NONLOCAL ELLIPTIC PROBLEMS IN INFINITE CYLINDER AND APPLICATIONS

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ABSTRACT. We consider a unique solvability of nonlocal elliptic problems in infinite cylinder in weighted spaces and in Hölder spaces. Using these results we prove the existence and uniqueness of classical solution for the Vlasov–Poisson equations with nonlocal conditions in infinite cylinder for sufficiently small initial data.

1. Introduction. An interest to nonlocal elliptic problems increased considerably during last years. In [5], it was studied the Laplace equation in a bounded domain with nonlocal boundary condition connecting the values of unknown function on a boundary with the values on some manifold inside domain. The solvability of general nonlocal elliptic problems of that type was formulated as an open problem [22], [17]. A solution of this problem and analysis of spectral properties and asymptotic formulas of solutions to nonlocal elliptic problems is presented in [24, 25]. For other investigations of nonlocal elliptic problems, see also [10], [11], [13], and [27].

In this paper we consider solvability of nonlocal elliptic problems in infinite cylinder in weighted spaces and in Hölder spaces. An interest to this problem is motivated by its application to a problem of plasma confinement in thermonuclear reactors having the shape of long cylinder (“mirror traps”), see [18], [23]. We note that local boundary value problems for elliptic differential equations in a cylinder were studied in [15], [19], and [20].

Section 2 is devoted to statement of problem and notation. We consider a second order elliptic differential equation in the cylinder \( Q = G \times \mathbb{R} \), where \( G \subset \mathbb{R}^n \) is a bounded domain with boundary \( \partial G \in C^\infty \). A solution of this equation \( u(x, \tau) \) \((x \in G, \tau \in \mathbb{R})\) satisfies a nonlocal boundary condition connected the values of solution on the boundary \( \partial Q = \partial G \times \mathbb{R} \) with its values on some cylindrical surface from \( Q \).

In Section 3, along with nonlocal elliptic boundary value problem we consider a linear bounded operator \( L \) acting in weighted spaces with the weight \( e^{\beta \tau}, \beta \in \mathbb{R} \), and exponent \( p = 2 \). Using the Fourier transform, we obtain an operator-valued function \( \lambda \mapsto \hat{L}(\lambda) \) acting in Sobolev spaces. It is shown that, for any \( h \in \mathbb{R} \), there exists \( \lambda_1 > 1 \) such that, for all \( \lambda \in \{ \text{Im} \lambda = h, |\text{Re} \lambda| \geq \lambda_1 \} \), the operator \( \hat{L}(\lambda) \) has a bounded inverse \( \hat{R}(\lambda) = \hat{L}^{-1}(\lambda) \). Moreover, the operator-valued function \( \lambda \mapsto \hat{R}(\lambda) \) is finitely meromorphic in \( \mathbb{C} \). This result together with a priori estimates

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of solutions of nonlocal elliptic boundary-value problems with a parameter allows to prove that the operator $L$ is an isomorphism of weighted spaces with the weight $e^{\beta t}$ and exponent $p = 2$ if and only if the line $\{\text{Im} \lambda = \beta\}$ does not contain eigenvalues of the operator-valued function $\lambda \mapsto \hat{L}(\lambda)$.

Section 4 deals with auxiliary results devoted to the Dirichlet problem for a second order elliptic equation in the cylinder $Q = G \times \mathbb{R}$. There we prove a unique solvability and a priori estimates of solutions in Hölder spaces.

In Section 5, we generalize results of Section 3 to the case $p \geq 2$. We prove that the operator $L$ is an isomorphism of weighted spaces with the weight $e^{\beta t}$ and exponent $p \geq 2$ if and only if the line $\{\text{Im} \lambda = \beta\}$ does not contain eigenvalues of the operator-valued function $\lambda \mapsto \hat{L}(\lambda)$.

In Section 6, we show that, if the operator-valued functions $\lambda \mapsto \hat{L}_0(\lambda)$ and $\lambda \mapsto \hat{L}(\lambda)$ have no real eigenvalues, then a nonlocal elliptic boundary-value problem in the cylinder $Q = G \times \mathbb{R}$ has a unique solution $u \in C^{2+\sigma}_0(\bar{Q})$ for all right hand sides of equation and nonlocal boundary conditions from corresponding Hölder spaces. Here $\lambda \mapsto \hat{L}_0(\lambda)$ is the operator-valued function with trivial nonlocal term, $C^{2+\sigma}_0(\bar{Q})$ is the closure of set of functions from $C^{2+\sigma}(\bar{Q})$ with compact supports in $\bar{Q}$, $0 < \sigma < 1$.

Section 7 deals with applications. We consider the Vlasov–Poisson equations in infinite cylinder with nonlocal boundary condition for the electric field potential and initial conditions for density distribution functions of charged particles. Applying Theorem 6.3 on a unique solvability of nonlocal elliptic problems in a cylinder in Hölder spaces and Theorem 5.1 in [26] on solvability of abstract Vlasov equations, we prove that there is a unique classical solution of the above problem for sufficiently small initial density distribution functions. Moreover, the supports of density distribution functions belong to some interior cylinder.

Problems of existence for classical solutions and generalized solutions to the Cauchy problem and for generalized solutions to mixed boundary value problems for Vlasov equations were studied in details, see [2], [3], [7], [16] and [21]. The question of existence of classical solutions to mixed problems for Vlasov equations was studied much less. In [22], [16] it was formulated as an open problem. An interest to classical solutions of mixed boundary value problems for Vlasov–Poisson equations is associated with the design of a controlled thermonuclear fusion reactor. One of the devices for thermonuclear fusion is the mirror trap, which has the shape of a long cylinder tapered at the ends, see [18], [23].

The production of a stable high-temperature plasma in a reactor requires that the so-called plasma column be strictly inside the domain during some time interval in order to keep it away from the vacuum container wall. In most of models of thermonuclear fusion reactors an external magnetic field is used as a control ensuring the existence of plasma confinement in the reactor [18], [23]. From mathematical point of view this means that one has to ensure existence of solutions of the Vlasov–Poisson equations for which the supports of the charged-particle density distributions do not intersect the boundary. This can be achieved by the influence of the external magnetic field. However in mathematical investigations devoted to classical solutions of mixed problems for the Vlasov–Poisson equations it was studied the behavior of the trajectories of particles near the boundary with reflection boundary conditions [9], [12]. The effect of the magnetic field was not taken into account. According to [18], [23], the presence of a considerable number of particles on the boundary can result in either destruction of the reactor walls or in cooling the high-temperature plasma due to its contact with the reactor walls. As a distinct
from other papers, we consider here the Vlasov–Poisson equations with external magnetic field and study solutions with supports at some distance from the boundary. We also examine a two-component plasma, since the word “plasma” is used in physics to designate a high-temperature state of an ionized gas with charge neutrality [18]. Another distinction, just as in [22], is connected with nonlocal boundary conditions for the potential of electric field.

2. Statement of problem. Some notation.

2.1. We consider the equation

$Au = A(x, \partial_x, \partial_\tau)u(x, \tau) = f_0(x, \tau) \quad ((x, \tau) \in Q)$

(2.1)

with nonlocal boundary condition

$u(x, \tau) + b(x)u(\omega(x), \tau + \chi) = f_1(x, \tau) \quad ((x, \tau) \in \partial Q).$

(2.2)

Here $Q = G \times \mathbb{R}$, $G \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial G \in C^\infty$ if $n \geq 2$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\tau \in \mathbb{R}$,

$A(x, \partial_x, \partial_\tau) = \sum_{|\alpha| + q \leq 2} a_{\alpha q}(x)\partial_\alpha^\alpha \partial_\tau^q,$

$b, a_{\alpha q} \in C^\infty(\mathbb{R}^n)$ are real-valued functions, $\partial_\alpha^\alpha = \left( \frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left( \frac{\partial}{\partial x_n} \right)^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $\alpha_j \geq 0$ are integers ($j = 1, \ldots, n$), $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial_\tau^q = \left( \frac{\partial}{\partial \tau} \right)^q$, $q \geq 0$ is an integer; $\omega$ is a $C^\infty$ diffeomorphism mapping some neighborhood $\Omega_0$ of the boundary $\partial G$ onto $\omega(\Omega_0)$ so that $\overline{\omega(\Omega_0)} \subset G$, $\chi \in \mathbb{R}$. If $n = 1$, then $Q = (a, b)$, $x \in \mathbb{R}$ is a scalar variable, $\alpha \geq 0$ is an integer, $\partial_x^\alpha = \left( \frac{\partial}{\partial x} \right)^\alpha$.

We suppose that the operator $A(x, D_x, D_\tau)$ is uniformly elliptic in $\mathcal{Q}$, i.e., the following condition holds.

Condition 2.1. $- \sum_{|\alpha| + q \leq 2} a_{\alpha q}(x)\xi^\alpha \lambda^q > 0$ for all $x \in \mathcal{G}$, $\xi \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$ such that $|\xi| + |\lambda| \neq 0$.

We denote $a_0(x) = a_{00}(x)$ and $\theta = \max_{x \in \mathcal{G}} |b(x)|$.

In some cases we also assume that the following conditions are fulfilled.

Condition 2.2. $a_0(x) \geq 0 \quad (x \in \mathcal{G})$

Condition 2.3. $\theta < 1$.

2.2. We introduce some function spaces, which will be necessary later on.

Let $\Omega \in \mathbb{R}^N$ be either a domain with boundary $\partial \Omega \in C^\infty$, or $\Omega = \mathbb{R}^N$. We denote by $C^k(\Omega)$, $k \in \mathbb{Z}$, $k \geq 0$, the space of continuous functions on $\Omega$, having all continuous derivatives in $\Omega$ up to the order $k$ with the norm

$\|u\|_k = \max_{|\alpha| \leq k} \sup_{y \in \Omega} |\partial_\alpha^\alpha u(y)|.$

(2.3)

We denote by $C^{k+\sigma}(\Omega)$, $k \in \mathbb{Z}$, $k \geq 0$, $0 < \sigma < 1$, the Hölder space of continuous functions on $\Omega$, having all continuous derivatives on $\Omega$ up to the order $k$ with the finite norm

$\|u\|_{k+\sigma} = \|u\|_k + |u|_\sigma,$

(2.4)
where
\[ |u|_\sigma = \max_{|\alpha|=k} \sup_{y,z \in \Omega: y \neq z} |y - z|^{-\sigma} |\partial^\alpha u(y) - \partial^\alpha u(z)|. \] (2.5)

Thus we have defined the space $C^\sigma(\bar{\Omega})$ for any $s \geq 0$.

Similarly one can define the spaces $C^k(\cdot)$ and $C^{k+\sigma}(\cdot)$ on an $(N-1)$-dimensional $C^\infty$-manifold, on a finite cylinder, or on a semi-infinite cylinder. Let $C(\Omega) = C^0(\Omega)$.

**Remark 2.4.** For any $k$ and $\sigma$, the spaces $C^k(\bar{\Omega})$ and $C^{k+\sigma}(\bar{\Omega})$ are Banach spaces. Moreover, $C^k(\Omega)$ is a separable space. However the space $C^{k+\sigma}(\Omega)$ is nonseparable, and the set of infinitely differentiable functions on $\bar{\Omega}$ is nondense in $C^{k+\sigma}(\Omega)$ see [4].

Denote by $C^0_0(Q)$ ($C^0_0(\partial Q)$), $s \geq 0$, the closure of the set of functions from $C^s(Q)$ ($C^s(\partial Q)$) with compact supports in $\bar{Q}$ ($\partial Q$).

The properties of spaces $C^k(\bar{\Omega})$ and $C^{k+\sigma}(\bar{\Omega})$ are presented in [28].

