NONEQUILIBRIUM DYNAMICS OF THE O(N) LINEAR SIGMA MODEL IN THE HARTREE APPROXIMATION

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We investigate the out-of-equilibrium evolution of a classical background field and its quantum fluctuations in the scalar O(N) model with spontaneous symmetry breaking \(^1\). We consider the 2-loop 2PI effective action in the Hartree approximation, i.e. including bubble resummation but without non-local contributions to the Dyson-Schwinger equation. We concentrate on the (nonequilibrium) phase structure of the model and observe a first-order transition between a spontaneously broken and a symmetric phase at low and high energy densities, respectively. So typical structures expected in thermal equilibrium are encountered in nonequilibrium dynamics even at early times before thermalization.

1. The model

1.1. Applications

Scalar models have a wide range of applications in quantum field theory. Normally they are parts of more complex models like e.g. the Standard Model or Grand Unified Theories but they often serve as toy models for a simplified description of complex phenomena such as inflationary cosmology or meson interactions in relativistic heavy ion collisions.

1.2. Nonequilibrium 2PI effective potential

We consider the O(N) model with spontaneous symmetry breaking whose classical action is

\[
S[\Phi] = \int d^4x \mathcal{L}[\Phi] = \int d^4x \left\{ \frac{1}{2} \partial_\mu \Phi \cdot \partial^\mu \Phi - \frac{\lambda}{4} \left( \Phi^2 - v^2 \right)^2 \right\}.
\]

Following Refs. \(^1,2\) we can compute the 2PI effective action \(^3\) in the Hartree approximation. Furthermore, we diagonalize the Green function
by an $O(N)$-symmetric ansatz and by restricting the background field to one direction. Since the Green function is local, it can be described by two 
time-dependent) mass parameters $M_{\sigma,\pi}^2$. For nonequilibrium purposes it
 can be factorized into mode functions
\begin{equation}
G_{\sigma,\pi}(x,t>;x',t<) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\sigma,\pi}} f_{\sigma,\pi}(k,t>;) \bar{f}_{\sigma,\pi}(k,t<) e^{ik(x-x')} , \tag{2}
\end{equation}
where $\omega_{\sigma,\pi} = \sqrt{k^2 + M_{\sigma,\pi}^2(0)}$. One constructs an expression for the total
(conserved) energy density of the system in the Hartree approximation
\[ E = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} M_{\sigma}^2 \phi^2 - \frac{\lambda}{2} \phi^4 - \frac{v^2}{2(N+2)} \left[ M_{\sigma}^2 + (N-1)M_{\pi}^2 \right] \]
\[ - \frac{1}{8\lambda(N+2)} \left[ (N+1)M_{\sigma}^4 + 3(N-1)M_{\pi}^4 - 2(N-1)M_{\sigma}^2 M_{\pi}^2 \right] 
+ 2N\lambda^2 v^4 \right) + E_{\sigma}^{\sigma}(t) + (N-1) E_{\pi}^{\sigma}(t) , \tag{3}
\]
where the fluctuation energy densities $E_{\sigma,\pi}^{\sigma}$ are the nonequilibrium analogs
of the one-loop “log det” terms expressed by mode functions $f(k,t)$
\[ E_{\sigma}^{\sigma}(t) = \frac{\hbar}{2} \int \frac{d^3k}{(2\pi)^3 2\omega_{\sigma}} \left[ |f_{\sigma}(k,t)|^2 + (k^2 + M_{\sigma}^2) |f_{\sigma}(k,t)|^2 \right] , * = \sigma, \pi . \tag{4}
\]

1.3. Equations of motion

The equations of motion follow from the conservation of the energy (3). The
background field obeys
\[ \ddot{\phi} + \left[ M_{\sigma}^2(t) - 2\lambda \phi^2(t) \right] \phi(t) = 0 , \tag{5}
\]
the mass parameters are solutions of the gap equations
\[ M_{\sigma}^2 = \lambda \left( 3\phi_0^2 - v^2 + 3\hbar \mathcal{F}_\sigma + (N-1)\hbar \mathcal{F}_\pi \right) \tag{6}
\]
\[ M_{\pi}^2 = \lambda \left( \phi_0^2 - v^2 + \hbar \mathcal{F}_\sigma + (N+1)\hbar \mathcal{F}_\pi \right) , \tag{7}
\]
where $\mathcal{F}_\sigma$ is the fluctuation integral
\[ \mathcal{F}_\sigma(t) = \int \frac{d^3k}{(2\pi)^3 2\omega_{\sigma}} |f_{\sigma}(k,t)|^2 \quad \text{with} \quad * = \sigma, \pi
\]
which equals the usual tadpole integral at $t = 0$ (cf. section 1.4). The
equation for the mode functions is
\[ \ddot{f}_{\sigma}(k,t) + \left[ k^2 + M_{\sigma}^2(t) \right] f_{\sigma}(k,t) = 0 . \tag{8}
\]
The fact that the mode equation is coupled to the gap equations (6) and (7) has an important influence on the dynamics. When a time-dependent mass square $M^2(t)$ acquires a negative value, eq. (8) will imply an exponential growth of the modes which reacts back on the mass squares via the gap equations by driving them back to positive values. In the one-loop approximation (with no gap equations) the system shows unphysical behavior because the modes never stop growing exponentially.

