Multipole moment and singular source in Newtonian gravity and in Einstein gravity

Yu-Zhu Chen,\textsuperscript{a,b,1} Yu-Jie Chen,\textsuperscript{b} Shi-Lin Li,\textsuperscript{b} Wu-Sheng Dai\textsuperscript{b,2}

\textsuperscript{a}Theoretical Physics Division, Chern Institute of Mathematics and LPMC, Nankai University, Tianjin 300071, P.R. China
\textsuperscript{b}Department of Physics, Tianjin University, Tianjin 300350, P.R. China

Abstract: The multipole moments are defined as the multipole expansion coefficients of the gravitational field at infinity. In Newtonian gravity, the multipole moments are determined by the source distribution — the multipole integrals of the source. In this paper, we show that the multipole moments in general relativity cannot be determined by the multipole integrals of the source. We provide the multipole integrals in static axial spacetimes, such as, the Curzon spacetime. The Curzon spacetime possesses the same multipole integrals of the source with the Schwarzschild spacetime, while they possess different multipole moments.

Keywords: multipole moment; multipole integral; singular source;

\textsuperscript{1}chenyuzhu@nankai.edu.cn
\textsuperscript{2}daiwusheng@tju.edu.cn
1 Introduction

In Newtonian gravity, the multipole moments of the gravitational field are determined by the multipole integrals of the source. However, in general relativity, the relation between the multipole moments of the gravitational field and the multipole integrals of the source is unclear.

The multipole moments in general relativity are first defined by Geroch for static space-times \cite{1,2}. The definition is then generalized to stationary spacetimes by Hansen \cite{3}. Multipole moments are also defined by Thorne \cite{4}, Beig \cite{5}, and others \cite{6}. In ref. \cite{7–10}, the authors provide serial calculations and discussions of multipole moments in stationary spacetimes. Multipole moments are related with source integrals in recent works \cite{11,12}. In ref. \cite{12}, the authors provide a pellucid review about the definition of multipole moments.

The multipole integrals of the source are rarely discussed in general relativity. One reason is that most exact solutions of the Einstein equation are vacuum solutions. Nevertheless, singular sources are discussed in gravitational collapse problems. A singularity is regarded as a singular source with infinite density while finite mass \cite{13}. In a previous work \cite{14}, we provide the quantitative source in the Schwarzschild spacetime and show that the NUT metric which is generally considered as non-curvature singularity metric possesses a Dirac-delta source.
In this paper, we show that there are different spacetimes possess the same multipole integrals of the source so that the multipole moments of the gravitational field cannot be determined by the multipole integrals of the source.

Spacetimes considered in this paper are vacuum solutions outside singularities. Nevertheless, they possess singular sources at singularities. We provide the singular sources in special static vacuum axial solutions of the Einstein equation, i.e. the Levi-Civita spacetime and the Curzon spacetime. The singular sources contribute to the multipole integrals of the source.

The multipole integrals contributed by the singular source can be calculated directly, so does the multipole expansion of the spacetime at infinity. That is, we obtain the multipole moments of the gravitational field and the multipole integrals of the source simultaneously in general relativity. The Curzon spacetime and the Schwarzschild spacetime possess same multipole integrals, while they possess different multipole expansions at infinity. This result shows that the multipole moments of the gravitational field in general relativity cannot determined by the multipole integrals of the source.

In section 2, we introduce the definition of multipole moments in general relativity and restate the multipole moments problem in general relativity. That is, whether the multipole moments of the gravitational field is determined by the multipole integrals of the source in general relativity. In section 3, we show that there exist singular sources at singularities in static axial spacetimes and calculate multipole integrals contributed by the singular sources. Specifically, we calculate the singular source and multipole integrals in Newtonian gravity, in Levi-Civita spacetime, and in the Curzon spacetime. These results are used to discuss the multipole moments problem in general relativity. In section 4, we provide the conclusion and other open questions concerning singular sources and multipole moments in general relativity.

2 Multipole moment in general relativity: the problem

In this section, we introduce the multipole moments problem in general relativity. Generally, the problem is whether the multipole moments of the gravitational field in general relativity is determined by the multipole integrals of the source or not.

