On Fast Implementation of Clenshaw-Curtis and Fejér-type Quadrature Rules

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Abstract. Based upon the fast computation of the coefficients of the interpolation polynomials at Chebyshev-type points by FFT, DCT and IDST, respectively, together with the efficient evaluation of the modified moments by forwards recursions or by the Oliver’s algorithm, this paper presents interpolating integration algorithms, by using the coefficients and modified moments, for Clenshaw-Curtis, Fejér’s first and second-type rules for Jacobi or Jacobi weights multiplied by a logarithmic function. The corresponding MATLAB codes are included. Numerical examples illustrate the stability, accuracy of the Clenshaw-Curtis, Fejér’s first and second rules, and show that the three quadratures have nearly the same convergence rates as Gauss-Jacobi quadrature for functions of finite regularities for Jacobi weights, and are more efficient upon the cpu time than the Gauss evaluated by fast computation of the weights and nodes by CHEBFUN.

Keywords. Clenshaw-Curtis-type quadrature, Fejér’s type rule, Jacobi weight, FFT, DCT, IDST.

AMS subject classifications. 65D32, 65D30

1 Introduction

The interpolation quadrature of Clenshaw-Curtis rules as well as of the Fejér-type formulas for

\[ I[f] = \int_{-1}^{1} f(x)w(x)dx \approx \sum_{k=0}^{N} w_k f(x_k) := I_N[f] \]  \hspace{1cm} (1.1)

have been extensively studied since Fejér \[7, 8\] in 1933 and Clenshaw-Curtis \[2\] in 1960, where the nodes \( \{x_k\} \) are of Chebyshev-type while the weights \( \{w_k\} \) are computed by sums of trigonometric functions.

- **Fejér’s first-type rule** uses the zeros of the Chebyshev polynomial \( T_{N+1}(x) \) of the first kind \n\[ y_j = \cos \left( \frac{2j + 1}{2N + 2} \pi \right), \quad w_j = \frac{1}{N+1} \left\{ M_0 + 2 \sum_{m=1}^{N} M_m \cos \left( \frac{m(2j + 1)}{2N + 2} \pi \right) \right\} \]  \hspace{1cm} for \( j = 0, 1, \ldots, N \), where \( \{y_j\} \) is called Chebyshev points of first kind and \( M_m = \int_{-1}^{1} w(x)T_m(x)dx \) \([23, Sommariva]\).

- **Fejér’s second-type rule** uses the zeros of the Chebyshev polynomial \( U_{N+1}(x) \) of the second kind \n\[ x_j = \cos \left( \frac{j + 1}{N + 2} \pi \right), \quad w_j = \frac{2\sin \left( \frac{j + 1}{N + 2} \pi \right)}{N + 2} \sum_{m=0}^{N} \tilde{M}_m \sin \left( (m + 1) \frac{j + 1}{N + 2} \pi \right) \]  \hspace{1cm} for \( j = 0, 1, \ldots, N \), where \( \{x_j\} \) is called Chebyshev points of second kind or Filippi points and \( \tilde{M}_m = \int_{-1}^{1} w(x)U_m(x)dx \) \([23, Sommariva]\).

- **Clenshaw-Curtis-type quadrature** is to use the Clenshaw-Curtis points \n\[ \tilde{x}_j = \cos \left( \frac{j\pi}{N} \right), \quad w_j = \frac{2}{N} \alpha_j \sum_{m=0}^{N} \tilde{M}_m \cos \left( \frac{j m \pi}{N} \right), \quad j = 0, 1, \ldots, N \]
where the double prime denotes a sum whose first and last terms are halved, \( \alpha_0 = \alpha_N = \frac{1}{2} \), and \( \alpha_j = 1 \) for \( 1 \leq j \leq N - 1 \) ([21] Sloan and Smith).

In the case \( w(x) \equiv 1 \), a connection between the Fejér and Clenshaw-Curtis quadrature rules and DFTs was given by Gentleman [9] in 1972, where the Clenshaw-Curtis rule is implemented with \( N + 1 \) nodes by means of a discrete cosine transformation. An independent approach along the same lines, unified algorithms based on DFTs of order \( n \) for generating the weights of the two Fejér rules and of the Clenshaw-Curtis rule, was presented in Waldvogel [27] in 2006. A streamlined Matlab code is given as well in [27]. In addition, Clenshaw and Curtis [2], Hara and Smith [12], Trefethen [24, 25], Xiang and Bornemann in [29], and Xiang [30, 31], etc., showed that the Gauss, Clenshaw-Curtis and Fejér quadrature rules are about equally accurate.

More recently, Sommariva [23], following Waldvogel [27], showed that for general weight function \( w \), the weights \( \{w_k\} \) corresponding to Clenshaw-Curtis, Fejér’s first- and second-type rules can be computed by IDCT (inverse discrete cosine transform) and DST (discrete sine transform) once the weighted modified moments of Chebyshev polynomials of the first and second kind are available, which generalized the techniques of [27] if the modified moments can be rapidly evaluated.

