PRESSURE GAPS, GEOMETRIC POTENTIALS, AND NONPOSITIVELY CURVED MANIFOLDS

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Dedicated to the memory of Todd Fisher

Abstract. In this paper, we derive a general pressure gap criterion for closed rank 1 manifolds with singular sets characterized by codimension 1 totally geodesic flat subtori. As an application, we demonstrate that under specific curvature constraints, potentials that decay faster than geometric potentials (towards the singular set) exhibit pressure gaps and lack phase transitions. Additionally, we prove that geometric potentials are Hölder continuous near singular sets.

1. Introduction.

This paper is dedicated to characterizing pressure gaps in nonuniformly hyperbolic dynamical systems originating from geometric contexts. One distinctive feature of these systems is the absence of pressure gaps. Broadly speaking, pressure gaps can be interpreted as indicating that the "magnitude" of "nonuniform hyperbolicity," as measured by the topological pressure from the perspective of a potential, is small. It has been demonstrated that the presence of a pressure gap is crucial for a potential and its equilibrium states to exhibit ergodic properties similar to those in uniformly hyperbolic systems, including uniqueness, equidistribution, and the Bernoulli property.

The work by Burns, Climenhaga, Fisher, and Thompson in [BCFT18] initiated the study of pressure gaps (see Historical remarks in this section). However, the characterization of pressure gaps remains incomplete. This paper aims to further investigate pressure gaps within natural and concrete geometric contexts.

Let $M$ be a closed, connected, smooth $n$-dimensional manifold, and $g$ be a $C^{m+2}$ ($m > 0$) rank 1 Riemannian metric on $M$. Let $\mathcal{F} = (f_t)_{t \in \mathbb{R}}$ be the geodesic flow on the unit tangent bundle of the Riemannian manifold $(M, g)$. The topological pressure $P(\varphi)$ of a potential $\varphi$ is the supremum of the free energy $h_\mu(\mathcal{F}) + \int \varphi \, d\mu$ over $\mathcal{F}$-invariant Borel probability measures, where $h_\mu(\mathcal{F})$ is the measure-theoretic entropy. A measure that achieves this supremum is called an equilibrium state of $\varphi$.

For the geodesic flow $\mathcal{F}$, "nonuniform hyperbolicity" arises from a geometric object on $T^1 M$, namely, the singular set $\text{Sing}$ (see Section 2 for the precise definition). A potential $\varphi$ is said to have a pressure gap if $P(\text{Sing}, \varphi) < P(\varphi)$, where $P(\text{Sing}, \varphi)$ is the pressure

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restricted to the $\mathcal{F}$-invariant set $\text{Sing}$. In this paper, we prove that if $\varphi$ decays rapidly enough near the singular set, then $\varphi$ has a pressure gap.

**Setting.** We shall introduce notations and the setup to contextualize our results. Throughout the paper, we assume the following conditions on the Riemannian manifold $M$:

1. $M$ is a closed rank 1 manifold with nonpositive sectional curvature.
2. $\text{Sing}$ is either $T^1T_0$ or the flat strip case $T^1T_0 \times [-1, 1]$, where $T_0$ is a codimension 1 totally geodesic flat torus.
3. $M$ is negatively curved outside $T_0$ or the flat strip $T_0 \times [-1, 1]$.

For these types of manifolds, it is convenient to study the behavior of geodesics near $T_0$ or $T_0 \times [-1, 1]$ using Fermi coordinates (see Section 2.3 for more details). For $p \in M$, let $x(p)$ be the signed distance from $p$ to $T_0$ or $T_0 \times [-1, 1]$, and $s(p)$ the closest point on $T_0$ or $T_0 \times [-1, 1]$ from $p$. The map $s : M \to T_0$ is a projection onto $T_0$, which induces a projection $ds : TM \to TT_0$.

For $v \in T_pM$, we consider two components from its Fermi coordinates: $x_v = x(p)$, and the signed angle $\phi_v$ between $v$ and the hypersurface $x = x(p)$. We define the radial curvature $K_\perp(v)$ at $v$ as the sectional curvature of the tangent plane spanned by $\{v, X\}$ where $X = \partial/\partial x$.

We are particularly interested in the following two types of manifolds:

- $M$ is **type 1** if $M$ has an order $m$ uniform curvature bound, i.e., $K_\perp$ vanishes uniformly to order $m-1$ over $T_0$ or the flat strip (see (4.2) for more details).
- $M$ is **type 2** if $\dim M = 2$ and $M$ has an order $m$ nonuniform curvature bound, i.e., the Gaussian curvature vanishes up to order $m-1$ over $T_0$ or the flat strip (see (5.1) and (5.2)).

In Examples subsection 1, we discuss manifolds satisfying the above hypotheses in more detail. For example, surfaces of genus greater than one with analytic Riemannian metrics are type 2 manifolds. See Figure 1.1 and Figure 1.2 for some illustrations.

![Type 1 manifolds](image)

**Figure 1.1.** Type 1 manifolds

In this paper, in addition to conditions (C1)-(C3), we assume the following condition on the continuous potential $\varphi : T^1M \to \mathbb{R}$:
(C4) If \( M \) is type 1, then \( \varphi \) is constant on \( \text{Sing} \). If \( M \) is type 2, then \( \varphi \) is transversally constant on \( \text{Sing} \), meaning \( \varphi \) depends only on the image of the projection \( \text{Sing} \to T^1 T_0 \).

Note that when \( M \) is type 2, we do not assume that \( \varphi \) is constant on \( \text{Sing} \).

Any potential function \( \varphi : T^1 M \to \mathbb{R} \) can be extended to \( \varphi : TM \to \mathbb{R} \) via

\[
\varphi(v) = |v| \varphi \left( \frac{v}{|v|} \right).
\]

With this extension, the integral of \( \varphi \) along any curve is independent of parametrization.

**General results for pressure gaps:** We are now ready to state our first result on obtaining the pressure gap. We denote by \( N_R(\text{Sing}) \) the radius \( R \) neighborhood of \( \text{Sing} \) in \( T^1 M \).

**Theorem A** (Pressure gap criterion). Suppose \( M \) and \( \varphi \) satisfy conditions (C1)-(C4), and there exist \( R, C, \varepsilon_1, \varepsilon_2 > 0 \) such that

\[
\varphi(v) - \varphi(ds(v)) \geq -C \left( |x_v| \frac{m}{m+1} + |\phi_v| \frac{m}{m+2} + \varepsilon_2 \right)
\]

for any \( v \in N_R(\text{Sing}) \). Then \( P(\text{Sing}, \varphi) < P(\varphi) \).

In particular, any potential that is Hölder continuous with sufficiently large exponents at \( \text{Sing} \) satisfies (1.2). We also note that it is sufficient to have a lower bound on \( \varphi(v) - \varphi(ds(v)) \) near \( \text{Sing} \), since when \( \varphi(v) \) is larger than \( \varphi(ds(v)) \), \( \varphi \) accumulates more pressure outside the singular set. When \( \varphi \) is locally constant, we achieve [BCFT18, Theorem B] in our setup:

**Corollary** (Locally constant potentials). Suppose \( M \) is type 1 or type 2, and \( \varphi \) is locally constant in a neighborhood of \( \text{Sing} \), then \( P(\varphi) > P(\text{Sing}, \varphi) \).

**Remark** 1.1. We briefly discuss the improvements and differences between Theorem A and [BCFT18, Theorem B]:

(1) We do not assume that \( \varphi \) is constant on \( \text{Sing} \) when \( M \) is type 2.
(2) One key difference is the construction of shadowing orbits. [BCFT18] uses stable and unstable manifolds to construct orbit segments that shadow those contained in the singular set. This method applies to all nonpositively curved manifolds but provides less control. Our approach requires more precise estimates of the shadowing orbits. To achieve this, we use bouncing orbits (see Figure 4.1) to shadow singular orbits, with curvature bounds providing additional control. See Section 3.1 for more details.

(3) Another difference is that the topological entropy of our singular set is zero, allowing us to better estimate \( P(\text{Sing}, \varphi) \). This is the primary reason we can dispense with the locally constant assumption in [BCFT18, Theorem B]. However, when \( M \) has a flat strip, the transversally constant condition in Theorem A is necessary because increasing \( \varphi \) in the middle of the strip could eliminate the pressure gap. See [BCFT18, Section 10.1] for more details.

Remark 1.2. For brevity and readability, we assume \( M \) contains only one flat torus \( T_0 \) or one flat strip. However, Theorem A holds when Sing is induced by finitely many codimension 1 totally geodesic flat tori or flat strips. Specifically, Theorem A is valid under the following assumptions:

1. \( \text{Sing} = \bigcup_{i=0}^{k} S_i \), where \( S_i = T^1T_i \) or \( T^1T_i \times [-1, 1] \) for a codimension 1 totally geodesic flat subtorus \( T_i \).
2. Without loss of generality, we assume \( P(\varphi|S_0) = \max\{P(\varphi|S_i) : i = 0, \ldots, k\} = P(\text{Sing}, \varphi) \). We only need \( \varphi \) to satisfy the assumptions of Theorem A near \( S_0 \).
3. \( M \) satisfies the curvature bounds near \( T_0 \) or \( T_0 \times [-1, 1] \) as a type 1 or type 2 manifold.

We say that a potential \( \varphi \) has a phase transition at \( q_0 \) if the pressure map \( q \mapsto P(q\varphi) \) fails to be differentiable at \( q_0 \). It is well known that the uniqueness of equilibrium states implies the differentiability of the pressure map; see [Rue78].

Our second main result is that no phase transition appears if \( \varphi \) decays rapidly near Sing:

Theorem B (No phase transition). Let \( M \) and \( \varphi \) satisfy conditions (C1)-(C4), and let \( \varphi \) be a H"older continuous potential such that

\[ |\varphi(v) - \varphi(d\sigma(v))| \leq C \left( |x_v|^{\frac{n}{m}+\varepsilon_1} + |\phi_v|^{\frac{m}{m+2}+\varepsilon_2} \right), \quad \forall v \in N_R(\text{Sing}). \]

Then \( q\varphi \) has a unique equilibrium state for each \( q \in \mathbb{R} \); thus, \( \varphi \) does not have phase transitions.

Remark 1.3. The argument for Theorem B remains valid even if \( \varphi \) is only controlled from below, i.e., satisfying (1.2). In this case, the conclusion is that \( q\varphi \) has a unique equilibrium state for \( q > 0 \).

The proof of Theorem A relies on an abstract pressure criterion (Proposition 7.2). For readability, we defer the detailed exposition of this abstract result to Section 7. In essence, Proposition 7.2 demonstrates that if the geodesic flow \( \mathcal{F} \) satisfies the following conditions: (1) a "strong" specification property; (2) the singular set can be shadowed by nearby vectors, and (3) the potential \( \varphi \) decays rapidly enough, then \( \varphi \) exhibits a pressure gap.
Results for Geometric Potentials. The second theme of this paper is dedicated to studying the behavior of the geometric potential $\varphi^u : T^1 M \to \mathbb{R}$ near $T_0$. Recall that the geometric potential $\varphi^u$ is defined as

$$\varphi^u(v) := -\lim_{t \to 0} \frac{1}{t} \log \det(\frac{df}{dt} |_{E^u(v)})$$

where $E^u(v)$ is the unstable subspace (see Section 2 for details).

For type 2 surfaces without flat strips, Gerber and Niţică [GN99, Theorem 3.1] and Gerber and Wilkinson [GW99, Lemma 3.3] provided Hölder continuity estimates for the geometric potential $\varphi^u$ at Sing. The following result shows that, under a natural Ricci curvature constraint, similar Hölder continuity estimates can be extended to higher-dimensional cases.

In what follows, we denote by $a(v) \approx b(v)$ near $S$, if there exists a neighborhood $N$ of $S$ and $C > 1$ such that $C^{-1}b(v) \leq a(v) \leq Cb(v)$ for $v \in N$. We say $M$ has order $m$ Ricci curvature bounds if the Ricci curvature $\text{Ric}(v)$ vanishes uniformly to order $m - 1$ over $T_0$ (see (6.3)).

**Theorem C (Geometric Potentials).** Let $M$ be a type 1 manifold without flat strips. Suppose $M$ has an order $m$ Ricci curvature bound, then near Sing we have

$$-\varphi^u(v) \approx |x_v|^\frac{m}{m+2} + |\phi_v|^\frac{m}{m+2}.$$

The no-flat-strip condition is necessary for the Hölder continuity. See Remark 6.5 for more details.

**Remark 1.4.** In general, radial curvature and Ricci curvature have no strong relationship. Only in the surface case are these two curvatures the same. Nevertheless, in the appendix, we show that if the Riemannian metric is a warped product, then the Ricci curvature bound and the radial curvature bound hypotheses are equivalent.

In most cases, the Hölder continuity of geometric potentials is unknown for nonuniformly hyperbolic systems, especially in higher dimensions. As an immediate consequence of Theorem C, we have the following partial result for higher-dimensional manifolds:

**Theorem D (Local Hölder Continuity).** Under the same assumptions as Theorem C, the geometric potential $\varphi^u$ is Hölder continuous at Sing.

We note that our method, inspired by [GW99], currently only establishes the Hölder continuity of the geometric potential $\varphi^u$ at Sing. Achieving global Hölder continuity of $\varphi^u$ may require the Hölder continuity of the unstable Jacobian tensor $U^u$, which is still unclear in higher-dimensional cases.

On the other hand, as a consequence of Theorem C, we know that $\varphi^u$ is a borderline case of the pressure gap criterion given in Theorem A (i.e., $\varepsilon_1 = \varepsilon_2 = 0$). Moreover, it is known that for manifolds (including surfaces) whose singular sets are unit tangent bundles of flat, totally geodesic codimension 1 tori, the geometric potential $\varphi^u$ exhibits a phase transition at $q = 1$ (see [BBFS21, p. 530] and [BCFT18, Theorem C]).
In other words, Theorem C shows that the pressure gap criterion given in Theorem A is optimal in the sense that there are examples at the boundary of our criterion that do not have pressure gaps (see Figure 1.3).

In Figure 1.3, each point corresponds to potentials \( \varphi \) satisfying \(-\varphi(v) \approx |x_v|^a + |\phi_v|^b\) near Sing. The shaded region represents potentials that have a pressure gap and no phase transitions by Theorem A. The geometric potential \( \varphi^u \) lies at the vertex \((m^2, mm+2)\) of the shaded region.

We conclude this subsection by posing an open question:

**Question.** Suppose \((a, b)\) lies on the boundary of the shaded region in Figure 1.3. Is there a potential \( \varphi \) satisfying \( |\varphi(v)| \approx |x_v|^a + |\phi_v|^b \) near Sing such that \( \varphi \) has a phase transition?

**Examples:** The prototype of type 1 manifolds is the surface of revolution with profile \( f(x) = 1 + |x|^r \), which is the main example discussed in Lima, Matheus, and Melbourne [LMM]. For higher dimensions, an important example is the Heintze example (see Ballman, Brin, and Eberlein [BBE85] or [BCFT18, Section 10.2]). The simplest version of the Heintze example starts with a finite volume hyperbolic 3-manifold with one cusp, then removes the cusp and flattens the region near the cross-section. Recall that the cross-section of the cusp is a codimension 1 totally geodesic flat torus. The Heintze example is obtained by gluing two identical copies of the above 3-manifold along the cross-section (see Figure 1.1).

Type 2 surfaces (see Figure 1.2) were introduced in Gerber and Nîtcă [GN99] and Gerber and Wilkinson [GW99]. The archetype is a rank 1 nonpositively curved surface with an analytic metric. In such cases, it is well-known that Sing consists of unit tangent bundles of finitely many closed geodesics (see [BCFT18, Section 10.1] for a sketched proof).

