PSEUDO-RIEMANNIAN METRICS IN MODELS BASED ON NONCOMMUTATIVE GEOMETRY

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Abstract
Several examples and models based on noncommutative differential calculi on commutative algebras indicate that a metric should be regarded as an element of the left-linear tensor product of the space of 1-forms with itself. We show how the metric compatibility condition with a linear connection generalizes to this framework.

1 Introduction
In ordinary differential geometry there are several equivalent ways to introduce the concept of a metric. It should not be surprising, however, that corresponding generalizations to the huge framework of noncommutative geometry in general lead to somehow inequivalent structures. Some definition of a metric in noncommutative geometry is easily chosen and calculations performed with it. What is often missing, however, is a serious application which demonstrates its usefulness outside the single point which corresponds to ordinary differential geometry in the set of noncommutative differential geometries. We do not believe that a particular definition of a metric will finally be singled out by general arguments. Rather, we expect that a convenient definition of a metric will depend on the area of applications which one has in mind, and the relations between different definitions may turn out to be extremely complicated.

In section 2 we briefly recall some metric definitions in noncommutative geometry and add to it a new one which, however, is restricted to the case of (noncommutative) differential calculi on commutative algebras, which includes the case of discrete spaces. Section 3 provides some examples and models in which this metric definition shows up. In section 4 we introduce compatibility of such a metric with a linear connection. Section 5 contains some conclusions.
2 General setting for noncommutative pseudo-Riemannian geometry

Let \( A \) be an associative algebra (over \( \mathbb{C} \) or \( \mathbb{R} \)) with unit element \( 1 \). A graded algebra over \( A \) is a \( \mathbb{Z} \)-graded associative algebra \( \Omega(A) = \bigoplus_{r \geq 0} \Omega^r(A) \) where \( \Omega^0(A) = A \). A differential calculus over \( A \) consists of a graded algebra \( \Omega(A) \) over \( A \) and a linear map \( d : \Omega^r(A) \to \Omega^{r+1}(A) \) with the properties

\[
d^2 = 0, \quad d(w w') = (dw) w' + (-1)^r w dw'
\]

where \( w \in \Omega^r(A) \) and \( w' \in \Omega(A) \). We also require \( 1 w = w 1 = w \) for all elements \( w \in \Omega(A) \). The identity \( 1 1 = 1 \) then implies \( d1 = 0 \). In the following, we simply write \( \Omega \) instead of \( \Omega(A) \).

A (left \( A \)-module) linear connection is a linear map \( \nabla : \Omega^1 \to \Omega^1 \otimes_A \Omega^1 \) such that

\[
\nabla(f \alpha) = df \otimes_A \alpha + f \nabla \alpha
\]

for \( f \in A \) and \( \alpha \in \Omega^1 \). It extends to a linear map \( \nabla : \Omega \otimes_A \Omega^1 \to \Omega \otimes_A \Omega^1 \) via

\[
\nabla(w \otimes_A \alpha) = dw \otimes_A \alpha + (-1)^r w \nabla \alpha \quad w \in \Omega^r, \alpha \in \Omega^1.
\]

The curvature of the linear connection \( \nabla \) is the map \( R = -\nabla^2 \) and the torsion \( \Theta : \Omega \otimes_A \Omega^1 \to \Omega \) is defined as \( \Theta = d \circ \pi - \pi \circ \nabla \) where \( \pi \) is the projection \( \Omega \otimes_A \Omega^1 \to \Omega \).

What about a generalization of the concept of a (pseudo-) Riemannian metric? In the literature we find the following suggestions.

- Connes’ distance formula [1] generalizes the Riemannian geodesic distance. This is an interesting new tool even in ordinary Riemannian geometry where, however, it is bound to the case of positive definite metrics (see [2] for an attempt to overcome this restriction). Its generalization to noncommutative geometry requires some more understanding what a convenient counterpart to the Riemannian Dirac operator should be. This makes model building quite complicated.

- In several papers a metric has been considered as an element of \( \Omega^1 \otimes_A \Omega^1 \) or as a map \( \Omega^1 \otimes_A \Omega^1 \to A \). In particular, a technical problem arose, namely the impossibility to extend a linear connection (on \( \Omega^1 \)) to a connection on \( \Omega^1 \otimes \Omega^1 \) in certain examples (see [3, 4]). Such an extension is needed in order to define metric compatibility in a straight way. Furthermore, according to our knowledge there have been no applications so far to really prove the usefulness of this definition of a metric in noncommutative geometry.

