Classification of states of single-\(j\) fermions with \(J\)-pairing interaction

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In this paper we show that a system of three fermions is exactly solvable for the case of a single-\(j\) in the presence of an angular momentum-\(J\) pairing interaction. On the basis of the solutions for this system, we obtain new sum rules for six-\(j\) symbols. When the Hamiltonian contains only an interaction between pairs of fermions coupled to spin \(J = J_{\text{max}} = 2j - 1\), the “non-integer” eigenvalues of three fermions with angular momentum \(I\) around the maximum appear as “non-integer” eigenvalues of four fermions if \(I\) is around (or larger than) \(J_{\text{max}}\). This pattern is also found in five and six fermion systems. A boson system with spin \(l\) exhibits a similar pattern.

**PACS number**: 21.60.Ev, 21.60.Fw, 24.60.Lz, 05.45.-a
1 Introduction

To obtain a simple solution of the many-body Schrödinger equation is a long dream of physicists. There have been numerous efforts to obtain analytical solutions which do not require diagonalization of the secular equation by using computers. Among the many efforts in nuclear physics, the Elliott Model [1], the seniority scheme [2], the $s$ and $d$ interacting boson model [3] and a similar model by using schematic $S$ and $D$ pairs, the fermion dynamical symmetry model [4] are successful examples along this line.

In Ref. [5] we showed that for a large array of states of four fermions in a single-$j$ shell with only the $J = J_{\text{max}}$ pairing interaction, the eigenvalues are asymptotically integers labeled by numbers of $J = J_{\text{max}}$ pairs in the wavefunction, and that those corresponding wavefunctions of these low $I^{(4)}$ ($I^{(4)}$ is the total angular momentum for a state of the four fermions) states are readily constructed in the nucleon pair basis. Besides the “integer” eigenvalues (as explained in Sec. 2), there are eigenvalues not close to integers when $I^{(4)}$ is around or larger than $2j$, and little was known about these states. It would be desirable to discuss both the “integer” and “non-integer” eigenvalues on a more general footing. This is one of the goals in this paper.

In Sec. II of this paper, we shall first study $n = 3$ systems ($n$ is the number of fermions), which are readily solvable for any $J$ pairing interaction only. The solutions of $n = 3$ provide an appropriate platform to explain the main idea of this paper. In Sec. III, we report relations between the eigenvalues of $n = 3$, 4-, 5-, and 6-particle systems with the $J_{\text{max}}$ pairing interaction. Using these relations one may obtain approximate values for both the eigenvalues and wave functions of low-lying states of these systems. We propose a hypothesis by which one readily obtains, to a high precision and without diagonalization of the Hamiltonian, the wavefunctions of states corresponding to some non-integer eigenvalues discussed in our earlier work [5]. A summary and discussion is given in Sec. IV. In the Appendix, we present a number of new sum rules for six-$j$ symbols, some of which were derived recently by Talmi [6].

In this paper we use a convention that $j$ ($j'$) is a half integer, and that $l$ ($l'$) is an integer. They correspond to the angular momenta of the single-particle levels of fermions and spin carried by boson, respectively. $J$ is used as the angular momentum coupled by two fermions in a single-$j$ shell or two bosons with spin $l$; the maximum of $J$, $J_{\text{max}}$, = $2j - 1$ for fermions and $2l$ for two bosons. We use superscript ($n$) to specify the particle number $n$ in the angular momentum $I^{(n)}$ and the eigenvalue $E_{I^{(n)},J(j)}^{(n)}$. 
2 Three fermions in the presence of $H_J$ only

The pairing interaction which couples two fermions to an angular momentum $J$ is

$$H_J = G_J \sum_{M=-J}^{J} A_M^J A_M^\dagger, \quad A_M^J = \frac{1}{\sqrt{2}} \left[ a_j^\dagger \times a_j^\dagger \right]^J_M,$$

$$A_M^J = -(-1)^M \frac{1}{\sqrt{2}} \left[ \tilde{a}_j \times \tilde{a}_j \right]^J_{M}, \quad \tilde{A}^J = -\frac{1}{\sqrt{2}} \left[ \tilde{a}_j \times \tilde{a}_j \right]^J.$$

(1)

where $[ \ ]^J_M$ means coupled to angular momentum $J$ and projection $M$. We take $G_J = -1$ in this paper.

We now consider the pair basis of three nucleons

$$|j^3[jJ]I^{(3)}, M\rangle = \frac{1}{\sqrt{N_{jJ}^{I^{(3)}}}} \left( a_j^\dagger \times A_J^I \right)^{I^{(3)}}_M |0\rangle,$$

(2)

where $N_{jJ}^{I^{(3)}}$ is the diagonal matrix element of the normalization matrix

$$N_{jJ}^{I^{(3)}} = \langle 0 | \left( a_j^\dagger \times A_J^I \right)^{I^{(3)}}_M \left( a_j^\dagger \times A_J^I \right)^{I^{(3)}}_M |0\rangle.$$

(3)

In general this basis is over complete and the normalization matrix may have zero eigenvalues for a given $I^{(3)}$.

We first rewrite the matrix elements of $H_J$ and $N_{jJ}^{I^{(3)}}$ as follows [7].

$$\langle j^3[jK']I^{(3)}, M|H_J|j^3[jK]I^{(3)}, M\rangle$$

$$= -\frac{1}{\sqrt{N_{jK'}^{I^{(3)}} N_{jK}^{I^{(3)}}}} \sum_{L} (-)^{I^{(3)}+J-L} \hat{L} \frac{\hat{L}}{2I^{(3)}+1} \times$$

$$\langle 0 | \left[ \left( \tilde{a}_j \times \tilde{A}^{K'} \right)^{I^{(3)}}_M, A_J^I \right]^L \times \left[ \tilde{A}^J, \left( a_j^\dagger \times A^{K'^I} \right)^{I^{(3)}}_M \right]^L \rangle |0\rangle,$$

$$N_{jJ'}^{I^{(3)}} = \frac{1}{I^{(3)}} \langle 0 | \left( \tilde{a}_j \times \left[ \tilde{A}^{J'}, \left( a_j^\dagger \times A^{J'^I} \right)^{I^{(3)}}_M \right] \right)^{I^{(3)}}_M |0\rangle,$$

(4)

where $\hat{L}$ is a short hand notation of $\sqrt{2L+1}$.

