BIRKHOFF POLYTOPES OF DIFFERENT TYPE AND THE ORTHANT-LATTICE PROPERTY

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ABSTRACT. The Birkhoff polytope, defined to be the convex hull of $n \times n$ permutation matrices, is a well studied polytope in the context of the Ehrhart theory. This polytope is known to have many desirable properties, such as the Gorenstein property and existence of regular, unimodular triangulations. In this paper, we study analogues of the Birkhoff polytope for finite irreducible Coxeter groups of other types. We focus on a type-$B$ Birkhoff polytope $BB(n)$ arising from signed permutation matrices and prove that it and its dual polytope are reflexive, and hence Gorenstein, and also possess regular, unimodular triangulations. Noting that our triangulation proofs do not rely on the combinatorial structure of $BB(n)$, we define the notion of an orthant-lattice property polytope and use this to prove more general results for the existence of regular, unimodular triangulations and unimodular covers for a significant family of reflexive polytopes. We conclude by remarking on some connections to Gale-duality, Birkhoff polytopes of other types, and possible applications of orthant-lattice property.

1. INTRODUCTION

Let $P \subset \mathbb{R}^d$ be a $d$-dimensional, convex lattice polytope. Reflexive polytopes, that is, polytopes whose (polar) dual polyhedra are lattice polytopes, are a frequent topic of study in Ehrhart theory. Reflexive polytopes were originally introduced by Batyrev [Bat94] in the context of mirror symmetry, and roughly speaking, a pair of reflexive polytopes gives rise to a mirror-dual pair of Calabi–Yau manifolds (c.f. [Cox15]). As thus, the results of Batyrev, and the subsequent connection with string theory, have stimulated interest in the classification of reflexive polytopes both among combinatorics and theoretical physics communities. As a consequence of a famous result of Lagarias and Ziegler [LZ91], there are only finitely many reflexive polytopes in each dimension, up to unimodular equivalence. In two dimensions, it is a straightforward exercise to verify that there are precisely 16 reflexive polygons, as depicted in Figure 1. While still finite, there are significantly more reflexive polytopes in higher dimensions. Kreuzer and Skarke [KS98, KS00] have completely classified reflexive polytopes in dimensions 3 and 4, noting that there are exactly 4319 reflexive polytopes in dimension 3 and 47380776 reflexive polytopes in dimension 4. The number of reflexive polytopes in dimension 5 is not known.

Another topic of great interest are polytopes which arise from $S_n$, the symmetric group on $[n] := \{1, 2, \ldots, n\}$. The Birkhoff polytope is the convex hull of all $n \times n$ permutation matrices in $\mathbb{R}^{n \times n}$, or equivalently the set of all doubly stochastic matrices [Bir46, vN53]. This...
polytope has been studied quite extensively and is known to have many properties of interest (see, e.g., [Ath05, BR97, BP03, CM09, Dav15, DLLY09, Paf15]). Given that the symmetric group $\mathfrak{S}_n$ is equal to the type-$A$ the Coxeter group $A_{n-1}$, one could create similar constructions for finite irreducible Coxeter groups of other types.

The structure of this paper is as follows. In Section 2, we provide a brief review of common topics for polytopes and Ehrhart theory. Section 3 is devoted to studying a type-$B$ analogue of the Birkhoff polytope, as well as its dual polytopes, and proving many similar results to those known in the type-$A$ case such as the existence of a regular unimodular triangulation. In Section 4, we define the orthant-lattice property and prove the existence of regular, unimodular triangulations and unimodular covers in a large class of reflexive polytopes and thus generalizing the results shown in the previous section. We conclude in Section 5 with some connections to Gale duality, a discussion of Birkhoff polytopes in other types, and some future directions.

2. LATTICE POLYTOPES AND EHHRART THEORY

In this section, we provide a brief introduction to Ehrhart theory and lattice polytopes. For additional background and details, we refer the reader to the excellent books [BR15, Zie95]. A lattice polytope $P \subseteq \mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{Z}^d$. That is,

$$P = \text{conv} \{v_1, v_2, \ldots, v_r : v_i \in \mathbb{Z}^d \} := \left\{ \sum_{i=1}^{r} \lambda_i v_i : \lambda_i \geq 0, \sum_{i=1}^{r} \lambda_i = 1 \right\}.$$

The inclusion-minimal set $V := V(P)$ such that $P = \text{conv}(V)$ is called the vertex set of $P$ and its elements are called the vertices of $P$. This description of $P$ is called the vertex description or V-description. The polytope $P$ can also be expressed as the solution set of $A x \leq b$ for some integral matrix $A$ and some integral vector $b$. This description of $P$ is called the half-space description or H-description.
The dimension of $P$ is defined to be the dimension of its affine span. A $d$-dimensional polytope has at least $d + 1$ vertices and a $d$-polytope with exactly $d + 1$ many vertices is called a $d$-simplex. A $d$-simplex $\Delta = \text{conv}\{v_0, v_1, \ldots, v_d\}$ is called unimodular if $v_1 - v_0,$ $v_2 - v_0, \ldots, v_d - v_0$ generate the lattice $\mathbb{Z}^d$, or equivalently if $\text{vol}(\Delta) = \frac{1}{d!}$, where $\text{vol}(P)$ is the Euclidean volume of a polytope $P$.

**Definition 2.1.** We say that two polytopes $P$ and $P'$ are unimodularly equivalent if

$$P' = f_U(P) + v,$$

where $f_U$ is the linear transformation defined by $U \in \text{GL}_d(\mathbb{Z})$ and $v \in \mathbb{Z}^d$.

Given a lattice $d$-polytope $P$ and $t \in \mathbb{Z}_{\geq 1}$, let $tP := \{t \cdot \alpha : \alpha \in P\}$ be the $t$th dilate of $P$. The lattice-point enumeration function for $P$ is called the Ehrhart polynomial $ehr_P$ and it is defined as

$$ehr_P(t) := \#(tP \cap \mathbb{Z}^d).$$

By a famous result of Ehrhart [Ehr62, Thm. 1], this function agrees with a polynomial in the variable $t$ of degree $d$ with leading coefficient $\text{vol}(P)$.

