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On a generalization of KU-algebras pseudo-KU algebras

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Received: 10 January 2020; Accepted: 1 June 2020; Published: 5 July 2020.

Abstract: As a generalization of KU-algebras, the notion of pseudo-KU algebras is introduced in 2020 by the author (D. A. Romano. Pseudo-UP algebras, An introduction. Bull. Int. Math. Virtual Inst., 10(2)(2020), 349-355). Some characterizations of pseudo-KU algebras are established in that article. In addition, it is shown that each pseudo-KU algebra is a pseudo-UP algebra. In this paper it is a concept developed of pseudo-KU algebras in more detail and it has identified some of the main features of this type of universal algebras such as the notions of pseudo-subalgebras, pseudo-ideals, pseudo-filters and pseudo homomorphisms. Also, it has been shown that every pseudo-KU algebra is a pseudo-BE algebra. In addition, a congruence was constructed on a pseudo-KU algebra generated by a pseudo-ideal and shown that the corresponding factor-structure is and pseudo-KU algebra as well.

Keywords: KU-algebra, Pseudo-KU algebra, pseudo-UP algebra, pseudo-BE algebra, pseudo-ideals, pseudo-homomorphism.

MSC: 62D05.

1. Introduction

The concept of pseudo-BCK algebras was introduce in [1] by Georgescu and Iorgulescu as an extension of BCK-algebras. The notion of pseudo-BCI algebras was introduced and analyzed in [2] by Dudek and Jun as a generalization of BCI-algebras. The concept of pseudo-BE algebras was introduced in 2013 and their properties were explored by Borzooei et al., in [3]. These algebraic structures has been in the focus of many authors (for example, see [4–10]). Pseudo BL-algebras are a non-commutative generalization of BL-algebras introduced in [11]. Pseudo BL-algebras are intensively studied by many authors (for example, [12–14]).

Prabpayak and Leerawat 2009 in [15,16] introduced a new algebraic structure which is called KU-algebras. They studied ideals and congruences in KU-algebras. They also introduced the concept of homomorphism of KU-algebras and investigated some related properties. Moreover, they derived some straightforward consequences of the relations between quotient KU-algebras and isomorphism. Many authors took part in the study of this algebraic structure (for example: [17,18]).

A detailed listing of the researchers and their contributions to these activities it can be found in [19]. Here, we will highlight the contribution of [20]. In [21], Kim and Kim introduced the concept of BE-algebras as a generalization of dual BCK-algebras. This class of algebra was also studied by Rezaei and Saeid 2012 in article [22]. In the article [20], the authors (Rezaei, Saeed and Borzooei) proved that a KU-algebra is equivalent to a commutative self-distributive BE-algebra. (A BE-algebra $A$ is a self-distributive if $x \cdot (y \cdot z) = (z \cdot y) \cdot (x \cdot z)$ for all $x, y, z \in A$.) Additionally, they proved that every KU-algebra is a BE-algebra ([20], Theorem 3.4), every Hilbert algebra is a KU-algebra ([20], Theorem 3.5) and a self-distributive KU-algebra is equivalent to a Hilbert algebra ([20]). Iampan constructed PU-algebra as a generalization of KU-algebra in [19] in 2017 and showed that each KU-algebra is a PU-algebra.

In article [23], the author designed the concepts of pseudo-UP ([23], Definition 3.1) and pseudo-KU-algebras ([23], Definition 4.1) and showed that each pseudo-KU algebra is a pseudo-UP algebra ([23], Theorem 4.1). However, the term ‘pseudo KU-algebra’ and mark ‘PKU’ has already been used in [24] for different purposes. It should be noted here that this term 2019 has been renamed to ‘JU-algebra’ ([25]). Although introducing the term ‘pseudo-KU algebra’ as a name for a structure constructed in the manner described here and using the abbreviation ‘pKU’ for this algebra can lead to confusion, we did it for needs of article [23] and of this paper.
In this paper we develop the concept in more detail of pseudo-KU algebras and we identify some of the main features of this type of universal algebras. The paper was designed as follows: After the Section 2, which outlines the necessary previous terms, Section 3 introduces the concept of pseudo-KU algebra and analyzes some of its important properties. In Section 4, the concept of pseudo-KU algebras is linked to the concepts of pseudo-UP and pseudo-BE algebras. Section 5 deals with some substructures of this class of algebras such as pseudo-subalgebras, pseudo-ideals and pseudo-filters. Finally, in Section 6, the concepts of pseudo-homomorphisms and congruences on pseudo-KU algebras are analyzed.

2. Preliminaries

In this section we will describe some elements of KU-algebras from the literature [15,16] necessary for our intentions in this text.

Definition 1. ([15]) An algebra \( A = (A, \cdot, 0) \) of type \((2,0)\) is called a KU-algebra where \( A \) is a nonempty set, \( \cdot, \) \( \cdot \) is a binary operation on \( A \), and \( 0 \) is a fixed element of \( A \) (i.e. a nullary operation) if it satisfies the following axioms:

- **(KU-1)** \((\forall x, y, z \in A)( (x \cdot y) \cdot (y \cdot z) = x \cdot (y \cdot z) ) = 0 \),
- **(KU-2)** \((\forall x \in A)(0 \cdot x = x) \),
- **(KU-3)** \((\forall x \in A)(x \cdot 0 = 0) \), and
- **(KU-4)** \((\forall x, y \in A)((x \cdot y = 0 \land y \cdot x = 0) \implies x = y) \).

