LONG TIME STRONG CONVERGENCE TO BOSE-EINSTEIN DISTRIBUTION FOR LOW TEMPERATURE

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Abstract. We study the long time behavior of measure-valued isotropic solutions \( F_t \) of the Boltzmann equation for Bose-Einstein particles for low temperature. The global in time existence of such solutions \( F_t \) that converge at least semi-strongly to equilibrium (the Bose-Einstein distribution) has been proven in previous work and it has been known that the long time strong convergence to equilibrium is equivalent to the long time convergence to the Bose-Einstein condensation. Here we show that if such a solution \( F_t \) as a family of Borel measures satisfies a uniform double-size condition (which is also necessary for the strong convergence), then \( F_t \) converges strongly to equilibrium as \( t \) tends to infinity. We also propose a new condition on the initial datum \( F_0 \) such that a corresponding solution \( F_t \) converges strongly to equilibrium.

1. Introduction. The quantum Boltzmann equations for Bose-Einstein particles and for Fermi-Dirac particles (which are also called Boltzmann-Nordheim equation, Uehling-Uhlenbeck equation, etc.) were first derived by Nordheim [20] and Uehling & Uhlenbeck [26] and then taken attention and developed by [4], [6], [7], [18]. For the case of Bose-Einstein particles and for the spatially homogeneous solutions, the equation under consideration is written

\[
\frac{\partial}{\partial t} f(v, t) = \int_{\mathbb{R}^3 \times S^2} B(v - v_*, \omega)[f'(1 + f') - f_*(1 + f')'] d\omega dv_*
\]

where \( (v, t) \in \mathbb{R}^3 \times (0, +\infty) \), \( f(v, t) \geq 0 \) is the number density of particles at time \( t \) with the velocity \( v \), \( B(v - v_*, \omega) \) is the collision kernel, and \( f_* = f(v_*, t) \) is the velocity of the other particle before the collision:

\[
v' = v - ((v - v_*) \cdot \omega)\omega, \quad v_*' = v_* + ((v - v_*) \cdot \omega)\omega, \quad \omega \in S^2
\]

which conserves the kinetic energy

\[
|v'|^2 + |v_*'|^2 = |v|^2 + |v_*|^2.
\]

According to [4] and [7], the collision kernel \( B(v - v_*, \omega) \) in the weak-coupling regime takes the form (after normalizing physical parameters)

\[
B(v - v_*, \omega) = \frac{|(v - v_*) \cdot \omega|}{(\phi(v' - v') + \phi(v - v_'))^2}
\]

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where \( \hat{\phi}(\xi) = \int_{\mathbb{R}^3} \phi(x)e^{-ix\cdot \xi}dx \) is the Fourier transform of an interacting potential \( \phi \) which is assumed to be real and even. In particular if \( \phi(x) = \frac{1}{8\pi}\delta(x) \), where \( \delta(x) \) is the three dimensional Dirac delta function concentrating at \( x = 0 \), then (2) becomes the hard sphere model:

\[
B(v - v_*, \omega) = \frac{1}{(4\pi)^2} |(v - v_*) \cdot \omega|
\]

which is the only model that has the same form as in the classical Boltzmann equation and has been mainly concerned in many papers about Eq.(1).

Mathematical results on Eq.(1) for anisotropic (i.e. non-radially symmetric) initial data are quite few, due to the higher nonlinearity of the collision integral operator and the condensation effect which bring big difficulties in proving the existence of solutions. Except for a recent result [5] that proves the existence and uniqueness of solutions in finite time interval for anisotropic initial data, all other results obtained so far are concerned with isotropic initial data hence isotropic solutions. Despite this shortage, results obtained so far have shown that the Eq.(1) can be used to describe the formation and evolution of the Bose-Einstein condensation of dilute Bose gases at low temperature, see for instance [12], [19], [23], [24], [25] for self-similar structure and deterministic numerical methods; [3], [9], [10], [11], [15], [16] for singular solutions and the formation of blow-up and condensation in finite time; [14], [17] for long time strong and weak convergence to the Bose-Einstein distribution, and [1], [2], [21] for general discussions and basic results for similar models on low temperature evolution of condensation. The present paper is a continuation of the previous work on the long time strong convergence of isotropic solutions to the Bose-Einstein distribution for low temperature.

Before stating our main results of the paper, let us introduce the isotropic version of Eq.(1). By velocity translation we can assume that the mean velocity is zero so that the isotropic solution \( f(v, t) \) can be written as \( f(|v|^2/2, t) \). Accordingly, let \( \epsilon, \epsilon_*, \epsilon', \epsilon'_* \) stand for \( |v|^2/2, |v_*|^2/2, |v|^2/2, |v_*|^2/2 \) respectively, then Eq.(1) for the hard sphere model (3) becomes (see e.g. [8], [15])

\[
\frac{\partial}{\partial t} f = \int_{\mathbb{R}_+^2} \min\{\sqrt{\epsilon}, \sqrt{\epsilon_*}, \sqrt{\epsilon'}, \sqrt{\epsilon'_*}\} (f' f_*(1 + f + f_*) - f f_*(1 + f' + f_*)) d\epsilon' d\epsilon_*
\]

with \( f = f(\epsilon, t), f' = f(\epsilon', t), f_* = f(\epsilon_*, t), f_* = f(\epsilon'_*, t) \) and \( \epsilon_* = (\epsilon' + \epsilon'_* - \epsilon)_+ \), where \( u_+ = \max\{u, 0\} \). In order to include the equilibrium for low temperature and to study the Bose-Einstein condensation, one has to use a weak form of Eq.(4) which is an equation for positive Borel measures \( F_t \) on \( \mathbb{R}_{\geq 0} \). First of all let us make a note on the notation: throughout this paper we use \([0, +\infty)\) to stand for the set of the time variable \( t \), whereas we use \( \mathbb{R}_{\geq 0} \) to stand for the set of the energy variables and, for convenience, the energy variables are denoted by

\[
x = \epsilon, \quad x_* = \epsilon_*, \quad y = \epsilon', \quad z = \epsilon'_*, \quad \text{etc.}
\]

Let \( \mathcal{B}(\mathbb{R}_{\geq 0}) \) be the class of signed real Borel measures \( F \) on \( \mathbb{R}_{\geq 0} \) satisfying \( \int_{\mathbb{R}_{\geq 0}} |F|(x) < +\infty \) where \( |F| \) is the total variation of \( F \). For any \( k \geq 0 \) let

\[
\mathcal{B}_k(\mathbb{R}_{\geq 0}) = \left\{ F \in \mathcal{B}(\mathbb{R}_{\geq 0}) \mid \int_{\mathbb{R}_{\geq 0}} (1 + x)^k |F|(x) < +\infty \right\},
\]

\[
\mathcal{B}_k^+(\mathbb{R}_{\geq 0}) = \left\{ F \in \mathcal{B}_k(\mathbb{R}_{\geq 0}) \mid F \geq 0 \right\}, \quad \mathcal{B}_0^+(\mathbb{R}_{\geq 0}) = \mathcal{B}_0^+(\mathbb{R}_{\geq 0}).
\]
The moment $M_p(F)$ of order $p \in [0,k]$ for $F \in \mathcal{B}^+_k(\mathbb{R}_{\geq 0})$ is defined by

$$M_p(F) = \int_{\mathbb{R}_{\geq 0}} x^p dF(x).$$

Moments of orders 0,1 correspond to the mass and energy and are particularly denoted as $N(F) = M_0(F), E(F) = M_1(F)$, i.e.

$$N(F) = \int_{\mathbb{R}_{\geq 0}} dF(x), \quad E(F) = \int_{\mathbb{R}_{\geq 0}} x dF(x).$$

A test function space for defining weak solutions is chosen

$$C^{1,1}_b(\mathbb{R}_{\geq 0}) = \left\{ \varphi \in C^1_b(\mathbb{R}_{\geq 0}) \mid \frac{d}{dx}\varphi \in \text{Lip}(\mathbb{R}_{\geq 0}) \right\}$$

where $C^k_b(\mathbb{R}_{\geq 0})$ with $k \in \mathbb{N}$ is the class of bounded continuous functions on $\mathbb{R}_{\geq 0}$ having bounded continuous derivatives on $\mathbb{R}_{\geq 0}$ up to the order $k$, and Lip($\mathbb{R}_{\geq 0}$) is the class of functions satisfying Lipschitz condition on $\mathbb{R}_{\geq 0}$.

On the basis of the existence results we introduce directly the concept of mass-energy conserved measure-valued solutions of Eq.(4) in the weak form as follows.

**Definition 1.1.** Let $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$. We say that a family $\{F_t\}_{t \geq 0} \subset \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$, or simply $F_t$, is a mass-energy conserved measure-valued solution of Eq.(4) on the time-interval $[0, +\infty)$ with the initial datum $F_t|_{t=0} = F_0$ if

(i) $N(F_t) = N(F_0), \quad E(F_t) = E(F_0)$ for all $t \in [0, +\infty)$,

(ii) for every $\varphi \in C^{1,1}_b(\mathbb{R}_{\geq 0})$, the function $t \mapsto \int_{\mathbb{R}_{\geq 0}} \varphi(x) dF_t(x)$ belongs to $C^1([0, +\infty))$,

