A GALOIS CONNECTION BETWEEN INTUITIONISTIC AND
CLASSICAL LOGICS. I: SYNTAX

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Abstract. In a 1985 commentary to his collected works [42], Kolmogorov remarked
that his 1932 paper [40] “was written in hope that with time, the logic of solution of
problems [i.e., intuitionistic logic] will become a permanent part of a [standard] course
of logic. A unified logical apparatus was intended to be created, which would deal
with objects of two types — propositions and problems.” We construct such a formal
system, as well as its predicate version QHC, which is a conservative extension of both
the intuitionistic predicate calculus QH and the classical predicate calculus QC.

The only new connectives ? and ! of QHC induce a Galois connection between the
Lindenbaum posets (i.e. the underlying posets of the Lindenbaum algebras) of QH and
QC. Kolmogorov’s double negation translation of propositions into problems extends
to a retraction of QHC onto QH; whereas Gödel’s provability translation of problems
into modal propositions extends to a retraction of QHC onto its QC+(??) fragment,
identified with the modal logic QS4. The QH+(??) fragment is an intuitionistic modal
logic, whose modality ?? is a “strict lax modality” in the sense of Aczel — and thus
resembles the squash/bracket operation in intuitionistic type theories.

The axioms of QHC attempt to give a fuller formalization (with respect to the axioms
of intuitionistic logic) to the two best known contentual interpretations of intuitionalistic
logic: Kolmogorov’s problem interpretation (incorporating standard refinements by
Heyting and Kreisel) and the proof interpretation by Orlov and Heyting (as clarified
by Gödel). While these two interpretations are often conflated, from the viewpoint of
the axioms of QHC neither of them reduces to the other one, although they do overlap.

1. Introduction

1.1. Problems versus propositions

The present series of papers (the sequels being [2] and [3]) belongs firmly to the field
of Logic, but is motivated primarily by considerations of mathematical practice rather
than any internal developments in the field of Logic. Therefore it is addressed not only
to logicians, but to other mathematicians as well. The reader who is not familiar with
any of the terms used can consult the treatise [1] as need arises; one of its main goals is
precisely to make the present series accessible to a general mathematical audience.

\footnote{Galois connection is a standard notion of order theory, whose eponymous example is the correspondence
between the poset of fixed fields and the poset of subgroups in Galois theory. It can be defined as a
pair of adjoint functors between two posets, regarded as categories. See [29; pp. 166–167] for a concise
introduction to Galois connections, and [18] for further details.
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This paper introduces a logical apparatus that enables one to study in a formal setting basic interdependencies between what can be called (cf. §6.1) two modes of knowledge: knowledge-that (or knowledge of truths) and knowledge-how (or knowledge of methods). In mathematical practice, these have been traditionally represented by propositions (i.e., assertions, such as theorems and conjectures) and problems (such as geometric construction problems and initial value problems). The English word “problem” is, in fact, somewhat imprecise; we will use it in the narrow sense of a request (or desire) to find a construction meeting specified criteria on output and permitted means (as in “chess problem”). This meaning is less ambiguously captured by the German Aufgabe (as opposed to the German Problem) and the Russian задача (as opposed to проблема). The closest English word is task (other words with related meanings include assignment, exercise, challenge, aim, mission), but as it is not normally used in mathematical contexts, we prefer to speak of problems.

To appreciate the difference between problems and propositions, let us note firstly that the problem requesting to find a proof of a proposition $P$ is closely related to both (i) the proposition asserting that $P$ is true; and (ii) the proposition asserting that $P$ is provable. These are not the same, of course, whenever “proofs” are taken to be in some formal theory $T$ and “truth” is taken according to some two-valued model $M$ of $T$, with respect to which $T$ is not complete.\footnote{For instance, $T$ could consist of the axioms of planar geometry except for the axiom of parallel lines, and $M$ could be the Euclidean planar geometry. One could object that it would be fair to compare truth according to Euclidean geometry with proofs in its complete theory; but then $T$ could be Peano Arithmetic or ZFC, which by Gödel’s theorem are not complete with respect to any models. (See [1; §3.6] for a more detailed discussion.)} There are other reasons why “true” should not be equated with “provable”; for instance, they differ also in the modal logic S4, where it is a simple consequence of the axioms that consistency is provable.\footnote{With respect to the internal notion of provability. As shown by Artemov [7], the latter can be modelled by the existence of proofs in Peano Arithmetic, where “proofs” have the usual meaning of formal proofs (except that Artemov needs one “proof” to be able to prove several formulas), but “existence” is understood in an explicit sense, not expressible internally in Peano Arithmetic.} Conversely, the proposition asserting that two groups $G$ and $H$ are isomorphic is closely related to both (i) the problem requesting to prove that $G$ and $H$ are isomorphic; and (ii) the problem requesting to construct an isomorphism between $G$ and $H$. These are generally not the same because one proof that an isomorphism exists might represent several distinct isomorphisms or no specific isomorphism.

The logical distinction between problems and theorems (as they appear, in particular, in Euclid’s Elements) has been articulated at length by a number of ancient Greek geometers in response to others who disputed it. A detailed review of what the ancients had to say on this matter is included in the third part of this paper [3]. In modern times, the distinction was emphasized by Kolmogorov [40]:
“On a par with theoretical logic, which systematizes schemes of proofs of theoretical truths, one can systematize schemes of solutions of problems — for example, of geometric construction problems. For instance, similarly to the principle of syllogism we have the following principle here: If we can reduce solving \( b \) to solving \( a \), and solving \( c \) to solving \( b \), then we can also reduce solving \( c \) to solving \( a \).

Upon introducing appropriate notation, one can specify the rules of a formal calculus that yield a symbolic construction of a system of such problem solving schemes. Thus, in addition to theoretical logic, a certain new calculus of problems arises. In this setting there is no need for any special, e.g. intuitionistic, epistemic presuppositions.

The following striking fact holds: The calculus of problems coincides in form with Brouwer’s intuitionistic logic, as recently formalized by Mr. Heyting.

In the second section we undertake a critical analysis of intuitionistic logic, accepting general intuitionistic presuppositions; and observe that intuitionistic logic should be replaced with the calculus of problems, since its objects are in reality not theoretical propositions but rather problems.”

A key difference between problems and propositions is that the notion of truth for propositions has no direct analogue for problems, so that problems cannot be asserted. For instance, let \( \Gamma \) be the problem Divide any given angle into three equal parts with compass and (unmarked) ruler. Then \( \Gamma \lor \neg \Gamma \) reads, Divide any given angle into three equal parts with compass and ruler or prove that it is impossible to do so (cf. [1; §3.8] and 3.10 below). This is not a trivial problem; indeed, its solution took a couple of millennia. Even now that a solution is well-known, the problem still makes perfect sense: the law of excluded middle would not help a student to solve this problem on an exam (in Galois theory). By citing the law of excluded middle she could solve another problem: Prove that either \( \Gamma \) has a solution or \( \Gamma \) has no solutions; in symbols,

\[
!(?\Gamma \lor \neg ?\Gamma),
\]

where \(?\Gamma\) denotes the proposition There exists a solution of the problem \( \Gamma \), and \(!P\) denotes the problem Prove the proposition \( P \).\(^3\) In fact, this problem is strictly easier than

\[
!?\Gamma \lor \neg !?\Gamma
\]

(in words, Prove or disprove that \( \Gamma \) has a solution), which requires a justified explicit choice. But the latter problem, which can be written equivalently as \(!?\Gamma \lor \neg !\Gamma\), is still strictly easier than the original problem, \( \Gamma \lor \neg \Gamma \), for it is generally easier to prove that some problem has a solution than to actually solve it.

1.2. A joint logic

The present paper is devoted to the study of the logical operators \(?\) and \(!\) in a formal setting. Like in the previous example, \(!\) is meant to refer to non-constructive proofs, whereas

\(^3\)Let us explain the notation. A proposition comes with a question whether it is true or false; whereas a problem comes with an urge to solve it. Thus \(?\) can serve as a concise typing symbol for propositions, and \(!\) for problems. By placing a typing symbol in front of a sentence we indicate its conversion into the corresponding type.
is understood to signify explicit existence. We extract axioms and rules governing the use of ? and ! essentially from two sources:

- The problem interpretation of intuitionistic logic. This is essentially Kolmogorov’s 1932 explanation of the intuitionistic connectives [40], which had some parallels with the independent writings of Heyting (1931), and was slightly refined by Heyting (1934). A disguised form of this explanation, often incorporating a further refinement by Kreisel, has come to be known as the BHK interpretation of intuitionistic logic (see [1; §3] for a detailed review and discussion). We include Kreisel’s addendum in the following form, also found in the ancient commentary by Proclus on Euclid’s Elements (see [3]): A solution of a problem must include not only a construction, but also the verification, i.e. a proof that the construction meets the requirements specified in the problem (see [1; §3.8] for a discussion of this principle).

- The proof interpretation of intuitionistic logic. This is essentially the meaning explanation of intuitionistic logic given independently by Orlov (1928) and Heyting (1930, 31) (see a detailed review in §6.2.1), which was partially formalized in Gödel’s 1933 translation of problems into modal propositions (see [1; §5.7.3]), and further clarified by Gödel’s proof-relevant analogue of S4 (see §2.4.2).

It then comes as a little surprise that the resulting axioms and rules harbor a great deal of unintended symmetries, and are also compatible with Kolmogorov’s double negation translation of propositions into problems (reviewed briefly in [1; §5.6]). (For a different connection between Kolmogorov’s and Gödel’s translations see [16].) What is most surprising, however, is that nobody seems to have studied the operators ? and ! before, apart from hints of an abandoned project aimed at a similar study, found in Kolmogorov’s own writings. In his 1931 letter to Heyting [41], Kolmogorov wrote:

Each ‘proposition’ in your framework belongs, in my view, to one of two sorts:

(α) p expresses hope that in prescribed circumstances, a certain experiment will always produce a specified result. (For example, that an attempt to represent an even number \( n \) as a sum of two primes will succeed upon exhausting all pairs \((p, q), p < n, q < n.\) Of course, every “experiment” must be realizable by a finite number of deterministic operations.

(β) p expresses intention to find a certain construction.

[... I prefer to keep the name proposition (Aussage) only for propositions of type (α) and to call “propositions” of type (β) simply problems (Aufgaben). Associated to a proposition p are the problems \( \sim p \) (to derive contradiction from p) and \( + p \) (to prove p).

The fact that the Orlov–Heyting–Gödel proof interpretation is substantially different from the Kolmogorov–Heyting–Kreisel problem interpretation seems to have been never properly recognized, except that Gödel’s paper formalizing the Orlov–Heyting interpretation begins with a reference to Kolmogorov’s “somewhat different interpretation ... [given] without, to be sure, specifying a precise formalism” [25] (see also [51; p. 235]). A certain precise formalism attempting to capture Kolmogorov’s interpretation alone is specified in [1; §5.1].

This \( p \) (prime number) is unrelated to the previous \( p \) (proposition).
Apart from this fragment and the quote in the abstract, there are only a few further hints at how Kolmogorov envisaged the connection between problems and propositions. Several problems consisting in proving a proposition are also mentioned in Kolmogorov’s paper [40]. There is also a bit more in Kolmogorov’s letters to Heyting, which will be thoroughly reviewed in §6.2.2. There we note, in particular, that while Kolmogorov’s propositions of type ($\beta$) seem to stand precisely for the objects of intuitionistic logic, his propositions of type ($\alpha$) could not be intended to exhaust all objects of classical logic; in fact, it appears that they can be identified with the “stable propositions” of §4. Another apparent divergence between Kolmogorov’s remarks and our approach is noted in 3.10 and discussed more thoroughly in [1; §3.8].

The joint logic of problems and propositions that is constructed in the present paper is presumably very unnatural in the standard constructivist paradigm (of Brouwer and Heyting) that views intuitionistic logic as an alternative to classical logic that criminalizes some of its principles. We work in the other paradigm (of Kolmogorov), which views intuitionistic logic as an extension package that upgrades classical logic without removing it. For us, the main purpose of this upgrade is solution-relevance (“proof-relevance”), or “categorification”. Thus from the viewpoint of the BHK semantics, topological (Tarski) models are in fact models of a “squashed” copy of intuitionistic logic — whose existence is only revealed with the aid of the new connectives ! and ? (see §5.2 below); whereas “true” models of the genuine intuitionistic logic are the (solution-relevant) sheaf-valued models of [1] (a special case of “categorical models” — not to be confused with the usual “sheaf models” of intuitionistic logic). Models of the joint logic of problems and propositions will be discussed in [3].

1.3. Double negation translation

Speaking of “intuitionistic logic as an extension package that upgrades classical logic without removing it”, we run into the natural question: “Wait, but what about the double negation translation?” Indeed, there is a version of the double negation translation that redefines classical connectives in terms of intuitionistic ones and introduces no other modifications to formulas (see [1; §5.6]). However, this syntactic translation fails to reflect actual mathematical practice. There are several levels at which this failure occurs:

(i) In the words of Kreisel [43], “there is a good reason why mathematicians neglect” the double negation translation, in the form of “replacing $\exists$ by $\neg\forall\neg$ and $p \lor q$ by $\neg(\neg p \lor \neg q)$”, “namely, this: For the sense in which mathematicians actually understand the propositions of mathematical practice, … the difference between $\exists$ and $\lor$ on the one hand and their translations on the other … is not significant”. “Put differently, they do not understand the intuitionistic meaning of $\lor$ and $\neg$ which makes the [double negation] translation significant.”

This is not merely a matter of mathematicians’ conventions, psychology or ignorance. For mathematicians to be serious about the intuitionistic meaning of propositions, in the tradition of Brouwer and Heyting, they would have to sacrifice their understanding of
mathematical objects as ideal entities existing independently of one’s knowledge about them. But most of them certainly do not want to be “expelled from the paradise that Cantor has created”, and for a good reason: the customary mental aid of Platonism does simplify their job immensely.

(ii) Kolmogorov’s problem interpretation of intuitionistic logic entirely avoids the issue of sacrificing platonist thinking. But then the double negation translation makes no sense, because, when understood in these terms, it conflates problems with propositions; and when corrected so as to respect their distinction, it is no longer a translation into plain intuitionistic logic. This “corrected” double negation translation (see §5.4) is, actually, quite meaningful from the viewpoint of mathematical practice; for instance, \( \exists x P(x) \), “there exists an \( x \) such that \( P(x) \)” is interpreted as \( \neg \neg \exists x !P(x) \), “it is impossible to derive a contradiction from a construction of an \( x \) along with a proof of \( P(x) \)”. The “corrected” double negation translation is essentially equivalent to Fitting’s translation of classical logic into the modal logic QS4.

(iii) Even though the “corrected” double negation translation is no longer a translation into plain intuitionistic logic, one might still ask if its effect is significant from the viewpoint of mathematical practice. The assertion that its effect is trivial is equivalent (see [2; 3.21]) to the so-called K-principle, \( \neg !P \rightarrow !\neg P \), an independent principle of the joint logic of problems and propositions. But the effect of the K-principle is drastic: it immediately rules out independent statements (see [1; §3.8.3] and [2; §3.1]).

1.4. Related work

Modern literature contains a number of attempts to blend classical and intuitionistic logics. On the one hand, there are the Linear Logic and the logics of Japaridze [35], [36], [37], [38] and Liang–Miller [44], [45], which all have something classical and something intuitionistic in them — albeit fused in far more elaborate ways than Kolmogorov could have possibly meant in his words: “A unified logical apparatus was intended to be created, which would deal with objects of two types — propositions and problems” [42].

On the other hand, there is Artëmov’s Logic of Proofs LP, which he actually meant to address these very words of Kolmogorov [7; p. 2]. It can be said to deal with objects of two types — propositions and their proofs; so, it does not exactly fit Kolmogorov’s description. In a paper in progress the author studies a proof-relevant extension of the joint logic of problems and propositions which includes a variation of Artëmov’s LP.

What is more obviously related to Kolmogorov’s research program is the “propositions-as-some-types” paradigm, and indeed our composite operator \(!?\) on problems is very similar to the squash/bracket operator in intuitionistic type theories (see §3 and 3.18). There are also similarities between our approach and some ideas behind the Calculus of Constructions [13] (see also [5] and [11]).

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6This is a literal translation; the meaning of “Предполагалось создание ...” is inherently ambiguous, and could well be either “I intended to create ...” or “We intended to create with my colleagues ...” or “I intended a student to create ...”.
A direct type-theoretic analogue of our \( \Box \) due to Aczel and Gambino [5; §1.3] is dissimilar to \( \Diamond \) in that it satisfies a reversible analogue of our schema \((?\_\_\_\_)\) (see §2.4). In contrast, the reversibility of our \((?\_\_\_\_)\) would amount to allowing the BHK interpretation to represent arbitrary, and not just constructive functions (see [1; §3.9]). But there is nothing surprising here, since Aczel and Gambino do not assume the principle of excluded middle on either the two sides.

A type-theoretic analogue of our \( \Diamond \) due to Coquand [13; §1] satisfies an analogue of our schema \((\forall \_\_\_\_)\) (see §3.6), which Coquand argues to express “Heyting’s semantics of the universal quantification”. This time we see a full agreement on the syntactic level; but it is remarkable that our formalization of the BHK clause for the universal quantification is not \((\forall \_\_\_\_)\), which is reversible just like its Coquand’s version, but \((\forall \_\_\_\_)\) (see §2.4), which is irreversible for the same reasons as \((?\_\_\_\_)\).

2. QHC calculus

In the present series of papers we work in first-order logic, but with some deviations from standard terminology, notation and conventions. Namely, our basic syntactic setup is the meta-logic of [1; §4], which is a slightly simplified and “mathematicized” version of the meta-logic used in the Isabelle proof-checker. The simplification is mostly concerned with omission of features that are not needed for dealing with first-order logics (without equality). To be precise, in the present series of papers we use the straightforward extension of the setup in [1; §4] to the case of first-order logic with many-sorted predicate variables.

The following includes a quick summary of [1; §4] which should suffice for the reader who is familiar with some conventional treatments of first-order logic as well as simply-typed \( \lambda \)-calculus and natural deduction.

