SUPERIZATION AND \((q,t)\)-SPECIALIZATION IN COMBINATORIAL HOPF ALGEBRAS

JEAN-CHRISTOPHE NOVELLI AND JEAN-YVES THIBON

Abstract. We extend a classical construction on symmetric functions, the superization process, to several combinatorial Hopf algebras, and obtain analogs of the hook-content formula for the \((q,t)\)-specializations of various bases. Exploiting the dendriform structures yields in particular \((q,t)\)-analogs of the Björner-Wachs \(q\)-hook-length formulas for binary trees, and similar formulas for plane trees.

1. Introduction

Combinatorial Hopf algebras are special graded and connected Hopf algebras based on certain classes of combinatorial objects. There is no general agreement of what their precise definition should be, but looking at their structure as well as to their existing applications, it is pretty clear that they are to be regarded as generalizations of the Hopf algebra of symmetric functions.

It is well known that one can define symmetric functions \(f(X - Y)\) of a formal difference of alphabets. This can be interpreted either as the image of the difference \(\sum_i x_i - \sum_j y_j\) by the operator \(f\) in the \(\lambda\)-ring generated by \(X\) and \(Y\), or, in Hopf-algebraic terms, as \((\text{Id} \otimes \tilde{\omega}) \circ \Delta(f)\), where \(\Delta\) is the coproduct and \(\tilde{\omega}\) the antipode.

And in slightly less pedantic terms, this just amounts to replacing the power-sums \(p_n(X)\) by \(p_n(X) - p_n(Y)\), a process already discussed at length in Littlewood’s book \([21\text{, p. 100}]\).

As is well known, the Schur functions \(s_\lambda(X)\) are the characters of the irreducible tensor representations of the general Lie algebra \(\mathfrak{gl}(n)\). Similarly, the \(s_\lambda(X - Y)\) are the characters of the irreducible tensor representations of the general Lie superalgebras \(\mathfrak{gl}(m|n)\) \([1]\). These symmetric functions are not positive sums of monomials, and for this reason, one often prefers to use as characters the so-called supersymmetric functions \(s_\lambda(X|Y)\), which are defined by \(p_n(X|Y) = p_n(X) + (-1)^n - 1 p_n(Y)\) (see \([36]\)), and are indeed positive sums of monomials: their complete homogeneous functions are given by

\[
\sigma_t(X|Y) = \sum_{n \geq 0} h_n(X|Y)t^n = \lambda_t(Y)\sigma_t(X) = \prod_{i,j} \frac{1 + ty_j}{1 - tx_i}.
\]

Another (not unrelated) classical result on Schur functions is the hook-content formula \([25\text{, I.3 Ex. 3}]\), which gives in closed form the specialization of a Schur
function at the virtual alphabet

\[
\frac{1-t}{1-q} = \frac{1}{1-q} - t \frac{1}{1-q} = 1 + q + q^2 + \cdots - (t + tq + tq^2 + \cdots).
\]

This specialization was first considered by Littlewood [21, Ch. VII], who obtained a factorized form for the result, but with possible simplifications. The improved version known as the hook-content formula

\[
s_{\lambda} \left( \frac{1-t}{1-q} \right) = q^{n(\lambda)} \prod_{x \in \lambda} \frac{1-tq^{c(x)}}{1-q^{h(x)}},
\]

which is a \((q,t)\)-analog of the famous hook-length formula of Frame-Robin-Thrall [8], is due to Stanley [38].

The first example of a combinatorial Hopf algebra generalizing symmetric functions is Gessel’s algebra of quasi-symmetric functions [10]. Its Hopf structure was further worked out in [26, 9], and later used in [19], where two different analogs of the hook-content formula for quasi-symmetric functions are given. Indeed, the notation

\[
F_I \left( \frac{1-t}{1-q} \right)
\]

is ambiguous. It can mean (at least) two different things:

either \(F_I \left( \frac{1}{1-q} \hat{\times} (1-t) \right)\) or \(F_I \left( (1-t) \hat{\times} \frac{1}{1-q} \right)\),

where \(\hat{\times}\) denotes the ordered product of alphabets. The second one is of the form \(F_I(X - Y)\) (in the sense of [19]), but the first one is not (cf. [19]).

In this article, we shall extend the notion of superization to several combinatorial Hopf algebras. We shall start with \(\text{FQSym}\) (Free quasi-symmetric functions, based on permutations), and our first result (Theorem 3.1) will allow us to give new expressions and combinatorial proofs of the \((q,t)\)-specializations of quasi-symmetric functions. Next, we extend these results to \(\text{PBT}\), the Loday-Ronco algebra of planar binary trees, and obtain a \((q,t)\)-analog of the Knuth and Björner-Wachs hook-length formulas for binary trees. These results rely on the dendriform structure of \(\text{PBT}\). Exploiting in a similar way the tridendriform structure of \(\text{WQSym}\) (Word quasi-symmetric functions, based on packed words, or set compositions), we arrive at a \((q,t)\) analog of the formula of [15] counting packed words according to the shape of their plane tree.

Acknowledgments.- This work has been partially supported by Agence Nationale de la Recherche, grant ANR-06-BLAN-0380. The authors would also like to thank the contributors of the MuPAD project, and especially those of the combinat package, for providing the development environment for this research (see [16] for an introduction to MuPAD-Combinat).
2. Background

2.1. Noncommutative symmetric functions. The reader is referred to [9] for the basic theory of noncommutative symmetric functions. The encoding of Hopf-algebraic operations by means of sums, differences, and products of virtual alphabets is fully explained in [11].

It is customary to reserve the letters \( A, B, \ldots \) for noncommutative alphabets, and \( X, Y, \ldots \) for commutative ones.

We recall that the Hopf algebra of noncommutative symmetric functions is denoted by \( \text{Sym} \), or by \( \text{Sym}(A) \) if we consider the realization in terms of an auxiliary alphabet. Bases of \( \text{Sym} \) are labelled by compositions \( I \) of \( n \). The noncommutative complete and elementary functions are denoted by \( S_n \) and \( \Lambda_n \), and the notation \( S_I \) means \( S_{i_1} \cdots S_{i_r} \). The ribbon basis is denoted by \( R_I \). The notation \( I \models n \) means that \( I \) is a composition of \( n \). The conjugate composition is denoted by \( \sim \). The descent set of \( I \) is \( \text{Des}(I) = \{i_1, i_1 + i_2, \ldots, i_1 + \cdots + i_{r-1}\} \). The descent composition of a permutation \( \sigma \in S_n \) is the composition \( I = D(\sigma) \) of \( n \) whose descent set is the descent set of \( \sigma \).

The graded dual of \( \text{Sym} \) is \( \text{QSym} \) (quasi-symmetric functions). The dual basis of \( (S^I) \) is \( (M_I) \) (monomial), and that of \( (R_I) \) is \( (F_I) \).

The evaluation \( \text{Ev}(w) \) of a word \( w \) over a totally ordered alphabet \( A \) is the sequence \( (|w|_a)_{a \in A} \) where \( |w|_a \) is the number of occurrences of \( a \) in \( w \). The packed evaluation \( I = \text{Ipack}(w) \) is the composition obtained by removing the zeros in \( \text{Ev}(w) \).

The Hopf structures on \( \text{Sym} \) and \( \text{QSym} \) allows one to mimic, up to a certain extent, the \( \lambda \)-ring notation. If \( A \) is a totally ordered alphabet,

\[
(5) \quad \sigma_t((1-q)A) := \lambda_{-qt}(A)\sigma_t(A),
\]

\[
(6) \quad \sigma_t \left( \frac{A}{1-q} \right) := \cdots \sigma_{qt}(A)\sigma_{qt}(A)\sigma_t(A).
\]

We usually consider that our auxiliary variable \( t \) is of rank one, so that \( \sigma_t(A) = \sigma_1(tA) \).

2.2. Free quasi-symmetric functions. The standardized word \( \text{std}(w) \) of a word \( w \in A^* \) is the permutation obtained by iteratively scanning \( w \) from left to right, and labelling \( 1, 2, \ldots \) the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. For example, \( \text{std}(bbacab) = 341625 \). For a word \( w \) on the alphabet \( \{1, 2, \ldots\} \), we denote by \( w[k] \) the word obtained by replacing each letter \( i \) by the integer \( i + k \).

Recall from [6] that for an infinite totally ordered alphabet \( A \), \( \text{FQSym}(A) \) is the subalgebra of \( \mathbb{K}(A) \) spanned by the polynomials

\[
(7) \quad G_\sigma(A) = \sum_{\text{std}(w) = \sigma} w
\]
the sum of all words in $A^n$ whose standardization is the permutation $\sigma \in S_n$. The multiplication rule is, for $\alpha \in S_k$ and $\beta \in S_l$,

$$G_\alpha G_\beta = \sum_{\gamma \in S_{k+l}, \gamma = uv, \text{std}(u) = \alpha, \text{std}(v) = \beta} G_\gamma. \tag{8}$$

The noncommutative ribbon Schur function $R_I \in \text{Sym}$ is then

$$R_I = \sum_{D(\sigma) = I} G_\sigma. \tag{9}$$

This defines a Hopf embedding $\text{Sym} \to \text{FQSym}$. As a Hopf algebra, $\text{FQSym}$ is self-dual. The scalar product materializing this duality is the one for which $(G_\sigma, G_\tau) = \delta_{\sigma, \tau} - 1$ (Kronecker symbol). Hence, $F_\sigma := G_{\sigma^{-1}}$ is the dual basis of $G$.

The internal product $\ast$ of $\text{FQSym}$ is induced by composition $\circ$ in $S_n$ in the basis $F$, that is,

$$F_\sigma \ast F_\tau = F_{\sigma \circ \tau} \quad \text{and} \quad G_\sigma \ast G_\tau = G_{\tau \circ \sigma}. \tag{10}$$

Each subspace $\text{Sym}_n$ is stable under this operation, and anti-isomorphic to the descent algebra $\Sigma_n$ of $S_n$.

The transpose of the Hopf embedding $\text{Sym} \to \text{FQSym}$ is the commutative image $F_\sigma \mapsto F_\sigma(X) = F_I(X)$, where $I$ is the descent composition of $\sigma$, and $F_I$ is Gessel’s fundamental basis of $QSym$.

