those of a photon mass. We discuss the result in Section 6, giving special emphasis to the important issues which are still open. A fuller discussion of what our result means physically can be found in another work [4].

## 2 Vacuum polarization in flat space
Figure 1: One loop contribution to the vacuum polarization from the 4-point (seagull) interaction.

Figure 2: One loop contribution to the vacuum polarization from two 3-point interactions.

Figure 3: Photon field strength renormalization counterterm.
2.1 Momentum space treatment

Although working in a curved background typically requires a position space treatment, it is desirable to begin our discussion of the vacuum polarization with the traditional momentum space treatment. The contribution from the 4-point interaction is depicted in Fig. 1. It actually vanishes for massless scalar QED in dimensional regularization,

\[
\left(\text{Fig. 1}\right)_{\mu\nu}^{\mu\nu} = -2ie^2\eta_{\mu\nu}^{\mu\nu}\int \frac{d^Dk}{(2\pi)^D} \frac{-i}{k^2 - i\epsilon} = 0. \tag{8}
\]

Fig. 2 shows the contribution from two 3-point interactions,

\[
\left(\text{Fig. 2}\right)_{\mu\nu}^{\mu\nu} = \int \frac{d^Dk}{(2\pi)^D} ie(p + 2k)^\mu \frac{-i}{(p + k)^2 - i\epsilon} ie(p + 2k)^\nu \frac{-i}{k^2 - i\epsilon} , \tag{9}
\]

\[
= \frac{i2e^2}{(4\pi)^2} \Gamma(1 - \frac{D}{2}) \Gamma^2(\frac{D}{2}) (p^2)^{\frac{D}{2} - 2}(p^2\eta_{\mu\nu}^{\mu\nu} - p^\mu p^\nu) . \tag{10}
\]

Since we have the result for any spacetime dimension \(D\) we can note that its vanishing on-shell (i.e., at \(p^2 = 0\) in the transverse direction) for \(D = 4\) indicates that the photon stays massless in that dimension. The fact that the vacuum polarization does not vanish on-shell for \(D = 2\) indicates that the massless scalars form a massive photon bound state in two dimensions. This is a scalar version of the Schwinger model [5] in which the photon develops a mass of \(m_\gamma \sim e\).

It is straightforward to extract the divergence as \(D \equiv 4 - \epsilon\) approaches 4,

\[
\left(\text{Fig. 2}\right)_{\mu\nu}^{\mu\nu} = -\frac{i2e^2}{24\pi^2} (p^2\eta_{\mu\nu}^{\mu\nu} - p^\mu p^\nu)^\frac{1}{\epsilon} + \text{finite} . \tag{11}
\]

It is of course removed by the field strength renormalization of Fig. 3,

\[
\left(\text{Fig. 3}\right)_{\mu\nu}^{\mu\nu} = -i\delta Z(p^2\eta_{\mu\nu}^{\mu\nu} - p^\mu p^\nu) . \tag{12}
\]

One sets \(\Delta Z = -e^2\mu^{-\epsilon}/(24\pi^2\epsilon)\) plus any convenient finite term.

2.2 Position space treatment

Let us now repeat the exercise in position space. The propagator is,

\[
i\Delta(x; x') = \frac{\Gamma(1 - \frac{D}{2})}{4\pi^{\frac{D}{2} - \frac{3}{2}}} \frac{1}{(\Delta x^2)^{\frac{1}{2} - \frac{D}{2}}} , \tag{13}
\]
where we define $\Delta x^2$ to be,

$$\Delta x^2 \equiv \| \vec{x} - \vec{x}' \|^2 - (|t - t'| - i\delta)^2 .$$  \hfill (14)

We will use the same figures for the position space diagrams. Of course the first diagram still gives zero,

$$\left( \text{Fig. 1} \right)^{\mu\nu} = \left[ 2e^2 \eta^{\mu\nu} \right] \delta \left(x - x' \right) = 0 .$$  \hfill (15)

Note the general rule for using dimensional regularization in position space: the parameter $\epsilon$ is assumed to lie in the range for which terms such as $\Delta x^{-N+\epsilon}$ vanish at coincidence. For example, to derive (15) we assume $\epsilon > 2$, even though we shall eventually take $\epsilon$ to zero.

The position space versions of the other two diagrams are,

$$\left( \text{Fig. 2} \right)^{\mu\nu} = \left[ 2e^2 \eta^{\mu\nu} \right] \partial_\rho i \Delta(x; x') \partial_\sigma i \Delta(x; x') - i \Delta(x; x') \partial_\rho \partial_\sigma i \Delta(x; x' \right), \hfill (16)$$

$$\left( \text{Fig. 3} \right)^{\mu\nu} = -i \delta Z(\eta^{\mu\nu} \partial' - \partial \partial' \eta^{\nu\sigma}) \delta D(x - x') .$$ \hfill (17)

Note that these particular diagrams involve no integrations when written in position space. At two loop order there are integrations.

The first step in evaluating (14) is to substitute the propagator and take the derivatives,

$$i \left[ \mu \Pi_{1+2}^\nu \right] (x; x') = 2e^2 \left( \frac{\Gamma(1 - \frac{\epsilon}{2})}{4\pi^{\frac{1}{2}} - \frac{\epsilon}{2}} \right)^2 \left[ -\left( 2 - \epsilon \right)^2 \frac{\Delta x^\mu \Delta x^\nu}{\Delta x^{8-2\epsilon}} - \left( 2 - \epsilon \right) \frac{\eta^{\mu\nu}}{\Delta x^{8-2\epsilon}} + \left( 2 - \epsilon \right) \left( 4 - \epsilon \right) \frac{\Delta x^\mu \Delta x^\nu}{\Delta x^{8-2\epsilon}} \right] , \hfill (18)$$

$$= \frac{e^2}{8\pi^2} \pi^2 (1 - \frac{\epsilon}{2})(\epsilon - \frac{\epsilon}{2}) \left[ -\eta^{\mu\nu} \frac{\Delta x^\mu \Delta x^\nu}{\Delta x^{8-2\epsilon}} + \frac{2\Delta x^\mu \Delta x^\nu}{\Delta x^{8-2\epsilon}} \right] . \hfill (19)$$

We anticipate the notation of curved space in denoting the vacuum polarization as a bi-tensor function, $\left[ \mu \Pi_{1+2}^\nu \right] (x; x')$. This means that the first index ($\mu$) transforms according to the tangent space at the first argument ($x$) and the same relative relation exists between the second index ($\nu$) transforms and the second argument ($x'$).
The next step is reaching manifestly transverse form. This is done by writing inverse powers of $\Delta x$ as derivatives of lower powers. For example, one can easily derive the identity,

$$\partial^\mu \partial^\nu \frac{1}{\Delta x^{4-2\epsilon}} = -(4 - 2\epsilon) \left[ \frac{-\eta^{\mu\nu}}{\Delta x^{6-2\epsilon}} + (6 - 2\epsilon) \frac{\Delta x^\mu \Delta x^\nu}{\Delta x^{8-2\epsilon}} \right].$$  \hspace{1cm} (20)

By combining with $\eta^{\mu\nu}$ times the trace it follows that,

$$\left[ \frac{-\eta^{\mu\nu}}{\Delta x^{6-2\epsilon}} + \frac{2\Delta x^\mu \Delta x^\nu}{\Delta x^{8-2\epsilon}} \right] = \frac{1}{2(2 - \epsilon)(3 - \epsilon)} \left[ \eta^{\mu\nu} \partial^\nu \partial^\mu - \partial^\mu \partial^\nu \right] \frac{1}{\Delta x^{4-2\epsilon}} .$$ \hspace{1cm} (21)

Substitution into (19) gives a manifestly transverse form,

$$i\left[ \mu \Pi_{1+2}^{\nu} \right] (x; x') = \frac{e^2}{16\pi^2} \frac{\pi^3 \Gamma^2(1 - \frac{\epsilon}{2})}{3 - \epsilon} \left[ \eta^{\mu\nu} \partial^\nu \partial^\mu - \partial^\mu \partial^\nu \right] \frac{1}{\Delta x^{4-2\epsilon}} .$$ \hspace{1cm} (22)

The next step is extracting the ultraviolet divergence. This typically comes from a term of the form $1/\Delta x^{4-2\epsilon}$ through the identity,

$$\partial^2 \frac{1}{\Delta x^{4-2\epsilon}} = \frac{-2\epsilon (1 - \epsilon)}{\Delta x^{4-2\epsilon}} .$$ \hspace{1cm} (23)

The reader will note that the various derivatives have so far failed to induce any delta functions. That was because there were either too few derivatives or because the power of $\Delta x$ was wrong. Getting a delta function in dimensional regularization requires two derivatives acting on precisely the power $\Delta x^{\epsilon-2}$,

$$\partial^2 \frac{1}{\Delta x^{2-\epsilon}} = \frac{4i\pi^{2-\frac{\epsilon}{2}}}{\Gamma(1 - \frac{\epsilon}{2})} \delta^D(x - x') .$$ \hspace{1cm} (24)

Combining (23) and (24) allows us to extract the ultraviolet divergence in a form which can be canceled by a local counterterm,

$$\frac{1}{\Delta x^{4-2\epsilon}} = \frac{-\partial^2}{2\epsilon(1 - \epsilon) \Delta x^{2-2\epsilon}} ,$$ \hspace{1cm} (25)

$$= - \frac{\partial^2}{2\epsilon(1 - \epsilon)} \left[ \frac{1}{\Delta x^{2-2\epsilon}} - \frac{\mu^{-\epsilon}}{\Delta x^{2-\epsilon}} \right] - \frac{2\pi^2i(\sqrt{\pi}\mu)^{-\epsilon}}{\epsilon(1 - \epsilon)\Gamma(1 - \frac{\epsilon}{2})} \delta^D(x - x'),$$ \hspace{1cm} (26)

$$\quad \longrightarrow - \frac{\partial^2}{4} \left[ \ln(\mu^2 \Delta x^2) \right] - \frac{2\pi^2i(\sqrt{\pi}\mu)^{-\epsilon}}{\epsilon(1 - \epsilon)\Gamma(1 - \frac{\epsilon}{2})} \delta^D(x - x') .$$ \hspace{1cm} (27)
As usual, dimensional regularization has resulted in a scale $\mu$. Substituting (27) reveals the divergence structure we already found in momentum space,

$$i[\Pi_{1+2}^\nu](x; x') \longrightarrow -\frac{e^2}{192\pi^4}[\eta^{\mu\nu}\partial' \cdot \partial - \partial'^\mu \partial'^\nu] \partial^2 \left[ \ln(\mu^2 \Delta x^2) \right] \Delta x^2$$

$$-\frac{ie^2}{8\pi^2} \left( \frac{\pi}{\mu^2} \right)^\frac{\epsilon}{2} \frac{\Gamma(1 - \frac{\epsilon}{2})}{\epsilon(1 - \epsilon)(3 - \epsilon)} [\eta^{\mu\nu}\partial' \cdot \partial - \partial'^\mu \partial'^\nu] \delta^D(x - x') , \quad (28)$$

$$\longrightarrow -\frac{ie^2}{24\pi^2} [\eta^{\mu\nu}\partial' \cdot \partial - \partial'^\mu \partial'^\nu] \delta^D(x - x') \frac{1}{\epsilon} + \text{finite} . \quad (29)$$