2.1. Simply-typed \( \lambda \)-calculus

The language in which our logic and its meta-logic are formulated is the simply-typed \( \lambda \)-calculus with (binary) products, with [1] and the present series of papers taking the following deviations from standard terminology, notation and conventions.

- The word “term” is used in the sense of first order logic, and consequently we speak of \( \lambda \)-expressions to refer to terms in the sense of \( \lambda \)-calculus. The word “arity” is used is the traditional sense (of logic and mathematics), and consequently we speak of \( \text{types} \) (rather than arities) of \( \lambda \)-expressions. The word “closed” (as e.g. in “closed formula”) is used in the sense of first-order logic, so we refer to \( \lambda \)-expressions that are closed in the sense of \( \lambda \)-calculus as \( \lambda \)-closed ones.
- Abstraction is written in the style of mathematics, as \( x \mapsto T \), and not in the style of logic and computer science, \( \lambda x. T \). Function application is normally written as \( F(T) \) and only in some cases abbreviated as \( FT \). The function type is denoted \( \Gamma \to \Delta \); no associativity conventions for \( \to \) and \( \mapsto \) are assumed.
We omit brackets in iterated products using standard isomorphisms, and use tuples \((T_1, \ldots, T_n)\), also written \(\vec{T}\), which are defined recursively in terms of pairs. Projection on the \(i\)th factor of a product is denoted \(p_i\). We also use multivariable abstraction \(x_1, \ldots, x_n \mapsto T\), which is defined recursively in terms of abstraction and tuples (not just up to \(\alpha\)-equivalence; see \([1; \S 4.2.4]\)).

Substitution is denoted \(S\mid_{x:=T}\) and is undefined whenever some variable is captured. The same goes for the simultaneous substitution \(S\mid_{\vec{x}:=\vec{T}}\). A \(\lambda\)-expression of the form \((\vec{x} \mapsto S)(\vec{T})\) may \(\beta\eta\)-reduce beyond \(S\mid_{\vec{x}:=\vec{T}}\); if such a \(\beta\eta\)-reduction involves no \(\alpha\)-conversions, and its result is in \(\beta\eta\)-normal form, then this resulting \(\lambda\)-expression is denoted \(S[\vec{x}/\vec{T}]\), and the tuple \(\vec{T}\) is called free for \(\vec{x}\) in \(S\) (see \([1; \S 4.2.5]\)).

The variables of a type \(\Gamma\) are denoted \(x^{\Gamma}_1, x^{\Gamma}_2, \ldots\). We generally use lowercase letters to write metavariables for variables and constants, and uppercase letters to write metavariables for arbitrary \(\lambda\)-expressions.

### 2.2. Language of QHC

QHC is a first-order logic without equality, whose predicate variables are of two sorts. To describe its language, we need three basic types:

- \(0\), the type of terms;
- \(1_i\), the type of \(i\)-formulas ("\(i\)" stands for "intuitionistic");
- \(1_c\), the type of \(c\)-formulas ("\(c\)" stands for "classical").

The language of QHC consists of the following sets of typed \(\lambda\)-expressions (variables and constants only), where \(n\) ranges over \(\mathbb{N} = \{0, 1, 2, \ldots\}\):

1. the set of variables of type \(0\), called individual variables;
2. the set of variables of type \(0^n \rightarrow 1_c\), called \(n\)-ary predicate variables;
3. the set of variables of type \(0^n \rightarrow 1_i\), called \(n\)-ary problem variables.

Each of the sets (1), (2_\(n\)), (3_\(n\)) is a countably infinite set. Nullary predicate variables are also called propositional variables. For reasons of readability we will also use the alternative spelling \(a, b, c, \ldots, x, y, z\) for the first 26 individual variables \(x^0_1, \ldots, x^0_{26}\), reserving an upright sans-serif font for this purpose. Similarly, we use the abbreviations \(a, b, c, \ldots, x, y, z\) for the first 26 predicate variables of each arity and \(\alpha, \beta, \gamma, \ldots, \chi, \psi, \omega\) for the first 24 problem variables of each arity, reserving a fancy (Euler) upright serif font for this purpose.

In using predicate and problem variables we follow the tradition of classic texts in first-order logic such as those by Hilbert–Ackermann, Hilbert–Bernays, Church and P. S. Novikov, who did include predicate variables in addition to predicate constants. Modern treatments of first-order logic usually do not include predicate variables in the language, and are content with predicate constants (even though they include propositional variables in the language of propositional logic). In fact, it is clear that the language of a logic in reality contains only predicate variables, whereas predicate constants are chosen.
differently for each theory over the logic, and so actually belong to the language of a theory and not to the language of the logic.

The sets (1)–(3) are common to any two-sorted first-order logic. Specific to QHC are the following constants. **Connectives:**

(4) truth and falsity \( \top, \bot : \mathbb{1}_c \);
(5) classical negation \( \sim : \mathbb{1}_c \rightarrow \mathbb{1}_c \);
(6) classical binary connectives \( \land, \lor, \rightarrow, \leftrightarrow : \mathbb{1}_c \times \mathbb{1}_c \rightarrow \mathbb{1}_c \);
(7) triviality and absurdity \( \checkmark, \times : \mathbb{1}_i \);
(8) intuitionistic negation \( \neg : \mathbb{1}_i \rightarrow \mathbb{1}_i \);
(9) intuitionistic binary connectives \( \land, \lor, \rightarrow, \leftrightarrow : \mathbb{1}_i \times \mathbb{1}_i \rightarrow \mathbb{1}_i \),

**quantifiers:**

(10) classical quantifiers \( \forall, \exists : (0 \rightarrow \mathbb{1}_c) \rightarrow \mathbb{1}_c \);
(11) intuitionistic quantifiers \( \forall, \exists : (0 \rightarrow \mathbb{1}_i) \rightarrow \mathbb{1}_i \),

and **conversion operators:**

(12) \( ! : \mathbb{1}_c \rightarrow \mathbb{1}_i \);
(13) \( ? : \mathbb{1}_i \rightarrow \mathbb{1}_c \).

Some of the connectives and quantifiers are "syntactic sugar", i.e. they should not really be on the above list as they are definable in terms of others. Namely, the intuitionistic \( \leftrightarrow, \sim \) and \( \checkmark \) are definable in terms of the intuitionistic \( \land, \lor, \rightarrow \) and \( \times \); and the classical \( \leftrightarrow, \land, \lor, \sim \) and \( \top \) are definable in terms of the classical \( \rightarrow \) and \( \bot \), and the classical \( \exists \) is definable in terms of the classical \( \forall \) and \( \rightarrow \). However it is convenient to regard all these symbols (4)–(11), including the redundant ones, as "connectives" and "quantifiers".

It should be noted that we do not differentiate graphically between classical connectives/quantifiers and intuitionistic ones, since they can be distinguished by the type of the \( \lambda \)-expressions that they act upon (\( \mathbb{1}_c \) or \( \mathbb{1}_i \)) — except for the nullary connectives, which we do take care to differentiate (classical: \( \top, \bot \); intuitionistic: \( \checkmark, \times \)). This is based on the observation that lowercase Greek letters, which we use to denote problem variables, are visually distinct from lowercase Roman letters, which we use to denote predicate variables. Note, however, the difference between \( \rightarrow \) (classical or intuitionistic implication) and \( \rightarrow \) (function type).

If \( q \) is a quantifier, \( A \) is a \( \lambda \)-expression of type \( \mathbb{1}_c \) or \( \mathbb{1}_i \), and \( x \) is an individual variable, then \( qxA \) abbreviates the \( \lambda \)-expression \( q(x \mapsto A) \). More generally, \( q\vec{x}A \) abbreviates \( q(\vec{x} \mapsto A) \). Due to this abbreviation, \( \lambda \)-abstraction is only implicit in formulas.

**Remark 2.1.** In the preceding paragraph, \( A \) is a metavariable that stands for an arbitrary unknown \( \lambda \)-expression of type \( \mathbb{1}_c \) or \( \mathbb{1}_i \). Accordingly, the symbol "\( A \)" can be read in two ways: as the first Roman uppercase letter or as the first Greek uppercase letter. We will use uppercase letters that are unambiguously Greek (from the viewpoint of \( \text{TeX} \)) to write metavariables that stand unambiguously for a \( \lambda \)-expression of type \( \mathbb{1}_i \), and those unambiguously Roman for \( \lambda \)-expressions of type \( \mathbb{1}_c \).
This completes the description of the pure language of QHC. However, the language $\mathcal{L}$ of a theory over QHC (such as the plane geometry of $[3]$) may additionally contain the following sets:

(14) a finite set of constants of type $0^n \rightarrow 0$, called $n$-ary function symbols;
(15) a finite set of constants of type $0^n \rightarrow 1_c$, called $n$-ary predicate constants;
(16) a finite set of constants of type $0^n \rightarrow 1_i$, called $n$-ary problem constants.

It should be noted that nullary predicate and problem constants are the same kind of $\lambda$-expressions as nullary connectives (i.e., constants of types $1_c$ and $1_i$). It is nevertheless convenient to distinguish them, since the latter belong to the pure language of QHC but the former do not.

Terms of the language $\mathcal{L}$ are defined inductively, as built out of individual variables using the function symbols. Thus not every $\lambda$-expression of type $0$ is a term (for example, no term involves $\lambda$-abstraction). An atomic $c$-formula of $\mathcal{L}$ is a $\lambda$-expression of type $1_c$ obtained by applying either an $n$-ary predicate constant or an $n$-ary predicate variable to an $n$-tuple of terms; an atomic $i$-formula is a $\lambda$-expression of type $1_i$ obtained by applying either an $n$-ary problem constant or an $n$-ary problem variable to an $n$-tuple of terms. A formula of $\mathcal{L}$ is a $\lambda$-expression built out of atomic $c$-formulas and $i$-formulas using the connectives, quantifiers and conversion operators.

A formula of type $1_c$ is called a $c$-formula and a formula of type $1_i$ is called an $i$-formula. (Clearly, every formula is either a $c$-formula or an $i$-formula.)

A purely classical formula is a $\lambda$-expression of type $1_c$ built out of atomic $c$-formulas using classical connectives and classical quantifiers only; a purely intuitionistic formula is a $\lambda$-expression of type $1_i$ built out of atomic $i$-formulas using intuitionistic connectives and intuitionistic quantifiers only.

A $\lambda$-expression of the form $x_1, \ldots, x_n \mapsto F$, where $F$ is a formula and $x_1, \ldots, x_n$ are pairwise distinct individual variables, is called an $n$-formula. It can also be called an $n$-$c$-formula or an $n$-$i$-formula if $F$ is a $c$-formula or an $i$-formula.

### 2.3. Meta-logic

#### 2.3.1. Introduction.

Any kind of literature on first-order logic constantly deals with meta-logical concepts and assertions, but usually only implicitly. Why would one want to make them explicit, and discuss a first-order logic in terms of a formal meta-logic? One reason is that a pedantic verbalist, who ignores the implicit, must perceive the hidden meta-logic as an ever-present conflation and ambiguity. Here are two examples.

**Example 2.2.** The literature on first-order classical and intuitionistic logics is accustomed to speaking of “the syntactic consequence”; but the syntactic consequence in the sense of e.g. the textbooks by Schoefield and Mendelson is inequivalent to the syntactic consequence in the sense of e.g. the textbooks by Church, Enderton, Kolmogorov–Dragalin and Troelstra–van Dalen. Moreover, Kleene and Avron have considered the two notions simultaneously, as well as the corresponding notions of semantic consequence, pointing out that both are commonly used in elementary mathematics.
Kleene’s textbook contains the following example: the arithmetical formula \((x + y)^2 = x^2 + 2xy + y^2\) begs to be understood as an identity (valid for all natural numbers \(x\)), whereas the arithmetical formula \(x^2 + 2 = 3x\) begs to be understood as an equation (i.e., as a condition on \(x\)). There is no special syntax to reflect this obvious distinction in meaning. Yet it is not illusory, as it is reflected in use. For, as noted by Avron, when “dealing with identities [...] the substitution rule is available, and one may infer \(\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}\) from the identity \(\sin 2x = 2 \sin x \cos x\). In contrast, [...] substituting \(\frac{x}{2}\) for \(x\) everywhere in an equation is an error” (see references in \([1; \S 4]\)).

In fact, the difference between the two variants of syntactic consequence is due to the implicit presence of a first-order meta-quantifier in one of them.

**Example 2.3.** In intuitionistic logic, the principle of excluded middle is derivable from the double negation principle (due to the derivability of the schema \(\neg \neg (\gamma \lor \neg \gamma)\)). Nevertheless, the schema \(\alpha \lor \neg \alpha\) expressing the principle of excluded middle is not derivable from the schema \(\neg \neg \alpha \rightarrow \alpha\) expressing the double negation principle (since for \(\alpha = \neg \beta\) the latter is derivable, and the former is not). Thus the widespread practice of expressing principles by schemata is sometimes misleading.

In fact, the difference between principles and schemata is due to the implicit presence of a second-order meta-quantifier in principles.

But, actually, the explicit use of the second-order meta-quantifier makes the whole concept of schemata (i.e., the formal use of metavariables for this purpose) superfluous. Let us recall that early textbooks on first-order logic, such as those of Hilbert–Ackermann, Hilbert–Bernays and P. S. Novikov did not speak of any schemata, but only of formulas; instead, their derivation systems included a substitution rule. Some problems with this early approach are that inference rules were anyway stated in schematic form, and also that the substitution rule is, in contrast to other inference rules, not structural (i.e. it is not preserved itself by substitution without anonymous variables). Non-structurality is a serious complication in trying to treat rules as fully formal objects.

In fact, the use of both first-order and second-order meta-quantifiers enables one to state (structural) rules without using metavariables; and one way to understand the substitution rule is that it is not an inference rule of the logic, but an inference meta-rule of the meta-logic. An advantage of this approach is that side conditions that normally occur in first-order logics, such as “provided that \(x\) is not free in \(\alpha\)” or “provided that \(t\) is free for \(x\) in \(\alpha(x)\)” effectively disappear (more precisely, they remain at the meta-level, but they disappear from what needs to be specified in order to state rules and principles). One consequence of not having to specify exactly which English phrases qualify as “side conditions” in rules and principles is that it becomes feasible to give actual formal definitions of these notions (a rule and a principle) as well as further notions such as a derivable rule, an admissible rule, a first-order logic, and (both variants of) syntactic consequence.
2.3.2. **Meta-formulas.** The language of the meta-logic\(^7\) of a two-sorted first-order logic involves, in addition to the basic types \(0, 1_i,\) and \(1_c,\) a fourth basic type:

- \(\mu,\) the type of meta-formulas;

and consists of the following constants (common to all two-sorted first-order logics).

**Reflection operators:**

- \(\text{!}_i : 1_i \rightarrow \mu,\) the \(i\)-reflection;
- \(\text{!}_c : 1_c \rightarrow \mu,\) the \(c\)-reflection,

**Meta-connectives:**

- \(\& : \mu \times \mu \rightarrow \mu,\) the meta-conjunction;
- \(\Rightarrow : \mu \times \mu \rightarrow \mu,\) the meta-implication,

and **meta-quantifiers**

- \(q : (0 \rightarrow \mu) \rightarrow \mu,\) the first-order (universal) meta-quantifier;
- \(q^n_i : ((0^n \rightarrow 1_i) \rightarrow \mu) \rightarrow \mu,\) the \(n\)-ary second-order (universal) \(i\)-meta-quantifier;
- \(q^n_c : ((0^n \rightarrow 1_c) \rightarrow \mu) \rightarrow \mu,\) the \(n\)-ary second-order (universal) \(c\)-meta-quantifier.

Here \(n\) ranges over \(\mathbb{N} = \{0, 1, 2, \ldots\}.\) In practice, meta-quantifiers are written like the old-style (early 20th century) universal quantifiers, but with fancy parentheses (\(⟮\)\)) so as to avoid visual confusion with the ordinary parentheses (\((\))): if \(q : (\Delta \rightarrow \mu) \rightarrow \mu\) is a meta-quantifier (either of them), \(F\) is a \(\lambda\)-expression of type \(\mu,\) and \(x\) is a variable of type \(\Delta,\) then \((x) F\) abbreviates the \(\lambda\)-expression \(q(x \mapsto F).\) More generally, \((x_0, \ldots, x_n) F\) abbreviates \((x_0)(x_1, \ldots, x_n) F.\)

An **atomic meta-formula** is a \(\lambda\)-expression of type \(\mu\) that is either of the form \(\text{!}_c F,\) where \(F\) is a \(c\)-formula, or of the form \(\text{!}_i \Phi,\) where \(\Phi\) is an \(i\)-formula. A **meta-formula** is a \(\lambda\)-expression of type \(\mu\) built out of atomic meta-formulas using meta-connectives and meta-quantifiers. We usually omit \(\text{!}_c\) and \(\text{!}_i\) in writing \(\lambda\)-expressions of type \(\mu;\) thus atomic meta-formulas are effectively identified with formulas, keeping in mind that meta-connectives and meta-quantifiers cannot be used inside of formulas.

