CLASSES OF SOME HYPERSURFACES IN THE GROTHENDIECK
RING OF VARIETIES

EMEL BILGIN

Abstract. Let $X$ be a projective hypersurface in $\mathbb{P}^n_k$ of degree $d \leq n$. In this paper we study the relation between the class $[X]$ in $K_0(\text{Var}_k)$ and the existence of $k$-rational points. Using elementary geometric methods we show, for some particular $X$, that $X(k) \neq \emptyset$ if and only if $[X] \equiv 1$ modulo $L$ in $K_0(\text{Var}_k)$. More precisely we consider the following cases: a union of hyperplanes, a quadric, a cubic hypersurface with a singular $k$-rational point, and a quartic which is a union of two quadrics one of which being smooth.

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1. Introduction

Let $k$ be a field, let $\text{Var}_k$ denote the category of varieties over $k$. The Grothendieck ring of varieties, denoted by $K_0(\text{Var}_k)$, is a ring whose first appearance in the literature dates back to 1964, in a letter of A. Grothendieck to J. P. Serre [11]. Even though this ring has been studied from many aspects since then, still only little is known about it. In the article [20], B. Poonen proves that if the base field $k$ is of characteristic zero, then $K_0(\text{Var}_k)$ is not a domain, by providing some example of zero divisors. Another example for the zero divisors of $K_0(\text{Var}_k)$ was given by J. Kollár in [14]. F. Heinloth [4] defined $K_0(\text{Var}_k)$ in a different way, introducing blow up relations on smooth projective varieties, and proved that this definition is equivalent to the classical one. M. Larsen and V. A. Lunts [15] established an interesting relation of $K_0(\text{Var}_k)$ with stable rationality, which is a notion weaker than rationality.

Let $X$ be a hypersurface of degree $d$ in $\mathbb{P}^n_k$, where $d \leq n$. Let $[X]$ denote its class in $K_0(\text{Var}_k)$. The main question that concerns us is whether it is true that $X(k)$ is nonempty if and only if

$$[X] \equiv 1 \mod ([A^1_k])$$

(1.0.1)

in $K_0(\text{Var}_k)$, or not. We explain and prove that for some certain types of hypersurfaces this question has a positive answer.
After the completion of this thesis, we came to know of an independent work submitted on Arxiv by X. Liao [16], which has a significant intersection with what we have done. Whereas our main question is based on the relation between the existence of a \( k \)-rational point and the class in \( K_0(\text{Var}_k) \), with \( k \) an arbitrary field, in [16] \( k \) is assumed to be algebraically closed of characteristic zero and the focus is on the "\( L \)-rationality". Other than this, the main parts of this work that differ from [16] are the following theorems where we prove that Question 3.7 is positively answered: Theorem 5.3, which is about a hypersurface \( X \) over a field \( k \) such that \( X \times_k L \) is a union of \( d \leq n \) hyperplanes where \( L/k \) is a finite Galois extension, and Theorem 7.1 where we study the union of two quadrics, one of which is assumed to be smooth, over an algebraically closed field \( k \) of characteristic zero.

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2. Grothendieck Ring of Varieties

Let us recall some definitions and properties that will be used.

**Definition 2.1.** [19, Definition 2.1] Let \( k \) be a field. Let \( \text{Var}_k \) denote the category of varieties over \( k \). Note that what we mean here by a variety over \( k \) is a reduced separated scheme of finite type over \( k \). The Grothendieck group \( K_0(\text{Var}_k) \) is the abelian group generated by isomorphism classes of varieties over \( k \), with the relation

\[
[X] = [X \setminus Y] + [Y],
\]

if \( Y \subset X \) is a closed subvariety. The product

\[
[X] \cdot [Z] := [X \times_k Z]
\]

defines a ring structure on \( K_0(\text{Var}_k) \).

Note that the zero of this ring is \( 0 := [\emptyset] \), and the multiplicative identity is \( 1 := [\text{Spec}(k)] \). We denote by \( L \) the class of \( \mathbb{A}^1_k \) in \( K_0(\text{Var}_k) \).

**Remark 2.2.** We could actually drop the requirement of reducedness in Definition 2.1. Let \( X \) be a separated scheme of finite type over \( k \), and let \( X_{\text{red}} \) denote the reduced scheme associated to \( X \). Applying the scissor relations given in Definition 2.1 to the natural closed immersion \( X_{\text{red}} \to X \) we obtain

\[
[X] = [X \setminus X_{\text{red}}] + [X_{\text{red}}] = [X_{\text{red}}].
\]

Let \( \pi : X \to Y \) be a morphism of varieties over \( k \). Recall that \( \pi \) is said to be a Zariski locally trivial fibration with fiber \( F \) if each closed point \( y \in Y \) has a Zariski open neighborhood \( U \) such that the pre-image \( \pi^{-1}(U) \) is isomorphic over \( k \) to \( F \times_k U \). We will frequently use the following remark for our calculations in \( K_0(\text{Var}_k) \).
Remark 2.3. A Zariski locally trivial fibration $\pi : X \to Y$ becomes trivial in the Grothendieck ring of varieties, i.e. one gets $[X] = [F] \cdot [Y]$. This follows from the defining relations of $K_0(Var_k)$, using induction on the number of open neighborhoods $U_i$ in $Y$ which has $\pi^{-1}(U_i) = F \times_k U_i$, that covers $Y$.

Consider the blow up of a smooth projective variety $X$ along a closed smooth subvariety $Z \subset X$ of codimension $r$. For such a blow up we also have

$$[\text{Bl}_Z X] - [E] = [X] - [Z]$$

in $K_0(Var_k)$. This equation actually yields another definition for the Grothendieck ring of varieties, which is proven to be an equivalent presentation of $K_0(Var_k)$, by F. Heinloth in her paper [4].

Remark 2.4. ([4, Theorem 3.1]) Let $k$ be a field of characteristic zero, and let $K^\text{bl}_0(Var_k)$ denote the abelian group generated by the isomorphism classes of smooth projective varieties with the relations $[\emptyset]_\text{bl} = 0$ and $[\text{Bl}_Z X]_\text{bl} - [E]_\text{bl} = [X]_\text{bl} - [Z]_\text{bl}$ whenever $\text{Bl}_Z X$ is the blow up of a smooth projective variety $X$ along a smooth closed subvariety $Z$ with the exceptional divisor $E$. Then F. Heinloth proves that the ring homomorphism

$$K^\text{bl}_0(Var_k) \to K_0(Var_k)$$

$$[X]_\text{bl} \mapsto [X]$$

is an isomorphism.

2.1. Stable Birationality.

Definition 2.5. Let $X, Y$ be irreducible varieties over a field $k$. Then $X$ and $Y$ are called stably birational if there exists integers $m, n \geq 0$ such that $X \times_k \mathbb{P}^n_k$ is birational to $Y \times_k \mathbb{P}^m_k$ over $k$.

Note that stable birationality defines an equivalence relation $\sim_{SB}$ on the set of smooth complete irreducible varieties over $k$. Let $SB$ denote the set of equivalence classes of this relation. We will denote by $\mathbb{Z}[SB]$ the free abelian group generated by $SB$.

In the article of M. Larsen and V. A. Lunts [15], for a smooth complete irreducible variety it is proven that stable rationality is a necessary and sufficient condition for being equivalent to 1 modulo the class of affine line in $K_0(Var_\mathbb{C})$. Let us recall this important result.

Theorem 2.6. ([15, Theorem 2.3]) Let $G$ be an abelian commutative monoid and $\mathbb{Z}[G]$ be the corresponding monoid ring. Denote by $\mathcal{M}$ the multiplicative monoid of isomorphism classes of smooth complete irreducible varieties over $\mathbb{C}$. Let

$$\Psi : \mathcal{M} \to G$$

be a homomorphism of monoids such that

1. $\Psi([X]) = \Psi([Y])$ if $X$ and $Y$ are birational;
2. $\Psi([\mathbb{P}_\mathbb{C}]) = 1$ for all $n \geq 0$.

Then there exists a unique ring homomorphism

$$\Phi : K_0(Var_\mathbb{C}) \to \mathbb{Z}[G]$$

such that $\Phi([X]) = \Psi([X])$ for $[X] \in \mathcal{M}$.

Now taking $G = SB$ in Theorem 2.6 one gets the ring homomorphism $\Phi_{SB} : K_0(Var_\mathbb{C}) \to \mathbb{Z}[SB]$, which is clearly surjective.
Proposition 2.7. [15] Proposition 2.7] With the assumptions and notations of Theorem 2.6, the kernel of the ring homomorphism $\Phi_{SB}$ defined above is the principal ideal generated by the class $L$.