In this paper, along the way [24] Trefethen], we consider interpolation approaches for Clenshaw-Curtis rules as well as of the Fejér’s first and second-type formulas, and present Matlab codes for

\[
I[f] = \int_{-1}^{1} f(x)w(x)dx
\]  

(1.2)

for \( w(x) = (1 - x)^{\alpha}(1 + x)^{\beta} \) or \( w(x) = (1 - x)^{\alpha}(1 + x)^{\beta} \ln \left( \frac{1 + x}{1 - x} \right) \), which can be efficiently calculated by FFT, DCT and IDST (inverse DST), respectively: Suppose \( Q_N[f](x) = \sum_{j=0}^{N} a_j T_j(x) \) is the interpolation polynomial at \( \{y_j\} \) or \( \{\tau_j\} \), then the coefficients \( a_j \) can be efficiently computed by FFT [9, 24] for Clenshaw-Curtis and by DCT for the Fejér’s first-type rule, respectively, and then \( I_N[f] = \sum_{j=0}^{N} a_j M_j(\alpha, \beta) \). So is the interpolation polynomial at \( \{x_j\} \) in the form of \( Q_N[f](x) = \sum_{j=0}^{N} a_j U_j(x) \) by IDST for the Fejér’s second-type rule with \( I_N[f] = \sum_{j=0}^{N} a_j \tilde{M}_j(\alpha, \beta) \). An elegant Matlab code on the coefficients \( a_j \) by FFT for Clenshaw-Curtis points can be found in [24]. Furthermore, here the modified moments \( M_j(\alpha, \beta) \) and \( \tilde{M}_j(\alpha, \beta) \) can be fast computed by forwards recursions or by Oliver’s algorithms with \( O(N) \) operations.

Notice that the fast implementation routine based on the weights \( \{w_k\} \) or the coefficient \( \{a_k\} \) both will involve in fast computation of the modified moments. In section 2, we will consider algorithms and present Matlab codes on the evaluation of the modified moments. Matlab codes for the three quadratures are presented in section 3, and illustrated by numerical examples in section 4.

## 2 Computation of the modified moments

Clenshaw-Curtis-type quadratures are extensively studied in a series of papers by Piessens [15, 16] and Piessens and Branders [17, 18, 19]. The modified moment \( \int_{-1}^{1} w(x)T_j(x)dx \) can be efficiently evaluated by recurrence formulae for Jacobi weights or Jacobi weights multiplied by \( \ln((x + 1)/2) \) [15] Piessens and Branders] in most cases.

- For \( w(x) = (1 - x)^{\alpha}(1 + x)^{\beta} \): The recurrence formula for the evaluation of the modified moments

\[
M_k(\alpha, \beta) = \int_{-1}^{1} w(x)T_k(x)dx, \quad w(x) = (1 - x)^{\alpha}(1 + x)^{\beta}
\]  

(2.3)

by using Fasenmyer’s technique is

\[
(\beta + \alpha + k + 2)M_{k+1}(\alpha, \beta) + 2(\alpha - \beta)M_k(\alpha, \beta) + (\beta + \alpha - k + 2)M_{k-1}(\alpha, \beta) = 0
\]  

(2.4)
Recursion is catastrophic particularly when $\alpha$ the cases (2.5) or (2.6) (see Table 1).

For this case, we use the Oliver’s method [14] with two starting values and one end value to compute approximately as

$$M_0(\alpha, \beta) = 2^{\alpha+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\beta+\alpha+2)}, \quad M_1(\alpha, \beta) = 2^{\alpha+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\beta+\alpha+2)} \frac{\beta - \alpha}{\beta + \alpha + 2}. $$

The forward recursion is numerically stable [15] Piessens and Branders, except in two cases:

$$\alpha > \beta \quad \text{and} \quad \beta = -1/2, 1/2, 3/2, \ldots$$  \quad (2.5)

$$\beta > \alpha \quad \text{and} \quad \alpha = -1/2, 1/2, 3/2, \ldots$$  \quad (2.6)

For $w(x) = \ln((x + 1)/2)(1 - x)^\alpha(1 + x)^\beta$: For

$$G_k(\alpha, \beta) = \int_{-1}^{1} \ln((x + 1)/2)(1 - x)^\alpha(1 + x)^\beta T_k(x) dx,$$  \quad (2.7)

the recurrence formula [15] is

$$(\beta + \alpha + k + 2)G_{k+1}(\alpha, \beta) + 2(\beta - \alpha)G_k(\alpha, \beta) + (\beta + \alpha - k + 2)G_{k-1}(\alpha, \beta) = 2M_k(\alpha, \beta) - M_{k-1}(\alpha, \beta) - M_{k+1}(\alpha, \beta)$$  \quad (2.8)

with

$$G_0(\alpha, \beta) = -2^{\alpha+1}\Phi(\alpha, \beta + 1), \quad G_1(\alpha, \beta) = -2^{\alpha+1}[2\Phi(\alpha, \beta + 2) - \Phi(\alpha, \beta + 1)],$$

where

$$\Phi(\alpha, \beta) = B(\alpha + 1, \beta)\Psi(\alpha + \beta + 1) - \Psi(\beta),$$

$B(x, y)$ is the Beta function and $\Psi(x)$ is the Psi function [1] Abramowitz and Stegun]. The forward recursion is numerically stable the same as for (2.4) except for (2.5) or (2.6) [15] Piessens and Branders.

Thus, the modified moments can be fast computed by the forward recursions (2.4) or (2.8) except the cases (2.5) or (2.6) (see Table 1).