According to Eq. (12a) of Ref. [8] and Eqs. (3.10a) and (3.10b) [9],

$$\left[ \tilde{a}_j, \left( a_j^\dagger \times \tilde{a}_j \right)^I \right]^t = -\delta_{I^{(3)},j} (-t)^t \hat{I} \frac{\hat{I}}{j} \tilde{a}_j,$$

$$\left[ \tilde{A}^r, A^s \right]^t = \hat{r} \delta_{r,s} \delta_{t,0} - 2\hat{r} \hat{s} \left\{ \begin{array}{ccc} r & s & t \\ j & j & j \end{array} \right\} \left( a_j^\dagger \times \tilde{a}_j \right)^t.$$

(5)
Using commutators in (5), we obtain
\[
\langle 0 | \left( \left( a_j \times A^\dagger \right) I^{(3)} , A^j \right) \rangle^L = \langle 0 | \left( -I^{(3)} - j \delta_{L,j} \delta_{K',J} \hat{J} \right) \left( a_j \times (a_j^\dagger \times a_j)^t \right) \rangle^L
\]
\[
-2\hat{J} \sum_j \langle 0 | \left( -I^{(3)} - j \delta_{L,j} \delta_{K',J} \hat{J} \right) \left( a_j \times (a_j^\dagger \times a_j)^t \right) \rangle^L
\]
\[
= (-)^{I^{(3)} - j} \delta_{L,j} \left( \delta_{K',J} + 2\hat{J} \left\{ \left\{ \hat{J} \right\} \right\} \right) \langle 0 | a_j \rangle.
\]
where a sum rule
\[
\sum_j (-)^{I^{(3)} + j} (2j + 1) \left\{ \left\{ \hat{J} \right\} \right\} \left\{ \left\{ \hat{J} \right\} \right\} = \left\{ \left\{ \hat{J} \right\} \right\}
\]
is used. The Hermitian conjugate of Eq. (6) yields
\[
\left[ \hat{A}^\dagger, (a_j^\dagger \times A^{K^\dagger}) I^{(3)} \right]^L \langle 0 | = (-)^{I^{(3)} + J - L - j} \delta_{L,j} \left( \delta_{K,J} + 2\hat{J} \left\{ \left\{ \hat{J} \right\} \right\} \right) \langle 0 | a_j \rangle^\dagger.
\]
Substituting Eqs. (6) and (8) into Eq. (4), we obtain
\[
N^{I^{(3)}}_{j,J,j,J} = \delta_{j',J} + 2\hat{J} \left\{ \left\{ \hat{J} \right\} \right\},
\]
\[
\langle j^3 [j K'] I^{(3)}, M | H_J | j^3 [j K] I^{(3)}, M \rangle = -\frac{1}{\sqrt{N^{I^{(3)}}_{j,K',J} N^{I^{(3)}}_{j,K,j} N^{I^{(3)}}_{j,K',J} N^{I^{(3)}}_{j,K,j} K}}.
\]
Below we explain that there is at most one non-zero eigenvalue for each \( I^{(3)} \) of \( n = 3 \) with \( H = H_J \). For a fixed \( J \) and for any \( I^{(3)} \), we construct the \( | j^3 J : I^{(3)} \rangle \) and other states \( | j^3 K : I^{(3)} \rangle \) (\( K \neq J \)) which are orthogonal to \( | j^3 J : I^{(3)} \rangle \) as follows.
\[
| j^3 J : I^{(3)} \rangle = | j^3 [j J] I^{(3)} \rangle,
\]
\[
| j^3 K : I^{(3)} \rangle = | j^3 [j K] I^{(3)} \rangle - \frac{N^{I^{(3)}}_{j,K',J} N^{I^{(3)}}_{j,K,j} N^{I^{(3)}}_{j,K',J} N^{I^{(3)}}_{j,K,j} K}}{\sqrt{N^{I^{(3)}}_{j,J,j,J} N^{I^{(3)}}_{j,K,j} N^{I^{(3)}}_{j,K',J} N^{I^{(3)}}_{j,K,j} K}} | j^3 [j J] I^{(3)} \rangle, \ (K \neq J).
\]
Using Eq. (9), we easily confirm that all matrix elements of the Hamiltonian in the basis (10), \( \langle j^3 K' : I^{(3)} | H_J | j^3 K : I^{(3)} \rangle \), are zero except \( \langle j^3 [j J] I^{(3)} | H_J | j^3 [j J] I^{(3)} \rangle = N^{I^{(3)}}_{j,J,j,J} \). Thus all the eigenvalues of \( n = 3 \) for a given \( I^{(3)} \) are zero for \( H = H_J \) except for the state with one pair with spin \( J \), with an eigenvalue \( E^{I^{(3)}}_{j,J,j,J} \) (the
number in superscript specify the particle number \( n \) given by \(-N_{jJ}^{(3)}\). This result was also proved recently by Talmi in terms of coefficients of fractional parentage [6].

Next we explain why the eigenvalues for the \( n = 3 \) cases are close to integers when \( H = H_{\text{max}} \). As shown above, the wave function of the lowest energy state for each \( I^{(3)} \) is given by \((a_j^{\dagger} \times a_j^{\dagger}) (J_{\text{max}}) \times a_j^{\dagger}) |0\rangle\). The eigenvalue \( E_{I^{(3)},J(j)}^{(3)} \) equals to \(-1\) subtracted by a six-\( j \) symbol (refer to Eq. (9)), this six-\( j \) symbol is in fact very close to zero when \( I^{(3)} \) is not close to \( I_{\text{max}}^{(3)} = 3j - 3 \). The lowest eigenvalue for each \( I^{(3)} \) (\( I^{(3)} \geq j - 1 \)) is thus very close to \(-1\) unless \( I^{(3)} \sim I_{\text{max}}^{(3)} \). All eigenvalues for \( I^{(3)} \leq j - 2 \) are zero. To show that the six-\( j \) symbols involved in Eq. (9) asymptotically vanishes, we list a few formulas of these six-\( j \) symbols:

\[
\begin{align*}
\{ j & \quad j - 1 \quad 2j - 1 \\
\} & \quad \{ j & \quad j \quad 2j - 1 \}
\end{align*}
\]

\[
\left\{ \begin{array}{c}
\frac{2j(2j - 1)!}{(4j - 3)!} \\
\frac{j(4j - 3)(2j - 1)!}{(4j - 1)!} \\
\frac{j(4j^2 - 4j - 1)(2j - 1)!}{(4j - 1)!}
\end{array} \right. ;
\]

which are less than \(10^{-14}\) in magnitude for \( j = 31/2\). Clearly, the approximate integer eigenvalues of \( n = 3 \) with \( H = H_{\text{max}} \) comes from the fact that the six-\( j \) symbol \( \left\{ \begin{array}{c}
j & \quad I^{(3)} \quad 2j - 1 \\
j & \quad j \quad 2j - 1
\end{array} \right. \) are negligible unless \( I^{(3)} \sim I_{\text{max}}^{(3)} \).

