Ladder Operators for the Rosen-Morse System
Through Classical Analogy

L Delisle-Doray and V Hussin
Centre de Recherches Mathématiques, Université de Montréal, Montréal H3C 3J7, QC, Canada
E-mail: hussin@crm.umontreal.ca

Abstract. In quantum mechanics, ladder operators allow to connect the eigenstates of a system together. Ladder functions are algebraically analog objects defined in the framework of classical Hamiltonian physics. For a class of exactly solvable one dimensional Hamiltonians, both the ladder operators and ladder functions take a simple form and there is a close similarity between them. In this work, we show how the analogy extends to the case of the Rosen-Morse Hamiltonian, for which the ladder functions have a more complicated structure. We compute a form for the ladder operators, based on the ladder functions of the system and we analyse the correspondences between both cases. Physical mean values are also obtained as a byproduct of the construction.

1. Introduction

In quantum mechanics, ladder operators are defined as operators connecting the eigen–spaces of a given operator, generally an observable of the system, e.g. its Hamiltonian \( \hat{H} \), or a component part of the angular momentum \( \hat{J}_i \) [1]. Since even the early days of quantum mechanics, they have served as a tool for solving the eigenvalue problem of the observable without explicitly obtaining its eigen–solutions [2]. They are related to an algebraic structure, the Generalized Heisenberg Algebra (GHA), in which the ladder operators can be independently defined [3, 4] and which takes part in a larger spectrum generating algebra (SGA), determining the spectrum of the system under consideration when the observable is the Hamiltonian [5], for example.

Analog objects have been defined in the context of classical Hamiltonian mechanics. For example, they correspond to constants of the motion with an explicit time dependence [5]. It also leads to a very similar algebraic structure to that of the quantum ladder operators, once we include the other constants of the motion. Functions of the phase space obeying such algebras can be called ladder functions. At the classical level, this construction can be thought as algebraically determining both the motion of the system and its frequency (in the case of bounded motion).

One feature of the recent study of ladder functions is their formal similarity with their quantum analogs, which has been first observed for a class of simple one dimensional solvable systems and extended more recently to the Kepler-Coulomb system [6,7].

In this work we seek to extend the analogy to more elaborated one dimensional systems. We will motivate and write a form for the ladder operators of the Rosen-Morse system in close correspondence with the structure of its classical counterpart, which was characterized...
recently [8]. The problem will be approached by factoring the ladder function using essentially two parts, which we call factor functions, associating operator to each of them, and combining the operators together to produce the ladder operators.

The paper is organized as follow: in Section 2, we recall the definition of ladder operators and ladder functions and we then discuss the form that they take for simple systems already treated in the literature. In Section 3, we give the form of the factor functions for the Rosen-Morse system. Then we obtain operator analogs for the factor functions and we describe their action on the Rosen-Morse eigenstates. In Section 4, we use these factor functions to construct the ladder operators in formal correspondence with their classical counterparts. In Section 5, we discuss how physical mean values and matrix elements might be derived from our construction. We conclude the paper in Section 6.

2. Ladder operators and functions

In quantum mechanics, for nondegenerate systems, the ladder operators $\hat{A}^+, \hat{A}^-$ can be defined by their action $\hat{A}^\pm |\Psi_n\rangle \propto |\Psi_{n\pm1}\rangle$ on the eigenstates $|\Psi_n\rangle$ of a system with Hamiltonian $\hat{H}$ where $n$ is a non negative integer, made explicit with the specification of the functions $k_{n\pm}$ of the quantum number $n$:

$$\hat{A}^\pm |\Psi_n\rangle = k_{n\pm}^\pm |\Psi_{n\pm1}\rangle. \quad (1)$$

We then get:

$$(\hat{H}\hat{A}^\pm - \hat{A}^\pm \hat{H})|\Psi_n\rangle = \hat{A}^\pm (E_{n\pm1} - E_n)|\Psi_n\rangle. \quad (2)$$

Introducing an operator $\Delta_\pm(\hat{H})$, diagonal with respect to the basis $|\Psi_n\rangle$:

$$\Delta_\pm(\hat{H})|\Psi_n\rangle = (E_{n\pm1} - E_n)|\Psi_n\rangle \quad (3)$$

and the commutator $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$, we may disconnect the operators from the action on the states and write the equation (2) in terms of operators only:

$$[\hat{H}, \hat{A}^\pm] = \hat{A}^\pm \Delta_\pm(\hat{H}). \quad (4)$$

Introducing more diagonal operators $\kappa(\hat{H})$ and $\Delta(\kappa(\hat{H}))$, defined by

$$\kappa(\hat{H})|\Psi_n\rangle = \kappa(E_n)|\Psi_n\rangle = k_{n\pm}^{\pm} k_{n-1\pm}^{-\pm} |\Psi_n\rangle, \quad (5)$$

$$\Delta(\kappa(\hat{H}))|\Psi_n\rangle = (\kappa(E_{n+1}) - \kappa(E_n))|\Psi_n\rangle \quad (6)$$

we further obtain:

$$\kappa(\hat{H}) = \hat{A}^+ \hat{A}^-, \quad (7)$$

and

$$[\hat{A}^-, \hat{A}^+] = \Delta(\kappa(\hat{H})). \quad (8)$$

The equations (4) and (8) constitute the Generalized Heisenberg Algebra (GHA) satisfied by the ladder operators of the system, and (7) is the associated factorization condition. We note that the diagonal operators appearing in the context of the GHA have a clear role. For example, $\Delta_\pm(\hat{H})$ are the operators giving the energy difference between states.

In the hamiltonian formalism of classical mechanics, a counterpart of the GHA can be introduced [4, 6, 7, 9] with the Poisson bracket $\{ A, B \} = \frac{\partial A}{\partial x} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial x}$ on a one dimensional system with Hamiltonian $H = H(x, p)$ and justified on the ground of the Dirac connection ($h = 1$) $[\hat{A}, \hat{B}] \to -i\{ A, B \}$:

$$\{ H, A^\pm \} = \mp i\alpha(H) A^\pm, \quad (9)$$
\[ \{A^-, A^+\} = -i\alpha(H) \frac{d\kappa}{dH}, \]  
(10)

where \(\kappa(H)\) is a function of the Hamiltonian, which is as in the quantum case factorized by \(A^\pm\):

\[ \kappa(H) = A^+ A^- . \]  
(11)

It is called the classical GHA of the system and the functions \(A^-, A^+\) of phase-space variables \((x, p)\) are called the ladder functions.