Remark 2.8. For $X$ connected smooth projective, Proposition 2.7 implies that $X$ is stably rational if and only if $[X] \equiv 1 \text{ mod } L$ in $K_0(Var_k)$. Note that however we have no longer this implication for $X$ singular or non-complete in general. This is because $\Phi_{SB}$, by construction, sends only the class of a smooth complete variety to its own equivalence class in $Z[SB]$, i.e. for $X$ not smooth complete, the equivalence $X \sim_{SB} Spec(\mathbb{C})$ does not necessarily imply that $\Phi_{SB}([X]) = 1$.

As it is pointed out by J. Kollár in [14], the result of Larsen and Lunts holds in fact over any field of characteristic zero: this follows from the presentation of $K_0(Var_k)$ by means of the blow up relations (see Remark 2.3).

The following proposition, which follows directly from Theorem 2.6, demonstrates a noteworthy connection between $K_0(Var_k)$ and $k$-rational points.

Proposition 2.9. Let $k$ be a field of characteristic zero, let $X$ be a smooth connected complete variety of dimension $m$ over $k$. If $[X] \equiv 1 \text{ mod } L$, then Proposition 2.7 implies that $X(k) \neq \emptyset$.

Proof. As it is mentioned in Remark 2.8, we know that $X$ is stably rational, i.e. there is an $a \geq 0$ such that $X \times \mathbb{P}^a_k$ is rational. Let $\phi : \mathbb{P}^{a+a}_k \to X \times \mathbb{P}^a_k$ be a birational map, and let $U \subset \mathbb{P}^{a+a}_k$ be the open subset on which the map $\phi$ is defined, and is an isomorphism. It is then guaranteed that $U(k) \neq \emptyset$ since we assume $\text{char}(k) = 0$, which implies that $k$ is an infinite field. Hence $X \times \mathbb{P}^a_k$ has a $k$-rational point, which implies that $X$ has a $k$-rational point. \qed

As an effort to make the point of Remark 2.8 clearer, let us illustrate some negative examples. For non-complete (stably) rational varieties, consider simply the affine space $\mathbb{A}^n_k$. Although it is rational, one has the equivalence $[\mathbb{A}^n_k] \equiv 0 \text{ mod } L$ in $K_0(Var_k)$, since $[\mathbb{A}^n_k] = L^n$, for all $n \geq 0$.

Below we give an easy example of singular varieties whose equivalence classes are 1 mod $L$ even though they are not stably rational.

Example 2.10. Let $X \subset \mathbb{P}^n_k$ be a nonrational cone over a hypersurface $Z \subset \mathbb{P}^{n-1}_k$. Note that a cone is always singular, and it always has a $k$-rational point. Therefore after a change of coordinates, we can assume that $x := [0 : \cdots : 0 : 1] \in X$. The projection

$$
\begin{array}{ccc}
\mathbb{P}^n_k \setminus \{x\} & \longrightarrow & \mathbb{P}^{n-1}_k \\
[x_0 : \cdots : x_n] & \longmapsto & [x_0 : \cdots : x_{n-1}]
\end{array}
$$

is a Zariski locally trivial affine fibration. The following is a cartesian diagram:

$$
\begin{array}{ccc}
\mathbb{P}^n_k \setminus \{x\} & \longrightarrow & \mathbb{P}^{n-1}_k \\
\mathbb{P}^n_k & \longrightarrow & \mathbb{P}^{n-1}_k \\
X \setminus \{x\} & \longrightarrow & Z
\end{array}
$$

Thus we get that $X \setminus \{x\} \to Z$ is also a Zariski locally trivial affine fibration, which implies by Remark 2.8 that $[X \setminus \{x\}] = L \cdot [Z]$. Hence $[X] = 1 + L \cdot [Z]$, and we see that $[X] \equiv 1 \text{ modulo } L$ in $K_0(Var_k)$. 

Now let us consider the case of a stably rational variety that is not smooth. In the below examples one sees that such a variety need not have its class in $K_0(\text{Var}_k)$ equivalent to 1 modulo $\mathbb{L}$.

**Example 2.11.** Let $X \subset \mathbb{P}^2_k$ be the projective nodal curve, more precisely the projective hypersurface given by the homogenization of the equation $y^2 = x^3 + x^2$.

It is well known that $X$ is a rational curve with a unique singular point at the origin $O$. Let $\pi : \tilde{X} \to X$ be the blow up of $X$ at $O$, and let $E := \pi^{-1}(O)$. Since the exceptional divisor $E$ consists of two disjoint $\mathbb{C}$-rational points, we have $[E] = 2$ in $K_0(\text{Var}_k)$. On the other hand, the blow up $\tilde{X}$ is the normalization of the curve $X$, and therefore it is a projective smooth rational irreducible variety. Hence Remark 2.8 implies that $[\tilde{X}] \equiv 1 \mod \mathbb{L}$ in $K_0(\text{Var}_k)$. Thus we obtain

$$[X] = [X \setminus O] + [O] = [\tilde{X} \setminus E] + [O] = [\tilde{X}] - [E] + [O] \equiv 0 \mod \mathbb{L}.$$

The following example illustrates that even a normal singular variety, i.e. a variety with singularities at least codimension 2, might have its class in $K_0(\text{Var}_k)$ not equivalent to 1 modulo $\mathbb{L}$.

**Example 2.12.** Let $X$ be a rational normal projective surface over an algebraically closed field $k$ with a unique singular point $x \in X$, and let $\tilde{X} \to X$ be the minimal resolution of singularities of $X$ where $\tilde{X}$ is the blow up of $X$ at $x$. Let moreover the exceptional divisor $E := \pi^{-1}(x)$ be a union of three lines in $\mathbb{P}^2_k$ which form a cycle, i.e. the intersection of all three lines is empty. Note that one has indeed such a surface, and below we will sketch its construction. But let us first show that $[X] \not\equiv 1 \mod \mathbb{L}$ in $K_0(\text{Var}_k)$. Being the blow up of a projective rational variety, $\tilde{X}$ is also rational and projective, since it is smooth we can conclude by Remark 2.8 that $[\tilde{X}] \equiv 1 \mod \mathbb{L}$ in $K_0(\text{Var}_k)$. Also note that

$$[E] = 3[\mathbb{P}^1_k] - 3 = 3\mathbb{L} \equiv 0 \mod \mathbb{L}$$

in $K_0(\text{Var}_k)$. Thus we get

$$[X] = [\tilde{X} \setminus E] + \{x\} = [\tilde{X}] - [E] + 1 \equiv 2 - [E] \mod \mathbb{L} \equiv 2 \mod \mathbb{L}$$

in $K_0(\text{Var}_k)$.

Let us now roughly describe the construction of $X$ of Example 2.12. Let $L_1, L_2, L_3 \subset \mathbb{P}^2_k$ be distinct lines such that the intersection $L_1 \cap L_2 \cap L_3$ is empty, so that they form a cycle. Now we mark four distinct points $\{p_{i1}, p_{i2}, p_{i3}, p_{i4}\} =: P_i$ on each $L_i$, where none of the $p_{ij}$’s is the intersection point of $L_i$ and $L_j$. Write

$$S := P_1 \cup P_2 \cup P_3.$$ 

Let $\tilde{X} := Bl_S \mathbb{P}^2_k \to \mathbb{P}^2_k$ be the blow up of $\mathbb{P}^2_k$ at $S$. We claim that

$$R := L_1 \cup L_2 \cup L_3 \subset \tilde{X}$$

...
is contractible, i.e. there exists a birational morphism
\[ f : \bar{X} \rightarrow X \]
such that \( f(R) = \{ x \} \), where \( x \) is a point of \( X \), and \( f : \bar{X} \setminus R \rightarrow X \setminus \{ x \} \) is an isomorphism. Recall that to be contractible a curve needs to be negative definite which means that the matrix of the irreducible components of the curve is negative definite: in our example the matrix is
\[
(L_i \cdot L_j)_{i,j} = \begin{pmatrix} -3 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & -3 \end{pmatrix}
\]
since \( L_i \cdot L_i = -3 \) for all \( i \), and \( L_i \cdot L_j = 1 \) for all \( i \neq j \). Thus \( R \) is negative definite.

