Vacuum, identical particles, and causal inequalities

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This work overviews the single-particle two-way communication protocol recently introduced by del Santo and Dakić (dSD), and analyses it using the process matrix formalism. We give a detailed account of the violation of causal inequalities via this protocol and discuss the importance of the vacuum state — in particular its role in the process matrix description. Motivated by this analysis, we extend the process matrix formalism to Fock space using the framework of second quantisation, in order to characterize protocols with an indefinite number of identical particles.

I. INTRODUCTION

In recent years there have been advances in quantum information theory related to new techniques for discussing quantum circuits and quantum computation. One of those techniques is the recently developed process matrix formalism [1]. This formalism is general enough to describe all known quantum processes, in particular the superposed orders of operations in a quantum circuit. Moreover, its most prominent feature is that it allows for a description of more general situations of indefinite causal orders of spacetime events. A formal example of such a process has been introduced and discussed in [1], leading to the violation of the so-called causal inequalities. The latter represent device-independent conditions that need to be satisfied in order for a given process to have a well-defined causal order. It is still an open question whether such a process is physical and can be realised in nature. Also, a lot of attention in the literature has been devoted to the quantum switch operation, which has been discussed through both theoretical descriptions [2–5] and experimental implementations [6–8].

One of the interesting aspects of the quantum switch is that it gives rise to a superposition of orders of quantum operations. In a recent work [9], the difference between the superposition of orders of quantum operations and the superposition of causal orders in spacetime was discussed in detail, and it was demonstrated that the latter can in principle be realised only in the context of quantum gravity. The detailed analysis of the causal structure of the quantum switch has revealed one important qualitative aspect of the process matrix description — in order to properly account for the causal structure of an arbitrary process, it is necessary to introduce the notion of the quantum vacuum as a possible physical state. Otherwise, the naive application of the process matrix formalism may suggest a misleading conclusion that quantum switch implementations in flat spacetime feature genuine superpositions of spacetime causal orders. This demonstrates the importance of the concept of vacuum in quantum information processing.

Simultaneously with these developments, another interesting quantum process has been recently proposed [10] by del Santo and Dakić (dSD protocol), in the context of causal inequalities and the so-called “Guess Your Neighbor’s Input” (GYNI) game [11]. As it turns out, while this process enables one to win one version of the GYNI game with certainty, it cannot be correctly described within the process matrix formalism without the introduction of the vacuum state. Thus, it represents an additional motivation to introduce the vacuum state into the process matrix formalism, independent of the reasons related to the description of the quantum switch process.

Moreover, while the dSD protocol employs only one photon, it is also relevant for multiphoton processes, which opens the question of the treatment of identical particles within the process matrix formalism. Also, taking into account the presence of the vacuum state, one is steered towards the generalisation of the process matrix formalism to systems with variable number of identical particles, to the second quantisation and ultimately quantum field theory (QFT).

In this work we give a detailed description and treatment of dSD protocol within the process matrix formalism. In addition, we make use of the protocol as an illuminative example to introduce the generalisation of the process matrix formalism to multipartite systems of identical particles.

The paper is organized as follows. In Section II we give a short overview of the process matrix formalism and the causal inequalities. Section III reviews the dSD protocol. Section IV is devoted to the process matrix formalism description of dSD protocol, and discuss the role and importance of the vacuum state for its description. In Section V we provide the basic rules for the generalisation of the process matrix formalism to identicalmultipartite systems. Section VI is devoted to the summary of our results, discussion and prospects for future research. The Appendix contains various technical details of the calculations.
II. STATE OF THE ART

In this section, we present an overview of the relevant background results. First, we give a short introduction to the process matrix formalism, then we discuss the causal inequalities, and finally we present the dSD protocol. This overview is not intended to be complete or self-contained, but merely of informative type. The reader should consult the literature for more details.

A. The process matrix formalism

The process matrix formalism is based on an idea of a set of laboratories, interacting with the outside world by exchanging quantum systems. Each laboratory is assumed to be spatially local in the sense that one can consider its size negligible for the problem under discussion. Inside the laboratory, it is assumed that the ordinary laws of quantum theory hold. The laboratory interacts with the outside world by receiving an input quantum system and by sending an output quantum system. Inside the laboratory, the input and output quantum systems are being manipulated using the notion of an instrument, denoting the most general operation one can perform over quantum systems. Each interaction is also assumed to be localised in time, such that each operation of a given laboratory has a separate spacetime point assigned to it. Thus, we introduce the notion of a gate, which represents the action of an instrument at a given spacetime point (see Section 2 of [3]); for simplicity, both the gate, as well as its corresponding spacetime point, we denote by the same symbol, $G$. By $G_I$ and $G_O$ we denote the Hilbert spaces of the input and the output quantum systems, respectively. These Hilbert spaces may be infinite-dimensional, finite-dimensional or trivial. The action of the instrument is described by an operator, $M_{x,a}^G : G_I \otimes G_I^* \rightarrow G_O \otimes G_O^*$, which may depend on some classical input information $a$ available to the gate $G$, and some readouts $x$ of eventual measurement results that may take place in $G$. Thus, the instrument maps a generic mixed input state $\rho_I$ into the output state $\rho_O = M_{x,a}^G(\rho_I)$.

Given such a setup, one defines a process, denoted $W$, as a functional over the instruments of all gates, as

$$p(x,y,\ldots|a,b,\ldots) = W(M_{x,a}^{G^{(1)}} \otimes M_{y,b}^{G^{(2)}} \otimes \ldots),$$

where $p(x,y,\ldots|a,b,\ldots)$ represents the probability of obtaining measurement results $x, y, \ldots$, given the inputs $a, b, \ldots$. In order for the right-hand side to be interpreted as a probability distribution, the process $W$ must satisfy three basic axioms,

$$W \geq 0, \quad \text{Tr} W = \prod_i \dim G_O^{(i)}, \quad W = \mathcal{P}_G(W),$$

where $\mathcal{P}_G$ is a certain projector onto a subspace of $\bigotimes_i (G_I^{(i)} \otimes G_O^{(i)})$ which, together with the second requirement, ensures the normalisation of the probability distribution (see [3] for details).