For the weight $(1 - x)^\alpha(1 + x)^\beta$ in the cases of (2.5) or (2.6): The accuracy of the forward recursion is catastrophic particularly when $\alpha - \beta \gg 1$ and $n \gg 1$ (also see Table 2): In case (2.5) the relative errors $\epsilon_n$ of the computed values $M_n(\alpha, \beta)$ obtained by the forward recursion behave approximately as

$$\epsilon_n \sim n^{2(\alpha - \beta)}, \quad n \to \infty$$

and in case (2.6) as

$$\epsilon_n \sim n^{2(\beta - \alpha)}, \quad n \to \infty.$$  

For this case, we use the Oliver’s method [14] with two starting values and one end value to compute the modified moments. Let

$$A_N := \begin{pmatrix}
2(\alpha - \beta) & \alpha + \beta + 2 + 0 \\
\alpha + \beta + 2 - 1 & 2(\alpha - \beta) & \alpha + \beta + 2 + 1 \\
& \ddots & \ddots & \ddots \\
& & \alpha + \beta + 2 - (N - 1) & 2(\alpha - \beta) & \alpha + \beta + 2 + (N - 1) \\
& & & \alpha + \beta + 2 - N & 2(\alpha - \beta)
\end{pmatrix},$$  \quad (2.9)

$$b_N := \begin{pmatrix}
2^{\alpha+1} \Gamma(\alpha+1) \Gamma(\beta+1) \frac{\Gamma(\alpha+\beta+2)}{} \end{pmatrix}^T,$$  \quad (2.10)

where $\cdot^T$ denotes the transpose, then the modified moments $M$ can be solved by

$$A_N M = b_N, \quad M = (M_0, M_1, \ldots, M_N)^T,$$  \quad (2.11)

where $M_{N+1}$ is computed by hypergeometric function [15] for $N \leq 2000$,

$$M_{N+1} = 2^{\alpha+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \begin{pmatrix}
\alpha - \beta \\
\alpha + \beta + 2 + 1 \\
\alpha + \beta + 2 - (N - 1) \\
\alpha + \beta + 2 - N
\end{pmatrix}.$$  \quad (2.12)
Particularly, if \( N > 2000 \), \( M_{N+1} \) is computed by the following asymptotic expression. Taking a change of variables \( x = \cos(\theta) \) for \( \theta \), it yields

\[
M_n(\alpha, \beta) = \int_0^\pi \varphi(\theta)\theta^{2\alpha+1}(\pi - \theta)^{2\beta+1}\cos(n\theta)d\theta,
\]

where

\[
\varphi(\theta) = \left( \frac{1 - \cos(\theta)}{\theta^2} \right)^{\alpha + \frac{1}{2}} \left( \frac{1 + \cos(\theta)}{(\pi - \theta)^2} \right)^{\beta + \frac{1}{2}},
\]

then it holds that

\[
M_n(\alpha, \beta) = 2^{\beta - \alpha} \sum_{k=0}^{m-1} a_k(\alpha, \beta)h(\alpha + k) + (-1)^n 2^{\alpha - \beta} \sum_{k=0}^{m-1} a_k(\beta, \alpha)h(\beta + k) + O(n^{-2m}) \quad (2.13)
\]

by means of the Theorem 3 in [5, Erdélyi], in which

\[
a_0(\alpha, \beta) = 1, \quad a_1(\alpha, \beta) = -\frac{\alpha}{12} - \frac{\beta}{4} - \frac{1}{6}, \quad a_2(\alpha, \beta) = \frac{1}{120} + \frac{19\alpha}{1440} + \frac{\alpha^2}{288} + \frac{\alpha \beta}{48} + \frac{\beta}{32} + \frac{\beta^2}{32},
\]

and

\[
a_3(\alpha, \beta) = -\frac{1}{3040} - \frac{\beta}{960} - \frac{107\alpha}{181440} - \frac{\beta^2}{384} - \frac{\alpha^2}{1920} - \frac{\beta^3}{384} - \frac{\alpha^3}{10368} - \frac{7\alpha \beta}{2880} - \frac{\alpha^2 \beta}{1152} - \frac{\alpha \beta^2}{384}.
\]

The Oliver’s algorithm can be fast implemented by applying LU factorization (chasing method) with \( O(N) \) operations.

In the case \((2.6)\), by \( x = -t \) and \( T_n(-x) = \begin{cases} T_n(x), & n \text{ even} \\ -T_n(x), & n \text{ odd} \end{cases} \), the computation of the moments can be transferred into the case \((2.5)\).

**In addition, for the weight** \( w(x) = \ln((x + 1)/2)(1 - x)^\alpha(1 + x)^\beta \), **in the case** \((2.5)\): The forward recursion \((2.8)\) is also perfectly numerically stable (see Table 5) even for \( \alpha \gg \beta \). However, in the case \((2.6)\), the forward recursion \((2.8)\) collapses, which can be fixed up by the Oliver’s algorithm similar to \((2.9)\) with two starting values \( G_0(\alpha, \beta), G_1(\alpha, \beta) \) and one end value \( G_{N+10^3}(\alpha, \beta) \), by solving an \( (N + 10^3 + 1) \times (N + 10^3 + 1) \) linear system for the first \( N + 1 \) moments. The end value can be calculated by its asymptotic formula, by a change of variables \( x = \cos(\theta) \) for \((2.7)\) and using \( \ln(\frac{1 + \cos(\theta)}{2}) = \ln(\frac{1 + \cos(\theta)}{2\pi - \theta}) + 2\ln(\pi - \theta) \), together with the Theorems in [5, Erdélyi], as

