Deformed Heisenberg algebra and minimal length

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Received 22 September 2011, in final form 9 January 2012
Published 3 February 2012
Online at stacks.iop.org/JPhysA/45/075309

Abstract

A one-dimensional deformed Heisenberg algebra \([X, P] = i f(P)\) is studied. We answer the question: for what function of deformation \(f(P)\) does there exist a nonzero minimal uncertainty in position (minimal length)? We also find an explicit expression for the minimal length in the case of an arbitrary function of deformation.

PACS numbers: 03.65.-w, 02.40.Gh

1. Introduction

Recently, much attention has been devoted to the study of deformed Heisenberg algebras of different kinds. In this paper, we focus on deformed algebras with minimal length. The history of this subject is very long. Snyder’s paper [1] was the first publication on this subject. In that paper, the Lorentz-covariant deformed Heisenberg algebra leading to quantized spacetime was proposed. For a long time, there were only a few papers on this subject. The interest in deformed algebras was renewed after investigations in string theory and quantum gravity, which suggested the existence of a nonzero minimal uncertainty in position following from the generalized uncertainty principle (GUP). In [2, 3], it was shown that the GUP and nonzero minimal uncertainty in position can be obtained from a modified Heisenberg algebra, on the right-hand side of which a term proportional to squared momentum is added. Subsequently, many papers were published where a different quantum system in space with a deformed Heisenberg algebra was studied. They are a one-dimensional harmonic oscillator with minimal uncertainty in position [2] and also with minimal uncertainty in position and momentum [4, 5], \(D\)-dimensional isotropic harmonic oscillator [6, 7], three-dimensional Dirac oscillator [8], one-dimensional Coulomb problem [9], \((1+1)\)-dimensional Dirac oscillator with a Lorentz-covariant deformed algebra [10], three-dimensional Coulomb problem with a deformed Heisenberg algebra in the frame of perturbation theory [11–15], singular inverse square potential with a minimal length [16, 17], ultracold neutrons in the gravitational field...
with minimal length [18–20], and composite system in deformed space with minimal length [21, 22].

In this paper, we study a general deformation of the one-dimensional Heisenberg algebra, the right-hand side of which is some function of momentum. As we know, up to now there has been no answer to the question about the existence of a minimal length in this general case. The aim of this paper is to fill this gap.

2. Minimal length

We consider a modified one-dimensional Heisenberg algebra generated by position \( X \) and momentum \( P \) Hermitian operators satisfying

\[
[X, P] = if(P),
\]

where \( f \) is called the function of deformation and we assume that it is strictly positive (\( f > 0 \)), even function (cf [2]).

In momentum representation both operators act on a square integrable function \( \phi(p) \in L^2(-a, a; f) (a \leq \infty) : \)

\[
P\phi(p) = p\phi(p),
\]

\[
X\phi(p) = if(p)\frac{d}{dp}\phi(p),
\]

where the norm of \( \phi \) is given by

\[
\| \phi \|^2 = \int_{-a}^{a} \frac{dp}{f(p)} |\phi(p)|^2.
\]

The Hermiticity of \( X \) demands \( \phi(-a) = \phi(a) = 0 \).

The aim of this paper is to answer the question: for what function of deformation \( f(p) \) does there exist a nonzero minimal uncertainty in position \( \Delta(X) \geq \Delta(X)_{\text{min}} \)? Nonzero minimal uncertainty in position \( \Delta(X)_{\text{min}} = l_0 \) is called the nonzero minimal length.

Further, we use the following definitions of the mean value \( \langle A \rangle_\phi \) and dispersion \( \Delta_\phi(A) \) of some operator \( A \) in the state \( \phi \in L^2(-a, a; f) \):

\[
\langle A \rangle_\phi = \int_{-a}^{a} \frac{dp}{f(p)} \phi^*(p) A \phi(p),
\]

\[
\Delta_\phi^2(A) = \langle A^2 \rangle_\phi - \langle A \rangle_\phi^2 = \int_{-a}^{a} \frac{dp}{f(p)} \phi^*(p) (A - \langle A \rangle_\phi)^2 \phi(p)
\]

for normed states \( \| \phi \|^2 = (I)_\phi = 1 \).

