Excitations and S-matrix for $su(3)$ spin chain combining $\{3\}$ and $\{3^*\}$ representations

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Abstract

The associated Hamiltonian for a $su(3)$ spin chain combining $\{3\}$ and $\{3^*\}$ representations is calculated. The ansatz equations for this chain are obtained and solved in the thermodynamic limit, and the ground state and excitations are described. Thus, relations between the number of roots and the number of holes in each level have been found. The excited states are characterized by means of these quantum numbers. Finally, the exact $S$ matrix for a state with two holes is found.

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1. Introduction

The Yang-Baxter equation (YBE) and the quantum inverse scattering method (QISM) have contributed to find and solve a lot of many body quantum systems. The best known system is the Heisemberg model, which was solved by Bethe. This model can be derived from the YBE using the $su(2)$ Lie algebra. Generalizations of this model have been obtained using other Lie algebras.

An interesting problem is to derive integrable models where the chain is formed by two kind of states. The original work, an alternating chain with $s = 1/2$ and $s = 1$, was presented in Ref. Later, several works, using several Lie algebras, have been studied. In this systems it is possible to solve the ansatz equations in the thermodynamic limit. This allows us to describe the system by means of the quantum numbers and so we would be able to one can find the ground state and the excited estates. Besides, the $S$ matrix for the scattering of excitations can be determined.

In this paper we use the $su(3)$ rational solutions of the YBE and we form a chain combining $\{3\}$ and $\{3^*\}$ representations. For the alternating chain we find the Hamiltonian, which contains a coupling of three neighboring site pieces. We solve the ansatz equations and we deduce the root and hole densities. The relations between these densities allow us to describe the ground and excited states. In the last section we calculate the exact $S$ matrix for the two-hole scattering.

2. The model and the Hamiltonian

In this section, we are going to construct an alternating chain that mixes the $\{3\}$ and $\{3^*\}$ representations of $su(3)$. We use the rational solutions of Yang-Baxter equation. If we take the $\{3\}$ representation as auxiliary space and $\{3\}$ as site space, we have the operator

$$L^{\{3\},\{3\}}(u) = (1 - iu) \sum_{j=1}^{3} e_{j,j} \otimes e_{j,j} - iu \sum_{j,k=1, j \neq k}^{3} e_{j,j} \otimes e_{k,k} + \sum_{j,k=1, j \neq k}^{3} e_{j,k} \otimes e_{k,j}. \quad (2.1)$$

For the $\{3\}$ as auxiliary and $\{3^*\}$ as site, the operator is

$$L^{\{3\},\{3^*\}}(u) = \left( \frac{1}{2} - iu \right) \sum_{j=1}^{3} e_{j,j} \otimes e_{j,j} - \left( \frac{3}{2} - iu \right) \sum_{j,k=1, j \neq k}^{3} e_{j,j} \otimes e_{k,k} - \sum_{j,k=1, j \neq k}^{3} e_{j,k} \otimes e_{j,k}. \quad (2.2)$$
with \((\epsilon_{l,m})_{i,j} = \delta_{l,i}\delta_{m,j}\).

We consider a chain with \(N\) sites (\(N\) even) in which the site spaces are alternating in the representations \(\{3\}\) and \(\{3^*\}\). The monodromy matrix, which describes the transportation along the chain, is defined by

\[
T_{a,b}(u, \alpha) = L_{a,a_1}^{\{3\},\{3\}}(u)L_{a_1,a_2}^{\{3\},\{3^*\}}(u + \alpha) \ldots L_{a_{N-2},a_{N-1}}^{\{3\},\{3\}}(u)L_{a_{N-1},b}^{\{3\},\{3^*\}}(u + \alpha),
\]  

(2.3)

where the indices are in the auxiliary space and \(\alpha\) is an arbitrary parameter.

Since \(L^{\{3\},\{3\}}(u)\) and \(L^{\{3\},\{3^*\}}(u)\) operators verify the YBE, then \(T(u, \alpha)\) also verifies it

\[
R(u - v) \cdot (T(u, \alpha) \otimes T(v, \alpha)) = (T(v, \alpha) \otimes T(u, \alpha)) \cdot R(u - v),
\]  

(2.4)

with

\[
R(u) = (1 - iu) \sum_{j=1}^{3} e_{j,j} \otimes e_{j,j} - iu \sum_{j,k=1}^{3, j\neq k} e_{j,k} \otimes e_{k,j} + \sum_{j,k=1}^{3, j\neq k} e_{j,j} \otimes e_{k,k}.
\]  

(2.5)

Following the standard procedure, we take the transfer matrix as the trace, on the auxiliary space, of the monodromy matrix

\[
F(u, \alpha) = \text{trace}[T(u, \alpha)].
\]  

(2.6)

Due to YBE, the transfer matrices commute for different values of the argument

\[
[F(u, \alpha), F(v, \alpha)] = 0.
\]  

(2.7)

