ON ALTERNATING QUASIPOSITIVE LINKS

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Abstract. We prove that if a quasipositive link can be represented by an alternating diagram satisfying the condition that no pair of Seifert circles is connected by a single crossing, then the diagram is positive and the link is strongly quasipositive.

1. Introduction

An $n$-braid is called quasipositive if it is a product of conjugates of the standard generators $\sigma_1, \ldots, \sigma_{n-1}$ of the braid group $B_n$. A braid is called strongly quasipositive if it is a product of braids of the form $\tau_{k,j}\sigma_j\tau_{k,j}^{-1}$ for $j \leq k$ where $\tau_{k,j} = \sigma_k\sigma_{k-1}\ldots\sigma_j$. All links in this paper are assumed to be oriented links in the 3-sphere $S^3$. A link is called (strongly) quasipositive if it is the braid closure of a (strongly) quasipositive braid (see [11, 12]).

S. Baader [1, p. 268, Question (4)] asked: Do quasipositive alternating links have positive diagrams? Note that positive diagrams represent strongly quasipositive links (see [8], [13]) and alternating strongly quasipositive links have positive alternating diagrams by [2, Cor. 7.3]. Notice also that positive alternating diagrams are special (a diagram is called special [7] if its Seifert circles bound disjoint disks).

In this note we give an affirmative answer for a rather large class of alternating links: those which have an alternating diagram whose number of Seifert circles is equal to the braid index of the link. We call such diagrams Diao–Hetyei–Liu or DHL diagrams (and the corresponding links DHL links) because these authors gave in [4] the following very nice and simple characterization for them.

Theorem 1. ([4, Thm. 1.1]) An alternating diagram is DHL if and only if there is no pair of Seifert circles connected by a single crossing.

Our main result is the following.

Theorem 2. Let $D$ be a DHL diagram of a quasipositive link. Then $D$ is positive.

The proof is an easy combination of results from [5], [6], [7], [14], and [15] (see Section 2). Theorems 1 and 2 allow to produce a lot of examples of non-quasipositive links without any computations.

Since any positive diagram represents a strongly quasipositive link (see [8], [13]), we obtain:

Corollary 3. Let $L$ be a DHL link. Then the following conditions are equivalent:

(i) $L$ is quasipositive;
(ii) $L$ is strongly quasipositive;
(iii) $L$ has a positive alternating diagram.
In Section 3 we generalize Theorem 2 to all alternating links whose braid index is computed in [3]; see Theorem 6 and Remark 7.

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2. Proof of the main theorem

Let $D$ be a connected link diagram. The Seifert graph of $D$ is the graph $G_D$ whose vertices correspond to Seifert circles and the edges correspond to the crossings. The sign of an edge is the sign of the corresponding crossing. A diagram $D$ is called reduced if $G_D$ does not have any edge whose removal disconnects $G_D$. Let $d(D)$ denote the sum of signs of all edges of a spanning tree of $G_D$, and let $w(D)$ be the writhe of $D$, i.e., the sum of signs of all crossings.

For a link $L$, let $\sigma(L)$ and $n(L)$ be its signature and nullity (the latter is the nullity of a symmetrized Seifert form on a connected Seifert surface).

Theorem 4. (Traczyk [14]) Let $D$ be a connected reduced alternating diagram of a link $L$. Then

$$\sigma(L) = d(D) - w(D)$$

and

$$n(L) = 0.$$  

This formula for $\sigma(L)$ is given in [14, Thm. 2(1)] (the factor $1/2$ is erroneous there). The fact that $n(L) = 0$ (equivalently, $\det(L) \neq 0$) is proven [7, Lem. 5.1] and in the appendix to [14]. Otherwise it can be easily derived from [14, Thm. 1].

Proof of Theorem 2. Let $D$ be a DHL diagram of a quasipositive link $L$. Then each connected component of $D$ is evidently a DHL diagram and it represents a quasipositive link by [10]. So, it is enough to consider the case when $D$ is connected.

Let $n$ be the braid index of $L$. By definition of DHL diagrams, $D$ has $n$ Seifert circles. Hence, by [15, Thm. 1] (see the discussion of this theorem in the introduction to [15]), $L$ can be represented by an $n$-braid $\beta_1$ with

$$w(\beta_1) = w(D).$$  \hspace{1cm} (1)

By [6, Thm. 1.2] $L$ can be represented by a quasipositive $n$-braid $\beta_2$. Then Murasugi–Tristram inequality [7] for quasipositive braids can be reformulated as follows (see [9, Cor. 3.2])

$$1 + n(L) \geq |\sigma(L)| + n - w(\beta_2).$$  \hspace{1cm} (2)

By Dynnikov–Prasolov Theorem [5] (Generalized Jones Conjecture) we have

$$w(\beta_1) = w(\beta_2)$$  \hspace{1cm} (3)

By combining (1) – (3) with Theorem 4 (note that any DHL diagram is reduced), we obtain $|d(D) - w(D)| \leq 1 - n + w(D)$ whence $w(D) - d(D) \leq 1 - n + w(D)$, i.e., $d(D) \geq n - 1$. Recall that $d(D)$ is the sum of signs of all edges of a spanning tree of $G_D$. Any spanning tree of $G_D$ has $n - 1$ edges, hence all its edges are positive. Since each edge of $G_D$ belongs to some spanning tree, we conclude that all crossings of $D$ are positive. Theorem 2 is proven.  \hspace{1cm} $\square$
3. A generalization of the main theorem

Let $D$ be an alternating diagram of a link $L$. Let $b = b(L)$ be the braid index of $L$ and $s = s(D)$ be the number of Seifert circles of $D$. Define $d^\pm = d^\pm(D)$ as the number of edges of this sign in a spanning tree of $G_D$, thus $d = d(D) = d^+ - d^-$.  

Let $\beta$ be a braid with $b$ strands realizing $L$. Due to Dynnikov – Prasolov Theorem [5], $w(\beta)$ does not depend on the choice of $\beta$, which allows us to define the numbers $r^\pm = r^\pm(D)$ from the system of equations

$$r^+ + r^- = s - b, \quad r^+ - r^- = w(D) - w(\beta).$$

**Remark 5.** The definition of the numbers $r^\pm$ in [3] is not quite clear but in all cases when they are computed in [3], they satisfy our definition; cf. [3, Rem. 3.1–3.3].

If $D$ is a DHL diagram, then $r^+ = r^- = 0$ (recall that in this case $w(D) = w(\beta)$ by [15, Thm. 1]), thus the following statement is a generalization of Theorem 2.

**Theorem 6.** Let $D$ be a reduced alternating diagram of a quasipositive link $L$, and

$$2r^-(D) \leq d^-(D). \quad (4)$$

Then $D$ is positive (and hence $L$ is strongly quasipositive by [8, 13]).

**Proof.** Since the arguments are almost the same as for Theorem 2, we just write down the final computation. So, we have $w(D) - d \leq |\sigma| \leq 1 + b + w(\beta)$, hence

$$d + 1 \geq w(D) - w(\beta) + b = (r^+ - r^-) + (s - (r^+ + r^-)) = s - 2r^- \geq s - d^-$$

whence $d^+ \geq s - 1$ and the result follows. \qed

**Remark 7.** In all cases when the braid index of a reduced alternating diagram is computed in [3], the inequality (4) holds, in particular it holds for minimal diagrams of two-bridge links and of alternating Montesinos links.

**Question 8.** Does (4) hold for any reduced alternating diagram?

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