2.1 Multipole moment in Newtonian gravity

We begin with the definition of the multipole moments in Newtonian gravity. In Newtonian gravity, the field $\phi$ satisfies the Possion equation

$$\nabla^2 \phi (x) = 4\pi \rho (x),$$  \hspace{1cm} (2.1)

where $\rho (x)$ is the density of the source and gravitational constant $G \equiv 1$. With the Green function method, $\phi$ can be expressed as

$$\phi (x) = -\int_V d^3x' \frac{\rho (x')}{|x - x'|}.$$  \hspace{1cm} (2.2)
With the expansion at \( x \to \infty \) or \( x' \to 0 \)

\[
\frac{1}{|x - x'|} = \frac{4\pi}{2l + 1} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r'^l}{r^{l+1}} \bar{Y}_{lm}(\theta, \phi) Y_{lm}^\ast(\theta', \phi') \quad (r > r') , \tag{2.3}
\]

where \( x = (r, \theta, \varphi) \quad x' = (r', \theta', \varphi') \), \( Y_{lm}(\theta, \phi) \) are the spherical harmonics and \( \ast \) represents the complex conjugate, we have

\[
\phi(x) = -\frac{4\pi}{2l + 1} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{Q^{(lm)}}{r^{l+1}} \bar{Y}_{lm}(\theta, \varphi) \tag{2.4}
\]

with

\[
Q^{(lm)} \equiv \int_V d^3x' \rho(x') x'^l \bar{Y}_{lm}(\theta', \varphi'). \tag{2.5}
\]

\( Q^{(lm)} \) are the multipole moments [15].

We can also expand \( \frac{1}{|x - x'|} \) in the cartesian coordinate

\[
\frac{1}{|x - x'|} = \sum_{n=0}^{\infty} \sum_{i_1=1}^{3} \cdots \sum_{i_n=1}^{3} \frac{(2n - 1)!!}{n!} \frac{1}{r^{2n+1}} x^{i_1 \cdots i_n} \tag{2.6}
\]

where \( x = (x^1, x^2, x^3) \), \( x' = (x'^1, x'^2, x'^3) \), \( r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \), \( (2n - 1)!! = (2n - 1)(2n - 3) \cdots 3 \cdot 1 \), and \( x^{i_1 \cdots i_n} \) is the traceless part of \( x^1 x^2 \cdots x^n \), for example,

\[
x^j = x^j, \quad x^{jk} = x^j x^k - \frac{1}{3} \delta^{jk} \delta_{mn} x^m x^n = x^j x^k - \frac{1}{3} \delta^{jk} r^2.
\]

With the expansion (2.6), we have

\[
\phi(x) = -\sum_{n=0}^{\infty} \frac{(2n - 1)!!}{n!} \frac{1}{r^{2n+1}} \sum_{i_1=1}^{3} \cdots \sum_{i_n=1}^{3} \int_V d^3x' \rho(x') x^{i_1 \cdots i_n} \bar{Y}_{i_1 \cdots i_n} \tag{2.7}
\]

with the multipole moments

\[
Q^{(i_1 \cdots i_n)} \equiv \int_V d^3x' \rho(x') x^{i_1 \cdots i_n}. \tag{2.8}
\]

\( \frac{1}{r^{2n+1}} x^{i_1 \cdots i_n} \) and \( Y_{lm}(\theta, \varphi) \) can be expressed by each other since they are two complete bases [4]. That is, the multipole moments defined in eq. (2.5) and eq. (2.8) are equivalent to each other.

The multipole moments play two roles in eqs. (2.4) and (2.5). On the one hand, the multipole moments are the expansion coefficients of the field at infinity. On the other hand, the multipole moments are the multipole integrals over the source. In this paper, the expansion coefficients of the gravitational field at infinity are called the multipole moments of the gravitational field or the multipole moments of the spacetime, and the multipole integrals of the source are called the multipole moments of the source. In Newtonian gravity, these two physical quantities coincide. That is, the gravitational field or the multipole moments of the spacetime are completely determined by the multipole integrals of the source or the multipole moments of the source.
2.2 Multipole moment in general relativity

In general relativity, the multipole integrals of the source and the multipole moments of the spacetime are two different physical quantities. We only consider the static and asymptotically flat metric with the following form

\[ ds^2 = -g_{00}\left(x^k\right)dt^2 + g_{ij}\left(x^k\right)dx^idx^j \]  

(2.9)

with \(i, j, k = 1, 2, 3\). The Einstein equation of the metric (2.9) can be expressed as

\[ D_iD_i\xi = 4\pi\xi\left(-T^0_0 + T^i_i\right) \equiv 4\pi\xi\rho_M, \]

\[ R_{ij} - \frac{1}{\xi}D_iD_j\xi = 8\pi\left(T_{ij} - \frac{1}{2}g_{ij}T^\alpha_\alpha\right), \]

\[ g_{00} \equiv -\xi^2 < 0 \]  

(2.10)

in the orthonormal frame, where \(D_i\) is the covariant derivative with respect to the space metric \(g_{ij}\), \(T^a_a = T^0_0 + T^1_1 + T^2_2 + T^3_3\), \(T^i_i = T^1_1 + T^2_2 + T^3_3\), and the subscript "M" of \(\rho_M\) means "matter".