The cleanest subtraction in position space is to absorb the entire local term by choosing $\delta Z$ to be,

$$\delta Z = -\frac{e^2}{8\pi^2} \left( \frac{\pi}{\mu^2} \right)^\frac{\epsilon}{2} \frac{\Gamma(1 - \frac{\epsilon}{2})}{\epsilon(1 - \epsilon)(3 - \epsilon)} . \quad (30)$$

With this choice the fully renormalized vacuum polarization becomes,

$$i[\Pi_{\text{ren}}^\nu](x; x') = -\frac{e^2}{192\pi^4}[\eta^{\mu\nu}\partial' \cdot \partial - \partial'^\mu \partial'^\nu] \partial^2 \left[ \ln(\mu^2 \Delta x^2) \right] . \quad (31)$$

### 2.3 Going on-shell in position space

The position space derivation we have just completed was simpler than its momentum space cousin because no integrations had to be performed. They occur when one goes on-shell. What we are really checking, when we take the vacuum polarization on-shell, is whether or not there are quantum corrections to the linearized wave functions. One might define these as the matrix element of the operator $A_\mu(x)$ between the vacuum and a plane wave photon state. The equations obeyed by this matrix element come from varying and linearizing the effective action,

$$\Gamma[A] = -\frac{1}{4} \int d^4 x F_{\mu\nu} F^{\mu\nu}$$

$$+\frac{1}{2} \int d^4 x \int d^4 x' A_\mu(x) [^{\mu}\Pi^\nu](x; x') A_\nu(x') + O(A^4) . \quad (32)$$
The associated field equations are,

\[ \frac{\delta \Gamma[A]}{\delta A_\mu(x)} = \partial_\nu F^{\nu\mu}(x) + \int d^4x' [^\mu \Pi^\nu](x; x') A_\nu(x') + O(A^3) = 0. \tag{33} \]

The general classical solution comes from superposing plane waves of the form,

\[ A_\mu^0(x) = \epsilon_\mu(\vec{k}) e^{ik \cdot x}, \tag{34} \]

where \( k^0 = \|\vec{k}\| \) and the Lorentz gauge polarization vectors obey \( \epsilon_0 = 0 = k \cdot \epsilon \).

To see if there are quantum corrections one merely expands the solution in powers of \( \bar{\hbar} \),

\[ A_\mu(x) = A_\mu^0(x) + A_\mu^1(x) + \ldots. \tag{35} \]

and then segregates all terms of the same order. Potential one loop corrections are determined by the equation,

\[ \left[ \partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu \right] A^1_\nu(x) = -\int d^4x' [^\mu \Pi^\nu](x; x') A^0_\nu(x'). \tag{36} \]

We therefore conclude that the necessary and sufficient condition for there to be one loop corrections to the photon wave function is a nonvanishing integral for the one loop vacuum polarization against a classical plane wave solution.

3 Scalar propagator in de Sitter D-space

The behavior of free, massless and minimally coupled scalars on a locally de Sitter background has been investigated extensively \cite{3, 6, 7, 8}. Among the curious properties of these particles are the absence of normalizable, de Sitter invariant states \cite{6} and the appearance of acausal infrared singularities when the Bunch-Davies vacuum is used with infinite spatial surfaces \cite{4, 8}. To regulate this infrared problem we work on the manifold \( T^{D-1} \times R \), with the spatial coordinates in the finite range, \(-H^{-1}/2 < x^i \leq H^{-1}/2\). Although the actual propagator is a mode sum on this manifold, the small possible variation in conformal coordinates renders the first term of the Euler-Maclaurin formula — just the integral — an excellent approximation. So the finite spatial range of \( T^{D-1} \) serves merely to cut off what would have been a logarithmic infrared divergence on \( R^{D-1} \). In \( D = 3 + 1 \) spacetime dimensions
the result is [9],

\[ i\Delta(x; x') \bigg|_{d=4} = \left( \frac{H}{2\pi} \right)^2 \left\{ \frac{1}{y(x; x')} - \frac{1}{2} \ln(y(x; x')) + \frac{1}{2} \ln\left( a(\eta)a(\eta') \right) \right\}, \tag{37} \]

where the modified de Sitter length function has the definition,

\[ y(x; x') \equiv a(\eta)a(\eta')H^2 \left[ \| \vec{x} - \vec{x}' \|^2 - (|\eta - \eta'| - i\delta)^2 \right]. \tag{38} \]

Neglecting the higher order Euler-Maclaurin terms does not prevent (37) from solving the correct differential equation. The higher terms also drop out of quite complicated, nonlinear relations such as the Ward identity for the one loop graviton self-energy [10]. We shall therefore regard the technique as valid and confine ourselves to finding the appropriate generalization of (37) to \( D \) spacetime dimensions.

We seek a function of \( y(x; x') \) and the two conformal factors which obeys,

\[ \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} a^{D-2}(\eta) \frac{\partial}{\partial x'^\nu} i\Delta(x; x') = i\delta^D(x - x'). \tag{39} \]

When the kinetic operator acts on a function of just \( y(x; x') \) one finds,

\[
\eta^{\mu\nu} \frac{\partial}{\partial x^\mu} a^{D-2}(\eta) \frac{\partial}{\partial x'^\nu} f\left(y(x; x')\right)
= H^2 a^D(\eta) \left\{ (4y - y^2)f''(y) + D(2 - y)f'(y) - 4i\delta(\eta - \eta')f'(y)\delta \right\}. \tag{40}
\]

The only symmetric function of \( a(\eta) \) and \( a(\eta') \) which can give the same prefactor of \( a^D(\eta) \) is a constant times the same logarithm that appears in (37). The \( D \)-dimensional propagator must therefore take the form,

\[ i\Delta(x; x') = f\left(y(x; x')\right) + b\ln\left(a(\eta)a(\eta')\right), \tag{41} \]

\footnote{What is termed “the de Sitter length function” in the literature is,

\[ z(x; x') = 1 - y(x; x'). \]

The geodesic length from \( x^\mu \) to \( x'^\mu \), \( \ell(x; x') \), is related to \( y(x; x') \) as follows,

\[ y(x; x') = \sin^2 \left( \frac{1}{2} H\ell(x; x') \right). \]

9
where the function \( f(y) \) obeys,

\[
H^2 a^D(\eta) \left\{ (4y - y^2) f''(y) + D(2 - y) f'(y) - 4i\delta(\eta - \eta') f'(y)\delta - b(D - 1) \right\} = i\delta^D(x - x').
\]

(42)

The delta function (for \( \delta \to 0 \)) obviously descends from a factor of \( y^1 \). Series solution of the equation then generates an infinite sum of higher powers. Defining \( D \equiv 4 - \epsilon \) and normalizing correctly gives,

\[
\frac{H^{2-\epsilon}}{4\pi^{2-\frac{\epsilon}{2}}} \Gamma \left( 1 - \frac{\epsilon}{2} \right)^{-\epsilon} \left\{ \frac{1}{y^{1-\frac{\epsilon}{2}}} + \left( 1 - \frac{\epsilon}{2} \right) \sum_{n=0}^{\infty} \frac{1}{n + \frac{\epsilon}{2}} \frac{\Gamma(3 + n - \frac{\epsilon}{2})}{(n + 1)!\Gamma(2 - \frac{\epsilon}{2})} \frac{y^{n+\frac{\epsilon}{2}}}{4^n+1} \right\}.
\]

(43)

This series solves (42) for \( b = 0 \), but it does not reduce to (37) for \( \epsilon = 0 \). The \( n = 0 \) term of the sum is not even finite in this limit! The resolution to both problems is a series of strictly nonnegative integer powers of \( y \), which cancels the divergence and the unwanted terms. This series obeys the homogeneous equation up to a constant which is canceled by the \( b(D - 1) \) term,

\[
i\Delta(x; x') = \left( \frac{H}{2\pi} \right)^2 \left( \frac{H}{\sqrt{n}} \right)^{-\epsilon} \Gamma \left( 1 - \frac{\epsilon}{2} \right) \left\{ \frac{1}{y^{1-\frac{\epsilon}{2}}} + \left( 1 - \frac{\epsilon}{2} \right) \left( 1 - \frac{\epsilon}{4} \right) \left( 1 - \frac{y^2}{\epsilon} \right) + \left( 1 - \frac{\epsilon}{2} \right) \sum_{n=1}^{\infty} \left[ \frac{1}{n\Gamma(2 + n - \frac{\epsilon}{2})} - \frac{1}{n + \frac{\epsilon}{2}} \frac{\Gamma(3 + n - \frac{\epsilon}{2})}{(n + 1)!\Gamma(2 - \frac{\epsilon}{2})} \right] \frac{y^n}{4^n+1} \right\}.
\]

(44)

The great advantage of this regularization is that it preserves general coordinate invariance (once \( \delta \) is taken to zero). One might think that the propagator is unwieldy but this is not so in practice. For example, this formalism has recently been used to compute and renormalize all two loop contributions to the stress-energy tensor of a real scalar with a \( \phi^4 \) self-interaction [11]. The really cumbersome part of (44) is the infinite sum on the second line. But these terms all vanish at coincidence \( (y(x; x) = 0) \) and they vanish for all \( y(x; x') \) at \( D = 4 \). So one need only retain them when they multiply something else that diverges like \( 1/\epsilon \). Note also that one need never worry about large \( y(x; x') \) on account of causality.