The “non-integer” eigenvalues with \( I^{(3)} \sim I_{\text{max}}^{(3)} \) are also readily obtained:

\[
\begin{align*}
-E_{I_{\text{max}}^{(3)},J_{\text{max}}^{(3)}}^{(3)} & = \frac{9}{4} + \frac{3}{4(4j - 3)}; \\
-E_{I_{\text{max}}^{(3)} - 2, J_{\text{max}}^{(3)}}^{(3)} & = \frac{27}{16} - \frac{15}{32(4j - 5)} - \frac{21}{32(4j - 3)}; \\
-E_{I_{\text{max}}^{(3)} - 3, J_{\text{max}}^{(3)}}^{(3)} & = \frac{9}{16} + \frac{15}{32(4j - 5)} + \frac{45}{32(4j - 3)}; \\
-E_{I_{\text{max}}^{(3)} - 4, J_{\text{max}}^{(3)}}^{(3)} & = \frac{81}{64} + \frac{105}{512(4j - 7)} - \frac{15}{256(4j - 5)} - \frac{1155}{512(4j - 3)}; \\
-E_{I_{\text{max}}^{(3)} - 5, J_{\text{max}}^{(3)}}^{(3)} & = \frac{27}{64} - \frac{105}{256(4j - 7)} - \frac{105}{128(4j - 5)} + \frac{819}{256(4j - 3)}; \\
-E_{I_{\text{max}}^{(3)} - 6, J_{\text{max}}^{(3)}}^{(3)} & = \frac{279}{256} - \frac{315}{4096(4j - 9)} + \frac{1785}{4096(4j - 7)} + \frac{9135}{4096(4j - 5)} - \frac{17325}{4096(4j - 3)};
\end{align*}
\]
\[-E^{(3)}_{I^{(3)}_{\text{max}}-7, J^{(3)}_{\text{max}}}(j) = \frac{243}{256} + \frac{945}{4096(4j - 9)} - \frac{315}{4096(4j - 7)} \]
\[\quad - \frac{17325}{4096(4j - 5)} + \frac{2187}{4096(4j - 3)};\]
\[-E^{(3)}_{I^{(3)}_{\text{max}}-8, J^{(3)}_{\text{max}}}(j) = \frac{1053}{1024} + \frac{3465}{131072(4j - 11)} - \frac{12915}{32768(4j - 9)} \]
\[\quad - \frac{65536(4j - 7)}{32768(4j - 5)} + \frac{225225}{131072(4j - 3)};\]
\[-E^{(3)}_{I^{(3)}_{\text{max}}-9, J^{(3)}_{\text{max}}}(j) = \frac{63}{64} - \frac{3465}{32768(4j - 11)} + \frac{8192(4j - 9)}{3465} \]
\[\quad + \frac{45045}{16384(4j - 7)} - \frac{83655}{8192(4j - 5)} + \frac{255225}{32768(4j - 3)};\] (12)

etc. We see that these above eigenvalues \(E^{(3)}_{I^{(3)}_{\text{max}}-J^{(3)}_{\text{max}}}(j)\) stagger and saturate at \(-1\) as \(I^{(3)}\) becomes smaller and smaller (but \(I^{(3)} \geq j - 1\)). In the large \(j\) limit, the non-zero eigenvalue for each \(I^{(3)}\) takes the first term; for a very small \(j\) value, e.g., \(j = 9/2\), the eigenvalue \(E^{(3)}_{I^{(3)}_{\text{max}}-J^{(3)}_{\text{max}}}(9/2)\) (i.e., \(I^{(3)} = 7/2\), \(E^{(3)}_{7/2,J^{(3)}_{\text{max}}}(9/2)\) equals to \(-\frac{744}{110}\) is already very close to \(-1\) (within a precision of \(10^{-2}\)). This explains why we frequently obtain asymptotic \(-1\) eigenvalues for \(H = H_{J_{\text{max}}}\) and \(n = 3\). For a state which has \(E^{(3)}_{I^{(3)}_{\text{max}}-J^{(3)}_{\text{max}}}(j) \sim -1\), the corresponding wavefunction can be understood as a single-\(j\) “spectator” coupled to one pair with spin \(J_{\text{max}}\). There is no such a spectator for \(I^{(3)} \sim I^{(3)}_{\text{max}}\) states, although their wave functions can be written as \(j^2(J_{\text{max}})j : I^{(3)}\).

We note that three bosons with spin \(l\) exhibit a similar pattern: there is up to one non-integer eigenvalue for each \(I\) in the presence of boson Hamiltonian \(H_J\),

\[E^{(3)}_{I^{(3)},J(l)} = -1 - 2(2J + 1) \left\{ J \atop J \atop l \atop l. \right\},\] (13)
i.e., this non-zero (zero) eigenstate are given by a pair with spin \(J\), i.e., \((b^\dagger_l \times b^\dagger_l)^{(J)}\), coupled with a single boson operator \(b^\dagger_l\).

### 3 Relations between states of \(n = 3\) and \(4\) for \(H = H_{J_{\text{max}}}\).

In this section, we first discuss the cases with \(n = 4\). In Ref. [5], it was found that the eigenvalues of \(n = 4\) are asymptotically 0, \(-1\) or \(-2\) for small \(I^{(4)}\). These states are constructed by coupling one or two pairs with spin \(J = J_{\text{max}}\). However, some
“non-integer” eigenvalues appear as $I^{(4)}$ is larger than $2j - 9$. These values are very stable for $2j - 8 \leq 4j - 12$, and the origin for these states was unknown.

Let us compare the eigenvalues of a system with $n = 3$ and $n = 4$ fermions with $H = H_{J_{\text{max}}}$ for $j = 31/2$. The distribution of all non-zero eigenvalues for $n = 3$ and 4 is plotted in Fig. 1(a)-(c)\(^1\), where (a), (b) and (c) correspond to the range of $|E|$ from 0 to 0.8, 0.8 to 1.5, 1.5 to 3.8, respectively.

From Fig. 1, we see that these eigenvalues are clustered at a few values but with exceptions. The “clustered” values are very close to the eigenvalues of $n = 3$. This indicates that the eigenstates of $n = 4$ are closely related to those of $n = 3$.

