The lognormal-like statistics of a stochastic squeeze process

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We analyze the full statistics of a stochastic squeeze process. The model’s two parameters are the bare stretching rate $w$, and the angular diffusion coefficient $D$. We carry out an exact analysis to determine the drift and the diffusion coefficient of $\log(r)$, where $r$ is the radial coordinate. The results go beyond the heuristic lognormal description that is implied by the central limit theorem. Contrary to the common “Quantum Zeno” approximation, the radial diffusion is not simply $D_r = (1/8)w^2/D$, but has a non-monotonic dependence on $w/D$. Furthermore, the calculation of the radial moments is dominated by the far non-Gaussian tails of the $\log(r)$ distribution.

I. INTRODUCTION

In this paper we analyze the full statistics of a physically-motivated stochastic squeeze process that is described by the Langevin (Stratonovich) equation

\[
\begin{align*}
\dot{x} &= wx - \omega(t)y \\
\dot{y} &= -wy + \omega(t)x
\end{align*}
\]

(1)

where the rotation frequency $\omega(t)$ is a zero mean white noise with fluctuations:

\[
\langle \omega(t')\omega(t'') \rangle = 2D\delta(t' - t'')
\]

(2)

Accordingly the model has two parameters: the angular diffusion coefficient $D$ of the polar phase, and the bare stretching rate $w$ of the radial coordinate $r = \sqrt{x^2 + y^2}$. In a physical context the noise arises due to the interaction with environmental degrees of freedom, typically modeled as an harmonic bath of “phonons”. Hence we can assume for it a Gaussian-like distribution with bounded moments. The white noise assumption means that the correlation time is very short, hence the Stratonovich interpretation of Eq.(1) is in order, as argued by Ref. [1].

The squeeze operation is of interest in many fields of science and engineering, but our main motivation originates from the quantum mechanical arena, where it is known as parametric amplification. In particular it describes the dynamics of a Bosonic Josephson Junction (BJJ) given that all the particles are initially condensed in the upper orbital. Such preparation is unstable [2, 3], but it can be stabilized by introducing frequent measurements or by introducing noise. This is the so-called “quantum Zeno effect” (QZE) [4–8]. The manifestation of the QZE in the BJJ context has been first considered in [9, 10], and later in [11].

The main idea of the QZE is usually explained as follows: The very short-time decay of an initial preparation due to a constant perturbation is described by the survival probability $P(t) = 1 - (vt)^2$, where $v$ is determined by pertinent couplings to the other eigenstates; Dividing the evolution into $\tau$-steps, and assuming a projective measurement at the end of each step one obtains

\[ P(t) \approx [P(\tau)]^{t/\tau} \approx [1 - (v\tau)^2]^{t/\tau} \approx \exp \left[ -(v^2\tau)t \right] \]

The common phrasing is that frequent measurements (small $\tau$) slow down the decay process due to repeated “collapse” of the wavefunction. Optionally one considers a system that is coupled to the environment. Such interaction is formally similar to a continuous measurement process, that is characterized by a dephasing time $\tau$. In the latter case the phrasing is that the introduction of “noise” leads to the slow-down of the decay process. Contrary to simple minded intuition, stronger noise leads to slower decay.

At this point one might get the impression that the QZE is a novel “quantum” effect, that has to do with mysterious collapses, and that such effect is not expected to arise in a “classical” reality. Such conclusion is in fact wrong: whenever the the system of interest has a meaningful classical limit, the same Zeno effect arises also in the classical analysis. This point has been emphasized by Ref. [11] in the context of the BJJ. It has been realized that the QZE is the outcome of the classical dynamics that is is generated by Eq.(1), where the $(x, y)$ are local canonical conjugate coordinates in the vicinity of an hyperbolic (unstable) fixed-point in phase space. The essence of the QZE in this context is the observation that the introduction of the noise via the phase-variable leads to slow-down of the radial spreading. For strong noise (large $D$ in Eq.(2)), the radial spreading due to $w$ is inhibited. Using quantum terminology this translates to suppression of the decoherence process.

From pedagogical point of view it is useful to note that the dynamics of the BJJ is formally similar to that of a mathematical pendulum. Condensation of all the particles in the upper orbital is formally the same as preparing the pendulum in the upper position. Such preparation is unstable. If we want to stabilize the pendulum in the upper position we have the following options: (i) Introducing periodic driving that leads to the Kapitza effect; (ii) Introducing noisy driving that leads to a Zeno effect. We note that the Kapitza effect in the BJJ context has been discussed in [12], while our interest here is in the semiclassical perspective of the QZE that has been illuminated in [11].

Experiments with cold atoms are state of the art [13, 14]. In such experiments it is common to perform a “fringe visibility” measurement, which indicates the condensate occupation. The latter is commonly quan-
For strong noise the following asymptotic results have been obtained:

\[ F(t) = \exp \left\{ -\frac{1}{N} S(t) \right\} \]  

where \( N \) is the number of condensed bosons, and

\[ S(t) = \left( \frac{w^2}{D} \right) t \]  

The key realizations of Ref.\[11\] is that \( S(t) \) is in fact the radial spreading in a stochastic process that is described by Eq.(1).

A practical question arises, whether the heuristic QZE expression for \( S(t) \) is useful in order to describe the actual decay of the one-body coherence. The answer of Ref.\[11\] was: (i) The heuristic result is correct only for a very strong noise (small \( w/D \)), and holds only during a very short time. (ii) Irrespective of correctness, it is unlikely to obtain a valid estimate for \( S(t) \) in a realistic measurement, because the statistics is log-normal, dominated by far tails.

On the quantitative side, Ref.\[11\] was unable to provide an analytical theory for the lognormal statistics of the spreading. Rather it has been argued that the \( \ln(r) \) distribution has some average \( \mu \propto t \), and some variance \( \sigma^2 \propto t \). The radial stretching rate \( w_r \), and a radial diffusion coefficient \( D_r \) were determined numerically from the assumed time dependence:

\[ \mu = w_r t \]  

\[ \sigma^2 = 2D_r t \]  

From the lognormal assumption it follows that

\[ S(t) = e^{D_r t + 2w_r t} - 1 \]  

For strong noise the following asymptotic results have been obtained:

\[ w_r \sim \frac{w^2}{4D} \]  

\[ D_r \sim \frac{w^2}{8D} \]  

These approximations are satisfactory for \( w/D \ll 1 \), but fail miserably otherwise. We also see that Eq.(7) reduces to Eq.(4) in this strong noise limit, for a limited duration of time. Note that Eq.(7) is not identical with the expression that has been advertised in \[11\] for reasons that will be discussed in the concluding section.

