On free group generated by two Heisenberg translations

SAGAR B KALANE1,* and DEVENDRA TIWARI2

1Indian Institute of Science Education and Research (IISER) Pune, Dr Homi Bhabha Road, Ward No. 8, NCL Colony, Pashan, Pune 411 008, India
2Bhaskaracharya Pratishthana, Erandwane, Damle Path, Off Law College Road, Pune 411 004, India
*Corresponding Author.
E-mail: sagark327@gmail.com; sagarkalane@iiserpune.ac.in; devendra9.dev@gmail.com

MS received 20 November 2021; revised 22 February 2022; accepted 27 March 2022

Abstract. Let A and B be two Heisenberg translations of PU(2, 1). In this paper, we will discuss the groups ⟨A, B⟩ generated by two non-commuting Heisenberg translations and determine when they are free. The main result of the paper improves an assertion made by Xie et al. (Canad. Math. Bull. 56(4) (2013) 881–889). We also extend the improved result for two Heisenberg translations in PSp(2, 1).

Keywords. Free group; Heisenberg translation; discreteness.

2000 Mathematics Subject Classification. 30F40, 22E40, 20H10.

1. Introduction

To study the discreteness of a group in PSL(2, ℝ) generated by prescribed elements is the central focus in the study of Fuchsian groups. Quasi-Fuchsian theory expands this question to representations of surface groups into PSL(2, ℂ) and similar low-dimensional Lie groups, and adds the question of faithfulness. In particular, Goldman and Parker [5] developed a complex hyperbolic quasi-Fuchsian theory, which started by examining representations of ideal triangle groups. This culminated in the proof of the Goldman–Parker conjecture by Schwartz [12] which states that a natural embedding of the ideal triangle group is discrete and faithful if and only if a certain “angular invariant” is below a particular threshold. Work continues on the discreteness question for more intricate complex hyperbolic quasi-Fuchsian groups and the higher Teichmüller space of representations into other Lie groups.

It is an interesting problem to study when a subgroup generated by two non-commuting parabolic elements is free. A similar problem in the case of Fuchsian and Kleinian groups has been studied by many authors [4,6,7,9,10]. They have explored the conditions for two elements in Fuchsian or Kleinian groups to generate a discrete free group. We suggest interested readers to consult with many works of Fine and Rosenberger on this topic, especially surveys [4,10] for more references on this topic.
A particular case of this classical problem is to find conditions when a subgroup \( G \) of \( \text{PSL}(2, \mathbb{C}) \) generated by two non-commuting, parabolic, linear fractional transformations

\[
A = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix}
\]

is free.

This problem can be reformulated to find the set of complex numbers \( \mu \) such that the corresponding group \( G \) is free. Sanov [11] and Brenner [1] in their fundamental works provided the condition for \( G \) to be free; if \( |\mu| \geq 1, |\mu - 1| \geq 1 \) and \( |\mu + 1| \geq 1 \), then \( G \) is free. This was further improved by Lyndon and Ullman [8], who gave very simple proof of results of Brenner [1] by using a theorem of Macbeath.

In the present work, we focus on the paper of Xie et al. [13] in which they studied the groups generated by two non-commuting parabolic transformations of \( \text{PU}(2, 1) \) and derived conditions to determine when they are free. They consider two variants on the question and resolve them, for their Theorem 1.1, by direct calculations and for their Theorem 1.2, by appealing to the “faithful” part of the Goldman–Parker–Schwartz theorem. We find that the main result in Theorem 1.2 of [13] does not work for all \( \mu \) as suggested by the theorem.

In this paper, we improve the work in [13] to obtain a condition for the resulting group to be free. As a by-product, we obtained similar results for groups generated by two Heisenberg translations of \( \text{PSp}(2, 1) \). We further provide, using a classical result of Lyndon and Ullman [7], a very simple proof of Theorem 1.1 of [13]. Now we will state the main result of our paper which is an improved version of the result proved in Theorem 1.2 of [13].

**Theorem 1.1.** Let \( \mu \in \mathbb{C} \) and assume that the subgroup \( G \subset \text{PU}(2, 1) \) is generated by

\[
A = \begin{pmatrix} 1 & 2\sqrt{2} & -4 \\ 0 & 1 & -2\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2\sqrt{2}\mu & 1 & 0 \\ -4|\mu|^2 & -2\sqrt{2}\bar{\mu} & 1 \end{pmatrix}.
\]

If

\[
|\mu|^2 = -\Re(\mu) \geq \frac{3}{128},
\]

then \( G \) is freely generated by \( A \) and \( B \).

