A proposal for the non-Abelian tensor multiplet

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Abstract

If one compactifies the Abelian (1,0) tensor multiplet on a circle, one finds 5d SYM for the zero modes. For the Kaluza-Klein modes one can likewise find a Lagrangian description in 5d \[3\]. Since in 5d we have an ordinary YM gauge potential, one may look for a non-Abelian generalization and indeed such a non-Abelian generalization was found in \[3\]. In this paper, we study this non-Abelian generalization for the (1,0) tensor multiplet in detail. We obtain the supersymmetry variations that we close on-shell. This way we get the fermionic equation of motion and a modified selfduality constraint.
1 Introduction

To address the 6d non-Abelian tensor multiplet theory, either with (1, 0) supersymmetry or (2, 0) supersymmetry, one may consider the theory compactified on a circle where one may relate it with the dimensionally reduced 5d SYM theory.

One proposal [5], [6] has been that 5d SYM theory is exactly the same thing as 6d (2,0) theory compactified on a circle, the instanton particles in the 5d SYM could play the role of Kaluza-Klein modes.

Another proposal [3] has been to incorporate Kaluza-Klein modes in the 5d description of the non-Abelian tensor multiplet.

Our original motivation to study the proposal of [3] was to try to find an inconsistency as it is unsatisfactory to have two different proposals for the 6d tensor multiplet which appear to be in conflict with each other. We did not find any inconsistency, at least not so far. In this paper we clarify part of the result in [3]. For the sake of simplicity, we restrict ourselves to the non-Abelian (1, 0) tensor multiplet. We make an ansatz based on [3] for the on-shell supersymmetry variations. We close these supersymmetry variations on-shell up to a gauge transformation. By doing this, we find closure relations on the following form

\[ \delta^2 \Phi^{(n)} = -i s^m \partial_m \Phi^{(n)} - \frac{ns}{r} \Phi^{(n)} \]

for all the fields in the non-Abelian (1, 0) tensor multiplet. The Kaluza-Klein mode number is \( n \). The 5d parameters that form a 5d vector \( s^m \) together with the 5d parameter \( s \) have an uplift to a 6d Lorentz vector \( S^M \), and the first line in the closure relation can be written in the 6d Lorentz covariant form as

\[ -i s^m \partial_m \Phi^{(n)} - \frac{ns}{r} \Phi^{(n)} = -i S^M \partial_M \Phi^{(n)} \]

where, if we compactify on a temporal circle of radius \( r \), we have \( \partial_0 \Phi^{(n)} = in \Phi^{(n)} \). This closure relation suggests that a hidden 6d Lorentz symmetry may emerge in the decompactification limit. The second thing we learn is what the fermionic equation of motion shall be and the third thing we learn is that we need to modify the selfduality constraint in nontrivial way in order to close the supersymmetry variations on-shell.

In principle it should be possible to derive the fermionic equation of motion and the modified selfduality constraint from the Lagrangian in [3]. To this end, we would need to eliminate the auxiliary field from that Lagrangian. The auxiliary field \( Y^I \) enters their
Lagrangian as \( \mathcal{L} = h_{IJ} Y^I Y^J + f_{IJK} \left( \phi^I Y^J Y^K + Y^I \chi^J \chi^K \right) + \ldots \) where \( f^{IJK} \) denote Lie algebra structure constants associated with the gauge group using their notation. Here we are suppressing the KK-mode numbers. If we would display the mode number, we would also see how \( Y^I \) mixes zero modes and KK modes. To eliminate \( Y^I \) from this Lagrangian, we need to invert the matrix \( h_{IJ} + f_{IJK} \phi^K \). This seems like a challenging problem and it was not carried out in full generality in [3]. Only two special cases where studied there; dimensional reduction resulting in the usual 5d SYM, and Abelian gauge group that puts \( f^{IJK} = 0 \).

2 The non-Abelian tensor multiplet

As we outlined in the introduction, we will consider the 6d \((1,0)\) tensor multiplet on a circle. The classical 6d tensor multiplet is best understood in Lorentzian signature. We will restrict ourselves to flat Lorentzian spacetime with the metric

\[
ds^2 = -r^2 dt^2 + \delta_{mn} dx^m dx^n
\]

We choose to compactify the time direction

\[
t \sim t + 2\pi
\]

and reduce to Euclidean 5d SYM with the YM coupling constant

\[
g^2 = 4\pi^2 r
\]

The 6d formulation of the non-Abelian tensor multiplet may not be known, but for the Abelian tensor multiplet we do know what it is. So we may start there. It consists of one scalar field \( \sigma \), one fermionic field \( \chi \) and a tensor field \( B_{MN} \) with selfdual field strength. When we put this theory on a circle, we may expand the fields in modes as follows,

\[
\sigma = \phi + \sum_{n \neq 0} \sigma^{(n)}
\]

\[
\chi = \psi + \sum_{n \neq 0} \chi^{(n)}
\]

where \( n \) is integer, and

\[
\partial_t \sigma^{(n)} = i n \sigma^{(n)}
\]

and similarly for the other fields. For the gauge field, we expand

\[
B_{m0} = a_m + \sum_{n \neq 0} A^{(n)}_m
\]
\[ B_{mn} = b_{mn} + \sum_{n \neq 0} B^{(n)}_{mn} \]

To get the standard SYM normalization, we shall relate the zero modes to the SYM fields as \( \phi_{SYM} = \phi/(2\pi r) \), \( \psi_{SYM} = \psi/(2\pi r) \) and \( (a_m)_{SYM} = a_m/(2\pi) \) but we will not make such a rescaling here.

For the Abelian theory we define
\[
\mathcal{F}^{(n)}_{mn} = \partial_m A^{(n)}_n - \partial_n A^{(n)}_m + i n B^{(n)}_{mn} \\
H^{(n)}_{mnp} = \partial_m B^{(n)}_{np} + \partial_p B^{(n)}_{mn} + \partial_n B^{(n)}_{pm}
\]

We have the selfduality constraints
\[ H^{(n)}_{mnp} = -\frac{1}{2r} \mathcal{E}_{mnp}^{rs} \mathcal{F}^{(n)}_{rs} \]
and the Bianchi identities
\[ 3\partial_{[n} \mathcal{F}^{(n)}_{np]} - inH^{(n)}_{mnp} = 0 \]

We can use these conditions to eliminate \( H^{(n)}_{mnp} \) and then we just need to work with \( \mathcal{F}^{(n)}_{mn} \) subject to a `selfdual Bianchi identity’
\[ 3\partial_{[m} \mathcal{F}^{(n)}_{np]} + \frac{in}{2r} \mathcal{E}_{mnp}^{rs} \mathcal{F}^{(n)}_{rs} = 0 \]

