Non-malleable encryption of quantum information

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We introduce the notion of non-malleability of a quantum state encryption scheme (in dimension \(d\)): in addition to the requirement that an adversary cannot learn information about the state, here we demand that no controlled modification of the encrypted state can be effected.

We show that such a scheme is equivalent to a unitary 2-design [Dankert et al.], as opposed to normal encryption which is a unitary 1-design. Our other main results include a new proof of the lower bound of \((d^2 - 1)^2 + 1\) on the number of unitaries in a 2-design [Gross et al.], which lends itself to a generalization to approximate 2-design. Furthermore, while in prime power dimension there is a unitary 2-design with \(\leq d^5\) elements, we show that there are always approximate 2-designs with \(O(\epsilon^{-2}d^4 \log d)\) elements.

INTRODUCTION

The ordinary (and in terms of secret key length, optimal) encryption of quantum states on \(n\) qubits is by applying a randomly chosen tensor product of Pauli operators (including the identity). This requires \(2n\) bits of shared secret randomness, corresponding to the \(4^n\) Pauli operators. (More generally, for states on a \(d\)-dimensional system, one can use the elements of the discrete Weyl group – up to global phases – of which there are \(d^2\).) This is perfectly secure in the sense that the state the adversary can intercept is, without her knowing the key, always the maximally mixed state. For perfectly secure encryption with random unitaries, it was shown in [2] that \(2n\) bits of secret key are also necessary for \(n\) qubits. The lower bound of 2 bits of key per qubit continues to hold even for \(\epsilon\)-approximate encryption (up to expressions in \(\epsilon\)), but there it becomes relevant how the approximation is defined — whether it randomizes entangled states or not [see Eq. (2”) and (2’) below]. In [16] it was shown that in the latter case one gets away with \(n + o(n)\) key bits for arbitrary \(n\)-qubit states; their construction was derandomized later in [3] and [11].

However, even perfectly secure encryption allows for a different sort of intervention by the adversary: she can, without ever attempting to learn the message, change the plaintext by effecting certain dynamics on the encrypted state. Consider briefly the classical one-time pad, i.e. an \(n\)-bit message XORed with a random \(n\)-bit string: by flipping a bit of the ciphertext, an adversary can effectively flip any bit of the recovered plaintext. In the quantum case, due to the (anti-)commutation relations of the Pauli operators, by applying to the ciphertext (encrypted state) some Pauli, she forces that the decrypted state is the plaintext modified by that Pauli: for an \(n\)-qubit state \(|\varphi\rangle\), any adversary’s Pauli operator \(Q\) and secret key Pauli \(P_k\), the decrypted state is

\[P_k^\dagger Q P_k|\varphi\rangle = \zeta Q|\varphi\rangle,\]

with some (unimportant) global phase \(\zeta = \zeta(P, Q)\).

This is evidently an undesirable property of a encryption scheme, and can be classically addressed e.g. by authenticating the message as well as encrypting it. Interestingly, in the above quantum message case, it was shown in [4] that authenticating quantum messages is at least as expensive as encrypting them (it actually encrypts the message as well): one needs 2 bits of shared
secret key for each qubit authenticated, even in the approximate setting considered in [4]. Classical non-malleable cryptosystems include both symmetric and asymmetric encryption schemes, bit commitment, zero knowledge proofs and others [12].

Here we will introduce a formal definition of perfect non-malleability of a quantum state encryption scheme (NMES), i.e. resistance against predictable modification of the plaintext, as well as of two notions of approximate encryption with approximate non-malleability. We show that a unitary non-malleable channel is equivalent to unitary $2$-design in the sense of Dankert et al. [9]. We use this fact to design an exact ideal non-malleable encryption scheme requiring $5 \log d$ bits of key. Also, the lower bound of Gross et al. [15] for unitary $2$-designs applies for perfect NMES; we give a new proof of their result that at least $(d^2 - 1)^2 + 1$ unitaries are required, which also yields a more general lower bound of $(4 - O(\epsilon)) \log d$ on the entropy of an approximate unitary $2$-design. Finally we demonstrate that approximate NMES (unitary $2$-designs) exist which require only $4 \log d + \log \log d + O(\log 1/\epsilon)$ bits of key.

