Stability analysis for the Implicit-Explicit discretization of the Cahn-Hilliard equation

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Abstract

Implicit-Explicit methods have been widely used for the efficient numerical simulation of phase field problems such as the Cahn-Hilliard equation or thin film type equations. Due to the lack of maximum principle and stiffness caused by the effect of small dissipation coefficient, most existing theoretical analysis relies on adding additional stabilization terms, mollifying the nonlinearity or introducing auxiliary variables which implicitly either changes the structure of the problem or trades accuracy for stability in a subtle way. In this work we introduce a robust theoretical framework to analyze directly the stability of the standard implicit-explicit approach without stabilization or any other modification. We take the Cahn-Hilliard equation as a model case and prove energy stability under natural time step constraints which are optimal with respect to energy scaling. These settle several questions which have been open since the work of Chen and Shen [4].

1 Introduction

The Cahn-Hilliard (CH) equation was first introduced by Cahn and Hilliard in [2] to describe the complicated phase separation and coarsening phenomena in non-uniform systems such as glasses, alloys and polymer mixtures. In this work we are concerned with the numerical solutions for the Cahn-Hilliard equation in nondimensionalized form as

\[
\begin{aligned}
    \partial_t u &= \Delta(-\nu \Delta u + f(u)), \quad (t, x) \in \Omega \times (0, \infty) \\
    u \big|_{t=0} &= u_0,
\end{aligned}
\]

(1.1)

where \( u = u(t, x) \) is a real-valued function which represents the concentration difference in a binary system, and \( \nu > 0 \) is usually called mobility coefficient. The function \( f(u) \) is taken as the derivative of a standard double well potential:

\[
f(u) = u^3 - u = F'(u), \quad F(u) = \frac{1}{4}(u^2 - 1)^2.
\]
Note that with this choice the equation (1.1) is invariant under the sign change $u \to -u$ which is natural since $u$ corresponds to the difference of concentrations. The minima of the double well potential are $u = \pm 1$ which typically correspond to the formation of domains. The length scale of the transition regions between domains is typically proportional to $\sqrt{\nu}$. For simplicity we shall fix the spatial domain $\Omega$ in (1.1) as the usual 1-periodic torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d = [-\frac{1}{2}, \frac{1}{2})^d$ in physical dimensions $d \leq 3$. With this it is convenient to work with the Fourier basis $(e^{2\pi ik \cdot \cdot \cdot})_{k \in \mathbb{Z}^d}$ and this is the convention we shall adopt for the Fourier inversion formulae to be used throughout this paper. For smooth solutions, the mass conservation law takes the form

$$\frac{d}{dt} M(t) = \frac{d}{dt} \int_{\Omega} u(t, x) dx = 0.$$  

In particular $M(t) \equiv 0$ if the initial total mass is zero. If $M(t) \equiv c$ for some $c$ nonzero, then by a change of variable $u \to u + c$ one can work with a modified nonlinearity $f(u + c)$ in (1.1) and the corresponding analysis can be adjusted suitably. Therefore throughout this work we will only consider mean zero initial data for simplicity. As is well known the system (1.1) is a gradient flow of a Ginzburg-Landau type energy functional $\mathcal{E}(u)$ in $H^{-1}$, i.e.

$$\partial_1 u = \left. \frac{\delta \mathcal{E}}{\delta u} \right|_{H^{-1}} = \Delta \left( \frac{\delta \mathcal{E}}{\delta u} \right),$$

where $\left. \frac{\delta \mathcal{E}}{\delta u} \right|_{H^{-1}}, \frac{\delta \mathcal{E}}{\delta u}$ denote the standard variational derivatives in $H^{-1}$ and $L^2$ respectively, and

$$\mathcal{E}(u) = \int_{\Omega} \left( \frac{1}{2} \nu |\nabla u|^2 + F(u) \right) dx = \int_{\Omega} \left( \frac{1}{2} \nu |\nabla u|^2 + \frac{1}{4} (u^2 - 1)^2 \right) dx.$$  

As such the basic energy conservation law takes the form

$$\frac{d}{dt} \mathcal{E}(u(t)) + \| |\nabla|^{-1} \partial_1 u \|^2_2 = \frac{d}{dt} \mathcal{E}(u(t)) + \int_{\Omega} |\nabla (-\nu \Delta u + f(u))|^2 dx = 0.$$  

Here one should note that $\partial_1 u$ has mean zero so that $|\nabla|^{-1}$ is certainly well-defined. From energy conservation one immediately deduces monotonic energy decay and a priori $H^1$ bound of the solution as

$$\mathcal{E}(u(t)) \leq \mathcal{E}(u(s)), \quad \forall t \geq s;$$

$$\|\nabla u(t)\|_2 \leq \sqrt{\frac{2}{\nu}} \mathcal{E}(u(t)) \leq \sqrt{\frac{2}{\nu}} \mathcal{E}(u_0), \quad \forall t > 0.$$  

By a scaling analysis one can identify the critical spaces for CH in 2D and 3D as $L^2$ and $\dot{H}^{\frac{d}{2}}$ respectively. Global wellposedness in $H^1$ and regularity of solutions then follow easily from the a priori $\dot{H}^1$ bound and standard arguments. As the primary goal of the CH model is to understand the physics and dynamics of spinodal decomposition (especially concerning the stages after quenching), a large body of existing mathematical analysis is naturally devoted to the investigation of asymptotic behavior of solutions concerning coarsening, pattern formation, evolution of interfaces, stability and instability in various time regimes. One should keep in mind that the dynamics of CH is quite complex and takes place on a myriad of time scales ranging from short time scales $t = O(\sqrt{\nu})$ on which the solution generally develops rich structures with many internal layers and sharp gradients to metastable time scales $t = O(e^{\text{const}/\sqrt{\nu}})$ on which the unstable or super slow
modes are fully developed and coarsening starts to dominate. As a result the numerical simulation of CH can be quite stiff as one need to resolve layers of order $\sqrt{\nu}$ especially when $\nu \ll 1$.

There exist some natural scaling transformations which lead to slightly different forms of the CH system in the literature. A sample case considered in Liu and Shen [23] can be expressed as

$$\partial_s \tilde{u} = \alpha \Delta (-\Delta \tilde{u} + \beta f(\tilde{u})), \quad (s, y) \in (0, \infty) \times \Omega_L,$$

where $\alpha > 0$, $\beta > 0$, $\tilde{u} = \tilde{u}(s, y)$, and $\Omega_L = [-\frac{L}{2}, \frac{L}{2}]^d$. Now make a change of variable

$$\tilde{u}(s, y) = u\left(\frac{s}{\lambda}, \frac{y}{L}\right) = u(t, x), \quad \lambda = \frac{L^2}{\alpha \beta},$$

then we can recast (1.2) into our standard form as

$$\partial_t u = \Delta (-\nu \Delta u + f(u)), \quad (t, x) \in (0, \infty) \times \mathbb{T}^d,$$

where $\nu = \frac{1}{\beta L^2}$. If we denote the typical time step for (1.2) as $\Delta s$ and for (1.3) as $\Delta t$, then apparently we have

$$\Delta t = \frac{\Delta s}{\lambda} = \Delta s \cdot \alpha \beta \cdot L^{-2}, \quad \nu = \frac{1}{\beta} \cdot L^{-2}.$$

We record these relations here so that one can translate many existing numerical analysis results in the literature about the system (1.2) in terms of the system (1.3) which is quite convenient for comparison purposes.

On the numerical side, there is by now an enormous body of literature on the simulation and analysis of the CH equation and related phase field models (cf. [1, 4, 6, 7, 16, 20, 30, 37, 36] and the references therein). A fundamental challenge is to design fast, efficient and accurate numerical schemes which are robust and energy stable especially in the computationally stiff small $\nu$ regime. Roughly speaking almost all existing numerical algorithms can be classified into five categories with some possible overlaps. To elucidate the discussion below, it is useful to recast (1.1) as a abstract and more general model:

$$\partial_t u = \mathcal{L} u + \mathcal{N}(u),$$

where $\mathcal{L}$ denotes the linear operator and $\mathcal{N}(u)$ collects the nonlinear part. In its most generality one should think of $\mathcal{N}(u)$ as a functional. For example we write $G(u, \nabla u, \nabla^2 u, \cdots)$ for some function $G$ as $\mathcal{N}(u)$. To ease the discussion we shall ignore completely the space discretization and focus momentarily on first order in time discretization. Now for $n = 0, 1, 2, \cdots$, denote by $u^n \approx u(n\Delta t)$ as the numerical solution at time $t = n\Delta t$ where $\Delta t$ is the time step size. Then the following are the prototypical schemes often considered in the literature:

$$\frac{u^{n+1} - u^n}{k} = \begin{cases}
\mathcal{L} u^n + \mathcal{N}(u^n), & \text{(Explicit)}, \\
\mathcal{L} u^{n+1} + \mathcal{N}(u^{n+1}), & \text{(Fully implicit)}, \\
\mathcal{L} u^{n+1} + \mathcal{N}(u^n), & \text{(Semi-implicit/Implicit-Explicit)}, \\
\mathcal{L} u^{n+1} + \mathcal{N}_{I}(u^n, u^{n+1}), & \text{(Partially implicit)}, \\
\mathcal{L} u^{n+1} + \tilde{\mathcal{N}}(u^n, u^{n+1}) + S(u^n, u^{n+1}), & \text{(Stabilization)}.
\end{cases}$$

In the above $\mathcal{N}_{I}$ represents a careful splitting/interpolation of the nonlinearity term using $u^n$ and $u^{n+1}$. To ensure consistency it should satisfy $\mathcal{N}_{I}(u, u) = \mathcal{N}(u)$. In practical algorithms such as
convex splitting one often convexify the problem and choose \( N_1(u^n, u^{n+1}) = \mathcal{N}_+(u) - \mathcal{N}_-(u) = \mathcal{N}(u) \) where \( \mathcal{N}_+ \) and \( \mathcal{N}_- \) are convex. The term \( S(u^n, u^{n+1}) \) represents certain carefully chosen additional stabilization terms which vanishes suitably fast as the time step \( k \to 0 \). In the last one we use the general notation \( N_1(u^n, u^{n+1}) \) to include \( \mathcal{N}(u^n) \) or \( N_1(u^n, u^{n+1}) \) as special cases.

As mentioned above, the first type in our classification is pure explicit methods such as forward Euler in time and explicit treatment of the linear dissipation and nonlinearity. For small systems and short time scales, these methods are speedy, efficient and relatively easy to implement. But due to the poor stability and low accuracy one often has to employ very small time step and spatial grid size which puts a serious limitation for large scale and long time simulations. The second is (fully) implicit schemes such as forward Euler in time and fully implicit of both the linear dissipation and the nonlinearity. The Crank-Nicolson (CN) and modified Crank-Nicolson type methods also fall into this category. A representative work for the CH equation in this direction dates back to Du and Nicolaides [1] which analyzed both a semi-discrete fully implicit in space with continuous time and a modified CN (see also the work of Elliott and Stuart [9] pp. 1644 for the idea of using secant approximation) fully discrete scheme for the 1D CH system with Dirichlet boundary conditions. The deficiency of fully implicit methods is the severe restriction on the time step in order to ensure solvability and the expense of Newton’s method for which efficient preconditioning is often needed in practice. Besides, for the CN type schemes the nonlinearity often has to be modified suitably in order to ensure energy decay.

The third category is the semi-implicit methods which treats the principal linear dissipation term implicitly and the nonlinear term explicitly. In the phase-field context such methods date back to the work of Chen and Shen [4] in which a semi-implicit Fourier spectral method was implemented on a Allen-Cahn system and a CH system. These methods are quite efficient and accurate and observed to have good stability properties in practical numerical simulations. However due to the lack of maximum principle and stiffness caused by small viscosity coefficient a rigorous stability and error analysis was a longstanding open problem. To get around this issue many stabilized methods have been developed over the past decades which we will discuss in more detail in the fifth category below.

The fourth group in our classification contains partially implicit methods. These are one of the most explored directions during the past decades. The most popular ones are the convex-splitting schemes (CSS) which have been developed in [10, 11, 17, 18, 19, 14, 6] for the CH model, higher order models and related nonlocal versions. The advantages of a typical CSS are two: 1) Unconditional energy stability with no stringent restriction on the time step; 2) Guaranteed convergence of the Newton iteration and relatively easy solvability of the associated nonlinear system. This is in stark contrast with a standard fully implicit scheme where very small time steps need to be taken in order to ensure energy stability.

However recently Xu et al in [36] discovered a surprising reformulation of many CSS and stabilized schemes as a version of the fully implicit scheme with a proper time re-parametrization/rescaling. As such it was argued that these methods implicitly trade numerical accuracy for stability. The fifth category in our classification consists of stabilized or mollified methods. The basic idea of stabilization is to introduce an additional \( O(\Delta t^p) \) (for a \( p^{th} \)-order method) term to the numerical scheme to alleviate the time step constraint. These methods were first developed in [37] for a Cahn-Hilliard-Cook equation, [20] for CH and [34] for epitaxial growth models. The work of [20] and [34] relies on some conditional \( L^\infty \)-bound of the numerical solution. In [30] Shen and Yang considered a modified/mollified CH system with suitable Lipschitz truncation and proved various stability results under such assumptions. Removing these conditional assumptions and proving the unconditional energy stability for such stabilized methods were known as the unconditional stability conjecture. Recently in a series of papers [24, 25, 26, 27] several new methods were de-
veloped to settle the unconditional stability conjecture for the 2D and 3D CH systems, including both first order and second order in time methods. Developing upon the second-order scheme in [26], Song and Shu [31] recently constructed a new unconditionally stable second order stabilized semi-implicit local discontinuous Galerkin method for the CH equation. In another direction Shen et al (see [21] and the references therein) fashioned another novel form of stabilization which is based on the introduction of an auxiliary variable. A nice feature of this novel workaround is that it can render unconditional energy stability more easily. However a new challenging issue is how to navigate properly the dynamics of the fictitious variable in order to minimize its deviation from the true dynamics. On the practical side in order for the fictitious variable to stay close to a constant value, one has to monitor very carefully the fluctuations of the auxiliary variable and even adaptively adjust time steps in practical simulations. For a more detailed account of this and other more recent algorithms and developments, we refer to [21, 35, 36] and the references therein.

The comprehensive stability and error analysis in [25, 26, 27] shows that the incorporation of additional stabilization terms in the numerical schemes does increase the stability of the algorithm, however it also introduces undesirable approximation errors which may deteriorate accuracy in the long run. This phenomenon is also inherently present in the auxiliary variable approach [21] and accords well with the point of view advocated by Xu et al in [36] which shows that there exists a subtle and fundamental balance between stability and accuracy. All these naturally lead us to wonder whether the sole pursuit of unconditional energy stability whilst losing accuracy is worthwhile, and perhaps one should look for some sort of conditional stability with affordable time step constraints, and more importantly without sacrificing accuracy too much. In this perspective a fundamental unsettled issue since the work of Chen and Shen [4] is the identification of optimal time step constraints and a rigorous stability analysis of the original semi-implicit scheme without any stabilization, mollification or auxiliary variables. Indeed the very purpose of this work is to settle this important problem in the affirmative. We now state the main results.

