ASYMPTOTIC EXPANSION OF SOLUTIONS TO THE WAVE EQUATION WITH SPACE-DEPENDENT DAMPING

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ABSTRACT. We study the large time behavior of solutions to the wave equation with space-dependent damping in an exterior domain. We show that if the damping is effective, then the solution is asymptotically expanded in terms of solutions of corresponding parabolic equations. The main idea to obtain the asymptotic expansion is the decomposition of the solution of the damped wave equation into the solution of the corresponding parabolic problem and the time derivative of the solution of the damped wave equation with certain inhomogeneous term and initial data. The estimate of the remainder term is an application of weighted energy methods with suitable supersolutions of the corresponding parabolic problem.

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1. Introduction

1.1. Problem and backgrounds. Let \( \Omega \) be an exterior domain with a smooth boundary \( \partial \Omega \) in \( \mathbb{R}^N \) with \( N \geq 2 \), or \( \Omega = \mathbb{R}^N \) with \( N \geq 1 \). We consider the initial-boundary value problem of the wave equation with space-dependent damping

\[
\begin{aligned}
\frac{\partial^2}{\partial t^2} u - \Delta u + a(x) \frac{\partial}{\partial t} u &= 0, & x \in \Omega, t > 0, \\
u(x, t) &= u_0(x), & x \in \partial \Omega, t > 0, \\
u(x, 0) &= u_0(x), \quad \frac{\partial}{\partial t} u(x, 0) = u_1(x), & x \in \Omega.
\end{aligned}
\]  

(1.1)

Here, \( u = u(x, t) \) is a real-valued unknown function, and \( a(x) \) denotes the coefficient of the damping term. We assume that \( a(x) \) is a smooth positive function on \( \mathbb{R}^N \) having bounded derivatives and satisfying

\[
\lim_{|x| \to \infty} |x|^{\alpha} a(x) = a_0
\]  

(1.2)

with some constants \( \alpha \in [0, 1) \) and \( a_0 > 0 \). Here, the precise meaning of (1.2) is \( \lim_{r \to \infty} \sup_{|x| > r} |x|^{\alpha} a(x) - a_0 = 0 \), that is, the convergence is uniform in the direction. In this case, the damping is called effective, and, as we will see later, the asymptotic behavior of the solution is closely related to a certain corresponding parabolic problem. Here, we remark that it is sufficient that \( a(x) \) is defined on \( \bar{\Omega} \), because we can extend it to \( \mathbb{R}^N \) so that it has the same property as above.

The initial data \((u_0, u_1)\) are assumed to belong to \((H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)\). Then, it is known that (1.1) admits a unique solution

\[
u \in C([0, \infty); H^2(\Omega)) \cap C^1([0, \infty); H^1_0(\Omega)) \cap C^2([0, \infty); L^2(\Omega))
\]

(see [7, Theorem 2]). In our main result, we shall put stronger assumptions on the data.

The aim of this paper is to prove the asymptotic expansion of the solution as time tends to infinity. In particular, we show that the solution \( u \) is asymptotically expanded in terms of a sequence of solutions to corresponding parabolic equations with certain inhomogeneous terms.

The asymptotic behavior of solutions to the damped wave equation has long history after a pioneering work by Matsumura [15]. He studied the Cauchy problem of the wave equation with constant damping

\[
\frac{\partial^2}{\partial t^2} u - \Delta u + \frac{\partial}{\partial t} u = 0, \quad (x, t) \in \mathbb{R}^N \times (0, \infty),
\]

(1.3)

and applied the Fourier transform to obtain the \( L^\infty \) and \( L^2 \) estimates of solutions. In particular, he showed that the decay rates are the same as those of the corresponding heat equation

\[
\frac{\partial}{\partial t} v - \Delta v = 0, \quad (x, t) \in \mathbb{R}^N \times (0, \infty).
\]

(1.4)

After that, the precise asymptotic profile of solutions were studied by Hsiao and Liu [6] for the hyperbolic conservation laws with damping, and by Karch [13] and by Yang and Milani [47] for (1.3), and the so-called diffusion phenomena was proved, that is, the solution \( u \) of (1.3) is asymptotically approximated by a solution of the heat equation (1.4) as time tends to infinity. More detailed asymptotic behavior was studied by many mathematicians, and we refer the reader to [3, 14, 20, 21, 29] for the asymptotic behavior involving the decomposition of solution into the heat-part and wave-part.
For the higher order asymptotic expansions of the Cauchy problem of (1.3), Gallay and Raugel [4] determined the second order expansion when \( N = 1 \) by the method of scaling-variables. Moreover, by Fourier transform, Takeda [37] studied the case \( N \leq 3 \) and obtained the expansion of any order in terms of the Gaussian. Michihisa [17] also gave another expression of expansion for any \( N \geq 1 \) by the Fourier transform method.

On the other hand, the asymptotic behavior of solutions to the initial-boundary value problem of the wave equation with constant damping in an exterior domain is also well-studied. This problem firstly studied by Ikehata [8], and he proved the diffusion phenomena, that is, the asymptotic profile of solution to the exterior problem is given by the exterior heat semigroup with Dirichlet boundary condition. After that, this result was extended by Ikehata and Nishihara [10], Chill and Haraux [3], and Radu, Todorova, and Yordanov [27] to the abstract problem

\[
    u''(t) + Au(t) + u'(t) = 0, \quad t > 0,
\]

(1.5)

where \( A \) is a nonnegative self-adjoint operator in a Hilbert space. Recently, the first author [31] proved the higher order asymptotic expansion of the solution to (1.5) in terms of the solutions of the corresponding first order equation. Radu, Todorova, and Yordanov [28] studied the diffusion phenomena for more general abstract equation

\[
    Cu''(t) + Bu(t) + u'(t) = 0, \quad t > 0
\]

with a nonnegative self-adjoint operator \( B \) and a positive bounded operator \( C \) by the method of diffusion approximation. Nishiyama [23] also studied a similar problem by the method of resolvent estimates.

We also refer the reader to Wirth [41, 42, 43, 44, 45], Yamazaki [46], and [40], for the diffusion phenomena of the wave equation with time dependent damping.

For the initial-boundary value problem of the wave equation with space-dependent damping (1.1), under the assumption of (1.2), it is expected that the damping is classified in the following way:

- (scattering) When \( \alpha > 1 \), the solution behaves like that of the wave equation without damping.
- (effective) When \( \alpha < 1 \), the solution behaves like that of the corresponding heat equation.
- (critical) When \( \alpha = 1 \), the equation is formally invariant under hyperbolic scaling and the behavior of the solution may also depend on the constant \( a_0 \).

The scattering case \( \alpha > 1 \) when \( \Omega = \mathbb{R}^N \) (\( N \neq 2 \)) or \( \Omega \subset \mathbb{R}^N \) is an exterior domain (\( N \geq 3 \)) was studied by Mochizuki [18], Mochizuki and Nakazawa [19], and Matsuyama [10], and they proved that there exist initial data such that the energy of the corresponding solution does not decay to zero, and it approached a solution of the wave equation without damping in the energy norm. The case \( N = 2 \) seems still open.

The critical case \( \alpha = 1 \) with the assumption on \( a(x) \) replaced by \( b_0 \langle x \rangle^{-1} \leq a(x) \leq b_1 \langle x \rangle^{-1} \) (\( b_0, b_1 > 0 \)) was studied by Ikehata, Todorova and Yordanov [11]. They proved that, when \( \Omega = \mathbb{R}^N \) with \( N \geq 3 \) and the initial data are in \( C^0_0(\mathbb{R}^N) \), the energy of the solution decays as \( O(t^{-b_0}) \) if \( 1 < b_0 < N \), and \( O(t^{-N+d}) \) with arbitrary small \( d > 0 \) if \( b_0 \geq N \). Moreover, the decay rate \( O(t^{-b_0}) \) when \( 1 < b_0 < N \) is optimal under some additional assumptions on \( a(x) \). A similar result was also
obtained for the case $N \leq 2$. This indicates that the behavior of the solution depends on the coefficient $b_0$.

For the effective case $\alpha < 1$, Todorova and Yordanov [38] developed a weighted energy method with an exponential-type weight function

$$\exp \left( m(a) \frac{A(x)}{t} \right),$$

which is a refinement of the method by Ikehata [39]. Here, $A(x)$ is a solution of the Poisson equation $\Delta A(x) = a(x)$ satisfying $A(x) \sim \langle x \rangle^{2-\alpha}$, and $m(a) = \liminf_{|x| \to \infty} |\nabla A(x)|^2$. They showed that if $\Omega = \mathbb{R}^N$ ($N \geq 1$), $a(x)$ is radially symmetric and satisfies (1.2) with some $\alpha \in [0, 1)$ and $a_0 > 0$, and the initial data belong to $C^\infty_0(\mathbb{R}^N)$, then we have $m(a) = \frac{N-\alpha}{2-\alpha}$, and the following estimates hold:

$$\|u(t)\|_{L^2} \leq C_\delta (1 + t)^{-\frac{N-\alpha}{2-\alpha} + \frac{\alpha}{2} + \delta},$$

$$\|\partial_t u(t), \nabla u(t)\|_{L^2} \leq C_\delta (1 + t)^{-\frac{N-\alpha}{2-\alpha} + \frac{\alpha}{2} + \delta},$$

where $\delta > 0$ is an arbitrary small loss of decay. Radu, Todorova, and Yordanov [25, 26] studied the energy decay of higher order derivatives and extended the result to more general second-order hyperbolic equations. The assumption of the radial symmetry on $a(x)$ was removed by the authors [33] by modifying the function $A(x)$ above. Moreover, the authors [34, 32, 31] proved the diffusion phenomena in the case of $\alpha \in (-\infty, 1)$ and exterior domains. The asymptotic profile of solution $u$ is given by the corresponding parabolic problem

\[
\begin{align*}
& a(x) \partial_t V_0 - \Delta V_0 = 0, \\
& V_0(x, t) = 0, \\
& V_0(x, 0) = u_0(x) + a(x)^{-1} u_1(x),
\end{align*}
\]

(1.6)

Recently, the authors [35, 36] developed a different kind of weighted energy method applicable to a wider class of initial data including polynomially decaying functions. Roughly speaking, the suggested weight functions form the inverse of the self-similar solutions $\Phi_\beta$ of the equation $|x|^{-\alpha} \partial_t \Phi - \Delta \Phi = 0$ given by

$$\Phi_\beta(x, t) = t^{-\beta} \varphi_\beta(\xi(x, t)),$$

where $\beta \in (0, \frac{N-\alpha}{2-\alpha})$ is a parameter,

$$\varphi_\beta(z) = e^{-z} M \left( \frac{N-\alpha}{2-\alpha} - \beta, \frac{N-\alpha}{2-\alpha}; z \right), \quad \xi(x, t) = \frac{|x|^{2-\alpha}}{(2-\alpha)t},$$

with the Kummer confluent hypergeometric function $M(b, c; z)$ (see Section 2 for the precise definition). Moreover, the relation between the order of the weight of initial data and the decay rates of the solution was revealed. It is worthy noticing that these weight functions have a polynomial growth which enables us to take initial data having a polynomial decay, and the endpoint $\beta = \frac{N-\alpha}{2-\alpha}$ provides the exponential type solution

$$\Phi_{\frac{N-\alpha}{2-\alpha}}(x, t) = t^{-\frac{N-\alpha}{2-\alpha}} \exp \left( \frac{-|x|^{2-\alpha}}{(2-\alpha)t} \right),$$

which corresponds to the exponential type weight function introduced in [38].

As mentioned above, the sharp decay estimates of solutions and the diffusion phenomena for the effective case $\alpha < 1$ is now known very well. In contrast, the higher order asymptotic expansion of the solution remains open.
such that the solution \( u \) coincides with the known result \( \tilde{V} \).

Remark 1.3. For each \( j \), the profiles \( \tilde{V}_j \) are successively determined as \( \tilde{V}_j = \partial_t^j \tilde{V} \) with the unique solutions \( V_j \) of

\[
\begin{cases}
  a(x)\partial_t V_j - \Delta V_j = -\partial_t V_{j-1}, & x \in \Omega, t > 0, \\
  V_j(x, t) = 0, & x \in \partial \Omega, t > 0, \\
  V_j(x, 0) = -(-a(x))^{-j-1} u_1(x), & x \in \Omega.
\end{cases}
\]

for \( j = 1, \ldots, n \).

Remark 1.2. For each \( V_j \) \( (j = 0, 1, \ldots, n) \), we also have

\[
\|\partial_t^j V_j(t)\|_{L^2(\Omega)} \leq C(1 + t)^{-\frac{\alpha}{2} - \frac{2(1-\alpha)}{2\alpha + \alpha - 1}}
\]

(see Section 6). We remark that when \( a(x) \equiv 1 \), the expansion in Theorem 1.1 coincides with the known result \([31]\).

Remark 1.3. One can also represent the profiles \( \tilde{V}_j \) for \( j = 1, \ldots, n \) in terms of the semigroup \( e^{tL} \) generated by \( L = a^{-1} \Delta \). For instance, the second profile \( \tilde{V}_1 \) can be written as

\[
\tilde{V}_1(t) = -Le^{tL}[a^{-2}u_1] - a^{-1}Le^{tL}[u_0+a^{-1}u_1] - \int_0^t Le^{(t-s)L}\left[a^{-1}Le^{sL}[u_0+a^{-1}u_1]\right] ds.
\]

If \( a \equiv 1 \), then \( e^{tL} \) and \( a^{-1} \) commute, and therefore, the above description can be simplified to \( \tilde{V}_1(t) = -a^{-1}Le^{t\Delta}u_1 - \Delta(1+t\Delta)e^{t\Delta}[u_0+u_1] \) as in \([31]\), but the semigroup \( e^{tL} \) and \( a^{-1} \) do not commute in general. To avoid such a complicated situation, we have chosen the parabolic equations \([1] \) for the determination of the profiles \( \tilde{V}_j \).
Remark 1.4. (i) About the explicit values of $s = s(n)$ and $m = m(n, \alpha, \lambda)$ in Theorem 1.1, a rough computation shows that we can take $s = 5(n + 1)$ and $m = (\lambda + 2n + 1)\frac{2n + 2}{n} + (6n^2 + 14n + 8)\alpha$. However, we omit the detailed computation, and do not discuss the optimality of them here.

