Quantum illumination via quantum-enhanced sensing

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Quantum-enhanced sensing has a goal of enhancing a parameter sensitivity with input quantum states, while quantum illumination has a goal of enhancing a target detection capability with input entangled states in a heavy noise environment. Here we propose a concatenation between quantum-enhanced sensing and quantum illumination that can take quantum advantage over the classical limit. Using quantum Fisher information formula, we connect quantum phase sensing in an interferometry with sensing a target reflectivity that we call a target sensitivity. Including thermal noise and loss, we put the target sensitivity into noisy quantum-enhanced sensing. Under the constraint of a total input state energy, for example, $N$-photon entangled states can exhibit better performance than a two-mode squeezed vacuum state and a separable coherent state. Incorporating a photon-number difference measurement that discriminates the presence and absence of the target distinctly, we enhance not only the target sensitivity but also the signal-to-noise ratio with increasing thermal noise.

INTRODUCTION

Quantum-enhanced sensing(QES) takes quantum advantage over classical strategies via input entanglement and squeezing[1, 2]. For a single-parameter sensing, a parameter sensitivity is lower bounded by the inverse of the quantum Fisher information which provides maximum information we can extract in a small change of the parameter, where the mean value of the parameter is equal to the true value of one. According to input probe states, the sensitivity of a phase is lower bounded by the standard quantum limit $(1/\sqrt{N})$ with coherent states or by the Heisenberg limit $(1/N)$ with NOON states and squeezed states, where $N$ is a mean photon number of an input probe state. In a noisy scenario, we can explore noisy quantum-enhanced sensing[3] as well as quantum illumination(QI)[4, 5] that discriminates the presence and absence of a target. For the QI which detects the target with entangled states in a heavy noise environment, the target is simply modelled by a beam splitter in a laboratory. Using entangled states, we can enhance the possibility of detecting the target even if there is no entanglement in the output modes. Specifically, two-mode squeezed vacuum(TMSV) states can exhibit quantum advantage over classical states in QI[6] with no output entanglement, where it was not shown about how to achieve the quantum advantage with any measurement setup.

For the QI using an input TMSV state, it was first implemented with direct photon counting on output modes[7], where the signal-to-noise ratio(SNR) of the TMSV state exhibited higher than the SNR of a correlated thermal state. Regarding a specific measurement setup, Guha and Erkmen proposed a measurement scheme with optical parametric amplifier that achieves a half of the bound[8]. It was implemented by Z.Zhang et al.[9], showing the 20% improvement of the SNR over optimal classical setup. Later, Q. Zhang, Z. Zhang, and J.H. Shapiro proposed another measurement scheme[10] that sum-frequency generation with feedforward can achieve the bound asymptotically. Except those works, there were several investigations on the QI both theoretically[11–25] and experimentally[26–30].

Both QI and QES take quantum advantage with input probe entangled states, but the former can achieve it with no entanglement in the output modes[22] and the latter can do it with entanglement in the output modes[31]. Moreover the QI is based on quantum discrimination and the QES is based on quantum estimation[32]. Although they exhibit different characteristics, the QI and the QES can be connected by a parameter sensitivity and an SNR. In Fig. 1, we draw the architecture of our scheme for concatenating QI and QES. In terms of quantum Fisher information(QFI) which determines the ultimate precision limit of quantum phase estimation[33], first, we derive that a phase sensitivity in an interferometer is being equivalent to a target sensitivity at low reflectivity $(\eta \ll 1)$. Including thermal noise and loss, we investigate noise-resilient QES about sensing the target reflectivity. With a specific measurement setup, the noise-resilient QES can be adapted for the QI which is based on a target detection with SNR. The target sensitivity is investigated with the error propagation relation of the target reflectivity, and correspondingly the SNR presents a similar behavior of the target sensitivity.
EQUIVALENCE BETWEEN DIFFERENT PARAMETER SENSITIVITIES