It is now straightforward to derive the supersymmetry variations for these modes from the 6d supersymmetry variations. To this end, we need to fix some spinor conventions. We collect all our spinor conventions in Appendix A. For the zero modes, one finds
\[
\delta a_m = i r \bar{\varepsilon} \gamma_m \psi \\
\delta \phi = -i \bar{\varepsilon} \Psi \\
\delta \psi = \frac{1}{2r} \gamma^{mn} \varepsilon f_{mn} + \gamma^m \bar{\varepsilon} \partial_m \sigma
\]
and for the non-zero KK modes one finds
\[
\delta \mathcal{F}^{(n)}_{mn} = -2ir\bar{\varepsilon} \gamma_m D_n \chi - n\bar{\varepsilon} \gamma_{mn} \chi \\
\delta \sigma = -i \bar{\varepsilon} \chi \\
\delta \chi = \frac{1}{2r} \gamma^{mn} \varepsilon \mathcal{F}_{mn} + \gamma^m \varepsilon \partial_m \sigma + \frac{in}{r} \varepsilon \sigma
\]

We have suppressed the mode number \((n)\) which is common for all the KK fields that appear in the supersymmetry variations \( (\mathcal{F}^{(n)}_{mn} = \mathcal{F}^{(n)}_{mn}, \chi^{(n)}, \sigma = \sigma^{(n)}) \). This is nothing but a 5d reformulation of the Abelian 6d tensor multiplet, as we show explicitly in Appendix B.
Now once having this reformulation, it is natural to try to find a non-Abelian generalization by promoting the 5d gauge field $a_m$ to a non-Abelian gauge field and letting the other fields and the KK modes transform in the adjoint representation of the gauge group. For the zero mode part, we have the supersymmetry variations of 5d SYM

$$\delta a_m = i r \bar{\varepsilon} \gamma_m \psi$$
$$\delta \phi = -i \bar{\varepsilon} \psi$$
$$\delta \psi = \frac{1}{2r} \gamma^{mn} \varepsilon f_{mn} + \gamma^m \varepsilon D_m \sigma \quad (2.1)$$

For the KK-modes we make the following ansatz [3]

$$\delta F_{mn} = -2i r \bar{\varepsilon} \gamma_m D_n \chi - n \bar{\varepsilon} \gamma_{mn} \chi$$
$$+ i C_1 [\phi, \bar{\varepsilon} \gamma_{mn} \chi] - i C_1 [\sigma, \bar{\varepsilon} \gamma_{mn} \psi]$$
$$\delta \sigma = -i \bar{\varepsilon} \chi$$
$$\delta \chi = \frac{1}{2r} \gamma^{mn} \varepsilon F_{mn} + \gamma^m \varepsilon D_m \sigma + \frac{in}{r} \varepsilon \sigma + \frac{C}{r} \varepsilon [\phi, \sigma] \quad (2.2)$$

for some parameters $C_1$ and $C$ that may depend on the mode number $n$. We define the non-Abelian YM field strength as

$$f_{mn} = \partial_m a_n - \partial_n a_m - i [a_m, a_n]$$

and the covariant derivative as

$$D_m \sigma = \partial_m \sigma - i [a_m, \sigma]$$

We now turn to the closure computation for these on-shell supersymmetry variations. Since the gauge parameter should agree for zero modes and nonzero modes, we will obtain the closure relations for the zero mode fields first.

### 2.1 Closure on the SYM fields

For the bosonic SYM fields, we get

$$\delta^2 \phi = -i s^m \partial_m \phi - i [\phi, \lambda_T]$$
$$\delta^2 A_m = -i s^a \partial_n A_m + D_m \lambda_T$$

where the gauge parameter is

$$\lambda_T = i s (a - r \phi)$$

Here we have introduced the vertical component of the gauge potential,

$$a := a_m s^m_s$$
If we keep the covariant derivative, the closure relations take the form
\[ \delta^2 \phi = -is^m D_m \phi - i[\phi, \lambda] \]
with
\[ \lambda = -isr \phi \] (2.3)
We have now determined the gauge parameter that should also appear in the closure for the KK modes, to which we now turn.

2.2 Closure on \( \sigma \)
\[ \delta^2 \sigma = -is^m D_m \sigma - s \frac{r n}{r} \sigma - i[\sigma, \lambda] \]
where
\[ \lambda = \frac{sc}{r} \phi \]
Matching this with (2.3) determines
\[ C = -imr^2 \]

2.3 Closure on \( \mathcal{F}_{mn} \)
\[ \delta^2 \mathcal{F}_{mn} = -2is^q D_n \mathcal{F}_{mq} - s^r \mathcal{E}_{mnr}^{pq} \left( \frac{n}{2r} \mathcal{F}_{pq} - \frac{iC_1}{2r} \left( [\sigma, f_{pq}] - [\phi, \mathcal{F}_{pq}] \right) \right) - \frac{ns}{r} \mathcal{F}_{mn} \\
+ \left( \frac{iC_1}{r} - r \right) s[f_{mn}, \sigma] \\
+ 2i (C_1 - C) s_m D_n ([\phi, \sigma]) \\
- i \left[ \mathcal{F}_{mn}, \frac{C_1 s}{r} \phi \right] \\
+ \text{fermionic bilinears} \]
By taking
\[ C_1 = C = -imr^2 \]
we get
\[ \delta^2 \mathcal{F}_{mn} = -2is^q D_n \mathcal{F}_{mq} - s^r \mathcal{E}_{mnr}^{pq} \left( \frac{n}{2r} \mathcal{F}_{pq} - \frac{iC_1}{2r} \left( [\sigma, f_{pq}] - [\phi, \mathcal{F}_{pq}] \right) \right) - \frac{ns}{r} \mathcal{F}_{mn} \]
\[-i \left[ F_{mn}, \frac{C_1 s}{r} \phi \right] \text{ + fermionic bilinears} \]

We now read off the bosonic part of the selfdual Bianchi identity

\[ 3D_r F_{mn} + \frac{in}{2r} E_{mnr} \left( F_{pq} - \frac{r}{2} \left[ [\sigma, f_{pq}] - [\phi, F_{pq}] \right] \right) \text{ + fermions} = 0 \]

and the gauge parameter

\[ \lambda = \frac{C_1 s}{r} \phi \]

Let us now look at the fermionic bilinears. These are

\[
\begin{align*}
\text{fermionic bilinears} &= -2ir^2 \{ \bar{\epsilon} \gamma_n \psi, \bar{\epsilon} \gamma_m \chi \} \\
&+ C_1 \{ \bar{\epsilon} \psi, \bar{\epsilon} \gamma_{mn} \chi \} \\
&- C_1 \{ \bar{\epsilon} \chi, \bar{\epsilon} \gamma_{mn} \psi \}
\end{align*}
\]

Now we use the identities

\[
\begin{align*}
\gamma_n \gamma_p \gamma_m - \gamma_m \gamma_p \gamma_n &= \{ \gamma_{mn}, \gamma_p \} \\
\gamma_n \gamma_{pq} \gamma_m - \gamma_m \gamma_{pq} \gamma_n &= -\{ \gamma_{mn}, \gamma_{pq} \}
\end{align*}
\]