I. GENERAL MODEL OF ENCRYPTION

Suppose Alice wants to send a secret quantum message to Bob, say an arbitrary state $\rho \in B(\mathcal{H})$, a Hilbert space of dimension $d$. For this purpose they will use an encryption scheme with pre-shared secret key $K$ as follows. $K$ is distributed according to some probability distribution $p_K(k)$ and for each $k$ there is a pair of c.p.t.p. (completely positive and trace preserving) maps $E_k : B(\mathcal{H}) \rightarrow B(\mathcal{H}')$ and $D_k : B(\mathcal{H}') \rightarrow B(\mathcal{H})$ for encryption and decryption. The combined effect of encryption, averaged over all keys, is described by a c.p.t.p. map (noisy quantum channel) $R : B(\mathcal{H}) \rightarrow B(\mathcal{H})$, acting on operators on $\mathcal{H}$ as

$$R(\rho) = \sum_k p_K(k) D_k(E_k(\rho)).$$

Similarly, for an adversary who intercepts the encrypted state but doesn’t know the secret key, we have an average channel $R' : B(\mathcal{H}) \rightarrow B(\mathcal{H}')$,

$$R'(\rho) = \sum_k p_K(k) E_k(\rho).$$

Loosely speaking, the quality of the scheme is described by two parameters: first, the reliability, i.e. how close $R$ is to the ideal channel; secondly, the secrecy, i.e. how close $R'$ is to a constant (meaning a map taking all input states to a fixed output state). In an ideal scheme, $R = \text{id}$ and $R' = \text{const.},$ i.e. there is a state $\xi_0$ on $\mathcal{H}'$, such that

$$\forall \rho \quad R(\rho) = \rho,$$

$$\forall \rho \quad R'(\rho) = \xi_0. \tag{1}$$

The issue of approximate performance is a little bit tricky: whereas for the reliability of communication there is essentially one notion, namely, for $\delta > 0$,

$$\forall \rho \quad \|\rho - R(\rho)\|_1 \leq \delta, \tag{1'}$$

there are two asymptotically radically different notions of secrecy. One is the “naive” one

$$\forall \rho \quad \|R'(\rho) - \xi_0\|_1 \leq \epsilon \tag{2'}$$
that does not randomize entangled states when applied locally. The “correct” (composable!) definition takes into account the possibility to apply \( R' \) to part of an entangled state:

\[
\forall \rho_{12} \quad \left\| (R' \otimes \text{id})\rho_{12} - \xi_0 \otimes \rho_2 \right\|_1 \leq \epsilon.
\]

We note that the two conditions coincide in the ideal case \( \epsilon = 0 \).

The minimal key length required for (approximate) encryption reflects whether Eq. (2') or Eq. (2'') is used. In the former case \( \log d \) bits of key are necessary, and \( \log d + o(\log d) \) bits of key are sufficient [3, 16] to randomize quantum system of dimension \( d \), while in the latter case the key length essentially coincides with the exact encryption case and equals \( (2 - O(\epsilon)) \log d \) [2].

II. NON-MALLEABILITY

There is, of course, a simple scheme of encryption that implements an ideal scheme: on \( n \) qubits, use a key of length \( 2^n \) and apply an independent random Pauli operator to each qubit. (More generally, in dimension \( d \), the key identifies one of the \( d^2 \) discrete Weyl operators made up of the basis shift and phase shift operators.) The adversary evidently cannot see any information about the plaintext state, but she can use the ciphertext in another way: by modulating the ciphertext with an arbitrary Pauli operation, she can effectively implement this Pauli transformation on the plaintext state.

We shall show that this is not at all a necessary feature of any encryption scheme. There are, however, always two possible actions for the adversary (and their arbitrary convex combination). Namely, not to interfere at all, resulting in correct decryption of the state \( \rho \) sent; or interception of the ciphertext and its replacement by a state \( \eta_0 \) on \( \mathcal{H}' \), resulting in Bob always decrypting the constant state \( \rho_0 = \sum_k p_K(k)D_k(\eta_0) \). In other words, assuming the adversary implements an arbitrary quantum channel, i.e. a completely positive and trace non-increasing (c.p.t. \( \leq \)) map \( \Lambda : \mathcal{B}(\mathcal{H}') \rightarrow \mathcal{B}(\mathcal{H}') \), the class of effective channels on the plaintext she can realize, namely all channels

\[
\tilde{\Lambda} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \text{ s.t.} \\
\rho \mapsto \sum_k p_K(k)D_k\left(\Lambda(E_k(\rho))\right),
\]

will include all convex combinations of the identity (up to approximation as specified by \( \epsilon \)) and the completely forgetful channels \( \langle \rho_0 \rangle \) mapping all inputs to the state \( \rho_0 = \sum_k p_K(k)D_k(\eta_0) \), with arbitrary \( \eta_0 \).