**Outline**— The QZE motivation for the analysis of Eq.(1) is introduced in Sections II. Numerical results for the radial spreading due to such process are presented in Section III. Our objective is to find explicit expression for \( w_r \) and \( D_r \), and also to characterize the full statistics of \( r(t) \) in terms of the bare model parameters \( (w, D) \). The first step is to analyze the phase randomization in Sections IV, and to discuss the implication of its non-isotropic distribution in Section V. Consequently the exact calculation of the \( \ln(r) \) distribution is presented in Sections VI and VII. In Sections VIII we clarify that the statistics of \( r(t) \) is in fact a bounded lognormal distribution. It follows that the \( r \) moments of the spreading, unlike the \( \ln(r) \) moments, cannot be deduced directly from our results for \( w_r \) and \( D_r \). Nevertheless, in Section IX we find the \( r \) moments using the equation of motion for the moments. Finally in Section X we come back to the discussion of the QZE context of our results. On the one hand we note that Eq.(7) should be replaced by a better version that takes into account the deviations from the lognormal statistics. But the formal result for \( S(t) \) has no experimental significance: the feasibility of experimental \( S(t) \) determination is questionable, because averages are sensitive to the far tails. Rather, in a realistic experiment it is feasible to accumulate statistics and to deduce what are \( w_r \) and \( D_r \), which can tested against our predictions. Some extra details regarding the QZE perspective and other technicalities are provided in the Appendices.

**II. SEMICLASSICAL PERSPECTIVE**

In the present section we clarify the semiclassical perspective for the QZE model, and motivate the detailed analysis of Eq.(1). The subsequent sections are written in a way that is independent of a specific physical context. We shall come back to the discussion of the QZE in the concluding section, where the implications of our results are summarized.

For a particular realization of \( \omega(t) \) the evolution that is generated by Eq.(1) is represented by a symplectic matrix

\[ \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = U \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \]  

The matrix is characterized by its trace \( a = \text{trace}(U) \). If \( |a| < 2 \) it means elliptic matrix (rotation). If \( |a| > 2 \) it means hyperbolic matrix. In the latter case, the radial coordinate \( r \) is stretched in one major direction by some factor \( \exp(a) \), while in the other major direction it is squeezed by factor \( \exp(-a) \). Hence \( a = \pm 2 \cosh(\alpha) \). If we operate with \( U \) on an initial isotropic cloud that has radius \( r_0 \), then we get a stretched cloud with \( \langle r^2 \rangle = A r_0^2 \), where \( A = \cosh(2\alpha) \). For more details see Appendix A. The numerical procedure of generating a stochastic process that is described by Eq.(1) is explained in Appendix B. Rarely the result is a rotation. So from now on we refer to it as “squeeze”.

The initial preparation can be formally described as a minimal wavepacket at the origin of phase-space. The local canonical coordinates are \( (x, y) \), or optionally one can use the polar coordinates \( (\varphi, r) \). The initial spread of the wavepacket is \( \langle r^2 \rangle = \hbar \). In the case of a BJJ the dimensionless Planck constant is related to the number of particles, namely \( \hbar = 2/N \). In the absence of noise \( (D = 0) \)
the wavepacket is stretched exponentially in the $x$ direction, which implies a very fast decay of the initial preparation. This decay can be described by functions $P(t)$ and $\mathcal{F}(t)$ that give the survival probability of the initial state, and the one-body coherence of the evolving state. For precise definitions see Appendix C. Note that $\mathcal{F}(t)$ is defined as the length of the Bloch vector, normalized such that $\mathcal{F}(t) = 1$ for the initial coherent state.

We now consider the implication of having a noisy dephasing term ($D > 0$). The common perspective is to say that this noise acts like a measurement of the $r$ coordinate, which randomizes the phase $\varphi$ over a time scale $\tau \sim 1/D$, hence introducing a “collapse” of the wavefunction. The succession of such interventions (see Appendix C) leads to a relatively slow exponential decay of the coherence, namely $\mathcal{F}(t) = \exp\{-(\hbar/2)S(t)\}$, where $S(t)$ is given by Eq.(4). The stronger the noise ($D$), the slower is the decay of $\mathcal{F}(t)$. Similar observation applies to $P(t)$. Using a semiclassical perspective [11] it has been realized that

$$S(t) = A(t) - A(0) \tag{11}$$

Note that by definition $\hbar A(t)$ is the spread $\langle r^2 \rangle$ of the evolving phase-space distribution, where $A(t)$ is normalized such that $A(0) = 1$.

The well known QZE expression Eq.(4), in spite of its popularity, poorly describes the decoherence process [11]. In fact, it agrees with numerical simulations only for extremely short times for which $(w^2/D)t \ll 1$. The semiclassical explanation is as follows: In each $\tau$-step of the evolution the phase-space distribution is stretched by a random factor $\lambda_n = \exp[\alpha_n]$, where the $\alpha_n$ are uncorrelated random variables. Hence by the central limit theorem the product $\lambda = \lambda_1 \ldots \lambda_2 \lambda_1$ has lognormal distribution, where $\log(\lambda)$ has some average $\mu \propto t$ and variance $\sigma^2 \propto t$ that determine an $A(t)$ and hence $S(t)$ that differs from the naive expression of Eq.(4). The essence of the QZE is that $\mu$ and $\sigma^2$ are inversely proportional to the intensity of the erratic driving. Consequently one has to distinguish between 3 time scales: the “classical” time for phase ergodization $\tau \sim D^{-1}$ which is related to the angular diffusion; the “classical” time for loss of isotropy $t_r \sim (w^2/D)^{-1}$ that characterizes the radial spreading; and the “quantum” coherence time $t_c \sim (1/\hbar)t_r$, after which $\mathcal{F}(t) \ll 1$.

In [11] the time dependence of $\mu$ and $\sigma$ has been determined numerically. Here we would like to work out a proper analytical theory. It turns out that a quantitative analysis of the stochastic squeezing process requires to go beyond the above heuristic description. The complication arises because what we have is not multiplication of random number, but multiplication of random matrices. Furthermore we shall see that the calculation of moments requires to go beyond central limit theorem, because they are dominated by the far tails of the distribution.

In the concluding section X we shall clarify that from an experimental point of view the formal expression $\mathcal{F}(t) = \exp\{-(\hbar/2)S(t)\}$ is not very useful. For practical purpose it is better to consider the full statistics of the Bloch vector, and to determine $\mu$ and $\sigma$ via a standard fitting procedure.