\(\xi\) or \(\xi - 1\) plays the role of a "Newtonian gravitational potential" [2]. The multipole moments of the spacetime (the multipole moments of the gravitational field) in general relativity are the expansion coefficients of \(\xi\) at infinity [2]. We remind that the multipole moments of the spacetime are only well defined for asymptotically flat spacetimes. \(g_{00}\) and \(g_{ij}\) are not independent variables since the expansion coefficients of \(g_{00}\) and \(g_{ij}\) at infinity can be uniquely determined by the multipole moments of the spacetime in "asymptotically Cartesian and mass centered" coordinates defined by Thorne [4].

In Newtonian gravity, the gravitational potential satisfies a linear equation. The expansion coefficients of gravitational potential and the multipole integrals coincides. That is, the multipole moments of the spacetime is uniquely determined by the multipole integrals of the source. In general relativity, \(\xi\) is related to the source \(\rho_M\) in nonlinear equations (2.10). The multipole moments problem is whether the multipole moments of the spacetime is uniquely determined by the multipole integrals of the source \(\rho_M\) or \(\xi\rho_M\).

The multipole integrals are defined similarly as in Newtonian gravity

\[ Q^{(lm)}_M = \int dV\rho_M rY^*_m(\theta, \varphi) \]  

(2.11)

where \(r\) is the radial coordinate, \(dV \equiv \sqrt{g}dx^1dx^2dx^3\) and \(g = \xi^2 det g_{ij}\). Actually, \(r\) is very difficult to define. Nevertheless, the strict definition of \(r\) will not influence the conclusion in this paper so that we will not concentrate on it. For convenience of comparison, we generalize the definition (2.11) in an orthonormal frame as a tensor form

\[ Q^{(lm)}_{\mu\nu} = \int dVT_{\mu\nu} rY^*_m. \]  

(2.12)

Strictly speaking, the tensor integrals \(Q^{(lm)}_{\mu\nu}\) in eq. (2.12) may be not well defined. Nevertheless, they give all information in \(Q^{(lm)}_M\) since \(Q^{(lm)}_M = -Q^{(lm)}_{00} + Q^{(lm)}_{11} + Q^{(lm)}_{22} + Q^{(lm)}_{33}\).
Now the multipole moments problem can be expressed as whether the multipole moments of the spacetime can be determined by $Q_{M}^{(lm)}$.

By the way, spacetimes considered in this paper are axisymmetric so that $T_{\mu\nu} = T_{\mu\nu} (r, \theta)$ (or $T_{\mu\nu} = T_{\mu\nu} (\rho, z)$) and $Q_{\mu\nu}^{(lm)} = 0$ if $m \neq 0$.

3 Singular source and multipole integral in static axial spacetimes

The multipole moments problem in general relativity is about the relation between the multipole expansion of the gravitational field at infinity and the multipole integrals of the source. The multipole expansion of the gravitational field at infinity is widely discussed, while the multipole integrals are rarely calculated.

In a previous work [14], we show that there exists a Dirac delta source in the Schwarzschild spacetime. In this section, we show that there exist singular sources at singularities of static axial spacetimes. Singular sources contribute to multipole integrals. We calculate the singular source and the multipole integrals in Levi-Civita spacetime and compare with the previous result to verify the validity of our method. We calculate the singular source and multipole integrals in the Curzon spacetime and point out that the Curzon spacetime possesses the same multipole integrals with the Schwarzschild spacetime. The Curzon spacetime and the Schwarzschild spacetime possess same multipole integrals, which is a counterexample of the conjecture that the multipole moments of the spacetime in general relativity is determined by the multipole integrals. That is, the multipole moments of the spacetime in general relativity cannot determined by the multipole integrals.

3.1 Singular source and multipole integral in Newtonian gravity

We take Newtonian gravity as an example to illustrate the singular source and the multipole moments.

Solving the source free equation

$$- \nabla^2 \phi = 0, \quad (3.1)$$

we obtain a special solution

$$\phi_1 = \frac{Q}{4\pi r}. \quad (3.2)$$

$\phi_1$ is singular at $r = 0$. The singularity at $r = 0$ means a singular source. For completeness, we provide the procedure of the analysis of the singularity. Replacing $r$ by $\sqrt{r^2 + \epsilon^2}$, we have

$$\phi_1 (\epsilon) = \frac{Q}{4\pi} \frac{1}{\sqrt{r^2 + \epsilon^2}}. \quad \text{(3.4)}$$

Substituting $\phi_1 (\epsilon)$ into eq. (3.1), we have

$$- \nabla^2 \phi_1 (\epsilon) = \frac{3Q}{4\pi} \frac{\epsilon^2}{(r^2 + \epsilon^2)^{\frac{3}{2}}}. \quad (3.3)$$

Taking the limit $\epsilon \to 0$ on both sides of eq. (3.3), we have

$$- \nabla^2 \phi_1 = Q \delta (r). \quad \text{(3.5)}$$
There is a Dirac delta source at $r = 0$. The corresponding monopole moment is

$$Q^{(00)} = \int_V d^3r = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dxdydz \delta(r) = Q,$$

with $r = \sqrt{x^2 + y^2 + z^2}$. We can also obtain other solutions from eq. (3.1), such as,

$$\phi_2 = \frac{Q}{4\pi r} + \frac{P}{r^2} \cos \theta.$$  

With the same procedure, we find that the second term in $\phi_2$ contributes a finite dipole moment $P$. That is, a solution may possess different multipole moments simultaneously.