Consider the following semi-implicit Fourier-spectral discretization of (1.1) on $T^d = [-\frac{1}{2}, \frac{1}{2}]^d$ ($d \leq 3$):

$$\begin{cases}
\frac{u^{n+1} - u^n}{\tau} = -\nu \Delta^2 u^{n+1} + \Delta \Pi_N(f(u^n)), & n \geq 0, \\
u^0 = \Pi_N u_0.
\end{cases} \quad (1.4)$$

where $\tau > 0$ is the time step. For each integer $N \geq 2$, define

$$X_N = \text{Span}\{\cos(2\pi k \cdot x), \sin(2\pi k \cdot x) : k = (k_1, \ldots, k_d) \in \mathbb{Z}^d, |k|_{\infty} = \max\{|k_1|, \ldots, |k_d|\} \leq N\}.$$ 

Note that the space $X_N$ includes the constant function. By a minor adjustment of the analysis one can also consider the following space

$$\tilde{X}_N = \text{Span}\{\cos(2\pi k \cdot x), \sin(2\pi k \cdot x) : k = (k_1, \ldots, k_d) \in \mathbb{Z}^d, -\frac{N}{2} \leq k_j \leq \frac{N}{2} - 1 \text{ for all } 1 \leq j \leq d\}.$$ 

which is more often used in practical computations especially when $N$ is a dyadic number so that FFT can be implemented. We define the $L^2$ projection operator $\Pi_N : L^2(\Omega) \to X_N$ by

$$(\Pi_N u - u, \phi) = 0, \quad \forall \phi \in X_N, \quad (1.5)$$

where $(\cdot, \cdot)$ denotes the usual $L^2$ inner product (for real-valued functions) on $\Omega$. In yet other words, the operator $\Pi_N$ is simply the truncation of Fourier modes to the frequency sector \{|$k|_{\infty} \leq N$\}. 

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Since \( u^0 = \Pi_N u_0 \in X_N \), by a simple induction one can check that \( u^n \in X_N \) for all \( n \geq 0 \). It is also possible to reformulate (1.4) in terms of the usual weak formulation, for example:

\[
\frac{u^{n+1} - u^n}{\tau}, v + (\nabla (f(u^n)), \nabla v) + \nu (\Delta u^{n+1}, \Delta v) = 0, \quad \forall v \in X_N.
\]

However in our analysis it is slightly more convenient to work with (1.4). Note that \( u^n \) has mean zero for all \( n \geq 0 \) since we assume \( u_0 \) has mean zero.

The following proposition albeit conditional is instrumental to understand the relationship between the time step and the \( L^\infty \)-norm of the numerical solution.

**Proposition 1.1** (Conditional stability for semi-implicit discretization, practical version). Let \( d \leq 3, \nu > 0, \tau > 0 \) and \( N \geq 2 \). Assume \( u_0 \in H^1(\mathbb{T}^d) \) and has mean zero. Suppose up to \( n = N_1 \) the time step \( \tau > 0 \) satisfies

\[
\sqrt{\frac{2\nu}{\tau}} \geq \frac{3}{2} \max_{0 \leq n \leq N_1} \| u^n \|^2_{L^\infty(\mathbb{T}^d)} - \frac{1}{2}.
\]

Then the semi-implicit scheme (1.4) is conditionally energy stable up to \( n = N_1 \), i.e.

\[
\mathcal{E}(u^{n+1}) \leq \mathcal{E}(u^n), \quad \forall 0 \leq n \leq N_1.
\]

**Proof of Proposition 1.1.** This follows directly from the discrete energy estimate Lemma 2.8. \( \square \)

In practical numerical simulations, it is observed that typical numerical solutions satisfy \( \| u^n \|_\infty = O(1) \) even for small \( \nu \ll 1 \). As such if we assume the boundedness of the numerical solution then by Proposition 1.1 the time step constraint for \( \tau \) is roughly \( \tau \leq \nu \). In this sense Proposition 1.1 is already useful for guiding practical numerical simulations. On the other hand, even for the PDE exact solution there is no \( \nu \)-independent \( L^\infty \) bound for the nonlinearity without any mollification. Therefore some trade-off must be made in order to obtain a stability result without assuming the boundedness of the numerical solution. Our next theorem is a first result in this direction.

**Theorem 1.2** (Conditional stability for semi-implicit discretization). Let \( d \leq 3, \nu > 0, \tau > 0 \) and \( N \geq 2 \). Assume \( u_0 \in H^1(\mathbb{T}^d) \) and has mean zero. Assume \( \| u^0 \|_{L^\infty(\mathbb{T}^d)} = L_0 < \infty \) (recall \( u^0 = \Pi_N u_0 \) and we may assume \( L_0 \neq 0 \)). Then the semi-implicit scheme (1.4) is conditionally energy stable, i.e.

\[
\mathcal{E}(u^{n+1}) \leq \mathcal{E}(u^n), \quad \forall n \geq 0,
\]

provided the following time step constraint is satisfied:

\[
0 < \tau \leq \tau_{\text{max}} = \min \left\{ \frac{8\nu}{9L_0^4}, \tau_{\text{max}}^{(1)} \right\},
\]

where

\[
\tau_{\text{max}}^{(1)} = \begin{cases} 
C_1 \nu^{\frac{2}{d}}, & d = 1; \\
C_2 \nu^{\frac{3}{d}}, & d = 2; \\
C_3 \nu^{\frac{3}{d}}, & d = 3,
\end{cases}
\]

where \( C_1, C_2, C_3 > 0 \) are constants depending only on the initial energy \( E_0 = \mathcal{E}(u^0) \).
Remark. By Proposition 2.1, we have \( E(u^0) \lesssim 1 + E(u_0) \) uniformly in \( N \) and \( \nu \).

Remark. Note that our \( L^\infty \)-assumption is made on \( u^0 \) instead of \( u_0 \). For \( d = 1 \), we have by Sobolev embedding \( \| u^0 \|_\infty \lesssim \| u_0 \|_{H^1} < \infty \). For dimension \( d = 2 \) and \( d = 3 \), one should note that the mere assumption \( u_0 \in L^\infty \) in general does not guarantee \( u^0 \in L^\infty \) since the spectral projection is a non-smooth cut-off in frequency space.

Remark 1.3. The constants \( C_j, 1 \leq j \leq 3 \) in Theorem 1.2 can be quantified explicitly. See in particular Theorems 3.2, 4.2, 5.3, 6.2 and 6.4 for more precise statements. Heuristically speaking in general the threshold time step \( \tau_{\text{max}} \) can be determined via the relation:

\[
\sqrt{\frac{2\nu}{\tau_{\text{max}}}} = \frac{3}{2} L^2 \Rightarrow \tau_{\text{max}} = \frac{8}{9} \nu \cdot \nu^{-\frac{d}{2}},
\]

where

\[
L = \max_{n \geq 0} \| u^n \|_{\infty}.
\]

In 2D the \( L^\infty \)-norm of \( u^n \) can almost be bounded by the \( H^1 \) norm of \( u^n \) which in turn is bounded by \( \nu^{-\frac{1}{2}} \sqrt{2E(u^n)} \) multiplied by some logarithm factors depending on \( \nu \). However in our analysis we shall remove this logarithm and obtain the optimal scaling. For general \( d \), we have the heuristic bound (below we neglect the dependence of the constants on energy and focus only on the \( \nu \)-dependence)

\[
L \lesssim \| P_{\lesssim (\nu \tau_{\text{max}})^{-\frac{1}{4}}} \|_{L^1 \to L^\infty} \lesssim (\nu \tau_{\text{max}})^{-\frac{d}{2}},
\]

where \( P_{\lesssim (\nu \tau_{\text{max}})^{-\frac{1}{4}}} \) is a frequency localization operator. From this one can roughly determine \( \tau_{\text{max}} \) as

\[
\tau_{\text{max}} \sim \nu^{\frac{4 + d}{4 - d}}.
\]

Our proof of Theorem 1.2 proceeds by a simple yet powerful Trade-Energy-For-\( L^\infty \) (TEFL) scheme which is a refinement of our earlier work [25, 26, 27]. In several cases we even manage to calculate explicit constants and identified nearly optimal parametric dependences which seem to be the first done in the literature. These will be instrumental for future refined analysis on these algorithms. It is expected that this new streamlined proof can be adapted to higher order cases and generalized to many other models and settings.

For the first order IMEX scheme 1.4, our TEFL recipe consists of three steps.

Step 1. Discrete energy estimate. We show that

\[
\mathcal{E}(u^{n+1}) - \mathcal{E}(u^n) + \left( \frac{1}{2} + \sqrt{\frac{2\nu}{\tau}} \right) \| u^{n+1} - u^n \|_2^2 \leq \| u^{n+1} - u^n \|_2^2 \cdot \frac{3}{2} \max\{\| u^n \|_{\infty}, \| u^{n+1} \|_{\infty}\}.
\]

Thus to show energy monotonicity it suffices to show

\[
\sqrt{\frac{2\nu}{\tau}} \geq \frac{3}{2} \max\{\| u^n \|_{\infty}, \| u^{n+1} \|_{\infty}\}.
\]

Step 2. Trade energy for \( L^\infty \). This is the key step. We split \( u^{n+1} \) as

\[
u^{n+1} = \mathcal{L}_1 u^n + \mathcal{L}_2 \left( \mathcal{N}(u^n) \right),
\]
where $\mathcal{L}_1$ and $\mathcal{L}_2$ are both linear operators mimicking the resolvent of elliptic type operators and $\mathcal{N}(u^n)$ denotes the nonlinear part. We then prove a direct $L^\infty$ estimate using only the energy conservation and certain smoothing properties of the operators $\mathcal{L}_1$ and $\mathcal{L}_2$. To achieve an “optimal trade” (i.e. optimal dependence on $\nu$) it is of some importance to use scaling-critical norms. In the end one obtains

$$\|u^{n+1}\|_\infty \leq C_{E(u^n)} \cdot h(\nu, \tau),$$

where $C_{E(u^n)} > 0$ depends only on the energy $E(u^n)$, and $h(\nu, \tau)$ typically has the form $h(\nu, \tau) = \nu^{-\alpha_\tau-\beta}$ for some exponents $\alpha > 0$, $\beta > 0$.

Step 3. Identification of the optimal time step constraint. Here we work on the inequality

$$\sqrt{\frac{2\nu}{\tau}} \geq \frac{3}{2} (C_{E(u^n)} \cdot h(\nu, \tau))^2$$

and determine an optimal $\tau_{\text{max}} = \tau_{\text{max}}(\nu, E)$. A suitable induction procedure then closes the needed estimates and yields the result.

The rest of this paper is organized as follows. In Section 2 we set up the notation and collect various preliminary materials. The discrete energy inequality is proved in Lemma (2.8). Some Sobolev inequalities on $\mathbb{T}^2$ with explicit constants are also presented here. Section 3 and 4 are devoted to the case $\nu = 1$ with slightly different two approaches. We showcase the proofs with explicit constants. In Section 5 we present the our streamlined TEFL proof for the general case $\nu > 0$ in dimension two. In Section 6 we explain the modifications needed for dimensions 1 and 3 respectively. In the last section we give concluding remarks.

## 2 Notation and preliminaries

For any two positive quantities $X$ and $Y$, we shall write $X \lesssim Y$ or $Y \gtrsim X$ if $X \leq CY$ for some constant $C > 0$ whose precise value is unimportant. We shall write $X \sim Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold. We write $X \lesssim_\alpha Y$ if the constant $C$ depends on some parameter $\alpha$. We shall write $X = O(Y)$ if $|X| \lesssim Y$ and $X = O_\alpha(Y)$ if $|X| \lesssim_\alpha Y$.

We shall denote $X \ll Y$ if $X \leq cY$ for some sufficiently small constant $c$. The smallness of the constant $c$ is usually clear from the context. The notation $X \gg Y$ is similarly defined. Note that our use of $\ll$ and $\gg$ here is different from the usual Vinogradov notation in number theory or asymptotic analysis.

For any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we denote

$$|x| = |x|_2 = \sqrt{x_1^2 + \cdots + x_d^2}, \quad |x|_\infty = \max_{1 \leq j \leq d} |x_j|.$$

Also occasionally we use the Japanese bracket notation:

$$\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}.$$

For $1 \leq p \leq \infty$ and any function $f : x \in \mathbb{T}^d \to \mathbb{R}$, we denote the Lebesgue $L^p$-norm of $f$ as

$$\|f\|_{L^p(\mathbb{T}^d)} = \|f\|_{L^p(\mathbb{R}^d)} = \|f\|_p.$$

If $(a_j)_{j \in I}$ is a sequence of complex numbers and $I$ is the index set, we denote the discrete $l^p$-norm as

$$\|(a_j)\|_{l^p(I)} = \|(a_j)\|_{l^p(I)} = \begin{cases} \left( \sum_{j \in I} |a_j|^p \right)^{\frac{1}{p}}, & 0 < p < \infty, \\ \sup_{j \in I} |a_j|, & p = \infty. \end{cases}$$
For example

\[ \| \hat{f}(k) \|_{l^2(Z^d)}^2 = \sum_{k \in Z^d} |\hat{f}(k)|^2. \]

If \( f = (f_1, \cdots, f_m) \) is a vector-valued function, we denote \(|f| = \sqrt{\sum_{j=1}^m |f_j|^2} \), and

\[ \|f\|_p = \| |f| \|_p = \| \left( \sum_{j=1}^m |f_j|^2 \right)^{\frac{1}{2}} \|_p. \]

We use similar convention for the corresponding discrete \( l^p \) norms for the vector-valued case.

We denote

\[ \text{sgn}(x) = \begin{cases} 1, & x > 0; \\ -1, & x < 0. \end{cases} \]

We use the following convention for the Fourier transform pair:

\[ \hat{f}(k) = \int_{T^d} f(x) e^{-2\pi i k \cdot x} dx, \quad f(x) = \sum_{k \in Z^d} \hat{f}(k) e^{2\pi i k \cdot x}. \]

We denote for \( 0 \leq s \in \mathbb{R} \),

\[ \| f \|_{H^s} = \| f \|_{H^s(T^d)} = \| |\nabla|^s f \|_{L^2(T^d)} = \| (2\pi |k|)^s \hat{f}(k) \|_{l^2(Z^d)}, \]

\[ \| f \|_{\tilde{H}^s} = \left( \| f \|_2^2 + \| f \|_{H^s}^2 \right)^{\frac{1}{2}} = \| (2\pi |k|)^s \hat{f}(k) \|_{l^2(Z^d)}. \]

To simplify the notation, in the later part of this paper we shall often denote

\[ E_n = \mathcal{E}(u^n) = \int_{T^d} \left( \frac{1}{2} \nu |\nabla u^n|^2 + \frac{1}{4} ((u^n)^4 - 2(u^n)^2 + 1) \right) dx, \]

where \( u^n \) is the discrete numerical solution computed according to the scheme (1.4). Note that \( E_0 = \mathcal{E}(u^0) \neq \mathcal{E}(u_0) \) in general since \( u^0 = \Pi_N u_0 \). The next proposition clarifies this point.