We remark that for nonpositively curved surfaces, Coudène and Schapira [CS14, Theorem 3.2] (inspired by the unpublished work of Cao and Xavier [CX08]) showed that flat strips for nonpositively curved manifolds close up. However, in general, it is unknown if the singular geodesics or the (higher dimensional) zero curvature strips close up. Nevertheless, in all known examples, to the best of the authors’ knowledge, the singular sets do close up, leading to our hypothesis on the existence of \( T_0 \).
Historical Remarks. There is no singular set when the dynamical system is uniformly hyperbolic. Hence, the pressure gap persists for a broad class of potentials, allowing one to derive ergodic properties for associated equilibrium states. The origin of this fact traces back to the work of Bowen [Bow74] for maps and Franco [Fra77] for flows. For nonuniformly hyperbolic systems, Climenhaga and Thompson [CT16] proposed using the pressure gap as a condition to obtain ergodic properties of equilibrium states, particularly uniqueness, similar to uniform hyperbolic cases.

Burns et al. [BCFT18] applied the argument from [CT16] and derived a necessary condition for the pressure gap. Specifically, they showed that for closed rank 1 nonpositively curved manifolds, if $\varphi$ is locally constant near Sing, then $\varphi$ has a pressure gap. This work was inspired by Knieper [Kni98], where the entropy gap $h_{\text{top}}(\text{Sing}) < h_{\text{top}}(\mathcal{F})$ was established as a consequence of the uniqueness of the measure of maximal entropy. Recall that the topological entropy $h_{\text{top}}(\mathcal{F})$ is the pressure of the zero potential.

Gelfert and Schapira [GS14] compared different notions of pressure for closed rank 1 nonpositively curved manifolds, such as topological pressure, Gurevich pressure (or periodic orbit pressure), and their restrictions on singular and regular sets. They pointed out that under certain conditions, these different notions of pressure are identical. Similar discussions can be found in [BCFT18, Propositions 2.8 and 6.4].

In geometry, the Liouville measure is an equilibrium state for the geometric potential. Ergodic properties of equilibrium states have been extensively studied. Several recent contributions have employed the Climenhaga-Thompson strategy (see [CT21] for a survey on the strategy). For example, Chen, Kao, and Park [CKP20, CKP21] worked on no focal points settings; Climenhaga, Knieper, and War focused on no conjugate points manifolds [CKW21]; Call, Constantine, Erchenko, and Sawyer [CCE+] discussed flat surfaces with singularities.

There are many other relevant discussions on the ergodic properties of equilibrium states. For example, uniqueness is discussed in [GR19], the Kolmogorov property in [CT22], the Bernoulli property in [Pes77, BG89, OW98, LLS16, CT22, ALP], and the central limit theorem in [TW21, LMM].

For geometric potentials of rank 1 surfaces, Burns and Gelfert [BG14] pointed out the existence of a phase transition. This was also confirmed in [BCFT18] using a different approach. A recent work by Burns, Buzzi, Fisher, and Sawyer [BBFS21] further investigated the edge case of $\varphi = q^u$ for $q = 1$. They showed that the Liouville measure is the only equilibrium state not supported on Sing. However, the Hölder continuity of $\varphi^u$ is even less known. It is only assured by Gerber and Wilkinson [GW99, Theorem I] for type 2 surfaces. Much less is known about $\varphi^u$ in higher dimensional cases.

Outline of the Paper. In Section 2, we recall some relevant background material from geometry and dynamics. In Section 3, we introduce the main shadowing technique and a key inequality to prove Theorem A. Sections 4 and 5 are devoted to technical estimates in type 1 and type 2 settings by analyzing the relevant Riccati equations. Section 6 focuses on the geometric potential and the proof of Theorem C. The proof of Theorem A is presented in Section 7 as a consequence of a more general pressure gap criterion, Proposition 7.2. The
proof draws inspiration from [BCFT18, Theorem B]. However, our specific setup allows us to circumvent several technicalities and arrive at a more straightforward proof than that presented in [BCFT18]. In the appendix, we show that for warped product metrics, radial curvature bounds and Ricci curvature bounds are equivalent, and we provide a proof of Peres’ lemma for flows.

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2. Preliminary

2.1. Geometry of nonpositively curved manifolds. This subsection will survey relevant geometric features of nonpositively curved manifolds. A more comprehensive survey of these results can be found in [Bal95, Ebe01].

Let $M$ be a closed nonpositively curved manifold, and $\{f_t\}_{t \in \mathbb{R}}$ the geodesic flow on the unit tangent bundle $T^1M$. The tangent bundle $TT^1M$ contains three $df_t$-invariant continuous bundles $E^s, E^u$, and $E^c$. The bundle $E^c$ is one-dimensional along the flow direction, and the other two bundles $E^s/u$, which are orthogonal to $E^c$ with respect to the Sasaki metric, can be described using Jacobi fields. If $M$ is negatively curved, these three bundles form a splitting of $TT^1M$.

A Jacobi field $J$ along a geodesic $\gamma$ is a vector field along $\gamma$ satisfying the Jacobi equation

$$J''(t) + R(J(t), \gamma'(t))\gamma'(t) = 0$$

where $R$ is a Riemannian curvature tensor. A Jacobi field $J$ is orthogonal if there exists $t_0 \in \mathbb{R}$ such that $J(t_0)$ and $J'(t_0)$ are perpendicular to $\gamma'(t_0)$. It is well known that when this orthogonal property holds at some $t_0 \in \mathbb{R}$, then it holds for all $t \in \mathbb{R}$. A Jacobi field $J$ is parallel if $J'(t) = 0$ for all $t \in \mathbb{R}$.

Denoting the space of orthogonal Jacobi fields along $\gamma$ by $J^\perp(\gamma)$, we can identify $T_vT^1M$ with $J^\perp(\gamma_v)$ as follows. Consider a vector $v \in T_pM$. Using the Levi-Civita connection the tangent space $T_vTM$ at $v$ may be identified with the direct sum $H_v \oplus V_v$ of horizontal and vertical subspace, respectively, each isomorphic to $T_pM$ equipped with the norm induced from the Riemannian metric on $M$. The Sasaki metric on $TM$ is defined by declaring $H_v$ and $V_v$ to be orthogonal. Restricted to $T^1M$, the tangent space $T_vT^1M$ corresponds to $H_v \oplus v^\perp$ under this identification. Then any vector $\xi \in T_vT^1M$ for $v \in T^1M$ may be written as $(\xi_h, \xi_v)$ and can be identified with an orthogonal Jacobi field $J_\xi \in J^\perp(\gamma_v)$ along $\gamma_v$ with the initial conditions $(J_\xi(0), J'_\xi(0)) = (\xi_h, \xi_v)$. Moreover, the Sasaki norm of $df_t(\xi)$ satisfies

$$\|df_t(\xi)\|^2 = \|J_\xi(t)\|^2 + \|J'_\xi(t)\|^2.$$
The stable subspace $E^s(v)$ at $v \in T^1M$ is then defined as

$$E^s(v) := \{ \xi \in T_vT^1M : \|J_\xi(t)\| \text{ is bounded for } t \geq 0 \}.$$ 

Similarly, the unstable subspace $E^u(v)$ consists of vectors $\xi \in T_vT^1M$ where $\|J_\xi(t)\|$ is bounded for $t \leq 0$. Notice that the subbundles $E^u(v)$ and $E^s(v)$ are integrable to the respective foliations $W^u(v) \subset T^1M$ and $W^s(v) \subset T^1M$. The footprints of $W^u(v)$ and $W^s(v)$ on $M$ are called the unstable and stable horospheres, which are denoted by $H^u(v)$ and $H^s(v)$, respectively.

The rank of a vector $v \in T^1M$ is the dimension of the space of parallel Jacobi fields along $\gamma_v$, which coincides with the number $1 + \dim(E^s \cap E^u)$. We say the manifold is rank 1 if it has at least one rank 1 vector. This paper will focus mainly on closed rank 1 nonpositively curved manifolds.

The singular set is a set of vectors on which the geodesic flow fails to display uniform hyperbolicity, and it is defined by

$$\text{Sing} := \{ v \in T^1M : E^s(v) \cap E^u(v) \neq 0 \}.$$ 

The singular set is closed and $\mathcal{F}$-invariant, and in the case where $M$ is a surface the singular set can be characterized as the set of vectors $v$ where the Gaussian curvature $K(\gamma_v(t))$ vanishes for all $t \in \mathbb{R}$. The complement of the singular set is the regular set

$$\text{Reg} := T^1M \setminus \text{Sing}.$$ 

The geodesic flow restricted to the regular set is hyperbolic, but the degree of hyperbolicity, which can be measured by the function

$$\lambda(v) := \min(\lambda^u(v), \lambda^s(v))$$

where $\lambda^u(v)$ is the minimum eigenvalue of the shape operator on the unstable horosphere $H^u(v)$ at $v$. Using $\lambda$ we can define nested compact subsets $\{\text{Reg}(\eta)\}_{\eta > 0}$ of Reg where

$$\text{Reg}(\eta) := \{ v \in \text{Reg} : \lambda(v) \geq \eta \}.$$ 

These subsets may be viewed as uniformity blocks in the sense of Pesin’s theory, where the hyperbolicity is uniform. More details and properties of the function $\lambda$ can be found in [BCFT18].

The geometric potential $\varphi^u : T^1M \to \mathbb{R}$ is an important potential that measures the infinitesimal volume growth in the unstable direction:

$$\varphi^u(v) := -\lim_{t \to 0} \frac{1}{t} \log \det(df_t | E^u(v)) = -\left. \frac{d}{dt} \right|_{t=0} \log \det(df_t | E^u(v)).$$

In order to study the geometric potential, it is convenient to study Riccati equations. Interestingly, the shape operator of unstable (and unstable) horosphere is a solution of a Riccati equation. To see this, we start by introducing terminologies.

Let $H \subset M$ be a hypersurface orthogonal to $\gamma_v$ at $\pi v$ where $\pi : T^1M \to M$ is the canonical projection. An orthogonal Jacobi field $J \in J^+(\gamma_v)$ is called a $H$-Jacobi field along $\gamma_v$, if $J$ comes from varying $\gamma_v$ through unit speed geodesics orthogonal to $H$. We denote the set of $H$-Jacobi fields by $J_H(\gamma_v)$. The shape operator on $H$ is the symmetric
linear operator $U : T_{\pi v}H \to T_{\pi v}H$ defined by $U(v) = \nabla_v N$, where $N$ is the unit normal vector field toward the same side as $v$.

We are particularly interested in the unstable horosphere $H = H^u(v)$ at $v$. In this case, $J_{H^u(v)}$ coincides with the space of unstable Jacobi field $J^u(v)$. For $t \in \mathbb{R}$, let $U^u_v(t) : T_{\pi v}H^u(f_tv) \to T_{\pi v}H^u(f_tv)$ be the shape operator of the unstable horosphere $H^u(f_tv)$. We know $U^u_v(t)$ is a positive semidefinite symmetric linear operator on $(f_tv)^{\perp}$, and for any unstable Jacobi field $J(t)$ it satisfies $J'(t) = U^u_v(t)J(t)$; see [BCFT18, Lemma 2.9].

For any vector $v \in T^1 M$, let $K(v) : v^{\perp} \to v^{\perp}$ be the symmetric linear map defined via $\langle K(v)X, Y \rangle := \langle R(X, v)v, Y \rangle$ for $X, Y \in v^{\perp}$. Using the Jacobi equation, for an unstable Jacobi field $J(t)$ we know $J''(t) + K(f_tv)J(t) = 0$ and $J'(t) = U^u_v(t)J(t)$. Thus, we get the operator-valued Riccati equation:

\[(U^u_v)'(t) + U^u_v(t)^2 + K(\gamma_v(t)) = 0;\]

see [BCFT18, (7.6)]. Using the above notation, the Ricci curvature $\text{Ric}(v)$ at $v$ is defined as the trace of the map $K(v)$.

### 2.2. Thermodynamic formalism

We now briefly survey relevant results in thermodynamic formalism. The general notion of topological entropy and pressure described in the following can be defined for an arbitrary flow $\mathcal{F} = \{f_t\}_{t \in \mathbb{R}}$ in a compact metric space $(X, d)$.

For any $t > 0$, we define a metric $d_t$ on $X$ via

\[d_t(x, y) := \max_{0 \leq \tau \leq t} d(f_{\tau}x, f_{\tau}y),\]

and the corresponding $\delta$-ball around $x \in X$ in $d_t$-metric will be denoted by $B_t(x, \delta)$. We say a subset $E$ of $X$ is $(t, \delta)$-separated if $d_{\tau}(x, y) \geq \delta$ for distinct $x, y \in E$. Moreover, we will identify $(x, t) \in X \times [0, \mathbb{R}^+]$ with the orbit segment of length $t$ starting at $x$.

Let $\varphi : X \to \mathbb{R}$ be a continuous function on $X$, which we often call a potential. We define $\Phi(x, t) := \int_0^t \varphi(f_{\tau}x) \, d\tau$ to be the integral of $\varphi$ along an orbit segment $(x, t)$. For any subset $C \subset X \times [0, \mathbb{R}^+]$, we let $C_t$ be the subset of $C$ consisting of orbit segments of length $t$. We define

\[\Lambda(C, \varphi, \delta, t) = \sup \left\{ \sum_{x \in E} e^{\Phi(x, t)} : E \subset C_t \text{ is } (t, \delta) - \text{separated} \right\} .\]

The topological pressure of $\varphi$ on $C$ is then defined by

\[P(C, \varphi) := \lim_{\delta \to 0} \limsup_{t \to \infty} \frac{1}{t} \log \Lambda(C, \varphi, \delta, t) .\]

When $C$ is the entire orbit space $X \times [0, \mathbb{R}^+]$, then we denote it by $P(\varphi)$ and call it the topological pressure of $\varphi$. In the case where $\varphi \equiv 0$, the resulting pressure $P(0)$ is called the topological entropy of the flow $\mathcal{F}$ denoted by $h_{\text{top}}(\mathcal{F})$.

Denoting by $\mathcal{M}(\mathcal{F})$ the set of all $\mathcal{F}$-invariant measures on $X$, the pressure $P(\varphi)$ satisfies the variational principle

\[P(\varphi) = \sup_{\mu \in \mathcal{M}(\mathcal{F})} \left\{ h_{\mu}(\mathcal{F}) + \int \varphi \, d\mu \right\} .\]
where $h_\mu(F)$ is the measure-theoretic entropy of $\mu$. Any invariant measure $\mu \in \mathcal{M}(F)$ attaining the supremum is called an equilibrium state for $\varphi$. Likewise, any invariant measure attaining the supremum when $\varphi \equiv 0$ is called a measure of maximal entropy.

2.3. Codimension 1 totally geodesic flat torus and Fermi coordinates. Let $M$ be an $n$-dimensional closed rank 1 nonpositively curved manifold and $T_0$ a totally geodesic $(n-1)$-torus in $M$ with $K \equiv 0$ on any $x \in T_0$. We further suppose that the complement of $T_0$ is negatively curved and that curvature away from a small neighborhood of $T_0$ admits a uniform upper bound strictly smaller than 0. A more precise control of the curvature of the neighborhood will be specified later, depending on the setting under consideration.

In what follows, we fix a fundamental domain in $\tilde{M}$ the universal covering of $M$ and (abusing the notation) continue denoting the lifts of $p \in M$ and $v \in T_p M$ by $p$ and $v$, respectively. Recall that the Fermi coordinate of $p$ is given by $(s, x)$ where $s$ is an $(n-1)$-dimensional coordinate on $\tilde{T}_0$ and $x$ measures the signed distance on $\tilde{M}$ to $\tilde{T}_0$.