In order to have a computationally manageable formalism, one often employs the Choi-Jamiolkowski (CJ) map over the instrument operations, such that a given operation $M_{x,a}^G$ is being represented by a matrix,

$$M_{x,a}^G = (I \otimes M_{x,a}^G) (1) \otimes 1 \otimes (G_I \otimes G_O \otimes G_I \otimes G_O)^*,$$

where

$$| \mathbf{1} \rangle = \sum_i |i\rangle \otimes |i\rangle \in G_I \otimes G_I$$

is the so-called transport vector, representing the non-normalised maximally entangled state, and $I$ is the identity operator. Then, one can describe the process $W$ using the process matrix $W$ to write

$$p(x,y,\ldots|a,b,\ldots) = \Tr \left[ (M_{x,a}^{G^{(1)}} \otimes M_{y,b}^{G^{(2)}} \otimes \ldots) W \right].$$

Finally, if an instrument $M_{x,a}^G$ is linear, one can also use a corresponding “vector” notation (see Appendix A.1 in [3]),

$$|(M_{x,a}^G)^* \rangle \equiv [I \otimes (M_{x,a}^G)^*] | \mathbf{1} \rangle \in G_I \otimes G_O,$$

so that

$$M_{x,a}^G = |(M_{x,a}^G)^* \rangle \langle (M_{x,a}^G)^*|.$$

In cases where all instruments are linear, and in addition the process matrix $W$ is a one-dimensional projector, one can introduce the the corresponding process vector $|W\rangle$, such that $W = |W\rangle \langle W|$, and rewrite (4) in the form:

$$p(x,y,\ldots|a,b,\ldots) =$$

$$\left\| \left( (M_{x,a}^{G^{(1)}})^* \otimes (M_{y,b}^{G^{(2)}})^* \otimes \ldots \right) |W\rangle \right\|^2.$$

B. Causal inequalities

In order to introduce the causal inequalities, we start by describing a game motivated by the GYNI game [11]. In its simplest version, two parties, Alice and Bob, are enclosed in separated laboratories, and each is allowed exactly once to receive and subsequently send a particle carrying one bit of information. In addition, upon receiving and before sending their corresponding particles, Alice and Bob each toss a coin, randomly generating one bit of information, $a$ and $b$, respectively. Afterwards, both are required to answer the question what bit was generated by the other party. We denote their answers as $x$ and $y$, respectively. Classically, the optimal probability that both Alice and Bob can correctly answer this question is $1/2$ [12]. For example, if Alice has generated her bit in the causal past of Bob’s coin toss, she can use
the particle to encode information about her input and send it to Bob, thus making sure that Bob will answer the question with certainty. However, Alice can then answer her own question (about Bob’s input) only with probability $1/2$, since the input bit she receives cannot carry information about Bob’s bit. The above rules of the game satisfy three general assumptions (causal structure (CS), free choice (FC) and closed laboratories (CL), see [1] for details), using which the probability of successful guessing can be rigorously formulated as a causal inequality,

$$p_{\text{success}} \equiv p(x = b \land y = a) \leq \frac{1}{2}.$$  

Another version of this game, sometimes called “Lazy Guess Your Neighbor’s Input” (LGYNI), is a similar game, where we only ask one of the parties to answer the question about the input of their neighbor, rather than both of them. For the LGYNI case, one can formulate another causal inequality,

$$p_{\text{success}} \equiv p(a(x \oplus b) = 0 \land b(y \oplus a) = 0) \leq \frac{3}{4}.$$  

It has been proven that any other causal inequality with two input bits can be understood as one of the above two inequalities [12].

The natural question that arises is whether the bounds given by causal inequalities can be violated beyond classical physics. To this end, one can employ the process matrix formalism, and a bipartite process matrix has been found which allows for the violation of the causal inequalities [1]. Nevertheless, it is still an open problem whether this process matrix can in fact be realised physically in nature.

In the next section, we present the dSD protocol, and subsequently analyse its process matrix formulation and the reasons behind the algebraically maximal violation of the causal inequality [1].

III. THE dSD PROTOCOL

In a recent paper [11], del Santo and Dakić have introduced a protocol which wins the above version of the GYNI game with certainty, thus maximally violating the above causal inequalities. The protocol goes as follows. Initially, a single particle is prepared in a superposition state of being sent to Alice and being sent to Bob. Upon receiving the particle, both Alice and Bob perform unitary operations on it, encoding their bits of information, $a$ and $b$, respectively, about the outcomes of their coin tosses. They do this by changing the local phase of the particle by $(-1)^a$ and $(-1)^b$. The particle is subsequently forwarded to a beam splitter, and after that again to Alice and Bob, who now measure the presence or absence of the particle.

This way, the state of the particle stays in coherent superposition of different paths in a Mach-Zehnder interferometer. The interference of its paths gives rise to deterministic outcome that depends on the relative phase $e^{i\phi} = (-1)^a \otimes b$ between the two branches: in case $\phi = 0$, the particle will end up in Alice’s laboratory, while otherwise it will end in Bob’s. Thus, knowing their own inputs and the outcomes of their local measurements, both agents can determine each other’s inputs, allowing for two-way communication using only one particle. This is clearly impossible in classical physics, demonstrating yet another example of the advantage of quantum over classical solutions.

The crucial aspect of the protocol lies in the fact that the absence of the particle represents a useful piece of information for an agent. This gives rise to the notion of the vacuum state as a carrier of information, playing the central role in the protocol. Thus, in order to describe the protocol using the process matrix formalism, one has to incorporate the notion of the vacuum in the formalism itself. We show this in detail in the next section.

