Post-Newtonian Hydrodynamic Equations Using the (3+1) Formalism in General Relativity

Hideki Asada\textsuperscript{1}, Masaru Shibata\textsuperscript{1} and Toshifumi Futamase\textsuperscript{2}

\textsuperscript{1} Department of Earth and Space Science, Faculty of Science, Osaka University, Toyonaka, Osaka 560, Japan

\textsuperscript{2} Astronomical Institute, Graduate School of Science, Tohoku University, Sendai 980-77, Japan

ABSTRACT

Using the (3+1) formalism in general relativity, we perform the post-Newtonian(PN) approximation to clarify what sort of gauge condition is suitable for numerical analysis of coalescing compact binary neutron stars and gravitational waves from them. We adopt a kind of transverse gauge condition to determine the shift vector. On the other hand, for determination of the time slice, we adopt three slice conditions (conformal slice, maximal slice and harmonic slice) and discuss their properties. Using these conditions, the PN hydrodynamic equations are obtained up through the 2.5PN order including the quadrupole gravitational radiation reaction. In particular, we describe methods to solve the 2PN tensor potential which arises from the spatial 3-metric. It is found that the conformal slice seems appropriate for analysis of gravitational waves in the wave zone and the maximal slice will be useful for describing the equilibrium configurations. The PN approximation in the (3+1) formalism will be also useful to perform numerical simulations using various slice conditions and, as a result, to provide an initial data for the final merging phase of coalescing binary neutron stars which can be treated only by fully general relativistic simulations.
1. Introduction

Kilometer-size interferometric gravitational wave detectors, such as LIGO\cite{1} and VIRGO\cite{2}, are now in construction aiming at direct detection of gravitational waves from relativistic astrophysical objects. Coalescing binary neutron stars are the most promising sources of gravitational waves for such detectors. The reasons are that (1) we expect to detect the signal of coalescence of binary neutron stars about several times per year\cite{3}, and (2) the wave form from coalescing binaries can be predicted with a high accuracy compared with other sources\cite{1}.

In the case when the orbital separation of each star is large compared with the radius of neutron stars, i.e., in the so-called inspiraling phase, binary neutron stars are evolving in the adiabatic manner due to gravitational radiation reaction with much longer time scale than the orbital period. As for the inspiraling phase, the theoretical investigation is usually done by the point particle approach using the PN approximation in general relativity\cite{4,5,6,7}. Since the separation is large compared with the neutron star radius, the hydrodynamic effect is small enough and we can regard each star of binary as a point particle. Theoretical studies for such a phase is potentially important because by comparing the observational signal with the theoretical prediction of the signal of inspiraling binary, we may be able to know not only the various parameters of binary\cite{8,9}, but also the cosmological parameters\cite{10}.

After a long time emission of gravitational waves, the orbital separation becomes comparable to the radius of the neutron star. Then, each star of binary neutron stars begins to behave as a hydrodynamic object, not as a point particle, because they are tidally coupled each other. Recently, Lai, Rasio and Shapiro\cite{11} have pointed out that such a tidal coupling of binary neutron stars is very important for their evolution in the final merging phase because the tidal effect causes the instability to the circular motion of them. Also important is the general relativistic gravity because in such a phase, the orbital separation is larger than $\sim 10\%$ of the Schwarzschild radius of the system. Thus, we need not only a hydrodynamic treatment, but also general relativistic one to study the final phase of binary neutron stars.

Fully general relativistic simulation is sure to be the best method, but it is also one of the most difficult ones. Although much effort has been focused and much progress can be expected there\cite{13}, it will take a long time until numerical relativistic calculations become reliable. One
of the main reasons is that we do not know the behavior of the geometric variables in the strong field around coalescing binary neutron stars. Owing to this, we do not know what sort of gauge condition is useful and how to give an appropriate general relativistic initial condition for coalescing binary neutron stars.

The other reason is a technical one: In numerical relativistic simulations, gravitational waves are generated, and in the case of coalescing binary neutron stars, the wavelength is the order of \( \lambda \sim \pi R^{3/2}M^{-1/2} \), where \( R \) and \( M \) are the orbital radius and the total mass of binary, respectively. Thus, we need to cover a region \( L > \lambda \propto R^{3/2} \) with numerical grids in order to perform accurate simulations. This is in contrast with the case of Newtonian and/or PN simulations, in which we only need to cover a region \( \lambda > L > R \). Since the circular orbit of binary neutron stars becomes unstable at \( R < \sim 10M \) owing to the tidal effects\(^{[11]}\) or the strong general relativistic gravity\(^{[12]}\), we must set an initial condition of binary at \( R > \sim 10M \). For such a case, to perform an accurate simulation, the grid must cover a region \( L > \lambda \sim 100M \) in numerical relativistic simulations. When we assume to cover each neutron star of its radius \( \sim 5M \) with \( \sim 30 \) homogeneous grid\(^{[14,15]}\), we need to take grids of at least \( \sim 500^3 \), but it seems impossible to take such a large amount of mesh points for the present power of supercomputer. At present, we had better search other methods to prepare an initial condition for binary neutron stars.

In the case of PN simulations, the situation is completely different because we do not have to treat gravitational waves explicitly in numerical simulations, and as the result, only need to cover a region at most \( L \sim 20 - 30M \). In this case, it seems that \( \sim 200^3 \) grid numbers are enough. Furthermore, we can take into account general relativistic effects with a good accuracy: In the case of coalescing binary neutron stars, the error will be at most \( \sim M/R \sim a few \times 10\% \) for the first PN approximation, and \( \sim (M/R)^2 \sim several \% \) for 2PN approximation. Hence, if we could take into account up through 2PN terms, we would be able to give a highly accurate initial condition( the error \( \lesssim several \% \)). For these reasons, we consider the 2.5PN hydrodynamic equations including 2.5PN radiation reaction potential in this paper.

The purpose of this paper is twofold: One is to establish the basic formulation of the 2.5PN hydrodynamic equation, and the other is to investigate what kind of gauge condition is appropriate for simulation of the coalescing binary neutron stars and extraction of gravitational waves.
from them. As for the PN hydrodynamic equation, Blanchet et al. have already obtained the 
(1+2.5)PN formula \[^{16}\]. Although their formula was very useful for PN hydrodynamic simulations including the radiation reaction \[^{14,15,17}\], they did not take into account 2PN terms. Also, in their formula, they fixed the gauge conditions to the ADM gauge, but in numerical relativity, it has not been known yet what sort of gauge condition is suitable for simulation of the coalescing binary neutron stars and estimation of gravitational waves from them. For these reasons, we shall investigate several gauge conditions using the (3+1) formalism in general relativity.

This paper is organized as follows. In section 2 we present the (3+1) formalism of the Einstein equation and the equations for the PN approximation. Several slice conditions are imposed in section 3. In section 4, the quadrupole radiation-reaction potential is calculated in combination of the conformal slice \[^{18}\] and the transverse gauge. It is found that this combination of the gauge conditions simplifies the calculation of the back reaction potential. The methods to solve the 2PN tensor potential are discussed in detail for the sake of actual numerical simulations in section 5. The conserved energy and conserved linear momentum are referred in section 6. Section 7 is devoted to summary.

We use the units of \(c = G = 1\) in this paper. Greek and Latin indices take 0, 1, 2, 3 and 1, 2, 3, respectively.

2. (3+1) Formalism for Post-Newtonian Approximation

2.1. (3+1) Formalism

We consider the (3+1) formalism to perform the PN approximation. In the (3+1) formalism, the metric is split as

\[
g_{\mu\nu} = \gamma_{\mu\nu} - \hat{n}_\mu \hat{n}_\nu, \tag{2.1}
\]

and

\[
\hat{n}_\mu = (-\alpha, 0),
\]

\[
\hat{n}^\mu = \left( \frac{1}{\alpha} - \frac{\beta^i}{\alpha} \right), \tag{2.2}
\]

where \(\alpha\), \(\beta^i\) and \(\gamma_{ij}\) are the lapse function, shift vector and metric on the 3D hypersurface.
respectively. Then the line element is written as
\[
\text{ds}^2 = -(\alpha^2 - \beta_i \beta^i)dt^2 + 2\beta_i dt dx^i + \gamma_{ij} dx^i dx^j. \tag{2.3}
\]

Using the (3+1) formalism, the Einstein equation
\[
G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.4}
\]
is split into the constraint equations and the evolution equations. The formers are the so-called Hamiltonian and momentum constraints which respectively become
\[
\text{tr} R - K_{ij} K^{ij} + K^2 = 16\pi \rho_H, \tag{2.5}
\]
\[
D_i K^i_j - D_j K = 8\pi J_j, \tag{2.6}
\]
where $K_{ij}$, $K$, tr$R$ and $D_i$ are the extrinsic curvature, the trace part of $K_{ij}$, the scalar curvature of 3D hypersurface and the covariant derivative with respect of $\gamma_{ij}$. $\rho_H$ and $J_j$ are defined as
\[
\rho_H = T_{\mu\nu} \hat{n}^\mu \hat{n}^\nu, \tag{2.7}
\]
\[
J_j = -T_{\mu\nu} \hat{n}^\mu \gamma^\nu_j.
\]

Evolution equations for the spatial metric and extrinsic curvature are respectively
\[
\frac{\partial}{\partial t} \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i, \tag{2.8}
\]
\[
\frac{\partial}{\partial t} K_{ij} = \alpha(R_{ij} + KK_{ij} - 2K^l_{ii} K_{lj}) - D_i D_j \alpha
\]
\[
+ (D_j \beta^m) K_{mi} + (D_i \beta^m) K_{mj} + \beta^m D_m K_{ij} - 8\pi\alpha \left(S_{ij} + \frac{1}{2} \gamma_{ij} (\rho_H - S^l_l) \right), \tag{2.9}
\]
\[
\frac{\partial}{\partial t} \gamma = 2\gamma(-\alpha K + D_i \beta^i), \tag{2.10}
\]
\[
\frac{\partial}{\partial t} K = \alpha(\text{tr}R + K^2) - D^i D_i \alpha + \beta^j D_j K + 4\pi\alpha (S^l_l - 3\rho_H), \tag{2.11}
\]
where $R_{ij}$, $\gamma$ and $S_{ij}$ are, respectively, the Ricci tensor with respect of $\gamma_{ij}$, determinant of $\gamma_{ij}$ and
\[
S_{ij} = T_{kl} \gamma^k_i \gamma^l_j. \tag{2.12}
\]

Hereafter we use the conformal factor $\psi = \gamma^{1/12}$ instead of $\gamma$ for simplicity.
To distinguish the wave part from the non-wave part (for example, Newtonian potential) in the metric, we use \( \tilde{\gamma}_{ij} = \psi^{-4} \gamma_{ij} \) instead of \( \gamma_{ij} \). Then \( \det(\tilde{\gamma}_{ij}) = 1 \) is satisfied. We also define \( \tilde{A}_{ij} \) as

\[
\tilde{A}_{ij} \equiv \psi^{-4} A_{ij} \equiv \psi^{-4} \left( K_{ij} - \frac{1}{3} \gamma_{ij} K \right). \tag{2.13}
\]

We should note that in our notation, indices of \( \tilde{A}_{ij} \) are raised and lowered by \( \tilde{\gamma}^{ij} \), so that the relations, \( \tilde{A}^i_j = A^i_j \) and \( \tilde{A}^{ij} = \psi^4 A^{ij} \), hold. Using these variables, the evolution equations (2.8)-(2.11) can be rewritten as follows;

\[
\frac{\partial}{\partial n} \tilde{\gamma}_{ij} = -2 \alpha \tilde{A}_{ij} + \tilde{\gamma}_{il} \frac{\partial \beta^l}{\partial x^i} + \tilde{\gamma}_{ji} \frac{\partial \beta^l}{\partial x^i} - \frac{2}{3} \tilde{\gamma}_{ij} \frac{\partial \beta^l}{\partial x^l}, \tag{2.14}
\]

\[
\frac{\partial}{\partial n} \tilde{A}_{ij} = \frac{1}{\psi^4} \left[ \alpha \left( R_{ij} - \frac{1}{3} \gamma_{ij} \text{tr} \tilde{R} \right) - \left( \tilde{D}_i \tilde{D}_j \alpha - \frac{1}{3} \tilde{\gamma}_{ij} \tilde{\Delta} \alpha \right) - \frac{2}{\psi} \left( \psi_i \alpha, j + \psi_j \alpha, i - \frac{2}{3} \tilde{\gamma}_{ij} \tilde{\gamma}^{kl} \tilde{\psi}_{k \alpha, l} \right) \right] + \alpha \left( K \tilde{A}_{ij} - 2 \tilde{A}_{il} \tilde{A}^l_j \right) + \frac{\partial \beta^m}{\partial x^i} \tilde{A}_{mj} + \frac{\partial \beta^m}{\partial x^j} \tilde{A}_{mi} - \frac{2}{3} \frac{\partial \beta^m}{\partial x^m} \tilde{A}_{ij} - 8 \pi \frac{\alpha}{\psi^4} \left( S_{ij} - \frac{1}{3} \gamma_{ij} S^l_l \right), \tag{2.15}
\]

\[
\frac{\partial}{\partial n} \psi = \psi \left( -\alpha K + \frac{\partial \beta^i}{\partial x^i} \right), \tag{2.16}
\]

\[
\frac{\partial}{\partial n} K = \alpha \left( \tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{3} K^2 \right) - \frac{1}{4} \tilde{\Delta} \alpha - \frac{2}{\psi^5} \tilde{\gamma}^{kl} \tilde{\psi}_{k \alpha, l} + 4 \pi \alpha (S^i_i + \rho_H), \tag{2.17}
\]

where \( \tilde{D}_i \) and \( \tilde{\Delta} \) are the covariant derivative and Laplacian with respect to \( \tilde{\gamma}_{ij} \) and

\[
\frac{\partial}{\partial n} = \frac{\partial}{\partial t} - \beta^i \frac{\partial}{\partial x^i}. \tag{2.18}
\]