All valid regularizations must reproduce the result of Vilenkin and Ford that the coincidence limit of the propagator contains a finite term which
grows like $\ln(a) = Ht$ \[3\]. To check this note that $y(x; x) = 0$ at coincidence. When a variable vanishes like this in dimensional regularization one must always assume $\epsilon$ to be large enough that the variable is raised to only nonnegative powers. We therefore find,

$$
\lim_{x' \to x} i\Delta(x; x') = \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\sqrt{\pi}}\right)^{-\epsilon} \left\{ \frac{1}{2\epsilon} \Gamma(3 - \frac{\epsilon}{2}) + \frac{1}{2} \Gamma(3 - \epsilon) \ln\left(a(\eta)\right) \right\}.
$$

(45)

Note that (45) is exact for arbitrary $\epsilon$ and indeed reduces to (5) in the limit when $\epsilon$ approaches zero.

### 4 Vacuum polarization in de Sitter

The diagrams which contribute to the vacuum polarization in de Sitter background are drawn exactly as those of flat space, so we shall use the same figures. Of course there are now some factors of the de Sitter metric! We express this in conformal coordinates and adopt the usual convention that indices are raised and lowered by the Lorentz metric,

$$
\left(\text{Fig. 1}\right)^{\mu\nu} = -2ie^2\sqrt{-g(x)}g^{\mu\nu}(x)i\Delta(x; x)\delta^D(x - x'),
$$

(46)

$$
= -2ie^2a^{D-2}\eta^{\mu\nu}i\Delta(x; x)\delta^D(x - x').
$$

(47)

$$
\left(\text{Fig. 2}\right)^{\mu\nu} = 2e^2\sqrt{-g(x)}g^{\mu\rho}(x)\sqrt{-g(x')}g^{\nu\sigma}(x')
\times \left[ \partial_{\rho}i\Delta(x; x')\partial'_{\sigma}i\Delta(x; x') - i\Delta(x; x')\partial_{\rho}\partial'_{\sigma}i\Delta(x; x') \right],
$$

(48)

$$
= 2e^2a^{D-2}a'^{D-2}\left[ \partial^\mu i\Delta(x; x')\partial'^\nu i\Delta(x; x') - i\Delta(x; x')\partial^\mu\partial'^\nu i\Delta(x; x') \right].
$$

(49)

$$
\left(\text{Fig. 3}\right)^{\mu\nu} = -i\delta Z\partial_{\rho}\left(\sqrt{-g} \left| g^{\mu\sigma}g^{\rho\nu} - g^{\mu\rho}g^{\nu\sigma}\right| \partial'_{\sigma}\delta^D(x - x') \right),
$$

(50)

$$
= -i\delta Z \left| \eta^{\mu\nu}\partial - \partial'^\mu\partial'^\nu \right| a^{D-4}\delta^D(x - x').
$$

(51)

In these and subsequent expressions we define $a \equiv a(\eta)$ and $a' \equiv a(\eta')$. Note as well that the scalar propagator is the de Sitter one \[14\].

It is convenient to subsume the complicated, $\epsilon$-dependent constants which appear in the propagator \[14\],

$$
i\Delta(x; x') \equiv a \left\{ \gamma\left(y(x; x')\right) + \beta \ln\left(a(\eta)a(\eta')\right) \right\}.
$$

(52)
Comparison with (44) reveals,
\[
\alpha \equiv \left( \frac{H}{2\pi} \right)^2 \left( \frac{\pi}{H^2} \right)^{\frac{1}{2}} \Gamma(2 - \frac{\xi}{2}) , \quad \beta \equiv \frac{1}{4} \frac{\Gamma(3 - \epsilon)}{\Gamma(2 - \frac{\xi}{2})} ,
\]
and,
\[
\gamma(y) \equiv \frac{1}{1 - \frac{\xi}{2} y^{1-\frac{\xi}{2}}} \frac{1}{1 - \frac{\xi}{4}} \left( \frac{1 - y^{\frac{\xi}{4}}}{e} \right) + \sum_{n=1}^{\infty} \left[ \frac{1}{n} \frac{\Gamma(3 + n - \epsilon)}{\Gamma(2 + n - \epsilon)} \frac{y^n}{4^{n+1}} - \frac{1}{n + \frac{\xi}{2}} \frac{\Gamma(3 + n - \frac{\xi}{2})}{\Gamma(2 - \frac{\xi}{2})} \frac{y^{n+\frac{\xi}{2}}}{4^{n+1}} \right] . \tag{54}
\]
In this notation derivatives of the propagator (52) can be written as,
\[
\partial_{x_i} \Delta(x; x') = \alpha \left\{ \gamma'(y) \frac{\partial y}{\partial x^\rho} + \beta H a(\eta) \delta_0^\rho \right\} , \tag{55}
\]
\[
\partial_{x_\sigma} \partial_{x_\sigma} \Delta(x; x') = \alpha \left\{ \gamma''(y) \frac{\partial y}{\partial x^\rho} \frac{\partial y}{\partial x^\sigma} + \gamma'(y) \frac{\partial^2 y}{\partial x^\rho \partial x^\sigma} \right\} , \tag{56}
\]
\[
\partial_{x_i} \partial_{x_\sigma} \Delta(x; x') = \alpha \left\{ \gamma'(y) \frac{\partial y}{\partial x^\rho} + \beta H a(\eta) \delta_0^\rho \right\} , \tag{57}
\]
We can therefore express the portion of (18) within the brackets as,
\[
\frac{1}{\alpha^2} \left[ \partial_{x_i} \Delta(x; x') \partial_{x_\sigma} \partial_{x_\sigma} \Delta(x; x') - i \Delta(x; x') \partial_{x_i} \partial_{x_\sigma} \Delta(x; x') \right] =
\[
\left[ \gamma'' - \gamma'' \left( \gamma + \beta \ln(aa') \right) \right] \frac{\partial y}{\partial x^\rho} \frac{\partial y}{\partial x^\sigma} - \gamma' \left[ \gamma + \beta \ln(aa') \right] \frac{\partial^2 y}{\partial x^\rho \partial x^\sigma} + \beta \gamma'H \left[ a\delta^0_\rho \frac{\partial y}{\partial x^\sigma} + a' \frac{\partial y}{\partial x^\rho} \delta^0_\sigma \right] + \beta^2 H^2 aa' \delta^0_\rho \delta^0_\sigma . \tag{58}
\]
It is straightforward to differentiate the de Sitter length function (38),
\[
\frac{\partial y}{\partial x^\rho} = a(\eta) H \left[ y \delta^0_\rho + 2a(\eta') H \Delta x_\rho + 2i H a(\eta') \text{sgn}(\eta - \eta') \delta^0_\rho \delta \right] , \tag{59}
\]
\[
\frac{\partial y}{\partial x^\sigma} = a(\eta') H \left[ y \delta^0_\sigma - 2a(\eta') H \Delta x_\sigma - 2i H a(\eta') \text{sgn}(\eta - \eta') \delta^0_\sigma \delta \right] , \tag{60}
\]
\[
\frac{\partial^2 y}{\partial x^\rho \partial x^\sigma} = aa' H^2 \left[ y \delta^0_\rho \delta^0_\sigma - 2a(\eta) \delta^0_\rho \Delta x_\sigma + 2a(\eta') H \Delta x_\rho \delta^0_\sigma - 2\eta \delta \right.
\]
\[
\left. - 2i a(\eta) a(\eta') H^2 [\eta - \eta'] \delta^0_\rho \delta^0_\sigma \delta - 4i \delta(\eta - \eta') \delta^0_\rho \delta^0_\sigma \delta \right] . \tag{61}
\]
Gauge invariance requires taking $\delta$ to zero. Only the final order $\delta$ term can contribute in this limit, and then only when it multiplies $\gamma'$,

$$-4iaa'H^2 \delta(\eta - \eta') \lim_{\delta \to 0} \gamma' \left( y(x; x') \right) \delta = \frac{i}{\alpha a^{D-2}} \delta^D (x - x'). \quad (62)$$

The net result is to subtract off the purely temporal components of (47). We accordingly combine this term with (Fig. 1) $^{\mu \nu}$ to form,

$$i [\mu \Pi_2^\nu] (x; x') \equiv -i e^2 \alpha a^{2-\epsilon} \eta^{\mu \nu} \left\{ \left( \frac{2 - \frac{\epsilon}{2}}{\epsilon} \right) + \frac{\Gamma(3 - \epsilon)}{\Gamma(2 - \frac{\epsilon}{2})} \ln(a) \right\} \delta^D (x - x'). \quad (63)$$

A bar over a tensor indicates its zero components have been removed,

$$\overline{\eta}^{\mu \nu} = \eta^{\mu \nu} + \delta^\mu_0 \delta^\nu_0, \quad \overline{\partial}^\mu = \partial^\mu - \delta^\mu_0 \delta^0. \quad (64)$$

The left-over portion of (Fig. 2) $^{\mu \nu}$ will be known as $i [\mu \Pi_2^\nu] (x; x').$

Expression (58) seems complicated due to the infinite sum in the definition of $\gamma(y)$. However, we need only retain terms which survive as $\epsilon \to 0$,

$$\gamma^2 - \gamma' \left[ \gamma + \beta \ln(a a') \right] \longrightarrow -\frac{1}{1 - \frac{\epsilon}{2}} \frac{1}{y^{4-\epsilon}} + \left( \frac{2 - \frac{\epsilon}{2}}{\epsilon} \right) \frac{1}{y^{3-\epsilon}}$$

$$-\frac{(2 - \frac{\epsilon}{2})^2}{2\epsilon} \frac{1}{y^{3-\epsilon}} - \left( \frac{2 - \frac{\epsilon}{2}}{4} \right) \frac{\Gamma(3 - \epsilon) \ln(a a')}{\Gamma(2 - \frac{\epsilon}{2})} \frac{1}{y^{3-\epsilon}} + \ln(H^2 \Delta x^2) + \frac{1}{4y^2}, \quad (65)$$

$$\gamma' \left[ \gamma + \beta \ln(a a') \right] \longrightarrow -\frac{1}{1 - \frac{\epsilon}{2}} \frac{1}{y^{4-\epsilon}} + \frac{(1 - \epsilon)(2 - \frac{\epsilon}{2})}{2\epsilon(1 - \frac{\epsilon}{2})} \frac{1}{y^{2-\epsilon}}$$

$$-\left( \frac{2 - \frac{\epsilon}{2}}{2\epsilon} \right) \frac{1}{y^{2-\epsilon}} - \frac{1}{4} \frac{\Gamma(3 - \epsilon) \ln(a a')}{\Gamma(2 - \frac{\epsilon}{2})} \frac{1}{y^{2-\epsilon}} + \frac{\ln(H^2 \Delta x^2)}{4y}. \quad (66)$$

