Modeling general-relativistic plasmas with collisionless moments and dissipative two-fluid magnetohydrodynamics

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ABSTRACT

Relativistic plasmas are central to the study of black hole accretion, jet physics, neutron star mergers, and compact object magnetospheres. Despite the need to accurately capture the dynamics of these plasmas and the implications for relativistic transients, their fluid modeling is typically done using a number of (overly) simplifying assumptions, which do not hold in general. This is especially true when the mean free path in the plasma is large compared to the system size, and kinetic effects start to become important. Going beyond common approaches used in the literature, we describe a fully relativistic covariant 14-moment based two-fluid system appropriate for the study of electron-ion or electron-positron plasmas. This generalized Israel-Stewart-like system of equations of motion is obtained directly from the relativistic Boltzmann-Vlasov equation. This new formulation can account for non-ideal effects, such as anisotropic pressures and heat fluxes, not present in previous formulations of two-fluid magnetohydrodynamics. We show that a relativistic two-fluid plasma can be recast as a single fluid coupled to electromagnetic fields with (potentially large) out-of-equilibrium corrections. We keep all electron degrees of freedom, which provide self-consistent evolution equations for electron temperature and momentum. The out-of-equilibrium corrections take the form of a collisional 14-moment closure previously described in the context of viscous single fluids. The equations outlined in this paper are able to capture the full two-fluid character of collisionless plasmas found in black hole accretion and flaring processes around compact objects, as well Braginskii-like two-fluid magnetohydrodynamics applicable to weakly collisional plasmas inside accretion disks.

Key words: MHD – plasmas – relativistic processes – methods: analytical – accretion

1 INTRODUCTION

Relativistic flows are ubiquitous in high-energy astrophysics. Accretion onto black holes can power electromagnetic transients associated with the formation of a highly relativistic jet, e.g. short gamma-ray bursts in the case of stellar mass black holes formed in neutron star collisions (Abbott et al. 2017), or in supermassive black hole accretion (Prieto et al. 2016). Supermassive black holes, such as M87, feature intriguing flaring activity (Berge et al. 2006), which has recently been related to large scale reconnection happening in the vicinity of the black hole (Porth et al. 2021; Ripperda et al. 2020; Nathanael et al. 2020; Ripperda et al. 2021a). Such reconnection events are, however, not limited to black holes, but will also occur in over twisted neutron star magnetospheres (Parfrey et al. 2013). For sufficiently large magnetic fields, the rotation of the star or orbital motion, if it is in a close binary, can create electric fields close to the surface that are large enough to accelerate surfaces charges to the pair-creation limit (Goldreich & Julian 1969). Such an electron-positron plasma, also present around supermassive black holes (Moscibrodzka et al. 2009; Wong et al. 2021), will have a large mean-free-path compared to the size of the compact object, and will therefore be collisionless (Cranquaud et al. 2020; Bransgrove et al. 2021). Reconnection events around black holes and neutron stars can trigger relativistic flaring events. These flares in turn can power associated X-ray and radio transients, and have been proposed as a mechanism to explain Fast Radio Bursts (Lyubarsky 2014; Beloborodov 2017; Metzger et al. 2019; Beloborodov 2020b; Yuan et al. 2020; Lyubarsky 2020), giant pulses in the Crab pulsar (Lyubarsky 2019; Philippov et al. 2019), and electromagnetic precursors in neutron star collisions (Most & Philippov 2020; Beloborodov 2020a).

Collisionless plasmas are also highly relevant for the dynamics of accretion disks around supermassive black holes. Temperature gradients can drive effective heat fluxes, and fluctuations in the magnetic field lead to pressure anisotropies (Chandra et al. 2015; Foucart et al. 2016), which could potentially impact the global dynamics of supermassive black-hole accretion flows (Foucart et al. 2017). Interestingly, regions where the magnetic pressure is small compared to the thermal pressure, become mirror- and firehose-unstable, increasing

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the effective collisionality of the plasma (Kunz et al. 2014). Such plasmas are then effectively weakly collisional (Braginskii 1965).
Yet, they still require an accurate treatment of both electrons and ions in order to correctly capture the heating of electrons, producing observable signatures for the accretion flows, as observed by the Event Horizon Telescope (Akiyama et al. 2019).

While there is an urgent need for the accurate modeling of relativistic plasmas in (weakly) collisionless scenarios, current numerical models are severely limited in several aspects. Firstly, in collisionless plasmas that are directly governed by solutions of the Boltzmann-Vlasov equation the characteristic rate at which magnetic fields reconnect is roughly given by $v_{\text{inflow}}/v_A \approx 0.1$, where $v_A$ is the Alfvén speed and $v_{\text{inflow}}$ is the inflow speed of reconnecting magnetic field lines into the current sheet (e.g., Sironi & Spitkovsky 2014). However, most simulations using general-relativistic magnetohydrodynamics (Porth et al. 2019) formally appropriate only for collisional plasmas, reproduce a reconnection rate that is too low by up to a factor 10 (Bransgrove et al. 2021). This has important consequences for setting the right time scale in the reconnection process and connecting it to observations (Ripperda et al. 2021a).

Secondly, in such simulations magnetic reconnection is typically captured only in terms of a scalar resistivity (e.g. Palenzuela et al. 2009; Ripperda et al. 2019), or even just grid dissipation in ideal MHD simulations. Detailed investigations in the context of space-physics plasmas indicate, however, that anisotropic pressure contributions to Ohm’s law might be the main driver of reconnection in collisionless systems (Bessho & Bhattacharjee 2005; Wang et al. 2015).

Thirdly, effective collisionalities leading to heat fluxes are typically not accounted for in simulations of black hole accretion, see Chandra et al. 2015; Foucart et al. 2016 for notable exceptions. Similarly, a consistent evolution of the electron temperature in a relativistic plasma is largely unavailable, with approximations being currently in use (Ressler et al. 2015). These require formulations that truly model the plasma as consisting of two interacting fluids.

Finally, we remark that a consistent relativistic description of viscous fluids is a complex task (see, e.g., review Romatschke & Romatschke 2019), with standard formulations e.g. (Eckart 1940) displaying issues with causality and stability (Hiscock & Lindblom 1985). Designing schemes that are causal, stable, and strongly hyperbolic (well-posed) is a highly nontrivial feat even for purely viscous hydrodynamical descriptions, although those properties have been recently established in recent years (Bemfica et al. 2018, 2019b,c; Bemfica et al. 2019a; Bemfica et al. 2021, 2020a,b). On the other hand, magneto-hydrodynamic formulations can also face a number of shortcomings, including the lack of strong hyperbolicity (Schoepe et al. 2018).

In this work, we introduce a new description appropriate for relativistic electron-ion and electron-positron plasmas, which goes beyond traditional approaches currently in use. Before giving a detailed description of our method, we review current approaches used in the literature, carefully highlighting which physics they miss out, in the context of the discussion above.

In effective single fluid approaches, several studies have incorporated scalar resistivity in relativistic MHD (e.g. Palenzuela et al. 2009; Zenitani et al. 2010; Bucciantini & Del Zanna 2013; Dionysopoulou et al. 2013; Qian et al. 2017; Ripperda et al. 2019; Wright & Hawke 2020). The incorporation of scalar resistivity is done using a Newtonian-type Ohm’s law, which inevitably breaks strong hyperbolicity of the system (Schoepe et al. 2018). Going beyond simple scalar resistivity, several works have suggested the inclusion of the Hall (Zanotti & Dumbser 2011) or sub-grid dynamo effects (Del Zanna & Bucciantini 2018; Tomei et al. 2020; Shibata et al. 2021). When modeling neutron matter, where weak interactions are relevant, it has also been suggested to add composition-dependent terms to Ohm’s law (Andersson et al. 2021).

Critically, these approaches are only valid in the regime where the plasma mean free path is short, i.e. where collisions dominate. As such, they do not capture collisionless reconnection correctly, and cannot account for anisotropic pressure effects, neither in reconnection or in viscous angular momentum transport in disks.

Instead of starting from phenomenological approaches, it is also possible to derive relativistic single-fluid dynamics beyond the (magneto-)hydrodynamical limit from the Boltzmann-(Vlasov) equation. Based on a moment reduction of the Boltzmann equation it is possible to derive general equations for out-of-equilibrium single fluid dynamics for non-resistive (Denicol et al. 2018; Panda et al. 2021a) and resistive (Denicol et al. 2019; Panda et al. 2021b) plasmas. In particular, the so-called 14-moment decomposition can be thought of as a relativistic extension of Grad’s 13-moment approach (Grad 1949), which accounts for heat fluxes and anisotropic stresses. Different from the original formulation of Israel & Stewart (1979), the approach of Denicol et al. (2012) allows for a systematic power counting expansion in terms of inverse Reynolds and Knudsen numbers, whose asymptotic regime agrees with the Chapman-Enskog expansion of the Boltzmann equation (Chapman & Cowling 1990; Denicol et al. 2012). The consistent derivation from the Boltzmann-Vlasov equations results in generalized Israel-Stewart-like equations that are likely to possess a causal regime for some range of values of transport coefficients and dissipative fluxes (Biswas et al. 2020), and those equations can account for anisotropic pressures as well as heat conduction and bulk viscosities. However, they are formally only valid in the regime of strong and weak collisionality.

While the studies above mainly concerned themselves with modeling a single fluid, charged plasmas strictly require two individual components to maintain charge quasi-neutrality. As such, modeling electron-ion plasmas in black hole accretion problems or electron-positron plasmas, appropriate for the study of neutron-star or black-hole magnetospheres, strictly requires the inclusion of two (interacting) plasma components. Several attempts to model relativistic multi-fluids have been taken in the literature. In the case of modeling reconnection in electron-positron plasmas, early work had tried to naively couple two separate perfect single fluids via a simple momentum exchange term, in non-covariant (Zenitani et al. 2009a) and covariant form (Zenitani et al. 2009b; Barkov et al. 2014). These approaches did not include further non-ideal effects such as heat fluxes or anisotropic pressure contributions, which might be significant for reconnection (Bessho & Bhattacharjee 2005; Wang et al. 2015). On the other hand, Koide (2009) proposed a simple set of equations to model relativistic two-component plasmas, under the assumption that each plasma can be modelled as a perfect fluid. While this model can account for the interaction of electrons and ions through electromagnetic fields, it fails to include heat fluxes and anisotropic electron pressure contributions to Ohm’s law. These equations miss out on important contributions to the electron temperature evolution (Ressler et al. 2015), governing the emission near accreting black holes (Moscibrodzka et al. 2009). Modeling the interior of neutron stars, several variational formalisms (e.g. Carter 1991; Andersson & Comer 2015; Rau & Wasserman 2020) have been proposed to model relativistic resistive and reactive plasmas (Andersson et al. 2017a,b).
2 MOTIVATION AND OUTLINE

Before we dive into the technical details of deriving a system of relativistic dissipative two-fluid magnetohydrodynamics equations, we want to sketch the overall picture and motivation of this work. To this end, we can summarize our discussion in Sec. 1 as follows. The goal of this paper is to describe a framework that can model dissipative effects in relativistic plasmas. These may be important in dense plasmas found in neutron star mergers (Alford et al. 2018; Most et al. 2021) or in accretion disks around supermassive black holes (Foucart et al. 2016, 2017). Moreover, the key effect of dissipation on a magnetized plasma is to allow for dissipation of electromagnetic energy, which happens in reconnecting current sheets (see, e.g., Ji et al. 2022 for a recent review). This form of dissipation is crucial in understanding reconnection powered transient produced in compact object magnetospheres (Beloborodov 2017; Lyubarsky 2020), precursor emission to compact object mergers (Most & Philippov 2020) and flaring activities around supermassive black holes (Ripperda et al. 2021a). Due to the collisionless nature of the electron-positron pair plasmas in these systems, their dynamics can in principle be obtained ab-initio as solutions of the Vlasov-Boltzmann equation. However, the vast physical separation of scales of the plasma (e.g. electron skin depth and Larmor radius) and of the macroscopic system (e.g., horizon size and orbital size) makes first principle approaches with correct separation of scales computationally unfeasible. Instead, what is needed from a practical point, is a framework that allows to simulate approximate scale separations, i.e., where the Larmor radius and skin depth enter as free parameters. Some of these scales (e.g., thermal Larmor radius) can also efficiently be overstepped numerically using implicit time integration methods (Most & Noronha 2021), or be locally resolved using adaptive mesh refinement techniques (Berger & Colella 1989), which have been extensively used for fluid equations (see, e.g., Plewa et al. 2005). However, these advantages come at the cost of loosing the ability to correctly capture the saturation phase of plasma instabilities arising from kinetic physics (e.g., Weibel 1959).