We call an encryption scheme (perfectly) non-malleable, if these are the only effective channels the adversary can realize, i.e. if for every \( \Lambda, \tilde{\Lambda} \) is in the semi-linear span of \( \text{id} \) and the \( \langle \rho_0 \rangle \),

\[
\tilde{\Lambda} \in \mathcal{C} := \text{semi-lin} \left( \{\text{id}\} \cup \left\{ \langle \rho_0 \rangle : \rho \mapsto \rho_0 = \sum_k p_K(k)D_k(\eta_0) \right\} \right),
\]

with \( \text{semi-lin} \) being the semi-linear hull, i.e. with any family of elements it also contains all their linear combinations, subject to complete positivity of the resulting operator. [Clearly, in the above the convex hull can be realized by an adversary; however, in general the full semi-linear hull is accessible; e.g. for the Haar measure on the unitary group – and infinite key – the only constant channel is \( \langle \tau \rangle \), with the maximally mixed state \( \tau = \frac{1}{d} \mathbb{1} \), cf. the beginning of the next section, in particular eqs. (4)–(6) On the other hand, any traceless unitary by the adversary results in the effective channel \( \Lambda(\rho) = \frac{1}{d^2-1} (d^2 \tau - \rho) \).]
Also, a word on why we demand this for all c.p.t. ≤ maps, which is a strictly larger class than c.p.t.p.: note that the adversary could implement an instrument \([10]\), which is a resolution of a c.p.t.p. map into c.p.t. ≤ ones. One of them will act randomly, but the adversary can learn which one, so could effectively correlate herself with the effective channel \(\Lambda\).

As before, this is to be understood up to approximations: for every effective channel \(\tilde{\Lambda}\) there is \(\Theta \in \mathcal{C}\) such that

\[
\forall \rho, \quad \|\tilde{\Lambda}(\rho) - \Theta(\rho)\|_1 \leq \theta. \tag{3'}
\]

However, again the “correct” (composable) definition has to take into account the possibility of applying the effective channels to part of an entangled state:

\[
\forall \rho_{12}, \quad \|(\tilde{\Lambda} \otimes \text{id})\rho_{12} - (\Theta \otimes \text{id})\rho_{12}\|_1 \leq \theta. \tag{3’’}
\]

We call the scheme strictly non-malleable, if Eq. (3) or (3’) or (3’’) holds for some set \(\mathcal{C}' = \text{semi-lin}\{\text{id}, \langle \tau \rangle\}\) instead of \(\mathcal{C}\). (In other words, there is essentially only one constant channel in \(\mathcal{C}\), independent of \(\eta_0\).) Perfect non-malleability then corresponds to \(\theta = 0\), in either Eq. (3’) or (3’’).

III. MAIN RESULTS

In this paper we restrict ourselves to the “minimal” case, when \(\mathcal{H}' = \mathcal{H}\) is a \(d\)-dimensional Hilbert space, and to perfect transmission, i.e. Eq. (1). This entails that \(E_k\) is conjugation by a unitary \(U_k\), while \(D_k\) is simply the inverse, i.e. conjugation by \(U_k^\dagger\):

\[
E_k(\rho) = U_k\rho U_k^\dagger, \quad D_k(\sigma) = U_k^\dagger\sigma U_k.
\]

Since convex combinations of unitary conjugation channels are unital, in an encryption scheme all input states are encrypted as the maximally mixed state \(\xi_0 = \tau := \frac{1}{d} \mathbb{1}\) in Eqs. (2), (2’) and (2’’). (For a more general discussion see [5].) This means that the adversary can always implement channels

\[
\Theta \in \mathcal{C}' = \text{semi-lin}\{\text{id}, \langle \tau \rangle\}, \tag{4}
\]

where \(\langle \tau \rangle\) is the completely depolarizing channel. Conversely, we demand that these are the only ones she can achieve: for every c.p.t. ≤ map \(\Lambda\), we demand that the effective channel \(\tilde{\Lambda} \in \mathcal{C}'\), with

\[
\tilde{\Lambda}(\rho) = \sum_k p_k(k)U_k^\dagger(\Lambda(U_k\rho U_k^\dagger))U_k.
\]

