Mass scale effects for the Sudakov form factors in theories with the broken gauge symmetry

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Abstract

The off-shell and the on-shell Sudakov form factors in theories with broken gauge symmetry are calculated in the double-logarithmic approximation. We have used different infrared cut-offs, i.e. different mass scales, for virtual photons and weak gauge bosons.

1 Introduction

In QED the electro-magnetic vertex function, $\Gamma_\mu$, can be written as:

$$\Gamma_\mu = \bar{u}(p_2) \left[ \gamma_\mu f(p_1, p_2) - (1/2m)\sigma_{\mu\nu} q_\nu g(p_1, p_2) \right] u(p_1),$$

where $f$ and $g$ are the form factors. In the fifties V.V. Sudakov showed that in the limit of large momentum transfer, i.e.,

$$q^2 = (p_2 - p_1)^2 \gg p_1^2, p_2^2,$$

the most important radiative corrections to the form factor $f(p_1, p_2)$ are the double-logarithmic ones (DL). The summing of these corrections to all orders in $\alpha$ leads to,

$$f = e^{-\left(\alpha/2\pi\right) \ln(q^2/m^2) \ln(q^2/p_2^2)},$$

for off-shell momenta $p_1$ and $p_2$ and to the formula

$$f = e^{-\left(\alpha/4\pi\right) \ln^2(q^2/m^2)},$$

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if $p_1^2 = p_2^2 = m^2$.

After this pioneer work, the Sudakov form factor $f$ was calculated in QCD \cite{2} and recently it has also been considered in the electroweak (EW) theory \cite{3,4,5}. In non-Abelian theories the direct graph-by-graph calculation to all orders in the couplings is a very complicated procedure, even when the double-logarithmic approximation (DLA) is used. Technically, it is more convenient to use some evolution equation. In particular, the infrared evolution equation (IREE) approach was used in ref. \cite{3} to calculate the “inclusive” EW Sudakov form factor, i.e., where summation over the left handed lepton flavour was assumed. The IREE method is based on the Gribov bremsstrahlung theorem \cite{6}. It was applied earlier, in ref. \cite{8}, to calculate the radiative form factors for $e^+e^-$ annihilation into quarks and gluons. It has been proved to be a very efficient and simple method. This theorem was formulated and proved first for QED and then generalised in refs.\cite{7,8,9,10} to QCD. Besides the calculation of the Sudakov form factor $f(q^2)$, the IREE turned out to be also useful \cite{11} in order to calculate the form factor $g(q^2)$ of electrons and quarks in the kinematic region specified by eq. (2).

The IREE exploits the evolution of the scattering amplitudes with respect to the infrared cut-off $\mu$ introduced in the space of the transverse momenta of virtual particles. This cut-off plays the role of a mass scale and with DL accuracy all other masses can be safely neglected. So, in theories with unbroken gauge symmetry it is enough to have one mass scale. On the other hand, for the electroweak theory, the $SU(2) \times U(1)$ symmetry is broken down to the $U_{EM}(1)$ symmetry. This introduces a second mass parameter, $M$, associated with the symmetry breaking scale. Therefore, besides the conventional DL contributions of the order of $\ln^{2n}(q^2/M^2)$ in $n$-th order of the perturbative expansion, there appear other corrections of the type

$$\sim a_k \ln^k(M^2/\mu^2) \ln^{(n-k)}(q^2/M^2),$$

where $a_k$ are numerical coefficients and $k$ runs from 1 to $n$. Since $\mu$ and $M$ can be widely different (we assume the value of $\mu$ to be equal or greater than masses of the involved fermions whereas $M$ is comparable with masses of the weak bosons), the impact of these two mass scales on the value of the inclusive Sudakov form factor can be important. To calculate these contributions to all orders in the electroweak couplings is the aim of the present work.
Sect. 2 we obtain the Sudakov form factor for a $U(1) \times U(1)$ model with two “photons”, using the graph-by-graph calculation. In Sect. 3 we derive and solve the IREE for the Sudakov form factor for the same model. Then, in Sect. 4 a similar derivation is made for the Sudakov form factor in the electroweak theory. Expressions for the off-shell electroweak Sudakov form factor are obtained in Sect. 5. Finally, in Sect. 6 we summarise and discuss our results.