2.1. Ladder operators and functions for simple systems

We now restrict further the discussion to Hamiltonians of the form \(H(x, p) = V(x) + p^2\) with the potential \(V = V(x)\). For the simplest potentials, the ladder operators can be put in the form:

\[ \hat{A}^\pm = a(\hat{x}, \hat{N}) \mp ib(\hat{x}, \hat{N})\hat{p}, \]  
(12)

where \(\hat{N}\) is the number operator closely related to the Hamiltonian \(\hat{H}\) and such that \(\hat{N}\vert \Psi_n \rangle = n\vert \Psi_n \rangle\).

A similar form has been investigated [6] for classical ladder functions:

\[ \hat{A}^\pm = a(x, H) \mp ib(x)p. \]  
(13)

In fact, the hypothesis \(b(x, H) = b(x)\) is extremely restrictive. It can be shown that all the potentials which ladder functions can be written in this way take the form:

\[ V(x) = \frac{-r_2 - c_2 h^2(x) - d_2 h(x)}{r_1 + c_1 h^2(x) + d_1 h(x)}, \quad h(x) = w_1 \sqrt{c_1} x + w_2 \cos \sqrt{c_1} x - \frac{d_1}{\sqrt{c_1}}, \]  
(14)

where \(c_1, c_2, d_1, d_2, r_1, r_2, w_1, w_2\) are a family of eight real parameters constrained by the relation:

\[ w_2^2 + w_1^2 \frac{c_1}{|c_1|} = -\frac{r_1}{c_1} + \frac{d_1^2}{4c_1^2}. \]  
(15)

The explicit form of the corresponding ladder functions is:

\[ A^\pm = \frac{h(x)\alpha(H)}{2} + \Phi(H) \mp ih'(x)p, \]  
(16)

with

\[ \alpha(H) = 2\sqrt{c_1 H + c_2}, \quad \Phi(H) = \frac{d_1 H + d_2}{2\sqrt{c_1 H + c_2}}. \]  
(17)

They satisfy the classical GHA in the form (9)-(10). The function of the Hamiltonian factorized by \(A^\pm\) is given as:

\[ \kappa(H) = \frac{(d_1 H + d_2)^2}{4(c_1 H + c_2)} - r_1 H - r_2. \]  
(18)

From these classical ladder functions, we can directly by applying a method developed in Section 3.2, obtain a form for the ladder operators of the same potentials:

\[ \hat{A}^\pm = \frac{h(\hat{x})\alpha(\hat{H})}{2} + \Phi(\hat{H}) \mp \frac{c_1 d_2 - c_2 d_1}{\alpha(\hat{H})\alpha(\hat{H}) \mp c_1} \mp ih'(\hat{x})\hat{p}. \]  
(19)

As it can be noticed the ladder operators differ from their classical counterparts by an additional term proportional to \(c_1 d_2 - c_2 d_1\).
All the potentials treated in [6] are included in (14) and their ladder functions have the form (16), as we have checked by computing all the specific cases, including those appearing as limiting values of the parameters. For example, the limit $d_1 = 0$ and $c_1 \to 0^+$ while we keep $w_2$ and $r_1$ fixed in the constraint (15), corresponds to the simple harmonic oscillator, whereas the limit $c_1 \to 0^+, w_2 > 0$, while fixing $d_1 \neq 0$, $w_1^2 c_1$ and $r_1$ in the constraint (15) (then we obtain $w_2 = \frac{d_1}{2c_1} - \frac{r_1}{d_1} - \frac{w_1^2 c_1}{2r_1} + O(c_1^2)$), corresponds to the singular oscillator.

For the quantum case (19) a partial list of such ladder operators is provided in [1].

3. Factor functions and operators for Rosen-Morse

The classical Hamiltonian of the Rosen-Morse system is defined as [10,11]:

$$H = p^2 + B \tanh x - \frac{b_0(b_0 + 1)}{\cosh^2 x}. \quad (20)$$

The quantum Hamiltonian will be denoted by $\hat{H}_B = -\hbar^2 \frac{d^2}{dx^2} + B \tanh \hat{x} - \frac{b_0(b_0 + 1)}{\cosh^2 \hat{x}}$, to make explicit the $B$ dependence ($b_0$ is considered to be fixed). We work in units with $\hbar = 1$ (and the mass $m = \frac{1}{2}$), but it should be mentioned that $\hbar$ can be restored whenever needed.

In the classical case, for bounded motion $H < -|B|$, it has been shown [8] that a set of particular functions plays a role in the construction of its ladder functions ($\epsilon = \pm 1$). They look similar to the ones introduced in the preceding section. They are called factor functions since they are not the ladder functions in such a case.

3.1. Factor functions

The first type of factor functions is:

$$f_\epsilon = \cosh x (\phi_\epsilon(H) \tanh x + \phi_{-\epsilon}(H) + ip), \quad (21)$$

where $\phi_\epsilon$ is introduced as a function of the Hamiltonian only:

$$\phi_\epsilon(H) = \sqrt{-H + B + \epsilon \sqrt{-H - B}}. \quad (22)$$

We also define the set $\pi_\epsilon$, related to $\phi_\epsilon$ through $\pi_\epsilon = \phi_1 + \epsilon \phi_{-1}$:

$$\pi_\epsilon(H) = \sqrt{-H + \epsilon B}. \quad (23)$$

The second type of functions are:

$$g_\epsilon = \frac{1}{\tanh x - \epsilon} \left( \frac{(B + 2b_0(b_0 + 1)) \tanh x + \epsilon B - 2(H + b_0(b_0 + 1))}{2\sqrt{-H + \epsilon B}} + ip \right). \quad (24)$$

Although these functions have the simple form (13), they do not satisfy the GHA brackets (9) and (10) and thus they are not themselves ladder functions. They nevertheless satisfy the partial property of factorizing the Hamiltonian in the sense of (11). Indeed, we get:

$$f_\epsilon f^*_\epsilon = \kappa f_\epsilon(H) = b_0(b_0 + 1) - \phi_\epsilon(H)^2, \quad (25)$$

$$g_\epsilon g^*_\epsilon = \kappa g_\epsilon(H) = \frac{B^2 + 4b_0(b_0 + 1)(b_0(b_0 + 1) + H)}{4(-H + \epsilon B)}. \quad (26)$$
If we denote \( f_{\epsilon}^- = f_{\epsilon}, g_{\epsilon}^- = g_{\epsilon} \) and \( f_{\epsilon}^+ = f_{\epsilon}, g_{\epsilon}^+ = g_{\epsilon} \), the ladder functions may be written in the following (alternative for \( \epsilon = \pm 1 \)) way:

\[
A_1^\pm = (f_{-1}^\mp)^{\gamma_{\epsilon}(H)} g_{1}^\pm, \quad A_{-1}^\pm = (f_{-1}^\mp)^{\gamma_{-1}(H)} g_{-1}^\pm,
\]

with

\[
\gamma_{\epsilon}(H) = \frac{2\phi_{-1}(H)\sqrt{-H + \epsilon B}}{B}.
\]