Another important aspect of the dSD protocol is the maximal violation of the causal inequalities. One may then ask the question which of the three assumptions in the proof of causal inequalities (CS, FC or CL) are violated by the protocol. However, it turns out that all three of them are satisfied. In fact, there is an additional implicit assumption that the causal inequality is formulated for the protocol featuring only two spacetime events, and this assumption is violated in the dSD protocol. This will also be discussed in detail within the process matrix formalism in the next section.

It is interesting to note that these two features — the crucial role of the vacuum and the temporal nonlocality of laboratories — play an important part not only in the dSD protocol, but also in a completely different setup that has been discussed a lot in recent literature, namely the quantum switch [2]. As analysed in detail in [3], if one takes care to distinguish the two temporal positions of a given laboratory and introduces the notion of a vacuum explicitly, one can demonstrate that the optical implementations of the quantum switch in flat spacetime do not feature any superposition of causal orders induced by the spacetime metric. Instead, it was argued that superpositions of spacetime causal orders can be present only within the context of a theory of quantum gravity. As we shall see below, the notions of the vacuum and of temporal locality of instrument activities in a laboratory will both prove essential to the process matrix description of the dSD protocol as well.

IV. PROCESS MATRIX DESCRIPTION OF THE dSD PROTOCOL

A. Preliminaries

In order to successfully formulate a process matrix for the dSD protocol, there are two crucial principles that must be upheld:

A: Given a space-localised laboratory and an instru-
ment in it, the activity of the instrument must be
time-localised, in effect making the action of the
gate in a quantum circuit localised both in space
and time.

B: The action of an instrument over the vacuum state
must be explicitly accounted for via the appropriate
gate in a quantum circuit.

Regarding these principles, there are two important com-
ments that need to be made. First, regarding the
principle A, the notions of “space-localised” and “time-
localised” gates are relative to the process under consid-
eration. An instrument $\mathcal{M}^G$ of gate $G$ may be composed
of several smaller instruments, and it can be considered
localised in space and time only as far as it does not in-
teract with its environment during its execution. In such
a case the space-time arrangement of its constituent in-
struments can be considered irrelevant for the process in
which $\mathcal{M}^G$ participates. On the other hand, the “pieces”
of an instrument which interact with the environment
should be considered as separate gates.

For example, consider spatially localised laboratory of
Alice in which at moment $t_i$ a unitary $U$ is applied to
an incoming system, followed by a measurement of an
observable $M$ at the later moment $t_f$. As long as between
the initial and the final moments $t_i$ and $t_f$ Alice’s lab,
together with the system in it, does not interact with the
environment, the two corresponding gates, $G_U$ and $G_M$,
can be considered as one gate $G_{M'}$, with $M' = U^\dagger MU$
(see FIG. 1).

![FIG. 1: Implementing two consecutive Alice’s gates $G_U$ and
$G_M$ (upper diagram) with a single Alice’s gate $G_{M'}$ (lower
diagram).]

But, if upon applying $U$, Alice sends the system to
Bob, located in a spatially distant laboratory, who per-
forms unitary $V$ to the system at time $t^* < t_f$, and sends
it back to Alice, to measure $M$ at the time $t_f$, then one
cannot consider $G_U$ and $G_M$ to be one gate (see FIG. 2).

Second, regarding the principle B, the action of a
(spacetime localised) instrument over the vacuum state
cannot in principle be distinguished from its “idle” state,
in which the instrument is perpetually “ready” and wait-
ing for a nontrivial input quantum system (say, a parti-
cle) to act on. In such a state, the instrument is continu-
ously acting on the vacuum state, performing the identity
operation over the vacuum as its input quantum system.
In this sense, representing the action of an instrument on

A vacuum via a gate in a quantum circuit, as per prin-
ciple B, would entail an infinite sequence of gates, one for
each spacetime point along the worldline of the instru-
ment. Some of these gates may or may not be relevant
for the description of the process itself. Deciding which
gates are relevant and which are not relevant depends
on the context of the process in which the instrument is
participating. Relevant gates are those which may have
more than one different input or output states, while all
others are irrelevant. For example, all gates between the
relevant gates $G_U$ and $G_M$ in FIG. 1 can only have one
input and output, namely the vacuum, and as such are
irrelevant.

With the conceptual points and principles A and B in
hand, we proceed to the construction of the process ma-
trix and the description of the relevant gates for the dSD
protocol. We begin by drawing the spacetime diagram
of the process corresponding to the dSD protocol (see FIG. 3).

![FIG. 2: Coupling two distinct Alice’s gates $G_U$ and $G_M$ with
Bob’s intermediate gate $G_V$.]

At the initial time $t_i$ the laser $L$ creates a photon and
shoots it towards the beam splitter $S$, which at time $t_1$
performs the Hadamard operation and entangles it with the incoming vacuum state (described by the dotted arrow from the grey gate \(V\)). The entangled state of the photon and the vacuum continues on towards Alice’s and Bob’s gates \(A\) and \(B\), respectively. At time \(t_2\), Alice and Bob generate their random bits \(a\) and \(b\), and encode them into the phase of the incoming photon-vacuum system. The system then proceeds to the beam splitter \(S’\) which again performs the Hadamard operation at time \(t_3\). The photon-vacuum system then proceeds to the gates \(A’\) and \(B’\), where it is measured at time \(t_f\) by Alice and Bob, respectively. Note that the spatial distance \(\Delta l\) between Alice and Bob is precisely equal to the time distance between the generation of the random bits and the final measurements,

\[
\Delta l = c(t_f - t_2),
\]

so that a single photon has time to traverse the space between Alice and Bob only once. Also, note that the gate \(V\), which generates the vacuum state, corresponds to a “trivial instrument”, since the vacuum does not require any physical device to be generated. Nevertheless, the vacuum is still a legitimate physical state of the EM field, so the appropriate gate \(V\) has to be formally introduced and accounted for in the process matrix formalism calculations.