2 The Sudakov form factor in the Abelian model

As a toy model it is instructive to consider a $U(1) \times U(1)$ gauge theory. The first group is the normal electro-magnetic gauge group and the second $U(1)$ is broken. The corresponding gauge boson, called $B$ meson has a mass $M$. Besides the photon and the $B$ meson the model has only one charged fermion, and a neutral scalar Higgs particle arising from the spontaneous breaking of the second $U(1)$.

At one loop order the Sudakov form factor is obtained from the diagrams in figure 1 where the dashed line represents the photon and the wavy line represents the $B$ meson. Summing both contributions in $DL$ approximation

\[ f^{(1)} = -\left(\frac{g_1^2}{16\pi^2}\right) \ln^2\left(\frac{q^2}{\mu^2}\right) - \left(\frac{g_2^2}{16\pi^2}\right) \ln^2\left(\frac{q^2}{M^2}\right) \]  

where $g_1, g_2$ are the gauge couplings corresponding to the unbroken and broken groups respectively. It is interesting to point out that there is also a
similar diagram with the higgs particle in loop. However this contribution vanishes as \((m/M)^2\) when the fermion mass, \(m\), goes to zero. This is obviously true in all orders of perturbation theory.

At two loop order the DL contributions stems from the diagrams of figure Fig. 2. The calculation of the first four diagrams in Fig. 2 is similar to the

\[
\begin{align*}
\mathcal{F}_{a+b+c+d} &= \frac{1}{2} \left[ \left( g_1^2 / 16 \pi^2 \right) \ln^2 \left( q^2 / \mu^2 \right) \right] + \frac{1}{2} \left[ \left( g_2^2 / 16 \pi^2 \right) \ln^2 \left( q^2 / M^2 \right) \right]. \quad (7)
\end{align*}
\]

The calculation of the remaining four diagrams is less trivial. Let us consider diagram \(e\), for instance. It is well known that in DLA one can perform the integration over the momenta \(k_i\), \(i = 1, 2\), replacing the boson propagators by \(-2\pi i \delta(k_i^2)\). Then, using the Sudakov parametrisation,

\[
k_i = \alpha_i p_2 + \beta_i p_1 + k_{i\perp},
\]

it is easy to integrate over \(k_{i\perp}\) and to obtain

\[
f_e^{(2)} = \frac{g_1^2 g_2^2}{(8\pi)^2} \int_{D_\perp} \frac{d\alpha_1 d\beta_1 d\alpha_2 d\beta_2}{\alpha_1 \beta_1 \alpha_2 \beta_2} \Theta(\alpha_1 \beta_1 - \lambda_1^2) \Theta(\alpha_2 \beta_2 - \lambda_2^2), \quad (9)
\]
where $\lambda_1^2 = M^2/s$, $\lambda_2^2 = \mu^2/s$, and $s = |q^2|$. Notice that the DL arises if one uses the approximation $\alpha_1 + \alpha_2 \approx \alpha_2$ and $\beta_1 + \beta_2 \approx \beta_2$. This implies $\alpha_2 \gg \alpha_1$ and $\beta_2 \gg \beta_1$. These conditions plus the boundaries stemming from the arguments of the $\Theta$ functions define the integration region, $D_e$. In Fig. 3 we show this region. The full curve corresponds to the equation $\alpha_2 \beta_2 = \lambda_2^2$.

