On Relativistic Entanglement and Localization of Particles and on their Comparison with the Non-Relativistic Theory

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Abstract

We make a critical comparison of relativistic and non-relativistic classical and quantum mechanics of particles in inertial frames as well as the open problems in particle localization at both levels. The solution of the problems of the relativistic center of mass, of the clock synchronization convention needed to define relativistic 3-spaces and of the elimination of the relative times in the relativistic bound states leads to a description with a decoupled non-local (non-measurable) relativistic center of mass and with only relative variables for the particles (single particle subsystems do not exist). We analyze the implications for entanglement of this relativistic spatial non-separability not existing in non-relativistic entanglement. Then we try to reconcile the two visions showing that also at the non-relativistic level in real experiments only relative variables are measured with their directions determined by the effective mean classical trajectories of particle beams present in the experiment. The existing results about the non-relativistic and relativistic localization of particles and atoms support the view that detectors only identify effective particles following this type of trajectories: these objects are the phenomenological emergent aspect of the notion of particle defined by means of the Fock spaces of quantum field theory.
I. INTRODUCTION

The classical mechanics (CM) of Newton gives a deterministic description of objects (particles, bodies) supposed to have a reality in an inertial frame of the Galilei space-time centered on an inertial mathematical observer playing no dynamical role beyond defining Cartesian coordinates. This space-time is assumed to be a given background container of the real objects, whose world-lines are described in terms of an absolute notion of time. At each instant there is an absolute Euclidean 3-space where the objects are localized. The inertial frames are connected by the transformations of the Galilei group. This description can be extended to non-inertial frames centered on mathematical accelerated observers.

This realistic description of the world-lines of particles is preserved in special relativity (SR). However, now they are described in the inertial frames of Minkowski space-time centered on inertial mathematical relativistic observers and the Poincaré group describes the transformations connecting the inertial frames. However, in SR there is no notion of absolute time and of absolute 3-space: only the whole Minkowski space-time is absolute and only the conformal structure (i.e. the light-cone describing the locus of incoming and outgoing radiation in every point) has an intrinsic meaning. As a consequence, we must introduce a convention of clock synchronization to define an instantaneous 3-space, whose definition is needed to formulate the Cauchy problem for wave equations like the Maxwell’s ones.

In this Introduction we give an outline of the main problems to be faced in the definition of relativistic classical mechanics (RCM) and relativistic quantum mechanics (RQM). This will clarify the context which gave origin to this paper and to its results and implications.

Usually RCM is formulated in inertial frames, whose Euclidean 3-spaces are defined by Einstein’s convention \(^1\). Only with this convention does the 1-way velocity of light between two observers (it depends on how their clocks are synchronized) coincide with the 2-way velocity of light \(c\) of an inertial observer (it replaces the unit of length in relativistic metrology [1]).

However this description of RCM is still incomplete for interacting systems due to the following problems:

1) There is not a unique notion of relativistic center of mass of a system of particles like in Newtonian mechanics;

2) There is the problem of the elimination of relative times in relativistic bound states (time-like excitations are not seen in spectroscopy);

3) It is highly non trivial to find the explicit form of the Poincaré generators (especially the Lorentz boosts) for interacting particles in the instant form of dynamics;

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\(^1\) The inertial observer A sends a ray of light at \(x^a_i\) towards the (in general accelerated) observer B; the ray is reflected towards A at a point \(P\) of B world-line and then reabsorbed by A at \(x^a_f\); by convention \(P\) is synchronous with the mid-point between emission and absorption on A’s world-line, i.e. \(x^a_p = x^a_i + \frac{1}{2} (x^a_f - x^a_i) = \frac{1}{2} (x^a_i + x^a_f)\). This convention selects the Euclidean instantaneous 3-spaces \(x^\alpha = ct = \text{const.}\) of the inertial frames centered on A. However, if the observer A is accelerated, the convention can breaks down due the possible appearance of coordinate singularities.
4) There is no accepted global formulation of non-inertial frames without the pathologies of the rotating disk and of Fermi coordinates.

Recently a solution to all these problems has been given in Refs.[2–4] (see also the review in Ref.[5]). As sketched in Section II, the main differences from non-relativistic CM are the non-local nature of the relativistic collective variables proposed for the relativistic center of mass (implying their non-measurability with local measurements) and a spatial non-separability of the particles, which must be described by means of suitable Wigner-covariant relative 3-variables.

This formulation of RCM allows one to get a consistent definition of RQM of particles with an associated notion of relativistic entanglement as an extension of non-relativistic quantum mechanics (NRQM) avoiding all the known relativistic pathologies. This was done in Ref.[6]. This framework for RCM has been extended to classical field theory (CFT) in Refs.[3, 5] both for classical fields and fluids, but the extension of the approach to quantum field theory (QFT) has still to be done.

Unlike from the transition from Galilei space-time to the Minkowski one, the transition from CM to NRQM can be done only in an operational way due to the big unsolved foundational problems of NRQM (see for instance Ref.[7]) 2. The main problem is that the notion of reality of a classical particle and of its properties cannot be extended to NRQM as shown

A) by the EPR experiment (see Ref.[9] for a review) and the violation of Bell’s inequalities (no local realistic hidden variable explanation; see Ref.[10] for the status of experiments);

B) by the Kochen-Specker theorem [11] (no non-contextual explanation of the properties of a quantum system);

C) by the probabilistic Born’s rule for the unique outcomes of measurements (in a random way a unique value is obtained for the observable, describing a property of the state of the quantum object under investigation; but a repeated measurement on an identical state can give other values, i.e. we cannot speak of a property of the quantum system but only of quantity which takes value depending on the context randomly).

As a consequence it is not clear which is the meaning of the localization of a quantum particle even having taken into account the Heisenberg uncertainty relations (see Ref.[12] for the problem of simultaneous measurements of position and momentum).

On one side we have the mathematical theory of NRQM with the unitary evolution of the wave function, but there is no consensus on whether this wave function describes the given quantum system or is only an information on a statistical ensemble of such systems. Moreover it could be that only the real-valued density matrix, i.e. the statistical operator determining the probabilities, makes sense and not the complex-valued wave function (only a mathematical tool). There is no accepted interpretation for the theory of measurements going beyond the non-unitary collapse of the wave function in an instantaneous idealized Von

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2 Let us remark that for many physicists the absence of experimental facts in contrast with QM is an indication that the foundational problems are fake problems of philosophical type. See for instance Ref.[8]. Our attitude is neither operational nor foundational: we try to understand some aspects of the transition from the quantum to the classical regime following Bohr’s viewpoint.
Neumann measurement of a self-adjoint operator describing some mathematical observable property of the quantum system.

In experiments we have macroscopic semi-classical objects as source and detectors of quantities named quantum particles (or atoms) and the results shown by the pointers of the detectors are the end point of a macroscopic (many-body) amplification of the interaction of the quantum object with some microscopic constituent of the detector (for instance an α particle interacting with a water molecule followed by the formation of a droplet as the amplification allowing detection of the particle trajectory in bubble chambers). Usually one invokes the theory of decoherence [13, 14] with its uncontrollable coupled environment for the emergence of robust classical aspects explaining the well defined position of the pointer in a measurement.

Also recently Haag [15] said that the deterministically propagating (due to the Schrödinger equation) pure state of a quantum particle has no objective significance and do not represent a real phenomenon. The only relevant notion are the events, namely a set of mutually exclusive possibilities with an associate probability assignment for the outcomes of operationally defined measurements. To identify events in particle physics we need not only the beams of incoming particles but also an effective microscopic description of the interaction of the particle with some constituent of the macroscopic detector. This means that we need a localization region in space-time and a description of the detector (what it is supposed to measure with a microscopic interaction then suitably amplified). Then we need a principle of random realization to justify the unique outcome of each measurement, whose repetition gives results distributed according to the Born rule. In this ensemble interpretation only the events, containing not only the quantum particle but also the measuring apparatus, have some kind of reality: due to decoherence things happens "as if" a semi-classical particle interacts with semi-classical constituents of the detector.

