ORBITS OF ACTIONS OF GROUP SUPERSCHEMES

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Abstract. Working over an algebraically closed field \( k \), we prove that all orbits of a left action of an algebraic group superscheme \( G \) on a superscheme \( X \) of finite type are locally closed. Moreover, such an orbit \( Gx \), where \( x \) is a \( k \)-point of \( X \), is closed if and only if \( G_{ev}x \) is closed in \( X_{ev} \), or equivalently, if and only if \( G_{res}x \) is closed in \( X_{res} \). Here \( G_{ev} \) is the largest purely even group super-subscheme of \( G \) and \( G_{res} \) is \( G_{ev} \) regarded as a group scheme. Similarly, \( X_{ev} \) is the largest purely even super-subscheme of \( X \) and \( X_{res} \) is \( X_{ev} \) regarded as a scheme. We also prove that \( \text{sdim}(Gx) = \text{sdim}(G) - \text{sdim}(G_x) \), where \( G_x \) is the stabilizer of \( x \).

Introduction

One of the fundamental properties of the action of a (not necessary affine) algebraic group scheme \( G \) on a scheme \( X \) of finite type is the existence of closed orbits (see \cite[Proposition 7.6]{13}, or \cite[II, §5, Proposition 3.2]{3}). In these notes we show that the similar property takes place for the action of a (not necessary affine) algebraic group superscheme \( G \) on a superscheme \( X \) of finite type.

Contrary to the purely even case we can not use directly the standard argument on orbits of minimal dimension. The difference is that for a closed super-subscheme \( Z \) of \( X \), even if its underlying topological space \( Z_{ev} \) is \( G(k) \)-stable, \( Z \) is not necessary \( G \)-stable. It is clear why, \( G(k) \) is a group of \( k \)-points of the largest purely even super-subgroup \( G_{ev} \) of \( G \). In general, \( G \) is a product of \( G_{ev} \) and some normal group subfunctor \( N(G) \) (in terms of \cite{10}, the latter is called the formal neighborhood of the identity in \( G \)), and the action of \( N(G) \) on \( X \) is not controlled by \( G(k) \) at all.

To overcome this difficulty, we first prove that for any point \( x \in X(k) \) the superscheme morphism \( \eta_x : G/G_x \to X \), induced by the orbit morphism \( a_x : G \to X, g \mapsto gx \), is an immersion. In other words, the orbit \( Gx \) is always a locally closed super-subscheme of \( X \). Therefore, \( Gx \) is closed if and only if \( (Gx)^c = (G_{ev}x)^c \) is.

The above two principal results are based on several auxiliary results that are interesting on their own. First, we introduce the super-dimension of certain Noetherian superschemes. This generalizes the previous definition from \cite{12}. We use the same notation \( \text{sdim}(X) \) as in \cite{12}. Then we prove that the super-dimension of a sheaf quotient \( G/H \), where \( G \) is an algebraic group superscheme and \( H \) is its (closed) group super-subscheme, is equal to \( \text{sdim}(G) - \text{sdim}(H) \). The proof of this result is made possible by recent progress in the study of sheaf quotients of algebraic group superschemes (cf. \cite[Theorem 14.1]{10}).

Recall that with any superscheme \( X \) one can associate a graded superscheme \( \text{gr}(X) \) as follows. The underlying topological space of \( \text{gr}(X) \) coincides with \( X^c \) and the superalgebra sheaf of \( \text{gr}(X) \) is isomorphic to the sheafification of the presheaf

\[
U \mapsto \bigoplus_{n \geq 0} \mathcal{I}_X(U)^n / \mathcal{I}_X(U)^{n+1},
\]
where $U$ runs over open subsets of $X^e$ and $\mathcal{L}_X$ is the superideal sheaf generated by $(O_X)_1$. If $G$ is a (locally) algebraic group superscheme, then $\text{gr}(G)$ has the natural structure of a (locally) algebraic group superscheme as well. Moreover, $G \to \text{gr}(G)$ is an endofunctor of both categories of locally algebraic and algebraic group superschemes. We show that if $G$ is algebraic and $H$ is a closed group subscheme of $G$, then $\text{gr}(G/H) \cong \text{gr}(G)/\text{gr}(H)$. Besides, $(G/H)_{ev} \cong G_{ev}/H_{ev}$. These results extend [9, Proposition 4.18] and [15, Corollary 6.23].

The article is organized as follows. In the first section we recall some elementary properties of super-commutative superalgebras. In the second section we recall the notion of Krull super-dimension of a Noetherian superalgebra, whose even component is a Noetherian algebra of finite Krull dimension. The definition of Krull super-dimension reflects the notion of longest system of odd parameters, which is more general and less restrictive than the long known notion of odd regular sequence (see [12, 14]).

The third section contains all the necessary facts about superschemes and group superschemes. In the fourth section we recall the definition of the category of Harish-Chandra pairs. It has been recently proven that this category is naturally equivalent to the category of locally algebraic group superschemes (see [10, Theorem 12.10], we also refer the reader to [11, §] for a better understanding of the development of this concept).

In the fifth section we recall the definition of left/right action of a group superscheme on a superscheme. For a given $k$-point $x$ we also define the orbit morphism $a_x : G \to X$ and the induced morphism $\bar{a}_x$. The sixth section is devoted to the proving of the above mentioned isomorphisms $\text{gr}(G/H) \cong \text{gr}(G)/\text{gr}(H)$ and $(G/H)_{ev} \cong G_{ev}/H_{ev}$. We also characterize locally algebraic group superschemes which isomorphic to a graded group superscheme. The content of seventh section is about the notion of super-dimension of certain Noetherian superschemes. In the eighth section we prove an analog of the well known formula for the dimension of a sheaf quotient $G/H$ in the category of algebraic group superschemes. In ninth section we prove the main Theorem 9.5. We conclude with some elementary example of an action of the odd unipotent group superscheme $G^{-a}$ with respect to which all orbits are closed.

1. Superalgebras

Throughout this article $k$ is a field of odd or zero characteristic. A $\mathbb{Z}_2$-graded $k$-algebra $A$ is said to be a superalgebra. The homogeneous components of $A$ are denoted by $A_0$ and $A_1$. The elements of $A_0$ are called even and the elements of $A_1$ are called odd. We have the parity function $(A_0 \sqcup A_1) \setminus 0 \to \mathbb{Z}_2, a \to |a|$, which maps $A_0 \setminus 0$ to 0 and $A_1 \setminus 0$ to 1, correspondingly.

A superalgebra $A$ is called super-commutative, provided $ab = (-1)^{|a||b|}ba$ for any couple of homogeneous elements $a$ and $b$. Throughout this article all superalgebras are supposed to be super-commutative, unless stated otherwise. The category of superalgebras (with graded morphisms) is denoted by $S\text{Alg}_k$.

Let $A$ be a superalgebra. Then a prime (maximal) superideal of $A$ has a form $\mathfrak{p} = \mathfrak{p} \oplus A_1$, where $\mathfrak{p}$ is a prime (respectively, maximal) ideal of $A_0$. In particular, $A$ is a local superalgebra if and only if $A_0$ is a local algebra. A localization $A_{\mathfrak{p}}$ is defined as $(A_0 \setminus \mathfrak{p})^{-1}A$. It is clear that $A_{\mathfrak{p}}$ is a local superalgebra. If $A$ and
B are local superalgebras with maximal superideals P and Ω, then a superalgebra morphism $\phi : A \to B$ is said to be local if $\phi(P) \subseteq \Omega$.

Let $k[x_1, \ldots, x_m \mid y_1, \ldots, y_n]$ denote a polynomial superalgebra, freely generated by even and odd indeterminants $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$.

A superalgebra $A$ is called graded, provided $A$ is $\mathbb{N}$-graded, say $A = \oplus_{n \geq 0} A(n)$, and $A_0 = \oplus_{n \geq 0} A(2n)$, $A_1 = \oplus_{n \geq 0} A(2n + 1)$. We can associate with arbitrary superalgebra $A$ a graded superalgebra $\text{gr}(A) = \oplus_{n \geq 0} I_A^n/I_A^{n+1}$, where $I_A = AA_1$. The 0-th component of $\text{gr}(A)$ is denoted by $\bar{A}$. We also say that a graded superalgebra $A$ is Grassman graded, provided $A(n) = A(1)^n$ for all $n \geq 1$. It is obvious that $A$ is Grassman graded if and only if $A \cong \text{gr}(A)$.

**Remark 1.1.** It is clear that $A \to \text{gr}(A)$ is an endofunctor of the category $\text{SAlg}_k$.

Moreover, if $f : A \to B$ is a morphism of Grassman graded superalgebras, i.e. $f(A(n)) \subseteq B(n)$ for each $n \geq 0$, then we have a commutative diagram

$$
\begin{array}{ccc}
gr(A) & \cong & A \\
\downarrow & & \downarrow \\
gr(B) & \cong & B 
\end{array}
$$

whose vertical arrows are $gr(f)$ and $f$ respectively.

