Geodesic Structure of Naked Singularities in AdS$_3$ Spacetime

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Abstract

We present a detailed study of the geodesics around naked singularities in AdS$_3$. These spacetimes are obtained through identifications along spacelike Killing vectors with a fixed point, and are in general interpreted as massive spinning point particles. The general solution to the geodesic equation is expressed in terms of elementary functions. We classify different geodesics in terms of their radial bounds, which depend on the constants of motion. Null geodesics always escape to infinity and can reach the singularity for some values of these constants. Timelike geodesics never escape to infinity and do not always fall into the singularity. The spatial projections of the geodesics (polar orbits) exhibit self-intersections, which are particularly simple for null geodesics. As a particular application, we also compute the lengths of fixed-time spacelike geodesics of the static naked singularity using two different regularizations.
1 Introduction

Shortly after the black hole in three-dimensional spacetime was discovered \[1\,2\], its geodesic structure was studied in detail \[3\,4\]. The geodesics were found to be qualitatively similar to the four-dimensional case, with a few differences. In general, the study of these geodesics not only gave insight into the geometrical properties of the BTZ spacetime, but also led to some interesting applications \[5\,8\].

Here we analyze geodesics of a geometry obtained by continuing the black hole of mass \(M\) and angular momentum \(J\) to negative values of \(M\). In the static case \(J = 0\), for \(0 > M > -1\) the resulting three-dimensional spacetime is a conical geometry with deficit angle. These are naked singularities (NS) that correspond to point particles, while for \(M < -1\) they are conical geometries with angular excesses and may be interpreted as antiparticles \[9\,11\]. The geodesic structure is shown to be surprisingly rich and in many ways quite different from the black hole case.

From a geometric perspective, the NS in three-dimensional AdS spacetime can be obtained by identifying points along a rotational Killing vector. More precisely, one identifies with rotations on two independent planes in \(\mathbb{R}^{(2,2)}\). The singularity thus obtained has been an object of extensive study in the past \[9\,15\]. In this paper we continue the path to unravelling these intriguing entities. Our analysis may contribute to an understanding of some aspects of current interests: specifically, these geodesics could be useful for computing entanglement entropy \[16\,17\], and for recent studies of quantum backreaction on naked singularities \[18\,21\].

This paper is organized as follows. In Section 2 we review the NSs and discuss how to obtain them by identifications on the covering AdS\(_3\) space embedded in \(\mathbb{R}^{2,2}\) (as a pseudosphere). In Section 3 we use conserved quantities along geodesics to find the first order geodesic equations in NS spacetimes, and then through re-scaling write the equations in a convenient form. In Section 4, we present solutions to the radial equations and corresponding bounds for the different types of geodesics. We note that all null geodesics escape to infinity or fall into the singularity, except for a special case which allows circular orbits. Similarly, spacelike geodesics either have both ends at infinity, or one end at infinity and the other at the singularity. Meanwhile, all timelike geodesics are bounded: they either orbit the NS at finite radius or fall into it. Section 5 deals with the spatial projections of the geodesics. We find exact analytic solutions, plot representative orbits and discuss their qualitative behavior. In Section 6, we analyze an interesting property of the geodesics arising from the results of Section 5: null, spacelike, and timelike geodesics can all intersect themselves, and we calculate the number of self-intersections. Section 7 presents two more properties: the time behavior of geodesics, and the lengths of spacelike geodesics. The last section contains a summary and discussion of the main results.

\(^1\)The intermediate state \(M = -1\) with zero angular momentum (AdS\(_3\)) corresponds to the vacuum.
2 The NS spacetime

Although all vacuum solutions of the three-dimensional Einstein equations with negative cosmological constant $\Lambda$ are constant curvature spacetimes locally isometric to $\text{AdS}_3$, there exist geometries globally distinct from $\text{AdS}_3$, including black holes. This is the case of the family of BTZ geometries described by the stationary line element

$$ds^2 = -\left(\frac{r^2}{\ell^2} - M\right)dt^2 - Jdtd\theta + \left(\frac{r^2}{\ell^2} - M + \frac{J^2}{4r^2}\right)^{-1}dr^2 + r^2d\theta^2,$$

(2.1)

where $\ell^2 = -\Lambda$, $-\infty < t < \infty$, $0 < r < \infty$, and $0 \leq \theta \leq 2\pi$. Here the mass $M$ and angular momentum $J$ are integration constants. Depending on the values of $M$ and $J$ various spacetimes emerge from the BTZ metric (2.1), which are summarized in Table 1.

| $M - J$ regions          | Geometries               |
|--------------------------|--------------------------|
| $M > 0$ and $|J| < M\ell$ | Black holes              |
| $M > 0$ and $|J| = M\ell$ | Extremal black holes     |
| $M < 0$ and $|J| < -M\ell$ | Naked singularities     |
| $M < 0$ and $|J| = -M\ell$ | Extremal naked singularities |
| $M = 0$ and $J = 0$      | Massless BTZ geometry    |
| $M = -1$ and $J = 0$     | $\text{AdS}_3$ vacuum   |

Here we are interested in the naked singularities (with $|J| \leq |M|\ell$), namely, the geometries without an event horizon, which correspond to spacetimes with nonpositive mass $M \leq 0$. The only exception is the case $M = -1, J = 0$, which is the $\text{AdS}_3$ spacetime. NSs can be obtained by identifications on the universal covering space $\text{CAdS}_3$. We present below a brief review of this construction.