Leveraging these benefits of a fluid-like approach, we will formulate a two-fluid approach to dissipative magnetohydrodynamics. In order to systematically incorporate these corrections, we will utilize the 14-moment approach of Denicol et al. (2012) (see also Denicol et al. 2018, 2019), which is a second-order Müller-Israel-Stewart-type system (Israel & Stewart 1979). Crucially, this system provides additional evolution equations for the anisotropic pressure \( \pi^{\mu\nu} \) and heat fluxes \( q^\mu \),

\[
\begin{align*}
    u^\alpha \partial_\alpha q^\mu &= \ldots, \\
    u^\alpha \partial_\alpha \pi^{\mu\nu} &= \ldots
\end{align*}
\]

(1) (2)

thereby making the system (likely) causal (Bemfica et al. 2021). Here \( u^\mu \) is the fluid four-velocity. In addition, the inclusion of effects of dissipation on the electromagnetic sector will require an effective Ohm’s law. Schematically, this will take the following form (Denicol et al. 2019),

\[
\begin{align*}
    \varepsilon^\mu &= \eta_1 j^\mu + \eta_2 b^{\mu\nu} j_\nu + \alpha \nabla_\nu \pi^{\mu\nu} + \beta u^\nu \nabla_\nu j^\mu + \ldots.
\end{align*}
\]

(3)

Here, \( \varepsilon^\mu \) is the electric field as seen by an observer comoving with the fluid four-velocity \( u^\mu \), \( j^\mu \) is the dissipative part of the electric current, and \( b^{\mu\nu} \) is a projector orthogonal to the comoving magnetic field, see (27). Terms with \( \eta_i \) coefficients are first-order in gradients.\(^1\)

\(^1\) We point out that in a two-component plasma gradients can either be on the right-hand side provides an effective time-evolution equation for the electric current. In this sense, the Ohm’s law becomes dynamical and takes a similar form to the Müller-Israel-Stewart-type equations discussed above. While the validity of such an approach might not be directly apparent, it has been extensively investigated both in the context of collisionless Newtonian plasmas (Wang et al. 2015), as well as relativistic pair plasmas (Bessho & Bhattacharjee 2012; Liu et al. 2015). By comparing with fully kinetic simulations, e.g., Bessho & Bhattacharjee (2012) and Liu et al. (2015) have directly shown that second order terms are crucial in relativistic reconnection processes. Building by similar findings in electron-ion reconnection, Newtonian ten-moment closures have been developed (Wang et al. 2015; Ng et al. 2020a). Motivated by their success in describing dissipation in planetary magnetospheres (Wang et al. 2018; Dong et al. 2019), we will construct a 14-moment two-fluid scheme featuring an effective Ohm’s law of the form (3). More specifically, we propose an alternative formulation for two-fluid (dissipative) magnetohydrodynamics appropriate for the study of collisionless, weakly collisional, and collisional resistive plasmas. Building on recent progress in the moment expansion of the relativistic Boltzmann-Vlasov equation (Tinti et al. 2019; Denicol et al. 2019), we will derive a set of collisionless 14-moment equations. These can be seen as the relativistic generalization of 10-moment approaches used in space-physics (Wang et al. 2015). The resulting equations are presented in Sec. 4. In the second part of this work, we show how the 14-moment representation of two fluids can be recast to resemble an effective single-fluid description with dissipative out-of-equilibrium corrections. These equations, as presented in Sec. 5, themselves resemble the 14-moment closures presented in Sec. 4. Specializing on the case of an electron-ion plasma, in Sec. 6 we discuss simplified cases appropriate for weakly collisional plasmas. Finally, we make a connection to a resistive dissipative magnetohydrodynamics description of a single-fluid plasma in Sec. 7. Throughout this work we adopt geometric units, \( G = c = k_B = 1 \), and a mostly plus signature for the spacetime metric \( g_{\mu\nu} \). Lorentz scalars constructed via the scalar product among vectors are denoted with a "*", i.e., \( p_\mu u^\mu = p \cdot u \).

3 RELATIVISTIC FLUIDS

In the following, we will give a brief description of a single relativistic fluid described in terms of 14 moments. While this section will introduce all basic concepts and expressions, they will lack the statement of a closure, the simplest of which being an equation of state. These will be discussed separately in Sec. 4.

A relativistic fluid can be described in terms of a symmetric energy-momentum tensor \( T^\mu_\nu \) and a conserved number density \( N_f^\mu \), i.e.,

\[
\nabla_\mu N_f^\mu = 0.
\]

(4)

The fluid energy-momentum tensor \( T^\mu_\nu \) is also subject to total energy-momentum conservation, i.e. \( \nabla_\mu T^\mu_\nu = 0 \). Introducing a characteristic timelike 4-velocity \( u^\mu \) of the fluid (with \( u_\mu u^\mu = -1 \)), we can split \( N_f^\mu \) in components along and across \( u^\mu \), i.e.,

\[
N_f^\mu = n_f u^\mu + \nabla_f^\mu.
\]

(5)

ion or electron scales, complicating the notion of the gradients in the above expressions. For clarity, we will drop this distinction for the most part of this paper.
where $V^\mu_f$ is typically referred to as particle diffusion current, as it refers to particle motion across the fluid current $n_f u^\mu$. By construction $V^\mu_f$ is orthogonal to $u^\mu$, i.e.

$$\Delta^\mu N^\nu_f := (\delta^\mu + u^\mu u_\nu) N^\nu_f = V^\mu_f, \quad V^\mu_f u_\nu = 0,$$

(6) (7)

where we have introduced the fluid frame projector $\Delta_{\mu \nu}$. While in many astrophysical situations the diffusion current $V^\mu_f = 0$, we will see in Sec. 5, that this depends on the choice of the fluid hydrodynamic frame. If the fluid particles have characteristic charge $q$, we can introduce an electric fluid current, proportional to the conserved number density current $N^\nu_f$,

$$J^\mu_f = q N^\mu_f.$$

From a fundamental point of view, gravity couples energy and momentum to the curvature of spacetime. Within the framework of general relativity, the Einstein equations impose that

$$G^\mu \nu = 8 \pi T^\mu \nu,$$

(9)

where $G^\mu \nu$ is the Einstein tensor and $T^\mu \nu$ is the energy-momentum tensor encompassing all matter and non-gravitational fields, e.g. electric and magnetic fields. As a consequence of the Bianchi identity, $\nabla_\mu G^\mu \nu = 0$, the total energy-momentum is conserved

$$\nabla_\mu T^\mu \nu = 0.$$

(10)

This implies that the ten independent components of the $T^\mu \nu_f$ and the four components of the number density current, $N^\mu_f$, are the natural variables describing a relativistic fluid. Those 14 degrees of freedom will be referred as the 14-moments in this work. Their construction in terms of kinetic theory will be discussed in Sec. 4.

Introducing an energy density $\varepsilon$ and equilibrium pressure $P = P(\varepsilon, n, \ldots)$, we can decompose the energy-momentum tensor $T^\mu \nu_f$ as follows

$$T^\mu \nu_f = \varepsilon u^\mu u^\nu + P \Delta^\mu \nu + q^\mu u^\nu + q^\nu u^\mu + \pi^\mu \nu.$$

(11)

In doing so, we have introduced the viscous bulk scalar $P$, the anisotropic stress tensor $\pi^\mu \nu$, and the energy diffusion 4-vector $q^\mu$. By construction these are orthogonal to the fluid four-velocity, i.e.

$$u^\mu u^\nu = 0, \quad u^\mu q^\nu = 0, \quad u^\mu \pi^\nu = 0,$$

(12) (13) (14)

and additionally $\pi^\mu _\nu = 0$. It is important to stress that this decomposition is purely algebraic. Any symmetric rank two tensor can be split along the direction of an arbitrary timelike vector $u^\mu$ in this way. It will turn out to be beneficial to also introduce the enthalpy $h$ per particle of the fluid,

$$h = m \bar{h} = \frac{\varepsilon + P}{n},$$

(15)

where $m$ is the mass of the fluid particle and $\bar{h}$ is the specific enthalpy.

In equilibrium, the viscous, or out-of-equilibrium, parts vanish and the baryon current and energy momentum current reduce to their equilibrium values, i.e.

$$N^\mu _{\text{eq}} := n_f u^\mu, \quad T^\mu \nu _{\text{eq}} := \varepsilon u^\mu u^\nu + P \Delta^\mu \nu, \quad \pi^\mu \nu = 0,$$

(16) (17)

which implies

$$u^\nu T^\mu \nu _{\text{eq}} = -q^\mu = -\frac{\varepsilon}{n_f} N^\mu _{\text{eq}}.$$

(18)

This means that in equilibrium the flow of energy aligns with the flow of particles. On the other hand, in the presence of the viscous corrections, these relations no longer hold. This can be seen as follows. Projecting the energy momentum tensor onto the fluid four-velocity, we find

$$-u^\mu T^\nu \mu_f = \varepsilon u^\mu + q^\mu.$$

(19)

For non-vanishing energy diffusion $q^\mu$, the flow of energy and particles is no longer aligned, since $q^\mu u^\mu = 0$. The same is also true for the current, see (5). The actual heat flux can be computed by removing the energy carried by particle diffusion,

$$Q^\mu = q^\mu - h_0 V^\mu_f,$$

(20)

where $h_0$ is the enthalpy per fluid particle in equilibrium.

It is important to remark that we have 17 independent variables in the set $\{\varepsilon, u^\mu, P, q^\mu, \pi^\mu \nu, n_f, V^\mu_f\}$ introduced in (5) and (11). Thus, to recover a 14-variable description, choices have to be made for the fields that appear in these equations. Each choice for the variables used in the description of the energy-momentum tensor and the conserved current defines the so-called hydrodynamic frame (Stewart 1972; Israel & Stewart 1979; Tsumura et al. 2007; Kovtun 2012), of which well-known choices are the Landau hydrodynamic frame (Landau & Lifshitz 1987), where $q^\mu = 0$ and the flow of energy aligns with the fluid four-velocity (no energy diffusion), and the Eckart frame (Eckart 1940), where particle diffusion is absent, i.e. $V^\mu_f = 0$. There is an infinite number of possible hydrodynamic frames, and we refer the reader to Refs. Bemfica et al. (2018); Kovtun (2019); Bemfica et al. (2019b); Hoult & Kovtun (2020); Bemfica et al. (2020a) for a discussion of such choices and their properties and consequences. For instance, one can rigorously prove (Bemfica et al. 2018, 2019b,c; Bemfica et al. 2019a; Bemfica et al. 2021, 2020a,b) that first-order viscous hydrodynamic theories can be causal and stable when constructed using general hydrodynamic frames. In this work it will be most natural to consider the Eckart frame. However, for the sake of generality, for now we will not further specify our choice and keep both dissipative components, $V^\mu_f$ and $q^\mu$, in the equations.

Before moving on to describe a set of closure relations for the out-of-equilibrium terms $P, q^\mu, V^\mu_f$ and $\pi^\mu \nu$, we need to introduce a description of electric and magnetic fields in general spacetimes. This is done in the next section.

