Exponentially-fitted forth-order explicit modified Runge-Kutta type method for solving third-order ODEs

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Abstract. In this paper exponentially-fitted explicit modified Runge-Kutta type method denoted as EFMRKT for solving \( y'''(x) = f(x, y, y') \) is derived. The idea presented is based on the Simos and Berghe approach which exactly integrates initial value problems whose solutions are linear combinations of the set functions \( e^{wx} \) and \( e^{-wx} \) with \( w \in \mathbb{R} \) the principal frequency of the problem. We developed the new EFMRKT three-stage fourth-order method called EFRKGTG4 for solving third-order initial value problems. The numerical results indicate that EFRKGTG4 method is more efficient than existing Runge-Kutta methods.

1. Introduction

This work deals with exponentially-fitted explicit modified Runge-Kutta type methods for solving third-order ordinary differential equations (ODEs)

\[
y'''(x) = f(x, y(x), y'(x)), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad y''(x_0) = y''_0, \quad x \geq x_0.
\] (1)

In the past and recent years many researchers constructed exponentially-fitted and trigonometrically-fitted explicit Runge-Kutta methods for solving first-order and second-order ordinary differential equations. Paternoster [1] developed Runge-Kutta-Nyström methods for ODEs with periodic solutions based on trigonometric polynomials. Vanden Berghe et al. [2] constructed exponentially-fitted Runge-Kutta methods. Simos [3] extended exponentially-fitted Runge-Kutta methods for the numerical solution of the Schrödinger equation and related problems. Kalogiratou et al. [4, 5] constructed trigonometrically and exponentially-fitted Runge-Kutta-Nyström methods for the numerical solution of the Schrödinger equation and related problems which is eighth algebraic order. Simos et al. [6] constructed exponentially-fitted Runge-Kutta-Nyström method for the numerical solution of initial-value problems with oscillating solutions. Sakas et al. [7] developed a fifth algebraic order trigonometrically-fitted modified Runge-Kutta Zonneveld method for the numerical solution of orbital problems. Van de Vyver [8] constructed Runge-Kutta-Nyström pair for the numerical integration of perturbed oscillators. Yang et al. [9] constructed trigonometrically-fitted adapted Runge-Kutta-Nyström methods for
perturbed oscillators. Demba et al. [10] constructed an explicit trigonometrically-fitted Runge-Kutta-Nyström method using Simos technique.

In this paper we construct exponentially-fitted explicit modified Runge-Kutta type three-stage fourth-order method denoted as EFRKTG4. In section 2 we give the necessary conditions for exponentially-fitted explicit modified Runge-Kutta type method for solving third-order ODEs. The derivation of new method is given in section 3. The effectiveness of the new EFMRKT method when compared with existing methods are given in section 4.

2. Exponentially-fitted MRKT method

In this section, we will construct exponentially-fitted MRKT method. In order to construct the exponentially fitted MRKT method we introduce the extra \( \gamma_i \) and \( \hat{\gamma}_i \) at each stage then the MRKT method is given by

\[
y_{n+1} = y_n + hy'_n + \frac{h^2}{2} y''_n + h^3 \sum_{i=1}^{s} b_i k_i,
\]

\[
y'_{n+1} = y'_n + hy''_n + h^2 \sum_{i=1}^{s} b'_i k_i,
\]

\[
y''_{n+1} = y''_n + h \sum_{i=1}^{s} b''_i k_i.
\]

where

\[
k_1 = f(x_n, y_n, y'_n),
\]

\[
k_i = f \left( x_n + c_i h, \gamma_i y_n + h c_i y'_n + \frac{h^2}{2} c^2_i y''_n + h^3 \sum_{j=1}^{i-1} a_{ij} k_j, y'_n + \hat{\gamma}_i h c_i y''_n + h^2 \sum_{j=1}^{i-1} \hat{a}_{ij} k_j \right)
\]

for \( i = 2, 3, \ldots, s \).

