String Theory and Quantum Spin Chains

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Abstract

The space-time light-cone hamiltonian $P^{-}$ of large-$N$ matrix models for dynamical triangulations may be viewed as that of a quantum spin chain and analysed in a mean field approximation. As $N \to \infty$, the properties of the groundstate as a function of the bare worldsheet cosmological constant exhibit a parton phase and a critical string phase, separated by a transition with non-trivial scaling at which $P^{-} \to -\infty$. 
1 Introduction

The quantization of non-critical relativistic string theories remains a challenging and important problem, both from the point of view of constructing superstring theories directly in four dimensions and as a useful description of strongly-coupled gauge theories. Non-critical strings differ from the critical ones in that physical degrees of freedom are associated with the longitudinal oscillations. Arguably the most powerful method for tackling this problem is the dynamical triangulation of the string worldsheet and subsequent reformulation as a matrix field theory [1, 2]. I.Klebanov and the author have suggested to capitalize on the well-known simplicity of string theories in light-cone formalism by performing light-cone quantisation of the matrix models [3]. A similar approach was described many years earlier by C.Thorn [4]. The light-cone quantisation has the advantage that observables such as the spectrum and scattering amplitudes can be directly calculated. Some initial numerical analysis of the 1+1-dimensional \((c = 2)\) matrix \(\phi^3\) theory in the large-N limit was carried out in refs.[3, 5]. In this paper a mean field theory will be used to determine the spectrum; these results agree with the numerical ones to a certain extent, but throw up more structure which must be carefully interpreted. In particular one finds two phases, a parton-like phase and a critical-string phase, these being separated by a kind of self-organising transition at which one can define a non-critical string theory. Although the 1+1-dimensional theory is studied principally, some of the conclusions will be seen to carry over directly to higher dimensions.

2 1+1-Dimensional \(\phi^3\) Theory

In \(c\) dimensions the field theory with action

\[
S = \int d^c x \text{Tr} \left( \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} \mu \phi^2 + \lambda V(\phi) \right),
\]

where \(\phi_{ij}(x)\) is an \(N \times N\) hermitian matrix field, is used to generate the \(1/N\) planar diagram expansion [8]. If \(V(\phi) = \phi^3/3\sqrt{N}\), the dual graphs are triangulations and the theory is finite after normal ordering in \(c = 2\) dimensions, the case which will now be discussed [8]. Only the longitudinal oscillations of strings can occur in this case. \(\log \lambda\) is the bare worldsheet cosmological constant conjugate to area, and the Feynman propagator \(\partial^2 + \mu\) is the simplest choice of link factor to specify the embedding in spacetime. In light-cone quantisation, the fourier modes of \(\phi\) at fixed light-cone time \(x^+\),

\[
\phi_{ij} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{dk^+}{\sqrt{2k^+}} \left( a_{ij}(k^+) e^{-ik^+x^-} + a^\dagger_{ji}(k^+) e^{ik^+x^-} \right),
\]

(2)
satisfy standard creation and annihilation commutators;

\[ [a_{ij}(k^+), a^\dagger_{lk}(\tilde{k}^+)] = \delta(k^+ - \tilde{k}^+) \delta_{il} \delta_{jk}. \]  

(3)

The normal-ordered light-cone energy \( P^- \) and momentum \( P^+ \) are

\[
: P^- := \frac{1}{2} \mu \int_0^\infty \frac{dk^+}{k^+} a^\dagger_{ij}(k^+) a_{ij}(k^+) 
- \frac{\lambda}{4\sqrt{\pi}} \int_0^\infty \frac{dk^+_1 dk^+_2}{\sqrt{k^+_1 k^+_2 (k^+_1 + k^+_2)}} \left\{ a^\dagger_{ij}(k^+_1 + k^+_2) a_{ik}(k^+_2) a_{kj}(k^+_1) + a^\dagger_{ik}(k^+_1) a^\dagger_{kj}(k^+_2) a_{ij}(k^+_1 + k^+_2) \right\} 
\]

(4)

\[
: P^+ := \int_0^\infty dk^+ k^+ a^\dagger_{ij}(k^+) a_{ij}(k^+) \]  

(5)

where repeated indices are summed over and \( \dagger \) does not act on indices but means quantum conjugate.

