NONRATIONALITY AND PRINCIPAL PERMUTATION
CLASSES: THE REMAINING CASE

MIKLÓS BÓNA

Abstract. We complete the proof of the fact that all principal permutation classes generated by a pattern longer than two have a nonrational generating function.

1. Introduction

We say that a permutation \( p \) contains the pattern \( q = q_1q_2 \cdots q_k \) if there is a \( k \)-element set of indices \( i_1 < i_2 < \cdots < i_k \) so that \( p_{i_r} < p_{i_s} \) if and only if \( q_r < q_s \). If \( p \) does not contain \( q \), then we say that \( p \) avoids \( q \). For example, \( p = 3752416 \) contains \( q = 2413 \), as the first, second, fourth, and seventh entries of \( p \) form the subsequence 3726, which is order-isomorphic to \( q = 2413 \). A recent survey on permutation patterns can be found in [17] and a book on the subject is [5]. Let \( \text{Av}_n(q) \) be the number of permutations of length \( n \) that avoid the pattern \( q \). In general, it is very difficult to compute, or even describe, the numbers \( \text{Av}_n(q) \), or their sequence as \( n \) goes to infinity.

As far as the generating function \( A_q(z) = \sum_{n \geq 0} \text{Av}_n(q)z^n \) goes, there are known examples when it is algebraic, (when \( q \) is of length three, or when \( q = 1342 \)), and known examples when it is not algebraic (when \( q \) is the monotone pattern \( 12 \cdots k \), where \( k \) is an even integer that is at least four). There are no known examples for a single pattern \( q \) for which \( A_q(z) \) is not differentiably finite, but Scott Garrabrant and Igor Pak [9, 10] constructed a large set \( S \) of patterns and proved that the generating function \( A_S(z) \) of the class of permutations avoiding all patterns in \( S \) is not differentiably finite.

In [7], the present author proved that for most patterns \( q \), the generating function \( A_q(z) \) was not rational. However, he was not able to prove that nonrationality result for patterns \( q = q_1q_2 \cdots q_k \) for which \( q_1 = 1 \) and \( q_k = k \), and which are not Wilf-equivalent to a skew indecomposable pattern \( q' \) that does not satisfy those equalities for \( q_1 \) and \( q_k \). A pattern is skew indecomposable if it cannot be cut into two parts so that each entry before the cut is larger than each entry after the cut. If \( p \) is not skew indecomposable, then there is a unique ways to cut it up into skew indecomposable permutations of consecutive entries, which we call the skew blocks of \( p \). For instance, the skew blocks of 645231 are 6, 45, 23, and 1.
In this paper, we handle the missing case by proving that $A_q(z)$ is not rational if $q$ is any pattern longer than two, and ends in its largest entry. Together with the results from [7] that we quoted in the preceding paragraph, this proves that $A_q(z)$ is not rational if $q$ is any pattern longer than two. Note that obviously, $A_{12}(z) = A_{21}(z) = 1/(1 - z)$, which is a rational function. The set of all finite permutations avoiding a single pattern $q$ is often called a principal permutation class, which explains the title of this paper.

Our main tools are the theory of supercritical sequences by Flajolet and Sedgewick [8], and a 2014 result of Atapour and Madras [11] on the ratio $A_{n+1}(q)/A_n(q)$. We will explain these tools in the next section. Our main strategy is to show that on average, the number of skew blocks of a $q$-avoiding permutation of length $n$ grows slower than any linear function of $n$, and that would not be possible if $A_q(z)$ were rational.

2. Background

2.1. Supercritical relations. We will place our results into the broader context of supercritical relations. Readers who are interested to learn more about this subject are invited to consult Sections V.2 and VI.9 of [8].

Definition 2.1. Let $F$ and $G$ be two generating functions with nonnegative real coefficients that are analytic at 0, and let us assume that $G(0) = 0$. Then the relation

$$F(z) = \frac{1}{1 - G(z)}$$

is called supercritical if $G(R_G) > 1$, where $R_G$ is the radius of convergence of $G$.