On a KU-algebra \( A = (A, \cdot, 0) \), we define the KU-ordering \( \leq \) on \( A \) as follows ([15], pp. 56):

\[
(\forall x, y \in A)(x \leq y \iff y \cdot x = 0).
\]

Lemma 1. In a KU-algebra \( A \), the following properties hold:

1. \((\forall x \in A)(x \leq x) \),
2. \((\forall x, y \in A)((x \leq y \land y \leq x) \implies x = y) \),
3. \((\forall x, y, z \in A)((x \leq y \land y \leq z) \implies x \leq z) \),
4. \((\forall x, y, z \in A)(x \leq y \implies z \cdot x \leq z \cdot y) \),
5. \((\forall x, y, z \in A)(x \leq y \implies y \cdot z \leq x \cdot z) \),
6. \((\forall x, y \in A)(x \cdot y \leq y) \) and
7. \((\forall x \in A)(0 \leq x) \).

Definition 2. ([15]) Let \( S \) be a non-empty subset of a KU-algebra \( A \).

(a) The subset \( S \) is said to be a KU-subalgebra of \( A \) if \((S, \cdot, 0)\) is a KU-algebra.

(b) The subset \( S \) is said to be an ideal of \( A \) if it satisfies the following conditions:

- \((I1)\ 0 \in S \)
- \((I2)\ (\forall x, y, z \in A)((x \cdot (y \cdot z) \in S \land y \in S) \implies x \cdot z \in S) \).

As shown in [18], this kind of algebra satisfies one specific equality.

Lemma 2 ([18]). In a KU-algebra \( A \), the following holds:

- **(KU-5)** \((\forall x, y, z \in A)(z \cdot (y \cdot x) = y \cdot (z \cdot x)) \).

In the light of the previous equality, condition \((I2)\) is transformed into condition:

- **(J2)** \((\forall x, y \in A)((x \cdot y \in S \land x \in S) \implies y \in S) \).

Indeed, if we put \( x = 0, y = x \) and \( z = y \) in \((J2)\), we immediately obtain \((J3)\) by \((KU-2)\). Conversely, let \((J3)\) be a valid formula and let \( x, y, z \in A \) be arbitrary elements such that \( x \cdot (y \cdot z) \in J \) and \( y \in J \). Then \( y \cdot (x \cdot z) \in J \) by \((KU-5)\). Thus \( x \cdot z \in J \) by \((J3)\).

From \((J3)\) it immediately follows:

Lemma 3. Let \( S \) be an ideal in a KU-algebra \( A \). Then

- **(J4)** \((\forall x, y \in A)((x \leq y \land y \in S) \implies x \in S) \).

We can introduce the concept of filters in KU-algebra if formula \((J3)\) serves as a motivation.
Definition 3. The subset $F$ is said to be a filter of $A$ if it satisfies the following conditions:

(F1) $0 \in F$, and

(F3) $(\forall x, y \in A)((x \cdot y \in F \land y \in F) \implies x \in F)$.

A filter in KU-algebra, designed in this way, has the following property:

Lemma 4. Let $F$ be a filter in a KU-algebra $A$. Then

(F4) $(\forall x, y \in A)((x \leq y \land x \in F) \implies y \in F)$.

3. Concept of pseudo-KU algebra

Definition 4. ([23]) An algebra $\mathfrak{A} = ((A, \leq), \cdot, *, 0)$ of type $(2, 2, 0)$ is called a pseudo-KU algebra if it satisfies the following axioms:

(pKU-1): $(\forall x, y, z \in A)((y \cdot x) \leq ((x \cdot z) \ast (y \cdot z)) \land (y \ast x) \leq ((x \ast z) \ast (y \ast z)))$,

(pKU-2): $(\forall x \in A)((0 \cdot x = x) \land (0 \ast x = x))$,

(pKU-3): $(\forall x \in A)(x \leq 0)$,

(pKU-4): $(\forall x, y \in A)((x \leq y \land y \leq x) \implies x = y)$, and

(pKU-5): $(\forall x, y \in A)((x \leq y \iff x \cdot y = 0) \land (x \leq y \iff x \ast y = 0))$.

Remark 1. We emphasize that in pseudo-KU algebra the relation of the order is determined inversely with respect to the definition of the order in the KU-algebra.

Lemma 5. If $\mathfrak{A}$ is a pseudo-KU algebra, then

(pKU-6) $(\forall x \in A)((x \cdot x = 0) \land (x \ast x = 0))$.

Proof. If we put $x = 0$, $y = 0$, and $z = x$ in the formula (pKU-1), we get

$$(0 \cdot 0) \ast ((0 \cdot x) \ast (0 \cdot x)) = 0 \land (0 \ast 0) \cdot ((0 \ast x) \cdot (0 \ast x)) = 0.$$ 

From where we get

$$x \cdot x = 0 \land x \ast x = 0$$

with respect to (pKU-2). □

Proposition 1. If $\mathfrak{A}$ is a pseudo-KU algebra, then

(11) $(\forall x, y, z \in A)(x \leq y \implies ((y \cdot z \leq x \cdot z) \land (y \ast z \leq x \ast z))$ and

(12) $(\forall x, y, z \in A)(x \leq y \implies ((z \cdot x \leq z \cdot y) \land (z \ast x \leq z \ast y))$.

Proof. Let $x, y, z \in A$ such that $x \leq y$. Then $x \ast y = 0 = x \ast y$. If we put $x = y$ and $y = x$ in (pKU-1), we get

$$0 = (x \cdot y) \ast ((y \cdot z) \ast (x \cdot z)) = 0 \ast ((y \cdot z) \ast (x \cdot z)) = (y \cdot z) \ast (x \cdot z).$$

So, we have $y \cdot z \leq x \cdot z$. Similarly, we have

$$0 = (x \ast y) \cdot ((y \ast z) \cdot (x \ast z)) = 0 \cdot ((y \ast z) \cdot (x \ast z)) = (y \ast z) \cdot (x \ast z)$$

and $y \ast z \leq z \ast x$.