Using quantum Fisher information (QFI) formula, we show that a phase($\phi$) sensitivity can be equivalent to a target sensitivity. For pure states, the quantum Fisher information is given by $H = 4 |\langle \partial_{\phi} (\psi_{a}) \rangle^{2} |^{2} (\psi_{a})_{ab} - |\langle \psi_{b} \rangle|^{2}$, where $|\langle \psi_{x} \rangle\rangle_{ab} = \hat{U}_{ab}(x) |\psi_{a}\rangle_{ab}$. First, we look into the target sensitivity which is replaced by a sensitivity of a beam splitter reflectivity($\eta$). A general beam splitting operation[34, 35] is represented by

$$\hat{B}_{ab}(\theta, \varphi) = \exp \left\{ \frac{\theta}{2}(\hat{a}^{\dagger} \hat{b}^{\dagger} e^{i\varphi} - \hat{a}^{\dagger} \hat{b} e^{-i\varphi}) \right\}$$

where $\eta = \sin(\theta/2)$ is the reflectivity of a beam splitter, and $\varphi$ is the phase difference between the transmitted and reflected fields. Assuming $\theta \ll 1$, the beam splitting operation is approximated as $\hat{B}_{ab}(\eta, \varphi)$. Applying the beam splitting operation on a two-mode input state as $|\psi_{a}\rangle_{ab} = \hat{B}_{ab}(\eta, \varphi)|\psi_{a}\rangle_{ab}$, we derive the following QFI

$$H = -4 \left[ |\langle \hat{a} | \hat{b} e^{i\varphi} - \hat{b}^{\dagger} \hat{a} e^{-i\varphi} \rangle | \langle \psi_{a} \rangle_{ab}^{2} \right]$$

At $\varphi = \pi/2$, the QFI is given by $H = 4((\hat{a} \hat{b}^{\dagger} + \hat{b}^{\dagger} \hat{a})^{2} - (\hat{a}^{\dagger} \hat{b} + \hat{b} \hat{a})^{2}^{2})$. We can derive the same QFI formula for sensing a phase in an interferometer, as shown in Fig. 2.

In an ideal state, the QFI becomes the same as the Eq. (2). In an ideal scenario, thus, sensing a phase in the interferometry is equivalent to sensing a reflectivity($\eta$) of the beam splitter by means of quantum Fisher information. Given a unitary operation $\hat{U}_{ab}(x) = e^{ix\hat{O}_{ab}}$, in general, we can derive the similar relation if the other unitary operation is transformed into $\hat{U}_{ab}(y) = \hat{A} e^{iy\hat{O}_{ab}}$, where $\hat{A}$ is independent of a parameter $y$. The sensitivity of $x$ is being equivalent to the sensitivity of $y$ by their QFI formula.

When one of the input modes is a coherent state as $|\psi_{a}\rangle_{ab} = |\alpha\rangle_{a} |\Psi_{b}\rangle$, the QFI of Eq. (2) is given by

$$H_{|\psi_{a}\rangle_{ab}} = 4 \left( |\langle \hat{b} | \hat{b} \rangle | 2|\Delta X_{\theta+\pi/2}^{2} \right)$$

where $\Delta X_{\theta+\pi/2}^{2} = \langle \hat{X}_{\theta+\pi/2}^{2} \rangle - |\langle \hat{X}_{\theta+\pi/2} \rangle|^{2}$ is the variance of a quadrature operator $\hat{X}_{\theta+\pi/2} = (be^{-i\theta} - \hat{b}^{\dagger} e^{i\Theta})/i\sqrt{2}$. Note that $\alpha = |\alpha| e^{i\theta}$ and $\Theta = \theta - \varphi$. Since the optimal condition of the other input mode is antisqueezed in the direction of $\Theta + \pi/2$, it is the best to inject a squeezed vacuum state in the input mode $b$[36]. When one of the input modes is a vacuum state as $|\psi_{a}\rangle_{ab} = |0\rangle_{a} |\Psi_{b}\rangle$, the QFI is given by $4 |\langle \hat{b} | \hat{b} \rangle |$ such that the optimal condition is proportional to the mean photon number of the other input mode. The best thing that we can do is to inject a coherent state in the input mode $b$.