A first step to face all these problems (always in an ensemble interpretation) is presented in Ref.[16]. It is an approach more general than decoherence but limited to spin systems (the only ones where many-body calculations can be explicitly made). One considers the interacting object as an open quantum subsystem [17] of a macroscopic many-body system (system plus detector plus environment) with the induced non-unitary stochastic behavior (even without going to the thermodynamic limit): now time scales for the various phases of the measurement and aspects of decoherence can be explicitly evaluated.

The described state of affairs is in accord with Bohr’s point of view according to which we need a classical description of the experimental apparatus. It seems that all the realizable experiments must admit a quasi-classical description not only of the apparatus but also of the quantum particles: they are present in the experimental area as classical effective particles with a mean trajectory and a mean value of 4-momentum (measured with time-of-flight methods).

As shown in Ref.[18], if one assumes that the wave function describes the given quantum system (no ensemble interpretation), the statement of Bohr can be justified by noting that the wave functions used in the preparation of particle beams (semi-classical objects with a mean classical trajectory and a classical mean momentum determined with time-of-flight methods) are a special subset of the wave functions solutions of the Schrödinger equation for the given particles. Their associated density matrix, pervading the whole 3-space, admits a multi-polar expansion around a classical trajectory having zero dipole. This implies that
in this case the equations of the Ehrenfest theorem give rise to the Newton equations for
the Newton trajectory (the monopole; it is not a Bohmian trajectory) with a classical force
augmented by forces of a quantum nature coming from the quadrupole and the higher
multipoles (they are proportional to powers of the Planck constant). As shown in Ref.[18]
the mean trajectories of the prepared beams of particles and of the particles revealed by the
detectors are just these classically emerging Newton trajectories implied by the Ehrenfest
theorem for wave functions with zero dipole. Also all the intuitive descriptions of experiments
in atomic physics are compatible with this emergence of classicality. In these descriptions an
atom is represented as a classical particle delocalized in a small sphere, whose origin can be
traced to the effect of the higher-multipole forces in the emerging Newton equations for the
atom trajectory. The wave functions without zero dipole do not seem to be implementable
in feasible experiments. No explanation is given of the probabilistic Born rule, but it is
suggested that the random unique outcomes have a quasi-classical localization given by
these Newton trajectories.

In this paper we want to focus on the problems connected with the localization of particles
both at the relativistic and non-relativistic levels and both at the classical and quantum
levels QFT included. We identify the existing proposals for position measurements and
we analyze the existing theoretical problems in this area (usually they are not well known
to many researchers). The comparison with the experimental status of particle and atom
localization will help to clarify whether it is possible to measure the non-relativistic center
of mass of a system or whether it is non-measurable like the relativistic collective variables.
Then the results on localization will be used to clarify the connection between relativistic
and non-relativistic entanglement and what can be seen in the experiments.

We hope that collecting and gluing together results on these topics coming from usually
non mutually interacting communities will be helpful for researchers approaching the quickly
developing areas of mesoscopic physics, atomic and molecular physics, atomic clocks and
space physics, quantum information, teleportation,...

In Section III we make a review of the problems in the localization of particles both
at the relativistic (Subsection A) and at the non-relativistic (Subsection B) level; then in
Subsection C we look at the notion of particle in QFT and to its problems. In Section
IV we show the differences between non-relativistic and relativistic entanglement in a two-
body case induced by the relativistic spatial non-separability forbidding the identification of
subsystems. In Section V we study the preparation and detection of particles in experiments
and we propose a reconciliation of the non-relativistic and relativistic visions valid for all
practical purposes. In the final Section there are concluding remarks and a list of open
problems.

II. REVIEW OF RELATIVISTIC CLASSICAL AND QUANTUM MECHANICS

The new formulation of RCM [2, 3, 5] and of a consistent RQM [6] makes use of the 3+1
point of view to build a theory of global non-inertial frames centered on arbitrary time-like
observers [2, 5]. This is done by giving the world-line of the time-like observer and a nice
foliation of Minkowski space-time with non-intersecting space-like Riemannian 3-spaces, all
tending to the same space-like hyper-plane at spatial infinity. Moreover, one uses the radar
4-coordinates (τ; σr), i.e. an arbitrary monotonically increasing function of the proper time
of the atomic clock carried by the observer and curvilinear 3-coordinates σr centered on
the observer for the 3-spaces. CFT may be reformulated in this framework by using fields knowing the clock synchronization convention.

Both the knowledge of the whole world-line of an arbitrary time-like observer and of nice foliation with 3-spaces of Minkowski space-time are non-factual notions. The observer is a purely mathematical entity carrying a clock, an idealization of a physical atomic clock carried by a dynamical observer. The foliation is the mathematical idealization of a physical protocol of clock synchronization. Actually the physical protocols (think of GPS) can establish a clock synchronization convention only inside future light-cone of the physical observer defining the local 3-spaces only inside it. However to be able to formulate the Cauchy problem for field equations and to have predictability of the future, due to the theorem on the existence and unicity of solution of partial differential equations we have to extend the convention outside the light-cone. Once we have given the Cauchy data on the initial Cauchy surface (an unphysical process), we can predict the future with every observer receiving the information only from his/her past light-cone (retarded information from inside it; electromagnetic signals on it).

For non-relativistic observers the situation is simpler, but the non-factual need of giving the Cauchy data on a whole initial absolute Euclidean 3-space is present also in this case for non-relativistic field equations like the Euler equation for fluids.

A. Relativistic Classical Mechanics

As shown in Ref.[2] in this framework the description of isolated systems can be done with an action principle (the parametrized Minkowski theories for particles, fields, strings, fluids) implying that the transition among non-inertial frames is described by gauge transformations (so that only the appearances of phenomena change, not the physics) and allowing one to define the energy-momentum tensor and then the Poincaré generators of the system.

Inertial frames are a special case of this theory having Euclidean 3-spaces. For isolated systems there is a special family of inertial systems, the intrinsic rest frames, in which the space-like 3-spaces are orthonormal to the conserved time-like 4-momentum of the isolated system. At the Hamiltonian level it turns out that every isolated system can be described by a decoupled canonical non-covariant relativistic center of mass (whose spatial part is the classical counterpart of the Newton-Wigner position operator). Such a system carries a pole-dipole structure, namely an internal 3-space with a well defined total invariant mass $M$ and a total rest spin $\vec{S}$ and a well defined realization of the Poincaré algebra (the external Poincaré group for a free point particle, which we identify with the external center of mass, whose mass and spin are its Casimir invariants describing the matter of the isolated system in a global way). The internal rest 3-space, named Wigner 3-space, is defined in such a way that it is the same for all the inertial rest frames and its 3-vectors are Wigner spin-1 3-vectors [19], so that the covariance under Poincaré transformations is under control. The particles of the isolated system, all having the same time of the given 3-space, are identified by this type of Wigner-covariant 3-vectors (see Refs.[3,5] for the description of fields).

