Abstract. We give Erdmann-Nakano type theorem for the finite quiver Hecke algebras \( R_{\Lambda_0}(\beta) \) of affine type \( A^{(1)}_{\ell} \). Note that each finite quiver Hecke algebra lies in one parameter family, and the original Erdmann-Nakano theorem studied the finite quiver Hecke algebra at a special parameter value. We study the general case in our paper. Our result shows in particular that their representation type does not depend on the parameter. Moreover, when the parameter value is nonzero, we show that finite quiver Hecke algebras of tame representation type are biserial algebras.

Introduction

This paper is the third of our series of papers on the representation type of finite quiver Hecke algebras. The affine type we treat in this paper is the affine type \( A \). Thus, it includes the original case of block algebras of the Hecke algebras associated with the symmetric group. The latter classical Hecke algebras appeared in many branches of mathematics: knot theory, mathematical physics, number theory, geometric representation theory and so on, but recent progress reveals new features of the algebras as we explain in the next paragraph.

Let \( q \in k^{\times} \) and \( e = \min\{k \mid 1 + q + \cdots + q^{k-1} = 0\} \). Block algebras of the Hecke algebra \( \mathcal{H}_n(q) \) associated with the symmetric group of degree \( n \) are labelled by \( e \)-cores \( \kappa \), and we denote them by \( B_\kappa(n,q) \). As is well-known, the Lascoux-Leclerc-Thibon conjecture on the decomposition numbers of the Hecke algebras inspired the first author and he introduced categorification scheme for integrable highest weight modules \( V(\Lambda) \) over the Kac-Moody Lie algebra of affine type \( A \). When \( \Lambda = \Lambda_0 \), the basic module \( V(\Lambda_0) \) is categorified by the \( B_\kappa(n,q) \)'s. It was vastly generalized and refined after the introduction of Khovanov-Lauda-Rouquier algebras, which we may call affine quiver Hecke algebras. Cyclotomic quotients of affine quiver Hecke algebras were introduced by Khovanov and Lauda, and the integrable module \( V(\Lambda) \) is categorified by the cyclotomic quiver Hecke algebras. The categorification

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scheme itself was strengthened by Rouquier from our weak form, by generalizing the Chuang-
Rouquier \( sl_2 \)-categorification. Indeed, these results together with Kang and Kashiwara’s
result on adjoint pairs of induction and restriction functors have been essentially used in our
series of papers. When \( \Lambda = \Lambda_0 \), we call cyclotomic quiver Hecke algebras finite quiver Hecke
algebras. Our interest lies in the study of finite quiver Hecke algebras. New features arising
from the above mentioned development are two fold:

(i) each \( B_\kappa(n,q) \) lies in a one-parameter family of finite quiver Hecke algebras,

(ii) the finite quiver Hecke algebras are graded algebras, i.e. they have the KLR grading.

The second point was already taken up by Brundan and Kleshchev [7]. They refined the
first author’s categorification theorem into graded version. See also Mathas’ proof in [18].

In this paper, we take up the first point. We consider finite quiver Hecke algebras \( R^{\Lambda_0}(\beta) \)
with parameters \( \lambda \in k \). They fall into various isomorphism classes. Nevertheless, we may
show that the representation type of the finite quiver Hecke algebra \( R^{\Lambda_0}(\beta) \) does not depend
on the parameter \( \lambda \) (Theorem 5.9). This is our first result. The proof follows our strategy
in [3] and [4], but we need extra work for the case \( \ell = 1 \). This generalized Erdmann-Nakano
theorem tells that \( R^{\Lambda_0}(\beta) \) has tame representation type only when \( \ell = 1 \). Thus, we give a
detailed study of tame finite quiver Hecke algebras. Recall that tame block algebras \( B_\kappa(n, -1) \)
are Morita equivalent to either \( B_{(0)}(4, -1) \) or \( B_{(1)}(5, -1) \) by Scopes’ equivalence. We may
only show that tame finite quiver Hecke algebras for \( \lambda \neq 0 \) are biserial algebras. However,
if it is special biserial then it is Morita equivalent to either \( B_{(0)}(4, -1) \) or \( B_{(1)}(5, -1) \). This
is our second result (Theorem 6.6). We can also describe possible form of tame finite quiver
Hecke algebras by concrete quiver presentation when \( \lambda = 0 \). For the proof of the second
result, we first classify two-point symmetric special biserial algebras (Theorem 6.1). It is a
bit surprise that this classification also seems to be new.

The paper is organized as follows. §1 and §2 are for preliminaries. We construct irreducible
\( R^{\Lambda_0}(\delta - \alpha_i) \)-modules, for \( 1 \leq i \leq \ell \), and irreducible \( R^{\Lambda_0}(\delta) \)-modules in §3. The proof in §3
follows the same strategy as [3] and [4]. Then, we need extra work for \( R^{\lambda_0}(2\delta) \) in \( \ell = 1 \) case,
which is the topic of §4. In §5, we prove the generalized Erdmann-Nakano theorem, the first
main result. In §6, we classify two-point symmetric special biserial algebras and then prove
the second main result. The appendix is for explaining some interesting results from [19].
Although main results in [19] are incorrect, we may use his setup to prove, at the end of
§6, that tame finite quiver Hecke algebras in affine type A, for parameter values \( \lambda \neq 0 \), are
biserial algebras.

1. Preliminaries

In this section, we briefly recall necessary materials.
1.1. Cartan datum. Let $I = \{0, 1, \ldots, \ell\}$ be an index set, and let $A$ be the affine Cartan matrix of type $A^{(1)}_\ell$ ($\ell \geq 2$)

$$A = (a_{ij})_{i,j \in I} = \begin{pmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & -1 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
-1 & 0 & 0 & \cdots & 0 & -1 & 2
\end{pmatrix}.$$  

When $\ell = 1$, the affine Cartan matrix is

$$A = (a_{ij})_{i,j \in I} = \begin{pmatrix}
2 & -2 \\
-2 & 2
\end{pmatrix}.$$  

We have an affine Cartan datum $(A, P, \Pi, \Pi^\vee)$, where

1. $A$ is the affine Cartan matrix given above.
2. $P$ is the weight lattice, a free abelian group of rank $\ell + 1$.
3. $\Pi = \{\alpha_i \mid i \in I\} \subset P$, the set of simple roots.
4. $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee := \text{Hom}(P, \mathbb{Z})$.

They satisfy the following properties:

(a) $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$.
(b) $\Pi$ and $\Pi^\vee$ are linearly independent sets.

We fix a scaling element $d$ which obeys the condition $\langle d, \alpha_i \rangle = \delta_{i0}$, and assume that $\Pi^\vee$ and $d$ form a $\mathbb{Z}$-basis of $P^\vee$. Then, the fundamental weight $\Lambda_0$ is defined by

$$\langle h_i, \Lambda_0 \rangle = \delta_{i0}, \quad \langle d, \Lambda_0 \rangle = 0.$$  

The free abelian group $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is the root lattice, and we denote $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$. If $\beta \in Q^+$, we denote the sum of coefficients by $|\beta|$. The Weyl group $W$ associated with $A$ is the affine symmetric group, which is generated by $\{r_i\}_{i \in I}$, where $r_i\Lambda = \Lambda - \langle h_i, \Lambda \rangle \alpha_i$, for $\Lambda \in P$. The null root of type $A^{(1)}_\ell$ is

$$\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_\ell.$$  

Note that $\langle h_i, \delta \rangle = 0$ and $w\delta = \delta$, for $i \in I$ and $w \in W$.  


1.2. Young diagrams. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l > 0)$ be a Young diagram of depth $l$. We denote the depth $l$ by $l(\lambda)$. If $\lambda$ has $n$ nodes, we write $\lambda \vdash n$. For each $\lambda$, $\text{ST}(\lambda)$ is the set of standard tableaux of shape $\lambda$. We consider the residue pattern which repeats

\[
0 \ 1 \ 2 \ \ldots \ \ell
\]

in the first row, and we shift the residue pattern to the right by one in the next row. Namely, the residue of the $(i,j)$-node of $\lambda$ is defined to be

$$\text{res}(i,j) \equiv j - i \mod (\ell + 1).$$

For example, if $\ell = 3$ and $\lambda = (10, 7, 3)$, the residues are given as follows:

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 0 & 1 & 2 \\
3 & 0 & 1 & 2 & 3 & 0 & 1 \\
2 & 3 & 0
\end{array}
\]

Then, for $T \in \text{ST}(\lambda)$, we define the residue sequence of $T$ by

$$\text{res}(T) = (\text{res}_1(T), \text{res}_2(T), \ldots, \text{res}_n(T)) \in \mathbb{I}^n,$$

where $\text{res}_k(T)$ is the residue of the box filled with $k$ in $T$, for $1 \leq k \leq n$.

1.3. The combinatorial Fock space. Let $\mathcal{F}$ be the complex vector space whose basis is given by $\{ |\lambda\rangle \mid \lambda : \text{Young diagrams} \}$. Then $\mathcal{F}$ is a $\mathfrak{g}$-module, where $\mathfrak{g}$ is the Kac-Moody algebra associated with $\mathbf{A}$. To describe the action, we use the following notation:

- if we may remove a box of residue $i$ from $\lambda$ and obtain a new Young diagram, then we write $\lambda \searrow \square$ for the resulting diagram,
- if we may add a box of residue $i$ to $\lambda$ and obtain a new Young diagram, then we write $\lambda \nearrow \square$ for the resulting diagram.

Then, the action of the Chevalley generators $f_i$ and $e_i$, for $i \in I$, is given as follows.

\[
e_i |\lambda\rangle = \sum_{\mu = \lambda \searrow \square} |\mu\rangle, \quad f_i |\lambda\rangle = \sum_{\mu = \lambda \nearrow \square} |\mu\rangle,
\]
2. Quiver Hecke algebras

2.1. Quiver Hecke algebras. Let $k$ be an algebraically closed field, and let $(A, P, \Pi, \Pi')$ be a Cartan datum. We assume that $A$ is symmetrizable. We denote the symmetrized bilinear form on $P$ by $(\ | )$.

We take polynomials $Q_{i,j}(u, v) \in k[u, v]$, for $i, j \in I$, of the form

$$Q_{i,j}(u, v) = \begin{cases} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \alpha_{i1}^{p+q} (\alpha_{ij} + \alpha_{ji})^{p+q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

where $t_{i,j,p,q} \in k$ are such that $t_{i,j,a_{ij},0} \neq 0$ and $Q_{i,j}(u, v) = Q_{j,i}(v, u)$. The symmetric group $\mathfrak{S}_n = \langle s_k \mid k = 1, \ldots, n-1 \rangle$ acts on $I^n$ by place permutations.

Definition 2.1. Let $\Lambda \in P^+$. The cyclotomic quiver Hecke algebra $R^\Lambda(n)$ associated with $(Q_{i,j}(u, v))_{i,j \in I}$ is the $\mathbb{Z}$-graded $k$-algebra defined by generators $e(\nu)$, for $\nu = (\nu_1, \ldots, \nu_n) \in I^n$, $x_1, \ldots, x_n$, $\psi_1, \ldots, \psi_{n-1}$ and the following relations.

$$e(\nu) e(\nu') = \delta_{\nu, \nu'} e(\nu), \quad \sum_{\nu \in I^n} e(\nu) = 1,$$

$$x_k e(\nu) = e(\nu) x_k, \quad x_k x_l = x_l x_k,$$

$$\psi_l e(\nu) = e(s_l(\nu)) \psi_l, \quad \psi_k \psi_l = \psi_l \psi_k \text{ if } |k-l| > 1,$$

$$\psi_k^2 e(\nu) = Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) e(\nu),$$

$$(\psi_k x_l - x_{s_k(l)} \psi_k) e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k \text{ and } \nu_k = \nu_{k+1}, \\ e(\nu) & \text{if } l = k+1 \text{ and } \nu_k = \nu_{k+1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\psi_{k+1} \psi_k \psi_{k+1} - \psi_k \psi_{k+1} \psi_k) e(\nu) = \begin{cases} Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_k, \nu_{k+1}}(x_{k+2}, x_{k+1}) e(\nu) & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise}, \end{cases}$$

$$x_1^{(b_1, \Lambda)} e(\nu) = 0.$$

The $\mathbb{Z}$-grading on $R^\Lambda(n)$ is given as follows:

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k} | \alpha_{\nu_k}), \quad \deg(\psi_l e(\nu)) = -(\alpha_{\nu_l} | \alpha_{\nu_{l+1}}).$$

Lemma 2.2 (see [21] p.25, for example) below shows that each cyclotomic quiver Hecke algebra for the affine type $A_t^{(1)}$ lies in one parameter family. Namely, we may assume without
loss of generality that
\[ Q_{0,1}(u, v) = u^2 + \lambda uv + v^2, \]
for the affine type \( A_1^{(1)} \), and
\[
\begin{aligned}
Q_{i,i+1}(u, v) &= u + v \quad (0 \leq i \leq \ell - 1), \\
Q_{\ell,0}(u, v) &= u + \lambda v, \\
Q_{i,j}(u, v) &= 1 \quad (j \not\equiv i \pm 1 \mod (\ell + 1)),
\end{aligned}
\]
for the affine type \( A_\ell^{(1)} \) with \( \ell \geq 2 \). Here \( \lambda \in k \) is a parameter. In the rest of the paper, we assume that \( Q_{i,j}(u, v) \) are of this form.

**Lemma 2.2.** Let \( C = (c_{ij})_{i,j \in I} \in \text{Mat}(I, I, k) \) be a symmetric matrix such that \( c_{ij} \neq 0 \), for all \( i \) and all \( j \). If we define \( Q'_{i,j}(u, v) = c_{ij}^2 Q_{i,j}(c_{ii} u, c_{jj} v) \), then the cyclotomic quiver Hecke algebra associated with \( \{Q'_{i,j}(u, v)\}_{i,j \in I} \) is isomorphic to the cyclotomic quiver Hecke algebra associated with \( \{Q_{i,j}(u, v)\}_{i,j \in I} \) by the algebra isomorphism
\[
e(\nu) \mapsto e(\nu), \quad x_k e(\nu) \mapsto c_{ij}^{-1} x_k e(\nu), \quad \psi_k e(\nu) \mapsto c_{ij} \psi_k e(\nu).
\]

**Remark 2.3.** We studied affine types \( A_2^{(2)} \) and \( D_\ell^{(2)} \) in [3] and [4]. Lemma 2.2 shows that \( R^\Lambda_0(n) \) does not depend on the polynomials \( \{Q_{i,j}(u, v)\}_{i,j \in I} \) in those types. This is the reason why the choice of the polynomials did not matter in [3] and [4].

For \( \beta \in \mathbb{Q}^+ \) with \( |\beta| = n \), we define the central idempotent \( e(\beta) \) of \( R^\Lambda(n) \) by
\[
e(\beta) = \sum_{\nu \in I^\beta} e(\nu), \quad \text{where} \quad I^\beta = \left\{ \nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \sum_{k=1}^n \alpha_{\nu_k} = \beta \right\}
\]
Then we denote \( R^\Lambda(\beta) = R^\Lambda(n) e(\beta) \). We will be interested in the case when \( \Lambda = \Lambda_0 \). We call \( R^\Lambda_0(\beta) \) finite quiver Hecke algebras of type \( A^{(1)}_\ell \). The finite quiver Hecke algebras categorify the highest weight \( g \)-module \( V(\Lambda_0) \). Thus, Chevalley generators \( E_i \) and \( F_i \) are categorified to exact functors, and irreducible modules over finite quiver Hecke algebras of various rank are labelled by the Kashiwara crystal \( B(\Lambda_0) \). We use standard notations (\( \text{wt}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i \)) for the crystal structure. As this is the third of our series of papers, we assume that the reader is familiar with the strategy used in our series. The following is a consequence of derived equivalence explained in [3].

**Proposition 2.4 ([3 Cor.4.8]).** For \( w \in W \) and \( k \in \mathbb{Z}_{\geq 0} \), \( R^\Lambda(k\delta) \) and \( R^\Lambda(\Lambda - w\Lambda + k\delta) \) have the same number of irreducible modules and the same representation type.
2.2. Dimension formula for $R^{\Lambda_0}(\beta)$. For $\lambda \vdash n$ and $\nu \in I^n$, define a non-negative integer $K(\lambda, \nu)$ by

$$K(\lambda, \nu) = |\{ T \in \text{ST}(\lambda) \mid \nu = \text{res}(T) \}|.$$ 

Then we may give a formula for $\dim R^{\Lambda_0}(\beta)$ in terms of $K(\lambda, \nu)$. The proof is entirely similar to the cases $A^{(2)}_{2\ell}$ and $D^{(2)}_{\ell+1}$ and we omit the proof. Note that $\text{wt}(\lambda) = \Lambda_0 - \sum_{(i,j) \in \lambda} \alpha_{i-j}$.

**Theorem 2.5.** For $\beta \in \mathbb{Q}^+$ with $|\beta| = n$ and $\nu, \nu' \in I^\beta$, we have

$$\dim e(\nu') R^{\Lambda_0}(n) e(\nu) = \sum_{\lambda \vdash n} K(\lambda, \nu') K(\lambda, \nu),$$

$$\dim R^{\Lambda_0}(\beta) = \sum_{\lambda \vdash n, \text{wt}(\lambda) = \Lambda_0 - \beta} |\text{ST}(\lambda)|^2,$$

$$\dim R^{\Lambda_0}(n) = \sum_{\lambda \vdash n} |\text{ST}(\lambda)|^2 = n!.$$

3. Irreducible representations of $R^{\Lambda_0}(\delta)$

By explicit computation, we have

$$r_{\ell-1} r_{\ell-2} \cdots r_1 r_0 \Lambda_0 = \Lambda_0 - \delta + \alpha_\ell,$$

$$r_k r_{k-1} (\Lambda_0 - \delta + \alpha_k) = \Lambda_0 - \delta + \alpha_{k-1}, \text{ for } 2 \leq k \leq \ell.$$

Hence $\Lambda_0 - \delta + \alpha_i \in \mathcal{W} \Lambda_0$, which implies that $R^{\Lambda_0}(\delta - \alpha_i)$, for $1 \leq i \leq \ell$, are simple algebras. In particular, $R^{\Lambda_0}(\delta - \alpha_i)$ has a unique irreducible module if $i \neq 0$.

3.1. Representations of $R^{\Lambda_0}(\delta - \alpha_i)$. In this section, we give explicit description of the irreducible $R^{\Lambda_0}(\delta - \alpha_i)$-module. We will use the result to determine the structure of $R^{\Lambda_0}(\delta)$.

We consider a hook partition $\lambda(i) = (i, 1^{\ell-i}) \vdash \ell$ of weight $\Lambda_0 - \delta + \alpha_i$, for $1 \leq i \leq \ell$. We shall define $R^{\Lambda_0}(\delta - \alpha_i)$-modules $\mathcal{L}_i$, for $1 \leq i \leq \ell$. Let

$$\mathcal{L}_i = \bigoplus_{T \in \text{ST}(\lambda(i))} kT.$$

**Lemma 3.1.** We may define an $R^{\Lambda_0}(\delta - \alpha_i)$-module structure on $\mathcal{L}_i$, for $1 \leq i \leq \ell$, by $x_k = 0$ and

$$e(\nu)T = \begin{cases} T & \text{if } \nu = \text{res}(T), \\ 0 & \text{otherwise}, \end{cases}$$

$$\psi_k T = \begin{cases} s_k T & \text{if } s_k T \text{ is standard}, \\ 0 & \text{otherwise.} \end{cases}$$

(3.1.1)
Proof. To check the defining relations on $T$, we may assume that
\[ \nu = (\nu_1, \nu_2, \ldots, \nu_l) = \text{res}(T). \]

The relation for \((\psi_k x_l - x_{s_k(l)} \psi_k) e(\nu)\) is clear because \(\nu_k = \nu_{k+1}\) does not occur. Next observe that one of the following holds.

(i) \(a_{\nu_k \nu_{k+1}} = 0\) and \(s_k T\) is standard.

(ii) \(s_k T\) is not standard and \(\text{deg} \ Q_{\nu_k \nu_{k+1}}(u, v) > 0\).

Further, \(x_k = 0, x_{k+1} = 0\) imply \(Q_{\nu_k \nu_{k+1}}(x_k, x_{k+1}) = 0\) in the latter case. Hence the relation for \(\psi^2 \ e(\nu)\) holds. Finally, we prove \(\psi_{k+1} \psi_k \psi_{k+1} T = \psi_k \psi_{k+1} \psi_k T\), because \(\nu_k = \nu_{k+2}\) does not occur. But it follows from the fact that \(s_{k+1} T, s_k s_{k+1} T, s_{k+1} s_k s_{k+1} T\) are all standard if and only if \(s_k T, s_{k+1} s_k T, s_{k+1} s_{k+1} s_k T\) are all standard. It is straightforward to check other relations. \(\square\)

Lemma 3.2. The \(R^{\Lambda_0} (\delta - \alpha_i)\)-module \(L_i\), for \(1 \leq i \leq \ell\), is irreducible of dimension \((\ell-1)\).

Proof. Note that \(\dim e(\nu) L_i \leq 1\). Then the standard argument shows that \(L_i\) is irreducible. Its dimension is \(|S T(\lambda(i))| = (\ell-1)!\). \(\square\)

3.2. Representations of \(R^{\Lambda_0} (\delta)\). In this section, we construct irreducible \(R^{\Lambda_0} (\delta)\)-modules by extending the modules \(L_i\) from Section 3.1.

Lemma 3.3. Let \(h = \ell + 1\). By declaring that \(x_h\) and \(\psi_{h-1}\) act as \(0\), and \(e(\nu)\), for \(\nu \in I^\delta\), as
\[
e(\nu) T = \begin{cases} T & \text{if } \nu = \text{res}(T) \ast i, \\ 0 & \text{otherwise}, \end{cases}
\]
where \(\text{res}(T) \ast i\) is the concatenation of \(\text{res}(T)\) and \((i)\), the irreducible \(R^{\Lambda_0} (\delta - \alpha_i)\)-module \(L_i\) extends to an irreducible \(R^{\Lambda_0} (\delta)\)-module, for \(1 \leq i \leq \ell\).

Proof. We may assume that \(\nu = \text{res}(T) \ast i\). As \((\nu_{h-1}, \nu_h) = (i \pm 1, i)\), \(\nu_{h-1} = \nu_h\) does not occur. Thus, the relation for \((\psi_{h-1} x_l - x_{s_k(l)} \psi_{h-1}) e(\nu)\) holds. Since \(\text{deg} Q_{\nu_{h-1}, \nu_h}(u, v) > 0\), it also follows \(Q_{\nu_{h-1}, \nu_h}(x_{h-1}, x_h) = 0\). Hence, we have also proved the relation for \(\psi^2 \ e(\nu)\). Finally, \(\nu_{h-2} \neq i = \nu_h\) if \(\ell \geq 2\) implies the relation for \((\psi_{h-1} \psi_{h-2} \psi_{h-1} - \psi_{h-2} \psi_{h-1} \psi_{h-2}) e(\nu)\). It is easy to check the remaining defining relations. \(\square\)

Definition 3.4. We denote the irreducible \(R^{\Lambda_0} (\delta)\)-module defined in Lemma 3.3 by \(S_i\), for \(i = 1, 2, \ldots, \ell\).

By construction, \(E_j (S_i) = \delta_{ij}\), for \(1 \leq i, j \leq \ell\). Hence \(S_1 \ldots S_\ell\) are pairwise non-isomorphic \(R^{\Lambda_0} (\delta)\)-modules. But we know from the categorication theorem that the number of irreducible \(R^{\Lambda_0} (\delta)\)-modules is \(\ell\). Hence we have the following lemma.
Lemma 3.5. The modules $S_i$ ($i = 1, 2, \ldots, \ell$) form a complete list of irreducible $\mathcal{R}^{\Lambda_0}(\delta)$-modules.