For $p \in \tilde{M}$ near $\tilde{T}_0$, by $x(p)$ we mean the $x$-coordinate of $p$. For any $v \in T_p \tilde{M}$ with $p$ near $\tilde{T}_0$, we define $x_v := x(p)$ and denote by $\phi_v$ the signed angle between $v$ and the hypersurface $x = x(p)$; we adopt the convention that $\phi_X = \pi/2$ when $X = \partial/\partial x$. We also define

$$x_v(\tau) := x(\gamma_v(\tau)) \text{ and } \phi_v(\tau) := \phi(\gamma_v(\tau)).$$

When there is no confusion, we may write $x_v$ and $\phi_v$ for $x_v(0)$ and $\phi_v(0)$, respectively.

Remark 2.1. With respect to Fermi coordinates $(s, x)$, the curve $s = \text{const.}$ is always a geodesic perpendicular to $\tilde{T}_0$, while $(s(t), x(t))$ with $x(t) \equiv x_0$ for some $x_0$ is not a geodesic unless $x_0 = 0$ and $s(t)$ is linear.

For $\varepsilon > 0$ small, the Riemannian metric near $\tilde{T}_0$ can be written as

$$g = dx^2 + g_x, \quad |x| \leq \varepsilon$$

where $g_x$ is the Riemannian metric on $\tilde{T}_x := \tilde{T}_0 \times \{x\}$. In particular, $g_0$ is the Euclidean metric on $\tilde{T}_0$.

Denoting by $X := \partial/\partial x$ the vertical vector field, the second fundamental form on $\tilde{T}_x$ is defined via

$$\Pi(v, w) := \langle \nabla_v X, w \rangle,$$

for any $v, w \in T_{(s, x)} \tilde{T}_x$. The shape operator $U(s, x) : T_{(s, x)} \tilde{T}_x \rightarrow T_{(s, x)} \tilde{T}_x$ is defined via

$$\Pi(v, w) = \langle U(s, x)v, w \rangle.$$

As $\Pi$ is bilinear and symmetric, the shape operator $U(s, x)$ is diagonalizable. Its eigenvalues

$$\lambda_1(s, x) \leq \lambda_2(s, x) \leq \cdots \leq \lambda_{n-1}(s, x)$$

are called principal curvatures at $(s, x)$.

For any geodesic $\gamma(t) = (s(t), x(t))$ near $\tilde{T}_0$, by the first variation formula, we have

$$x' = \sin \phi,$$

where $\phi(t) := \phi(\gamma'(t))$, which then gives $x'' = \phi' \cos \phi$.
We denote by
\[ \gamma'_\perp(t) := \gamma'(t) - \langle \gamma'(t), X \rangle X \]
the component of \( \gamma'(t) \) that is orthogonal to \( X \). Then \( |\gamma'_\perp| = \cos \phi \).

**Lemma 2.2.** If \( \gamma(t) = (s(t), x(t)) \) is a geodesic on \( \tilde{M} \) near \( \tilde{T}_0 \), we have
\[ x'' = \Pi(\gamma'_\perp, \gamma'_\perp) \in [\lambda_1(s, x) \cos^2 \phi, \lambda_{n-1}(s, x) \cos^2 \phi]. \]

**Proof.** By (2.4), we have \( x' = \sin \phi = \langle X, \gamma'(t) \rangle \). Thus
\[ x'' = \frac{d}{dt}(X, \gamma'(t)) = \langle \nabla_{\gamma'(t)} X, \gamma'(t) \rangle = \langle \nabla_{\gamma'_\perp(t)} X, \gamma'_\perp(t) \rangle = \Pi(\gamma'_\perp, \gamma'_\perp). \]
Note that the third equality uses the fact \( \nabla_X X = 0 \); see Remark 2.1. Since \( |\gamma'_\perp| = \cos \phi \), we have
\[ \Pi(\gamma'_\perp, \gamma'_\perp) \geq \lambda_1(s, x)|\gamma'_\perp|^2 = \lambda_1(s, x) \cos^2 \phi \]
and
\[ \Pi(\gamma'_\perp, \gamma'_\perp) \leq \lambda_{n-1}(s, x) \cos^2 \phi. \]
This completes the proof. \( \square \)

**Remark 2.3.** When \( M \) is a surface, then \( T_0 \) is a closed geodesic. In this case the Riemannian metric (2.3) near \( \tilde{T}_0 \) may be written as
\[ g = dx^2 + G(s, x)^2 ds^2 \]
with \( G(s, 0) \equiv 1 \). The Gaussian curvature is given by \( K = -G_{xx}/G \), and the second fundamental form at \( \tilde{T}_x \) is \( G_x/G \). In particular, by Lemma 2.2, \( x'' \) admits the following expression
\[ x'' = \lambda(s, x) \cos^2 \phi = \frac{G_x}{G} \cos^2 \phi. \]

3. Preparation and Outline for Pressure Gaps Results

**3.1. Shadowing map.** To distinguish vectors in Sing and generic vectors in \( T^1 M \), we will use different fonts to denote them; more precisely, we will write \( v \in \text{Sing} \) and \( v \in T^1 M \). Given an orbit segment \( (v, t) \in T^1 M \times \mathbb{R}^+ \) with \( v \in \text{Sing} \), we now describe a method for constructing a new orbit segment that shadows \( (v, t) \). Though simple, this construction will be crucial in proving Theorem A and Theorem D.

For any \( v \in \text{Sing} \), suppose \( \pi(v) = (s_0, 0) \) for some \( s_0 \). For any \( t > 0 \) and any \( R > 0 \) such that the Fermi coordinates \( (s, x) \) are well-defined for \( |x| < R \), there exists \( s_1 \in \mathbb{R} \) such that the distance on \( \tilde{M} \) between \( (s_0, R) \) and \( ((s \circ \pi)(f_{s_1} v), R) \) is equal to \( t \); see Figure 3.1. From the triangle inequality we know that \( |t - s_1| < 2R \).

Denoting by \( \gamma \) the geodesic connecting these two points, we define
\[ \Pi_{t, R}(v) := \gamma'(0). \]
Throughout the paper, we will often write \( w_v \), or simply \( w \), to denote \( \Pi_{t, R}(v) \) whenever the context is clear.

The next few lemmas establish a few properties on the map \( \Pi_{t, R} \).
Lemma 3.1. For any $v \in \text{Sing}$ and $t > 0$, the following statements hold:

1. For any $R > 0$, the function $x_w(\tau)$ is convex for any $\tau \in [0, t]$.
2. There exists $R_0 > 0$ such that for any $R \in (0, R_0)$,
   \[
   |\phi_w(\tau)| < \frac{\pi}{4}
   \]
   for all $\tau \in [0, t]$.
3. For any $R > 0$, there exists $\eta = \eta(R) > 0$ such that $w, f_tw \in \text{Reg}(\eta)$.

Proof. For (1), it is not hard to see from the construction of $\Pi_{t,R}$ that $|\phi_w(\tau)| \leq \pi/2$ and $\phi'_w(\tau) \geq 0$ for all $\tau \in [0, t]$. The statement then is a consequence of $x''_w(\tau) = \phi'_w(\tau) \cos \phi_w(\tau)$ from (2.4).

For (2), it is clear that
\[
|\phi_w(\tau)| \leq \max\{|\phi_w(0)|, |\phi_w(t)|\}
\]
for all $\tau \in [0, t]$. We then observe that $|\phi_w(0)| \leq |\phi(\gamma'_{w,s}(0))|$ where $\gamma_{w,s}$ is the geodesic with the same initial point as $\gamma_w$ that is forward asymptotic to $\overrightarrow{T_0}$. The statement then follows as the function
\[
R \mapsto \sup\{|\phi(u)| : u \in W^s(v) \text{ for some } v \in \text{Sing} \text{ and } x(\pi u) = R\}
\]
vanishes when $R = 0$ and varies continuously in $R$. By repeating the same argument with the unstable manifolds $W^u(v)$ for $v \in \text{Sing}$, we can find $R_0 > 0$ with the desired property.

For (3), any unit vector $v \in T^1M$ satisfies $x(\pi(v)) \neq 0$ belongs to $\text{Reg}$. Since $\text{Reg}$ is exhausted by compact subsets $\text{Reg}(\eta)$ and the flat torus $T_0$ in $M$ is compact, the statement follows. \hfill $\square$

Remark 3.2. From here on, we will assume that $R$ belongs to $[0, \min(R_0, R_1)]$ with $R_0, R_1$ defined as in above lemmas. In particular, we will often evoke Lemma 3.1 to use the inequality
\[
\cos \phi_w(\tau) \leq \left[\frac{\sqrt{2}}{2}, 1\right]
\]
for any $w = \Pi_{t,R}(v)$ and $\tau \in [0, t]$. 
3.2. Key inequality and the outline of Theorem A. In this subsection, we provide a brief outline of what consists of the remaining sections. Recall from Lemma 3.1 that \( x_w(\tau) \) is a convex function on \( \tau \in [0, t] \) for any \( w = \Pi_{t,R}(v) \), and hence there exists a well-defined number \( \tilde{t} \in [0, t] \) such that

\[
(3.2) \quad x_w(\tilde{t}) = \min_{\tau \in [0, t]} x_w(\tau)
\]

and that \( \tilde{t} \) is the smallest among all such numbers. In the case where \( x_w(\tau) \) is strictly convex, there is a unique \( \tilde{t} \) which attains the minimum of \( x_w(\tau) \). Note that \( x_w'(\tau) \leq 0 \) and \( \phi_w(\tau) \leq 0 \) for \( \tau \in [0, \tilde{t}] \).

The goal of the next few sections is to establish bounds on the distance \( x_w(\tau) \) and the angle \( \phi_w(\tau) \) under the assumption that the radial curvature \( K_\perp \) vanishes to the order of \( m - 1 \) at \( T_0 \) and that the curvature near \( T_0 \) is controlled; see Section 4 and 5 for the precise description of the setting. In particular, we will show that for suitable \( R > 0 \), there exists \( Q = Q(R) > 1 \) independent of \( t \) such that the shadowing vector \( w = \Pi_{t,R}(v) \) for any \( v \in \text{Sing} \) and any \( t > 0 \) satisfies

\[
(3.3) \quad Q^{-1}(\tau + 1)^{-\frac{2}{m}} \leq x_w(\tau) \leq Q(\tau + 1)^{-\frac{2}{m}}, \quad |\phi_w(\tau)| \leq Q[(\tau + 1)^{-\frac{m+2}{m}} - (\tilde{t} + 1)^{-\frac{m+2}{m}}]
\]

for any \( \tau \in [0, \tilde{t}] \). From its derivation in Proposition 4.2 and 5.3, it will be clear that the analogous inequality holds for \( \tau \in [\tilde{t}, t] \) by simply applying the symmetric argument starting from \( \tau = t \) instead of \( \tau = 0 \):

\[
(3.4) \quad Q^{-1}(t - \tau + 1)^{-\frac{2}{m}} \leq x_w(\tau) \leq Q(t - \tau + 1)^{-\frac{2}{m}}, \quad |\phi_w(\tau)| \leq Q[(t - \tau + 1)^{-\frac{m+2}{m}} - (\tilde{t} + 1)^{-\frac{m+2}{m}}].
\]

In the setting considered in Section 4 where a uniform control on the curvature \( K_\perp \) is assumed on the entire \( \overline{T}_0 \), the angle \( |\phi_w(\tau)| \) from (3.3) and (3.4) admits the corresponding lower bound also; see Proposition 4.2.

In the context considered by Gerber and Wilkinson [GW99] where \( M \) is a surface (see Section 5 for details) fits into the assumption of \( K_\perp \) described in the above paragraph, and \( \varphi \) satisfying (1.2) is related to the geometric potential, as elaborated in more detail in the next subsection.

Assuming the estimates (3.3) and (3.4), we now derive useful consequences from them when considering potentials \( \varphi \) satisfying (1.2). We will see in Section 7 that, together with certain properties of the geodesic flow, these results serve as sufficient criteria for the potential \( \varphi \) to have the pressure gap. From direct integration using the estimates (3.3) and (3.4) we immediately get

**Proposition 3.3** (Key inequality). For some \( R > 0 \), suppose that the shadowing vector \( w = \Pi_{t,R}(v) \) satisfies (3.3) and (3.4) and \( \varphi \) satisfies the assumption of Theorem A. Then
there exists \( Q = Q(R, C, \varepsilon_1, \varepsilon_2, \delta) > 0 \) such that
\[
\int_0^t \varphi(f_\tau u)d\tau \geq \int_0^{s_1} \varphi(f_\tau v)d\tau - Q
\]
for any \( u \in B_t(w, \delta) \) where \( w = \Pi_{t,R}(v) \) for some \( v \in \text{Sing} \).

Proof. We firstly prove the inequality for \( u = w \). For type 1 manifold is easier as \( \varphi \equiv c \) on \( \text{Sing} \). By (1.2), (3.3) and (3.4), we have
\[
\int_0^t \varphi(f_\tau u)d\tau \geq ct - Q \geq cs_1 - Q
\]
as \( |t - s_1| \leq 2R \).

For Type 2 manifolds, use (1.2), (3.3) and (3.4) again we have
\[
\int_0^t \varphi(f_\tau w)d\tau \geq \int_0^t \varphi(f_\tau ds(w))d\tau - Q = \int_0^{s_1} \varphi(f_\tau v)d\tau - Q.
\]
For general \( u \in B_t(w, \delta) \), we begin by intersecting the geodesic \( \gamma_u(\tau) \) with the hyperplane \( x = R \) so that there are exactly two intersection points, each near \( \pi(u) \) and \( \pi(f_t u) \); see Figure 3.2. We denote by \((u_0, t_0)\) the orbit segment connecting such intersection points. Since two orbit segments \((u, t)\) and \((u_0, t_0)\) differ only at either ends by length at most \( 2\delta \), there exists a constant \( C_\delta \) depending only on \( \delta \) and \( \|\varphi\| \) such that
\[
\left| \int_0^t \varphi(f_\tau u)d\tau - \int_0^{t_0} \varphi(f_\tau u_0)d\tau \right| < C_\delta.
\]
Notice from its construction that \( u_0 \) is equal to \( \Pi_{t_0,R}(v_0) \) for some \( v_0 \in \text{Sing} \) near \( v \), and

\[\text{Figure 3.2. Figure for Proposition 3.3}\]

hence the integral \( \int_0^{t_0} \varphi(f_\tau u_0)d\tau \) admits a uniform lower bound independent of \( t_0 \) of the above proposition. Therefore, the same is true for \( \int_0^t \varphi(f_\tau u)d\tau \) for \( u \in B_t(w, \delta) \). \( \square \)
4. Estimates for Type 1 Manifolds

Recall that $X = \partial/\partial x$ is the vertical vector field. For any $v \in T\tilde{M}$ that is not collinear with $X$ we define the radial curvature of $v$ by

$$K_\perp(v) := K_\sigma,$$

that is, the sectional curvature of the plane $\sigma := \text{span}\{v, X\}$.

In this section, we will consider the first of the two settings in which $K_\perp$ vanishes uniformly to the order $m - 1$. Namely, if $(s, x)$ are the Fermi coordinates along $\tilde{T}_0$, there exists $C_1, C_2, \varepsilon > 0$ such that

$$-C_1|x(v)|^m \leq K_\perp(v) \leq -C_2|x(v)|^m$$

for any $v \perp X$ with $|x(v)| < \varepsilon$.

As outlined in Subsection 3.2, the main goal of this section is to prove that under the above assumption on $K_\perp$, the shadowing vector $w = \Pi_{t,R}(v)$ for any $v \in \text{Sing}$ satisfies the estimates on $x_w$ and $\phi_w$ claimed in (3.3).