### B. Formulation of the process matrix

Based on the spacetime diagram, we formulate the process matrix description as follows. All spacetime points, where interaction between the EM field and some apparatus may happen, are assigned a gate and an operation which describes the interaction. Each gate has an input and output Hilbert space, as follows:

\[
L : \mathbb{C} \rightarrow \mathcal{H}_L, \quad A : \mathcal{H}_A \rightarrow \mathcal{H}_O, \\
V : \mathbb{C} \rightarrow \mathcal{H}_V, \quad B : \mathcal{H}_O \rightarrow \mathcal{H}_B, \\
S : \mathcal{H}_S \rightarrow \mathcal{H}_O, \quad A' : \mathcal{H}_A \rightarrow \mathcal{C}, \\
S' : \mathcal{H}_S' \rightarrow \mathcal{H}_O, \quad B' : \mathcal{H}_B \rightarrow \mathcal{C}.
\]

The initial gates, \(L\) and \(V\), have trivial input spaces and nontrivial output spaces. The final gates, \(A’\) and \(B’\), have trivial output spaces and nontrivial input spaces. The gates \(A\) and \(B\) have nontrivial input and output spaces. Each nontrivial space is isomorphic to \(\mathcal{H}_1 \oplus \mathcal{H}_0 \subset \mathcal{F}\), where \(\mathcal{H}_0\) and \(\mathcal{H}_1\) are the vacuum and single-excitation subspaces of the Fock space \(\mathcal{F}\) in perturbative QED. Namely, by design of the dSD protocol, Alice and Bob may exchange at most one photon, which means that multiparticle Hilbert subspaces of the Fock space can be omitted. Moreover, the resulting probability distribution of the experiment outcomes does not in principle depend on the frequency or the polarisation of the photon in use, so we can approximate the single-excitation space as a one-dimensional Hilbert space \(\mathcal{H}_0\) is by definition one-dimensional, we can write

\[
\mathcal{H}_0 = \text{span}\{|0\rangle\} \equiv \mathbb{C}, \quad \mathcal{H}_1 = \text{span}\{|1\rangle\} \equiv \mathbb{C},
\]

so that \(\mathcal{H}_0 \oplus \mathcal{H}_1 \equiv \mathbb{C} \oplus \mathbb{C}\). Here, \(|0\rangle\) and \(|1\rangle\) denote the states of the vacuum and the photon in the occupation number basis of the Fock space. Therefore, we have

\[
L_O \equiv V_O \equiv A_I \equiv A_O \equiv B_I \equiv B_O \equiv A'_I \equiv B'_I \equiv \mathbb{C} \oplus \mathbb{C}.
\]

Finally, the input and output spaces of beam splitters \(S\) and \(S'\) are “doubled”, since a beam splitter operates over two inputs to produce two outputs. In particular,

\[
S_I = S_I^L \otimes S_I^V, \quad S'_I = S'_I^A \otimes S'_I^B, \\
S_O = S_O^L \otimes S_O^B, \quad S'_O = S'_O^A \otimes S'_O^B,
\]

where again

\[
S_I^L \equiv S_I^V \equiv S_O^A \equiv S_O^B \equiv S'_I^A \equiv S'_I^B \equiv S'_O^A \equiv S'_O^B \equiv \mathbb{C} \oplus \mathbb{C}.
\]

With all relevant Hilbert spaces defined, we formulate the action of each gate, using the CJ map in the form (5). The gates \(L\) and \(V\) simply generate the photon and the vacuum,

\[
|L^*\rangle^L_O = |1\rangle^L_O, \quad |V^*\rangle^V_O = |0\rangle^V_O, \quad (8)
\]

where * is the complex conjugation. The action of the beam splitters is

\[
|S^*\rangle^S_O = [I^S_I^A \otimes (H^*)^S_O S^I_I^A |1\rangle^S_I^A, \quad (9)
\]

and

\[
|S'^*\rangle^S'_O = [I^{S'}_I^{S'}_I^A \otimes (H^*)^{S'}_O S^{S'}_I^A |1\rangle^{S'}_I^{S'}_I^A, \quad (10)
\]

where the Hadamard operator for \(S\) is defined as

\[
H^{S_O S_I} = \frac{1}{\sqrt{2}} (|1\rangle^{S_O} S^B_0 |0\rangle^{S_B} + |0\rangle^{S_O} |1\rangle^{S_B} S^B_0 |0\rangle^{S_O} S^B_0 |1\rangle^{S_B}) |1\rangle^{S_I^I} |0\rangle^{S_I^I},
\]

and analogously for \(H^{S'_O S'_I}\). The unit operator is denoted as \(I\). Next, in the gates \(A\) and \(B\), Alice and Bob generate their random bits \(a\) and \(b\), and encode them into the phase of the photon. The corresponding actions are defined as

\[
|A^*\rangle^A_I A_O = [I^{A_I A_I} \otimes (A^*)^{A_O A_I} |1\rangle^A_I A_I, \quad (11)
\]

and

\[
|B^*\rangle^B_I B_O = [I^{B_I B_I} \otimes (B^*)^{B_O B_I} |1\rangle^B_I B_I, \quad (12)
\]

where

\[
A^{A_O A_I} = (-1)^a |1\rangle^{A_O} |1\rangle^{A_I} \oplus |0\rangle^{A_O} |0\rangle^{A_I},
\]
and
\[ B^{BO}_{B1} = (-1)^b |1 \rangle^{BO} \langle 1 |^{B1} \otimes |0 \rangle^{BO} |0 \rangle^{B1}. \]

Finally, the gates \( A' \) and \( B' \) describe Alice’s and Bob’s measurement of the incoming state in the occupation number basis,
\[ |A'^{\alpha} \rangle^{A_i} = |a'\rangle, \quad |B'^{\beta} \rangle^{B_i} = |b'\rangle, \quad \text{(13)} \]
where their respective measurement outcomes \( a' \) and \( b' \) take values from the set \( \{0, 1\} \), depending on whether the vacuum or the photon has been measured, respectively.