For $j = 31/2$ and $n = 4$ the total number of states is 790. The number of states with non-zero eigenvalues is 380. Within a precision $10^{-2}$, the eigenvalues of these 308 states are located at the eigenvalues of $n = 3$, and 21 states have eigenvalues closely at $-2$. We note that almost all the “non-integer” eigenvalues of $n = 4$ can be rather accurately given by one of three-particle clusters with $I^{(3)} \sim I_{\text{max}}^{(3)}$ coupled to a single-$j$ particle. In this example only four states with $I^{(4)} = 48$, two states with $I^{(4)} = 46$, and two states with $I^{(4)} = 44$ cannot be understood by either one of three-particle clusters with $I^{(3)} \sim I_{\text{max}}^{(3)}$ coupled to a single-$j$ particle or two pairs with one or two spins being $J_{\text{max}}$.

For example, the peak for $n = 4$ in Fig. 1(c) near 2.25 is very close to the energy of $|E_{I_{\text{max}}^{(3)},J_{\text{max}}^{(j)}}^{(3)}|$ of $n = 3$. For $j = 31/2$, the maximum angular momentum $I_{\text{max}}^{(3)}$ of three fermions is $\frac{87}{2}$. The $E_{I_{\text{max}}^{(3)},J_{\text{max}}^{(j)}}^{(3)} = -\frac{267}{18} = -2.26271186440677966$. The minimum $I^{(4)}$ obtained by coupling a three-body cluster with $I^{(3)} = I_{\text{max}}^{(3)}$ to a single-$j$ particle is given by $3j - 3 - j = 2j - 3$ (the triangle relation for vector couplings) and here 28. We find that the lowest eigenvalue of $I^{(4)} = 28$ for $n = 4$ obtained from a shell model diagonalization is $-2.26271186440689$. The $E_{I_{\text{max}}^{(3)},J_{\text{max}}^{(j)}}^{(4)}$ (close to $-2.26$) with $I^{(4)}$ between 28 to 56 are listed in Table I. Two observations can be made: (1) the lowest state of each $I^{(4)}$ are well separated from the second lowest one, and (2), there is no eigenvalue which is smaller than -2.0 when $I^{(4)}$ is lower than $2j - 3$ for $n = 4$.\(^3\)

We also see that the overlap of the wave function obtained by the exact shell

\(^1\)The inset in Fig. 1(b) is re-scaled in order to see more clearly the exceptions of energies which are not close to those of $n = 3$.

\(^2\)There is one peak at 2.0 which was explained using two pairs with spin $J = 2j - 1$ in Ref. [5].

\(^3\)We also note that the above nearly equality is asymptotic for a rather large $j$, not exact. For examples, when $j$ is very small, for $j = 7/2$, the energy of $E_{I_{\text{max}}^{(3)},J_{\text{max}}^{(j)}}^{(3)}$ is $-\frac{51}{22} = 2.31818182$ while the lowest energies of $I^{(4)} = 2j - 3 = 4$ for four fermions obtained by diagonalization is $-\frac{7}{2} = 2.66666667$; for $j = 9/2$, the energy of $E_{I_{\text{max}}^{(3)},J_{\text{max}}^{(j)}}^{(3)}$ is $-\frac{29}{11}$ while the lowest energies of $I^{(4)} = 2j - 3 = 6$ for four fermions obtained by diagonalization is $-2.34965034965038$.\(^3\)
model with the state constructed as the \( I_{\text{max}}^{(3)} \) state coupled to a single-\( j \) particle, \([I_{\text{max}}^{(3)} \times j]^{I^{(4)}}\), is very close to 1. This may be argued as follows. The eigenvalue for the lowest state of \( n = 4 \) for \( I^{(4)} \geq 28 \) is very close to the matrix element of the Hamiltonian for the state \([I_{\text{max}}^{(3)} \times j]^{I^{(4)}}\). Suppose that the lowest spin \( I^{(4)} \) state is not degenerate. Then a certain state which produces the same energy as the lowest spin \( I^{(4)} \) state will have the same wavefunction. In Table I most of the energies obtained by diagonalizing \( H_{J_{\text{max}}}^{J_{\text{max}}} \) for \( n = 4 \) are close to the matrix element for the pure configuration of the \([I_{\text{max}}^{(3)} \times j]^{I^{(4)}}\) state (also close to \( E_{I_{\text{max}}^{(3)}}^{(3)} \)). Overlaps of states having other “non-integer” eigenvalues near \(-2.25\) for \( n = 4 \) with those given by the \([I_{\text{max}}^{(3)} \times j]^{I^{(4)}}\) are close to 1, except three cases (two of them can be approximated by other three-particle clusters with \( I^{(3)} \sim I_{\text{max}}^{(3)} \) (but \( I^{(3)} \neq I_{\text{max}}^{(3)} \) coupled to a single particle).

We have calculated all overlaps between states of \( n = 4 \) which have energies close to the peaks and those of simple wavefunctions obtained by coupling a single particle to a non-zero energy cluster with \( I_{\text{max}}^{(3)} \sim I_{\text{max}}^{(3)} \) of three fermions. These show a similar situation as Table I. Therefore, we conclude that those stable “non-integer” eigenvalues of \( n = 4 \) with \( H = H_{J_{\text{max}}} \) in Fig. 1 are given to a high precision by a three-particle cluster (nonzero energy) coupled to a single-\( j \) particle.

One may ask which picture is more relevant to the states of \( n = 4 \) with eigenvalues close to integers; one in which a three-particle cluster (nonzero energy) is coupled to a single-\( j \) particle, \([I^{(3)} \times j]^{I^{(4)}}\) with \( I^{(3)} \sim I_{\text{max}}^{(3)} \), as proposed in this paper, or one in which four particles are coupled pairwise with one or two spins being \( J_{\text{max}} \), as proposed in Ref. [5]?

First we note that for \( I^{(4)} \geq 53 \) only a single state is possible and these two pictures are therefore equivalent and exact; for \( I^{(4)} = 0 \) (or 3) the number of states is the largest integer not exceeding \((2j+3)/6\) (or \((2j-3)/6\)) which is larger than 1 in most cases [10] but there is only one state which gives a non-zero eigenvalue for the \( J_{\text{max}} \) pairing interaction [5]. Also in this case the two pictures are therefore equivalent and exact.