3.2 Static axial spacetime and Einstein tensor

The static axial metric can be generally expressed as

$$ds^2 = -e^{2\psi} dt^2 + e^{2\gamma} (d\rho^2 + dz^2) + e^{2\psi} \rho^2 d\varphi^2,$$

where $\gamma = \gamma(\rho, z)$ and $\psi = \psi(\rho, z)$. When we calculate the multipole integrals of the source, spherical coordinates $(r, \theta, \varphi)$ are also used which is defined as

$$\rho = r \sin \theta,$$
$$z = r \cos \theta.$$  

The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} R$ with $\eta_{\mu\nu} = \text{diag} (-1, 1, 1, 1)$ in the orthonormal frame are

$$G_{tt} = 2e^{\psi-2\gamma} \left( \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} \right) - G_{\varphi\varphi},$$
$$G_{\rho\rho} = -G_{zz} = e^{2\psi-2\gamma} \left[ \frac{1}{\rho} \frac{\partial \gamma}{\partial \rho} - \left( \frac{\partial \psi}{\partial \rho} \right)^2 + \left( \frac{\partial \psi}{\partial z} \right)^2 \right],$$
$$G_{\rho z} = G_{z\rho} = e^{2\psi-2\gamma} \left[ \frac{1}{\rho} \frac{\partial \gamma}{\partial z} - 2 \frac{\partial \psi}{\partial \rho} \frac{\partial \psi}{\partial z} \right],$$
$$G_{\varphi\varphi} = e^{2\psi-2\gamma} \left[ \left( \frac{\partial \psi}{\partial \rho} \right)^2 + \left( \frac{\partial \psi}{\partial z} \right)^2 + \frac{\partial^2 \gamma}{\partial \rho^2} + \frac{\partial^2 \gamma}{\partial z^2} \right].$$

For vacuum spacetime, the equation of $\psi$ reads

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = 0.$$  

3.3 Singular source and multipole integral in Levi-Civita spacetime

In this section, we calculate the singular source and multipole integrals in the Levi-Civita metric.
The Levi-Civita metric [16] is a static cylindrical spacetime which reads
\[ ds^2 = -\rho^{2\alpha} dt^2 + \rho^2 (d\rho^2 + dz^2) + \rho^{2-2\alpha} d\varphi^2 \]  
with
\[ \psi = \alpha \ln \rho, \] \[ \gamma = \alpha^2 \ln \rho. \] (3.14) (3.15)

When \( \alpha = -1 \), the metric (3.13) becomes
\[ ds^2 = \frac{1}{\rho^2} dt^2 + \rho^4 (d\rho^2 + dz^2) + \rho^4 d\varphi^2. \] (3.16)

Redefining the coordinates \( \rho^2 \rightarrow (\frac{M}{\rho})^{1/3} r, \) \( t \rightarrow 2\frac{M}{r} t, \) \( z \rightarrow (\frac{M}{\rho})^{1/3} z, \) and \( \varphi \rightarrow (\frac{M}{\rho})^{1/3} \varphi, \) the metric (3.16) becomes the Kasner metric [18] which reads
\[ ds^2 = -\frac{2M}{r} dt^2 + \frac{r}{2M} dr^2 + r^2 (dz^2 + d\varphi^2). \] (3.17)

When \( \alpha = 0 \) or \( 1 \), the metric (3.13) becomes the Minkowski metric.

Now we calculate the singular source in the Levi-Civita metric (3.13). Replacing \( \rho \) by \( \sqrt{\rho^2 + \epsilon^2} \) in \( \psi \) (3.14) and \( \gamma \) (3.15) and substituting into the Einstein tensor \( G_{\mu\nu} \), we obtain
\[ G_{tt}(\epsilon) = -\frac{\alpha(\alpha - 4)}{\sqrt{g}} \frac{\rho \epsilon^2}{(\rho^2 + \epsilon^2)^2}, \]
\[ G_{\rho\rho}(\epsilon) = -G_{zz}(\epsilon) = \frac{\alpha^2}{\sqrt{g}} \frac{\rho \epsilon^2}{(\rho^2 + \epsilon^2)^2}, \]
\[ G_{\varphi\varphi}(\epsilon) = \frac{\alpha^2}{\sqrt{g}} \frac{\rho \epsilon^2}{(\rho^2 + \epsilon^2)^2} \] (3.18)

with \( \frac{1}{\sqrt{g}} = (\rho^2 + \epsilon^2)^{-\alpha^2 + \alpha} \rho. \) The energy-momentum tensor is given by
\[ T_{\mu\nu} = \frac{1}{8\pi} G_{\mu\nu} = \frac{1}{8\pi} \lim_{\epsilon \to 0} G_{\mu\nu}(\epsilon). \] (3.19)