2. **Krull super-dimension**

Let $A$ be a Noetherian superalgebra. In other words, $A_0$ is a Noetherian algebra and $A_1$ is a finitely generated $A_0$-module (cf. [12] Lemma 1.4). We also assume that the Krull dimension $K\text{dim}(A_0)$ of $A_0$ is finite. Set $K\text{dim}(A_0) = n$.

A collection of odd elements $y_1, \ldots, y_k \in A_1$ is said to be a system of odd parameters of $A$, provided there is a longest prime chain $p_0 \subseteq \cdots \subseteq p_n$ in $A_0$ such that $\text{Ann}_{A_0}(y_1 \cdots y_k) \subseteq p_0$, or equivalently, $K\text{dim}(A_0/\text{Ann}_{A_0}(y_1 \cdots y_k)) = K\text{dim}(A_0)$. The odd Krull dimension of $A$ is defined as the length of a longest system of odd parameters of $A$ (see [12] Section 4). It is denoted by $K\text{sdim}_0(A)$, as well as $K\text{dim}_0(A)$ and called the even Krull dimension of $A$. The couple of nonnegative integers $K\text{sdim}_0(A) \mid K\text{sdim}_1(A)$ is called just Krull super-dimension of $A$ and denoted by $K\text{sdim}(A)$.

There is a more restrictive notion of an odd regular sequence. More precisely, the odd elements $y_1, \ldots, y_k \in A_1$ form an odd regular sequence, provided $\text{Ann}_{A_0}(y_1 \cdots y_k) = A_0y_1 + \cdots + A_0y_k$ (see [14] Corollary 3.1.2)). It is clear that any odd regular sequence is a system of odd parameters, but it is not necessary a part of some longest system of odd parameters (see [16] Section 3).

Let $A$ be local superalgebra with maximal superideal $\mathfrak{M}$. A is said to be oddly regular if one of the following equivalent conditions hold ([14] Corollary 3.3]):

(i) $I_A$ is generated by an odd regular sequence;

(ii) Every minimal base of $I_A$ is an regular sequence.

Following [14] we denote $A_1/\mathfrak{m}_A A_1$ by $\Phi_A$.

**Lemma 2.1.** Let $k(A)$ denote the residue field $A/\mathfrak{M}$. If $A$ is oddly regular, then $K\text{sdim}_1(A) = \dim_{k(A)}(\Phi_A)$.

**Proof.** Note that $\dim_{k(A)}(\Phi_A)$ equals the minimal number of generators of $A_0$-module $A_1$, and the minimal number of (odd) generators of $I_A$ as well. On the other hand, $K\text{sdim}_1(A) \leq \dim_{k(A)}(\Phi_A)$ by [12] Lemma 5.1. Thus if $A$ is oddly regular, then any minimal base of $I_A$ is the longest system of odd parameters. $\square$
Recall that a geometric superspace $X$ consists of a topological space $X^e$ and a sheaf of super-commutative superalgebras $\mathcal{O}_X$ on $X^e$, such that all stalks $\mathcal{O}_{X,x}$ for $x \in X^e$ are local superalgebras, whose maximal ideals are denoted by $\mathfrak{m}_x$. Define a residue field at a point $x$ as $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$.

A morphism of superspaces $f: X \to Y$ is a pair $(f^e, f^s)$, where $f^e: X^e \to Y^e$ is a morphism of topological spaces and $f^s: \mathcal{O}_Y \to f^*_s\mathcal{O}_X$ is a morphism of sheaves such that $f^*_e: \mathcal{O}_{Y,f^e(x)} \to \mathcal{O}_{X,x}$ is a local morphism for any $x \in X^e$. For any couple of open subsets $U \subseteq V \subseteq X^e$ let $\text{res}_{\mathcal{O}}$ denote the corresponding superalgebra morphism $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$.

Let $R$ be a superalgebra. An affine geometric superscheme $\text{SSpec}(R)$ can be defined as follows. The underlying topological space of $\text{SSpec}(R)$ coincides with the prime spectrum of $R_0$, endowed with the Zariski topology. For any open subset $U \subseteq (\text{SSpec}(R))^e$, the super-ring $\mathcal{O}_{\text{SSpec}(R)}(U)$ consists of all locally constant functions $h: U \to \bigcup_{\mathfrak{p} \in U} R_\mathfrak{p}$ such that $h(\mathfrak{p}) \in R_\mathfrak{p}$ and $\mathfrak{p} \in U$.

A superspace $X$ is called a (geometric) superscheme if there is an open covering $X^e = \bigcup_{i \in I} U_i$, such that each open super-subspace $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine superscheme $\text{SSpec}(R_i)$. Superschemes form a full subcategory of the category of geometric superspaces, denoted by $\mathcal{S}\mathcal{V}$.

A superscheme $X$ is said to be locally of finite type over $k$, if it can be covered by open affine super-subschemes $\text{SSpec}(R_i)$ with each superalgebra $R_i$ to be finitely generated. If additionally this covering is finite, then $X$ is said to be of finite type (over $k$). Finally, a superscheme $X$ is said to be Noetherian, if it can be covered by finitely many open affine super-subschemes $\text{SSpec}(R_i)$, where each $R_i$ is a Noetherian superalgebra.

With any superscheme $X$ we can associate a largest purely even closed super-subscheme $X_{ev} = (X^e, \mathcal{O}_X/\mathcal{I}_X)$ and a graded superscheme $\text{gr}(X) = (X^e, \text{gr}(\mathcal{O}_X))$, where $\text{gr}(\mathcal{O}_X)$ is a sheafification of the presheaf

$$U \mapsto \oplus_{n \geq 0} \mathcal{I}_X(U)^n/\mathcal{I}_X(U)^{n+1}, \quad U \subseteq X^e,$$

and $\mathcal{I}_X$ is a subsheaf of the sheaf $\mathcal{O}_X$ generated by $(\mathcal{O}_X)_1$. In more detail, $\text{gr}(\mathcal{O}_X)$ is a subsheaf of the (superalgebra) sheaf $\prod_{n \geq 0} \mathcal{T}_X^n/\mathcal{T}_X^{n+1}$ such that a section $s \in \prod_{n \geq 0} (\mathcal{T}_X^n/\mathcal{T}_X^{n+1})(U)$ belongs to $\text{gr}(\mathcal{O}_X)(U)$ if and only if there is a covering of $U$ by open super-subschemes $U_1$, with $\text{res}_{U_1}(s) \in \oplus_{n \geq 0} (\mathcal{T}_X^n/\mathcal{T}_X^{n+1})(U_i)$ for each index $i$ (cf. [10 Section 10]). It is easy to see that $\text{gr}(\text{SSpec}(A)) \simeq \text{SSpec}(\text{gr}(A))$.

The superscheme $X_{ev}$ can be regarded as an ordinary scheme, in which case it is denoted by $X_{res}$. Also, one can associate with $X$ a scheme $X_0 = (X^e, (\mathcal{O}_X)_0)$. So, we have two endofunctors $X \to \text{gr}(X)$ and $X \to X_{ev}$ of the category $\mathcal{S}\mathcal{V}$, and two functors $X \to X_{res}$ and $X \to X_0$ from $\mathcal{S}\mathcal{V}$ to the category of schemes. For any superscheme morphism $f: X \to Y$, the induced (super)scheme morphisms $X_{ev} \to Y_{ev}$, $\text{gr}(X) \to \text{gr}(Y)$, $X_{res} \to Y_{res}$ and $X_0 \to Y_0$ are denoted by $f_{ev}$, $\text{gr}(f)$, $f_{res}$ and $f_0$, respectively.

Recall that a superscheme morphism $f: X \to Y$ is called an immersion if there is an open super-subscheme $U$ of $Y$ such that $f^e(X^e) \subseteq U^e$ and $f$ induces an isomorphism of $X$ onto a closed super-subscheme of $U$. This superscheme is said to be a locally closed super-subscheme of $X$.

Equivalently, $f$ is an immersion if and only if $f^e$ is an homeomorphism of $X^e$ onto a closed subset of $U^e$ and the induced morphism of sheaves $f^*: \mathcal{O}_Y|_U \to f_*\mathcal{O}_X$
is surjective (cf. [10] Section 1]). The surjectivity of the sheaf morphism $f^*$ is equivalent to the condition that for any $x \in X^c$ the local superalgebra morphism $\mathcal{O}_{Y,f'(x)} \to \mathcal{O}_{X,x}$ is surjective.