At a fixed \( x^+ \) time-slice one considers the singlet states under \( \phi \to \Omega^\dagger \phi \Omega \), \( \Omega \) unitary, as closed strings; the single string states are those elements of Fock space with all indices contracted by one Trace,

\[ N^{-q/2} \text{Tr}[a^\dagger_{(k^+_1)} \cdots a^\dagger_{(k^+_q)}]|0> , \sum_{i=1}^q k^+_i = P^+ . \]  

(6)

This represents a boundary of length \( q \) in the dynamical triangulation, each \( a^\dagger_{ij}(k^+) \) creating a parton of momentum \( k^+ \). If one lets \( N \to \infty \) the multi-string states, corresponding to more than one Trace, can be neglected since \( 1/N \) is the string coupling constant, and \( P^- \) propagates strings without splitting or joining them. Also \( (a^\dagger a + a^\dagger a^\dagger a) \) acts locally on the string, coalescing two neighbouring partons in the Trace or performing the inverse process. Since the states [1] are already diagonal in \( P^+ \), by finding the linear combinations which diagonalise also \( P^- \) one determines the mass spectrum \( M^2 = 2P^+ P^- \) of the free \( c = 2 \) string. Fixing \( P^+ \), it is useful to discretise the momenta [4], \( k^+ = nP^+/K \), for positive integer \( n \) and some fixed large integer \( K \) which plays the role of cutoff. This renders the total number of states [4] finite. Numerical analysis of the resulting eigenvalue problem was done in refs. [3, 5]; here an approximate analytic approach will be given which allows the \( K \to \infty \) continuum limit. The spectrum at finite \( K \) is given by

\[ \frac{2P^+ P^-}{\mu} = K(V - yT) ; \quad y = \frac{\lambda}{2\mu \sqrt{\pi}}. \]  

(7)

where \( V \) and \( T \) are discretised versions of terms appearing in [4] and \( \mu \) sets the string tension scale. The eigenfunctions of \( 2P^+ P^- \) give directly the structure functions of the string in terms of the Bjorken scaling variable \( n/K \).

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1One excludes \( n = 0 \) because \( a^\dagger(0) \) modes are presumably associated with non-perturbative vacuum structure, whereas here one is only interested in the Feynman diagrams obtained by perturbation theory about \( \phi = 0 \) as a fishnet approximation to string worldsheets.
It is helpful to consider the states (6) as possible spin configurations on a $K$-site periodic chain, each site representing a smallest fraction $1/K$ of the momentum of the string. The sites are partitioned into $q \leq K$ partons by placing a down spin on any bond between neighbouring sites belonging to different partons, while an up spin on bonds between sites belonging to the same parton. The action of $P^-$ on this spin chain is in general very complicated, involving interactions with a range that depends upon the state of the system itself. In a mean field approximation however one may replace each momentum variable occurring in $P^-$ (4) by some typical momentum per parton $K/\beta$, $1 \leq \beta \leq K$, where $\beta(y)$ is to be determined self-consistently. In this approximation $V$ and $T$ become single bond operators on the spin chain; namely,

$$a^\dagger a \equiv \frac{1 - \sigma_3}{2}$$
$$a^\dagger a^\dagger a + a^\dagger a a \equiv \sigma_+ + \sigma_-$$

and one has the following 2x2 mean field hamiltonian at each bond;

$$\begin{pmatrix}
0 & \frac{y^{3/2}}{\sqrt{K}} \\
\frac{y^{3/2}}{\sqrt{K}} & \beta
\end{pmatrix}$$

Thus the effect of the mean-field approximation is to replace the true string structure function by a $\delta$-function at the average parton momentum. This is a good approximation if the variance is small such as when low momentum “sea partons” dominate. Strictly speaking, one should also project onto the spin states invariant under translations along the chain for exact equivalence with the states (3), since Trace is cyclically invariant. However in the mean field approximation the translation operator is trivial and all its representations are degenerate.