The constraint equations are also written as

\[
\tilde{\Delta} \psi = \frac{1}{8} \text{tr} \tilde{R} \psi - 2 \pi \rho_H \psi^5 - \frac{\psi^5}{8} \left( \tilde{A}_{ij} \tilde{A}^{ij} - \frac{2}{3} K^2 \right), \tag{2.19}
\]

and

\[
\tilde{D}_j (\psi^6 \tilde{A}^i_i) - \frac{2}{3} \psi^6 \tilde{D}_i K = 8 \pi \psi^6 J_i, \tag{2.20}
\]

where \( \text{tr} \tilde{R} \) is the scalar curvature with respect to \( \tilde{\gamma}_{ij} \).
Now let us consider $R_{ij}$ in Eq. (2.15), which is one of the main source terms of the evolution equation for $\tilde{A}_{ij}$. First we split $R_{ij}$ into two parts as

$$R_{ij} = \tilde{R}_{ij} + R^\psi_{ij}, \quad (2.21)$$

where $\tilde{R}_{ij}$ is the Ricci tensor with respect to $\tilde{\gamma}_{ij}$,

$$R^\psi_{ij} = -\frac{2}{\psi} \tilde{D}_i \tilde{D}_j \psi - \frac{2}{\psi} \tilde{\gamma}_{ij} \tilde{D}^k \psi + \frac{6}{\psi^2} (\tilde{D}_i \psi) (\tilde{D}_j \psi) - \frac{2}{\psi^2} \tilde{\gamma}_{ij} (\tilde{D}_k \psi) (\tilde{D}^k \psi). \quad (2.22)$$

Using the property of $\text{det}(\tilde{\gamma}_{ij}) = 1$, $\tilde{R}_{ij}$ is written as

$$\tilde{R}_{ij} = \frac{1}{2} \left[ \tilde{\gamma}^{kl} (\tilde{\gamma}_{ij,kl} + \tilde{\gamma}_{li,kj} - \tilde{\gamma}_{ij,kl}) + \tilde{\gamma}^{kl}_{,k} (\tilde{\gamma}_{l,j,i} + \tilde{\gamma}_{li,j} - \tilde{\gamma}_{ij,l}) \right] - \tilde{\Gamma}^l_{kj} \tilde{\Gamma}^k_{li}, \quad (2.23)$$

where $,i$ denotes $\partial/\partial x^i$ and $\tilde{\Gamma}^k_{ij}$ is the Christoffel symbol with respect to $\tilde{\gamma}_{ij}$. We split $\tilde{\gamma}_{ij}$ and $\tilde{\gamma}^{ij}$ as $\delta_{ij} + h_{ij}$ and $\delta^{ij} + f^{ij}$, where $\delta_{ij}$ denotes the flat metric, and rewrite $\tilde{R}_{ij}$ as

$$\tilde{R}_{ij} = \frac{1}{2} \left[ -h_{ij,kk} + h_{jl,i} + h_{il,j} + f^{kl}_{,k} (h_{ij,i} + h_{li,j} - h_{ij,l}) ight. \\
+ \left. f^{kl} (h_{kj,il} + h_{ki,jl} - h_{ij,kl}) \right] - \tilde{\Gamma}^l_{kj} \tilde{\Gamma}^k_{li}. \quad (2.24)$$

In this paper, we consider only the linear order in $h_{ij}$ and $f_{ij}$ assuming $|h_{ij}|, |f_{ij}| \ll 1$. (So that, $h_{ij} = -f^{ij}$.) Such an assumption is justified because in this paper, we choose a gauge condition, in which $h_{ij}$ is a 2PN quantity (see below). This implies that we neglect higher PN effects such as the non-linear coupling between gravitational waves themselves, but does not imply that we neglect the non-linear coupling between the Newtonian potentials themselves and between gravitational waves and the Newtonian potentials. In other words, although we can not see the non-linear memory of gravitational waves \cite{21}, we can see the tail term of gravitational waves and can derive the exact quadrupole formula (see below). Here, to guarantee the wave property of $\tilde{\gamma}_{ij}$, we impose a kind of transverse gauge* to $h_{ij}$ as

$$h_{ij,j} = 0. \quad (2.25)$$

*Hereafter, we call this condition merely the transverse gauge.
This condition is guaranteed by $\beta^i$ which satisfies

$$
-\beta^k_{,j} \tilde{\gamma}_{ij,k} = \left(-2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{il} \beta^l_{,j} + \tilde{\gamma}_{jl} \beta^l_{,i} - \frac{2}{3} \tilde{\gamma}_{ij} \beta^l_{,l}\right)_{,} j.
$$

(2.26)

Using the above conditions, Eq. (2.24) becomes

$$
\tilde{R}_{ij} = -\frac{1}{2} \Delta_{flat} h_{ij} + O(h^2),
$$

(2.27)

where $\Delta_{flat}$ is the Laplacian with respect to $\delta_{ij}$. Note that $\text{tr} \tilde{R} = O(h^2)$ is guaranteed in the transverse gauge because the traceless property of $h_{ij}$ holds in the linear order.

Finally, we show the equations for the perfect fluid. The energy momentum tensor for the perfect fluid is written as

$$
T^{\mu\nu} = (\rho + \rho\varepsilon + P) u^\mu u^\nu + P g^{\mu\nu},
$$

(2.28)

where $u^\mu$, $\rho$, $\varepsilon$ and $P$ are the four velocity, the mass density, the specific internal energy and the pressure. The mass density obeys the continuity equation

$$
\nabla_\mu (\rho u^\mu) = 0,
$$

(2.29)

where $\nabla_\mu$ is the covariant derivative with respect to $g_{\mu\nu}$. The explicit form is

$$
\frac{\partial \rho_*}{\partial t} + \frac{\partial (\rho_* v^i)}{\partial x^i} = 0,
$$

(2.30)

where $\rho_*$ is the conserved density defined as

$$
\rho_* = \alpha \psi^6 \rho u^0.
$$

(2.31)

The equations of motion and the energy equation are derived from

$$
\nabla_\mu T^{\mu\nu} = 0.
$$

(2.32)

Explicit forms of them become

$$
\frac{\partial S_i}{\partial t} + \frac{\partial (S_i v^j)}{\partial x^j} = -\alpha \psi^6 P_{,i} - \alpha \alpha_{,i} S^0 + S^j_{,j} \beta^i_{,i} - \frac{1}{3\alpha} S_j S_k \tilde{\gamma}_{jk,i},
$$

(2.33)
and
\[
\frac{\partial H}{\partial t} + \frac{\partial (Hv^j)}{\partial x^j} = -P \left( \frac{\partial (\alpha \psi^6 u^0)}{\partial t} + \frac{\partial (\alpha \psi^6 u^0 v^j)}{\partial x^j} \right),
\]

where
\[
S_i = \alpha \psi^6 (\rho + \rho \varepsilon + P) u^0 u_i = \rho_* \left( 1 + \varepsilon + \frac{P}{\rho} \right) u_i (= \psi^6 J_i),
\]
\[
S^0 = \alpha \psi^6 (\rho + \rho \varepsilon + P) (u^0)^2 \left( = \frac{(\rho H + P) \psi^6}{\alpha} \right),
\]
\[
H = \alpha \psi^6 \rho \varepsilon u^0 = \rho_* \varepsilon,
\]
\[
v^i \equiv \frac{u^i}{u^0} = -\beta^i + \frac{\gamma^{ij} S_j}{S^0}.
\]

Finally, we note that in the above equations, only $\beta^i$ appears, and $\beta_i$ does not, so that, in the subsequent section, we only consider the PN expansion of $\beta^i$, not of $\beta_i$.

### 2.2. Post-Newtonian approximation

Next, we consider the PN approximation of the above equations. First of all, we review the PN expansion of the variables. Each metric variable, which is relevant to the present paper, is expanded as
\[
\psi = 1 + (2) \psi + (4) \psi + (6) \psi + (7) \psi + \ldots,
\]
\[
\alpha = 1 + (2) \alpha + (4) \alpha + (6) \alpha + (7) \alpha + \ldots,
\]
\[
= 1 - U + \left( \frac{U^2}{2} + X \right) + (6) \alpha + (7) \alpha + \ldots,
\]
\[
\beta^i = (3) \beta_i + (5) \beta_i + (6) \beta_i + (7) \beta_i + (8) \beta_i + \ldots,
\]
\[
h_{ij} = (4) h_{ij} + (5) h_{ij} + \ldots,
\]
\[
\tilde{A}_{ij} = (3) \tilde{A}_{ij} + (5) \tilde{A}_{ij} + (6) \tilde{A}_{ij} + \ldots,
\]
\[
K = (3) K + (5) K + (6) K + \ldots,
\]

where subscripts denote the PN order($c^{-n}$) and $U$ is the Newtonian potential satisfying
\[
\Delta_{\text{flat}} U = -4\pi \rho.
\]
which satisfies
\[ \Delta_{\text{flat}} \Phi = -4\pi \rho \left( v^2 + U + \frac{1}{2} \epsilon + \frac{3}{2} \frac{P}{\rho} \right). \]  
(2.38)

Note that the terms relevant to the radiation reaction appear in (7)\(\psi\), (7)\(\alpha\), (8)\(\beta_i\) and (5)\(h_{ij}\), and the quadrupole formula is derived from (7)\(\alpha\) and (5)\(h_{ij}\).

The four velocity is expanded as
\[
\begin{align*}
\rho \quad u^0 &= 1 + \left( \frac{1}{2} v^2 + U \right) + \left( \frac{3}{8} v^4 + \frac{5}{2} v^2 U + \frac{1}{2} U^2 + (3)\beta_i v^i - X \right) + O(c^{-6}), \\
u_0 &= - \left[ 1 + \left( \frac{1}{2} v^2 - U \right) + \left( \frac{3}{8} v^4 + \frac{5}{2} v^2 U + \frac{1}{2} U^2 + X \right) \right] + O(c^{-6}), \\
u^i &= v^i \left[ 1 + \left( \frac{1}{2} v^2 + U \right) + \left( \frac{3}{8} v^4 + \frac{5}{2} v^2 U + \frac{1}{2} U^2 + (3)\beta_i v^i - X \right) \right] + O(c^{-7}), \\
u_i &= v^i \left\{ \left(3\beta_i + v^i \left( \frac{1}{2} v^2 + 3U \right) \right) + \left( (5)\beta_i + (3)\beta_i \left( \frac{1}{2} v^2 + 3U \right) \right) \right\} + O(c^{-7}), \\

\end{align*}
\]

where \(v^2 = v^i v^i\). The PN expansion of the relation \(u^\mu u_\mu = -1\) becomes
\[
(\alpha u^0)^2 = 1 + \gamma^{ij} u_i u_j \\
= 1 + v^2 + v^4 + 4v^2 U + 2(3)\beta_i v^i + O(c^{-6}).
\]  
(2.40)

Thus \(\rho_H\), \(J_i\) and \(S_{ij}\) are respectively expanded as
\[
\begin{align*}
\rho_H &= \rho \left[ 1 + \left( v^2 + \epsilon \right) + \left\{ v^4 + v^2 \left( 4U + \epsilon + \frac{P}{\rho} \right) + 2(3)\beta_i v^i \right\} + O(c^{-6}) \right], \\
J_i &= \rho \left[ v^i \left( 1 + v^2 + 3U + \epsilon + \frac{P}{\rho} \right) + (3)\beta_i + O(c^{-5}) \right], \\
S_{ij} &= \rho \left[ \left( v^i v^j + \frac{P}{\rho} \delta_{ij} \right) + \left\{ \left( v^2 + 6U + \epsilon + \frac{P}{\rho} \right) v^i v^j + v^i (3)\beta_j + v^j (3)\beta_i + 2 \frac{UP}{\rho} \delta_{ij} \right\} + O(c^{-6}) \right], \\
S^l_{\ell} &= \rho \left[ v^2 + 3\frac{P}{\rho} + \left\{ 2(3)\beta_i v^i + v^2 \left( 2v^2 + 4U + \epsilon + \frac{P}{\rho} \right) \right\} + O(c^{-6}) \right].
\end{align*}
\]  
(2.41)

\(\psi\) (and \(\alpha\) in the conformal slice) is determined by the Hamiltonian constraint. At the lowest order, it becomes
\[
\Delta_{\text{flat}(2)} \psi = -2\pi \rho.
\]  
(2.42)

Thus, \(\alpha c_{\text{fr}} = -2(\psi) = -U\) is satisfied in this paper. At the 2PN and 3PN orders, the Hamiltonian constraint is
\[
\Delta_{\text{flat}(2)} \psi = -2\pi \rho.
\]  
(2.42)

Thus, \(\alpha c_{\text{fr}} = -2(\psi) = -U\) is satisfied in this paper. At the 2PN and 3PN orders, the Hamiltonian constraint is
\[
\Delta_{\text{flat}(2)} \psi = -2\pi \rho.
\]  
(2.42)

Thus, \(\alpha c_{\text{fr}} = -2(\psi) = -U\) is satisfied in this paper. At the 2PN and 3PN orders, the Hamiltonian constraint is
\[
\Delta_{\text{flat}(2)} \psi = -2\pi \rho.
\]  
(2.42)
nian constraint equation becomes, respectively,

$$\Delta_{flat(4)}\psi = -2\pi \rho \left( v^2 + \varepsilon + \frac{5}{2} U \right), \quad (2.43)$$

and

$$\Delta_{flat(6)}\psi = -2\pi \rho \left\{ v^4 + v^2 \left( \varepsilon + \frac{P}{\rho} + \frac{13}{2} U \right) + 2(3)\beta_i v^i + \frac{5}{2} \varepsilon U + \frac{5}{2} U^2 + 5(4)\psi \right\} + \frac{1}{2}(4)h_{ij}U_{,ij} - \frac{1}{8} (3)\tilde{A}_{ij}(3)\tilde{A}_{ij} - \frac{2}{3}(3)K^2. \quad (2.44)$$

The term relevant to the radiation reaction first appears in (7)ψ and the equation for it becomes

$$\Delta_{flat(7)}\psi = \frac{1}{2}(5)h_{ij}U_{,ij}. \quad (2.45)$$

Hence, (7)\(\alpha\) may be also relevant to the radiation reaction and whether it may or not depends on the slice condition.