We then use the following flipping rules,

\[
\begin{align*}
\bar{\epsilon} \gamma_m \psi &= \bar{\psi} \gamma_m \epsilon \\
\bar{\epsilon} \psi &= \bar{\psi} \epsilon
\end{align*}
\]

where we recall that \( \bar{\epsilon}^I \) and \( \bar{\psi}^I \) have raised index \( I \) by default, and as that we rise that index by \( \epsilon^{IJ} \) which is antisymmetric. We then get

\[
\begin{align*}
\text{fermionic bilinears} &= -2ir^2 \{ \bar{\psi} \gamma_n \epsilon \bar{\epsilon} \gamma_m \chi \} \\
&+ C_1 \{ \bar{\psi} \epsilon \bar{\epsilon} \gamma_{mn} \chi \} \\
&- C_1 \{ \bar{\chi} \epsilon \bar{\epsilon} \gamma_{mn} \psi \}
\end{align*}
\]

Then we expand

\[
\epsilon \bar{\epsilon} = c + c^p \gamma_p + c^{pq} \gamma_{pq}
\]

where the first two coefficients are related to \( s \) and \( s^p \) by some factor. Here we just need the relation

\[
c^p = \frac{s^p}{8}
\]
We get

\[
\text{fermionic bilinears} = -2ir^2 \{ \bar{\psi} \gamma_m (c+c^p \gamma_p + c^{pq} \gamma_{pq}) \gamma_m \chi \} \\
+ C_1 \{ \bar{\psi} (c + c^p \gamma_p + c^{pq} \gamma_{pq}) \gamma_m \chi \} \\
- C_1 \{ \bar{\chi} (c + c^p \gamma_p + c^{pq} \gamma_{pq}) \gamma_m \chi \}
\]

We now need to apply flipping rules to the last line. The curly bracket is a reminder that we have antisymmetry due to Lie algebra commutator. Also the fermionic fields are anticommuting, there is a hidden index \( I \) that is contracted by antisymmetric \( c \) and \( c^p \) and symmetric \( c^{pq} \). Taking everything into account, we find the flipping rule

\[
\{ \bar{\chi} (c + c^p \gamma_p + c^{pq} \gamma_{pq}) \gamma_m \chi \} = \{ \bar{\psi} \gamma_m (-c - c^p \gamma_p - c^{pq} \gamma_{pq}) \chi \}
\]

and so we get

\[
\text{fermionic bilinears} = -ir^2 \{ \bar{\psi} (-2c \gamma_m + c^p \{ \gamma_p, \gamma_m \} - c^{pq} \{ \gamma_{pq}, \gamma_m \}) \chi \} \\
+ C_1 \{ \bar{\psi} (2c \gamma_m + c^p \{ \gamma_p, \gamma_m \} + c^{pq} \{ \gamma_{pq}, \gamma_m \}) \chi \}
\]

By using that

\[
C_1 = -ir^2
\]

we get

\[
\text{fermionic bilinears} = -ir^2 \frac{s^p}{2} \{ \bar{\psi} \gamma_{mnp} \chi \}
\]

The selfdual Bianchi identity therefore becomes

\[
3D [r \mathcal{F}_{mn}] + \frac{in}{2r} \mathcal{E}_{mnr}^{pq} \mathcal{F}_{pq} - \frac{ir}{2} \mathcal{E}_{mnr}^{pq} ([\sigma, f_{pq}] - [\phi, \mathcal{F}_{pq}]) + \frac{r^2}{2} (\bar{\psi} \gamma_{mnr} \chi) = 0
\]

This can be viewed as the standard Bianchi identity

\[
3D [r \mathcal{F}_{mn}] + in \mathcal{H}_{mnr} = 0
\]

but with a modified selfduality condition

\[
\mathcal{H}_{mnr} = -\frac{1}{2r} \mathcal{E}_{mnr}^{pq} \left( \mathcal{F}_{pq} - \frac{r^2}{n} ([\sigma, f_{pq}] - [\phi, \mathcal{F}_{pq}]) - \frac{ir^3}{2n} \{ \bar{\psi} \gamma_{pq} \chi \} \right)
\]

where terms proportional to \( 1/n \) are nonlocal.
2.4 Closure on $\chi$

We carry out the closure computation on $\chi$ for each chiral component separately. We decompose the gamma matrices into horizontal and vertical components,

$$\gamma^m = \gamma'^m + \frac{s^m}{s}\gamma$$

$$\gamma = \frac{s_m^m}{s}\gamma$$

and we put a prime on anything that is traceless (has vanishing contraction with $s^m$).

The fields are decomposed into horizontal and vertical components as well,

$$F_{mn} = F'_{mn} + 2F'_{[m} s_{n]}$$

$$F'_m = F_m s^n s$$

In the Appendix we show that

$$\gamma\varepsilon = -\varepsilon$$

Let us decompose the fermion field decomposes into its two chiral components

$$\chi = \chi^+ + \chi^-$$

$$\gamma\chi^\pm = \pm\chi^\pm$$

These then will have the supersymmetry variations

$$\delta\chi^- = \frac{1}{2r}\gamma'^{mn}\varepsilon F'_{mn} - \frac{s^m}{s}\varepsilon D_m\sigma + \frac{i n}{r}\varepsilon\sigma + \frac{1}{r}C\varepsilon[\sigma^{(0)}, \sigma]$$

$$\delta\chi^+ = -\frac{1}{r}\gamma'^{mn}\varepsilon F'_m + \gamma'^{mn}\varepsilon D'_m\sigma$$

We also make a new ansatz for the variation of the field strength,

$$\delta F'_{mn} = -2ir\varepsilon\gamma'^m D'_n\chi^+ - n\varepsilon\gamma'_m\chi^- + \frac{i}{s}\delta_{[m}\gamma'_{n]}\chi^+$$

$$- iC'_1[\phi, \varepsilon\gamma'_m\chi^-] - iC'_1[\sigma, \varepsilon\gamma'_m\psi^-]$$

$$\delta F'_m = -ir\varepsilon\gamma'_m D_n\chi^+ - n\varepsilon\gamma'_m\chi^- - ir\varepsilon D'_m\chi^- + C'[\phi, \varepsilon\gamma'_m\chi^+] - C'[\sigma, \varepsilon\gamma'_m\psi^+]$$

that reduces to the above ansatz when we take $C' = r^2$, but it can be nice to see how this happens by closing supersymmetry on the fermions, so we keep $C'$ general for the moment.
2.4.1 Closure on $\chi^+$

$$\delta^2 \chi^+ = -is^m D_m \chi^+ - \frac{ns}{r} \chi^+$$

$$+ \left( \frac{C'}{r} - r \right) \gamma^{mn} \varepsilon \bar{\varepsilon} \gamma'_m [\sigma, \psi^+]$$

$$- \frac{C'}{r} \gamma^{mn} \varepsilon \bar{\varepsilon} \gamma'_{mn} [\phi, \chi^+]$$

For the second line to vanish we need to take

$$C' = r^2$$

We use the Fierz identity

$$\varepsilon \bar{\varepsilon} = -\frac{s}{8} (1 - \gamma) + \frac{1}{8} \Theta^{mn} \gamma'_{mn}$$

