Deep Gaussian networks for function approximation on data defined manifolds

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Abstract

In much of the literature on function approximation by deep networks, the function is assumed to be defined on some known domain, such as a cube or sphere. In practice, the data might not be dense on these domains, and therefore, the approximation theory results are observed to be too conservative. In manifold learning, one assumes instead that the data is sampled from an unknown manifold; i.e., the manifold is defined by the data itself. Function approximation on this unknown manifold is then a two stage procedure: first, one approximates the Laplace-Beltrami operator (and its eigen-decomposition) on this manifold using a graph Laplacian, and next, approximates the target function using the eigen-functions.

In this paper, we propose a more direct approach to function approximation on unknown, data defined manifolds without computing the eigen-decomposition of some operator, and estimate the degree of approximation in terms of the manifold dimension. This leads to similar results in function approximation using deep networks where each channel evaluates a Gaussian network on a possibly unknown manifold.

1 Introduction

The main problem of machine learning is the following. Given data \{ (x_j,f(x_j) + \epsilon_j) \}_{j=1}^M, where f is an unknown function, x_j’s are sampled randomly from a probability distribution \mu, and \epsilon_j’s are realizations of a mean zero random variable, find an approximation to f. The approximations are usually neural networks or radial basis function networks or some other classes based on some kernels. An important question is then to estimate the degree of approximation to f in terms of the model complexity in a suitable norm. Typically, x_j \in \mathbb{R}^Q for a large Q, and one assumes some “prior” on f, such as that it belongs to some reproducing kernel Hilbert space with a known kernel, or that it has a certain number of derivatives, or that it satisfies some conditions on its Fourier transform. Following the practice in classical approximation theory, we will refer to the prior as the smoothness class to which f is supposed to belong. The only information one is allowed to assume is this prior on f.

A vexing problem in the theory is the so called curse of dimensionality. In approximation theory parlance, this is the nonlinear width of the smoothness class. If the only information about f is that it has a certain number of derivatives, then the number of parameters required to approximate f in a robust way up to an accuracy of \epsilon is at least cc^{-Q/r}; i.e., grows exponentially in Q. However, this estimate is sometimes too conservative to be applicable directly. For example, if the data is 784 dimensional (e.g., the MNIST data), and we assume that the target function is once differentiable, then the suggested number of neurons is 2^{1568} to get an accuracy of 1/4, orders of magnitude more than the estimated number of atoms (2^{272}) in the observable universe.

There are several remedies in the literature for this problem. One is to assume a different prior on f. A well known example is the result of Barron [4], where a condition on f is prescribed in terms of its Fourier transform, and it shown that the L^2 error in the approximation by f by neural networks with n neurons is O(1/\sqrt{n}) independently of Q. Similar results are known also in the case when f admits an integral representation of the form f(x) = \int_{\mathbb{R}^Q} K(x,y) d\mu_f(y) for some (signed) measure \mu_f. The smoothness class is described by stipulating that \mu_f should have a bounded variation, and the class of approximants has the form \sum_{k=1}^M a_k K(x,y_k). It is known that the degree of approximation by such kernel based approximation is then typically of the form O(\sqrt{\log n/n}), where also the constants involved in the O are either independent of Q or grow at most polynomially in Q [16 17 22]. Sometimes, this seems to contradict the result about widths, but this is explained by the fact that the parameters a_k, y_k may depend upon f in a discontinuous manner.

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We have observed recently in [28] that if the target function has a compositional structure described by a directed acyclic graph (DAG), then deep networks with the same structure in their architecture overcome the curse of dimensionality. For example, if the target function is
\[ f(x_1, x_2, x_3, x_4) = F(f_1(x_1, x_2), f_2(x_3, x_4)) \]
then we may approximate the bivariate functions \( f_1, f_2, F \) by \( P_1, P_2, P \) respectively, and obtain a “deep network” \( P(P_1(x_1, x_2), P_2(x_3, x_4)) \) to approximate \( f \). Since we are doing only an approximation of bivariate functions at each stage, the curse is only \( \epsilon c^{-2/r} \) rather than \( \epsilon^{-2/r} \) if we were to ignore the compositionality and treat \( f \) as a function of 4 variables. We have called this the “blessing of compositionality”. Clearly, a shallow network cannot be adapted to follow the DAG architecture, and hence, cannot take advantage of this blessing.

It is impossible to say whether a given function has a compositional structure, even if we prescribe the DAG, and whether such a structure is unique. A simple, perhaps funny, example even in the univariate case is the constant function 1 on \([0, 1]\), which can be expressed also as a fairly deep compositional function:
\[ x \mapsto \frac{x + 1}{2} \cosh \left( \log \left( \frac{2 + \sqrt{3 - 2x - x^2}}{2} \right) \right), \quad x \in [0, 1]. \]
It is easy to create many such examples just by taking into account all kinds of identities. Nevertheless, it is reasonable to assume that there is a natural compositional structure in the target function, based on the domain knowledge. Thus, for example, it is argued in [18] that many examples in cosmology as well as image classification can be viewed as causal hierarchies that involve a Markov chain, where the probability distribution \( p(y_i) \) at the \( i \)-th level is determined by its causal successor alone. In these examples, the goal is then to reverse this generative hierarchy, starting with \( y_n = x \). Then Bayes theorem can be applied to view this problem as the problem of determining a compositional function, where each constituent function represents the conditional probability of \( y_i \) given a sufficient statistic for \( y_{i+1} \) in terms of \( x \). Similarly, our visual cortex appears to engage in hierarchical learning with various areas of the cortex receiving only a small number of inputs [2, 15, 30].

One consequence of the blessing of compositionality is that the classical paradigm of measuring the degree of approximation in some \( L^2 \) norm does not work anymore. The inputs \( P_1(x_1, x_2), P_2(x_3, x_4) \) to \( P \) are different from the inputs \( f_1(x_1, x_2), f_2(x_3, x_4) \) to \( F \), and may be distributed differently. So, there is no natural measure to define the \( L^2 \) norm with for the approximation at the next level that reflects the compositional structure. We have therefore proposed to do the approximation in the uniform norm, not as a “poor man’s alternative” as is often done in the literature, but because the \( L^2 \) space cannot be defined in a natural way to take into account the important compositional structure.

There is a deluge of papers on function approximation by deep networks. As far as we are aware, they all deal with function approximation on known domains - interval, cube, sphere, Euclidean space, etc. While this is of theoretical interest, and can even be justified using Stein’s extension theorems, such an extension leads necessarily to non-constructive results.

A different direction to overcome the curse of dimensionality is to assume that the data \( x_j \) is sampled from some unknown manifold, uniformly with respect to the Riemannian volume element of that manifold. There is again a great deal of research in the direction of figuring out the geometry of this unknown manifold, and even in the direction of function approximation on the same. One of the fundamental results in this direction is the following theorem of Belkin and Niyogi [6].

**Theorem 1.1** Let \( X \) be a smooth, compact, \( q \)-dimensional sub-manifold of \( \mathbb{R}^Q \), \( \mu^* \) be its Riemannian volume measure, normalized by \( \mu^*(X) = 1 \), and \( \Delta \) denote the Laplace-Beltrami operator on \( X \). Then for a smooth function \( f : X \to \mathbb{R} \),
\[
\lim_{t \to 0} \frac{1}{t^{(4\pi)^{q/2}}} \int_X \exp \left( -\frac{\|x - y\|_{2,Q}^2}{t} \right) (f(y) - f(x)) \mu^*(y) = \Delta(f)(x) \quad (1.1)
\]
uniformly for \( x \in X \), where \( \| \cdot \|_{2,Q} \) denotes the \( \ell^2 \) norm on \( \mathbb{R}^Q \).

The entire areas of diffusion geometry and spectral clustering seek to estimate the Laplace-Beltrami operator and its eigen-decomposition, typically using a graph Laplacian, and lead to several impressive results both in theory and practice. An early introduction to this topic is in the special issue [7] of Applied and Computational Harmonic Analysis, edited by Chui and Donoho.

From an approximation theory point of view, the theorem is more of a saturation theorem for approximating \( f \) on \( X \), analogous to the Voronowskaja estimates for Bernstein polynomials ([19 Section 1.6.1]). This is to be
expected because the Gaussian kernel involved is a positive operator. Moreover, the curvature of the manifold contributes to the saturation as well. The Gaussian kernel has many advantages, invariance under translations and rotations is one of the them. This plays a major role in the proof of Theorem 1.1. Nevertheless, it is natural to ask whether another kernel can be found that leads directly to the approximation of the target function \( f \) on the manifold from the data without knowing the manifold itself and without having to go through an expensive eigen-decomposition. The curvature of the manifold will still contribute to the saturation, but when the manifold happens to be an affine space, such a construction should lead to approximation without any saturation, without knowing what the affine space is.

The main objective of this paper is to demonstrate such a construction using certain localized kernels based on Hermite polynomials (Theorem 6.1). We will obtain an analogue of Theorem 1.1 to obtain function approximation on an unknown manifold based only on noise-corrupted samples on the manifold, and give estimates on the degree of approximation (Theorem 6.2). In the case when the manifold happens to be an affine space, our construction is free of any saturation, and does not need to know what the affine space is (Theorem 5.1).