We may also write the ladder functions using only the set \( f_{\epsilon} \), as:

\[
A^\pm = (f_{-1}^\mp)^{\nu(H)} f_{1}^\pm,
\]

with

\[
\nu(H) = \frac{\phi_{-1}(H)}{\phi_1(H)}.
\]

These functions indeed satisfy the brackets (9), (10) and the frequency \( \alpha(H) \) is given by:

\[
\alpha(H) = \frac{4\sqrt{H^2 - B^2}}{\sqrt{-H + B + \sqrt{-H - B}}},
\]

There are thus many valid ways to write the ladder functions with the set \( f_{\pm}^\pm, g_{\pm}^\pm \). They all have to be related by a function of \( H \), and thus \( f_{\pm}^\pm \) should be related with \( g_{\pm}^\pm \). The exact relation is found to be:

\[
g_{1}^\pm = -f_{1}^\pm \frac{1}{\pi_1} f_{-1}^\mp, \quad g_{-1}^\pm = -f_{-1}^\pm \frac{1}{\pi_{-1}} f_{1}^\mp.
\]

The relation between the alternative forms of the ladder functions can be understood because we have \( \gamma_{\epsilon}(H) = 1 + \epsilon \nu(H) \). In the quantum case, we will see that the situation is similar.

### 3.2. Transition to factor operators

Our goal is now to introduce operators playing a similar role as \( f_{\epsilon} \) that will help in the construction of ladder operators. To do this we notice that the functions can be written as \( f_{\epsilon}(x,p) = f_{\phi, \phi_{\epsilon}}(x,p) \) with the help of a family of functions of \( (x,p) \), \( \{ f_{K,K'} \} \), indexed by real parameters \((K,K')\):

\[
f_{K,K'}(x,p) = \cosh x(K \tanh x + K' + ip).
\]

We notice two things about the functions \( f_{K,K'} \). First along with their complex conjugate \( f_{K,K'}^* \), we can form the quantity

\[
I_{K,K'} = f_{K,K'}^* f_{K,K'} = \cosh^2 x((K \tanh x + K')^2 + p^2),
\]

which, when evaluated at \((K,K') = (\phi_{\epsilon}, \phi_{-\epsilon})\), gives a function of the Hamiltonian (see (25)).

\[
I_{\phi_{\epsilon}, \phi_{-\epsilon}} = \kappa_{f_{\epsilon}}(H, B) = b_0(b_0 + 1) - \phi_{\epsilon}^2(H).
\]

The r.h.s. of this equation will define a function of \((K,K')\), denoted by \( \kappa(K,K') = b_0(b_0 + 1) - K^2 \), which gives to \( \kappa_{f_{\epsilon}}(H, B) \) when we set \((K,K') = (\phi_{\epsilon}, \phi_{-\epsilon})\). We stress that \( I_{K,K'} \) and \( \kappa(K,K') \) are distinct, as they are only equal when we restore the \( H \) dependency via \((K,K') = (\phi_{\epsilon}, \phi_{-\epsilon})\).

Second, they satisfy the following Poisson bracket:

\[
\{ f_{K,K'}, f_{K,K'}^* \} = -i \frac{\partial (I_{K,K'} - \kappa(K,K'))}{\partial K}.
\]
We now proceed as follow: we obtain operator analogs of $\hat{f}_{K,K'}$. $\hat{f}'_{K,K'}$ as a family of operators indexed by $(K, K')$, denoted by $\hat{f}'_{K,K'}$, $\hat{f}'_{K+1,K'}$, which together factorize the operator $\hat{I}_{K,K'}$, replacing $x \to \tilde{x}$ and $p \to \tilde{p} = -\frac{d}{dx}$, with no order change in (34):

$$\hat{I}_{K,K'} = \cosh^2 x((K \tanh x + K')^2 - \frac{d^2}{dx^2}),$$  

$$\hat{I}_{K,K'} = \hat{f}'_{K+1,K'} \hat{f}'_{K,K'} + \rho(K, K').$$  

They will be constructed so as to satisfy a commutator counterpart of (36):

$$[\hat{f}'_{K+1,K'}, \hat{f}'_{K,K'}] = \Delta_K (\hat{I}_{K,K'} - \kappa(K, K'))$$

with the definition $\Delta_K (\hat{I}_{K,K'} - \kappa(K, K')) = \hat{I}_{K+1,K'} - \kappa(K + 1, K') - (\hat{I}_{K,K'} - \kappa(K, K'))$.

Now, given the form $\hat{f}_{K,K'} = a_{K,K'} + ibp$, the operators $\hat{f}'_{K,K'}, \hat{f}'_{K+1,K'}$ may be simply constructed by writing the ansatz $\hat{f}'_{K,K'} = a^- + b \frac{d}{dx}$ and $\hat{f}'_{K+1,K'} = a^+ - b \frac{d}{dx}$ and requiring (38) to be satisfied for some function $\rho(K, K')$. Then the following equations constraining $a^+, a^-, \rho$ are obtained:

$$a^+ = b' + a^-,$$

$$\rho_{K,K'} + (a^-)^2 + \mathcal{W}(a^-, b) = a^2_{K,K'},$$

with the Wronskian $\mathcal{W}(f, g) = fg' - f'g$. Here we also notice that $\mathcal{W}(a_{K,K'}, b) = -K$ so that we can simply take $\rho_{K,K'} = -\mathcal{W}(a_{K,K'}, b) = K$ and $a^- = a_{K,K'}$. The result is that we can write the operators $\hat{f}'_{K,K'}$ in the same form as in the classical case:

$$\hat{f}'_{K,K'} = \cosh x(K \tanh x + K') \pm \frac{d}{dx}.$$  

They are seen to satisfy (38) and (39), in analogy with the classical case (34) and (36). We note that $a^- = -a_{K,K'}$ is also possible as a solution to (41) (it is in fact associated with $\hat{f}'_{K,K'}$), but it leads to another commutator than (39).