Here $\Theta$ is horizontal and selfdual. We also use the identities

$$\gamma^{mn} \gamma'_{pq} \gamma'_{m} = 0$$

$$\gamma^{mn} \gamma'_{m} = 4$$

and then we get

$$\delta^2 \chi^+ = -is^m D_m \chi^+ - \frac{ns}{r} \chi^+ - i[\chi^+, \lambda]$$

where

$$\lambda = -irs \phi$$

2.4.2 Closure on $\chi^-$

$$\delta^2 \chi^- = -i \left( \gamma^{mn} \varepsilon \bar{\varepsilon} \gamma'_{mn} - \frac{s^n}{s} \varepsilon \bar{\varepsilon} \right) D_n \chi^-$$

$$- \frac{n}{r} \left( \frac{1}{2} \gamma^{mn} \varepsilon \bar{\varepsilon} \gamma'_{mn} - \varepsilon \bar{\varepsilon} \right) \chi^-$$

$$+ \left( \frac{iC_1}{2r} \gamma^{mn} \varepsilon \bar{\varepsilon} \gamma'_{mn} - \frac{iC}{r} \varepsilon \bar{\varepsilon} \right) [\phi, \chi^-]$$

$$+ \left( \frac{iC_1}{2r} \gamma^{mn} \varepsilon \bar{\varepsilon} \gamma'_{mn} + \left( r - \frac{iC}{r} \right) \varepsilon \bar{\varepsilon} \right) [\psi^-, \sigma]$$

We now use the following identity

$$\gamma^{mn} \gamma'_{pq} \gamma'_{m} = \gamma'_{pq} \gamma'_{mn}$$
that implies
\[
\gamma^{mn} \varepsilon I \varepsilon^J \gamma'_m = \frac{3s}{4} \gamma^m (P_+) I^J + \frac{1}{8} \Theta^{pq} I^J \gamma'_p \gamma'^m
\]
\[
\gamma^{mn} \varepsilon I \varepsilon^J \gamma'_m = 3s (P_-) I^J + \frac{1}{2} \Theta^{pq} I^J \gamma'_p
\]

One can see that these are mutually consistent by contracting the first relation by \(\gamma'_n\) from the right. Anything involving \(P_+\) goes into the equation of motion. We get
\[
\delta^2 \chi^- = -is^n D_n \chi^- - \frac{ns}{r} \chi^- \\
- \frac{3is}{4} \left( \gamma^m D'_n \chi^+ - \frac{s^n}{s} D_n \chi^- - \frac{in}{r} \chi^- \right) \\
- \frac{is}{r} \left( \frac{3C_1}{2} + \frac{C'}{4} \right) [\chi^-, \phi] \\
+ \frac{is}{r} \left( \frac{3C_1}{2} + \frac{C'}{4} + \frac{ir^2}{4} \right) [\psi^-, \sigma] \\
- \frac{i}{8} \Theta^{pq} \gamma_{pq} \left( \gamma^n D_n \chi - \frac{s^n}{s} D_n \chi - \frac{in}{r} \chi \right) \\
- \frac{i}{8r} \Theta^{pq} \gamma_{pq} (2C_1 - C) [\chi^-, \phi] \\
+ \frac{i}{8r} \Theta^{pq} \gamma_{pq} (2C_1 - C - ir^2) [\psi^-, \sigma]
\]

Here, in the third and fourth lines, we put \(C = C_1\) to get
\[
- \frac{7isC_1}{4r} [\chi^-, \phi] \\
+ \frac{7isC_1 - sir^2}{4r} [\psi^-, \sigma]
\]

In the third line, we then decompose
\[
7 = 4 + 3
\]

Then 4 goes into the gauge transformation and 3 goes into the equation of motion. Thus we get
\[
\delta_{\text{gauge}} \chi^- = -i [\chi^-, \lambda]
\]
with
\[
\lambda = \frac{sC_1}{r} \phi
\]
and we get the equation of motion
\[
\gamma^m D'_n \chi^+ - \frac{s^n}{s} D_n \chi^- - \frac{in}{r} \chi^- + \frac{C_1}{r} [\chi^-, \phi] + \frac{7C_1 + sir^2}{3r} [\psi^-, \sigma] = 0
\]
The contributions that are proportional to $\Theta^{pq}$ are given by

$$-\frac{i}{8} \Theta_{pq} \gamma_{pq} \left( \gamma^n D_n \chi - \frac{s}{s} D_n \chi - \frac{in}{r} \chi + \frac{C_1}{r} [\chi, \phi] + \frac{C_1 - ir^2}{r} [\psi, \sigma] \right)$$

If we plug in the values

$$C = C_1 = -ir^2$$

the two fermionic equations of motion agree, since then

$$\frac{7C_1 + ir^2}{3} = C_1 - ir^2$$

3 The result

The following 5d supersymmetry variations, which consist of a SYM part (zero modes part),

$$\delta a_m = i r \bar{\epsilon} \gamma_m \psi$$
$$\delta \phi = -i \bar{\epsilon} \psi$$
$$\delta \psi = \frac{1}{2r} \gamma^{mn} \varepsilon f_{mn} + \gamma^m \varepsilon D_m \sigma$$

and a KK-mode part (nonzero mode part),

$$\delta F_{mn} = -2i r \bar{\epsilon} \gamma_m D_n \chi - n \bar{\epsilon} \gamma_{mn} \chi + r^2 [\phi, \bar{\epsilon} \gamma_{mn} \chi] - r^2 [\sigma, \bar{\epsilon} \gamma_{mn} \psi]$$
$$\delta \sigma = -i \bar{\epsilon} \chi$$
$$\delta \chi = \frac{1}{2r} \gamma^{mn} \varepsilon f_{mn} + \gamma^m \varepsilon D_m \sigma + \frac{in}{r} \varepsilon \sigma - i r \bar{\epsilon} [\phi, \sigma]$$

(3.1)

close on-shell on the fermionic equation of motion

$$\gamma^n D_n \chi - \frac{in}{r} \chi - ir [\chi, \phi] - 2ir [\psi, \sigma] = 0$$

the Bianchi identity

$$3D_r F_{mn} - in H_{mnr} = 0$$

and a modified selfduality condition

$$H_{mnr} = -\frac{1}{2r} \varepsilon_{mnr}pq \left( F_{pq} - \frac{r^2}{n} ([\sigma, f_{pq}] - [\phi, F_{pq}]) - \frac{ir^3}{2n} \{\bar{\psi} \gamma_{pq} \chi\} \right)$$

(3.2)
4 The non-Abelian gerbe

The gauge symmetry in 5d has an interesting non-Abelian gerbe structure [1], [2], [3]. In the simplest set-up where all the fields transform in the adjoint representation, the gauge symmetry variations are

\[ \delta \sigma = -i[\sigma, \lambda] \]
\[ \delta a_m = D_m \lambda \]

for the 5d SYM fields, and

\[ \delta A_m = D_m \Lambda_0 - in \Lambda_m - i[A_m, \lambda] \]
\[ \delta B_{mn} = 2D_{[m} \Lambda_{n]} - i[B_{mn}, \lambda] - ic[f_{mn}, \Lambda_0] \]

for the 5d KK modes. Here the coefficient \( c \) (that may depend on the mode number \( n \)) is not fixed by demanding closure of these gauge transformations alone. Instead this coefficient will be fixed below in a different way. The closure relation for these gauge variations reads

\[ [\delta \Lambda', \delta \Lambda] = \delta \Lambda'' \]

where the new gauge parameters are given by

\[ \Lambda''_m = -i ([\Lambda', \Lambda_m] - [\lambda, \Lambda'_m]) \]
\[ \Lambda''_0 = -i ([\Lambda', \Lambda_0] - [\lambda, \Lambda'_0]) \]
\[ \Lambda' = -i [\lambda', \lambda] \]