### III. PRELIMINARY CONSIDERATIONS

Below we are not using a matrix language, but address directly the statistical properties of an evolving distribution. In $(\varphi, r)$ polar coordinates Eq.(1) takes the form

$$\dot{\varphi} = -w \sin(2\varphi) + \omega(t) \tag{12}$$

$$\dot{r} = [w \cos(2\varphi)] \: r \tag{13}$$

We see the equation for the phase decouples, while for the radius

$$\frac{d}{dt} \ln(r(t)) = w \cos(2\varphi) \tag{14}$$

![FIG. 1. Scaled stretching rate $w_r/w$ versus $w/D$.](image1)

The numerical results (black symbols) are based on simulations with 2000 realizations. The lines are for the naive result Eq.(8) (green dotted); the exact result Eq.(23) (red solid); and its practical approximation Eq.(24) (blue dashed-dotted). For large values of $w/D$ we get $w_r/w = 1$, as for a pure stretch.

![FIG. 2. Scaled diffusion coefficient $D_r/w$ versus $w/D$.](image2)

The numerical results (black symbols) are based on simulations with 2000 realizations. The lines are for the naive result Eq.(9) (green dotted); the exact result Eq.(33) (red solid); and the approximation Eq.(28) with $\tau = 1/(2D)$ (blue dashed-dotted), and with Eq.(34) (dashed orange line).
The RHS has some finite correlation time \( \tau \approx 1/D \), and therefore \( \ln(r) \) is like a sum of \( t/\tau \) uncorrelated random variables. It follows from the central limit theorem that for long time the main body of the \( \ln(r) \) distribution can be approximated by a normal distribution, with some average \( \mu \propto t \), and some variance \( \sigma^2 \propto t \). Consequently we can define a radial stretching rate \( w_r \), and a radial diffusion coefficient \( D_r \) via Eq. (6).

Our objective is to find explicit expression for \( w_r \) and \( D_r \), and also to characterize the full statistics of \( r(t) \) in terms of the bare model parameters \((w, D)\). We shall see that the statistics of \( r(t) \) is described by a bounded lognormal distribution.

Some rough estimates are in order. For large \( D \) one naively assumes that due to ergodization of the phase \( \mu = \langle \cos(2\varphi) \rangle w \) is zero, while \( \sigma^2 \approx (w \tau)^2/t \). Hence one deduces that \( w_r \to 0 \) while \( D_r \propto w^2/D \). A more careful approach [11] that takes into account the non-isotropic distribution of the phase gives the asymptotic results Eq. (8) and Eq. (9). The dimensionless parameter that controls the accuracy of this result is \( w/D \). These approximations are satisfactory for \( w/D \ll 1 \), and fails otherwise, see Fig.1 and Fig.2. For large \( w/D \) we get \( w_r \to w \), while \( D_r \to 0 \).

**IV. PHASE ERGODIZATION**

The Fokker-Planck equation (FPE) that is associated with Eq. (12) is

\[
\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial \varphi} \left[ \left(D \frac{\partial}{\partial \varphi} + w \sin(2\varphi) \right) \rho \right]
\] (15)

It has the canonical steady state solution

\[
\rho_\infty(\varphi) \propto \exp \left[ \frac{w}{2D} \cos(2\varphi) \right]
\] (16)

If we neglect the cosine potential in Eq. (15) then the time for ergodization is \( \tau_{\text{erg}} \sim 1/D \). But if \( w/D \) is large we have to incorporate an activation factor, accordingly

\[
\tau_{\text{erg}} = \frac{1}{D} \exp \left[ \frac{w}{D} \right]
\] (17)

Fig.3(a) shows the distribution of the phase for two different initial conditions, as obtained by a finite time numerical simulation. It is compared with the steady state solution. The dynamics of \( r \) depends only on \( 2\varphi \), and is dominated by the distribution at the vicinity of \( \cos(2\varphi) \sim 1 \). We therefore display in Fig.3(b) the distribution of \( \varphi \) modulo \( \pi \). We deduce that the transient time of the \( \ln(r) \) spreading is much shorter than \( \tau_{\text{erg}} \).

For the later calculation of \( w_r \) we have to know also the moments of the angular distribution. From Eq. (16) we obtain:

\[
X_n = \langle \cos(2n\varphi) \rangle_\infty = \frac{I_n \left( \frac{w}{2D} \right)}{I_0 \left( \frac{w}{2D} \right)}
\] (18)

Here \( I_n(z) \) are the modified Bessel functions. For small \( z \) we have \( I_n(z) \approx \frac{1}{n!}(z/2)^n \), while for large \( z \) we have \( I_n(z) \approx (2\pi z)^{-1/2}e^z \). The dependence of the \( X_n \) on \( n \) for representative values of \( w/D \) is illustrated in the upper panel of Fig.4.

For the later calculation of \( D_r \) we have to know also the temporal correlations. We define

\[
C_n(t) = \langle \cos(2n\varphi(t)) \cos(2\varphi) \rangle_\infty - X_n X_1
\] (19)

where a constant is subtracted such that \( C_n(\infty) = 0 \). We use the notations

\[
c_n = \int_0^\infty C_n(t) dt
\] (20)

and

\[
\Delta_n = C_n(0) = \frac{1}{2} (X_{n+1} + X_{n-1}) - X_n X_1
\] (21)

In order to find an asymptotic expression we use

\[
I_n(z) \approx \frac{e^z}{\sqrt{2\pi z}} \left[ 1 - \frac{4n^2 - 1}{(8z)} + \frac{(4n^2 - 1)(4n^2 - 9)}{2(8z)^2} \right]
\]

and get

\[
\Delta_n \approx 2 \left( \frac{w}{D} \right)^{-2} n^2 \quad \text{for} \quad \left( \frac{w}{D} \right) \gg 1
\] (22)

The dependence of the \( \Delta_n \) on \( n \) for representative values of \( w/D \) is illustrated in the lower panel of Fig.4.

**V. RADIAL SPREADING**

If follows from Eq. (14) that the radial stretching rate is

\[
w_r = w \langle \cos(2\varphi) \rangle_\infty = X_1 w
\] (23)
A rough interpolation for $X_1$ that is based on the asymptotic expressions for the Bessel functions in Eq.(18) leads to the following approximation

$$w_r \approx w \left[ 1 - \exp \left( -\frac{w}{4D} \right) \right]$$

(24)

The exact result as well as the approximation are illustrated in Fig.1 and compared with the results of numerical simulations.