This also implies that the formal generating function

$$\text{Ehr}_P(z) := 1 + \sum_{t \geq 1} ehr_P(t) z^t = \frac{h_0^* + h_1^* z + \cdots + h_d^* z^d}{(1 - z)^{d+1}}$$

is a rational function with denominator $(1 - z)^{d+1}$ and that the degree of the numerator is at most $d$ (see, e.g., [BR15, Lem 3.9] [Ehr62, Thm. 1]). We call $\text{Ehr}_P$ the Ehrhart series of $P$ and the numerator is called the $h^*$-polynomial of $P$. The coefficient vector $(h_0^*, h_1^*, \ldots, h_d^*)$ is called the $h^*$-vector of $P$. If we want to emphasize the corresponding polytope $P$, we will write $h^*_P$. One should note that both the Ehrhart polynomial and series are invariant under unimodular transformations. Though it is equivalent, it often more convenient to study the Ehrhart series of a polytope instead of the Ehrhart polynomial, in large part due to the following result of Richard Stanley:

**Theorem 2.2** ([Sta80, Thm. 2.1]). *Suppose $P$ is as above. Then

$$\text{Ehr}_P(z) = \frac{h_0^* + h_1^* z + \cdots + h_d^* z^d}{(1 - z)^{d+1}}$$

and the coefficients $h_i^*$ are nonnegative integers.*

This $h^*$-vector encodes a lot of information about the underlying polytope. This is nicely illustrated in the case of reflexive polytopes:

**Definition 2.3.** Let $P \subset \mathbb{R}^d$ be a $d$-dimensional lattice polytope that contains the origin in its interior. We say that $P$ is reflexive if it has half-space description

$$P = \{x \in \mathbb{R}^d : A x \leq 1\},$$

where $A$ is an integral matrix. Equivalently, $P$ is reflexive if its (polar) dual polytope

$$P^* := \{y \in \mathbb{R}^d : \langle x, y \rangle \leq 1 \text{ for all } x \in P\}$$

is a lattice polytope.

Quite surprisingly, reflexivity can be completely characterized by enumerative data of the $h^*$-vector.
Theorem 2.4 ([Hib92, Thm. 2.1]). Let $P$ be a $d$-dimensional lattice polytope with Ehrhart series

$$\text{Ehr}_P(z) = \frac{h^*_0 + h^*_1 z + \cdots + h^*_d z^d}{(1 - z)^{d+1}}.$$ 

Then $P$ is unimodularly equivalent to a reflexive polytope if and only if $h^*_k = h^*_{d-k}$ for all $0 \leq k \leq \frac{d}{2}$.

The reflexivity property is also deeply related to commutative algebra. A polytope $P$ is reflexive if the canonical module of an associated graded algebra $k[P]$ is (up to a shift in grading) isomorphic to $k[P]$ and its minimal generator has degree 1. If one allows the unique minimal generator to have arbitrary degree, one arrives at the notion of Gorenstein rings, for details we refer to [BG09, Sec 6.C]. We say that $P$ is Gorenstein if there exists a $c \in \mathbb{Z}_{\geq 1}$ such that $cP$ is unimodularly equivalent to a reflexive polytope. This is equivalent to saying that $k[P]$ is Gorenstein. The dilation factor $c$ is often called the codegree.

In particular, reflexive polytopes are Gorenstein of codegree 1. By combining results of Stanley [Sta78] and De Negri–Hibi [DNH97], we have a characterization of the Gorenstein property in terms of the $h^*$-vector. Namely, $P$ is Gorenstein if and only if $h^*_i = h^*_{d-c+1-i}$ for all $i$.

Aside from examining algebraic properties of lattice polytopes, one can also investigate discrete geometric properties. Every lattice polytope admits a subdivision into lattice simplices. Even more, one can guarantee that every lattice point contained in a polytope corresponds to a vertex of such a subdivision. However, one cannot guarantee the existence of a subdivision where all simplices are unimodular when the dimension is greater than 2. This leads us to our next definition:

Definition 2.5. A triangulation of a lattice polytope $P$ is a subdivision into lattice simplices such that the intersection of any two simplices is a (possibly empty) face of both. A unimodular triangulation is a triangulation into simplices, where every full-dimensional simplex is unimodular. A triangulation is regular if there is a convex function $P \to \mathbb{R}$ whose domains of linearity are exactly the maximal simplices in the triangulation.

Given a lattice polytope $P$, a pulling triangulation is a triangulation obtained by a sequence of pulling refinements. Given $v \in P \cap \mathbb{Z}^d$ and a lattice subdivision $S$, pull$_v P$ is the refined lattice subdivision induced by replacing every face $F \in S$ such that $v \in F$ with the pyramids $\text{conv}(v, F')$, for each face $F'$ of $F$ that does not contain $v$. Such refinements preserve regularity and thus a triangulation constructed by a sequence of pulling refinements is a regular triangulation. The reader should consult [DLRS10, HPPS14] for more details.

A special class of polytopes which possess regular, unimodular triangulations are compressed polytopes. A polytope $P$ is compressed if every pulling triangulation is unimodular [Sta80]. In the interest of providing a useful characterization of compressed polytopes, we must define the notion of width. The (lattice) width of a polytope $P$ with respect to a linear functional $\ell \in (\mathbb{R}^d)^*$ is

$$\text{width}_\ell(P) := \max_{p, q \in P} |\ell(p) - \ell(q)|.$$ 

The following theorem attributed independently to Ohsugi–Hibi and Sullivant gives additional equivalent definitions.
Theorem 2.6 ([OH01, Thm. 1.1] [Sul06, Thm. 2.4]). Let \( P \) be a lattice polytope. The following are equivalent:

1. \( P \) is compressed;
2. \( P \) has width one with respect to all its facet-defining linear functionals;
3. \( P \) is unimodularly equivalent to the intersection of a unit cube with an affine space.