(iii) for every $\varphi \in C^{1,1}_b(\mathbb{R}_{\geq 0})$

$$\frac{d}{dt} \int_{\mathbb{R}_{\geq 0}} \varphi dF_t = \int_{\mathbb{R}_{\geq 0}} \mathcal{J}[\varphi] d^2F_t + \int_{\mathbb{R}_{\geq 0}} \mathcal{K}[\varphi] d^3F_t \quad \forall t \in [0, +\infty) \quad (5)$$

where $d^2F = dF(y) dF(z), d^3F = dF(x) dF(y) dF(z),

$$\mathcal{J}[\varphi](y, z) = \frac{1}{2} \int_{\mathbb{R}_{\geq 0}} \mathcal{K}[\varphi](x, y, z) \sqrt{x} dx, \quad \mathcal{K}[\varphi](x, y, z) = W(x, y, z) \Delta \varphi(x, y, z)$$

and

$$W(x, y, z) = \begin{cases} \frac{\min\{\sqrt{x}, \sqrt{y}, \sqrt{z}\}}{\sqrt{xyz}} & \text{if } x, y, z > 0 \\ \frac{1}{\max\{\sqrt{y}, \sqrt{z}, \sqrt{xyz}\}} & \text{if } xyz = 0, x > 0, \max\{y, z, x\} > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Delta \varphi(x, y, z) = \varphi(x) + \varphi(x_+) - \varphi(y) - \varphi(z), \quad x_+ = (y + z - x)_+.$$
Kinetic Temperature. Let $F \in B^+_1(\mathbb{R}_{\geq 0})$, $N = N(F)$, $E = E(F)$ and suppose $N > 0$. If $m$ is the mass of one particle, then $m4\pi\sqrt{2N}$, $m4\pi\sqrt{2E}$ are total mass and kinetic energy of the particle system per unite space volume. Keeping in mind the number $m4\pi\sqrt{2}$, there will be no confusion if we call $N$ and $E$ the mass and energy of a particle system. The kinetic temperature $T$ and the kinetic critical temperature $T_c$ are defined by (see e.g.[13] and references therein)
\[ T = \frac{2m}{3k_B} \frac{E}{N}, \quad T_c = \frac{\zeta(5/2)}{(2\pi)^{1/3}[\zeta(3/2)]^{5/3}} \frac{2m}{k_B} N^{2/3} \]
where $k_B$ is the Boltzmann constant, $\zeta(\cdot)$ is the Riemann zeta function. Some properties involving temperature effect, for instance the Bose-Einstein condensation at low temperature, are often expressed in terms of the ratio
\[ \frac{T}{T_c} = c_0 \frac{E}{N^{5/3}}, \quad c_0 = \frac{(2\pi)^{1/3}[\zeta(3/2)]^{5/3}}{3\zeta(5/2)} \approx 2.2720. \]

Regular-Singular Decomposition. Let $F \in B^+(\mathbb{R}_{\geq 0})$. According to measure theory (see e.g.[22]), $F$ can be uniquely decomposed as the regular part $0 \leq f \in L^1(\mathbb{R}_+, \sqrt{x}dx)$ and the singular part $\nu \in B^+(\mathbb{R}_{\geq 0})$ with respect to the Lebesgue measure, i.e. there is a Borel null set $Z \subseteq \mathbb{R}_{\geq 0}$ (i.e. $\text{mes}(Z) = 0$) such that
\[ dF(x) = f(x)\sqrt{x}dx + d\nu(x), \quad \nu(\mathbb{R}_{\geq 0} \setminus Z) = 0. \]
If the singular part is zero, i.e. if $dF(x) = f(x)\sqrt{x}dx$, then we say that $F$ is regular; if the regular part $f$ of $F$ is non-zero, i.e., if $\int_{\mathbb{R}_{\geq 0}} f(x)\sqrt{x}dx > 0$, then we say that $F$ is non-singular. In this paper we are mainly interested in such solutions $F_i$ whose initial data $F_0$ are non-singular.

Bose-Einstein Distribution. According to Theorem 5 of [13] and its equivalent version proved in the Appendix of [15] we know that for any $N > 0$, $E > 0$ there exists a unique triple $(A, \kappa, N_0)$ of constants $A \geq 1, \kappa > 0, N_0 \geq 0$ such that the measure $F_{\text{be}} \in B^+_1(\mathbb{R}_{\geq 0})$ defined by
\[ dF_{\text{be}}(x) = \frac{1}{Ae^{x/\kappa} - 1} \sqrt{x}dx + N_0 d\nu_0(x) \] (6)
is the equilibrium solution of Eq.(5) satisfying $N(F_{\text{be}}) = N, E(F_{\text{be}}) = E$, where $\nu_0$ is the Dirac measure concentrated at $x = 0$. Moreover $A$ and $N_0$ have the following relation:
\[ \begin{cases} A > 1, \quad N_0 = 0 & \text{if } \frac{T}{T_c} > 1, \\ A = 1, \quad N_0 = (1 - (T/T_c)^{3/5})N & \text{if } \frac{T}{T_c} \leq 1. \end{cases} \] (7)
The equilibrium $F_{\text{be}}$ is also called the Bose-Einstein distribution with the mass $N$ and energy $E$. From (6) and (7) one sees that
\[ \frac{T}{T_c} < 1 \implies F_{\text{be}}(\{0\}) = N_0 = (1 - (T/T_c)^{3/5})N > 0. \]
The positive number $(1 - (T/T_c)^{3/5})N$ is called the Bose-Einstein condensation (BEC) of the equilibrium state of Bose-Einstein particles at low temperature.

According to mathematical and physical results on the formation and occurrence of the Bose-Einstein condensation, it is naturally proposed the following problem on the long time convergence of condensation:
Problem. Let $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ be a non-singular measure with $N = N(F_0) > 0, E = E(F_0) > 0$ satisfying the low temperature condition $\frac{T}{T_c} = \frac{E}{\sqrt{N}} < 1$. Prove (or disprove) that there exists a mass-energy conserved measure-valued solution $F_t$ of Eq. (5) on $[0, +\infty)$ with the initial datum $F_0$ such that $F_t(\{0\})$ converges to BEC as $t \to +\infty$, i.e.

$$\lim_{t \to +\infty} F_t(\{0\}) = (1 - (\frac{T}{T_c})^{3/5})N. \tag{8}$$

It should be noted that in this Problem 1 there is no any local or microscopic condition on the initial data; the only condition is the macroscopic condition $\frac{T}{T_c} < 1$. Note also that in the statement of Problem 1 we used “there exists a ... solution” which is because there has been no uniqueness result for mass-energy conserved measure-valued solutions. In fact as is well known that the problem of uniqueness of weak solutions is also very difficult.

In the investigation of the Problem 1, we have obtained the following basic results which will be also used in this paper. Before stating these results we need to introduce the norm $\| \cdot \|_1$ and the semi-norm $\| \cdot \|_1^\circ$ for the measure space $\mathcal{B}_1(\mathbb{R}_{\geq 0})$:

$$\|F\|_1 = \int_{\mathbb{R}_{\geq 0}} (1 + x)d|F|(x), \quad \|F\|_1^\circ = \int_{\mathbb{R}_{\geq 0}} xd|F|(x), \quad F \in \mathcal{B}_1(\mathbb{R}_{\geq 0}).$$

Theorem 1.2 ([13],[14],[15],[17]). Let $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ be a non-singular measure with $N = N(F_0) > 0, E = E(F_0) > 0$ satisfying the low temperature condition $\frac{T}{T_c} = \frac{E}{\sqrt{N}} < 1$. Let $F_{be}$ be the unique Bose-Einstein distribution with the mass $N$ and energy $E$. Then there exists a mass-energy conserved measure-valued solution $F_t$ of Eq. (5) on $[0, +\infty)$ with the initial datum $F_0$, such that $F_t$ is non-singular for all $t \geq 0$ and has the following properties:

(a) $F_t$ converges at least semi-strongly to equilibrium as $t \to +\infty$, i.e.

$$\lim_{t \to +\infty} \|F_t - F_{be}\|_1^\circ = 0. \tag{9}$$

(b) $F_t(\{0\})$ converges as $t \to +\infty$ and

either $\lim_{t \to +\infty} F_t(\{0\}) = 0$ or $\lim_{t \to +\infty} F_t(\{0\}) = (1 - (\frac{T}{T_c})^{3/5})N.$

(c) With the semi-strong convergence (9), the long time convergence to BEC determines the strong convergence to equilibrium, i.e.

$$\lim_{t \to +\infty} \|F_t - F_{be}\|_1 = 0 \iff \lim_{t \to +\infty} F_t(\{0\}) = (1 - (\frac{T}{T_c})^{3/5})N.$$

Theorem 1.3 ([17]). Let $F_0 \in \mathcal{B}_1^+(\mathbb{R}_{\geq 0})$ be a non-singular measure with $N = N(F_0) > 0, E = E(F_0) > 0$ satisfying

$$\frac{T}{T_c} = \frac{E}{\sqrt{N}} \leq \frac{c_0}{974} \quad (< 1)$$

and there is $0 < \alpha < 1$ such that

$$\frac{F_0([0, \varepsilon])}{\varepsilon^\alpha} \geq C_\alpha \left( \frac{E}{N} \right)^{\frac{3}{2} - \alpha} \quad \text{for all } \varepsilon \in \left( 0, 3\alpha \frac{E}{N} \right). \tag{10}$$

where

$$C_\alpha = e^{3/5}540 \left( \frac{2 + \alpha}{3\alpha} \right)^{5} \left( \frac{3 + 2\alpha}{1 - \alpha} \right)^{3+2\alpha}.$$
Let \( F_t \) be a mass-energy conserved measure-valued solution of Eq. (5) on \([0, +\infty)\) obtained by Theorem 1.2 with the initial datum \( F_0 \). Then
\[
\lim_{t \to +\infty} F_t(\{0\}) = (1 - (\mathcal{T}/\mathcal{T}_c)^{3/5})N.
\]

**Remark 1.** Part (a) of Theorem 1.2 is a part of Theorem 2 in [14] where the assumption that \( F_0 \) is non-singular is mainly used for defining and using the entropy. Although the fact that “\( F_t \) is non-singular for all \( t \geq 0 \)” stated in Theorem 1.2 is not stated in Theorem 2 of [14], it is easily deduced from the proof of that theorem. In fact from (7.11) and pages 1064-1065 of [14] one sees that the entropy functional defined on the regular part of \( F_t \) is strictly positive for all \( t \geq 0 \) and thus \( F_t \) is non-singular for all \( t \geq 0 \).

**Remark 2.** Part (c) of Theorem 1.2 shows that in order to prove the long time strong convergence to equilibrium \( \lim_{t \to +\infty} \|F_t - F_{\text{be}}\|_1 = 0 \), one needs only to prove the long time strong convergence to BEC (8). Theorem 1.3 gives a partial positive answer to the Problem 1 by adding a local condition (10) on the initial datum \( F_0 \). For a regular initial datum \( dF_0(x) = f_0(x) \sqrt{\mathcal{T}} \, dx \), the local condition (10) means that \( f_0(x) \) is unbounded near the origin and has a certain rate of growing to infinity as \( x \to 0^+ \). In view of theory and numerical simulation, we hope to prove that the long time convergence to BEC (8) holds for such a class of regular initial data \( dF_0(x) = f_0(x) \sqrt{\mathcal{T}} \, dx \) where \( f_0(x) \) are bounded near the origin or grow to infinity as slowly as possible.