Let $\mathcal{F}$ denote the category of $k$-functors from $\text{SAlg}_k$ to $\text{Sets}$. There is a natural functor from $\mathcal{S}$ to $\mathcal{F}$ defined as

$$
X \to \mathcal{X}, \quad \text{where } \mathcal{X}(A) = \text{Mor}_\mathcal{S}(\text{SSpec}(A), X), \quad A \in \text{SAlg}_k.
$$

The $k$-functor $\mathcal{X}$ is called a functor of points of $X$. For example, if $X \simeq \text{SSpec}(R)$, then $\mathcal{X} \simeq \text{SSp}(R)$, where

$$
\text{SSp}(R)(A) = \text{Mor}_{\text{SAlg}}(R, A), \quad A \in \text{SAlg}_k.
$$

The $k$-functor $\text{SSp}(R)$ is called an affine superscheme.

The functor $X \to \mathcal{X}$ is an equivalence between the category $\mathcal{S}$ and a full subcategory of $\mathcal{F}$. The latter subcategory consists of all local $k$-functors covered by open subfunctors isomorphic to affine superschemes (for more details see [2 Theorem 10.3.7], or [10] Theorem 5.14]). It is is denoted by $\mathcal{SF}$ and the objects of $\mathcal{SF}$ are called just superschemes.

In what follows we use the term superscheme for both geometric superschemes and just superschemes, but we distinguish them by notations, i.e. geometric superschemes and their morphisms are denoted by $X,Y, \ldots$, and $f,g,\ldots$, and just superschemes and their morphisms (natural transformations of $k$-functors) are denoted by $\mathcal{X}, \mathcal{Y}, \ldots$, and $\mathcal{f}, \mathcal{g}, \ldots$, respectively. Similarly, a superscheme $\mathcal{X}$ is of (locally) finite type over $k$ or Noetherian if and only if its geometric counterpart $X$ is. Throughout this article what is proven for geometric superschemes can be naturally translated to the category of superschemes and vice versa.

Note that $f : X \to Y$ is an open (closed) immersion if and only if $\mathcal{f} : \mathcal{X} \to \mathcal{Y}$ is an open (closed) embedding (see [10] Proposition 5.12 and [10] Proposition 1.6]). Therefore, $f : X \to Y$ is an immersion if and only if $f : \mathcal{X} \to \mathcal{Y}$ is a natural isomorphism of $\mathcal{X}$ onto a closed subfunctor of an open subfunctor of $\mathcal{Y}$. As above, we call $f$ an immersion and $\mathcal{X}$ a locally closed super-subscheme of $\mathcal{Y}$.

For example, the closed immersion $X_{ev} \to X$ corresponds to the closed embedding $X_{ev} \to \mathcal{X}$, where

$$
\mathcal{X}(A) = \mathcal{X}(\iota)(\mathcal{X}(A_0)) \simeq \mathcal{X}(A_0), \quad \text{for } A \in \text{SAlg}_k,
$$

and $\iota : A_0 \to A$ is the natural (super)algebra monomorphism.

Let $\text{gr}(\mathcal{X})$ denote the functor of points of $\text{gr}(X)$. As above, $\text{gr}(\text{SSp}(A)) \simeq \text{SSp}(\text{gr}(A))$.

Let $X$ be a superscheme and $U$ be an open super-subscheme of $X$. By [10 Proposition 5.12] we have $\mathcal{O}_X(U) \simeq \text{Mor}_{\mathcal{SF}}(\mathbb{A}^1, \mathbb{A}^1)$, where $\mathbb{A}^1 \simeq \text{SSp}(k[x \mid y])$. If it does not lead to confusion, we denote $\mathcal{O}_X(U)$ just by $\mathcal{O}(U)$, or by $\mathcal{O}(\mathcal{U})$ respectively.

A group object in the category of superschemes of locally finite type over $k$ is called a locally algebraic group superscheme. Respectively, a group object in the category of superschemes of finite type over $k$ is called an algebraic group superscheme. If $G$ (respectively, $\mathcal{G}$) is a (locally) algebraic group superscheme, then $\text{gr}(G)$ (respectively, $\text{gr}(\mathcal{G})$) is a (locally) algebraic group superscheme as well.

Recall that with any group superscheme $G$ one can associate a largest affine factor-group superscheme $G^{aff}$ such that any group superscheme morphism $G \to H$, with $H$ to be affine, factors through $G \to G^{aff}$ (cf. [10 Corollary 5.3]). By
the equivalence of categories, the functor of points of $G^{aff}$ is the largest affine factor-group superscheme of $G$, and it is denoted by $G^{aff}$. If $G$ (respectively, $\mathbb{G}$) is algebraic, then $G^{aff} \simeq \text{SSpec}(\mathcal{O}(G))$ (respectively, $\mathbb{G}^{aff} \simeq \text{Sp}(\mathcal{O}(\mathbb{G}))$).

For any $g \in G(k)$ we have a natural transformation $l_g : G \to G$, defined as $x \mapsto gx$, where $x \in G(A)$ and $A \in \text{SAlg}_k$. We call it a left shift transformation (by $g$). The right shift transformation $r_g$ is defined symmetrically. Since both left and right shifts are natural isomorphisms of $G$ onto itself, they take closed (open) subfunctors to closed (open) subfunctors as well. In particular, for any closed group super-subscheme $H$ of $G$ the left and right cosets $l_g(H)$ and $r_g(H)$ are closed super-subschemas in $G$. We denote them by $gH$ and $HG$ respectively.

4. Harish-Chandra pairs and the fundamental equivalence

For the content of this section we refer to [10].

Recall that a couple $(G, V)$ is called a Harish-Chandra pair, provided $G$ is a purely even locally algebraic group superscheme (which can be regarded as group scheme as well), $V$ is a finite dimensional $G$-(super)module and the following conditions hold:

(a) there is a symmetric bilinear map $V \times V \to \text{Lie}(G)$, $(v, w) \mapsto [v, w]$, $v, w \in V$, that is $G$-equivariant with respect to the diagonal action of $G$ on $V \times V$ and the adjoint action of $G$ on $\text{Lie}(G)$;

(b) the induced action of $\text{Lie}(G)$ on $V$ satisfies $[v, v] \cdot v = 0$ for any $v \in V$.

Harish-Chandra pairs form a category with morphisms $(f, u) : (G, V) \to (H, W)$, where $f : G \to H$ is a morphism of group schemes and $u : V \to W$ is a morphism of $G$-modules ($W$ is regarded as a $G$-module via $f$). Besides, the diagram

$$
\begin{array}{ccc}
V \times V & \to & \text{Lie}(G) \\
\downarrow & & \downarrow \\
W \times W & \to & \text{Lie}(H)
\end{array}
$$

is commutative, where the first vertical map is $u \times u$ and the second vertical map is the differential of $f$. The category of Harish-Chandra pairs is denoted by $\text{HCP}$.

There is a functor from the category of locally algebraic group superschemes to the category of Harish-Chandra pairs

$$
G \to (G_{ev}, \text{Lie}(G)_1)
$$
or

$$
\mathbb{G} \to (\mathbb{G}_{ev}, \text{Lie}(\mathbb{G})_1).
$$

The action of $G_{ev}$ (respectively, of $\mathbb{G}_{ev}$) on $\text{Lie}(G)_1$ (respectively, on $\text{Lie}(\mathbb{G})_1$) is the restriction of the adjoint action of $G$ (respectively, of $\mathbb{G}$) on $\text{Lie}(G)$ (respectively, on $\text{Lie}(\mathbb{G})$). This functor is called Harish-Chandra functor and denoted by $\Phi$.

**Theorem 4.1.** (see [10] Theorem 12.10) The Harish-Chandra functor is an equivalence.

The quasi-inverse of $\Phi$ is denoted by $\Psi$. For any Harish-Chandra pair $(G, V)$ the group superscheme $G = \Psi((G, V))$ is constructed as follows.

Let $k[\varepsilon]$ denote the algebra of dual numbers, i.e. this is the algebra generated by the element $\varepsilon$ subject to the relation $\varepsilon^2 = 0$. Then we have an exact sequence of groups

$$
1 \to \text{Lie}(G) \to G(k[\varepsilon]) \xrightarrow{G(p)} G(k) \to 1,
$$
where $p : \mathbb{k}[\varepsilon] \to \mathbb{k}$ is the algebra morphism such that $p(\varepsilon) = 0$ (cf. [3] II, §4). The image of $x \in \text{Lie}(G)$ in $G(\mathbb{k}[\varepsilon])$ is denoted by $e^{\varepsilon x}$. For any superalgebra $A$ and $b \in A_0$ such that $b^2 = 0$ there is the unique algebra morphism $\alpha_b : \mathbb{k}[\varepsilon] \to A_0$ that takes $\varepsilon$ to $b$. The element $G(\alpha_b)(e^{\varepsilon x}) \in G(A_0)$ is denoted by $f(b, x)$. Let $F(A)$ denote a free group, freely generated by the formal elements $e(a, v)$, where $a \in A_1$, $v \in V$. Then the group $G(A)$ is isomorphic to the factor-group of the free product $G(A_0) * F(A)$ modulo the relations

(i) $[e(a, v), e(a', v')] = f(-aa', [v, v'])$;
(ii) $e(a, v) f(b, x) = e(ab, [v, x])$;
(iii) $f(b, x) f(b', x') = f(bb', [x, x'])$;
(iv) $e(a, v) e(a', v) = f(-aa', [v, v]) e(a + a', v)$;
(v) $e(a, v)^g = e(a, g \cdot v)$, $f(b, x)^g = f(b, g \cdot x)$ for any $g \in G(A_0)$;
(vi) $e(a, c) = e(c, a)$ for any scalar $c \in \mathbb{k}$.

