CHERN CLASSES OF CONFORMAL BLOCKS ON $\overline{M}_{0,n}$

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Abstract. We derive a formula for the Chern classes of the bundles of conformal blocks on $\overline{M}_{0,n}$ and explicitly compute some special cases.

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1.1. Introduction. The mathematical theory of conformal blocks of Tsuchiya–Kanie [11] and Tsuchiya–Ueno–Yamada [12] gives rise to a family of vector bundles on $\overline{M}_{g,n}$, the moduli stack of stable $n$-pointed curves of genus $g$, parametrised by a simple Lie algebra $\mathfrak{g}$, a non-negative integer $\ell$ called the level, and an $n$-tuple of dominant weights for $\mathfrak{g}$ of “level $\ell$”. The ranks of these bundles are given by the celebrated Verlinde formula (see [10] for a survey); the purpose of this article is to investigate the other Chern classes of these bundles, the rank being $c_0$, with an emphasis on $c_1$ or the determinant bundle.

The bundles of conformal blocks have been objects of interest for algebraic geometers ever since it was realised that they can be described in terms of sections of natural line bundles on suitable moduli stacks of parabolic principal bundles on curves. However, our motivation for studying these bundles is the hope that we will get some insight into the geometry of the moduli stacks $\overline{M}_{g,n}$ themselves. Our main result, Theorem 3.1, is a formula for the Chern classes of the bundles of conformal blocks on $\overline{M}_{0,n}$. The main tool that we use for this is the KZ connection on the restriction of these bundles to $\overline{M}_{0,n}$. The rest of the paper consists in obtaining consequences of this formula in some special cases, principally for $\mathfrak{g} = \mathfrak{sl}_2$ and any level and for arbitrary $\mathfrak{g}$ and level 1. For $\mathfrak{g} = \mathfrak{sl}_2$, we show that the non-zero determinants of conformal blocks of level 1 form a basis of Pic$(\overline{M}_{0,n})$$_\mathbb{Q}$ (Theorem 4.2). For what we call the critical level, we show that the bundles are pullbacks from suitable GIT quotients $(\mathbb{P}^1)^n//SL_2$ (Theorem 4.5); in general they are pulled back from suitable moduli spaces of weighted stable curves constructed by Hassett (Proposition 4.7).

In the final section we discuss some questions that we have not been able to answer.

1.2. Notation. We will work over an algebraically closed field $K$ of characteristic zero. $\mathfrak{g}$ will always denote a finite dimensional simple Lie algebra over $K$, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, $\Delta \subset \mathfrak{h}^*$ the corresponding root system and $\alpha_1, \alpha_2, \ldots, \alpha_r$ a basis of the root system inducing a partition $\Delta = \Delta^+ \cup \Delta^-$. Let $P \subset \mathfrak{h}^*$ be the weight lattice and $P_+ \subset P$ be the set of dominant weights. For $\lambda \in P_+$ we let $V_\lambda$ denote the corresponding irreducible representation of $\mathfrak{g}$. Let $\theta \in \Delta^+$ be the highest root and let $\mathfrak{s} \cong \mathfrak{sl}_2$ be the subalgebra of $\mathfrak{g}$ generated by $H_\theta$ and $H_{-\theta}$, where $H_\alpha$, for a root $\alpha$, is the corresponding coroot.

We normalize the Killing form $\langle \rangle$ on $\mathfrak{g}$ so that $\langle \theta | \theta \rangle = 2$ and use the same notation for the induced forms on $\mathfrak{h}$ and $\mathfrak{h}^*$. We let $\{X_\kappa\}_\kappa$ be an orthonormal basis of $\mathfrak{g}$ with respect to this form. Let $h^*$ be the dual Coxeter number of $\mathfrak{g}$; this is half the scalar by which the normalised Casimir element $c := \sum_\kappa X_\kappa \otimes X_\kappa$ acts on the adjoint representation of $\mathfrak{g}$. For
any $\lambda \in P_+$, we let $c(\lambda)$ be the scalar by which $c$ acts on $V_\lambda$. For $\ell$ a non-negative integer, let $P_\ell = \{\lambda \in P_+ \mid (\lambda|\theta) \leq \ell\}$. For $\lambda \in P_\ell$, let $\lambda^*$ be the highest weight of $(V_\lambda)^*$; if $\lambda \in P_\ell$ then $\lambda^* \in P_\ell$ as well.

2. Conformal blocks

In this section we recall without proofs some of the basic results in the theory of conformal blocks.

2.1. Let $S$ be a smooth variety over $K$ and $\pi : \mathcal{C} \to S$ be a flat proper family of curves with only ordinary double point singularities; we do not assume that the fibres are connected. Let $\mathcal{P} = (p_1, \ldots, p_n)$, with $p_i : S \to \mathcal{C}$ sections of $\pi$ whose images are disjoint and contained in the smooth locus of $\pi$. We also assume that $\mathcal{C} \setminus \bigcup_i p_i(S)$ is affine over $S$. Let $\overline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ be an $n$-tuple of elements of $P_\ell$. To this data is attached a canonically defined locally free sheaf $V_\mathcal{C}(\mathcal{P}, \overline{\lambda})$ on $S$, called the sheaf of covacua [12], [13], [10] or conformal blocks [9]. Note that these sheaves depend on $\ell$ which we have omitted from the notation.

Let $\mathcal{P} = (q_1, \ldots, q_m)$ with the $q_j$ also sections of $\pi$ with images which are disjoint, disjoint from all the $p_i(S)$ and contained in the smooth locus of $\pi$. Let $\mathcal{P}^0 = (p_1, \ldots, p_n, q_1, \ldots, q_m)$, $\overline{\lambda}_0 = (\lambda_1, \ldots, \lambda_n, 0, \ldots, 0)$, where we have $m$ zeros. We then have the following [10, 2.3.2], [9, Proposition 3.4]:

**Proposition 2.1** (Propagation of vacua). There is a natural isomorphism $V_\mathcal{C}(\mathcal{P}, \overline{\lambda}) \sim V_\mathcal{C}(\mathcal{P}^0, \overline{\lambda}_0)$. Moreover, these isomorphisms are compatible with composition.

Using this proposition and descent, for any $n$-tuple $\overline{\lambda}$ as above one may define the bundle of conformal blocks on $\overline{\mathcal{M}}_{g,n}$, the moduli stack of stable $n$-pointed curves of genus $g$. We shall be mostly concerned with the case $g = 0$ and shall denote the corresponding bundle on $\mathcal{M}_{0,n} = \overline{\mathcal{M}}_{0,n}$ simply by $\mathcal{V}_\lambda$ and its determinant line bundle by $\mathcal{D}_\lambda$.

The bundles of conformal blocks are preserved by base change, so it follows from Proposition 2.1 that if $\lambda_i = 0$ for some $i$ then $\mathcal{V}_\lambda = \pi_i^*(\mathcal{V}_\lambda)$ where $\pi_i : \mathcal{M}_{0,n} \to \mathcal{M}_{0,n-1}$ is the morphism given by forgetting the $i$’th marked point and $\overline{\lambda}$ is the $n - 1$-tuple obtained from $\overline{\lambda}$ by omitting $\lambda_i(= 0)$.

**Lemma 2.2.** All the $\mathcal{V}_\lambda$ are generated by global sections, therefore so are all the $\mathcal{D}_\lambda$. In particular, the $\mathcal{D}_\lambda$ are nef line bundles.

**Proof.** With notation as in the beginning of this section, there is always a natural map from the constant vector bundle on $S$ with fibre $(\otimes_i V_{\lambda_i})_\theta$ to $V_\mathcal{C}(\mathcal{P}, \overline{\lambda})$. It follows from [13, Proposition 3.5.1] that this map is surjective if the fibres of the family are smooth curves of genus 0. The proof only uses the fact that given any point $x \in \mathbb{P}^1(K)$ there is a rational function $f \in K(\mathbb{P}^1)$ which has a simple pole at $x$. If $C$ is a semi-stable curve of genus 0 and $x \in C(K)$ is a smooth point, then there exists $f \in \mathcal{O}_{C,x}$ having simple pole at $x$, so the proof of [13, Proposition 3.5.1] extends to the case where the fibres are semi-stable curves of genus 0.

