REGULARITY RESULTS FOR NONLOCAL EQUATIONS AND APPLICATIONS

MOUHAMED MOUSTAPHA FALL

Abstract. We introduce the concept of $C^{m,\alpha}$-nonlocal operators, extending the notion second order elliptic operator in divergence form with $C^{m,\alpha}$-coefficients. We then derive the nonlocal analogue of the key existing results for elliptic equations in divergence form, notably the Hölder continuity of the gradient of the solutions in the case of $C^{0,\alpha}$-coefficients and the classical Shauder estimates for $C^{m+1,\alpha}$-coefficients. We further apply the regularity results for $C^{m,\alpha}$-nonlocal operators to derive optimal higher order regularity estimates of Lipschitz graphs with prescribed Nonlocal Mean Curvature. Applications to nonlocal equation on manifolds are also provided.

1. Introduction

We are concerned with a class of (not necessarily translation invariant) elliptic equations driven by nonlocal operators of fractional order. We extend in the nonlocal setting some key existing results for elliptic equations in divergence form with $C^{m,\alpha}$-coefficients. For a better description of how far the results in this paper extend to the fractional setting those available in the classical case, we start by recalling some main results of the classical local theory. We consider a weak solution $u \in H^1(\Omega)$ to the equation

$$\sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j u) = f \quad \text{in } \Omega,$$

where $\Omega$ is an open subset of $\mathbb{R}^N$, $f \in L^p_{\text{loc}}(\Omega)$, $p > N/2$, and the matrix coefficients $a_{ij}$ are measurable functions and satisfy, for every $x \in \Omega$, the following properties:

(i) $a_{ij}(x) = a_{ji}(x)$ for all $i,j = 1, \ldots, N$,

(ii) $\kappa \delta_{ij} \leq a_{ij}(x) \leq \frac{1}{\kappa} \delta_{ij}$ for all $i,j = 1, \ldots, N$.

In the regularity theory for elliptic equations in divergence form with measurable coefficients, the De Giorgi-Nash-Moser theory provides a priori $C^{0,\alpha_0}(\Omega)$ estimates for weak solutions to (1.1), for some $\alpha_0 = \alpha_0(N,p,\kappa)$, see e.g. [37]. The range or value of the largest Hölder exponent $\alpha_0$ is known in general once the coefficients are sufficiently regular. For instance, if $a_{ij} \in C(\Omega)$ then $u \in C^{0,\beta}(\Omega)$ for all $\beta < \min(2 - N/p, 1)$. Now Hölder continuous coefficients $a_{ij}$ yields Hölder continuity of the gradient of $u$. Namely, if $a_{ij} \in C^{0,\alpha}(\Omega)$, for some $\alpha \in (0,1)$, then $u \in C^{1,\min(1-\frac{N}{p},\alpha)}(\Omega)$, provided $2 - N/p > 1$. Moreover the Schauder theory states that if $a_{ij} \in C^{m+1,\alpha}(\Omega)$ and $f \in C^{m,\alpha}(\Omega)$, then $u \in C^{m+2,\alpha}(\Omega)$ for $m \in \mathbb{N}$. We refer the reader to [37][59].

The aim of this paper is to extend all the above regularity results to equations driven by $C^{m,\alpha}$-nonlocal operators of fractional order which we describe below. Our notion of $C^{m,\alpha}$-nonlocal operators can be seen as nonlocal version of second order partial differential equations in divergence form. On the other hand, as in the local case, since our notion of $C^{m,\alpha}$-nonlocal operators is stable under $C^{m,1}$ local change of coordinates, our results apply to nonlocal equations on manifolds. As a matter of fact, our results apply to nonlocal geometric problems such as the prescribed nonlocal mean curvature problems.

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We consider $s \in (0,1)$, $N \geq 1$ and $K : \mathbb{R}^N \times \mathbb{R}^N \to [\frac{1}{s}, \infty)$ such that

$$\begin{align*}
(i) & \quad K(x,y) = K(y,x) \quad \text{for all } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N, \\
(ii) & \quad |K(x,y)| \leq \frac{1}{\kappa}|x-y|^{N-2s} \quad \text{for all } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N, \\
(iii) & \quad \kappa|x-y|^{N-2s} \leq K(x,y) \quad \text{for all } (x,y) \in B_\delta \times B_\delta,
\end{align*}$$

for some constants $\kappa, \delta > 0$. We introduce the space of function $u \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that $\|u\|_{L^1(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |u(y)|(1 + |y|)^{-N-2s} dy < \infty$. A kernel $K$ satisfying $(1.3)$ induces a linear nonlocal operator $L_K : H^s(\Omega) \cap L_s(\mathbb{R}^N) \to \mathcal{D}'(\Omega)$ given by

$$\langle L_K u, \psi \rangle := \frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} (u(x) - u(y))(\psi(x) - \psi(y))K(x,y)dxdy \quad \text{for all } \psi \in C^\infty_c(\Omega).$$

The weight in the definition of the space $L_s(\mathbb{R}^N)$ is determined by $(1.3)$ and can be modified accordingly. Given $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, we say that $u \in H^s(\Omega) \cap L_s(\mathbb{R}^N)$ is a (weak) solution to the equation

$$L_K u = f \quad \text{in } \Omega,$$

if $L_K u = f$ in $\mathcal{D}'(\Omega)$.

The class of operators $L_K$ corresponding to the kernels $K$ satisfying $(1.3)$ are the nonlocal analogue of second order elliptic operators in divergence form with measurable coefficients on $B_\delta$. In this case the de Giorgi-Nash-Moser a priori Hölder estimates is well developed, see [21, 23, 26, 45, 46, 48, 49]. In particular, it follows from [26] that, if $f \in L^p(B_\delta)$, for some $p > N/(2s)$, then $u \in C^0_{\text{loc}}(\mathbb{R}^N)$, for some $\alpha_0 = \alpha_0(N,s,p) > 0$.

Following [29], we now introduce the notion of $C^{m,\alpha}$-nonlocal (or fractional order) operators which, in particular, are the object of study in the present paper. For $\delta > 0$, we define $Q_\delta := B_\delta \times [0,\delta)$. Let $\alpha \in (0,1)$, $m \in \mathbb{N}$ and $K$ satisfy $(1.3)$. We say that the kernel $K$ defines a $C^{m,\alpha}$-nonlocal operator in $Q_\delta$, if the function

$$B_\delta \times (0,\delta) \times S^{N-1} \to \mathbb{R}, \quad (x,r,\theta) \mapsto r^{N+2s}K(x,x+r\theta)$$

extends to a map $\mathcal{A}_K : Q_\delta \times S^{N-1} \to \mathbb{R}$ satisfying, for some $\kappa > 0$, the following properties:

$$\begin{align*}
(iii) & \quad \|\mathcal{A}_K\|_{C^{m,\alpha}(Q_\delta \times S^{N-1})} \leq \frac{1}{\kappa}, \\
(iv) & \quad \partial^2_\theta \mathcal{A}_K(x,0,\theta) = \partial^2_\theta \mathcal{A}_K(x,0,-\theta) \quad \text{for all } (x,\theta) \in B_\delta \times S^{N-1}, \gamma \in \mathbb{N}^N, |\gamma| \leq m.
\end{align*}$$

The class of kernels $K$ satisfying $(1.3)$ and $(1.5)$ is denoted by $\mathcal{K}^s(\kappa, m + \alpha, Q_\delta)$. We observe that $C^{m,\alpha}$-nonlocal operators can be seen as an extension of second order elliptic operators with $C^{m,\alpha}$-coefficients. Indeed, provided $(1-s)\mathcal{A}_K$ has a limit as $s \to 1$, the computations in [8] Section 5] show, for all $\psi \in C^1_c(\mathbb{R})$, that

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} (\psi(x) - \psi(y))^2 K(x,y)dxdy \to \sum_{i,j=1}^N \int_{\mathbb{R}^N} a^K_{ij}(x) \partial_i \psi(x) \partial_j \psi(x) dx \quad \text{as } s \to 1,$$

where $a^K_{ij}(x) = \lim_{s \to 1} \int_{S^{N-1}} (1-s)\mathcal{A}_K(x,0,\theta)\theta_i\theta_j d\theta$. In $(1.5)$-$(iii)$, we impose the regularity of $\mathcal{A}_K$ in the angular variable $\theta$. However, this is typically not necessary to derive the accurate local behavior of solutions to $(1.4)$ which parallels those solving $(1.1)$ as stated above. In fact, nonlocal operators provide a wider framework than their local counterpart, since translation invariant nonlocal operators are those given by kernels $K$ of the form $K(x,y) = J(x-y)$, for some even function $J$. In addition, only in this translation invariant setting, regularity theory is already rich enough to incude fully nonlinear problems. [17, 18, 19, 20, 21, 22, 23]. This issue on the possible anisotropic regularity of $\mathcal{A}_K$ in its variables will be taken into account in our main results stated in Section 1.3 below.

We note that for translation invariant nonlocal operators $K(x,y) = J(x-y)$, condition $(1.5)$-$(iii)$ always implies $(1.5)$-$(iv)$, since $\mathcal{A}_K(x,r,\theta) = r^{N+2s}J(r\theta)$ for all $(x,r) \in B_\delta \times (0,\delta)$ and $\theta \in S^{N-1}$. 


Beyond their appearances in the mathematical modeling of real-world phenomenon, $C^{m,\alpha}$-nonlocal operators appear naturally in geometric problems. Indeed, we are naturally confronted with nonlocal equation resulting from an initial one after a change coordinates. For instance, consider $K(x,y) = |x-y|^{-N-2s}$ (the kernel of the fractional Laplacian) and $K_{\Phi}(x,y) = |\Phi(x) - \Phi(y)|^{-N-2s}$, for some diffeomorphism $\Phi \in C^{m+1,\alpha}(\mathbb{R}^N;\mathbb{R}^N)$ with $D\Phi$ close to the identity matrix, so that \ref{1.3} holds. In this case, apart in dimension $N = 1$, may not have any regularity of $z \mapsto |z|^{N+2s} K_{\Phi}(x,x+z)$ at $z = 0$. However, using polar coordinates, we easily see that the map

$$(x,r) \mapsto r^{N+2s} K_{\Phi}(x,x+r) = \left| \int_0^1 D\Phi(x + tr\theta) \theta dt \right|^{-N-2s}$$

extends to a $C^{m,\alpha}$ map on $\mathbb{R}^N \times [0,\infty) \times S^{N-1}$ satisfying \ref{1.5}-(ii), so that $K_{\Phi}$ defines a $C^{m,\alpha}$-nonlocal operator. This fact motivates, in particular, the splitting in polar coordinates in our definition of $C^{m,\alpha}$-nonlocal operators. Moreover, it turns out to be useful in the study of prescribed nonlocal mean curvature problems and nonlocal equations on hypersurfaces, see Section \ref{1.1} and Section \ref{1.2} respectively. In particular, with the nonlocal version of the de Giorgi-Nash-Moser a priori Hölder estimate at hand and our gradient estimates, we shall show that Lipschitz graphs with nonlocal (or fractional) mean curvature are of class $C^{m+2s+\alpha}$, and with quantitative estimates. On the other hand, we remark that in some interesting non-translation invariant cases, the map $z \mapsto |z|^{N+2s} K(x,x+z)$ can be smooth at $z = 0$, and a first nontrivial example is given by the \textit{censored fractional Laplacian} or the $\Omega$-regional fractional Laplacian, where the kernel is given by $K(x,y) = 1_{\Omega}(x) 1_{\Omega}(y) |x-y|^{-N-2s}$, see e.g. Mou and Yi \cite{50}. An other example arises in problems from image processing, see e.g. Gilboa Osher \cite{35} and Caffarelli, Chan and Vasseur \cite{13}, where the kernel depends on the solution $u \in C^{1,\alpha}$ and, for simplicity, reads as $K(x,y) = 1_{\Omega}(x) 1_{\Omega}(y) \phi''(u(x) - u(y)) |x-y|^{-N-2s}$, for some even and convex function $\phi$. This is also the case for sign changing kernels e.g. $K(x,y) = |x-y|^{-N-2s+1} - |x-y|^{-N-2s+2}$, with $s_1 \in (0,1)$ and $s_2 < s_1$. However the conditions \ref{1.3} and \ref{1.5} are flexible enough to include such cases.

We now start by stating the main results concerning $C^{m,\alpha}$-nonlocal operators. Their generalization are contained in Section \ref{1.3} below. Our first main result is the following.

**Theorem 1.1.** Let $s \in (0,1)$, $N \geq 1$, $\kappa > 0$ and $\alpha \in (0,1)$. Let $K \in \mathcal{H}^s(\kappa,\alpha,Q_2)$, $u \in H^s(B_2) \cap L_s(\mathbb{R}^N)$ and $V, f \in L^p(B_2)$, for some $p > N/(2s)$, satisfy

$$L_K u + V u = f \quad \text{in } B_2.$$ 

(i) If $2s \leq 1$, then there exists $C = C(s,N,\kappa,\alpha,p,\|V\|_{L^p(B_2)}) > 0$ such that

$$\|u\|_{C^{0,2s-\frac{N}{p},\alpha}(B_2)} \leq C(\|u\|_{L^2(B_2)} + \|u\|_{L_s(\mathbb{R}^N)} + \|f\|_{L^p(B_2)}).$$

\hspace{1cm} (1.7)

(ii) If $2s - 1 > \max\left(\frac{N}{p},\alpha\right)$, then there exists $C = C(s,N,\kappa,\alpha,p,\|V\|_{L^p(B_2)}) > 0$ such that

$$\|u\|_{C^{1,\min(2s-\frac{N}{p},\alpha)}(B_2)} \leq C(\|u\|_{L^2(B_2)} + \|u\|_{L_s(\mathbb{R}^N)} + \|f\|_{L^p(B_2)}).$$

\hspace{1cm} (1.8)

The Hölder continuity of the gradient in \ref{1.7} is the main novelty in the above result. Theorem \ref{1.1} was only know in the translation invariant case, i.e. $K(x,y) = J(x-y)$, see \cite{29}. We mention that the regularity estimate in \ref{1.7} remains valid if $\alpha = 0$, see \cite{29}, where it was proven that if $K \in \mathcal{H}^s(0,0,Q_2)$ (and for all $s \in (0,1)$), then $u \in C^{0,\beta}(B_1)$ for all $\beta < \min(2s - \frac{N}{p},1)$. In view of \ref{1.3}, it will be apparent from the proof that the estimate in \ref{1.8} remains stable as $s \to 1$ once we replace $L_K$ by $(1-s)L_K$ and provided $(1-s)A_K$ has a limit as $s$ tends to 1. We recall that Hölder continuity of the gradient of solutions to fully nonlinear and non translation invariant integro-differential equations, in the spirit of Cordes and Nirenberg for elliptic equations in nondivergence form, has been first established by Caffarelli and Silvestre in \cite{14}, see also \cite{17,54} for higher order regularity estimates in nonlocal problems corresponding to elliptic equations in non-divergence form. Our next results is concerned with $C^{m+2s+\alpha}$ regularity estimates for solutions to equations driven
by \( C^{m+(2s-1)+\alpha} \) nonlocal operators, provided \( 2s + \alpha \notin \mathbb{N} \). Here and in the following, we put \( \ell_+ = \max(\ell, 0) \) for \( \ell \in \mathbb{R} \).

**Theorem 1.2.** Let \( N \geq 1, s \in (0, 1) \) and \( \kappa > 0 \). Let \( m \in \mathbb{N} \) and \( \alpha \in (0, 1) \), with \( 2s + \alpha \notin \mathbb{N} \). Let \( K \in \mathcal{K}(\kappa, m + \alpha + (2s-1)_+, Q_2) \), \( u \in H^s(B_2) \cap L^\infty(\mathbb{R}^N) \) and \( f \in C^{m,\alpha}(B_2) \) such that

\[
L_K u = f \quad \text{in } B_2.
\]

(i) If \( 2s + \alpha < 1 \), then

\[
\|u\|_{C^{m,2s+\alpha}(B_1)} \leq C(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{C^{m,\alpha}(B_2)}).
\]

(ii) If \( 1 < 2s + \alpha < 2 \) and \( 2s \neq 1 \), then

\[
\|u\|_{C^{m+1,2s+\alpha-1}(B_1)} \leq C(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{C^{m,\alpha}(B_2)}).
\]

(iii) If \( 2 < 2s + \alpha \), then

\[
\|u\|_{C^{m+2,2s+\alpha-2}(B_1)} \leq C(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{C^{m,\alpha}(B_2)}).
\]

(iv) If \( 2s = 1 \), then for all \( \beta \in (0, \alpha) \),

\[
\|u\|_{C^{m+1,\beta}(B_1)} \leq C(\|u\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{C^{m,\beta}(B_2)}).
\]

Here \( C = C(N, s, \kappa, \alpha, \beta, m) \).

We note that Theorem 1.2 includes the fractional Laplacian \( L_K = (-\Delta)^s \), for which it was proved in [24][35][51][56].

The following two paragraphs are devoted to the application of the above regularity estimates in nonlocal geometric problems.

1.1. **Application I: Graphs with prescribed nonlocal mean curvature.** In this section, we assume that \( s \in (1/2, 1) \). Recall that for a set \( E \subset \mathbb{R}^{N+1} \) of class \( C^{1,2s-1+\alpha} \), with \( \alpha > 0 \), near a point \( X \in \partial E \), the nonlocal (or fractional) mean curvature of the set \( E \) (or the hypersurface \( \partial E \)) at the point \( X \in \partial E \) is defined as

\[
H_s(\partial E; X) := PV \int_{\mathbb{R}^{N+1}} \frac{1_{E^c}(Y) - 1_E(Y)}{|Y - X|^{N+2s}} \, dY, \tag{1.9}
\]

where \( E^c := \mathbb{R}^{N+1} \setminus E \) and \( 1_D \) denotes the characteristic function of a set \( D \subset \mathbb{R}^{N+1} \). Recall that the notion of nonlocal mean curvature appeared first in the work of Caffarelli and Souganidis in [19] and first studied by Caffarelli, Roquejoffre, and Savin in [14]. As first discovered in [14] (see also [22][35]), the nonlocal mean curvature arises as the first variation of the fractional perimeter. For the convergence of fractional curvature to the classical one as \( s \to 1 \), see [22][27].

Suppose that \( \partial E \) is the graph of a function \( u \in C^{1,2s-1+\alpha}(\Omega) \cap C^{0,1}_{loc}(\mathbb{R}^N) \), then see e.g. [27], by a change of variable, for all \( x \in \Omega \), we have

\[
H_s(\partial E; (x,u(x))) = PV \int_{\mathbb{R}^N} \frac{F_s(p_u(x,y)) - F_s(p_u(y,x))}{|x - y|^{N+2s-1}} \, dy, \tag{1.10}
\]

where

\[
F_s(p) := \int_0^{\infty} \left(1 + \tau^2\right)^{-\frac{(N+2s)}{2}} \, d\tau \tag{1.11}
\]

and for a measurable function \( w : \mathbb{R}^N \to \mathbb{R} \), we put

\[
p_w(x,y) = \frac{w(y) - w(x)}{|x - y|}. \tag{1.12}
\]

By the fundamental theorem of calculus and (1.10) and noting that \( F_s(p_u(y,x)) = F_s(-p_u(x,y)) \), we have

\[
H_s(\partial E; (x,u(x))) = PV \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} q_u(x,y) \, dy, \tag{1.13}
\]
where for a measurable function $u : \mathbb{R}^N \to \mathbb{R}$,

$$q_u(x, y) := -\int_{-1}^{1} F'_s(t p_u(x, y)) \, dt = \int_{-1}^{1} \left(1 + t^2 p_u(x, y)^2\right)^{-\frac{(N+2s)}{2}} \, dt.$$ 

For the following, we define the nonlocal mean curvature kernel by

$$K_u(x, y) := \frac{1}{|x - y|^{N+2s}} q_u(x, y) \quad \text{for all } x \neq y \in \mathbb{R}^N.$$ 

Letting $\Omega$ be an open set of $\mathbb{R}^N$ and $f \in L^1_{\text{loc}}(\Omega)$, we are interested in the regularity of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ with $u \in H^s(\Omega)$ and satisfying

$$\mathcal{L}_{K_u} u = f \quad \text{in } \Omega,$$  \hspace{1cm} (1.14)

or equivalently,

$$\frac{1}{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} (u(x) - u(y))(\psi(x) - \psi(y))K_u(x, y) \, dx dy = \int_{\mathbb{R}^N} f(x)\psi(x) \, dx \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^N).$$  \hspace{1cm} (1.15)

We note that, even if we do not make any assumption on $u$ in $\mathbb{R}^N \setminus \Omega$, the left hand side of the above equation is finite. This follows from the fact that $F'_s \in L^\infty(\mathbb{R}$), $q_u \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, $2s > 1$ and that

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} (u(x) - u(y))(\psi(x) - \psi(y))K_u(x, y) \, dx dy$$

$$= \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{F'_s(p_u(x, y)) - F'_s(p_u(y, x))}{|x - y|^{N+2s-1}} (\psi(x) - \psi(y)) \, dx dy$$

$$= \int_{\Omega \times \Omega} (u(x) - u(y))(\psi(x) - \psi(y))K_u(x, y) \, dx dy$$

$$+ 2 \int_{\Omega} \psi(x) \int_{\mathbb{R}^N \setminus \Omega} \frac{F'_s(p_u(x, y)) - F'_s(p_u(y, x))}{|x - y|^{N+2s-1}} \, dy dx. \hspace{1cm} (1.16)$$

We observe that if $u \in C^{1,2s-1+\alpha}(\Omega) \cap L^1_{\text{loc}}(\mathbb{R}^N)$ solves (1.14), then the set $E_u := \{(x, t) \in \mathbb{R}^N \times \mathbb{R} : u(x) < t\}$, satisfies $H_s(\partial E_u, (x, u(x))) = f(x)$, for all $x \in \Omega$, provided the $(N+1)$-dimension Lebesgue measure $\partial E_u$ is equal to zero. This follows by approximating $u$ by a sequence of smooth functions. We consider next locally Lipschitz graphs with prescribed nonlocal mean curvature in the weak sense of (1.15), and we prove that they are of class $C^1$ in $\Omega$ as long as $f$ is $C^\infty$ in $\Omega$, with quantitative estimates. In the classical case, this is a consequence of the de Giorgi-Nash theorem and the Shauder theory for uniformly elliptic equations in divergence form with $C^{m,\alpha}$-coefficients. See e.g. Figalli-Valdinoci [34], it was hardly believed that the same strategy can be carried out in the nonlocal setting. In [34], the authors used geometric arguments to prove that Lipschitz sets, locally minimizing fractional perimeter are of class $C^\infty$. However there argument does not provide quantitative estimates. Here, we shall show that it is indeed possible to proceed as in the prescribed mean curvature problem, thanks to our regularity estimates for $C^{m,\alpha}$-nonlocal operators. It is important to note, in the theorem below, that we do not require any integrability of $u$ in $\mathbb{R}^N \setminus B_2$. We have the following result.

**Theorem 1.3.** Let $f \in L^1_{\text{loc}}(B_2)$ and $u : \mathbb{R}^N \to \mathbb{R}$ be a measurable function, with $\|u\|_{C^0(B_2)} \leq c_0$, such that

$$\mathcal{L}_{K_u} u = f \quad \text{in } B_2,$$ 

in the sense of (1.15). Then the following statements hold.

(i) If $f \in C^{0,1}(B_2)$, then

$$\|u\|_{C^{1,\alpha}(B_2)} \leq C(1 + \|f\|_{C^{0,1}(B_2)}),$$ \hspace{1cm} (1.17)

for some constants $\alpha_0, C > 0$, only depending on $N, s$ and $c_0$. Moreover, for all $\beta \in (0, 2s-1)$,

$$\|u\|_{C^{2,\beta}(B_1)} \leq C,$$

for some constant $C$, only depending on $N, s, \beta, c_0$ and $\|f\|_{C^{0,1}(B_2)}$. 


(ii) If \( f \in C^{m,\alpha}(B_2) \), for some \( \alpha \in (0,1) \) and \( m \geq 1 \), then
\[
\|u\|_{C^{m+1,2+\alpha-1}(B_1)} \leq C \quad \text{if } 2s + \alpha < 2,
\|u\|_{C^{m+2,2+\alpha-2}(B_1)} \leq C \quad \text{if } 2s + \alpha > 2,
\]
for some constant \( C \), only depending on \( N, s, \alpha, m, c_0 \) and \( \|f\|_{C^{m,\alpha}(B_2)} \).

The first qualitative estimates for nonlocal minimal graphs was found recently by Cabré and Cozzi in [10]. Indeed, they provide, in [10], quantitative gradient estimates for global graphs that locally minimize the fractional area functional in a cylinder \( B_R \times \mathbb{R} \), in the spirit of Finn [26] and Bombieri-de Giorgi-Miranda [7]. In this case \( f \equiv 0 \). Therefore combining their result and Theorem 1.3 we get quantitative estimates of all partial derivatives of such graphs. This follows from the fact that Theorem 1.3 remains valid if we replace the nonlocal mean curvature kernel \( K_u(x,y) \) with the truncate kernel \( 1_{B_R}(x)1_{B_R}(y)K_u(x,y) \), for some \( R \geq 2 \).