\[
\begin{align*}
\text{Figure 3: Integration regions in the } \alpha_2 \beta_2 \text{ plane.}
\end{align*}
\]

and the dashed curve represents the condition $\alpha_1 \beta_1 = \lambda_1^2$. Because the lower limits of the $\alpha_2$ and $\beta_2$ integrals are $\alpha_1$ and $\beta_1$, respectively, it is easy to see that we are integrating over the rectangle $E$. So we obtain

\[

def^{(2)} = \frac{g_1^2 \ g_2^2}{8\pi^2 \ 8\pi^2} \int_{\lambda_1^2}^{1} \frac{d\alpha_1}{\alpha_1} \int_{\lambda_1^2/\alpha_1}^{1} \frac{d\beta_1}{\beta_1} \int_{\alpha_1}^{1} \frac{d\alpha_2}{\alpha_2} \int_{\beta_1}^{1} \frac{d\beta_2}{\beta_2}.
\]

The integrals are now trivial. However, rather than doing this it is better to realize that the remaining diagrams give rise to similar integrals but with
the integration regions $F$, $G$ and $H$ of Fig. 3, respectively. Then, the sum of the diagrams $e$, $f$, $g$ and $h$ corresponds to

$$f^{(2)}_{e+f+g+h} = \frac{g_1^2}{8\pi^2} \frac{g_2^2}{8\pi^2} \int_{x_1^2}^1 \frac{d\alpha_1}{\alpha_1} \int_{x_2^2/\alpha_1}^1 \frac{d\beta_1}{\beta_1} \int_{x_2^2/\alpha_2}^1 \frac{d\alpha_2}{\alpha_2} \int_{x_2^2/\beta_2}^1 \frac{d\beta_2}{\beta_2}. \quad (11)$$

which immediately leads to

$$f^{(2)}_{e+f+g+h} = \frac{g_1^2}{16\pi^2} \ln^2(q^2/\mu^2) \frac{g_2^2}{16\pi^2} \ln^2(q^2/M^2). \quad (12)$$

Adding these results to eq.(11) one obtains the total two-loop contribution to the form factor, namely:

$$f^{(2)} = (1/2)[(g_1^2/16\pi^2) \ln^2(q^2/\mu^2) + (g_2^2/16\pi^2) \ln^2(q^2/M^2)]^2. \quad (13)$$

Repeating the same analyses in higher orders in $g_1$ and $g_2$, we would arrive at the simple exponentiation of the double-logarithmic corrections to $\Gamma_\mu$, i.e.,

$$\Gamma^{DL}_\mu = \Gamma^{Born}_\mu f(q^2, \mu^2, M^2), \quad (14)$$

with

$$f(q^2, \mu^2, M^2) = e^{-[(g_1^2/16\pi^2) \ln^2(q^2/\mu^2) + (g_2^2/16\pi^2) \ln^2(q^2/M^2)].} \quad (15)$$

This equation accounts for all DL contributions described by Eq. (5).

3 Infrared evolution equations with two mass scales

In order to avoid the direct graph-by-graph summation of the DL contributions, it is possible to obtain the Sudakov form factor $f(q^2, \mu^2, M^2)$ as a solution of some integral equation. The method of obtaining this infrared evolution equation (IREE) can be extended in order to include two mass scales, $\mu$ and $M$.

Let us first notice that the boson mass $M$ in virtual propagators can be regarded in DLA as the infrared cut-off for integrating over transverse momentum space for $B$-bosons. In fact, with logarithmic accuracy we have

$$\int_0^s \frac{dk_1^2}{k_1^2 + M^2} = \int_M^s \frac{dk_1^2}{k_1^2}. \quad (16)$$
In DLA the integrals over the longitudinal momenta have the transverse momenta as the lowest limit. So, after introducing the cut-off shown in the previous equation one can neglect the mass $M$ in the $B$-propagators and still be free of infra-red singularities. On the other hand, $\mu$ is the IR cut-off in the transverse momentum space for photons. Therefore, with the DL accuracy, we have QED with two kinds of photons, each one has an independent infra-red cut-off. We remind the reader that we have assumed that $\mu \ll M$. According to the generalisation\[8, 10\] of the Gribov bremsstrahlung theorem\[6\], the boson with the minimal $k_\perp$ can be factorized, i.e. the main contribution from integrating over its momentum comes from the graphs where its propagator is attached, in all possible ways, to the external charged lines whereas its $k_\perp$ acts as a new IR cut-off for the integrations over the remaining loop momenta and becomes, in DLA, a new effective mass scale. Therefore the blob in Fig.4 is on-shell. It depends on the new infrared cut-off $k_\perp$ and does