**Remark 1.1.** We note here that Theorem 1.2 of [13] does not work for all \( \mu \) as claimed there. The reason for this is a delicate error that happened due to their computation of \( \Im(z) \) instead of \( \arg(z) \). As a result, \( AB \) was not unipotent and hence it is necessary to apply the Goldman–Parker–Schwartz theorem. We have given one such example where it does not hold in Section refs3.

As an immediate corollary, we have the following result in the case of \( \text{PSp}(2, 1) \).
COROLLARY 1.1

Let $\mu \in \mathbb{H}$ and assume that the subgroup $G \subset \text{PSp}(2, 1)$ is generated by

$$A = \begin{pmatrix} 1 & 2\sqrt{2} & -4 \\ 0 & 1 & 2\sqrt{2} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 2\sqrt{2}\mu & 1 & 0 \\ -4|\mu|^2 & -2\sqrt{2}\bar{\mu} & 1 \end{pmatrix}.$$

If

$$|\mu|^2 = -\Re(\mu) \geq \frac{3}{128},$$

then $G$ is freely generated by $A$ and $B$.

After discussing some background material in Section 2, we prove Theorem 1.1 and Corollary 1.1 in Section 3.

2. Preliminaries

2.1 The quaternions

Let $\mathbb{H}$ denote the division ring of real quaternions. Every element in $\mathbb{H}$ can be uniquely expressed as $q = r_0 + r_1i + r_2j + r_3k$, where $r_0, r_1, r_2, r_3 \in \mathbb{R}$, and $i, j, k$ satisfy relations: $i^2 = j^2 = k^2 = ijk = -1$. The real part of $q$ is denoted by $\Re(q)$, and defined as $\Re(q) = r_0$. The imaginary part of $q$ is given by $\Im(q) = r_1i + r_2j + r_3k$. The conjugate $\bar{q}$ of $q$ is defined as $\bar{q} = r_0 - r_1i - r_2j - r_3k$. The norm of $q$ is $|q| = \sqrt{r_0^2 + r_1^2 + r_2^2 + r_3^2}$.

We will denote the set of all unit quaternions by $\text{Sp}(1)$.

2.2 Hyperbolic space

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{H}$ and suppose $V = \mathbb{K}^{2,1}$ be the three dimensional right vector space over $\mathbb{K}$ equipped with the Hermitian form of signature $(2, 1)$. We will use the following form,

$$\langle z, w \rangle = w^*Hz = \bar{w}_3z_1 + \bar{w}_2z_2 + \bar{w}_1z_3,$$

where $z, w$ are the column vectors in $\mathbb{K}^3$ and $*$ denotes conjugate transpose. The matrix of the Hermitian form is given by

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Define the following subsets of $\mathbb{K}^{2,1}$:

$$V_- = \{z \in \mathbb{K}^{2,1} : \langle z, z \rangle < 0\}, \quad V_+ = \{z \in \mathbb{K}^{2,1} : \langle z, z \rangle > 0\}, \quad V_0 = \{z - \{0\} \in \mathbb{K}^{2,1} : \langle z, z \rangle = 0\}.$$

There are two distinguished points in $V_0$ which we denote by $o$ and $\infty$, given by

$$o = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \infty = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$
Let $\mathbb{P} : \mathbb{K}^{2,1} - \{0\} \rightarrow \mathbb{K}^{2,2}$ be the right projection onto the respective projective space. We will denote by $z$, the image of a vector $z$, under this map. The respective two-dimensional $\mathbb{K}$-hyperbolic space is $H^2_{\mathbb{K}} = \mathbb{P}(V_-)$. The ideal boundary of $H^2_{\mathbb{K}}$ is $\partial H^2_{\mathbb{K}} = \mathbb{P}(V_0)$. For a point $z = [z_1 \ z_2 \ z_3]^T \in V_- \cup V_0$, projective coordinates are given by $(w_1, w_2)$, where $w_i = z_i z_3^{-1}$ for $i = 1, 2$. In projective coordinates, we have

$$H^2_{\mathbb{K}} = \{(w_1, w_2) \in \mathbb{K}^2 : 2\Re(w_1) + |w_2|^2 < 0\},$$

$$\partial H^2_{\mathbb{K}} = \mathbb{P}(V_0) - \infty = \{(w_1, w_2) \in \mathbb{K}^2 : 2\Re(w_1) + |w_2|^2 = 0\}.$$

This is the well-known Siegel domain model of two-dimensional $\mathbb{K}$-hyperbolic space $H^2_{\mathbb{K}}$. In the same way, one can define the ball model by replacing $H$ with a equivalent Hermitian form given by $H_1 = \text{Diag}(1, 1, -1)$. In this work, we shall mostly use the Siegel domain model. Let $U(2, 1)$ be the isometry group of the Hermitian form. We also mention that $g \in U(2, 1)$ acts on $H^2_{\mathbb{K}} \cup \partial H^2_{\mathbb{K}}$ as $g(z) = \mathbb{P} g \mathbb{P}^{-1}(z)$. For more details, we refer to [3].

