Cohomology of the infinite-order jet space

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Abstract De Rham cohomology, $d_V$- and $d_H$-cohomology of the differential algebra of locally pull-back exterior forms on the infinite-order jet manifold of a smooth fibre bundle are calculated.

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1 Introduction

Let $Y \rightarrow X$ be a smooth fibre bundle (throughout the paper, smooth manifolds are assumed to be real, finite-dimensional, Hausdorff, paracompact, and connected). We study cohomology of differential algebras of exterior forms on the infinite-order jet space $J^\infty Y$ of $Y \rightarrow X$. This cohomology plays an important role in some physical models, e.g., in the calculus of variations in Lagrangian field theory [1-4] and in the field-antifield BRST formalism for constructing the descent equations [5-8].

Recall that the infinite-order jet space of a smooth fibre bundle $Y \rightarrow X$ is defined as a projective limit $(J^\infty Y, \pi^\infty)$ of the surjective inverse system

\begin{equation}
X \leftarrow \pi_0 Y \leftarrow \pi_1 Y \leftarrow \ldots \leftarrow J^r Y \leftarrow \ldots \leftarrow J^{r-1} Y \leftarrow \ldots
\end{equation}

of finite-order jet manifolds $J^r Y$. Provided with the projective limit topology, $J^\infty Y$ is a paracompact Fréchet (but not Banach) manifold. Given a bundle coordinate chart $(\pi^{-1}(U_X); x^\lambda, y^i)$ on the fibre bundle $Y \rightarrow X$, we have the coordinate chart $((\pi^\infty)^{-1}(U_X); x^\lambda, y^i_{\Lambda})$, $0 \leq |\Lambda|$, on $J^\infty Y$, together with the transition functions

\begin{equation}
y^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d^\mu y^i_{\Lambda},
\end{equation}

where $\Lambda = (\lambda_k \ldots \lambda_1)$, $|\Lambda| = k$, is a multi-index, $\lambda + \Lambda$ is the multi-index $(\lambda \lambda_k \ldots \lambda_1)$ and $d_\lambda$ are the total derivatives

\begin{equation}
d_\lambda = \partial_\lambda + \sum_{|\Lambda|=0} y^i_{\lambda+\Lambda} \partial_i^\Lambda.
\end{equation}

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In studying cohomology of differential algebras on the infinite-order jet manifold, the key point lies in the following fact [3, 4] (see Appendix A).

**Lemma 1.** A smooth fibre bundle \( Y \) is a strong deformation retract of its infinite-order jet space \( J^\infty Y \).

Since \( J^\infty Y \) is paracompact, it follows that there is an isomorphism

\[
H^*(J^\infty Y, \mathbb{R}) = H^*(Y, \mathbb{R})
\]

(3)

of the cohomology groups of the infinite-order jet space \( J^\infty Y \) with coefficients in the constant sheaf \( \mathbb{R} \) and those \( H^*(Y, \mathbb{R}) \) of the fibre bundle \( Y \).

The goal is to construct a resolution of the constant sheaf \( \mathbb{R} \) on \( J^\infty Y \) by acyclic sheaves of some differential algebra on \( J^\infty Y \). Then the well-known theorem on cohomology of global sections of these sheaves ([12], Theorem 2.12.1) can be called into play. For the sake of convenience, we will agree to call it the general De Rham theorem.

Given the surjective inverse system (1), we have the direct system

\[
\begin{align*}
O^*_X & \xrightarrow{\pi^*} O^*_Y \xrightarrow{\pi^*_1} O^*_1 \xrightarrow{\pi^*_2} \cdots \xrightarrow{\pi^*_r} O^*_r \xrightarrow{\pi^*_r} \cdots \\
\end{align*}
\]

(4)

of ringed spaces \( (J^r Y, O^*_r) \) whose structure sheaves \( O^*_r \) are sheaves of differential \( \mathbb{R} \)-algebras of exterior forms on finite-order jet manifolds \( J^r Y \), and \( \pi^*_{r-1} \) are the pull-back morphisms. We follow the terminology of Ref. [12] where by a sheaf is meant a sheaf bundle. The direct system (4) admits a direct limit \( O^*_\infty \) which is a sheaf of differential exterior \( \mathbb{R} \)-algebras on the infinite-order jet space \( J^\infty Y \).