Denote by \( E_{ij} \) the exceptional divisor of \( p_{ij} \) in \( \bar{X} \). Let \( H \) be a general line in \( \mathbb{P}_k^2 \). Consider the divisor \( D = 4H - \sum_{i=1}^{3} \sum_{j=1}^{4} E_{ij} \) on \( \bar{X} \). One can prove that the complete linear system \( |mD| \) is base point free for a sufficiently large integer \( m \gg 0 \), and therefore \( |mD| \) defines a morphism
\[ f : \bar{X} \rightarrow \mathbb{P}_k^N. \]
Note that we have \( D \cdot L_i = 4H \cdot L_i - \sum_{j=1}^{4} E_{ij} \cdot L_i = 0 \) for all \( i \). Besides \( D^2 = 16 - 12 = 4 > 0 \), and \( D \cdot E_{ij} = -E_{ij} \cdot E_{ij} = 1 \). Let \( C \subset Bl_5 \mathbb{P}_k^2 \) be any irreducible curve with \( C \notin \{ L_i, E_{ij} \} \) for all \( i, j \). Then we can identify it with the irreducible curve it comes from in the projective plane \( \mathbb{P}_k^2 \). Hence one gets
\[
D \cdot C = 4\text{deg}(C) - \sum_{i=1}^{3} \sum_{j=1}^{4} \text{mult}_{p_{ij}}(C)
\]
where \( \text{mult}_{p_{ij}}(C) \) denotes the multiplicity of the curve \( C \) at the point \( p_{ij} \). By the definition of multiplicity one has \( \text{mult}_{p_{ij}}(C \cap L_i) \geq \text{mult}_{p_{ij}}(C) \), which yields
\[
D \cdot C = 4\text{deg}(C) - \sum_{i=1}^{3} \sum_{j=1}^{4} \text{mult}_{p_{ij}}(C)
\geq 4\text{deg}(C) - 3\text{deg}(C)
= \text{deg}(C) > 0.
\]
Thus \( f \) contracts precisely \( R \), and the image \( X := f(\bar{X}) \) has the desired properties of Example 2.12.

3. Introduction to The General Conjecture

Let \( k \) be a field of characteristic zero, let \( X \) be a hypersurface over \( k \) of degree \( d \) in \( \mathbb{P}_k^n \), where \( d \leq n \).

Remark 3.1. Let us recall that a smooth hypersurface \( X \subset \mathbb{P}_k^n \) of degree \( d \) is a Fano variety, i.e. its canonical bundle is anti-ample if and only if \( d \leq n \).
3.1. Category of Motives. [2] Chapter 4] Let \( k \) be a field, and let \( \mathcal{V}_k \) denote the category of smooth projective schemes over \( k \). For \( X \in \mathcal{V}_k \), and for a non-negative integer \( d \leq \dim X \), recall that the group of codimension \( d \) cycles \( Z^d(X) \) is the free abelian group generated by the codimension \( d \) subvarieties of \( X \). Then the Chow group of \( X \) of codimension \( d \) is defined as \( CH^d(X) := Z^d(X)/\sim \), where \( \sim \) denotes the rational equivalence. Note that we consider \( CH^d(X) \) to be with \( \mathbb{Q} \)-coefficients, unless otherwise stated.

**Definition 3.2.** Let \( X, Y \in \mathcal{V}_k \), and let \( X_i \) be the connected components of \( X \). The group of correspondences of degree \( r \) from \( X \) to \( Y \) is defined to be

\[
\text{Hom}^r(X, Y) := \bigoplus_i CH^{\dim X_i + r}(X_i \times Y).
\]

We will simply denote by \( \text{Hom}(X, Y) \) the group of the correspondences of degree zero.

Let \( P \in \text{Hom}(X, Y), Q \in \text{Hom}(Y, Z) \). Then the composition is given by

\[
Q \circ P := (p_{13})_i (p_{12}^*(P) \cdot p_{23}^*(Q))
\]

where \( p_{ij} \) are the projection maps from \( X \times Y \times Z \) to the product of the \( i \)-th and \( j \)-th factors, and \( (p_{ij})_i, p_{ij}^* \) are the pull-back and push-forward of \( p_{ij} \) in the Chow groups, respectively. Note also that \( \cdot \) denotes the intersection product in \( CH^{\dim X} (X \times Y \times Z) \). This composition gives \( \text{Hom}(X, Y) \) a \( \mathbb{Q} \)-algebra structure.

**Definition 3.3.** A pair \( (X, P) \) with \( X \in \mathcal{V}_k \) and \( P \in \text{Hom}(X, X) \) a projector, i.e. \( P = P \circ P \), is called a motive.

Motives form a category denoted by \( \mathcal{M}_k \) with morphism groups

\[
\text{Hom}_{\mathcal{M}_k}((X, P), (Y, Q)) := Q \circ \text{Hom}(X, Y) \circ P \subset \text{Hom}(X, Y).
\]

Note that the identity morphism of a motive \( (X, P) \) is the projector \( P \). The sum and the product in \( \mathcal{M}_k \) are defined by disjoint union and product:

\[
(X, P) \oplus (Y, Q) = (X \cup Y, P + Q),
\]

\[
(X, P) \otimes (Y, Q) = (X \times Y, P \times Q).
\]

The motive associated with \( X \in \mathcal{V}_k \) is the motive \( (X, id_X) \) where \( id_X \in \text{Hom}(X, X) \) is the class of the diagonal \( \Delta_X \) in the Chow group. There is a functor

\[
h : \mathcal{V}_k^{op} \rightarrow \mathcal{M}_k
\]

given on objects by \( h(X) := (X, id_X) \), and on morphisms by \( h(\varphi) := [\Gamma_{\varphi}] \), the class of the transpose of the graph of \( \varphi : Y \rightarrow X \). Note that \( \mathbb{Q}(0) := h(\text{Spec}(k)) \) is the identity for the product and it is called the unit motive. Consider the motive \( h(\mathbb{P}^1_k) \). For any \( x \in \mathbb{P}^1_k(\mathbb{P}^1_k) \) we have

\[
|\Delta_{\mathbb{P}^1_k}| = \{x \times \mathbb{P}^1_k\} + \{|\mathbb{P}^1_k \times \{x\}\} \tag{3.1.1}
\]

in \( CH^1(\mathbb{P}^1_k \times \mathbb{P}^1_k) \). Hence

\[
h(\mathbb{P}^1_k) = (\mathbb{P}^1_k,\{x \times \mathbb{P}^1_k\}) \oplus (\mathbb{P}^1_k,\mathbb{P}^1_k \times \{x\}). \tag{3.1.2}
\]

The first summand of the decomposition \( 3.1.2 \) is isomorphic to \( \mathbb{Q}(0) \), and the latter summand

\[
(\mathbb{P}^1_k,\mathbb{P}^1_k \times \{x\}) =: \mathbb{Q}(-1)
\]

is isomorphic to \((-1)\)-twist of \( \mathbb{Q}(0) \). By a \((-1)\)-twist, we mean that for all \( (X, P) \in \mathcal{M}_k \) one has

\[
\text{Hom}_{\mathcal{M}_k}((X, P), \mathbb{Q}(-1)) = id_{\text{Spec}(k)} \circ \text{Hom}^{\dim^{-1}}(X, \text{Spec}(k)) \circ P.
\]
Then we get \( \alpha \) for all \( M, N \) decomposition where both summands are projectors. This yields a decomposition and

dated by the isomorphism classes of motives with the relation

\[ P \]

\[ \kappa \]

This motive \( Q(-1) \) is called the Lefschetz motive. Now let

\[ Q(a) := Q(-1)^{-a} \]

for \( a < 0 \), and let \( X \in V_k \) be connected. Then

\[ Hom_{\mathcal{M}_k}((X, P) \otimes Q(a), (Y, Q) \otimes Q(b)) = Q \circ CH^{dim_X-a+b}(X \times Y) \circ P. \]

Similar to \( h(\mathbb{P}_k^1) \), one can give a decomposition of \( h(x) \) for any \( X \in V_k \). Let \( x \in X \) be a closed point with residue field \( \kappa(x) \). Consider the degree one cycles

\[ \alpha := \text{deg}(\kappa(x)/k)^{-1} \{ [x] \times \text{Spec}(k) \} \in Hom_{\mathcal{M}_k}(h(X), Q(0)) \]

and

\[ \beta := [\text{Spec}(k) \times X] \in Hom_{\mathcal{M}_k}(Q(0), h(X)). \]

Then we get \( \alpha \circ \beta = id_{Q(0)} \). Therefore \( id_X = id_{Q(0)} + (id_X - id_{Q(0)}) \) is an orthogonal decomposition where both summands are projectors. This yields a decomposition

\[ h(X) = Q(0) \oplus \tilde{h}(X) \quad (3.1.3) \]

in \( \mathcal{M}_k \), where \( \tilde{h}(X) \) denotes the motive \( (X, id_X - id_{Q(0)}) \). Note that this decomposition in general depends on \( x \). However the decomposition \( h(\mathbb{P}_k^1) = Q(0) \oplus Q(-1) \) is canonical since the decomposition \( 3.1.1 \) of \( \Delta^1_{\mathbb{P}_k^1} \) in \( CH^1(\mathbb{P}_k^1 \times \mathbb{P}_k^1) \) is independent of the choice of \( x \in \mathbb{P}_k^1(k) \).