The parameters of the MRKT methods are \( c_i, a_{ij}, \hat{a}_{ij}, b_i, b'_i, b''_i, \gamma_i \) and \( \hat{\gamma}_i \) for \( i = 1, 2, \ldots, s \) and \( j = 1, 2, \ldots, s \) are assumed to be real. If \( a_{ij} = 0 \) and \( \hat{a}_{ij} = 0 \) for \( i \leq j \), it is an explicit method and otherwise implicit method.

The MRKT method can be expressed in Butcher notation using the table of coefficients as follows (see Table 1).

| Table 1. The Butcher tableau MRKT method |
|---|---|---|---|
| \( c \) | \( \gamma \) | \( \hat{\gamma} \) | \( A \) |
| \( A \) | \( \hat{A} \) | \( b \) | \( b' \) | \( b'' \) |

3. Exponentially-fitted MRKT method

To construct the exponentially-fitted Runge-Kutta type three-stage fourth-order method needs at each stage to integrate exactly the function \( exp(wx) \) and \( exp(-wx) \), therefore the following four equations are obtained.

\[
e^{\pm c_i v} = \gamma_i \pm c_i v + \frac{1}{2} c^2_i v^2 \pm v^3 \sum_{j=1}^{s} a_{ij} e^{ \pm c_j v}, \tag{2}
\]
\[ e^{\pm c_i v} = 1 \pm \hat{\gamma}_i c_i v \pm v^2 \sum_{j=1}^{s} \hat{a}_{ij} e^{\pm c_j v}, \] (3)

and six more equations corresponding to \( y, y' \) and \( y'' \)

\[ e^{\pm v} = 1 \pm \frac{1}{2} v^2 \pm v^3 \sum_{i=1}^{s} b_i e^{\pm c_i v}, \] (4)

\[ e^{\pm v} = 1 \pm v^2 \sum_{i=1}^{s} b_i e^{\pm c_i v}, \] (5)

\[ e^{\pm v} = 1 \pm v \sum_{i=1}^{s} b_i e^{\pm c_i v}. \] (6)

where \( v = wh, w \in \mathbb{R} \). Using relations: \( \cosh(x) = \frac{e^x + e^{-x}}{2}, \ \sinh(x) = \frac{e^x - e^{-x}}{2} \).

The following order conditions are obtained:

\[ \cosh(v c_i) = \gamma_i + \frac{1}{2} v^2 c_i^2 + v^3 \sum_{j=1}^{i-1} a_{ij} \sinh(v c_j), \] (7)

\[ \sinh(v c_i) = v c_i + v^3 \sum_{j=1}^{i-1} a_{ij} \cosh(v c_j), \] (8)

\[ \cosh(v c_i) = 1 + v^2 \sum_{j=1}^{i-1} \hat{a}_{ij} \cosh(v c_j), \] (9)

\[ \sinh(v c_i) = \gamma_i c_i v + v^2 \sum_{j=1}^{i-1} \hat{a}_{ij} \sinh(v c_j), \] (10)

and six equations corresponding to \( y, y' \) and \( y'' \).

\[ \cosh(v) = 1 + \frac{1}{2} v^2 + v^3 \sum_{i=1}^{s} b_i \sinh(v c_i), \] (11)

\[ \sinh(v) = v + v^3 \sum_{i=1}^{s} b_i \cosh(v c_i), \] (12)

\[ \cosh(v) = 1 + v^2 \sum_{i=1}^{s} b_i \cosh(v c_i), \] (13)

\[ \sinh(v) = v + v^3 \sum_{i=1}^{s} b_i \sinh(v c_i), \] (14)

\[ \cosh(v) = 1 + v \sum_{i=1}^{s} b_i'' \sinh(v c_i), \] (15)

\[ \sinh(v) = v \sum_{i=1}^{s} b_i'' \cosh(v c_i). \] (16)

