Performance Analysis of Raptor Codes under Maximum-Likelihood (ML) Decoding

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Abstract

Raptor codes have been widely used in many multimedia broadcast/multicast applications. However, our understanding of Raptor codes is still incomplete due to the insufficient amount of theoretical work on the performance analysis of Raptor codes, particularly under maximum-likelihood (ML) decoding, which provides an optimal benchmark on the system performance for the other decoding schemes to compare against. For the first time, this paper provides an upper bound and a lower bound, on the packet error performance of Raptor codes under ML decoding, which is measured by the probability that all source packets can be successfully decoded by a receiver with a given number of successfully received coded packets. Simulations are conducted to validate the accuracy of the analysis. More specifically, Raptor codes with different degree distribution and pre-coders, are evaluated using the derived bounds with high accuracy.

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I. INTRODUCTION

Recent work has shown that, by applying rateless codes, wireless transmission efficiency and reliability can be dramatically improved [1], [2], [3]. Rateless codes are a class of forward error correction (FEC) codes with special properties. Compared with other FEC codes with finite length, such as the Reed-Solomon codes, Block codes and Convolutional codes, rateless codes have numerous advantages. Firstly, this class of codes can be implemented with far less complicated encoding and decoding algorithms, making such codes easy to be employed in modern communication systems. Secondly, they can automatically adapt to instantaneous channel states and avoid the need for feedback channels [3], [1], [4]. This is because rateless codes can generate a potentially limitless stream of coded packets, and when a sufficient number of coded packets are successfully received, all source packets can be correctly decoded. Hence, for certain channels, such as erasure multicast or broadcast channels whose real-time channel erasure probability estimation might be nearly impossible to obtain, and non-uniform channels or time-varying channels whose channel states are unknown or difficult to capture due to fast variation, rateless codes are desirable means for data transmission. Because of the above mentioned advantages, rateless codes have the potential to replace the conventional automatic repeat request (ARQ) mechanism as a new mechanism of transmission control protocol (TCP) [5].

Among the known rateless codes, two codes stand out. One is the LT codes, which is the first practical digital fountain code with the average decoding cost in the order of $O(k \log(k))$ [1]. The other one is the Raptor codes, which are the first class of fountain codes with linear time encoding and decoding complexities. Raptor codes are concatenated codes, which combines a traditional FEC with an LT code to relax the condition that all input symbols need to be recovered in an LT decoder. Moreover, Raptor codes only require $O(1)$ time to generate an encoding symbol
Note that Raptor codes have already been standardized in 3GPP to efficiently disseminate data over a broadcast/multicast network to provide MBMS service [6]. Despite the successful application of Raptor codes in 3GPP, our understanding of Raptor codes is still incomplete due to the insufficient amount of theoretical work on the performance analysis of Raptor codes. Without analytical results, the optimization of the degree distribution as well as the parameters for Raptor codes would be extremely difficult, if not impossible. In [1], Shokrollahi provided a decoding error probability analysis of Raptor codes with finite length under the assumption of the belief propagation (BP) decoding. The analysis relies on the exact calculation of the error probability of the LT codes under the BP decoding, which was derived in [7]. The maximum-likelihood (ML) decoding, on the other hand, is more computational demanding than the BP decoding for codes with large length. Nevertheless, the derivation of bounds of decoding error probability for the ML decoding is still meaningful, because it provides an optimal benchmark on the system performance for the other decoding schemes to compare against. In this light, for Raptor codes with limited lengths, i.e. in the order of a few thousands, Shokrollahi proposed a decoding algorithm based on the maximum-likelihood (ML) criterion in [8]. Furthermore, in [5], the authors proposed a method to compute the upper and lower bounds on the bit error rate (BER) of Raptor codes under the assumption of the ML decoding. However, the work in [5] needs to be improved or re-examined in some aspects. Firstly, the pre-coder assumed in [5] is impractical. In more detail, all the entries of the parity check matrix of the pre-coder are assumed to be independent and identically distributed (i.i.d) Bernoulli random variables, so it is possible that the parity check matrix of the pre-coder may become ill-conditioned, rendering no generator matrix working with the parity check matrix. Hence, the analytical bounds proposed in [5] cannot be verified via simulation. Secondly, the derived bit error probability of Raptor codes under ML decoding in [5] is for the intermediate bits of Raptor codes rather than the source bits of Raptor codes. So the decoding error performance of Raptor codes still needs further investigation. In our previous work [9], we proposed a wireless broadcast scheme based on network coding in a single tier cellular network. In this paper, we further treat Raptor codes by analyzing the performance of the source bits of Raptor codes, i.e., all source bits can be
successfully decoded with ML decoding by a receiver with a given number of successfully received coded bits, and verifying the derived results via simulations. The contributions of this work are summarized in the following:

- This paper, for the first time, provides the analytical result, i.e., an upper bound and a lower bound, on the packet error performance of Raptor codes under maximum-likelihood (ML) decoding, which is measured by the probability that all source packets can be successfully decoded by a receiver with a given number of successfully received coded packets.

- Simulations are conducted to validate the accuracy of the analysis. More specifically, Raptor codes with different degree distribution and pre-coders, are evaluated using the derived bounds with high accuracy. According to our study, we conclude that Raptor codes with the binomial distribution achieve the best performance among the investigated ones.

The rest of the paper is organized as follows. In Section II, a brief review of the encoding and decoding process of Raptor codes is given. In Section III, performance analysis of Raptor code is conducted by deriving an upper bound and a lower bound on the probability that all source packets can be successfully decoded by a receiver with a given number of successfully received coded packets. Section IV validates the analytical results through simulations, followed by concluding remarks in Section V.

II. AN INTRODUCTION TO RAPTOR CODES

This section is provided to familiarize the readers with the basic idea of Raptor codes, their efficient encoding and decoding algorithms.

The encoding process of Raptor codes is carried out in two phases: a) Encode $k$ source packets with an $(n, k)$ error correcting code referred as pre-code $C$ to form $n$ intermediate packets; b) Encode the $n$ intermediate packets with an LT code. Each coded packet is generated by the following encoding rules of LT code. Firstly, a positive integer $d$ (often referred to as the “degree” [4] of coded packets) is drawn from the set of integers $\{1, \ldots, n\}$ according to a probability distribution $\Omega = (\Omega_1, \ldots, \Omega_n)$, where $\Omega_d$ is the probability that $d$ is picked and $\sum_{d=1}^{k} \Omega_d = 1$. Then, $d$ distinct source packets are selected randomly and independently from the
n intermediate packets to form the coded packet to be transmitted using the XOR operation \([1], [4]\), where each source packet is selected with equal probability. A Raptor code with parameters \((k, C, \Omega)\) is an LT code with distribution \(\Omega = (\Omega_1, \ldots, \Omega_n)\) on \(n\) packets that are the coded packets of the pre-code \(C\). An illustration of a Raptor code is given in Figure 1. In this study, we assume that the pre-code is an \((n, k)\) systematic LDPC code whose generator matrix, \(G_{n \times k}^{\text{pre}}\), can always be written as \(G_{n \times k}^{\text{pre}} = [I_k | P_{k \times (n-k)}]^T\), where \(I_k\) is an identity matrix of size \(k\), and \(P_{k \times (n-k)}\) is a \(k\) by \((n-k)\) matrix with its entries being independent and identically distributed (i.i.d) Bernoulli random variables with parameter \(\eta\). Such code is denoted as \((n, k, \eta)\) LDPC code \([5]\). Further, we can obtain the parity check matrix of this LDPC code as \(H_{(n-k) \times n} = [P_{(n-k) \times k} | I_{(n-k)}](n-k) \times n\) and \(H_{(n-k) \times n} \times G_{n \times k}^{\text{pre}} = 0\). In case where generator matrix of pre-code is a deterministic matrix, i.e., a typical \((n, k)\) error correction code, there are well known methods to handle the situation and actually make our analysis easier.