As far as we know the theorem on the existence and unicity of solutions has not yet been extended starting from data given only on the past light-cone.
As shown in Refs.[6, 20] the canonical non-covariant (a pseudo 4-vector) relativistic center of mass, the non-canonical covariant (a 4-vector) Fokker-Price center of inertia and the non-canonical non-covariant (a pseudo 4-vector) Møller center of energy are the only three relativistic collective variables which can be built only in terms of the Poincaré generators of an isolated system so that they depend only on the system and nothing external to it. All of them collapse onto the Newton center of mass of the system in the non-relativistic limit 4.

As shown in Refs.[2, 4] in the Wigner 3-space there is another realization of the Poincaré algebra (the internal Poincaré group) built with the rest 3-coordinates and 3-momenta of the matter of the isolated system starting from its energy-momentum tensor: the internal energy is the invariant mass $M$ (the Hamiltonian inside the rest 3-space) and the internal angular momentum is the rest spin $\vec{S}$. Since we are in rest frames the internal 3-momentum must vanish. Moreover, to avoid a double counting of the center of mass, the internal center of mass, conjugate to the vanishing 3-momentum, has to be eliminated: this is done by fixing the value of the internal Poincaré boost. If we put it equal to zero, this implies that the time-like observer has to be an inertial observer coinciding with the non-canonical 4-vector describing the Fokker-Price center of inertia of the isolated system. Therefore the internal realization of the Poincaré algebra is unfaithful and inside the Wigner rest 3-spaces the matter is described by relative 3-positions and 3-momenta.

The world-lines of the particles (and their 4-momenta) are derived notions, which can be rebuilt given the relative 3-coordinates, the time-like observer (for instance the Fokker-Price center of inertia) and the axes of the inertial rest frame [4]. They are described by 4-vectors $x^\mu(\tau)$, which however are not canonical like in most of the approaches: there is a classical non-commutative structure induced by the Lorentz signature of Minkowski space-time [4, 6].

As shown in Ref. [3] these three variables can be expressed as known functions of the Lorentz scalar rest time $\tau$, of canonically conjugate Jacobi data (frozen (fixed $\tau = 0$) Cauchy data) $\vec{z} = Mc\vec{x}_{NW}(0)$, $\vec{h} = \vec{P}/Mc$, ($\vec{x}_{NW}(\tau) = \vec{x}(\tau)$ is the standard Newton-Wigner 3-position; $P^\mu$ is the external 4-momentum), and of the invariant mass $M$ and rest spin $\vec{S}$. The external Poincaré generators are then expressed in terms of these variables.

As said in Ref.[6], since the three relativistic collective variables depend on the internal Poincaré generators $M$ and $\vec{S}$, which are conserved integrals of suitable components of the energy-momentum tensor of the isolated system over the whole rest 3-space, they are non-local quantities which cannot be determined with local measurements.

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4 It is of interest that the three properties of the non-relativistic center of mass, namely i) a position associated with the spatial mass distribution of the constituents ii) it transformation under rotations as a three vector and iii) together with the total momentum being canonical variables, have their respective relativistic counterparts taken up by the Moller non-covariant, non-canonical center of energy $R^\mu(\tau)$, the covariant but non-canonical Fokker Pryce center of inertia $Y^\mu(\tau)$ and the canonical, but non-covariant center of mass $\ddot{x}^\mu(\tau)$. 

B. Relativistic Quantum Mechanics

The use of $\vec{z}$ avoids taking into account the mass spectrum of the isolated system at the quantum kinematical level and allows one to avoid the Hegerfeldt theorem (the instantaneous spreading of wave packets with violation of relativistic causality) in RQM [6].

Besides these non-local features in RQM there is an intrinsic spatial non-separability forbidding the identification of subsystems at the physical level and generating a notion of relativistic entanglement very different from the non-relativistic one.

In order to exhibit these two properties, let us consider a quantum two-body system. In non-relativistic quantum mechanics (NRQM) its Hilbert space can be described in the three following unitarily equivalent ways [6]: A) as the tensor product $H = H_1 \otimes H_2$, where $H_i$ are the Hilbert spaces of the two particles (separability of the two subsystems as the zeroth postulate of NRQM); B) as the tensor product $H = H_{\text{com}} \otimes H_{\text{rel}}$, where $H_{\text{com}}$ is the Hilbert space of the decoupled free Newton center of mass and $H_{\text{rel}}$ the Hilbert space of the relative motion (in the interacting case only this presentation implies the separation of variables in the Schrödinger equation); C) as the tensor product $H = H_{J,\text{com}} \otimes H_{\text{rel}}$, where $H_{J,\text{com}}$ is the Hilbert space of the frozen Jacobi data of the Newton center of mass (use is made of the Hamilton-Jacobi transformation).

Each of these three presentations gives rise to a different notion of entanglement due to the different notion of separable subsystems. As shown in Ref. [19] other presentations are possible in NRQM: in each presentation there is a different notion of separable or entangled pure state (the same is true in the mixed case).

As shown in Ref. [6], at the relativistic level the elimination of the relative times of the particles (they are defined in a 3-space with a definite value of time) and the treatment of the relativistic collective variables allows only the presentation C), i.e. $H = H_{J,\text{com}} \otimes H_{\text{rel}}$ with $H_{J,\text{com}}$ being the Hilbert space associated to the quantization of the canonically conjugate frozen Jacobi data $\vec{z}$ and $h$ and $H_{\text{rel}}$ is the Hilbert space of the Wigner-covariant relative 3-coordinates and 3-momenta. Therefore only the frozen relativistic 3-center of mass and the set of all the relative variables are the admissible separable relativistic subsystems in RQM. Already at the classical level the subsystems particle 1 and particle 2 (without relative times) are only defined in the un-physical rest 3-space, which is how one terms the 3-space before adding the rest-frame conditions eliminating the internal 3-center of mass and its 3-momentum. In contrast, the physical space is the one with these constraints imposed. The rest-frame conditions, defining the physical variables, destroy the separability of the particles leaving only relative variables. In this framework there are no problems with the treatment of relativistic bound states.

Let us remark that instead of starting from the physical Hilbert space containing the frozen Jacobi data, one could first define an un-physical Hilbert space containing the Jacobi data and the 3-position and 3-momenta of the particles (in it we have the same kind of

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5 If one considers the tensor product $H_1 \otimes H_2$ of two massive Klein-Gordon particles most of the states will have one particle allowed to be the absolute future of the other due to the lack of restrictions on the relative times. Only in S-matrix theory is this irrelevant since one takes the limit for infinite future and past times.
separability as in the presentation A) of NRQM) and then define the physical Hilbert space by imposing the rest-frame conditions at the quantum level with the Gupta-Bleuler method. However there is the risk to get an inequivalent quantum theory due to the complex form of the internal boosts.

C. Classical and Quantum Field Theory

Given a 3+1 splitting associated with a time-like observer using radar 4-coordinates $\sigma^A = (\tau, \sigma^r)$ we can rebuild the Cartesian coordinates of an inertial observer of Minkowski space-time with a coordinate transformation $\sigma^A \rightarrow x^\mu = z^\mu(\tau, \sigma^r)$ with $z^\mu(\tau, 0) = x^\mu(\tau)$ being the world-line of the time-like observer. The functions $z^\mu(\tau, \sigma^r)$ describe the embedding of the instantaneous 3-spaces $\Sigma_\tau$ in Minkowski space-time.

Given a classical field, for instance the Klein-Gordon field $\tilde{\phi}(x^\mu)$, its reformulation as a field knowing the clock synchronization convention is $\phi(\tau, \sigma^r) = \tilde{\phi}(x^\mu = z^\mu(\tau, \sigma^r))$. These are the fields used in parametrized Minkowski theories [2].