We show that $\mathcal{R}^{\Lambda_0}(\delta) \simeq k[x]/(x^2)$ if $\ell = 1$. As the Young diagrams (2) and (1,1) are all for contributing to dim $\mathcal{R}^{\Lambda_0}(\delta)$ by Theorem 2.5, we have dim $\mathcal{R}^{\Lambda_0}(\delta) = 2$. Then, the map $\mathcal{R}^{\Lambda_0}(\delta) \rightarrow k[x]/(x^2)$ given by

$$e(\nu) \mapsto \begin{cases} 1 & \text{if } \nu = (0,1), \\ 0 & \text{otherwise,} \end{cases} \quad x_k \mapsto \begin{cases} 0 & \text{if } k = 1, \\ x & \text{if } k = 2, \end{cases} \quad \psi_1 \mapsto 0,$$

defines a surjective algebra homomorphism, which is an isomorphism by dim $\mathcal{R}^{\Lambda_0}(\delta) = 2$.

4. Representations of $\mathcal{R}^{\Lambda_0}(\beta)$ when $\ell = 1$

4.1. Representations of $\mathcal{R}^{\Lambda_0}(2\delta - \alpha_i)$ for $\ell = 1$. For the case $\ell = 1$, we have chosen the parameter $Q_{0,1}(u,v) = u^2 + \lambda uv + v^2$ in Section 2.1. Note that $Q_{i,j}(u,v) - Q_{i,j}(w,v) = u - w$ for $i \neq j \in I$.

As the Young diagram (2,1) is a unique one to contribute to dim $\mathcal{R}^{\Lambda_0}(2\delta - \alpha_0)$ by Theorem 2.5, we have

$$\dim \mathcal{R}^{\Lambda_0}(2\delta - \alpha_0) = 4.$$  

Proposition 2.4 and $r_1r_0(\Lambda_0) = \Lambda_0 - 2\delta + \alpha_0$ imply $\mathcal{R}^{\Lambda_0}(2\delta - \alpha_0) \simeq \text{Mat}(2, k)$. More explicitly, it is easy to check that the correspondence

$$e(\nu) \mapsto \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } \nu = (0,1,1), \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{otherwise,} \end{cases} \quad x_k \mapsto \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } k = 1, \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{if } k = 2, \\ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} & \text{if } k = 3, \end{cases} \quad \psi_l \mapsto \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } l = 1, \\ \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} & \text{if } l = 2, \end{cases}$$

defines an irreducible representation of $\mathcal{R}^{\Lambda_0}(2\delta - \alpha_0)$ and dimension counting shows that it gives an isomorphism between $\mathcal{R}^{\Lambda_0}(2\delta - \alpha_0)$ and $\text{Mat}(2, k)$. Let

$$\mathcal{M}_0 := \text{Span}_k \{v_1 := (1, 0), v_2 := (\frac{1}{0})\}$$

be the corresponding irreducible $\mathcal{R}^{\Lambda_0}(2\delta - \alpha_0)$-module.

Similarly, the Young diagrams (3) and (1,1,1) are all for contributing to dim $\mathcal{R}^{\Lambda_0}(2\delta - \alpha_1)$ and we have

$$\dim \mathcal{R}^{\Lambda_0}(2\delta - \alpha_1) = 2.$$
It is straightforward to check that
\[
e(\nu) \mapsto \begin{cases} 
(1, 0, 0) & \text{if } \nu = (0, 1, 0), \\
(0, 1, 0) & \text{otherwise},
\end{cases}
\]
\[
x_k \mapsto \begin{cases} 
(0, 0, 0) & \text{if } k = 1, \\
(0, 0, 0) & \text{if } k = 2, \\
(0, -\lambda) & \text{if } k = 3,
\end{cases}
\]

\[
\psi_1, \psi_2 \mapsto (0, 0, 0),
\]

is a well-defined representation of \(R^\Lambda(2\delta - \alpha_1)\). Hence, we have an algebra isomorphism \(R^\Lambda(2\delta - \alpha_1) \cong k[x]/(x^2)\). Let

\[
\hat{M}_1 := \text{Span}_k\{w_1 := (0, 0), w_2 := (0, 1)\}
\]

be the corresponding \(R^\Lambda(2\delta - \alpha_1)\)-module and let \(\hat{M}_1\) be the irreducible quotient of \(\hat{M}_1\). The module \(\hat{M}_1\) is uniserial of length 2 and the composition factors are \(M_1\). Thus, \(\hat{M}_1\) is an indecomposable projective \(R^\Lambda(2\delta - \alpha_1)\)-module. Further,

\[
(4.1.1) \quad E_0M_0 = 0, \quad E_1\hat{M}_1 = 0, \quad E_1M_0 \cong E_0\hat{M}_1,
\]

and \(E_1M_0 \cong E_0\hat{M}_1\) is the regular representation of \(R^\Lambda(\delta) \cong k[x]/(x^2)\) via \(x \mapsto x_2\).

4.2. Representations of \(R^\Lambda(2\delta)\) for \(\ell = 1\).

Lemma 4.1. (1) By declaring that \(x_4\) and \(\psi_3\) act as 0, and \(e(\nu)\), for \(\nu \in I^4\), as

\[
e(\nu)v_i = \begin{cases} 
v_i & \text{if } \nu = (0110), \\
0 & \text{otherwise},
\end{cases}
\]

the irreducible \(R^\Lambda(2\delta - \alpha_0)\)-module \(M_0\) in Section 4.1 extends to an irreducible \(R^\Lambda(2\delta)\)-module.

(2) By declaring that \(\psi_3\) acts as 0, and \(x_4\), \(e(\nu)\), for \(\nu \in I^4\), act as

\[
x_4w_i = \begin{cases} 
0 & \text{if } i = 1, \\
(\lambda^2 - 1)w_1 & \text{if } i = 2,
\end{cases}
\]

\[
e(\nu)w_i = \begin{cases} 
w_i & \text{if } \nu = (0101), \\
0 & \text{otherwise},
\end{cases}
\]

the \(R^\Lambda(2\delta - \alpha_1)\)-module \(\hat{M}_1\) in Section 4.1 extends to an \(R^\Lambda(2\delta)\)-module.

Proof. (1) It is straightforward to check the defining relations except for \((\psi_3x_k - x_{s_3(k)}\psi_3)e(0110), (\psi_3\psi_2\psi_3 - \psi_2\psi_3\psi_2)e(0110)\) and \(\psi_3^2e(0110)\). For the remaining relations, we have

\[
(\psi_3x_k - x_{s_3(k)}\psi_3)e(0110)v_i = 0,
\]

\[
(\psi_3\psi_2\psi_3 - \psi_2\psi_3\psi_2)e(0110)v_i = 0,
\]

\[
\psi_3^2e(0110)v_i = 0 = (x_3^2 + \lambda x_3x_4 + x_4^2)e(0110)v_i, \quad \text{for } i = 1, 2,
\]

by direct computation, the action is well-defined.
(2) Similarly, it is easy to verify the defining relations except for $(\psi_3 x_k - x_{s_3(k)} \psi_3)e(0101)$, $(\psi_3 \psi_2 \psi_3 - \psi_2 \psi_3 \psi_2)e(0101)$ and $\psi_3^2 e(0101)$, and we have

$$(\psi_3 x_k - x_{s_3(k)} \psi_3)e(0101)w_i = 0,$$

$$(\psi_3 \psi_2 \psi_3 - \psi_2 \psi_3 \psi_2)e(0101)w_i = 0 = (x_2 + \lambda x_3 + x_4)e(0101)w_i,$$

$$\psi_3^2 e(0101)w_i = 0 = (x_3^2 + \lambda x_3 x_4 + x_4^2)e(0101)w_i.$$ 

Hence the action is well-defined. 

We denote by $N_0$ (resp. $\widehat{N}_1$) the $R^{\Lambda_0}(2\delta)$-module defined in Lemma 4.1(1) (resp. (2)), and let $N_1$ be the irreducible quotient of $\widehat{N}_1$. Note that $M_1$ extends to $N_1$ by declaring that $\psi_3$ and $x_4$ and $e(\nu)$, for $\nu \neq (0101)$, act as 0 and $e(0101)$ acts as 1. Hence, $\widehat{N}_1$ is uniserial of length 2 whose composition factors are $N_1$. By construction, $N_0$ and $N_1$ are irreducible and they are non-isomorphic since

$$E_i(N_j) \simeq \delta_{ij} M_i.$$ 

As the categorification theorem tells that the number of irreducible $R^{\Lambda_0}(2\delta)$-modules is 2, we have the following lemma.

**Lemma 4.2.** The modules $N_0$ and $N_1$ form a complete list of irreducible $R^{\Lambda_0}(2\delta)$-modules.

We now construct some $R^{\Lambda_0}(2\delta)$-modules which will be used for proving that $R^{\Lambda_0}(2\delta)$ has tame representation type.

**Lemma 4.3.** If $\lambda = 0$, there exists a uniserial $R^{\Lambda_0}(2\delta)$-module $T_0$ whose radical series is

$$T_0/\text{Rad}(T_0) \simeq N_0, \quad \text{Rad}(T_0)/\text{Rad}^2(T_0) \simeq N_1, \quad \text{Rad}^2(T_0) \simeq N_1.$$ 

**Proof.** We change the action of $\psi_3$ on

$$T_0 = N_0 \oplus \widehat{N}_1$$

to $\psi_3 v_i = w_i$, for $i = 1, 2$. Then, $\lambda = 0$ in mind, direct computation shows

$$\psi_3^2 e(0110)v_i = \psi_3 e(0101)w_i = 0 = (x_3^2 + \lambda x_3 x_4 + x_4^2)e(0110)v_i,$$

$$(\psi_3 \psi_2 \psi_3 - \psi_2 \psi_3 \psi_2)e(0110)v_1 = \psi_3 \psi_2 w_1 - \psi_2 \psi_3 (-v_2) = \psi_2 w_2 = 0,$$

$$(\psi_3 \psi_2 \psi_3 - \psi_2 \psi_3 \psi_2)e(0110)v_2 = \psi_3 \psi_2 w_2 = 0,$$

$$(\psi_3 x_3 - x_4 \psi_3)e(0110)v_1 = -x_4 w_1 = 0,$$

$$(\psi_3 x_3 - x_4 \psi_3)e(0110)v_2 = \psi_3(-v_1) - x_4 w_2 = -w_1 - (\lambda^2 - 1)w_1 = -\lambda^2 w_1 = 0,$$

$$(\psi_3 x_4 - x_3 \psi_3)e(0110)v_1 = -x_3 w_1 = 0,$$

$$(\psi_3 x_4 - x_3 \psi_3)e(0110)v_2 = -x_3 w_2 = \lambda w_1 = 0,$$
which implies that $T_0$ is well-defined. As $N_0$ is irreducible, $\text{Rad}(T_0) \subseteq \hat{N}_1$. Thus, we have either $\text{Rad}(T_0) = \text{Soc}(\hat{N}_1)$ or $\text{Rad}(T_0) = \hat{N}_1$. On the other hand, $\psi_3 v_2 = w_2 \notin \text{Soc}(\hat{N}_1)$ implies that $T_0 / \text{Soc}(\hat{N}_1)$ is not a semisimple module. Thus, we conclude that $\text{Rad}(T_0) \cong \hat{N}_1$ and $\text{Rad}^2(T_0) \cong N_1$.

Remark 4.4. If $\lambda \neq 0$, then $T_0$ is not well-defined.

Lemma 4.5. If $\lambda \neq 0$, there exists a uniserial $R^{\lambda_0}(2\delta)$-module $T_1$ whose radical series is

$$T_1 / \text{Rad}(T_1) \cong N_0, \quad \text{Rad}(T_1) / \text{Rad}^2(T_1) \cong N_1, \quad \text{Rad}^2(T_1) \cong N_0.$$ 

Proof. We choose the basis $\{v_1, v_2\}$ of $N_0$ as in the definition, and we denote the basis of $N_1$ by $\{w\}$. Let

$$T_1 = N_0 \oplus N_1 \oplus N_0.$$ 

To distinguish the third direct summand $N_0$ of $T_1$ from the first direct summand $N_0$, we rename the basis $\{v_1, v_2\}$ for the third direct summand $N_0$ to $\{\tilde{v}_1, \tilde{v}_2\}$. Thus, we have

(4.2.3) 

$$T_1 = \text{Span}_k \{v_1, v_2, w, \tilde{v}_1, \tilde{v}_2\}.$$ 

We keep the action of $e(\nu)$, $x_k$ and $\psi_1$ on $T_1$ unchanged except

$$x_4 v_i = \tilde{v}_i \quad (i = 1, 2), \quad \psi_3 v_1 = 0, \quad \psi_3 v_2 = -\lambda w, \quad \psi_3 w = \tilde{v}_1.$$ 

To show that $T_1$ is well-defined, it suffices to check $x_k x_4 = x_4 x_k$ and the defining relations for $\psi_3^2$, $\psi_3 x_4 - x_3 \psi_3$, $\psi_3 x_3 - x_4 \psi_3$, $\psi_3 x_2 - x_2 \psi_3$, and $\psi_3 \psi_4 \psi_3 - \psi_2 \psi_3 \psi_2$ on $\{v_1, v_2, w\}$. It is easy to check $x_k x_4 = x_4 x_k$. By direct computation, we obtain

$$\psi_3^2 e(0110) v_1 = 0 = \lambda x_3 \tilde{v}_1 + x_4 \tilde{v}_1 = (x_3^2 + \lambda x_3 x_4 + x_4^2) v_1,$$
$$\psi_3^2 e(0110) v_2 = \psi_3(-\lambda w) = -\lambda \tilde{v}_1 = x_3(-v_1) + \lambda x_3 \tilde{v}_2 + x_4 \tilde{v}_2 = (x_3^2 + \lambda x_3 x_4 + x_4^2) v_2,$$
$$\psi_3^2 e(0101) w = \psi_3 \tilde{v}_1 = 0 = (x_3^2 + \lambda x_3 x_4 + x_4^2) w.$$ 

Hence the relation for $\psi_3^2$ holds. The equations

$$(\psi_3 x_4 - x_3 \psi_3)e(0110) v_1 = \psi_3 \tilde{v}_1 = 0,$$
$$(\psi_3 x_3 - x_4 \psi_3)e(0110) v_1 = 0,$$
$$(\psi_3 x_4 - x_3 \psi_3)e(0110) v_2 = \psi_3 \tilde{v}_2 - x_3(-\lambda w) = 0,$$
$$(\psi_3 x_3 - x_4 \psi_3)e(0110) v_2 = \psi_3(-v_1) - x_4(-\lambda w) = 0,$$
$$(\psi_3 x_4 - x_3 \psi_3)e(0101) w = -x_3 \tilde{v}_1 = 0,$$
$$(\psi_3 x_3 - x_4 \psi_3)e(0101) w = -x_4 \tilde{v}_1 = 0,$$
$$(\psi_3 x_2 - x_2 \psi_3)e(0101) v_1 = 0,$$
$$(\psi_3 x_2 - x_2 \psi_3)e(0101) v_2 = \psi_3 v_1 - x_2(-\lambda w) = 0,$$
imply that the relations for $\psi_3 x_4 - x_3 \psi_3$, $\psi_3 x_3 - x_4 \psi_3$ and $\psi_3 x_2 - x_2 \psi_3$ hold, and

$$(\psi_3 \psi_2 \psi_3 - \psi_2 \psi_3 \psi_2) e(0110) v_1 = -\psi_2 \psi_3 (-v_2) = \psi_2 (-\lambda w) = 0,$$

$$(\psi_3 \psi_2 \psi_3 - \psi_2 \psi_3 \psi_2) e(0110) v_2 = \psi_3 \psi_2 (-\lambda w) = 0,$$

$$(\psi_3 \psi_2 \psi_3 - \psi_2 \psi_3 \psi_2) e(0110) w = \psi_3 \psi_2 \tilde{v}_1 = \psi_3 (-\tilde{v}_2) = 0 = (x_2 + \lambda x_3 + x_4) e(0101) w$$

verify the relation for $\psi_3 \psi_2 \psi_3 - \psi_2 \psi_3 \psi_2$. Thus, $T_1$ is well-defined.

We show that $T_1$ is uniserial. It is clear that $\text{Rad}(T_1) \subseteq \text{Span}_k \{w, \tilde{v}_1, \tilde{v}_2\}$. Then $\psi_3 w = \tilde{v}_1$ implies that $\text{Span}_k \{w, \tilde{v}_1, \tilde{v}_2\}$ is not semisimple, and we have either

$$\text{Rad}(T_1) = \text{Span}_k \{w, \tilde{v}_1, \tilde{v}_2\} \quad \text{or} \quad \text{Rad}(T_1) = \text{Span}_k \{\tilde{v}_1, \tilde{v}_2\}.$$

Since $T_1 / \text{Span}_k \{\tilde{v}_1, \tilde{v}_2\}$ is not semisimple by $\psi_3 v_2 = -\lambda w$, we have

$$\text{Rad}(T_1) = \text{Span}_k \{w, \tilde{v}_1, \tilde{v}_2\}, \quad \text{Rad}^2(T_1) = \text{Span}_k \{\tilde{v}_1, \tilde{v}_2\},$$

which completes the proof. \qed

**Remark 4.6.** If $\lambda = 0$, then $\text{Rad}(T_1) \simeq N_0$ in the proof of Lemma 4.5.

**Lemma 4.7.** If $\lambda \neq 0$, there exists a uniserial $R^{\Lambda_0}(2\delta)$-module $\tilde{T}_1$ whose radical series is $\tilde{T}_1 / \text{Rad}(\tilde{T}_1) \simeq N_1$, $\text{Rad}(\tilde{T}_1) / \text{Rad}^2(\tilde{T}_1) \simeq N_0$, $\text{Rad}^2(\tilde{T}_1) / \text{Rad}^3(\tilde{T}_1) \simeq N_1$, $\text{Rad}^3(\tilde{T}_1) \simeq N_0$.

**Proof.** Recall the $R^{\Lambda_0}(2\delta)$-module $T_1 = \text{Span}_k \{v_1, v_2, w, \tilde{v}_1, \tilde{v}_2\}$ given in (4.2.3). Let

$$\tilde{T}_1 = T_1 \oplus ku,$$

where $T_1$ is its submodule, and define

$$e(\nu) u = \begin{cases} u & \text{if } \nu = (0101), \\ 0 & \text{otherwise}, \end{cases} \quad x_k u = \begin{cases} 0 & \text{if } k = 1, \\ w & \text{if } k = 2, \\ -\lambda w & \text{if } k = 3, \\ -w & \text{if } k = 4, \end{cases} \quad \psi_l u = \begin{cases} 0 & \text{if } l = 1, 2, \\ -\lambda v_1 + \tilde{v}_2 & \text{if } l = 3. \end{cases}$$

We show that $\tilde{T}_1$ is well-defined. It is straightforward to check that

$$x_j x_k e(0101) u = 0 = x_k x_j e(0101) u,$$

for $1 \leq j, k \leq 4$. Then $(x_k^2 + \lambda x_k x_{k+1} + x_{k+1}^2) u = 0$, for $k = 1, 2, 3$, and

$$\psi_k^2 e(0101) u = \psi_k^2 e(0101) u = 0, \quad \psi_3^2 e(0101) u = \psi_3 (-\lambda v_1 + \tilde{v}_2) = 0$$

imply that the relation for $\psi_k^2$ holds. Next, using $\psi_1 w = 0$ and $\psi_2 w = 0$, we have

$$(\psi_k x_l - x_{s_k(l)} \psi_k) e(0101) u = 0,$$
for \( k = 1, 2 \). When \( k = 3 \), we have

\[
(\psi_3 x_4 - x_3 \psi_3) e(0101) u = \psi_3 (-w) - x_3 (-\lambda v_1 + \tilde{v}_2) = -\tilde{v}_1 - (-\tilde{v}_1) = 0,
\]

\[
(\psi_3 x_3 - x_4 \psi_3) e(0101) u = \psi_3 (-\lambda w) - x_4 (-\lambda v_1 + \tilde{v}_2) = -\lambda \tilde{v}_1 - (-\lambda \tilde{v}_1) = 0,
\]

\[
(\psi_3 x_2 - x_2 \psi_3) e(0101) u = \psi_3 w - x_2 (-\lambda v_1 + \tilde{v}_2) = \tilde{v}_1 - \tilde{v}_1 = 0.
\]

Thus the relation for \( \psi_k x_l - x_{sk(l)} \psi_k \) holds. Finally, we compute

\[
(x_1 + \lambda x_2 + x_3) e(0101) u = 0, \quad (x_2 + \lambda x_3 + x_4) e(0101) u = -\lambda^2 w,
\]

\[
(\psi_2 \psi_1 \psi_2 - \psi_1 \psi_2 \psi_1) e(0101) u = 0,
\]

\[
(\psi_3 \psi_2 \psi_3 - \psi_2 \psi_3 \psi_2) e(0101) u = \psi_3 \psi_2 (-\lambda v_1 + \tilde{v}_2) = \psi_3 (\lambda v_2) = -\lambda^2 w.
\]

They show that the relation for \( \psi_{k+1} \psi_k \psi_{k+1} - \psi_k \psi_k \psi_k \) holds, and \( \hat{T}_1 \) is well-defined.

We consider the radical series. For any \( y = t + u \) with \( t \in \hat{T}_1 \), we have

\[
\psi_3 u \not\in \text{Span}_k \{ w, \tilde{v}_1, \tilde{v}_2 \} = \text{Rad}(\hat{T}_1),
\]

which implies that \( \hat{T}_1 / \text{Rad}(\hat{T}_1) \) is not semisimple. Hence \( \text{Rad}(\hat{T}_1) = T_1 \) and the assertion follows from Lemma 4.5.

4.3. Representations of \( R^{\Lambda_0}(2\delta + \alpha_0) \) for \( \ell = 1 \). We extend the \( R^{\Lambda_0}(2\delta) \)-module \( \hat{T}_1 \) described in the proof of Lemma 4.7 to \( R^{\Lambda_0}(2\delta + \alpha_0) \). Note that \( \hat{T}_1 \) is well-defined regardless of the choice of \( \lambda \). We write

\[
\hat{T}_1 = \text{Span}_k \{ v_1, v_2, w, \tilde{v}_1, \tilde{v}_2, u \}
\]

as \( 4.2.4 \) in the proof of Lemma 4.7. Let us declare that \( x_5, \psi_4 \) and \( e(\nu) \), for \( \nu \in I^{2\delta + \alpha_0} \), act as follows:

\[
\begin{align*}
x_5 v_k &= -\tilde{v}_k, & x_5 \tilde{v}_k &= 0, & x_5 w &= 0, & x_5 u &= \lambda w, \\
\psi_4 v_k &= 0, & \psi_4 \tilde{v}_k &= -v_k, & \psi_4 w &= 0, & \psi_4 u &= 0, \\
e(\nu) v_k &= \begin{cases} v_k & \text{if } \nu = (01010), \\ 0 & \text{otherwise,} \end{cases} & e(\nu) \tilde{v}_k &= \begin{cases} \tilde{v}_k & \text{if } \nu = (01000), \\ 0 & \text{otherwise,} \end{cases} \\
e(\nu) w &= \begin{cases} w & \text{if } \nu = (01010), \\ 0 & \text{otherwise,} \end{cases} & e(\nu) u &= \begin{cases} u & \text{if } \nu = (01010), \\ 0 & \text{otherwise.} \end{cases}
\end{align*}
\]
Thus, $x_1 = \psi_1 = 0$, $e(01100) + e(01010) = 1$, where

$$
e(01100) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

and the other generators in matrix form are,

$$
x_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad x_3 = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\lambda \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$
x_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad x_5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

$$
\psi_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \psi_3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -\lambda \\
0 & -\lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \psi_4 = \begin{pmatrix}
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

We check that the action is well-defined. It is easy to verify the defining relations except for $x_k x_5 - x_5 x_k$, $\psi_k^2$, $\psi_k x_1 - x_{sk(t)} \psi_k$ and $\psi_4 \psi_3 \psi_4 - \psi_3 \psi_4 \psi_3$.