Recall that the smoothness character for fractional perimeter minimizing sets was known, but without quantitative bounds. Indeed, the seminal paper [14] established the first existence and \( C^{1,\gamma} \) (except a closed set of zero \((N-3)\)-Hausdorff measure) regularity for fractional perimeter minimizing sets. In [5], Barrios, Figalli and Valdinoci, proved that fractional perimeter minimizing sets which are of class \( C^{1,(2s-1)/2-\epsilon} \), are of class \( C^{\infty} \). On the other hand Caffarelli and Valdinoci showed, in [20], that, for \( s \) close to 1, these sets possess the smoothness property of the classical perimeter minimizing regions.

It is proven in [25], by Dipierro, Savin and Valdinoci, that the boundary of a fractional perimeter minimizing set, in a reference smooth set \( \Omega \), which coincide with a continuous graph \( \mathbb{R}^N \setminus \Omega \) is in fact a global graphs that is continuous in \( \Omega \).

The fact that we do not require any integrability of \( u \) in \( \mathbb{R}^N \setminus B_2 \) makes the proof of Theorem 1.3 particularly nontrivial. In view of the decomposition in (1.16), the proof resides on the regularity of the map
\[
\Omega' \to \mathbb{R}, \quad x \mapsto \int_{\mathbb{R}^N \setminus \Omega} \frac{F_s(u(x,y)) - F_s(u(x,y))}{|x-y|^{N+2s-1}} dy,
\]
for \( \Omega' \subset \subset \Omega \). Surprisingly, the local behavior of this map is completely determined by the one of \( u \) only in \( \Omega' \). In fact we will show, in Lemma 6.2 below, that this function is indeed as smooth as \( u \) in \( \Omega' \). This is proved, once this is proved, the above function is sent in the right hand side, so that we can use the argument as in the classical case. Indeed, we apply first the nonlocal de Giorgi-Nash a priori Hölder estimate to the function \( u(x+y)-u(x) \) which satisfies a nonlocal equation of the form (1.13), driven by a kernel \( K^u \) satisfying (1.13), to deduce that \( \nabla u \in C^{0,\alpha} \). This will imply that \( K^u \in \mathcal{K}^{\kappa}(\kappa,\alpha,Q_\delta) \), for some \( \kappa, \delta > 0 \). Now Theorem 1.3(i) and Theorem 1.2(ii) kick in and yield the result, since \( \mathcal{A}_{K^u} \) will be, locally, as regular as \( \nabla u \).

1.2. Application II: Nonlocal equations on manifolds. Let \( \Sigma \) be a Lipschitz hypersurface of \( \mathbb{R}^{N+1} \), with \( 0 \in \Sigma \). We define the space \( L_s(\Sigma) \) given by the set of functions \( u \in L^1_{loc}(\Sigma) \) such that
\[
\|u\|_{L_s(\Sigma)} := \int_\Sigma |u(y)|(1+|y|)^{-N-2s} d\sigma(y) < \infty,
\]
where \( d\sigma \) denote the volume element on \( \Sigma \). We assume that
\[
\|1\|_{L_s(\Sigma)} = \int_\Sigma (1+|y|)^{-N-2s} d\sigma(y) < \infty. \tag{1.18}
\]
We note that this condition always holds when \( \Sigma \) has finite diameter. In this section we are interested in the regularity estimates of functions \( u \in H^s_{loc}(\Sigma) \cap L_s(\Sigma) \) satisfying, for all \( \Psi \in C^\infty_c(\Sigma) \),
\[
\frac{1}{2} \int_\Sigma \int_\Sigma \frac{(u(x) - u(y))(\Psi(x) - \Psi(y))}{|x-y|^{N+2s}} d\sigma(x)d\sigma(y) + \int_\Sigma V(x)u(x)\Psi(x) d\sigma(x) = \int_\Sigma f(x)\Psi(x) d\sigma(x), \tag{1.19}
\]
where \( f, V \in L^1_{loc}(\Sigma) \) and \( uV \in L^1_{loc}(\Sigma) \).
**Theorem 1.4.** Let \( s, \gamma \in (0, 1) \), \( N \geq 1 \) and \( \Sigma \) be a \( C^{1, \gamma} \)-hypersurface of \( \mathbb{R}^{N+1} \) as above satisfying (1.18). Let \( f, V \in L^p(\Sigma) \), for some \( p > \frac{N}{s} \) and \( u \in H^s_{loc}(\Sigma) \cap L_s(\Sigma) \) satisfy (1.19). Then the following estimates holds.

(i) If \( 2s \leq 1 \), then
\[
\|u\|_{C^{2s-N/p} (B_{\rho} \cap \Sigma)} \leq C(\|u\|_{L^2(B_{2\rho} \cap \Sigma)} + \|u\|_{L_s(\Sigma)} + \|f\|_{L^p(\Sigma)}).
\]

(ii) If \( 2s - 1 > \max(N, p) \), then
\[
\|u\|_{C^{1, \min(2s-N/p-1, \gamma)} (B_{\rho} \cap \Sigma)} \leq C(\|u\|_{L^2(B_{2\rho} \cap \Sigma)} + \|u\|_{L_s(\Sigma)} + \|f\|_{L^p(\Sigma)}),
\]
where \( C, \rho > 0 \) are constants only depending on \( N, s, \gamma, p \).

In the case of higher order regularity, we obtain the

**Theorem 1.5.** Let \( s, \alpha, \gamma \in (0, 1) \), \( N \geq 1 \) and \( \Sigma \) be a \( C^{1, \gamma} \)-hypersurface of \( \mathbb{R}^{N+1} \) as above satisfying (1.18). Let \( f, V \in C^{0,\alpha}(\Sigma) \) and \( u \in H^s_{loc}(\Sigma) \cap L_s(\Sigma) \) satisfy (1.19).\n
(i) If \( 2s > 1 \) and \( \gamma \geq \alpha + 2s - 1 \), then
\[
\|u\|_{C^{2s+1-\alpha} (B_{\rho} \cap \Sigma)} \leq C(\|u\|_{L^2(B_{2\rho} \cap \Sigma)} + \|u\|_{L_s(\Sigma)} + \|f\|_{C^{0,\alpha}(\Sigma)}).
\]

(ii) If \( 2s + \alpha < 1 \) and \( \gamma \geq \alpha \), then
\[
\|u\|_{C^{2s+\alpha} (B_{\rho} \cap \Sigma)} \leq C(\|u\|_{L^2(B_{2\rho} \cap \Sigma)} + \|u\|_{L_s(\Sigma)} + \|f\|_{C^{0,\alpha}(\Sigma)}).
\]

(iii) If \( 2s = 1 \) and \( \gamma > \alpha \), then
\[
\|u\|_{C^{1, \alpha} (B_{\rho} \cap \Sigma)} \leq C(\|u\|_{L^2(B_{2\rho} \cap \Sigma)} + \|u\|_{L_s(\Sigma)} + \|f\|_{C^{0,\alpha}(\Sigma)}).
\]

Here \( C, \rho > 0 \) are constants only depending on \( N, s, \gamma, \alpha, \|V\|_{C^{0,\alpha}(\Sigma)}, \|u\|_{C^{0,\alpha}(\Sigma)}, \|f\|_{C^{0,\alpha}(\Sigma)} \) and the bound of the local geometry of \( \Sigma \) near 0.

Here, by the bound of the local geometry of \( \Sigma \) near 0, we mean the \( C^{1, \gamma} \) norm of a local parameterization of \( \Sigma \) flattening \( B_{\rho_0} \cap \Sigma \), for some \( \rho_0 > 0 \). If \( \Sigma \) is of class \( C^{m+1, \gamma} \) and \( f, V \in C^{0,\alpha}_{loc}(\Sigma) \), then under the same assumptions on \( \gamma \) in Theorem 1.5 we have the estimates of \( C^{m+2s+\alpha} \)-norm of \( u \) as long as \( 2s + \alpha \notin \mathbb{N} \), thanks to Theorem 1.2.

Theorems 1.4 and 1.5 are consequences of Theorem 1.1 and Theorem 1.2, respectively, after using a local change of variables locally that flattens \( \Sigma \).

For \( 2s > 1 \), Theorem 1.4 and Theorem 1.5 provide regularity estimates for solution to some nonlocal equation driven by the linearized nonlocal mean curvature operator (i.e. the nonlocal or fractional Jacobi operator) of a set \( E \) with constant nonlocal mean curvature (not necessarily bounded). Indeed, consider \( \Sigma := \partial E \) is a \( C^2 \)-hypersurface of \( \mathbb{R}^{N+1} \) with constant nonlocal mean curvature such that \( 0 \in \partial E \) and \( \|u\|_{L_s(\Sigma)} < \infty \). See e.g. [22,33], the second variation of fractional perimeter yields the bilinear form \( D_{\Sigma} : H^s(\Sigma) \times H^s(\Sigma) \to \mathbb{R} \), given by
\[
D_{\Sigma}(u, v) := \frac{1}{2} \int_{\Sigma} \int_{\Sigma} \frac{(u(\theta) - u(\bar{\theta}))(v(\theta) - v(\bar{\theta}))}{|\theta - \bar{\theta}|^{N+2s}} d\sigma(\theta)d\sigma(\bar{\theta}) - \frac{1}{2} \int_{\Sigma} V_{\Sigma}(\theta) u(\theta)v(\theta)d\sigma(\theta),
\]
where, letting \( \nu_{\Sigma} \) be the unit exterior normal vectorfield of \( \Sigma := \partial E \),
\[
V_{\Sigma}(\theta) := \frac{1}{2} \int_{\Sigma} \frac{|\nu_{\Sigma}(\theta) - \nu_{\Sigma}(\bar{\theta})|^2}{|\theta - \bar{\theta}|^{N+2s}} d\sigma(\theta).
\]
One then defines the fractional Jacobi operator as
\[
J_{\Sigma} := L_{\Sigma} - V_{\Sigma},
\]
where, for \( u \in C^{1,2s-1+\alpha}_{loc}(\Sigma) \cap L_s(\Sigma) \),
\[
L_{\Sigma} u(\theta) := PV \int_{\Sigma} \frac{u(\theta) - u(\bar{\theta})}{|\theta - \bar{\theta}|^{N+2s}} d\sigma(\theta).
\]
The fractional Jacobi fields are solutions to $\mathcal{J}_\Sigma u = 0$, and they play an important role in the study of stability of constant nonlocal mean curvature surfaces or fractional area estimates of such surfaces.

We observe that if $\Sigma$ is a $C^{1,\gamma}$-hypersurface for some $\gamma > s$, then $V_2 \subset C^1_{loc}(\Sigma)$. Moreover we may consider weak solution $u \in H^s_{loc}(\Omega) \cap L^s(\Sigma)$ to the equation $\mathcal{J}_\Sigma u = f$ on open subsets $\Omega$ of $\Sigma$, in the sense of (1.19). Hence Theorem 1.4 and Theorem 1.3 can be used to obtain regularity estimates of $u$.

When $\Sigma = S^{N-1}$, then Theorem 1.1(iii) was proved in [11], using the regularity theory of the fractional Laplacian and the Fredholm theory. Recall that besides the nonlocal minimal surfaces, there exist several nontrivial hypersurfaces with nonzero constant nonlocal mean curvature, see e.g. the survey paper [27].

1.3. Anisotropic $C^{m,\alpha}$-nonlocal operators. As mentioned earlier, in many situations, nonlocal equations provide a wider framework than their local counterpart, since $A_K$ may have anisotropic regularity in its variables. Namely, the spacial variable $x$, the singular variable $r$ and the angular variable might have different qualitative properties. This affects the local behavior of the solutions.

First note that the class of operators $L_K$ falls in the class of nonlocal operators generated by a Lévy measure $\nu_x$. In particular, the map $z \mapsto K(x,z)$ is the density of a Lévy measure $\nu_x$ and thus does not necessarily possess any regularity. If the Lévy measure is symmetric and stable, then see [51].

Let $\nu_x(rE) = r^{N-1}d\nu(E)$ for $E \subset S^{N-1}$. Under fairly general assumptions on the spectral measure $a$ on $S^{N-1}$ (not depending on $x$), optimal interior and boundary regularity were proved by Ros-Oton and Serra in [51]. The papers [20][25][40] obtained also regularity estimates provided $a$ is absolutely continuous with respect to the Lebesgue measure on $S^{N-1}$ only on an open set of positive measure. To capture this possible anisotropic regularity of $A_K$ in its variables, we introduce a new class of fractional order nonlocal operators which are much larger than the class of $C^{m,\alpha}$-nonlocal operators introduced above.

In the following, for $\delta > 0$, we define

$$Q_\delta := B_\delta \times [0, \delta) \quad \text{and} \quad Q_\infty := \mathbb{R}^N \times [0, \infty).$$

We define the space $C^m_{1,\alpha}(Q_\delta)$ by the set functions $A \in L^\infty(Q_\delta \times S^{N-1})$ such that, for every $\theta \in S^{N-1}$, the map $(x,r) \mapsto A(x,r,\theta)$ belongs to $C^{m,\alpha}(Q_\delta)$ and

$$\|A\|_{C^m_{1,\alpha}(Q_\delta)} := \sup_{\theta \in S^{N-1}} \|A(\cdot,\cdot,\theta)\|_{C^{m,\alpha}(Q_\delta)} < \infty. \tag{1.21}$$

For $\tau \geq 0$, the space $C^m_\tau(Q_\delta)$ is given by the the set of function $A \in L^\infty(Q_\delta \times S^{N-1})$ such that

$$\|A\|_{L^\infty_\tau(Q_\delta)} := \sup_{\theta \in S^{N-1}} \sup_{x \in B_\delta, r \in (0,\delta)} \frac{|A(x,r,\theta)|}{r^\tau} < \infty$$

and

$$[A]_{C^m_\tau(Q_\delta)} := \sup_{\theta \in S^{N-1}} \sup_{x \in B_\delta, r \in (0,\delta)} \frac{|A(x,r,\theta) - A(y,r,\theta)|}{\min(r, |x-y|)^\tau} < \infty.$$

The space $C^m_\tau(Q_\delta)$ is defined as the set of functions $A \in C^{m,\theta}(Q_\delta)$ such that

$$\|A\|_{C^m_\tau(Q_\delta)} := \sup_{\gamma \in \mathbb{N}^N, |\gamma| \leq m} \|\partial_\gamma A\|_{L^\infty_\tau(Q_\delta)} + \sup_{\gamma \in \mathbb{N}^N, |\gamma| \leq m} \|[\partial_\gamma A]\|_{C^m_\tau(Q_\delta)} < \infty. \tag{1.22}$$

This section is concerned with optimal Hölder estimates for nonlocal equation driven by the operator $L_K$ with coefficient $A_K$ in the spaces defined above.

**Definition 1.6.** Let $\alpha \in [0,1]$, $\tau \geq 0$, $m \in \mathbb{N}$ and $\kappa > 0$. For $\delta \in (0,1]$, we define $\mathcal{K}_\delta^\tau(\kappa, m+\alpha, Q_\delta)$ by the set of kernels $K : \mathbb{R}^N \times \mathbb{R}^N \to [-\infty, +\infty]$ satisfying (1.3) and

\begin{align*}
(iii)& \|A_K\|_{C^{m,\alpha}_1(Q_\delta)} + \|A_{o,K}\|_{C^m_\tau(Q_\delta)} \leq \frac{1}{\kappa}, \\
(iv)& A_{o,K}(x,0,\theta) = 0 \quad \text{for all } (x,\theta) \in B_\delta \times S^{N-1},
\end{align*}

where

$$A_{o,K}(x,r,\theta) := \frac{1}{2} (A_K(x,r,\theta) - A_K(x,r,-\theta)). \tag{1.23}$$
and $\mathcal{A}_K(\cdot, \cdot, \theta)$ is a continuous extension of $(x, r) \mapsto r^{N+2s}K(x, x + r\theta)$ on $Q_\delta$ for all $\theta \in S^{N-1}$.

The model case for the class of operators in Definition 1.6 is the anisotropic fractional Laplace operator, with kernel $K(x, y) = a((x - y)/(x - y))/(x - y)|x - y|^{-N-2s}$. In this case, $\mathcal{A}_K(x, r, \theta) = a(\theta)$.

We observe that $\mathcal{X}^s(\kappa, m + \alpha, Q_\delta) \subset \mathcal{X}^s(\kappa, m + \alpha, Q_\delta)$ for all $\tau > 0$. Moreover, we have the following interesting propriety on the set $\mathcal{X}^s(\kappa, m + \alpha, Q_\delta)$ concerning scaling and translations. Indeed, for $K \in \mathcal{X}^s(\kappa, m + \alpha, Q_\delta)$, $\rho \in (0, 1)$ and $z \in \mathbb{R}^N$, we let $K_{z, \rho}(x, y) := \rho^{N+2s}K(\rho x + z, \rho y + z)$. We then have that $\mathcal{A}_K_{z, \rho}(x, r, \theta) = \mathcal{A}_K(\rho x + z, pr, \theta)$ and thus $K_{z, \rho} \in \mathcal{X}^s(\kappa, m + \alpha, Q_\delta)$.

The kernels in $\mathcal{X}^s(\kappa, m + \alpha, Q_\delta)$ yield, in many cases, similar regularity estimates as those in $\mathcal{X}^s(\kappa, m + \alpha, Q_\delta)$, stated above, provided some global regularity/behavior of $u$ is a priori known.

Our first main result in this section is the following:

**Theorem 1.7.** Let $s \in (0, 1)$, $N \geq 1$, $\kappa > 0$ and $\alpha \in (0, 1)$. Let $K \in \mathcal{X}_0^s(\kappa, \alpha, Q_2)$, $u \in H^s(B_2) \cap L_\alpha(\mathbb{R}^N)$ and $V, f \in L^p(B_2)$, for some $p > N/(2s)$, satisfy

$$\mathcal{L}_Ku + Vu = f \quad \text{in } B_2.$$

(i) If $2s \leq 1$, then there exists $C = C(s, N, \kappa, \alpha, p, \|V\|_{L^p(B_2)}) > 0$ such that

$$\|u\|_{C^{0,2s-2\alpha}_p(B_1)} \leq C\left(\|u\|_{L^2(B_2)} + \|u\|_{L_\alpha(\mathbb{R}^N)} + \|f\|_{L^p(B_2)}\right).$$

(ii) If $2s - 1 > \max\left(\frac{N}{p}, \alpha\right)$, then there exists $C = C(s, N, \kappa, \alpha, p, \|V\|_{L^p(B_2)}) > 0$ such that

$$\|u\|_{C^{0,\min(2s-2\alpha,1-\alpha)}_p(B_1)} \leq C\left(\|u\|_{L^2(B_2)} + \|u\|_{L_\alpha(\mathbb{R}^N)} + \|f\|_{L^p(B_2)}\right).$$

Our next result reads as follows.

**Theorem 1.8.** Let $s \in (0, 1/2)$, $N \geq 1$, $\kappa > 0$ and $\alpha \in (0, 1)$. Let $K \in \mathcal{X}_0^s(\kappa, \alpha, Q_2)$, $u \in H^s(B_2) \cap C^{0,\alpha}(\mathbb{R}^N)$ and $f \in C^{0,\alpha}(B_2)$, satisfy

$$\mathcal{L}_Ku = f \quad \text{in } B_2.$$

We then have the following estimates.

- If $2s + \alpha < 1$, then for all $\beta \in (0, \alpha)$, there exists $C = C(N, s, \kappa, \alpha, \beta) > 0$ such that

$$\|u\|_{C^{0,2s+\beta}_p(B_1)} \leq C\left(\|u\|_{C^{0,\alpha}(\mathbb{R}^N)} + \|f\|_{C^{0,\alpha}(B_2)}\right).$$

- If $2s = 1$, then there exists $C = C(N, s, \kappa, \alpha) > 0$ such that

$$\|u\|_{C^{0,1}_p(B_1)} \leq C\left(\|u\|_{C^{0,\alpha}(\mathbb{R}^N)} + \|f\|_{C^{0,\alpha}(B_2)}\right).$$

If moreover $\|\mathcal{A}_K\|_{C^{0,\alpha}(Q_2 \times S^{N-1})} \leq \frac{1}{2}$, then we can replace $\|u\|_{C^{0,\alpha}(\mathbb{R}^N)}$ with $\|u\|_{L^{\infty}(\mathbb{R}^N)}$.

Our next result is concerned with $C^{m+2s+\alpha}$ Shauder estimates.

**Theorem 1.9.** Let $N \geq 1$ and $s \in (0, 1)$. Let $\kappa > 0$, $\alpha \in (0, 1)$ and $m \in \mathbb{N}$. Let $K \in \mathcal{X}_0^s(\kappa, m + \alpha, Q_2)$, $u \in H^s(B_2) \cap L_\alpha(\mathbb{R}^N)$ and $f \in C^{m,\alpha}(B_2)$ such that

$$\mathcal{L}_Ku = f \quad \text{in } B_2.$$

(i) If $u \in C^{m,\alpha}(\mathbb{R}^N)$ and $2s + \alpha < 1$, then

$$\|u\|_{C^{m,2s+\alpha}_p(B_1)} \leq C\left(\|u\|_{C^{m,\alpha}(\mathbb{R}^N)} + \|f\|_{C^{m,\alpha}(B_2)}\right).$$

(ii) If $u \in C^{m,\alpha}(\mathbb{R}^N)$, $2s + \alpha < 1$ and $1 < 2s + \alpha < 2$, then

$$\|u\|_{C^{m+1,2s+\alpha-1}_p(B_1)} \leq C\left(\|u\|_{C^{m,\alpha}(\mathbb{R}^N)} + \|f\|_{C^{m,\alpha}(B_2)}\right).$$

(iii) If $u \in C^{m,\alpha}(\mathbb{R}^N)$ and $2 < 2s + \alpha$, then

$$\|u\|_{C^{m+2,2s+\alpha-2}_p(B_1)} \leq C\left(\|u\|_{C^{m,\alpha}(\mathbb{R}^N)} + \|f\|_{C^{m,\alpha}(B_2)}\right).$$

(iv) If $u \in C^{m,\alpha}(\mathbb{R}^N)$, $2s = 1$ and $K \in \mathcal{X}_0^s(\kappa, m + \alpha, Q_2)$, for some $\tau > \alpha$, then

$$\|u\|_{C^{m+1,\alpha}_p(B_1)} \leq C\left(\|u\|_{C^{m,\alpha}(\mathbb{R}^N)} + \|f\|_{C^{m,\alpha}(B_2)}\right).$$
If moreover \( \| A_K \|_{C^{m,\alpha}(Q_2 \times S^{N-1})} \leq \frac{1}{r} \), then we can replace \( \| u \|_{C^{m,\alpha}(\mathbb{R}^N)} \) with \( \| u \|_{L^\infty(\mathbb{R}^N)} \). Here \( C = C(N, s, \kappa, \alpha, m, \tau) \).

We point out the remarkable differences between the last assertion in Theorem 1.9 and the results in Theorem 1.2. Indeed, in the former, \( A_K \) is only required to be in \( C^{m,\alpha}(Q_2 \times S^{N-1}) \) instead of \( C^{m,\alpha+(2s-1)}(Q_2 \times S^{N-1}) \) which was assumed in the latter. Moreover, Theorem 1.3-(ii), for \( s = 1/2 \), provides the optimal estimate which covers the case \( L_\kappa := (-\Delta)^s \), the anisotropic fractional Laplacian i.e. when \( K(x, y) = a((x - y)/|x - y|)|x - y|^{N-2s} \), while Theorem 1.2 does not if \( a \) is not smooth enough. In fact the results in Theorem 1.9 were known for the anisotropic fractional Laplacian when \( a \) is a measure on the unit sphere \( S^{N-1} \), see Ros-Oton and Serra [51] and when \( a \in C^\infty(S^{N-1}) \), see Grubb [40].

Interior regularity and Harnack inequality for linear and fully nonlinear nonlocal equations have been intensively investigated in last decades by many authors, see e.g. [1, 3, 5, 6, 15–17, 26, 31, 41, 43, 44, 47, 53, 55, 57] and the references therein.

Next, we observe that Theorem 1.1 and 1.2 are immediate consequences of Theorem 1.7 and 1.9. Indeed, in the former, \( K_1 = K \) and we show, under mild regularity assumptions on \( K, K_1 \), that we can replace

\[
K_{h,\alpha}(x, y) = |h|^{-\alpha} \{ K(x + h, y + h) - K(x, y) \},
\]

with \( K \in \mathcal{F}_0^\alpha(\kappa, \alpha, Q_\infty) \), we immediately see that \( |K_{h,\alpha}(x, y)| \leq \frac{1}{\kappa} |x - y|^{-N-2s} \). This leads us to consider more general equations of the form

\[
\mathcal{L}_K u + \mathcal{L}_{K'} U = F \tag{1.24}
\]

and we show, under mild regularity assumptions on \( K, K', U \) and \( F \), that \( v \) is Hölder continuous. Furthermore, using the a priori estimates for solutions to (1.24), we carry then out the proof of Theorem 1.7-(ii) and also Theorem 1.9. Some advantages in working with incremental quotients are to keep the limit of the blowing up solutions in the distributional domain \( L_s(\mathbb{R}^N) \) and also to decrease the degree of the subtracted polynomials so to stay in \( L_s(\mathbb{R}^N) \).