From Eq.(2.26), the relation between (3)\(\tilde{A}_{ij}\) and (3)\(\beta_i\) becomes

$$-2(3)\tilde{A}_{ij} + (3)\beta_{i,j} + (3)\beta_{j,i} - \frac{2}{3}\delta_{ij}(3)\beta_{l,l} = 0. \quad (2.46)$$

(3)\(\tilde{A}_{ij}\) must also satisfy the momentum constraint. Since (3)\(\tilde{A}_{ij}\) does not contain the transverse-traceless(TT) part and only contains the longitudinal part, it can be written as

$$(3)\tilde{A}_{ij} = (3)W_{i,j} + (3)W_{j,i} - \frac{2}{3}\delta_{ij}(3)W_{k,k}, \quad (2.47)$$

where (3)\(W_i\) is a vector on the 3D hypersurface and satisfies the momentum constraint at the first PN order as follows;

$$\Delta_{flat(3)}W_i + \frac{1}{3}(3)W_{j,ji} - \frac{2}{3}(3)K_{,i} = 8\pi \rho v^i. \quad (2.48)$$

From Eq.(2.46), the relation,

$$(3)\beta_i = 2(3)W_i, \quad (2.49)$$
holds and at the first PN order, Eq.(2.16) becomes

$$3\dot{U} = -(3)K + (3)\beta_{t,t}, \quad (2.50)$$

where $\dot{U}$ denotes the derivative of $U$ with respect to time, so that Eq.(2.48) is rewritten as

$$\Delta_{flat}(3)\beta_i = 16\pi\rho v^i + (3)K, \quad (2.51)$$

This is the equation for the vector potential at the first PN order.

From the next order, $(n)\beta_i$ is determined by the gauge condition, $h_{ij,j} = 0$. Making use of the momentum constraint and the 2PN order of Eq.(2.16),

$$6(4)\dot{\psi} - 3(3)\beta_t U,t - \frac{1}{2}U(2(3)K + 3\dot{U}) + (5)K = (5)\beta_{t,t}, \quad (2.52)$$

the equation for $(5)\beta_i$ is written as

$$\Delta_{flat}(5)\beta_i = 16\pi\rho \left[ v^i (v^2 + 2U + \varepsilon + \frac{P}{\rho}) + (3)\beta_i \right] - 8U_{(3)}\tilde{A}_{ij}$$

$$+ (5)K_{,i} - U_{(3)}K_i + \frac{1}{3}U_{,i(3)}K - 2(4)\dot{\psi}_{,i} + \frac{1}{2}(U\dot{U})_{,i} + (3)\beta_{tU,ij} \right). \quad (2.53)$$

Since $J_i$ at the 1.5PN order vanishes, the merely geometrical equation for $(6)\beta_i$ is given by

$$\Delta_{flat}(6)\beta_i = (6)K_{,i}. \quad (2.54)$$

Then, let us consider the wave equation for $h_{ij}$. From Eqs.(2.14), (2.15), (2.21) and (2.27), the wave equation for $h_{ij}$ is written as

$$h_{ij} = \left(1 - \frac{\alpha^2}{\psi^4}\right)\Delta_{flat}h_{ij} + \left(\frac{\partial^2}{\partial n^2} - \frac{\partial^2}{\partial t^2}\right)h_{ij}$$

$$+ 2\alpha^2 \left[ -\frac{2\alpha}{\psi} \left( \tilde{D}_i \tilde{D}_j - \frac{1}{3} \tilde{\gamma}_{ij} \tilde{D}_k \tilde{D}_k \right) \psi + \frac{6\alpha}{\psi^2} \left( \tilde{D}_i \psi \tilde{D}_j \psi - \frac{1}{3} \tilde{\gamma}_{ij} \tilde{D}_k \psi \tilde{D}_k \psi \right)$$

$$- \left( \tilde{D}_i \tilde{D}_j - \frac{1}{3} \tilde{\gamma}_{ij} \tilde{D}_k \tilde{D}_k \right) \alpha - \frac{2}{\psi} \left( \tilde{D}_i \psi \tilde{D}_j \alpha + \tilde{D}_j \psi \tilde{D}_i \alpha - \frac{2}{3} \tilde{\gamma}_{ij} \tilde{D}_k \psi \tilde{D}_k \alpha \right) \right]$$

$$+ 2\alpha^2 \left( K \tilde{A}_{ij} - 2 \tilde{A}_{il} \tilde{A}^l_{ij} \right) + 2\alpha \left( \beta_{m,i} \tilde{A}_{mj} + \beta_{m,j} \tilde{A}_{mi} - \frac{2}{3} \beta_{m,m} \tilde{A}_{ij} \right)$$

$$- 16\pi \alpha^2 \left( S_{ij} - \frac{1}{3} \tilde{\gamma}_{ij} S_l^l \right) - \frac{\partial}{\partial n} \left( \beta_{*,i} \tilde{\gamma}_{mj} + \beta_{*,j} \tilde{\gamma}_{mi} - \frac{2}{3} \beta_{*,m} \tilde{\gamma}_{ij} \right) + 2\frac{\partial}{\partial n} \tilde{A}_{ij} \equiv \tau_{ij}. \quad (2.55)$$
where
\[ \square = -\frac{\partial^2}{\partial t^2} + \Delta_{\text{flat}}. \] (2.56)

We should note that \( (4)\tau_{ij} \) has the TT property, i.e., \( (4)\tau_{ij,j} = 0 \) and \( (4)\tau_{ii} = 0 \). This is a natural consequence of the transverse gauge, \( h_{ij,j} = 0 \) and \( h_{ii} = O(h^2) \). Thus \( (4)h_{ij} \) is determined from
\[ \Delta_{\text{flat}}(4)h_{ij} = (4)\tau_{ij}. \] (2.57)

\((5)h_{ij}\) is derived from
\[ (5)h_{ij}(t) = \frac{1}{4\pi} \frac{\partial}{\partial t} \int (4)\tau_{ij}(t,y)d^3y, \] (2.58)

and the quadrupole mode of gravitational waves in the wave zone is written as
\[ h_{ij}^{\text{rad}} = -\frac{1}{4\pi} \lim_{|x|\to\infty} \int \frac{(4)\tau_{ij}(t - |x - y|,y)}{|x - y|} d^3y. \] (2.59)

In the subsequent section, we derive the quadrupole radiation-reaction metric in the near zone using Eq. (2.58).

Finally, we show the evolution equation for \( K \). Since we adopt slice conditions which do not satisfy \( K = 0 \)(i.e., maximal slice condition), the evolution equation for \( K \) is necessary. The evolution equations appear at the 1PN, 2PN and 2.5PN orders which become respectively
\[ \frac{\partial}{\partial t}(3)K = 4\pi \rho \left( 2v^2 + \varepsilon + 2U + 3\frac{P}{\rho} \right) - \Delta_{\text{flat}}X, \] (2.60)
\[ \frac{\partial}{\partial t}(5)K = 4\pi \rho \left[ 2v^4 + v^2 \left( 6U + 2\varepsilon + 2\frac{P}{\rho} \right) - \left( \varepsilon + 3\frac{P}{\rho} \right) U - 4U^2 + 4(4)\psi + X + 4(3)\beta_i v^i \right] 
+ (3)\tilde{A}_{ij}(3)\tilde{A}_{ij} + \frac{1}{3}(3)\alpha^2 - (4)h_{ij}U_{ij} + (3)\beta_i(3)K_{,i} 
- \frac{3}{2} UU_{,k}U_{,k} - U_{,k}X_{,k} + 2U_{,k}(4)\psi_{,k} - \Delta_{\text{flat}}(6)\alpha + 2U\Delta_{\text{flat}}X, \] (2.61)
\[ \frac{\partial}{\partial t}(6)K = -\Delta_{\text{flat}}(7)\alpha - (5)h_{ij}U_{,ij}. \] (2.62)

We note that for the PN equations of motion up to the 2.5PN order, we need \( (2)\alpha, (4)\alpha, (6)\alpha, (7)\alpha, (2)\psi, (4)\psi, (3)\beta_i, (5)\beta_i, (6)\beta_i, (4)h_{ij}, (5)h_{ij}, (3)K, (5)K \) and \( (6)K \). Therefore, if we solve

* Since \( O(h^2) \) turns out to be \( O(c^{-8}) \), it is enough to consider only the linear order of \( h_{ij} \) in the case when we perform the PN approximation up to the 3.5PN order.
the above set of the equations, we can obtain the 2.5 PN equations of motion. Up to the 2.5PN
order, the hydrodynamic equations become
\[
\frac{\partial S_i}{\partial t} + \frac{\partial (S_i v^j)}{\partial x^j} = -\left(1 + 2U + \frac{5}{4} U^2 + 6(4) \psi + X\right) P_{,i}
+ \rho_* \left[U_i \left\{1 + \varepsilon + \frac{P}{\rho} + \frac{3}{2} v^2 - U + \frac{5}{8} v^4 + 4v^2 U + \left(\frac{3}{2} v^2 - U\right) \left(\varepsilon + \frac{P}{\rho}\right) + 3(3) \beta_j v^j\right\}
- X_{,i} \left(1 + \varepsilon + \frac{P}{\rho} + \frac{v^2}{2}\right) + 2v^2 (4) \psi_{,i} - (6) \alpha_{,i} - (7) \alpha_{,i}
+ v^j \left\{(3) \beta_{j,i} \left(1 + \varepsilon + \frac{P}{\rho} + \frac{v^2}{2} + 3U\right) + (5) \beta_{j,i} + (6) \beta_{j,i}\right\} + (3) \beta_{j(3) \beta_{j,i}}
+ \frac{1}{2} v^j v^k (4) h_{jk,i} + (5) h_{jk,i} + O(c^{-8}),\right)
\]
(2.63)
\[
\frac{\partial H}{\partial t} + \frac{\partial (H v^j)}{\partial x^j} = -P \left[v^j \frac{\partial \left(\frac{1}{2} v^2 + 3U\right)}{\partial x^j} + \frac{\partial}{\partial x^j} \left\{\left(\frac{1}{2} v^2 + 3U\right) v^j\right\} + O(c^{-5})\right],
\]
(2.64)
where we make use of relations
\[
\alpha S^0 = \rho_* \left[1 + \varepsilon + \frac{P}{\rho} + \frac{v^2}{2} + \frac{v^2}{\rho} \left(\varepsilon + \frac{P}{\rho}\right) + \frac{3}{8} v^4 + 2v^2 U + (3) \beta_j v^j + +O(c^{-6})\right],
\]
(2.65)
\[
S_i = \rho_* \left[v^i \left(1 + \varepsilon + \frac{P}{\rho} + \frac{v^2}{2} + 3U\right) + (3) \beta_i + O(c^{-5})\right].
\]

3. Slice Conditions

In this section, we perform the PN analysis using the conformal slice\(^{18}\), maximal slice and
harmonic slice\(^{20}\) which are often used in 3D numerical relativity. Among them, we find that the
conformal slice seems most tractable and useful to estimate gravitational waves in the far zone,
while the maximal slice is suitable for describing the equilibrium configurations. Hence, first
of all we describe the property of the conformal slice and then mention the properties of other
slices.
3.1. Conformal Slice

The conformal slice is defined\(^{[18]}\) as

\[
\alpha = \exp \left(-2\epsilon - \frac{2}{3}\epsilon^3 - \frac{2}{5}\epsilon^5\right),
\]

(3.1)

where \(\epsilon = \psi - 1\). In the conformal slice, \((n)\alpha\) becomes

\[
(2)\alpha = -2(2)\psi,
\]

\[
(4)\alpha = 2((2)\psi)^2 - 2(4)\psi,
\]

\[
(6)\alpha = -2((2)\psi)^3 + 4(2)\psi(4)\psi - 2(6)\psi,
\]

\[
(7)\alpha = -2(7)\psi,
\]

(3.2)

Although in the usual PN approximation we need to solve the Poisson equation for the lapse function, this slicing saves solving it.