Definition 2.7. A lattice polytope \( P \) has the integer decomposition property if for any positive integer \( t \) and for all \( x \in tP \cap \mathbb{Z}^d \), there exists \( v_1, \ldots, v_t \in P \cap \mathbb{Z}^d \) such that \( x = v_1 + \cdots + v_t \).

For brevity, we will say that such a \( P \) has the IDP.

One should note that if \( P \) has a (regular, unimodular) triangulation, then \( P \) has the IDP. However, there are examples of polytopes which have the IDP, yet do not even admit a unimodular cover, that is, a covering of \( P \) by unimodular simplices, see [BG99, Sec. 3]. A more complete hierarchy of covering properties can be found in [HPPS14].

We say that \( h^*_P \) is unimodal if there exists a \( k \) such that \( h^*_0 \leq h^*_1 \leq \cdots \leq h^*_k \geq \cdots \geq h^*_{d-1} \geq h^*_d \). Unimodality appears frequently in combinatorial settings and it often hints at a deeper underlying algebraic structure, see [AHK18, Bre94, Sta89]. One famous instance is given by Gorenstein polytopes that admit a regular unimodular triangulation.

Theorem 2.8 ([BR07, Thm. 1]). If \( P \) is Gorenstein and has a regular, unimodular triangulation, then \( h^*_P \) is unimodal.

The following conjecture is commonly attributed to Ohsugi and Hibi [OH06]:

Conjecture 2.9. If \( P \) is Gorenstein and has the IDP, then \( h^*_P \) is unimodal.

3. THE TYPE-B BIRKHOFF POLYTOPE

Recall that the Birkhoff polytope \( B(n) \) is defined as the convex hull of all \( n \times n \) permutation matrices or equivalently as the set of all doubly stochastic matrices, that is, matrices \( M \) with row and column sums equal to 1 and entries \( 0 \leq M_{i,j} \leq 1 \), by work of Birkhoff [Bir46] and von Neumann [vN53] independently. This polytope is known to have many nice properties; it is known to be Gorenstein, to be compressed [Sta80], and to be \( h^* \)-unimodal [Ath05]. In this section, we will introduce a type-B analogue of this polytope corresponding to signed permutation matrices and verify many similar properties already known for \( B(n) \).

3.1. The type-B-Birkhoff polytope and its dual. The hyperoctahedral group is defined to by \( B_n := \mathbb{Z}/2\mathbb{Z} \wr S_n \), which is the Coxeter group of type-B (or type-C). Elements of this group can be thought of as permutations from \( S_n \) expressed in one-line notation \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \), where we also associate a sign \( \text{sgn}(\sigma_i) \) to each \( \sigma_i \). To each signed permutation \( \sigma \in B_n \), we associate a matrix \( M_\sigma \) defined as \( (M_\sigma)_{i,\sigma_i} = \text{sgn}(\sigma_i) \) and \( (M_\sigma)_{i,j} = 0 \) otherwise. If every entry of \( \sigma \) is positive, then \( M_\sigma \) is simply a permutation matrix. This leads to the following definition:

Definition 3.1. The type-B Birkhoff polytope (or signed Birkhoff polytope) is

\[
BB(n) := \text{conv} \{ M_\sigma : \sigma \in B_n \} \subset \mathbb{R}^{n \times n}.
\]

That is, \( BB(n) \) is the convex hull of all \( n \times n \) signed permutation matrices.
This polytope was previously studied in [MOSZ02], though the emphasis was not on Ehrhart-theoretic questions. Let us first show that every signed permutation matrix is indeed a vertex.

**Proposition 3.2.** For every $\sigma \in B_n$, $M_{\sigma}$ is a vertex of $BB(n)$.

**Proof.** It is sufficient to show that for any $\sigma \in B_n$, we can find a hyperplane which separates $M_{\sigma}$ from $M_\pi$ for all $\pi \in B_n$, $\pi \neq \sigma$. Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in B_n$ and consider the hyperplane

$$\text{sgn}(\sigma_1)x_{1,\sigma_1} + \text{sgn}(\sigma_2)x_{2,\sigma_2} + \cdots + \text{sgn}(\sigma_n)x_{n,\sigma_n} = n.$$ 

By construction, evaluating $\sum_{i=1}^n \text{sgn}(\sigma_i)x_{i,\sigma_i}$ at $M_\sigma$ yields $n$ and evaluating at $M_\pi$ yields an integer strictly less than $n$ whenever $\pi \neq \sigma$. Since $\sigma$ is arbitrary, we must have that each $M_\sigma$ is a vertex. \qed

The $H$-description of this polytope is known due to work of McCarthy et al. [MOSZ02].

**Theorem 3.3 ([MOSZ02, Thm. 4.6]).** The $H$-description of $BB(n)$ is given by the intersection of the following $2^{n+1}n$ half-spaces:

$$\begin{aligned}
&\sum_{j=1}^n a_{i,j}x_{i,j} \leq 1, \quad \text{for each } 1 \leq i \leq n, \text{ and} \\
&\sum_{i=1}^n b_{i,j}x_{i,j} \leq 1, \quad \text{for each } 1 \leq j \leq n,
\end{aligned}$$

where $a_{i,j}, b_{i,j} \in \{\pm 1\}$.

The proof in [MOSZ02] uses the notion of Birkhoff tensors and other techniques common in Coxeter theory. One can also prove this statement using elementary techniques, but for brevity we will merely state the strategy. First, show that all vertices satisfy the given inequalities and then for each inequality find $n^2$ affinely independent vertices that satisfy the inequality with equality.