Direct computation shows that we have, for $1 \leq k \leq 4$ and $1 \leq i \leq 2$, $x_k x_5 \tilde{\psi}_i = x_5 x_k \tilde{\psi}_i = 0$, $x_k x_5 w = x_5 x_k w = 0$, $x_k x_5 u = x_5 x_k u = 0$ and

$$x_k x_5 \psi_i = x_5 x_k \psi_i = \begin{cases} 
-\tilde{\psi}_1 & \text{if } i = 2, k = 2, \\
\tilde{\psi}_1 & \text{if } i = 2, k = 3, \\
0 & \text{otherwise},
\end{cases}$$
which means that the relation for $x_kx_5 - x_5x_k$ holds. On the other hand,
\[ \psi_1^2 e(01100)v_i = 0, \quad \psi_1^2 e(01100)\tilde{v}_i = \psi_4(-v_i) = 0, \]
for $1 \leq i \leq 2$, and
\[ \psi_1^2 e(01010)w = 0 = (x_4^2 + \lambda x_4 x_5 + x_5^2)w, \quad \psi_1^2 e(01010)u = 0 = (x_4^2 + \lambda x_4 x_5 + x_5^2)u, \]
verify the relation for $\psi_1^2$. Direct computation shows
\[
\begin{align*}
(\psi_4 x_5 - x_4 \psi_4)e(01100)v_i &= \psi_4(-\tilde{v}_i) = v_i, \\
(\psi_4 x_5 - x_4 \psi_4)e(01100)\tilde{v}_i &= -x_4(-v_i) = \tilde{v}_i, \\
(\psi_4 x_5 - x_4 \psi_4)e(01010)w &= 0, \\
(\psi_4 x_5 - x_4 \psi_4)e(01010)u &= \psi_4(\lambda w) = 0,
\end{align*}
\]
and
\[
\begin{align*}
(\psi_3 x_5 - x_5 \psi_3)e(01100)v_1 &= \psi_3(-\tilde{v}_1) = 0, \\
(\psi_3 x_5 - x_5 \psi_3)e(01100)v_2 &= \psi_3(-\tilde{v}_2) - x_5(-\lambda w) = 0, \\
(\psi_3 x_5 - x_5 \psi_3)e(01010)\tilde{v}_i &= 0, \\
(\psi_3 x_5 - x_5 \psi_3)e(01010)w &= -x_5\tilde{v}_1 = 0, \\
(\psi_3 x_5 - x_5 \psi_3)e(01010)u &= \psi_3(\lambda w) - x_5(-\lambda v_1 + \tilde{v}_2) = \lambda \tilde{v}_1 - \lambda \tilde{v}_1 = 0.
\end{align*}
\]
It is straightforward to verify the remaining $\psi_kx_i - x_{s_k(i)}\psi_k$. Lastly, we have
\[
\begin{align*}
(\psi_4 \psi_3 \psi_4 - \psi_3 \psi_4 \psi_3)e(01100)v_1 &= 0, \\
(\psi_4 \psi_3 \psi_4 - \psi_3 \psi_4 \psi_3)e(01100)v_2 &= -\psi_3 \psi_4(-\lambda w) = 0, \\
(\psi_4 \psi_3 \psi_4 - \psi_3 \psi_4 \psi_3)e(01100)\tilde{v}_1 &= \psi_4 \psi_3(-v_1) = 0, \\
(\psi_4 \psi_3 \psi_4 - \psi_3 \psi_4 \psi_3)e(01100)\tilde{v}_2 &= \psi_4 \psi_3(-v_2) = \psi_4(\lambda w) = 0, \\
(\psi_4 \psi_3 \psi_4 - \psi_3 \psi_4 \psi_3)e(01010)w &= -\psi_3 \psi_4(\tilde{v}_1) = -\psi_3(-v_1) = 0 = (x_3 + \lambda x_4 + x_5)w, \\
(\psi_4 \psi_3 \psi_4 - \psi_3 \psi_4 \psi_3)e(01010)u &= -\psi_3 \psi_4(-\lambda v_1 + \tilde{v}_2) = -\psi_3(-v_2) = -\lambda w = (x_3 + \lambda x_4 + x_5)u.
\end{align*}
\]
Thus, the relation for $\psi_4 \psi_3 \psi_4 - \psi_3 \psi_4 \psi_3$ holds. Therefore, it is well-defined as an $R^{\lambda_0}(2\delta + \alpha_0)$-module. We denote this $R^{\lambda_0}(2\delta + \alpha_0)$-module by $\mathcal{V}$.

Let
\[
\mathcal{O}_0 = \begin{cases} 
\text{Span}_k\{v_1, v_2, \tilde{v}_1, \tilde{v}_2\} & \text{if } \lambda = 0, \\
\text{Span}_k\{v_1, v_2, w, \tilde{v}_1, \tilde{v}_2\} & \text{if } \lambda \neq 0.
\end{cases}
\]

It is easy to check that $\mathcal{O}_0$ is a submodule of $\mathcal{V}$. If $\lambda \neq 0$ then $\mathcal{O}_0$ viewed as an $R^{\lambda_0}(2\delta)$-module is $\mathcal{T}_1$ and it has the simple socle $\text{Span}\{v_1, v_2\}$. We know that the same is true for
\( \lambda = 0 \) by considering the action of \( \psi_4 \). Thus, any \( R^{\lambda_0}(2\delta + \alpha_0) \)-submodule of \( \mathcal{O}_0 \) contains \( \text{Span}\{v_1, v_2\} \), which shows that \( \mathcal{O}_0 \) is irreducible. Observe that

\[
\mathcal{U} = \text{Span}_k\{v_1, v_2, w, \tilde{v}_1, \tilde{v}_2\}
\]

is a submodule of \( \mathcal{V} \). Let

\[
\mathcal{O}_1 = \mathcal{V}/\mathcal{U}.
\]

Note that the module \( \mathcal{O}_1 \) is a 1-dimensional module on which \( x_1, \ldots, x_5 \) and \( \psi_1, \psi_2, \psi_3, \psi_4 \) act as 0. From the categorification theorem, we know that the number of irreducible \( R^{\lambda_0}(2\delta + \alpha_0) \)-modules is 2. Combining (4.2.1) with

\[
\varepsilon_0(\mathcal{O}_0) = 2, \quad \varepsilon_0(\mathcal{O}_1) = 1,
\]

we have the following lemma.

**Lemma 4.8.** The module \( \mathcal{O}_0 \) and \( \mathcal{O}_1 \) form a complete list of irreducible \( R^{\lambda_0}(2\delta + \alpha_0) \)-modules. Moreover, \( \mathcal{O}_0 = \tilde{f}_0\mathcal{N}_0 \) and \( \mathcal{O}_1 = \tilde{f}_0\mathcal{N}_1 \).

The \( R^{\lambda_0}(2\delta) \)-module \( \tilde{\mathcal{N}}_1 = \text{Span}_k\{w_1, w_2\} \) in Section 4.2 can also be extended to an \( R^{\lambda_0}(2\delta + \alpha_0) \)-module. Indeed, by declaring that \( \psi_4 \) acts as 0, and \( x_5, e(\nu) \) act as

\[
x_5w_i = \begin{cases} 
0 & \text{if } i = 1, \\
(2\lambda - \lambda^3)w_1 & \text{if } i = 2,
\end{cases}
\]

\[
e(\nu)w_i = \begin{cases} 
w_i & \text{if } \nu = (01010), \\
0 & \text{otherwise},
\end{cases}
\]

we have a well-defined action. We denote this module by \( \tilde{\mathcal{O}}_1 \). By construction, \( \tilde{\mathcal{O}}_1 \) is uniserial of length 2 whose composition factors are \( \mathcal{O}_1 \).

On the other hand, when \( \lambda = 0 \), using \( \psi_3w = \tilde{v}_1 \), we know that the exact sequence

\[
0 \to \mathcal{O}_0 \to \mathcal{U} \to \mathcal{O}_1 \to 0
\]

is non-split. When \( \lambda \neq 0 \), we use \( x_5u = \lambda w \) to show that the exact sequence

\[
0 \to \mathcal{O}_0 \to \mathcal{V} \to \mathcal{O}_1 \to 0
\]

does not split. Thus, we have the following lemma.

**Lemma 4.9.** There exist uniserial \( R^{\lambda_0}(2\delta + \alpha_0) \)-modules whose radical series are

\[
\mathcal{O}_1 \quad \text{and} \quad \mathcal{O}_1 \quad \text{and} \quad \mathcal{O}_1.
\]
5. Representation type

5.1. The algebra $R^{\Lambda_0}(\delta)$. In this section, we show that $R^{\Lambda_0}(\delta)$ is the Brauer tree algebra whose Brauer tree is the straight line without exceptional vertex.

Recall the irreducible $R^{\Lambda_0}(\delta - \alpha_i)$-modules $L_i$ defined in Section 3.1. We define

$$P_i = F_i L_i, \quad \text{for } 1 \leq i \leq \ell.$$

As $R^{\Lambda_0}(\delta - \alpha_i)$ is a simple algebra, $L_i$ is a projective $R^{\Lambda_0}(\delta - \alpha_i)$-module. It follows that $P_i$ are projective $R^{\Lambda_0}(\delta)$-modules, since the functor $F_i$ sends projective objects to projective objects.

Recall the irreducible $R^{\Lambda_0}(\delta)$-modules $S_i$ from Lemma 3.3 and the biadjointness of the functors $E_i$ and $F_i$ [15, Thm.3.5]. Then, we have

$$\text{Hom}(S_j, P_i) \simeq \text{Hom}(E_i S_j, L_i) \simeq \begin{cases} k & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases}$$

$$\text{Hom}(P_i, S_j) \simeq \text{Hom}(L_i, E_i S_j) \simeq \begin{cases} k & \text{if } j = i, \\ 0 & \text{if } j \neq i. \end{cases}$$

Thus, $P_i$ is the projective cover of $S_i$, for $1 \leq i \leq \ell$.

**Theorem 5.1.** If $\ell = 1$ then $P_1$ is uniserial of length 2 whose composition factors are $S_1$. If $\ell \geq 2$, then the radical series of $P_i$ are given as follows:

$$P_1 \simeq S_2, \quad P_i \simeq S_{i-1} S_{i+1} \quad (i \neq 1, \ell), \quad P_\ell \simeq S_{\ell-1}$$

**Proof.** We compute $\text{dim Hom}(P_i, P_j)$. Suppose that $i \neq j$. Then $\ell \geq 2$ and the isomorphism of functors $E_j F_i \simeq F_i E_j$ implies

$$\text{Hom}(P_i, P_j) \simeq \text{Hom}(E_j F_i L_i, L_j) \simeq \text{Hom}(F_i E_j L_i, L_j) \simeq \text{Hom}(E_j L_i, E_i L_j).$$

Hence, if $i \neq j$ then

$$\text{dim Hom}(P_i, P_j) = \begin{cases} 1 & \text{if } j = i \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $i = j$. Then $E_i L_i = 0$ and $\langle h_i, \Lambda_0 - \delta + \alpha_i \rangle = 2$, for $1 \leq i \leq \ell$, imply $E_i F_i L_i \simeq L_i \oplus L_i$. Thus, we have

$$\text{dim Hom}(P_i, P_i) = \text{dim Hom}(L_i, E_i F_i L_i) = 2.$$
Therefore, \([\mathcal{P}_i] = 2[\mathcal{S}_1]\) if \(\ell = 1\), and if \(\ell \geq 2\) then

\[
[\mathcal{P}_i] = 2[\mathcal{S}_1] + [\mathcal{S}_2],
\]

\[
[\mathcal{P}_i] = [\mathcal{S}_{i-1}] + 2[\mathcal{S}_i] + [\mathcal{S}_{i+1}] \quad (2 \leq i \leq \ell - 1),
\]

\[
[\mathcal{P}_\ell] = [\mathcal{S}_{\ell-1}] + 2[\mathcal{S}_\ell],
\]

in the Grothendieck group \(K_0(R^{\Lambda_0}(\delta)\text{-mod})\).

Recall that the algebra \(R^{\Lambda_0}(\beta)\) has an anti-involution which fixes all the defining generators elementwise, and we have the corresponding duality on the category of \(R^{\Lambda_0}(\beta)\)-modules:

\[ M \mapsto M^\vee = \text{Hom}_k(M, k). \]

It is straightforward to check that \(\mathcal{S}_i\) are self-dual, so that the heart of \(\mathcal{P}_i\) is self-dual. Then the self-duality implies that the heart of \(\mathcal{P}_i\) is semi-simple. \(\square\)

**Corollary 5.2.** The algebra \(R^{\Lambda_0}(\delta)\) is the Brauer tree algebra whose tree is the straight line of length \(\ell\) and without exceptional vertex. In particular, \(R^{\Lambda_0}(\delta)\) is representation-finite.

### 5.2. The algebra \(R^{\Lambda_0}(2\delta)\)

In this section, we show that \(R^{\Lambda_0}(2\delta)\) is of wild type if \(\ell \geq 2\). Let \(\nu = (\nu_1, \ldots, \nu_{\ell+1}) = (0, 1, 2, \ldots, \ell)\) and

\[ e_1 = e(\nu * \nu), \quad e_2 = e(s_\ell(\nu) * s_\ell(\nu)). \]

It follows from Theorem 2.5 that

\[ \dim e_i R^{\Lambda_0}(2\delta) e_i = 4, \quad \text{for } i = 1, 2, \]

because Young diagrams which contribute to \(\dim e_1 R^{\Lambda_0}(2\delta) e_1\) are

\[
(2\ell + 2), \quad (2\ell + 1, 1), \quad (\ell + 1, \ell, 1), \quad (\ell, \ell, 2),
\]

and Young diagrams which contribute to \(\dim e_2 R^{\Lambda_0}(2\delta) e_2\) are

\[
(\ell + 1, \ell + 1), \quad (\ell + 1, \ell, 1), \quad (\ell, \ell - 1, 2, 1), \quad (\ell - 1, \ell - 1, 2, 2).
\]

**Lemma 5.3.**

1. There exists an algebra isomorphism

\[ e_1 R^{\Lambda_0}(2\delta) e_1 \simeq k[x, y]/(x^2, y^2 - axy) \]

for some \(a \in k\). Under this isomorphism, \(x_{\ell+1}, x_{2\ell+2}\) correspond to \(x, y\) respectively, and \((x_{\ell} + x_{\ell+1})(x_{2\ell+1} + x_{2\ell+2}) \neq 0\) holds in \(e_1 R^{\Lambda_0}(2\delta) e_1\).

2. There exists an algebra isomorphism

\[ e_2 R^{\Lambda_0}(2\delta) e_2 \simeq k[z, w]/(z^2, w^2 - bwz) \]

for some \(b \in k\). Under this isomorphism, \(x_{\ell+1}, x_{2\ell+2}\) correspond to \(z, w\) respectively, and \((x_{\ell} + x_{\ell+1})(x_{2\ell+1} + x_{2\ell+2}) \neq 0\) holds in \(e_2 R^{\Lambda_0}(2\delta) e_2\).
Proof. (1) For \( t \geq 0 \) and \( 0 \leq k \leq \ell \), we set
\[
e^{t,k} = e(\nu + \cdots + \nu)(0) \cdots (k - 1)), \quad \beta^{t,k} = t\delta + \alpha_0 + \alpha_1 + \cdots + \alpha_{k-1}.
\]
Note that \( e^{2,0} = e_1 \). By Theorem 2.5, we have
\[
\dim e^{t,k} R^\Lambda_0(\beta^{t,k}) e^{t,k} = \begin{cases} 1 & \text{if } t = 0 \text{ and } 0 \leq k \leq \ell, \\ 2 & \text{if } t = 1 \text{ and } 0 \leq k \leq \ell, \\ 4 & \text{if } (t, k) = (2, 0). \end{cases}
\]
(5.2.3)

Note that Young diagram which contributes to \( \dim e^{0,k} R^\Lambda_0(\beta^{0,k}) e^{0,k} \) is \((k + 1)\), and Young diagrams which contribute to \( \dim e^{1,k} R^\Lambda_0(\beta^{1,k}) e^{1,k} \) are \((\ell + k + 2)\) and \((\ell, k + 2)\), and if \((t, k) = (2, 0)\) then Young diagrams which contribute to \( \dim e^{2,0} R^\Lambda_0(\beta^{2,0}) e^{2,0} \) are \((2\ell + 2), (2\ell + 1, 1), (\ell + 1, \ell, 1)\) and \((\ell, \ell, 2)\).

As
\[
\langle h_{k+1}, \Lambda_0 - \beta^{t,k} \rangle = \langle h_k, \Lambda_0 - \beta^{t,k} \rangle = \begin{cases} 1 & \text{if } 0 \leq k \leq \ell - 1, \\ 2 & \text{if } k = \ell, \end{cases}
\]
we have \((e^{t,k} R^\Lambda_0(\beta^{t,k}) e^{t,k}, e^{t,k} R^\Lambda_0(\beta^{t,k}) e^{t,k})\)-bimodule monomorphisms
\[
e^{t,k} R^\Lambda_0(\beta^{t,k}) e^{t,k} \mapsto e^{t,k+1} R^\Lambda_0(\beta^{t,k+1}) e^{t,k+1},
\]
\[
e^{t,\ell} R^\Lambda_0(\beta^{t,\ell}) e^{t,\ell} \otimes (k1 \oplus kx) \mapsto e^{t+1,0} R^\Lambda_0(\beta^{t+1,0}) e^{t+1,0}.
\]
for \( t = 0, 1 \) and \( 0 \leq k \leq \ell - 1 \). Then (5.2.3) shows that the above monomorphisms are isomorphisms. Thus \( k \cong e^{0,0} R^\Lambda_0(\beta^{0,0}) e^{0,0} \cong e^{0,\ell} R^\Lambda_0(\beta^{0,\ell}) e^{0,\ell} \). The bimodule isomorphism
\[
e^{0,\ell} R^\Lambda_0(\beta^{0,\ell}) e^{0,\ell} \otimes (k1 \oplus kx) \sim e^{1,0} R^\Lambda_0(\beta^{1,0}) e^{1,0}
\]
is given by \( x \mapsto x_{\ell+1} e^{1,0} \) [15 Thm.3.4], and it induces an algebra isomorphism
\[
k[x]/(x^2) \sim e^{1,0} R^\Lambda_0(\beta^{1,0}) e^{1,0}.
\]
Similarly, \( k[x]/(x^2) \cong e^{1,0} R^\Lambda_0(\beta^{1,0}) e^{1,0} \cong e^{1,\ell} R^\Lambda_0(\beta^{1,\ell}) e^{1,\ell} \) and the bimodule isomorphism
\[
k[x]/(x^2) \otimes (k1 \oplus ky) \cong e^{1,\ell} R^\Lambda_0(\beta^{1,\ell}) e^{1,\ell} \otimes (k1 \oplus ky) \sim e^{2,0} R^\Lambda_0(\beta^{2,0}) e^{2,0}
\]
given by \( x \mapsto x_{\ell+1} e^{2,0}, y \mapsto x_{2\ell+2} e^{2,0} \) induces an algebra isomorphism
\[
k[x,y]/(x^2, y^2 - axy) \sim e_1 R^\Lambda_0(2\delta) e_1,
\]
for some \( a \in k \). Taking the grading into consideration, the above argument also implies that \( x_\ell = 0 \) and \( x_{2\ell+1} \) is a scalar multiple of \( x_{\ell+1} \) in \( e_1 R^\Lambda_0(2\delta) e_1 \). Thus, \( xy \neq 0 \) maps to \((x_\ell + x_{\ell+1})(x_{2\ell+1} + x_{2\ell+2})\) under the isomorphism.
(2) We replace $\nu$ with $s_\ell(\nu)$ and follow the argument in (1). Then $\alpha_t, \beta_t$ are similarly defined, and $\alpha_2 = e_2$. Moreover, we have $\langle h_k, \Lambda_0 - \beta_t \rangle = 1$, for $0 \leq k \leq \ell - 2$, and

$$\langle h_\ell, \Lambda_0 - \beta_t \rangle = 1, \quad \langle h_{\ell-1}, \Lambda_0 - \beta_t \rangle = 2.$$  

Thus, Theorem 2.5 gives

$$\dim e_t R^\Lambda(\beta_t) e_t = \begin{cases} 1 & \text{if } t = 0 \text{ and } 0 \leq k \leq \ell, \\ 2 & \text{if } t = 1 \text{ and } 0 \leq k \leq \ell, \\ 4 & \text{if } (t, k) = (2, 0), \end{cases}$$

and we can conclude that $z \mapsto x_{\ell+1}e^{2,0}, w \mapsto x_{2\ell+2}e^{2,0}$ defines an algebra isomorphism

$$k[z, w]/(z^2, w^2 - bw) \sim e_2 R^\Lambda(2\delta)e_2,$$

for some $b \in k$. It also follows $(x_{\ell} + x_{\ell+1})(x_{\ell+1} + x_{2\ell+2}) \neq 0$ in $e_2 R^\Lambda(2\delta)e_2$. \hfill $\square$

**Proposition 5.4.** Suppose $\ell \geq 2$. Then $R^\Lambda(2\delta)$ has wild representation type.

**Proof.** Let $e = e_1 + e_2$. Combining Theorem 2.5 with (5.2.1) and (5.2.2), we have

$$\dim e R^\Lambda(2\delta)e = 10.$$

It follows from Lemma 5.3 that

$$(\psi_\ell \psi_{2\ell+1} e_2)(\psi_\ell \psi_{2\ell+1} e_1) = e_1 \psi_\ell^2 \psi_{2\ell+1}^2 e_1 = (x_\ell + x_{\ell+1})(x_{\ell+1} + x_{2\ell+2})e_1$$

is nonzero, which implies that $\psi_\ell \psi_{2\ell+1} e_2$ and $\psi_\ell \psi_{2\ell+1} e_1$ are nonzero in $e R^\Lambda(2\delta)e$. Hence,

$$\{e_1, e_2, x_{\ell+1} e_1, x_{\ell+1} e_2, x_{2\ell+2} e_1, x_{2\ell+2} e_2, x_{\ell+1} x_{2\ell+2} e_1, x_{\ell+1} x_{2\ell+2} e_2, \psi_\ell \psi_{2\ell+1} e_1, \psi_\ell \psi_{2\ell+1} e_2\}$$

forms a basis of $e R^\Lambda(2\delta)e$. As $\dim e_1 R^\Lambda(2\delta)e_2 = 1$ and $\dim e_2 R^\Lambda(2\delta)e_1 = 1$, the degree consideration shows that $\psi_\ell \psi_{2\ell+1} e_1$ and $\psi_\ell \psi_{2\ell+1} e_2$ are annihilated by both $x_{\ell+1}$ and $x_{2\ell+2}$.