In the conformal slice, equations (2.15) and (2.17) are rewritten as

\[
\frac{\partial}{\partial n}\tilde{A}_{ij} = -\frac{1}{2}\frac{\alpha}{\psi^4}\Delta_{flat}h_{ij} + \frac{2\alpha}{\psi^4}\left[\left(\tilde{D}_i\tilde{D}_j\psi - \frac{\gamma_{ij}}{3}\tilde{D}_k\tilde{D}^k\psi\right)\frac{\epsilon}{\psi}(1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4) + \frac{1}{\psi^2}\left(\tilde{D}_i\psi\tilde{D}_j\psi - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{D}^k\psi\tilde{D}_k\psi\right)
\right]
\]

\[
\left((3 + 6\epsilon + 6\epsilon^2 + 6\epsilon^3 + 6\epsilon^4 + 12\epsilon^5 + 10\epsilon^6 + 8\epsilon^7 + 6\epsilon^8 + 4\epsilon^9 + 2\epsilon^{10}\right)\]

\[
+ \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_{\ j}) + \beta_{m,i}\tilde{A}_{mj} + \beta_{m,j}\tilde{A}_{mi} - \frac{2}{3}\beta_{,m}\tilde{A}_{ij}
\]

\[
- 8\pi\frac{\alpha}{\psi^4}\left(S_{ij} - \frac{1}{3}\gamma_{ij}S^l_l\right),
\]

(3.3)

and

\[
\frac{\partial K}{\partial n} = 2\frac{\alpha}{\psi^4}\left[\tilde{\Delta}\psi(1 + \epsilon^2 + \epsilon^4) - \frac{2}{\psi^2}\tilde{D}_k\psi\tilde{D}^k\psi(3\epsilon^5 + 2\epsilon^6 + 2\epsilon^7 + \epsilon^8 + \epsilon^9)\right]
\]

\[
+ \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) + 4\pi\alpha(S^l_l + \rho_H),
\]

(3.4)

where we use the TT property as well as the linear approximation for \(h_{ij}\) in the above equation.
Then Eq. (2.55) is written as

\[
\begin{align*}
\Box h_{ij} &= -\left(\frac{\alpha^2}{\psi^4} - 1\right) \Delta_{\text{flat}} h_{ij} + \left(\frac{\partial^2}{\partial n^2} - \frac{\partial^2}{\partial t^2}\right) h_{ij} \\
&\quad + \frac{4\alpha^2}{\psi^4} \left[ (\tilde{\nabla}_i \tilde{\nabla}_j \psi - \frac{7}{3} \tilde{\nabla}_k \tilde{D}_k \psi) \frac{\epsilon}{\psi^2} (1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4) \right. \\
&\quad \left. \quad - \frac{1}{\psi^2} \left( \tilde{D}_i \psi \tilde{D}_j \psi - \frac{1}{3} \tilde{\gamma}_{ij} \tilde{D}_k \psi \tilde{D}_k \psi \right) \right] \\
&\quad + (3 + 6\epsilon + 6\epsilon^2 + 6\epsilon^3 + 12\epsilon^5 + 10\epsilon^6 + 8\epsilon^7 + 6\epsilon^8 + 4\epsilon^9 + 2\epsilon^{10}) \\
&\quad + 2\alpha^2 (K \tilde{A}_{ij} - 2 \tilde{A}_i \tilde{A}_j) + 2\alpha \left( \beta_{m,i} \tilde{A}_{mj} + \beta_{m,j} \tilde{A}_{mi} - \frac{2}{3} \beta_{m,m} \tilde{A}_{ij} \right) \\
&\quad - 16\pi \frac{\alpha^2}{\psi^4} \left( S_{ij} - \frac{1}{3} \tilde{\gamma}_{ij} S_{il} \right) - \frac{d}{dt} \left( \beta_{m,i} \tilde{\gamma}_{mj} + \beta_{m,j} \tilde{\gamma}_{mi} - \frac{2}{3} \beta_{m,m} \tilde{\gamma}_{ij} \right) + 2 \frac{\partial \alpha}{\partial n} \tilde{A}_{ij} \\
&\equiv \tau_{ij},
\end{align*}
\]

where we use \( \epsilon = \psi - 1 \) and \( \psi \) satisfies

\[
\tilde{\Delta} \psi = -2\pi \rho_H \psi^5 - \frac{\psi^5}{8} (\tilde{A}_{ij} \tilde{A}^{ij} - \frac{2}{3} K^2). \tag{3.6}
\]

Eq. (3.5) is expanded as follows;

\[
\Box h_{ij} = \left( UU_{,ij} - \frac{1}{3} \delta_{ij} U \Delta_{\text{flat}} U - 3U_{,i} U_{,j} + \delta_{ij} U_{,k} U_{,k} \right) \\
- 16\pi \left( \rho v^i v^j - \frac{1}{3} \delta_{ij} \rho v^2 \right) - \left( (3) \hat{\beta}_{i,j} + (3) \hat{\beta}_{j,i} - \frac{2}{3} \delta_{ij} (3) \hat{\beta}_{k,k} \right) + O(c^{-6}) \tag{3.7}
\]

\[
= (4) \tau_{ij} + O(c^{-6}),
\]

where we use \( \epsilon = 2U \) which holds in the PN approximation.

In the conformal slice, the asymptotic form of \( \epsilon \) becomes

\[
\epsilon \sim \frac{M_{\text{ADM}}}{2r}. \tag{3.8}
\]

Thus \( \alpha \) behaves as \( 1 - M_{\text{ADM}}/r \) at spatial infinity. This means that in the conformal slice, the metric at spatial infinity becomes the static Schwarzschild’s one. This property seems helpful for discerning the wave part from the non-wave part in the wave zone in numerical relativity.


Also, we have an advantage to derive the radiation reaction potential in this slice; From a relation $(\gamma)\alpha = -2(\gamma)\psi$ and Eq.(2.45), we have

$$\Delta_{\text{flat}}(\gamma)\alpha = -(5)h_{ij}(t)U_{,ij}. \quad (3.9)$$

Thus, the radiation reaction potential $(\gamma)\alpha$ is derived as

$$(\gamma)\alpha = \frac{(5)h_{ij}(t)}{4\pi} \int U_{,ij}d^3x. \quad (3.10)$$

Finally, we comment on the following weak point of the conformal slice; in the conformal slice, the evolution equation for $(3)K$ becomes

$$(3)\dot{K} = 4\pi\rho \left( v^2 + 3\frac{P}{\rho} - \frac{1}{2}U \right). \quad (3.11)$$

Since $\dot{K}$ does not vanish, $K$ continues to change even in the case of a stationary spacetime. Thus, it seems inconvenient to describe equilibrium configurations of stars and binary systems in the conformal slice. To describe equilibrium configurations, we had better use the slice, such as the maximal slice, where $\dot{K} = 0$ is satisfied.

### 3.2. Maximal Slice

The maximal slice is given by

$$K = 0, \quad (3.12)$$

and this equation leads to the equation for $\alpha$ as

$$D_kD^k\alpha = \alpha \left( \tilde{A}_{ij}\tilde{A}^{ij} + 4\pi(E + S^l) \right). \quad (3.13)$$

At the first PN order, the equation becomes

$$\Delta_{\text{flat}(4)}X_{MS} = 4\pi\rho \left( 2v^2 + \varepsilon + \frac{3P}{\rho} + 2U \right), \quad (3.14)$$

where the subscript $MS$ denotes “maximal slice”. In the case of the conformal slice, the following...
relation holds;

\[ X_{CS} = -2(4)\psi, \quad (3.15) \]

where CS similarly denotes “conformal slice”. Using the above equation, we rewrite \( X_{MS} \) as

\[ X_{MS} = -2(4)\psi + Y. \quad (3.16) \]

Then the equation for \( Y \) becomes

\[ \Delta_{\text{flat}} Y = 4\pi \left( \rho v^2 + 3P - \frac{1}{2} \rho U \right). \quad (3.17) \]

We should also note that by means of the virial theorem\(^{[23]}\), the integration of the source term for \( Y \) can be written as

\[ \int \left( \rho v^2 + 3P - \frac{1}{2} \rho U \right) d^3x = \frac{1}{2} \ddot{I}_{ll}, \quad (3.18) \]

where

\[ I_{ij}(t) = \int \rho x^i x^j d^3x. \quad (3.19) \]

Hence, the behavior of \( Y \) far from the matter becomes

\[ Y \sim -\frac{1}{2r} \ddot{I}_{ll}. \quad (3.20) \]

In total, the behavior of \( \alpha \) in the wave zone becomes

\[ \alpha \sim 1 - \frac{1}{r} \left( M + \frac{1}{2} \ddot{I}_{ll} \right). \quad (3.21) \]

Therefore, contrary to the conformal slice, in the maximal slice, the spurious time-dependent term is included in \( \alpha \) in the wave zone. Since the metric does not approach the static Schwarzschild metric even in spatial infinity, the maximal slice is inconvenient to distinguish a wave part from non-wave parts such as the Newtonian potential.
In the case of the maximal slice, the equations for the shift vector are obtained by simply taking \( K = 0 \) in Eqs.\( (2.51) \) and \( (2.53) \). Also, it is found that the equation for \( (7) \alpha \) is the same as that in the conformal slice: The right-hand side of Eq.\( (3.13) \) has no \( O(c^{-7}) \) terms. Therefore Eq.\( (3.13) \) becomes

\[
\Delta_{\text{flat}(7)} \alpha = -(5) h_{ij} U_{,ij} . \tag{3.22}
\]

Finally, we show the wave equation for \( h_{ij} \) in the maximal slice as

\[
\Box h_{ij} = -2 \left( Y_{,ij} - \frac{1}{3} \delta_{ij} \Delta_{\text{flat}} Y \right) + \left( U U_{,ij} - \frac{1}{3} \delta_{ij} U \Delta_{\text{flat}} U - 3 U_{,i} U_{,j} + \delta_{ij} U_{,k} U_{,k} \right) - 16 \pi \left( \rho v^{i} v^{j} - \frac{1}{3} \delta_{ij} \rho v^{2} \right) - \left( (3) \dot{\beta}_{i,j} + (3) \dot{\beta}_{j,i} - \frac{2}{3} \delta_{ij} \dot{\beta}_{k,k} \right) + O(c^{-6}). \tag{3.23}
\]

### 3.3. Harmonic Slice

The condition for the harmonic slice is

\[
\Box t = 0, \tag{3.24}
\]

which becomes in the \( (3+1) \) terminology,

\[
\dot{\alpha} + \alpha^2 K - \beta^i \alpha_{,i} = 0. \tag{3.25}
\]

Differentiating this equation with respect to time, the wave equation for the lapse function is derived as

\[
\Box \alpha = 4 \pi \alpha^3 \left( S^i_l - 3 \rho H \right) - \left( \frac{\alpha^2}{\psi^4} \Delta - \Delta_{\text{flat}} \right) \alpha - \frac{2 \alpha^2}{\psi^5} \tilde{D}^{i} \psi \tilde{D}_{i} \alpha - \frac{8 \alpha^3}{\psi^5} \Delta \psi
\]

\[
+ 2 \alpha \dot{\alpha} \tilde{K} + \frac{\alpha^3}{\psi^4} \tilde{R} + \alpha^2 \beta^i \tilde{D}_{i} \tilde{K} - \beta^i \alpha_{,i} - \beta^i \dot{\alpha}_{,i} \tag{3.26}
\]

\[
\equiv \Lambda_{\alpha},
\]

where \( \Lambda_{\alpha} \) is expanded as follows,

\[
\Lambda_{\alpha} = 4 \pi \rho \left[ 1 + \left( v^2 + 3 \frac{P}{\rho} - \frac{U}{2} \right) \right] + \Delta_{\text{flat}} \left( \frac{U^2}{2} - 2 \langle 4 \rangle \psi \right) + O(c^{-6}). \tag{3.27}
\]

This equation is formally solved by using the retarded Green function and the Taylor expansion. For example, we obtain the Newtonian and first PN order lapse function

\[
(2) \alpha = - \int d^3 y \frac{\rho(t, y)}{|\mathbf{x} - \mathbf{y}|} = - \frac{1}{\rho} \int \rho d^3 y + O(r^{-2}), \tag{3.28}
\]
\[
(4)\alpha = -\frac{1}{2} \int d^3 y \rho |x - y| - \int d^3 y \frac{(\rho v^2 + 3P - \frac{4}{3}\rho U)}{|x - y|} + \frac{1}{2} U^2 - 2(4)\psi. \tag{3.29}
\]

Thus, at the spatial infinity, we find the following behavior

\[
(4)\alpha + 2(4)\psi \sim -\frac{3}{4} r \left( I_{kk} - \frac{1}{3} n^k n^l I_{kl} \right), \tag{3.30}
\]

where \( n^i = x^i / r \). From these equations we find that at the spatial infinity the lapse function does not behave as \( 1 - M/r + O(r^{-2}) \) unlike in the conformal slice, but behaves as \( 1 - (M + T(t))/r + O(r^{-2}) \). Thus the harmonic slice is also inconvenient to distinguish a wave part from non-wave parts.

The quadrupole radiation reaction potential takes the following rather lengthy form.

\[
(7)\alpha = \frac{1}{480\pi} \frac{\partial^5}{\partial t^5} \int d^3 y \rho |x - y|^4 + \frac{1}{24\pi} \frac{\partial^3}{\partial t^3} \int d^3 y (4)\Lambda_\alpha |x - y|^2 + \frac{1}{4\pi} \frac{\partial}{\partial t} \int d^3 y (6)\Lambda_\alpha + \frac{1}{4\pi} \int d^3 y (5)h_{ij} U_{ij} \tag{3.31}
\]

This expression is similar to Chandrasekhar’s one in the harmonic gauge\(^{[24]}\) and indicates that the fifth time derivative of the quadrupole moment appears in the reaction force, which is not convenient to treat in numerical calculations.

4. The Radiation Reaction due to Quadrupole Radiation

This topic has been already investigated by using some gauge conditions in previous papers\(^{[24,19,16]}\). However, if we use the combination of the conformal slice and the transverse gauge, calculations are simplified. This is why we briefly mention the derivation of the radiation reaction potential in this section.