![Figure 4: Factorisation of boson with the minimal $k_\perp$.](image)

not depend on the longitudinal components of momentum $k$. However, in contrast to the usual QED situation, now we have two options: the factorized particle with minimal $k_\perp$ can be either a photon or a $B$-boson. Applying the Feynman rules to the graph in Fig.4, we obtain for the first possibility

\[
M^\gamma = -\frac{g_1^2}{8\pi^2} \left[ \int_{\mu^2}^{M^2} \frac{dk_1^2}{k_1^2} \ln(s/k_1^2)f(s/k_1^2, s/M^2) + \int_{M^2}^{s} \frac{dk_1^2}{k_1^2} \ln(s/k_1^2)f(s/k_1^2, s/k_1^2) \right],
\]

where $\ln(s/k_1^2)$ appears as a result of integrating over the longitudinal momentum of the factorized photon quite similarly to the way it appears in the
first loop calculation of $f$ in QED. Note that the form factors $f$ in the right hand side (rhs) of eq. (17) have the same first argument, $s/k_2^2$, but different second arguments, $s/M_2^2$, in the first integral and $s/k_2^2$ in the second one. The reason for this is that the propagators of the $B$-bosons depend on $M$ through $(k_2^2 + M^2)$ and in DLA they are approximated by $\max(k_2^2, M^2)$. The second possibility is that the factorized particle is the $B$-meson. Now, we can regard $M$ as the infra-red cut-off for the integration over the $k_2$ of the $B$-meson. Hence the result is

\[ M^B = -\frac{g_2^2}{8\pi^2} \int_{M^2}^s \frac{dk_2^2}{k_2^2} \ln(s/k_2^2) f(s/k_2^2, s/k_2^2). \] (18)

Adding eqs. (17) and (18) and including the Born contribution, $f_{\text{Born}} = 1$, one obtains the integral IREE for the form factor $f(s/\mu^2, s/m^2)$, i.e.

\[ f(s/\mu^2, s/M^2) = 1 + M^\gamma + M^B. \] (19)

The solution of Eq. (19) can be obtained, for example, by iterations: substituting $f_{\text{Born}} = 1$ into the integrand of the rhs of eq. (19) we obtain $f(s/\mu^2, s/m^2)$ in the one loop approximation and so on. It is easy to see that, after summing up contributions to all orders, the solution to eq. (19) coincides with Eq. (15) obtained by the direct graph-by-graph summation. Alternatively, the integral IREE given in eq. (19) can be rewritten in the differential form (cf. [3]). Indeed, differentiating Eq. (19) with respect to $\mu$ yields

\[ \frac{\partial f(\rho_1, \rho_2)}{\partial \rho_1} = -\frac{g_2^2}{8\pi^2} \rho_1 f(\rho_1, \rho_2) \] (20)

where $\rho_1 = \ln(s/\mu^2)$ and $\rho_2 = \ln(s/M^2)$. Obviously the solution is

\[ f(s/\mu^2, s/M^2) = G(\rho_2)e^{-g_2^2/16\pi^2\rho_2^2}. \] (21)

Substituting it into Eq. (20) and differentiating with respect to $\rho_2$ gives

\[ \frac{dG(\rho_2)}{d\rho_2} = -\frac{g_2^2}{8\pi^2} \rho_2 G(\rho_2). \] (22)

With the boundary condition $G(0) = 1$ it is easy to solve this equation. Again one obtains eq.(15)
4 The Sudakov form factor in the electroweak theory