- A generalization of the Hodge \( \ast \)-operator to noncommutative geometry appeared to be a useful structure in some applications based on noncommutative geometry of commutative algebras [5, 6, 7]. It has been generalized to noncommutative algebras in [8].

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\(^1\) If \( A \) is an algebra over \( \mathbb{C} \) (\( \mathbb{R} \)), a linear map is linear over \( \mathbb{C} \) (\( \mathbb{R} \)).
Departing from all of these definitions, for (noncommutative) differential calculi on commutative algebras we propose that a metric should be taken to be an element of $\Omega^1 \otimes \Omega^1$ where $\otimes$ is the left-linear tensor product which satisfies

$$(f \alpha) \otimes_L (h \beta) = f h \alpha \otimes_L \beta$$

for $f, h \in \mathcal{A}$ and $\alpha, \beta \in \Omega^1$ (see also [9]). The next section provides some examples through which we were led to this definition of a metric. It should be noticed that there is no direct way to extend this definition to the case of noncommutative algebras $\mathcal{A}$. In some sense, however, the Hodge operator mentioned above may be regarded as such an extension.

In the following we make use of the fact that, given a differential calculus on a commutative algebra $\mathcal{A}$, there is a unique associative and commutative product $\bullet$ in the space of 1-forms such that $\alpha \bullet df = [\alpha, f]$ and

$$(f ah) \bullet (f'h'h') = f f' (\alpha \bullet \beta) hh' \quad \forall f, f', h, h', h' \in \mathcal{A}, \alpha, \beta \in \Omega^1$$

We call it the **canonical product** in $\Omega^1$. It measures the deviation from the ordinary differential calculus where $\alpha \bullet \beta = 0$ for all $\alpha, \beta \in \Omega^1$.

### 3 Where left-linear pseudo-Riemannian metrics show up

#### 3.1 Metrics on finite sets

Let $\mathcal{A}$ be the algebra of functions on a finite set $M$. It has been shown in [11] that first order differential calculi on $\mathcal{A}$ are in one-to-one correspondence with digraphs the vertices of which are the elements of $M$. Given such a digraph, we associate with an arrow from $i \in M$ to $j \in M$ an algebraic object $e^{ij}$. Let $\Omega^1$ be the linear space (over $\mathbb{C}$) generated by all these $e^{ij}$. An $\mathcal{A}$-bimodule structure can then be introduced via

$$f e^{ij} h = f (i) e^{ij} h(j) \quad \forall f, h \in \mathcal{A}. \quad (6)$$

With

$$df = \sum_{i,j} [f(j) - f(i)] e^{ij} \quad (7)$$

we obtain a first order differential calculus. It is natural to regard the set of outgoing arrows at some point $i \in M$ as the analogue of the cotangent space in ordinary continuum differential geometry.

As a candidate for a metric let us consider $g \in \Omega^1 \otimes_L \Omega^1$. Using the above formulas and the properties of the tensor product over $\mathcal{A}$, we obtain

$$g = \sum_{i,j,k} g_{ijk} e^{ij} \otimes_L e^{jk} \quad (8)$$
with constants $g_{ijk}$. Here $e^{ij}$ and $e^{ik}$ live in different cotangent spaces and it would be quite unnatural for a metric to compare vectors located at different points. In contrast, if we take $g \in \Omega^1 \otimes L \Omega^1$, then

$$ g = \sum_{i,j,k} g(i)_{jk} e^{ij} \otimes_L e^{ik} $$

(9)

with constants $g(i)_{jk}$. Here $e^{ij}$ and $e^{ik}$ live in the same cotangent space and this enables us to make contact with classical geometry [9].