(ii) If $u_0, u_1 \in C_0^\infty(\Omega)$, then the assumptions on the initial data of Theorem 1.1 are automatically fulfilled.

1.3. A rough description of strategy. By the previous studies [31, 32, 35], the solution of (1.1) is known to be the first asymptotic profile of the solution of (1.1).

To investigate the asymptotic behavior of solution to (1.1), we follow the idea of 1.3. First, the fact that $V_0$ is the first asymptotic profile implies that $u - V_0$ is a remainder term. In [31], it is found that the remainder term $u - V_0$ can be expressed as the time derivative of the solution of the damped wave equation with a certain inhomogeneous term. More precisely, let $U_1$ be the solution of

\[
\begin{cases}
\partial_t^2 U_1 - \Delta U_1 + a(x)\partial_t U_1 = -\partial_t V_0, & x \in \Omega, t > 0, \\
U_1(x, t) = 0, & x \in \partial\Omega, t > 0, \\
U_1(x, 0) = 0, & \partial_t U_1(x, 0) = -a(x)^{-1}u_1(x), & x \in \Omega.
\end{cases}
\]

Next, we have the decomposition $u = V_0 + \partial_t U_1$ (see Lemma 3.9). Then we further consider the asymptotic profile of $U_1$. By experience, it is natural to choose $V_1$ via (1.3) with $n = 1$ ($V_1$ and $U_1$ has the same inhomogeneous term in respective equations). Then, in a similar way $U_1$ can also be also decomposed as $U_1 = V_1 + \partial_t U_2$ with the second auxiliary function $U_2$ via

\[
\begin{cases}
\partial_t^2 U_2 - \Delta U_2 + a(x)\partial_t U_2 = -\partial_t V_1, & x \in \Omega, t > 0, \\
U_2(x, t) = 0, & x \in \partial\Omega, t > 0, \\
U_2(x, 0) = 0, & \partial_t U_2(x, 0) = (-a(x))^{-2}u_1(x), & x \in \Omega.
\end{cases}
\]

The relation

\[ u = V_0 + \partial_t U_1 = V_0 + \partial_t V_1 + \partial_t^2 U_2 \]

can be expected to determine the second expansion. Continuously, using the $(n+1)$-th auxiliary function $U_{n+1}$ given by

\[
\begin{cases}
\partial_t^2 U_{n+1} - \Delta U_{n+1} + a(x)\partial_t U_{n+1} = -\partial_t V_n, & x \in \Omega, t > 0, \\
U_{n+1}(x, t) = 0, & x \in \partial\Omega, t > 0, \\
U_{n+1}(x, 0) = 0, & \partial_t U_{n+1}(x, 0) = (-a(x))^{-n-1}u_1(x), & x \in \Omega,
\end{cases}
\]

one can obtain the relation

\[ u = V_0 + \partial_t V_1 + \partial_t^2 V_2 + \cdots + \partial_t^n V_n + \partial_t^{n+1} U_{n+1}. \]  (1.11)

More precise discussion will be given in Section 3. Note that even if the initial data $(u_0, u_1)$ are compactly supported, $V_0, \ldots, V_n$ and $U_{n+1}$ do not have compact supports in general. Therefore the finite propagation property does not work in this situation. Applying a weighted energy method developed by the authors' previous papers [30, 33], we prove that $\partial_t^{n+1} U_{n+1}$ decays faster than the other terms in (1.11), and this implies that the solution $u$ is asymptotically expanded by the sum of $V_0, \partial_t V_1, \ldots, \partial_t^n V_n$.

1.4. Construction of the paper. This paper is constructed as follows. In the next section, we prepare the weight functions used in the energy method in subsequent sections. In section 3, we state the well-posedness and regularity of solutions of the problem (1.11) and discuss the validity of the decomposition (1.11) (formally explained in Subsection 1.3) in a suitable weighted Sobolev space. In Section 4, we
discuss the weighted energy estimates for the corresponding parabolic equations. In Section 5, we prove the weighted energy estimates for the damped wave equation (1.1) with an inhomogeneous term. Finally, in Section 6, we complete the proof of Theorem 1.1 by adapting the energy estimates prepared in Sections 4 and 5 to the original problem (1.1).

1.5. Notations. We finish this section with some notations used throughout this paper. The letter $C$ indicates a generic positive constant, which may change from line to line. We also express constants by $C(\ast, \ldots, \ast)$, which means this constant depends on the parameters in the parenthesis. The symbol $f \lesssim g$ stands for $f \leq Cg$ holds with some constant $C > 0$, and $f \sim g$ means both $f \lesssim g$ and $g \lesssim f$ hold.

We denote $\langle x \rangle = \sqrt{1 + |x|^2}$ for $x \in \mathbb{R}^n$. Let $L^2(\Omega)$ be the usual Lebesgue space with the norm

$$\|f\|_{L^2(\Omega)} = \left( \int_{\Omega} |f(x)|^2 \, dx \right)^{1/2},$$

and $C^\infty_0(\Omega)$ stands for the space of infinitely differentiable functions with compact support in $\Omega$. For a nonnegative integer $k$ and $m \in \mathbb{R}$, we introduce the weighted Sobolev spaces by

$$H^{k,m}(\Omega) = \left\{ f : \Omega \to \mathbb{R}; (x)^m \partial_x^\alpha f \in L^2(\Omega) \text{ for any } \alpha \in \mathbb{Z}^N_+ \text{ with } |\alpha| \leq k \right\},$$

$$\|f\|_{H^{k,m}(\Omega)} = \sum_{|\alpha| \leq k} \|\langle x \rangle^m \partial_x^\alpha f\|_{L^2(\Omega)},$$

where we used the notion of multi-index and the derivatives are in the sense of distribution. When $m = 0$, we denote $H^k(\Omega) = H^{k,0}(\Omega)$ for short. Also, $H^{k,m}_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ with respect to the norm $\|f\|_{H^{k,m}(\Omega)}$.

2. Preliminaries

2.1. Weight functions. Throughout this section, we slightly generalize the conditions on $a(x)$ and assume that $a(x)$ is a smooth positive function on $\mathbb{R}^N$ satisfying

$$\lim_{|x| \to \infty} |x|^\alpha a(x) = a_0$$

with some constants $\alpha \in (-\infty, \min\{2, N\})$ and $a_0 > 0$.

We prepare weight functions constructed in [36]. First, we introduce a suitable approximate solution of the Poisson equation $\Delta A(x) = a(x)$.

Lemma 2.1 ([33, Lemma 2.1], [36, Lemma 3.2]). Let $a(x)$ be a smooth positive function on $\mathbb{R}^N$ satisfying the condition (2.1) with some constants $\alpha \in (-\infty, \min\{2, N\})$ and $a_0 > 0$. Then for every $\varepsilon \in (0, 1)$, there exist a function $A_\varepsilon \in C^2(\mathbb{R}^N)$ and positive constants $c_\varepsilon$ and $C_\varepsilon$ such that

$$(1 - \varepsilon)a(x) \leq \Delta A_\varepsilon(x) \leq (1 + \varepsilon)a(x),$$

$$c_\varepsilon \langle x \rangle^{2-\alpha} \leq A_\varepsilon(x) \leq C_\varepsilon \langle x \rangle^{2-\alpha},$$

$$\frac{|\nabla A_\varepsilon(x)|^2}{a(x)A_\varepsilon(x)} \leq \frac{2 - \alpha}{N - \alpha} + \varepsilon$$

hold for $x \in \mathbb{R}^N$. 
Remark 2.2. The above type function $A_\varepsilon(x)$ was firstly introduced by Ikehata [9], Todorova and Yordanov [38], and Nishihara [22]. In particular, in [38], a solution of $\Delta A_\varepsilon(x) = a(x)$, that is, the equation obtained by taking $\varepsilon = 0$ in (2.2), was applied for weighted energy estimates for the damped wave equation (1.1) with radially symmetric $a(x)$. Lemma 2.1 is a refinement of the method of [38] to remove the assumption of radial symmetry on $a(x)$.

The following definitions are connected to the supersolution of $a(x)v_t - \Delta v = 0$ constructed in [36], which plays a crucial role to obtain several estimates verifying asymptotic expansion.

Definition 2.3 (Kummer’s confluent hypergeometric functions). For $b, c \in \mathbb{R}$ with $-c \notin \mathbb{N} \cup \{0\}$, Kummer’s confluent hypergeometric function of first kind is defined by

$$M(b, c; s) = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} s^n n!, \quad s \in [0, \infty),$$

where $(d)_n$ is the Pochhammer symbol defined by $(d)_0 = 1$ and $(d)_n = \prod_{k=1}^{n} (d + k - 1)$ for $n \in \mathbb{N}$; note that when $b = c$, $M(b, b; s)$ coincides with $e^s$.

Definition 2.4. (i) For $\varepsilon \in (0, 1/2)$, we define

$$\tilde{\gamma}_\varepsilon = \left(2 - \frac{\alpha}{N - \alpha} + 2\varepsilon\right)^{-1}, \quad \gamma_\varepsilon = (1 - 2\varepsilon)\tilde{\gamma}_\varepsilon. \quad (2.4)$$

(ii) For $\beta \geq 0$ and $\varepsilon \in (0, 1/2)$, define

$$\varphi_{\beta, \varepsilon}(s) = e^{-s}M(\gamma_\varepsilon - \beta, \gamma_\varepsilon; s), \quad s \geq 0.$$

Remark 2.5. (i) We slightly modify the definition of $\tilde{\gamma}_\varepsilon$ and $\gamma_\varepsilon$ from those of [36] in order to gain a positive term in the right-hand side of Proposition 2.8 (iv). This modification enables us to unify the proof of energy estimates for the case $N = 1$ and $N \geq 2$ (see Sections 4 and 5).

(ii) We note that $\varphi_{\beta, \varepsilon}(s)$ is a unique (modulo constant multiple) solution of

$$s\varphi''(s) + (\gamma_\varepsilon + s)\varphi'(s) + \beta\varphi(s) = 0 \quad (2.5)$$

with bounded derivative near $s = 0$.

Lemma 2.6. The function $\varphi_{\beta, \varepsilon}$ defined in Definition 2.3 satisfies the following properties.

(i) If $0 \leq \beta < \gamma_\varepsilon$, then $\varphi_{\beta, \varepsilon}(s)$ satisfies the estimates

$$k_{\beta, \varepsilon}(1 + s)^{-\beta} \leq \varphi_{\beta, \varepsilon}(s) \leq K_{\beta, \varepsilon}(1 + s)^{-\beta}$$

with some constants $k_{\beta, \varepsilon}, K_{\beta, \varepsilon} > 0$.

(ii) For every $\beta \geq 0$, the estimate

$$|\varphi_{\beta, \varepsilon}(s)| \leq K_{\beta, \varepsilon}(1 + s)^{-\beta}$$

holds with some constant $K_{\beta, \varepsilon} > 0$.

(iii) For every $\beta \geq 0$, $\varphi_{\beta, \varepsilon}(s)$ and $\varphi_{\beta + 1, \varepsilon}(s)$ satisfy the recurrence relation

$$\beta\varphi_{\beta, \varepsilon}(s) + s\varphi'_{\beta, \varepsilon}(s) = \beta\varphi_{\beta + 1, \varepsilon}(s).$$
(iv) If $0 \leq \beta < \gamma_\varepsilon$, then we have
\[
\varphi_{\beta,\varepsilon}'(s) = -\frac{\beta}{\gamma_\varepsilon}e^{-s}M(\gamma_\varepsilon - \beta, \gamma_\varepsilon + 1; s) \leq 0,
\]
\[
\varphi_{\beta,\varepsilon}''(s) = \frac{\beta(\beta + 1)}{\gamma_\varepsilon(\gamma_\varepsilon + 1)}e^{-s}M(\gamma_\varepsilon - \beta, \gamma_\varepsilon + 2; s) \geq 0.
\]

(v) If $0 \leq \beta < \gamma_\varepsilon$, then $\varphi_{\beta,\varepsilon}'$ satisfies
\[
-\varphi_{\beta,\varepsilon}'(s) \geq k_{\beta,\varepsilon}(1 + s)^{-\beta - 1}
\]
holds with some constant $k_{\beta,\varepsilon} > 0$.

**Proof.** The proof of the assertions (i)–(iv) are completely the same as that of Lemma 3.5, and we omit the detail. The property (v) follows from the expression in (vi) and the fact $M(\gamma_\varepsilon - \beta, \gamma_\varepsilon + 1; s) \sim \frac{1}{(\gamma_\varepsilon - \beta)}s^{-\beta - 1}e^s$ as $s \to \infty$ (see, for example, Lemma 2.2 (ii) or p.192, (6.1.8)).

Here, we give a family of supersolutions of $a(x)v_t - \Delta v = 0$, which we use later.

**Definition 2.7.** For $\beta \geq 0$ and $(x, t) \in \mathbb{R}^N \times [0, \infty)$, we define
\[
\Phi_{\beta,\varepsilon}(x, t; t_0) = (t_0 + t)^{-\beta}\varphi_{\beta,\varepsilon}(z), \quad z = \frac{\gamma_\varepsilon A_\varepsilon(x)}{t_0 + t},
\]
where $\varepsilon \in (0, 1/2)$, $\gamma_\varepsilon$ is the constant given in Lemma 3.1, $t_0 \geq 1$, $\varphi_{\beta,\varepsilon}$ is the function defined by Definition 2.7, and $A_\varepsilon(x)$ is the function constructed in Lemma 2.1.