Accordingly, we have the direct system

\[
\begin{align*}
O^*(X) & \xrightarrow{\pi^*} O^*(Y) \xrightarrow{\pi^*_1} O^*_1 \xrightarrow{\pi^*_2} \cdots \xrightarrow{\pi^*_r} O^*_r \xrightarrow{\pi^*_r} \cdots \\
\end{align*}
\]

(5)

of the structure algebras \( O^*_r = \Gamma(J^r Y, O^*_r) \) of global sections of the sheaves \( O^*_r \), i.e., \( O^*_r \) are differential \( \mathbb{R} \)-algebras of (global) exterior forms on finite-order jet manifolds \( J^r Y \). The direct limit \( (O^*_\infty, \pi^*_r) \) of the direct system (5) is a differential \( \mathbb{R} \)-algebra of all exterior forms on finite-order jet manifolds modulo the pull-back identification. Therefore, one usually thinks of elements of \( O^*_\infty \) as being the pull-back onto \( J^\infty Y \) of exterior forms on finite-order jet manifolds.

It should be emphasized that the direct limit \( O^*_\infty \) of the direct system (4) of structure algebras of sheaves \( O^*_r \) fails to coincide with the structure algebra \( Q^*_\infty = \Gamma(J^\infty Y, O^*_\infty) \) of the direct limit \( O^*_\infty \) of these sheaves. By definition, \( O^*_\infty \) is the sheaf of germs of local exterior forms on finite-order jet manifolds. These local forms constitute a presheaf \( O^*_\infty \) from which the sheaf \( O^*_\infty \) is constructed. It means that, given a section \( \phi \in \Gamma(O^*_\infty) \) of \( O^*_\infty \) over an open subset \( U \subseteq J^\infty Y \) and any point \( q \in U \), there exists a neighbourhood \( U_q \) of \( q \)
such that \( \phi|_{U_q} \) is the pull-back of a local exterior form on some finite-order jet manifold. However, \( Q^*_\infty \) does not coincide with the canonical presheaf \( \Gamma(U, Q^*_\infty) \) of sections of the sheaf \( Q^*_\infty \). There are obvious monomorphisms \( O^*_\infty \to Q^*_\infty \) and \( O^*_\infty \to \Gamma(Q^*_\infty) \).

For short, we agree to call \( Q^*_\infty \) and \( Q^*_\infty \) the sheaf and algebra of locally pull-back exterior forms on \( J^\infty Y \). Being restricted to a coordinate chart \( (\pi^\infty)^{-1}(U_X) \) on \( J^\infty Y \), elements of \( Q^*_\infty \) can be written in the familiar coordinate form, where basic forms \( \{dx^\lambda\} \) and contact 1-forms \( \{\theta^i_\Lambda = dy^i_\Lambda - y^i_{\Lambda+\lambda} dx^\lambda\} \) provide the local generators of the algebra \( Q^*_\infty \).

There is the canonical splitting of the space of \( m \)-forms

\[
Q^m_\infty = Q^{0,m}_\infty \oplus Q^{1,m-1}_\infty \oplus \cdots \oplus Q^{m,0}_\infty
\]

into spaces \( Q^{k,m-k}_\infty \) of \( k \)-contact forms. Accordingly, the exterior differential on \( Q^*_\infty \) is decomposed into the sum \( d = d_H + d_V \) of horizontal and vertical differentials

\[
d_H : Q^{k,s}_\infty \to Q^{k,s+1}_\infty, \quad d_V : Q^{k,s}_\infty \to Q^{k+1,s}_\infty
\]

which obey the nilpotency rule

\[
d_H \circ d_H = 0, \quad d_V \circ d_V = 0, \quad d_V \circ d_H + d_H \circ d_V = 0. \tag{6}
\]

Traditionally, one has tried to introduce the algebra of locally pull-back forms on \( J^\infty Y \) in a standard geometric way [1, 2, 3]. The difficulty lies in the geometric interpretation of derivations of the \( \mathbb{R} \)-ring \( Q^0_\infty \) of locally pull-back functions as vector fields on the Fréchet manifold \( J^\infty Y \) and their duals as differential forms on \( J^\infty Y \). Therefore, one usually considers the subalgebra \( O^*_\infty \) of pull-back exterior forms on \( J^\infty Y \), given as the direct limit of the direct system (5). Accordingly, the infinite-order De Rham complex of these forms

\[
0 \to \mathbb{R} \to O^0_\infty \xrightarrow{d} O^1_\infty \xrightarrow{d} \cdots \tag{7}
\]

is the direct limit of the De Rham complexes of exterior forms on finite-order jet manifolds. Hence, as was repeatedly proved, the cohomology groups \( H^*(O^*_\infty) \) of the complex (7) are equal to the De Rham cohomology groups \( H^*(Y) \) of the fibre bundle \( Y \) [1-4]. At the same time, \( d_H \) - and \( d_V \)-cohomology of the differential algebra \( O^*_\infty \) remains unknown. The algebraic Poincaré lemma on the local exactness of differentials \( d_H \) and \( d_V \) only has been repeatedly proved (see, e.g., [13-15]).