**Proposition 2.1** (Relation between \( \mathcal{E}(\Pi_N f) \) and \( \mathcal{E}(f) \)). Let \( d \leq 3 \). For any \( f \in H^1(T^d) \), we have

1) \( \sup_{N \geq 1} \mathcal{E}(\Pi_N f) \leq \beta_1 \mathcal{E}(f) + \beta_2; \)

2) \( \lim_{N \to \infty} \mathcal{E}(\Pi_N f) = \mathcal{E}(f). \)

Here \( \beta_1 > 0, \beta_2 > 0 \) are constants depending only on \( d \).

**Remark.** One may wonder whether it is possible to get rid of \( \beta_2 \) and prove a perfect inequality of the form

\[ \sup_{N \geq 2} \int ((\Pi_N f)^2 - 1)^2 dx \lesssim \int (f^2 - 1)^2 dx. \]  

(2.6)

This is in general not valid. For simplicity consider 1D and \( T = [-\frac{1}{2}, \frac{1}{2}] \). Note that for \( f = \text{sgn}(x) \), \( \Pi_2 f \) or \( \Pi_{N_0} f \) for any finite \( N_0 \) clearly does not vanish. One can then take a suitable mollification \( f^\epsilon \) of \( f = \text{sgn}(x) \) to disprove the inequality in this case. By taking \( \nu > 0 \) sufficiently small, one can then show \( \mathcal{E}(\Pi_2 f^\epsilon) \gg \mathcal{E}(f^\epsilon) \).
Proof. We first note that $\Pi_N$ can be expressed as the product of one-dimensional Hilbert-type transforms and
\[
\sup_{N \geq 2} \|\Pi_N f\|_{L^4(T^d)} \leq c_1 \|f\|_{L^4(T^d)},
\]
where $c_1 > 0$ is some constant depending only on $d$. Clearly then
\[
\int_{T^d} ((\Pi_N f)^4 - 2(\Pi_N f)^2 + 1) dx \leq \int_{T^d} (c_1^4 |f|^4 + 1) dx \leq \int_{T^d} (2c_1^4(f^2 - 1)^2 + 2c_1^4 + 1) dx.
\]
Since $\|\nabla \Pi_N f\|_2 \leq \|\nabla f\|_2$, it follows easily that 1) holds.

2) Since $\lim_{N \to \infty} \|\nabla \Pi_N f\|_2 = \|\nabla f\|_2$, we only need to check the double well energy part. Now denoting $\Pi_N^\perp = \text{Id} - \Pi_N$, we have
\[
|\int (|\Pi_N f|^2 - 1)^2 dx - \int (f^2 - 1)^2 dx| \lesssim \|\Pi_N f - f\|_4 \cdot (\|\Pi_N f\|_4 + \|f\|_4) \cdot (\|\Pi_N f\|_3^2 + \|f\|_3^2 + 1)
\]
\[
\lesssim \|\Pi_N f\|_{H^1} \cdot (\|f\|_4 + \|f\|_3^2) \to 0, \quad \text{as } N \to \infty.
\]
Thus 2) holds.

\begin{lemma}
Let $0 \leq \epsilon < 1$. For any $\omega \in \mathbb{R}^2$ with $|\omega| = 1$, we have
\[
F(\omega) = \int_{[-\frac{1}{2}, \frac{1}{2}]^2} (1 + \epsilon^2 |k|^2 + 2\epsilon \omega \cdot k)^{-1} dk \geq 1 + \frac{1}{6} \epsilon^2 - \frac{7}{30} \epsilon^4.
\]
In particular if $\epsilon^2 \leq \frac{5}{7}$, then $F(\omega) \geq 1$.
\end{lemma}

Proof. Denote the integrand as $g(k)$. By symmetry we have
\[
F(\omega) = \int_{[-\frac{1}{2}, \frac{1}{2}]^2} \frac{1}{2} (g(k) + g(-k)) dk.
\]
Denote $a = \epsilon^2$. We shall slightly abuse the notation and write $k^2 = |k|^2$. Clearly (note that $|k|^2 \leq \frac{1}{2}$)
\[
\frac{1}{2} (g(k) + g(-k)) = (1 + ak^2)^{-1} \cdot \left(1 - \frac{4a(\omega \cdot k)^2}{(1 + ak^2)^2}\right)^{-1}
\]
\[
\geq (1 + ak^2)^{-1} \cdot \left(1 + \frac{4a(\omega \cdot k)^2}{(1 + ak^2)^2}\right)
\]
\[
\geq 1 - ak^2 + (1 - ak^2)^2 \cdot 4a(\omega \cdot k)^2.
\]
By using symmetry (under the swapping of variables $k_1 \leftrightarrow k_2$) and the fact that $|\omega| = 1$, we have for any $f$,
\[
\int_{[-\frac{1}{2}, \frac{1}{2}]^2} f(k^2)(\omega \cdot k)^2 dk_1 dk_2 = \frac{1}{2} \int_{[-\frac{1}{2}, \frac{1}{2}]^2} f(k^2)k^2 dk_1 dk_2.
\]
Here we also used the fact
\[
\int_{[-\frac{1}{2}, \frac{1}{2}]^2} f(k^2)k_1k_2dk_1dk_2 = 0.
\]
Thus we only need to work with the integrand
\[
1 - ak^2 + (1 - ak^2)^3 \cdot 2ak^2.
\]
An explicit calculation then yields the result
\[
1 + \frac{a}{6} - \frac{7a^2}{30} + \frac{9a^3}{140} - \frac{83a^4}{12600} \geq 1 + \frac{a}{6} - \frac{7a^2}{30}.
\]

Remark. There is some subtle dependence of the parameters when we consider the general inequality
\[
\int_{[-\frac{1}{2}, \frac{1}{2}]^2} (1 + \epsilon^2 |k|^2 + 2\epsilon \omega \cdot k)^{-s} dk > 1 \text{ or } \leq 1
\]
for \(s > 0\) and \(0 < \epsilon \ll 1\). Note that for \(0 < \epsilon \ll 1\), we have (below \(a = \epsilon^2\), \(r = a|k|^2 = \epsilon^2|k|^2\))
\[
X = 4a(\omega \cdot k)^2(1 + r)^{-2} = 4a(\omega \cdot k)^2(1 - 2r + 3r^2) + O(a^4);
\]
\[
\ln(1 + r) + \frac{1}{2} \ln(1 - X) = r - \frac{1}{2}r^2 + \frac{1}{2}(-X - \frac{1}{2}X^2) + O(a^3).
\]
By an explicit computation, we have
\[
\int_{[-\frac{1}{2}, \frac{1}{2}]^2} (r - \frac{1}{2}r^2 - \frac{1}{2}X)dk_1dk_2 = \frac{7}{120}a^2 + O(a^3);
\]
\[
\int_{[-\frac{1}{2}, \frac{1}{2}]^2} (-\frac{1}{4}X^2)dk_1dk_2 = -4a^2 \int_{[-\frac{1}{2}, \frac{1}{2}]^2} (\omega \cdot k)^4dk_1dk_2 + O(a^3)
\]
\[
= -\frac{a^2}{60}(3\omega_1^4 + 10\omega_1^2\omega_2^2 + 3\omega_2^4) + O(a^3)
\]
\[
= -\frac{a^2}{60}(3 + 4\omega_1^2\omega_2^2) + O(a^3),
\]
where in the last equality we used the fact that \(\omega_1^2 + \omega_2^2 = 1\).
Then clearly
\[
H(\epsilon, \omega) = \int_{[-\frac{1}{2}, \frac{1}{2}]^2} \ln(1 + \epsilon^2 |k|^2 + 2\epsilon \omega \cdot k)dk
\]
\[
= \int_{[-\frac{1}{2}, \frac{1}{2}]^2} (\ln(1 + \epsilon^2 |k|^2) + \frac{1}{2} \ln(1 - \frac{4\epsilon^2(\omega \cdot k)^2}{(1 + \epsilon^2 |k|^2)^2}))dk
\]
\[
= \int_{[-\frac{1}{2}, \frac{1}{2}]^2} (r - \frac{1}{2}r^2 + \frac{1}{2}(-X - \frac{1}{2}X^2))dk + O(a^3)
\]
\[
= \frac{1}{60}a^2 \left(0.5 - 4\omega_1^2\omega_2^2\right) + O(a^3)
\]
Note that in the above calculation in the main order the integral is dependent on \( \omega \). In particular its sign is dependent on the choice of \( \omega \). If \( \omega = \omega^{(1)} = (1, 0) \), then apparently
\[
H(\epsilon, \omega^{(1)}) = \frac{1}{120} a^2 + O(a^3) \geq \frac{1}{200} \cdot \epsilon^4 > 0.
\]
If \( \omega = \omega^{(2)} = \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \), then
\[
H(\epsilon, \omega^{(2)}) = -\frac{1}{120} a^2 + O(a^3) \leq -\frac{1}{200} \cdot \epsilon^4 < 0.
\]

Now denote for \( s > 0 \), \( 0 < \epsilon \leq 1/4 \),
\[
I(s, \omega) = \int_{[-\frac{1}{2}, \frac{1}{2})^2} e^{-s \ln(1+\epsilon^2|k|^2+2\epsilon \omega \cdot k)} dk = \int_{[-\frac{1}{2}, \frac{1}{2})^2} (1 + \epsilon^2|k|^2 + 2\epsilon \omega \cdot k)^{-s} dk.
\]

Obviously
\[
(\partial_s I)(0, \omega) = -H(\epsilon, \omega).
\]

It is not difficult to check that for some absolute constant \( C_1 > 0 \) we have \( \sup_\omega |\partial_{ss} I(s, \omega)| \leq C_1 \) for all \( 0 < s \leq 1/4 \), \( 0 < \epsilon \leq 1/4 \). It follows that for some \( s_0(\epsilon) \) depending on \( \epsilon \), we have
\[
|\partial_s I(s, \omega^{(j)}) - \partial_s I(0, \omega^{(j)})| \leq \frac{1}{10} |\partial_s I(0, \omega^{(j)})| = \frac{1}{10} |H(\epsilon, \omega^{(j)})|, \quad j = 1, 2, \text{ for all } 0 \leq s \leq s_0(\epsilon).
\]

Consequently
\[
I(s, \omega^{(1)}) \leq 1 - \frac{1}{400} s \epsilon^4;
I(s, \omega^{(2)}) \geq 1 + \frac{1}{400} s \epsilon^4,
\]
for \( 0 < \epsilon \ll 1 \) and \( 0 \leq s \leq s_0(\epsilon) \). In other words one should take \( s \) moderately large in order to exceed 1. Alternatively for small \( \epsilon \) one can directly expand the integrand \((1 + \epsilon^2|k|^2 + 2\epsilon \omega \cdot k)^{-s}\) in binomial series and obtain
\[
I(s, \omega^{(1)}) = 1 + \frac{1}{360} \epsilon^2 s (60s + \epsilon^2 (-3 - 2s + 4s^2 + 3s^3)) + O(\epsilon^6).
\]

Clearly then \( I(s, \omega^{(1)}) < 1 \) if \( s \ll \epsilon^2 \). Similarly
\[
I(s, \omega^{(2)}) = 1 + \frac{1}{360} \epsilon^2 s (60s + \epsilon^2 (3 + 9s + 10s^2 + 4s^3)) + O(\epsilon^6).
\]

Clearly \( I(s, \omega^{(2)}) > 1 \) for \( 0 < \epsilon \ll 1 \).

Remark. Note that \( \ln | \cdot | \) is harmonic and \( | \cdot |^{-s} \) (\( s > 0 \)) is subharmonic on \( \mathbb{R}^2 \setminus \{0\} \). The preceding computations show that a subharmonic approach is not suitable. This is hardly surprising since we are working with a square.

Lemma 2.3. Let
\[
I(n) = \int_{[-\frac{1}{2}, \frac{1}{2})^2} \frac{1}{|k+n|^2} dk - \frac{1}{|n|^2}.
\]
By symmetry, we have for any \( n = (n_1, n_2) \), \( I(n) = I(\pm n_1, \pm n_2) \). Furthermore
\[
I(1, 0) \geq 0.1731, \quad I(1, 1) \geq 0.0525,
I(2, 0) \geq 0.0105, \quad I(2, 1) \geq 0.007, \quad I(2, 2) \geq 0.0027
I(3, 0) \geq 0.0020, \quad I(3, 1) \geq 0.0016, \quad I(3, 2) \geq 0.0010, \quad I(3, 3) \geq 0.0005.
\]
Proof. Direct numerical verification.