**Remark 3.4.** The matrices described in Theorem 3.3 are sometimes called doubly bistochastic matrices (c.f. [MOSZ02]). Hence this theorem is a type-B version of the well-known Birkhoff–von Neumann Theorem [Bir46, vN53].

The half-space description of $BB(n)$ directly implies that $BB(n)$ is reflexive. We record this in the following corollary:

**Corollary 3.5.** $BB(n)$ is reflexive.

Since $BB(n)$ is a reflexive polytope, it is natural to ask what its dual polytope is. Luckily, the $H$-representation of $BB(n)$ gives us the vertex description of the dual polytope.
Definition 3.6. Let $n \in \mathbb{Z}_{\geq 1}$. We define the $n$th dual type-$B$ polytope $\mathcal{C}(n)$ as the convex hull of the following matrices

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 \\
\end{pmatrix}
\cup
\begin{pmatrix}
a_{i1} & a_{i2} & \ldots & a_{in} \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 \\
\end{pmatrix} : i \in [n]
\cup
\begin{pmatrix}
0 & \ldots & 0 & a_{1j} & 0 & \ldots & 0 \\
0 & \ldots & 0 & a_{1j} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & a_{1j} & 0 & \ldots & 0 \\
\end{pmatrix} : j \in [n],
\]

where $a_{ij} \in \{\pm 1\}$.

Corollary 3.7. $\mathcal{C}(n) = BB(n)^*$.

3.2. Triangulations of the type-$B$ Birkhoff polytope. In this subsection, we will prove that $BB(n)$ admits a regular, unimodular triangulation. We will construct such a triangulation as follows:

1. Restrict $BB(n)$ to the positive orthant.
2. Verify that the subpolytope obtained by this restriction is a compressed lattice polytope.
3. Observing that, by symmetry, the subpolytope obtained by restricting to any orthant is compressed, we may obtain a regular, unimodular triangulation of $BB(n)$ by a composition of pulling refinements.

Remark 3.8. These steps will not rely on the combinatorial structure of $BB(n)$. They apply to a broader family of reflexive polytopes and we will record these more general results in Section 4.

The restriction of $BB(n)$ to the positive orthant will reappear frequently, so it deserves a special name:

Definition 3.9. Let $n \in \mathbb{Z}_{\geq 1}$. We define the positive type-$B$ Birkhoff polytope, $BB_+(n)$, to be the polytope

\[
BB_+(n) := BB(n) \cap \{x_{ij} \geq 0 : 1 \leq i, j \leq n\}.
\]

We will first give the half-space description, which will then directly imply that $BB_+(n)$ is a compressed lattice polytope.

Lemma 3.10. Let $n \in \mathbb{Z}_{\geq 1}$. Then

\[
BB_+(n) = \{x \in \mathbb{R}^{n \times n} : x \geq 0, Ax \leq 1\},
\]

where

\[
A = \begin{pmatrix}
1 & \ldots & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 1 & \ldots & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
1 & \ldots & 0 & 1 & \ldots & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 1 & 0 & \ldots & 1 & 0 & \ldots & 1 \\
\end{pmatrix} \subset \mathbb{R}^{2n \times n^2},
\]

i.e., $A$ encodes the inequalities coming from the row and column sums.
Proof. We will prove the claim in two steps:

1. The inequalities $x_{i,j} \geq 0$ indeed give rise to facets.
2. If an inequality of the form $\sum_i a_{i,j} x_{i,j} \leq 1$ or of the form $\sum_j b_{i,j} x_{i,j} \leq 1$ for $a_{i,j}, b_{i,j} \in \{\pm 1\}$ has at least one negative sign, then the points on the corresponding hyperplane are also in a hyperplane of the form $x_{ij} \geq 0$.

We prove claim (1) by giving $n^2$ affinely independent points on this hyperplane. Let $i, j$ be given and let

$$e^{(k,l)}_{r,s} := \begin{cases} 1, & \text{if } (r, s) = (k, l) \\ 0, & \text{otherwise.} \end{cases}$$

Then the points 0 and $e^{(k,l)}$ for $(k, l) \neq (i, j)$ are affinely independent and they lie on the hyperplane $x_{i,j} = 0$.

To prove claim (2), we will show that if $\sum_j a_{i,j} x_{i,j} = 1$ for any row $j$ has a minus at any spot $k$ this forces that $c_{k,j} = 0$ for all $c \in BB(n) \cap \{\sum_i a_{i,j} x_{i,j} = 1\}$. Without loss of generality, assume that this minus sign appears for $k = 1$, i.e., we have

$$-x_{1,j} + a_{2,j} x_{2,j} + \cdots + a_{n,j} x_{n,j} = 1.$$ 

Let $c \in BB(n)$ be a point on this hyperplane. Since all coordinates of $c$ need to be nonnegative, we have that $c_{1,j} \geq 0$. However, if $c_{1,j} > 0$, then the inequality

$$+x_{1,j} + a_{2,j} x_{2,j} + \cdots + a_{n,j} x_{n,j} \leq 1$$

is violated. Hence we have that $c_{1,j} = 0$ and thus we are on the hyperplane given by $x_{1,j} = 0$. The case for hyperplane of the form $\sum_j a_{i,j} x_{i,j} = 1$ follows similarly. \hfill $\square$

The following result is likely well-known. However, we will provide a proof for completeness.

**Proposition 3.11.** Let $A$ be as above. Then $A$ is a totally unimodular matrix, i.e., all possible subminors are in $\{0, \pm 1\}$. Furthermore, appending the rows from the inequalities $x_{ij} \geq 0$ does not change total unimodularity.

In order to prove that $A$ is a totally unimodular matrix, we need the following result:

**Theorem 3.12** ([Sch86, Ex. 1 in Sec 19.3]). The incidence matrix of a graph $G$ is totally unimodular if and only if $G$ is bipartite.