These closure relations hold for any value of \( c \). The YM field strength

\[ f_{mn} = \partial_m a_n - \partial_n a_m - i[a_m, a_n] \]

transforms homogeneously

\[ \delta f_{mn} = -i[f_{mn}, \lambda] \]

For the KK modes, we shall define the field strengths as

\[ H_{mnp} = 3D_mB_{np} - \frac{3}{n} [f_{mn}, A_p] \]
\[ F_{mn} = inB_{mn} + 2D_mA_n \]

These transform homogeneously

\[ \delta H_{mnp} = -i[H_{mnp}, \lambda] \]
\[ \delta F_{mn} = -i [F_{mn}, \lambda] \]

only for one particular value

\[ c = -\frac{1}{in} \]

and so this is what ultimately determines the value of this parameter in the gauge variations. One may now check that the Bianchi identity

\[ 3D_v F_{mn} - inH_{mnr} = 0 \]

is satisfied.

By using the modified selfduality condition (3.2), one can show that the supersymmetry variations

\[ \begin{align*}
\delta \sigma &= -i \bar{\varepsilon} \chi \\
\delta A_m &= i r \bar{\varepsilon} \gamma_m \chi \\
\delta B_{mn} &= i \bar{\varepsilon} \gamma_{mn} \chi - \frac{ir^2}{n} ([\sigma, \bar{\varepsilon} \gamma_{mn} \psi] - [\phi, \bar{\varepsilon} \gamma_{mn} \chi]) - \frac{2ir}{n} [A_n, \bar{\varepsilon} \gamma_m \psi] \\
\delta \chi &= \frac{1}{2r} \gamma^m \bar{\varepsilon} F_{mn} + \gamma^m \bar{\varepsilon} D_m \sigma + \frac{in}{r} \varepsilon \sigma - ir \varepsilon [\phi, \sigma]
\end{align*} \]

close on-shell up to a gerbe gauge transformation for each field \( \Phi = (\sigma, A_m, B_{mn}, \chi) \),

\[ \delta^2 \Phi = -is^m \partial_m \Phi - \frac{sn}{r} \Phi + \delta_{\text{gauge}} \Phi \]

with the gerbe gauge transformation parameters

\[ \begin{align*}
\lambda &= is^m \left( a_m - r \frac{s_m}{s} \phi \right) \\
\Lambda_m &= \frac{is}{r} \left( A_m - r \frac{s_m}{s} \sigma \right) - iB_{mn}s^n + \frac{ir}{n} \left[ A_m - r \frac{s_m}{s} \sigma, \phi \right] \\
\Lambda_0 &= is^m \left( A_m - r \frac{s_m}{s} \sigma \right)
\end{align*} \]

and that the supersymmetry variations (3.1) is a consequence of the above. What gets clarified when we express the supersymmetry variations in terms of the gauge potentials, is the underlying gerbe gauge structure and the fact that the modified selfdual Bianchi identity really shall be viewed as an ordinary Bianchi identity with the modified selfduality condition (3.2). This is in particular needed in order to obtain closure on \( B_{mn} \).

5 Discussion

One may obtain off-shell supersymmetry variations, include 6d hypermultiplets, and try to enhance to \((2,0)\) superconformal symmetry. This is work in progress.
Finally we should probably discuss the two proposals, \cite{5}, \cite{6} versus \cite{3}. Can both proposals be valid at the same time, or does one of them have to be wrong? For the proposal in \cite{5}, \cite{6}, few checks have been concerning the non-BPS sector of the theory. Maybe the conjecture is valid for the BPS sector, but as we go beyond the BPS sector maybe we will need another description for the non-Abelian tensor multiplet?

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A Spinor conventions

The 5d spinors have been studied in for example \cite{7}, \cite{8}. We have four-component spinors in 5d, whose spinor indices we usually suppress. But in this appendix, we will display all the indices. The spinor indices are denoted $\alpha, \beta, \ldots$. We use the NW-SE convention to rise and lower indices,

\begin{align*}
v^\alpha &= C^\alpha{}\beta v_\beta \\
v_\alpha &= v^\beta C^\beta{}\alpha
\end{align*}

where $C_{\alpha\beta}$ is the charge conjugation matrix. It is antisymmetric. We define

\begin{align*}
C^{\alpha\beta} &= C^{\alpha\gamma} C^{\beta\gamma'} C_{\gamma'\delta'} \\
C^{\alpha\beta} &= C^{\alpha'\beta'} C_{\alpha'\beta'}
\end{align*}

Consistency requires that

\begin{align*}
C^{\alpha\beta} C_{\beta\gamma} &= -\delta^\alpha_{\beta}
\end{align*}

There are two supercharges, labeled by an index $I = 1, 2$. These two supercharges form a doublet of the $SU(2)$ R symmetry. We use the same type of NW-SE convention for the index $I$,

\begin{align*}
v^I &= \epsilon^{IJ} v_J \\
v_I &= v^J \epsilon_{IJ}
\end{align*}

and

\begin{align*}
\epsilon^{IJ} \epsilon_{JK} &= -\delta^I_K
\end{align*}
where $\epsilon_{IJ}$ is the antisymmetric tensor of $SU(2)$. Throughout this paper we use commuting supersymmetry parameters $\epsilon_I^\alpha$. We have the following Fierz identity

$$\epsilon^I_\alpha \epsilon^\beta_J = \frac{1}{8} \epsilon_{IJ} \left( C^{\alpha \beta} s + \gamma_m^{\alpha \beta} s^m \right) + \frac{1}{8} \gamma_m^{\alpha \beta} \theta_{IJ}^{mn}$$

Using the NW-SE convention, and

$$C^{\alpha \beta} = -C^{\beta \alpha}$$
$$\gamma_m^{\alpha \beta} = -\gamma_m^{\beta \alpha}$$
$$\gamma_m^{\alpha \beta} = \gamma_m^{\beta \alpha}$$

we then get

$$s = \epsilon_{IJ} \epsilon^\alpha_I C^{\alpha \beta} \epsilon^\beta_J$$
$$s^m = \epsilon_{IJ} \epsilon^\alpha_I \gamma_m^{\alpha \beta} \epsilon^\beta_J$$

and

$$\theta_{IJ}^{mn} = \epsilon^\alpha_I \gamma_{\alpha \beta}^{pq} \epsilon^\beta_J$$