Indeed, in this subsection, we will derive estimates on $x_v(t)$ for generic vectors $v \in T^1M$ near $\tilde{T}_0$; namely, bouncing, asymptotic, and crossing vectors (see Definition 4.1). In Section 6, we will discuss behaviors of geometric potentials $\varphi^u$ with respect to bouncing, asymptotic, and crossing vectors. Notice that by Definition 4.1, shadowing vectors $w = \Pi_{t,R}(v)$ are bouncing vectors.

Let $N_R(\tilde{T}_0) := \{(s, x) : s \in \tilde{T}_0, |x| < R\}$ be a neighborhood of $\tilde{T}_0$, $v \in T^1M$ and $\gamma_v(t) = (s_v(t), x_v(t))$ in the Fermi coordinates.

**Definition 4.1.** Suppose $v \in T^1M$ such that $\gamma_v(0) \in N_R(\tilde{T}_0)$. Let $-T_1 = \inf\{t : \gamma_v(t) \in N_R(\tilde{T}_0)\}$ and $T_2 = \sup\{t : \gamma_v(t) \in N_R(\tilde{T}_0)\}$. We say that $v$ (relative to $N_R(\tilde{T}_0)$) is

1. **bouncing** if $T_1, T_2 < \infty$ and $x_v(t) > 0$ for $t \in (-T_1, T_2)$,
2. **asymptotic** if $T_1 = \infty$ or $T_2 = \infty$,
3. **crossing** if $T_1, T_2 < \infty$ and $x_v(t_0) = 0$ for $t_0 \in (-T_1, T_2)$.

Please see Fig 4.1 for examples of these vectors. The definition and study of these vectors are inspired by [LMM]. Notice that according to Lemma 5.5 the above definition is well-defined when $R$ is sufficiently small. Moreover, by definition, all shadowing vectors are bouncing vectors and asymptotic vectors are limiting cases of the bouncing vectors. More precisely, one can regard asymptotic vectors as bouncing vectors $v$ with the minimal of $x_v$ (when $\phi(v) < 0$) occurring at $\hat{t} = \infty$ where, recalling from (3.2), $\hat{t}$ is the the time (unique in this case) in which $x_w(\hat{t})$ attains its minimum.

For type 1 manifolds, the condition (3.3) is established in the proposition below (as for bouncing vectors):

**Proposition 4.2.** For any $R > 0$ sufficiently small (see Lemma 4.5 for the domain $R$ can take), there exists $Q = Q(R) > 1$ independent of $t$ such that for any $v \in T^1M$ with $x_v(0) = R$ and $\phi_v(0) < 0$, 


Remark 4.3. Proposition 4.2 is more general than (3.3). The lower bound of $|\phi_v(\tau)|$ is not necessary for deriving Proposition 3.3.

We begin by collecting relevant lemmas to prove this proposition, the first of which concerns the general property of Riccati solutions.

Lemma 4.4. There exists $R_0 = R_0(C,m)$ such that for any $R \in [0,R_0]$, and any solution $U : [0,R] \to M_{n-1}(\mathbb{R})$ of the following Riccati equation

$$U' + U^2 - Cx^mI_{n-1} = 0, \quad U(0) = 0,$$

we have

$$\frac{C^m}{2(m+1)}I_{n-1} \leq U(x) \leq \frac{C^{m+1}}{m+1}I_{n-1}.$$
Proof. Let $\lambda : [0, R] \rightarrow \mathbb{R}$ be the solution of
\[ \lambda' + \lambda^2 - Cx^m = 0, \quad \lambda(0) = 0. \]
Then $\lambda I_{n-1}$ satisfies (4.3). Since the solution of any first-order ODE is unique; we have $U = \lambda I_{n-1}$.

Now we estimate $\lambda$. Since $\lambda' = Cx^m - \lambda^2 \leq Cx^m$, we have
\[ \lambda(x) = \int_0^x \lambda'(s)ds \leq \int_0^x C s^m ds = \frac{C}{m+1} x^{m+1}. \]
establishing the required upper bound on $U(x)$.

For the lower bound, let $R_0 := \left( \frac{(m+1)^2}{2C} \right)^{\frac{1}{m+2}}$. Then for any $x \in [0, R_0]$ the upper bound for $\lambda(x)$ gives
\[ \lambda' = Cx^m - \lambda^2 \geq Cx^m - \frac{C^2}{(m+1)^2} x^{2m+2} \geq \frac{C}{2} x^m \]
which then gives $\lambda(x) = \int_0^x \lambda'(s)ds \geq \frac{C}{2(m+1)} x^{m+1}$ as required. \(\square\)

Recalling the notations $x_v(\tau) := x(\gamma_v(\tau))$ and $\phi_v(\tau) := \phi(\gamma_v(\tau))$ from (2.2), the following lemma uses the curvature bound (4.2) to compare $x''_v$ with $x^{m+1}_v$ for any shadowing vector $v$.

**Lemma 4.5.** There exists $R_1 = R_1(C_1, C_2, m) > 0$ such that for any $R \in [0, R_1]$ and any $v$ with $x_v(0) < R$, we have
\[ \frac{C_2}{4m+4} x^{m+1}_v \leq x''_v \leq \frac{C_1}{m+1} x^{m+1}_v, \]
as long as $|x_v| < R$. In particular, $x_v$ is strictly convex and positive.

**Proof.** Since $-C_1 x(v)^m \leq K_1(v) \leq -C_2 x(v)^m$, we have
\[ -C_1 x^m I_{n-1} \leq K(X) \leq -C_2 x^m I_{n-1} \]
where $K : v^\perp \rightarrow v^\perp$ is the symmetric linear map such that $\langle K(v)X, Y \rangle = \langle R(X, v), Y \rangle$ for $X, Y \in v^\perp$. Using $R_0$ from the previous lemma, we claim that we can take
\[ R_1 := \min\{ R_0(C_1, m), R_0(C_2, m) \}. \]

For $i = 1, 2$ and $R \in (0, R_1)$, let $U_i : [0, R] \rightarrow M_{n-1}(\mathbb{R})$ be the solution of
\[ U_i' + U_i^2 - C_i x^m I_{n-1} = 0, \quad U_i(0) = 0. \]
By the main theorem in [EH90], the solutions satisfy
\[ U_2 \leq U \leq U_1 \]
on $[0, R]$. By Lemmas 2.2 and 4.4 and Remark 3.2, we have
\[ x''_v(x) \geq \frac{\lambda_1(s, x_v)}{2} \geq \frac{C_2}{4m+4} x^{m+1}_v \]
and
\[ x''_v(x) \leq \lambda_{n-1}(s, x_v) \leq \frac{C_1}{m+1} x'^{m+1}_v. \]

If \( x_v \) vanishes somewhere in \((0, t)\), and assume \( \bar{a} \) is the smallest zero of \( x_w \) in \((0, t)\). It is clear that \( x'_v(\bar{a}) \leq 0 \) because \( x'_v(\bar{a}) = 0 \), then it would imply \( x_v \subseteq \bar{T}_0 \) which is impossible. Thus \( x'_v(\bar{a}) < 0 \), and denote by \( \tilde{b} \) the next zero of \( x_v \). Then \( x_v \) is a geodesic connecting two distinct points on a totally geodesic submanifold \( \bar{T}_0 \), which would imply \( x_v \subseteq \bar{T}_0 \), again resulting in a contradiction. Therefore, \( x_v(\tau) \) is positive for all \( \tau \in [0, t] \).

We also need the following auxiliary lemma.

**Lemma 4.6.** Let \( f : [a, b] \to \mathbb{R} \) be a piecewise smooth, strictly decreasing function with finitely many discontinuities. Assume that \( f(b) > 0 \) and that there exists \( 0 < Q_1 < Q_2, \alpha > 0, \beta \in (0, 1) \) with \( \alpha \beta > 1 \) such that
\[ -Q_2(f(\tau) - f(b))\beta \leq f'(\tau) \leq -Q_1(f(\tau) - f(b))\beta \]
when \( f \) is smooth at \( \tau \). Then exists a constant \( Q_0 = Q_0(\alpha, \beta) > 0 \) independent of \( a, b, f \) such that
\[ (f(a)^{1-\alpha \beta} + Q_2(\alpha \beta - 1)(\tau - a))^{\frac{1}{1-\alpha \beta}} \leq f(\tau) \leq Q_0(f(a)^{1-\alpha \beta} + Q_1(\alpha \beta - 1)(\tau - a))^{\frac{1}{1-\alpha \beta}} \]
for all \( \tau \in [a, b] \).

**Proof.** We first consider the lower bound of \( f \). Firstly we have
\[ \frac{d}{d\tau} f(\tau)^{1-\alpha \beta} = (\alpha \beta - 1)(-f') f(\tau)^{-\alpha \beta} \leq Q_2(\alpha \beta - 1), \]
whenever \( f \) is smooth. Thus,
\[ f(\tau)^{1-\alpha \beta} \leq f(a)^{1-\alpha \beta} + Q_2(\alpha \beta - 1)(\tau - a). \]
Hence,
\[ f(\tau) \geq (f(a)^{1-\alpha \beta} + Q_2(\alpha \beta - 1)(\tau - a))^{\frac{1}{1-\alpha \beta}}. \]
Now, we compute the upper bound. Similar to the lower bound, we get
\[ \frac{d}{d\tau} f(\tau)^{1-\alpha \beta} = (\alpha \beta - 1)(-f') f(\tau)^{-\alpha \beta} \geq Q_1(\alpha \beta - 1) \left( 1 - \left( \frac{f(b)}{f(\tau)} \right)^\alpha \right)^\beta, \]
whenever \( f \) is smooth.

We then define an auxiliary piecewise smooth function
\[ g(\tau) := \left( \frac{f(\tau)}{f(b)} \right)^{1-\alpha \beta} \]
which is strictly increasing on \([a, b]\) with \( g(a) \in (0, 1) \) and \( g(b) = 1 \). Moreover,
\[ \frac{dg}{d\tau} \geq Q_1(\alpha \beta - 1)f(b)^{\alpha \beta - 1} \left( 1 - g^{\alpha \beta - 1} \right)^\beta. \]
Let \( \tilde{g} \) be the solution of the ODE
\[
\frac{d\tilde{g}}{d\tau} = Q_1(\alpha \beta - 1) f(b)^{\alpha \beta - 1} \left( 1 - \tilde{g}^{1/\beta} \right)^\beta, \quad \tilde{g}(a) = g(a).
\]

Since \( \frac{\alpha}{\alpha \beta - 1} > \frac{1}{\beta} \), from (4.5) and (4.6) we know that \( \tilde{g}'(a) < g'(a) \), thus \( \tilde{g}(\tau) < g(\tau) \) when \( \tau \) is slightly larger than \( a \). In fact, we have \( \tilde{g}(\tau) < g(\tau) \) for all \( \tau \in (a, b) \). This is because if \( \tilde{g}(\tau) \geq g(\tau) \) for some \( \tau \), then we can define \( \tau_0 \in (a, b) \) to be the smallest \( \tau \) with \( \tilde{g}(\tau) = g(\tau) \). Since \( \tilde{g}(\tau) < g(\tau) \) on \( (a, \tau_0) \), the condition \( \tilde{g}(\tau_0) = g(\tau_0) \) implies \( \tilde{g}'(\tau_0) \geq g'(\tau_0) \). On the other hand, the condition \( \tilde{g}(\tau_0) = g(\tau_0) \) considered with (4.5) and (4.6) implies \( \tilde{g}'(\tau_0) < g'(\tau_0) \), deriving a contradiction. Thus \( \tilde{g} < g \) on \( (a, b) \).

By (4.6), we have
\[
\left( 1 - \tilde{g}^{1/\beta} \right)^{-\beta} \frac{d\tilde{g}}{d\tau} = Q_1(\alpha \beta - 1) f(b)^{\alpha \beta - 1}.
\]
Thus, for any \( \tau \in [a, b] \),
\[
F_\beta(\tilde{g}(\tau)) - F_\beta(\tilde{g}(a)) = Q_1(\alpha \beta - 1) f(b)^{\alpha \beta - 1}(\tau - a),
\]
where
\[
F_\beta(x) := \int_0^x (1 - y^{1/\beta})^{-\beta} dy.
\]
It is clear that \( F_\beta \) is convex on \( [0, 1] \), thus \( F_\beta'(x) \geq F_\beta'(0) = 1 \) for \( x \in [0, 1] \). Hence, for any \( x \in [0, 1] \),
\[
F_\beta(x) \geq x.
\]
On the other hand, since \( F_\beta \) is convex and increasing, \( F_\beta(x)/x \) is also increasing on \( [0, 1] \). Thus, for any \( x \in (0, 1) \)
\[
\frac{F_\beta(x)}{x} \leq F_\beta(1) = \int_0^1 t^{\beta - 1} (1 - t)^{-\beta} dt = B(\beta, 1 - \beta) < \infty,
\]
where \( B \) is the beta function. By combining (4.7), (4.8), and (4.9), we get
\[
B(\beta, 1 - \beta) \tilde{g}(\tau) - \tilde{g}(a) \geq Q_1(\alpha \beta - 1) f(b)^{\alpha \beta - 1}(\tau - a).
\]
Hence
\[
B(\beta, 1 - \beta) f(\tau)^{1-\alpha \beta} - f(a)^{1-\alpha \beta} \geq Q_1(\alpha \beta - 1)(\tau - a).
\]
Setting \( Q_0 := B(\beta, 1 - \beta)^{\frac{1}{\alpha \beta - 1}} \), we have
\[
f(\tau) \leq Q_0(f(a)^{1-\alpha \beta} + Q_1(\alpha \beta - 1)(\tau - a))^{\frac{1}{1-\alpha \beta}}.
\]
This completes the proof.

We are ready to prove Proposition 4.2.

**Proof of Proposition 4.2.** We will use \( x \) to denote \( x_v \) for simplicity. We will also use \( Q \) to denote a generic constant that may need to be updated; this will be made clearer as they show up in the proof.
Case 1: $v$ is a bouncing vector. We will first prove the lower bound for $x(\tau)$ when $\tau \in [0, \tilde{t}]$. Noting that $x'(\tilde{t}) = 0$, by Lemma 4.5, we have

$$x'(\tau)^2 = -\int_{\tau}^{\tilde{t}} 2x'x'' ds \leq \frac{2C_1}{m+1} \int_{\tau}^{\tilde{t}} -x'x^{m+1} ds \leq \frac{2C_1}{(m+1)^2} (x(\tau)^{m+2} - x(\tilde{t})^{m+2})$$

and

$$x'(\tau)^2 = -\int_{\tau}^{\tilde{t}} 2x'x'' ds \geq \frac{C_2}{2m+2} \int_{\tau}^{\tilde{t}} -x'x^{m+1} ds \geq \frac{C_2}{2(2m+2)} (x(\tau)^{m+2} - x(\tilde{t})^{m+2}).$$

By taking $\alpha = m + 2, \beta = 1/2, f(a) = R$ in Lemma 4.6, we know that there exists $Q$ independent of $v, t$ such that for any $\tau \in [0, \tilde{t}]$,

$$Q^{-1}(\tau + 1)^{-\frac{2}{m}} \leq x(\tau) \leq Q(\tau + 1)^{-\frac{2}{m}}.$$

For $\phi$, recall that $x' = \sin \phi$. Thus $(\sin \phi)' = x''$. By Lemma 4.5 we have

$$Q^{-1}(\tau + 1)^{-\frac{2(m+1)}{m}} \leq \frac{C_2}{4m+4} x^{m+1} \leq (\sin \phi)' \leq \frac{C_1}{m+1} x^{m+1} \leq Q(\tau + 1)^{-\frac{2(m+1)}{m}}.$$

Taking the integral on $[\tau, \tilde{t}]$, we get

$$Q^{-1}[(\tau + 1)^{-\frac{m+2}{m}} - (\tilde{t} + 1)^{-\frac{m+2}{m}}] \leq \sin \phi(\tau) \leq Q[(\tau + 1)^{-\frac{m+2}{m}} - (\tilde{t} + 1)^{-\frac{m+2}{m}}].$$

Since $\sin \phi/\phi \in [2/\pi, 1]$, we have the same bounds for $\phi$.