The Hamiltonian of that system is defined by the first derivative of the transfer matrix,

\[
H(\alpha) = \frac{d}{du} \ln(F(u, \alpha)) \bigg|_{u=0}.
\]  

(2.8)

Collecting the diverse terms, the Hamiltonian becomes

\[
H(\alpha) = \frac{i}{\bar{\rho}(\alpha)} \sum_{j=1 \text{ even}}^{N-1} h_{j,j+1}^{[1]} + \frac{i}{c_1\rho(\alpha)} \sum_{j=1 \text{ even}}^{N-1} h_{j,j+1,j+2}^{[2]},
\]  

(2.9)

with

\[
(h_{j,j+1}^{[1]})_{a,b;j,\gamma} = [L_{a,c}^{\{3\},\{3^*\}}(\alpha)]_{\beta,\delta} [L_{\delta,\gamma}^{\{3^*\},\{3\}}(-\alpha)]_{c,b} \]  

(2.10a)

\[
(h_{j,j+1,j+2}^{[2]})_{a,b;\gamma;c,d} = [L_{a,c}^{\{3\},\{3^*\}}(\alpha)]_{\beta,\delta} [L_{\delta,\gamma}^{\{3\},\{3\}}(\alpha)]_{c,f} [L_{\delta,\gamma}^{\{3^*\},\{3\}}(-\alpha)]_{f,b} \]  

(2.10b)
and

\[ R(0) = c_1 I \] (2.11a)

\[ [L_{a,b}^{\{3\},\{3^*\}}(u)]_{\alpha,\beta}[L_{\beta,\gamma}^{\{3\},\{3\}}(-u)]_{b,c} = \bar{\rho}(u)\delta_{a,c}\delta_{\alpha,\gamma}. \] (2.11b)

Thus, we find

\[
H(\alpha) = \frac{2}{9 + 4\alpha^2} \left\{ \sum_{i=1}^{N-1} \sum_{a=1}^{8} \lambda_i^a \otimes \bar{\lambda}_{i+1}^a + \sum_{a=1}^{N} \sum_{i=2}^{8} \lambda_i^a \otimes \lambda_{i+1}^a \right. \\
+ \sum_{i=1}^{N-1} \sum_{a,b,c=1}^{8} \left( \frac{3}{2} d_{a,b,c} - \alpha f_{a,b,c} \right) \lambda_i^a \otimes \lambda_{i+1}^b \otimes \lambda_{i+2}^c \right. \\
+ \frac{5 + 4\alpha^2}{4} \sum_{i=1}^{N-1} \sum_{a=1}^{8} \lambda_i^a \otimes \bar{\lambda}_{i+1}^a \otimes \lambda_{i+2}^a \left. \right\} + \frac{41 + 12\alpha(\alpha + i)}{9(9 + 4\alpha^2)} I, \]

(2.12)

where we have used the Gell-Mann matrices \( \lambda \) and \( \bar{\lambda} \) for the \{3\} and \{3*\} representations respectively, being \( d_{a,b,c} \) and \( f_{a,b,c} \) the structure constants of \( SU(3) \). When \( \alpha = 0 \), we obtain the simplest case.

3. Diagonalization and ansatz equations

We have solved the chain that mixes the \{3\} and \{3*\} representation of \( su(3) \), using the method given in [9]. The eigenvalue of the transfer matrix is

\[
\Lambda(u) = [a(u)]^{N_3} [\bar{b}(u)]^{N_3^*} \prod_{j=1}^{r} g(\mu_j - u) + [b(u)]^{N_3} \prod_{i=1}^{r} g(u - \mu_i) \times \\
\left\{ \bar{b}(u)]^{N_3^*} \prod_{l=1}^{s} g(\lambda_l - u) + [\bar{a}(u)]^{N_3^*} \prod_{k=1}^{s} g(u - \lambda_k) \prod_{n=1}^{r} \frac{1}{g(u - \mu_n)} \right\}, \] (3.1)

and the coupled Bethe equations are

\[ [g(\mu_k)]^{N_3} = \prod_{j=1}^{r} \frac{g(\mu_k - \mu_j)}{g(\mu_j - \mu_k)} \prod_{i=1}^{s} g(\lambda_i - \mu_k) \] (3.2a)

\[ [\bar{g}(\lambda_l)]^{N_3^*} = \prod_{j=1}^{r} g(\lambda_l - \mu_j) \prod_{i=1}^{s} \frac{g(\lambda_i - \lambda_l)}{g(\lambda_l - \lambda_i)} \] (3.2b)

\[ k = 1, \ldots, r \quad ; \quad l = 1, \ldots, s, \]
with
\[
\begin{align*}
a(u) &= 1 - iu \\
b(u) &= -iu \\
\bar{a}(u) &= \frac{1}{2} - iu \\
\bar{b}(u) &= \frac{3}{2} - iu \\
g(u) &= \frac{a(u)}{b(u)} \\
g(u) &= \frac{\bar{a}(u)}{\bar{b}(u)}. 
\end{align*}
\]