\[
G_n(\alpha, \beta) = 2^{\beta - \alpha} \sum_{k=0}^{m-1} c_k h(\alpha + k) + (-1)^n 2^{\alpha - \beta} \sum_{k=0}^{m-1} h(\beta + k) \left( 2a_k(\beta, \alpha)\phi(\beta + k) + b_k \right) + O(n^{-2m}),
\]

where

\[
\phi(\beta) = \Psi(2\beta + 2) - \ln(2n) - \frac{\pi}{2} \tan(\pi \beta),
\]

and

\[
\begin{cases}
    b_0 = 0, & b_1 = -\frac{\beta}{96} - \frac{\alpha}{384} + \frac{\beta}{192} + \frac{\alpha}{384}, \\
b_2 = \frac{7\alpha}{2880} - \frac{\beta}{960} - \frac{\alpha^2}{3360} - \frac{\beta^2}{1920} + \frac{\alpha \beta}{384} - \frac{\alpha^2}{10368} - \frac{\alpha \beta}{1152} - \frac{\beta^2}{384}, \\
b_3 = -\frac{7\alpha}{2880} - \frac{\beta}{960} - \frac{\alpha^2}{3360} - \frac{\beta^2}{1920} + \frac{\alpha \beta}{384} - \frac{\alpha^2}{10368} - \frac{\alpha \beta}{1152} - \frac{\beta^2}{384}.
\end{cases}
\]

Tables 3-6 show the accuracy of the Oliver’s algorithm for different \((\alpha, \beta)\), and Table 7 shows the cpu time for implementation of the two Oliver’s algorithms. Here, Oliver-1 means that the Oliver’s algorithms with the end value computed by one term of asymptotic expansions, while Oliver-4 signifies that the end value is calculated by four terms of asymptotic expansions. The Oliver-4 can also be applied to the case \((2.5)\) for the Jacobi weight multiplied by \( \ln((x + 1)/2) \), which can be seen from Table 5 (the Oliver-4 is better than the forward recursion \((2.8)\) in the case \((2.5)\)).
The MATLAB codes on the Oliver’s algorithms and all the MATLAB codes in this paper can be downloaded from [32]. The all codes and numerical experiments in this paper are implemented in a Lenovo computer with Intel Core 3.20GHz and 3.47GB Ram.

Table 1: Computation of $M_n(\alpha, \beta)$ and $G_n(\alpha, \beta)$ with different $n$ and $(\alpha, \beta)$ by the forward recursion (2.4) and (2.8) respectively

| n   | $M_n(-0.6, -0.5)$ | $M_n(-0.6, -0.5)$ | $M_n(-0.6, -0.5)$ | $M_n(-0.6, -0.5)$ |
|-----|------------------|------------------|------------------|------------------|
| 10  | 0.061104330977316 | 0.061104330977316 | 0.00881657781753  | 0.00881657781753  |
| 100 | 0.01535055343264  | 0.01535055343264  | 0.000881657781753 | 0.000881657781753 |
| 1000| 0.000881657781753 | 0.000881657781753 | 0.000881657781753 | 0.000881657781753 |
| 2000| 0.000881657781753 | 0.000881657781753 | 0.000881657781753 | 0.000881657781753 |

Table 2: Computation of $M_n(\alpha, \beta) = \int_1^{-1} (1-x)^\alpha (1+x)^\beta T_n(x) dx$ with different $n$ and $(\alpha, \beta)$

| n   | 5      | 10     | 100    |
|-----|--------|--------|--------|
| Exact value for (20,-0.5) | -1.73481054604316e+05 | 4.04900366168904e+03 | -3.083991348593134e-41 |
| Approximation by (2.4) for (20,-0.5) | -1.734810546043088e+05 | 4.0490366190983e+03 | 1.787242305340324e-11 |
| Exact value for (100,-0.5) | -2.471295049468578e+29 | 1.174275526131312e+29 | 2.805165440968788e+13 |
| Approximation by (2.4) for (100,-0.5) | -2.471295049468764e+29 | 1.174275526131312e+29 | -1.380038973213404e+13 |

Table 3: Computation of $M_n(\alpha, \beta) = \int_1^{-1} (1-x)^\alpha (1+x)^\beta T_n(x) dx$ with $(\alpha, \beta) = (100, -0.5)$ and different $n$ by the Oliver’s algorithm

| n   | 2000   | 4000   | 8000   |
|-----|--------|--------|--------|
| Exact value for (0.6,-0.5) | 9.55168420184334e-12 | 1.039402748103725e-12 | 1.13106574497495c-13 |
| Oliver-4 for (0.6,-0.5) | 9.55168420184334e-12 | 1.039402748103725e-12 | 1.13106574497495c-13 |
| Oliver-1 for (0.6,-0.5) | 9.55168420184334e-12 | 1.039402748103725e-12 | 1.13106574497495c-13 |
| Exact value for (10,-0.5) | -8.41234592129556e-57 | -2.005493070382270e-63 | -4.78136884895069e-70 |
| Oliver-4 for (10,-0.5) | -8.41234592129556e-57 | -2.005493070382270e-63 | -4.78136884895069e-70 |
| Oliver-1 for (10,-0.5) | -8.41234592129556e-57 | -2.005493070382270e-63 | -4.78136884895069e-70 |