When a coded packet is received by a MU, we use a \(1 \times k\) binary row vector \(g_{i}^{\text{LT}}G_{k}^{\text{pre}}\) to represent the coding information contained in the coded packet, where \(g_{i}^{\text{LT}}\) is a \(1 \times n\) binary row vector and \(G_{k}^{\text{pre}}\) is a \(n \times k\) binary matrix. Let \([G]_{i,j}\) denote the entry in the \(i^{th}\) row and the \(j^{th}\) column of a matrix \(G\). Particularly, \([g_{i}^{\text{LT}}]_{1,j}\) is 1 if the coded packet is a result of the XOR operation on the \(j^{th}\) intermediate packet (and other intermediate packets); otherwise \([g_{i}^{\text{LT}}]_{1,j}\) equals 0. For \([G_{k}^{\text{pre}}]_{i,j}\), it is 1 if the \(i^{th}\) intermediate packet is a result of the XOR operation on the \(j^{th}\) source packet (and other source packets); otherwise \([G_{k}^{\text{pre}}]_{i,j}\) equals 0. Therefore, a

Figure 1. Two-stage structure of a Raptor code with a systematic pre-code.
random row vector in this paper refers to the row vector of a randomly chosen coded packet where the coded packet is generated using the Raptor encoding process described above. Recall that \( s = (s_1, s_2, \ldots, s_k) \) represents the \( k \) equal-length source packets to be transmission. The coded packet can be expressed as: \( y_i = g_i^{\text{LT}}G^{\text{pre}}s^T \), where \( s^T \) is transpose of \( s \).

Raptor codes can be decoded by using a variety of decoding algorithms. A typically used decoding algorithm for Raptor Codes is the so-called “LT process” \([4]\), but it is well known that the LT process is unable to decode all source packets which can be possibly recovered from information contained in the received coded packets. For example, LT process relies on the existence of at least one degree-one coded packet to be received in order to start the decoding process. For Raptor codes with limited lengths, i.e. on the order of a few thousand, maximum-likelihood (ML) decoding has been proposed to replace LT process. Therefore, in this paper we use a different decoding algorithm called the inactivation decoding algorithm \([8]\) to decode the source packets. This decoding algorithm combines the optimality of Gaussian elimination with the efficiency of the “LT process” algorithm. Specifically, let \( m, (m \geq k) \), be the number of coded packets that have already been successfully received by a MU. The performance of inactivation algorithm is the same as Gaussian elimination. One way to apply Gaussian elimination on raptor code is to solve a system of linear equations given in the following \([10]\).

\[
\begin{pmatrix}
G^{\text{LT}}_{m \times n} \\
H_{(n-k)\times n}
\end{pmatrix}
\begin{pmatrix}
G^{\text{pre}}_{n \times k}s^T_{k \times 1}
\end{pmatrix}
= \begin{pmatrix}
Y_{m \times 1} \\
0_{(n-k) \times 1}
\end{pmatrix}
\]

where \( Y_{m \times 1} = (y_1, y_2, \ldots, y_m)^T \). Additionally, we can obtain the following Lemma:

**Lemma 1.** A MU can recover all \( k \) source packets from the \( m \) coded packets using the inactivation decoding algorithm if and only if \( (G^{\text{LT}}; H_{(n-k)\times n})_{(m+n-k) \times n} \) is a full rank matrix, i.e. its rank equals \( n \), which is equivalent to the event that \( (G^{\text{LT}}G^{\text{pre}})_{m \times k} \) is a full rank matrix.

**Proof:** The proof of this statement is provided in Appendix A.

Note that in this paper, all algebraic operations and the associated analysis are conducted in
a binary field. Obviously the event that \((G^{LT}G^{pre})_{m\times k}\) is a full rank matrix is equivalent to the event \(A^k_m\) that a MU can successfully decode all \(k\) source packets using inactivation decoding algorithm provided the event that the MU has successfully received \(m\) coded packets. The main result of this paper is summarized in Theorems [2] and [3].

### III. Performance Analysis of Raptor Codes

Denote by \(A^k_m\) the event that a receiver can successfully decode all \(k\) source packets conditioned on the event that the receiver has successfully received \(m\) coded packets which is encoded with Raptor code from the BS. In this section, we shall analyze the probability of \(A^k_m\).

Because of the equivalence between the event \(A^k_m\) and the event that \((G^{LT}G^{pre})_{m\times k}\) is a full rank matrix, the analysis of \(\text{Pr}(A^k_m)\) is conducted by analyzing the probability that the rank of \((G^{LT}G^{pre})_{m\times k}\) is \(k\).

#### A. Lower Bound on the Decoding Success Probability of Raptor Codes

In this subsection, we will derive a lower bound on the decoding success probability of Raptor codes with systematic pre-code, which is presented in the following theorem:

**Theorem 2.** When the BS generates coded packets using the Raptor code \((k,C,\Omega(x))\) where \(C\) is \((n,k,\eta)\) LDPC code and the coded packets received at a mobile user (MU) are decoded using the inactivation decoding algorithm, the probability that a MU can successfully decode all \(k\) source packets from \(m\) received coded packets with \(m \geq k\), denoted by \(\text{Pr}(A^k_m)\), is lower bounded by

\[
\text{Pr}(A^k_m) \geq 1 - \sum_{i=1}^{k} \binom{k}{i} \sum_{r=i}^{n-k+i} (J(r)) m \ D(i,r)
\]

where

\[
J(r) = \sum_{d=1}^{n} \Omega_d \frac{\sum_{s=0,2,...,2\left\lfloor \frac{n-r}{d} \right\rfloor}^{\binom{n-r}{d-s}} \binom{n-r}{d-s}}{\binom{n}{d}}
\]
and
\[
D(i, r) = \binom{n-k}{i-r} \left[ \frac{1 + (1 - 2\eta)^i}{2} \right]^{n-k-r+i} \times \left[ \frac{1 - (1 - 2\eta)^i}{2} \right]^{r-i}
\]
and \(\Omega_d\) is the degree distribution of LT codes.

**Proof:** Our proof relies on the use of the union bound of the independent events that vectors in the column vector space of \(G_{n \times k}^{pre}\) are in the null space of \(G_{m \times n}^{LT}\).

According to the property of the matrix product [11, Eq. (4.5.1)], we have
\[
\text{rank}(G_{m \times n}^{LT} G_{n \times k}^{pre}) = \text{rank}(G_{n \times k}^{pre}) - \dim\{N(G_{m \times n}^{LT}) \cap R(G_{n \times k}^{pre})\}
\]
where \(N(\bullet)\) is the right-hand null space of a matrix, \(R(\bullet)\) is the column vector space generated by a matrix and \(\dim\{\mathcal{V}\}\) represents number of vectors in any basis for a vector space \(\mathcal{V}\). It follows from the definition of \(G_{n \times k}^{pre}\) given earlier that the rank of \(G_{n \times k}^{pre}\) surely is \(k\). It can be readily obtained that:
\[
\Pr[\text{rank}(G_{m \times n}^{LT} G_{n \times k}^{pre}) = k] = \Pr[\dim\{N(G_{m \times n}^{LT}) \cap R(G_{n \times k}^{pre})\} = 0]
\]
(2)

For convenience let \(W_{m,n,k}\) represent the event that \(\dim\{N(G_{m \times n}^{LT}) \cap R(G_{n \times k}^{pre})\} = 0\). Now we need to analyze \(\Pr[W_{m,n,k}]\). Provided that \(G_{n \times k}^{pre}\) is a \((n, k, \rho)\) systematic LDPC code, the event \(\dim\{N(G_{m \times n}^{LT}) \cap R(G_{n \times k}^{pre})\} \neq 0\), i.e., \(\overline{W}_{m,n,k}\), is equivalent to the event that at least one column vector from \(R(G_{n \times k}^{pre})\) is among \(N(G_{m \times n}^{LT})\), i.e., \(\bigcup_{x \in R(G_{n \times k}^{pre})} G_{m \times n}^{LT} x = 0\), where \(x\) is a column vector of \(R(G_{n \times k}^{pre})\). It can be readily shown that:
\[
\Pr[\overline{W}_{m,n,k}] = \Pr[\bigcup_{x \in R(G_{n \times k}^{pre})} G_{m \times n}^{LT} x = 0]
\]
The column vector space $R(G_{n\times k}^{pre})$ is partitioned into $k$ subspaces $(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k)$ and $\mathcal{V}_i$ is the subspace that contains all the column vectors which are summation of $i$ column vectors of $G_{n\times k}^{pre}$.