For states with \( I^{(4)} < J_{\text{max}} \) and energy close to \(-2.0\), it was proven in Ref. [5] that a description by using two pairs with two spins being \( J_{\text{max}} \) is very good. For these states one may ask whether a description by using a single-\( j \) particle coupled to one of three-particle clusters with \( I^{(3)} \sim I_{\text{max}}^{(3)} \) is also relevant. To see whether or not this is true, we calculate the overlaps of the states of four fermions with \( E_{I^{(4)},J_{\text{max}}(j)}^{(4)} \sim -2.0 \) and \( I^{(4)} < J_{\text{max}} \) which were obtained by the shell model diagonalization with \textit{all} possible three-particle clusters coupled with a single-\( j \) particle. These overlaps are between around 0.6-0.8. Thus the picture using a single-\( j \) particle coupled to the three-particle cluster for states with \( I^{(4)} < J_{\text{max}} \) and energy close to \(-2.0\) is not appropriate.
Then, how does the picture of two pairs with spin \( J = J_{\text{max}} \) work as \( I^{(4)} \) increases? We calculate the overlaps of states with energy around \(-2\). We see that for \( I^{(4)} = 42 \) \((-I_{\text{max}}^{(4)} - 14\) the overlap is still 0.9962, showing that the pair picture is still very good.

The next question is related to the states of four fermions with energies near \(-1\). There are about 100 states for \( n = 4 \) and \( j = 31/2 \). Both pictures can give eigenvalues at \(-1\). The number of states with \( E \sim -1 \) is not unique. As was shown in Ref. [5], for states with small \( I^{(4)} \), the number of states with \( E^{(4)}_{I^{(4)}, J_{\text{max}}(j)} \sim -1 \) states is the largest integer not exceeding \( I^{(4)}/2 \), which is larger than 1 except for \( I^{(4)} = 0, 2 \) and 3. Because these eigenvalues are very close to but not exactly \(-1\), the mixing of these configurations can be large. Coupling two pairs, one with spin \( J_{\text{max}} \) and the other spin \( J' \neq J_{\text{max}} \) picture gives a very good classification of states, but not the exact wavefunctions. On the other hand, the picture using a three-particle cluster of nonzero energy coupled with a single-\( j \) particle provides us with better wavefunction than the pair picture, but it does not provide us the number of states with \( E^{(4)}_{I^{(4)}, J_{\text{max}}(j)} \sim -1 \). These two pictures are therefore complementary in describing the states for \( n = 4 \) with \( E^{(4)}_{I^{(4)}, J_{\text{max}}(j)} \sim -1 \).

4 States of five particles and those of six particles with \( H = H_{J_{\text{max}}} \)

In this section, we proceed to more particle systems. Although we did not find simple descriptions for them, we are able to find some relations between states of different \( n \) systems with an attractive \( J_{\text{max}} \) pairing interaction.

The picture using clusters of \( \mathcal{N} \) \((\mathcal{N} < n) \) particles coupled to \((n - \mathcal{N}) \) single-\( j \) is also found in states of systems with \( n > 4 \). We study in this section the \( j = 19/2 \) shell for both \( n = 5 \) and 6. The cases with larger \( j \)-shells yield a similar picture with higher accuracy.

Asymptotic integers appear in the eigenvalues \( E^{(5)}_{I^{(5)}, J_{\text{max}}(j)} \) when \( I^{(5)} \) is not very large. They are either zero, or very close to \(-1 \) and \(-2 \). The number of states for \( I^{(5)} = 1/2 \) is three, among which there is one with zero eigenvalue, one with eigenvalues \( \sim -1 \) (within a precision of 0.01) and one with eigenvalue \( \sim -2 \) (within a precision of 0.01). The number of states for \( I^{(5)} = 3/2 \) is seven, among which there are two with zero eigenvalues, three with eigenvalues \( \sim -1 \) (within a precision of 0.01) and two with eigenvalues \( \sim -2 \) (within a precision of 0.01). A similar situation holds for larger \( I^{(5)} \) states except that eigenvalues \( \sim (E^{(3)}_{I^{(3)}, J_{\text{max}}(j)} - 1) \) \((I^{(3)} \sim I^{(3)}_{\text{max}}) \) or \( E^{(3)}_{I^{(3)}, J_{\text{max}}(j)} \) appear. Corresponding to each three-body cluster with \( I^{(3)} \sim I^{(3)}_{\text{max}} \), the
minimum of $I$ which gives “non-integer” eigenvalues $E_{I(5),J_{\text{max}}(j)}^{(5)} \sim (E_{I(3),J_{\text{max}}(j)}^{(3)} - 1)$ and $E_{I(5),J_{\text{max}}(j)}^{(5)} \sim E_{I(3),J_{\text{max}}(j)}^{(3)}$ is $I^{(3)} = (2j - 1)$ and $I^{(3)} = (2j - 3)$, respectively. For examples, $E_{I(5),J_{\text{max}}(j)}^{(5)} = -1 = \frac{6499}{3410} = -1.89707$ appears in states with $I^{(5)} \geq I_{\text{max}}^{(3)} - 5 - (2j - 1) = j - 7 = \frac{5}{2}$; $E_{I(5),J_{\text{max}}(j)}^{(5)} \sim E_{I_{\text{max}}^{(3)} - 5,J_{\text{max}}(j)}^{(3)} = \frac{3059}{3410} = -0.89707$ appears in states with $I^{(5)} \geq I_{\text{max}}^{(3)} - 5 - (2j - 3) = j - 5 = \frac{9}{2}$, etc. The non-zero eigenvalues for $n = 5$ are equal to or concentrated around 0, $-1$, $-2$, $\sim E_{I(3),J_{\text{max}}(j)}^{(3)}$ and $\sim (E_{I(3),J_{\text{max}}(j)}^{(3)} - 1)$ with $I^{(3)} \sim I_{\text{max}}^{(3)}$. The above regularities survive unless $I^{(5)} \sim I_{\text{max}}^{(5)}$.

Now let us look at the case with $n = 6$ in the same shell $j = 19/2$. Below we consider the case with $I^{(6)} = 0$ as an example because other low $I^{(6)}$ states behave similarly. There are ten states with $I^{(6)} = 0$. Among them there are two with zero eigenvalues. Non-zero eigenvalues are more complicated than systems with smaller $n$, because $n = 6$ can be divided into more sets of clusters. We first divide $n = 6$ into two clusters with $n = 3$ and $I^{(3)} \sim I_{\text{max}}^{(3)}$ for each three-body cluster. Then we obtain eigenstates with eigenvalues: $-4.54286$, $-3.31055$, $-1.23269$, $-2.42027$, $-1.75573$, and $-2.10284$, which are approximately equal to twice those of $E_{I(3),J_{\text{max}}(j)}^{(3)}$ ($I^{(3)} \sim I_{\text{max}}^{(3)}$): $\sim -\frac{139}{35}$, $-\frac{182}{35}$, $-\frac{95}{17}$, $-\frac{164}{682}$, $\frac{3059}{1705}$, $\frac{20727}{6829}$, respectively. We can also divide 6 into three two-body pairs. Here we take $I^{(2)} = J_{\text{max}}$ for these three pairs which lead to an eigenvalue very close to $-3$ ($-3.01537$). Besides these eigenvalues, there are one eigenvalues which are close to $-1.0$.