On the other hand, if we put $z = y$ and $y = z$ in (pKU-1), we have

$$0 = (z \cdot x) \ast ((x \cdot y) \ast (z \cdot y)) = (z \cdot x) \ast ((0 \ast (z \cdot y)) = (z \cdot x) \ast (z \cdot y).$$

This means $z \cdot x \leq z \cdot y$. It can be similarly proved that it is $z \ast x \leq z \ast y$. □

In 2011, Mostafa, Naby and Yousef proved Lemma 2.2 in [18]. In the following Proposition, we show that analogous equality is also valid in pseudo-KU algebras.

Proposition 2. In pseudo-KU algebra $\mathfrak{A}$, then

(pKU) $(\forall x, y, z \in A)(x \ast (y \cdot z) = y \ast (x \cdot z) \land x \cdot (y \ast z) = y \cdot (x \ast z))$
is valid formula.

**Proof.** If we put \( y = 0 \) in (pKU-1), we have

\[
0 \cdot x \leq (x \cdot z) * (0 \cdot z).
\]

Then, we have \( x \leq (x \cdot z) * z \). From here it follows

\[
((x \cdot z) * z) \cdot (y * z) \leq x \cdot (y * z)
\]

by (11). On the other hand, if we put \( x = z \cdot z \) in (pKU-1), we get

\[
y * (x \cdot z) \leq ((x \cdot z) * z) \cdot (y * z) \leq x \cdot (y * z).
\]

Since the variables \( x, y, z \in A \) are free variables, if we put \( x = y \) and \( y = x \), we get an inverse inequality. From here it follows (pKU) by (pKU-4).

The other equality can be proved in an analogous way. \( \square \)

4. Correlation of pseudo-KU algebras with other types of pseudo algebras

The notion of pseudo-UP algebra as a generalization of the concept of UP-algebras was introduced and analyzed in [23].

**Definition 5.** ([23]) A pseudo-UP algebra is a structure \( \mathfrak{A} = (A, \leq, \cdot, *, 0) \), where \( \leq \) is a binary relation on a set \( A \), \( \cdot \) and \( * \) are internal binary operations on \( A \) and \( 0 \) is an element of \( A \), verifying the following axioms:

- (pUP-1) \( (\forall x, y, z \in A) (y \cdot z \leq (x \cdot y) \cdot (z \cdot y) \land y \cdot z \leq (x \cdot y) \cdot (z \cdot y)) \);
- (pUP-4) \( (\forall x, y \in A) (x \leq y \Rightarrow x = y) \);
- (pUP-5) \( (\forall x, y \in A) (y = 0 \Rightarrow x = y = 0) \); and
- (pUP-6) \( (\forall x, y \in A) (x \leq y \Leftrightarrow x = y = 0) \).

The following theorem is an important result of pseudo-KU algebras for study in the connections between pseudo-UP algebras and pseudo-KU algebras.

**Theorem 1.** Any pseudo-KU algebra is a pseudo-UP algebra.

**Proof.** It only needs to show (pUP-1). By Proposition 2, we have that any pseudo-KU algebra satisfies (pUP-1). \( \square \)

Pseudo-BE algebra is defined by the follows:

**Definition 6.** ([3]) An algebra \( A = (A, \cdot, *, 1) \) of type \( (2, 2, 0) \) is called a pseudo BE-algebra if satisfies in the following axioms:

- (pBE-1) \( (\forall x \in A) (x \cdot x = 1 \land x \ast x = 1) \);
- (pBE-2) \( (\forall x \in A) (x \cdot 1 = 1 \land x \ast 1 = 1) \);
- (pBE-3) \( (\forall x \in A) (1 \cdot x = x \land 1 \ast x = x) \);
- (pBE-4) \( (\forall x, y, z \in A) (x \cdot (y \ast z) = y \ast (x \cdot z)) \); and
- (pBE-5) \( (\forall x, y \in A) (x \cdot y = 1 \iff x \ast y = 1) \).

If we replace 1 with 0 in (BE-1), (BE-2), (BE-3) and (BE-5) and prove that the formula (pBE-4) is a valid formula in a pseudo-KU algebra \( A \), we have proved that every pseudo-KU algebra \( A \) is a pseudo-BE algebra.

**Theorem 2.** Any pseudo-KU algebra is a pseudo-BE algebra.

**Proof.** It is sufficient to prove that the formula (pBE-4) is a valid formula in any pseudo-KU algebra. If we put \( y = 0 \) in the left-hand side of the formula (pKU-1), we get \( 0 \cdot x \leq ((x \cdot z) \ast (0 \cdot z). \) It means \( x \leq (x \cdot z) \ast z. \) From here follows

\[
((x \cdot z) \ast z) \cdot (y \ast z) \leq x \cdot (y * z).
\]
by the left part of formula (11). On the other hand, if we put \( x = x \cdot z \) in the right-hand side of the formula (pKU-1), we get
\[
y \ast (x \cdot z) \leq ((x \cdot z) \ast z) \cdot (y \ast z).
\]

Which together with the previous inequality gives
\[
y \ast (x \cdot z) \leq x \cdot (y \ast z).
\]

From this inequality by substituting the variables \( x \) and \( y \), we obtain the necessary reverse inequality
\[
x \cdot (y \ast z) \leq y \ast (x \cdot z).
\]

From these two inequalities follows the validity of the formula (pBE-4) in any pseudo-KU algebra by the axiom (pKU-4).

Since the formula previously proven is important below, we point it out in particular.