We denote by $W^k_p(\Omega)$, $k \in \mathbb{N}$, $p \geq 2$, the Sobolev space of functions $v \in L_p(\Omega)$ having all generalized derivatives $\partial^\alpha v \in L_p(\Omega)$, $|\alpha| \leq k$, with the norm
\[
\|v\|_{W^k_p(\Omega)} = \left\{ \sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha v(x)|^p \, dx \right\}^{1/p}.
\] (2.6)

Similarly one can define the spaces $W^k_p(\cdot)$ on a finite cylinder or on a semi-infinite cylinder. Let $M \subset \bar{\Omega}$ be an $(N-1)$-dimensional $C^\infty$-manifold. Let $W^{k-1/p}_p(M)$ be the space of traces on $M$ for functions from $W^k_p(\Omega)$ with the norm
\[
\|\psi\|_{W^{k-1/p}_p(M)} = \inf_{v} \|v\|_{W^k_p(\Omega)} \quad (v|_M = \psi).
\] (2.7)

If $n = 1$, then by virtue of the Sobolev imbedding theorem $W^k_p(a,b) \subset C[a,b]$. Thus for any $v \in W^k_p(a,b)$ we can define the value $v(d)$, where $d \in [a,b]$. For the theory of Sobolev spaces including imbedding theorems, extension theorems, and spaces of traces, see [28].

We introduce the weighted Kondrat’ev space $W^{k}_{p,\beta}(Q)$ as the completion of the set $C^\infty_0(Q)$ with respect to the norm
\[
\|w\|_{W^{k}_{p,\beta}(Q)} = \left\{ \sum_{|\alpha|+q \leq k} \int_Q e^{p\beta \tau} |\partial^\alpha \partial^\beta w(x,\tau)|^p \, dx \, d\tau \right\}^{1/p},
\] (2.8)

where $C^\infty_0(Q)$ is the set of infinitely differentiable in $Q$ functions with compact supports in $Q$, $0 \leq k \in \mathbb{Z}$, $\beta \in \mathbb{R}$, $p \geq 2$. If $\beta = 0$, then the space $W^{k}_{p,\beta}(Q)$ coincides with the Sobolev space $W^k_p(Q)$.

We denote by $W^{k-1/p}_p(\partial Q)$, $k \geq 1$, the space of traces on $\partial Q$ for functions from $W^k_p(\cdot)$ with the norm
\[
\|g\|_{W^{k-1/p}_p(\partial Q)} = \inf_{u} \|u\|_{W^k_p(\cdot)} \quad (u|_{\partial Q} = g).
\] (2.9)

For the first time the theory of weighted spaces $W^{k}_{2,\beta}(Q)$ was created in [14] for investigation of elliptic equations in domains with conical or angular points. For further research on the spaces $W^{k}_{p,\beta}(Q)$, $p > 1$, and bibliography, see [19].
3. Nonlocal elliptic problems in $W^2_{2,\beta}(Q)$.

3.1. First we consider a nonlocal elliptic problem with a parameter associated with problem (2.1), (2.2).

We introduce a real-valued function $\eta \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leq \eta(x) \leq 1$ ($x \in \mathbb{R}^n$), $\eta(x) = 1$ ($x \in \omega(\Omega_1)$), and $\eta(x) = 0$ ($x \notin \omega(\Omega_2)$), where $\Omega_1$ and $\Omega_2$ are some open neighborhoods of $\partial G$, $\Omega_1 \subset \Omega_2$, and $\Omega_2 \subset \Omega_0$. We denote $\kappa = \text{dist}(\omega(\Omega_2), \partial G)$. Clearly, $\kappa > 0$. We define a linear bounded operator $B^1: W^2_{2,\beta}(Q) \rightarrow W^2_{2,\beta}(Q)$, $j = 0, 1, 2$, by the formula

$$
(B^1u)(x, \tau) = b(x)\eta(\omega(x))u(\omega(x), \tau + \lambda) \quad ((x, \tau) \in (\Omega_0 \times \mathbb{R}) \cap Q),
$$

$$
(B^1u)(x, \tau) = 0 \quad ((x, \tau) \in Q \setminus (\Omega_0 \times \mathbb{R})).
$$

We introduce a linear bounded operator

$$
L: W^2_{2,\beta}(Q) \rightarrow \mathcal{V}^0_{2,\beta}(Q, \partial Q) = W^0_{2,\beta}(Q) \times W^{3/2}_{2,\beta}(\partial Q),
$$

associated with problem (2.1), (2.2) by the formula

$$
Lu = (Au, u|_{\partial Q} + B^1u|_{\partial Q}).
$$

Using the Fourier transform with respect to $\tau$, from (2.1), (2.2) we obtain

$$
\hat{A}(\lambda)\hat{u} = A(x, \partial_x, i\lambda)\hat{u}(x, \lambda) = \hat{f}_0(x, \lambda) \quad (x \in G).
$$

$$
\hat{u}(x, \lambda) + b(x)e^{i\lambda \chi}\hat{u}(\omega(x), \lambda) = \hat{f}_1(x, \lambda) \quad (x \in \partial G).
$$

Here $\hat{u}(x, \lambda) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} u(x, \tau)e^{-i\lambda \tau}d\tau$ is the Fourier transform of $u(x, \tau)$ with respect to $\tau$.

We now define an operator-valued function $\mathbb{C} \ni \lambda \mapsto \hat{L}(\lambda)$ corresponding to problem (3.3), (3.4). The values of this function are linear bounded operators $\hat{L}(\lambda): W^2_{2}(G) \rightarrow \mathcal{V}^0_{2}(G, \partial G) = L_2(G) \times W^{3/2}_{2}(\partial G)$ given by

$$
\hat{L}(\lambda)v = (\hat{A}(\lambda)v, v|_{\partial G} + e^{i\lambda \chi}\hat{B}^1v|_{\partial G}).
$$

Here $\hat{B}^1: W^2_{2}(G) \rightarrow W^2_{2}(G)$, $j = 0, 1, 2$, is a linear bounded operator having the form

$$
(\hat{B}^1v)(x) = b(x)\eta(\omega(x))v(\omega(x)) \quad (x \in \Omega_0 \cap G),
$$

$$
(\hat{B}^1v)(x) = 0 \quad (x \in G \setminus \Omega_0).
$$

We also introduce an operator-valued function $\mathbb{C} \ni \lambda \mapsto \hat{L}_\sigma(\lambda), 0 \leq \sigma \leq 1$. The values of this function are linear bounded operators $\hat{L}_\sigma(\lambda): W^2_{2}(G) \rightarrow \mathcal{V}^0_{2}(G, \partial G)$ of the form

$$
\hat{L}_\sigma(\lambda)v = (\hat{A}(\lambda)v, v|_{\partial G} + \sigma e^{i\lambda \chi}\hat{B}^1v|_{\partial G}).
$$

Clearly, the operators $\hat{L}_0(\lambda)$ and $\hat{L}_1(\lambda) = \hat{L}(\lambda)$ correspond to the Dirichlet problem for equation (2.1) and to nonlocal problem (2.1), (2.2), respectively.

In the Hilbert spaces $W^2_{\alpha}(\Omega)$ ($\alpha \geq 0$, $G = \Omega$, $\partial G$) and $\mathcal{V}^0_{2}(G, \partial G)$ we introduce the equivalent norms depending on $\lambda$:

$$
|||v|||_{W^2_{2}(\Omega)} = \left\{ ||v||^2_{W^2_{2}(\Omega)} + |\lambda|^{2\alpha}||v||^2_{L_2(\Omega)} \right\}^{1/2},
$$

$$
|||f|||_{W^2_{2}(\Omega, \partial G)} = \left\{ ||f_0||^2_{L_2(\Omega)} + ||f_1||^2_{W^{3/2}_{2}(\partial G)} \right\}^{1/2},
$$

where $f = (f_0, f_1)$. 

NONLOCAL ELLIPTIC PROBLEMS IN INFINITE CYLINDER 851
Lemma 3.1. Let Condition 2.1 hold. Then for any $h \in \mathbb{R}$ there exists $\lambda_1 > 1$ such that for all $\lambda \in S_{h,\lambda_1} = \{ \lambda \in \mathbb{C}: \Im \lambda = h, |\Re \lambda| \geq \lambda_1 \}$, $0 \leq \sigma \leq 1$, and $v \in W^2_2(G)$ the following estimate is fulfilled:

$$c_1 \| \hat{L}_\sigma(\lambda)v \|_{W^2_2(G,\partial G)} \leq \| v \|_{W^2_2(G)} \leq c_2 \| \hat{L}_\sigma(\lambda)v \|_{W^2_2(G,\partial G)},$$

(3.10)

where $c_1, c_2 > 0$ do not depend on $\lambda$, $\sigma$, and $v$.

Proof. We denote $\hat{L}_\sigma(\lambda)v = f$, where $f = (f_0, f_1)$. Then

$$\hat{L}_0(\lambda)v = f + \Phi,$$

(3.11)

where $\Phi = (0, -\sigma e^{i\lambda x} \hat{B} v |_{\partial G})$.

By virtue of Theorem 4.1 from [1, Chapter 1] there are $\varepsilon, 0 < \varepsilon < \pi/2$, and $\lambda_0 > 1$ such that for $\lambda \in \omega_{\varepsilon, \lambda_0}$ a solution of “local” problem (3.11) satisfies the inequality

$$\| v \|_{W^2_2(G)} \leq k_1 \| f + \Phi \|_{W^2_2(G,\partial G)},$$

(3.12)

where $\omega_{\varepsilon, \lambda_0} = \{ \lambda \in \omega_{\varepsilon}: |\lambda| \geq \lambda_0 \}$, $\omega_{\varepsilon} = \{ \lambda \in \mathbb{C}: |\arg \lambda| \leq \varepsilon \} \cup \{ \lambda \in \mathbb{C}: |\arg \lambda - \pi| \leq \varepsilon \}$, $k_1 > 0$ does not depend on $\lambda$, $\sigma$, and $v$.

Later on the following interpolational inequalities will be needed:

$$|\lambda|^{k-s} \| u \|_{W^2_2(G)} \leq k_2 \left( \| u \|_{W^2_2(G)} + |\lambda|^k \| u \|_{L^2_2(G)} \right)$$

(3.13)

for any $u \in W^2_2(G)$ and $\lambda \in \mathbb{C}$, where $k, s \in \mathbb{N}$, $0 < s < k$, and $k_2 > 0$ does not depend on $u$ and $\lambda$;

$$|\lambda|^{1/2} \| u \|_{L^2_2(\partial G)} \leq k_3 \left( \| u \|_{W^2_2(G)} + |\lambda| \cdot \| u \|_{L^2_2(G)} \right),$$

(3.14)

for any $u \in W^2_2(G)$ and $\lambda \in \mathbb{C}$, where $k_3 > 0$ does not depend on $u$ and $\lambda$.

For a proof, see [1].

From formula (3.1) it follows that for any $u \in W^2_2(G)$

$$\| \hat{B}^1 u \|_{W^2_2(G)} \leq k_4 \| u \|_{W^2_2(G, \omega)} \quad (k = 0, 2),$$

(3.15)

where $k_4 > 0$ does not depend on $u$, $G_\omega = \{ x \in G: \text{dist}(x, \partial G) > \omega \}$.

By virtue of (3.8), (3.14), and (3.13), we have

$$I(\lambda) = \| \sigma e^{i\lambda x} \hat{B}^1 v |_{\partial G} \|_{W^{2,2}_2(\partial G)} \leq k_5 e^{-h\varepsilon} \left( \| \hat{B}^1 v \|_{W^2_2(G)} + |\lambda| (\| \hat{B}^1 v \|_{W^2_2(G)} + |\lambda| \cdot \| \hat{B}^1 v \|_{L^2_2(G)}) \right) \leq k_6 e^{-h\varepsilon} \left( \| \hat{B}^1 v \|_{W^2_2(G)} + |\lambda|^2 \| \hat{B}^1 v \|_{L^2_2(G)} \right) \leq k_7 e^{-h\varepsilon} \| v \|_{W^2_2(G, \omega)},$$

(3.16)

where $k_5, k_6, k_7 > 0$ do not depend on $v$, $\sigma$ and $\lambda$.