As usual, \(F \iff G\) abbreviates \((F \Rightarrow G) \& (G \Rightarrow F);\) “\(\iff\)” is called **meta-equivalence.** We stick to the following order of precedence of logical and meta-logical symbols (in groups of equal priority, starting with higher precedence/stronger binding):

1. \(!, ?, \neg, \exists \) and \(\forall;\)
2. \(\land\) and \(\lor;\)
3. \(\rightarrow\) and \(\iff;\)
4. \(();\)
5. \(\&;\)
6. \(\Rightarrow\) and \(\iff.\)

2.3.3. **Meta-rules.** The inference meta-rules (i.e., the inference rules of the meta-logic) are the \(\alpha\)-conversion rule for meta-formulas:

\(^7\)Not to be confused with the meta-language of a logic. (This one would have to be formalized if we were to give a formal treatment of schemata.)
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\[ \frac{\mathcal{F}}{\mathcal{G}}, \text{if } \mathcal{F} \text{ is } \alpha\text{-equivalent to } \mathcal{G}, \]

and the usual introduction/elimination rules of natural deduction for \&, \( \Rightarrow \) and the meta-quantifiers:

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathcal{F} & \mathcal{G} & \mathcal{F} \& \mathcal{G} & \mathcal{F} \& \mathcal{G} & \mathcal{F} \Rightarrow \mathcal{G} & \mathcal{G} \\
\hline
\mathcal{F} \& \mathcal{G} & \mathcal{F} & \mathcal{G} & \mathcal{G} & \mathcal{F} \Rightarrow \mathcal{G} & \vdots
\end{array}
\]

where \( \mathcal{F} \) and \( \mathcal{G} \) are meta-formulas;

\[
\vdots \\
\frac{\mathcal{F}}{(x)\mathcal{F}}, \text{provided that } x \text{ does not occur freely in any of the assumptions;}
\]

\[
\vdots \\
\frac{(x)\mathcal{F}}{\mathcal{F}[x/T]}, \text{provided that } T \text{ is free for } x \text{ in } \mathcal{F},
\]

where \( \mathcal{F} \) is a meta-formula, and there are three ways to read \( x \) and \( T \):

1. \( x \) is an individual variable and \( T \) is a term;
2. \( x \) is an \( n \)-ary problem variable and \( T \) is an \( n \)-i-formula;
3. \( x \) is an \( n \)-ary predicate variable, \( T \) is an \( n \)-c-formula.

It should be noted that \( \mathcal{F}[x/T] \) boils down to the ordinary substitution \( \mathcal{F}|_{x:=T} \) of \( \lambda \)-calculus in the case (1), but not in the cases (2), (3) (see §2.1 above).

Let us note that by using a meta-specialization (=meta-quantifier elimination meta-rule) immediately after the corresponding meta-generalization (=meta-quantifier introduction meta-rule), we get the meta-rules of substitution:

\[
\vdots \\
\frac{\mathcal{F}}{\mathcal{F}[x/T]}, \text{as long as } x \text{ does not occur freely in the assumptions and } T \text{ is free for } x \text{ in } \mathcal{F}.
\]

A meta-formula \( \mathcal{F} \) is called deducible if using the meta-rules one can obtain (from the trivial deductions, in which a meta-formula is deduced from itself) a deduction of \( \mathcal{F} \) from no assumptions.
2.3.4. **Syntactic meta-sugar.** The first-order meta-closure (1) $\mathcal{F}$ of the meta-formula $\mathcal{F}$ is $(\vec{x})\mathcal{F}$, where $\vec{x}$ is the tuple of all individual variables occurring freely in $\mathcal{F}$. The second-order meta-closure (2) $\mathcal{F}$ is $(\vec{\gamma})\mathcal{F}$, where $\vec{\gamma}$ is the tuple of all predicate and problem variables occurring freely in $\mathcal{F}$.

A rule, written $A_1, \ldots, A_m / B$, or, in more detail,

$$\frac{A_1, \ldots, A_m}{B},$$

where $A_1, \ldots, A_m$ and $B$ are formulas, is an abbreviation for the meta-formula

$$\langle 2 \rangle (1) A_1 \& \cdots \& (1) A_m \Rightarrow (1) B).$$

The formulas $A_1, \ldots, A_m$ are called the premisses of the rule, and $B$ its conclusion.

If $B$ is a formula (and only in this case) we abbreviate (2)(1) $B$ by $\cdot B$. A meta-formula of the form $\cdot B$, where $B$ is a formula, is called a principle. In other words, a principle is a formula that is meta-quantified over all its free (individual, predicate and problem) variables. Rules with no premisses can be identified with principles, in the sense that each meta-formula of the form $(/B) \Leftrightarrow \cdot B$ is deducible, as long as the empty meta-conjunction is defined as an abbreviation of some deducible meta-formula (for example, $(\gamma)\gamma \Rightarrow (\gamma)\gamma$).

The difference between formulas and principles is clear from Example 2.3: in (the meta-logical extension of) intuitionistic logic, the meta-formula

$$\cdot \neg\neg\alpha \Rightarrow \neg\alpha \land \alpha$$

is deducible, whereas the meta-formula

$$\neg\neg\alpha \Rightarrow \alpha \Rightarrow \alpha \lor \neg\alpha$$

is not deducible.

A derivation system\(^8\) $\mathcal{D}$ is a meta-formula of the form

$$\mathcal{H}_1 \& \cdots \& \mathcal{H}_k,$$

where each $\mathcal{H}_i$ is a rule (possibly with no premisses) in the pure language of QHC. The $\mathcal{H}_i$ with no premisses, or rather the corresponding principles, are called the laws, and the $\mathcal{H}_i$ with at least one premise are called the inference rules.

A logic is a meta-equivalence class of derivation systems. In other words, derivation systems $\mathcal{D}$ and $\mathcal{D}'$ are said to determine the same logic if the meta-formula $\mathcal{D} \Leftrightarrow \mathcal{D}'$ is deducible.

A meta-formula $\mathcal{F}$ is called derivable in the logic determined by a derivation system $\mathcal{D}$ if the meta-formula $\mathcal{D} \Rightarrow \mathcal{F}$ is deducible (in the meta-logic). Clearly, adding a derivable principle or rule to a derivation system $\mathcal{D}$ does not affect derivability of principles and rules in the logic determined by $\mathcal{D}$.

\(^8\)Also called a “deductive system” in the literature. For our purposes it is convenient to distinguish meta-logical deductions from derivations in a specific logic.
If $L$ is the logic determined by a derivation system $\mathcal{D}$, we denote by $\vdash F$, or in more detail $\vdash_L F$, the judgement that the meta-formula $F$ is derivable in the logic. The meta-meta-logical symbol $\vdash$ is set to have lower priority than all logical and meta-logical symbols. The judgement $\vdash F_1 \land \cdots \land F_m \Rightarrow G$ is also abbreviated by $F_1,\ldots,F_m \vdash G$.

When this judgement is true, we also say that $G$ is a (syntactic) consequence of the $F_i$. This yields two notions of syntactic consequence for formulas: $A_1,\ldots,A_m \vdash B$ is the traditional “fixed variables” one, as in the textbooks by Church, Troelstra and van Dalen; whereas $(\forall) A_1,\ldots,(\forall) A_m \vdash (\forall) B$ is the traditional “varied variables” one, as in the textbooks by Schoenfield and Mendelson. There seems to be no standard notation for the judgement of interderivability for formulas:

$$\vdash A \iff B$$

so we will keep it in this form. Let us note that, due to the absence of the deduction theorem in QHC, it is weaker than the (object-level) equivalence (which makes sense when both if $A$ and $B$ are either i-formulas or c-formulas),

$$\vdash A \leftrightarrow B,$$

but stronger than the equivalence of principles,

$$\vdash \top A \leftrightarrow \top B,$$

which is in turn stronger than the equivalence of judgements:

$$\vdash A \text{ if and only if } \vdash B.$$ 

2.4. Derivation system

When writing down a derivation system for a new logic, one has to engage in informal considerations, or else risk the new logic being entirely unmotivated.

To provide an informal mathematical meaning to the judgements of QHC, we interpret c-formulas by propositions/predicates and i-formulas by problems. More precisely, we instantiate predicate variables and problem variables by particular mathematical predicates and problems. Upon such instantiation, classical connectives and quantifiers are interpreted according to the usual truth tables; intuitionistic connectives and quantifiers according to the BHK interpretation, in Kolmogorov’s problem solving terminology (see below); and the conversion operators $!$ and $?$ are interpreted as in §1. The interpretation of the meta-logical constants and judgements will be discussed in part II.

Some laws and inference rules of the QHC calculus are immediate:

- All laws and inference rules of classical predicate logic (see [1; §4.6]).
- All laws and inference rules of intuitionistic first-order logic (see [1; §4.6]).

Let us note that by using the substitution (i.e., meta-generalization followed by meta-specialization) we can apply the classical laws and inference rules to arbitrary i-formulas.
(possibly involving ? and !) and the intuitionistic laws and inference rules to all c-formulas (possibly involving ? and !).

We will now discuss the remaining part of the derivation system.

2.4.1. From the problem interpretation. Let us recall Kolmogorov’s problem interpretation of intuitionistic logic \[40\] (with minor improvements largely due to Heyting; see \[1; \S 3.7, \S 3.8\] for further details).\(^9\)

We fix a prescribed class of specific problems, which may have parameters that run over a fixed domain \(D\). These are our primitive problems, and we assume that it is known what is a solution of each primitive problem for each value of the parameters. For instance, Euclid’s first three postulates are the following primitive problems:

1. draw a straight line segment from a given point to a given point;
2. extend any given straight line segment continuously to a longer one;
3. draw a circle with a given center and a given radius.

We may thus stipulate that each of (1) and (3) has a unique solution, and describe all possible solutions of (2). (Euclid’s Elements is discussed in some detail in part III of the present series, \[3\].)

Composite problems are obtained from the primitive ones by using connectives \(\land, \lor, \to, \neg, \times\) and quantifiers \(\forall, \exists\). What it is a solution of a composite problem is explained as follows:

- a solution of \(\Gamma \land \Delta\) consists of a solution of \(\Gamma\) and a solution of \(\Delta\);
- a solution of \(\Gamma \lor \Delta\) consists of an explicit choice between \(\Gamma\) and \(\Delta\) along with a solution of the chosen problem;
- a solution of \(\Gamma \to \Delta\) is a reduction of \(\Delta\) to \(\Gamma\); that is, a general method of solving \(\Delta\) on the basis of any given solution of \(\Gamma\);
- the absurdity \(\times\) has no solutions; \(\neg \Gamma\) is an abbreviation for \(\Gamma \to \times\);
- a solution of \(\exists x \Theta(x)\) is a solution of \(\Theta(x_0)\) for some explicitly chosen \(x_0 \in D\);
- a solution of \(\forall x \Theta(x)\) is a general method of solving \(\Theta(x_0)\) for all \(x_0 \in D\).

A key element here is the notion of a general method (roughly corresponding to the notion of “construction” advocated by Brouwer and Heyting), which Kolmogorov further explains as follows. If \(\Gamma(\mathcal{X})\) is a problem depending on the parameter \(\mathcal{X}\) “of any sort”, then “to present a general method of solving \(\Gamma(\mathcal{X})\) for every particular value of \(\mathcal{X}\)” should be understood as “to be able to solve \(\Gamma(\mathcal{X}_0)\) for every given specific value of \(\mathcal{X}_0\) of the variable \(\mathcal{X}\) by a finite sequence of steps, known in advance (i.e. before the choice of \(\mathcal{X}_0\))”.

Let us observe that if \(|\Gamma|\) denotes the set of solutions of the problem \(\Gamma\), then the above clauses guarantee that:

- \(|\Gamma \land \Delta|\) is the product \(|\Gamma| \times |\Delta|\);

\(^9\)This can also be understood as the BHK interpretation presented in Kolmogorov’s language. However, given that Heyting’s early ideas are often conflated with the BHK interpretation in the literature, but will be understood in a very different way below, as providing a complement to the BHK interpretation, one must be very careful here about exactly what is meant by the “BHK interpretation”.

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- \(|\Gamma \lor \Delta|\) is the disjoint union \(|\Gamma| \sqcup |\Delta|\);
- there is a map \(\mathcal{F} : |\Gamma| \to |\Delta|\) into the set of all maps;
- \(|\times| = \emptyset\);
- \(|\exists x \Theta(x)|\) is the disjoint union \(\bigsqcup_{d \in D} |\Theta(d)|\);
- there is a map \(\mathcal{F} : |\forall x \Theta(x)| \to \prod_{d \in D} |\Theta(d)|\) into the product.

Now the proposition “\(\Gamma\) has a solution” can be rephrased as “\(|\Gamma| \neq \emptyset\)”. It follows that the following propositions must be true for any contentful problems \(\Gamma, \Delta\) and any contentful parametric problem \(\Theta\):

- \(\neg ?(\Gamma \land \Delta) \iff ?(\Gamma \land ?\Delta)\);
- \(\neg ?(\Gamma \lor \Delta) \iff ?(\Gamma \lor ?\Delta)\);
- \(\neg ?(\Gamma \rightarrow \Delta) \iff (\?\Gamma \rightarrow ?\Delta)\);
- \(\neg ?\times\);
- \(\neg \exists x \Theta(x) \iff \exists x ?\Theta(x)\);
- \(\neg \forall x \Theta(x) \iff \forall x ?\Theta(x)\).

See [1; §3.8] for a more thorough discussion of these propositions.

This motivates some laws of QHC (beware that some of these will turn out to be redundant):

\[
(\bot \land) \cdot ?(\gamma \land \delta) \iff ?(\gamma \land ?\delta);
(\bot \lor) \cdot ?(\gamma \lor \delta) \iff ?(\gamma \lor ?\delta);
(\bot \rightarrow) \cdot ?(\gamma \rightarrow \delta) \iff (?\gamma \rightarrow ?\delta);
\]

It should be noted that formulas with almost same appearance and motivation, but somewhat different meaning appear in [1; §5.1].

Informally, \(\bot\) is saying that \(\times\) is not just the hardest problem (as guaranteed by the explosion principle, \(\bullet \times \rightarrow \gamma\)), but a problem that has no solutions whatsoever. This is just the first example of how some content found in the BHK interpretation and not entirely captured in the usual formalization of intuitionistic logic is more fully captured in QHC.

Some versions of the BHK interpretation include the well-known principle (see [1; §3.8]), that every solution of a problem \(\Gamma\) must be supplied with a proof that is it indeed a solution of \(\Gamma\). This principle was emphasized by G. Kreisel in connection with interpreting intuitionistic logic (in a somewhat different form) and also by the ancient Greeks, particularly Proclus, in the context of geometric construction problems, which as we now know can be seen as a model of intuitionistic logic (see [3]). This Proclus–Kreisel principle is usually considered to be relevant when one tries to make sense out the BHK interpretation in the context of first-order logic, rather than a constructive type theory (see references in [1; §3.8]).
A consequence of this Proclus–Kreisel principle is that a solution of a problem $\Gamma$ yields a proof of the existence of a solution of $\Gamma$. This is expressible in the language of QHC:

\[ (!?) \cdot \gamma \rightarrow !?\gamma. \]

2.4.2. From the proof interpretation. The remaining part of the derivation system is motivated by the proof interpretation of intuitionistic logic, given independently by Orlov and Heyting (see details in §6.2.1) and partially formalized in Gödel’s translation of intuitionistic logic into classical modal logic S4 (see [1; §5.7.3]). A remarkable attempt to clarify the informal notion of “proof” used by Orlov and Heyting occurs in Gödel’s sketch of a proof-relevant analogue of S4, which is found in his outline of a 1938 lecture, published posthumously in his collected works [24].

Gödel’s proposal is based on a ternary relation “$zBp, q$, that is, $z$ is a derivation of $q$ from $p$”. But as a matter of fact he also uses a binary relation “$aBq$” which is presumably meant to abbreviate $aB\top, q$. Here $B$ stands for German Beweis (proof), and apparently refers to proofs “understood not in a particular system, but in the absolute sense (that is, one can make it evident)” (these words of Gödel appears earlier on the same page). Gödel’s axioms for $B$ are as follows (literally):

1. “$zB\varphi(x, y) \rightarrow \varphi(x, y)$”;
2. “$uBv \rightarrow u'B(uBv)$”;
3. “$zBp, q \& uBq, r \rightarrow f(z, u)Bp, r$”;
4. “if $q$ has been proved and $a$ is the proof, [then] $aBq$ is to be written down”.

Instead of attempting to clarify the meaning of this in Gödel’s original terms, let us consider a very similar but more clearly described logic. Namely, let S4pr be the extension of classical predicate logic with the following additional elements of the language:

- an operator $\cdot$ associating to every formula $F$ and every term $t$ a formula $t:F$;
- a unary function $'$ that associates to every term $t$ a term $t'$;
- a binary function $[\cdot]$ that associates to every two terms $s, t$ a term $s[t]$;
- an operator $\ast$ that associates to every formula $F$ a term $\ast F$,

and the following additional laws and inference rules:

(i) $\cdot t:p \rightarrow p$;
(ii) $\cdot t:p \rightarrow t':(t:p)$;
(iii) $s:(p \rightarrow q) \rightarrow (t:p \rightarrow s[t]:q)$;
(iv) $p \ast p :p$.

S. Artemov discovered that a further extension of S4pr by an additional function (“sum of proofs”) and an additional law (not hinted at in any way by Gödel) is indeed a proof-relevant analogue of S4 in a sense one could expect [7]. But we do not need these for our motivational purposes.

---

In fact, the rule $p/\ast p$ is only applied to axioms in Artemov’s logic. The reason why one cannot do without the “sum of proofs” is clear from [7; Example 5.6].
The logic $S4pr$ has the following derived principles and rules:

(i') $\neg (t: \bot)$;
(ii') $\cdot \exists t \; t:p \rightarrow p$;
(iii) $\frac{t:p}{p}$;
(iii') $\cdot t:p \rightarrow \tilde{t}:(\exists t \; t:p)$.

Here (i') is just the special case of (i) with $p$ substituted by the classical falsity $\bot$. Next, (i'') is derived from (i) by using two inference rules of classical logic: $q(t) / \forall t q(t)$ and $\forall t (r(t) \rightarrow p) / \exists t r(t) \rightarrow p$. Of course, (i''') is derived from (i) using the modus ponens rule. To establish (ii'), let us first note that from the classical law $q(t) \rightarrow \exists t q(t)$ we get $t:p \rightarrow \exists t t:p$, and if $i$ denotes the latter formula, then by (iv) we get $*_{\delta} : (t : p \rightarrow \exists t t : p)$. Now from (iii) and the modus ponens rule we get $t' : (t : p \rightarrow *_{\delta}[t'] : (\exists t t : p))$. Finally, (ii') follows from this and (ii), if we set $\tilde{t} = *_{\delta}[t']$.

