Perturbative approaches in relativistic kinetic theory and the emergence of first-order hydrodynamics

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Introduction

- Fluid dynamics: effective theory for collective behavior in systems out of equilibrium in the regime where there is a wide separation of scales $L_{\text{micro}} \ll L_{\text{macro}}$

(Credit: Jorge Porto
https://www.waves.com.br/expedicao/arraial-do-cabo-de-gala/)
Hydrodynamic variables and EoMs

- Basic hydrodynamics EoMs: local conservation of net charge, energy and momentum
  \[ \partial_\mu N^\mu = 0 \quad \partial_\mu T^{\mu\nu} = 0 \]

- Simplest model: ideal fluid (all cells in local equilibrium)
  \[ N_E^\mu = n_0 u^\mu, \]
  \[ T_E^{\mu\nu} = \varepsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu} \]

- Dissipative fluids: Fictitious local equilibrium state
  \[ N^\mu = N_E^\mu + \tilde{N}^\mu = (n_0 + \delta n)u^\mu + \nu^\mu \]
  \[ T^{\mu\nu} = T_E^{\mu\nu} + \tilde{T}^{\mu\nu} = (\varepsilon_0 + \delta \varepsilon)u^\mu u^\nu - (P_0 + \Pi)\Delta^{\mu\nu} + h^\mu u^\nu + h^\nu u^\mu + \pi^{\mu\nu} \]

Credit: Chun Shen
https://youtu.be/G-Fbon0YQak
Matching conditions

- The separation is not unique – $(\alpha, \beta, u^\mu)$ are not uniquely defined out of equilibrium.

\[ \frac{\mu}{T}, \frac{1}{T} \]

- Prescriptions are used to make the definitions – the matching conditions. They usually restrict the conserved tensors.

- The most used in HIC is the Landau one: “the comoving observer should see no heat flux”

Define a fictitious equilibrium state so that we can define $T$ and $\mu$ with equilibrium EoS.

\[ \delta n = \delta \varepsilon = 0 \]
\[ \varepsilon = \varepsilon_0(\mu, T) \]
\[ n = n_0(\mu, T) \]

Kovtun JPA: Math. and Th., 45(47):473001, 2012

Landau and Lifshitz: Fluid Mechanics - Volume 6 (Course of Theoretical Physics), 1987
Matching conditions

- The separation is not unique – \((\alpha, \beta, u^{\mu})\) are not uniquely defined out of equilibrium.

\[
\begin{align*}
\mu & \quad \frac{1}{T} \quad \frac{1}{T}
\end{align*}
\]

- Prescriptions are used to make the definitions – the matching conditions. They usually restrict the conserved tensors.

- The astrophysics community uses mostly Eckart matching: “\(u^{\mu}\) is the velocity of [one of the] matter currents.”

Define a fictitious equilibrium state so that we can define \(T\) and \(\mu\) as before:

\[
\begin{align*}
\delta n &= \delta \varepsilon = 0 \\
\varepsilon &= \varepsilon_0(\mu, T) \\
n &= n_0(\mu, T) \\
\nu^{\mu} &= 0
\end{align*}
\]
Hydrodynamic variables and EoMs

- \( \partial_\mu N^\mu = 0 \) \( \partial_\mu T^{\mu\nu} = 0 \) 5 Eqs for 14 variables (Landau)

- Constitutive relations/further dynamical equations must be derived
  - Navier-Stokes (Landau):
    - \( \Pi = -\zeta \theta \)
    - \( \theta \equiv \partial_\mu u^\mu \)
    - \( \nu^\mu = \kappa \nabla^\mu \alpha \)
    - \( \nabla^\mu = \Delta^{\mu\nu} \partial_\nu \)
    - \( \pi^{\mu\nu} \equiv 2\eta \sigma^{\mu\nu} \)
    - \( \sigma^{\mu\nu} = \Delta^{\mu\nu\alpha\beta} \partial_\alpha u_\beta \)

Only space-like derivatives!

