Post-Selection Inference via Algorithmic Stability

Tijana Zrnic  
tijana.zrnic@berkeley.edu  
University of California, Berkeley

Michael I. Jordan  
jordan@stat.berkeley.edu  
University of California, Berkeley

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Abstract

When the target of statistical inference is chosen in a data-driven manner, the guarantees pro-
vided by classical theories vanish. We propose a solution to the problem of inference after selection
by building on the framework of algorithmic stability, in particular its branch with origins in the field
of differential privacy. Stability is achieved via randomization of selection and it serves as a quantita-
tive measure that is sufficient to obtain non-trivial post-selection corrections for classical confidence
intervals. Importantly, the underpinnings of algorithmic stability translate directly into computa-
tional efficiency—our method computes simple corrections for selective inference without recourse to
Markov chain Monte Carlo sampling.

1 Introduction

Classical statistical theory provides tools for valid inference under the assumption that the statistical
question is determined before observing any data. In practice, however, the choice of question is typically
guided by exploring the same data that is used for inference. This coupling between the statistical
question and the data used for inference induces dependencies that invalidate guarantees derived from
classical theories.

While traditional wisdom might deem this coupling unacceptable, recent literature embraces this
coupled approach to statistical investigation and grants novel ways of thinking about validity. Indeed,
data-driven model selection is widely taught and practiced, and even stands as a research area of its own.
Sometimes model selection is even unavoidable; in the canonical setting of linear regression, the statistician
often starts with a pool of candidate variables large enough that it makes the solution unidentifiable
without additional constraints, and when those constraints are data-dependent the solution depends on
the data in two ways.

This coupling of the problem formulation and inference stages of statistical analysis has been thor-
oughly studied in a line of work called selective, or post-selection, inference [9, 50, 7]. To this day, however,
there are few general principles that enable both statistically powerful and computationally tractable in-
fERENCE after selection. Most existing solutions are either tailored to specific selection strategies [e.g.,
33, 66, 34], and as such do not generalize to all popular selection methods, or are valid for arbitrary
selections at the cost of increased conservativeness [e.g., 9, 2].

In the current paper, we build on concepts from the field of differential privacy [18, 17] to derive
selective confidence intervals that are both tractable computationally and powerful statistically. Our the-
oretical framework delivers intervals of tunable width, a useful consequence of the fact that our confidence
intervals derive from a quantitative measure of the algorithmic stability of the selection procedure. More
precisely, we provide a valid correction to classical, non-selective confidence intervals simultaneously for
all procedures that have the same level of algorithmic stability. Informally, a selection being stable means
that it is not too sensitive to the particular realization of the data, and the more stable the selection is,
the smaller the resulting intervals are. In particular, if the selection is “perfectly stable” in the sense that
the inferential target is fixed up front and does not depend on the data at hand, the confidence intervals resulting from our approach smoothly recover classical confidence intervals.

We sketch our main result. Let \( \hat{S} \) denote a data-dependent outcome involving selection. For example, \( \hat{S} \) could be subset of \( \{1, \ldots, d\} \) corresponding to the variables selected for inclusion in a linear regression model. For every possible selection \( S \), let \( \beta_S \) denote the resulting inferential target. In the linear regression context, \( \beta_S \) could be the population-level least-squares solution within the model determined by \( S \).

Imagine that there is an oracle that guesses \( \hat{S} \), only knowing the method used to arrive at the selection together with the distribution of the data, but not its realization. Denote by \( \hat{S}_0 \) the oracle’s guess. We say that a selection procedure is \( \eta \)-stable for some \( \eta > 0 \) if there exists an oracle such that, with high probability over the distribution of the data, the likelihood of any selection under \( \hat{S} \) and the likelihood of the same selection under \( \hat{S}_0 \) can differ by at most a multiplicative factor of \( e^\eta \). Intuitively, \( \eta \) quantifies how much the selection can vary across different realizations of the data; \( \eta = 0 \) essentially means that the selection cannot depend on the data and hence \( \hat{S} \) is fixed, while as \( \eta \) grows the selection is allowed to be increasingly data-adaptive. Note that the magnitude of stability depends not only on the selection method, but also on the distribution of the data.

Our main result provides a post-selection-valid correction to classical, non-selective confidence intervals for stable selection procedures. We state an informal version of our key inference tool.

**Theorem 1 (Informal).** For every fixed selection \( S \), suppose that \( \text{CI}^{(α)}_S \) are confidence intervals with valid coverage,

\[
P\{ \beta_S \not\in \text{CI}^{(α)}_S \} \leq α.
\]

Let \( \hat{S} \) be an \( \eta \)-stable selection. Then,

\[
P\{ \hat{β}_S \not\in \text{CI}^{(αe^{-\eta})}_{\hat{S}} \} \leq α.
\]

Theorem 1 is valid simultaneously across all possible selection methods which are \( \eta \)-stable. In other words, under the computational notion of stability we consider, the stability parameter of a selection method alone is sufficient to correct for selective inferences.

Our stability designs are based on explicit randomization schemes which calibrate the level of randomization to a pre-specified algorithmic stability requirement. Together with Theorem 1 this allows the statistician to choose the confidence interval width, obtaining a perturbation of a selection algorithm (e.g., the LASSO), to obtain a target interval width and a guarantee of valid coverage. Since the derived perturbation is an explicit function of the target interval width, this provides a way to understand the loss in utility due to randomization; for example, expressing how “far” the perturbed LASSO solution is from the standard, non-randomized LASSO solution, in some appropriate sense. With this methodology in hand, one can explicitly analyze the inherent tradeoff between the post-selection correction and loss in utility due to randomization for any stable procedure.

We note that the use of randomization in selective inference is by no means a new idea [see, e.g., [52, 53, 54, 30, 40, 39, 10]. The main difference between our work and previous work is the use of stability as an analysis tool, which, on the one hand, leads to a computationally efficient, sampling-free approach to constructing selective confidence intervals with strict coverage, and on the other hand, explicitly connects the level of randomization to the resulting interval width. We elaborate on the comparisons to related work in Section 8.

**Organization.** In the following section we present two motivating vignettes together with our solutions based on stability. In Section 3 we discuss related work. In Section 4 we introduce the notion of algorithmic stability at the focus of our study and in Section 5 we give theory for statistical inference under this definition. Then, in Section 6 we instantiate our theory in the context of model selection in linear regression. In Section 7 we draw connections to conditional post-selection inference. In Section 8 we discuss the design of stable algorithms and give stable versions of the LASSO and marginal screening. In Section 9 we study the performance of our procedures empirically. We end with a brief discussion in Section 10.
2 Motivating vignettes

To illustrate our framework, we present two motivating examples together with solutions implied by our theory, deferring the proofs of validity of the solutions to later sections. In addition, we compare our correction to some relevant baselines.

Vignette 1: Winner’s curse. The first vignette considers the problem of selecting the largest observed effect. Suppose that we observe an $n$-dimensional vector $y \sim \mathcal{N}(\mu, \sigma^2 I)$; for example, each entry in this vector could be an observed treatment effect for a separate treatment. We are interested in doing inference on the most significant effect. More formally, denoting $i_* = \arg\max_i y_i$, we want to construct a confidence interval for $\mu_{i_*}$. Note that this is a random inferential target because $i_*$ is a function of the data.

One simple way of providing valid inference for $\mu_{i_*}$ is to apply the Bonferroni correction:

$$\Pr\{\mu_{i_*} \in (y_{i_*} \pm z_{1-\alpha/(2n)}\sigma)\} \geq 1 - \alpha,$$

where $z_q$ is the $q$ quantile of the standard normal distribution.

Benjamini et al. [8] show that a tighter correction is valid, namely

$$\Pr\{\mu_{i_*} \in (y_{i_*} \pm z_{1-\alpha/(n+1)}\sigma)\} \geq 1 - \alpha.$$

We show that, if we randomize the selection step, rather than select $i_*$ exactly, the intervals can be made even tighter. Furthermore, the reduction in interval width is directly related to the amount of randomization.

**Claim 1.** Suppose that we select $i_* = \arg\max_i(y_i + \xi_i)$, where $\xi_i \sim \text{Lap}\left(\frac{2\pi_{\eta_2} \xi_i}{2n}\right)$, for user-chosen parameters $\eta > 0, \delta \in (0, 1)$. Then,

$$\Pr\{\mu_{i_*} \in (y_{i_*} \pm z_{1-\alpha(1-\delta)e^{-n/2\sigma}})\} \geq 1 - \alpha.$$

The proof of validity of this construction relies on our notion of stability, introduced in later sections; we defer the analysis of Claim 1 to the Supplement. Note that, as $\eta$ and $\delta$ decrease toward zero, the noise level increases and the intervals approach classical, non-selective intervals. In general, $\eta$ is the key parameter that trades off the information used for selection versus inference: small $\eta$ corresponds to using more information for inference, while large $\eta$ corresponds to prioritizing selection quality. Figure 1 illustrates how the interval width changes with $\eta$, in comparison to baselines, for $\sigma = 1, \delta = 0.5$, and varying $n$.

Note that if there is significant separation between $\mu_{i_*}$ and the other effects, $\hat{i}_*$ will likely be equal to $i_*$ even for small $\eta$. If, on the other hand, there are multiple effects of similar magnitude, the randomization will smooth out the selection and place non-negligible probability on all the competitive effects. In particular, to ensure $\hat{i}_* = i_*$ with high probability, it suffices to have $\eta$ inversely proportional to the gap between the largest and second largest observed effect, $\Delta = y_{i_*} - \max_{j \neq i_*} y_j$. 

![Figure 1](image_url)
Vignette 2: Feature selection. In the second example we look at inference after data-driven feature selection. Suppose we have a fixed design matrix, \( X \in \mathbb{R}^{n \times d} \), with \( n \) observations and \( d \) features and a corresponding outcome vector \( y \sim \mathcal{N}(\mu, \sigma^2 I) \in \mathbb{R}^n \). Denote by \( X_i \) the columns of \( X \), for \( i \in [d] \). We would like to select a model corresponding to a subset of the \( d \) features, and perform valid inference on the least-squares target after regressing \( y \) on the selected features only. This problem is discussed in depth by Berk et al. [9].

We set this problem up more generally in later sections; to keep this illustration light, assume that the features are normalized so that \( \|X_i\|_2 = 1 \) and we are selecting a single feature. Then, this problem amounts to doing inference on \( X_i^\top \mu \), where \( i^\ast \) is the selected feature. Again, we note that this is a random inferential target since \( i^\ast \) is data-dependent.

Suppose that the goal of selection is to simply maximize the absolute correlation of the selected feature with \( y \): \( i^\ast = \arg \max_i |X_i^\top y| \). Then, our results imply the following.

Claim 3. Suppose that we select \( \hat{i^\ast} = \arg \max_i |X_i^\top y + \xi_i| \), where \( \xi_i \overset{\text{i.i.d.}}{\sim} \text{Lap} \left( \frac{2z_{1-\alpha/2}/(2d)}{\gamma} \right) \), for user-chosen parameters \( \eta > 0, \delta \in (0, 1) \). Then,

\[
P \left\{ X_i^\top \mu \in (X_i^\top y \pm z_{1-\alpha(1-\delta)}/\sqrt{2}\sigma) \right\} \geq 1 - \alpha.
\]

Again, as \( \eta \) and \( \delta \) tend toward zero, the intervals approach non-selective intervals, and the relationship between \( \eta \) and the gap between the largest and second largest correlation, \( \Delta = |X_i^\top y| - \max_{j \neq i, \Delta} |X_j^\top y| \), drives the accuracy of selection.

Claim 4. If \( \eta \geq \frac{4 \log(d/\delta')}{2 \max_{\delta' \in (0, 1)} \Delta} \), then \( \hat{i^\ast} = i^\ast \) with probability at least \( 1 - \delta' \) over the randomness in the selection.

An alternative, equally simple solution to inference after feature selection is data splitting: we use a fraction \( f \in (0, 1) \) of the data for selection and the remaining \( 1 - f \) fraction for inference. In Section 5.1, we present a more detailed comparison with data splitting. For now we remark that, given that both data splitting and stability lead to intervals that look like classical intervals with an additional correction factor, every parameter \( \eta \) has a corresponding parameter \( f \equiv f(\eta) \) that yields data-splitting intervals of

Figure 2. Average error, defined as \( \max_i |X_i^\top y| - |X_i^\top y| \), achieved by data splitting and our stability-based approach. We use one of the experimental setups in Section 9. The rows of \( X \) are drawn i.i.d. from an equicorrelated multivariate Gaussian distribution with pairwise correlation \( \rho = 0.5 \). The outcome is generated as \( y = X \beta + \epsilon \), where \( \epsilon \) is isotropic Gaussian noise and half of the entries in \( \beta \) are zero and half are sampled independently from \( \text{Exp}(0.2) \). From left to right, we increase the value of \( d \in \{50, 100, 200\} \) and keep \( n = 50 \) fixed.
roughly the same width as stability-based intervals (given a fixed \( \delta \) such as 0.5). However, we observe that stability can be a significantly more powerful solution. Figure 2 compares the error of data splitting with the error of the stability solution described in Claim 3 in a simple simulation setting.

Finally, we emphasize that stability can be applied to the problem of feature selection even when data splitting is not an option, such as when there are spatial or temporal dependencies in the data.

3 Related work

In this section, we elaborate on the comparisons between our work and existing work in post-selection inference, and additionally discuss relevant work in the algorithmic stability literature.

Simultaneous coverage. In the formulation of post-selection inference by Berk et al. [9], the goal is to construct simultaneous confidence intervals (as per Eq. (3)) that are valid for any model selection method \( \hat{M} : y \rightarrow \mathcal{M} \), for a pre-specified model class \( \mathcal{M} \). The framework of Berk et al. was subsequently generalized by Bachoc et al. [2] to handle distributions beyond the homoscedastic Gaussian, as initially assumed. These proposals are computationally infeasible in high dimensions as they essentially require looking for the “worst possible” model \( M \in \mathcal{M} \), one that implies the largest so-called PoSI (Post-Selection Inference) constant, an analog of which we introduce and characterize in the context of stability. More recent work has proposed computationally efficient confidence regions via UPoSI [32].

Another approach to valid post-selection inferences that applies to general selection rules is data splitting [42]: split the data into two disjoint subsets, then use one subset to select the inferential target and the other subset to perform inference. Data splitting is appealing because, if the two subsets of the data are independent, classical inferences will be valid regardless of the selection procedure. However, data splitting is not universally applicable as one cannot always obtain two independent data sets, and even if applicable, it can suffer a significant loss in power, such as when only a few samples capture some relevant information. Our stability-based approach does not rely on any independence assumption between different observations, and, as illustrated in Figure 2, it can be a more powerful solution than data splitting when the latter is applicable. We give a further discussion of data splitting and its relationship to stability in Section 5.1.

All the aforementioned strategies strive for robustness: they protect against all selection procedures. For specific selection procedures, however, the intervals computed by simultaneous methods and related approaches are unnecessarily wide, as they do not exploit any knowledge of how the analyst arrives at the selected target. Recent work aims to address this issue in the framework of simultaneous coverage over the selected variables (SoS) by constructing SoS-controlling confidence intervals for \( k \) seemingly largest effects [8]. Our work likewise implies SoS intervals, by putting forward a general stability perspective and analyzing the relationship between stability and interval width for arbitrary stable procedures.

Conditional coverage. Conditional methods [24] exploit properties of the selection procedure. However, they control a different error criterion than simultaneous methods. In particular, the goal of conditional post-selection inference is to design \( \text{CI}_S \) such that

\[
P\{ \beta_S \in \text{CI}_S \mid \hat{S} = S \} \geq 1 - \alpha,
\]

for all fixed selections \( S \).

For a fixed selection procedure, conditional post-selection inference aims to characterize the distribution of the data given \( \hat{S} = S \), and then using the knowledge of this conditional distribution it computes \( \text{CI}_S \). This approach is tailored to the selection method at hand, and existing work has derived intervals for model selection via methods such as the LASSO [35], marginal screening, orthogonal matching pursuit [34], forward stepwise, and LARS [56].