We introduce a function $\xi \in C_0^\infty(\mathbb{R}^n)$ such that $\xi(x) = 1$ ($x \in G_\omega$), $\xi(x) = 0$ ($x \notin G_{\omega/2}$), $0 \leq \xi(x) \leq 1$ ($x \in \mathbb{R}^n$). From inequality (3.16), Theorem 4.1 from [1, Chapter 1], and inequality (3.13) we obtain

$$I(\lambda) \leq k_7 e^{-h\varepsilon} \| \xi v \|_{W^2_2(G)} \leq k_8 e^{-h\varepsilon} \| v \|_{L^2_2(G)} \leq k_9 e^{-h\varepsilon} \left( \| \hat{A}(\lambda)v \|_{L^2_2(G)} + \| v \|_{W^2_2(G)} \right) \leq k_{10} e^{-h\varepsilon} \left( \| \hat{A}(\lambda)v \|_{L^2_2(G)} + |\lambda|^{-1} \| v \|_{W^2_2(G)} \right),$$

(3.17)

Inequalities (3.12) and (3.17) imply that

$$\| v \|_{W^2_2(G)} \leq k_1 (1 + k_{10} e^{-h\varepsilon}) \left( \| \hat{L}_\sigma(\lambda)v \|_{W^2_2(G, \partial G)} + k_1 k_{10} e^{-h\varepsilon} |\lambda|^{-1} \| v \|_{W^2_2(G)} \right),$$

(3.18)
where \(k_8, k_9, k_{10} > 0\) do not depend on \(v\) and \(\lambda\).

Clearly, there exists a \(\lambda_2, \lambda_2 \geq \lambda_0\), such that \(S_{h, \lambda_2} \subset \omega_{\varepsilon, \lambda_0}\). Choosing \(\lambda_1 > \lambda_2\) so that \(k_1k_10\lambda_1^{-1}e^{-\beta x} < 1/2\), we obtain the right hand side of inequality (3.10) with the constant \(c_2 = 2k_1(1 + k_{10}e^{-\beta x})\) that does not depend on \(v, \sigma, \) and \(\lambda\).

The left hand side of (3.10) follows from (3.16).

\textbf{Definition 3.2.} Let \(H\) and \(\mathcal{H}\) be Hilbert spaces. A linear bounded operator \(K: H \to \mathcal{H}\) is called a Fredholm operator (or is said to possess the Fredholm property) if \(\mathcal{R}(K)\) is closed in \(\mathcal{H}\) and \(\dim \mathcal{N}(K) < \infty, \ \text{codim} \mathcal{R}(K) < \infty\), where \(\mathcal{N}(K)\) and \(\mathcal{R}(K)\) are the kernel and the range of the operator \(K\), respectively. The index of the Fredholm operator is defined by \(\text{ind} K = \dim \mathcal{N}(K) - \text{codim} \mathcal{R}(K)\).

Below we shall prove that the operator \(\hat{L}(\lambda): W^2_2(G) \to W^0_2(G, \partial G)\) is Fredholm and \(\text{ind} \hat{L}(\lambda) = 0\) for all \(\lambda \in \mathbb{C}\). For the proof of this statement it will be needed some auxiliary result concerning Fredholm operators.

Let \(H, H_1,\) and \(H_2\) be Hilbert spaces, and let \(\mathcal{H} = H_1 \times H_2\). Let \(A: H \to H_1\) and \(B^0, B^1: H \to H_2\) be linear bounded operators. We introduce the operators \(L_0, L_1: H \to \mathcal{H}\) by the formulas \(L_0 = \{A, B^0\}\) and \(L = \{A, B\}\), where \(B = B^0 + B^1\). Denote by \(\Phi_i\) and \(\Phi\) the restrictions of the operators \(B^i\) and \(B\) to \(\mathcal{N}(A)\), \(i = 0, 1\).

\textbf{Lemma 3.3.} Let the operator \(L_0: H \to \mathcal{H}\) possess the Fredholm property. Then the operator \(\Phi_0: \mathcal{N}(A) \to H_2\) also possesses the Fredholm property. Moreover, if \(\Phi: \mathcal{N}(A) \to H_2\) is a Fredholm operator, and \(\text{ind} \Phi = \text{ind} \Phi_0\), then \(L: H \to \mathcal{H}\) is a Fredholm operator, and \(\text{ind} L = \text{ind} L_0\).

For a proof, see Lemma 2.2.4 in [24, Chapter 2, Section 2.2]

\textbf{Definition 3.4.} Let \(H\) and \(\mathcal{H}\) be Hilbert spaces. The operator-valued function \(\mathbb{C} \ni \lambda \mapsto R(\lambda)\) is said to be finitely meromorphic at \(\lambda_0 \in \mathbb{C}\), if for some \(\varepsilon > 0\), it can be expanded into the Laurent series

\[ R(\lambda) = \sum_{j=-r}^{\infty} R_j(\lambda - \lambda_0)^j \quad (\lambda \in \mathbb{C}: 0 < |\lambda - \lambda_0| < \varepsilon) \]

converging with respect to operator norm, where \(R(\lambda), R_j: H \to \mathcal{H}, j = -r, -r + 1, \ldots,\) are linear bounded operators, and the operators \(R_k, -r \leq k < 0,\) are finite dimensional.

\textbf{Theorem 3.5.} Let Condition 2.1 hold. Then

(a) The operator \(\hat{L}(\lambda): W^2_2(G) \to W^0_2(G, \partial G)\) is Fredholm, and \(\text{ind} \hat{L}(\lambda) = 0\) for all \(\lambda \in \mathbb{C}\).

(b) For every \(h \in \mathbb{R}\), there exists \(\lambda_1 > 1\) such that for \(\lambda \in S_{h, \lambda_1}\), the operator \(\hat{L}(\lambda)\) has a bounded inverse \(\hat{R}(\lambda) = \hat{L}^{-1}(\lambda): W^0_2(G, \partial G) \to W^2_2(G)\) in the norm \(\|\cdot\|\).

(c) The operator-valued function \(\lambda \mapsto \hat{R}(\lambda)\) is finitely meromorphic on \(\mathbb{C}\).

(d) There exists \(\gamma > 1\) such that the set \(\tilde{\omega}_{\delta, \gamma} = \{\lambda \in \mathbb{C}: |\text{Im} \lambda| \leq \delta |\text{Re} \lambda|, |\lambda| \geq \gamma\}\) does not contain eigenvalues of operator-valued function \(\lambda \mapsto \hat{L}(\lambda),\) where \(0 < \delta < |\lambda|^{-1}\).

\textit{Proof.} 1. We prove statement (b). From Theorem 5.1 in [1, Chapter 1] and Lemma 3.1 it follows that, for \(\lambda \in S_{h, \lambda_1} \subset \omega_{\varepsilon, \lambda_0}\), there exists a linear bounded inverse
operator \( \hat{L}_0^{-1}(\lambda) \) in the norm \( \| \cdot \| \). Moreover, for all \( \lambda \in S_{\delta,\gamma_1} \), \( v \in W^2_2(G) \), and \( 0 \leq \sigma \leq 1 \), estimate (3.10) is fulfilled. Hence, using the well-known method of continuation with respect to a parameter, we conclude that there is a bounded inverse operator \( \hat{L}^{-1}(\lambda) \) in the norm \( \| \cdot \| \) for all \( \lambda \in S_{\delta,\gamma_1} \).

2. Now we can prove statement (a). Let \( \lambda \in \mathbb{C} \) be an arbitrary fixed number. Let \( H = W^2_2(G) \), \( H_1 = L_2(G) \), \( H_2 = W^{3/2}_{2}(\partial G) \), \( \mathcal{H} = W^0_2(G, \partial G) \), \( A_0 = \hat{A}(\lambda)v \), \( B^0_1v = e^{i\lambda x}B^1v|_{\partial G} \), \( L_0(\lambda)v = \hat{L}_0(\lambda)v = \{ \hat{A}(\lambda)v, v|_{\partial G} \} \), and \( L_0v = \hat{L}(\lambda)v = \{ \hat{A}(\lambda)v, v|_{\partial G} + e^{i\lambda x}B^1v|_{\partial G} \} \) (\( v \in W^2_2(G) \)). By virtue of Theorem 5.1 in [1, Chapter 1], the operator \( L_0 : W^2_2(G) \to W^1_2(G, \partial G) \) is Fredholm, and \( \text{ind} L_0 = 0 \). Then from Lemma 3.3 it follows that the operator \( \Phi_0 : \mathcal{N}(A) \to W^{3/2}_2(\partial G) \) is also Fredholm, and \( \text{ind} \Phi_0 = 0 \), where \( \Phi_1 \) and \( \Phi \) are restrictions of the operators \( B^1 \) and \( B \) to \( \mathcal{N}(A) \), \( i = 0, 1 \). On the other hand, by virtue of (3.17) we have

\[
\| \Phi_1 v \|_{H_2} = \| B^1 v \|_{W^{3/2}_2(\partial G)} \leq k_9 e^{-h_1} \| v \|_{W^1_2(G)} \quad (v \in \mathcal{N}(A)).
\]

Hence from the compactness of the embedding of \( W^2_2(G) \) into \( W^1_2(G) \) it follows that the operator \( \Phi_1 : \mathcal{N}(A) \to W^{3/2}_2(\partial G) \) is compact. By virtue of theorem concerning compact perturbations of Fredholm operators, the operator \( \Phi : \mathcal{N}(A) \to W^{3/2}_2(\partial G) \) is also Fredholm, and \( \text{ind} \Phi = \text{ind} \Phi_0 = 0 \). Therefore, by Lemma 3.3, the operator \( \hat{L}(\lambda) : W^2_2(G) \to W^1_2(G, \partial G) \) is Fredholm, and \( \text{ind} \hat{L}(\lambda) = 1 \).

3. Statement (c) follows from statements (a), (b) and Theorem 1 in [6].

4. Finally we shall prove statement (d). We choose \( \gamma_1 > 1 \) so that \( \omega_{\delta,\gamma_1} \subset \omega_{\epsilon,\gamma_1} \). For \( \lambda \in \omega_{\delta,\gamma_1} \), we have

\[
e^{\delta|\lambda| \ln|\text{Re} \lambda|} \leq |\lambda\delta| |\lambda|.
\]

From inequality (3.18) with \( \sigma = 1 \) and (3.19) it follows that

\[
\| v \|_{W^2_2(G)} \leq k_1(1 + k_{10} e^{-h_1}) \| \hat{L}(\lambda)v \|_{W^1_2(G, \partial G)} + k_1 k_{10} |\lambda| \delta - 1 \| v \|_{W^2_2(G)}.
\]

Choosing \( \gamma > \gamma_1 \) such that \( k_1 k_{10} \delta \| |\lambda| - 1 < 1/2 \), we obtain

\[
\| v \|_{W^2_2(G)} \leq 2k_1(1 + k_{10} e^{-h_1}) \| \hat{L}(\lambda)v \|_{W^1_2(G, \partial G)}
\]

for \( \lambda \in \omega_{\delta,\gamma} \). Hence the set \( \omega_{\delta,\gamma} \) does not contain eigenvalues of the operator-valued function \( \lambda \mapsto \hat{L}(\lambda) \).

\[\square\]

3.2. Using Theorem 3.5, we can obtain a statement concerning a unique solvability of problem (2.1), (2.2) in weighted spaces. A proof is realized by standard methods, see Theorem 1.1 in [14, Section 1] and Theorem 1.1 in [19, Chapter 3, Section 1]. However, the presence of nonlocal term leads to some additional arguments. Therefore, for a convenience of reader, we present a full proof.

**Theorem 3.6.** Let Condition 2.1 hold. Then the operator \( L : W^2_{2,\beta}(Q) \to W^0_{2,\beta}(Q, \partial Q) \) is an isomorphism if and only if the line \( \{ \lambda \in \mathbb{C} : \text{Im} \lambda = \beta \} \) does not contain eigenvalues of the operator-valued function \( \lambda \mapsto \hat{L}(\lambda) \).