This can be conveniently re-expressed using the Choi-Jamiołkowski operators [8, 17]: for the maximally entangled state \(\Phi_d = \frac{1}{d} \sum_{i,j=0}^{d-1} |ii\rangle\langle jj|\) on two systems labelled 1 and 2, let \(\omega = J_\Lambda := (\Lambda \otimes \text{id})\Phi_d\). Note that \(\text{Tr} J_\Lambda \leq 1\) and that \(\Lambda\) can be be recovered from the Choi-Jamiołkowski operator as follows:

\[
\Lambda(\rho) = d \text{Tr}_2((\mathbb{1} \otimes \rho^\top)J_\Lambda), \tag{5}
\]

where \(\rho^\top\) is the transpose operator of \(\rho\) with respect to the basis \(\{|i\rangle\}_{i=0}^{d-1}\). The image of the set \(\mathcal{C}'\) under the Choi-Jamiołkowski isomorphism is the set of bipartite positive operators

\[
(\mathcal{C}' \otimes \text{id})\Phi_d = \text{semi-lin}\{\Phi_d, \tau \otimes \tau\} = \mathbb{R}_{\geq 0}\Phi_d + \mathbb{R}_{\geq 0}(\mathbb{1} - \Phi_d) =: \mathcal{I}, \tag{6}
\]
which are (up to normalization) just the so-called isotropic states. Note that these are exactly the (semidefinite) operators invariant under conjugation with $U \otimes U$, and that integration over the Haar measure $dU$ implements the projection into $\mathcal{I}$: for every operator $X$,

$$
\int dU(U \otimes U)X(U \otimes U)^\dagger = \alpha \Phi_d + \beta(1 - \Phi_d), \quad \text{with } \alpha = \text{Tr}X\Phi_d, \quad \beta = \frac{1}{d^2 - 1} \text{Tr}X(1 - \Phi_d). \quad (7)
$$

The c.p.t.p. mapping from $X$ to the above average is known as the $U \otimes U$-twirl, denoted $\mathcal{T}_{U \otimes U}$.

On the other hand, exploiting the symmetry $\Phi_d = (U \otimes U)\Phi_d(U \otimes U)^\dagger$, we can write the Choi-Jamiołkowski operator of the effective channel,

$$
\tilde{\omega} = (\tilde{\Lambda} \otimes \text{id})\Phi_d
= \sum_k p_K(k)(U_k \otimes 1)\left[(\Lambda \otimes \text{id})(U \otimes 1)\Phi_d(U \otimes 1)^\dagger\right](U_k \otimes 1)
= \sum_k p_K(k)(U_k \otimes 1)^\dagger\left[(\Lambda \otimes \text{id})(1 \otimes U_k^\dagger)\Phi_d(1 \otimes U_k^\dagger)^\dagger\right](U_k \otimes 1)
= \sum_k p_K(k)(U_k \otimes \overline{U}_k)^\dagger\left[(\Lambda \otimes \text{id})\Phi_d\right](U_k \otimes \overline{U}_k)
= \sum_k p_K(k)(U_k \otimes \overline{U}_k)^\dagger\omega(U_k \otimes \overline{U}_k) =: T(\omega),
$$

where $T$ is manifestly a c.p.t.p. map. The condition that $\{p_K(k), U_k\}$ forms a perfect NMES is now concisely expressed as $T = \mathcal{T}_{U \otimes U}$.

This is precisely the condition for a so-called unitary 2-design [9], see also [15]. Note that modulo a partial transpose, the $U \otimes U$-twirl is equivalent to the more familiar $U \otimes U$-twirl

$$
\mathcal{T}_{U \otimes U}(X) = \int dU(U \otimes U)X(U \otimes U)^\dagger = \alpha F + \beta(1 - F),
$$

with the swap (or flip) operator $F = \sum_{i,j=0}^{d-1} |ij\rangle\langle ji|$, mapping density operators to Werner states [19]. Thus we have proved,

**Theorem 1** Every perfect non-malleable encryption scheme is a unitary 2-design. \qed

**Corollary 2** Any perfect non-malleable encryption scheme, i.e., an ensemble of unitaries $\{p_K(k), U_k\}$ satisfying $\tilde{\Lambda} \in \mathcal{C}'$, is automatically an ideal encryption scheme, i.e. Eq. (2) holds.