A point $z$ in the boundary of the Siegel domain is called finite if $z \neq \mathbb{P}(\infty)$. Given a finite point $z$, we may lift $z = (z_1, z_2)$ to a point $z$ in $V_0$, called the standard lift of $z$, where

$$z = \begin{pmatrix} \frac{z_1}{\sqrt{2}} \\ \frac{z_2}{\sqrt{2}} \\ 1 \end{pmatrix}, \text{ where } 2\Re(z_1) + |z_2|^2 = 0.$$

If we put $\zeta = \frac{z_1}{\sqrt{2}} \in \mathbb{K}$ in this condition then we get $2\Re(z_1) = -2|\zeta|^2$. Hence we may write $z_1 = -|\zeta|^2 + \nu$ for $\nu \in \mathcal{I}(\mathbb{K})$. Therefore, for $\zeta \in \mathbb{K}$ and $\nu \in \mathcal{I}(\mathbb{K})$,

$$z = \begin{pmatrix} -|\zeta|^2 + \nu \\ \sqrt{2} \zeta \\ 1 \end{pmatrix}.$$

The finite points in the boundary of the Siegel domain naturally carry the structure of the generalised Heisenberg group $\mathcal{H}$, which is defined to be $\mathcal{H} = \mathbb{K} \times \mathcal{I}(\mathbb{K})$, with the group law

$$(\zeta_1, \nu_1) \ast (\zeta_2, \nu_2) = (\zeta_1 + \zeta_2, \nu_1 + \nu_2 + 2\mathcal{I}(\bar{\zeta}_2 \zeta_1)). \quad (1)$$

### 2.3 Heisenberg translation

Heisenberg translation is one type of parabolic element fixing $\infty$. The matrix $T_{(\zeta_0, \nu_0)}$ is given by

$$T_{(\zeta_0, \nu_0)} = \begin{pmatrix} 1 - \sqrt{2} \bar{\zeta}_0 - |\zeta_0|^2 + \nu_0 \\ 0 & 1 & \sqrt{2} \zeta_0 \\ 0 & 0 & 1 \end{pmatrix}.$$ \quad (2)

When $\zeta_0 = 0$, $T_{(0, \nu_0)}$ is called the vertical translation; otherwise, $T_{(\zeta_0, \nu_0)}$ is called the non-vertical translation. The Heisenberg group acts on itself by Heisenberg translation. This is given as follows: for $(\zeta_0, \nu_0), (\zeta, \nu) \in \mathcal{H}$, we have

$$(\zeta_0, \nu_0) \ast (\zeta, \nu) = (\zeta_0 + \zeta, \nu + \nu_0 + 2\mathcal{I}(\bar{\zeta} \zeta_0) = T_{(\zeta_0, \nu_0)}(\zeta, \nu).$$
3. Main result

In this section, we improve the results of [13] for general Heisenberg translations in \( \mathrm{PU}(2, 1) \). If we take \( \mu = -\frac{3}{4} \), then it satisfies the following condition of Theorem 1.2 in [13]:

\[
\frac{1}{\sqrt{1 + \left(\tan^{-1}\left(\frac{\sqrt{125}}{3}\right)\right)^2}} \leq |\mu| = \frac{3}{4}.
\]

On the other hand, for \( \mu = -\frac{3}{4} \), we get

\[
AB = \begin{pmatrix}
4 & -4\sqrt{2} & -4 \\
3\sqrt{2} & -5 & -2\sqrt{2} \\
-9/4 & 3/\sqrt{2} & 1
\end{pmatrix}.
\]

Here \( \text{tr}(AB) = 0 \) and thus \( AB \) has order 3, which implies that the subgroup generated by \( A \) and \( B \) is not free.