**Proof.** 1. Let the line \( \{ \lambda \in \mathbb{C} : \text{Im} \lambda = \beta \} \) does not contain eigenvalues of the operator-valued function \( \lambda \mapsto \hat{L}(\lambda) \). Then, by virtue of Lemma 3.1 and Theorem 3.5, for all \( \lambda \in \{ \lambda \in \mathbb{C} : \text{Im} \lambda = \beta \} \) and \( f = (\hat{f}_0, \hat{f}_1) \in W^0_2(G, \partial G) \) there exists a unique solution of problem (3.3), (3.4) and

\[
\| \hat{u} \|_{W^2_2(G)} \leq k_1 \| \hat{f} \|_{W^0_2(G, \partial G)},
\]

(3.21)
where $k_1 > 0$ does not depend on $\hat{f}$.

Hence from the complex analog of the Parseval equality

$$
\int_{-\infty}^{+\infty} e^{2\beta\tau}|w(\tau)|^2d\tau = \int_{-\infty}^{+\infty+i\beta} |\hat{w}(\lambda)|^2d\lambda
$$

and interpolational inequalities (3.13), (3.14) it follows that, for every vector-valued function $f = (f_0, f_1) \in W^0_2(Q, \partial Q)$, there exists a unique solution $u \in W^2_{2, \beta}(Q)$ of problem (2.1), (2.2) and inequality (3.22) holds.

where $k_2 > 0$ does not depend on $f$.

We have proved that the operator $L: W^2_{2, \beta}(Q) \to W^0_2(Q, \partial Q)$ is an isomorphism.

2. Now we assume that the operator-valued function $\lambda \mapsto \hat{L}(\lambda)$ has an eigenvalue $\lambda_0$ on the line $\{\lambda \in \mathbb{C}: \text{Im} \lambda = \beta\}$. Then we prove that the operator $L: W^2_{2, \beta}(Q) \to W^0_2(Q, \partial Q)$ is not an isomorphism. Assume to the contrary that the operator $L$ is an isomorphism. Hence for any $f \in W^0_2(Q, \partial Q)$ there exists a unique solution $u \in W^2_{2, \beta}(Q)$ of problem (3.1), (3.2) and inequality (3.22) holds.

Let $\varphi_0(x)$ be an eigenfunction of the operator-valued function $\lambda \mapsto \hat{L}(\lambda)$ corresponding to the eigenvalue $\lambda_0$. We introduce a truncation function $\zeta \in C^\infty(\mathbb{R})$ such that $0 \leq \zeta(x) \leq 1$ ($x \in \mathbb{R}$), $\zeta(x) = 1$ ($x \leq 0$), $\zeta(x) = 0$ ($x \geq 1$). Let $u_N(x, \tau) = \zeta(|\tau| - N)e^{i\lambda_0\tau}\varphi_0(\tau)$. We shall prove that inequality (3.22) is violated for $f = Lu_N$, $u = u_N$.

Since $\hat{L}(\lambda_0)\varphi_0 = 0$, we have

$$
Lu_N = \zeta(|\tau| - N)e^{i\lambda_0\tau}\hat{L}(\lambda_0)\varphi_0(x) + \left[L, \zeta(|\tau| - N)\right](e^{i\lambda_0\tau}\varphi_0(x))
$$

$$
= \left[L, \zeta(|\tau| - N)\right](e^{i\lambda_0\tau}\varphi_0(x)).
$$

Hence supp $Au_N \subset \hat{G} \times \{[-N - 1, -N] \cup [N, N + 1]\}$. Therefore, since $|e^{i\lambda_0\tau}| = e^{-\beta\tau}$, we obtain

$$
\|Au_N\|_{W^2_{2, \beta}(G)} \leq k_3\|\varphi_0\|_{W^2_2(G)} \leq k_4.
$$

In addition, it is clear that supp $(u_N|_{\partial Q} + B^1u_N|_{\partial Q}) \subset \partial G \times \text{supp}(\zeta(|\tau| + \chi|\tau| - N) - \zeta(|\tau| - N))$. Therefore we have

$$
supp (u_N|_{\partial Q} + B^1u_N|_{\partial Q}) \subset \partial G \times \{[-N - 1, -N - \chi] \cup [N, N + 1]\} \quad \text{if} \quad \chi \geq 0,
$$

$$
supp (u_N|_{\partial Q} + B^1u_N|_{\partial Q}) \subset \partial G \times \{[-N - 1, -N - \chi] \cup [N, N + 1 - \chi]\} \quad \text{if} \quad \chi < 0.
$$

Hence the relation $|e^{i\lambda_0\tau}| = e^{-\beta\tau}$ implies that

$$
\|u_N|_{\partial Q} + B^1u_N|_{\partial Q}\|_{W^{3/2}_{2, \beta}(\partial Q)} \leq k_5\|\varphi_0\|_{W^2_2(G)}^2 \leq k_6.
$$

Here constants $k_3, \ldots, k_6 > 0$ do not depend on $N$. Thus for $u = u_N$, the right hand side of (3.22) is bounded by a constant that does not depend on $N$.

On the other hand, using again the relation $|e^{i\lambda_0\tau}| = e^{-\beta\tau}$, we have

$$
\|u_N\|_{W^2_{2, \beta}(Q)} \geq \int_N dt \int_G |\varphi_0(x)|^2dx \to +\infty \quad \text{as} \quad N \to \infty.
$$
We obtained a contradiction with inequality (3.22). Thus, if the operator $L: W^2_{2,\beta}(Q) \to \mathcal{V}_{2,\beta}^0(Q, \partial Q)$ is an isomorphism, then the line $\text{Im } \lambda = \beta$ does not contain eigenvalues of the operator-valued function $\lambda \mapsto \hat{L}(\lambda)$. \hfill \Box

**Remark 3.7.** Using part 2 of the proof of Theorem 3.6, one can show that, if the line $\{ \lambda \in \mathbb{C} : \text{Im } \lambda = \beta \}$ contains eigenvalues of the operator-valued function $\lambda \mapsto \hat{L}(\lambda)$, then the range of the operator $L$ is not closed in $\mathcal{V}_{2,\beta}^0(Q, \partial Q)$.

Further for investigation of solvability of problem (3.1), (3.2) in Hölder spaces we shall assume that the operator-valued function $\lambda \mapsto \hat{L}(\lambda)$ has no real eigenvalues. Let us demonstrate some examples, in which the realization of this assumption will be analyzed.

**Example 3.8.** We consider the nonlocal elliptic problem in a strip

$$\begin{align*}
- u_{xx} - u_{\tau \tau} &= f_0(x, \tau) \quad ((x, \tau) \in Q = (0, d) \times \mathbb{R}), \\
u(x, \tau)|_{x=0} - b_1 u \left( x + \frac{d}{2}, \tau \right) |_{x=0} &= f_1(\tau) \quad (\tau \in \mathbb{R}), \\
u(x, \tau)|_{x=d} - b_2 u \left( x - \frac{d}{2}, \tau \right) |_{x=d} &= f_2(\tau) \quad (\tau \in \mathbb{R}),
\end{align*}$$

(3.24) (3.25)

where $b_1, b_2 \in \mathbb{R}$. Nonlocal boundary conditions (3.25) mean that the values of unknown function $u(x, \tau)$ on the lines $\gamma_1 = \{(x, \tau) : x = 0\}$ and $\gamma_2 = \{(x, \tau) : x = d\}$ are connected with the values of $u(x, \tau)$ on the line $\{(x, \tau) : x = d/2\}$.

We define the operator $L: W^2_{2,\beta}(Q) \to \mathcal{V}_{2,\beta}^0(Q, \gamma) = W^0_{2,\beta}(Q) \times \prod_{j=1,2} W^{3/2}_{2,\beta}(\gamma_j)$ by the formula

$$L u = \left( -\Delta u, u(x, \tau)|_{x=0} - b_1 u \left( x + \frac{d}{2}, \tau \right) |_{x=0}, u(x, \tau)|_{x=d} - b_2 u \left( x - \frac{d}{2}, \tau \right) |_{x=d} \right).$$

The operator $\hat{L}(\lambda): W^2_{2}(0, d) \to \mathcal{V}_{2}^0[0, d] = L_2(0, d) \times \mathbb{C}^2$ is given by

$$\hat{L}(\lambda) v = \left( -v_{xx} + \lambda^2 v, v|_{x=0} - b_1 v|_{x=d}, v|_{x=d} - b_2 v|_{x=d} \right).$$

In [25, Section 4.1] it was shown that the operator-valued function $\lambda \mapsto \hat{L}(\lambda)$ has no real eigenvalues if and only if $b_1 + b_2 < 2$. In particular, this is true if $b_1 = b_2 = 0$. Therefore the operator-valued function $\lambda \mapsto \hat{L}_0(\lambda)$ has no real eigenvalues.

**Example 3.9.** We consider nonlocal elliptic boundary value problem (2.1), (2.2) assuming that

$$A(x, \partial_x, \partial_\tau) = A_0(x, \partial_x) - \frac{\partial^2}{\partial \tau^2},$$

(3.26)

where

$$A_0(x, \partial_x) = -\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} + a_0(x),$$

$$\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j > 0 \quad (0 \neq \xi \in \mathbb{R}^n, \ x \in \bar{G})$$

and

$$a_0(x) \geq 0 \quad (x \in \bar{G}), \quad a_{ij}, a_i, a_0 \in C^\infty(\mathbb{R}^n)$$

are real-valued functions. We also assume that $\theta < 1$. We shall prove that in this case the operator-valued function $\lambda \mapsto \hat{L}(\lambda)$ has no real eigenvalues.
Assume to the contrary that there exists a $\lambda_0 \in \mathbb{R}$ and a complex-valued function $0 \neq u_0(x) \in W^2_2(G)$ such that
\begin{align}
A_0(x, \partial x)u_0(x) + \lambda^2_0 u_0(x) &= 0 \quad (x \in G), \quad (3.27) \\
u_0(x) + b(x)e^{i\lambda_0 x}u_0(\omega(x)) &= 0 \quad (x \in \partial G). \quad (3.28)
\end{align}

By virtue of the theorem on interior smoothness of eigenfunctions of elliptic equations, we have $u_0 \in C^\infty(G)$. On the other hand, nonlocal boundary condition (3.28) implies that $u_0|_{\partial G} \in C^\infty(\partial G)$. Therefore from the theorem on smoothness of generalized solutions of elliptic problems near a boundary and Sobolev imbedding theorem it follows that $u_0 \in C^\infty(G)$.

We write the function $u_0(x)$ in the form $u_0(x) = v_0(x) + iv_0(x)$, where $v_0(x)$ and $w_0(x)$ are real-valued functions. Since equation (3.27) has real coefficients, the functions $v_0(x)$ and $w_0(x)$ are also the solutions of this equation. Denote $A_1 = -A_0(x, \partial x) - \lambda^2_0$, $G_0 = \{x \in G: u_0(x) = 0\}$, and $G_1 = G \setminus G_0$. The set $G_0$ is closed, and the set $G_1 \neq \emptyset$ is open.

Taking into account the equalities $A_1 v_0(x) = 0$ and $A_1 w_0(x) = 0$ ($x \in G$) and collecting similar terms, we obtain
\begin{align}
A_1|u_0| &= \frac{1}{|u_0|^3} \sum_{i,j} a_{ij} \left\{ (v_{0x_i}v_{0x_j} + w_{0x_i}w_{0x_j}) (v_0^2 + w_0^2) \\
&+ (v_0 v_{0x_i} + w_0 w_{0x_i}) (v_0^2 + w_0^2) - (v_0 v_{0x_j} + w_0 w_{0x_j}) (v_0 v_{0x_j} + w_0 w_{0x_j}) \right\} \\
&- \frac{1}{|u_0|^2} \sum_i a_i (v_0 v_{0x_i} + w_0 w_{0x_i}) - (a_0 + \lambda^2_0) |u_0| \\
&= \frac{1}{|u_0|^2} \sum_{i,j} a_{ij} (w_0 v_{0x_i} - v_0 w_{0x_i}) (w_0 v_{0x_j} - v_0 w_{0x_j}) \geq 0 \quad (3.29)
\end{align}

($x \in G_1$).