Case 2: $v$ is an asymptotic vector. It is not hard to see that the above argument is valid when $\tilde{t} = \infty$.

Case 3: $v$ is a crossing vector. Denote by $a(t) := -\sin \phi(t)$. By Lemma 4.5, we know that whenever $|x| \leq R$,

$$\frac{C_2}{4m+4} x^{m+1} \leq x'' \leq \frac{C_1}{m+1} x^{m+1}.$$

Thus

$$\frac{C_2}{4m+4} (-x'x^{m+1}) \leq -x'x'' \leq \frac{C_1}{m+1} (-x'x^{m+1}).$$

Taking integral, we get

$$\frac{C_2}{(2m+4)^2} x^{m+2} \leq a^2 - a(t_0)^2 \leq \frac{C_1}{(m+1)^2} x^{m+2}. \tag{4.10}$$

Hence

$$\frac{(m+1)^2}{C_1} (a^2 - a(t_0)^2)^{\frac{m+2}{m+1}} \leq x^{m+2} \leq \frac{(2m+4)^2}{C_2} (a^2 - a(t_0)^2)^{\frac{m+1}{m+2}}.$$

Since $x' = a$, we have $a' = -x''$. Thus there exists $C > c > 0$ independent of $v, t$ such that

$$-C(a^2 - a(t_0)^2)^{\frac{m+2}{m+1}} \leq a' \leq -c(a^2 - a(t_0)^2)^{\frac{m+1}{m+2}}.$$

Setting $\alpha = 2, \beta = \frac{m+1}{m+2}$ in Lemma 4.6, we have

$$\frac{a(\tau)}{a(\tau)}^\frac{m}{m+2} + C(\tau - a))^{-\frac{m+2}{m}} \leq a(\tau) \leq Q_0(a(\tau) - \frac{m}{m+2} + c(\tau - a))^{-\frac{m+2}{m}} \tag{4.11}$$
Since \( x_v(0) = R \) and \( a(0) > 0 \), by compactness, \( a(0) \) has a uniform lower bound depending on \( R \). Thus, we have

\[
Q^{-1}(\tau + 1)^{-\frac{m+2}{m}} \leq |\phi(\tau)| \leq Q(\tau + 1)^{-\frac{m+2}{m}}.
\]

Similar to Case 1, the bound of \( x \) comes from \( x(\tau) = \int_0^\tau a(s)\,ds \) and (4.11).

\[\Box\]

**Remark 4.7.** Note that we did not use the full strength of Lemma 4.6 in the above proof; that is, \( x(\tau) \) had no discontinuities. The next section will make similar use of Lemma 4.6 applied to a function with discontinuities.

## 5. Estimates for Type 2 Surfaces

In this section, we consider a different setting considered by Gerber and Niţică [GN99] as well as Gerber and Wilkinson [GW99] where \( M = S \) is a complete nonpositively curved surface, and \( T_0 \) is a closed geodesic of some length \( \gamma_0 \) on which the Gaussian curvature \( K \) vanishes to order \( m - 1 \). Namely, if \((s,x)\) are the Fermi coordinates along \( T_0 \), there exists \( C_1, C_2, \varepsilon > 0 \) and an interval \( L = [0, \gamma_1] \) for some \( \gamma_1 \in (0, \gamma_0) \) such that

\[
-C_1|x|^m \leq K(s, x) \leq 0 \quad \text{for all } |x| < \varepsilon \text{ and for all } s \in \mathbb{R} \text{ and}
\]

\[
-C_1|x|^m \leq K(s, x) \leq -C_2|x|^m \quad \text{for all } |x| < \varepsilon \text{ and } s \in \mathcal{L}.
\]

To simplify the argument, whenever applicable, we will adopt the notation for the Riemannian metric \( g \) specified for a surface introduced in Remark 2.3.

**Remark 5.1.** Compared to the assumption in the previous section, the underlying manifold considered in this section is 2-dimensional, and the curvature assumption near \( T_0 \) is weakened: the neighborhood of only a small subset \( L \) of \( T_0 \) is assumed to satisfy the uniform curvature bound as in (4.2).

On the complement of \( L \) in \( T_0 \) and its neighborhood, only the trivial upper bound (i.e., zero) is imposed on the curvature. Despite the weaker assumption on the curvature, the low dimensionality of the manifold enables us to do a finer analysis to prove the similar estimates (3.3) on \( x_w \) and \( \phi_w \) for \( w = \Pi_{t,R}(v) \). Furthermore, unlike in the previous section where \( R \) from the definition (3.1) of the shadowing map \( \Pi_{t,R} \) had to be carefully chosen, this setting is less sensitive to the choice of \( R \).

**Remark 5.2.** One can continue using techniques developed in the previous section to study bouncing, asymptotic, and crossing vectors. However, the geometric potential estimates were well studied in the surface setting in [GW99]. Without deviating from the main goal and to simplify the argument in this section, we will only focus on shadowing vectors \( w = \Pi_{t,R}(v) \).

The goal of this section is to show that under this different set of assumptions, the shadowing vector \( w = \Pi_{t,R}(v) \) satisfies the estimates in \( x_w \) and \( \phi_w \) as claimed in (3.3). Recalling that \( \gamma_0 \) is the length of \( T_0 \), we state it as a proposition below, which is the analog of Proposition 4.2.
Proposition 5.3. There exists \( Q = Q(R) > 1 \) independent of \( t \) such that for any shadowing vector \( w = \Pi_{t,R}(v) \) and \( \tau \in [0, \hat{t}] \) we have
\[
Q^{-1}(\tau + 1)^{-\frac{m}{m+2}} \leq x_w(\tau) \leq Q(\tau + 1)^{-\frac{m}{m+2}},
\]
and
\[
|\phi_w(\tau)| \leq Q[(\tau + 1)^{-\frac{m+2}{m}} - (\hat{t} + 1)^{-\frac{m+2}{m}}],
\]
and for any \( \tau \in [0, \hat{t} - 2\sqrt{2}\gamma_0] \),
\[
|\phi_w(\tau)| \geq Q^{-1}[(\tau + 1)^{-\frac{m+2}{m}} - (\hat{t} + 1)^{-\frac{m+2}{m}}].
\]

Remark 5.4. We do not expect the lower bound of \( |\phi_w(\tau)| \) to hold for all \( \tau \in [0, \hat{t}] \) since we have little control of the metric near \( \gamma(\hat{t}) \). For instance, when the metric near \( \gamma(\hat{t}) \) is isometric to the surface of revolution of \( 1 + x^{2m} \), by Proposition 3.1 we know that \( \phi_w(\tau) \) is of the same scale as \( (\tau + 1)^{-\frac{m+2}{m}} - (\hat{t} + 1)^{-\frac{m+2}{m}} \) when \( \tau \) is near \( \hat{t} \).

To prove the proposition, we need to exploit the assumptions on \( K(s,x) \) and establish a few auxiliary lemmas. Consider any \( w = \Pi_{t,R}(v) \) for some \( t,R > 0 \) and \( v \in \text{Sing} \). Recall that \( \hat{t} \in [0, \hat{t}] \) is the smallest number in which \( x_w(\tau) \) attains the minimum. We can decompose \([0, \hat{t}]\) into subintervals \( I_i = [t_i, t_{i+1}] \) and \( I'_i = (t'_i, t_{i+1}) \) for \( i = 0, 1, \ldots, n \) so that
\[
s(\gamma_w(\tau)) \in \hat{L} \text{ when } \tau \in I_i
\]
and
\[
s(\gamma_w(\tau)) \notin \hat{L} \text{ when } \tau \in I'_i;
\]
see Figure 5.1. Notice that \( \hat{t} \geq t_n \), thus \( I_n \) is not empty (though it may be arbitrarily short), but \( I'_n \) may be empty.

![Figure 5.1. Division into \( I_i \) and \( I'_i \).](image)

Since the angle \( \phi_w(\tau) \) satisfies \( |\phi_w(\tau)| \in [0, \pi/4] \) for any \( \tau \in [0, \hat{t}] \) from Lemma 3.1, there exists \( N \in \mathbb{N} \) so that
\[
\min_i |I_i| \geq \frac{1}{N} \max_i |I'_i|
\]
where the maximum is taken over all possible \( i \) and the minimum is taken over \( i \) in \( \{1, \ldots, n\} \) if \( t_0 \in I_n' \) (i.e. \( I_n' \) is nonempty) or else (i.e. \( \tilde{t} \in I_n \) and \( I_n' \) is empty) in \( \{1, \ldots, n - 1\} \) in order to exclude \( |I_n| \) which could be arbitrarily small.

The following lemma shows \( x''_w(\tau) \) admits a similar bound as in Lemma 4.5 when \( \tau \) belongs to \( I_i \) for some \( i \).

**Lemma 5.5.** There exists \( C_0 > 0 \) independent of \( w \) such that

\[
0 \leq x''_w \leq C_0 x_{w}^{m+1}
\]

for all \( \tau \in [0, \tilde{t}] \). Moreover, there exists \( c_0 \in (0, C_0) \) such that whenever \( \tau \in I_i \) for some \( i \), we also have the lower bound

\[
x''_w \geq c_0 x_{w}^{m+1}.
\]

**Proof.** Since \( G_{xx} = -KG \leq 2C_1 x_w^m \) with \( G_x(s, 0) = 0 \) for all \( s \), thus \( G_x(s, x_w) \leq 2C_1(m + 1)^{-1} x_w^{m+1} \). Let \( C_0 := 2C_1(m + 1)^{-1} \). By (2.6),

\[
x''_w = \frac{G_x}{G} \cos^2 \phi_w \leq \frac{G_x}{G} \leq C_0 x_{w}^{m+1}.
\]

For \( t \in I_i \), we have \( G_{xx} = -KG \geq C_2 x_w^m \), thus \( G_x(s, x_w) \geq C_2(m + 1)^{-1} x_w^{m+1} \). Let \( c_0 := C_2(2m + 2)^{-1} \). By Lemma 3.1 (1), we have

\[
x''_w = \frac{G_x}{G} \cos^2 \phi_w \geq \frac{G_x}{\sqrt{2G}} \geq c_0 x_{w}^{m+1}.
\]

This completes the proof. \( \square \)

The following lemma we let \( t''_i := \frac{t_i + t_i'}{2} \). For simplicity, in the remaining part of this section, we abbreviate \( x = x_w \) and \( \phi = \phi_w \) where \( w = \Pi_{t,R}(v) \) for any \( v \in \text{Sing} \) and \( t, R > 0 \).

**Lemma 5.6.** For any \( i \) with \( |I_i| \geq \gamma_1 \) (namely, those \( I_i \) not containing \( \theta \) or \( t_0 \), we have

\[
(1) \int_{t_i''}^{t_i'} -x' x^{m+1} \, ds \geq \frac{1}{2N + 1} \int_{t_i''}^{t_i' + 1} -x' x^{m+1} \, ds.
\]

\[
(2) \int_{t_i''}^{t_i'} -x' x^{m+1} \, ds \geq \frac{1}{2N + 1} \int_{\tau}^{t_i' + 1} -x' x^{m+1} \, ds \text{ for any } \tau \in [t_i, t_i''].
\]

**Proof.** For (1) consider any \( s \in [t_i'', t_i'] \) and \( s' \in [t_i', t_i' + 1] \). Since \( x \) is convex and decreasing, we have \( -x'(s) \geq -x'(s') \) and \( x(s) \geq x(s') \). In particular,

\[
-x'(s)x(s)^{m+1} \geq -x'(s')x(s')^{m+1}.
\]

Since \( \min_{i} |I_i| \geq \frac{1}{N} \max_{i} |I'_i| \), we can divide \( I'_i \) into \( 2N \) subintervals of the same length. The integral on each subinterval, whose length is at most \( |I_i|/2 \), is no more than that on \( [t_i'', t_i'] \).
Hence

\[
\int_{t_i'}^{t_{i+1}} -x'x^{m+1}ds = \int_{t_i'}^{t_i''} -x'x^{m+1}ds + \int_{t_i''}^{t_{i+1}} -x'x^{m+1}ds \\
\leq \int_{t_i'}^{t_i''} -x'x^{m+1}ds + 2N \int_{t_i''}^{t_{i+1}} -x'x^{m+1}ds \\
= (1 + 2N) \int_{t_i'}^{t_i''} -x'x^{m+1}ds.
\]

For (2), by (1), we have

\[
\int_{\tau}^{t_i'} -x'x^{m+1}ds = \int_{\tau}^{t_i''} -x'x^{m+1}ds + \int_{t_i''}^{t_i'} -x'x^{m+1}ds \\
\geq \int_{\tau}^{t_i''} -x'x^{m+1}ds + \frac{1}{2N + 1} \int_{t_i''}^{t_{i+1}} -x'x^{m+1}ds \\
\geq \frac{1}{2N + 1} \int_{\tau}^{t_{i+1}} -x'x^{m+1}ds.
\]

\[\boxdot\]

**Lemma 5.7.** There exists \(C_3 > 0\) such that for any \(i\) and \(\tau \in [t_i, t_{i}''],\)

\[x'(\tau) \leq -2C_3 \sqrt{x(\tau)^{m+2} - x(t_i)'^{m+2}}.\]

**Proof.** If \(\tilde{t} \in I_n',\) by Lemmas 5.5 and 5.6, we have

\[x'(\tau)^2 = \int_{\tau}^{\tilde{t}} -2x'x''ds \geq 2c_0 \left( \int_{\tau}^{t_i''} -x'x^{m+1}ds + \sum_{k=i+1}^{n} \int_{t_k}^{t_{k+1}} -x'x^{m+1}ds \right) \geq \frac{2c_0}{2N + 1} \left( \int_{\tau}^{t_{i+1}} -x'x^{m+1}ds + \sum_{k=i+1}^{n-1} \int_{t_k}^{t_{k+1}} -x'x^{m+1}ds + \int_{t_{i+1}}^{\tilde{t}} -x'x^{m+1}ds \right) \geq \frac{2c_0}{2N + 1} \int_{\tau}^{t_0} -x'x^{m+1}ds \geq \frac{2c_0}{(m + 2)(2N + 1)} (x(\tau)^{m+2} - x(t_i)'^{m+2}).\]
Similarly, if \( \tilde{t} \in I_n \), we have

\[
\begin{align*}
x'(\tau)^2 &= \int_{\tau}^{\tilde{t}} -2x'x''\,ds \\
&\geq 2c_0 \left( \int_{\tau}^{t_{i+1}} -x'x^{m+1}\,ds + \sum_{k=i+1}^{n-1} \int_{t_k}^{t_{k+1}} -x'x^{m+1}\,ds + \int_{t_{n}}^{\tilde{t}} -x'x^{m+1}\,ds \right) \\
&\geq \frac{2c_0}{2N+1} \left( \int_{\tau}^{t_{i+1}} -x'x^{m+1}\,ds + \sum_{k=i+1}^{n-1} \int_{t_k}^{t_{k+1}} -x'x^{m+1}\,ds + \int_{t_{n}}^{\tilde{t}} -x'x^{m+1}\,ds \right) \\
&\geq \frac{2c_0}{2N+1} \int_{\tau}^{\tilde{t}} -x'x^{m+1}\,ds \geq \frac{2c_0}{(m+2)(2N+1)} (x(\tau)^{m+2} - x(\tilde{t})^{m+2}).
\end{align*}
\]

We finish the proof by taking \( C_3 := 2^{-1/2}c_0^{1/2}(m+2)^{-1/2}(2N+1)^{-1/2} \).