For the second moment it follows from Eq.(14) that the radial diffusion coefficient is

$$D_r = w^2 \int_0^\infty C_1(t) dt = c_1 w^2$$

(25)

If we assume that the ergodic angular distribution is isotropic, the calculation of $C_1(t)$ becomes very simple, namely,

$$C_1(t) = \frac{1}{2} \langle \cos (2\phi_t - \phi_0) \rangle = \frac{1}{2} e^{-4Dt|t|}$$

(26)

This expression implies a correlation time $\tau = 1/(2D)$, such that $c_1 = (1/2)\Delta_1 \tau$ is half the “area” of the correlation function whose “height” is $\Delta_1 = 1/2$. Thus we get for the radial diffusion coefficient $D_r = w^2/(8D)$.

But in fact the ergodic angular distribution is not isotropic, meaning that $X_1$ is not zero, and $\Delta_1 < 1/2$. If $w$ is not too large we may assume that the correlation time $\tau$ is not affected. Then it follows that a reasonable approximation for the correlation function is

$$C_1(t) \approx \Delta_1 e^{-2|t|/\tau}$$

(27)

leading to

$$D_r \approx \frac{1}{2} \Delta_1 \tau w^2 = \Delta_1 \frac{w^2}{4D}$$

(28)

This approximation is compared to the exact result that we derive later in Fig.2. Unlike the rough approximation $D_r = w^2/(8D)$, it captures the observed non-monotonic dependence of $D_r$ versus $w$, but qualitatively it is an over-estimate.

VI. THE EXACT CALCULATION OF THE DIFFUSION COEFFICIENT

We now turn to find an exact expression for the diffusion coefficient Eq.(25) by calculating $c_1$ of Eq.(20). Propagating an initial distribution $\rho_0(\phi)$ with the FPE Eq.(15) we define the moments:

$$x_n = \langle \cos(2n \phi_t) \rangle_0 = \langle \cos(2n \phi) \rangle_t$$

$$= \int \cos(2n \phi) \rho_t(\phi) d\phi$$

(29)

The moments equation of motion resulting from the FPE is [15]:

$$\frac{dx_n}{dt} = -\Lambda_n x_n + W_n (x_{n-1} - x_{n+1})$$

(30)

where $\Lambda_n = 4Dn^2$ and $W_n = w_n$. Due to $\Lambda_0 = W_0 = 0$ the zeroth moment $x_0 = 1$ does not change in time. Thus the rank of Eq.(30) is less than its dimension reflecting the existence of a zero mode $x_n = X_n$ that corresponds to the steady state of the FPE. We shall use the subscript “$\infty$” to indicate the steady state distribution. Any other solution $x_n(t)$ goes to $X_n$ in the long time limit, while all the other modes are decaying. To find $X_n$ the equation should be solved with the boundary condition $X_\infty = 0$, and normalized such that $X_0 = 1$. Clearly this is not required in practice: because we already know the steady state solution Eq.(15), hence Eq.(18).

We define $x_n(t; \phi_0)$ as the time-dependent solution for an initial preparation $\rho_0(\phi) = \delta(\phi - \phi_0)$. Then we can express the correlation function of Eq.(19) as follows:

$$C_n(t) = \langle x_n(t; \phi) \cos(2\phi) \rangle_\infty - X_n X_1$$

(31)

By linearity the $C_n(t)$ obey the same equation of motion as that of the $x_n(t)$, but with the special initial conditions $C_n(0) = \Delta_n$. Note that $C_0(t) = 0$ at any time. In the infinite time limit $C_n(\infty) = 0$ for any $n$.

Our interest is in the area $c_n$ as defined in Eq.(20). Writing Eq.(30) for $C_n(t)$, and integrating it over time we get

$$\Lambda_n c_n - W_n (c_{n-1} - c_{n+1}) = \Delta_n$$

(32)

This equation should be solved with the boundary conditions $c_0 = 0$ and $c_\infty = 0$. The solution is unique because the $n = 0$ site has been effectively removed, and the truncated matrix is no longer with zero mode. One possible numerical procedure is to start iterating with $c_1$ as initial condition, and to adjust it such that the solution will go to zero at infinity. An optional procedure is to integrate the recursion backwards as explained in the next section.

The bottom line is the following expression

$$D_r = c_1 w^2 = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \Delta_n X_n w$$

(33)

where $X_n$ and $\Delta_n$ are given by Eq.(18) and Eq.(21) respectively.
The leading term approximation \( D_r \approx \Delta_1 X_1 w \) is consistent with the heuristic expression \( D_r \approx (1/2) \Delta_1 \tau w^2 \) of Eq.(28) upon the identification

\[
\tau = \frac{2}{w} \left[ 1 - \exp \left( - \frac{w}{4D} \right) \right]
\]  

(34)

This expression reflects the crossover from diffusion-limited \((\tau \propto 1/D)\) to drift-limited \((\tau \propto 1/w)\) spreading. Fig. 2 compares the approximation that is based on Eq.(28) with Eq.(34) to the exact result Eq.(33).

In the limit \((w/D) \to 0\) the asymptotic result for the radial diffusion coefficient is \( D_r = w^2/(8D) \). We now turn to figure out what is the asymptotic result in the other extreme limit \((w/D) \to \infty\). The large \( w/D \) approximation that is based on the first term of Eq.(33), with the limiting value \( X_1 = 1 \), provides the asymptotic estimate \( D_r \approx 2D^2/w \). This expression is based on the asymptotic result Eq.(22) for \( \Delta_n \) with \( n = 1 \). In fact we can do better and add all the higher order terms. Using Abel summation we get

\[
D_r = \frac{2}{w} \left[ \frac{D^2}{w} \sum_{n=1}^{\infty} (-1)^{n-1} n \left( \frac{D^2}{w} \right)^n \right] = \frac{1}{2} \frac{D^2}{w} \]  

(35)

Thus the higher order terms merely add a factor 1/4 to the asymptotic result. If we used Eq.(28), we would have obtained the wrong prediction \( D_r \approx D/2 \) that ignores the \( \tau \) dependence of Eq.(34).

### VII. DERIVATION OF THE RECURSIVE SOLUTION

In this section we provide the details of the derivation that leads from Eq.(32) to Eq.(33). We define \( W_n^\pm = \mp W_n \) and rewrite the equation in the more general form

\[
-W_n^+ c_{n+1} + \Lambda_n c_n - W_n^- c_{n-1} = \Delta_n
\]  

(36)

A similar problem was solved in [16], while here we present a much simpler treatment. First we solve the associated homogeneous equation. The solution \( c_n = X_n \) satisfies

\[
-W_n^+ X_{n+1} + \Lambda_n X_n - W_n^- X_{n-1} = 0
\]  

(37)

and one can define the ratios \( R_n = X_n/X_{n-1} \). Note that these ratios satisfies a simple first-order recursive relation. However we bypass this stage because we can extract the solution from the steady state distribution.