To recuperate the advantage of the Gaussian kernel, we will study approximation by Gaussian networks. A (shallow) Gaussian network with \( n \) neurons has the form \( \sum_{k=1}^{n} a_k \exp(-|x-y_k|^2/Q) \). A deep network is constructed following a DAG structure, where each node (referred to as “channel” in the literature on deep learning) evaluates a Gaussian network. Using the close connection between Hermite polynomials and Gaussian networks (cf. [21][23][9]), we can translate the result about approximation on the manifold into a result on approximation by shallow Gaussian networks, where the input is assumed to lie on an unknown low dimensional manifold of the nominal high dimensional ambient space (Theorem 6.4). In turn, using a property called “good propagation of errors” (Theorem 7.1), we will “lift” this theorem to estimate the degree of approximation by deep Gaussian networks, where each channel evaluates a Gaussian network on a similarly manifold-based data (Theorem 7.2). The networks themselves are constructed from certain pre-fabricated networks in the ambient space to approximate the Hermite functions with a correspondingly high number of neurons. However, we will give explicit formula for such networks (Proposition 3.1), so that there is no training required here. The amount of information used in the final synthesis of the network will depend only on the dimension of the manifold on which the input lives. We consider this to be a step in bringing approximation theory of deep networks closer to the practice, so that the results are proved in the setting of approximation on unknown manifolds analogous to diffusion geometry rather than on known domains.

The statement of the main results in this paper mentioned above require a good deal of background information on the theory of weighted polynomial approximation. Accordingly, we start by summarizing the relevant properties of multivariate Hermite polynomials in Section 2. Of particular interest is the Mehler identity (2.2), which has many consequences, described in Section 3. The results on function approximation in Euclidean spaces are described in Section 4. Our main theorem in the context of approximation on unknown affine spaces is stated and proved in Section 5. The main theorems on approximation by weighted polynomials and Gaussian networks on unknown manifolds are stated and proved in Section 6. Section 7 contains a discussion of our viewpoint on deep networks, the statement and proof of the good propagation of errors, and its consequence for deep Gaussian network approximation based on data on unknown manifolds.

2 Hermite polynomials

A good preliminary source of many identities regarding Hermite polynomials is the book [31] of Szegö or the Bateman manuscript [5].

We denote the class of all univariate algebraic polynomials of degree \( < n \) by \( \mathbb{P}_n \). The orthonormalized Hermite polynomial \( h_k \) of degree \( k \) is defined by the Rodrigues’ formula:

\[
h_k(x) = \frac{(-1)^k}{\pi^{1/4} 2^{k/2} \sqrt{k!}} \exp(x^2) \left( \frac{d^k}{dx^k} \right) \exp(-x^2) .
\]

Writing \( \psi_k(x) = h_k(x) \exp(-x^2/2) \), one has the orthogonality relation for \( k, j \in \mathbb{Z}_+ \),

\[
\int_{\mathbb{R}} \psi_k(x) \psi_j(x) dx = \begin{cases} 
1, & \text{if } k = j, \\
0, & \text{if } k \neq j. 
\end{cases}
\]

The Hermite polynomial \( h_m \) has \( m \) real and simple zeros \( x_{k,m} \). Writing

\[
\lambda_{k,m} = \left( \sum_{j=0}^{m-1} h_j(x_{k,m})^2 \right)^{-1},
\]

(2.3)
it is well known (cf. [31, Section 3.4]) that
\[ \sum_{k=1}^{m} \lambda_{k,m} P(x_{k,m}) = \int_{\mathbb{R}} P(x) \exp(-x^2) dx, \quad P \in \mathbb{P}_{2m}. \] (2.4)
It is also known (cf. [20, Theorem 8.2.7]), applied with \( p = 2, b = 0 \) that
\[ \sum_{k=1}^{m} \lambda_{k,m} \exp(x_{k,m}^2) \leq cm^{1/2}. \] (2.5)

The Mehler formula [11, Formula (6.1.13)] states that
\[ \sum_{j=0}^{\infty} \psi_j(y) \psi_j(z) r^j = \frac{1}{\sqrt{\pi(1-r^2)}} \exp \left( \frac{2y z r - (y^2 + z^2) r^2}{1 - r^2} \right) \exp(-(y^2 + z^2)/2), \quad y, z \in \mathbb{R}, \ |r| < 1. \] (2.6)

Next, we introduce and review the properties of Hermite polynomials in the multivariate setting. We will need to use spaces with many different dimensions. Therefore, in this section and Sections 3, 4, we will use the symbol \( d \) to denote a generic dimension, which will be replaced later by \( q, Q, d(v) \), etc.

If \( d \geq 2 \) is an integer, we define Hermite polynomials on \( \mathbb{R}^d \) using tensor products. We adopt the notation \( x = (x_1, \cdots, x_d) \). The orthonormalized Hermite function is defined by
\[ \psi_k(x) = \prod_{j=1}^{d} \psi_{k_j}(x_j). \] (2.7)

In general, when univariate notation is used in multivariate context, it is to be understood in the tensor product sense as above; e.g., \( k! = \prod_{j=1}^{d} (k_j)! \), \( x^k = \prod_{j=1}^{d} x_j^{k_j} \), etc. The notation \( | \cdot |_p,d \) will denote the \( \ell^p \) norm on \( \mathbb{R}^d \).

For any set \( A \subset \mathbb{R}^d \) and \( f : A \rightarrow \mathbb{R} \), we denote by \( C(A) \) the space of all uniformly continuous and bounded functions on \( A \), with the norm \( \| f \|_A = \sup_{x \in A} |f(x)| \). The space \( C_0(A) \) is the subspace of all \( f \in C(A) \) vanishing at infinity.

**Constant convention:**

*In the sequel, \( c, c_1, \cdots \) will denote generic positive constants depending upon the dimension and other fixed quantities in the discussion, such as the norm. Their values may be different at different occurrences, even within a single formula. The notation \( A \sim B \) means \( c_1 A \leq B \leq c_2 B \). ■*

We will often use (without mentioning it explicitly) the fact deduced from the univariate bounds proved in [33] that
\[ |\psi_k(x)| \leq c. \] (2.8)
We will denote by \( \Pi_n^d \) the span of \( \{\psi_k : \sqrt{|k|}_d < n\} \) and by \( \mathbb{P}_n^d \) the space of all algebraic polynomials of total degree \( < n \). Thus, if \( P \in \Pi_n^d \) then \( P(x) = R(x) \exp(-|x|_2^2/2) \) for some \( R \in \mathbb{P}_n^d \). The following proposition lists a few important properties of these spaces (cf. [20, 24, 25]).

**Proposition 2.1.** Let \( n > 0, \ P \in \Pi_n^d \).
(a) (Infinite-finite range inequality) For any \( \delta > 0 \), there exists \( c = c(\delta) \) such that
\[ \|P\|_{\mathbb{R}^d \backslash [-\sqrt{2n(1+\delta)}, \sqrt{2n(1+\delta)}]^d} \leq c_1 e^{-cn^2} \|P\|_{[-\sqrt{2n(1+\delta)}, \sqrt{2n(1+\delta)}]^d} \] (2.9)
(b) (MRS identity) We have
\[ \|P\|_{\mathbb{R}^d} = \|P\|_{[-\sqrt{2n}, \sqrt{2n}]^d}. \] (2.10)
(c) (Bernstein inequality) There is a positive constant \( B \) depending only on \( d \) such that
\[ \|\nabla P\|_{\mathbb{R}^d} \leq B n \|P\|_{\mathbb{R}^d}. \] (2.11)

We note the following corollary, but omit the straightforward proof.

**Corollary 2.1.** Let \( n \geq 1 \). There exists \( C = C_n \subset \mathbb{R}^d \) with \( |C| \sim n^{2d} \) such that for any \( P \in \Pi_n^d \),
\[ \|P\|_{\mathbb{R}^d} \leq 2 \max_{x \in C} |P(x)|. \] (2.12)
Let $m \geq 1$. For a multi-integer $j$, $1 \leq j \leq m$, we write $\mathbf{x}_{j,m,d} := (x_{j,1,m}, \ldots, x_{j,d,m})$, and $\lambda_{j,m,d} := \prod_{k=1}^{d} \lambda_{j,k}$.

We observe further that if $P_1, P_2 \in \Pi_{m}^d$, then $P_1(\mathbf{x})P_2(\mathbf{x}) = R(\mathbf{x})\exp(-|\mathbf{x}|^2)$ for some $R \in \mathbb{R}^{d^2}$. Therefore, (2.4) and (2.5) lead to the following fact, which we formulate as a proposition.