### 3.1 Electromagnetic fields in relativity

In covariant form, the electromagnetic field is described by a totally antisymmetric field strength tensor $F^{\mu \nu}$, and its dual $^{*}F^{\mu \nu} = -\epsilon^{\mu \nu \alpha \beta} F_{\alpha \beta}$. The dynamics of the electric and magnetic fields is then governed by the Maxwell equations

$$\nabla_\mu F^{\mu \nu} = 0,$$

(21)

$$\nabla_\mu F^{\nu \mu} = 4 \pi J^\nu,$$

(22)

where $J^\nu$ is the total electric current. Due to the antisymmetric property of the field strength tensor $F^{\mu \nu}$, the Maxwell equations (21) imply global conservation of electric charge.

$$\nabla_\nu J^\nu = 0.$$

The electric and magnetic fields further give rise to an energy-momentum tensor

$$T^{\mu \nu}_{\text{EM}} = \frac{1}{4 \pi} \left( F^{\mu \alpha} F_{\alpha \beta} - \frac{1}{4} \eta^{\alpha \beta} F_{\alpha \beta} F_{\mu \nu} \right),$$

(23)
which obeys the following conservation law
\[ \nabla_{\mu} T_{\mathrm{EM}}^{\mu \nu} = -F^{\nu \lambda} j_{\lambda}. \] 

(24)

Using the fluid four-velocity \( u^\mu \), we can project the field strength tensor \( F^{\mu \nu} \) and the corresponding electric and magnetic fields obtained in this procedure are then those seen by an observer comoving with the fluid. Thus, we can define the comoving fields via
\[ F^{\mu \nu}_{\text{c}} = \bar{e}^\mu, \]
\[ \bar{F}^{\mu \nu} u_\nu = b^\mu. \]

(25)
(26)

We can furthermore introduce the following tensors
\[ b^{\mu} = e^{\mu \alpha \beta} b_{\alpha} u_\beta, \]
\[ e^{\mu} = e^{\mu \alpha \beta} e_{\alpha} u_\beta, \]
which allow us to re-express the field strength tensor entirely in terms of the comoving fields. It is then given by,
\[ F^{\mu \nu} = u^\mu e_\nu - u_\nu e^\mu + F^{\mu \nu}, \]
\[ \bar{F}^{\mu \nu} = u^\mu b_\nu - u_\nu b^\mu - e^{\mu \nu}. \]

(29)
(30)

The electromagnetic energy-momentum tensor (23) then takes the convenient form
\[ 4 \pi T^{\mu \nu}_{\text{EM}} = \frac{1}{2} \left( b^2 + e^2 \right) g^{\mu \nu} + \left( b^2 + e^2 \right) \Delta^{\mu \nu} - b^\mu b_\nu - e^\mu e_\nu - \epsilon_{\alpha \beta} u^\mu b^{\alpha \beta} - e_{\alpha} u^\mu b^{\alpha \beta}. \]

This completes our short review of the covariant formulation of Maxwell equations used in this work.

4 MOMENT EQUATIONS AND CLOSURE RELATIONS

Having discussed the general framework of relativistic fluids, we need to specify closure relations for the out-of-equilibrium corrections. From the fundamental point of view of kinetic theory, these should be described as reductions of the Boltzmann-Vlasov equation. Such an approach has been taken by Tinti et al. (2019), which we summarize in the following.

A microscopic kinetic theory is described in terms of a distribution function \( f(x^\mu, p^\nu) \), which describes a distribution of particles at point \( x^\mu \) with momenta \( p^\nu \). Given a single species of particles with mass \( m \), we have the on-shell relation \( p_\mu p^{\mu} = -m^2 \). This distribution function obeys the Boltzmann-Vlasov equation,
\[ p^{\mu} \partial_\mu f + \left\{ q^{\mu \nu} p_\mu + \Gamma^\alpha_{\mu \nu} p^\mu p^\nu \right\} \partial_\alpha f = C \left[ f \right], \]

(32)

where we have included couplings to electromagnetic fields and a suitable collision operator \( C \left[ f \right] \), see also Denicol et al. (2019). It should be pointed out that the appearance of the Christoffel symbol \( \Gamma^\alpha_{\mu \nu} \) is associated with ensuring general covariance of the Boltzmann equation. In all what follows we will, without loss of generality, perform our calculations in flat Minkowski spacetime. The resulting equations are, however, manifestly covariant and correct also in general-relativistic spacetimes as long as long as the full covariant derivative is used.

In order to obtain a fluid description we need to remove the momentum dependence, which is replaced by an average velocity \( u^\mu \) describing the bulk motion of the fluid. It turns out to be beneficial to systematically introduce an irreducible basis of particle momenta relative to the fluid four-velocity \( u^\mu \). That is, we are going to assemble products of the form \( \langle -p \cdot u \rangle, p^{\mu} u_\mu, p^{\mu} u_\nu p^{\nu}, \ldots \), where we have introduced the notation \( p^{\mu \nu} = \Delta^{\mu \nu}, \) with \( \Delta^{\mu \nu} = g^{\mu \nu} + u^{\mu} u_\nu \). This basis can then be used to expand the distribution function \( f \). We further introduce the notion of a generalized covariant moment
\[ f_r^{\mu_1 \cdots \mu_n} = \int_p \left( -p \cdot u \right)^r p^{\langle \mu_1 \rangle} \cdots p^{\langle \mu_n \rangle} f, \]

(33)

for which the particle momentum dependencies have been integrated out. Note that these moments are orthogonal to the 4-velocity. Since the momenta are on-shell, we have split out the momentum component along \( u^\mu \), such that in the fluid rest-frame the Lorentz invariant measure in the integral can be written as
\[ \int_p := \int \frac{d^3 p}{2 p^0}, \]

(34)

where the integral runs over all spatial momenta. Making use of the on-shell relation \( p_\mu p^{\mu} = -m^2 \), we can write
\[ \left( p_\mu p^{\mu} \right)^2 = m^2 + p^{\langle \mu \rangle} p^{\langle \nu \rangle}, \]

(35)

and, thus, simplify the moments
\[ f_r^{\mu_1 \cdots \mu_n} = \int_p \left( m^2 + p^{\langle \mu \rangle} p^{\langle \nu \rangle} \right)^r p^{\langle \mu_1 \rangle} \cdots p^{\langle \mu_n \rangle} f, \]

having again used that the integral over the momentum is subject to the on-shell constraint. This decomposition also ensures that we can directly identify the hydrodynamic degrees of freedom, as will be shown in the following.

In particular, the particle current \( N^{\mu}_f \) and the energy momentum tensor \( T^{\mu \nu}_f \) can be expressed as the first and second moment of the distribution function, respectively,
\[ N^{\mu}_f = \frac{1}{m} \int_p p^{\mu} f, \]
\[ T^{\mu \nu}_f = \int_p p^{\mu} p^{\nu} f. \]

(37)
(38)

Using the above definition, we can readily identify some of the irreducible moments with our hydrodynamic variables,
\[ n_f = u_\mu N^{\mu} = \frac{1}{m} f_1, \]
\[ e = u_\mu u_\nu T^{\mu \nu} = f_2, \]
\[ P + \Pi = \frac{1}{3} \Delta^{\mu \nu} T^{\mu \nu} = \frac{1}{3} \Delta^{\mu \nu} f_1^{\mu \nu}, \]
\[ V^{\mu} = \Delta^{\mu \nu} n^{\nu} = \frac{1}{m} f_0^{\mu}, \]
\[ q^{\mu} = -\Delta^{\mu \nu} u_\nu T^{\alpha \beta} = f_1^{\mu \nu}, \]
\[ \Pi^{\mu \nu} = \Delta^{\mu \nu} T^{\alpha \beta} = f_0^{\mu \nu}. \]

(39)
(40)
(41)
(42)
(43)
(44)

In writing these expressions we have introduced the following notations for the projection of a vector and rank-two tensor into the fluid frame, \( A^{\langle \mu \rangle} B^{\langle \nu \rangle} = A^{\mu} B^{\nu} - \Delta^{\mu \nu} A B / 2 - \Delta^{\alpha \beta} A^{\langle \alpha} B^{\langle \beta} A^{\mu} / 3 \) is the symmetric trace-free projector. In writing the above, we have implicitly split the total pressure tensor,
\[ \Pi^{\mu \nu} = \pi^{\mu \nu} + \left( P + \Pi \right) \Delta^{\mu \nu} = f_0^{\mu \nu}, \]

(45)

into isotropic \((P + \Pi)\) and anisotropic \((\pi^{\mu \nu})\) components. It is important to understand the split of the isotropic pressure into an equilibrium, \( P \), and dissipative, \( \Pi \), part is not always well defined. In the absence of an equilibrium equation of state, these two contributions cannot be separated. To keep a consistent notation through this work, we retain this split explicitly in this section. As mentioned before, it
is important to realize that particle and energy diffusion are related phenomena. In line with the discussion of (19), expanding Eq. (43) in the small velocity limit gives

$$q^\mu = m V_f + \frac{1}{2m^2} \int d^3 p \ p_{<\nu>} p_{<\mu>} + \ldots.$$  

(46)

This demonstrates that the energy diffusion 4-vector contains a diffusion contribution plus energy flux corrections orthogonal to the 4-velocity. The latter should be compared with the heat-flux in the Newtonian version of Grad’s 13-moment equations (Grad 1949),

$$q^\mu_{\text{Grad}} = \int d^3 v \ (v_j - u_j) (v^i - u^i) f.$$  

(47)

In this sense, a 14-moment based closure in relativistic systems is the natural extension of Grad’s 13-moment expansion.

In the following we will consider two limiting cases for the collision operator. The simplest one being the absence of collisions, $C[f] = 0$, and the strongly collisional limit described by Denicol et al. (2012, 2018, 2019).

### 4.1 Collisional fluids

Although the specific choice of closure will highly depend on the properties of the system, the general form of the closure is known in the collisional limit (Denicol et al. 2012, 2018, 2019). Starting from a perturbative description in inverse Reynolds and Knudsen numbers, $\text{Re}^{-1}$ and $\text{Kn}$, respectively, Denicol et al. (2012) introduced an effective power counting scheme in order to expand the collision operator, in addition to identifying (and keeping) only the most relevant time scale associated with binary collisions. More precisely, the expansion of the collision is done up to second order in

$$\text{Re}_{\Pi}^{-1} := \frac{\Pi}{P}, \quad \text{Re}_{\varphi}^{-1} := \frac{\varphi}{n_f}, \quad \text{Re}_\Pi := \frac{\pi^{\mu\nu}}{P}, \quad \text{Kn} := \ell_{\text{micro}}/L,$$  

(48)

where $\ell_{\text{micro}}$ is the mass scale set by microscopically time scales associated with interactions, and $L$ is the typical system size associated with gradients of the hydrodynamic variables, such as the flow velocity. Expressing the closure only in the 14 moments, in a near-equilibrium expansion one finds that higher-rank moments relax to higher derivatives of those moments. Hence, in this limit the higher order moments have to diffuse the following scaling in terms of Knudsen numbers

$$f_{\mu}^{i \mu \gamma \ldots} \sim O \left(\text{Kn}^3 \right), \quad i > 0.$$  

(49)

This directly implies that as long as we are in the limit of high collisionality, $\text{Kn} \ll 1$, neglecting all higher-rank moments with negative $r < 0$ is expected to give a good approximation of the dynamics. Using a general treatment of the collisionl operator $C[f]$, up to second order in $O \left(\text{Re}^{-1} \text{Kn} \right)$ the closure relations read (Denicol et al. 2019),

$$u^\alpha \nabla_\alpha q^{\langle \mu \nu \rangle} = -\frac{1}{\tau_q} q^{\mu} - \kappa \nabla^{\mu} q^{\langle \nu \rangle} a_0 + \frac{1}{m} q^{\nu} \omega^{\nu} q^{\mu} + \delta_{q q} q^{\mu} \theta + \lambda_{q q} q^{\nu} \sigma^{\nu} q^{\mu} + \ell_q \Pi \nabla^{\mu} q^{\langle \nu \rangle} - \frac{\tau_q}{m} \Pi^{\mu} q^{\nu} + \lambda_{\Pi q} \Pi^{\mu} q^{\nu} a_0 - \lambda_q \Pi \nabla^{\mu} q^{\langle \nu \rangle} a_0 + \lambda_q \Pi \nabla^{\mu} q^{\langle \nu \rangle} a_0 + \delta_{q q} q^{\nu} \delta_{q q} q^{\mu} q^{\nu}.$$  