Table 4: Computation of $G_n(\alpha, \beta) = \int_1^{-1} (1-x)^\alpha (1+x)^\beta \ln(1+x) T_n(x) dx$ with different $n$ and $(\alpha, \beta)$ by the Oliver’s algorithm

| n   | 10     | 100    | 500    |
|-----|--------|--------|--------|
| Exact value for (-0.4999,-0.5) | -0.314181354550401 | -0.031418104511487 | -0.006283620842004 |
| Oliver-4 for (-0.4999,-0.5) | -0.314181354550401 | -0.031418104511487 | -0.006283620842004 |
| Oliver-1 for (-0.4999,-0.5) | -0.314181354550401 | -0.031418104511487 | -0.006283620842004 |
| Exact value for (0.9999,-0.5) | -0.895286620533541 | -0.08885816406923 | -0.01777035274330 |
| Oliver-4 for (0.9999,-0.5) | -0.895286620533541 | -0.08885816406923 | -0.01777035274330 |
| Oliver-1 for (0.9999,-0.5) | -0.895286620533541 | -0.08885816406923 | -0.01777035274330 |
Table 5: Computation of $G_n(\alpha, \beta) = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \ln((1+x)/2) T_n(x) \, dx$ with $(\alpha, \beta) = (100, -0.5)$ and different $n$ by the Oliver’s algorithm

| $n$ | 100                  | 500                  | 1000                  |
|-----|----------------------|----------------------|-----------------------|
| Exact value for $(100,-0.5)$ | $-5.660760361182362e+28$ | $-1.126631188200461e+28$ | $-5.632306274999927e+27$ |
| Oliver-4 for $(100,-0.5)$ | $-5.660760361182364e+28$ | $-1.126631188200460e+28$ | $-5.632306274999938e+27$ |
| Oliver-1 for $(100,-0.5)$ | $-5.660525683370006e+28$ | $-1.12660659170211e+28$ | $-5.632235588089685e+27$ |
| (2.8) for $(100,-0.5)$ | $-5.660760361182770e+28$ | $-1.126631188200544e+28$ | $-5.632306275000348e+27$ |

Table 6: Computation of $G_n(\alpha, \beta) = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \ln((1+x)/2) T_n(x) \, dx$ with $(\alpha, \beta) = (-0.5, 100)$ and different $n$ by the Oliver’s algorithm compared with that computed by the forward recursion (2.8)

| $n$ | 100                  | 500                  | 1000                  |
|-----|----------------------|----------------------|-----------------------|
| Exact value for $(-0.5,100)$ | $1.089944378602585e-28$ | $7.222157005510196e-198$ | $5.715301877322031e-259$ |
| Oliver-4 for $(-0.5,100)$ | $1.089944378602615e-28$ | $7.222157005510282e-198$ | $5.715301877322160e-259$ |
| Oliver-1 for $(-0.5,100)$ | $1.089944378602615e-28$ | $7.222157005510282e-198$ | $5.715301877322160e-259$ |
| (2.8) for $(-0.5,100)$ | $-5.331299059334499e+14$ | $-1.061058894110758e+14$ | $-5.304494050667818e+13$ |

Table 7: The cpu time for calculation of the modified moments by the Oliver-4 method for $\alpha = -0.5$ and $\beta = 100$

| modified moments | $N = 10^3$ | $N = 10^4$ | $N = 10^5$ | $N = 10^6$ |
|------------------|------------|------------|------------|------------|
| $\{M_n(\alpha, \beta)\}_{n=0}^N$ | 0.004129s  | 0.012204s  | 0.120747s  | 1.119029s  |
| $\{G_n(\alpha, \beta)\}_{n=0}^N$ | 0.006026s  | 0.029010s  | 0.295988s  | 2.902172s  |

The MATLAB codes for weights $M_n(\alpha, \beta)$ and $G_n(\alpha, \beta)$ are as follows:

- **A MATLAB code for weight** $M_n(\alpha, \beta) = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} T_n(x) \, dx$

  ```matlab
  function M=momentsJacobiT(N,alpha,beta) % (N+1) modified moments on T
  f(1)=1;f(2)=(beta-alpha)/(2+beta+alpha); % initial values
  for k=1:N-1
    f(k+2)=1/(beta+alpha+2+k)*(2*(beta-alpha)*f(k+1)-(beta+alpha-k+2)*f(k));
  end;
  M=2^(beta+alpha+1)*gamma(alpha+1)*gamma(beta+1)/gamma(alpha+beta+2)*f;
  end
  ```