Let $p = \psi_\ell \psi_{2\ell+1} e_1$ and $q = \psi_\ell \psi_{2\ell+1} e_2$. Using the isomorphisms in Lemma 5.3, we have the following quiver presentation of $e R^\Lambda(2\delta)e$:

$$\begin{array}{c}
\begin{tikzpicture}
\node (1) at (0,0) {$x$};
\node (2) at (1,1) {$q$};
\node (3) at (2,0) {$z$};
\node (4) at (1,-1) {$p$};
\node (5) at (0,-1) {$w$};
\draw[->] (1) edge (2);
\draw[->] (2) edge (3);
\draw[->] (3) edge (4);
\draw[->] (4) edge (5);
\draw[->] (5) edge (1);
\end{tikzpicture}
\end{array}$$

with relations

$$x^2 = 0, \quad y^2 = axy, \quad xy = yx, \quad z^2 = 0, \quad w^2 = bwz, \quad zw = wz,$$

$$pq = xy, \quad qp = zw, \quad xp = yp = pz = pw = 0, \quad zq = wq = qx = qy = 0,$$

where $a, b \in k$ are given in Lemma 5.3. Then, by [1] 1.10.8, $e R^\Lambda(2\delta)e$ has wild representation type, and so does $R^\Lambda(2\delta)$. \hfill $\square$
5.3. The algebras for $\ell = 1$. Using the $R^{\Lambda_0}(2\delta - \alpha_0)$-module $\mathcal{M}_0$ and the $R^{\Lambda_0}(2\delta - \alpha_1)$-module $\hat{\mathcal{M}}_1$ given in Section 4.1, we define

$$Q_0 = F_0 \mathcal{M}_0, \quad Q_1 = F_1 \hat{\mathcal{M}}_1.$$ 

As the modules $\mathcal{M}_0$ and $\hat{\mathcal{M}}_1$ are projective, the $R^{\Lambda_0}(2\delta)$-modules $Q_0$ and $Q_1$ are projective.

Recall the irreducible $R^{\Lambda_0}(2\delta)$-modules $\mathcal{N}_0$ and $\mathcal{N}_1$ given in Lemma 4.2. By (4.2.1) and the biadjointness [15, Thm.3.5] of the functors $E_i$ and $F_i$, we have

$$\text{Hom}(\mathcal{N}_i, Q_0) \simeq \text{Hom}(E_0 \mathcal{N}_i, \mathcal{M}_0) \simeq \begin{cases} k & \text{if } i = 0, \\ 0 & \text{if } i = 1, \end{cases}$$

$$\text{Hom}(Q_0, \mathcal{N}_i) \simeq \text{Hom}(\mathcal{M}_0, E_0 \mathcal{N}_i) \simeq \begin{cases} k & \text{if } i = 0, \\ 0 & \text{if } i = 1, \end{cases}$$

$$\text{Hom}(\mathcal{N}_i, Q_1) \simeq \text{Hom}(E_1 \mathcal{N}_i, \hat{\mathcal{M}}_1) \simeq \begin{cases} 0 & \text{if } i = 0, \\ k & \text{if } i = 1, \end{cases}$$

$$\text{Hom}(Q_1, \mathcal{N}_i) \simeq \text{Hom}(\hat{\mathcal{M}}_1, E_1 \mathcal{N}_i) \simeq \begin{cases} 0 & \text{if } i = 0, \\ k & \text{if } i = 1. \end{cases}$$

Hence, the modules $Q_0$ and $Q_1$ are projective cover of $\mathcal{N}_0$ and $\mathcal{N}_1$ respectively. Moreover, since $E_i F_j \simeq F_j E_i$ for $i \neq j$ and

$$E_0 F_0 \mathcal{M}_0 \simeq F_0 E_0 \mathcal{M}_0 \oplus \mathcal{M}_0^\oplus(h_0, \alpha_0 - 2\delta + \alpha_0),$$

$$E_1 F_1 \hat{\mathcal{M}}_1 \simeq F_1 E_1 \hat{\mathcal{M}}_1 \oplus \hat{\mathcal{M}}_1^\oplus(h_1, \alpha_0 - 2\delta + \alpha_1),$$

it follows from (1.11) that

$$\dim \text{Hom}(Q_0, Q_0) = \dim \text{Hom}(\mathcal{M}_0, E_0 F_0 \mathcal{M}_0) = \dim \text{Hom}(\mathcal{M}_0, \mathcal{M}_0^\oplus) = 3,$$

$$\dim \text{Hom}(Q_1, Q_1) = \dim \text{Hom}(\hat{\mathcal{M}}_1, E_1 F_1 \hat{\mathcal{M}}_1) = \dim \text{Hom}(\hat{\mathcal{M}}_1, \hat{\mathcal{M}}_1^\oplus) = 4,$$

$$\dim \text{Hom}(Q_0, Q_1) = \dim \text{Hom}(E_1 \mathcal{M}_0, E_0 \hat{\mathcal{M}}_1) = 2,$$

$$\dim \text{Hom}(Q_1, Q_0) = \dim \text{Hom}(E_0 \hat{\mathcal{M}}_1, E_1 \mathcal{M}_0) = 2.$$

Thus, we have

$$[Q_0] = 3[\mathcal{N}_0] + 2[\mathcal{N}_1], \quad [Q_1] = 2[\mathcal{N}_0] + 4[\mathcal{N}_1].$$

in the Grothendieck group $K_0(R^{\Lambda_0}(2\delta)\text{-mod}).$

**Proposition 5.5.** Suppose $\ell = 1$. 

(1) If $\lambda = 0$, then the radical series of $Q_0$ and $Q_1$ are given as follows:

$$Q_0 \simeq \begin{array}{c} N_0 \\ N_1 \\ N_0 \end{array}, \quad Q_1 \simeq \begin{array}{c} N_1 \\ N_0 \\ N_1 \\ N_0 \\ N_1 \end{array}.$$ 

(2) If $\lambda \neq 0$, then the radical series of $Q_0$ and $Q_1$ are given as follows:

$$Q_0 \simeq \begin{array}{c} N_0 \\ N_1 \\ N_0 \\ N_1 \\ N_0 \end{array}, \quad Q_1 \simeq \begin{array}{c} N_1 \\ N_0 \\ N_1 \end{array}.$$ 

**Proof.** Recall the anti-involution of $R^\Lambda_0(\beta)$ fixing all defining generators elementwise, which yields the duality $M \mapsto M^\vee = \text{Hom}_k(M, k)$ on the category of $R^\Lambda_0(\beta)$-modules. As $Q_i$ are indecomposable projective-injective modules, $Q_i^\vee$ is isomorphic to $Q_0$ or $Q_1$. Then, (5.3.1) implies that $Q_i$ are self-dual. Thus, $\text{Soc}(Q_i) \simeq \text{Top}(Q_i) \simeq N_i$ and the heart of $Q_i$ is self-dual.

Suppose $\lambda = 0$. Then $\text{Rad}(Q_0)/\text{Soc}(Q_0)$ has the quotient module

$$\mathcal{N}' = \begin{array}{c} N_1 \\ N_1 \end{array}$$

by Lemma 4.3. Since $[\text{Rad}(Q_0)/\text{Soc}(Q_0)] = [N_0] + 2[N_1]$, $N_0$ appears in $\text{Soc}(\text{Rad}(Q_0)/\text{Soc}(Q_0))$. Taking its dual, we know that $N_0$ appears in $\text{Top}(\text{Rad}(Q_0)/\text{Soc}(Q_0))$. Hence,

$$0 \to N_0 \to \text{Rad}(Q_0)/\text{Soc}(Q_0) \to \mathcal{N}' \to 0$$

splits, and we conclude that $Q_0$ is as in the assertion. Similarly, $\text{Rad}(Q_1)/\text{Soc}(Q_1)$ has the quotient module

$$\mathcal{N}'' = \begin{array}{c} N_1 \\ N_0 \end{array}$$

and its dual as its submodule. Then $\text{Ext}^1(N_1, N_i) \neq 0$, for $i = 0, 1$, implies that

$$\text{Top}(\text{Rad}(Q_1)/\text{Soc}(Q_1)) \simeq N_0 \oplus N_1.$$ 

Thus, it suffices to show that $\mathcal{N}''$ appears as a submodule. The definition of $Q_1$ implies

$$0 \to F_1 M_1 \to Q_1 \to F_1 M_1 \to 0$$

and $[F_1 M_1] = [N_0] + 2[N_1]$. In particular, we have $\text{Soc}(F_1 M_1) \simeq N_1$, $\text{Top}(F_1 M_1) \simeq N_1$, and $F_1 M_1$ is uniserial. Hence, $\text{Rad}(Q_1)/\text{Soc}(Q_1)$ has a submodule which is isomorphic to $\mathcal{N}''$. We conclude that $Q_1$ is as in the assertion.
Suppose that $\lambda \neq 0$. Lemma 4.7 implies that we have uniserial modules

$$\tilde{T}_1 = N_1 \quad \tilde{T}_1^\vee = N_0,$$

and epimorphisms $Q_0 \to \tilde{T}_1^\vee$ and $Q_1 \to \tilde{T}_1$. Therefore, $Q_0$ is as in the assertion and the assertion for $Q_1$ follows from (5.3.1) and the self-duality of $\text{Rad}(Q_1)/\text{Soc}(Q_1)$. □

**Proposition 5.6.** If $\ell = 1$, then $R^{\Lambda_0}(2\delta)$ is a symmetric algebra of tame representation type.

**Proof.** Suppose that $\lambda = 0$. By Proposition 5.5(1), the basic algebra of $R^{\Lambda_0}(2\delta)$ is

$$
\begin{array}{c}
\gamma \\
\alpha \\
\beta \\
1 \\
\delta
\end{array}
\quad
\begin{array}{c}
0 \\
\beta \\
\alpha \\
1 \\
\gamma
\end{array}
$$

with relations

$$\alpha \beta = \beta \gamma = \gamma \alpha = \delta^2 = 0, \quad \gamma^2 = \alpha \delta \beta, \quad \beta \alpha \delta = c \delta \beta \alpha,$$

for some $c \in k^\times$.

Here, we choose $\delta$ to be $Q_1 \to Q_1/F_1 M_1 \simeq F_1 M_1 \hookrightarrow Q_1$.

As $e(0101)N_1 \neq 0$ and $R^{\Lambda_0}(2\delta)e(0101)$ is a projective $R^{\Lambda_0}(2\delta)$-module, we have a surjective $R^{\Lambda_0}(2\delta)$-module homomorphism $R^{\Lambda_0}(2\delta)e(0101) \to Q_1$. Then,

$$\dim R^{\Lambda_0}(2\delta)e(0101) = 8 = \dim Q_1$$

implies that $R^{\Lambda_0}(2\delta)e(0101) \simeq Q_1$ and we have $\text{End}(Q_1) \simeq e(0101)R^{\Lambda_0}(2\delta)e(0101)$. Now we observe that Lemma 5.3(1) is valid for $\ell = 1$. In particular, $\text{End}(Q_1)$ is commutative. As the paths $\beta \alpha$ and $\delta$ can be regarded as elements in $\text{End}(Q_1)$, we have $c = 1$, i.e.,

$$\beta \alpha \delta = \delta \beta \alpha.$$

Thus, $R^{\Lambda_0}(2\delta)$ is a symmetric algebra. The number of outgoing arrows and the number of incoming arrows are 2 at each vertex, and $\alpha \beta = \beta \gamma = \gamma \alpha = \delta^2 = 0$ implies that $R^{\Lambda_0}(2\delta)$ is a special biserial algebra. Further, we may define a surjective algebra homomorphism from $\text{End}(Q_1)$ to $k[x, y]/(x^2, xy, y^2)$ by $\delta \mapsto x, \beta \alpha \mapsto y$. Thus, $R^{\Lambda_0}(2\delta)$ is tame if $\lambda = 0$.

Suppose that $\lambda \neq 0$. By Proposition 5.5(2), the basic algebra of $R^{\Lambda_0}(2\delta)$ is

$$
\begin{array}{c}
0 \\
\beta \\
\alpha \\
1 \\
\gamma
\end{array}
$$

with relations $\alpha \gamma = \gamma \beta = 0, \quad (\beta \alpha)^2 = \gamma^2$. It is a special biserial algebra. Indeed, it is the algebra given in [11, Prop.(A)]. Thus, it is a symmetric algebra of tame type. □
The reason we need extra task for $\ell = 1$ is that the two algebras $e(0101)R^{\Lambda_0}(2\delta)e(0101)$ and $e(01010)R^{\Lambda_0}(2\delta + \alpha_0)e(01010)$ are different, which we know from

$$\dim e(0101)R^{\Lambda_0}(2\delta)e(0101) = 4, \quad \dim e(01010)R^{\Lambda_0}(2\delta + \alpha_0)e(01010) = 8.$$ 

Thus, the previous argument used for showing wildness is not valid when we show the wildness of $R^{\Lambda_0}(3\delta)$ for $\ell = 1$. Hence, we argue as in the next proposition.

**Proposition 5.7.** If $\ell = 1$, then $R^{\Lambda_0}(3\delta)$ has wild representation type.

**Proof.** It follows from Theorem 2.5 that $\dim e(\nu, 1)R^{\Lambda_0}(2\delta + \alpha_0) = 0$ for any $\nu \in I^{2\delta}$. Here, $e(\nu, 1)$ is the idempotent corresponding to the concatenation of $\nu$ and (1). Thus, if we define

$$e = \sum_{\nu \in I^{2\delta}} e(\nu, 1)$$

then we have $E_1R^{\Lambda_0}(2\delta + \alpha_0) = eR^{\Lambda_0}(2\delta + \alpha_0) = 0$. Since $\langle h_1, \Lambda_0 - 2\delta - \alpha_0 \rangle = 2$, we have an bimodule isomorphism

$$E_1F_1R^{\Lambda_0}(2\delta + \alpha_0) \simeq F_1E_1R^{\Lambda_0}(2\delta + \alpha_0) \oplus R^{\Lambda_0}(2\delta + \alpha_0) \otimes_k k[t]/(t^2) \simeq R^{\Lambda_0}(2\delta + \alpha_0) \otimes_k k[t]/(t^2),$$

by [13, Theorem 5.2], and it induces an algebra isomorphism

$$eR^{\Lambda_0}(3\delta)e/\text{Rad}^2(eR^{\Lambda_0}(3\delta)e) \simeq R^{\Lambda_0}(2\delta + \alpha_0) \otimes_k k[t]/(t^2, t\text{Rad}(R^{\Lambda_0}(2\delta + \alpha_0)), \text{Rad}^2(R^{\Lambda_0}(2\delta + \alpha_0))).$$

Thus, if we denote the irreducible $k[t]/(t^2)$-module by $S$, then $eR^{\Lambda_0}(3\delta)e/\text{Rad}^2(eR^{\Lambda_0}(3\delta)e)$ has the irreducible modules $O_0 \otimes S$ and $O_1 \otimes S$, where $O_0$ and $O_1$ are irreducible $R^{\Lambda_0}(2\delta + \alpha_0)$-modules in Lemma 4.8. By Lemma 4.9, the projective cover of $O_1 \otimes S$ has the following radical series:

$$O_1 \otimes S \quad O_1 \otimes S \quad O_1 \otimes S \quad O_0 \otimes S.$$ 

It implies that the quiver of $eR^{\Lambda_0}(3\delta)e$ has

![Diagram](image.png)

as a proper subquiver. It follows from [9, I.10.8] that $eR^{\Lambda_0}(3\delta)e$ is wild, and so is $R^{\Lambda_0}(3\delta)$. \qed
5.4. **Representation type of** $R^{\Lambda_0}(\beta)$. In this section, we prove the Erdmann-Nakano type theorem for $R^{\Lambda_0}(\beta)$ with arbitrary parameter value $\lambda \in k$.

Let $A$ and $B$ be finite dimensional $k$-algebras. If there exists a constant $C > 0$ and functors $F : A\text{-mod} \to B\text{-mod}$, $G : B\text{-mod} \to A\text{-mod}$ such that, for any $A$-module $M$,

1. $M$ is a direct summand of $GF(M)$ as an $A$-module,
2. $\dim F(M) \leq C \dim M$,

then wildness of $A$ implies wildness of $B$ [11, Prop.2.3]. As a corollary, we have the following lemma.

**Lemma 5.8.** If $R^{\Lambda_0}(k\delta + \alpha_0 \cdots + \alpha_{i-1})$ is wild, so is $R^{\Lambda_0}(k\delta + \alpha_0 \cdots + \alpha_i)$.

**Proof.** For $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq \ell$, we have

$$\langle h_i, \Lambda_0 - k\delta - \alpha_0 \cdots - \alpha_{i-1} \rangle = \begin{cases} 1 & \text{if } 0 \leq i \leq \ell - 1, \\ 2 & \text{if } i = \ell. \end{cases}$$

Thus, the functor $F_i : R^{\Lambda_0}(k\delta + \alpha_0 \cdots + \alpha_{i-1})\text{-mod} \to R^{\Lambda_0}(k\delta + \alpha_0 \cdots + \alpha_i)\text{-mod}$ satisfies the assumptions (1) and (2) above. $\square$

Recall that a weight $\mu$ with $V(\Lambda_0)_\mu \neq 0$ can be written as

$$\mu = \kappa - k\delta$$

for some $\kappa \in \mathcal{W} \Lambda_0$ and $k \in \mathbb{Z}_{\geq 0}$ and a weight $\mu$ of the above form always satisfies $V(\Lambda_0)_\mu \neq 0$. Note that the pair $(\kappa, k)$ is determined uniquely by $\mu$. Then, the following Erdmann-Nakano type theorem follows from Corollary 5.2, Propositions 5.4, 5.6 and 5.7.

**Theorem 5.9.** For $\kappa \in \mathcal{W} \Lambda_0$ and $k \in \mathbb{Z}_{\geq 0}$, the finite quiver Hecke algebra $R^{\Lambda_0}(\Lambda_0 - \kappa + k\delta)$ of type $A^{(1)}_{\ell}$ ($\ell \geq 1$) is

1. simple if $k = 0$,
2. of finite representation type but not semisimple if $k = 1$,
3. of tame representation type if $\ell = 1$ and $k = 2$,
4. of wild representation type otherwise.

When $\ell = 1$ and $k = 2$, we may study $R^{\Lambda_0}(\Lambda_0 - \kappa + \delta)$ in more detail. When $\lambda \neq 0$, they are all biserial algebras. If they are special biserial algebras, we may classify them and they already appeared as tame block algebras of the Hecke algebras associated with the symmetric group. This is the topic of the next section.
6. Symmetric quiver Hecke algebras of tame type

We first classify two-point symmetric special biserial algebras up to Morita equivalence. For those who are not familiar with special biserial algebras, see [9, II].

**Theorem 6.1.** Suppose that a symmetric \( k \)-algebra \( A = kQ/I \), where \( Q \) is a connected quiver and \( I \) is an admissible ideal, has the following properties.

(a) \( Q \) has two vertices, which we denote \( Q_0 = \{0, 1\} \).

(b) \( A \) is a special biserial algebra.

Then, \( A \) is one of the following algebras.

1. \( Q = \begin{array}{cc}
\alpha & \\
\beta & 
\end{array} \), \( (\alpha\beta)^m\alpha = (\beta\alpha)^m\beta = 0 \).

2. \( Q = \begin{array}{cc}
\alpha & \\
\beta & 
\end{array} \) such that the relations are either (2a) or (2b).

   (2a) \( \beta\gamma = \gamma\alpha = 0, \quad \gamma^p = (\alpha\beta)^q \).

   (2b) \( \beta\alpha = \gamma^2 = 0, \quad (\gamma\alpha\beta)^m = (\alpha\beta\gamma)^m \).

3. \( Q = \begin{array}{cc}
\alpha,\alpha' & \\
\beta,\beta' & 
\end{array} \) such that the relations are either (3a) or (3b).

   (3a) \( \alpha\beta = \beta'\alpha = \alpha'\beta = \beta\alpha' = 0, \quad (\alpha\beta)^p = (\alpha'\beta')^q, \quad (\beta\alpha)^p = (\beta'\alpha')^q \).

   (3b) \( \alpha\beta = \beta\alpha = \alpha'\beta = \beta'\alpha' = 0, \quad (\alpha\beta\alpha'\beta')^m = (\alpha'\beta'\alpha\beta)^m, \quad (\beta\alpha'\beta')^m = (\beta'\alpha\beta)^m \).

4. \( Q = \begin{array}{cc}
\alpha & \\
\beta & 
\end{array} \) such that the relations are either (4a) or (4b) or (4c).

   (4a) \( \beta\gamma = \gamma\alpha = \alpha\delta = \delta\beta = 0, \quad (\alpha\beta)^p = \gamma^q, \quad (\beta\alpha)^p = \delta^q \).

   (4b) \( \beta\alpha = \gamma^2 = \alpha\delta = \delta\beta = 0, \quad (\gamma\alpha\beta)^p = (\alpha\beta\gamma)^p, \quad (\beta\gamma\alpha)^p = \delta^q \).

   (4c) \( \alpha\beta = \beta\alpha = \gamma^2 = \delta^2 = 0, \quad (\beta\gamma\alpha\delta)^m = (\delta\beta\gamma\alpha)^m, \quad (\gamma\alpha\delta\beta)^m = (\alpha\delta\beta\gamma)^m \).

**Remark 6.2.** There are algebras from (3a) that appear in a different context: the principal blocks of restricted Lie algebras [13].

When we asked her comment on our paper, Professor Erdmann informed us Donovan’s work [8]. In page 189 of the paper, he claimed classification of Brauer graph algebras and its
twisted forms with one or two irreducible modules, without the definition of twisted Brauer graph algebras and without the proof of his classification. Then his list is given in the same page. After excluding algebras which are not special biserial, there remain $A_{200}$ with $u = 2$, $A_{210}, A_{220}, A_{310}, A_{320}$. The first algebra is (2a) with $(p, q) = (2, 1)$ from our list. $A_{210}, A_{220}, A_{230}$ are (4a), (4c) and (4b) respectively. (But the last exponent $\nu$ in $A_{230}$ should be $\mu$.) If we allow $\nu = 1$ in $A_{210}$ and $A_{230}$, and eliminate a generator from the relations then we obtain (2a) and (2b). Note that the algebras from (2b) and (4b) are algebras of dihedral type: see [9, Thm.VI.8.1(i)] for (2b) and [9, Thm.VI.8.2(i)] for (4b). $A_{310}$ is (3a) with $p = q$, and $A_{320}$ is (3b). (Thus, last two $\mu$’s in $A_{320}$ seem to be $\nu$.) In any case, it is worth mentioning that our algebras are Brauer graph algebras. See [17] or [20] for general statement in this direction.

A Brauer graph is a finite graph which may have loops and multiple edges such that

(i) for each vertex, a natural number, the multiplicity of the vertex, is assigned,
(ii) for each vertex, a cyclic ordering of the edges connected to the vertex is specified.

In (ii), a loop is considered as two edges whose other ends are closed to the loop. Thus, a loop appears twice in the cyclic ordering. To define a Brauer graph algebra, we consider a quiver whose vertices and arrows are given by the edges of the Brauer graph and the arrows of the cyclic orderings, respectively. Note that if the number of arrows connected to a vertex is one then the cyclic ordering defines a loop of the quiver. The relations are defined as follows.

(1) If $\beta$ does not appear immediately after $\alpha$ in any of the cyclic orderings, then $\alpha\beta = 0$.
(2) For each edge of the Brauer graph, let $\alpha_1 \cdots \alpha_p$ and $\beta_1 \cdots \beta_q$ be two cyclic orderings starting at the edge and let $m, n$ be the multiplicities of the two ends of the edge. Then $(\alpha_1 \cdots \alpha_p)^m = (\beta_1 \cdots \beta_q)^n$.

In particular, if the number of edges connected to a vertex is 1 and the vertex has multiplicity 1 then we may eliminate the loop which the cyclic ordering defines from the defining relations.