**Remark 2.3.** The map from $(\otimes_i V_{\lambda_i})_\theta$ to $V_\mathcal{C}(\mathcal{P}, \overline{\lambda})$ is in general not surjective if the fibres have genus $g > 0$ and we do not know whether the determinants of conformal blocks on $\overline{\mathcal{M}}_{g,n}$ are always nef. However, it follows from [4, Theorem 0.3], the above and the factorisation formula (Proposition 2.4) that this is true if it is so for $\overline{\mathcal{M}}_{1,1}$.

\[1\] This is work in progress.
2.2. Let $\pi : C \to S$, $\overline{\lambda}$ be as above and let $p', p'' : S \to C$ be two other sections with images which are disjoint, disjoint from the $p_i(S)$ and contained in the smooth locus of $\pi$. Let $\pi_\varphi : \mathcal{D} \to S$ be the curve obtained by gluing $C$ along the sections $p'$ and $p''$. The sections $p_i$ induce sections of $\pi_\varphi : \mathcal{D} \to S$ which we also denote by $p_i$. We then have [9, Proposition 4.1], [10, 2.4.2]

**Proposition 2.4** (Factorisation formula). There are natural isomorphisms
\[ \bigoplus_{\lambda \in P} V_\varphi((p_1, \ldots, p_n, p', p''), (\lambda_1, \ldots, \lambda_n, \lambda, \lambda^*)) \sim V_\varphi(\overline{\pi}, \overline{\lambda}) . \]

Recall that the Chow groups of $\overline{M}_{0,n}$ are generated by the stratification induced by the boundary divisors i.e. the irreducible components of $\overline{M}_{0,n} \setminus M_{0,n}$ and the Chow groups are equal to the Chow groups modulo numerical equivalence [7]. Thus, one may compute the class of $\overline{\lambda}$ in $\text{Pic}(\overline{M}_{0,n})$ by computing the degree of $V_\varphi$ restricted to the one dimensional strata, the so called vital curves.

Let $r_{\overline{\lambda}}$ denote the rank of $V_\varphi$. Classes of vital curves modulo numerical equivalence correspond to partitions $\{1, \ldots, n\} = N_1 \cup N_2 \cup N_3 \cup N_4$ with $|N_k| = n_k > 0$. Given such a partition, let $F$ be the family of $n$-pointed genus 0 curves given by gluing fixed $n_k + 1$-pointed curves $C_k$ of genus 0 at the last marked point along the sections of the universal family over $\overline{M}_{0,4}$. If $n_k = 1$ for some $k$, we do not glue any curve at the $k$'th section. This gives rise to a family of stable $n$-pointed curves of genus 0 such that the class in the Chow group of the image of $\overline{M}_{0,4}$ in $\overline{M}_{0,n}$ by the classifying map for this family is independent of the choice of the glued curves.

Given a partition as above and $\overline{\mu} = (\mu_1, \mu_2, \mu_3, \mu_4) \in P_4^4$, let $\overline{\lambda}_{\mu_k}$ be the $n_k + 1$-tuple $(\lambda_{i_1}, \ldots, \lambda_{i_{n_k}}, \mu_k)$ where $N_k = \{i_1, i_2, \ldots, i_{n_k}\}$. Since in the construction of $F$ the attached curves do not vary in moduli, it follows from the factorisation formula applied four times that we have the following

**Proposition 2.5.**
\[ \deg(\overline{V}_{\varphi}|_F) = \sum_{\overline{\pi} \in P_4^4} \deg(\overline{V}_{\varphi}) \prod_{k=1}^4 n_{\mu_k} . \]

The ranks of the bundles of conformal blocks can be computed from the Verlinde formula or inductively from the case $n = 3$, so it follows that to compute the determinant of $\overline{V}_{\varphi}$, equivalently its degree on any vital curve, it suffices to consider the case when $n = 4$.

3. **The KZ connection on $\overline{M}_{0,n}$**

In this section we shall derive an expression for an explicit divisor representing the first Chern class of the bundles of conformal blocks. This will be done by computing the residue of the KZ connection along the boundary divisors.

We assume that the reader is familiar with the construction of conformal blocks using representations of affine Lie algebras as in say [9]. In particular, we shall use without explanation standard notation for affine Lie algebras, the Sugawara representation, etc. We shall also use without mention the fact that the Sugawara action does not change under change of coordinates by a fractional linear transformation; this is a consequence of the transformation formula for the stress-energy tensor [3, 8.2.2].
3.1. We identify $M_{0,n}$ with the open subset of $\mathbb{A}^{n-3}$ given by
\[
\{(z_1, z_2, \ldots, z_{n-3}) \mid z_i \neq 0, 1 \text{ for all } i \text{ and } z_i \neq z_j \text{ for } i \neq j\}.
\]
The universal family of marked curves is given by $M_{0,n} \times \mathbb{P}^1$ with the $n$ ordered sections given by the $n$ maps $M_{0,n} \to \mathbb{P}^1$, $(z_1, z_2, \ldots, z_{n-3}) \mapsto z_1, z_2, \ldots, z_{n-3}, 0, 1, \infty$. Letting $x$ be the coordinate on $\mathbb{P}^1$, the sections are given by the equations $x = z_1, z_2, \ldots, z_{n-3}, 0, 1$ and $1/x = 0$.

Fix a simple Lie algebra $\mathfrak{g}$, a level $\ell$ and $\mathfrak{X} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in P^n$. Following [9], we now recall (in a slightly different form) the construction of the KZ connection which is a flat connection on the constant bundle with fibre $\otimes_i V_{\lambda_i}$ which induces a connection on $\mathbb{V}_X$ via the quotient map
\[
\otimes_i V_{\lambda_i} \otimes_k \Theta_{M_{0,n}} \to (\otimes_i V_{\lambda_i})_{\mathfrak{g}} \otimes_k \Theta_{M_{0,n}} \to \mathbb{V}_X.
\]
We shall describe the connection by computing the actions of the vector fields $\partial_i := \partial/\partial z_i$, $i = 1, 2, \ldots, n - 3$. Consider $\partial_i$ as a vector field on $M_{0,n} \times \mathbb{P}^1$ with trivial component along the second factor. It acts trivially on the local equations of all the sections except for the $i$th section. We have $\partial_i = (\partial_x + \partial_i) - \partial_x$ (where $\partial_x := \partial/\partial x$) with the first term being tangent to the $i$th section and the second term being vertical. The first term acts as $\partial_i$ on the second factor of $(\otimes_i V_{\lambda_i}) \otimes_k \Theta_{M_{0,n}}$ whereas the second acts by the Sugawara representation. This is given by
\[
\frac{-1}{\ell + \hbar} \sum_{\kappa} X_\kappa (x - z_i)^{-1} \circ X_\kappa
\]
acting on the $i$th factor and the identity on the others. This does not induce an action on the constant bundle with fibre $\otimes_i V_{\lambda_i}$ but does so if we add the term
\[
\frac{1}{\ell + \hbar} \sum_{\kappa} \sum_{j=1}^{n} (X_\kappa (x - z_i)^{-1})^{(j)} \circ X^{(j)}_\kappa,
\]
where the superscript in ( ) indicates the factor on which the corresponding element acts. This term acts trivially on $\mathbb{V}_X$ since the action of $\sum_{j=1}^{n} (X_\kappa (x - z_i)^{-1})^{(j)}$ is the same (by definition) as the action of $X_\kappa (x - z_i)$. The resulting action is then given by
\[
\frac{1}{\ell + \hbar} \sum_{\kappa} \sum_{j \neq i} (X_\kappa (x - z_i)^{-1})^{(j)} \circ X^{(j)}_\kappa.
\]
Since $(x - z_i)^{-1}$ is regular along all except the $i$th section and vanishes at the $n$th section (corresponding to $\infty$) it follows that the Sugawara action of $\partial_i$ is given by
\[
\frac{-1}{\ell + \hbar} \sum_{\kappa} \sum_{j \neq i} \frac{1}{z_i - z_j} X^{(j)}_\kappa \circ X^{(j)}_\kappa,
\]
where we set $z_{n-2} = 0$ and $z_{n-1} = 1$ for ease of notation. Thus the connection matrix for the KZ connection on $\otimes_i V_{\lambda_i} \otimes_k \Theta_{M_{0,n}}$ is given by
\[
\frac{-1}{\ell + \hbar} \sum_{i=1}^{n-3} \sum_{\kappa} \sum_{j \neq i,n} \frac{dz_i}{z_i - z_j} X^{(j)}_\kappa \circ X^{(j)}_\kappa.
\]
3.2. Our aim in this section is to compute the residues of the KZ connection along the boundary divisors in \( \overline{M}_{0,n} \). Recall that these are parametrized by partitions \( \{1, 2, \ldots, n\} = A \cup B \) with \( |A|, |B| \geq 2 \). In the coordinates above, they correspond to the exceptional divisors in the blowup of the following loci:

1. For \( \emptyset \neq S \subset \{1, 2, \ldots, n-3\} \), the locus in \( \mathbb{A}^{n-3} \supset M_{0,n} \) given by the equations \( \{z_i = 0\}_{i \in S} \).
2. For \( \emptyset \neq S \subset \{1, 2, \ldots, n-3\} \), the locus in \( \mathbb{A}^{n-3} \supset M_{0,n} \) given by the equations \( \{z_i = 1\}_{i \in S} \).
3. For \( \emptyset \neq S \subset \{1, 2, \ldots, n-3\} \), the locus in \( (\mathbb{P}^1)^{n-3} \supset \mathbb{A}^{n-3} \supset M_{0,n} \) given by the equations \( \{1/z_i = 0\}_{i \in S} \).
4. For \( S \subset \{1, 2, \ldots, n-3\} \), with \( |S| \geq 2 \), the locus in \( \mathbb{A}^{n-3} \supset M_{0,n} \) given by the equations \( \{z_i = z_j\}_{i,j \in S} \).