**Proposition 2.2** For $m \geq 1$, we have

$$
\sum_{1 \leq j \leq m} \lambda_{j,m^2,d} \exp(|\mathbf{x}_{j,m^2,d}|^2_{2,d}) P_1(\mathbf{x}_{j,m^2,d})P_2(\mathbf{x}_{j,m^2,d}) = \int_{\mathbb{R}^d} P_1(\mathbf{x})P_2(\mathbf{x})d\mathbf{x}, \quad P_1, P_2 \in \Pi_{m}^d,
$$

and

$$
\sum_{1 \leq j \leq m} \lambda_{j,m^2,d} \exp(|\mathbf{x}_{j,m^2,d}|^2_{2,d}) \leq cm^d.
$$

### 3 Applications of Mehler identity

The Mehler identity for multivariate Hermite polynomials is expressed conveniently by writing

$$
\text{Proj}_{m,d}(\mathbf{x}, \mathbf{y}) = \sum_{|\mathbf{k}|_{1,d} = m} \psi_{\mathbf{k}}(\mathbf{x})\psi_{\mathbf{k}}(\mathbf{y}).
$$

Using the univariate Mehler identity (2.6), it is then easy to deduce that for $w \in \mathbb{C}$, $|w| < 1$,

$$
\sum_{\mathbf{k} \in \mathbb{Z}^d} \psi_{\mathbf{k}}(\mathbf{x})\psi_{\mathbf{k}}(\mathbf{y}) w^{|\mathbf{k}|_{1,d}} = \sum_{m=0}^{\infty} w^m \text{Proj}_{m,d}(\mathbf{x}, \mathbf{y})
$$

$$
= \frac{1}{(\pi(1-w^2))^{d/2}} \exp\left(\frac{4w\mathbf{x} \cdot \mathbf{y} - (1+w^2)(|\mathbf{x}|^2_{2,d} + |\mathbf{y}|^2_{2,d})}{2(1-w^2)}\right)
$$

$$
= \frac{1}{(\pi(1-w^2))^{d/2}} \exp\left(\frac{-1+w|\mathbf{x}-\mathbf{y}|^2_{2,d}}{1-w} - \frac{1-w|\mathbf{x}+\mathbf{y}|^2_{2,d}}{1+w}\right)
$$

$$
= \frac{1}{(\pi(1-w^2))^{d/2}} \exp\left(\frac{-1+w^2}{2(1-w^2)}|\mathbf{x} - \frac{2w}{1+w^2}\mathbf{y}|^2_{2,d}\right) \exp\left(-\frac{1-w^2}{2(1-w^2)}|\mathbf{y}|^2_{2,d}\right).
$$

In this section, we note some immediate consequences of the Mehler identity: Approximation of polynomials by Gaussian networks and invariance and localization properties of certain kernels.

We discuss first the close connection between Hermite polynomials and Gaussian networks. We denote by $\nu_{m,d}$ the measure that associates the mass $\lambda_{j,2m^2,d}$ with the point $\mathbf{x}_{j,2m^2,d}$ for $1 \leq j_1, \ldots, j_d \leq 2m^2$.

**Proposition 3.1** Let $m \geq 1$, $\mathbf{k} \in \mathbb{Z}^d_+$, and for $|\mathbf{k}|_{1,d} < m^2$,

$$
G_{\mathbf{k},m,d}(\mathbf{x}) = \left(\frac{3}{2\pi}\right)^{d/2} 3^{|\mathbf{k}|_{1,d}/2} \iint_{\mathbb{R}^d} \exp\left(-\left|\mathbf{x} - \frac{\sqrt{3}}{2}\mathbf{u}\right|^2_{2,d}\right) \psi_{\mathbf{k}}(\mathbf{u})\exp(-|\mathbf{u}|^2_{2,d}/4)d\nu_{m,d}^*(\mathbf{u})
$$

$$
= \left(\frac{3}{2\pi}\right)^{d/2} 3^{|\mathbf{k}|_{1,d}/2} \sum_{1 \leq j \leq m^2} \lambda_{j,2m^2,d} \exp(|\mathbf{x}_{j,2m^2,d}|^2_{2,d}) \psi_{\mathbf{k}}(\mathbf{x}_{j,2m^2,d}) \exp(-|\mathbf{x}_{j,2m^2,d}|^2_{2,d}/4)
$$

$$
\times \exp\left(-\left|\mathbf{x} - \frac{\sqrt{3}}{2}\mathbf{x}_{j,2m^2,d}\right|^2_{2,d}\right).
$$

Then

$$
\max_{|\mathbf{k}|_{1,d} < m} \|\psi_{\mathbf{k}} - G_{\mathbf{k},m,d}\|_{\mathbb{R}^d} \leq cm^{d-2}3^{-m^2/2}.
$$

Clearly, the number of neurons in the network $G_{\mathbf{k},m}$ is $O(m^{2d})$. 
Proof. This proof is the same as that in [9] Lemma 4.2 and [23] Lemma 4.1. Using the last expression in (3.2) with \( w = 1/\sqrt{3} \), we obtain
\[
\sum_{\mathbf{k} \in \mathbb{Z}^d} \psi_k(\mathbf{x}) \psi_k(\mathbf{u}) 3^{-|\mathbf{k}|_{1, d}/2} \left( \frac{3}{2\pi} \right)^{d/2} \exp \left( -\|\mathbf{x} - \frac{\sqrt{3}}{2} \mathbf{u}\|^2 \right) \exp(-\|\mathbf{u}\|^2/4).
\]

Therefore, using Proposition 2.2 with \( m\sqrt{2} \) in place of \( m \), we obtain
\[
\psi_k(\mathbf{x}) = 3^{|\mathbf{k}|_{1, d}/2} \left( \frac{3}{2\pi} \right)^{d/2} \int_{\mathbb{R}^d} \exp \left( -\|\mathbf{x} - \frac{\sqrt{3}}{2} \mathbf{u}\|^2 \right) \psi_k(\mathbf{u}) \exp(-\|\mathbf{u}\|^2/4) d\nu^*_{m, d}(\mathbf{u})
\]
\[+ 3^{|\mathbf{k}|_{1, d}/2} \int_{\mathbb{R}^d} \sum_{|\mathbf{j}|_{1, d} \geq 2m^2} \psi_k(\mathbf{u}) \psi_j(\mathbf{x}) \psi_j(\mathbf{u}) 3^{-|\mathbf{j}|_{1, d}/2} d\nu^*_{m, d}(\mathbf{u}).
\]

The first term on the right hand side above is \( \mathcal{G}_{k, m, d} \). The second term is estimated using \( 2.8 \) and \( 2.14 \) (applied with \( m\sqrt{2} \) in place of \( m \)) exactly as in the proof of [9] Lemma 4.2. We omit the details. \( \blacksquare \)

For \( m \geq 1 \), \( P = \sum_{|\mathbf{k}|_{1, d} < m^2} b_k \psi_k \in \Pi^d_m \), we define
\[
\mathcal{G}_d(P)(\mathbf{x}) = \sum_{|\mathbf{k}|_{1, d} < m^2} b_k \mathcal{G}_{k, m, d}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d.
\]

(3.5)

We note that the centers and the number of neurons in the network \( \sum_{|\mathbf{k}|_{1, d} < m^2} b_k \mathcal{G}_{k, m, d} \) are independent of \( P \). In particular, the number of neurons is \( O(m^{2d}) \). The following corollary is easy to deduce (cf. [23] Proposition 4.1).

Corollary 3.1 Let \( m \geq 1 \), \( P \in \Pi^d_m \). Then
\[
||P - \mathcal{G}_d(P)||_{\mathbb{R}^d} \leq c_1 m^{-3m^2/2}.
\]

(3.6)

The second interesting consequence of the Mehler identity is that the projection \( \text{Proj}_{m, d} \) is invariant under rotations. For any \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^d \), we may therefore use an appropriate rotation to write
\[
\text{Proj}_{m, d}(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{m} \text{Proj}_{j, 2}((|\mathbf{x}|_{2, d}, 0), (|\mathbf{y}|_{2, d} \cos \theta, |\mathbf{y}|_{2, d} \sin \theta)) \sum_{|\mathbf{k}|_{1, d-2} \leq m-j} |\psi_k(\mathbf{0})|^2,
\]

(3.7)

where \( \cos \theta = \mathbf{x} \cdot \mathbf{y}/(|\mathbf{x}| |\mathbf{y}|) \). Using the Mehler identity (3.2), we deduce that
\[
\sum_{r=0}^{\infty} w^{2r} \sum_{|\mathbf{k}|_{1, d-2} = 2r} |\psi_k(\mathbf{0})|^2 = (\pi(1 - w^2))^{-(d-2)/2} = \pi^{1-d/2} \sum_{r=0}^{\infty} \frac{\Gamma(d/2 + r - 1)}{\Gamma(d/2 - 1)r!} w^{2r}.
\]

Hence, for \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^d \),
\[
\text{Proj}_{m, d}(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{m} \text{Proj}_{j, 2}((|\mathbf{x}|_{2, d}, 0), (|\mathbf{y}|_{2, d} \cos \theta, |\mathbf{y}|_{2, d} \sin \theta)) D_{d-2, m-j-\ell},
\]

(3.8)

where
\[
\psi_\ell(0) = \begin{cases} \pi^{-1/4} (-1)^{\ell/2} \sqrt{\pi} \frac{\sqrt{\pi}}{2^{\ell/2}(\ell/2)!}, & \text{if } \ell \text{ is even}, \\ 0, & \text{if } \ell \text{ is odd}, \end{cases}
\]

(3.9)

and
\[
D_{d-2, r} = \begin{cases} 1, & \text{if } r \text{ is even, } d \leq 2, \\ \pi^{1-d/2} \frac{\Gamma(d/2 + r/2 - 1)}{\Gamma(d/2 - 1)(r/2)!}, & \text{if } r \text{ is even, } d \geq 3, \\ 0, & \text{if } r \text{ is odd}. \end{cases}
\]

(3.10)
The third interesting consequence of the Mehler identity is the localization of certain kernels. In the sequel, \( H : [0, \infty) \to [0, 1] \) is a fixed, infinitely differentiable function, with \( H(t) = 1 \) if \( 0 \leq t \leq 1/2 \), \( H(t) = 0 \) if \( t \geq 1 \). All constants may depend upon \( H \) as well. We define

\[
\Phi_{n,d}(H; x, y) = \Phi_{n,d}(x, y) = \sum_{k \in \mathbb{Z}^d_+} H\left(\frac{\sqrt{|k|_1 d}}{n}\right) \psi_k(x) \psi_k(y) = \sum_{m=0}^{\infty} H\left(\frac{\sqrt{m}}{n}\right) \text{Proj}_{m,d}(x, y), \quad x, y \in \mathbb{R}^d.
\] (3.11)

Using Mehler identity and the Tauberian theorem in [26, Theorem 4.3], we proved in [8, Lemma 4.1] the following proposition.