We denote $\Gamma_i$ as the set of indices of the column vectors in $\mathcal{V}_i$ and there are $(\binom{k}{i})$ indices in $\Gamma_i$. Let $x_a^i$ represent the $a^{th}$, $a \in \Gamma_i$ column vector in $\mathcal{V}_i$. It can be readily shown that:

$$\sum_{x \in R(G_{n\times k}^{pre})} \Pr [G_{m\times n}^{LT} x = 0] = \sum_{i=1}^{k} \sum_{a \in \Gamma_i} \Pr [G_{m\times n}^{LT} x_a^i = 0]$$

We can observe that $x_a^i = G_{n\times i}^{a} \cdot 1_i$, where $G_{n\times i}^{a}$ is the matrix formed by $i$ selected column vectors from $k$ column vectors of $G_{n\times k}^{pre}$ and $1_i$ represent the $i \times 1$ all one column vector. Let $|x_a^i|$ represent the weight of column vector $x_a^i$, considering the law of total probability, we have

$$\Pr [G_{m\times n}^{LT} x_a^i = 0] = \sum_{r=0}^{n} \Pr \left[ G_{m\times n}^{LT} x_a^i = 0 \bigg| |x_a^i| = r \right] \Pr [|x_a^i| = r]$$

Firstly, we need to calculate $\Pr [|x_a^i| = r]$. Provided $G_{n\times k}^{pre} = [I_k | P_{k\times(n-k)}]^T$, in the first $k$ entries of $G_{n\times i}^{a} \cdot 1_i$ there are $i$ ones. If $|x_a^i| = r$, then there are $r - i$ ones in the last $n - k$ entries of $G_{n\times i}^{a}$, i.e., $P_{(n-k)\times i}^{a} \cdot 1_i$. Hence we can obtain that

$$\Pr [|x_a^i| = r] = \Pr [||P_{(n-k)\times i}^{a} \cdot 1_i|| = (r - i)]$$

and $i \leq r \leq n - k + i$. The rows of $P_{(n-k)\times i}^{a}$, i.e., $p_j, 1 \leq j \leq (n - k)$, are random binary row vectors, which are generated independently. Each entry of $P_{(n-k)\times i}^{a}$ is independent and identically distributed (i.i.d) Bernoulli random variable with parameter $\rho$. Therefore, $\Pr [p_j 1_i = 0] = \Pr [p_{k,k\neq j} 1_i = 0]$. The event that the $j^{th}$ entry in $x_a^i$ is zero is equivalent to the event that there are even number of ones in row vector $p_j$. We have

$$\Pr [p_j 1_i = 0] = \Pr [||p_j|| \text{ is even} ]$$
\[
\sum_{s=0,2,...,2^i} (\binom{i}{s}) \eta^s (1 - \eta)^{(i-s)}
\]
\[
= \frac{((\eta + (1 - \eta))^i + (-\eta + (1 - \eta))^i)}{2}
\]
\[
= \frac{1 + (1 - 2\eta)^i}{2}
\]
\[\text{(7)}\]

There are \(\binom{n-k}{r-i}\) possible combinations for \(r-i\) ones in the last \(n-k\) entries. It can be readily shown that:

\[
\Pr \left[ \left| \sum_{i=0}^{2^i} \eta^s (1 - \eta)^{(i-s)} \right| = (r-i) \right]
\]
\[
= \binom{n-k}{r-i} \{\Pr[\eta^i = 0]\}^{n-k-r+i} \times \{1 - \Pr[\eta^i = 0]\}^{r-i}
\]
\[\text{(8)}\]

Combining equations (6), (7) and (8), we can obtain that

\[
D(i, r) = \Pr \left[ \left| x_a^i \right| = r \right]
\]
\[
= \binom{n-k}{r-i} \left[ \frac{1 + (1 - 2\eta)^i}{2} \right]^{n-k-r+i} \times \left[ \frac{1 - (1 - 2\eta)^i}{2} \right]^{r-i}
\]
\[\text{(9)}\]

For \(x_a^i, x_b^i, b \neq a \in V_i\), \(P_a^{(n-k) \times i}\), and \(P_b^{(n-k) \times i}\) have the same probability to form the same matrix formation. So we can obtain that \(\Pr \left[ \left| \sum_{i=0}^{2^i} \eta^s (1 - \eta)^{(i-s)} \right| = (r-i) \right] = \Pr \left[ \left| \sum_{i=0}^{2^i} \eta^s (1 - \eta)^{(i-s)} \right| = (r-i) \right]\), in turn \(\Pr[\left| x_a^i \right| = r] = \Pr[\left| x_b^i \right| = r]\). Now, we calculate \(\Pr \left[ G_{m \times n}^{LT} x_a^i = 0 \mid \left| x_a^i \right| = r \right]\). The rows of \(G_{m \times n}^{LT}\), i.e., \(g_j^{LT}, 1 \leq j \leq m\), are random binary row vectors, which are generated independently. We have

\[
\Pr \left[ G_{m \times n}^{LT} x_a^i = 0 \mid \left| x_a^i \right| = r \right] = \left\{ \Pr \left[ g_j^{LT} x_a^i = 0 \mid \left| x_a^i \right| = r \right] \right\}^m
\]
\[\text{(10)}\]
and uniformly into $g^\text{LT}_j$. It can be readily obtain that

$$
\Pr \left[ g^\text{LT}_j x^i_a = 0 \mid |x^i_a| = r \right] = \sum_{d=1}^{n} \Omega_d \Pr \left[ g^\text{LT}_j x^i_a = 0 \mid |x^i_a| = r, |g^\text{LT}_j| = d \right]
$$

Let $r^i_j = (g^\text{LT}_j x^i_{a1}, g^\text{LT}_j x^i_{a2}, ..., g^\text{LT}_j x^i_{an})$, where $g^\text{LT}_j$ is $[g^\text{LT}_j]_{1,k}$ and $x^i_{ak}$ is $[x^i_a]_{k,1}$. Then, we can obtain that

$$
\Pr \left[ g^\text{LT}_j x^i_a = 0 \mid |x^i_a| = r, |g^\text{LT}_j| = d \right] = \Pr \left[ |r^i_j| \text{ is even} \mid |x^i_a| = r, |g^\text{LT}_j| = d \right] = \frac{\sum_{s=0,2,...,2\left\lfloor \frac{d}{2} \right\rfloor}^{\left( \frac{r}{2} \right) \left( \frac{n-r}{d-s} \right)}}{\binom{n}{d}}
$$

Combining equations (11) and (12), we can obtain that

$$
J(r) = \Pr \left[ g^\text{LT}_j x^i_a = 0 \mid |x^i_a| = r \right] = \sum_{d=1}^{n} \Omega_d \sum_{s=0,2,...,2\left\lfloor \frac{d}{2} \right\rfloor}^{\left( \frac{r}{2} \right) \left( \frac{n-r}{d-s} \right)} \binom{n}{d}
$$