**Proof** By Theorem 1 a perfect NMES is a unitary 2-design. But then it is automatically a unitary 1-design, meaning that for all $\rho$, $\sum_k p_K(k)U_k\rho U_k^\dagger = \tau$, which is precisely Eq. (2). \qed

**Theorem 3** Every perfect non-malleable encryption scheme $\{p_K(k), U_k\}$ requires at least $(d^2 - 1)^2 + 1$ unitaries. Furthermore, every $\theta$–NMES as in Eq. (3”) with $\theta \leq 1/e$ satisfies

$$
H(p_K) \geq H_2\left(\frac{1}{d^2}\right) + 2\left(1 - \frac{1}{d^2}\right)\log(d^2 - 1) - 4\theta \log d - H_2(\theta) \geq (4 - O(\theta)) \log d,
$$

where $H_2(x) = -x \log x - (1 - x) \log(1 - x)$ is the binary entropy.

**Remark** In the light of Theorem 1 the first part amounts to a demonstration that 2-designs have to have at least $(d^2 - 1)^2 + 1$ unitaries; this was proved by Gross et al. [15], but we give a different, direct, proof below. It seems that it is conjectured that in fact the better lower bound $d^2(d^2 - 1)$
holds in general – which is true for so-called “Clifford twirls”, and tight in some dimensions \[7, 15\].

**Proof** Consider the Choi-Jamiolkowski operator of \( T \), labeling the systems 1, 2, 1’ and 2’, and with the maximally entangled state understood between systems 12 and 1’2’:

\[
\Omega_{U \otimes U} := (T_{12} \otimes \text{id}^{1/2'}) \Phi_{d^2} = \frac{1}{d^2} \Phi_d^{12} \otimes \Phi_d^{1/2'} + \frac{1}{d^2(d^2 - 1)} (\mathbb{1} - \Phi_d)^{12} \otimes (\mathbb{1} - \Phi_d)^{1/2'}.
\]

On the other hand, for the first part of the theorem this has to be equal to

\[
\Omega := (T^{12} \otimes \text{id}^{1/2'}) \Phi_{d^2} = \sum_{k=1}^{N} p_k(k) (U_k^1 \otimes U_k^2 \otimes 1^{1/2'}) \Phi_{d^2} (U_k^1 \otimes U_k^2 \otimes 1^{1/2'})^{\dagger}.
\]

Comparing ranks of the two right hand side expressions reveals immediately \( N \geq (d^2 - 1)^2 + 1 \).

For the entropy statement in the approximate case, we note that by Eq. (3’), \( \|\Omega - \Omega_{U \otimes U}\|_1 \leq \theta \), so by Fannes’ inequality [13] and Schur concavity of the entropy [13],

\[
H(p_k) \geq S(\Omega) \geq S(\Omega_{U \otimes U}) - \theta \log d^4 - H_2(\theta),
\]

and we are done. \( \square \)

**Theorem 4 (Chau [7], Gross et al. [15])** If \( d = p^n \) is a prime power, then there exists a perfect non-malleable encryption scheme with \( d^5 - d^3 \) unitaries, meaning that the key length is \( \leq 5 \log d \). In fact, such a scheme is obtained as the uniform ensemble over a particular subgroup of the Clifford group (i.e., the normalizer) of the \( n \)-th power Heisenberg-Weyl (aka generalised Pauli) group \( \mathcal{P}_p^\otimes n \), where \( \mathcal{P}_p \) is the group generated by the discrete Weyl operators

\[
X_p = \sum_{j=0}^{p-1} |j+1 \mod p \rangle \langle j|, \quad Z_p = \sum_{k=0}^{p-1} e^{2\pi i k/p} |k\rangle \langle k|.
\]

**Proof** Apart from Chau [7] see Gross et al. [15], as well as the crisp presentation of Grassl [14]. \( \square \)

**Remark** We note that in even prime power dimension, the cardinality of the subgroup can be reduced to \((d^5 - d^3)/8 \). Furthermore, Chau [7] showed that for several small dimensions the minimum \( d^5 - d^3 \) is attainable; see also Gross et al. [13] for another example of \( 2(d^4 - d^2) \).