**Proposition 3.** In any pseudo-KU algebra \( A \),
\[
(pKU-7) \ (\forall x, y, z \in A)(x \cdot (y \ast z) = y \ast (x \cdot z))
\]
is a valid formula.

5. Some substructures in pseudo-KU algebras

5.1. Concept of pseudo-subalgebras

**Definition 7.** A nonempty subset \( S \) of a pseudo-KU algebra \( A \) is a pseudo-subalgebra in \( A \) if
\[
(\forall x, y \in A)((x \in S \wedge y \in S) \implies (x \cdot y \in S \wedge x \ast y \in S)).
\]

holds.

Putting \( y = x \) in the previous definition, it immediately follows:

**Lemma 6.** If \( S \) is a pseudo-subalgebra of a pseudo-KU algebra \( A \), then \( 0 \in S \).

**Proof.** Let \( S \) be a pseudo-subalgebra of a pseudo-KU algebra \( A \). It means that \( S \) is a nonempty subset of \( A \). Then there exists an element \( y \in S \). Thus \( 0 = y \cdot y = y \ast y \in S \) by Definition 7.

It is clear that subsets \( \{0\} \) and \( A \) are pseudo-subalgebras of a pseudo-KU algebras \( A \). So, the family \( S(A) \) of all pseudo-subalgebras of a pseudo-KU algebra \( A \) is not empty. Without major difficulties, the following theorem can be proved.

**Theorem 3.** The family \( S(A) \) of all pseudo-subalgebras of a pseudo-KU algebra \( A \) forms a complete lattice.

5.2. Concept of pseudo-ideals

**Definition 8.** The subset \( J \) is said to be a pseudo-ideal of a pseudo-KU algebra \( A \) if it satisfies the following conditions:
\[
(pJ1) \ 0 \in J,
(pJ3a) \ (\forall x, y \in A)((x \cdot y \in J \wedge x \in J) \implies y \in J) \text{ and }
(pJ3b) \ (\forall x, y \in A)((x \ast y \in J \wedge x \in J) \implies y \in J).
\]

**Proposition 4.** Let \( J \) be a nonempty subset of a pseudo-KU algebra \( A \). Then the condition \((pJ3a)\) is equivalent to the condition:
\[
(pJ4a) \ (\forall x, y, z \in A)((x \ast (y \cdot z) \in J \wedge y \in J) \implies x \ast z \in J).
\]

**Proof.** Putting \( x = y \) and \( y = x \ast z \) in the condition \((pJ3a)\), it immediately follows
\[
(\forall x, y, z \in A)((y \cdot (x \ast z) \in J \wedge y \in J) \implies x \ast z \in J).
\]
Thus

\[(\forall x,y,z \in A)((y \ast (x \cdot z) \in J \land y \in J) \implies x \ast z \in J)\]

by (pKU-7).

Conversely, let (pJ4a) be. Let us choose \(x = 0, y = x\) and \(z = y\) in (pJ4a). We get \((0 \ast (x \cdot y) \in J \land x \in J) \implies 0 \ast y \in J\). Thus (pJ3a) by (pKU-2). \(\square\)

**Corollary 4.** Let \(J\) be a pseudo-ideal in a pseudo-KU-algebra \(A\). Then

\[(\forall x,y,z \in A)((x \cdot (y \ast z) \in J \land y \in J) \implies x \cdot z \in J).\]

**Proof.** Putting \(z = y\) in (pJ4a), with respect to (pKU-6), (pKU-3) and (pJ1), we obtain (13). \(\square\)

**Proposition 5.** Let \(J\) be a nonempty subset of a pseudo-KLI algebra \(A\). Then the condition (pJ3b) is equivalent to the condition

\[(\forall x,y,z \in A)((x \cdot (y \ast z) \in J \land y \in J) \implies x \cdot z).\]

**Proof.** If we put \(x = y\) and \(y = x \cdot z\) in (pJ3b), we get

\[(y \ast (x \cdot z) \in J \land y \in J) \implies x \cdot z \in J.\]

Hence

\[(x \cdot (y \ast z) \in J \land y \in J) \implies x \cdot z \in J.\]

by (pKU-7).

Conversely, if we put \(x = 0, y = x\), and \(z = y\) in (pJ4b), we get

\[(0 \cdot (x \ast y) \in J \land x \in J) \implies 0 \cdot y \in J.\]

Thus (pJ3b) with respect to (pKU-2). \(\square\)

**Corollary 5.** Let \(J\) be a pseudo-ideal in a pseudo-KLI-algebra \(A\). Then

\[(\forall x,y,z \in A)((x \cdot (y \ast z) \in J \land z \in J) \implies x \cdot y \in J).\]

**Proof.** Putting \(z = y\) in (pJ4b), with respect to (pKU-6), (pKU-3) and (pJ1), we obtain (14). \(\square\)

The following important statement describes the connection between conditions (pJ3a) and (pJ3b).

**Proposition 6.** Let \(J\) be a pseudo-ideal of a pseudo-KLI algebra \(A\). Then

\[(pJ3a) \iff (pJ3b).\]

**Proof.** \((pJ3a) \iff (pJ3b)\). Suppose (pJ3a) holds and let \(x \ast y \in J\) and \(x \in J\). How obvious it is that the following

\[x \ast ((x \cdot y) \ast y) = 0 \iff x \cdot ((x \cdot y) \ast x) = 0 \iff (x \cdot y) \ast (x \cdot y) = 0\]

is valid, we have

\[(x \in J \land x \cdot ((x \ast u) \cdot y) = 0 \in J) \implies (x \ast y) \cdot (x \cdot y) = 0\]

Now

\[(x \ast y \in J \land (x \ast y) \cdot y \in J) \implies y \in J.\]

We have proved that (pJ3b) is a valid implication.