**Lemma 2.4.** For any $\frac{1}{\sqrt{2}} < r_1 < r_2$, we have

$$
\sum_{\substack{|n| \geq r_2 \cap n \in \mathbb{Z}^2}} \frac{1}{|n|^2} \leq 2\pi \left( \ln(r_2 + \frac{1}{\sqrt{2}}) - \ln(r_1 - \frac{1}{\sqrt{2}}) \right).
$$

Also for $r_2 > \frac{1}{\sqrt{2}}$,

$$
\sum_{\substack{|n| \geq r_2 \cap n \in \mathbb{Z}^2}} \frac{1}{|n|^4} \leq \pi(r_2 - \frac{1}{\sqrt{2}})^{-2},
$$

and more generally for any $s > 2$,

$$
\sum_{\substack{|n| \geq r_2 \cap n \in \mathbb{Z}^2}} \frac{1}{|n|^s} \leq 2\pi \cdot \frac{1}{s-2} (r_2 - \frac{1}{\sqrt{2}})^{-(s-2)}.
$$

**Remark.** Consequently for any $r \geq 2$, we have

$$
\sum_{\substack{|n| \leq r \cap n \in \mathbb{Z}^2}} |k|^{-2} = 6 + \sum_{2 \leq |k| \leq r} |k|^{-2} \leq 6 + 2\pi (\ln(r + \frac{1}{\sqrt{2}}) - \ln(2 - \frac{1}{\sqrt{2}}))
$$

$$
\leq 4.4 + 2\pi \ln(r + \frac{1}{\sqrt{2}}). \quad (2.7)
$$

**Proof.** By Lemma 2.2 and 2.3 we have

$$
\sum_{r_1 \leq |n| \leq r_2} |n|^{-2} \leq \sum_{r_1 \leq |n| \leq r_2} \int_{[-\frac{1}{2}, \frac{1}{2}]^2} |n+k|^{-2} \, dk.
$$

Now for each $n \in \mathbb{Z}^2$, define $I_n = \{z \in \mathbb{R}^2 : |z-n| < \frac{1}{2} \}$. Clearly $I_n$ and $I_{n'}$ are disjoint if $n \neq n'$. Obviously

$$
\bigcup_{r_1 \leq |n| \leq r_2} I_n \subset \{z \in \mathbb{R}^2 : r_1 - \frac{1}{\sqrt{2}} \leq |z| \leq r_2 + \frac{1}{\sqrt{2}} \}.
$$

Thus

$$
\sum_{r_1 \leq |n| \leq r_2} |n|^{-2} \leq \int_{r_1 - \frac{1}{\sqrt{2}}}^{r_2 + \frac{1}{\sqrt{2}}} \int_{[-\frac{1}{2}, \frac{1}{2}]^2} |z|^{-2} \, dz = 2\pi \left( \ln(r_2 + \frac{1}{\sqrt{2}}) - \ln(r_1 - \frac{1}{\sqrt{2}}) \right).
$$

For the second inequality, we note that

$$
|n|^{-4} = (|n|^{-2})^2 \leq (\int_{[-\frac{1}{2}, \frac{1}{2}]^2} |n+k|^{-2} \, dk)^2 \leq \int_{[-\frac{1}{2}, \frac{1}{2}]^2} |n+k|^{-4} \, dk.
$$

Thus

$$
\sum_{|n| \geq r_2} |n|^{-4} \leq \int_{|z| \geq r_2 - \frac{1}{\sqrt{2}}} |z|^{-4} \, dz = \pi (r_2 - \frac{1}{\sqrt{2}})^{-2}.
$$
Lemma 2.5. Let \( f \in H^2(\mathbb{T}^2) \) with mean zero. Then for any \( r \geq 2 \), we have
\[
\|f\|_{L^\infty(\mathbb{T}^2)} \leq \|f\|_{\dot{H}^1} \cdot \frac{1}{2\pi} \cdot (4.386 + 2\pi (\ln(r + \frac{1}{\sqrt{2}})))^{\frac{1}{2}} + \|f\|_{\dot{H}^2} \cdot \frac{1}{4\pi^2} (r - \frac{1}{\sqrt{2}})^{-1}.
\]
Proof. Splitting \( f \) into low and high frequencies, we have
\[
\|f\|_\infty \leq \|f\|_{\dot{H}^1} \cdot \left( \sum_{0 < |k| \leq r} \frac{1}{(2\pi |k|)^2} \right)^{\frac{1}{2}} + \|f\|_{\dot{H}^2} \cdot \left( \sum_{|k| > r} \frac{1}{(2\pi |k|)^4} \right)^{\frac{1}{2}}.
\]
By Lemma 2.4, we have
\[
\sum_{0 < |k| \leq r} |k|^{-2} = 6 + \sum_{2 \leq |k| \leq r} |k|^{-2} \leq 6 + 2\pi (\ln(r + \frac{1}{\sqrt{2}}) - \ln(2 - \frac{1}{\sqrt{2}})) \leq 4.386 + 2\pi \ln(r + \frac{1}{\sqrt{2}}),
\]
\[
\sum_{|k| > r} |k|^{-4} \leq \pi (r - \frac{1}{\sqrt{2}})^{-2}.
\]
\( \square \)

Lemma 2.6. Denote \( \mathbb{T} \) the one-periodic torus on \( \mathbb{R} \) which we identify as \( \mathbb{T} = [-\frac{1}{2}, \frac{1}{2}] \). For any smooth \( f : \mathbb{T} \rightarrow \mathbb{R} \), we have
\[
\|f - \bar{f}\|_{L^\infty(\mathbb{T})} \leq \frac{1}{2} \|f'\|_{L^1(\mathbb{T})},
\]
where \( \bar{f} \) denotes the average of \( f \) on \( \mathbb{T} \). On \( \mathbb{T}^2 = [-\frac{1}{2}, \frac{1}{2}]^2 \), we have
\[
\|f - \bar{f}\|_{L^2(\mathbb{T}^2)} \leq \frac{1}{2} \|(|\partial_1 f| + |\partial_2 f|)\|_{L^1(\mathbb{T}^2)} \\
\leq \frac{1}{\sqrt{2}} \|\nabla f\|_{L^1(\mathbb{T}^2)} = \frac{1}{\sqrt{2}} \sqrt{\|\partial_1 f\|^2 + \|\partial_2 f\|^2}_{L^1(\mathbb{T}^2)}.
\]
Note that we use the convention \( |\nabla f| = \sqrt{\|\partial_1 f\|^2 + \|\partial_2 f\|^2} \).
Remark. The first inequality can be achieved by a smooth approximation of \( f = \frac{1}{2} \text{sgn}(x) \) on \( [-\frac{1}{2}, \frac{1}{2}] \). The constant in the second inequality is not sharp.
Proof. The first inequality can be proved in two ways. Without loss of generality one may assume \( \bar{f} = 0 \). One can then choose some \( x_0 \in \mathbb{T} \) such that \( f(x_0) = 0 \). Writing \( f(x) = \int_{x_0}^x f'(s)ds = -\int_{x_0}^{x_0+1} f'(s)ds \) then yields the result. Alternatively one can resort to Fourier analysis and show that
\[
K(x) = \mathcal{F}^{-1}(\frac{1}{2\pi ik}1_{k \neq 0}) = \frac{1}{2} \text{sgn}(x) - x = \begin{cases} \frac{1}{2} - x, & 0 < x \leq \frac{1}{2}; \\
-\frac{1}{2} - x, & -\frac{1}{2} \leq x < 0. \end{cases}
\]
Obviously \( \|K\|_{\infty} \leq \frac{1}{2} \) and the desired inequality follows.
For the second inequality we may also assume \( \bar{f} = 0 \). Observe
\[
|f(x_1, x_2) - \int_{\text{:=B}(x_2)} f(y_1, x_2)dy_1| \leq \frac{1}{2} \int |\partial_1 f(y_1, x_2)|dy_1,
\]
\[
|f(x_1, x_2) - \int_{\text{:=A}(x_1)} f(x_1, y_2)dy_2| \leq \frac{1}{2} \int |\partial_2 f(x_1, y_2)|dy_2.
\]
Note that since $\bar{f} = 0$ we have $\int A(x_1)dx_1 = 0 = \int B(x_2)dx_2$. Clearly then

$$\|A(x_1)\|_{L_1^\infty} \leq \frac{1}{2} \int |\partial_1 A|dx_1 \leq \frac{1}{2} \|\partial_1 f\|_{L^1}, \quad \|B(x_2)\|_{L_1^\infty} \leq \frac{1}{2} \|\partial_2 f\|_{L^1}. $$

It follows that

$$f^2 \leq -A(x_1)B(x_2) + (A(x_1) + B(x_2))f + \frac{1}{4} \int |\partial_1 f(y_1, x_2)||dy_1| \cdot \int |\partial_2 f(x_1, y_2)|dy_2$$

$$\leq \frac{1}{2} f^2 + \frac{1}{2} (A(x_1) + B(x_2))^2 - A(x_1)B(x_2) + \frac{1}{4} \int |\partial_1 f(y_1, x_2)|dy_1 \cdot \int |\partial_2 f(x_1, y_2)|dy_2. $$

We then obtain

$$\|f\|^2_{L^2_1} \leq \int A(x_1)^2 dx_1 + \int B(x_2)^2 dx_2 + \frac{1}{2} \|\partial_1 f\|_{L^1} \|\partial_2 f\|_{L^1}$$

$$\leq \frac{1}{4} (\|\partial_1 f\|_{L^1} + \|\partial_2 f\|_{L^1})^2, $$

where in the last inequality we used $\|A(x_1)\|_{\infty} \leq \frac{1}{2} \|\partial_1 f\|_{L^1}, \quad \|B(x_2)\|_{\infty} \leq \frac{1}{2} \|\partial_2 f\|_{L^1}$. \hfill \Box

**Lemma 2.7.**

$$\|f - \bar{f}\|_{L^\infty(\mathbb{T}^2)} \leq \frac{\sqrt{6.05}}{(2\pi)^2} \|f\|_{H^2(\mathbb{T}^2)}. $$

**Proof.** Note that by Lemma 2.4 for any $r > \frac{1}{\sqrt{2}}$, we have

$$\sum_{0 \neq k \in \mathbb{Z}^2} \frac{1}{(2\pi|k|)^4} \leq \frac{1}{(2\pi)^4} \left( \sum_{0 < |k| \leq r} \frac{1}{|k|^4} + \pi (r - \frac{1}{\sqrt{2}})^{-2} \right). $$

Choosing $r = 10$ then yields the result. One should note that

$$\sum_{0 < |k| \leq 10} \frac{1}{|k|^{-4}} \leq \sum_{0 < |k| \leq 10} \frac{1}{|k|^{-4}} \approx 6.00355 < 6.0036;$$

$$\pi (10 - \frac{1}{\sqrt{2}})^{-2} \approx 0.0363788 < 0.037. $$

**Lemma 2.8** (Discrete energy estimate). *For any $n \geq 0$,*

$$E_{n+1} - E_n + \left( \frac{1}{2} + \sqrt{\frac{2\nu}{\tau}} \right) \|u^{n+1} - u^n\|^2_2 \leq \|u^{n+1} - u^n\|^2_2 \cdot \frac{3}{2} \max\{\|u^n\|^2_{\infty}, \|u^{n+1}\|^2_{\infty}\}. \quad (2.8) $$

**Proof.** In this proof we denote by $(\cdot, \cdot)$ the usual $L^2$ inner product. Recall

$$\frac{u^{n+1} - u^n}{\tau} = -\nu \Delta^2 u^{n+1} + \Delta \Pi_N f(u^n). $$

Taking the $L^2$ inner product with $(-\Delta)^{-1}(u^{n+1} - u^n)$ on both sides and applying the identity

$$b \cdot (b - a) = \frac{1}{2}(|b|^2 - |a|^2 + |b - a|^2), \quad \forall a, b \in \mathbb{R}^d,$$
we get
\[
\frac{1}{\tau} \| \nabla^{-1}(u^{n+1} - u^n) \|_2^2 + \frac{\nu}{2} (\| \nabla u^{n+1} \|_2^2 - \| \nabla u^n \|_2^2 + \| \nabla (u^{n+1} - u^n) \|_2^2)
\]
\[
= (\Delta \Pi_N f(u^n), (-\Delta)^{-1}(u^{n+1} - u^n)).
\]
(2.9)

Since \(u^n\) and \(u^{n+1}\) have Fourier modes trapped in the sector \(\{ k : |k|_\infty \leq N \}\), we have
\[
(\Delta \Pi_N f(u^n), (-\Delta)^{-1}(u^{n+1} - u^n)) = -(f(u^n), u^{n+1} - u^n).
\]

By using the auxiliary function \(g(s) = F(u^n + s(u^{n+1} - u^n))\) (recall \(f = F'\)) and the Taylor expansion
\[
g(1) = g(0) + g'(0) + \int_0^1 g''(s)(1-s)ds,
\]
we get
\[
F(u^{n+1}) = F(u^n) + f(u^n)(u^{n+1} - u^n) - \frac{1}{2}(u^{n+1} - u^n)^2
\]
\[
+ (u^{n+1} - u^n)^2 \int_0^1 f'(u^n + s(u^{n+1} - u^n))(1-s)ds,
\]
where \(\tilde{f}(z) = z^3\) and \(\tilde{f}'(z) = 3z^2\) (for \(z \in \mathbb{R}\)). From this it is easy to see that
\[
-(f(u^n), u^{n+1} - u^n) \leq F(u^n) - F(u^{n+1}) - \frac{1}{2}\| u^{n+1} - u^n \|_2^2 + \| u^{n+1} - u^n \|_2^2 \cdot \frac{3}{2}\max\{\| u^n \|_\infty, \| u^{n+1} \|_\infty\}.
\]

Thus
\[
E_{n+1} - E_n + \frac{1}{\tau} \| \nabla^{-1}(u^{n+1} - u^n) \|_2^2 + \frac{\nu}{2} \| \nabla (u^{n+1} - u^n) \|_2^2 + \frac{1}{2}\| u^{n+1} - u^n \|_2^2
\]
\[
\leq \| u^{n+1} - u^n \|_2^2 \cdot \frac{3}{2}\max\{\| u^n \|_\infty, \| u^{n+1} \|_\infty\}.
\]
(2.10)

Finally observe
\[
\frac{1}{\tau} \| \nabla^{-1}(u^{n+1} - u^n) \|_2^2 + \frac{\nu}{2} \| \nabla (u^{n+1} - u^n) \|_2^2
\]
\[
\geq \sqrt{\frac{2\nu}{\tau}} \| \nabla^{-1}(u^{n+1} - u^n) \|_2 \| \nabla (u^{n+1} - u^n) \|_2 \geq \sqrt{\frac{2\nu}{\tau}}\| u^{n+1} - u^n \|_2.
\]

The desired inequality then follows easily.

\[\square\]

**Lemma 2.9.** Let \(d \leq 3\) and \(\beta > 0\). Consider on the torus \(\mathbb{T}^d\),
\[
K(x) = \mathcal{F}^{-1}((1 + \beta(2\pi|k|)^4)^{-1}) = (1 + \beta d^2)^{-1}\delta_0,
\]
where \(\delta_0\) is the periodic Dirac comb. Then for any \(1 \leq p \leq \infty\),
\[
\| K \|_{L^p(\mathbb{T}^d)} \leq c_{d,p}(1 + \beta^{-d(\frac{4}{d} - \frac{1}{d'})}),
\]
where \(c_{d,p} > 0\) depends only on \(d\) and \(p\). Define
\[
\tilde{K} = \mathcal{F}^{-1}((1 + \beta(2\pi|k|)^4)^{-1}1_{k \neq 0}).
\]

Then
\[
\| \tilde{K} \|_{L^p(\mathbb{T}^d)} \leq \tilde{c}_{d,p}\beta^{-d(\frac{4}{d} - \frac{1}{d'})},
\]
where \(\tilde{c}_{d,p} > 0\) depends only on \(d\) and \(p\).
Remark.

\[ \max_{1 \leq p \leq \infty} (c_{d,p} + \tilde{c}_{d,p}) \leq B_d < \infty, \]

where \( B_d \) depends only on \( d \).

Remark. By examining the \( L^2 \) Fourier coefficients of \( K \) one can see that the constant \( 1 \) needs to be present in the \( L^p \) upper bound. This also follows from the fact that \( \mathcal{K} = \int K = 1 \) and \( \| K \|_{L^p(T^d)} \geq \| K \|_{L^1(T^d)} \geq \mathcal{K} = 1 \).