The energy-momentum tensor is
\[ T_{tt} = \frac{1}{8} \frac{\alpha(\alpha - 4)}{\sqrt{g}} \delta(\rho) \delta(\varphi), \]
\[ T_{\rho\rho} = -T_{zz} = \frac{1}{8} \frac{\alpha^2}{\sqrt{g}} \delta(\rho) \delta(\varphi), \]
\[ T_{\varphi\varphi} = \frac{1}{8} \frac{\alpha^2}{\sqrt{g}} \delta(\rho) \delta(\varphi). \] (3.20)

In above calculations, \( \lim_{\epsilon \to 0} \frac{\epsilon^2}{(x^2 + y^2 + z^2)} = \pi \delta(x) \delta(y) = \frac{x}{\rho^3} \delta(\rho) \delta(\varphi) [19] \) is used. With eq. (2.12), the multipole integrals of the source can be obtained directly. The monopole
integrals of the source in the Levi-Civita spacetime is

\[
Q^{(00)}_{tt} = -\frac{L_z}{2} \alpha^2 + \frac{L_z}{8} \alpha,
\]
\[
Q^{(00)}_{\rho\rho} = -Q^{(00)}_{zz} = \frac{L_z}{8} \alpha^2,
\]
\[
Q^{(00)}_{\varphi\varphi} = \frac{L_z}{2} \alpha^2,
\]
\[
Q^{(00)}_M = \frac{L_z}{2} \alpha,
\]

(3.21)

where \(L_z \equiv \int_{z_1}^{z_2} dz\). Other multipole integrals of the Levi-Civita spacetime vanish. That is, there exists a line source in the Levi-Civita metric. In ref. [20], Israel confirms that \(\alpha\) may be interpreted as the effective gravitational mass per unit proper length along the \(z\) direction [16]. We provide the exact result in this paper.

The above results are valid only for \(\alpha \in (0, 1)\) since \(\varphi\) can be regarded as a angular coordinate only for \(\alpha \in (0, 1)\). More details can be found in ref. [16].

3.4 Multipole integral in Weyl spacetime

In this section, we calculate the multipole integrals in the Weyl spacetime. The Weyl solution is the general asymptotically flat vacuum solution of the metric (3.6). In coordinates \((r, \theta, \varphi)\), the Weyl solution is [16, 17]

\[
\psi = -\sum_{n=0}^{\infty} \frac{a_n}{r^{n+1}} P_n (\cos \theta),
\]

(3.22)

\[
\gamma = -\sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(l+1)(k+1)}{(l+k+2)} \frac{a_la_k}{r^{l+k+2}} [P_l (\cos \theta) P_k (\cos \theta) - P_{l+1} (\cos \theta) P_{k+1} (\cos \theta)],
\]

(3.23)

where \(a_n\) are expansion coefficients and \(P_n\) are Legendre polynomials.

The singular source is difficult to calculate. We only calculate the multipole integrals in the Weyl solution. With the Einstein equation \(T_{\mu\nu} = \frac{1}{8\pi} G_{\mu\nu}\) and eqs. (2.10), we have

\[
\rho_M = \frac{1}{8\pi} (G_{tt} + G_{rr} + G_{\theta\theta} + G_{\varphi\varphi}) = \frac{e^{2\psi-2\gamma}}{\sqrt{r^2+\epsilon^2}} \left[ \frac{2 \psi^2}{r^2} + 2r \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial \theta^2} + \cot \theta \frac{\partial \psi}{\partial \theta} \right].
\]

(3.24)

Replacing \(r\) by \(\sqrt{r^2+\epsilon^2}\) in eqs. (3.22) and (3.23) and substituting into eq. (3.24), we have

\[
\rho_M (\epsilon) = \frac{e^{2\psi-2k}}{4\pi r^2} \sum_{n=0}^{\infty} a_n P_n (\cos \theta) (r^2 + \epsilon^2)^{-\frac{a+k}{2}} [(2n+3) (n+1) r^2 e^2 + n (n+1) e^4].
\]

(3.25)

The multipole integrals of the source reads

\[
Q^{(00)}_M = \lim_{\epsilon \to 0} \int dV \rho_M (\epsilon) r^4 Y^{*}_{lm} (\theta, \varphi) = a_l.
\]

(3.26)

That is, the expansion coefficients \(a_n\) in the Weyl solution are the multipole integrals of the source defined in eq. (2.11).
3.5 Singular source and multipole integral in Curzon spacetime

In this section, we calculate the singular source and multipole integrals in the Curzon spacetime.