From (3.29) and weak maximum principle (see Corollary 3.2 in [8, Chapter 3]) it follows that
\begin{align}
sup_{x \in G_1} |u_0(x)| \leq \sup_{x \in \partial G_1} |u_0(x)|.
\end{align}

Hence
\begin{align}
sup_{x \in G_1} |u_0(x)| \leq |u_0(x^0)|, \quad (3.30)
\end{align}

where $|u_0(x^0)| = \sup_{x \in \partial G} |u_0(x)|$, $x_0 \in \partial G$.

On the other hand, by virtue of inequality (3.30) and condition $\theta < 1$, we have
\begin{align}
|b(x)e^{i\lambda_0 x}u_0(\omega(x^0))| \leq \theta |u_0(\omega(x^0))| < |u_0(x^0)|.
\end{align}

This inequality contradicts to nonlocal boundary condition (3.28). Thus we have proved that the operator-valued function $\lambda \mapsto \hat{L}(\lambda)$ has no real eigenvalues. Assuming that $\theta = 0$, we obtain the Dirichlet problem for equation (2.1). Therefore the operator-valued function $\lambda \mapsto \hat{L}(\lambda)$ has no real eigenvalues.

In Example 3.8 it was shown that in the case when $n = 1$ violation of condition $\theta < 1$ can lead to the appearance of real eigenvalues for the operator-valued function $\lambda \mapsto \hat{L}(\lambda)$. It is easy to make sure that this phenomenon takes place also for $n > 1$.

**Example 3.10.** We consider nonlocal elliptic boundary value problem (2.1), (2.2), assuming that the operator $A(x, \partial x, \partial_r)$ has form (3.26), $a_0(x) \equiv 0$ ($x \in \hat{G}$), and $b(x) \equiv -1$. Then the function $u_0(x) \equiv 1$ is the eigenfunction of the operator-valued
function \( \lambda \mapsto \hat{L}(\lambda) \), corresponding to the eigenvalue \( \lambda_0 = 0 \). Thus, under the assumptions of this example, problem (2.1), (2.2) has properties that are similar to properties of the Neumann problem for equation (2.1).

4. Some auxiliary results for the Dirichlet problem.

4.1. We consider the equation

\[
Au = A(x, \partial_x, \partial_{\tau})u_0(x, \tau) = f_0(x, \tau) \quad ((x, \tau) \in Q)
\]

with the Dirichlet boundary condition

\[
u_0(x, \tau) = 0 \quad ((x, \tau) \in \partial Q).
\]

Here \( Q = G \times \mathbb{R} \), \( G \subset \mathbb{R}^n \) is a bounded domain with boundary \( \partial G \in C^\infty \); the operator \( A(x, \partial_x, \partial_{\tau}) \) is defined in Section 2 and satisfies Conditions 2.1, 2.2.

Let \( C_0(\bar{Q}) = \{ w \in C(\bar{Q}) : w(x, \tau) \to 0 \text{ as } |\tau| \to \infty \text{ uniformly with respect to } x \in G \} \). Clearly, \( C_0(\bar{Q}) \subset C_0(Q) \).

**Lemma 4.1.** Let Conditions 2.1, 2.2 hold. Then, for any function \( f_0 \in C(\bar{Q}) \) such that there exists a solution \( u_0 \in C^2(Q) \cap C_0(Q) \) of problem (4.1), (4.2), the following estimate is fulfilled

\[
||u_0||_{C(\bar{Q})} \leq c_1||f_0||_{C(\bar{Q})},
\]

where \( c_1 > 0 \) does not depend on \( f_0 \).

**Proof.** By virtue of the weak maximum principle, for \( f_0 \equiv 0 \ (x \in \bar{Q}) \), there is a unique trivial solution of problem (4.1), (4.2) in \( C^2(Q) \cap C_0(Q) \). Clearly, this solution satisfies inequality (4.3). Therefore we can assume that \( f_0(x) \neq 0 \ (x \in \bar{Q}) \).

We consider the auxiliary problem

\[
-A(x, \partial_x, 0)\psi(x) = -1 \quad (x \in G),
\]

\[
\psi(x) = 0 \quad (x \in \partial G).
\]

Problem (4.4), (4.5) has a unique solution \( \psi \in C^\infty(\bar{G}) \). From the maximum principle it follows that \( \psi(x) \geq 0 \ (x \in \bar{G}) \).

Clearly, the function \( v(x, \tau) = a\psi(x) \) is a solution of the problem

\[
-A(x, \partial_x, \partial_{\tau})v(x, \tau) = -a \quad (x \in Q),
\]

\[
v(x, \tau) = 0 \quad ((x, t) \in \partial Q).
\]

where \( a \in \mathbb{R} \).

Let \( a = 2||f_0||_{C(\bar{Q})} \). We shall prove that \( u_0(x, \tau) \leq a\psi(x) \) for all \((x, \tau) \in \bar{Q}\).

Assume to the contrary: there exists \((x^0, \tau^0) \in Q\) such that \( u_0(x^0, \tau^0) > v(x^0, \tau^0) \).

From the relation \( u_0 \in C_0(\bar{Q}) \) it follows the existence of a number \( N > 0 \) such that \( u_0(x, \tau) < u_0(x^0, \tau^0) - v(x^0, \tau^0) \) for \( |\tau| \geq N \), \( x \in \bar{G} \), and \((x^0, \tau^0) \in G \times (-N, N) \). Since \( v(x, \tau) \geq 0 \), then \( u_0(x, \tau) - v(x, \tau) < u_0(x^0, \tau^0) - v(x^0, \tau^0) \) for \( |\tau| \geq N \), \( x \in \bar{G} \). We denote \( Q_{N,f_0} = \{(x, \tau) \in Q : |\tau| < N, u_0(x, \tau) > v(x, \tau)\} \) and \( w(x, \tau) = v(x, \tau) - u_0(x, \tau) \). By construction, \((x^0, \tau^0) \in Q_{N,f_0}\). Therefore \( Q_{N,f_0} \neq \emptyset \). Clearly, \( w(x, \tau) = 0 \) for \((x, \tau) \in \partial G \times [-N, N] \) and \( w(x, \tau) > v(x^0, \tau^0) \) for \((x, \tau) \in (G \times \{-N\}) \cup \{G \times \{N\}\} \). Since \( w(x^0, \tau^0) < 0 \), the function \( w(x, \tau) \) takes a negative minimum on \( Q_{N,f_0} \) at the point \((x^1, \tau^1) \in Q_{N,f_0} \). Therefore, from the maximum principle it follows that \( -A(x^1, \partial_x, \partial_{\tau})w(x^1, \tau^1) \geq 0 \). On the other hand, since \( a > ||f_0||_{C(\bar{Q})} \), we obtain \( -A(x^1, \partial_x, \partial_{\tau})w(x^1, \tau^1) < 0 \). This contradiction proves that \( u_0(x, \tau) \leq a\psi(x) \). Similarly one can show that \( u_0(x, \tau) \geq -a\psi(x) \). Thus \( |u_0(x, \tau)| \leq a\psi(x) \). Hence we obtain inequality (4.3). \(\square\)
4.2. We now prove that problem (4.1), (4.2) has a unique solution in the space $C^{2+\sigma}_0(Q)$ for any $f_0 \in C^\sigma_0(Q)$.

**Lemma 4.2.** Let Conditions 2.1, 2.2 hold, and let the operator-valued function $\lambda \mapsto \hat{L}_0(\lambda)$ has no real eigenvalues. Then for any $f_0 \in C^\sigma_0(Q)$, there exists a unique solution of problem (4.1), (4.2) $u_0 \in C^{2+\sigma}_0(Q)$, and

$$
\|u_0\|_{C^{2+\sigma}_0(Q)} \leq c_2 \|f_0\|_{C^\sigma_0(Q)},
$$

(4.8)

where $c_2 > 0$ does not depend on $f_0$.

**Proof.** 1. First, we shall assume that a function $f_0 \in C^\sigma_0(\hat{Q})$ has a compact support.

We prove that if $u_0 \in C^2(\hat{Q}) \cap C_0(\hat{Q})$ is a solution of problem (4.1), (4.2) then $u_0 \in C^{2+\sigma}(\hat{Q})$ and estimate (4.8) holds.

Let $Q_N^c = \{(x, \tau) \in \hat{Q} : |\tau| < N\}$. By virtue of Lemma 6.18 in [8, Chapter 6], we have $u \in C^{2+\sigma}(\bar{Q}_N^c)$ for any $N > 0$. Hence from Corollary 6.7 in [8, Chapter 6] it follows that for any $N > 0$ the following inequality is fulfilled

$$
\|u_0\|_{C^{2+\sigma}(\bar{Q}_N^c)} \leq k_1 (\|u_0\|_{C(\bar{Q}_{N+1}^c)} + \|f_0\|_{C^\sigma(\bar{Q}_{N+1}^c)}),
$$

(4.9)

where $k_1 > 0$ does not depend on $N$ and $f_0$. From (4.9) and (4.3) it follows that

$$
\|u_0\|_{C^{2+\sigma}(\bar{Q}_N^c)} \leq k_1 (\|u_0\|_{C(\bar{Q})} + \|f_0\|_{C^\sigma(\bar{Q})}) \leq k_2 \|f_0\|_{C^\sigma(\bar{Q})},
$$

(4.10)

where $k_2 > 0$ does not depend on $N$ and $f_0$.

2. We now prove that, for every function $f_0 \in C^\sigma(\hat{Q})$ with compact support there exists a solution $u_0 \in C^2(\hat{Q}) \cap C_0(\hat{Q})$ of problem (4.1), (4.2).

By assumption, the operator-valued function $\lambda \mapsto \hat{L}_0(\lambda)$ has no real eigenvalues. Hence from Theorem 3.4 in [19, Chapter 3, Section 6] it follows that for any $f_0 \in W_{p,0}^n(\hat{Q}) = L_p(\hat{Q})$ there is a unique solution of problem (4.1), (4.2) $u_0 \in W_p^2(\hat{Q}) = W_p^2(\hat{Q})$, where $p > n/2$. In particular, this is true for any compactly supported function $f_0 \in C^\sigma(\hat{Q})$. Hence, by virtue of Theorem 9.19 in [8, Section 9], $u_0 \in C^{2+\sigma}(\bar{Q}_N^c)$ for every $N > 0$. In addition, the relation $u_0 \in W^2_p(\hat{Q})$ implies that $\|u_0\|_{W^2_p(\hat{Q}\backslash \bar{Q}_N^c)} \to 0$ as $N \to \infty$. From Sobolev embedding theorem it follows that $u \in C(\bar{Q})$ and

$$
\|u_0\|_{C(\bar{Q}\backslash \bar{Q}_N^c)} \to 0 \quad \text{as} \quad N \to \infty,
$$

(4.11)

i.e., $u_0 \in C_0(\hat{Q})$. Thus we have proved that, for any compactly supported function $f_0 \in C^\sigma(\hat{Q})$, there exits a solution $u_0 \in C^2(\hat{Q}) \cap C_0(\hat{Q})$ of problem (4.1), (4.2). Hence from part 1 of the proof it follows that, for any function $f_0 \in C^\sigma(\hat{Q})$ with compact support, there exists a unique solution $u_0 \in C^{2+\sigma}(\hat{Q}) \cap C_0(\hat{Q})$ of problem (4.1), (4.2), and estimate (4.8) holds.

3. Now we prove that $u_0 \in C^{2+\sigma}_0(\hat{Q})$. For this, it is sufficient to show that $\xi_N u_0 \to u_0$ in $C^{2+\sigma}_0(\hat{Q})$ as $N \to \infty$. Here $\xi_N = \xi_N(\cdot) \in C^\infty(\mathbb{R})$ is an even function, $0 \leq \xi_N(\tau) \leq 1$ for $\tau \in \mathbb{R}$, $\xi_N(\tau) = 1$ for $|\tau| \leq N$, $\xi_N(\tau) = 0$ for $|\tau| \geq N + 1$, and $|\xi_N^{(i)}(\tau)| \leq k_3$ for $\tau \in \mathbb{R}$ ($i = 1, 2, 3$), $k_3 > 0$ does not depend on $\tau$ and $N$.