**Main Results.** Our first main result of this paper is Theorem 1.4, which is an “if theorem” since it assumes a priori double-size condition (11). It makes us believe the truth of the long time convergence to BEC (Problem 1).

**Theorem 1.4.** Let \( F_0 \in \mathcal{B}_+^1(\mathbb{R}_{\geq 0}) \) be a non-singular measure with \( N = N(F_0) > 0, \ E = E(F_0) > 0 \) satisfying the low temperature condition \( \mathcal{T}/\mathcal{T}_c = c_0 \frac{E}{N^{3/5}} < 1 \). Let \( F_t \) be a mass-energy conserved measure-valued solution of Eq. (5) on \([0, +\infty)\) obtained by Theorem 1.2 with the initial datum \( F_0 \). Assume in addition that \( F_t \) satisfies a uniform double-size condition: there exist finite constants \( C > 1, \varepsilon_0 > 0 \) and \( t_0 > 0 \) such that
\[
F_t([0, 2\varepsilon]) \leq CF_t([0, \varepsilon]) \quad \forall \varepsilon \in (0, \varepsilon_0], \ \forall \ t \geq t_0.
\]

Then
\[
\lim_{t \to +\infty} F_t(\{0\}) = (1 - (\mathcal{T}/\mathcal{T}_c)^{3/5})N.
\]

Note that the double-size condition (11) is necessary for the convergence to BEC (8). In fact, if (8) holds, then there exists \( t_0 > 0 \) such that \( F_t(\{0\}) > aN \) for all \( t \geq t_0 \) where \( a = \frac{1}{2}(1 - (\mathcal{T}/\mathcal{T}_c)^{3/5}) > 0 \). This gives
\[
F_t([0, 2\varepsilon]) \leq N \leq \frac{1}{a} F_t(\{0\}) \leq \frac{1}{a} F_t([0, \varepsilon]) \quad \forall \varepsilon > 0, \ \forall \ t \geq t_0.
\]

Our second main result Theorem 1.5 below is an improvement of Theorem 1.3; it provides an optimal sufficient condition on the initial data for the long time convergence to BEC (8). Introduce
\[
\overline{D^{-1}}(F, \varepsilon) = \sup_{0 < \delta \leq \varepsilon} \frac{\delta}{F([0, \delta])}, \quad \varepsilon > 0, \quad F \in \mathcal{B}_+(\mathbb{R}_{\geq 0}).
\]
Note. In the set \( \mathbb{R}_{\geq 0} \) we define the arithmetic operation
\[
\frac{b}{a} = +\infty \quad \text{if} \quad a = 0, b > 0.
\]

**Theorem 1.5.** Let \( F_0 \in \mathcal{B}^+_1(\mathbb{R}_{\geq 0}) \) be a non-singular measure with \( N = N(F_0) > 0, E = E(F_0) > 0 \) satisfying
\[
\int_0^{\frac{2E}{3}} \frac{(D^{-1}(F_0, \varepsilon))^{1/5}}{\varepsilon} \, d\varepsilon \leq \frac{1}{5} \left( \frac{N}{E} \right)^{1/10}
\]
(12)
(which implies \( T/T_c = c_0 \frac{E}{N \lambda^T} << 1 \)). Let \( F_t \) be a mass-energy conserved measure-valued solution of Eq. (5) on \([0, +\infty)\) obtained by Theorem 1.2 with the initial datum \( F_0 \). Then
\[
\lim_{t \to +\infty} F_t(\{0\}) = (1 - (T/T_c)^{3/5})N.
\]

**Remark 3.** Compare the new Theorem 1.5 with the previous Theorem 1.3 for regular initial data \( dF_0(x) = f_0(x) \sqrt{\lambda} \, dx \): the condition (10) in Theorem 1.3 implies that there is a constant \( C_0^\alpha > 0 \) such that
\[
\sup_{0 < x \leq \varepsilon} f_0(x) \geq C_0^\alpha \varepsilon^{-\beta}
\]
for all \( 0 < \varepsilon << 1 \), where \( \beta = \frac{9}{2} - \alpha > \frac{1}{2} \) and, in many cases, \( C_0^\alpha \to +\infty \) as \( \alpha \to 1^- \) (see e.g. part (1) of Remark 1.3 in [17]), while the condition (12) in Theorem 1.5 includes such initial data \( F_0 \) whose density functions \( f_0 \) grow to infinity with a slower speed, \( f_0(\varepsilon) \sim C \varepsilon^{-\frac{5}{2}} (\log(1/\varepsilon))^{10} \) for all \( 0 < \varepsilon << 1 \), and they do not satisfy the condition (10) in Theorem 1.3 (see below for a construction of such \( F_0 \)). It is in this sense that we say that Theorem 1.5 is an improvement of Theorem 1.3.

To construct \( F_0 \) mentioned above, we consider \( F_0 = \lambda G_0 \) where \( \lambda > 0 \) is a constant, \( G_0 \in \mathcal{B}^+_1(\mathbb{R}_{\geq 0}) \) is defined by \( dG_0(x) = g_0(x) \sqrt{x} \, dx \) with a function \( 0 \leq g_0 \in L^1(\mathbb{R}_{\geq 0}, (1 + x) \sqrt{x} \, dx) \) which is expressed on \((0, e^{-10}]\) by
\[
g_0(x) \sqrt{x} = \frac{d}{dx} \left( x (\log(1/x))^{10} \right) = (\log(1/x))^{9} (\log(1/x) - 10), \quad x \in (0, e^{-10}].
\]
It is easily seen that \( \frac{5}{2} x^{-\frac{3}{2}} (\log(1/x))^{10} \leq g_0(x) \leq x^{-\frac{1}{2}} (\log(1/x))^{10} \) for all \( x \in (0, e^{-20}] \) and \( G_0([0, \varepsilon]) = \varepsilon (\log(1/\varepsilon))^{10} \) for all \( 0 < \varepsilon \leq e^{-10} \) and so
\[
\frac{D^{-1}(G_0, \varepsilon)}{\varepsilon} = \begin{cases} 
\frac{1}{(\log(1/\varepsilon))^{10}} & \text{if } 0 < \varepsilon \leq e^{-10}, \\
\frac{1}{(e/10)^{10}} & \text{if } \varepsilon > e^{-10}.
\end{cases}
\]
Since
\[
\int_0^R \frac{\varepsilon (\log(1/\varepsilon))^{10}}{\log(1/R)} \, d\varepsilon < +\infty \quad \forall \ 0 < R < +\infty,
\]
Now from \( F_0 = \lambda G_0 \) and \( dF_0(x) = f_0(x) \sqrt{x} \, dx \) we have \( f_0(x) = \lambda g_0(x) \sim \lambda x^{-\frac{5}{2}} (\log(1/x))^{10} \) for all \( 0 < x \leq e^{-20} \) and
\[
E(F_0) = \frac{E(G_0)}{N(G_0)}, \quad D^{-1}(F_0, \varepsilon) = \frac{1}{\lambda} D^{-1}(G_0, \varepsilon),
\]
\[
\int_0^{2E(F_0)/N(F_0)} \frac{(D^{-1}(F_0, \varepsilon))^{1/5}}{\varepsilon} \, d\varepsilon = \frac{1}{\lambda^{1/5}} \int_0^{2E(G_0)/N(G_0)} \frac{(D^{-1}(G_0, \varepsilon))^{1/5}}{\varepsilon} \, d\varepsilon.
\]
If $\lambda > 0$ is chosen large enough such that
\[
\frac{1}{\lambda^{1/5}} \int_0^{2\varepsilon/(N(\epsilon))} \frac{D^{-T}(G_0, \epsilon)^{1/5}}{\epsilon} \, d\epsilon \leq \frac{1}{5} \left( \frac{N(G_0)}{E(G_0)} \right)^{1/10}
\]
then $F_0$ satisfies the condition (12). On the other hand for any $0 < \alpha < 1$ we have
\[
\frac{F_0([0, \epsilon])}{\epsilon^\alpha} = \lambda \epsilon^{1-\alpha} (\log(1/\epsilon))^{10} \to 0 \quad \text{as} \quad \epsilon \to 0^+
\]
which means that $F_0$ does not satisfy the condition (10) in Theorem 1.3.

**Remark 4.** In the proof of Theorem 1.5 we will show that the assumption (12) implies a very low temperature condition: $T/T_c = c_0 N^{-\alpha} \leq c_0 (5^{10} \cdot 2)^{-2/3} \ll 1$.

Theorem 1.5 can be easily extended as follows: let $F_0 \in \mathcal{B}^+(\mathbb{R}_\geq 0)$ be a non-singular measure and let $F_t$ be a mass-energy conserved measure-valued solution of Eq. (5) on $[0, +\infty)$ obtained by Theorem 1.2 with the initial datum $F_0$. Suppose that there exists $t_0 \geq 0$ such that $F_{t_0}$ as a new initial datum of $F_t$ satisfying (12), then the conclusion of Theorem 1.5 still holds true. In view of this extended version, it is easily seen that the condition (12) is also necessary for the long time convergence to BEC (8) for the case of very low temperature. In fact suppose that $T/T_c < 1$ and (8) holds, then there is $\tau_0 > 0$ such that for any $t_0 \geq \tau_0$, $F_{t_0}([0]) \geq \frac{1}{2}(1-(T/T_c)^{3/5})N$.

This gives
\[
\frac{D^{-T}(F_{t_0}, \epsilon)}{\epsilon^{1/5}} \leq \frac{2\epsilon}{(1-(T/T_c)^{3/5})N},
\]
\[
\int_0^{2\epsilon} \frac{D^{-T}(F_{t_0}, \epsilon)^{1/5}}{\epsilon} \, d\epsilon \leq \left( \frac{2}{1-(T/T_c)^{3/5}} \right)^{1/5} \int_0^{2\epsilon} \epsilon^{-1+\delta/5} \, d\epsilon = 5 \left( \frac{4}{1-(T/T_c)^{3/5}} \right)^{1/5} \left( \frac{E}{N^{\alpha}} \right)^{3/10} \left( \frac{N}{E} \right)^{1/10}
\]
\[
= 5 \left( \frac{4}{1-(T/T_c)^{3/5}} \right)^{1/5} \left( \frac{1}{c_0 T_c} \right)^{3/10} \left( \frac{N}{E} \right)^{1/10}.\]

Therefore if $T/T_c$ is small enough such that
\[
5 \left( \frac{4}{1-(T/T_c)^{3/5}} \right)^{1/5} \left( \frac{1}{c_0 T_c} \right)^{3/10} \leq \frac{1}{5}
\]
then $F_{t_0}$ satisfies the condition (12).