In combination of the conformal slice and the transverse gauge, Eq.(2.58) becomes

\[
(5)h_{ij}(t) = \frac{1}{4\pi} \frac{\partial}{\partial t} \int \left[ -16\pi \left( \rho v^i v^j - \frac{1}{3} \delta_{ij} \rho v^2 \right) \right. \\
+ \left. \left( UU_{,ij} - \frac{1}{3} \delta_{ij} U \Delta_{\text{flat}} U - 3U_{,i} U_{,j} + \delta_{ij} U_{,k} U_{,k} \right) \right] d^3 y \\
+ \frac{1}{4\pi} \frac{\partial}{\partial t} \int \left( (3)\hat{\beta}_{i,j} + (3)\hat{\beta}_{j,i} - \frac{2}{3} \delta_{ij(3)} \hat{\beta}_{k,k} \right) d^3 y. \tag{4.1}
\]

From a straightforward calculation, we find that the sum of the first and second lines becomes \(-2(3)\hat{F}\), and the third line becomes \(6(3)\hat{F} / 5\), where \( (3)\hat{F} = \frac{d^3 F_{ij}}{dt^3} \). (This calculation is
replaced by a fairly simple one noticing the transverse property of $(4)\tau_{ij}$. It is described in the appendix A.) Thus, $(5)h_{ij}$ in the near zone becomes

$$(5)h_{ij} = -\frac{4}{5} I^{(3)}_{ij},$$

where

$$I_{ij} = I_{ij} - \frac{1}{3} \delta_{ij} I_{kk}. \quad (4.3)$$

Since $h_{ij}$ has the transverse and traceless property, it is likely that $(5)h_{ij}$ remains the same for other slices. However it is not clear whether the TT property of $h_{ij}$ is satisfied even after the PN expansion is taken in the near zone and, as a result, whether $(5)h_{ij}$ is independent of slicing conditions or not. The fact that slicing conditions never affect $(5)h_{ij}$ is understood on the ground that $(4)\tau_{ij}$ does not depend on slices, which will be shown in the section 5.

Then the Hamiltonian constraint at the 2.5PN order turns out to be

$$\Delta_{flat}(7)\psi = -\frac{2}{5} I^{(3)}_{ij} U_{,ij} = \frac{1}{5} I^{(3)}_{ij} \Delta_{flat} \chi_{,ij},$$

where $\chi$ is the superpotential and defined as

$$\chi = -\int \rho |x - y| d^3y, \quad (4.5)$$

which satisfies the relation $\Delta_{flat} \chi = -2U$. From this, we find $(7)\psi$ takes the following form,

$$(7)\psi = -\frac{1}{5} I^{(3)}_{ij} \int \rho, \frac{(x^j - y^j)}{|x - y|} d^3y$$

$$= \frac{1}{5} I^{(3)}_{ij} \left(-x^j U_{,i} + \int \frac{\rho_i y^j}{|x - y|} d^3y\right). \quad (4.6)$$

Therefore, the lapse function at the 2.5PN order, $(7)\alpha = -2(7)\psi$, is derived from $U$ and $U_r$, where $U_r$ satisfies

$$\Delta U_r = -4\pi I^{(3)}_{ij} \rho, x^j. \quad (4.7)$$

Since the right-hand side of Eq.(2.62) cancels out, $(6)K$ disappears if the $(6)K$ does not exist on the initial hypersurface, which seems reasonable under the condition that there are no initial gravitational waves. Also, $(6)\beta_i$ vanishes according to Eq.(2.54). Hence, the quadrupole radiation reaction metric has the same form as that derived in the case of the maximal slice.
From Eq. (2.33), the PN equations of motion becomes

\[ \dot{v}^i + v^j v_j = \frac{P_i}{\rho} + U_i + F^{1\text{PN}}_i + F^{2\text{PN}}_i + F^{2.5\text{PN}}_i + O(c^{-8}), \]  

(4.8)

where \( F^{1\text{PN}}_i \) and \( F^{2\text{PN}}_i \) are, respectively, the 1PN and 2PN forces and conservative ones. Since the radiation reaction potentials, \((5)h_{ij}\) and \((7)\alpha\), are the same as those by Schäfer [19] and Blanchet, Damour and Schäfer [16] in which they use the ADM gauge, the radiation reaction force per unit mass, \( F^{2.5\text{PN}}_i \equiv F^r_i \), is the same as their force and

\[
F^r_i = - \left( (5)h_{ij} v^j \right)' + v^k v^j \partial_{i} (5)h_{kj} + (7)\alpha_i
\]

\[
= \left[ \frac{4}{5} \tau_{ij}^{(3)} + \frac{4}{5} \tau_{ij}^{(5)} v^j \partial_{i} + \frac{2}{5} \tau_{kl}^{(3)} \frac{\partial}{\partial x^i} \int \rho(t, \mathbf{y}) \frac{(x^k - y^k)(x^l - y^l)}{|\mathbf{x} - \mathbf{y}|^3} d^3 \mathbf{y} \right].
\]

(4.9)

Since the work done by the force (4.9) is given by

\[
W \equiv \int \rho F^r_i v^i d^3 x
\]

\[
= \frac{4}{5} \partial_t \left( \tau_{ij}^{(3)} \int \rho v^i v^j d^3 x \right) - \frac{1}{5} \tau_{ij}^{(3)} \tau_{ij}^{(3)},
\]

(4.10)

we obtain the so-called quadrupole formula of the energy loss by averaging Eq.(4.10) with respect to time as

\[
\left\langle \frac{dE_N}{dt} \right\rangle = -\frac{1}{5} \left\langle \tau_{ij}^{(3)} \tau_{ij}^{(3)} \right\rangle + O(c^{-6}).
\]

(4.11)

5. Strategy to obtain 2PN tensor potential

In this section, we describe methods to solve the equation for the 2PN tensor potential \( (4)h_{ij} \). Although Eq.(2.57) is formally solved as

\[
(4)h_{ij}(t, \mathbf{x}) = -\frac{1}{4\pi} \int \frac{\tau_{ij}(t, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} d^3 \mathbf{y},
\]

(5.1)

it seems difficult to estimate this integral in practice since \( (4)\tau_{ij} \to O(r^{-3}) \) for \( r \to \infty \) and the integral is taken all over the space. Thus it is desirable to replace this equation by some tractable forms in numerical evaluation. In the following, we show two approaches: In section 5.1, we change Eq.(5.1) into the form in which the integration is performed only over the matter distribution like as in the Newtonian potential. In section 5.2, we propose a method to solve Eq.(2.57) as the boundary value problem.
5.1. Direct integration method

The explicit form of \((4)\tau_{ij}\) is

\[
(4)\tau_{ij} = -2\hat{\partial}_{ij}(X + 2(4)\psi) + U\hat{\partial}_{ij}U - 3U_iU_{,j} + \delta_{ij}U_{,k}U_{,k} - 16\pi\left(\rho v^i v^j - \frac{1}{3}\delta_{ij}\rho v^2\right)
- \left(3\hat{\beta}_{i,j} + 3\hat{\beta}_{j,i} - \frac{2}{3}\delta_{ij}\hat{\beta}_{k,k}\right),
\]

(5.2)

where

\[
\hat{\partial}_{ij} \equiv \frac{\partial^2}{\partial x^i \partial x^j} - \frac{1}{3}\delta_{ij}\Delta_{flat}.
\]

Although \((4)\tau_{ij}\) looks as if it depends on the slice condition, the independence is shown as follows. Eq.(2.51) is rewritten as

\[
\Delta_{flat}(3)\beta_i = \Delta_{flat}p_i + (3)K_{,i},
\]

(5.3)

where

\[
p_i = -4\int \frac{\rho v^i}{|x - y|} d^3 y - \frac{1}{2}\left(\int \hat{\rho}|x - y|d^3 y\right)_{,i}.
\]

(5.4)

This is solved as

\[
(3)\beta_i = p_i - \frac{1}{4\pi}\int \frac{(3)K}{|x - y|} d^3 y_{,i}.
\]

(5.5)

From Eqs.(2.43) and (2.60), we obtain

\[
(3)\dot{K} = -\Delta_{flat}(X + 2(4)\psi) + 4\pi\rho\left(v^2 + 3\frac{P}{\rho} - \frac{U}{2}\right).
\]

(5.6)

Combining Eq.(5.5) with Eq.(5.6), the equation for \((3)\dot{\beta}_i\) is written as

\[
(3)\dot{\beta}_i = \dot{p}_i - (X + 2(4)\psi)_{,i} - \left\{\int \frac{\left(\rho v^2 + 3P - \rho U/2\right)}{|x - y|} d^3 y\right\}_{,i}.
\]

(5.7)

Using this relation, the source term, \((4)\tau_{ij}\), is split into five parts

\[
(4)\tau_{ij} = (4)\tau_{ij}^{(S)} + (4)\tau_{ij}^{(U)} + (4)\tau_{ij}^{(C)} + (4)\tau_{ij}^{(\psi)} + (4)\tau_{ij}^{(V)}.
\]

(5.8)
where we introduce

\[ (4)\tau_{ij}^{(S)} = - 16\pi \left( \rho v^i v^j - \frac{1}{3} \delta_{ij} \rho v^2 \right), \]

\[ (4)\tau_{ij}^{(U)} = U U_{,ij} - \frac{1}{3} \delta_{ij} U \Delta_{\text{flat}} U - 3 U_{,i} U_{,j} + \delta_{ij} U_{,k} U_{,k}, \]

\[ (4)\tau_{ij}^{(C)} = 4 \frac{\partial}{\partial x^j} \int \frac{(\rho v^i)}{|x-y|} d^3 y + 4 \frac{\partial}{\partial x^i} \int \frac{(\rho v^j)}{|x-y|} d^3 y - \frac{8}{3} \delta_{ij} \frac{\partial}{\partial x^k} \int \frac{(\rho v^k)}{|x-y|} d^3 y, \]

\[ (4)\tau_{ij}^{(\rho)} = \hat{\delta}_{ij} \int \rho |x-y| d^3 y, \]

\[ (4)\tau_{ij}^{(V)} = 2 \hat{\delta}_{ij} \int \frac{(\rho v^2 + 3P - \rho U/2)}{|x-y|} d^3 y. \]

Thus it becomes clear that \((4)h_{ij}\) and \((5)h_{ij}\) as well as \((4)\tau_{ij}\) are expressed in terms of matter variables only and thus do not depend on slicing conditions.

Then, we define \(\Delta_{\text{flat}} (4)h_{ij}^{(S)} = (4)\tau_{ij}^{(S)}\), \(\Delta_{\text{flat}} (4)h_{ij}^{(U)} = (4)\tau_{ij}^{(U)}\), \(\Delta_{\text{flat}} (4)h_{ij}^{(C)} = (4)\tau_{ij}^{(C)}\), \(\Delta_{\text{flat}} (4)h_{ij}^{(\rho)} = (4)\tau_{ij}^{(\rho)}\) and \(\Delta_{\text{flat}} (4)h_{ij}^{(V)} = (4)\tau_{ij}^{(V)}\), and consider each term separately. First, since \((4)\tau_{ij}^{(S)}\) is a compact source, we immediately obtain

\[ (4)h_{ij}^{(S)} = 4 \int \frac{\left( \rho v^i v^j - \frac{1}{3} \delta_{ij} \rho v^2 \right)}{|x-y|} d^3 y. \]

Second, we consider the following equation

\[ \Delta_{\text{flat}} G(x, y_1, y_2) = \frac{1}{|x-y_1||x-y_2|}. \]

(5.10)

It is possible to write \((4)h_{ij}^{(U)}\) using integrals over the matter if this function, \(G\), is used. Eq.(5.10) has solutions \(^{[25]}\),

\[ G(x, y_1, y_2) = \ln(r_1 + r_2 \pm r_{12}), \]

(5.11)

where

\[ r_1 = |x-y_1|, \]

\[ r_2 = |x-y_2|, \]

\[ r_{12} = |y_1-y_2|. \]

(5.12)

Note that \(\ln(r_1 + r_2 - r_{12})\) is not regular on the interval between \(y_1\) and \(y_2\), while \(\ln(r_1 + r_2 + r_{12})\) is regular on the matter. Thus we use \(\ln(r_1 + r_2 + r_{12})\) as a Green function. Using this function
$UU_{ij}$ and $U_j U_j$ are rewritten as

$$
UU_{ij} = \left[ \frac{\partial^2}{\partial x^i \partial x^j} \left( \int \frac{\rho(y_1)}{|x-y_1|} d^3y_1 \right) \left( \int \frac{\rho(y_2)}{|x-y_2|} d^3y_2 \right) \right] = \int d^3y_1 d^3y_2 \rho(y_1) \rho(y_2) \frac{\partial^2}{\partial y_1^i \partial y_1^j} \left( \frac{1}{|x-y_1| |x-y_2|} \right) 
$$

$$
= \Delta_{flat} \int d^3y_1 d^3y_2 \rho(y_1) \rho(y_2) \frac{\partial^2}{\partial y_1^i \partial y_1^j} \ln(r_1 + r_2 + r_{12}), 
$$

$$
U_j U_j = \left( \frac{\partial}{\partial x^i} \int \frac{\rho(y_1)}{|x-y_1|} d^3y_1 \right) \left( \frac{\partial}{\partial x^j} \int \frac{\rho(y_2)}{|x-y_2|} d^3y_2 \right) 
$$

$$
= \int d^3y_1 d^3y_2 \rho(y_1) \rho(y_2) \frac{\partial^2}{\partial y_1^i \partial y_2^j} \left( \frac{1}{|x-y_1| |x-y_2|} \right) 
$$

$$
= \Delta_{flat} \int d^3y_1 d^3y_2 \rho(y_1) \rho(y_2) \frac{\partial^2}{\partial y_1^i \partial y_2^j} \ln(r_1 + r_2 + r_{12}). \tag{5.13}
$$