Let \( K_0(\mathcal{M}_k) \) denote the Grothendieck ring of the category \( \mathcal{M}_k \), this is the group generated by the isomorphism classes of motives with the relation

\[ [M \oplus N] - [M] - [N] \]

for all \( M, N \in \mathcal{M}_k \). The product is given by

\[ [M] \cdot [N] := [M \otimes N]. \]

Let \( X \in V_k \), and let \( Z \) be a closed smooth subvariety of \( X \). Then one has the following canonical isomorphism of motives

\[ h(Bl_Z X) \oplus h(Z) \cong h(X) \oplus h(E) \]

where \( Bl_Z X \) is the blow up of \( X \) at \( Z \), and \( E \) is the exceptional divisor \([17] \S 9\). Hence

\[ [h(Bl_Z X)] - [h(E)] = [h(X)] - [h(Z)] \]

in \( K_0(\mathcal{M}_k) \). By Remark \([24]\), we know also that \( K_0(Var_k) \) is generated by smooth projective varieties with the blow up relations. This implies that the functor \( h \) induces a ring homomorphism \( \chi : K_0(Var_k) \to K_0(\mathcal{M}_k) \) given by \( \chi([X]) = [h(X)] \) for \( X \in V_k \). Note that one has

\[ \chi(\mathbb{P}^1) = [h(\mathbb{P}^1)] = [Q(0)] = [Q(-1)]. \]

Let us now give the definition for the Chow groups of a motive \( M \in \mathcal{M}_k \) by

\[ CH^i(M) := Hom_{\mathcal{M}_k}(Q(-i), M), \quad CH_i(M) := Hom_{\mathcal{M}_k}(M, Q(-i)) \]

for \( i \geq 0 \) and \( CH^0(M) = 0 = CH_i(M) \) for \( i < 0 \). Note that if \( M = h(X) \) for a \( X \in V_k \) then we get simply \( CH^0(M) = CH^0(X) \) and \( CH_i(M) = CH_i(X) \).

Consider a field extension \( k \subset L \). There is a functor between \( \mathcal{M}_k \) and \( \mathcal{M}_L \) given by base change:

\[ \times_k L : \mathcal{M}_k \longrightarrow \mathcal{M}_L \]

\[ (X, P) \longmapsto (X \times_k L, P \times_k L). \]

In his article \([7]\), using the argumentation of \([6]\), Chatzistamatiou proves the following proposition for the motives of which the degree zero Chow groups are zero:
Proposition 3.4. [7] Proposition 1.2 Let $k$ be a perfect field, and $X \in V_k$ be connected.

1. A motive $M = (X, P)$ can be written as $M \cong N \otimes \mathbb{Q}(-1)$ with some motive $N$ if and only if $CH_0(M \times_k L) = 0$ for some field extension $L$ of the function field $k(X)$ of $X$.

2. There exists an isomorphism $M \cong N \otimes \mathbb{Q}(a)$ with some motive $N$ and $a < 0$ if and only if $CH_i(M \times_k L) = 0$ for all $i < -a$ and all field extensions $k \subset L$.

We now explain that the following theorem, which is due to A. Roitman [21], implies for $X$ a hypersurface of degree $d \leq n$ that one gets $CH_0(\tilde{h}(X)) = 0$.

Theorem 3.5. [21] Theorem 2] Let $k$ be an algebraically closed field, let $X \subset \mathbb{P}_k^n$ be a hypersurface of degree $d$ with $d \leq n$. Then the subgroup of $CH_0(X)$ of degree zero cycles over $\mathbb{Z}$, denoted by $CH_0^0(X)$ is zero.

Hence the degree map

$$CH_0(X) \otimes \mathbb{Q} \rightarrow \mathbb{Q}$$

$$\sum_i n_i [Z_i] \mapsto \sum_i n_i$$

is an isomorphism.

Note that the degree map given above is still an isomorphism over an algebraically non-closed field (cf. [5, Lecture 1, Appendix, Lemma 3]). Let $k$ be any field. Recall that

$$CH_0(X_E) \otimes \mathbb{Q} = \varinjlim CH_0(X_E) \otimes \mathbb{Q},$$

where the direct limit is taken over all finite extensions of $k$, and where $\overline{k}$ denotes the algebraic closure of $k$. Let $\pi : X_E \rightarrow X$ be the base change map of a hypersurface $X$ as given in Theorem 3.5 over a finite extension $E$ of $k$. Then one has the pull-back $\pi^* : CH_0(X) \rightarrow CH_0(X_E)$ and the push-forward homomorphism $\pi_* : CH_0(X_E) \rightarrow CH_0(X)$, and one gets that $\pi_* \circ \pi^*$ is multiplication by $[E : k]$ since $E/k$ is finite. Therefore after tensoring with $\mathbb{Q}$, $\pi_* \circ \pi^*$ becomes injective. By passing to the direct limits, this gives an injection $CH_0(X) \otimes \mathbb{Q} \rightarrow CH_0(X_E) \otimes \mathbb{Q}$.

Therefore one gets that the degree map $CH_0(X) \otimes \mathbb{Q} \rightarrow \mathbb{Q}$ is an isomorphism, since the degree map over the algebraic closure is an isomorphism by Theorem 3.5.

Now let $X \subset \mathbb{P}_k^n$ be a hypersurface of degree $d \leq n$ over a perfect field $k$, not necessarily algebraically closed. Then one has

$$\mathbb{Q} \cong CH_0(X) = Hom_{\mathcal{M}_k}(h(X), \mathbb{Q}(0))$$

$$= Hom_{\mathcal{M}_k}(\mathbb{Q}(0) \oplus \tilde{h}(X), \mathbb{Q}(0))$$

$$= \mathbb{Q} \oplus CH_0(\tilde{h}(X)),$$

hence $CH_0(\tilde{h}(X)) = 0$. Therefore applying Proposition 3.4 to the motive $\tilde{h}(X)$ of Equation 3.1.3 one gets the following decomposition

$$h(X) = \mathbb{Q}(0) \oplus N \otimes \mathbb{Q}(-1)$$

with some motive $N \in \mathcal{M}_k$.

Remark 3.6. If one considers the category of motives over $\mathbb{Z}$, i.e. if the correspondences are Chow groups with integer coefficients, then it is not known if having $CH_0(\tilde{h}(X)) = 0$ implies a decomposition of the form

$$h(X) = \mathbb{Z}(0) \oplus N \otimes \mathbb{Z}(-1)$$

with some integral motive $N$. 
It is also unknown if one in general gets a corresponding decomposition to \(3.1.5\) of the class of \(X\) in the Grothendieck ring of varieties, note that this is again an integral question. Our main concern will be to search for such decomposition of the classes of hypersurfaces of degree \(d \leq n\) in \(K_0(Var_k)\). Let us formally ask our main question which is due to H. Esnault. Let \([X]\) denote the equivalence class of \(X\) in \(K_0(Var_k)\) from now on, unless otherwise stated.

**Question 3.7.** For a projective hypersurface \(X \subset \mathbb{P}_k^n\) of degree \(d \leq n\), does one have \(X(k) \neq \emptyset\) if and only if \([X] \equiv 1 \mod L\) in \(K_0(Var_k)\)?

Now clearly the spirit of Question 3.7 varies with the degree of \(X\) and also with the base field \(k\). Recall that for a field \(k\) to be \(C_1\) means that any homogeneous polynomial \(F(x_0, \ldots, x_n) \in k[x_0, \ldots, x_n]\) of degree \(d \leq n\) has a nontrivial solution over \(k\). Thus over a \(C_1\) field Question 3.7 takes the following form

**Question 3.8.** Let \(k\) be a \(C_1\) field, and let \(X\) be as in Question 3.7. Is \([X] \equiv 1 \mod L\)?

In other words, having a positive answer to Question 3.7 over a \(C_1\) field \(k\) implies that for every projective hypersurface \(X \subset \mathbb{P}_k^n\) of degree \(d \leq n\), one has \([X] \equiv 1 \mod L\) in \(K_0(Var_k)\).

In the remaining sections we study some particular cases which give a positive answer to Question 3.7 using elementary geometric methods. We show that for the union of \(d\) hyperplanes in \(\mathbb{P}_k^n\) over any field \(k\) with \(d \leq n\), and for quadrics over a field of characteristic zero the answer is affirmative. In the case of a nonsingular cubic, the setting is already far more complicated: recall that a variety \(X\) of dimension \(m\) over a field \(k\) is called unirational when there is a rational map \(\varphi: \mathbb{P}_k^m \rightarrow X\) such that \(\varphi(\mathbb{P}_k^m)\) is dense in \(X\) and the function field \(k(\mathbb{P}_k^m)\) is a separable extension of \(k(X)\). It is well known that all smooth cubic hypersurfaces over an algebraically closed field are unirational. More generally, for any cubic hypersurface which is not a cone over a smaller dimensional cubic, J. Kollár [13, Theorem 1.2] proved that having a \(k\)-rational point is equivalent to being unirational over \(k\). The nonsingular cubic hypersurfaces in \(\mathbb{P}_k^4\), which are in particular unirational, are proven by C. H. Clemens and P. Griffiths to be nonrational varieties [8, Theorem 13.12]. However, it is not known whether it is stably birational or not, therefore it would be interesting to study this particular instance. Unfortunately we are not able to answer Question 3.7 for this difficult case. In the higher dimensions questions of the rationality and stable rationality of nonsingular cubic hypersurfaces are open in general.
4. The Class of a Quadric in $K_0(\text{Var}_k)$

In this section we will consider the quadrics, i.e. degree two hypersurfaces, since these hypersurfaces would be the next natural example after hyperplanes to look for the answer to Question 3.7.