NOISY QUANTUM-ENHANCED SENSING

Including thermal noise and loss, we manipulate the target sensitivity in noisy quantum-enhanced sensing. For mixed states, we utilize the QFI formula[37] that is given by $H = 2 \sum_{n} \rho_{n} |\langle \partial_{\eta} |\phi_{n}\rangle_{\eta=0} |^{2}$, where $\rho_{n}$ is the derivative of the output state $\rho_{n}$, $\lambda_{n}$, and $|\phi_{m}\rangle$ are the eigenvalues and the eigenstates of $\rho_{\eta=0}$, respectively. Note that a higher QFI represents a better target sensitivity. In Fig. 3 (a), we assume that the input mode $a$ experiences photon loss, where the input mode $b$ is in vacuum state and one of the output modes is discarded. Using the QFI formula, we obtain the maximum value of the QFI as $4N$, where $N$ is the mean photon number of the input mode $a$. In spite of preparing entangled states in the input modes $a$ and $c$, the non-interacting input mode $c$ does not contribute to the performance of the sensing. Thus, it is useless to prepare input entangled states in the input photon loss scenario.

Instead of the vacuum state in an input mode $b$, we insert a thermal state into the input mode $b$ as a thermal noise effect. Based on the QFI of $|\eta\rangle_{\eta}$, we may consider a maximally entangled state $(1/\sqrt{N}) \sum_{n=0}^{N-1} |n, n\rangle_{ac}$ but the maximally entangled state cannot beat the performance of a coherent state[16]. In Fig. 3 (a) here we consider another type of entangled states, i.e., $N$-photon entangled states[38] which are given by the formula, $\sum_{n=0}^{N} |n\rangle_{N} - n, n\rangle_{ac}$, where $\sum_{n=0}^{N} |n\rangle_{N}^{2} = 1$. In the constraint of a total input state energy, we show

![FIG. 2. Equivalence between sensing a phase($\phi$) and sensing a reflectivity($\eta \ll 1$) of a beam splitter via quantum Fisher information.](image-url)
and (c), we observe that the QFI of a 4-photon entangled state\((\sum_{n=0}^{4} a_n |4-n, n\rangle_{ac})\) is always larger than the QFI of the two-mode squeezed state with increasing \(N_b\) at \(\langle \tilde{n}_a + \tilde{n}_c \rangle = 4\), while both the entangled states outperform a separable coherent state that is considered as a reference state. The corresponding coefficients have a descending order: \(|a_0| > |a_1| > |a_2| > |a_3| > |a_4|\). With increasing thermal noise, the absolute value of \(a_0\) decreases but the absolute values of the other coefficients increase. It implies that the portion of the signal photon decreases whereas the portion of the idler photon increases. Although the less portion of signal photon is sent to a target, the optimized \(N\)-photon entangled state is more noise-resilient than the two-mode squeezed vacuum state in the noisy quantum-enhanced sensing.

**SENSING A TARGET WITH A SPECIFIC MEASUREMENT SETUP**

Since the QFI is based on the corresponding correlated measurement that is derived with the symmetric logarithmic derivative of \(\rho_{th}\) calculated at \(\eta = 0\) \cite{37}, it is hard to implement in a laboratory. Proposing an implementable measurement setup to detect a target in a noisy environment, we investigate not only the target sensitivity but also the signal-to-noise ratio (SNR) for the output modes. The target sensitivity is simply evaluated with the error propagation relation \(\Delta \eta = \sqrt{\Delta M(\eta) / |\partial \eta M(\eta)|}\), where \(\Delta M(\eta)^2 = (\dot{M})^2 - \langle \dot{M} \rangle^2\) and \(\langle \dot{M} \rangle = M(\eta)\). The SNR is given by the formula \(\text{SNR} = M(\eta) / \sqrt{\Delta M(\eta)^2}\). Combining both formulas, we derive the following relation

\[
\text{SNR} = \frac{M(\eta)}{\Delta \eta \left| \frac{\partial \eta M(\eta)}{\partial \eta} \right|}.
\]

In the constraint of a function of \(M(\eta)\), the SNR is inversely proportional to the target sensitivity. We can infer that the better the target sensitivity is, the higher the SNR is.

We assume that an input thermal noise is separately distributed after a beam splitting operation as \(\rho_{th}^{(ia)}(\eta^2 N_b) \otimes \rho_{th}^{(ib)}((1 - \eta^2)N_b)\) in the condition of \(\eta^2 N_b \ll 1\) which satisfies the Gaussian Rényi-2 mutual information \cite{39} of the output thermal noise being \(I_2 = \ln(1 + 2\eta^2(1 - \eta^2)) N_b^2(1 + 2N_b) \approx 0\).