Upon contractions, we get

$$\epsilon_{IJ} \epsilon^\alpha_I C^{\alpha \beta} \epsilon^\beta_J = \frac{s}{2} \theta_{IJ}^{\alpha \beta}$$
$$\epsilon^\alpha_I C^{\alpha \beta} \epsilon^\beta_J = \frac{s}{2} \epsilon_{IJ}$$

where

$$\theta_{IJ}^{\alpha \beta} = \frac{1}{2} \left( C^{\alpha \beta} + \gamma_m^{\alpha \beta} s^m - \frac{s}{8} \right)$$

Now here is a subtlety. When we lower $\beta$ with $C_{\beta \gamma}$, we get

$$\theta_{\alpha \beta} = -\theta_{\beta \alpha}$$

where we define

$$\theta_{\alpha \beta} = \frac{1}{2} \left( \delta_{\alpha \beta} - \gamma_{\alpha \gamma}^\beta \right)$$
$$\gamma_{\alpha \beta}^\gamma = (\gamma_m)^\alpha_{\beta \gamma} \frac{s^m}{s}$$

Using this, we get

$$\theta_{\alpha \beta} \epsilon^\beta_I = \epsilon^\alpha_I$$
or
\[
\gamma^\alpha_\beta \varepsilon^\beta_I = -\varepsilon^\alpha_I \tag{A.1}
\]

Let us check this is really consistent with our definitions. We have
\[
s^m s_m = \epsilon^{IJ} \gamma_m^\alpha \gamma_m^\beta s_m \varepsilon^\alpha_I \varepsilon^\beta_J
\]

In order for the right-hand side to be equal to \(s^2\), we must have
\[
\gamma_m^\alpha s_m \varepsilon^\alpha_J = s C_\alpha^\beta \varepsilon^\beta_J
\]

Now rising \(\alpha\) and using \(C_\alpha^\beta = -\delta^\alpha_\beta\) we get the Weyl projection (A.1).

We have the Fierz identity
\[
(\gamma^m)_{\alpha \beta} (\gamma_m)_{\gamma \delta} = 2 C_\delta^\alpha C_{\gamma \beta} - 2 C_{\delta \beta} C_{\alpha \gamma} - C_{\alpha \beta} C_{\gamma \delta}
\]

Using this, we get
\[
s^m s_m = s^2
\]

We have
\[
\gamma_m = \gamma'_m + \gamma \frac{s_m}{s}
\]
\[
\gamma_{mn} = \gamma'_{mn} + 2 \gamma'_m \gamma \frac{s_n}{s}
\]

subject to the Clifford algebra
\[
\{\gamma, \gamma'_m\} = 0
\]
\[
\{\gamma'_{mn}, \gamma'_{n}\} = 2 \left( G_{mn} - \frac{s_m s_n}{s^2} \right)
\]

Using the identities
\[
\gamma_{mnqr} \gamma_{pq} = -2 \gamma_{mnr}
\]
\[
\gamma_{mnqr} = \varepsilon_{mnqr}
\]

we get upon contraction with \(s^r\) the new identity
\[
\gamma'_{mn} \gamma_s = -\frac{1}{2} \varepsilon_{mnqr} \gamma_{pq} s^r
\]

By noting that
\[
\gamma_{mn} \frac{s^n}{s} = \gamma'_m \gamma
\]
When we suppress indices, we use the following conventions, $\varepsilon = \varepsilon^I_i$ and $\bar{\varepsilon} = \bar{\varepsilon}_I^\alpha = \varepsilon^T C = \varepsilon^I_j \varepsilon^\beta C_{\beta\alpha}$. For instance, in this notation, we have

$$\gamma \varepsilon = -\varepsilon$$

and by taking the transpose, we get

$$\varepsilon^T C \gamma = -\varepsilon^T C$$

Since we use the NW-SE rule, we have

$$\gamma^m_{\alpha\beta} = (\gamma^m)^{\gamma}_\beta C_{\gamma\alpha} = -C_{\alpha\gamma} (\gamma^m)^{\gamma}_\beta$$

with an extra minus sign. This in turn means that

$$s^m = \bar{\varepsilon}^I \gamma^m_{\gamma\alpha}$$
$$s = -\bar{\varepsilon}^I \varepsilon_I$$

This extra minus sign could have been avoided if we had chosen to instead using the SW-NE rule for the $\alpha$ index. In this paper, we will stick to the NW-SE rule for both the $\alpha$ and $I$ indices.

## B The circle bundle

Here we collect some circle bundle equations and relate 5d with 6d language following \[9\] which was concerned with the SYM part, but here we also consider the KK modes.

The 11d gamma matrices are denoted $\Gamma_M$ for $M = 0, 1, 2, 3, 4, 5$ and $\Gamma_A$ for $A = 1, 2, 3, 4, 5$. The 6d tensor multiplet has the following supersymmetry variations

$$\delta \sigma = -i \bar{\epsilon} \chi$$
$$\delta B_{MN} = i \bar{\epsilon} \Gamma_{MN} \chi$$
$$\delta \chi = \frac{1}{12} \Gamma_{MNP} \varepsilon H_{MNP} + \Gamma^M \varepsilon \partial_M \sigma + 4 \eta \sigma$$

where the supersymmetry parameter satisfies the conformal Killing spinor equation

$$D_M \epsilon = \Gamma_M \eta$$

The 6d supersymmetry parameter and the spinor field are subject to the Weyl projections

$$\Gamma \epsilon = -\varepsilon$$
$$\Gamma \chi = \chi$$
where we define the 6d chirality matrix as
\[ \Gamma = \Gamma^{012345} \]

We define \( H_{MNP} = 3 \partial_{[M} B_{NP]} \). We find the following closure relations
\[
\begin{align*}
\delta^2 \sigma &= -i \mathcal{L}_S \sigma - \frac{i}{3} (D^M S_M) \sigma \\
\delta^2 H_{MNP} &= -i \mathcal{L}_S H_{MNP} \\
\delta^2 \psi &= -i \mathcal{L}_S \psi - \frac{5i}{12} (D^M S_M) \psi \\
&\quad + \frac{i}{4} S^M \Gamma_M \Gamma^N D_N \psi
\end{align*}
\]
where \( \mathcal{L}_S \) denotes the Lie derivative along \( S^M = \bar{e} \Gamma^M e \). Upon expanding these equations in flat space, we get closure relations that realize the superconformal algebra.

We now consider the 6d tensor multiplet on a circle bundle with the metric
\[
ds^2 = -r^2 (dt + \kappa_m dx^m)^2 + G_{mn} dx^m dx^n
\]
The vielbein is
\[
e^\hat{M}M = \begin{pmatrix} r & r \kappa_m \\ 0 & E^\hat{m}m \end{pmatrix}
\]
with the inverse
\[
e^M \hat{M} = \begin{pmatrix} 1 & -\kappa_\hat{m} \\ r & E^m \hat{m} \end{pmatrix}
\]
From these we get the 6d metric components as
\[
g_{mn} = G_{mn} - r^2 \kappa_m \kappa_n \\
g_{m0} = -r^2 \kappa_m
\]
and
\[
g^{mm} = G^{mm} \\
g^{m0} = -\kappa^m \\
g^{00} = -\frac{1}{r^2} + \kappa_m \kappa^m
\]
where we define \( \kappa^m = G^{nm} \kappa_n \) and \( G_{mn} \) denote the 5d metric components.