Consider $\text{CAdS}_3$ as the set of points $X^a = (X^0, X^1, X^2, X^3)$ of the pseudo-sphere em-

\[2\text{We set the three-dimensional Newton constant as } G = 1/8.\]
bedded in $\mathbb{R}^{2,2}$ defined by

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 - (X^3)^2 = -\ell^2. \quad (2.2)$$

This embedding can be parametrized with coordinates $(t, r, \theta)$ on the algebraic curve (2.2), which yields the induced metric (2.1). The Killing vector $\Theta = \partial_\theta$ is chosen as the identification vector and is written as a linear combination of the $so(2, 2)$ generators $J_{ab} := X_b \partial_a - X_a \partial_b$:

$$\Theta = \frac{1}{2} \omega^{ab} J_{ab} = \frac{\partial X^a}{\partial \theta} \partial_a = \partial_\theta, \quad (2.3)$$

where the antisymmetric matrix $\omega^{ab}$ characterizes the identification in terms of the $so(2, 2)$ generators. Then, the action of the matrix $H = e^{2\pi \Theta}$ on the coordinates of the embedding space is

$$H^a_b X^b(t, r, \theta) = X^a(t, r, \theta + 2\pi). \quad (2.4)$$

The explicit form of the embeddings for the different geometries and the corresponding identification matrices $H$ can be found in Refs. [11,21]. The different identification vectors $\Theta$ are shown in Table 2, where we have defined

$$b_\pm = \frac{1}{2} \left( \sqrt{-M + J/\ell} \pm \sqrt{-M - J/\ell} \right). \quad (2.5)$$

| Killing vector $\Theta$ | Geometry               |
|------------------------|------------------------|
| $b_+ J_{21} + b_- J_{30}$ | Generic NS             |
| $\sqrt{-M/2}(J_{03} - J_{12}) - \frac{1}{2}(J_{01} + J_{03} + J_{12} - J_{23})$ | Extremal NS           |
| $J_{12} - J_{13}$ | Massless BTZ geometry |

Table 2: Identification Killing vectors $\Theta$ in terms of $so(2, 2)$ generators for different NS spacetimes

As Table 2 displays, the non-extremal NS is obtained by an identification by a Killing vector formed by two rotations. Note that for the extremal and massless cases, the Killing vectors contain rotations and boosts that are not limiting cases of the generic form.

$^3$Here $X_a = \eta_{ab} X^b$, with $\eta_{ab} = diag(-, +, +, -)$. 

3
3 Geodesic equations

The NS spacetimes have two Killing vectors $\xi = \partial_t$ and $\Theta = \partial_\theta$. They provide two conserved quantities along the geodesic motion, $E = -\xi_\mu \dot{x}^\mu$ and $L = \Theta_\mu \dot{x}^\mu$, respectively, where $\dot{x}^\mu = dx^\mu / d\lambda$ is tangent to the geodesic with affine parameter $\lambda$. This allows obtaining the first integrals

$$\dot{t} = \frac{Er^2 - JL/2}{r^2 \left( \frac{r^2}{\ell^2} - M + \frac{J^2}{4r^2} \right)}, \quad (3.1)$$

$$\dot{\theta} = \frac{(r^2/\ell^2 - M)L + JE/2}{r^2 \left( \frac{r^2}{\ell^2} - M + \frac{J^2}{4r^2} \right)}. \quad (3.2)$$

Since the velocity can be normalized as $v^\mu v_\mu = -\varepsilon$ -- with $\varepsilon = 0$ for null geodesics, $\varepsilon > 0$ for timelike geodesics, and $\varepsilon < 0$ for spacelike geodesics --, one gets

$$r^2 \dot{r}^2 = -\varepsilon r^2 \left( \frac{r^2}{\ell^2} - M + \frac{J^2}{4r^2} \right) + \left( E^2 - \frac{L^2}{\ell^2} \right) r^2 + L^2 M - JEL. \quad (3.3)$$

Equations (3.1-3.3) are exactly the same for black holes ($M > 0$) [4] and naked singularities ($M < 0$). However, the denominators of $\dot{t}$ and $\dot{\theta}$ vanish at the BH horizons, while for NS they are positive definite. Consequently, the geodesics around a NS are drastically different from those in the BH case.

Using (2.5) it is convenient to write

$$r^2 \left( \frac{r^2}{\ell^2} - M + \frac{J^2}{4r^2} \right) = \ell^2 \left( \frac{r^2}{\ell^2} + b_+^2 \right) \left( \frac{r^2}{\ell^2} + b_-^2 \right), \quad (3.4)$$

and since $M < 0$ and $\ell \neq 0$, it is useful to introduce the following quantities

$$u = r^2/(-\ell^2), \quad a = J/(-\ell^2). \quad (3.5)$$

Furthermore, for $M \neq 0$ we use the rescale quantities\(^4\)

$$\bar{L} = L/(-M\ell), \quad \bar{E} = E/(-M), \quad \bar{\varepsilon} = \varepsilon/(-M), \quad \bar{\lambda} = \lambda/\ell,$$

$$b_{\pm} = b_{\pm}/\sqrt{-M} = \sqrt{1 + a \pm \sqrt{1 - a^2}}/2, \quad (3.6)$$

with $a^2 \leq 1$. Then, omitting the tildes hereafter, the geodesic equations read

$$\dot{t} = \frac{Eu - b_+ b_- L}{(u + b_+^2)(u + b_-^2)}, \quad (3.7)$$

$$\dot{\theta} = \frac{(u + 1)L + b_+ b_- E}{(u + b_+^2)(u + b_-^2)}, \quad (3.8)$$

$$\frac{\dot{u}^2}{4(-M)} = -\varepsilon(u + b_+^2)(u + b_-^2) + (E^2 - L^2) u - L^2 - 2b_+ b_- EL. \quad (3.9)$$