The paper is organize as follows. In Section 2, we collect some preliminary result and notations. Section 3 contains the regularity estimates for solutions to (1.24). Now Theorem 1.7-(ii) is proved in Section 4 and Theorem 1.9 in Section 5. Finally the proof of the main results are gathered in Section 6.
Throughout this paper, for the seminorm of the fractional Sobolev spaces, we adopt the notation
\[ [u]_{H^s(\Omega)} := \left( \int_{\Omega \times \Omega} |u(x) - u(y)|^2 \mu_1(x, y) \, dx \, dy \right)^{1/2}. \]
We will, sometimes use the notation
\[ [u]_{H^s_0(\Omega)} := \left( \int_{\Omega \times \Omega} |u(x) - u(y)|^2 |K(x, y)| \, dx \, dy \right)^{1/2}, \]
for a function \( K : \Omega \times \Omega \to [-\infty, +\infty] \). For the Hölder and Lipschitz seminorm, we write
\[ [u]_{C^{\alpha,\alpha}(\Omega)} := \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}}, \]
for \( \alpha \in (0, 1] \). If there is no ambiguity, when \( \alpha \in (0, 1) \), we will write \([u]_{C^{\alpha,\alpha}(\Omega)}\) instead of \([u]_{C^{\alpha,\alpha}(\Omega)}\).

If \( m \in \mathbb{N} \) and \( \alpha \in (0, 1) \), the Hölder space \([u]_{C^{m,\alpha}(\Omega)}\) is given by the set of functions in \( C^m(\Omega) \) such that
\[ \|u\|_{C^{m,\alpha}(\Omega)} := \sup_{\gamma \in \mathbb{N}^N, |\gamma| \leq m} \|\partial^\gamma u\|_{L^\infty(\Omega)} + \sup_{\gamma \in \mathbb{N}^N, |\gamma| = m} \|\partial^\gamma u\|_{C^\alpha(\Omega)} < \infty. \]

Letting \( u \in L^1_{loc}(\mathbb{R}^N) \), the mean value of \( u \) in \( B_r(z) \) is denoted by
\[ u_{B_r(z)} = \frac{1}{|B_r|} \int_{B_r(z)} u(x) \, dx. \]
For \( \alpha \in [0, 1] \), \( h \in \mathbb{R}^N \setminus \{0\} \) and \( f \in C^{0,\alpha}_{loc}(\mathbb{R}^N) \), we define
\[ f_{h,\alpha}(x) := \frac{f(x + h) - f(x)}{|h|^\alpha}. \]

### 2.2. Preliminary results

We gather in this paragraph some results which we will frequently use in the following of the paper. Let \( K : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty] \) satisfy the following properties:

(i) \( K(x, y) = K(y, x) \) for all \( x, y \in \mathbb{R}^N \),

(ii) \( k \mu_1(x, y) \leq K(x, y) \leq \frac{1}{k} \mu_1(x, y) \) for all \( x, y \in \mathbb{R}^N \). \hspace{1cm} (2.2)

For \( \alpha' \geq 0 \), let \( K' : \mathbb{R}^N \times \mathbb{R}^N \to [-\infty, +\infty] \) satisfy

(i) \( K'(x, y) = K'(y, x) \) for all \( x, y \in \mathbb{R}^N \),

(ii) \( |K'(x, y)| \leq \frac{1}{k} [(|x| + |y| + 1)^{\alpha'} \mu_1(x, y) \) for all \( x, y \in \mathbb{R}^N \). \hspace{1cm} (2.3)

Let \( U \in H^s_{loc}(\Omega) \cap L^{(\alpha'+2\alpha)/2}(\mathbb{R}^N) \) and \( f \in L^1_{loc}(\mathbb{R}^N) \). We say that \( u \in H^s_{loc}(\Omega) \cap L^s(\mathbb{R}^N) \) is a (weak) solution to
\[ \mathcal{L}_K u + \mathcal{L}_{K'} U = f \quad \text{in} \; \Omega, \]
if, for every \( \psi \in C^\infty_c(\Omega) \),
\[ \int_{\mathbb{R}^{2N}} (u(x) - u(y))(\psi(x) - \psi(y)) K(x, y) \, dx \, dy + \int_{\mathbb{R}^{2N}} (U(x) - U(y))(\psi(x) - \psi(y)) K'(x, y) \, dx \, dy \]
\[ = \int_{\mathbb{R}^N} f(x) \psi(x) \, dx. \]

We note that each of the terms in the above identity is finite. For \( \beta \in [0, 2s) \), we define the Morrey space \( \mathcal{M}_\beta \) by the set of functions \( f \in L^1_{loc}(\mathbb{R}^N) \) such that
\[ \|f\|_{\mathcal{M}_\beta} := \sup_{x \in \mathbb{R}^N} \sup_{r \in (0,1)} r^{\beta - N} \int_{B_r(x)} |f(y)| \, dy < \infty, \]
with $\mathcal{M}_0 := L^\infty(\mathbb{R}^N)$, and we note that $\|f\|_{\mathcal{M}_N\mathcal{P}} \leq C(N,p)\|f\|_{L^p(\mathbb{R}^N)}$. We have the following coercivity property, see [29],

$$\|f|^{1/2}v\|_{L^2(\mathbb{R}^N)}^2 \leq C(N,s,\beta)\|f\|_{\mathcal{M}_\beta}\|v\|_{H^s(\mathbb{R}^N)}$$

for all $v \in H^s(\mathbb{R}^N)$. (2.5)

We prove our a priori estimates for right hand in $\mathcal{M}_\beta$. Recall that $\mathcal{M}_N\mathcal{P}$ contains strictly $\|f\|_{L^p(\mathbb{R}^N)}$. The following energy estimate can be seen as a nonlocal Caccioppoli inequality.

**Lemma 2.1.** Let $N \geq 1$, $s \in (0,1)$ and $\kappa > 0$. We consider $K$ satisfying (2.2) and $K'$ satisfying (2.3), for some $\alpha' \geq 0$. Let $v \in H^s(\mathbb{R}^N)$ and $U \in H^s_{loc}(\mathbb{R}^N) \cap L^{(\alpha'+2s)/2}(\mathbb{R}^N)$ and $f \in \mathcal{M}_\beta$ satisfy

$$\mathcal{L}_Kv + \mathcal{L}_{K'}U = f \quad \text{in } B_{2R}.$$ (2.6)

Then for every $\varepsilon > 0$, there exist $\overline{C} = \overline{C}(s,N,\kappa,R)$ and $C = C(\varepsilon, s, N, \kappa, R)$ such that

$$\{\kappa - \varepsilon|f|_{\mathcal{M}_\beta}\} \int_{\mathbb{R}^N} (v(x) - v(y))^2 \varphi_R^2(y) \mu_1(x,y) dxdy \leq C(\|f\|_{\mathcal{M}_\beta} + 1)\|v\|_{L^2(\mathbb{R}^N)}^2 + C|f|_{\mathcal{M}_\beta}\|\varphi\|_{H^s(\mathbb{R}^N)}^2$$

$$+ C[U]_{H^s_{K'}(B_{4R})}^2 + C \int_{\mathbb{R}^N} \varphi_R^2(y) dv(y) \left( \int_{\mathbb{R}^N \setminus B_{4R}} |U(x) - U(y)||K'(x,y)| dx \right) dy. \quad (2.7)$$

**Proof.** Applying [29] Lemma 9.1, we get

$$(\kappa - \varepsilon) \int_{\mathbb{R}^N} (v(x) - v(y))^2 \varphi_R^2(y) \mu_1(x,y) dxdy \leq \int_{\mathbb{R}^N} |f(x)||v(x)||\varphi_R^2(x) dxdy$$

$$+ C \int_{\mathbb{R}^N} (\varphi_R(x) - \varphi_R(y))^2\varphi_R^2(y) \mu_1(x,y) dxdy$$

$$+ \int_{\mathbb{R}^N} |U(x) - U(y)||\varphi_R^2(x)v(x) - \varphi_R^2(y)v(y)||K'(x,y)| dxdy. \quad (2.7)$$

We now estimate

$$\int_{\mathbb{R}^N} |U(x) - U(y)||\varphi_R^2(x)v(x) - \varphi_R^2(y)v(y)||K'(x,y)| dxdy$$

$$= \int_{B_{4R} \times B_{4R}} |U(x) - U(y)||\varphi_R^2(x)v(x) - \varphi_R^2(y)v(y)||K'(x,y)| dxdy$$

$$+ 2 \int_{\mathbb{R}^N} \varphi_R^2(y) dv(y) \left( \int_{\mathbb{R}^N \setminus B_{4R}} |U(x) - U(y)||K'(x,y)| dx \right) dy$$

$$\leq \varepsilon/\kappa(4R) + 1)\alpha' |\varphi_R^2(x)v(x)||K'(x,y)| dxdy.$$ (2.8)

We recall that

$$\int_{\mathbb{R}^N} (\varphi_1(x) - \varphi_1(y))^2 \mu_1(x,y) dy \leq C(N,s)(1 + |x|^{-N-2s}) \quad \text{for every } x \in \mathbb{R}^N. \quad (2.9)$$

Therefore

$$[\varphi_R^2]_{H^s_{K'}(B_{4R})} \leq 2 \int_{\mathbb{R}^N} (v(x) - v(y))^2 \varphi_R^2(y) \mu_1(x,y) dxdy + 2 \int_{\mathbb{R}^N} (\varphi_R^2(x) - \varphi_R^2(y))^2 \varphi_R^2(y) \mu_1(x,y) dxdy$$

$$\leq 2 \int_{\mathbb{R}^N} (v(x) - v(y))^2 \varphi_R^2(y) \mu_1(x,y) dxdy + \|v\|_{L^2(\mathbb{R}^N)}^2.$$
Next, from (2.5), Young’s inequality and (2.9), we deduce that

\[
\int_{\mathbb{R}^N \times \mathbb{R}^N} |U(x) - U(y)||\varphi_R^2(x)v(x) - \varphi_R^2(y)v(y)||K'(x,y)| \,dx \,dy \\
\leq C \int_{\mathbb{R}^N} (v(x) - v(y))^2 \varphi_R^2(y)\mu_1(x,y) \,dx \,dy + C\|v\|^2_{L^2(\mathbb{R}^N)} \\
+ C[U]_{H_{K,v}(B_{4R})}^2 + C \int_{\mathbb{R}^N} \varphi_R^2(y)\left(\int_{\mathbb{R}^N \setminus B_{4R}} |U(x) - U(y)||K'(x,y)| \,dx\right) \,dy. \tag{2.10}
\]

Next, from (2.5), Young’s inequality and (2.9), we deduce that

\[
\int_{\mathbb{R}^N} |f(x)||\varphi_R(x)|^2 v(x) \,dx \leq C\|f\|_{M_\beta} \int_{\mathbb{R}^N} (v(x) - v(y))^2 \varphi_R^2(y)\mu_1(x,y) \,dx \,dy \\
+ C\|f\|_{M_\beta}\|v\|^2_{L^2(\mathbb{R}^N)} + C\|f\|_{M_\beta}\|\varphi_R\|^2_{H^s(\mathbb{R}^N)}.
\]

Using this and (2.10) in (2.7), we get the result. \qed

We state the following result.

**Lemma 2.2.** Let \( N \geq 1, s \in (0,1) \) and \( \kappa > 0 \). We consider \( K \) satisfying (2.2) and \( K' \) satisfying (2.3), for some \( \alpha' \geq 0 \). Let \( v \in H^s(\mathbb{R}^N) \) and \( U \in H^s_{loc}(\mathbb{R}^N) \cap L^{(\alpha'+2\alpha)/2}(\mathbb{R}^N) \) and \( f \in M_\beta \) satisfy

\[
\mathcal{L}_Kv + \mathcal{L}_K U = f \quad \text{in} \, \mathbb{R}^N.
\]

Then there exists \( C = C(N,s,\kappa,\alpha',R) \) such that for every \( \psi \in C^\infty(\mathbb{R}^N) \), we have

\[
\left|\int_{\mathbb{R}^N} (v(x) - v(y))\psi(x) - \psi(y)|K(x,y)| \,dx \,dy\right| \leq C\|f\|_{M_\beta} \left(1 + \|\psi\|^2_{H^s(\mathbb{R}^N)}\right) \tag{2.11}
\]

\[
+ C[U]_{H_{K,v}(B_{4R})}^2 + C \int_{\mathbb{R}^N} |\psi(y)| \left(\int_{\mathbb{R}^N \setminus B_{4R}} |U(x) - U(y)||K'(x,y)| \,dx\right) \,dy. \tag{2.12}
\]

**Proof.** Using the weak formulation of the equation and (2.5), we get the expression on the left hand side in (2.11). Now expression (2.12) appears after decomposing the domain of integration and using Hölder’s inequality as in the beginning of the proof of Lemma 2.1. \qed

We close this section with the following result.

**Lemma 2.3.** Let \( K \) satisfy (1.3)(i)-(ii). Let \( v \in H^s_{loc}(B_{2R}) \cap L^s(\mathbb{R}^N) \) and \( f \in L^1_{loc}(\mathbb{R}^N) \) satisfy

\[
\mathcal{L}_Kv = f \quad \text{in} \, B_{2R},
\]

for some \( R > 0 \). We let \( v_R := \varphi_R v \). Then

\[
\mathcal{L}_K v_R = f + G_{K,v,R} \quad \text{in} \, B_{R/2}, \tag{2.13}
\]

where

\[
G_{K,v,R}(x) = \int_{\mathbb{R}^N} v(y)(\varphi_R(x) - \varphi_R(y))K(x,y) \,dy. \tag{2.14}
\]

Moreover, \( G_{K,v,R} \) satisfies the following properties.

(i) There exists \( C = C(N,s,R) \) such that

\[
\|G_{K,v,R}\|_{L^\infty(B_{R/2})} \leq C \sup_{x \in B_{R/2}} \int_{|y| \geq R} |v(y)||K(x,y)| \,dy.
\]

(ii) If \( v \in C^{k+\alpha}(\mathbb{R}^N) \) and \( \|A_K\|_{C^{k+\alpha}(Q^\infty)} \leq c_0 \), then there exists \( C = C(N,s,\alpha,c_0,R,k) \) such that

\[
\|G_{K,v,R}\|_{C^{k+\alpha}(B_{R/2})} \leq C\|v\|_{C^{k+\alpha}(\mathbb{R}^N)}.
\]

(iii) If \( \|A_K\|_{C^{k+\alpha}(Q^\infty \times S^{N-1})} \leq c_0 \), then there exists \( C = C(N,s,\alpha,c_0,R,k) \) such that

\[
\|G_{K,v,R}\|_{C^{k+\alpha}(B_{R/4})} \leq C\|v\|_{L^1(\mathbb{R}^N)}.
\]
Then there exist \( \varepsilon \) Proposition 3.3. A priori estimates and consequences. Lemma 3.1. Let we now recording two results from [29] that will be need in the following of the paper.

Lemma 3.2. Proof. For (2.13), see [29, Lemma 9.2]. Statement (14) MOUHAMED MOUSTAPHA FALL

(3.1)

\[ a \] is an affine function.

\[ a \] satisfies (3.1) and such that \( K \) satisfies

\[ \lambda < 0 \] and \( \kappa \) satisfies \( \kappa \) is close to the translation invariant operator \( L_{\mu} \), with \( a : S^{N-1} \rightarrow \mathbb{R} \) satisfies

\[ a(-\theta) = a(\theta) \quad \text{and} \quad \kappa \leq a(\theta) \leq \frac{1}{\kappa} \quad \text{for all } \theta \in S^{N-1}. \quad (3.1) \]

We now recording two results from [29] that will be need in the following of the paper.

Lemma 3.1. Let \( b \in L^\infty(S^{N-1}) \). Suppose that there exists a sequence of functions \( (a_n)_n \) satisfying 3.1 and such that \( a_n \sim b \) in \( L^\infty(S^{N-1}) \). Let \( \lambda_n : \mathbb{R}^N \times \mathbb{R}^N \rightarrow [0, \kappa^{-1}] \), with \( \lambda_n \rightarrow 0 \) pointwise on \( \mathbb{R}^N \times \mathbb{R}^N \). Let \( (K_n)_n \) be sequence of symmetric kernels satisfying

\[ |K_n(x, y) - a_n(x, y)| \leq \lambda_n(x, y) \mu_n(x, y) \quad \text{for all } x \neq y \in \mathbb{R}^N \text{ and for all } n \in \mathbb{N}. \]

If \( (v_n)_n \) is a bounded sequence in \( L_a(\mathbb{R}^N) \cap H^s_{\text{loc}}(\mathbb{R}^N) \) such that \( v_n \rightarrow v \) in \( L_a(\mathbb{R}^N) \), then

\[ \int_{\mathbb{R}^N} v(x)L_{\mu}v(x) \, dx = \frac{1}{2} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^2N} (v_n(x) - v_n(y))(\psi(x) - \psi(y))K_n(x, y) \, dy \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^N). \]

Lemma 3.2. Let \( b \in L^\infty(S^{N-1}) \). Suppose that there exists a sequence of functions \( (a_n)_n \) satisfying 3.1 and such that \( a_n \sim b \) in \( L^\infty(S^{N-1}) \). Consider \( u \in H^s_{\text{loc}}(\mathbb{R}^N) \) satisfying

\[ \|u\|^2_{L^2(B_R)} \leq R^{N+2s} \quad \text{for some } \gamma < 2s \text{ and for every } R \geq 1. \]

Then \( u \) is an affine function.

3.1. A priori estimates and consequences. We now state the main result of the present section.

Proposition 3.3. Let \( s \in (0, 1) \), \( \beta \in [0, 2s) \), \( \sigma \in (s, 1) \) and \( \kappa > 0 \). Let \( \alpha' \geq 0 \), with \( \alpha' + \sigma \in (0, 2s) \). Pick

\[ \gamma \in (0, 1) \cap (0, \sigma) \cap (0, 2s - \beta). \]

Then there exist \( \varepsilon_0 > 0 \) and \( C > 0 \) such that if

\[ a \] satisfies 3.1, \( K_n \) satisfies 2.2 with

\[ |K_n - a_n| < \varepsilon_0 \quad \text{on } B_2 \times B_2 \setminus \{x = y\}, \]
• $K'$ satisfies (2.3),
• $f \in \mathcal{M}_\beta$, $g \in H^s(\mathbb{R}^N)$, $U \in C^0(\mathbb{R}^N) \cap L_\infty(\mathbb{R}^N)$ are such that
\[
\mathcal{L}_{K_n} g + \mathcal{L}_{K} U = f \quad \text{in } B_2,
\]
then
\[
\sup_{r > 0} r^{-2\gamma - N} \| g - g_{B_r} \|_{L^2(B_r)}^2 \leq C (\|g\|_{L^2(\mathbb{R}^N)} + [U]_{C^0(\mathbb{R}^N)} + \|f\|_{\mathcal{M}_\beta})^2.
\]

Proof. Assume that the assertion in the proposition does not hold, then for every $n \in \mathbb{N}$, there exist:
• $a_n$ and $K_{a_n}$ satisfying (3.1) and (2.2) respectively, with
\[
|K_{a_n} - \mu_{a_n}| < \frac{1}{n} \quad \text{on } B_2 \times B_2 \setminus \{x = y\},
\]
• $K'$ satisfying (2.3), $f_n \in \mathcal{M}_\beta$, $U_n \in C^0(\mathbb{R}^N) \cap L_\infty(\mathbb{R}^N)$ and $g_n \in H^s(\mathbb{R}^N)$, with
\[
\|g_n\|_{L^2(\mathbb{R}^N)} + \|f_n\|_{\mathcal{M}_\beta} + \|U_n\|_{C^0(\mathbb{R}^N)} \leq 1,
\]
\[
\mathcal{L}_{K_{a_n}} g_n + \mathcal{L}_{K'} U_n = f_n \quad \text{in } B_2,
\]
with the property that
\[
\sup_{r > 0} r^{-N - 2\gamma} \| g_n - (g_n)_{B_r} \|_{L^2(B_r)}^2 > n.
\]
Consequently, there exists $\tau_n > 0$ such that
\[
\tau_n^{-N - 2\gamma} \| g_n - (g_n)_{B\tau_n} \|_{L^2(B\tau_n)}^2 > n/2.
\]
We consider the (well defined, because $\|g_n\|_{L^2(\mathbb{R}^N)} \leq 1$) nonincreasing function $\Theta_n : (0, \infty) \to [0, \infty)$ given by
\[
\Theta_n(\tau) = \sup_{r \in [\tau, \infty)} r^{-N - 2\gamma} \| g_n - (g_n)_{B_r} \|_{L^2(B_r)}^2.
\]
Obviously, for $n \geq 2$, by (3.4),
\[
\Theta_n(\tau_n) > n/2 \geq 1.
\]
Hence, provided $n \geq 2$, there exists $r_n \in [\tau_n, \infty)$ such that
\[
\Theta_n(r_n) \geq r_n^{-N - 2\gamma} \| g_n - (g_n)_{B_{r_n}} \|_{L^2(B_{r_n})}^2 \geq \Theta_n(\tau_n) - 1/2 \geq (1 - 1/2)\Theta_n(\tau_n) \geq \frac{1}{2}\Theta_n(r_n),
\]
where we used the monotonicity of $\Theta_n$ for the last inequality, while the first inequality comes from the definition of $\Theta_n$. In particular, thanks to (3.5), $\Theta_n(r_n) \geq n/4$. Now since $\|g_n\|_{L^2(\mathbb{R}^N)} \leq 1$, we have that $r_n^{-N - 2\gamma} \geq n/4$, so that $r_n \to 0$ as $n \to \infty$. We now define the sequence of functions
\[
w_n(x) = \Theta_n(r_n)^{-1/2} r_n^{-\gamma} \left\{ g_n(r_n x) - \frac{1}{|B_1|} \int_{B_1} g_n(r_n x) \, dx \right\},
\]
which satisfies
\[
\|w_n\|_{L^2(B_1)}^2 \geq \frac{1}{2}, \quad \int_{B_1} w_n(x) \, dx = 0 \quad \text{for every } n \geq 2.
\]
Using that, for every $r > 0$, $\|g_n - (g_n)_{B_r}\|_{L^2(B_r)} \leq r^{N + 2\gamma} \Theta_n(r)$ and the monotonicity of $\Theta_n$, by (29) Lemma 3.1, we find that
\[
\|w_n\|_{L^2(B_R)}^2 \leq C R^{N + 2\gamma} \quad \text{for every } R \geq 1 \text{ and } n \geq 2,
\]
for some constant $C = C(N, \gamma) > 0$.
We define
\[
\underline{K}_n(x, y) = r_n^{N + 2s} K_{a_n}(r_n x, r_n y), \quad \overline{K}_n(x, y) = r_n^{N + 2s} K'_n(r_n x, r_n y)
\]
and
\[
\underline{U}_n(x) = U_n(r_n x), \quad \overline{U}_n(x) = r_n^{2s} f_n(r_n x).
\]
It is plain that
\[
\mathcal{L}_{\underline{K}_n} w_n + r_n^{-\gamma} \Theta_n(r_n)^{-1/2} \mathcal{L}_{\underline{K}_n} \underline{U}_n = r_n^{-\gamma} \Theta_n(r_n)^{-1/2} \underline{J}_n \quad \text{in } B_{2/r_n}.
\]
We fix $M > 1$ and let $n \geq 2$ large, so that $1 < M < \frac{1}{\sqrt{n}}$. Therefore, letting $w_{n,M} := \varphi_{4M} w_n \in H^s(\mathbb{R}^N)$, we apply Lemma 2.3(i) to get

$$\mathcal{L}_{K_n} w_{n,M} + r_n^{-\gamma} \Theta_n(r_n)^{1-\gamma} \mathcal{L}_{K_n} \mathcal{U}_n = r_n^{-\gamma} \Theta_n(r_n)^{1-\gamma} \mathcal{U}_n + G_{K_n,w_{n,M}} \quad \text{in } B_{2M},$$

(3.9)

with $\|G_{K_n,w_{n,M}}\|_{L^\infty(B_{M/2})} \leq C \|w_n\|_{L^\infty(\mathbb{R}^N)} \leq C$, by (3.7). We also note that

$$\|\mathcal{U}_n\|_{L^\infty(B_M)} \leq r_n^{2s-\gamma}.$$  

(3.10)

Clearly $K_n$ satisfies (3.3). Applying Lemma 2.1 to the equation (3.9) and using (3.7) together with (3.13), we get

$$C(\Theta_n(r_n)^{-1} - 2s-\gamma + 1) + C r_n^{-\gamma} \Theta_n(r_n)^{-1/2} \mathcal{U}_n^2(\mathcal{K}_n(B_{4M}))$$

$$+ C r_n^{1-\gamma} \Theta_n(r_n)^{-1/2} \int_{\mathbb{R}^N} \varphi_{K_n}^2(y) |w_n(y)| \left( \int_{\mathbb{R}^N \setminus B_{4M}} |\mathcal{U}_n(x) - \mathcal{U}_n(y)| |\mathcal{K}_n(x,y)| \right) dy.$$  

(3.11)

We observe that

$$|K_n(x,y)| \leq \frac{1}{K} (r_n |x| + r_n |y| + 1) \phi_1(x,y).$$

From this and the fact that $|\mathcal{U}_n| \leq r_n^\sigma$, we have the following estimate:

$$\int_{B_{4M} \setminus B_{4M}} (\mathcal{U}_n(x) - \mathcal{U}_n(y))^2 |\mathcal{K}_n(x,y)| \, dy$$

$$\leq C(M) \int_{B_{4M} \setminus B_{4M}} (|x|^{\sigma} + |y|^{\sigma}) \mu_1(x,y) \, dy$$

$$\leq C(M) r_n^\sigma \int_{B_{4M} \setminus B_{4M}} |x-y|^{-N-2s+2\sigma} \, dx$$

$$\leq C(M) r_n^\sigma.$$  

(3.12)

because $\sigma > s$. In addition, since $\alpha' + \sigma < 2s$, we get

$$\sup_{y \in B_M} \int_{\mathbb{R}^N \setminus B_{4M}} |U_n(x) - U_n(y)| |K_n(x,y)| \, dx \leq C(M) r_n^\sigma \int_{|x| \geq 2M} (1 + |x|^{\alpha'}) |x|^{-N-2s+\sigma} \, dx$$

$$\leq C(M) r_n^\sigma.$$  

(3.13)