Using an estimate of norm for a product of two functions in a Hölder space and inequality similar to (4.9), we obtain

$$
\|(1 - \xi_N)u_0\|_{C^{2+\sigma}(\hat{Q}\backslash \bar{Q}_N^c)} = \|(1 - \xi_N)u_0\|_{C^{2+\sigma}(\hat{Q}\backslash \bar{Q}_N^c)} \leq k_4 \|1 - \xi_N\|_{C^{2+\sigma}(\hat{Q}\backslash \bar{Q}_N^c)}
$$

$$
\times \|u_0\|_{C^{2+\sigma}(\hat{Q}\backslash \bar{Q}_N^c)} \leq k_5 \|u_0\|_{C^{2+\sigma}(\hat{Q}\backslash \bar{Q}_N^c)} \leq k_6 \|u_0\|_{C(\hat{Q}\backslash \bar{Q}_{N-1}^c)}
$$

$$
+ \|f_0\|_{C(\hat{Q}\backslash \bar{Q}_{N-1}^c)} = k_6 \|u_0\|_{C(\hat{Q}\backslash \bar{Q}_{N-1}^c)}.
$$

(4.12)
Here $N$ is such that $f(x, \tau) \equiv 0$ for $(x, \tau) \in \overline{Q} \setminus Q_{N-1}'$, $k_3, k_5, k_6 > 0$ do not depend on $N$ and $f$.

To complete the proof we note that, by definition, the set of compactly supported functions from $C^p(\overline{Q})$ is dense in $C^p_0(\overline{Q})$.

5. Nonlocal elliptic problems in weighted space $W_{p,\beta}^2(Q)$.

5.1. We introduce the linear bounded operators $L, L_0 : W_{p,\beta}^2(Q) \to W_{p,\beta}^0(Q, \partial Q) = W_{p,\beta}^0(Q) \times W_{p,\beta}^{2-1/p}(\partial Q)$ by the formulas

$$Lu = (Au, u|_{\partial Q}) + B^1 u|_{\partial Q},$$

(5.1)

$$L_0 u = (Au, u|_{\partial Q}).$$

(5.2)

The operator $L$ corresponds to problem (2.1), (2.2), while the operator $L_0$ is related with problem (4.1), (4.2).

For a proof of theorem on the necessary and sufficient conditions of a unique solvability to problem (2.1), (2.2) in weighted space $W_{p,\beta}^2(Q)$, we need some auxiliary results.

We introduce a partition of unity $\{\chi_k\}$ on $\mathbb{R}$ subordinated to the covering of $\mathbb{R}$ by the intervals $(k - 1, k + 1)$ such that $|\chi_k(i)(\tau)| \leq c_1$ ($i = 1, 2, k \in \mathbb{Z}, \tau \in \mathbb{R}$).

We also define functions $\xi_k \in C^\infty(\mathbb{R})$ so that $0 \leq \xi_k(\tau) \leq 1$ ($\tau \in \mathbb{R}$), $\xi_k(\tau) = 1$ ($\tau \in (k - 3 - |\chi|, k + 3 + |\chi|)$), supp $\xi_k \subset (k - 4 - |\chi|, k + 4 + |\chi|)$, $|\xi_k(i)(\tau)| \leq c_1$ ($i = 1, 2, k \in \mathbb{Z}, \tau \in \mathbb{R}$). Here $c_1 > 0$ does not depend on $\tau$ and $k$.

Lemma 5.1. Let Condition 2.1 hold. Then for any function $u \in W_{p,\beta}^2(Q)$, $p > 1$, we have

$$\|\chi_k u\|_{W_{p}^2(Q)} \leq c_2 (\|\xi_k Lu\|_{W_{p}^0(Q, \partial Q)} + \|\xi_k u\|_{L_2(Q)}),$$

(5.3)

where $c_2 > 0$ does not depend on $u$ and $k$.

Proof. From a priori estimates of solutions of elliptic problems near a boundary and inside domain and estimate of norm for trace of nonlocal term it follows that

$$\|\chi_k u\|_{W_{p}^2(Q)} \leq k_1 (\|\xi_k L_0 u\|_{W_{p}^0(Q, \partial Q)} + \|\xi_k u\|_{L_2(Q)})$$

$$\leq k_1 (\|\xi_k L_0 u\|_{W_{p}^0(Q, \partial Q)} + \|\xi_k B^1 u|_{\partial Q}\|_{W_{p}^{2-1/p}(\partial Q)} + \|\xi_k u\|_{L_2(Q)})$$

$$\leq k_2 (\|\xi_k L_0 u\|_{W_{p}^0(Q, \partial Q)} + \|u\|_{W_{p}^2(D_k)} + \|\xi_k u\|_{L_2(Q)})$$

$$\leq k_3 (\|\xi_k L_0 u\|_{W_{p}^0(Q, \partial Q)} + \|\xi_k L u\|_{L_p(Q)} + \|\xi_k u\|_{L_2(Q)}),$$

where $k_1, k_2, k_3 > 0$, $\xi_k = \chi_{k-1} + \chi_k + \chi_{k+1}$, $D_k = \omega(\Omega_0) \times (-k - 2 - |\chi|, k + 2 + |\chi|)$.

Lemma 5.2. Let Condition 2.1 hold, and let the line $\{\lambda \in \mathbb{C} : \text{Im} \lambda = \beta\}$ do not contain eigenvalues of the operator-function $\lambda \mapsto \tilde{L}(\lambda)$. Then for any function $u \in W_{2,\beta}^2(Q)$ such that supp $Lu \subset (m - 1, m + 1)$ and $Lu \in W_{p,\beta}^0(Q, \partial Q)$ the following estimate takes place

$$\|e^{\beta x} \chi_k u\|_{L_p(Q)} \leq c_3 e^{-|k-m|\beta}\|Lu\|_{W_{p,\beta}^0(Q, \partial Q)},$$

(5.4)

where $c_3, \alpha > 0$ do not depend on $k, m \in \mathbb{Z}$ and $u$; $p \geq 2$.

Proof. By virtue of Theorem 3.5, there exits $\alpha > 0$ such that the strip $\{\lambda \in \mathbb{C} : \text{Im} \lambda \in [\beta - \alpha, \beta + \alpha]\}$ does not contain eigenvalues of the operator-valued function $\lambda \mapsto \tilde{L}(\lambda)$. Therefore from Theorem 3.6 and condition $p \geq 2$ it follows that for all $\gamma \in [\beta - \alpha, \beta + \alpha]$ and $u \in W_{2,\beta}^2(Q)$ we have
Lemma 5.3. For any \( k \), where \( k_1, k_2, k_3 > 0 \) do not depend on \( k, m, \gamma, \) and \( u \).
Let \( |k - m| < 5 + |\chi| \). Then \( e^{-\gamma(m-k)} \leq k_4 \), where \( k_4 \) does not depend on \( k \) and \( m \). Hence from Lemma 5.1 it follows that

\[
\left\{ \int_{k-4-|\chi|}^{k+4+|\chi|} \|e^{\beta \tau} u(\cdot, \tau)\|_{L^2(G)}^2 \, d\tau \right\}^{1/2} \leq k_1 e^{(\beta-\gamma)k} \|L u\|_{W^0_{p, \beta}(Q, \partial Q)}
\]

\[
\leq k_2 e^{(\beta-\gamma)k} \|L u\|_{W^0_{p, \beta}(Q, \partial Q)} \leq k_3 e^{(\beta-\gamma)(k-m)} \|L u\|_{W^0_{p, \beta}(Q, \partial Q)},
\]

where \( k_1, k_2, k_3 > 0 \) do not depend on \( k, m, \gamma, \) and \( u \).

Now let \( |k - m| \geq 5 + |\chi| \). Then \( \mathcal{L}(\cdot, \tau) = 0 \) for \( \tau \in (k - 4 - |\chi|, k + 4 + |\chi|) \). Therefore Lemma 5.1 implies that

\[
\|e^{\beta \tau} \xi_k u\|_{L^p(Q)} \leq k_5 \left( e^{(\beta-\gamma)(k-m)} \|L u\|_{W^0_{p, \beta}(Q, \partial Q)} \right)
\]

\[
+ \left( \int_{k-4-|\chi|}^{k+4+|\chi|} \|e^{\beta \tau} u(\cdot, \tau)\|_{L^2(G)}^2 \, d\tau \right)^{1/2},
\]

(5.6)

where \( k_5 \) does not depend on \( k, m, \gamma, \) and \( u \).

From (5.5)–(5.7) it follows that

\[
\|e^{\beta \tau} \xi_k u\|_{L^p(Q)} \leq k_7 e^{(\beta-\gamma)(k-m)} \|L u\|_{W^0_{p, \beta}(Q, \partial Q)},
\]

(5.8)

where \( k_6, k_7 > 0 \) do not depend on \( k, m, \gamma, \) and \( u \).

Let \( \gamma = \beta + \kappa \), if \( k > m \), and let \( \gamma = \beta - \kappa \), if \( k \leq m \). Then (5.8) implies (5.4).

It is easy to prove the following statement.

Lemma 5.3. For any \( u \in W^0_{p, \beta}(Q) \), \( v \in W^2_{p, \beta}(Q) \), and \( f \in \mathcal{W}^0_{p, \beta}(Q, \partial Q) \), the following inequalities are fulfilled:

\[
c_4 \|u\|_{W^0_{p, \beta}(Q)} \leq \left( \sum_k \|\xi_k u\|^p_{W^0_{p, \beta}(Q)} \right)^{1/p},
\]

(5.9)

\[
\left( \sum_k \|\xi_k u\|^p_{W^0_{p, \beta}(Q)} \right)^{1/p} \leq c_5 \|u\|_{W^0_{p, \beta}(Q)} ,
\]

(5.10)

\[
\|v\|_{W^2_{p, \beta}(Q)} \leq c_6 \left( \sum_k \|\xi_k v\|^p_{W^2_{p, \beta}(Q)} \right)^{1/p},
\]

(5.11)

\[
\left( \sum_k \|\xi_k f\|^p_{W^0_{p, \beta}(Q, \partial Q)} \right)^{1/p} \leq c_7 \|f\|_{W^0_{p, \beta}(Q, \partial Q)},
\]

(5.12)

where \( p \geq 2, c_4, \ldots, c_7 > 0 \) do not depend on \( u, v, \) and \( f \).

Lemma 5.4. Let Condition 2.1 hold, and let the line \( \{ \lambda \in \mathbb{C}: \text{Im} \lambda = \beta \} \) do not contain eigenvalues of the operator-valued function \( \lambda \mapsto \mathcal{L}(\lambda) \). Then, for all \( f \in W^0_{p, \beta}(Q, \partial Q) \) with compact supports, the following estimate takes place:

\[
\|L^{-1} f\|_{W^0_{p, \beta}(Q)} \leq c_8 \|f\|_{W^0_{p, \beta}(Q, \partial Q)} ,
\]

(5.13)
where \( c_8 > 0 \) does not depend on \( f \).

Since the vector-valued function \( f \in W^0_{p,\beta}(Q, \partial Q) \) has a compact support and \( p \geq 2 \), from the Hölder inequality it follows that \( f \in W^2_{p,\beta}(Q, \partial Q) \). Therefore, by Theorem 3.6, for the indicated vector-valued function \( f \), we have \( L^{-1}f \in W^2_{2,\beta}(Q) \). It remains to show that inequality (5.13) holds. A proof of this inequality is similar to the proof of Lemma 6.2 in [19, Chapter 3, Section 6]. It is based on Lemma 5.2, Minkowski inequality, and inequalities (5.9), (5.12).

**Lemma 5.5.** Let Condition 2.1 hold, and let the line \( \{ \lambda \in \mathbb{C} : \text{Im} \lambda = \beta \} \) do not contain eigenvalues of the operator-valued function \( \lambda \mapsto \tilde{L}(\lambda) \). Then, for all \( f \in W^0_{p,\beta}(Q, \partial Q) \) with compact supports, the following inequality takes place:

\[
\|L^{-1}f\|_{W^2_{p,\beta}(Q)} \leq c_0\|f\|_{W^0_{p,\beta}(Q, \partial Q)},
\]

where \( c_0 > 0 \) does not depend on \( f \).