Note that the functional equation $F(z) = 1/(1 - G(z))$ is very common in combinatorics. It occurs when the objects counted by the generating function $F(z)$ are sequences of nonempty objects counted by $G(z)$. In our case, if $q$ is a skew-indecomposable pattern, and $G(z)$ is the ordinary generating function for the number of skew indecomposable permutations of a given length that avoid $q$, then

$$A_q(z) = \frac{1}{1 - G(z)}.$$  

Indeed, a permutation avoids a skew indecomposable pattern $q$ if and only if all its skew blocks do.

In general, we $G(z)$ counts the components, sometimes called the $G$-components, of the composite structures counted by $F(z)$. In case of our $q$-avoiding permutations, these components are the skew blocks.

Theorem 2.2. Let $G(z)$ be a rational power series with nonnegative real coefficients that satisfies $G(0) = 0$. Then the relation

$$F(z) = \frac{1}{1 - G(z)}$$

is supercritical.
Proof. If \( G(z) \) is a polynomial, then \( R_G = \infty \), so \( G(R_G) = \infty > 1 \), and our claim is proved. Otherwise, \( G(z) \) is a rational function that has at least one singularity, and all its singularities are poles. Let \( R_G \) be a singularity of smallest modulus. Then \( G(R_G) = \infty > 1 \), completing our proof. \( \square \)

Therefore, if we want to prove that \( F(z) \) is not rational, it suffices to prove that the relation \( F(z) = 1/(1 - G(z)) \) is not supercritical. The following theorem is a useful tool in proving that non-supercritical property.

**Theorem 2.3.** Let the relation between \( F \) and \( G \) be supercritical as given in Definition 2.1. Let \( X_n \) denote the number of \( G \)-components in a random \( F \)-structure of length \( n \). Then there exists a positive constant \( C \) so that

\[
E(X_n) \simeq Cn.
\]

Here \( E(X_n) \) denotes the expectation of \( X_n \) taken over all \( F \)-structures of size \( n \). For our permutations, the size is simply the number of entries. See Theorem V.1. in [8] for a more detailed version of this theorem. In particular, the exact value of \( C \) is known, but we will not need that information in our argument.

In other words, in order to prove that a combinatorial generating function \( F(z) = 1/(1 - G(z)) \) is not rational, it suffices to prove that the expected number of \( G \)-components in a random \( F \)-structure is not linear. Let

\[
\Delta E(X_n) = E(X_{n+1}) - E(X_n).
\]

**Corollary 2.4.** Let us keep the notation of Theorem 2.3. If \( F(z) \) is rational, then there exists a positive constant \( C \) so that

\[
\Delta E(X_n) \simeq C.
\]

2.2. The sequence \( \text{Av}_{n+1}(q)/\text{Av}_n(q) \). It is well known that for any pattern \( q \), the limit

\[
L(q) = \lim_{n \to \infty} \sqrt[n]{\text{Av}_n(q)}
\]

exists. The limit \( L(q) \) is often called the growth rate of \( q \). However, the question whether the limit

\[
\lim_{n \to \infty} \frac{\text{Av}_{n+1}(q)}{\text{Av}_n(q)}
\]

exists is more difficult and not completely settled. It is clear that if this limit exists, then it is equal to the limit \( L(q) \) defined in (1). The following result will be crucial in our argument in the next section.

**Theorem 2.5.** (Theorem 6.4. and Remark 2. in [11]) Let \( q = q_1q_2 \cdots q_k \) so that \( q_k = k \). Then

\[
\lim_{n \to \infty} \frac{\text{Av}_{n+1}(q)}{\text{Av}_n(q)}
\]

exists and (trivially) equals \( L(q) \).
3. Expected number of skew blocks

For brevity, let \(a_n = Av_n(q)\) when the pattern \(q\) is fixed in advance. Let \(q\) be a pattern of length \(k > 2\) that ends in its largest entry \(k\). Note that such patterns are automatically skew indecomposable, and so a permutation \(p\) avoids \(q\) if and only if all the skew blocks of \(p\) avoid \(q\). Theorem 2.5 tells us that the limit \(L = \lim_{n \to \infty} a_{n+1}/a_n\) exists.