The proofs of Theorem 1.4 and Theorem 1.5 are given in Section 3 after making sufficient preparation in Section 2.

2. **Some lemmas.** This section collects and proves some technical lemmas; some of them are easy improvements of those in [15], [17]. Let us introduce notations for some integrals of Borel measures $F \in \mathcal{B}^+(\mathbb{R}_\geq 0)$. For $p > 0$, $\varepsilon > 0$, define
\[
N_{0,p}(F, \varepsilon) = \int_{\mathbb{R}_\geq 0} \left( 1 - \frac{y}{\varepsilon} \right)_+^p \, dF(x) = \int_{[0, \varepsilon]} \left( 1 - \frac{y}{\varepsilon} \right)_+^p \, dF(x), \quad (13)
\]
where $y_+ = \max\{y, 0\}$, $y \in \mathbb{R}$. For the case $\varepsilon = 0$, we define $N_{0,p}(F, 0)$ by the limit:
\[
N_{0,p}(F, 0) := \lim_{\varepsilon \to 0^+} N_{0,p}(F, \varepsilon) = F([0]). \quad (14)
\]
We also define for \( \alpha \geq 0, \varepsilon > 0 \)
\[
N_{\alpha, p}(F, \varepsilon) = \frac{1}{\varepsilon^\alpha} N_{0, p}(F, \varepsilon), \quad N_{N, p}(F, \varepsilon) = \inf_{0 < \delta \leq \varepsilon} N_{\alpha, p}(F, \delta),
\]
(15)
and
\[
A_{0, p}(F, \varepsilon) = \int_{[0, \varepsilon]} \left( \frac{x}{\varepsilon} \right)^p dF(x).
\]
(16)

The first lemma provides a way of obtaining a positive lower bound of \( F_t(\{0\}) \) through “smooth” terms. This can be seen in the proofs of Lemma 2.5, Theorem 1.4 and Theorem 1.5.

Lemma 2.1. Let \( F \in \mathcal{B}^+(\mathbb{R}_{\geq 0}), 1 < p < +\infty \). Then
\[
N_{0, p}(F, \varepsilon) \leq F(\{0\}) + \left( \int_0^\varepsilon [A_{0, p}(F, \varepsilon)]^1/p \frac{d\varepsilon}{\varepsilon} \right)^p, \quad \varepsilon > 0.
\]
(17)
In general, for any bounded nonnegative Borel measurable function \( \psi \) on \( \mathbb{R}_{\geq 0} \) we have
\[
N_{0, p}^\psi(F, \varepsilon) \leq \psi(0) F(\{0\}) + \left( \int_0^\varepsilon [A_{0, p}^\psi(F, \varepsilon)]^1/p \frac{d\varepsilon}{\varepsilon} \right)^p, \quad \varepsilon > 0
\]
where
\[
N_{0, p}^\psi(F, \varepsilon) = \int_{\mathbb{R}_{\geq 0}} [(1 - x/\varepsilon)_+]^p \psi(x) dF(x), \quad A_{0, p}^\psi(F, \varepsilon) = \int_{[0, \varepsilon]} (x/\varepsilon)^p \psi(x) dF(x).
\]
Proof. We need only to prove the general inequality (18). Let
\[
\hat{N}_{0, p}^\psi(F, \varepsilon) = \int_{\mathbb{R}_{\geq 0}} [(1 - x/\varepsilon)_+]^p \psi(x) dF(x), \quad \varepsilon > 0.
\]
Then
\[
N_{0, p}^\psi(F, \varepsilon) = \psi(0) F(\{0\}) + \hat{N}_{0, p}^\psi(F, \varepsilon), \quad \varepsilon > 0.
\]
Using differentiation under the integral and Hölder inequality we compute
\[
\frac{d}{d\varepsilon} \hat{N}_{0, p}^\psi(F, \varepsilon) = \frac{p}{\varepsilon} \int_{\mathbb{R}_{\geq 0}} \left( \frac{1 - x}{\varepsilon} \right)_+^{p-1} x \frac{\psi(x) dF(x)}{\varepsilon} \leq \frac{p}{\varepsilon} [\hat{N}_{0, p}^\psi(F, \varepsilon)]^{1-1/p} [A_{0, p}^\psi(F, \varepsilon)]^{1/p}, \quad \varepsilon > 0
\]
and so for any \( \delta > 0 \) we have
\[
\frac{d}{d\varepsilon} [\hat{N}_{0, p}^\psi(F, \varepsilon) + \delta]^{1/p} \leq \frac{1}{\varepsilon} [A_{0, p}^\psi(F, \varepsilon)]^{1/p}, \quad \varepsilon > 0.
\]
(20)
Since \( \varepsilon \to [\hat{N}_{0, p}^\psi(F, \varepsilon) + \delta]^{1/p} \) belongs to \( C^1((0, +\infty)) \), integrating both sides of (20) over \( \varepsilon \in [\eta, \varepsilon] \) (with \( 0 < \eta < \varepsilon \) ) gives
\[
[\hat{N}_{0, p}^\psi(F, \varepsilon) + \delta]^{1/p} \leq [\hat{N}_{0, p}^\psi(F, \eta) + \delta]^{1/p} + \int_{\eta}^{\varepsilon} [A_{0, p}^\psi(F, \varepsilon)]^{1/p} \frac{d\varepsilon}{\varepsilon}.
\]
(21)
Note that the second term in the right hand side of (21) is not larger than \( \int_{0}^{\varepsilon} [A_{0, p}^\psi(F, \varepsilon)]^{1/p} \frac{d\varepsilon}{\varepsilon} \). By first letting \( \delta \to 0^+ \) and then letting \( \eta \to 0^+ \) leads to
\[
[\hat{N}_{0, p}^\psi(F, \varepsilon)]^{1/p} \leq \int_{0}^{\varepsilon} [A_{0, p}^\psi(F, \varepsilon)]^{1/p} \frac{d\varepsilon}{\varepsilon}, \quad \varepsilon > 0.
\]
(22)
Here we have used Lebesgue dominated convergence to get
\[
\lim_{\eta \to 0^+} \hat{N}_{0, p}^\psi(F, \eta) = \lim_{\eta \to 0^+} \int_{\mathbb{R}_{\geq 0}} \left( \frac{1 - x}{\eta} \right)_+^{p} \psi(x) dF(x) = 0.
\]
From (19) we see that (18) is equivalent to (22). This proves the lemma. □

**Remark 5.** Since \( p > 1 \), the integral in the right hand side of (17) may be divergent for all \( \epsilon > 0 \) even for regular measure in \( B_1^+(\mathbb{R}) \). For instance let \( F \) be given by 
\[
dF(x) = f(x)\sqrt{x}dx \text{ with a function } f \in L^1(\mathbb{R}, (1 + |x|)\sqrt{x}dx) \text{ whose expression on } (0, 1/2) \text{ is given by}
\]
\[
f(x)\sqrt{x} = \frac{1}{x^p} \frac{d}{dx} \left( x^p (\log(1/x))^{-p} \right) = \frac{p}{x} \left( (\log(1/x))^{-p} + (\log(1/x))^{-p-1} \right),
\]
\( x \in (0, 1/2] \).

Then 
\[
A_{0,p}(F, \epsilon) = (\log(1/\epsilon))^{-p}, \quad \epsilon \in (0, 1/2]
\]

hence for any \( \epsilon > 0 \)
\[
\int_0^\epsilon \left[ A_{0,p}(F, \epsilon) \right]^{1/p} \frac{d\epsilon}{\epsilon} \geq \int_0^{\min\{\epsilon, 1/2\}} (\log(1/\epsilon))^{-1} \frac{d\epsilon}{\epsilon} = +\infty.
\]

This possibility of divergence reminds us that the application of Lemma 13 for \( F_t \) should be combined with other properties of \( F_t \) as shown in other lemmas.

**Lemma 2.2.** Let \( F \in B_1^+(\mathbb{R}, \mathbb{N} = N(F), E = E(F), \text{ and let } \varphi \in C_0^{1,1}(\mathbb{R}) \text{ be a nonnegative convex function on } \mathbb{R} \). Then

\[
\int_{\mathbb{R}^2_0} J[\varphi]d^2F \geq -\sqrt{NE} \int_{\mathbb{R}^2_0} \varphi dF;
\]

\[
\int_{\mathbb{R}^2_0} \mathcal{K}[\varphi]d^3F \geq \int_{0 \leq x < y \leq z} \chi_{y,z} \frac{1}{\sqrt{y^2}} \Delta_{sym} \varphi(x, y, z) d^3F
\]

where

\[
\chi_{y,z} = \begin{cases} 
2 & \text{if } y < z, \\
1 & \text{if } y = z
\end{cases}
\]

\( \Delta_{sym} \varphi(x, y, z) = (y - x)^2 \int_0^1 \int_0^1 \varphi''(z + (s - t)(y - x)) ds dt, \quad 0 \leq x \leq y \leq z. \)

**Proof.** By Cauchy-Schwarz inequality we have \( M_{1/2}(F) \leq \sqrt{M_0(F) M_1(F)} = \sqrt{NE} \). Also we have proved in [16] that

\[
\int_{\mathbb{R}^2_0} J[\varphi]d^2F \geq -M_{1/2}(F) \int_{\mathbb{R}_0^2} \varphi dF.
\]

These together with the nonnegativity of \( \varphi \) imply (23). The inequality (24) follows directly from Proposition 2.2 (convex positivity) of [15]. □

Our next lemma shows that if a convex function \( \varphi \) is chosen suitably, the cubic term \( \int_{\mathbb{R}^3_0} \mathcal{K}[\varphi]d^3F \) contributes a useful lower bound that connects everything.