Solving Eqs (7) to (10) and we find \( a_{i,i-1}, \hat{a}_{i,i-1}, \gamma_i \) and \( \hat{\gamma}_i \).
\[
\gamma_i = \cosh(v c_i) - \frac{1}{2} v^2 c_i^2 - v^3 \sum_{j=1}^{i-1} a_{i,j} \sinh(v c_j), \quad (17)
\]
\[
a_{i,i-1} = \frac{\sinh(v c_i) - v c_i - v^3 \sum_{j=1}^{i-2} a_{i,j} \cosh(v c_j)}{v^2 \cosh(v c_{i-1})}, \quad (18)
\]
\[
\hat{a}_{i,i-1} = \frac{\cosh(v c_i) - 1 - v^2 \sum_{j=1}^{i-2} \hat{a}_{i,j} \cosh(v c_j)}{v^2 \cosh(v c_{i-1})}, \quad (19)
\]
\[
\hat{\gamma}_i = \frac{\sinh(v c_i) - v^2 \sum_{j=1}^{i-2} \hat{a}_{i,j} \sinh(v c_j)}{v c_i}, \quad i = 2, \ldots, s. \quad (20)
\]

4. The derivation of the new exponentially-fitted modified Runge-Kutta type method

In this section we set the values of parameters \( c_1 = 0 \), \( c_2 = \frac{1}{2} \), \( c_3 = 1 \), \( a_{21} = \frac{3}{100} \), \( a_{31} = 0 \), \( \hat{a}_{31} = 0 \), \( b_3 = -\frac{1}{120} \), \( b_3' = 0 \), \( b_3'' = \frac{1}{8} \), \( b_3''' = \frac{1}{2} \).

Which fourth-order three stages method developed by Firas et al[14], into above equations.

Solving the eqs (17) to (20) by Maple and letting \( a_{32} \), \( \hat{a}_{21} \), \( a_{32} \), \( \gamma_2 \), \( \gamma_3 \), \( \hat{\gamma}_2 \) and \( \hat{\gamma}_3 \) as free parameters yield.

\[
\hat{a}_{21} = \frac{\cosh\left(\frac{v}{2}\right)-1}{v^2 \cosh\left(\frac{v}{2}\right)}, \quad \hat{a}_{32} = \frac{\cosh(v)-1}{v^2 \cosh\left(\frac{v}{2}\right)}, \quad a_{32} = \frac{\sinh(v)-v}{v^3 \cosh\left(\frac{v}{2}\right)}.
\]

\[
\gamma_2 = \cosh\left(\frac{v}{2}\right) - \frac{v^2}{8}, \quad \gamma_3 = \cosh\left(v\right) - \frac{v^2}{2} - \frac{13 v^3}{100} \sinh\left(\frac{v}{2}\right),
\]
\[
\hat{\gamma}_2 = 2 \frac{\sinh\left(\frac{v}{2}\right)}{v}, \quad \hat{\gamma}_3 = \frac{\sinh(v)-\frac{1}{2} \sinh\left(\frac{v}{2}\right)v^2}{v}.
\]

Solving the eqs (11) to (16) and using the above coefficients to find \( b_1 \), \( b_2 \), \( b_1' \), \( b_2' \) and \( b_2'' \).