It is often remarked that the conditional approach leads to overconditioning, thus leading to wide intervals [29]. Informally, overconditioning refers to the phenomenon of overstating the cost of selection,
thus leaving little information for inference. Surprisingly, it has even been observed that simultaneous approaches can in some cases yield smaller intervals, due to the intervals being unconditional rather than conditional [2]. One attempt at narrowing down the intervals involves choosing a better event on which to condition [37]. Another solution to overconditioning which is relevant to the present context is the idea of randomizing the selection procedure [32, 53, 54, 30, 40, 39, 10]. Notably, the pioneering work in this direction due to Tian and Taylor [52] proves a central limit theorem that asymptotically relates the validity of statistical inferences without selection to their selective counterparts, a result similar in flavor to our Theorem 2. However, existing randomization proposals suffer several drawbacks. One is that they give little insight into the tradeoff between confidence interval width and the loss in utility from the additional noise. Another issue is that inference is based on a selective pivot which, unlike in exact conditional approaches, lacks closed-form expressions. As a result, to approximate the pivot, existing work resorts to computationally expensive sampling [51, 52], which is generally infeasible in high dimensions. There are other, computationally-efficient approaches which aim to approximate the pivot [10, 39], although these are only approximate and the general theory applies to restricted classes of selection problems.

Although our primary goal is to provide unconditional guarantees, in Section 7 we will also discuss implications of stability for conditional inference. We also point out the work of Andrews et al. [1], who propose a hybrid approach that interpolates between unconditional and conditional post-selection inference to obtain smaller confidence intervals relative to a purely conditional approach.

Algorithmic stability. The technical tools of this paper are rooted in the theory of differential privacy [18, 17] and its extensions [20, 4]. Initially, differential privacy was developed as a standard for private data analysis. A more recent line of work, typically referred to as adaptive data analysis [see, e.g., 21, 20, 5], has recognized that a stability concept can be extracted from differential privacy and exploited to obtain perturbation-based generalization guarantees in learning theory. Superficially, adaptive data analysis has the same goal as post-selection inference—developing statistical tools for valid inference when hypotheses about the data are also data-driven—but the typical formalization of this problem is not directly comparable to that of the canonical post-selection inference setup in regression. The conceptual connection between the two areas has, however, been recognized (dating back to at least the seminal work of Tian and Taylor [52]), and several existing works discuss selective hypothesis tests and stability-based corrections to arbitrary selective p-values [43, 45]. Our work does not aim to contribute to adaptive data analysis per se; rather, we build on and adapt existing tools in this literature for the purpose of providing post-selection corrections for common selection problems, such as within the framework put forward by Berk et al. [9]. Finally, we note that connections between stability and generalization are not new [11], and stability ideas have been utilized to construct predictive confidence intervals [47, 3].

4 Algorithmic stability and selection

The formal theory of algorithmic stability characterizes how the output of an algorithm changes when the input is perturbed. Randomized algorithms have as output a random variable; therefore, to study the stability of a randomized algorithm, an appropriate notion of closeness of two random variables is required. The particular notion of closeness considered in differential privacy and related work is known as indistinguishability, or max-divergence.

Definition 1 (Indistinguishability). We say that a random variable Q is \((\eta, \tau)\)-indistinguishable from W, denoted \(Q \approx_{\eta, \tau} W\), if for all measurable sets \(O\),

\[
P\{Q \in O\} \leq e^\eta P\{W \in O\} + \tau.
\]

Roughly speaking, \(\tau\) bounds the probability of the event that Q and W are “very different.” For fixed \(\tau \in [0, 1]\), the parameter \(\eta\) is meant to capture how similar the distributions of Q and W are—the larger
is the larger the divergence between \( Q \) and \( W \) can be. One should think of \( \tau \) as being at most a small factor proportional to the miscoverage probability \( \alpha \).

We now formally introduce the main notion of algorithmic stability considered in this paper. The algorithm whose stability we analyze will usually be a selection algorithm.

**Definition 2 (Stability).** Let \( \mathcal{A} : \mathbb{R}^n \to S \) be a randomized algorithm. We say that \( \mathcal{A} \) is \((\eta, \tau, \nu)\)-stable with respect to a distribution \( \mathcal{P} \) supported on \( \mathbb{R}^n \) if there exists a random variable \( A_0 \), possibly dependent on \( \mathcal{P} \), such that

\[
\mathcal{P} \{ \omega \in \mathbb{R}^n : \mathcal{A}(\omega) \approx_{\eta, \tau} A_0 \} \geq 1 - \nu.
\]

This notion is a special case of *typical stability* introduced by Bassily and Freund [4]. It is closely related to the notions of perfect generalization [15] and max-information [20]. Unless stated otherwise, whenever we use the term stability we will assume stability in the sense of Definition 2. The parameter \( \nu \) can in principle take on any value in \([0, 1]\) but in practice we will set it to be proportional to \( \alpha \).

We will only invoke stability with respect to the data distribution, which we will denote \( \mathcal{P}_y \). Thus, for simplicity, when we say that \( \mathcal{A} \) is \((\eta, \tau, \nu)\)-stable we are implicitly assuming that \( \mathcal{P}_y \) is the reference distribution.

Throughout we will use \( \hat{S}(\cdot) \) to denote a possibly randomized selection algorithm, which takes as input the data \( y \) and outputs a selection that determines the inferential target. For example, \( \hat{S} \) could be a model selection algorithm such as in the second vignette, or it could be an algorithm that selects itself, \( \hat{y} \), an effect that is the focus of subsequent inference, such as in the first vignette. With a slight abuse of notation, we will use \( \hat{S} \) to denote both the mapping from the data to the selection as well as the selection itself, \( \hat{S}(y) \equiv S \).

Intuitively, \( \hat{S} \) will be stable if we can “guess” the distribution of \( \hat{S}(y) \) (conditional on \( y \)) with only knowledge of the data distribution \( \mathcal{P}_y \), not the data \( y \) itself.

**Example 1.** To provide intuition for Definition 2 we present one simple mechanism for achieving stability. Although basic, this mechanism will be a fundamental building block in our stability proofs. Suppose that we wish to compute \( w^\top y \), for some fixed vector \( w \), and suppose that we take \( \mathcal{P}_y \) to be \( \mathcal{N}(\mu, \sigma^2 I) \) with known \( \sigma > 0 \). Let \( \mathcal{A}(y) = w^\top y + \xi \), where \( \xi \sim \text{Lap} \left( \frac{z_{1-\nu/2}\sigma\|w\|_2}{\eta} \right) \), for user-specified parameters \( \eta > 0, \nu \in (0, 1) \). Here, Lap(b) denotes a draw from the zero-mean Laplace distribution with parameter \( b \), independent of \( y \). We argue that this mechanism is \((\eta, 0, \nu)\)-stable. First, we know

\[
\mathbb{P} \{ |w^\top y - w^\top \mu| \geq z_{1-\nu/2}\sigma\|w\|_2 \} = \mathbb{P} \{ |\mathcal{N}(0, \sigma^2\|w\|_2^2)| \geq z_{1-\nu/2}\sigma\|w\|_2 \} = \nu.
\]

Denote \( Y_\nu = \{ \omega \in \mathbb{R}^n : |w^\top \omega - w^\top \mu| \leq z_{1-\nu/2}\sigma\|w\|_2 \} \), and notice that we have shown that \( \mathbb{P} \{ y \in Y_\nu \} = 1 - \nu \).

Now let \( A_0 = \mathcal{A}(\mu) \). Since the ratio of densities of \( \xi \sim \text{Lap}(b) \) and its shifted counterpart \( x + \xi \) is upper bounded by \( e^{z_i/b} \), we can conclude that for all \( \omega \in Y_\nu \) and measurable sets \( \mathcal{O} \),

\[
\frac{\mathbb{P} \{ \mathcal{A}(\omega) \in \mathcal{O} \}}{\mathbb{P} \{ \mathcal{A}(\mu) \in \mathcal{O} \}} \leq e^{\nu};
\]

that is, we have \( \mathcal{A}(\omega) \approx_{\nu, 0} A_0 \) for all \( \omega \in Y_\nu \). Putting everything together, we see that \( \mathcal{A}(\cdot) \) is \((\eta, 0, \nu)\)-stable with respect to \( \mathcal{P}_y \).

### 5 Confidence intervals after stable selection

Given the assumption of \((\eta, \tau, \nu)\)-stability, we now show how a simple modification to classical confidence intervals suffices to correct for selective inferences. This correction is valid *regardless* of any additional property of the selection criterion.
The main intuition behind this assertion is the following. If the selection algorithm is stable, then by Definition 2 there must exist an oracle selection \( \hat{S}_0 \) such that \( \hat{S}(y) \) and \( \hat{S}_0 \) are indistinguishable. Note that \( \hat{S}_0 \) is independent of \( y \). Since \( \hat{S}_0 \) is indistinguishable from \( \hat{S}(y) \), we can pretend that \( \hat{S}_0 \) is the selection of interest. Furthermore, since \( \hat{S}_0 \) and \( y \) are independent, we are free to use \( y \) for inference. Stability ensures that, despite data reuse, inference behaves almost like with data splitting, in which we perform selection on one batch of data and then use independent data for constructing intervals.

We state a technical lemma, similar to Lemma 3.3 by Bassily and Freund [4], that we use to prove our main theorem. We include a proof of Lemma 1 in the Supplement.

**Lemma 1.** Let \( \hat{S} : \mathbb{R}^n \rightarrow S \) be an \((\eta, \tau, \nu)\)-stable selection algorithm and let \( \hat{S}_0 \) be the corresponding oracle selection. Then, it holds that

\[
(y, \hat{S}(y)) \approx_{\eta, \tau + \nu} (y, \hat{S}_0).
\]  

(1)

Equipped with Lemma 1, we can now describe how to construct post-selection-valid confidence intervals after stable selection.

Suppose that, under selection \( S \), our target of inference is \( \beta_S \). Moreover, suppose that \( \text{CI}^{(a)}_S \) are valid confidence intervals at level \( 1 - \alpha \) for any fixed \( S \), meaning that

\[
P\{ \beta_S \not\in \text{CI}^{(a)}_S \} \leq \alpha.
\]

Such intervals are provided by classical theory.

Theorem 2 formally states how to construct confidence intervals for an adaptive target \( \hat{\beta}_S \), when \( \hat{S} \) is selected in a stable way. This is the key result of our paper.

**Theorem 2.** Fix \( \delta \in (0, 1) \), and let \( \hat{S} \) be an \((\eta, \tau, \nu)\)-stable selection algorithm. Then,

\[
P\{ \hat{\beta}_S \not\in \text{CI}^{(\delta e^{-\eta})}_S \} \leq \delta + \tau + \nu.
\]

In words, if \( \hat{S} \) is \((\eta, \tau, \nu)\)-stable, we can pretend that there is no selection bias and simply construct classical intervals, albeit at a more conservative level, to achieve validity. If we set the target error level to be \( \delta e^{-\eta} \), then the realized error level will be at most \( \delta + \tau + \nu \). For example, if we let \( \tau = \nu = \alpha/3 \), then to get coverage at level \( 1 - \alpha \) we can set the target coverage level to be \( \alpha/3 \cdot e^{-\eta} \).

### 5.1 Comparison with data splitting

In many scenarios it is possible to split the data into two independent chunks, one to be used for selection and the other to be reserved for inference. Classical inferences are then valid because the inferential target is determined before seeing any of the data used in the inference step. This simple baseline for valid inference after selection is called data splitting. In this section, we illuminate the relationship between our approach via stability and data splitting.

First we want to emphasize that the stability principle is applicable even with dependent samples: Theorem 2 can be applied even when it is not clear how to create two independent subsets of the data. Moreover, in some selection problems data splitting makes little conceptual sense, such as in our first motivating vignette about inference on the winning effect.

The appeal of data splitting lies in its broad applicability. As long as the data can be split into two independent components, the criteria for choosing the inferential target can be arbitrary. Therefore, data splitting provides a selection-agnostic correction, universally valid across all possible selection strategies.

Conceptually, stability lies somewhere between data splitting and conditional post-selection inference. It computes a correction level as a function of how adaptive the selection is to the data, thereby adapting to some properties of the selection rule like conditional inference methods. However, at the same time it provides a correction that is universally valid across all possible selection strategies with the same level of stability, which can be seen as a refinement of the principle of data splitting.
To illustrate the conceptual difference between the stability principle and the data splitting principle, suppose that in the latter case we allocate \( f \)-fraction of the data to selection, and \((1 - f)\)-fraction to inference. Then, the resulting intervals will roughly look like classical intervals augmented by a factor of \( \sqrt{\frac{1}{n}} \) regardless of how the selection is performed.

In contrast, the stability approach augments classical intervals as a function of the adaptivity of the selection algorithm. Suppose for concreteness that \( y \sim \mathcal{N}(\mu, I) \) and we are considering doing inference on one of two targets, \( v_0^T \mu \) or \( v_1^T \mu \), where the selection \( \tilde{S} = \{0, 1\} \) depends on the data \( y \). Consider three different selection methods:

- \( \tilde{S} = 1 \) no matter what the data vector is.
- \( \tilde{S} = 1 \) if \( \hat{y} := \frac{1}{n} \sum_{i=1}^{n} y_i \geq 0 \), and \( \tilde{S} = 0 \) otherwise.
- \( \tilde{S} = 1 \) if \( X_1^T y \geq 0 \) for some unit vector \( X_1 \), and \( \tilde{S} = 0 \) otherwise.

We can write all three procedures as \( \tilde{S} = 1 \{ w^T y \geq 0 \} \); in the first case \( w = 0 \), in the second case \( w = \frac{1}{n} 1 \), and in the third case \( w = X_1 \).

Let us fix the noise level \( \gamma > 0 \) and select \( \tilde{S} = 1 \{ w^T y + \xi \geq 0 \} \), where \( \xi \sim \text{Lap}(\gamma) \). The first method is trivially \((0, 0, 0)\)-stable for any level \( \gamma \), hence we can simply use \( y \) for inference without any correction. Based on the same analysis as in the example in Section \ref{sec:example} the second selection method is \((\sqrt{2 \log(2/\nu)}/(\gamma \sqrt{m}), 0, 0)\)-stable for all \( \nu > 0 \); i.e., it is \((\sqrt{2 \log(4/\alpha)}/(\gamma \sqrt{m}), 0, \alpha/2)\)-stable. Similarly, the third selection method is \((\sqrt{2 \log(4/\alpha)}/\gamma, 0, \alpha/2)\)-stable.

We can thus observe that, even though in all three examples we perturb the selection by the same constant level of noise, the stability approach exploits the fact that some selection criteria are more stable than others and this is reflected in the resulting stability parameter. By Theorem \ref{thm:stability} this stability parameter, in turn, directly determines the correction factor, i.e., how conservative we need to make classical inferences for them to be valid post selection.

While data splitting and stability come with conceptual differences, they also have technical similarities. In particular, each one has a leading parameter—\( f \in (0, 1) \) in the case of data splitting and \( \eta > 0 \) in the case of stability—and this parameter interpolates between two extremes. One extreme is when all information is reserved for inference (attained when \( f = 0 \) and \( \eta = 0 \) respectively) and the other is when all information is used for selection (attained when \( f = 1 \) and \( \eta \to \infty \) respectively). Therefore, it might make sense to ask how the two interpolations relate.

For every \( \eta \), there is an \( f(\eta) \) such that, if we used \( f(\eta) \)-fraction of the data for selection and \( 1 - f(\eta) \) for inference, we would approximately get the same interval correction. We sketch the derivation of \( f(\eta) \) in the case of normal intervals for simplicity, however this calculation can be generalized to other distributions. We will assume that \( \nu + \tau \leq \delta \alpha \) for some \( \delta \in (0, 1) \); then, the intervals resulting from \((\eta, \tau, \nu)\)-stability are of width proportional to \( z_1 - (1 - \delta) \frac{2}{\nu} \). The intervals resulting from data splitting are of width proportional to \( z_1 - \frac{2}{\nu} (1 - f(\eta))^{-1/2} \). By equating the two expressions to achieve the same width and simplifying, we obtain

\[
f(\eta) = 1 - \left( \frac{z_1 - \frac{2}{\nu}}{z_1 - (1 - \delta) \frac{2}{\nu} \eta} \right)^2 \approx \frac{\log \frac{1}{\nu} + \eta}{\log \frac{2}{(1 - \delta) \alpha \eta} + \eta},
\]

where the approximation on the right-hand side follows by a subgaussian approximation.