A proof is similar to the proof of Lemma 6.3 in [19, Chapter 3, §6]. It is based on Lemmas 5.1, 5.4 and inequalities (5.10)–(5.12).

**Theorem 5.6.** Let Condition 2.1 hold. The operator \( L: W^2_{p,\beta}(Q) \to W^0_{p,\beta}(Q, \partial Q) \) corresponding to problem (2.1), (2.2) is an isomorphism if and only if the line \( \{ \lambda \in \mathbb{C} : \text{Im} \lambda = \beta \} \) does not contain eigenvalues of the operator-valued function \( \lambda \mapsto \tilde{L}(\lambda) \).

**Proof.** 1. Let the line \( \{ \lambda \in \mathbb{C} : \text{Im} \lambda = \beta \} \) does not contain eigenvalues of the operator-valued function \( \lambda \mapsto \tilde{L}(\lambda) \). Then we prove that the operator \( L: W^2_{p,\beta}(Q) \to W^0_{p,\beta}(Q, \partial Q) \) is an isomorphism. By Theorem 3.6, there exists a bounded inverse operator \( L^{-1}: W^0_{2,\beta}(Q, \partial Q) \to W^2_{2,\beta}(Q) \). Clearly, this operator is defined on the set of infinitely differentiable vector-valued functions with compact supports. Moreover, by virtue of Lemma 5.5, \( L^{-1}f \in W^2_{p,\beta}(Q) \) and estimate (5.14) holds. Since the set of compactly supported infinitely differentiable vector-valued functions is dense in \( W^0_{p,\beta}(Q, \partial Q) \), it follows that the operator \( L \) maps the space \( W^2_{p,\beta}(Q) \) onto \( W^0_{p,\beta}(Q, \partial Q) \).

We prove that the kernel of the operator \( L: W^2_{p,\beta}(Q) \to W^0_{p,\beta}(Q, \partial Q) \) is trivial. Let \( u_0 \in \mathcal{N}(L) \subset W^2_{p,\beta}(Q) \). We introduce a cutoff function \( \zeta \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \zeta(\tau) \leq 1 \) (\( \tau \in \mathbb{R} \)), \( \zeta(\tau) = 1 \) (\( \tau \leq 0 \)), \( \zeta(\tau) = 0 \) (\( \tau \geq 1 \)). We set \( u_N(x, \tau) = \zeta(|\tau| - N)u_0(x, \tau) \), where \( N > |\chi| \). Since \( u_0 \in \mathcal{N}(L) \), then

\[
\supp u_N \subset \tilde{G} \times \{[-N - 1, -N] \cup [N, N + 1]\}. \tag{5.15}
\]

In addition, by virtue of (3.23), we have

\[
\supp (u_N|_{\partial Q} + B_1u_N|_{\partial Q}) \subset \partial G \times \{[-N - 1 - |\chi|, -N + |\chi|] \\
\cup [N - |\chi|, N + 1 + |\chi|]\} \tag{5.16}
\]

From (5.14)–(5.16) and from the Leibniz formula we obtain

\[
\|\zeta(|\tau| - N)u_0\|_{W^2_{p,\beta}(Q)} \leq k_1\|L(\zeta(|\tau| - N)u_0)\|_{W^0_{p,\beta}(Q, \partial Q)} \leq k_2\|u_0\|_{W^2_{p,\beta}(\Omega_N)}, \tag{5.17}
\]

where \( k_1, k_2 > 0 \) do not depend on \( N, \Omega_N = Q_{N+1+|\chi|} \setminus \bar{Q}_{N-|\chi|} \).

The right hand side of (5.17) tends to zero as \( N \to \infty \). Hence \( u_0(x, t) \equiv 0 \) \((x, t) \in Q \).
2. It remains to prove that, if the line \( \{ \lambda \in \mathbb{C} : \text{Im} \lambda = \beta \} \) contains eigenvalues of the operator-valued function \( \lambda \mapsto \hat{L}(\lambda) \), then the operator \( L : W_{p,\beta}^2(Q) \rightarrow \mathcal{V}_{p,\beta}^n(Q, \partial Q) \) is not an isomorphism. A proof of this statement is similar to part 2 in the proof of Theorem 3.6. \( \square \)

6. Nonlocal problems in Hölder spaces.

6.1. For a proof of theorem on a unique solvability of nonlocal elliptic problem in Hölder space, we need two auxiliary statements.

**Lemma 6.1.** Let Conditions 2.1–2.3 hold, and let the operator-valued function \( \lambda \mapsto \hat{L}_0(\lambda) \) has no eigenvalues on the line \( \{ \lambda \in \mathbb{C} : \text{Im} \lambda = \beta \} \). Then, for all \( f_0 \in C_0^\sigma(Q) \) and \( f_1 \in C_0^{2+\sigma}(\partial Q) \) such that there is a solution \( u \in C^2(Q) \cap C_0(Q) \) of problem (2.1), and (2.2), the following estimate holds:

\[
\|u\|_{C(Q)} \leq c_1 (\|f_0\|_{C(Q)} + \|f_1\|_{C(\partial Q)}), \tag{6.1}
\]

where \( c_1 > 0 \) does not depend on \( f_0 \) and \( f_1 \).

**Proof.** By virtue of Lemmas 4.1 and 4.2, for any function \( f_0 \in C_0^\sigma(Q) \), there exists a unique solution of problem (4.1), (4.2) \( u_0 \in C_0^{2+\sigma}(Q) \), and

\[
\|u_0\|_{C(Q)} \leq k_1 \|f_0\|_{C(\partial Q)}. \tag{6.2}
\]

Let \( v = u - u_0 \). Clearly, \( v \in C^2(Q) \cap C_0(Q) \). Then the function \( v \) is a solution of nonlocal boundary value problem

\[
A(x, \partial_x, \partial_x) v(x, \tau) = 0 \quad ((x, \tau) \in Q), \tag{6.3}
\]

\[
v(x, \tau) + b(x) v(\omega(x), \tau + \chi) = -b(x) u_0(\omega(x), \tau + \chi) + f_1(x, \tau) \quad ((x, \tau) \in \partial Q). \tag{6.4}
\]

By virtue of the weak maximum principle

\[
\|v\|_{C(Q)} = \|v\|_{C(\partial Q)} \tag{6.5}
\]

From equality (6.5), nonlocal boundary condition (6.4), and a priori estimate (6.2) we obtain

\[
(1 - \theta)\|v\|_{C(Q)} = \|v\|_{C(\partial Q)} - \theta \|v\|_{C(Q)} \leq \|v(x, \tau) + b(x) v(\omega(x), \tau + \chi)\|_{C(\partial Q)} \leq \|u_0\|_{C(Q)} + \|f_1\|_{C(\partial Q)} \leq k_1 \|f_0\|_{C(\partial Q)} + \|f_1\|_{C(\partial Q)}. \tag{6.6}
\]

Thus

\[
\|v\|_{C(Q)} \leq k_1 (1 - \theta)^{-1} \|f_0\|_{C(\partial Q)} + (1 - \theta)^{-1} \|f_1\|_{C(\partial Q)}. \tag{6.7}
\]

Adding term by term (6.2) and (6.7), we derive inequality (6.1) with

\[
c_1 = \max \left\{ (1 - \theta)^{-1}, k_1 (1 + (1 - \theta)^{-1}) \right\}. \tag*{□}
\]

**Lemma 6.2.** Assume that Conditions 2.1–2.3 hold, and that the operator-valued function \( \lambda \mapsto \hat{L}_0(\lambda) \) has no eigenvalues on the line \( \{ \lambda \in \mathbb{C} : \text{Im} \lambda = \beta \} \). Let \( f_0 \in C_0^\sigma(Q) \), \( f_1 \in C_0^{2+\sigma}(\partial Q) \), and let \( u \in W_{p,\beta}^2(Q) \) be a solution of nonlocal boundary value problem (2.1), (2.2), where \( p > n/2 \). Then \( u \in C_0^{2+\sigma}(Q) \).

Moreover the following a priori estimate holds:

\[
\|u\|_{C^{2+\sigma}(Q)} \leq C_2 (\|f_0\|_{C^\sigma(Q)} + \|f_1\|_{C^{2+\sigma}(\partial Q)}) \tag{6.8}
\]

where \( c_2 > 0 \) does not depend on \( f_0 \) and \( f_1 \).
Proof. 1. Let \( f_0 \in C^\sigma(\overline{Q}) \) and \( f_1 \in C^{2+\sigma}(\partial Q) \) have compact supports. First we prove that \( u \in C^{2+\sigma}(Q) \cap C_0(\overline{Q}) \) and estimate (6.8) holds.

By virtue of the Sobolev imbedding theorem, \( u \in C(Q) \) and

\[
\|u\|_{C(Q,\overline{Q}_N)} \leq k_1 \|u\|_{W_2^2(\overline{Q}_N)},
\]

where \( k_1 > 0 \) does not depend on \( u \) and \( N \).

Since \( u \in W_2^2(Q) \), the right hand side of this inequality tends to 0 as \( N \to \infty \). Hence

\[
\|u\|_{C(Q,\overline{Q}_N)} \to 0 \quad \text{as} \quad N \to \infty.
\]

Thus \( u \in C_0(\overline{Q}) \).

On the other hand, by virtue of Theorem 9.19 in [8, Chapter 9], we have \( u \in C^{2+\sigma}(\overline{Q}_N) \) for any \( N > 0 \). From Corollary 6.3 in [8, Chapter 6] on interior Schauder estimates for solutions of elliptic equations we obtain

\[
\|u\|_{C^{2+\sigma}(\overline{Q}_N)} \leq k_2 (\|u\|_{C(\overline{Q})} + \|f_0\|_{C^\sigma(\overline{Q})}),
\]

where \( Q''_{N+1} = \{x \in \omega(\Omega_0): |r| < N + 1 + \chi\}, k_3 > 0 \) does not depend on \( f_0 \) and \( N \).

Nonlocal boundary condition (2.2) can be written in the form

\[
u(x, \tau) = \psi(x, \tau) \quad ((x, \tau) \in \partial Q),
\]

where \( \psi(x, \tau) = -b(x)u(\omega(x), \tau + \chi) + f_1(x, \tau) \in C^{2+\sigma}(\partial Q \cap \overline{Q''_{N+1}}). \)

Clearly,

\[
\|\psi\|_{C^{2+\sigma}(\partial Q \cap \overline{Q''_{N+1}})} \leq k_3 (\|u\|_{C^{2+\sigma}(\overline{Q''_{N+1}})} + \|f_1\|_{C^{2+\sigma}(\partial Q \cap \overline{Q''_{N+1}})}),
\]

where \( k_3 > 0 \) does not depend on \( f_1, u, \) and \( N \).

Using Corollary 6.7 in [8, Chapter 6] on Schauder estimates for solutions of elliptic equations near a boundary and inequalities (6.10), (6.12), and (6.1), we have

\[
\|u\|_{C^{2+\sigma}(\overline{Q}_N'' \cap \partial Q)} \leq k_4 (\|u\|_{C(\overline{Q}_N'')} + \|f_0\|_{C^\sigma(\overline{Q}_N'')} + \|\psi\|_{C^{2+\sigma}(\partial Q \cap \overline{Q''_{N+1}})}) + k_5 (\|f_0\|_{C^\sigma(\overline{Q})} + \|f_1\|_{C^{2+\sigma}(\partial Q)}),
\]

where \( k_4, k_5 > 0 \) do not depend on \( f_0, f_1 \) and \( N \).

Hence \( u \in C^{2+\sigma}(Q) \cap C_0(\overline{Q}) \), and estimate (6.8) holds.

2. From part 3 in the proof of Lemma 4.2 and inequality (6.9) it follows that for all \( f_0 \in C^\sigma(\overline{Q}) \) and \( f_1 \in C^{2+\sigma}(\partial Q) \) with compact supports \( u \in C^{2+\sigma}(\overline{Q}) \). To complete the proof it remains to note that, by definition, the set of functions from \( C^\sigma(\overline{Q}) \) (\( C^{2+\sigma}(\partial Q) \)) with compact supports is dense in \( C^\sigma_0(\overline{Q}) \) (\( C^{2+\sigma}_0(\partial Q) \)).