Now that we have our operators $\hat{f}_{K,K'}$ we need one last fact to be able to make deductions about them:

$$\hat{I}_{K,K'}\Psi = \kappa(K, K')\Psi \iff \hat{H}_{2KK'}\Psi = (-K^2 - K'^2)\Psi.$$  

(43)

It means that the eigenstate of $\hat{I}_{K,K'}$ with eigenvalue $\kappa(K, K')$ is an eigensate of the Hamiltonian $\hat{H}_{2KK'}$ with energy $E = -K^2 - K'^2$. If such a state exists, we call it $|K, K'\rangle$. From (38), (39) and (43), it is then readily shown that the action of the couple $\hat{f}'_{K,K'}, \hat{f}'_{K+1,K'}$ on $|K, K'\rangle$ is:

$$\hat{f}'_{K,K'}|K, K'\rangle \propto |K + 1, K'\rangle.$$  

(44)

We have to adjoint conditions under which $\hat{f}'_{K,K'}$ preserves the normalizability of $|K, K'\rangle$. Assuming for our needs $0 \leq K$ and $|K'| < K$, an asymptotic analysis of the Schrödinger equation shows that $\hat{f}'_{K,K'}$ always preserves the normalizability of a state $|K, K'\rangle$ (if it exists), whereas $\hat{f}'_{K,K'}$ preserves the normalizability of $|K, K'\rangle$ if and only if:

$$K \pm K' > 1.$$  

(45)
At this point, all the bounded eigenstates of the system may be derived in a classical manner [2]. First for any bounded eigenstate $|\Psi\rangle$ of the Hamiltonian $(B, E)$ such that $\hat{H}_B |\Psi\rangle = E |\Psi\rangle$, we may set up two variables, using the same functions $\phi_{\pm 1}$ as the classical case:

$$\phi_{\pm 1}(B, E) = \frac{\sqrt{-E + B} \pm \sqrt{-E - B}}{2} \iff E = -\phi_1^2 - \phi_{-1}^2, \quad B = 2\phi_1\phi_{-1}, \quad (46)$$

so that all the eigenstates of the Hamiltonian $\hat{H}_B$ may be identified by the variables $(\phi_1, \phi_{-1})$, $|\Psi\rangle = |\phi_1, \phi_{-1}\rangle$, taking (43) into account.

If any such state exists, the operators $\hat{f}_{\phi_1, \phi_{-1}}^+, \hat{f}_{\phi_1+1, \phi_{-1}}^+, ...$ may be applied to it until 0 is reached (because $\phi_1 \leq b_0$), which happens when $\phi_1 = b_0$. Thus we start from all states annihilated by $\hat{f}_{b_0, \phi_{-1}}^-$, which in spatial representation are then given by solving the corresponding first order differential equation:

$$\hat{f}_{b_0, \phi_{-1}}^- |b_0, \phi_{-1}\rangle = 0 \implies \langle x | b_0, \phi_{-1} \rangle \propto e^{-\phi_{-1}x} \cosh^{-b_0} x.$$  \hspace{1cm} (47)

Starting from the states (47), all the other states may be generated by using $\hat{f}_{b_0, \phi_{-1}}^+, \hat{f}_{b_1, \phi_{-1}}^+, ...$ in succession, until we reach a not-normalizable function. Because of (44) and (45), the procedure results in the (yet to be normalized) states:

$$|b_0 - n, \phi_{-1}\rangle \propto \hat{f}_{b_0 - n + 1, \phi_{-1}}^+ \hat{f}_{b_0 + 1, \phi_{-1}}^+ \hat{f}_{b_0, \phi_{-1}}^- |b_0, \phi_{-1}\rangle, \quad n = 0, 1, ... < b_0 - |\phi_{-1}|.$$  \hspace{1cm} (48)

Considering (46), this results in the known eigenenergies [10]:

$$E_n = -(b_0 - n)^2 - \left( \frac{B}{2(b_0 - n)} \right)^2, \quad n = 0, 1, ... < b_0 - \sqrt{\frac{|B|}{2}}.$$  \hspace{1cm} (49)

We give in our notation the resulting functions as were determined in the literature in normalized form [12]. To obtain the bounded eigenstates of a given Hamiltonian $\hat{H}_B$ from them, we set $\phi_1 = b_0 - n$ and $\phi_{-1} = \frac{B}{2\phi_1}$ in these expressions:

$$\langle x | \phi_1, \phi_{-1} \rangle = N_{\phi_1, \phi_{-1}} 2F_1(\phi_1 - b_0, 1 + b_0 + \phi_1; \phi_1 + 1 - \phi_{-1}; \frac{1 + \tanh x}{2}) e^{-\phi_{-1}x} \cosh^{-\phi_1} x,$$  \hspace{1cm} (50)

with $2F_1(a, b; c; z)$ the gauss hypergeometric function and $N_{\phi_1, \phi_{-1}}$ is the normalization constant given in terms of gamma functions $\Gamma(z)$:

$$N_{\phi_1, \phi_{-1}} = \frac{2^{-\phi_1}}{\Gamma(\phi_1 - \phi_{-1} + 1)} \sqrt{\frac{(\phi_1 - \phi_{-1})(\phi_1 + \phi_{-1})\Gamma(b_0 + \phi_1 + 1)\Gamma(b_0 - \phi_{-1} + 1)}{\phi_1\Gamma(b_0 - \phi_1 + 1)\Gamma(b_0 + \phi_{-1} + 1)}}.$$  \hspace{1cm} (51)

Finally we mention that the method of this section was used to obtain the operators associated the other set of functions $g_r$, (24). We get:

$$g_{r, K, K'}(x, p) = \frac{(2(4K^2 - K'^2) + 2b_0|b_0 + 1|) \tan x + 2(4K^2 - K'^2) - 2b_0|b_0 + 1| + 1)|K^2 + K'^2|}{4|K| \tanh x - \epsilon} + ip.$$  \hspace{1cm} (52)

This time $I_{K, K'} = g_{r, K, K'} g_{r, K, K'}^*$ evaluates to a function of $H$ when $(K, K') = (\frac{\pi}{2} \pm \frac{\phi_{-1}}{2})$. Then, proceeding with the same steps the bracket (36) is obtained. The operators $\hat{g}_{r, K, K'}$ are computed so as to satisfy (41) and (39). They will then factorize $I_{K, K'}$ as in (38). At the end of the calculation, we set $(K, K') = (\frac{\pi(B, E)}{2}, \frac{\pi(-B, E)}{2})$. The result is given in the next section.
3.3. Hierarchy