Now using (3.12) and (3.13) in (3.11) and the fact that $\gamma \leq \min(2s - \beta, \sigma)$, we find that

$$\left\{ \kappa - \varepsilon \int \Theta_n(r_n)^{-1/2} \right\} \|w_n\|^2_{H^s(B_{4M}/s)} \leq C(\Theta_n(r_n)^{-1} + 1).$$

Therefore, since $\Theta_n(r_n)^{-1} \leq 1$, then provided $\varepsilon$ is small enough, by (3.7), we deduce that $w_n$ is bounded in $H^s(\mathbb{R}^N)$. Hence by Sobolev embedding and (3.7), there exists $w \in H^s(\mathbb{R}^N)$ such that, up to a subsequence, $w_n \to w$ in $L^2(\mathbb{R}^N)$ with the same constant. Moreover, by (3.7) we deduce that

$$\|w\|^2_{L^2(B_1)} \geq \frac{1}{2} \quad \text{and} \quad \|w\|_{L^2(B_1)} = 0.$$  

(3.14)

In addition by (3.7), we have

$$\|w\|^2_{L^2(B_1)} \leq CR^{N+2\gamma} \quad \text{for every } R \geq 1.$$  

(3.15)

Now applying Lemma 2.3 to the equation (3.8) and using (3.12) together with (3.13), we get

$$\left| \int_{\mathbb{R}^N} (w_n(x) - w_n(y))(\psi(x) - \psi(y)) K_n(x,y) \, dx dy \right| \leq C \Theta_n(r_n)^{-1/2} \quad \text{for all } \psi \in C^\infty_c(B_M).$$

Since $|K_n - \mu_{n_\infty}| \leq \frac{1}{n}$ almost everywhere in $B_{1/r_n} \times B_{1/r_n}$ and $\Theta_n(r_n) \to \infty$ as $n \to \infty$, we can apply Lemma 3.1 to deduce that $\mathcal{L}_{\mu_n} w = 0$ in $\mathbb{R}^N$, where $b$ is the weak-star limit of $a_n$ (which satisfies
and thus, for every

\[ \text{Corollary 3.5.} \]

Let \( s \in (0, 1), \ N \geq 1, \ \beta \in [0, 1), \ \sigma \in (s, 1], \ \alpha' \geq 0, \ \text{with } \alpha' + \sigma \in (0, 2s), \ \text{and } \kappa > 0. \) Let \( a \) satisfy \( \text{(3.1)} \). Consider \( K \) and \( K' \) satisfying \( \text{(2.2)} \) and \( \text{(2.3)} \), respectively. Let \( g \in H^s(B_2) \cap L_s(\mathbb{R}^N), \ U \in C^{0,\sigma}_{\text{loc}}(\mathbb{R}^N) \cap L(\alpha' + 2s)/2(\mathbb{R}^N) \) and \( f \in \mathcal{M}_\beta \) satisfy

\[ \mathcal{L}_K g + \mathcal{L}_{K'} U = f \quad \text{in } B_2. \]  

Then, for \( \gamma \in (0, 1) \cap (0, \sigma] \), there exist \( \varepsilon_0, C > 0 \), such that if \( |K - \mu_a| < \varepsilon_0 \) in \( B_2 \times B_2 \setminus \{ x = y \} \), we have

\[ \| g \|_{C^{0,\gamma}(B_1/2)} \leq C(\| g \|_{L^2(B_2)} + \| g \|_{L_s(\mathbb{R}^N)} + \| U \|_{C^{0,\sigma}(\mathbb{R}^N)} + \| f \|_{\mathcal{M}_\beta}), \]

with \( C, \varepsilon_0 > 0 \) depend only on \( s, N, \beta, \sigma, \alpha', \kappa \) and \( \gamma \).

\[ \text{Proof.} \] Without loss of generality, we may assume that

\[ \| g \|_{L^2(B_2)} + \| g \|_{L_s(\mathbb{R}^N)} + \| U \|_{C^{0,\sigma}(\mathbb{R}^N)} + \| f \|_{\mathcal{M}_\beta} \leq 1. \]  

Let \( z \in B_{1/2} \) and define \( g_z := g(x + z), \ K_z(x, y) = K(x + z, y + z), \ K'_z(x, y) = K(x + z, y + z), \ f_z(x) = f(x + z), \ f_z(x) = f(x + z), \ \text{and } U_z(x) = U(x + z). \) Then we have

\[ \mathcal{L}_K g_z + \mathcal{L}_{K'} U_z = f_z \quad \text{in } B_1. \]

On the other hand by Lemma \[ \text{Lemma 3.1}, \]

\[ \mathcal{L}_K (\varphi_1 g_z) + \mathcal{L}_{K'} U_z (\varphi_1 g_z) = \tilde{f}_z \quad \text{in } B_{1/2}, \]

for some function \( \tilde{f}_z \) satisfying

\[ \| \tilde{f}_z \|_{\mathcal{M}_\beta} \leq C(\| \varphi_1 \|_{\mathcal{M}_\beta} + \| g_z \|_{L_s(\mathbb{R}^N)}) \leq C, \]

where we used \[ \text{(3.2)} \) for the last inequality. By \[ \text{(3.1)} \) and Proposition \[ \text{Proposition 3.3}, \]

there exist \( \varepsilon_0, C > 0 \), only depending on \( s, N, \beta, \sigma, \alpha', \kappa, \sigma \) and \( \gamma \), such that if \( |K_z - \mu_a| < \varepsilon_0 \) in \( B_2 \times B_2 \setminus \{ x = y \} \), we get

\[ \| g_z - (g_z)_{B_r} \|_{L^2(B_r)} = \| g - (g)_{B_r(z)} \|_{L^2(B_r(z))} \leq C r^{N/2+\gamma} \quad \text{for every } r > 0. \]

It then follows, from \[ \text{Lemma 3.1}, \] that

\[ \| g - g(z) \|_{L^2(B_{r}(z))} \leq C r^{N/2+\gamma} \quad \text{for every } z \in B_1 \text{ and } r \in (0, 1). \]

This implies that \( \| g \|_{C^{0,\gamma}(B_1/4)} \leq C. \) The proof is thus finished. \[ \square \]

By scaling and covering, we have the

\[ \text{Corollary 3.5.} \]

Let \( s \in (0, 1), \ N \geq 1, \ \beta \in [0, 2s), \ \sigma \in (s, 1], \ \alpha' \geq 0, \ \text{with } \alpha' + \sigma \in (0, 2s), \ \kappa > 0 \) and \( \gamma \in (0, 1) \cap (0, \sigma] \cap (0, 2s - \beta). \) Consider \( K \in \mathcal{K}^s_0(\kappa, 0, Q_\infty) \) and \( K' \) satisfies \( \text{(2.3)} \). Let \( g \in H^s(B_2) \cap L_s(\mathbb{R}^N), \ U \in C^{0,\sigma}_{\text{loc}}(\mathbb{R}^N) \cap L(\alpha' + 2s)/2(\mathbb{R}^N) \) and \( f \in \mathcal{M}_\beta \) satisfy

\[ \mathcal{L}_K g + \mathcal{L}_{K'} U = f \quad \text{in } B_2. \]  

Then there exists \( C > 0, \) only depending on \( N, s, \alpha', \beta, \kappa \) and \( \gamma \), such that

\[ \| g \|_{C^{0,\gamma}(B_1)} \leq C(\| g \|_{L^2(B_2)} + \| g \|_{L_s(\mathbb{R}^N)} + \| U \|_{C^{0,\sigma}(\mathbb{R}^N)} + \| f \|_{\mathcal{M}_\beta}). \]

\[ \text{Proof.} \] Pick \( x_0 \in B_{3/2} \). By the continuity of \( \mathcal{A}_K(\cdot, \cdot, \theta) \) (uniformly with respect to \( \theta \)), for every \( \varepsilon > 0 \) there exists \( \delta = \delta_{x_0, \varepsilon} \in (0, 1/100) \) such that, for every \( x \in B_{4\delta}(x_0), \ r \in (0, 4\delta) \) and \( \theta \in S^{N-1}, \) we have

\[ |K(x, x + r\theta) - \mathcal{A}_K(x_0, 0, \theta)r^{-N-2s}| \leq \varepsilon r^{-N-2s}. \]

Therefore, for every \( x \in B_{4\delta}(x_0) \) and \( 0 < |z| < 4\delta \),

\[ |K(x, x + z) - \mathcal{A}_K(x_0, 0, z/|z|)|z|^{-N-2s}| \leq \varepsilon |z|^{-N-2s} \]

and thus, for every \( x, y \in B_{2\delta}(x_0), \) with \( x \neq y, \)

\[ |K(x, y) - \mu_a(x, y)| \leq \varepsilon \mu_1(x, y). \]
Consider Theorem 3.7. Let \( K \in \mathcal{K}_0^*(\kappa, Q_\infty) \) and \( K_i' \) satisfying (2.3), for \( i = 1, \ldots, \ell \), and for some, \( \alpha_i' \geq 0 \), with \( \alpha_i' + \sigma_i \in (0, 2s) \). Let \( g \in H^s(B_2) \cap L^{\infty}(\mathbb{R}^N) \), \( U_i \in C_{0,\alpha_i}(\mathbb{R}^N) \cap L^{(\alpha_i' + 2s)/2}(\mathbb{R}^N) \) and \( f \in \mathcal{M}_\beta \) satisfy

\[
\mathcal{L}_{Kg} + \sum_{i=1}^{\ell} \mathcal{L}_{K_i'}U_i = f \quad \text{in } B_2. \tag{3.23}
\]

Then, there exists \( C > 0 \), only depending on \( s, N, \beta, \alpha_i, \sigma_i, \kappa, \ell \text{ and } \gamma \), such that

\[
\|g\|_{C^{0,\gamma}(B_1)} \leq C(\|g\|_{L^\infty(\mathbb{R}^N)} + \sum_{i=1}^{\ell} \|U_i\|_{C^{0,\sigma_i}(\mathbb{R}^N)} + \|f\|_{\mathcal{M}_\beta}).
\]

We now state the following result from which we will derive the first part of Theorem 1.17.

Theorem 3.7. Let \( s \in (0, 1/2] \) and \( \alpha \in (0, 1) \). Suppose that \( K \in \mathcal{K}_0^*(\kappa, \alpha, Q_\infty) \) and let \( u \in H^s(B_2) \cap L^{\infty}(\mathbb{R}^N) \) and \( f \in C^\alpha(\mathbb{R}^N) \), satisfy

\[
\mathcal{L}_Ku = f \quad \text{in } B_2.
\]

Then there exists \( C = C(s, N, \kappa, \beta, g) > 0 \) with the following properties.

(i) If \( u \in C^\alpha(\mathbb{R}^N) \), then for all \( g \in (0, \min(s, \alpha)) \),

\[
\|u\|_{C^{0,\min(2s+\alpha-\beta, 1)}(B_1)} \leq C(\|u\|_{C^\alpha(\mathbb{R}^N)} + \|f\|_{C^\alpha(\mathbb{R}^N)}).
\]

(ii) If \( \|A_K\|_{C^{0,\alpha}(Q_\infty \times S^{N-1})} \leq \frac{1}{R} \), then

\[
\|u\|_{C^{0,\min(2s+\alpha-\beta, 1)}(B_1)} \leq C(\|u\|_{L^{\infty}(\mathbb{R}^N)} + \|f\|_{C^\alpha(\mathbb{R}^N)}).
\]
Proof. We apply first Corollary \textbf{3.5} (with $U \equiv 0$ and $\sigma = 1$) to find that for all $g \in (0, \min(s, \alpha))$, there exists $C = C(N, s, k, g, \alpha)$ such that
\[
\| u \|_{C^{2s-\varepsilon}(B_1)} \leq C(\| u \|_{L^\infty(\mathbb{R}^N)} + \| f \|_{L^\infty(\mathbb{R}^N)}).
\]
In particular, letting $v := \varphi_{1/2} u \in C^{2s-\varepsilon}(\mathbb{R}^N)$, then by Lemma \textbf{2.3}(ii), we have
\[
\mathcal{L}_K v = \tilde{f} \quad \text{in } B_{1/4},
\]
for some function $\tilde{f}$ satisfying
\[
\| \tilde{f} \|_{C^\alpha(\mathbb{R}^N)} \leq C(\| u \|_{C^\alpha(\mathbb{R}^N)} + \| f \|_{C^\alpha(\mathbb{R}^N)}).
\]
Let $h \in B_{1/8} \setminus \{0\}$, then we have (recalling \textbf{2.41})
\[
\mathcal{L}_K v_{h, \alpha} + \mathcal{L}_{K_{h, \alpha}} v(\cdot + h) = \tilde{f}_{h, \alpha} \quad \text{in } B_{1/8},
\]
where $K_{h, \alpha}(x, y) = |h|^{-\alpha}[K(x + h, y + h) - K(x, y)]$. Hence
\[
\mathcal{A}_{K_{h, \alpha}}(x, r, \theta) = |h|^{-\alpha}[\mathcal{A}_K(x + h, r, \theta) - \mathcal{A}_K(x, r, \theta)].
\]
Therefore, since $K \in \tilde{\mathcal{K}}^\alpha_\infty(k, \alpha, Q_\infty)$, we find that $\| \mathcal{A}_{K_{h, \alpha}} \|_{L^\infty(Q_\infty \times S^{N-1})} \leq \frac{1}{r}$, so that $K_{h, \alpha}$ satisfies \textbf{2.33}, with $\alpha' = 0$. By Corollary \textbf{3.3} and since $2s - \sigma > s$, we get
\[
\| v_{h, \alpha} \|_{C^{2s-\varepsilon}(B_{1/6})} \leq C(\| v_{h, \alpha} \|_{L^\infty(\mathbb{R}^N)} + \| v(\cdot + h) \|_{C^{2s-\varepsilon}(\mathbb{R}^N)} + \| \tilde{f}_{h, \alpha} \|_{L^\infty(\mathbb{R}^N)}).
\]
Hence by \textbf{3.26} and \textbf{3.24}, we get
\[
\| v_{h, \alpha} \|_{C^{2s-\varepsilon}(B_{2/5})} \leq C(\| u \|_{C^\alpha(\mathbb{R}^N)} + \| f \|_{C^\alpha(\mathbb{R}^N)}).
\]
Since $h$ was arbitrary, \textbf{12} Proposition 5.5 then implies, for $2s + \alpha - \sigma < 1$, that
\[
\| u \|_{C^{\alpha,2s+\alpha-\varepsilon}(B_{2/5})} \leq C(\| u \|_{C^\alpha(\mathbb{R}^N)} + \| f \|_{C^\alpha(\mathbb{R}^N)})
\]
and, for $2s + \alpha - \sigma > 1$, that
\[
\| u \|_{C^{\alpha,1}(B_{2/5})} \leq C(\| u \|_{C^\alpha(\mathbb{R}^N)} + \| f \|_{C^\alpha(\mathbb{R}^N)}).
\]
By a covering and scaling argument, we get (i).
To prove (ii), we can proceed as above, but instead of applying Lemma \textbf{2.3}(ii), we use Lemma \textbf{2.3}(iii) to get
\[
\| \tilde{f} \|_{C^\alpha(\mathbb{R}^N)} \leq C(\| u \|_{L^\infty(\mathbb{R}^N)} + \| f \|_{C^\alpha(\mathbb{R}^N)}).
\]
It follows that
\[
\| u \|_{C^{\alpha,\min(2s+\alpha-\varepsilon, 1)}(B_{1/32})} \leq C(\| u \|_{C^{\alpha,\sigma}(B_{2})} + \| u \|_{L^\infty(\mathbb{R}^N)} + \| f \|_{C^\alpha(\mathbb{R}^N)}).
\]
To absorb the the $C^{\alpha,\sigma}(B_2)$-norm of $u$, we use the classical adimensional Hölder norms and interpolation (see e.g. \textbf{3.37}) to get
\[
\| u \|_{C^{\alpha,\min(2s+\alpha-\varepsilon, 1)}(B_{1/32})} \leq C(\| u \|_{L^\infty(\mathbb{R}^N)} + \| f \|_{C^\alpha(\mathbb{R}^N)}).
\]
Now by a covering and scaling argument, we get the conclusion of the theorem. \hfill \qed

4. Gradient estimates

In this section, we consider the fractional parameter $s \in (1/2, 1)$ and we prove Hölder estimates of $\nabla u$. For $g \in L^1_{loc}(\mathbb{R}^N)$ and $r > 0$, we define
\[
P_{r,g}(x) = g_{B_r} + T^{r,g} \cdot x = g_{B_r} + \sum_{i=1}^N T_i^{r,g} x_i,
\]
where
\[
T_i^{r,g} = \frac{\langle g, x_i \rangle_{L^2(B_r)}}{\| x_i \|_{L^2(B_r)}}
\]
Note that $P_{r,g}$ is the $L^2(B_r)$-projection of $g$ on the space of affine functions.

In view of Corollary 3.3 we know that the solutions $u$ to $\mathcal{L}_K u = f$ in $B_2$ are of class $C^{1-\varepsilon}(B_1)$ for every small $\varepsilon > 0$, provided $K \in \mathcal{X}_0^s(\kappa, 0, Q_\infty)$ and $f \in \mathcal{M}_\beta$, with $\beta \in [0, 2s)$. In particular $|T^{r,u-u(0)}| \leq C r^{-\varepsilon}$. The result below improves this to Hölder regularity estimates of the gradient of $u$ when $2s - \beta > 1$ and $K \in \mathcal{X}_0^s(\kappa, \alpha, Q_\infty)$, with $\alpha > 0$.

**Proposition 4.1.** Let $N \geq 1$, $s \in (1/2, 1)$, $\kappa, c_0 > 0$. Consider $\alpha \in (0, 2s - 1)$, $\beta \in [0, 2s)$, $\varrho \in [0, 1)$ such that

$$\gamma := \min(1 - \varrho + \alpha, 2s - \beta) > 1.$$ 

Then there exists $C = C(N, s, \alpha, \beta, \kappa, c_0, \varrho) > 0$ such that if:

- $K \in \mathcal{X}_0^s(\kappa, \alpha, Q_\infty)$,
- $g \in H^s(B_2) \cap L^\infty(\mathbb{R}^N)$ and $f \in \mathcal{M}_\beta$ with

$$\|g\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{\mathcal{M}_\beta} \leq 1,$$

$$|T^{r,g}| \leq c_0 r^{-\varrho} \quad \text{for all } r > 0$$

are such that

$$\mathcal{L}_K g = f \quad \text{in } B_2,$$

then we have

$$\sup_{r > 0} r^{-\gamma} \|g - P_{r,g}\|_{L^\infty(B_r)} \leq C.$$

**Proof.** Suppose on contrary that the assertion in the proposition does not hold. Then as in the proof of Proposition 3.3 for all $n \geq 2$, there exist

- $r_n > 0$, $K_n \in \mathcal{X}_0^s(\kappa, \alpha, Q_\infty)$,
- $g_n \in H^s(B_2) \cap L^\infty(\mathbb{R}^N)$ and $f_n \in \mathcal{M}_\beta$ satisfying

$$\|g_n\|_{L^\infty(\mathbb{R}^N)} + \|f_n\|_{\mathcal{M}_\beta} \leq 1, \quad |T^{r,g_n}| \leq c_0 r^{-\varrho} \quad \text{for all } r \in (0, \infty),$$

$$\mathcal{L}_{K_n} g_n = f_n \quad \text{in } B_2, \quad (4.3)$$

- a nonincreasing function $\Theta_n : (0, \infty) \to (0, \infty)$ satisfying

$$\Theta_n(r) \geq r^{-\gamma} \|g_n - P_{r,g_n}\|_{L^\infty(B_r)} \quad \text{for every } r \in (0, \infty) \text{ and } n \geq 2, \quad (4.4)$$

with the properties that $r_n \to 0$ as $n \to \infty$ and

$$r_n^{-\gamma} \|g_n - P_{r_n,g_n}\|_{L^\infty(B_{r_n})} \geq \frac{1}{2} \Theta_n(r_n) \geq \frac{n}{4}.$$ 

We define

$$v_n(x) = \Theta_n(r_n)^{-1} r_n^{-\gamma} [g_n(r_n x) - P_{r_n,g_n}(r_n x)],$$

so that

$$\|v_n\|_{L^\infty(B_1)} \geq \frac{1}{2}. \quad (4.6)$$

In addition, by a change of variable, we get

$$\int_{B_1} v_n(x) dx = \int_{B_1} v_n(x) x_i dx = 0 \quad \text{for every } i \in \{1, \ldots, N\}. \quad (4.7)$$

Since $\Theta_n$ is nonincreasing and $\gamma > 1$ then see e.g. 29, 55, inequality always implies that

$$\|v_n\|_{L^\infty(B_R)} \leq C R^{-\gamma} \quad \text{for every } R \geq 1, \quad (4.8)$$

for some constant $C = C(N, \gamma)$. From 4.4, we deduce that

$$\mathcal{L}_{K_n} v_n + \Theta_n(r_n)^{-1} r_n^{-\gamma} T^{r_n,g_n} \cdot \mathcal{L}_{K_n} x = \overline{f}_n \quad \text{in } B_2/r_n, \quad (4.9)$$

where $\overline{K}_n(x,y) := r_n^{N+2s} K_n(r_n x, r_n y)$ and $\overline{f}_n(x) = r_n^{2s} f_n(r_n x)$. Then, since $\mathcal{A}_{K_n}(x,r,\theta) = A_{K_n}(r_n x, r_n r, \theta)$ and $A_{K_n} \in C^{0,\alpha}_{1,2}(Q_\infty)$, we get

$$|A_{K_n}(x,r,\theta) - A_{K_n}(0,0,\theta)| \leq C \min(r_n^\alpha(|x| + r)^\alpha, 1) \quad \text{for all } x \in \mathbb{R}^N, \theta \in S^{N-1} \text{ and } r > 0.$$
Moreover, recalling Definition \[1.6\] we have \( A_{\alpha,K_n}(0,0,\theta) = 0 \) for all \( \theta \in S^{N-1} \). Letting \( a_n(\theta) := A_{K_n}(0,0,\theta) \) and \( K_{n}(x,y) := r_n^{-\alpha} (K_n(x,y) - \mu_{a_n}(x,y)) \), we immediately see that
\[
|K_{n}(x,y) - \mu_{a_n}(x,y)| \leq C \min(r_n^{\alpha}(|x|+|y|)^{\alpha},1) \mu_1(x,y)
\]
and
\[
|K_{n}(x,y)| \leq C(|x|+|y|)^{\alpha} \mu_1(x,y).
\]
Since \( L_{\mu_{a_n}} x_i = 0 \) on \( \mathbb{R}^N \), we can rewrite \[4.9\] as
\[
\mathcal{L}_{K_{n}} v_n + \Theta_n(r_n)^{-1} \sum_{i=1}^{N} \nabla x_i = r_n^{-\gamma} \Theta_n(r_n)^{-1} \mathcal{T}_{n} x_i \quad \text{in } B_{2/n},
\]
where (recall \[4.1\]) \( \mathcal{T}_{n} := r_n^{1+\alpha-\gamma} \mathcal{T}_{n}^{\gamma} \). Note that \( x_i \in L_{(\alpha+2s)/2}(\mathbb{R}^N) \), provided \( \alpha \in (0,2s-1) \) and \( [x_i]_{C^{0,1}(\mathbb{R}^N)} \leq 1 \). Clearly by \[4.3\],
\[
\|\mathcal{T}_{n}\|_{\mathcal{M}_s} \leq r_n^{2s-\beta} \quad \text{and} \quad |\mathcal{T}_{n}| \leq c_0 r_n^{1+\alpha-\gamma-\epsilon} \leq c_0.
\]
Since \( K_{n} \in \mathcal{A}_{\alpha}^{\epsilon}(\kappa,0,Q_{\infty}) \) and \( \Theta_n(r_n) \to \infty \) as \( n \to \infty \), applying Corollary \[8.6\] to \[4.12\] and using \[4.15\] together with \[4.11\], we find that \( v_n \) is bounded in \( C_{loc}^{1-\delta}(\mathbb{R}^N) \), for all \( \delta \in (0,1) \). Hence, provided \( \delta \) is small, there exists \( v \in C_{loc}^{1-\delta}(\mathbb{R}^N) \) such that, up to a subsequence, \( v_n \to v \) in \( C_{loc}^{0}(\mathbb{R}^N) \). Hence by \[4.8\], up to a subsequence, \( v_n \) converges strongly, in \( L_{s}(\mathbb{R}^N) \), to \( v \in H_{s}^{1}(\mathbb{R}^N) \cap L_{s}(\mathbb{R}^N) \). Moreover, by \[4.6\], we deduce that
\[
\|v\|_{L^{\infty}(B_{1})} \geq \frac{1}{2} \quad \text{and} \quad \int_{B_{1}} v(x) \, dx = \int_{B_{1}} v(x) x_i \, dx = 0 \quad \text{for every } i \in \{1,\ldots,N\}.
\]
In addition, passing to the limit in \[4.8\], we have
\[
\|v\|_{L^{\infty}(B_{1})} \leq R \gamma \quad \text{for every } R \geq 1.
\]
We observe that \( a_n \) satisfies \[3.1\] for all \( n \). By \[4.10\] and \[3.1\], we can pass to the limit in \[4.12\], to get \( \mathcal{L}_{\mu} v = 0 \) in \( \mathbb{R}^N \), where \( b \) is the weak-star limit of \( a_n \). Now, since \( v \in H_{s}^{1}(\mathbb{R}^N) \) and satisfies \[4.15\], by Lemma \[3.2\], we deduce that \( v \) is an affine function, because \( \gamma < 2s \). This is clearly in contradiction with \[4.14\].