The reason for this error is the following. In order to apply the result of Schwartz [12], one needs \( AB \) to be unipotent. Observe that \( \text{tr}(AB) = 3 + 16\Re(\mu) + 16|\mu|^2 \) and hence we need

\[
\Re(\mu) + |\mu|^2 = 0.
\]

If we take two vertical Heisenberg translations \( A \) and \( B \) which fixes \( \infty \) and \( o \), respectively, then we will get a simple proof of Theorem 1.1 in [13] by using the following result in [8].

Lemma 3.1 [8]. If \( m, n \in \mathbb{C} \) and \( |mn| \geq 4 \), then

\[
A_o = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_o = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]

freely generate a free group.

As the proof of the following theorem is very simple, we include it here for completeness.

Theorem 3.1. Suppose the subgroup \( G \subset \mathrm{PU}(2, 1) \) is generated by

\[
A_1 = \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n & 0 & 1 \end{pmatrix}.
\]

If \( |mn| \geq 4 \), then \( G \) is free.

Proof. Consider the following embedding \( \phi \) of \( \mathrm{GL}(2, \mathbb{C}) \) in \( \mathrm{GL}(3, \mathbb{C}) \) defined by

\[
\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}.
\]

Thus under this embedding, image of \( \langle A_0, B_0 \rangle \) is \( G = \langle A_1, B_1 \rangle \). Hence by using Lemma 3.1, \( G \) is free if \( |mn| \geq 4 \). \( \square \)
We obtain the following corollary of Theorem 3.1 for the vertical Heisenberg translations in $\text{PSp}(2, 1)$.

**COROLLARY 3.1**

*Suppose the subgroup $G \subset \text{PSp}(2, 1)$ is generated by*

\[
A = \begin{pmatrix} 1 & 0 & \tau \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \tau & 0 & 1 \end{pmatrix},
\]

*where $\tau$ is purely imaginary quaternion, i.e., $\tau = t_1 i + t_2 j + t_3 k$. If $|\tau| \geq 2$, then $G$ is free.*

**Proof.** $A$ and $B$ can be simultaneously conjugated to $A_1$ and $B_1$, respectively, by the conjugation element $C = \text{diag}(\alpha, 1, \alpha)$, where $\alpha \tau \alpha^{-1} = ti$. Here $\alpha$ is a unit quaternion and $t = |\tau| = \sqrt{t_1^2 + t_2^2 + t_3^2}$. Now the result follows from Theorem 3.1. □

Before proving the main theorem, we state some lemmas used in our proof of the main theorem.

**Lemma 3.2.** Let $G = \langle a \rangle * \langle b \rangle * \langle c \rangle$ be the free product of the three cyclic subgroups of order two and $H = \langle ab, bc \rangle$, then $H$ is free.

The above lemma is well-known in the study of discreteness of groups generated by two transformations.

Let $(p_0, p_1, p_2)$ be a triple of distinct points in $\partial H^2_C$. The points $p_0, p_1, p_2$ determine complex geodesics $C_0, C_1, C_2$ that are fixed by the inversions $i_0, i_1, i_2$ respectively. For $p = (p_0, p_1, p_2)$, Cartan angular invariant is defined by

\[
\Lambda(p) = \Lambda((p_0, p_1, p_2)) = \arg \left( -\langle p_0, \mathbf{p}_1 \rangle \langle \mathbf{p}_1, \mathbf{p}_2 \rangle \langle \mathbf{p}_2, p_0 \rangle \right) \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right],
\]

where $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ are lifts of $p_0, p_1, p_2$, respectively. Define

\[
\rho_\Lambda : \mathbb{Z}/2 * \mathbb{Z}/2 * \mathbb{Z}/2 \to \text{PU}(2, 1),
\]

\[(a, b, c) \mapsto (i_0, i_1, i_2).
\]

In this regard, we have the following result by Goldman–Parker–Schwartz, in [12].

**Lemma 3.3** (Goldman–Parker–Schwartz theorem). Let $i_0, i_1, i_2$ be the inversions of complex geodesics defined above. Then subgroup generated by $i_0, i_1, i_2$ is discrete if and only if $|\Lambda| \leq \tan^{-1}(\sqrt{\frac{125}{3}})$. Furthermore, the group $\langle i_0, i_1, i_2 \rangle$ freely generates the free product $\langle i_0, \rangle * \langle i_1 \rangle * \langle i_2 \rangle$ of the three cyclic groups of order 2.