3. **Free super-quasi-symmetric functions**

3.1. **Supersymmetric functions.** As already mentioned in the introduction, in the $\lambda$-ring notation, the definition of supersymmetric functions is transparent. If $X$ and $\bar{X}$ are two independent infinite alphabets, the superization $f^\#$ of $f \in \text{Sym}$ is

$$f^\# := f(X \mid \bar{X}) = f(X - q\bar{X})|_{q = -1}, \tag{11}$$

where $f(X - qX)$ is interpreted in the $\lambda$-ring sense ($p_n(X - q\bar{X}) := p_n(X) - q^n p_n(\bar{X})$), $q$ being of rank one, so that $p_n(X|\bar{X}) = p_n(X) - (-1)^n p_n(\bar{X})$. This can also be written as an internal product

$$f^\# := f \ast \sigma_1^\#, \tag{12}$$

where $\sigma_1^\# = \sigma_1(X - q\bar{X})|_{q = -1} = \lambda_1(\bar{X}) \sigma_1(X)$, and the internal product is extended to the algebra generated by $\text{Sym}(X)$ and $\text{Sym}(\bar{X})$ by means of the splitting formula

$$(f_1 \cdots f_r) \ast g = \mu_r \cdot (f_1 \otimes \cdots \otimes f_r) \ast, \Delta^r g, \tag{13}$$

and the rules

$$\sigma_1 \ast f = f \ast \sigma_1, \quad \bar{\sigma}_1 \ast \sigma_1 = \sigma_1. \tag{14}$$
3.2. Noncommutative supersymmetric functions. The same can be done with noncommutative symmetric functions. We need two independent infinite totally ordered alphabets \( A \) and \( \bar{A} \) and we define \( \text{Sym}(A|\bar{A}) \) as the subalgebra of the free product \( \text{Sym}^{(2)} := \text{Sym}(A) \ast \text{Sym}(\bar{A}) \) generated by \( S_i^\# \) where
\[
\sigma^\#_1 = \bar{x}_1 \sigma_1 = \sum_{I = (i_1, \ldots, i_r)} (-1)^{i_1 + \cdots + i_{r-1}} S_{i_1}^{\bar{x}_1} \cdots S_{i_{r-1}}^{\bar{x}_{i_{r-1}}} S_{i_r}^{\bar{x}_{i_r}}.
\]

For example,
\[
S_1^\# = S_1^1 + S_1^{\bar{x}_1}, \quad S_2^\# = S_2^2 + S_2^{\bar{x}_1} - S_1^{\bar{x}_2} + S_2^{\bar{x}_1},
\]
\[
S_3^\# = S_3^3 + S_3^{\bar{x}_2} + S_2^{\bar{x}_3} - S_1^{\bar{x}_3} + S_3^{\bar{x}_2} - S_2^{\bar{x}_3} + S_2^{\bar{x}_1}.
\]

We shall denote the generators of \( \text{Sym}^{(2)} \) by \( S_{(i,\epsilon)} \) where \( \epsilon = \{\pm 1\} \), so that \( S_{(i,1)} = S_i \) and \( S_{(i,-1)} = S_\bar{i} \).

The corresponding basis of \( \text{Sym}^{(2)} \) is then written
\[
S^{(I,\epsilon)} = S_{((i_1,\epsilon_1),(i_2,\epsilon_2),\ldots,(i_r,\epsilon_r))} := S_{(i_1,\epsilon_1)} S_{(i_2,\epsilon_2)} \cdots S_{(i_r,\epsilon_r)}.
\]

where \( I = (i_1, \ldots, i_r) \) is a composition and \( \epsilon = (\epsilon_1, \ldots, \epsilon_r) \in \{\pm 1\}^r \) is a vector of signs.

Again, we extend the internal product by formulas (13) and (14) where, now, \( f_1, \ldots, f_r, g \in \text{Sym}^{(2)} \), and \( \sigma_1 = \sigma_1(A), \sigma_\bar{1} = \sigma_1(\bar{A}) \). The resulting algebra is isomorphic to the Mantaci-Reutenauer algebra of type \( B \) \cite{27}. We define the superization of \( f \in \text{Sym} \) by
\[
f^\# := f \ast \sigma_1^\# = f(A - q\bar{A})|_{q = -1} = f \ast (\bar{x}_1 \sigma_1).
\]

3.3. Super quasi-symmetric functions. There are two natural and nonequivalent choices for defining super quasi-symmetric functions. The first one is to set \( F(X|\bar{X}) = F(X - q\bar{X})|_{q = -1} \) as in \cite{14}. The second one is obtained by commutative image from the free super quasi-symmetric functions to be defined below. Let us note that super quasi-symmetric functions have been recently interpreted as characters of certain abstract crystals of the Lie superalgebras \( \mathfrak{gl}(m|n) \) \cite{20}.

3.4. Free super quasi-symmetric functions. The expressions (19) are still well-defined for an arbitrary \( f \in \text{FQSym} \). We can define \( \text{FQSym}(A|\bar{A}) \) as the subalgebra of the free product \( \text{FQSym}(A) \ast \text{FQSym}(\bar{A}) \) spanned by
\[
G_{\sigma^\#} := G_\sigma(A|\bar{A}) = G_\sigma \ast \sigma_1^\#.
\]

Again, \( \ast \) is extended to the free product by conditions (13) (valid only if \( g \in \text{Sym}^{(2)} \), which is enough), and (14). This free product can be interpreted as \( \text{FQSym}^{(2)} \), the algebra of free quasi-symmetric functions of level 2, as defined in \cite{29}. Let us set
\[
A^{(0)} = A = \{a_1 < a_2 < \ldots < a_n \},
\]
\[
A^{(1)} = \bar{A} = \{\ldots < \bar{a}_n < \ldots < \bar{a}_2 < \bar{a}_1 \},
\]

order \( A = \bar{A} \cup A \) by \( \bar{a}_i < a_j \) for all \( i, j \), and denote by \( \text{std} \) the standardization of signed words with respect to this order. We also need the signed standardization.
Std, defined as follows. Represent a signed word $w \in A^n$ by a pair $(w, \epsilon)$, where $w \in A^n$ is the underlying unsigned word, and $\epsilon \in \{\pm 1\}^n$ is the vector of signs. Then $\text{Std}(w, \epsilon) = (\text{std}(w), \epsilon)$.

We denote by $m(\epsilon)$ the number of entries $-1$ in $\epsilon$.

A basis of $\mathbf{FQSym}^{(2)}$ is given by

$$G_{\sigma, \epsilon} := \sum_{\text{Std}(w) = (\sigma, \epsilon)} w \in \mathbb{Z}\langle A \rangle.$$ 

and the internal product obtained from (13-14) coincides with the one of [29], so that it is in fact always well-defined. In particular, viewing signed permutations as elements of the group $\{\pm 1\} \wr S_n$,

$$G_{\alpha, \epsilon} \ast G_{\beta, \eta} = G_{\beta \circ \alpha, (\eta \alpha) \cdot \epsilon}$$

with $\eta \alpha = (\eta \alpha(1), \ldots, \eta \alpha(n))$ and $\epsilon \cdot \eta = (\epsilon_1 \eta_1, \ldots, \epsilon_n \eta_n)$.

**Theorem 3.1.** The expansion of $G_\sigma(A|\bar{A})$ on the basis $G_{\tau, \epsilon}$ is

$$G_\sigma(A|\bar{A}) = \sum_{\text{std}(\tau, \epsilon) = \sigma} G_{\tau, \epsilon}.$$ 

**Proof –** This is clear for $\sigma = 12 \ldots n$:

$$\sum_n G_{12 \ldots n}(A|\bar{A}) = \bar{\lambda}_1 \cdot \sigma_1 = \sum_{i_1 < i_2 < \ldots < i_k \atop j_1 \leq j_2 \leq \ldots \leq j_l} \bar{a}_{i_1} \bar{a}_{i_2} \cdots \bar{a}_{i_k} a_{j_1} a_{j_2} \cdots a_{j_l},$$

and writing

$$G_\sigma(A|\bar{A}) = G_\sigma \ast (\bar{\lambda}_1 \cdot \sigma_1) = \sum_{\text{std}(\tau, \epsilon) = 12 \ldots n} G_{\tau, \epsilon, \sigma} = \sum_{\text{std}(\tau, \epsilon) = \sigma} G_{\tau, \epsilon},$$

we obtain (25). 

3.5. **The canonical projection.** We have an obvious projection

$$\mathbf{FQSym}(A|\bar{A}) \to \mathbf{FQSym}(A)$$

consisting in setting $\bar{A} = A$. One can even describe the refined map

$$\eta_t(G_\sigma^\#) = G_\sigma(A|tA).$$

**Corollary 3.2.** In the special case $\bar{A} = tA$, one gets

$$G_\sigma(A|tA) = \sum_{\text{std}(\tau, \epsilon) = \sigma} t^{m(\epsilon)} G_{\tau}(A).$$

**Proof –** This follows from (25).
Example 3.3. We have

\[(31)\quad G_{12}(A|tA) = (1 + t)(G_{12} + tG_{21}), \quad G_{21}(A|tA) = (1 + t)(G_{21} + tG_{12})\]
\[(32)\quad G_{4132}(A|tA) = (1 + t)(G_{4132} + tG_{3421} + tG_{4231} + tG_{4321} + t^2G_{2413} + t^2G_{3412} + t^2G_{4312} + t^3G_{1423}).\]

Indeed, (32) is obtained from the 16 signed permutations whose standardized word is 4132:

\[4132, \quad 4132, \quad 3421, \quad 3421, \quad 4231, \quad 4231, \quad 4321, \quad 4321, \quad 4321, \quad 4321, \quad 4321, \quad 4321, \quad 4321, \quad 4321, \quad 4321,\]
\[2413, \quad 2413, \quad 3412, \quad 3412, \quad 3412, \quad 3412, \quad 3412, \quad 3412, \quad 3412, \quad 3412, \quad 3412, \quad 3412, \quad 3412, \quad 3412, \quad 3412, \quad 3412.\]

Summing over a descent class, we obtain

Corollary 3.4.

\[(34)\quad R_I(A|\bar{A}) = \sum_{C(J,\epsilon) = I} R_{J,\epsilon},\]

where \(R_{I,\epsilon}\) is the signed ribbon Schur function defined as in [29] and \(C(J,\epsilon)\) is the composition whose descents are the descents of any signed permutation \((\sigma,\epsilon)\) where \(\sigma\) is of shape \(J\).

Substituting \(\bar{A} = tA\) yields

\[(35)\quad R_I(A|tA) = \sum_{C(J,\epsilon) = I} t^{m(\epsilon)} R_{J}(A),\]

which allows us to recover a formula of [19] (in [19], the exponent \(b(I, J)\) is incorrectly stated). Recall that a peak of a composition is a cell of its diagram having no cell to its right or on its top and that a valley is a cell having no cell to its left or at its bottom.

Corollary 3.5 ([19], (121)).

\[(36)\quad R_I(A|tA) = \sum_{J} (1 + t)^{v(J)} t^{b(I,J)} R_I(A),\]

where the sum is over all compositions \(I\) which have either a peak or a valley at each peak of \(J\). The power of \(1 + t\) is given by the number of valleys \(v(J)\) of \(J\) and the power of \(t\) is the number of descents of \(J\) that are not descents of \(I\) plus the number of descents \(d\) of \(I\) such that neither \(d\) nor \(d - 1\) are descents of \(J\).

