Risk Assessment of Dam Overtopping Based on The Extended Three-parameter Distribution and Stochastic Differential Equation

GAO Hai-feng1,∗, ZHANG Gui-jin2, LEI Rui-li3, ZENG Cheng-jin4
1Zhejiang Institute of Hydraulics and Estuary, Hangzhou 310020, China;
2School of Water Conservancy, Changsha University of Science and Technology, Changsha 410000, China;
3Yongzhou Hydro& Power Design Institute, Yongzhou 425000, China;
4Zhejiang Provincial Hydrological Management Center, Hangzhou 310020, China;
5Zhejiang Provincial Key Laboratory of Hydraulics Disaster Prevention and Mitigation, Hangzhou 310020, China

*Corresponding author’s e-mail: 120220989@qq.com

Abstract. For earth and rockfill dam, overtopping is intolerable especially for those dams where cascade hydropower stations exist in the river basin. The flood safety of control reservoir directly relates to the safety of the dam. It is necessary to carry out an accurate analysis for flood risk. Based on the extended three-parameter Burr XII distribution of the maximum reservoir inflow design flood computation, the results are larger than the design standard using analysis method which is also based on Person III distribution. The flood risk analysis of uncertainty factors is considered based on the stochastic differential equation. Combination of these two methods, dam overtopping risk is calculated, greater risk is obtained than the traditional JC method, and the conclusion is much more rational. When determining the crest elevation in water conservancy engineering, the design of dam safety evaluation and risk analysis should be considered using more uncertain factors involved, and the new method is more feasible and reasonable than traditional methods.

1. Introduction

Dam overtopping risk is the probability of the crest overflow risk when dam undergoes designing flood and checking flood. For earth and rockfill dam, overtopping is intolerable, especially for those dams where cascade hydropower stations exist in the river basin. The flood safety of control reservoir directly relates to the safety of the dam. It is necessary to carry out an accurate analysis for flood risk. Recently, more and more studies devote to the risk of flood relieving. Dam overtopping risk is associated with reservoir flood routing process. In the whole routing process, there are lots of uncertainties, such as hydrological conditions of flood into reservoir, hydraulic conditions of outflow discharge capacity, boundary conditions of the relationship between storage capacity and water level, the initial condition of limit levels for flood control, etc[1]. But none of the related studies has yet combining these uncertain factors together to calculate the overtopping risk.

The frequency analysis, domestic and international studies provide many specific methods, theories and references. Burr XII distribution is one of the important distributions in the reliability analysis.
Shao[2~5] studies and compares various distributions, based on his conclusion, and the extended three-parameter Burr XII distribution is suggested to flood frequency analysis. It is much closer to the maximum point in the fitting curve. The above distribution is a special case of Burr XII distribution (Burr, 1942). A mathematic model based on the stochastic differential equation and probability theory is applied to describing and analyzing the random phenomena and their behaviors in the reservoir flood regulation procedures. A combination of random input processes and initial conditions will affect the flood risk. It provides scientific basis for economic and reasonable selections of the water release structure and scheduling mode. In this study, the dam overtopping risk is studied based on the extended three-parameter Burr XII distribution and stochastic differential equation, expecting more reasonable results.

2. Calculation of the maximum reservoir inflow design flood based on the Burr XII theory

In this study, the hydrological frequency calculation generally uses Pearson III frequency curve. National regulation SL44-93 《Regulation for calculating design flood of water resources and hydropower projects》 indicates that the general frequency curve line type should adopt Pearson III curve. If there are special cases, other lines can be used after demonstration[6].

Pearson III frequency curve is a curve of asymmetry, which is unimodal and positive-skewed. It is one end limited, one end unlimited (Fig. 1), which is given as:

\[ f(x) = \frac{\beta^a}{\Gamma(a)} (x-a_0)^{a-1} e^{-\beta(x-a_0)} \]  (1)

where \( \Gamma(a) \) is the gamma function of \( a \), \( a \), \( \beta \), \( a_0 \) represent respectively the shape, scale and position parameters of Pearson III distribution, \( a > 0, \beta > 0 \).

There are some disadvantages during frequency analysis, for instance, all the statistical parameters need repeated adjustment and pilot calculation. Moreover, the final results are obtained by inquiring corresponding tables. The calculation progress is more complex, and experience frequency points are selected by ourselves[7]. For experience points fitting, three sections of Pearson III curve (namely head, central and tail) can’t completely fit the experience points although most of the rest points can be fitted, there is no guarantee that the curve can approach the maximum point. Therefore, underestimating maximum flood is highly possible.