(a) Let us consider a vertex with multiplicity $m$ and draw 4 open end edges which connects to the vertex in a clockwise manner, and declare that this is the cyclic ordering. If we close two adjacent edges to a loop, and the remaining two edges to a loop, we obtain a Brauer graph with one vertex of multiplicity $m$ and 2 loops. Its Brauer graph algebra is (4c). If we close two pairs of diagonal open end edges, we obtain another Brauer graph with one vertex of multiplicity $m$ and 2 loops. Its Brauer graph algebra is (3b).

(b) If we consider two vertices of multiplicity $p$ and $q$ and a loop on the vertex of multiplicity $p$, we obtain (4b) if $q \geq 2$ and (2b) if $q = 1$.

(c) If we consider two vertices of multiplicity $p$ and $q$ and connect the two vertices with two edges, we obtain (3a).
(d) If we consider three vertices of multiplicity \( p, q, r \) and connect the vertices of multiplicity \( p \) and \( q \), the vertices of multiplicity \( p \) and \( r \), then we obtain (4a) if \( r \geq 2 \) and (2a) if \( r = 1 \).

The proof of Theorem 6.1 will be given in subsequent sections. Note that the number of outgoing arrows and incoming arrows at each vertex is at most 4, so that the number in total is 8 because the quiver has two vertices. But each arrow is counted twice and we have that the number of arrows is at most 4. The following lemma is useful.

**Lemma 6.3.** Suppose that \( A = kQ/I \), where \( Q \) is a connected quiver and \( I \) is an admissible ideal, is a symmetric special biserial algebra. If \( u = u_1 \cdots u_r \neq 0 \), where \( u_1, \ldots, u_r \) are arrows, satisfies \( u\delta = 0 \) in \( A \), for any arrow \( \delta \), then the following hold.

1. \( \delta u = 0 \), for any arrow \( \delta \).
2. \( u \) is a loop or a cycle.
3. If \( u = u_1 \cdots u_r \neq 0 \) and \( v = v_1 \cdots v_s \neq 0 \) are such that
   - (i) \( u\delta = 0 \) and \( v\delta = 0 \), for any arrow \( \delta \),
   - (ii) \( u_1 = v_1 \),
   then \( r = s \) and \( u_i = v_i \), for all \( i \).
4. If \( u = u_1 \cdots u_r \neq 0 \) and \( v = v_1 \cdots v_s \neq 0 \) are such that
   - (i) \( u\delta = 0 \) and \( v\delta = 0 \), for any arrow \( \delta \),
   - (ii) \( u \) and \( v \) share their initial point.
   then \( u = cv \), for some \( c \in k^\times \).
5. Let \( Tr : A \to k \) be a non-degenerate trace map. If \( u = u_1 \cdots u_r \neq 0 \) is such that \( u\delta = 0 \), for any arrow \( \delta \), then \( Tr(u) \neq 0 \).

**Proof.** (1) follows from the fact that \( \text{Soc}(A) \) is the socle of the right and the left regular representations for self-injective algebras [5, Thm.(58.12)]. To see (2), let \( i \) be the initial point of \( u_1 \). Then \( u \) spans \( \text{Soc}(e_iA) \simeq \text{Top}(e_iA) \), which implies that \( we_i = u \neq 0 \) and \( we_j = 0 \), for \( j \neq i \). Hence, \( i \) is the endpoint of \( u_r \). To prove (3), observe that \( \{u_1, u_1u_2, \ldots, u\} \) span a uniserial \( A \)-submodule of \( \text{Rad}(e_iA) \). Thus, if \( u_1 = v_1 \) then \( \{v_1, v_1v_2, \ldots, v\} \) span the same uniserial \( A \)-submodule. To prove (4), observe that \( u, v \in \text{Soc}(e_iA) \), for the common initial point \( i \). Then \( \dim \text{Soc}(e_iA) = 1 \) implies the result. (5) is clear because if \( Tr(u) = 0 \) then \( Tr(ux) = 0 \), for all \( x \in A \), which contradicts the assumptions that the trace map is non-degenerate and \( u \neq 0 \).

**Definition 6.4.** If \( u = u_1 \cdots u_r \neq 0 \), where \( u_1, \ldots, u_r \) are arrows, satisfies \( u\delta = 0 \) in \( kQ/I \), for any arrow \( \delta \), we call \( u \) a **maximal path which extends \( u_1 \)**.

Note that if \( u \) is a maximal path, then \( \delta u = 0 \) in \( kQ/I \), for any arrow \( \delta \), by Lemma 6.3(1).
Because of Lemma 6.3(2), we may exclude the case of one arrow since the unique arrow must be a loop and the quiver $Q$ cannot be connected.

6.1. **The case of two arrows.** This case is easy and we omit the proof. We obtain a symmetric Nakayama algebra, which is the case (1) of Theorem 6.1. Note that it cannot be derived equivalent to $R^\Lambda_0(2\delta)$ with $\ell = 1$, since symmetric Nakayama algebras are of finite representation type.

6.2. **The case of three arrows.** If there is no loop, then we may assume that two arrows, say $\alpha$ and $\alpha'$, start at the vertex 0 and end at the vertex 1, and the other arrow, say $\beta$, starts at the vertex 1 and ends at the vertex 0. Since $A$ is special biserial, we may assume $\alpha' \beta = 0$ without loss of generality. Then, $\alpha'$ is a maximal path which is not a cycle, contradicting Lemma 6.3(2). On the other hand, if there are two loops, then we extend the remaining arrow, say $\beta$, to a maximal path $u = u_1 \cdots u_r$ with $u_1 = \alpha$. But $u$ cannot be a cycle, which contradicts Lemma 6.3(2) again. As $Q$ is connected, it does not have three loops. Thus, $Q$ is as in the case (2) of Theorem 6.1. We show that the relations are either (2a) or (2b).

Suppose that $\beta \alpha \neq 0$. Then $\beta \gamma = \gamma \alpha = 0$ since $A$ is special biserial. We extend $\beta$ to a maximal path. Then it is of the form $(\beta \alpha)^n$, for some $n \geq 1$. Next we extend $\gamma$ and $\alpha$. They are of the form $\gamma^p$ and $(\alpha \beta)^q$, for some $p,q \geq 1$, respectively. Then Lemma 6.3(4) implies that $\gamma^p = c(\alpha \beta)^q$, for some $c \in k^\times$. If $n > q$ then

$$(\beta \alpha)^n = \beta \cdot (\alpha \beta)^q \cdot \alpha(\beta \alpha)^{n-q-1} = 0$$

by the maximality of $(\alpha \beta)^q$, which contradicts $(\beta \alpha)^n \neq 0$. Similarly, $n < q$ leads to a contradiction and we have $n = q$. We may list basis elements of $A$ as follows.

$$\{e_0, \gamma, \gamma^2, \ldots, \gamma^{p-1}, \beta, \alpha \beta, \beta \alpha \beta, \ldots, (\alpha \beta)^q; e_1, \alpha, \beta \alpha, \alpha \beta \alpha, \ldots, (\alpha \beta)^q\}$$

We may check that the defining relations may be chosen as in (2a). We define its trace map by the values on the basis elements. Let the values be 0 except

$$\text{Tr}((\alpha \beta)^q) = \text{Tr}((\beta \alpha)^q) = 1.$$ 

Suppose that $\text{Tr}(xy) \neq 0$, for two basis elements $x$ and $y$. Then we may show that $(x, y)$ is one of the following.

$$((\alpha \beta)^q, e_0), ((\beta \alpha)^q, e_0),$$

$$((\alpha \beta)^i, \gamma^{p-i}),$$

$$((\alpha \beta)^j, (\alpha \beta)^{q-j-1}, (\beta, (\alpha \beta)^{q-1}), (\beta(\alpha \beta)^j, (\alpha \beta)^{q-j-1} \alpha),$$

$$((\beta \alpha)^j, (\beta \alpha)^{q-j}), (\alpha, \gamma^{p-1}), (\alpha \beta, (\beta \alpha)^{q-j}), (\alpha(\beta \alpha)^j, (\beta \alpha)^{q-j-1} \beta),$$

$$((\alpha \beta)^q, e_0), ((\beta \alpha)^q, e_1),$$
for $1 \leq i \leq p-1$ and $1 \leq j \leq q-1$. Thus the trace map is well-defined and non-degenerate. We have proved that $A$ is a symmetric algebra in case (2a).

Next suppose that $\beta\alpha = 0$. We extend $\beta$ to a maximal path. As the maximal path is a cycle, it has the form $(\beta\gamma\alpha)^n$, for some $n \geq 1$. It then follows that $\beta\gamma \neq 0$, $\gamma\alpha \neq 0$ and we have $\gamma^2 = 0$. If we extend $\alpha$ to a maximal path, it is either $\alpha \beta$ or $(\alpha\beta\gamma)^m$ or $(\alpha\beta\gamma)^m\alpha\beta$, for some $m \geq 1$. But our assumption is that the algebra $A$ is symmetric. Thus we have a non-degenerate trace map $\text{Tr} : A \to k$ and both

$$\text{Tr}(\alpha\beta) = \text{Tr}(\beta\alpha) = 0, \quad \text{Tr}((\alpha\beta\gamma)^m\alpha\beta) = \text{Tr}(\beta(\alpha\beta\gamma)^m\alpha) = 0$$

contradicts Lemma 6.3(5). Hence, $(\alpha\beta\gamma)^m$ is the maximal path which extends $\alpha$.

Moreover, if $m \geq n + 1$ then

$$(\alpha\beta\gamma)^m = \alpha \cdot (\beta\gamma\alpha)^n \cdot (\beta\gamma\alpha)^{m-n-1}\beta\gamma = 0,$$

which is a contradiction. If $m \leq n - 1$ then

$$(\beta\gamma\alpha)^n = \beta \cdot (\alpha\beta\gamma)^m \cdot (\alpha\beta\gamma)^{n-m-1}\alpha = 0,$$

which is a contradiction again. Thus, $m = n$ follows. Next we extend $\gamma$ to a maximal path. Then it is either $\gamma$ or $(\gamma\alpha\beta)^l$ or $(\gamma\alpha\beta)^l\gamma$, for some $l \geq 1$. But we may exclude $(\gamma\alpha\beta)^l\gamma$ by $\text{Tr}((\gamma\alpha\beta)^l\gamma) = \text{Tr}(\gamma(\gamma\alpha\beta)^l) = 0$. If $(\gamma\alpha\beta)^l$ is a maximal path then $l = n$ because

$$(\gamma\alpha\beta)^l = \gamma\alpha \cdot (\beta\gamma\alpha)^n \cdot (\beta\gamma\alpha)^{l-n-1}\beta = 0, \quad \text{if } l \geq n + 1,$$

$$(\beta\gamma\alpha)^n = \beta \cdot (\alpha\beta\gamma)^l \cdot (\alpha\beta\gamma)^{n-l-1}\gamma\alpha = 0. \quad \text{if } l \leq n - 1.$$

Thus, if we extend $\gamma$ to a maximal path, then it is either $\gamma$ or $(\gamma\alpha\beta)^n$.

Recall that maximal paths which extends $\alpha$ and $\gamma$ coincide up to a nonzero scalar multiple by Lemma 6.3(4). If $\gamma$ is a nonzero scalar multiple of $(\alpha\beta\gamma)^n$, then we have $\gamma = 0$, which contradicts $\gamma \neq 0$, $\gamma\alpha \neq 0$. Therefore, we may exclude $\gamma$ and $(\gamma\alpha\beta)^n$ is the maximal path which extends $\gamma$. Now we write $(\alpha\beta\gamma)^n = c(\gamma\alpha\beta)^n$, for $c \in k^\times$. Then

$$\text{Tr}((\alpha\beta\gamma)^n) = \text{Tr}((\gamma\alpha\beta)^n) \neq 0$$

by Lemma 6.3(5) and we deduce $c = 1$. As a result, the following elements form a basis of $A$ and we may choose (2b) as the defining relations:

$$e_0, \beta, \alpha\beta, \beta(\gamma\alpha\beta)^i, \alpha\beta(\gamma\alpha\beta)^i, (\gamma\alpha\beta)^i, \gamma, \beta\gamma, \gamma(\alpha\beta\gamma)^i, \beta\gamma(\alpha\beta\gamma)^i, (\alpha\beta\gamma)^i, (\gamma\alpha\beta)^n,$$
$$e_1, \alpha(\beta\gamma\alpha)^i, \gamma(\beta\gamma\alpha)^i, (\beta\gamma\alpha)^{i+1},$$
for $1 \leq i \leq n-1$ and $0 \leq j \leq n-1$. To show that this algebra is indeed a symmetric algebra, we define its trace map by the values on basis elements. Let the values be 0 except

$$\text{Tr}((\gamma \alpha \beta)^n) = \text{Tr}((\beta \gamma \alpha)^n) = 1.$$ 

Suppose that $\text{Tr}(xy) \neq 0$, for basis elements $x$ and $y$. If $x = e_0$ or $e_1$, then $y = (\gamma \alpha \beta)^n$ and $(\beta \gamma \alpha)^n$, respectively. Otherwise, $xy$ coincides with $(\alpha \beta \gamma)^n$ or $(\beta \gamma \alpha)^n$ or $(\gamma \alpha \beta)^n$ as words in alphabet $\{\alpha, \beta, \gamma\}$, by Lemma 6.3(3). Hence, it is easy to check that the Gram matrix is a non-singular symmetric block diagonal matrix whose block size is 1 or 2. Hence $A$ is symmetric in case (2b) as desired.

6.3. The case of four arrows. Suppose that $Q$ has no loop. Then, since $A$ is special biserial, we have two arrows, say $\alpha$ and $\alpha'$, which starts at the vertex 0 and ends at the vertex 1, and two arrows, say $\beta$ and $\beta'$, which starts at the vertex 1 and ends at the vertex 0. If $\alpha \beta = \alpha' \beta = \alpha \beta' = \alpha' \beta' = 0$ then we can not have a cycle which starts at the vertex 0, a contradiction. Thus, we may assume that $\alpha \beta \neq 0$ without loss of generality. Then, we have $\alpha \beta' = 0$ and $\alpha' \beta = 0$. We consider the cases $\beta \alpha \neq 0$ and $\alpha \beta = 0$ separately.

Suppose that $\beta \alpha \neq 0$. Then we have $\alpha \beta' = \beta' \alpha = \alpha' \beta = \beta \alpha' = 0$. We are going to show that we are in case (3a). If we extend $\alpha$ to a maximal path then it is of the form $(\alpha \beta)^p$, for some $p \geq 1$. If we extend $\alpha'$ to a maximal path then it is of the form $(\alpha' \beta')^q$, for some $q \geq 1$. $(\alpha \beta)^p$ is a nonzero scalar multiple of $(\alpha' \beta')^q$. Renormalizing $\alpha'$ or $\beta'$, we may assume that $(\alpha \beta)^p = (\alpha' \beta')^q$. Similarly, let $(\beta \alpha)^{p'}$ and $(\beta \alpha')^{q'}$ be maximal paths which extends $\beta$ and $\beta'$ respectively. Then we have $p' = p$ by

$$(\alpha \beta)^p = \alpha (\beta \alpha)^{p'} (\beta \alpha)^{p-p'-1} \beta, \quad \text{if} \ p' \leq p - 1,$$

$$(\beta \alpha)^{p'} = \beta (\alpha \beta)^{p}(\alpha \beta)^{p'-1} \alpha, \quad \text{if} \ p' \geq p + 1,$$

and $q = q'$ by the similar argument. As $\text{Tr}((\beta \alpha)^p) = \text{Tr}((\beta' \alpha')^{q'}) \neq 0$, we may conclude $(\beta \alpha)^p = (\beta' \alpha')^{q'}$ as well. Thus, we are in case (3a) and we may give a set of paths which form a basis of $A$. Defining $\text{Tr} : A \to k$ by $\text{Tr}(x) = 0$, for basis elements $x$, except for

$$\text{Tr}((\alpha \beta)^p) = \text{Tr}((\beta \alpha)^p) = \text{Tr}((\alpha' \beta')^q) = \text{Tr}((\beta' \alpha')^{q'}) = 1,$$

we may show that $A$ is a symmetric algebra in case (3a).

Next suppose that $\beta \alpha = 0$. If we extends $\beta$ to a maximal path, we deduce that $\beta \alpha' \neq 0$. Thus, we have $\alpha \beta' = \beta \alpha = \alpha' \beta = \beta' \alpha' = 0$. We are going to show that we are in (3b). We
extend four arrows to get maximal paths

\((αβα′)^a\) or \((αβα′)^{a-1}αβ\),
\((βα′β′α)^b\) or \((βα′β′α)^{b-1}βα′\),
\((α′β′αβ)^c\) or \((α′β′αβ)^{c-1}α′β′\),
\((β′αβα′)^d\) or \((β′αβα′)^{d-1}β′α\).

However, the trace values at the elements on the right hand side are 0, and they can not be maximal paths. We may also deduce \(a = b = c = d\) by the similar argument as above. Denote \(a = b = c = d\) by \(m\). Then we are in case (3b). We may give a set of path which is a basis of \(A\) and we may define \(Tr : A \to k\) by \(Tr(x) = 0\), for basis elements \(x\), except for

\(Tr((αβα′β′)^m) = Tr((βα′β′α)^m) = Tr((α′β′αβ)^m) = Tr((β′αβα′)^m) = 1\).

Then the Gram matrix is a non-singular symmetric block diagonal matrix whose block size is 1 or 2. Hence, \(A\) is a symmetric algebra in case (3b). No loop cases are classified.

Suppose that \(Q\) has one loop. We may assume that the loop is at the vertex 0. Then, the other three arrows connect the vertices 0 and 1. Hence, at the vertex 0, we have either two outgoing arrows which ends at the vertex 1 or two incoming arrows which starts at the vertex 1. If there are two outgoing arrows, say \(α\) and \(α′\), then \(αβ = 0\) or \(α′β = 0\) holds, for the remaining arrow \(β\), since \(A\) is special biserial. But then the maximal path which extends \(α\) or \(α′\) is not a cycle, a contradiction. If there are two incoming arrows, the similar argument leads to a contradiction. Thus, \(Q\) must have more than one loop.

Suppose that \(Q\) has three loops, then the remaining arrow, say \(α\), is the only arrow which connects the vertices 0 and 1. Then, a maximal path which extends \(α\) can not be a cycle.

As \(Q\) is connected, it can not have four loops. Hence, we are left with two loop cases.

The remaining part is for classifying two loop cases.

Since \(Q\) is connected, each vertex must have one loop. The remaining two arrows, which we denote \(α\) and \(β\), connect the vertices 0 and 1, and they must have opposite direction. Otherwise, a maximal path which extends \(α\) and \(β\) would not be a cycle. Hence the quiver \(Q\) is as in Theorem 6.1(4).

Suppose that \(αβ ≠ 0\) or \(βα ≠ 0\). We may assume \(αβ ≠ 0\) without loss of generality. Then \(αδ = 0\) and \(δβ = 0\) hold. We consider the cases \(βα ≠ 0\) and \(βα = 0\) separately. We are going to show that we are in case (4a) in the former case, and case (4b) in the latter case.

If \(βα ≠ 0\) then \(βγ = γα = αδ = δβ = 0\). Thus, a maximal path which extends \(α\) has the form \((αβ)^p\), for some \(p ≥ 1\), and a maximal path which extends \(γ\) has the form \(γ^q\), for some \(q ≥ 1\). Thus, we may assume \((αβ)^p = γ^q\). On the other hand, a maximal path which extends \(β\) has the form \((βα)^{p′}\), for some \(p′ ≥ 1\), and a maximal path which extends \(δ\) has the form
\(\delta^r,\) for some \(r \geq 1,\) and we may assume \((\beta\alpha)^p' = \delta^r.\) Then,

\[
(\alpha\beta)^p = \alpha \cdot (\beta\alpha)^p' \cdot (\beta\alpha)^{p-p'-1}\beta, \quad \text{if } p' \leq p - 1,
\]

\[
(\beta\alpha)^{p'} = \beta (\alpha\beta)^{p'-p-1} \cdot (\alpha\beta)^p \cdot \alpha, \quad \text{if } p' \geq p + 1,
\]

implies \(p' = p.\) Thus we are in case (4a). We may show that \(A\) is symmetric in this case.

If \(\beta\alpha = 0,\) then we extend \(\beta\) to a maximal path and obtain \(\beta\gamma \neq 0\) and \(\gamma\alpha \neq 0.\) Thus, we have \(\beta\alpha = \gamma^2 = \alpha\delta = \delta\beta = 0.\) The maximal path which extends \(\beta\) has the form \((\beta\gamma\alpha)^p,\) for some \(p \geq 1,\) and a maximal path which extends \(\delta\) has the form \(\delta^q,\) for some \(q \geq 1,\) and we may assume \((\beta\gamma\alpha)^p = \delta^q.\) We extend \(\alpha\) to a maximal path. It has the form \((\alpha\beta\gamma)^m,\) for some \(m \geq 1,\) and we may prove \(m = p.\) Similarly, We extend \(\gamma\) to a maximal path. It has the form \((\gamma\alpha\beta)^p,\) for some \(n \geq 1,\) and we may prove \(n = p.\) Since \((\alpha\beta\gamma)^p\) is a nonzero scalar multiple of \((\gamma\alpha\beta)^p\) and \(\text{Tr}((\alpha\beta\gamma)^p) = \text{Tr}((\gamma\alpha\beta)^p) \neq 0,\) \((\gamma\alpha\beta)^p = (\alpha\beta\gamma)^p\) holds, and we are in case (4b). We may show that \(A\) is symmetric in this case.

Finally, we consider the case \(\alpha\beta = \beta\alpha = 0.\) We extend \(\alpha\) to a maximal path and obtain \(\alpha\delta \neq 0\) and \(\delta\beta \neq 0.\) We extend \(\beta\) to a maximal path and obtain \(\beta\gamma \neq 0\) and \(\gamma\alpha \neq 0.\) Thus, we have \(\alpha\beta = \beta\alpha = \gamma^2 = \delta^2 = 0.\)

Our task is to show that we are in case (4c). Note that the possibilities for maximal paths which extend four arrows are

\[
(\alpha\delta\beta\gamma)^a \quad \text{or} \quad (\alpha\delta\beta\gamma)^{a-1}\alpha\delta\beta,
\]

\[
(\beta\gamma\alpha\delta)^b \quad \text{or} \quad (\beta\gamma\alpha\delta)^{b-1}\beta\gamma\alpha,
\]

\[
(\gamma\alpha\delta\beta)^c \quad \text{or} \quad (\gamma\alpha\delta\beta)^{c-1}\gamma,
\]

\[
(\delta\beta\gamma\alpha)^d \quad \text{or} \quad (\delta\beta\gamma\alpha)^{d-1}\delta.
\]

Then, we may exclude the elements on the right hand side because their trace values are 0. We have \(c = a\) by

\[
(\gamma\alpha\delta\beta)^c = \gamma \cdot (\alpha\delta\beta\gamma)^a \cdot (\alpha\delta\beta\gamma)^{c-a-1}\alpha\delta\beta, \quad \text{if } c \geq a + 1,
\]

\[
(\alpha\delta\beta\gamma)^a = \alpha\delta\beta \cdot (\gamma\alpha\delta\beta)^c \cdot (\gamma\alpha\delta\beta)^{a-c-1}\gamma, \quad \text{if } c \leq a - 1.
\]

It follows that \((\alpha\delta\beta\gamma)^a = (\gamma\alpha\delta\beta)^a.\) \((\delta\beta\gamma\alpha)^d\) is the maximal path which extends \(\delta,\) and it is easy to prove \(d = a.\) Similarly, \((\beta\gamma\alpha\delta)^b\) is the maximal path which extends \(\beta,\) and it is easy to prove \(b = a\) as well. It follows that \((\beta\gamma\alpha\delta)^a = (\delta\beta\gamma\alpha)^a,\) and if we denote \(a = b = c = d\) by \(m,\) we are in case (4c).