Remark. For \( \beta \geq 1 \), one has the stronger bound on \( \tilde{K} \) as

\[ \| \tilde{K} \|_{L_1^1(T^d)} \leq \| \tilde{K} \|_{L_\infty^\beta(T^d)} \lesssim d \beta. \]

By using the identity

\[ \frac{\beta(2\pi)^4}{1 + \beta(2\pi|k|)^4} = |k|^{-4} - \frac{1}{(2\pi)^4 \beta} \cdot \frac{1}{|k|^4 (|k|^4 + \frac{1}{(2\pi)^4 \beta})}, \]

we also have for \( \beta \geq 1 \),

\[ \| \tilde{K} \|_{L_1^1(T^d)} \gtrsim d \beta. \]

Proof. Define

\[ K_1(x) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} \cdot \frac{1}{1 + (2\pi|\xi|)^4} d\xi. \]

It is easy to check that \( |K_1(x)| \lesssim \langle x \rangle^{-10} \) and \( K_1 \in L_1^1(\mathbb{R}^d) \) for \( d \leq 3 \).

Now note that for \( d \leq 3 \), if \( |x|_{\infty} \leq \frac{1}{2} \), then \( |x| \leq \sqrt{d} \cdot \frac{1}{2} \leq \frac{\sqrt{3}}{2} \). Thus if \( |l| \geq 4 \), then

\[ \frac{1}{2} |l| \leq |x + l| \leq 2 |l|, \quad \forall |x|_{\infty} \leq \frac{1}{2}. \]

It follows that for all \( 1 \leq p \leq \infty \) and \( |l| \geq 4 \),

\[ \| \langle \beta^{-\frac{1}{4}} (x + l) \rangle^{-10} \|_{L_p^\beta(|x|_{\infty} < \frac{1}{2})} \lesssim \langle \beta^{-\frac{1}{4}} \frac{1}{2} |l| \rangle^{-10} \lesssim \| \langle \beta^{-\frac{1}{4}} \frac{1}{4} |x + l| \rangle^{-10} \|_{L_1^1(|x|_{\infty} < \frac{1}{2})}. \]

Clearly then

\[ \| K \|_{L_p^\beta(T^d)} \lesssim \beta^{-\frac{d}{4}} \sum_{l \in \mathbb{Z}^d} \| K_1 (\beta^{-\frac{1}{4}} (x + l)) \|_{L_p^\beta(|x|_{\infty} < \frac{1}{2})} \]

\[ \lesssim \beta^{-\frac{d}{4}} \sum_{|l| \leq 4} \| K_1 (\beta^{-\frac{1}{4}} (x + l)) \|_p + \beta^{-\frac{d}{4}} \sum_{|l| > 4} \| \langle \beta^{-\frac{1}{4}} \frac{1}{4} |x + l| \rangle^{-10} \|_{L_1^1(|x|_{\infty} < \frac{1}{2})} \]

\[ \lesssim \beta^{-d(\frac{1}{4} - \frac{1}{4p})} + 1. \quad (2.11) \]

Now we consider the estimate for \( \tilde{K}(x) = K(x) - 1 \). Obviously by using the previous bound we have \( \| \tilde{K} \|_1 \lesssim \| K \|_1 + 1 \lesssim 1 \). Alternatively one can compute

\[ \| \tilde{K} \|_{L_1^1(T^d)} \leq 1 + \sum_{l \in \mathbb{Z}^d} \beta^{-\frac{d}{4}} \| K_1 (\beta^{-\frac{1}{4}} (x + l)) \|_{L_1^1(T^d)} \leq 1 + \| K_1 \|_{L_1^1(\mathbb{R}^d)} \lesssim 1. \]

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We then bound the $L^2$ norm as

$$\|\tilde{K}\|_{L^2(T^d)} = \frac{1}{1 + \beta(2\pi|k|)^4} \|l^4_k(0 \neq k \in \mathbb{Z}^d) \lesssim \begin{cases} \beta^{-1}, & \text{if } \beta \geq 1; \\ \beta^{-\frac{d}{4}}, & \text{if } 0 < \beta < 1. \end{cases}$$

Since $d \leq 3$, we have the uniform bound for all $\beta > 0$ as

$$\|\tilde{K}\|_{L^2(T^d)} \lesssim \beta^{-\frac{d}{4}}.$$

Similarly

$$\|\tilde{K}\|_{L^\infty(T^d)} \leq \frac{1}{1 + \beta(2\pi|k|)^4} \|l^4_k(0 \neq k \in \mathbb{Z}^d) \lesssim \begin{cases} \beta^{-1}, & \text{if } \beta \geq 1; \\ \beta^{-\frac{d}{4}}, & \text{if } 0 < \beta < 1; \end{cases}$$

$$\lesssim \beta^{-\frac{d}{4}}, \quad \text{for all } \beta > 0 \ (\text{since } d \leq 3 \Rightarrow \frac{d}{4} < 1).$$

By using interpolation we then get the $L^p$ estimate.

**Remark.** For $2 \leq p \leq \infty$, one can also use

$$\|f\|_{L^p(T^d)} \leq \|\hat{f}\|_{L^\frac{p}{p-1}(\mathbb{Z}^d)}$$

which is a variant of Hausdorff-Young. Note the inequality itself also follows from interpolation.

**Remark.** Interestingly it is also possible to bound $\|\tilde{K}\|_p$ directly on the real side without using interpolation or Fourier transform by modifying the argument in (2.11). Note that we only need to treat the case $\beta \gg 1$. For simplicity consider 1D and $p = \infty$. Denote $\epsilon = \beta^{-\frac{1}{4}}$. It suffices to check that

$$\|\left(\sum_{n \in \mathbb{Z}} \epsilon K_1(\epsilon(x + n))\right) - 1\|_{L^\infty(|x| \leq \frac{1}{2})} \lesssim \epsilon^4.$$

To simplify we shall just consider the point $x = 0$. The argument is similar and estimates are uniform for all $|x| < \frac{1}{2}$. Note that $K_1^{(4)}(x) = -K_1(x) + \delta(x)$. One can then apply the Euler-MacLaurin formula to the function $g(x) = \epsilon K_1(\epsilon x)$. The tail term is then given by

$$-\frac{1}{24} \int_{\mathbb{R}} g^{(4)}(x) B_4(x - [x]) dx,$$

where $B_4$ is the Bernoulli polynomial:

$$B_4(x) = x^2(x-1)^2 - \frac{1}{30},$$

and $[x]$ denotes the smallest integer less than or equal to $x$. It is easy to check that $B_4(x - [x])$ is continuous at the origin so that it can be paired with the Dirac delta function. It follows easily that

$$|\int_{\mathbb{R}} g^{(4)}(x) B_4(x - [x]) dx| \lesssim \epsilon^4.$$

Note that one can also use truncation and mollification to make the whole argument rigorous. We omit the details.
3 Proof for 2D and $\nu = 1$: first approach

Consider the semi-implicit scheme:

$$\frac{u^{n+1} - u^n}{\tau} = -\Delta^2 u^{n+1} + \Delta \Pi_N(f(u^n)), \quad n \geq 0. \quad (3.12)$$

where $\tau > 0$ is the time step. Then

$$(1 + \tau \Delta^2) u^{n+1} = u^n + \tau \Delta \Pi_N(f(u^n)).$$

Lemma 3.1. Denote $\mathcal{E}(u^n) = E_n$. Then

$$\|\Delta \frac{\tau \Delta}{1 + \tau \Delta^2} \Pi_N((u^n)^3 - 3u^n)\|_{L^2(\mathbb{T}^2)} \leq 3E_n;$$

$$\|u^{n+1}\|_{H^2(\mathbb{T}^2)} \leq (2 + \frac{1}{\tau}) \sqrt{E_n} + 3E_n;$$

$$\|u^{n+1}\|_{H^1(\mathbb{T}^2)} \leq 2\sqrt{2}\sqrt{E_n} + 3E_n.$$

Proof. For simplicity of notation we denote $v = u^{n+1}$ and $u = u^n$. We shall prove the first two inequalities at one stroke. Rewrite

$$v = \frac{1 + 2\tau \Delta}{1 + \tau \Delta^2} u + \frac{\tau \Delta}{1 + \tau \Delta^2} \Pi_N(u^3 - 3u).$$

The special splitting here is to make the bound for nonlinear part easier to express in terms of the energy.

Then by Lemma 2.6 we have (below $\bar{u}^3$ denotes the average of $u^3$ on $\mathbb{T}^2$)

$$\|v\|_{H^2} \leq (2 + \frac{1}{\tau}) \|u\|_2 + \|u^3 - \bar{u}^3 - 3u\|_2$$

$$\leq (2 + \frac{1}{\tau}) \sqrt{2} \|
abla u\|_1 + \frac{1}{\sqrt{2}} \|3\nabla u(u^2 - 1)\|_1$$

$$\leq (2 + \frac{1}{\tau}) \sqrt{2} \|
abla u\|_2 + 3\|2 \cdot \frac{1}{\sqrt{2}} \nabla u \cdot \frac{1}{2}(u^2 - 1)\|_1$$

$$\leq (2 + \frac{1}{\tau}) \sqrt{E_n} + 3E_n.$$

The third inequality is similarly proved. We omit details.

Theorem 3.2 (Conditional energy stability for 2D $\nu = 1$, first approach). Let $d = 2$, $\nu = 1$ and $N \geq 2$. Assume $u_0 \in H^1(\mathbb{T}^2)$ and has zero mean. Recall $u^0 = \Pi_N u_0$ and assume $\|u^0\|_\infty \leq 1.25$. Take

$$\tau_{\text{max}} = \begin{cases} \frac{1}{2}, & \text{if } F_0 \leq 1; \\ \min\left\{ \frac{1}{2}, \frac{1}{(F_0(1 + \ln F_0))^2} \right\}, & \text{if } F_0 > 1, \end{cases}$$

where $F_0 = (2\sqrt{2E_0} + 3E_0)^2$. Then for any $0 < \tau \leq \tau_{\text{max}}$, the scheme (3.12) is energy stable, i.e.

$$E(u^{n+1}) \leq E(u^n), \quad \forall n \geq 0.$$

Remark. We chose the assumption $\|u^0\|_\infty \leq 1.25$ for simplicity. It can be replaced by a general upper $L_0$ and $\tau_{\text{max}}$ can be adjusted accordingly with some simple changes in numerology.
Proof. We use induction. By Lemma 2.8, in order to have $E_{n+1} \leq E_n$, the main condition to verify is the inequality

$$\frac{1}{2} + \sqrt{\frac{2}{\tau}} \geq \frac{3}{2} \max\{\|u^n\|_\infty^2, \|u^{n+1}\|_\infty^2\}. \quad (3.13)$$

Step 1. Base step $n = 0$. Since by assumption $\|u^0\|_\infty \leq 1.25$ and $0 < \tau \leq \frac{1}{2}$, it is clear that

$$\frac{1}{2} + \sqrt{\frac{2}{\tau}} \geq 2.5 \geq \frac{3}{2} \cdot 1.25^2 \geq \frac{3}{2}\|u^0\|_\infty^2.$$

Thus to have $E_1 \leq E_0$ we only need to verify

$$\frac{1}{2} + \sqrt{\frac{2}{\tau}} \geq \frac{3}{2}\|u^1\|_\infty^2. \quad (3.14)$$

We first note that if

$$(2 + \frac{1}{\tau})\sqrt{E_0} + 3E_0 \leq \frac{8}{3}\pi^\frac{3}{2},$$

then by Lemma 2.7 and 3.1, we have

$$\|u^1\|_\infty^2 \leq \frac{6.05}{(2\pi)^4}((2 + \frac{1}{\tau})\sqrt{E_0} + 3E_0)^2 \leq \frac{6.05}{(2\pi)^4} \cdot \frac{64\pi^3}{9} = \frac{605}{9} \approx 69.45.$$ 

Since we assume $\tau \leq \frac{1}{2}$, the inequality (3.14) then clearly holds in this case. Thus in the following we may assume that we are given the condition

$$(2 + \frac{1}{\tau})\sqrt{E_0} + 3E_0 > \frac{8}{3}\pi^\frac{3}{2}. \quad (3.15)$$

We stress that there is no need to deduce from (3.15) any constraint on $\tau$.

By Lemma 2.5 and 3.1 we have for any $r \geq 2$,

$$\|u^1\|_{L^\infty(\mathbb{T}^2)} \leq \|u^1\|_{H_1} \cdot \frac{1}{2\pi} \cdot (4.386 + 2\pi(\ln(r + \frac{1}{\sqrt{2}})))^{\frac{1}{2}} + \|u^1\|_{H^2} \cdot \frac{1}{4\pi^{\frac{3}{2}}} (r - \frac{1}{\sqrt{2}})^{-1} \leq (2\sqrt{2}\sqrt{E_0} + 3E_0) \cdot \frac{1}{2\pi} \cdot (4.386 + 2\pi(\ln(r + \frac{1}{\sqrt{2}})))^{\frac{1}{2}} + ((2 + \frac{1}{\tau})\sqrt{E_0} + 3E_0) \cdot \frac{1}{4\pi^{\frac{3}{2}}} (r - \frac{1}{\sqrt{2}})^{-1}.$$ 

Now we take

$$r = \frac{3}{4\pi^{\frac{3}{2}}} ((2 + \frac{1}{\tau})\sqrt{E_0} + 3E_0) + \frac{1}{\sqrt{2}} > 2. \quad (\text{by (3.15)})$$