### 3.1 Proof of Theorem 1.1

**Proof.** Let $A$ and $B$ be the given generators having fixed points $p_0 = 0$ and $p_1 = \infty$, respectively. Let $p_2 = (\xi, \nu)$ be an arbitrary point from $\partial H^2_C$, which is distinct from $p_0$.
and \( p_1 \). Consider \( p = (p_0, p_1, p_2) \) to be a triple of points in \( (\partial H^2_{\mathbb{C}})^3 \) and let \( C_0 \) be the complex geodesic \( \overrightarrow{p_1 p_2} \) spanned by \( p_1 \) and \( p_2 \). Similarly, let \( C_1 = \overrightarrow{p_0 p_2} \) and \( C_2 = \overrightarrow{p_0 p_1} \). Here the complex geodesics \( C_j \) are given by

\[
C_j = \mathbb{P}(\{ z \in \mathbb{C}^{2,1} | \langle z, \mathbf{c}_j \rangle = 0 \}),
\]

for \( j = 0, 1, 2 \), where \( \mathbf{c}_j \) is the polar vector to \( C_j \).

Define the coordinates of lifts of \( p_0, p_1, p_2 \) into \( \mathbb{C}^{2,1} \) as

\[
p_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad p_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad p_2 = \begin{pmatrix} -|\zeta|^2 + vi \\ \sqrt{2}\zeta \\ 1 \end{pmatrix}.
\]

Also, we can define the polar vectors corresponding to the above geodesics as follows:

\[
\mathbf{c}_0 = \left( \frac{\sqrt{2}\zeta}{-1} \right), \quad \mathbf{c}_1 = \left( \frac{-|\zeta|^2 - iv}{\sqrt{|\zeta|^4 + |v|^2}}, \frac{-\sqrt{2}\zeta}{-\sqrt{|\zeta|^4 + |v|^2}} \right), \quad \mathbf{c}_2 = \left( 0, 1 \right).
\]

Thus the inversions around \( C_j \) are given by

\[
i_j(z) = -z + \frac{2 \langle z, \mathbf{c}_j \rangle}{\langle \mathbf{c}_j, \mathbf{c}_j \rangle} \mathbf{c}_j, \quad j = 0, 1, 2.
\]

We can write the inversion generators in PU(2, 1) as

\[
i_0 = \begin{pmatrix} -1 & -2\sqrt{2}\zeta & 4|\zeta|^2 \\ 0 & 1 & -2\sqrt{2}\zeta \\ 0 & 0 & -1 \end{pmatrix},
\]

\[
i_1 = \begin{pmatrix} -1 & 0 & 0 \\ \frac{2\sqrt{2}\zeta|\zeta|^2 + iv}{|\zeta|^4 + |v|^2} & 1 & 0 \\ \frac{-2\sqrt{2}\zeta(-|\zeta|^2 + iv)}{|\zeta|^4 + |v|^2} & 0 & -1 \end{pmatrix}
\]

and

\[
i_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Now we will determine the coordinates of \( p_2 \) so that

\[
A = i_0i_2, \quad B = i_2i_1.
\]
By comparing the entries of $i_0i_2$ with $A$, we get

$$2\sqrt{2}\zeta = -2\sqrt{2}.$$  

Therefore, we have $\zeta = -1$.

Similarly, after comparing the entries of $i_2i_1$ with $B$, we obtained as follows:

$$\frac{-4|\zeta|^2}{|\zeta|^4 + v^2} = -4|\mu|^2, \quad 2\sqrt{2}\mu = \frac{2\sqrt{2}\zeta(|\zeta|^2 + iv)}{|\zeta|^4 + v^2}.$$  

Substituting $\zeta = -1$ above, we have

$$\mu = \frac{-1 - i v}{1 + v^2}, \quad |\mu|^2 = \frac{1}{1 + v^2}.$$  

Hence $\mu$ satisfies the following condition:

$$\mathcal{R}(\mu) + |\mu|^2 = 0. \quad (3)$$

Now, if $\mu = a + bi$ then we get $a^2 + b^2 + a = 0$. That is, $a^2 + a + 1/4 + b^2 = 1/4$. This restricts $\mu$ to lie on the circle $|\mu + \frac{1}{2}| = \frac{1}{2}$.