**Theorem 4.1.** ([9, Theorem 1.11]) Let $k$ be a field of characteristic zero. Let $X \subset \mathbb{P}_k^n$ be an irreducible quadric hypersurface. Then $X$ is rational if and only if it has a smooth $k$-rational point.

By Remark 2.8 for a smooth quadric $X$, we have $X$ stably rational if and only if $[X] \equiv 1 \text{ modulo } \mathbb{L}$ over a field of characteristic zero. This together with Theorem 4.1 and Proposition 2.9 give the positive answer to Question 3.7 for a smooth quadric $X$, one has $X(k) \neq \emptyset$ if and only if $[X] \equiv 1 \text{ modulo } \mathbb{L}$. We like to give a proof to show that for a not necessarily smooth quadric hypersurface over a characteristic zero field, the answer to the Question 3.7 is positive. We will use the following well known lemma from linear algebra, which is about the nondegenerate quadratic forms with a nontrivial solution over the base field, first for writing down such an affine fibration for a smooth quadric, and then later in Section 7 for proving Theorem 7.1.

**Lemma 4.2.** Let $k$ be a field with char$(k) = 0$, and let $X = V(q)$ be a smooth quadric hypersurface in $\mathbb{P}_k^n$. Let $x \in X(k)$ be a point. Then we can choose coordinates in $\mathbb{P}_k^n$ such that

$$x_i(x) = 0, \text{ for all } i \neq 1,$$

and

$$q(x_0, \ldots, x_n) = x_0x_1 + q'(x_2, \ldots, x_n),$$

where $q' \in k[x_2, \ldots, x_n]$ is a quadratic form.

**Remark 4.3.** With the new coordinates from Lemma 4.2, the point $x$ corresponds to the point $[0 : 1 : 0 : \cdots : 0]$. This will be a convenient choice for Proposition 7.1.

**Theorem 4.4.** Let $k$ be a field with char$(k) = 0$. Let $X \subset \mathbb{P}_k^n$ be a hypersurface of degree 2, then $[X] \equiv 1 \text{ mod } \mathbb{L}$ in $K_0(\text{Var}_k)$ if and only if $X(k) \neq \emptyset$.

**Proof.** In the case that $X$ is singular, it always has a $k$-rational point. If $[X] \equiv 1 \text{ mod } \mathbb{L}$ for $X$ smooth, then we know by Proposition 2.9 that $X(k) \neq \emptyset$.

Now let us consider a quadric $X = V(q)$ where $q$ is a quadratic form. First we consider the case $X$ is smooth, that is $q$ is nondegenerate. We make the change of coordinates as given in Lemma 4.2 and denote by $x_0, \ldots, x_n$ these new coordinates. Let $U_0 := \{[x_0 : \cdots : x_n] \in \mathbb{P}_k^n : x_0 \neq 0\} \subset \mathbb{P}_k^n$, then

$$U_0 \cap X = V\left(\frac{x_1}{x_0} + q'\left(\frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right)\right) \cong \mathbb{A}_k^{n-1}.$$

Let $Y := V(q') \subset \mathbb{P}_k^{n-2}$. Now let us consider the Zariski locally trivial affine fibration

$$p : \mathbb{P}_k^n \setminus (U_0 \cup \{[0 : 1 : 0 : \cdots : 0]\}) \rightarrow \mathbb{P}_k^{n-2},$$

$$[x_0 : \cdots : x_n] \mapsto [x_2 : \cdots : x_n].$$

Then the following is a cartesian diagram:

$$\begin{array}{ccc}
\mathbb{P}_k^n \setminus (U_0 \cup \{[0 : 1 : 0 : \cdots : 0]\}) & \rightarrow & \mathbb{P}_k^{n-2} \\
\downarrow \downarrow & & \downarrow \\
X \setminus (X \cap U_0) \setminus \{[0 : 1 : 0 : \cdots : 0]\} & \rightarrow & Y
\end{array}$$
Hence
\[ p|_X : X \setminus ((X \cap U_0) \cup \{[0 : 1 : 0 : \cdots : 0]\}) \rightarrow Y \]

is also Zariski locally trivial with fibres isomorphic to \( \mathbb{A}^1_k \). Then by Remark 2.3 we have
\[
[X] = [X \setminus ((X \cap U_0) \cup \{[0 : 1 : 0 : \cdots : 0]\})] + ([X \cap U_0] \cup \{[0 : 1 : 0 : \cdots : 0]\})
\]
\[
= L \cdot [Y] + [X \cap U_0] + 1
\]
\[
= 1 + L \cdot [Y] + \mathbb{L}^{n-1}
\]

and \([X] \equiv 1 \mod \mathbb{L}\) in \( K_0(Var_k) \).

Now let us consider the case where \( X \) is singular. Since one can diagonalize every quadratic form, after a change of coordinates we can write \( q(x_0, \ldots, x_n) = a_0x_0^2 + \cdots + a_rx_r^2 \), with \( r < n \) since \( q \) is degenerate. Thus we have
\[
P := \{[0 : \cdots : 0 : x_{r+1} : \cdots : x_n] | [x_{r+1} : \cdots : x_n] \in \mathbb{P}^{n-r-1}_k \} \subset X = V(q)
\]

Let \( Y := V(a_0x_0^2 + \cdots + a_rx_r^2) \subset \mathbb{P}^r_k \). Consider the projection
\[
p : \mathbb{P}^n_k \setminus P \rightarrow \mathbb{P}^r_k
\]
\[
[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_r].
\]

Observe that \( x \in X \setminus P \) if and only if \( p(x) \in Y \): indeed \( p(x) \in Y \) if and only if \( x \notin P \) and \( q(x) = 0 \). Therefore the following is a cartesian diagram
\[
\begin{array}{ccc}
\mathbb{P}^n_k \setminus P & \longrightarrow & \mathbb{P}^r_k \\
\downarrow & & \downarrow \\
X \setminus P & \longrightarrow & Y
\end{array}
\]

and the projection \( p \) is Zariski locally trivial, hence we see that the map \( X \setminus P \) is also Zariski locally trivial fibration with fibres isomorphic to \( \mathbb{A}^{n-r}_k \), and by Remark 2.3 we obtain
\[
[X] = [X \setminus P] + [P]
\]
\[
= L^{n-r} \cdot [Y] + [\mathbb{P}^{n-r-1}_k]
\]
\[
= 1 + L + \cdots + L^{n-r-1} + L^{n-r} \cdot [Y].
\]

Thus \([X] \equiv 1 \mod \mathbb{L}\) in \( K_0(Var_k) \). \(
\)

**Corollary 4.5.** For quadric hypersurfaces Question 3.7 has a positive answer.

**5. The Class of a Union of Hyperplanes**

For a hypersurface of degree greater than two, the simplest case to consider is the union of hyperplanes. Such a variety always has \( k \)-rational points. Therefore in this case Question 3.7 asks if the class \([X]\) is always equivalent to 1 modulo \( \mathbb{L} \).

**Theorem 5.1.** Let \( k \) be a field, and let \( X := V(h_1 \cdots h_d) \subset \mathbb{P}^n_k \) be a hypersurface of degree \( d \leq n \) where \( h_i \in k[x_0, \ldots, x_n] \) are homogeneous polynomials of degree 1, for \( 1 \leq i \leq d \), then \([X] \equiv 1 \mod \mathbb{L}\) in \( K_0(Var_k) \).

**Proof.** Let \( Y := \bigcap_{i=1}^d V(h_i) \), let \( r \) be the codimension of \( Y \) in \( \mathbb{P}^n_k \). Since \( Y \) is an intersection of hyperplanes, it is isomorphic to the projective space \( \mathbb{P}^{n-r}_k \). Note that \( n - r \geq 0 \) since we
Indeed, after a coordinate change we get
\[ Y = \bigcap_{i=1}^{d} V(h_i) = \{ [0 : \cdots : 0 : x_r : \cdots : x_n] \mid [x_r : \cdots : x_n] \in \mathbb{P}^{n-r}_k \} \cong \mathbb{P}^{n-r}_k. \]

Since \( h_i(x) = 0 \) for all \( x \in Y \) and for all \( i = 1, \ldots, d \), we find that \( h_i \in k[x_0, \ldots, x_{r-1}] \) for all \( i = 1, \ldots, d \). Thus we get \( f \in k[x_0, \ldots, x_{r-1}] \). Let \( Z := V(f) \subset \mathbb{P}^{r-1}_k \), and let us consider the Zariski locally trivial affine fibration
\[ p : \mathbb{P}^{n}_k \setminus Y \rightarrow \mathbb{P}^{r-1}_k \]
\[ [x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_{r-1}]. \]

Note that then we have \( x \in X \setminus Y \) if and only if \( p(x) \in Z \). Hence the following is a cartesian diagram:
\[
\begin{array}{ccc}
\mathbb{P}^{n}_k \setminus Y & \xrightarrow{p} & \mathbb{P}^{r-1}_k \\
\uparrow & & \uparrow \\
X \setminus Y & \xrightarrow{} & Z
\end{array}
\]

Thus
\[ p|_X : X \setminus Y \rightarrow Z \]
is also Zariski locally trivial with fibres isomorphic to \( \mathbb{A}^{n-r+1}_k \). Thus we get
\[ [X \setminus Y] = L^{n-r+1} \cdot [Z]. \]

Hence by Remark 2.3 we obtain
\[ [X] = [X \setminus Y] + [Y] = L^{n-r+1} \cdot [Z] + [\mathbb{P}^{n-r}_k] = 1 + L + \cdots + L^{n-r} + L^{n-r+1} \cdot [Z] \]
and \([X] \equiv 1 \mod L\) in \( K_0(Var_k) \).