For $t_0 \geq 1$ and $(x, t) \in \mathbb{R}^N \times [0, \infty)$, we also define
\[
\Psi(x, t; t_0) := t_0 + t + A_\varepsilon(x).
\]

**Proposition 2.8.** The function $\Phi_{\beta,\varepsilon}(x, t; t_0)$ defined in Definition 2.7 satisfies the following properties:

(i) For every $\beta \geq 0$, we have
\[
\partial_t \Phi_{\beta,\varepsilon}(x, t; t_0) = -\beta \Phi_{\beta + 1,\varepsilon}(x, t; t_0)
\]
for any $(x, t) \in \mathbb{R}^N \times [0, \infty)$.

(ii) If $\beta \geq 0$, then there exists a constant $C_{\alpha,\beta,\varepsilon} > 0$ such that
\[
|\Phi_{\beta,\varepsilon}(x, t; t_0)| \leq C_{\alpha,\beta,\varepsilon}\Psi(x, t; t_0)^{-\beta}
\]
for any $(x, t) \in \mathbb{R}^N \times [0, \infty)$.

(iii) If $\beta \in [0, \gamma_\varepsilon)$, then there exists a constant $c_{\alpha,\beta,\varepsilon} > 0$ such that
\[
\Phi_{\beta,\varepsilon}(x, t; t_0) \geq c_{\alpha,\beta,\varepsilon}\Psi(x, t; t_0)^{-\beta}
\]
for any $(x, t) \in \mathbb{R}^N \times [0, \infty)$.

(iv) For every $\beta \geq 0$, there exists a constant $c_{\alpha,\beta,\varepsilon} > 0$ such that
\[
a(x)\partial_t \Phi_{\beta,\varepsilon}(x, t; t_0) - \Delta \Phi_{\beta,\varepsilon}(x, t; t_0) \geq c_{\alpha,\beta,\varepsilon}a(x)\Psi(x, t; t_0)^{-\beta - 1}
\]
for any $(x, t) \in \mathbb{R}^N \times [0, \infty)$. 

Proof: The properties (i)-(iii) are the same as [36] Lemma 3.8 and [36] Lemma 5.1. Thus, we omit the detail. For (iv), we put \( z = \tilde{\gamma}_\varepsilon A_x(x)/(t_0 + t) \) and compute

\[
(t_0 + t)^{\beta+1} (a(x) \partial_t \Phi_{\beta,\varepsilon}(x, t; t_0) - \Delta \Phi_{\beta,\varepsilon}(x, t; t_0))
\]

\[
= -a(x) \left( \beta \varphi_{\beta,\varepsilon}(z) + z \varphi'_{\beta,\varepsilon}(z) + \gamma_\varepsilon \frac{\Delta A_x(x)}{a(x)} \varphi'_{\beta,\varepsilon}(z) + \gamma_\varepsilon \frac{\nabla A_x(x)}{a(x) A_x(x)} \varphi''_{\beta,\varepsilon}(z) \right).
\]

Using the equation (2.5) with (2.4), we rewrite the right-hand side as

\[
\tilde{\gamma}_\varepsilon a(x) \left( 1 - 2\varepsilon - \frac{\Delta A_x(x)}{a(x)} \right) \varphi'_{\beta,\varepsilon}(z) + a(x) \left( 1 - \tilde{\gamma}_\varepsilon \frac{\nabla A_x(x)}{a(x) A_x(x)} \right) \varphi''_{\beta,\varepsilon}(z).
\]

By (2.2) and (2.3) in Lemma 2.1, we have

\[
1 - 2\varepsilon - \frac{\Delta A_x(x)}{a(x)} \leq -\varepsilon,
\]

\[
1 - \tilde{\gamma}_\varepsilon \frac{\nabla A_x(x)}{a(x) A_x(x)} \geq \varepsilon \left( \frac{2 - \alpha}{N - \alpha} + 2\varepsilon \right)^{-1} > 0.
\]

Combining them to the properties (iv) and (v) in Lemma 2.6, we conclude

\[
a(x) \partial_t \Phi_{\beta,\varepsilon}(x, t; t_0) - \Delta \Phi_{\beta,\varepsilon}(x, t; t_0) \geq -\varepsilon \tilde{\gamma}_\varepsilon a(x)(t_0 + t)^{-\beta-1} \varphi'_{\beta,\varepsilon} \left( \frac{\tilde{\gamma}_\varepsilon A_x(x)}{t_0 + t} \right) \geq Ca(x)(t_0 + t)^{-\beta-1} \left( 1 + \frac{\tilde{\gamma}_\varepsilon A_x(x)}{t_0 + t} \right)^{-\beta-1} \geq Ca(x)(t_0 + t + A_x(x))^{-\beta-1},
\]

which completes the proof. \( \square \)

Finally, we prepare a useful lemma for our weighted energy method.

Lemma 2.9 ([30] Lemma 2.5). Let \( \Phi \in C^2(\overline{\Omega}) \) be a positive function and let \( \delta \in (0, 1/2) \). Then, for any \( u \in H^2(\Omega) \cap H^1_0(\Omega) \), we have

\[
\int_{\Omega} u \Delta u \Phi^{-1+2\delta} \, dx \leq -\frac{\delta}{1-\delta} \int_{\Omega} |\nabla u|^2 \Phi^{-1+2\delta} \, dx + \frac{1 - 2\delta}{2} \int_{\Omega} u^2 (\Delta \Phi) \Phi^{-2+2\delta} \, dx,
\]

provided that the right-hand side is finite.

3. Justification of the decomposition

In this section, we justify the decomposition

\[
u = \sum_{j=0}^n \partial_t^j V_j + \partial_t^{n+1} U_{n+1}
\]

which is explained in Subsection 1.3. Here we need to clarify existence, uniqueness and also an expected regularity of respective components \( V_0, \ldots, V_n \) and \( U_{n+1} \). Therefore we discuss it in the following way: we first prepare the well-posedness of the initial-boundary value problem of the damped wave equation. Next, we show a key decomposition lemma which states that a solution of the damped wave equation can be decomposed into a solution of the corresponding parabolic equation and the derivative of a solution of the damped wave equation with another inhomogeneous
term. Finally, using the decomposition lemma repeatedly, we explain how the higher order asymptotic profiles are determined.

3.1. Well-posedness and regularity of solutions for the damped wave equation. We consider the initial-boundary value problem of the damped wave equation with a general inhomogeneous term

\[
\begin{cases}
\partial_t^2 w - \Delta w + a(x)\partial_t w = F, & x \in \Omega, t > 0, \\
w(x, t) = 0, & x \in \partial\Omega, t > 0, \\
w(x, 0) = w_0(x), \ \partial_tw(x, 0) = w_1(x), & x \in \Omega.
\end{cases}
\tag{3.1}
\]

We first prepare the well-posedness and the regularity of solutions for (3.1).

We recall the following well-posedness result by Ikawa [7].

**Theorem 3.1** ([7 Theorem 1]). For any \((w_0, w_1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)\) and \(F \in C^1([0, \infty); L^2(\Omega))\), there exists a unique solution

\[w \in C([0, \infty); H^2(\Omega)) \cap C^1([0, \infty); H_0^1(\Omega)) \cap C^2([0, \infty); L^2(\Omega))\]

of (3.1).

By applying the above theorem to \(\langle x \rangle^m w, ((\langle x \rangle^m w_0, (\langle x \rangle^m w_1), and \langle x \rangle^m F, we have the well-posedness of the problem (3.1) in weighted Sobolev spaces.

**Theorem 3.2.** For any \((w_0, w_1) \in (H^{2,m}(\Omega) \cap H_0^{1,m}(\Omega)) \times H_0^{1,m}(\Omega)\) and \(F \in C^1([0, \infty); H^{m,m}(\Omega))\), there exists a unique solution

\[w \in C([0, \infty); H^{2,m}(\Omega)) \cap C^1([0, \infty); H_0^{1,m}(\Omega)) \cap C^2([0, \infty); H^{m,m}(\Omega))\]

of (3.1).

Next, we discuss the regularity of the solution. We first recall the following regularity theorem by Ikawa [7]:

**Theorem 3.3** ([7 Theorem 2]). Let \(k \geq 1\) be an integer and let \(m \geq 0\), \(w_0 \in H^{k+2}(\Omega), w_1 \in H^{k+1}(\Omega),\) and \(F \in \bigcap_{j=0}^{k+1}C^{j+1}([0, \infty); H^{k-j}(\Omega))\). We successively define

\[w_p = \Delta w_{p-2} - a(x)w_{p-1} + \partial_t^{p-2}F(x, 0)\]

for \(p = 2, \ldots, k+1\), and assume the \(k\)-th order compatibility condition

\[(w_p, w_{p+1}) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)\]

for \(p = 0, 1, \ldots, k\). Then, the solution \(w\) to (3.1) obtained by Theorem 3.1 belongs to

\[C([0, \infty); H^{k+2}(\Omega) \cap \left(\bigcap_{j=1}^{k+1}C^{k+2-j}([0, \infty); H_0^1(\Omega))\right) \cap C^{k+2}([0, \infty); L^2(\Omega))].\]

From the above theorem and the same argument as Theorem 3.2, we have the following regularity theorem in weighted Sobolev spaces.

**Theorem 3.4.** Let \(k \geq 1\) be an integer and let \(m \geq 0\), \(w_0 \in H^{k+2,m}(\Omega), w_1 \in H^{k+1,m}(\Omega),\) and \(F \in \bigcap_{j=0}^{k}C^{j+1}([0, \infty); H^{k-j,m}(\Omega))\). We successively define

\[w_p = \Delta w_{p-2} - a(x)w_{p-1} + \partial_t^{p-2}F(x, 0)\]
for $p = 2, \ldots, k + 1$, and assume the $k$-th order compatibility condition

$$
(w_p, w_{p+1}) \in (H^{2,m}_0(\Omega) \cap H_0^{1,m}(\Omega)) \times H_0^{1,m}(\Omega)
$$

for $p = 0, 1, \ldots, k$. Then, the solution $w$ to (3.1) obtained by Theorem 3.2 belongs to

$$
C([0, \infty); H^{k+2,m}(\Omega)) \cap \bigcap_{j=1}^{k+1} C^{k+2-j}([0, \infty); H_0^{j,m}(\Omega)) \cap C^{k+2}([0, \infty); H^{0,m}(\Omega)).
$$

### 3.2. Regularity of solutions for the corresponding heat equation.

Following our previous study [32, section 2], we prepare the well-posedness and regularity of solutions for the initial-boundary problem of the corresponding heat equation with a general inhomogeneous term

$$
\begin{cases}
  a(x)\partial_t v - \Delta v = G, & x \in \Omega, t > 0, \\
  v(x, t) = 0, & x \in \partial\Omega, t > 0, \\
  v(x, 0) = v_0(x), & x \in \Omega.
\end{cases}
$$

Let $d\mu = a(x)\,dx$ and we define

$$
L^2_{d\mu}(\Omega) = \left\{ f \in L^2_{\text{loc}}(\Omega); \|f\|_{L^2_{d\mu}} = \left( \int_{\Omega} |f(x)|^2 \,d\mu \right)^{1/2} < \infty \right\},
$$

$$(f, g)_{L^2_{d\mu}} := \int_{\Omega} f(x)g(x) \,d\mu.$$

The operator $-a(x)^{-1}\Delta$ is formally symmetric in $L^2_{d\mu}(\Omega)$, and its bilinear closed form is defined by

$$
a(u, v) = \int_{\Omega} \nabla u(x) \cdot \nabla v(x) \,dx,
$$

$$
D(a) = \left\{ u \in L^2_{d\mu}(\Omega) \cap \dot{H}^1(\Omega); \int_{\Omega} \frac{\partial u}{\partial x_j} \varphi \,dx = -\int_{\Omega} u \frac{\partial \varphi}{\partial x_j} \,dx \text{ for all } \varphi \in C_0^\infty(\mathbb{R}^N) \right\}.
$$

From [32], we have the Friedrichs extension $-L$ of the operator $-a(x)^{-1}\Delta$ in $L^2_{d\mu}(\Omega)$.

**Lemma 3.5 ([32 Lemma 2.2]).** The operator $-L$ in $L^2_{d\mu}(\Omega)$ defined by

$$
D(L) = \left\{ u \in D(a); \exists f \in L^2_{d\mu}(\Omega) \text{ s.t. } a(u, v) = (f, v)_{L^2_{d\mu}} \text{ for any } v \in D(a) \right\},
$$

$$
-Lu = f
$$

is nonnegative and selfadjoint in $L^2_{d\mu}(\Omega)$. Therefore, $L$ generates an analytic semigroup $T(t)$ on $L^2_{d\mu}(\Omega)$ satisfying

$$
\|T(t)f\|_{L^2_{d\mu}} \leq \|f\|_{L^2_{d\mu}}, \quad \|LT(t)f\|_{L^2_{d\mu}} \leq \frac{1}{t} \|f\|_{L^2_{d\mu}}
$$

for any $f \in L^2_{d\mu}(\Omega)$.

We also recall the following property proved in [32], which will be used in Section 6:
Lemma 3.6 ([22] Lemma 2.3). We have
\[ \{ u \in H^2(\Omega) \cap H^1_0(\Omega); \ a(x)^{-1/2} \Delta u \in L^2(\Omega) \} \subset D(L). \]

For the inhomogeneous problem [22], applying [22] Lemma 4.1.1, Proposition 4.1.6, we have the following well-posedness result.