It is convenient to set the parameterization
\[
\mu_j = v_j^{(1)} - \frac{i}{2} \\
\lambda_j = v_j^{(2)} - i.
\]

Using such parameterization, the Bethe equations can be written
\[
\left[ \frac{v_k^{(1)} - \frac{i}{2}}{v_k^{(1)} + i} \right]^{N_3} = -\prod_{j=1}^{r} \frac{v_k^{(1)} - v_j^{(1)} - i}{v_k^{(1)} - v_j^{(1)} + i} \prod_{l=1}^{s} \frac{v_l^{(2)} - v_k^{(1)} - \frac{i}{2}}{v_l^{(2)} - v_k^{(1)} + \frac{i}{2}}. 
\]

We define the function
\[
\phi(x) = \ln \frac{1 + ix}{1 - ix} \equiv 2i \arctan x,
\]

and taking logarithms in (3.5a, b) we obtain
\[
\begin{align*}
N_3 \phi(2v_k^{(1)}) - \sum_{j=1}^{r} \phi(v_k^{(1)} - v_j^{(1)}) + \sum_{l=1}^{s} \phi(2v_k^{(1)} - 2v_l^{(2)}) &= 2\pi I_k^{(1)}, \\
1 \leq k \leq r, \\
N_3 \phi(2v_k^{(2)}) + \sum_{j=1}^{r} \phi(2v_k^{(2)} - 2v_j^{(1)}) - \sum_{l=1}^{s} \phi(v_k^{(2)} - v_l^{(2)}, \gamma) &= 2\pi I_k^{(2)}, \quad 1 \leq k \leq s.
\end{align*}
\]

where $I_k^{(1)}$ and $I_k^{(2)}$ are half-integers.

In the thermodynamic limit $N \to \infty$, the roots tend to have continuous distributions. Unlike what happens in other cases, we cannot distinguish between the roots coming from
the different types of representations, this is noted by simple inspection of the equations of the ansatz. Due to that, we define two root densities, one by each level, 
\[ \rho_l(v_j^{(l)}) = \lim_{N_3 \to \infty} \frac{1}{N_3(v_{j+1}^{(l)} - v_j^{(l)})}, \quad l = 1, 2. \] (3.8)

Let it be
\[ Z_{N_3}(v) = \frac{1}{2\pi} \left[ \phi(2v) - \frac{1}{N_3} \sum_{j=1}^{r} \phi(v - v_j^{(1)}) + \frac{1}{N_3} \sum_{j=1}^{s} \phi(2v - 2v_j^{(2)}) \right], \] (3.9a)
\[ Z_{N_3^*}(v) = \frac{1}{2\pi} \left[ \phi(2v) - \frac{1}{N_3^*} \sum_{j=1}^{s} \phi(v - v_j^{(2)}) + \frac{1}{N_3^*} \sum_{j=1}^{r} \phi(2v - 2v_j^{(1)}) \right]. \] (3.9b)

In the thermodynamic limit, the derivative of these functions are
\[ \sigma_1(v) \equiv \frac{d}{dv} Z_{N_3}(v) \approx \frac{N}{N_3} \rho_1(v) + \frac{1}{N_3} \sum_{h=1}^{N_h^{(1)}} \delta(v - \theta_h^{1}), \] (3.10a)
\[ \sigma_2(v) \equiv \frac{d}{dv} Z_{N_3^*}(v) = \frac{N}{N_3^*} \rho_2(v) + \frac{1}{N_3^*} \sum_{h=1}^{N_h^{(2)}} \delta(v - \theta_h^{2}). \] (3.10b)

Using the approximation
\[ \lim_{N \to \infty} \frac{1}{N} \sum_{j} f(v_j^{(k)}) \approx \int d\lambda f(\lambda) \rho_k(\lambda), \] (3.11)
and doing the Fourier transform, we can solve the system of equations. Thus, we write
\[ \sigma_1(v) = \sigma_1^{(o)}(v) + \frac{1}{N_3} \sigma_1^{(h)}(v) \] (3.12a)
\[ \sigma_2(v) = \sigma_2^{(o)}(v) + \frac{1}{N_3^*} \sigma_2^{(h)}(v), \] (3.12b)