Proof – This is best understood at the level of permutations. First, the coefficient of \(R_I(A)\) is equal, by definition, to the number of signed permutations of shape \(I\) whose underlying (unsigned) permutation is of shape \(J\). Now, on the ribbon diagram of a permutation of shape \(J\), in order to obtain a signed permutation of shape \(I\), we distinguish three kinds of cells: those which must have a plus sign, those which must have a minus sign, and those which can have both signs. The valleys of \(J\) can get any sign without changing their final shape whereas all other cells have a fixed plus or minus sign, depending on \(I\) and \(J\), thus explaining the coefficient \((1 + t)^{v(J)}\). The cells which must have a minus sign are either the descents of \(J\) that are not descents
of \( I \) or the descents \( d \) (plus one) of \( I \) such that neither \( d \) nor \( d - 1 \) are descents of \( J \), whence the power of \( t \) in the middle cell for all pairs of compositions of 3, since it depends only on the relative positions of their adjacent cells in \( I \) and \( J \).

3.6. The dual transformation. Corollary 3.2 is equivalent, up to substituting \(-t\) to \( t \), to a combinatorial description of

\[
G_s((1 - t)A) = G_s(A) * \sigma_1((1 - t)A).
\]

Let \( \eta^*_t \) be the adjoint of \( \eta_t \). We can consistently set

\[
F_s(A \cdot (1 - t)) := \eta^*_t(F_s(A)),
\]

since the noncommutative Cauchy formula reads

\[
\sigma_1(A \cdot (1 - t) \cdot B) = \sum_{\alpha} F_\alpha(A \cdot (1 - t))G_\alpha(B)
\]

\[
= \sum_{\beta} F_\beta(A)G_\beta((1 - t)B).
\]

Writing

\[
G_\beta((1 - t)B) = G_\beta(B) * S_n((1 - t)B)
\]

\[
= \sum_{k=0}^{n-1} (1 - t)(-t)^k G_\beta * R_{(1^k,n-k)}
\]

\[
= \sum_{k=0}^{n-1} (1 - t)(-t)^k \sum_{\text{Des}(\tau) = \{1,\ldots,k\}} G_{\tau_{o\beta}}(B),
\]

we have, setting \( \sigma = \tau \circ \beta \)

\[
\sigma_1(A \cdot (1 - t) \cdot B) = 1 + \sum_{|\sigma| \geq 1} \left( \sum_{\text{Des}(\tau) = \{1,\ldots,k\}} (1 - t)(-t)^k F_{\tau^{-1}o\sigma}(A) \right) G_\sigma(B),
\]

so that

\[
F_s(A \cdot (1 - t)) = \sum_{k=0}^{n-1} (1 - t)(-t)^k \sum_{\text{Des}(\tau) = \{1,\ldots,k\}} F_{\tau^{-1}o\beta}(A).
\]

Theorem 3.6. In terms of signed permutations, we have

\[
F_s(A \cdot (1 - t)) = \sum_{\epsilon \in \{\pm 1\}^n} (-t)^{m(\epsilon)}F_{\text{std}(\sigma,\epsilon)}(A).
\]

Proof – This follows from the above discussion and Corollary 3.2.
Corollary 3.7. Specializing \( A = \frac{1}{1-q} \), we obtain

\[
F_\sigma(X) = F_{D(\sigma)}(X)
\]

in the notation of [19]

\[
(44) \quad = \frac{1}{(q)_n} \sum_{\epsilon \in \{\pm1\}^n} (-t)^{m(\epsilon)} q^{maj(\sigma, \epsilon)},
\]

where \( maj(w) = \sum_{i \in \text{Des}(w)} i \), and \( \text{Des}(w) = \{i \mid w_i > w_{i+1}\} \).

3.7. A hook-content formula in FQSym. Let us denote by \( SP_i \) the set of words \( \epsilon \in \{\pm1\}^n \) where \( \epsilon_i = 1 \) and by \( SM_i \) the set of words \( \epsilon \in \{\pm1\}^n \) where \( \epsilon_i = -1 \).

Let \( \phi_i \) be the involution on signed permutations \( (\sigma, \epsilon) \) which changes the sign of \( \epsilon_i \) and leaves the rest unchanged.

Lemma 3.8. Let \( (\sigma, \epsilon) \) be a signed permutation such that \( \epsilon_i = 1 \) and let \( (\sigma, \epsilon') = \phi_i(\sigma, \epsilon) \). Then

\[
(45) \quad (-t)^{m(\epsilon')} q^{maj(\sigma, \epsilon')} = (-t) \frac{q^{(i-1)x_i}}{q^{iy_i}} (-t)^{m(\epsilon)} q^{maj(\sigma, \epsilon)},
\]

where \( x_i = 0 \) if \( \sigma_{i-1} > \sigma_i \) and \( x_i = 1 \) otherwise, and \( y_i = 0 \) if \( \sigma_i < \sigma_{i+1} \) and \( y_i = 1 \) otherwise. By convention, \( x_1 = 0 \) and \( y_n = 0 \), which is equivalent to fix \( \sigma_0 = \sigma_{n+1} = +\infty \).

Proof – The factor \((-t)\) is obvious. The difference between the \( q \)-statistics of both words depends only on the descents at position \( i-1 \) and position \( i \). Let us discuss position \( i-1 \) (value of \( x_i \)). If \( \sigma_{i-1} > \sigma_i \), we have

\[
(46) \quad -\sigma_{i-1} < -\sigma_i < \sigma_i < \sigma_{i-1},
\]

so that there is a descent at position \( i-1 \) in \( (\sigma, \epsilon) \) iff there is a descent at the same position in \( (\sigma, \epsilon') \). This proves the case \( x_i = 0 \).

Now, if \( \sigma_{i-1} > \sigma_i \), we have

\[
(47) \quad -\sigma_i < -\sigma_{i-1} < \sigma_{i-1} < \sigma_i,
\]

so that there is no descent at position \( i-1 \) in \( (\sigma, \epsilon) \) and there is a descent at the same position in \( (\sigma, \epsilon') \). This proves the case \( x_i = 1 \). The discussion of position \( i \) is similar.

Theorem 3.9. Let \( \sigma \in \mathfrak{S}_n \). Then

\[
(48) \quad F_\sigma(X) = q^{maj(\sigma)} \prod_{i=1}^{n} \frac{1 - q^{(i-1)x_i - iy_i} t}{1 - q^i} = \prod_{i=1}^{n} \frac{q^{iy_i} - q^{(i-1)x_i} t}{1 - q^i},
\]

where \( x_i \) and \( y_i \) are as in Lemma 3.8.

This gives an analog of the hook-content formula, where the hook-length of cell \#\( i \) is its “ribbon length” \( i \), and its “content” is \( c_i = (i - 1)x_i - iy_i \).

Proof – Thanks to Lemma 3.8 we have

\[
(49) \quad (q)_n F_\sigma(X) = \sum_{\epsilon \in \{\pm1\}^n} (-t)^{m(\epsilon)} q^{maj(\sigma, \epsilon)} = \left(1 - t \frac{q^{(i-1)x_i}}{q^{iy_i}} \right) \sum_{\epsilon \in SP_i} (-t)^{m(\epsilon)} q^{maj(\sigma, \epsilon)},
\]
since each signed permutation \((\sigma, \epsilon')\) with \(\epsilon'_i = -1\) gives the same contribution as \(\phi_i(\sigma, \epsilon')\) up to the factor involving \(x_i\) and \(y_i\). The same can be done for signed permutations such that \(\epsilon'_i = 1\) and \(\epsilon'_j = -1\), so that the whole expression factors and gives the first formula of (48). The second expression is clearly equivalent.

3.8. Graphical representations. We shall see later that (48) is the special case of formula (86) for binary trees, when the tree is a zig-zag line. This is why we have chosen to represent graphically \(F_{\sigma}(X)\) with hook-content type factors in the following way: let the mirror shape of a permutation \(\sigma\) be the mirror image of its descent composition. We represent it as the binary tree in which each internal node has only one subtree, depending on whether the corresponding cell of the composition is followed by a cell to its right or to its bottom. For example, with \(\sigma = (5, 6, 7, 4, 3, 2, 8, 9, 10, 1, 11)\), the shape is \((3, 1, 1, 4)\), the mirror shape is \((2, 4, 1, 1)\), and its binary tree is shown on Figure 1. Theorem 3.9 can be visualized by placing into the \(i\)-th node (from bottom to top) the \(i\)-th factor of \(F_{\sigma}(X)\) in Equation (48). For example, the first tree of Figure 2 shows the expansion of \(F_{\sigma}(X)\) with the hook-content factors of \(\sigma = (5, 6, 7, 4, 3, 2, 8, 9, 10, 1, 11)\). We shall see two alternative hook-content formulas for \(F_{\sigma}(X)\). The first one is obtained from an induction formula expressing \(F_{\sigma}(X)\) from \(F_{\text{Std}(\sigma_1, \ldots, \sigma_{n-1})}(X)\), and follows directly from Theorem 3.9.

**Corollary 3.10.** Let \(\partial F_{\sigma}(X) := F_{\text{Std}(\sigma_1, \ldots, \sigma_{n-1})}(X)\) as in [15]. Then,

\[
F_{\sigma}(X) = \partial F_{\sigma}(X) \times \begin{cases} \frac{1 - q^{-n+1}t}{1 - q^n} & \text{if } \sigma_{n-1} < \sigma_n, \\ \frac{1 - q^{-n+1} - t}{1 - q^n} & \text{if } \sigma_{n-2} > \sigma_{n-1} > \sigma_n, \\ \frac{(q^{n-1} - q^{n-2}t)(1 - t)}{(1 - q^{n-2}t)(1 - q^n)} & \text{if } \sigma_{n-2} < \sigma_{n-1} > \sigma_n, \end{cases}
\]

or, equivalently

\[
F_{\sigma}(X) = \partial F_{\sigma}(X) \cdot \frac{q^{(n-1)\alpha} - q^{(n-2)\beta}t}{1 - q^{(n-2)\beta}t} \cdot \frac{1 - q^{(n-1)(1-\alpha)}t}{1 - q^n},
\]
Figure 2. The three hook-content formulas for the permutation 
$(5, 6, 7, 4, 3, 2, 8, 9, 10, 1, 11)$: signed permutations (left diagram), induction (middle diagram), and simplification of the induction (right diagram).

where $a = 1$ if $\sigma_{n-1} > \sigma_n$ and $a = 0$ otherwise, and $b = 1$ if $\sigma_{n-2} < \sigma_{n-1} > \sigma_n$ and $b = 0$ otherwise.