We define the 6d and 5d spin connections from
\[
de^\hat{M} + \omega^{\hat{M} \hat{N}} \wedge e^{\hat{N}} = 0
\]
\[ dE^\hat{m} + \Omega_{\hat{m}\hat{n}} \wedge E^\hat{n} = 0 \]

where

\[
\begin{align*}
e^\hat{\delta} &= r (dt + \kappa) \\
e^\hat{m} &= E^\hat{m}
\end{align*}
\]

We find

\[
\begin{align*}
\omega_{\hat{p}\hat{n}} &= \Omega_{\hat{p}\hat{n}} \\
\omega_{\hat{0}\hat{n}} &= \frac{1}{r} D^\hat{m} r \\
\omega_{\hat{n}\hat{0}} &= \frac{-r}{2} w^\hat{m} \hat{n} \\
\omega_{\hat{0}\hat{0}} &= \omega_{\hat{n}\hat{n}}
\end{align*}
\]

Alternatively

\[
\begin{align*}
\omega_{\hat{p}\hat{n}} &= \Omega_{\hat{p}\hat{n}} - \frac{r^2}{2} \kappa_p w^\hat{m} \hat{n} \\
\omega_{\hat{0}\hat{n}} &= \frac{-r^2}{2} w^\hat{m} \hat{n} \\
\omega_{\hat{0}\hat{0}} &= D^\hat{m} r \\
\omega_{\hat{p}\hat{0}} &= \frac{-r}{2} w_p \hat{n} + \kappa_p D^\hat{m} r
\end{align*}
\]

Hence the covariant derivatives are

\[
D_M \chi = \partial_M \chi + \frac{1}{4} \omega^\hat{M} \hat{N} \Gamma^\hat{N} \hat{M} \chi
\]

We get

\[
\begin{align*}
D_0 \chi &= \partial_0 \chi - \frac{r^2}{8} w_{mn} \Gamma^{mn} \chi + \frac{1}{2} (D_m r) \Gamma^\hat{m} \hat{0} \chi \\
D_m \chi &= \tilde{D}_m \chi - \frac{r^2}{8} \kappa_m w_{pq} \Gamma^{pq} \chi + \frac{1}{2} \kappa_m D_m r \Gamma^\hat{m} \hat{0} \chi + \frac{r}{4} w_{mn} \Gamma^\hat{m} \hat{0} \chi \\
&= \tilde{D}_m \chi + \kappa_m (D_0 - \partial_0) \chi + \frac{r}{4} w_{mn} \Gamma^\hat{m} \hat{0} \chi
\end{align*}
\]

where we define

\[
\tilde{D}_m \chi = \tilde{D}_m \chi - \kappa_m \partial_0 \chi
\]

and we put tilde on 5d quantities, so for example

\[
\tilde{D}_m = \partial_m + \frac{1}{4} \Omega^\hat{m} \hat{n} \Gamma^\hat{n}
\]

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On the other hand, the Lie derivative is given by

$$L_V \chi = V^M D_M \chi + \frac{1}{8} W_{MN} \Gamma^{MN} \chi,$$

$$W_{MN} = \partial_M V_N - \partial_N V_M$$

Then

$$L_V \chi = D_0 \chi + \frac{r^2}{8} w_{mn} \Gamma^{mn} \chi - \frac{1}{2} (D_m r) \Gamma^m \hat{\chi}$$

so that when we expand out the spin connection, we get

$$L_V \chi = \partial_0 \chi$$

We get

$$\chi \Gamma^M D_M \chi = \frac{1}{r} \chi \Gamma^0 \partial_0 \chi + \chi \Gamma^m \hat{D}_m \chi + \frac{r}{8} w_{mn} \chi \Gamma^{mn} \Gamma^0 \chi$$

We define 6d and 5d gamma matrices as

$$\Gamma_M = \Gamma_M^\hat{M} e_\hat{M}$$
$$\hat{\Gamma}_m = \Gamma_m^\hat{m} e_\hat{m}$$

These are related as

$$\Gamma_m = \hat{\Gamma}_m + r \kappa_m \Gamma_0$$
$$\Gamma_0 = r \Gamma_\hat{0}$$

and

$$\Gamma^m = \hat{\Gamma}^m$$
$$\Gamma^0 = \frac{1}{r} \Gamma^\hat{0} - \kappa_m \hat{\Gamma}^m$$

We define

$$\Gamma^{MNPQRS} = \varepsilon^{MNPQRS} \Gamma$$

We now use the gamma matrix identity

$$\Gamma^{MNPQRS} \Gamma_{QRS} = -6 \Gamma^{MNP}$$

together with the above definition, to get

$$\frac{1}{6} \varepsilon^{MNP} \Gamma_{QRS} \Gamma^{QRS} = \Gamma^{MNP} \Gamma$$
Now let us define

$$H_{MNP}^\pm = \frac{1}{2} \left( H_{MNP} \pm \frac{1}{6} \varepsilon_{MNP}^{QRS} H_{QRS} \right)$$

These are subject to

$$H_{MNP}^\pm = \frac{1}{6} \varepsilon_{MNP}^{QRS} H_{QRS}^\pm$$

We have

$$\Gamma^{MNP} \varepsilon H_{MNP} = \Gamma^{MNP} \varepsilon H_{MNP}^\pm$$

and thus the relevant piece of the field strength is the selfdual part that belongs to the tensor multiplet. It is subject to the selfduality constraint

$$H_{MNP}^\pm = \frac{1}{6} \varepsilon_{MNP}^{QRS} H_{QRS}^\pm$$

Now we will reduce this selfduality constraint to 5d. We start by defining the 6d covariant epsilon tensors

$$\varepsilon_{M_1 \cdots M_6} = \epsilon_{M_1}^{\hat{M}_1} \cdots \epsilon_{M_6}^{\hat{M}_6} \epsilon_{\hat{M}_1 \cdots \hat{M}_6}$$

$$\varepsilon_{M_1 \cdots M_6} = \sqrt{-g} \varepsilon_{M_1 \cdots M_6}$$

where

$$\varepsilon_{012345} = 1$$

$$\varepsilon_{012345} = -1$$

Likewise, we define the 5d covariant epsilon tensors as

$$\mathcal{E}_{m_1 \cdots m_5} = E_{m_1}^{\hat{m}_1} \cdots E_{m_5}^{\hat{m}_5} \epsilon_{\hat{m}_1 \cdots \hat{m}_5}$$

$$\mathcal{E}_{m_1 \cdots m_5} = \sqrt{G} \epsilon_{m_1 \cdots m_5}$$

$$\mathcal{E}_{m_1 \cdots m_5} = E_{m_1}^{\hat{m}_1} \cdots E_{m_5}^{\hat{m}_5} \epsilon_{\hat{m}_1 \cdots \hat{m}_5}$$

$$\mathcal{E}_{m_1 \cdots m_5} = \frac{1}{\sqrt{G}} \epsilon_{m_1 \cdots m_5}$$

where

$$\epsilon_{12345} = 1$$
By noting that
\[ \sqrt{-g} = r \sqrt{G} \]
we get the following relations
\[ \varepsilon_{0m_1 \cdots m_5} = -r \mathcal{E}_{m_1 \cdots m_5} \]
\[ \varepsilon^0 _{m_1 \cdots m_5} = - \frac{1}{r} \mathcal{E}^{m_1 \cdots m_5} \]