where \( \sigma_1^{(o)}(v) \) and \( \sigma_1^{(h)}(v) \) show the root contribution and hole contribution respectively for \( k \)-level. One finds
\[ \sigma_1^{(o)}(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{\sinh \alpha}{\sinh(\frac{3\alpha}{2})} + \frac{N_3^*}{N_3} \frac{\sinh(\frac{\alpha}{2})}{\sinh(\frac{3\alpha}{2})} \right) e^{iav} d\alpha \] (3.13a)
\[ \sigma_2^{(o)}(v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \frac{\sinh \alpha}{\sinh(\frac{3\alpha}{2})} + \frac{N_3}{N_3^*} \frac{\sinh(\frac{\alpha}{2})}{\sinh(\frac{3\alpha}{2})} \right) e^{iav} d\alpha \] (3.13b)
\[ \sigma_1^{(h)}(v) = \frac{1}{2\pi} \left\{ \sum_{h=1}^{N_h^{(1)}} r_a(v - \theta_h^{1}) - \sum_{h=1}^{N_h^{(2)}} r_b(v - \theta_h^{2}) \right\} \] (3.13c)
\[ \sigma_2^{(h)}(v) = \frac{1}{2\pi} \left\{ \sum_{h=1}^{N_h^{(2)}} r_a(v - \theta_h^{2}) - \sum_{h=1}^{N_h^{(1)}} r_b(v - \theta_h^{1}) \right\}, \] (3.13d)
with

\[ r_a(x) = \int_{-\infty}^{+\infty} \frac{\sinh\left(\frac{\alpha}{2}\right)}{\sinh\left(\frac{3\alpha}{2}\right)} e^{i\alpha x - |\alpha|} d\alpha \]  \hspace{1cm} (3.14a)

\[ r_b(x) = \int_{-\infty}^{+\infty} \frac{\sinh\left(\frac{\alpha}{2}\right)}{\sinh\left(\frac{3\alpha}{2}\right)} e^{i\alpha x + |\alpha|} d\alpha. \]  \hspace{1cm} (3.14b)

4. Ground state and excitations

Any physical state is characterized by two sets of roots satisfying the Bethe equations (3.5a,b). These roots can be complex and besides we can find some modifications of the distribution of roots (holes). So, we write the solutions of Bethe equations as strings

\[ v^{(1)}_{k,(m)} = v^{(1)}_{k,M} + im \quad ; \quad m = -M, \ldots + M \]  \hspace{1cm} (4.1a)

\[ v^{(2)}_{k,(m)} = v^{(2)}_{k,M'} + im \quad ; \quad m = -M', \ldots + M'. \]  \hspace{1cm} (4.1b)

A \( M \)-string has \( M \) length and contains \( M \) roots, which share the same real part. The 0-strings are real numbers. In order to find the equations for the center of strings we introduce (4.1a,b) into (3.5a,b) and multiply the Bethe equations for all the roots of the same string. Using the appendix A, we get

\[ 2N_3 \arctan \frac{v^{(1)}_{k,M}}{M + \frac{i}{2}} = 2\pi Q^{(1)}_{k,M} + \sum_{M'} \sum_{j=1}^{\nu^{(1)}_{M'}} \psi_{M,M'}(v^{(1)}_{k,M} - v^{(1)}_{j,M'}) \]

\[ - \sum_{M''} \sum_{l=1}^{\nu^{(2)}_{M''}} \phi_{M,M''}(v^{(1)}_{k,M} - v^{(2)}_{l,M''}), \]  \hspace{1cm} (4.2a)

\[ -2N_3^* \arctan \frac{v^{(2)}_{k,M}}{M + \frac{i}{2}} = -2\pi Q^{(2)}_{k,M} - \sum_{M'} \sum_{j=1}^{\nu^{(2)}_{M'}} \psi_{M,M'}(v^{(2)}_{k,M} - v^{(2)}_{j,M'}) \]

\[ + \sum_{M''} \sum_{l=1}^{\nu^{(1)}_{M''}} \phi_{M,M''}(v^{(2)}_{k,M} - v^{(1)}_{l,M''}), \]  \hspace{1cm} (4.2b)

where \( \nu^{(i)}_M \) is the number of \( M \)-strings at level \( i \), and the numbers \( Q^{(1)}_{k,M} \) and \( Q^{(2)}_{k,M} \) are integers or half-odd. They vary in the intervals \( |Q^{(1)}_{k,M}| \leq Q_{\max,M}^{(1)} \) and \( |Q^{(2)}_{k,M}| \leq Q_{\max,M}^{(2)}. \)
In order to obtain $Q_{\text{max},M}^{(1)}$ and $Q_{\text{max},M}^{(2)}$, we define the functions

$$F_{\lambda}^{(1)}(\lambda) = \frac{N_3}{\pi} \arctan \frac{\lambda}{M + \frac{1}{2}} - \frac{1}{2\pi} \sum_{M'} \sum_{j=1}^{\nu_{M,M'}^{(1)}} \psi_{M,M'}(\lambda - v_{j,M'}^{(1)}) + \frac{1}{2\pi} \sum_{M''} \sum_{l=1}^{\nu_{M,M''}^{(2)}} \phi_{M,M''}(\lambda - v_{l,M''}^{(2)}), \quad (4.3a)$$

$$F_{\lambda}^{(2)}(\lambda) = \frac{N_3^*}{\pi} \arctan \frac{\lambda}{M + \frac{1}{2}} - \frac{1}{2\pi} \sum_{M'} \sum_{j=1}^{\nu_{M,M'}^{(2)}} \psi_{M,M'}(\lambda - v_{j,M'}^{(2)}) + \frac{1}{2\pi} \sum_{M''} \sum_{l=1}^{\nu_{M,M''}^{(1)}} \phi_{M,M''}(\lambda - v_{l,M''}^{(1)}). \quad (4.3b)$$