As shown in Ref.[3] for the case of particles plus the electro-magnetic field, at the classical level one can define the relativistic external center of mass and the relative variables for these fields and find the rest-frame conditions eliminating the internal center of mass. In atomic physics this allows to avoid pathologies like the Haag theorem (non existence of the interaction picture in QFT) and to follow the evolution of atoms in the interaction region for finite times taking into account the relativistic properties of non-separability and non-locality. The extension of these results to QFT is highly non-trivial, because at the classical level one uses variables of the action-angle type for which no consistent quantization exists. The alternative is to quantize the standard variables and to try to impose the quantum rest-frame conditions with the Gupta-Bleuler methods. In any case a consistent quantization along these lines would lead to a non-local QFT due to the relativistic properties of non-separability and non-locality.

III. LOCALIZATION OF PARTICLES

Both NRQM and RQM are defined on a fixed space-time structure, the Galilei and Minkowski space-times respectively. More exactly they are defined in the inertial frames of these space-times, because the extension to non-inertial frames is still an open problem 6.

This spatio-temporal point of view is presupposed to the postulates of quantum mechanics (QM) and to each possible interpretation of it. It is only at the level of Einstein general relativity, where the metric structure of space-time and space-time itself become dynamical, that this scheme breaks down opening the basic problem of getting a consistent theory of quantum gravity conciliating QM and gravitation (such a problem does not exist for Newtonian gravity, which is defined in Galilei space-time).

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6 At the classical level, the framework described in Section II for the definition of parametrized Minkowski theories leads to the theory of global non-inertial frames (see Ref.[2]). However till now only the quantization of particles in relativistic rotating frames and its non-relativistic limit have been studied (see Refs.[22]).
If we start with this space-time oriented point of view, QM is defined in the Euclidean 3-spaces of the inertial frames of either Galilei or Minkowski space-time. This implies that the coordinate representation has a privileged kinematical and descriptive status among all possible bases in the Hilbert space of quantum systems. Since all the experiments are localized in space-time, it is important to consider always the trajectories of the carriers of quantum properties (like spin or qubits or other quantum numbers) and not to treat the quantum systems independently from their localization in the space-time. See the second part of the review paper [23] for the relevance of Lorentz transformations for creating entanglement between spin and momentum degrees of freedom.

This privileged status of the coordinate representation coming from the spatio-temporal interpretation is different (but is reinforced) by the existence of a natural selection of robust positional bases of pointer states for the apparatuses appearing in the de-coherence approach to QM with its dominant role of the environment in the description of entanglement (see Refs. [13, 14]).

Since the relativistic spatial non-separability forbidding the identification of the subsystems of the given quantum system is a consequence of defining relativistic collective position variables, one has to face the open problem of position measurements in QM. While most of the mathematical properties of quantum systems are based on instantaneous precise measurements of self-adjoint bounded operators (the observables) with a discrete spectrum, whose treatment requires projection operators (or projection valued measures, PVM), position operators are usually described by self-adjoint unbounded operators without normalized position eigenvalues (usually one uses the improper Dirac kets |\vec{x}>, sharp eigenstates of the position operator, satisfying \langle \vec{x}|\vec{y}> = \delta^3(\vec{x} - \vec{y})). However, as noted in Ref.[13], to be able to describe the standard (even if questionable) postulate of the non-unitary collapse of the wave function one needs position wave functions with a finite support (inside the apparatus) to avoid the necessity of using arbitrary strong couplings and arbitrarily large amounts of energy to make an arbitrarily precise measurement of position. As a consequence, the notion of unsharp positions with bad localization has emerged (see Refs.[24] for the theory of projection operator valued measures, POVM). The results of measurements of a POVM give imprecise information of stochastic type on the localization of particles (see for instance Ref.[25] for continuous quantum position weak measurements).

We will now describe some of the existing problems with the notions of position and localization both at the relativistic and non-relativistic levels, having in mind the following question: "Is the center-of-mass position measurable"?

A. The Relativistic Case

In the relativistic case there are two types of problems, one at the classical level, the other at the quantum level.

α) Möller non-covariance world-tube [26]. As we have said, in each relativistic inertial frame one has the world-line of the Fokker-Price non-canonical, covariant center of inertia $Y^\mu(\tau)$ of the isolated system and different pseudo-world-lines for the non-covariant, canonical 4-center of mass $\tilde{x}^\mu(\tau)$ and for the non-covariant, non-canonical Möller center of energy $R^\mu(\tau)$. If in a given inertial frame we consider the positions of $\tilde{x}^\mu(\tau)$ and $R^\mu(\tau)$ corresponding to every possible inertial frame, we get a tube centered on $Y^\mu(\tau)$ (with $\tilde{x}^\mu(\tau)$ always lying
between $Y^\mu(\tau)$ and $R^\mu(\tau))$. The invariant radius of the tube is determined by the two Casimirs invariant mass $M$ and rest spin $\vec{S}$: $\rho = |\vec{S}|/Mc$. As said in Ref.[3], this classical intrinsic radius is a non-local effect of the Lorentz signature of Minkowski space-time absent in Euclidean spaces and delimits the place of the non-covariant effects (the pseudo-worldlines) connected with the relativistic collective variables. These effects are not classically detectable because the Møller radius is of the order of the Compton wavelength of the isolated system: an attempt to test its interior would mean to enter in the quantum regime of pair production. The Møller radius $\rho$ is also a remnant of the energy-conditions of general relativity in the flat Minkowski space-time: if a body has its material radius less than its Møller one, then there is some inertial frame in which the energy density of the body is not positive definite even if the total energy is positive [26].

Therefore the Compton wavelength is the best theoretical approximation for the localization of a classical massive particle.

\( \beta \) Newton-Wigner operator. As found in Ref.[6], at the quantum level the spatial component $\vec{x}$ of the canonical non-covariant center of mass $\vec{x}^\mu = (\vec{x}^0; \vec{x})$ becomes the Newton-Wigner position operator [27], whose eigenfunctions are wave functions with infinite tail and a mean width around the eigenvalue of the order of the Compton wavelength. Therefore also at the quantum level there is bad localization.

In Refs.[28] it is said that we cannot consider the Newton-Wigner operator a self-adjoint operator (in the framework of quantum field theory it is neither a local nor a quasi-local operator) but at best a symmetric operator \(^7\). See Refs.[29] for an approach to fuzzy localization based on the use of certain types of symmetric operators.

Let us remark that, whichever point of view is chosen for the position operator, the generators of the Poincaré algebra (in particular the Lorentz ones) of isolated systems must be described by self-adjoint operators as it is usually assumed. This implies that the 4-momentum operators must be self-adjoint operators.

In the approach presented in Section II these problems appear only for the external non-covariant canonical center of mass described by the Jacobi data $\vec{z}$. While its conjugate variable $\vec{h}$ must be taken as a self-adjoint operator, it is a totally open problem how to quantize $\vec{z}$ and whether one has to introduce super-selection rules [30] either forbidding its measurability or at least forbidding the possibility of making center-of-mass wave packets (only plane waves with fixed eigenvalue of $\vec{h}$ allowed).

However, particle physics experiments utilize beams of particles with a mean 4-momentum and localized around a classical trajectory pointing to the experimental area. Therefore the description of particle beams requires well picked wave packets in momentum space with also a good localization in the 3-space.

\(^7\) An operator $A$ in an infinite dimensional Hilbert spaces is said to be symmetric if $\langle Ay|x \rangle = \langle y|Ax \rangle$. Such operators are not diagonalizable and therefore describe real degrees of freedom which display a form of "unsharpness" or "fuzzyness".
B. Non-Relativistic Case

In standard NRQM the position of particles are usually described by self-adjoint unbounded operators and usually one says that there are only experimental problems with localization of particles and atoms.