Taking \( n = 0 \) \((n \geq 2)\) in the Weyl solution, we get the Curzon solution

\[
\psi = -\frac{M}{r}, \\
\gamma = -\frac{M^2 \sin^2 \theta}{2r^2}.
\]  

(3.27)  

(3.28)

The Curzon metric in coordinates \((t, r, \theta, \varphi)\) reads

\[
ds^2 = -e^{-\frac{2M}{r}} dt^2 + e^{-\frac{M^2 \sin^2 \theta}{r^2}} \left( dr^2 + r^2 d\theta^2 \right) + e^{\frac{2M}{r}} r^2 \sin^2 \theta d\varphi^2.
\]

(3.29)

The Curzon metric has same asymptotics with the Schwarzschild metric at \( r \rightarrow \infty \). That is, when \( r \rightarrow \infty \), the metric (3.29) becomes

\[
ds^2 \sim -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right).
\]

(3.30)

We remind that

\[
G(r) = -\frac{M}{r}
\]

is the Green function of the equation (3.12)

\[
\frac{\partial^2 \psi}{\partial z^2} + \frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} = 0.
\]

(3.32)

That is

\[
\left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \right) G(r) = 4\pi M \delta(x) \delta(y) \delta(z)
\]

(3.33)

with \( r^2 = \rho^2 + z^2 = x^2 + y^2 + z^2 \). This fact indicates that the Curzon metric is not a vacuum solution at \( \rho = 0 \). Nevertheless, the source in eq. (3.33) is not the energy-momentum in the Einstein equation.

Now we calculate the source in the Curzon solution. Replacing \( \rho \) by \( \sqrt{\rho^2 + \epsilon^2} \) in \( \psi \) (3.27) and \( \gamma \) (3.28) and substituting into the Einstein tensor \( G_{\mu \nu} \), we obtain

\[
G_{tt}(\epsilon) = \frac{1}{\sqrt{g}} M^2 \epsilon^2 \frac{r^2 \sin \theta}{(r^2 + \epsilon^2)^2} - G_{\varphi \varphi}(\epsilon),
\]

\[
G_{rr}(\epsilon) = -G_{\theta \theta}(\epsilon) = \frac{1}{\sqrt{g}} M^2 \epsilon^2 \left[ \frac{\cos^2 \theta \sin \theta}{(r^2 + \epsilon^2)^2} + \frac{r^2 \sin \theta}{(r^2 + \epsilon^2)^3} \right],
\]

\[
G_{r \theta}(\epsilon) = -\frac{1}{\sqrt{g}} M^2 \epsilon^2 \frac{\cos \theta \sin^2 \theta}{(r^2 + \epsilon^2)^2},
\]

\[
G_{\varphi \varphi}(\epsilon) = \frac{1}{\sqrt{g}} M^2 \epsilon^2 \left[ -\frac{\sin \theta \cos (2\theta)}{(r^2 + \epsilon^2)^2} - \frac{2r^2 \sin \theta \cos (2\theta)}{(r^2 + \epsilon^2)^3} + \frac{r^2 \sin \theta}{(r^2 + \epsilon^2)^3} \right].
\]

(3.34)
Taking the limit $\epsilon \to 0$, we obtain the energy-momentum tensor

\[
T_{tt} = \frac{M}{\sqrt{g}} \delta(r) \delta(\theta) \delta(\varphi) - T_{\varphi\varphi},
\]

\[
T_{rr} = -T_{\theta\theta} = \frac{1}{\sqrt{g}} M^2 \frac{1}{6r} (1 + 3 \cos^2 \theta) \delta(r) \delta(\theta) \delta(\varphi),
\]

\[
T_{r\theta} = -\frac{1}{\sqrt{g}} M^2 \frac{r}{2r} \sin \theta \cos \theta \delta(r) \delta(\theta) \delta(\varphi),
\]

\[
T_{\varphi\varphi} = -\frac{1}{\sqrt{g}} M^2 \frac{r}{3r} (-3 + 5 \cos^2 \theta) \delta(r) \delta(\theta) \delta(\varphi).
\] (3.35)

with $r^2 = x^2 + y^2 + z^2$. In above calculations, following results are used [14]

\[
\lim_{\epsilon \to 0} \frac{\epsilon^2}{(r^2 + \epsilon^2)^{\frac{3}{2}}} = \frac{4\pi}{3} \delta(x) \delta(y) \delta(z),
\]

\[
\lim_{\epsilon \to 0} \frac{\epsilon^2}{r^2 (r^2 + \epsilon^2)^{\frac{3}{2}}} = 4\pi \delta(x) \delta(y) \delta(z),
\]

\[
\lim_{\epsilon \to 0} \frac{r}{(r^2 + \epsilon^2)^{\frac{1}{2}}} = 1,
\]

\[
\delta(x) \delta(y) \delta(z) = \frac{1}{r^2 \sin \theta} \delta(r) \delta(\theta) \delta(\varphi).
\]