Thus we can express $(4)h^{(U)}_{ij}$ using the integral over the matter as

$$
(4)h^{(U)}_{ij} = \int d^3y_1 d^3y_2 \rho(y_1) \rho(y_2) 
$$

$$
\left[ \left( \frac{\partial^2}{\partial y_1^i \partial y_1^j} - \frac{1}{3} \delta_{ij} \Delta_1 \right) - 3 \left( \frac{\partial^2}{\partial y_1^i \partial y_2^j} - \frac{1}{3} \delta_{ij} \Delta_{12} \right) \right] \ln(r_1 + r_2 + r_{12}), \tag{5.14}
$$

where we introduce

$$
\Delta_1 = \frac{\partial^2}{\partial y_1^k \partial y_1^k}, 
$$

$$
\Delta_{12} = \frac{\partial^2}{\partial y_1^k \partial y_2^k}. \tag{5.15}
$$

Using relations $\Delta_{flat}|x-y| = 2/|x-y|$ and $\Delta_{flat}|x-y|^3 = 12|x-y|$, $(4)h^{(C)}_{ij}$, $(4)h^{(p)}_{ij}$ and $(4)h^{(V)}_{ij}$ are solved as

$$
(4)h^{(C)}_{ij} = 2 \frac{\partial}{\partial x^i} \int (\rho v^i) |x-y| d^3y + 2 \frac{\partial}{\partial x^j} \int (\rho v^j) |x-y| d^3y + \frac{4}{3} \delta_{ij} \int \tilde{\rho} |x-y| d^3y, \tag{5.16}
$$

$$
(4)h^{(p)}_{ij} = \frac{1}{12} \frac{\partial^2}{\partial x^i \partial x^j} \int \tilde{\rho} |x-y| d^3y - \frac{1}{3} \delta_{ij} \int \tilde{\rho} |x-y| d^3y, \tag{5.17}
$$

$$
(4)h^{(V)}_{ij} = \frac{\partial^2}{\partial x^i \partial x^j} \int \left( (\rho v^2 + 3P - \rho U/2) |x-y| d^3y - 2 \delta_{ij} \int \frac{\rho v^2 + 3P - \rho U/2}{|x-y|^3} d^3y \right). \tag{5.18}
$$
In total, we obtain

\[
(4) h_{ij} = (4) h_{ij}^{(S)} + (4) h_{ij}^{(U)} + (4) h_{ij}^{(C)} + (4) h_{ij}^{(\rho)} + (4) h_{ij}^{(V)}.
\] (5.19)

5.2. Treatment as a boundary value problem

The above expression for \((4) h_{ij}\) is quite interesting because it only consists of integrals over the matter. However, in actual numerical simulations, it will take a very long time to perform the direct integration. Therefore, we also propose other strategies where Eq.\((2.57)\) is solved as the boundary value problem. Here, we would like to emphasize that the boundary condition should be imposed at \(r = |\mathbf{x}| \gg |\mathbf{y}_1|, |\mathbf{y}_2|\), but \(r\) does not have to be greater than \(\lambda\), where \(\lambda\) is a typical wave length of gravitational waves. We only need to impose \(r > R\) (a typical size of matter). This means that we do not need a large amount of grid numbers compared with the case of fully general relativistic simulations, in which we require \(r > \lambda \gg R\).

First of all, we consider the equation

\[
\Delta_{\text{flat}} \left((4) h_{ij}^{(S)} + (4) h_{ij}^{(U)}\right) = (4) \tau_{ij}^{(S)} + (4) \tau_{ij}^{(U)}.
\] (5.20)

Since its source term behaves as \(O(r^{-6})\) at \(r \to \infty\), this equation can be accurately solved under the boundary condition at \(r > R\) as

\[
(4) h_{ij}^{(S)} + (4) h_{ij}^{(U)} = \frac{2}{r} \left(I_{ij} - \frac{1}{3} \delta_{ij} I_{kk}\right)
+ \frac{2}{3r^2} \left(n^k I_{ijk} - \frac{1}{3} \delta_{ij} n^k I_{lk} + 2n^k (\dot{S}_{ikj} + \dot{S}_{jki}) - \frac{4}{3} \delta_{ij} n^k \dot{S}_{lkl}\right) + O(r^{-3}),
\] (5.21)

where

\[
I_{ijk} = \int \rho x^i x^j x^k d^3 x,
\]

\[
S_{ijk} = \int \rho (v^i x^j - v^j x^i) x^k d^3 x.
\] (5.22)

Next, we consider the equations for \((4) h_{ij}^{(C)}\), \((4) h_{ij}^{(\rho)}\) and \((4) h_{ij}^{(V)}\). Noting the identity,

\[
\ddot{\rho} = -(\rho v^i)_i = (\rho v^i v^j)_{ij} + \Delta_{\text{flat}} P - (\rho U)_{,i}.
\] (5.23)
we find the following relations;

\[
\int \frac{\rho}{|x - y|} d^3 y = - \int d^3 y \frac{x^i - y^i}{|x - y|} (\rho v^i), \\
\int \frac{\rho}{|x - y|} d^3 y = 3 \int d^3 y \left[ \rho v^i v^j (x^i - y^i)(x^j - y^j) \right] + \left( 4P + \rho v^2 - \rho U, i(x^i - y^i) \right) |x - y|.
\]

Using Eqs.(5.24), (4)\(h_{ij}^{(C)}\), (4)\(h_{ij}^{(p)}\) and (4)\(h_{ij}^{(V)}\) in Eqs.(5.16-18) can be rewritten as

\[
(4)h_{ij}^{(C)} = 2 \int \frac{(\rho v^i) x^i - y^i}{|x - y|} d^3 y + 2 \int \frac{(\rho v^i) x^j - y^j}{|x - y|} d^3 y - \frac{4}{3} \delta_{ij} \int \frac{(\rho v^k) x^k - y^k}{|x - y|} d^3 y,
\]

\[
(4)h_{ij}^{(p)} = \frac{1}{4} \frac{\partial^2}{\partial x^i \partial x^j} \left\{ (\rho v^k) (x^i - y^i) (x^j - y^j) \right\} + \frac{1}{4} \frac{\partial}{\partial x^i} \left\{ (\rho v^k) (x^j - y^j) \right\} + \frac{1}{4} \frac{\partial}{\partial x^j} \left\{ (\rho v^k) (x^i - y^i) \right\} + \frac{1}{4} \frac{\partial}{\partial x^j} \left\{ (\rho v^k) (x^i - y^i) \right\} + \frac{1}{4} \frac{\partial}{\partial x^i} \left\{ (\rho v^k) (x^j - y^j) \right\},
\]

\[
(4)h_{ij}^{(V)} = \frac{1}{2} \left\{ \frac{\partial}{\partial x^i} \left\{ (\rho v^2 + 3P - \rho U/2) x^i - y^i \right\} \right\} + \frac{1}{2} \left\{ \frac{\partial}{\partial x^j} \left\{ (\rho v^2 + 3P - \rho U/2) x^j - y^j \right\} \right\} - \frac{2}{3} \delta_{ij} \int \frac{(\rho v^2 + 3P - \rho U/2)}{|x - y|} d^3 y,
\]

where \(P' = P + \rho v^2/4 + \rho U, l y^l/4\). From the above relations, (4)\(h_{ij}^{(C)}\), (4)\(h_{ij}^{(p)}\) and (4)\(h_{ij}^{(V)}\) become

\[
(4)h_{ij}^{(C)} = 2(x^i(3) \hat{P}^j + x^j(3) \hat{P}^i - Q_{ij}) + \frac{4}{3} \delta_{ij} \left( \frac{Q_{kk}}{2} - x^k(3) \hat{P}^k \right),
\]

\[
(4)h_{ij}^{(p)} = \frac{1}{4} \frac{\partial^2}{\partial x^i \partial x^j} \left\{ (V_{kl}^{(P)}) x^k x^l - 2 V_k^{(P)} x^k + V^{(P)} \right\} + \frac{1}{3} \delta_{ij} \left( x^k(3) \hat{P}^k - \frac{Q_{kk}}{2} \right)
\]

\[
- \frac{1}{2} \left\{ \frac{\partial}{\partial x^i} \left\{ (V^{(P)} x^i - V^{(P)}_i) \right\} + \frac{\partial}{\partial x^j} \left\{ (V^{(P)} x^j - V^{(P)}_j) \right\} \right\}
\]

\[
+ \frac{1}{8} \left\{ 2 \left( x^i V_j^{(P)} + x^j V_i^{(P)} - V_{ij}^{(P)} \right) + x^k V_k^{(P)} - V_{kj}^{(P)} \right\},
\]

\[
(4)h_{ij}^{(V)} = \frac{1}{3} \left( Q_{ij}^{(I)} x^i + Q_{ij}^{(I)} x^j - Q_{ij}^{(I)} \right),
\]

\[
+ \frac{1}{3} Q_{ij}^{(I)} \delta_{ij}.
\]
where

\[
\Delta_{\text{flat}(3)} P_i = -4\pi \rho v^i,
\]
\[
\Delta_{\text{flat}} Q_{ij} = -4\pi \left\{ x^i (\rho v^j) + x^j (\rho v^i) \right\},
\]
\[
\Delta_{\text{flat}} Q^{(I)} = -4\pi \left( \rho v^2 + 3P - \frac{1}{2} \rho U \right),
\]
\[
\Delta_{\text{flat}} Q^{(I)} = -4\pi \left( \rho v^2 + 3P - \frac{1}{2} \rho U \right) x^i,
\]
\[
\Delta_{\text{flat}} V^{i(\rho v)} = -4\pi \rho v^i v^j,
\]
\[
\Delta_{\text{flat}} V^{i(\rho v)} = -4\pi \rho v^i v^j x^j,
\]
\[
\Delta_{\text{flat}} V^{(P)} = -4\pi P',
\]
\[
\Delta_{\text{flat}} V^{i(P)} = -4\pi P' x^i,
\]
\[
\Delta_{\text{flat}} V^{i(\rho U)} = -4\pi \rho U, i,
\]
\[
\Delta_{\text{flat}} V^{i(\rho U)} = -4\pi \rho U, i x^j.
\]

Therefore, \((4) h_{ij}^{(C)}\), \((4) h_{ij}^{(\rho)}\) and \((4) h_{ij}^{(V)}\) can be derived from the above potentials which satisfy the Poisson equations with compact sources.

We note that instead of the above procedure, we may solve the Poisson equation for \((4) h_{ij}\) carefully imposing the boundary condition for \(r \gg R\) as

\[
(4) h_{ij} = \frac{1}{r} \left\{ \frac{1}{4} I_{ij}^{(2)} + \frac{3}{4} n^k \left( n^i I_{kj}^{(2)} + n^j I_{ki}^{(2)} \right) \right. \\
- \frac{5}{8} n^i n^j I_{kk}^{(2)} + \frac{3}{8} n^i n^j n^k n^l I_{kl}^{(2)} + \frac{1}{8} \delta_{ij} I_{kk}^{(2)} - \frac{5}{8} \delta_{ij} n^k n^l I_{kl}^{(2)} \} \\
+ \frac{1}{r^2} \left\{ \frac{5}{12} n^k I_{ijk}^{(2)} - \frac{1}{24} (n^i I_{jkk}^{(2)} + n^j I_{ikk}^{(2)}) + \frac{5}{8} n^i n^j n^k n^l I_{klm}^{(2)} + \frac{11}{24} \delta_{ij} n^k n^l I_{klm}^{(2)} - \frac{5}{8} \delta_{ij} n^k n^l n^m I_{klm}^{(2)} \right. \\
- \frac{7}{8} n^i n^j n^k I_{kl}^{(2)} + \frac{5}{8} n^i n^j n^k n^l n^m I_{klm}^{(2)} + \frac{11}{24} \delta_{ij} n^k n^l n^m I_{klm}^{(2)} \} \\
+ \left\{ \frac{2}{3} n^k (\dot{S}_{ikj} + \dot{S}_{jki}) - \frac{4}{3} (n^i \dot{S}_{jkk} + n^j \dot{S}_{ikk}) \right\} + O(r^{-3}).
\]

It is verified that \(O(r^{-1})\) and \(O(r^{-2})\) parts satisfy the traceless and divergence-free conditions respectively. It should be noticed that \((4) h_{ij}\) obtained in this way becomes meaningless at the far zone because Eq.(2.57), from which \((4) h_{ij}\) is derived, is valid only in the near zone.
6. Conserved quantities

The conserved quantities are gauge-invariant so that, in general relativity, they play important roles because we are able to compare various systems described in different gauge conditions using them. From the practical view, these are also useful for checking the numerical accuracy in simulations. Thus, in this section, we show several conserved quantities in the 2PN approximation.

6.1. Conserved Mass And Energy

In general relativity, the volume integral of the mass density \( \rho \) does not conserve, and instead we have the following conserved mass;

\[
M_* = \int \rho_* d^3 x.
\] (6.1)

In the PN approximation, \( \rho_* \) is expanded as

\[
\rho_* = \rho \left[ 1 + \left( \frac{1}{2}v^2 + 3U \right) \right.
\]
\[
+ \left( \frac{3}{8}v^4 + \frac{7}{2}v^2u + \frac{15}{4}U^2 + 6(4)\psi + (3)\beta_i v^i \right) + (6)\delta_* + O(c^{-7}) \right],
\] (6.2)

where \( (6)\delta_* \) denotes the 3PN contribution to \( \rho_* \).