**Corollary 5.2.** Let \( X \) be as in Theorem 5.1, then \([X] \equiv 1 \mod L\) in \( K_0(Var_k) \) if and only if \( X(k) \neq \emptyset \).

**Proof.** This follows from the fact that any point of the form
\[ [0 : \cdots : 0 : x_r : \cdots : x_n] \in \mathbb{P}^{n}_k \]
is a point of \( X \). Hence \( X \) has always \( k \)-rational points. \( \square \)

After checking union of hyperplanes, next example we consider will be the hypersurfaces which become a union of hyperplanes after a finite Galois base change.

**Theorem 5.3.** Let \( L \) be a finite Galois extension of \( k \), let \( X \) be a hypersurface in \( \mathbb{P}^{n}_k \) of degree \( d \leq n \) such that \( X \times_k L \) is a union of \( d \) hyperplanes over \( L \), that is \( X \times_k L = V(h_1 \cdots h_d) \) where \( h_i \in L[x_0, \ldots, x_n] \) are homogeneous of degree 1 for \( 1 \leq i \leq d \). Then \([X] \equiv 1 \mod L\) in \( K_0(Var_k) \), and \( X(k) \neq \emptyset \).

**Proof.** Let \( Y := \bigcap_{i=1}^{d} V(h_i) \) in \( \mathbb{P}^{n}_k \), and let \( G := Gal(L/k) \). Then \( G \) acts on \( X \times_k L \), and also on \( Y \) since one has \( g \cdot y \in Y \) for all \( y \in Y \). Therefore we see that \( Y/G \subset X \) is a closed immersion.
Claim. $Y/G$ is a linear subspace of $\mathbb{P}_k^n$.

In order to prove this claim, let us consider the $L$-vector subspace

$$V := L < h_1, \ldots, h_d > \subset \mathbb{L}^{n+1}.$$ 

Since $G$ acts on $V(h_1 \cdots h_d) = X \times_k L$, it also acts on $V$. Therefore one can use Lemma 5.4 below to deduce that there exists homogeneous polynomials \( \{ h'_i \}_{i=1}^d \) in $k[x_0, \ldots, x_n]$ of degree 1 such that $L < h'_1, \ldots, h'_d > \subset V$, where $r := \dim L$. Hence, $Y/G = \bigcap_{i=1}^r V(h'_i)$ and $Y = (Y/G) \times_k L$. Now that $Y/G$ is a linear subspace of $\mathbb{P}_k^n$ of codimension $r$, we can choose the coordinates so that we have

$$Y/G = \{ [0 : \cdots : 0 : x_r : \cdots : x_n] \mid [x_r : \cdots : x_n] \in \mathbb{P}_k^{n-r} \}.$$

Consider the projection map

$$p : \mathbb{P}_k^n \setminus (Y/G) \longrightarrow \mathbb{P}_k^{r-1}$$

$$[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_{r-1}].$$

Observe that

$$h_i \in L[x_0, \ldots, x_{r-1}]$$

and

$$f := \prod_{i=1}^d h_i \in k[x_0, \ldots, x_{r-1}]$$

in a similar manner to the polynomials $h_i$ of Theorem 5.1 (see proof of Theorem 5.1). Let $Z := V(f) \subset \mathbb{P}_k^{r-1}$. With this setting we obtain that $x \in X \setminus (Y/G)$ if and only if $p(x) \in Z$. Therefore the following diagram

$$\begin{array}{ccc}
\mathbb{P}_k^n \setminus (Y/G) & \xrightarrow{p} & \mathbb{P}_k^{r-1} \\
\downarrow & & \downarrow \\
X \setminus (Y/G) & \xrightarrow{} & Z
\end{array}$$

is cartesian, $p|_X : X \setminus (Y/G) \longrightarrow Z$ is Zariski locally trivial affine fibration with fibres isomorphic to $\mathbb{A}_k^{n-r+1}$. Thus we are now again in the same situation as in Theorem 5.1, i.e., $[X] \equiv 1 \mod L$ in $K_0(Var_k)$. Moreover $X(k) \neq \emptyset$ since $\emptyset \neq (Y/G)(k) \subset X$ is a closed immersion. \qed

Lemma 5.4 (Hilbert 90). Let $k$ be a field, and let $L$ be a finite Galois extension of $k$. We denote by $G := \text{Gal}(L/k)$ the Galois group of the extension $L/k$. Let $V \subset L^n$ be a $G$-invariant $L$-vector subspace, i.e., $g(V) \subset V$ for all $g \in G$, then there exists a $k$-vector space $V' \subset k^n$ such that $V' \otimes_k L = V$ where $V' \otimes_k L$ is considered as an $L$-vector subspace of $L^n$ by the extension of the scalars.

Proof. Let $r := \dim L$, and let $\{ t_i \}_{i=1}^r$ be an $L$-basis of $V$. Since $g(V) \subset V$ for all $g \in G$, $G$ acts on $V$, as below

$$g \cdot t_i = \sum_{j=1}^r m_{ji}(g)t_j$$

where \( m_{ij} \) is the matrix of $g$ with respect to the basis $\{ t_i \}_{i=1}^r$. Then $V' := \{ [v_1, \ldots, v_r] \mid v_i \in k \}$ is a $k$-vector space such that $V' \otimes_k L = V$. \qed
where \( m_{ij}(g) \in L \) not all simultaneously zero, for all \( t_i \). We need to prove that there exists an \( L \)-basis \( \{ t'_i \}_{i=1}^r \) of \( V \) such that \( g \cdot t'_i = t'_i \) for all \( i \), because this means that \( \{ t'_i \}_{i=1}^r \) consist a \( k \)-basis for a vector space \( V' \) such that \( V' \otimes_k L = V \). For this aim, let us define the map

\[
\alpha : \ G \longrightarrow GL_r(L)\]

\[
g \longrightarrow \alpha_g := (m_{ji}(g))_{i,j}^{-1}.
\]

Recall that a 1-cocycle with values in \( GL(V) \) is a map \( \rho : G \longrightarrow GL(V) \) such that \( \rho(g_1g_2) = \rho(g_1)\rho(g_2) \) for all \( g_1, g_2 \in G \).

Claim: \( \alpha : G \longrightarrow GL_r(L) \) is a 1-cocycle. Indeed, for all \( g_1, g_2 \in G \) we have

\[
g_1g_2 \cdot t_i = g_1 \cdot (g_2 \cdot t_i) = \sum_{j=1}^r \sum_{k=1}^r g_1(m_{ki}(g_2))m_{jk}(g_1)t_j
\]

Thus we have

\[
\alpha_{g_1g_2} = ((m_{ji}(g_1g_2))_{i,j})^{-1} = [(g_1(m_{ji}(g_2))(m_{ji}(g_1)))_{i,j}]^{-1} = (m_{ji}(g_1))_{i,j}^{-1}(g_1(m_{ji}(g_2))_{i,j})^{-1} = \alpha_{g_1}(\alpha_{g_2}).
\]

By the Hilbert 90 theorem [23, Chapter X, Proposition 3], we know that

\[
H^1(G, GL_r(L)) = \{1\},
\]

i.e., every 1-cocycle with values in \( GL_r(L) \) is cohomologous to the trivial 1-cocycle that maps every element of \( G \) to \( id \in GL_r(L) \). Hence \( \alpha \) is cohomologous to 1, i.e. there exists a \( B \in GL_r(L) \) such that \( \alpha_g = Bg(B^{-1}) \) for all \( g \in G \). Now the following claim will complete the proof of the lemma:

Claim: \( \{ B^{-1} \cdot t_i \}_{i=1}^r \) is an \( L \)-basis of \( V \) such that \( g(B^{-1} \cdot t_i) = B^{-1} \cdot t_i \) for all \( i = 1, \ldots, r \). Indeed, we have

\[
g(B^{-1} \cdot t_i) = g(B^{-1})g \cdot t_i = g(B)^{-1} \alpha_g^{-1} \cdot t_i = g(B)^{-1} g(B) \cdot B^{-1} \cdot t_i = B^{-1} \cdot t_i.
\]

\( \square \)

**Corollary 5.5.** For hypersurfaces of the type given in Theorem 5.4 and 5.8, [27] Question 3.7 has a positive answer.

6. Cubic Hypersurfaces

In this section we will consider the case of degree three hypersurfaces for Question 3.7. Therefore we are only interested in the hypersurfaces that live in at least three dimensional projective space. Note that over an algebraically closed field \( k \), it is known that a smooth cubic surface is rational [18, Chapter IV, Theorem 24.1]. For a smooth projective hypersurface \( X \) over a field of characteristic zero, (stable) rationality implies that the class \([X]\) is equivalent to 1 modulo \( L \) in \( K_0(Var_k) \), by Proposition 2.7. On the other hand there are smooth cubic hypersurfaces that are not rational, like smooth cubic threefolds which are proven to be non-rational by Clemens and Griffiths in their paper [8]. It is still unknown
whether smooth cubic threefolds are stably rational or not. If an irreducible projective cubic hypersurface that is not a cone over a cubic of lower dimension has a $k$-rational singular point, then it is rational [9, Chapter 1, Section 5, Example 1.28]. However in the singular case rational varieties do not necessarily have class 1 modulo $\mathbb{L}$, but it is possible to show that a singular cubic hypersurface with a rational singular point, actually has this property. This is what we are going to prove next.

**Theorem 6.1.** Let $k$ be a $C1$ field, and let $X$ be a hypersurface of degree 3 in $\mathbb{P}_k^n$. Denote by $X_{sing}(k)$ the set of $k$-rational points of the singular locus of $X$. If $X_{sing}(k) \neq \emptyset$, then $[X] \equiv 1 \mod \mathbb{L}$ in $K_0(Var_k)$.

**Proof.** Let $x \in X_{sing}(k)$. After a change of coordinates, we get $x = [0 : \cdots : 0 : 1]$. Let $X = V(f)$. Since $x = [0 : \cdots : 0 : 1] \in X(k)$, $f$ has the following form with these new coordinates

$$f(x_0, \ldots, x_n) = x_n^2f_1(x_0, \ldots, x_{n-1}) + x_nf_2(x_0, \ldots, x_{n-1}) + f_3(x_0, \ldots, x_{n-1})$$

where $f_i$ are homogeneous polynomials of degree $i$, $i = 1, 2, 3$. Moreover $x$ is a singular point of $X$, i.e.

$$\frac{\partial f}{\partial x_i}|_x = 0, \text{ for all } 0 \leq i \leq n.$$

Let us note that we have in particular

$$\frac{\partial f_1}{\partial x_i}|_x = 0, \text{ for all } 0 \leq i \leq n - 1.$$

Hence we get $f_1 = 0$. Now let us consider the following Zariski locally trivial affine fibration:

$$\pi : \mathbb{P}_k^n \setminus \{x\} \longrightarrow \mathbb{P}_k^{n-1}$$

$$[x_0 : \cdots : x_n] \longmapsto [x_0 : \cdots : x_{n-1}].$$

Note that

$$\pi^{-1}(p) = \{[p_0 : \cdots : p_{n-1} : \gamma] \mid \gamma \in k\} \cong \mathbb{A}_k^1$$

and that $\pi^{-1}(p) \cup \{x\} \cong \mathbb{P}_k^1$ for all $p = [p_0 : \cdots : p_{n-1}] \in \mathbb{P}_k^{n-1}$. Let us denote by $\pi_X$ the restriction of $\pi$ to $X$. Therefore we get that

$$[X] = [X \setminus \{x\}] + [[x]] = 1 + [\pi^{-1}_X(\mathbb{P}_k^{n-1})] \quad (6.0.1)$$

by Remark 22. For any $p \in \mathbb{P}_k^{n-1}$ we have one of the following possibilities for the fibre of $\pi_X$ at $p$

$$\pi_X^{-1}(p) = \begin{cases} y \in X \setminus \{x\} \\ \pi^{-1}(p) \\ \emptyset \end{cases}$$

First of all, let us observe that

$$Y := \{p \in \mathbb{P}_k^{n-1} \mid \pi_X^{-1}(p) = y \in X \setminus \{x\}\} \cong \pi_X^{-1}(Y)$$

$$= \{p \in \mathbb{P}_k^{n-1} \mid \pi_X^{-1}(p) = [p_0 : \cdots : p_{n-1} : -f_3(p)/f_2(p)]\}$$

$$= \{p \in \mathbb{P}_k^{n-1} \mid f_2(p) \neq 0\}$$

$$= \mathbb{P}_k^{n-1} \setminus V(f_2).$$

Note that $k$ being a $C1$ field assures that

$$V(f_2)(k) \neq \emptyset,$$
Let us denote by $Z$ the set of points of which the pre-images under $\pi$ is completely contained in $X$:

$$Z := \{ p \in \mathbb{P}^{n-1}_k \mid \pi^{-1}(p) \subset X \setminus \{x\} \}$$

Hence the diagram

$$\begin{array}{ccc}
\mathbb{P}^{n-1}_k \setminus \{x\} & \xrightarrow{\pi} & \mathbb{P}^{n-1}_k \\
\downarrow \pi^{-1}(Z) & & \downarrow \pi_X \\
\pi^{-1}(Z) & \xrightarrow{\pi_X} & Z
\end{array}$$

is cartesian, and $\pi_X^{-1}(Z) \to Z$ is a Zariski locally trivial $\mathbb{A}^1_k$-fibration. This yields by Remark 2.3 that

$$[\pi_X^{-1}(Z)] = [\pi^{-1}(Z)] = \mathbb{L} \cdot [V(f_2, f_3)] \quad (6.0.3)$$

Thus we obtain

$$[X] = 1 + [\pi_X^{-1}(\mathbb{P}^{n-1}_k)] \quad \text{by Equation 6.0.1}$$

$$= 1 + [\pi_X^{-1}(Y) \cup \pi_X^{-1}(Z)]$$

$$= 1 + [Y] + \mathbb{L} \cdot [V(f_2, f_3)] \quad \text{by Equation 6.0.3}$$

$$= 1 \mod \mathbb{L} \quad \text{by Equation 6.0.2}.\]$$

in $K_0(\text{Var}_k)$.

Corollary 6.2. For cubic hypersurfaces over $k$ with a singular $k$-rational point, Question 3.7 is positively answered.

7. Union of Two Quadric Hypersurfaces

In this section we consider a particular example of quartic hypersurfaces, namely ones in $\mathbb{P}^n_k$ for any $n \geq 4$, which consist of the union of two quadric hypersurfaces, one of which is smooth. We need to assume that $k$ is algebraically closed to conclude with the proof.

We will compute the class in $K_0(\text{Var}_k)$ modulo the class $\mathbb{L}$.

Theorem 7.1. Let $k$ be an algebraically closed field of characteristic zero, let $X$ be a union of two quadric hypersurfaces $Q_1, Q_2 \subset \mathbb{P}^n_k$, where $Q_1$ is smooth and $n \geq 4$. Then 

$$[X] \equiv 1 \mod \mathbb{L} \quad \text{in } K_0(\text{Var}_k).$$

Proof. If $Q_1 = Q_2$, then $[Q_1 \cup Q_2] = [Q_1 \cup Q_1] = [Q_1]$. Hence it reduces to the case of Theorem 4.4 and we are done. For distinct $Q_1, Q_2$ we have

$$[X] = [Q_1 \cup Q_2] = [Q_1] + [Q_2] - [Q_1 \cap Q_2]$$

in $K_0(\text{Var}_k)$. Since $k$ is algebraically closed, both $Q_1$ and $Q_2$ have $k$-rational points. Hence, by Theorem 4.4 we get

$$[Q_i] \equiv 1 \mod \mathbb{L}, \quad i = 1,2, \quad (7.0.1)$$

in $K_0(\text{Var}_k)$.
and therefore \([X] \equiv 1 \text{ mod } \mathbb{L}\) if and only if \([Q_1 \cap Q_2] \equiv 1 \text{ mod } K_0(Var_k)\). Let

\[Q_1 := V(y_i), i = 1, 2.\]