Proof of Proposition 3.11. We will show that $A$ is the vertex-edge incidence matrix of $K_{n,n}$. Let us label the vertices of $K_{n,n}$ by $r_1, r_2, \ldots, r_n, c_1, c_2, \ldots, c_n$ such that there are no edges $r_i r_j$ and $c_i c_j$. Therefore, for all $r_i$ and $c_j$, there is an edge $r_i c_j$ which we will denote $(r_i, c_j)$. We label these edges lexicographically so that $(r_i, c_j) < (r_k, c_l)$ means either $i < k$ or if $i = k$ then $j < l$. With this labeling it is immediate that $A$ equals the incidence matrix of $K_{n,n}$. The claim now follows from Theorem 3.12. The last fact follows from Laplace expansion of determinants. \hfill $\square$

This result has several immediate nice consequences, as it for instance shows that $BB_+(n)$ is a lattice polytope. Before we give the vertex description, we need one more definition.

**Definition 3.13.** A **partial permutation matrix** is a matrix $M \in \{0, 1\}^{n \times n}$ such that $M$ has at most one 1 per row and per column.
Proposition 3.14. Let $n \in \mathbb{Z}_{\geq 1}$. Then
\[
BB_+(n) = \text{conv}\{M \in \mathbb{R}^{n \times n} : M \text{ is a partial } n \times n \text{ permutation matrix}\}.
\]
In particular, $BB_+(n)$ is a lattice polytope.

Proof. The fact that any facet unimodular polytope is a lattice polytope is well-known, see for instance [HPPS14, Sch86]. Moreover, the structure of $A$ shows that $BB_+(n)$ has to be a $0/1$ polytope and thus every integer point has to be a vertex, and vice versa. However, the inequalities description gives $M \in BB_+(n) \cap \mathbb{Z}^{n \times n}$ if and only if $M$ is a partial permutation matrix. \hfill \Box

Knowing that $BB_+(n)$ is lattice, as well as its half-space description, enables us to deduce that $BB_+(n)$ is Gorenstein.

Proposition 3.15. Let $n \in \mathbb{Z}_{\geq 1}$. Then $BB_+(n)$ is Gorenstein.

Proof. The point
\[
1_{n \times n} := \begin{pmatrix}
1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \ldots & 1 
\end{pmatrix}
\]
has lattice distance 1 to all facets of the $(n + 1)BB_+(n)$. Hence, by [BG09, Thm. 6.33] $BB_+(n)$ is a Gorenstein polytope of codegree $n + 1$. \hfill \Box

Proposition 3.16. $BB_+(n)$ is compressed.

Proof. We will show that every vertex that does not lie on a given facet has lattice distance one to it. By Proposition 3.14, the vertices of $BB_+(n)$ are given by the partial permutation matrices. By Lemma 3.10, the half-space description is determined by the row and column sums being one and by the entries all being nonnegative. It directly follows that all vertices have lattice distance at most one from the facet-defining inequalities. \hfill \Box

By definition, if we restrict the polytope $BB(n)$ to any orthant, we get a reflection of $BB_+(n)$, which by the previous result all are compressed. These triangulations glue together nicely:

Theorem 3.17. For every $n \in \mathbb{Z}_{\geq 1}$, the polytope $BB(n)$ has a regular, unimodular triangulation.

Proof. Let us order the orthants $O_1, O_2, \ldots, O_{2^n}$. Now pick a pulling triangulation first pulling all integer points in $O_1$, then pull all integer points in $O_2 \setminus O_1$, etc. Since all orthant pieces are compressed, this results in a regular, unimodular triangulation of $BB(n)$. \hfill \Box

Remark 3.18. It is worth noting an alternative way of viewing this proof. The hyperplane arrangement defined by $x_i = 0$ for all $i$ induces a lattice subdivision of $BB(n)$, where each cell is unimodularly equivalent to $BB_+(n)$. Since each cell is compressed, any pulling refinement to a triangulation is a regular unimodular triangulation.

This geometric result has immediate consequences for the Ehrhart series. Namely, the combinatorics of a unimodular triangulation fully determines the $h^*$-vector of the polytope, see for instance [BR15, Thm. 10.3]. Moreover, since $BB_+(n)$ is Gorenstein, we have that its $h^*$ vector is unimodal by Theorem 2.8. We record this in the following proposition:

Proposition 3.19. Let $n \in \mathbb{Z}_{\geq 1}$. Then $h^*_{BB_+(n)}$ is unimodal.
As before, Theorem 2.8 implies that the corresponding $h^*$-vector is unimodal.

**Proposition 3.20.** Let $n \in \mathbb{Z}_{\geq 1}$. Then $h^*_{BB(n)}$ is unimodal.

We conclude our discussion of $BB(n)$ and $BB^+(n)$ by considering some enumerative data as displayed in Table 1 and Table 2. Given the dimension and volumes of these polytopes, our computational resources are quite quickly exhausted. That being said, computations for $BB^+(n)$, which are less complex, allow us to compute the normalized volume for larger $BB(n)$.

### 3.3. Triangulations of the type-$B$ dual polytope.

We now turn our attention to the dual type-$B$ Birkhoff polytope $C(n)$. We conspire to prove that $C(n)$, much like $BB(n)$, has a regular, unimodular triangulation. To do so, we will use a similar strategy. Observe that if we subdivide $C(n)$ by the coordinate axes so that each cell of the subdivision in $C(n)$ intersected with the appropriately chosen orthant, which is a regular subdivision, all cells of the subdivision are unimodularly equivalent. Thus, we will first consider only the positive orthant.