The fermions in the electroweak theory are such that the left-handed fields are doublets of the weak isospin group and the right handed fields are singlets of $SU(2)\times U_Y(1)$. So one has a left, $F_L$, and a right, $F_R$, Sudakov form factors. In line with the approximation of massless particles there is no chirality flip amplitude. Because the right-handed fermions couple to the $U_Y(1)$ boson, $F_R$ only gets contributions from $Z$ and photon exchange. Hence, borrowing directly from the results of the previous section, one can easily obtain:

$$F_R(s/\mu^2, s/M^2) = \exp(-\psi_R),$$

with

$$\psi_R = \frac{\alpha Q^2}{4\pi} \left[ \ln^2(s/\mu^2) + \tan^2\theta \ln^2(s/M^2) \right],$$

where $M = M_Z$ and $\theta_W$ is the Weinberg angle.

Now, let us derive the IREE for $F_L(s/\mu^2, s/M^2)$, neglecting the mass difference between the $Z$ and the $W$ boson, i.e., $M_W = M_Z = M$. To do this, applying the Gribov bremsstrahlung theorem, one should factorize the boson with the minimal $k_\perp$. If the integration over the $k_\perp$ of the factorized boson is done in the region $M < k_\perp < \sqrt{s}$ this particle can be any of the four bosons present in the theory, the $W^\pm$, the $Z$ and the photon. This yields the following contribution to the IREE:

$$\tilde{M}^{WZ} = -\frac{e^2 Q^2}{8\pi^2} \int_{M^2}^{s} \frac{dk_\perp^2}{k_\perp^2} \ln(s/k_\perp^2) F(s/k_\perp^2, s/M^2),$$

where we have explicitly separated the photon from the $WZ$ contributions. The latter are proportional to

$$C_{WZ} = g^2[(t_1^2 + t_2^2) + (1/\cos^2\theta_W)(t_3 - \sin^2\theta_W Q)^2].$$

The second DL region is when $\mu < k_\perp < M$, but, now, the integration over the $k_\perp$ of the factorized boson only gives a DL contribution if this boson is a photon. So we obtain

$$\tilde{M}^\gamma = -\frac{e^2 Q^2}{8\pi^2} \int_{\mu^2}^{M^2} \frac{dk_\perp^2}{k_\perp^2} \ln(s/k_\perp^2) F(s/k_\perp^2, s/M^2).$$
The sum of the factorized contributions given by eqs. (25) and (27) together with the Born value \( F_{\text{Born}} = 1 \) leads to the IREE for the Sudakov form factor \( F_{L}(s/\mu^{2}, s/M^{2}) \) in the integral form,

\[
F(s/\mu^{2}, s/M^{2}) = 1 - \frac{e^{2}Q^{2}}{8\pi^{2}} \int_{\mu^{2}}^{M^{2}} \frac{dk_{1}^{2}}{k_{1}^{2}} \ln(s/k_{1}^{2})F(s/k_{1}^{2}, s/M^{2}) - \left( C_{WZ} + \frac{e^{2}Q^{2}}{8\pi^{2}} \right) \int_{M^{2}}^{s} \frac{dk_{1}^{2}}{k_{1}^{2}} \ln(s/k_{1}^{2})F(s/k_{1}^{2}, s/k_{1}^{2}).
\] (28)

This equation is similar to Eq. (19) where \( g_{1} \) is replaced by \( eQ \) and \( g_{2} \) is replaced by \( C_{WZ} \). Differentiating Eq. (28) first with respect to \( \mu \) and then with respect to \( M \) we obtain the IREE in the differential form (cf eqs. (20),(22)):

\[
\frac{\partial F_{L}(\rho_{1}, \rho_{2})}{\partial \rho_{1}} = -\frac{e^{2}Q^{2}}{8\pi^{2}} \rho_{1}F_{L}(\rho_{1}, \rho_{2}) ,
\] (29)

\[
\frac{d\tilde{G}(\rho_{2})}{d\rho_{2}} = -\frac{C_{WZ}}{8\pi^{2}} \rho_{2}\tilde{G}(\rho_{2}) ,
\] (30)