\(^4\)The massless BTZ spacetime will be considered without this rescaling.
4 Radial bounds

The equation for the radial motion (3.9) is conveniently written as

\[
\frac{\dot{u}^2}{4(-M)} = -\varepsilon u^2 + Bu - C \equiv h(u),
\]

with

\[
B = E^2 - L^2 - \varepsilon \quad \text{and} \quad C = \varepsilon a^2 / 4 + L^2 + aEL.
\]

Geodesics exist in the regions \( u \geq 0 \) where \( h(u) \) is non-negative. With this criterion one can find radial bounds for the different geodesics. It is important to note that for timelike and null geodesics, \( E^2 \leq L^2 \) would imply \( B < 0 \) and \( C > 0 \), which in turn implies \( h(u) < 0 \). Hence, \( E^2 > L^2 \) is a necessary condition for the existence of timelike and null geodesics, although not for spacelike ones. For each of the following geodesics, we begin by analyzing the non-extremal case \( (M < -|J|) \), and finish with comments specific to the extremal case \( (M = -|J|) \).

4.1 Null geodesics

For non-extremal NSs, the existence of null geodesics \( (\varepsilon = 0) \) requires \( E^2 > L^2 \), and since \( E \neq 0 \), it is useful to define

\[
\eta = \frac{L}{E},
\]

which verifies \( \eta^2 < 1 \). Under this condition on \( \eta \) the region where \( h(u) \) is non-negative depends on the sign of \( C = E^2\eta(\eta + a) \). In the case \( \eta(\eta + a) \geq 0 \) null geodesics are allowed for \( \eta(\eta + a) / (1 - \eta^2) \leq u < \infty \). Otherwise, if \( \eta(\eta + a) < 0 \) the null geodesics are permitted in the half line \( 0 \leq u < \infty \). Table 3 summarizes the possible ranges of \( r \) for null geodesics around a non-extremal naked singularity.

| Range of \( \eta \) and \( a \) | Range of \( r \) |
|---------------------------------|----------------|
| \( \eta(\eta + a) \leq 0 \)    | \( 0 \leq r^2 < \infty \) |
| \( \eta(\eta + a) \geq 0 \)    | \( -M\ell^2 \eta(\eta + a) / (1 - \eta^2) \leq r^2 < \infty \) |
Integrating Eq. (4.1) with $\varepsilon = 0$, we obtain

$$u(\lambda) = \frac{\eta(\eta + a)}{(1 - \eta^2)} - ME^2(1 - \eta^2)(\lambda - \lambda_0)^2,$$  \hspace{1cm} (4.4)

where $\lambda_0$ is an arbitrary integration constant. Note that for $\eta(\eta + a) < 0$ the minimum of the parabola $u(\lambda)$ (4.4) is negative, which implies that any null geodesic coming from a finite radius reaches $u = 0$ at a finite value of $\lambda$. Hence, these null geodesics have no turning point. Meanwhile, in the case $\eta(\eta + a) \geq 0$ there is a non-zero turning point given by

$$u_{\text{min}} = \frac{\eta(\eta + a)}{1 - \eta^2},$$  \hspace{1cm} (4.5)

as shown in Table 3. In short, when the specific angular momentum $a$ of the NS is larger than and of opposite sign to $\eta$, null geodesics fall into the singularity. In all other cases, null geodesics do not reach $r = 0$.

For the extremal naked singularity ($M = -|J|$), the cases $\eta^2 > 1$ and $\eta a = 1$ are not allowed for null geodesics. Remarkably, for $\eta a = -1$, Eq. (4.1) provides the circular null geodesics $u(\lambda) = \text{constant}$, where any radius is permissible. For $\eta^2 < 1$ we have two cases:

(a) If $\eta a < 0$ or $\eta = 0$, then $\eta(\eta \pm 1) \leq 0$, which implies that the null geodesics are allowed for $0 \leq u < \infty$;

(b) If $\eta a > 0$ then $\eta(\eta \pm 1) > 0$, which implies a lower bound $u_{\text{min}} = \eta(\eta \pm 1)/(1 - \eta^2)$ for the null geodesics and now the allowed interval is $|\eta|/(1 - |\eta|) \leq u < \infty$. In fact, this lower bound is a turning point.

This is also summarized in Table 3 upon substituting $a = \pm 1$.

### 4.2 Timelike geodesics

Timelike geodesics exist in the regions where the quadratic function $h(u)$ in (4.1), defined in the domain $u \geq 0$, is non-negative. In cases (a) $C < 0$ and (b) $C = 0, B > 0$, the function $h(u)$ is non-negative in the interval $0 \leq u \leq u_+$, where

$$u_\pm = \frac{B \pm \sqrt{\Delta}}{2\varepsilon}, \quad \text{with} \quad \Delta = B^2 - 4\varepsilon C.$$  \hspace{1cm} (4.6)

Note that under conditions (a) or (b), the discriminant $\Delta$ is always positive. A third case is defined by the condition (c) $C > 0, B > 0, \Delta > 0$. Here $h(u) \geq 0$ in the interval $[u_-, u_+]$. On the other hand, timelike geodesics are not possible for the cases $C > 0, \Delta \leq 0$ or $B \leq 0, C \geq 0$. 