Here the group commutator $[u, v]$ is defined as $uvu^{-1}v^{-1}$ and $u'$ denotes $vu'w^{-1}$. A superalgebra morphism $\beta : A \to A'$ induces the natural group morphism $G(A_0) * F(A) \to G(A'_0) * F(A')$ that takes $e(a, v)$ to $e(\beta(a), v)$, hence it preserves the relations (i)-(v). Thus $\mathcal{G}$ is obviously a group functor. Moreover, by [10] Corollary 12.4 and Theorem 12.5 any element of $G(A)$ can be uniquely expressed as the product

$$ge(a_1, v_1) \cdots e(a_t, v_t),$$

where $g \in G(A_0)$ and $v_1, \ldots, v_t$ is a fixed basis of $V$. In particular, $G$ is isomorphic to $G \times \text{SSAlg}(\Lambda(V^*))$ as a superscheme.

5. Actions, Orbits and Quotients

Let $\mathcal{G}$ be a group superscheme and $\mathcal{X}$ be a superscheme. The functor morphism $f : \mathcal{X} \times \mathcal{G} \to \mathcal{X}$ defines a right action of $\mathcal{G}$ on $\mathcal{X}$, provided the following properties hold:

(1) $f(A)(x, 1) = x$;
(2) $f(A)(f(A)(x, g), h) = f(A)(x, gh)$, $x \in \mathcal{X}(A)$, $g, h \in G(A)$, $A \in \text{SAlg}_k$.

A left action is defined similarly.

In what follows we denote $f(A)(x, g)$ just by $xg$ (respectively, $gx$ for a left action), if it does not lead to confusion. A given action is called free, whenever $xg = x$ (respectively, $gx = x$) implies $g = e$ for any $x \in \mathcal{X}(A)$, $g \in G(A)$.

With each (right) free action of $\mathcal{G}$ on $\mathcal{X}$ one can associate the functor

$$A \mapsto \mathcal{X}(A)/G(A), \quad A \in \text{SAlg}_k,$$

which is called a naive quotient and denoted by $(\mathcal{X}/\mathcal{G})_{(n)}$. The sheafification of this functor in the Grothendieck topology of fppf coverings is denoted by $\mathcal{X}/\mathcal{G}$ (see [11] for more details). The functor $(\mathcal{X}/\mathcal{G})_{(n)}$ is a dense subfunctor of $\mathcal{X}/\mathcal{G}$ in the following sense. For any superalgebra $A$ and any $x \in (\mathcal{X}/\mathcal{G})(A)$ there is a finitely presented $A$-superalgebra $A'$, which is a faithfully flat $A$-supermodule, such that $(\mathcal{X}/\mathcal{G})_{(n)}(x)$ belongs to $(\mathcal{X}/\mathcal{G})(A')$. Here $\iota_A : A \to A'$ is the corresponding (injective) superalgebra morphism. We call such superalgebra $A'$ a fppf covering of $A$ as well.

In general, $\mathcal{X}/\mathcal{G}$ is no longer superscheme. But if $\mathcal{X}$ is an algebraic group superscheme and $\mathcal{G}$ is its closed group super-subscheme, naturally acting on $\mathcal{X}$ on
the right, then \( \mathcal{X}/G \) is a superscheme of finite type over \( k \). Moreover, the quotient morphism \( \mathcal{X} \to \mathcal{X}/G \) is faithfully flat (see [10, Theorem 14.1]).

Another case when \( \mathcal{X}/G \) is a superscheme is the following. Let \( X \) be an affine superscheme and \( G \) is a finite (hence affine) group superscheme, acting freely on \( X \). Then \( \mathcal{X}/G \) is an affine superscheme (see [13]). This statement is partially covered by the recent remarkable result [7, Theorem 1.8]. The authors proved that if \( X \) is a locally Noetherian superscheme and \( G \) is affine, then \( \mathcal{X}/G \) is a superscheme if and only if \( X_{ev}/G_{ev} \) is a scheme.

Assume that \( G \) acts on \( X \) on the left. Take \( x \in X(k) \) and consider the orbit morphism \( a_x : G \to X, \ g \to gx, \) where \( g \in G(A) \) and \( A = \text{SAlg}_k \). The stabilizer \( G_x \) is a group subfunctor of \( G \) defined as

\[
G_x(A) = \{ g \in G(A) | gx = x \}, \quad A = \text{SAlg}_k.
\]

The subfunctor \( G_x \) is a closed group super-subscheme of \( G \). In fact, let \( Y \) denote the subfunctor of \( X \) such that \( Y(A) = \{ x \} \) for any superalgebra \( A \). By [10, Lemma 1.1] \( Y \) is closed in \( X \), hence \( G_x = a^{-1}_x(Y) \) is closed in \( G \).

Further, \( (G/G_x)_{(n)} \) is a subfunctor of \( X \). Its sheafification \( G/G_x \) can be naturally identified with a sub-faisceau of the faisceau \( X \) (see [17, Section 2] for more details).

In other words, \( a_x \) factors through the embedding \( G/G_x \to X \). The image of \( G/G_x \) in \( X \) is regarded as the \( G \)-orbit of \( x \). We also denote it by \( Gx \). Note that \( G(A)x \) is a proper subset of \( (Gx)(A) \) in general. If \( G \) is an algebraic group superscheme, then \( Gx \simeq G/G_x \) is a superscheme of finite type.

In terms of geometric superschemes, \( x \in \mathcal{X}(k) \) corresponds to a superscheme morphism \( \text{SSpec}(k) \to X \). The latter morphism is uniquely defined by a point in \( X^{c} \), which can be denoted by the same symbol \( x \), such that \( \kappa(x) = k \). By the equivalence of categories, \( G \) acts on \( X \) on the left and we have the orbit morphism \( a_x : G \to X \) that coincides with the composition \( G \to G \times \text{SSpec}(k) \to G \times X \to X \) of superscheme morphisms. Finally, if \( G \) is an algebraic group superscheme, then \( a_x \) factors through the corresponding morphism \( G/G_x \to X \) of superschemes of finite type.

6. Graded group superschemes

A locally algebraic group superscheme \( G \) is called graded if there is a locally algebraic group superscheme \( H \) such that \( G \simeq \text{gr}(H) \). The following proposition refines [10, Proposition 11.1] and extends [9, Lemma-Definition 3.9].

**Proposition 6.1.** Let \( G \) be a locally algebraic group superscheme. Set \( \Phi(G) = (G, V) \) or, equivalently, \( \Psi((G, V)) \simeq G \). The following statements are equivalent:

(i) \( G \) is graded;
(ii) \( G_{ev} \to G \) is split in the category of group superschemes;
(iii) The bilinear map \( V \times V \to \text{Lie}(G) \) is zero;
(iv) The Lie super-bracket on \( \text{Lie}(G) \), restricted to \( \text{Lie}(G)_1 \times \text{Lie}(G)_1 \), vanishes.

**Proof.** The equivalence (iii)\( \iff \) (iv) is obvious.

If \( G \simeq \text{gr}(H) \), then \( G_{ev} \simeq \text{gr}(H)_{ev} \simeq H_{ev} \). Moreover, there is a group superscheme morphism \( q : \text{gr}(H) \to H_{ev} \) whose composition with the embedding \( H_{ev} \to \text{gr}(H) \) is the identity morphism (see [10, Section 11]). This implies (i)\( \iff \) (ii).

In the language of Harish-Chandra pairs \( G_{ev} \to G \) is split if and only if there is a morphism of pairs \( (f, 0) : (G, V) \to (G, 0) \) such that its composition with
Let \( \text{Remark 1.1 combined with [10, Proposition 10.3(1)] } \) conclude the proof.

If (iii) holds, then \( \mathbb{G} \) is a semi-direct product of \( \mathbb{G}_{ev} \) and a normal abelian purely odd unipotent group super-subscheme \( \mathbb{G}_{odd} \) (cf. [10 Proposition 11.1]). More precisely, for any \( A \in \text{SAlg}_k \) the group \( \mathbb{G}_{odd}(A) \) is generated by the elements \( e(a, v) \), which are pairwise commuting and satisfy \( e(a,v)e(a',v) = e(a+a',v) \). In other words, \( \mathbb{G}_{odd} \simeq \text{SSp}(\Lambda(V^*)) \), where all elements of \( V^* \) are supposed to be primitive in the Hopf superalgebra \( \Lambda(V^*) \).