However recently in the framework of the theory of measurements based on POVM there was a revisit of the problem by extending the Wigner-Araki-Yanase theorem [31] from bounded (like angular momentum) to unbounded (position) operators. The theorem says that given a conserved quantity (additive over the system plus apparatus), then a discrete self-adjoint operator non-commuting with the conserved quantity does not admit perfectly accurate and repeatable measurements. In Refs.[32] it was shown that generically in an isolated two-body system with conserved momentum the conjugate center-of-mass operator (and also the absolute positions of the two particles) are unsharp. Unsharp positions are different from the un-determination of symmetric operators, but the final result is the same. On the other hand, there is no problem with the relative position variable.

At the experimental level the previous statements have been confirmed also in Refs.[33], where it is shown that it is only possible to measure mutual relative positions of atoms. Regarding their absolute positions the best localization of atoms which can be realized is at the level of hundred of nanometers [34, 35], much higher than the atom Compton wavelength.

In conclusion at every level we have indications that the absolute position of massive particles can be determined only with a precision most probably much greater that the Compton wavelength of the particle, as it happens with the radius of the macroscopic tracks of particles in bubble chambers. Therefore also in the non-relativistic case it seems that there are problems with the localization of the center of mass of isolated quantum systems: one can only say that effective atoms are inside the size of the apparatus.

As a consequence this state of affairs together with the results of Ref.[33] point to the same picture as in RQM: a non-measurable center of mass plus non-separable relative motions.

Let us also remark that the standard treatment of non-relativistic particles, with its notion of separability of subsystems, ignores the fact that to take into account electromagnetic interactions one has to use a $1/c$ approximation of QED below the threshold of pair production. Only in the limit $c \to \infty$ one has an irreversible contraction of the Poincaré algebra to the Galilei one. Therefore atomic physics needs a relativistic formulation like the one of Ref.[5] even when the particles have non-relativistic velocities: as just said this picture is emerging already in NRQM. However the quantization of the electromagnetic field in the rest frame, with the rest-frame conditions implemented, has still to be done.

Finally notions like the Planck length, dimensionally relevant when one takes into account gravity, are completely outside the existing experimental level.

C. The Notion of Particle in Quantum Field Theory

Often it is said that RQM of particles is an irrelevant theory, because relativistic particles have to be described by QFT, which solves the problem of negative energies with anti-particles and allows pair production. This is an ambiguous statement. Firstly the description
of relativistic bound states requires a transition from exact QFT equations (like the Bethe-Salpeter one) to effective RQM ones, valid below the threshold of pair production. Secondly the existing notion of particle in QFT can be done only for free fields and is subject to criticism as it can be seen from the (philosophically oriented, but mathematically relevant) papers of Ref. [36] 8. Finally the standard definition of particles by means of the Fock space gives rise to completely delocalized objects (plane waves to be used in the in- and out-states of scattering theory). Moreover, the definition of positive- and negative-energy particles requires the existence of a time-like Killing vector of the space-time and of a suitable vacuum state (so that in curved space-times without Killing vectors one has to use algebraic QFT, where no sound definition of particle exists [37]).

Here we will consider only the massive Klein-Gordon uncharged quantum field in an inertial frame of Minkowski space-time re-expressed in the radar 4-coordinates of an inertial 3+1 splitting with Euclidean 3-spaces. It has the expression

\[
\hat{\phi}(\tau, \vec{\sigma}) = \int \frac{d^3k}{(2\pi)^3 \sqrt{2\omega_k}} [e^{-i(\omega_k \tau - \vec{k} \cdot \vec{\sigma})} \hat{a}(\vec{k}) + e^{i(\omega_k \tau - \vec{k} \cdot \vec{\sigma})} \hat{a}^\dagger(\vec{k})],
\]

with \(\omega_k = \sqrt{\vec{k}^2 + m^2 c^2}\). Instead of the plane waves \(e^{\pm i(\omega_k \tau - \vec{k} \cdot \vec{\sigma})}\) one can use any other basis of positive- and negative-energy solutions of the classical Klein-Gordon equation. By using the creation operators \(\hat{a}^\dagger(\vec{k})\) one can build the standard Fock space starting from the vacuum (defined by \(\hat{a}(\vec{k})|0 \rangle = 0\)): it describes the particle (or better quanta) states of the theory.

Let us consider a 1-particle state \(\hat{a}^\dagger(\vec{k})|0 \rangle\) with an associated positive-energy solution \(g(\tau, \vec{\sigma}) = \langle \vec{\sigma} | g(\tau) \rangle\) of the classical Klein-Gordon equation. Therefore this wave function satisfies \(i\hbar \frac{\partial}{\partial \tau} g(\tau, \vec{\sigma}) = +\sqrt{m^2 c^2 - \hbar^2} \frac{\partial}{\partial \vec{\sigma}} g(\tau, \vec{\sigma})\). Let \(< g(\tau) | \vec{\sigma} | g(\tau) \rangle\) and \(< g(\tau) | \vec{\sigma} | g(\tau) \rangle\) denote the expectation values of the position and momentum operators in this state.

As shown in Ref. [18] 9 we can consider the multipolar expansion of the wave function \(g(\tau, \vec{\sigma})\) around a classical trajectory \(\vec{\sigma}_c(\tau)\). For all the wave functions with vanishing dipole moment with respect to the classical trajectory we get \(< g(\tau) | \vec{\sigma} | g(\tau) \rangle = \vec{\sigma}_c(\tau)\) and the Ehrenfest theorem implies \(\frac{d}{d\tau} < g(\tau) | \vec{\sigma} | g(\tau) \rangle = \frac{d}{d\tau} \vec{\sigma}_c(\tau)\) and \(\frac{d}{d\tau} < g(\tau) | \vec{\sigma}_c(\tau) | g(\tau) \rangle = 0\), so that the classical trajectory is determined by the equation \(\frac{d^2}{d\tau^2} \vec{\sigma}_c(\tau) = 0\). Therefore it is possible to associate an effective particle following an effective mean trajectory only to all the 1-particle states whose wave function has a vanishing dipole.

Already in Minkowski space-time, without going to curved space-times and remaining in the area of condensed matter, the definition of an interacting theory governed by a

---

8 In the interacting case one looses the control on the mass shell condition of the interacting particles. Let us remark that in formulation of RCM of Refs. [2–4] the mass-shell condition is a derived property and depends upon the interactions.

9 See the expanded version 1 of the arXiv paper.
unitary time evolution is a non-trivial problem: see Ref.[38] for the difficulties to define a self-adjoint Hamiltonian operator bounded from below in free and non-free QFT. Even when this can be done, like in some cases with Hamiltonians bilinear in the creation and annihilation operators, the time evolution implies a unitary (i.e. of the Hilbert-Schmidt type) Bogoliubov transformation leading to new creation and annihilation operators linear combination of the old ones. At each time the instantaneous annihilation operator defines a new instantaneous vacuum, from which a new instantaneous Fock space with a different notion of particle can be created. The new 1-particle states are a superposition of all the states (with every possible particle number) of the initial Fock space. The big open problem is what kind of quanta (either the initial or the final ones) materialize as effective particles detected by the measuring apparatus (it too can be either inertial or accelerated).

In the free case in Minkowski space-time one can consider uniformly accelerated observers (the Rindler ones used for obtaining the Unruh effect [39]): they use a different time-like Killing vector for defining the notion of positive energy and their description of the free Klein-Gordon quantum field is connected with the standard description given by an inertial observer by a Bogoliubov transformation leading to a representation of the free field unitarily inequivalent to the inertial one. Again which one of the unitarily inequivalent quanta give rise to an effective particle to be detected [40] ? The use of Rindler observers for studying the entanglement of modes of the electro-magnetic field in moving cavities in the framework of quantum optics is a quickly developing sector of relativistic quantum information even if the basic interpretational problems are unsolved.