We introduce operators acting on the space (formally) spanned by the kets $|\phi_1, \phi_{-1}\rangle$ (eventually including some unbounded states), which we might consider as some part of the direct sum of the Hilbert spaces of the Hamiltonians $\hat{H}_B$. We then define the (mutually commuting) operators $\hat{\phi}_{\pm 1}$, extending linearly from:

$$\hat{\phi}_{\pm 1}|\phi_1, \phi_{-1}\rangle = \phi_{\pm 1}|\phi_1, \phi_{-1}\rangle.$$  

(53)

In such a context there are operators $\hat{B} = 2\hat{\phi}_{-1}\hat{\phi}_1$ and $\hat{H} = -\hat{\phi}_{1}^{2} - \hat{\phi}_{-1}^{2}$ extracting the values of the $B$ and $E$ parameters associated to a state $|\phi_1, \phi_{-1}\rangle$. The factor operators give rise to the following operators, acting on all the states $|\phi_1, \phi_{-1}\rangle$:

$$f^\pm_\epsilon = \cosh \hat{x}(\tanh \hat{x}\hat{\phi}_\epsilon + \hat{\phi}_{\epsilon \mp} \mp \frac{d}{dx}).$$

(54)

Their action is then clear from (44), because they reduce to the operators $f^\pm_\epsilon$ when we apply to a given state. The exact relations are:

$$f^+_1|\phi_1, \phi_{-1}\rangle = \kappa^+_1|\phi_1 \mp 1, \phi_{-1}\rangle,$$

(55)

$$f^-_1|\phi_1, \phi_{-1}\rangle = \nu^-_1|\phi_1, \phi_{-1} \mp 1\rangle,$$

(56)

where we have computed the factors $\kappa^+_1, \nu^-_1$ from the underlying theory of hypergeometric functions and the normalization constants (51):

$$\kappa^+_1|\phi_1, \phi_{-1}\rangle = -\sqrt{b_0(b_0 + 1)}|\phi_1 + 1\rangle \sqrt{\frac{(1 - \phi_{-1})(\phi_1 + \phi_{-1})(1 + 1)}{(\phi_1 + 1)(\phi_1 + \phi_{-1} + 1)}},$$

$$\nu^-_1|\phi_1, \phi_{-1}\rangle = \sqrt{b_0(b_0 + 1)}|\phi_{-1} - 1\rangle \sqrt{\frac{(1 - \phi_{-1})(\phi_1 + \phi_{-1})(1)}{(\phi_1 - 1)(\phi_1 + 1 + \phi_{-1} + 1)}},$$

(57)

(58)

From these values, we then find, after applying $\hat{f}^+_\epsilon \hat{f}^-_\epsilon$ and $\hat{f}^-_\epsilon \hat{f}^+_\epsilon$ to a state $|\phi_1, \phi_{-1}\rangle$, the following brackets:

$$[\hat{f}^\pm_\epsilon, \hat{f}^\pm_\epsilon] = f^\pm_\epsilon,$$

(59)

$$[\hat{f}^+_\epsilon, \hat{f}^-_\epsilon] = -2\hat{\phi}_\epsilon.$$

(60)

We recognize this as the brackets satisfied by two representations ($\epsilon = \pm 1$) of the Lie algebra so(3).

Finally the operators corresponding to $g_\epsilon$ computed from the last section with $\tilde{\pi}_\epsilon = \hat{\phi}_\epsilon + \epsilon\hat{\phi}_{-1}$ are:

$$\hat{g}^\pm_\epsilon = \frac{1}{\tanh \hat{x} - \epsilon} \left( \frac{\epsilon \tilde{\pi}_\epsilon}{\mp \tilde{\pi}_\epsilon + \epsilon} + \tanh \hat{x} \frac{\hat{B} + 2\epsilon b_0(b_0 + 1)}{2\tilde{\pi}_\epsilon \mp 2\epsilon} \mp \frac{d}{dx} \right).$$

(61)

We can obtain their action on the states as $\hat{g}^+_1|\phi_1, \phi_{-1}\rangle \propto |\phi_1 \pm 1, \phi_{-1} \pm 1\rangle$ and $\hat{g}^-_1|\phi_1, \phi_{-1}\rangle \propto |\phi_1 \mp 1, \phi_{-1} \mp 1\rangle$ by the methods of the last section. An alternative is to observe (by applying to any ket $|\phi_1, \phi_{-1}\rangle$ and using $\frac{d^2}{dx^2}|\phi_1, \phi_{-1}\rangle = (V_B - E)|\phi_1, \phi_{-1}\rangle$ that we have, up to ordering, the same relations as the classical case (32):

$$\hat{g}^-_1 = -\hat{f}^-_1 \frac{1}{\mp 1} \hat{f}^-_1, \quad \hat{g}^+_1 = -\hat{f}^+_1 \frac{1}{\mp 1} \hat{f}^+_1;$$

$$\hat{g}^-_{-1} = -\hat{f}^-_1 \frac{1}{\mp 1} \hat{f}^-_1, \quad \hat{g}^+_{-1} = -\hat{f}^+_1 \frac{1}{\mp 1} \hat{f}^+_1.$$

(62)
4. Ladder operators

4.1. Exponentiation of operators

We now have operator analogs for the two functions $f_{\pm}$ appearing in the ladder functions as presented by (29), but we still have to find to what corresponds the exponent $\nu$ on the $f_{-1}$ part of the function. As its operator analog $\hat{f}_{-1}^{\pm}$ makes the change $|\phi_{1}, \phi_{-1}\rangle \rightarrow |\phi_{1}, \phi_{-1} + 1\rangle$ in the hierarchy, a natural definition of $(\hat{f}_{-1}^{\pm})^{\nu}$ if $\nu$ is an integer is simply $\nu$ applications of the same operator, which has the action:

$$ (\hat{f}_{-1}^{\pm})^{\nu}|\phi_{1}, \phi_{-1}\rangle \propto |\phi_{1}, \phi_{-1} + \nu\rangle. \quad (63) $$