### 3.2 A class of noncommutative differential calculi on $\mathbb{R}^n$

In terms of coordinates $x^\mu$, $\mu = 1, \ldots, n$, on $\mathbb{R}^n$ a class of first order differential calculi is determined by the commutation relations

$$ \left[ dx^\mu, x^\nu \right] = \ell C^{\mu\nu\kappa} dx^\kappa $$

(10)

where $\ell$ is a constant and $C^{\mu\nu\kappa}$ are functions of the coordinates which have to satisfy certain consistency conditions [5, 10]. In terms of the canonical product in $\Omega^1$ this becomes $dx^\mu \cdot dx^\nu = \ell C^{\mu\nu\kappa} dx^\kappa$. We assume that $dx^\mu$ forms a basis of $\Omega^1$ as a left- and as a right $\mathcal{A}$-module. Generalized partial (left- and right-) derivatives can then be introduced via

$$ df = (\partial_{+\mu} f) dx^\mu = dx^\mu (\partial_{-\mu} f) $$

(11)

A coordinate transformation is a bijection $x'^\mu (x^\nu)$ such that $\partial_{+\nu} x'^\mu$ is invertible. Then

$$ \left[ dx'^\mu, x'^\nu \right] = \partial_{+\kappa} x'^\mu \left[ dx^\kappa, x'^\nu \right] = \partial_{+\kappa} x'^\mu \left[ dx'^\kappa, x^\nu \right] $$

$$ = \partial_{+\kappa} x'^\mu \partial_{+\lambda} x'^\nu \left[ dx^\lambda, x^\nu \right] = \ell \partial_{+\kappa} x'^\mu \partial_{+\lambda} x'^\nu C^{\kappa\lambda\sigma} dx^\sigma $$

(12)

using the commutativity of $\mathcal{A}$ and the derivation property of $d$. Hence,

$$ C^{\mu'\nu'\kappa'} = \ell \partial_{+\kappa} x'^\mu \partial_{+\lambda} x'^\nu C^{\kappa\lambda\sigma} \partial_{+\kappa'} x'^\sigma $$

(13)

If we define

$$ g^{\mu\nu} = C^{\mu\kappa}_\lambda C^{\lambda\nu}_\kappa $$

(14)

we obtain the transformation rule

$$ g^{\mu'\nu'} = \partial_{+\kappa} x'^\mu \partial_{+\lambda} x'^\nu g^{\kappa\lambda} $$

(15)

Suppose an inverse $g_{\mu\nu}$ exists. Then

$$ g_{\mu'\nu'} = \partial_{+\mu} x^\eps \partial_{+\nu} x^\lambda g_{\kappa\lambda} $$

(16)

This is *not* compatible with $g = g_{\mu\nu} dx^\mu \otimes_A dx^\nu$, but rather with

$$ g = g_{\mu\nu} dx'^\mu \otimes_L dx'^\nu $$

(17)

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2 A first order differential calculus $d : \mathcal{A} \to \Omega^1$ always extends to higher orders via the rules of differential calculus.

3 On the rhs we use the summation convention (summation over $\kappa$).
We mention that one can prove that if $g^{\mu \nu}$ is invertible, then there is an $A$-module basis $\theta^\mu$ of $\Omega^1$ such that

$$ \theta^\mu \bullet \theta^\nu = \delta^\mu_\kappa \delta^\nu_\kappa \theta^\kappa . $$

(18)

If the $\theta^\mu$ are holonomic, then we have $C^{\mu \nu \kappa} = \delta^\mu_\kappa \delta^\nu_\kappa$ and therefore the lattice differential calculus considered in [3].

The basis of an interesting physical model. Think of the $x^\mu$ as space-time coordinates. If the functions $C^{\mu \nu \kappa}$ and the $g^{\mu \nu}$ derived from them are dimensionless, then $\ell$ should have the dimension of a length and a natural candidate for it would be the Planck length. The kinematical structure of space-time is then modified at the Planck scale.

3.3 From the Hodge operator to the metric tensor

Let $A$ be a commutative algebra and $(\Omega(A), d)$ a differential calculus over $A$ which admits linear and invertible maps

$$ \star : \Omega^r \to \Omega^{n-r} \quad r = 0, \ldots, n $$

(19)

for some $n \in \mathbb{N}$ such that

$$ \star (w f) = f \star w \quad \forall f \in A, \ w \in \Omega . $$

(20)

As a consequence, $\star^{-1}(f w) = (\star^{-1} w) f$. The set of maps $\star$ is called a (generalized) Hodge operator. It induces an inner product in $\Omega^1$ as follows,

$$ (\alpha, \beta) = \star^{-1}(\alpha \star \beta) . $$

(21)

As a consequence of (20), it satisfies

$$ (\alpha, \beta f) = (\alpha f, \beta), \quad (f \alpha, \beta) = f (\alpha, \beta) . $$