(50)

$$u^\alpha \nabla_\alpha q^{\langle \mu \nu \rangle} = -\frac{1}{\tau_q} q^{\mu} - \kappa \nabla^{\mu} q^{\langle \nu \rangle} a_0 + \frac{1}{m} q^{\nu} \omega^{\nu} q^{\mu} + \delta_{q q} q^{\mu} \theta + \lambda_{q q} q^{\nu} \sigma^{\nu} q^{\mu} + \ell_q \Pi \nabla^{\mu} q^{\langle \nu \rangle} - \frac{\tau_q}{m} \Pi^{\mu} q^{\nu} + \lambda_{\Pi q} \Pi^{\mu} q^{\nu} a_0 - \lambda_q \Pi \nabla^{\mu} q^{\langle \nu \rangle} a_0 + \lambda_q \Pi \nabla^{\mu} q^{\langle \nu \rangle} a_0 + \delta_{q q} q^{\nu} \delta_{q q} q^{\mu} q^{\nu}.$$  

(51)

$$u^\alpha \nabla_\alpha q^{\langle \mu \nu \rangle} = -\frac{1}{\tau_q} q^{\mu} - \kappa \nabla^{\mu} q^{\langle \nu \rangle} a_0 + \frac{1}{m} q^{\nu} \omega^{\nu} q^{\mu} + \delta_{q q} q^{\mu} \theta + \lambda_{q q} q^{\nu} \sigma^{\nu} q^{\mu} + \ell_q \Pi \nabla^{\mu} q^{\langle \nu \rangle} - \frac{\tau_q}{m} \Pi^{\mu} q^{\nu} + \lambda_{\Pi q} \Pi^{\mu} q^{\nu} a_0 - \lambda_q \Pi \nabla^{\mu} q^{\langle \nu \rangle} a_0 + \lambda_q \Pi \nabla^{\mu} q^{\langle \nu \rangle} a_0 + \delta_{q q} q^{\nu} \delta_{q q} q^{\mu} q^{\nu}.$$  

(52)

Here $\delta_i$, $\ell_i$, $\tau_i$ are transport coefficients that appear in this approach. It is important to stress that while the functional form of the closure is known, the precise values of the transport coefficients need to be specified, as they depend on the microscopic properties of the fluid. For the case of a massless ultra-relativistic gas interacting via a constant cross section, they can be found in Denicol et al. (2019). Some of these transport coefficients have further been computed for Coulomb collisions in Newtonian plasmas (e.g., Kulsrud 2020).

### 4.2 Collisionless fluids

Although introducing the 14-moment decomposition might first glance imply that the system may be in a hydrodynamic regime, it is important to understand that if effective collisions are only mediated by the collectively created mean electromagnetic fields an equilibrium might be reached but its form is not known a priori. As a direct consequence, the non-perfect fluid equivalents, $V_f^{\mu}$, $q^{\mu}$ and $\pi^{\mu\nu}$, can not only be large, but there is not necessarily a well-defined isotropic diagonal pressure component $P = P \left(\epsilon, n_f, \ldots\right)$. Instead, $f_0^{\mu\nu}$ contains all components of the pressure with off-diagonal elements that can be as large or even exceed the diagonal ones.

It is trivial to show that Eq. (32) implies conservation of particle number and energy-momentum, i.e.,

$$\nabla_\mu \left( n_f u^\mu + V_F^{\mu} \right) = 0,$$  

(53)

$$\nabla_\mu \left( \epsilon u^\mu u^\nu + \left( P + \Pi \right) \Delta^{\mu} \Delta^{\nu} + q^{\mu} q^{\nu} + q^{\mu} u^\nu + q^{\nu} u^\mu + \pi^{\mu\nu} \right) = -q F^{\mu\nu} N_{\nu},$$  

(54)

The missing evolution equations for the diffusion current and the stresses have to be derived from the Boltzmann equation (32).

Assuming that the rest-mass of the particles is larger than other scales, which includes the effective temperature of the plasma in the nearly-collisionless limit, one may consider $p^{<\mu>} p^{<\nu>} \ll m^2$ and Taylor-expand the square-root in Eq. (36), so as to obtain the following recursion relation

$$f^{\mu_1 \ldots \mu_n}_{\Pi + \Pi} = m^x \left( f^{\mu_1 \ldots \mu_n}_{\Pi} + \frac{s}{2m^2} f^{\alpha \mu_1 \ldots \mu_n}_{\Pi} \Delta_{\alpha \beta} + O \left( \frac{1}{m^4} \right) \right).$$  

(55)

It is important to understand that this assumption, which is akin to considering non-relativistic effective temperatures in the $\epsilon$-closure relation (55), does not break the covariance of the moment equations. However, this approximation begins to fail higher momenta (equivalently, this would be the case when the effective plasma temperature $T \sim m$). While determining the implicit limitations

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introduced by this assumptions is beyond the scope of this work, future studies of ultrarelativistic plasmas might require different closure relations, see also the discussion in Tinti et al. (2019).

In order to obtain the moment reduction, we note that
\[ p^{\mu} \partial_\mu f = -(p_{\alpha} u^{\alpha}) u^{\mu} \nabla_\mu f + p^{(\mu)} \nabla_\mu f, \]  \tag{56} \]
implies that an irreducible moment decomposition will naturally lead to an Israel-Stewart-type equation (Israel & Stewart 1979), i.e. equations of motion of relaxation-type containing \( u^{\mu} \nabla_\mu a^{\alpha B} \), \( u^{\mu} \nabla_\mu \Pi \), \( u^{\mu} \nabla_\mu V^{\mu}_f \), and \( u^{\mu} \nabla_\mu q^{\mu} \) terms. Since the total heat flux combines particle diffusion and energy diffusion, it is only natural to consider the evolution of
\[ G^{\mu} = q^{\mu} - m \nabla^\mu f. \]  \tag{57} \]

for a collisionless plasma (this is also in line with Denicol et al. 2019).

Tinti et al. (2019) have shown that the moment equations obey,
\[ u^{\alpha} \nabla_\alpha G^{<\mu>} = -\frac{q}{m} (f_1 - f_0) - q e^{\alpha} a^{ab\mu} + \frac{q}{m} g^{\alpha} g^{ab\beta} \left( \nabla^\mu f_0 - \nabla^\nu f_1 \right) \]
\[ - \nabla^a \left( \Pi^{\alpha} - m f_1^{\alpha} \right) - \nabla^a u^{\mu} G^{a} - \nabla^a u^{\mu} f_1^{a\beta\nu}, \]  \tag{58} \]
\[ u^{\alpha} \nabla_\alpha G^{<\mu\nu>} = -\frac{q^2}{m} (f_1 f_{-2}^{\mu\nu} + q e^{\alpha} f_1^{a\mu\nu} + q \frac{q}{m} g^{\alpha} g^{ab\beta} \left( f_0 f_{-1}^{\mu\nu} + m \nabla^\mu f_0 - m \nabla^\nu f_1 \right) \]
\[ - \nabla^a \left( \Pi^{\alpha} - m f_1^{\alpha} \right) - \nabla^a u^{\mu} G^{a} - \nabla^a u^{\mu} f_1^{a\beta\nu}. \]  \tag{59} \]

We have also used that \( u^{\alpha} = u^{\mu} \nabla_\mu \alpha^{\alpha} \). We can see that the evolution of the hydrodynamic moments depends on higher order moments in \( r < 0 \). Using our proposed closure relation valid in the low and intermediate temperature regime, Eq. (56), we can re-express them as follows,
\[ u^{\alpha} \nabla_\alpha G^{<\mu>} = -\frac{q}{m} (e - mn) - \frac{q}{m} e^{\alpha} \nabla^{\mu\nu} \]
\[ + \frac{q^{2}}{m} g^{\alpha} g^{ab\beta} \left( \nabla^\mu f_0 - \nabla^\nu f_1 \right) + \nabla^{\mu\nu} (e - mn) \]
\[ - u^{\alpha} \nabla_\alpha \nabla_0 ^{\mu\nu} - \nabla^a \nabla^0 ^{\mu\nu} - \nabla^a u^{\mu} G^{a} - \nabla^a u^{\mu} f_1^{a\beta\nu}, \]  \tag{60} \]
\[ u^{\alpha} \nabla_\alpha G^{<\mu\nu>} = -\frac{q^2}{m} (f_1 f_{-2}^{\mu\nu} + q e^{\alpha} f_1^{a\mu\nu} + q \frac{q}{m} g^{\alpha} g^{ab\beta} \left( f_0 f_{-1}^{\mu\nu} + m \nabla^\mu f_0 - m \nabla^\nu f_1 \right) \]
\[ - \nabla^a \left( \Pi^{\alpha} - m f_1^{\alpha} \right) - \nabla^a u^{\mu} G^{a} - \nabla^a u^{\mu} f_1^{a\beta\nu}. \]  \tag{61} \]

In accordance with our previous discussion, we can further split the last equation into its trace and trace-free part. This then reads,
\[ u^{\alpha} \nabla_\alpha G^{<\mu \nu>} = -\frac{2}{3} \frac{q}{m} \nabla^\mu f_0 + \frac{2}{3} \frac{q}{m} e^{\alpha} f_1^{a\mu\nu} + \frac{2}{3} m^2 \nabla^\mu V_f \]
\[ - \frac{5}{3} m \nabla^a f_1^{a\mu\nu} - \frac{5}{3} \nabla^a u^{\mu} \Pi - \frac{3}{2} \nabla^a u^{\mu} \nabla_{-2} \mu + \frac{2}{3} m^2 \nabla^a V_f \]  \tag{62} \]
\[ u^{\alpha} \nabla_\alpha G^{<\mu \nu>} = -\frac{2}{3} \frac{q}{m} \nabla^\mu f_0 + \frac{2}{3} \frac{q}{m} e^{\alpha} f_1^{a\mu\nu} + \frac{2}{3} m^2 \nabla^\mu V_f \]
\[ - \frac{5}{3} m \nabla^a f_1^{a\mu\nu} - \frac{5}{3} \nabla^a u^{\mu} \Pi - \frac{3}{2} \nabla^a u^{\mu} \nabla_{-2} \mu + \frac{2}{3} m^2 \nabla^a V_f \]  \tag{63} \]

While these equations are fully generic and self-consistent, they do require the specification of additional closure relations for the higher than second moments, with negative values for \( r \). We will provide one such a potential closure in Sec. 4.3.