Of course, this sketch only gives intuition for when data splitting and stability imply equally powerful inference; it does not say anything about which selection is more accurate—one where we select on \( f(\eta) \)-fraction of the data, or one where we select on the whole data set in an \( \eta \)-stable way. We will tackle this question empirically, as the notion of “more accurate” varies greatly depending on the context. In Figure \ref{fig:empirical} we used the splitting fraction in Eq. (2) and observed that stability outperforms data splitting. We provide further empirical comparisons in Section \ref{sec:empirical}.

Finally, we mention another proposal that is conceptually closely related to data splitting, namely the \((U, V)\) decomposition of Rasines and Young \cite{RasinesYoung}. Like stability, the \((U, V)\) decomposition allows the statistician to see all data points—more precisely, noisy versions thereof—both in the selection step and in the inference step. This is an important advantage over data splitting when there are only a few
samples that capture information about certain directions. In contrast with stability, performing the 
$(U,V)$ decomposition does not rely on any properties of the selection method. However, finite-sample 
guarantees of this approach crucially rely on the data being Gaussian with known covariance, while 
the stability principle is applicable beyond Gaussianity and is robust to only having an estimate of the 
covariance.

6 Model selection in linear regression

In this section, we discuss an application of our stability tools to the problem of model selection in linear 
regression. We focus on the framework presented in the seminal work of Berk et al. [9]. We begin by 
reviewing the model and introduce the necessary notation.

Let $X \in \mathbb{R}^{n \times d}$ denote a fixed design matrix, and let $X_i \in \mathbb{R}^n$ denote the $i$-th column of $X$, for $i \in [d]$. We refer to vectors $X_i$ as variables or features. For a subset $M \subseteq [d]$, we denote by $X_M \in \mathbb{R}^{n \times |M|}$ the 
submatrix of $X$ given by selecting the columns indexed by $M$. We make no assumptions about how $n$ 
and $d$ relate; in particular, we could have $d \gg n$.

By $y \in \mathbb{R}^n$ we denote the random vector of outcomes corresponding to $X$. Importantly, we do not 
assume knowledge of a true data-generating process; for example, we do not assume that 
$\mu := \mathbb{E}[y]$ can be expressed as a linear combination of $\{X_i\}_{i=1}^d$. The vector $\mu \in \mathbb{R}^n$ 
is unconstrained and need not reside in the column space of $X$. Rather, different subsets of $\{X_i\}_{i=1}^d$ provide different approximations to $\mu$, some better than others.

The statistician wishes to let the data decide how the initial pool of features should be reduced to 
a smaller set of seemingly relevant features, and then run linear regression on this smaller set. That is, 
letting $\hat{X}_M := \text{arg min}_{M \subseteq \mathbb{R}^{|M|}} \left\{ \|y - X_M \beta_M\|_2^2 \right\} = X_M^+ y,$ 

where $X_M^+ := (X_M^T X_M)^{-1} X_M^T$ to be the pseudoinverse of $X_M$. For a fixed model $M$, the target 
estimand of $\beta_M$ is 
$\beta_M := \text{arg min}_{\beta \in \mathbb{R}^{|M|}} \mathbb{E} \left[ \|y - X_M \beta\|_2^2 \right] = X_M^+ \mu,$

and hence for a random model $\hat{M}$, this implies a random target $\beta_{\hat{M}} = X_{\hat{M}}^+ \mu.$

We denote by $\beta_{j,\hat{M}}$ the entry of $\beta_\hat{M}$ corresponding to feature $X_j$, for all $j \in \hat{M}$. Note that $\beta_{j,\hat{M}}$ is not defined for $j \notin \hat{M}$. We adopt similar notation for the entries of $\beta_\hat{M}$.

Our goal is to construct simultaneous confidence intervals for the target of inference $\beta_{\hat{M}}$. More 
precisely, we wish to design $\text{CI}_{j,\hat{M}}(\alpha)$ such that 

$$\mathbb{P} \left\{ \beta_{j,\hat{M}} \in \text{CI}_{j,\hat{M}}(\alpha), \forall j \in \hat{M} \right\} \geq 1 - \alpha,$$

for a fixed $\alpha \in (0,1)$ and a fixed selection procedure $\hat{M}$. Note that the work of Berk et al. and various 
extensions [9, 2, 32] provide simultaneity both over the selected variables and over all selection methods, 
while we keep the selection method fixed. Our guarantees are simultaneous over the selected [cf. 8].

The confidence intervals resulting from our approach take the usual form, 

$$\text{CI}_{j,\hat{M}}(K) := \left( \hat{\beta}_{j,\hat{M}} \pm K \hat{\sigma}_{j,\hat{M}} \right),$$

where $\hat{\sigma}_{j,\hat{M}}^2$ is an estimator of variance for the OLS estimate $\hat{\beta}_{j,\hat{M}}$; e.g., the “sandwich” variance estimator [12]. Our goal is to find a suitable value of $K$ such that $\text{CI}_{j,\hat{M}}(K)$ are valid $(1 - \alpha)$-confidence
intervals, as per Eq. (3). By analogy with Berk et al. [9], we refer to the minimal such valid K as the PoSI constant. It is important to remember that, unlike in Berk et al., our PoSI constant depends on the selection procedure, rather than a family of all possible models.

The PoSI constant is well characterized when the model is fixed rather than determined in a data-driven fashion. For a fixed model M and given \( \alpha \in (0, 1) \), we define \( K_{M, \alpha} \) to be the minimum value of \( K \) such that

\[
P\left\{ \max_{j \in \hat{M}} \left| \frac{\hat{\beta}_{j, \hat{M}} - \beta_{j, M}}{\hat{\sigma}_{j, \hat{M}}} \right| \geq K \right\} \leq \alpha.
\]

In other words, \( K_{M, \alpha} \) defines the PoSI constant when the model is fixed rather than determined in a data-driven fashion. For a fixed model M and given \( \alpha \in (0, 1) \), we assume that \( \sigma > 0 \) is known but there exists an estimate, \( \hat{\sigma}_{j, M} \). If the full model is assumed to be correct, that is \( j \in \hat{M} \), then this assumption is satisfied for \( \hat{\beta}_{j, \hat{M}} \). Let:

\[
\text{CI}_{j, \hat{M}}(K_{M, \delta, \alpha}) = \left( \hat{\beta}_{j, \hat{M}} \pm K_{M, \delta, \alpha} \hat{\sigma}_{j, \hat{M}} \right).
\]

Then,

\[
P\left\{ \exists j \in \hat{M} : \beta_{j, \hat{M}} \notin \text{CI}_{j, \hat{M}}(K_{M, \delta, \alpha}) \right\} \leq \delta + \tau + \nu.
\]

We are now ready to state a corollary of Theorem 2 that focuses on the problem of model selection in linear regression.

**Corollary 1.** Fix \( \delta \in (0, 1) \). Let \( \hat{M} \) be an \( (\eta, \tau, \nu) \)-stable model selection algorithm. For all \( j \in \hat{M} \), let:

\[
\text{CI}_{j, \hat{M}}(K_{M, \delta, \alpha}) = \left( \hat{\beta}_{j, \hat{M}} \pm K_{M, \delta, \alpha} \hat{\sigma}_{j, \hat{M}} \right).
\]

Then,

\[
P\left\{ \exists j \in \hat{M} : \beta_{j, \hat{M}} \notin \text{CI}_{j, \hat{M}}(K_{M, \delta, \alpha}) \right\} \leq \delta + \tau + \nu.
\]

We denote by \( z_{1-\alpha} \) the \( 1 - \alpha \) quantile of the standard normal distribution, and by \( t_{r, 1-\alpha} \) the \( 1 - \alpha \) quantile of the \( r \)-distribution with \( r \) degrees of freedom.

**Corollary 2.** Fix \( \delta \in (0, 1) \), and suppose \( y \sim N(\mu, \sigma^2 I) \). Further, let \( \hat{M} \) be an \( (\eta, \tau, \nu) \)-stable model selection algorithm. If \( \sigma \) is known, let:

\[
\text{CI}_{j, \hat{M}} = \left( \hat{\beta}_{j, \hat{M}} \pm z_{1-\delta/(2|\hat{M}|\alpha)} \sigma \sqrt{((X_M^T X_M)^{-1})_{jj}} \right).
\]

If, on the other hand, \( \sigma \) is not known but there exists an estimate, \( \hat{\sigma} \), independent of the OLS estimates, let:

\[
\text{CI}_{j, \hat{M}} = \left( \hat{\beta}_{j, \hat{M}} \pm t_{r, 1-\delta/(2|\hat{M}|\alpha)} \hat{\sigma} \sqrt{((X_M^T X_M)^{-1})_{jj}} \right).
\]

In either case, we have

\[
P\left\{ \exists j \in \hat{M} : \beta_{j, \hat{M}} \notin \text{CI}_{j, \hat{M}} \right\} \leq \delta + \tau + \nu.
\]

The proof follows by a direct application of Corollary 1 together with a Bonferroni correction over \( j \in \hat{M} \) when computing \( K_{M, \delta, \alpha} \). Approximating Gaussian quantiles by subgaussian concentration, we observe that the PoSI constant in Corollary 2 scales roughly as \( \sqrt{2 \left( \log(2|\hat{M}|\delta) + \eta \right)} \) (when \( \sigma \) is known, or as \( r \to \infty \) when \( \sigma \) is estimated from data).
Recovering the Scheffé rate. Our main technical step in deriving selective confidence intervals is Lemma 1 which argues that the joint distribution of \( (y, \hat{S}) \) cannot be too different from the joint distribution of \( (y, \hat{S}_0) \), where \( \hat{S}_0 \) is the oracle from the definition of stability, in the indistinguishability metric. In the context of model selection in linear regression, we verify that the confidence intervals resulting from this approach are not vacuously wide in the two most extreme settings: the first, in which the model selection is independent of the data, and the second, in which the model selection is arbitrarily complex and dependent on the data.

Suppose that \( \hat{M} \) is independent of \( y \). Then, the distribution of \( \hat{M}(y) \), conditional on \( y \), is equal to the distribution of \( \hat{M}(\omega) \) for any point \( \omega \), hence \( \hat{M}(\omega) \) is an oracle which trivially implies \((0,0,0)\)-stability. In this case, the intervals in Corollary 1 reduce to \( \text{CI}_{j,\hat{M}}(K_{\hat{M},\Delta}) \) and are valid at level \( 1 - \delta \), as expected.

Now suppose that \( \hat{M} \) is allowed to have arbitrary dependence on \( y \); in particular, it can attain the “significant triviality bound” of Berk et al. [9]. While arguing stability in the sense of Definition 2 would require additional assumptions, the only property of stability used to prove Theorem 2—the indistinguishability bound in Eq. 1—can be obtained. This allows for the proof of Theorem 2 to go through, thus recovering the tight rate of existing analyses.

**Proposition 1.** Let \( \hat{M} \) be an arbitrary, possibly randomized model selection procedure, such that \(|\hat{M}| \leq s\) almost surely. Then, for any \( \mathcal{P}_y \), there exists an oracle selection \( \hat{M}_0 \) such that for any \( \tau \in (0,1) \),

\[
(y, \hat{M}(y)) \approx_{\eta,\tau} (y, \hat{M}_0), \text{ for some } \eta = O(s \log(d/s)) + \log(1/\tau).
\]

Consequently, there exists a value \( \eta = O(s \log(d/s)) + \log(1/\tau) \) such that the intervals \( \text{CI}_{j,\hat{M}}(K_{\hat{M},\Delta e^{-\eta}}) = (\hat{\beta}_{j,\hat{M}} \pm K_{\hat{M},\Delta e^{-\eta}} \hat{\sigma}_{j,\hat{M}}) \) satisfy

\[
\Pr\{ \exists j \in \hat{M} : \hat{\beta}_{j,\hat{M}} \not\in \text{CI}_{j,\hat{M}}(K_{\hat{M},\Delta e^{-\eta}}) \} \leq \delta + \tau.
\]

By approximating Gaussian quantiles via subgaussian concentration, we obtain confidence intervals which are universally valid for all \( s \)-sparse selections under Gaussian outcomes and scale as \( O(\sqrt{n}) = O(\sqrt{s \log(d/s)}) \). This rate is in general tight [31], and as \( s \) approaches \( d \), it matches the rate given by the Scheffé protection [48, 49].

7 Conditional coverage

So far all results we have presented have been about marginal coverage. Sometimes it is desirable to provide conditional coverage, whereby we condition on the event that a given selection was made. We discuss how stability can provide guarantees that closely resemble those of conditional post-selection inference.

We start by stating an implication of Lemma 1 for a specific oracle selection \( \hat{S}_0 \). In what follows, let \( E \) denote the subset of \( \text{supp}(\mathcal{P}_y) \) over which a selection \( \hat{S} \) is indistinguishable from \( \hat{S}_0 \), i.e., if \( \hat{S} \) is \((\eta,\tau,\nu)\)-stable, we let

\[
E = \{ \omega : \hat{S}(\omega) \approx_{\eta,\tau} \hat{S}_0 \}.
\]

Note that we know \( \Pr\{y \in E\} \geq 1 - \nu \) by definition.

**Lemma 2.** Suppose that \( \hat{S} \) is \((\eta,0,\nu)\)-stable with respect to oracle \( \hat{S}_0 = \hat{S}(y'_E) \), where \( y'_E \) is a sample from \( \mathcal{P}_y \) truncated to \( E \). Then, it holds that

\[
\Pr\{ y \in \mathcal{O}_S \mid \hat{S}(y) = S, y \in E \} \leq e^\nu \Pr\{ y \in \mathcal{O}_S \mid y \in E \},
\]

for all selections \( S \) and measurable sets \( \mathcal{O}_S \).
As suggested by Lemma 2, the main difference between conditional post-selection inference and the conditional guarantees implied by stability is that in the latter case we additionally truncate the distribution of \( y \) to a high-probability set \( E \). Note that on the right-hand side of Eq. (4) there is no dependence on the selection event, which makes inference, despite selection, essentially as easy as classical inference.

We illustrate the conditional properties of stability with an example.

Example: publication bias. We consider an illustration of the publication bias problem, also known as the file-drawer problem \[44, 24, 52\]. Suppose we observe an effect \( y \sim \mathcal{N}(\mu, \sigma^2) \) with \( \mathbb{E}[y] = \mu \), \( \text{supp}(\mathcal{P}_y) \subseteq \mathbb{R} \). We are interested in constructing an interval for \( \mu \) given that we observe effect \( y \) on the selection event, which makes inference, despite selection, essentially as easy as classical inference.

One approach to this problem is to evaluate the distribution of the data \( \text{conditional} \) on the selection event. For example, we could find \( K \) such that \( \mathbb{P}\{ |y - \mu| > K \mid \text{report}(y) \} \leq \alpha \), and report \( \text{CI}(K) = (y \pm K) \) on the event \( \text{report}(y) \). Importantly, this approach generally requires an explicit characterization of the event \( \text{report}(y) \).

Our theory suggests a criterion-agnostic solution based on randomizing the selection, whose validity we prove in the Supplement.

**Claim 5.** Let \( y \sim \mathcal{N}(\mu, \sigma^2) \). Suppose that we apply the selection criterion to \( y + \xi \), where \( \xi \sim \text{Lap}(b) \) for some user-chosen parameter \( b > 0 \); that is, we report the confidence interval on the event \( \text{report}(y + \xi) \).