**Proposition 3.2** For \( n \geq 1 \), \( x, y \in \mathbb{R}^d \), we have

\[
|\Phi_{n,d}(x, y)| \leq \frac{c n^d}{\max(1, (n|x - y|_2^2)^3)}.
\] (3.12)

In particular,

\[
|\Phi_{n,d}(x, y)| \leq c n^d,
\] (3.13)

and

\[
\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\Phi_{n,d}(x, y)| dy \leq c.
\] (3.14)

### 4 Function approximation

In this section, we describe some results on approximation of functions on \( \mathbb{R}^d \). If \( f \in C_0(\mathbb{R}^d) \), we define its degree of approximation by

\[
E_n(\mathbb{R}^d; f) = \min_{P \in \mathbb{P}_n^{\mathbb{R}^d}} \|f - P\|_{\mathbb{R}^d}.
\] (4.1)

For \( \gamma > 0 \), the smoothness class \( W_\gamma(\mathbb{R}^d) \) comprises \( f \in C_0(\mathbb{R}^d) \) for which

\[
\|f\|_{W_\gamma(\mathbb{R}^d)} = \|f\|_{\mathbb{R}^d} + \sup_{n \geq 0} 2^{n\gamma} E_{2^n}(\mathbb{R}^d; f) < \infty.
\] (4.2)

We need some results from [20] [21], in the form stated in Theorem 4.1 below. To state this theorem, we need some notation first. For integer \( k = 1, \ldots, d \), we define

\[
\Delta_{t,k}f(x) = f(x_1, \ldots, x_{k-1}, x_k + t, x_{k+1}, \ldots, x_d) - f(x), \quad Q_{t,k}(x) = \min(\delta^{-1}, (1 + x_k^2)^{1/2}),
\] (4.3)

and

\[
\omega_k(f, \delta) = \sup_{|t| \leq \delta} \|\Delta_{t,k}f\|_{\mathbb{R}^d} + \delta \|Q_{t,k}f\|_{\mathbb{R}^d}.
\] (4.4)

**Theorem 4.1** Let \( f \in C_0(\mathbb{R}^d) \), \( 0 < \gamma < 1 \). Then \( f \in W_\gamma(\mathbb{R}^d) \) if and only if

\[
\sum_{k=1}^{d} \omega_k(f, \delta) = O(\delta^\gamma), \quad 0 < \delta \leq 1.
\] (4.5)

*Proof.* The theorem is already contained in the results in [21], but we need to reconcile notation and explain why. In [21] Formulas (42),(43)] we have defined a univariate \( K \)-functional and a pre-modulus of smoothness, which we need to apply with \( r = 1 \), for \( q_k(x) = \exp(|x|^2/2)f(x) \) applied to the \( k \)-th component of \( x \), \( k = 1, \ldots, d \). The \( K \)-functional obtained in this way is denoted in [21, Formula (21)] by \( K_{1,k} \). Likewise, the quantity denoted by \( \omega_1 \) in [21] is the \( k \)-summand of the left hand side of (4.5). Therefore, [21, Theorem 5.1, Proposition 4.5] lead to the statement of this theorem. ■
Remark 4.1 If $f \in C_0(\mathbb{R}^d)$ satisfies
\[
\sup_{|u|_{2,d} \leq \delta} \|f(\cdot + u) - f\|_{\mathbb{R}^d} + \delta \|\min(\delta^{-1}, 1 + |\cdot|^2_{2,d})f\|_{\mathbb{R}^d} \leq c(f)\delta^\gamma,
\] (4.7)
then the condition (4.5) is satisfied. ■

We define
\[
\sigma_n(\mathbb{R}^d; f)(x) = \int_{\mathbb{R}^d} \Phi_n,d(x, y)f(y)dy, \quad f \in C_0(\mathbb{R}^d), \ n > 0, \ x \in \mathbb{R}^d.
\] (4.8)

The following proposition is routine to prove using Proposition 3.2.

Proposition 4.1 (a) If $n > 0$ and $P \in \Pi_{n/\sqrt{2}}^d$, then $\sigma_n(\mathbb{R}^d; P) = P$.
(b) If $f \in C_0(\mathbb{R}^d)$, $n > 0$, then
\[
||\sigma_n(\mathbb{R}^d; f)||_{\mathbb{R}^d} \leq c||f||_{\mathbb{R}^d}, \quad E_n(\mathbb{R}^d; f) \leq ||f - \sigma_n(\mathbb{R}^d; f)||_{\mathbb{R}^d} \leq cE_{n/\sqrt{2}}(\mathbb{R}^d; f).
\] (4.9)

Remark 4.2 In view of the rotation invariance of $\Pi^d$ and $\sigma_n(\mathbb{R}^d; f)$, the results in Theorem 4.1 and Proposition 4.1 remain valid $\mathbb{R}^d$ is substituted by any subspace $Y$ of dimension $d$ embedded in a higher dimensional Euclidean space. ■

5 Approximation on affine spaces

In the sequel, we fix integers $Q \geq q \geq 1$, and assume for simplicity that $Q - q$ is an even integer.

Let $\mathbb{X}$ be a $q$-dimensional affine subspace of $\mathbb{R}^Q$, passing through a point $x_0 \in \mathbb{R}^Q$, $\hat{e}_1, \cdots, \hat{e}_q$ be an orthonormal basis of $\mathbb{X}$. If $f : \mathbb{X} \to \mathbb{R}$, and the function $F : \mathbb{R}^q \to \mathbb{R}$ is defined by
\[
F(u_1, \cdots, u_q) = f(x_0 + \sum_{k=1}^q u_k\hat{e}_k),
\] (5.1)

we define
\[
E_n(\mathbb{X}; f) = E_n(\mathbb{R}^d; F).
\] (5.2)

In terms of the points $x = x_0 + \sum_{k=1}^q u_k\hat{e}_k \in \mathbb{X}$, the class of approximants of functions on $\mathbb{X}$ have the form $x \mapsto P(x)\exp(-|x - x_0|^2/2)$, where $P \in \Pi_{n/2}^Q$. The invariance of this space under rotations that leave $x_0$ fixed, shows that $E_n(\mathbb{X}; f)$ is independent of the choice of the basis $\hat{e}_1, \cdots, \hat{e}_q$. However, $E_n(\mathbb{X}; f)$ obviously depends upon the choice of the point $x_0$. If we are interested only in approximation on $\mathbb{X}$, we may decide to use some standard point, such as the best approximation to $0 \in \mathbb{R}^Q$ from $\mathbb{X}$. Our main goal however, is to use the results in this section with $\mathbb{X}$ replaced by the tangent space to a manifold. In that context, our definition is more natural. We note that if $f$ is supported on a compact neighborhood of $x_0$, then $F$ is supported on a compact neighborhood of $0 \in \mathbb{R}^q$. Therefore, for approximation of functions $f$ satisfying a Lipschitz condition, we may use Theorem 4.1 with $F$ and get the estimates where the constants do not depend upon $x_0$, although the space of approximants does.

Our goal in this section is to study the analogue of Proposition 4.1 in the context of approximation on $\mathbb{X}$. Toward this end, we extend the notation $\text{Proj}_{m,q}(x, y)$ for $x, y \in \mathbb{R}^Q$ by setting
\[
\text{Proj}_{m,q,Q}(x, y) = \sum_{j=0}^m \text{Proj}_{j, 2}((|x|_{2,Q}, 0), (|y|_{2,Q} \cos \theta, |y|_{2,Q} \sin \theta))D_{q-2, m-j-\ell}, \quad x, y \in \mathbb{R}^Q.
\] (5.3)

Clearly, if $x, y$ are in a $q$-dimensional subspace of $\mathbb{R}^Q$, then $\text{Proj}_{m,q,Q}$ is a reproducing kernel on that subspace.