**Theorem 5** For \( 0 < \theta \leq 1/2 \) there exists a \( \theta \)-NMES with \( O(\theta^{-2} d^4 \log d) \) unitaries, i.e. with key requirement of \( 4 \log d + \log \log d + O(\log \frac{1}{\theta}) \) bits. In fact, Eq. (3”) holds in the stronger form

\[
(1 - \theta)\Theta \leq \bar{\Lambda} \leq (1 + \theta)\Theta.
\]

**Proof** Start from any exact unitary 2-design, such as the unitary group with Haar measure, or the Clifford group or one of its admissible subgroups. We shall select \( U_1, \ldots, U_N \) independently at random from that chosen 2-design, and show that Eq. (3”) is true with high probability as soon as \( N \gg \theta^{-2} d^4 \log d \); which of course implies that there exist a particular selection of an ensemble \( \{1/N, U_k\}_{k=1}^N \) satisfying (3”).

In fact, it is sufficient to show that for \( T(\omega) = \frac{1}{N} \sum_{k=1}^N (U_k \otimes \bar{U}_k) \omega(U_k \otimes \bar{U}_k)^\dagger \),

\[
(1 - \theta)T_{U \otimes U} \leq T \leq (1 + \theta)T_{U \otimes U},
\]
which in turn is equivalent to the corresponding statement for the Choi-Jamiołkowski states – compare Eq. (5):

\[(1 - \theta) \Omega_{U \otimes U} \leq \Omega \leq (1 + \theta) \Omega_{U \otimes U},\]

where

\[\Omega_{U \otimes U} = (T_{U \otimes U}^{12} \otimes \mathbb{1}^{1/2}) \Phi_{d^2} = \frac{1}{d^2} \Phi_d^{12} \otimes \Phi_d^{1/2} + \frac{1}{d^2(d^2 - 1)} (\mathbb{1} - \Phi_d)^{12} \otimes (\mathbb{1} - \Phi_d)^{1/2},\]

\[\Omega = (T^{12} \otimes \mathbb{1}^{1/2}) \Phi_{d^2} = \frac{1}{N} \sum_{k=1}^{N} (U_k^{1} \otimes U_k^{2} \otimes \mathbb{1}^{1/2}) \Phi_{d^2}(U_k^{1} \otimes U_k^{2} \otimes \mathbb{1}^{1/2})^\dagger.\]

Now \(\Omega\) is a random variable, in fact an average of \(N\) independent, identically distributed terms \(X_k := (U_k^{1} \otimes U_k^{2} \otimes \mathbb{1}^{1/2}) \Phi_{d^2}(U_k^{1} \otimes U_k^{2} \otimes \mathbb{1}^{1/2})^\dagger\) with expectation \(\mathbb{E}X_k = \mathbb{E}\Omega = \Omega_{U \otimes U}\). All \(X_k\) are bounded between 0 and \(\mathbb{1}\), so the technical result from [1] applies, the operator Chernoff bound, yielding (with a universal constant \(c > 0\))

\[\Pr\left\{ (1 - \theta) \Omega_{U \otimes U} \leq \Omega \leq (1 + \theta) \Omega_{U \otimes U} \right\} \geq 1 - 2d^4 e^{-cd^2 N/d^4},\]

which implies the claim. \(\square\)

IV. DISCUSSION

We have introduced the cryptographic primitive of a non-malleable quantum state encryption scheme. While many questions remain open, we have shown that every such scheme based on random unitaries is a unitary 2-design, showing in particular that every such scheme must use \(4 \log d\) bits of key, as opposed to the well-known \(2 \log d\) necessary and sufficient for quantum state encryption [2].

This situation essentially persists even if we relax the non-malleability to being approximate. On the other hand, there exists an exact construction based on the Jacobi subgroup of the Clifford group in dimension \(d\), which requires \(5 \log d\) bits of key, and we show a new randomized construction requiring only \((4 + o(1)) \log d\) bits of key. We leave open the question of finding an explicit description of such a scheme, as well as that of finding an exact unitary 2-design with only \(O(d^4)\) elements.

What we also leave open is the perhaps more pressing problem of relaxing the condition that encryption is done by unitaries. Giving up this restriction results in an advantage in key size, see the work of Barnum et al. [3]. More precisely, these authors show how using \(2n + O(s)\) bits of secret key to encrypt \(n - s\) qubits into \(n\) qubits results in a \(\theta\)-NMES with \(\theta = 2^{-O(s)}\). In our setting this can be understood as only using \(d_0 < d\) of the Hilbert space dimensions for quantum information. Then, to transmit a state in the \(d_0\)-dimensional space \(\mathcal{H}_0 \subset \mathcal{H}\), first \(s\) key bits are used to specify a unitary rotation \(V_\ell\) of \(\mathcal{H}\), and then the familiar further \(2 \log d\) bits of key are used to encrypt \(\mathcal{H}\). If the \(V_\ell (\ell = 1, \ldots, 2^s)\) are “sufficiently random” and \(2^s \geq d/d_0\) then it can be shown that while the adversary can implement certain effective channels on \(\mathcal{H}\), for most \(\ell\) this will map the state significantly outside of \(\mathcal{H}_\ell := V_\ell \mathcal{H}_0\).