**Definition 3.21.** Let $n \in \mathbb{Z}_{\geq 1}$. We define the positive dual type-$B$ Birkhoff polytope, $C^+(n)$, to be the polytope

$$C^+(n) := C(n) \cap \{ x_{ij} \geq 0 : 1 \leq i, j \leq n \}.$$

**Lemma 3.22.** Let $n \in \mathbb{Z}_{\geq 1}$. Then

$$C^+(n) = \{ x \in \mathbb{R}^{n \times n} : x \geq 0, B x \leq 1 \}$$

where $B \in \mathbb{R}^{n! \times n^2}$ such that

$$b_{\sigma(i,j)} = \begin{cases} 1 & \text{if } \sigma(i) = j, \\ 0 & \text{otherwise.} \end{cases}$$

That is, think of rows of $B$ as arising from permutation matrices written as a single row vector.
| $n$ | $\text{Vol}(\mathcal{C}_+(n))$ | $h^*_\mathcal{C}_+(n)$ |
|-----|-------------------|-----------------|
| 1   | 1                 | 1               |
| 2   | 6                 | (1, 4, 1)       |
| 3   | 642               | (1, 24, 156, 280, 156, 24, 1) |
| 4   | 2389248           | (1, 88, 2656, 34568, 201215, 562112, 787968, 562112, 201215, 34568, 2656, 88, 1) |

Table 3. $\mathcal{C}_+(n)$.

**Proof.** Analogous proof of that for $BB(n)_+$.

A brief computation gives the next result:

**Proposition 3.23.** Let $n \in \mathbb{Z}_{\geq 1}$. Then

$$\mathcal{C}_+(n) = \text{conv}\{M \in \mathbb{R}^{n \times n} : M \text{ is an } n \times n \text{ partial row or column 0/1 matrix}\},$$

i.e., 0/1 matrices where at most one row (column resp.) has non-zero entries.

**Proposition 3.24.** $\mathcal{C}_+(n)$ is Gorenstein.

**Proof.** We will show that the dilation $(n + 1)\mathcal{C}_+(n)$ has an integer point with distance 1 to all of its facets. The hyperplane description of this dilation is given by

$$\mathcal{C}_+(n) = \{x \in \mathbb{R}^{n \times n} : x \geq 0, Bx \leq (n + 1)1\}$$

where $B \in \mathbb{R}^{n! \times n^2}$ such that

$$b_{\sigma(i,j)} = \begin{cases} 1 & \text{if } \sigma(i) = j, \\ 0 & \text{otherwise}. \end{cases}$$

This directly follows from the observation that the all 1’s matrix $1_{n \times n}$ has facet distance one to all facets which implies the claim by [BG09, Thm. 6.33].

**Theorem 3.25.** Let $n \in \mathbb{Z}_{\geq 1}$. Then $\mathcal{C}(n)$ has a regular, unimodular triangulation.

We should note that we can construct a similar proof to Theorem 3.17 after first proving that $\mathcal{C}_+(n)$ is compressed. However, we will omit such arguments for the time being as this result, as well as Theorem 3.17, are immediate corollaries of Theorem 4.4 in the coming section.

These corollaries follows from Theorem 2.8 and Theorem 3.25.

**Corollary 3.26.** Let $n \in \mathbb{Z}_{\geq 1}$. Then $h^*_{\mathcal{C}(n)}$ is unimodal.

**Corollary 3.27.** Let $n \in \mathbb{Z}_{\geq 1}$. Then $h^*_{\mathcal{C}_+(n)}$ is unimodal.

**Remark 3.28.** As noted in Table 2 and Table 4, $BB(3)$ and $\mathcal{C}(3)$ have precisely the same Ehrhart data and normalized volume. This is a consequence of a much stronger coincidence which occurs when $n = 3$, specifically that $BB(3) \cong \mathcal{C}(3)$ which is a straightforward computation.
In the previous section, we did not use any of the underlying combinatorial structure. Instead, we used the fact that the restriction to any given orthant was compressed. In this section, we will continue this line of thought and derive some basic conditions that exactly guarantee that these orthant restrictions are all compressed. We then mention some well-known polytopes that have this nice structure, as well as some computational results indicating how rare/common these properties are among reflexive polytopes.

**Definition 4.1.** Let \( x \subset \mathbb{R}^d \) be a given vector. We define the sign vector \( \sigma(x) \subset \{\pm 1\}^d \) by setting

\[
\sigma_i(x) = \begin{cases} 
1 & \text{if } x_i > 0, \\
-1 & \text{if } x_i < 0, \\
0 & \text{otherwise.}
\end{cases}
\]

The sign vector \( \sigma(O) \) of an orthant \( O \) is defined to be the sign vector \( \sigma(x) \) for any \( x \in O \setminus \bigcup_i \{x_i = 0\} \).

**Definition 4.2.** We say that \( P \) has the orthant-lattice property if the restriction \( P|_O \) to any orthant \( O \) is either empty or a lattice polytope. For brevity, we will say that \( P \) satisfying this is an OLP polytope.

**Remark 4.3.** We should note that if \( P \) is a reflexive OLP polytope, then \( P|_O \) is a full dimensional lattice polytope for all \( O \).

**Theorem 4.4.** Let \( P = \{x : Ax \leq 1\} \subset \mathbb{R}^d \) be a reflexive d-polytope with the origin in its interior. Furthermore, let \( P \) satisfy the following conditions:

1. \( P \) is an OLP polytope.
2. \( P \) is a 0/1/-1 polytope.
3. For any facet-defining inequality \( a \) of \( P|_O \), we have that \( a_i \sigma_i(x) \geq 0 \) for all \( i \) and for any \( x \in O \setminus \bigcup_i \{x_i = 0\} \).

Then \( P \) has a regular, unimodular triangulation.