\((pJ3b) \implies (pJ3a)\). Let (pJ3b) be a valid formula and let \(x,y \in A\) be such that \(x \in J\) and \(x \cdot y \in J\). As above, from

\[x \ast ((x \cdot y) \ast y) = 0 \iff x \cdot ((x \cdot y) \ast y) = 0 \iff (x \cdot y) \ast (x \cdot y) = 0\]

it follows

\[(x \in J \land x \ast ((x \cdot y) \ast y) = 0 \in J) \implies (x \cdot y) \ast y \in J.\]

Now, \(x \cdot y \in J\) and \((x \cdot y) \ast y\) it follows \(y \in J\). This proves the validity of the formula (pJ3a). \(\square\)
Proposition 7. Any pseudo-ideal in a pseudo-KU-algebra $\mathfrak{A}$ is a pseudo-subalgebra in $\mathfrak{A}$.

Proof. The proof of this proposition follows from (13) and (14). □

Theorem 6. The family $\mathcal{I}(A)$ of all pseudo-ideals in a pseudo-KU algebra $\mathfrak{A}$ forms a complete lattice and $\mathcal{I}(A) \subseteq \mathcal{S}(A)$ holds.

Proof. Let $\{I_i\}_{i \in I}$ be a family of pseudo-ideals in a pseudo-KU algebra $\mathfrak{A}$. Clearly $0 \in \bigcap_{i \in I} I_i$ is valid. Let $x, y \in A$ be elements such that $x \cdot y \in \bigcap_{i \in I} I_i$, $x \cdot y \in \bigcap_{i \in I} I_i$, and $x \in \bigcap_{i \in I} I_i$. Then $x \cdot y \in I_i$, $x \cdot y \in I_i$, and $x \in F_i$ for any $i \in I$. Thus $y \in I_i$ because $I_i$ is a pseudo-ideal in $\mathfrak{A}$ and $x \in \bigcap_{i \in I} I_i$. So, $\bigcap_{i \in I} I_i$ is a pseudo-ideal in $\mathfrak{A}$.

If $\mathcal{X}$ is the family of all pseudo-ideals of $\mathfrak{A}$ that contain the union $\bigcup_{i \in I} I_i$, then $\bigcap \mathcal{X}$ is also a pseudo-ideal in $\mathfrak{A}$ that contains $\bigcup_{i \in I} I_i$ by previous evidence.

If we put $\bigcap_{i \in I} I_i = \bigcap_{i \in I} I_i$ and $\bigcup_{i \in I} I_i = \bigcap \mathcal{X}$, then $(\mathcal{I}(A), \bigcap, \bigcup)$ is a complete lattice. □

To round out this subsection we need the following lemma.

Lemma 7. Let $J$ be a pseudo-ideal in a pseudo-KU algebra $\mathfrak{A}$. Then

$$(15) \ (\forall x, y \in A)((x \leq y \land x \in J) \implies y \in J).$$

Proof. The proof of this proposition follows from (pJ3a) or (pJ3b) with respect to (pKU-6) and (p1). □

Theorem 7. Let $J$ be a subset of a pseudo-KLU algebra $\mathfrak{A}$ such that $0 \notin J$. Then, $J$ is a pseudo-ideal in $\mathfrak{A}$ if and only if the following holds

$$(pJ5) \ (\forall x, y, z \in A)((x \in A \land y \in A \land x \leq y \cdot z) \implies z \in J).$$

Proof. Let $J$ be a pseudo-ideal in $\mathfrak{A}$ and let $x, y, z \in A$ such that $x \in J$ and $x \cdot y \in J$. Then $x \cdot (y \cdot z) = 0 \in J$.

Thus $y \cdot z \in J$ by (pJ3a) and again, from here and $y \in J$ it follows $z \in J$. So, we have shown that (pJ5) is a valid formula.

Opposite, suppose that (pJ5) is a valid in $\mathfrak{A}$. Let us show that $J$ is a pseudo-ideal and $\mathfrak{A}$. Let $x, y \in A$ be such that $x \in J$ and $x \cdot y \in J$. Then $x \cdot (y \cdot z) = 0 \in J$.

From inequality (pKU-1) in the form $x \cdot y \leq (y \cdot z) \cdot (x \cdot z)$ and $x \cdot y \in J$ it follows $(y \cdot z) \cdot (x \cdot z) \in J$ according to (15). From here and from $y \cdot z \in J$ it follows $x \cdot z \in J$ according to (pJ3a). Hence, the relation $' \leq \cdot \cdot \cdot'$ is transitive. So, this relation is a quasi-order in $\mathfrak{A}$.

Let $x, y, z \in A$ be such $x \leq y$. Then $x \cdot y \in J$ and $x \cdot y \in J$.

(i) If we put $x = y$ and $y = x$ in the left part of the formula (pKU-1), we get $x \cdot y \leq (y \cdot z) \cdot (x \cdot z)$. Now, from here and from $x \cdot y \in J$ it follows $(y \cdot z) \cdot (x \cdot z) \in J$ by (15). Thus $(y \cdot z) \cdot (x \cdot z) \in J$ by Proposition 6. Finally, we have $y \cdot z \leq x \cdot z$. So, the relation $' \leq \cdot \cdot \cdot$ is reverse right compatible with the internal operation $' \cdot \cdot \cdot$ in $\mathfrak{A}$.

(ii) If we put $x = y$ and $y = x$ in the right part of the formula (pKU-1), we get $x \cdot y \leq (y \cdot z) \cdot (x \cdot z)$. Then $(y \cdot z) \cdot (x \cdot z) \in J$ by (15). Thus $y \cdot z \leq x \cdot z$. Therefore, the relation $' \leq \cdot \cdot \cdot$ is reverse right compatible with the internal operation $' \cdot \cdot \cdot$ in $\mathfrak{A}$.