4.3 Local relaxation closure for collisionless fluids

As a particular application of a collisional closure to a collisionless fluid, we follow the approach of Wang et al. (2015) and adopt an isotropic pressure closure. The idea is to damp anisotropic pressure contributions over a damping time \( \tau_p \). Since we evolve \( \Pi \) and \( \pi^{\mu\nu} \) separately, this only results in the addition of a damping term to Eq. (63). Fully consistent with the application of a collisional closure in the Newtonian approach, we further apply the additional collisional assumption that higher-rank moments can be neglected according to Eq. (49). Although this assumption might appear ad-hoc at first, in particular when the Knudsen number becomes comparable to unity, comparisons with particle-in-cell simulations of the Vlasov equation for Newtonian plasmas have shown reasonable agreement with such a closure approach in a 10-moment formulation (Wang et al. 2015). For convenience, we will also introduce collisional damping times for the particle diffusion and the energy diffusion current. Furthermore, we will work within the Eckart frame and drop the particle diffusion current \( V^\mu_f \) in favor of the heat flux vector \( q^\mu \). We then find that expressions (60), (62) and (63) reduce to
\[ u^{\alpha} \nabla_\alpha q^{<\mu>} = -\frac{q}{m} (e - mn + P + \Pi) e^{\mu} - \frac{q}{m} e^{\alpha} \nabla^{\mu\nu} - \frac{q}{m} g^{\alpha} g^{ab\beta} \left( f_0 f_{-1}^{\mu\nu} + m \nabla^\mu f_0 - m \nabla^\nu f_1 \right) \]
\[ - \nabla^a \nabla^0 ^{\mu\nu} - \nabla^a u^{\mu} G^{a} - \nabla^a u^{\mu} f_1^{a\beta\nu} = -\frac{1}{\tau_p} q^{\mu}. \]  \tag{64} \]
\[ u^{\alpha} \nabla_\alpha (P + \Pi) = -\frac{2}{3} \frac{q}{m} e^{\mu} q^{\nu} + \frac{2}{3} m \nabla^\mu q^{\nu} - \frac{1}{3} (P + \Pi) q^{\mu} \]
\[ - \frac{2}{3} m q^{\mu} \tau_p + \frac{2}{3} \sigma_{\mu\nu} \pi^{\mu\nu}, \]  \tag{65} \]
\[ u^{\alpha} \nabla_\alpha \pi^{<\mu\nu>} = -\frac{q}{m} e^{\mu} q^{\nu} + \frac{2}{3} \frac{q}{m} e^{\alpha} \nabla^{\mu\nu} \]
\[ + 2 m \nabla^\mu q^{\nu} - \frac{1}{3} m \nabla^\mu q^{\nu} \]
\[ - 2 \nabla^{\mu\nu} \pi^{<\mu\nu>} + (P + \Pi) \nabla^{\mu\nu} = -\frac{1}{\tau_p} \pi^{\mu\nu}. \]  \tag{66} \]

Comparing the above expressions with the generic collisional closure discussed in Sec. 4.1, we can see that the collisionless system just becomes a particular variant of a collisional system, consistent with the assumption of neglecting higher order moments and introducing relaxation times. Most importantly, the collisionless moment expansion together with the higher-moment truncation fixes the coefficients of the closure exactly. We note that while other closure relations are in use in the Newtonian plasma community (e.g., Hammett & Perkins...
their non-local nature renders them unsuitable for use with relativistic approaches. Thus, for such an isotropic closure we can easily use the same tools that will be developed in the rest of this paper allowing us to describe a unified system of equations for two-fluid collisional and collisionless plasmas.

5 RELATIVISTIC TWO-FLUID MAGNETOHYDRODYNAMICS

Having discussed how to model out-of-equilibrium single component fluids, we now want to recast those equations in a form suitable to describe multi-fluid plasmas. More precisely, we aim to describe a two-component plasma consisting of electrons ($e$) and ions ($p$), with charge number $Z$, i.e. $q_p = Z q_e$, in the calculation. Here $q_e$ refers to the electron and $q_p$ to the ion characteristic charge, respectively. While we adopt the convention of referring to the second species as ions, all results would equally be valid also for positrons. Instead of evolving each component separately, it will turn out to be most beneficial to introduce a single fluid reference frame. This will allow us to replace one of the species with an effective single fluid description. Having discussed how to model out-of-equilibrium single component fluids, we now want to recast those equations in a form suitable to describe multi-fluid plasmas. More precisely, we aim to describe a two-component plasma consisting of electrons ($e$) and ions ($p$), with charge number $Z$, i.e. $q_p = Z q_e$, in the calculation. Here $q_e$ refers to the electron and $q_p$ to the ion characteristic charge, respectively. While we adopt the convention of referring to the second species as ions, all results would equally be valid also for positrons. Instead of evolving each component separately, it will turn out to be most beneficial to introduce a single fluid reference frame. This will allow us to replace one of the species with an effective single fluid description. Such single fluid descriptions have been discussed, e.g., by Koide (2009); Andersson et al. (2021), and Cercignani & Kremer (2002), whose notation we follow.

We start out by providing a brief description of both fluids using the language of transient (magneto-) hydrodynamics. Each species, electrons and ions, will have separate particle four-currents and energy-momentum tensors, viz.

$$N_e^\mu, T_e^{\mu\nu}(electrons),$$
$$N_p^\mu, T_p^{\mu\nu}(ions/positrons).$$

Within this description, each species has its own rest-frame associated with $h_{\epsilon/p}^\mu$. Within the description of a single fluid, it turns out to be more useful to introduce a mass-weighted average velocity to be used as the rest-frame of the joint single fluid. This corresponds to the center of mass frame in which collisions between the two species take place. We define this via

$$N^\mu = \chi_e N_e^\mu + \chi_p N_p^\mu =: n U^\mu,$$

where the joint number density $n$ is defined by demanding that $U^\mu U_{\mu} = -1$. The coefficients can be chosen freely depending on the particular fluid and equilibrium being described. Example choices include averaging $\chi_e = \chi_p = 1$, or mass-weighted averages ($\chi_e = m_e/(m_e + m_p), \chi_p = m_p/(m_e + m_p)$), where $m_e$ and $m_p$ are the electron and ion masses, respectively. Adopting $U^\mu$ as the primary velocity frame, we can recast (67) and (68) into

$$N_e^\mu = n_e U^\mu + V_e^\mu,$$
$$N_p^\mu = n_p U^\mu + V_p^\mu.$$  

As a consequence of using a joint reference frame that does not align with the comoving frame of either species, we can see that now both species develop diffusion currents $V_{e/p}^\mu$. By means of (69), these will not be independent but are related via

$$V_p^\mu = \frac{\chi_e}{\chi_p} V_e^\mu.$$

Furthermore, we may split the electric 4-current into its individual contributions

$$J^\mu = q_e \left( Z N_p^\mu - N_e^\mu \right),$$
$$= q_e \left( Z n_p - n_e \right) U^\mu - \left( 1 + \frac{\chi_e}{\chi_p} \right) V_e^\mu,$$

where $q_e$ denotes the electric charge. We have further made use of (72) when re-expressing this equation.

Each species, can be described my means of their energy momentum tensors, $T_{e/p}^{\mu\nu}$ and $T_p^{\mu\nu}$. Decomposing each of these using the common fluid velocity $U^\mu$, these are given by (Cercignani & Kremer 2002)

$$T_{X}^{\mu\nu} = e X U^\mu U^\nu + U^\mu \left( q_X^\nu + h_X V_X^\nu \right) + U^\nu \left( q_X^\mu + h_X V_X^\mu \right) + \left( p_X + \Pi_X \right) \Delta^{\mu\nu} + \pi_{X}^{\mu\nu},$$

Here, $X = e, p$ refers to the individual species. This decomposition differs from (11) in several important ways. First, the constraints (13) apply with respect to $U^\mu$ for each species $X$. Second, because we have adopted a frame that does not align with the individual Eckart frame of each species, an additional contribution to the heat flux, $h_X V_X^\mu$, appears, corresponding to the diffusion of that species in the frame described by $U^\mu$, see also the discussion around Eq. (19).

The combined single-fluid energy momentum tensor, can then be expressed as

$$T^{\mu\nu}_f = T_{e}^{\mu\nu} + T_{p}^{\mu\nu},$$

Using the decomposition lined out in (11),

$$T^{\mu\nu}_f = e U^\mu U^\nu + \left( P + \Pi \right) \Delta^{\mu\nu} + \pi_f^{\mu\nu} = e U^\mu U^\nu + \pi_f^{\mu\nu},$$

we can easily identify

$$e = e_e + e_p,$$
$$p = p_e + p_p,$$
$$\Pi = \Pi_e + \Pi_p,$$

$$\pi_f^{\mu\nu} = \pi_e^{\mu\nu} + \pi_p^{\mu\nu}.$$  

We provide a more detailed discussion of how the single fluid frame relates to the component frames in Appendix A. Overall, conservation of total energy-momentum (10) then implies

$$\nabla_{\mu} \left( T_{f}^{\mu\nu} + T_{EM}^{\mu\nu} \right) = 0.$$  

We point out when solving these equations no electromagnetic source terms are present, and energy and momentum are exactly conserved. As such, this description is really that of a single fluid coupled to electromagnetism. However, the two fluid nature of the system is fundamentally encoded in the heat fluxes and anisotropic stresses present in the system.

5.1 Two-fluid interactions and electron diffusion

Following the discussion of how to combine a relativistic two-fluid plasma into a single fluid description, we are left to specify evolution equations for the out-of-equilibrium variables. In particular, we will need to specify internal heat fluxes $\dot{q}_X^{\mu}$ of the plasma, the bulk viscous scalar $\Pi$ and the anisotropic stresses $n_{X}^{\mu\nu}$. In addition, we strictly require prescriptions of the electron diffusion current $V_e^\mu$ and the electron enthalpy $h_e$, as these are required in order to determine the
total heat flux $q^\mu$, see Eq. (80). Different from the other variables, it is straightforward to determine evolution equations for $V_e^\mu$ and $h_e$.

When writing the electron evolution equation, we want to account for potential electron-ion collisions. We denote such a term as $C_e^\nu$.

In the absence of explicit collisions $C_e^\nu = 0$.

Overall, the evolution equations for the electron read

$$\nabla_\mu T_e^{\mu\nu} + q_e F^{\mu\nu} N_e^\mu = C_e^\nu. \quad (83)$$

We can alternatively write expression (83) as

$$\nabla_\mu T_e^{\mu\nu} = -q_e n_e e^\nu - q_e b^{\mu\nu} V_{e\nu} - q_e U^\nu V_e^\mu + C_e^\nu. \quad (84)$$

These equations describe the evolution of the electron fluid. However, using the decomposition with respect to the single fluid frame, see Eq. (74) for $X = e$, it does not describe the evolution of the single fluid frame velocity $U^\mu$ or energy density $e$, as would be the case for a single fluid. Instead, the conservation of electron energy and momentum determines the evolution of the electron diffusion current $V_e^\mu$ and the electron enthalpy $h_e$ in the single-fluid frame.

We can make this more explicit by splitting (84) along the joint fluid velocity $U^\mu$. More specifically, by contracting (84) with $U^\mu$, we find that

$$U^\mu \nabla_\mu e = - (e + P_e + P_e + \Pi_e) \nabla_\mu U^\mu + \frac{1}{2} \sigma^{\mu\nu} \sigma_{\mu\nu} + \nabla_\mu Q_e^\mu + q_e e^\mu + q_e b^{\mu\nu} V_{e\nu},$$

where we have introduced the shear tensor $\sigma_{\mu\nu} = \Delta^{\mu\nu}_{\sigma\beta} \nabla_\sigma U_\beta$. At the same time, we can obtain an evolution equation for $V_e^\mu$, by contracting (84) with $\Delta^{\mu\nu}$,

$$-\Delta^{\mu\nu}_{\sigma\beta} \nabla_\sigma Q_e^\beta = (e + P_e + P_e + \Pi_e) \nabla_\mu U^\mu + \Delta^{\mu\nu}_{\sigma\beta} \nabla_\sigma e^\nu + \frac{1}{2} Q_e e^\nu + \frac{1}{2} Q_e \sigma^{\nu\mu} + Q_e b^{\mu\nu} V_{e\nu} + \nabla_\mu \Delta^{\mu\nu} V_{e\nu} + q_e n_e e^\mu + q_e b^{\mu\nu} V_{e\nu} + C_e^\nu, \quad (85)$$

where we have introduced the vorticity tensor $\omega_{\mu\nu} = \frac{1}{2} \Delta^{\mu\nu}_{\sigma\beta} (\nabla_\sigma U_\beta - \nabla_\beta U_\sigma)$ and the total electron energy diffusion vector

$$Q_e^\mu = q_e^\mu + h_e V_e^\mu. \quad (87)$$

It is important to stress at this point that Eqs. (85) and (86) have exactly the same form as dissipative bulk pressures (50) and energy diffusion fluxes (51). As such, the electron fluid contributions appear as 14-moment dissipative corrections to the effective single fluid. This fundamentally implies that relativistic two fluid system can be handled in exactly the same way as dissipative single fluids (Most & Noronha 2021). Moreover, expression (86) is nothing but the general Ohm’s law we set out to derive in (3). Compared to a Newtonian Ohm’s law, the main difference is that (86) provides a time evolution equation for the dissipative electric current $V_e^\mu$. This completes the description of a two-fluid plasma within the framework of a single fluid with dissipative corrections. Together with the single fluid evolution equation (82), the electron fluid equations (85) and (86) provide evolution equations for the diffusive heat flux and electron energy.