It remains to show that \( \mathbb{G} \simeq \text{gr}(\mathbb{G}) \), or, equivalently, \( \mathbb{G} \simeq \text{gr}(\mathbb{G}) \). The super-scheme \( \mathbb{G} \simeq \mathbb{G} \times \text{SSp}(\Lambda(V^*)) \) can be covered by open affine super-subschemas of the form \( \mathbb{U} = \mathbb{U} \times \text{SSp}(\Lambda(V^*)) \), where \( \mathbb{U} \) runs over open affine subschemas of \( \mathbb{G} \).

The multiplication morphism \( m : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G} \) is defined as

\[
(g,e) \times (g',e') \mapsto (gg') \times (ee'^{-1}e'), \quad g,g' \in \mathbb{G}_{ev}, e,e' \in \mathbb{G}_{odd}.
\]


The conjugation action of \( \mathbb{G}_{ev} \) on \( \mathbb{G}_{odd} \simeq \text{SSp}(\Lambda(V^*)) \) is uniquely defined by the induced action of \( \mathbb{G}_{ev} \simeq \mathbb{G} \) on \( V^* \), hence it factors through \( \mathbb{G}_{ev} \rightarrow \mathbb{G} \simeq \text{SSp}(\mathbb{O}(\mathbb{G})) \) (cf. [10 Corollary 5.3 and Lemma 8.1]). Let \( \tau \) be the corresponding \( \mathbb{O}(\mathbb{G}) \)-comodule map \( V^* \rightarrow V^* \otimes \mathbb{O}(\mathbb{G}), \tau(v^*) = v^*_1 \otimes v^*_2, v^* \in V^* \), where we omit the symbol of summation in Sweedler's notation for \( \tau \).

Let \( m \) denote the multiplication morphism \( \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G} \). For any triple \( \mathbb{U}, \mathbb{U}', \mathbb{U}'' \) of open affine subschemas of \( \mathbb{G} \), the open subschema \( m^{-1}(\mathbb{U}'') \cap (\mathbb{U} \times \mathbb{U}') \cap \mathbb{G} \) can be covered by open subschemas of the form \( (\mathbb{U} \times \mathbb{U}')_f \simeq \text{Sp}(\mathbb{O}(\mathbb{U}) \otimes \mathbb{O}(\mathbb{U}')), f \in \mathbb{O}(\mathbb{U}) \otimes \mathbb{O}(\mathbb{U}'). \) Since \( m \) can be glued from all \( m|_{(\mathbb{U} \times \mathbb{U}')_f} : (\mathbb{U} \times \mathbb{U}')_f \rightarrow \mathbb{U}'' \), the multiplication map \( m : \mathbb{G} \times \mathbb{G} \rightarrow \mathbb{G} \) can be glued from all \( (\mathbb{U} \times \mathbb{U}')_f \rightarrow \mathbb{U}'' \). More precisely, each morphism \( (\mathbb{U} \times \mathbb{U}')_f \rightarrow \mathbb{U}'' \) is dual to the superalgebra morphism \( \mathbb{O}((\mathbb{U} \times \mathbb{U}')_f) \rightarrow \mathbb{O}(\mathbb{U}'') \), defined by

\[
x \otimes y \mapsto x \otimes y = \frac{x(1) \otimes x(2) \otimes (y(1))(1) \otimes y(2)},
\]

where \( x \in \mathbb{O}(\mathbb{U}''), y \in \Lambda(V^*) \), the algebra morphism

\[
x \mapsto \frac{x(1) \otimes x(2)}{s}, \quad (s \text{ depends on } x)
\]

is dual to \( m|_{(\mathbb{U} \times \mathbb{U}')_f} \), and

\[
y \mapsto y(1) \otimes y(2), \quad (y(1))(1) \otimes (y(1))(2)
\]

are the comultiplication in \( \Lambda(V^*) \) and \((\text{id}_{\Lambda(V^*)} \otimes \text{res}_{\mathbb{G},U'}) \wedge (\tau) \) respectively. Thus obviously follows that \( \mathbb{O}((\mathbb{U} \times \mathbb{U}')_f) \rightarrow \mathbb{O}(\mathbb{U}'') \) is a morphism of graded superalgebras. Remark 1.1 combined with [10 Proposition 10.3(1)] conclude the proof. \( \square \)

The following proposition extends [9 Proposition 4.18] and [15 Corollary 6.23].

**Proposition 6.2.** Let \( \mathbb{G} \) be an algebraic group superscheme and \( H \) be its closed group super-subscheme. Then \( \text{gr}(\mathbb{G}/H) \simeq \text{gr}(\mathbb{G})/\text{gr}(H) \) and \( (\mathbb{G}/H)_{ev} \simeq \mathbb{G}_{ev}/\mathbb{H}_{ev} \).

**Proof.** We use again the language of \( k \)-functors and follow the steps in the proof of [10 Theorem 14.1]. Set also \( \Phi(\mathbb{H}) = (H,W) \).
Observe that if $\Phi(G) = (G, V)$, then $\Phi(gr(G)) = (G, V)$ as well, with the only difference that the bilinear map $V \times V \to \text{Gr}(G)$ is the zero map. Besides, for any group superschemes morphism $f : G \to H$ there is $\Phi(f) = \Phi(gr(f))$. By [10, Theorem 12.11] our proposition follows for any normal $H$. If $L$ is a normal closed group super-sub-scheme of $G$, such that $L \subseteq H$ and our proposition holds for $G/L$ and $H/L$, then

$$gr(G/H) \simeq gr((G/L)/(H/L)) \simeq gr(G/L)/gr(H/L) \simeq (gr(G)/gr(L))/(gr(H)/gr(L)) \simeq gr(G)/gr(H).$$

Arguing as in [10, Lemma 14.2], one can reduce the general case to the case when $G$ and $H$ satisfy the following conditions:

(a) $G = M \times A(G)$, where $A(G) = \ker(G \to G^{aff})$ is a central anti-affine group subscheme of $G$ (cf. [13, Corollary 8.14 and Proposition 8.37]);

(b) $M$ is an affine group subscheme of $G$ such that $H \leq M$.

On can choose a covering of $G$ by open affine $H$-saturated subschemes $U = U_{aff} \times U_{ab}$, such that $U_{aff}$ form an open covering of $M$ and $U_{aff}/H$ form an covering of $M/H$ by open affine subschemes. Thus $U/H$ form an open covering of $G/H$ by open affine subschemes. Next, each $V = U \times SSpec(V^*)$ is an open affine $H$-stable (or $H$-saturated) super-sub-scheme of $G$ (see [10, Lemma 14.5]). The superscheme $G/H$ is covered by open affine super-sub-schemes $V/H$ and similarly, $gr(G)/gr(H)$ is covered by open affine super-sub-schemes $gr(V)/gr(H)$.

As a right $O(H)$-coideal superalgebra $O(V)$ is isomorphic to the cotensor product $(O(U) \otimes \Lambda(Z))\boxtimes O(H)$ of the right and left $O(H)$-coideal superalgebras $O(U) \otimes \Lambda(Z)$ and $O(H)$ respectively, where $Z = \ker(V^* \to W^*)$. In particular, the superalgebra $O(V/H) \simeq O(V)^{O(H)}$ is isomorphic to $(O(U) \otimes \Lambda(Z))^{O(H)}$. For another open affine subscheme $U = U_{aff} \times U_{ab}$ and for the corresponding open affine $H$-saturated super-sub-scheme $V = U \times SSpec(V^*)$ we have

$$(V \cap V')/H = V/H \cap V'/H$$

This open super-sub-scheme is glued from the open affine super-sub-schemes $T/H$, where $T = T \times SSpec(V^*)$ and the $H$-saturated affine subschemes $T$ form an open covering of $U \cup U'$. In other words, the superscheme structure of $G/H$ is uniquely defined by the family of superalgebras $(O(U) \otimes \Lambda(Z))^{O(H)}$ and by the family of superalgebra morphisms $(O(U) \otimes \Lambda(Z))^{O(H)} \to (O(T) \otimes \Lambda(Z))^{O(H)}$, which are dual to the open embeddings $T/H \to V/H$.

Similarly, $O(gr(V)) \simeq (O(U) \otimes \Lambda(Z))\boxtimes gr(H)$ and $O(gr(V)/gr(H)) \simeq (O(U) \otimes \Lambda(Z))^{O(H)}$ (see [10, Lemma 14.11]). It remains to note that all superalgebras $(O(U) \otimes \Lambda(Z))^{O(H)}$ and $(O(T) \otimes \Lambda(Z))^{O(H)}$ are Grassmann graded and use Remark 1.1.