6
For the static NS \((a = 0)\), timelike geodesics with \(L \neq 0\) satisfy condition (c), since
\[C = L^2 > 0,\]
and thus do not fall into the singularity. The radial timelike geodesics \((a = 0, L = 0)\) satisfy condition (b).

### Table 4: Radial bounds for timelike geodesics

| Cases | Range of \(r\) |
|-------|----------------|
| (a), (b) | \(0 \leq r^2 \leq -M\ell^2u_+\) |
| (c) | \(-M\ell^2u_- \leq r^2 \leq -M\ell^2u_+\) |

The radial equation (4.1) for \(\varepsilon > 0\) is integrated as

\[
u(\lambda) = \frac{B + \sqrt{\Delta} \sin[2\sqrt{-M\varepsilon}(\lambda - \lambda_0)]}{2\varepsilon},
\]

which agrees with the cases (a), (b) and (c) previously discussed. In the extremal NS, the cases \(\eta^2 > 1\) or \(\eta a = \pm 1\), are also not allowed for timelike geodesics. For \(\eta^2 < 1\) the bounds shown in Table 4 are obtained under the condition \(a = \pm 1\). Equation (4.7) also holds for the extremal NS.

### 4.3 Spacelike geodesics

For spacelike geodesics, \(h(u)\) becomes a convex parabola. In the analysis of radial bounds there are three cases to consider:

(a) For \(B \geq 0\) and \(C > 0\), the geodesics stretch from infinity to a minimum radius \(r_{\min}^2 = -M\ell^2u_-\).

(b) For \(B \geq 0\) and \(C \leq 0\), all geodesics end at the singularity and there is no minimum radius.

(c) For \(B < 0\) and \(C \geq 0\), once again the geodesics can be in the region \(u_- \leq u < \infty\). It can be shown that \(B < 0\) is incompatible with \(C < 0\), so we do not consider this case.

The solution for Eq. (4.1) with \(\varepsilon < 0\) is

\[
u(\lambda) = \frac{1}{4(-\varepsilon)} \left( e^{2\sqrt{\varepsilon M}(\lambda - \lambda_0)} + \Delta e^{-2\sqrt{\varepsilon M}(\lambda - \lambda_0)} - 2B \right).
\]
We can verify that this solution satisfies the bounds mentioned earlier, and the results are summarized in Table 5. This table also provides radial bounds for the extremal case $a^2 = 1$. The solution (4.8) holds as well in this case.

Table 5: Radial bounds for spacelike geodesics

| Cases   | Range of $r$     |
|---------|------------------|
| (b)     | $0 \leq r^2 < \infty$ |
| (a), (c)| $-M\ell^2u_\perp \leq r^2 < \infty$ |

A qualitative summary of the results for radial bounds is that null and spacelike geodesics approach the NS from infinity, and either fall into the singularity or wind around and go back to infinity (with the exception of the extremal case for which a null geodesic with $\eta a = -1$ has a circular orbit). Meanwhile timelike geodesics either orbit continually bounded between two radii, or fall into the NS.

4.4 Massless BTZ Geodesics

In the case $M = J = 0$ (massless BTZ spacetime) whose geodesic equations written in terms of the original variables of Eqs. (3.1)–(3.3), are

$$
\dot{t} = \frac{E\ell^2}{r^2}, \quad \dot{\theta} = \frac{L}{r^2}, \quad \dot{r}^2 = -\varepsilon \frac{r^2}{\ell^2} + E^2 - \frac{L^2}{\ell^2}.
$$

(4.9)

Note that $L = 0$ provides radial geodesics and the case $E^2 < L^2/\ell^2$ is not allowed for null and timelike geodesics. The integration of these equations is straightforward and Table 6 summarizes the solutions of the radial equation and bounds. Note that the bounds match those for $M \neq 0$ in the limit $M \to 0$, c.f. [4].

In the null case when $E^2 > L^2/\ell^2$,

$$
r(\lambda) = \sqrt{E^2 - \frac{L^2}{\ell^2}}|\lambda - \lambda_0|
$$

(4.10)

while for $E^2 = L^2/\ell^2$, $r(\lambda)$ is constant. For the timelike case, the solution of the radial equation is given by

$$
r(\lambda) = \frac{\ell}{\sqrt{\varepsilon}} \sqrt{E^2 - \frac{L^2}{\ell^2}} \sin \frac{\sqrt{\varepsilon}}{\ell}|\lambda - \lambda_0|
$$