After specifying the actions of the gates, the last step is the construction of the process vector \( |W_{dSD}\rangle \) itself. The dSD protocol assumes that the state of the photon remains unchanged during its travel between the gates. Therefore, the process vector will be a tensor product of transport vectors, one for each line connecting two gates in the spacetime diagram. The input and output spaces of the gates connected by the line determine the spaces of the corresponding transport vectors. Thus, the total process vector is:
\[ |W_{dSD}\rangle = |1\rangle^{A0} S^L |\rangle^{BO} V^I |1\rangle^{A1} S^A |\rangle^{BO} B_1 |1\rangle^{AO} S^I |\rangle^{BO} S^{A_i} B_1 |1\rangle^{BO} S^B B_i. \quad \text{(14)} \]

C. Evaluation of the probability distribution

Now that the process vector and the operations of all gates have been specified in detail, we can evaluate the probability distribution
\[ p(a', b'|a, b) = \| \mathcal{M} \|^2, \quad \text{(15)} \]
where the probability amplitude \( \mathcal{M} \) is obtained by taking the scalar product of \( |W_{dSD}\rangle \) with the tensor product of all gates, see (6). It is most instructive to perform the computation iteratively, taking the partial scalar product of \( |W_{dSD}\rangle \) with each gate, one by one. The explicit calculation of each step is based on two lemmas from Appendix A.

We begin by taking the partial scalar product of (14) and the preparation gates (8). Using Lemma 1 from Appendix A we obtain:
\[ \langle L^* |^{LO} \langle V^* |^{VO} |W_{dSD}\rangle = \]
\[ |1\rangle^{A0} S^L |0\rangle^{I} |1\rangle^{AO} S^I |\rangle^{BO} S^{A_i} B_1 |1\rangle^{BO} S^B B_i. \]
Next we take the partial scalar product with the beam splitter S gate operation (9). Using Lemma 2 from Appendix A we obtain:
\[ \langle S^* |^{S1S0} \langle L^* |^{LO} \langle V^* |^{VO} |W_{dSD}\rangle = \]
\[ \frac{1}{\sqrt{2}} (|1\rangle^{A1} |0\rangle^{B1} + |0\rangle^{A1} |1\rangle^{B1}) \]
\[ |1\rangle^{AO} S^A |\rangle^{BO} S^B |1\rangle^{AO} S^A |\rangle^{BO} S^B B_i. \]

Now we apply the Alice’s gate operation (11) to obtain:
\[ \langle A^* |^{A1} \langle A_0 \rangle^{AO} \langle S^* |^{S1S0} \langle L^* |^{LO} \langle V^* |^{VO} |W_{dSD}\rangle = \]
\[ \frac{1}{\sqrt{2}} ((-1)^a |1\rangle^{S1} |0\rangle^{B1} + |0\rangle^{S1} |1\rangle^{B1}) \]
\[ |1\rangle^{AO} S^A |\rangle^{BO} S^B |1\rangle^{AO} S^A |\rangle^{BO} S^B B_i. \]

Similarly, applying Bob’s gate (12) we get:
\[ \langle B^* |^{B1} \langle B_0 \rangle^{BO} \langle A^* |^{A1} \langle A_0 \rangle^{AO} \langle S^* |^{S1S0} \langle V^* |^{VO} |W_{dSD}\rangle = \]
\[ \frac{1}{\sqrt{2}} ((-1)^a |1\rangle^{S1} |0\rangle^{B1} + |0\rangle^{S1} |1\rangle^{B1}) \]
\[ |1\rangle^{AO} S^A |\rangle^{BO} S^B |1\rangle^{AO} S^A |\rangle^{BO} S^B B_i. \]

The next step is the application of the second beam splitter gate (10). After a little bit of algebra, the result is:
\[ \langle S^* |^{S1S0} \langle B^* |^{B1} \langle B_0 \rangle^{BO} \langle A^* |^{A1} \langle A_0 \rangle^{AO} \langle S^* |^{S1S0} \langle V^* |^{VO} |W_{dSD}\rangle = \]
\[ \frac{(-1)^a + (-1)^b}{2} |1\rangle^{A1} |0\rangle^{B1} + \frac{(-1)^a - (-1)^b}{2} |0\rangle^{A1} |1\rangle^{B1}. \]

Finally, applying the measurement gates (13), we obtain the complete probability amplitude,
\[ \mathcal{M} = \frac{(-1)^a + (-1)^b}{2} \delta_{a'0} \delta_{b'0} + \frac{(-1)^a - (-1)^b}{2} \delta_{a'0} \delta_{b'1}, \]
and substituting this into (14), we obtain the desired probability distribution of the dSD process:
\[ p(a', b'|a, b) = \frac{1 + (-1)^{a+b}}{2} \delta_{a'0} \delta_{b'0} + \frac{1 - (-1)^{a+b}}{2} \delta_{a'0} \delta_{b'1}. \]

From the probability distribution we can now conclude that there are two distinct possibilities: either Alice detects the photon and Bob does not, \( a' = 1, b' = 0 \), or vice versa, \( a' = 0, b' = 1 \). In the first case, because total probability must be equal to one, we have
\[ \frac{1 + (-1)^{a+b}}{2} = 1, \quad \frac{1 - (-1)^{a+b}}{2} = 0. \]

The only solution to these equations is \( a = b \), which means that Alice and Bob have initially generated equal bits. Since both know the probability distribution and their own bit, they both know each other’s bit as well, with certainty. In the second case, when Bob detects the photon, we instead have
\[ \frac{1 + (-1)^{a+b}}{2} = 0, \quad \frac{1 - (-1)^{a+b}}{2} = 1, \]
and the only solution is \( a \neq b \), meaning that Alice and Bob have initially generated opposite bits. Again, both
parties know the probability distribution and their own bit, and therefore each other’s bit as well, with certainty.