Then, we consider the ADM mass which is also the conserved quantity. Since the asymptotic behavior of the conformal factor becomes

\[
\psi = 1 + \frac{M_{ADM}}{2r} + O\left(\frac{1}{r^2}\right),
\] (6.3)

the ADM mass in the PN approximation becomes

\[
M_{ADM} = -\frac{1}{2\pi} \int \Delta_{flat} \psi d^3 x
\]
\[
= \int d^3 x \rho \left[ \left\{ 1 + \left( \frac{v^2 + \varepsilon}{2} + \frac{5}{2}U \right) \right. \right.
\]
\[
\left. + \left( \frac{3}{8}v^4 + \frac{7}{2}v^2u + \frac{15}{4}U^2 + 6(4)\psi + (3)\beta_i v^i \right) \right. \right.
\]
\[
\left. + \frac{5}{2}U\varepsilon + \frac{5}{2}U^2 \right] + \frac{1}{16\pi \rho} \left( (3)\hat{A}_{ij}(3)\hat{A}_{ij} - \frac{2}{3}(3)K^2 \right) + (6)\delta_{ADM} + O(c^{-7}) \right],
\] (6.4)

where \( (6)\delta_{ADM} \) denotes the 3PN contribution.
Using these two conserved quantities, we can define the conserved energy as follows;

\[ E \equiv M_{\text{ADM}} - M_* \]

\[ = \int d^3 x \rho \left\{ \left( \frac{1}{2} v^2 + \varepsilon - \frac{1}{2} U \right) + \left( \frac{5}{8} v^4 \right) + 3v^2 U + v^2 \varepsilon + \frac{P}{\rho} v^2 + \frac{5}{2} U \varepsilon - \frac{5}{4} U^2 - (4) \psi + (3) \beta_i v^i \right\} + \frac{1}{16 \pi \rho} \left( \beta_{ij}(3) \beta_{ij} - \frac{2}{3} (3) K^2 \right) + \left( (6) \delta_{\text{ADM}} - (6) \delta_* \right) + O(\epsilon^{-7}) \]

\[ \equiv E_N + E_{1\text{PN}} + E_{2\text{PN}} + \cdots. \]

We should notice that the following equation holds

\[ \int (3) \beta_{i,j}(3) \beta_{i,j} d^3 x = -8\pi \int \rho v^i (3) \beta^i d^3 x + \int \left( \frac{2}{3} (3) K^2 + 2 \dot{U} (3) K \right) d^3 x, \]

where we use the identities derived from Eqs.(2.50) and (2.51)

\[ \int (3) \beta_{i,j}(3) \beta_{i,j} d^3 x = -16\pi \int \rho v^i (3) \beta^i d^3 x + \int (3) K^2 + 2 \dot{U} (3) K - 3 \dot{U}^2 d^3 x, \]

\[ \int (3) \beta_{i,j}(3) \beta_{j,i} d^3 x = \int (3) K^2 + 6 \dot{U} (3) K + 9 \dot{U}^2 d^3 x. \]

Using these relations, we obtain the Newtonian and the first PN energies as

\[ E_N = \int \rho \left( \frac{1}{2} v^2 + \varepsilon - \frac{1}{2} U \right) d^3 x, \]

and

\[ E_{1\text{PN}} = \int d^3 x \left[ \rho \left( \frac{5}{8} v^4 + \frac{5}{2} v^2 U + v^2 \varepsilon + \frac{P}{\rho} v^2 + 2 U \varepsilon - \frac{5}{2} U^2 + \frac{1}{2} (3) \beta_i v^i \right) + \frac{1}{8\pi} \dot{U} (3) K \right]. \]

\[ E_{1\text{PN}} \]

can be rewritten immediately in the following form used by Chandrasekhar \[^{[26]}\];

\[ E_{1\text{PN}} = \int d^3 x \rho \left[ \frac{5}{8} v^4 + \frac{5}{2} v^2 U + v^2 \left( \varepsilon + \frac{P}{\rho} \right) + 2 U \varepsilon - \frac{5}{2} U^2 - \frac{1}{2} v^i q_i \right], \]

where \( q_i \) is the first PN shift vector in the standard PN gauge (i.e., \( (3) K = 0 \)) and satisfies

\[ \Delta_{3i} q_i = -16\pi \rho v^i + \dot{U}_i. \]
$E_{2PN}$ is calculated from the 3PN quantities $(6)\delta_*$ and $(6)\delta_{ADM}$. $(6)\delta_*$ becomes

\[
(6)\delta_* = \frac{5}{16}v^6 + \frac{33}{8}v^4U + v^2\left(5(4)\psi + \frac{93}{8}U^2 + \frac{3}{2}(3)\beta_iv^i - X\right) \\
+ 6(6)\psi + 15U(4)\psi + \frac{5}{2}U^3 + 7(3)\beta_iv^iU + \frac{1}{2}(4)h_{ij}v^iv^j + \frac{1}{2}(3)\beta_i(3)\beta_i + (5)\beta_iv^i,
\]

and we obtain

\[
(6)M_* = \int \rho(6)\delta_* d^3x.
\]

The Hamiltonian constraint at $O(c^{-8})$ becomes

\[
\Delta_{flat}(8)\psi - (4)h_{ij}(4)\psi_{,ij} - \frac{1}{2}(6)h_{ij}U_{,ij} \\
= -\frac{1}{32}(2(4)h_{kl,m(4)h_{km,l}} + (4)h_{kl,m(4)}h_{kl,m}) \\
- 2\pi(6)\rho\psi - \frac{1}{4}\left(3(\bar{A}_{ij}(5)\bar{A}_{ij} - \frac{2}{3}(3)K(5)K) - \frac{1}{16}U(3(\bar{A}_{ij}(3)\bar{A}_{ij} - \frac{2}{3}(3)K^2)\right),
\]

where we define $(6)\rho\psi$ as

\[
(6)\rho\psi = \rho\left[v^6 + v^4\left(\varepsilon + \frac{P}{\rho} + \frac{21}{2}U\right) + v^2\left\{ \frac{13}{2}U\left(\varepsilon + \frac{P}{\rho} \right) + 9(4)\psi - 2X + 20U^2 \right\} \\
+ \varepsilon\left(5(4)\psi + \frac{5}{2}U^2\right) + 5(6)\psi + 10U(4)\psi + \frac{5}{4}U^3 \\
+ (4)h_{ij}v^iv^j + 2(3)\beta_iv^i\left\{2v^2 + \varepsilon + \frac{P}{\rho} + \frac{13}{2}U\right\} + 2(5)\beta_i(3)\beta_i + (5)\beta_i(3)\beta_i\right].
\]

Making use of relations $(4)h_{ij,j} = 0$ and $(6)h_{ij,j} = 0$, we obtain

\[
(6)M_{ADM} = \int d^3x(6)\rho\psi \\
+ \frac{1}{8\pi} \int d^3y\left(3(\bar{A}_{ij}(5)\bar{A}_{ij} - \frac{2}{3}(3)K(5)K)\right) \\
+ \frac{1}{32\pi} \int d^3yU(3(\bar{A}_{ij}(3)\bar{A}_{ij} - \frac{2}{3}(3)K^2),
\]

where we assume $(6)h_{ij} \to O(r^{-1})$ as $r \to \infty$. Although this assumption must be verified by performing the 3PN expansions which have not been done here, it seems reasonable in the asymptotically flat spacetime. From $(6)M_{ADM}$ and $(6)M_*$, we obtain the conserved energy at
the 2PN order

\[
E_{2PN} = (6)M_{\text{ADM}} - (6)M_*
\]

\[
= \int d^3x \rho \left[ \frac{11}{16} \psi^6 + v^4 \left( \varepsilon + \frac{P}{\rho} + \frac{51}{8} U \right) \\
+ v^2 \left\{ 4(4)\psi - X + \frac{13}{2} U \left( \varepsilon + \frac{P}{\rho} \right) + \frac{67}{8} U^2 + \frac{5}{2} \beta_i v^i \right\} \\
+ \varepsilon \left( 5(4)\psi + \frac{5}{2} U^2 \right) - (6)\psi - 5U(4)\psi - \frac{5}{4} U^3 \\
+ \frac{1}{2} (4)h_{ij} v^i v^j + 2(3)\beta_i v^i \left( \varepsilon + \frac{P}{\rho} + 3U \right) + (5)\beta_i v^i + \frac{1}{2} (3)\beta_i (3)\beta_i \right\}
\]

\[
+ \frac{1}{8\pi} \int d^3y \left( (3)\tilde{A}_{ij}(5)\tilde{A}_{ij} - \frac{2}{3} (3)K(5)K \right) \\
+ \frac{1}{32\pi} \int d^3y U \left( (3)\tilde{A}_{ij}(3)\tilde{A}_{ij} - \frac{2}{3} (3)K^2 \right).
\]

When we use the relation, \( \int d^3x \rho(6)\psi = -\frac{1}{4\pi} \int d^3x U \Delta(6)\psi \), we obtain

\[
E_{2PN} = \int d^3x \rho \left[ \frac{11}{16} \psi^6 + v^4 \left( \varepsilon + \frac{P}{\rho} + \frac{47}{8} U \right) \\
+ v^2 \left\{ 4(4)\psi - X + 6U \left( \varepsilon + \frac{P}{\rho} \right) + \frac{41}{8} U^2 + \frac{5}{2} \beta_i v^i \right\} \\
+ \varepsilon \left( 5(4)\psi + \frac{5}{2} U^2 \right) - \frac{15}{2} U(4)\psi - \frac{5}{2} U^3 \\
+ \frac{1}{2} (4)h_{ij} v^i v^j + 2(3)\beta_i v^i \left( \varepsilon + \frac{P}{\rho} \right) + 5U \left\{ (5)\beta_i v^i + \frac{1}{2} (3)\beta_i (3)\beta_i \right\}
\]

\[
+ \frac{1}{8\pi} \int d^3y \left( (4)h_{ij} U U_{ij} + (3)\tilde{A}_{ij}(5)\tilde{A}_{ij} - \frac{2}{3} (3)K(5)K \right).
\]

### 6.2. Conserved linear momentum

When we use the center of mass system as usual, the linear momentum of the system should vanish. However, it may arise from numerical errors in numerical calculations. Since it is useful for investigation of the numerical accuracy, we mention the linear momentum derived from

\[
P_i = \frac{1}{8\pi} \lim_{r \to \infty} \oint \left( K_{ijn} - Kn^i \right) dS
\]

\[
= \frac{1}{8\pi} \lim_{r \to \infty} \oint \left( \psi^4 \tilde{A}_{ij} n^j - \frac{2}{3} Kn^i \right) dS,
\]

where the surface integrals are taken over a sphere of constant \( r \). Since the asymptotic behavior
of $\tilde{A}_{ij}$ is determined by

$$(3)\tilde{A}_{ij} = \frac{1}{2} ((3)\beta_{i,j} + (3)\beta_{j,i} - \frac{2}{3} \delta_{ij}(3)\beta_{l,l}) + O(r^{-3}), \quad (6.20)$$

and

$$(5)\tilde{A}_{ij} = \frac{1}{2} ((5)\beta_{i,j} + (5)\beta_{j,i} - \frac{2}{3} \delta_{ij}(5)\beta_{l,l}) + (5)\tilde{A}_{TT} + O(r^{-3}), \quad (6.21)$$

the leading term of the shift vector is necessary. Using the asymptotic behavior

$$(3)\beta_i = -\frac{7}{2} \frac{l_i}{r} - \frac{1}{2} \frac{l_j}{r} + O(r^{-2}), \quad (6.22)$$

the following relation is obtained

$$\int ((3)\beta_{i,j} + (3)\beta_{j,i} - \frac{2}{3} \delta_{ij}(3)\beta_{l,l}) n^j dS = 16\pi l_i, \quad (6.23)$$

where $l_i = \int \rho v^i d^3x$. Therefore the Newtonian linear momentum is

$$P^i_N = \int d^3x \rho v^i. \quad (6.24)$$

Similarly the first PN linear momentum is obtained as follows;

$$P_{1PN}^i = \int d^3x \rho \left[ v^i \left( v^2 + \varepsilon + 6U + \frac{P}{\rho} \right) + (3)\beta_i \right]. \quad (6.25)$$

$P_{2PN}^i$ is obtained by the similar procedure.
7. Summary

In this paper, we have developed the PN approximation in the (3+1) formalism of general relativity. In this formalism, it is clarified what kind of gauge condition is suitable for each problem such as how to extract the waveforms of gravitational waves and how to describe equilibrium configurations. It was found that the combination of the conformal slice and the transverse gauge is useful to separate the wave part and the non-wave part in the metric variables such as $h_{ij}$ and $\psi$. We also found that, in order to describe the equilibrium configuration, the conformal slice is not useful and instead we had better use the maximal slice. Although we restricted ourselves within some gauge conditions in this paper, we may try to use any gauge condition and investigate its property relatively easily in the (3+1) formalism, compared with in the standard PN approximation performed so far.[24]

We have also developed a formalism for the hydrodynamic equation accurate up to 2.5PN order. For the sake of an actual numerical simulation, we carefully consider methods to solve the various metric quantities, especially, the 2PN tensor potential $(4)h_{ij}$. We found it possible to solve them by using standard numerical methods. Thus, the formalism developed in this paper will be useful also in numerical calculations.