Just like Gödel's proofs "in the absolute sense", the "proofs" of propositions referred to in the intended reading of the problem $!P$, Find a proof of $P$, are not supposed to be formal proofs. In the language of QHC, we have the following direct analogues of (i'), (i''), (ii'), (iii), (iv) and (i''):

$$
\begin{align*}
(l_\bot) & \rightarrow l_\bot; \\
(?!') & \cdot ?!p \rightarrow p; \\
(?!') & \cdot p \rightarrow ?!p; \\
(l_\bot) & \cdot !(p \rightarrow q) \rightarrow (l p \rightarrow !q); \\
(l_{\top}) & \frac{p}{l p}; \\
(l'_{\top}) & \frac{!p}{p}.
\end{align*}
$$

Here $(l_\bot)$ is a kind of internal soundness: a proof of falsity leads to absurdity. Semantically (informally), this is pretty much like in Gödel's system; but let us note that (i') is a c-formula ($t : \bot \rightarrow \bot$), whereas $(l_\bot)$ is an i-formula $(!_\bot \rightarrow \times)$. In contrast, $(?_\bot)$ is a c-formula $(? \times \rightarrow \bot)$. Note that by the explosion principle, the reverse implications to $(?_\bot)$ and $(l_\bot)$ are trivial. Thus $(?_\bot)$ identifies the classical falsity, $\bot$, with the proposition "$\times$ has a solution"; and $(l_\bot)$ identifies the intuitionistic absurdity, $\times$, with the problem "Prove $\bot$".

This completes the list of additional inference rules and laws of QHC. Let us note that $(?!')$ can be dropped from this list since it follows immediately from $(?!), (l_{\top})$ and $(l_\bot)$. Some other laws will be shown to be redundant in 3.6.

### 3. Symmetries and redundancy

#### 3.1. Galois connection

**Proposition 3.1.** The inference rule $(l'_{\top})$ is equivalent to the following inference rule:
We will see in [2] that the converse rule, ?\gamma / \gamma, is not derivable in QHC.

\textbf{Proof.} Given (? p), we can derive (? p) using (?): \gamma, \gamma \rightarrow !? \gamma / !? \gamma and !? \gamma / ? \gamma. Conversely, given (? p), we can derive (? p) using (?): !p / ?!p and ?!p, ?!p \rightarrow p / p. \Box

The equivalence relations \vdash \Phi \leftrightarrow \Psi on i-formulas and \vdash F \leftrightarrow G on c-formulas yield the “Lindenbaum” poset of equivalence classes of i-formulas, ordered by [\Phi] \geq [\Psi] if \vdash \Phi \rightarrow \Psi, and the “Lindenbaum” poset of equivalence classes of c-formulas, ordered by [F] \geq [G] if \vdash F \rightarrow G. By (? p) and (? p), and respectively (! p) and (! p), we have:

\begin{itemize}
  \item \vdash \Phi \rightarrow \Psi implies \vdash ?\Phi \rightarrow ?\Psi;
  \item \vdash F \rightarrow G implies \vdash !F \rightarrow !G.
\end{itemize}

Thus ? and ! descend to monotone maps between the two posets. Using the monotonicity of ? and ! and substitution, from (?!) and (!?) we also obtain:

\begin{itemize}
  \item \vdash ?!F \leftrightarrow !G;
  \item \vdash ??!\Phi \leftrightarrow ?!\Psi.
\end{itemize}

These identities resemble well-known properties of a Galois connection. Indeed, it turns out that our two monotone maps do form a Galois connection between the two Lindenbaum posets:

\textbf{Theorem 3.2.} For an i-formula \Phi and a c-formula F, \vdash ?\Phi \rightarrow F if and only if \vdash \Phi \rightarrow !F.

The same argument works to prove a slightly stronger assertion, \vdash ?\alpha \rightarrow p \leftrightarrow \alpha \rightarrow !p.

\textbf{Proof.} If \vdash \Phi \rightarrow !F, then \vdash ?\Phi \rightarrow ?!F. So from (?!) we get \vdash ?\Phi \rightarrow F.

Conversely, if \vdash ?\Phi \rightarrow F, then \vdash ?!\Phi \rightarrow !F. So from (!?) we get \vdash \Phi \rightarrow !F. \Box

Another standard fact on Galois connections takes the following form in our situation.

\textbf{Corollary 3.3.} Let F denote a c-formula and let \Phi denote an i-formula.

(a) [!F] is the least among all [\Phi] such that [?\Phi] is an upper bound of [F]; and [?\Phi] is the greatest among all [F] such that [!F] is a lower bound of [F].

(b) [?!F] is the least of all upper bounds of [F] of the form [?\Phi]; and [?!\Phi] is the greatest of all lower bounds of [\Phi] of the form [!F].

\textbf{Proof.} The first assertion of (a) says that \vdash ?!F \rightarrow F, and if \vdash ?\Phi \rightarrow F, then \vdash \Phi \rightarrow !F. This is indeed so by (?!) and by 3.2. The first assertion of (b) says that \vdash ?!F \rightarrow F, and if \vdash ?\Phi \rightarrow F, then \vdash ?\Phi \rightarrow ?!F. This follows similarly, using additionally the monotonicity of ?. The second assertions of (a) and (b) are proved similarly. \Box

3.2. Modalities

Let us write \Box F for the c-formula ?!F, and \nabla \Phi for the i-formula !?\Phi. Upon substituting problems and propositions for the atoms of F and \Phi, these are interpreted by the
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Proposition \( \Box P \), “There exists a proof of \( P \)”, and the problem \( \nabla \Gamma \), “Prove that \( \Gamma \) has a solution”. By another standard fact on Galois connections, the “provability” operator \( \Box = ?! \) descends to an interior operator (in the sense of order theory) on the poset of equivalence classes of c-formulas, whereas the “solubility” operator \( \nabla = !? \) descends to a closure operator (in the same sense) on the poset of equivalence classes of i-formulas. In the case of \( \Box \), this amounts to (i) the derivability in QHC of the principles

\[
(1) \cdot \Box p \rightarrow p; \\
(2) \cdot \Box p \rightarrow \Box \Box p;
\]

and (ii) the judgement

\( \ast \vdash F \rightarrow G \) implies \( \vdash \Box F \rightarrow \Box G \).

These are easy to verify directly: (1) is the same as (?!); (2) follows from (!?) and the monotonicity of ?; and (\( \ast \)) follows from the monotonicity of ! and ?.

In fact, (\( \ast \)) is a consequence of the derivability in QHC of the following principle and rule:

\[
(3) p / \Box p; \\
(4) \cdot \Box (p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q);
\]

Here (3) follows from (!\( \top \)) and (?\( \top \)), and (4) from (!\( \bot \)) and (?\( \bot \)). We have proved

**Proposition 3.4.** Sending \( \Box \) to ?! yields a syntactic interpretation of QS4 in QHC, which is the identity on QC.

We will see in §5.1 that this interpretation is exact. Before we get there, we need to distinguish two roles of the symbol “\( \Box \)”: the modality of QS4 and an abbreviation for ?!

Similarly, that \( \nabla \) induces a closure operator on the poset of equivalence classes of i-formulas translates to (i) the derivability in QHC of the principles

\[
(1) \alpha \rightarrow \nabla \alpha; \\
(2) \cdot \nabla \nabla \alpha \rightarrow \nabla \alpha,
\]

and (ii) the judgement

\( \ast \vdash \Phi \rightarrow \Psi \) implies \( \vdash \nabla \Phi \rightarrow \nabla \Psi \).

Here (\( \ast \)) is a consequence of the derivability in QHC of the principle

\[
(4) \cdot \nabla (\alpha \rightarrow \beta) \rightarrow (\nabla \alpha \rightarrow \nabla \beta),
\]

which follows from (!\( \bot \)) and (?\( \bot \)). We also note that the following consequence of (?\( \bot \)) and (!\( \bot \)),

\[
(3) \nabla \times \rightarrow \times,
\]

is equivalent (modulo (1) and (4)) to \( \neg \alpha / \neg \nabla \alpha \) (cf. §3.4 below), which can be considered to be dual to (3). We define QH4 to be the logic obtained from QH by adding a new unary connective \( \nabla \) and additional laws (1)–(4). We have thus proved:

**Proposition 3.5.** Sending \( \nabla \) to ?! yields a syntactic interpretation of QH4 in QHC, which is the identity on QC.
It should be noted that the laws of $\neg\neg$ mimic some properties of $\neg\neg$. In fact, by substituting $\neg\neg$ for $\nabla$ we get an interpretation of $\neg\neg$ in $\text{QH}$. Indeed, under this substitution, $(1^\nabla)$ holds by ([1; §3.14, (1)]), $(2^\nabla)$ and $(3^\nabla)$ follow from ([1; §3.14, (3)]), and $(4^\nabla)$ holds by ([1; §3.14, (4)]). Let us note that since the purely intuitionistic fragment of $\neg\neg$ is fixed under this interpretation, this fragment is precisely $\text{QH}$ (in other words, $\neg\neg$ is a conservative extension of $\text{QH}$). We will see in §5.3 that the constructed interpretation of $\neg\neg$ in $\text{QH}$ factors through the interpretation of §3.5.

It is not clear to the author whether the interpretation of §3.5 is faithful (in other words, whether $\text{QHC}$ is a conservative extension of $\neg\neg$).

Thus one should not conflate two potentially distinct roles of the symbol “$\nabla$”: the modality of $\neg\neg$ and an abbreviation for $!?!$.

The modal logic $\neg\neg$ was studied by Fairtlough–Walton [20], who called it $\text{QLL}^+$, and Aczel [4], who called it the logic of a strict lax modality (see also [19; Theorem 4.5]). Later the zero-order fragment $\neg\neg$ was also studied by Artëmov and Protopopescu, who showed its completeness with respect to some Kripke models [9] (beware that $\neg\neg$, which is called $\text{IEL}^+$ in [9], disappeared from the published version of the preprint [9]). The intuitionistic modal logic given by the laws $(1^\nabla)$, $(2^\nabla)$ and $(4^\nabla)$ was studied as early as 1950 by H. Curry [14; p. 120] (see also [15; §5]), and later by Goldblatt [26; §14.5] and many others. In particular, categorical models of $\neg\neg$ related to the sheaf-valued models of $\text{QH}$ in [1] are known; see [19], [27; §7.6], [6].

The properties of $\nabla$ are also similar to those of the squash/bracket operator in dependent type theory (see [10] and references there).

3.3. Simplification

**Proposition 3.6.** (a) The laws $(\nabla \land)$, $(\nabla \lor)$, $(\nabla \perp)$ and $(\nabla \exists)$ are redundant.

(b) The following holds in $\text{QHC}$:

$$(!\lambda) \vdash !p \land !q \iff ![p \land q];$$

$$(!\lor) \vdash !p \lor !q \iff ![p \lor q];$$

$$(!\forall) \vdash \forall x !p(x) \iff !\forall x p(x);$$

$$(!\exists) \vdash \exists x !p(x) \iff !\exists x p(x).$$

**Remark 3.7.** From the informal semantic viewpoint, the implication $\vdash !p \lor !q \rightarrow ![p \lor q]$ cannot be reversed. Indeed, let $P$ be the proposition $i^i$ is a rational number and $Q$ the proposition $i^i$ is an irrational real number. The problem $![P \lor Q]$ amounts to showing that $i^i$ is a real number. This problem is trivial: $i^i = (e^{i\pi/2})^i = e^{-\pi/2}$. On the other hand, the problem $!P \lor !Q$ amounts to $![P \lor Q] \land \Gamma$, where $\Gamma$ is the problem Determine whether $e^{-\pi/2}$ is rational or irrational. This is not an easy problem. This shows incidentally that one cannot get an exact interpretation of classical logic in intuitionistic logic by just looking at problems of the form $!P$, where $P$ is a proposition.

---

11Recently A. Onoprienko [47], [48] affirmatively answered this question (which appeared already in the first arXiv version of the present paper).

12In fact, $i^i$ is transcendental by the Gelfond–Schneider theorem.
Proof. Redundancy of \( (?_\land) \). By an intuitionistic law, \( \vdash \forall \alpha(x) \rightarrow \alpha(t) \). Then by \( (?_\top) \) and \( (?_\bot) \), we get \( \vdash ?\forall \alpha(x) \rightarrow ?\alpha(t) \). By the classical generalization rule, we obtain \( \vdash \forall t [?\forall \alpha(x) \rightarrow ?\alpha(t)] \). By another intuitionistic rule, we infer that \( \vdash ?\forall \alpha(x) \rightarrow \forall t ?\alpha(t) \). Now the variable can be renamed. \( \square \)

Redundancy of \( (?_\lor) \). The \( \rightarrow \) implication in \( (?_\lor) \) is redundant similarly to the redundancy of \( (?_\land) \). Conversely, the intuitionistic validity \( \alpha \land \beta \rightarrow \alpha \land \beta \) can be rewritten, by the exponential law, as \( \alpha \rightarrow (\beta \rightarrow (\alpha \land \beta)) \). Then by \( (?_\top) \) and \( (?_\bot) \) it follows that \( \vdash ?\alpha \rightarrow (?\beta \rightarrow (?\alpha \land \beta)) \). Again applying the exponential law, this time regarded as an inference rule of classical logic, we obtain \( \vdash ?\alpha \land ?\beta \rightarrow (?\alpha \land \beta) \). \( \square \)

Proof of \( (!_\exists) \) and \( (!_\forall) \). This is parallel to the redundancy of \( (?_\land) \). In more detail, by a classical principle, \( \vdash p(t) \rightarrow \exists x p(x) \). Then by \( (!_\top) \) and \( (!_\bot) \), we get \( \vdash p(t) \rightarrow !\exists x p(x) \). By the generalization rule, we obtain \( \vdash \forall t [p(t) \rightarrow !\exists x p(x)] \). By another classical rule, we get \( \vdash \exists t p(t) \rightarrow !\exists x p(x) \). Now the variable can be renamed. The case of \( (!_\forall) \) is similar. \( \square \)

Redundancy of \( (?_\exists) \) and \( (?_\forall) \). The \( \leftarrow \) implication in \( (?_\exists) \) is redundant similarly to the proof of \( (!_\exists) \) or to the redundancy of \( (?_\land) \). Conversely, by the proof of \( (!_\exists) \) we have shown that \( \vdash \exists x !p(x) \rightarrow !\exists x p(x) \) using only \( (!_\top) \), \( (!_\bot) \) and classical logic. Substituting, we get \( \vdash ?\exists !?\alpha(x) \rightarrow !\exists x ?\alpha(x) \). On the other hand, from \( (?_\top) \) it follows that \( \vdash ?\exists \alpha(x) \rightarrow ?\exists !?\alpha(x) \). By combining the two implications we get \( \vdash ?\exists \alpha(x) \rightarrow !\exists x ?\alpha(x) \). By the proof of 3.2, we obtain from this the \( \rightarrow \) implication in \( (?_\exists) \), using only \( (?_\top) \), \( (?_\bot) \) and \( (?!) \). The case of \( (?_\forall) \) is similar. \( \square \)

Proof of \( (?_\forall) \) and \( (?_\land) \). The \( \leftarrow \) implication in \( (?_\forall) \) is proved similarly to the redundancy of \( (?_\lor) \). The converse implication is parallel to the redundancy of \( (?_\exists) \). In more detail, \( (?_\forall) \) implies \( \vdash ?\forall x !p(x) \rightarrow \forall x ?p(x) \), and it follows from \( (?!) \) that \( \vdash \forall x ?!p(x) \rightarrow \forall x p(x) \). Thus \( \vdash \forall x !p(x) \rightarrow \forall x p(x) \), hence by 3.2 \( \vdash \forall x !p(x) \rightarrow !\forall x p(x) \). The case of \( (?_\land) \) is similar, or alternatively can be treated similarly to the redundancy of \( (?_\lor) \). \( \square \)

Redundancy of \( (?_\bot) \). By the explosion principle, we have \( \vdash x \rightarrow !\bot \). Then by \( (?_\top) \) and \( (?_\bot) \) we get \( \vdash ?x \rightarrow ?!\bot \). On the other hand, by \( (?!) \) we have \( \vdash ?!\bot \rightarrow \bot \). Composing the two implications, we obtain \( \vdash ?x \rightarrow \bot \). \( \square \)

**Corollary 3.8.** The meta-conjunction of the following meta-formulas is a deductive system for QHC:

- A deductive system for intuitionistic logic;
- A deductive system for classical logic;

\[
!_\top \vdash p; \\
!_\bot \vdash \neg \alpha; \\
?_\top \vdash \forall \alpha; \\
?_\bot \vdash ?!p \rightarrow p; \\
?!_\top \vdash \alpha \rightarrow !!}\alpha;
\]
\[(\bot) \cdot !(p \rightarrow q) \rightarrow (!p \rightarrow !q);\]
\[(? \bot) \cdot ?(\alpha \rightarrow \beta) \rightarrow (?\alpha \rightarrow ?\beta);\]
\[(\bot \bot) \rightarrow !\bot.\]

### 3.4. Negation

**Proposition 3.9.** Some laws of QHC can be rewritten as follows.

(a) \((? \bot)\) is equivalent, modulo \((? \bot)\) and \((? \top)\), to \(\cdot ?\neg\alpha \rightarrow ?\neg\alpha\) and to \(\neg\alpha / ?\neg\alpha\);

(b) \((! \bot)\) is equivalent, modulo \((! \bot)\) and \((!!)\), to \(\cdot !p \rightarrow !p \rightarrow !p\) and to \(p / \neg !p\).

**Proof.** (a). By \((? \bot)\), we have \(\vdash ?(?\alpha \rightarrow \bot) \rightarrow (?)\neg\alpha \rightarrow ?\bot\). Assuming \((? \bot)\), we also have \(\vdash ?\bot \rightarrow \bot\). Hence \(\vdash (?)\neg\alpha \rightarrow ?\neg\alpha\). That is, \(\vdash ?\neg\alpha \rightarrow ?\neg\alpha\).

By \((? \top)\) we have \(\neg\neg\alpha \vdash \neg\alpha?\alpha\). Assuming \(\neg\alpha\), by *modus ponens* we have \(?\neg\alpha \vdash \neg\neg\alpha\). Combining these yields \(\neg\neg\alpha \vdash ?\neg\alpha\).