Here, we show that the problem of cohomology of differential forms on the infinite-order jet space \( J^\infty Y \) has a comprehensive solution by enlarging the algebra \( O^*_\infty \) to the algebra \( Q^*_\infty \). From the physical viewpoint, it enables one also to study effective field theories whose Lagrangians involve derivatives of arbitrary high order. The key point is that the Fréchet manifold \( J^\infty Y \) admits a partition of unity performed by elements of the ring \( Q^0_\infty \) of locally pull-back functions on \( J^\infty Y \) [1, 2]. It follows that the sheaf \( Q^*_\infty \)
of $\mathcal{Q}_\infty^0$-modules on $J^\infty Y$ is fine and, consequently, acyclic. Therefore, studying different resolutions performed by subsheaves of the sheaf $\mathcal{Q}_\infty^*$, one may hope to get a complete picture of cohomology of the differential algebra $\mathcal{Q}_\infty^*$.

Given a smooth fibre bundle $Y \to X$, we will show the following.

1. The De Rham cohomology groups of the differential algebra $\mathcal{Q}_\infty^*$ are isomorphic to those of the fibre bundle $Y$.

2. Its $d_V$-cohomology groups are related to the cohomology groups of the fibre bundle $Y$ with coefficients in the sheaf $\mathcal{O}_X^*$ of exterior forms on the base $X$.

3. The $d_H$-cohomology groups of contact elements $\phi \in \mathcal{Q}_\infty^{0<,*}$ of the algebra $\mathcal{Q}_\infty^*$ are trivial.

4. In degrees $r < n = \dim X$, the $d_H$-cohomology groups of its horizontal elements $\phi \in \mathcal{Q}_\infty^{0,*}$ coincide with the De Rham cohomology groups of the fibre bundle $Y$.

Note that, as was mentioned above, the result (i) is also true for the differential algebra $\mathcal{O}_\infty^*$. The result (iii) recovers that in Refs. [3, 4], obtained by means of the Mayer-Vietoris sequence.

Point out the particular case of an affine bundle $Y \to X$, interesting for physical applications, e.g., to BRST theory. In this case, $X$ is a strong deformation retract of $Y$ and the cohomology of $Y$ under consideration is equal to that of $X$. Then the above results are reformulated as follows.

1. Any closed form $\phi \in \mathcal{Q}_\infty^*$ is decomposed into the sum $\phi = \varphi + d\xi$ where $\varphi \in \mathcal{O}_X^*(X)$ is a closed form on the base $X$.

2. Any $d_V$-closed form $\phi \in \mathcal{Q}_\infty^*$ is $d_V$-exact.

3. Any $d_H$-closed form $\phi \in \mathcal{Q}_\infty^*$ is decomposed into the sum $\phi = \varphi + d_H\xi$ where $\varphi \in \mathcal{O}_X^*(X)$ is a closed form on the base $X$.

The results (i) and (ii) are also true for the differential algebra $\mathcal{O}_\infty^*$.

2 De Rham cohomology

Let us start from De Rham cohomology of the differential algebra $\mathcal{Q}_\infty^*$ of locally pull-back exterior forms on the infinite-order jet manifold $J^\infty Y$. We consider the complex of sheaves of $\mathcal{Q}_\infty^0$-modules

$$0 \to \mathbb{R} \to \mathcal{Q}_\infty^0 \xrightarrow{d} \mathcal{Q}_\infty^1 \xrightarrow{d} \cdots$$ (8)
on the infinite-order jet space $J^\infty Y$ and the corresponding infinite-order De Rham complex
\[ 0 \to \mathbb{R} \to \mathcal{Q}_\infty^0 \xrightarrow{d} \mathcal{Q}_\infty^1 \xrightarrow{d} \cdots \] (9)
of algebras of locally pull-back exterior forms on $J^\infty Y$.