With eq. (2.12), the multipole integrals of the source can be obtained directly. The multipole integrals in the Curzon spacetime read

\[
Q_{\mu \nu}^{(00)} = M,
\]

\[
Q_{\mu \nu}^{(10)} = 0,
\]

\[
Q_{M}^{(00)} = M.
\] (3.36)

Other multipole integrals vanish.

The first term in $T_{00}$ is a same Dirac delta source with the Schwarzschild metric [14], which is the reason that the Curzon metric has the same asymptotics with the Schwarzschild metric at infinity. The energy-momentum tensor in the Curzon spacetime possess other polarized terms diverging stronger that the Dirac delta source, which is the reason that the Curzon metric is not spherically symmetric and the singularity in Curzon metric have different asymptotics along different directions. The asymptotics of the Curzon spacetime are analyzed in ref [21]. In ref. [22], the author mentioned that the far-field of the Curzon metric is of a mass at $r = 0$ with the multipole moments on it. We provide the quantitative results.

### 3.6 Singular source and source integral in Schwarzschild spacetime

In a previous work [14], we provide the singular source in the Schwarzschild spacetime. The metric of the Schwarzschild spacetime reads

\[
ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + d\phi^2).
\]
For convenience of discussion, we present the singular source and the multipole integrals of the Schwarzschild spacetime here. The singular source in the Schwarzschild spacetime reads

\[
T_{tt} = \frac{M}{r^2 \sin \theta} \delta (r) \delta (\theta) \delta (\phi),
\]

\[
T_{rr} = -\frac{M}{r^2 \sin \theta} \delta (r) \delta (\theta) \delta (\phi),
\]

\[
T_{\theta \theta} = T_{\phi \phi} = \frac{M}{2r^2 \sin \theta} \delta (r) \delta (\theta) \delta (\phi). \tag{3.37}
\]

The monopole integrals of the source in the Schwarzschild spacetime are

\[
Q_{tt}^{(00)} = M, \\
Q_{rr}^{(00)} = -M, \\
Q_{\theta \theta}^{(00)} = Q_{\phi \phi}^{(00)} = \frac{M}{2}, \\
Q_{M}^{(00)} = M. \tag{3.38}
\]

All above results about the multipole moments of the source are used to discuss the multipole moments problem in general relativity.

### 3.7 Multipole moment problem in general relativity

In this section, we illustrate that the multipole moments of the gravitational field in general relativity cannot be determined by the multipole integrals of the source.

In above sections, we show that the Levi-Civita spacetime, the Curzon spacetime and the Schwarzschild spacetime only have nonvanishing monopole moments of the source. However, the energy-momentum tensor \(T_{\mu \nu}\) in these spacetimes are quite different. The Levi-Civita spacetime possesses a line-like source, while the Levi-Civita spacetime and the Schwarzschild spacetime possess point-like sources. Furthermore, the Levi-Civita spacetime and the Schwarzschild spacetime possess the same multipole integrals of the sources. Nevertheless, they have different expansions at infinity. The above results show that the multipole moments of the spacetime cannot be determined by multipole integrals of the source.

Generally, the radial coordinate \(r\) in curved spacetimes is difficult to define so that the multipole integrals of the source is difficult to define. Nevertheless, the curzon spacetime and the Schwarzschild spacetime only possess nonvanishing \(Q_{M}^{(00)}\) which is independent of the definition of \(r\). That is, the conclusion in this paper is independent of the definition of the radial coordinate \(r\) and the results is invariant under the space coordinates transform.

### 4 Conclusion and outlook

In Newtonian gravity, the multipole moments of the gravitational field is determined by the source distribution — a series of multipole integrals. In this paper, with a counterexample, we show that the multipole moments of the gravitational field cannot be determined
by multipole integrals of the source in general relativity. The Curzon spacetime and the Schwarzschild spacetime possess same multipole integrals of the source, while they possess different multipole moments of the spacetime.