![Fig. 1 Pearson III and Burr XII distribution probability density curve](image)

Shao[2] (2004) systematically studied several kinds of flood frequency analysis focusing the conversation on the extended three-parameter Burr XII distribution. The probability density distribution curve is controlled by three parameters \( a \), \( c \), \( k \) (Fig.1). Different parameters lead to different curves. Typical shapes such as L-shaped, J-shaped and others are also concluded in Shao’s study. This method overcomes the disadvantages of experience frequency points selected by human and repeated pilot calculation of statistical parameters. For experience points fitting, the curve does not only fit most of points but also provides a much closer maximum point than P-III curve.

The three-parameter Burr XII distribution:
\[ F_{EBX} (x; c, \beta, b) = 1 - \left(1 + \frac{x}{b}\right)^{-\beta} (b, c, \beta > 0) \] (2)

The extended three-parameter Burr XII distribution (EBXII distribution for short) can be defined by the cumulative distribution, which is given as:

\[ F_{EBXII} (x; c, k, \lambda) = 1 - \left(1 - k \left(\frac{x}{\lambda}\right)^c\right)^{1/k} \quad k \neq 0 \]
\[ = 1 - e^{-(s+1)\lambda} \quad k = 0 \] (3)

Additionally, the density function is given by:

\[ f_{EBXII} (x; c, k, \lambda) = c\lambda^{-c-1} \left(1 - k \left(\frac{x}{\lambda}\right)^c\right)^{1/k} \quad k \neq 0 \]
\[ = c\lambda^{-c-1} e^{-(s+1)\lambda} \quad k = 0 \] (4)

where \( \lambda \) is the scale parameter, \( c \) is the shape parameter and \( k \) is the inequality parameter.

Under the reparameterization \( k = -\frac{1}{\beta} \) and \( \lambda = -\frac{b}{\beta^{1/c}} \), equation (3) can be derived from equation (2). The extension is real because the value of parameter \( k \) can be any real number \((-\infty < k < +\infty)\), compared with \( 0 < \beta = -1/k \) in the equation (2).

The Burr \( \text{III} \) and its reciprocal type \( \text{III} \) distributions have been used in many applications such as actuarial mathematics, iatriology, electrical informatics, actuarial science, forestry, ecotoxicology, reliability and survival analysis. In the design flood forecasting, the fitting effects between distribution and points are better, and the results are larger than the P-III frequency analysis method given by domestic standard, namely, it is more actual[2,8].

Applications in China
(1) Dan Jiang Kou River

This example includes the flood peak data for Dan Jiang Kou River in northeastern China. Daily flows over 30 years were collected. The GPD and EB\( \text{III} \) to the data are applied and compared, then, the T-year return levels were estimated. Detailed results show that the estimation of T-year return levels by GPD have decreased as the threshold increases. However, the estimation by EB\( \text{III} \) distribution don’t show particular trends. For the GPD, the variations of T-year return levels are small as the threshold increases. For EB\( \text{III} \) distribution, such variations are large, especially for higher thresholds. The larger variations are expected because the extra parameter \( c \) introduces more uncertainties during the model fitting.

(2) Pearl River Basin

Another example is the annual flow data from the Pearl River (Zhujiang) basin, which is the third largest river system in China after the Yangtze and Yellow rivers. Twelve stations are used for demonstration. Annual maximum flows from 1900 to 1997 are used for each station. Maximum likelihood estimation method is used to estimate three parameters \( \hat{\lambda}, \hat{c}, \hat{k} \) of EBXII distribution, the results are the compared with generalized Pareto and log-Pearson III distribution. It can be obtained that the fitting effects of EBXII distribution with experience points are better than that of Pearson III distribution. It is much closer to the maximum point, behaves more realistic.
3. Risk analysis for flood relief based on the stochastic differential equation

3.1 The stochastic differential equation of flood routing

According to the simple equilibrium relationship of reservoir storage capacity, and the traditional deterministic method of flood regulating, differential equation is established as:

$$\frac{dw(h)}{dt} = Q(t) - q(h, c)$$  \hspace{1cm} (5)

Under the reparameterization \( \frac{dh}{dh} = G(h) \), and considering the initial conditions, yields:

$$\frac{dh}{dt} = \frac{[Q(t) - q(h, c)]}{G(h)}, \ h(t_0) = h_0$$  \hspace{1cm} (6)

where \( h(t) \) is the reservoir level, \( h_0 \) is the initial level; \( Q(t) \) is any moment incoming flow of routing progress; \( q(h, c) \) is the flood discharge flow of corresponding time, when the scale of the water release structure is determined, it is can be described as the function of upstream water level \( h \) and discharge coefficient \( c \) and other hydraulic parameters; \( w(h) \) is the reservoir storage capacity. The functions above are certain quantities. In order to figure out the influence of many uncertain factors during the flood routing progress, stochastic differential equation is established which contains random elements.