(4.11)

\[5\]In this case the rescalings (3.6) are no longer valid
Table 6: Radial bounds for geodesics of the massless BTZ spacetime

| Cases                        | Range of $r$                      |
|------------------------------|-----------------------------------|
| $\varepsilon = 0, E^2 > L^2/\ell^2$ | $0 \leq r < \infty$               |
| $\varepsilon = 0, E^2 = L^2/\ell^2$ | $r$ constant and arbitrary        |
| $\varepsilon > 0, E^2 > L^2/\ell^2$ | $0 \leq r \leq \ell \sqrt{\frac{E^2 - L^2/\ell^2}{\varepsilon}}$ |
| $\varepsilon < 0, E^2 \geq L^2/\ell^2$ | $0 \leq r < \infty$               |
| $\varepsilon < 0, E^2 < L^2/\ell^2$ | $\ell \sqrt{\frac{E^2 - L^2/\ell^2}{\varepsilon}} \leq r < \infty$ |

with $\lambda \in [\lambda_0 - (\ell\pi/2\sqrt{\varepsilon}), \lambda_0 + (\ell\pi/2\sqrt{\varepsilon})]$ (i.e., $\lambda$ is bounded). This is further discussed in the following section. Finally for the spacelike case,

$$r(\lambda) = \frac{\ell}{2(-\varepsilon)} \left(e^{\frac{\sqrt{- \varepsilon}}{2}(\lambda - \lambda_0)} + \varepsilon(E^2 - L^2/\ell^2)e^{-\frac{\sqrt{- \varepsilon}}{2}(\lambda - \lambda_0)}\right).$$

(4.12)

5 Orbits in the $r$-$\theta$ plane

Once again we begin with the non-extremal case. The orbit equation, i.e. $r(\theta)$ or $\theta(r)$, can be obtained from (3.8) and (3.9). Integration the orbit equation yields

$$2\sqrt{-M}(\theta - \theta_0) = A_+I_+ + A_-I_-,$$

(5.1)

with

$$A_\pm = \mp \frac{b_\pm}{b_+^2 - b_-^2}$$

(5.2)

and the values of $I_\pm$ given in Table 7 assuming $C_\pm \neq 0$.

There remain the cases where $C_+ = 0$ or $C_- = 0$. For null and timelike geodesics, $E^2 > L^2$ so $C_+ \neq 0$. Then we only have the case $C_+ \neq 0, C_- = 0$. Here the decomposition in partial fractions indicates $A_-I_- = 0$, so the solution is simply $2\sqrt{-M}(\theta - \theta_0) = A_+I_+$. 9
Table 7: $I_{\pm}$ for orbit integrals ($C_{\pm} \neq 0$). Here $B_{\pm} = \pm \varepsilon (b_{\pm}^2 - b_{\mp}^2) + E^2 - L^2$ and $C_{\pm} = b_{\pm}E + b_{\mp}L$.

| Type of geodesic | $I_{\pm}$ |
|------------------|-----------|
| Timelike/Spacelike | $\arctan \frac{B_{\pm}(u + b_{\pm}^2) - 2C_{\pm}^2}{2C_{\pm}\sqrt{-\varepsilon u^2 + Bu - C}}$ |
| Null             | $2\arctan \frac{\sqrt{(1 - \eta^2)u - \eta(\eta + a)}}{b_{\pm} + \eta b_{\mp}}$ |

On the other hand, $C_+ = 0, C_- \neq 0$ is allowed for a spacelike geodesic. Analogously, here the decomposition in partial fractions indicates that $2\sqrt{-M}(\theta - \theta_0) = A_- I_-$. In the static case $a = 0$ we obtain from (3.8) the radial geodesic $\theta = \theta_0$ for $L = 0$. When $L \neq 0$, for null geodesics we have the simple expression

$$r^2(\theta) = \frac{-M \ell^2 \eta^2}{(1 - \eta^2) \cos^2 \sqrt{-M}(\theta - \theta_0)}, \quad (5.3)$$

which has a turning point at $r_{\text{min}} = \ell \sqrt{\frac{-M \eta^2}{1 - \eta^2}}$. For timelike and spacelike geodesics,

$$r^2(\theta) = \frac{-2M \ell^2 L^2}{B + \sqrt{B^2 - 4\varepsilon L^2 \cos 2\sqrt{-M}(\theta - \theta_0)}}, \quad (5.4)$$

in agreement with the radial bounds shown for condition (c) in Table 4. In the limit $\varepsilon \to 0$, Eq. (5.4) matches the orbit for the static null geodesics (5.3). Note that the integration constant $\theta_0$ here is different from the one in (5.1), chosen to make the expression simpler.

We include plots that help visualize the different geodesics. Note in particular the freedom of null geodesics to escape to infinity. Additionally, the winding of geodesics near the NS can either be increased by the rotation of the NS, or decreased, depending on the relative signs of $L$ and $J$. That is, the geodesics are dragged by the rotation of the NS, as illustrated in Figures 1a and 1b.

As we adjust different parameters (in particular, $M$), the number of times geodesics wind around the NS and intersect themselves changes. This winding phenomenon is studied in detail in the following section.

In agreement with the radial bounds and solutions discussed in the previous section, timelike geodesics follow bounded orbits around the NS. For $C > 0$ and rational values of
\[
\sqrt{-M}, \text{ the orbits are closed. As the denominator of } \sqrt{-M} \text{ (expressed as an irreducible fraction) grows, timelike geodesics take more winds to close. Figure 2c is an example of the failure of timelike geodesics to close when } \sqrt{-M} \text{ is irrational. In this figure } \lambda \text{ is not allowed to cover its entire range, for otherwise nothing would be visible.}
\]

![Diagram of null geodesics](image1)

(a) \( \eta a > 0 \)  
(b) \( \eta a < 0 \)

Figure 1: Null geodesics with equal \(|\eta a|\)

![Diagram of timelike geodesics](image2)

(a) \( \sqrt{-M} = 1/5 \)  
(b) \( \sqrt{-M} = 1/17 \)  
(c) \( \sqrt{-M} = 1/\sqrt{15} \)