Theorem 3.7. Assume that \( v_0 \in D(L) \) and \( a(x)^{-1} G \in C^1([0, \infty); L^2_{d\mu}(\Omega)) \). Then, the function \( v \) defined by
\[ v(t) = T(t)v_0 + \int_0^t T(t-s)(a(x)^{-1} G(s)) \, ds \]
is the unique solution to the problem \( \text{(3.2)} \) satisfying
\[ v \in C([0, \infty); D(L)) \cap C^1([0, \infty); L^2_{d\mu}(\Omega)). \]

Next, we discuss the higher order regularity in time for the solution of \( \text{(3.2)} \). We note that, by a formal straightforward computation, the initial values of \( \partial_t^j v \) for \( j \geq 1 \) are given by
\[ \partial_t^j v(x, 0) = [a(x)^{-1} \Delta]^j v_0(x) + \sum_{l=0}^{j-1} [a(x)^{-1} \Delta]^j a(x)^{-1} \partial_t^{j-l} G(x, 0). \]

Theorem 3.8. Let \( k \geq 1 \) be an integer, \( v_0 \in D(L) \), and \( a(x)^{-1} G \in C^{k+1}([0, \infty); L^2_{d\mu}(\Omega)). \) Assume that \( \partial_t^j v(x, 0) \) defined by the right-hand side of \( \text{(3.3)} \) satisfies \( \partial_t^j v(x, 0) \in D(L) \) for \( j = 1, \ldots, k \). Then, the solution \( v \) to \( \text{(3.2)} \) obtained by Theorem 3.7 belongs to
\[ C^k([0, \infty); D(L)) \cap C^{k+1}([0, \infty); L^2_{d\mu}(\Omega)). \]

Proof. When \( k = 1 \), let \( \psi = \psi(x, t) \) be the solution of \( \text{(3.2)} \) with the inhomogeneous term \( \partial_t G \) and the initial data \( \psi(x, 0) = a(x)^{-1} \Delta v_0(x) + a(x)^{-1} G(x, 0) \). Then, by Theorem 3.7 \( \psi \) is given by
\[ \psi(t) = T(t)[a(x)^{-1} \Delta v_0(x) + a(x)^{-1} G(x, 0)] + \int_0^t T(t-s)(a(x)^{-1} \partial_s G(s)) \, ds \]
\[ = \partial_t T(t)v_0 + T(t)[a(x)^{-1} G(x, 0)] \]
\[ + [T(t-s)(a(x)^{-1} G(s))]_0^t + \int_0^t \partial_t T(t-s)(a(x)^{-1} G(s)) \, ds \]
\[ = \partial_t T(t)v_0 + a(x)^{-1} G(t) \]
\[ + \partial_t \int_0^t T(t-s)(a(x)^{-1} G(s)) \, ds - a(x)^{-1} G(t) \]
\[ = \partial_t v(t). \]
Since \( \psi \in C([0, \infty); D(L)) \cap C^1([0, \infty); L^2_{d\mu}(\Omega)), \) we have
\[ v \in C^1([0, \infty); D(L)) \cap C^2([0, \infty); L^2_{d\mu}(\Omega)), \]
that is, the assertion when \( k = 1 \) is proved. The general case \( k \geq 1 \) can be proved in the same way with induction, and we omit the detail. \( \square \)
3.3. A decomposition lemma. In the following two subsections, we give the idea of the asymptotic expansion of the solution of the damped wave equation (3.1). To simplify the discussion, we only give formal computation here. The justification and the complete proof of the asymptotic expansion will be given in Section 6.

Related to the initial-boundary value problem of the damped wave equation (3.1), we consider the parabolic problem with the same inhomogeneous term \( F \) and the initial data \( w_0(x) + a(x)^{-1}w_1(x) \):

\[
\begin{aligned}
\begin{cases}
  a(x)\partial_t V - \Delta V = F, & x \in \Omega, \ t > 0, \\
  V(x,t) = 0, & x \in \partial \Omega, \ t > 0, \\
  V(x,0) = w_0(x) + a(x)^{-1}w_1(x), & x \in \Omega.
\end{cases}
\end{aligned}
\]

(3.5)

For the solution \( V \) of the above problem, we further consider the following initial-boundary value problem of the damped wave equation with the inhomogeneous term \( -\partial_t V \) and the initial data \((U, \partial_t U)(x,0) = (0, -a(x)^{-1}w_1(x))\):

\[
\begin{aligned}
\begin{cases}
  \partial_t^2 U - \Delta U + a(x)\partial_t U = -\partial_t V, & x \in \Omega, \ t > 0, \\
  U(x,t) = 0, & x \in \partial \Omega, \ t > 0, \\
  U(x,0) = 0, \ \partial_t U(x,0) = -a(x)^{-1}w_1(x), & x \in \Omega.
\end{cases}
\end{aligned}
\]

(3.6)

Then, we have the following decomposition of the solution \( w \) to (3.1).

**Lemma 3.9.** Let \( w \) be the solution of the damped wave equation (3.1) with the inhomogeneous term \( F \) and the initial data \((w_0, w_1)\). Let \( V \) be the solution of the parabolic problem (3.5), and let \( U \) be the solution of the problem (3.6). Then, we have

\[
w = V + \partial_t U.
\]

**Proof.** Let \( \tilde{w} = V + \partial_t U \). Then, we have

\[
\tilde{w}(x,0) = V(x,0) + \partial_t U(x,0) = w_0(x) + a(x)^{-1}w_1(x) - a(x)^{-1}w_1(x) = w_0(x).
\]

Also, by (3.6), we obtain

\[
\partial_t \tilde{w} = \partial_t V + \partial_t^2 U = \Delta U - a(x)\partial_t U.
\]

(3.7)

This implies \( \partial_t \tilde{w}(x,0) = \Delta U(x,0) - a(x)\partial_t U(x,0) = w_1(x) \). Finally, differentiating (3.7) again, and using the relation \( \partial_t^2 U = \partial_t \tilde{w} - \partial_t V \), we deduce

\[
\partial_t^2 \tilde{w} = \Delta \partial_t U - a(x)\partial_t^2 U
\]

\[
= \Delta(\tilde{w} - V) - a(x)(\partial_t \tilde{w} - \partial_t V)
\]

\[
= \Delta \tilde{w} - a(x)\partial_t \tilde{w} + F.
\]

Consequently, \( \tilde{w} \) is the solution of (3.1), and hence, the uniqueness shows \( \tilde{w} = w \). This completes the proof. \( \square \)

3.4. Derivation of the asymptotic expansion. Let \( u \) be the solution of (1.1). To expand \( u \) in terms of solutions of the corresponding parabolic problem, we consider functions \( V_0, V_1, \ldots, V_n \) and the remainder terms \( U_1, U_2, \ldots, U_{n+1} \) successively defined in the following way: first, we define \( V_0 \) by

\[
\begin{aligned}
\begin{cases}
  a(x)\partial_t V_0 - \Delta V_0 = 0, & x \in \Omega, \ t > 0, \\
  V_0(x,t) = 0, & x \in \partial \Omega, \ t > 0, \\
  V_0(x,0) = u_0(x) + a(x)^{-1}u_1(x), & x \in \Omega.
\end{cases}
\end{aligned}
\]

(3.8)

Let \( \tilde{V} \) be the solution of (3.8), and let \( V \) be the solution of the parabolic problem (3.5). Then, we have

\[
\tilde{V}(x,0) = V(x,0).
\]

Then, we have the decomposition of the solution \( u \).

**Lemma 3.10.** Let \( u \) be the solution of the damped wave equation (1.1). Then, we have

\[
\begin{aligned}
\begin{cases}
  a(x)\partial_t V - \Delta V = F, & x \in \Omega, \ t > 0, \\
  V(x,t) = 0, & x \in \partial \Omega, \ t > 0, \\
  V(x,0) = u_0(x) + a(x)^{-1}u_1(x), & x \in \Omega.
\end{cases}
\end{aligned}
\]

(3.9)

Then, we have the following decomposition of the solution \( u \) to (1.1).
Then, we can have the expected higher order decomposition $U_n$ to justify that (3.9) actually gives the parabolic problem.

By Theorems 4.2 and 4.4, the existence, uniqueness, and regularity of the solution $U_n$ to (3.8) can be obtained from the assumptions on the initial data of Theorem 4.1. The detail will be discussed in Section 6.

In the following sections, we give energy estimates for $V_0, V_1, \ldots, V_n$ and $U_n$ to justify that (3.9) actually gives the $n$-th order asymptotic expansion of $u$.

4. Energy estimates for the heat equation

We apply the weighted energy method to obtain the decay estimate of the parabolic problem

$$
\begin{cases}
  a(x)\partial_t v - \Delta v = G, & x \in \Omega, t > 0, \\
  v(x, t) = 0, & x \in \partial\Omega, t > 0, \\
  v(x, 0) = v_0(x), & x \in \Omega.
\end{cases}
$$

(4.1)
The goal of this section is the following weighted energy estimates for higher order derivatives of solutions to (4.1).

**Theorem 4.1.** Let $k \geq 0$ be an integer, $\delta \in (0, 1/2)$, $\varepsilon \in (0, 1/2)$, $\lambda \in [0, (1-2\delta)\gamma_\varepsilon)$ (see (2.4) for the definition of $\gamma_\varepsilon$), $t_0 \geq 1$ and $\beta = \lambda/(1-2\delta)$. Let $v_0 \in D(L)$, $a(x)^{-1}G \in C^{k+1}([0, \infty); L^2_0(\Omega))$, and let $v$ be the corresponding solution of (4.1). Moreover, we assume that $\partial_t^j v(x, 0)$ for $j = 0, 1, \ldots, k$, and for $j=0, 1, \ldots, k$. Then, we have

\[
\int_\Omega a(x)^{-1} |\partial_t^j G(x, t)|^2 \Psi(x, t, t_0)^{\lambda+1+2j} \, dx \in L^1(0, \infty)
\]

for $j = 0, 1, \ldots, k$.

We note that Theorem 3.3 ensures the regularity property (3.4) for the solution $v$. Thus, it suffices to show the estimates (4.2) – (4.5). The proof of Theorem 4.1 is based on an induction argument. The following lemma is the first step.

**Lemma 4.2.** Under the assumptions on Theorem 4.1 with $k = 0$, we have (4.2) – (4.5) for $k = 0$.

**Proof.** By Lemma 2.9 Proposition 2.8 (iii), and the Schwarz inequality, we calculate

\[
\frac{d}{dt} \int_\Omega a(x)v^2 \phi_{\beta, \varepsilon}^1 \, dx = 2\int_\Omega \frac{\nabla v}{\phi_{\beta, \varepsilon}^1} \, dx - (1 - 2\delta) \int_\Omega \frac{a(x)v^2 \partial_t \phi_{\beta, \varepsilon}}{\phi_{\beta, \varepsilon}^2} \, dx + 2\int_\Omega \frac{vG}{\phi_{\beta, \varepsilon}} \, dx \\
\leq -\frac{2\delta}{1 - \delta} \int_\Omega |\nabla v|^2 \phi_{\beta, \varepsilon}^1 \, dx + (1 - 2\delta) \int_\Omega \frac{a(x)\partial_t \phi_{\beta, \varepsilon} - \Delta \phi_{\beta, \varepsilon}v^2}{\phi_{\beta, \varepsilon}^2} \, dx \\
\quad + C \left( \int_\Omega a(x)v^2 \phi_{\beta, \varepsilon}^1 \, dx \right)^{1/2} \left( \int_\Omega a(x)^{-1}G^2 \phi_{\beta, \varepsilon}^1 \, dx \right)^{1/2}.
\]

From the Young inequality and Proposition 2.8 (iv), we obtain

\[
\frac{d}{dt} \int_\Omega a(x)v^2 \phi_{\beta, \varepsilon}^1 \, dx = -\frac{2\delta}{1 - \delta} \int_\Omega |\nabla v|^2 \phi_{\beta, \varepsilon}^1 \, dx + C \int_\Omega a(x)^{-1}G^2 \phi_{\beta, \varepsilon}^1 \, dx.
\]
Integrating it over $[0, t]$ and using Proposition \[\text{Proposition 2.3}\] (iii), we deduce

\[
\int_{\Omega} a(x)v(x,t)^2 \Psi(x,t; t_0)^{\lambda} \, dx \in L^\infty(0, \infty),
\]
\[
\int_{\Omega} |\nabla v(x,t)|^2 \Psi(x,t; t_0)^{\lambda} \, dx \in L^1(0, \infty), \tag{4.6}
\]

Thus, we have \[\text{(4.2)}\] and \[\text{(4.3)}\] in the case $j = 0$. We next compute

\[
\frac{d}{dt} \int_{\Omega} |\nabla v|^2 \Psi^{\lambda+1} \, dx = (\lambda + 1) \int_{\Omega} |\nabla v|^2 \Psi^{\lambda} \, dx + 2 \int_{\Omega} (\nabla \partial_t v \cdot \nabla v) \Psi^{\lambda+1} \, dx. \tag{4.7}
\]

By \[\text{Remark 4.3}\], the first term of the right-hand side belongs to $L^1(0, \infty)$, and by integration by parts, the second term is estimated as

\[
2 \int_{\Omega} (\nabla \partial_t v \cdot \nabla v) \Psi^{\lambda+1} \, dx
\]
\[
= -2 \int_{\Omega} (a(x) \partial_t v - G) \partial_t v \Psi^{\lambda+1} \, dx - (\lambda + 1) \int_{\Omega} \partial_t v (\nabla v \cdot \nabla A_\varepsilon(x)) \Psi^\lambda \, dx
\]
\[
\leq -2 \int_{\Omega} a(x) \partial_t v^2 \Psi^{\lambda+1} \, dx
\]
\[
+ \frac{1}{2} \int_{\Omega} a(x) |\partial_t v|^2 \Psi^{\lambda+1} \, dx + 2 \int_{\Omega} a(x)^{-1} G^2 \Psi^{\lambda+1} \, dx
\]
\[
+ \frac{1}{2} \int_{\Omega} a(x) |\partial_t v|^2 A_\varepsilon(x) \Psi^\lambda \, dx + 2(\lambda + 1)^2 \int_{\Omega} \frac{|\nabla A_\varepsilon(x)|^2}{a(x) A_\varepsilon(x)} |\nabla v|^2 \Psi^\lambda \, dx
\]
\[
\leq - \int_{\Omega} a(x) |\partial_t v|^2 \Psi^{\lambda+1} \, dx
\]
\[
+ C \int_{\Omega} |\nabla v|^2 \Psi^\lambda \, dx + 2 \int_{\Omega} a(x)^{-1} G^2 \Psi^{\lambda+1} \, dx. \tag{4.8}
\]

Here, we have used the property \[\text{Proposition 2.3}\] in Lemma \[\text{Lemma 2.1}\] and the relation $A_\varepsilon(x) \leq \Psi(x, t; t_0)$, which follows from the definition $\Psi$ (see \[\text{Proposition 2.3}\]). By \[\text{Proposition 2.3}\] and the assumption on $G$, the last two terms of the right-hand side of above are in $L^1(0, \infty)$. Thus, integrating \[\text{(4.7)}\] over $[0, t]$, we conclude

\[
\int_{\Omega} |\nabla v(x, t)|^2 \Psi(x,t; t_0)^{\lambda+1} \, dx \in L^\infty(0, \infty),
\]
\[
\int_{\Omega} a(x) |\partial_t v(x,t)|^2 \Psi(x,t; t_0)^{\lambda+1} \, dx \in L^1(0, \infty),
\]

that is, \[\text{(4.4)}\] and \[\text{(4.5)}\] in the case $j = 0$, and the proof is now complete. \[\square\]

**Remark 4.3.** It should be noted that the integration by parts in the above proof can be justified completely by the approximation argument in the same as \[\text{Remark 4.3}\].