5.2 Electron-ion collisions

To model inter-species collisions, we proceed as follows. Instead of parameterizing them directly, we adopt an effective relaxation time approach, meaning that collisions will drive the electron energy-momentum tensor $T_e^{\mu\nu}$ towards its equilibrium value $T_{e,eq}^{\mu\nu}$, see Eq. (16), on a time scale $\tau$. Following Cercignani & Kremer (2002), we can write the collision term as follows,

$$C_e^\nu = -\frac{1}{\tau} \left( T_e^{\mu\nu} - T_{e,eq}^{\mu\nu} \right) U_\mu. \quad (88)$$

Since the equilibrium solution $T_{e,eq}^{\mu\nu}$ is defined in the absence of heat fluxes, and the energy density $e$ is the same in the given frame, the collision term is given by the heat flux only. We can alternatively write expression (88) as (Cercignani & Kremer 2002)

$$C_e^\nu = -\frac{1}{\tau} e^\nu - \frac{h_e b^{\mu\nu} V_{e\nu}}{h}. \quad (89)$$

Further work is needed when modeling this effective relaxation time approach when considering the place where the relaxation scale $\tau$ is momentum dependent. In this case, a modified version of the collision term in the relaxation time approximation has to be used to ensure agreement with general properties of the Boltzmann equation, see Rocha et al. (2021).

5.3 After discussing how to recast two relativistic fluids into a single fluid form with potentially large out-of-equilibrium corrections, we want to briefly summarize the main equations as follows. The electromagnetic fields are evolved using the Maxwell equation coupled to the current provided by both species,

$$\nabla_\mu F^{\mu\nu} = 4\pi q_e \left( (Zn - n_e) U^\mu - \left[ 1 + \frac{Ze}{\chi_p} \right] V_e^\mu \right). \quad (90)$$

Total conservation of energy and momentum then implies that

$$\nabla_\mu \left[ (e + p) U^\mu \right] U^\mu = (P_e + P_p + \Pi_e + \Pi_p) \Delta^{\mu\nu} + \left[ Q_e + q_p \frac{b^{\mu\nu} V_{p\nu}}{\chi_p} \right] U^\nu + \left[ Q_e + q_p \frac{b^{\mu\nu} V_{p\nu}}{\chi_p} \right] U^\mu + \pi^{\mu\nu} + \Pi^{\mu\nu}_{EM} = 0. \quad (91)$$

It is important to stress that the single fluid energy-momentum and the electromagnetic fields are conserved together. This equation, thus has the character of a single fluid coupled to electromagnetism, where two-fluid corrections appear as dissipative heat fluxes and pressures. These obey a 14-moment like evolution equation consistent with the collisional closure outlined in Sec. 4.1. More specifically, in analogy with a bulk scalar pressure, the evolution electron energy $e$ is governed by

$$U^\mu \nabla_\mu e = - (e + P_e + \Pi_e) \nabla_\mu U^\mu + \frac{1}{2} \sigma^{\mu\nu} \sigma_{\mu\nu} + \nabla_\mu Q_e^\mu - q_e e^\mu + Q_e^{\mu\nu} \nabla_\mu V_{e\nu}. \quad (92)$$

Similarly, the electron heat fluxes take the following form

$$-\Delta^{\mu\nu}_{\sigma\beta} \nabla_\sigma e^\nu = (e + P_e + \Pi_e) U^\mu \nabla_\sigma Q_e^\beta + \frac{1}{2} Q_e e^\nu + \frac{1}{2} Q_e \sigma^{\nu\mu} + Q_e b^{\mu\nu} V_{e\nu} + \nabla_\mu \Delta^{\mu\nu} V_{e\nu} + q_e n_e e^\mu + q_e b^{\mu\nu} V_{e\nu} + C_e^\nu. \quad (93)$$

Similar to their non-relativistic equivalent, a choice of closure for the individual species is still needed. That is, constitutive relations for the anisotropic stresses, $(\Pi^{\mu\nu}_{e}, \Pi^{\mu\nu}_{p})$, heat fluxes $(q_e^\mu, q_p^\mu)$ and pressures $(P_e, P_p, \Pi_e, \Pi_p)$ need to be provided. These are different from dissipative corrections arising purely by the presence of a second fluid and its interaction through the electromagnetic field.
6 TWO-FLUID MAGNETOHYDRODYNAMICS FOR RELATIVISTIC ELECTRON-ION PLASMAS

Having recast the two fluid equations into an effective single fluid description with (potentially large) dissipative corrections, we want to apply the formalism to an electron-ion plasma. Such a plasma is characterized by the ion mass greatly exceeding the electron mass, \( m_p \gg m_e \).

We begin by choosing \( \chi_p = m_p/\bar{m} \) and \( \chi_e = m_e/\bar{m} \), with \( \bar{m} = (m_e + m_p) \). This mass weighted averaging is a common choice in Newtonian plasma physics (e.g., Sturrock 1994). In practical terms, since \( m_p \gg m_e \), the single-fluid frame almost aligns with the ion-frame, where the difference is given by electron contributions. Hence the dissipative correction of the presence of electrons onto the ions is small, since the effective inverse Reynolds number associated with out-of-frame corrections will be of order \( m_e/m_p \ll 1 \).

The effective particle number then is

\[
N^\mu = \left( m_e \rho_e + \frac{m_p}{m} \rho_p \right) U^\mu, \tag{94}
\]

\[
v_p^\mu = \frac{m_e}{m_p} v_e^\mu. \tag{95}
\]

This choice also has implications for the total heat flux in the single fluid frame. From Eq. (80) we find

\[
q^\mu = q_e^\mu + q_p^\mu + \left( 1 - \frac{m_e}{m_p} \frac{b_p}{b_e} \right) h_e V_e^\mu, \tag{96}
\]

\[
\approx q_e^\mu + q_p^\mu + \left( 1 - \frac{b_p}{b_e} \right) h_e V_e^\mu, \tag{97}
\]

where the last line is independent of the particle mass ratio, \( m_e/m_p \). Since \( h_e \approx m_e \), we find that the diffusion contribution is on the order of the electron mass, while the total energy of the single fluid, scalers like \( h \sim m \approx m_p \). Unless very large diffusion currents are present, the diffusive heat flux will be suppressed by the electron-ion mass ratio.

On the other hand, the diffusion part of the total electric current (73),

\[
\mathcal{J}^\mu = q_e \left[ (Z n_p - n_e) U^\mu - \left( 1 + \frac{m_e}{m_p} \right) V_e^\mu \right]. \tag{98}
\]

is entirely dominated by the diffusion of electrons in the single fluid frame, as \( n_p \approx n_p \) in collisional equilibrium.

Since the effective single fluid frame and the ion frame now almost coincide we can, to a very good approximation, truncate our expressions to linear order in \( (m_e/m_p) \). That is

\[
n_p \approx \bar{n}_p, \tag{99}
\]

\[
\epsilon_p \approx \bar{\epsilon}_p + \frac{2 m_e}{m_p} \frac{m_e}{m} \bar{q}_e^\mu \bar{q}_e^\mu, \tag{100}
\]

\[
\bar{q}_e^\mu \approx \Delta \bar{q}_e^\mu = \frac{m_e}{m_p} V_e^\alpha \Delta \bar{\Pi}^A \bar{\Pi}_{\alpha \beta} \left( \bar{h}_p + \frac{1}{n_p} \bar{\Pi}_p \right) \left( \frac{m_e}{m_p} \bar{V}_e^\mu \right), \tag{101}
\]

\[
\Delta \bar{q}_p^\mu \approx \Delta \bar{q}_e^\mu = \frac{m_e}{m_p} \bar{V}_e^\alpha \Delta \bar{\Pi}^A \bar{\Pi}_{\alpha \beta} \left( \bar{h}_p + \frac{1}{n_p} \bar{\Pi}_p \right) \left( \frac{m_e}{m_p} \bar{V}_e^\mu \right). \tag{102}
\]

Here we have used \( \Delta \bar{\Pi}^A \) to denote quantities in the fluid rest frame of species \( X \). For a detailed discussion see Appendix A. As expected, the quantities in the ion fluid frame and in the single fluid frame almost coincide. The small discrepancies scale exclusively with the electron number diffusion current \( \bar{V}_e^\mu \). It is, hence, crucial to understand how the evolution of the diffusion current progresses relative to the evolution of the ion system.

To this end, what remains to be specified are internal closure relations for the plasmas. In what follows, we will assume that electrons and ions only interact via electromagnetic fields and, potentially, via the collision term \( \bar{Q}_{i} \) in the joint frame, which was introduced in Eq. (83). In practical terms, this implies that the remaining equations can be closed for each species individually. That is, we need to specify relations for the single fluid dissipative quantities \( \bar{\Pi}_X, \bar{q}_X^\mu, \) and \( \bar{\Pi}_X^{\nu \mu} \) in the component fluid rest-frames. These will depend on the physical state of the system. In particular, we will present two systems describing, in particular, collisionless, weakly collisional, and highly collisional electron-ion plasmas.

6.1 Collisionless electron-ion plasmas

In the case of a collisionless two-fluid plasma we can solve the local relaxation closure equations, (65), (64), and (66), in the individual fluid frames. For completeness we repeat these equations below.

\[
\bar{u}_X^\alpha \nabla_{\alpha} (\bar{P}_X + \bar{\Pi}_X) = -\frac{q}{m_X} (\bar{e}_X - m_X \bar{h}_e \bar{\Pi}_X + \bar{P} + \bar{\Pi}_X) \bar{e}_X - \frac{q}{m_X} \bar{e}_X \bar{a}_X^{\alpha \mu}, \tag{103}
\]

\[
\bar{u}_X^\alpha \nabla_{\alpha} \bar{a}_X^{\nu \mu} = -\frac{q}{m_X} (\bar{e}_X - m_X \bar{h}_e \bar{\Pi}_X + \bar{P} + \bar{\Pi}_X) \bar{e}_X - \frac{q}{m_X} \bar{e}_X \bar{a}_X^{\alpha \mu}, \tag{104}
\]

\[
\bar{u}_X^\alpha \nabla_{\alpha} \bar{\Pi}_X^{\nu \mu} = \frac{q}{m_X} \bar{e}_X \bar{a}_X^{\nu \mu} + \frac{2 q}{m_X} \bar{a}_X^{\nu \mu}, \tag{105}
\]

Solving these equations would provide the most complete and consistent fluid-type description of collisionless plasmas in curved spacetimes. In particular, this description retains all electron inertia terms, resistive and Hall terms. Moreover, all anisotropic pressure contributions are retained, with all transport coefficients apart from the relaxation times being fixed, in principle, by first-principle kinetic theory calculations.

6.2 Weakly collisional systems

For some black hole accretion problems collisions become important (Ressler et al. 2015). If we want to account for Coulomb collisions inside the plasma (i.e. electron-electron) collisions, we need to include effective collisionality in these equations. Such a situation might arise inside accretion disks around supermassive black holes, where electron heat fluxes and anisotropic viscosities would affect the flow structure and electron heating.

Different from the previous case, we proceed by introducing an equilibrium state for each fluid. That is, we specify equations of state for the pressures \( \bar{P}_X \). A sensible, and simple choice, would be a simple gamma-law as is commonly done in black hole accretion (Porth et al. 2019),

\[
P_e = \bar{n}_e T_e = m_e \bar{n}_e \bar{e}_e (\gamma_e - 1), \tag{106}
\]

\[
P_p = \bar{n}_p T_p = m_p \bar{n}_p \bar{e}_p (\gamma_p - 1). \tag{107}
\]
where $\gamma_e$ and $\gamma_p$ are the adiabatic coefficients of the plasmas.