Finally, by [10, Theorem 12.11] the second statement obviously holds, whenever $H$ is normal. Again, arguing as in [10, Lemma 14.2], one can assume that $G$ and $H$ satisfy the above conditions (a) and (b). In particular, one can construct a covering of $G/H$ by open affine super-sub-schemes $V/H$ as above.

Thus $$(V/H)_{ev} \simeq SSp(O(U) \otimes \Lambda(Z))^{O(H)} \simeq SSp(SU(U) \otimes O(H))$$ and each open embedding $(T/H)_{ev} \to (V/H)_{ev}$ is dual to the algebra morphism $O(U) \otimes O(H) \to O(T) \otimes O(H)$. In other words, gluing together the purely even superschemes $(V/H)_{ev}$ one obtains $G_{ev}/H_{ev}$. Proposition is proven. $\square$
7. Super-dimension of certain Noetherian superschemes

Recall that if \( X \) is an irreducible superscheme of finite type over a field \( k \), then its super-dimension is defined as the Krull super-dimension \( \text{Ksdim}(\mathcal{O}(U)) \) of the superalgebra of global sections (coordinate superalgebra) of any open (nonempty) affine super-subscheme \( U \) in \( X \) (see [12], Section 6.1). By analogy with this, we define the super-dimension of a Noetherian superscheme \( X \) with \( \dim(X_{res}) < \infty \) as

\[
\text{sdim}(X) = \sup_{U} \text{Ksdim}(\mathcal{O}(U)),
\]

where \( U \) runs over a covering of \( X \) by open (nonempty) affine super-subschemas and the supremum is given with respect to the lexicographical order on couples of nonnegative integers (see [12], Section 5.3).

**Lemma 7.1.** If \( X \) is affine, then for any covering of \( X \) by open (nonempty) affine super-subschemas \( U \) we have

\[
\sup_{U} \text{Ksdim}(\mathcal{O}(U)) = \text{Ksdim}(\mathcal{O}(X)).
\]

**Proof.** Let \( A \) denote \( \mathcal{O}(X) \) and let \( d \) denote \( \dim(X_{res}) \). Note that any affine open super-subschema \( U \) of \( X \) has a form \( \bigcup_{i \in I} \text{SSpec}(A_{a_i}) \), where \( I \) is a finite set, each element \( a_i \) belongs to \( A_0 \) and \( \sum_{i \in I} \mathcal{O}(U)_0 a_i|_U = \mathcal{O}(U)_0 \) (cf. [11, Lemma 3.5]). Thus all we need is to prove our lemma for any such finite covering of \( X \).

So, let \( X = \bigcup_{1 \leq i \leq k} \text{SSpec}(A_{a_i}) \), where \( \sum_{1 \leq i \leq k} A_0 a_i = A_0 \). Since the underlying topological space of \( X \) coincides with the underlying topological space of the scheme \( X_{res} = \text{Spec}(\mathcal{A}) \), [11 Exercise I.1.10] implies

\[
\max_{1 \leq i \leq k} \text{Ksdim}_0(A_{a_i}) = \text{Ksdim}_0(A) = d.
\]

Let \( J \) denote the set \( \{ i \mid \text{Ksdim}_0(A_{a_i}) = d \} \). It remains to show that

\[
\max_{i \in J} \text{Ksdim}_1(A_{a_i}) = \text{Ksdim}_1(A).
\]

If \( \text{Ksdim}_1(A) = t > 0 \), then let \( y_1, \ldots, y_t \in A_1 \) be a corresponding longest system of odd parameters. Choose a longest prime chain \( p_0 \subseteq \cdots \subseteq p_d \) in \( A_0 \) such that \( \text{Ann}_{A_{a_i}}(y_1 \cdots y_t) \subseteq p_0 \). There is an index \( i \) such that \( a = a_i \) does not belong to \( p_d \), hence

\[
(p_0)_a \subseteq \cdots \subseteq (p_d)_a
\]

is the longest prime chain in \( A_a \), and thus \( i \in J \). Moreover, there is

\[
\text{Ann}_{(A_0)_a}(y_1 \cdots y_t) = (\text{Ann}_{A_0}(y_1 \cdots y_t))_a \subseteq (p_0)_a
\]

and the latter inclusion implies \( \text{Ksdim}_1(A_a) \geq \text{Ksdim}_1(A) \).

Conversely, for any \( i \in J \) there is a longest prime chain \( q_0 \subseteq \cdots \subseteq q_d \) in \( A_0 \) such that \( (q_0)_a \subseteq \cdots \subseteq (q_d)_a \) is the longest prime chain in \( A_{a_i} \). If

\[
y'_1 = \frac{z_1}{a_1}, \quad \ldots, \quad y'_s = \frac{z_s}{a_s}
\]

is the longest system of odd parameters in \( A_{a_i} \) that subordinates \( (q_0)_a \), then again

\[
\text{Ann}_{(A_0)_a}(y'_1 \cdots y'_s) = (\text{Ann}_{A_0}(z_1 \cdots z_s))_a \subseteq (q_0)_a
\]

implies \( \text{Ann}_{A_{a_i}}(z_1 \cdots z_s) \subseteq q_0 \) and \( \text{Ksdim}_1(A_{a_i}) \leq \text{Ksdim}_1(A) \).

Finally, \( \text{Ksdim}_1(A) = 0 \) if and only if \( \text{Ann}_{A_0}(A_1) \) do not subordinate the first member of any longest prime chain in \( A \). The same arguments as above show that \( \text{Ksdim}_1(A_{a_i}) = 0 \) for any \( i \in J \). Lemma is proven. \( \square \)
Proposition 7.2. The definition of $\text{sdim}(X)$ does not depend on the choice of covering of $X$ by affine open super-subschemes.

Proof. Assume that $X$ has a covering by open affine super-subschemes $U_i, i \in I$, and by open affine super-subschemes $V_j, j \in J$, as well. For any couple of indices $(i,j) \in I \times J$ we choose a finite covering by open affine super-subschemes $Z_{ijt}, t \in T_{ij}$. Then Lemma 7.1 implies

$$\sup_{(i,j) \in I \times J, t \in T_{ij}} K\text{sdim}(O(Z_{ijt})) = \sup_{i \in I} \sup_{j \in J, t \in T_{ij}} K\text{sdim}(O(U_i))$$

and symmetrically

$$\sup_{(i,j) \in I \times J, t \in T_{ij}} K\text{sdim}(O(Z_{ijt})) = \sup_{j \in J} \sup_{i \in I, t \in T_{ij}} K\text{sdim}(O(V_j)).$$

\[\square\]

Example 7.3. Let $G$ be an algebraic group superscheme. Then

$$\text{sdim}(G) = (\text{dim}(G_{res}), \text{dim}(\text{Lie}(G)_1)).$$

As it has been already mentioned, the superscheme $G$ is isomorphic to $G_{res} \times \text{SSpec}(\Lambda(V^*))$, where $V = \text{Lie}(G)_1$. Furthermore, if the open affine subschemes $U$ form a covering of $G_{res}$, then the open affine super-subschemes $W = U \times \text{SSpec}(\Lambda(V^*))$ form a covering of $G$. It remains to note that any basis of $V^*$ form a longest system of odd parameters of $O(W) \cong O(U) \otimes \Lambda(V^*)$, hence

$$K\text{sdim}(O(W)) = (K\text{sdim}(O(U)), \text{dim}(\text{Lie}(G)_1)).$$

Remark 7.4. If $X$ is an Noetherian superscheme with $\text{dim}(X_{res}) < \infty$, then for any field extension $k \subseteq k'$ there is

$$K\text{sdim}(X) = K\text{sdim}(X_{k'})$$

where $X_{k'} \cong X \times \text{SSpec}(k) \text{SSpec}(k')$. Indeed, without loss of a generality one can suppose that $X$ is affine, say $X \cong \text{SSpec}(A)$. Then $X_{k'} \cong \text{SSpec}(A \otimes_k k')$ and it is obvious that

$$K\text{sdim}(A) = K\text{sdim}(A \otimes_k k').$$

8. SUPER-DIMENSION OF A SHEAF QUOTIENT

Let $G$ be an algebraic group superscheme and $H$ be its closed group super-subscheme. Recall that $G$ corresponds to the Harish-Chandra pair $(G_{ev}, V)$, where $V = \text{Lie}(G)_1$ is regarded as a $G_{ev}$-module with respect to the adjoint action, and the bilinear $G_{ev}$-invariant map $V \times V \to \text{Lie}(G_{ev}) = \text{Lie}(G)_0$ is induced by the Lie superalgebra bracket.

Theorem 8.1. We have $\text{sdim}(G/H) = \text{sdim}(G) - \text{sdim}(H)$.