A first consequence of the previous result is the

**Corollary 4.2.** Let \( s \in (1/2,1) \), \( \beta \in [0,2s-1) \), \( N \geq 1 \) and \( \kappa > 0 \). Let \( \alpha \in (0,2s-1) \) and \( g \in [0,\alpha) \). Let \( K \in \mathcal{A}_{\alpha}^{\epsilon}(\kappa,0,Q_{\infty}) \), \( g \in H_{s}(B_{2}) \cap C_{loc}^{1-\epsilon}(\mathbb{R}^N) \) and \( f \in \mathcal{M}_{\beta} \) satisfy
\[
\mathcal{L}_{K} g = f \quad \text{in } B_{2}.
\]

Then, there exists \( C > 0 \), only depending on \( s,N,\beta,\alpha,\kappa,g \), such that
\[
\|g\|_{C_{s}^{1}(\min(1+\alpha-g,2s-\beta)-1)(B_{1})} \leq C(\|g\|_{C_{s}^{0,1-\epsilon}(\mathbb{R}^N)} + \|f\|_{\mathcal{M}_{\beta}}).
\]

**Proof.** Put \( A := \|g\|_{C_{s}^{0,1-\epsilon}(\mathbb{R}^N)} + \|f\|_{\mathcal{M}_{\beta}} \). We define \( G_{z}(x) := g(x+z) - g(z) \), for \( z \in B_{1} \). Since \( G_{z}(0) = 0 \), we have \( |T_{r}^{\gamma} G_{z}| \leq C(\|g\|_{C_{s}^{0,1-\epsilon}(\mathbb{R}^N)} \leq CA \), for \( r > 0 \). Obviously, \( \mathcal{L}_{K_{r}} G_{z} = f(\cdot + z) \) in \( B_{1} \), where \( K_{r}(x,y) = K(x+z,y+z) \). We then apply Proposition \[4.1\] to get a constant \( C > 0 \), only depending on \( s,N,\beta,\alpha,\kappa,g \), such that
\[
\sup_{r > 0} r^{-\gamma} \|G_{z} - P_{r} G_{z}\|_{L^{\infty}(B_{1})} \leq CA,
\]
where \( \gamma := \min(1+\alpha-g,2s-\beta) > 1 \). By a well known iteration argument (see e.g \[55\]), we find that
\[
|g(x) - g(z) - T(z)(x-z)| \leq CA|x-z|^{\gamma} \quad \text{for every } x,z \in B_{1/2},
\]
for some \( T \), satisfying \( |T|_{L^{\infty}(B_{1})} \leq CA \). Since \( \gamma > 1 \), then \( \nabla u(z) = T(z) \). By a classical extension theorem (see e.g. \[58\][Page 177], we deduce that \( u \in C_{s}^{1-\gamma-1}(B_{1/2}) \). Moreover
\[
\|u\|_{C_{s}^{1-\gamma-1}(B_{1/2})} \leq CA.
\]
By a bootstrap argument, we have the following result.

**Theorem 4.3.** Let $s \in (1/2, 1)$, $\beta \in [0, 2s - 1)$, $\alpha \in (0, 2s - 1)$ and $\kappa > 0$. Let $K \in \mathcal{K}_{0}^{s}(\kappa, \alpha, Q_{\infty})$, $u \in H^{s}(B_{2}) \cap L_{s}(\mathbb{R}^{N})$ and $V, f \in \mathcal{M}_{\beta}$ such that

$$L_{K}u + Vu = f \quad \text{in } B_{2}.$$ 

Then

$$\|u\|_{C^{1, \min(\alpha, 2s - \beta - 1)}(B_{1})} \leq C(\|u\|_{L^{2}(B_{2})} + \|u\|_{L_{s}(\mathbb{R}^{N})} + \|f\|_{\mathcal{M}_{\beta}}),$$

where $C > 0$ only depends on $s, N, \kappa, \alpha, \beta$ and $\|V\|_{\mathcal{M}_{\beta}}$.

**Proof.** Since $2s - \beta > 1$, by [29], for every $\varrho \in (0, 1)$, there exists $C = C(N, s, \beta, \alpha, \|V\|_{\mathcal{M}_{\beta}}, \varrho) > 0$ such that

$$\|u\|_{C^{1, \varrho}(B_{1})} \leq C(\|u\|_{L^{2}(B_{2})} + \|u\|_{L_{s}(\mathbb{R}^{N})} + \|f\|_{\mathcal{M}_{\beta}}).$$

Using Lemma 2.3(i), we apply first Corollary 1.2 to get

$$\|\varphi_{1/2}u\|_{C^{0,1}(B_{1/s})} \leq C(\|\varphi_{1/2}u\|_{L^{2}(\mathbb{R}^{N})} + \|\varphi_{1/2}u\|_{C^{1-\varrho}(\mathbb{R}^{N})} + \|u\|_{L_{s}(\mathbb{R}^{N})} + \|\varphi_{2}Vu\|_{\mathcal{M}_{\beta}} + \|f\|_{\mathcal{M}_{\beta}}).$$

Therefore,

$$\|u\|_{C^{0,1}(B_{2-\varepsilon})} \leq C(\|u\|_{L^{2}(B_{2})} + \|u\|_{L_{s}(\mathbb{R}^{N})} + \|f\|_{\mathcal{M}_{\beta}}).$$

We then apply once more Corollary 1.2 (with $\varrho = 0$) and use Lemma 2.3(i) to obtain

$$\|\varphi_{2^{-4}}u\|_{C^{1, \min(\alpha, 2s - \beta - 1)}(B_{2-\varepsilon})} \leq C(\|\varphi_{2^{-4}}u\|_{L^{2}(\mathbb{R}^{N})} + \|\varphi_{2^{-4}}u\|_{C^{0,1}(\mathbb{R}^{N})} + \|u\|_{L_{s}(\mathbb{R}^{N})} + \|\varphi_{2^{-4}}Vu\|_{\mathcal{M}_{\beta}} + \|f\|_{\mathcal{M}_{\beta}}),$$

so that

$$\|u\|_{C^{1, \min(\alpha, 2s - \beta - 1)}(B_{2} - \varepsilon)} \leq C(\|u\|_{L^{2}(B_{2})} + \|u\|_{L_{s}(\mathbb{R}^{N})} + \|f\|_{\mathcal{M}_{\beta}}),$$

with $C$ as in the statement of the theorem. After a covering argument, we obtain the result. \( \square \)

### 5. Shauder estimates

Here and in the following, given $u \in C^{2s+\alpha}_{loc}(\mathbb{R}^{N})$, with $\alpha > 0$, we let

$$\delta^{\alpha}u(x, r, \theta) := \frac{1}{2}(2u(x) - u(x + r\theta) - u(x - r\theta)), \quad \delta^{\alpha}u(x, r, \theta) := \frac{1}{2}(u(x + r\theta) - u(x - r\theta)). \quad (5.1)$$

For $A \in C^{m,\alpha}_{1,2}(Q_{\infty})$, we define

$$\mathcal{E}_{A, \alpha}^{s}(x) := \int_{S^{N-1}} \int_{0}^{\infty} \delta^{\alpha}u(x, r, \theta) A(x, r, \theta) r^{-1-2s} drd\theta \quad (5.2)$$

and for $B \in C^{m}_{\alpha + (2s - 1)_+}(Q_{\infty})$, we define

$$\mathcal{O}_{B, \alpha}^{s}(x) := \int_{S^{N-1}} \int_{0}^{\infty} \delta^{\alpha}u(x, r, \theta) B(x, r, \theta) r^{-1-2s} drd\theta. \quad (5.3)$$

We observe that, using the symmetry of $K \in \mathcal{K}_{0}^{\alpha + (2s - 1)_+}(\kappa, \alpha, Q_{\infty})$ and a change of variables, we get

$$\frac{1}{2} \int_{\mathbb{R}^{2N}} (u(x) - u(y))(\psi(x) - \psi(y)) K(x, y) dx dy = \int_{\mathbb{R}^{N}} \psi(x) \mathcal{E}_{A_{e, K}, \alpha}^{s}(u)(x) dx + \int_{\mathbb{R}^{N}} \psi(x) \mathcal{O}_{A_{e, K}, \alpha}^{s}(u)(x) dx, \quad (5.4)$$

where

$$A_{e, K}(x, r, \theta) := \frac{1}{2}(A_{K}(x, r, \theta) + A_{K}(x, r, -\theta)), \quad A_{o, K}(x, r, \theta) := \frac{1}{2}(A_{K}(x, r, \theta) - A_{e, K}(x, r, -\theta)).$$

We have the following result which will be proved in Section 7.

**Lemma 5.1.** Let $s \in (0, 1)$ and $\alpha \in (0, 1)$. Let $\kappa > 0$, $m \in \mathbb{N}$, $A \in C^{m,\alpha}_{1,2}(Q_{\infty})$ and $B \in C^{m}_{\tau}(Q_{\infty})$, with $\tau := \alpha + (2s - 1)_+$. 
• Let \( u \in C^{2s+\alpha+m}(\mathbb{R}^N) \). If \( 2s + \alpha < 1 \) or \( 2s > 1 \), then
\[
\| \mathcal{E}_{A,u} \|_{C^{\alpha}(\mathbb{R}^N)} \leq C \| A \|_{C^{m,\alpha}_{1,2}(Q_\infty)} \| u \|_{C^{2s+\alpha+m}(\mathbb{R}^N)}
\]
and
\[
\| \mathcal{O}_{B,u} \|_{C^{\alpha}(\mathbb{R}^N)} \leq C \| B \|_{C^m(Q_\infty)} \| u \|_{C^{2s+\alpha+m}(\mathbb{R}^N)},
\]
with \( C = C(N,s,\alpha,m) \).

• Let \( u \in C^{1+\alpha+m+\varepsilon}(\mathbb{R}^N) \), for some \( \varepsilon \in (0,1-\alpha) \). If \( 2s = 1 \) and \( B \in C^m(\mathbb{R}^N) \), then
\[
\| \mathcal{E}_{A,u} \|_{C^{\alpha}(\mathbb{R}^N)} \leq C \| A \|_{C^{m,\alpha}_{1,2}(Q_\infty)} \| u \|_{C^{2s+\alpha+m}(\mathbb{R}^N)}
\]
and
\[
\| \mathcal{O}_{B,u} \|_{C^{\alpha}(\mathbb{R}^N)} \leq C \| B \|_{C^m(\mathbb{R}^N)} \| u \|_{C^{2s+\alpha+m}(\mathbb{R}^N)},
\]
with \( C = C(N,\alpha,m,\varepsilon) \).

We remark that under the assumptions on \( A \) and \( B \), for \( 2s = 1 \), the first assertion of Lemma 5.1 does not in general hold.

5.1. Shauder estimates. The following result is intended to the \( C^{2s+\alpha} \) regularity estimates, for \( 2s + \alpha \notin \mathbb{N} \). To deal with the case \( 2s + \alpha > 2 \), we look for optimal growth estimate of the difference between \( u \) a second order polynomial that is close to \( u \) in the \( L^2 \)-norm. For \( g \in L^2(B_r) \) and \( i,j = 1,\ldots,N \), we define
\[
T_{ij}^{r,g} = \frac{1}{\| y_i y_j \|_{L^2(B_r)}} \int_{B_r} y_i y_j g(y) \, dy
\]
and
\[
Q_{r,g}(x) = \sum_{i,j=1}^N T_{ij}^{r,g} x_i x_j.
\]
We note that \( Q_{r,g} \) is nothing but the \( L^2(B_r) \)-projection of \( g \) on the space of homogeneous quadratic polynomials. We now state the main result of this section.

Proposition 5.2. Let \( N \geq 1 \), \( s \in (0,1) \), \( \kappa > 0 \), \( \alpha \in (0,1) \), \( \beta \in (0,\alpha) \) and \( \varepsilon > 0 \). Let \( K \in \mathcal{K}^{a}_{\alpha+2s-1,\kappa}(\kappa,\alpha,Q_\infty) \). Let \( f \in C^\alpha(\mathbb{R}^N) \) and \( g \in C^{2s+\beta}(\mathbb{R}^N) \), for \( 2s + \beta \notin \mathbb{N} \), such that
\[
\mathcal{L}_K g = f \quad \text{in } B_2,
\]
(i) If \( 1 < 2s + \alpha < 2 \) and \( 2s \neq 1 \), then there exists \( C = C(N,s,\kappa,\alpha,\beta) > 0 \) such that
\[
\sup_{r>0} r^{-2s-1+\alpha-\beta} \| \nabla g - \nabla Q_{r,g} \|_{C^{\beta}(B_r)} \leq C \| g \|_{C^{2s+\beta}(\mathbb{R}^N)} + \| f \|_{C^\alpha(\mathbb{R}^N)}.
\]
(ii) If \( 2s + \alpha > 2, 2s + \beta > 1 + \alpha \) and \( g(0) = | \nabla g(0) | = 0 \), then there exists \( C = C(N,s,\kappa,\alpha,\beta) > 0 \) such that
\[
\sup_{r>0} r^{-2s-1+\alpha-\beta} \| \nabla g - \nabla Q_{r,g} \|_{C^{\beta}(B_r)} \leq C \| g \|_{C^{2s+\beta}(\mathbb{R}^N)} + \| f \|_{C^\alpha(\mathbb{R}^N)}.
\]
(iii) If \( 2s = 1 \) and \( K \in \mathcal{K}^{\alpha}_{\alpha+\varepsilon}(\kappa,\alpha,Q_\infty) \), then there exists \( C = C(N,s,\kappa,\alpha,\beta,\varepsilon) > 0 \) such that
\[
\sup_{r>0} r^{-2s-1+\alpha-\beta} \| \nabla g - \nabla Q_{r,g} \|_{C^{\beta}(B_r)} \leq C \| g \|_{C^{2s+\beta}(\mathbb{R}^N)} + \| f \|_{C^\alpha(\mathbb{R}^N)}.
\]
(iv) If \( 2s + \alpha < 1 \), then there exists \( C = C(N,s,\kappa,\alpha,\beta) > 0 \) such that
\[
\sup_{r>0} r^{-\alpha-\beta} \| g \|_{C^{2s+\beta}(B_r)} \leq C \| g \|_{C^{2s+\beta}(\mathbb{R}^N)} + \| f \|_{C^\alpha(\mathbb{R}^N)}.
\]

Proof. We start with (i). Assume that the assertion in (i) does not hold, then arguing as in the proof of Proposition 5.3 we can find sequences
\[
\bullet \, r_n > 0, \, K_n \in \mathcal{K}^{a}_{\alpha+2s-1,\kappa}(\kappa,\alpha,Q_\infty) \quad \{ g_n \in C^{2s+\beta}(\mathbb{R}^N) \} \quad \{ f_n \in C^\alpha(\mathbb{R}^N) \} \quad \| g_n \|_{C^{2s+\beta}(\mathbb{R}^N)} + \| f_n \|_{C^\alpha(\mathbb{R}^N)} \leq 1,
\]
\[
\mathcal{L}_{K_n} g_n = f_n \quad \text{in } B_2, \quad (5.5)
\]
Moreover by (5.11) and (5.10), there exists $e$ for some $C$ a subsequence, and combining this with (5.10), we get, for all $R > 0$ and $n \geq 2$

$$\Theta_n(r) \geq r^{-2s+1-\alpha-\beta}|\nabla g_n|_{C^\beta(B_r)}$$

for every $r > 0$ and $n \geq 2$.

and

$$r^{-2s+1-\alpha-\beta}|\nabla g_n|_{C^\beta(B_r)} \geq \frac{1}{2} \Theta_n(r_n) \geq \frac{n}{4}.$$ (5.7)

We define

$$u_n(x) := \frac{1}{r^{2s+\alpha}\Theta_n(r_n)}g(r_n x).$$

By (5.6), for $R \geq 1$, we have

$$|\nabla u_n|_{C^\beta(B_R)} \leq \frac{1}{r^{2s+\alpha}\Theta_n(r_n)}|\nabla g_n|_{C^\beta(B_{r_n} n)} \leq R^{2s+1+\alpha-\beta} \frac{\Theta_n(R r_n)}{\Theta_n(r_n)}.$$ (5.8)

Hence by the monotonicity of $\Theta_n$, we have

$$|\nabla u_n|_{C^\beta(B_R)} \leq R^{2s+1+\alpha-\beta}$$

for all $R \geq 1$.

In addition, by (5.7), we get

$$|\nabla u_n|_{C^\beta(B_1)} \geq \frac{1}{2}.$$ (5.9)

This then implies that there exists $x_n, h_n \in B_1$, with $h_n \neq 0$, such that

$$|h_n|^{-\beta}|\nabla u_n(x_n + h_n) - \nabla u_n(x_n)| \geq \frac{1}{4}.$$ (5.10)

We define the new sequence

$$v_n(x) := \frac{u_n(x_n + |h_n|x) - u_n(x_n) - |h_n|\nabla u_n(x_n) \cdot x}{|h_n|^{1+\beta}}.$$ (5.11)

By construction, we have that

$$v_n(0) = |\nabla v_n(0)| = 0.$$ (5.12)

and by (5.11),

$$|\nabla v_n(h_n/|h_n|)| \geq \frac{1}{4}.$$ (5.13)

Moreover by (5.8), for $R \geq 1$ and $x, y \in B_R$,

$$|\nabla v_n(x) - \nabla v_n(y)| = |h_n|^{-\beta}|\nabla u_n(x_n + |h_n|x) - \nabla u_n(x_n + |h_n|y)| \leq |h_n|^{-\beta}|h_n|\beta|y - x|^\beta(1 + R)^{2s-1+\alpha-\beta} \leq 2^{2s-1+\alpha-\beta}|x - y|^\beta R^{2s-1+\alpha-\beta}.$$ (5.14)

Combining this with (5.10), we get, for all $R \geq 1$,

$$|\nabla v_n|_{L^\infty(B_R)} \leq CR^{2s-1+\alpha}$$ (5.15)

and

$$|v_n|_{C^{1,\beta}(B_R)} \leq CR^{2s+\alpha},$$

for some $C = C(s, \alpha, \beta)$. This latter estimate implies that there exists $v \in C^1_{loc}(\mathbb{R}^N)$ such that, up to a subsequence,

$$v_n \to v$$

in $C^1_{loc}(\mathbb{R}^N).$ (5.16)

Moreover by (5.11) and (5.10), there exists $e \in S^{N-1}$ such that

$$|\nabla v(e)| \geq \frac{1}{4}$$

and

$$\nabla v(0) = 0.$$ (5.17)

We shall show that $\nabla v \equiv 0$ on $\mathbb{R}^N$, which leads to a contradiction. Indeed, given $h \in \mathbb{R}^N$, we define $w_n(x) = (v_n)_{h,0}(x) = v_n(x + h) - v_n(x)$. It follows from (5.12) and the fundamental theorem of calculus that

$$|w_n|_{L^\infty(B_R)} \leq CR^{2s-1+\alpha}$$

for every $R \geq 1$. (5.18)
where here and in the following of the proof, the letter $C$ is a positive constant only depending on $h,N,s,\beta$ and $\alpha$. We put $\rho_n := r_n|n|$, $z_n := r_n x_n$ and we define

$$K_n(x,y) = \rho_n^{N+2s} K_n(z_n + \rho_n x, z_n + \rho_n y),$$

$$C = \rho_n^N C_n.$$ 

We recall that $C$ and $\rho_n$ are defined by $C_n := r_n^{N+2s} C_n(z_n + \rho_n x, z_n + \rho_n y)$ and $\rho_n := r_n h$.

Moreover by (5.18), for all $x \in \mathbb{R}^N \setminus \{0\}$, we let $n$ large so that $z_n + h \in B_{1/2r_n}$, by changing variables and using (5.5), we then have that

$$\mathcal{L}_{K_n} w_n + \frac{\Theta_n(r_n)}{\Theta_n(r_n)} \mathcal{L}_{K_n} U_n = \frac{1}{\Theta_n(r_n)} F_n \quad \text{in } B_{1/2r_n}.$$ 

Therefore by (5.16),

$$\mathcal{L}_{K_n} w_n = F_n \quad \text{in } B_{1/2r_n},$$ 

where

$$F_n(x) := \frac{1}{\Theta_n(r_n)} F_n - \frac{1}{\Theta_n(r_n)} \mathcal{E}_{A_{+\pi n},U_n} \quad \text{and} \quad \mathcal{O}_{A_{+\pi n},U_n}.$$ 

By a change of variable, we get

$$\mathcal{E}_{A_{+\pi n},U_n} = |h_n|^{2s-1} \int_{S^{N-1}} \int_0^\infty \delta^\alpha g_n(z_n + \rho_n x + \rho_n h, t, \theta) A_{+\pi n}(x,t/\rho_n,\theta)^{-1-2s} dt d\theta$$ 

and

$$\mathcal{O}_{A_{+\pi n},U_n} = |h_n|^{2s-1} \int_{S^{N-1}} \int_0^\infty \delta^\alpha g_n(z_n + \rho_n x + \rho_n h, t, \theta) A_{+\pi n}(x,t/\rho_n,\theta)^{-1-2s} dt d\theta.$$ 

We recall that

$$A_{+\pi n}(x,r,\theta) := A_{+\pi n}(z_n + \rho_n x, \rho_n r, \theta)$$ 

and

$$A_{+\pi n}(x,r,\theta) := r_n^{-\alpha} |h_n|^{-\beta} \{ A_{+\pi n}(z_n + \rho_n x, \rho_n h, \rho_n r, \theta) - A_{+\pi n}(z_n + \rho_n x, \rho_n r, \theta) \}.$$ 

Since $K_n \in \mathcal{K}_\alpha^{s+1} (\kappa, \alpha, Q_\infty)$ (recall Definition 1.6), for all $x, y \in \mathbb{R}^N$, $\theta \in S^{N-1}$ and $r > 0$,

$$|A_{+\pi n}(x,r,\theta)| \leq \frac{1}{\kappa} \min(r,1)^{2s-1+\alpha},$$ 

and

$$|A_{+\pi n}(x,r,\theta) - A_{+\pi n}(y,r,\theta)| \leq \frac{1}{\kappa} \min(1,|x-y|)^{2s-1+\alpha}.$$ 

Therefore,

$$|A_{+\pi n}(x,r,\theta)| \leq C |h_n|^\beta \leq C \quad \text{for all } x \in \mathbb{R}^N, r > 0 \text{ and } \theta \in S^{N-1}.$$ 

Consequently, since $\|g_n\|_{C^{2s+2}} \leq 1$ and recalling (5.11), we have that

$$|\delta^\alpha g_n(z_n + \rho_n x + \rho_n h, t, \theta)| \leq C \min(1,|t|^{2s+\beta})$$ 

and thus

$$\| \mathcal{E}_{A_{+\pi n},U_n} \|_{L^\infty(\mathbb{R}^N)} \leq C.$$ 

Moreover by (5.18), for all $x \in \mathbb{R}^N$, $r > 0$ and $\theta \in S^{N-1}$, we have

$$|A_{+\pi n}(x,r,\theta)| \leq C r_n^{-\alpha} |h_n|^{-\beta} \min(\rho_n, \rho_n r)^{2s-1+\alpha}.$$
Since $|h_n| \leq 1$ and $\|g_n\|_{C^{0,1}(\mathbb{R}^N)} \leq 1$ (recalling (5.11)), the above estimate implies that

$$|\mathcal{O}^n_{A_n, \mathcal{F}_n} U_n(x)| \leq C r_n^{-\alpha} |h_n|^{-\beta} \int_0^\infty \min(1, t) \min(\rho_n, t)^{2s-1+\alpha} t^{-1-2s} dt$$

$$\leq C r_n^{-\alpha} |h_n|^{-\beta} \int_0^{\rho_n} t^{2s+\alpha} t^{-1-2s} dt$$

$$+ C r_n^{-\alpha} |h_n|^{-\beta} \rho_n^{2s-1+\alpha} \int_{\rho_n}^1 t^{-2s} dt + C r_n^{-\alpha} |h_n|^{-\beta} \rho_n^{2s+1+\alpha} \int_1^\infty t^{-1-2s} dt.$$}

Using that $2s > 1$ and recalling that $\rho_n = r_n |h_n|$, we then conclude that

$$\|\mathcal{O}^n_{A_n, \mathcal{F}_n} U_n\|_{L^\infty(\mathbb{R}^N)} \leq C.$$  (5.20)

Because $[f_n]_{C^{0}(\mathbb{R}^N)} \leq 1$, it is plain that

$$\|f_n\|_{L^\infty(\mathbb{R}^N)} \leq C.$$  (5.21)

Recalling (5.14), it follow from (5.17), (5.19) and (5.20) that

$$\|F_n\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{\Theta_n(r_n)}.$$  (5.22)

In view of (5.13) and (5.14), for every $h \in \mathbb{R}^N$, we have that

$$w_n = \delta_N(n_{h,0} \mapsto v_{h,0} \quad \text{in} \quad L_2(\mathbb{R}^N) \cap H^s_{\text{loc}}(\mathbb{R}^N).$$  (5.23)

Letting $a_n(\theta) := \mathcal{A}_{A_n, \mathcal{F}_n}(z_n, 0, \theta)$, we have that

$$|\mathcal{A}_{A_n, \mathcal{F}_n}(x, r, \theta) - a_n(\theta)| \leq C \min(\rho_n^\alpha (|x| + r)^\alpha, 1) \quad \text{for all} \quad x \in \mathbb{R}^N, \; r > 0 \; \text{and} \; \theta \in \mathbb{S}^{N-1},$$

so that

$$[\mathcal{K}_n(x, y) - \mu_{a_n}(x, y)] \leq C \min(\rho_n^\alpha (|x| + |y|)^\alpha, 1) \mu_1(x, y) \quad \text{for all} \quad x \neq y \in \mathbb{R}^N.$$  (5.24)

Moreover $a_n$ satisfies (5.1) for all $n$. Therefore in view of Lemma 3.4 Lemma 2.2, (5.23) and (5.22), passing to the limit in (5.10), we deduce that

$$\mathcal{L}_{\mu_b} v_{h,0} = 0 \quad \text{in} \quad \mathbb{R}^N,$$  (5.25)

were $b$ is the weak-star limit of $a_n$ in $L^\infty(S^{N-1})$. Furthermore by (5.12),

$$\|v_{h,0}\|_{L^\infty(B_R)} \leq C R^{2s-1+\alpha},$$

Thanks to (5.25) and since $2s - 1 + \alpha < 1$, we can apply Lemma 3.2 to get a constant $c = c(h, \alpha, \beta, N, s, \kappa)$ such that $v_{h,0}(x) = v(x + h) - v(x) = c$ for all $x, h \in \mathbb{R}^N$. Hence, since $\nabla v(0) = 0$, we find that $\nabla v(h) = 0$ for all $h \in \mathbb{R}^N$. This contradicts the first inequality in (5.14). The proof of (i) is thus finished.