The equations (4.2a, b) can be written as follows

$$F_{\lambda}^{(1)}(v_{j,M}^{(1)}) = Q_{k,M}^{(1)} \quad (4.4a)$$

$$F_{\lambda}^{(2)}(v_{j,M}^{(2)}) = Q_{k,M}^{(2)}. \quad (4.4b)$$

Note that $F_{\lambda}^{(1)}(\lambda)$ and $F_{\lambda}^{(2)}(\lambda)$ are increasing functions of $\lambda$, so we deduce

$$F_{\lambda}^{(1)}(-\infty) \leq Q_{k,M}^{(1)} \leq F_{\lambda}^{(1)}(+\infty), \quad (4.5a)$$

$$F_{\lambda}^{(2)}(-\infty) \leq Q_{k,M}^{(2)} \leq F_{\lambda}^{(2)}(+\infty). \quad (4.5b)$$

The total number of allowed $Q_{\text{max},M}^{(i)}$ will be

$$2Q_{\text{max},M}^{(i)} + 1 = 2F_{\lambda}^{(i)}(+\infty). \quad (4.6)$$

If we denote by $H_{\lambda}^{(i)}$ the number of holes in the sea of $M$-strings at level $i$, then we have

$$2Q_{\text{max},M}^{(i)} + 1 = \nu_{M}^{(i)} + H_{\lambda}^{(i)}, \quad (4.7)$$

because the total number of allowed $Q_{\text{max},M}^{(i)}$ corresponds to the sum of roots and holes.

Taking the limit when $\lambda$ tends to infinity in (4.3a, b) and using (4.6) and (4.7) we get

$$\nu_{M}^{(1)} + H_{\lambda}^{(1)} = N_3 - 2 \sum_{M'\geq 0} J(M,M')\nu_{M'}^{(1)} + 2 \sum_{M''\geq 0} K(M,M'')\nu_{M''}^{(2)}, \quad (4.8a)$$

$$\nu_{M}^{(2)} + H_{\lambda}^{(2)} = N_3^* - 2 \sum_{M'\geq 0} J(M,M')\nu_{M'}^{(2)} + 2 \sum_{M''\geq 0} K(M,M'')\nu_{M''}^{(1)}, \quad (4.8b)$$

7
with
\[ J(M_1, M_2) = \begin{cases} 
2M_1 + \frac{1}{2} & \text{if } M_1 = M_2 \\
2 \min(M_1, M_2) + 1 & \text{if } M_1 \neq M_2
\end{cases} \] (4.9a)
\[ K(M_1, M_2) = \begin{cases} 
M_2 + \frac{1}{2} & \text{if } M_2 + \frac{1}{2} \leq M_1 \\
M_1 + \frac{1}{2} & \text{if } M_2 + \frac{1}{2} > M_1
\end{cases} \] (4.9a)

If \( N_\rho \) is the number of states \( \rho \) in the chain, then, as we have shown, in a recent paper, \[22\],

\[ N_u - N_{\bar{a}} = N_3 - r \] (4.10a)
\[ N_d - N_{\bar{d}} = r - s \] (4.10b)
\[ N_s - N_{\bar{s}} = s - N_3^* \] (4.10c)

On the other hand, the total number of strings is \( r \) and \( s \) at first and second level respectively, that is

\[ r = \sum_{M \geq 0} (2M + 1) \nu_M^{(1)} \] (4.11a)
\[ s = \sum_{M \geq 0} (2M + 1) \nu_M^{(2)} \] (4.11b)

where \( M \) is integer or half-odd. Applying (4.8a, b) for the real roots, and using (4.9a, b), we get

\[ H_0^{(1)} = N_3 - 2 \sum_{M' \geq 0} \nu_{M'}^{(1)} + \sum_{M'' \geq 0} \nu_{M''}^{(2)} \] (4.12a)
\[ H_0^{(2)} = N_3^* - 2 \sum_{M' \geq 0} \nu_{M'}^{(2)} + \sum_{M'' \geq 0} \nu_{M''}^{(1)} \] (4.12b)

For the general case this relation can be written as

\[ \nu_n^{(1)} - \frac{\nu_n^{(2)}}{2} + H_n^{(1)} = \frac{H_{n-1/2}^{(1)} + H_{n+1/2}^{(1)}}{2} \] (4.13a)
\[ \nu_n^{(2)} - \frac{\nu_n^{(1)}}{2} + H_n^{(2)} = \frac{H_{n-1/2}^{(2)} + H_{n+1/2}^{(2)}}{2} \] (4.13b)

\[ n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \]

where we have used \( H_{-1/2}^{(1)} \equiv N_3 \) and \( H_{-1/2}^{(2)} \equiv N_3^* \).
At this moment, we can characterize the ground state and the excited states:

Firstly, **Ground state.** In the ground state we have no holes and only real roots. That is

\[
\nu_0^{(1)} = \frac{2N_3 + N^*_3}{3}, \quad (4.14a)
\]
\[
\nu_0^{(2)} = \frac{N_3 + 2N^*_3}{3}, \quad (4.14b)
\]
\[
\nu_{M>0}^{(1)} = \nu_{M>0}^{(2)} = H_{M>0}^{(1)} = H_{M>0}^{(2)} = 0, \quad (4.14c)
\]

and the quantum numbers are

\[
N_u - N_\bar{u} = N_d - N_\bar{d} = N_s - N_\bar{s} = \frac{N_3 - N^*_3}{3}. \quad (4.15)
\]

Then, the ground state is formed by pairs \(u\bar{u}, d\bar{d}\) and \(s\bar{s}\).

Secondly, **Excited state with the same quantum numbers.** This is characterized by one hole and one two-string in each level,

\[
\nu_0^{(1)} = \frac{2N_3 + N^*_3}{3} - 2, \quad (4.16a)
\]
\[
\nu_0^{(2)} = \frac{N_3 + 2N^*_3}{3} - 2, \quad (4.16b)
\]
\[
\nu_{1/2}^{(1)} = \nu_{1/2}^{(2)} = H_0^{(1)} = H_0^{(2)} = 1, \quad (4.16c)
\]
\[
\nu_{M>1/2}^{(1)} = \nu_{M>1/2}^{(2)} = H_{M>0}^{(1)} = H_{M>0}^{(2)} = 0, \quad (4.16d)
\]

and the quantum numbers as in \((4.15)\).

Thirdly, **Excited state with other quantum numbers.** We can find an excited state characterized by two holes, one in each level, and real number, that is

\[
\nu_0^{(1)} = \frac{2N_3 + N^*_3}{3} - 1, \quad (4.17a)
\]
\[
\nu_0^{(2)} = \frac{N_3 + 2N^*_3}{3} - 1, \quad (4.17b)
\]
\[
H_0^{(1)} = H_0^{(2)} = 1, \quad (4.17c)
\]
\[
\nu_{M>0}^{(1)} = \nu_{M>0}^{(2)} = H_{M>0}^{(1)} = H_{M>0}^{(2)} = 0. \quad (4.17d)
\]

Here, the quantum numbers are

\[
N_u - N_\bar{u} = \frac{N_3 - N^*_3}{3} + 1 \quad (4.18a)
\]
\[
N_d - N_\bar{d} = \frac{N_3 - N^*_3}{3} \quad (4.18b)
\]
\[
N_s - N_\bar{s} = \frac{N_3 - N^*_3}{3} - 1. \quad (4.18c)
\]
There are four ways to get this state from the ground state. The first is when a \(d\)-site changes to an \(u\)-site and a \(\bar{d}\)-site to a \(\bar{s}\)-site. The second when it changes from an \(\bar{u}\) to a \(\bar{d}\) and a \(s\) to \(d\). The third from a \(s\) to \(u\), and the fourth is a \(\bar{u}\) to a \(\bar{s}\).

The next excited states have more than two holes, and they can be found by using (4.11) to (4.13).

5. \(S\)-matrix

In this section we are going to find the \(S\)-matrix for the excitations over the ground state. For this purpose we will use the transfer matrix \(\bar{F}(u, \gamma)\), which is obtained by taking the \(\{3^*\}\)-representation in the auxiliary space. Thus, we define

\[
\bar{T}_{\alpha,\beta}(u, \gamma) = L_{\alpha,\alpha_1}^{\{3^*\}}(u + \gamma)L_{\alpha_1,\alpha_2}^{\{3^*\}}(u) \ldots L_{\alpha_{N-2},\alpha_{N-1}}^{\{3^*\}}(u + \gamma)L_{\alpha_{N-1},\alpha}^{\{3^*\}}(u),
\]

and the transfer matrix is

\[
\bar{F}(u, \gamma) = \text{trace}[\bar{T}(u, \gamma)].
\]

From the Yang-Baxter equation we have that

\[
[T(u, \gamma), \bar{T}(v, -\gamma)] = 0.
\]

We take the simplest case, that is \(\gamma = 0\).