\( \Box \)

**Proof of Proposition 5.3.** As did in Proposition 4.2, we will use \( Q \) to denote a generic constant. The desired lower bound for \( x(\tau) \) can be established just as in Proposition 4.2. This is because the upper bound \( 0 \leq x'' \leq C_0 x^{m+1} \) from Lemma 5.5 holds for all \( \tau \in [0, \tilde{t}] \), and this is the only ingredient needed for the lower bound on \( x(\tau) \) in Proposition 4.2. In particular, we have \( x(\tau) \geq (R^{-\frac{m}{2}} + \sqrt{C_0} \tau)^{-\frac{m}{2}} \) for all \( \tau \in [0, \tilde{t}] \).

On the other hand, the desired upper bound for \( x(\tau) \) is more difficult to obtain. The reason for introducing \( t_{i+1}' \) and establishing Lemma 5.7 was to obtain the upper bound. We will prove the case where \( 0 \in I_0' \) and \( \tilde{t} \in I_n \). Other cases are similar, and we will comment on them at the end of the proof. We let \( R_{t_1} > 0 \) be the lower bound on \( x(t_1) \) from the above paragraph.

First, define a sequence

\[
S_n := \left( \sum_{k=0}^{n-1} |t_k'' - t_k| \right) + |\tilde{t} - t_n|, \quad S_0 = 0, \quad \text{and} \quad S_i = \sum_{k=1}^{i} |t_k'' - t_k|.
\]

Then we have \( 0 = S_0 < S_1 < \cdots < S_{n-1} < S_n \). Define a function \( f : [0, S_n] \to \mathbb{R} \) via

\[
f(\tau) := x(\tau - S_i + t_{i+1}) \quad \text{when} \quad \tau \in (S_i, S_{i+1}].
\]

Then \( f \) is a piecewise smooth function with discontinuities at each \( S_i \). Moreover, Lemma 5.7 shows that \( f \) is strictly decreasing function satisfying the assumption of Lemma 4.6 with \( Q = C_3 \). Thus by Lemma 4.6, there exists \( Q_0 > 1 \) such that

\[
f(\tau) \leq Q_0(f(0)^{-\frac{m}{2}} + C_3 m \tau)^{-\frac{m}{2}} \leq Q_0(R_{t_1}^{-\frac{m}{2}} + C_3 m \tau)^{-\frac{m}{2}}
\]

for all \( \tau \in [0, S_n] \). In particular, this inequality provides an upper bound for \( x(\tau) \) for \( \tau \in [t_i, t_{i+1}'] \) for \( 1 \leq i \leq n \); here \( t_{i+1}' \) should be replaced by \( \tilde{t} \) when \( i = n \).
For $\tau \in [t'_i, t_{i+1}]$, we have $\frac{S_i}{\tau} \geq \frac{1}{2N+2}$ from the choice of $N$. Thus for $\tau \in [t''_i, t_{i+1}]$, 
\[ x(\tau) \leq x(t''_i) = f(S_i) \leq Q_0(f(0)^{-\frac{m}{2}} + C_3mS_i)^{-\frac{2}{m}} \leq Q_0 \left( R_{t_i}^{-m/2} + \frac{C_3m}{2N+2} \right)^{-\frac{2}{m}}. \]

For the last remaining subset of the domain when $\tau \in [0, t_1]$, which is due to the assumption that $0 \in I'_0$, we have $t_1 \leq \sqrt{2}\gamma_0$. Therefore,
\[ x(\tau) \leq x(0) = R \leq R(\sqrt{2}\gamma_0) + 1)^{\frac{2}{m}}(\tau + 1)^{-\frac{2}{m}}. \]

In sum, we can find $Q > 0$ such that
\[ x(\tau) \leq Q(\tau + 1)^{-\frac{2}{m}} \]
for all $\tau \in [0, \tilde{t}]$.

For $\phi$, using (2.4) and Lemmas 3.1 (1) and 5.5, there exists $Q > 0$ such that
\[ \phi' \leq \sqrt{2}\phi \cos \phi = \sqrt{2}\phi'' \leq \sqrt{2}C_0x^{m+1} \leq Q(\tau + 1)^{\frac{2(m+1)}{m}}. \]

Thus we get the required upper bound for $\phi$:
\[ |\phi(\tau)| = \int_{\tau}^{\tilde{t}} |\phi'|d\tau \leq \int_{\tau}^{\tilde{t}} |\phi'|d\tau \leq Q \int_{\tau}^{\tilde{t}} (s + 1)^{\frac{2(m+1)}{m}}d\tau \leq Q[\tau + 1]^{-\frac{m}{2}} - (\tilde{t} + 1)^{-\frac{m+2}{m}}. \]

This completes the proof when $0 \in I'_0$ and $\tilde{t} \in I_n$.

Other remaining cases can be dealt with similarly. When $0 \in I_0$ and $\tilde{t} \in I_n$, then exactly the same proof works; in fact, there is no need to separately consider $\tau \in [0, t_1]$ as we did above. In the case where $\tilde{t} \in I'_n$, we can use proceed just as we did above by bounding $x(\tau)$ above by $x(t''_n)$ for $\tau \in [t''_n, \tilde{t}]$.

Now, we consider the lower bound of $|\phi|$. For any $\tau \in [0, \tilde{t} - 2\sqrt{2}\gamma_0]$, the interval $[\tau, \tilde{t}]$ contains at least one $[t_i, t'_i]$. Let $l$ (resp. $L$) be the minimal (resp. maximal) $i$ with $[t_i, t'_i] \subset [\tau, \tilde{t}]$. We firstly compare the integrals of $x(s)^{m+1}$ on $[\tau, t_i]$ and $[t_l, t_{l+1}]$. Since $|\phi| < \pi/4$,
\[ t_{l+1} - t_l \geq \gamma_0 \geq \frac{t_l - \tau}{\sqrt{2}}. \]

Moreover, we have
\[ \frac{t_{l+1} + 1}{\tau + 1} = 1 + \frac{t_{l+1} - \tau}{\tau + 1} \leq 1 + 2\sqrt{2}\gamma_0. \]

Hence
\[ x(t_{l+1}) \geq Q(t_{l+1} + 1)^{-2/m} \geq Q(\tau + 1)^{-2/m} \geq Qx(\tau). \]

Together with (5.3), since $x$ is non-increasing, we get
\[ \int_{t_l}^{t_{l+1}} x(s)^{m+1}ds \geq (t_{l+1} - t_l)x(t_{l+1})^{m+1} \geq Q\frac{t_l - \tau}{\sqrt{2}}x(\tau)^{m+1} \geq Q \int_{\tau}^{t_l} x(s)^{m+1}ds. \]
Notice that $\phi' = x'' / \cos \phi \geq x''$, and $|I_i|/|I_j| \in [1/\sqrt{2}, \sqrt{2}]$ for any $i, j$. Thus, by (5.4),

$$|\phi(\tau)| = \int_\tau^{\tilde{\tau}} \phi'(s) ds \geq \int_\tau^{\tilde{\tau}} x''(s) ds \geq c_0 \sum_{i=1}^{L} \int_{t_i}^{t_{i+1}} x(s)^{m+1} ds$$

$$\geq \frac{c_0}{N+1} \sum_{i=1}^{L-1} \int_{t_i}^{t_{i+1}} x(s)^{m+1} ds + \frac{c_0}{N+1+\sqrt{2}} \int_{t_L}^{\tilde{\tau}} x(s)^{m+1} ds$$

$$\geq Q \int_\tau^{\tilde{\tau}} x(s)^{m+1} ds \geq Q \int_\tau^{\tilde{\tau}} (s+1)^{-2(m+1)/m} ds$$

$$\geq Q[(\tau + 1)^{-\frac{m+2}{m}} - (\tilde{\tau} + 1)^{-\frac{m+2}{m}}].$$

$\square$

### 6. Geometric potentials

This section aims to prove Theorem C. Let $M$ be a closed rank 1 nonpositively curved manifold, and $\mathcal{F} = (f_t)_{t \in \mathbb{R}}$ denote the geodesic flow on $T^1 M$. Recall that the geodesic potential is defined via

$$\varphi^u(v) := - \lim_{t \to 0} \frac{1}{t} \log \det(df_t | E^u(v)) = - \frac{d}{dt} \bigg|_{t=0} \log \det(df_t | E^u(v)).$$

As indicated in [BCFT18, Section 7.2], it is convenient to consider the following auxiliary function whose time evolution is governed by a Riccati equation:

$$\psi^u(v) := - \lim_{t \to 0} \frac{1}{t} \log \det(J^u_{v,t}) = - \frac{d}{dt} \bigg|_{t=0} \log \det(J^u_{v,t}),$$

where $J^u_{v,t} : w \in v^+ \mapsto J(t) \in (f_t v)^+$ and $J(t)$ is a unstable Jacobi field along $\gamma_v$ such that $J(0) = w$. We also have $\psi^u(v) = - \text{tr} U^u_v(0)$ where $U^u_v(t)$ is the shape operator of the unstable horoshpere $H^u(f_t v)$.

Let $\pi : T^1 M \to M$ be the canonical projection. Its derivative $d\pi_v : T_v T^1 M \to T_{\pi v} M$ sends $E^u(v)$ onto $v^+$. We have $df_t = d\pi_v \circ J^u_{v,t} \circ d\pi_v$, and thus

$$\det(df_t | E^u(v)) = \det(d\pi_v)^{-1} \det(J^u_{v,t}) \det(d\pi_v).$$

Thus

$$\varphi^u(v) - \psi^u(v) = \frac{d}{dt} \bigg|_{t=0} \log \det(d\pi_v).$$

(6.1)

For any $t$, since $U^u_v(t)$ is symmetric, we can take an orthonormal basis $\{e_i(t)\}_{i=1}^{n-1}$ of $v^+$ so that $U^u_v(t)e_i(t) = \lambda_i(t)e_i(t)$ with $\lambda_i(t) \geq 0$. Since $||df_t(\xi)||_S = ||J_\xi(t)||^2 + ||J'_\xi(t)||^2$ for $\xi \in E^u_v$, for any $t$, we have an orthonormal basis $\{\xi_i(t)\}_{i=1}^{n-1}$ of $E^u_{f_t v}$, where $\xi_i(t)$ is determined by

$$J_{\xi_i}(t) = \frac{e_i(t)}{\sqrt{1 + \lambda_i(t)^2}}, \quad J'_{\xi_i}(t) = \frac{\lambda_i(t)e_i(t)}{\sqrt{1 + \lambda_i(t)^2}}.$$
Thus $d\pi_{f,v}(\xi(t)) = e_i(t)/\sqrt{1 + \lambda_i(t)^2}$ and the matrix of $d\pi_{f,v}$ with respect to these two orthonormal basis is diag($(1 + \lambda_i(t)^2)^{-1/2}, \ldots, (1 + \lambda_i(t)^2)^{-1/2})$. Hence

\begin{equation}
\log \det(d\pi_{f,v}) = \log \prod_{i=1}^{n-1} (1 + \lambda_i(t)^2)^{-1/2} = -\frac{1}{2} \log \det(I_{n-1} + U^u_v(t)^2).
\end{equation}

Now we use the following Jacobi formula for $A : \mathbb{R} \to M_{n-1}$:

$$\frac{d}{dt} \det A(t) = \det A(t) \tr(A(t)^{-1} \frac{dA(t)}{dt}).$$

For simplicity, denote by $U = U^u_v$ and $I = I_{n-1}$. By (6.1) and (6.2), we have

$$\varphi^u(v) - \psi^u(v) = -\frac{1}{2} \tr[(I + U(0)^2)^{-1}(U'(0)U(0) + U(0)U'(0))]
\leq -\frac{1}{2} \tr[(U(0)(I + U(0)^2)^{-1} + (I + U(0)^2)^{-1}U(0))U'(0)]
= -\tr[U(0)(I + U(0)^2)^{-1}U'(0)].$$

Since $U' + U^2 + \kappa = 0$, $U$ is positive semidefinite, and $|\tr(AB)| \leq |\tr(A)||\tr(B)|$ if $A, B$ are positive semidefinite, we have

$$|\varphi^u(v) - \psi^u(v)| \leq \tr[U^3(0)(I + U(0)^2)^{-1}] + \tr[U(0)(I + U(0)^2)^{-1}(-\kappa)]
\leq \tr(U(0)) + \tr[U(0)(I + U(0)^2)^{-1}] \tr(-\kappa)
= -\psi^u(v)^2 - \psi^u(v)(-\tric(v)) \leq -\psi^u(v)(\psi^u(v)^2 - \tric(v)).$$

When $v$ is sufficiently close to Sing, $-\psi^u(v)$ and $-\tric(v)$ are small nonnegative numbers, thus we have $\varphi^u(v) \approx \psi^u(v)$ near Sing. We summarize the above discussion below:

**Proposition 6.1.** Suppose $M$ is a closed rank 1 nonpositively curved manifold. Then we have

$$|\varphi^u(v) - \psi^u(v)| \leq -\psi^u(v)(\psi^u(v)^2 - \tric(v)).$$

In particular, we have $\varphi^u(v) \approx \psi^u(v)$ near Sing.

### 6.1. The proof of Theorem C.

The strategy of the proof is to study the auxiliary function $\psi^u$ through the associated Riccati equation. We establish a version of Theorem C for $\psi^u$. Then Theorem C follows Proposition 6.1.

We remark that the additional Ricci curvature constraint is essential in our argument. In the higher dimension scenario, only having radial curvature controlled is insufficient. Nevertheless, for some special Riemannian metrics, namely, warped products, the radial curvature $K_\perp(v)$ and Ricci curvature $\tric(v)$ are comparable. Since this observation is not in the mainstream of the current paper, we leave the proof in Appendix A.

Let $M$ be a type 1 manifold with order $m - 1$ Ricci curvature bounds, that is, there exists $k_0, K_0 > 0$ such that

$$-K_0|x_v|^m \leq \tric(v) \leq -k_0|x_v|^m$$

for all $v$ with $|x_v| \leq \varepsilon$. 


Proposition 6.2. Assume $M$ satisfies the assumption of Theorem C, then
\[ -\psi^u(v) \approx |x_v|^{m/2} + |\phi_v|^{m/(m+2)}, \]
for any $v$ near $\text{Sing}$.

In particular, we have the same scale estimation for $-\varphi^u$, and Theorem C follows. To prove Proposition 6.2, we need the following lemma.

Lemma 6.3. Assume there exist $K_1 > k_1 > 0$ so that
1. $-K_1^2 T^{-2} \leq \text{Ric}(\gamma'_0(t)) \leq -k_1^2 T^{-2}$ for all $t \in [-T, 0]$, then there exists $K_2 > k_2 > 0$ depending on $k_1, K_1$ so that
   \[ k_2 T^{-1} \leq -\psi^u(v) \leq K_2 T^{-1}. \]
2. $-K_1^2 T^{-2} \leq \text{Ric}(\gamma'_0(t)) \leq 0$ for all $t \in [0, k_1 T]$, and $k_3 T^{-1} \leq -\psi^u(v) \leq K_3 T^{-1}$, then there exist $K_4 > k_4 > 0$ depending on $k_1, K_1, k_3, K_3$ so that
   \[ k_4 T^{-1} \leq -\psi^u(\gamma'_0(t)) \leq K_4 T^{-1}, \]
for all $t \in [0, k_1 T]$.