Let \(n\) be large enough so that \(|(a_{n+1}/a_n) - L| < \epsilon\), where \(\epsilon\) will be determined later.

We obtain all permutations counted by \(a_{n+1}\) by taking each permutation \(p\) counted by \(a_n\), and inserting the entry \(n + 1\) to a gap position close to the front end of \(p\). Indeed, as \(q\) ends in its largest entry \(k\), the gap positions into which \(n + 1\) can be inserted without forming a copy of \(q\) constitute an initial segment of all gap positions in \(p\). Because the limit \(L\) exists, on average, there will be \(L\) gap positions where we can insert the entry \(n + 1\). (The front end of \(p\) is also considered a gap position.) We would like to find an upper bound for \(\Delta E(X_n)\); therefore, we will analyze how these insertions change the number of skew blocks in \(p\). There are three cases as follows. Let \(p = p_1 p_2 \cdots p_n\), and let us insert \(n + 1\) immediately to the left of the entry \(p_i\), to get the new permutation \(p'\).

1. If \(i = 1\), then \(p'\) has one more skew blocks than \(p\), the new skew block being the one-entry skew block consisting of \(n + 1\).
2. If \(i\) is such that \(t > 1\) skew blocks are at least partially contained in the initial segment \(p_1 p_2 \cdots p_{i-1}\), then \(p'\) will have \(t - 1\) fewer skew blocks than \(p\), because all those \(t\) skew blocks will be absorbed by the skew block containing \(n + 1\). For instance, if \(p = 5|4|123\), and \(i = 3\), then \(t = 2\), and \(p' = 54|6|123\) has one fewer skew blocks than \(p\). If \(i = 4\), then \(t = 3\), and \(p' = 54|6|23\) has two fewer skew blocks than \(p\). Note that it is here that we need the condition that \(k > 2\). If \(k = 2\), then \(q = 12\), and the only \(q\)-avoiding permutation is the decreasing one, and so \(n + 1\) can only be inserted into the first position. So this second case never occurs.
3. Keeping the notation of the previous case, if \(t = 1\), then \(p'\) has the same number of skew blocks as \(p\). For instance, if \(p = 456123\), and \(i = 3\), then \(p' = 4576123\) has the same number of skew blocks as \(p\).

Insertions discussed in the first case contribute a positive number to \(\Delta E(X_n)\), insertions in the second case will have a negative contribution, and insertions in the last case will have a zero contribution. We are in the first case exactly \(a_n\) times, since we can affix \(n + 1\) to the front of each permutation counted by \(a_n\). Therefore, the contribution of these insertions to \(\Delta E(X_n)\) is

\[
\frac{a_n}{a_{n+1}} \simeq \frac{1}{L}.
\]
Now let us consider permutations $p = p_1 p_2 \cdots p_n$ that avoid $q$ and for which $p_1 = n$. There are $a_{n-1}$ such permutations, and $R = p_2 p_3 \cdots p_n$ can be any permutation of length $n-1$ that avoids $q$. Then, on average, we can insert $n+1$ into $L$ gap positions of $R$, and $L-1$ of them will decrease the number of skew blocks of $p'$ by at least one. (The only insertion that does not result in a decrease in the number of skew blocks is when we insert $n+1$ to the right of $R$, to get the permutation $n(n+1)R$.) So this decrease will occur at $(L-1)a_{n-1}$ times, resulting in a total decrease of $\Delta(E(X_n))$ by at least

$$\frac{(L-1)a_{n-1}}{a_{n+1}} \approx \frac{L-1}{L^2} = \frac{1}{L} - \frac{1}{L^2}.$$  

