A Note on Pointwise Dimensions

Neil Lutz*

Department of Computer Science, Rutgers University
Piscataway, NJ 08854, USA
njlutz@rutgers.edu

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Abstract

This short note describes a connection between algorithmic dimensions of individual points and classical pointwise dimensions of measures.

1 Introduction

Effective dimensions are pointwise notions of dimension that quantify the density of algorithmic information in individual points in continuous domains. This note aims to clarify their relationship to the classical notion of pointwise dimensions of measures, which is central to the study of fractals and dynamics. This connection is then used to compare algorithmic and classical characterizations of Hausdorff and packing dimensions. See [15, 17] for surveys of the strong ties between algorithmic information and fractal dimensions.

2 Pointwise Notions of Dimension

We begin by defining the two most well-studied formulations of algorithmic dimension [5], the effective Hausdorff and packing dimensions. Given a point $x \in \mathbb{R}^n$, a precision parameter $r \in \mathbb{N}$, and an oracle set $A \subseteq \mathbb{N}$, let

$$K^A_r(x) = \min\{K^A(q) : q \in \mathbb{Q}^n \cap B_{2^{-r}}(x)\},$$

where $K^A(q)$ is the prefix-free Kolmogorov complexity of $q$ relative to the oracle $A$, as defined in [10], and $B_{2^{-r}}(x)$ is the closed ball of radius $2^{-r}$ around $x$.

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**Definition.** The effective Hausdorff dimension and effective packing dimension of \(x \in \mathbb{R}^n\) relative to \(A\) are

\[
\dim^A(x) = \liminf_{r \to \infty} \frac{K^A_r(x)}{r},
\]

\[
\text{Dim}^A(x) = \limsup_{r \to \infty} \frac{K^A_r(x)}{r},
\]

respectively [11, 14, 1].

Intuitively, these quantities are the (lower and upper asymptotic) density of algorithmic information in \(x\). In the classical setting, pointwise dimensions are defined for a given measure according to its (lower and upper asymptotic) rate of decay around \(x\).

**Definition.** For any locally finite measure \(\mu\) on \(\mathbb{R}^n\), the lower and upper pointwise dimension of \(\mu\) at \(x \in \mathbb{R}^n\) are

\[
\dim_\mu(x) = \liminf_{\rho \to 0} \frac{\log \mu(B_\rho(x))}{\log \rho},
\]

\[
\text{Dim}_\mu(x) = \limsup_{\rho \to 0} \frac{\log \mu(B_\rho(x))}{\log \rho},
\]

respectively [6].

As Young [19] notes, these limits are unchanged if \(\rho\) is replaced by any sequence \(\{\rho_r\}_{r \in \mathbb{N}}\) satisfying \(\rho_r \downarrow 0\) and \(\log \rho_{r+1}/\log \rho_r \to 1\). In particular, the sequence \(\{2^{-r}\}_{r \in \mathbb{N}}\) may be used. Also, this definition in no way relies on additivity, so it applies equally well to outer measures and semimeasures.

We relate these two notions of pointwise dimension by defining an outer measure \(\kappa^A\) on \(\mathbb{R}^n\) for any given oracle set \(A \subseteq \mathbb{N}\). For every \(E \subseteq \mathbb{R}^n\),

\[
\kappa^A(E) = 2^{-K^A(E)},
\]

where, following Shen and Vereschagin [18],

\[
K^A(E) = \min_{q \in E \cap \mathbb{Q}^n} K^A(q).
\]

This minimum is taken to be infinite when \(F \cap \mathbb{Q}^n = \emptyset\). It is easy to see that \(\kappa\) is also subadditive and monotonic, and that \(\kappa^A(\emptyset) = 0\). Since \(K^A\) is non-negative, \(\kappa^A\) is finite.

**Observation 1.** For every oracle set \(A \subseteq \mathbb{N}\) and all \(x \in \mathbb{R}^n\),

\[
\dim_{\kappa^A}(x) = \dim^A(x).
\]

\[
\text{Dim}_{\kappa^A}(x) = \text{Dim}^A(x).
\]
This fact is closely related to (and was observed independently of) an unpublished remark by Reimann stating that \( \dim(x) \) is equal to the pointwise dimension at \( x \) of Levin’s universal lower semicomputable continuous semimeasure.

Pointwise dimensions of measures give rise to global dimensions of measures, which we now briefly comment on. In classical fractal geometry, the global dimensions of Borel measures play a substantial role in studying the interplay between local and global properties of fractal sets and measures.

**Definition.** For any locally finite Borel measure \( \mu \) on \( \mathbb{R}^n \) and \( x \in \mathbb{R}^n \), the lower and upper Hausdorff and packing dimension of \( \mu \) are

\[
\dim_H(\mu) = \sup \{ \alpha : \mu(\{x : \dim(\mu(x) < \alpha)\}) = 0 \}
\]

\[
\operatorname{Dim}_H(\mu) = \inf \{ \alpha : \mu(\{x : \dim(\mu(x) > \alpha)\}) = 0 \}
\]

\[
\dim_P(\mu) = \sup \{ \alpha : \mu(\{x : \operatorname{Dim}(\mu(x) < \alpha)\}) = 0 \}
\]

\[
\operatorname{Dim}_P(\mu) = \inf \{ \alpha : \mu(\{x : \operatorname{Dim}(\mu(x) > \alpha)\}) = 0 \},
\]

respectively.

Extending these definitions to outer measures, we may consider global dimensions of the outer measures \( \kappa^A \). For every \( A \subseteq \mathbb{N} \), \( \kappa^A \) is supported on \( Q^n \) and \( \dim^A(p) = 0 \) for all \( p \in Q^n \), which implies the following.