After taking $\epsilon$ to zero in other non-divergent terms the result is,

$$i [\mu \Pi_2^\nu] (x; x') = 2a^2 a^{2-\epsilon} a^{2-\epsilon} \left\{ 4a^2 a^2 H^4 \Delta x^\mu \Delta x^\nu \left[ \frac{1}{1 - \frac{\epsilon}{2}} \frac{1}{y^{4-\epsilon}} - \left( \frac{2 - \frac{\epsilon}{2}}{\epsilon} \right) \frac{1}{y^{3-\epsilon}} \right] \right.$$

$$+ \frac{(2 - \frac{\epsilon}{2})^2}{2\epsilon} \frac{1}{y^{3-\epsilon}} + \left( \frac{2 - \frac{\epsilon}{2}}{4} \right) \frac{\Gamma(3 - \epsilon) \ln(a a')}{\Gamma(2 - \frac{\epsilon}{2})} \frac{1}{y^{3-\epsilon}} - \frac{\ln(H^2 \Delta x^2)}{4y^2} - \frac{1}{4y^2} \right\}$$

$$-2aa'H^2 \eta^{\mu \nu} \left[ \frac{1}{1 - \frac{\epsilon}{2}} \frac{1}{y^{4-\epsilon}} - \frac{(1 - \epsilon)(2 - \frac{\epsilon}{2})}{2\epsilon(1 - \frac{\epsilon}{2})} \frac{1}{y^{2-\epsilon}} + \left( \frac{2 - \frac{\epsilon}{2}}{2\epsilon} \right) \frac{1}{y^{2-\epsilon}} \right].$$
\[ + a a' H^2 \left[ - a' H \Delta x^\mu \delta_0^\nu + a \delta_0^\mu H \Delta x^\nu \right] \left[ \ln(H^2 \Delta x^2) + 1 \right] \frac{\ln(aa')}{y^2} + a a' H^2 \delta_0^\mu \delta_0^\nu \ln(H^2 \Delta x^2) \frac{1}{2y} \right] , \] (67)

\[ = - \frac{4 e^2 a^2 H^{2-4}}{1 - \frac{5}{2}} \left[ \eta^{\mu \nu} - 2 \frac{\Delta x^\mu \Delta x^\nu}{\Delta x^2} \right] \frac{1}{\Delta x^{4-2\epsilon}} \]

\[ + a e^2 a^2 H^{2-\frac{1}{2}} \left( \frac{2 - \frac{4}{3}}{1 - \frac{5}{2}} \right) \right] \left[ (1 - \epsilon) \eta^{\mu \nu} - (4 - 2\epsilon) \frac{\Delta x^\mu \Delta x^\nu}{\Delta x^2} \right] \frac{1}{\Delta x^{4-2\epsilon}} \]

\[ + 2 e^2 a^2 H^{2-2 \frac{1}{2}} a^{1-\frac{1}{2}} a^1 \frac{1}{\epsilon} \left( \frac{2 - \frac{4}{3}}{1 - \frac{5}{2}} \right) \right] \left[ \ln(H^2 \Delta x^2) + 1 \right] \frac{1}{\Delta x^{4-\epsilon}} \]

\[ + \frac{4 e^2 H^2}{16 \pi^4} a^2 a^2 \left[ - a^{-1} H \Delta x^\mu \delta_0^\nu + a' \delta_0^\mu H \Delta x^\nu - H^2 \Delta x^\mu \Delta x^\nu \right] \times \left[ \eta^{\mu \nu} - (4 - \epsilon) \frac{\Delta x^\mu \Delta x^\nu}{\Delta x^2} \right] \frac{1}{\Delta x^{4-\epsilon}} . \] (68)

The next step is reducing to manifestly transverse form. The first term in (68) is exactly the same as the flat space result (19) and its reduction gives (28), as before. The second term is reduced with the identity,

\[ \left[ (1 - \epsilon) \eta^{\mu \nu} - (4 - 2\epsilon) \frac{\Delta x^\mu \Delta x^\nu}{\Delta x^2} \right] \frac{1}{\Delta x^{4-2\epsilon}} = \]

\[ - \frac{1}{2 - 2\epsilon} [\eta^{\mu \nu} \partial^\nu \partial^\mu - \partial^\mu \partial^\nu] \frac{1}{\Delta x^{2-2\epsilon}} . \] (69)

The fact that the third term goes like \( \Delta x^{\epsilon-4} \) means that its reduction produces a local term,

\[ \left[ \eta^{\mu \nu} - (4 - \epsilon) \frac{\Delta x^\mu \Delta x^\nu}{\Delta x^2} \right] \frac{1}{\Delta x^{4-\epsilon}} = \]

\[ - \frac{1}{2 - \epsilon} [\eta^{\mu \nu} \partial^\nu \partial^\mu - \partial^\mu \partial^\nu] \frac{1}{\Delta x^{2-\epsilon}} - \frac{2 e^2}{\Gamma(2 - \frac{5}{2})} \eta^{\mu \nu} \delta^D(x - x') . \] (70)
This local term completely cancels (63). The sum of all terms from Figures 1 and 2 is therefore,

\[ i [\mu \Pi^\nu_{\mu+2}] (x; x') = \frac{e^2}{8\pi^4} \left\{ \pi^4 \Gamma^2 (1 - \frac{\mu}{2}) \left[ \eta^{\mu\nu} \partial' \cdot \partial - \partial^{\mu} \partial^{\nu} \right] \frac{1}{\Delta x^{4-2\epsilon}} \right. \]

\[-\frac{1}{\eta^{\mu\nu}} \left[ \eta^{\mu\nu} \partial' \cdot \partial - \partial^{\mu} \partial^{\nu} \right] \left[ \frac{1}{2} \ln (H^2 \Delta x^2) + 1 \right] \frac{1}{\Delta x^{2}} + \frac{\eta^{\mu\nu} \ln (H^2 \Delta x^2)}{2\eta^{\mu\nu} \Delta x^{2}} \]

\[ + \left[ \eta \Delta x^{\mu} \delta^{\nu}_0 - \eta' \delta^{\mu}_0 \Delta x^{\nu} - \Delta x^{\mu} \Delta x^{\nu} \right] \left[ \frac{1}{2} \ln (H^2 \Delta x^2) + 1 \right] \left\{ \frac{1}{2} \ln (H^2 \Delta x^2) + 1 \right\} \].

(71)

Note that only the first term — the one that survives in the flat space limit — still requires regularization.

Although expression (71) is transverse this is not yet quite manifest owing to the factor of \(1/(\eta^{\mu\nu})\) still standing to the left of the derivatives in the second term. After some tedious tensor algebra one finds,

\[ i [\mu \Pi^\nu_{\mu+2}] (x; x') = \frac{e^2}{8\pi^4} \left\{ \pi^4 \Gamma^2 (1 - \frac{\mu}{2}) \left[ \eta^{\mu\nu} \partial' \cdot \partial - \partial^{\mu} \partial^{\nu} \right] \frac{1}{\Delta x^{4-2\epsilon}} \right. \]

\[-\left[ \eta^{\mu\nu} \partial' \cdot \partial - \partial^{\mu} \partial^{\nu} \right] \left[ \frac{1}{2} \ln (H^2 \Delta x^2) + 1 \right] \frac{1}{\eta' \Delta x^{2}} \]

\[ + \left[ \eta' \mathbf{\nabla}' \cdot \mathbf{\nabla} - \partial' \partial^{\nu} \right] \left[ \frac{1}{2} \ln (H^2 \Delta x^2) + 1 \right] \left\{ \frac{1}{2} \ln (H^2 \Delta x^2) + 1 \right\} \].

(72)

It remains only to extract the divergence and subtract it with Fig. 3. Although the divergence is exactly the same as in flat space (28) the counterterm is not, owing to the factor of \(a^{D-4}\) in (51). The incomplete cancellation gives rise to a finite factor of \(\ln (a)\) times a delta function. Of course this is the conformal anomaly [12, 13]. The renormalized vacuum polarization is,

\[ i [\mu \Pi^\nu_{\text{ren}}] (x; x') = \frac{e^2}{8\pi^4} \left\{ \left[ \eta^{\mu\nu} \partial' \cdot \partial - \partial^{\mu} \partial^{\nu} \right] \left[ \frac{\partial^2}{\Delta x^{2}} \left( \frac{\ln (\mu^2 \Delta x^2)}{24 \Delta x^5} \right) \right] \right. \]

\[ + \left. \left( \frac{\pi}{2} \ln (H^2 \Delta x^2) + 1 \right) \frac{1}{\eta' \Delta x^2} + \frac{i\pi}{3} \ln (a) \delta^4 (x - x') \right] \]

\[ + \left[ \eta' \mathbf{\nabla}' \cdot \mathbf{\nabla} - \partial' \partial^{\nu} \right] \left[ \frac{1}{2} \ln (H^2 \Delta x^2) + 1 \right] \left\{ \frac{1}{2} \ln (H^2 \Delta x^2) + 1 \right\} \].

(73)

Note that it is completely integrable and gauge invariant, and that we have everywhere taken \(D = 4\).
5 Going on-shell

Taking the vacuum polarization “on-shell” in a locally de Sitter background is complicated by the fact that the “in” vacuum is not equal to the “out” vacuum. This is obvious from the fact that there is particle creation. The photon wave function is therefore not the matrix element of \( A_\mu(x) \) between a 1-photon in-state and the out-vacuum. It is rather the matrix element of \( A_\mu(x) \) between a 1-photon state and the Bunch-Davies vacuum, both prepared at \( t = 0 \). The field equations obeyed by this matrix element are given by varying the Schwinger-Keldysh effective action \([15, 16]\).