\[
b_1 = -\left(\cosh\left(\frac{v}{2}\right)\sinh\left(v\right) - \cosh\left(v\right)\sinh\left(\frac{v}{2}\right)\right) \frac{1}{120 \sinh\left(\frac{v}{2}\right)}
\]
\[
+ \frac{-2 \cosh\left(\frac{v}{2}\right) \cosh\left(v\right) + 2 \cosh\left(\frac{v}{2}\right) v^2 + 2 \sinh\left(v\right) \sinh\left(\frac{v}{2}\right) - 2 v \sinh\left(\frac{v}{2}\right)}{2 v^3 \sinh\left(\frac{v}{2}\right)},
\]
\[
b_2 = \frac{1}{120 \sinh\left(\frac{v}{2}\right)} \frac{\sinh\left(v\right) - 1 - 2 \cosh\left(v\right) + 2 + v^2}{-\frac{1}{2} v^3 \sinh\left(\frac{v}{2}\right)},
\]
\[
b_2' = -\frac{\cosh\left(\frac{v}{2}\right) \sinh\left(v\right) - \cosh\left(v\right) \sinh\left(\frac{v}{2}\right) + \sinh\left(\frac{v}{2}\right)}{v^2 \sinh\left(\frac{v}{2}\right)},
\]
\[
b_2'' = \frac{1}{6} \frac{\cosh\left(\frac{v}{2}\right) v \sinh\left(v\right) - \cosh\left(\frac{v}{2}\right) \cosh\left(v\right) + \cosh\left(\frac{v}{2}\right) - \frac{1}{6} \cosh\left(v\right) \sinh\left(\frac{v}{2}\right) + \sinh\left(v\right) \sinh\left(\frac{v}{2}\right)}{v \sinh\left(\frac{v}{2}\right)},
\]
\[
b_2''' = -\frac{1 + \frac{v}{6} \sinh\left(v\right) - \cosh\left(v\right)}{v \sinh\left(\frac{v}{2}\right)}
\]

These lead to our new method exponentially-fitted explicit modified Runge-Kutta type method three-stage fourth-order method denoted as (EFRKTEG4) .
The corresponding Taylor series expansion of the solution is given by

\[ b_1 = \frac{3}{40} - \frac{1}{720} v^2 + \frac{1}{30240} v^4 - \frac{1}{1209600} v^6 + \frac{1}{47900160} v^8 - \frac{691}{1307674368000} v^{10} + \ldots, \]

\[ b_2 = \frac{1}{10} + \frac{1}{720} v^2 + \frac{19}{21920} v^4 - \frac{13}{30370400} v^6 + \frac{689}{3056102400} v^8 - \frac{35059}{66992928641600} v^{10} + \ldots, \]

\[ b_1' = \frac{1}{6} - \frac{1}{360} v^2 + \frac{1}{15120} v^4 - \frac{1}{604800} v^6 + \frac{1}{23950080} v^8 - \frac{691}{653837184000} v^{10} + \ldots, \]

\[ b_2' = \frac{1}{3} + \frac{1}{360} v^2 + \frac{12}{120960} v^4 - \frac{1}{691200} v^6 + \frac{647}{15328031200} v^8 - \frac{176639}{16738219104000} v^{10} + \ldots, \]

\[ b_2'' = \frac{2}{3} - \frac{1}{2880} v^4 - \frac{1}{241920} v^6 - \frac{1}{464868640} v^8 - \frac{1}{15328051200} v^{10} + \ldots, \]

\[ \hat{a}_{21} = \frac{5}{6} + \frac{1}{384} v^2 + \frac{1}{30240} v^4 + \frac{1}{10321920} v^6 + \frac{8779}{3715891200} v^8 + \frac{1}{4241998053600} v^{10} + \ldots, \]

\[ \hat{a}_{32} = \frac{5}{48} v^2 + \frac{31}{11520} v^4 - \frac{147}{645120} v^6 + \frac{25261}{928972800} v^8 - \frac{675691}{239248819200} v^{10} + \ldots, \]

\[ a_{32} = \frac{1}{6} + \frac{1}{80} v^2 + \frac{107}{80640} v^4 - \frac{779}{5806080} v^6 + \frac{15437}{1135411200} v^8 - \frac{4391993}{3188234649600} v^{10} + \ldots, \]

\[ \gamma_2 = 1 + \frac{1}{384} v^4 + \frac{1}{46080} v^6 + \frac{1}{10321920} v^8 + \frac{1}{3715891200} v^{10} + \ldots, \]

\[ \gamma_3 = 1 - \frac{7}{384} v^4 - \frac{19}{14400} v^6 - \frac{73}{8900000} v^8 + \frac{84}{58060800} v^{10} + \ldots, \]

\[ \gamma_4 = 1 - \frac{1}{24} v^2 + \frac{1}{920} v^4 + \frac{1}{722500} v^6 + \frac{1}{92897280} v^8 + \frac{40874803200}{20437401600} v^{10} + \ldots, \]

where \( \gamma_1 = 1, \gamma_1' = 1 \).