Figure 2: Timelike geodesics

The extremal naked singularity produces different solutions for polar orbits. For null geodesics the orbit equation is defined only if \( \eta^2 < 1 \) and the solutions are

\[
2 \text{ sgn}(E)\sqrt{-M}(\theta - \theta_0) = \frac{\sqrt{(1 - \eta^2)u - \eta(\eta \pm 1)}}{(\eta \pm 1)(u + 1/2)} \\
+ \sqrt{2} \arctan \frac{\sqrt{2} \sqrt{(1 - \eta^2)u - \eta(\eta \pm 1)}}{(\eta \pm 1)}. \tag{5.5}
\]
As for timelike and spacelike geodesics, we have

\[
2 \text{sgn}(E) \sqrt{-M(\theta - \theta_0)} = \frac{\sqrt{-\varepsilon(u + 1/2)^2/E^2 + (1 - \eta^2)u - \eta(\eta \pm 1)}}{(\eta \pm 1)(u + 1/2)} + \frac{1}{\sqrt{2}} \arctan \frac{(1 - \eta^2)(u + 1/2) - (1 \pm \eta)^2}{\sqrt{2(\eta - 1)} \sqrt{-\varepsilon(u + 1/2)^2/E^2 + (1 - \eta^2)u - \eta(\eta \pm 1)}}. 
\]

We note that sgn($E$) can always be chosen to be positive for the null and timelike cases, as will be discussed in Section 7.

### 5.1 Massless BTZ Orbits

Finally we consider orbits in the massless BTZ spacetime. The non-radial ($L \neq 0$) and non-circular ($E^2 \neq L^2/\ell^2$) orbits for all geodesics are given by

\[
r^2(\theta) = \frac{L^2}{(E^2 - L^2/\ell^2)(\theta - \theta_0)^2 + \frac{\varepsilon L^2}{E^2 - L^2}}. 
\]

For $E^2 > L^2/\ell^2$, this equation describes a spiral for small $r$, and is symmetric when $(\theta - \theta_0) \rightarrow -(\theta - \theta_0)$. The result is then two symmetric spirals, each of which has an infinite number of turns before reaching the origin. In Figure 3 we illustrate each type of geodesic. Globally they are quite different, though close to the singularity all geodesics simply look like the aforementioned spirals. Once a particle reaches $r = 0$, it remains there. However, the plots indicate all possible trajectories given fixed $E$ and $L$ (i.e., they display all possible choices of $\theta_0$) which is why it naively looks like particles could escape the singularity.

The case $E^2 < L^2/\ell^2$ for spacelike geodesics is also considered in (5.7), but here the geodesics do not reach the singularity. Instead they have a finite minimum radius as indicated in the previous section. Qualitatively the behavior is similar to what was found in the $M \neq 0$ case.

From the orbit equation we also see that the parameter $\lambda$ is bounded for timelike geodesics of the massless BTZ spacetime. Though $r(\lambda)$ for timelike geodesics (4.11) is sinusoidal, it does not mean that it oscillates near the singularity. Rather, it has the spiral orbit mentioned above. This point was incorrectly interpreted in [5].
Null and spacelike geodesics that have a turning point intersect themselves. These geodesics start at infinity and wind around the singularity a finite number of turns, reach the turning point $r_{\text{min}}$ and go back to infinity after the same number of turns. The number of self-intersections is the integer number of times that $2\pi$ is contained in the angle swept by the geodesic as $r$ goes from $r = \infty$ to $r = r_{\text{min}}$ and comes back to $r = \infty$. Equivalently, it is possible to compute just the absolute value of the angle from $r = r_{\text{min}}$ to $r = \infty$ and divide it by $\pi$. Here we count self-intersections for $M \neq 0$ (for the massless case there are infinitely many).

Let us consider null geodesics with a turning point $r_{\text{min}}$, which require $\eta^2 < 1$ and $\eta(\eta + a) > 0$. Setting $\theta_0 = 0$, the angle at $r_{\text{min}}$ is $0$ and at $r = \infty$ it is

$$|\theta(\infty)| = \frac{\pi}{2\sqrt{-M} (b_+^2 - b_-^2)} |b_+ \text{sgn}(b_- + \eta b_+) - b_- \text{sgn}(b_+ + \eta b_-)|.$$  \hfill (6.1)

Since $b_+ > b_-$ and $\eta^2 < 1$, we have $\text{sgn}(b_+ + \eta b_-) = 1$. Moreover, if $\eta(\eta + a) > 0$ we have $\text{sgn}(b_- + \eta b_+) = \text{sgn}(\eta)$. Thus, dividing by $\pi$,

$$N := \left|\frac{\theta(\infty)}{\pi}\right| = \frac{1}{2\sqrt{-M} + \text{sgn}(\eta) J/\ell}.$$  \hfill (6.2)

The number of self-intersections is then

$$N_\pm = \left\lceil\frac{1}{2\sqrt{-M} \pm J/\ell}\right\rceil - 1,$$  \hfill (6.3)

where the ceiling function $\lceil x \rceil$ is the least integer greater than or equal to $x$, and $\pm = \text{sgn}(\eta)$. $N_\pm$ is the integer part of $N$ except for $N$ integer, in which case we must subtract 1, which
would correspond to a self-intersection at \( r = \infty \). The difference between \( N_+ \) and \( N_- \) for a given sign of \( J \) is due to the fact that the rotating background breaks the clockwise/counter-clockwise symmetry of the null geodesic. Once again, we see that the rotation of the NS can either increase or decrease the winding of the geodesic. For the extremal case, we get

\[
N = \left\lceil \frac{1}{2\sqrt{-2M}} \right\rceil - 1. \tag{6.4}
\]