The proof of (ii) is similar to the one of (i), we therefore give a sketch below, emphasizing the main differences. Indeed, following the proof, we put

$$\overline{\pi}_n(x) = \frac{1}{r_n^{2s+\alpha} \Theta_n(r_n)} \left\{ g_n(r_n x) - Q_{r_n, g_n}(r_n x) \right\},$$

with $\Theta_n(r)$ is a nonincreasing function as above, with $g_n$ replaced with $g_n - Q_{r_n, g_n}$. From the definition of $Q_{r_n, g_n}$, the monotonicity of $\Theta_n$ and the fact that $2s - 1 + \alpha > 1$, we then get $\|\nabla \overline{\pi}_n\|_{C^0(B_R)} \leq C R^{2s-1+\alpha}$ for all $R \geq 1$. On the other hand, there are $x_n \in B_1$ and $h_n \in B_1 \setminus \{0\}$ such that $|\nabla \overline{\pi}_n(x_n + h_n) - \nabla \overline{\pi}_n(x_n)||h_n|^{-\beta} \geq \frac{1}{4}$. Similarly as above, we define

$$\overline{\eta}_n(x) := \frac{\overline{\eta}_n(x_n + h_n) - \overline{\pi}_n(x_n) - |h_n| \nabla \overline{\pi}_n(x_n) \cdot x}{|h_n|^{1+\beta}},$$
so that \( \|\nabla \nu_n\|_{L^\infty(B_R)} \leq CR^{2s-1+\alpha} \) for all \( R \geq 1 \). Moreover, \( \varpi_n \) is bounded in \( C^{1,\beta}_{loc}(\mathbb{R}^N) \), so that \( \varpi_n \to \varpi \) in \( C^{1,\beta}_{loc}(\mathbb{R}^N) \). In addition,

\[
\nabla \varpi(0) = 0 \quad \text{and} \quad |\nabla \varpi(\varepsilon)| \geq \frac{1}{4} \quad \text{for some} \ \varepsilon \in S^{N-1}.
\]

(5.26)

and \( \|\nabla(\cdot + h) - \nabla\|_{L^\infty(B_R)} \leq R^{2s-1+\alpha} \) for all \( R \geq 1 \). Next, letting, \( \varpi_n = (\varpi_n)_{h,0}(x) = \varpi_n(x + h) - \varpi_n(x) \), we find that

\[
L_{\mathcal{K}} \varpi_n + \frac{2\rho_n^2}{r_n^2 |h_n|^{1+\beta}} \Theta_n(r_n) \sum_{i,j=1}^N T^{\alpha}_{ij} r_n h_j L_{\mathcal{K}} x_i = F_n \quad \text{in} \ B_{1/2r_n},
\]

where \( F_n \) is given by (5.17). By (5.3) and the fact that \( \mathcal{E}_{\alpha,\mathcal{K}} x_{i} \equiv 0 \) on \( \mathbb{R}^N \), we then get

\[
L_{\mathcal{K}} \varpi_n = F_n - \frac{2\rho_n^2}{r_n^{2s+\alpha}|h_n|^{1+\beta}} \Theta_n(r_n) \sum_{i,j=1}^N T^{\alpha}_{ij} r_n h_j \mathcal{O}^s_{\alpha,\mathcal{K}} x_i \quad \text{in} \ B_{1/2r_n}.
\]

(5.27)

Since \( g_n(0) = |\nabla g_n(0)| = 0 \) and \( \|g_n\|_{C^{2s+\beta}(\mathbb{R}^N)} \leq 1 \), we then have that \( |T^{\alpha}_{ij} r_n h_j\| \leq C R^{-5+2s+\beta} \). On the other hand by the first inequality in (5.18),

\[
|\mathcal{O}^s_{\alpha,\mathcal{K}}(x_i)(x)| \leq C \int_0^\infty \min(\rho_n, r, 1)r^{-2s} dr \leq C r_n^{2s-1}.
\]

Now, since \( 2s + \beta \geq 1 + \alpha \), by (5.22), the right hand in (5.27) is bounded by \( \frac{C}{\Theta_n(r_n)} \), which tends to zero as \( n \to \infty \). By Lemma (5.22), we can pass to the limit in (5.27), to get \( L_{\mathcal{K}} \varpi_{n,0} = 0 \) in \( \mathbb{R}^N \). Hence, since \( 2s - 1 + \alpha < 2s \), Lemma (5.2) implies that, there exist \( \varpi \in \mathbb{R} \) and \( \tilde{d} \in \mathbb{R}^N \), only depending on \( h, \alpha, \beta, N, s \) and \( k \), such that \( \varpi_{h,0}(x) = \varpi(x + h) - \varpi(x) = \tilde{d} \cdot x + \tilde{d} \) for all \( x, h \in \mathbb{R}^N \). Now, since \( \nabla \varpi(0) = 0 \), we find \( \nabla \varpi \equiv 0 \) on \( \mathbb{R}^N \). This contradicts (5.26) and the proof of (ii) is finished.

The proof of (iii) follows (in verbatim) the same argument as the one of (i). The fact that \( K_n \in \mathcal{X}^{1/2}_{\alpha+\varepsilon}(\kappa, \alpha, Q_\infty) \) is only needed to deduce the uniform bound \( \|\mathcal{O}^{1/2}_{\alpha,\mathcal{K}} \mathcal{U}_n\|_{L^\infty(\mathbb{R}^N)} \leq C \) from the uniform estimate \( \|\mathcal{A}_{\alpha,\mathcal{K}}(x, r, \theta)\| \leq C r^{-\alpha} |h_n|^{-\alpha/2} \min(\rho_n, \rho_n r)^{\alpha+\varepsilon} \).

The proof of (iv) does not differ much from the one of (i). We skip the details. \( \square \)

As a consequence of the previous result, we have the following

**Theorem 5.3.** Let \( s \in (0, 1), \ N \geq 1, \ \kappa > 0 \) and \( \alpha \in (0, 1) \). Let \( K \in \mathcal{X}^s_{\alpha+\varepsilon}(\kappa, s, Q_\infty) \) and \( f \in C^\alpha(B_2) \) satisfy

\[
L_{\mathcal{K}} u = f \quad \text{in} \ B_2.
\]

(i) If \( 2s + \alpha > 1, \ 2s + \alpha \neq 2 \) and \( 2s \neq 1 \), then there exists \( C = C(N, s, k, \alpha) > 0 \) such that

\[
\|u\|_{C^{2s+\alpha}(B_1)} \leq C(\|u\|_{C^\alpha(\mathbb{R}^N)} + \|f\|_{C^\alpha(\mathbb{R}^N)}).
\]

(ii) If \( 2s = 1 \) and \( K \in \mathcal{X}^s_{\alpha+\varepsilon}(\kappa, s, Q_\infty) \), for some \( \varepsilon > 0 \), then there exists \( C = C(N, s, \kappa, \alpha, \varepsilon) > 0 \) such that

\[
\|u\|_{C^{1+\alpha}(B_1)} \leq C(\|u\|_{C^\alpha(\mathbb{R}^N)} + \|f\|_{C^\alpha(\mathbb{R}^N)}).
\]

(iii) If \( 2s + \alpha < 1 \), then there exists \( C = C(N, s, \kappa, \alpha) > 0 \) such that

\[
\|u\|_{C^{2s+\alpha}(B_1)} \leq C(\|u\|_{C^\alpha(\mathbb{R}^N)} + \|f\|_{C^\alpha(\mathbb{R}^N)}).
\]

Proof. From (29), there exists \( \beta \in (0, \alpha) \), only depending on \( N, s, \kappa \) and \( \alpha \) such that, if \( 2s + \beta \notin \mathbb{N} \),

\[
\|u\|_{C^{2s+\beta}(B_1)} \leq C(\|u\|_{C^\alpha(\mathbb{R}^N)} + \|f\|_{C^\alpha(\mathbb{R}^N)}),
\]

(5.28)

for some \( C = C(N, s, \alpha, \beta, \kappa) \).

**Case 1:** \( 1 < 2s + \alpha < 2 \). For \( z \in B_1 \), we define

\[
g_z(x) = u(x + z) - u(z) - \varphi_z(x) \nabla u(z) \cdot x
\]

(5.29)
which satisfies \( g_z(0) = |\nabla g_z(0)| = 0 \). We introduce the cut-off function \( \varphi_4 \) only because the functions \( x \mapsto x_i \), for \( i = 1, \ldots, N \), do not belong to \( L_2(\mathbb{R}^N) \) when \( 2s = 1 \). By construction and (5.28), we have
\[
\|g_z\|_{C^{2s+\beta}([0,1])} \leq C\|u\|_{C^{2s+\beta}(B_1)} \leq C(\|u\|_{C^0(\mathbb{R}^N)} + \|f\|_{C^0(\mathbb{R}^N)}).
\]
(5.30)

In addition, by Lemma 2.3 (iii),
\[
L_{K_z}g_z = f(\cdot + z) - L_{K_z}U \quad \text{in } B_1,
\]
where \( K_z(x,y) = K(x + z, y + z), U(x) := \varphi_4(x)\nabla u(z) \cdot x \). From Lemma 2.3 (iii), we get
\[
L_{K_z}(\varphi_1/2g_z) = \tilde{f} - L_{K_z}U \quad \text{in } B_{1/8},
\]
for some function \( \tilde{f} \), satisfying
\[
\|\tilde{f}\|_{C^0(\mathbb{R}^N)} \leq C(\|u\|_{C^0(\mathbb{R}^N)} + \|f\|_{C^0(\mathbb{R}^N)}).
\]
(5.31)

Using (5.41), we then have that
\[
L_{K_z}(\varphi_1/2g_z) = F \quad \text{in } B_{1/8},
\]
(5.32)

where \( F := \tilde{f} - E_{\alpha_s,K_z}U - O_{\alpha_s,K_z}^sU \). Because \( K_z \in \mathcal{H}_{\alpha+2s-1}(\kappa, \alpha, Q_\infty) \) for \( 2s > 1 \) and \( K_z \in \mathcal{H}_{\alpha+\varepsilon}(\kappa, \alpha, Q_\infty) \) for \( 2s = 1 \) and since \( U \in C^2(\mathbb{R}^N) \), we deduce from Lemma 5.1 that
\[
\|E_{\alpha_s,K_z}^sU\|_{C^0(\mathbb{R}^N)} + \|O_{\alpha_s,K_z}^sU\|_{C^0(\mathbb{R}^N)} \leq C(N, s, \alpha)\|\nabla u\|_{L^\infty(B_1)}.
\]

This with (5.31) and (5.32) imply that
\[
\|F\|_{C^0(\mathbb{R}^N)} \leq C(\|u\|_{C^0(\mathbb{R}^N)} + \|f\|_{C^0(\mathbb{R}^N)}).
\]
(5.33)

Thanks to (5.32), applying Proposition 5.2 (i) and (iii) and using (5.33), provided \( 1 < 2s + \alpha < 2 \), we get
\[
\|\nabla(\varphi_1/2g_z)\|_{L^\infty(B_1)} \leq C\|\varphi_1/2g_z\|_{C^{2s+\beta}(\mathbb{R}^N)} + \|u\|_{C^0(\mathbb{R}^N)} + \|f\|_{C^0(\mathbb{R}^N)}.
\]

As a consequence, by (5.33), for all \( r \in (0,1/2) \) and all \( z \in B_1 \),
\[
\|\nabla u - \nabla u(z)\|_{L^\infty(B_r(z))} \leq C\|\varphi_1/2g_z\|_{C^{2s+\beta}(\mathbb{R}^N)} + \|f\|_{C^0(\mathbb{R}^N)}.
\]

for some \( C = C(N, s, \alpha, \kappa) \). We then conclude that
\[
\|\nabla u\|_{C^{2s-1+\alpha}(B_{1/r})} \leq C \left( \|u\|_{C^0(\mathbb{R}^N)} + \|f\|_{C^0(\mathbb{R}^N)} \right).
\]

Therefore (i) and (iii) follow from a covering and scaling argument.

\textbf{Case 2:} \( 2s + \alpha > 2 \). We know from Case 1 that for all \( \beta \in (0, 2 - 2s) \),
\[
\|u\|_{C^{2s+\beta}(B_1)} \leq C(\|u\|_{C^0(\mathbb{R}^N)} + \|f\|_{C^0(\mathbb{R}^N)}).
\]
(5.34)

We then consider the function \( g_z \) defined in (5.24). Hence, thanks to (5.32), by Proposition 5.2 (ii), (5.31) and (5.33), we get
\[
\|\nabla(\varphi_1/2g_z) - \nabla Q_{\alpha_s/2g_z}\|_{L^\infty(B_r)} \leq C\|\varphi_1/2g_z\|_{C^{2s+\beta}(\mathbb{R}^N)} + \|f\|_{C^0(\mathbb{R}^N)}.
\]
(5.35)

provided \( 2s + \beta \geq 1 + \alpha \). In view of (5.33), we can use an iteration argument to obtain, for all \( r \in (0, 1/2) \),
\[
\|\nabla u - \nabla u(z) - M_z(\cdot - z)\|_{L^\infty(B_r(z))} \leq C\|\varphi_1/2g_z\|_{C^{2s+\beta}(\mathbb{R}^N)} + \|f\|_{C^0(\mathbb{R}^N)}.
\]

for some \((N \times N)\)-matrix \( M_z \) satisfying \( |M_z| \leq C(\|u\|_{C^0(\mathbb{R}^N)} + \|f\|_{C^0(\mathbb{R}^N)}) \). Since \( 2s - 1 + \alpha > 1 \), we deduce that \( M_z = D^2u(z) \). Using now an extension theorem, see e.g. [58] Page 177, we conclude that
\[
\|\nabla u\|_{C^{1,2s-1+\alpha}(B_{1/8})} \leq C \left( \|u\|_{C^0(\mathbb{R}^N)} + \|f\|_{C^0(\mathbb{R}^N)} \right).
\]

We thus get (ii) after a covering and scaling argument.

\textbf{Case 3:} \( 2s + \alpha < 1 \). Here, we argue as in Case 1, by applying Proposition 5.2 (iii) to the function \( g_z(x) = \varphi_1/2(x)\{u(x + z) - u(z)\} \). We skip the details. \(\square\)

By an induction argument, we have the following result.
Theorem 5.4. Let $N \geq 1$, $s \in (0,1)$ and $\alpha \in (0,1)$. Let $\kappa > 0$, $k \in \mathbb{N}$ and $K \in \mathcal{F}^{\kappa}_{\alpha+(2s-1)_+}(\kappa,k+\alpha,Q_{\infty})$. Let $u \in H^s(B_2) \cap C^{k+\alpha}(\mathbb{R}^N)$ and $f \in C^{k+\alpha}(\mathbb{R}^N)$ such that

$$L_K u = f \quad \text{in } B_2.$$ 

(i) If $2s \neq 1$ and $2s + \alpha \not\in \mathbb{N}$, then there exists $C = C(N,s,k,\kappa,\alpha)$ such that

$$\|u\|_{C^{k+2s+(\kappa,\alpha)}(B_1)} \leq C(\|u\|_{C^{k+\alpha}(\mathbb{R}^N)} + \|f\|_{C^{k+\alpha}(\mathbb{R}^N)}).$$

(ii) If $2s = 1$ and $K \in \mathcal{F}^{1/2}_{\alpha+\varepsilon}((\kappa,k+\alpha,Q_{\infty})$, for some $\varepsilon > 0$, then there exists $C = C(N,k,\kappa,\alpha,\varepsilon)$ such that

$$\|u\|_{C^{k+1+\alpha}(B_1)} \leq C(\|u\|_{C^{k+\alpha}(\mathbb{R}^N)} + \|f\|_{C^{k+\alpha}(\mathbb{R}^N)}).$$

Proof. The case $k = 0$, that is $u \in C^{2s+\alpha}(B_1)$, is proved in Theorem 5.3. We prove the statement first for $k = 1$. By Lemma 2.3(ii), we have that $u^1 := \varphi_1 u \in C^{2s+\alpha}(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N)$ and

$$L_K u^1 = f^1 \quad \text{in } B_{2^{-2}},$$

for some function $f^1$ satisfying

$$\|f^1\|_{C^{1+\alpha}(B_{2^{-2}})} \leq C(\|u\|_{C^{1+\alpha}(\mathbb{R}^N)} + \|f\|_{C^{1+\alpha}(\mathbb{R}^N)}).$$

Let $h \in B_{2^{-4}}$, with $h \neq 0$. Then (recalling (2.1))

$$L_K u^1_{h,1} - L_K h_{h,1} u^1(\cdot + h) = f^1_{h,1} \quad \text{in } B_{2^{-4}}.$$ 

We note that $K_{h,1}$ satisfies (2.2), with $\alpha' = 0$. By Corollary 5.5, we obtain $u^1_{h,1}$ is uniformly bounded in $C^\sigma(B_{2^{-5}})$, for some $\sigma \in (s,2s)$. Therefore $\nabla u^1 \in C^{0,\sigma}(B_{2^{-5}}) \subset H^s(B_{2^{-5}})$. Using (5.3), we then have that

$$L_K u^1_{h,1} = \mathcal{E}_{A_{h,1}}^s u^1_{h,1} - \mathcal{O}_{A_{h,1}}^s u^1_{h,1} = f^1_{h,1} \quad \text{in } B_{2^{-5}}.$$ 

Letting $h \to 0$, we see that, for all $i_1 \in \{1,\ldots,N\}$,

$$L_K \partial_{i_1} u^1 = \mathcal{E}_{\partial_{i_1} A_{h,1}}^s u^1 + \mathcal{O}_{\partial_{i_1} A_{h,1}}^s u^1 + \partial_{i_1} f^1 = f^1_{i_1} \quad \text{in } B_{2^{-5}}.$$ 

By Lemma 5.1 and (5.39), if $2s \neq 1$, the right hand-side in the above display belongs to $C^\alpha(\mathbb{R}^N)$ and satisfies

$$\|f^2_{i_1}\|_{C^{\alpha}(B_{2^{-5}})} \leq C(\|u\|_{C^{1+\alpha}(\mathbb{R}^N)} + \|f\|_{C^{\alpha}(B_2)}).$$

On the other hand if $2s = 1$, then Lemma 5.1 and (5.39) yield

$$\|f^2_{i_1}\|_{C^{0}\left(\mathbb{R}^N\right)} \leq C(\|u\|_{C^{1+\alpha}(\mathbb{R}^N)} + \|f\|_{C^{\alpha}(B_2)}),$$

for all $\delta \in (0,1-\alpha)$. It follows from Theorem 5.3 that if $2s \neq 1$, then

$$\|\partial_{i_1} u^1\|_{C^{2s+\alpha}(B_{2^{-6}})} \leq C(\|u\|_{C^{1+\alpha}(\mathbb{R}^N)} + \|f\|_{C^{\alpha}(B_2)})$$

and for $2s = 1$,

$$\|\partial_{i_1} u^1\|_{C^{2s+\alpha}(B_{2^{-6}})} \leq C(\|u\|_{C^{1+\alpha}(\mathbb{R}^N)} + \|f\|_{C^{\alpha}(B_2)}).$$

We now remove the $\delta$ in the above estimate (for $2s = 1$). Indeed, we define $\overline{u}^1 := \varphi_{2^{-s}} u^1 \in C^{2s+\alpha+1-\delta}(\mathbb{R}^N)$ which, by Lemma 2.3(iii), satisfies

$$L_K \overline{u}^1 = \overline{f}^1 \quad \text{in } B_{2^{-6}},$$ 

with

$$\|\overline{f}^1\|_{C^{\alpha}(\mathbb{R}^N)} \leq C(\|u\|_{C^{1+\alpha}(\mathbb{R}^N)} + \|f\|_{C^{\alpha}(\mathbb{R}^N)}).$$

Therefore proceeding as above, we have

$$L_K \partial_{i_1} \overline{u}^1 = \mathcal{E}_{\partial_{i_1} A_{h,1}}^s \overline{u}^1 + \mathcal{O}_{\partial_{i_1} A_{h,1}}^s \overline{u}^1 + \partial_{i_1} \overline{f}^1 =: \overline{f}^1 \quad \text{in } B_{2^{-7}}.$$ 

Since $K \in \mathcal{F}^{1/2}_{\alpha+\varepsilon}(\kappa,\alpha,Q_{\infty})$ and $\overline{u}^1 := \varphi_{2^{-s}} u^1 \in C^{2s+\alpha+1-\delta}(\mathbb{R}^N)$, Lemma 5.1 yields

$$\|E_{\partial_{i_1} A_{h,1}}^s \overline{u}^1\|_{C^{\alpha}(\mathbb{R}^N)} + \|O_{\partial_{i_1} A_{h,1}}^s \overline{u}^1\|_{C^{\alpha}(\mathbb{R}^N)} \leq C(\|\overline{u}^1\|_{C^{2s+\alpha+1-\delta}(\mathbb{R}^N)}).$$
Applying Theorem 5.3 to the equation (5.39), we then get

\[ \| \partial_1 \mathbf{u}^k \|_{C^{1+\alpha}(B_{2^{-1/16}})} \leq C(\| \partial_1 \mathbf{u}^k \|_{C^{1+\alpha}(\mathbb{R}^N)} + \| \mathbf{f}_{\delta}^k \|_{C^{\alpha}(B_2)}). \]

The theorem is thus proved for \( k = 2 \).

Let \( k > 2 \). We now prove by induction that for every \((i_1, i_2, \ldots, i_k)\) \(\in\{1, \ldots, N\}^k\) there exist a constant \( r_k \), only depending on \( k \), and a constant \( C_k > 0 \), only depending on \( N, s, \kappa, \alpha \) and \( k \), such that

\[ \| \partial_{i_1 i_2 \ldots i_k} u \|_{C^{2s+\alpha+k} (B_{r_k})} \leq C_k (\| u \|_{C^{k+\alpha}(\mathbb{R}^N)} + \| f \|_{C^{k+\alpha}(B_2)}). \]  

We assume, as induction hypothesis that, the result is true up order \( k - 1 \). That is, there exist \( r_{k-1}, C_{k-1} > 0 \), as above, such that

\[ \| u \|_{C^{2s+k-1+\alpha} (B_{r_{k-1}})} \leq C_{k-1} (\| u \|_{C^{k-1+\alpha}(\mathbb{R}^N)} + \| f \|_{C^{k-1+\alpha}(B_2)}). \]  

We then consider \( u^k := \varphi_{r_{k-1}/2} u \in C^{2s+\alpha+k-1}(\mathbb{R}^N) \cap C^{k,\alpha}(\mathbb{R}^N) \).