Globally, each of these divisors is the image of the embedding of \( \overline{M}_{0,r+2} \times \overline{M}_{0,n-r} \) into \( \overline{M}_{0,n} \) by a suitable gluing map, where \( r = |S| \) in the first three cases and \( r = |S| - 1 \) for the last case. By Proposition 2.4, the restriction of \( \nabla_X \) to each of these divisors is a sum \( \bigoplus_{\mu \in P_t} \mathcal{V}_\mu^\lor \otimes \mathcal{V}_\mu^{\lor^*} \), where \( \mathcal{V}_\mu^\lor \) and \( \mathcal{V}_\mu^{\lor^*} \) are obtained by restricting \( \mathcal{V}_\mu \) and attaching \( \mu \) (resp. \( \mu^* \)) at the glued point of the first (resp. second) factor. The residue of the KZ connection along these divisors—this is an endomorphism of the restricted bundle—preserves this direct sum decomposition and moreover acts on each summand by a scalar which we shall now determine.

3.2.1. Consider a boundary divisor of type (1). For ease of notation we shall assume \( S = \{1, 2, \ldots, r\} \), since the general case follows from this by permuting coordinates. An open set in the blowup is given by \( U \cong \mathbb{A}^{n-3} \) with coordinates \( t, w_2, \ldots, w_r, z_{r+1}, \ldots, z_{n-3} \) with the map to \( M_{0,n} \) given by \( (t, w_2, \ldots, w_r, z_{r+1}, \ldots, z_{n-3}) \mapsto (t, tw_2, \ldots, tw_r, z_{r+1}, \ldots, z_{n-3}) \) and the exceptional divisor \( B \) is given by \( t = 0 \). The universal family in a neighbourhood of the generic point of exceptional divisor is given by blowing up the locus given by \( t = x = 0 \) in \( U \times \mathbb{P}^1 \), so over the generic point there are two components isomorphic to \( \mathbb{P}^1 \) meeting transversally in a single point.

Let \( y := t/x \). The \( n \) sections defined over \( M_{0,n} \) extend to sections of this family as follows, where \( w_1 := 1 \):

- The sections given by \( x = z_{r+1}, \ldots, z_n, x = 1 \) and \( 1/x = 0 \) are given by the same equations.
- The section given by \( x = 0 \) is given by \( 1/y = 0 \).
- The sections given by \( x = z_i, 1 \leq i \leq r \), are given by \( y = w_i^{-1} \).

Replacing \( K \) by \( K(w_2, \ldots, w_r, z_{r+1}, \ldots, z_{n-1}) \), it follows from [9, Theorem 4.5] that the residue of the connection obtained using the above coordinates along this divisor and the new equations for the sections is given by the endomorphism of \( \nabla_X|B \) which acts by multiplication by \( c(\mu)/2(\ell + h^\lor) \) on the summand \( \mathcal{V}_{\mu^*} \otimes \mathcal{V}_{\mu^*}^\lor \).

To compute the residue of the KZ connection defined in §3.1 we must take into account the fact that in the above we have used different equations for the sections: Let \( \partial_t := \partial/\partial t \) and lift it to a derivation on the universal family with trivial action in the fibre direction. To compare the connection in the new coordinates as above with the KZ connection we must decompose \( \partial_t \) locally along each section as \( D_{\text{vert}} + D_{\text{hor}} \) and consider the Sugawara action of \( D_{\text{vert}} \). This decomposition depends on the local equation for the sections; we use the above
notation for the original equations used in the construction of the KZ connection and write \( \partial_t = D'_{\text{vert}} + D'_{\text{hor}} \) for the decomposition for the new coordinates defined above.

- For the sections given by \( x = z_{r+1}, \ldots, z_n, x = 1 \) and \( 1/x = 0 \), \( D_{\text{vert}} = D'_{\text{vert}} = 0 \).
- For the section given by \( x = 0 \) (this is the \( n-2 \)'nd according to our numbering), we have \( D_{\text{vert}} = 0 \) but \( D'_{\text{vert}} = -(x/t)\partial_x \). This contributes an extra term

\[
\frac{1}{2(\ell + h^\vee)} \sum_{\kappa} t^{-1} X_{\kappa}^{(n-1)} \circ X_{\kappa}^{(n-1)}.
\]

- For the sections with equations \( x = z_i, 1 \leq i \leq r \), by substituting \( z_i = tw_i \) we see that \( D_{\text{vert}} = -w_i \) whereas \( (\partial_t + (x/t)\partial_x)((t/x) - w_i^{-1}) = 0 \) so \( D'_{\text{vert}} = -(x/t)\partial_x = -(\frac{x}{t}z_i + w_i) \). It follows that the extra term is given by

\[
\frac{1}{2(\ell + h^\vee)} \sum_{\kappa} t^{-1} X_{\kappa}^{(i)} \circ X_{\kappa}^{(i)}.
\]

Adding up all the terms, we see that the residue of the KZ connection along this divisor is the endomorphism of \( \mathcal{V}'_X|_B \) which acts by multiplication by

\[
\frac{c(\mu) - c(\lambda_{n-2}) - \sum_{i \in S} c(\lambda_i)}{2(\ell + h^\vee)}
\]

on the summand \( \mathcal{V}'_{X_{\mu}} \otimes \mathcal{V}'_{X_{\mu}^*} \) for each \( \mu \in P_\ell \).

**3.2.2.** We consider a boundary divisor of type (2). The change of coordinates given by \( z_i \mapsto 1 - z_i, i = 1, 2, \ldots, n - 3 \) and \( x \mapsto 1 - x \) on \( M_{0,n} \times \mathbb{P}^1 \) preserves the equations of all the sections except for \( x = 0 \) and \( x = 1 \) which it interchanges and \( 1/x = 0 \) which becomes \( 1/(1 - x) = 0 \). Moreover, it sends the locus given by \( \{z_i = 1\}_{i \in S} \) to the locus given by \( \{z_i = 1\}_{i \in S} \). Since \( \partial_t(1/x) = \partial_t(1/(1 - x)) = 0 \) for all \( i \), it follows from (3.3) that the residue of the KZ connection along a divisor \( B \) of type (2) is the endomorphism of \( \mathcal{V}'_X|_B \) which acts by multiplication by

\[
\frac{c(\mu) - c(\lambda_{n-1}) - \sum_{i \in S} c(\lambda_i)}{2(\ell + h^\vee)}
\]

on the summand \( \mathcal{V}'_{X_{\mu}} \otimes \mathcal{V}'_{X_{\mu}^*} \) for each \( \mu \in P_\ell \).

**3.2.3.** We consider a boundary divisor of type (3). The change of coordinates given by \( z_i \mapsto 1/z_i, i = 1, 2, \ldots, n - 3 \) and \( x \mapsto 1/x \), preserves the sections given by \( x = z_i, i = 1, 2, \ldots, n - 3 \) and \( x = 1 \) and switches the sections given by \( x = 0 \) and \( 1/x = 0 \). However, the equations of the sections are not preserved so we must compute the change in the connection caused by this transformation.

For any \( i = 1, 2, \ldots, n - 3 \) the new equations for all the sections except for the \( i \)'th one are killed by \( \partial_{i} \), so \( D_{\text{vert}} = D'_{\text{vert}} = 0 \) for them, where the decomposition here is with respect to \( \partial_{i} \). The new equation for the \( i \)'th section is \( 1/x - 1/z_i = 0 \). We have \( D_{\text{vert}} = -\partial_{x} \) and since \( (\partial_{i} - (x^2/z_i^2)\partial_{x})(1/x - 1/z_i) = 0 \) it follows that \( D'_{\text{vert}} = -x^2/z_i^2 \partial_{x} \). We have

\[
\frac{x^2}{z_i} = -(x - z_i)^2 + 2z_i(x - z_i) + z_i^2 - 1
\]
so the difference of the two actions is by the Sugawara action of
\[ \frac{(x - z_i)^2 + 2z_i(x - z_i)}{z_i^2} \partial_x. \]

This is equal to
\[ \frac{1}{2(\ell + h^\vee)} \sum_{\kappa} \sum_{r \in \mathbb{Z}} : X_\kappa \left( \frac{(x - z_i)^2}{z_i^2} \right) + 2 \left( \frac{x - z_i}{z_i} \right) (x - z_i)^{-r-1} \circ X_\kappa (x - z_i)^r :. \]

The first term is given by
\[ \frac{1}{2(\ell + h^\vee)} \sum_{\kappa} \sum_{r \in \mathbb{Z}} : X_\kappa \left( \frac{x - z_i}{z_i^2} \right) (x - z_i)^{-r+1} \circ X_\kappa (x - z_i)^r :. \]

which by the normal ordering rules acts by 0. Again using the normal ordering rules we see that the second term is equal to
\[ \frac{1}{\ell + h^\vee} \sum_{\kappa} z_i^{-1} X_\kappa^{(i)} \circ X_\kappa^{(i)}. \]

It then follows from (3.3) that the residue of the KZ connection along this divisor is the endomorphism of $\mathbb{V}_{X_\mu} \otimes \mathbb{V}_{X_\mu^*}$ for each $\mu \in P_t$.