Next, we prove the following lemma, which is the main part of the induction argument of the proof of Theorem \[\text{Theorem 4.1}\].

**Lemma 4.4.** Assume $a(x)$ satisfies \[\text{Proposition 4.2}\]. Let $\sigma \geq 0$, $t_0 \geq 1$, $v_0 \in D(L)$ and $a(x)^{-1}G \in C^1([0, \infty); L^2_{\text{dual}}(\Omega))$. Let $v$ be the corresponding solution of \[\text{Equation 4.1}\]. We
further assume

\[
\begin{align*}
  v_0 \in H^{0, \frac{2}{\sigma} - \frac{2}{\sigma} - \frac{2}{\alpha} (\Omega)}, \\
  \nabla v_0 \in H^{0, \frac{2}{\sigma} (\sigma + 1)}(\Omega), \\
  \int_{\Omega} a(x)^{-1}[G(x, t)]^2 \Psi(x, t; t_0)^{\sigma + 1} \, dx \in L^1(0, \infty),
\end{align*}
\]

and also

\[
\int_{\Omega} a(x)|v(x, t)|^2 \Psi(x, t; t_0)^{\sigma - 1} \, dx \in L^1(0, \infty).
\]

Then, we have

\[
\begin{align*}
  \int_{\Omega} a(x)|v(x, t)|^2 \Psi(x, t; t_0)^{\sigma} \, dx & \in L^\infty(0, \infty), \\
  \int_{\Omega} |\nabla v(x, t)|^2 \Psi(x, t; t_0)^{\sigma} \, dx & \in L^1(0, \infty), \\
  \int_{\Omega} |\nabla v(x, t)|^2 \Psi(x, t; t_0)^{\sigma + 1} \, dx & \in L^\infty(0, \infty), \\
  \int_{\Omega} a(x)|\partial_t v(x, t)|^2 \Psi(x, t; t_0)^{\sigma + 1} \, dx & \in L^1(0, \infty).
\end{align*}
\]

Remark 4.5. In the proof of Theorem 4.1, we will choose \( \sigma = \lambda + 2j \) for \( j \in \mathbb{N} \).

Proof of Lemma 4.4. Suppose (4.9) and (4.10). Similarly as the proof of Lemma 4.2, we compute

\[
\frac{d}{dt} \int_{\Omega} a(x)|v(x, t)|^2 \Psi(x, t; t_0)^{\sigma} \, dx = 2 \int_{\Omega} v(\Delta v) \Psi^\sigma \, dx + \sigma \int_{\Omega} a(x)|v|^2 \Psi^{\sigma - 1} \, dx
\]

\[
+ 2 \int_{\Omega} vG \Psi^\sigma \, dx.
\] (4.15)

The second term of the right-hand side is in \( L^1(0, \infty) \) due to the assumption (4.10). Noting the relation \( 2v\Delta v = -2|\nabla v|^2 + \Delta(|v|^2) \) and using the integration by parts, we calculate the first term of the right-hand side as

\[
2 \int_{\Omega} v(\Delta v) \Psi^\sigma \, dx = -2 \int_{\Omega} |\nabla v|^2 \Psi^\sigma \, dx + \int_{\Omega} |v|^2 \Delta(\Psi^\sigma) \, dx.
\]

The last term of the above is further estimated by

\[
\int_{\Omega} |v|^2 |\Delta(\Psi^\sigma)| \, dx = \sigma \int_{\Omega} |v|^2 \left| \Delta A_\varepsilon \Psi + (\sigma - 1)|\nabla A_\varepsilon|^2 \right| \Psi^{\sigma - 2} \, dx
\]

\[
\leq C \int_{\Omega} a(x)|v|^2 \Psi^{\sigma - 1} \, dx,
\]

where we have used (2.2), (2.3), and \( A_\varepsilon \leq \Psi \). Therefore, this term also belongs to \( L^1(0, \infty) \) by the assumption (4.10). Finally, we apply the Schwarz inequality to the last term of (4.15) and obtain

\[
2 \int_{\Omega} vG \Psi^\sigma \, dx \leq \int_{\Omega} a(x)|v|^2 \Psi^{\sigma - 1} \, dx + \int_{\Omega} a(x)^{-1}|G|^2 \Psi^{\sigma + 1} \, dx
\]
and these are in $L^1(0, \infty)$ due to the assumptions (4.9) and (4.10). Consequently, integrating (4.15) over $[0, t]$, we have
\[
\int_{\Omega} a(x)|v(x, t)|^2 \Psi(x, t; t_0)^\sigma \, dx \in L^\infty(0, \infty),
\]
\[
\int_{\Omega} |\nabla v(x, t)|^2 \Psi(x, t; t_0)^\sigma \, dx \in L^1(0, \infty).
\]

Thus, we have (4.11) and (4.12).

Next, we compute
\[
\frac{d}{dt} \int_{\Omega} |\nabla v(x, t)|^2 \Psi(x, t; t_0)^{\sigma + 1} \, dx = (\sigma + 1) \int_{\Omega} |\nabla v|^2 \Psi^\sigma \, dx
\]
\[
+ 2 \int_{\Omega} (\nabla \partial_t v \cdot \nabla v) \Psi^{\sigma + 1} \, dx. \tag{4.16}
\]

The first term of the right-hand side belongs to $L^1(0, \infty)$ by (4.12). The second term can be estimated in completely the same way as (4.8), and we have
\[
2 \int_{\Omega} (\nabla \partial_t v \cdot \nabla v) \Psi^{\sigma + 1} \, dx \leq - \int_{\Omega} a(x)|\partial_t v|^2 \Psi^{\sigma + 1} \, dx
\]
\[
+ C \int_{\Omega} |\nabla v|^2 \Psi^\sigma \, dx
\]
\[
+ C \int_{\Omega} a(x)^{-1} |\nabla G|^2 \Psi^{\sigma + 1} \, dx.
\]

By using (4.9), and (4.12), the last two terms of above belong to $L^1(0, \infty)$. Finally, integrating (4.16) over $[0, t]$, we conclude
\[
\int_{\Omega} |\nabla v(x, t)|^2 \Psi(x, t; t_0)^{\sigma + 1} \, dx \in L^\infty(0, \infty),
\]
\[
\int_{\Omega} a(x)|\partial_t v(x, t)|^2 \Psi(x, t; t_0)^{\sigma + 1} \, dx \in L^1(0, \infty).
\]

This completes the proof of (4.11) and (4.12). \qed

**Proof of Theorem 4.1**. We note that the case $k = 0$ has been already proved by Lemma 4.2. Let $k \geq 1$ be an integer. Then, by Lemma 4.2, we have (4.12)–(4.15) in the case $j = 0$.

Next, for $j = 1$, we apply Lemma 4.3 with $\sigma = \lambda + 2j$ and with the replacement of $v$ and $G$ by $\partial_t v$ and $\partial_t G$, respectively. We remark that the condition (4.10) with $\sigma = \lambda + 2j$ is fulfilled by virtue of (4.5) with $j = 0$. Then, we obtain (4.11)–(4.14) for $\sigma = \lambda + 2j$ with the replacement of $v$ by $\partial_t v$, namely, we reach the conclusions (4.2)–(4.5) for $j = 1$.

The properties (4.2)–(4.5) for $j = 1$ allow us to apply again Lemma 4.3 with $\sigma = \lambda + 2j$, $j = 2$ and with the replacement of $v$ and $G$ by $\partial^2_t v$ and $\partial^2_t G$, respectively. Then, we can see that (4.2)–(4.5) for $j = 2$ hold. Repeating this argument until $j = k$, we complete the proof of Theorem 4.1. \qed

5. **Energy estimates for the damped wave equation**

5.1. **First order energy estimates.** In this section, we discuss the energy estimate for the general damped wave equation (5.1).
The results of this section will be used in the next section by putting \( w = U_{n+1} \), \( F = -\partial_x V_n \), \( w_0 = 0 \), and \( w_1 = (-a(x))^{-n-1}u_1(x) \) (see (3.5)) to derive the energy estimate of \( \partial_x^{n+1}U_{n+1} \).

We start with the definition of the weighted energy of \( w \).

**Definition 5.1.** For \( \delta \in (0, 1/2) \), \( \varepsilon \in (0, 1/2) \), \( \lambda \in [0, (1 - 2\delta)\gamma_c] \) (see (2.4) for the definition of \( \gamma_c \)), \( \beta = \lambda/(1 - 2\delta) \), \( t_0 \geq 1 \), and \( \nu > 0 \), we define

\[
E_1[w](t; t_0, \lambda) := \int_{\Omega} (|\nabla w(x, t)|^2 + |\partial_t w(x, t)|^2) \Psi(x, t; t_0)^{\lambda+1} \, dx,
\]

\[
E_0[w](t; t_0, \lambda) := \int_{\Omega} (2w(x, t)\partial_t w(x, t) + a(x)|w(x, t)|^2) \Phi_{\beta, \varepsilon}(x, t; t_0)^{-1+2\delta} \, dx,
\]

\[
E[w](t; t_0, \lambda, \nu) := \nu E_1[w](t; t_0, \lambda) + E_0[w](t; t_0, \lambda)
\]

for \( t \geq 0 \).

We note that, for any \( \nu > 0 \), there exists \( t_1 > 0 \) such that

\[
E[w](t; t_0, \lambda, \nu) \sim E_1[w](t; t_0, \lambda) + \int_{\Omega} a(x)|w(x, t)|^2 \Psi(x, t; t_0)^{\lambda} \, dx
\]

holds for any \( t_0 \geq t_1 \). Indeed, the Schwarz inequality implies

\[
|2w\partial_t w| \leq \frac{a(x)}{2}|w|^2 + 2a(x)|\partial_t w|^2,
\]

and Proposition 2.8 (iii) and (iv) lead to

\[
2a(x)^{-1} |\partial_t w|^2 \Phi_{\beta, \varepsilon}^{-1+2\delta} \leq C \Psi^{\frac{\beta}{\lambda-\nu}} |\partial_t w|^2 \Psi^\lambda \leq C \lambda_{0}\frac{\beta}{\lambda-\nu} |\partial_t w|^2 \Psi^\lambda + 1 \leq \frac{\nu}{2} |\partial_t w|^2 \Psi^\lambda + 1
\]

for sufficiently large \( t_0 \).

The main theorem of this subsection is the following:

**Theorem 5.2.** Assume that \( a(x) \) satisfies (1.2). Let \( \delta \in (0, 1/2) \), \( \varepsilon \in (0, 1/2) \), \( \lambda \in [0, (1-2\delta)\gamma_c] \), and \( \beta = \lambda/(1-2\delta) \). Then, there exist constants \( \nu = \nu(N, \alpha, \delta, \varepsilon, \lambda) > 0 \) and \( t_* = t_*(N, \alpha, \delta, \varepsilon, \lambda, \nu) \geq 1 \) such that for any \( t_0 \geq t_* \), the following holds: Let \( m = (\lambda + 1)\frac{\beta}{\lambda-\nu} \) and assume \( F \in C^1([0, \infty); H^{0, m}(\Omega)) \) satisfies

\[
\int_{\Omega} a(x)^{-1} |F(x, t)|^2 \Psi(x, t; t_0)^{\lambda+1} \, dx \in L^1(0, \infty).
\]

Let \( w \) be the solution of (3.1) with the initial data \( (w_0, w_1) \in (H^{2, m}(\Omega) \cap H_0^{1, m}(\Omega)) \times H_0^{1, m}(\Omega) \) given in Theorem 3.2. Then, we have

\[
E[w](t; t_0, \lambda, \nu) \in L^\infty(0, \infty),
\]

\[
\int_{\Omega} |\nabla w(x, t)|^2 \Psi(x, t; t_0)^{\lambda} \, dx \in L^1(0, \infty),
\]

\[
\int_{\Omega} \int_{\Omega} a(x)|\partial_t w(x, t)|^2 \Psi(x, t; t_0)^{\lambda+1} \, dx \in L^1(0, \infty).
\]

The proof of Theorem 5.2 is a bit lengthy, but the outline is as follows. We shall derive good terms \(- \int_{\Omega} |\nabla w|^2 \Psi^\lambda \, dx \) and \(- \int_{\Omega} a(x)|\partial_t w|^2 \Psi^{\lambda+1} \, dx \) from the computations of \( \frac{d}{dt}E_0[w](t; t_0, \lambda) \) and \( \frac{d}{dt}E_1[w](t; t_0, \lambda) \), respectively (see the right-hand sides of Lemmas 5.3 and 5.4). Then, we sum up them with sufficiently small
\( \nu \) and sufficiently large \( t_0 \) so that the other bad terms are absorbed by these good terms.