Here, we would like to emphasize that from the 2PN order, the tensor part of the 3-metric, $\tilde{\gamma}_{ij}$, cannot be neglected even if we ignore gravitational waves. Recently, Wilson and Mathews[27] presented numerical equilibrium configurations of binary neutron stars using a semi-relativistic approximation, in which they assume the spatially conformal flat metric as the spatial 3-metric, i.e., $\tilde{\gamma}_{ij} = \delta_{ij}$. Thus, in their method, a 2PN term, $h_{ij}$, was completely neglected. This means that their results unavoidably have an error of the 2PN order which will become $\sim (M/R)^2 \sim 1 - 10\%$. If we hope to obtain a general relativistic equilibrium configuration of binary neutron stars with a better accuracy(say less than 1%), we should take into account the tensor part of the 3-metric.

In section 3, we used several slice conditions and investigated their properties, but, as for the spatial gauge condition, we fix it to the transverse gauge for the sake of convenience. It is not clear, however, whether this is the best gauge condition in numerical relativity. In numerical relativity, the shift vector plays a very important role to reduce the coordinate shear. If we fail
to choose the appropriate condition, the coordinate shear in the spatial metric will continue to grow, and as a result, the simulation will break down. The minimal distortion gauge which was proposed by Smarr and York\textsuperscript{[28]} is a candidate which may efficiently reduce the coordinate shear. Even if we use this gauge condition in the PN analysis, equations for $h_{ij}$, $\beta_{i}$, and $\beta_{i}$ remain unchanged, but higher order terms of $h_{ij}$ and $\beta_{i}$ may slightly change. If we investigate the effects due to the difference, we may be able to give some important suggestions about the gauge condition appropriate for numerical relativity.

In this paper, we mainly payed attention to the PN hydrodynamic equation in the (3+1) formalism to describe the final hydrodynamic merging phase of coalescing binary neutron stars. This formalism, however, may be useful also for gravitational wave generation problems in the inspiraling phase, which is usually performed using the harmonic gauge\textsuperscript{[4,5]}. Since we can choose any gauge conditions in the (3+1) formalism, it may be possible to reduce some lengthy calculations which appear in previous works using the harmonic gauge\textsuperscript{[4,5]}. This problem is interesting and important as a future work.

Acknowledgment

We thank T. Nakamura for his suggestion to pursue this problem and useful discussions. We also thank M. Sasaki, T. Tanaka and K. Nakao for helpful discussion. H. A. thanks Prof. S. Ikeuchi and Prof. M. Sasaki for encouragements. This work was in part supported by the Japanese Grant-in-Aid for Scientific Research of the Ministry of Education, Science, and Culture, No. 07740355.

APPENDIX A

Calculation of $h_{ij}$

We make use of the transverse property of $\tau_{ij}$, which is guaranteed by the transverse gauge condition, in order to obtain $h_{ij}$. Using the following identity

$$\tau_{ij} = \left(\tau^{ik} \tau_{kj}\right)_{i},$$  \hspace{1cm} (A.1)
Eq.(2.58) can be rewritten in the surface integral form

\[ (5) h_{ij} = \frac{1}{4\pi} \frac{\partial}{\partial t} \int (4) \tau_{ik} x^j n^k dS. \]  

(A.2)

Thus, we only need to estimate terms of \( O(r^{-3}) \) in \((4) \tau_{ij}\), which come only from the shift vector in the conformal slice as

\[ (3) \beta_i = \frac{1}{r^2} \left( n^j Z_{ij} + \frac{1}{2} n^j \ddot{I}_{ij} + \frac{1}{4} n_i n^k n^j \dddot{I}_{jk} + \frac{n^i}{4\pi} \int (3) K d^3x \right) + O(r^{-3}), \]  

(A.3)

where

\[ Z_{ij}(t) = -4 \int \rho v^i y^j d^3x. \]  

(A.4)

Here, note the following relations as

\[ \frac{\partial}{\partial t} Z_{ij} = -2 \dddot{I}_{ij}, \]  

(A.5)

and

\[ \int (3) K d^3x = 2\pi \dddot{I}_{kk}. \]  

(A.6)

Therefore, the relevant terms of \((4) \tau_{ij}\) for the surface integral become

\[ (4) \tau_{ij} \to (3) \dot{\beta}_{i,j} + (3) \dot{\beta}_{j,i} - \frac{2}{3} \delta_{ij}(3) \dot{\beta}_{l,l} = \frac{1}{r^3} \left\{ -3 \dddot{I}_{ij} + 3 \left( \dddot{I}_{ik} n^k n^j + \dddot{I}_{jk} n^k n^i \right) - \frac{9}{2} \dddot{I}_{kk} n^i n^j + \frac{15}{2} \dddot{I}_{kl} n^i n^j n^k n^l \right\} - \frac{1}{3} \delta_{ij} \left\{ -\frac{15}{2} \dddot{I}_{kk} + \frac{27}{2} \dddot{I}_{kl} n^k n^l \right\} + O(r^{-4}). \]  

(A.7)

Thus we obtain \( h_{ij} \) at the 2.5PN order,

\[ (5) h_{ij} = -\frac{1}{4\pi} \frac{\partial}{\partial t} \int \left( (3) \dot{\beta}_{i,k} + (3) \dot{\beta}_{k,i} - \frac{2}{3} \delta_{ik}(3) \dot{\beta}_{l,l} \right) x^j n^k dS = -\frac{4}{5} \dot{I}^{(3)}_{ij}(t). \]  

(A.8)

This derivation seems fairly simple owing to the gauge condition. Thus, it is expected that higher order calculations, say at 3.5PN order, may become easier in this gauge condition.
APPENDIX B

In this appendix, we briefly comment on a method to solve the Poisson equations for \((3)\beta_i\) and \((5)\beta_i\), i.e., Eqs.(2.51) and (2.53). Since the source terms of them have terms such as \(-\dot{U}_i\) and \(-2\dot{\psi}_i\) which behaves as \(O(r^{-2})\) at \(r \to \infty\), it seems that a technical problem arises in solving these equations in numerical calculation. However, this is easily overcome in a simple manner. We consider the case of the maximal slice for simplicity, but other cases may be treated similarly.

First of all, we write \((3)\beta_i\) and \((5)\beta_i\) as,

\[
(3)\beta_i = -4(3)P_i + \frac{1}{2}\dot{\chi}_i, \tag{B.1}
\]

\[
(5)\beta_i = -4(5)P_i + \frac{1}{2}(4)\dot{\chi}_i,
\]

where \(\chi\) and \((3)P_i\) satisfy Eq.(4.4) and the first equation of Eqs.(5.29), respectively. \((5)P_i\) and \((4)\chi\) satisfy the following Poisson equations;

\[
\Delta_{\text{flat}}(5)P_i = -4\pi \rho \left[ v^i \left( v^2 + 2U + \varepsilon + \frac{P}{\rho} \right) + (3)\beta_i \right] + 2U_{j(3)} \bar{A}_{ij} - \frac{1}{8}(\dot{U}U)_i - \frac{1}{4}(3)\beta_l U_l)_i ,
\]

\[
\Delta_{\text{flat}}(4)\chi = -4(4)\psi. \tag{B.2}
\]

\((4)\chi\) can be written as

\[
(4)\chi = -\int \rho_4 |\mathbf{x} - \mathbf{y}| d^3 y, \tag{B.3}
\]

where

\[
\rho_4 = \rho \left( v^2 + \varepsilon + \frac{5}{2}U \right). \tag{B.4}
\]

From Eqs.(4.4) and (B.3), \(\chi_i\) and \((4)\chi_i\) become

\[
\chi_i = -\int d^3 y \rho \frac{x^i - y^i}{|\mathbf{x} - \mathbf{y}|} = -x^i U + \eta_i, \tag{B.5}
\]

\[
(4)\chi_i = -\int d^3 y \rho_4 \frac{x^i - y^i}{|\mathbf{x} - \mathbf{y}|} = -2x^i (4)\psi + (4)\eta_i,
\]

† Note that in the Newtonian limit, \(-\dot{U}_i\) is \(O(r^{-4})\), but at the 1PN order, it becomes \(O(r^{-2})\) because of \(\int \dot{\rho} dV \neq 0\).
where $\eta_i$ and $(4)\eta_i$ satisfy

$$\Delta_{\text{flat}} \eta_i = -4\pi \rho x^i,$$

$$\Delta_{\text{flat}}(4)\eta_i = -4\pi \rho_4 x^i.$$  \hspace{1cm} (B.6)

Hence,

$$(3)\beta_i = -4(3)P_i - \frac{1}{2} \left( x^i \dot{U} - \dot{\eta}_i \right),$$

$$(5)\beta_i = -4(5)P_i - \frac{1}{2} \left( 2x^i(4)\dot{\psi} - (4)\dot{\eta}_i \right).$$ \hspace{1cm} (B.7)

Since the source terms of the Poisson equations for $(3)P_i$, $(5)P_i$, $\eta_i$ and $(4)\eta_i$ behaves as $O(r^{-n})$, where $n \geq 5$, at $r \to \infty^*$, these vector potentials can be accurately obtained by solving the Poisson equations for them under appropriate boundary conditions. Thus, there is no difficulty to obtain $(3)\beta_i$ and $(5)\beta_i$.

Finally, we note that the above method is not unique prescription. For example, $(3)\beta_i$ in the first PN approximation\(^\dagger\) may be expressed as

$$(3)\beta_i = -4(3)P_i + \frac{1}{2} \left\{ (3)P_kx^k)_i - \chi_{2,i} \right\},$$ \hspace{1cm} (B.8)

where $\chi_2$ satisfies

$$\Delta_{\text{flat}} \chi_2 = -4\pi \rho v^i x^i.$$ \hspace{1cm} (B.9)

\* Note that the non-compact sources of the Poisson equation for $(5)P_i$ may be regarded as $O(r^{-5})$ in the 2PN approximation because $\dot{U}$ is $O(r^{-3})$ in the Newtonian order.

\^ See ref. [16].
REFERENCES

1. A. Abramovici et al. Science, 256(1992), 325;
   K. S. Thorne, in proceedings of the eighth Nishinomiya-Yukawa memorial symposium
   on Relativistic Cosmology, edited by M. Sasaki (Universal Academy Press, Tokyo, 1994),
   pp.67.

2. C. Bradaschia et al., Nucl. Instrum. Method Phys. Res. Sect. A289(1990), 518.

3. E. S. Phinney, Astrophys. J. Lett. 380(1991), 17.

4. For example, C. M. Will, in proceedings of the eighth Nishinomiya-Yukawa memorial
   symposium on Relativistic Cosmology (ref.1), pp.83; See also references cited therein.

5. L. Blanchet et al., Phys. Rev. Lett. 74(1995), 3515; as for detailed description of their
   calculations, see references cited therein:
   L. E. Kidder, Phys. Rev. D52(1995), 821.

6. L. Blanchet: He has already completed the energy luminosity accurate up to 2.5PN
   order (without spin dependent terms) extending the work by Blanchet et al. (ref.5).

7. H. Tagoshi and T. Nakamura, Phys. Rev. D49(1994), 4016;
   H. Tagoshi and M. Sasaki, Prog. Theor. Phys. 92(1994), 745;
   M. Shibata, M. Sasaki, H. Tagoshi and T. Tanaka, Phys. Rev. D51(1995), 1646;
   H. Tagoshi, M. Shibata, T. Tanaka and M. Sasaki, submitted to Phys. Rev. D.

8. C. Cutler, Phys. Rev. Lett. 70(1993), 2984.

9. C. Cutler and E. E. Flanagan, Phys. Rev. D49(1994), 2658.

10. B. F. Schutz, Nature, 323(1986), 210;
    D. Markovic, Phys. Rev. D48(1993), 4738.

11. D. Lai, F. Rasio and S. L. Shapiro, Astrophys. J. 420 (1994), 811; Astrophys. J. supplement
    88(1993), 205.

12. L. E. Kidder, C. M. Will and A. G. Wiseman, Phys. Rev. D47(1993), 3281.

13. T. Nakamura, in the proceedings of the eighth Nishinomiya-Yukawa memorial symposium
    on Relativistic Cosmology (ref.1), pp.155.
14. K. Oohara and T. Nakamura, Prog. Theor. Phys. 83(1990), 906; 86(1991), 73; 88(1992), 307.

15. M. Shibata, T. Nakamura and K. Oohara, Prog. Theor. Phys. 88(1992), 1079; 89(1993), 809.

16. L. Blanchet, T. Damour and G. Schäfer, Mon. Not. R. Astr. Soc. 242(1990), 289.

17. M. Reffert, H-T. Janka and G. Schäfer, Max-Planck-Institute preprint.

18. M. Shibata and T. Nakamura, Prog. Theor. Phys. 88(1992), 317.

19. G. Schäfer, Ann. Phys. 161(1985), 81.

20. C. Bona and J. Masso, Phys. Rev. Lett. 68(1992), 1097.

21. Christodoulou, Phys. Rev. Lett. 67(1991), 1486;
A. G. Wiseman and C. M. Will, Phys. Rev. 44(1991), 2945;
K. S. Thorne, Phys. Rev. 45(1992), 520.

22. S. Chandrasekhar, Astrophys. J. 142(1965), 1488.

23. S. Chandrasekhar, Ellipsoidal Figures of Equilibrium(Dover, 1969).

24. S. Chandrasekhar and F. P. Esposito, Astrophys. J. 160(1970), 55.

25. T. Ohta. H. Okamura, T. Kimura and K. Hiida, Prog. Thor. Phys. 51(1974), 1598.

26. S. Chandrasekhar, Astrophys. J. 158(1969), 45.

27. J. R. Wilson and G. J. Mathews, Phys. Rev. Lett. 75(1995), 4161.

28. L. Smarr and J. W. York, Jr., Phys. Rev. D17(1978), 1945; 2529.