We did not succeed in setting up a simple scenario of the distribution for all eigenvalues of systems with $n = 5$ or 6 and $H = H_{J_{\text{max}}}$. This is partly because the number of states for each $I^{(n)}$ is not analytically known. The number of combinations for different clusters is also much larger than the cases with $n = 3$ and $n = 4$.

Based on these relations we suggest that the low-lying states of each $I^{(n)}$ of fermions (bosons) in a single-$j$ shell (with spin $l$) interacting by an attractive $H = H_{J_{\text{max}}}$ favor a cluster structure, where each cluster has a maximum (or close to maximum) angular momentum. The coupling between the constituent clusters (including pairs and spectators) are very weak and negligible, therefore we can obtain both their approximate wave functions and eigenvalues, which are simple summation of those of the clusters.

In Fig. 2, we showed the distribution of all non-zero eigenvalues for systems with $n$ ranging from 2 to 6. It is easy to notice that the eigenvalues are concentrated around some values for $n = 2$ to 5. This pattern becomes less striking for $n = 6$. 
5 Discussion and summary

In this paper, we first show that a system of three fermions in a single-$j$ shell in the presence of $H = H_J$ is solvable. We prove that there is at most one state with a non-zero eigenvalue for each $I^{(3)}$. We can analytically construct both the eigenvalues and corresponding wave functions. A similar remark applies to three bosons with spin $l$ in the presence of $H_J$. On the basis of the above results for $n = 3$ a series of new sum rules of six-$j$ symbols can be found.

We show that the eigenvalues of three fermions in a single-$j$ shell with $H = H_{J_{\text{max}}}$ are very close to 0 or $-1$ unless $I^{(3)} \sim I^{(3)}_{\text{max}} = 3j - 3$. This kind of situation is very similar to the case of $n = 4$, as studied in Ref. [5].

We also find that the “non-integer” eigenvalues of $I^{(3)} \sim I^{(3)}_{\text{max}}$ for $n = 3$ appear as “non-integer” eigenvalues for $n = 4$ when $I^{(4)}$ is around or larger than $J_{\text{max}}$. The overlaps between the wavefunction of these “non-integer” eigenvalues of $n = 4$ and that of $I^{(3)} \sim I^{(3)}_{\text{max}}$ state coupled to a single-$j$ particle is very close to 1. This finding allows us to construct approximately the states of $n = 4$ by using results of $n = 3$ as we have shown. We confirmed that this is also true for five and six fermions in a single-$j$ shell in the presence of $J_{\text{max}}$ pairing interaction. Bosons with spin $l$ exhibit a similar pattern. Similar regularity was found for $n = 5$ and 6, although we did not succeed in setting up a simple rule for all states.

The relations between $E_{I^{(2)}_{\text{max}}}, J_{\text{max}}(j)$; $E_{I^{(3)}_{\text{max}}}, J_{\text{max}}(j)$; $E_{I^{(4)}_{\text{max}}}, J_{\text{max}}(j)$; $\cdots$ indicate the following pattern: the attractive $J_{\text{max}}$ pairing interaction favors clusters (including pairs and spectators), where the angular momentum of each cluster is close to the maximum. One thus explains the “integer” eigenvalues and “non-integer” eigenvalues proposed in Ref. [5] by using a picture of the clusters for fermions in a single-$j$ shell or bosons with spin $l$.

As is well known, the existence of degeneracy indicates that the Hamiltonian has a certain symmetry. The degeneracy for the $J_{\text{max}}$ pairing interaction, however, is not exact. It would be interesting to explore the broken symmetry hidden in the $J_{\text{max}}$ pairing interaction discussed in this paper. It would be also interesting to discuss the modification of the $J_{\text{max}}$ pairing interaction in order to recover the exact degeneracy.