Since locally pull-back exterior forms fulfill the Poincaré lemma, the complex of sheaves \( (8) \) is exact. Since the paracompact space $J^\infty Y$ admits a partition of unity performed by elements of $\mathcal{Q}_\infty^0$, the sheaves $\mathcal{Q}_\infty^r$ of $\mathcal{Q}_\infty^0$-modules are fine for all $r \geq 0$. Then they are acyclic, i.e., the cohomology groups $H^r(\mathcal{Q}_\infty^*, \mathcal{Q}_\infty^0)$ of the paracompact space $J^\infty Y$ with coefficients in sheaves $\mathcal{Q}_\infty^r$ vanish. Consequently, the exact sequence (8) is a fine resolution of the constant sheaf $\mathbb{R}$ on $J^\infty Y$. Then, in accordance with the above mentioned general De Rham theorem, we have an isomorphism
\[ H^*(\mathcal{Q}_\infty^*) = H^*(J^\infty Y, \mathbb{R}) \] (10)
of the De Rham cohomology groups $H^*(\mathcal{Q}_\infty^*)$ of the differential algebra $\mathcal{Q}_\infty^*$ and the cohomology groups $H^*(J^\infty Y, \mathbb{R})$ of the infinite-order jet space $J^\infty Y$ with coefficients in the constant sheaf $\mathbb{R}$. Combining isomorphisms (3) and (10), we come to the manifested assertion.

**PROPOSITION 2.** There is an isomorphism
\[ H^*(\mathcal{Q}_\infty^*) = H^*(Y) \]
of the De Rham cohomology groups $H^*(\mathcal{Q}_\infty^*)$ of the differential algebra $\mathcal{Q}_\infty^*$ to the De Rham cohomology groups $H^*(Y)$ of the fibre bundle $Y$.

### 3 Cohomology of $d_V$

Due to the nilpotency rule (4), the vertical and horizontal differentials $d_V$ and $d_H$ define the bicomplex of sheaves
\[
\begin{array}{cccccc}
& d_V & d_V & d_V & d_V & d_V \\
0 \to & \mathcal{Q}_\infty^k & d_H & \mathcal{Q}_\infty^k & d_H & \cdots & \mathcal{Q}_\infty^k & d_H & \cdots & \mathcal{Q}_\infty^k & d_H & \cdots \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 \to \mathbb{R} \to & \mathcal{Q}_\infty^0 & d_H & \mathcal{Q}_\infty^0 & d_H & \cdots & \mathcal{Q}_\infty^0 & d_H & \cdots & \mathcal{Q}_\infty^0 & d_H & \cdots \\
& \pi^\infty & \pi^\infty & \pi^\infty & \pi^\infty & \pi^\infty & \pi^\infty & \pi^\infty & \pi^\infty & \pi^\infty & \pi^\infty & \pi^\infty & \pi^\infty \\
0 \to \mathbb{R} \to & \mathcal{O}_X^0 & d & \mathcal{O}_X^0 & d & \cdots & \mathcal{O}_X^0 & d & \cdots & \mathcal{O}_X^0 & d & \cdots \\
& 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\] (11)
The rows and columns of these bicomplex are horizontal and vertical complexes. Moreover, the above mentioned algebraic Poincaré lemma is obviously extended to elements of $Q^\infty$ and leads to the following.

**Lemma 3.** The columns and rows of the bicomplex (11) are exact sequences of sheaves.

It follows that, since all sheaves $Q_{k,m}^\infty$ are fine, the columns and rows of the bicomplex (11) are fine resolutions of their first terms. Then the general De Rham theorem on a resolution of a sheaf on a paracompact manifold can be used again.

Let us consider a vertical exact sequence of sheaves

$$0 \rightarrow \mathcal{D}_X^m \xrightarrow{\pi^*} \mathcal{D}_\infty^0 \xrightarrow{dv} \cdots \xrightarrow{dv} \mathcal{D}_\infty^k \xrightarrow{dv} \cdots, \quad m \leq n,$$

(12)

and the corresponding complex of their structure algebras

$$0 \rightarrow Q^m(X) \xrightarrow{\pi^*} Q_\infty^0 \xrightarrow{dv} Q_\infty^k \xrightarrow{dv} \cdots, \quad m \leq n.$$

(13)