- Linearly acausal and unstable EoMs
- Possible solutions: IS-like theory; \textit{BDNK theory}

Pichon, Ann. de l'I.H.P. Phys. théo. 2, 21 (1965)
Hiscock, Lindblom PRD 31, 725 (1985)
BDN - PRD, 98(10):104064, (2018); PRD 100(10):104020, (2019); PRX 12 2 021044 (2022)
K - JHEP 1910 (2019) 034
Bemfica-Disconzi-Noronha-Kovtun hydro

Idea: Modified constitutive relations now with *time-like* derivatives

\[ D = u \cdot \partial \]

General matching:

\[ \Pi = \zeta^{(\alpha)} D\alpha - \zeta^{(\beta)} \frac{D\beta}{\beta} - \zeta^{(\theta)} \theta, \]

\[ \delta n = \xi^{(\alpha)} D\alpha - \xi^{(\beta)} \frac{D\beta}{\beta} - \xi^{(\theta)} \theta, \]

\[ \delta \varepsilon = \chi^{(\alpha)} D\alpha - \chi^{(\beta)} \frac{D\beta}{\beta} - \chi^{(\theta)} \theta, \]

\[ \nu^\mu = \kappa^{(\alpha)} \nabla^\mu \alpha - \kappa^{(\beta)} \left( \frac{1}{\beta} \nabla^\mu \beta + D u^\mu \right), \]

\[ h^\mu = \lambda^{(\alpha)} \nabla^\mu \alpha - \lambda^{(\beta)} \left( \frac{1}{\beta} \nabla^\mu \beta + D u^\mu \right), \]

\[ \pi^{\mu\nu} = 2 \eta \sigma^{\mu\nu}, \quad \nabla^\mu = \Delta^{\mu \nu} \partial_\nu; \quad \sigma^{\mu \nu} = \Delta^{\mu \nu \alpha \beta} \partial_\alpha u_\beta \]

- EoMs are causal, hyperbolic and have non-negative entropy production even w/ GR. [BDN PRX 12 2 021044]
- Non-linear causality requires

\[ \delta \varepsilon \neq 0 \quad h^\mu \neq 0 \]

Unusual definitions of the equilibrium state!

How do the coeffs. depend on \( \mu, T \)?

BDN - PRD, 98(10):104064, (2018); PRD 100(10):104020, (2019); PRX 12 2 021044 (2022)
K - JHEP 1910 (2019) 034
Microscopic derivation of BDNK from the Boltzmann eqn.

- Chapman-Enskog (Navier-Stokes): compatibility conditions imply $D \rightarrow \nabla_\mu$

\[ \epsilon p^\mu \partial_\mu f_p = C[f_p] \]

\[ f_p = \sum_{i=0}^{\infty} \epsilon^i f_p^{(i)}. \quad Df_p = \sum_{i=0}^{\infty} \epsilon^i (Df_p)^{(i)} \]

"Inhomogeneity must be orthogonal to zero-mode space"

At $O(1)$ \[ \int dPp^\nu \left( p^\mu \partial_\mu f_0 p = f_0 p \hat{L}_\phi \right) \Rightarrow \text{Euler eqns} \]

- To derive BDNK, we propose a novel perturbation method

\[ \epsilon \int dPP_n^{(\ell)}(\beta E_p) p_{\mu_1} \cdots p_{\mu_\ell} \partial_\mu f_p = \int dPP_n^{(\ell)}(\beta E_p) p_{\mu_1} \cdots p_{\mu_\ell} C[f_p], \]

Basis, not necessarily orthogonal, conserved quantities **explicitly excluded** from perturbation procedure, when present.
Microscopic derivation of BDNK from the Boltzmann eqn.

- Using \( P_n^{(\ell)}(\beta E_p) = (\beta E_p)^n, \quad n = 0, 1, \ldots \)

\[
\int dP p^\mu \partial_\mu f_p = 0, \quad \int dPE_p p^\mu \partial_\mu f_p = 0, \quad \int dP p_{\langle \mu \rangle} p^\alpha \partial_\alpha f_p = 0, \\
\epsilon \int dP (\beta E_p)^n p^\mu \partial_\mu f_p = \int dP (\beta E_p)^n C[f_p], \quad n = 2, 3, 4, \ldots , \\
\epsilon \int dP (\beta E_p)^n p_{\langle \mu \rangle} p^\alpha \partial_\alpha f_p = \int dP (\beta E_p)^n p_{\langle \mu \rangle} C[f_p], \quad n = 1, 2, 3, \ldots , \\
\epsilon \int dP (\beta E_p)^n p_{\langle \mu_1 \cdots \mu_\ell \rangle} p^\mu \partial_\mu f_p = \int dP (\beta E_p)^n p_{\langle \mu_1 \cdots \mu_\ell \rangle} C[f_p], \\
\quad n = 0, 1, 2, \ldots , \text{ for } \ell \geq 2,
\]

- Another choice \( P_m^{(\ell)}(x) = \frac{x^{m-m_\ell}}{(1+x)^{N-n_\ell}}, \quad m = 1, \ldots N \)