We define
\[
\Phi_{n,q,Q}(x, y) = \sum_{m=0}^\infty H \left(\frac{\sqrt{m}}{n}\right) \text{Proj}_{m,q,Q}(x, y), \quad x, y \in \mathbb{R}^Q.
\] (5.4)
Proposition 5.1 Let $m \geq 0$, and $x, y \in \mathbb{R}^Q$.
(a) We have
\[
\text{Proj}_{m,q,Q}(x, y) = \pi^{(q-q)/2} \sum_{\ell=0}^{[m/2]} \left( (Q - q)/\ell + q - 1 \right) \text{Proj}_{m-2\ell,q,Q}(x, y).
\] (5.5)
(b) We have
\[
\text{Proj}_{m,q,Q}(x, y) = \pi^{(q-q)/2} \sum_{\ell=0}^{[m/2]} (-1)^{\ell} \left( (Q - q)/\ell \right) \text{Proj}_{m-2\ell,q,Q}(x, y).
\] (5.6)
Hence, $\text{Proj}_{m,q,Q}(x, y)$ is a weighted polynomial in $\Pi_m^Q$ as a function of $x$ and $y$. Therefore, $\Phi_{n,q,Q}(x, y) \in \Pi_m^Q$ as a function of $x$ and $y$.
(c) Let $S > q$ be an integer. For $x, y \in \mathbb{R}^Q$, we have
\[
|\Phi_{n,q,Q}(x, y)| \leq \frac{c_n^q}{\max(1, (n|x-y|_{2,Q})^S)}.
\] (5.7)
In particular,
\[
|\Phi_{n,q,Q}(x, y)| \leq c q^q.
\] (5.8)

Proof. In this proof, let $x' = (|x|_2, 0, \ldots, 0)$, $y' = (|y|_2, 0, \ldots, 0)$. In view of (5.7), we observe that
\[
\text{Proj}_{m,q,Q}(x, y) = \text{Proj}_{m,q,Q}(x', y').
\] (5.9)
Further, $|x - y|_{2,Q} = |x' - y'|_{2,Q}$, and $|x + y|_{2,Q} = |x' + y'|_{2,Q}$. Therefore, the Mehler identity shows that
\[
\sum_{m=0}^{\infty} w^m \text{Proj}_{m,q,Q}(x, y) = \frac{1}{(\pi(1-w^2))^{Q/2}} \exp \left( -\frac{1}{1+w} \frac{|x - y|_{2,Q}^2}{4} - \frac{1}{1-w} \frac{|x + y|_{2,Q}^2}{4} \right)
\] (5.10)
\[
= \frac{1}{(\pi(1-w^2))^{(Q-q)/2}} \sum_{m=0}^{\infty} w^m \text{Proj}_{m,q,Q}(x', y')
\] (5.11)
\[
= \frac{1}{(\pi(1-w^2))^{(Q-q)/2}} \sum_{m=0}^{\infty} w^m \text{Proj}_{m,q,Q}(x, y).
\]
We now recall the McClaurin expansion for $(1 - w^2)^{-Q/2}$, multiply the two power series using the Cauchy-Leibnitz formula, and compare the coefficients to arrive at (5.5). Part (b) is proved similarly by observing that
\[
\sum_{m=0}^{\infty} w^m \text{Proj}_{m,q,Q}(x, y) = \pi^{(q-q)/2}(1-w^2)^{(Q-q)/2} \sum_{m=0}^{\infty} w^m \text{Proj}_{m,q,Q}(x, y).
\] (5.11)
Since $\Phi_{n,q,Q}(x, y) = \Phi_{n,q,Q}(x', y')$ and $|x - y|_{2,Q} = |x' - y'|_{2,Q}$, part (c) follows directly from Proposition 3.2.

We denote the volume measure of $X$ by $\mu_X$, and for $f \in C_0(X)$,
\[
\sigma_n(X; f)(x) = \int_X \Phi_{n,q,Q}(x, y) f(y) d\mu_X(y), \quad x \in \mathbb{R}^Q.
\] (5.12)

Theorem 5.1 Let $Q \geq q \geq 1$ be integers, $Q - q$ be an even integer, $X$ be a $q$-dimensional affine subspace of $\mathbb{R}^Q$, passing through $x_0 \in \mathbb{R}^Q$, $f \in C_0(X)$. Then
\[
\|\sigma_n(X; f) - f\|_{X} \leq c E_{n/\sqrt{2}}(X; f).
\] (5.13)

Proof. Let $\tilde{e}_1, \ldots, \tilde{e}_q$ be an orthonormal basis of $X$, and for $x \in X$, $x = x_0 + \sum_{k=1}^{q} u_k(x) \tilde{e}_k$. We define $F$ as in [5.1]. Since the kernel $\Phi_{n,q,Q}$ is invariant under rotations preserving $x_0$, it is easy to verify that
\[
\sigma_n(X; f)(x) = \sigma_n(\mathbb{R}^q; F)(u_1(x), \ldots, u_q(x)), \quad x \in X.
\]
Hence, the theorem follows from Proposition 4.3.
6 Approximation on manifolds

Let $\mathcal{X}$ be a $q$ dimensional, compact, connected, sub-manifold of $\mathbb{R}^Q$ (without boundary), with geodesic distance $\rho$ and volume measure $\mu^*$, normalized so that $\mu^*(\mathcal{X}) = 1$. We will identify the tangent space at $x \in \mathcal{X}$ with an affine space $T_x(\mathcal{X})$ in $\mathbb{R}^Q$ passing through $x$. With this convention, the exponential map $\mathcal{E}_x$ at $x \in \mathcal{X}$ (based on the definition in \cite[Proposition 2.9]{[13]}) is a diffeomorphism of a ball centered at $x$ in $T_x(\mathcal{X})$ onto its image in $\mathcal{X}$ such that $\rho(x, \mathcal{E}_x(u)) = |u - x|_{2,Q}$. The smallest radius of all such balls is the injectivity radius of $\mathcal{X}$, denoted in this paper by $\iota^*$. For any $x \in \mathcal{X}$, we need to consider three kinds of balls.

$$B_Q(x, r) = \{ y \in \mathbb{R}^Q : |x - y|_{2,Q} \leq r \},$$
$$B_T(x, r) = T_x(\mathcal{X}) \cap B_Q(x, r)$$
$$B(x, r) = \{ y \in \mathcal{X} : \rho(x, y) \leq r \}. \quad (6.1)$$

Clearly, if $r < \iota^*$, then $B(x, r) = \mathcal{E}_x(B_T(x, r))$.

The following proposition is not difficult to prove using definitions and Taylor expansions (cf. \cite{[6]}).

**Proposition 6.1** We have

$$||x - \mathcal{E}_x(u)||_{2,Q} - \rho(x, \mathcal{E}_x(u)) = ||x - \mathcal{E}_x(u)||_{2,Q} - (x - u, q, Q) \leq c \rho(x, \mathcal{E}_x(u))^2, \quad \mathcal{E}_x(u) \in B(x, \iota^*). \quad (6.2)$$

If $\delta \leq \iota^*$ then

$$||\mathcal{E}_x(u) - u||_{2,Q} \leq c\delta^2, \quad \mathcal{E}_x(u) \in B(x, \delta),$$
$$d\mu^*(\mathcal{E}_x(u)) = (1 + O(\delta^2))d\mu, \quad \mathcal{E}_x(u) \in B(x, \delta). \quad (6.3)$$

*The constant involved in $O$ is uniform in $x$.*

In the sequel, we assume the following polynomial growth condition:

$$\sup_{x \in \mathcal{X}, r > 0} \frac{\mu^*(B(x, r))}{r^q} \leq c. \quad (6.5)$$

Our first theorem is the analogue of Theorem 3.1 when $\mathcal{X}$ is a manifold instead of an affine space. The curvature of the manifold puts a limitation on how well one can approximate using our method. Therefore, we restrict our attention to the approximation of functions satisfying a Lipschitz condition.

In the sequel, $\text{Lip}(\mathcal{X})$ denotes the class of all $f \in C(\mathcal{X})$ with the property that there exists $L = L(f) > 0$ such that

$$|f(x) - f(y)| \leq L\rho(x, y), \quad x, y \in \mathcal{X}. \quad (6.6)$$

For $f \in \text{Lip}(\mathcal{X})$, we denote

$$\|f\|_{\text{Lip}(\mathcal{X})} = \|f\|_{\mathcal{X}} + \sup_{x, y \in \mathcal{X}, x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)}. \quad (6.7)$$

**Theorem 6.1** Let $\mathcal{X}$ be a $q$ dimensional, compact, connected, sub-manifold of $\mathbb{R}^Q$, (6.5) be satisfied, and $Q - q$ be an even integer. Let $L > 0$, and $f \in \text{Lip}(\mathcal{X})$. Let $0 < \gamma < 1$. Then for $n \geq 1$,

$$\|f - \sigma_n(\mathcal{X}; f)\|_{\mathcal{X}} \leq c(\|f\|_{\text{Lip}(\mathcal{X})}, \gamma)n^{-\gamma}. \quad (6.8)$$

It is convenient to summarize some details of the proof of this theorem in the form of the following lemma.