The exact sequence (12) is a resolution of the sheaf $\pi^*\mathcal{D}_X^m$ of the pull-back onto $J^\infty Y$ of exterior forms on the base $X$. Then, by virtue of the general De Rham theorem, we have an isomorphism

$$H^*(dv, m) = H^*(J^\infty Y, \pi^*\mathcal{D}_X^m)$$

(14)

of the cohomology groups $H^*(dv, m)$ of the complex of differential algebras (13) to the cohomology groups $H^*(J^\infty Y, \pi^*\mathcal{D}_X^m)$ of the infinite-order jet manifold $J^\infty Y$ with coefficients in the sheaf $\pi^*\mathcal{D}_X^m$. Since $J^\infty Y$ and $Y$ are homotopic, there is an isomorphism of cohomology groups

$$H^*(J^\infty Y, \pi^*\mathcal{D}_X^m) = H^*(Y, \pi^*\mathcal{D}_X^m).$$

Thus, it is stated the following.

**Proposition 4.** There is an isomorphism of the cohomology groups

$$H^*(dv, m) = H^*(Y, \pi^*\mathcal{D}_X^m).$$

In particular, if $Y \rightarrow X$ is an affine bundle, we have

$$H^*(dv, m) = H^*(Y, \pi^*\mathcal{D}_X^m) = H^*(X, \mathcal{D}_X^m) = 0$$

because the sheaf $\mathcal{D}_X^m$ on $X$ is fine. This result also follows directly from the expression for the corresponding homotopy operator and, therefore, remains true for to the differential algebra $\mathcal{O}^\infty_X$ (see Appendix B).
4 Cohomology of $d_H$

Turn now to the rows of the bicomplex (11) (excluding the bottom one which is obviously the De Rham complex on the base $X$). We have the exact sequences of sheaves

$$0 \to \mathcal{Q}^k,0 \to \mathcal{Q}^k,1 \to \cdots \to \mathcal{Q}^k,\infty, \quad k > 0,$$

(15)

$$0 \to \mathcal{R} \to \mathcal{Q}_0^0 \to \mathcal{Q}_0^1 \to \cdots \to \mathcal{Q}_0^\infty.$$

(16)

Speaking rigorously, the exact sequences (15) and (16) fail to be fine resolutions of the sheaves $\mathcal{Q}^k,0$ and $\mathcal{R}$, respectively, because of their last terms. At the same time, following directly the proof of the above mentioned Theorem 2.12.1 in Ref. [12] till these terms, one can show the following.

PROPOSITION 5. The cohomology groups $H^r(d_H,k)$, $r < n$, of the complex

$$0 \to \mathcal{Q}^k,0 \to \mathcal{Q}^k,1 \to \cdots \to \mathcal{Q}^k,\infty$$

are isomorphic to the cohomology groups $H^r(J^\infty Y,\mathcal{Q}^k,0)$ of $J^\infty Y$ with coefficients in the sheaf $\mathcal{Q}^k,0$ and, consequently, are trivial because the sheaf $\mathcal{Q}^k,0$ is fine.

PROPOSITION 6. The cohomology groups $H^r(d_H)$, $r < n$, of the complex

$$0 \to \mathcal{R} \to \mathcal{Q}_0^0 \to \mathcal{Q}_0^1 \to \cdots \to \mathcal{Q}_0^\infty$$

are isomorphic to the cohomology groups $H^r(J^\infty Y,\mathcal{R})$ of $J^\infty Y$ with coefficients in the constant sheaf $\mathcal{R}$ and, consequently, to the cohomology groups $H^r(Y,\mathcal{R})$.

Note that one can also study the exact sequences of presheaves

$$0 \to \mathcal{Q}^k,0 \to \mathcal{Q}^k,1 \to \cdots \to \mathcal{Q}^k,\infty, \quad k > 0,$$

$$0 \to \mathcal{R} \to \mathcal{Q}_0^0 \to \mathcal{Q}_0^1 \to \cdots \to \mathcal{Q}_0^\infty,$$

but comes again to the results of Propositions 5, 6. Because $J^\infty Y$ is paracompact, the cohomology groups $H^*(J^\infty Y,\mathcal{Q}^k,0)$ of $J^\infty Y$ with coefficients in a sheaf $\mathcal{Q}^k,0$ and those $H^*(J^\infty Y,\mathcal{Q}^k,\infty)$ with coefficients in a presheaf $\mathcal{Q}^k,\infty$ are isomorphic. It follows that the cohomology group $H^0(J^\infty Y,\mathcal{Q}^k,0)$ of a presheaf $\mathcal{Q}^k,0$ is isomorphic to the $\mathcal{R}$-module $\mathcal{Q}^k,0 = H^0(J^\infty Y,\mathcal{Q}^k,0)$, but not $\mathcal{O}^k,\infty$. 