**Lemma 2.3.** Let \( F \in B_1^+(\mathbb{R}_0), \varphi_\epsilon(x) = [(1 - x/\epsilon_+)^2, \epsilon > 0. \text{ Then}

\[
\int_{\mathbb{R}^3_0} \mathcal{K}[\varphi_\epsilon]d^3F \geq \frac{1}{\epsilon^{1-\alpha}} N_{\alpha,2}(F, \epsilon) |A_{0,\alpha}(F, \epsilon)|^2, \quad \alpha \geq 0, \quad p = \frac{3}{2} + \alpha,
\]

\[
N_{0,2}(F, \epsilon) \int_{\mathbb{R}^3_0} \mathcal{K}[\varphi_\epsilon]d^3F \geq \frac{1}{\epsilon} |A_{0,3/2}(F, \epsilon)|^2
\]
where
\[ A^{(2)}_{0,3/2}(F, \varepsilon) = \int_{[0,\varepsilon]} \left( \frac{2}{\varepsilon} \right)^{3/2} N_{0,2}(F, x) \, dF(x). \] (27)

**Proof.** Using Lemma 2.2 to the convex function \( \varphi_\varepsilon(x) = [(1-x/\varepsilon)_+]^2 \) (which belongs to \( C^1_{0,1}(\mathbb{R}_{\geq 0}) \)) and using Fubini’s theorem we have
\[ \int_{\mathbb{R}_{\geq 0}^3} \mathcal{K}[\varphi_\varepsilon] d^3 F \geq \frac{1}{\varepsilon^2} \int_{0 \leq z < y \leq \varepsilon} \frac{y^2}{\sqrt{z}} N_{0,2}(F, y) \, dF(y) \, dF(z) \]
\[ = \frac{1}{\varepsilon^2} \int_{0 < y \leq \varepsilon} \frac{y^{3/2}}{\sqrt{z}} N_{0,2}(F, y) \, dF(y) \, dF(z). \] (28)

Next we compute
\[ \frac{1}{\varepsilon^2} \int_{0 \leq y \leq \varepsilon} \chi_{y, z} \frac{y^2}{\sqrt{z}} N_{0,2}(F, y) \, dF(y) \, dF(z) \]
\[ = \frac{1}{\varepsilon^2} \int_{0 \leq y \leq \varepsilon} \chi_{y, z} \frac{y^p z^p}{z^{2+\alpha}} N_{0,2}(F, y) \, dF(y) \, dF(z) \]
\[ \geq N_{0,2}(F, \varepsilon) \frac{1}{\varepsilon^{4+\alpha}} \int_{0 \leq y \leq \varepsilon} \chi_{y, z} y^p z^p \, dF(y) \, dF(z) \]
\[ = \frac{1}{\varepsilon^{4+\alpha}} \left( \int_{[0,\varepsilon]} \psi(y) \, dF(y) \right)^2 = \frac{1}{\varepsilon^{4+\alpha}} N_{0,2}(F, \varepsilon) |A_{0,p}(F, \varepsilon)|^2. \]

This together with (28) gives (25). Here we have used the identity
\[ \left( \int_{[0,\varepsilon]} \psi(y) \, dF(y) \right)^2 = \int_{0 < y \leq z \leq \varepsilon} \chi_{y, z} \psi(y) \psi(z) \, dF(y) \, dF(z). \] (29)

Note that the function \( \varepsilon \mapsto N_{0,2}(F, \varepsilon) \) is non-decreasing on \([0, +\infty)\). We have
\[ N_{0,2}(F, y) N_{0,2}(F, \varepsilon) \geq N_{0,2}(F, y) N_{0,2}(F, z) \quad \forall 0 < y \leq z \leq \varepsilon \]
which together with (28),(29) gives

\[ N_{0,2}(F, \varepsilon) \int_{\mathbb{R}_{\geq 0}^3} \mathcal{K}[\varphi_\varepsilon] d^3 F \]
\[ \geq \frac{1}{\varepsilon^4} \int_{0 \leq y \leq \varepsilon} \chi_{y, z} y^2 z^2 N_{0,2}(F, y) N_{0,2}(F, z) \, dF(y) \, dF(z) \]
\[ = \frac{1}{\varepsilon^4} \left( \int_{0 < y \leq \varepsilon} y^2 N_{0,2}(F, y) \, dF(y) \right)^2 = \frac{1}{\varepsilon} |A^{(2)}_{0,3/2}(F, \varepsilon)|^2. \]

\[ \square \]

**Lemma 2.4.** Let \( F_0 \in B^+_{1,1}(\mathbb{R}_{\geq 0}) \) with \( N = N(F_0) > 0 \), \( E = E(F_0) > 0 \) and let \( F_t \) be a mass-energy conserved measure-valued solution of Eq. (5) on \([0, +\infty)\) with the initial datum \( F_0 \). Let \( c = \sqrt{NE} \). Then
(a) For any convex function $0 \leq \varphi \in C^{1,1}_{b} (\mathbb{R}_{\geq 0})$, the function $t \mapsto e^{ct} \int_{\mathbb{R}_{\geq 0}} \varphi(x) \, dF_t(x)$ is non-decreasing on $[0, +\infty)$ and for all $t \geq 0$, $h > 0$ we have
\[
\frac{d}{dt} \int_{\mathbb{R}_{\geq 0}} \varphi \, dF_t \geq -c \int_{\mathbb{R}_{\geq 0}} \varphi \, dF_t + \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}[\varphi] \, d^3 F_t,
\] (30)
\[
e^{ch} \int_{\mathbb{R}_{\geq 0}} \varphi \, dF_{t+h} - \int_{\mathbb{R}_{\geq 0}} \varphi \, dF_t \geq e^{c(t-h)} \left( \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}[\varphi] \, d^3 F_s \right) ds.
\] (31)

(b) The function $t \mapsto e^{ct} F_t(\{0\})$ is non-decreasing on $[0, +\infty)$, i.e.
\[
F_{t_1}(\{0\}) \leq F_{t_2}(\{0\}) e^{c(t_2 - t_1)}, \quad 0 \leq t_1 < t_2.
\] (32)

Proof. (a): From the differential equation (5), Lemma 2.3 and the conservation of mass and energy we have
\[
\frac{d}{dt} \int_{\mathbb{R}_{\geq 0}} \varphi \, dF_t = \int_{\mathbb{R}_{\geq 0}} J[\varphi] d^2 F_t + \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}[\varphi] \, d^3 F_t
\]
\[
\geq -\sqrt{N(F_t)} E(\mathcal{F}_t) \int_{\mathbb{R}_{\geq 0}} \varphi \, dF_t + \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}[\varphi] \, d^3 F_t
\]
\[
= -c \int_{\mathbb{R}_{\geq 0}} \varphi \, dF_t + \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}[\varphi] \, d^3 F_t \quad \forall \ t \in [0, +\infty).
\]

Thus
\[
\frac{d}{dt} \left( e^{ct} \int_{\mathbb{R}_{\geq 0}} \varphi \, dF_t \right) \geq e^{ct} \int_{\mathbb{R}^3_{\geq 0}} \mathcal{K}[\varphi] \, d^3 F_t \geq 0 \quad \forall \ t \in [0, +\infty).
\] (33)

This proves part (a).

(b): Applying part (a) to the convex function $\varphi_c(x) = [(1 - x/\varepsilon)^2]$ and recalling that $N_{0,2}(F_t, \varepsilon) = \int_{\mathbb{R}_{\geq 0}} \varphi_c(x) \, dF_t(x)$ we have
\[
e^{ct_1} N_{0,2}(F_{t_1}, \varepsilon) \leq e^{ct_2} N_{0,2}(F_{t_2}, \varepsilon) \quad \forall \ 0 \leq t_1 < t_2.
\]
Letting $\varepsilon \to 0^+$ gives
\[
e^{ct_1} F_{t_1}(\{0\}) \leq e^{ct_2} F_{t_2}(\{0\}) \quad \forall \ 0 \leq t_1 < t_2.
\]

The following lemma is an important technical preparation for estimating $F_t(\{0\})$.

Lemma 2.5. Let $F_0 \in B^+_c (\mathbb{R}_{\geq 0})$ satisfy $N = N(F_0) > 0$, $E = E(F_0) > 0$ and let $F_t$ be a mass-energy conserved measure-valued solution of Eq.(5) with the initial datum $F_0$. Then for any $\alpha \geq 0$, $h > 0$, $\varepsilon > 0$ and $t \geq 0$ we have
\[
F_{t+h}(\{0\}) \geq e^{-2ch} N_{0,p}(F_t, \varepsilon) - \left( \frac{e^{-ch} N}{h} \right)^{1/2} \left( \int_0^\varepsilon \left( \frac{e^{1-\alpha} \left( \frac{\varepsilon}{N_{0,2}(F_t, \varepsilon)} \right)^{1/2} \beta}{\varepsilon} \right)^p \right)
\] (34)

where $c = \sqrt{NE}$, $p = \frac{3}{2} + \alpha$. In particular for $\alpha = 0$, i.e. $p = 3/2$, we have
\[
F_{t+h}(\{0\}) \geq e^{-2ch} N_{0,3/2}(F_t, \varepsilon) - \left( e^{-ch} \frac{27}{h} \right)^{1/2} \left( \frac{N}{F_t(\{0\})} \right)^{1/2} \varepsilon^{1/2}.
\] (35)
Proof. In the following we will use the convention that \( \frac{b}{a} = +\infty \) for \( a = 0 < b \).