**Proof.** First, note that if \( P \) is reflexive and OLP, then \( \pm e_i \in P \cap \mathbb{Z}^d \) for all \( 1 \leq i \leq d \). Particularly, these are lattice points on the boundary of \( P \) and hence optimize some facet-defining inequality of \( P \). This implies that \( A_{i,j} \in \{0, \pm 1\} \) for all \( i, j \). This observation in combination with condition (3) implies that each \( P|_O \) has lattice width one with respect to all facet-defining linear functionals. Thus by Theorem 2.6 \( P|_O \) is compressed for each orthant. Given that the subdivision induced by the hyperplanes \( x_i = 0 \) for \( 1 \leq i \leq d \).
\[ \begin{align*}
\text{Corollary 4.5.} & \quad \text{If } P \subset \mathbb{R}^d \text{ is as in Theorem 4.4, then } h^*_P \text{ is unimodal.} \\
\text{Remark 4.6.} & \quad \text{Regarding the applicability of Theorem 4.4:} \\
& \quad \text{(1) There are several well-known polytopes that fall within this category. For instance,} \\
& \quad \text{the hypercube } [-1,1]^d, \text{ its dual the cross polytope } \text{conv}\{ \pm e_i : i \in [d] \}, \text{ the type-B} \\
& \quad \text{Birkhoff polytope } BB(n) \text{ and its dual } C(n), \text{ and the Legendre polytope or full type-A} \\
& \quad \text{root polytope } \frac{1}{2} P_d, \text{ which is defined the convex hull of } e_i - e_j \text{ and } e_j - e_i \text{ for all} \\
& \quad 1 \leq i < j \leq d \text{ in } \mathbb{R}^d. \\
& \quad \text{(2) Using Normaliz } [\text{BIR}^+] \text{ and the Kreuzer–Skarke database for reflexive polytopes} \\
& \quad [\text{KS98, KS00}], \text{ we were able to verify that these conditions are satisfied by 72 3-} \\
& \quad \text{dimensional reflexive polytopes and by at least 407 of the 4-dimensional reflexive} \\
& \quad \text{polytopes with at most 12 vertices. Unfortunately, our computational resources} \\
& \quad \text{were too limited to test most of the 4-dimensional polytopes.} \\
\end{align*} \\
\text{While Theorem 4.4 is perhaps more applicable than suspected as exhibited in Remark} \\
\text{4.6, it is still rather restrictive. In the interest of applicability to a larger family of poly-} \\
topes, \text{we give the following weaker result.} \\
\text{Theorem 4.7. Let } P = \{ x : A x \leq 1 \} \text{ be a 0/1/~1 reflexive polytope with the origin} \\
in its interior. \text{Suppose that } P \text{ is an OLP polytope and hence } A_{i,j} \in \{ 0, \pm 1 \} \text{ for all } i, j. \text{ For any orthant } O, \text{ let} \\
A^O \text{ be the relevant inequalities. The polytope } P \text{ has a unimodular cover and hence the IDP if for} \\
each orthant } O, \text{ at least one of the following conditions holds:} \\
\begin{align*}
& (1) \text{ For any facet-defining inequality } a \text{ of } P|_O, \text{ we have that } a_i \sigma_i(x) \geq 0 \text{ for all } i \text{ and for any} \\
& \quad x \in O \setminus \bigcup_i \{ x_i = 0 \}; \\
& (2) \quad A^O \text{ is a totally unimodular matrix;} \\
& (3) \quad A^O \text{ consists of of rows which are } B_d \text{ roots;} \\
& (4) \quad P|_O \text{ is the product of unimodular simplices;} \\
& (5) \quad \text{There exists a projection } \pi : \mathbb{R}^d \to \mathbb{R}^{d-1} \text{ such that } \pi(P|_O) \text{ has a regular,} \\
& \quad \text{unimodular triangulation } T \text{ which } \pi^*(T). \\
\end{align*} \\
\text{Proof.} \text{To show that } P \text{ exhibits a unimodular cover and hence the IDP, it is sufficient to} \\
\text{verify that each of these conditions ensures that the ortrangulation } P|_O \text{ has a regular,} \\
\text{unimodular triangulation. Condition (1), ensures that } P|_O \text{ is compressed by the argument} \\
given for Theorem 4.4. \text{ Condition (2) follows from } [\text{HPPS14, Thm. 2.4}]. \text{ Condition (3) gives the existence of regular,} \\
\text{unimodular triangulation by } [\text{HPPS14, Prop. 3.4}]. \text{ Condition} \\
(4) \text{ follows from } [\text{DLRS10, Sec. 6.2.3}]. \text{ Condition (5) follows from } [\text{HPPS14, Thm. 2.8}]. \\
\text{Remark 4.8.} \text{ Note that, since many of the known triangulations are not constructive, we} \\
can not ensure that the resulting unimodular cover given by Theorem 4.7 is actually a} \\
\text{triangulation. Subsequently, such polytopes may be a fruitful location in searching for a} \\
counterexample to Conjecture 2.9.}
We say that a polytope $P \subset \mathbb{R}^d$ is $B_d$-equivariant if $P = \sigma P$ for any $\sigma \in B_d$. In this special case, we can prove a stronger result.

**Theorem 4.9.** Let $P = \{x : Ax \leq 1\}$ be a $0/1$ or $-1$ $B_d$-equivariant reflexive polytope with the origin in its interior such that $P$ is an OLP polytope and hence $A_{i,j} \in \{0, \pm 1\}$ for all $i, j$. Let $P_+$ be the intersection of $P$ with the positive orthant and let $A^+$ be the relevant inequalities for $P_+$. Then $P$ has a regular, unimodular triangulation if one of the following occurs:

1. $A^+$ is a $0/1$ matrix;
2. $A^+$ is a totally unimodular matrix;
3. $A^+$ consists of of rows which are $B_d$ roots;
4. $P_+$ is the product of unimodular simplices;
5. There exists a projection $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ such that $\pi(P_+)$ has a regular, unimodular triangulation $T$ which $\pi^*(T)$.