During the routing process, the reservoir storage capacity is critical, which restricts the random growth of water level, meanwhile, it is restricted by flooding actions of input and output stochastic processes. The stochastic processes includes three aspects: flood into reservoir \( Q(t) \) (including lots of uncertain factors like the hydrologic data and the solution method of design flood hydrograph), the outflow discharge capacity \( q(h, c) \) (when the scale of the water release structure is determined, it contains \( h \) and \( c \) uncertainties), the capacity and water level \( w(h) \) (involving the field measurement and plot calculation error and the sediment deposition for many years, etc.) [1,9–11].

Stochastic differential equation contained random elements is deduced as [1]:

$$\frac{dH(t)}{dt} = \frac{\bar{Q}(t) - \bar{q}(h, c)}{G(h)} + \frac{dB(t)}{G(h)} , \ H(t_0) = H_0$$  \hspace{1cm} (7)

There is an additional term \( \frac{dB(t)}{dt} \) for Equation (7) than (6). It means that random factors are introduced. Equation (7) can be simplified as:

$$dH(t) = \varphi(t, H(t))dt + g(t, H(t))dB(t), \ H(t_0) = H_0$$  \hspace{1cm} (8)

This is a typical Itô equation with a random term (input term) and random initial conditions, the solution is the Markov process. \( B(t) \) is the Wiener process in equation (7), \( E[B(t)] = 0 \), and the variance \( D[B(t)] = \sigma^2 t \). Among them, \( \sigma^2 \) is a constant, it is so-called intensity of process, which depends on these three stochastic processes variability, namely, flood into reservoir, outflow discharge capacity and storage capacity. The investigation and data analysis give the standard deviations \( \sigma_Q(t), \ \sigma_q(t), \ \text{and} \ \sigma_w(t) \) of three stochastic processes, assuming that three stochastic input processes are irrelevant. Then the variance \( D[B(t)] \) of output progress can be yield:

$$D[B(t)] = \sigma^2 t = [\sigma_Q(t)^2 + \sigma_q(t)^2] \Delta t^2 + \sigma_w(t)^2$$  \hspace{1cm} (9)
3.2 Probability distribution of water level process
For the typical Ito equation (8), the square solution can be obtained by mean square integral calculus and Ito stochastic integral method. Some statistical properties are relevant to solution process, which can be simply determined through certain formula. But for flood relief risk, random distribution of water level needs to be paid attention at any moment during flood routing. Under the circumstance of equation (8), forward equation is deduced as [1]:

$$\frac{\partial f(h,t)}{\partial t} = -\frac{\partial}{\partial h}[f(h,t)\phi(t,H)] + \frac{\partial}{\partial h^2}[f(h,t)g(t,H)^2\sigma^2], \quad f(h,t_0) = f_0(h)$$

Equation (10) belongs to nonlinear Ito equation, which is difficult to obtain the precise theory solution. But approximate solution can be gradually got by the adopting iterative. During the solving process, the routing process each time \(f(h,t)\) and the corresponding value discharge \(\bar{q}(t)\) can be synchronously calculated:

$$\bar{q}(t) = \int_{h_{min}}^{h_{max}} q(h,c)f(h,t)dh$$

3.3 The flood relief risk analysis of routing process
The flood relief risk limit state of the reservoir flood routing process, the mark and limit value can be defined as the reservoir water level \(H(t)\) less than crest elevation \(Z\). Flood relief risk \(P_f\) is the incident risk of various possible reservoir natural, project and running conditions during the period of flood overflowing crest and under the water release structure designing scale.

$$P_f = P_f[H \geq Z]$$

where \(Z\) follows Gaussian distribution, the mean \(\mu_Z\) can be taken as crest elevation, the standard deviation \(\sigma_Z\) is relatively minor. Different times of the flood routing process \(P_f(t)\) can be obtained by JC method.