In order to formalize this result, one can also introduce the parity $\pi \equiv a \oplus b$ and rewrite the probability distribution in the form

$$p(a', b' | \pi) = \frac{1 + (-1)^\pi}{2} \delta_{a'1} \delta_{b'0} + \frac{1 - (-1)^\pi}{2} \delta_{a'0} \delta_{b'1}. \quad (16)$$

Thus, if Alice detects the photon, then $\pi$ is even, while if Bob detects the photon, $\pi$ must be odd. In both cases, they can “guess” each other’s bits with certainty by calculating

$$x = \pi \oplus a, \quad y = \pi \oplus b,$$

where $x$ is Alice’s prediction of the value of Bob’s bit, and $y$ is Bob’s prediction of the value of Alice’s bit. Therefore, the probability of winning the GYNI game is

$$p_{\text{success}} \equiv p(x = b \land y = a) = 1 > \frac{1}{2}, \quad (17)$$

thus maximally violating the causal inequality $\square$.

D. Analysis of the process matrix description

After we have given the detailed process matrix description of the dSD protocol and derived the main result $\square$, we must reflect on two important points.

First, the obtained result $\square$ maximally violates the causal inequality $\square$, which is stated as a no-go theorem $\square$. Therefore, one of the assumptions of the theorem must be violated by the dSD protocol. As we have mentioned in Section $\square$ the proof of the causal inequality is based on three main assumptions — causal structure (CS), free choice (FC) and closed laboratories (CL). Looking at the spacetime diagram of the dSD protocol, Figure $\square$ it is straightforward to conclude that the CS assumption is not violated, since all causally related events are within, or on, their corresponding future and past light cones in Minkowski spacetime. The FC assumption also does not seem to be violated, since the dSD protocol does not impose any restrictions on the Alice’s and Bob’s generated bits $a$ and $b$. Either bit can take any value from the set $\{0, 1\}$ and the causal inequality is maximally violated for all choices of values.

Therefore, the CL assumption is seemingly the only one left which has to be violated by the dSD protocol. The formal definition of the CL assumption introduced in $\square$ says: Alice’s guess $x$ can be correlated with Bob’s bit $b$ only if $b$ has been generated in the causal past of the photon entering Alice’s laboratory (the latter denoted as event $A_1$). Also, Bob’s guess $y$ can be correlated with Alice’s bit $a$ only if $a$ has been generated in the causal past of the photon entering Bob’s laboratory (denoted as event $B_1$). Simply put, and without referring to events $A_1$ and $B_1$, the two classical variables $x$ and $b$ from the above described game can be correlated only if signalling from the latter to the former is possible (and the same for $y$ and $a$).

However, this definition of the CL assumption, as written in $\square$, is in fact not directly applicable to the dSD protocol, as it implicitly assumes only two spacetime points, $A_1$ and $B_1$. Indeed, in the proof of the theorem, between the events (and the corresponding gates) $A_1$, $a$ and $x$, that happen in Alice’s lab, no interaction occurs with the outside world. Therefore, according to the discussion from Subsection $\square$, they can all be approximated as a single gate and the corresponding single spacetime point (and the same for Bob). Hence the name “closed laboratories” for the assumption of the causal inequality. On the other hand, in the dSD protocol, there exists nontrivial interaction between Alice’s lab and the outside world after Alice’s coin flip $a$ (which happens at $A$) and before her guess $x$ (which happens at $A'$), since Alice sends and receives a photon in between. Thus, in the dSD case it is impossible to approximate events $A_1$, $a$ and $x$ as a single gate. This allows for two-way signalling (both $x$ and $y$ are correlated with $b$ and $a$, respectively) and maximal algebraic violation of the causal inequality, established in the protocol defined in a fixed causal order. In other words, the causal inequality derived under the assumption of protocols featuring only two spacetime points cannot serve to probe the existence of a fixed causal order for protocols featuring different number, say four, spacetime points. On the other hand, when a protocol is indeed performed featuring only two spacetime points, violation of the causal inequality may indeed signal the existence of the indefinite causal order, as shown in $\square$. In $\square$, it was argued that such indefinite causal orders of spacetime points may only be achieved in the context of quantum gravity, by coherently superposing two or more gravitational fields.

Second, we see from $\square$ that either Alice or Bob deduces the value of the parity $\pi$ precisely from the nondetection of the photon. In other words, detecting the absence of the photon is crucial for the successful execution of the protocol. In this sense, the vacuum state $|0\rangle$ plays a physically relevant role, and cannot be ignored. From the point of view of QFT this is a perfectly natural state of affairs, but from the point of view of quantum mechanics (QM) it is not, since the notion of vacuum as a physical state does not exist in QM. We understand this as a conceptual deficiency of QM as compared to QFT, since it is apparently impossible to provide a description of the dSD protocol within the formalism of QM alone. Hence the necessity of the principle B as spelled out at the beginning of Section $\square$. This advantage of QFT over QM is the main motivation for the generalisation of the process matrix formalism to QFT, which is further explored in Section $\square$. 
V. IDENTICAL PARTICLES

In this section, we give basic elements of the process matrix formalism, when applied to systems of identical particles. In order to avoid working with (anti-)symmetrised vectors of multi-particle states that contain nonphysical entanglement whenever two or more identical particles are fully distinguishable (say, one photon is in Alice’s, and another in Bob’s lab), we will use the representation of the second quantisation in which the effects of particle statistics are governed by the creation and annihilation (anti-)commutation rules. First, we need to move from the single-particle Hilbert spaces associated to the gates and the process matrix to the corresponding Fock spaces.