Acknowledgement: We wish to extend our special thanks to Prof. I. Talmi for his valuable comments concerning the sum rules of six-$j$ symbols. We also thank Dr. N. Yoshinaga for discussions in the early stage of this work, and Drs. I. Talmi and O. Scholten for their reading of this manuscript.
Table I  The lowest eigenvalues of the $I^{(4)}$ states in a single-$j$ ($j = 31/2$) shell with $I^{(4)}$ between 28 to $I^{(4)}_{\text{max}} = 56$. When $I^{(4)}$ is smaller than 48 there is no eigenvalue lower than $-2$. The eigenvalue of the $I^{(3)}_{\text{max}}$ state with three fermions in the same single-$j$ shell is $-\frac{267}{118} = -2.26271186440677966$. The column “(SM)” is obtained by a shell model diagonalization, and the column “$E_I$” is matrix element of $H_{J_{\text{max}}}$ for the state constructed by three-fermion with $I^{(3)} = I^{(3)}_{\text{max}}$ coupled to a spectator. The column “error” presents the difference between $E_I$ and $E_I$ (two effective digits). The column “overlap” is the overlap between the lowest eigenstates of $n = 4$ and the states obtained by coupling single fermion $a_j^\dagger$ to the $I^{(3)}_{\text{max}}$ state. Italic font is used for three cases for which the overlap is not close to 1. We note that the case of $I^{(4)} = 50 (52)$ can be approximated rather accurately ($10^{-3}$) as a three-particle cluster with $I^{(3)} = I^{(3)}_{\text{max}} - 8 (I^{(3)}_{\text{max}} - 6)$ coupled to a single-$j$ spectator.
| \( I \) | \( E_I \) (SM) | \( \mathcal{E}_I \) (coupled) | “error” | overlap |
|---|---|---|---|---|
| 28 | -2.26271186440689 | -2.262711864406782 | \( 1.1 \times 10^{-13} \) | 1.0000000000000000 |
| 29 | -2.26271186440682 | -2.262711864406777 | \( 4 \times 10^{-14} \) | 1.0000000000000000 |
| 30 | -2.26271186440678 | -2.262711864406780 | \( 1 \times 10^{-14} \) | 1.0000000000000000 |
| 31 | -2.26271186440669 | -2.262711864406782 | \( 0.9 \times 10^{-13} \) | 0.9999999999999999 |
| 32 | -2.26271186440692 | -2.2627118644066805 | \( 1.1 \times 10^{-13} \) | 0.9999999999999965 |
| 33 | -2.2627118644040700 | -2.262711864406981 | \( 2 \times 10^{-14} \) | 1.0000000000000000 |
| 34 | -2.26271186442899 | -2.262711864409884 | \( 1.9 \times 10^{-11} \) | 0.999999999963606 |
| 35 | -2.26271186442233 | -2.262711864422405 | \( 8 \times 10^{-14} \) | 0.9999999999999996 |
| 36 | -2.26271186573172 | -2.262711864593695 | \( 1.1 \times 10^{-9} \) | 0.9999999997833635 |
| 37 | -2.26271186512903 | -2.262711865128374 | \( 6.6 \times 10^{-13} \) | 0.9999999999999758 |
| 38 | -2.2627119191546116 | -2.262711871692375 | \( 4.3 \times 10^{-8} \) | 0.999999916591887 |
| 39 | -2.26271188689249 | -2.262711886864667 | \( 2.8 \times 10^{-11} \) | 0.9999999999888818 |
| 40 | -2.26271325181426 | -2.262712064607266 | \( 1.2 \times 10^{-6} \) | 0.999997726566892 |
| 41 | -2.26271236805292 | -2.262712367094411 | \( 9.6 \times 10^{-10} \) | 0.9999999598199 |
| 42 | -2.26274016611845 | -2.262715960012236 | \( 2.4 \times 10^{-5} \) | 0.999952666087478 |
| 43 | -2.2627203287460 | -2.26272086322401 | \( 2.7 \times 10^{-8} \) | 0.999999987392690 |
| 44 | -2.26317530567842 | -2.2627641261782 | \( 4.0 \times 10^{-7} \) | 0.999151747579904 |
| 45 | -2.26282037299297 | -2.262819747102017 | \( 6.2 \times 10^{-7} \) | 0.99999632523561 |
| 46 | -2.26963309159052 | -2.263514816015588 | \( 6.1 \times 10^{-3} \) | 0.982828211942919 |
| 47 | -2.26378385186917 | -2.263772302947436 | \( 1.2 \times 10^{-5} \) | 0.999992036003522 |
| 48 | -2.34719850307215 | -2.270625142453812 | \( 7.7 \times 10^{-2} \) | 0.780582505446094 |
| 49 | -2.27068252318197 | -2.270571840272616 | \( 1.1 \times 10^{-4} \) | 0.9999221753443 |
| 50 | -2.57872589150598 | -2.323429204525185 | \( 2.6 \times 10^{-1} \) | 0.70685989289674 |
| 51 | -2.30488200470359 | -2.304882004703592 | \( 0 \) | 1.00000000000000 |
| 52 | -2.89017281282010 | -2.592166600952603 | \( 3.0 \times 10^{-1} \) | 0.87317071305796 |
| 53 | -2.41926851025870 | -2.419268510258698 | \( 0 \) | 1.00000000000000 |
| 54 | -3.24511394047522 | -3.245113940475225 | \( 0 \) | 1.00000000000000 |
| 55 | -3.60693131132918 | -3.663693131132918 | \( 0 \) | 1.00000000000000 |
Appendix A New sum rules of six-\( j \) symbols

The solution of \( H_J \) for \( n = 3 \) gives new sum rules. The procedure to obtain these sum rules is straightforward. As is well known, the summation of all eigenvalues with a fixed \( I \) is equal to \( \frac{n(n-1)}{2} \) times the number of \( I \) states, where \( n \) is the particle number. For \( n = 3 \), the number of states can be empirically expressed in a compact formula [10].

In Ref. [5] we applied this idea and obtained that

\[
\sum_{J \text{ even}} (2J + 1) \left\{ \begin{array}{c} j \ j \ J \\ j \ j \ J \end{array} \right\} = \frac{3}{2} \left[ \frac{2j + 3}{6} \right] - \frac{2j + 1}{4} = \begin{cases} \frac{1}{2} & \text{if } 2j = 3k, \\ 0 & \text{if } 2j = 3k + 1, \\ -\frac{1}{2} & \text{if } 2j = 3k + 2, \end{cases} \tag{14}
\]

where \( j \) is a half integer, and \( \lfloor x \rfloor \) means to take the largest integer not exceeding \( x \).

we derive a similar sum rule using the \( I = 0 \) states of four bosons with spin \( l \):

\[
\sum_{J \text{ even}} (2J + 1) \left\{ \begin{array}{c} l \ l \ J \\ l \ l \ J \end{array} \right\} = \frac{3}{2} \left[ \frac{l}{3} \right] + 1 - \left( \frac{l}{2} \right) = \begin{cases} \frac{1}{2} & \text{if } l = 3k, \\ \frac{1}{2} & \text{if } l = 3k + 1, \\ 0 & \text{if } l = 3k + 2, \end{cases} \tag{15}
\]

Below we give other sum rules of six-\( j \) symbols. For a half integer \( j \),

\[
\sum_{J \text{ even}} 2(2J + 1) \left\{ \begin{array}{c} j \ I \ J \\ j \ j \ J \end{array} \right\} = \begin{cases} 3 \left[ \frac{2j+3}{6} \right] - I - \frac{1}{2} + 3\delta_l^{j} - \left[ \frac{3j+1-l}{2} \right] & \text{if } I \leq j; \\ 3 \left[ \frac{3j-3-I}{6} \right] + 3\delta_l^{j} & \text{if } I \geq j. \end{cases} \tag{16}
\]

where

\[
\delta_l^{j} = \begin{cases} 0 & \text{if } (3j - 3 - I) \text{ mod } 6 = 1 \\ 1 & \text{otherwise}. \end{cases}
\]

For integer \( l \), we obtain similar sum rules given as follows. For \( I \leq l \) (\( l \) is an integer),

\[
\sum_{J \text{ even}} 2(2J + 1) \left\{ \begin{array}{c} l \ I \ J \\ l \ l \ J \end{array} \right\} = \begin{cases} 3 \left[ \frac{l}{3} \right] - I + 1 + (-1)^{I+l} & \text{if } I \leq l; \\ 3 \left[ \frac{3l-I}{6} \right] + 3\delta_l^{l} - \left[ \frac{3l-I+2}{2} \right] & \text{if } I \geq l. \end{cases} \tag{17}
\]

where

\[
\delta_l^{l} = \begin{cases} 0 & \text{if } (3l - I) \text{ mod } 6 = 1 \\ 1 & \text{otherwise}. \end{cases}
\]