Proof. Proposition 6.2 and [13] Proposition 5.23 imply

$$\text{sdim}_0(G/H) = \text{dim}((G/H)_{ev}) = \text{dim}(G_{ev}/H_{ev})$$

$$= \text{dim}(G_{ev}) - \text{dim}(H_{ev}) = \text{sdim}_0(G) - \text{sdim}_0(H).$$
If $H$ is normal, then by [10] Theorem 12.11] the statement of our proposition is obvious. Using the same reduction as in Proposition 6.2 based on arguments from [10] Lemma 14.2], one can assume that $G$ and $H$ satisfy the conditions (a) and (b) from Proposition 6.2. It remains to show that for any affine super-subscheme $V$ constructed therein, we have $\text{ssdim}_1(V/H) = \dim(\text{Lie}(G)_1) - \dim(\text{Lie}(H)_1)$. Let $B$ and $A$ denote $\mathcal{O}(V)$ and $\mathcal{O}(V/H)$ respectively.

Without loss of a generality one can assume that $\mathbb{k}$ is algebraically closed. In fact, for any field extension $\mathbb{k} \subseteq \mathbb{k}'$ the naive quotient $A \to \mathbb{G}(A)/\mathbb{H}(A), A \in S\text{Alg}_{\mathbb{k}}$, is apparently dense in $\mathbb{G}(\mathbb{H})_{\mathbb{k}'}$ with respect the Grothendieck topology of fpf coverings, that is $\mathbb{G}_{\mathbb{k}'}/\mathbb{H}_{\mathbb{k}'} \simeq (\mathbb{G}/\mathbb{H})_{\mathbb{k}'}$. Remark 7.3 implies the required.

By [10] Theorem 14.1] the quotient morphism $V \to V/H$ is faithfully flat. Since $V$ and $V/H$ are affine, it is equivalent to the condition that $A \to B$ is faithfully flat. In particular, for any prime superideal $\mathfrak{P}$ of $A$ there is a prime superideal $\mathfrak{Q}$ of $B$ such that $\mathfrak{Q} \cap A = \mathfrak{P}$. Moreover, the induced local superalgebra morphism $A_{\mathfrak{P}} \to B_{\mathfrak{Q}}$ is flat (cf. [10] Corollary 3.2]).

We choose $\mathfrak{P}$ so that its even component $\mathfrak{P}_0 = \mathfrak{p}$ is the largest member of a longest prime chain of $A_0$ (in particular, $\mathfrak{p}$ is a maximal ideal), and longest system of odd parameter of $A$ subordinates the smallest prime ideal of this chain. Thus $\text{Ksdim}_1(A) = \text{Ksdim}_1(A_0)$. The superalgebra $B_{\mathfrak{Q}} \simeq \mathcal{O}(U_0) \otimes \Lambda(V^*)$ is obviously oddly regular, thus by [12] Proposition 3.6.1(iii)] the superalgebra $A_{\mathfrak{Q}}$ is oddly regular as well. Next, by [12] Proposition 3.6.3(ii)] we have

$$\dim_{\mathbb{k}}(\Phi_{B_{\mathfrak{Q}}}) = \dim_{\mathbb{k}}(\Phi_{A_{\mathfrak{Q}}}) + \dim_{\mathbb{k}}(B_{\mathfrak{Q}}/\mathfrak{P}B_{\mathfrak{Q}}),$$

or

$$\text{Ksdim}_1(V) = \text{Ksdim}_1(V/H) + \dim_{\mathbb{k}}(B_{\mathfrak{Q}}/\mathfrak{P}B_{\mathfrak{Q}}).$$

Recall that the superideals $\mathfrak{Q}$ and $\mathfrak{P}$ can be interpreted as the (closed) points $x \in V(\mathbb{k})$ and $y \in (V/H)(\mathbb{k})$. As it has been observed in [10] Theorem 7.5], the superalgebra $B_{\mathfrak{Q}}/\mathfrak{P}B_{\mathfrak{Q}}$ is isomorphic to the stalk of the fiber $V_y \simeq V \times_{V/H} S\text{Sp}(\mathbb{k})$ at the point $x$, where, by virtue of Yoneda lemma, the morphism $S\text{Sp}(\mathbb{k}) \to V/H$ is uniquely defined by the point $y$. It is easy to see that $V_y$ is naturally isomorphic to the left coset $x\mathbb{H}$, whence to $\mathbb{H}$. Since any stalk of $\mathbb{H}$ is oddly regular with odd Krull dimension $\dim(\text{Lie}(H)_1)$, our theorem follows.

9. Orbits

Let $G$ be an algebraic group superscheme that acts on a superscheme $X$ of finite type (on the left). Take $x \in X(\mathbb{k})$ and consider the orbit $Gx \simeq G/G_x$. Below we prove that $Gx$ is a locally closed super-subscheme of $X$, provided $\mathbb{k}$ is algebraically closed.

A superscheme morphism $f : X \to Y$ is called a monomorphism, provided $f(A) : X(A) \to Y(A)$ is injective for any superalgebra $A$. For example, the superscheme morphism $a_x : G/G_x \to X$, induced by the orbit morphism $a_x$, is a monomorphism. Also note that [11] Lemma 5.5] implies that $f^*$ is injective, whenever $f$ is a monomorphism.

Lemma 9.1. The orbit $Gx$ is a $G$-stable (or, $G$-saturated) subfunctor of $X$.

Proof. An element $y \in X(A)$ belongs to $(Gx)(A)$ if and only if there is a fpf covering $A'$ of $A$ such that $y' = X(\iota_A)(y)$ belongs to $G(A')x$. Thus for any $g \in G(A)$ we have $X(\iota_A)(gy) = g'y' \in G(A')x$, hence $gy \in (Gx)(A)$. \hfill $\Box$
Lemma 9.2. If $SSpec(B) \to SSpec(A)$ is a monomorphism of superschemes of finite type, induced by a superalgebra homomorphism $\phi : A \to B$, then $B_1 = B_0\phi(A_1)$.

Proof. Let $C$ denote $B_0 \oplus B_0\phi(A_1)$. The monomorphism $SSpec(\phi)$ is a composition of $SSpec(B) \to SSpec(C)$ and $SSpec(C) \to SSpec(A)$. Thus the first morphism is a monomorphism as well.

Let $m$ be a maximal ideal in $B_0$. Then $B/Bm = L \oplus V$, where $L = B_0/m$ is a field extension of $k$ and $V = B_1/B_1m$ is a finite-dimensional purely odd $L$-superspace such that $V^2 = 0$. If $W = (C_1 + B_1m)/B_1\phi$ is a proper subspace of $V$, then choose a subspace $W'$ of $V$ of codimension 1, such that $W \subseteq W'$. Observe that any (super)subspace of $V$ is a superideal of $B/Bm$. Thus $R = L \oplus V/W'$ is a superalgebra and any linear map of $L$-superspaces $B/Bm \to R$ is a superalgebra morphism.

Let $v$ be a basis vector of a complement of $W'$. Then there are at least $|L| \geq 2$ different superalgebra morphisms from $B/Bm$ to a superalgebra $R$, which are the same being restricted on the super-subalgebra $(C + Bm)/Bm$. They are $\phi_v(W') = 0, \phi_v(v) = av + W', a \in L$. Furthermore, it also follows that the map $SSp B(R) \to SSp C(R)$ is not injective. This contradiction infers that $B_1 = C_1 + B_1m$, and hence, by Nakayama’s lemma, $(B_1)_m = (C_1)_m$ for any maximal ideal $m$. Theorem 1 from [1] Chapter II, §3, implies $C_1 = B_1$. □

Proposition 9.3. Let $f : X \to Y$ be a monomorphism of superschemes of finite type. Then there is an open dense subset $V \subseteq Y^e$ such that $U = (f^e)^{-1}(V) \neq \emptyset$ and $f|_U : U \to V$ is an immersion.

Proof. By [10] Lemma 1.4 and Lemma 1.5] one needs to consider the case $X = SSpec(B)$ and $Y = SSpec(A)$ only. Then $f = SSpec(\phi)$, where $\phi : A \to B$ is a superalgebra morphism.