To each gate $G$, we assign the input/output Fock spaces, $G_{I/O}$, given in terms of the vacuum state $|0\rangle$ and the single-particle Hilbert spaces $G_{I/O}$. The single-particle input Hilbert space is given as

$$G_I = \text{span} \left\{ |i\rangle = a_i^\dagger |0\rangle \mid i = 1, 2, \ldots, d_I \right\},$$

such that its creation and annihilation operators satisfy the standard (anti-)commutation relations,

$$[a_i^\dagger, a_j] = [a_i, a_j] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}, \quad (18)$$

where $[,]^\dagger$ stands for anti-commutator, and $[,]$ for commutator. The overall bosonic input Fock space is then

$$G_I = \bigoplus_{\ell=0}^{\infty} G_I(\ell), \quad (19)$$

where $G_I(0) = \text{span} \{|0\rangle\}$ is the zero-particle, $G_I(1) = G_I$ the single-particle, and

$$G_I(\ell) = \{|(a_i^\dagger)^{s_1} \ldots (a_d^\dagger)^{s_d}|0\rangle \mid s_1 + \ldots + s_d = \ell \}$$

are the $\ell$-particle orthogonal subspaces of the input Fock space. For fermions, each $s_i \in \{0, 1\}$, and the orthogonal sum in Equation (19) goes until $d_I$, instead of $\infty$.

For a given gate, the output Fock space $G_O$ is defined analogously, and we denote its creation and annihilation operators as $\tilde{a}_i^\dagger$ and $\tilde{a}_i$, respectively, in order to distinguish them from the corresponding operators in $G_I$.

Our formalism is constructed for quantum circuits which consist of finite number of gates. This means that we work in the approximation of a finite number of spacetime points, as opposed to the standard QFT where one works with an uncountably infinitely many spacetime points. Thus, given the algebra $\{18\}$ for the creation and annihilation operators at a single gate, the full algebra across all gates is normalised to a Kronecker delta, instead of the standard Dirac delta function. Moreover, the operators in $\{18\}$ are operators in coordinate space, as opposed to the momentum space operators which are standard in QFT, since they create and annihilate modes at a given gate (i.e., a given spacetime point), instead of modes with a given momentum. Since the gates are distinguishable, the modes assigned to different gates always (anti-)commute.

We restrict ourselves to the Minkowski spacetime, so that the global Poincaré symmetry implies that the vacuum state $|0\rangle$ is identical across different gates, as well as between input and output Fock spaces for a given gate. In this sense, each gate is assumed to be stationary in some inertial reference frame, since the Fock spaces of non-inertial gates would be subject to the Unruh effect. We leave the discussion of non-inertial gates and spacetimes with more general geometries for future work.

Once the Fock spaces have been defined, we pass on to the process matrix description of gate operations. A gate operation is represented via a CJ isomorphism of the corresponding operator between the input and the output Fock spaces, defined as a generalisation of $\{12\}$,

$$M = \sum_{k=0}^{\infty} \left( [I \otimes \mathcal{M}_k] \langle I_k | \right)^T,$$

where the $k$-transport vectors are defined as

$$|I_k\rangle = \sum_{s_1 + \ldots + s_d = k} \left[ \prod_{i=1}^{d} \left( a_i^\dagger \right)^{s_i} \right] \otimes \left[ \prod_{i=1}^{d} \left( a_i \right)^{s_i} \right] |0\rangle, \quad (20)$$

and $\mathcal{M}_k$ represents the $k$-particle operator for the gate.

In case of a linear gate operation, one can analogously use the “vector” formalism, and the generalisation of the CJ vector $[15]$, where now we can write

$$\mathcal{M} = \sum_{k=0}^{\infty} \mathcal{M}_k,$$

and

$$|I\rangle = \sum_{k=0}^{\infty} |I_k\rangle. \quad (21)$$

Here, it is assumed that by definition

$$\mathcal{M}_k |I_{k'}\rangle = 0, \quad k \neq k'.$$

For example, one can consider a single-particle unitary operator

$$U = \sum_{i,j} u_{ij} \tilde{a}_i^\dagger a_j,$$

and then its Fock-space generalisation is given as

$$\mathcal{M} = |0\rangle \langle 0| + \sum_{k=1}^{\infty} \frac{1}{k!} : U^{\otimes k}:,$$

where $: U^{\otimes k} :$ is the normal ordering of $U^{\otimes k}$.

Given the Fock spaces and actions of instruments in all gates, a process matrix is defined in the same way.
as in Section $11$ according to Eq. $3$. In the field theory case, the process matrix maps the tensor product of output Fock spaces for all gates into the tensor product of input Fock spaces for all gates. For example, if the process under consideration is a quantum circuit (see Section 2 of $[1]$), the corresponding process matrix can be represented as a tensor product of transport vectors, each corresponding to a wire connecting two gates. Transport vectors are defined in the same way as $24$, where in $20$ the first set of creation operators corresponds to the input Fock space of the wire, while the second set corresponds to its output Fock space. Given that a wire is connecting two gates, its input and output Fock spaces correspond to the output and input Fock subspaces of the two gates, respectively. A gate can in general have multiple incoming or outgoing wires attached to it. Therefore, its input (output) Fock space is a tensor product of all output (input) Fock spaces of the corresponding wires.

The final step in generalising the process matrix formalism to the second quantisation framework consists of formulating the axioms that a process matrix is required to satisfy, namely the generalisations of $1$. The first and third of these axioms have the same form as in the case of finite number of dimensions. The second axiom also has the same form in cases when the number of particles is known to be bounded from above for a given process (such as the discussed case of dSD protocol), since one can truncate the full Fock spaces to a subspace of finite number of degrees of freedom. However, in the general case (for example when discussing coherent states), the second axiom requires a much more careful formulation, since the issue of normalisation and renormalisation in QFT is highly nontrivial. As such, this topic is outside the scope of the current paper, and is left for future work.