We first give estimates of \( E_0[w](t; t_0, \lambda) \).

**Lemma 5.3.** Under the assumptions on Theorem 5.2 for any \( t_0 \geq 1 \) and \( t > 0 \), we have

\[
\frac{d}{dt} E_0[w](t; t_0, \lambda) \leq -\eta \int_{\Omega} |\nabla w(x, t)|^2 \Psi(x, t; t_0)^\lambda \, dx \\
- \eta \int_{\Omega} a(x)|w(x, t)|^2 \Psi(x, t; t_0)^{\lambda-1} \, dx \\
+ C \int_{\Omega} |\partial_t w(x, t)|^2 \Psi(x, t; t_0)^\lambda \, dx \\
+ C \int_{\Omega} a(x)^{-1} |F(x, t)|^2 \Psi(x, t; t_0)^{\lambda+1} \, dx
\]

with some constants \( \eta_0 = \eta_0(\varepsilon, \delta) > 0 \) and \( C = C(N, \alpha, \delta, \varepsilon, \lambda) > 0 \).

**Proof.** By the definition of \( E_0[w](t; t_0, \lambda) \) and using the equation (5.1), we calculate

\[
\frac{d}{dt} E_0[w](t; t_0, \lambda) = 2 \int_{\Omega} |\partial_t w|^2 \Phi_{\beta, \varepsilon}^{-1+2\delta} \, dx + 2 \int_{\Omega} w (\partial_t^2 w + a(x)\partial_t w) \Phi_{\beta, \varepsilon}^{-1+2\delta} \, dx \\
- (1 - 2\delta) \int_{\Omega} (2w\partial_t w + a(x)|w|^2) (\partial_t \Phi_{\beta, \varepsilon}) \Phi_{\beta, \varepsilon}^{-2+2\delta} \, dx
\]

\[
= 2 \int_{\Omega} |\partial_t w|^2 \Phi_{\beta, \varepsilon}^{-1+2\delta} \, dx + 2 \int_{\Omega} w (\Delta w + F) \Phi_{\beta, \varepsilon}^{-1+2\delta} \, dx \\
- (1 - 2\delta) \int_{\Omega} (2w\partial_t w + a(x)|w|^2) (\partial_t \Phi_{\beta, \varepsilon}) \Phi_{\beta, \varepsilon}^{-2+2\delta} \, dx.
\]

Applying Lemma 2.9 we have

\[
\frac{d}{dt} E_0[w](t; t_0, \lambda) \leq 2 \int_{\Omega} |\partial_t w|^2 \Phi_{\beta, \varepsilon}^{-1+2\delta} \, dx - \frac{2\delta}{1 - \delta} \int_{\Omega} |\nabla w|^2 \Phi_{\beta, \varepsilon}^{-1+2\delta} \, dx \\
- (1 - 2\delta) \int_{\Omega} |w|^2 (a(x)\partial_t \Phi_{\beta, \varepsilon} - \Delta \Phi_{\beta, \varepsilon}) \Phi_{\beta, \varepsilon}^{-2+2\delta} \, dx \\
- 2(1 - 2\delta) \int_{\Omega} w \partial_t w (\partial_t \Phi_{\beta, \varepsilon}) \Phi_{\beta, \varepsilon}^{-2+2\delta} \, dx \\
+ 2 \int_{\Omega} w F \Phi_{\beta, \varepsilon}^{-1+2\delta} \, dx.
\]  

(5.1)

By Proposition 2.8 (iii) and (iv), the third term of the right-hand side of (5.1) is estimated as

\[
-(1 - 2\delta) \int_{\Omega} |w|^2 (a(x)\partial_t \Phi_{\beta, \varepsilon} - \Delta \Phi_{\beta, \varepsilon}) \Phi_{\beta, \varepsilon}^{-2+2\delta} \, dx \leq -\eta \int_{\Omega} a(x)|w|^2 \Psi^{\lambda-1} \, dx
\]

with some \( \eta > 0 \). Moreover, by Proposition 2.8 (iii), the second term of the right-hand side of (5.1) is estimated as

\[
- \frac{2\delta}{1 - \delta} \int_{\Omega} |\nabla w|^2 \Phi_{\beta, \varepsilon}^{-1+2\delta} \, dx \leq -\eta' \int_{\Omega} |\nabla w|^2 \Psi^{\lambda} \, dx
\]  

(5.2)
with some \( \eta' > 0 \). Moreover, we can drop the third term of the right-hand side, and we also have \(|\partial_t \Phi_{\beta, \varepsilon}| = |\beta \Phi_{\beta+1, \varepsilon}| \leq C \Psi^{-\beta-1} \), which implies

\[
|w \partial_t w(\partial_t \Phi_{\beta, \varepsilon}) \Phi_{\beta, \varepsilon}^{-2+2\delta}| \leq C |w| |\partial_t w| \Psi^{-1}.
\]

Therefore, from the above inequality with the Schwarz inequality, we estimate the fourth term of the right-hand side of (5.1) as

\[
\int_{\Omega} w \partial_t w(\partial_t \Phi_{\beta, \varepsilon}) \Phi_{\beta, \varepsilon}^{-2+2\delta} \, dx \leq C \int_{\Omega} |w| |\partial_t w| \Psi^{-1} \, dx \\
\leq \left( \int_{\Omega} a(x)|w|^2 \Psi^{\lambda-1} \, dx \right)^{1/2} \left( \int_{\Omega} a(x)^{-1} |\partial_t w|^2 \Psi^{\lambda-1} \, dx \right)^{1/2} \\
\leq \eta \int_{\Omega} a(x)|w|^2 \Psi^{\lambda-1} \, dx + C(\eta) \int_{\Omega} |\partial_t w|^2 \Psi \, dx
\]

for any \( \eta > 0 \). Similarly, the last term of the right-hand side of (5.1) is estimated as

\[
2 \int_{\Omega} w F \Phi_{\beta, \varepsilon}^{-1+2\delta} \, dx \leq C \int_{\Omega} |w| |F| \Psi \, dx \\
\leq \left( \int_{\Omega} a(x)|w|^2 \Psi^{\lambda-1} \, dx \right)^{1/2} \left( \int_{\Omega} a(x)^{-1} |F|^2 \Psi^{\lambda+1} \, dx \right)^{1/2} \\
\leq \eta \int_{\Omega} a(x)|w|^2 \Psi^{\lambda-1} \, dx + C \int_{\Omega} a(x)^{-1} |F|^2 \Psi^{\lambda+1} \, dx \quad (5.3)
\]

for any \( \eta > 0 \). Therefore, by applying (5.2)–(5.3) to (5.1), we have the desired estimate.

\[\square\]

**Lemma 5.4.** Under the assumptions on Theorem 5.2, there exists \( t_2 \geq 1 \) such that for any \( t_0 \geq t_2 \) and \( t > 0 \), we have

\[
\frac{d}{dt} E_1(w(t); t_0, \lambda) \leq - \int_{\Omega} a(x)|\partial_t w(x,t)|^2 \Psi(x,t; t_0)^{\lambda+1} \, dx + C \int_{\Omega} |\nabla w(x,t)|^2 \Psi(x,t; t_0)^{\lambda} \, dx \\
+ C \int_{\Omega} a(x)^{-1} |F(x,t)|^2 \Psi(x,t; t_0)^{\lambda+1} \, dx
\]

with some constant \( C = C(N, \alpha, \delta, \varepsilon, \lambda, t_2) > 0 \).
Proof. By the definition of $E_1[w](t; t_0, \lambda)$ and the equation (3.1), we calculate
\begin{align*}
\frac{d}{dt} E_1[w](t; t_0, \lambda) &= 2 \int_\Omega (\nabla \partial_t w \cdot \nabla w + \partial_t w \partial_t^2 w) \Psi^{\lambda + 1} \, dx \\
&\quad + (\lambda + 1) \int_\Omega (|\nabla w|^2 + |\partial_t w|^2) \Psi^\lambda \, dx \\
&= 2 \int_\Omega \partial_t w (-\Delta w + \partial_t^2 w) \Psi^{\lambda + 1} \, dx \\
&\quad - 2(\lambda + 1) \int_\Omega \partial_t w (\nabla w \cdot \nabla \Psi) \Psi^\lambda \, dx \\
&\quad + (\lambda + 1) \int_\Omega (|\nabla w|^2 + |\partial_t w|^2) \Psi^\lambda \, dx \\
= -2 \int_\Omega a(x) |\partial_t w|^2 \Psi^{\lambda + 1} \, dx + 2 \int_\Omega \partial_t w F \Psi^{\lambda + 1} \, dx \\
&\quad - 2(\lambda + 1) \int_\Omega \partial_t w (\nabla w \cdot \nabla \Psi) \Psi^\lambda \, dx \\
&\quad + (\lambda + 1) \int_\Omega (|\nabla w|^2 + |\partial_t w|^2) \Psi^\lambda \, dx. \quad (5.4)
\end{align*}

Here, we note that the integration by parts in the second identity is justified, since $\partial_t w \nabla \Psi \lambda + 1 \in L^1(\Omega)$ for each $t \geq 0$. For the second term of the right-hand side of (5.4), we apply the Schwarz inequality to obtain
\begin{align*}
|2\partial_t w F| &\leq \frac{a(x)}{4} |\partial_t w|^2 + 4a(x)^{-1} |F|^2. \quad (5.5)
\end{align*}

Next, by the Schwarz inequality, the third term of the right-hand side of (5.4) is estimated as
\begin{align*}
| - 2(\lambda + 1) \partial_t w (\nabla w \cdot \nabla \Psi) | &\leq \frac{a(x)}{4} |\partial_t w|^2 \Psi + C|\nabla w|^2 \frac{|\nabla \Psi|^2}{a(x) \Psi} \\
&\leq \frac{a(x)}{4} |\partial_t w|^2 \Psi + C|\nabla w|^2,
\end{align*}
where we have also used
\begin{align*}
\frac{|\nabla \Psi|^2}{a(x) \Psi} &\leq \frac{|\nabla A_{\varepsilon}(x)|^2}{a(x) A_{\varepsilon}(x)} \leq \frac{2 - \alpha}{N - \alpha} + \varepsilon,
\end{align*}
which follows from (2.3). Moreover, for the last term of the right-hand side of (5.4), we note that $\Psi^{-1} \leq t_0^{-1 + \frac{\alpha}{2N}} A_{\varepsilon}(x)^{-\frac{\alpha}{2N}} \leq Ct_0^{-1 + \frac{\alpha}{2N}} a(x) \leq \frac{a(x)}{2(\lambda + 1)}$ holds for $t_0 \geq t_2$, provided that $t_2$ is sufficiently large. Thus, we have
\begin{align*}
| (\lambda + 1) \int_\Omega |\partial_t w|^2 \Psi^\lambda \, dx | &\leq \frac{1}{2} \int_\Omega a(x) |\partial_t w|^2 \Psi^{\lambda + 1} \, dx. \quad (5.6)
\end{align*}

Finally, applying (5.5)–(5.6) to (5.4), we have the desired estimate. \[\square\]

Now we are in the position to prove Theorem 5.2.
Proof of Theorem 5.2. Let \( t_2 \) be the constant given in Lemma 5.4. For \( t_0 \geq t_2 \), by Lemmas 5.3 and 5.4 we calculate
\[
\frac{d}{dt} E[w](t; t_0, \lambda, \nu) \leq \int_{\Omega} \left( -\nu a(x) + C \Psi^{-1} \right) |\partial_t w|^2 \Psi^{\lambda+1} \, dx \\
\quad + (\nu C + \eta_0) \int_{\Omega} |\nabla w|^2 \Psi^{\lambda} \, dx \\
\quad + C \int_{\Omega} a(x)^{-1} |F|^2 \Psi^{\lambda+1} \, dx.
\] (5.7)
By taking \( \nu > 0 \) sufficiently small so that \( \nu C - \eta_0 < 0 \). After that, taking \( t_* \geq t_2 \) sufficiently large so that
\[-\nu a(x) + C \Psi^{-1} \leq -\nu a(x) + C t_0^{-\frac{\alpha}{1-\alpha}} a(x) \leq -\frac{\nu}{2} a(x)\]
holds for \( t_0 \geq t_* \). Therefore, integrating (5.7) over \([0, t]\), we have
\[
E[w](t; t_0, \lambda, \nu) + \eta_* \int_0^t \int_{\Omega} a(x) |\partial_t w|^2 \Psi^{\lambda+1} \, dx \, d\tau \\
\quad + \eta_* \int_0^t \int_{\Omega} |\nabla w|^2 \Psi^{\lambda} \, dx \, d\tau \\
\leq E[w](0; t_0, \lambda, \nu) + C \int_0^t \int_{\Omega} a(x)^{-1} |F|^2 \Psi^{\lambda+1} \, dx \, d\tau
\]
with some constant \( \eta_* > 0 \). By the assumptions of the theorem, the right-hand side is bounded with respect to \( t > 0 \). Thus, we complete the proof. \( \square \)

5.2. Higher order energy estimates.

Definition 5.5. Let \( \delta \in (0, 1/2) \), \( \varepsilon \in (0, 1) \), \( \lambda \in \[0, (1-2\delta)\gamma_{\varepsilon}\) \), where \( \gamma_{\varepsilon} \) is defined in (2.4), and let \( \nu > 0 \). For an integer \( j \geq 1 \), we define
\[
E_1^{(j)}[w](t; t_0, \lambda) := \int_{\Omega} \left( (|\nabla w(x, t)|^2 + |\partial_t w(x, t)|^2) \Psi(x, t; t_0)^{\lambda+1+2j} \right) \, dx,
\]
\[
E_0^{(j)}[w](t; t_0, \lambda) := \int_{\Omega} \left( 2w(x, t) \partial_t w(x, t) + a(x)|w(x, t)|^2 \right) \Psi(x, t; t_0)^{\lambda+2j} \, dx,
\]
\[
E^{(j)}[w](t; t_0, \lambda, \nu) := \nu E_1^{(j)}[w](t; t_0, \lambda) + E_0^{(j)}[w](t; t_0, \lambda)
\]
for \( t \geq 0 \).