(iii) Let us put $y = z$ and $z = y$ in the left part of the formula (pKU-1). We get $(z \cdot x) \cdot ((x \cdot y) \cdot (z \cdot y)) = 0 \in J$. From here and from $x \cdot y \in J$ it follows $(z \cdot x) \cdot (z \cdot y) \in J$ by (pJ4a). Thus $z \cdot x \leq z \cdot y$. So, the relation $' \leq \cdot \cdot \cdot$ is left compatible with the operation $' \cdot \cdot \cdot$.
Remark 2. Note that if in $A \ast B$, it is obvious that $0 \ast y \in A$ if and only if $y \ast 0 \in B$. From here and from $x \ast y \in A$ it follows $(z \ast x) \cdot (z \ast y) \in J$ by (pKU-1). Thus $z \ast x \preceq z \ast y$. So, the relation $' \preceq ' $ is left compatible with the operation $' \ast ' $.

5.3. Concept of pseudo-filters

Definition 9. A non-empty subset $F$ of a pseudo-KU algebra $\mathfrak{A}$ is called a pseudo-filter of $A$ if it satisfies in the following axioms:

(pF1) $0 \in F$;

(pF3) $(\forall x, y \in A)((x \cdot y \in F \land x \ast y \in F \land y \in F) \implies x \in F)$.

Lemma 8. Let $F$ be a pseudo-filter in a pseudo-KU algebra $\mathfrak{A}$. Then

\[
(16) (\forall x, y \in A)((x \ast y \land y \in F) \implies x \in F).
\]

Theorem 9. The family $\mathfrak{F}(A)$ of all pseudo-ideals in a pseudo-KU algebra $\mathfrak{A}$ forms a complete lattice.

Proof. Let $\{F_i\}_{i \in I}$ be a family of pseudo-filters in a pseudo-KU algebra $\mathfrak{A}$. Clearly $0 \in \bigcap_{i \in I} F_i$ is valid. Let $x, y \in A$ be elements such that $x \cdot y \in \bigcap_{i \in I} F_i$, $x \ast y \in \bigcap_{i \in I} F_i$ and $y \in \bigcap_{i \in I} F_i$. Then $x \cdot y \in F_i$, $x \ast y \in F_i$ and $y \in F_i$ for any $i \in I$. Thus $x \in F_i$ because $F_i$ is a pseudo-filter in $\mathfrak{A}$ and $x \in \bigcap_{i \in I} F_i$. So, $\bigcap_{i \in I} F_i$ is a pseudo-filter in $\mathfrak{A}$.

If $\mathfrak{X}$ is the family of all pseudo-filters of $\mathfrak{A}$ that contain the union $\bigcup_{i \in I} F_i$, then $\cap \mathfrak{X}$ is also a pseudo-filter in $\mathfrak{A}$ that contains $\bigcup_{i \in I} F_i$ by previous evidence.

If we put $\bigcap_{i \in I} F_i = \bigcap_{i \in I} F_i$ and $\bigcup_{i \in I} F_i = \cap \mathfrak{X}$, then $(\mathfrak{F}(A), \cap, \cup)$ is a complete lattice.

6. Concept of pseudo-homomorphisms

Definition 10. $(A, \preceq A, \ast_A, 0_A)$ and $(B, \preceq B, \ast_B, 0_B)$ be pseudo-KU algebras. A mapping $f : A \rightarrow B$ of pseudo-KU algebras is called a pseudo-homomorphism if

\[
(\forall x, y \in A)((f(x) \cdot B f(y)) \land f(x \ast B f(y)) = \ast_B f(x) \cdot f(y)).
\]

Remark 2. Note that if $f : A \rightarrow B$ is a pseudo homomorphism, then $f(0_A) = 0_B$. Indeed, if we chose $y = x$, from the previous formula we immediately get $f(0_A) = \ast_B 0_B$ with respect (pKU-6).

From here it immediately follows:

Lemma 9. Any pseudo-homomorphism between pseudo-KU algebras is isotone mapping.

Proof. Let $f : A \rightarrow B$ be a pseudo-homomorphism between pseudo-KU algebras and let $x, y \in A$ be such $x \preceq A y$. Then $x \cdot A y = \preceq A 0_A$. Thus $0_B = \ast_B f(x) \ast_B f(y)$ and $0_B = \ast_B f(x) \cdot B f(y)$. This means $f(x) \preceq B f(y)$.

Lemma 10. Let $f : A \rightarrow B$ be a pseudo-homomorphism between pseudo-KU algebras. Then the set $\text{Ker}(f) = \{x \in A : f(x) = B 0_B\}$ is a pseudo-ideal in $\mathfrak{A}$.

Proof. It is obvious $0_A \in \text{Ker}(f)$.

Let $x, y \in A$ be such $x \cdot A y \in \text{Ker}(f)$ and $x \in \text{Ker}(f)$. Then $f(x) = B 0_B$ and $0 = \ast_B f(x) \ast_B f(y) = \ast_B f(x) \cdot B f(y)$. Thus $y \in \text{Ker}(f)$.

The implication of $x \ast A y \in \text{Ker}(f) \land x \in \text{Ker}(f) \implies y \in \text{Ker}(f)$ can be proved by analogy with the previous proof.

The following statement is easy to prove:

Lemma 11. If $f : A \rightarrow B$ is a pseudo-homomorphism between pseudo-KU algebras, then $f(A)$ is a pseudo-subalgebra in $B$. 