For these equations of state, we can solve Eq. (A4) for the temperature $T_X$ of each species,

$$T_X = \left[ E_X \sqrt{n_X^2 - V_X^2} - m_X n_X^2 \right] \frac{\Gamma_X - 1}{n_X^2 + (\Gamma_X - 1) V_X^2},$$

where

$$E_X = e_X + \frac{2}{\sqrt{n_X^2 - V_X^2}} V_X^{\mu \nu} q_X^{\mu \nu} - \frac{1}{n_X^2 - V_X^2} V_X^{\mu \nu} g_{\mu \nu} \Gamma_X - 1.$$ (109)

Within the mass hierarchy adopted in this section, the ion temperature reduces to

$$T_p = \frac{P_p}{n_p} = \bar{T}_p = \frac{m_e}{m_p} \frac{\Gamma_p - 1}{\tilde{\varepsilon}_p \tilde{q}_p \mu_p}.$$ (110)

We can see that in the absence of intrinsic ion heat fluxes, the ion temperature in the ion frame and in the single fluid frame begin to coincide. We can similarly recover the electron temperature from Eq. (108), however, no simplification a priori is possible.

In the following we will consider the case of ion-ion and electron-electron collisions. This scenario has first been considered by Braginskii (1965) in the non-relativistic case. Additionally, relative diffusion between electrons and ions is possible due to the coupling the electromagnetic field in Eq. (83). Following Kulsrud (2020), we assume that ion-ion collisions lead to an effective anisotropic shear stress. In the limit of vanishing Larmor radius, this will have to approach the Braginskii limit (Braginskii 1965). Following Most & Noronha (2021), this can be achieved by imposing the following closure relations of the ion pressure tensor, see also Denicol et al. (2018),

$$\tilde{\alpha}_e \nabla \tilde{\alpha}_e \tilde{q}_e^{\mu \nu} = -\frac{2}{\tau_p} \tilde{\alpha}_e \nabla \tilde{\alpha}_e \tilde{p}_e + \delta^{\nu}_{\alpha} \tilde{p}_e \tilde{q}_e^{\mu \nu} - \frac{1}{\tau_p} \tilde{\varepsilon}_e \tilde{q}_e^{\mu \nu},$$ (111)

$$\tilde{\pi}_e^{\mu \nu} = 0.$$ (112)

Additionally, we assume that intra-species collisions will drive a heat flux. Again, inspired by Most & Noronha (2021), we can assume that to within first-order this leads to a closure of the following form,

$$\tilde{\alpha}_p \nabla \tilde{\alpha}_p \tilde{q}_p^{\mu \nu} = -\frac{\tilde{m}_e}{\tau_q} \tilde{q}_e^{\mu \nu} - \kappa_p \nabla \tilde{\alpha}_p \tilde{q}_e^{\mu \nu} - \tilde{\alpha}_e \nabla \tilde{\alpha}_e \tilde{p}_e + \delta^{\nu}_{\alpha} \tilde{p}_e \tilde{q}_e^{\mu \nu} - \frac{1}{\tau_p} \tilde{\varepsilon}_e \tilde{q}_e^{\mu \nu},$$ (113)

$$\tilde{\pi}_e^{\mu \nu} = 0.$$ (114)

We further assume that bulk scalar pressures in their respective local frames vanish

$$\Pi_X = 0,$$ (115)

since we do not take into account bulk viscosities. Coulomb collisions arising from electron-electron and ion-ion interactions give rise to effective transport coefficients. These have been derived in the weakly collisional limit of large gyrofrequencies, $\Omega_e = q_e B / m_e$ and $\Omega_p = Z q_p B / m_p$. A list of all transport coefficients can be found, e.g. in Kulsrud (2020).

7 DISSIPATIVE MAGNETOHYDRODYNAMICS FOR RESISTIVE RELATIVISTIC SINGLE-FLUID PLASMAS

So far we have considered closures that explicitly keep all two-fluid degrees of freedom. Current state-of-the-art simulations of ideal and resistive relativistic magnetohydrodynamics, however, normally treat only single (ion-)fluid plasmas (see Marti & Müller 2015 for a review). In the following, we would like to derive this limit from the two-fluid equations considered in this work.

We start by considering the single fluid plasma described by Eq. (82). We now neglect also all terms linear in $m_e / m_p$ in Eqs. (99)-(102), that is

$$n_p \approx \tilde{n}_p, \quad n \approx n_p, \quad e_p \approx \tilde{e}_p, \quad e \approx e_p.$$ (116)

Consistent with the assumption of a single ion-fluid we also neglect anisotropic electron pressures,

$$\tilde{\pi}_e^{\mu \nu} \approx 0,$$ (117)

but keep the electron heat flux, in line with the discussion of weakly collisional plasmas in Sec. 6.2.

If we neglect electron shear viscosity and heat conduction, to lowest order in $m_e / m_p$, the electric current reduces to

$$\mathcal{J}^{\mu} = q_e \left[ (Z n - n_e) U^{\mu} - V_e^{\mu} \right],$$ (118)

which will be given in terms of the electron Ohm’s law. The heat flux (97) remains unchanged. Furthermore, if we neglect all direct couplings between the electron energy diffusion vector $Q_e^{\mu}$ and the single fluid velocity $U^{\nu}$ and its gradients, we arrive at

$$U^{\alpha} \nabla \alpha Q_e^{\mu} = -q_e n_e e^{\mu} + q_e b^{\nu} Q_e^{\nu} - \nabla^{<\mu>} P_e$$ (119)

$$- P_e U^{\alpha} \nabla \alpha U^{\mu} + \frac{1}{\tau} Q_e^{\mu}.$$ (120)

At this point, it remains to specify the electron enthalpy, which effectively amounts to fixing the electron temperature. This could either be done by evolving the electron energy or by algebraically fixing the electron temperature $T_e$. The latter is commonly done in the post-processing of black-hole accretion simulations, where the electron temperature governs part of the emission process (e.g., Ressler et al. 2015). Different from such ad-hoc assumptions, the explicit inclusion of dissipative terms allows us to explicitly track the change in electron temperature in a self-consistent manner. More specifically, using the same assumption as for the electron momentum equation (119) we find,

$$\nabla^{\mu} (e^{\nu} U^{\mu}) = -P_e \nabla \mu U^{\mu} + \nabla \mu Q_e^{\mu} - q_e e^{\mu} V_e^{\mu}.$$ (121)

We can now see that in this minimalistic model the electron energy only changes via diffusion, Ohmic heating, and adiabatic compression. Furthermore, since we retain ion heat fluxes and anisotropic stresses, these can be closed using the 14-moment relations given in Sec. 4.1. Keeping only terms that are first-order in gradients, we
arive at

\[ U^{\mu} \nabla_{\mu} q_{p}(\nu) = -\kappa_{p} T^{(\mu)} T_{\nu} - \kappa_{p} T_{p} U^{\mu} \nabla_{\nu} U^{\mu} \]

\[ = \frac{\sigma}{\tau_{q}} b^{\nu} q_{p} \nu - \frac{1}{\tau_{q}} \nabla_{\nu} \rho_{p}, \quad (121) \]

\[ \tilde{\rho} \nabla_{\nu} \tilde{\rho} q_{e}(\mu) = -\kappa_{e} T^{(\mu)} \tilde{T}_{\nu} - \kappa_{e} \tilde{T}_{e} \nabla_{\nu} \tilde{\rho}_{e} \tilde{\rho}_{e} \]

\[ - \frac{\delta_{e}}{\tau_{q}} b^{\nu} \tilde{\rho}_{e} \nu - \frac{\tilde{b}_{e}}{\tau_{q}} \tilde{b}_{e} \tilde{b}_{e}. \quad (122) \]

\[ U^{\mu} \nabla_{\mu} \tilde{\rho}_{e}(\nu) = -\frac{\tilde{\gamma}}{\tau_{p}} \nabla_{\nu} (U^{\mu} U_{\nu}) = \frac{1}{\tau_{p}} \tilde{\rho}_{p}^{\nu} \]

\[ + q_{e} \delta_{e} b_{B} \tilde{b}_{A} \nabla_{\nu} \tilde{b}_{e} \tilde{b}_{e} \frac{\tau_{p}}{2} \tilde{b}_{e}^{\nu} \equiv \tilde{b}_{p}(\nu). \quad (123) \]

Together with Eqs. (119) and (120), the system described here is the most complete form of dissipative MHD for a resistive single component plasma with dissipative corrections from the secondary fluid. Crucially, for consistency with charge neutrality, a minimal number of degrees of freedom of the second species needs to be retained. In particular these are the electron number density and momentum. In addition, we also retain electron temperature evolution, which provides a consistent way to extract electron temperature beyond previously adopted ad-hoc approaches (Ressler et al. 2015; Chael et al. 2018).

### 7.1 Single-fluid dissipative magnetohydrodynamics

Instead of solving the electron energy equation (120), one could alternatively specify the electron temperature via an effective prescription for the electron specific enthalpy

\[ \tilde{h}_{e} = \tilde{h}_{e}(\tilde{b}_{p}, n, e). \quad (124) \]

Such relations have been proposed in Howes (2010); Kawazura et al. (2019); Rowan et al. (2017). A simple choice, commonly done for single fluid plasmas is to neglect the electron temperature all together. In that limit, we may simply set \( \tilde{h}_{e} \approx 1 \), assuming that thermal contributions are negligible compared to the rest-mass energy. It is important to understand that under this assumption the electron temperature can no longer be recovered from the evolution system. On the other hand, such a simplification allows us to rewrite Eq. (119) to read,

\[ U^{\mu} \nabla_{\mu} V_{e}(\nu) = -\frac{q_{e}}{m_{e}} \sigma e^{\mu} + \frac{q_{e}}{m_{e}} b_{e} \tilde{b}_{e} \nabla_{\nu} V_{e} \nu - \frac{1}{\tau} V_{e}^{\mu}. \quad (125) \]

This is the most minimalistic form of a self-consistent electron momentum equation. In this case, both electron inertia terms, as well as resistivity and the Hall term, are kept. Together with the conservation of electric charge (118) and (22), this completes the electron closure relations. If we further neglect electron inertia terms on the LHS, we arrive at the standard Ohm’s law

\[ V_{e}^{\mu} \approx \sigma e^{\mu} - \tau \frac{q_{e}}{m_{e}} b^{\nu} V_{e} \nu. \quad (126) \]

where \( \sigma = \tau_{e} \frac{q_{e}}{m_{e}} \) is the electric conductivity. The second term will give rise to a Hall effect with varying degree of anisotropy. It is important to note that dropping the advection operator in Eq. (125) breaks the strong hyperbolicity of the system, as has been shown for previously used resistive relativistic MHD systems (Schoepe et al. 2018). Hence, the consistent evolution of \( V_{e}^{\mu} \), see Eq. (125), is crucial to the causality and stability of the dissipative MHD system.

### 7.2 Force-free limit

When modeling compact object magnetospheres the copious production of electron-positron pairs will lead to an efficient screening of the electric field component parallel to the magnetic field (Goldreich & Julian 1969). This limit has been extensively studied in global magnetospheric models (e.g., Spitzkovsky 2006; Alic et al. 2012; Palenzuela 2013; Parfrey et al. 2013; Carrasco et al. 2018; Most & Philippov 2020), although one is not always able to meaningfully capture reconnection physics in current sheets (e.g., Mahnmann et al. 2021; Ripperda et al. 2021b). In the following, we will give a brief outline of the force-free limit in the dissipative MHD system. As we will show, this system naturally contains the force-free limit, where the transition between the two is not ad-hoc (Palenzuela 2013) but physically motivated in terms of conductivities and gyration frequencies.