Let us recall two well-known facts that follow from the Heisenberg uncertainty relation

\[
\Delta_\phi^2(X) \Delta_\phi^2(P) \geq \frac{1}{4} (f(P))^2_\phi.
\]

The first one is that for the non-deformed case when \( f(p) = 1 \) the minimal length is zero. The second one states that if in the case of the function of deformation \( f(p) = 1 + \beta p^2 \) the minimal length is nonzero and reads [2, 3]

\[
l_0 = \Delta(X)_{\text{min}} = \sqrt{\beta}.
\]

For these two cases the momentum \( p \) is given on the full line \(-\infty < p < \infty\).

In the case of some other functions of deformation it is also possible to obtain the minimal length using a Heisenberg uncertainty relation. But in the general case of arbitrary \( f(p) \) it is
difficult to find the minimal length using (7) and to give an answer about the existence of a minimal length.

The aim of this paper is to relate the deformed algebra characterized by $f(P)$ to one of these two algebras, namely either to the non-deformed ($f(P) = 1$) one or to the deformed one characterized by $f(P) = 1 + \beta P^2$. We find that the minimal length in the first case is zero and in the second is nonzero. Moreover, we also find the value of the minimal length.

One can also consider a nonlinearly transformed momentum operator $Q = h(P)$, where a function $q = h(p)$ is continuous, strictly increasing on the interval $[-a, a]$, and the operator $X$ is the same for both algebras. Under this mapping, we obtain a new deformed algebra related to (1) and satisfying the relation

$$[X, Q] = i g(q), \quad g(q) = f(p) \frac{dq}{dp},$$

and we assume that the function of deformation $g$, similar to $f$, is a positive even function $^3$.

Using the function $q = h(p)$, we can change the variable $\tilde{\phi}(h(p)) = \phi(p)$ and obtain the realization of $Q, X$ in the space of square integrable functions $\tilde{\phi}(q) \in L^2(-b, b; g)$:

$$Q \tilde{\phi}(q) = q \tilde{\phi}(q), \quad X \tilde{\phi}(q) = i g(q) \frac{d\tilde{\phi}}{dq},$$

and the norms of state in both spaces are equal:

$$\| \phi \|^2 = \int_{-a}^{a} \frac{dp}{f(p)} | \phi(p) |^2 = \int_{-b}^{b} \frac{dq}{g(q)} | \tilde{\phi}(q) |^2 = \| \tilde{\phi} \|^2,$$

which follows from (9). The same holds for dispersions

$$\Delta_{\phi}^2(A) = \Delta_{\tilde{\phi}}^2(A).$$

From the second equation in (9) we find a relation between $p$ and $q$,

$$\int_{0}^{p} \frac{dp'}{f(p')} = \int_{0}^{q} \frac{dq'}{g(q')},$$

which implicitly defines the transformation $q = h(p)$. In this case, the function $h(p)$ maps the domain $-a \leq p \leq a$ onto $-b = h(-a) \leq q \leq h(a) = b$. When such mapping is possible the minimal length will be the same for the two algebras (1) and (9).

From (13) it follows that

$$\int_{0}^{a} \frac{dp}{f(p)} = \int_{0}^{b} \frac{dq}{g(q)},$$

which is equivalent to saying that a mapping $q = h(p)$ from the domain $-a \leq p \leq a$ to the domain $-b \leq q \leq b$ is possible.

We consider two cases. In the first case,

$$\int_{0}^{a} \frac{dp}{f(p)} = \infty.$$  (15)

In order to fulfill this condition, we can put $g = 1$ with $q$ given on the full line,

$$\int_{0}^{\infty} \frac{dp}{f(p)} = \int_{0}^{\infty} dq = \infty.$$  (16)

So, in this case algebra (1) is mapped to a non-deformed one ($g = 1$) on the full line and therefore the minimal length is zero.