**Lemma 6.1** Let $x \in \mathcal{X}$, $f \in C(\mathcal{X})$ be supported on $B(x, \iota^*/8)$, $F(u) = f(\mathcal{E}_x(u))$, and $0 < \gamma < 1$. Then for $n \geq 1$,

$$\left| \int_{\mathcal{X}} \Phi_{n,q,Q}(x, y)f(y)d\mu^*(y) - \int_{T_x(\mathcal{X})} \Phi_{n,q,Q}(x, u)F(u)d\mu \right| \leq cn^{-\gamma}\|f\|_{\mathcal{X}}, \quad (6.9)$$

where $F$ is extended outside $\{u : |x - u|_{2,q} \leq \iota^*/8\}$ as a zero function.
Proof. In this proof only, let $\delta = n^{-(q+1+\gamma)/(q+2)}$, and we assume without loss of generality that $\|f\|_{L^p(\mathbb{X})} = 1$. Without loss of generality, we may assume $n \geq c$ so that $\delta \leq \iota^*/2$ and the conclusions of Proposition 6.1 hold. In particular, we have for $\rho(x, E_x(u)) \leq \delta$ (equivalently, $u \in B_T(x, \delta)$),

$$
(1/2)\rho(x, E_x(u)) \leq |x - u|_{2,Q} \leq \rho(x, E_x(u)).
$$

(6.10)

In view of the localization property (5.7), used with

$$
S \geq \frac{(q+2)(q+\gamma)}{1-\gamma},
$$

(6.11)

we obtain $n\delta = n^{1-\gamma}/(q+2)$ and, since $f$ is supported on $\mathbb{B}(x, \iota^*/8)$,

$$
\int_{\mathbb{X} \setminus B_Q(x, \delta)} |\Phi_{n,q,Q}(x, y)f(y)|d\mu^*(y) = \int_{\mathbb{B}(x, \iota^*/8) \setminus B_Q(x, \delta)} |\Phi_{n,q,Q}(x, y)f(y)|d\mu^*(y) \leq cn^q(n\delta)^{-S} \leq cn^{-\gamma}.
$$

(6.12)

We now assume $n$ to be large enough so that with $c$ as in (6.3), $c\delta^2 \leq \delta/2$. If $y \in \mathbb{B}(x, \iota^*/8) \cap B_Q(x, \delta)$, $u \in T_x(\mathbb{X})$, $y = E_x(u)$, then (6.3), (6.2) show that

$$
|x - u|_{2,Q} \leq |x - y|_{2,Q} + |E_x(u) - u|_{2,Q} \leq \delta + c\delta^2 \leq (3/2)\delta,
$$

(6.13)

thus,

$$
E_\delta := E^{-1}_x(\mathbb{B}(x, \iota^*/8) \cap B_Q(x, \delta)) \subseteq B_T(x, 3\delta/2).
$$

(6.14)

Since $\Phi_{n,q,Q}$ is a weighted polynomial of order $n$ in both $x$ and $y$ ($\in \mathbb{R}^Q$), the Bernstein inequality (2.11) and (6.10) together show that for $y \in B_Q(x, \delta)$ and any $v \in \mathbb{R}^Q$,

$$
|\Phi_{n,q,Q}(x, y) - \Phi_{n,q,Q}(x, v)| \leq cn^{q+1}|y - v|_{2,Q}.
$$

In particular, (cf. (6.3)) for $y = E_x(u) \in \mathbb{B}(x, \iota^*/8) \cap B_Q(x, \delta)$,

$$
|\Phi_{n,q,Q}(x, y) - \Phi_{n,q,Q}(x, u)| \leq cn^{q+1}\delta^2.
$$

(6.15)

The estimates (6.13) and (6.15) lead further to

$$
\left| \int_{\mathbb{B}(x, \iota^*/8) \cap B_Q(x, \delta)} d\mu^*(y) - \int_{E_\delta} d\mu(u) \right| \leq c\delta^{q+2}.
$$

(6.16)

In view of (6.15) and (6.16), we deduce that

$$
\left| \int_{\mathbb{B}(x, \iota^*/8) \cap B_Q(x, \delta)} \Phi_{n,q,Q}(x, y)f(y)d\mu^*(y) - \int_{E_\delta} \Phi_{n,q,Q}(x, u)f(u)d\mu(u) \right| \leq cn^{q+1}\delta^{q+2} = cn^{-\gamma}.
$$

(6.17)

The localization estimate (5.7) shows that

$$
\left| \int_{T_x(\mathbb{X}) \setminus B_T(x, \iota^*/8)} \Phi_{n,q,Q}(x, u)f(u)d\mu(u) \right| \leq cn^q(n\delta)^{-S} \leq cn^{-\gamma}.
$$

(6.18)

If $u \in B_T(x, \iota^*/8)$ then $E_x(u)$ is well defined. If $u \in B_T(x, \iota^*/8) \setminus E_\delta$, then (6.3), (6.2) show that

$$
|x - u|_{2,Q} \geq |x - E_x(u)|_{2,Q} - |E_x(u) - u|_{2,Q} \geq \delta - c\delta^2 \geq \delta/2.
$$

Invoking the localization estimate (5.7) again, we deduce that

$$
\left| \int_{B_T(x, \iota^*/8) \setminus E_\delta} \Phi_{n,q,Q}(x, u)f(u)d\mu(u) \right| \leq cn^{-\gamma}.
$$

(6.19)
The estimates (6.12), (6.17), (6.18) and (6.19) lead to (6.9). ■

We are now in a position to prove Theorem 6.1.

**Proof of Theorem 6.1**

Let \( x \in X \) and \( \phi \in C^\infty(X) \) be chosen so that \( \phi(y) = 1 \) if \( y \in B(x, r^*/16) \) and \( \phi(y) = 0 \) if \( y \in X \setminus B(x, r^*/8) \). We assume without loss of generality that \( \|f\|_{\text{Lip}(X)} = 1 \). We note that the function \( F : T_X(X) \to \mathbb{R} \) defined by

\[
F(u) = \begin{cases} 
  f(\mathcal{E}_X(u))\phi(\mathcal{E}_X(u)), & \text{if } |x - u| \leq r^*/2, \\
  0, & \text{otherwise},
\end{cases}
\]

is in \( \text{Lip}(T_X(X)) \) and \( \|F\|_{\text{Lip}(T_X(X))} \leq \|f\|_{\text{Lip}(X)} \).

We choose \( S \) as in (6.13). Then, the localization property (5.7) shows that

\[
\left| \int_X \Phi_{n,q,Q}(x,y)(1 - \phi(y))f(y)\mu^*(y) \right| = \left| \int_{X \setminus B(x,r^*/8)} \Phi_{n,q,Q}(x,y)(1 - \phi(y))f(y)\mu^*(y) \right| \leq cn^{-S} \leq cn^{-\gamma}. \tag{6.21}
\]

In view of Lemma 6.1

\[
\left| \int_X \Phi_{n,q,Q}(x,y)\phi(y)f(y)\mu^*(y) \right| \to \int_{T_X(X)} \Phi_{n,q,Q}(x,u)F(u)\mu(u) \leq cn^{-\gamma}, \tag{6.22}
\]

so that

\[
\left| \int_X \Phi_{n,q,Q}(x,y)f(y)\mu^*(y) \right| \leq c\{E_{n/\sqrt{2}}(T_X(X);F) + n^{-\gamma}\}. \tag{6.24}
\]

Since \( F(x) = f(x) \), Theorem 5.1 now shows that

\[
\left| \int_X \Phi_{n,q,Q}(x,y)f(y)\mu^*(y) - f(x) \right| \leq c\{E_{n/\sqrt{2}}(T_X(X);F) + n^{-\gamma}\}. \tag{6.24}
\]

Our next objective in this section is to obtain the following discretization of Theorem 6.1 based on noise-corrupted random samples of \( f \). We assume noisy data of the form \((y, \epsilon)\), with a joint probability distribution \( \tau \) and with \( \mu^* \) being the marginal distribution of \( y \) with respect to \( \tau \). In place of \( f(y) \), we consider a noisy variant \( F(y, \epsilon) \), and denote

\[
f(y) = \mathbb{E}_\tau(F(y, \epsilon)|y). \tag{6.25}
\]

It is easy to verify using Fubini’s theorem that if \( F \) is integrable with respect to \( \tau \) then for any \( x \in \mathbb{R}^Q \),

\[
\mathbb{E}_\tau(F(y, \epsilon)|\Phi_{n,q,Q}(x,y)) = \sigma_n(X; f)(x). \tag{6.26}
\]

**Theorem 6.2** Let (6.5) be satisfied, \( \tau \) be a probability distribution on \( X \times \Omega \) for some sample space \( \Omega \) such the marginal distribution of \( \tau \) restricted to \( X \) is \( \mu^* \). Let \( F : X \times \Omega \to \mathbb{R} \) be a bounded function, and \( f \) defined by (6.25) be in \( \text{Lip}(X) \). Let \( 0 < \gamma, \delta < 1 \). Let \( M \geq 1 \), \( Y = \{ (y_1, \epsilon_1), \cdots, (y_M, \epsilon_M) \} \) be a set of random samples chosen i.i.d. from \( \tau \), and define

\[
\hat{F}_n(x) = \hat{F}(Y; x) = \frac{1}{M} \sum_{j=1}^{M} F(y_j, \epsilon_j)\Phi_{n,q,Q}(x,y_j), \quad x \in \mathbb{R}^Q. \tag{6.27}
\]

Then there exists \( c_1 > 0 \) such that for every \( n \geq 1 \)

\[
\text{Prob}_\tau \left( \left\| \hat{F}_n(Y; \cdot) - \sigma_n(X; f) \right\|_{\mathbb{R}^Q} \geq c_2\|F\|_{\mathbb{R}^Q}n^q \sqrt{\log(cn^{2Q}/\delta)} \right) \leq \delta. \tag{6.28}
\]

In particular, with

\[
n \sim \left( \frac{M}{\log(M/\delta)} \right)^{1/(2q+2\gamma)}, \tag{6.29}
\]

we have

\[
\text{Prob}_\tau \left( \left\| \hat{F}_n(Y; \cdot) - f \right\|_X \geq c_3\|f\|_{\mathbb{R}^Q}c_4\|\sigma_n(X; f)\|_{\mathbb{R}^Q}/n^\gamma \right) \leq \delta. \tag{6.30}
\]
The proof of Theorem 6.2 is almost verbatim the same as that of [27 Theorem 3.1]. The starting point is the following consequence of Hoeffding inequality ([27 Theorem 6.1]).