Moreover it has been shown [41] that if one describes the free massive Klein-Gordon field in non-inertial frames (like it is done in the Tomonaga-Schwinger formulation on arbitrary space-like hyper-surfaces of Minkowski space-time) then generically the time evolution is not unitarily implementable (the implied Bogoliubov transformation is not of the Hilbert-Schmidt type).

In conclusion the notion of particle in QFT is essentially valid for the in- and out-states of the S matrix in inertial frames, a framework relying upon a perturbation expansion with suitable ultra-violet and infra-red cutoffs.

IV. RELATIVISTIC ENTANGLEMENT VERSUS NON-RELATIVISTIC ENTANGLEMENT

After the localization problem let us now look at a simple two-body problem to display the changes in its separability and entanglement properties going from the non-relativistic case to the relativistic one. This example will also show the explicit construction of the relativistic collective variables in the two-body case.

As shown in Ref.[42], the electron-proton system (with masses $m_e$ and $m_p$ respectively) in the hydrogen atom, governed by the Hamiltonian $H = \frac{\vec{p}_e^2}{2m_e} + \frac{\vec{p}_p^2}{2m_p} - \frac{e^2}{|\vec{x}_e - \vec{x}_p|} = H_{\text{com}} + H_{\text{rel}}$ with $H_{\text{com}} = \frac{\vec{p}_e^2}{2M}$ and $H_{\text{rel}} = \frac{\vec{p}_p^2}{2\mu} - \frac{e^2}{r}$ ($M = m_e + m_p$, $\mu = \frac{m_em_p}{M}$), can be presented in two ways. Either it is composed by the subsystems electron $m_e$ and proton $m_p$ (with coordinates and momenta $\vec{x}_e$, $\vec{p}_e$ and $\vec{x}_p$, $\vec{p}_p$, respectively) or by the subsystems center of mass $M$ (with coordinate and momentum $\vec{x} = \frac{m_e\vec{x}_e + m_p\vec{x}_p}{M}$, $\vec{p} = \vec{p}_e + \vec{p}_p$) and relative motion $\mu$ (with
coordinate and momentum \( \vec{r}, \vec{p} \). At the quantum level all the positions and momenta become self-adjoint operators.

Since both in scattering and bound state theories one describes the dynamics in the preferred momentum basis with given conserved total momentum \( \vec{p} \), the center-of-mass position \( \vec{x} \) is undetermined. In the theoretical description of these theories one never considers wave packets in \( \vec{p} \) with defined localization properties of the center of mass, but only plane waves with the given value of \( \vec{p} \) following the standard approach without considering the problems of unsharp states exposed in the previous Section.

Therefore a stationary solution of the Schroedinger equation for the hydrogen atom (it factorizes only in the center-of-mass and relative variables) in the coordinate representation is

\[
\psi(\vec{x}, \vec{r}) = \phi_{\text{int}}(\vec{r}) e^{i \vec{p} \cdot \vec{x}} = \phi_{\text{int}}(\vec{x}_e - \vec{x}_p) e^{\frac{i}{\hbar} \vec{p} \cdot \vec{x}_e + m_e \vec{p} \cdot \vec{x}_p} \Xi(\vec{x}_e, \vec{x}_p),
\]

where \( \phi_{\text{int}}(\vec{r}) \) is one of the energy levels of the atom. Our presentation here is in the spirit of formal scattering theory done with plane waves in virtually all text books. In actuality the effective particle beams must be described with Gaussian wave packets with a "classical mean momentum" obtained with flight time methods.

If we now trace out the center of mass, we get the reduced density matrix \( \rho_{\text{rel}}(\vec{r}, \vec{r}') = \phi_{\text{int}}(\vec{r}) \phi_{\text{int}}^*(\vec{r}') \) with the associated entanglement properties of the subsystem relative motion. If instead we trace out the proton, we get the reduced density matrix for the entanglement properties of the subsystem electron.

To avoid the complications of the full particle and field configurations discussed in Ref.[3], we will consider the simple two-body system studied in Ref.[4], which is described in the framework explained in the Introduction. If \( \vec{z}, \vec{h} \), are the frozen Jacobi data of the relativistic center of mass, at the classical level the rest frame is defined by the embedding of the intrinsic rest 3-spaces of the 3+1 foliation into Minkowski space-time (see Refs.[4, 6]; \( \tau \) and \( \sigma^r \) are radar coordinates and the \( W \) index on the embedding refers to the role of the Wigner rest frame)

\[
\begin{align*}
\tilde{z}_\mu(\tau, \vec{\sigma}) &= Y^\mu(\tau) + \epsilon_\nu^\mu(\vec{h}) \sigma^r, \\
\epsilon_\nu^\mu(\vec{h}) &= \left(h_r; \delta_\nu^i + \frac{h^i h_r}{1 + \sqrt{1 + \vec{h}^2}}\right), \\
Y^\mu(\tau) &= \left(\sqrt{1 + \vec{h}^2} (\tau + \frac{\vec{h} \cdot \vec{z}}{M c}) \frac{\vec{z}}{M c} + (\tau + \frac{\vec{h} \cdot \vec{z}}{M c}) \vec{h} + \frac{\vec{S} \times \vec{h}}{M c (1 + \sqrt{1 + \vec{h}^2})}\right), \\
\tilde{x}^\mu(\tau) &= Y^\mu(\tau) + \left(0; \frac{-\vec{S} \times \vec{h}}{M c (1 + \sqrt{1 + \vec{h}^2})}\right),
\end{align*}
\]

(4.3)
where $Y^\mu(\tau)$ is the Fokker-Price center of inertia, $\tilde{x}^\mu(\tau)$ the canonical center of mass, $M_c$ the invariant mass and $\vec{S}$ the rest spin of the two-body system. The external Poincaré group has the generators $P^\mu = M_c c h^\mu = M c \left( \sqrt{1 + \vec{h}^2}; \vec{h} \right)$, $J^{ij} = z^i h^j - z^j h^i + \epsilon^{ijk} S^k$, $K^i = J^{oi} = -\sqrt{1 + \vec{h}^2} z^i + \frac{(\vec{S} \times \vec{h})}{1 + \sqrt{1 + \vec{h}^2}}$ (the last term in the boost is responsible for the Wigner covariance of the 3-vectors in the rest Wigner 3-space $\tau = \text{const.}$).