6.2. Now we can formulate the theorem on a unique solvability of nonlocal elliptic problem in a Hölder space.

Theorem 6.3. Let Conditions 2.1–2.3 hold, and let the operator-valued functions \( \lambda \mapsto L_0(\lambda) \) and \( \lambda \mapsto L(\lambda) \) have no real eigenvalues. Then, for all \( f_0 \in C^\sigma(\overline{Q}) \) and \( f_1 \in C^{2+\sigma}(\partial Q) \), there exists a unique solution of problem (2.1), (2.2) \( u \in C^\sigma_0(\overline{Q}) \), and the following estimate takes place:

\[
\|u\|_{C^{2+\sigma}(\overline{Q})} \leq c_3 (\|f_0\|_{C^\sigma(\overline{Q})} + \|f_1\|_{C^{2+\sigma}(\partial Q)}),
\]

where \( c_3 > 0 \) does not depend on \( f_0 \) and \( f_1 \).
Proof. Let \( f_0 \in C^\sigma(\bar{Q}) \) and \( f_1 \in C^{2+\sigma}(\partial Q) \) are functions with compact supports. Then, by virtue of Theorem 5.6, there is a unique solution of problem (2.1), (2.2) \( u \in W_{p,0}^2(Q) = W_{p}^2(\bar{Q}) \), where \( p > n/2 \). By Lemma 6.2, \( u \in C^\sigma(\bar{Q}) \) and estimate (6.13) holds. It remains to note that the set of functions from \( C^\sigma(\bar{Q}) \) \( (C^{2+\sigma}(\partial Q)) \) with compact supports is dense in \( C^\sigma_0(\bar{Q}) \) \( (C^{2+\sigma}_0(\partial Q)) \).

In Examples 3.8 and 3.9, it was considered the nonlocal elliptic problems, in which the operator-valued functions \( \lambda \mapsto \hat{L}_0(\lambda) \) and \( \lambda \mapsto \hat{L}(\lambda) \) have no real eigenvalues.

Remark 6.4. The following problems are unsolved:

1. Is it possible to omit condition 2.3 in Theorem 6.3?
2. Can we omit the assumption in Theorem 6.3 that the operator-valued function \( \lambda \mapsto \hat{L}_0(\lambda) \) has no real eigenvalues?

7. Nonlocal problem for Vlasov–Poisson equations.

7.1. We consider the Vlasov–Poisson system in an infinite cylinder

\[
- \Delta \varphi(x,t) = 4\pi e \int_{\mathbb{R}^3} \sum_{\beta} \beta f^\beta(x,v,t) \, dv \quad (x \in Q, \ 0 < t < T),
\]

(7.1)

\[
\frac{\partial f^\beta}{\partial t} + (v, \nabla_x f^\beta) + \frac{\beta e}{m_\beta} \left( - \nabla_x \varphi + \frac{1}{c} [v, B] \nabla_x f^\beta \right) = 0
\]

(7.2)

with the initial conditions

\[
f^\beta(x,v,t)|_{t=0} = f^\beta_0(x,v) \quad (x \in \bar{Q}, \ v \in \mathbb{R}^3, \ \beta = \pm 1)
\]

(7.3)

and the boundary condition

\[
\varphi(x,t) + b(x') \varphi(\omega(x'), x_3 + \chi, t) = 0 \quad (x \in \partial Q, \ 0 \leq t < T).
\]

(7.4)

Here \( Q = G \times \mathbb{R} \), \( G \subset \mathbb{R}^2 \) is a bounded domain with boundary \( \partial G \in C^\infty \), \( \partial Q = \partial G \times \mathbb{R} \), \( f^\beta = f^\beta(x,v,t) \) is the density distribution function of positively charged ions (for \( \beta = +1 \)) or of electrons (for \( \beta = -1 \)) at a point \( x \) with velocity \( v \) at a time \( t \); \( \varphi = \varphi(x,t) \) is the potential of the self-consistent electric field; \( \nabla_x \) and \( \nabla_v \) are, respectively, the gradients with respect to \( x \) and \( v \); \( m_{+1} \) and \( m_{-1} \) are the ion and electron masses; \( e \) is the electron charge; \( c \) is the velocity of light; \( B \) is the external magnetic field induction; \( \langle \cdot, \cdot \rangle \) is the inner product in \( \mathbb{R}^3 \); \( [\cdot, \cdot] \) is the vector product in \( \mathbb{R}^3 \); \( \omega \) is a \( C^\infty \)-diffeomorphism mapping some neighborhood \( \Omega_0 \) of the boundary \( \partial G \) onto \( \omega(\Omega_0) \subset G, \ \chi \in \mathbb{R}; \ x = (x', x_3) \in \mathbb{R}^2; \ b \in C^\infty(\mathbb{R}^2) \) is a real-valued function.

Assume that Condition 2.3 holds, i.e., \( \theta = \max_{x' \in \bar{G}} |b(x')| < 1 \).

We introduce the Banach space \( C([0,T], C^\sigma(\bar{Q})) \), \( s > 0 \), of continuous functions \( [0,T] \ni t \mapsto \varphi(\cdot, t) \in C^\sigma(\bar{Q}) \) with the norm

\[
\|\varphi\|_{s,T} = \sup_{0 \leq t \leq T} \|\varphi(\cdot, t)\|_s.
\]

The space \( C([0,T], C^\sigma_0(\bar{Q})) \) is defined similarly.

To state the theorem on a unique solvability of problem (7.1)–(7.4), we first give the definition of a classical solution of this problem.
Definition 7.1. A vector-valued function \( \{ \varphi, f^\beta \} \) with \( \varphi \in C([0, T], C_0^{2+\sigma}(\bar{Q})) \) and \( f^\beta \in C^1(\bar{Q} \times \mathbb{R}^3 \times [0, T]) \) is called a classical solution of problem (7.1)–(7.4) if \( \{ \varphi, f^\beta \} \) satisfies equations (7.1), (7.2), initial conditions (7.3), and nonlocal boundary condition (7.4).

We denote \( G_\delta = \{ x' \in G : \text{dist}(x', \partial G) > \delta \} \) and \( Q_\delta = \{ x \in Q : \text{dist}(x, \partial Q) > \delta \} \), where \( \delta > 0 \).

We now formulate the conditions which the magnetic field \( B \) and the initial charged-particle density distributions \( f^\beta(x, v) \) must satisfy.

Condition 7.2. Let \( B = (0, 0, h) \) for \( x \in \bar{Q} \), where \( h > 0 \) is independent of \( x \) and
\[
32 \frac{c m_{\rho} + 1}{c \delta} < h. \tag{7.5}
\]

Condition 7.3. Let \( f_0^\beta \in C^\infty(\bar{Q} \times \mathbb{R}^3) \) be nonnegative functions, and let \( \text{supp} f_0^\beta \subset Q_{2\delta} \) be such that for all initial functions \( f^\beta \) satisfying relations
\[
\text{supp} f_0^\beta \subset (Q_{2\delta} \cap Q'_N) \times B_{\rho/4}, \quad \| f_0^\beta \|_2 < \varepsilon \quad (\beta = \pm 1), \tag{7.6}
\]
where \( Q'_N = \{ x \in Q : |x_3| < N \} \) for a number \( N > 0 \), there is a unique classical solution of problem (7.1)–(7.4). Furthermore,
\[
\text{supp} f^\beta(\cdot, \cdot, t) \subset Q_{5\delta/4} \times B_{\rho} \quad \text{for all} \quad t \in [0, T]. \tag{7.7}
\]

To prove Theorem 7.4, we consider the abstract Vlasov equations
\[
\frac{\partial f^\beta}{\partial t} + (v, \nabla_x f^\beta) + \frac{\beta c}{m_\beta} \frac{e}{\sigma} \left( \int_{\mathbb{R}^3} \beta f^\beta(x, v, t) dv \right) = 0 \quad (x \in Q, \quad v \in \mathbb{R}^3, \quad 0 < t < T, \quad \beta = \pm 1) \tag{7.8}
\]
with the initial conditions
\[
f^\beta(x, v, t)|_{t=0} = f_0^\beta(x, v) \quad (x \in Q, \quad v \in \mathbb{R}^3, \quad \beta = \pm 1). \tag{7.9}
\]
Here \( P : C_0^2(\bar{Q}) \to C_0^{2+\sigma}(\bar{Q}) \) is a linear bounded operator.

Definition 7.5. A vector-valued function \( \{ f^\beta \} \) with \( f^\beta \in C^1(\bar{Q} \times \mathbb{R}^3 \times [0, T]) \) for \( \beta = \pm 1 \) is called a classical solution of problem (7.8), (7.9) if
\[
\int_{\mathbb{R}^3} \sum_{\beta} \beta f^\beta(\cdot, v, t) dv \in C([0, T], C_0^\sigma(\bar{Q}))
\]
and \( \{ f^\beta \} \) satisfies (7.8) and initial condition (7.9).

In [26], it was proved the following statement.

Theorem 7.6. Let \( \delta > 0 \) be such that \( G_{2\delta} \neq \emptyset \), and let Conditions 7.2, 7.3 hold for this \( \delta > 0 \) and some \( h, \rho > 0 \). Then there exists an \( \varepsilon = \varepsilon(T, \delta, \rho, h, \sigma) > 0 \) such that for all initial functions \( f_0^\beta \) satisfying relations (7.6) there is a unique classical solution of problem (7.8), (7.9). Furthermore, formula (7.7) takes place.

Now we can prove Theorem 7.4.
Proof of Theorem 7.4. A proof is based on reduction of problem (7.1)–(7.4) to problem (7.8), (7.9) and application of Theorem 7.6.

1. We consider the nonlocal elliptic boundary value problem

\[
-\Delta u(x) = f_0(x) \quad (x \in Q) \tag{7.10}
\]

\[
u(x) + b(x')\varphi(\omega(x'), x_3 + \chi) = 0 \quad (x \in \partial Q). \tag{7.11}
\]

Clearly, the operator \( A = -\Delta \) satisfies Conditions 2.1, 2.2 with \( n = 2 \) and \( \tau = x_3 \). Therefore from Condition 2.3 and Example 3.9 it follows that the operator-valued functions \( \lambda \mapsto \hat{L}_0(\lambda), \lambda \mapsto \hat{L}(\lambda) \) have no real eigenvalues. Hence, by virtue of Theorem 6.3, for any \( f_0 \in C^\sigma_0(\bar{Q}) \), there exists a unique solution of problem (7.10), (7.11) \( u \in C^{2+\sigma}_0(\bar{Q}) \). Moreover,

\[
\|u\|_{C^{2+\sigma}_0(\bar{Q})} \leq k_1\|f_0\|_{C^\sigma_0(\bar{Q})}, \tag{7.12}
\]

where \( k_1 > 0 \) does not depend on \( f_0 \).

Thus the relation \( u = Pf_0 \) defines a linear bounded operator \( P : C^\sigma_0(\bar{Q}) \to C^{2+\sigma}_0(\bar{Q}) \). This statement implies that, for any function \( F \in C([0, T], C^\sigma_0(\bar{Q})) \), we have \( PF \in C([0, T], C^{2+\sigma}_0(\bar{Q})) \), and

\[
\|PF\|_{2+\sigma,T} \leq k_2\|F\|_{\sigma,T}, \tag{7.13}
\]

where \( k_2 > 0 \) does not depend on \( F \).

2. From part 1 of the proof it follows that the vector-valued function \( \{\varphi, f^g\} \),

\[
\varphi = P \left( 4\pi e \int_{\mathbb{R}^3} \sum_{\gamma = \pm 1} \gamma f^g(x, v, t, ) \, dt \right)
\]

is a classical solution of problem (7.1)–(7.4) if and only if the vector-valued function \( \{f^g\} \) is a classical solution of problem (7.8), (7.9). Hence from Theorem 7.6 we derive Theorem 7.4.

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