- **A MATLAB code for weight** $G_n(\alpha, \beta) = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} \ln((1+x)/2) T_n(x) \, dx$

  ```matlab
  function G=momentslogJacobiT(N,alpha,beta) % (N+1) modified moments on T
  M=momentsJacobiT(N+1,alpha,beta); % modified moments on T
  Phi=inline('beta(x+1,y)*(psi(x+y+1)-psi(y))','x','y');
  G(1)=-2^(alpha+beta+1)*Phi(alpha,beta+1);
  G(2)=-2^(alpha+beta+2)*Phi(alpha+beta+2)-G(1);
  for k=1:N-1
    G(k+2)=1/(beta+alpha+2+k)*2*(beta-alpha)*G(k+1)-(beta+alpha-k+2)*G(k)+2*M(k+1)-M(k)-M(k+2);
  end;
  ```
The modified moments \( \tilde{M}_k(\alpha, \beta) = \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta U_k(x) dx \) on Chebyshev polynomials of second kind \( U_k \) were considered in Sommariva [23] by using the formulas

\[
U_n(x) = \begin{cases} 
2 \sum_{j=0}^{n} \text{odd } T_j(x), & n \text{ odd} \\
2 \sum_{j=0}^{n} \text{even } T_j(x) - 1, & n \text{ even},
\end{cases}
\]

which takes \( O(N^2) \) operations for the \( N \) moments if \( M_k(\alpha, \beta) \) are available. The modified moments \( \tilde{M}_k(\alpha, \beta) \) can be efficiently calculated with \( O(N) \) operations by using

\[
(1 - x^2) U_n' = -kxU_k + (k+1)U_{k-1}
\]

(see Abramowitz and Stegun [1, pp. 783]) and integrating by parts as

\[
(\beta + \alpha + k + 2) \tilde{M}_{k+1}(\alpha, \beta) + 2(\alpha - \beta) \tilde{M}_k(\alpha, \beta) + (\beta + \alpha - k) \tilde{M}_{k-1}(\alpha, \beta) = 0
\]

(2.15)

with

\[
\tilde{M}_0(\alpha, \beta) = M_0(\alpha, \beta), \quad \tilde{M}_1(\alpha, \beta) = 2M_1(\alpha, \beta),
\]

while for \( \tilde{G}_k(\alpha, \beta) = \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta \ln((x+1)/2) U_k(x) dx \) as

\[
(\beta + \alpha + k + 2) \tilde{G}_{k+1}(\alpha, \beta) + 2(\alpha - \beta) \tilde{G}_k(\alpha, \beta) + (\beta + \alpha - k) \tilde{G}_{k-1}(\alpha, \beta) = 2\tilde{M}_k(\alpha, \beta) - \tilde{M}_{k-1}(\alpha, \beta) - \tilde{M}_{k+1}(\alpha, \beta)
\]

(2.16)

with

\[
\tilde{G}_0(\alpha, \beta) = G_0(\alpha, \beta), \quad \tilde{G}_1(\alpha, \beta) = 2G_1(\alpha, \beta).
\]

To keep the stability of the algorithms, here we use the following simple equation

\[
U_{k+2} = 2T_{k+2} + U_k \quad \text{(see [1] pp. 778)}
\]

(2.17)

to derive the modified moments with \( O(N) \) operations.

• A MATLAB code for weight \( \tilde{M}_n(\alpha, \beta) = \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta U_n(x) dx \)

```matlab
function U=momentsJacobiU(N,alpha,beta) % modified moments on 1
U(1)=M(1); U(2)=2*M(2); % initial moments
for k=1:N-1, U(k+2)=2*M(k+2)+U(k); end
```

• A MATLAB code for weight \( \tilde{G}_n(\alpha, \beta) = \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta \ln((1 + x)/2) U_n(x) dx \)

```matlab
function U=momentslogJacobiU(N,alpha,beta) % modified moments on 1
U(1)=G(1); U(2)=2*G(2); % initial moments
for k=1:N-1, U(k+2)=2*G(k+2)+U(k); end
```

3 MATLAB codes for Clenshaw-Curtis and Fejér-type quadrature rules

The coefficients \( a_j \) for the interpolation polynomial at \( \{\tau_j\} \) can be efficiently computed by FFT [21].

For the Clenshaw-Curtis, we shall not give details but just offer the following MATLAB functions.

• For \( I[f] = \int_{-1}^{1} (1 - x)^\alpha (1 + x)^\beta f(x) dx \)

A MATLAB code for \( I_{NC}^{C}[f] \):

```matlab
function I=clenshaw_curtis(f,N,alpha,beta) % (N+1)-pt C-C quadrature
x=cos(pi*(0:N)'/N); % C-C points
fx=feval(f,x)/(2*N); % f evaluated at these points
G=fft(fx([1:N+1 N:-1:2])); % FFT
a=[g(1); g(2:N)+g(2*N:-1:N+2); g(N+1)]; % Chebyshev coefficients
I=momentsJacobiT(N,alpha,beta)*a; % the integral
```

```matlab
function I=clenshaw_curtis(f,N,alpha,beta) % (N+1)-pt C-C quadrature
x=cos(pi*(0:N)'/N); % C-C points
fx=feval(f,x)/(2*N); % f evaluated at these points
G=fft(fx([1:N+1 N:-1:2])); % FFT
a=[g(1); g(2:N)+g(2*N:-1:N+2); g(N+1)]; % Chebyshev coefficients
I=momentsJacobiT(N,alpha,beta)*a; % the integral
```
For $I[f] = \int_{-1}^{1} (1-x)^a(1+x)^b \ln((1+x)/2)f(x)\,dx$

A Matlab code for $I_N^{C-C}[f]$:

 function I=clenshaw_curtislogJacobi(f,N,alpha,beta) \% (N+1)-pt C-C quadrature
 x=cos(pi*(0:N)/N); \% C-C points
 fx=feval(f,x)/(2*N); \% f evaluated at the points
 g=fft(fx([1:N+1 N:-1:2])); \% FFT
 a=[g(1); g(2:N)+g(2*N:-1:N+2); g(N+1)]; \% Chebyshev coefficients
 I=momentslogJacobiT(N,alpha,beta)*a; \% the integral

The discrete cosine transform DCT denoted by $Y = \text{dct}(X)$ is closely related to the discrete Fourier transform but using purely real numbers, and takes $O(N \log N)$ operations for

$$Y(k) = w(k) \sum_{s=1}^{N} X(s) \cos \left( \frac{(k-1)\pi(2s-1)}{2N} \right)$$
with $w(1) = \frac{1}{\sqrt{N}}$ and $w(k) = \sqrt{2} \frac{1}{\sqrt{N}}$ for $2 \leq k \leq N$.