By Lemma 2.3(ii), we then have that

\[ \mathcal{L}_K u^k = f^k \quad \text{in} \quad B_{r_{k-1}/4}, \]

for some function

\[ \| f^k \|_{C^{k+\alpha}(B_{r_{k-1}/4})} \leq C'_k (\| u \|_{C^{k+\alpha}(\mathbb{R}^N)} + \| f \|_{C^{k+\alpha}(B_2)}), \]

where, unless otherwise stated, \( C'_k \) denotes a positive constant, only depending on \( N, s, \kappa, \alpha \) and \( k \). Proceeding as above, we can differentiate the equation \( k \) times to deduce that for all \((i_1, i_2, \ldots, i_k)\) \(\in\{1, \ldots, N\}^k\),

\[ \mathcal{L}_K \partial_{i_1 i_2 \ldots i_k}^k u^k = g^k + \partial_{i_1 i_2 \ldots i_k}^k f^k \quad \text{in} \quad B_{r_k}, \]

for constant \( r_k < r_k \), only depending on \( r_k \) and \( k \), and for some function \( g^k := \sum_{j=1}^{m} c_j^k \phi_{a_j^k, w_j} \) where \( c_j^k \), \( a_j^k \), \( v_j \) and \( w_j \) are respectively given by the partial derivatives in \( x \) of \( \mathcal{A}_{i_1 K}, \mathcal{A}_{a_i K} \) up to order \( k \) and \( v_j \) together with \( w_j \) are given by partial derivatives of \( u^k \) up to order \( k - 1 \). Therefore, provided \( 2s \neq 1 \), by Lemma 5.41,

\[ \| g^k \|_{C^{0}(\mathbb{R}^N)} \leq C'_k \| u^k \|_{C^{2s+k-1+\alpha}(\mathbb{R}^N)} \leq C(\| u \|_{C^{k-1+\alpha}(\mathbb{R}^N)} + \| f^k \|_{C^{k,\alpha}(\mathbb{R}^N)}). \]

Now Theorem 5.3 implies that, for \( 2s \neq 1 \),

\[ \| \partial_{i_1 i_2 \ldots i_k}^k u^k \|_{C^{2s+\alpha+k} (B_{r_k/2})} \leq C_k (\| \partial_{i_1 i_2 \ldots i_k}^k u^k \|_{C^{0}(\mathbb{R}^N)} + \| g^k \|_{C^{0}(\mathbb{R}^N)} + \| f^k \|_{C^{k,\alpha}(\mathbb{R}^N)}). \]

By (5.42), we get (5.40) in the case \( 2s \neq 1 \). Therefore (i) follows by a covering and scaling argument.

Now when \( 2s = 1 \), then we can argue similarly as above, noticing that, under the induction hypothesis (5.41), by Lemma 5.41, the function \( g^k \) in the right hand side of (5.40) belongs to \( C^{0-\delta}(\mathbb{R}^N) \), for all \( \delta \in (0, \alpha) \). Hence Theorem 5.3 implies that, for all \( \delta \in (0, \alpha) \),

\[ \| \partial_{i_1 i_2 \ldots i_k}^k u^k \|_{C^{2s+\alpha-\delta} (B_{r_k/2})} \leq C_k (\| \partial_{i_1 i_2 \ldots i_k}^k u^k \|_{C^{0}(\mathbb{R}^N)} + \| g^k \|_{C^{0-\delta}(\mathbb{R}^N)} + \| f^k \|_{C^{k,\alpha}(\mathbb{R}^N)}), \]

where, unless otherwise stated, \( C_k'' \) denotes a positive constant, only depending on \( N, \kappa, \alpha, \delta, \epsilon \) and \( k \). To remove the parameter \( \delta \), we consider

\[ \mathbf{u}^k := \varphi_{r_k'/4} u \in C^{k+1+\alpha-\delta}(\mathbb{R}^N), \]

which satisfies

\[ \| \mathbf{u}^k \|_{C^{k+\alpha+1-\delta}(\mathbb{R}^N)} \leq C_k'' (\| u \|_{C^{k+\alpha}(\mathbb{R}^N)} + \| f \|_{C^{k+\alpha}(\mathbb{R}^N)}). \]

By Lemma 2.3(ii), we then have that

\[ \mathcal{L}_K \mathbf{u}^k = \mathbf{f}^k \quad \text{in} \quad B_{r_k'/8}, \]

for some function

\[ \| \mathbf{f}^k \|_{C^{k+\alpha}(B_{r_k'/8})} \leq C_k'' (\| u \|_{C^{k+\alpha}(\mathbb{R}^N)} + \| f \|_{C^{k+\alpha}(\mathbb{R}^N)}). \]
As above, we then differentiate the equation \( k \) times to deduce that for all \( (i_1, i_2, \ldots, i_k) \in \{1, \ldots, N\}^k \),

\[
\mathcal{L}_k \partial_{i_1i_2\ldots i_k} u^k = \bar{g}^k + \partial_{i_1i_2\ldots i_k} F^k \quad \text{in } B_{r''/2},
\]

(5.46)

for constant \( r'' \), only depending on \( k \), and for some function \( \bar{g}^k(x) := \sum_{j=1}^m \epsilon_j^k \xi_j^{1/2} + \sum_{j=1}^m \epsilon_j^k \eta_j^{1/2} \),

where \( \epsilon_j^k, \xi_j^k \) are real numbers and \( \epsilon_j^k, \eta_j^k \) (resp. \( v_j \) and \( w_j \)) are respectively given by the partial derivatives in \( x \)-variable of \( \mathcal{A}_e, \mathcal{A}_o \) up to order \( k \) (resp. \( \mathcal{A}_e \) and \( \mathcal{A}_o \) together with \( \mathcal{A}_e \) respectively given by partial derivatives of \( \bar{g}^k \) up to order \( k-1 \)). Therefore by Lemma 5.1, Lemma 6.1, and since \( \mathcal{A}_e, \mathcal{A}_o \) are given by \( \mathcal{L}_k \)

\[
\mathcal{A}_e, \mathcal{A}_o \in \mathcal{X}^{1/2}_{\alpha+} (\kappa, k+\alpha, \mathbb{R}^N),
\]

we get the estimates. \( \square \)

6. PROOF OF THE MAIN RESULTS

We start this section with the following result which shows how to pass from a nonlocal equation with kernels in \( \mathcal{X}^\alpha_+ (\kappa, m+\alpha, \mathcal{Q}_\infty) \) to a nonlocal equation driven by kernels in \( \mathcal{X}^\alpha_+ (\kappa, m+\alpha, \mathcal{Q}_\infty) \).

**Lemma 6.1.** Let \( K \in \mathcal{X}^\alpha_+ (\kappa, m+\alpha, \mathcal{Q}_\infty) \), \( f \in \mathcal{L}^1_{\text{loc}}(B_{4R}), \) and \( f \in \mathcal{L}^1_{\text{loc}}(B_{4R}) \), satisfy

\[
\mathcal{L}_K v = f \quad \text{in } B_{4R}.
\]

Let

\[
K(x, y) = \varphi_R(x) \varphi_R(y) K(x, y) + (2 - \varphi_R(x) - \varphi_R(y)) \mu_1(x, y).
\]

Then

\[
\mathcal{L}_k v + \nabla v = f + \bar{f} \quad \text{in } B_{R/4},
\]

(6.1)

where, for \( x \in B_{R/4} \),

\[
\nabla v(x) = G_{1,K,2R}(x) - G_{1,\mu_1,R}(x), \quad \bar{f}(x) = G_{v,K,2R}(x) - G_{v,\mu_1,R}(x),
\]

and \( G_{v,K,2R} \) is given by (2.4). In particular, \( K \in \mathcal{X}^\alpha_+ (\kappa, m+\alpha, \mathcal{Q}_\infty) \), for some constant \( \kappa = \bar{\kappa}(\kappa, \alpha, m, R, \tau, \mathcal{Q}_\infty) \).

**Proof.** The proof of (6.1) is elementary, and we skip it. Next, we observe that

\[
\mathcal{A}_k(x, r, \theta) = \varphi_{2R}(x) \varphi_{2R}(x+r\theta) \mathcal{A}_k(x, r, \theta) + (2 - \varphi_{2R}(x) - \varphi_{2R}(x+r\theta)).
\]

Recalling the definition of the cut-off function \( \varphi_R \) in the beginning of Section 1, we easily deduce that

\[
\min(k, 1) \leq \mathcal{A}_k(x, r, \theta) \leq \max(1/k, 4) \quad \text{for all } x \in \mathbb{R}^N, r \geq 0, \theta \in S^{N-1}.
\]

This in particular implies that \( \bar{K} \) satisfies (2.2). Moreover, it is also not difficult to check that

\[
\|\mathcal{A}_e \|_{C^{2,0}_1 (Q_\infty)} = \|\mathcal{A}_o \|_{C^0 (Q_\infty)} \leq C(k, m, \alpha, \tau, \mathcal{Q}_\infty) \leq C(k, m, \alpha, \tau, R).
\]

\( \square \)

**Proof of Theorem 1.7.** As mentioned in the first section, the case \( 2s \leq 1 \) was already proved in [29]. Now the case \( 2s > 1 \) follows from Theorem 1.3, Lemma 6.1, Lemma 2.3 and the fact that \( L^p(\mathbb{R}^N) \hookrightarrow \mathcal{M}_{N/p} \).

(6.1)

**Proof of Theorem 1.8.** It follows from Theorem 3.1, Lemma 6.1 and Lemma 2.3.

**Proof of Theorem 1.9.** First applying Theorem 5.3 and using Lemma 6.1 together with Lemma 2.3 we get the estimates.

**Proof of Theorem 1.10.** It follows from Theorem 1.7.
Proof of Theorem 1.2. By Lemma 6.1 we have that
\[ \mathcal{L}V u = f + \mathcal{J} - \nabla u \quad \text{in } B_{1/2}, \] (6.2)
with \( \mathcal{K} \in \mathcal{K}_{\alpha + (2s-1)}(\mathcal{R}, m + \alpha, Q_\infty) \). In addition, by Lemma 2.3(iii), we have
\[ \|\nabla\|_{C^{\alpha}(B_{1/2})} \leq C \] (6.3)
and
\[ \|\mathcal{J}\|_{C^{\alpha}(B_{1/2})} \leq C\|u\|_{L_s(\mathcal{R}^N)}. \] (6.4)
We consider first the case \( 2s \neq 1 \). Since \( u \) satisfies (6.2), applying Theorem 5.4 and using Lemma 2.3(iii), we get
\[ \|\varphi_{1/2}u\|_{C^{2s+m-\alpha}(B_{1/2})} \leq C(\|\varphi_{1/2}u\|_{C^{\alpha}(\mathcal{R}^N)} + \|u\|_{L_2(\mathcal{R}^N)} + \|F\|_{C^{\alpha}(B_{1/2})}), \]
where \( F := f + \mathcal{J} - \nabla u \). Consequently, by (6.3) and (6.4)
\[ \|u\|_{C^{2s+m-\alpha}(B_{1/2})} \leq C(\|u\|_{C^{\alpha}(\mathcal{R}^N)} + \|u\|_{L_2(\mathcal{R}^N)} + \|F\|_{C^{\alpha}(B_{1/2})}). \]
Using now adimensional Hölder norms and interpolation (see e.g. [137]), we can absorb the \( C^{\alpha}(B_1) \)-norm of \( u \) to deduce that
\[ \|u\|_{C^{2s+m-\alpha}(B_{2-s})} \leq C(\|u\|_{L_2(\mathcal{R}^N)} + \|F\|_{C^{\alpha}(B_{2})}). \]
If now \( 2s = 1 \), then since \( \mathcal{K} \in \mathcal{K}_{3/2}(\mathcal{R}, m + \alpha, Q_\infty) \) and in view of Theorem 5.4, the same arguments as above yield
\[ \|u\|_{C^{1+m-\alpha}(B_{2-s})} \leq C(\|u\|_{L_2(\mathcal{R}^N)} + \|F\|_{C^{\alpha}(B_{2})}), \]
for all \( \varepsilon \in (0, \alpha) \). Now by scaling and covering, we get the result. \( \square \)

6.1. Proof of Theorem 1.3 The following fundamental lemma allows, in particular, to consider truncation of the nonlocal mean curvature kernel \( 1_{B_r}(x) \times 1_{B_r}(y)K_u(x, y) \) without any assumption on \( u \) in the exterior of \( B_r \).

Lemma 6.2. Let \( u : \mathcal{R}^N \to \mathcal{R} \) be a measurable function and \( \Gamma^{u,R} : B_{R/2} \to \mathcal{R} \) be given by
\[ \Gamma^{u,R}(x) := \int_{\mathcal{R}^N} (1 - 1_{B_R}(y)) \frac{f_s(p_u(x, y)) - f_s(-p_u(x, y))}{|x - y|^{N+2s-1}} \, dy. \] (6.5)
If \( u \in C^{k,\alpha}(B_{R/2}), \) for \( k \geq 1 \) and \( \alpha \in [0, 1], \) then, there exists a constant \( C = C(N, s, k, \alpha, R) \) such that
\[ \|\Gamma^{u,R}\|_{C^{k,\alpha}(B_{R/2})} \leq C(1 + \|u\|_{C^{k,\alpha}(B_{R/2})})^{2k}. \] (6.6)
If \( u \in C^{0,1}(B_{R/2}) \) then, there exists a constant \( C = C(N, s, R) \) such that
\[ \|\Gamma^{u,R}\|_{C^{0,1}(B_{R/2})} \leq C(1 + \|u\|_{C^{0,1}(B_{R/2})}). \] (6.7)
Proof. For simplicity, we assume that \( R = 2 \), and to alleviate the notations, we put \( \Gamma^u := \Gamma^{u,R} \). We first observe, from (1.11), that \( f'_s(p) = -(1 + p^2)^{(-N-2s)/2}, \) so that for all \( j \in \mathcal{N}, \)
\[ |p|^j |f'_s(j)(p)| \leq C(N, s, j) \quad \text{for all } p \in \mathcal{R}. \] (6.8)
In particular, since \( 2s > 1, \)
\[ \|\Gamma^u\|_{L_\infty(B_1)} \leq C(N, s). \] (6.9)
Next, for all \( (x, y) \in B_1 \times \mathcal{R}^N \setminus B_2, \) we have
\[ |\partial_\mu u(x, y)| \leq C(k)(|u(y)| + \|u\|_{C^{k-1,\alpha}(B_1)}) \quad \text{for } \mu \in \mathcal{N} \text{ with } |\mu| \leq k. \] (6.10)
On the other hand, by writing \( u(y) = (u(y) - u(z)) + u(z), \) we easily deduce that
\[ |u(y)| \leq C(|p_u(z, y)| + \|u\|_{L_\infty(B_1)}) \quad \text{for all } z \in B_1 \text{ and } y \in \mathcal{R}^N \setminus B_2. \]
Using this in (6.11), we see that, for \( \mu \in \mathcal{N} \) with \( |\mu| \leq k, \)
\[ |\partial_\mu u(x, y)| \leq C(k)(|p_u(z, y)| + \|u\|_{C^{k-1,\alpha}(B_1)}) \quad \text{for all } x, z \in B_1 \text{ and } y \in \mathcal{R}^N \setminus B_2. \] (6.11)
By the Faà de Bruno formula (see e.g. [12]), for $|\gamma| = k$ and $(x, y) \in B_1 \times \mathbb{R}^N \setminus B_2$, we get
\[
\partial_x^\gamma F_s(p_u(x, y)) = \sum_{\pi \in \mathcal{P}_k} F_s^{(\pi)}(p_u(x, y)) \prod_{\mu \in \pi} \partial_x^\mu p_u(x, y),
\] (6.12)
where $\mathcal{P}_k$ denotes the set of all partitions of $\{1, \ldots, k\}$. Hence, for $x \in B_1$ and $y \in \mathbb{R}^N \setminus B_2$, by (6.11), we have that
\[
|\partial_x^\gamma F_s(p_u(x, y))| \leq C \sum_{\pi \in \mathcal{P}_k} 2^{|\pi|-1} \left( |p_u(x, y)|^{\|\|} \left| F_s^{(\pi)}(p_u(x, y)) \right| + \|u\|_{C^{k-1,1}(B_1)} \left| F_s^{(\pi)}(p_u(x, y)) \right| \right)
\]
\[
\leq C \left( 1 + \|u\|_{C^{k-1,1}(B_1)}^k \right) + \sum_{\pi \in \mathcal{P}_k} 2^{|\pi|-1} |p_u(x, y)|^{\|\|} |F_s^{(\pi)}(p_u(x, y))|.
\]
From this and (6.8), we deduce that, for all $\gamma \in \mathbb{N}^N$ with $|\gamma| = k$,
\[
\sup_{(x, y) \in B_1 \times \mathbb{R}^N \setminus B_2} |\partial_x^\gamma F_s(p_u(x, y))| + \sup_{(x, y) \in B_1 \times \mathbb{R}^N \setminus B_2} |\partial_x^\gamma F_s(-p_u(x, y))| \leq C(1 + \|u\|_{C^{k-1,1}(B_1)})^k,
\] (6.13)
with $C = C(s, N, k)$. Since $2s > 1$, from the above estimate, (6.9) and the dominated convergence theorem, we can differentiate under the integral sign in (6.5) to deduce that
\[
\|\Gamma_u\|_{C^{k-1,1}(B_1)} \leq C(1 + \|u\|_{C^{k-1,1}(B_1)})^k.
\] (6.14)
Moreover, to see (6.7), we note that if $u \in C^{0,1}(B_1)$, then Redemarcher’s theorem implies that $u$ is equivalent to a differentiable function. Therefore (6.13) holds (with $k = 1$) and replacing “sup” with “esssup”. Now by the dominated convergence theorem, we get (6.7).

Let us now fix $x_1, x_2 \in B_1$ and $y \in \mathbb{R}^N \setminus B_2$. Direct computations yield
\[
|\partial_x^\mu p_u(x_1, y) - \partial_x^\mu p_u(x_2, y)| \leq C|x_1 - x_2|^a \{ |u(y)| + \|u\|_{C^{k-1,1}(B_1)} \}
\]
for $\mu \in \mathbb{N}^N$ with $|\mu| \leq k$.

Note that, (6.10) implies that
\[
|u(y)| \leq C\{\min(|p_u(x_1, y)|, |p_u(x_2, y)|) + \|u\|_{L^1(B_1)}\}.
\]
Therefore, for all $\mu \in \mathbb{N}^N$ with $|\mu| \leq k$, we get
\[
|\partial_x^\mu p_u(x_1, y) - \partial_x^\mu p_u(x_2, y)| \leq C|x_1 - x_2|^a \{ \min\{ |p_u(x_1, y)|, |p_u(x_2, y)| \} + \|u\|_{C^{k-1,1}(B_1)} \}
\] (6.15)
and, by (6.11),
\[
|\partial_x^\mu p_u(x_1, y)| \leq C\{ \min\{ |p_u(x_1, y)|, |p_u(x_2, y)| \} + \|u\|_{C^{k-1,1}(B_1)} \}.
\] (6.16)

Next, we define
\[
g_s \in C^\infty(\mathbb{R}^+, \mathbb{R}), \quad g_s(r) = -r^{-(N+2s-1)/2},
\]
so that $F_s(p) = g_s(1 + p^2)$ for all $p \in \mathbb{R}$. Moreover, for $r > 0$,
\[
g_s^{(j)}(r) = (-1)^{j+1}2^{-j} \prod_{i=0}^{j-1}(N + 2s - 2i)r^{-(N+2s-1+2i)/2}.
\] (6.17)

From this and the generalized chain rule for higher derivatives, we get
\[
F_s^{(j+1)}(p) = \sum_{(m_1, m_2) \in \mathcal{N}_j} \tau_j(m_1, m_2)p^{m_1}g_s^{(m_1+m_2)}(1 + p^2),
\] (6.18)
where $\tau_j(m_1, m_2) = \frac{\partial^{m_1}g_s}{\partial m_1^{m_2}}$ and $\mathcal{N}_j := \{(m_1, m_2) \in \mathbb{N}^2 : m_1 + 2m_2m_2 = j\}$. Hence, for all $p_1, p_2 \in \mathbb{R}$,
\[
|F_s^{(j+1)}(p_1) - F_s^{(j+1)}(p_2)| \leq \sum_{(m_1, m_2) \in \mathcal{N}_j} \tau_j(m_1, m_2)|p_1^{m_1} - p_2^{m_1}|g_s^{(m_1+m_2)}(1 + p_1^2) + |p_2^{m_1} - p_2^{m_2}| \int_0^1 |g_s^{(m_1+m_2+1)}(1 + tp_1^2 + (1-t)p_2^2)| dt.
\]
It then follows from (6.15) and (6.16), that
\[
\left| F_s^{(j+1)}(p_u(x_1,y)) - F_s^{(j+1)}(p_u(x_2,y)) \right|
\leq C|x_1 - x_2|^\alpha \sum_{(m_1,m_2)\in N_j} \tau_j(m_1,m_2) \frac{\left( \min(|p_u(x_1,y)|, |p_u(x_2,y)|) + \|u\|_{C^{\kappa}(B_1)} \right)^{m_1}}{(1 + \min(|p_u(x_1,y)|, |p_u(x_2,y)|)^2) N + 2s - 1}.
\]
\[
+ C|x_1 - x_2|^\alpha \sum_{(m_1,m_2)\in N_j} \tau_j(m_1,m_2) \frac{\left( \min(|p_u(x_1,y)|, |p_u(x_2,y)|) + \|u\|_{C^{\kappa}(B_1)} \right)^{m_1+2}}{(1 + \min(|p_u(x_1,y)|, |p_u(x_2,y)|)^2) N + 2s - 1}.
\]

(6.19)

On the other hand, it is immediate, from (6.17) and (6.18), that
\[
\text{sup}_{x \in \mathbb{R}^N} \left| \prod_{\mu \in \Pi} \partial_{x_\mu}^\mu u(x_1,y) - \prod_{\mu \in \Pi} \partial_{x_\mu}^\mu p_u(x_1,y) \right| \leq C|x_1 - x_2|^\alpha \left\{ \min(|p_u(x_1,y)|, |p_u(x_2,y)|) + \|u\|_{C^{\kappa}(B_1)} \right\}^{|\Pi|}.
\]

(6.20)

Moreover (6.16) yields
\[
\text{sup}_{x \in \mathbb{R}^N} \left| \prod_{\mu \in \Pi} \partial_{x_\mu}^\mu p_u(x_2,y) \right| \leq C\left( \min(|p_u(x_1,y)|, |p_u(x_2,y)|) + \|u\|_{C^{\kappa}(B_1)} \right)^{|\Pi|}.
\]

(6.21)

We have, from (6.12), that
\[
\left| \partial_{x_\mu}^\mu F_s(p_u(x_1,y)) - \partial_{x_\mu}^\mu F_s(p_u(x_2,y)) \right| \leq \sum_{\Pi \in \mathcal{P}_k} \left| F_s^{(\Pi)}(p_u(x_1,y)) - F_s^{(\Pi)}(p_u(x_2,y)) \right| \prod_{\mu \in \Pi} \partial_{x_\mu}^\mu p_u(x_1,y)
\]
\[
+ \sum_{\Pi \in \mathcal{P}_k} \left| F_s^{(\Pi)}(p_u(x_2,y)) \right| \prod_{\mu \in \Pi} \partial_{x_\mu}^\mu p_u(x_1,y) - \prod_{\mu \in \Pi} \partial_{x_\mu}^\mu p_u(x_2,y).\]

(6.22)

Next, we observe that for $(m_1,m_2) \in N_{|\Pi|-1}$, then
\[
|\Pi| + m_1 - 2(m_1 + m_2) - (N + 2s - 1) < 0.
\]

Now from this, (6.19), (6.20), (6.21), (6.22) and (6.23), we deduce that, for all $x_1, x_2 \in B_1$ and
\[
\text{sup}_{y \in \mathbb{R}^N \setminus B_2} \left| \partial_{x_\mu}^\mu F_s(p_u(x_1,y)) - \partial_{x_\mu}^\mu F_s(p_u(x_2,y)) \right| \leq C|x_1 - x_2|^\alpha (1 + \|u\|_{C^{\kappa}(B_1)})^{2k}.
\]

Combining this with (6.13), we get
\[
\text{sup}_{y \in \mathbb{R}^N \setminus B_2} \left| F_s(p_u(\cdot,y)) \right|_{C^{\kappa}(B_1)} \leq C(1 + \|u\|_{C^{\kappa}(B_1)})^{2k}.
\]

Since the same estimates remains valid when $p_u$ is replaced with $-p_u$, then (6.16) follows.

We will need the following elementary result which follows from the fact that $F_s$ is even on $\mathbb{R}$ and the fundamental theorem of calculus.