**Theorem 6.3** Let \( \mathcal{Y} \) be a topological space, \( W \) be a linear subspace of \( C_0(\mathcal{Y}) \). We assume that there is a finite set \( C \subseteq \mathcal{Y} \) such that there exists a constant \( n(W, C) > 0 \) with the property that

\[
\|P\|_{\mathcal{Y}} \leq n(W, C) \max_{y \in \mathcal{Y}} |P(y)|, \quad P \in W.
\] (6.31)

Let \((\Omega, B, \mu)\) be a probability space, and \( Z : \Omega \to W \). We assume further that for any \( x \in \mathcal{Y}, \omega \in \Omega, |Z(\omega)(x)| \leq R \) for some \( R > 0 \). Then for any \( \delta > 0 \), integer \( M \geq 1 \), and independent sample \( \omega_1, \cdots, \omega_M \), we have

\[
\text{Prob}_{\mu} \left( \sup_{x \in \mathcal{Y}} \left| \frac{1}{M} \sum_{j=1}^{M} Z(\omega_j)(x) - \mathbb{E}_{\mu}(Z(\cdot)(x)) \right| \geq 4n(W, C)R \sqrt{\frac{\log(2|C|/\delta)}{M}} \right) \leq \delta.
\] (6.32)

Next, we recall the Lambert function defined by

\[
\mathcal{W}(ze^z) = z, \quad W(z) > 1 \text{ if } z \geq 1.
\] (6.33)

It is known that

\[
\mathcal{W}(x) = \log x - \log \log x + o(1), \quad x \to \infty.
\] (6.34)

The following lemma [27 Lemma 6.1] is a version of (6.34) that is convenient for our use.

**Lemma 6.2** Let \( y, \alpha, \beta, A, B > 0 \). The solution of the equation

\[
Ax^\alpha \log(Bx^\beta) = y
\] (6.35)

is given by

\[
x = B^{-1/\beta} \exp \left( \frac{1}{\alpha} W \left( \frac{B^{\alpha/\beta} \alpha}{A \beta y} \right) \right) \approx \left( \frac{\alpha}{\beta A} \right)^{1/\alpha} \left\{ \frac{y}{\log y + \log \left( \frac{\alpha B^{\alpha/\beta}}{\beta A} \right)} \right\}^{1/\alpha},
\] (6.36)

where \( \approx \) denotes an asymptotic relationship.

With this preparation, we are in a position to prove Theorem 6.2.

**Proof of Theorem 6.2** Let \( x_0 \in \mathbb{R}^q, n \geq 1 \). We apply Theorem 6.3 with \( \mathcal{Y} = \mathbb{R}^Q, \Pi_n^Q \) in place of \( W, \mathcal{X} \times \Omega \) in place of \( \mathcal{Y} \), \( \tau \) as in Theorem 6.2. We take

\[
Z(z, \epsilon)(x) = F_n(z, \epsilon)\Phi_n(z, \epsilon)(x, z),
\]

where \((z, \epsilon)\) is a random sample from \( \tau \) (Necessarily, \( z \in \mathcal{X} \)). Then the quantity \( R \) in Theorem 6.3 can be chosen to be \( c_1\|F\|_{\mathcal{X} \times \Omega}n^q \). Moreover, (6.26) shows that

\[
\mathbb{E}_{\tau}(Z(x)) = \sigma_n(\mathcal{X}; f)(x).
\]

In view of Corollary 2.1 there exists a set \( C \subseteq \mathbb{R}^Q \) for \( \Pi_n^Q \) with \(|C| \sim n^{2Q} \) satisfying (6.31) with \( n(C, \Pi_n^Q) = 2 \). Theorem 6.3 shows that

\[
\text{Prob}_{\tau} \left( \left\| \frac{1}{M} \sum_{j=1}^{M} F(y_j, \epsilon_j)\Phi_n(\cdot, y_j) - \sigma_n(\mathcal{X}; f) \right\|_{\mathbb{R}^Q} \geq c_1\|F\|_{\mathcal{X} \times \Omega} n^{q/2} \sqrt{\frac{\log(c_2n^{2Q}/\delta)}{M}} \right) \leq \delta.
\] (6.37)

This proves (6.28).

Since \( f \in \text{Lip}(\mathcal{X}) \), Theorem 6.1 shows that

\[
\|f - \sigma_n(\mathcal{X}; f)\|_{\mathcal{X}} \leq c\|f\|_{\text{Lip}(\mathcal{X})} n^{-\gamma}.
\] (6.38)
Together with (6.37), this yields
\[
\Pr_{\tau} \left( \left\| \frac{1}{M} \sum_{j=1}^{M} F(y_j, \epsilon_j) \Phi_n(o, y_j) - f \right\|_X \geq c_1 \|F\|_{X^\Theta} n^6 \sqrt{\log(\frac{c_2 n^{2Q}/\delta}{M})} + c(f, \gamma)n^{-\gamma} \right) \leq \delta. \tag{6.39}
\]
To prove (6.30), we choose \( n \) using Lemma 6.2 so that
\[
c_1 \|F\|_{X^\Theta} n^6 \sqrt{\log(\frac{c_2 n^{2Q}/\delta}{M})} = c(\|f\|_{\text{Lip}(X)}, \gamma)n^{-\gamma}.
\]

**Theorem 6.4** Let (6.5) be satisfied, \( \tau \) be a probability distribution on \( X \times \Omega \) for some sample space \( \Omega \) such that the marginal distribution of \( \tau \) restricted to \( X \) is \( \mu^* \). Let \( F : X \times \Omega \to \mathbb{R} \) be a bounded function, and \( f \) defined by (6.25) be in \( \text{Lip}(X) \). Let \( 0 < \gamma, \delta < 1 \). Let \( M \geq 1, Y = \{(y_1, \epsilon_1), \cdots, (y_M, \epsilon_M)\} \) be a set of random samples chosen i.i.d. from \( \tau \), and define (cf. (3.3))
\[
\mathcal{G}_{n,q,Q}(Y; F)(x) = \frac{1}{M} \sum_{j=1}^{M} F(y_j, \epsilon_j) \Phi_n(\Phi_{n,q,Q}(o, y_j))(x), \quad x \in \mathbb{R}^Q. \tag{6.40}
\]
Then with
\[
n \sim \left( \frac{M}{\log(\frac{M}{\delta})} \right)^{\frac{1}{2+2\gamma}}, \tag{6.41}
\]
we have
\[
\Pr_{\tau} \left( \left\| \mathcal{G}_{n,q,Q}(Y; F) - f \right\|_X \geq c_3(\|F\|_{X^\Theta}, c(\|f\|_{\text{Lip}(X)}, \gamma)) \right) \leq \delta. \tag{6.42}
\]
**Proof.** The theorem follows easily from Theorem 6.2 and Proposition 3.1. \(
\]
**Remark 6.1** Even though \( \mathcal{G}_{n,q,Q}(Y; F) \) contains \( \mathcal{O}(n^{2Q}) \) neurons, their centers are fixed, and the network is constructed using pre-fabricated networks \( \Phi_n(\Phi_{n,q,Q}(o, y_j)) \). There is no “training” involved, and the number of samples of \( F \) used is \( M \). \( \blacksquare \)

## 7 Deep networks

The following discussion about the terminology about the deep networks, is based on the discussion in [28, 29], and elaborates upon the same. In particular, Figure 1 is taken from the arxiv version of [28].

A commonly used definition of a deep network is the following. Let \( \phi : \mathbb{R} \to \mathbb{R} \) be an activation function; applied to a vector \( x = (x_1, \cdots, x_q) \), \( \phi(x) = (\phi(x_1), \cdots, \phi(x_q)) \). Let \( L \geq 2 \) be an integer, for \( \ell = 0, \cdots, L \), let \( q_\ell \geq 1 \) be an integer (\( q_0 = q \)), \( T_\ell : \mathbb{R}^{q_\ell} \to \mathbb{R}^{q_{\ell+1}} \) be an affine transform, where \( q_{L+1} = 1 \). A deep network with \( L - 1 \) hidden layers is defined as the compositional function
\[
x \mapsto T_L(\phi(T_{L-1}(\phi(T_{L-2} \cdots \phi(T_0(x)) \cdots))). \tag{7.1}
\]
This definition has several shortcomings. First, it does not distinguish between a function and the network architecture. As we have seen, a function may have more than one compositional representation, so that the affine transforms and \( L \) are not determined uniquely by the function itself. Second, this notion does not capture the connection between the nature of the target function and its approximation. Third, the affine transforms \( T_\ell \) define a special directed acyclic graph (DAG). It is cumbersome to describe notions of weight sharing, convolutions, sparsity, skipping of layers, etc. in terms of these transforms. Therefore, we have proposed in [28] to separate the architecture from the function itself, and describe a deep network more generally as a directed acyclic graph (DAG) architecture.