Proposition 8. Let \( f : A \rightarrow B \) be a pseudo homomorphism between pseudo-KU algebras \( \mathfrak{A} \) and \( \mathfrak{B} \).
(i) If \( K \) is a pseudo-ideal in \( \mathfrak{B} \), then \( f^{-1}(K) \) is a pseudo-ideal in \( \mathfrak{A} \).
(ii) If \( G \) is a pseudo-filter in \( \mathfrak{B} \), then \( f^{-1}(G) \) is a pseudo-filter in \( \mathfrak{A} \).

Proof. (i) Assume that \( K \) is a pseudo-filter of \( \mathfrak{B} \). Obviously \( 0_{\mathfrak{A}} \in f^{-1}(K) \). Let \( x, y \in A \) be such \( x \cdot y \in f^{-1}(K) \) and \( x \in f^{-1}(K) \). Then \( f(x) \cdot_B f(y) =_B f(x \cdot_A y) \in K \) and \( f(x) \in K \). It follows that \( f(y) \in K \) by (pJ3a) since \( K \) is a pseudo-ideal in \( \mathfrak{B} \). Therefore, \( y \in f^{-1}(K) \). Thus, the set \( f^{-1}(K) \) satisfies the implication (pJ3a). That the set \( f^{-1}(K) \) satisfies the implication (pJ3b) can be proved in an analogous way. Therefore, the set \( f^{-1}(K) \) is a pseudo-ideal in \( \mathfrak{A} \).

(ii) It is obvious \( 0_{\mathfrak{A}} \in f^{-1}(G) \) again. Let \( x, y \in A \) be elements such that \( x \cdot_A y \in f^{-1}(G) \), \( x \cdot_A y \in f^{-1}(G) \) and \( y \in f^{-1}(G) \). Then \( f(x) \cdot_B f(y) =_B f(x \cdot_A y) \in G \), \( f(x) \cdot_B f(y) =_B f(x \cdot_A y) \in G \) and \( f(y) \in G \). Thus \( f(x) \in G \) because \( G \) is a pseudo-filter in \( \mathfrak{B} \). This means \( x \in f^{-1}(G) \). So, the set \( f^{-1}(G) \) is a pseudo-filter in \( \mathfrak{A} \). \( \square \)

In the following definition, we will introduce the concept of congruence on pseudo-KU algebras. Since we have two unitary operations on this algebra, it is possible to determine three different types of congruences.

Definition 11. Let \( \mathfrak{A} = ((A, \leq), \cdot, +, 0) \) be a pseudo-KU algebra.
For the equivalence relation \( q \) on the set \( A \) we say that it is a \textit{congruence of type }\( \cdot' \cdot' \) on \( \mathfrak{A} \) if it is compatible with the operations \( \cdot' \cdot' \) in \( \mathfrak{A} \) in the following sense
\[
(\forall x, y, z \in A)((x, y) \in q \Rightarrow (x \cdot z, y \cdot z) \in q \land (z \cdot x, z \cdot y) \in q))
\]
For the equivalence relation \( q \) on the set \( A \) we say that it is a \textit{congruence of type }\( \cdot' \cdot' \) on \( \mathfrak{A} \) if it is compatible with the operations \( \cdot' \cdot' \) in \( \mathfrak{A} \) in the following sense
\[
(\forall x, y, z \in A)((x, y) \in q \Rightarrow (x \cdot z, y \cdot z) \in q \land (z \cdot x, z \cdot y) \in q))
\]
For the equivalence relation \( q \) on the set \( A \) we say that it is a \textit{congruence of common type }on \( \mathfrak{A} \) if it is compatible with both operations in \( \mathfrak{A} \).

Lemma 12. Let \( q \) be a relation on a pseudo-KU algebra \( \mathfrak{A} \). Then:
(i) The condition (17) is equivalent to the condition
\[
(\forall x, y, u, v \in A)((x, y) \in q \land (u, v) \in q) \Rightarrow (x \cdot u, y \cdot v) \in q).
\]
(ii) The condition (18) is equivalent to the condition
\[
(\forall x, y, u, v \in A)((x, y) \in q \land (u, v) \in q) \Rightarrow (x \cdot u, y \cdot v) \in q).
\]

Proof. (17a) \( \Rightarrow \) (17). If we choose \( v = z \) in (17a), we get the implication \( (x, y) \in q \Rightarrow (x \cdot z, y \cdot z) \in q \). On the other hand, if we put \( u = y = z, u = x \) and \( v = y \) in (17a), we get the implication \( (x, y) \in q \Rightarrow (z \cdot x, z \cdot y) \). (17) \( \Rightarrow \) (17a). Suppose (17) and let \( x, y, u, v \in A \) such that \( (x, y) \in q \) and \( (u, v) \in q \). Thus \( (x \cdot u, x \cdot v) \in q \) and \( (x \cdot v, y \cdot v) \in q \) by (16). Hence \( (x \cdot u, y \cdot v) \in q \) by transitivity of \( q \).
Equivalence (18) \( \iff \) (18a) can be proved analogous to the previous proof. \( \square \)

Let \( f : A \rightarrow B \) be a pseudo homomorphism between pseudo-KU algebras. By direct check without difficulty, it can be proved that the relation \( q_{f} \), defined by
\[
(\forall x, y \in A)((z, y) \in q_{f} \iff f(x) =_{B} f(y)),
\]
is a congruence (all three types) on \( \mathfrak{A} \).

Theorem 10. The relation \( q_{f} \) is a congruence of type \( \cdot' \cdot' \) (type \( \cdot' \cdot' \), common type) on the pseudo-KU algebra \( \mathfrak{A} \).