**Observation 2.** For every \( A \subseteq \mathbb{N} \),

\[
\dim_H(\kappa^A) = \operatorname{Dim}_H(\kappa^A) = \dim_P(\kappa^A) = \operatorname{Dim}_P(\kappa^A) = 0.
\]

### 3 Pointwise Principles for Dimensions of Sets

The point-to-set principle of J. Lutz and N. Lutz expresses classical Hausdorff and packing dimensions in terms of relativized effective Hausdorff and packing dimensions.

**Theorem 3** (J. Lutz and N. Lutz [12]). For every nonempty \( E \subseteq \mathbb{R}^n \),

(a) \( \dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x) \).

(b) \( \dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \operatorname{Dim}^A(x) \).

In light of Observation 1, this principle may be considered a member of the family of results, such as Billingsley’s lemma [2] and Frostman’s lemma [7], that relate the local decay of measures to global properties of measure and dimension. Useful references on such results include [4, 8, 13].

Among classical results, this principle is most directly comparable to the weak duality principle of Cutler [4] (see also [6]), which expresses Hausdorff and packing dimensions in terms of lower and upper pointwise dimensions of measures. For nonempty \( E \subseteq \mathbb{R}^n \), let \( \mathcal{P}(E) \) be the collection of Borel probability measures on \( \mathbb{R}^n \) such that the \( E \) is measurable and has measure 1, and let \( \overline{E} \) be the closure of \( E \).
Theorem 4 (Cutler [4]). For every nonempty $E \subseteq \mathbb{R}^n$,

(a) $\dim_H(E) = \inf_{\mu \in \mathcal{P}(E)} \sup_{x \in E} \dim_\mu(x)$.

(b) $\dim_P(E) = \inf_{\mu \in \mathcal{P}(E)} \sup_{x \in E} \dim_\mu(x)$.

By letting $\mathcal{A} = \{\kappa^A : A \subseteq \mathbb{N}\}$, Theorem 3 can be restated even more similarly as $\dim_H(E) = \inf_{\mu \in \mathcal{A}} \sup_{x \in E} \dim_\mu(x)$ and $\dim_P(E) = \inf_{\mu \in \mathcal{A}} \sup_{x \in E} \dim_\mu(x)$. Notice, however, that the collections over which the infima are taken in these two results, $\mathcal{A}$ and $\mathcal{P}(E)$, are disjoint and qualitatively very different. In particular, $\mathcal{A}$ does not depend on $E$. Whereas the global dimensions of the measures in $\mathcal{P}(E)$ are closely tied to the dimensions of $E$ [6], Observation 2 shows that the outer measures in $\mathcal{A}$ all have trivial global dimensions.

References

[1] Krishna B. Athreya, John M. Hitchcock, Jack H. Lutz, and Elvira Mayordomo. Effective strong dimension in algorithmic information and computational complexity. *SIAM J. Comput.*, 37(3):671–705, 2007.

[2] P. Billingsley. Hausdorff dimension in probability theory II. *Illinois Journal of Mathematics*, 5(2):291–298, 1961.

[3] Christopher J. Bishop and Yuval Peres. *Fractals in Probability and Analysis*. Cambridge University Press, 2017.

[4] Colleen D. Cutler. Strong and weak duality principles for fractal dimension in Euclidean space. *Math. Proc. Camb. Phil. Soc.*, 118:393–410, 1995.

[5] Rod Downey and Denis Hirschfeldt. *Algorithmic Randomness and Complexity*. Springer-Verlag, 2010.

[6] K. J. Falconer. *Techniques in Fractal Geometry*. Wiley, 1997.

[7] O. Frostman. Potential d’équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. *Meddel. Lunds Univ. Math. Sen.*, 3:1–118, 1935.

[8] Michael Hochman. Lectures on dynamics, fractal geometry, and metric number theory. *Journal of Modern Dynamics*, 8(3–4):437–497, 2014.

[9] Leonid A. Levin. Laws of information conservation (nongrowth) and aspects of the foundation of probability theory. *Problemy Peredachi Informatsii*, 10(3):30–35, 1974.

[10] Ming Li and Paul M.B. Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications*. Springer, third edition, 2008.
[11] Jack H. Lutz. The dimensions of individual strings and sequences. *Inf. Comput.*, 187(1):49–79, 2003.

[12] Jack H. Lutz and Neil Lutz. Algorithmic information, plane kakeya sets, and conditional dimension. *Symposium on Theoretical Aspects of Computer Science*, 2017.

[13] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces: fractals and rectifiability*. Cambridge University Press, 1995.

[14] Elvira Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Inf. Process. Lett.*, 84(1):1–3, 2002.

[15] Elvira Mayordomo. Effective fractal dimension in algorithmic information theory. In S. Barry Cooper, Benedikt Löwe, and Andrea Sorbi, editors, *New Computational Paradigms: Changing Conceptions of What is Computable*, pages 259–285. Springer, 2008.

[16] Jan Reimann. Effective multifractal spectra. [http://www.personal.psu.edu/jsr25/Lectures/CFTM_2014_talk_Reimann.pdf](http://www.personal.psu.edu/jsr25/Lectures/CFTM_2014_talk_Reimann.pdf). Lecture slides.

[17] Jan Reimann. *Computability and fractal dimension*. PhD thesis, Heidelberg University, 2004.

[18] Alexander Shen and Nikolai K. Vereshchagin. Logical operations and Kolmogorov complexity. *Theoretical Computer Science*, 271(1–2):125–129, 2002.

[19] Lai-Sang Young. Dimension, entropy and Lyapunov exponents. *Ergodic Theory and Dynamical Systems*, 2:109–124, 1982.