### 3.2.4. Boundary divisors of type (4)

Again, for ease of notation we shall suppose $S = \{1, 2, \ldots, r\}$ for some $r$, $2 \leq r \leq n - 3$. Then an open subset of the blowup may be identified with $\mathbb{A}^{n-3}$ with coordinates $s, t, w_3, \ldots, w_r, z_{r+1}, \ldots, z_{n-3}$ so that the map to $M_0, n$ is given by $(s, t, w_3, \ldots, w_r, z_{r+1}, \ldots, z_{n-3}) \mapsto (s, s+t, s+t+w_3, \ldots, s+t+w_r, z_{r+1}, \ldots, z_{n-3})$ and the exceptional divisor $B$ is given by $t = 0$. The universal family in a neighbourhood of $B$ is then given by blowing up the locus given by $t = x - s = 0$, where $x$ is the coordinate on $\mathbb{P}^1$ as before.

Let $y = t/(x - s)$. All $n$ sections extend to sections of the universal family in a neighbourhood of the generic point of $T$ with equations given as follows, where $w_2 := 1$:

- $x - z_1 = 0$ is replaced by $1/y = 0$.
- $x - z_i = 0$ is replaced by $y - w_i^{-1} = 0$ for $2 \leq i \leq r$.
- The equations for the remaining sections are unchanged.

As in §3.2.1 we now decompose $\partial_t$ as $D_{\text{vert}} + D_{\text{hor}}$ and $D'_{\text{vert}} + D'_{\text{hor}}$ and compare the two:

- For $x - z_1 = 0$, $D_{\text{vert}} = 0$ and $D'_{\text{vert}} = (x - s)/t \partial_x$ so we have an extra term of the form
  \[ \frac{1}{2(\ell + h^\vee)} \sum_{\kappa} t^{-1} X_\kappa^{(1)} \circ X_\kappa^{(1)}. \]

- For $x - z_i = 0, 2 \leq i \leq r$, $D_{\text{vert}} = -w_i \partial_x$ and $D'_{\text{vert}} = -(x - s)/t \partial_x = -[(x - (s + tw))/(t + w)] \partial_x$ so we get an extra term of the form
  \[ \frac{1}{2(\ell + h^\vee)} \sum_{\kappa} t^{-1} X_\kappa^{(i)} \circ X_\kappa^{(i)}. \]

- There are no extra terms for the remaining sections.
As before, it follows from the above computations that the residue along this divisor is the endomorphism of $V_\lambda|_B$ which acts by multiplication by
\begin{equation}
(3.6) \quad \frac{c(\mu) - \sum_{i \in S} c(\lambda_i)}{2(\ell + h^\vee)}
\end{equation}
on the summand $V_\lambda \otimes V_\lambda^*$ for each $\mu \in P_\ell$.

3.3. The KZ connection on $V_\lambda$ has logarithmic singularities, so using the computations of the residues of the KZ connection along the boundary divisors of $\overline{M}_{0,n}$ of the previous section and a formula of Esnault and Verdier [2, Appendix B, Corollary], one may deduce an expression for all the Chern classes of $V_\lambda$ in de Rham cohomology or, equivalently, the rational Chow groups of $\overline{M}_{0,n}$. The result is the following

**Theorem 3.1.** Let $\mathfrak{g}$ be a simple Lie algebra, $\ell \geq 0$ an integer and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in P^n_\ell$. Then
\begin{equation}
(3.7) \quad N_p(V_\lambda) = (-1)^p \sum_{\alpha_1 + \cdots + \alpha_s = p} \text{Tr}(\Gamma_1^{\alpha_1} \cdots \Gamma_s^{\alpha_s})[B_1]^{\alpha_1} \cdots [B_s]^{\alpha_s}
\end{equation}
in $CH^p(\overline{M}_{0,n})_Q$. Here $N_p$ denotes the $p$'th Newton class, the $B_i$, $i = 1, \ldots, s$, are the irreducible components of $\overline{M}_{0,n} \setminus M_{0,n}$ and $\Gamma_i$ denotes the residue of the KZ connection along $B_i$ given by one of (3.3), (3.4), (3.5), and (3.6).

**Remarks 3.2.**

1. Keel [7] has determined the Chow ring of $\overline{M}_{0,n}$, so all intersections involved in (3.7) may be computed explicitly.

2. To compute the traces one needs to know the ranks of the bundles of conformal blocks. These are given in closed form by the Verlinde formula for the classical groups and $G_2$ [10] or can be derived inductively from the 3-point ranks using [13, Corollary 3.5.2].

3. The traces appearing in (3.7) are rational numbers but not, in general, integers.

Since the KZ connection depends on the choice of coordinates so do the residues, hence the representing cycle for $c_1(V_\lambda) = N_1(V_\lambda)$ in Theorem 3.1. However, by averaging over all choices we obtain a canonical representative:

**Corollary 3.3.** Let $\mathfrak{g}$ be a simple Lie algebra, $\ell \geq 0$ an integer and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in P^n_\ell$. Then
\begin{equation}
(3.8) \quad c_1(V_\lambda) = \\
\frac{1}{2(\ell + h^\vee)} \sum_{i=2}^{[n]} \epsilon_i \left\{ \sum_{A \subset \{1,2,\ldots,n\}} \left\{ \frac{r_\lambda}{(n-1)(n-2)} \left( (n-i)(n-i-1) \sum_{a \in A} c(\lambda_a) + i(i-1) \sum_{a' \in A^c} c(\lambda_{a'}) \right) \\
- \left\{ \sum_{\mu \in P_\ell} c(\mu) \cdot r_\lambda \cdot r_{\lambda^\vee,\mu} \cdot r_{\lambda^\vee,\mu^*} \right\} \cdot [D_{A,A^c}] \right\}
\end{equation}
in $\text{Pic}(\overline{M}_{0,n})_Q$, where $D_{A,A^c}$ is the irreducible boundary divisor corresponding to the partition $\{1,2,\ldots,n\} = A \cup A^c$ and $\epsilon_i = 1/2$ if $i = n/2$ and 1 otherwise.
Proof. The choices involved in §3.1 are the labelling of the last three points as \( p_0, p_1 \) and \( p_\infty \). We consider all \( n(n-1)(n-2) \) ways of choosing these labels and compute the cycle obtained by averaging the coefficients of \( D_{A,A^e} \) for each possible choice.

We consider the four types of boundary divisors considered in 3.2 and consider the coefficient of \( c(\lambda_a) \) for \( a \in A \) coming from each of these divisor types. Let \( i = |A| \).

1. \( p_0 \in A \) and \( p_1, p_\infty \in A^e \). There are \( i(n-i)(n-i-1) \) such cases and from (3.3) each one of these gives a contribution of \(-1\), for a total contribution of \(-i(n-i)(n-i-1)\).
2. \( p_1 \in A \) and \( p_0, p_\infty \in A^e \). There are again \( i(n-i)(n-i-1) \) such cases and from (3.4) each one of these gives a contribution of \(-1\), for a total contribution of \(-i(n-i)(n-i-1)\).
3. \( p_\infty \in A \) and \( p_0, p_1 \in A^e \). There are \( i(n-i)(n-i-1) \) such cases. From (3.5) it follows that if \( a = p_\infty \) then we get a contribution of \(-1\) and otherwise we get a coefficient of \( 1 \) so the total contribution is \((i-2)(n-i)(n-i-1)\).
4. \( p_0, p_1, p_\infty \in A^e \). There are \( (n-i)(n-i-1)(n-i-2) \) such cases and from (3.6) it follows that each gives a contribution of \(-1\) for a total contribution of \(-(n-i)(n-i-1)(n-i-2)\).