Inserting equation (10) into (13), it can be obtained that

$$
\Pr \left[ G^\text{LT}_{m \times n} x^i_a = 0 \mid |x^i_a| = r \right] = [J(r)]^m
$$

We can obtain that $\Pr[G^\text{LT}_{m \times n} x^i_a = 0 \mid |x^i_a| = r]$ is only determined by the weight of $x^i_a$ rather than which $i$ column vectors is chosen from $G^\text{pre}_{n \times k}$ to obtain the summation $x^i_a$. So we can conclude that $\Pr[G^\text{LT}_{m \times n} x^i_a = 0] = \Pr[G^\text{LT}_{m \times n} x^i_b = 0]$. Recall that there are $\binom{k}{i}$ indices in $\Gamma_i$. Inserting equations (9) and (14) into (5) and combining with equation (4), yields the following results

$$
\Pr[\overline{W}_{m,n,k}] \leq \sum_{i=1}^{k} \sum_{a \in \Gamma_i} \Pr \left[ G^\text{LT}_{m \times n} x^i_a = 0 \right]
$$
\[
\sum_{i=1}^{k} \binom{k}{i} \sum_{r=i}^{n-k+i} \left[ \sum_{d=1}^{n} \Omega_d \binom{n}{d} \right]^{m} \times \binom{n-k-r+i}{r-i} \left[ \frac{1 + (1 - 2\eta)^i}{2} \right]^{n-k-r+i} \left[ \frac{1 - (1 - 2\eta)^i}{2} \right]^{r-i}
\]

which proves the assertion.

B. Upper Bound on the Decoding Success Probability of Raptor Codes

In addition to the above lower bound, we can also derive an upper bound on the decoding success probability of Raptor codes with systematic pre-code, which is presented in the following theorem:

**Theorem 3.** When the BS generates coded packets using the Raptor code \((k, C, \Omega(x))\) and the coded packets received at a mobile user (MU) are decoded using the inactivation decoding algorithm, the probability that a MU can successfully decode all \(k\) source packets from \(m\) received coded packets with \(m \geq k\), denoted by \(\Pr(A^k_m)\), is upper bounded by:

\[
\Pr(A^k_m) \leq 1 - \sum_{i=1}^{k} \binom{k}{i} \sum_{r=i}^{n-k+i} (J(r))^m D(i, r)
\]

\[
+ \frac{1}{2} \sum_{i=1}^{k} \binom{k}{i} \sum_{w_0=0}^{n-k} \sum_{w_1-i}^{n-k-w_0} \sum_{w_2=0}^{n-k-w_0} 1(w_0 + w_2)1(w_1 + w_2)
\]

\[
\times \binom{i}{w_0} \binom{k-i}{w_2} \sum_{r_0=w_0}^{n-k+w_0} \sum_{r_1=w_1}^{n-k+w_1} \sum_{r_2=w_2}^{n-k+w_2} D(w_0, r_0)D(w_1, r_1)
\]

\[
\times D(w_2, r_2)\{J(r_0)J(r_1)J(r_2) + \tilde{J}(r_0)\tilde{J}(r_1)\tilde{J}(r_2)\}^m
\]

where

\[
1(x) := \begin{cases} 
0 & \text{if } x = 0 \\
1 & \text{otherwise}
\end{cases}
\]

\[
\tilde{J}() = 1 - J(), \text{ } D(w_0, r_0) \text{ is defined in equation (9) and } J(r_0) \text{ is defined in equation (13).}
\]
Proof: By using the Bonferroni inequality [?], we can obtain a lower bound of \( \Pr[\overline{W}_{m,n,k}] \) as:

\[
\Pr[\overline{W}_{m,n,k}] = \Pr[\bigcup_{x \in R(G_{n \times k}^{\text{pre}})} G_{m \times n}^{\text{LT}} x = 0] \\
\geq \sum_{x \in R(G_{n \times k}^{\text{pre}})} \Pr[G_{m \times n}^{\text{LT}} x = 0] \\
- \frac{1}{2} \sum_{x,y \in R(G_{n \times k}^{\text{pre}}), x \neq y} \Pr[G_{m \times n}^{\text{LT}} x = 0 \& G_{m \times n}^{\text{LT}} y = 0]
\]

(17)

where \( x = G_{n \times k}^{\text{pre}} a, a \in GF(2)^k \) and \( y = G_{n \times k}^{\text{pre}} b, b \in GF(2)^k \setminus a \). The first term can be calculated by using Theorem 2. Recall that \( \mathcal{V}_i \) is subspace that contain all the column vectors which are summation of \( i \) column vectors of \( G_{n \times k}^{\text{pre}} \), \( \Gamma_i \) is the set of indices of the column vectors in \( \mathcal{V}_i \) and \( x_a^i \) represents the \( a^{th}, a \in \Gamma_i \) column vectors in \( \mathcal{V}_i \). It can be readily shown that:

\[
\sum_{x,y \in R(G_{n \times k}^{\text{pre}}), x \neq y} \Pr[G_{m \times n}^{\text{LT}} x = 0 \& G_{m \times n}^{\text{LT}} y = 0] \\
= \sum_{x \in R(G_{n \times k}^{\text{pre}})} \sum_{y \in R(G_{n \times k}^{\text{pre}}) \setminus x} \Pr[G_{m \times n}^{\text{LT}} x = 0 \& G_{m \times n}^{\text{LT}} y = 0] \\
= \sum_{i=1}^{k} \sum_{a \in \Gamma_i} \sum_{y \in R(G_{n \times k}^{\text{pre}}) \setminus x_a^i} \Pr[G_{m \times n}^{\text{LT}} x_a^i = 0 \& G_{m \times n}^{\text{LT}} y = 0]
\]

(18)

where \( x_a^i = G_{n \times k}^{\text{pre}} a, |a| = i \). Recall that \( y = G_{n \times k}^{\text{pre}} b, b \in GF(2)^k \). We define three binary vectors \( z_0, z_1, \) and \( z_2 \in GF(2)^k \) such that for \( t = 1, \ldots, k, z_0(t) = 1 \) if and only if \( a(t) = 1 \) and \( b(t) = 1, z_1(t) = 1 \) if and only if \( a(t) = 1 \) and \( b(t) = 0, \) and \( z_2(t) = 1 \) if and only if \( a(t) = 0 \) and \( b(t) = 1 \). Let \( w_0, w_1, \) and \( w_2 \) be the weights of vectors \( z_0, z_1, \) and \( z_2, \) respectively. For \( x_a^i, \) we have \( z_0 + z_1 = a \) and \( z_0 + z_2 = b \). Hence we can obtain:

\[
\Pr[G_{m \times n}^{\text{LT}} x_a^i = 0 \& G_{m \times n}^{\text{LT}} y = 0] \\
= \Pr[G_{m \times n}^{\text{LT}} G_{n \times k}^{\text{pre}} z_0 = G_{m \times n}^{\text{LT}} G_{n \times k}^{\text{pre}} z_1 = G_{m \times n}^{\text{LT}} G_{n \times k}^{\text{pre}} z_2 \mid |z_0| = w_0 \& |z_1| = w_1 \& |z_2| = w_2]
\]