6.4. Classification. Not all the algebras in the list of Theorem 6.1 appear as a finite quiver Hecke algebra \(R^{\Lambda_0}(\Lambda_0 - \kappa + 2\delta),\) for \(\kappa \in \mathcal{W}\Lambda_0.\) In this section, we compute the center and the stable Auslander-Reiten quiver for those algebras from the list and give classification when
In the special case when we consider weight two blocks of the symmetric group or its Hecke algebra, Scopes’ equivalence shows that weight two blocks with 2-core \((k, k-1, \ldots, 1)\) with \(k \geq 1\) are Morita equivalent to each other, and we have only two Morita equivalent classes among them. We may generalize this fact. Namely, if \(\lambda \neq 0\) then any special biserial finite quiver Hecke algebras \(R^{\Lambda_0}(\beta)\) of tame type is Morita equivalent to either the Hecke algebra \(H_4(q)|_{q=-1}\) associated with the symmetric group \(S_4\), or the principal block of the Hecke algebra \(H_5(q)|_{q=-1}\) associated with the symmetric group \(S_5\). If \(\lambda = 0\) then a special biserial finite quiver Hecke algebra of tame type must be Morita equivalent to one of the algebras from case (4b) of Theorem 6.1 with \(q = 2\), but we do not pursue further to check if they actually occur as finite quiver Hecke algebras.

**Lemma 6.5.** Suppose that \(\ell = 1\).

1. If \(\lambda = 0\) then \(R^{\Lambda_0}(2\delta)\) is not of polynomial growth and its stable Auslander-Reiten quiver has
   - (i) the unique component of \(\mathbb{Z}A_{\infty}/\langle \tau^3 \rangle\),
   - (ii) infinitely many components of \(\mathbb{Z}A_{\infty}\),
   - (iii) infinitely many components of homogeneous tubes.
   Further, its center is 5 dimensional commutative local algebra.

2. If \(\lambda \neq 0\) then \(R^{\Lambda_0}(2\delta)\) is domestic and its stable Auslander-Reiten quiver has
   - (i) the unique component of \(\mathbb{Z}\tilde{A}_{2,2}\),
   - (ii) two components of \(\mathbb{Z}A_{\infty}/\langle \tau^2 \rangle\),
   - (iii) infinitely many components of homogeneous tubes.
   Further, its center is 5 dimensional commutative local algebra.

**Proof.** (1) We follow the argument in [4, Prop.5.6]. Let \(A\) be the basic algebra of \(R^{\Lambda_0}(2\delta)\). Recall that if \(\lambda = 0\) then the quiver presentation of \(A\) is

\[
\begin{array}{ccc}
\gamma & 0 & 1 \\
\alpha & \beta & \delta
\end{array}
\]

with relations \(\alpha \beta = \beta \gamma = \gamma \alpha = \delta^2 = 0\), \(\gamma^2 = \alpha \delta \beta\), \(\beta \alpha \delta = \delta \beta \alpha\). Let \(a = \alpha \delta^{-1} \beta \gamma^{-1}\) and \(b = \alpha \delta^{-1} \beta\). Then, for each prime \(q\),

\[
\{x_1 x_2 \cdots x_q \mid x_i = a \text{ or } b\} \setminus \{a^q, b^q\}
\]

defines \((2^q - 2)/q\) equivalence classes of bands, for the string algebra \(A/\text{Rad}(A)\). Hence, it suffices to compute the \(\tau\)-orbits of string modules \(Ae_1/A\alpha\), \(Ae_0/A\beta\), \(Ae_0/A\gamma\) and \(Ae_1/A\delta\), in order to know the components of the stable Auslander-Reiten quiver.
of $A$, by the general result [12 Thm.2.2]. Then, explicit computation shows
\[
\tau(Ae_1/A\alpha) = Ae_0/A\beta, \quad \tau(Ae_0/A\beta) = Ae_0/A\gamma, \quad \tau(Ae_0/A\gamma) = Ae_1/A\alpha,
\]
\[
\tau(Ae_1/A\delta) = Ae_1/A\delta,
\]
where $Ae_0 = \text{Span}\{e_0, \beta, \gamma, \delta\}$ and $Ae_1 = \text{Span}\{e_1, \alpha, \beta, \alpha\delta, \beta\alpha\delta\}$.

The elements that commute with $e_0$ and $e_1$ are
\[
e_0Ae_0 \oplus e_1Ae_1 = \text{Span}\{e_0, \gamma, \gamma^2, e_1, \delta, \beta\alpha, \beta\alpha\delta\}.
\]

It is clear that $\text{Soc}(A) = \text{Span}\{\gamma^2, \beta\alpha\delta\}$ is contained in the center. As the center is a local algebra, it suffices to find central elements in $\text{Span}\{\gamma, \delta, \beta\alpha\}$. Then, $\gamma$ and $\beta\alpha$ are central and $\delta$ is not central. We have $Z(A) = \text{Span}\{1, \gamma, \beta\alpha, \gamma^2, \beta\alpha\delta\}$.

(2) Let $A$ be the basic algebra of $R^{\Lambda_0}(2\delta)$ as above and recall that if $\lambda \neq 0$ then the quiver presentation of $A$ is

\[
0 \xrightarrow{\alpha} 1 \xrightarrow{\beta} \gamma
\]

with relations $\alpha\gamma = \gamma \beta = 0, (\beta\alpha)^2 = \gamma^2$. Then $\beta\alpha\gamma^{-1}$ is the unique band, for the string algebra $A/\text{Rad}(A)$. Hence, [12 Thm.2.1] tells that there are positive integers $m, p, q$ such that the stable Auslander-Reiten quiver has $m$ components of tubes $ZA_\infty/\langle \tau^p \rangle$, $m$ components of tubes $ZA_\infty/\langle \tau^q \rangle$, $m$ components of $ZA_{p,q}$, and the remaining components are infinitely many homogeneous tubes. Hence, it suffices to compute the $\tau$-orbits of string modules $Ae_1/A\alpha$, $Ae_0/A\beta$, $Ae_1/A\gamma$, and $Ae_0/\text{Soc}(Ae_0)$ because $\text{Rad}(Ae_0)/\text{Soc}(Ae_0)$ is indecomposable. $Ae_0/A\beta$ is a simple $A$-module and explicit computation shows
\[
\tau(Ae_1/A\alpha) = Ae_0/A\beta, \quad \tau(Ae_0/A\beta) = Ae_1/A\alpha.
\]

On the other hand, $\tau(Ae_1/A\gamma) = Ae_0/\text{Soc}(Ae_0)$ by explicit computation, and the almost split sequence
\[
0 \to \text{Rad}(Ae_0) \to Ae_0 \oplus \text{Rad}(Ae_0)/\text{Soc}(Ae_0) \to Ae_0/\text{Soc}(Ae_0) \to 0
\]
shows that $\tau(Ae_0/\text{Soc}(Ae_0)) = \text{Rad}(Ae_0) = Ae_1/A\gamma$. Thus, we may conclude that
\[
m = 1 \text{ and } p = q = 2.
\]

The computation of the center is similar to (1) and we obtain $Z(A) = \text{Span}\{1, \alpha\beta + \beta\alpha, (\alpha\beta)^2, \gamma, \gamma^2\}$. \hfill \Box

**Theorem 6.6.** Any special biserial finite quiver Hecke algebra $R^{\Lambda_0}(\Lambda_0 - \kappa + 2\delta)$ in affine type $A_1^{(1)}$ is Morita equivalent to

(1) one of the algebras from Theorem 6.1(4b) with $q = 2$ if $\lambda = 0$.  

(2) the algebra from Theorem 6.1(2a) with \(p = q = 2\), or (4a) with \(p = 1\) and \(q = r = 2\) if \(\lambda \neq 0\).

**Proof.** We compute the stable Auslander-Reiten quivers for the algebras from Theorem 6.1.

Suppose that we are in case (2a) and denote the algebra by \(A\). The relations are

\[
\beta \gamma = \gamma \alpha = 0, \quad \gamma^p = (\alpha \beta)^q.
\]

Then, as \(A\) is symmetric, we may compute \(\tau(Ae_1/A\alpha)\) by computing the kernel of \(Ae_0 \to Ae_1\) given by right multiplication by \(\alpha\).

\[
\begin{array}{ccc}
\gamma & \beta & \alpha \\
\gamma^2 & \alpha \beta & \xrightarrow{\alpha} \beta \alpha \\
\vdots & \vdots & \vdots \\
(\alpha \beta)^q & & (\beta \alpha)^q
\end{array}
\]

We may compute \(\tau(Ae_0/A\beta)\) and \(\tau(Ae_0/A\gamma)\) in the similar way. We obtain

\[
\tau(Ae_1/A\alpha) \simeq \text{Ker}(Ae_0 \xrightarrow{\alpha} Ae_1) \simeq Ae_0/A\beta,
\]

\[
\tau(Ae_0/A\beta) \simeq \text{Ker}(Ae_1 \xrightarrow{\beta} Ae_0) \simeq Ae_1/A\alpha,
\]

\[
\tau(Ae_0/A\gamma) \simeq \text{Ker}(Ae_0 \xrightarrow{\gamma} Ae_0) \simeq Ae_1/\text{Soc}(Ae_1),
\]

and \(\tau(Ae_1/\text{Soc}(Ae_1)) \simeq Ae_0/A\gamma\). Thus, if it is the basic algebra of a finite quiver Hecke algebra of tame representation type, we must have \(\lambda \neq 0\) by Lemma 6.5 and \(A\) must be domestic. However, if \(p \geq 3\) then we may use \(a = \alpha \beta \gamma^{-1}\) and \(b = \alpha \beta \gamma^{-2}\) to show that \(A\) is not of polynomial growth. Similarly, if \(q \geq 3\) then we may use \(a = \alpha \beta \gamma^{-1}\) and \(b = (\alpha \beta)^2 \gamma^{-1}\) to show that \(A\) is not of polynomial growth. Thus, we have either \((p, q) = (2, 1)\) or \((2, 2)\). But if \((p, q) = (2, 1)\) then \(A\) is a Brauer tree algebra because

\[
\begin{array}{c}
S_0 \\
Ae_0 \simeq S_0 \oplus S_1 \\
S_0 \\
S_1
\end{array} \quad \begin{array}{c}
S_1 \\
Ae_1 \simeq S_0 \\
S_1
\end{array}
\]

Hence, we may exclude this case. The case \(p = q = 2\) is nothing but the basic algebra of \(R^{\Lambda_0}(2\delta)\), for \(\lambda \neq 0\). We conclude that this is the only possibility in case (2a). Note that it is Morita equivalent to the Hecke algebra \(H_4(q)|_{q = -1}\) associated with the symmetric group \(S_4\).

Suppose that we are in case (2b). The relations are

\[
\beta \alpha = \gamma^2 = 0, \quad (\gamma \alpha \beta)^m = (\alpha \beta \gamma)^m.
\]
Then we consider the following map $Ae_0 \to Ae_1$, for computing $\tau(Ae_1/A\alpha)$,

$$e_0 \begin{array}{l} \beta \\ \alpha \beta \\ \gamma \alpha \\ \gamma \alpha \beta \\ \alpha \beta \gamma \\ : \\ (\alpha \beta \gamma)^m \end{array} \xrightarrow{\alpha} e_1 \begin{array}{l} \alpha \\ \beta \gamma \alpha \\ \beta \alpha \beta \\ \beta \gamma \alpha \beta \\ \beta \gamma \alpha \beta \gamma \\ : \\ (\beta \gamma \alpha)^m \end{array}$$

and consider the similar maps for other arrows to obtain

$$\tau(Ae_1/A\alpha) \simeq Ae_1/Soc(Ae_1), \quad \tau(Ae_1/Soc(Ae_1)) \simeq \text{Rad}(Ae_1) \simeq Ae_0/A\beta,$$

$$\tau(Ae_0/A\beta) \simeq Ae_1/A\alpha, \quad \tau(Ae_0/A\gamma) \simeq Ae_0/A\gamma.$$ 

As there is a tube of period 3, if $A$ is the basic algebra of a finite quiver Hecke algebra of tame representation type, we must have $\lambda = 0$ by Lemma 6.5 and $A$ is not of polynomial growth. Hence its center is 5 dimensional. However, we have

$$Z(A) = \text{Span}\{1, \alpha \beta (\gamma \alpha \beta)^{m-1}, (\alpha \beta \gamma)^m, (\beta \gamma \alpha)^m\}$$

by explicit computation as in (1), and we conclude that case (2b) can not occur.

Suppose that we are in case (3a). The relations are

$$\alpha \beta' = \beta' \alpha = \alpha' \beta = \beta \alpha' = 0, \quad (\alpha \beta)^p = (\alpha' \beta')^q, \quad (\beta \alpha)^p = (\beta' \alpha')^q.$$ 

Then, we have

$$\tau(Ae_1/A\alpha) \simeq Ae_1/A\alpha, \quad \tau(Ae_1/A\alpha') \simeq Ae_1/A\alpha',$$

$$\tau(Ae_0/A\beta) \simeq Ae_0/A\beta, \quad \tau(Ae_0/A\beta') \simeq Ae_0/A\beta'.$$

There is no tube of period greater than 1, and case (3a) can not occur by Lemma 6.5.

Suppose that we are in case (3b). The relations are

$$\alpha \beta' = \beta \alpha = \alpha' \beta = \beta' \alpha' = 0, \quad (\alpha \beta \alpha' \beta')^m = (\alpha' \beta' \alpha \beta)^m, \quad (\beta \alpha \beta' \alpha)^m = (\beta' \alpha \beta \alpha')^m.$$ 

Then, the radical series of $Ae_0$ and $Ae_1$ are as follows.

$$e_0 \begin{array}{l} \beta \\ \alpha \beta \\ \beta' \alpha \beta \\ : \\ \beta' \alpha \beta \alpha \end{array} \xrightarrow{\alpha} e_1 \begin{array}{l} \alpha \\ \beta' \alpha \\ \beta' \alpha \beta' \alpha \\ : \\ \beta' \alpha \beta \alpha \beta' \alpha \\ \beta' \alpha \beta' \alpha \beta \alpha' \end{array}$$
It follows that
\[
\tau(Ae_1/A\alpha) \simeq Ae_1/A\alpha', \quad \tau(Ae_1/A\alpha') \simeq Ae_1/A\alpha, \\
\tau(Ae_0/A\beta) \simeq Ae_0/A\beta', \quad \tau(Ae_0/A\beta') \simeq Ae_0/A\beta.
\]
Thus, Lemma 6.5 implies that we must have \( \lambda \neq 0 \) and \( A \) is domestic. However, we may use \( a = \alpha\beta(\alpha'\beta')^{-1} \) and \( b = \beta'\alpha(\beta\alpha')^{-1} \) to show that \( A \) is not of polynomial growth. We conclude that case (3b) cannot occur.

Suppose that we are in case (4a). The relations are
\[
\beta\gamma = \gamma\alpha = \alpha\delta = \delta\beta = 0, \quad (\alpha\beta)^p = \gamma^q, \quad (\beta\alpha)^p = \delta^r.
\]
As before, we compute that the Auslander-Reiten translate swaps \( Ae_1/A\alpha \) and \( Ae_0/A\beta \), \( Ae_0/A\gamma \) and \( Ae_1/A\delta \), respectively. Thus, Lemma 6.5 implies that we must have \( \lambda \neq 0 \) and \( A \) is domestic. If \( q \geq 3 \) then we may use \( a = \alpha\beta\gamma^{-1} \) and \( b = \alpha\beta\gamma^{-2} \) to show that \( A \) is not of polynomial growth. If \( r \geq 3 \) then we may use \( a = \beta\alpha\delta^{-1} \) and \( b = \beta\alpha\delta^{-2} \) to show that \( A \) is not of polynomial growth. Thus, \( q = r = 2 \) follows. Similarly, if \( p \geq 3 \) then we may use \( a = \alpha\beta\gamma^{-1} \) and \( b = (\alpha\beta)^2\gamma^{-1} \) to show that \( A \) is not of polynomial growth. Hence, we have either \( p = 1 \) or \( p = 2 \). If \( p = 2 \) and \( q = r = 2 \) then
\[
Ae_0 = \text{Span}\{e_0, \beta, \alpha\beta, \beta\alpha\beta, \gamma, \gamma^2\}, \quad Ae_1 = \text{Span}\{e_1, \alpha, \beta\alpha, \alpha\beta\alpha, \delta, \delta^2\},
\]
and the center is \( Z(A) = \text{Span}\{1, \alpha\beta + \beta\alpha, \gamma, \gamma^2, \delta, \delta^2\} \), which is not 5 dimensional. We conclude that \( p = 1 \) and \( q = r = 2 \) is the only possibility in case (4a). Note that it is Morita equivalent to the principal block of the Hecke algebra \( H_5(q)|_{q=-1} \) associated with the symmetric group \( S_5 \).

Suppose that we are in case (4b). The relations are
\[
\beta\alpha = \gamma^2 = \alpha\delta = \delta\beta = 0, \quad (\gamma\alpha\beta)^p = (\alpha\beta\gamma)^p, \quad (\beta\gamma\alpha)^p = \delta^q.
\]
Then, the radical series of \( Ae_0 \) and \( Ae_1 \) are as follows.

\[
\begin{array}{cccc}
\beta & \gamma & \alpha & \delta \\
\alpha\beta & \beta\gamma & \gamma\alpha & \delta^2 \\
\gamma\alpha\beta & \alpha\beta\gamma & \beta\gamma\alpha & \delta^3 \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

It follows that
\[
\tau(Ae_1/A\alpha) \simeq Ae_1/A\alpha, \quad \tau(Ae_1/A\alpha') \simeq Ae_1/A\alpha', \\
\tau(Ae_0/A\beta) \simeq Ae_1/A\alpha, \quad \tau(Ae_0/A\beta') \simeq Ae_1/A\alpha, \\
\tau(Ae_0/A\gamma) \simeq Ae_1/A\alpha, \quad \tau(Ae_0/A\gamma') \simeq Ae_1/A\alpha.
\]
Thus, Lemma 6.5 implies that we must have $\lambda = 0$. Further, we have

$$Z(A) = \text{Span}\{1, \alpha \beta (\gamma \alpha \beta)^{p-1}, (\gamma \alpha \beta)^p, \delta, \delta^2, \ldots, \delta^q\},$$

which forces $q = 2$. Here, we do not pursue further to determine which $p$ actually occur as a finite quiver Hecke algebra. We have computed the first Hochshild cohomology group and the answer is $\dim HH^1(A) = q$, which does not determine $p$. But higher Hochshild cohomology groups might be helpful: use the method in [10] to compute them.

Finally, we suppose that we are in case (4c). The relations are

$$\alpha \beta = \beta \alpha = \gamma^2 = \delta^2 = 0, \ (\beta \gamma \alpha \delta)^m = (\delta \beta \gamma \alpha)^m, \ (\gamma \alpha \delta \beta)^m = (\alpha \delta \beta \gamma)^m.$$  

Then, the radical series of $Ae_0$ and $Ae_1$ are as follows.

|   | $e_0$ |   | $e_1$ |
|---|-------|---|-------|
|  $\beta$ | $\gamma$ | $\alpha$ | $\delta$ |
|  $\delta \beta$ | $\beta \gamma$ | $\gamma \alpha$ | $\alpha \delta$ |
|  $\alpha \delta \beta$ | $\delta \beta \gamma$ | $\beta \gamma \alpha$ | $\gamma \alpha \delta$ |
|   |   |   |   |

It follows that

$$\tau(Ae_1/A\alpha) \simeq Ae_1/A\alpha, \ \tau(Ae_0/A\beta) \simeq Ae_0/A\beta,$$

$$\tau(Ae_0/A\gamma) \simeq Ae_0/A\gamma, \ \tau(Ae_1/A\delta) \simeq Ae_1/A\delta.$$  

There is no tube of period greater than 1, and case (4c) can not occur by Lemma 6.5. □

6.5. **Tame finite quiver Hecke algebras.** In the previous subsection, we classified special biserial finite quiver Hecke algebras. In this subsection, we show that if $\lambda \neq 0$ then any tame finite quiver Hecke algebras are biserial algebras. As finite quiver Hecke algebras of tame type is derived equivalent to $R^{\Lambda_0}(2\delta)$, they are stably equivalent to the special biserial algebra given by the quiver

$$\begin{array}{c}
0 \xrightarrow{\beta} 1 \\
\alpha \end{array}$$

with relations $\alpha \gamma = \gamma \beta = 0, \ (\beta \alpha)^2 = \gamma^2$. We denote the algebra by $A$. Let $S_0 = ke_0$ and $S_1 = ke_1$ be the irreducible $A$-modules and $P_0, P_1$ their projective covers, respectively. For a string $C$, we denote the corresponding string module by $M(C)$. See [4, Def.5.2].

In the terminology of the appendix, the stable equivalence to $R^{\Lambda_0}(2\delta)$ defines a maximal system of orthogonal stable bricks for $A$. Hence, its classification allows us to determine the quivers of all tame finite quiver Hecke algebras for $\lambda \neq 0$.  

Lemma 6.7. A system of orthogonal stable bricks for $A$ is one of the following pairs of string modules.

1. $X_0 = S_0, X_1 = S_1$.
2. $X_0 = M(\beta), X_1 = M(\alpha\beta\alpha)$.
3. $X_0 = M(\beta\alpha\beta), X_1 = M(\alpha)$. 
4. $X_0 = M(\beta\alpha\beta), X_1 = M(\alpha\beta\alpha)$.

Proof. Maximal paths in the quiver are $\alpha\beta, \beta\alpha\beta, \gamma$. Suppose that $\gamma$ appears in the string of $X_i$. Then, as $\gamma$ is a maximal path, $S_1$ appears in both $\text{Top}(X_i)$ and $\text{Soc}(X_i)$ and we have $r \in \text{Rad}(\text{End}(X_i))$ defined by

$$r : X_i \to S_1 \to X_i.$$ 

If it factors through a projective module, it factors through $P_1$. Let $f : X_i \to P_1, p_1 : P_1 \to S_1$ be such that $p : X_i \to \text{Im}(r) \cong S_1$ is $fp_1$. Then, there exists $g : P_1 \to X_i$ with $gp = p_1$ so that $p_1 = gfp_1$. It follows that $1 - gf \in \text{Rad}(\text{End}(P_i))$ is nilpotent and $gf$ is invertible. Then $P_i$ is a direct summand of $X_i$, a contradiction. Thus, $\text{End}(X_i)$ contains $1$ and $\mathbf{0}$, which are linearly independent. Since $\text{End}(X_i) = k$, this is impossible. Hence the string does not contain $\gamma$. It implies that the string cannot contain a substring of the form $uv^{-1}, u^{-1}v$, for arrows $u, v$. Hence, the string is one of

$$e_0, e_1, \alpha, \beta, \alpha\beta, \beta\alpha, \alpha\beta\alpha, \beta\alpha\beta.$$ 

We may delete the possibility of $\alpha\beta$ and $\beta\alpha$ because $S_0$ or $S_1$ appears in both $\text{Top}(X_i)$ and $\text{Soc}(X_i)$. On the other hand, $r \in \text{End}(M(\alpha\beta\alpha))$ with $\text{Im}(r) = \text{Rad}^2(M(\alpha\beta\alpha))$ factors through $P_1$ and $\text{End}(M(\alpha\beta\alpha)) = k$. Similarly, we have $\text{End}(M(\beta\alpha\beta)) = k$.