Then clearly

$$((2 + \frac{1}{\tau})\sqrt{E_0} + 3E_0) \cdot \frac{1}{4\pi^{\frac{3}{2}}} (r - \frac{1}{\sqrt{2}})^{-1} \leq \frac{1}{3}.$$
We then only need to check the inequality
\[
\sqrt{\frac{2}{3} \left( \frac{1}{2} + \sqrt{\frac{2}{\tau}} \right)} \geq \frac{1}{3} + (2\sqrt{2E_0} + 3E_0) \cdot \frac{1}{2\pi} \cdot (4.386 + 2\pi(\ln(r + \frac{1}{\sqrt{2}})))^{\frac{1}{2}}.
\]
By using the inequality \(\sqrt{a+b} \geq \sqrt{\frac{a}{\sqrt{2}} + \sqrt{b}}\) for any \(a \geq 0, b \geq 0\), we have
\[
\sqrt{\frac{2}{3} \left( \frac{1}{2} + \sqrt{\frac{2}{\tau}} \right)} \geq \sqrt{\frac{1}{6} + \frac{1}{3} \sqrt{\frac{2}{\tau}}} > \frac{1}{3} + \sqrt{\frac{1}{5} \sqrt{\frac{2}{\tau}}}.
\]
Thus we need to verify
\[
\frac{1}{3} \sqrt{\frac{2}{\tau}} \geq (2\sqrt{2E_0} + 3E_0)^2 \cdot \frac{1}{4\pi^2} \cdot (4.386 + 2\pi(\ln(r + \frac{1}{\sqrt{2}}))).
\]
Now note that \(\ln(\frac{3}{4\pi^2}) \approx -2.00478 < -2\) and \(\sqrt{\frac{2\pi}{3}} \approx 10.49974 < 10.5\), and
\[
\ln(r + \frac{1}{\sqrt{2}}) = \ln(\sqrt{2} + \frac{3}{4\pi^2}((2 + \frac{1}{\tau})\sqrt{E_0} + 3E_0))
\]
\[
< -2 + \ln(10.5 + 2\sqrt{E_0} + 3E_0 + \frac{1}{\tau}\sqrt{E_0})
\]
\[
\leq -2 + \ln\left(\frac{10.5}{6} + \ln(2\sqrt{E_0} + 3E_0 + \frac{1}{\tau}\sqrt{E_0})\right), \quad \text{(by (3.15) and (3.10))}
\]
\[
\leq -1.44 + \frac{1}{2} \ln E_0 + \ln(2 + \frac{1}{\tau} + 3\sqrt{E_0})
\]
\[
\leq -1.44 + \frac{1}{2} \ln E_0 + \ln(2 + 3\sqrt{E_0}) + \ln \frac{1}{\tau}. \quad \text{(since } \frac{1}{\tau} \geq 2)\]
Here in the second inequality above, we have used the simple inequality
\[
10.5 + b \leq \frac{10.5}{6} b, \quad \text{if } b \geq \frac{8}{3\pi^2}. \quad (3.16)
\]
It then suffices for us to prove
\[
\frac{1}{3} \sqrt{\frac{2}{\tau}} \geq (2\sqrt{2E_0} + 3E_0)^2 \cdot \frac{1}{4\pi^2} \cdot (4.386 + 2\pi\left(-1.44 + \frac{1}{2} \ln E_0 + \ln(2 + 3\sqrt{E_0}) + \ln \frac{1}{\tau}\right)).
\]
Denote \(F_0 = (2\sqrt{2E_0} + 3E_0)^2\). Note that
\[
\frac{4.386 - 2\pi \cdot 1.44}{\pi} \approx -1.48389 < -1.45,
\]
\[
\frac{1}{2} \ln E_0 + \ln(2 + 3\sqrt{E_0}) = \ln(2\sqrt{E_0} + 3E_0) < \frac{1}{2} \ln F_0.
\]
We then need to show
\[
\frac{1}{3} \sqrt{\frac{2}{\tau}} \geq F_0 \cdot \frac{1}{4\pi^2} \left(-1.45 + \ln F_0 - 2\ln \tau\right),
\]
or equivalently
\[
\frac{4\pi}{3} \sqrt{2} \geq F_0(-1.45 + \ln F_0)\sqrt{\tau} - 2F_0\sqrt{\tau} \ln \tau.
\]
Note that \( \frac{4\pi \sqrt{2}}{3} \approx 5.92384 \). Now we discuss two cases.

Case A: \( 0 < F_0 \leq 1 \). It is not hard to check that

\[
\sup_{0 < x \leq \frac{1}{2}} \sqrt{2} \ln \left( \frac{1}{x} \right) \leq 0.8.
\]

Clearly then for \( 0 < \tau \leq \frac{1}{2} \), we have

\[
2\sqrt{\tau} \ln \left( \frac{1}{\tau} \right) \leq 1.6 < 5.9
\]

which is clearly ok for us.

Case B: \( F_0 > 1 \). In this case set

\[
\sqrt{\tau} = \frac{\delta}{F_0(1 + \ln F_0)}.
\]

Then

\[
-2F_0 \sqrt{\tau} \ln \tau = \frac{4\delta}{1 + \ln F_0} (-\ln \delta + \ln F_0 + \ln(1 + \ln F_0)).
\]

It is not difficult to check that

\[
\sup_{x > 1} \frac{\ln x + \ln(1 + \ln x)}{1 + \ln x} \leq 1.2.
\]

Thus for \( 0 < \delta \leq 1 \),

\[
-2F_0 \sqrt{\tau} \ln \tau \leq -4\delta \ln \delta + 4.8\delta
\]

Then

\[
F_0(-1.45 + \ln F_0)\sqrt{\tau} - 2F_0 \sqrt{\tau} \ln \tau \leq -4\delta \ln \delta + 5.8\delta \leq 5.8, \quad \text{for any } 0 < \delta \leq 1.
\]

Concluding from all cases, we obtain that it suffices to take

\[
\tau_{\max} = \begin{cases} 
\frac{1}{2}, & \text{if } F_0 \leq 1; \\
\min\left\{ \frac{1}{2}, \left( \frac{1}{F_0(1 + \ln F_0)} \right)^2 \right\}, & \text{if } F_0 > 1,
\end{cases}
\]

where \( F_0 = (2\sqrt{2E_0} + 3E_0)^2 \). Thus (3.14) holds and this completes the base step.

Step 2. Induction step. The main induction hypothesis is that for \( n \geq 1 \),

\[
E_n \leq E_{n-1},
\]

\[
\frac{3}{2} \left\| u^n \right\|_\infty \leq \frac{1}{2} + \sqrt{\frac{2}{\tau}}.
\]

Clearly by using similar estimates as in Step 1 for \( u^1 \), one can check that \( u^{n+1} \) satisfies the same inequality and \( E_{n+1} \leq E_n \). This then completes the induction step.
4 Proof for 2D and \( \nu = 1 \): second approach

Recall

\[
\frac{u^{n+1} - u^n}{\tau} = -\Delta^2 u^{n+1} + \Delta \Pi_N(f(u^n)), \quad n \geq 0.
\]  

(4.17)

We rewrite it as

\[
u^{n+1} = \frac{1 + 2\tau \Delta}{1 + \tau \Delta^2} u^n + \frac{\tau \Delta}{1 + \tau \Delta^2} \Pi_N((u^n)^3 - 3u^n).
\]

Lemma 4.1.

\[
\left( \sum_{0 \neq k \in \mathbb{Z}^2} \left( \frac{1}{1 + \tau (2\pi |k|)^2} \right)^4 \frac{1}{(2\pi |k|)^2} \right)^{1/2} \leq \begin{cases} \left(\frac{1}{8\pi} \ln \tau + 0.52 \right)^{1/2}, & \text{if } 0 < \tau \leq (4\pi)^{-4}; \\ 0.8803, & \text{if } \tau > (4\pi)^{-4}. \end{cases}
\]

\[
\left( \sum_{0 \neq k \in \mathbb{Z}^2} \left( \frac{1 - 2\tau (2\pi |k|)^2}{1 + \tau (2\pi |k|)^2} \right)^4 \frac{1}{(2\pi |k|)^2} \right)^{1/2} \leq h(\tau) = \begin{cases} \left(\frac{1}{8\pi} \ln \tau + 0.52 \right)^{1/2} + 0.00872, & \text{if } 0 < \tau \leq (4\pi)^{-4}; \\ 0.8988, & \text{if } \tau > (4\pi)^{-4}. \end{cases}
\]

Proof. The natural cut-off is \( k_0 = k_0(\tau) = \frac{1}{2\pi \tau} \frac{1}{4}. \)

Case 1: \( k_0 \geq 2 \). This is equivalent to \( 0 < \tau \leq (4\pi)^{-4} < 4.02 \times 10^{-5} \). Then by (2.7), we have

\[
\sum_{0 < |k| \leq k_0} (1 + \tau (2\pi |k|)^4)^{-2} |k|^{-2} \leq \sum_{0 < |k| \leq k_0} |k|^{-2} \leq 4.4 + 2\pi \ln(k_0 + \frac{1}{\sqrt{2}}) \\
= 4.4 + 2\pi \ln(k_0 + \frac{1}{\sqrt{2}}) \leq -\frac{\pi}{2} \ln \tau - 2\pi \ln(2\pi) + 4.4 + 2\pi \ln(1 + \frac{1}{2\sqrt{2}}) \\
\leq -\frac{\pi}{2} \ln \tau - 5.24.
\]

Also by Lemma [2.4] we have

\[
\sum_{|k| > k_0} (1 + \tau (2\pi |k|)^4)^{-2} |k|^{-2} \leq \tau^{-2}(2\pi)^{-8} \sum_{|k| > k_0} |k|^{-10} \leq \tau^{-2}(2\pi)^{-7} \cdot \frac{1}{8} (k_0 - \frac{1}{\sqrt{2}})^{-8} \\
= 2\pi \cdot \frac{1}{8} \cdot (1 - \frac{1}{k_0 \sqrt{2}})^{-8} \leq 25.76.
\]

Thus

\[
\left( \sum_{0 \neq k \in \mathbb{Z}^2} (1 + \tau (2\pi |k|)^4)^{-2} (2\pi |k|)^{-2} \right)^{1/2} \leq \frac{1}{2\pi} \cdot (-\frac{\pi}{2} \ln \tau + 20.52)^{1/2} \leq \left(\frac{1}{8\pi} \ln \tau + 0.52 \right)^{1/2}.
\]

Now observe (for the first inequality we use \( 1 + \tau (2\pi |k|)^4 \geq 2\sqrt{\tau} (2\pi |k|)^2 \))

\[
\sum_{0 < |k| \leq k_0} \left( \frac{2\tau (2\pi |k|)^2}{1 + \tau (2\pi |k|)^4} \right)^2 |k|^{-2} \leq \tau(-\frac{\pi}{2} \ln \tau - 5.24) < 0.0015, \quad \text{for any } 0 < \tau < 4.02 \times 10^{-5};
\]

\[
\sum_{|k| > k_0} \left( \frac{2\tau (2\pi |k|)^2}{1 + \tau (2\pi |k|)^4} \right)^2 |k|^{-2} \leq 4 \cdot (2\pi)^{-4} \sum_{|k| > k_0} |k|^{-6} \leq (2\pi)^{-3} \cdot (2 - \frac{1}{\sqrt{2}})^{-4} < 0.0015.
\]
Thus
\[ \left( \sum_{0 \neq k \in \mathbb{Z}^2} \left( \frac{2\tau(2\pi|k|)^2}{1 + \tau(2\pi|k|)^4} \right)^2 (2\pi|k|)^{-2} \right)^{\frac{1}{2}} \leq \frac{1}{2\pi} (0.0015 + 0.0015)^{\frac{1}{2}} < 0.00872. \]

Case 2: \(0 < k_0 < 2\). In this regime \(\tau > (4\pi)^{-4}\).

\[ \sum_{0 < |k| < 2} (1 + \tau(2\pi|k|)^4)^{-2} |k|^{-2} = 4(1 + \tau(2\pi)^4)^{-2} + 4 \cdot \frac{1}{2} \cdot (1 + \tau(2\pi\sqrt{2})^4)^{-2} \leq 4(1 + \frac{1}{16})^{-2} + 2 \cdot (1 + \frac{1}{4})^{-2} \leq 4.83. \]

\[ \sum_{|k| \geq 2} (1 + \tau(2\pi|k|)^4)^{-2} |k|^{-2} \leq \tau^{-2}(2\pi)^{-8} \sum_{|k| \geq 2} |k|^{-10} \leq (4\pi)^{8}(2\pi)^{-2} \cdot \frac{1}{8}(2 - \frac{1}{\sqrt{2}})^{-8} = 2\pi \cdot \frac{1}{8} \cdot (1 - \frac{1}{2\sqrt{2}})^{-8} \leq 25.76. \]

Thus
\[ \left( \sum_{0 \neq k \in \mathbb{Z}^2} (1 + \tau(2\pi|k|)^4)^{-2} (2\pi|k|)^{-2} \right)^{\frac{1}{2}} \leq \frac{1}{2\pi} \sqrt{4.83 + 25.76} < 0.8803. \]

Now observe
\[ \sum_{0 < |k| < 2} \left( \frac{2\tau(2\pi|k|)^2}{1 + \tau(2\pi|k|)^4} \right)^2 |k|^{-2} = 4\tau^2(2\pi)^4 \left( (1 + \tau(2\pi)^4)^{-2} + 8(1 + \tau(2\pi\sqrt{2})^4)^{-2} \right) = \pi^{-4}x^2((1 + x)^{-2} + 2(1 + 4x)^{-2}) < 0.012, \quad \text{for any } x > \frac{1}{16}, \]

where we have denoted \(x = (2\pi)^4\tau > \frac{1}{16}\).

On the other hand, we have
\[ \sum_{|k| \geq 2} \left( \frac{2\tau(2\pi|k|)^2}{1 + \tau(2\pi|k|)^4} \right)^2 |k|^{-2} \leq 4 \cdot (2\pi)^{-4} \sum_{|k| \geq 2} |k|^{-6} \leq (2\pi)^{-3} \cdot (2 - \frac{1}{\sqrt{2}})^{-4} < 0.0015. \]

Thus
\[ \left( \sum_{0 \neq k \in \mathbb{Z}^2} \left( \frac{2\tau(2\pi|k|)^2}{1 + \tau(2\pi|k|)^4} \right)^2 (2\pi|k|)^{-2} \right)^{\frac{1}{2}} \leq \frac{1}{2\pi} (0.012 + 0.0015)^{\frac{1}{2}} < 0.0185. \]

Finally to estimate \(\left( \sum_{0 \neq k \in \mathbb{Z}^2} \left( \frac{1 - 2\tau(2\pi|k|)^2}{1 + \tau(2\pi|k|)^4} \right)^2 \frac{1}{(2\pi|k|)^2} \right)^{\frac{1}{2}}\), we just use the triangle inequality
\[ \|A + B\|_{l^2_k} \leq \|A\|_{l^2_k} + \|B\|_{l^2_k}. \]

□
Theorem 4.2 (Conditional energy stability for 2D $\nu = 1$, second approach). Let $d = 2$, $\nu = 1$ and $N \geq 2$. Assume $u_0 \in H^1(\mathbb{T}^2)$ and has zero mean. Recall $u^0 = \Pi_N u_0$ and assume $\|u^0\|_\infty \leq 1.25$. Take

$$\tau_{\text{max}} = \min\left\{ \frac{1}{2}, \frac{1}{B_1 (\ln B_1)^2}, \frac{8}{9} \frac{1}{B_2} \right\},$$

where

$$B_1 = \frac{3.1}{99.8} \left( \sqrt{2E_0} + 0.19676E_0 \right)^4, \quad B_2 = (0.8988\sqrt{2E_0} + 0.18692E_0)^4.$$

Then for any $0 < \tau \leq \tau_{\text{max}}$, the scheme (4.17) is energy stable, i.e.

$$E(u^{n+1}) \leq E(u^n), \quad \forall n \geq 0.$$

Proof. The proof is similar to that in Theorem 3.2 and we only sketch the needed modifications. Note that since by assumption $0 < \tau \leq \frac{1}{2}$, we have

$$\frac{1}{2} + \sqrt{\frac{2}{\tau}} \geq 2.5 \geq \frac{3}{2}\|u^0\|_\infty^2.$$

We shall focus on the induction step (since the estimate for $u^1$ is the same as the estimate for $u^{n+1}$ below). Our inductive hypothesis is for $n \geq 1$,

$$E_n \leq E_{n-1}; \quad \|u^n\|_\infty \leq \sqrt{2E_0} h(\tau) + 0.18692E_0,$$

where $h(\tau)$ is the same as in Lemma 4.1. By Lemma 4.1, we have

$$\|1 + 2\tau \frac{\Delta}{1 + \tau \Delta^2} u^n\|_\infty \leq h(\tau)\|u^n\|_{H^1} \leq \sqrt{2E_0} h(\tau).$$