The groundstate wavefunction (the “tachyon”), which need not be tachyonic in general, has one parton, therefore one spin down, at $y = 0$: $\mathrm{Tr}[a^\dagger(K)]|0 \rangle$. Diagonalising the mean-field hamiltonian one obtains a consistency condition for $\beta$, the average number of down spins;

$$\beta = \frac{2y^2\beta}{K + 4y^2\beta - K\sqrt{1 + 4y^2}\beta/\sqrt{K}} + (K - 1)\left(1 - \frac{2y^2\beta}{K + 4y^2\beta - K\sqrt{1 + 4y^2}\beta/\sqrt{K}}\right),$$

which for $K \to \infty$ and $\beta$ finite becomes

$$\beta = \frac{1}{1 - y^2}.$$

The mass squared of this state is

$$M^2 = \mu\beta \left(\frac{K}{2} + \left(1 - \frac{K}{2}\right)\sqrt{1 + 4y^2}\beta/\sqrt{K}\right)$$
$$= \mu\beta(1 - y^2\beta); \beta \ll K$$
$$= \mu \left(\frac{1 - 2y^2}{(1 - y^2)^2}\right).$$
Thus as $y$ is increased the average number of partons increases until there is a critical point at $y = 1$ where $\beta \to \infty$. At this point $M^2 \to -\infty$. A comparison with the numerical result of ref. [3] is made in fig.1. having adjusted the horizontal axis so that both curves become tachyonic at the same point. (It is difficult to compare $y$ used here with the analogous dimensionless variable employed in ref. [3] because one could reasonably include some coefficient of $O(1)$ in the off-diagonals of the mean-field Hamiltonian as a refinement of the approximation made.) The numerical results are expected to be accurate when few partons dominate ($\beta = 2$ at $M = 0$) while the mean field argument should be better when there are many partons. Thus the two methods give complementary results. In particular one confirms the conjecture of ref. [3] that the critical point is characterised by $M^2 \to -\infty$. One further confirms that the critical point signals a transition to a long wavelength regime for longitudinal string oscillations, since $\beta \to \infty$. Using (9) one can determine in more detail how this happens. An ansatz $\beta = \alpha K^\gamma$ at $y = 1$ yields $\gamma = 1/2$, $\alpha = 1/\sqrt{3}$. For $y < 1$, $\beta$ is finite (10), while for $y > 1$ only $\gamma = 1$ is consistent, with $\alpha$ satisfying

$$\alpha^2 y^2 - 2\alpha y^2 + (\alpha - 1)(1 - \sqrt{1 + 4\alpha y^2}) = 0 \quad (13)$$

for which $\alpha \to 1/2$ as $y \to \infty$. From this one deduces that there is a non-trivial scaling law operating at the transition point $y = 1$, separating a parton phase with $\beta$ finite from a “critical string” phase in which the spin dynamics are essentially those of the $\beta = K$ critical string for which there are no longitudinal oscillations i.e. it is analogous to the massive phase of a conventional spin chain. The neighborhood of $y = 1$ may allow one to define a continuum string theory in non-critical spacetime background. Indeed if one defines a renormalised worldsheet cosmological constant through $y = 1 - \epsilon/\sqrt{K}$, then the groundstate satisfies $\beta = \alpha \sqrt{K}$ with

$$\alpha = \frac{-\epsilon + \sqrt{\epsilon^2 + 3}}{3} \quad (14)$$

$$M^2 = -K \mu \alpha^2 + O(\sqrt{K}) \quad (15)$$

This would imply that renormalised worldsheet area scales like string length and that polymerised surfaces dominate the functional integral as in Euclidean space [1]. The divergence of $M^2$ is similar to that of critical string theory in light-cone gauge but it is not clear at this stage how one should renormalise it to extract the physically significant part. In critical string theory one can do it by examining the $M^2$ of excited states.