Finally, assuming \(\neg\alpha / \neg\alpha\), we have, in particular, \(\neg\alpha \vdash ?\neg\alpha\). Since \(\neg\alpha = \neg\alpha \rightarrow \bot\) is an intuitionistic validity, we get \(\vdash ?\neg\alpha\). □

(b). By \((! \bot)\), we have \(\vdash !(p \rightarrow \bot) \rightarrow (p \rightarrow \bot)\). Assuming \((! \bot)\), we also have \(\vdash !\bot \rightarrow \bot\).

Hence \(\vdash !(p \rightarrow \bot) \rightarrow (p \rightarrow \bot)\); that is, \(\vdash !p \rightarrow !p\).

By \((!!)\) we have \(\neg p \vdash !p\). Assuming \(!p \rightarrow !p\), by *modus ponens* we have \(\neg p \vdash \neg !p\). Combining these yields \(\neg p \vdash \neg !p\).

Finally, assuming \(\neg p / \neg !p\), we have, in particular, \(\bot \vdash !\bot\). Since \(\bot = \bot \rightarrow \bot\) is a classical validity, we get \(\vdash !\bot\). □

**Proposition 3.10.** \(\vdash \neg\alpha \leftrightarrow !?\alpha\).

This yields a definition of intuitionistic negation in terms of classical one. As discussed in detail in [1; §3.8], this fully agrees with the BHK interpretation (and with a remark by Heyting; but disagrees with a remark by Kolmogorov). Thus, this is yet another feature of the BHK interpretation that is captured in QHC but not in the usual formalization of intuitionistic logic.

**Proof.** By \((!!)\), \(\vdash \neg\alpha \rightarrow !?\neg\alpha\), from 3.9 we get \(\vdash !?\neg\alpha \rightarrow !\neg\alpha\) and \(\vdash !?\neg\alpha \rightarrow !?\alpha\), and by the contrapositive of \((!!)\), \(\vdash !?\neg\alpha \rightarrow \neg\alpha\). □

**Remark 3.11.** Since \(\vdash \neg\neg\alpha \rightarrow \alpha\) (see [1; (32)]), by 3.10 and by the converse of \((!!)\), we have \(\vdash \neg\neg\alpha \rightarrow \alpha\); thus it is impossible to prove that \(\alpha \vee \neg\alpha\) has no solutions.

**Corollary 3.12.** \(\vdash ?\neg\alpha\) if and only if \(\vdash ?\neg\alpha\).

Here the “only if” part is a consequence of 3.9(a). The “if” part can also be stated in a stronger form: \(\neg\alpha \vdash ?\neg\alpha\).

**Proof.** Indeed, we have \(\neg\alpha \vdash ?\neg\alpha\) by \((!!)\) and \((!!)\), and \(\vdash !?\neg\alpha \leftrightarrow ?\neg\alpha\) by 3.10. □

**Corollary 3.13.** \(\vdash \neg\neg\alpha \leftrightarrow \neg\alpha\) and \(\vdash \neg\alpha \leftrightarrow \neg\alpha\).
This follows from the proof of 3.10.

We note that Corollary 3.13 implies that \( \vdash \neg\alpha \iff \neg\neg\alpha \) and \( \vdash \neg\neg\alpha \iff \neg(\neg\neg\neg\alpha) \), which is in contrast with \( \vdash (\neg\square\neg)(\neg\alpha) \leftrightarrow \square p \). 

**Corollary 3.14.** \( \vdash \neg\square\alpha \iff \neg\neg\alpha \).

**Proof.** By 3.13, \( \vdash \neg\alpha \iff \neg\neg\alpha \). Then \([1; \S 3.14, (33)]\) yields \( \vdash \neg\square\alpha \iff \neg\neg\alpha \). \( \square \)

In fact, \( \neg\square\alpha \iff \neg\neg\alpha \) is yet another equivalent form of the law \((\neg\alpha)\), since \( \neg\square\alpha \iff \neg\neg\alpha \) implies \( \neg\square\alpha \iff \neg\neg\alpha \), or \( \neg\alpha \iff \neg\neg\alpha \). Moreover, as observed in \([4]\), \( \neg\square\alpha \iff \neg\neg\alpha \) is also an equivalent form of the law \((3^V)\) of QH4.

**Remark 3.15.** Using 3.10, the following consequence can be drawn from the fact that the implication of \((\neg\alpha)\) goes, in a sense, in the opposite direction with respect to that of \((\neg\alpha)\) and with respect to one of the implications of \((\neg\alpha)\). The intuitionistic implications \( \vdash \alpha \lor \beta \iff \neg(\neg\alpha \land \neg\beta) \) and \( \vdash \alpha \lor \beta \iff \neg\alpha \iff \beta \) (cf. \([1; (1), (29)\) and (7)])], when specialized to the image of \( \llcorner \), i.e., in the form \( \vdash \llcorner p \lor \llcorner q \iff \neg(\neg\llcorner p \lor \neg\llcorner q) \), each factor into two irreversible (as we will see in \([2]\) implications in QHC: \( \vdash \llcorner \llcorner p \lor \llcorner q \iff (\neg\llcorner p \lor \llcorner q) \) and \( \vdash \llcorner \llcorner p \lor \llcorner q \iff (\neg\llcorner p \lor \llcorner q) \). Consequently, \( \vdash \llcorner \llcorner p \lor \llcorner q \iff (\neg\llcorner p \lor \llcorner q) \) and \( \vdash \llcorner \llcorner p \lor \llcorner q \iff (\neg\llcorner p \lor \llcorner q) \).

### 3.5. Implication

**Proposition 3.16.**

(a) \( \vdash \llcorner \llcorner p \iff \llcorner \llcorner q \iff ?\neg\neg(\llcorner \llcorner p \lor \llcorner q) \iff ?(\llcorner \llcorner p \lor \llcorner q) \).

(b) \( \vdash \llcorner p \iff \llcorner q \iff \neg(\llcorner p \lor \llcorner q) \iff \neg(\llcorner p \lor \llcorner q) \).

We will actually prove stronger assertions:

(a) \( \vdash \llcorner \llcorner p \iff \llcorner \llcorner q \iff \neg\neg(\llcorner p \lor \llcorner q) \iff \neg\neg(\llcorner p \lor \llcorner q) \).

(b) \( \vdash \llcorner \llcorner p \iff \llcorner \llcorner q \iff \neg(\llcorner p \lor \llcorner q) \iff \neg(\llcorner p \lor \llcorner q) \).

(Their converses follow from \((\neg\alpha)\) and \((\neg\alpha)\), respectively.)

**Proof.** (a). By \((\neg\alpha)\), \( \vdash \llcorner \llcorner p \iff \llcorner \llcorner q \iff \neg\neg(\llcorner p \lor \llcorner q) \iff \neg\neg(\llcorner p \lor \llcorner q) \).

(b). By \((\neg\alpha)\), \( \vdash \llcorner \llcorner p \iff \llcorner \llcorner q \iff \neg(\llcorner p \lor \llcorner q) \iff \neg(\llcorner p \lor \llcorner q) \).

Finally, by \((\neg\alpha)\), \( \vdash \llcorner \llcorner p \iff \llcorner \llcorner q \iff \neg(\llcorner p \lor \llcorner q) \iff \neg(\llcorner p \lor \llcorner q) \).

The following proposition strengthens \(3.3(b)\). In addition, its part (a) along with part (a) of the preceding proposition generalize \(3.10\) and \(3.13\).

**Proposition 3.17.** We have

(a) \( \vdash \neg\neg(\llcorner \llcorner p \lor \llcorner q) \iff (\llcorner \llcorner p \lor \llcorner q) \iff \neg(\llcorner \llcorner p \lor \llcorner q) \iff \neg(\llcorner \llcorner p \lor \llcorner q) \).

(b) \( \vdash \neg\neg(\llcorner \llcorner p \lor \llcorner q) \iff (\llcorner \llcorner p \lor \llcorner q) \iff \neg(\llcorner \llcorner p \lor \llcorner q) \iff \neg(\llcorner \llcorner p \lor \llcorner q) \).

(a). Since \( \vdash \llcorner \llcorner p \iff \llcorner \llcorner q \iff \neg\neg(\llcorner \llcorner p \lor \llcorner q) \iff \neg\neg(\llcorner \llcorner p \lor \llcorner q) \).

(b). Since \( \vdash \llcorner \llcorner p \iff \llcorner \llcorner q \iff \neg\neg(\llcorner \llcorner p \lor \llcorner q) \iff \neg\neg(\llcorner \llcorner p \lor \llcorner q) \).

(\text{Finally, } \vdash \llcorner \llcorner p \iff \llcorner \llcorner q \iff \neg\neg(\llcorner \llcorner p \lor \llcorner q) \iff \neg\neg(\llcorner \llcorner p \lor \llcorner q) \text{ by } (4^V) \text{ and } (2^V). \) \( \square \)
(b). By $(1)?$, $\vdash (\Box p \rightarrow \Box q) \rightarrow (\Box p \rightarrow q)$. By $(3)?$, $\Box p \rightarrow q \vdash (\Box p \rightarrow q)$. Finally, by $(4)?$ and $(2)?$, $\vdash (\Box p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$.

**Corollary 3.18.** If $\Phi$ is an $i$-formula, $[\nabla \Phi]$ is the least upper bound of all classes $[(\Phi \rightarrow !F) \rightarrow !F]$, where $F$ is a $c$-formula.

Let us note that $(\Phi \rightarrow !F) \rightarrow !F$ specializes to $\neg \neg \Phi$ when $F = \bot$.

**Proof.** Let us observe that $[\Psi]$ is an upper bound of all $[(\Phi \rightarrow !F) \rightarrow !F]$ if and only if $\vdash \Psi \rightarrow (\Phi \rightarrow !F) \rightarrow !F$ for all $c$-formulas $F$. By the exponential law, the latter is equivalent to $\vdash (\Phi \rightarrow !F) \rightarrow (\Psi \rightarrow !F)$. Now by 3.17(a) we do have $\vdash (\Phi \rightarrow !F) \rightarrow (\nabla \Phi \rightarrow !F)$ for all $c$-formulas $F$. It remains to show that if $\vdash (\Phi \rightarrow !F) \rightarrow (\Psi \rightarrow !F)$ for all $c$-formulas $F$, then $\vdash \Psi \rightarrow \nabla \Phi$. Indeed, this follows by setting $F = ?\Phi$.

A variation of 3.18 can be formulated within the meta-logic, similarly to [1; 4.54]:

**Corollary 3.19.** $\vdash \nabla \alpha \iff (p)(\alpha \rightarrow !p) \rightarrow !p$.

Of course, if we replace $\iff$ by $\leftrightarrow$ here, we will get a meaningless expression (i.e., not a well-typed $\lambda$-expression) since the right hand side contains a meta-quantifier. But if we could do this, then 3.19 would be saying that $\nabla$ is a “Russell–Prawitz modality” in the terminology of Aczel [4] (see also [13]).

**Proof.** Since $p$ does not occur in $\nabla \alpha$, to show that $\nabla \alpha \vdash (p)(\alpha \rightarrow !p) \rightarrow !p$, it suffices to show that $\nabla \alpha \vdash (\alpha \rightarrow !p) \rightarrow !p$ (by the generalization meta-rule). This in turn reduces to deriving $\nabla \alpha \rightarrow ((\alpha \rightarrow !p) \rightarrow !p)$. By the exponential law the latter formula is equivalent to $(\alpha \rightarrow !p) \rightarrow (\alpha \rightarrow !p)$, which was derived in 3.17(a).

Conversely, by the specialization meta-rule, $(p)(\alpha \rightarrow !p) \rightarrow !p \vdash (\alpha \rightarrow !\alpha) \rightarrow !\alpha$. But the latter formula is equivalent to $\nabla \alpha$ due to $!\alpha$.

**3.6. Distributivity properties**

The following is a direct consequence of 3.6(b).

**Proposition 3.20.** $\vdash \Box (p \wedge q) \iff \Box p \wedge \Box q$ and $\vdash \nabla (\alpha \wedge \beta) \iff \nabla \alpha \wedge \nabla \beta$.

**Proposition 3.21.** The following holds in QHC.

(a) $\vdash \Box (?\alpha \wedge ?\beta) \iff ?\alpha \wedge ?\beta$;

(b) $\vdash \Box (?\alpha \vee ?\beta) \iff ?\alpha \vee ?\beta$;

(c) $\vdash \Box ?\exists x ?\alpha(x) \iff ?\exists x ?\alpha(x)$;

(d) $\vdash \nabla (?p \wedge q) \iff ?p \wedge q$;

(e) $\vdash \nabla (?p \rightarrow q) \iff ?p \rightarrow q$;

(f) $\vdash \nabla \forall x !p(x) \iff \forall x !p(x)$.

It is easy to see that these assertions are equivalent to their special cases for $i$-formulas in the image of $!$ and for $c$-formulas in the image of $?$. Those special cases are in turn parallel to [1; 5.21 and 5.18].

**Proof.** (a,d). These follow from 3.20 using $\vdash ?!\alpha \leftrightarrow ?\alpha$ or $\vdash ?!!p \leftrightarrow !p$. □
(a,d,b,c,f). Let us check (b). We have \( \vdash \alpha \lor \beta \iff (\forall \alpha \land \forall \beta) \) and \( \vdash \neg (\forall \alpha \lor \forall \beta) \iff \neg (!\alpha \lor \forall \beta). \]

(e). \( \rightarrow \) follows from (4\( \forall \)), (4?) and (4!!). The converse follows from (4?).

**Proposition 3.22.** The following holds in QHC.

\[
\begin{align*}
(a) &\vdash ?(\alpha \land \beta) \iff ?(\forall \alpha \land \forall \beta); \\
(b) &\vdash ?(\alpha \lor \beta) \iff ?(\forall \alpha \lor \forall \beta); \\
(c) &\vdash \exists x ?(x \alpha(x)) \iff ?\exists x ?(x \alpha(x)); \\
(d) &\vdash !?(p \land q) \iff !?(\Box p \land \Box q); \\
(e) &\vdash \forall x !?(p(x)) \iff \forall x !?(p(x)).
\end{align*}
\]

Applying \( ! \) to both sides in (a), (b), (c), and \( ? \) to both sides in (d), (e) leads to no loss of generality, but makes the validities parallel to [1; 5.23 and 5.19] — with the exception of one “missing validity”, \( ?(\alpha \rightarrow \beta) \iff ?(\forall \alpha \rightarrow \forall \beta) \), which will turn out to be an independent principle [2; 3.15(a) and 4.11].

**Proof.** Assertions (a,b,c) follow from (\( ?_\wedge \), (\( ?_\lor \)) and (\( ?_\exists \)) using that \( \vdash ?\alpha \iff ?!\alpha \). Assertions (d,e) follow from (4\( _\forall \)) and (4\( _\lor \)) using that \( \vdash !p \iff !?p \). \( \Box \)

**Proposition 3.23.** The following holds in QHC.

\[
\begin{align*}
(a) &\vdash !?(\forall \alpha \lor \forall \beta) \iff !?(\forall \alpha \lor \forall \beta); \\
(b) &\vdash !?(\forall \alpha \land \forall \beta) \iff !?(\forall \alpha \land \forall \beta); \\
(c) &\vdash !?\exists x ?(x \alpha(x)) \iff !?\exists x ?(x \alpha(x)); \\
(d) &\vdash !?\forall x ?(?p(x)) \iff !?\forall x ?(?p(x)).
\end{align*}
\]

**Proof.** (a,b). On applying (\( ?_\lor \)) or (\( ?_\exists \)) to the right hand side, these reduce to 3.21(b,c). \( \Box \)

(c,d). On applying 3.16(a) or (\( ?_\forall \)) to the left hand side, these reduce to 3.21(e,f). \( \Box \)

4. Stability and decidability

4.1. Stable and decidable c-formulas

Let us recall that a i-formula \( \Phi \) is called decidable if \( \vdash \Phi \lor \neg \Phi \), and stable if \( \vdash \neg \neg \Phi \rightarrow \Phi \); decidable i-formulas are stable (see [1; (8)]). Let us call a c-formula \( F \) decidable if \( \vdash !F \lor \neg !F \), and stable if \( \vdash \neg \neg !F \rightarrow !F \); decidable c-formulas are stable (using the intuitionistic law \( \alpha \lor \beta \rightarrow \neg \beta \rightarrow \alpha \), cf. [1; (7)]). Let us note that by 3.2, \( \vdash \neg \neg !F \rightarrow !F \) is equivalent to \( \vdash \neg \neg !F \rightarrow F \), which by 3.10 is in turn equivalent to \( \vdash ?(\neg \neg !F \rightarrow F) \), that is, \( \vdash \neg \neg \neg !F \rightarrow F \). In words, “if \( F \) is provably irrefutable, then it is true”.

Usually the notions of stability and decidability are considered relatively to a theory over intuitionistic logic. Instead of doing this we will consider internalizations of stability and decidability as operators.

Thus we define \( D, S : \mathbb{I}_i \rightarrow \mathbb{I}_i \) as \( \alpha \mapsto \alpha \lor \neg \alpha \) and \( \alpha \mapsto \neg \neg \alpha \rightarrow \alpha \) respectively; and \( D, S : \mathbb{I}_c \rightarrow \mathbb{I}_i \) as \( p \mapsto \neg \neg !p \lor \neg p \) and \( p \mapsto \neg \neg \neg \neg \neg !p \rightarrow \neg \neg \neg \neg !p \) respectively.
Proposition 4.1. (a) If a c-formula $F$ is stable or decidable, then so is the i-formula $\neg \Phi$. The converse holds for c-formulas $F$ of the form $\neg \Phi$.

(b) If an i-formula $\Phi$ is stable or decidable, then so is the c-formula $\neg \Phi$. The converse holds for i-formulas $\Phi$ of the form $\neg F$.

These judgements about the QHC calculus follow from their internalized versions, which will be proved below:

(a) $\vdash S(p) \rightarrow S(\neg p)$ and $\vdash S(p) \rightarrow S(\neg p)$.