Proof. Denote by $u(t) = \frac{1}{n-1} \text{tr}(U^u_v(t))$. Since $U^u_v$ is diagonalizable and all eigenvalues $\lambda_i(t)$ are nonnegative, by Cauchy-Schwartz we have
\[ \text{tr}((U^u_v)^2) \leq (n-1)^2 u^2 \leq (n-1) \text{tr}((U^u_v)^2). \]
Thus, by the Riccati equation,
\[ u' = \frac{\text{tr}((U^u_v)^2)}{n-1} = -\frac{\text{tr}((U^u_v)^2)}{n-1} - \frac{\text{Ric}(\gamma)}{n-1} \leq -u^2 - \frac{\text{Ric}(\gamma)}{n-1}. \]
On the other hand, denote by $w(t) = (n-1)u(t) = \text{tr}(U^u_v(t))$. We have
\[ w' = -\text{tr}((U^u_v)^2) - \text{Ric}(\gamma) \geq -w^2 - \text{Ric}(\gamma). \]

1. Compare $u$ with the solution of
   \[ \ddot{u} + \ddot{u}^2 - K_1^2 T^{-2} = 0, \quad \ddot{u}(-T) = +\infty \quad \Rightarrow \quad \ddot{u}(0) = K_1 T^{-1} \coth K_1. \]
   By the main theorem in [EH90], we have
   \[ -\psi^u(v) = (n-1)u(0) \leq (n-1)\ddot{u}(0) =: K_2 T^{-1}. \]
   Compare $w$ with the solution of
   \[ \ddot{w} + \ddot{w}^2 - k_1^2 T^{-2} = 0, \quad \ddot{w}(-T) = 0 \quad \Rightarrow \quad \ddot{w}(0) = k_1 T^{-1} \tanh k_1. \]
   We have
   \[ -\psi^u(v) = w(0) \geq \ddot{w}(0) =: k_2 T^{-1}. \]

2. Compare $u$ with the solution of
   \[ \ddot{u} + \ddot{u}^2 - K_1^2 T^{-2} = 0, \quad \ddot{u}(0) = K_3 T^{-1}. \]
   We have
   \[ \ddot{u}(t) = K_1 T^{-1} \coth(K_1 T^{-1} t + \coth^{-1}(K_3/K_1)). \]
Since \( \coth \) is decreasing, so does \( \bar{u} \). For \( t \in [0, k_1 T] \), we get
\[
-\psi^u(\gamma'_v(t)) = (n-1)u(t) \leq (n-1)\bar{u}(t) \leq (n-1)\bar{u}(0) =: K_4 T^{-1}.
\]
Compare \( w \) with the solution of
\[
\bar{w}' + w^2 = 0, \quad \bar{w}(0) = k_3 T^{-1} \quad \Rightarrow \quad \bar{w}(t) = (t + k_3^{-1} T)^{-1}.
\]
For \( t \in [0, k_1 T] \),
\[
-\psi^u(\gamma'_v(t)) = w(t) \geq \bar{w}(t) \geq (k_1 T + k_3^{-1} T)^{-1} =: k_4 T^{-1}.
\]

\[\square\]

**Proof of Proposition 6.2.** We follow the main steps in the proof of [GW99, Lemma 3.3]. We use \( x, \phi \) instead of \( x_v, \phi_v \) for simplicity.

**Case 1:** \( v \) is bouncing or asymptotic. See Figure 6.1a. Since the asymptotic case is the bouncing case with \( \ell = \infty \), we only have to consider the bouncing \( v \). We may assume \( 0 < x(0) \leq \varepsilon/2 \). Denote by
\[
T_v := \max \{ T > 0 : x(t) \in [x(0)/2, 2x(0)], \forall -T \leq t \leq 0 \}.
\]
For any \( t \in [-T_v, 0] \), we have
\[
-K_5 x(0)^m \leq \text{Ric}(\gamma(t)) \leq -k_5 x(0)^m.
\]
Moreover we have

**Lemma 6.4.** \( T_v \geq k_6 x(0)^{-m/2} \) for some \( k_6 > 0 \) independent of \( v \).

**Proof.** Case 1: \( x(-T_v) = 2x(0) \), and \( x(t) \) is decreasing on \( [-T_v, 0] \). By (4.4) with \( \alpha = m + 2, \beta = 1/2 \), we have
\[
x(0)^{-m/2} \leq (2x(0))^{-m/2} + K_6 T_v.
\]
Thus \( T_v \geq k_6 x(0)^{-m/2} \).

Case 2: \( x(-T_v) = x(0)/2 \), similar to Case 1.
We prove (6.4) in the following two cases: k
\[k(6.4)\]
Thus, it suffices to prove Case 2.
\[\text{Thus, we finish the proof of Proposition 6.2 in this case.}\]

Case 3: \(x(-T_v) = 2x(0)\), and \(x(t)\) first decreases, then increases on \([-T_v, 0]\). Assume \(T_3 \in (0, T_v)\) satisfies \(x(-T_3) = x(0)\). By Case 1 we know that \(T_v > T_v - T_3 \geq k_6x(0)^{-m/2}\).

Take \(T = k_6x(0)^{-m/2}\) in Lemma 6.3(1), we get
\[k_7x(0)^{m/2} \leq -\psi^n(v) \leq K_7x(0)^{m/2}.\]
By (3.6), we know that
\[|\phi(0)|^{m/(m+2)} \leq |2\sin \phi(0)|^{m/(m+2)} = (-2x'(0))^{m/(m+2)} \leq Qx(0)^{m/2}.\]
Thus, we finish the proof of Proposition 6.2 in this case.

**Case 2:** \(v\) is crossing. See Figure 6.1b. Recall that \(x(t_0) = 0\), and \(a(t) = -\sin \phi(t)\). Denote by \(A := a(t_0)\). Since \(v\) is close to Sing, we may assume that \(2A < \varepsilon^{(m+2)/2}\) and \(0 \leq x(0) < A^{2/(m+2)}\). Since \(x' = \sin \phi\), by (4.10) we have
\[x(0)^{m/2} \leq A^{m/(m+2)} \leq |x'(0)|^{m/(m+2)} \leq |\phi(0)|^{m/(m+2)},\]
and
\[A \leq |x'(0)| = |\sin \phi(0)| \leq \sqrt{A^2 + K_8x(0)^{m+2}} \leq A\sqrt{1 + K_8}.
Thus, it suffices to prove
\[(6.4)\]
\[k_9A^{m/(m+2)} \leq -\psi^n(v) \leq K_9A^{m/(m+2)}.\]
We prove (6.4) in the following two cases:

- **Case 2a:** \(\phi(0) < 0\). In this case, \(t_0 > 0\) and \(x' < 0\) for \(t \in [0, t_0]\). Let \(T_v, t_v > 0\) be the minimal solutions of
\[x(-T_v) = (2A)^{2/(m+2)}, \quad x(-t_v) = A^{2/(m+2)}.\]
By the choice of \(A\), we have \(x(-t_v) < x(-T_v) < \varepsilon\). For \(t \in [-T_v, -t_v]\), by (4.10) and (4.4) with \(\alpha = 2, \beta = \frac{m+1}{m+2}\), we have
\[A^{-\frac{m}{m+2}} \leq (2A)^{-\frac{m}{m+2}} + K_{10}(T_v - t_v).
Thus \(T_v - t_v > k_{10}A^{-m/(m+2)}\). Take \(T = k_{10}A^{-m/(m+2)}\) in Lemma 6.3(1), we have
\[k_{11}A^{m/(m+2)} \leq -\psi^n(x'(0)) \leq K_{11}A^{m/(m+2)}.\]
Since \(x' = \sin \phi\), we have \(\phi' = x'' \sec \phi \geq 0\), thus \(|x'| = |\sin \phi| \geq |\sin \phi(t_0)| = A\) for all \(t \in [-t_v, t_0]\). Thus,
\[(6.6)\]
\[(t_v + t_0)A \leq \int_{-t_v}^{t_0} -x'(s)ds = -x(t_0) + x(-t_v) = A^{2/(m+2)}.
Thus \(t_v \leq A^{-m/(m+2)}\). By (6.5), take \(T = A^{-m/(m+2)}\) in Lemma 6.3(2), we get
\[(6.7)\]
\[k_{12}A^{m/(m+2)} \leq -\psi^n(v) \leq K_{12}A^{m/(m+2)}.\]
Case 2b: \( \phi(0) > 0 \). In this case, we have \( t_0 < 0 \). Since \( \gamma_v(t_0) \) crosses \( T_0 \) and we do not have flat strips, by the result of Case 2a, we know that

\[(6.8) \quad k_{12} A^{m/(m+2)} \leq -\psi^u(\gamma_v(t_0)) \leq K_{12} A^{m/(m+2)}, \]

and \( |t_0| \leq A^{-m/(m+2)} \) by (6.6). Again take \( T = A^{-m/(m+2)} \) in Lemma 6.3(2), we have

\[k_{13} A^{m/(m+2)} \leq -\psi^u(v) \leq K_{13} A^{m/(m+2)}. \]

\[\square\]

Remark 6.5. The absence of flat strips is crucial in Proposition 6.2 and Theorem C; otherwise, the Hölder continuity does not hold. Here is a counterexample: consider a surface of revolution generated by

\[ f(x) = \begin{cases} 
|x + 0.5|^{m+2} + 1, & x < -0.5, \\
1, & -0.5 \leq x \leq 0.5, \\
(x - 0.5)^{m+2} + 1, & x > 0.5.
\end{cases} \]

The flat strip is the part with \(-0.5 \leq x \leq 0.5\), and the metric satisfies both curvature conditions by Lemma A.1. Let \( v \) be a unit vector with \( x = 0.5 \) and angle \( \phi > 0 \), meaning that \( v \) is a vector exiting the flat strip. At time \( t = -\csc \phi \), the geodesic \( \gamma_v \) enters the strip with vector \( v' = \gamma_v'(-\csc \phi) \). By symmetry, the angle of \( v' \) is \(-\phi\). Assume that the Hölder continuity in Proposition 6.2 holds for both \( v \) and \( v' \), namely, \( \psi^u(v), \psi^u(v') \approx -\phi^{m/(m+2)} \).

As \( \gamma_v(t) \) is in the flat strip for \( t \in [-\csc \phi, 0] \), \( \psi^u(f_tv) \) satisfies the Riccati equation \( u' + u^2 = 0 \), and the solution is

\[ \psi^u(f_tv) = (t + \psi^u(v)^{-1})^{-1}. \]

Plug in \( \psi^u(v) \approx -\phi^{m/(m+2)} \) and \( t = -\csc \phi \), we have

\[ \psi^u(v') = (-\csc \phi + \psi^u(v)^{-1})^{-1} \approx -\phi. \]

Contradictory to \( \psi^u(v') \approx -\phi^{m/(m+2)} \).

The Hölder continuity of \( \varphi^u \) is an important, yet still open, question in nonpositively curved geometry. Only some partial results are known for surfaces under certain conditions, including [GW99, Lemma 3.3] where Gerber and Wilkinson show the Hölder continuity of \( \varphi^u \) for type 2 surfaces. Since Ricci curvature and Gaussian curvature are the same thing for surfaces, using Theorem C we obtain a partial generalization of [GW99, Lemma 3.3]:

Corollary 6.6. Under the same assumptions as Theorem C, \( \psi^u \) and \( \varphi^u \) are Hölder continuous in a small neighborhood of Sing.

7. Sufficient criteria for the pressure gap

Let \( M \) be a closed Riemannian manifold and \( \{f_t\}_{t \in \mathbb{R}} \) the geodesic flow on \( T^1 M \). In this section, we will describe an abstract result to establish the pressure gap for a given potential \( \varphi: T^1 M \to \mathbb{R} \). But first, we need to introduce the notion of specification in the following subsection.
7.1. Specification. While there are various definitions for it in the literature, roughly speaking specification is a property that allows one to find an orbit segment that shadows any given finite number of orbit segments at a desired scale with controlled transition time. It was introduced by Bowen [Bow74] as one of the conditions to establish the uniqueness of the equilibrium states for potentials over uniformly hyperbolic maps. The specification still plays a vital role in many generalizations of this result [BCFT18, CKP20, CKP21]. The following version of the specification is from [BCFT18, Theorem 4.1].

**Definition 7.1** (Specification). We say a set of orbit segments $C$ satisfies the specification at scale $\rho > 0$ if there exists $T > 0$ such that given finite orbit segments $(v_1, t_1), \ldots, (v_k, t_k) \in C$ and $T_1, \ldots, T_k \in \mathbb{R}$ with $T_{j+1} - T_j \geq t_j + T$ for all $1 \leq j \leq k - 1$, there is $w \in T^1 M$ such that $f_{T_j} w \in B_{t_j}(v_j, \rho)$ for all $1 \leq j \leq k - 1$.

This is a stronger version of the specification that appears in [BCFT18] providing flexibility in the transition time. However, in practice, we will always take $T_j$’s such that $T_{j+1} - T_j = t_j + T$; that is, the transition time is exactly equal to $T$.

7.2. Abstract result on the pressure gap. We now list the conditions together which establish the pressure gap. Let $M$ be a closed rank 1 manifold with a codimension 1 flat subtorus, and $\text{Sing}$ are induced by the subtorus. As mentioned in the introduction, one can easily extend results in this section to multiple subtori scenarios. However, for brevity, we stick to this simpler assumption.

By setting 
\[ C(\eta) = \{(v, t) \in T^1 M \times \mathbb{R}^+: v, f_t v \in \text{Reg}(\eta)\} \]
to be the set of orbit segments with endpoints in $\text{Reg}(\eta)$, we require that the geodesic flow \{f_t\} and the potential $\varphi: T^1 M \to \mathbb{R}$ satisfy the following property:

1. Singular set zero entropy property: $h_{\text{top}}(\mathcal{F}|_{\text{Sing}}) = 0$.
2. Specification property: For any $\eta > 0$ and $\rho > 0$, the orbit segments $C(\eta)$ satisfies the specification at scale $\rho$.
3. Shadowing property: There exists $R > 0$ such that for every $t > 0$ there exists a map 
\[ \Pi_{t, R}: \text{Sing} \to T^1 M \]
with the following properties: denoting by $w_v := \Pi_{t, R}(v)$ the shadowing vector of an arbitrary $v \in \text{Sing}$,
   - (a) The conclusion of Lemma 3.1 hold for $\Pi_{t, R}$.
   - (b) For any $\varepsilon > 0$, there exists $L := L(\varepsilon, R)$ such that for any $t > 2L$, the vector $w_v$ satisfies 
     \[ d(f_\tau w_v, \text{Sing}) < \varepsilon \]
     for all $\tau \in [L, t - L]$.
4. Special Bowen property: For any $\delta > 0$, there exists $C = C(\delta, R) > 0$ independent of $t$ such that 
\[ \int_0^t \varphi(f_\tau u) d\tau - \int_0^t \varphi(f_\tau v) d\tau > -C \]
for any $u \in B_t(w_v, \delta)$. 


Proposition 7.2. Suppose the geodesic flow \( \{f_t\}_{t \in \mathbb{R}} \) and the potential \( \varphi \) satisfy the above listed conditions (1), (2) and (3). Then, \( \varphi \) has a pressure gap.

Assume we have Proposition 7.2, we finish the proof of Theorem A:

Proof of Theorem A. Each condition listed above can be verified as follows. By design, we know \( h_{\text{top}}(\mathcal{F}|_{\text{Sing}}) = 0 \). The specification property (2) is already established in [BCFT18] for the geodesic flow \( \{f_t\} \) over rank 1 nonpositively curved manifold. With \( \Pi_{t,R} \) defined as in (3.1) via the Fermi coordinates, (3a) is immediate as we have already proved Lemma 3.1. For (3b), we can take \( L \) to be \( (Q/\varepsilon)^{m/2} \) where \( Q \) is the constant from Proposition 4.2 and 5.3. Lastly, (4) follows from Proposition 3.3. Hence, \( \varphi \) has the pressure gap by the above proposition.

We note that the proof of Proposition 7.2 draws inspiration from [BCFT18, Theorem B]. However, leveraging the singular set’s zero entropy property, Peres’ Lemma allows us to circumvent several technicalities and arrive at a more straightforward proof than that presented in [BCFT18]. See Remark 7.4 for more details.