Fix any \( h > 0, \varepsilon > 0, t \geq 0 \). Using Lemma 2.4 to the convex function \( \varphi_{\varepsilon}(x) = \left((1 - x/\varepsilon)\right)^{\alpha} \) and recalling that \( N_{0,2}(F_t, \varepsilon) = \int_{\mathbb{R}^d} \varphi_{\varepsilon}(x) dF_t(x) \) and then using Lemma 2.3 (see (25)) we have

\[
e^{ch} N_{0,2}(F_{t+h}, \varepsilon) - N_{0,2}(F_t, \varepsilon) \geq \frac{1}{\varepsilon^{1-\alpha}} \int_t^{t+h} e^{c(s-t)} N_{\alpha,2}(F_s, \varepsilon)[A_{0,p}(F_s, \varepsilon)]^2 ds. \tag{36}
\]

Next for any \( \delta \in (0, \varepsilon] \), using the monotone property in part (a) of Lemma 2.4 to the convex function \( x \mapsto \frac{1}{\delta} \varphi_{\delta}(x) \) gives

\[
e^{c(s-t)} N_{\alpha,2}(F_s, \delta) \geq N_{\alpha,2}(F_t, \delta), \quad s \geq t.
\]

Taking \( \inf_{0<\delta\leq\varepsilon} \) leads to the same inequality

\[
e^{c(s-t)} N_{\alpha,2}(F_s, \varepsilon) \geq N_{\alpha,2}(F_t, \varepsilon), \quad s \geq t. \tag{37}
\]

Also from \( N_{0,2}(F_{t+h}, \varepsilon) \leq N \) we have \( e^{ch} N_{0,2}(F_{t+h}, \varepsilon) - N_{0,2}(F_t, \varepsilon) \leq e^{ch} N \). Thus from (36), (37) we obtain

\[
e^{ch} N \geq \frac{1}{\varepsilon^{1-\alpha}} N_{\alpha,2}(F_t, \varepsilon) \int_t^{t+h} [A_{0,p}(F_s, \varepsilon)]^2 ds
\]
i.e.

\[
\frac{1}{h} \int_t^{t+h} [A_{0,p}(F_s, \varepsilon)]^2 ds \leq \frac{e^{ch} N}{h} \cdot \frac{\varepsilon^{1-\alpha}}{N_{\alpha,2}(F_t, \varepsilon)}. \tag{38}
\]

On the other hand from Lemma 2.1 we have

\[
N_{0,p}(F_s, \varepsilon) \leq F_s(\{0\}) + \left( \int_0^\varepsilon [A_{0,p}(F_s, \varepsilon)]^{1/p} \frac{de}{e} \right)^p, \quad \varepsilon > 0.
\]

Taking integration and using Minkowski inequality ("the norm of sum \( \leq \) the sum of norm")

\[
\frac{1}{h} \int_t^{t+h} \left( \int_0^\varepsilon [A_{0,p}(F_s, \varepsilon)]^{1/p} \frac{de}{e} \right)^p ds \leq \left( \int_0^\varepsilon \left( \frac{1}{h} \int_t^{t+h} A_{0,p}(F_s, \varepsilon) ds \right)^{1/p} \frac{de}{e} \right)^p
\]

and Cauchy-Schwarz inequality and (38) we obtain

\[
\frac{1}{h} \int_t^{t+h} N_{0,p}(F_s, \varepsilon) ds \leq \frac{1}{h} \int_t^{t+h} F_s(\{0\}) ds + \left( \int_0^\varepsilon \left( \frac{1}{h} \int_t^{t+h} [A_{0,p}(F_s, \varepsilon)]^2 ds \right)^{\frac{1}{p}} \frac{de}{e} \right)^p
\]

\[
\leq \frac{1}{h} \int_t^{t+h} F_s(\{0\}) ds + \left( \frac{e^{ch} N}{h} \right)^{1/2} \left( \int_0^\varepsilon \left( \frac{e^{1-\alpha}}{N_{\alpha,2}(F_t, \varepsilon)} \right)^{\frac{1}{p}} \frac{de}{e} \right)^p. \tag{39}
\]

Next using monotonicity in Lemma 2.4 to the convex function \( \varphi(x) = [(1 - x/\varepsilon)\]^{\alpha} \) we have

\[
N_{0,p}(F_s, \varepsilon) \geq e^{-c(s-t)} N_{0,p}(F_t, \varepsilon) \geq e^{-ch} N_{0,p}(F_t, \varepsilon), \quad t \leq s \leq t + h,
\]

\[
F_s(\{0\}) \leq e^{c(t+h-s)} F_{t+h}(\{0\}) \leq e^{ch} F_{t+h}(\{0\}), \quad t \leq s \leq t + h.
\]
Thus
\[ e^{-cb}N_{0,p}(F_t, \varepsilon) \leq \frac{1}{h} \int_t^{t+h} N_{0,p}(F_s, \varepsilon)ds, \quad \frac{1}{h} \int_t^{t+h} F_s(\{0\})ds \leq e^{cb}F_{t+h}(\{0\}). \]

This together with (39) proves (34).

Finally for the case \( \alpha = 0 \), we have
\[ N_{0,2}(F_t, \varepsilon) = \inf_{0 \leq \theta \leq \varepsilon} \int_{[0,\theta]} \left( 1 - \frac{x}{\varepsilon} \right)^2 dF_t(x) = F_t(\{0\}) \text{ for all } \varepsilon > 0. \]

If \( F_t(\{0\}) = 0 \), then (35) holds trivially since the right hand side of (35) is \(-\infty\). If \( F_t(\{0\}) > 0 \), then
\[ \left( \int_0^\varepsilon \left( \frac{\varepsilon}{N_{0,2}(F_t, \varepsilon)} \right)^{3/2} \frac{dx}{\varepsilon} \right)^{3/2} \leq \left( \frac{1}{F_t(\{0\})} \right)^{1/2} (3\varepsilon^{1/3})^{3/2} = \left( \frac{27\varepsilon}{F_t(\{0\})} \right)^{1/2}. \]

Inserting it into (34) gives (35) and the proof is complete.

**Lemma 2.6.** Let \( F_0 \in B_1^+(\mathbb{R}_{\geq 0}) \) be a non-singular measure with \( N = N(F_0) > 0 \), \( E = E(F_0) > 0 \) satisfying the low temperature condition \( T/T_c < 1 \), let \( F_{be} \) be the unique Bose-Einstein distribution with the mass \( N \) and energy \( E \), and let \( F_t \) be a mass-energy conserved measure-valued solution of Eq.(5) on \([0, +\infty)\) obtained in Theorem 1.2 with the initial datum \( F_0 \). Then
\[ \lim_{t \to +\infty} \int_{\mathbb{R}_{\geq 0}} \varphi(x)dF_t(x) = \lim_{t \to +\infty} \int_{\mathbb{R}_{\geq 0}} \varphi(x)dF_{be}(x) \quad \forall \varphi \in C_b(\mathbb{R}_{\geq 0}) \tag{40} \]

\[ \lim_{t \to +\infty} F_t(\{0\}) \geq \lim_{\varepsilon \to +\infty} \sqrt{\lim_{t \to +\infty} N_{0,2}(F_t, \varepsilon)} \tag{41} \]

where
\[ N_{0,2}(F_t, \varepsilon) = \int_{\mathbb{R}_{\geq 0}} [(1 - x/\varepsilon)_+]N_{0,2}(F_t, x)dF_t(x). \tag{42} \]

**Proof.** Let \( \varphi \in C_b(\mathbb{R}_{\geq 0}) \). By conservation of mass we have for any \( \varepsilon > 0 \)
\[ \left| \int_{\mathbb{R}_{\geq 0}} \varphi dF_t - \int_{\mathbb{R}_{\geq 0}} \varphi dF_{be} \right| \leq \int_{\mathbb{R}_{\geq 0}} |\varphi(x) - \varphi(0)|d|F_t - F_{be}| (x) \]
\[ \leq \sup_{0 \leq x \leq \varepsilon} |\varphi(x) - \varphi(0)|2N + \frac{2||\varphi||_\infty}{\varepsilon} \|F_t - F_{be}\|_1^c \]

where \( ||\varphi||_\infty = \sup_{x \in \mathbb{R}_{\geq 0}} |\varphi(x)| \). Since, by the assumption on \( F_t, \|F_t - F_{be}\|_1^c \to 0 \) as \( t \to +\infty \), it follows by first letting \( t \to +\infty \) and then letting \( \varepsilon \to 0^+ \) that (40) holds true.

To prove (41), we first prove that for all \( \varepsilon > 0, t \geq 0, \)
\[ \int_t^{t+1} F_s(\{0\})ds \geq \inf_{s \geq t} N_{0,2}^{(2)}(F_s, \varepsilon) - \left( \frac{2\varepsilon N^2}{2} \right)^{1/4} 4\varepsilon^{1/4} \tag{43} \]