Then, for any user-chosen parameter \( \nu \in (0, 1) \), we have

\[
\mathbb{P}\left\{ \mu \not\in (y \pm z_{1-\nu/2} (1-\nu)e^{-\eta} \sigma) \mid \text{report}(y + \xi), y \in E \right\} \leq \alpha,
\]

where

\[
\eta = \frac{z_{1-\nu/2}^2}{b} - \frac{\sigma^2}{2b^2} + \log \left( \frac{1 - \nu}{2(\Phi(z_{1-\nu/2} + \frac{\sigma}{b}) - \Phi(z_{1-\nu/2}))} \right)
\]

and \( E \) is an event such that \( \mathbb{P}\{y \in E\} \geq 1 - \nu \).

Although in the main body of the paper we state the result when \( \mathcal{P}_y = \mathcal{N}(\mu, \sigma^2) \) for simplicity, the Laplace noise addition strategy is valid for arbitrary \( \mathcal{P}_y \); only the expression for \( \eta \) changes as a function of \( \mathcal{P}_y \). In the Supplement we state a general version of Claim 5.

The proof of Claim 5 relies on showing that the selection to report is stable with respect to the oracle \( \hat{S}(y_{\nu}) \), and hence we can invoke Lemma 2.

We can see that, by choosing \( \nu \to 0, b \to \infty \), we recover the non-selective confidence intervals, albeit at the cost of making the decision to report virtually independent of \( y \). As we decrease \( b \) (and keep \( \nu \) bounded away from zero), the decision to report becomes more reflective of the event \( \text{report}(y) \) and the inference level smoothly becomes more stringent. In Figure 3 we plot the normalized interval width, \( z_{1-\nu/2} (1-\nu)e^{-\eta} \), together with the unadjusted normalized width \( z_{1-\nu/2} \), for several different noise levels \( b \) and error levels \( \alpha \) and \( \nu = 0.05 \).

![Figure 3](image_url)

**Figure 3.** Normalized interval width implied by stability solution and unadjusted width for different noise levels \( b \) and error levels \( \alpha \). We fix \( \nu = 0.05 \).
As a concrete example, suppose that \( \text{report}(y) = \{ y > T \} \), for some threshold \( T \). Then, as in Claim 2 and Claim 3, we can conclude that \( 1 \{ \text{report}(y) \} = 1 \{ \text{report}(y + \xi) \} \) with probability \( 1 - \delta \) over the choice of \( \xi \) as long as \( b \leq \frac{|y - T|}{\log(1/(2\delta))} \).

8 The design of stable selection algorithms

We discuss general tools for designing stable selection methods and present an application of these tools to variable selection in linear regression. We begin with an overview of the basic properties of stability, which are key to efficient design of stable selections.

8.1 Properties of stability

Stability satisfies two key algorithmic properties: \textit{closure under post-processing} and \textit{composition}. We provide precise definitions of the two shortly. The reason why these properties enable efficient stability designs is that many selection rules can be written as post-processing and composition of simple computations, such as linear functions of the data or finding maxima of a sequence. As long as we know how to stabilize the necessary simple computations, closure under post-processing and composition provide rules for computing the overall stability parameter of the whole algorithm efficiently.

Post-processing. First, stability is \textit{closed under post-processing}: if \( A : \mathbb{R}^n \rightarrow S \) is \( (\eta, \tau, \nu) \)-stable, then for any (possibly randomized) map \( B : S \rightarrow G \), the composition \( B \circ A \) is also \( (\eta, \tau, \nu) \)-stable. While the proof of this fact is a straightforward consequence of the definition of stability, the implications are significant. Suppose for the moment that the statistician is given a stable version of the LASSO algorithm, and denote its solution by \( \hat{\theta}_{\text{LASSO}} \). Since \( \hat{\theta}_{\text{LASSO}} \) is stable, then so is \( \hat{M} = \{ j \in [d] : \hat{\theta}_{\text{LASSO}} \neq \emptyset \} \).

In fact, the statistician need not necessarily choose the model corresponding \textit{exactly} to the support of \( \hat{\theta}_{\text{LASSO}} \); for example, they could choose \( \hat{M} = \{ j \in [d] : |\hat{\theta}_{\text{LASSO}}| \geq \epsilon \} \), for some constant threshold \( \epsilon \), or they could pick \( d_{\text{sel}} \leq d \) entries with the maximum absolute value. More generally, any model chosen solely as a function of \( \hat{\theta}_{\text{LASSO}} \) inherits the same stability parameters as \( \hat{\theta}_{\text{LASSO}} \). And, according to Corollary 4 the same PoSI constant suffices to correct the confidence intervals resulting from any such model.

Composition. The second important property is \textit{composition}. In Algorithm 1 we define adaptive composition, after which we discuss simpler, non-adaptive composition.

**Algorithm 1 Adaptive composition**

\[
\text{input: } \text{data } y \in \mathbb{R}^n, \text{ sequence of algorithms } A_t : S_1 \times \cdots \times S_{t-1} \times \mathbb{R}^n \rightarrow S_t, \ t \in [k] \\
\text{output: } \ (a_1, \ldots, a_k) \in S_1 \times \cdots \times S_k \\
\text{for } t = 1, 2, \ldots, k \text{ do} \\
\quad \text{Compute } a_t = A_t(a_1, \ldots, a_{t-1}, y) \in S_t \\
\text{end} \\
\text{Return } (a_1, \ldots, a_k)
\]

Adaptive composition consists of \( k \) sequential rounds in which the analyst observes the outcomes of all previous computations and selects the next computation \textit{adaptively}—as a function of the previous evaluations. The adaptive composition property bounds the stability parameters of Algorithm 1 in terms of the stability parameters of \( A_t \). In its simplest form, it says that Algorithm 1 is \( (k\eta, 0, 0) \)-stable if for all \( t \in [k] \), \( A_t(a_1, \ldots, a_{t-1}, \cdot) \) is \( (\eta, 0, 0) \)-stable for all fixed \( a_1, \ldots, a_{t-1} \). For example, for
some selection algorithms such as forward stepwise, it is clear to see how they can be represented using adaptive composition. In forward stepwise, $A_t$ outputs an index $i_t \in [d]$, which corresponds to the variable $i$ that minimizes the squared error resulting from adding $i$ to the current pool of selected features; $i_t = A_t(i_1, \ldots, i_{t-1}, y)$. It suffices to prove that any given step of forward stepwise selection is stable, in order to infer that the overall algorithm is stable as well.

Our proofs will only require adaptive composition for algorithms with $\nu = 0$; such results follow from classical theory on differential privacy. More advanced (and naturally more conservative) adaptive composition theorems which allow $\nu > 0$ can be found in the context of typical stability [4]. In the Supplement we formally state the adaptive composition results we will need in our proofs.

A simpler kind of composition is non-adaptive composition. Here, the algorithms $A_t$ have no dependence on the past computations. Non-adaptive composition can capture a protocol that involves running multiple selection methods and choosing a final selection target as an arbitrary function of all the outputs. As we state formally in the Supplement, the resulting stability parameters simply add up. This is a rather appealing property of stability, as it suggests that the statistician only needs to keep track of the stability parameters of each selection algorithm they run, in order to derive valid selective confidence intervals. An analogous combination of the results of different selection methods was considered by Markovic and Taylor [38]; their approach, however, relies on a sophisticated Monte Carlo sampling scheme.

8.2 Model selection algorithms: examples

We now consider several algorithms for variable selection in linear regression through the lens of stability. While many of the principles presented in this section can be adapted to different distributional assumptions, for the sake of clarity and interpretability we assume that $y \sim N(\mu, \sigma^2 I)$, where $\sigma^2$ is unknown but we have access to an estimate $\hat{\sigma}^2 \sim \sigma^2 \frac{2}{r}$, independent of $y$. This is the setup studied by Berk et al. [9]. More generally, we only need to know the decay of the tail of the distribution of $y$ in order to enforce stability. In the Supplement, we extend the algorithms in this section to the case of outcome vectors with a known bound on their Orlicz norm, for any Orlicz function. This includes cases such as general subgaussian and subexponential outcome vectors.

Model selection via the LASSO. We begin by considering the canonical example of the LASSO estimator [55]. The LASSO estimate is the solution to the usual least-squares problem with an additional $\ell_1$-constraint on the regression coefficients:

$$\hat{\theta}_{\text{LASSO}} \in \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{2} \|y - X\theta\|_2^2 \quad \text{s.t.} \quad \|\theta\|_1 \leq C_1,$$

where $C_1 > 0$ is a tuning parameter. This problem is sometimes referred to as the LASSO in constrained/bound form, to contrast it with the LASSO in penalized form:

$$\hat{\theta}_{\text{LASSO}}^\lambda \in \arg\min_{\theta \in \mathbb{R}^d} \frac{1}{2} \|y - X\theta\|_2^2 + \lambda \|\theta\|_1,$$

where $\lambda > 0$ is now the tuning parameter. These two problems are equivalent: for any $C_1 > 0$, there exists a corresponding $\lambda > 0$ such that $\hat{\theta}_{\text{LASSO}}$ is an optimal solution for the problem in Eq. (6), and vice versa. In our analysis we focus on the formulation (5).

The LASSO objective induces sparse solutions, and a common way of declaring that a feature is relevant is to check for a corresponding non-zero entry in the LASSO solution vector. That is, the model “selected” by the LASSO is:

$$\hat{M} = \{ j \in [d] : \hat{\theta}_{\text{LASSO}, j} \neq 0 \}.$$

Model selection via the LASSO has been of great interest in prior work on selective inference, starting with Lee et al. [35]. While this work provides exact confidence intervals, it has been observed that these intervals (which do not make use of randomization) have infinite expected length [29]. Subsequent work
has improved upon these often large confidence intervals by choosing a better event to condition on \[37\], or by applying randomization \[54, 53, 52, 30, 40, 39\].

We now formulate a stable version of the LASSO algorithm. It is inspired by the differentially private LASSO algorithm of Talwar et al. \[49\], although the noise variables are calibrated somewhat differently due to different modeling assumptions.

We use \(e_i\) to denote the \(i\)-th standard basis vector in \(\mathbb{R}^d\), and \(\{\pm e_i\}_{i=1}^d\) to denote the set of \(2d\) standard basis vectors, multiplied by \(1\) and \(-1\). We also let \(\|X\|_{2,\infty}\) denote the \(L_{2,\infty}\) norm of \(X\), \(\|X\|_{2,\infty} := \max_{i \in [d]} \|X_i\|_2\).

### Algorithm 2 Stable LASSO algorithm

**input:** design matrix \(X \in \mathbb{R}^{n \times d}\), outcome vector \(y \in \mathbb{R}^n\), variance estimate \(\hat{\sigma}^2 \sim \sigma^2 \frac{3}{\eta}\), \(\ell_1\)-constraint \(C_1\), number of steps \(k\), parameters \(\delta \in (0, 1), \eta > 0\)

**output:** LASSO solution \(\hat{\theta}_{\text{LASSO}} \in \mathbb{R}^d\)

Initialize \(\hat{\theta}_1 = 0\)

for \(t = 1, 2, \ldots, k\) do

\(\forall \phi \in C_1 \cdot \{\pm e_i\}_{i=1}^d\), sample \(\xi_{t, \phi} \sim \text{Lap}(\frac{4t \epsilon_{t-1} \epsilon_t/\sqrt{d} C_1}{\eta n})\)

\(\forall \phi \in C_1 \cdot \{\pm e_i\}_{i=1}^d\), let \(\alpha_\phi = -\frac{2 \hat{\sigma}^2}{\eta} \phi^\top X (y - X \theta_t) + \xi_{t, \phi}\)

Set \(\phi_t = \arg \min_{\phi \in C_1 \cdot \{\pm e_i\}_{i=1}^d} \alpha_\phi\)

Set \(\theta_{t+1} = (1 - \Delta_t) \theta_t + \Delta_t \phi_t\), where \(\Delta_t = \frac{2}{t+1}\)

end

Return \(\hat{\theta}_{\text{LASSO}} = \theta_{k+1}\)

In essence, Algorithm 2 is a randomized version of the classical Frank-Wolfe algorithm from constrained optimization \[25\].

We now argue that \(\hat{\theta}_{\text{LASSO}}\) is stable. The proof is based on a composition argument: namely, we can view \(\hat{\theta}_{\text{LASSO}}\) as the result of a composition of \(k\) subroutines, each given by one optimization step which produces \(\theta_t\). The stability of each subroutine is proved by extending an argument related to the “report noisy max” mechanism from differential privacy \[17\]. The full proof of Proposition 2 can be found in the Supplement.

**Proposition 2 (LASSO stability).** Algorithm 2 is both

(a) \(\left(\frac{1}{2} k \eta^2 + \sqrt{2 k \log(1/\delta)} \eta, \delta, \delta\right)\)-stable, and

(b) \((k \eta, 0, \delta)\)-stable.

We state two rates because there exist parameter regimes where either rate leads to tighter confidence intervals than the other (the first rate being tighter when \(\eta\) is small).

By the post-processing property, Proposition 2 implies stability of any model \(\hat{M}\) obtained as a function of \(\hat{\theta}_{\text{LASSO}}\), such as the model corresponding to its non-zero entries.

Notice that the noise level in Algorithm 2 is an explicit function of \(\eta\). This allows the statistician to understand the loss in utility—that is, how much worse \(\hat{\theta}_{\text{LASSO}}\) is relative to an exact LASSO solution—due to randomization. In fact, building on work by Jaggi \[27\] and Talwar et al. \[49\], we can upper bound the excess risk resulting from randomization.

**Proposition 3 (LASSO utility).** Suppose we run Algorithm 2 for \(k = \left\lceil \frac{n \|X\|_{2,\infty}^2 C_1 \eta}{\sigma \|X\|_{2,\infty}} \right\rceil\) steps. Then,

\[
\frac{1}{n} \mathbb{E}[\|y - X \hat{\theta}_{\text{LASSO}}\|_2^2 \mid y] - \min_{\theta : \|\theta\| \leq C_1} \frac{1}{n} \|y - X \theta\|_2^2 = \tilde{O} \left( \frac{C_1 \|X\|_{2,\infty} \log(d) t_{r, 1 - \delta/(2d)} \sigma}{n \eta} \right).
\]

The proof is deferred to the Supplement.
Model selection via marginal screening. One of the most commonly used model selection methods involves simply picking a constant number of the features with the largest absolute inner product with the outcome $y$ \cite{LeeT20}. That is, one selects features $i$ corresponding to the top $k$ values of $|X_i^\top y|$, for a pre-specified parameter $k$. This strategy is known as marginal screening, and it was first analyzed in the context of selective inference by Lee and Taylor \cite{LeeT20}.

In Algorithm 3 we state a stable version of marginal screening. Notice that the randomization scheme is similar to that of the stable LASSO method. As before, we let $\|X\|_{2,\infty}$ denote the $L_{2,\infty}$ norm of $X$.

**Algorithm 3** Stable marginal screening algorithm

**input:** design matrix $X \in \mathbb{R}^{n \times d}$, outcome vector $y \in \mathbb{R}^n$, variance estimate $\hat{\sigma}^2 \sim \sigma^2 \frac{\lambda^2}{r}$, model size $k$, parameters $\delta \in (0, 1)$, $\eta > 0$

**output:** $\tilde{M} = \{i_1, \ldots, i_k\}$

Compute $(c_1, \ldots, c_d) = \frac{1}{n\sigma}X^\top y \in \mathbb{R}^d$

res$_1 = [d]$

for $t = 1, 2, \ldots, k$

\[
\forall i \in \text{res}_t, \text{ sample } \xi_{t,i} \overset{\text{i.i.d.}}{\sim} \text{Lap}\left(\frac{2t_r, 1-\delta/(2d)}{n\eta}\|X\|_{2,\infty}\right)
\]

\[
i_t = \arg \max_{i \in \text{res}_t} |c_i + \xi_{t,i}|
\]

res$_{t+1} = \text{res}_t \setminus \{i_t\}$

end

Return $\tilde{M} = \{i_1, \ldots, i_k\}$

The high-level idea behind the proof of stability of Algorithm 3 is similar to that of Algorithm 2 and we present it in the Supplement.

**Proposition 4** (Marginal screening stability). Algorithm 3 is both

(a) $\left(\frac{1}{2}k\eta^2 + \sqrt{2k\log(1/\eta)}\eta, \delta, \hat{\sigma}\right)$-stable, and

(b) $(k\eta, 0, \delta)$-stable.