As before, this result can be represented graphically with analogs of the hook-content factors, by placing into node $i$ (from bottom to top) the $i$-th factor of $F_\sigma(X)$ of (50). For example, the second tree of Figure 2 shows the expansion of $F_\sigma(X)$ with our second hook-contents of $\sigma = (5, 6, 7, 4, 3, 2, 8, 9, 10, 1, 11)$.

The hook-content factors described in Corollary 3.10 can have either two or four terms. But one easily checks that, if a factor has four terms, those terms simplify with the factors associated to the preceding letter in the permutation. We recover in this way the partial factors of [19] and obtain a third version of the hook-content formula:
Corollary 3.11 ([19], (152)).

\[
F_\sigma(X) = \prod_{i=1}^{n} \frac{1}{1 - q^i} \begin{cases} 
1 - q^{-1}t & \text{if } \sigma_{i-1} < \sigma_i < \sigma_{i+1}, \\
1 - t & \text{if } \sigma_{i-1} < \sigma_i > \sigma_{i+1}, \\
q^{-1} - t & \text{if } \sigma_{i-2} > \sigma_{i-1} > \sigma_i, \\
q^{-1} - q^{-2}t & \text{if } \sigma_{i-2} < \sigma_{i-1} > \sigma_i,
\end{cases}
\]

with the conventions \(\sigma_0 = 0\) and \(\sigma_{n+1} = +\infty\).

The third tree of Figure 2 shows the resulting expansion of \(F_\sigma(X)\) for the permutation \(\sigma = (5, 6, 7, 4, 3, 2, 8, 9, 10, 1, 11)\). Note that it is obtained by permuting cyclically the numerators of the first formula among right branches.

4. Compatibility between the dendriform operations and specialization of the alphabet

4.1. Dendriform algebras. A dendriform algebra [22] is an associative algebra whose multiplication \(\cdot\) splits into two operations

\[
a \cdot b = a \prec b + a \succ b
\]

satisfying

\[
\begin{cases} 
(x \prec y) \prec z = x \prec (y \cdot z), \\
(x \succ y) \prec z = x \succ (y \prec z), \\
(x \cdot y) \succ z = x \succ (y \succ z).
\end{cases}
\]

For example, \(\text{FQSym}\) is dendriform with the following rules

\[
G_\alpha \prec G_\beta = \sum_{\gamma = u(|\alpha|) \cdot |\beta|; \max(v) \prec \max(u)} G_\gamma,
\]

\[
G_\alpha \succ G_\beta = \sum_{\gamma = u(|\alpha|) \cdot |\beta|; \max(v) \geq \max(u)} G_\gamma.
\]

Note that \(x = G_1 = F_1\) generates a free dendriform dialgebra in \(\text{FQSym}\), isomorphic to \(\text{PBT}\), the Loday-Ronco algebra of planar binary trees [23].

4.2. The half-products and the specialization.

4.2.1. Descent statistics on half-shuffles. On the basis \(F_\sigma\), the half-products are shifted half-shuffles. Recall that the half-shuffles are the two terms of the recursive definition of the shuffle product. For an alphabet \(A\), and two words \(u = u'a, v = v'b, a, b \in A\), one has

\[
u \uplus v = u \prec v + u \succ v,
\]

where

\[
u \prec v = (u' \uplus v)a \text{ and } u \succ v = (u \uplus v')b.
\]
Assuming now that $A$ is totally ordered, we want to investigate the distribution of descents on half-shuffles. To this aim we introduce a linear map

$$\langle w \rangle = F_{D(w)}(X) = \langle w|\sigma_1(XA) \rangle$$

from $\mathbb{K}\langle A \rangle$ to $QSym(X)$, the scalar product on $\mathbb{K}\langle A \rangle$ being defined by $\langle u|v \rangle = \delta_{u,v}$.

For $w \in A^*$, let $\alpha(w) \subseteq A$ be the set of letters occuring in $w$.

**Lemma 4.1.** If $\alpha(u) \cap \alpha(v) = \emptyset$, then

$$\langle u \shuffle v \rangle = \langle u \rangle \langle v \rangle .$$

In particular, the descents of the elements of a shuffle on disjoint alphabets depend only on the descents of the initial elements.

**Proof –** Denote by $\Delta$ the canonical (unshuffle) coproduct of $\mathbb{K}\langle A \rangle$, and write $u''$ for $u \otimes v$, so that $\Delta(a) = 1 \otimes a + a \otimes 1 = a' + a''$ for $a \in A$. Then,

$$\langle u \shuffle v \rangle = \langle u \shuffle v |\sigma_1(XA) \rangle = \left( u'' | \prod_{x \in X} \Delta \sigma_x(A) \right)$$

$$= \left( \prod_{x \in X} \prod_{a \in A} (1 - x(a' + a''))^{-1} \right)$$

$$= \left( \prod_{x \in X} \prod_{a' \in \alpha(u')} (1 - xa')^{-1} \prod_{a'' \in \alpha(v'')} (1 - xa'')^{-1} \right)$$

$$= \langle u|\sigma_1(XA) \rangle \langle v|\sigma_1(XA) \rangle = \langle u \rangle \langle v \rangle .$$

There is an equivalent statement for the dendriform half-products.

**Theorem 4.2.** Let $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$ of respective lengths $k$ and $l$. If $\alpha(u) \cap \alpha(v) = \emptyset$, then

$$\langle u \prec v \rangle = \langle \sigma \prec \tau \rangle$$

where $\sigma = \text{std}(u)$ and $\tau = \text{std}(v)[k]$ if $u_k < v_l$, and $\sigma = \text{std}(u)[l]$ and $\tau = \text{std}(v)$ if $u_k > v_l$.

**Proof –** It is enough to check the first case, so we assume $u_k < v_l$. The proof proceeds by induction on $n = k + l$. Let us set $u = u'a'a$ and $\text{std}(u) = u'_1a'_1a_1$.

If $a' > a'$, since $u \prec v = (u'a' \shuffle v) a$, we have

$$\langle u \prec v \rangle = \sum_{w \in u'a' \shuffle v} F_{D(w)-1} = \langle (u'_1a'_1 \shuffle \tau) \cdot a_1 \rangle$$

with $\tau = \text{std}(v)[k]$, according to Lemma 4.1.

If $a' < a$, write $u \prec v = (u'a' \prec v) \cdot a + (u'a' \succ v) \cdot a$. From the induction hypothesis, we have, with $\tau$ as above, $\langle u'a' \prec v \rangle = \langle u'_1a'_1 \prec \tau \rangle$ and $\langle u'a' \succ v \rangle = \langle u'_1a'_1 \succ \tau \rangle$, so that

$$\langle u \prec v \rangle = \sum_{w \in u'_1a'_1 \succ \tau} F_{D(w)-1} + \sum_{w \in u'_1a'_1 \prec \tau} F_{D(w)-1} .$$
as required.

For example,
\[
\langle 634 \prec 125 \rangle = \langle 631254 + 613254 + 612354 + 612534 + 163254 + 162354 + 162534 + 126354 + 126534 \rangle,
\]
\[
\langle 312 \prec 456 \rangle = \langle 314562 + 341562 + 345162 + 345612 + 431562 + 435162 + 435612 + 453162 + 453612 + 456312 \rangle,
\]
and one can check that both expressions are equal to
\[
F_{132} + F_{141} + F_{1131} + F_{1221} + F_{2231} + F_{2121} + F_{312} + F_{321} + F_{42}.
\]

**Corollary 4.3.** Let \( u \) and \( v \) be two words of respective lengths \( k \) and \( l \). Then, if \( \text{alph}(u) \cap \text{alph}(v) = \emptyset \),
\[
\sum_{x \in u \prec v} q^{\text{maj}(x)} = \sum_{y \in \sigma \prec \tau} q^{\text{maj}(y)},
\]
where \( \sigma = \text{std}(u) \) and \( \tau = \text{std}(v)[k] \) if \( u_k < v_l \), and \( \sigma = \text{std}(u)[l] \) and \( \tau = \text{std}(v) \) if \( u_k > v_l \).

**4.2.2. \((q, t)\)-specialization.** We shall now see that Theorem 4.2 implies a hook-content formula for half-products evaluated over \( \mathcal{X} \). Let \( \sigma \in \mathfrak{S}_n \) and \( \tau \in \mathfrak{S}_m \). Recall that \( \tau[n] \) denotes the word \( \tau_1 + n, \tau_2 + n, \ldots, \tau + m + n \). We have
\[
(q)_{n+m} (F_\sigma \prec F_\tau) (\mathcal{X}) = \sum_{\epsilon \in \{\pm 1\}^{n+m}} \sum_{\mu \in \sigma \prec \tau[n]} (-t)^{m(\epsilon)} q^{\text{maj}(\mu, \epsilon)}
\]
\[
= \sum_{\epsilon_1 \in \{\pm 1\}^n} \sum_{\epsilon_2 \in \{\pm 1\}^m} (-t)^{m(\epsilon)} q^{\text{maj}(\mu')},
\]
where \( (\sigma, \epsilon_1), (\tau[n], \epsilon_2) \), and \( \mu' \) are signed words. Then, thanks to Theorem 4.2, the inner sum is the generating function of the maj statistic on the left dendriform product of two permutations. Its value is known (see [15], Equation (34)), and is
\[
q^{\text{maj}(\sigma, \epsilon_1)} q^{\text{maj}(\tau, \epsilon_2)} C(q),
\]
where \( C(q) \) only depends on the sizes of \( \sigma \) and \( \tau \).

This implies that, if \( \epsilon_1 \) and \( \epsilon'_1 \) are two sign words differing only on one entry,
\[
\sum_{\mu' \in (\sigma, \epsilon_1) \prec (\tau[n], \epsilon_2)} q^{\text{maj}(\mu')} \quad \text{and} \quad \sum_{\mu' \in (\sigma, \epsilon'_1) \prec (\tau[n], \epsilon_2)} q^{\text{maj}(\mu')}
\]
are equal up to a power of \( q \). Moreover, thanks to Lemma 3.8, this factor depends only on \( \sigma \) and is the same as in the Lemma. The same holds for two sign words \( \epsilon_2 \) and \( \epsilon'_2 \) differing on one entry, except for the last value of \( \tau[n] \). In that special case, the contribution of the letter is not given by Lemma 3.8 but by a similar statement where the convention \( y_m = 0 \) is replaced by \( y_m = 1 \). Hence, we have, deducing the second formula from the first one, since their sum is \( F_\sigma(\mathcal{X}) F_\tau(\mathcal{X}) \):
Corollary 4.4. Let \( \sigma \in S_n \) and \( \tau \in S_m \). Then
\[
(F_\sigma \prec F_\tau)(X) = \frac{1 - q^n}{1 - q^{n+m}} \frac{q^m - q^{(m-1)d}t}{1 - q^{(m-1)d}t} F_\sigma(X) F_\tau(X),
\]
and
\[
(F_\sigma \succ F_\tau)(X) = \frac{1 - q^m}{1 - q^{n+m}} \frac{1 - q^{n+(m-1)d}t}{1 - q^{(m-1)d}t} F_\sigma(X) F_\tau(X),
\]
where \( d \) is 1 if \( \tau_m - 1 < \tau_m \) and 0 otherwise.