We now propose to extend this to any real $\nu$: we will define an operator $(\hat{f}_{-1}^{\pm})^{\nu}$ that realizes the same connection across the hierarchy for any $\nu$. As it turns out, it suffices to make a definition for the ground states $|b_0, \phi_{-1}\rangle$. For then we will extend the definition by formally writing:

$$ (\hat{f}_{-1}^{\pm})^{\nu} = (\hat{f}_{1}^{\pm})^{b_0, \phi_{-1}}(\hat{f}_{-1}^{\pm})^{\nu}(\hat{f}_{1}^{\pm})^{b_0, \phi_{-1}} \Sigma(\phi_{1}, \phi_{-1}, \nu), \quad (64) $$

where the operators $(\hat{f}_{1}^{\pm})^{b_0, \phi_{-1}}$ are defined as:

$$ (\hat{f}_{1}^{\pm})^{b_0, \phi_{-1}}|\phi_{1}, \phi_{-1}\rangle = (\hat{f}_{1}^{\pm})^{b_0, \phi_{-1}}|\phi_{1}, \phi_{-1}\rangle \quad (65) $$

and $\Sigma(\phi_{1}, \phi_{-1}, \nu)$ is a scalar operator $\Sigma(\phi_{1}, \phi_{-1}, \nu)|\phi_{1}, \phi_{-1}\rangle = \Sigma(\phi_{1}, \phi_{-1}, \nu)|\phi_{1}, \phi_{-1}\rangle$ defined through (57), so as to make the definition for $\nu = 1$ coincide with $\hat{f}_{-1}^{\pm}$:

$$ \Sigma(\phi_{1}, \phi_{-1}, \nu) = (\kappa^{-}_{\phi_{1}, \phi_{-1}}\kappa^{+}_{\phi_{1}+1, \phi_{-1}+\nu}\kappa^{-}_{\phi_{1}+1, \phi_{-1}+\nu}\kappa^{+}_{\phi_{1}+2, \phi_{-1}+\nu}...\kappa^{-}_{b_0-1, \phi_{-1}+\nu}\kappa^{+}_{b_0, \phi_{-1}+\nu})^{-1}. \quad (66) $$

Thus the effect of (64) is to reduce $|\phi_{1}, \phi_{-1}\rangle$ to $|b_0, \phi_{-1}\rangle$, apply the ground state definition $|b_0, \phi_{-1}\rangle \rightarrow |b_0, \phi_{-1} + \nu\rangle$, and then restore $\phi_{1}$ to its original value $|b_0, \phi_{-1} + \nu\rangle \rightarrow |\phi_{1}, \phi_{-1} + \nu\rangle$.

For the ground states, we have:

$$ \hat{f}_{-1}|b_0, \phi_{-1}\rangle = (b_0 - \phi_{-1})e^{-\hat{x}}|b_0, \phi_{-1}\rangle. \quad (67) $$

By extension we define:

$$ (\hat{f}_{-1}^{\pm})^{\nu}|b_0, \phi_{-1}\rangle = (b_0 - \phi_{-1})^{\nu}e^{-\nu\hat{x}}|b_0, \phi_{-1}\rangle. \quad (68) $$

It is then easy to show, from the explicit form (47) of the ground states, that we have:

$$ (\hat{f}_{-1}^{\pm})^{\nu}|b_0, \phi_{-1}\rangle = \frac{(b_0 - \phi_{-1})^{\nu}N_{b_0, \phi_{-1}}}{N_{b_0, \phi_{-1}+\nu}}|b_0, \phi_{-1} + \nu\rangle, \quad (69) $$

which is enough, along with the extension (64) to show that the definition implements the desired transition $(\hat{f}_{-1}^{\pm})^{\nu} : |\phi_{1}, \phi_{-1}\rangle \rightarrow |\phi_{1}, \phi_{-1} + \nu\rangle$.

The same reasoning can be made for $(\hat{f}_{1}^{\pm})^{\nu}$ and we can write the result for both cases in a compact form:

$$ (\hat{f}_{1}^{\pm})^{\nu} = (\hat{f}_{1}^{\pm}{b_0, \phi_{-1}})^{\nu}(\hat{f}_{1}^{\pm}{b_0, \phi_{-1}})^{\nu}(\hat{f}_{1}^{\pm}{b_0, \phi_{-1}})^{\nu}\Sigma(\phi_{1}, \phi_{-1}, \nu) \quad (70) $$

and their effects across the hierarchy are:

$$ (\hat{f}_{1}^{\pm})^{\nu}|\phi_{1}, \phi_{-1}\rangle = \frac{(b_0 + \phi_{-1})^{\nu}N_{b_0, \phi_{-1}}}{N_{b_0, \phi_{-1}+\nu}}|\phi_{1}, \phi_{-1} + \nu\rangle. \quad (71) $$

As a technical note, we mention that to make the definition really work we have to carefully consider how we treat unbounded states in the set defined in Section 3.3 as the r.h.s. of this equation might not be normalizable.
4.2. Construction of ladder operators

We are now ready to give an expression for the ladder operator in analogy with the classical form (29). We again allow exponents of operators to (formally) take on "operator values" in the same manner as in (65). Thus once a function of \((\phi_1, \phi_{-1})\) is specified as \(\nu(\phi_1, \phi_{-1})\), we define an operator \((\hat{f}_{-1}^\pm)^{\nu(\phi_1, \phi_{-1})}\) as:

\[
(\hat{f}_{-1}^\pm)^{\nu(\phi_1, \phi_{-1})}|\phi_1, \phi_{-1}\rangle = (\hat{f}_{-1}^\pm)^{\nu(\phi_1, \phi_{-1})}|\phi_1, \phi_{-1}\rangle
\]  

(72)

and the expression is well defined because the expression (70) was defined for any real \(\nu\).

Using the same function as in the classical case \(\nu(\phi_1, \phi_{-1}) = \frac{\phi_1}{\phi_1}\) we propose the following definition for the ladder operators, which is, up to ordering, in the same form as (29):

\[
\hat{A}^- = (\hat{f}_{-1}^-)^{\frac{\phi_{-1}}{\phi_1}} f_1^-.
\]  

(73)

We first show that it works as a definition of ladder operators. We first have to show that the operator changes the energy \(E = -\phi_1^2 - \phi_{-1}^2\) of a state \(|\phi_1, \phi_{-1}\rangle\) in a way coherent with the form of the spectrum (49) without changing the value of the potential parameter \(B = 2\phi_1\phi_{-1}\). Now according to (70) and (55) the action of the operators is:

\[
\hat{A}^-|\phi_1, \phi_{-1}\rangle = \kappa_{\phi_1, \phi_{-1}}^- \left(\frac{\phi_1}{\phi_{-1}}\right)^{\frac{\phi_{-1}}{\phi_1}} N_{\phi_{0}, \phi_{-1}} |\phi_1 + 1, \phi_{-1} - \frac{\phi_{-1}}{\phi_1 + 1}\rangle,
\]

