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UNIFORM EXPONENTIAL GROWTH FOR SOME $SL(2, \mathbb{R})$ MATRIX PRODUCTS

ARTUR AVILA AND THOMAS ROBLIN

Abstract. Given a hyperbolic matrix $H \in SL(2, \mathbb{R})$, we prove that for almost every $R \in SL(2, \mathbb{R})$, any product of length $n$ of $H$ and $R$ grows exponentially fast with $n$ provided the matrix $R$ occurs less than $o\left( \frac{n}{\log n \log \log n} \right)$ times.

1. Introduction

For $t, \theta \in \mathbb{R}$, let $H = H(t)$ be the hyperbolic matrix $\begin{pmatrix} \exp \frac{1}{2}t & 0 \\ 0 & \exp -\frac{1}{2}t \end{pmatrix}$ and let $R = R(\theta)$ be the rotation matrix $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$. For a finite word $w = w_n \ldots w_1$ on the symbols $H$ and $R$, we let $|w|$ denote its length and we let $m(w)$ denote the number of occurrences of $R$ in $w$. For any such word, and for any choice of parameters $t$ and $\theta$, we let $A_w(t, \theta)$ denote the corresponding matrix product in $SL(2, \mathbb{R})$, and denote by $\|A_w(t, \theta)\|$ its norm.

By the Oseledets Theorem, for a typical large word $w$ on $H$ and $R$, the size of the matrix product is given up to subexponential error, by $e^{L(t, \theta)|w|}$, where $L(t, \theta)$ is the Lyapunov exponent of the Bernoulli product giving equal probabilities for $H$ and $R$. By Furstenberg’s Theorem (cf [3]), $L(t, \theta) > 0$ unless $t = 0$ or $\theta = \pi/2 \mod \pi$, thus hyperbolic behavior prevails under a very mild “transversality condition” on the pair $(H, R)$.

Here we are interested in the following subtler question: Assuming some stronger transversality condition on the pair $(H, R)$, can one ensure hyperbolic behavior just by limiting the frequency of rotation elements in the word? A basic question in this direction, raised by Bochi and Fayad in [1], is whether for almost every $t$ and $\theta$, a condition of the type $C(t, \theta)m(w) \leq |w|$ implies that $\|A_w(t, \theta)\|$ grows exponentially. While this question is still open, in [2], Fayad and Krikorian showed that for almost every $t$ and $\theta$, one has exponential growth provided $m(w) \leq |w|^\alpha$ with $0 < \alpha < 1/2$. Our goal in this paper will be to show that the weaker condition $C(t, \theta)m(w) \log m(w) \log \log m(w) \leq |w|$ suffices.

**Theorem 1.** For every $t > 0$, $0 < \gamma < \frac{t}{2}$ and almost every $\theta \in \mathbb{R}$, there exists $\epsilon > 0$ such that for any word $w$ on $H$ and $R$, if $m(w) \leq \epsilon |w|(|\log |w| \log \log |w|)|^{-1}$, then the spectral radius of $A_w(t, \theta)$ is at least $e^{\epsilon |w|^{\gamma}}$.

In fact, our proof allows us to take for $R$ a general matrix of $SL(2, \mathbb{R})$, presented in its Cartan decomposition form, as follows.

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Theorem 2. For every $t > 0$, $s > 0$, $\alpha \in \mathbb{R}$, $0 < \gamma < \frac{t}{2}$ and almost every $\theta \in \mathbb{R}$, there exists $\epsilon > 0$ such that for any word $w$ on $H = H(t)$ and $R = R(\theta) H(s) R(\alpha)$, if $m(w) \leq \epsilon |w| (\log |w| \log \log |w|)^{-1}$, then the spectral radius of $A_w$ is at least $\epsilon |w|^\gamma$.

Corollary. For every $t > 0$, $0 < \gamma < \frac{t}{2}$ and almost every $R \in SL(2, \mathbb{R})$ with respect to the Haar measure, there exists $\epsilon > 0$ such that for any word $w$ on $H = H(t)$ and $R$, if $m(w) \leq \epsilon |w| (\log |w| \log \log |w|)^{-1}$, then the spectral radius of $A_w$ is at least $\epsilon |w|^\gamma$.

2. Proof of the theorems

We now give a detailed proof of theorem 1. Then we shall indicate how theorem 2 is obtained following the same lines.

From now on we fix $t > 0$, and drop the dependence on $t$ from the notation.