Since we are over an algebraically closed field, the intersection \(Q_1 \cap Q_2\) also has a \(k\)-rational point \(x\). Since \(Q_1\) is smooth, we can make a change of coordinates in order to get \(q_1(x_0, \ldots, x_n) = x_0x_1 - h(x_2, \ldots, x_n)\). To do this, we apply Lemma 4.2 to \(q_1\) and \(x\). Hence we get coordinates in which \(q_1\) is of the desired form, and the point \(x\) becomes the point \([0 : 1 : 0 : \cdots : 0]\). Now since \([0 : 1 : 0 : \cdots : 0]\) \(\in Q_2\), the monomial \(x_1^2\) does not appear in \(q_2\). Thus we can write

\[q_2(x_0, \ldots, x_n) = x_0L_0(x_0, x_2, \ldots, x_n) + x_1L_1(x_0, x_2, \ldots, x_n) + R(x_2, \ldots, x_n)\]

where \(L_0, L_1, R\) are homogeneous polynomials, \(L_0, L_1\) are linear and \(R\) is of degree 2. Let \(U_0 := \{[x_0 : \cdots : x_n] \mid x_0 \neq 0\}\). We will calculate the class of the intersection of \(Q_1 \cap Q_2\) with \(U_0\). Let \(Q_1^{\{x_0 \neq 0\}} := Q_1 \cap U_0 = V(q_1|_{U_0})\). We have

\[
q_1|_{U_0}(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) = \frac{x_1}{x_0} - h(\frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0})
\]

and \(Q_1^{\{x_0 \neq 0\}} \cong \mathbb{A}^{n-1}_k\) where the isomorphism is given by

\[
\varphi: \quad Q_1^{\{x_0 \neq 0\}} \longrightarrow \mathbb{A}^{n-1}_k
\]

\[
\begin{pmatrix}
\frac{x_1}{x_0} \\
\ldots \\
\frac{x_n}{x_0}
\end{pmatrix}
\]

\[
\longrightarrow
\begin{pmatrix}
\frac{x_2}{x_0} \\
\ldots \\
\frac{x_n}{x_0}
\end{pmatrix}
\]

Let us now consider the intersection \(Q_1^{\{x_0 \neq 0\}} \cap Q_2\). Via the isomorphism \(\varphi\), it is defined by the following polynomial:

\[
g(\frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}) := L_0(1, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}) + h(\frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0})L_1(1, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}) + R(\frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0})
\]

i.e.,

\[
\varphi(Q_1^{\{x_0 \neq 0\}} \cap Q_2) = V(g) \subset \mathbb{A}^{n-1}_k.
\]

We embed \(\mathbb{A}^{n-1}_k \subset \mathbb{P}^{n-1}_k\) as \(\{y_1 \neq 0\}\) with homogeneous coordinates \(\{y_1, \ldots, y_n\}\) for \(\mathbb{P}^{n-1}_k\). Then the closure

\[
Y := \overline{\varphi(Q_1^{\{x_0 \neq 0\}} \cap Q_2)} \subset \mathbb{P}^{n-1}_k
\]

is defined by the homogenization of \(g\):

\[
\overline{g}(y_1, \ldots, y_n) = y_1^2L_0(y_1, \ldots, y_n) + h(y_2, \ldots, y_n)L_1(y_1, \ldots, y_n) + y_1R(y_2, \ldots, y_n).
\]

Thus

\[
\varphi(Q_1^{\{x_0 \neq 0\}} \cap Q_2) = V(g) = Y \setminus (Y \cap V(y_1)),
\]

and we get

\[
[\varphi(Q_1^{\{x_0 \neq 0\}} \cap Q_2)] = [Y] - [Y \cap V(y_1)]. \tag{7.0.2}
\]

The intersection \(Y \cap V(y_1)\) is the vanishing locus of

\[
h(y_2, \ldots, y_n)L_1(0, y_2, \ldots, y_n).
\]

Hence

\[
[Y \cap V(y_1)] = \left[V(hL_1) \cap V(y_1)\right] - \left[V(h) \cap V(y_1)\right] + \left[V(L_1) \cap V(y_1)\right] - \left[V(h) \cap V(L_1) \cap V(y_1)\right]
\]
Consider the projection map \( \pi : \mathbb{P}^{n-1}_k \to \mathbb{P}^{n-2}_k \) where \( p := [1 : 0 : \cdots : 0] \). Now this projection map induces an isomorphism

\[
(Q_1^{x_0=0} \cap Q_2) \setminus ((V(L_1) \cap V(x_0)) \cup \{p\}) \cong (V(h) \cap V(x_0)) \setminus ((V(h) \cap V(L_1)) \cap V(x_0))
\]

and thus

\[
[Q_1^{x_0=0} \cap Q_2 \setminus ((V(L_1) \cap V(x_0)) \cup \{p\})] = [V(h) \cap V(x_0)] - [V(h) \cap V(L_1) \cap V(x_0)]
\]  

(7.0.4)

in \( K_0(Var_k) \).

Since \( n - 2 \geq 2 \), we have \([V(h) \cap V(x_0)] \equiv 1 \) mod \( L \) in \( K_0(Var_k) \). Besides, the projection map \( \pi \) induces a Zariski locally trivial \( \mathbb{A}^1_k \)-fibration

\[
Q_1^{x_0=0} \cap Q_2 \cap V(L_1) \setminus \{p\} \to V(L_1) \cap V(R) \cap V(h) \cap V(x_0).
\]

Hence we have

\[
[Q_1^{x_0=0} \cap Q_2 \cap V(L_1) \setminus \{p\}] = L \cdot [V(L_1) \cap V(R) \cap V(h) \cap V(x_0)]
\]  

(7.0.5)

in \( K_0(Var_k) \). By Equality 7.0.3 and Equality 7.0.5 we obtain

\[
[Q_1^{x_0=0} \cap Q_2] = [Q_1^{x_0=0} \cap Q_2 \setminus ((V(L_1) \cap V(x_0)) \cup \{p\})] + [Q_1^{x_0=0} \cap Q_2 \cap V(L_1) \setminus \{p\}] + 1
\]

\[
= [V(h) \cap V(x_0)] - [V(h) \cap V(L_1) \cap V(x_0)] + L \cdot [V(L_1) \cap V(R) \cap V(h) \cap V(x_0)] + 1
\]

\[
= 2 - [V(h) \cap V(L_1) \cap V(x_0)] \mod L
\]  

(7.0.6)
in $K_0(\text{Var}_k)$. Here let us note that $V(L_1) \cap V(y_1) = V(L_1) \cap V(x_0)$. Therefore, putting Congruence 7.0.3 and Congruence 7.0.4 together, we get

$$[Q_1 \cap Q_2] = [Q_1^{x_0 \neq 0} \cap Q_2] + [Q_1^{x_0 = 0} \cap Q_2]$$

$$= [Y] - [Y \cap V(y_1)] + [Q_1^{x_0 = 0} \cap Q_2]$$

$$\equiv [Y] \mod L$$

(7.0.7)

in $K_0(\text{Var}_k)$. Now let us examine the class of $Y$ in $K_0(\text{Var}_k)$. We consider the following subvariety of $Y$

$$S := \{[y_1 : \cdots : y_n] \in \mathbb{P}_k^{n-1} \mid y_1 = h(y_2, \ldots, y_n) = L_1(y_1, \ldots, y_n) = R(y_2, \ldots, y_n) = 0\}.$$

For each $s := [s_1 : \cdots : s_n] \in S$, we have

$$\frac{\partial \sigma}{\partial y_1}(s) = 2s_1L_0(s) + s_1^2\frac{\partial L_0}{\partial y_1}(s) + h(s)\frac{\partial L_1}{\partial y_1}(s) = 0,$$

and for $2 \leq i \leq n$,

$$\frac{\partial \sigma}{\partial y_i}(s) = s_1^2\frac{\partial L_0}{\partial y_i}(s) + \frac{\partial h}{\partial y_i}(s)L_1(s) + h(s)\frac{\partial L_1}{\partial y_i}(s) + s_1\frac{\partial R}{\partial y_i}(s) = 0.$$

Hence $S \subset \text{Sing}(Y)$. Now for $n \geq 5$, $S \neq \emptyset$, therefore $\text{Sing}(Y) \neq \emptyset$, which implies that $[Y] \equiv 1 \mod L$ in $K_0(\text{Var}_k)$, by Theorem 6.1. In the case that $n = 4$, $Y \subset \mathbb{P}_k^3$ is in general smooth. However, in $\mathbb{P}_k^3$ a smooth cubic surface is always rational [18, Chapter IV, Theorem 24.1]. Thus

$$[Y] \equiv 1 \mod L \text{ in } K_0(\text{Var}_k)$$

(7.0.8)

for all $n \geq 4$. Hence we have

$$[X] = [Q_1] + [Q_2] - [Q_1 \cap Q_2]$$

$$= 2 - [Q_1 \cap Q_2], \quad \text{by Congruence 7.0.1}$$

$$\equiv 2 - [Y] \mod L, \quad \text{by Congruence 7.0.4}$$

$$\equiv 1 \mod L, \quad \text{by Congruence 7.0.8}$$

in $K_0(\text{Var}_k)$. 

\[\square\]

**Corollary 7.2.** Quartic hypersurfaces of the form described in Theorem 7.1 gives a positive answer to Question 7.7.

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