Summing all these we get that the coefficient of \( c(\lambda_a) \) for \( a \in A \) is \(-n(n-i)(n-i-1)\). By symmetry it follows that the coefficient of \( c(\lambda_a') \) for \( a' \in A^e \) is \(-ni(i-1)\). The claimed formula then follows from Theorem 3.1. \( \square \)

Specialising Corollary 3.3 to the case \( n = 4 \), we get the following formula which we state here for ease of reference later:

**Corollary 3.4.** Let \( \mathfrak{g} \) be a simple Lie algebra, \( \ell \geq 0 \) an integer and \( \overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in P_\ell^4 \). Then

\[
\deg(\mathcal{V}_{\overline{\lambda}}) = \frac{1}{2(\ell + h^\vee)} \times \left\{ r_{\overline{\lambda}} \sum_{i=1}^{4} c(\lambda_i) \right\} - \left\{ \sum_{\lambda \in P_\ell} c(\lambda) \{ r_{(\lambda_1,\lambda_2,\lambda)} \cdot r_{(\lambda_3,\lambda_4,\lambda^*)} + r_{(\lambda_1,\lambda_3,\lambda)} \cdot r_{(\lambda_2,\lambda_4,\lambda^*)} + r_{(\lambda_1,\lambda_4,\lambda)} \cdot r_{(\lambda_2,\lambda_3,\lambda^*)} \} \right\}.
\]

Inserting (3.9) into Proposition 2.5, one obtains a formula for \( \deg(\mathcal{V}_{\overline{\lambda}}|_F) \) for any vital curve \( F \). This gives a dual expression for \( c_1(\mathcal{V}_{\overline{\lambda}}) \) (since the vital curves generate \( CH_1(\overline{M}_{0,n}) \)) which will be useful to us below. Similar expressions can also be obtained for the other Chern classes.

For \( 2 \leq i \leq n/2 \), let \( D_i := \epsilon_i \sum_A D_{A,A^e} \) where the sum is over all \( A \subset \{1, 2, \ldots, n\} \) with \( |A| = i \) and \( D_{A,A^e} \), \( \epsilon_i \) are as above. For \( \overline{\lambda} \in P_\ell^4 \) and \( \mu \in P_\ell \), let \( \overline{\lambda}_{A,\mu} \) be the \( i + 1 \)-tuple \((\lambda_{a_1}, \ldots, \lambda_{a_i}, \mu)\) where \( A = \{a_1, \ldots, a_i\} \).
Corollary 3.5. For the action of the symmetric group $S_n$ on $\overline{M}_{0,n}$ by permutation of the marked points we have

$$
(3.10) \quad \sum_{\sigma \in S_n} \sigma^*(c_1(V_\lambda)) = \frac{1}{2(\ell + h^\vee)} \sum_{2 \leq i \leq n/2} i!(n - i)! \left\{ \left( \binom{n-3}{i-1} + \binom{n-3}{n-i-1} \right) \cdot r_{\lambda} \sum_{j=1}^{n} c(\lambda_j) \right\} \\
- \left\{ \sum_{|A|=i} \sum_{\mu \in P_i} c(\mu) \cdot r_{\lambda A,\mu} \cdot r_{\lambda A^c,\mu^*} \right\} \cdot [D_i]
$$

Proof. This follows from Theorem 3.1 and a simple counting argument. \qed

Remark 3.6. The set $\{[D_i]\}_{1 \leq i \leq n/2}$ is a basis of $\text{Pic}(\overline{M}_{0,n})_{\mathbb{Q}} \cong \text{Pic}(\overline{M}_{0,n}/S_n)_{\mathbb{Q}}$, so the RHS of (3.10) is independent of all choices. Keel and McKernan [8, Theorem 1.3] have proved that the $[D_i]$ generate the cone of effective divisors of $\overline{M}_{0,n}/S_n$; since $c_1(V_\lambda)$ is always nef, it follows that the coefficients of $[D_i]$ in (3.10) are always non-negative. This is not the case for the coefficients of $[D_{A,A^c}]$ in Corollary 3.3.

4. The case $g = sl_2$

4.1. In this section we consider the case $g = sl_2$. We identify $P$ with $\mathbb{Z}$ so that $P_+$ is identified with $\mathbb{Z}_{\geq 0}$. Then for any $\lambda \in P_+$, $c(\lambda) = \lambda^2/2 + \lambda$. Furthermore, $h^\vee = 2$.

The following lemma follows from [13, Corollary 3.5.2] by using elementary facts about the representation theory of $sl_2$.

Lemma 4.1.

1. For any $\ell$ and $\lambda$, $r_\lambda = 0$ if $\sum_i \lambda_i$ is odd, and $V_\lambda$ is a trivial bundle, hence has trivial determinant, if $\sum_i \lambda_i \leq 2\ell$.
2. For $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, with $\lambda_1 \leq \lambda_2 \leq \lambda_3$ and all $\lambda_i$ even we have

$$
r_\lambda = \begin{cases} 
1 & \text{if } \lambda_3 \leq \lambda_1 + \lambda_2 \text{ and } 2\ell \geq \sum_i \lambda_i/2, \\
0 & \text{otherwise}.
\end{cases}
$$

4.2. The case $\ell = 1$. If $\ell = 1$, $P_1 = \{0,1\}$ so there are $2^n$ bundles of conformal blocks on $\overline{M}_{0,n}$. However, of these the ones corresponding to an odd number of 1’s are 0 by Lemma 4.1 and the ones with two 1’s are trivial bundles of rank 1. Thus the maximal number of non-trivial determinants that one can possibly get is $2^{n-1} - \binom{n}{2} - 1$, which is the same as the rank of $\text{Pic}(\overline{M}_{0,n})$. In fact, one has the following

Theorem 4.2. For any $n \geq 4$, the set of non-trivial determinants of conformal blocks of level $\ell = 1$ for $sl_2$ form a basis of $\text{Pic}(\overline{M}_{0,n})_{\mathbb{Q}}$.

Proof. We use induction on $n$.

Suppose $n = 4$. Then the only possibly non-trivial determinant is for $\lambda = (1,1,1,1)$. From factorisation, and the fact that $r_{(1,1,0)} = 1$, it follows that $r_\lambda = 1$. Moreover, $c(0) = 0$, $c(1) = 3/2$ and $r_{(1,1,1)} = 0$, so applying the formula (3.9) we get that $\deg(V_\lambda) = \frac{1}{2(1+2)}(4 \times \frac{3}{2} - 0) = 1$, proving the first step of the induction.
Now let $n > 4$. Let $Q$ be the quotient of $\text{Pic}(\overline{M}_{0,n})$ by the subspace $P'$ generated by $p_i^*(\text{Pic}(\overline{M}_{0,n-1}))$, $i = 1, 2, \ldots, n$, where the $p_i$ are forgetting maps as before.

Recall from [7] that for any $n \geq 4$, $\text{Pic}(\overline{M}_{0,n})$ is generated by the classes of the boundary divisors which are parametrised by partitions $\{1, \ldots, n\} = A \cup B$ with $|A|, |B| \geq 2$. We denote the corresponding divisor by $D_{A,B}$, so we have $D_{A,B} = D_{B,A}$. For a partition $\{1, 2, \ldots, \hat{i}, \ldots, n\} = A' \cup B'$ corresponding to the boundary divisor $D_{A',B'}$ on $\overline{M}_{0,n-1}$ (with points labelled by elements of $\{1, \ldots, \hat{i}, \ldots, n\}$), we have

$$
(4.1) \quad p_i^*(D_{A',B'}) = D_{A' \cup \{i\},B'} + D_{A',B' \cup \{i\}}.
$$

Let $A = \{1, 2, \ldots, r\}$ and $B = \{r + 1, r + 2, \ldots, n\}$ with $2 \leq r \leq n/2$. By switching elements of $A$ and the first $r$ elements of $B$ in pairs using equation (4.1) we get

$$D_{\{1,2,\ldots,r\},\{r+1,r+2,\ldots,n\}} = D_{\{r+1,r+2,\ldots,2r\},\{1,2,\ldots,r+1,\ldots,n\}}$$

in $Q$. Then moving $2r + 2r + 2, \ldots, n$ to the first set in the new partition and using the same equation we get

$$D_{\{r+1,r+2,\ldots,2r\},\{1,2,\ldots,r+1,\ldots,n\}} = (-1)^{n-2r}D_{\{r+1,r+2,\ldots,n\},\{1,2,\ldots,r\}} = (-1)^nD_{\{1,2,\ldots,r\},\{r+1,r+1,\ldots,n\}}$$

in $Q$. If $n$ is odd, it follows that $D_{A,B} = 0$ in $Q$. Since the symmetric group $S_n$ acts on $Q$, it follows that $Q = 0$ in this case. If $n$ is even, a similar argument shows that $Q$ has rank 1, generated by $D_{\{1,2,\ldots,n/2\},\{n/2+1,n/2+2,\ldots,n\}}$.

As we observed earlier, if some $\lambda_i = 0$ then the bundle of conformal blocks, hence also the determinant, is in $P'$. To complete the proof it remains to show that $\mathbb{D}_{(1,1,\ldots,1)}$ generates $Q$ if $n$ is even.

