Exact solitary wave solution for the fractional and classical GEW-Burgers equations: an application of Kudryashov method

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ABSTRACT
The fractional partial differential equations have wide applications in science and engineering. In this paper, the Kudryashov techniques were utilized to obtain an exact solution of both fractional generalized equal width (GEW)-Burgers and classical GEW-Burgers equations. The general analytical solutions of the two partial differential equations are constructed for \(n > 1\). The graphical representation of the solutions is given in comparison with some previous results in the literature. The advantages and disadvantages of the method were listed.

1. Introduction
The non-linear partial differential equations of fractional order are important class of differential equations used in modelling many science and engineering applications. This is strongly connected to their various applications and abilities in describing lots of complicated physical situations.

In this paper, we discuss the analytical solutions of fractional generalized equal width (GEW)-Burgers equation, i.e.

\[
D_t^n u + pD_x^n u^{n+1} - qD_t^2 u - rD_x^3 u = 0, \quad n > 1, \quad (1)
\]

and generalized non-linear equal width (EW) classical GEW-Burgers equation by Morrison et al. [1], i.e.

\[
u_t + pu^2 \nu_x - q \nu_{xx} - r \nu_{xxx} = 0, \quad n > 1, \quad (2)
\]

that is used in one-dimensional wave propagation attached with dispersive waves together with dissipative effects in non-linear media where \(p, q\) and \(r\) are real constants. Several methods have been used in the literature to construct the solutions of fractional order partial differential equations, ranging from semi-analytical methods to numerical techniques [1–38]. However, little work is available in the literature in respect to the analytical methods especially for the GEW-Burgers equation. Hamdi et al. [6] coupled some techniques to construct hyperbolic tangent exact solution of the GEW-Burgers equation [6].

In this study, we employ the Kudryashov method [7] to analytically treat and examine the fractional GEW-Burgers equation (1) and classical GEW-Burgers equation (2). The fractional GEW-Burgers equation considered features fractional order derivatives in both the spatial and temporal variables, defined in the sense of the new conformable fractional derivative. We have successfully constructed the general solutions for the two equations for any \(n > 1\) and later establish a comparison between the result found and that of the classical GEW-Burgers equation obtained by Hamdi et al. [6]. Three-dimensional graphical representations for some solutions and two-dimensional plots for the sake of comparisons were also presented. The compared solutions are found to be in good agreement.

The main advantages of the Kudryashov method in comparison to others are: the method is simple in application and it can be used to find all solitary wave solutions and all single periodic solutions, when we get the expansion of the general solutions of non-linear differential equations in the Laurent series. Second, exact solution can be obtained by considering different functions. Finally, the method can use a unique formulae for all non-linear differential equations in the polynomial form [7].

This paper is organized as follows: in Section 2, we give some basic definitions and properties of the conformable fractional derivatives. While in Section 3, we introduce the Kudryashov method [7] and the
application of the method to the space–time fractional
GEW-Burgers equation and classical GEW-Burgers
equation, respectively, followed by conclusion in
Section 4.

2. Conformable fractional derivative and some properties

In this section, we provide the reader basic properties
and definition of conformable fractional derivatives.

If \( u : [0, \infty) \to \mathbb{R} \), the \( \alpha \)'s order conformable
derivative of \( u \) is defined by

\[
D^\alpha_t u(t) = \lim_{\varepsilon \to 0} \frac{u(t + \varepsilon t^{1 - \alpha}) - u(t)}{\varepsilon},
\]

where \( 0 < \alpha < 1 \).

Some important properties of the conformable fractional
derivative are summarized in Theorems 2.1 and 2.2.

**Theorem 2.1:** Suppose \( u(t) \) and \( v(t) \) are \( \alpha \)-differentiable
at \( t > 0 \) and \( \alpha \in (0, 1] \), then

(a) \( D^\alpha_t (v(t)^c) = cv(t)^{c-\alpha} \), for all \( c \in \mathbb{R} \).
(b) \( D^\alpha_t (a) = 0 \), for all constant function \( u(t) = a \).
(c) \( D^\alpha_t (au(t)) = a D^\alpha_t (u(t)) \), for all a constant.
(d) \( D^\alpha_t (au(t) + bv(t)) = a D^\alpha_t (u(t)) + b D^\alpha_t (v(t)) \),
for all \( a, b \in \mathbb{R} \).
(e) \( D^\alpha_t (v(t)u(t)) = D^\alpha_t (v(t))u(t) + u(t)D^\alpha_t (v(t)) \).
(f) \( D^\alpha_t (u(t)v(t)) = (v(t)D^\alpha_t (u(t)) - u(t)D^\alpha_t (v(t)))/v^2(t) \),
\( v(t) \neq 0 \).
(g) If in addition to \( u(t) \) is differentiable, then
\( D^\alpha_t u(t) = t^{1-\alpha}(du/dt) \).