(22)

In applications of the formalism in the context of completely integrable models (and in particular generalized principal chiral models) [3], a symmetric Hodge operator was needed, i.e.,

$$ \alpha \star \beta = \beta \star \alpha \quad \forall \alpha, \beta \in \Omega^1 . $$

(23)

As a consequence of this rather restrictive condition, we have

$$ (\alpha, f \beta) = (f \beta, \alpha) = f (\alpha, \beta) . $$

(24)

For a differential calculus of the kind considered in the previous subsection, one can introduce metric components

$$ g^{\mu \nu} = (dx^\mu, dx^\nu) . $$

(25)

Using the above formulas, the effect of a coordinate transformation is

$$ g^{\mu' \nu'} = \partial_\kappa x^\mu \partial_\lambda x^{\nu'} g^{\kappa \lambda} . $$

(26)

As a consequence, if $g^{\mu \nu}$ has an inverse $g_{\mu \nu}$, then

$$ g = g_{\mu \nu} dx^\mu \otimes_A dx^\nu $$

(27)

is a tensor (but not $g = g_{\mu \nu} dx^\mu \otimes_A dx^\nu$).

4In order to generalize this to a noncommutative algebra $A$, an involution $*$ on $A$ is needed and the rhs has to be replaced by $f^* \star w$ [8].
4 The metric compatibility condition

Let \((\Omega(\mathcal{A}), d)\) be a differential calculus over a commutative algebra \(\mathcal{A}\) and \(\nabla\) a linear connection. Let us introduce the twist map
\[
\tau(\alpha \otimes_L \beta) = \beta \otimes_L \alpha
\]
and the map
\[
\bullet ((\alpha \otimes_A \gamma) \otimes_L (\beta \otimes_A \delta)) = (\alpha \bullet \beta) \otimes_A (\gamma \otimes_L \delta)
\]
where the canonical product in \(\Omega^1\) enters on the rhs. Now we define
\[
\nabla(\alpha \otimes_L \beta) = \nabla \alpha \otimes_L \beta + (\text{id} \otimes_A \tau)((\nabla \beta \otimes_L \alpha)) - \bullet((\nabla \alpha \otimes_L \nabla \beta)).
\]
It is easy to verify that this defines a left \(\mathcal{A}\)-module connection on \(\Omega^1 \otimes_L \Omega^1\). Now we can impose the condition
\[
\nabla g = 0
\]
on an element \(g \in \Omega^1 \otimes_L \Omega^1\). If \(g\) is a candidate for a metric, this condition generalizes the familiar metric compatibility condition of ordinary differential geometry.

5 Conclusions

We have proposed a definition of a metric tensor as an element of \(\Omega^1 \otimes_L \Omega^1\) for (noncommutative) differential calculi on commutative algebras and presented examples in which such a structure appears naturally. Furthermore, a corresponding compatibility condition with a linear connection has been formulated. In the particular case of differential calculi on discrete sets, these structures have been explored in \[9\].

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References

[1] Connes A.: Noncommutative Geometry, Academic Press, San Diego, 1994.
[2] Parfionov G.N. and Zapatrin R.R.: gr-qc/9803090.
[3] Mourad J.: Class. Quantum Grav. 12 (1995) 965.
[4] Bresser K., Dimakis A., Müller-Hoissen F. and Sitarz A.: J. Phys. A 29 (1996) 2705.
[5] Dimakis A., Müller-Hoissen F. and Striker T.: J. Phys. A 26 (1993) 1927.
[6] Dimakis A. and Müller-Hoissen F.: J. Phys. A 29 (1996) 5007.
[7] Dimakis A. and Müller-Hoissen F.: Lett. Math. Phys. 39 (1997) 69.

\(^{5}\)An expression like \(\alpha \otimes_A \beta \otimes_L \gamma\) has to be read as \(\alpha \otimes_A (\beta \otimes_L \gamma)\).
[8] Dimakis A. and Müller-Hoissen F.: Czech. J. Phys. 48 (1998) 1319.
[9] Dimakis A. and Müller-Hoissen F.: J. Math. Phys. 40 (1999) 1518.
[10] Baehr H. C., Dimakis A. and Müller-Hoissen F.: J. Phys. A 28 (1995) 3197.
[11] Dimakis A. and Müller-Hoissen F.: J. Math. Phys. 35 (1994) 6703.