The picture of the excited states coming from the mean field argument is much less clear and does not agree well with the numerical solution. Excited states are very badly represented even at $y = 0$ since the $q$-parton continuum, for example, is compressed into one infinitely degenerate
level at the threshold $M^2 = \mu q^2$. If the preceding analysis is formally repeated for the states having $q$ partons at $y = 0$ one gets

$$\beta = \frac{q}{1 - y^2}, \quad M^2 = \frac{\mu q^2 (1 - 2y^2)}{(1 - y^2)^2}. \quad (16)$$

According to this, $y = 1/\sqrt{2}$ is an inflexion point through which the spectrum for $q$ finite inverts itself (giving continuous spectrum in the neighbourhood). In particular the large $q$ states flip from $+\infty$ to $-\infty$, which seems particularly unphysical. Comparison with the numerical solution (fig.1.) is not good, yet the latter are expected to be accurate at $y \sim 1/\sqrt{2}$. Therefore this tentatively suggests that the mean-field argument is not a very reliable guide to the excitation spectrum. Of course it could still be that the spectrum is continuous at $y = 1$, but at the moment this question, and the calculation of string loop corrections, cannot be addressed without some reliable scheme for the free-string excitation spectrum.

**3 $\phi^4$ Theory**

Using a $\phi^4$ theory should lead to similar results, being a quadrangulation rather than triangulation of the worldsheet. In this case an important variational argument can be used, even in the presence of some transverse dimensions e.g. $c = 4$. Using a transverse lattice action to regulate the propagation in the transverse directions

$$S = \int d^2x \text{Tr} \left( \sum_a \frac{1}{2} (\partial \phi_a)^2 - \frac{1}{2} \mu \phi_a^2 + \frac{\lambda}{4N} \phi_a^4 + \sum_{<ab>} \phi_a \phi_b \right) \quad (17)$$

($<ab>$ indicates nearest neighbour sites $a$ and $b$ on the transverse lattice) the light-cone hamiltonian is

$$: P^- : = -\sum_{<ab>} \int_0^\infty \frac{dk_+}{2k_+} a_+^\dagger(k_+)a_-(k_+) + \sum_a \frac{1}{2} \mu \int_0^\infty \frac{dk_+}{k_+} a_+^\dagger(k_+)a_-(k_+)$$

$$- \frac{\lambda}{8N\pi} \int_0^\infty \frac{dk_+^1 dk_+^2 dk_+^3}{\sqrt{k_+^1 k_+^2 k_+^3 k_+^4}} \left\{ a_+^\dagger(k_+^1) a_+^\dagger(k_+^2) a_-(k_+^3) a_-(k_+^4) \right\} \delta(k_+^1 + k_+^2 - k_+^3 - k_+^4)$$

$$+ \left\{ a_+^\dagger(k_+^1) a_-(k_+^2) a_-(k_+^3) a_+^\dagger(k_+^4) + a_+^\dagger(k_+^2) a_+^\dagger(k_+^3) a_-(k_+^4) a_-(k_+^1) \right\} \delta(k_+^1 + k_+^2 - k_+^3 - k_+^4) \quad (18)$$

(The full index structure has been suppressed for clarity). This again acts locally on the string when $N \to \infty$, the kinetic terms taking three neighbouring partons in the Trace into one, one
into three, or redistributing the momenta between two neighboring partons. Discretising $k^+$ and choosing as variational state

$$|\Psi > = N^{-K/2} \text{Tr}(a_1 \dagger (1) a_1 \dagger (1) \ldots a_1 \dagger (1)) |0 >,$$

i.e. the maximum length string embedded at one particular site, the light-cone hamiltonian satisfies

$$\frac{\langle \Psi | P^- | \Psi \rangle}{\langle \Psi | \Psi \rangle} = \frac{K}{2P^+} \left( \mu - \frac{\lambda}{4\pi} \right).$$

Thus in the continuum limit $K \to \infty$ the groundstate mass squared tends to $-\infty$ for some $\lambda = \lambda_c < 4\pi \mu$. This further confirms the existence and nature of the critical point found in the previous section. It is natural to associate the phenomena with the divergence of planar graphs, which is expected to occur on general grounds.