The discrete sine transform DST denoted by $Y = \text{dst}(X)$ and its inverse The inverse discrete sine transform IDST denoted by $X = \text{idst}(Y)$ both takes $O(N \log N)$ operations for

$$Y(k) = \sum_{s=1}^{N} X(s) \sin \left( \frac{k\pi s}{N+1} \right).$$

Note that the coefficients $a_j$ for the interpolation polynomial $Q_N(x) = \sum_{j=1}^{N} a_{j-1} T_{j-1}(x)$ at $\cos \left( \frac{(2k-1)\pi}{2N} \right)$ are represented by

$$a_{j-1} = \frac{2}{N} \sum_{s=1}^{N} f \left( \cos \left( \frac{(2s-1)\pi}{2N} \right) \cos \left( \frac{(2s-1)(j-1)\pi}{2N} \right) \right), \quad j = 1, 2, \ldots, N,$$

and $a_j$ for the interpolation polynomial $Q_N(x) = \sum_{j=1}^{N} a_{j-1} U_{j-1}(x)$ at $\cos \left( \frac{k\pi}{N+1} \right)$ satisfies

$$f \left( \cos \left( \frac{j\pi}{N+1} \right) \sin \left( \frac{j\pi}{N+1} \right) \right) = \sum_{s=1}^{N} a_{s-1} \sin \left( \frac{s\pi}{N+1} \right), \quad j = 1, 2, \ldots, N.$$

Then both can be efficiently calculated by DCT and IDST respectively.

For $I[f] = \int_{-1}^{1} (1-x)^a(1+x)^b f(x)\,dx$

A Matlab code for $I_N^{F-1}[f]$:

 function I=fejer1Jacobi(f,N,alpha,beta) \% (N+1)-pt Fejér’s first rule
 x=cos(pi*(2*(0:N)'+1)/(2*N+2)); \% Chebyshev points of 1st kind
 fx=feval(f,x); \% f evaluated at these points
 a=dct(fx)*sqrt(2/(N+1));a(1)=a(1)/sqrt(2); \% Chebyshev coefficients
 I=momentsJacobiT(N,alpha,beta)*a; \% the integral

A Matlab code for $I_N^{F-2}[f]$:

 function I=fejer2Jacobi(f,N,alpha,beta) \% (N+1)-pt Fejér’s second rule
 x=cos(pi*(1:N+1)'/(N+2)); \% Chebyshev points of 2nd kind
 fx=feval(f,x).*sin(pi*(1:N+1)'/(N+2)); \% f evaluated at these points
 a=idst(fx); \% Chebyshev coefficients
 I=momentsJacobiU(N,alpha,beta)*a; \% the integral
For \( I[f] = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta \ln((1+x)/2) f(x) \, dx \)

A Matlab code for \( I_N^F[f] \):

```matlab
function I=fejer1logJacobi(f,N,alpha,beta) % (N+1)-pt Féjér’s first rule
x=cos(pi*(2*(0:N)’+1)/(2*N+2)); % Chebyshev points of 1st kind
fx=feval(f,x); % f evaluated at these points
a=dct(fx)*sqrt(2/(N+1));a(1)=a(1)/sqrt(2); % Chebyshev coefficients
I=momentslogJacobiT(N,alpha,beta)*a; % the integral
```

A Matlab code for \( I_N^F[f] \):

```matlab
function I=fejer2logJacobi(f,N,alpha,beta) % (N+1)-pt Féjér’s second rule
x=cos(pi*(1:N+1)’/(N+2)); % Chebyshev points of 2nd kind
fx=feval(f,x).*sin(pi*(1:N+1)’/(N+2)); % f evaluated at these points
a=idst(fx); % Chebyshev coefficients
I=momentslogJacobiU(N,alpha,beta)*a; % the integral
```

**Remark 3.1** The coefficients \( \{a_j\}_{j=0}^N \) for Clenshaw-Curtis can also be computed by idst, while the coefficients for Féjér’s rules can be computed by FFT. The following table shows the total time for calculation of the coefficients for \( N = 10^2 : 10^4 \).

| Clenshaw-Curtis | Féjér first | Féjér second |
|-----------------|-------------|--------------|
| FFT: 10.539741s | FFT: 16.127888s | FFT: 9.608675s |
| idst: 12.570079s | dct: 10.449258s | idst: 10.256482s |

From Table 8, we see that the coefficients computed by the FFT is more efficient than that by the idst for Clenshaw-Curtis, the coefficients computed by the dct more efficient than that by the FFT for Féjér first rule, and the coefficients of the interpolant for the second kind of Chebyshev polynomials \( U_n \) computed by the idst nearly equal to that for the first kind of Chebyshev polynomials \( T_n \) by the FFT for Féjér second rule. Notice that the FFT’s for Féjér’s rules involves computation of complex numbers. Here we adopt dct and idst for the two rules.