For a given word $w$ we shall use the notations $w_{[i,j]} = w_i \ldots w_j$ for $1 \leq i \leq j \leq |w|$. We also let $a_w, b_w, c_w, d_w : \mathbb{R} \to \mathbb{R}$ be defined so that $A_w(\theta) = \begin{pmatrix} a_w(\theta) & b_w(\theta) \\ c_w(\theta) & d_w(\theta) \end{pmatrix}$.

Let us say that a function $\psi : \mathbb{Z}_+ \to \mathbb{R}_+$ is good if

$$\forall k,l \geq 1, \psi(k) + \psi(l) \leq \psi(k + l) - \log 2. \tag{1}$$

We will mostly work with multiples (by reals greater than 1) of the functions $\psi_1(m) = m(1 + \log^2 m)$ and $\psi_2(m) = m(1 + \log m)(1 + \log \log \max\{e, m\})$ (with $0 \log 0 = 0$). Both $\psi_1$ and $\psi_2$ are easily seen to be good.

Given a good function $\psi$ and $0 < \gamma \leq \frac{t}{2}$, for any word $w$ of length $n$, we let $F_w(\psi, \gamma) = F_w$ be the set of all $\theta \in [0, \pi)$ such that

$$\log |a_w| \geq k\gamma - \psi(m(w[k+1])) \quad \text{and} \quad \log |a_w| \geq (n-k)\gamma - \psi(m(w[k+1]))$$

for all $0 < k < n$, but $\log |a_w| < n\gamma - \psi(m(w))$. Notice that if $F_w$ is not empty, necessarily $w_1 = w_n = R$. In view of (1), it follows that on the set $F_w$,

$$|a_w| \leq \frac{1}{2} |a_{w[k+1]} a_{w[k+1]}|, \quad \forall 0 < k < n. \tag{3}$$

Lemma 1. For every $w$ we have, writing $|w| = n$ and $m(w) = m$:

$$|F_w| \leq 8 n^2 e^{\psi(m)} \log m. \tag{4}$$

Proof. Since $a_w$ is in general a polynomial of degree $m(\omega)$ in $\cos \theta$, as is easily checked, the set $F_w$ is the union of at most $4mn$ intervals. Now, in order to bound the size of such an interval, we show that the derivative of $a_w$ with respect to $\theta$ at any (fixed) point of $F_w$ is not too small.

Since the derivative of $R(\theta)$ is $R(\frac{\pi}{2}) R(\theta)$, using the product rule, it is easy to derive the following formula for the derivative of $a_w$:

$$a'_w = \sum_{k, w_k = R} c_{w[k]} a_{w[k+1]} - a_{w[k]} b_{w[k+1]}, \tag{5}$$

On the one hand, we have, for all $0 < k < n$,

$$a_w = a_{w[k]} a_{w[k+1]} + c_{w[k]} b_{w[k+1]}. \tag{6}$$

In view of (3), this shows that

$$\frac{1}{2} \leq - \frac{c_{w[k]} b_{w[k+1]}}{a_{w[k]} a_{w[k+1]}} \leq \frac{3}{2}. \tag{7}$$
In particular, for each $0 < k < n$, $c_{w^1[k]}a_{w^{k+1}[n]}$ and $-a_{w^1[k]}b_{w^{k+1}[n]}$ have the same sign.

On the other hand, one easily sees that $\forall 1 < k < n$, the upper left entry of the matrix $A_{w^{k+1}[n]} R(\frac{\pi}{2}) A_{w^1[k]}$ is $c_{w^1[k]}a_{w^{k+1}[n]} - a_{w^1[k]}b_{w^{k+1}[n]} = c_{w^1[k]}a_{w^{k+1}[n]} - a_{w^1[k-1]}b_{w[i]}$ if $w_k = R$ and $c_{w^1[k]}a_{w^{k+1}[n]} - a_{w^1[k]}b_{w^{k+1}[n]} = e^{-\epsilon}c_{w^1[k]}a_{w[n]} - e^{\epsilon}a_{w^1[k-1]}b_{w[n]}$ if $w_k = H$ (indeed, $R(\frac{\pi}{2})H(t) = H(-t)R(\frac{\pi}{2}) = H(t)H(-2t)R(\frac{\pi}{2})$).