Example 4.5. Let us present all possible cases on the left dendriform product.
\[
(F_{3421} \prec F_{132})(X) = q^2 \frac{(q - t)^2(1 - t)^3(q^3 - t)^2}{(1 - q^7)(1 - q^3)^2(1 - q^2)^2(1 - q)^2} = \frac{1 - q^4}{1 - q^7} \frac{q^3 - t}{1 - t} F_{3421}(X) F_{132}(X).
\]
\[
(F_{3421} \prec F_{213})(X) = \frac{1 - q^4}{1 - q^7} \frac{q^3 - q^2 t}{1 - q^2 t} F_{3421}(X) F_{213}(X).
\]
\[
(F_{25134} \prec F_{3421})(X) = \frac{1 - q^5}{1 - q^9} \frac{q^4 - t}{1 - t} F_{25134}(X) F_{3421}(X).
\]

5. A hook-content formula for binary trees

5.1. Subalgebras of FQSym. Recall that PBT, the Loday-Ronco algebra of planar binary trees [23], is naturally a subalgebra of FQSym, the embedding being
\[
P_T(A) = \sum_{P(\sigma) = T} F_\sigma(A),
\]
where \( P(\sigma) \) is the shape of the binary search tree associated with \( \sigma \) [14]. Hence, \( P_T(X) \) is well defined.

It was originally defined [23] as the free dendriform algebra on one generator as follows: if \( T \) is a binary tree \( T_1 \) (resp. \( T_2 \)) be its left (resp. right) subtree, then
\[
P_T = P_{T_1} \succ P_1 \prec P_{T_2}.
\]

5.2. Hook-content formulas in PBT. Note first that Corollary 4.4 implies that the left and right dendriform half-products factorize in the \( X \)-specialization. Because of the different expressions on signed permutations, it also proves that the same property holds for trees, thanks to \( (78) \).

Then, as a corollary of the definition of \( P_T \) and Corollary 3.7 we have

Corollary 5.1. Let \( T \) be a binary tree. Then
\[
P_T(X) = \frac{1}{(q)_n} \sum_{(\sigma, \epsilon) | P(\sigma) = T} (-t)^{m(\epsilon)} q^{maj(\sigma, \epsilon)}.
\]
Recall that any binary tree has a unique standard labelling that makes it a binary search tree. We then define the hook-content of a given node as the contribution of its label among all signed permutations having this tree as binary search tree. Thanks to Corollary 4.4, we get a two-parameter version of the $q$-hook-length formulas of Björner and Wachs [3, 4] (see also [15]):

**Theorem 5.2.** Let $T$ be a tree and $s$ a node of $T$. Let $n$ be the size of the subtree whose root is $s$. The $(q, t)$-hook-content factor of $s$ into $T$ is given by the following rules:

\[
\frac{1}{1-q^n} \begin{cases} 
q^n - q^{n'} t & \text{if } s \text{ is the right son of its father}, \\
1 - q^{n'} t & \text{otherwise},
\end{cases}
\]

where $n'$ is the size of the left subtree of $s$.

As in the case of $\mathbf{FQSym}$, this can be represented graphically by placing into each node the fraction appearing in Equation (80). For example, the first tree of Figure 2 shows the expansion of $F_\sigma(X)$ with the first hook-contents of a zig-zag tree. Figure 3 gives another example of this construction.

**Figure 3.** A binary tree (left diagram) labelled as a standard binary search tree and the first hook-content formula on trees (right diagram).

In particular, replacing $t$ by $-t$ in all formulas, this gives the following combinatorial interpretation of the $(q, t)$ hook-length formula (recall that $P(\sigma) = T(\sigma^{-1})$, where $T(\tau)$ denotes the decreasing tree of $\tau$):

**Corollary 5.3.** Let $T$ be a binary tree. Then the generating function of signed permutations of shape $T$ by major index and number of signs is:

\[
(q)_n P_T(X) |_{t=-t} = \sum_{(\sigma, e) \mid P(\sigma) = T} t^{\ell(e)} q^{\text{maj}(\sigma, e)}.
\]
Example 5.4. For example, with

\[ T = \begin{array}{c}
\text{3} \\
\text{2} \\
\text{1} \\
\text{5} \\
\text{4} \\
\text{6}
\end{array} \]

one has:

\[ \sum_{(\sigma, e) \mid P(\sigma) = T} t^{m(e)} q^{\text{maj}(\sigma, e)} = (q)_6 \frac{(q + t)^2(1 + t)^2(1 + q^2 t)(q^3 + qt)}{(1 - q)^3(1 - q^3)(1 - q^6)(1 - q^6)} \]

\[ = (q + t)^2(1 + t)^2(1 + q^2 t)[4]_q [5]_q \]

Here are the analogs of the other two hook-content formulas of FQSym.

Theorem 5.5. Let \( T \) be a binary tree and \( T_1 \) (resp. \( T_2 \)) be its left (resp. right) subtree. Let \( T'_2 \) be the left subtree of \( T_2 \). We then have

\[ \mathbf{P}_T(X) = \frac{(q^{|T_2|} - q^{|T'_2|} t)(1 - q^{|T_1|})}{(1 - q^{|T'_2|} t)(1 - q^n)} \mathbf{P}_{T_1}(X) \mathbf{P}_{T_2}(X). \]

Proof – This is a direct consequence of the dendriform specializations in FQSym thanks to (78).

As in the case of FQSym, it is possible to simplify the product so as to obtain a single quotient at each node.

Corollary 5.6. Let \( T \) be a tree and \( s \) a node of \( T \). Let \( n \) be the size of the subtree whose root is \( s \). The \((q, t)\)-hook-content factor of \( s \) into \( T \) is given by the following rules:

\[ \frac{1}{1 - q^n} \begin{cases} 
q^n' - q^n'' t & \text{if } s \text{ has a right son}, \\
1 - q^{n-1} t & \text{if } s \text{ has no right son and is not the right son of its father}, \\
1 - q^d t & \text{if } s \text{ has no right son and is the right son of its father},
\end{cases} \]

where \( n' \) is the size of the right subtree of \( s \), \( n'' \) is the size of the left subtree of the right subtree of \( s \), and \( d \) is the size where of the left subtree of the topmost ancestor of \( s \) leading to \( s \) only by right branches.

For example, on Figure 4, the rightmost node of the second tree has coefficient \( \frac{1 - q^2 t}{1 - q^2} \): its topmost ancestor is the root of the tree and the left subtree of the root is of size 2. Note that it is obtained by permuting cyclically the numerators of the first formula among right branches, as it was already the case in FQSym.

6. Word Super-quasi-symmetric functions

6.1. Word quasi-symmetric functions. Recall that a word \( u \) over \( \mathbb{N}^* \) is packed if the set of letters appearing in \( u \) is an interval of \( \mathbb{N}^* \) containing 1. Recall also
that \( \mathsf{WQSym}(A) \) is defined as the subalgebra of \( \mathbb{K}(A) \) indexed by packed words and spanned by the elements

\[
\mathbf{M}_u(A) := \sum_{\text{pack}(w) = u} w,
\]

where \( \text{pack}(w) \) is the packed word of \( w \), that is, the word obtained by replacing all occurrences of the \( k \)-th smallest letter of \( w \) by \( k \). For example,

\[
\text{pack}(871883319) = 431442215.
\]

Let \( \mathbf{N}_u = \mathbf{M}_u^* \) be the dual basis of \( (\mathbf{M}_n) \). It is known that \( \mathsf{WQSym} \) is a self-dual Hopf algebra \([13, 32]\) and that on the dual \( \mathsf{WQSym}^* \), an internal product * may be defined by

\[
\mathbf{N}_u * \mathbf{N}_v = \mathbf{N}_{\text{pack}(u,v)},
\]

where the packing of biwords is defined with respect to the lexicographic order, so that, for example,

\[
\text{pack} \begin{pmatrix}42412253 \\ 53154323\end{pmatrix} = 62513274.
\]

This product is induced from the internal product of parking functions \([33, 28, 34]\) and allows one to identify the homogeneous components \( \mathsf{WQSym}_n \) with the (opposite) Solomon-Tits algebras, in the sense of \([35]\).
The (opposite) Solomon descent algebra, realized as $\text{Sym}_n$, is embedded in the (opposite) Solomon-Tits algebra realized as $\text{WQSym}_n^*$ by

$$S^I = \sum_{ev(u)=I} N_u.$$  

6.2. **An algebra on signed packed words.** Let us define $\text{WQSym}^{(2)}$ as the space spanned by the $M_{u,\epsilon}$, where

$$M_{u,\epsilon}(A) := \sum_{(w,\epsilon) \text{ regular}} \text{pack}(w) = u (w, \epsilon).$$

This is a Hopf algebra for the standard operations. We denote by $N_{u,\epsilon}$ the dual basis of $M$. This algebra contains $\text{Sym}^{(2)}$, the Mantaci-Reutenauer algebra of type $B$. To show this, let us describe the embedding.

A signed word is said to be regular if all occurrences of any unsigned letter have same sign. For example, $12231$ is regular, but $11\overline{1}2$ and $112\overline{1}2$ are not.

The signed evaluation $\text{sev}(w, \epsilon)$ of a regular word is the signed composition $(I, \mu)$ where $i_j$ is the number of occurrences of the (unsigned) letter $j$ and $\mu_j$ is the sign of $j$ in $(w, \epsilon)$.

Let $\phi$ be the morphism from $\text{Sym}^{(2)}$ into $\text{WQSym}^{(2)}$ defined by

$$\phi(S_n) = N_{1^n}, \quad \phi(S_{\pi}) = N_{\pi^n}.$$  

We then have:

**Lemma 6.1.**

$$\phi(S^{(I,\epsilon)}) = \sum_{(u,\epsilon') \text{ regular} \atop \text{sev}(u,\epsilon')=(I,\epsilon)} N_{u,\epsilon'}.$$  

*Proof –* This follows from the product formula of the $N$, which is a special case of the multiplication of signed parking functions [29].

The image of $\text{Sym}^{(2)}$ by this embedding is contained in the Hopf subalgebra $BW$ of $\text{WQSym}^{(2)}$ generated by the $N_{u,\epsilon}$ indexed by regular signed packed words. The dimensions of the homogeneous components $BW_n$ are given by Sequence A004123 of [37] whose first values are

$1, 2, 10, 74, 730, 9002, 133210.$

Note in particular that $\sigma^\#_1$ has a simple expression in terms of $N_{u,\epsilon}$.