Returning to two dimensions, it would be nice to carry out a similar mean field analysis of $\phi^4$ theory. Unfortunately $P^-$ does not factorise between bonds in this case. The best one can do is to construct an effective $\phi^4$ theory by “squaring” $\phi^3$ amplitudes, i.e. decomposing the parton number-changing process $1 \to 3$ into $1 \to 2$ then $2 \to 3$, which should capture the features for large parton numbers. This motivates the single bond hamiltonian

$$
\begin{pmatrix}
0 & \frac{i \sqrt{\beta^3/2}}{\sqrt{K}} \\
\frac{-i \sqrt{\beta^3/2}}{\sqrt{K}} & \beta (1-x)
\end{pmatrix}
$$

where $x = \lambda/4\pi \mu$. This form has been chosen because: $a_1 \dagger a_2 \dagger a a$ can move a down spin any distance along the chain up to $\sim K/\beta$, so in mean field can be approximated by a single bond term $(K/\beta) x (-x \beta^2 / K)$ on the diagonal; $a_1 \dagger a_1 \dagger a_1 a$ creates two down spins, with amplitude $A^2$ say, where $A$ is the amplitude to create one down spin, and since the second spin must be created within distance $K/\beta$ of the first there is a compensating factor of $1/\beta$ if $A$ creates a down spin at any bond, i.e. $(A^2/\beta) = (-x \beta^2 / K)$. The groundstate analysis then proceeds as before, giving

$$
\beta = \frac{1}{1 - \frac{x}{(1-x)^2}}
$$

and the same critical behaviour. Beyond the critical point at $x = (3 - \sqrt{5})/2$ one finds $\alpha \to 1$ at $x \to \infty$. Thus in the strong coupling limit one recovers exactly the critical string.

The mean-field hamiltonian can also be considered at $x < 0$, corresponding to the “right-sign” $\phi^4$ theory. A more appropriate parameterization in this case is to let $\lambda/4\pi$ set the energy scale
and $z = -4\pi\mu/\lambda$ be the dimensionless parameter;

$$
\beta = \frac{1}{1 - \frac{1}{(1+z)^2}} \quad (23)
$$

$$
M^2 = \frac{\lambda(1+z)^3(z^2 + 2z - 1)}{(z^2 + 2z)^2} \quad (24)
$$

showing that tachyons appear for small enough parton mass squared $z$. Indeed this appears to occur at $z > 0$ indicating dynamical symmetry breaking to a true groundstate which requires a more careful treatment of zero modes to elucidate. The naive continuation of the Fock vacuum $|0\rangle$ into its tachyonic phase again takes one into a critical string region eventually. Transverse lattice QCD at large $N$ is very similar to the $\phi^4$ theory, although the coupling $\lambda$ can be both attractive and repulsive, and it is natural to expect a similar parton/string phase structure as the parton mass is varied as has been suggested in the past by Klebanov and Susskind.

4 Conclusions

The free non-critical bosonic string theory described by dynamical triangulations has been shown to possess a transition separating a parton phase from a critical string phase. Introducing string interactions, it would be interesting to see if the non-critical string theory defined at the transition point is consistent in any spacetime dimension. It is perhaps more appropriate, at least eventually, to study the non-critical Green-Schwarz superstring described by supertriangulations. There appears to be no obstacle to the numerical light-cone quantisation of this case, which always has a tachyon-free true groundstate. The spin-chain idea and its investigation by mean field, variational, and exact solution, has provided a partially satisfactory analytic approach to complement the available numerical solutions of light-cone quantisation of large-$N$ theories. In most cases one is only interested in the lowest few energy levels and so use of appropriate algorithms could surely push the attainable numerical results well beyond the present set. It is hoped to address this in future work.

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2Parton, i.e. transverse gluon, mass terms can appear on renormalisation of the light-cone gauge QCD.
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FIGURE CAPTION

Fig.1. – A comparison of mean field results with the numerical results of ref.[5] for the mass squared ($\mu = 1$). The solid lines are the mean field answer for the states corresponding to one and two partons at $y = 0$. The dotted lines are obtained by exact solution at each $y$ for $K$ up to 15, then extrapolation to $K = \infty$ by fitting to a ratio of two polynomials in $K$. The numerical data contain the lowest two states of the two-parton continuum at $y = 0$, which split at $y > 0$ (they are always degenerate in mean field). The variable $y$ used in this paper has been rescaled in this graph by $1/0.46\sqrt{2}$ so that the groundstates of mean field and numerical solution become massless at the same point.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9404058v1