(74)

and

\[
\hat{A}^+|\phi_1, \phi_{-1}\rangle = \kappa_{\phi_1, \phi_{-1}}^+ \left(\frac{\phi_1}{\phi_{-1}}\right)^{\frac{\phi_{-1}}{\phi_1}} N_{\phi_{0}, \phi_{-1}} |\phi_1 - 1, \phi_{-1} + \frac{\phi_{-1}}{\phi_1 - 1}\rangle.
\]  

(75)

It is thus easy to show that the state \(|\phi_1 + 1, \phi_{-1} - \frac{\phi_{-1}}{\phi_1 + 1}\rangle\) has the parameter \(B = 2\phi_1\phi_{-1}\), which is exactly the same as \(|\phi_1, \phi_{-1}\rangle\). Moreover, this state has the energy \(E = -(\phi_1 + 1)^2 - \phi_{-1}^2\left(\frac{\phi_{-1}}{\phi_1 + 1}\right)^2\). If we set \(\phi_1 = b_0 - n\) and \(\phi_{-1} = \frac{B}{2(b_0 - n)}\), we see that the change is \(E_n \to E_{n-1}\) in (49), as it should be for the annihilation operator. The same reasoning applied to \(\hat{A}^+\) shows that \(B\) is unchanged and it implements \(E_n \to E_{n+1}\). We also mention alternative forms of the ladder operators, similar to the classical case (27). Indeed from (73) and (62), the following also works:

\[
\hat{A}^- = (\hat{f}_{-1}^-)^{\nu(\hat{B})} \hat{f}_{-1}^- g_1^- , \quad \hat{A}^+ = (\hat{f}_{-1}^-)^{\nu(\hat{B})} g_1^- \hat{f}_{-1}^-.
\]  

(76)

4.3. Ladder operators labeled by \((B, n)\)

The definition of the previous section unwinds into operator functions of \((\hat{x}, \hat{p})\) well defined in the context of the Hilbert space of a given Hamiltonian \(\hat{H}_B\) in the hierarchy by applying the ladder operator to a given eigenstate that could be labelled as \(|B, n\rangle = |b_0 - n, \frac{B}{2(b_0 - n)}\rangle\), and writing \(\phi_1 = b_0 - n\) and \(\phi_{-1} = \frac{B}{2(b_0 - n)}\) in the resulting expression in terms of \((\hat{x}, \hat{p})\):

\[
\hat{A}^-|B, n\rangle \propto \prod_{i=n-2}^{0} \hat{f}^+_i \hat{g}_i \exp \frac{B\hat{x}}{2(b_0 - n)(b_0 - n + 1)} \prod_{i=1}^{n} \hat{f}_{-i} \hat{g}_i |B, n\rangle.
\]  

(77)

This gives a family of operators \(\hat{A}^-_{B, n}\) written with the help of the first order in \(\hat{p}\) operators \(\hat{f}^\pm_{K,K'}\), as defined in Section 3.2. As it is manifest, the whole operator has finite \((2n - 1)th\)
order in $\hat{p}$. It can also be reduced to an operator of order 1 in $\hat{p}$ with the help of the relation
\[-\frac{d^2}{dx^2}|B, n\rangle = (-V_B(\hat{x}) + E_{B,n})|B, n\rangle,\]
successively applying the relation on the r.h.s. of (77).

In Figure 1, we give an example of the resulting action of $\hat{A}_{B,2}$ explicitly given as:
\[\hat{A}_{B,2} = \hat{f}_0^+ \prod_i^{n-1} \exp \frac{B\hat{x}}{2(b_0 - 1)(b_0 - 2)} \hat{f}_{b_0-1,2(b_0-2)}^{-}\hat{f}_{b_0-2,2(b_0-2)}^{-} \hat{f}_{b_0-1,2(b_0-2)}^{-}.\] (78)

Application of the rightmost operator changes $n$ by 1, but the state is that of a different Hamiltonian $\hat{H}_{b_0-1,B}$. To compensate this, we apply in succession operators changing both $(B, n)$ to restore the original Hamiltonian $\hat{H}_B$.

Finally we can do the same for $\hat{A}^+$ and the resulting expression (this time $(2n + 1)$th order in $\hat{p}$) is:
\[\hat{A}^+_{B,n} = \prod_{i=n}^{0} \hat{f}_0^+ \prod_i^{n-1} \exp \frac{B\hat{x}}{2(b_0 - n)(b_0 - n - 1)} \hat{f}_{b_0-i,2(b_0-n-1)}^{-},\] (79)

5. Matrix elements and mean values

The form of the ladder operator as given by (77) can difficultly lead to physical results like matrix elements between states $|B, n\rangle$, except for the lowest quantum numbers. We can still solve the eigenvalue problem but, as indicated in Section 3.2, the problem is already solved by considering the action of the factor operators. Similarly the factor operators can also themselves be used to extract mean values rather effortlessly. The cost is that we have to consider matrix elements between non-orthogonal states $|\phi_1, \phi_{-1}\rangle$. One such matrix element is:
\[\Gamma_{\phi_1,\phi_{-1}} = \langle \phi_1 + 1, \phi_{-1} | \sinh \hat{x} | \phi_1, \phi_{-1} \rangle.\] (80)

It can be computed once we realize that the hermitian conjugate of $\hat{f}_{\phi_1,\phi_{-1}}^\pm$ is the displaced operator:
\[(\hat{f}_{\phi_1,\phi_{-1}}^\pm)^\dagger = \hat{f}_{\phi_1,\phi_{-1}}^-\] (81)

which implies, along with $\hat{f}_{\phi_1,\phi_{-1}}^\pm = 2 \sinh \hat{x} + \hat{f}_{\phi_1,\phi_{-1}}^\pm$:
\[\langle \phi_1 + 1, \phi_{-1} | (\hat{f}_{\phi_1+1,\phi_{-1}}^\pm)^\dagger | \phi_1, \phi_{-1} \rangle = \langle \phi_1 + 1, \phi_{-1} | 2 \sinh \hat{x} + \hat{f}_{\phi_1,\phi_{-1}}^- | \phi_1, \phi_{-1} \rangle.\] (82)
After some calculations, we get:

$$\Gamma_{\phi_1, \phi_{-1}} = \frac{\kappa_{\phi_1,1, \phi_{-1}} - \kappa_{\phi_1, \phi_{-1}}}{2}. \quad (83)$$