On the other hand by Lemma 2.7 and 3.1, we have

$$\|1 + \tau \frac{\Delta}{1 + \tau \Delta^2} \Pi_N ((u^n)^3 - 3u^n)\|_\infty \leq \frac{\sqrt{6.05}}{(2\pi)^2} \cdot 3E_0 \leq 0.18692E_0.$$

Clearly then

$$\|u^{n+1}\|_\infty \leq \sqrt{2E_0} h(\tau) + 0.18692E_0.$$

It remains for us to verify $E_{n+1} \leq E_n$. This amounts to checking

$$\frac{1}{2} + \sqrt{\frac{2}{\tau}} \geq \frac{3}{2} \max\{\|u^n\|_\infty^2, \|u^{n+1}\|_\infty^2\}.$$

It suffices for us to prove

$$\sqrt{\left( \frac{1}{2} + \sqrt{\frac{2}{\tau}} \right) \cdot \frac{2}{3}} \geq \sqrt{2E_0} h(\tau) + 0.18692E_0.$$
Case 1: $\tau > (4\pi)^{-4}$. We only need to show
\[
\sqrt{\left(\frac{1}{2} + \sqrt{\frac{2}{\tau}}\right) \cdot \frac{2}{3}} \geq 0.8988\sqrt{2E_0} + 0.18692E_0,
\]
or in a slightly simpler form:
\[
\sqrt{\frac{2}{\tau}} \geq (0.8988\sqrt{2E_0} + 0.18692E_0)^2 \cdot \frac{3}{2} - \frac{1}{2}.
\]
Thus it is sufficient to require
\[
\tau \leq \frac{8}{9} \frac{1}{A_1},
\]
where
\[
A_1 = (0.8988\sqrt{2E_0} + 0.18692E_0)^4.
\]

Case 2: $0 < \tau \leq (4\pi)^{-4}$. Then we need to show
\[
\sqrt{\left(\frac{1}{2} + \sqrt{\frac{2}{\tau}}\right) \cdot \frac{2}{3}} \geq \sqrt{2E_0} \left[\left(-\frac{1}{8\pi}\ln\tau + 0.52\right)^{\frac{1}{2}} + 0.00872\right] + 0.18692E_0.
\]
It is not difficult to check that
\[
f_0(\tau) \geq 0.95, \quad \forall \, 0 < \tau \leq (4\pi)^{-4}.
\]
Also $0.18692/0.95 < 0.19676$. Thus it suffices for us to show
\[
\sqrt{\left(\frac{1}{2} + \sqrt{\frac{2}{\tau}}\right) \cdot \frac{2}{3}} \frac{1}{f_0(\tau)} \geq (\sqrt{2E_0} + 0.19676E_0).
\]
We only need to show
\[
\frac{8}{9} \cdot (\sqrt{2E_0} + 0.19676E_0)^{-4} \geq \tau f_0(\tau)^4.
\]
It is not difficult to check that
\[
f_0(\tau) \leq 1.54(-\frac{1}{8\pi}\ln\tau)^{\frac{1}{2}}, \quad \forall \, 0 < \tau \leq (4\pi)^{-4}.
\]
We then only need to prove
\[
99.8(\sqrt{2E_0} + 0.19676E_0)^{-4} \geq \tau (\ln\tau)^2.
\]
Now discuss two cases.

Case a): $E_0 \geq 4.57916$. In this case we have
\[
99.8(\sqrt{2E_0} + 0.19676E_0)^{-4} \leq 0.419538 < 3.1e^{-2} \approx 0.419539,
\]
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By using Lemma 4.3 below, it suffices for us to require

\[ 0 < \tau \leq \frac{1}{B_1 (\ln B_1)^2}, \]

where

\[ B_1 = \frac{3.1}{99.8} (\sqrt{2E_0} + 0.19676E_0)^4. \]

Case b): \( E_0 < 4.57916 \). In this case we have

\[ 99.8(\sqrt{2E_0} + 0.19676E_0)^{-4} \geq 0.418. \]

It suffices to require \( 0 < \tau \leq 0.04 \) since

\[ \sup_{0 < \tau \leq 0.04} \tau (\ln \tau)^2 < 0.41445 < 0.418. \]

Now recall that we are in the sub-case \( 0 < \tau < (2\pi)^{-4} < 0.04 \), so this condition is certainly satisfied.

**Lemma 4.3.** Consider \( h(x) = x(\ln x)^2 \) for \( 0 < x \leq 1 \). If \( 0 < x \leq e^{-2} \), then \( h'(x) \geq 0 \). If \( A \geq e^2/3.1 \approx 2.38357 \) and

\[ 0 < x \leq \frac{1}{B(\ln B)^2}, \quad B = 3.1A, \]

then

\[ h(x) \leq \frac{1}{A}. \]

**Proof.** The monotonicity of \( h \) is easy to check. It is also not difficult to check that if \( B \geq e^2 \), then

\[ \left( \frac{\ln(B(\ln B)^2)}{\ln B} \right)^2 = \left( 1 + \frac{2\ln B}{\ln B} \right)^2 \leq 3.1. \]

By monotonicity of \( h \), we then have for any \( 0 < x \leq \frac{1}{B(\ln B)^2} \),

\[ h(x) \leq h\left( \frac{1}{B(\ln B)^2} \right) = \frac{1}{B} \left( 1 + \frac{2\ln B}{\ln B} \right)^2 \leq \frac{3.1}{B}. \]

\[ \square \]

### 5 Proof for general \( \nu > 0 \): 2D case

In this section we consider the general case \( \nu > 0 \) in 2D. Recall

\[ \frac{u^{n+1} - u^n}{\tau} = -\nu \Delta^2 u^{n+1} + \Delta \Pi_N(f(u^n)), \quad n \geq 0. \tag{5.18} \]

We rewrite it as

\[ (1 + \nu \tau \Delta^2)u^{n+1} = u^n + \tau \Delta \Pi_N(f(u^n)). \]
Lemma 5.1. Let $N \geq 2$, $d = 2$ and $\nu > 0$. Let $\tau > 0$. Then for any $g \in L^4(\mathbb{T}^2)$ with zero mean, we have

$$
\|(1 + \nu \tau \Delta^2)^{-1}g\|_{\infty} \leq C_1(\nu\tau)^{-\frac{1}{4}}\|g\|_4;
$$

For any $g_1 \in L^\frac{4}{3}(\mathbb{T}^2)$, we have

$$
\|\tau \Delta (1 + \nu \tau \Delta^2)^{-1}\Pi_N g_1\|_{\infty} \leq C_2 \tau (\nu\tau)^{-\frac{7}{8}}\|g_1\|_\frac{4}{3}.
$$

In the above $C_1 > 0$, $C_2 > 0$ are absolute constants.

Remark. For the first estimate a similar estimate holds if the spectral projection $\Pi_N$ is present. In our application later we do not need it since $u^n$ is already spectrally localized. The operator $\Pi_N$ can also be replaced by more general projection operators.

Remark. Remarkably if we use the $\dot{H}^1$-norm which seems to be stronger, it will incur a logarithmic loss for $\nu$. The adoption of $L^4$ (for the homogeneous term) and $L^\frac{4}{3}$ (for the inhomogeneous term) removes this divergence. A further refinement is possible by using $\dot{H}^1$ for the high frequency piece. This will lower some of the exponents on the constants such as $\alpha_1$, $\alpha_2$ in the proof of Theorem 5.3 below. However for simplicity of presentation we shall not dwell on this issue here.

Proof. Denote $\beta = \nu\tau$. The first inequality follows from Lemma 2.9 (see the bound for $\tilde{K}$ therein). For the second inequality denote

$$
K_\beta = \mathcal{F}^{-1}(\frac{\beta^{\frac{1}{2}}(2\pi|k|)^2}{1 + \beta(2\pi|k|)^2})1_{|k|\leq N}).
$$

We then have

$$
\|K_\beta\|_4 \leq \|\hat{K}_\beta\|_{l^\frac{4}{3}} \lesssim \beta^{-\frac{3}{8}}.
$$

Lemma 5.2. Let $d \geq 1$. If $E_p = \int_{\mathbb{T}^d} \frac{1}{4}(v^2 - 1)^2 dx$, then

$$
\|v\|_{L^4(\mathbb{T}^d)} \leq \sqrt{1 + 2\sqrt{E_p}},
$$

$$
\|v^3 - v\|_{L^\frac{4}{3}(\mathbb{T}^d)} \leq 2\sqrt{E_p}\sqrt{1 + 2\sqrt{E_p}}.
$$

Proof. Obvious. For the second inequality, note that $\|(v^2 - 1)v\|_{\frac{4}{3}} \leq \|v^2 - 1\|_2\|v\|_4$. 

Theorem 5.3 (Conditional energy stability for 2D $\nu > 0$). Let $d = 2$, $\nu > 0$, $N \geq 2$. Assume $u_0 \in H^1(\mathbb{T}^2)$ and has zero mean. Assume $\|u_0\|_\infty = L_0 < \infty$. Take

$$
\tau_{\text{max}} = \min\left\{\frac{8\nu}{9L_0^4}, \tau^{(1)}_{\text{max}}\right\},
$$

where

$$
\tau^{(1)}_{\text{max}} = 0.04\nu^3 \min\{\alpha_1^{-8}, \alpha_2^{-\frac{8}{3}}\},
$$
and
\[
\alpha_1 = C_1 \sqrt{1 + 2 \sqrt{E_0}};
\]
\[
\alpha_2 = 2C_2 \sqrt{E_0} \cdot \sqrt{1 + 2 \sqrt{E_0}}.
\]

In the above \(C_1, C_2\) are the same constants in Lemma 5.1. Then for any \(0 < \tau \leq \tau_{\text{max}}\), the scheme (5.18) is energy stable, i.e.
\[
E(u^{n+1}) \leq E(u^n), \quad \forall n \geq 0.
\]

\textbf{Proof.} We shall use induction.

Step 1. The base step \(n = 0\). Thanks to our choice of \(\tau\), we clearly have for \(n = 0\),
\[
\sqrt{\frac{2\nu}{\tau}} \geq \frac{3}{2} \|u^0\|_\infty^2.
\]
To ensure \(E_1 \leq E_0\), we need to check
\[
\sqrt{\frac{2\nu}{\tau}} \geq \frac{3}{2} \|u^1\|_\infty^2.
\]

By Lemma 5.1 and Lemma 5.2, we have
\[
\|u^1\|_\infty \leq \alpha_1 (\nu \tau)^{-\frac{1}{8}} + \alpha_2 \tau (\nu \tau)^{-\frac{7}{8}},
\]
where
\[
\alpha_1 = C_1 \sqrt{1 + 2 \sqrt{E_0}};
\]
\[
\alpha_2 = 2C_2 \sqrt{E_0} \cdot \sqrt{1 + 2 \sqrt{E_0}}.
\]

By Lemma 2.8 to show \(E_1 \leq E_0\), it suffices for us to check the inequality
\[
\left(\frac{2}{3} \sqrt{\frac{2\nu}{\tau}}\right)^{\frac{1}{2}} \geq \alpha_1 (\nu \tau)^{-\frac{1}{8}} + \alpha_2 \tau (\nu \tau)^{-\frac{7}{8}}.
\]
Now set \(\tau = \nu^3 r\) and for \(r\) we need to check
\[
2^\frac{3}{4} 3^{-\frac{1}{4}} \geq \alpha_1 r^{\frac{1}{8}} + \alpha_2 r^{\frac{3}{8}}.
\]
Let \(r = \alpha_1^4 \alpha_2^{-4} z\). Then we need
\[
2^\frac{3}{4} 3^{-\frac{1}{4}} \geq \alpha_1^{\frac{3}{8}} \alpha_2^{-\frac{1}{8}} (z^{\frac{1}{8}} + z^{\frac{3}{8}}).
\]
Now we choose \(\eta \in (0, 1)\), such that
\[
z^{\frac{1}{8}} \leq \eta 2^\frac{3}{4} 3^{-\frac{1}{4}} \alpha_1^{\frac{3}{8}} \alpha_2^{-\frac{1}{8}};
\]
\[
z^{\frac{3}{8}} \leq (1 - \eta) 2^\frac{3}{4} 3^{-\frac{1}{4}} \alpha_1^{\frac{3}{8}} \alpha_2^{-\frac{1}{8}}.
\]
A nearly optimal choice is \( \eta = 0.690119 \) for which
\[
(\eta^2 3^{-\frac{1}{2}})^8 \approx 0.0406525 > 0.04;
\]
\[
((1 - \eta) 3^{-\frac{1}{2}})^{\frac{8}{16}} \approx 0.0406523 > 0.04.
\]
Thus it suffices to require
\[
z \leq 0.04 \min \{ \alpha_1^{-12} \alpha_2^4, \alpha_1^{-4} \alpha_2^4 \}.
\]
Thus for \( \tau \) we need
\[
0 < \tau \leq \tau_{(1)}^{\max} = 0.04 \nu^3 \min \{ \alpha_1^{-8}, \alpha_2^{-8} \}.
\]
This in turn guarantees that \( E_1 \leq E_0 \).

Step 2. Induction. For \( n \geq 1 \), the induction hypothesis is that
\[
E_n \leq E_{n-1},
\]
\[
\| u^n \|_\infty \leq \alpha_1 (\nu \tau)^{-\frac{1}{8}} + \alpha_2 \tau (\nu \tau)^{-\frac{2}{8}}.
\]
Clearly by using similar estimates as in Step 1 for \( u^1 \), one can check that
\[
\| u^{n+1} \|_\infty \leq \alpha_1 (\nu \tau)^{-\frac{1}{8}} + \alpha_2 \tau (\nu \tau)^{-\frac{2}{8}},
\]
and
\[
\sqrt{\frac{2\nu}{\tau}} \geq \frac{3}{2} \max \{ \| u^n \|_\infty^2, \| u^{n+1} \|_\infty^2 \}.
\]
Then by Lemma 2.8 we obtain \( E_{n+1} \leq E_n \) which completes the induction step.

\[ \square \]

6 Proof for general \( \nu > 0 \): 1D and 3D case

In this section we sketch the needed modifications for the 1D and 3D cases. The 1D case will be similar to the 2D case. On the other hand the analysis for the 3D case will be slightly different since \( \tilde{H}^1 \) is no longer a critical case.

We first consider the 1D case.