The rules for computing in the Schwinger-Keldysh formalism are simple. The diagrams have the same topology as those of Feynman but the endpoints of lines bear either a “+” or a “−” polarity. All external lines are +, whereas vertices can be either all + or all −. The + vertices are the same as those of the standard Feynman rules; the − vertices are conjugated. There are ++, +−, −+ and −− propagators. All of them are the same function \((44)\) of the appropriate version of the modified de Sitter length function

\[
\Delta x^2_{++} (x; x') \equiv \| \vec{x} - \vec{x}' \|^2 - (|\eta - \eta'| - i\delta)^2 , \quad (74)
\]

\[
\Delta x^2_{+−} (x; x') \equiv \| \vec{x} - \vec{x}' \|^2 - (|\eta - \eta'| + i\delta)^2 , \quad (75)
\]

\[
\Delta x^2_{−+} (x; x') \equiv \| \vec{x} - \vec{x}' \|^2 - (|\eta - \eta'| - i\delta)^2 , \quad (76)
\]

\[
\Delta x^2_{−−} (x; x') \equiv \| \vec{x} - \vec{x}' \|^2 - (|\eta - \eta'| + i\delta)^2 . \quad (77)
\]

\( i\Delta_{++}(x; x') \) and \( i\Delta_{+−}(x; x') \) are equal for \( \eta' > \eta \), hence the ++ and +− contributions cancel whenever \( \eta' > \eta \). When the \( x^\mu \) and \( x'^\mu \) are spacelike related, the real part of \( y(x; x') \) is positive; when they are timelike, the real part of \( y(x; x') \) is negative. Therefore meromorphic functions of \( y_{±±}(x; x') \) agree for spacelike separation for \( \delta \to 0 \). That is why the ++ and +− contributions cancel when \( x'^\mu \) strays outside the past lightcone of \( x^\mu \). Inside the past lightcone the ++ and +− propagators are conjugate.

The Schwinger-Keldysh effective action involves background fields \( A^\pm_\mu(x) \) for the + lines and \( A^-_\mu(x) \) for the − lines,

\[
\Gamma[A^+; A^-] = -\frac{1}{4} \int d^4x \left\{ F^+_{\mu\nu} F^+_{\rho\sigma} - F^-_{\mu\nu} F^-_{\rho\sigma} \right\} g^{\mu\rho} g^{\nu\sigma} \sqrt{-g} \\
+ \frac{1}{2} \int d^4x \, d^4x' \left\{ A^+_\mu(x) \left[ \Pi^\nu_{++} \right](x; x') A^+_\nu(x') + A^+_\mu(x) \left[ \Pi^\nu_{+−} \right](x; x') A^-_\nu(x') \right\}
\]
$$+ A_\mu^-(x) \left[ \mu \Pi_{--}^\nu \right] (x; x') A_\nu^+ (x') + A_\mu^-(x) \left[ \mu \Pi_{-+}^\nu \right] (x; x') A_\nu^- (x') \right) + O(A^4).$$

The various ± permutations of the vacuum polarization are all the same function \( (73) \) with the appropriate ± permutation \( (74-77) \) substituted for “\( \Delta x^2 \).” The Schwinger-Keldysh field equations are obtained by varying the effective action with respect to either background and then equating the + and − fields,

$$\frac{\delta \Gamma[A^+; A^-]}{\delta A_\mu^+(x)} \bigg|_{A_\mu^+ = A_\mu} = \partial_\nu \left( \sqrt{-g} g^{\mu\rho} g^{\mu\sigma} F_{\rho\sigma} \right)$$

$$+ \int d^4 x' \left\{ \left[ \mu \Pi_{++}^\nu \right] (x; x') + \left[ \mu \Pi_{+-}^\nu \right] (x; x') \right\} A_\nu (x') + O(A^3) = 0 .$$

Note that we have exploited the relation \( \left[ \mu \Pi_{--}^\nu \right] (x; x') = \left[ \mu \Pi_{-+}^\nu \right] (x; x') \).

Because electromagnetism is conformally invariant for \( D = 4 \) the order \( \hbar^0 \) field equations are the same (in conformal coordinates) as those of flat space. The general classical solution is therefore a superposition of plane waves,

$$A_\mu^0 (x) = \epsilon_\mu (\vec{k}) e^{i \vec{k} \cdot x} ,$$

where \( k^0 = \| \vec{k} \| \) and the Lorentz gauge polarization vectors obey \( \epsilon_0 = 0 = k \cdot \epsilon. \) As in flat space we check for quantum corrections by expanding the solution in powers of \( \hbar, \)

$$A_\mu (x) = A_\mu^0 (x) + A_\mu^1 (x) + \ldots .$$

and then segregating all terms of the same order in the field equations. Potential one loop corrections are determined by the equation,

$$\left[ \partial^2 \eta^{\mu\nu} - \partial_\mu \partial_\nu \right] A_\nu^1 (x) = - \int d^4 x' \left\{ \left[ \mu \Pi_{++}^\nu \right] (x; x') + \left[ \mu \Pi_{+-}^\nu \right] (x; x') \right\} A_\nu^0 (x') .$$

This may seem surprising because there are no ++ or −− seagull graphs or counterterms, so the only diagram topology contributing to \( \left[ \mu \Pi_{++}^\nu \right] (x; x') \) or its conjugate is Fig. 2. The apparent paradox is resolved by noting that the mixed permutations also fail to produce local terms coming from partial integration. Owing to the absence of the temporal absolute value in \( \Delta x^2_{+-} \) we have,

$$\frac{1}{\Delta x^2_{+-}} = - \frac{\partial^2}{2e(1 - \epsilon)} \left[ \frac{1}{\Delta x^2_{+-} - \mu^{-\epsilon}} \right] \rightarrow - \frac{\partial^2}{4} \left[ \ln(\mu^2 \Delta x^2_{+-}) \right].$$
The temporal integration in this equation begins at the initial time of \( \eta' = -H^{-1} \). Its upper limit is irrelevant as long as it comes later that \( \eta \) because the cancellation between ++ and +− contributions eliminates contributions from any point \( x^\mu \) which is outside the past lightcone of \( x^\mu \).

From (73) we see that the vacuum polarization can be expressed as the sum of four distinct terms,

\[
[\mu \Pi^\nu_{\pm \pm}(x; x')] = \frac{ie^2}{8\pi^4} \left\{ \left[ \eta^{\mu \nu} \partial' \cdot \partial - \partial'^\mu \partial' \nu \right] F(\Delta x^2_{\pm \pm}) + \frac{G(\Delta x^2_{\pm \pm})}{\eta'^2 \eta} \right\} .
\]

Note that the conformal anomaly only contributes for the ++ and −− cases. The functions of \( \Delta x^2_{\pm \pm} \) in the other three terms are,

\[
F(\Delta x^2) \equiv \partial^2 \left[ \frac{\ln(\mu^2 \Delta x^2)}{24 \Delta x^2} \right] = \partial^4 \left[ \frac{1}{192} \ln^2(\mu^2 \Delta x^2) - \frac{1}{96} \ln(\mu^2 \Delta x^2) \right] ,
\]

\[
G(\Delta x^2) \equiv \frac{1}{2} \ln(\mu^2 \Delta x^2) + 1 = \partial^2 \left[ \frac{1}{16} \ln^2(\mu^2 \Delta x^2) + \frac{1}{8} \ln(\mu^2 \Delta x^2) \right] ,
\]

\[
K(\Delta x^2) \equiv \frac{1}{8} \ln^2(\mu^2 \Delta x^2) + \frac{1}{2} \ln(H^2 \Delta x^2) .
\]

It is useful to make the additional definitions, \( F(\Delta x^2) \equiv \partial^4 f(\Delta x^2) \) and \( G(\Delta x^2) \equiv \partial^2 g(\Delta x^2) \).

Of course it is easy to compute the conformal anomaly’s contribution to the right hand side of (82). Since this term is local we can actually derive it for an arbitrary vector potential and then specialize to the classical solution,

\[
C^\mu(x) \equiv \frac{-e^2}{24\pi^2} \int d^4x' \left\{ \left[ \eta^{\mu \nu} \partial' \cdot \partial - \partial'^\mu \partial' \nu \right] \ln(a) \delta^4(x - x') \right\} A_\nu(x') ,
\]

\[
= \frac{e^2}{24\pi^2} \partial_\nu \left( \ln(a) F^\nu(\mu(x)) \right) ,
\]

\[
\rightarrow \frac{ie^2}{24\pi^2} H k a(\eta) e^\mu(\vec{k}) e^{ik \cdot x} .
\]

\( C^\mu(x) \) completely dominates the classical term by virtue of the factor of \( a(\eta) \), but it is much less significant than the \( a^2(\eta) \) associated with a true photon mass. Interestingly, the conformal contribution is also purely dispersive.
Since the anomaly contribution (88) is local, it is instructive to combine it with the classical action $S_0$ into an effective action,

$$S_0 + \delta S_{\text{anom}} = \frac{-1}{2} \int d^4x d^4x' A_\mu(x) \left\{ \partial_\mu \partial' \cdot \partial - \partial \partial' \partial \right\} \{ (\eta^{\mu\nu} \partial' \cdot \partial - \partial \partial' \partial) - \text{ln} \left( \frac{a}{a_0} \right) \} \delta^4(x - x') \delta A_\nu(x'). \tag{90}$$

More generally, for a nonabelian gauge theory $G$ coupled to $N_f$ Dirac fermions and $N_s$ complex scalars in representation $r$, we have \[13, 12, 14\]

$$S_0 + \delta S_{\text{anom}} = \frac{-1}{4} \int d^4x \times \left[ 1 + \frac{\alpha g}{3\pi} \left( N_s C(r) + 4N_f C(r) - 11C_2(G) \right) \text{ln} \left( \frac{a}{a_0} \right) \right] \eta^{\mu\rho} \eta^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, \tag{91}$$

where $g$ is the gauge coupling constant, $\alpha_g = g^2/4\pi$, $C_2(G)$ is the quadratic Casimir of the adjoint representation of $G$, which can be defined in terms of the generators of the adjoint representation $T^a$ and the group structure constants $f^{acd}$ as $\text{tr}[T^a T^b] = -f^{acd} f^{bcd} = -C_2(G) \delta^{ab}$, and $C(r)$ is defined in terms of the generators $t^a(r)$ of representation $r$ as $\text{tr}[t^a(r) t^b(r)] = C(r) \delta^{ab}$. For example, for SU($N$), $C_2 = N$, and for fermions and scalars in the fundamental representation of SU($N$), $C(r = \text{fundamental}) = 1/2$. The effect of the anomaly (91) on the photon dynamics is quite moderate \[14\]. Since the photon mass contribution is parametrically much larger, we expect the photon dynamics to be affected much more by the photon mass \[1\].