**Lemma 6.2.** Let $PW$ denote the set of packed words, and $\text{max}(u)$ the maximal letter of $u$. Then

$$\sigma^\#_1 = \sum_{u \in PW} (-1)^{n-\text{max}(u)} N_{u,(-1)^n} + (-1)^{m(\epsilon')-(\text{max}(u)-1)} N_{u,\epsilon'},$$

where $(u, \epsilon')$ is such that all letters but the maximal one are signed.
Example 6.3.

\[(97) \quad S^2 = -N_{11} + N_{12} + N_{21} + N_{22} + N_{3T} + N_{3T}.
\]

\[(98) \quad S^3 = + N_{111} - N_{112} + N_{121} + N_{211} - N_{212} + N_{311} + N_{312} + N_{321} + N_{32T}.
\]

6.3. An internal product on signed packed words. The internal product of $WQSym^*$ \[89\] can be extended to $WQSym^{(2)*}$ by

\[(99) \quad N_{u,\epsilon} \star N_{v,\rho} = N_{\text{pack}(u,v),\epsilon \rho},
\]

where $\epsilon \rho$ is the componentwise product. One obtains in this way the (opposite) Solomon-Tits algebra of type $B$. This product is induced from the internal product of signed parking functions \[29\] and can be shown to coincide with the one introduced by Hsiao \[17\].

From this definition, we immediately have:

**Proposition 6.4.** $BW$ is a subalgebra of $WQSym^{(2)*}$ for the internal product.

Since $\sigma_1^\#$ belongs to $WQSym^{(2)*}$, we can define

\[(100) \quad N_1^\# := N_u(A|\bar{A}) = N_u \star \sigma_1^\#.
\]

Example 6.5. Let us compute the first $N_u(A|\bar{A})$.

\[(101) \quad N_{11}^\# = -N_{11} + N_{12} + N_{21} + N_{22} + N_{3T} + N_{3T}.
\]

\[(102) \quad N_{12}^\# = N_{121} + N_{122} + N_{211} + N_{212} + N_{311} + N_{312} + N_{321} + N_{32T}.
\]

\[(103) \quad N_{21}^\# = N_{2T} + N_{211} + N_{212} + N_{3T} + N_{312} + N_{31T} + N_{32T}.
\]

\[(104) \quad N_{112}^\# = -N_{111} - N_{112} + N_{121} + N_{122} + N_{123} + N_{12T} + N_{21T} + N_{22T} + N_{23T} + N_{31T} + N_{32T} + N_{33T}.
\]

\[(105) \quad N_{121}^\# = -N_{121} - N_{122} + N_{123} + N_{12T} + N_{123} + N_{12T} + N_{21T} + N_{22T} + N_{23T} + N_{31T} + N_{32T} + N_{33T}.
\]

In the light of the previous examples, let us say that a packed word $v$ is finer than a packed $u$, and write $v \geq u$ if $u$ can be obtained from $v$ by application of a nondecreasing map from $N^*$ to $N^*$. Note that this definition is easy to describe on set compositions: $u$ is then obtained by gluing together consecutive parts of $v$. For example, the words finer than 121 are 121, 132, and 231.
Theorem 6.6. Let $u$ be a packed word. Then
\begin{equation}
\mathbf{N}_u^\# = \sum_{v \geq u} \sum_{\epsilon} (-1)^{m(\epsilon) + m'(v, \epsilon)} \mathbf{N}_{v, \epsilon}
\end{equation}
where $m'(v, \epsilon)$ is equal to the number of different signed letters of $v$ and where the sum on $\epsilon$ is such that the words $(v, \epsilon)$ are regular and such that if more than two letters of $v$ go to the same letter of $u$, all letters but the greatest are signed (the greatest can be either signed or not). In particular, the number of such $\epsilon$ for a given $v$ is equal to $2^{\max(u)}$, so is independent on $v$.

Proof – From the definitions of $\sigma_1^\#$ and of the packing algorithm, it is clear that the words appearing on the expansion of $\mathbf{N}_u^\#$ are exactly the words given in the previous statement.

Moreover, the coefficient of a signed word $(w, \epsilon)$ in $\sigma_1^\#$ is equal to the coefficient of any of its rearrangements (where the signs follow their letter). Now, given a permutation $\sigma$ and two words $u$ and $u'$ having a word $v$ as packed word, the packed word of $u \cdot \sigma$ and $u' \cdot \sigma$ is $v \cdot \sigma$. So we can restrict ourselves to compute $\mathbf{N}_u^\#$ for all nondecreasing words $u$ since all the other ones are obtained by permutation of the entries.

Assume now that $u$ is a nondecreasing word, and let us show that the coefficient of $(v, \epsilon)$ in $\mathbf{N}_u^\#$ is either 1 or $-1$. The only terms $\mathbf{N}$ in $\sigma_1^\#$ that can yield $(v, \epsilon)$ when multiplied on the left by $\mathbf{N}_u$ are the signed words with negative entries exactly as in $\epsilon$. Let $T_\epsilon$ denote this set. Thanks to Lemma 6.2, the $\mathbf{N}$ appearing in the expansion of $\sigma_1^\#$ with negative signs at $k$ given slots are the following packed words: all the elements of $\text{PW}_k$ at the negative slots and one letter greater than all the others at the remaining slots. In particular, the cardinality of $T_\epsilon$ depends only on $k$ and is equal to $|\text{PW}_k|$. Since there is only one positive value for each element, two words $w$ and $w'$ of $T_\epsilon$ give the same result by packing $(u, w)$ and $(u, w')$ if they coincide on the negative slots.

This means that we can restrict ourselves to the special case where $\epsilon = (-1)^n$ since the positive slot do not change the way of regrouping the elements of $T_\epsilon$ to obtain $(v, \epsilon)$. Now, the sign has been disposed of and we can concentrate on the packing algorithm. The previous discussion shows that we only need to prove that, given a word $v$ finer than a word $u$, the set $T$ of packed words $w$ such that pack$(u, w) = v$ satisfies the following property: if $t_d$ is the number of elements of $T$ with maximum $d$, then
\begin{equation}
\sum_d (-1)^d t_d = \pm 1.
\end{equation}

Thanks to the packing algorithm, we see that $T$ is the set of packed words with (in)equalities coming from the values of $v$ at the places where $u$ have equal letters. So $T$ is a set of packed words with (in)equalities between adjacent places with no other relations. Hence, if $u$ has $l$ different letters, $T$ is obtained as the product of $l$ quasi-monomial functions $M_w$. The conclusion of the proof comes from the following lemma. 

\end{document}
Lemma 6.7. Let \( w_1, \ldots, w_k \) be \( k \) packed words with respective maximum letters \( a_1, \ldots, a_k \). Let \( T \) be the set of packed words appearing in the expansion of
\[
\text{M}_{w_1} \cdots \text{M}_{w_k}.
\]
Then, if \( t_d \) is the number of elements of \( T \) with maximum \( d \), then
\[
\sum_d (-1)^d t_d = (-1)^{a_1 + \cdots + a_k}.
\]

Proof – We only need to prove the result for \( k = 2 \) since the other cases follow by induction: compute \( \text{M}_{w_1} \cdots \text{M}_{w_{k-1}} \) and multiply this by \( \text{M}_{w_k} \) to get the result.

Let us compute \( \text{M}_{w_1} \text{M}_{w_2} \). The number of words with maximum \( a_1 + a_2 - d \) in this product is equal to
\[
\binom{a_1}{d} \binom{a_1 + a_2 - d}{a_1}.
\]
Indeed, a word in \( \text{M}_{w_1} \text{M}_{w_2} \) with maximum \( a_1 + a_2 - d \) is completely characterized by the \( d \) integers between 1 and \( a_1 + a_2 - d \) common to the prefix of size \( |w_1| \) and the suffix of size \( |w_2| \) of \( w \), by the \( (a_1 - d) \) integers only appearing in the prefix, and the \( (a_2 - d) \) integers only appearing in the suffix, which hence gives the enumeration formula
\[
t_{a_1 + a_2 - d} = \frac{(a_1 + a_2 - d)!}{d!(a_1 - d)!(a_2 - d)!},
\]
equivalent to the previous one.

It remains to compute
\[
\sum_d (-1)^{a_1 + a_2 - d} \binom{a_1}{d} \binom{a_1 + a_2 - d}{a_1},
\]
which is, with the usual notation for elementary and complete homogeneous symmetric functions, understood as operators of the \( \lambda \)-ring \( \mathbb{Z} \),
\[
(-1)^{a_1 + a_2} \sum_d (-1)^d e_d(a_1) h_{a_2 - d}(a_1 + 1)
\]
\[
= (-1)^{a_1 + a_2} \sum_d h_d(-a_1) h_{a_2 - d}(a_1 + 1)
\]
\[
= (-1)^{a_1 + a_2} h_{a_2}(-a_1 + a_1 + 1)
\]
\[
= (-1)^{a_1 + a_2} h_{a_2}(1) = (-1)^{a_1 + a_2}.
\]

This combinatorial interpretation of (110) gives back in particular one interpretation of the Delannoy numbers (sequence A001850 of [37]) and of their usual refinement (sequence A008288 of [37]).
6.4. Specializations. The internal product of $\text{WQSym}^*$ allows in particular to define
\[ N_u((1-t)A) := N_u(A) * \sigma_1((1-t)A) = \eta_t(N_u), \]
so that we have
\[ S'((1-t)A) = \sum_{\text{Ev}(u)=1} N_u((1-t)A). \]

Example 6.8. Taking the same five examples as in Example 6.5, we get
\[ N_{11}((1-t)A) = (1-t^2)N_{11} - t(1-t)N_{12} - t(1-t)N_{21}. \]
\[ N_{12}((1-t)A) = (1-t)^2N_{12} \quad \text{and} \quad N_{21}((1-t)A) = (1-t)^2N_{21}. \]
\[ N_{112}((1-t)A) = (1-t)(1-t^2)N_{112} - t(1-t)^2N_{123} - t(1-t)^2N_{213} - t(1-t)^2N_{132} - t(1-t)^2N_{231}. \]

Theorem 6.9. Let $u$ be a packed word. Then
\[ N_u((1-t)A) = \sum_{v \geq u} (-1)^{\max(v) - \max(u)} \prod_{k=1}^{\max(u)} (1 - t^{g(u,v,k)}) \ N_v(A). \]
where, if one writes
\[ \text{Ev}(u) = (i_1, \ldots, i_p) \quad \text{and} \quad \text{Ev}(v) = ((i_1^{(1)}, \ldots, i_1^{(q_1)}), \ldots, (i_p^{(1)}, \ldots, i_p^{(q_p)})), \]
then
\[ f(u,v) := \sum_{k=1}^{p} \sum_{j=1}^{q_k-1} i_k^{(j)} \quad \text{and} \quad g(u,v,k) := i_k^{(q_k)}. \]

Proof – This is a direct consequence of Theorem 6.6.