This quantity is useful in order to compute some mean values. One example is the mean value of sinh² x. To show that we first note that:

$$\langle \phi_1, \phi_{-1}| \sinh^2 x| \phi_1, \phi_{-1} \rangle = -\frac{1}{2} - \langle \phi_1, \phi_{-1}| \cosh x \sinh x \frac{d}{dx}| \phi_1, \phi_{-1} \rangle. \quad (84)$$

The second term in the r.h.s. can be computed in terms of the mean values of \(\hat{f}_{\phi_1, \phi_{-1}}^{-}\hat{f}_{\phi_1, \phi_{-1}}^{+}\) and \(\hat{f}_{\phi_1, \phi_{-1}}^{+}\hat{f}_{\phi_1, \phi_{-1}}^{-}\), which in turn can be computed from (83). The final result is:

$$\langle \phi_1, \phi_{-1}| \sinh^2(x)| \phi_1, \phi_{-1} \rangle = -\frac{1}{2} + b_0(b_0 + 1) - \phi^2_1 + \frac{(\kappa_{\phi_1, \phi_{-1}}^{+})^2 + (\kappa_{\phi_1, \phi_{-1}}^{-})^2}{4}. \quad (85)$$

For comparison we have also computed the classical analog, the mean value of sinh²(x) averaged over one cycle back and forth between the classical turning points V(x±) = H, with momentum \(p(x, H) = \sqrt{H - V(x)}\):

$$\langle \sinh^2(x) \rangle = \frac{\int_{x(H)}^{x_{+}(H)} \sinh^2(x) dx}{\int_{x(H)}^{x_{+}(H)} \frac{dx}{p(x,H)}}. \quad (86)$$

Its value can be computed say by integration in the complex plane and we find:

$$\langle \sinh^2(x) \rangle = -\frac{1}{2} + \frac{1}{4(\sqrt{H-B} + \sqrt{H+B})}\left(\frac{B + 2b_0(b_0 + 1)}{(-H + B)^{3/2}} - \frac{B - 2b_0(b_0 + 1)}{(-H - B)^{3/2}}\right). \quad (87)$$

As a consistency check we can also obtain the same result by restoring \(\hbar\) in (85) and performing the classical limit \(\hbar \to 0\), with \(n\hbar\) maintained constant to preserve the given value of the energy. The resulting expression, evaluated with (22), will be seen to be equal to (87).

6. Conclusion

In this paper, we have discussed how to connect classical ladder functions with quantum ladder operators for many one dimensional systems. For the simplest systems which were all already treated in the literature, the connection appears as a simple modification of the classical functions involved, represented by the transition from formula (16) to (19).

In the Rosen-Morse case, we have shown that the ladder operators are a product of factor operators similar to the factor functions used to construct the ladder functions. They are first order differential operators in \(\tilde{x}\) having a dependence in the number operator \(\tilde{N}\) (closely related to the Hamiltonian \(\hat{H}\)). Thus they are truly 1st order in \(\hat{p}\) only when we apply them to a given eigenstate of a system.

Such an expression for the ladder operators has proven useful in connection to many problems, such as getting explicit expressions for physical mean values and matrix elements of the functions of \((\hat{x}, \hat{p})\) involved in the definition of the ladder operators, or obtaining information on the eigenstates and eigen-energies [1,13]. When comparing with the classical ladder functions, they also suggest relevant functions for comparison between classical motion and average motion of the coherent states defined using the ladder operators [14]. This relies here on the observation that the ladder operators and ladder functions coincide in some cases \((c_1d_2 - c_2d_1 = 0\) in (16)), and we suggest that for more general settings further structure is added in the comparison by
the formal differences (for example, $c_1d_2 - c_2d_1 \neq 0$) between the ladder operators and their classical counterparts. For more elaborated potentials, the analogy takes a more complicated form [7]. The ladder functions must be split into component parts (factor functions) which can be separately compared with appropriate quantum counterparts at a proper classical limit.

Finally we mention that because of the mathematical similarity between the Rosen-Morse and the curved Kepler-coulomb systems [8], the same construction should apply to the later. We have verified that it is the case. We note that the construction should also apply to the trigonometric version of the same potentials.

**Acknowledgment**

V.H. acknowledges the support of research grant from NSERC of Canada.

**References**

[1] Dong S H 2007 *Factorization method in quantum mechanics* (Springer, Dordrecht)
[2] Schrödinger E 1940 A method of determining quantum-mechanical eigenvalues and eigenfunctions *Proc. R. Irish Acad. A* **46** 9
[3] Curado E M F, Hassouni Y, Rego-Monteiro LMA and Rodrigues M C S 2008 Generalized Heisenberg algebra and algebraic method: The example of an infinite square-well potential *Phys. Lett. A* **372** 3350
[4] Hussin V and Marquette I 2011 Generalized Heisenberg Algebras, SUSYQM and Degeneracies: Infinite Well and Morse Potential *SIGMA* **7** 024
[5] Dothan Y 1970 Finite-dimensional spectrum-generating algebras *Phys. Rev. D* **2** 2944
[6] Kuru Ş and Negro J 2008 Factorizations of one-dimensional classical systems *Ann. Phys.* **323** 413
[7] Kuru Ş and Negro J 2012 Classical spectrum generating algebra of the Kepler-Coulomb system and action-angle variables *Phys. Lett. A* **376** 260
[8] Delisle-Doray L, Hussin V, Kuru Ş and Negro J 2019 Classical ladder functions for Rosen-Morse and curved Kepler-Coulomb systems *Ann. Phys.* **405** 69
[9] Campoamor-Stursberg R, Gadella M, Kuru Ş and Negro J 2012 Action-angle variables, ladder operators and coherent states *Phys. Lett. A* **376** 2515
[10] Rosen N and Morse P 1932 On the Vibrations of Polyatomic Molecules *Phys. Rev.* **42** 210
[11] Gadella M, Kuru Ş and Negro J 2017 The hyperbolic step potential: Anti-bound states, SUSY partners and Wigner time delays *Ann. Phys.* **379** 86
[12] Nieto M M 1978 Exact wave-function normalization constants for the $B_0 \tanh z - U_0 \cosh^2 z$ anti Pošchl-Teller potentials *Phys. Rev. A* **17** 1273
[13] Infeld L and Hull TE 1951 The Factorization Method. *Rev. Mod. Phys.* **23** 21
[14] Cruz y Cruz S, Kuru Ş and Negro J 2008 Classical motion and coherent states for Pöschl Teller potentials *Phys. Lett. A* **372** 1391