Our strategy for evaluating the other three contributions is to exploit the fact that the range of $x'{}^\mu$ is independent of $x^\mu$ to first pull the derivatives outside the integration. We then combine the ++ and +− terms to obtain an integrand which is only nonzero for $x'{}^\mu$ within the past lightcone of $x^\mu$. The final step is a sometimes lengthy series of asymptotic expansions under the assumption that the mode under study went super-horizon long ago (hence $k \ll H a(\eta) = -1/\eta$) after a long period of inflation (hence $H \ll k$). We shall organize these expansions in terms of two dimensionless parameters,

$$y \equiv -k \eta \quad \text{and} \quad w \equiv \frac{k}{H}. \tag{92}$$

The physically interesting region is $0 < y \ll 1 \ll w \ll a$. At horizon crossing one would have $y \approx 1$ and $w \approx a$. Evolution of the physical scales
in de Sitter inflation is illustrated in figure 4. The physical wave length \( \lambda_{\text{phys}} \equiv 2\pi a/k = 2\pi/H y \) grows with time, while the Hubble radius remains constant.

Figure 4: Evolution of the physical scales in de Sitter inflation. Horizon crossing occurs at \( \eta_x \).

Although it is the least important of the four terms, an exact result can be obtained for the “F” contribution to the right hand side of (82). We begin by reflecting the \( x'\mu \) derivatives (\( \partial'_\mu F(\Delta x^2) = -\partial_\mu F(\Delta x^2) \)) and then pulling all derivatives outside the integration,

\[
F^\mu(x) = -\frac{ie^2}{8\pi^4} \int d^4x' \left\{ \left[ \eta^{\mu\nu} \partial' \cdot \partial - \partial'^\mu \partial'\nu \right] \left[ F(\Delta x_{++}^2) - F(\Delta x_{+-}^2) \right] \right\} \epsilon_\nu e^{ik \cdot x'} , \tag{93}
\]

\[
= \frac{ie^2}{8\pi^4} \left[ \epsilon^\mu \partial^2 - \epsilon \cdot \partial \eta^\mu \right] \partial^4 \int d^4x' \left\{ f(\Delta x_{++}^2) - f(\Delta x_{+-}^2) \right\} e^{ik \cdot x'} . \tag{94}
\]

The difference between the ++ and +- contributions vanishes for \( x'\mu \) outside the past lightcone of \( x^\mu \),

\[
f(\Delta x_{++}^2) - f(\Delta x_{+-}^2) = \frac{i\pi}{48} \left\{ \ln \left[ \mu^2(\Delta \eta^2 - ||\vec{x}||^2) \right] - 1 \right\} \theta(\Delta \eta) \theta(\Delta \eta - ||\vec{x}||) . \tag{95}
\]

We next change variables to \( \Delta x^\mu \equiv x^\mu - x'^\mu \) and perform the angular integrations,

\[
F^\mu(x) = \frac{ie^2}{8\pi^4} \left[ \epsilon^\mu \partial^2 - \epsilon \cdot \partial \eta^\mu \right] \partial^4 e^{ik \cdot x} \int_0^{\eta_{+H-1}} d\Delta \eta e^{ik \Delta \eta} .
\]
\[
\times 4\pi \int_0^{\Delta \eta} drr \frac{2\sin(\eta r) i\pi}{kr} \left\{ \ln\left[ \mu^2 (\Delta \eta^2 - r^2) \right] - 1 \right\}.
\] (96)

The radial integration involves a combination of special functions \( \xi(k\Delta \eta) \) that we will meet again,

\[
\int_0^{\Delta \eta} drr \sin(\eta r) \left\{ \ln\left[ \mu^2 (\Delta \eta^2 - r^2) \right] - 1 \right\}
= \Delta \eta^2 \int_0^{1} dx x \sin(k\Delta \eta x) \left\{ 2 \ln(\mu \Delta \eta) - 1 + \ln(1 - x^2) \right\},
\] (97)

\[
= \frac{1}{k^2} \left[ \sin(k\Delta \eta) - k\Delta \eta \cos(k\Delta \eta) \right] \left[ 2 \ln(\mu \Delta \eta) - 1 \right] + \Delta \eta^2 \xi(k\Delta \eta) .
\] (98)

Our definition for \( \xi \) is,\(^3\)

\[
\xi(\alpha) \equiv \int_0^{1} dx x \sin(\alpha x) \ln(1 - x^2),
\] (99)

\[
= \frac{2}{\alpha^2} \sin(\alpha) - \frac{1}{\alpha^2} \left[ \cos(\alpha) + \alpha \sin(\alpha) \right] \left[ \text{si}(2\alpha) + \frac{\pi}{2} \right]
+ \frac{1}{\alpha^2} \left[ \sin(\alpha) - \alpha \cos(\alpha) \right] \left[ \text{ci}(2\alpha) - \gamma - \ln\left( \frac{\alpha}{2} \right) \right].
\] (100)

Here \( \gamma \approx .577 \) is Euler’s constant. The following small and large \( \alpha \) expansions are sometimes useful,

\[
\xi(\alpha) = - \left[ \frac{8}{9} - \frac{2}{3} \ln(2) \right] \alpha + O(\alpha^3),
\] (101)

\[
= \frac{\ln(\alpha)}{\alpha} \cos(\alpha) + \frac{1}{\alpha} \left[ \gamma \cos(\alpha) - \frac{\pi}{2} \sin(\alpha) \right] + O \left( \frac{\ln(\alpha)}{\alpha^2} \right).
\] (102)

Combining the previous results gives

\[
F^\mu(x) = \left[ \epsilon^\mu \partial^2 - \epsilon \cdot \partial \partial^\mu \right] \partial^4 e^{ik.x} I(\eta)
\]

where,

\[
I(\eta) \equiv - \frac{e^2}{96\pi^2 k^3} \int_0^{\eta + \mu - 1} d\Delta \eta e^{ik\Delta \eta}
\]

\(^3\)We use the following notation for the sine and cosine integrals:

\[
si(x) \equiv - \int_x^\infty dt \frac{\sin(t)}{t} = - \frac{\pi}{2} + \int_0^x dt \frac{\sin(t)}{t} ,
\]

\[
ci(x) \equiv - \int_x^\infty dt \frac{\cos(t)}{t} = \gamma + \ln(x) + \int_0^x dt \frac{\cos(t) - 1}{t} .
\]
\[ \times \left\{ \left[ \sin(k\Delta\eta) - k\Delta\eta \cos(k\Delta\eta) \right] \left[ 2 \ln(\mu\Delta\eta) - 1 \right] + k^2 \Delta\eta^2 \xi(k\Delta\eta) \right\}. \tag{103} \]

The only \( \vec{x} \) dependence in this expression resides in the outer exponential factor. We can act with the spatial derivatives and then commute the temporal derivatives through the exponential to obtain,

\[ \left[ e^\mu \partial^2 - \epsilon \cdot \partial \partial^\mu \right] \partial^4 e^{ik \cdot x} I(\eta) = e^\mu (-\partial_0^2 - k^2)^3 e^{ik \cdot x} I(\eta), \quad \tag{104} \]

\[ = -e^\mu e^{ik \cdot x} (\partial_0^2 - 2ik \partial_0)^3 I(\eta). \tag{105} \]

After commutation the only \( \eta \) dependence to the right of the derivatives resides in the upper limit of the \( \Delta\eta \) integration. So one of the \( \partial_0 \)'s acts to undo the integral and we are left with the messy task of taking the remaining derivatives. The final result is,

\[ F^\mu(\eta) = -\frac{e^2}{24\pi^2} \frac{kH}{1 - a^{-1}} \left( i + \frac{H}{k} \left[ e^{\frac{k}{\eta}(1 - a^{-1})} \right] \right) \epsilon^\mu e^{ik \cdot x}. \tag{106} \]

Because it lacks positive powers of the scale factor, \( F^\mu(x) \) is far weaker than \( C^\mu(x) \). In fact it is only nonzero because the process begins at \( t = 0 (\eta = -H^{-1}) \), rather than at \( t = -\infty (\eta = -\infty) \). This is as it should be since the “F” contributions to the vacuum polarization are the same as those of flat space, which vanish when exactly on-shell. We get a nonzero result because beginning at a finite time precludes one from going precisely on-shell. This is an important check on the accuracy and consistency of the process.