Moreover, $\vdash S(\neg p) \leftrightarrow S(\neg p)$ and $\vdash S(\neg p) \leftrightarrow S(\neg p)$.

(b) $\vdash S(p) \rightarrow S(\neg p)$ and $\vdash S(p) \rightarrow S(\neg p)$.

Moreover, $\vdash S(p) \rightarrow S(\neg p)$ and $\vdash S(p) \rightarrow S(\neg p)$.

Proof. (a). The first assertion follows since $\vdash \neg \neg p \rightarrow \neg p$ by 3.9(b). The moreover assertion follows since $\vdash \neg \neg \neg p \leftrightarrow \neg p$ by 3.10 and 3.13.

(b). By 3.13, $\vdash \neg \neg p \leftrightarrow \neg p$. From this and (1$\neg$) or (4$\neg$) it follows that $\vdash S(p) \rightarrow S(\neg p)$ and $\vdash S(p) \rightarrow S(\neg p)$. Now the first assertion of (b) follows from the moreover assertion of (a). The moreover assertion of (b) follows from the first assertion of (a).

Proposition 4.2. (a) C-formulas of the form $\neg \neg \Phi$ are stable.

(b) If $\Phi$ is stable, then $\neg \neg \Phi$ is decidable if and only if $\Phi$ is decidable.

(c) If $\neg F$ is stable, then $\neg F$ is decidable if and only if $F$ is decidable.

We will prove the internalizations: (a) $\vdash S(\neg \neg p)$;

(b) $\vdash S(\neg p) \rightarrow (S(\neg p) \leftrightarrow S(\neg p))$;

(c) $\vdash S(\neg p) \rightarrow (S(\neg p) \leftrightarrow S(\neg p))$.

Proof. (a). By the classical double negation law, $\vdash \neg \neg p \rightarrow \neg p$ and by 3.10 and 3.13, also $\vdash \neg \neg \neg p \leftrightarrow \neg \neg p$. Thus $\vdash \neg \neg \neg p \rightarrow \neg \neg p$.

(b). Assuming $\neg \neg p \leftrightarrow \neg \neg p$, and writing $\neg \neg p = \neg p$, from $\vdash \neg \neg p \rightarrow \neg \neg p \leftrightarrow \neg \neg p \leftrightarrow \neg \neg p$ we get $\neg \neg p \rightarrow \neg \neg p \leftrightarrow \neg \neg p \leftrightarrow \neg \neg p$. This shows that $\vdash S(\neg p) \rightarrow (S(\neg p) \leftrightarrow S(\neg p))$, and the assertion now follows from 4.1(b).

(c). Clearly, $\vdash S(\neg p) \leftrightarrow (\neg p \leftrightarrow \neg p)$, and the assertion follows.

Proposition 4.3. $\vdash \neg \neg \Phi \leftrightarrow \neg \Phi$ if and only if $\Phi$ is stable.

We will prove the internalization: $\vdash (\neg \neg \Phi \leftrightarrow \neg \Phi) \leftrightarrow S(\neg \Phi)$.

Proof. $\neg \Phi \rightarrow \neg \Phi$ is derivable 3.14. The converse implication, $\neg \Phi \rightarrow \neg \Phi$, is equivalent by 3.13 to $S(\neg \Phi)$, which by 4.1(b) is in turn equivalent to $S(\neg \Phi)$.

4.2. Semi-stability and semi-decidability

Let us call an i-formula $\Phi$ semi-decidable if $\vdash S(\Phi)$, and semi-stable if $\vdash D(\Phi)$. Similarly, we call a c-formula $F$ semi-decidable if $\vdash S(\Phi)$, and semi-stable if $\vdash D(\Phi)$.

Here each “$\neg$” can be replaced by “$\neg$” due to $\vdash \neg \neg \neg p \leftrightarrow \neg p$. Stability or decidability implies semi-stability or semi-decidability (both for i-formulas and for c-formulas) due to...
Remark 4.4. Since 4.1 holds in the internalized form, we can apply (?\top) and (?\bot) to obtain the literal analogue of 4.1 for semi-stability and semi-decidability, in the internalized form.

Proposition 4.5. (a) An i-formula \( \Phi \) is semi-decidable if and only if \( \vdash ?\neg \Phi \leftrightarrow ?\neg \Phi \).
(b) A c-formula \( \neg F \) is stable if and only if \( \vdash !\neg F \leftrightarrow !\neg F \).

These hold internally:
(a) \( \vdash ?\mathcal{D}(\alpha) \leftrightarrow (\neg \alpha \leftrightarrow ?\neg \alpha) \);
(b) \( \vdash \mathcal{G}(\neg p) \leftrightarrow (!p \leftrightarrow !p) \).

Part (b) is trivial.

Proof of (a). \( ?\neg \alpha \rightarrow ?\neg \alpha \) is classically equivalent to \( \vdash ?\alpha \lor ?\neg \alpha \), which is in turn equivalent to \( \vdash (?\alpha \lor \neg \alpha) \).

Proposition 4.6. (a) A c-formula \( F \) is semi-stable if and only if it is stable.
(b) A i-formula \( \Phi \) is semi-decidable if and only if the c-formula \( ?\Phi \) is.

These hold internally:
(a) \( \vdash ?\mathcal{G}(p) \leftrightarrow \mathcal{G}(p) \);
(b) \( \vdash ?\mathcal{D}(\neg \alpha) \leftrightarrow \mathcal{D}(\alpha) \).

Proof. (a). By (!\top), \( ?\mathcal{G}(p) \vdash \nabla \mathcal{G}(p) \). On the other hand, by (4\top) and 3.13 we also have \( \vdash \nabla (\neg !\neg p \rightarrow !p) \rightarrow (\neg !p \rightarrow !p) \), that is, \( \vdash \nabla \mathcal{G}(p) \rightarrow \mathcal{G}(p) \).

(b). Using (?\lor) and 3.10, we get \( \vdash (?\neg \alpha \lor ?\alpha) \leftrightarrow (?\alpha \lor ?\neg \alpha) \), and using (?\lor) again, we get \( \vdash (?\alpha \lor \neg ?\alpha) \leftrightarrow (?\alpha \lor \neg \alpha) \), as desired.

Corollary 4.7. \( \vdash \nabla \mathcal{D}(p) \rightarrow \mathcal{G}(p) \).

This is a strengthening of “decidability implies stability” for c-formulas.

Proof. Decidability does imply stability: \( \vdash \mathcal{D}(p) \rightarrow \mathcal{G}(p) \). Hence \( \vdash \nabla \mathcal{D}(p) \rightarrow \nabla \mathcal{G}(p) \). On the other hand, by 4.6(a), \( \vdash \nabla \mathcal{G}(p) \rightarrow \mathcal{G}(p) \).

Corollary 4.8. \( \vdash \nabla (\nabla \alpha \lor \neg \nabla \alpha) \rightarrow (\nabla \alpha \leftrightarrow \neg \nabla \alpha) \).

This will be used in §5.3 to show that the classical \( \neg \neg \)-translation of QC into QH cannot be improved in a certain sense.

Proof. By 4.1(a), \( \vdash \nabla \mathcal{D}(\nabla \alpha) \leftrightarrow \nabla \mathcal{D}(?\alpha) \). By 4.7, \( \vdash \nabla \mathcal{D}(?\alpha) \rightarrow \mathcal{G}(?\alpha) \). By 4.3, \( \vdash \mathcal{G}(?\alpha) \leftrightarrow (\neg \alpha \leftrightarrow \nabla \alpha) \).
5. Syntactic interpretations

5.1. □-interpretation

The classical provability translation of QH in QS4 (see [1; §5.7.3]) extends to the following syntactic □-interpretation of QHC in QS4, denoted by \( A \mapsto \Box A \):

- Atomic c-formulas and classical connectives remain unchanged;
- Atomic i-formulas are re-typed as atomic c-formulas and are prefixed by \( \Box \);
- Intuitionistic \( \land, \lor \) and \( \exists \) become classical, and \( \times, \checkmark \) are replaced by \( \perp, \top \);
- Intuitionistic \( \rightarrow \) and \( \forall \) become classical and are prefixed by \( \Box \);
- ? is erased, and ! is replaced by \( \Box \).

Indeed, let us write out the images of the laws and inference rules in 3.8 under the □-interpretation:

\[
\begin{align*}
\text{(□)} \quad & \alpha / ? \alpha \text{ becomes } \Box \alpha / \Box \alpha; \\
\text{(□)} \quad & p / !p \text{ becomes } p / \Box p; \\
\text{(!□)} \quad & !p \rightarrow p \text{ becomes } \Box(p \rightarrow p); \\
\text{(!□)} \quad & \alpha \rightarrow \nabla \alpha \text{ becomes } \Box(\Box a \rightarrow \Box a); \\
\text{(!□)} \quad & (p \rightarrow q) \rightarrow (p \rightarrow q) \text{ becomes } \Box(\Box(p \rightarrow q) \rightarrow \Box(p \rightarrow q)); \\
\text{(!□)} \quad & (\alpha \rightarrow \beta) \rightarrow (?) \alpha \rightarrow (?) \beta \text{ becomes } \Box(\Box a \rightarrow \Box b) \rightarrow \Box(a \rightarrow \Box b); \\
\text{(!□)} \quad & !\perp \rightarrow \perp \text{ becomes } \Box(\Box \perp \rightarrow \perp).
\end{align*}
\]

The resulting rules and principles are easily derivable in QS4, including the last one, which is the principle of internal consistency (see [1; §5.7.3]).

The classical laws and inference rules of QHC hold under the □-interpretation since it does nothing to classical connectives and quantifiers and to atomic c-formulas. The intuitionistic laws and inference rules of QHC hold under the □-interpretation since the restriction of the □-interpretation to QH is known to be an interpretation (see [1; §5.7.3]).

Finally, let us note that by an inductive argument based on [1; 5.21], \( \vdash \Phi \leftrightarrow \Box \Phi \) for any i-formula \( \Phi \). It follows that the second-order meta-specialization of the intuitionistic type holds under the □-interpretation. The other meta-rules hold under the □-interpretation for trivial reasons.

We have proved

**Theorem 5.1.** (a) If \( A_1, \ldots, A_n \vdash_{QHC} A \), then \( (A_1)_\Box, \ldots, (A_n)_\Box \vdash_{QS4} A_\Box \), and the converse holds (trivially) if \( A_1, \ldots, A_n, A \) are formulas of QS4.

(b) If a formula \( A \) is derivable in QHC from \( \cdot A_1, \ldots, \cdot A_n \), then \( A_\Box \) is derivable in QS4 from \( \cdot (A_1)_\Box, \ldots, (A_n)_\Box \).

Of course, by [1; 5.21], all of the intuitionistic connectives and quantifiers (and not only \( \rightarrow \) and \( \forall \)) could be prefixed by a □ in the definition of the □-interpretation. Consequently, by [1; 5.23], one could alternatively formulate the □-translation as an extension of Gödel’s original provability translation: postfix by □ ‘es the intuitionistic \( \lor, \exists \) and \( \rightarrow \), erase every !, and replace every ? by a □. Like before, all intuitionistic connectives and quantifiers become classical, and all atomic i-formulas are re-typed as atomic c-formulas.
Theorem 5.2. The QHC calculus is:
(a) a strongly conservative extension of classical predicate calculus QC;
(b) a strongly conservative extension of QS4, via □ ↦ → ?!

Here a formula of QHC is regarded as a formula of QS4 if it involves only classical atoms, connectives and quantifiers, as well as the combination □ = ?! (but not ? and ! alone).

Strong conservativity in (b) means that if a derivable rule of QHC is expressed in the language of QS4, then it is derivable in QS4.

Proof. Since the standard interpretation of QS4 in QHC (see 3.2) composed with the □-interpretation is the identity, we get (b). Omitting each □ is clearly an interpretation of QS4 in QC that restricts to the identity on QC. Thus QS4 is a strongly conservative extension of QC, and we obtain (a).

5.2. ∇-interpretation

By Theorem 5.2(b), the □-interpretation of QHC in QS4 can be regarded as an interpretation of QHC in itself. This does not preserve the types of formulas (i.e., i-formulas versus c-formulas), but can be amended to do so. This results in the following ∇-interpretation of QHC in itself, which restricts to an unintended embedding of QH in QHC:

Theorem 5.3. If A is a formula of QHC, let A∇ be the formula of QHC obtained from A by prefixing atomic i-formulas and the intuitionistic ∨ and ∃ by ∇ := ?!

(a) ⊢ A implies ⊢ A∇, and the converse holds when A is a formula of QH or (trivially) of QS4;
(b) if A₁, . . . , Aₙ ⊨ A, then (A₁)∇, . . . , (Aₙ)∇ ⊨ A∇;
(c) if a formula A is derivable in QHC from · A₁, . . . , · Aₙ, then A∇ is derivable in QHC from · (A₁)∇, . . . , · (Aₙ)∇.

Let us note that (a) is only a meta-judgement, that is, it does not claim that A ⊨ A∇ in QHC, nor the converse when A is a formula of QH. In fact these claims are false as we will see in [2; Remark 4.9].

Of course, by 3.21(d,e,f) we may redefine the ∇-interpretation A → A∇, without changing its effect, so as to prefix all intuitionistic connectives and quantifiers of A (not just ∨ and ∃) and all atomic i-formulas by ∇. Alternatively, by 3.22(a,b,c) we might redefine the ∇-interpretation A → A∇, without changing its effect, so as to prefix the entire formula A, if it represents a i-formula, by ∇, and postfix every intuitionistic → and ∀ by ∇’s (atomic subformulas are now kept intact).

Proof. (a). Let A□ be the □-interpretation of A regarded as a formula of QHC, by identifying □ with ?!. Thus A□ can be obtained from A by first erasing every ? and replacing every ! by ?!, then prefixing all atomic i-formulas and all intuitionistic connectives and quantifiers by ?!, and finally retyping all atomic i-formulas as c-formulas and replacing...
all intuitionistic connectives and quantifiers by the corresponding classical ones. By 5.1, $\vdash A$ implies $\vdash A^\square$, and (since the classical provability translation of QH in QS4 is faithful) the converse holds when $A$ is a formula of QH. Let $A'_\square$ denote the formula of QHC obtained from $A$ by first erasing every $?$ and replacing every $!$ by $?!$, then prefixing all atomic i-formulas and all occurrences of $\times$ and $\checkmark$ by $?$ and all other intuitionistic connectives and quantifiers by $?!$, and finally replacing all intuitionistic connectives and quantifiers except $\times$ and $\checkmark$ by the corresponding classical ones. Then $\vdash A'_\square$ implies $\vdash A^\square$ by substituting the $!$-images of atomic c-formulas for the atomic i-formulas of $A'_\square$. The converse implication follows by substituting $?!$-images of atomic i-formulas for the atomic c-formulas of $A^\square$ and using $\vdash ?!?\alpha \leftrightarrow ?\alpha$.

On the other hand, as observed above, we may assume $A \leftrightarrow A^\triangledown$ to prefix all intuitionistic connectives and quantifiers of $A$ (not just $\lor$ and $\exists$) and all atomic i-formulas by $!\otimes$. Let $A'_\triangledown$ denote $A^\triangledown$ if $A$ is a c-formula, and $?A^\triangledown$ if $A$ is an i-formula. If $\Phi$ is an i-formula, then $\Phi^\triangledown$ is an i-formula of the form $?!\Psi$. In this case, we have $\Phi^\triangledown \vdash ?\Phi^\triangledown$ by ($\vdash_\top$), and conversely $\Phi^\triangledown \vdash \Phi^\triangledown$ since $?\Psi \vdash !?\Psi$ by ($\vdash_\top$). Thus $\vdash A^\triangledown \leftrightarrow A'_\triangledown$ for any formula $A$.

Each judgement of QHC of the form $\vdash A$ corresponds to a rooted tree whose root is labelled with $\vdash$, whose leaves are labelled with the atomic subformulas of $A$ or with nullary connectives, and whose other vertices are labelled with the unary and binary connectives and the quantifiers of $A$. We can draw this tree on the plane so that every connective or quantifier is drawn above those in the subformulas that it applies to. (Thus the root is at the top, and the leaves are in the bottom.) Then $?$’s and $!$’s alternate along every path that is vertical (in the sense that its projection to the vertical axis is a monotone function). Hence $?$’s and $!$’s partition the tree into intuitionistic fragments, bounded below by $!$’s or atomic i-formulas or $\times$ or $\checkmark$, and above by a $?$ or by the $\vdash$; and classical fragments, bounded below by $?$’s or atomic c-formulas or $\bot$ or $\top$, and above by an $!$ or by the $\vdash$.

To analyze the difference between $A'_\square$ and $A'_\triangledown$, we can use the tree of $A$ and write any prefix added to a connective or quantifier of $A$ on the edge just above the vertex that it labels, in respective order. Then the difference is confined to the intuitionistic fragments of the tree of $A$, and is that $A'_\square$ has an extra $?$ (with respect to $A'_\triangledown$) just below each intuitionistic connective and quantifier and a missing $?$ just above it. Then we may push the extra $?$’s up using ($?_\alpha$), ($?_\lor$) and ($?_\exists$) as well as 3.23(c,d), thus obtaining that $\vdash A'_\square \leftrightarrow A'_\triangledown$. (Alternatively, one can push $!$’s down, using ($!_\alpha$) and ($!_\lor$) as well as 3.16(a) and 3.23(a,b)).

(b,c). By (a) all laws of QHC hold under the $\nabla$-interpretation. The inference rules of QHC: $\frac{\alpha, \alpha \rightarrow \beta}{\forall x \alpha(x)}, \frac{p, p \rightarrow q}{q}, \frac{p(x)}{\forall x p(x)}, \frac{\alpha, p}{\forall x (\alpha)}, \frac{?\alpha}{!p}$ are easily seen to hold under the $\nabla$-interpretation, as they do not involve intuitionistic $\lor$ and $\exists$.