Before adding the rest-frame conditions the world-lines and the 4-momenta of the two particles are ($V$ is an arbitrary action-at-a-distance potential $^{10}$)

$$
\begin{align*}
  x_i^\mu(\tau) &= z_i^\mu(\tau, \vec{\eta}(\tau)) = Y^\mu(\tau) + c_i^\mu(\tau) \eta_i(\tau), \\
  p_i^\mu(\tau) &= h^\mu \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} + V((\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau))^2) - c_i^\mu(\vec{h}) \kappa_i(\tau), \\
  |p_i^2| &= m_i^2 c^2 + V((\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau))^2).
\end{align*}
$$

This equations imply that the un-physical Wigner-covariant 3-positions and 3-momenta inside the rest Wigner 3-space are $\vec{\eta}_i(\tau), \vec{\kappa}_i(\tau), i = 1, 2$. The conserved internal Poincaré generators are ($M_c$ is the Hamiltonian for the motion inside the Wigner 3-space)

$$
\begin{align*}
  M c &= \sum_{i=1}^{2} \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} + V((\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau))^2), \\
  \vec{\bar{P}} &= \sum_{i=1}^{2} \vec{\kappa}_i(\tau) \approx 0, \\
  \vec{\bar{S}} &= \sum_{i=1}^{2} \vec{\eta}_i(\tau) \times \vec{\kappa}_i(\tau), \\
  \vec{\bar{K}} &= -\sum_{i=1}^{2} \vec{\eta}_i(\tau) \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} + V((\vec{\eta}_1(\tau) - \vec{\eta}_2(\tau))^2) \approx 0.
\end{align*}
$$

The rest-frame conditions $\vec{\bar{P}} \approx 0, \vec{\bar{K}} \approx 0$, imply that the physical canonical variables in the rest 3-space are $\vec{r}(\tau) = \vec{\eta}_1(\tau) - \vec{\eta}_2(\tau)$ and $\vec{\pi}(\tau) = \frac{m_1}{M} \vec{\kappa}_1(\tau) - \frac{m_2}{M} \vec{\kappa}_2(\tau), (M = m_1 + m_2)$. Using these relative variables and imposing the rest frame condition gives the following expressions for the internal center of mass $\vec{\eta}(\tau)$ (conjugate to $\vec{\bar{P}} \approx 0$) and for the world-lines

$$
\begin{align*}
  \vec{\eta}(\tau) &= \frac{m_1 \vec{\eta}_1(\tau) + m_2 \vec{\eta}_2(\tau)}{M} \approx \frac{m_1 \sqrt{m_2^2 c^2 + H(\tau)} - m_2 \sqrt{m_1^2 c^2 + H(\tau)}}{M (\sqrt{m_1^2 c^2 + H(\tau)} + \sqrt{m_2^2 c^2 + H(\tau)})} \vec{\rho}(\tau), \\
  H(\tau) &= \vec{\pi}^2(\tau) + V(\vec{\rho}^2(\tau)),
\end{align*}
$$

$^{10}$ In the electromagnetic case of Ref.[3] the Coulomb potential plus the Darwin one are outside of the square root.

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\[
M c \approx \sqrt{m_1^2 c^2 + H(\tau)} + \sqrt{m_2^2 c^2 + H(\tau)}, \quad \vec{S} \approx \vec{\rho}(\tau) \times \vec{\pi}(\tau),
\]
\[
x_1^\mu(\tau) \approx Y^\mu(\tau) + \epsilon_\mu^\nu(\vec{h}) \sqrt{m_2^2 c^2 + H(\tau)} \rho^\nu(\tau),
\]
\[
x_2^\mu(\tau) \approx Y^\mu(\tau) - \epsilon_\mu^\nu(\vec{h}) \sqrt{m_1^2 c^2 + H(\tau)} \rho^\nu(\tau).
\]

(4.6)

Let us remark that only in the global inertial frame defined by \( \vec{h} = 0 \) (which we designate as the center-of-mass frame) one has \( x_1^0(\tau) = x_2^0(\tau) = \tau \). For any other value of \( \vec{h} \) (for instance in the laboratory frame \( \vec{p}_2(\tau) = 0 \), so that \( \vec{\pi} \) becomes parallel to \( \vec{h} \)) the time variables of the world-lines do not coincide, so that we cannot make equal time statements by using them (for more details see the relativistic kinetic theory of fluids developed in Ref. [43]).

The quantization of the model is done in the preferred \( \vec{h} \)-base. The variables \( \vec{h} \), \( \vec{\rho} \), \( \vec{\pi} \), are replaced by self-adjoint operators. The open problems are whether we replace \( \vec{z} \) with either a self-adjoint or a symmetric operator (see the discussion on the Newton-Wigner operator in Section III) and whether we accept either only momentum plane waves or also wave packets with some localization of the center of mass (the particle beams with mean classical trajectory). If \( \vec{z} \) becomes a self-adjoint operator, then also the external Lorentz generators can be made self-adjoint after a suitable ordering. As a consequence, the operators corresponding to the world-lines of the particles become complicated objects needing non-trivial orderings except in the case \( \vec{h} = 0 \), the only one in which \( \vec{\rho} = \vec{\eta}_1 - \vec{\eta}_2 = \vec{x}_1 - \vec{x}_2 \).

If we work in the \( \vec{h} \), \( \vec{\rho} \), basis of the Hilbert space and we fix \( \vec{h} = \vec{k} \), then the wave function is \( \psi(\vec{h}, \vec{\rho}, \tau) = \delta^3(\vec{h} - \vec{k}) \phi(\vec{\rho}, \tau) \) with \( \phi(\vec{\rho}, \tau) \) satisfying the Schroedinger equation \( i \hbar \frac{\partial}{\partial \tau} \phi(\vec{\rho}, \tau) = M c \phi(\vec{\rho}, \tau) \). By putting \( \phi(\vec{\rho}, \tau) = \exp(-\frac{i}{\hbar} \vec{z} \cdot \vec{k} \tau) \phi(\vec{\rho}) \), the stationary solutions \( \phi_{n1m}(\vec{\rho}) \) satisfy the equations

\[
\check{M} c \phi_{n1m}(\vec{\rho}) = \epsilon_n \phi_{n1m}(\vec{\rho}),
\]
\[
\hat{S}^2 \phi_{n1m}(\vec{\rho}) = l (l + 1) \phi_{n1m}(\vec{\rho}), \quad \hat{S}_3 \phi_{n1m}(\vec{\rho}) = m \phi_{n1m}(\vec{\rho}),
\]

and the external 4-momentum becomes \( P_n^\mu = \frac{1}{c} \left( \epsilon_n \sqrt{1 + k^2}; \epsilon_n k \right) \). In the \( \vec{z} \) basis of the Hilbert space we get a plane wave \( e^{i \vec{k} \cdot \vec{z}} \) for the delocalized center of mass.

Regarding entanglement we can trace out the center of mass and find the reduced density matrix of the subsystem ”relative motion”: it is of the type \( \rho_{rel}(\vec{\rho}, \vec{\rho}^*) = \phi(\vec{\rho}) \phi^*(\vec{\rho}) \) like in the non-relativistic case.

In the center-of-mass frame \( \vec{h} = 0 \), where \( \vec{\rho} = \vec{\eta}_1 - \vec{\eta}_2 = \vec{x}_1 - \vec{x}_2 \), for \( \vec{\rho} = \vec{\rho}^* \) we get \( \rho_{rel}(\vec{\rho}, \vec{\rho}) = |\phi(\vec{\rho})|^2 = \rho_{int}(\vec{x}_1 - \vec{x}_2) \) to be compared with the non-relativistic equation (4.2) with \( \vec{p} = 0 \).

However we cannot study the subsystem 1 tracing out the subsystem 2: this is the spatial non-separability of relativistic entanglement discussed in the Introduction and in Ref. [6].

The same problems would appear in the study of the entanglement in scattering processes: see Refs. [44] for some of the existing results.
V. RECONCILIATION OF THE RELATIVISTIC AND NON-RELATIVISTIC WORLDS TAKING INTO ACCOUNT THE PREPARATION AND THE DETECTION OF PARTICLES IN EXPERIMENTS

We have seen that the non-relativistic notion of separability, according to which the Hilbert space of a quantum system composed of subsystems is the tensor product of the Hilbert spaces of the subsystems (zeroth postulate of NRQM), is destroyed in special relativity where clock synchronization is needed to define 3-space and to avoid relative times in bound states. This fact, together with the non-local nature of relativistic collective variables, induces a spatial non-separability implying that we can speak of subsystems only at an un-physical level, that existing before adding the rest-frame conditions. After their imposition we describe the overall system by a non-measurable external canonical non-covariant decoupled center of mass and by an internal world of Wigner-covariant relative variables. Only the frozen Jacobi data of the center of mass condition $\vec{h} = 0$ and the relative variables can be quantized consistently at the physical level.