We remark that there exists \( t_1 > 0 \) such that for any \( t_0 \geq t_1 \) and \( t \geq 0 \), we have
\[
E^{(j)}[w](t; t_0, \lambda, \nu_1) \sim \int_{\Omega} \left[ (|\nabla w|^2 + |\partial_t w|^2) \Psi^{\lambda+1+2j} + a(x)|w|^2 \Psi^{\lambda+2j} \right] \, dx.
\]

The main result of this subsection is the following energy estimates for higher order derivatives of the solution of the damped wave equation (3.4).

Theorem 5.6. Let \( k \geq 1 \) be an integer. Assume that \( a(x) \) satisfies (1.2). Let \( \delta \in (0, 1/2) \), \( \varepsilon \in (0, 1/2) \), and \( \lambda \in \[0, (1-2\delta)\gamma_{\varepsilon}\) \). Then, there exist constants \( \nu^{(j)} = \nu^{(j)}(N, \alpha, \lambda, j) > 0 \) and \( t_*^{(j)} = t_*^{(j)}(N, \alpha, \varepsilon, \lambda, \nu^{(j)}, j) \geq 1 \) for \( j = 1, \ldots, k \)
such that for any $t_0 \geq \max_{1 \leq j \leq k} t^{(j)}_*$, the following holds: let $m = (\lambda + 1 + 2k)/2$, and assume $F \in \cap_{j=0}^k C^{j+1}(0, \infty); H^{k-j,m}(\Omega)$ satisfies
\[
\int_{\Omega} a(x)^{-1} |\partial_t^j F(x, t)|^2 \Psi(x, t; t_0)^{\lambda+1+2j} \, dx \in L^1(0, \infty)
\]
for $j = 0, 1, \ldots, k$. Let $w$ be the solution of (3.1) in Theorem 3.4 with the initial data $(w_0, w_1) \in H^{k+2,m}(\Omega) \times H^{k+1,m}(\Omega)$ satisfying the k-th order compatibility condition in the sense of Theorem 3.4. Then, we have
\[
E^{(j)}|\partial_t^j w|(t; t_0, \lambda, \nu^{(j)}) \in L^\infty(0, \infty),
\]
\[
\int_{\Omega} a(x)|\partial_t^{j+1} w(x, t)|^2 \Psi(x, t; t_0)^{\lambda+1+2j} \, dx \in L^1(0, \infty)
\]
for $j = 1, \ldots, k$.

The proof of Theorem 5.6 is based on an induction argument, which is similar to that of Lemma 4.4. The main part of the induction argument is the following lemma.

**Lemma 5.7.** Let $j \in \mathbb{N}$. Assume $a(x)$ satisfies (1.2). Let $\lambda \geq 0$ and $m = (\lambda + 1 + 2j)/(2 - \alpha)/2$. Then, there exist constants $\nu^{(j)} = \nu^{(j)}(N, \alpha, \lambda, j) > 0$ and $t_*^{(j)} = t_*^{(j)}(N, \alpha, \lambda, \nu^{(j)}, j) \geq 1$ such that for any $t_0 \geq t_*^{(j)}$, the following holds: Assume $F \in C^1([0, \infty); H^{0,m}(\Omega))$ and let $w$ be the solution of (3.1) with initial data $(w_0, w_1) \in (H^2,m(\Omega) \cap H^1,m(\Omega)) \times H^1,m(\Omega)$. If
\[
\int_{\Omega} a(x)^{-1} |F(x, t)|^2 \Psi(x, t; t_0)^{\lambda+1+2j} \, dx \in L^1(0, \infty),
\]
\[
\int_{\Omega} a(x)|w(x, t)|^2 \Psi(x, t; t_0)^{\lambda-1+2j} \, dx \in L^1(0, \infty)
\]
are satisfied, then
\[
E^{(j)}|w|(t; t_0, \lambda) \in L^\infty(0, \infty),
\]
\[
\int_{\Omega} a(x)|\partial_t w(x, t)|^2 \Psi(x, t; t_0)^{\lambda+1+2j} \, dx \in L^1(0, \infty)
\]
hold.

For the proof of Lemma 5.7, we further prepare the following two lemmas.

**Lemma 5.8.** Under the assumptions of Lemma 5.7, we have
\[
\frac{d}{dt} E_0^{(j)}|w|(t; t_0, \lambda) \leq C \int_{\Omega} |\partial_t w|^2 \Psi^{\lambda+2j} \, dx - \int_{\Omega} |\nabla w(x, t)|^2 \Psi(x, t; t_0)^{\lambda+2j} \, dx + C \int_{\Omega} a(x)|w(x, t)|^2 \Psi(x, t; t_0)^{\lambda-1+2j} \, dx + C \int_{\Omega} a(x)^{-1} |F(x, t)|^2 \Psi(x, t; t_0)^{\lambda+1+2j} \, dx
\]
with some constant $C > 0$. 

**Proof.** We compute

\[
\frac{d}{dt} E_0^{(j)}[w](t; t_0, \lambda) = 2 \int_\Omega |\partial_t w|^2 \Psi^{\lambda+2j} \, dx + 2 \int_\Omega w (\partial_t^2 w + a(x) \partial_t w) \Psi^{\lambda+2j} \, dx \\
+ 2(\lambda + 2j) \int_\Omega (2w \partial_t w + a(x)|w|^2) \Psi^{\lambda-1+2j} \, dx \\
= 2 \int_\Omega |\partial_t w|^2 \Psi^{\lambda+2j} \, dx + 2 \int_\Omega w (\Delta w + F) \Psi^{\lambda+2j} \, dx \\
+ 2(\lambda + 2j) \int_\Omega (2w \partial_t w + a(x)|w|^2) \Psi^{\lambda-1+2j} \, dx \\
= 2 \int_\Omega |\partial_t w|^2 \Psi^{\lambda+2j} \, dx - 2 \int_\Omega |\nabla w|^2 \Psi^{\lambda+2j} \, dx \\
- 2(\lambda + 2j) \int_\Omega w(\nabla w \cdot \nabla \Psi) \Psi^{\lambda-1+2j} \, dx \\
+ 2(\lambda + 2j) \int_\Omega (2w \partial_t w + a(x)|w|^2) \Psi^{\lambda-1+2j} \, dx \\
+ 2 \int_\Omega w F \Psi^{\lambda+2j} \, dx.
\]

The Schwarz inequality implies

\[
\left| 2(\lambda + 2j) \int_\Omega w(\nabla w \cdot \nabla \Psi) \Psi^{\lambda-1+2j} \, dx \right| \\
\leq \int_\Omega |\nabla w|^2 \Psi^{\lambda+2j} \, dx + C \int_\Omega a(x)|w|^2 \frac{|\nabla \Psi|^2}{a(x)} \Psi^{\lambda-1+2j} \, dx \\
\leq \int_\Omega |\nabla w|^2 \Psi^{\lambda+2j} \, dx + C \int_\Omega a(x)|w|^2 \Psi^{\lambda-1+2j} \, dx,
\]

where we have used \( \nabla \Psi = \nabla A_\varepsilon(x) \), \( \Psi(x, t; t_0) \geq A_\varepsilon(x) \), and (2.3) in Lemma 2.1.

Similarly, we have

\[
\left| 2(\lambda + 2j) \int_\Omega 2w \partial_t w \Psi^{\lambda-1+2j} \, dx \right| \\
\leq C \int_\Omega |\partial_t w|^2 \Psi^{\lambda+2j} \, dx + C \int_\Omega a(x)|w|^2 a(x)^{-1} \Psi^{\lambda-2+2j} \, dx \\
\leq C \int_\Omega |\partial_t w|^2 \Psi^{\lambda+2j} \, dx + C \int_\Omega a(x)|w|^2 \Psi^{\lambda-1+2j} \, dx,
\]

and

\[
\left| 2 \int_\Omega w F \Psi^{\lambda+2j} \, dx \right| \\
\leq C \int_\Omega a(x)|w|^2 a(x)^{-1} \Psi^{\lambda-1+2j} \, dx + C \int_\Omega a(x)^{-1} |F|^2 \Psi^{\lambda+1+2j} \, dx.
\]

This completes the proof. \( \square \)
Lemma 5.9. Under the assumptions of Lemma [5.7] there exists \( t_2 \geq 1 \) such that for any \( t_0 \geq t_2 \) and \( t > 0 \), we have

\[
E_1^{(j)}[w](t; t_0, \lambda) \leq -\int_{\Omega} a(x)|\partial_t w(x, t)|^2 \Psi(x, t; t_0)^{\lambda+1+2j} \, dx \\
+ C \int_{\Omega} |\nabla w(x, t)|^2 \Psi(x, t; t_0)^{\lambda+2j} \, dx \\
+ C \int_{\Omega} a(x)^{-1}|F(x, t)|^2 \Psi(x, t; t_0)^{\lambda+1+2j} \, dx
\]

with some constant \( C = C(N, \alpha, \delta, \varepsilon, \lambda, j, t_2) > 0 \).

The proof is completely the same as that of Lemma [5.4] by replacing \( \lambda \) by \( \lambda + 2j \). Thus, we omit the detail.

Proof of Lemma 5.7. By Lemmas 5.8 and 5.9, taking \( \nu^{(j)} > 0 \) sufficiently small, and then, taking \( t^{(j)}_* \geq t_2 \) sufficiently large depending on \( \nu^{(j)} \), we have

\[
\frac{d}{dt} E^{(j)}[w](t; t_0, \lambda, \nu^{(j)}) = \nu^{(j)} \frac{d}{dt} E_1^{(j)}[w](t; t_0, \lambda) + \frac{d}{dt} E_0^{(j)}[w](t; t_0, \lambda) \\
\leq -\eta_1 \int_{\Omega} a(x)|\partial_t w|^2 \Psi^{\lambda+1+2j} \, dx - \eta_2 \int_{\Omega} |\nabla w|^2 \Psi^{\lambda+2j} \, dx \\
+ C \int_{\Omega} a(x)|w|^2 \Psi^{\lambda-1+2j} \, dx + C \int_{\Omega} a(x)^{-1}|F|^2 \Psi^{\lambda+1+2j} \, dx
\]

for \( t \geq t^{(j)}_* \) and \( t > 0 \) with some constants \( \eta_1, \eta_2 > 0 \). By integrating the above inequality on \([0, t]\) and using the assumptions, we conclude

\[
E^{(j)}[w](t; t_0, \lambda, \nu^{(j)}) + \int_0^t \int_{\Omega} a(x)|\partial_t w|^2 \Psi^{\lambda+1+2j} \, dx \, d\tau + \int_0^t \int_{\Omega} |\nabla w|^2 \Psi^{\lambda+2j} \, dx \, d\tau \\
\leq E^{(j)}[w](0; t_0, \lambda, \nu^{(j)}) + C \int_0^t \int_{\Omega} a(x)|w|^2 \Psi^{\lambda-1+2j} \, dx \, d\tau \\
+ C \int_0^t \int_{\Omega} a(x)^{-1}|F|^2 \Psi^{\lambda+1+2j} \, dx \, d\tau,
\]

and the proof is complete. \( \square \)

Proof of Theorem 5.6. By Theorem 5.2, there exist \( \nu > 0 \) and \( t_* \geq 1 \) such that

\[
E[w](t; t_0, \lambda, \nu) \in L^\infty(0, \infty), \\
\int_{\Omega} a(x)|\partial_t w(x, t)|^2 \Psi(x, t; t_0)^{\lambda+1} \, dx \in L^1(0, \infty)
\]

hold for \( t_0 \geq t_* \). Now, thanks to the property (5.8), we apply Lemma 5.7 with \( j = 1 \) and the replacement of \( w \) and \( F \) by \( \partial_t w \) and \( \partial_t F \), respectively. Then, there exist \( \nu^{(1)} > 0 \) and \( t^{(1)}_* \geq 1 \) such that

\[
E^{(1)}[\partial_t w](t; t_0, \lambda, \nu^{(1)}) \in L^\infty(0, \infty), \\
\int_{\Omega} a(x)|\partial_t^2 w(x, t)|^2 \Psi(x, t; t_0)^{\lambda+1+2} \, dx \in L^1(0, \infty)
\]
Similarly as before, if we take \( s_j \) the solution of (1.9) with \( j = 2 \) and the replacement of \( w \) and \( F \) by \( \partial_t^2 w \) and \( \partial_t^2 F \), respectively. Repeating this argument until \( j = k \), we reach the conclusion of Theorem 6.6 \( \square \)

6. PROOF OF THE ASYMPTOTIC EXPANSION

In this section, we give the estimates of the right-hand side of (3.9) and complete the proof of Theorem 1.1.

Let \( n \in \mathbb{N} \) be fixed, and let \( \delta \in (0, 1/2) \), \( \varepsilon \in (0, 1/2) \), \( \lambda \in (0, (1 - 2\delta) \gamma \varepsilon) \), and \( t_0 \geq 1 \). Moreover, we define \( \lambda_j = \lambda - \frac{2\alpha}{2-\alpha}j \) for \( j = 1, \ldots, n+1 \), and we assume that \( \lambda_{n+1} \in [0, (1 - 2\delta) \gamma \varepsilon) \), that is, \( \lambda \in \left( \frac{2\alpha}{2-\alpha}(n+1), (1 - 2\delta) \gamma \varepsilon \right) \). Here, we note that the assumption \( n+1 < \frac{N-2}{2-\alpha} \) ensures that this interval is not empty, provided that \( \delta \) and \( \varepsilon \) are sufficiently small. We also put \( \tilde{m} := (\lambda_{n+1} + 1 + 2n) \frac{2-\alpha}{2} \).

