Casimir Energy due to a Semi-Infinite Plane Boundary

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Abstract

Following the derivation of the Green function for the massless scalar field satisfying the Dirichlet boundary condition on the Plane \((x \geq 0, y = 0)\), we calculate the Casimir energy.

1. Introduction

Thinking the open space Casimir energy calculations with plane boundaries we can name three essential examples: Single infinite plane [1], the original parallel planes configuration [2], and the wedge problem, that is the two inclined planes geometry [3]. In this note we study a single semi-infinite plane boundary.

In the following section we derive the exact propagator for massless scalar field with Dirichlet boundary condition on the semi-infinite plane.

Section III is devoted to the calculation of the Casimir energy.

II. Green Function

We need the Green function for massless scalar field satisfying Dirichlet boundary condition on the semi-infinite plane given by

\[ P : \quad x \geq 0, \quad y = 0. \] (1)

Since the time and coordinate section of the massless Klein-Gordon field is trivial we can write the Green function as (with \(x\) standing for the four dimensional vector \(x_\mu\))

\[ G(x, x') = \frac{1}{(2\pi)^2} \int dp_0 dp_z e^{-ip_0(t-t')} e^{-ip_z(z-z')} G_p(\vec{x}, \vec{x}') \] (2)

Here \(\vec{x} = (x, y)\) and \(p = \sqrt{p_0^2 + p_z^2}\). \(G_p\) satisfies

\[ (p^2 - \partial^2_x - \partial^2_y)G_p(\vec{x}, \vec{x}') = \delta(x - x')\delta(y - y') \] (3)

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Note that we used Euclidean metric instead of Minkowski one. This is always possible if the propagation of the fields in the direction of time is free. We now employ a variable change

\[ x = u^2 - v^2, \quad y = 2uv, \quad v \geq 0 \]  

which maps the plane \( P \) onto \( \bar{P} : u = 0 \).

The full \( (x, y) \) plane is mapped on the half plane \( v \geq 0 \).

In \( u, v \) coordinates (3) becomes

\[ (4p^2(u^2 + v^2) - \partial_u^2 - \partial_v^2)G_p(\vec{u}, \vec{u}') = \delta(u - u')\delta(v - v') \]  

which is the equation for two noninteracting harmonic oscillators, with \( v \geq 0 \) and Dirichlet boundary condition at \( v = 0 \). The Green function \( G_p(\vec{u}, \vec{u}') \) corresponds to the Kernel satisfying

\[ \left( \frac{d}{dS} + \left( -\frac{1}{2} \partial_u^2 + \omega^2 u^2 \right) + \left( -\frac{1}{2} \partial_v^2 + \omega^2 v^2 \right) \right)K(\vec{u}, \vec{u}') = \delta(S)\delta(u - u')\delta(v - v') \]  

where \( S \) is the Euclidean "time" interval \( s - s' \) and \( \omega \equiv 2p \). The Euclidean "time" oscillator Kernel is given by the well known formula [4]:

\[ K(\vec{u}, \vec{u}'; S) = \Theta(S)K_1(u, u'; S)K_2(v, v'; S) \]  

where

\[ K_1(u, u'; S) = \sqrt{\frac{\omega}{2\pi \sinh \omega S}}\exp\left[-\frac{\omega}{2\sinh \omega S}((u^2 + u'^2) \cosh \omega S - 2uu')\right] \]  

\[ K_2(v, v'; S) = \sqrt{\frac{\omega}{2\pi \sinh \omega S}}\left(\exp\left[-\frac{\omega}{2\sinh \omega S}((v^2 + v'^2) \cosh \omega S - 2vv')\right] - \exp\left[-\frac{\omega}{2\sinh \omega S}((v^2 + v'^2) \cosh \omega S + 2vv')\right]\right) \]  