This picture is strongly different from the standard non-relativistic framework, where there is unitary equivalence between the presentation with separable subsystems and the one with Newton center of mass and relative variables. However we have seen in Section III that there are problems also with the localization of the Newton center of mass (unsharp positions).

Let us consider a non-relativistic (but the same is true at the relativistic level) experiment testing a quantum system, let us say a two-particle system. There are dynamical observers (replaced with mathematical ones like Alice and Bob) using some apparatus for the preparation of the beam of particles to be used to define the system, having the particles interacting in a well (classically defined) way and using some detectors to extract information from the process. As said in the Introduction all these steps, realized by the observers, are imagined and realized by using a strongly classical intuition, in agreement with Bohr’s point of view according to which every feasible experiment must admit a classical description.

Coming back to the experiment, we see that the two incoming particles of our system are in special states (mean trajectory and mean momentum) prepared by the apparatus and that the outgoing particles can be detected only if they have a mean trajectory collineated with the detectors.

This means that at the experimental level an isolated two-particle system is a mathematical idealization. As said in the Introduction at best it is a (non-unitary evolving) open quantum system [17] always with some interaction first with the preparing apparatus and then with the detectors, both of which should be considered as quantum many-body systems.

Moreover, as noted in Ref.[45], the observers are not able to define a perfect classical mathematical reference frame, but only a bounded one determined by the level of precision of every instrument used.

As a consequence, in a realistic description of an experiment in NRQM one should consider as an idealized isolated quantum system at least the union of our two-particle system plus an environment composed by a quantum many-body description of the preparing apparatus and of the detecting one (they admit an effective classical description in terms of emerging effective notions like a pointer). The presence of interactions forces us to work in the position...
basis of the overall Hilbert space by using the decoupled Newton center of mass of the whole system and relative variables. Even if the non-relativistic center-of-mass position operator would be measurable (and this is an open problem due to unsharp positions), it is out of reach of the experiment (it pertains to the reference frame of the observer). What is measured of the particles are their relative variables with respect to the preparing apparatus and, after the interaction in the experimental area, with respect to the detectors (the effective mean trajectories of the incoming and outgoing particles) plus the relative variable between the two particles (it controls their mutual interaction). This is in accord with the experimental results of Ref.[33]: only the relative positions of atoms are measurable.

For all practical purposes (FAPP) this description is the same that we have at the relativistic level, now both for the isolated two-body problem and for the system particles plus experimental apparatus.

Therefore at the experimental level there is not a drastic difference between the relativistic and the non-relativistic frameworks induced by the Lorentz signature of Minkowski space-time below the threshold of pair production.

VI. CONCLUDING REMARKS

We have emphasized the differences between relativistic and non-relativistic quantum mechanics and the associated notions of entanglement in inertial frames and at the same time revealed unexpected common features.

Due to the Lorentz signature of Minkowski space-time, creating the problems of clock synchronization to define 3-space, of the elimination of the relative times in bound states and of the non-uniqueness of the relativistic collective variables, at the relativistic level we have global spatial non-separability limiting the existence of subsystems to an un-physical level before adding the rest-frame conditions to eliminate the internal collective variable in the Wigner 3-space. The external decoupled canonical non-covariant relativistic center of mass is a non-local, and therefore non-measurable, quantity already at the classical level.

These non-separability and non-locality both at the classical and quantum levels reduce the relevance of the still debated quantum non-locality of NRQM. As shown in Ref.[3] these properties are present also in CFT and their extension to QFT is a difficult open problem, which adds to the existing problems with the notion of particle in QFT [36] (instability under either unitary or non-unitary Bogoliubov transformations). The relativistic non-separability and non-locality point towards non-local QFT and rise the problem of the validity of micro-causality (quantum operators at space-like distances commute) at the relativistic level. This issue has already been raised by Busch [46] with the notion of unsharp observables (a local operator not measurable with local actions in a given 3-region). According to him sharp spatial localization is an operationally meaningless idealization (it requires an infinite amount of energy with unavoidable pair production; the quantum nature of the constituents of the detectors should be taken into account; and so on).

We have shown the status of the particle localization problem both at the relativistic and non-relativistic levels showing the existing theoretical problems and some of the experimental limitations. One would need the identification of some relevant (well mathematically defined) position basis of wave functions with compact support (or like the over-complete coherent state basis for the harmonic oscillator) for the position operator to be used for every type of localization problem.
By taking into account the quantum nature of both the preparing apparatus and of
the detectors we have shown that also at the non-relativistic level the real quasi-isolated
system with a decoupled center of mass is the full set of ”apparatus + detectors + observed
system” and that the prepared and detected particles, moving along mean classical Newton
trajectories, are described by relative variables.

Since, in any case, electromagnetism is always present, this implies that the relativistic
picture is valid at every level in experiments.

We have shown the non-factual nature of the mathematical time-like observers and
of their mathematical synchronization conventions for building reference frames (see also
Ref.[47] in the framework of quantum information). It seems quite difficult to develop a the-
ory in which Alice and Bob are dynamical observers 11 and exchange information by using a
dynamical electromagnetic field! Instead Alice and Bob are present in every protocol for rel-
avtivistic quantum information (see Ref.[49] for an old review emphasizing the problems with
special relativity), a quickly developing theory, in which the study of the spatio-temporal
trajectories of the investigated quantum systems is nearly always lacking.

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trajectories of the investigated quantum systems is nearly always lacking.

We have nothing to say about the probabilistic Born rule (randomness of the unique
outcomes) and on the nature of quantum reality (objective, subjective, mixture of classical
reality and information theory,...). However, if the wave function of a quantum system de-
dcribes its properties (no ensemble interpretation), then the experimentally observable wave
functions have an associated emergent classical description in terms of Newton trajectories.
Moreover, the spatial non-separability, introduced by SR already at the classical level, gives
rise to many non-ignorable problems: A) the world-lines of the observers and of the macro-
scopic apparatuses are inter-wined with the investigated relativistic (classical or quantum)
system and must be taken into account; B) as already said this implies that the observers
can no longer be treated as mathematical decoupled entities but must be macroscopic bod-
ies localized in the space-time; C) the non-separability together with Busch unsharpness
show that causality problems can no longer be solved by saying that systems in disjoined
regions with space-like distance are un-related; D) therefore foundational statements like the
freedom to choose the measurements settings independently from the investigated quantum
system (see for instance Refs.[50]) are no longer meaningful.

Finally in this paper we have only considered inertial frames. The extension of our
results to non-inertial frames is now under investigation, with an attempt to avoid uniformly
accelerated Rindler observers (they disappear with the light-cone in the non-relativistic
limit) but taking into account Unruh-DeWitt detectors 12 (see the review paper [22] and
the bibliography of Ref.[52]). The extension of these ideas to classical general relativity to
include gravity can be done along the lines described in the review paper [5], but the totally
open problem is to find a consistent theory of quantum gravity [37].

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11 First steps in this direction are done with quantum metrology (see Ref.[48], ch.7), in which a quantum
system is described by means of the relative variables with respect to another quantum system (the ob-
server) without using variables defined with respect to an external classical reference frame (this relational
approach is consistent with relativistic non-separability). It is under investigation what is the change in
the information and in the entanglement if one goes from the description with respect to a quantum
observer to the one with respect to another quantum observer.

12 See Ref.[51] for the relativistic description of the two-level atoms used in Unruh-DeWitt detectors.
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