First, we may consider a direct photon counting on both output modes, which was already implemented in experimental quantum illumination \cite{7}. However, the direct photon counting setup cannot discriminate an input pure entangled state from its mixture, such as \(|\psi\rangle_{ac} = a_1|12\rangle_{ac} + a_2|21\rangle_{ac}\) and \(\rho_{ac} = |a_1|^2|12\rangle_{ac}\langle 12| + |a_2|^2|21\rangle_{ac}\langle 21|\). It does not demonstrate the contribution of the interference terms so that we cannot observe
of the target, the output state \((\rho_{\text{th}}^{(b)} \otimes \text{tr}_a[|\psi\rangle_{ac} \langle \psi|])\) is always observed as \(M(\eta) = \langle \hat{b}^{\dagger} e^{-i\varphi} \hat{c}^{\dagger} be^{i\varphi} \rangle = 0\), in which the measurement observable \(M = \hat{n}_d - \hat{n}_e\) is transformed into \(\langle \hat{b}^{\dagger} e^{-i\varphi} + \hat{c}^\dagger be^{i\varphi} \rangle\) by the reverse 50:50 beam splitting operation. In the presence of the target, we measure the interference terms in the modes \(b\) and \(c\). However, there is an exception that \(M(\eta) = 0\) can indicate the presence of the target with an input pure two-mode Gaussian state having zero first moments, since the off-diagonal elements of the output covariance matrix have the relation \([40]\), such as \(\langle \hat{X}_b \hat{X}_c \rangle = -\langle \hat{P}_b \hat{P}_c \rangle\) and \(\langle \hat{X}_b \hat{P}_c \rangle = \langle \hat{P}_b \hat{X}_c \rangle = 0\). The measurement observable is reformulated with the corresponding position and momentum operators, \(\langle \hat{b}^{\dagger} e^{-i\varphi} + \hat{c}^\dagger be^{i\varphi} \rangle = (\langle \hat{X}_b X_c + P_b P_c \rangle) \cos \varphi + (\langle \hat{X}_b P_c - \hat{P}_b X_c \rangle) \sin \varphi\). Due to that reason, we do not consider a two-mode squeezed vacuum state under the photon-number difference measurement.

Based on the photon-number difference measurement, we explore whether entangled states can exhibit better performance than a separable coherent state. For example, we consider a 4-photon entangled state in comparison with the separable coherent state, at the low reflectivity of \(\eta = 10^{-3}\). For the target sensitivity, the sensitivity with the 4-photon entangled state initially deteriorates with \(N_b\) but it is getting improved at \(N_b \approx 0.5\), as shown in Fig. 4 (b). For the SNR of the output mode, correspondingly, the SNR with the 4-photon entangled state initially deteriorates with \(N_b\) but it is getting improved at \(N_b \approx 0.5\), as shown in Fig. 4 (c). The corresponding coefficients of the 4-photon entangled state have a descending order as \(|\langle a_0 \rangle > |\langle a_1 \rangle > |\langle a_2 \rangle > |\langle a_3 \rangle > |\langle a_4 \rangle |\). With increasing thermal noise, the portion of the signal photon decreases whereas the portion of the idler photon increases. It contributes to decrease the standard deviation \(\sqrt{\Delta M(\eta)^2}\) of the photon-number difference measure with increasing thermal noise, in which the other terms \(M(\eta)\) and \(\frac{\partial M(\eta)}{\partial \eta}\) are not significant. Both of them present a counterintuitive phenomena whereas the performance of the separable coherent state deteriorates with \(N_b\) for the target sensitivity and the SNR. Thus, the transmitted thermal noise contributes to enhance the sensitivity and the SNR in the regime of \(N_b \gtrsim 0.5\). Therefore, entangled states can enhance not only the target sensitivity but also the target detection with increasing thermal noise. Note that there is a similar behavior of enhancing phase sensitivity with the increasing thermal noise [11, 12].