We have
\[
H_{mn0} = \frac{1}{6} \varepsilon_{mn0}^{\quad qrs} H_{qrs} + \frac{1}{2} \varepsilon_{mn0}^{\quad qr0} H_{qr0} \\
= - \frac{r}{6} \varepsilon_{mn}^{\quad qrs} H_{qrs} + \frac{r}{2} \varepsilon_{mn}^{\quad qr0} H_{qr0} \kappa_s \\
= - \frac{r}{6} \varepsilon_{mn}^{\quad qrs} \left( H_{qrs} - 3 H_{qr0} \kappa_s \right)
\]
that we can invert,
\[
H_{qrs} = - \frac{1}{2r} \varepsilon_{qrs}^{\quad mn} H_{mn0} + 3 H_{[qr0] \kappa_s} 
\]

We have
\[ \Gamma^{mn0} = \frac{1}{r} \tilde{\Gamma}^{mn} \Gamma^0 \kappa_p \\
\Gamma^{mnp} = \Gamma^{mnp} \]
and also
\[ \Gamma^{mn} \Gamma^0 + \Gamma^0 \Gamma^{mn} + \Gamma^m \Gamma^0 \Gamma^m = \frac{3}{r} \tilde{\Gamma}^{mn} \tilde{\Gamma}^0 \kappa_p - 3 \tilde{\Gamma}^{mnp} \kappa_p \]
which leads to
\[
3 \Gamma^{mn0} \varepsilon H_{mn0} = \frac{3}{r} \tilde{\Gamma}^{mn} \tilde{\Gamma}^0 \varepsilon H_{mn0} - 3 \tilde{\Gamma}^{mnp} \varepsilon H_{mn0} \kappa_p \\
\Gamma^{mnp} \varepsilon H_{mnp} = - \frac{1}{2r} \tilde{\Gamma}^{mnp} \varepsilon \varepsilon H_{qrs} + 3 \tilde{\Gamma}^{mnp} \varepsilon H_{mn0} \kappa_p 
\]
Adding these, we find a nice cancelation,
\[
\frac{1}{6} \Gamma^{MNP} \varepsilon H_{MNP} = \frac{1}{r} \tilde{\Gamma}^{mn} \tilde{\Gamma}^0 \varepsilon H_{mn0} 
\]

We have
\[ \Gamma^M \varepsilon D_M \sigma = \tilde{\Gamma}^m \varepsilon (D_m \sigma - \kappa_m D_0 \sigma) + \frac{1}{r} \tilde{\Gamma}^0 \varepsilon D_0 \sigma \]

We have
\[ \Gamma_{mn} = \tilde{\Gamma}_{mn} + 2 r \kappa_{[m} \Gamma_{n]} \]
\[ \Gamma_{m0} = r \tilde{\Gamma}_m \Gamma_0 \Gamma^0 - r^2 \kappa_m \]

We are now ready to decompose the Abelian supersymmetry variations into one zero mode part. We define

\[ a_m = B_{m0}^{(0)} \]

Then we get

\[ \delta \sigma^{(0)} = -i \bar{\epsilon} \chi^{(0)} \]
\[ \delta a_m = i r \bar{\epsilon} \tilde{\Gamma}_m \Gamma_0 \chi^{(0)} - i r^2 \kappa_m \bar{\epsilon} \chi^{(0)} \]
\[ \delta \chi^{(0)} = \frac{1}{2r} \tilde{\Gamma}^{mn} \Gamma_0 \bar{\epsilon} F_{mn} + \tilde{\Gamma}^m \bar{\epsilon} D_m \sigma^{(0)} + 4 \eta \sigma^{(0)} \]

For the KK-modes, we get

\[ \delta \sigma^{(n)} = -i \bar{\epsilon} \chi^{(n)} \]
\[ \delta B_{m0}^{(n)} = i r \bar{\epsilon} \tilde{\Gamma}_m \Gamma_0 \chi^{(n)} - i r^2 \kappa_m \bar{\epsilon} \chi^{(n)} \]
\[ \delta B_{mn}^{(n)} = i \bar{\epsilon} \tilde{\Gamma}_{mn} \chi^{(n)} + 2 i r \kappa_m \bar{\epsilon} \tilde{\Gamma}_0 \tilde{\Gamma}_n \chi^{(n)} \]
\[ \delta \chi^{(n)} = \frac{1}{2r} \tilde{\Gamma}^{mn} \Gamma_0 \bar{\epsilon} H_{m0}^{(n)} + \tilde{\Gamma}^m \bar{\epsilon} \left( D_m \sigma^{(n)} - \kappa_m D_0 \sigma^{(n)} \right) + \frac{1}{r} \Gamma_0 \bar{\epsilon} D_0 \sigma^{(n)} + 4 \eta \sigma^{(n)} \]

Now we use

\[ \Gamma^0 = i \sigma^2 \otimes 1 \otimes 1 \]
\[ \Gamma^m = \sigma^1 \otimes \gamma^m \otimes 1 \]

We have

\[ \Gamma = \sigma^3 \otimes 1 \otimes 1 \]

We have \( \Gamma \chi = \chi \), so we get

\[ \Gamma^0 \chi = -\chi \]
\[ \tilde{\Gamma}^m \chi = \gamma^m \chi \]

We get

\[ \delta \sigma^{(0)} = -i \bar{\epsilon} \chi^{(0)} \]
\[ \delta a_m = i r \bar{\epsilon} \gamma_m \chi^{(0)} - i r^2 \kappa_m \bar{\epsilon} \chi^{(0)} \]
\[ \delta \chi^{(0)} = \frac{1}{2r} \gamma^{mn} \bar{\epsilon} f_{mn} + \gamma^m \bar{\epsilon} D_m \sigma^{(0)} + 4 \eta \sigma^{(0)} \]

and

\[ \delta \sigma^{(n)} = -i \bar{\epsilon} \chi^{(n)} \]

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\[
\delta B^{(n)}_{m0} = i r \tilde{\epsilon} \gamma_m \chi^{(n)} - i r^2 \kappa_m \tilde{\epsilon} \chi^{(n)} \\
\delta B^{(n)}_{mn} = i \tilde{\epsilon} \gamma_{mn} \chi^{(n)} - 2 i r \kappa_m \tilde{\epsilon} \gamma_n \chi^{(n)} \\
\delta \chi^{(n)} = \frac{1}{2r} \gamma^{mn} \varepsilon H^{(n)}_{mn0} + \gamma^m \varepsilon \left( D_m \sigma^{(n)} - i n \kappa_m \sigma^{(n)} \right) + \frac{i n}{r} \varepsilon \sigma^{(n)} + 4 \eta \sigma^{(n)}
\]

By putting $\kappa_m$ to zero, these supersymmetry variations reduce to those that appear in the main text for Abelian gauge group.

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