Now, we will calculate the angular invariant $\mathbb{A}(p_0, p_1, p_2)$. Since,

$$\langle p_0, p_1 \rangle = 1, \quad \langle p_1, p_2 \rangle = 1 \quad \text{and} \quad \langle p_2, p_0 \rangle = -1 + iv,$$

we have

$$\mathbb{A}(p_0, p_1, p_2) = \arg(1 - iv). \quad (4)$$

Observe that

$$\tan(\mathbb{A}(p)) = \frac{3(1 - iv)}{\mathcal{R}(1 - iv)} = -v.$$

Thus by Lemma 3.3, we see that the group $\langle i_0, i_1, i_2 \rangle$ freely generates the free product $\langle i_0 \rangle * \langle i_1 \rangle * \langle i_2 \rangle$ when

$$|\tan \mathbb{A}(p)| \leq \sqrt{\frac{125}{3}}.$$  

So we get that

$$| - v | = |v| \leq \sqrt{\frac{125}{3}}.$$

By applying Lemma 3.2, we see that $\langle i_0i_2, i_1i_2 \rangle = \langle A, B \rangle$ is free if

$$|v|^2 \leq \frac{125}{3}.$$
Since, $|μ|^2 = \frac{1}{1+|v|^2}$, we have $|v|^2 = -1 + \frac{1}{|μ|^2}$, i.e., $|μ|^2 \geq \frac{3}{128}$. Now using equation (3), we have

$$|μ|^2 = -R(μ) \geq \frac{3}{128}.$$

3.2 Proof of Corollary 1.1

Proof. Let

$$A_0 = A \quad \text{and} \quad B_0 = \begin{pmatrix} 1 & 0 & 0 \\ 2\sqrt{2}τ & 1 & 0 \\ -4|τ|^2 & -2\sqrt{2}¯τ & 1 \end{pmatrix},$$

where

$$τ = R(μ) + \sqrt{|μ|^2 - R(μ)^2} i.$$ 

By conjugating $A_0$ and $B_0$ by diagonal matrix $D = \text{diag}(μ, μ, μ)$, where $μ \in \text{Sp}(1)$, we get $A = DA_0D^{-1}$ and $B = DB_0D^{-1}$, respectively. Now applying Theorem refgh on $⟨A_0, B_0⟩$, we see that $⟨A_0, B_0⟩$ is free if

$$\frac{3}{128} \leq |τ|^2 = -R(τ).$$

Consequently, if $|μ|^2 = -R(μ) \geq \frac{3}{128}$, then $G$ is freely generated by $A$ and $B$. □

Acknowledgements

The authors thank the anonymous referee for many helpful comments and suggestions. They also thank Krishnendu Gongopadhyay for suggesting this problem and discussions. The second author (DT) acknowledges the support of ARSI-Foundation.

References

[1] Brenner J L, Quelques groupes libres de matrices, C. R. Acad. Sci. Paris 241 (1955) 1689–1691
[2] Chang B, Jenning S A and Ree R, On certain pairs of matrices which generate free groups, Canad. J. Math. 10 (1958) 279–284
[3] Chen S S and Greenberg L, Hyperbolic spaces, in: Contributions to Analysis (A Collection of Papers Dedicated to Lipman Bers) (1974) (New York: Academic Press) pp. 49–87
[4] Fine B and Rosenberger G, Classification of All Generating Pairs of Two Generator Fuchsian Groups, from Groups ’93 Galway/St. Andrews, Vol. 1 (Galway, 1993), London Math. Soc. Lecture Note Ser., 211 (1995) (Cambridge: Cambridge Univ. Press) pp. 205–232
[5] Goldman W M and Parker J R, Complex hyperbolic ideal triangle groups, J. Reine Angew. Math. 425 (1992) 71–86
[6] Knapp A W, Doubly generated Fuchsian groups, Michigan Math. J. 15 (1969) 289–304
[7] Lyndon R C and Ullman J L, Groups generated by two parabolic linear fractional transformations, *Canad. J. Math.* 21 (1969) 1388–1403
[8] Lyndon R C and Ullman J L, Pairs of real 2-by-2 matrices that generate free products, *Michigan Math. J.* 15 (1968) 161–166
[9] Purzitsky N, Two generator discrete free products, *Math. Z.* 126 (1992) 209–223
[10] Rosenberger G, All generating pairs of all two-generator Fuchsian groups, *Arch. Math.* 46 (1986) 198–204
[11] Sanov L N, A property of a representation of a free group, *Doklady Akad. Nauk SSSR* 57 (1947) 657–659 (Russian)
[12] Schwartz R E, Ideal triangle groups, dented tori, and numerical analysis, *Ann. Math.* 153(3) (2001) 533–598
[13] Xie B-H, Wang J-Y and Jiang Y-P, Free Groups Generated by Two Heisenberg Translations, *Canad. Math. Bull.* 56(4) (2013) 881–889

**Communicating Editor: B Sury**