We can also obtain the winding number for spatial geodesics, which start at infinity and reach a minimum radius, as \( |\theta(\infty) - \theta(r_{\min})|/\pi \). Initially we assume \( C_{\pm} \neq 0 \). At the turning point \( r_{\min} \), the angle is

\[
\theta(r_{\min}) = \frac{\pi}{4\sqrt{-M(b_+^2 - b_-^2)}} [-b_- \text{ sgn}(C_+(B_+(u_{\min} + b_+^2) - 2C_-^2)) + b_+ \text{ sgn}(C_-(B_-(u_{\min} + b_-^2) - 2C_+^2))]. \tag{6.5}
\]

From this it is relatively straightforward to show that

\[
\theta(r_{\min}) = \frac{\pi}{4\sqrt{-M(b_+^2 - b_-^2)}} [b_- \text{ sgn}(C_+) - b_+ \text{ sgn}(C_-)], \tag{6.6}
\]

Meanwhile at \( r = \infty \) we have

\[
\theta(\infty) = \frac{1}{2\sqrt{-M(b_+^2 - b_-^2)}} \left(-b_- \arctan \frac{B_+}{2C_+\sqrt{-\xi}} + b_+ \arctan \frac{B_-}{2C_-\sqrt{-\xi}}\right). \tag{6.7}
\]

As explained in Section 5, if \( C_+ = 0 \) only the second terms in (6.6) and (6.7) must be considered, while if \( C_- = 0 \) just the first ones appear in those expressions.

Self-intersections are also present in timelike orbits. They occur in case (c) of Table 4, where timelike geodesics are bounded by two radii. However, in general orbits do not close which leads to an infinite number of intersections (see for instance Figure 2c).

It is worth noting that the number of self-intersections for null geodesics depends only on the \textit{spacetime quantities} (i.e., \( M \) and \( J \)), while self-intersections for spacelike and timelike geodesics also depend on the motion constants \( E \) and \( L \).

7 Additional features

7.1 Behavior of the time coordinate

So far we have not discussed the behavior of the time coordinate for geodesics in the NS spacetimes. We note, firstly, that the equation of motion (3.7) for \( t \) closely resembles
equation (3.8) for $\theta$. Indeed, the integration gives a very similar expression to (5.1),

$$2\sqrt{-M(t - t_0)} = \tilde{A}_+ I_+ + \tilde{A}_- I_-,$$

where

$$\tilde{A}_\pm = A_\mp = \pm \frac{b_\pm}{b_+^2 - b_-^2},$$

and $I_\pm$ remain the same (and can be found in Table 7).

Another aspect of the time coordinate worth noting is the sign of $\dot{t}(\lambda)$ throughout a trajectory. We can see from the geodesic equation (3.7) that $\dot{t}$ could change sign at $u_c \equiv \eta b_+ b_-$. However, this does not occur for timelike or null geodesics. This can be understood as follows. If there was a value of $r$ for which $\dot{t} = 0$, this would require $\eta a > 0$. From the radial bounds for geodesics this implies that null geodesics do not reach the singularity. It is straightforward to show that they have a turning point at $u_{\min} > u_c$. Then massless particles never cross the $u_c$ point, and the sign of $\dot{t}$ for their geodesics is constant. Additionally, from Eq. (3.7) note that as $u \to \infty$, $\text{sgn}(\dot{t}) = \text{sgn}(E)$. Thus for null geodesics we can choose $E > 0$ without loss of generality.

Similarly, for timelike geodesics $\dot{t}$ does not change sign. It can be shown that $h(u_c) < 0$ in Eq. (4.1), and therefore $u_c$ lies outside of the allowed range for $u$. Hence, $\text{sgn}(\dot{t}) = \text{sgn}(E)$ so we can also choose $E > 0$. This choice is simply a statement that our parameter $\lambda$ moves forward with time. This agrees with the fact that there are no closed timelike geodesics in a NS spacetime in AdS$_3$.

### 7.2 Lengths of spacelike geodesics

The lengths of spacelike geodesics have been relevant for some time now due to the Ryu-Takayanagi (RT) prescription for computing entanglement entropy from the area of minimal surfaces [16]. The length of a geodesic that sweeps out an angle $2\alpha$ (where $0 < \alpha < \pi/2$) in going from $r = \infty$ back to $r = \infty$ is

$$\lambda = 2 \int_0^\alpha \frac{d\theta}{\dot{\theta}},$$

where $\theta = 0$ corresponds to the minimum radius. We set $E = 0$ to analyze fixed-time geodesics. Carrying out the integration for the static case $a = 0$ (the general case is analogous but considerably more cumbersome),

$$\lambda = \ell \log \left( \frac{L^2 - M - (L^2 + M) \cos 2\sqrt{-M} \theta + 2L\sqrt{-M} \sin 2\sqrt{-M} \theta}{(L^2 - M) \cos 2\sqrt{-M} \theta - (L^2 + M)} \right) \bigg|_0^\alpha$$