7
5 Appendix A

Here, we construct a desired homotopy from the infinite order jet manifold $J^\infty Y$ to a fibre bundle $Y$ in an explicit form. Given a coordinate chart $((\pi^\infty)^{-1}(U_X); x^\lambda, y_\Lambda^i)$ on $J^\infty Y$, let us consider the map

$$[0, 1] \times J^\infty Y \ni (t; x^\lambda, y^i, y_\Lambda^i) \rightarrow (x^\lambda, y^i, y_\Lambda^i) \in J^\infty Y, \quad 0 < |\Lambda|,$$

$$y_\Lambda^i = f_k(t)y_\Lambda^i + (1 - f_k(t))\Gamma_{(k)}^i, \quad |\Lambda| = k > 0,$$

where $\Gamma_{(k)}$ is a section of the affine jet bundle $J^k Y \rightarrow J^{k-1} Y$ and $f_k(t)$ is a continuous (smooth) real function on $[0, 1]$ such that

$$f_k(t) = \begin{cases} 0, & t \leq 1 - 2^{-k}, \\ 1, & t \geq 1 - 2^{-(k+1)}. \end{cases}$$

A glance at the transition functions (2) shows that, given in a coordinate form, this map is well-defined. It is a desired homotopy from $J^\infty Y$ to $Y$ which is identified with its image under the global section $\gamma: Y \rightarrow J^\infty Y$.

6 Appendix B

We start from a remark that, studying cohomology of exterior forms on the infinite-order jet space of an affine bundle, we can restrict our consideration to vector bundles $Y \rightarrow X$ without loss of generality as follows. Let $Y \rightarrow X$ be a smooth affine bundle modelled over a smooth vector bundle $Y \rightarrow X$. A glance at the transformation law (2) shows that $J^\infty Y \rightarrow X$ is an affine topological bundle modelled on the vector bundle $J^\infty Y \rightarrow X$. This affine bundle admits a global section $J^\infty s$ which is the infinite-order jet prolongation of a global section $s$ of $Y \rightarrow X$. With $J^\infty s$, we have a homeomorphism

$$\tilde{s}_\infty: J^\infty Y \ni q \mapsto q - (J^\infty s)(\pi^\infty(q)) \in J^\infty Y,$$

of the topological spaces $J^\infty Y$ and $J^\infty Y$, together with an exterior algebra isomorphism $\tilde{s}_\infty^*: \mathcal{O}_\infty \rightarrow \mathcal{O}_\infty^*$. Moreover, it is readily observed that the pull-back morphism $\tilde{s}_\infty^*$ commutes with the differentials $d$, $d_V$ and $d_H$. Therefore, the differential algebras $\mathcal{Q}_\infty^*$ and $\mathcal{Q}_\infty$ (as like as $\mathcal{Q}_\infty^*$ and $\mathcal{Q}_\infty$) have the same $d$-, $d_V$- and $d_H$-cohomology.

Let $Y \rightarrow X$ be a vector bundle. Let us consider the vertical complex

$$0 \rightarrow \mathcal{O}^m(X) \rightarrow \mathcal{O}^{0,m} \rightarrow \mathcal{O}^{k,m} \rightarrow \cdots,$$

where $m \leq n$. 

of differential algebras of pull-back exterior forms on $J^\infty Y$. Its local exactness on a coordinate chart $((\pi^\infty)^{-1}(U_X); x^\lambda, y^\Lambda_i)$, $0 \leq |\Lambda|$, on $J^\infty Y$ follows from a version of the Poincaré lemma with parameters (see, e.g., [13]). We have the corresponding homotopy operator

$$\sigma = \int_0^1 k[y]\phi(x^\lambda, ty^\Lambda_i)dt, \quad \phi \in \Omega^k_m,$$

where $y^\Lambda_i = y^i_\Lambda \partial^\Lambda_i$. Since $Y \to X$ is a vector bundle, it is readily observed that, given in a coordinate form, this homotopy operator is globally defined on $J^\infty Y$, and so is the exterior form $\sigma$.

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