We write the solution of the non-homogeneous equation as

\[
c_n := X_n \tilde{c}_n
\]  

(38)

and we get the equation

\[
-W_n^+ X_{n+1} \tilde{c}_{n+1} + \Lambda_n X_n \tilde{c}_n - W_n^- X_{n-1} \tilde{c}_{n-1} = \Delta_n
\]  

(39)

We define the discrete derivative

\[
\tilde{a}_n := \tilde{c}_n - \tilde{c}_{n-1}
\]  

(30)

And obtain a reduction to a first-order equation:

\[
-W_n^+ X_{n+1} \tilde{a}_{n+1} + W_n^- X_{n-1} \tilde{a}_n = \Delta_n
\]  

(40)

This can be re-written in a simpler way by appropriate definition of scaled variables. Namely, we define the notations

\[
\tilde{R}_n = \frac{W_n^+}{W_n^-} R_n \quad \tilde{\Delta}_n = \frac{\Delta_n}{W_n^-}
\]  

(41)

and the rescaled variable

\[
a_n := X_n \tilde{a}_n
\]  

(42)

and then solve the \( a_n \) recursion in the backwards direction:

\[
a_\infty = 0; \quad a_n = \tilde{R}_n \left[ \tilde{\Delta}_n + a_{n+1} \right]
\]  

(43)

If all the \( R_n \) were unity it would imply that \( a_1 - a_\infty \) equals \( \sum \tilde{\Delta}_n \). So it is important to verify that the \( \tilde{\Delta}_n \) converges. Next we can solve in the forward direction the \( c_n \) recursion for the non-homogeneous equation, namely,

\[
c_0 = 0; \quad c_n = \tilde{R}_n c_{n-1} + a_n
\]  

(44)

In fact we are only interested in

\[
c_1 = a_1 = \tilde{R}_1 \tilde{\Delta}_1 + \tilde{R}_1 \tilde{R}_2 \tilde{\Delta}_2 + \ldots
\]  

(45)

Note that in our calculation the \( \tilde{R}_n = -R_n \), and therefore \( \tilde{R}_1 \cdots \tilde{R}_n = (-1)^n X_n \).

### VIII. THE MOMENTS OF THE RADIAL SPREADING

The moments of a lognormal distribution are given by the following expression

\[
\ln \langle r^n \rangle = \mu n + \frac{1}{2} \sigma^2 n^2
\]  

(46)

On the basis of the discussion after Eq.(14), if one assumed that the radial spreading at time \( t \) could be globally approximated by the lognormal distribution (tails included), it would follow that

\[
\frac{d}{dt} \ln \langle r^n \rangle = nw_r + n^2 D_r
\]  

(47)

In Fig.5 we plot the lognormal-based expected growth-rate of the 2nd and the 4th moments as a function of \( w/D \). For small \( w/D \) there is a good agreement with the expected results, which are \( w^2/D \) and \( 3w^2/D \) respectively. For large \( w/D \) the dynamics is dominated by the stretching, meaning that \( w_r \approx w \), while \( D_r \to 0 \), so again we have a trivial agreement. But for intermediate values of \( w/D \) the lognormal moments constitute an overestimate when compared with the exact analytical results.
that we derive in the next section. In fact also the exact analytical result looks like an overestimate when compared with the results of numerical simulations. But the latter is clearly a sampling issue that is explained in Appendix D.

The deviation of the lognormal moments from the exact results indicates that the statistics of large deviations is not captured by the central limit theorem. This point is illuminated in Fig.6. The Gaussian approximation constitutes a good approximation for the body of the distribution but not for the tails that dominate the moment-calculation. Clearly, the actual distribution can be described as a bounded lognormal distribution, meaning that it has a natural cutoff which is implied by the strict inequality $w_r < w$. The stretching rate cannot be faster than $w$. But in fact, as observed in Fig.6b, the deviation from the lognormal distribution happens even before the cutoff is reached.

Below we carry out an exact calculation for the 2nd and 4th moments. In the former case we show that

$$ \frac{d}{dt} \ln \langle r^2 \rangle \sim 2 \left( (w^2 + D^2)^{1/2} - D \right) $$

(48)

This agrees with the lognormal-based prediction $w^2/D$ for $(w/D) \ll 1$, and goes to $2w$ for $(w/D) \gg 1$, as could be anticipated.

Before we go the derivation of this result we would like to illuminates its main features by considering a simple-minded reasoning. Let us ask ourselves what would be the result if the spreading was isotropic ($w_r = 0$). In such case the moments of spreading can be calculated as if we are dealing with the multiplication of random numbers. Namely, assuming that the duration of each step is $\tau = 1/(2D)$, and treating $t$ as a discrete index, Eq.(13) implies that the spreading is obtained by multiplication of uncorrelated stretching factors $\exp[w rt \cos(\varphi)]$.

Each stretching exponent has zero mean and dispersion $\sigma^2 = (1/2) [w t]^2$, which implies $D_r = \sigma^2 / (2 \tau)$. Consequently we get for the moments

$$ \langle r^n \rangle = \left[ \left( e^{n w t \cos(2\varphi)} \right)^{1/\tau} \right] r_0^n $$

leading to

$$ \frac{d}{dt} \ln \langle r^n \rangle = \frac{1}{\tau} \ln \left[ I_0 \left( \sqrt{2n \sigma_1} \right) \right] $$

(49)

(50)

This gives a crossover from $n^2 D_r$ for $\sigma_1 \ll 1$ to $nw$ for $\sigma_1 \gg 1$, reflecting isotropic lognormal spreading in the former case, and pure stretching in the latter case. So again we see that the asymptotic limits are easily understood, but for the derivation of the correct interpolation, say Eq.(48), further effort is required.

**IX. THE EXACT CALCULATION OF THE MOMENTS**

We turn to perform an exact calculation of the moments. One can associate with the Langevin equation Eq.(1) an FPE for the distribution, and from that to derive the equation of motion for the moments. The procedure is explained and summarized in Appendix E.