Theorem 7.3. [Per88, Lemma 2] Let \( \mathcal{F} = \{f_t\} \) be a continuous flow on a compact space \( X \), and \( \mu \) be an \( \mathcal{F} \)-invariant probability measure. Then for every potential \( \varphi : X \to \mathbb{R} \) there exists some \( v \in X \)

\[
\frac{1}{T} \int_0^T \varphi(f_\tau v) d\tau \geq \int_X \varphi d\mu
\]

for all \( T > 0 \).

The authors believe this Peres’ result is known among the experts; however, we cannot find proof of the about flow version in the literature. For the completeness, we give a proof in the appendix; see Theorem B.1.

Proof of Proposition 7.2. This proof has three steps. The first step is using the singular set zero entropy property (1) and Theorem 7.3 to bound \( P(\varphi, \text{Sing}) \) by the integration of some special \( v \) in Sing along the flow. The second step is using the specification property (2) and shadowing property (3) to create a \( (t, \delta) \)-separated set that bounds \( P(\varphi, \text{Sing}) \). The last step is using the special Bowen property (4) to estimate the pressure.

Notice that without lost of generality, we may assume \( P(\varphi, \text{Sing}) = 0 \); otherwise, we consider \( \varphi - P(\varphi, \text{Sing}) \). Hence, it is sufficient to show that there exists a \( (t, \delta) \)-separated set \( F_t \) on \( T^1 M \) such that

\[
(7.1) \lim_{t \to \infty} \frac{1}{t} \sum_{w \in F_t} e^{\int_0^t \varphi(f_\tau w) d\tau} > 0.
\]

The first step is to find a special singular vector \( v \) for shadowing. Since \( \mathcal{F} \) on Sing is entropy-expansive, we know there exists an equilibrium state \( \mu \in \mathcal{M}(\mathcal{F}|_{\text{Sing}}) \) of \( \varphi \). That is

\[
0 = P(\varphi, \text{Sing}) = h_\mu(\mathcal{F}) + \int_{\text{Sing}} \varphi d\mu.
\]
By the singular set zero entropy property (1), we know \( h_\mu(\mathcal{F}) = 0 \); and thus, \( \int_{\text{Sing}} \varphi \, d\mu = 0 \). By Peres’ Lemma (Theorem 7.3), there exists \( \nu \in \text{Sing} \) such that for all \( T > 0 \)

\[
\int_0^T \varphi(f_t \nu) \, dt \geq T \int_{\text{Sing}} \varphi \, d\mu = 0.
\]  

The second step is to create such a \((t, \delta)\)–separated set \( F_t \) on \( T^1 M \). For \( R > 0 \) given in the shadowing property (3), from Lemma 3.1 we get \( \eta > 0 \) where \( \omega, f_t \omega \in \text{Reg}(\eta) \) for any \( \omega = \Pi_{L,R}(\nu) \). We also obtain the constant \( L \) from (3b) corresponding to \( \varepsilon = R/2 \).

We now set \( \xi \) be a constant such that \( \xi > T + 2L \), and for each \( N \in \mathbb{N} \) consider

\[
\mathcal{J} = \{ \xi, 2\xi, \ldots, (N-1)\xi \} \subseteq [0, N\xi].
\]

For any small \( \alpha > 0 \) such that \( \alpha N \in \mathbb{N} \), consider any size \((\alpha N - 1)\) subset

\[ J = \{ N_1 \xi, \ldots, N_{\alpha N - 1} \xi \} \subset \mathcal{A} \]

with \( N_i \in \{1, \ldots, N - 1\} \). Setting \( N_0 = 0 \) and \( N_{\alpha N} = N \), such a subset can be viewed as a partition of the interval \([0, N\xi]\) into \( \alpha N \) subintervals, each of length \( n_i \xi \) for \( n_i = N_i - N_{i-1} \).

We denote \( \mathcal{J}^0_N = \{ J \subset \mathcal{A} : \#J = \alpha N - 1 \} \), and we know \( \#\mathcal{J}_N^0 = (N-1)^{\alpha N-1} \). Given \( J \in \mathcal{J}^0_N \), we consider singular vectors \( \nu_i^j := f_{N_i \xi}(\nu) \) and the corresponding regular orbits \( \{(w_i^j, t_i)\} \) where \( w_i^j := \Pi_{n_i \xi - T,R}(\nu_i^j) \) and \( t_i = n_i \xi - T \) for \( i = 1, \ldots, \alpha N \). Using the specification property (2), one can find a vector \( w_J \in T^1 M \) which shadows orbits \( \{(w_i^j, t_i)\}_{i=1}^{\alpha N-1} \) (see Fig. 7.1), that is, for \( i = 1, \ldots, \alpha N \)

\[
f_{N_{i-1} \xi}w_J \in B_t(w_i^j, \rho).
\]

We claim that \( F_{N\xi} := \{(w_J, N\xi) : J \in \mathcal{J}^0_N \} \) is a \((N\xi, R/2 - 2\rho)\)-separated set. To see this, let \( J_j = \{ N_j^1 \xi, \ldots, N_j^\alpha N_{-1} \xi \} \) for \( j = 1, 2 \), and \( m = \min\{n : N_1^j \neq N_1^k\} \). Without loss

**Figure 7.1.** Shadowing orbit
of generality, suppose $N_1 < N_2$, then the claim follows the inequalities below (see Fig. 7.2):

$$d(f_{N_1} - T(v_{J_1}), \text{Sing}) > R - \rho \text{ and } d(f_{N_2} - T(v_{J_2}), \text{Sing}) < \frac{R}{2} + \rho.$$  

The last step is using the special Bowen property (4) to bound the pressure from below. For $(w, N) \in \mathcal{F}$, the integral of $\varphi(f_{\tau}v_{J})$ during each transition period is bounded below by $-T\|\varphi\|$, by the special Bowen property (4) and (7.2) we get

$$\int_0^{N} \varphi(f_{\tau}v_{J})d\tau = \sum_{i=1}^{N} \int_{N_i-1}^{N_i} \varphi(f_{\tau}v_{J})d\tau$$

$$\geq \sum_{i=1}^{N} \int_{0}^{t_i} \varphi(f_{\tau}v_{J})d\tau - \alpha N C - 2\alpha N\|\varphi\|T$$

$$\geq \sum_{i=1}^{N} \int_{N_i-1}^{N_i} \varphi(f_{\tau}v_{J})d\tau - \alpha N(3T\|\varphi\| + C)$$

$$= \int_0^{N} \varphi(f_{\tau}v_{J})d\tau - \alpha N(3T\|\varphi\| + C) \geq -\alpha N(3T\|\varphi\| + C).$$

Recall that $\#\mathcal{F} = \#\mathcal{F}_{N} = \binom{N-1}{\alpha N-1}$, summing the above inequality over all possible subsets $J$ gives

$$\sum_{w,J \in \mathcal{F}_{N}} e^{N\varphi(f_{\tau}v_{J})d\tau} \geq \left(\frac{N-1}{\alpha N-1}\right) e^{-\alpha N(3T\|\varphi\| + C)}.$$  

As $\mathcal{F}_{N}$ is a $(N,\delta)$-separated set (where $\delta < R/2 - \rho$ with $R$ and $\rho$ are arbitrary), this implies that

$$P(\varphi) \geq \lim_{N \to \infty} \frac{1}{N_{\xi}} \log \left( \sum_{w,J \in \mathcal{F}_{N}} e^{N\varphi(f_{\tau}v_{J})d\tau} \right).$$
Combining the last two inequalities with \( \binom{N-1}{N-1} \geq \alpha e^{-\alpha \log \alpha N} \), one establishes the pressure gap for \( \varphi \) by taking \( \alpha \) sufficiently small. \( \Box \)

**Remark 7.4.** We list below several main differences between this proof and the proof of [BCFT18, Theorem B].

1. Our proof utilize \( h_{\text{top}}(F|_{\text{Sing}}) = 0 \) fact and Peres’ Lemma (Theorem 7.3) to simplify two steps (that is [BCFT18, Sec. 8.2 & 8.3] in the [BCFT18, Theorem B]. Precisely, using \( h_{\text{top}}(F|_{\text{Sing}}) = 0 \) and Peres’ Lemma, we can easily find a vector in the singular set to shadow regular orbit segments with good control of the integration of potentials.

2. The equation (7.3) is another major difference. The extra constant \( C \) in (7.3) comes from the condition (4) given in Section 7.2. This constant \( C \) allows us to have some wiggling room for the potential without losing the pressure gap.

**Appendix A. Ricci curvature bound and radial curvature bound comparison**

Recall that \( M \) is a \( n \)-dimensional manifold with nonpositive sectional curvature, and \( T_0 \subset M \) is a totally geodesic \((n - 1) \)-subtorus. We assume that \( K(\sigma) = 0 \) for any \( x \in T_0 \) and any 2-plane \( \sigma \subset T_x M \). On the universal cover \( \tilde{M} \), we define the Fermi coordinate \((s, x)\) near \( \tilde{T}_0 \) in the following way: \( s \) is the coordinate on \( \tilde{T}_0 \), and \( x \) measures the signed distance on \( \tilde{M} \) to \( \tilde{T}_0 \). \( s = \text{const.} \) is always a geodesic perpendicular to \( \tilde{T}_0 \). The Riemannian metric near \( \tilde{T}_0 \) is

\[
g = dx^2 + g_x, \quad |x| \leq \varepsilon
\]

where \( g_x \) is the Riemannian metric on \( \tilde{T}_x := \tilde{T}_0 \times \{ x \} \). In particular, \( g_0 \) is the Euclidean metric on \( \tilde{T}_0 \).

If \( g_x \) is a warped product, namely, \( g = dx^2 + f(x)^2 g_0 \). We have \( f(0) = 1 \). Since \( T_0 \) is totally geodesic, we have \( f'(0) = 0 \).

**Lemma A.1.** If \( g = dx^2 + f(x)^2 g_0 \), then the following conditions are equivalent:

1. There exists \( C_1, C_2, \varepsilon > 0 \) such that
   \[-C_1 |x_v|^m \leq K_\perp(v) \leq -C_2 |x_v|^m, \text{ for any } v \perp X \text{ with } |x_v| < \varepsilon.\]

2. There exists \( C_1, C_2, \varepsilon > 0 \) so that
   \[-C_1 |x_v|^m \leq \text{Ric}(v) \leq -C_2 |x_v|^m, \text{ for any } v \text{ with } |x_v| < \varepsilon.\]

3. There exists \( C_1, C_2, \varepsilon > 0 \) so that
   \[C_1 |x|^{m+2} \leq f(x) - 1 \leq C_2 |x|^{m+2}.\]

**Proof.** Since \( g = dx^2 + f(x)^2 g_0 \), the radial curvature is

\[K_\perp(v) = -\frac{f''(x_v)}{f(x_v)},\]
where \( \theta \) be the angle between \( X \) and \( \sigma \).

Now we compute the Ricci curvature. If \( v \) is normal, then Ric\((v) = (n - 1)K_\perp(v) \) and we are done. Otherwise, denote by \( v^\perp \) the perpendicular complement in \( T_pM \), \( \theta \) the angle between \( v \) and \( X \), and \( H(v) \) the horizontal subspace at \( v \). Since both \( v^\perp \) and \( H(v) \) have codimension 1, \( v^\perp \cap H(v) \) has dimension \( n - 2 \). We can construct an orthonormal basis \( \{e_i\}_{i=1}^n \) such that \( e_1 = v \), \( e_3, \ldots, e_n \in v^\perp \cap H(v) \), and the section spanned by \( e_1, e_2 \) is normal. Then we have

\[
\text{(A.2)} \quad \text{Ric}(v) = -\frac{f''}{f} + (n - 2) \left( -\frac{f''}{f} \cos^2 \theta - \left( \frac{f'}{f} \right)^2 \sin^2 \theta \right)
\]

When \( \theta = 0 \), we get Ric\((v) = (n - 1)K_\perp(v) \). Thus, for any \( v \), the Ricci curvature can be calculated using (A.2).

\[
(1) \Rightarrow (2) : \text{Since } K_\perp = -\frac{f''}{f} \text{ and } f(0) = 1, \text{ we have } f'' \approx |x|^m. \text{ Since } f'(0) = 0, f' \approx |x|^{m+1}. \text{ Therefore, Ric}(v) \approx -|x|^m \text{ by (A.2)}. \]

\[
(2) \Rightarrow (3) : \text{Assume } f - 1 \approx |x|^k \text{ for some } k > 2. \text{ We have } (f'/f)^2 \approx |x|^{2k-2} \text{ and } f''/f \approx |x|^{k-2}. \text{ Thus, } f''/f \text{ is the dominant term in (A.2). Therefore, } k = m + 2. \]

\[
(3) \Rightarrow (1) : \text{Since } f - 1 \approx x^{m+2} \text{ and } f'(0) = 1, \text{ we have } f' \approx x^{m+1} \text{ and } f'' \approx x^m, \text{ thus } K_\perp = -\frac{f''}{f} \approx -|x|^m. \]

**APPENDIX B. A LEMMA OF PERES**

In this section, we prove a proof of Theorem 7.3 as Peres’ original theorem [Per88, Lemma 2] is for transformations.

**Theorem B.1.** [Per88, Lemma 2] Let \( \mathcal{F} = \{f_t\} \) be a continuous flow on a compact space \( X \), and \( \mu \) be an \( \mathcal{F} \)-invariant probability measure. Then for every potential \( \varphi : X \to \mathbb{R} \) there exists some \( v \in X \)

\[
\frac{1}{T} \int_0^T \varphi(f_t v) d\tau \geq \int_X \varphi d\mu
\]

for all \( T > 0 \).

Peres’ proof is based on the Maximal Ergodic Theorem, and it works almost line by line in the flow case. A version of the Maximal Ergodic Theorem for flows can be found in [Pet83, Theorem 1.2, P.76].

**Theorem B.2** (Maximal Ergodic Theorem). Let \( (X, \mathcal{B}, \mu) \) be a probability space and \( \mathcal{F} = \{f_t\} \) be a measure-preserving flow on \( X \). If \( \varphi \in L^1(\mu) \) and \( \alpha \in \mathbb{R} \), then

\[
\int_{\{\varphi^* > \alpha\}} \varphi d\mu \geq \alpha \cdot \mu(\{\varphi^* > \alpha\})
\]
where $\varphi^*(x) = \sup_{T > 0} \frac{1}{T} \int_0^T \varphi(f_\tau(x)) d\tau$.

Proof of Theorem B.1. For $\epsilon > 0$, we define

$$E_\epsilon := \{ x \in X : \forall t \geq 0, \frac{1}{t} \int_0^t \varphi(f_\tau x) d\tau > \int_X \varphi d\mu - \epsilon \}$$

and

$$\Psi(x) := \int_X \varphi d\mu - \varphi - \epsilon.$$

It is enough to show

$$\bigcap_{\epsilon > 0} E_\epsilon \neq \emptyset.$$

To see this, we first notice that $E_\epsilon = \{ x : \Psi^*(x) \leq 0 \}$. We then apply the Maximal Ergodic Theorem on $\Psi$ and $X \setminus E_\epsilon = \{ x : \Psi^*(x) > 0 \}$, i.e., $\alpha = 0$, and get

$$\int_{X \setminus E_\epsilon} \Psi d\mu \geq 0.$$

Observe that $\int_X \Psi d\mu = -\epsilon$, and which guarantees $E_\epsilon \neq \emptyset$ for all $\epsilon > 0$. Since $\varphi$ is continuous and $X$ is compact, we know $E_\epsilon$’s are nested compact sets, and thus

$$\bigcap_{\epsilon > 0} E_\epsilon \neq \emptyset.$$  

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