Suppose not, so $\mathbb{D}_{(1,1,\ldots,1)}$ lies in $P'$. For even $r$, $4 \leq r \leq n$, let $\mathbb{D}_r$ be the sum of all the determinants of conformal blocks of level 1 with $r$ of the $\lambda_i$ equal to 1. Since $\mathbb{D}_n$ is preserved by the action of $S_n$, it follows by averaging that we must have a linear relation

$$\sum_{i=2}^{n/2} a_i \mathbb{D}_{2i} = 0$$

with $a_i \in \mathbb{Q}$.

Let $\iota : M_{0,n-1} \to \overline{M}_{0,n}$ be the morphism corresponding to attaching a 3-pointed $\mathbb{P}^1$ to the last marked point. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-2}, 0)$ with $\lambda_i \in \{0,1\}$ and $\sum \lambda_i = 2r \geq 4$. The coefficient of $\mathbb{D}_\lambda$ in $\iota^*(\sum_{i=2}^{n/2} a_i \mathbb{D}_{2i})$, which is well defined by induction, must be zero. By the factorisation formula this coefficient is equal to $a_i + a_{i+1}$ so we must have

$$a_i + a_{i+1} = 0, 2 \leq i \leq n/2 - 1.$$
It follows that $\mathbb{D}_r \cdot F = \binom{n-3}{r-3}$ for $r$ even, $4 \leq r \leq n$. Since $\sum_{i=2}^{n/2} a_i \mathbb{D}_{2i} \cdot F = 0$, by putting together the two relations obtained so far we get

$$f(n) := \binom{n-3}{n-3} - \binom{n-3}{n-5} + \binom{n-3}{n-7} \cdots + (-1)^{n/2} \binom{n-3}{1} = 0.$$  

However, by the binomial theorem, $2f(n) = (1 + \sqrt{-1})^n + (1 - \sqrt{-1})^n$, which is clearly non-zero. This contradiction completes the proof of the theorem. \hfill $\Box$

Remark 4.3. The basis of Theorem has several nice properties which are clear from the construction: All the elements are nef line bundles and the basis is preserved by the action of the symmetric group. In fact, the basis is compatible with all natural morphisms among the $\overline{M}_{0,n}$'s including the forgetting and gluing morphisms. However, even though our basis is contained in Pic($\overline{M}_{0,n}$), it does not form an integral basis for $n > 4$ as may be seen by an explicit computation.

4.3. The critical level and GIT quotients. For $\overline{\lambda} = (\lambda_1, \ldots, \lambda_n)$ we call $\ell = (\sum_i \lambda_i/2) - 1$ the critical level associate to $\overline{\lambda}$. This is the largest possible value for the level so that the bundle $\mathbb{V}_{\overline{\lambda}}$ is not trivial. In this section we identify the line bundles $\mathbb{D}_{\overline{\lambda}}$ for the critical level and with all $\lambda_i > 0$ with pullbacks of ample line bundles on the GIT quotients $(\mathbb{P}^1)^n \sslash SL_2$, where the polarisation on $(\mathbb{P}^1)^n$ is given by the line bundle $\mathcal{O}(\lambda_1) \boxtimes \cdots \boxtimes \mathcal{O}(\lambda_n)$. We will do this by comparing the degrees of the $\mathbb{D}_{\overline{\lambda}}$ on vital curves with the degrees of the GIT bundles for which a formula has been given by Alexeev–Swinarski [1].

We first prove the following

Lemma 4.4. Suppose $n = 4$, $\overline{\lambda} = (\lambda_1, \ldots, \lambda_4)$ with $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ and $\ell$ the critical level. Then

$$\deg(\mathbb{D}_{\overline{\lambda}}) = \begin{cases} \ell + 1 - \lambda_4 & \text{if } \lambda_1 + \lambda_4 \geq \lambda_2 + \lambda_3, \\ \lambda_1 & \text{if } \lambda_1 + \lambda_4 \leq \lambda_2 + \lambda_3. \end{cases}$$

Proof. We will use our key formula (3.9) to compare the degrees at the critical level and the level $\ell' = \sum_i \lambda_i/2$. For $\ell'$ the bundle is trivial, so the degree is zero. Thus for this level

$$(4.2) \quad r_{\overline{\lambda}}\left( \sum_i c(\lambda_i) \right) - \sum_{\lambda \in P_{\ell'}} c(\lambda)(r_{(\lambda_1,\lambda_2,\lambda)} \cdot r_{(\lambda_3,\lambda_4,\lambda^*)} + r_{(\lambda_1,\lambda_3,\lambda)} \cdot r_{(\lambda_2,\lambda_4,\lambda^*)} + r_{(\lambda_1,\lambda_4,\lambda)} \cdot r_{(\lambda_2,\lambda_3,\lambda^*)}) = 0.$$  

Note that for $sl_2$, $\lambda = \lambda^*$.

We compare the terms occurring in (4.2) with the corresponding terms for level $\ell$. By the factorisation formula, one sees that $r_{\overline{\lambda}_{\ell'}} = r_{\overline{\lambda}_{\ell}} + 1$. From the properties of the 3-point ranks, it follows that for each of the three partitions there is exactly one extra non-zero summand in the second sum for level $\ell'$ compared to the critical level. Thus the new terms in level $\ell'$ must correspond to a case when $\lambda_1 + \lambda_j + \lambda = 2\ell'$ i.e. $\ell' = \lambda_k + \lambda_l$ for some $k, l$.

Consider the first case. It follows from the inequalities that here we must have $\lambda = \lambda_1 + \lambda_2$ for the first term, $\lambda = \lambda_1 + \lambda_3$ for the second term and $\lambda = \lambda_2 + \lambda_3$ for the third term. Since
the sum for $\ell'$ is 0, it follows that the sum for the critical level is equal to
\begin{align}
(4.3) \quad c(\lambda_1 + \lambda_2) + c(\lambda_1 + \lambda_3) + c(\lambda_2 + \lambda_3) - \sum_i c(\lambda_i) \\
= (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)/2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + 2(l_1 + l_2 + l_3) - \lambda_4^2 - \lambda_4 \\
= 1/2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2)(\lambda_1 + \lambda_2 + \lambda_3 - \lambda_4).
\end{align}

To get the degree, we must divide this by $2(\ell + h^\vee)$. Since $\ell = \sum_i \lambda_i/2 - 1$ and $h^\vee = 2$ we get the desired formula.

The second case is entirely analogous, so we omit the details.

For $\bar{\lambda}$, with none of the $\lambda_i = 0$, let $p_{\bar{\lambda}} : \overline{M}_{0,n} \to (\mathbb{P}^1)^n // SL_2$ denote the morphism constructed by Kapranov [6], where the GIT quotient is constructed using the polarisation corresponding to $\bar{\lambda}$.

**Theorem 4.5.** For the critical level $\ell = \sum_i \lambda_i/2 - 1$, $\mathbb{D}_{\bar{\lambda}}$ is a multiple of the pullback by $p_{\bar{\lambda}}$ of the canonical ample line bundle on $(\mathbb{P}^1)^n // SL_2$. Furthermore, $\mathbb{V}_{\bar{\lambda}}$ is the pullback of a vector bundle on $(\mathbb{P}^1)^n // SL_2$.

**Proof.** We first show that the degree of $\mathbb{D}_{\bar{\lambda}}$ on any vital curve $F$ is a fixed multiple of the degree of the GIT bundle on this curve. This suffices for the first part since Pic$(\overline{M}_{0,n})$ is finitely generated and torsion free.

Let $F$ correspond to the partition $\{1, \ldots, n\} = N_1 \cup N_2 \cup N_3 \cup N_4$. For $j = 1, \ldots, 4$, let $\nu_j = \sum_{k \in N_j} \lambda_k$. Let $\nu_{\max} = \max_j \{\nu_j\}$ and $\nu_{\min} = \min_j \{\nu_j\}$. From Proposition 2.5 we have
\[
\deg(\mathbb{D}_{\bar{\lambda}})|_F = \sum_{\pi \in P_i^4} \deg(\mathbb{V}_{\pi}) \prod_{j=1}^4 r_{\nu_j}^*.
\]
Note that $\mu_j = \mu_j^*$ since $g = sl_2$. To get a non-zero summand, each of its factors must be non-zero. Considering the ranks, we see from Lemma 4.1 that this implies that for each non-zero summand we must have $\mu_j \leq \nu_j$ for all $k$. Now considering the term $\deg(\mathbb{V}_{\pi})$ and applying Lemma 4.1 again, we see that we must have $\sum_j \mu_j \geq 2\ell + 2$. Since $2\ell = \sum_i \lambda_i - 2 = \sum_j \nu_j - 2$, it follows that all the inequalities must be equalities. Thus there is only one possibly non-zero summand corresponding to $\mu_j = \nu_j$, $j = 1, \ldots, 4$. In this summand, since $\ell \geq 2\nu_j$ for all $j$, by applying the factorisation formula one sees that $r_{\nu_j} = 1$ for all $j$.