In the original variables, save for a re-scaling of $L$ by $\ell$. Here, $B = E^2 - L^2 - M$, and with $\varepsilon = -1$. 15
where $\alpha$ satisfies
\[ \cos 2\sqrt{-M} \alpha = (L^2 + M)/(L^2 - M). \] (7.5)
Evaluation at 0 yields 0, and evaluation at $\alpha$ yields a divergent quantity, as should be expected. To give physical meaning to this length, one can regularize it by subtracting off another infinite length. In this case, a reasonable choice is to compare with the corresponding geodesic length in AdS$_3$ ($M = -1$), which in a sense corresponds to the vacuum. A problem is that there are different “corresponding geodesics” between different spacetimes. One can choose geodesics that have the same parameters $L$ and $E$, or that sweep the same angle $\alpha$. These two options lead to different results. Indeed, comparing geodesics of the same $L$ (and $E = 0$) leads to
\[ \Delta \lambda|_L \equiv \lambda|_L - \lambda_{\text{AdS}}|_L = 0, \] (7.6)
i.e., the length of a spacelike geodesic with $E = 0$ and angular momentum $L$ is the same in a NS spacetime as in AdS. Meanwhile, comparing geodesics that sweep the same angle $\alpha$ gives
\[ \Delta \lambda|_{\alpha} \equiv \lambda|_{\alpha} - \lambda_{\text{AdS}}|_{\alpha} = \ell \log \left( \frac{1}{\sqrt{-M}} \frac{\sin 2\sqrt{-M} \alpha}{\sin 2\alpha} \right). \] (7.7)
This result is different from the previous one because in the current setup a different quantity, $\alpha$, is held fixed. The two situations could be thought of as different “ensembles”. To arrive at these expressions, the limits must be taken with care. For instance,
\[ \Delta \lambda|_L = \lim_{\theta \to \alpha(L)} \lambda|_L - \lim_{\theta \to \tilde{\alpha}(L)} \lambda_{\text{AdS}}|_L \] (7.8)
where $\alpha$ is defined by Eq. (7.5), and $\tilde{\alpha}$ is defined analogously but with $M = -1$. Thus in order to combine the limits, one must make an appropriate change of variables first.

Note that our result for $\Delta \lambda$ differs from the one found by other authors. For instance, Eq. (2.5) from [17] (where $n = 1/\sqrt{-M}$) in our notation reads
\[ \Delta \lambda = 2\ell \log \left( \frac{2\ell}{\mu} \frac{\sin \sqrt{-M} \alpha}{\sin \sqrt{-M} \alpha} \right), \] (7.9)
where $\mu$ is a regulator. The discrepancy is possibly due to a different regularization choice. Since the result depends so strongly on the regularization procedure, perhaps one should exercise caution in the interpretation of $\Delta \lambda$. 

8 Conclusions and discussion

We have investigated the geodesic structure of NS in three-dimensional anti-de Sitter spacetime. We found that null and spacelike geodesics have self-intersections. This occurs when
they do not reach the singularity, but rather start and end at infinity. The number of self-intersections is finite, and for null geodesics it is a simple expression dependent only on properties of the spacetime ($M$ and $J$), not the constants $E, L$ along the geodesics. On the other hand, the number of self-intersections of spacelike geodesics also depends on the motion constants. Timelike geodesics have bounded orbits and also intersect themselves, but this behavior is more complicated. For example, for the static NS the orbits do not close unless $1/\sqrt{-M}$ is a rational number. When they do not close, they have an infinite number of self-intersections.

It is interesting to point out differences and similarities with the $M > 0$ case. The orbits are quite dissimilar; for the black hole, the dependence of $\theta$ on $r$ is logarithmic and no self-intersections are present. For the black hole, massive particles always fall into the singularity, while for the NS this is not the case (see Figure 2). Null geodesics can escape the singularity for both black holes and NS, but radial bounds are complementary. More precisely, if a null geodesic has parameters $E, L$ such that it reaches the NS, that same geodesic will not reach the black hole singularity. Analogously, a null geodesic that reaches the black hole singularity does not fall into the NS.

An interesting application of our results is in recent developments relating to cosmic censorship [22]. For instance, using a conformally coupled real scalar field, the naked singularity in $2 + 1$ dimensions has been covered up by a “quantum dress” [18, 20]. In the static case, the necessary computation of Green functions in this spacetime can be carried out through the method of images, since the conical singularity is obtained by identifications in AdS$_3$. The number of images to be summed over is in correspondence with the number of self-intersections of null geodesics calculated here. This last statement can be understood pictorially as follows: to compute Green functions one needs the geodesic distance between two points. If null geodesics in a spacetime have $N$ self-intersections, any two infinitesimally close points can be joined by $N$ topologically distinct null geodesics. To compute the two-point function, one must sum over all of these geodesics. There is a correspondence between this number $N$ and the number of images of a point under the identification used to get a NS from AdS$_3$.

Another application lies in the study of entanglement entropy of $1 + 1$ dimensional CFT’s, where according to the RT conjecture the lengths of minimal spatial geodesics play a central role [16]. In the second part of Section 7 we compute these lengths, and find somewhat different results to previous studies.

There are related questions that we have not touched upon, but that could be considered in the future. For instance, a similarly detailed study of multiple conical singularities could be carried out. Moreover, if one considers other identifications of AdS$_3$, different to the one considered here using a spacelike Killing vector, it is possible to build other spacetimes [14, 15] and study their corresponding geodesics.
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