**Lemma 6.3.** For all $a, b \in \mathbb{R}$, we have
\[
[F_s(a) - F_s(b)] - [F_s(-a) - F_s(-b)] = 2(a - b) \int_0^1 F_s'(b + \rho(a - b)) d\rho.
\]

We now complete the
Proof of Theorem 1.3. In view of (1.10), we have
\[ \mathcal{L}_{K_u} u = f - \Gamma^u \quad \text{in } B_{1/2}, \]  
(6.24)
where
\[ \tilde{K}_u(x, y) := 1_{B_1(x)} 1_{B_1(y)} K_u(x, y) \quad \text{for all } x \neq y \in \mathbb{R}^N. \]
and, for \( x \in B_{1/2}, \)
\[ \Gamma^u(x) := \int_{\mathbb{R}^N} (1 - 1_{B_1(y)}) \frac{F_s(p_u(x, y)) - F_s(-p_u(x, y))}{|x - y|^{N+2s-1}} \, dy. \]
We recall from the fundamental theorem of calculus that
\[ (u(x) - u(y)) K_u(x, y) = [F_s(p_u(x, y)) - F_s(-p_u(x, y))]|x - y|^{-(N+2s-1)}, \]  
(6.25)
Let \( h \in B_{1/4} \) with \( h \neq 0. \) Then, recalling the notation in (2.1), by Lemma 6.3, (6.26) and (6.24),
\[ \mathcal{L}_{K_h} u_{h,1} = f_{h,1} + \Gamma^u_{h,1} \quad \text{in } B_{1/4}, \]
where
\[ K^u_h(x, y) := 1_{B_1(x)} 1_{B_1(y)} \frac{1}{|x - y|^{N+2s}} q^u_h(x, y) \]  
(6.26)
and
\[ q^u_h(x, y) := -2 \int_0^h F'_s(p_{u+h}(x, y) + \rho p_{u-h}(x, y)) \, d\rho. \]
Since \( F'_s \) is even and \( p_u(x, y) = -p_u(y, x), \) we see that \( K^u_h(x, y) = K^u_h(y, x). \) Moreover
\[ K^u_h(x, y) \geq C|x - y|^{-N-2s} \quad x \neq y \in B_1, \]  
(6.27)
for some constant \( C > 0, \) only depending on \( N, s \) and \( \|u\|_{C^0_1(B_1)}. \) Letting \( v := \varphi_{1/8} u_{h,1} \) and using Lemma 2.3, we have that
\[ \mathcal{L}_{K^u_h} v = f_{h,1} + \Gamma^u_{h,1} + G_h, \quad \text{in } B_{2^{-4}}, \]  
(6.28)
with \( G_h := G_{K^u_h, u_{h,1}, 1/4} \) satisfying (note that \( K^u_h \) is supported in \( B_1 \times B_1 \) and \( |q^u_h| \leq 2) \)
\[ \|G_h\|_{L^\infty(B_{1/5})} \leq C(N, s) \|u_{h,1}\|_{L^\infty(B_1)} \leq C\|\nabla u\|_{L^\infty(B_1)}. \]  
(6.29)
We would like to apply [21, Theorem 2.4] to get the \( C^{0\alpha} \) bound of \( v, \) but our kernel \( K^u_h, \) which is compactly supported might vanish at some diagonal points \( \{x = y\}. \) A way out to such difficulty, is to use the argument in [29, Remark 2.1] (see also Lemma 6.1) by considering
\[ K^u_h(x, y) = K^u_h(x, y) + (2 - 1_{B_{1/2}}(x) - 1_{B_{1/2}}(y))|x - y|^{-N-2s}. \]
We then deduce, from (6.28), that
\[ \mathcal{L}_{K^u_h} v + \nabla v = f_{h,1} + \Gamma^u_{h,1} + G_h + f, \quad \text{in } B_{2^{-4}}, \]  
(6.30)
for some functions \( \|\nabla v\|_{L^\infty(B_{2^{-4}})} \leq C(N, s) \|v\|_{L^\infty(B_1)} \) and \( \|f\|_{L^\infty(B_2)} \leq C(N, s) \|v\|_{L^\infty(B_1)}. \) Since \( K^u_h(x, y) \) satisfies (6.22), we find that \( K^u_h \) satisfies (6.22). Therefore, since \( v \in H^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \) by [21, Theorem 2.4], we have that
\[ \|v\|_{C^{0,\alpha}_0(B_{2^{-5}})} \leq C(\|v\|_{L^\infty(\mathbb{R}^N)} + \|f_{h,1}\|_{L^\infty(\mathbb{R}^N)} + \|\Gamma^u_{h,1}\|_{L^\infty(B_{2^{-4}})} + \|G_h\|_{L^\infty(B_{2^{-4}})}), \]
for some \( \alpha > 0 \) and \( C > 0, \) only depending on \( N, s \) and \( \|u\|_{C^0_1(B_1)}. \) From (6.29) and the fact that \( v = u_{h,1} \) on \( B_{1/8}, \) we get
\[ \|u_{h,1}\|_{C^{0,\alpha}_0(B_{2^{-5}})} \leq C(\|u_{h,1}\|_{L^\infty(B_1)} + \|f_{h,1}\|_{L^\infty(\mathbb{R}^N)} + \|\Gamma^u_{h,1}\|_{L^\infty(B_{2^{-4}})}). \]
This and Lemma 6.2 imply that
\[ \|u\|_{C^{1,\alpha}_0(B_{2^{-5}})} \leq C(1 + \|u\|_{C^0_1(B_1)} + \|f\|_{C^0_1(\mathbb{R}^N)}), \]  
(6.31)
which proves \( 1.13 \).
To obtain the gradient estimate of \( v \) from Theorem \( 1.1 \), we check that \( \mathcal{L}_{K_h^u} \) is a \( C^{0,\alpha_0} \)-nonlocal operator. To this scope, for every \( w \in C^{0,1}(B_1) \), we define \( Z_w : B_{1/2} \times [0,1/2] \times S^{N-1} \to \mathbb{R} \) by

\[
Z_w(x,r,\theta) := -\int_0^1 \nabla w(x + rt\theta) \cdot \theta \, dt \quad \text{for } r \in [0,1/2), \ x \in B_{1/2} \text{ and } \theta \in S^{N-1}.
\]

Clearly \( Z_w \) is as smooth as \( \nabla w \) and \( Z_w(x,r,\theta) := p_w(x,x + r\theta) \) for \( r > 0 \). We then define \( A_{K_h^u} : B_{1/4} \times [0,1/4] \times S^{N-1} \to \mathbb{R} \) by

\[
A_{K_h^u}(x,r,\theta) := 1_{B_2}(x)1_{B_2}(x + r\theta) \int_0^1 \frac{2\rho}{(1 + (Z_{u+h}(x,r,\theta) + pZ_{u-h}(x,r,\theta))^2)^{(N+2s)/2}} \, dt,
\]

(6.32)

which by \( 6.26 \), satisfies \( A_{K_h^u}(x,r,\theta) = r^{N+2s}K_h^u(x,x + r\theta) \) for all \( (x, r, \theta) \in B_{1/4} \times [0,1/4] \times S^{N-1} \). Moreover,

\[
A_{K_h^u}(x,0,\theta) - A_{K_h^u}(x,0,\theta) = 0 \quad \text{for all } (x, \theta) \in B_{1/4} \times S^{N-1}.
\]

In addition from, \( 6.31 \) together with \( 6.32 \), we have that

\[
\|A_{K_h^u}\|_{C^{0,\alpha}(Q_{2^{-7}} \times S^{N-1})} \leq C,
\]

with \( C \), only depending on \( N, s, \|u\|_{C^{0,1}(B_2)}, \alpha_0 \) and \( \|f\|_{C^{0,1}(B_2)} \). We then conclude that \( K_h^u \in \mathcal{K}^{s}(\kappa, \alpha_0, Q_{2^{-7}}) \), for some \( \kappa \), only depending on \( N, s, \|u\|_{C^{0,1}(B_2)}, \alpha_0 \) and \( \|f\|_{C^{0,1}(B_2)} \). Therefore applying Theorem \( 1.1 \) (iii) to \( 6.25 \), we deduce that

\[
\|\nabla v\|_{C^{0,\alpha_1}(B_{2^{-9}})} \leq C(\|v\|_{L^\infty(B_1)} + \|f_{h,1}\|_{L^\infty(B_1)}),
\]

for all \( \varepsilon \in (0,2s-1) \) and \( C \) a constant, only depending on \( N, s, \|u\|_{C^{0,1}(B_2)}, \alpha_0, \varepsilon \) and \( \|f\|_{C^{0,1}(B_2)} \). Hence, recalling that \( v = u_{h,1} \) in \( B_{1/8} \), we get

\[
\|\nabla u\|_{C^{1,\alpha_1}(B_{2^{-9}})} \leq C,
\]

with \( \alpha_1 := \min(2s - 1 - \varepsilon, \alpha_0) \). Hence, for all \( h \in B_{2^{-10}} \), we have \( K_h^u \in \mathcal{K}^{s}(\kappa, 1, Q_{2^{-10}}) \), for some \( \kappa \), only depending on \( N, s, \|u\|_{C^{0,1}(B_2)}, \alpha_1 \) and \( \|f\|_{C^{0,1}(B_2)} \). We apply once more Theorem \( 1.1 \) (ii) to \( 6.25 \), to get \( \|v\|_{C^{1,2s-1-\varepsilon}(B_{2^{-11}})} \leq C \), so that

\[
\|u\|_{C^{2s-(2s-1-\varepsilon)}(B_{2^{-12}})} \leq C.
\]

(6.33)

This finishes the proof of \( (i) \) after a scaling and covering.

For \( (ii) \), we consider first the case \( m = 1 \). Clearly \( 6.33 \) and \( 6.32 \) imply that \( K_h^u \in \mathcal{K}^{s}(\kappa, 2s-1+\alpha, Q_{2^{-13}}) \), for all \( h \in B_{2^{-13}} \) and \( \alpha \in (0,1) \). In particular, by Lemma \( 6.3 \) (iii), we have \( \|G_h\|_{C^{0,\alpha}(B_{2^{-13}})} \leq C\|uh_{h,1}\|_{L^\infty(B_1)} \). Now by \( 6.33 \) and Lemma \( 6.2 \) for all \( h \in B_{2^{-13}} \), we have \( \|\Gamma_{h,1}\|_{C^{1,2s-1-\varepsilon}(B_{2^{-13}})} \leq C \). Therefore, applying Theorem \( 1.2 \) to the equation \( 6.28 \), we get \( \|v\|_{C^{2s+\alpha}(B_{2^{-15}})} \leq C \), provided \( 2s + \alpha \notin \mathbb{N} \). Hence

\[
\|u\|_{C^{1+2s+\alpha}(B_{2^{-16}})} \leq C.
\]

If now \( m \geq 2 \), then the above estimate implies that \( K_h^u \in \mathcal{K}^{s}(\kappa, 2s+\alpha, Q_{2^{-18}}) \) for all \( h \in B_{2^{-18}} \). Hence, Lemma \( 6.4 \) (iii) implies that \( \|G_h\|_{C^{1,\alpha}(B_{2^{-18}})} \leq C\|uh_{h,1}\|_{L^\infty(B_1)} \). On the other hand, by Lemma \( 6.2 \) \( \Gamma_{h,1} \in C^{2s+\alpha}(B_{2^{-18}}) \subset C^{1,\alpha}(B_{2^{-18}}) \), because \( 2s > 1 \). It then follows, from \( 6.28 \) and Theorem \( 1.2 \) that \( \|\partial_{x_k} v\|_{C^{2s+\alpha}(B_{2^{-19}})} \leq C \). This yields \( \|\partial_{x_k} u\|_{C^{1+2s+\alpha}(B_{2^{-19}})} \leq C \), because \( v = \varphi_{1/8}uh_{1} \). Now iterating the above argument, then for all \( k \in \{1, \ldots, m\} \) and \( i \in \{1, \ldots, N\} \), we can find two constants \( r_k \), only depending on \( k \), and a constant \( C_k > 0 \), only depending on \( N, s, \|u\|_{C^{0,1}(B_2)}, k, \alpha \) and \( \|f\|_{C^{0,\alpha}(B_2)} \), such that

\[
\|\partial_{x_k} u_{h,1}\|_{C^{1+2s+\alpha}(B_{r_k})} \leq C_k
\]

for all \( h \in B_{r_k/2} \). A covering and scaling argument yields \( (iii) \). \( \square \)
6.2. Proof of Theorem 1.4 and Theorem 1.5

Up to a change of coordinates and a scaling, we may assume that a neighborhood of $0 \in \Sigma$ is parameterized by a $C^{1,\gamma}$-diffeomorphism $\Phi : B_2 \to \Sigma$, for some $\gamma \in (0,1)$, satisfying $\Phi(0) = 0$ and

$$|D\Phi(x) - id| \leq \frac{1}{2} \quad \text{for all } x \in B_2.$$  \hfill (6.34)

We consider the following open sets in $\Sigma$ given by

$$B_r := \Phi(B_{r'}) \quad \text{for } r \in (0,2]$$

and we define $\eta_r(\overline{x}) = \varphi_r(\Phi^{-1}(\overline{x}))$. For $\Psi \in C_c^{\infty}(B_{1/2})$, we then have

$$\int_{B_2} \int_{B_2} \frac{(u(x) - u(y))(\Psi(x) - \Psi(y))}{|x-y|^{N+2s}} \eta_2(x) \eta_2(y) \, d\sigma(x) d\sigma(y) + \int_{\Sigma} V_1(\overline{x}) u(\overline{x}) \Psi(\overline{x}) \, d\sigma(\overline{x}) = \int_{\Sigma} f_1(\overline{x}) \Psi(\overline{x}) \, d\sigma(\overline{x}),$$

where

$$V_1(\overline{x}) := V(\overline{x}) + \int_{\Sigma} (1 - \eta_2(y)) |x-y|^{-N-2s} \, d\sigma(y) \quad \text{and} \quad f_1(\overline{x}) := f(\overline{x}) + \int_{\Sigma} (1 - \eta_2(y)) u(y) |x-y|^{-N-2s} \, d\sigma(y).$$  \hfill (6.35, 6.36)

We denote by $Jac \Phi$ the Jacobian determinant of $\Phi$. Let $\psi(x) = \Psi(\Phi(x))$, $v(x) = u(\Phi(x))$, $\overline{V}(x) = V_1(x) Jac \Phi(x)$ and $\overline{f}(x) = f_1(x) Jac \Phi(x)$. Then by the changes of variables $\overline{x} = \Phi(x)$ and $\overline{y} = \Phi(y)$, in (1.14), we get

$$\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (v(x) - v(y))(\psi(x) - \psi(y)) K(x,y) \, dx \, dy + \int_{B_1} \overline{V}(x) u(x) \psi(x) \, dx = \int_{B_1} \overline{f}(x) \psi(x) \, dx,$$

where

$$K(x,y) = \varphi_2(x) \varphi_2(y) Jac \Phi(x) Jac \Phi(y) |\Phi(x) - \Phi(y)|^{-N-2s}. \quad \text{(6.37)}$$

We further consider $w = \varphi_{1/4} v \in H^s(\mathbb{R}^N)$, so that by Lemma 2.3

$$L_K w + \overline{V} w = \overline{f} + G \quad \text{in } B_{1/16},$$  \hfill (6.38)

where

$$G(x) = \int_{B_2} (1 - \varphi_{1/4}(y)) v(y) K(x,y) \, dy. \quad \text{(6.39)}$$

Next, we observe that the function $A_K : B_1 \times [0,1] \times S^{N-1} \to \mathbb{R}^N$, given by

$$A_K(x,r,\theta) = \varphi_2(x) \varphi_2(x + r\theta) Jac \Phi(x) Jac \Phi(x + r\theta) \left( \int_0^1 D\Phi(x + r\theta) \, d\theta \right)^{-N-2s}$$

is an extension of $(x,r,\theta) \mapsto r^{N+2s} K(x,r\theta)$ on $B_1 \times [0,1] \times S^{N-1}$. Moreover, since $\Phi \in C^{1,\gamma}(B_2)$, we see that

$$\|A_K\|_{C^{\gamma}(B_1 \times [0,1/2] \times S^{N-1})} \leq \frac{1}{\kappa}, \quad \text{for all } \kappa > 0,$$

$$A_K(x,0,\theta) = A_K(x,0,-\theta) \quad \text{and} \quad A_K(x,\theta,0) = A_K(x,-\theta,0), \quad \text{for all } x, \theta \in \mathbb{R}^N \times S^{N-1}, \quad \text{for some } \kappa > 0,$$

$$\|\overline{f}\|_{L^p(B_{1/16})} + \|G\|_{L^p(B_{1/16})} \leq C(\|u\|_{L^1(\Sigma)} + \|f\|_{L^p(B_2)}) \quad \text{and} \quad \|\overline{V}\|_{L^p(B_{1/2})} \leq C,$$  \hfill (6.41)

where $C$ is a constant only depending on $N, s, p, \gamma$, $\|\nabla \Phi\|_{C^{1,\gamma}(B_2)}$, $|V|_{L^p(B_2)}$ and $\|1\|_{L^1(\Sigma)}$.

Proof of Theorem 1.4 (completed). From the computations above, we have that $w = \varphi_{1/2} u \circ \Phi \in H^s(\mathbb{R}^N)$ satisfies (6.38) with $K \in \mathcal{K}^s(\kappa,\gamma,Q_{1/2})$. Thanks to (6.41), we can apply Theorem 1.3 to get the result. \qed
Proof of Theorem 1.3 (completed). Since \( w = \varphi_{1/2} \omega \Phi \in H^s(\mathbb{R}^N) \) solves (6.38) and \( K \in \mathcal{K}(\kappa, \gamma, Q_{1/2}) \), it suffices to prove that \( \tilde{f}, V, w, G \in C^0(B_{1/8}) \). However, in view of (6.35) and (6.36), we can use similar arguments as in the proof of Lemma 2.3(iv) to deduce that
\[
\| \tilde{f} \|_{C^0(B_{1/16})} \leq C(\| u \|_{L^2(\Sigma)} + \| f \|_{C^0(B_2)})
\]
and, using also (1.18), we get
\[
\| \tilde{V} \|_{C^0(B_{1/16})} \leq C(\| V \|_{C^0(B_2)} + \| 1 \|_{L^2(\Sigma)}) \leq C,
\]
where here and below, the letter \( C \) denotes a positive constant which may vary from line to line but only depends on \( N, s, \alpha, \gamma, \| V \|_{C^0(B_2)}, \| \nabla \Phi \|_{C^{1, \gamma}(B_2)} \) and \( \| 1 \|_{L^2(\Sigma)} \). Moreover, recalling (6.39), applying Lemma 2.3(iii), we have that
\[
\| G \|_{C^0(B_{1/16})} \leq C\| w \|_{L^2(B_2)} \leq C\| u \|_{L^2(\Sigma)}.
\]
In view of (6.35), from Theorem 3.7 (to get \( w \in C^0(B_{2^{-5}}) \) if \( 2s < 1 \)) and Theorem 1.1 we deduce that
\[
\| w \|_{C^0(B_{2^{-5}})} \leq C(\| w \|_{L^\infty(B_{2^{-2}})} + \| f \|_{C^0(B_2)}).
\]
This with Theorem 1.1 imply that
\[
\| w \|_{C^0(B_{2^{-5}})} \leq C(\| u \|_{L^2(B_2)} + \| u \|_{L^2(\Sigma)} + \| f \|_{C^0(B_2)}).
\]
Therefore, thanks to Theorem 1.2 and using (6.42), (6.43), (6.44), we find that
\[
\| w \|_{C^{2s+\alpha}(B_{2^{-6}})} \leq C(\| w \|_{L^\infty(\mathbb{R}^N)} + \| \tilde{V} w \|_{C^0(B_{2^{-5}})} + \| \tilde{f} w \|_{C^0(B_2)} + \| G \|_{C^0(B_{1/16})})
\]
\[
\leq C(\| u \|_{L^2(B_2)} + \| u \|_{L^2(\Sigma)} + \| f \|_{C^0(B_2)}).
\]
The proof is thus completed by scaling, covering and a change of variables.

7. Appendix

Proof of Lemma 5.7. For simplicity, recalling (1.21) and (1.22), we assume that
\[
\| A \|_{C^{k+2s+\alpha}(Q_\infty)} + \| B \|_{C^{k}(Q_\infty)} \leq 1,
\]
where \( \tau_\alpha := \alpha + (2s - 1)_+ \) if \( 2s \neq 1 \) and \( \tau_{1/2} := \alpha + \varepsilon \) if \( 2s = 1 \). We also assume that
\[
\| u \|_{C^{k+2s+\alpha+\varepsilon}(\mathbb{R}^N)} \leq 1,
\]
where \( \varepsilon_\alpha := 0 \) if \( 2s \neq 1 \) and \( \varepsilon_\alpha := \varepsilon \) if \( 2s = 1 \).
We consider the case \( k = 0 \). Since \( \| u \|_{L^\infty(\mathbb{R}^N)} \leq 1 \), we have
\[
| \delta^\alpha u(x, r, \theta) | \leq C \min(1, r^{2s+\alpha}).
\]
Here, for \( 2s \geq 1 \), we use the fact that \( \delta^\alpha u(x, r) = r^{1/2} \int_0^1 (\nabla u(x + tr\theta) - \nabla u(x - tr\theta)) \cdot \theta \, dt \). Moreover for \( x_1, x_2 \in \mathbb{R}^N \) and \( r > 0 \), then for \( 2s + \alpha < 1 \), we have
\[
| \delta^\alpha u(x_1, r, \theta) - \delta^\alpha u(x_2, r, \theta) | \leq C \min(r^{2s+\alpha}, |x_1 - x_2|^{2s+\alpha})
\]
and if \( 2s \geq 1 \), we have
\[
| \delta^\alpha u(x_1, r, \theta) - \delta^\alpha u(x_2, r, \theta) | \leq C \min(r^{2s+\alpha}, r|x_1 - x_2|). 
\]
On the other hand, for all \( s \in (0, 1) \),
\[
| \delta^\alpha u(x, r, \theta) | \leq C \min(1, r)^{\min(2s+\alpha, 1)},
\]
and
\[
| \delta^\alpha u(x_1, r, \theta) - \delta^\alpha u(x_2, r, \theta) | \leq C \min(r, |x_1 - x_2|)^{\min(2s+\alpha, 1)}.
\]
Using (7.4), for $s \in (0,1)$, we estimate
\[
|O_{B,u}^s(x)| \leq C \int_0^\infty \min(r,1)^{\min(2s+\alpha,1)} \min(r,1)^{(2s-1)+\alpha} r^{1-2s} dr
\leq C \int_0^1 r^{\min(2s+\alpha,1)} r^{2s-1} + C \int_1^\infty r^{1-2s} dr,
\]
so that,
\[
\|O_{B,u}^s\|_{L^\infty(\mathbb{R}^N)} \leq C.
\] (7.6)

We consider next $E_{A,u}^s$. For all $x \in \mathbb{R}^N$ and for all $s \in (0,1)$, by (7.3), we have
\[
|E_{A,u}^s(x)| \leq C \int_0^\infty \min(r^{2s+\alpha},1) r^{1-2s} dr \leq C \int_0^1 r^{\alpha-1} dr + C \int_1^\infty r^{1-2s} dr,
\]
yielding
\[
\|E_{A,u}^s\|_{L^\infty(\mathbb{R}^N)} \leq C.
\] (7.7)

Let $x_1, x_2 \in \mathbb{R}^N$ with $|x_1 - x_2| \leq 1$. Using (7.3), for $s \in (0,1)$ we have
\[
|O_{B,u}^s(x_1) - O_{B,u}^s(x_2)| \leq C \int_0^\infty \min(r,|x_1 - x_2|)^{\min(2s+\alpha,1)} \min(r,1)^{\tau_s} r^{1-2s} dr
\leq C \int_0^{|x_1 - x_2|^2} \min(r^{2s+\alpha},1) r^{1-2s} dr + C|x_1 - x_2|^2 \min(r^{2s+\alpha},1) \int_1^{|x_1 - x_2|} r^{\tau_s} r^{1-2s} dr
\leq C|x_1 - x_2|^\alpha.
\]

In the above estimate, it is used that $\tau_s = \alpha + \varepsilon$, for $s = 1/2$. This together with (7.6) imply that $\|O_{B,u}^s\|_{C^{0,\alpha}(\mathbb{R}^N)} \leq C$, for all $s \in (0,1)$.

Now for $2s \geq 1$, let $x_1 \neq x_2 \in \mathbb{R}^N$ with $|x_1 - x_2| \leq 1$. Using (7.3) and (7.4) we have
\[
|E_{A,u}^s(x_1) - E_{A,u}^s(x_2)|
\leq C \int_0^\infty \min(r^{2s+\alpha},r|x_1 - x_2|^\tau_s) r^{1-2s} dr + C|x_1 - x_2|^\alpha \int_0^\infty \min(r^{2s+\alpha},1) r^{1-2s} dr
\leq C \int_0^{|x_1 - x_2|^2} r^{\alpha-1} dr + C|x_1 - x_2|^\tau_s \int_1^\infty r^{2s} dr + C|x_1 - x_2|^\alpha \leq C|x_1 - x_2|^\alpha.
\]

Hence using (7.7), for $2s \geq 1$, we get $\|E_{A,u}^s\|_{C^{0,\alpha}(\mathbb{R}^N)} \leq C$.

We now consider the case $2s + \alpha < 1$. For $x_1, x_2 \in \mathbb{R}^N$, $|x_1 - x_2| \leq 1$, by (7.2), we estimate
\[
|E_{A,u}^s(x_1) - E_{A,u}^s(x_2)|
\leq C \int_0^\infty \min(r,|x_1 - x_2|)^{2s+\alpha} r^{1-2s} dr + C|x_1 - x_2|^\alpha \int_0^\infty \min(r^{2s+\alpha},1) r^{1-2s} dr
\leq C \int_0^{|x_1 - x_2|^2} r^{-1+\alpha} dr + C|x_1 - x_2|^{2s+\alpha} \int_1^\infty r^{-1-2s} dr + C|x_1 - x_2|^\alpha \leq C|x_1 - x_2|^\alpha.
\]

We then conclude from this and (7.7) that $\|E_{A,u}^s\|_{C^{0,\alpha}(\mathbb{R}^N)} \leq C$, provided $2s + \alpha < 1$.

If $k > 1$, we can use the Leibniz formula for the derivatives of the product of two functions. Note that for all $\gamma \in \mathbb{N}^N$ with $|\gamma| \leq k$, we have that $\delta^\gamma \partial^\gamma u$ (resp. $\delta^\gamma \partial^\gamma u$) satisfies (7.1) and (7.2) (resp. (7.3) and (7.4)).
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M.M.F.: AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES IN SENEGAL, KM 2, ROUTE DE JOAL, B.P. 14 18, MIIBOUR, SÉNÉGAL.

E-mail address: mouhamed.m.fall@aims-senegal.org, mouhamed.m.fall@gmail.com