where \( \tilde{G} \) is the general solution of eq. (29). Finally, solving this equation in the same way that we have solved eq. (22) we obtain

\[
F_{L}(s/\mu^{2}, s/M^{2}) = \exp(-\Psi_{L}) ,
\] (31)

with

\[
\Psi_{L} = \frac{e^{2}Q^{2}}{16\pi^{2}} \ln^{2}(s/\mu^{2})
+ \frac{g^{2}}{16\pi^{2}}[(t_{1}^{2} + t_{2}^{2}) + \frac{1}{\cos^{2}\theta_{W}}(t_{3} - \sin^{2}\theta_{W}Q)^{2}] \ln^{2}(s/M^{2}) .
\] (32)

5 The off-shell Sudakov electroweak form factor

Eqs. (3) and (4) show that even in the simplest QED case the on-shell Sudakov form factor cannot be obtained from the expression for the off-shell form factor with the simple replacement of the electron \( p_{1}^{2} \) and \( p_{2}^{2} \) by the
electron mass or by the infrared cut-off. Both form factors have to be calculated independently. Obviously, the same is true for the off-shell electroweak Sudakov form factor $\tilde{F}_L$. Before we calculate $\tilde{F}_L$, it is instructive to demonstrate how the IREE for the off-shell QED form factor can be obtained. This form factor $\tilde{f}$ depends on $q^2$, $p_1^2$, $p_2^2$ and also depend on the infrared cut-off $\mu$, i.e., $\tilde{f} = \tilde{f}((q^2/\mu^2, p_1^2/\mu^2, p_2^2/\mu^2))$.

Similarly to the on-shell case, factorizing the contribution of the virtual photon with the minimal $k_\perp$ leads to the following IREE:

$$\tilde{f}((q^2/\mu^2, p_1^2/\mu^2, p_2^2/\mu^2)) = 1 - \frac{\alpha}{2\pi} \int_D \frac{dk_\perp^2 d\beta}{k_\perp^2} \tilde{f}((q^2/k_\perp^2, p_1^2/k_\perp^2, p_2^2/k_\perp^2)).$$

(33)

However, in contrast to the IREE for the on-shell form factor, the region $D$ in the previous equation now depends on $p_1^2$ and $p_2^2$. The region $D$ is different for the particular case when the virtualities $p_1^2, p_2^2$ are small enough such that

$$p_1^2 p_2^2 < q^2 \mu^2$$

(34)
or large enough so that

$$p_1^2 p_2^2 > q^2 \mu^2.$$  

(35)

In the former case we denote the integrating region $D_1$ and call it $D_2$ in the latter case. In Fig. 3 we show $D_1$. In the plane $k_\perp^2 = \alpha \beta$, $\beta$, $D_1$ is bounded by the “on-shell” curves $s\beta = k_\perp^2$ and $s\beta = s$ and also by the new curves $s\beta = p_2^2$ and $p_1^2 p_2^2 = q^2 \mu^2$. From now on we call $\tilde{f}_1$ the contribution to $\tilde{f}$ from the region $D_1$. Performing the $\beta$ integration in eq. (33) and after that differentiating with respect to $\mu^2$ we obtain

$$\frac{\partial \tilde{f}_1}{\partial x} + \frac{\partial \tilde{f}_1}{z_1} + \frac{\partial \tilde{f}_1}{z_2} = -\frac{\alpha}{2\pi} [x - z_1 - z_2] \tilde{f}_1,$$

(36)

where $x = \ln(q^2/\mu^2)$, $z_1 = \ln(p_1^2/\mu^2)$ and $z_2 = \ln(p_2^2/\mu^2)$. Obviously, the solution to this equation is,

$$\tilde{f}_1(q^2, p_1^2, p_2^2, \mu^2) = e^{-[(\alpha/4\pi)(x^2 - z_1^2 - z_2^2)]}.$$  

(37)