6.5. Duality. By duality, one defines
\[ M_u(A \cdot (1-t)) := \eta_t^*(M_u(A)), \]
since
\[ \sum_u M_u(A \cdot (1-t)) \otimes N_u(B) = \sum_u M_u(A) \otimes N_u((1-t)B). \]

Example 6.10.
\[ M_{11}((1-t)A) = (1-t^2)M_{11}(A). \]
\[ M_{12}((1-t)A) = -t(1-t)M_{11}(A) + (1-t)^2M_{12}(A). \]
\[ M_{21}((1-t)A) = -t(1-t)M_{11}(A) + (1-t)^2M_{21}(A). \]
\[ M_{112}((1-t)A) = (1-t)(1-t^2)M_{112}(A) - t^2(1-t)M_{111}(A). \]
\[ M_{121}((1-t)A) = (1-t)(1-t^2)M_{121}(A) - t^2(1-t)M_{111}(A). \]
\[ M_{123}(A \cdot (1 - t)) = (1 - t)^3 M_{123}(A) - t(1 - t)^2 M_{112}(A) - t(1 - t)^2 M_{122}(A) + t^2(1 - t) M_{111}(A). \]

Since the transition matrix from \( M(A \cdot (1 - t)) \) to \( M(A) \) is the transpose of the transition matrix from \( N((1 - t)A) \) to \( N(A) \), we can obtain a simple combinatorial interpretation of \( M(A \cdot (1 - t)) \).

First, let us define the super-packed word \( v := \text{spack}(u, \epsilon) \) associated with a regular signed word \( (u, \epsilon) \). Let \( f_{\epsilon} \) be the nondecreasing function sending 1 to 1 and each value \( i \) either to \( f_{\epsilon}(i - 1) \) if the value \( i - 1 \) is signed in \( \epsilon \) or to \( 1 + f_{\epsilon}(i - 1) \) if not. Extend \( f_{\epsilon} \) to a morphism of \( A^* \). Then \( v = f_{\epsilon}(u) \).

For example,

\[ \text{spack}(5121354461) = 211122231. \]

Let \([v, u]\) be the interval for the refinement order on words, that is, the set of packed words \( w \) such that \( u \geq w \geq v \).

**Proposition 6.11.** Let \( u \) be a word. Then

\[ M_u(A \cdot (1 - t)) = \sum_{(u, \epsilon) \text{regular}} (-1)^{m'(u, \epsilon)} t^{m(\epsilon)} \sum_{w \in [\text{spack}(u, \epsilon), u]} M_w(A). \]

**Proof** – Observe that if a signed word \( (u, \epsilon) \) appears in \( N_\# w \) then it also appears in \( N_\# v \) for all \( v \in [u, w] \). The rest comes directly from Theorem 6.6 and from the fact that \( N(u, \epsilon) \) is sent to \( (-t)^{m(\epsilon)} N_u \) when sending \( A \) to \( -tA \).

**Example 6.12.**

\[ M_{21}(A \cdot (1 - t)) = (-t + t^2)(M_{11} + M_{21}) + (1 - t) M_{21}. \]

\[ M_{112}(A \cdot (1 - t)) = (-t^2 + t^3)(M_{111} + M_{112}) + (1 - t) M_{112}. \]

\[ M_{123}(A \cdot (1 - t)) = (t^2 - t^3)(M_{111} + M_{112} + M_{122} + M_{123}) + (-t + t^2)(M_{112} + M_{123}) + (-t + t^2)(M_{122} + M_{123}) + (1 - t) M_{123}. \]

When \( A \) is a commutative alphabet \( X \), this specializes to \( M_I(X(1 - t)) \) where \( I = \text{Ev}(u) \) and in particular, for \( X = \frac{1}{1 - q} \), we recover a result of [19]:

**Theorem 6.13 ([19]).** Let \( u \) be a packed word of size \( n \).

\[ M_u(X) = M_I(X) = \frac{1 - t^n}{1 - q^n} \prod_{k=1}^{n-1} \frac{(q^{i_1 + \cdots + i_k} - t^{i_k})}{1 - q^{i_1 + \cdots + i_k}}. \]

where the composition \( I = (i_1, \ldots, i_n) \) is the evaluation of \( u \).
Proof – From Proposition 6.11 giving a combinatorial interpretation of $M_u(A \cdot (1-t))$, we have:

\begin{equation}
M_u(X) = \sum_{(u, \epsilon) \text{regular}} (-1)^{m'(u, \epsilon)} t^{m(\epsilon)} \sum_{w \in \text{spack}(u, \epsilon, u)} M_w(1/(1-q)) \tag{137}
\end{equation}

We now have to evaluate the sum of $M_w(1/(1-q))$ over an interval of the composition lattice. Thanks to Lemma 6.14 below, it is equal to

\begin{equation}
q^{\text{maj}(I)} \cdot \\
(1-q^{k_1})(1-q^{k_1+k_2}) \cdots (1-q^{k_1+k_2+\cdots+k_s}),
\end{equation}

where $I = \text{Ev}(\text{spack}(u, \epsilon))$ and $K = \text{Ev}(u)$, which implies the result. \hfill \blacksquare

Lemma 6.14. Let $I$ and $K$ be two compositions of $n$ such that $K \geq I$. Then

\begin{equation}
\sum_{J \in [I, K]} M_J(1/(1-q)) = \frac{1}{1-q^n} \prod_{d \in \text{Des}(K)} \frac{q^d}{1-q^d} \tag{139}
\end{equation}

Proof – We have

\begin{equation}
M_J(1/(1-q)) = \frac{1}{1-q^n} \prod_{d \in \text{Des}(J)} \frac{q^d}{1-q^d} \tag{140}
\end{equation}

Factorizing by the common denominator of all these elements and by $q^{\text{maj}(K)}$, we have to evaluate

\begin{equation}
\sum_{D \subseteq \text{Des}(K) \setminus \text{Des}(I)} \prod_{d \in D} (1-q^d)q^{-d} \tag{141}
\end{equation}

which is equal to

\begin{equation}
\prod_{d \in \text{Des}(K)/\text{Des}(I)} (1 - 1 + q^{-d}) = q^{-\text{maj}(K)-\text{maj}(I)}. \tag{142}
\end{equation}

Putting together Proposition 6.11 and Lemma 6.14 one obtains:

Corollary 6.15. Let $u$ be a word of size $n$. Then

\begin{equation}
(q_{\text{Ev}(u)})M_u(X) = \sum_{(u, \epsilon) \text{regular}} (-1)^{m'(u, \epsilon)} t^{m(\epsilon)} q^{\text{maj}(\text{spack}(u, \epsilon))}, \tag{143}
\end{equation}

where $(q)_i$ is defined as $(1 - q^n) \prod_{d \in \text{Des}(I)} (1 - q^d)$.

Corollary 6.16. Let $u$ be a word of size $n$. Then the generating function of signed permutations of unsigned part $u$ by major index of their super-packed word and number of signs is:

\begin{equation}
\sum_{(u, \epsilon) \text{regular}} t^{m(\epsilon)} q^{\text{maj}(\text{spack}(u, \epsilon))} = (1 + t^p) \prod_{k=1}^{p-1} (q^{i_1+\cdots+i_k} + t^{i_k}). \tag{144}
\end{equation}
Example 6.17. For example, with $u = (1, 1, 2, 3, 3, 3, 4, 4)$, one has:
\[
\sum_{(u, c) \text{ regular}} t^{m(c)} q^{\text{maj}(\text{spack}(u, c))} = (1 + t^2)(q^2 + t^2)(q^3 + t)(q^7 + t^4).
\]

7. Tridendriform operations and the specialization of alphabet

7.1. Tridendriform structure of WQSym. A dendriform trialgebra \cite{24} is an associative algebra whose multiplication $\cdot$ splits into three pieces
\[
x \cdot y = x \prec y + x \circ y + x \succ y,
\]
where $\circ$ is associative, and
\[
(x \prec y) \prec z = x \prec (y \cdot z), \quad (x \succ y) \prec z = x \succ (y \prec z), \quad (x \cdot y) \succ z = x \succ (y \succ z),
\]
\[
(x \succ y) \circ z = x \circ (y \cdot z), \quad (x \prec y) \circ z = x \circ (y \succ z), \quad (x \circ y) \prec z = x \circ (y \prec z).
\]

It has been shown in \cite{31} that the augmentation ideal $\mathbb{K}\langle A \rangle^+$ has a natural structure of dendriform trialgebra: for two non empty words $u, v \in A^*$, we set
\[
u \prec v = \begin{cases} uv & \text{if } \max(u) > \max(v) \\ 0 & \text{otherwise}, \end{cases}
\]
\[
u \circ v = \begin{cases} uv & \text{if } \max(u) = \max(v) \\ 0 & \text{otherwise}, \end{cases}
\]
\[
u \succ v = \begin{cases} uv & \text{if } \max(u) < \max(v) \\ 0 & \text{otherwise}. \end{cases}
\]

WQSym$^+$ is a sub-dendriform trialgebra of $\mathbb{K}\langle A \rangle^+$, the partial products being given by
\[
M_{w'} \prec M_{w''} = \sum_{u = u' \cdot v \in \mathcal{W} \setminus W, |u| = |w'|; \max(v) < \max(u)} M_w,
\]
\[
M_{w'} \circ M_{w''} = \sum_{u = u' \cdot v \in \mathcal{W} \setminus W, |u| = |w'|; \max(v) = \max(u)} M_w,
\]
\[
M_{w'} \succ M_{w''} = \sum_{u = u' \cdot v \in \mathcal{W} \setminus W, |u| = |w'|; \max(v) > \max(u)} M_w,
\]
where the convolution $u' \ast_W u''$ of two packed words is defined as
\[
u' \ast_W u'' = \sum_{v, w; u = v \cdot w \in \mathcal{P}_W, \text{pack}(v) = u', \text{pack}(w) = u''} u.
\]
7.2. Specialization of the partial products. If $w$ is a packed word, let $\text{NbMax}(w)$ be the number of maximal letters of $w$ in $w$.