The eigenvalue of \(\bar{F}(u, 0)\) is

\[
\bar{\Lambda}(u) = [a(u)]^{N_3} [b(u)]^{N_3} \prod_{j=1}^{\bar{r}} g(\mu_j - u) + [b(u)]^{N_3^*} \prod_{i=1}^{\bar{r}} g(u - \bar{\mu}_i) \times
\]

\[
\left\{ [b(u)]^{N_3} \prod_{l=1}^{\bar{s}} g(\bar{\lambda}_l - u) + [a(u)]^{N_3^*} \prod_{k=1}^{\bar{s}} g(u - \bar{\lambda}_k) \prod_{n=1}^{\bar{r}} \frac{1}{g(u - \bar{\mu}_n)} \right\},
\]

and, using the parameterization (3.4), the ansatz equations can be written as

\[
\begin{align*}
\begin{bmatrix}
\bar{v}_k^{(1)} + \frac{i}{2} \\
\bar{v}_k^{(2)} - i
\end{bmatrix}^{N_3^*} & = - \prod_{j=1}^{\bar{r}} \frac{\bar{v}_k^{(1)} - \bar{v}_j^{(1)} - i}{\bar{v}_k^{(2)} - \bar{v}_j^{(2)} - i} \prod_{l=1}^{\bar{s}} \frac{\bar{v}_l^{(2)} - \bar{v}_k^{(2)} - i}{\bar{v}_l^{(1)} - \bar{v}_k^{(1)} + i}, \\
\begin{bmatrix}
\bar{v}_k^{(1)} - i \\
\bar{v}_k^{(2)} + \frac{i}{2}
\end{bmatrix}^{N_3} & = - \prod_{j=1}^{\bar{r}} \frac{\bar{v}_k^{(1)} - \bar{v}_j^{(1)} + i}{\bar{v}_k^{(2)} - \bar{v}_j^{(2)} - i} \prod_{l=1}^{\bar{s}} \frac{\bar{v}_l^{(2)} - \bar{v}_k^{(2)} + i}{\bar{v}_l^{(1)} - \bar{v}_k^{(1)} - i}.
\end{align*}
\]
But the commutation of the transfer matrices in (5.3) requires that the Bethe equations (3.5) and (5.5) are the same. Thus, we have

\[
\begin{align*}
\bar{v}_j^{(1)} &= v_{j+1}^{(2)} & j = 1, \ldots, \bar{r} = s, \\
\bar{v}_k^{(2)} &= v_{k-1}^{(1)} & k = 1, \ldots, \bar{s} = r.
\end{align*}
\] (5.6a, 5.6b)

In order to calculate the momentum of the chain, we consider an alternating chain, that is \( N_3 = N_3^* = \frac{N}{2} \). Then, the momentum is

\[
P = i \ln \left[ \rho(0)^{-N/2} \Lambda(0) \bar{\Lambda}(0) \right]. \tag{5.7}
\]

With (3.1), (3.4a, b), (5.4) and (5.6a, b) we have

\[
P = i \sum_{j=1}^{r} \ln \frac{v_j^{(1)}}{v_j^{(2)}} + i \sum_{k=1}^{s} \ln \frac{v_k^{(2)}}{v_k^{(1)}}. \tag{5.8}
\]

Using the approximation (3.11) and the root densities (3.13), we get

\[
P = P_0 + \sum_{h=1}^{N_h^{(1)}} p(\theta_h^{(1)}) + \sum_{h=1}^{N_h^{(2)}} p(\theta_h^{(2)}), \tag{5.9}
\]

where \( P_0 \) is the momentum of the ground state and the other terms are the hole contributions, with

\[
p(\theta) = \frac{1}{2} \int_{-\infty}^{+\infty} dx \frac{\sinh x + \sinh(\frac{x}{2})}{ix \sinh(\frac{3x}{2})} e^{ix\theta}. \tag{5.10}
\]

We calculate the \( S \) matrix for the scattering of two holes. Here we follow the Korepin-Andrei-Destri method [23]- [25]. For a state of two holes with rapidities \( \theta_1 \) and \( \theta_2 \), the momentum \( p(\theta_1) \) verifies the quantization condition

\[
e^{ip(\theta_1)N} S = 1. \tag{5.11}
\]

The \( S \) matrix can be written as \( S = e^{i\Phi} \), so we have

\[
p(\theta_1) + \frac{1}{N} \Phi = \frac{2\pi}{N} n, \tag{5.12}
\]

where \( n \) is an integer.

One can prove by direct calculation that

\[
p(\theta) = \pi \int_{-\infty}^{\theta} \sigma_1^{(o)}(\lambda) d\lambda + c_1, \tag{5.13}
\]
where $c_1$ is a constant, which will be irrelevant for our problem. From (5.10a) we can write

$$Z_{N/2}(\theta) = \int_{-\infty}^{\theta} \sigma_1(\lambda) d\lambda + c_2,$$

(5.14)

where $c_2$ is an irrelevant constant.