**Proof.** Note that given that $P$ is $B_d$-equivariant, $P|_O \simeq P_+$ for any orthant $O$. Similarly to Theorem 4.7, each condition guarantees the existent of a regular unimodular triangulation on $P_+$. To produce a triangulation of $P$, first produce an appropriate regular, unimodular triangulation for $P_+$, say $T_+$. Note that we can produce the analogous triangulation for $P|_O$ for any $O$, denoted $T_O$, such that $T_O = \tau \cdot T_+$ where $\tau$ is the appropriate reflections to map the the positive orthant to $O$. The unimodular cover this produces is in fact a triangulation as it by construction agrees on the boundaries of the orthant cells. Moreover, this is a composition of regular subdivisions and is hence regular. \hfill \Box

5. **Concluding remarks**

5.1. **A connection to Gale Duality.** Given two polytopes $P, Q \subset \mathbb{R}^d$, we say that these polytopes are a **Gale-dual** pair if

$$P = \mathbb{R}_{\geq 0}^d \cap \{x \in \mathbb{R}^d : \langle x, y \rangle = 1 \text{ for } y \in V(Q)\}$$

$$Q = \mathbb{R}_{\geq 0}^d \cap \{y \in \mathbb{R}^d : \langle x, y \rangle = 1 \text{ for } x \in V(P)\}.$$

A recent paper [FHSS], Fritsch, Heuer, Sanyal, and Schulz introduce the **Martin Gardner polytope**, $G_n$ which is defined to be convex hull of $n \times n$ matrices with a single row or column of all ones and zeros elsewhere. They remark that the Birkhoff polytope $B(n)$ and $G_n$ are a Gale-dual pair. In some sense, $C(n)$ is a signed **Martin Gardner polytope**. We note that the Gale-dual pair $B(n)$ and $G_n$ appear as faces of a pair of dual reflexive polytopes $BB(n)$ and $C(n)$ respectively. This phenomenon motivates the following question:

**Question 5.1.** Given two $0/1$ polytopes $P$ and $Q$ which are Gale duals, do there exist polytopes $P'$ and $Q'$ which are reflexive, contain $P$ and $Q$ as faces respectively, and $(P')^* \simeq Q'$?

5.2. **Birkhoff polytopes of other types.** It is only natural to look at Birkhoff-type polytopes of other finite irreducible Coxeter groups. Since the type-$B$ and the type-$C$ Coxeter groups are equal, we get the same polytope. Recall that the type-$D$ Coxeter group $D_n$ is the subgroup of $B_n$ with permutations with an even number of negatives. We can construct the **type-$D$ Birkhoff polytope**, $BD(n)$, to be the convex hull of signed permutation matrices with an even number of negative entries. As one may suspect from this construction, the omission of all lattice points in various orthants which occurs in $BD(n)$ ensures that it cannot be an OLP polytope and is thus not subject to any of our general theorems.
When $n = 2$ and $n = 3$, $BD(n)$ is a reflexive polytope, but $BD(3)$ does not have the IDP. Moreover, $BD(4)$ fails to be reflexive.

Additionally, one could consider Birkhoff constructions for Coxeter groups of exceptional type, in particular $E_6$, $E_7$, and $E_8$ (see, e.g., [BB05]). While we did not consider these polytopes in our investigation, we do raise the following question:

**Question 5.2.** Do the Birkhoff polytope constructions for $E_6$, $E_7$, and $E_8$ have the IDP? Are these polytopes reflexive? Do they have other interesting properties?

### 5.3. Future directions.

In addition to considering Birkhoff polytopes of other types and connections to Gale duality as discussed above, there are several (two?) immediate avenues for further research. First, one could consider additional triangulation questions for $BB(n)$ and $BB_+(n)$. Recall that a lattice triangulation $T$ is called **flag** if all of the minimal non-faces of $T$ are 1-dimensional. This leads to the following question:

**Question 5.3.** For what $n$ do $BB(n)$ or $BB_+(n)$ admit a flag, regular, unimodular triangulation?

This question is particularly of interest because the Birkhoff polytope does not always admit such a triangulation as demonstrated by $B(3)$ (see, e.g., [HPPS14, Sec. 3.3]). However, as noted with a Gröbner basis argument, $B(n)$ admits a regular, unimodular triangulation where all of the minimal non-faces are 1 or 2 dimensional [YOT14, Thm. 1]. Subsequently, studying the Gröbner bases for the defining toric ideals of $BB(n)$ or $BB_+(n)$ would also be a problem of interest to the community. Given that many properties of $B(n)$ are also exhibited by $BB(n)$, we state the following question:

**Question 5.4.** Is there a term order such that the Gröbner bases for the toric ideals of $BB(n)$ or $BB_+(n)$ are generated in degree 2 and degree 3?

Coxeter groups of great interest in the broader community of algebraic and geometric combinatorics (see, e.g., [BB05]). Subsequently, it is natural to consider how the Ehrhart-theoretic study of the type-$B$ Birkhoff polytope informs research area. This leads to the following question:

**Question 5.5.** Does the convex structure of $BB(n)$ encode combinatorial or group theoretic structure of interest in Coxeter combinatorics?

An additional future direction is the consider applications of the orthant-lattice property, particularly those of Theorem 4.4 and Theorem 4.7. One potentially fruitful avenue is application to reflexive smooth polytopes. Recall that a lattice polytope $P \subset \mathbb{R}^d$ is **simple** if every vertex of $P$ is contained in exactly $d$ edges (see, e.g., [Zie95]). A simple polytope $P$ is called **smooth** if the primitive edge direction generate $\mathbb{Z}^d$ at every vertex of $P$. Smooth polytopes are particularly of interest due to a conjecture commonly attributed to Oda [Oda]:

**Conjecture 5.6 (Oda).** If $P$ is a smooth polytope, then $P$ has the IDP.

This conjecture is not only of interest in the context of Ehrhart theory, but also in toric geometry. One potential strategy is to consider similar constructions to OLP polytopes for smooth reflexive polytopes to make progress towards this problem. As a first step, we pose the following question:

**Question 5.7.** Are all smooth reflexive polytopes OLP polytopes?
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