We now apply Lemma 4.4. It follows that
\begin{align}
(4.4) \quad \deg(\mathbb{D}_{\bar{\lambda}})|_F = \begin{cases} 
0 & \text{if } \nu_{\max} \geq \ell + 1, \\
\ell + 1 - \nu_{\max} & \text{if } \nu_{\max} \leq \ell + 1 \text{ and } \nu_{\max} + \nu_{\min} \geq \ell + 1, \\
\nu_{\min} & \text{if } \nu_{\max} \leq \ell + 1 \text{ and } \nu_{\max} + \nu_{\min} \leq \ell + 1. 
\end{cases}
\end{align}
This is exactly the same, up to scaling, as the formula of Alexeev–Swinarski [1, Lemma 2.2]. Since the scaling factor is independent of the specific vital curve $F$, the statement about $\mathbb{D}_{\bar{\lambda}}$ follows.

Since $\mathbb{V}_{\bar{\lambda}}$ is generated by its global sections (Lemma 2.2), there exists a morphism $f_{\bar{\lambda}} : \overline{M}_{0,n} \to \text{Gr}_{\bar{\lambda}}$, where $\text{Gr}_{\bar{\lambda}}$ is a grassmannian, such that $\mathbb{V}_{\bar{\lambda}}$ is isomorphic to $f_{\bar{\lambda}}^*$ of the tautological vector bundle on $\text{Gr}_{\bar{\lambda}}$. Since $\mathbb{D}_{\bar{\lambda}}$ is the determinant of $\mathbb{V}_{\bar{\lambda}}$ and the GIT quotient $(\mathbb{P}^1)^n // SL_2$ is normal, it follows that we have $f_{\bar{\lambda}} = \phi_{\bar{\lambda}} \circ p_{\bar{\lambda}}$ for some morphism $\phi_{\bar{\lambda}} : (\mathbb{P}^1)^n // SL_2 \to \text{Gr}_{\bar{\lambda}}$. Thus $\mathbb{V}_{\bar{\lambda}}$ is $p_{\lambda}^*$ of $q_{\lambda}$ of the tautological vector bundle. □
4.4. Relationship with the moduli spaces of weighted pointed curves. For any $g \geq 0$, Hassett [5] has constructed moduli spaces of weighted pointed stable curves. For $g = 0$, the spaces $\overline{M}_{0,\A}$ depend on a choice of weight data $\A = (a_1, a_2, \ldots, a_n)$ satisfying $0 < a_i \leq 1$ for all $i$ and $\sum_i a_i > 2$. There are canonical birational morphisms $p_\A : \overline{M}_{0,n} \to \overline{M}_{0,\A}$.

**Lemma 4.6.** The morphisms $p_\A$ are compositions of extremal contractions (in fact smooth blowdowns) corresponding to images of classes of vital curves.

The following proof was communicated to us by Valery Alexeev.

**Proof.** For any $S \subset \{1, 2, \ldots, n\}$ such that $2 < |S| < n - 2$ Hassett defines an associated wall in the space $\mathcal{D}_{0,n}$ of allowable weight data given by the equation $\sum_{i \in S} a_i = 1$. The set of all such walls induces a decomposition of $\mathcal{D}_{0,n}$ called the coarse chamber decomposition.

For each $i$ let $\epsilon_i$ be such that $0 < \epsilon_i << 1$, let $a'_i = a_i - \epsilon_i$ and $\A' = (a'_1, a'_2, \ldots, a'_n)$. Then there is a natural morphism $p_{\A,\A'} : \overline{M}_{0,\A} \to \overline{M}_{0,\A'}$ such that $p_{\A,\A'} \circ p_\A = p_\A'$ which is an isomorphism if the $\epsilon_i$ are sufficiently small. Thus we may assume that the $a_i$'s are general $i.e.$ the line segment joining $(a_1, a_2, \ldots, a_n)$ and $(1, 1, \ldots, 1)$ intersects the walls of the coarse chamber decomposition in only codimension one faces. It therefore suffices to show that each such crossing corresponds to the smooth blowdown associated to the image of a vital curve. This follows from Proposition 4.5 of [5], the vital curve may be taken to be weight data in the sense of Hassett, thus giving rise to a moduli space $\overline{M}_{0,\A}$ and birational morphism $p_\A : \overline{M}_{0,n} \to \overline{M}_{0,\A}$ as above.

For non-critical levels, the bundles $\mathbb{D}_\A$ are often ample, so are not pulled back from the GIT quotients. However, we have

**Proposition 4.7.** $\mathbb{V}_\A$ is the pullback by $p_{\A,\A}$ of a vector bundle on $\overline{M}_{0,\A}$.

**Proof.** We first show that $\mathbb{D}_\A$ is the pullback by $p_{\A,\A'}$ of a line bundle on $\overline{M}_{0,\A'}$. By Lemma 4.6 it suffices to show that $\text{deg}(\mathbb{D}_\A)|_F = 0$ for any vital curve $F$ contracted by $p_{\A,\A'}$. Let $F$ corresponds to the partition $\{1, 2, \ldots, n\} = N_1 \cup N_2 \cup N_3 \cup N_4$ and let $b_j = \sum_{k \in N_j} a_k$. The morphism $p_{\A,\A'}$ collapses $F$ if and only if $(\sum b_j) - b_j < 1$ for some $j \in \{1, 2, 3, 4\}$. Equivalently, setting $\nu_j = \sum_{k \in N_j} \lambda_k$, if and only if

$$\sum_j \nu_j - \nu_j' \leq \ell \text{ for some } j' \in \{1, 2, 3, 4\}. \quad (4.5)$$

We now apply Proposition 2.5. If a tuple $\overline{\nu} \in P^4_{\ell}$ is to contribute a non-zero summand, all the ranks have to be non-zero so we must have $\mu_j \leq \nu_j$ for all $j = 1, \ldots, 4$. Thus equation (4.5) implies that

$$\sum_j \mu_j - \mu_j' \leq \ell \text{ for some } j' \in \{1, 2, 3, 4\}. \quad (4.6)$$

Since $\mu_j \leq \ell$ as well, we get $\sum_j \mu_j \leq 2\ell$. It follows from Lemma 4.1 that $\text{deg}(\mathbb{D}_\overline{\nu}) = 0$.

Thus all the summands in the formula for $\text{deg}(\mathbb{D}_\overline{\nu}|_F)$ are 0, hence $\text{deg}(\mathbb{D}_\overline{\nu}|_F) = 0$. 


The statement about $V^*_\chi$ now follows in the same way as the corresponding statement in Theorem 4.5 since $\overline{M}_{0,n}^\chi$ is smooth, hence also normal. \hfill \Box

Remark 4.8. It is not true in general that $D_\chi$ is the pullback by $p_{\chi}$ of an ample line bundle on $\overline{M}_{0,\chi}$: Let $n = 6$, $\ell = 1$ and $\chi = (1, 1, 1, 1, 1, 1)$. In this case $D_\chi$ has degree 0 on any vital curve corresponding to the partition $6 = 1 + 1 + 2 + 2$ and the corresponding morphism is the well known birational morphism from $\overline{M}_{0,6}$ to the Igusa quartic. However, the morphism $p_{\chi}: \overline{M}_{0,6} \to \overline{M}_{0,\chi}$ is an isomorphism since all $a_i = 1/2$.

4.5. The case $n = 4$.

Proposition 4.9. Suppose $n = 4$, $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4$ and $\sum_i \lambda_i$ is even. Then for any level $\ell \geq \lambda_4$ we have

$$\deg(D_\chi) = \begin{cases} 
\max\{0, (\ell + 1 - \lambda_4)(\sum_i \lambda_i/2 - \ell)\} & \text{if } \lambda_1 + \lambda_4 \geq \lambda_2 + \lambda_3, \\
\max\{0, (\ell + 1 + \lambda_1 - \sum_i \lambda_i/2)(\sum_i \lambda_i/2 - \ell)\} & \text{if } \lambda_1 + \lambda_4 \leq \lambda_2 + \lambda_3.
\end{cases}$$

Proof. This can be proved by a tedious and unenlightening computation using the formula (3.9). We skip it (for now). \hfill \Box

5. Level 1 for any $\mathfrak{g}$

In this section we make some remarks about the level 1 conformal blocks for any simple Lie algebra $\mathfrak{g}$.

5.1. The following is the list of level 1 representations for the simple Lie algebras. We use the notation of Bourbaki.

- $A_\ell$: all fundamental weights; $\varpi_i$ is dual to $\varpi_{\ell+1-i}$.
- $B_\ell$: $\varpi_1$ and $\varpi_\ell$ i.e. the standard representation and the spin representation; these representations are self dual.
- $C_\ell$: all fundamental weights; these are all self dual.
- $D_\ell$: $\varpi_1$, $\varpi_{\ell-1}$ and $\varpi_\ell$ i.e. the standard representation and both the spin representations; the first representation is self dual and the other two are self dual if $\ell$ is even and dual if $\ell$ is odd.
- $E_6$: $\varpi_1$ and $\varpi_6$; these representations are dual.
- $E_7$: $\varpi_7$; this is self dual.
- $E_8$: There are no level 1 representations.
- $F_4$: $\varpi_4$; this is self dual.
- $G_2$: $\varpi_1$; this is self dual.