As for the LASSO, we aim to quantify the loss in utility due to randomization. Given that the goal of marginal screening is to detect the largest $k$ values $|c_i| = |X_i^\top y|$, a reasonable notion of utility loss is the difference between the values $c_i$ corresponding to the variables in $\tilde{M}$, and the actual largest values of $c_i$.

**Proposition 5** (Marginal screening utility). Let $m_i$ denote the index of the $i$-th largest value $c_j$ in absolute value, so that $(|c_{m_1}|, \ldots, |c_{m_d}|)$ is the decreasing order statistic of $\{|c_i|\}_{i=1}^d$. Then, for any $\delta' \in (0, 1)$, Algorithm 3 satisfies:

\[
\mathbb{P}\left\{\max_{j \in [k]} |c_{m_j}| - |c_j| \leq \frac{4t_r, 1-\delta/(2d)}{n\eta} \log(dk/\delta')\|X\|_{2,\infty} \mid y\right\} \geq 1 - \delta'.
\]

9 Experimental results

In this section, we evaluate our selective intervals for the LASSO and marginal screening and compare our solution with data splitting.

For a fixed sample size $n$ we vary the number of features $d$. We consider two different data-generating processes for the design matrix: one in which the rows of $X$ are drawn independently from an equicorrelated multivariate Gaussian distribution with pairwise correlation $\rho = 0.5$, and the second one in which all entries of $X$ are drawn as independent Bernoulli random variables with parameter $0.1$. In the former case, $X$ is normalized to have columns of unit norm. The outcome is generated as $y = X\beta + \epsilon$, where
We first state the results for the Gaussian design case.

\[ \epsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, 1), i \in [n], \] and the entries of \( \beta \) are sampled according to

\[ \beta_i = \begin{cases} \text{Exp}(\rho), & i \in \{1, \ldots, sd\}, \\ 0, & i \in \{sd + 1, \ldots, d\}, \end{cases} \]

for a signal parameter \( \rho > 0 \) and a sparsity parameter \( s \in (0, 1) \), which we vary. In the Supplement we provide additional experiments when the errors are drawn from a heavier-tailed, Laplace distribution. We fix the target miscoverage level to be \( \alpha = 0.1 \). In all experiments we vary \( \eta \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \). For the comparison with data splitting, we use the splitting fraction derived in Section 5.1. Further experimental details are given in the Supplement.

### 9.1 Gaussian design

We first state the results for the Gaussian design case.

Figure 4. Comparison of FDR after stable LASSO and LASSO with data splitting, with varying dimension and signal strength, in the Gaussian design case. In addition, we plot the average interval width (at \( \rho = 0.2 \) only, however the width varies minimally with \( \rho \)) and the average unadjusted width.

Figure 5. Comparison of FDR after stable LASSO and LASSO with data splitting, with varying sample size, in the Gaussian design case. In addition, we plot the average interval width at \( n = 200 \) and the average unadjusted width.
In Figure 4 we compare the false discovery rate (FDR) of the stable LASSO algorithm and the LASSO algorithm with data splitting. In all plots for different values of we increase based on how the intervals obtained via data splitting have essentially the same width (and are hence not plotted), the average width of stable intervals against the average width of naive, unadjusted intervals. Note that splitting fraction lying in \{3, 5, 7\}, and we fix \(s = 0.5\).

We observe that stability generally outperforms data splitting as \(\eta\) grows, equivalently when the splitting fraction \(f(\eta)\) grows, as well as when the signal strength grows. In Figure 4 we additionally plot the average width of stable intervals against the average width of naive, unadjusted intervals. Note that the intervals obtained via data splitting have essentially the same width (and are hence not plotted), based on how \(f(\eta)\) is chosen. We only plot interval width for \(\rho = 0.2\) since the width varies minimally for different values of \(\rho\). For completeness we include all plots of interval width in the Supplement.

In Figure 6 we compare the stable LASSO algorithm and the LASSO with data splitting in a sparse design case. In addition, we plot the average interval width (at \(\rho = 0.2\) only, however the width varies minimally with \(\rho\)), together with the average unadjusted width and the width implied by the conditional correction of Lee and Taylor [34]. We also plot the 90% quantile of the conditional width because it varies greatly across realizations.

Figure 6. Comparison of average error after stable marginal screening and marginal screening with data splitting, with varying dimension and signal strength, in the Gaussian design case. In addition, we plot the average interval width (at \(\rho = 0.2\) only, however the width varies minimally with \(\rho\)), together with the average unadjusted width and the width implied by the conditional correction of Lee and Taylor [34].
In Figure 6 we compare the average error of stable marginal screening and marginal screening. We include the plots of all interval widths in the Supplement.

Figure 8. Comparison of FDR after stable LASSO and LASSO with data splitting, with varying dimension and signal strength, in the Bernoulli design case. In addition, we plot the average interval width (at \( \rho = 0.2 \) only, however the width varies minimally with \( \rho \)) and the average unadjusted width.

Marginal screening. In Figure 8 we compare the average error of stable marginal screening and marginal screening with data splitting. Since marginal screening explicitly aims to maximize the values \( |X_i^\top y| \) for selected variables \( X_i \), we quantify the error as \( \frac{1}{k} \sum_{t=1}^{k} (|X_{i_t}^\top y| - |X_{i_t^*}^\top y|) \), where \( i_t \) is the estimated index of the \( t \)-th largest absolute inner product (based on a subsample in the case of data splitting, or based on a randomized sample in the case of stability), and \( i_t^* \) is the true index of the \( t \)-th largest absolute inner product in the data set. We vary the parameters as in the LASSO comparison in Figure 4, only instead of varying \( C_1 \) we vary \( k \in \{5, 10, 20\} \). We also plot the average interval width with stability, together with the unadjusted interval width and the average width obtained via the conditional method of Lee and Taylor [34] with no randomization. For the conditional method, since the intervals are sometimes high-dimensional setting with \( d = 500, s = 0.1 \), and we vary the sample size \( n \in \{100, 200, 300\} \). We fix \( \rho = 0.1 \). We observe that stability consistently outperforms data splitting for large enough \( \eta \) and this gap grows with \( n \). In addition, we plot the average interval width implied by stability against the average unadjusted interval width at \( n = 200 \) (again we do not plot the interval width given by data splitting for the same reason as in Figure 4). We include the plots of all interval widths in the Supplement.

Figure 9. Comparison of FDR after stable LASSO and LASSO with data splitting, with varying sample size, in the Bernoulli design case. In addition, we plot the average interval width at \( n = 200 \) and the average unadjusted width.
orders of magnitude larger than the average width, we also plot the 90\% quantile of interval width. We see that stability typically outperforms data splitting in terms of the average error, and this benefit is more pronounced for larger $\eta$ and signal strength. In terms of interval width, we observe that stability leads to significantly smaller intervals than the conditional approach. We only plot interval width when $\rho = 0.2$, and defer the remaining plots to the Supplement.

In Figure 7 we consider a setting analogous to that of Figure 5 and we analogously vary the sample size $n$. We again see that stability generally dominates data splitting. Moreover, the gap between the intervals obtained via stability and those of Lee and Taylor \[34\] is even more pronounced than in Figure 6. We provide the plots of all interval widths in the Supplement.

9.2 Bernoulli design

Now we consider the Bernoulli design case. The motivation for considering a sparse Bernoulli design lies in the fact that certain directions in the column space of $X$ are captured by only a few samples, hence missing out on them—as is possible with data splitting—can significantly affect the quality of selection.

**LASSO.** In Figure 8 and Figure 9 we provide comparisons analogous to those of Figure 4 and Figure 5 using the same parameter configurations. We observe a larger gap between data splitting and stability than in the Gaussian design case, and observe the same trends: as $\eta$ and the signal strength grow, the performance gap increases. As before, we defer the remaining plots of interval widths to the Supplement.

**Marginal screening.** In Figure 10 and Figure 11 we provide comparisons analogous to those of Figure 6 and Figure 7 using the same parameter configurations. We observe a larger gap between data splitting
and stability both than in the Gaussian design case, as well as in the LASSO experiments using the Bernoulli design. In addition, we observe an even more pronounced gap between stable confidence interval widths and widths of intervals obtained via a conditional correction [34]. For this reason, the y-axis in the widths plots is logarithmic. We defer the remaining plots of interval widths to the Supplement.

10 Discussion

Building on concepts from algorithmic stability, as originally developed for applications in differential privacy, we have provided general theory for designing post-selection confidence intervals when the selection procedure is stable. The stability principle is broadly applicable, ranging from inference on the winning effect to model selection in linear regression. In particular, stability is applicable even when data splitting is not, such as when there are dependencies between observations.

Performing inference after a stable selection is simple: it merely requires discounting the type I error based on the level of stability. Moreover, stability comes with several practically appealing properties, namely robustness to post-processing and composition. Thus, for example, the statistician can run various selection methods, and essentially only needs to keep track of the stability parameters of each in order to obtain valid confidence intervals for the final target, which could combine the results of all the selections in an arbitrary way.

There are numerous other potential applications of algorithmic stability to the problem of post-selection inference that would be worthwhile to explore. For example, it would be valuable to understand bootstrapping [42] from the perspective of stability, due to its conceptual relations to the “privacy amplification by subsampling” principle in differential privacy, which argues that privacy is amplified when run on a random subsample of the entire data set [25] [6]. More broadly, selective mechanisms have been long analyzed in the context of differential privacy [51] [36] [48] [10] [22], and we believe that some of these developments could be imported to selective inference via stability.

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A Proofs

A.1 Composition of stability

Here we summarize prior work on adaptive composition theorems for differential privacy, which will be crucial in our proofs. To facilitate readability, we restate the definition of adaptive composition.

**Algorithm 4** Adaptive composition

```plaintext
input: data \( y \in \mathbb{R}^n \), sequence of algorithms \( A_i : \mathcal{F}_1 \times \cdots \times \mathcal{F}_{i-1} \times \mathbb{R}^n \to \mathcal{F}_i \), \( i \in [k] \)
output: \((a_1, \ldots, a_k) \in \mathcal{F}_1 \times \cdots \times \mathcal{F}_k\)

for \( i = 1, 2, \ldots, k \) do
  Compute \( a_i = A_i(a_1, \ldots, a_{i-1}, y) \in \mathcal{F}_i \)
end
Return \( A^{(k)}(y) = (a_1, \ldots, a_k) \)
```

The first statement below is a reformulation of the “simple” composition property of differential privacy [18]. The second statement is a slightly stronger reformulation [13] of the so-called “advanced” composition theorem for differential privacy [19].

**Lemma 3** [19,13]. Fix two vectors \( y, y' \in \mathbb{R}^n \) and suppose that \( A_i(a_1, \ldots, a_{i-1}, y) \approx_{\eta, \tau} A_i(a_1, \ldots, a_{i-1}, y') \), for every fixed sequence \( a_1, \ldots, a_{i-1} \), and all \( t \in [k] \). Then,

(a) \( A^{(k)}(y) \approx_{kn,kr} A^{(k)}(y') \),

(b) \( A^{(k)}(y) \approx_{\frac{1}{2}kn^2 + \frac{1}{2} \log(1/\delta)kn + \frac{1}{2} \log(1/\delta) \eta} A^{(k)}(y') \), for all \( \delta \in (0,1) \).

We also state a non-adaptive composition property of stability, which is relevant when running multiple model selection algorithms. It says that the privacy parameters of all outputs combined simply add up.

**Lemma 4.** Suppose \( A_i : \mathbb{R}^n \to \mathcal{F}_i \) is \((\eta_i, \tau_i, \nu_i)\)-stable, for all \( i \in [k] \). Then, \( A^{(k)} : \mathbb{R}^n \to \mathcal{F}_1 \times \cdots \times \mathcal{F}_k \)
defined as \( A^{(k)}(\cdot) = (A_1(\cdot), \ldots, A_k(\cdot)) \) is \((\sum_{i=1}^k \eta_i, \sum_{i=1}^k \tau_i, \sum_{i=1}^k \nu_i)\)-stable.

The proof of Lemma 4 follows by the simple composition theorem for differential privacy [19], together with a union bound over all failure events with probability \( \nu_i \).

A.2 Proof of Claim 1

Let \( E = \{ \omega \in \mathbb{R}^n : \| \omega - \mu \|_\infty \leq z_{1-\alpha\delta/(2n)} \} \). Then, by a Bonferroni correction, we know that \( \mathbb{P}\{y \notin E\} \geq 1 - \alpha\delta \).

For any \( \omega \in \mathbb{R}^n \), let \( \hat{i}_s(\omega) = \arg \max_i (\omega_i + \xi_i) \), where \( \xi_i \overset{\text{i.i.d.}}{\sim} \text{Lap}\left(\frac{2z_{1-\alpha\delta/(2n)}}{\eta}\right) \). Define

\[
\xi^* = \arg \min_{i} y_i + \xi_i > y_j + \xi_j, \forall j \neq i.
\]

Then, \( \hat{i}_s(y) = i \) if and only if \( \xi_i \geq \xi^* \). By the definition of \( E \) we have that for all \( y \in E \)

\[
z_{1-\alpha\delta/(2n)} + \mu_i + \xi^* \geq y_i + \xi^* > y_j + \xi_j \geq \mu_j + \xi_j - z_{1-\alpha\delta/(2n)},
\]
for all \( j \neq i \). Rearranging the terms, we get

\[
2z_{1-\alpha \delta/(2n)} + \mu_i + \xi_i \geq \mu_j + \xi_j.
\]

Thus, if \( \xi_i \geq \xi^* + 2z_{1-\alpha \delta/(2n)} \), then \( \hat{i}_*(\mu) = i \) if the noise levels are \((\xi_1, \ldots, \xi_i, \ldots, \xi_n)\). Putting everything together, for fixed \( y \in E \), we have

\[
\mathbb{P}\{\hat{i}_*(\mu) = i \mid \{\xi_{t,j}\}_{j \neq i}\} \geq \mathbb{P}\{\xi_i \geq \xi^* + 2z_{1-\alpha \delta/(2n)} \mid \{\xi_{t,j}\}_{j \neq i}\}
\]

\[
\geq e^{-\eta}\mathbb{P}\{\xi_i \geq \xi^* \mid \{\xi_{t,j}\}_{j \neq i}\} = e^{-\eta}\mathbb{P}\{\hat{i}_*(y) = i \mid \{\xi_{t,j}\}_{j \neq i}\}.
\]

Multiplying by \( e^\eta \) and applying the law of iterated expectations completes the proof that \( \hat{i}_*(y) \approx \hat{i}_*(\mu) \) for all \( y \in E \), which implies that \( \hat{i}_*(\mu) \) is a valid oracle and \( \hat{i}_*(\cdot) \) is \((\eta, 0, \alpha \delta)\)-stable.

Finally, we invoke Theorem 2 to complete the proof.

### A.3 Proof of Claim 3

A subsequent proof of Proposition 4 implies as a special case that \( \hat{i}_* \) is \((\eta, 0, \alpha \delta)\)-stable. Putting this fact together with Theorem 2 completes the proof.

### A.4 Proof of Claim 2 and Claim 4

We prove Claim 2; the proof of Claim 4 is essentially identical. The event \( \{\hat{i}_* \neq i_*\} \) must imply that at least one of two events: \( \{\xi_{i_*} \leq -\frac{\Delta}{2}\} \) or \( \{\exists j \neq i_* : \xi_j \geq \frac{\Delta}{2}\} \). By a union bound and concentration of Laplace random variables, we can thus conclude

\[
\mathbb{P}\{\hat{i}_* \neq i_*\} \leq \frac{d}{2} \exp\left(-\frac{\Delta \eta}{4z_{1-\alpha \delta/(2d)}}\right).
\]

Setting this expression to \( \delta' \) and substituting for \( \eta \) completes the proof.