Lemma 6.1 (1D case). Let \( N \geq 2, d = 1 \) and \( \nu > 0 \). Let \( \tau > 0 \). Then for any \( g \in L^4(\mathbb{T}) \) with mean zero, we have
\[
\|(1 + \nu \tau \Delta^2)^{-1} g\|_{L^\infty(\mathbb{T})} \leq B_1 (\nu \tau)^{-\frac{1}{16}} \| g \|_{L^4(\mathbb{T})}.
\]
For any \( g_1 \in L^4(\mathbb{T}) \), we have
\[
\| \tau \Delta (1 + \nu \tau \Delta^2)^{-1} \Pi_N g_1 \|_{L^\infty(\mathbb{T})} \leq B_2 \tau (\nu \tau)^{-\frac{11}{16}} \| g_1 \|_{L^4(\mathbb{T})}.
\]
In the above \( B_1 > 0, B_2 > 0 \) are absolute constants.
Proof. Denote $\beta = \nu \tau$. The first inequality follows from the bound of $\tilde{K}$ in Lemma 2.9. For the second inequality denote (since we are in 1D, $|k| = |k|_\infty$)

$$K_\beta = F^{-1}(\beta^{-1/2}(2\pi|k|)^2 1_{|k| \leq N}).$$

We then have

$$\|K_\beta\|_4 \leq \|\hat{K}_\beta\|_{\ell_4} \lesssim \beta^{-1/4}.$$

\[\square\]

**Theorem 6.2** (Conditional energy stability for 1D $\nu > 0$). Let $d = 1$, $\nu > 0$, $N \geq 2$. Assume $u_0 \in H^1(\mathbb{T})$ and has zero mean. Assume $\|u_0\|_\infty = L_0 < \infty$. Take

$$\tau_{\max} = \min\left\{ \frac{8\nu}{9F_0^4}, \frac{\tau^{(1)}_{\max}}{9F_0^4} \right\},$$

where

$$\tau^{(1)}_{\max} = 0.118\nu^{\frac{2}{9}} \min\{\beta_1^{-\frac{10}{9}}, \beta_2^{-\frac{10}{9}}\},$$

and

$$\beta_1 = B_1 \sqrt{1 + 2\sqrt{E_0}},$$

$$\beta_2 = 2B_2 \sqrt{E_0} \cdot \sqrt{1 + 2\sqrt{E_0}}.$$

Here $B_1, B_2$ are the same constants in Lemma 6.7. Then for any $0 < \tau \leq \tau_{\max}$, the scheme (5.18) is energy stable, i.e.

$$E(u^{n+1}) \leq E(u^n), \quad \forall n \geq 0.$$

Proof. The induction procedure is similar to that in the proof of Theorem 5.3 and therefore we shall only sketch the needed modifications. The main inequality to verify is

$$\left(\frac{2}{3} \sqrt{\frac{2\nu}{\tau}}\right)^{\frac{3}{2}} \geq \max\{\|u^n\|_\infty, \|u^{n+1}\|_\infty\}.$$

The estimate of $\|u^n\|_\infty$ uses induction hypothesis. For $u^{n+1}$ we use Lemma 6.1 and this gives

$$\|u^{n+1}\|_\infty \leq B_1 (\nu \tau)^{-\frac{3}{16}} \|u^n\|_{L^1(\mathbb{T})} + B_2 \tau (\nu \tau)^{-\frac{11}{16}} \|f(u^n)\|_{L^\frac{4}{3}(\mathbb{T})} \leq \beta_1 (\nu \tau)^{-\frac{3}{16}} + \beta_2 \tau (\nu \tau)^{-\frac{11}{16}},$$

where in the second inequality we used Lemma 5.2 and

$$\beta_1 = B_1 \sqrt{1 + 2\sqrt{E_0}},$$

$$\beta_2 = 2B_2 \sqrt{E_0} \cdot \sqrt{1 + 2\sqrt{E_0}}.$$
Set $\tau = \nu^{\frac{5}{3}} r$, and for $r$ we need to check the inequality

$$2^{\frac{1}{4}} 3^{-\frac{1}{2}} \geq \beta_1 r^{\frac{3}{16}} + \beta_2 r^{\frac{9}{16}}.$$ 

We shall choose $r$ such that

$$r \leq z \cdot \min\{\beta_1^{-\frac{16}{3}}, \beta_2^{-\frac{16}{9}}\},$$

and

$$z^{\frac{3}{16}} \leq \eta 2^{\frac{3}{4}} 3^{-\frac{1}{2}},$$

$$z^{\frac{9}{16}} \leq (1 - \eta) 2^{\frac{3}{4}} 3^{-\frac{1}{2}}.$$ 

A nearly optimal choice is $\eta = 0.690119$ for which

$$(\eta 2^{\frac{3}{4}} 3^{-\frac{1}{2}})^{\frac{16}{3}} \approx 0.118229 > 0.118;$$

$$(1 - \eta) 2^{\frac{3}{4}} 3^{-\frac{1}{2}})^{\frac{16}{3}} \approx 0.118229 > 0.118.$$ 

Thus it suffices to require

$$r \leq 0.118 \min\{\beta_1^{-\frac{16}{3}}, \beta_2^{-\frac{16}{9}}\}.$$

The next lemma is for the 3D case. Note that the argument is slightly different from 2D and in some sense simpler.

**Lemma 6.3** (3D case). Let $N \geq 2$, $d = 3$, $\nu > 0$ and $\tau > 0$. Assume $g \in H^1(T^3)$ has mean zero on $T^3$. Then

$$\|(1 + \nu \tau \Delta^2)^{-1} g\|_{L^\infty(T^3)} \leq B_3 (\nu \tau)^{-\frac{7}{8}} \|g\|_{H^1(T^3)},$$

Let $N \geq 2$. For any $g_1 \in L^2(T^3)$, we have

$$\|\tau \Delta (1 + \nu \tau \Delta^2)^{-1} \Pi_N g_1\|_{L^\infty(T^3)} \leq B_4 \tau (\nu \tau)^{-\frac{7}{8}} \|g_1\|_{L^2(T^3)},$$

In the above $B_3 > 0$, $B_4 > 0$ are absolute constants. Also we have

$$\|\tau \Delta (1 + \nu \tau \Delta^2)^{-1} \Pi_N g_1\|_{L^\infty(T^3)} \leq B_4 \tau (\nu \tau)^{-\frac{7}{8}} \|g_1 - \overline{g_1}\|_{L^2(T^3)},$$

where $\overline{g_1}$ denotes the average of $g_1$ on $T^3$.

**Remark.** The second estimate also holds when $\Pi_N$ is not present.

**Remark.** Compared with 2D, here the argument is slightly simpler since $\dot{H}^1$ is no longer a critical space (for $L^\infty$) we can make use of $L^2$ techniques.

**Remark.** For the inhomogeneous estimate, one should note that

$$\|\Delta (1 + \Delta^2)^{-1} \delta_0\|_{L^4(T^3)} = \infty$$

and this is why we have to proceed differently from the 2D case.
Remark. In the proof below we do not consider the refined bounds for $\beta = \nu \tau > 1$ since one is primarily interested in the case $0 < \nu \ll 1$ and $0 < \tau \lesssim 1$.

Proof. Denote $\beta = \nu \tau$. For the first inequality it suffices to check that
\[
\left(\frac{1}{1 + \beta (2\pi |k|)^4}, \frac{1}{2\pi |k|}\right)_k (0 \neq k \in \mathbb{Z}^3) \lesssim \beta^{-\frac{1}{8}}.
\]
If $\beta \geq 1$ the inequality is obvious since we have the stronger bound $\beta^{-1}$ in this case. If $0 < \beta < 1$, one can then split into regimes $|k| \leq \beta^{-\frac{1}{8}}$ and $|k| > \beta^{-\frac{1}{8}}$ and estimate separately the contributions. The bound $\beta^{-\frac{3}{8}}$ is then immediate. Note that we can even calculate explicit constants here but we shall not dwell on this issue here.

The proof of the second inequality is similar. We use
\[
\left(\frac{\beta^\frac{1}{2} (2\pi |k|)^2}{1 + \beta (2\pi |k|)^4}\right)_k R^3 (\mathbb{R}^3) \lesssim \beta^{-\frac{3}{8}}.
\]

\[\Box\]

**Theorem 6.4** (Conditional energy stability for 3D $\nu > 0$). Let $d = 3$, $\nu > 0$, $N \geq 2$. Assume $u_0 \in H^1 (\mathbb{T}^3)$ and has zero mean. Assume $\|u^0\|_{\infty} = L_0 < \infty$. Take
\[
\tau_{\text{max}} = \min \left\{ \frac{8\nu}{9L_0^2}, \frac{\nu}{\nu_{\text{max}}} \right\}.
\]
Here
\[
\tau_{\text{max}}^{(1)} = 0.0007\nu^7 \min \{\beta_3^{-8}, \beta_4^{-\frac{8}{3}}, \beta_5^{-8}\},
\]
where
\[
\beta_3 = B_3' \sqrt{E_0}; \quad \beta_4 = B_4' E_0^\frac{3}{2}; \quad \beta_5 = B_5' (1 + E_0)^\frac{1}{4},
\]
and $B_3' > 0$, $B_4' > 0$, $B_5' > 0$ are some absolute constants. Then for any $0 < \tau \leq \tau_{\text{max}}$, the scheme (5.18) is energy stable, i.e.
\[
E(u^{n+1}) \leq E(u^n), \quad \forall n \geq 0.
\]

Proof. The induction procedure is similar to that in the proof of Theorem 5.3 and therefore we shall only sketch the needed modifications. By Lemma 2.8 the main inequality to verify is
\[
\left(\frac{1}{3} + \frac{2}{3} \sqrt{\frac{2\nu}{\tau}} \right)^\frac{7}{2} \geq \max \{\|u^n\|_{\infty}, \|u^{n+1}\|_{\infty}\}.
\]
The estimate of $\|u^n\|_{\infty}$ uses induction hypothesis. For $u^{n+1}$ we use Lemma 6.3 and this gives
\[
\|u^{n+1}\|_{\infty} \leq B_3 (\nu \tau)^{-\frac{1}{2}} \|\nabla u^n\|_{L^2 (\mathbb{T}^3)} + B_2 (\nu \tau)^{-\frac{7}{2}} \|f(u^n)\|_{L^2 (\mathbb{T}^3)}
\leq \beta_3 \nu^{-\frac{1}{8}} (\nu \tau)^{-\frac{7}{8}} + \beta_4 \tau (\nu \tau)^{-\frac{7}{8}} \nu^{-\frac{2}{3}} + \beta_5 (\nu \tau)^{-\frac{3}{8}},
\]

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where in the second inequality we used the fact that
\[
\| (u^n)^3 \|_{L^2(T^3)} \lesssim \| \nabla u^n \|_{L^2(T^3)}^3 \lesssim \nu^{-\frac{10}{9}} E_0^{\frac{1}{3}};
\]
\[
\| u^n \|_{L^2(T^3)} \lesssim (1 + E_0)^{\frac{3}{4}},
\]
and for some absolute constant \( \tilde{B}_4 > 0, B_5 > 0, \)
\[
\beta_3 = B_3 \sqrt{2E_0}; \quad \beta_4 = \tilde{B}_4 E_0^{\frac{3}{4}}; \quad \beta_5 = B_5 (1 + E_0)^{\frac{3}{4}}.
\]
Set \( \tau = \nu^7 r, \) and for \( r \) we need to check the inequality
\[
(\frac{1}{3} \nu^3 r^{\frac{1}{2}} + \frac{2}{3} \sqrt{2})^{\frac{1}{2}} \geq \beta_3 r^{\frac{1}{3}} + \beta_4 r^{\frac{3}{8}} + \beta_5 r^{\frac{3}{8}} \nu^{\frac{3}{2}}.
\]
Now by using the inequality
\[
\sqrt{a + b} \geq \frac{\sqrt{a} + \sqrt{b}}{\sqrt{2}}, \quad \forall a, b \geq 0,
\]
it suffices for us to choose \( r \) such that
\[
(\frac{\sqrt{2}}{3})^{\frac{1}{2}} \geq \beta_3 r^{\frac{1}{3}} + \beta_4 r^{\frac{3}{8}};
\]
\[
(\frac{1}{6} \nu^3 r^{\frac{1}{2}})^{\frac{1}{2}} \geq \beta_5 r^{\frac{3}{8}} \nu^{\frac{3}{2}}.
\]
The second inequality requires that
\[
r \leq 6^{-4} \beta_5^{-8}.
\]
Note that \( 6^{-4} \approx 0.000771605 \) \( \approx 0.0007. \) For the first inequality we shall choose \( r \) such that
\[
r \leq z \cdot \min\{ \beta_3^{-8}, \beta_4^{-\frac{8}{3}} \},
\]
and
\[
z^{\frac{1}{8}} \leq \eta 2^{\frac{1}{4}} 3^{-\frac{1}{2}},
\]
\[
z^{\frac{3}{8}} \leq (1 - \eta) 2^{\frac{1}{4}} 3^{-\frac{1}{2}}.
\]
A nearly optimal choice is \( \eta = 0.778006 \) for which
\[
(\eta 2^{\frac{1}{4}} 3^{-\frac{1}{2}})^{\frac{8}{3}} \approx 0.0066288 > 0.006;
\]
\[
((1 - \eta) 2^{\frac{1}{4}} 3^{-\frac{1}{2}})^{\frac{8}{3}} \approx 0.0066288 > 0.006.
\]
Thus it suffices to require
\[
r \leq 0.0007 \min\{ \beta_3^{-8}, \beta_4^{-\frac{8}{3}}, \beta_5^{-8} \}.
\]
7 Concluding remarks

Implicit-Explicit (IMEX) methods can simulate efficiently many phase field models such as the Cahn-Hilliard equation or thin film type equations. Compared with pure explicit methods, IMEX is more stable with larger allowable time steps whilst being efficient and accurate. In contrast with implicit methods and partially implicit methods, IMEX does not require solving a nonlinear system at each time step and is much more efficient. In numerical experiments IMEX is often observed to be energy stable provided the time step is not taken too large. Due to the difficulties caused by the lack of maximum principle and stiffness caused by the effect of small viscosity, the rigorous stability analysis of IMEX methods was a long standing open problem. In this work we analyzed a model IMEX scheme introduced by Chen and Shen [4] for the Cahn-Hilliard equation and gave a first rigorous proof of conditional energy stability with mild time step constraints. Our analysis does not rely on adding additional stabilization terms, truncating the nonlinearity or introducing auxiliary variables. To deal with the aforementioned difficulties caused by the lack of maximum principle and stiffness of small viscosity, we introduce a Trade-Energy-For-$L^\infty$ (TEFL) method which is a refinement of our earlier work [25, 20, 27]. In the course of the proof we computed explicitly (and nearly optimal in terms of energy scaling) time step constraints in several model cases which seem to be the first done in the literature. All these developments are pivotal for future refined analysis on these algorithms. Our theoretical analysis shows that IMEX is a robust algorithm for large scale and long-time simulations, due to its simplicity and guaranteed conditional energy stability with affordable time step constraints. It is expected that this new streamlined TEFL proof can be further refined and adapted to higher order cases and generalized to many other phase field models and settings.

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