Suppose that $X_0 = S_0$. If $X_1 = S_1$, it satisfies $\text{Hom}(X_0, X_1) = \text{Hom}(X_1, X_0) = 0$, which gives case (1). If $X_1 = M(\beta)$ or $M(\beta\alpha\beta)$, we may find a nonzero element in $\text{Hom}(X_1, X_0)$. If $X_1 = M(\alpha)$ or $M(\alpha\beta\alpha)$, we also have $\text{Hom}(X_0, X_1) \neq 0$. Suppose that $X_1 = S_1$ and we check the possibilities for

$$X_0 = M(\alpha), M(\beta), M(\alpha\beta\alpha), M(\beta\alpha\beta).$$

However, we have $\text{Hom}(X_1, X_0) \neq 0$ for $X_0 = M(\beta)$ or $M(\beta\alpha\beta)$, and $\text{Hom}(X_0, X_1) \neq 0$ for $X_0 = M(\alpha)$ or $M(\alpha\beta\alpha)$. Next suppose that $X_0 = M(\beta)$ and check the possibilities for $X_1 = M(\alpha), M(\alpha\beta\alpha), M(\beta\alpha\beta)$. We have $\text{Hom}(X_0, X_1) \neq 0$ for $M(\alpha)$ and $M(\beta\alpha\beta)$ on the one hand, $X_1 = M(\alpha\beta\alpha)$ gives case (2). If $X_1 = M(\alpha)$, $\text{Hom}(X_1, X_0) \neq 0$ for $X_0 = M(\alpha\beta\alpha)$, while $X_0 = M(\beta\alpha\beta)$ gives case (3). Finally, we see that $\text{Hom}(X_0, X_1) = \text{Hom}(X_1, X_0) = 0$ for the remaining case (4). \qed
The case (4) from Lemma \ref{lem:6.7} does not occur: the projective resolutions for $X_0$ and $X_1$ are

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_0 \rightarrow X_0 \rightarrow 0$$
$$\cdots \rightarrow P_0 \rightarrow P_1 \rightarrow P_1 \rightarrow X_1 \rightarrow 0$$

and we have $\text{Ext}^1(X_i, X_i) = 0$, for $i = 0, 1$, which implies that the finite quiver Hecke algebra is of finite type and not of tame type.

Similarly, the case (3) does not occur: $0 \rightarrow M(\gamma^{-1} \beta \alpha) \rightarrow P_1 \rightarrow X_1 \rightarrow 0$ gives

$$\text{Hom}(P_1, X_0) \rightarrow \text{Hom}(M(\gamma^{-1} \beta \alpha), X_0) \rightarrow \text{Ext}^1(X_1, X_0) \rightarrow 0.$$ 

Then $\text{Hom}(M(\gamma^{-1} \beta \alpha), X_0) = k$ is given by $M(\gamma^{-1} \beta \alpha) \rightarrow \text{Soc}(X_0)$ and it is the image of $P_1 \rightarrow \text{Rad}(X_0)$. Thus, we have $\text{Ext}^1(X_1, X_0) = 0$. On the other hand, direct computation using the projective resolution of $X_0$ shows that $\text{Ext}^1(X_0, X_1) = k$. However, as irreducible modules for finite quiver Hecke algebras are self-dual with respect to the anti-involution fixing the generators, we must have $\text{Ext}^1(X_0, X_1) \simeq \text{Ext}^1(X_1, X_0)$.

\begin{lemma}
\textbf{Lemma 6.8.} Let $A$ be as above, and we consider maximal systems of orthogonal stable bricks corresponding to tame finite quiver Hecke algebras. Then one of the following holds.

1. $X_0 \simeq S_0, X_1 \simeq S_1$ and
   \[ \text{Ext}^1(X_0, X_0) = 0, \quad \text{Ext}^1(X_1, X_1) = k, \quad \text{Ext}^1(X_0, X_1) = \text{Ext}^1(X_1, X_0) = k. \]

2. $X_0 \simeq M(\beta), X_1 \simeq M(\alpha \beta \alpha)$ and
   \[ \text{Ext}^1(X_0, X_0) = k, \quad \text{Ext}^1(X_1, X_1) = 0, \quad \text{Ext}^1(X_0, X_1) = \text{Ext}^1(X_1, X_0) = k. \]

\textit{Proof.} (1) is clear. (2) follows from direct computation using resolutions of $X_0$ and $X_1$. \hfill \square
\end{lemma}

\begin{proposition}
\textbf{Proposition 6.9.} Let $A$ be as above. Suppose that the pair $X_0 = M(\alpha), X_1 = M(\beta \alpha \beta)$ is a maximal s.o.s.b. and let $M_0$ and $M_1$ be the corresponding s-projective $A$-modules. Then

1. $M_0 \simeq M(\beta \alpha \gamma^{-1})$ and $M(\beta) \oplus M(\alpha \beta \alpha \gamma^{-1}) \rightarrow M_0$ is minimal right almost split.

2. $M_1 \simeq S_0$ and $M(\beta^{-1} \gamma) \rightarrow M_1$ is minimal right almost split.

\textit{Proof.} As $M_i \simeq \tau^{-1} \Omega(X_i)$ by Proposition \ref{prop:7.13} we use the combinatorial rule to give almost split sequences for string modules to obtain the result. \hfill \square
\end{proposition}

\begin{proposition}
\textbf{Proposition 6.10.} Let $W$ be the Weyl group of type $A_1^{(1)}$. If $\lambda \neq 0$ then tame finite quiver Hecke algebras $R^{\lambda_0}(\Lambda_0 - w\Lambda_0 + 2\delta)$, for $w \in W$, are biserial.

\textit{Proof.} We check the conditions (a) (b) (c) from Proposition \ref{prop:7.8} and use Corollary \ref{cor:7.6}. By Proposition \ref{prop:7.12} we may check them by using a maximal system of orthogonal stable bricks $\{X_0, X_1\}$. If $\{X_0, X_1\}$ is $\{S_0, S_1\}$, they are clearly satisfied. Thus, we may assume that

The case (4) from Lemma \ref{lem:6.7} does not occur: the projective resolutions for $X_0$ and $X_1$ are

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_0 \rightarrow X_0 \rightarrow 0$$
$$\cdots \rightarrow P_0 \rightarrow P_1 \rightarrow P_1 \rightarrow X_1 \rightarrow 0$$

and we have $\text{Ext}^1(X_i, X_i) = 0$, for $i = 0, 1$, which implies that the finite quiver Hecke algebra is of finite type and not of tame type.

Similarly, the case (3) does not occur: $0 \rightarrow M(\gamma^{-1} \beta \alpha) \rightarrow P_1 \rightarrow X_1 \rightarrow 0$ gives

$$\text{Hom}(P_1, X_0) \rightarrow \text{Hom}(M(\gamma^{-1} \beta \alpha), X_0) \rightarrow \text{Ext}^1(X_1, X_0) \rightarrow 0.$$ 

Then $\text{Hom}(M(\gamma^{-1} \beta \alpha), X_0) = k$ is given by $M(\gamma^{-1} \beta \alpha) \rightarrow \text{Soc}(X_0)$ and it is the image of $P_1 \rightarrow \text{Rad}(X_0)$. Thus, we have $\text{Ext}^1(X_1, X_0) = 0$. On the other hand, direct computation using the projective resolution of $X_0$ shows that $\text{Ext}^1(X_0, X_1) = k$. However, as irreducible modules for finite quiver Hecke algebras are self-dual with respect to the anti-involution fixing the generators, we must have $\text{Ext}^1(X_0, X_1) \simeq \text{Ext}^1(X_1, X_0)$.

\begin{lemma}
\textbf{Lemma 6.8.} Let $A$ be as above, and we consider maximal systems of orthogonal stable bricks corresponding to tame finite quiver Hecke algebras. Then one of the following holds.

1. $X_0 \simeq S_0, X_1 \simeq S_1$ and
   \[ \text{Ext}^1(X_0, X_0) = 0, \quad \text{Ext}^1(X_1, X_1) = k, \quad \text{Ext}^1(X_0, X_1) = \text{Ext}^1(X_1, X_0) = k. \]

2. $X_0 \simeq M(\beta), X_1 \simeq M(\alpha \beta \alpha)$ and
   \[ \text{Ext}^1(X_0, X_0) = k, \quad \text{Ext}^1(X_1, X_1) = 0, \quad \text{Ext}^1(X_0, X_1) = \text{Ext}^1(X_1, X_0) = k. \]

\textit{Proof.} (1) is clear. (2) follows from direct computation using resolutions of $X_0$ and $X_1$. \hfill \square
\end{lemma}

\begin{proposition}
\textbf{Proposition 6.9.} Let $A$ be as above. Suppose that the pair $X_0 = M(\alpha), X_1 = M(\beta \alpha \beta)$ is a maximal s.o.s.b. and let $M_0$ and $M_1$ be the corresponding s-projective $A$-modules. Then

1. $M_0 \simeq M(\beta \alpha \gamma^{-1})$ and $M(\beta) \oplus M(\alpha \beta \alpha \gamma^{-1}) \rightarrow M_0$ is minimal right almost split.

2. $M_1 \simeq S_0$ and $M(\beta^{-1} \gamma) \rightarrow M_1$ is minimal right almost split.

\textit{Proof.} As $M_i \simeq \tau^{-1} \Omega(X_i)$ by Proposition \ref{prop:7.13} we use the combinatorial rule to give almost split sequences for string modules to obtain the result. \hfill \square
\end{proposition}

\begin{proposition}
\textbf{Proposition 6.10.} Let $W$ be the Weyl group of type $A_1^{(1)}$. If $\lambda \neq 0$ then tame finite quiver Hecke algebras $R^{\lambda_0}(\Lambda_0 - w\Lambda_0 + 2\delta)$, for $w \in W$, are biserial.

\textit{Proof.} We check the conditions (a) (b) (c) from Proposition \ref{prop:7.8} and use Corollary \ref{cor:7.6}. By Proposition \ref{prop:7.12} we may check them by using a maximal system of orthogonal stable bricks $\{X_0, X_1\}$. If $\{X_0, X_1\}$ is $\{S_0, S_1\}$, they are clearly satisfied. Thus, we may assume that
$X_0 = M(\beta), X_1 = M(\alpha \beta \alpha)$. First, the following (i) (ii) (iii) shows that the condition (a) holds.

(i) If $N = M(\beta)$ then
\[
\text{Hom}(N, X_0) = \text{Hom}(X_0, N) = k, \quad \text{Hom}(N, X_1) = \text{Hom}(X_1, N) = 0.
\]

(ii) If $N = M(\alpha \beta \alpha \gamma^{-1})$ then
\[
\text{Hom}(N, X_0) = \text{Hom}(X_0, N) = 0, \quad \text{Hom}(N, X_1) = \text{Hom}(X_1, N) = k.
\]

(iii) If $N = M(\beta^{-1} \gamma)$ then
\[
\text{Hom}(N, X_0) = \text{Hom}(X_0, N) = k, \quad \text{Hom}(N, X_1) = \text{Hom}(X_1, N) = 0.
\]

We set $Y_1 = M(\beta), Y_2 = M(\alpha \beta \alpha \gamma^{-1})$ and denote $w_1 : Y_1 \hookrightarrow M_0$ and $w_2 : Y_2 \twoheadrightarrow M_0$. Let $p$ be either $M_0 = M(\alpha \beta \alpha \gamma^{-1}) \twoheadrightarrow M(\beta)$ or $M(\beta^{-1} \gamma)$. We have to show that $w_1 p$ or $w_2 p$ factors through a projective module. By inspection, we see that $w_1 p = 0$. Hence the condition (b) holds. To prove the condition (c), we set $Y_0 = M(\beta)$ and $Y_1 = M(\beta^{-1} \gamma)$, because the pair is the unique pair which satisfies $\text{Hom}(Y_1, X_i) = k$ and $\text{Hom}(Y_2, X_i) = k$, for some $i$. We choose $p_1 : M_0 \rightarrow Y_i$ in such a way that its composition with $Y_i \rightarrow X_0$ is nonzero in $\text{Hom}(M_0, X_0)$. We have to show that $w p_0$ or $w p_1$ factors through a projective module, for $w : M(\beta) \hookrightarrow M_0$ and $w : M(\alpha \beta \alpha \gamma^{-1}) \twoheadrightarrow M_0$. If $w : M(\beta) \hookrightarrow M_0$ then $w p_1 = 0$. If $w : M(\alpha \beta \alpha \gamma^{-1}) \twoheadrightarrow M_0$ then $w p_0$ factors through $P_0$, proving the condition (c). □

7. Appendix

This section is for explaining some results from [19]. As the main results in [19] are incorrect and the proofs for many parts in the paper are left to the reader, we explain the proofs of necessary materials.

7.1. Stable biserial algebras. We start by introducing stably biserial algebras. Our goal is Proposition 7.8.

**Definition 7.1.** Let $Q$ be a quiver, $I$ an admissible ideal of $kQ$. The algebra $A = kQ/I$ is called stably biserial if the following conditions are satisfied.

(a) $A$ is a self-injective $k$-algebra. In particular, the socle of the right regular representation and the left regular representation coincide, which we denote by $\text{Soc}(A)$.

(b) For each vertex $i \in Q_0$, the number of outgoing arrows and the number of incoming arrows are less than or equal to 2.

(c) For each arrow $\alpha \in Q_1$, there is at most one arrow $\beta$ that satisfies $\alpha \beta \not\in \alpha \text{Rad}(A) \alpha + \text{Soc}(A)$. 


(d) For each arrow $\alpha \in Q_1$, there is at most one arrow $\beta$ that satisfies 
\[ \beta \alpha \notin \beta \text{Rad}(A) \alpha + \text{Soc}(A). \]

The following is clear.

**Lemma 7.2.** Suppose that $A = kQ/I$ is stably biserial. Then the following hold.

1. If arrows $\alpha, \beta, \gamma$ satisfy $\alpha \beta \neq 0, \alpha \gamma \neq 0, \beta \neq \gamma$ then either 
   \[ \alpha \beta \in \alpha \text{Rad}(A) \beta + \text{Soc}(A) \quad \text{or} \quad \alpha \gamma \in \alpha \text{Rad}(A) \gamma + \text{Soc}(A). \]

2. If arrows $\alpha, \beta, \gamma$ satisfy $\beta \alpha \neq 0, \gamma \alpha \neq 0, \beta \neq \gamma$ then either 
   \[ \beta \alpha \in \beta \text{Rad}(A) \alpha + \text{Soc}(A) \quad \text{or} \quad \gamma \alpha \in \gamma \text{Rad}(A) \alpha + \text{Soc}(A). \]

**Lemma 7.3.** Suppose that $A = kQ/I$ is stably biserial. If two arrows $\beta$ and $\gamma$ start from the endpoint of an arrow $\alpha$ such that

(a) $\alpha \beta \notin \text{Soc}(A)$,
(b) there is $r \in \text{Rad}(A)$ such that $\alpha (1 - r) \beta \in \text{Soc}(A)$,

then the following hold.

1. We have $\alpha (1 - r') \gamma \notin \text{Soc}(A)$, for all $r' \in \text{Rad}(A)$.
2. $\alpha \gamma \notin \text{Soc}(A)$.

**Proof.** (1) The assumption (a) implies that there exists an arrow $\delta$ such that $\alpha \beta \delta \neq 0$. On the other hand, (b) implies $\alpha \beta \delta = \alpha r \beta \delta$, and we may assume that $r$ is a linear combination of loops and cycles that starts and ends at the endpoint of $\alpha$. As the number of outgoing arrows is at most two, we may write
\[ r = \beta a_1 + \gamma a_2 \quad (a_1, a_2 \in A). \]

Suppose that $\alpha (1 - r') \gamma \in \text{Soc}(A)$, for some $r' \in \text{Rad}(A)$. Then $\alpha \gamma \delta = \alpha r' \gamma \delta$ and
\[ r' = \beta a'_1 + \gamma a'_2 \quad (a'_1, a'_2 \in A). \]
Now, the following repeated use of $r = \beta a_1 + \gamma a_2$ and $r' = \beta a'_1 + \gamma a'_2$ makes the length of paths that appear on the right hand side longer and longer, so that we may conclude that $\alpha \beta \delta = 0$, which is a contradiction.

\[ \alpha \beta \delta = \alpha r \beta \delta \]
\[ = \alpha \beta a_1 \beta \delta + \alpha \gamma a_2 \beta \delta \]
\[ = \alpha r \beta a_1 \beta \delta + \alpha r' \gamma a_2 \beta \delta \]
\[ = \ldots \]

Hence, $\alpha (1 - r') \gamma \notin \text{Soc}(A)$, for all $r' \in \text{Rad}(A)$. 

(2) We assume that $\alpha \gamma \in \text{Soc}(A)$. Then $\alpha \gamma \text{Rad}(A) = 0$ and the similar argument shows

$$\alpha(1 - r)\gamma \delta = -\alpha r \gamma \delta = -\alpha (\beta a_1) \gamma \delta$$

$$= -\alpha r \beta a_1 \gamma \delta = -\alpha (\beta a_1) \beta a_1 \gamma \delta$$

$$= -\alpha r \beta a_1 \beta a_1 \gamma \delta = \cdots$$

$$= 0.$$ Thus, we have $\alpha(1 - r)\gamma \in \text{Soc}(A)$. But (1) says that $\alpha(1 - r)\gamma \notin \text{Soc}(A)$ and we conclude that $\alpha \gamma \notin \text{Soc}(A)$. □

We may prove the following lemma by the same proof.

**Lemma 7.4.** Suppose that $A = kQ/I$ is stably biserial. If two arrows $\beta$ and $\gamma$ end at the initial point of an arrow $\alpha$ such that

(a) $\beta \alpha \notin \text{Soc}(A)$,

(b) there is $r \in \text{Rad}(A)$ such that $\beta(1 - r)\alpha \in \text{Soc}(A)$,

then the following hold.

(1) We have $\gamma(1 - r')\alpha \notin \text{Soc}(A)$, for all $r' \in \text{Rad}(A)$.

(2) $\gamma \alpha \notin \text{Soc}(A)$.

**Proposition 7.5.** If $A = kQ/I$ is stably biserial then we may choose the presentation of $A$ in such a way that the following (1) and (2) hold.

(1) If $\alpha \beta \neq 0, \alpha \gamma \neq 0, \beta \neq \gamma$, for arrows $\alpha, \beta, \gamma$, then either $\alpha \beta \in \text{Soc}(A)$ or $\alpha \gamma \in \text{Soc}(A)$.

(2) If $\beta \alpha \neq 0, \gamma \alpha \neq 0, \beta \neq \gamma$, for arrows $\alpha, \beta, \gamma$, then either $\beta \alpha \in \text{Soc}(A)$ or $\gamma \alpha \in \text{Soc}(A)$.

**Proof.** Suppose that arrows $\alpha, \beta, \gamma$ are such that $\beta \neq \gamma, \alpha \beta \notin \text{Soc}(A), \alpha \gamma \notin \text{Soc}(A)$. Then, Lemma 7.2(1) shows that there is $r \in \text{Rad}(A)$ such that $\alpha(1 - r)\beta$ or $\alpha(1 - r)\gamma$ belongs to $\text{Soc}(A)$. As the argument is the same, we assume that $\alpha(1 - r)\beta \in \text{Soc}(A)$. We may also assume that $r$ is a linear combination of loops and cycles which start at the endpoint of $\alpha$. Thus, if $i$ is the initial point of $\alpha$ and $j$ is the endpoint of $\alpha$, then we have

$$e_k \alpha(1 - r) = \delta_{ik} \alpha(1 - r), \quad \alpha(1 - r)e_k = \delta_{jk} \alpha(1 - r).$$

It implies that we have a well-defined algebra homomorphism $p : kQ \to A$ defined by $\eta \mapsto \eta$, for arrows $\eta \neq \alpha$, and $\alpha \mapsto \alpha(1 - r)$. Let $I' = \text{Ker}(p)$ and $A' = kQ/I'$. If arrows $\rho, \kappa, \eta$ are such that $\rho \neq \alpha, \rho \kappa \in \text{Soc}(A)$ or $\rho \eta \in \text{Soc}(A)$ then $\rho \kappa \in \text{Soc}(A')$ or $\rho \eta \in \text{Soc}(A')$ holds because $\text{Soc}(A)(1 - r) = \text{Soc}(A)$. In this way, we may decrease

$$\sharp\{((\alpha, \beta, \gamma) \mid \alpha \beta, \alpha \gamma \notin \text{Soc}(A), \beta \neq \gamma\} + \sharp\{((\alpha, \beta, \gamma) \mid \beta \alpha, \gamma \alpha \notin \text{Soc}(A), \beta \neq \gamma\}$$

to zero. □
Corollary 7.6. Suppose that $A$ has an anti-involution which makes irreducible modules self-dual. If $A$ is stably biserial then $A$ is biserial and $A/\operatorname{Soc}(A)$ is special biserial.

Proof. $A/\operatorname{Soc}(A)$ is special biserial by Proposition 7.5. Let $P$ be an indecomposable projective $A$-module. Since $\operatorname{Rad}(P)/\operatorname{Soc}(P)$ is union of two uniserial submodules, so is $\operatorname{Rad}(P)$ and the assumption implies that $A$ is biserial.

In [19, Thm.2.6], the author asserts that stably biserial algebras are special biserial. As we show in the next example, this assertion fails even for symmetric two-point stably biserial algebras which has an anti-involution making two irreducible modules self-dual. We thank the referee for pointing out the failure of [19, Thm.2.6] by providing us with the local algebra with two loops $\alpha, \beta$ obeying the relations $\alpha^2 = (\alpha\beta)^2 = (\beta\alpha)^2$ and $\beta^2 = 0$.

Example 7.7. We consider the quiver $Q = \begin{array}{c} \gamma \\
\alpha \downarrow \\
\downarrow \beta \end{array}$ with relations $\gamma^2 = \gamma\alpha\beta = \alpha\beta\gamma$, $\beta\gamma\alpha = \beta\alpha$, $\alpha\beta\alpha = \beta\alpha\beta = 0$. It has the following basis. $\{e_1, e_2, \alpha, \beta, \gamma, \gamma^2, \gamma\alpha, \beta\gamma\}$ Then, it is stably biserial but not special biserial. Defining the trace map by $\operatorname{Tr}(x) = 1$ for $x \in \{e_1, e_2, \alpha\beta, \beta\alpha, \gamma^2\}$, $\operatorname{Tr}(x) = 0$ for $x \in \{\alpha, \beta, \gamma, \gamma\alpha, \beta\gamma\}$, we know that it is symmetric. It has the anti-involution which fixes $e_1, e_2, \gamma$ elementwise and swaps $\alpha$ and $\beta$. The anti-involution makes two irreducible modules $ke_1$ and $ke_2$ self-dual.

The following proposition is one of the key observations by Pogorzaly in [19], and we have used this result to show that tame finite quiver Hecke algebras in affine type A, for parameter values $\lambda \neq 0$, are biserial algebras.

Proposition 7.8. If a self-injective algebra $B$ satisfies the following three conditions, then $B$ is Morita equivalent to a stably biserial algebra.

(a) For each indecomposable projective module $P$, we have $\operatorname{Rad}(P)/\operatorname{Soc}(P) = X' \oplus X''$, where $X' \neq 0$, such that $\operatorname{Top}(X'), \operatorname{Top}(X''), \operatorname{Soc}(X'), \operatorname{Soc}(X'')$ are simple modules.
(b) Let $X = X'$ or $X''$, and let $Q$ be the projective cover of $X$. Then $X$ is non-projective and we denote $p : Q/\operatorname{Soc}(Q) \to X$. Suppose that $\operatorname{Rad}(Q)/\operatorname{Soc}(Q) = Y_1 \oplus Y_2$, where $Y_1$ and $Y_2$ are indecomposable modules. Then, for irreducible homomorphisms $w_1 : Y_1 \to Q/\operatorname{Soc}(Q), w_2 : Y_2 \to Q/\operatorname{Soc}(Q)$,
w_1 p or w_2 p factors through a projective module.