Finally, let us note that by an inductive argument based on 3.21(d,e,f), $\vdash \Phi^\triangledown \leftrightarrow \nabla \Phi^\triangledown$ for any i-formula $\Phi$. It follows that the second-order meta-specialization of the intuitionistic type holds under the $\nabla$-interpretation. The other meta-rules hold under the $\nabla$-interpretation for trivial reasons.
5.3. \(\neg\neg\)-interpretation

The classical \(\neg\neg\)-translation of QC in QH (see [1; §5.6]) extends to the following syntactic \(\neg\neg\)-interpretation of QHC in QH, denoted by \(A \mapsto A_{\neg\neg}\):

- Atomic i-formulas and intuitionistic connectives remain unchanged;
- Atomic c-formulas are re-typed as atomic i-formulas and are prefixed by \(\neg\neg\);
- Classical \(\land\), \(\rightarrow\) and \(\forall\) become intuitionistic, and \(\bot\), \(\top\) are replaced by \(\times\), \(\checkmark\);
- Classical \(\lor\) and \(\exists\) become intuitionistic and are prefixed by \(\neg\neg\);
- \(!\) is erased, and \(?\) is replaced by \(\neg\neg\).

Indeed, let us write out the images of the laws and inference rules in 3.8 under the \(\neg\neg\)-interpretation:

\((!\top) \; p / !p\) becomes \(\neg\neg\pi / \neg\neg\pi\);
\((?\top) \; \alpha / ?\alpha\) becomes \(\alpha / \neg\neg\alpha\);
\((?!) \; \cdot\alpha \rightarrow \nabla\alpha\) becomes \(\cdot\alpha \rightarrow \neg\neg\alpha\);
\((?!?) \; ?!p \rightarrow p\) becomes \(\neg\neg\neg\neg\pi \rightarrow \neg\neg\pi\);
\((!\bot) \; !\bot \rightarrow \bot\) becomes \(\times \rightarrow \times\);
\((!\leftarrow) \; !(p \rightarrow q) \rightarrow !(p \rightarrow q)\) becomes \(\neg\neg\pi \rightarrow \neg\neg\rho \rightarrow \neg\neg\pi \rightarrow \neg\neg\rho\);

The resulting formulas are easily derivable in intuitionistic logic, including the last one (see [1; §3.14, (44)]).

Intuitionistic laws and inference rules of QHC hold under the \(\neg\neg\)-interpretation since it does nothing to intuitionistic connectives and quantifiers and to atomic i-formulas. Classical laws and inference rules of QHC hold under the \(\neg\neg\)-interpretation since the restriction of the \(\neg\neg\)-interpretation to QC is known to be an interpretation (see [1; §5.6]).

Finally, let us note that by an inductive argument based on [1; 5.18], \(\vdash F_{\neg\neg} \leftrightarrow \neg\neg F_{\neg\neg}\) for every c-formula \(F\). It follows that the second-order meta-specialization of the classical type holds under the \(\neg\neg\)-interpretation. The other meta-rules hold under the \(\neg\neg\)-interpretation for trivial reasons.

We have proved

**Theorem 5.4.** (a) If \(A_1, \ldots, A_n \vdash_{\text{QHC}} A\), then \((A_1)_{\neg\neg}, \ldots, (A_n)_{\neg\neg} \vdash_{\text{QH}} A_{\neg\neg}\).

(b) If a formula \(A\) is derivable in QHC from \(\cdot A_1, \ldots, A_n\), then \(A_{\neg\neg}\) is derivable in QH from \(\cdot (A_1)_{\neg\neg}, \ldots, (A_n)_{\neg\neg}\).

Of course, by [1; 5.18], all of the classical connectives and quantifiers (and not only \(\lor\) and \(\exists\)) could be prefixed by a \(\neg\neg\) in the definition of the \(\neg\neg\)-interpretation. Consequently, by [1; 5.19] one could formulate the \(\neg\neg\)-interpretation as an extension of Kuroda’s translation: prefix by a \(\neg\neg\) the entire formula if it is a c-formula, postfix by \(\neg\neg\)’s every classical \(\forall\), replace every \(!\) by a \(\neg\neg\), and erase all \(?\)’s. Like before, all classical connectives and quantifiers become intuitionistic, and all atomic c-formulas are re-typed as atomic i-formulas.

As in [1], we also get an “essentially local” version of the \(\neg\neg\)-interpretation:

- postfix by a \(\neg\neg\) every classical \(\forall\) and \(\land\);
• prefix by a \( \neg \neg \) every classical \( \exists \) and \( \lor \);
• postfix by a \( \neg \neg \) every ! that is followed by a ? or by an atomic c-formula;
• prefix by a \( \neg \neg \) the entire formula if it is an atomic c-formula or starts with ?;
• now erase all ?’s and !’s;
• all classical connectives and quantifiers become intuitionistic, and all atomic c-formulas are re-typed as atomic i-formulas.

Since the restriction of the \( \neg \neg \)-interpretation to QH is the identity, we obtain

**Theorem 5.5.** The QHC calculus is a strongly conservative extension of QH.

### 5.4. \( \Diamond \)-interpretation

By Theorem 5.5, the \( \neg \neg \)-interpretation of QHC in QH can be regarded as an interpretation of QHC in itself which does not preserve the types of formulas. In this sense it can be improved, so as to preserve the typing. This results in the following \( \Diamond \)-interpretation of QHC in itself, which restricts to an unintended embedding of QC:

**Theorem 5.6.** If \( A \) is a formula of QHC, let \( A_{\Diamond} \) be the formula of QHC obtained from \( A \) by prefixing

- classical \( \rightarrow \) and \( \forall \) by \( \Box \); and
- atomic c-formulas, ?, and classical \( \lor \) and \( \exists \) by \( \Box \Diamond \),

where \( \Box = ?! \) and \( \Diamond = \neg \Box \neg \). Then

(a) \( \vdash A \) implies \( \vdash A_{\Diamond} \), and the converse holds when \( A \) is a formula of QC or (trivially) of QH;
(b) if \( A_1, \ldots, A_n \vdash A \), then \( (A_1)_{\Diamond}, \ldots, (A_n)_{\Diamond} \vdash A_{\Diamond} \);
(c) if a formula \( A \) is derivable in QHC from \( \cdot A_1, \ldots, \cdot A_n \), then \( A_{\Diamond} \) is derivable in QHC from \( \cdot (A_1)_{\Diamond}, \ldots, (A_n)_{\Diamond} \).

The conclusion of (a) is to be read as a meta-judgement, that is, it does not claim that \( A \vdash A_{\Diamond} \) in QHC, nor the converse when \( A \) is a formula of QC. In fact the former claim is false as we will see in [2; Remark 4.3].

The proof of Theorem 5.6 is similar to that of Theorem 5.3, using additionally diagram \((*)\) below. Let us only note that the combination \( \Box \Diamond \) arises by applying the \( \Box \)-interpretation (in the prefixing version) to a \( \neg \neg \). When this \( \neg \neg \) is in front of an atomic c-formula, that atomic c-formula would have to been prefixed by \( \Box \Diamond \Box \); but the last \( \Box \) is easily seen to be redundant.

By starting from different versions of the \( \neg \neg \)-interpretation, and applying different versions of the \( \Box \)-interpretation, one gets a few equivalent forms of the \( \Diamond \)-translation. For example, by starting from Kolmogorov’s original form of the \( \neg \neg \)-translation, we get the following succinct version of the \( \Diamond \)-interpretation:

- Prefix all classical connectives and quantifiers, all atomic c-formulas, and all ?’s by \( \Box \Diamond \).

This interpretation extends Fitting’s translation of classical logic in QS4 [22].
On the other hand, by starting with the “essentially local” form of the ¬¬-interpretation, and applying the prefix form of the □-translation, we get the following version of the ♦-interpretation: Prefix classical → and ∀, and atomic c-formulas by a □; prefix classical ∨ and ∃ by a □♦; postfix classical ∀ and ∧ by a □♦; prefix by a ♦ every ! that is followed by a ? or by an atomic c-formula; and if the entire formula is an atomic c-formula or starts with ?, prefix it by a ♦. Now atomic c-formulas do not really need to be prefixed by □’es, since they are anyway effectively prefixed by double negations in the form □♦, which clearly suffices. Next, the prefix □♦ of classical ∨’s and ∃’s can be reduced to a mere ♦ by the price of postfixing classical →’s, ∨’s and ∃’s by □’es. Finally, by [1; 5.21(a)], it does not hurt to also prefix classical ∧’s by □’es; and given that, by [1; 5.23], the postfix □♦ of classical ∀’s and ∧’s can be reduced to a mere ♦.

To summarize, we get the following “essentially local” form of the ♦-interpretation:

- Prefix and postfix every classical → by a □;
- prefix classical ∀’s and ∧’s by □’es, and postfix them by ♦’s;
- prefix classical ∨’s and ∃’s by ♦’s, and postfix them by □’es;
- postfix by a ♦ every ! that is followed by a ? or by an atomic c-formula;
- prefix by a ♦ the entire formula if it is an atomic c-formula or starts with ?.

Using the usual identities, this can be further reformulated in a more economical way in terms of ! and ?:

- Prefix every classical → by a ? and postfix it by an !;
- prefix every classical ∀ and ∧ by a ?, and postfix it by ¬!¬;
- prefix every classical ∨ and ∃ by ¬?¬, and postfix it by an !;
- replace every ∇ by a ¬¬;
- replace by ¬!¬ every ! that is followed by an atomic c-formula;
- if the formula starts with ?, replace that ? by ¬¬;
- if the entire formula is an atomic c-formula, prefix it by a ♦.

This can be regarded as an alternative form of Kolmogorov’s original ¬¬-translation, since it has the effect of expressing all classical connectives and quantifiers in terms of the intuitionistic ones along with ?, ! and the classical negation. In particular,

\[ \exists x p(x) \] is translated as \[ ¬?¬\exists x !p(x) \).

In words, there exists an x such that p(x) if and only if it is impossible to derive a contradiction from a construction of an x along with a proof of p(x).

5.5. Applications to QH4

The ∇-interpretation is easily seen to lift to the interpretation of QH4 in itself described by Aczel [4]:
The vertical arrows of this diagram commute with the inclusions of \( \text{QH} \) into \( \text{QH4} \) and into \( \text{QHC} \), so they are faithful on \( \text{QH} \) (since \( \text{QHC} \) is a conservative extension of \( \text{QH} \)). On the other hand, the \( \nabla \)-interpretation was shown to be faithful on \( \text{QH} \), so we conclude that Aczel’s interpretation restricts to an unintended embedding of \( \text{QH} \) into \( \text{QH4} \), which we will call the \( \nabla \)-translation.

Since \( \vdash \nabla \neg \neg \alpha \leftrightarrow \neg \neg \alpha \) not only in \( \text{QHC} \), but also in \( \text{QH4} \) (see [4]), we have the commutative diagram

\[
\begin{array}{ccc}
\text{QC} & \xrightarrow{\neg \neg \text{-translation}} & \text{QH} \\
\downarrow{\neg \neg \text{-translation}} & & \downarrow{\text{inclusion}} \\
\text{QH} & \xrightarrow{\nabla \text{-translation}} & \text{QH4}.
\end{array}
\]

The \( \neg \neg \)-interpretation of \( \text{QH4} \) in \( \text{QH} \) replaces every occurrence of \( \nabla \) by \( \neg \neg \). On the other hand, the \( \nabla \)-translation of \( \text{QH} \) into \( \text{QH4} \) and the \( \neg \neg \)-translation of \( \text{QC} \) into \( \text{QH} \) are defined using similar formulas \( A_\nabla \) and \( A_{\neg \neg} \), which, as discussed above, can both be written out by prefixing all connectives, quantifiers and atomic subformulas with either \( \nabla \) or \( \neg \neg \). Hence \( A_\nabla \) and \( A_{\neg \neg} \) become equivalent upon substituting \( \nabla \) by \( \neg \neg \), and we get the following commutative diagram.

\[
\begin{array}{ccc}
\text{QC} & \xrightarrow{\neg \neg \text{-translation}} & \text{QH} \\
\downarrow{\neg \neg \text{-translation}} & & \downarrow{\nabla \rightarrow \neg \neg} \\
\text{QH} & \xrightarrow{\nabla \text{-translation}} & \text{QH4}.
\end{array}
\]

By 4.8, which can in fact be proved in \( \text{QH4} \) and not just in \( \text{QHC} \) (we leave this for the reader to check), this is a pushout diagram, that is, the \( \neg \neg \)-translation of \( \text{QC} \) into \( \text{QH} \) does not factor through any logic obtained by adding a set of laws to \( \text{QH4} \) that are collectively strictly weaker than \( \cdot \nabla \alpha \leftrightarrow \neg \neg \alpha \). In this sense, the classical \( \neg \neg \)-translation of \( \text{QC} \) into \( \text{QH} \) cannot be improved.
6. Discussion

6.1. Knowledge-that vs. knowledge-how

Kolmogorov has summarized his philosophical views on intuitionism in his foreword\textsuperscript{13} to a 1936 translation of Heyting’s book [33] (translated from Russian):

“We cannot agree with intuitionists when they say that mathematical objects are products of constructive activity of our spirit. For us, mathematical objects are abstractions of actually existing forms of reality, which is independent of our spirit. But we know how essential in mathematics is, in addition to pure proof of theoretical propositions, constructive solution of posed problems. This second, constructive side of mathematics does not eclipse for us its first and foremost side: the cognitive one. However, the laws of mathematical construction, discovered by Brouwer and systematized by Heyting under the guise of a new intuitionistic logic, keep their fundamental importance for us, in their present understanding.”

This is quite in line with a passage from Kolmogorov’s 1929 survey [39]:

“We could distinguish two sides in this concept of mathematics. On the one side, there are theories postulating the existence of infinite systems of objects satisfying certain axioms and formally deriving from the axioms the properties of the system being studied. On the other side, construction of the corresponding objects, based either on positive integers or on some other resource of elementary objects, is also recognized as necessary. Experience of the last years shows that no stable balance was attained between these two sides. The standpoints that came to light in recent times may be roughly formulated as follows. Hilbert proposed to keep only the former, formal part of mathematics, while having set us free, by means of his theory of consistency, from the necessity to construct. On the contrary, Brouwer values mainly the constructive part, but thinks that construction is unable to give us the ultimate existence of infinite collections that is needed for a free use of the ways of reasoning that have became common to mathematics; and therefore he demands a radical revision of the methods of a mathematical proof.

The emergence of these extreme viewpoints is explained by the fact that joining of the two sides of the set-theoretic mathematics has led to great difficulties and even contradictions.” (There follows a discussion of Russell’s paradox, Weyl’s predicativist restrictions, and non-measurable sets.)

The two sides of mathematics referred to by Kolmogorov can be seen as representing two modes of knowledge (including formalized mathematical knowledge, but also keeping in mind subjects such as common knowledge and collective intelligence):

- knowledge-that (or knowledge of truths), and
- knowledge-how (or knowledge of methods).

This dichotomy is also noted, from a slightly different perspective, in [21]:

\textsuperscript{13}Which must have been addressed in part to the Soviet censor, as it included the obligatory denunciation of subjective idealism. This could well have implications for the wording and emphasis chosen, but hardly for the sincerity of Kolmogorov’s words (as one can judge from his published correspondence with Alexandrov and from the transcript of Luzin’s trial).
“In contrast with the structural (platonistic) point of view, intuitionistic mathematics focuses primarily on the subject (the creative mathematician) and his ability to perform certain mathematical operations by applying his previously designed constructions (knowing how). Hence a notion such as ‘proof’, which refers to the successful completion of a human action, appears to be more suitable than that of ‘truth’. On the other hand, classical mathematics focuses essentially on the object: eternal pre-existing mathematical structures (knowing that); and for this reason, the notion of ‘truth’, with its prominent descriptive untensed character, is more appropriate.”

There is, however, hardly any connection with the distinction made in philosophy between “knowledge how” and “knowledge that” — in the tradition originating with G. Ryle, whose “knowledge how” is an unconscious, non-articulable ability. The same can be said of procedural vs. declarative knowledge of cognitive psychology. Somewhat closer to our concern here are the distinction between declarative and imperative programming languages, and another distinction made in philosophy, starting with B. Russell: knowledge of objects (including mathematical objects) “by acquaintance” vs. “by description”. Hilbert’s distinction between formal mathematics (subject only to freedom from contradiction) and intuitively justifiable, “finitistic” methods (especially as reinterpreted in [54]) is also to the point. Mathematically most relevant is, of course, Lawvere’s adjunction between the Formal and the Conceptual.

Our connectives ! and ? amount to two “conversion” operators between the two modes of knowledge. These give rise to compound types of knowledge:

1. knowledge-that there-exists a knowledge-how (or knowledge of the possible);
2. knowledge-how to-acquire the knowledge-that (or knowledge of reasons).

Here knowledge-how to-acquire the knowledge-that some mathematical assertion is true means, of course, knowledge-how to prove that assertion (cf. [46; pp. 28–29]). In general, (2) could be dubbed “knowledge-why” or even “understanding”.

Now, (1) occurs most distinctively whenever one applies a non-constructive existence theorem. For instance, for those who feel at home with ZFC, presumably one is supposed to have the knowledge-that there-exists a knowledge-how to well-order the reals (without being aware of any specific well-ordering). For those who feel more at home with constructive mathematics, a more down-to-earth example is provided by constructive proof-checkers, such as Coq. If you know that Coq works correctly on your computer and you acquired a file with a fully Coq-formalized proof of, say, the Four Color Theorem, then by running Coq to certify this proof you would presumably acquire the knowledge-that there-exists a knowledge-how to color any given planar map in four colors (without getting any clue how to do the actual coloring).

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14Martin-Löf argued of “knowledge how in Ryle’s terminology” that “the distinction between knowledge how and knowledge that evaporates on the intuitionistic analysis of the notion of truth.” [46; p. 36].

15For instance, if you have manually verified the code of its rather small kernel (which in turn verifies all the needed extensions), and if you believe that one can neglect potential bugs in the operating system and in the design of the microprocessor, as well as possibilities of malfunctioning due to a manufacturing defect, heat or irregularities of power supply (or just because of a microscopic meteorite).