4. Risk of dam overtopping based on the EBXII distribution and stochastic differential equation

4.1 Project profile
The Sanbanxi hydropower station located in the midstream and downstream of Qingshui River, which is upstream of Yuanshui River. It is a bibcock hydropower station, which is the pluriennial regulation performance in Yuanshui cascade. The normal reservoir pool level is 475m, with a total capacity of 40.95 million m³, generating plant installed capacity is 1000MW. The maximum height of dam is 185.5m. It belongs to 200m grade A-one big (1) type engineering. The crest elevation of key dam is 482.5m.

4.2 The ascertainment of reservoir inflow flood hydrograph
The maximum frequency flood of Shidong and Jinping stations are calculated according to the EBXII theory, calculated parameters and the maximum flood results are listed in Table 1:

| Station Name | Statistical Parameters | P(%)  |
|--------------|------------------------|-------|
|              | \(c\) \(k\) \(\lambda\) | 0.01  | 0.2   |
| Shidong      | 11.98 -6.80 120        | 16400 | 11700 |
| Sanbanxi     | / / /               | 27900 | 17400 |
| Jinping      | 18.96 -14.86 98       | 28600 | 20500 |
Considering the relationship between maximum frequency flow and the basin areas of Jinping, Shidong stations. Sanbanxi design flood value can be got, as seen in Table 1.

Take July, 1970 flood of Jinping station as an example, according to the flood peak, flow with the same frequency and sub-period can be controlled and enlarged. P-III distribution of frequency analysis method and the EBXII theory are introduced to get the frequency flood hydrograph, and the reservoir inflow design flood results can be seen in Table 2, the comparison of the standard results and the results obtained by P-III distribution of frequency analysis method with EBXII distribution, is shown in Fig.2 and Fig.3.

Table 2 The plant reservoir inflow design flood hydrograph (July, 1970 Typical)  Flow: m³/s

| Month | Day | Time   | P(%) (EBXII Distribution) | P(%) (P-III Distribution) |
|-------|-----|--------|---------------------------|---------------------------|
| 7     | 11  | 13:00  | 1345                      | 880                       | 1110                       | 922                         |
|       |     | 17:00  | 4185                      | 3060                      | 3460                       | 2760                        |
|       |     | 21:00  | 6540                      | 4670                      | 5400                       | 4090                        |
|       | 12  | 1:00   | 7405                      | 5360                      | 6100                       | 4660                        |
|       |     | 5:00   | 8525                      | 6150                      | 7010                       | 5310                        |
|       |     | 9:00   | 15990                     | 11270                     | 13100                      | 9500                        |
|       |     | 13:00  | 24750                     | 17100                     | 20400                      | 14300                       |
|       |     | 17:00  | 30500                     | 20100                     | 27000                      | 18800                       |
|       |     | 21:00  | 25220                     | 17340                     | 20900                      | 14600                       |
|       | 13  | 1:00   | 18775                     | 12900                     | 15400                      | 10800                       |
|       |     | 5:00   | 15820                     | 11000                     | 13000                      | 9290                        |
|       |     | 9:00   | 13220                     | 9410                      | 10900                      | 8030                        |
|       |     | 13:00  | 10430                     | 7400                      | 8590                       | 6370                        |
|       |     | 17:00  | 8515                      | 6040                      | 7010                       | 5230                        |
|       |     | 21:00  | 7275                      | 5170                      | 5990                       | 4500                        |
|       | 14  | 1:00   | 6535                      | 4620                      | 5380                       | 4050                        |
|       |     | 5:00   | 6025                      | 4290                      | 4960                       | 3760                        |
|       |     | 9:00   | 5810                      | 4150                      | 4780                       | 3640                        |
|       |     | 13:00  | 5935                      | 4250                      | 4890                       | 3730                        |
|       |     | 17:00  | 5995                      | 4260                      | 4940                       | 3770                        |
|       |     | 21:00  | 5710                      | 4100                      | 4710                       | 3610                        |
|       | 15  | 1:00   | 5550                      | 3990                      | 4580                       | 3520                        |
|       |     | 5:00   | 5510                      | 3980                      | 4550                       | 3520                        |
|       |     | 9:00   | 5440                      | 3945                      | 4490                       | 3490                        |
|       |     | 13:00  | 5280                      | 3825                      | 4360                       | 3390                        |
|       |     | 17:00  | 5100                      | 3705                      | 4210                       | 3290                        |
|       |     | 21:00  | 4830                      | 3500                      | 3990                       | 3120                        |
|       | 16  | 1:00   | 4480                      | 3250                      | 3700                       | 2910                        |
|       |     | 5:00   | 4155                      | 3000                      | 3430                       | 2700                        |
|       |     | 9:00   | 3885                      | 2795                      | 3210                       | 2530                        |
|       |     | 13:00  | 3595                      | 2560                      | 2970                       | 2330                        |
|       |     | 17:00  | 3210                      | 2275                      | 2650                       | 2090                        |
|       |     | 21:00  | 2905                      | 2035                      | 2400                       | 1890                        |
|       | 17  | 1:00   | 2710                      | 1895                      | 2240                       | 1770                        |
|       |     | 5:00   | 2530                      | 1755                      | 2090                       | 1650                        |
|       |     | 9:00   | 2340                      | 1610                      | 1930                       | 1530                        |
4.3 Flood routing