Let \( G \) be a DAG, with the set of nodes \( V \cup S \), where \( S \) is the set of source nodes, and \( V \) that of non-source nodes. For each node \( v \in V \cup S \), we denote its in-degree by \( d(v) \). Associated with each \( v \in V \cup S \) is a compact,
connected, Riemmanian submanifold $\mathbb{X}_v$ of $\mathbb{R}^{d(v)}$ with dimension $q_v$, metric $\rho_v$, and volume element $\mu^*_v$. We assume further that $[6, 5]$ is satisfied with $q_v$ in place of $q$. Each of the in-edges to each node in $V \cup S$ represents an input real variable. If $v \in V$, $u \in V \cup S$, $u$ is called the child of $v$ if there is an edge from $u$ to $v$. The notion of the level of a node is defined as follows. The level of a source node is 0. The level of $v \in V$ is the length of the longest path from the nodes in $S$ to $v$.

Each node $v$ is supposed to evaluate a function $f_v$ on its input variables, supplied via the in-edges for $v$. The value of this function is propagated along the out-edges of $v$. Each of the source nodes obtains an input from some Euclidean space. Other nodes can also obtain such an input, but by introducing dummy nodes, it is convenient to assume that only the source nodes obtain an input from the Euclidean space.

Intuitively, we wish to say that the DAG structure implies a compositional structure for the functions involved; for example, if $u_1, \cdots, u_{d(v)}$ are children of $v$, then the function evaluated at $v$ is $f_v(f_{u_1}, \cdots, f_{u_{d(v)}})$. To make this meaningful, we have assumed some “pooling” operation on the input variables to make sure that the output of the vector valued function $(f_{u_1}, \cdots, f_{u_{d(v)}})$ belongs to $\mathbb{X}_v$. Thus, for example, if the domain of $f_v$ is the cube $[-1, 1]^{d(v)}$, some clipping operation is required; if the domain is the torus in $d(v)$ dimensions then some standard substitutions need to be made (e.g., [29]). We assume that this pooling operation $\pi_v : \mathbb{R}^{d(v)} \rightarrow \mathbb{X}_v$ has the following property: For any set of functions $\{f_v \in C(\mathbb{X}_v), \{g_v \in C(\mathbb{X}_v)\}$,

$$\rho_v \left(\pi_v(f_{u_1}(x_{u_1}), \cdots, f_{u_{d(v)}}(x_{u_{d(v)}})), \pi_v(g_{u_1}(x_{u_1}), \cdots, g_{u_{d(v)}}(x_{u_{d(v)}}))\right) \leq c(v) \sum_{k=1}^{d(v)} \|f_{u_k} - g_{u_k}\|_{\mathbb{X}_v}. \quad (7.2)$$

A $\mathcal{G}$-function is defined to be a set of functions $\{f_v\}_{v \in V \cup S}$ such that each $f_v \in C(\mathbb{X}_v)$, and if $v \in V$, $u_1, \cdots, u_{d(v)}$ are children of $v$, then the function evaluated at $v$ is $f_v(\pi_v(f_{u_1}, \cdots, f_{u_{d(v)}}))$. The individual functions $f_v$ will be called constituent functions.

For example, the DAG $\mathcal{G}$ in Figure 1 (28) represents the compositional function

$$f^*(x_1, \cdots, x_9) = h_{19}(h_{17}(h_{13}(h_{10}(x_1, x_2, x_3, h_{16}(h_{12}(x_6, x_7, x_8, x_9)))), h_{11}(x_4, x_5)),$$

$$h_{14}(h_{10}, h_{11}), h_{16}, h_{18}(h_{15}(h_{11}, h_{12}), h_{16}). \quad (7.3)$$

The $\mathcal{G}$-function is $\{h_{10}, \cdots, h_{19} = f^*\}$.

We assume that there is only one sink node, $v^*$ (or $v^*(\mathcal{G})$) whose output is denoted by $f_{v^*}$ (the target function). Technically, there are two functions involved here: one is the final output as a function of all the inputs to all source nodes, the other is the final output as a function of the inputs to the node $v^*$. We will use the symbol $f_{v^*}$ to denote both with comments on which meaning is intended when we feel that it may not be clear from the context. A similar convention is followed with respect to each of the constituent functions as well. For example, in the DAG of Figure 1 the function $h_{14}$ can be thought of both as a function of two variables, namely the outputs of $h_{10}$ and $h_{11}$ as well as a function of five variables $x_1, \cdots, x_5$. In particular, if each constituent function is a neural network, $h_{14}$ is a shallow network receiving two inputs.

We define the notion of the variables “seen” by a node. If $u \in S$, then these are the variables input to $u$. Let $v \in V$, and $u_1, \cdots, u_{d(v)}$ be the children of $v$. If $x_1, \cdots, x_{d(v)}$ are the inputs seen by $u_1, \cdots, u_{d(v)}$, then the inputs seen by $v$ are $(x_1, \cdots, x_{d(v)})$, where the order is respected. For example, consider the function

$$f^*(x_1, x_2, x_3, x_4) = f(f_1(x_1, x_2), f_2(x_4, x_2), f_3(x_3, x_1)).$$

The inputs seen by the leaves $f_1$, $f_2$, $f_3$ are $(x_1, x_2), (x_4, x_2), (x_3, x_2)$ respectively (not $(x_1, x_2), (x_2, x_4), (x_2, x_3)$). The inputs seen by $f^*$ are $(x_1, x_2, x_3, x_4)$.

The following theorem enables us to “lift” a theorem about shallow networks to that about deep networks.

**Theorem 7.1** Let $\mathcal{G}$ be a DAG as described above, $\{f_v\}_{v \in V \cup S}, \{g_v\}_{v \in V \cup S}$ be $\mathcal{G}$-functions, and

$$\|f_v - g_v\|_{\mathbb{X}_v} \leq \varepsilon, \quad v \in V \cup S. \quad (7.4)$$

Further assume that for each $v \in V$, $f_v \in \text{Lip}(\mathbb{X}_v)$, with $L = \max_{v \in V} \|f_v\|_{\text{Lip}(\mathbb{X}_v)}$. Then for the target function, thought of as a compositional function of all the input variables $x$ to all the nodes in $S$, we have

$$|f_{v^*}(x) - g_{v^*}(x)| \leq c(L, \mathcal{G})\varepsilon. \quad (7.5)$$

**Proof.** Let $v \in V$, and $u_1, \cdots, u_{d(v)}$ be the children of $v$, and $x_1, \cdots, x_{d(v)}$ be the inputs seen by these in that order. Let $x$ be the corresponding input seen by $v$. Then using the Lipschitz condition on $f_v$ and the property

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Figure 1: An example of a $G$-function ($f^*$ given in (7.3)). The vertices of the DAG $G$ are the channels of the network. The input to the various channels is indicated by the in–edges of the nodes, and the output of the sink node $h_{19}$ indicates the output value of the $G$-function, $f^*$ in this example.

We now use induction on the level of $v$. Thus, if $v^* \in S$, then the “shallow network” estimate (7.4) is already the one which we want. Suppose the theorem is proved for the DAGs for which the sink node is at level $\ell \geq 0$. If $v \in V$, so that its level $\ell \geq 1$, then its children are at level $\ell - 1 \geq 0$. For each of the children, say $u$, we consider the subgraph $G_u$ of $G$ comprising only those nodes and edges that culminate in $u$ as the sink node. We then apply the theorem to each of these subgraphs, and then use (7.6) to conclude that the statement is true for the subgraph $G_v$ of $G$ comprising only those nodes and edges that culminate in $v$ as the sink node.

**Theorem 7.2** Let $G$ be a DAG as described above, $\{f_v\}$ be a $G$-function, and we assume that each of the constituent functions $f_v \in \Lip(X_v)$. Let $0 < \gamma < 1$, and $n \geq 1$. Then there exists a $G$-function $\{g_v\}$ such that each $g_v$ is a Gaussian network constructed using $O(n^{2\gamma + 2\gamma})$ samples of its inputs, such that for any $x$ seen by $v^*$,

$$|f_v(x) - g_v(x)| \leq c(L, G)n^{-\gamma}. \quad (7.7)$$

**Remark 7.1** Suppose we consider a shallow Gaussian network acting on a $2^* \cdot n$ dimensional manifold of $\mathbb{R}^Q$. The number of samples required to obtain an accuracy of $n^{-\gamma}$ predicted by Theorem 6.4 is $O(n^{2^{n-1} + 2\gamma}) \log n$. On the other hand, suppose the target function has a compositional structure according to a binary tree, but in addition,
for any \( v \in V \) with children \( u_1, u_2 \), the image of \((f_{u_1}, f_{u_2})\) forms a curve in \( \mathbb{R}^2 \). Then the number of samples required to get the same accuracy with the corresponding network is only \( O(n^{2+2\gamma} \log n) \) at each level. In fact, it seems likely that this is the number of samples in the original submanifold of \( \mathbb{R}^Q \) itself, since the input variables external to the machine are given only at the source nodes.

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