After finite iteration, we deduce from these observations that the quantities $c_{w^1[k]}a_{w^{k+1}[n]}$ and $-a_{w^1[k]}b_{w^{k+1}[n]}$ for $k$ varying from 1 to $n-1$ all have the same sign; among them, the summands in (5). Therefore, taking $k$ with $w_k = R$ so that $m(w_{[1,k]}) = \frac{n}{2}$ where $m = m(w)$, we have

$$|a'_{w_k}| \geq |c_{w^1[k]}a_{w^{k+1}[n]}| + |a_{w^1[k]}b_{w^{k+1}[n]}| \geq 2|a_{w^1[k]}a_{w^{k+1}[n]}c_{w^1[k]}b_{w^{k+1}[n]}|^\frac{1}{2}.$$  

From (7) and (2), we get (at any point $\theta \in F_w$):

$$|a'_{w_k}| \geq |a_{w^1[k]}a_{w^{k+1}[n]}| \geq e^{t\omega - \psi(\frac{|\pi|}{2})+\psi(m-\frac{|\pi|}{2})}$$

From the above minoration, we deduce that any interval in $F_w$ as defined by (2) is of length less than $2e^{\psi(\frac{|\pi|}{2})+\psi(m-\frac{|\pi|}{2})-\psi(m)}$. Since $F_w$ is the union of at most $4nm$ such intervals, the result follows.

**Lemma 2.** If $F_w \neq \emptyset$ then

$$n \leq m(1 + \frac{1}{t} \psi(m)),$$

where $n = |w|$ and $m = m(w)$.

**Proof.** Let us fix some $\theta \in F_w$, and write $w = w_{[k+r+1,n]} H^r w_{[1,k]}$ with $r$ maximal. Since $w_1 = w_n = R$, as we have already observed, one has $0 < k < n-r$, $m(w_{[1,k]})$, $m(w_{[k+r+1,n]}) \geq 1$, and

$$r \geq \frac{n-m}{m-1}.$$  

(8)

We have

$$a_{w} = e^{\frac{\pi}{4}}c_{w^1[k]}a_{w^{k+r+1}[n]} + e^{-\frac{\pi}{4}}c_{w^1[k]}b_{w^{k+r+1}[n]}.$$  

(9)

Observe that in general $\max(c_{w^1[k]}^2 c_{w^{k+1}[n]}^2 c_{w^{k+1}[n]}^2 + d_{w^1[k]}^2 c_{w^{k+1}[n]}^2 c_{w^{k+1}[n]}^2) \leq e^{n|\pi|}$, so that here

$$|c_{w^1[k]}b_{w^{k+r+1}[n]}| \leq e^{(n-r)^\frac{1}{2}}.$$  

From (1),(2) and (9), we get

$$2e^{n(\omega - \psi(m))} \leq e^{\frac{n}{2} - \psi(m(w_{[1,k]})) - \psi(m(w_{[k+r+1,n]}))}$$

$$\leq e^{\frac{n}{2} |a_{w^1[k]}a_{w^{k+r+1}[n]}|}$$

$$\leq e^{\frac{n}{2} - \psi(m) + e^{(n-2r)^\frac{1}{2}}}. $$

Hence $rt < \psi(m)$, which combined with (8) gives the result.

From now on, let $E(\psi, \gamma)$ denote the set of all $\theta \in [0, \pi)$ such that

$$\log |a_{w}(\theta)| < |w|\gamma - \psi(m(w))$$  

for some word $w$.

**Lemma 3.** There exists some constant $c > 0$ such that $|E(\lambda \psi, \frac{1}{2})| = O_{\lambda>1}(e^{-c\lambda})$. 


Proof. Let $E_n = E_n(\lambda\psi_1, 1/2) \subset E = E(\lambda\psi_1, 1/2)$ be the set of $\theta$ such that $n$ is the minimal length of a word $w$ such that (10) holds. Clearly $E$ is the disjoint union of the $E_n$’s and each $E_n$ is covered by the $F_w$’s with $|w| = n$.

We then apply lemmas 1 and 2 to estimate $|E_n|$ for $n \geq 2$ as follows:

$$|E_n| \leq \sum_{|w|=n} |F_w| \leq 8n^2 \sum_m \left( \frac{n}{m} \right) e^{\lambda\psi_1(m - \lfloor \frac{m}{n} \rfloor) + \psi_1(\lfloor \frac{m}{n} \rfloor) - \psi_1(m)},$$

where the sum runs over the $2 \leq m \leq n$ such that $n \leq m(1 + \frac{1}{2}\lambda\psi_1(m))$, which implies $n \leq C_0\lambda m^2 \log^2 m$. Here and in the sequel, $C_0, C_1, \ldots$ stand for positive constants independent of $m$, $n$ or $\lambda$.

For $n = 1$, notice that $E_1 = \{ \theta \mid |\cos\theta| < e^{\frac{1}{2} - \lambda} \}$.

It is readily seen that $\forall m \geq 2$, $\psi_1(m - \lfloor \frac{m}{2} \rfloor) + \psi_1(\lfloor \frac{m}{2} \rfloor) - \psi_1(m) \leq -C_1 m \log m$.