As in (1.7), we assume that the initial data \( u_0 \) and \( u_1 \) satisfy

\[ u_0 \in H^{s+1,m}(\Omega) \cap H_0^{s,m}(\Omega), \quad u_1 \in H_0^{s,m}(\Omega) \]

with sufficiently large \( s \) and \( m \).

Note that, in what follows, we retake the parameter \( t_0 \geq 1 \) suitably larger from line to line.

**Step 0: Estimates of \( V_0 \):** We first give the estimates of \( V_0 \), which is the solution of (1.6). We apply Theorem 4.1 with \( k = n \), \( v = V_0 \), \( G = 0 \), \( v_0 = u_0 + a(x)^{-1}u_1 \). Noting Lemma 4.10 and taking \( s \) and \( m \) sufficiently large, we have

\[
\begin{align*}
\partial_t^j V_0(x, 0) &\in D(L) \cap H^{0, (\lambda + 1 + 2\alpha)\frac{2-\alpha}{2}}(\Omega), \\
\nabla \partial_t^j V_0(x, 0) &\in H^{0, (\lambda + 1 + 2\alpha)\frac{2-\alpha}{2}}(\Omega)
\end{align*}
\]

for \( j = 0, 1, \ldots, n \), where \( \partial_t^j V_0(x, 0) \) is defined by the right-hand side of (3.3) with \( v_0 = u_0 + a(x)^{-1}u_1 \) and \( G = 0 \). Therefore, the assumptions of Theorem 4.1 are fulfilled, and we have

\[
\begin{align*}
\int_{\Omega} a(x) |\partial_t^j V_0(x, t)|^2 \Psi(x, t; t_0)^{\lambda + 2j} \, dx &\in L^{\infty}(0, \infty), \\
\int_{\Omega} a(x) |\partial_t^{j+1} V_0(x, t)|^2 \Psi(x, t; t_0)^{\lambda + 1 + 2j} \, dx &\in L^1(0, \infty)
\end{align*}
\]

for \( j = 0, 1, \ldots, n \). In particular, (6.1) with \( j = 0 \) implies the \( L^2 \)-estimate of \( V_0 \):

\[ \|V_0(t)\|_{L^2(\Omega)} \leq C(1 + t)^{-\frac{1}{2} + \frac{\alpha}{2(2-\alpha)}}. \]

**Step 1: Estimates of \( V_1 \):** Next, we consider the estimate for \( V_1 \), which is the solution of (1.9) with \( j = 1 \). We apply Theorem 4.1 with \( k = n \), \( v = V_1 \), \( v_0 = -a(x)^{-2}u_1(x) \), \( G = -\partial_t V_0 \), and the replacement of \( \lambda \) by \( \lambda_1 = \lambda - \frac{2\alpha}{2-\alpha} \). Similarly as before, If we take \( s \) and \( m \) sufficiently large, then we have

\[
\begin{align*}
\partial_t^j V_1(x, 0) &\in D(L) \cap H^{0, (\lambda_1 + 2\alpha)\frac{2-\alpha}{2}}(\Omega), \\
\nabla \partial_t^j V_1(x, 0) &\in H^{0, (\lambda_1 + 1 + 2\alpha)\frac{2-\alpha}{2}}(\Omega)
\end{align*}
\]
for $j = 0, 1, \ldots, n$, where $\partial_t^j V_1(x, 0)$ is defined by the right-hand side of (3.3) with $v_0 = -(-a(x))^{-2}$ and $G = -\partial_t V_0$. Moreover, from (3.2), one obtains

$$\int_{\Omega} a(x)^{-1} |\partial_t^j (-\partial_t V_0)(x, t)|^2 \Psi(x, t; t_0) \lambda_1 + 1 + 2j \, dx$$

$$\leq \int_{\Omega} a(x) |\partial_t^{j+1} V_0(x, t)|^2 \Psi(x, t; t_0) \lambda_1 + 1 + 2j \, dx \in L^1(0, \infty)$$

for $j = 0, 1, \ldots, n$ (this is the reason why we define $\lambda_1 = \lambda - \frac{2m}{2-\alpha}$). Therefore, the assumptions of Theorem 4.1 are fulfilled, and we deduce

$$\int_{\Omega} a(x) |\partial_t^j V_1(x, t)|^2 \Psi(x, t; t_0) \lambda_1 + 1 + 2j \, dx \in L^1(0, \infty)$$

(6.3)

for $j = 0, 1, \ldots, n$. In particular, (6.3) with $j = 1$ implies the $L^2$-estimate of $\partial_t V_1$:

$$\|\partial_t V_1\|_{L^2} \leq C(1 + t)^{-\frac{\alpha}{2} - 1 + \frac{\alpha}{2(2-\alpha)}} \leq C(1 + t)^{-\frac{\alpha}{2} - \frac{2(1-n)(2-\alpha)}{2(2-\alpha)}}.$$

**Step n:** Estimates of $V_n$. Continuing this argument until $j = n$, we can estimate $V_n$. Indeed, we apply Theorem 4.1 with $k = n, v = V_n, v_0 = -(-a(x))^{-n-1}u_1(x), G = -\partial_t V_{n-1}$, and the replacement of $\lambda$ by $\lambda_n = \lambda - \frac{2m}{2-\alpha}$. Similarly as before, if we take $s$ and $m$ sufficiently large, then we have

$$\partial_t^n V_n(x, 0) \in D(L) \cap H^{0, (\lambda_n + 1 - 2s)}(\Omega)$$

for $j = 0, 1, \ldots, n$, where $\partial_t^n V_n(x, 0)$ is defined by the right-hand side of (3.3) with $v_0 = -(-a(x))^{-n-1}$ and $G = -\partial_t V_{n-1}$. Furthermore, by the $(n - 1)$-th step, we have the estimate of inhomogeneous term $-\partial_t V_{n-1}$:

$$\int_{\Omega} a(x)^{-1} |\partial_t^j (-\partial_t V_{n-1})(x, t)|^2 \Psi(x, t; t_0) \lambda_n + 1 + 2j \, dx$$

$$\leq \int_{\Omega} a(x) |\partial_t^{j+1} V_{n-1}(x, t)|^2 \Psi(x, t; t_0) \lambda_{n-1} + 1 + 2j \, dx \in L^1(0, \infty).$$

Thus, the assumptions of Theorem 4.1 are fulfilled, and we have

$$\int_{\Omega} a(x) |\partial_t^j V_n(x, t)|^2 \Psi(x, t; t_0) \lambda_n + 1 + 2j \, dx \in L^\infty(0, \infty),$$

(6.4)

$$\int_{\Omega} a(x) |\partial_t^{j+1} V_n(x, t)|^2 \Psi(x, t; t_0) \lambda_n + 1 + 2j \, dx \in L^1(0, \infty)$$

(6.5)

for $j = 0, 1, \ldots, n$. In particular, (6.4) with $j = n$ implies the $L^2$-estimate of $\partial_t^n V_n$:

$$\|\partial_t^n V_n(t)\|_{L^2(\Omega)} \leq C(1 + t)^{-\frac{\alpha}{2} - n + \frac{\alpha}{2(2-\alpha)}} = C(1 + t)^{-\frac{\alpha}{2} - \frac{2(n-\alpha)}{2(2-\alpha)}}.$$

Moreover, we shall check that $V_n$ has the regularity

$$\partial_t V_n \in \bigcap_{j=0}^n C^{j+1}([0, \infty); H^{n-j, \lambda_1}(\Omega))$$

(6.6)

which will be required in the next step. This follows if both the initial value $\partial_t V_n(x, 0)$ and the inhomogeneous term $-\partial_t V_{n-1}$ have enough regularity and belong to a suitable weighted Sobolev space. Note that $\partial_t V_n(x, 0)$ can be computed from
the following lemma: \[ (1.10) \] with \( j = n \). Thus, we can obtain (1.10) if \( s \) and \( m \) are sufficiently large.

**Final step: Estimates of** \( U_{n+1} \): Finally, we estimate \( \partial_t^{n+1} U_{n+1} \) defined by (1.10) with \( j = n \). First, to check the compatibility condition on \( U_{n+1} \), we prepare the following lemma:

**Lemma 6.1.** Let \( U_{n+1}^{(0)} = 0 \), \( U_{n+1}^{(1)} = -(a(x))^{-n-1}u_1(x) \), and let \( U_{n+1}^{(p)} \) for \( p = 2, \ldots, n+1 \) be successively defined by

\[
U_{n+1}^{(p)}(x) = \Delta U_{n+1}^{(p-2)} - a(x)U_{n+1}^{(p-1)} - \partial_t^{p-1}V_n(x,0).
\]

Then, we have, for \( p = 2, \ldots, n+1 \),

\[
U_{n+1}^{(p)}(x) = -(a(x))^{-(n-p+2)}u_1(x) - \partial_t V_{n-p+2}(x,0) - \cdots - \partial_t^{p-1}V_n(x,0).
\]

**Proof.** When \( p = 2 \), the conclusion is obvious. Suppose that the conclusion is true up to \( p-1 \), and consider the case \( p \). Using the assumption of the induction, we have

\[
U_{n+1}^{(p)}(x) = \Delta U_{n+1}^{(p-2)} - a(x)U_{n+1}^{(p-1)} - \partial_t^{p-1}V_n(x,0)
= \Delta (-(a(x))^{-(n-p+3)}u_1(x) - \partial_t V_{n-p+3}(x,0) - \cdots - \partial_t^{p-3}V_n(x,0))
- a(x) (-(a(x))^{-(n-p+3)}u_1(x) - \partial_t V_{n-p+3}(x,0) - \cdots - \partial_t^{p-2}V_n(x,0))
- \partial_t^{p-1}V_n(x,0).
\]

Recalling \( a(x)\partial_t V_j - \Delta V_j = -\partial_t V_{j-1} \) and \( V_j(x,0) = -(a(x))^{-j-1}u_1(x) \), we see that the right-hand side becomes

\[
-(a(x))^{-(n-p+2)}u_1(x) - \partial_t V_{n-p+2}(x,0) - \cdots - \partial_t^{p-1}V_n(x,0),
\]

which completes the proof. \( \square \)

Since \( u_1 \in H_0^{2m}(\Omega) \) with sufficiently large \( s \) and \( m \), we easily see \( U_{n+1}^{(1)} = -(a(x))^{-n-1}u_1 \in H^{n+1, \tilde{m}}(\Omega) \) and

\[
(a(x))^{-(n-p+2)}u_1 \in H^{2, \tilde{m}}(\Omega) \cap H_0^{1, \tilde{m}}(\Omega) \quad (p = 0, \ldots, n),
\]

\[
(a(x))^{-1}u_1 \in H_0^{1, \tilde{m}}(\Omega).
\]

Moreover, from (3.3) and a similar induction argument, we can see that the functions \( \partial_t^p V_j(x,0) \) satisfy the following conditions, provided that \( s \) and \( m \) are sufficiently large: for \( V_j \) (\( j = 2, \ldots, n \)),

\[
\partial_t^p V_j(x,0) \in H^{2, \tilde{m}}(\Omega) \cap H_0^{1, \tilde{m}}(\Omega) \quad (p = 1, \ldots, j-1),
\]

\[
\partial_t^j V_j(x,0) \in H_0^{1, \tilde{m}}(\Omega);
\]

for \( V_1 \),

\[
\partial_t V_1(x,0) \in H_0^{1, \tilde{m}}(\Omega).
\]

Therefore, by applying Lemma 6.1 we see that the data \( U_{n+1}^{(p)} \) for \( p = 0, 1, \ldots, n+1 \) satisfy the \( n \)-th order compatibility condition

\[
(U_{n+1}^{(p)}, U_{n+1}^{(p+1)}) \in (H^{2, \tilde{m}}(\Omega) \cap H_0^{1, \tilde{m}}(\Omega)) \times H_0^{1, \tilde{m}}(\Omega)
\]
for $p = 0, \ldots, n$. Combining this with the fact (6.6), we can apply Theorem 3.3 with $k = n, m = \bar{m}$ to obtain the regularity of $U_{n+1}$ and the weighted energy of $\partial_t^1 U_{n+1}$ is well-defined. Moreover, it follows from (6.5) that
\[
\int_{\Omega} a(x)^{-1} |\partial_t^j(-\partial_t V_n(x, t))|^2 \Psi(x, t; t_0)^{\lambda_{n+1}+1+2j} dx \in L^1(0, \infty)
\]
for $j = 0, 1, \ldots, n$. Hence, we can apply Theorems 5.2 and 5.6 and there exist $\nu^{(n)} > 0$ and $\tau^{(n)}_* \geq 1$ such that for any $t_0 \geq \tau^{(n)}_*$ we have
\[
E^{(n)}[\partial_t^1 U_{n+1}](t; t_0, \lambda_{n+1}, \nu^{(n)}) \in L^\infty(0, \infty).
\]
In particular, the above bound yields
\[
\int_{\Omega} |\partial_t^{n+1} U_{n+1}(x, t)|^2 dx \leq C(1 + t)^{-\lambda_{n+1}-1-2n}.
\]
Namely, we conclude
\[
\left\| u(t) - \sum_{j=0}^n \partial_t^j V_j(t) \right\|_{L^2(\Omega)} = \left\| \partial_t^{n+1} U_{n+1}(t) \right\|_{L^2(\Omega)} \leq C(1 + t)^{-\frac{1}{2} - \frac{1}{2n+1} \frac{(1-n)}{n-1}}
\]
which completes the proof of Theorem 1.1.

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