![Image](image-url)
For the second moments
\[ \frac{d}{dt} \langle x \rangle = w \langle x \rangle - D \langle x \rangle \quad (51) \]
\[ \frac{d}{dt} \langle y \rangle = -w \langle y \rangle - D \langle y \rangle \quad (52) \]
with the solution
\[ \langle x \rangle = x_0 \exp(-(D-w)t) \quad (53) \]
\[ \langle y \rangle = y_0 \exp(-(D+w)t) \quad (54) \]
For the second moments
\[ \frac{d}{dt} \left( \begin{array}{c} \langle x^2 \rangle \\ \langle y^2 \rangle \\ \langle xy \rangle \end{array} \right) = \left[ \begin{array}{ccc} -2D + 2D\sigma_1 + 2w\sigma_3 \\ 0 \\ e^{-4Dt} \end{array} \right] \left( \begin{array}{c} \langle x^2 \rangle \\ \langle y^2 \rangle \\ \langle xy \rangle \end{array} \right) \quad (55) \]
\[ \frac{d}{dt} \langle xy \rangle = -4D \langle xy \rangle \quad (56) \]
where \( \sigma \) are Pauli matrices. The solution is:
\[ \left( \begin{array}{c} \langle x^2 \rangle \\ \langle y^2 \rangle \\ \langle xy \rangle \end{array} \right) = \left[ \begin{array}{ccc} e^{-2Dt}M & 0 & 0 \\ 0 & e^{-4Dt} & 0 \\ x_0 y_0 & 0 & x_0 y_0 \end{array} \right] \left( \begin{array}{c} x_0^2 \\ y_0^2 \\ xy_0 \end{array} \right) \quad (57) \]
where \( M \) is the following matrix:
\[
cosh[2(w^2 + D^2)^{1/2}t] + \sinh[2(w^2 + D^2)^{1/2}t] \frac{D\sigma_1 + w\sigma_3}{\sqrt{w^2 + D^2}}
\]
For an initial isotropic distribution we get \( \langle r^2 \rangle_t = Mr_0^2 \), where
\[
M = e^{-2Dt} \cosh[2(w^2 + D^2)^{1/2}t] + \frac{D}{\sqrt{w^2 + D^2}} e^{-2Dt} \sinh[2(w^2 + D^2)^{1/2}t]
\quad (58)
\]
The short time \( t \) dependence is quadratic, reflecting “ballistic” spreading, while for long times
\[
\langle r^2 \rangle_t \approx \frac{r_0^2}{2} \left( 1 + \frac{D}{\sqrt{w^2 + D^2}} \right) \times \exp \left[ 2 \left( (w^2 + D^2)^{1/2} - D \right) t \right] \quad (59)
\]
From here we get Eq.(48). For the 4th moments the equations are separated into two blocks of even-even powers and odd-odd powers in \( x \) and \( y \). For the even block:
\[ \frac{d}{dt} \left( \begin{array}{c} \langle x^4 \rangle \\ \langle x^2 y^2 \rangle \\ \langle y^4 \rangle \end{array} \right) = 2M \left( \begin{array}{c} \langle x^4 \rangle \\ \langle x^2 y^2 \rangle \\ \langle y^4 \rangle \end{array} \right) \quad (60) \]
where
\[
M = \begin{pmatrix} 2(w-D) & 6D & 0 \\ D & -6D & D \\ 0 & 6D & -2(w+D) \end{pmatrix}
\quad (61)
\]
The eigenvalues of this matrix are the solution of \( \lambda^3 + 10D\lambda^2 + (16D^2 - 4w^2)\lambda - 24Dw^2 = 0 \). There are two negative roots, and one positive root. For small \( w/D \) the latter is \( \lambda \approx (3/2)(w^3/D) \), and we get that the growth-rate is \( 3w^2/D \) as expected from the log-normal statistics.

X. DISCUSSION

In this work we have studied the statistics of a stochastic squeeze process, defined by Eq. (1). Consequently we are able to provide a quantitatively valid theory for the description of the noise-affected decoherence process in bimodal Bose-Einstein condensates, aka QZE. As the ratio \( w/D \) is increased, the radial diffusion coefficient of \( \ln(r) \) changes in a non-monotonic way from \( D_r = w^2/(8D) \) to \( D_r = D^2/(2w) \), and the non-isotropy is enhanced, namely the average stretching rate increases from \( w_r = w^2/(4D) \) to the bare value \( w_r = w \). The analytical results Eq.(23) and Eq.(33) are illustrated in Fig.1 and Fig.2.

Additionally we have solved for the moments of \( r \). One observes that the central limit theorem is not enough for this calculation, because the moments are predominated by the non-Gaussian tails of the \( \ln(r) \) distribution. In particular we have derived for the second moment the expression \( \langle r^2 \rangle_t = Mr_0^2 \) with \( M \) that is given by Eq.(58), or optionally one can use the practical approximation Eq.(48).

The main motivation for our work comes form the interest in the BJJ. Form mathematical point of view the BJJ can be regarded as a quantum pendulum. It has both stable and unstable fixed points. Its dynamics has been explored by numerous experiments. We mention for example Ref.[17] who observed both Josephson oscillations (“liberations”) and self trapping (“rotations”), and Ref.[18] who observed the a.c. and the d.c. Josephson effects. The phase-space of the device is spherical, known as the Bloch sphere. A quantum state corresponds to a quasi-distribution (Wigner function) on that sphere, and can be characterized by the Bloch vector \( \vec{S} \). The length \( F = |\vec{S}| \) of the Bloch vector reflects the one-body coherence, and has to do with the “fringe visibility” in a “time-of-flight” measurement. If all the particles are initially condensed in the upper orbital of the BJJ, it corresponds to a coherent \( F = 1 \) wavepacket that is positioned on top of the hyperbolic point, which corresponds to the upper position of the pendulum. The dynamics has been thoroughly analyzed in [2] and experimentally demonstrated in [3].

To the best of our knowledge neither the Kapitza effect [12] nor the Zeno effect have been demonstrated experimentally in the BJJ context. We expect the decay of \( F \) to be suppressed due to the periodic or the noisy driving, respectively. Let us clarify the experimental significance of our results for the full statistics of the radial spreading in the latter case. In order to simplify the discussion, let us assume that the definition of \( F \) is associated with the measurement of a single coordinate \( \hat{x} \). Measurement of \( \hat{x} \) is essentially the same as probing an occupation difference. In a semiclassical perspective (Wigner function picture) the phase-space coordinate \( x \) satisfies Eq.(1), where \( \omega(t) \) arises from frequent interventions, or measurements, or noise that comes from the surround-
We define the canonical form of the squeeze operation as
\[
A = \begin{pmatrix} \exp(\alpha) & 0 \\ 0 & \exp(-\alpha) \end{pmatrix}
\]
(A4)

Then we can obtain any general squeeze operation via similarity transformation that involves re-scaling of the axes and rotation, and on top an optional reflection.