1.4. Initial conditions

At the beginning of the nonequilibrium evolution we fix the classical background field to a certain value $\phi(0) = \phi_0$. The mode functions are those of free fields: $f_i(k,0) = 1$, $\dot{f}_i(k,0) = -i\omega_i$, and the mass parameters $M_\sigma$ and $M_\pi$ are solutions of the gap equations (6) and (7) at $t = 0$.

2. Results and discussion

2.1. Time evolutions

Here we will present the results for $N = 4$, $\lambda = 1$ and for the renormalization scale set equal to the tree level sigma mass $\mu^2 = 2\lambda v^2$. We have only considered initial values $\phi_0 > v$ for the background field because for smaller values the initial value of the parameter $M_\pi(0)$ is imaginary. This means that the region $-v < \phi < v$ can only be explored dynamically. We display time evolutions of the background field for two different initial conditions in Fig. 1

![Time evolutions of the background field for two different initial conditions](image)

Figure 1. Time evolutions of the background field for $\phi_0 = 1.3v$, $\phi_0 = 1.6v$. Amplitudes are in units of $v$ and time is measured in units of $(\sqrt{\lambda v})^{-1}$.

It can be seen that there are two phases depending on the value of
\( \phi_0 \): a symmetric phase when the field \( \phi(t) \) is oscillating about zero, and a phase of broken symmetry when \( \phi(t) \) is oscillating about a finite value. The “critical” value of \( \phi_0 \) seems to be close to the classically expected value \( \sqrt{2\lambda v} \) where the total energy is equal to the height of the barrier.

### 2.2. Phase structure

In order to analyze the phase structure of the model in the Hartree approximation we define \( \phi_\infty \) as a time averaged amplitude at late times, i.e. the value about which the field oscillates. \( \phi_\infty \) plays the role of an order parameter whose dependence on the total energy of the system is investigated. The total energy of the system is given here by the initial value \( \phi_0 \) which is analogous to the temperature in equilibrium field theory.

Fig. 2 clearly shows a discontinuous jump of \( \phi_\infty \) at \( \phi_0 \simeq \sqrt{2v} \) — a typical sign of a first order phase transition as found in equilibrium (see e.g. Refs. \(^4,^2\)).

### 2.3. A dynamical effective potential

This nonequilibrium system shows a phase structure which is comparable to a system in thermal equilibrium, so it would be nice if there was another correspondence. We define a dynamical, i.e. time-dependent, potential which can be compared to the finite temperature potential in equilibrium

\[
V_{\text{pot}}(t) = E - \frac{1}{2} \dot{\phi}^2(t) .
\] (9)

This potential can only be measured within the oscillation range of the background field \( \phi(t) \). For two different initial conditions it is shown in
Fig. 3. In the broken symmetry phase the minimum at $\phi = v$ moves to smaller values but eventually remains different from zero, so the system settles at a finite expectation value. In the symmetric phase the two minima entirely disappear after a few oscillations and a new (symmetric) minimum at $\phi = 0$ appears.

3. Conclusions and summary

The analysis of the nonequilibrium dynamics of the $O(N)$ model in the Hartree approximation allowed us to study new features of the system which are not accessible in the one-loop or infinite component ($N \to \infty$) approximations. Though thermalization is only expected at approximations beyond the Hartree level, i.e., when including nonlocal corrections, the nonequilibrium system at late times shows striking similarities to a system in thermal equilibrium. One can define an order parameter which is dependent on the total energy of the system, given by the initial conditions. Analyzing the dependence of the order parameter on the initial conditions one finds a first order phase transition.

References

1. J. Baacke and S. Michalski, Phys. Rev. D 65, 065019 (2002) [arXiv:hep-ph/0109137].
2. Y. Nemoto, K. Naito and M. Oka, Eur. Phys. J. A 9, 245 (2000) [arXiv:hep-ph/9911431].
3. J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D 10, 2428 (1974).
4. H. Verschelde and J. de Pessemier, Eur. Phys. J. C 22, 771 (2002) [arXiv:hep-th/0009241].
5. J. Baacke, K. Heitmann and C. Pätzold, Phys. Rev. D 55, 2320 (1997) [arXiv:hep-th/9608006].