**Theorem 2.2:** Let \( u(t) \) be a differentiable function and
suppose \( v(t) \) is also differentiable defined in the range
of \( u(t) \), with \( \alpha \in (0, 1] \), then

\[
D^\alpha_t (u(t) \circ v(t)) = t^{1-\alpha}v'(t)u'(v(t)) \text{,}
\]

where \( D^\alpha_t \) stands for the conformable fractional
derivative with respect to temporal variable \( \alpha \).

3. The Kudryashov method of solution

In this section, steps needed to apply the method are
presented. We start by considering the conformable
fractional differential equation,

\[
P(u, D^\alpha_t u, D^\alpha_t x u, D^\alpha_t x^2 u, D^\alpha_t x^3 u, D^\alpha_t x^4 u, \ldots) = 0 \quad \text{for} \quad 0 < \alpha < 1.
\]

Kudryashov method follows by reducing (4) to ordinary
differential equation using the wave transformation, i.e.

\[
u(x, t) = U(\xi), \quad \xi = \frac{x^\alpha}{\alpha} - \frac{t^\alpha}{\alpha}.
\]

where \( \alpha \) is a fractional order and \( c \) is a non-zero constant.
Substituting the transformation (4) into (5), we get a reduced
ordinary differential equation of the polynomial form, i.e.

\[
Q(U(\xi), \frac{d}{d\xi}U(\xi), \frac{d^2}{d\xi^2}U(\xi), \ldots) = 0.
\]

Kudryashov method offers a truncated finite series
of the form:

\[
U(\xi) = a_0 + \sum_{i=1}^{N} a_i \Phi^i(\xi),
\]

which satisfies the ordinary differential equation,

\[
\Phi' = \Phi^2 - \Phi.
\]

where \( a_0, a_1, a_2, \ldots, a_N \) are constants; \( N \) is a positive
integer determined by the homogeneous balancing method.

Substituting (7) into (6) yields a polynomial in \( \Phi(\xi) \). The
resultant algebraic system of equations is then solved
for \( a_0, a_n, (n = 1, 2, \ldots) \) and \( c \).

3.1. Application

Finally, we apply the method presented in the previous
section to construct the analytical solutions of conformable
space–time fractional GEW-Burgers and generalized
classical GEW-Burgers partial differential equations.

**Examples 3.1:** Consider the conformable space–time
fractional GEW-Burgers equation of the form:

\[
D^\alpha_t u + pD^\alpha_t u^{n+1} - qD^{2\alpha}_t u - rD^{3\alpha}_t x u = 0, \quad n \geq 1.
\]

using the wave transformation,

\[
u(x, t) = U(\xi), \quad \xi = \frac{x^\alpha}{\alpha} - \frac{t^\alpha}{\alpha}.
\]

Equation (10) is reduced to an ordinary differential
equation,

\[
-cu' + pu^{n+1} - qu'' + cu'' = 0.
\]

Integrating Equation (11) once, and assuming the constant
of integration zero, we get

\[
-cu + pu^{n+1} - qu'' + cu'' = 0.
\]

Balancing the highest order derivative with the highest
non-linear order \( (u^{n+1}; u') \) in (11), we get \( N = 2/n \).

Applying the transformation \( u = v^{2/n} \), Equation (12)
becomes

\[
-cn^2 v^2 + n^2 p v^{2/n} - 2nq v' + (4cr - 2cnr) v^{2/n} + 2cnr v'' = 0.
\]
by balancing the highest non-linear order \((\nu^2; \nu^3)\) in (13), we get \(N = 1\), which implies using \((7)\) that
\[ \nu(\xi) = a_0 + a_1 \Phi(\xi). \] (14)
Substituting \((14)\) into \((13)\) and by equating the coefficients of various degrees of \(\Phi(\xi)\) to zero, we obtain the following algebraic system equations:
\[
\begin{align*}
2nqa_0a_1 + 6nra_0a_1 + cn^2a_1^2 - 2nqa_1^2 - 4ra_1^2 - 6n^3pa_0a_1^2 &= 0, \\
4nqa_0a_1 - 2nqa_1^2 - 8ra_1^2 - 2nra_1^2 + 4n^3pa_0a_1^3 &= 0, \\
2n^2p_0a_1a_1 - 2nqa_0a_1 - 2nra_0a_1 &= 0, \\
-4n^3pa_0^3a_1 &= 0, \\
0, cra_1^2 + 2nra_1^2 + n^2pa_1^2 &= 0, \\
0, cn^2a_0^2 - n^2pa_0^2 &= 0.
\end{align*}
\] (15)
Solving the system \((15)\), we obtain the following cases:

**Case I:** For \(n = 2\),
\[
a_0 = a_0, \quad a_1 = -a_0, \quad p = \frac{c}{a_1^2}, \quad q = -\frac{3c}{2},
\] (16)
\[
r = -\frac{1}{2}.
\]
From which, using \((14)\) and \((16)\), we have
\[
\nu(\xi) = a_0 - \frac{a_0}{1 + de^\xi},
\] (17)
which subsequently led to
\[
u_n(x, t) = a_0 - \frac{a_0}{1 + de^{\xi-ct/(\alpha)}},
\] (18)

**Case II:** In general for any value of \(n\), solving the systems \((15)\), we obtain the following arbitrary constants:
\[
a_0 = a_0, \quad a_1 = -a_0, \quad p = \frac{c}{a_1^2}, \quad q = -\frac{cn(4 + n)}{2(2 + n)},
\]
\[
r = -\frac{n^2}{2(2 + n)},
\]
which implies from \((14)\),
\[
\nu(\xi) = a_0 - \frac{a_0}{1 + de^\xi},
\] (19)
therefore, the general solution has the form:
\[
u_n(x, t) = \left( a_0 - \frac{a_0}{1 + de^{\xi-ct/(\alpha)}} \right)^{2/n}.
\] (20)
Thus for values of \(n = 3, 4, 5, 6, \ldots\), we get from \((20)\) the following solutions:
\[
u_n=3(x, t) = \left( a_0 - \frac{a_0}{1 + de^{\xi-ct/(\alpha)}} \right)^{2/3}
\]
\[
u_n=4(x, t) = \left( a_0 - \frac{a_0}{1 + de^{\xi-ct/(\alpha)}} \right)^{1/2}
\]
\[
u_n=5(x, t) = \left( a_0 - \frac{a_0}{1 + de^{\xi-ct/(\alpha)}} \right)^{1/5}
\]
\[
u_n=6(x, t) = \left( a_0 - \frac{a_0}{1 + de^{\xi-ct/(\alpha)}} \right)^{1/3}
\] (24)

The graphical representations of some solutions are presented in Figures 1–3.

**Examples 3.2:** Consider the GEW-Burgers equation, i.e.
\[
t + pu^n u_x - qu_{xx} - ru_{xt} = 0, \quad n > 1,
\] (25)
where \(p, q, r\) are real constants. Applying the Kudryashov method, we obtain the solution of the GEW-Burgers equation \((25)\) as
\[
u(x, t) = \left( a_0 - \frac{a_0}{1 + de^{\xi-ct/(\alpha)}} \right)^{2/n}.
\] (26)
The solution of \((2)\) obtained here, i.e. \((26)\), is exactly the same with the fractional version of GEW-Burgers equation in \((10)\) for \(\alpha = 1\), with little discrepancy in \(p\) that is found to be \(c(1 + n)/a_1^2\). Nevertheless, this does not change the solution in \((25)\), since \(p, q, r\) are not reflecting in the solution. Also, see Figures 4–6 for the graphical representations of some solutions.

**Remark 3.3:** It is clear from the graphical representations of the solutions presented above, the behaviour of the solutions depend on the values of the constants and the fractional order.
In Hamdi et al. [6], the classical GEW-Burgers equation (2) was studied coupling some techniques and found the following exact solution was obtained [6]:

\[
u(x, t) = \left( \frac{\phi}{2} (1 - \tanh(k(x - ct - x_0))) \right)^{2/n}
= \left( \frac{\phi}{e^{2k(x-ct-x_0)} + 1} \right)^{2/n}.
\] (27)

Thus we feel it is imperative to establish a comparison between our solution (25) and that of Hamdi et al. (27) [6].

3.2. Comparison of results

In Hamdi et al. [6], the classical GEW-Burgers equation (2) was studied coupling some techniques and found the following exact solution was obtained [6]:

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Thus we feel it is imperative to establish a comparison between our solution (25) and that of Hamdi et al. (27) [6].
Finally, the two results are found to be in good agreement; also one can obtain distinct results by maintaining the above parameters with \( d = 1 \). See Figures 7–10 for the graphical representations of some cases.

4. Conclusion

In conclusion, the conformable space–time fractional and classical GEW-Burgers equations are extensively examined by employing the powerful Kudryashov method. The generalized solutions for the two equations are established for any given \( n > 1 \). A kind of comparison analysis between the classical GEW-Burgers equation solution obtained and that of Hamdi et al. [6] is also presented and the two results are found to be in agreement. The comparison was established through graphical representations for the solutions. Thus the Kudryashov method is highly recommended for generalized and non-generalized problems and in this direction, respectively.

Disclosure statement

No potential conflict of interest was reported by the authors.

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