where \( c = \sqrt{NE} \). As did above, applying the differential equation (5) and inequality (30) to the convex function \( \varphi(x) = \varphi_\varepsilon(x) = [(1 - x/\varepsilon)_+]^2 \) and recalling that
Next using Hölder’s inequality and (44) we have
\[ N_{0,2}(F_t, \varepsilon) = \int_{\mathbb{R}^3} \varphi e \, dF_t \] we have
\[
\frac{d}{dt} [N_{0,2}(F_t, \varepsilon)]^2 = 2N_{0,2}(F_t, \varepsilon) \frac{d}{dt} N_{0,2}(F_t, \varepsilon)
\geq -2\varepsilon [N_{0,2}(F_t, \varepsilon)]^2 + 2N_{0,2}(F_t, \varepsilon) \int_{\mathbb{R}^3} K[\varphi_e] \, d^3F_t, \quad t \geq 0
\]
that is
\[
\frac{d}{dt} \left( e^{2\varepsilon t} [N_{0,2}(F_t, \varepsilon)]^2 \right) \geq 2e^{2\varepsilon t} N_{0,2}(F_t, \varepsilon) \int_{\mathbb{R}^3} K[\varphi_e] \, d^3F_t, \quad t \geq 0
\]
hence
\[
e^{2\varepsilon(t+1)} [N_{0,2}(F_{t+1}, \varepsilon)]^2 - e^{2\varepsilon t} [N_{0,2}(F_t, \varepsilon)]^2 \geq \int_t^{t+1} 2e^{2\varepsilon s} N_{0,2}(F_s, \varepsilon) \int_{\mathbb{R}^3} K[\varphi_e] \, d^3F_s ds, \quad t \geq 0.
\]
On the other hand, from Lemma 2.3 we have
\[
N_{0,2}(F_s, \varepsilon) \int_{\mathbb{R}^3} K[\varphi_e] \, d^3F_s \geq \frac{1}{\varepsilon} [A_{0,3/2}^{(2)}(F_s, \varepsilon)]^2
\]
and so
\[
\int_t^{t+1} e^{2\varepsilon s} [A_{0,3/2}^{(2)}(s, \varepsilon)]^2 ds \leq \frac{\varepsilon}{2} \left( e^{2\varepsilon(t+1)} [N_{0,2}(F_{t+1}, \varepsilon)]^2 - e^{2\varepsilon t} [N_{0,2}(F_t, \varepsilon)]^2 \right)
\]
\[
\leq \frac{\varepsilon}{2} e^{2\varepsilon(t+1)} N^2
\]
hence
\[
\int_t^{t+1} [A_{0,3/2}^{(2)}(s, \varepsilon)]^2 ds \leq \frac{\varepsilon}{2} e^{2\varepsilon t} N^2, \quad t \geq 0.
\tag{44}
\]
Now applying Lemma 2.1 to the measure $F_s$ with $p = 2$ and $\psi(x) = N_{0,2}(F_s, x)$, and recalling the definition of $A_{0,3/2}^{(2)}(F_s, \varepsilon)$ (see (27)), we have
\[
N_{0,2}^{(2)}(F_s, \varepsilon) \leq [F_s(\{0\})]^2 + \left( \int_0^\varepsilon [A_{0,3/2}^{(2)}(F_s, \varepsilon)]^{1/2} \frac{de}{\epsilon} \right)^2
\]
where we used the fact $\psi(0) = N_{0,2}(F_s, 0) = F_s(\{0\})$. Thus
\[
\sqrt{N_{0,2}^{(2)}(F_s, \varepsilon)} \leq F_s(\{0\}) + \int_0^\varepsilon [A_{0,3/2}^{(2)}(F_s, \varepsilon)]^{1/2} \frac{de}{\epsilon},
\]
\[
\int_t^{t+1} \sqrt{N_{0,2}^{(2)}(F_s, \varepsilon)} ds \leq \int_t^{t+1} F_s(\{0\}) ds + \int_t^{t+1} \int_0^\varepsilon [A_{0,3/2}^{(2)}(F_s, \varepsilon)]^{1/2} \frac{de}{\epsilon} ds. \tag{45}
\]
Next using Hölder’s inequality and (44) we have
\[
\int_t^{t+1} \int_0^\varepsilon [A_{0,3/2}^{(2)}(F_s, \varepsilon)]^{1/2} \frac{de}{\epsilon} ds = \int_0^\varepsilon \left( \int_t^{t+1} [A_{0,3/2}^{(2)}(F_s, \varepsilon)]^{1/2} ds \right) \frac{de}{\epsilon}
\leq \int_0^\varepsilon \left( \int_t^{t+1} [A_{0,3/2}^{(2)}(F_s, \varepsilon)]^{2} ds \right)^{1/4} \frac{de}{\epsilon} \leq \int_0^\varepsilon \left( \frac{\varepsilon e^{2\varepsilon N^2}}{2} \right)^{1/4} \frac{de}{\epsilon} = \left( \frac{e^{2\varepsilon N^2}}{2} \right)^{1/4} 4\varepsilon^{1/4}.
\]
Inserting this into the above inequality (45) gives
\[
\sqrt{\inf_{t \geq t} N_{0,2}^{(2)}(F_t, \varepsilon)} \leq \int_t^{t+1} F_s(\{0\}) ds + \left(\frac{e^{2\varepsilon N^2}}{2}\right)^{1/4} 4\varepsilon^{1/4}.
\]
This proves (43).
Since \( F_t(\{0\}) \) converges as \( t \to +\infty \), it follows from (43) that
\[
\lim_{t \to +\infty} F_t(\{0\}) \geq \liminf_{t \to +\infty} N_{0,2}^{(2)}(F_t, \varepsilon) - \left(\frac{e^{2\varepsilon N^2}}{2}\right)^{1/4} 4\varepsilon^{1/4} \quad \forall \varepsilon > 0.
\]
Therefore letting \( \varepsilon \to 0^+ \) leads to (41). Note that here we have used the fact that the function \( \varepsilon \to \liminf_{t \to +\infty} N_{0,2}^{(2)}(F_t, \varepsilon) \) is monotone non-decreasing on \((0, +\infty)\) and so the limit \( \lim_{\varepsilon \to 0^+} \left(\liminf_{t \to +\infty} N_{0,2}^{(2)}(F_t, \varepsilon)\right)^{1/2} \) exists. In fact, for any \( x \geq 0 \) the function \( \varepsilon \to [(1 - x/\varepsilon)^+]^2 \) is monotone non-decreasing on \((0, +\infty)\). This implies that the function \( \varepsilon \to N_{0,2}^{(2)}(F_t, \varepsilon) \) is monotone non-decreasing on \((0, +\infty)\) and thus the function \( \varepsilon \to \inf_{s \geq t} N_{0,2}^{(2)}(F_s, \varepsilon) \) hence the function \( \varepsilon \to \lim_{t \to +\infty} \inf_{s \geq t} N_{0,2}^{(2)}(F_s, \varepsilon) \) are also monotone non-decreasing on \((0, +\infty)\).

\[\Box\]

3. Proof of Theorem 1.4 and Theorem 1.5.

Proof of Theorem 1.4. According to part (b) of Theorem 1.2, we need only to prove that \( \lim_{t \to +\infty} F_t(\{0\}) > 0 \). From the double-size condition (11) we have for any \( t \geq t_0 \) and any \( \varepsilon \in (0, \varepsilon_0] \) that
\[
F_t([0, x/2]) \geq \frac{1}{C} F_t([0, x]) \quad \forall x \in (0, \varepsilon]
\]
and so
\[
N_{0,2}(F_t, x) \geq \int_{[0, x/2]} \left(1 - \frac{y}{x}\right)^2 dF_t(y) \geq \frac{1}{4} F_t([0, x/2]) \\
\geq \frac{1}{4C} F_t([0, x]) \geq \frac{1}{4C} \int_{[0, x]} \left(1 - \frac{y}{\varepsilon}\right)^2 dF_t(y).
\]
Note that for the case \( x = 0 \) we also have
\[
N_{0,2}(F_t, 0) = F_t(\{0\}) \geq \frac{1}{4C} F_t(\{0\}) = \frac{1}{4C} \int_{[0, x]} \left(1 - \frac{y}{\varepsilon}\right)^2 dF_t(y) \bigg|_{x=0}.
\]
Thus
\[
N_{0,2}(F_t, x) \geq \frac{1}{4C} \int_{[0, x]} \left(1 - \frac{y}{\varepsilon}\right)^2 dF_t(y) \quad \forall x \in [0, \varepsilon].
\]
From this we obtain
\[
N_{0,2}^{(2)}(F_t, \varepsilon) = \int_{[0, \varepsilon]} \left(1 - \frac{x}{\varepsilon}\right)^2 N_{0,2}(F_t, x) dF_t(x) \\
\geq \frac{1}{4C} \int_{[0, \varepsilon]} \left(1 - \frac{x}{\varepsilon}\right)^2 \int_{[0, x]} \left(1 - \frac{y}{\varepsilon}\right)^2 dF_t(y) dF_t(x) \\
= \frac{1}{4C} \int_{0 \leq y \leq x \leq \varepsilon} \left(1 - \frac{y}{\varepsilon}\right)^2 \left(1 - \frac{x}{\varepsilon}\right)^2 dF_t(y) dF_t(x) \\
\geq \frac{1}{8C} \left(\int_{[0, \varepsilon]} \left(1 - \frac{x}{\varepsilon}\right)^2 dF_t(x)\right)^2 = \frac{1}{8C} [N_{0,2}(F_t, \varepsilon)]^2.
\]
This is where the double-size condition (11) comes into play. Thus, with \( a = \frac{1}{\sqrt{8C}} \),
\[
\sqrt{N_{0,2}^{(2)}(F_t, \varepsilon)} \geq a N_{0,2}(F_t, \varepsilon) \quad \forall t \geq t_0, \forall \varepsilon \in (0, \varepsilon_0].
\] (46)
Taking the lower limit \( \liminf \) to both sides of (46) with \( \varepsilon \in (0, \varepsilon_0] \) fixed and applying the weak convergence (40) to \( \varphi = \varphi_\varepsilon \) we obtain
\[
\sqrt{\liminf_{t \to +\infty} N_{0,2}^{(2)}(F_t, \varepsilon)} \geq \liminf_{t \to +\infty} N_{0,2}(F_t, \varepsilon) = \lim_{t \to +\infty} N_{0,2}(F_t, \varepsilon)
\]
\[
= N_{0,2}(F_{\text{be}}, \varepsilon) \quad \forall \varepsilon \in (0, \varepsilon_0].
\]
This together with Lemma 2.6 implies that
\[
\lim_{t \to +\infty} F_1(\{0\}) \geq \lim_{\varepsilon \to 0^+} \sqrt{\liminf_{t \to +\infty} N_{0,2}^{(2)}(F_t, \varepsilon)} \geq a \lim_{\varepsilon \to 0^+} N_{0,2}(F_{\text{be}}, \varepsilon) = aF_{\text{be}}(\{0\}) > 0.
\]
This proves Theorem 1.4. \( \square \)

Proof of Theorem 1.5. We first prove that the assumption (12) implies a very low temperature condition:
\[
\frac{T}{T_c} = c_0 \frac{E}{N^{5/3}} \leq c_0 \left( \frac{1}{5^{10} \cdot 2} \right)^{2/3}.
\] (47)
In fact for any \( \varepsilon > 0 \) we have \( F_0([0, \varepsilon]) \leq N(F_0) = N \) and so
\[
\frac{\varepsilon}{F_0([0, \varepsilon])} \geq \frac{\varepsilon}{N} \quad \text{hence} \quad D^{-1}(F_0, \varepsilon) \geq \frac{\varepsilon}{N} \quad \text{for all} \quad \varepsilon > 0.
\]
This gives
\[
\int_0^{2E} \frac{D^{-1}(F_0, \varepsilon)^{1/5}}{\varepsilon} \, d\varepsilon \geq \int_0^{2E} \frac{(N^{-1} \varepsilon)^{1/5}}{\varepsilon} \, d\varepsilon = N^{-1/5} \left( \frac{2E}{N} \right)^{1/5}.
\]
Combining this with (12) we obtain
\[
N^{-1/5} \left( \frac{2E}{N} \right)^{1/5} \leq \int_0^{2E} \frac{D^{-1}(F_0, \varepsilon)^{1/5}}{\varepsilon} \, d\varepsilon \leq \frac{1}{5} \left( \frac{E}{N} \right)^{1/10}
\]
and so
\[
\frac{E}{N^{5/3}} \leq \left( \frac{1}{5^{10} \cdot 2} \right)^{2/3}.
\] (48)
This proves (47).