$^3$ This assumption means that $h(p)$ is an odd function.
In the second case,
\[
\int_{0}^{a} \frac{dp}{f(p)} = \text{const} < \infty.
\tag{17}
\]
Now in order to fulfill (17), we can choose \(g(q) = 1 + \beta q^2\) and \(b = h(a) = \infty\). Then, \(\beta\) can be found from the equation
\[
\int_{0}^{a} \frac{dp}{f(p)} = \int_{0}^{\infty} \frac{dq}{1 + \beta q^2} = \frac{\pi}{2\sqrt{\beta}}.
\tag{18}
\]
In this case, algebra (1) is mapped to the deformed algebra proposed by Kempf [2, 3] and according to (8) the minimal length is
\[
l_0 = \frac{\pi}{2} \left( \int_{0}^{a} \frac{dp}{f(p)} \right)^{-1}.
\tag{19}
\]
Let us consider more explicitly a few examples.

**Example 1.**
\[
f(p) = e^{\alpha p^2},
\tag{20}
\]
where \(-\infty < p < \infty\). This function of deformation in the case \(\alpha > 0\) was recently considered in [23].

For \(\alpha = \lambda^2 > 0\) using (19), we find that the minimal length is
\[
l_0 = \lambda \sqrt{\pi}.
\tag{21}
\]
In the case \(\alpha \leq 0\), the minimal length is zero.

For the case of \(\alpha = \lambda^2\), it is also possible to find the minimal length in another way using the fact that in this case the function of deformation as a function of \(p^2\) is convex. As a result, the Heisenberg uncertainty relation (7) reads [23]
\[
\Delta_{\phi}(X) \Delta_{\phi}(P) \gtrsim \frac{1}{4} \langle e^{2\lambda P} \rangle_{\phi} \gtrsim \frac{1}{4} e^{2\lambda^2 P} \langle P^2 \rangle_{\phi},
\tag{22}
\]
and one can find that
\[
l_0 = \lambda \sqrt{\frac{e}{2}}.
\tag{23}
\]
As we see our method gives a better result for minimal length (21) in comparison with (23).

**Example 2.**
\[
f(p) = \left(1 + \lambda^2 p^2\right)^{\alpha},
\tag{24}
\]
where \(-\infty < p < \infty\). For \(\alpha \leq 1/2\) the minimal length is zero, and in the case of \(\alpha > 1/2\) the minimal length reads
\[
l_0 = \lambda \sqrt{\frac{\pi \Gamma(\alpha)}{\Gamma(\alpha - 1/2)}}.
\tag{25}
\]
It is worth noting that in the case of \(\alpha > 1\), the function of deformation as a function of \(p^2\) is convex, and using this fact it is possible to find some result for the minimal length similarly as in the first example. For other \(\alpha\) the function of deformation is not convex; nevertheless, our method gives a possibility to obtain the result for the minimal length.
Example 3. To make example 2 complete, let us consider

\[ f(p) = (1 - \lambda^2 p^2)^\alpha, \]  

where \(-1/\lambda \leq p \leq 1/\lambda\).

We find that for \(\alpha \geq 1\) the minimal length is zero, and for \(\alpha < 1\) we obtain

\[ l_0 = \lambda \frac{\sqrt{\pi} \Gamma(3/2 - \alpha)}{\Gamma(1 - \alpha)}. \]  

3. Conclusion

In this paper, we have studied deformed algebras (1) with a symmetric function of deformation and answered the following question: for what function of deformation \(f(p)\) is the minimal length nonzero? The answer to this question is given by equation (19), which presents the minimal length in the case of an arbitrary function of deformation and is the main result of the paper. When \(\int_0^a dp f(p)\) is finite the minimal length is nonzero, and when this integral diverges the minimal length is zero. Using (19) we can calculate an explicit expression for the minimal length for different functions of deformation, which is demonstrated in this paper by several examples.

Acknowledgment

VMT thanks for warm hospitality the University of Zielona Góra where this paper was written.

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