(19)
Let \( I_z = \{i_{z1}, i_{z2}, \ldots, i_{zt}\} \) be the set of indices such that \( t \in I_z \) for \( z(t) = 1 \), we can obtain the sets of indices of vectors \( z_0, z_1, \) and \( z_2 \) as \( I_{z0}, I_{z1} \) and \( I_{z2} \). Corresponding to the three sets \( I_{z0}, I_{z1} \) and \( I_{z2} \), each column of the matrix \( G_{m \times n}^{pre} \), \( g_i^{pre}, 1 \leq i \leq k \), can be divided into four mutually exclusive parts, \( g_z, g_z1, g_z2 \) and \( \cup_{1 \leq i \leq k} g_i^{pre} \setminus (g_z \cup g_z1 \cup g_z2) \), i.e., \( g_z \cap g_z1 = \{0\} \). Let \( g_z \) be the subset of \( \cup_{1 \leq i \leq k} g_i^{pre} \) such that all the elements of this subset are selected from \( \cup_{1 \leq i \leq k} g_i^{pre} \) according to the indices in set \( I_{z0} \) and \( G_{z0}^{pre} \) be the matrix whose columns are elements of \( g_z \). The length of \( g_z \) is \( w_0 \). The same operation is applied to the formation of \( g_z1 \) and \( g_z2 \), in which the elements are selected according to the indices in set \( I_{z1} \) and \( I_{z2} \), and have length \( w_1 \) and \( w_2 \), respectively. Let \( x^{w_0} = G_{z0}^{pre}1_{w_0}, x^{w_1} = G_{z1}^{pre}1_{w_1} \) and \( x^{w_2} = G_{z2}^{pre}1_{w_2} \). Equivalently, equation (25) can be rewritten as,

\[
\Pr \left[ G_{m \times n}^{LT} G_{n \times k}^{pre} t_0 = G_{m \times n}^{LT} G_{n \times k}^{pre} t_1 = G_{m \times n}^{LT} G_{n \times k}^{pre} t_2 \right] = \Pr \left[ G_{m \times n}^{LT} x^{w_0} = G_{m \times n}^{LT} x^{w_1} = G_{m \times n}^{LT} x^{w_2} \right]
\]

According to the law of total probability, we have

\[
\Pr \left[ G_{m \times n}^{LT} x^{w_0} = G_{m \times n}^{LT} x^{w_1} = G_{m \times n}^{LT} x^{w_2} \right] = \sum_{r_0=w_0}^{n-k+w_0} \sum_{r_1=w_1}^{n-k+w_1} \sum_{r_2=w_2}^{n-k+w_2} \Pr[|x| = r_0] \times \Pr[|x| = r_1] \times \Pr[|x| = r_2]
\]

For \( \Pr[|x| = r_0] \), this can be calculated by using equation (9). Recall that the rows of \( G_{m \times n}^{LT} \), i.e., \( g_j^{LT}, 1 \leq j \leq m \), are random binary row vectors, which are generated independently. We have

\[
\Pr \left[ G_{m \times n}^{LT} x^{w_0} = G_{m \times n}^{LT} x^{w_1} = G_{m \times n}^{LT} x^{w_2} \right]
\]
Because all algebraic operations are conducted in a binary field, $g_j^{LT}x^{w0}$ can only be 1 or 0. Equation (22) can be further written as:

\[
\begin{align*}
\Pr \left[ g_j^{LT} x^{w0} = g_j^{LT} x^{u1} = g_j^{LT} x^{u2} \right] \\
&= \Pr \left[ x^{w0} = r_0, x^{u1} = r_1, x^{u2} = r_2 \right] \\
&= \Pr \left[ g_j^{LT} x^{w0} = 0, g_j^{LT} x^{u1} = 0, g_j^{LT} x^{u2} = 0 \right] \\
&\quad + \Pr \left[ g_j^{LT} x^{w0} = 1, g_j^{LT} x^{u1} = 1, g_j^{LT} x^{u2} = 1 \right] \\
&= \left\{ \Pr \left[ g_j^{LT} x^{w0} = r_0, x^{u1} = r_1, x^{u2} = r_2 \right] \right\}^m
\end{align*}
\] (23)

Recall that $x^{w0} = G^{pre}_{z_0}1_{w0}$, $x^{u1} = G^{pre}_{z_1}1_{u1}$, $x^{u2} = G^{pre}_{z_2}1_{u2}$ and the columns of $G^{pre}_{z_0}$, $G^{pre}_{z_1}$, $G^{pre}_{z_2}$ are mutually exclusive to each other. So event that $|x^{w0}| = r_0$ is independent of event that $|x^{u1}| = r_1$ or $|x^{u2}| = r_2$ and the event that $g_j^{LT} x^{w0} = 1$ is independent of event that $g_j^{LT} x^{u1} = 1$ or $g_j^{LT} x^{u2} = 1$. Conditioned on $|x^{w0}| = r_0, |x^{u1}| = r_1, |x^{u2}| = r_2$, the first part in equation (23) can be expressed as:

\[
\begin{align*}
\Pr \left[ g_j^{LT} x^{w0} = 0, g_j^{LT} x^{u1} = 0, g_j^{LT} x^{u2} = 0 \right] \\
&= \Pr \left[ g_j^{LT} x^{w0} = 0 \mid |x^{w0}| = r_0 \right] \Pr \left[ g_j^{LT} x^{u1} = 0 \mid |x^{u1}| = r_1 \right] \\
&\quad \times \Pr \left[ g_j^{LT} x^{u2} = 0 \mid |x^{u2}| = r_2 \right]
\end{align*}
\] (24)

Based on the previous analysis, we know that $\Pr[g_j^{LT} x^{w0} = 0 \mid |x^{w0}| = r_0]$ only relates to parameter $r_0$. Let $D(w_0, r_0) = \Pr[|x^{w0}| = r_0]$ and $J(r_0) = \Pr[g_j^{LT} x^{w0} = 0 \mid |x^{w0}| = r_0]$. For $J(r_0)$, it can be calculated by using equations (11) and (12). Based on the previous analysis,
we know that $J(r_0)$ only relates to parameter $r_0$ and $D(w_0, r_0)$ is affected by parameter $r_0$ and $w_0$. Hence for the same parameters $w_0$, $w_1$ and $w_2$, equations (20) has the same result. Because $x^i_a \neq y$, we can obtain that $w_1 + w_2 \neq 0$ and $w_0 + w_2 \neq 0$. For $x^i_a$, when $|z_0| = w_0$, we have $w_1 = i - w_0$ and there are $(i_0)$ possible combinations of $z_0$. For $z_2$, there are $(k-i)$ possible combination of $z_2$ when $|z_2| = w_2$. Inserting equation (20), (21), (22), (23) and (24) into (19), we can obtain:

$$\sum_{y \in R(G_{m \times n}^{\text{LT}} \setminus x^i_a)} \Pr[G_{m \times n}^{\text{LT}} x^i_a = 0 \& G_{m \times n}^{\text{LT}} y = 0] = \sum_{i=0}^{k} \sum_{w_0=0}^{n-k+w_0} \sum_{w_1=i-w_0}^{n-k+w_1} \sum_{w_2=0}^{n-k+w_2} 1(w_0 + w_2)1(w_1 + w_2) \times D(w_0, r_0)D(w_1, r_1)D(w_2, r_2) \times \{J(r_0)J(r_1)J(r_2) + J(r_0)\overline{J(r_1)\overline{J(r_2)}}\}^m$$