The multipole moments of the spacetime are widely discussed, while the multipole integrals are not. In this paper, we show that there exist singular sources at singularities in static axial spacetimes. Outside the singularities, the spacetimes are vacuum solutions of the Einstein equation. Nevertheless, point-like or line-like sources exist at singularities. These singular sources contribute to multipole integrals which allow us to discuss the multipole moments problem in general relativity. Besides, we provide a useful method to analyze the singular source quantitatively. The validity of the method is verified by the singular source and multipole integrals in the Levi-Civita spacetime and our previous work [14].

In the follow contents, we discuss some open questions concerning the singular source and the multipole moments in general relativity.

A singular source indicates whether the corresponding singularity is physically acceptable or not. We define a delta function in curved spacetime

$$\delta^3(x) = \frac{1}{\sqrt{g}} \delta(x_1 - x'_1) \delta(x_2 - x'_2) \delta(x_3 - x'_3),$$

where $x_i$ and $x'_i$ ($i = 1, 2, 3$) are space coordinates. The physically acceptable mass at one point should be finite in total. That is, only singularities diverging slower than $\delta^3(x)$ are physically acceptable, while singularities diverging stronger than $\delta^3(x)$ are not physically acceptable. In this paper, the singularity in the Levi-Civita spacetime and the Schwarzschild spacetime are physically acceptable, while the singularity in the Curzon spacetime are not physically acceptable.

There are other viewpoint about the multipole moments of the spacetime. We illustrate with the electrostatic field since the electrostatic field satisfies the same equation with Newtonian gravity. In the electrostatic field, the multipole moments can be defined as the multipole integrals. The fact that the multipole moments coincides with the multipole integrals can be explained as that the electrostatic field itself does contribute to the multipole moments. The multipole integrals is about the integrals over the electric charge. The electrostatic field does not carry any electric charge so that the electrostatic field does not contribute to the multipole moments. However, general relativity is different with the electrostatic field. The charge of general relativity is the mass. The gravitational field must contribute to the multipole moments of the spacetime if we suppose that the gravitational field possesses the energy. In this viewpoint, the definition of the energy of the gravitational field is a subproblem of the definition of the multipole moments of the spacetime. In Thorne’s work [4], the multipole moments of the spacetime involves the definition of the energy-moment of the gravitational field — the Landau pseudo-tensor. With this viewpoint, we provide another method to calculate the multipole integrals of the source in general relativity.

We demonstrate that the gravitational field itself contributes to the multipole moments of the spacetime in above paragraph. Another problem is whether the gravitational
field itself without any source or any singularity (the gravitational wave) collapses into a asymptotically flat spacetime or not.

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5 Appendix

In the appendix, we provide another method to calculate the multipole integrals of the source in the spacetime.

The multipole moments of the spacetime are also expressed as volume integrals

\[
M^{(lm)} = \int dV \rho_M r^l Y^*_l m (\theta, \varphi) - \frac{1}{4\pi} \int dV D_i D^i \left( r^l Y^*_l m (\theta, \varphi) \right) = Q^{(lm)}_M + Q^{(lm)}_F
\]  

(5.1)

where \(Q^{(lm)}_F = -\frac{1}{4\pi} \int dV D_i D^i \left( r^l Y^*_l m (\theta, \varphi) \right)\) with the subscript "F" of \(Q^{(lm)}_F\) means "field", and \( dV = \sqrt{g} dx^1 dx^2 dx^3 \) with \( g = \xi^2 \det g_{ij} \) (2.10). The multipole moments of the spacetime in eq. (5.1) coincide, at least in the axial case, with the multipole moments of the spacetime defined by Thorne or Geroch \[12\]. The multipole moments of the spacetime in eq. (5.1) are exact differentials so that they can be converted to boundary integrals \[12\]

\[
M^{(lm)} = \frac{1}{4\pi} \int_{\partial V} dS \left[ D^i \xi r^l Y^*_l m (\theta, \varphi) - \xi D^i \left( r^l Y^*_l m (\theta, \varphi) \right) \right].
\]  

(5.2)

There are two parts in \(M^{(lm)}\): multipole integrals \(Q^{(lm)}_M\) and source free integrals \(Q^{(lm)}_F\). With the viewpoint in the conclusion, \(Q^{(lm)}_M\) and \(Q^{(lm)}_F\) can be regarded as the multipole moments contribute by the source and the gravitational field respectively. When we take the first order of the asymptotically flat approximation, \(D_i\) is replaced by \(\partial_i\). In this case, \(Q^{(lm)}_F\) vanishes, which means that \(M^{(lm)}\) is totally contributed by \(Q^{(lm)}_M\). That is, we replace \(D_i\) with \(\partial_i\), eq. (5.2) and obtain the source integrals \(Q^{(lm)}_M\). In ref. \[12\], the author shows that in the Weyl solution, replacing \(D_i\) with \(\partial_i\) in eq. (5.2) provides the same multipole integrals with the results we calculated in section 3.

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