In the kinematical region specified by eq. (35), the integration of eq. (33) is over the domain $D_2$. This region does not involve $\mu$ so we have

$$-\mu^2 \frac{\partial \tilde{f}_2}{\partial \mu^2} = \frac{\partial \tilde{f}_2}{\partial x} + \frac{\partial \tilde{f}_2}{z_1} + \frac{\partial \tilde{f}_2}{z_2} = 0$$

(38)

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Figure 5: The integration region $D_1$
and its general solution is

\[ \tilde{f}_2 = \Phi(x - z_2, x - z_2) \] (39)

where \( \Phi \) is an arbitrary function. The matching condition \( \tilde{f}_1 = \tilde{f}_2 \) when \( p_1^2 p_2^2 = q^2 \mu^2 \), which is equivalent to \( x = z_1 + z_2 \), leads to

\[ \tilde{f}_2 = \exp \left[ -\left( \frac{\alpha}{2\pi} \right) \ln(q^2/p_1^2) \ln(q^2/p_2^2) \right]. \] (40)

This is exactly the expression that we had anticipated in the introduction (cf. eq.(3)).

Now, it should be clear that following a similar prescription one obtains the IREE for \( \tilde{F}_L \), namely

\[ \tilde{F}(s/\mu^2, s/M^2) = 1 - \frac{e^2}{8\pi^2} \int_D \frac{dk_1^2}{k_1^2} \ln(s/k_1^2) \tilde{F}(s/k_1^2, s/M^2) - \]

\[ -\frac{(C_{WZ} + e^2 Q^2)}{8\pi^2} \int_{D'} \frac{dk_1^2}{k_1^2} \ln(s/k_1^2) \tilde{F}(s/k_1^2, s/k_1^2). \] (41)

The regions \( D \) and \( D' \) are bounded, in addition to the “on-shell” requirements, by the relation between \( q^2 \mu^2, q^2 M^2 \) and \( p_1^2 p_2^2 \). We specify the following basic off-shell kinematic regions:

- **R1**: \( \mu^2 < p_1^2, p_2^2 < M^2 \), \( p_1^2 p_2^2 < q^2 \mu^2 \);
- **R2**: \( \mu^2 < p_1^2, p_2^2 < M^2 \), \( q^2 \mu^2 < p_1^2 p_2^2 < q^2 M^2 \);
- **R3**: \( p_1^2 p_2^2 > M^2 \), \( p_1^2 p_2^2 < q^2 M^2 \);
- **R3**: \( p_1^2 p_2^2 > M^2 \), \( p_1^2 p_2^2 > q^2 M^2 \).

For each region, \( R_i \) (\( i = 1 \ldots 4 \)), we obtain

\[ \tilde{F}_L = \exp(-\psi_i), \] (42)

with

\[ \psi_1 = \frac{e^2 Q^2}{16\pi^2} \ln^2(q^2/\mu^2) - \ln^2(p_1^2/\mu^2) - \ln^2(p_2^2/\mu^2) + \frac{g^2}{16\pi^2} C_{WZ} \ln^2(s/M^2) \] (43)
for the region $R_1$,

$$\psi_2 = \frac{e^2 Q^2}{8\pi^2} \ln(q^2/p_1^2) \ln(q^2/p_2^2) + \frac{g^2}{16\pi^2} C_{WZ} \ln^2(s/M^2) \tag{44}$$

for the region $R_2$,

$$\psi_3 = \frac{e^2 Q^2}{8\pi^2} \ln(q^2/p_1^2) \ln(q^2/p_2^2) + \frac{g^2}{16\pi^2} C_{WZ} [\ln^2(q^2/M^2) - \ln^2(p_1^2/M^2) - \ln^2(p_2^2/M^2)] \tag{45}$$

for the region $R_3$, and finally

$$\psi_4 = \frac{e^2 Q^2 + C_{WZ}}{8\pi^2} \ln(q^2/p_1^2) \ln(q^2/p_2^2) \tag{46}$$

for the region $R_4$.