Evaluating for a hole in the first level $\theta_1$ we have

$$Z_{N/2}(\theta_1) = \frac{I^{(h)}}{N/2}.$$

(5.15)

With the relations (5.13), (5.14) and (5.15) we deduce

$$p(\theta_1) = \frac{2\pi}{N} I^{(h)} - \frac{2\pi}{N} \int_{-\infty}^{\theta_1} \sigma_1^{(h)}(\lambda) d\lambda + \text{const.}$$

(5.16)

Comparing (5.12) with (5.16), we conclude

$$\Phi = 2\pi \int_{-\infty}^{\theta_1} \sigma_1^{(h)}(\lambda) d\lambda + \text{const.}$$

(5.17)

We remove the constants because they contribute as a rapidity-independent phase factor. Thus we have to calculate

$$\Phi(\theta) = -2 \int_{0}^{\infty} d\alpha \frac{\sinh \frac{\alpha}{2}}{\sinh \frac{3\alpha}{2}} e^\Psi \sin(\alpha \theta),$$

(5.18)

where $\theta = \theta_1 - \theta_2$ is the difference of rapidities of the two holes. This integral can be solved by means of the $\Psi$-function

$$\Psi(x) = \frac{d}{dx} \ln \Gamma(x),$$

(5.19)

and we get

$$\frac{d\Phi}{d\theta} = \frac{1}{3} \left[ -\Psi\left(\frac{1}{6} - i\frac{\theta}{3}\right) + \Psi\left(\frac{1}{2} - i\frac{\theta}{3}\right) - \Psi\left(\frac{1}{6} + i\frac{\theta}{3}\right) + \Psi\left(\frac{1}{2} + i\frac{\theta}{3}\right) \right].$$

(5.20)

For the corresponding $S$ matrix we have

$$S_1(\theta) = \frac{\Gamma\left(\frac{1}{6} - i\frac{\theta}{3}\right) \Gamma\left(\frac{1}{2} + i\frac{\theta}{3}\right)}{\Gamma\left(\frac{1}{6} + i\frac{\theta}{3}\right) \Gamma\left(\frac{1}{2} - i\frac{\theta}{3}\right)}.$$ 

(5.21)

For the state with one hole and one two-string in each level we find the scattering matrix to be

$$S_2(\theta) = \frac{\left(\frac{1}{2} - i\theta\right) \Gamma\left(\frac{1}{6} - i\frac{\theta}{3}\right) \Gamma\left(\frac{1}{2} + i\frac{\theta}{3}\right)}{\left(\frac{1}{2} + i\theta\right) \Gamma\left(\frac{1}{6} + i\frac{\theta}{3}\right) \Gamma\left(\frac{1}{2} - i\frac{\theta}{3}\right)}.$$ 

(5.22)
In order to calculate the $S$ matrix for holes in the same level we consider states with, at least, four holes (two in each level). Following the same procedure we find for two holes in the first (second) level

$$S_3(\theta) = \frac{\Gamma(\frac{2}{3} - i \frac{\theta}{3}) \Gamma(1 + i \frac{\theta}{3})}{\Gamma(\frac{2}{3} + i \frac{\theta}{3}) \Gamma(1 - i \frac{\theta}{3})}, \quad (5.23)$$

where $\theta = \theta_1^{(1)} - \theta_2^{(1)} \quad (\theta = \theta_1^{(2)} - \theta_2^{(2)})$. The $S_1$ and $S_3$ matrices coincide with those for the no-alternating chain. This shows that the scattering is the same in the alternating and the non alternating chain [6].

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**Appendix A.**

We define the function

$$V_0(x) = \frac{x - i}{x + i}. \quad (A.1)$$

It is easy to prove the relations

$$\prod_{m=-M}^{M} V_0(2x + im) = \prod_{m=-M}^{M} V_0(\frac{x}{M + \frac{i}{2}}) \equiv V_M(x) \quad (A.2a)$$

$$\prod_{m=-M}^{M} V_0(x + im) = \prod_{m=-M}^{M} V_0(\frac{x}{M}) V_0(\frac{x}{M + 1}) \equiv V_M(x) \quad (A.2b)$$

$$\prod_{m_1=-M_1}^{M_1} \prod_{m_2=-M_2}^{M_2} V_0(x + i(m_1 + m_2)) = \prod_{L=|M_2-M_1|}^{M_1+M_2} V_L(x) \equiv V_{M_1,M_2}(x) \quad (A.2c)$$

$$\prod_{m_1=-M_1}^{M_1} \prod_{m_2=-M_2}^{M_2} V_0(2x + 2(m_1 + m_2)) = \prod_{m_1=-M_1}^{M_1} V_0(\frac{x + im_1}{M_2 + \frac{i}{2}}) \equiv W_{M_1,M_2}(x). \quad (A.2d)$$

It is convenient to define the functions

$$\psi_{M_1,M_2}(x) = 2 \sum_{L=|M_2-M_1|}^{M_1+M_2} \left( \arctan \frac{x}{L} + \arctan \frac{x}{L + 1} \right) \quad (A.3a)$$

$$\phi_{M_1,M_2}(x) = 2 \sum_{L=-M_1}^{M_1} \arctan \left( \frac{x + iL}{M_2 + \frac{i}{2}} \right), \quad (A.3b)$$

which are connected with (A.2c,d) by the logarithm.
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