Next we prove that
\[
F_h(\{0\}) > \frac{1}{50} N \quad \text{with} \quad h = \frac{1}{\sqrt{NE}}.
\] (49)
To do this we first use the elementary inequality (for \( p > 1 \))
\[
\left( 1 - \frac{x}{\varepsilon} \right)^p \geq 1 - \frac{p}{\varepsilon} x \quad \forall x \in \mathbb{R}_{\geq 0}
\]
and the conservation of mass and energy to get
\[
N_{0, p}(F_t, \varepsilon) \geq N - \frac{p}{\varepsilon} E \quad \forall t \in [0, +\infty).
\] (50)
Inserting this inequality into (34) with \( \alpha = 1, p = 5/2, h = \frac{1}{\varepsilon} = 1/\sqrt{NE} \), and \( t = 0 \) we have for any \( \varepsilon > 0 \)
\[
F_h(\{0\}) \geq e^{-\frac{1}{2} \left( N - \frac{5}{2\varepsilon} E \right)} - \left( \frac{N\sqrt{NE}}{\varepsilon} \right)^{1/2} \left( \int_0^\varepsilon \left( \frac{1}{N_{1,2}(F_0, \varepsilon)} \right)^{1/5} \frac{d\varepsilon}{\varepsilon} \right)^{5/2}. \] (51)
Since 
\[ N_{1,2}(F_0, \delta) \geq \frac{1}{\delta} \int_{[0, \delta/2]} \left(1 - \frac{x}{\delta}\right)^2 dF_0(x) \geq \frac{1}{8} \cdot \frac{1}{\delta^2} F_0([0, \delta/2]) \]
which implies
\[ \frac{1}{N_{1,2}(F_0, \delta)} \leq 8 \cdot \frac{\delta/2}{F_0([0, \delta/2])} \quad \forall \delta > 0 \]
it follows that
\[ \frac{1}{N_{1,2}(F_0, \epsilon)} = \sup_{0 < \epsilon \leq \delta} \frac{1}{N_{1,2}(F_0, \delta)} \leq 8 \sup_{0 < \epsilon \leq \delta} \frac{\delta/2}{F_0([0, \delta/2])} = 8D^{-1}(F_0, \epsilon/2) \quad \forall \epsilon > 0. \]
Thus
\[ \left( \int_0^\epsilon \left( \frac{1}{N_{1,2}(F_0, \epsilon)} \right)^{1/5} d\epsilon \right)^{5/2} \leq 8^{1/2} \left( \int_0^{\epsilon/2} \frac{[D^{-1}(F_0, \epsilon)]^{1/5}}{\epsilon} d\epsilon \right)^{5/2}. \]
Inserting this inequality into (51) and taking \( \epsilon = \frac{4\epsilon}{N} \) we obtain with the condition (12) that
\[ F_h(\{0\}) \geq e^{-3/8} N - \left(8N \sqrt{N^E} \over e \right)^{1/2} \left( \int_0^{2\epsilon} [D^{-1}(F_0, \epsilon)]^{1/5} d\epsilon \right)^{5/2} \]
\[ \geq e^{-3/8} N - \left(8N \sqrt{N^E} \over e \right)^{1/2} \left( \frac{1}{5} \right)^{5/2} (N \over E)^{1/4} \]
\[ = \left( \frac{3}{e^2} - \frac{8}{e} \right)^{1/2} N > \frac{1}{50} N. \]
Next we prove the following implication with \( h = 1/\sqrt{N^E} \):
\[ t > 0 \quad \text{and} \quad F_t(\{0\}) > \frac{1}{50} N \quad \implies \quad F_{t+h}(\{0\}) > \frac{1}{50} N. \quad (52) \]
Suppose \( t > 0 \) and \( F_t(\{0\}) > \frac{1}{50} N \). Then using (35) in Lemma 2.5 with \( h = 1/c = 1/\sqrt{N^E} \) and \( \epsilon = 6E/N \) and using (50) with \( p = 3/2 \) and (48) we compute
\[ F_{t+h}(\{0\}) \geq \left( e^{-2} \left(N - \frac{3}{2}E \over \epsilon \right) - \left(27 \sqrt{N^E} \over e \right)^{1/2} \left( \frac{N}{F_t(\{0\})} \right)^{1/2} \epsilon^{1/2} \right) \bigg|_{\epsilon = \frac{6E}{N}} \]
\[ \geq e^{-2} \frac{3}{4} N - \left(27 \sqrt{N^E} \over e \right)^{1/2} (50)^{1/2} \left( E \over N \right)^{1/2} \]
\[ = \left( \frac{3}{e^2} - \frac{27 \cdot 50 \cdot 6}{e} \right)^{1/2} \left( E \over N^{5/3} \right)^{3/4} N \]
\[ \geq \left( \frac{3}{e^2} - \frac{27 \cdot 50 \cdot 6}{e} \right)^{1/2} \left( \frac{1}{5^{10/3}} \right)^{1/2} N > \frac{1}{50} N. \]
Thus (52) holds true. From (52) and (50) we deduce
\[ F_{nh}(\{0\}) > \frac{1}{50} N, \quad n = 1, 2, 3, \ldots \quad (53) \]
Finally for any $t \in [h, +\infty)$ there is $n \in \mathbb{N}$ such that $nh \leq t < (n+1)h$ and so from (32) and (53) we conclude

$$F_t(\{0\}) \geq F_{nh}(\{0\})e^{-ch} > \frac{1}{50e}N.$$

Thanks to part (b) of Theorem 1.2, this positivity of lower bound of $F_t(\{0\})$ on $[h, +\infty)$ implies the convergence (8), i.e. $\lim_{t \to +\infty} F_t(\{0\}) = (1 - (T/T_c)^{3/5})N$. \hfill \Box

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**REFERENCES**

[1] L. Arkeryd, On low temperature kinetic theory; spin diffusion, Bose Einstein condensates, anyons, *J. Stat. Phys.*, **150** (2013), 1063–1079.

[2] L. Arkeryd and A. Nouri, Bose condensates in interaction with excitations: A kinetic model, *Comm. Math. Phys.*, **310** (2012), 765–788.

[3] J. Banerji and J. J. L. Velázquez, Blow-up rate estimates for the solutions of the bosonic Boltzmann-Nordheim equation, *J. Math. Phys.*, **56** (2015), 063302, 27 pp.

[4] D. Benedetto, M. Pulvirenti, F. Castella and R. Esposito, On the weak-coupling limit for bosons and fermions, *Math. Models Methods Appl. Sci.*, **15** (2005), 1811–1843.

[5] M. Briant and A. Einav, On the Cauchy problem for the homogeneous Boltzmann-Nordheim equation for bosons: local existence, uniqueness and creation of moments, *J. Stat. Phys.*, **163** (2016), 1108–1156.

[6] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, Third Edition, Cambridge University Press, Cambridge, 1939.

[7] L. Erdős, M. Salmhofer and H.-T. Yau, On the quantum Boltzmann equation, *J. Stat. Phys.*, **116** (2004), 367–380.

[8] M. Escobedo, S. Mischler and M. A. Valle, *Homogeneous Boltzmann Equation in Quantum Relativistic Kinetic Theory*, Electronic Journal of Differential Equations, Monograph, 4. Southwest Texas State University, San Marcos, TX, 2003. 85 pp.

[9] M. Escobedo, S. Mischler and J. J. L. Velázquez, Singular solutions for the Uehling-Uhlenbeck equation, *Proc. Roy. Soc. Edinburgh*, **138** (2008), 67–107.

[10] M. Escobedo and J. J. L. Velázquez, On the blow up and condensation of supercritical solutions of the Nordheim equation for bosons, *Comm. Math. Phys.*, **330** (2014), 331–365.

[11] M. Escobedo and J. J. L. Velázquez, Finite time blow-up and condensation for the bosonic Nordheim equation, *Invent. Math.*, **200** (2015), 761–847.

[12] C. Josserand, Y. Pomeau and S. Rica, Self-similar singularities in the kinetics of condensation, *J. Low Temp. Phys.*, **145** (2006), 231–265 (2006).

[13] X. Lu, On isotropic distributional solutions to the Boltzmann equation for Bose-Einstein particles, *J. Statist. Phys.*, **116** (2004), 1597–1649.

[14] X. Lu, The Boltzmann equation for Bose-Einstein particles: Velocity concentration and convergence to equilibrium, *J. Statist. Phys.*, **119** (2005), 1027–1067.

[15] X. Lu, The Boltzmann equation for Bose-Einstein particles: Condensation in finite time, *J. Stat. Phys.*, **150** (2013), 1138–1176.

[16] X. Lu, The Boltzmann equation for Bose-Einstein particles: Regularity and condensation, *J. Stat. Phys.*, **156** (2014), 493–545.

[17] X. Lu, Long time convergence of the Bose-Einstein condensation, *J. Stat. Phys.*, **162** (2016), 652–670.

[18] J. Lukkarinen and H. Spohn, Not to normal order–notes on the kinetic limit for weakly interacting quantum fluids, *J. Stat. Phys.*, **134** (2009), 1133–1172.

[19] P. A. Markowich and L. Pareschi, Fast conservative and entropic numerical methods for the boson Boltzmann equation, *Numer. Math.*, **99** (2005), 509–532.

[20] L. W. Nordheim, On the kinetic methods in the new statistics and its applications in the electron theory of conductivity, *Proc. Roy. Soc. London Ser. A*, **119** (1928), 689–698.

[21] A. Nouri, Bose-Einstein condensates at very low temperatures: A mathematical result in the isotropic case, *Bull. Inst. Math. Acad. Sin. (N.S.)*, **2** (2007), 649–666.
[22] W. Rudin, *Real and Complex Analysis*, Third edition. McGraw-Hill Book Co., New York, 1987. xiv+416 pp. ISBN: 0-07-054234-1 00A05 (26-01 30-01 46-01).

[23] D. V. Semikov and I. I. Tkachev, Kinetics of Bose condensation, *Phys. Rev. Lett.*, **74** (1995), 3093–3097.

[24] D. V. Semikov and I. I. Tkachev, Condensation of Bose in the kinetic regime, *Phys. Rev. D*, **55** (1997), 489–502.

[25] H. Spohn, Kinetics of the Bose-Einstein condensation, *Physica D*, **239** (2010), 627–634.

[26] E. A. Uehling and G. E. Uhlenbeck, Transport phenomena in Einstein-Bose and Fermi-Dirac gases, I, *Phys. Rev.*, **43** (1933), 552–561.

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