**Theorem 7.1.** Let $u_1 \in \text{PW}(n)$ and $u_2 \in \text{PW}(m)$. Then

$$
(M_{u_1} \prec M_{u_2})(X) = \frac{1 - q^n}{1 - q^{n+m}} \frac{q^m - t^{\text{NbMax}(u_2)}}{1 - t^{\text{NbMax}(u_2)}} M_{u_1}(X)M_{u_2}(X),
$$

$$
(M_{u_1} \circ M_{u_2})(X) = \frac{(1 - q^n)(1 - q^m)}{1 - q^{n+m}} \frac{1 - t^{\text{NbMax}(u_1)+\text{NbMax}(u_2)}}{(1 - t^{\text{NbMax}(u_1)})(1 - t^{\text{NbMax}(u_2)})} M_{u_1}(X)M_{u_2}(X),
$$

and

$$
(M_{u_1} \succ M_{u_2})(X) = \frac{1 - q^m}{1 - q^{n+m}} \frac{q^n - t^{\text{NbMax}(u_1)}}{1 - t^{\text{NbMax}(u_1)}} M_{u_1}(X)M_{u_2}(X).
$$

**Proof.** Thanks to the combinatorial interpretation of $M_u(X)$ in terms of signed words (Proposition 6.11 and Lemma 6.14), one only has to check what happens to the major index of the evaluation of signed words in the cases of the left, middle, or right tridendriform products. The analysis is similar to that done for $\text{FQSym}$ in the previous sections.

**Example 7.2.** Note that the left tridendriform product does not depend on the actual values of $w_1$ but only on its length. Indeed, one can check that

$$
(M_{111} \prec M_{2122})(X) = \frac{1 - q^3}{1 - q^7} \frac{q^4 - t^3}{1 - t^3} M_{111}(X)M_{2122}(X)
$$

$$
(M_{132} \prec M_{2122})(X) = \frac{1 - q^3}{1 - q^7} \frac{q^4 - t^3}{1 - t^3} M_{132}(X)M_{2122}(X)
$$

But the result depends on the number of maximum of $w_2$:

$$
(M_{121} \prec M_{3122})(X) = \frac{1 - q^3}{1 - q^7} \frac{q^4 - t}{1 - t} M_{121}(X)M_{3122}(X)
$$

One can check on these examples the relation of dendriform trialgebras: $M_u M_v = M_u \prec M_v + M_u \circ M_v + M_u \succ M_v$:

$$
(M_{1212} \prec M_{33231})(X) = \frac{1 - q^4}{1 - q^9} \frac{q^5 - t^3}{1 - t^3} M_{1212}(X)M_{33231}(X)
$$

$$
(M_{1212} \circ M_{33231})(X) = \frac{(1 - q^4)(1 - q^5)}{1 - q^9} \frac{1 - t^5}{(1 - t^2)(1 - t^3)} M_{1212}(X)M_{33231}(X)
$$

$$
(M_{1212} \succ M_{33231})(X) = \frac{1 - q^6}{1 - q^9} \frac{q^4 - t^2}{1 - t^2} M_{1212}(X)M_{33231}(X).
$$
8. The free dendriform trialgebra

8.1. A subalgebra of WQSym. Recall that $TD$, the Loday-Ronco algebra of plane trees [24], is naturally a subalgebra of WQSym [32], the embedding being

$$M_T(A) = \sum_{T(u) = T} M_u(A),$$

where $T(u)$ is the decreasing plane tree associated with $u$ [32]. Hence, $M_T(X)$ is well-defined.

$TD$ was originally defined [24] as the free tridendriform algebra on one generator as follows: if $T$ is a planar tree and $T_1, \ldots, T_k$ are its subtrees, then

$$M_T = (M_{T_1} \triangleright M_1 \triangleleft M_{T_2}) \circ (M_1 \triangleleft M_{T_3}) \circ \ldots (M_1 \triangleleft M_{T_k}).$$

8.2. $(q, t)$-hooks. Let $T$ be a plane tree. Let Int$(T)$ denote all internal nodes of $T$ except the root. Let us define a region of $T$ as any part of the plane between two edges coming from the same vertex. The regions are the places where one writes the values of a packed word when inserting it (see [32]). For example, with $w = 243411$, one gets

Theorem 8.1. Let $T$ be a plane tree with $n$ regions. Then

$$M_T(X) = \frac{1 - t^{a(r)-1}}{1 - q^n} \prod_{i \in \text{Int}(T)} \frac{q^{r(i)} - t^{a(i)-1}}{1 - q^{s(i)}},$$

where $a(i)$ is the arity of $i$ and $r(i)$ the number of regions of $T$ below $i$.

Proof – This is obtained by applying the tridendriform operations in WQSym, thanks to (166). $\blacksquare$

Writing for each node the numerator of its $(q, t)$ contribution, one has, for example:

$$M_T(X) = \frac{1 - t^2}{q - t} \frac{1 - t^2}{q - t} \frac{q^2 - t^2}{q - t},$$

$$(q - t) \quad \begin{array}{c} q - t \quad \begin{array}{c} q - t \quad \begin{array}{c} 1 \quad \begin{array}{c} 1 - t^2 \end{array} \end{array} \end{array} \end{array} \end{array}$$

$$\begin{array}{c} (q - t) \quad \begin{array}{c} q^3 - t^3 \quad q^4 - t \end{array} \end{array}$$
References

[1] A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Adv. Math. 64:2 (1987), 118–175.

[2] N. Bergeron, F. Hivert and J.-Y. Thibon, *The peak algebra and the Hecke-Clifford algebra at q = 0*, J. Comb. Theory A 107 (2004), 1–19.

[3] A. Björner and M. Wachs, *q-Hook length formulas for forests*, J. Combin. Theory Ser. A 52 (1989), 165–187.

[4] A. Björner and M. Wachs, *Permutation statistics and linear extensions of posets*, J. Combin. Theory Ser. A 86 (1999), 153–175.

[5] F. Chapoton, *Alg`ebres de Hopf des permuto`edres, associa`edres et hypercubes*, Adv. in Math. 150 (2000), 264–275.

[6] G. Duchamp, F. Hivert and J.-Y. Thibon, *The peak algebra and the Hecke-Clifford algebra at q = 0*, J. Comb. Theory A 109 (2004), 1–19.

[7] D. Foata and M. P. Schützenberger, *Major index and inversion number of permutations*, Math. Nachr. 83 (1970), 143–159.

[8] S. Frame, G. de B. Robinson and R. M. Thrall, *The hook graphs of the symmetric group*, Canad. J. Math. 6 (1954), 316–324.

[9] I.M. Gelfand, D. Krob, A. Lascoux, V.S. Retakh and J.-Y. Thibon, *Noncommutative symmetric functions*, Adv. in Math. 112, 1995, p. 218–348.

[10] I. Gessel, *Multipartite P-partitions and inner product of skew Schur functions*, Contemp. Math., 34 (1984), 289–301.

[11] M. Haiman, J. Haglund and N. Loehr, *A Combinatorial Formula for Macdonald Polynomials*, J. Amer. Math. Soc. 18 (2005), 735–761.

[12] J.-H. Kwon, *Crystal graphs for general linear Lie superalgebras and quasi-symmetric functions*, preprint, arXiv:0710.0253.

[13] D. E. Knuth, *The art of computer programming*, vol. 3: Sorting and searching, Addison-Wesley, 1973.

[14] F. Hivert, J.-C. Novelli and J.-Y. Thibon, *The algebra of binary search trees*, Theoret. Comput. Sci. 339 (2005), 129–165.

[15] F. Hivert, J.-C. Novelli and J.-Y. Thibon, *Trees, functional equations, and combinatorial Hopf algebras*, Europ. J. Combin. (to appear) [arXiv:math/0701539]

[16] F. Hivert and N. Thiéry, *MuPAD-Combinat, an open-source package for research in algebraic combinatorics*, Sémin. Lothar. Combin. 51 (2004), 70p. (electronic).

[17] S. K. Hsiao, *A semigroup approach to wreath-product extensions of Solomon’s descent algebras*, arXiv:0710.2081

[18] D. E. Knuth, *The art of computer programming*, vol. 3: Sorting and searching, Addison-Wesley, 1973.

[19] J.-L. Loday, *The theory of group characters and matrix representations of groups*, 2nd ed., Clarendon Press, Oxford, 1950.

[20] J.-L. Loday, *Scindement d’associativité et algèbres de Hopf*, Actes des Journées Mathématiques à la Mémoire de Jean Leray, Sémin. Congr. Soc. Math. France 9 (2004), 155–172.

[21] J.-L. Loday and M. O. Ronco, *Hopf algebra of the planar binary trees*, Adv. Math. 139 (1998) n. 2, 293–309.

[22] J.-L. Loday and M. O. Ronco, *Trialgebras and families of polytopes*, Contemporary Mathematics 346 (2004).

[23] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995.
[26] C. Malvenuto and C. Reutenauer, *Duality between quasi-symmetric functions and the Solomon descent algebra*, J. Algebra 177 (1995), 967–982.

[27] R. Mantaci and C. Reutenauer, *A generalization of Solomon’s descent algebra for hyper-octahedral groups and wreath products*, Comm. Algebra 23 (1995), 27–56.

[28] J.-C. Novelli and J.-Y. Thibon, *A Hopf algebra of parking functions*, FPSAC’04, Vancouver, 2004.

[29] J.-C. Novelli and J.-Y. Thibon, *Free quasi-symmetric functions of arbitrary level*, arXiv:math/0405597v2.

[30] J.-C. Novelli and J.-Y. Thibon, *Noncommutative Symmetric Functions and Lagrange Inversion*, preprint [ArXiv:math.CO/0512570]

[31] J.-C. Novelli and J.-Y. Thibon, *Construction de trigèbres dendriformes*, C. R. Acad. Sci, Paris, Sér. I, 342, (2006), 365–369.

[32] J.-C. Novelli and J.-Y. Thibon, *Polynomial realizations of some trialgebras*, FPSAC’06. Also preprint [ArXiv:math.CO/0605061]

[33] J.-C. Novelli and J.-Y. Thibon, *Parking functions and descent algebras*, Annals of Comb., 11 (2007), 59–68.

[34] J.-C. Novelli and J.-Y. Thibon, *Hopf algebras and dendriform structures arising from parking functions*, Fund. Math., 193 (2007), 189–241.

[35] F. Patras and M. Schocker, *Twisted Descent Algebras and the Solomon-Tits Algebra*, Adv. in Math. 199 (2006), 151–184.

[36] J. Stembridge, *A characterization of supersymmetric polynomials*, J. Algebra 95:22 (1985), 439–444.

[37] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences* (electronic), http://www.research.att.com/~njas/sequences/

[38] R. P. Stanley, *Theory and applications of plane partitions: Part 2*, Stud. Appl. Math. 50 (1971), 259–279.

(Novelli and Thibon) Université Paris-Est, Institut Gaspard Monge, 5, Boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, FRANCE

E-mail address, Jean-Christophe Novelli: novelli@univ-mlv.fr

E-mail address, Jean-Yves Thibon: jyt@univ-mlv.fr