(c) Let X = X’ or X”, and let Y_1 and Y_2 be an indecomposable direct summand of Rad(Q_1)/Soc(Q_1) and Rad(Q_2)/Soc(Q_2), for indecomposable projective modules Q_1 and Q_2, respectively. Suppose that both Y_1 and Y_2 have P as their projective covers and we denote p_i : P/Soc(P) → Y_i, for i = 1, 2. Then, for an irreducible homomorphism w : X → P/Soc(P), w p_1 or w p_2 factors through a projective module.

Proof. For each vertex i ∈ Q_0, (a) implies that the number of incoming arrows is
\[ \dim \text{Soc}^2(I_i)/\text{Soc}(I_i) = \dim \text{Soc}(\text{Rad}(I_i)/\text{Soc}(I_i)) \leq 2. \]
Similarly, the number of outgoing arrows is
\[ \dim \text{Rad}(P_i)/\text{Rad}^2(P_i) = \dim \text{Top}(\text{Rad}(P_i)/\text{Soc}(P_i)) \leq 2. \]
For each arrow α ∈ Q_1, we have to show that there is at most one arrow β such that
\[ \alpha \beta \notin \alpha \text{Rad}(B) \beta + \text{Soc}(B). \]
We shall prove that if α β_1 ≠ 0 and α β_2 ≠ 0 in B/Soc(B), then either α β_1 ∈ α Rad(B/Soc(B)) β_1 or α β_2 ∈ α Rad(B/Soc(B)) β_2.
Let i and j be the initial point and the endpoint of α, respectively, and let P_i and P_j be the corresponding indecomposable projective modules. We write
\[ X' \oplus X'' = \text{Rad}(P_j)/\text{Soc}(P_j) \]
as in (a). Then, we have α = p w, for p : P_i/\text{Soc}(P_i) → X, where X = X’ or X”, and w : X ↦ P_j/\text{Soc}(P_j). Note that w is an irreducible homomorphism as it is a direct summand of the right minimal almost split homomorphism Rad(P_j)/Soc(P_j) ⊕ P_j → P_j/\text{Soc}(P_j).
Similarly, we may write β_1 and β_2 as
\[ \beta_1 : P_j/\text{Soc}(P_j) \xrightarrow{p_1} Y_1 \hookrightarrow Q_1/\text{Soc}(Q_1), \]
\[ \beta_2 : P_j/\text{Soc}(P_j) \xrightarrow{p_2} Y_2 \hookrightarrow Q_2/\text{Soc}(Q_2), \]
where Q_i is indecomposable projective and Y_i is an indecomposable direct summand of Rad(Q_i)/Soc(Q_i), for i = 1, 2. Then, by (c), we may assume that w p_1 factors through a projective-injective B-module, say Q. Thus, we have q : X → Q and r : Q → Y_1 such that qr = wp_1. Then we have q = wt and r = sp_1 as follows.

\[ \begin{array}{c}
X \xrightarrow{w} P_j/\text{Soc}(P_j) \xrightarrow{p_1} Y_1 \\
\downarrow q \hspace{1cm} \exists \hspace{1cm} \exists \hspace{1cm} \downarrow r \\
Q \\
\end{array} \]
If $ts$ was an isomorphism, then $P_j/Soc(P_j)$ would be a direct summand of $Q$, a contradiction. Thus, $ts \in \text{RadEnd}_B(P_j/Soc(P_j))$, and it follows that $\alpha \beta_1 \in \alpha \text{Rad}(B/Soc(B))\beta_1$.

To show that there is at most one arrow $\beta$ such that $\beta \alpha \not\in \beta \text{Rad}(B)\alpha + \text{Soc}(B)$, we prove that if $\beta_1 \alpha \neq 0$ and $\beta_2 \alpha \neq 0$ in $B/Soc(B)$, then either $\beta_1 \alpha \in \beta_1 \text{Rad}(B/Soc(B))\alpha$ or $\beta_2 \alpha \in \beta_2 \text{Rad}(B/Soc(B))\alpha$. Let

$$\alpha : P_i/Soc(P_i) \xrightarrow{p} X \hookrightarrow P_j/Soc(P_j)$$

as before, and we write $\beta_1$ and $\beta_2$ as

$$\beta_1 : Q_1/Soc(Q_1) \rightarrow Y_1 \hookrightarrow P_i/Soc(P_i),$$

$$\beta_2 : Q_2/Soc(Q_2) \rightarrow Y_2 \hookrightarrow P_i/Soc(P_i),$$

where $Q_1$ and $Q_2$ are indecomposable projective modules, and $\text{Rad}(P_i)/\text{Soc}(P_i) = Y_1 \oplus Y_2$.

We denote $w_1 : Y_1 \hookrightarrow P_i/Soc(P_i)$ and $w_2 : Y_2 \hookrightarrow P_i/Soc(P_i)$. Then, they are irreducible homomorphisms, and by (b), we may assume that $w_1p$ factors through a projective-injective module, say $Q$ again. Thus,

$$\xymatrix{ Y_1 \ar[rr]^{w_1} \ar[dr]_{\exists t} & & P_i/Soc(P_i) \ar[rr]^{p} \ar[dr]_{\exists s} & & X \\
 & Q & & & \text{Y_1} \oplus \text{Y_2} & \text{Y_1} \oplus \text{Y_2} }$$

and $ts \in \text{RadEnd}_B(P_i/Soc(P_i))$. It follows that $\beta_1 \alpha \in \beta_1 \text{Rad}(B/Soc(B))\alpha$. \hfill \Box

7.2. **Notions for stable module categories.** We introduce several useful notions for stable module categories of self-injective algebras. They are, systems of orthogonal stable bricks, $s$-top and $s$-socle of non-projective modules, $s$-projective and $s$-injective modules.

**Definition 7.9.** Let $A$ be a self-injective algebra. A collection of indecomposable $A$-modules $\mathcal{M} = \{M_i\}_{i \in I}$ is a system of orthogonal stable bricks, or s.o.s.b. for short, if $\tau(M_i) \not\cong M_i$, for all $i$, and the following holds in the stable category of $A$-modules.

$$\text{Hom}_A(M_i, M_j) = \begin{cases} k & (i = j) \\ 0 & (i \neq j) \end{cases}$$

We call the cardinality of $I$ the stable rank of the s.o.s.b. $\mathcal{M} = \{M_i\}_{i \in I}$, and we denote it by $\text{rank}(\mathcal{M})$. A s.o.s.b. $\mathcal{M}$ is called a maximal s.o.s.b. if we have

$$\text{Hom}_A(\oplus_{i \in I} M_i, N) \neq 0 \quad \text{and} \quad \text{Hom}_A(N, \oplus_{i \in I} M_i) \neq 0,$$

for all indecomposable non-projective $A$-modules $N$ such that $\tau(N) \not\cong N$. 

Let $\mathcal{M} = \{M_i\}_{i \in I}$ be a maximal s.o.s.b.. If $\text{Hom}_A(N, \oplus_{i \in I} M_i) = k$, there is a unique $i \in I$ such that $\text{Hom}_A(N, M_i) \neq 0$. If this is the case, we write $s\text{-Top}(N) = M_i$. Similarly, if $\text{Hom}_A(\oplus_{i \in I} M_i, N) = k$, there is a unique $i \in I$ such that $\text{Hom}_A(M_i, N) \neq 0$. If this is the case, we write $s\text{-Soc}(N) = M_i$.

**Definition 7.10.** Let $\mathcal{M} = \{M_i\}_{i \in I}$ be a maximal s.o.s.b.. An indecomposable non-projective $A$-module $N$ is $s$-projective with respect to $\mathcal{M}$ if

(i) $\tau(N) \neq N$.
(ii) $\text{Hom}_A(N, \oplus_{i \in I} M_i) = k$.
(iii) For any indecomposable non-projective $A$-module $X$ and $0 \neq f \in \text{Hom}_A(X, s\text{-Top}(N))$, there exists $0 \neq g \in \text{Hom}_A(N, X)$ such that $gf \neq 0$.

Dually, an indecomposable non-projective $A$-module $N$ is $s$-injective with respect to $\mathcal{M}$ if

(i) $\tau(N) \neq N$.
(ii) $\text{Hom}_A(\oplus_{i \in I} M_i, N) = k$.
(iii) For any indecomposable non-projective $A$-module $X$ and $0 \neq f \in \text{Hom}_A(s\text{-Soc}(N), X)$, there exists $0 \neq g \in \text{Hom}_A(X, N)$ such that $fg \neq 0$.

**Proposition 7.11.** Let $B$ be an indecomposable self-injective algebra which is not a local Nakayama algebra. Then, we have the following.

1. If $P$ is indecomposable projective, then $\tau(P/\text{Soc}(P)) \neq P/\text{Soc}(P)$.
2. If $S$ is irreducible, then $S$ is non-projective and $\tau(S) \neq S$.

**Proof.** (1) If $\tau(P/\text{Soc}(P)) \simeq P/\text{Soc}(P)$ then the almost split sequence

$$0 \to \text{Rad}(P) \to \text{Rad}(P)/\text{Soc}(P) \oplus P \to P/\text{Soc}(P) \to 0$$

tells $P/\text{Soc}(P) \simeq \text{Rad}(P)$ and we have a surjective homomorphism $P \to \text{Rad}(P)$. Hence we have surjective homomorphisms $\text{Rad}^i(P) \to \text{Rad}^{i+1}(P)$, for $i \geq 1$. As a result, $P$ is uniserial and all of the composition factors are isomorphic to $S = \text{Top}(P)$. If there is another indecomposable projective module $Q$, then the indecomposability of $B$ implies that we have a uniserial module of length two with composition factors $S$ and $T = \text{Top}(Q)$. But then it is either a submodule of $P$ or a quotient module of $P$, which contradicts $[P : T] = 0$. Therefore, $B$ is a local Nakayama algebra, which we have excluded in the assumption.

(2) If $S$ was projective, $B$ would be a local Nakayama algebra. Thus, $S$ is non-projective. Suppose that $\tau(S) \simeq S$. We set $X_0 = S$. Then we have an almost split sequence

$$0 \to X_0 \to X_1 \to X_0 \to 0.$$ 

$X_1$ is indecomposable as it is uniserial. If $X_1$ was projective, then $B$ would be a local Nakayama algebra. Thus, $X_1$ is non-projective. Let $X_1 \to M_1$ be a left minimal almost split
homomorphism. Then, the irreducible homomorphism \( X_1 \to X_0 \) is a direct summand and we may write \( M_1 = X_0 \oplus X_2 \), for some module \( X_2 \). Thus, \( X_0 \to \tau^{-1}(X_1) \) is an irreducible homomorphism and it is a direct summand of the left minimal almost split homomorphism \( X_0 \to X_1 \). We conclude that \( \tau^{-1}(X_1) \simeq X_1 \) and we have the following almost split sequence

\[
0 \to X_1 \to X_0 \oplus X_2 \to X_1 \to 0.
\]

Suppose that we have \( B \)-modules \( X_0, \ldots, X_{i+1} \) such that

(i) \( \text{Top}(X_k) \simeq S \), for \( 0 \leq k \leq i \).

(ii) We have almost split sequences

\[
0 \to X_k \to X_{k-1} \oplus X_{k+1} \to X_k \to 0,
\]

for \( 0 \leq k \leq i \), where we understand \( X_{-1} = 0 \).

(iii) All the composition factors of \( X_k \) are \( S \) and \( [X_k : S] = k + 1 \), for \( 0 \leq k \leq i \).

Note that (i)-(iii) hold if \( i = 1 \). We show that (i)-(iii) imply \( \text{Top}(X_{i+1}) \simeq S \). Consider

\[
0 \to \text{Hom}_B(X_i, S) \to \text{Hom}_B(X_{i-1}, S) \oplus \text{Hom}_B(X_{i+1}, S) \to \text{Hom}_B(X_i, S) = k.
\]

Then, \( \text{Hom}_B(X_{i+1}, S) = k \) and, noting that (iii) holds for \( k = i + 1 \), \( \text{Top}(X_{i+1}) \simeq S \) follows. Therefore, (i) and (iii) hold for \( k = i + 1 \). Next we show that if \( X_{i+1} \) is non-projective then we may increment \( i \). Indeed, if \( X_{i+1} \) is non-projective then we may take a left minimal almost split homomorphism \( X_{i+1} \to M_{i+1} \), and the irreducible homomorphism \( X_{i+1} \to X_i \) is a direct summand. Thus, we may write \( M_{i+1} = X_i \oplus X_{i+2} \), for some \( B \)-module \( X_{i+2} \), and we have the almost split sequence

\[
0 \to X_{i+1} \to X_i \oplus X_{i+2} \to \tau^{-1}(X_{i+1}) \to 0.
\]

Then, the irreducible homomorphism \( X_i \to \tau^{-1}(X_{i+1}) \) is a direct summand of the left minimal almost split homomorphism \( X_i \to X_{i-1} \oplus X_{i+1} \), and we have either \( \tau^{-1}(X_{i+1}) \simeq X_{i-1} \) or \( \tau^{-1}(X_{i+1}) \simeq X_{i+1} \). But \( \tau^{-1}(X_{i+1}) \simeq X_{i-1} \) implies that \( X_{i+1} \simeq \tau(X_{i-1}) \simeq X_{i-1} \), which contradicts \( [X_k : S] = k + 1 \), for \( k = i \pm 1 \). Thus, we have \( \tau(X_{i+1}) \simeq X_{i+1} \) and (ii) for \( k = i + 1 \) holds. As \( \dim X_i \) grows, \( X_{i+1} \) becomes a projective \( B \)-module at some \( i \), and we conclude that the projective cover \( P \) of \( S \) is uniserial and all the composition factors of \( P \) are \( S \). It follows that \( B \) is a local Nakayama algebra. Therefore, \( \tau(S) \neq S \) as desired.

\[\square\]

**Proposition 7.12.** Let \( A \) and \( B \) be indecomposable self-injective \( k \)-algebras which are not Nakayama algebras. Let \( P_1, \ldots, P_n \) be a complete set of indecomposable projective \( B \)-modules, and we denote \( S_i = \text{Top}(P_i) \), for \( 1 \leq i \leq n \). Suppose that a functor \( \Psi : B\text{-mod} \to A\text{-mod} \) gives stable equivalence. Then, we have the following.

1. Let \( M_i = \Psi(S_i) \), for \( 1 \leq i \leq n \). Then \( \mathcal{M} = \{M_i\}_{1 \leq i \leq n} \) is a maximal s.o.s.b..

2. Let \( N_i = \Psi(P_i/\text{Soc}(P_i)) \). Then, \( N_i \) is s-projective and \( \text{s-Top}(N_i) \simeq M_i \), for \( 1 \leq i \leq n \).
Proposition 7.11. (i) is clear and\[\Phi^\varepsilon(M_i) \neq M_i \iff \tau(M_i) \neq M_i.\]

Proof. (1) The stable Auslander-Reiten quivers of $A$ and $B$ coincide by [2, X. Cor.1.9] and it follows from Proposition 7.11(2) that $\tau(M_i) \neq M_i$, for $1 \leq i \leq n$. Since
\[
\text{Hom}_A(M_i, M_j) \simeq \text{Hom}_B(S_i, S_j) = \begin{cases}
k & (i = j) \\
0 & (i \neq j)
\end{cases}
\]
\[\mathcal{M}\] is a system of orthogonal stable bricks. As it is not difficult to prove\[\text{Hom}_A(\oplus_{i=1}^n M_i, N) \neq 0, \quad \text{Hom}_A(N, \oplus_{i=1}^n M_i) \neq 0,
\]
for indecomposable non-projective $A$-modules $N$ such that $\tau(N) \neq N$, $\mathcal{M}$ is maximal.

(2) We check the conditions (i)(ii)(iii) from the definition of $s$-projectivity. (i) follows from Proposition 7.11(1). (ii) is clear and $s$-Top$(N_i) \simeq M_i$. Let $0 \neq \ell: X \to s$-Top$(N_i)$, for an indecomposable non-projective $A$-module $X$. Then, we have a surjective homomorphism $\Psi^{-1}(X) \to S_i$, and $g: P_i \to \Psi^{-1}(X)$ such that their composition equals the surjective homomorphism $p_i: P_i \to S_i$.

If $g(\text{Soc}(P_i)) \neq 0$ then $g(P_i) \simeq P_i$ and we obtain $P_i \simeq \Psi^{-1}(X)$, a contradiction. Thus, $g$ induces $P_i/\text{Soc}(P_i) \to X$, and it follows (iii). \hfill $\Box$

**Proposition 7.13.** Let $A$ be a self-injective $k$-algebra, $\mathcal{M} = \{M_i\}_{i \in I}$ a maximal s.o.s.b. Then, we have the following.

1. $\tau^{-1}\Omega(M_i)$ is $s$-projective, for all $i \in I$.
2. If $N$ is $s$-projective such that $s$-Top$(N) \simeq M_i$, then $N \simeq \tau^{-1}\Omega(M_i)$.

Proof. (1) Note that $\tau^{-1}\Omega(M_i) \simeq \Omega(M_i)$ if and only if $\tau(M_i) \simeq M_i$. Thus, $\tau(N_i) \neq N_i$, for $N_i = \tau^{-1}\Omega(M_i)$, and the condition (i) is satisfied. Let $P$ be a projective $A$-module such that
\[
0 \to \Omega(M_i) \to P \to M_i \to 0.
\]
If $M = \oplus_{i \in I} M_i$ or $M = M_i$, then we have
\[
0 \to \text{Hom}_A(M, \Omega(M_i)) \to \text{Hom}_A(M, P) \to \text{Hom}_A(M, M_i) \to \text{Ext}^1_A(M, \Omega(M_i)) \to 0.
\]
Thus, $k = \text{Hom}_A(M, M_i) \simeq \text{Ext}^1_A(M, \Omega(M_i))$ and it follows that
\[
\text{Hom}_A(N_i, M) = \text{Hom}_A(\tau^{-1}\Omega(M_i), M) \simeq D\text{Ext}^1_A(M, \Omega(M_i)) = k,
\]
where $D = \text{Hom}_k(-, k)$. Hence, the condition (ii) is satisfied and $s$-Top$(N_i) \simeq M_i$.\[\square\]
For $0 \neq f \in \text{Hom}_A(X, M_i)$, for an indecomposable non-projective $A$-module $X$, we find $g : N_i = \tau^{-1}\Omega(M_i) \to X$ such that $gf \neq 0$. Let $w : P \to X \oplus P$ be the natural inclusion, and we define a homomorphism $j : \Omega(M_i) \to Y$ by the following commutative diagram.

\[
\begin{diagram}
0 & \to & \Omega(M_i) & \xrightarrow{\ell} & M_i & \to & 0 \\
\downarrow{j} & & \downarrow{w} & \uparrow{j} & \uparrow{w} & \downarrow{j} & \uparrow{w} \\
0 & \to & Y & \xrightarrow{(f,\ell)} & X \oplus P & \to & 0 \\
\end{diagram}
\]

If $j$ is split mono, then $Y = \Omega(M_i) \oplus Y'$, for an $A$-submodule $Y'$ of $Y$. Then

\[
0 \to Y/\Omega(M_i) \xrightarrow{\iota} X \oplus M_i \xrightarrow{(f,\text{id}_{M_i})} M_i \to 0
\]
gives $\iota : Y' \simeq Y/\Omega(M_i) \simeq X' = \{(x, -f(x)) \mid x \in M_i\}$. Therefore, we have the following commutative diagram where $X \oplus M_i \to M_i$ is the projection to the second factor.

\[
\begin{diagram}
X & \xrightarrow{\iota} & X \oplus P & \xrightarrow{\text{incl}} & X \oplus M_i \\
\downarrow{\iota} & & \downarrow{\text{incl}} & \downarrow{\text{incl}} & \downarrow{\text{incl}} \\
X' & \xrightarrow{\text{incl}} & X \oplus M_i & \xrightarrow{\text{incl}} & M_i \\
\end{diagram}
\]

In the diagram, $X \to M_i$ is $-f$ and the vertical homomorphism $X \oplus P \to M_i$ factors through $P$. Thus, $f$ factors through a projective module and it contradicts $f \neq 0$. We conclude that $j$ is not split mono. On the other hand, the snake lemma implies

\[
0 = \text{Ker} (\text{id}_{M_i}) \to \text{Coker}(j) \to \text{Coker}(w) \to \text{Coker}(\text{id}_{M_i}) = 0
\]

Hence, $\text{Coker}(j) \simeq X$ and we have the exact sequence

\[
0 \to \Omega(M_i) \xrightarrow{j} Y \to X \to 0.
\]

We consider the almost split sequence $0 \to \Omega(M_i) \to Z \to \tau^{-1}\Omega(M_i) \to 0$. Then we may define $t : Z \to Y$ and $g : N_i = \tau^{-1}\Omega(M_i) \to X$ as follows, because $j$ is not split mono.

\[
\begin{diagram}
0 & \to & \Omega(M_i) & \xrightarrow{i} & Z & \xrightarrow{p} & N_i & \to & 0 \\
\downarrow{j} & & \downarrow{t} & \downarrow{g} & \downarrow{\iota} & \downarrow{\text{id}_{M_i}} & \downarrow{w} & \downarrow{j} & \downarrow{w} \\
0 & \to & Y & \xrightarrow{s} & X & \to & 0 \\
\end{diagram}
\]
We shall prove that $gf \neq 0$. Suppose that $gf$ factors through a projective $A$-module. Then, it factors through $\ell : P \to M_i$ and we may write $gf = h\ell$, for some $h : N_i \to P$. Thus, $(g, -h) : N_i \to X \oplus P$ factors through $Y \to X \oplus P$, because $(g, -h)(f, \ell) = gf - h\ell = 0$.

It follows that $g = h's$ and $(t - ph')s = ts - pg = 0$. In particular, $\text{Im}(t - ph') \subseteq \Omega(M_i)$. On the other hand, $ip = 0$ implies $i(t - ph') = it = j$ and $\text{Im}(t - ph') = \Omega(M_i)$. Thus, $i(t - ph')$ is an isomorphism of $\Omega(M_i)$ and it implies that $0 \to \Omega(M_i) \to Z \to N_i \to 0$ splits, which is a contradiction. Therefore, $gf \neq 0$ and we have proved that $N_i$ is $s$-projective.

(2) Let $N$ be $s$-projective such that $s\text{-Top}(N) = M_i$. Then, for the homomorphism $f : \tau^{-1}\Omega(M_i) \to M_i$ such that $f \neq 0$, there exists $g : N \to \tau^{-1}\Omega(M_i)$ such that $gf \neq 0$. Suppose that $g$ is not an isomorphism. Then, we may define $h : Z \to P$ and $f' : \tau^{-1}\Omega(M_i) \to M_i$ as follows.

If $f' = 0$ then it factors through $\ell : P \to M_i$ and we have $f'' : \tau^{-1}\Omega(M_i) \to P$ such that $f' = f''\ell$. Then, $(pf'' - h)\ell = pf' - h\ell = 0$ implies $pf'' - h : Z \to \Omega(M_i)$ and $i(pf'' - h) = -ih$ implies $\text{Im}(pf'' - h) = \Omega(M_i)$. Thus, $0 \to \Omega(M_i) \to Z \to N_i \to 0$ splits, a contradiction. Therefore, $f' \neq 0$ and it is a scalar multiple of $f$. But the above diagram shows that $gf' = 0$ and we have $gf = 0$, which contradicts $gf \neq 0$. We have proved $g : N \simeq \tau^{-1}\Omega(M_i)$. 

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