Now let us consider permutations $p = p_1 p_2 \cdots p_n$ that avoid $q$ and for which $p_1 = n$, and $p_2 = n-1$, that is, permutations of the form $n(n-1)R$. If we insert $n+1$ into $R$, we will decrease the number of components by at least 2. One of them was already counted in the previous paragraph, so we will decrease the number of components by at least one more. This happens $(L-1)a_{n-2}$ times. Dividing this by $a_{n+1}$, we see that the additional decrease in $\Delta E(X_n)$ is

$$\frac{(L-1)a_{n-2}}{a_{n+1}} \approx \frac{L-1}{L^3} = \frac{1}{L^2} - \frac{1}{L^3}.$$  

Continuing in this way, we see that for each positive integer $k$, considering permutations starting with the string $n(n-1) \cdots (n-k+1)$ results in an additional decrease of

$$\frac{(L-1)a_{n-k}}{a_{n+1}} \approx \frac{L-1}{L^{k+1}} = \frac{1}{L^k} - \frac{1}{L^{k+1}}.$$  

Therefore, we can state and prove the following result.

**Theorem 3.1.** For all positive integers $k$, there exists a positive integer $N$ so that if $n > N$, then

$$\Delta E(X_n) < \frac{1}{L^k}.$$  

In particular, $\Delta E(X_n) \to 0$ as $n$ goes to infinity.

**Proof.** The argument we have just made shows that

$$\Delta E(X_n) \leq \frac{a_n}{a_{n+1}} - \sum_{i=1}^{k} \frac{(L-1)a_{n-k}}{a_{n+1}}.$$  

Now let us choose $n$ so large that for $i \leq k$, the inequality

$$\left| \frac{(L-1)a_{n-i}}{a_{n+1}} - \left( \frac{1}{L^i} - \frac{1}{L^{i+1}} \right) \right| \leq \frac{(1/L^k) - (1/L^{k+1})}{k}$$  

holds. We can do this because of inequality (5).
Comparing the last two inequalities, we see that if $n$ is large enough, then

$$\Delta E(X_n) \leq \frac{1}{L} - \sum_{i=1}^{k} \left( \frac{1}{L^i} - \frac{1}{L^{i+1}} \right) + k \cdot \frac{(1/L^k) - (1/L^{k+1})}{k}$$

$$\leq \frac{1}{L^{k+1}} + \frac{1}{L^k} - \frac{1}{L^{k+1}}$$

$$\leq \frac{1}{L^k}.$$  □

Our main results are now immediate.

**Theorem 3.2.** Let $q = q_1q_2 \cdots q_k$ be a permutation pattern so that $q_k = k > 2$. Then $A_q(z)$ is not rational.

**Proof.** If $A_q(z)$ were rational, then by Theorem 2.2 and 2.3, the expected number $E(X_n)$ of skew blocks in all $q$-avoiding permutations of length $n$ would by asymptotically equal to $Cn$ for some constant $C$, and Corollary 2.4, the asymptotic equality $\Delta E(X_n) \simeq C$ would hold. However, Theorem 3.1 shows that that is not the case. □

**Theorem 3.3.** Let $q$ be any permutation pattern longer than two. Then $A_q(z)$ is not rational.

**Proof.** It suffices to consider skew indecomposable patterns, because if $q$ is not skew indecomposable, its reverse is. If $q = q_1q_2 \cdots q_k$ is skew indecomposable, and $q_k \neq k$, then Theorem 4.3 in [7] shows that $A_q(z)$ is not rational. If $q_k = k$, and $k > 2$, then Theorem 3.2 shows that $A_q(z)$ is not rational, completing the proof. □

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, 358 LITTLE HALL, PO BOX 118105, GAINESVILLE, FL, 32611 – 8105 (USA)

Email address: bona@ufl.edu