(25)

where $1(x) := \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}$. For $x^i_a, x^i_{b, b \neq a} \in V_i$, the probability $\sum_{x^i \neq y} \Pr[G_{m \times n}^{\text{LT}} x^i_a = 0 \& G_{m \times n}^{\text{LT}} y = 0]$ is affected by parameter $i$. So we can obtain that $\sum_{x^i \neq y} \Pr[G_{m \times n}^{\text{LT}} x^i_a = 0 \& G_{m \times n}^{\text{LT}} y = 0] = \sum_{x^i \neq y} \Pr[G_{m \times n}^{\text{LT}} x^i_a = 0 \& G_{m \times n}^{\text{LT}} y = 0] = \sum_{x^i \neq y} \Pr[G_{m \times n}^{\text{LT}} x^i_a = 0 \& G_{m \times n}^{\text{LT}} y = 0]$. Recall that there are $(k)$ indices in $\Gamma^i$. We can get that

$$\sum_{x, y \in R(G_{m \times n}^{\text{LT}} \setminus x \neq y)} \Pr[G_{m \times n}^{\text{LT}} x = 0 \& G_{m \times n}^{\text{LT}} y = 0]$$

$$= \sum_{i=1}^{k} \sum_{a \in \Gamma^i} \sum_{y \in R(G_{m \times n}^{\text{LT}} \setminus x^i_a)} \Pr[G_{m \times n}^{\text{LT}} x^i_a = 0 \& G_{m \times n}^{\text{LT}} y = 0]$$

$$= \sum_{i=1}^{k} \binom{k}{i} \sum_{w_0=0}^{n-k+w_0} \sum_{w_1=i-w_0}^{n-k+w_1} \sum_{w_2=0}^{n-k+w_2} 1(w_0 + w_2)1(w_1 + w_2) \times D(w_0, r_0)D(w_1, r_1) \times D(w_2, r_2) \times \{J(r_0)J(r_1)J(r_2) + J(r_0)\overline{J(r_1)\overline{J(r_2)}}\}^m$$

(26)

The proof of Theorem 3 is completed. 

\[\square\]
C. A Special Case of the Derived Bounds

When we apply a special degree distribution—binomial degree distribution, i.e., \( \Omega_d = \binom{k_d}{(2^k-1)} \), \( 1 \leq d \leq k \), into Theorem 2, we can simplify equation (11) into a far less complex expression. The simplification procedure is shown in the following Corollary.

**Corollary 4.** When the BS generates coded packets using the Raptor code \((k, \mathcal{C}, \Omega_d)\) where \( \mathcal{C} \) is \((n, k, \eta)\) LDPC code, \( \Omega_d = \binom{k_d}{(2^k-1)} \), \( 1 \leq d \leq k \) and the coded packets received at a mobile user (MU) are decoded using the inactivation decoding algorithm, the probability that a MU can successfully decode all \( k \) source packets from \( m \) received coded packets with \( m \geq k \), denoted by \( \Pr(A_m^k) \), satisfies

\[
\Pr(A_m^k) \geq 1 - (2^k - 1) \left( \frac{2^{n-1} - 1}{2^n - 1} \right)^m
\]

(27)

**Proof:** When new degree distribution, i.e., \( \Omega_d = \binom{n_d}{(2^n-1)} \), \( 1 \leq d \leq n \), is inserted into equation (11), we can obtain that

\[
\Pr[g_j^{LT} x_a^i = 0 \mid \lvert x_a^i \rvert = r] = (2^n - 1)^{-1} \sum_{d=1}^{n} \sum_{s=0,2,\ldots,\lceil \frac{d}{2} \rceil} \binom{n_r}{d-s} \binom{n-d-s}{s} \]

(28)

When the upper limit of the inner summation is changed from \( 2 \left\lfloor \frac{d}{2} \right\rfloor \) to \( 2 \left\lfloor \frac{n}{2} \right\rfloor \), it will not affect the result of equation (28). This is because that \( \binom{n-r}{d-s} \) with \( s > 2 \left\lfloor \frac{d}{2} \right\rfloor \) equals 0. The inner summation variable \( s \) is now independent of the outer summation variable \( d \) and thus the order of the two summations can be exchanged:

\[
\Pr[g_j^{LT} x_a^i = 0 \mid \lvert x_a^i \rvert = r] = (2^n - 1)^{-1} \sum_{s=0,2,\ldots,\lceil \frac{n}{2} \rceil} \sum_{d=0}^{n} \binom{n-r}{d-s} - \binom{n-r}{d-s} s = d = 0 \]

(29)
The terms $\binom{n-r}{d-s}$ restricts $d$ to $s \leq d \leq n - r + s$, such that

$$\sum_{d=0}^{n} \binom{n-r}{d-s} = \sum_{d=s}^{n-r+s} \binom{n-r}{d-s} = \sum_{d=0}^{n-r} \binom{n-r}{d} = 2^{n-r}$$  \hspace{1cm} (30)

Combining this term with the last expression for $\Pr[g^LT_jx^i_a = 0 \mid |x^i_a| = r]$ yields

$$\left[ g^LT_jx^i_a = 0 \mid |x^i_a| = r \right] = (2^n - 1)^{-1} \left( 2^{n-r} \sum_{s=0}^{r} \binom{r}{s} - 1 \right) = (2^n - 1)^{-1} (2^{n-r} 2^{-1} - 1) \hspace{1cm} (31)$$

$$= \frac{2^{n-1} - 1}{2^n - 1} \hspace{1cm} (32)$$

where we have used identity $\sum_{s \text{ even}} \binom{r}{s} = 2^{r-1}$. We can observe that $\Pr[g^LT_jx^i_a = 0 \mid |x^i_a| = r]$ is independent from the weight of $x^i_a$, hence $\Pr[G^LT_{m\times n}x^i_a = 0 \mid |x^i_a| = r] = \Pr[G^LT_{m\times n}x^i_a = 0]$.

Combining equation (10), (32), (4) and (2), we can obtain that

$$\Pr[\Pr[W_{m,n,k}]] \\
= \Pr \left[ \bigcup_{x \in R(G^\text{pre}\!_{m\times n})} G^LT_{m\times n} x = 0 \right] \\
\leq \sum_{x \in R(G^\text{pre}\!_{m\times n})} \Pr [G^LT_{m\times n} x = 0] \\
= (2^k - 1) \Pr [G^LT_{m\times n} x = 0 \mid |x| = r] \\
= (2^k - 1) \left( \frac{2^{n-1} - 1}{2^n - 1} \right)^m \hspace{1cm} (33)$$

As for Theorem 3, we can simplify the upper bound into a far less complex expression as well. This is summarized in the following Corollary.

**Corollary 5.** When the BS generates coded packets using the Raptor code $(k, C, \Omega_d)$ where $C$ is $(n, k, \eta)$ LDPC code, $\Omega_d = \left( \frac{k}{2^k - 1} \right)$, $1 \leq d \leq k$ and the coded packets received at a mobile user (MU) are decoded using the inactivation decoding algorithm [8], the probability that a MU can
successfully decode all \( k \) source packets from \( m \) received coded packets with \( m \geq k \), denoted by \( \Pr(A^k_m) \), satisfies

\[
\Pr(A^k_m) \leq 1 - (2^k - 1) \left[ \frac{(2^{n-1} - 1)}{(2^n - 1)} \right]^m + (2^k - 1)(2^{k-1} - 1) \\
\times \left\{ \left[ \frac{(2^{n-1} - 1)}{(2^n - 1)} \right]^3 + \left[ 1 - \frac{(2^{n-1} - 1)}{(2^n - 1)} \right]^3 \right\}^m
\]

(34)

**Proof:** The new degree distribution, i.e., \( \Omega_d = \binom{n}{d} \), \( 1 \leq d \leq n \), is inserted into equation (7), by using the result of equation (32), we can obtain that

\[
J(r_0) = \Pr[g^L_j x^w_0 = 0 | x^w_0 = r_0] = \frac{(2^{n-1} - 1)}{(2^n - 1)}
\]

(35)

Insert equation (35) into equation (20), we can obtain that

\[
\Pr[G^L_{m \times n}^T G^{pre}_{n \times k} z_0 = G^L_{m \times n}^T G^{pre}_{n \times k} z_1 = G^L_{m \times n}^T G^{pre}_{n \times k} z_2 \\
|z_0| = w_0 \& |z_1| = w_1 \& |z_2| = w_2] = \sum_{r_0=w_0}^{n-k+w_0} \sum_{r_1=w_1}^{n-k+w_1} \sum_{r_2=w_2}^{n-k+w_2} D(w_0, r_0) D(w_1, r_1) D(w_2, r_2) \\
\times \left\{ \left[ \frac{(2^{n-1} - 1)}{(2^n - 1)} \right]^3 + \left[ 1 - \frac{(2^{n-1} - 1)}{(2^n - 1)} \right]^3 \right\}^m
\]