We can operate with \(U\) on an initial isotropic cloud that has radius \(r_0\) = 1. Then we get a stretched cloud that has spread \(\langle r^2 \rangle = A r_0^2\), where
\[
A \equiv \langle r^2 \rangle |_{r_0=1} = \cosh(2\alpha)
\]
(A5)

We also define the “spreading” as
\[
S = A - 1 = 2 \sinh^2(\alpha)
\]
(A6)

The notation \(\alpha\) has no meaning for a stochastic squeeze process, while the notation \(A \equiv \langle r^2 \rangle\) still can be used. In the latter case the average is over the initial conditions and also over realizations of \(\omega(t)\), implying that in Eq.(A5) the \(\cosh(2\alpha)\) should be averaged over \(\alpha\).

Appendix B: Numerical simulations

There are numerous numerical schemes that allow the simulation of a Langevin Equation. For example, the Milstein, the Runge-Kutta, and higher-order approximations such as the truncated Taylor expansion \[19\]. These schemes are based on iterative integration of the Langevin equations, then Taylor expand the solution in small \(dt\). The dynamics generated by Eq.(1) is symmetric, however the numerical methods listed above do not respect this constraint. Instead one can exploit the linear nature of the problem. Namely, Eq.(1) is re-written as
\[
\dot{r}_t = H(t) r_t
\]
(B1)
\[
H = H_s + H_a(t)
\]
(B2)

Where \(H_s\) and \(H_a\) are the generators of the stretching and the angular diffusion, respectively, while \(r_t = (x_t, y_t)\). If \(H_a\) were constant, the solution of Eq.(B1) would be obtained by simple exponentiation of \(H\), namely \(r_{tf} = U r_0\), with \(U = \exp[(H_s + H_a) t_f]\). Choosing a small enough time interval \(dt\) and using the Suzuki-Trotter formula, the latter equation is approximated by
\[
U = U_{t_f} \cdots U_{3dt} U_{2dt} U_{dt}
\]
(B3)
\[
U_t = \exp(H_s dt) \exp(H_a dt)
\]
(B4)

Where \(U_t\) gives the evolution of the vector \(r_t\) for small time \(dt\), namely, \(r_t = U_t r_{t-dt}\). Eq. (B3) is valid also for time dependent \(H\), where the small step evolution Eq.(B4) takes the form
\[
U_t = \begin{pmatrix} e^{w dt} & 0 \\ 0 & e^{-w dt} \end{pmatrix} \begin{pmatrix} \cos \alpha_t - \sin \alpha_t \\ \sin \alpha_t \cos \alpha_t \end{pmatrix}
\]
(B5)

The uncorrelated random variables \(\alpha_t\) have zero mean, and are taken from a box distribution of width \(\sqrt{24D dt}\),
such that their variance is $2D \, dt$. As a side note we remark that by Taylor expanding Eq. (B5) to second order in $dt$, the Milstein scheme is recovered. The radial coordinate $r$ is calculated under the assumption that the the preparation is $(x_0, y_0) = (0, 0)$. Accordingly, what we calculate for each realization is

$$r = \sqrt{U_{xx}^2 + U_{yx}^2}. \tag{B6}$$

In Fig. 7a we display the distribution of the trace $a$ for many realizations of such stochastic squeeze process. Rarely the result is a rotation, and therefore in the main text we refer to it as “squeeze”. From the trace we get the squeeze exponent $\alpha$, and from Eq. (B6) we get the radial coordinate $r$. The correlation between these two squeeze measures is illustrated in Fig. 7b. For the long time simulations that we perform in order to extract various moments, we observe full correlation (not shown). In order to extract the various moments, we perform the simulation for a maximum time of $wt = 7500$, with the initial condition $r_0 = (1, 0)$.

We note that the results of Section IX for the evolution of the moments can be recovered by averaging over product of the evolution matrices. For the first moments we get the linear relation $(r_t) = \langle U \rangle \, r_0$, where

$$\langle U \rangle = \langle \ldots, U_{t_3}, U_{t_2}, U_{t_1}, \rangle = [\langle U_t \rangle]^{t/dt} = \begin{pmatrix} 0 & e^{-(D+\omega)t} \\ e^{-(D-\omega)t} & 0 \end{pmatrix}. \tag{B7}$$

Similar procedure can be applied for the calculation of the higher moments.

### Appendix C: Relation to QZE

It is common to represent the quantum state of the bosonic Josephson junction by a Wigner function on the Bloch sphere, see [2] for details. A coherent state is represented by a Gaussian-like distribution, namely

$$\rho^{(0)}(x, y) \approx 2 \exp \left[ -\frac{1}{\hbar} (x^2 + y^2) \right] \tag{C1}$$

where $x$ and $y$ are local conjugate coordinates. The Wigner function is properly normalized with integration measure $dxdy/(2\pi \hbar)$. The dimensionless Plank constant is related to the number $N$ of Bosons, namely $\hbar = (N/2)^{-1}$. After a squeeze operation one obtains a new state $\rho^{(t)}(x, y)$. The survival probability is

$$P(t) = \text{Tr} \left[ \rho^{(0)} \rho^{(t)} \right] = \frac{1}{\cosh(\alpha)} = \frac{1}{1 + \frac{1}{2}S(t)} \tag{C2}$$

However it is more common, both theoretically and experimentally to quantify the decay of the initial state via the length of the Bloch vector, namely $F(t) = |\vec{S}(t)|$. It has been explained in [11] that

$$F(t) \approx \exp \left\{-\hbar \sinh^2(\alpha)\right\} = \exp \left\{-\frac{\hbar}{2}S(t)\right\} \tag{C3}$$

Comparing with the short time approximation of Eq. (C2), namely $P \approx \exp\left\{-\frac{1}{2}S(t)\right\}$, note the additional $\hbar = 2/N$ factor in Eq. (C3). This should be expected: the survival probability drops to zero even if a single particle leaves the condensate. Contrary to that, the fringe visibility reflects the expectation value of the condensate occupation, and hence its decay is much slower. Still both depend on the spreading $S(t)$.

The dynamics that is generated by Eq. (1) does not change the direction of the Bloch vector, but rather shortens its length, meaning that the one-body coherence is diminished, reflecting the decay of the initial preparation. Using the same coordinates as in [11] the Bloch vector is $\vec{S}(t) = (S, 0, 0)$, hence all the information is contained in the measurement of a single observable, aka fringe visibility measurement.