This result in the new method called EFRKTG4. As \( v \to 0 \), the coefficients \( b_1, b_2, b_1', b_2', b_3, a_{32}, a_{21}, a_{32}, a_{3}, a_3, a_3' \) and \( \gamma_3 \) the new method EFRKTG4 reduces to the coefficients of the original method RKTG4. That is to say \( b_1(0), b_2(0), b_1'(0), b_2'(0), b_3(0), a_{32}(0), a_{21}(0), a_{32}(0), a_{32}'(0), a_{32}'(0), \gamma_3(0), \gamma_3(0), \gamma_2(0) \) and \( \gamma_3(0) \) are identical to \( b_1, b_2, b_1', b_2', a_{32}, a_{21}, a_{32}, a_3, a_3' \) and \( \gamma_3 \) of RKTG4 method. Other than that, \( v \to 0 \), as EFRKTG4 method will have the same error constant as RKTG4 method.

5. Problems Tested and Numerical Results

In this section, a set of test problems are solved in order to investigate the effectiveness of the new EFMRKTG4 method compared to the existing RKTG4, RK4 and EFRKS4 methods in the scientific literature. The following methods are chosen for comparison:

- **In**: Step sizes.
- **EFRKTG4**: The three-stage fourth-order exponentially-fitted RK type method derived in this paper.
- **RKTG4**: The three-stage fourth-order RK type method given by Fawzi et.al in [14].
- **RK4**: The fourth -order classical RK method as given in Butcher [11].
- **EFRKS4**: Exponentially-fitted four-stage fourth-order RK method given in Simos [3].
Problem 1 (Homogeneous linear system)

\[ y_1'''(x) = y_3'(x) - y_2'(x), \quad y_1(0) = 1, \quad y_1'(0) = 0, \quad y_1''(0) = 1, \]
\[ y_2'''(x) = -y_1'(x) + y_3'(x), \quad y_2(0) = 0, \quad y_2'(0) = 1, \quad y_2''(0) = 0, \]
\[ y_3'''(x) = y_2'(x) + y_1'(x), \quad y_3(0) = 1, \quad y_3'(0) = 1, \quad y_3''(0) = 1, \]

Exact solution are

\[ y_1(x) = \cosh(x), \quad y_2(x) = \sinh(x), \quad y_3(x) = \cosh(x) + \sinh(x). \]

estimated frequency, \( \omega = 1, \ 0 \leq x \leq 1 \)

Problem 2 (Inhomogeneous linear problem)

\[ y'''(x) = 5y'(x) + \sinh(x), \quad y(0) = -\frac{1}{4}, \quad y'(0) = 0, \quad y''(0) = -\frac{1}{4}, \]

Exact solution is

\[ y(x) = -\frac{e^x}{8} - \frac{e^{-x}}{8}. \]

estimated frequency, \( \omega = 1, \ 0 \leq x \leq 4 \)

Figure 1. The efficiency curve for EFRKTG4, RKTG4, RK4 and EFRKS4 for Problem 1 with \( X_{end} = 1 \) and \( h = 0.005, 0.025, 0.1, 0.5 \)
6. Discussion and Conclusion

In this research, we have presented exponentially-fitted explicit modified Runge-Kutta type method for solving third-order ODEs in the form of \( y'''(x) = f(x, y, y') \). Consequently, we constructed exponentially-fitted explicit MRKT method a three-stage four-order denoted as EFRKTG4 method from Figure 1 and 2. The numerical results obtained showed clearly that the global error of the new exponentially-fitted explicit MRKT method is smaller than the other existing methods. The new EFRKTG4 method is much more efficient than the other existing methods. This is consistent with the results displayed in Tables 1 and 2. Subsequently, our new method is more accurate and suitable for solving third-order ODEs in the form \( y'''(x) = f(x, y, y') \) than the RKTG4, RK4 and EFRKS4 methods in the literature.

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