### A.5 Proof of Lemma 1

Denote by \( \hat{S}_0 \) the oracle from Definition 2 and let \( E = \{\omega \in \mathbb{R}^n : \hat{S}(\omega) \approx_{\eta, \tau} \hat{S}_0\} \). Fix an event \( \mathcal{O} \subseteq \mathbb{R}^n \times \mathcal{S} \), and let \( \mathcal{O}_y = \{S \in \mathcal{S} : \omega, S \in \mathcal{O}\} \). Notice that \( 1\{y, \hat{S}(y) \in \mathcal{O}\} = 1\{\hat{S}(y) \in \mathcal{O}_y\} \), and hence \( \mathbb{E}[1\{y, \hat{S}(y) \in \mathcal{O}\}|y] = \mathbb{E}[1\{\hat{S}(y) \in \mathcal{O}_y\}|y] \).

With this, we can write:

\[
\mathbb{P}\{(y, \hat{S}(y)) \in \mathcal{O}, y \in E\} = \mathbb{E}[\mathbb{E}[1\{\hat{S}(y) \in \mathcal{O}_y\}|y]1\{y \in E\}]
\]

\[
= \mathbb{E}[\mathbb{E}[\hat{S}(y) \in \mathcal{O}_y \mid y]1\{y \in E\}]
\]

\[
\leq \mathbb{E}[(e^\eta\mathbb{P}\{\hat{S}_0 \in \mathcal{O}_y \mid y\} + \tau)1\{y \in E\}]
\]

\[
= \mathbb{E}[(e^\eta1\{\hat{S}_0 \in \mathcal{O}_y\} + \tau)1\{y \in E\}]
\]

\[
\leq e^\eta\mathbb{P}\{(y, \hat{S}_0) \in \mathcal{O}, y \in E\} + \tau.
\]

Since \( \mathbb{P}\{y \in E\} \geq 1 - \nu \), we can conclude:

\[
\mathbb{P}\{(y, \hat{S}(y)) \in \mathcal{O}\} = \mathbb{P}\{(y, \hat{S}(y)) \in \mathcal{O}, y \in E\} + \mathbb{P}\{(y, \hat{S}(y)) \in \mathcal{O}, y \notin E\}
\]

\[
\leq \mathbb{P}\{(y, \hat{S}(y)) \in \mathcal{O}, y \in E\} + \nu
\]

\[
\leq e^\eta\mathbb{P}\{(y, \hat{S}_0) \in \mathcal{O}, y \in E\} + \tau + \nu
\]

\[
\leq e^\eta\mathbb{P}\{(y, \hat{S}_0) \in \mathcal{O}\} + \tau + \nu.
\]
A.6 Proof of Theorem 2

By Lemma 1, we know that
\[
\mathbb{P}\left\{ \hat{\beta}_S \not\in \text{CI}_S^{(\delta e^{-n})} \right\} \leq e^n \mathbb{P}\left\{ \beta_{S_0} \not\in \text{CI}_{S_0}^{(\delta e^{-n})} \right\} + \tau + \nu
= e^n \mathbb{E}\left[ \mathbb{P}\left\{ \beta_{S_0} \not\in \text{CI}_{S_0}^{(\delta e^{-n})} \mid \hat{S}_0 \right\} \right] + \tau + \nu,
\]
where CI_{S_0}^{(\delta e^{-n})} are confidence intervals computed on y and \( \hat{S}_0 \) is an oracle selection independent of y. By the construction of CI_{S_0}^{(\delta e^{-n})}, we know \( \mathbb{P}\left\{ \beta_{S_0} \not\in \text{CI}_{S_0}^{(\delta e^{-n})} \mid \hat{S}_0 \right\} \leq \delta e^{-\eta} \), and therefore
\[
\mathbb{P}\left\{ \hat{\beta}_S \not\in \text{CI}_S^{(\delta e^{-n})} \right\} \leq e^n e^{-\eta} \delta + \tau + \nu = \delta + \tau + \nu.
\]

A.7 Proof of Proposition 1

Denote by \( M_s \) the set of all models of size at most s and fix any \( \tau \in (0, 1) \). Let \( y' \sim \mathcal{P}_y \) be an i.i.d. copy of y. Define the set of bad models to be
\[
M^* = \left\{ M \in M_s : \exists \omega^* \in \text{supp}(\mathcal{P}_y) \text{ such that } \frac{\mathbb{P}\left\{ \hat{M}(\omega^*) = M \right\}}{\mathbb{P}\left\{ \hat{M}(y') = M \right\}} \geq \frac{s_{k=1}^{s_{d/s}}(\frac{d_k}{\tau})}{s}ight\}.
\]

By definition, we see
\[
\mathbb{P}\left\{ \hat{M}(y') \in M^* \right\} \leq \sum_{M \in M^*} \mathbb{P}\left\{ \hat{M}(y') = M \right\} \leq \tau,
\]
which follows by taking a union bound over all \( \sum_{k=1}^{s_{d/s}}(\frac{d_k}{\tau}) \) possible models. Consequently, for any event \( \mathcal{O} \subseteq \mathbb{R}^n \times M_s \) such that \( \{M : \exists \omega \text{ s.t. } (\omega, M) \in \mathcal{O}\} \subseteq M^* \), we have
\[
\mathbb{P}\left\{ (y, \hat{M}(y)) \in \mathcal{O} \right\} \leq \mathbb{P}\left\{ \hat{M}(y) \in M^* \right\} = \mathbb{P}\left\{ \hat{M}(y') \in M^* \right\} \leq \tau.
\]

Now denote \( \mathcal{O}_\omega = \{M \in M_s : (\omega, M) \in \mathcal{O}\}\), and notice that \( \{(y, \hat{M}(y)) \in \mathcal{O} \} = \{\hat{M}(y) \in \mathcal{O}_y\} \). Then, for all \( \mathcal{O} \subseteq \mathbb{R}^n \times M_s \) such that \( \{M : \exists \omega \text{ s.t. } (\omega, M) \in \mathcal{O}\} \cap M^* = \emptyset \), we know
\[
\mathbb{P}\left\{ (y, \hat{M}(y)) \in \mathcal{O} \right\} = \mathbb{P}\left\{ \hat{M}(y) \in \mathcal{O}_y \right\} = \mathbb{E}\left[ \mathbb{P}\left\{ \hat{M}(y) \in \mathcal{O}_y \mid y \right\} \right] = \sum_{k=1}^{s_{d/s}}(\frac{d_k}{\tau}) \mathbb{P}\left\{ (y, \hat{M}(y')) \in \mathcal{O} \right\}.
\]

Finally, take an arbitrary \( \mathcal{O} \subseteq \mathbb{R}^n \times M_s \), and partition it as follows:
\[
\mathcal{O}_{\text{bad}} = \{(\omega, M) : M \in M^*\}, \quad \mathcal{O}_{\text{good}} = \{(\omega, M) : M \notin M^*\}.
\]

Putting everything together, we have shown
\[
\mathbb{P}\left\{ (y, \hat{M}(y)) \in \mathcal{O} \right\} = \mathbb{P}\left\{ (y, \hat{M}(y)) \in \mathcal{O}_{\text{bad}} \right\} + \mathbb{P}\left\{ (y, \hat{M}(y)) \in \mathcal{O}_{\text{good}} \right\}
\leq \tau + \sum_{k=1}^{s_{d/s}}(\frac{d_k}{\tau}) \mathbb{P}\left\{ (y, \hat{M}(y')) \in \mathcal{O} \right\}.
\]

In other words, we can conclude that \( (y, \hat{M}(y)) \approx_{\eta, \tau} (y, \hat{M}(y')) \), with \( \eta = \log \left( \sum_{k=1}^{s_{d/s}}(\frac{d_k}{\tau}) \right) = O(s \log(d/s)) + \log(1/\tau) \), as desired.

Applying the same steps as in Theorem 2 allows us to conclude that CI_{\hat{M}}(K_{\hat{M},\delta e^{-n}}) = \left( \hat{\beta}_J \hat{M} \pm K_{\hat{M},\delta e^{-n}} \delta_J \hat{M} \right),
where \( \eta = O(s \log(d/s)) + \log(1/\tau) \), are valid confidence intervals at level \( \delta + \tau \).
A related argument is given in Theorem 6 of Dwork et al. [20].
A.8 Proof of Lemma

An intermediate result in the proof of Lemma says:

\[ P\{ (y, \hat{S}(y)) \in \mathcal{O}, y \in E \} \leq e^{\eta} P\{ (y, \hat{S}(y_E)) \in \mathcal{O}, y \in E \} + \tau, \]

where we set \( \hat{S}_0 = \hat{S}(y_E) \) by assumption. If we set \( \tau = 0 \) and normalize both sides by \( P\{ y \in E \} \), we get

\[ P\{ (y, \hat{S}(y)) \in \mathcal{O} \mid y \in \mathcal{E} \} \leq e^{\eta} P\{ (y, \hat{S}(y_E)) \in \mathcal{O} \mid y \in \mathcal{E} \}. \quad (7) \]

Now we choose \( \mathcal{O} = \mathcal{O}_S \times \{ \mathcal{S} \} \). The left-hand side above can be rewritten as

\[ P\{ y \in \mathcal{O}_S, \hat{S}(y) = S \mid y \in \mathcal{E} \} = P\{ y \in \mathcal{O}_S \mid \hat{S}(y) = S, y \in \mathcal{E} \} \ P\{ \hat{S}(y) = S \mid y \in \mathcal{E} \}, \]

while the right-hand side is equal to

\[ P\{ y \in \mathcal{O}_S, \hat{S}(y_E) = S \mid y \in \mathcal{E} \} = P\{ \hat{S}(y_E) = S \mid \hat{S}(y) = S, y \in \mathcal{E} \} \ P\{ \hat{S}(y) = S \mid y \in \mathcal{E} \} \]

\[ = P\{ y \in \mathcal{O}_S \mid y \in \mathcal{E} \} \ P\{ \hat{S}(y_E) = S \}, \]

where we use the fact that \( y \) and \( y_E \) are independent.

Since \( P\{ \hat{S}(y_E) = S \} = P\{ \hat{S}(y) = S \mid y \in \mathcal{E} \} \), Eq. (7) is equivalent to

\[ P\{ y \in \mathcal{O}_S \mid \hat{S}(y) = S, y \in \mathcal{E} \} \leq e^{\eta} P\{ y \in \mathcal{O}_S \mid y \in \mathcal{E} \}, \]

which completes the proof.

A.9 Generalization and proof of Claim

First we state a generalization of Claim 5 for general \( P_\mu \). In the following let \( \{ \mathcal{P}(\mu) \}_{\mu \in \mathbb{R}} \) be a location family obtained by translating a zero-mean distribution \( \mathcal{P}_0 \) by \( \mu \). Furthermore, denote by \( q_\delta \) the \( \delta \) quantile of \( |Q|, Q \sim \mathcal{P}_0 \) and by \( \mathcal{P}^\nu_0 \) the distribution \( \mathcal{P}_0 \) restricted to \( (-q_{1-\nu}, q_{1-\nu}) \):

\[ \mathcal{P}^\nu_0(\mathcal{O}) = \frac{\mathcal{P}_0(\mathcal{O} \cap (-q_{1-\nu}, q_{1-\nu}))}{1-\nu}, \]

for all measurable sets \( \mathcal{O} \). Finally, we let \( |\mathcal{P}^\nu_0| \) be the distribution of \( |Q|, Q \sim \mathcal{P}^\nu_0 \).

**Claim 6.** Let \( y \sim \mathcal{P}_0 = \mathcal{P}(\mu) \) for some \( \mu \in \mathbb{R} \). Suppose that we apply the selection criterion to \( y + \xi \), where \( \xi \sim \text{Lap}(b) \) for some user-chosen parameter \( b > 0 \); that is, we report the confidence interval on the event report\((y + \xi)\). Then, for any user-chosen parameter \( \nu \in (0, 1) \), we have

\[ P\{ \mu \not\in (y \pm q_{1-\alpha(1-\nu)}b) \mid \text{report}(y + \xi), y \in \mathcal{E} \} \leq \alpha, \]

where

\[ \eta = \frac{q_{1-\nu}}{b} - \log \text{MGF}_{|\mathcal{P}^\nu_0|}\left(\frac{1}{b}\right), \]

MGF_{|\mathcal{P}^\nu_0|} is the moment-generating function of \( |\mathcal{P}^\nu_0| \) and \( \mathcal{E} \) is an event such that \( P\{ y \in \mathcal{E} \} \geq 1-\nu. \)
Proof. To prove this, first let \( E = \{ \omega \in \mathbb{R}^n : |\omega - \mu| \leq q_1 - \nu \} \) and note that \( \mathbb{P}\{y \in E\} \geq 1 - \nu \). Denote by \( p_Q(a) \) the density of a random variable \( Q \) at \( a \). Letting \( Z \sim P'_0 \), we have for all \( \omega \in E \),

\[
\frac{p_{\omega + \xi}(a)}{p_{\mu + Z + \xi}(a)} = \frac{\frac{1}{2\pi} \exp \left(-\frac{\omega - \mu}{b}\right)}{\int_{-\infty}^{\infty} \frac{1}{2\pi} \exp \left(-\frac{u - \mu}{b}\right) p_Z(u) du} \leq \frac{\exp \left(\frac{\omega - \mu}{b}\right)}{\int_{-\infty}^{\infty} \exp \left(-\frac{u - \mu}{b}\right) p_Z(u) du} \leq \frac{\exp \left(\frac{\omega - \mu}{b}\right)}{\int_{-\infty}^{\infty} \exp \left(-\frac{|u|}{b}\right) p_Z(u) du} \leq \frac{\exp \left(\frac{2\omega - \mu}{b}\right)}{\text{MGF}_{|\mathcal{P}'_0|}(\frac{-1}{b})}.
\]

The value of \( \eta \) follows by taking the logarithm of this expression, and this proves that \( \text{report}(\cdot + \xi) \) is \((\eta, 0, \nu)\)-stable with respect to an oracle that reports on the event \( \text{report}(\mu + Z + \xi) \). The final claim follows by invoking Lemma 2.

In Claim 5, we instantiate the value of \( \eta \) when \( P_0 = \mathcal{N}(0, \sigma^2) \). In that case, the moment-generating function of the truncated normal \( |\mathcal{P}'_0| \) is equal to

\[
\text{MGF}_{|\mathcal{P}'_0|}(\frac{-1}{b}) = \frac{2}{1-\nu} e^{\frac{2\sigma^2}{b^2}} \left( \Phi \left( \frac{z_1 - \nu/2 + \sigma}{b} \right) - \Phi \left( \frac{\sigma}{b} \right) \right).
\]

**A.10 Proof of Proposition 2 (LASSO stability)**

For the sake of readability, we denote the squared loss, rescaled by \( \hat{\sigma} \), by \( L(\theta; X, y) := \frac{1}{n\sigma} \|y - X\theta\|_2^2 \); hence, \( \nabla L(\theta; X, y) = \frac{2}{n\sigma} X^\top(y - X\theta) \). Also, we denote by \( S_{C_1} := C_1 \cdot \{\pm e_i\}_{i=1}^d \) the set of \( 2d \) extreme points of the \( \ell_1 \)-ball in \( \mathbb{R}^d \), scaled by the LASSO constraint \( C_1 \). Similarly, we let \( S_{C_1}^d := C_1 \cdot \{e_i\}_{i=1}^d \) denote half of the points in \( S_{C_1} \) that correspond to the extreme points with non-negative coordinates.