16Without any tricks smuggling in the law of excluded middle as in [28]
6.2. Understanding historic writings

6.2.1. Orlov–Heyting interpretation. Gödel’s provability translation, as well as his sketch of a proof-relevant S4 that we relied upon in motivating our formulation of QHC (see §2.4) were anticipated by informal provability interpretations of intuitionistic logic in the papers by Heyting [30], [31] and, independently, Orlov [49; §§6,7] (see also [17] for a discussion of Orlov’s work in English). We will now briefly review these papers, which will also prepare us for a discussion of Kolmogorov’s letter to Heyting [41].

All three papers focus mainly on a pair of operators, which we will denote by + and ~ throughout, following [30] and [41]. (Orlov [49] writes Φ and X; and in his second paper [31], Heyting writes + and ¬. For consistency, we will alter these to + and ~, respectively, when quoting from these papers.) The meaning of these operators will be discussed in a moment.

Heyting and Kolmogorov also use the symbol ⊢ to mean something quite different from both the modern meaning of this symbol and its original meaning as used by Frege and Russell. According to Heyting [30]:

“To satisfy the intuitionistic demands, the assertion must be the observation of an empirical fact, that is, of the realization of the expectation expressed by the proposition p. Here, then, is the Brouwerian assertion of p: It is known how to prove p. We will denote this by ⊢ p. The words ‘to prove’ must be taken in the sense of ‘to prove by construction’.”

It is not really clear to the author exactly what this may mean from the viewpoint of classical meta-logic. But if letters used for unknown propositions (such as p in Heyting’s words above) are understood as propositional variables (or meta-variables for propositional variables), then, with an appropriate interpretation of + and ~ (discussed below), ⊢ may be read in a usual way, as asserting provability in the modal logic S4. With this in mind, we will follow Heyting et al. in using lowercase letters for propositions in the present section (in contrast to the notation elsewhere in the present series of papers).

If p is a proposition, both Heyting and Orlov interpret +p as p is provable. Orlov states unambiguously that + is constrained precisely by what turns out to be the modal axioms of S4;¹⁷ Heyting says only that “A logic that would treat properties of the function + would [...] be purely hypothetical; [...] one cannot ask [the intuitionistic mathematicians] to develop this logic” [30].

Heyting interprets ~p as “p implies a contradiction” and calls ~ “the Brouwerian negation”; whereas Orlov says that it “has the same meaning” as Brouwer’s notion of “absurdity” of a proposition in [12]. At the same time, Orlov is able to identify ~p as + ¬p, where ¬ is the classical negation; in this connection, Heyting only says: “the negation of a proposition always refers to a proof procedure which leads to the contradiction, even if the original proposition mentions no proof procedure” [31].

¹⁷Orlov’s own system of axioms is weaker than S4 in that it is based not on classical logic, but on a weaker system now known as relevant logic, which satisfies ⊢ p ↔ ¬¬p but neither of ⊢ ¬p → (p → q), ⊢ q → (p → q), ⊢ p ∨ ¬p (see [17]). Orlov erroneously believed that the use of full classical logic would trivialize the + operator.
In his second paper [31], Heyting also states:

“[I]ntuitionist logic, insofar as it has been developed up to now without using the function +, must be understood [... in the sense of] treating only propositions of the form ‘p is provable’ or, to put it another way, by regarding every intention as having the intention of a construction for its fulfillment added to it.”

In practical terms this means, in particular, that Brouwer’s theorem on triple absurdity [12] should be interpreted as \( \vdash \neg \neg \neg p \leftrightarrow \neg p \). Thus we consider Heyting’s earlier claim in [30] that “Mr. Brouwer has proved that \( \neg \neg \neg p \) is identical to \( \neg p \)” to be in error, as pointed out essentially by Heyting himself. This fully agrees with Orlov’s independent analysis, which contains valid proofs of \( \vdash p \rightarrow \neg \neg \neg p \) and \( \vdash \neg \neg \neg p \leftrightarrow \neg p \) and informal arguments that \( \vdash p \not\rightarrow \neg \neg p \) and \( \vdash \neg \neg \neg p \not\rightarrow \neg p \). Orlov also supported these judgements by an analysis of Brouwer’s writings:

“Brouwer often resorts to the following method of defining notions: ‘We call a real number \( g \) rational if two whole numbers \( p \) and \( q \) can be specified so that \( g = p/q \); and irrational if one can make the assumption of the rationality of \( g \) to lead to absurdity.’ [[12]]

Here it is evident that a rational number is defined via provability of the existence of the two integers, in other words, by a function of the form \(+a\). If the assumption that the existence of \( p \) and \( q \) is provable leads to absurdity, then \( g \) is irrational. Therefore, irrationality is defined by means of \( \neg +a \).”

Nevertheless, Orlov was only partially aware of Heyting’s principle quoted above, for he interpreted the intuitionistic understanding of the principle of excluded middle as \( + p \lor \neg p \), rather than \( + p \lor \neg + p \).

Apart from these oddities, Heyting’s both papers and Orlov’s paper seem to be compatible with each other and with Gödel’s provability translation. In particular, both Heyting and Orlov mention the equivalence of \(+ + p\) with \(+ p\) and of \(+ \neg p\) with \(\neg p\); and of the judgements \(\vdash + p\) and \(\vdash p\). We should mention, however, yet another oddity found in Heyting’s letter to Freudenthal, where he first gave an interpretation of the intuitionistic negation (see [52]):

“I believe that also \( a \rightarrow b \), like the negation, should refer to a proof procedure: ‘I possess a construction that derives from every proof of \( a \) a proof of \( b \).’ In the following, I will keep to this interpretation. There is therefore no difference between \( a \rightarrow b \) and \(+ a \rightarrow + b \).”

Note that this is at odds already with Heyting’s own distinction between \( \neg a \) and \( \neg + a \), which he was clear about in [30].

We will thus refer to the standard interpretation of \(+\) and \(\neg\), where “propositions” are formalized as formulas of S4, \(+\) is identified with the modality \(\Box\) of S4, and \(\neg\) is regarded as an abbreviation for \(\Box \neg\). The latter abbreviation is, in fact, very convenient also from a technical viewpoint, for it gives a more intelligible form to judgements that correspond to basic properties of a subset of a topological space under the topological interpretation of S4:

1. \( \vdash + F \) (or \( \vdash F \)): Entire space
2. \( \vdash \neg F \) (or \( \vdash \neg F \)): Empty set
(3) \( \vdash \sim + F \) (or \( \vdash \neg \Box F \)): Boundary set (\( \Leftrightarrow \) interior is empty \( \Leftrightarrow \) complement is dense)

(4) \( \vdash \sim \sim F \) (or \( \vdash \neg \neg \Box F \)): Dense set (\( \Leftrightarrow \) closure is the entire space)

(5) \( \vdash \sim \sim + F \) (or \( \vdash \neg \neg \neg \Box F \)): Complement is nowhere dense (\( \Leftrightarrow \) interior is dense)

(6) \( \vdash \sim \sim \sim F \) (or \( \vdash \neg \neg \neg \neg \Box F \)): Nowhere dense set (\( \Leftrightarrow \) closure is a boundary set)

Note that by Brouwer’s theorem that \( \vdash \sim \sim \sim + F \leftrightarrow \sim + F \), this list cannot be continued any further. It is immediate from \( \vdash \Box F \rightarrow F \) that

- (1) implies (5), which in turn implies (4); and
- (2) implies (6), which in turn implies (3).

Also, it is immediate from the necessitation rule that the following pairs of judgements are contradictory:

- (6) contradicts (4);
- (4) contradicts (2);
- (2) contradicts (1);
- (1) contradicts (3);
- (3) contradicts (5).

Heyting [30] overlooked only the last entry of the list (1)–(6). Accordingly, in his discussion of possible combinations of these judgements he missed precisely those that involve (6). His “possible combinations” consist of judgements that neither imply nor contradict one another. It is easy to check that there are just nine of them: the empty combination; the six singleton combinations; and two pairs: (3)+(4) and (5)+(6). Of these, only (1), (2), (3)+(4) and (5)+(6) are “definitive” in Heyting’s terminology, that is, cannot be extended to a larger combination.

6.2.2. Kolmogorov’s letters to Heyting. The combination (3)+(4) was discussed by Heyting in detail [30]:

“[L]et us consider the proposition ‘Every even number is a sum of two primes’ (Goldbach’s conjecture). Then \( p \) means simply that in taking an even number at random, one expects to be able to find two primes of which it is the sum. (This possibility is decided after a finite number of attempts.) \( +p \) on the contrary requires a construction that gives us this decomposition for all even numbers at once. [...] In order to be able to assert \( \vdash \sim + p \), it suffices to reduce to a contradiction the assumption that one can find a construction proving \( p \); by that one will not yet have proved that the assumption \( p \) itself implies a contradiction. If we appeal to the example of Goldbach’s conjecture, we find: \( \vdash \sim + p \) means that one will never be able to find a rule that effects in advance the decomposition of all even numbers; this does not mean that there is a contradiction when one supposes that in taking an even number at random, one will always be able to divide it into two prime numbers. It is even conceivable that it could one day be proved that this last supposition cannot lead to a contradiction; then one would have at the same time \( \vdash \sim + p \) and \( \vdash \sim \sim p \). One should abandon every hope of ever settling the question; the problem would be unresolvable.”
Heyting’s second paper [31] contains a virtually identical discussion but with a different choice of \( p \), namely, the one asserting that a given rational number lies within every interval with rational endpoints that contains Euler’s constant \( C \).

Heyting’s claims are confirmed rigorously in the standard interpretation, since there exist dense boundary sets (for instance, \( \mathbb{Q} \) viewed as a subset of \( \mathbb{R} \)). This is in contrast with Kolmogorov’s claims in his first letter to Heyting [41]:

1. You consider as an example (in [[30]]) the proposition ‘Every even number is a sum of two primes’. But it is known that the formula \( \vdash \sim p \rightarrow p \) is true in this case from either classical or intuitionistic viewpoint. If one asserts \( \vdash \sim p \), then automatically there is ‘a construction, which gives us this decomposition for all even numbers at once’. Hence \( \vdash \sim p \rightarrow +p \), and the case \( \vdash \sim p \land \sim +p \) is impossible.

2. It seems to me that the point is not in a defect of this particular example. Each ‘proposition’ in your framework belongs, in my view, to one of two sorts:

\((\alpha)\) \( p \) expresses hope [l’esperance] that in prescribed circumstances, a certain experiment will always produce a specified result. (For example, that an attempt to represent an even number \( n \) as a sum of two primes will succeed upon exhausting all pairs \((p,q), p < n, q < n\).)

\((\beta)\) \( p \) expresses the intention to find a construction.

3. We agree that in the case \((\beta)\), the difference between \( p \) and \(+p\) is not essential, but the proposition \( \sim p \rightarrow p \) should not be regarded as evident. In the first case \((\alpha)\), on the contrary, \( p \) and \(+p\) have distinct meanings, but we have \( \vdash \sim p \rightarrow p \) and \( \vdash \sim p \rightarrow +p \). This is why \( \vdash \sim p \land \sim +p \) is always impossible, both in the case \((\alpha)\) and in the case \((\beta)\).

4. I prefer to keep the name proposition (Aussage) only for propositions of type \((\alpha)\) and to call “propositions” of type \((\beta)\) simply problems (Aufgaben). Associated to a proposition \( p \) are the problems \( \sim p \) (to derive contradiction from \( p \)) and \( +p \) (to prove \( p \)).”

Kolmogorov insists that every proposition \( p \) of type \((\alpha)\) satisfies \( \vdash \sim p \rightarrow +p \), apparently because \( p \) comes endowed with a constructive procedure of verification of the validity of every particular instance of \( p \); from \( \sim p \) we infer that this procedure actually returns a positive result on all inputs; thus it yields a constructive proof of \( p \).

This applies to the example of \( p \) cited by Kolmogorov, Every even number is a sum of two primes, since it can be verified constructively whether a given specific number \( n \) is a sum of two primes (by exhausting all primes \(< n\)). The same applies to the other example of Heyting, and in general to every proposition of the form \( \forall x_1 \ldots \forall x_n q(x_1, \ldots, x_n) \), where the validity of \( q \) is verifiable by a finite procedure.\(^\text{18}\)

However, already the classical negation \( \neg p \) of such a proposition \( p \), for instance, the proposition There exists an even number that is not a sum of two primes, is presumably neither of type \((\alpha)\) nor of type \((\beta)\) — since it does not assert that such a number can be constructed explicitly. (In contrast, \( \sim p \) must be of type \((\beta)\), according to Kolmogorov’s

\(^{18}\)These so-called \( \Pi^0_1 \) propositions are sometimes claimed to be precisely the propositions accessible to Hilbert’s finitistic reasoning (see [23; p. 191]).
4. In his second letter to Heyting, Kolmogorov himself speaks of propositions that are neither of type (α) nor of type (β):

“In the meantime, I have thought about your example of the proposition ‘For all i we have \(a_i < b_i\).’ Let, in general, \(x\) be a variable and \(P(x)\) a problem depending on \(x\). The ‘hope’ [Hoffnung] to find for each \(x\) a solution of the problem \(P(x)\) is neither a problem nor a proposition in my terminology. It would be very interesting to know if with this hope you associate a positive expectation [Erwartung] that for each \(x\) the problem \(P(x)\) will really be solved (by whom and when)? If this expectation is not intended, then I am afraid that we will arrive at the naive non-intuitionistic understanding of the statement ‘\(P(x)\) is solvable for each \(x\).’”

If the \(a_i\) and \(b_i\) are assumed to be real numbers (Heyting’s reply to Kolmogorov’s first letter, which would clarify this matter, is not available; see, however, a fragment of Heyting’s letter to Becker below) then the proposition \(a_i < b_i\) amounts to an existentially quantified proposition about rational numbers (or integers). In this case, Kolmogorov’s statement ‘\(P(x)\) is solvable for each \(x\)’, where the problem \(P(x)\) is instantiated as “Prove that \(a_x < b_x\)” will be of the form \(∀x?!∃y q(x, y)\), to use the notation of QHC. As observed by Troelstra [60],

“In the second letter [to Heyting] Kolmogorov observes that the distinction between ‘\(P(x)\) can be solved for each \(x\)’ and ‘there is a uniform method for solving \(P(x)\) for each \(x\)’ is non-intuitionistic [that is, “does not fit into an intuitionistic point of view’]; the point was accepted by Heyting, as the fragment of his letter to Becker, reproduced above, shows.”

Here is the relevant part of the said fragment of Heyting’s letter to Becker [60] (translated from German):

“Another matter is that the application of my logic is restricted to constructive questions. What I mean by this may be illuminated by the following example. Let two sequences of real numbers \(\{a_i\}\) and \(\{b_i\}\) be given. The proposition ‘For each \(i\), \(a_i = b_i\)’ admits two interpretations.

a) It can mean the problem of finding a general proof that upon the choice of a particular index \(i\) specializes to a proof of \(a_i = b_i\);

b) one can understand it the expectation [Erwartung] that if one keeps choosing an index \(i\) arbitrarily, he will succeed in proving \(a_i = b_i\) every time.”

The difference is clear if one applies the negation to a) and b). It is conceivable that the assumption of a proof as requested in a) could be proved contradictory, without this contradiction affecting the assumption of b). My logic applies if each proposition is understood as in a); the logic of the non-constructive expectations b) would be much more involved; I do not consider its development to be very fruitful.”

6.2.3. Interpreting Kolmogorov’s letters. To summarize our reading of the quoted writings, Heyting is right in that one can have at the same time \(\vdash \sim p\) and \(\vdash \sim \sim p\); and Kolmogorov is right in that one cannot have these two at the same time if \(p\) is as in any of Heyting’s two examples, or more generally if \(p\) is a proposition either of type (α) or of type (β).
If we try extract precise definitions from Kolmogorov’s letter, propositions of type (α) satisfy ⊢ ∼∼p → +p and presumably no other identities (we assume, following Heyting, that all propositions satisfy ⊢ +p → p, so Kolmogorov’s ⊢ ∼∼p → p is automatic); whereas propositions of type (β) satisfy ⊢ p → +p and presumably no other identities. On the standard interpretation of + and ∼, propositions of type (β) should then correspond to all open sets, and propositions of type (α) to all sets S such that Int Cl S ⊂ Int S. This includes, in particular, all closed sets and all regular open sets, and no other open sets.

Although this is clearly not what Kolmogorov could have meant regarding the judgement ⊢ ∼+p ∧ ∼∼p, which under the standard interpretation of ∼ and + in QS4 is identified with ⊢ ¬□F ∧ ¬□¬F, it is worth observing that ⊢ ¬□p ∧ ¬□¬p / × is an admissible rule for S4 (see [1; Example 5.20]).

Kolmogorov concluded his first letter to Heyting with the proposal of two operators from propositions of type (α) to problems. The first operator, Prove the given proposition, resembles the restriction of our operator “!”, which is defined on all propositions. The second operator, Derive a contradiction from the given proposition, resembles what we denote by !∼. (Note, however, that to interpret Kolmogorov’s first operator ∼ as + ∼, one has to extend the domain of his second operator + to the classical negations of propositions of type (α).) In this setup, Kolmogorov’s propositions of type (α) are precisely those propositions that satisfy ⊢ ¬!¬p → !p (note that one of the two negations is classical and the other one is intuitionistic).

Heyting’s 1932 paper [32] contains an elaboration of some ideas in Kolmogorov’s first letter, which appears to agree with our conclusions. For each formula F of a logical calculus that contains classical propositional calculus, Heyting introduces the problem βF of proving F; and discusses the significance of the problems ¬βF → β¬F and ¬β¬F → βF.

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Disclaimers

1. Some translations quoted in this paper were edited in order to improve syntactic and semantic fidelity. When emphasis is present in quoted text, it is always original.

2. I oppose all wars, including those wars that are initiated by governments at the time when they directly or indirectly support my research. The latter type of wars include all wars waged by the Russian state in the last 25 years (in Chechnya, Georgia, Syria and Ukraine) as well as the USA-led invasions of Afghanistan and Iraq.
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