**Inspired in**
Arnold, Moore, and Yaffe, JHEP 11, 001 (2000);
Arnold, Moore, and Yaffe, JHEP 01, 030 (2003)
Transport coefficients

- First order: two classes of matching conditions

(i) \[ \delta n = 0 \quad \rho_s = \int dP E^s_p \delta f_p \equiv 0 \quad \nu^\mu \equiv 0 \]

\[ \delta \varepsilon = \chi^{(\alpha)} D \alpha - \chi^{(\beta)} \left( \frac{D \beta}{\beta} - \frac{1}{3} \theta \right), \]

\[ h^\mu = \chi^{(\alpha)} \nabla^\mu \alpha - \chi^{(\beta)} \left( \frac{1}{\beta} \nabla^\mu \beta + D u^\mu \right), \quad \pi^{\mu\nu} = 2\eta \sigma^{\mu\nu}, \]

(ii) \[ \delta \varepsilon = 0 \quad \rho_s = \int dP E^s_p \delta f_p \equiv 0 \quad \nu^\mu \equiv 0 \]

\[ \delta n = \xi^{(\alpha)} D \alpha - \xi^{(\beta)} \left( \frac{D \beta}{\beta} - \frac{1}{3} \theta \right), \]

Massless limit, relaxation time approximation, constant relaxation time

GSR, Denicol, Noronha PRL 127, 042301 (2021)

| Trsp. cff. / N | 10 |
|----------------|----|
| \chi^{(\alpha)}/(P_0 \tau_R) (s = 3) | 1.50 |
| \chi^{(\beta)}/(P_0 \tau_R) (s = 3) | 7.50 |
| \chi^{(\alpha)}/(P_0 \tau_R) (s = 4) | 1.00 |
| \chi^{(\beta)}/(P_0 \tau_R) (s = 4) | 6.00 |
| \xi^{(\alpha)}/(P_0 \tau_R) (s = 3) | -1.00 |
| \xi^{(\beta)}/(P_0 \tau_R) (s = 3) | -5.00 |
| \xi^{(\alpha)}/(P_0 \tau_R) (s = 4) | -0.50 |
| \xi^{(\beta)}/(P_0 \tau_R) (s = 4) | -3.00 |

\( \eta/(P_0 \tau_R) = 0.80 \)
\( \chi^{(\alpha)}/(P_0 \tau_R) = 1.33 \)
\( \chi^{(\beta)}/(P_0 \tau_R) = 4.00 \)
Attractor structure of BDNK theory in Bjorken flow

- $\delta n \equiv 0, \delta \varepsilon \neq 0$ – both hydro and early time attractors

Dynamics independent of ‘s’

$$\rho_s = \left\langle E_p^s \phi_p \right\rangle_0 \equiv 0$$

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**Inspired in** BDN PRD 100(10):104020, (2019)

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Late time attractor

Pullback attractor

A. Behtash, S. Kamata, M. Martinez, and H. Shi, Phys. Rev. D 99, 116012 (2019)
Attractor structure of BDNK theory in Bjorken flow

- $\delta n \neq 0, \delta \varepsilon \equiv 0$ No pullback attractor

(ii) 's'-dependent dynamics

$$\rho_s = \langle E_p^s \phi_p \rangle_0 \equiv 0$$
Comparison of attractor structures in Bjorken flow

- We compare the evolution under the Boltzmann moment equation, IS, and BDNK in Bjorken flow for the alternative matching conditions

\[ \delta n \equiv 0, \quad \delta \varepsilon \neq 0 \quad \rho_s = \left\langle E_p^s \phi_p \right\rangle_0 \equiv 0 \]

\[ s=3 \quad s=4 \]
Comparison of attractor structures in Bjorken flow

- We compare the evolution under the Boltzmann moment equation, IS, and BDNK in Bjorken flow for the alternative matching conditions

\[ \delta n \neq 0, \delta \varepsilon \equiv 0 \quad \rho_s = \langle E_p^s \phi_p \rangle_0 \equiv 0 \]
Conclusions

- We proposed a novel perturbative procedure to derive BDNK hydrodynamics from Kinetic Theory;
- We analytically compute the attractors of BDNK theory and compared it with Boltzmann moments and IS EoMs in Bjorken flow;

- We intend to generalize to other backgrounds and for momentum-dependent relaxation times;
- Improve hydro IS-like truncation;
- Application in thermal mass models;

THANK YOU FOR THE ATTENTION!

https://dailynewshungary.com/19-photos-that-will-make-you-regret-you-did-not-visit-budapest-in-the-winter/