It is noted that the sum rules (14) and (15) are special cases of the sum rules (16) and (17).
Starting from Eq. (10.14) of Ref. [11] for \( J' = J'' \), \( J_1 = J_3 = j \), \( J_2 = J_4 = j' \), we multiply \((2J + 1)\) and sum over \( J' \). Using Eq. (10.13), we obtain

\[
\sum_j (2J + 1) \left\{ \begin{array}{c} j \\ j' \\ J \\ J \\ J' \\ \end{array} \right\} = 0 ;
\]
\[
\sum_j (2J + 1) \left\{ \begin{array}{c} l \\ l' \\ J \\ J \\ J' \end{array} \right\} = 1 .
\] (18)

Similarly, we obtain

\[
\sum_j (2J + 1) \left\{ \begin{array}{c} l \\ j \\ J \\ J \\ J' \end{array} \right\} = \begin{cases} -1 & \text{if } l < j , \\ 0 & \text{otherwise} . \end{cases}
\] (19)

and

\[
\sum_j 2(2J + 1) \left\{ \begin{array}{c} j \\ j \\ I \\ J \\ J' \end{array} \right\} = \begin{cases} 0 & \text{if } I \leq j , \\ (-)^{I-j} - 1 & \text{if } I \geq j . \end{cases}
\] (20)

Using Eqs. (20) and (16), we obtain

\[
\sum_{J=\text{odd}} 2(2J + 1) \left\{ \begin{array}{c} j \\ j \\ I \\ J \\ J' \end{array} \right\} = I + \frac{1}{2} - 3 \left[ \frac{2j + 3}{6} \right] = \begin{cases} -1 & \text{if } 2j = 3k , \\ 0 & \text{if } 2j = 3k + 1 , \\ 1 & \text{if } 2j = 3k + 2 . \end{cases}
\] (21)

and for \( I \geq j \),

\[
\sum_{J=\text{odd}} 2(2J + 1) \left\{ \begin{array}{c} j \\ j \\ I \\ J \\ J' \end{array} \right\} = (-)^{I-j} - 1 - 3 \left[ \frac{3j - 3 - I}{6} \right] - 3\delta_j + \left[ \frac{3j + 1 - I}{2} \right] \] (22)

Similar to Eq. (20), we obtain

\[
\sum_j 2(2J + 1) \left\{ \begin{array}{c} l \\ l \\ I \\ J \\ J' \end{array} \right\} = \begin{cases} 2(-)^{I+l} & \text{if } I \leq l , \\ 1 + (-)^{I+l} & \text{if } I \geq l . \end{cases}
\] (23)

Using Eqs. (23) and (17), we obtain that

\[
\sum_{J=\text{odd}} 2(2J + 1) \left\{ \begin{array}{c} l \\ l \\ I \\ J \\ J' \end{array} \right\} = \begin{cases} (-)^{I+l} + I - 3 \left[ \frac{I}{3} \right] - 1 & \text{if } I \leq l ; \\ (-)^{I+l} - 3 \left[ \frac{3l-I}{6} \right] - 3\delta_l - \left[ \frac{3l-I}{2} \right] + 2 & \text{if } I \geq l . \end{cases}
\] (24)
References

[1] J. P. Elliott, Proc. Roy. Soc. (London), Ser. A 245, 128 (1958); 245, 562 (1958).

[2] G. Racah, Phys. Rev. 63, 367(1943); B. H. Flowers, Proc. Roy. Soc. (London) A212, 248(1952); I Talmi, Nucl. Phys. A 172, 1(1971).

[3] A. Arima and F. Iachello, Ann. Phys. 99, 253 (1976); ibid. 111, 209(1978); ibid. 123, 468 (1979); for a review, see F. Iachello and A. Arima, the Interacting Boson Model (Cambridge University press, Cambridge, 1987).

[4] J. N. Ginocchio, Ann. Phys. 126, 234(1980); C. L. Wu, D. H. Feng, X. G. Chen, J. Q. Chen, M. W. Guidry, Phys. Rev. C36, 1157 (1987).

[5] Y. M. Zhao, A. Arima, J. N. Ginocchio, and N. Yoshinaga, Phys. Rev. C 68, 044320 (2003).

[6] I. Talmi, private communication.

[7] N. Yoshinaga, T. Mizusaki, A. Arima, and Y. D. Devi, Prog. Theor. Phys. 125 (suppl.), 65(1996).

[8] J. Q. Chen, B. Q. Chen, and A. Klein, Nucl. Phys. A554, 61 (1993).

[9] J. Q. Chen, Nucl. Phys. A562, 218 (1993).

[10] Y. M. Zhao and A. Arima, Phys. Rev. C 68, 044310 (2003); J. N. Ginocchio and W. C. Haxton, Symmetries in Science VI, Edited by B. Gruber and M. Ramek (Plenum Press, New York, 1993), p. 263.

[11] I. Talmi, Simple Models of Nuclear Shell Theory, Harwood (1993).
Fig. 1 Detailed distribution of all non-zero eigenvalues for $n = 4$. The inset in Fig. 1(b) is rescaled to distinguish a few exceptional cases which energies are not close to those of $n = 3$. (a), (b) and (c) corresponds to different range of $|E_{I(J)}^{(4)}|$. We see that the eigenvalues for $n = 4$ are “clustered” at those of $n = 3$ with few exceptions.

Fig. 2 Distribution of non-zero eigenvalues $|E_{I(J)}^{(n)}|$ for systems with $n$ ranging from 2 to 6 and $j = 19/2$. One sees that the non-zero eigenvalues are highly concentrated. The concentration of eigenvalues for the case of $n = 6$ is less striking. The distribution is plotted using the number of counts for each $|E_{I(J)}^{(n)}|$ (with the step length being 0.01) divided by the total number of non-zero eigenvalues. For $H = H_{J_{\text{max}}}$ with $j = 19/2$, the number of non-zero eigenvalues is 1, 17, 122, 472, 1224 (in comparison with the number of the shell model space: 10, 45, 177, 521, 1242) for $n = 2, 3, \cdots, 6$, respectively.
Eigenvalues of $n=4$

distribution of eigenvalues of $n=4$

eigenvalues of $n=3$

Count of different eigenvalues for $n=4$

Figure 1  Nov 20th/2003
distribution of eigenvalues for $H=H_{J_{\text{max}}}$

$n=2$

$n=3$

$n=4$

$n=5$

$n=6$