Since the induced morphism $f_{res} : X_{res} = Spec(B) \to Y_{res} = Spec(A)$ is also a monomorphism (in the category of schemes), [3] I, §3, Corollary 4.7 implies that there is an open dense subset $V \subseteq Y^e = Y_{res}^e$ such that $U = (f^e)^{-1}(V) \neq \emptyset$ and for any $x \in U$ the induced algebra morphism

$$O_{Y,y}/\mathcal{J}_y \to O_{X,x}/\mathcal{J}_x,$$

where $\mathcal{J}_y = O_{Y,y}(O_{Y,y})_1$ and $\mathcal{J}_x = O_{X,x}(O_{X,x})_1$, $y = f^e(x)$, is surjective. Let $R, S$ and $\psi$ denote $O_{Y,y}, O_{X,x}$ and the induced local morphism $O_{Y,y} \to O_{X,x}$ respectively. By Lemma 9.2 there is $S_1 = S_0\psi(R_1)$. As it has been observed, the algebra morphism

$$\overline{R} \cong R_0/R_1^2 \to S = S_0/S_1^2 = S_0/S_0\psi(R_1)^2$$

is surjective. The induction on $n$ infers that all space morphisms

$$R_1^{2n}/R_1^{2n+2} \to S_1^{2n}/S_1^{2n+2} = S_0\psi(R_1)^{2n}/S_0\psi(R_1)^{2n+2}$$

are also surjective for any $n \geq 1$. Since the $R_0$-ideal $R_1^2$ is nilpotent, $\psi|R_0 : R_0 \to S_0$ is surjective, hence $\psi$ is. Thus our proposition follows. □

The following lemma is a folklore.

Lemma 9.4. (see [13] A.12) Let $X$ be a superscheme of finite type, then $X(\overline{k})$ is dense in $X^e$. In particular, if $f : X \to Y$ is a morphism of superschemes, then $f^e$ is surjective if and only if $f(\overline{k}) : X(\overline{k}) \to Y(\overline{k})$ is.
Proof. For any field extension $k \subseteq L$ the set $X(L)$ consists of all points $x \in X^e$ such that $\kappa(x)$ is a subfield of $L$. Without loss of a generality, one can assume that $X \simeq \SSp(A)$ and $x$ corresponds to a prime superideal $\mathfrak{p}$ of $A$. Any open neighborhood $U$ of $x$ has a form $\{ \mathfrak{q} \mid \mathfrak{q} \not\supseteq I \}$ for a certain superideal $I$ of $A$. Since $A_0$ is a finitely generated $k$-algebra and the ideal $I_0$ is not nilpotent, Hilbert Nullstellensatz implies that there is a maximal superideal $\mathfrak{m}$ such that $\mathfrak{m} \not\supseteq I$. If $y$ is the corresponding point in $X^e$, then $\kappa(y)$ is an algebraic extension of $k$, whence $y \in X(k) \cap U$. The second statement is now obvious.

Theorem 9.5. Let $X$ be a superscheme of finite type and let an algebraic group superscheme $G$ act on $X$ on the left. If $k$ is algebraically closed, then the orbit $G \cdot x$ is a locally closed super-subscheme of $X$ for any $x \in X(k)$. Moreover, if $G$ is a smooth algebraic group superscheme (i.e. $G_{res}$ is a smooth scheme), then there are always closed orbits.

Proof. All we need is to prove that the morphism $\tilde{a}_x : G/G_x \to X$ is an immersion. By Proposition 3.3 there is a dense open super-subscheme $V \subseteq X$ such that the induced morphism $U = \tilde{a}_x^{-1}(V) \to V$ is an immersion.

Let $U$ and $V$ denote the corresponding open super-subschemes of $G/G_x$ and $X$, respectively. Then $U = \cup_{g \in G(k)} gU$ is an open super-subscheme of $G/G_x$ and by Lemma 1.4 the restriction of $\tilde{a}_x$ on $U$ is an immersion. It remains to show that $W = G/G_x$. Let $W'$ be the full inverse image of $W$ in $G$. Since $W$ is obviously $G(k)$-stable, we have $W'(k) = G(k)$, and Lemma 9.4 infers $W' = G$. Thus $(G/G_x)_{(n)} \subseteq W$ and since any open subfunctor is closed with respect to the Grothendieck topology of fppf coverings (just superize [6, 1.1.7(6)]), it follows that $W = G/G_x$.

Since $X(k) = X_{ev}(k)$ and the super-subscheme $X_{ev}$ is obviously $G_{ev}$-stable, the restriction of $\tilde{a}_x$ on $G_{ev}$ coincides with the orbit morphism $G_{ev} \to X_{ev}$, $g \mapsto gx$, where $g \in G_{ev}(A)$, $A \in \text{SAlg}_k$. Furthermore, we have a commutative diagram

$$
\begin{array}{ccc}
(G/G_x)_{ev} & \simeq & G_{ev}/(G_x)_{ev} \\
\downarrow & & \downarrow \\
X_{ev} & = & X_{ev}
\end{array}
$$

in which $(G_x)_{ev} = (G_{ev})_x$, the left vertical arrow is $(\tilde{a}_x)_{ev}$ and the right vertical arrow is unduced by $\tilde{a}_x$. It remains to note that an immersion $Y \to Z$ is a closed immersion if and only if the image of $Y^e$ is closed in $Z^e$, hence if and only if $Y_{ev} \to Z_{ev}$ is a closed immersion. The geometric counterpart of the above diagram is

$$
\begin{array}{ccc}
(G/G_x)_{ev} & \simeq & G_{ev}/(G_x)_{ev} \\
\downarrow & & \downarrow \\
X_{ev} & = & X_{ev}
\end{array}
$$

where $(G_x)_{ev} = (G_{ev})_x$, the left vertical arrow is $(\tilde{a}_x)_{ev}$ and the right vertical arrow is unduced by $\tilde{a}_x$. Then [3, II, §5, Proposition 3.3] concludes the proof.

Corollary 9.6. In the conditions of the above theorem an orbit $G \cdot x$ (respectively, $G \cdot x$) is closed, whenever $G_{ev} \cdot x$ (respectively, $G_{ev} \cdot x$) is, or equivalently, whenever $G_{res} \cdot x$ is. The latter takes place if $\dim(G_{res}x) = \text{sdim}(G) = \text{sdim}(G_{res}x)$ is minimal.

Corollary 9.7. Theorem 8.7 implies that $\text{sdim}(Gx) = \text{sdim}(G) - \text{sdim}(G_x)$. 


10. An Example

We still assume that $k$ is algebraically closed.

Let $X$ be an affine superscheme and let $G$ be an affine algebraic group superscheme. Then (left) actions of $G$ on $X$ are in one-to-one correspondence with superalgebra morphisms $O(X) \to O(G) \otimes O(X)$ so that $O(X)$ becomes a $O(G)$-supercomodule.

Corollary 9.6 infers that if $G_{ev} = 1$, then any orbit is closed. Moreover, the underlying topological space of the orbit of a point $x$ is just $\{x\}$. Below we illustrate this fact by the following elementary example.

The condition $G_{ev} = 1$ implies that $G$ is isomorphic to the direct product of several copies of the odd unipotent group superscheme $G_a$ of super-dimension $0|1$. For the sake of simplicity we assume that $G = G_a$. The Hopf superalgebra $O(G)$ has a basis 1, $z$, where $z$ is an odd primitive element. The corresponding supercomodule map $\tau: O(X) \to O(G) \otimes O(X)$ is defined as

$$f \mapsto 1 \otimes f + z \otimes \phi(f), \quad f \in O(X),$$

where $\phi: O(X) \to O(X)$ is an odd (left) superderivation with $\phi^2 = 0$.

Choose $x \in X(k)$. For any superalgebra $A$ and arbitrary element $g \in G(A)$ the element $gx$ is a superalgebra morphism $O(X) \to A$ such that

$$(\ast) \quad (gx)(f) = x(f) + g(z)x(\phi(f)), \quad f \in O(X).$$

Recall that $\ker x = \mathfrak{M} = m \oplus O(X)_1$ is a maximal superideal in $O(X)$ with $O(X)_0/m = k$. Set $I = m \oplus \phi^{-1}(m)$.

**Lemma 10.1.** The subspace $I$ is a superideal in $O(X)$. Furthermore, the orbit $Gx$ is closed and isomorphic to $SSp(O(X)/I)$.

**Proof.** All we need is to prove that $O(X)_1m \subseteq \phi^{-1}(m)$ and $O(X)_0\phi^{-1}(m) \subseteq \phi^{-1}(m)$. We have

$$\phi(O(X)_1m) \subseteq O(X)_0m + O(X)^{2}_0 \subseteq m.$$ 

The proof of the second statement is similar. The equation $(\ast)$ implies $(gx)(I) = 0$. In other words, $Gx$ is a subfunctor of the closed super-subscheme $Y$ of $X$, which is isomorphic to $SSp(O(X)/I)$. Since $\tau(I) \subseteq O(G) \otimes I$, $Y$ is $G$-stable and $x \in Y(k)$.

If $\phi^{-1}(m) = O(X)_1$, then $Gx = G$ and $Y = Gx = \{x\}$. Otherwise, $O(X)_1/\phi^{-1}(m)$ is one dimensional. Let $v$ be a basis vector of $O(X)_1/\phi^{-1}(m)$ such that $\phi$ takes $v$ to $1 \in k = O(X)_0/m$. Then for any $\psi \in Y(A)$ there is $\psi = gx$, where $g(z) = \psi(v)$. Lemma is proven. \qed

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