On the other hand, by the use of Stirling’s formula, we find that

$$\left( \frac{n}{m} \right) \leq e^{m \log n - m \log m + C_2 m}.$$ 

So, summing over $n$ in (11) and then reversing the order of summation yields

$$|E| \leq |E_1| + \sum_{m \geq 2} e^{(C_3 - C_1)\lambda m \log m} \sum_{n \leq C_0\lambda m^2 \log^2 m} n^{(m+2)} \leq C_4 e^{-\lambda} + \sum_{m \geq 2} e^{(C_3 - C_1)\lambda m \log m + (m+3) \log \lambda}.$$ 

For large $\lambda$, this sum is finite and less than $e^{-\lambda}$. □

Lemma 4. Let $0 < \gamma < \frac{1}{2}$. There exists some constant $c > 0$ such that $|E(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{1}{2})| = O_{\lambda \geq 1}(e^{-c\lambda})$.

Proof. We first notice that if $F_w(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{1}{2}) \neq \emptyset$, then $\lambda\psi_1(m(w)) \geq (\frac{1}{2} - \gamma)|w|$. Thus, proceeding as in the previous lemma, we get (even for $n = 1$)

$$|E_n(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{1}{2})| \leq 8n^2 \sum_{\lambda\psi_1(m) \geq (\frac{1}{2} - \gamma)n} \left( \frac{n}{m} \right) e^{\lambda\psi_1(m - \lfloor \frac{m}{n} \rfloor) + \psi_1(\lfloor \frac{m}{n} \rfloor) - \psi_2(m)}.$$ 

Here $\forall m \geq 2$, $\psi_2(m - \lfloor \frac{m}{2} \rfloor) + \psi_1(\lfloor \frac{m}{2} \rfloor) - \psi_2(m) \leq -C_0 m (1 + \log \log \max\{e, m\})$.

Using again (12), we obtain

$$|E(\lambda\psi_2, \gamma) \setminus E(\lambda\psi_1, \frac{1}{2})| \leq \sum_{m \geq 2} \sum_{n \leq C_0\lambda m^2 \log^2 m} n^{(m+2)} e^{(C_3 - C_6)\lambda m (1 + \log \log \max\{e, m\}) - m \log m} \sum_{n \leq C_0\lambda m^2 \log^2 m} n^{(m+2)} e^{(C_3 - C_6)\lambda m (1 + \log \log \max\{e, m\}) + (m+3) \log \lambda}.$$ 

We conclude as before. □

The lemmata 3 and 4 show that for $0 < \gamma < \frac{1}{2}$, the sum $\sum_{\lambda \in \mathbb{N}} |E(\lambda\psi_2, \gamma)|$ converges. By the Borel-Cantelli lemma, we conclude that for almost every $\theta$, there exists $\lambda \geq 1$ such that for all word $w$, $\log |a_w(\theta)| \geq |w|\gamma - \lambda\psi_2(m(w))$. It follows that for almost every $\theta$, if $|w|$ is large and $m(w)$ is much smaller than $|w|(\log |w| \log \log |w|)^{-1}$, then $\frac{1}{|w|} \log ||A_w(\theta)||$ is close to $\frac{1}{2}$, as well as $\frac{1}{|w|^2} \log ||A_w(\theta)||$. But

$$A_{ww}(\theta) - A_w w A_w + id = A_w(\theta)^2 - A_w w A_w + id = 0,$$
since $A_w \in \text{SL}(2, \mathbb{R})$, which shows that \( \frac{1}{|w|} \log |\text{tr} A_w| \) is close to \( \frac{1}{2} \), yielding the estimate on the spectral radius in theorem 1.

In order to prove theorem 2 by the same method, we consider, instead of the words on $H$ and $R$, words $w = w_n \ldots w_1$ on $H(t)$, $R(\theta)$, $H(s)$ and $R(\alpha)$ such that the last three ones always appear consecutively, except maybe at the ends of the word, and $m(w)$ is now the number of these occurring in $w$. Then the proof goes the same way, notably the considerations of sign in lemma 1.

References

[1] Bochi, Jairo; Fayad, Bassam, Dichotomies between uniform hyperbolicity and zero Lyapunov exponents for $SL(2, \mathbb{R})$ cocycles. Bull. Braz. Math. Soc. (N. S.) 37 (2006), no 3, 307-349
[2] Fayad, Bassam ; Krikorian, Raphaël, Exponential growth of product of matrices in $SL(2, \mathbb{R})$. Nonlinearity 21 (2008), no. 2, 319323.
[3] Furstenberg, Harry Noncommuting random products. Trans. Amer. Math. Soc. 108 (1963), 377-428