Proof. We will only demonstrate the proof that \( q_{f} \) is a congruence of type \( \cdot' \cdot' \) on \( \mathfrak{A} \) because the evidence that \( q_{f} \) is a congruence of type \( \cdot' \cdot' \) can obtain by analogy with the previous one, and the proof of common type is obtained by combining this two evidences.

Clearly, \( q_{f} \) is an equivalence relation on the set \( A \). It remains to verify that (16) is a valid formula in \( \mathfrak{A} \). Let \( x, y, u, v \in A \) be such that \( (x, y) \in q_{f} \) and \( (u, v) \in q_{f} \). Then \( f(x) =_{B} f(y) \) and \( f(u) =_{B} f(v) \). Thus
\[
f(x \cdot_A u) =_{B} f(x) \cdot_B f(u) =_{B} f(y) \cdot_B f(v) =_{B} f(y \cdot_A u).
\]
Hence, \((x \cdot_A u, y \cdot_A v) \in q_f\). We proved that (17a) is a valid formula. So \(q_f\) is a congruence of type \(\cdot, \cdot\) on \(\mathfrak{A}\). \(\square\)

**Theorem 11.** Let \(J\) be a pseudo-ideal in a pseudo-KU algebra \(\mathfrak{A}\). Then the relation \(q_J\), defined by \(q_J = \triangleq \cap \triangleq^{-1}\), is a congruence of common type in \(\mathfrak{A}\).

**Proof.** The relation \(q\) is an equivalence relation on the set \(A\). It is sufficient to prove that \(q\) is compatible with operations in \(\mathfrak{A}\). Since the relation \(\triangleq\) is left compatible and right reverse compatible with the internal operations in \(\mathfrak{A}\), by Theorem 8, it is clear that the relation \(q_J\) is a congruence on \(\mathfrak{A}\). \(\square\)

For a congruence \(q\) on a pseudo-KU algebra \(\mathfrak{A}\) we denote \(q(x) = \{y \in A : (x, y) \in q\} = [x]\). Let’s define \(\cdot', \cdot^\ast\) and \(\cdot^\ast'\) in \(A/q\) on this way

\[
(\forall x, y \in A) ([x] \cdot [y] = [x \cdot y]) \quad \text{and} \quad (\forall x, y \in A) ([x] \cdot^\ast [y] = [x \cdot^\ast y]).
\]

Without much difficulty it can be verified that the functions \(\cdot, \cdot', \cdot^\ast, \cdot^\ast'\), defined in this way, are well-defined internal binary operations in \(A/q\). Also, one can check that the set \(A/q\) with the operations \(\cdot', \cdot^\ast, \cdot^\ast'\), determined as above, satisfies all the axioms of Definition 4 except the axiom (pKU-4). However, if we take the relation \(q_f\), defined by an pseudo-ideal \(J\) of a pseudo-KU algebra \(\mathfrak{A}\), then we have

**Theorem 12.** Let \(J\) be a pseudo-ideal in a pseudo-KU algebra \(\mathfrak{A}\). Then the structure \(((A/q, \triangleq, \cdot', \cdot^\ast, [0])\), where \(\triangleq\) is defined by

\[
(\forall x, y \in A) ([x] \triangleq [y] \iff x \triangleq y),
\]

is a pseudo-KU algebra, too.

**Proof.** According to the commentary preceding this theorem, to prove this theorem it suffices to show that the structure \(((A/q, \triangleq), \cdot', \cdot^\ast, [0])\) satisfies the axiom (pKU-4).

Let \(x, y \in A\) be such \([x] \triangleq [y]\) and \([y] \triangleq [x]\). Then \(x \triangleq y\) and \(y \triangleq x\) by definition. Thus \((x, y) \in q_f\) and \([x] = [y]\). \(\square\)

Let \(f : A \rightarrow B\) be pseudo-homomorphism between pseudo-KU algebras \(((A, \sqsubseteq_A), \cdot_A, \cdot^\ast_A, 0_A)\) and \(((B, \sqsubseteq_B), \cdot_B, \cdot^\ast_B, 0_B)\). Then the set \(f(A)\) is a pseudo-subalgebra of \(B\) by and the set \(f = \text{Ker}(f)\) is a pseudo-ideal in \(\mathfrak{A}\) by Lemma 10 and the relation \(q_f\) is a congruence on \(\mathfrak{A}\) by Theorem 10. If \((x, y) \in q_f\) holds soe some \(x, y \in A\), we have \(f(x) = B f(y)\). Thus \(f(x \cdot_A y) = B f(x) \cdot^\ast_B f(y) = B f(x) \cdot^\ast_B f(x) = B 0_B\), i.e. \(x \cdot_A y \in J\). Analogous to the previous one may be shown that \(y \cdot_A x \in f\) holds. Thus, \((x, y) \in q_f \implies (x, y) \in q_f\) is valid.

We end this section with the following theorem. Since this theorem can be proven by direct verification, we will omit evidence for it.

**Theorem 13.** Let \(f : A \rightarrow B\) be pseudo-homomorphism between pseudo-KU algebras \(((A, \sqsubseteq_A), \cdot_A, \cdot^\ast_A, 0_A)\) and \(((B, \sqsubseteq_B), \cdot_B, \cdot^\ast_B, 0_B)\). Then there exists the unique epimorphism \(\pi : A \rightarrow A/q_f\), defined by \(\pi(x) = [x]\) for any \(x \in A\), and the unique monomorphism \(g : A/q_f \rightarrow B\), defined by \(g([x]) = B f(x)\) for any \(x \in A\) such that \(f = g \circ \pi\).

**Acknowledgments:** The author thanks the reviewers for helpful suggestions. The author also thanks the editors of the journal for their assistance in the technical preparation of the manuscript.

**Conflicts of Interest:** The author declares no conflict of interest.

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