6 Discussion

Expression \((31)\) for the electroweak Sudakov form factor, $F_L$, accounts for the mass difference between the photon and the weak bosons. On the other hand, it neglects the difference between the masses of the $W$ and the $Z$. It has been obtained introducing different infrared cut-offs for the integration over transverse momenta of different gauge bosons: cut-off $\mu$ for photons and cut-off $M$ for the $W$ and the $Z$. Expanding Eq. \((31)\) into series, one can easily extract the first-loop and the second-loop DL contributions. The first-loop contribution (save the minus sign) is given by Eq. \((32)\). The DL contributions to $F_L$ in two loops were also calculated in \[12\]. Before comparing our results with results of \[12\], let us notice that besides the DL contributions we account for, the DL contributions in \[12\] account also for the double logarithms of $m^2/\mu^2$, where $m$ stands for fermion masses. Such contributions are absent if the photon cut-off is equal or greater than masses of involved fermions as we assume in this paper. Having dropped them, we arrive at agreement with the two-loop results of \[12\]. Usually, DL calculations involve only one mass scale. Using one mass scale, for instance $M$, for all DL terms in $\Psi_L$ of Eq. \((32)\) allows us to rewrite it as

$$\Psi_L = \frac{g^2}{16\pi^2} (t_i^2 + g^2 (Y/2)^2) \ln^2(s/M^2) + \frac{e^2 Q^2}{16\pi^2} \left[ 2 \ln(s/M^2) \ln(M^2/\mu^2) + \ln^2(M^2/\mu^2) \right]. \tag{47}$$
The first term in the rhs of this equation is the DL contribution with the same scale $M$ for both the electro-magnetic and the weak interactions. The second term is formally single-logarithmic and therefore it is beyond control of the IREE with one mass scale. Finally, the third term, $\ln^2(M^2/\mu^2)$, does not depend on $s$. It is usually dropped in the IREE with one mass scale.

An expression for $\Psi_L$ similar to this one was obtained earlier in ref. [3] using a similar approach of writing an IREE with two infrared cut-offs. However, there are certain differences between our result and the one given in eq. (28) of ref. [3]. Besides the cut-offs $\mu$ and $M$, Fadin et al. [3] have another mass scale $m$ defined such that $m < M$. Setting $m = M$ the result derived by Fadin et al. agrees with ours except for an over all factor $1/2$ which we don’t have. We believe that the origin of this disagreement could be traced back to the use of Fadin et al. of the axial gauge. In fact, in this gauge, the DL contributions arise from fermion self-energy diagrams. Then, it could be that the authors of ref. [3] give the one electron self-energy contribution rather than the Sudakov form factor, which is the double of it. There is another difference between our IREE for $F_L$ (see eq.28) and the corresponding equation in ref [3]. In the work of Fadin et al. the rhs of the evolution equation contains an extra logarithmic dependence on the fermion mass $m$. The $m$ dependence comes from the fermion propagators. But, with DL accuracy, one can write the propagators in terms of $\alpha$ and $\beta$ as $(p_2 - k)^2 - m^2 = k^2 - 2p_2k \approx -s\beta - k_\perp^2$. To obtain a log term from the $\beta$ integration one has to require that $s\beta \gg k_\perp^2$. Then, this condition fixes the lower limit of integration as $k_\perp^2$ and the remaining integral is

$$\int_{\mu^2}^{s} \frac{dk_\perp^2}{k_\perp^2} \ln \left(\frac{s}{k_\perp^2}\right)$$

with no logarithmic contribution depending on $m$.

As a final remark, we would like to stress that the exponentiation of the DL contributions to the EW Sudakov form factor, $F_L$, takes place when both the initial and the final state are not specified and summation over their weak isospin is done. This means that, in contrast to QCD, such form factors should be regarded as a theoretical object with rather limited applications. In principle the off-shell version of $F_L$ could be considered as an ingredient in the calculation of some more complicated physical processes.
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