By construction the Kernel \( K_2 \) vanishes at \( v = 0 \). The Green function \( \bar{G}_p(\vec{u}, \vec{u}') \) is simply the Fourier transform of the Kernel of [S]

\[ \bar{G}_p(\vec{u}, \vec{u}') = \frac{1}{2} \int_0^\infty dSK(\vec{u}, \vec{u}'; S). \]

After dropping the first term in [H] which corresponds to the free Minkowski space term, we get the regularized Green function

\[ \bar{G}_p^{R}(\vec{u}, \vec{u}') = -\frac{1}{2} \int_0^\infty dS \frac{\omega}{2\pi \sinh \omega S}\exp\left[-\frac{\omega}{2\sinh \omega S}(\xi \cosh \omega S - \eta)\right] \]  

where

\[ \xi = u^2 + u'^2 + v^2 + v'^2, \quad \eta = 2(uu' - vv'). \]
After inserting the above expression into (2) in place of $G_p(\vec{x}, \vec{x}')$ we arrive at the regularized Green function $G_R(x, x')$ which vanishes on the semi-infinite plane P of (1).

III. Casimir Energy density

To obtain the Casimir energy density we employ the well known coincidence limit formula [5] which in the Euclidean metric reads

$$T(x) = \frac{1}{2} \lim_{x \to x'} (-\partial_t \partial_{t'} + \partial_z \partial_{z'} + \frac{1}{4(u^2 + v^2)}(\partial_x \partial_{x'} + \partial_y \partial_{y'}))G_R(x, x')$$

or in $u, v$ coordinates

$$T(x) = \frac{1}{2} \lim_{x \to x'} (-\partial_t \partial_{t'} + \partial_z \partial_{z'} + \frac{1}{4(u^2 + v^2)}(\partial_x \partial_{x'} + \partial_y \partial_{y'}))G_R(x, x')$$

Since the system is invariant under the interchange of $t$ and $z$ after integration over $dp_0$ and $dp_z$ the first two terms cancel each other. This fact gives an opportunity to simplify our expression further. Instead of $G_R(x, x')$ we can work with an effective Green function

$$G_{eff}^R(\vec{x}, \vec{x}') = G_R(x, x') \mid t = t', z = z'$$

which is equal to

$$G_{eff}^R(\vec{x}, \vec{x}') = \frac{1}{2\pi} \int_0^\infty dp \overline{G_R}(\vec{u}, \vec{u}')$$

Inserting (12) in the above equation in place of $G_R^p(\vec{x}, \vec{x}')$ we can first integrate over $dp$ ($\omega$ in (12) is $\omega = 2\rho$), then over $dS$. We then obtain

$$G_{eff}^R(\vec{x}, \vec{x}') = -\frac{1}{8\pi^2 \xi (u - u')^2 + (v + v')^2}$$

Using this simple formula in (15) (with first two terms dropped) as the Green function we get expression for the energy density

$$T(x) = -\frac{1}{128\pi^2 \xi (u^2 + v^2)^2} \left( \frac{1}{2\nu^2} + \frac{1}{(u^2 + v^2)} \right)$$

or in terms of the original $x, y$ coordinates (with $r = \sqrt{x^2 + y^2}$)

$$T(x) = -\frac{1}{64\pi^2 r^2 (r - x)} \left( \frac{1}{r - x} + \frac{1}{r} \right),$$

which in terms of polar coordinates $x = r \cos \theta, y = r \sin \theta$ is

$$T(x) = -\frac{1}{64\pi^2 r^4 (1 - \cos \theta)} \left( \frac{1}{1 - \cos \theta} + 1 \right).$$
Inspecting (20) we see that at \( x > 0 \) and \( y \ll x \) we obtain the infinite plane result [5]

\[
T \simeq -\frac{1}{16\pi^2y^4} ; \quad x > 0, y \ll x
\]

as expected.

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[5] See for example N.D. Birrel and P. C. W. Davies, "Quantum Fields in Curved Spaces", Cambridge Univ. Press (1982).