5.2.

5.2.1. Recall (see e.g. [13, Corollary 3.5.2]) that for a simple Lie algebra $\mathfrak{g}$, any level $\ell$ and $\lambda, \mu \in P_\ell$, we have $r_{(\lambda, \mu, 0)} = 1$ if $\mu = \lambda^*$ and 0 otherwise. Moreover, for $\lambda_1, \lambda_2, \lambda_3$ of level 1, $V_{(\lambda_1, \lambda_2, \lambda_3)}$ is the quotient of $(V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3})_{\mathfrak{g}}$ by the image of the subspace $V_{\lambda_1}^{(1)} \otimes V_{\lambda_2}^{(1)} \otimes V_{\lambda_3}^{(1)}$, where $V_{\lambda_i}^{(1)}$ is the $\mathfrak{sl}$-submodule of $V_{\lambda_i}$ which is the direct sum of all the non-trivial irreducible $\mathfrak{sl}$-submodules. Since $H_{\theta}$ has no covariants on this subspace, it follows that the image of this subspace in $(V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3})_{\mathfrak{g}}$ is $\{0\}$. Therefore $V_{(\lambda_1, \lambda_2, \lambda_3)} = (V_{\lambda_1} \otimes V_{\lambda_2} \otimes V_{\lambda_3})_{\mathfrak{g}}$. \hfill \Box
5.2.2. $E_7$. In this case, $(V_{\omega_7} \otimes V_{\omega_7} \otimes V_{\omega_7})_{E_7} = 0$,\textsuperscript{8} so the corresponding conformal block is also trivial. It follows that for $E_7$ the determinants of conformal blocks of level 1 are the same, up to scaling, as those for $sl_2$.

5.2.3. $E_6$. In this case, $r(\lambda,\lambda,\lambda) = 1$ for $\lambda = \omega_1$ or $\omega_6$. Since the representations are not self-dual, it follows that the conformal blocks of level 1 for $E_6$ are the same, again up to scaling, as those for $sl_3$.

5.2.4. $F_4$ and $G_2$. In both these cases the rank of the conformal blocks for $\lambda = (\omega_4, \omega_4, \omega_4)$ for $F_4$ and $\lambda = (\omega_1, \omega_1, \omega_1)$ for $G_2$ is 1. It follows that the determinants of the conformal blocks at level 1 for both these cases are equal (up to perhaps a global scalar). Moreover, for $\lambda = (\omega_4, \omega_4, \omega_4, \omega_4)$ for $F_4$ and $\lambda = (\omega_1, \omega_1, \omega_1, \omega_1)$ for $G_2$, $\deg(\mathcal{D}_\lambda) > 0$.

For any $n$ and $\lambda = (\lambda, \lambda, \ldots, \lambda)$, $\lambda = \omega_4$ or $\omega_1$ as $g = F_4$ or $G_2$, it follows from factorisation and the above that $\mathcal{V}_\lambda$ has rank $\text{Fib}(n - 1)$, where $\text{Fib}(i)$ denotes the $i$th Fibonacci number. Moreover, Proposition 2.5 implies that if a vital curve $F$ corresponds to a partition \{1, 2, \ldots, n\} = \bigcup_{k=1}^{4} N_k$, then $\deg(\mathcal{V}_\lambda)(\mathcal{V}_\lambda) = c \prod_{k=1}^{4} \text{Fib}(|N_k|)$ where $c$ is a constant (which can be determined).

5.2.5. $A_\ell$. In this case given $\lambda_1, \lambda_2$ of level 1, there exists a unique $\lambda_3$ of level 1 such that $r(\lambda_1, \lambda_2, \lambda_3) = 1$. By the factorisation formula, it follows that for $\lambda_1, \lambda_2, \ldots, \lambda_n$ of level 1, there exists a unique $\lambda_n$ of level 1 such that for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $r_\lambda \neq 0$. Moreover, it then follows that $r_\lambda = 1$. Because of this there is only one non-zero summand in the formula in Proposition 2.5 and so the computation of the determinants reduces to the computation of the degree for the case $n = 4$.

Note that
\[
c(\omega_i) = (\omega_i|\omega_i) + 2(\omega_i|\rho) = i(m - i)/m + i(m - i) = i(m - i)(m + 1)/m.
\]
where $m = \ell + 1$. Using this, the fact that $h^\nu$ for $sl_m$ is $m$ and formula (3.9), one easily checks the following:

Lemma 5.1. Let $\omega_i, \omega_j, \omega_k, \omega_l$ be fundamental dominant weights of $sl_m$ and suppose that $i \leq j \leq k \leq l$. For $\lambda = (\omega_i, \omega_j, \omega_k, \omega_l)$, we have
\[
\deg(\mathcal{D}_\lambda) = \begin{cases} 
  i & \text{if } i + j + k + l = 2m \text{ and } j + k \geq i + l, \\
  m - l & \text{if } i + j + k + l = 2m \text{ and } j + k \leq i + l, \\
  0 & \text{otherwise}.
\end{cases}
\]

From this and Proposition 2.5 we immediately get the following:

Proposition 5.2. Let $\lambda = (\omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_n})$ with $0 \leq i_j < m$ for $j = 1, 2, \ldots, n$, where $\omega_0 := 0$. Let $F$ be a vital curve in $M_{0,n}$ corresponding to a partition \{0, 1, \ldots, n\} = $\bigcup_{k=1}^{n} N_k$. Let $\nu_k$ be the representative in \{0, 1, \ldots, m - 1\} of $\sum_{j \in N_k} i_j$ modulo $m$. Let $\nu_{\max} = \max_k \{\nu_k\}$ and $\nu_{\min} = \min_k \{\nu_k\}$. Then
\[
\deg(\mathcal{D}_\lambda|_F) = \begin{cases} 
  \nu_{\min} & \text{if } \sum_k \nu_k = 2m \text{ and } \nu_{\max} + \nu_{\min} \leq m, \\
  m - \nu_{\max} & \text{if } \sum_k \nu_k = 2m \text{ and } \nu_{\max} + \nu_{\min} \geq m, \\
  0 & \text{otherwise}.
\end{cases}
\]

\textsuperscript{8}This and other such computations mentioned below have been done using GAP.
Remark 5.3. Comparing this formula with the formula for the degrees for $sl_2$ and the critical level (4.4), we see that the critical level $\ell$ determinants for $sl_2$ are a special case for the level 1 determinants for $sl_{\ell+1}$. More precisely, given an $n$-tuple of non-negative integers $(i_1, i_2, \ldots, i_n)$ such that $\sum_j i_j$ is even, we let $\ell = \sum_j i_j/2 - 1$. Then the determinant of the bundle of conformal blocks on $\overline{M}_{0,n}$ associated to $sl_2$ with weights $(i_1, i_2, \ldots, i_n)$ and level $\ell$ is isomorphic to the bundle of conformal blocks (it is already of rank 1) on $\overline{M}_{0,n}$ associated to $sl_{\ell+1}$ with weights $(\overline{w}_{i_1}, \overline{w}_{i_2}, \ldots, \overline{w}_{i_n})$ and level 1.

We conclude this paper by discussing a couple of natural questions concerning the bundles of conformal blocks.

Question 6.1. Given a simple Lie algebra $g$ and an integer $n \geq 4$, is the closure of the subcone of $N^1(\overline{M}_{0,n})$ generated by determinants of conformal blocks for $g$ and all levels $\ell$ finitely generated? If so, is there an algorithm for computing this cone?

We do not know what to expect. For $n = 5$ the determinants of conformal blocks for $sl_2$ generate the nef cone. This does not appear to hold for $n = 6$, in which case computer calculations suggest that cone generated by conformal blocks for $sl_2$ has 128 vertices, 127 coming from the critical level and the remaining vertex corresponding to $\lambda = (1, 1, 1, 1, 1, 1)$ and level 1.

Question 6.2. Given an integer $n$, do the determinants of conformal blocks for all simple Lie algebras and all levels $\ell$ generate the nef cone of $\overline{M}_{0,n}$?

We do not know if this is true for any $n \geq 6$. However, for $n = 6$ the cone generated by conformal blocks for both $sl_2$ and $sl_3$ appears to strictly contain the cone for only $sl_2$.

One may ask similar questions for $\overline{M}_{0,n}/S_n$ and also for $\overline{M}_{g,n}$ for $g > 0$ (in which case we do not yet know whether the determinants are nef c.f. Remark 2.3).

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