For a noiseless canonical squeeze operation we have $D = 0$ and $\alpha = \omega t$, hence one obtains $S(t) = 2 \sinh^2(\omega t)$ which is quadratic for short times. In contrast to that, for a stochastic squeeze process Eq. (C3) should be averaged over realizations of $\omega(t)$. Thus $F(t)$ is determined by the full statistics that we have studied in this paper.

At this point we would like to remind the reader what is the common QZE argument that leads to the estimate of Eq. (4). One assumes that for strong $D$ the time for phase randomization is $\tau = 1/(2D)$. Dividing the evolution into $\tau$-steps, and assuming that at the end of each step the phase is totally randomized (as in projective measurement) one obtains

$$\overline{A(t)} \approx \left[ \overline{A(\tau)} \right]^{t/\tau} \approx [1 - 2(w\tau)^2]^{t/\tau} \tag{C4}$$

$$\approx \exp \left[ -\left(w^2/D\right)t \right] \tag{C5}$$
of generality we write the Langevin equation as follows:

\[ x_j'(t) = v_j(t) + g_j(t) + \omega(t) \equiv f_j(\omega(t)) \quad (E1) \]

\[ \langle \omega(t) \rangle(t') = 2D\delta_r(t-t) \quad (E2) \]

The \( v_j \) and the \( g_j \) are some functions of the \( x_i \).
Eq.(1) is obtained upon the identification \( x_i = (x,y) \) and \( v_j = (wx,-wy) \), and \( g_j = (-gy,x) \). The “noise” has zero average, namely \( \langle \omega(t) \rangle = 0 \), and is characterized by a correlation time \( \tau \). Accordingly the \( \delta_r(t-t') \) has a short but finite width, which is later taken to be zero.

For a particular realization of the noise, the continuity equation for the Liouville distribution \( \rho(x) \) reads:

\[ \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_j} (f_j \rho) \quad (E3) \]

We are interested in \( \rho(x) \) averaged over many realizations of the noise \( \omega \). In its current form Eq.(E3) cannot be averaged, because \( \rho \) and \( f \) are not independent variables. To overcome this issue Eq.(E3) is integrated iteratively. To second order one obtains

\[ \rho(t+dt) = \rho(t) - \int_t^{t+dt} dt' \frac{\partial}{\partial x_j} f_j(t') \left[ \rho(t) - \int_t^{t'} dt'' \frac{\partial}{\partial x_k} f_k(t'') \rho(t) \right] \quad (E4) \]

Performing the average over realizations of the noise, non-vanishing noise-related term arise from the correlator of Eq.(E2). Then performing the \( dt'' \) integral over the broadened delta one obtains a 1/2 factor. Dividing both sides by \( dt \), and taking the limit \( dt \to \tau \to 0 \), one obtains:

\[ \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_j} \left[ v_j \rho - g_j D \frac{\partial}{\partial x_i} (g_i \rho) \right] \quad (E5) \]

Terms that originate from higher order iterations or moment are \( O(dt) \) or vanish in the \( \tau \to 0 \) limit. Eq.(E5) is the FPE that is associated with the Stratonovich interpretation of Eq.(E1), see Eq.4.3.45 in p.100 of [20].

An observable \( X \) is a function of the \( x \) variables. In order to obtain an equation of motion for \( \langle X \rangle \), we multiply both sides of Eq.(E5) by \( X \), and integrate over \( x \). Using integration by parts, and dropping the boundary terms, we get the desired equation:

\[ \frac{d}{dt} \langle X \rangle = \left\langle \frac{\partial X}{\partial x_j} \left( v_j + \frac{\partial g_j}{\partial x_i} D g_i \right) \right\rangle + \left\langle \frac{\partial^2 X}{\partial x_j \partial x_i} g_j D g_i \right\rangle \quad (E6) \]

In the main text we use this equation for the moments of the distribution \( (x,y,x^2,xy,y^2,x^4,x^2y^2,y^4) \).

Remark concerning various interpretation of the Langevin equation.— The Langevin equation defined by Eq.(E1) and Eq.(E2), with \( \tau \to 0 \), can be written as an integral equation:

\[ x_j(t) - x_j(0) = \int_0^t v_j dt' + \int_0^t g_j dW(t) \quad (E7) \]
where
\[ W(t) = \int_0^t \omega(t')dt' \quad \text{(E8)} \]
\[ dW(t) = W(t + dt) - W(t) \quad \text{(E9)} \]
The second integral in Eq. (E7), is interpreted as a Riemann–Stieltjes like integral [21]:
\[ \int_0^t g_j dW(t) = \lim_{N \to \infty} \sum_{n} g_j(\bar{x})[W(t_n) - W(t_{n-1})] \]
where
\[ \bar{x} = \lambda x_i(t_{n-1}) + (1 - \lambda)x_i(t_n) \quad \text{(E10)} \]
with \(0 < \lambda < 1\), and \(0 = t_0 < ... < t_N = t\). Because of the singular nature of the stochastic process \(W(t)\), the final result of this integral depends on the chosen value of \(\lambda\). Each choice provides a different “interpretation” of the Langevin equation [22]: for \(\lambda = 1\), the equation is interpreted as “Itô”; for \(\lambda = 1/2\) it is interpreted as “Stratonovich”; and for \(\lambda = 0\) it is interpreted as the “Hänggi–Klimontovich”. Each interpretation produces a different FPE. The Stratonovich interpretation leads to Eq.(E5), while for the other interpretations the RHS of Eq.(E5) is replaced with:
\[ -\frac{\partial}{\partial x_j} \left[ v_{ji} \rho - D \frac{\partial}{\partial x_i} (g_{ji} \rho) \right] \quad \text{(Itô) (E11)} \]
\[ -\frac{\partial}{\partial x_j} \left[ v_{ji} - g_{ji} D \frac{\partial}{\partial x_i} (\rho) \right] \quad \text{(Hänggi) (E12)} \]
In the specific case of Eq.(1) with \(g = (-y,x)\), we have \(\partial_t \rho = g_i \partial_i \rho\). Consequently the same FPE is obtained for both the Stratonovich and the Hänggi interpretations. We note that turning off the squeeze in Eq. (1) \((w = 0)\), and using either of these interpretations, the FPE becomes:
\[ \frac{\partial}{\partial t^{\prime}} \rho(x,y,t) = D \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)^2 = D \frac{\partial^2}{\partial x^2} \rho \quad \text{(E13)} \]
Which is clearly the required equation. However if one uses the Itô prescription, an additional term appears in the FPE, namely, \(-D\partial_x(x\rho) - D\partial_y(y\rho)\).

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