### 4 Numerical examples

The convergence rates of the Clenshaw-Curtis, Féjér’s first and second rules have been extensively studied in Clenshaw and Curtis [2], Hara and Smith [12], Riess and Johnson [20], Sloan and Smith [21, 22], Trefethen [21, 25], Xiang and Bornemann in [29], and Xiang [30, 31], etc. In this section, we illustrate the accuracy and efficiency of the Clenshaw-Curtis, Féjér’s first and second-type rules for the two functions \( \tan|x| \) and \( |x-0.5|^{0.6} \) by the algorithms presented in this paper, comparing with those by the Gauss-Jacobi quadrature used \( [x, w] = jacpts(n, \alpha, \beta) \) in CHEBFUN v4.2 [25] (see Figure 1), where the Gauss weights and nodes are fast computed with \( O(N) \) operations by Hale and Townsend [11] based on Glaser, Liu and V. Rokhlin [10]. The first column computed by Gauss-Jacobi quadrature in Figure 1 takes 51.959797 seconds and the others totally take 2.357053 seconds. Additionally, the Gauss-Jacobi quadrature completely fails to compute \( I[f] = \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta T_n(x) \, dx \) for \( \alpha \gg 1 \) and \( n \gg 1 \), e.g., \( \alpha = 100, \beta = -0.5 \) and \( n = 100 \) (see Table 9). Figure 2 shows the convergence errors by the three quadrature, which takes 7.336958 seconds.

Sommariva [26] showed the efficiency of the computation of the weights \( \{w_k\} \) corresponding to Clenshaw-Curtis, Féjér’s first and second-type rules can be computed by IDCT and DST for the
Figure 1: The absolute errors compared with Gauss quadrature, $n^{-2}$ and $n^{-1.6}$, respectively, for $\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x)\,dx$ evaluated by the Clenshaw-Curtis, Fejér’s first and second-type rules with $n$ nodes: $f(x) = \tan |x|$ or $|x - 0.5|^{0.6}$ with different $\alpha$ and $\beta$ and $n = 10 : 1000$.

Figure 2: The absolute errors compared with $n^{-2} \ln(n)$ and $n^{-1.6}$, respectively, for $\int_{-1}^{1} f(x)\,dx$ evaluated by the Clenshaw-Curtis, Fejér’s first and second-type rules with $n$ nodes: $f(x) = \tan |x|$ or $|x - 0.5|^{0.6}$ with different $\alpha$ and $\beta$ and $n = 10 : 1000$. 

Table 9: Gauss-Jacobi quadrature $I_n[f]$ for $\int_{-1}^{1} (1-x)^{100} (1+x)^{-0.5} T_{100}(x) \, dx$ with $n$ nodes

| Exact value      | $n = 10^2$ | $n = 10^3$ | $n = 10^4$ | $n = 10^5$ |
|------------------|------------|------------|------------|------------|
| $2.805165440968788 \times 10^{-29}$ | $5.428948613067786 \times 10^{16}$ | $3.412774141539268 \times 10^{16}$ | $8.907453940922673 \times 10^{17}$ | NaN |

Gegenbauer weight function

$$w(x) = (1-x)^{\lambda-1/2}, \quad \lambda > -1/2$$

with $\lambda = 0.75$ and $N = 2^k$ for $k = 1, \ldots, 20$. Here the modified moments $M_n(\lambda - 1/2, \lambda - 1/2)$ are available (see [13, Hunter and Smith]). Table 10 illustrates the cpu time of the computation of the weights $\{w_k\}$ for the computation of Clenshaw-Curtis, Fejér’s first and second-type rules by the algorithms given in [23], compared with the cpu time of the computation of the coefficients $\{a_k\}$ for the three quadrature by the FFT, DCT and IDST in section 3.

Table 10: The cpu time for calculation of the weight $\{w_k\}_{k=0}^N$ by the algorithms given in [23] and the coefficients $\{a_k\}_{k=0}^N$ by the FFT, DCT and IDST in section 3

| $\{w_k\}_{k=0}^N$ | C-C | Fejér I | Fejér II |
|-------------------|-----|---------|----------|
| $N = 2^{10}$      | 0.7152e-3s | 0.4199e-3s | 0.3785e-3s |
| $N = 2^{15}$      | 0.0053s   | 0.0071s  | 0.0087s  |
| $N = 2^{18}$      | 0.0725s   | 0.0871s  | 0.1394s  |
| $N = 2^{20}$      | 0.2170s   | 0.2821s  | 0.2830s  |

| $\{a_k\}_{k=0}^N$ | C-C | Fejér I | Fejér II |
|-------------------|-----|---------|----------|
| $N = 2^{10}$      | 0.2847e-3s | 0.3710e-3s | 0.2905e-3s |
| $N = 2^{15}$      | 0.0052s   | 0.0061s  | 0.0072s  |
| $N = 2^{18}$      | 0.0609s   | 0.0604s  | 0.0567s  |
| $N = 2^{20}$      | 0.2066s   | 0.2477s  | 0.2345s  |

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