The SNR can be associated with a minimum error probability of distinguishing between the presence and absence of a target. In Gaussian regime, the minimum error probability is given by \(e^{-MR_G^2/2\pi M R_G}\) [8], where \(M(\eta)\) is the number of pairs for returned and idler modes, and \(R_G = (\overline{n}_1 - \overline{n}_0)^2/(2\sigma_0 + \sigma_1)^2\) is the error exponent. \(\overline{n}_1(\overline{n}_0)\) is the mean photon number of the presence(absence) of the target. \(\sigma_1(\sigma_0)\) is the standard

FIG. 4. (a) Measurement setup, (b) Target sensitivity, and (c) Signal-to-Noise Ratio for the reflected signal and the idler modes, as a function of the mean photon number of thermal noise: A 4-photon entangled state (blue curve) and a separable coherent state (red curve) at \(\eta = 10^{-3}\) and \(\langle \hat{n}_a \rangle = 4\). The 50:50 beam splitter has the transformation of \(\hat{d}^\dagger \rightarrow \frac{1}{\sqrt{2}}(\hat{b}^{\dagger} + \hat{c}^{\dagger})\) and \(\hat{c}^\dagger \rightarrow \frac{1}{\sqrt{2}}(\hat{c}^\dagger - \hat{b}^{\dagger})\).
deviation of the presence(absence) of the target. In our non-Gaussian regime, we simply define the error exponent as \( R_{QG} \equiv \frac{(\eta_1 - \eta_0)^2}{(\sigma_0 + \sigma_1)^2} \equiv \text{SNR}_Q^2 \), where \( \text{SNR}_Q = \sqrt{R_{QG}} \) we call an effective SNR. Due to \( \eta_0 = 0 \), the SNR \( \eta \) which is represented by \( \frac{\eta_1}{(\sigma_0 + \sigma_1)} \) exhibits a similar behavior of the SNR, \( \eta_1/\sigma_1 \), while keeping the tendency with thermal noise. Thus, in the minimum error probability, the SNR is just entirely degraded by the amount of \( \sigma_0 \).

**SUMMARY AND DISCUSSION**

We made a connection between quantum-enhanced sensing and quantum illumination by initiating with a scenario that the phase sensitivity is equivalent to the sensitivity of the target reflectivity (\( \eta \ll 1 \)), in terms of QFI. Including thermal noise and loss, we showed that noise-resilient quantum sensing for the target reflectivity can be more enhanced with input N-photon entangled states than with a two-mode squeezed vacuum state. By using a specific measurement setup, we found a link that the target sensitivity can be proportional to the possibility of the target detection, in which we utilized the error propagation relation and the SNR of the reflected signal together with the idler. In terms of the photon-number difference measurement after a 50 : 50 beam splitting operation, which discriminates the presence and absence of the target clearly, we observed that both the target sensitivity and the SNR of the output mode can be enhanced with increasing thermal noise. The SNR is associated with a minimum error probability of distinguishing the presence and absence of the target.

For the target sensitivity, there was a discrepancy between the QFI and the sensitivity of a target reflectivity. The former determines the lowest bound of a parameter sensitivity, and the latter is based on the error propagation relation which should be worse than the lowest bound. It is supposed to be that both target sensitivities should get worse with increasing thermal noise. However, the sensitivity with the error propagation formula enhances with increasing thermal noise whereas the QFI decreases with increasing thermal noise. In the QFI formula, only the first-order field operation of the beam splitter was counted on the derivation\([10]\) as \( (\partial_\eta \rho_\eta)|_{\eta = 0} \approx \text{tr}_a[(\hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a})|\psi\rangle a \langle \psi | \otimes \rho_\eta^{(b)}] \). Due to ignoring the high-order field operations of the beam splitter, we could not observe the effect of the high-order interference in an output state. However, the error propagation relation counted the higher-order field operations of the beam splitter, such that we could observe the effect of the high-order interference in the output state. For a separable coherent state, there was no discrepancy between the QFI and the target sensitivity, due to the fact that a coherent state is fully described with first moment fields.

Our SNR results show that thermal noise in a target detection can be beneficial when \( \eta \ll 1 \). It is therefore interesting to apply the SNR scenario to quantum ghost imaging\([43]\) and quantum-limited loss sensing\([44]\) which make use of input entangled states in heavy noise environment. As a further study, our measurement scheme can be modified even to be effective for input two-mode squeezed vacuum states.

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