In this force-free electrodynamics limit, the Lorentz force will vanish, i.e.

\[ \nabla \times \tilde{F}_{\mu}^{\nu} = -F_{\mu}^{\nu} J_{\mu} \approx 0, \quad (127) \]

where the electromagnetic fields act upon the fluid via Lorentz forces, viz.

\[ F_{\mu e}^{\nu} J_{\nu} = (q_{e} b_{\mu e}^{\nu} + U^{\mu} V_{e}^{\nu}) e_{\nu} - b_{e}^{\mu} V_{e}^{\nu}. \quad (128) \]

In writing the above we have made use of the short-hand \( q_{e} = q_{e} (Z_{e} - n_{e}). \) In order to obtain the force-free limit the Lorentz force

\[ F_{\mu e}^{\nu} J_{\nu} \approx 0, \quad (129) \]

needs to vanish. From Eq. (128) this can generally only be achieved if the comoving electric field and the diffusion current perpendicular to the (comoving) magnetic field vanish simultaneously. That is

\[ e_{\mu} \approx 0, \quad (130) \]

\[ b_{\mu e}^{\nu} V_{e}^{\nu} \approx 0. \quad (131) \]

The latter condition directly implies that

\[ V_{e}^{\mu} \approx V_{\parallel}^{\mu} \frac{1}{\sqrt{\tilde{b}_{\parallel}}} b^{\mu}, \quad (132) \]

where \( V_{\parallel} \) is the diffusion current parallel to the comoving magnetic field \( b^{\mu}. \) As such, the diffusion current will naturally maintain the \( F_{\mu e}^{\nu} F_{\mu e}^{\nu} = 0 \) condition, see also the discussions in Palenzuela (2013) and Paschalidis & Shapiro (2013). From Eq. (125), we can now see that the equations presented in this work naturally recover the force-free limit when both the electrical conductivity \( \sigma \) and gyration frequency \( \Omega_{e} \) are large, as the diffusion current \( V_{e}^{\mu} \) needs to remain finite. Since \( \Omega_{e} \sim \sqrt{\tilde{b}_{\parallel}} \), in this limit magnetic field strength is large. Hence, this system of equations naturally contains the force-free limit and can also achieve it self-consistently in the limit of a strongly magnetized perfectly conducting plasma.

In other words, the system presented here is well suited to study neutron star magnetospheres with self-consistent non-ideal effects, allowing to accurately capture reconnection processes.

### 7.3 Non-resistive dissipative magnetohydrodynamics

As a final reduction, we can further assume that conductivity is infinite \( \sigma \to \infty \). In this case, the comoving electric field \( e^{\mu} \) vanishes. Since the out-of-frame diffusion current has no effective source-term in Eq. (125), any initial condition in \( V_{e} \) will only be (anisotropically) advected and decay on a time scale \( \tau \). Hence, we may then set

\[ V_{e}^{\mu} = 0. \quad (133) \]
Because of the divergence constraint on the electric field, 
\[ U_\mu \nabla_\nu F^{\mu\nu} = 4\pi U_\mu J^\mu, \]  
this choice uniquely fixes the electron number density via \( 21, \)
\[ n_e = Z \rho - \frac{1}{4\pi q_e} \left( \nabla_\mu e^{\mu} - b^{\mu\nu} \nabla_\nu \right). \]  
Since the electron number density and temperature are now fixed, we can drop the evolution equation for the electron heat flux \( 125, \)
\[ \text{At this point, all electron degrees of freedom have been stripped from the system, and we are left with a single perfectly conducting ion fluid. We then only retain first-order out-of-equilibrium corrections of the ion fluid,} \]
\[ U^{\mu} \nabla_\mu q^{(\mu)} = -\frac{k}{\tau_q} \nabla^{(\mu)} T - \frac{k}{\tau_q} T U^{\mu} \nabla_\mu U^{\mu} + \delta_q b^{\mu\nu} q^{(\nu)} - \frac{1}{\tau_q} q^{(\mu)}, \]  
where for simplicity we have dropped all subscripts, as only one fluid is present. These equations now describe the evolution of a dissipative single fluid coupled to magnetic fields in the non-resistive limit \( 18, \) which have recently been studied by Most & Noronha \( 2021. \)

8 CONCLUSIONS

In this work we have systematically investigated the description of a general-relativistic plasma consisting of two charged components interacting through electromagnetic fields. To include non-ideal out-of-equilibrium effects, we have gone beyond the traditional magnetohydrodynamical approach, and have adopted a 14-moment formulation \( 12, 18, 19, \) which is a generalization of Grad’s 13-moment approach in Newtonian plasma physics. As a moment expansion of the general-relativistic Boltzmann equation, such an approach always requires the specification of a closure relation, expressing higher moments in terms of the main 14-moments used here. Building on closure relations devised in the context of nuclear physics applications, we present the general 14-moment decomposition of the Boltzmann equation in the absence of collisions. By comparing these equations to the general form of a collisional closure in the presence of electromagnetic fields \( 19, \) we show that these equations are consistent with the collisional limit, except for the presence of additional heat flux terms. Extending a similar closure from Newtonian 10-moment descriptions of collisionless plasmas \( 15, \) we propose a simple closure relation for the collisionless plasma. Although formally collisional in nature, such a closure has compared favourably to full numerical solutions of the Newtonian Vlasov equation \( 15. \) In deriving this closure we have formally assumed that the thermal energies of the fluid do not exceed their rest-mass energies. While it is a priori not clear how inaccurate these equations become in the limit of large effective temperatures, it might be necessary to resort to different resummation techniques for the moments \( 2019. \) for a discussion.

In order to recast the system into a form typically adopted in the numerical study of relativistic plasmas \( 09, 19, \) we rewrite the two-fluid equations as a single fluid system with potentially large out-of-equilibrium corrections. While being exactly equivalent to modeling two separate general fluids, this description is particularly useful for the case of two-component plasmas with large particle mass ratios. In this case the system naturally takes the form of a single fluid with dissipative corrections. These split into internal out-of-equilibrium processes \( e.g. \) electron-electron, ion-iron collisions \( as \) well as inter-species contributions \( electron-ion \). Remarkably, the latter part takes the form of a 14-moment \( \text{collisional} \) closure. This fundamentally illustrates that the entire out-of-equilibrium sector can be solved using recently developed methods for 14-moment dissipative relativistic MHD \( 2021. \)

Another feature of recasting the two-fluid system in dissipative single-fluid form is the ability to systematically approximate and simplify the dissipative sector. By expanding all expressions in the particle mass ratio, we are able to derive an effective Ohm’s law including only first-order dissipation and anisotropy effects. This most general form of dissipative MHD retains a minimal amount of electron degrees of freedom. In particular, it allows for the consistent evolution of electron temperature, which is driven by Ohmic and viscous heating. Such an approach is potentially relevant for current simulations of black-hole accretion which can either model electron temperature only in post-processing \( 10, 17, 19, \) or using approximate calculation that rely on non-convergent grid dissipation \( 15. \) It is important to stress at this point that the consistent treatment of electron momentum naturally requires a dissipative heat-flux correction to the effective single ion-fluid. In order to allow for causal evolutions in a second-order formulation, the heat flux cannot be algebraically related to the fluid and its gradients \( 79, \) and instead requires separate evolution equations. Fundamentally, this implies that in a relativistic setting the electric current entering the Maxwell equation will require a separate evolution equation, where the time derivatives cannot be dropped unlike in Newtonian contexts. While we postpone the issue of deriving strict causality and, in turn, strong hyperbolicity conditions for the system presented here, our results might explain why previous approaches to resistive relativistic MHD \( 09, 13, 19 \) have been found to be only weakly hyperbolic \( 18. \) Further studies are clearly needed in this subject, especially given that only very recently the causal properties of Israel-Stewart-like theories \( without \) electromagnetic field effects \( have \) been understood in the nonlinear regime \( 19c, 21. \)

Apart from a numerical assessment of this system and a potential comparison with collisionless relativistic particle-in-cell simulations \( 15, 20b, \) several extensions seem possible. Motivated by the need to model relativistic reconnection in neutron star and black hole magnetospheres, we have mainly focused on a two-component plasma that could either be electron-ion or electron-positron. In the context of black hole magnetospheres a potential coupling to photon radiation transport would allow to model dynamical pair processes in the jet regions of supermassive black holes \( 09. \) In the context of neutron star interiors, the inclusion of weak-interaction out-of-equilibrium effects coupled to neutrino radiation would allow for the self-consistent study of dissipative effects in neutron star mergers \( 2021. \) Finally, neutron star interiors are believed to consist of superfluid phases \( for \) a review, see Chamel \( 2017. \) Recent models seem to indicate that these can be modeled using similar Israel-Stewart like approaches \( 21. \) It thus seems natural to extend this 14-moment closure to a general multi-fluid approach \( see \) also Andersson et al. \( 2017b, a. \)
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DATA AVAILABILITY
No new data was produced in this study.

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APPENDIX A: TWO-FLUID TRANSFORMATIONS

Having derived evolution equations for the effective single fluid frame, we need to relate the heat fluxes and anisotropic stresses described in Eq. (74). To this end, we consider the rest-frames of each fluid, for which we can define individual closure relations, see Sec. 4.

Adopting an Eckart frame in the rest-frame of each fluid, we may write

\[ \hat{\Lambda}_e = \hat{\eta}_e u_e^\mu, \]
\[ \hat{\Pi}^\mu_P = \hat{\eta}_P u_P^\mu, \]

where we have introduced particle number densities \( \hat{\eta}_e/p \) and velocities \( u_e^\mu/p \). We can then relate these number densities to the one seen in the single fluid frame via

\[ \hat{n}_X^2 = n_X^2 - g_{\mu\nu} V^\mu_X V^\nu_X, \]

where \( X \) denotes either electrons (\( e \)) or ions/positrons (\( p \)). This implies that the number density as seen by an observer comoving with the single-fluid frame, will be different from the one in the component fluids rest-frame. This difference is given by the diffusion current. In the following, we will denote all quantities in the electron or ion frame with as \( \tilde{X} \), Similarly, we can relate the energy density and pressure in the two frames. From Eq. (11) we then find that

\[ e_X = \tilde{e}_X \left[ 1 + \frac{V^2_X}{n_X - V^\nu_X} \right] + (\tilde{P}_X + \tilde{\Pi}_X) \frac{V^2_X}{n_X - V^\nu_X}, \]
\[ = \frac{2}{n_X} V^\mu_X g_{\mu\nu} + \frac{1}{n_X} V^\mu_X V^\nu_X \tilde{n}_X^{\mu\nu}, \]

\[ P_X + \tilde{\Pi}_X = \tilde{P}_X + \tilde{\Pi}_X + \frac{1}{3} (e_X - \tilde{e}_X). \]

It can be seen that also the enthalpy receives (multiplicative) corrections related to the magnitude of the diffusion current \( V^\mu_X \). Similarly, we can write

\[ q_{XX} = \frac{n_X}{n_X} \Lambda_{\mu\nu} q_{XX} \Lambda_{\mu\nu} \tilde{n}_X^{\alpha\beta} + \tilde{e}_X + \tilde{P}_X + \tilde{\Pi}_X \frac{1}{n_X} g_{\mu\nu} \frac{V^\mu_X}{n_X}, \]
\[ = \frac{1}{n_X} g_{\mu\nu} V^\mu_X g_{\mu\nu} \frac{1}{n_X} \tilde{q}_{XX} \tilde{n}_X^{\alpha\beta} \frac{V^\mu_X}{n_X} + \frac{1}{n_X} \tilde{q}_{XX} \tilde{n}_X^{\alpha\beta} \frac{V^\mu_X}{n_X} \frac{V^\alpha_X}{n_X} \frac{V^\beta_X}{n_X} \Lambda_{\mu\nu} \tilde{n}_X^{\alpha\beta} \Lambda_{\mu\nu} \tilde{n}_X^{\alpha\beta}, \]

\[ \frac{1}{n_X} \tilde{q}_{XX} \tilde{n}_X^{\alpha\beta} \tilde{n}_X^{\alpha\beta} \frac{V^\mu_X}{n_X} + \frac{1}{n_X} \tilde{q}_{XX} \tilde{n}_X^{\alpha\beta} \tilde{n}_X^{\alpha\beta} \frac{V^\mu_X}{n_X} \frac{V^\alpha_X}{n_X} \frac{V^\beta_X}{n_X} \Lambda_{\mu\nu} \tilde{n}_X^{\alpha\beta} \Lambda_{\mu\nu} \tilde{n}_X^{\alpha\beta}, \]

Using those relations, we can easily translate between the individual component frames and the joint single fluid frame. This turns out to be particularly useful when relating individual component fluid dissipation (e.g. ion-ion collisions) to the joint single fluid frame.