Let \( y \sim \mathcal{N}(\mu, \sigma^2 I) \). Fix \( t \in [k] \) and \( \theta \) such that \( \|\theta\|_1 \leq C_1 \). For all \( \phi \in S_{C_1} \), we have

\[
\phi^\top(\nabla L(\theta; X, y) - \nabla L(\theta; X, \mu)) = \phi^\top \left( \frac{2}{n\sigma} X^\top(y - X\theta) - \frac{2}{n\sigma} X^\top(\mu - X\theta) \right) = \frac{2}{n\sigma} \phi^\top X^\top(y - \mu).
\]

Notice that \( \|X\phi\|_2 \leq C_1 \|X\|_{2,\infty} = C_1 \max_{i \in [d]} \|X_i\|_2 \) for all \( \phi \in S_{C_1} \). By a union bound, we can write:

\[
\mathbb{P}\left\{ \frac{2}{n\sigma} \max_{\phi \in S_{C_1}} |\phi^\top X^\top(y - \mu)| \geq s \right\} = \mathbb{P}\left\{ \frac{2}{n\sigma} \max_{\phi \in S_{C_1}^d} |\phi^\top X^\top(y - \mu)| \geq s \right\} \leq \sum_{\phi \in S_{C_1}^d} \mathbb{P}\left\{ \frac{2}{n\sigma} |\phi^\top X^\top(y - \mu)| \geq s \right\}.
\]
Since $\frac{2}{n\sigma} \theta^T X^\top (y - \mu)$ follows a rescaled $t$-distribution with $r$ degrees of freedom and there are $d$ terms in the sum on the right-hand side, for $s = s^* := \frac{2r-1-t/(2d)}{n\sigma} \|X\|_{2,\infty}$, the probability above is at most $\delta$. Denote $Y_3 = \{\omega : \max_{\phi \in S_{C_1}} \frac{2}{n\sigma} \phi^T X^\top (\omega - \mu) \leq s\}$; we have thus shown $\Pr\{y \in Y_3\} \geq 1 - \delta$.

We now show that, whenever $y \in Y_3$, stable LASSO with input $y$ is indistinguishable from stable LASSO with input $\mu$. From here on, we fix $y \in Y_3$ and only consider the randomness of the algorithm.

The output of Algorithm 2 can be written as a function of $\theta_1, \ldots, \theta_{k+1}$, and hence proving that $(\theta_1, \ldots, \theta_{k+1})$ is indistinguishable when computed on $y$ and $\mu$ is sufficient to argue that $\hat{\theta}^\text{LASSO}$ is indistinguishable on the two inputs, by the post-processing property.

For all $t \leq k$, we can write $\theta_{t+1} = g_t(\theta_t, y)$ for some randomized function $g_t$; in Algorithm 5 we express $g_t$ as an algorithm. If we show $g_t(\theta, y) \approx_{\eta,0} g_t(\theta, \mu)$ for every fixed $\theta$ such that $\|\theta\|_1 \leq C_1$, then we can apply Lemma 3 to conclude indistinguishability of the whole sequence $(\theta_1, \ldots, \theta_{k+1})$.

**Algorithm 5** The $g_t$ subroutine of the stable LASSO algorithm

**input:** $\theta_t, y$

**output:** $\theta_{t+1}$

$\forall \phi \in C_1 \cdot \{\pm e_i\}^{d}_{i=1}$, sample $\xi_{t, \phi} \sim \text{Lap}\left(\frac{4t+1-\frac{t}{(2d)}C_1\|X\|_{2,\infty}}{n\eta}\right)$

$\forall \phi \in C_1 \cdot \{\pm e_i\}^{d}_{i=1}$, let $\alpha_{\phi} = -\frac{2}{n\sigma} \phi^T X^\top (y - X\theta_t) + \xi_{t, \phi}$

Set $\phi_t = \arg\min_{\phi \in S_{C_1}} \alpha_{\phi}$

Set $\theta_{t+1} = (1 - \Delta_t) \theta_t + \Delta_t \phi_t$, where $\Delta_t = \frac{2}{t+1}$

Return $\theta_{t+1}$

Let $\phi_t$ and $\phi_t^\mu$ denote the minimizers of $\alpha_{\phi}$ when the input is $y$ and $\mu$, respectively, and fix an arbitrary point $\phi^* \in S_{C_1}$. Let $\{\xi_{t, \phi}\}_{\phi \in S_{C_1}}$ be independent samples from $\text{Lap}\left(\frac{2s^*}{n}\right)$. Denote

$$\xi^* = \arg\max_{\xi} \nabla L(\theta; X, y)^\top \phi^* + \xi \leq \nabla L(\theta; X, y)^\top \phi + \xi, \forall \phi \in S_{C_1} \setminus \{\phi^*\}.$$ 

Conditional on $\xi_{t, \phi}$, $\phi \in S_{C_1} \setminus \{\phi^*\}$, we get $\phi_t = \phi^*$ if and only if $\xi_{t, \phi}^* \leq \xi^*$.

By the definition of $Y_3$, we have:

$$(\phi_t)^\top \nabla L(\theta; X, \mu) - s^* + \xi^* \leq (\phi_t^\mu)^\top \nabla L(\theta; X, y) + \xi^*$$

$$\leq \phi_t^\top \nabla L(\theta; X, y) + \xi_{t, \phi} \leq \phi^\top \nabla L(\theta; X, \mu) + s^* + \xi_{t, \phi},$$

for all $\phi \in S_{C_1} \setminus \{\phi^*\}$. As a result, conditional on $\xi_{t, \phi}$, $\phi \in S_{C_1} \setminus \{\phi^*\}$, the event $\xi_{t, \phi}^* \leq \xi^* - 2s^*$ implies $\phi_t^\mu = \phi^*$. Thus, we get:

$$\Pr\{\phi_t^\mu = \phi^* \mid \xi_{t, \phi}, \phi \in S_{C_1} \setminus \{\phi^*\}\} \geq \Pr\{\xi_{t, \phi} \leq \xi^* - 2s^* \mid \xi_{t, \phi}, \phi \in S_{C_1} \setminus \{\phi^*\}\}$$

$$\geq e^{-\eta} \Pr\{\xi_{t, \phi} \leq \xi^* \mid \xi_{t, \phi}, \phi \in S_{C_1} \setminus \{\phi^*\}\}$$

$$= e^{-\eta} \Pr\{\phi_t = \phi^* \mid \xi_{t, \phi}, \phi \in S_{C_1} \setminus \{\phi^*\}\}.$$ 

Applying an expectation to both sides yields

$$\Pr\{\phi_t = \phi^*\} \leq e^{\eta} \Pr\{\phi_t^\mu = \phi^*\},$$

and this is true for all $\phi^* \in S_{C_1}$. Therefore, for all $y \in Y_3$, $\phi_t \approx_{\eta,0} \phi_t^\mu$. By post-processing, this also implies $g_t(\theta, y) \approx_{\eta,0} g_t(\theta, \mu)$, for all $\theta$.

By Lemma 3, we finally conclude that, for all $y \in Y_3$, the output of the stable LASSO algorithm when applied to $y$ is $(\frac{1}{2}k_n^2 + \sqrt{2k} \log(1/\delta) \eta, \delta)$-indistinguishable from the output implied by the oracle input $\mu$, for all $\delta \in (0, 1)$, or alternatively it is $(k_n, 0)$-indistinguishable. Since this holds with $1 - \delta$ probability over the choice of $y$, we see that Algorithm 2 is stable with the desired parameters.

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A.11 Proof of Proposition 3 (LASSO utility)

As in the proof of Proposition 2, we denote the rescaled squared loss by \( L(\theta; X, y) := \frac{1}{2n} \| y - X\theta \|_2^2 \), and by \( S_{C_1} := C_1 \cdot \{ \pm \epsilon_i \}_{i=1}^d \) we denote the set of 2d extreme points of the \( \ell_1 \)-ball in \( \mathbb{R}^d \), scaled by the constraint \( C_1 \).

We begin by stating a convergence result for the Frank-Wolfe algorithm due to Jaggi [27], which forms the core of our analysis.

Lemma 5 ([27]). Fix \( s > 0 \) and \( \theta_1 \in D \subseteq \mathbb{R}^d \). Let \( (\phi_1, \ldots, \phi_k) \) be a sequence of vectors from \( D \) and let \( \theta_{t+1} = (1 - \Delta_t)\theta_t + \Delta_t \phi_t \), for arbitrary \( \Delta_t \in [0, 1] \). Define the curvature constant of \( L \) as

\[
C_L := \sup_{\theta_1, \theta_2 \in D, \gamma \in [0, 1], \theta_3 = (1 - \gamma)\theta_1 + \gamma \theta_2} \frac{2}{\gamma^2}(L(\theta_3) - L(\theta_1) - (\theta_3 - \theta_1)^\top \nabla L(\theta_1)).
\]

Suppose that for all \( t \in [k] \), it holds that: \( \phi_t^\top \nabla L(\theta_t) \leq \min_{\phi \in S_{C_1}} \phi^\top \nabla L(\theta_t) + \frac{s\Delta_t C_L}{2} \). Then,

\[
L(\theta_{k+1}) - \min_{\theta : \|\theta\|_1 \leq C_1} L(\theta) \leq \frac{2C_L}{k + 2} \left( 1 + s \right).
\]

Denote by \( b := \frac{4t_{r,1} - s/2)1 \|X\|_2 \infty nq}{nq} \) the parameter of the Laplace noise in Algorithm 2. Fix \( s > 0 \). Denoting by \( C_L \) the curvature constant of \( L \), as defined in Lemma 5, and by applying subexponential concentration of the Laplace distribution, we know:

\[
P\left\{ \exists t \in [k] : \phi_t^\top \nabla L(\theta_t; X, y) > \min_{\phi \in S_{C_1}} \phi^\top \nabla L(\theta_t; X, y) + \frac{s\Delta_t C_L}{2} \right\}
\]

\[
\leq P\left\{ \exists t \in [k] : \max_{\phi \in S_{C_1}} |\xi_t,\phi| > \frac{s\Delta_t C_L}{4} \right\}
\]

\[
\leq P\left\{ \max_{t \in [k], \phi \in S_{C_1}} |\xi_t,\phi| > \frac{s\Delta_t C_L}{4} \right\}
\]

\[
\leq k|S_{C_1}| \exp\left( -\frac{s\Delta_t C_L}{4b} \right),
\]

where the last step follows by a union bound. Setting \( s = \frac{4b}{|S_{C_1}|} \log(k|S_{C_1}|/\zeta) \) controls this probability to be at most \( \zeta \).

We use a standard fact from convex geometry: for any set \( S_D \) such that its convex hull is equal to \( D \), it holds that \( \min_{\phi \in S_D} \phi^\top \nabla L(\theta_t; X, y) = \min_{\phi \in S_D} \phi^\top \nabla L(\theta_t; X, y) \). In our setting, \( D = \{ \theta : \|\theta\|_1 \leq C_1 \} \), and it can be obtained as the convex hull of \( S_{C_1} \).

With this, we can apply Lemma 5 as well as the fact that \( |S_{C_1}| = 2d \), to get that with probability \( 1 - \zeta \) over the Laplace noise variables:

\[
L(\theta_{k+1}; X, y) - \min_{\theta : \|\theta\|_1 \leq C_1} L(\theta; X, y) \leq \frac{2C_L}{k + 2} + \frac{8C_L b \log(2kd/\zeta)}{(k + 2)\Delta_k C_L}.
\]

By the curvature characterization for quadratics due to Clarkson [14], we can bound the curvature constant as

\[
C_L \leq \frac{1}{n\sigma} \max_{\theta, \theta' : \|\theta\|_1 \leq C_1, \|\theta'\|_1 \leq C_1} \| X(\theta - \theta') \|_2^2 \leq \frac{1}{n\sigma} \max_{\varphi : \|\varphi\|_1 \leq 2C_1} \| X\varphi \|_2^2 \leq \frac{4}{\sigma} \| X \|_{\infty}^2 C_1^2.
\]

Therefore, we can conclude

\[
L(\theta_{k+1}; X, y) - \min_{\theta : \|\theta\|_1 \leq C_1} L(\theta; X, y) \leq \frac{8\| X \|_{\infty}^2 C_1^2}{\sigma(k + 2)} + 4b \log(2kd/\zeta).
\]
Further, notice that for all $\theta, \theta'$ such that $\max(\|\theta\|_1, \|\theta\|_1) \le C_1$, by Hölder’s inequality we have:

$$|L(\theta; X, y) - L(\theta'; X, y)| = \left| \frac{1}{n\sigma} \|y - X\theta\|_2^2 - \frac{1}{n\sigma} \|y - X\theta'\|_2^2 \right|$$

$$\le \frac{2}{\sigma} \|X\|_\infty (\|X\|_\infty C_1 + \|y\|_\infty) \|\theta' - \theta\|_1 := L_1 \|\theta' - \theta\|_1 \le 2L_1 C_1,$$

where by $L_1$ we denote the $\ell_1$-Lipschitz constant of the squared loss restricted to the LASSO domain. Now we pick $\zeta = \frac{\gamma}{2C_1 L_1}$ for some constant $\gamma > 0$, which gives:

$$E[L(\theta_{k+1}; X, y) | y, \hat{\sigma}] - \min_{\theta : \|\theta\|_1 \le C_1} L(\theta; X, y) \le \gamma + \frac{8\|X\|_\infty^2 C_1}{\sigma(k + 2)} + 4b \log(4kdC_1 L_1 / \gamma)$$

$$= \gamma + \frac{8\|X\|_\infty^2 C_1^2}{\sigma(k + 2)} + \frac{16t_{r,1-\delta/(2d)} C_1 \|X\|_2,\infty \log(4kdC_1 L_1 / \gamma)}{n\eta},$$

where in the last step we use the noise level from Algorithm 2. Now we set $k = \left[ \frac{n\|X\|_2,\infty \gamma}{\sigma \|X\|_2,\infty} \right]$, and get the following utility upper bound:

$$E[L(\theta_{k+1}; X, y) | y, \hat{\sigma}] - \min_{\theta : \|\theta\|_1 \le C_1} L(\theta; X, y) \le \gamma + \frac{8C_1 \|X\|_2,\infty}{n\eta} + \frac{16t_{r,1-\delta/(2d)} C_1 \|X\|_2,\infty \log(4kdC_1 L_1 / \gamma)}{n\eta}.$$

Note that the above inequality is true for all $\gamma > 0$. After optimizing over $\gamma$, the right-hand side reduces to

$$\frac{8C_1 \|X\|_2,\infty}{n\eta} + \frac{16t_{r,1-\delta/(2d)} C_1 \|X\|_2,\infty (1 + \log(kdL_1 n\eta / (4t_{r,1-\delta/(2d)} \|X\|_2,\infty)))}{n\eta}.$$
Now we appeal to a similar composition argument as in Proposition 2. From here on, fix $y \in Y_d$. We will show that the output of stable marginal screening, when applied to $y$, is indistinguishable from the output of stable marginal screening given the oracle input $\mu$.

The selected model $M$ can be written as the output of a composition of $k$ functions $g_t(i_1, \ldots, i_{t-1}, y)$, $t \in [k]$. In particular, the feature “peeled off” at time $t$, $i_t$, is equal to $g_t(i_1, \ldots, i_{t-1}, y)$. We show that $g_t(i_1, \ldots, i_{t-1}, y) \approx_{\eta,0} g_t(i_1, \ldots, i_{t-1}, \mu)$ holds true for all fixed $i_1, \ldots, i_{t-1}$. By Lemma 3 that will imply that the overall selected model under input $y$ and under input $\mu$ is indistinguishable as well.