The “K” contributions are reduced in almost the same way. They have zero temporal component by definition. Their spatial components are,

\[ K^i(x) \]

\[ \equiv \frac{i e^2}{8\pi^4} \int d^4 x' \left[ \delta^{ij} \vec{\nabla}' \cdot \vec{\nabla} - \partial^0 \partial^j \right] \left[ H(\Delta x^2_{++}) - H(\Delta x^2_{+-}) \right] \frac{H(\Delta x^2_{++}) - H(\Delta x^2_{+-})}{\eta^2 \eta'} \epsilon_j e^{i k \cdot x'}, \tag{107} \]

\[ = -\frac{i e^2}{8\pi^4 \eta^2} \left[ \epsilon^i \nabla^2 - \epsilon^i \nabla \partial^j \right] \int d^4 x' \epsilon^{i k \cdot x'} \left[ H(\Delta x^2_{++}) - H(\Delta x^2_{+-}) \right], \quad \tag{108} \]

\[ = -\frac{i e^2 H^2}{8\pi^4} \frac{a^2}{\eta^2} \left[ \epsilon^i \nabla^2 - \epsilon^i \nabla \partial^j \right] e^{i k \cdot x} \int_0^{\eta + H^{-1}} d\Delta\eta e^{i k \Delta\eta} \frac{e^{i k \Delta\eta}}{(\Delta\eta - \eta)^2} \times 4\pi \int_0^{\Delta\eta} dr r^2 \sin(kr) \frac{i\pi}{2} \left\{ \ln \left[ H^2(\Delta\eta^2 - r^2) \right] + 2 \right\}, \tag{109} \]

\[ 22 \]
\[= \frac{e^2 H^2}{4\pi^2 k^3} a^2 \left[ \epsilon \nabla^2 - \epsilon \cdot \tilde{\nabla} \partial^i \right] e^{ik \cdot x} \int_0^{\eta + H^{-1}} d\Delta \eta \frac{e^{ik \Delta \eta}}{(\Delta \eta - \eta)^2} \]
\[\times \left\{ \left[ \sin(k \Delta \eta) - k \Delta \eta \cos(k \Delta \eta) \right] \left[ 2 \ln(H \Delta \eta) + 2 \right] + k^2 \Delta \eta^2 \xi(k \Delta \eta) \right\}, \quad (110)\]
\[= -\frac{e^2 H^2}{4\pi^2} a^2 e^{ik \cdot x} \int_0^{y-k} dz \frac{e^{iz}}{(z+y)^2} \]
\[\times \left\{ \left[ \sin(z) - z \cos(z) \right] \left[ 2 \ln(z) - 2 \ln(w) + 2 \right] + z^2 \xi(z) \right\}. \quad (111)\]

(Recall that we define \( y \equiv -k \eta \ll 1 \) and \( w \equiv k/H \gg 1 \).) The next step would be making an asymptotic expansion of the integral but this would be wasted effort because almost the same integral occurs with the opposite sign in the “G” contributions.

The contributions from \( G^{(85)} \) to the right hand side of (82) are the most difficult to evaluate owing to the combination of temporal derivatives and the factor of \( 1/\eta \eta' \). Because of this it is desirable to act with some of the derivatives and exploit the transversality of the polarization vector at an earlier stage than with the other terms,

\[G^\mu(x)\]
\[\equiv -\frac{ie^2}{8\pi^4} \int d^4x' \left\{ \eta^{\mu \nu} \partial' \cdot \partial - \eta^{\mu \nu} \partial' \right\} \left[ \frac{G(\Delta x^2_{++}) - G(\Delta x^2_{+-})}{\eta \eta'} \right] \epsilon \nu e^{ik \cdot x'}, \quad (112)\]
\[= -\frac{ie^2}{8\pi^4} \left[ \epsilon^\mu \partial' - \eta^{\mu \nu} \epsilon \cdot \partial \right] \frac{1}{\eta} \partial^2 e^{ik \cdot x} \]
\[\times \int d^4x' e^{-ik \cdot x} \left[ g(\Delta x^2_{++}) - g(\Delta x^2_{+-}) \right] \eta', \quad (113)\]
\[= \frac{ie^2}{8\pi^4} H^2 a^2 \epsilon^\mu e^{ik \cdot x} \left[ \delta^\nu_0 + \eta(-\delta^\nu_0 \partial_0 + ik^\nu) \right] (\partial_0 - 2ik) \partial_0 \]
\[\times \int d^4x' e^{-ik \cdot x} \left[ g(\Delta x^2_{++}) - g(\Delta x^2_{+-}) \right] \eta'. \quad (114)\]

At this point it is best to partially integrate the \( \partial'_\nu \). Owing to the cancellation between \( g(\Delta x^2_{++}) \) and \( g(\Delta x^2_{+-}) \) outside the past light cone, only the lower, temporal surface term can survive,

\[\int d^4x' e^{-ik \cdot x} \left[ g(\Delta x^2_{++}) - g(\Delta x^2_{+-}) \right] \eta'. \]
\[-i k_\nu \int d^4 x' e^{-i k \cdot \Delta x} \left[ g(\Delta x_{++}^2) - g(\Delta x_{+-}^2) \right] + \delta^0_\nu H \int d^3 x' e^{-i k \cdot \Delta x} \left[ g(\Delta x_{++}^2) - g(\Delta x_{+-}^2) \right] \eta' \bigg|_{\eta' = -H^{-1}}. \quad (115)\]

Since \( k^\nu k_\nu = 0 \) we see that only the \( \nu = 0 \) component survives.

The spatial integral can be evaluated by familiar techniques,
\[
\int d^3 x' e^{-i k \cdot \Delta x} \left[ g(\Delta x_{++}^2) + g(\Delta x_{+-}^2) \right] = e^{ik \Delta \eta} \frac{1}{4\pi} \int_0^\Delta \eta dr r^2 \sin(kr) \frac{i\pi}{4} \left\{ \ln[H^2(\Delta \eta^2 - r^2)] + 1 \right\}, \quad (116)\]
\[
= \frac{\pi^2 i}{k^3} e^{ik \Delta \eta} \left\{ [\sin(k \Delta \eta - k \Delta \eta \cos(k \Delta \eta)] \right. \nonumber \times \left[2 \ln(k \Delta \eta) - 2 \ln(w) + 1\right] + k^2 \Delta \eta^2 \xi(k \Delta \eta) \bigg\}, \quad (117)\]
\[
= \frac{\pi^2 i}{k^3} \Xi(k \Delta \eta, w). \quad (118)\]

The function \( \Xi(x, w) \) has the following asymptotic expansions for small and large \( x \) respectively,
\[
\Xi(x, w) = \frac{2}{3} x^3 \left[ \ln\left(\frac{2x}{w}\right) - \frac{5}{6} \right] + O\left(x^5 \ln(x)\right), \quad (119)\]
\[
= xe^{ix} \left[ \left(2 \ln(w) - \ln(2x) - (1 - \gamma)\right) \cos(x) - \frac{\pi}{2} \sin(x) \right] + O\left(\ln(xw)\right). \quad (120)\]

Substituting the spatial integral into (113), and inserting the result into (114) gives,
\[
G^\mu(x) = -\frac{e^2 H^2}{8\pi^2} a^2 e^\mu e^{ik \cdot x} (1 - y \partial_y) (2 - i \partial_y) \partial_y \int_0^{w-y} dz \frac{\Xi(z, w)}{z + y} \quad (121)\]
\[
= \frac{e^2 H^2}{8\pi^2} a^2 e^\mu e^{ik \cdot x} (2 - i \partial_y - 2y \partial_y + iy \partial_y^2) \int_0^{w-y} dz \frac{\Xi(z, w)}{(z + y)^2} \quad (122)\]
(Recall that we define \( y \equiv -k\eta \ll 1 \) and \( w \equiv k/H \gg 1 \).) Now note that the first term on the top line almost exactly cancels \( K^\mu(x) \),

\[
K^\mu(x) + \frac{e^2 H^2}{8\pi^2} a^2 e^{i k \cdot x} \times 2 \int_0^{w-y} dz \frac{\Xi(z, w)}{(z+y)^2}
\]

\[
= -\frac{e^2 H^2}{4\pi^2} a^2 e^{i k \cdot x} \int_0^{w-y} dz \frac{e^{iz}}{(z+y)^2} \left[ \sin(z) - z \cos(z) \right], \quad (123)
\]

\[
= -\frac{e^2 H^2}{4\pi^2} a^2 e^{i k \cdot x} \left\{ \int_0^w dz \frac{e^{iz}}{z^2} \left[ \sin(z) - z \cos(z) \right] + O(y) \right\}, \quad (124)
\]

\[
= \frac{e^2 H^2}{8\pi^2} a^2 e^{i k \cdot x} \left\{ \ln(w) + O(1) \right\}. \quad (125)
\]

This turns out to be the magnitude of the leading order contribution for super-horizon modes late during inflation.

Another leading order contribution comes from the second term on the top line of (122). To save space we suppress the prefactor of \( \frac{e^2 H^2}{8\pi^2} a^2 e^{i k \cdot x} \),

\[-i \partial_y \int_0^{w-y} dz \frac{\Xi(z, w)}{(z+y)^2} = \frac{i}{w^2} \Xi(w - y, w) + 2i \int_0^{w-y} dz \frac{\Xi(z, w)}{(z+y)^3}. \quad (126)\]

The first term on the left hand side is of order \( \ln(w)/w \) but the integral can make a leading order contribution. Since the integrand converges for \( y = 0 \) and \( w \to \infty \), the desired leading term comes from the explicit factor of \( \ln(w) \) in the function \( \Xi(z, w) \),

\[
2i \int_0^{w-y} dz \frac{\Xi(z, w)}{(z+y)^3}
\]

\[
= -4i \ln(w) \int_0^\infty dz \frac{e^{iz}}{z^3} \left[ \sin(z) - z \cos(z) \right] + O(1), \quad (127)
\]

\[= -4i \ln(w) \times \frac{i}{4z^2} \left[ e^{2iz} - 1 - 2iz \right] \Big|_0^\infty + O(1), \quad (128)\]

\[= 2 \ln(w) + O(1). \quad (129)\]

The final leading order contribution comes from the first term on the bottom line of (122). It is useful to first note the exact identity,

\[
(2 + i\partial_x) \Xi(x, w) =
\]

25
\[
-x\left\{\text{Ci}(2x) + \ln(2x) + 1 - \gamma - 2 \ln(w)\right\} + i\left\{\text{Si}(2x) + \frac{\pi}{2}\right\} + e^{ix}\left\{\sin(x) \times \text{Ci}(2x) + \ln(2x) + 3 - \gamma - 2 \ln(w)\right\} - i\cos(x)\left[\text{Si}(2x) + \frac{\pi}{2}\right]\}. \tag{130}
\]

Setting \(x = w - y\) we conclude,

\[
(2 - i\partial_y)\frac{\Xi(w - y, w)}{w} = \ln(w) + O(1). \tag{131}
\]

Combining all of the leading terms we obtain the following result for the right hand side of Eq. (82),

\[
C^\mu(x) + F^\mu(x) + G^\mu(x) + K^\mu(x) = a^2(\eta)\frac{e^2H^2}{2\pi^2}\left\{\ln(w) + O(1)\right\}A^\mu_0(x). \tag{132}
\]

Note that the contributions from \(C^\mu(x)\) and \(F^\mu(x)\) are down by factors of \(w/a \equiv k_{\text{phys}}/H \ll 1\) and \((w/a)^2\), respectively. Equation (132) is consistent with a photon mass of,

\[
m^2_\gamma = \frac{e^2H^2}{2\pi^2} \ln\left(\frac{k}{H}\right). \tag{133}
\]

It is important to keep in mind that this result is perturbative. A full non-perturbative analysis of (82) is thus required in order to calculate reliably the photon mass.