The spillway of this reservoir is open channel type, the elevation of the spillway crest is 456m, the width of spillway crest B=20m. The results of stochastic differential equation for dam flood routing by equation (10) are compared with the traditional flood routing method, which is shown in Table 3.

Table 3 The comparison of reservoir flood routing calculation

| Operating Condition | Traditional flood routing method | Stochastic differential equation method |
|---------------------|----------------------------------|----------------------------------------|
|                     | Elevation of water level/m       | Corresponding discharge flow/(m^3/s)   |
|                     |                                  | Elevation of water level/m             | Corresponding discharge flow/(m^3/s) |
| Check flood (P=0.01%) | 479.21                           | 16411                                  |
| Design flood (P=0.2%)  | 476.24                           | 13813                                  |

4.4 The calculation of overtopping risk

Considering the action of wind and wave, the highest water level is calculated when dam encountering flood, and the results comparison with domestic standard method are shown in Table 4.

Table 4 The highest water level comparison of dam encountered flood (considering parapet wall action) m

| The maximum high-water level of corresponding condition | Normative approach | Based on EB\(\lambda\)II distribution and stochastic differential equation method | Dam crest elevation+ parapet wall height |
|--------------------------------------------------------|--------------------|--------------------------------------------------------------------------------|----------------------------------------|
| Check flood (P=0.01%)                                  | 481.33             | 481.66                                                                          | 482.5+1.4                              |
| Design flood (P=0.2%)                                  | 483.38             | 483.47                                                                          | 482.5+1.4                              |

Considering the effect of parapet wall, the risk of overtopping is calculated under the condition of design flood and check flood. Meanwhile, the risk of the traditional JC method and the calculated results are compared\[12\], as shown in the Table 5.

Table 5 The overtopping risk comparison between EBXII theory and stochastic differential equation and traditional JC way

| Operating Condition | Risk of overtopping based on EB\(\lambda\)II distribution and stochastic differential equation method | Risk of overtopping based on traditional JC method |
|---------------------|-------------------------------------------------------------------------------------------------|--------------------------------------------------|
| Check flood (P=0.01%) | 3.45×10^{-5}                                                                                   | 3.12×10^{-6}                                    |
| Design flood (P=0.2%) | 8.7×10^{-4}                                                                                     | 8.97×10^{-4}                                    |
5. Conclusion
(1) The quantitative calculation of the risk of dam overtopping is made based on the EBXII distribution and stochastic differential equation method. This integrated method considers various uncertain factors, such as the effects of hydrological and hydraulic stochastic processes to flood relief risk. At the same time, dynamic factors of reservoir water level are introduced, and the action of controlling flood and flood for storage capacity to flood regulation are considered.

(2) The maximum reservoir inflow design flood is calculated based on the EBXII distribution. For experience points fitting, the maximum point is much closer to the curve than domestic standard P-III frequency analysis method. According to the measured data of other projects, it behaves much more realistic. More uncertainties are taken into account in the flood relief risk analysis based on stochastic differential equation. Combination of these two methods is used to calculate the risk of dam overtopping problems, and the results are much larger but more rational than other methods.

(3) In the dam design progress, EBXII theory is applied in correlation calculation. The crest elevation obtained from this method is safer than the standard applied P-III curve frequency analysis. During the dam flood control safety evaluation, the result is more realistic based on the combination of EBXII distribution and stochastic differential equation method. It is of great significance for promotion.

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