Fix a round $t \in [k]$, as well as an index $i \in \text{res}_t$. Suppose that we add independent draws $\xi_{t,j} \sim \text{Lap}\left(\frac{2t^{r,1-\delta/(2d)}\|X\|_2}{n^2}\right)$ to each value $c_j$, where $j \in \text{res}_t$. Define

$$
\xi^*_+ = \arg\min_{\xi \geq -c^*_j} c^*_j + \xi > |c^*_j| + \xi_{t,j}, \quad \xi^- = \arg\max_{\xi < -c^*_j} -c^*_j - \xi > |c^*_j| + \xi_{t,j}, \quad \forall j \neq i.
$$

Then, $g_t(i_1, \ldots, i_{t-1}, y) = i$ if and only if $\xi_{t,i} \geq \xi^*_+$ or $\xi_{t,i} \leq \xi^-$. Moreover, since $y \in Y_d$, we have

$$
\frac{t^{r,1-\delta/(2d)}}{n} \|X\|_2 + \xi^*_+ \geq c^*_j + \xi_{t,j}, \quad \frac{t^{r,1-\delta/(2d)}}{n} \|X\|_2 - \xi^- \geq c^*_j + \xi_{t,j}.
$$

Rearranging the terms, we get

$$
\frac{2t^{r,1-\delta/(2d)}}{n} \|X\|_2 + c^*_j + \xi^*_+ \geq |c^*_j| + \xi_{t,j}, \quad \frac{2t^{r,1-\delta/(2d)}}{n} \|X\|_2 - c^*_j - \xi^- \geq |c^*_j| + \xi_{t,j}.
$$

Thus, if $\xi_{t,i} \geq \xi^*_+ + \frac{2t^{r,1-\delta/(2d)}}{n} \|X\|_2$ or $\xi_{t,i} \leq \xi^- - \frac{2t^{r,1-\delta/(2d)}}{n} \|X\|_2$, then $i = g_t(i_1, \ldots, i_{t-1}, \mu)$ if the noise levels are $(\xi_{t,1}, \ldots, \xi_{t,i}, \ldots, \xi_{t,d})$. Finally, for fixed $y \in Y_d$, we have

$$
P\{g_t(i_1, \ldots, i_{t-1}, \mu) = i \mid \{\xi_{t,j}\}_{j \neq i}\} \geq P\left\{\xi_{t,i} \geq \xi^*_+ + \frac{2t^{r,1-\delta/(2d)}}{n} \|X\|_2 \mid \{\xi_{t,j}\}_{j \neq i}\right\} + P\left\{\xi_{t,i} \leq \xi^- - \frac{2t^{r,1-\delta/(2d)}}{n} \|X\|_2 \mid \{\xi_{t,j}\}_{j \neq i}\right\} \geq e^{-\eta n} P\{\xi_{t,i} \geq \xi^*_+ \mid \{\xi_{t,j}\}_{j \neq i}\} + e^{-\eta n} P\{\xi_{t,i} \leq \xi^- \mid \{\xi_{t,j}\}_{j \neq i}\}.
$$

Multiplying by $e^\eta$ and applying the law of iterated expectations completes the proof that $g_t(i_1, \ldots, i_{t-1}, y) \approx_{\eta,0} g_t(i_1, \ldots, i_{t-1}, \mu)$ for all $y \in Y_d$.

Finally, by Lemma 3 we conclude that for all fixed $y \in Y_d$, the output of stable marginal screening under input $y$ and under the oracle input $\mu$ is $(\frac{1}{2}k\eta^2 + \sqrt{2k\log(1/\delta)}\eta, \delta)$-indistinguishable for all $\delta \in (0, 1)$, or alternatively $(k\eta, 0)$-indistinguishable. Since this holds with $1 - \delta$ probability over the choice of $y$, we see that stable marginal screening satisfies stability with the desired parameters.

### A.13 Proof of Proposition 5 (marginal screening utility)

Fix $s > 0$. Taking a union bound, we get:

$$
P\left\{\max_{j \in [k]} |c_{m_j} - c_{i_j}| \geq s \mid y\right\} \leq \sum_{j=1}^k P\{c_{m_j} - c_{i_j} \geq s \mid y\}.
$$

At the time when $i_j$ is chosen, exactly $j - 1$ items have been selected; therefore, at least one of $m_1, \ldots, m_j$ has still not been selected. The event that $|c_{m_j} - c_{i_j}| \geq s$ implies that $i_j$ “beat” one of $m_1, \ldots, m_j$, so
which further implies that $\max_{i \in [d]} |\xi_{j,i}| \geq \frac{s}{2}$. By a union bound, this happens with probability at most $d \exp\left(-sn\eta/(4t_{r,1-\delta/(2d)} \|X\|_{2,\infty})\right)$. Putting everything together, we get

$$\sum_{j=1}^k P\{|c_{m_j}| - |c_{i_j}| \geq s \mid y\} \leq kd \exp\left(-\frac{sn\eta}{4t_{r,1-\delta/(2d)} \|X\|_{2,\infty}}\right).$$

Plugging in $s = \frac{4t_{r,1-\delta/(2d)} \log(dk/\delta') \|X\|_{2,\infty}}{n \eta}$ completes the proof.

## B Deferred numerical results and simulation details

**Deferred plots.** We include plots of confidence interval widths, which were deferred from Section 9.

Figure 12. Plots of average confidence interval widths of stable LASSO, with varying dimension and signal strength, in the Gaussian design case.

Figure 13. Plots of average confidence interval widths of stable LASSO, with varying sample size, in the Gaussian design case.
Figure 14. Plots of average confidence interval widths of stable marginal screening and exact marginal screening with a conditional correction, with varying dimension and signal strength, in the Gaussian design case. We also plot the 90% quantile of the conditional width because it varies greatly across realizations.

For the Gaussian design case, in Figure 12 we plot the widths corresponding to the experiments in Figure 4; in Figure 13 we plot the widths corresponding to the experiments in Figure 5; in Figure 14 we plot the widths corresponding to the experiments in Figure 6; in Figure 15 we plot the widths corresponding to the experiments in Figure 7.

For the Bernoulli design case, in Figure 16 we plot the widths corresponding to the experiments in Figure 8; in Figure 17 we plot the widths corresponding to the experiments in Figure 9; in Figure 18 we plot the widths corresponding to the experiments in Figure 10; in Figure 19 we plot the widths corresponding to the experiments in Figure 11. In Figure 18 and Figure 19 since the conditional widths are of a higher order of magnitude, the scale on the y-axis in the widths plots is logarithmic.

Figure 15. Plots of average confidence interval widths of stable marginal screening and exact marginal screening with a conditional correction, with varying sample size, in the Gaussian design case. In addition, we plot the 90% quantile of the conditional width because it varies greatly across realizations.
Figure 16. Plots of average confidence interval widths of stable LASSO, with varying dimension and signal strength, in the Bernoulli design case.

Experiments with Laplace errors. We include additional experiments when the errors have an exponential, rather than Gaussian, tail: the outcome is generated as $y = X\beta + \epsilon$, where $\epsilon_i \sim \text{Lap}\left(\frac{1}{\sqrt{2}}\right), i \in [n]$. The Laplace parameter is chosen so that the errors have variance equal to one.

For the Gaussian design case, Figure 20 shows plots analogous to those in Figure 4 and Figure 5 in this setting, and Figure 21 is the analogue of Figure 6 and Figure 7. Similarly, for the Bernoulli design case, Figure 22 is analogous to Figure 8 and Figure 9, and Figure 23 is analogous to Figure 10 and Figure 11.

Simulation details. All plots of FDR and average error are averaged over 10000 trials. Width plots are averaged over 500 trials because the stability correction implies interval widths of low variability. In each trial we generate a new triplet $(X, \beta, y)$, and run stable LASSO (respectively, marginal screening) on this data for all $\eta$, as well as LASSO (respectively, marginal screening) with data splitting. For both

Figure 17. Plots of average confidence interval widths of stable LASSO, with varying sample size, in the Bernoulli design case.
stability and data splitting we take a Bonferroni correction to provide simultaneous coverage over all selected variables. Moreover, we note that $\eta$ in the plots refers to the noise parameter in Algorithm 2 and Algorithm 3; the actual stability parameter used to compute the corresponding splitting fraction for data splitting is multiplied by the number of steps of composition, i.e. we use $f(k\eta)$ as the splitting fraction. Finally, the $\delta$ parameter in Algorithm 2 and Algorithm 3 is set to $\delta = 0.5$ throughout.

Figure 18. Plots of average confidence interval widths of stable marginal screening and exact marginal screening with a conditional correction, with varying dimension and signal strength, in the Bernoulli design case. We also plot the 90% quantile of the conditional width because it varies greatly across realizations.

Figure 19. Plots of average confidence interval widths of stable marginal screening and exact marginal screening with a conditional correction, with varying sample size, in the Bernoulli design case. We also plot the 90% quantile of the conditional width because it varies greatly across realizations.
Figure 20. Comparison of FDR after stable LASSO and LASSO with data splitting, with varying dimension and signal strength (first three columns) and sample size (last column), in the Gaussian design case. The errors are sampled from a Laplace distribution.

Figure 21. Comparison of average error after stable marginal screening and marginal screening with data splitting, with varying dimension and signal strength (first three columns) and sample size (last column), in the Gaussian design case. The errors are sampled from a Laplace distribution.
Figure 22. Comparison of FDR after stable LASSO and LASSO with data splitting, with varying dimension and signal strength (first three columns) and sample size (last column), in the Bernoulli design case. The errors are sampled from a Laplace distribution.

Figure 23. Comparison of FDR after stable marginal screening and marginal screening with data splitting, with varying dimension and signal strength (first three columns) and sample size (last column), in the Bernoulli design case. The errors are sampled from a Laplace distribution.

C Generalizations of stable algorithms

We show how our stable algorithms can be generalized beyond the setting of Gaussianity. Our proofs exploited Gaussianity of the outcome vector only in terms of the decay of its tails. In general we only need
to know how tightly \( y \) concentrates around \( \mu \) in order to reproduce the stable versions of the LASSO and marginal screening. To show this, we generalize our approach to all outcome vectors with bounded Orlicz norm. This includes other important cases such as general subgaussian and subexponential vectors \( y \).

**Definition 3** (Orlicz norm). A function \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) is an Orlicz function if \( \psi \) is convex, non-decreasing, and satisfies \( \psi(0) = 0 \), \( \psi(x) \to \infty \) as \( x \to \infty \). For an Orlicz function \( \psi \), the Orlicz norm or \( \psi \)-norm of a random variable \( W \) is defined as

\[
\|W\|_{\psi} = \inf \left\{ s > 0 : \mathbb{E} \left[ \frac{|W|}{s} \right] \leq 1 \right\}.
\]

This definition immediately implies a tail bound by Markov’s inequality:

\[
P\{|W| \geq s\|W\|_{\psi}\} \leq \frac{\mathbb{E} \left[ \psi \left( \frac{|W|}{\|W\|_{\psi}} \right) \right]}{\psi(s)} \leq \frac{1}{s}.
\]  

**A natural extension of Definition 3 to random vectors is to consider all one-dimensional projections.**

**Definition 4** (Orlicz norm in \( \mathbb{R}^n \)). For a random vector \( W \in \mathbb{R}^n \) and Orlicz function \( \psi \), we define the Orlicz norm of \( W \) as

\[
\|W\|_{\psi} = \inf \left\{ s > 0 : \sup_{v \in \mathbb{R}^n: \|v\|_2 \leq 1} \|W^Tv\|_{\psi} \leq s \right\}.
\]

We state a corollary of Theorem 2 where we construct confidence intervals for stable selection methods as long as the outcomes have bounded \( \psi \)-norm, for some Orlicz function \( \psi \).

**Corollary 3.** Suppose \( \|y - \mu\|_{\psi} \leq G \), for some known \( G > 0 \), and fix \( \delta \in (0,1) \). Let \( \hat{M} \) be an \((\eta, \tau, \nu)\)-stable model selection algorithm. For all \( j \in \hat{M} \), let

\[
\text{CI}_{j,\hat{M}} = \left( \hat{\beta}_{j,\hat{M}} \pm \psi^{-1}(\|M|e^n/\delta) \sqrt{G \|((X^T_MX_M)^{-1})_{jj}\|} \right).
\]

Then

\[
P\left\{ \exists j \in \hat{M} : \beta_{j,M} \not\in \text{CI}_{j,\hat{M}} \right\} \leq \delta + \tau + \nu.
\]

**Proof.** Fix a model \( M \). We only need to argue that

\[
P\left\{ \max_{j \in M} \left| \frac{\hat{\beta}_{j,M} - \beta_{j,M}}{G \sqrt{((X^T_MX_M)^{-1})_{jj}}} \right| \geq \psi^{-1}(\|M|e^n/\delta) \right\} \leq \delta e^{-\eta}.
\]

Invoking Theorem 2 then completes the proof.

Denote by \( X_{j,M} \) the residual vector when \( X_j \) is regressed onto all other variables in \( M \); that is, \( X_{j,M} = P_{X_{M\setminus j}}^T X_j \), where \( P_{X_{M\setminus j}}^T \) denotes the projection matrix onto the orthocomplement of \( X_{M\setminus j} \). With this notation, we can express the least-squares solution as

\[
\hat{\beta}_{j,M} = \frac{X_{j,M}^Ty}{\|X_{j,M}\|_2^2}, \quad \beta_{j,M} = \frac{X_{j,M}^T\mu}{\|X_{j,M}\|_2^2}.
\]

Moreover, \( (X_{j,M}^T X_{j,M})_{jj}^{-1} = \frac{1}{\|X_{j,M}\|_2^2} \). Using this fact, we have

\[
P\left\{ \max_{j \in M} \left| \frac{\hat{\beta}_{j,M} - \beta_{j,M}}{G \sqrt{((X^T_MX_M)^{-1})_{jj}}} \right| \geq \psi^{-1}(\|M|e^n/\delta) \right\}
\]

\[
= P\left\{ \max_{j \in M} |v_{j,M}^T(y - \mu)| \geq G\psi^{-1}(\|M|e^n/\delta) \right\},
\]

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where we define $v_{j,M}$ to be the random unit vector $\frac{X_{j,M}}{\|X_{j,M}\|_2}$. By applying the tail bound given by Eq. (8) together with a union bound, we get
\[
P\left\{ \max_{j \in M} |v_{j,M}^\top (y - \mu)| \geq G \psi^{-1} (|M| e^\eta / \delta) \right\} \leq \sum_{j \in M} P\left\{ |v_{j,M}^\top (y - \mu)| \geq \psi^{-1} (|M| e^\eta / \delta) \right\} \\
\leq |M| \left( \psi \left( \psi^{-1} (|M| e^\eta / \delta) \right) \right)^{-1} \\
= \delta e^{-\eta}.
\]

Now we can generalize the stable versions of the LASSO and marginal screening. We state counterparts of Algorithm 2 and Algorithm 3 which ensure stability as long as we know a bound $G$ on $\|y - \mu\|_\psi$.

**Algorithm 6** Stable LASSO algorithm under general Orlicz norm $\psi$

**input:** design matrix $X \in \mathbb{R}^{n \times d}$, outcome vector $y \in \mathbb{R}^n$, $\ell_1$-constraint $C_1$, number of optimization steps $k$, typical stability parameters $\delta \in (0, 1), \eta > 0$

**output:** LASSO solution $\hat{\theta}_{LASSO} \in \mathbb{R}^d$

Initialize $\theta_1 = 0$

for $t = 1, 2, \ldots, k$ do

\[ \forall \phi \in C_1 \cdot \{ \pm e_i \}_{i = 1}^d, \text{ sample } \xi_{t,\phi} \sim \text{Lap} \left( \frac{4 \psi^{-1} (1/\delta) C_1 \|X\|_2 \infty G}{\eta} \right) \]

\[ \forall \phi \in C_1 \cdot \{ \pm e_i \}_{i = 1}^d, \text{ let } \alpha_\phi = \frac{2}{n} \phi^\top X^\top (y - X \theta_t) + \xi_{t,\phi} \]

Set $\phi_t = \arg \min_{\phi \in C_1 \cdot \{ \pm e_i \}_{i = 1}^d} \alpha_\phi$

Set $\theta_{t+1} = (1 - \Delta_t) \theta_t + \Delta_t \phi_t$, where $\Delta_t = \frac{2}{\eta^2}$

end

Return $\hat{\theta}_{LASSO} = \theta_{k+1}$

**Algorithm 7** Stable marginal screening algorithm under general Orlicz norm $\psi$

**input:** design matrix $X \in \mathbb{R}^{n \times d}$, outcome vector $y \in \mathbb{R}^n$, model size $k$

**output:** $\hat{M} = \{i_1, \ldots, i_k\}$

Compute $(c_1, \ldots, c_d) = \frac{1}{d} X^\top y \in \mathbb{R}^d$

res$_1 = [d]$

for $t = 1, 2, \ldots, k$ do

\[ \forall i \in \text{res}_t, \text{ sample } \xi_{t,i} \sim \text{Lap} \left( \frac{4 \psi^{-1} (1/\delta) \|X\|_2 \infty G}{\eta} \right) \]

\[ i_t = \arg \max_{i \in \text{res}_t} |c_i + \xi_{t,i}| \]

res$_{t+1} = \text{res}_t \setminus i_t$

end

Return $\hat{M} = \{i_1, \ldots, i_k\}$