**VI. CONCLUSIONS**

**A. Summary of the results**

In this work we have presented a detailed account of the dSD protocol, formulating it within the process matrix formalism. We also gave a detailed analysis of the violation of causal inequalities via the dSD protocol, devoting special attention to the importance of the vacuum state, in particular to its role in the process matrix description. Furthermore, we generalised the process matrix formalism to Fock space using the second quantisation framework, allowing the description of systems of identical particles.

**B. Discussion**

The process matrix analysis presented in this work illuminates the reasons behind the maximal algebraic violation of the causal inequalities achieved by the dSD protocol. As it turns out, the assumptions for the discussed causal inequalities are incompatible with protocols featuring more than two relevant spacetime events. This illustrates the importance of the relationship between the circuit description of the protocol and its spacetime counterpart, emphasising their essential equivalence (see also Theorem from $[9]$). In particular, the circuit and the spacetime diagram for the dSD protocol are in one-to-one correspondence, and this is necessary for the proper process matrix analysis.

The second important point is the necessity of explicitly introducing the vacuum as a legitimate physical state in the dSD protocol, on equal footing with any other physical state. Indeed, the very lack of detection of the particle in the protocol provides an equal amount of information as its detection (explicit interaction). As a consequence, instead of interpreting the absence of particle as noninteraction, one should instead interpret it as the interaction between the vacuum and the apparatus.

The introduction of the vacuum into the process matrix formalism gives a natural motivation to extend the latter to the case of identical particles, both bosons and fermions. However, while employing the formalism of second quantisation, our construction still features only a discrete number of gates. This discreteness means that we still work in particle ontology (i.e., mechanics). Nevertheless, our construction is an important first step towards defining the process matrix formalism in field ontology, i.e., fully fledged QFT.

**C. Future lines of investigation**

As mentioned in the discussion, a natural next line of investigation would be a generalisation of the process matrix formalism to full QFT. This would include an analysis of non-inertial gates and the corresponding Uruh effect. In addition, the analysis of causal inequalities in the context of indefinite number of particles and in particular the scaling of their violation with the number of degrees of freedom would also be an interesting future line of research. Finally, a mathematically rigorous formulation of the axioms for the process matrix description in Fock spaces, as discussed at the end of Section $11$ is an important topic to be addressed.

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Calculation goes as follows:

Proof. Again using the expansion of the transport vectors as unnormalized maximally entangled states, the explicit calculation of the two is

\[ U \rightarrow \text{gate which performs the operation } U \]

Appendix A: Two lemmas for the process matrix evaluation

**Lemma 1.** Let \( |\Psi^*\rangle^{X_O} = |\Psi^*\rangle^{X_O} \) represent a gate which has no input, while it prepares the state \( |\Psi\rangle \in X_O \) as its output. Then, the scalar product of that vector and the transport vector \( \langle \Psi^* | \Psi \rangle^{X_O Y_I} \) is given as:

\[ X_O \langle \Psi^* | \Psi \rangle^{X_O Y_I} = |\Psi\rangle^{Y_I}. \]

**Proof.** Using the fact that the transport vector is an unnormalized maximally entangled state, the explicit calculation goes as follows:

\[ X_O \langle \Psi^* | \Psi \rangle^{X_O Y_I} = \langle \Psi^* | X_O \sum_k |k\rangle^{X_O} |k\rangle^{Y_I} = \sum_k \langle \Psi^* | k \rangle^{Y_I} = \sum_k \langle k | \Psi \rangle^{Y_I} = |\Psi\rangle^{Y_I}, \]

where we have used the unit decomposition \( I = \sum_k |k\rangle \langle k| \) and the fact that \( \langle \Psi^* | k \rangle = \langle \Psi | k \rangle = \langle k | \Psi \rangle \).

**Lemma 2.** Let

\[ |U^*\rangle^{X_I X_O} = [I^{X_I X_I} \otimes (U^*)^{X_O X_I}] |\Psi\rangle^{X_I X_I} \]

represent a gate which performs the operation \( U : X_I \rightarrow X_O \), and let \( |W\rangle = |\Psi\rangle^{X_I} |\Psi\rangle^{X_O Y_I} \). Then the scalar product of the two is

\[ X_I X_O \langle U^* | W \rangle = \left( U |\Psi\rangle \right)^{Y_I}. \]

**Proof.** Again using the expansion of the transport vectors as unnormalized maximally entangled states, the explicit calculation goes as follows:

\[ X_I X_O \langle U^* | W \rangle = \langle I | X_I X_I [I^{X_I X_I} \otimes (U^*)^{X_O X_I}] |\Psi\rangle^{X_I} |\Psi\rangle^{X_O Y_I} = \sum_k \langle k | X_I | k \rangle^{X_I} [I^{X_I X_I} \otimes (U^*)^{X_O X_I}] |\Psi\rangle^{X_I} \sum_m |m\rangle^{X_O} |m\rangle^{Y_I} = \sum_{k,m} \langle k | X_I | m \rangle^{X_I} (|k\rangle^{X_I} (U^*)^{X_O X_I} |m\rangle^{X_O}) |m\rangle^{Y_I} = \sum_{k,m} \langle k | \Psi \rangle (|m\rangle U |k\rangle) |m\rangle^{Y_I} = \sum_m \langle m | U |\Psi\rangle |m\rangle^{Y_I} = \left( U |\Psi\rangle \right)^{Y_I}, \]
where we have again used the unit decomposition and the fact that $\langle k|U^T|m \rangle = \langle m|U|k \rangle$.

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