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[1] R. Ahlswede, A. Winter. Strong Converse for Identification via Quantum Channels. *IEEE Trans. Inf. Theory* 48(3):569-579, 2002.
[2] A. Ambainis, M. Mosca, A. Tapp, R. de Wolf. Private quantum channels. In *FOCS 2000*, pp. 547-553, 2000. [arXiv:quant-ph/0003101](http://arxiv.org/abs/quant-ph/0003101)
[3] A. Ambainis, A. Smith. Small pseudo-random families of matrices: derandomizing approximative quantum encryption. In *RANDOM*, pp. 249-260, 2004. [arXiv:quant-ph/0404075](http://arxiv.org/abs/quant-ph/0404075)
[4] H. Barnum, C. Crépeau, D. Gottesman, A. Smith, A. Tapp. Authentication of quantum messages. In *FOCS 2002*, 2002. [arXiv:quant-ph/0205128](http://arxiv.org/abs/quant-ph/0205128)
[5] J. Bouda, M. Ziman. Optimality of private quantum channels. *J. Phys. A: Math. Gen.*, 40:5415-5426, 2007.
[6] A. R. Calderbank, E. M. Rains, P. W. Shor, and N. J. A. Sloane. Quantum error correction via codes over GF(4). *IEEE Trans. Inf. Theory*, 44:1369-1387, 1998. [arXiv:quant-ph/9608006](http://arxiv.org/abs/quant-ph/9608006)
[7] H. F. Chau. Unconditionally secure key distribution in higher dimensions by depolarization. *IEEE Trans. Inf. Theory*, 51:1451-1468, 2005. [arXiv:quant-ph/0405016](http://arxiv.org/abs/quant-ph/0405016)
[8] M.-D. Choi. Completely positive linear maps on complex matrices. *Lin. Alg. Appl.* 10:285-290, 1975.
[9] Ch. Dankert, R. Cleve, J. Emerson, E. Livine. Exact and approximate unitary 2-designs: Constructions and applications. [arXiv:quant-ph/0606161](http://arxiv.org/abs/quant-ph/0606161) 2006.
[10] E. B. Davies, J. T. Lewis. An Operational Approach to Quantum Probability. *Comm. Math. Phys.* 17:239-260, 1970.
[11] P. A. Dickinson, A. Nayak. Approximate randomization of quantum states with fewer bits of key. [arXiv:quant-ph/0611033](http://arxiv.org/abs/quant-ph/0611033) 2006.
[12] D. Dolev, C. Dwork, M. Naor Nonmalleable Cryptography. *SIAM J. Comput.* 30:391-437, 2001.
[13] M. Fannes. A continuity property of the entropy density for spin lattice systems. *Commun. Math. Phys.* 31:291-294, 1973
[14] M. Grassl. On SIC-POVMs and MUBs in Dimension 6. [arXiv:quant-ph/0406175](http://arxiv.org/abs/quant-ph/0406175) 2004.
[15] D. Gross, K. Audenaert, and J. Eisert. Evenly distributed unitaries: On the structure of unitary designs. *J. Math. Phys.*, 48:052104, 2007. [arXiv:quant-ph/0611002](http://arxiv.org/abs/quant-ph/0611002)
[16] P. Hayden, D. W. Leung, P. W. Shor, and A. Winter. Randomizing quantum states: Constructions and applications. *Commun. Math. Phys.*, 250:371–391, 2004. [arXiv:quant-ph/0307014](http://arxiv.org/abs/quant-ph/0307014)
[17] A. Jamiołkowski. Linear transformations which preserve trace and positive semidefiniteness of operators. *Rep. Math. Phys.*, 3:275-278, 1972.
[18] A. Wehrl. General properties of entropy. *Rev. Mod. Phys.* 50(2):221-260, 1978.
[19] R. F. Werner. Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. *Phys. Rev. A*, 40:4277-4281, 1989.