(36)

Insert equation (36) into equation (25), we can obtain that

\[
\sum_{x_0^i \neq y} \Pr[G^L_{m \times n}^T x_0^i = 0 \& G^L_{m \times n}^T y = 0] = \sum_{i \leq \frac{k}{w_0}} \sum_{w_1 = i - w_0}^{k-i} \sum_{w_2 = 0}^{w_1} 1(w_0 + w_2) 1(w_1 + w_2) \binom{i}{w_0} (k-i) \\
\times \left\{ \left[ \frac{(2^{n-1} - 1)}{(2^n - 1)} \right]^3 + \left[ 1 - \frac{(2^{n-1} - 1)}{(2^n - 1)} \right]^3 \right\}^m
\]
Combining equation (37), (18) and (17), we can obtain that

\[
\Pr[W_{m,n,k}] \geq (2^k - 2) \left\{ \left[ \frac{(2^{n-1} - 1)}{(2^n - 1)} \right]^3 + \left[ 1 - \frac{(2^{n-1} - 1)}{(2^n - 1)} \right]^3 \right\}^m
\]

(37)

Compared with the general expressions in Theorems 2 and 3 in the simplified expression of Corollaries 4 and 5, we can easily observe the relationship between the decoding success probability and the parameter of the encoding rules, i.e., \( k, n \) and \( m \).

IV. SIMULATION RESULTS

In this section, we use MATLAB based simulations to validate the accuracy of the analytical results and the tightness of the proposed performance bounds. Each point shown in the figures is the average result obtained from 100,000 simulations. The 95% confidence interval is also shown in each figure. For clarity, the simulation parameters adopted in this section are summarized in Table I.
Table I
SIMULATION PARAMETERS

| Rateless Codes encoding parameters |  |
|-----------------------------------|--|
| Number of source packets $k$      | 20, 40 |
| Number of intermediate packets $n$| 21, 41 |
| Parameter for bernoulli random variables $\eta$| 0.3, 0.7 |
| Pre-code $C$                      | $(n, k, \eta)$ LDPC codes |

| LT codes degree distribution |  |
|------------------------------|--|
| Standard degree distribution | $\Omega_3^{GFP}(x)$ [6, Annex B] |
| Binomial degree distribution | $\Omega_d = \binom{n}{d}, 1 \leq d \leq n$ |
| Ideal soliton degree distribution | $\Omega_d = \frac{1}{d(d-1)}, 2 \leq d \leq n$ and $\Omega_1 = \frac{1}{n}$ |
| Robust soliton degree distribution | $c = 0.04, \delta = 0.01$ |

Figure 2. The probabilities of successfully decoding all 20 source packets by the MU as a function of overhead $\gamma$ of transmission by the BS

A. Verification of the Derived Bounds

In this sub-section, the number of source packets is set to be $k = 20$, and the degree distribution of Raptor codes follows the widely used ideal soliton degree distribution [4]. Besides, the pre-code $C$ is assumed to be $(21, 20, 0.3)$ and $(21, 20, 0.7)$ LDPC codes.

In Figs. 2, our analytical and simulation results are presented in terms of the probability $\Pr[A_m^k]$ that the MU successfully decode all $k = 20$ source packets as a function of overhead $\gamma = m/k$ of transmission by the BSs. As shown in Fig. 2, our analytical results, i.e., the upper and lower bound match the simulation results very well, which validates the accuracy of the
analysis in this paper. However, when overhead \( \gamma \) is small, there is still a gap between the upper (lower) bounds and simulation results in Fig. 2. The gap between the exact value and the upper (lower) bound is caused by the approximation used in equation (3), and the gap between the exact value and the lower bound is caused by equation (2).

B. Investigation of the Impact of Degree Distribution on the Decoding Success Probability

When we fix the Pre-code \( C \) as \((21, 20, 0.7)\), the degree distributionS of Raptor codes are chosen as the widely used ideal soliton degree distribution, the robust soliton degree distribution [4], the standardized degree distribution in 3GPP [6, Annex B]:

\[
\Omega^{\text{3GPP}}(x) = 0.0099x + 0.4663x^2 + 0.2144x^3 + 0.1152x^4 + 0.1131x^{10} + 0.0811x^{11}
\]

and a Binomial degree distribution proposed in this paper (see Table I). As shown in Fig. 3(a) and 3(b) for different degree distributions, our analytical bounds are also corroborated by simulation results. Moreover, the performance of Raptor codes with the binomial degree distribution outperforms those with other three degree distributions. Additionally, the expression of decoding success probability of Raptor code with binomial degree distribution in Corollaries 5 and 4 has their computation superiority compared with the expression in Theorems 2 and 3. Therefore, we will focus on Raptor codes with the binomial degree distribution in the following simulations.

C. Investigation of the Impact of \( k \) on the Decoding Success Probability

When the number of source packets \( k \) increases from 20 to 40, our analytical results still tightly match the simulation ones. As can be seen from Fig. 4(a) and 4(b), comparing transmitting 20 source packets with transmitting 40 ones, the BS can reduce the overhead \( \gamma = m/k \) of transmission that required to achieve the same performance, which leads to reduced transmission latency and energy consumption.
D. Comparison of the Successful Transmission Probability for Raptor codes and Ideal Fountain codes

In Fig. 5(a) and 5(b), we compare the Raptor code to an ideal fountain code together with the baseline transmission without coding. As can be observed from Fig. 5(a) and 5(b), the performance gap between the Raptor code and the ideal fountain code is non-negligible because in the ideal fountain code the number of received symbols needed to decode the source symbols is exactly the number of source symbols, no matter which symbols are received. Hence, the decoding success probability of an ideal fountain code is as high as \( \Pr(A_m^k) = 1, m \geq k \). Besides, the
coding gain of Raptor codes compared with the baseline transmission without coding is shown to be tremendous. We apply Raptor codes and an ideal fountain code into a single BEC channel with different erasure probability $p$. The probability that the receiver can decode all $k$ source packets based on the successfully received coded packets, denoted as $P_{\text{suc}}$, can be expressed as:

$$P_{\text{suc}}(T) = \sum_{m=k}^{T} \binom{T}{m} \Pr(A^k_m) p^{T-m} (1-p)^m$$

As demonstrated in Fig. 5(a) and 5(b) for different erasure probability $p$, transmission without coding can significantly reduce the overhead $\gamma = m/k$ of transmitting that required to achieve the same performance. When the target performance, e.g., the probability of successful delivery...
is set to 0.95 for $p = 0.1$, the ratio of the number of packets transmitted without using coding to that using Raptor code equals 2.069; for $p = 0.3$, the ratio increases to 2.564. It seems that the ratio increases as the channel condition become worse. As for the comparison between the ideal fountain code and Raptor code, for the same target performance of 0.95, for $p = 0.1$, the ratio of the number of packets transmitted with Raptor code to that using ideal fountain code equals 1.16; for $p = 0.3$, the ratio decreases to 1.11. It seems that the performance of Raptor code converges to that of the ideal fountain code when the channel condition become worse.

![Graph](image)

(a) $p=0.1$

![Graph](image)

(b) $p=0.3$

Figure 5. The probabilities of successfully decoding all 20 source packets by the MU as a function of overhead $\gamma$ of transmission by the BS
V. CONCLUSION

In this paper we focus on finite-length Raptor codes and derive upper and lower bounds on packet error performance of Raptor codes under maximum-likelihood (ML) decoding, which is measured by the probability that all source packets can be successfully decoded by a receiver with a given number of successfully received coded packets. ML decoding ensures successful decoding when a full-rank matrix is received. Due to the concatenated coding structure of Raptor codes, we have analyzed the rank behavior of product of two random matrix.