6 Discussion

We have presented a long calculation in perturbative quantum field theory and it is worth commenting on the matter of reliability. Our result (73) for the vacuum polarization passes many important consistency checks. The first of these is gauge invariance. This is not trivial even in flat space. In a locally de Sitter background it requires a horrifying series of seemingly unrelated terms to combine into transverse projection operators. Yet they do.

Another important accuracy check is that there are no new ultraviolet divergences. There should not be if the theory is to stay renormalizable because there are no new counterterms in de Sitter background. Related to this is the fact that our result has the correct flat space limit. A final check is that the conformal anomaly term agrees with standard results [12, 13].
The structure of the self energy \( \Pi^{\mu\nu}(k) \) is very similar to that of the photon self energy in thermal QED. The self energy (in momentum space) is usually parametrized by two functions, \( \Pi^{\mu\nu}(k) = P_T^{\mu\nu}\Pi_t(k) + P_L^{\mu\nu}\Pi_l(k) \), where \( P_T^{\mu\nu} = \eta^\mu\eta^\nu(\delta_{ij} - \frac{k_i k_j}{k^2}) \) and \( P_L^{\mu\nu} = \eta^\mu\eta^\nu - k^\mu k^\nu/k^2 - P_T^{\mu\nu} \) are the (spatially) transverse and ‘longitudinal’ (in fact time-like transverse) projectors. Our result (73) can be viewed as the space-time generalization of the transverse and ‘logitudinal’ vacuum polarizations,

\[
\Pi_t(x, x') = \partial' \cdot \partial \Pi^{(1)}(x, x') + \nabla' \cdot \nabla \Pi^{(2)}(x, x'), \\
\Pi_l(x, x') = \partial' \cdot \partial \Pi^{(1)}(x, x'),
\]

where \( \Pi^{(1)} \) and \( \Pi^{(2)} \) are the transverse and spatially transverse contributions to (73) (cf. also Eq. (135)). In thermal QED vacuum polarization modifies the photon dynamics. In the static limit ‘longitudinal’ photons are Debye-screened by the fermionic plasma, with the Debye mass, \( m_D = eT/\sqrt{3} \), while transverse photons are screened only dynamically. At high momenta \( k \gg T \), the ‘longitudinal’ modes become unphysical, while the transverse photons propagate as massive particles with the thermally induced mass \( m_T = eT/\sqrt{6} \). Based on the above mentioned similarities, we expect that a more detailed study of the vacuum polarization (73) should reveal analogous physical effects on the photon dynamics in inflation.

We are very confident about our result (73) for \( [\mu \Pi^\nu](x, x') \). It has to be admitted that the process of going on-shell is much less well checked. This is also the most complicated part of the calculation. One important point is that (106), the flat space contribution \( F^{\mu}(x) \), vanishes in the limit that the initial time is taken to negative infinity. Although \( F^{\mu}(x) \) makes the weakest of the various contributions, its reduction is quite similar to that of the crucial \( G^{\mu}(x) \) and \( K^{\mu}(x) \) contributions. So the fact that \( F^{\mu}(x) \) obeys an important correspondence limit partially checks them as well.

Our result is consistent with a photon mass of \( m_\gamma^2 = \frac{e^2 H^2}{2\pi^2} \ln(k/H) \). Interestingly, this is precisely what follows from the Hartree-Fock estimate (4) if one replaces the time dependent factor of \( \ln(a) \) by its value at horizon crossing, \( \ln(k/H) \). It is premature to make too much of this coincidence. Although our 1-loop vacuum polarization is exact, all the work of taking it on-shell really demonstrates is that one loop corrections to the classical photon wave function become non-perturbatively large.

To actually solve for the photon wave function and show that it approaches that of a massive photon requires two extensions of the current
work. First, we must establish control over higher loop corrections. If the one loop correction becomes large then why does the two loop correction not give an even bigger effect? There is a curious parallel between what we must do and the problem that Schwinger faced for $D = 2$, massless QED in flat space [5]. Just as it was possible to use the one loop result in that context, so we believe it will prove to be in this case.

To understand how this can be, note that we know quite a lot about the general structure of the vacuum polarization. As a consequence of gauge invariance, spatial translation invariance and spatial rotational invariance, it must have the form,

$$\left[ \mu \Pi^\nu \right](x; x') = \left[ \eta^\mu \partial^\nu \cdot \partial - \delta^\mu \delta^\nu \right] A(\eta, \eta'; \Delta x^2) + \left[ \eta^\mu \vec{\nabla}' \cdot \vec{\nabla} - \vec{\partial}' \vec{\partial} \right] B(\eta, \eta'; \Delta x^2).$$ (135)

The functions $A(\eta, \eta'; \Delta x^2)$ and $B(\eta, \eta'; \Delta x^2)$ must be symmetric under interchange of $\eta$ and $\eta'$, and they must have the dimensions of inverse length to the fourth power. The full flat space result must reside in $A(\eta, \eta'; \Delta x^2)$, but it will be negligible in de Sitter background. Terms that matter for de Sitter are those with factors of $1/\eta$ and $1/\eta'$. We believe quite strongly that $A(\eta, \eta'; \Delta x^2)$ can contain at most $1/(\eta \eta')$ and that $B(\eta, \eta'; \Delta x^2)$ can contain at most $1/(\eta^2 \eta'^2)$.

Obviously each loop will contribute a factor of $e^2$. The only really difficult thing to guess is the number of logarithms. We believe — though less strongly — that the general result at $\ell$ loops is $A(\eta, \eta'; \Delta x^2)$ contains up to $\ell$ logarithms whereas $B(\eta, \eta'; \Delta x^2)$ contains up to $\ell + 1$. We further believe that in each case these logarithms can translate, after going on-shell, into up to one factor of $\ln(k/H)$ for each loop order. Since there is also a factor of $\alpha \equiv e^2/4\pi$ for each extra loop, this suggests that retaining only the one loop part would be reliable for modes which obey $\alpha \ln(k/H) < 1$. That is, the one loop term would dominate the classical one by a factor of $\alpha^2(\eta)$, but the two loop correction to it would be down by a factor of $\alpha \ln(k/H)$.

This seems a reasonable and probably provable conjecture. If it is true then we can essentially get the full photon wave function by solving the integral-differential equation obtained from just the classical term and the one loop vacuum polarization in (79). Which brings us to the second necessary extension of the current work: solving such an equation. Note that spatial
translation invariance implies we can still use spatial plane waves. But then
the spatial integrations are identical to the ones we already did in Section 5!
We can therefore reduce the problem to one of solving for the multiplicative
function of $\eta$. This is probably tractable analytically in some reasonable
approximation. If not, it can certainly be done numerically.

Another issue is the extent to which our process repeats the Higgs mecha-
nism of flat space. We do not have Poincare invariance during inflation
but one’s expectation is still that a massive photon has three polarizations.
Since no new degrees of freedom have been granted to the vector potential
one might suspect that the third polarization must come from the deriva-
tive of the scalar phase. It would be interesting to check this.

Finally, we comment on the possibility of important stochastic effects.
Although we have defined the photon wave function as the matrix element of
$A_\mu(x)$ between Bunch-Davies vacuum and a simultaneously prepared plane
wave photon state, one should bear in mind that the quantum averaging
implicit in matrix elements may give misleading results. The actual charge
density induced by inflationary particle production is not smooth but rather
stochastic [17], corresponding to a highly squeezed state. About one massless,
charged scalar exists per Hubble volume, moving at the speed of light in a
random direction. We do not expect that this distinction amounts to a
significant difference for super-horizon photons, which are affected by many
different Hubble volumes. This can and should be checked.

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References

[1] A. C. Davis, K. Dimopoulos, T. Prokopec and O. Törnkvist, Phys. Lett.
B501, 165 (2001), arXiv: astro-ph/0007214.
[2] A. C. Davis, K. Dimopoulos, T. Prokopec and O. Törnkvist, Phys. Rev. D65, 063505 (2002), arXiv: astro-ph/0108093.

[3] A. Vilenkin and L. H. Ford, Phys. Rev. D26, 1231 (1982).

[4] T. Prokopec, O. Törnkvist and R. P. Woodard, “Photon mass from inflation,” arXiv: astro-ph/0205331.

[5] J. Schwinger, Phys. Rev. 162, 2425 (1962).

[6] B. Allen and A. Folacci, Phys. Rev. D35, 3771 (1987).

[7] L. H. Ford and L. Parker, Phys. Rev. D16, 245 (1977).

[8] S. A. Fulling, M. Sweeny and R. M. Wald, Commun. Math. Phys. 63, 257 (1978).

[9] N. C. Tsamis and R. P. Woodard, Class. Quant. Grav. 11, 2969 (1994).

[10] N. C. Tsamis and R. P. Woodard, Phys. Rev. D54, 2621 (1996), arXiv: hep-ph/9602317.

[11] V. K. Onemli and R. P. Woodard, “Super-Acceleration from Massless, Minimally Coupled $\phi^4$”, arXiv: gr-qc/0204065.

[12] A. D. Dolgov, Sov. Phys. JETP 54, 223 (1981); Zh. Eksp. Teor. Fiz. 81, 417 (1981).

[13] F. D. Mazzitelli and F. M. Spedalieri, Phys. Rev. D52, 6694 (1995), arXiv: astro-ph/9505140.

[14] T. Prokopec, “Cosmological magnetic fields from photon coupling to fermions and bosons in inflation,” arXiv: astro-ph/0106247.

[15] J. Schwinger, J. Math. Phys. 2, 407 (1961).

[16] R. D. Jordan, Phys. Rev. D33, 444 (1986).

[17] A. D. Linde, D. A. Linde and A. Mezh lumian, Phys. Rev. D49, 1783 (1994).