On the basis of the results presented in the paper, in the future, we plan to explore the optimum degree distribution and optimal parameter of Raptor codes in different channels.

APPENDIX A: PROOF OF LEMMA 1

The event that \((G^{LT}; H^{(n-k)\times n})_{(m+n-k)\times n}\) is a full rank matrix, i.e. its rank equals \(n\), is equivalent to the event that \((G^{LT}G^{pre})_{m\times k}\) is a full rank matrix.

Firstly we prove that the event that \((G^{LT}G^{pre})_{m\times k}\) is a full rank matrix is a sufficient condition for the event that \((G^{LT}; H^{(n-k)\times n})_{(m+n-k)\times n}\) is a full rank matrix. Recall that \(H^{(n-k)\times n} = [P^{(n-k)\times k}I^{(n-k)}]^{(n-k)\times n}G^{pre}_{n\times k} = [I_k|P^{k\times (n-k)}]^{T}\) and \(H^{(n-k)\times n}G^{pre}_{n\times k} = 0\). If \((G^{LT}; H^{(n-k)\times n})_{(m+n-k)\times n}\) is a full rank matrix, we can obtain that \((G^{LT}; H^{(n-k)\times n})_{(m+n-k)\times n}G^{pre}_{n\times k} = (G^{LT}G^{pre}, 0^{(n-k)\times k})\) is of full rank, i.e., \((G^{LT}G^{pre})_{m\times k}\) is a full rank matrix.

\[
\begin{pmatrix}
G^{LT}_{m\times n} \\
H^{(n-k)\times n}
\end{pmatrix}
\begin{pmatrix}
G^{pre}s^T \\
0_{(n-k)\times 1}
\end{pmatrix}
= 
\begin{pmatrix}
Y_{m\times 1}
\end{pmatrix}
\]

Then we prove that the event that \((G^{LT}G^{pre})_{m\times k}\) is a full rank matrix is a necessary condition for the event that \((G^{LT}; H^{(n-k)\times n})_{(m+n-k)\times n}\) is a full rank matrix. Since \(H^{(n-k)\times n}G^{pre}_{n\times k} = 0\), we can observe that row vector space of \(H^{(n-k)\times n}\) span the left null space of \(G^{pre}_{n\times k}\), i.e., \(R(H^{T}_{(n-k)\times n}) = N((G^{pre}_{n\times k})^T)\).
As we can see that rank of \(\begin{pmatrix} G_{n \times k}^{\text{pre}} & H_{n \times (n-k)}^T \end{pmatrix}\) is \(n\), the span of \(\begin{pmatrix} G_{n \times k}^{\text{pre}} & H_{n \times (n-k)}^T \end{pmatrix}\) is \(\mathbb{Z}_2^n\). Hence we can obtain that \(\text{basis}\{\text{nullity}(G_{n \times k}^{\text{pre}})\} = \text{basis}\{\text{nullity}(H_{n \times (n-k)}^T)\} = \text{basis}\{\mathbb{Z}_2^n\}\). By using the same idea, we can obtain that \(\text{basis}\{\text{nullity}(G_{n \times k}^{\text{pre}})\} \cup \text{basis}\{\text{nullity}(G_{n \times k}^{\text{pre}})\} = \text{basis}\{\mathbb{Z}_2^n\}\). Once in binary field, no matter the formation of matrix, we have \(\text{basis}\{\text{nullity}(G_{n \times k}^{\text{pre}})\} = \text{basis}\{\mathbb{Z}_2^n\}\). For \(G_{m \times n}^{LT}\), given that \(\dim\{\text{nullity}(G_{m \times n}^{LT}) \cap \text{nullity}(G_{n \times k}^{\text{pre}})\} = 0\), we can get

\[
\text{basis}\{\text{nullity}(G_{m \times n}^{LT})\} = \text{basis}\{\mathbb{Z}_2^n\}\backslash \text{basis}\{\text{nullity}(G_{m \times n}^{LT})\} \subseteq \text{basis}\{\mathbb{Z}_2^n\}\backslash \text{basis}\{\text{nullity}(G_{n \times k}^{\text{pre}})\}
\]

\[
\implies \text{basis}\{\text{nullity}(G_{m \times n}^{LT})\} \supseteq \text{basis}\{\text{nullity}(G_{n \times k}^{\text{pre}})\}
\]

Because \(\text{basis}\{\text{nullity}(H_{n \times (n-k)}^T)\} = \text{basis}\{\mathbb{Z}_2^n\}\backslash \text{basis}\{\text{nullity}(G_{n \times k}^{\text{pre}})\}\), \(\text{basis}\{\text{nullity}(H_{n \times (n-k)}^T)\} = \text{basis}\{\text{nullity}(G_{m \times n}^{LT})\} \cup \text{basis}\{\text{nullity}(H_{n \times (n-k)}^T)\} = \text{basis}\{\mathbb{Z}_2^n\}\). That is \((G_{n \times k}^{LT}; H_{(n-k) \times n})_{m+n-k \times n}\) is a full rank matrix, i.e. its rank equals \(n\).

**APPENDIX B**

The event that \(Y = G^{LT}X\), where \(X = G^{pre}s^T\) can be decoded by using ML decoding is equivalent to the event that \((G^{LT}G^{pre})_{m \times k}\) is a full rank matrix.

Firstly when using a BEC channel, a encoded bit is either correctly received or lost. We consider \(Y' = G'^{LT}G'^{pre}s^T\) as the encoded bits generated by Raptor encoder. After transmission, the coded bits a receiver correctly received can be expressed as \(Y = G^{LT}X\), where \(G^{LT}\) is a part of \(G'^{LT}\). Provided the rank of \(G^{LT}\) is \(r\), so the nullity of \(G^{LT}\) is \(n - r\). Using ML decoding to decode \(X\) from \(Y\) is equivalent to solve the linear equation \(Y = G^{LT}X\) by using Gaussian Elimination method. The set of solutions to \(Y = G^{LT}X\) is an affine set. It has the form \(X = X_0 + N(G^{LT}) = \{X_0 + x, x \in N(G^{LT})\}\) where \(X_0 = G^{pre}s^T\) and \(X_0 \in R(G^{pre})\). If we prove
that the event that \((G^{LT} G^{pre})_{m \times k}\) is a full rank matrix is a sufficient condition for the event that \((G^{LT}; H_{(n-k) \times n}(m+n-k) \times n)\) is a full rank matrix. Recall that \(H_{(n-k) \times n} = [P_{(n-k) \times k} | I_{(n-k)}]_{(n-k) \times n}\), \(G^{pre}_{n \times k} = [I_k | P_{k \times (n-k)}]^T\) and \(H_{(n-k) \times n} \times G^{pre}_{n \times k} = 0\). If \(\dim \{ N(G^{LT}_{m \times n}) \cap R(G^{pre}_{n \times k}) \} = 0\), i.e., \(N(G^{LT}_{m \times n}) \cap R(G^{pre}_{n \times k}) = \{ \emptyset \}\) \(X_0 + N(G^{LT}) \notin R(G^{pre}_{n \times k})\). So \(X_0 + N(G^{LT})\) is not the final solution when using ML decoding. \(X_0 = Z\) is the unique solution left. So the condition that the ML decoding can decode \(X\) correctly, i.e., \(X\) has the unique solution, is that \(\dim \{ N(G^{LT}_{m \times n}) \cap R(G^{pre}_{n \times k}) \} = 0\), which is equivalent to the condition that \((G^{LT} G^{pre})_{m \times k}\) is a full rank matrix.

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