The functional determinant multiplicative anomaly, or defect, is more closely investigated and explicit forms for products of the particular operators, $L - \alpha_i$, are produced. I also present formulae for the defect of products of $L^2 - \alpha_i^2$ in terms of that for just two factors and discuss the specific cases of the sphere and hemisphere. The difference of Neumann and Dirichlet quantities on the hemisphere is equal to that for spin–1/2 on the rim. This is proved generally.
1. Introduction

I am interested in the functional determinant of a product of operators. As is well known, this is, generally, not the product of the determinants, the discrepancy being termed the ‘multiplicative anomaly’. An expression was given by Wodzicki in 1987, but the general phenomenon seems to have been noticed earlier in a physical context, Allen [1], Chodos and Myers, [2]. I do not give later references but just note that calculations, relevant to some of those here, have been done recently by Cognola, Elizalde and Zerbini, [3] using the Wodzicki approach.

The functional determinant of products of second order elliptic operators (on spheres) has appeared lately in connection with the AdS/CFT correspondence, e.g. [4], but the possibility of a defect is not considered, perhaps because it could be absorbed in renormalisation.

Therefore, I thought it might be useful to present some (possibly known) results for defects but derived in an elementary, spectral way. The operators will be restricted but have occurred in various contexts. My formulae may not be as general as they could be but they are, at least, explicit.

2. The set up and the defects

I consider a set of operators \( D_i \), \( i = 1, \ldots, k \), and ask for the difference in logdets,

\[
\log \det \prod_i D_i - \sum_i \log \det D_i \equiv M_k(D_1, \ldots, D_k)
\]

which I will refer to as a multiplicative anomaly, or, sometimes, defect. All the \( D_i \) commute and all the logdets are defined from the \( \zeta \)-function of the relevant operator, assuming this makes sense.

I make a special choice of operator,

\[
D_i = L^2 - \alpha_i^2 = (L + \alpha_i)(L - \alpha_i) \equiv D_i^{(+)} D_i^{(-)}
\]

where the \( \alpha_i \) are constants. \( L \) is a pseudo–differential operator, in particular the square–root of a shifted Laplace–Beltrami operator in \( d \)--dimensions,

\[
L = (-\Delta_2 + c)^{1/2},
\]

where \( c \) is a constant, to be suitably chosen.\(^2\)

\(^2\) On the sphere, for a particular \( c \), this operator is strictly Zoll, up to an additional constant. This means the spectrum is \( 1, 2, 3, \ldots \) (with degeneracies).
The $D_i$ are, therefore, second order. It is a general result that the defect vanishes for closed odd–dimensional manifolds.

This choice covers a number of relevant cases. For example, for an $O(n)$ scalar field, the functional part of the determinant takes the form,

$$\log \det \left[ (-\Delta_2 + m_1^2) \ldots (-\Delta_2 + m_n^2) \right],$$

and a similar product structure arises in the theory of massless, conformal higher spins on a sphere, which is a more interesting case and the one I have in mind. In field theory computations it is commonly assumed that the defect, $M_k$, is zero, even for even $d$. Although the physical relevance of the defect is doubtful, because of renormalisation, its computation is still an interesting mathematical question.

To this end I proceed by firstly saying that I do not wish to give any preliminary derivations beyond those that already exist. This is the reason I have introduced the factorisation of the second order operator into linear factors since the defects after a complete linear factorisation have already been evaluated. I therefore define a further anomaly for complete linear factorisation by,

$$\log \det \prod_i D_i - \sum_i \log \det D_i^{(+)} - \sum_i \log \det D_i^{(-)} \equiv L_k(D_1, \ldots, D_k). \quad (4)$$

In particular, applying this to a single $D_i$ and summing,

$$\sum_i \log \det D_i - \sum_i \log \det D_i^{(+)} - \sum_i \log \det D_i^{(-)} = \sum_i L_1(D_i) \quad (5)$$

Subtracting these equations, and referring to the definition, (1), I get

$$M_k(D_1, \ldots, D_k) = L_k(D_1, \ldots, D_k) - \sum_{i=1}^k L_1(D_i) \quad (6)$$

the point now being that the quantities on the right–hand side have been given elsewhere. A direct spectral means of finding the second term in (6) was given in [5] and extended to the $k$–fold product in [6]. I can, therefore, just write down the resulting combination in (6),

$$M_k(D_1, \ldots, D_k) = \frac{k-1}{2k} \sum_{r=1}^n \left( \frac{H(r-1)}{r} \frac{N_{2r}(d)}{r} \sum_{j=1}^k \alpha_i^{2r} \right)$$

$$- \frac{1}{2k} \sum_{r=1}^n \frac{1}{r} \sum_{t=1}^{n-r} \frac{N_{2r+2t}(d)}{t} \sum_{i<j=1}^k \alpha_i^{2r} \alpha_j^{2t}. \quad (7)$$
The upper limit $u$ equals $d/2$ for even dimensions, and $(d - 1)/2$ for odd.\(^3\)

In this formula $H$ is the harmonic series, $H(r) = \sum_{n=1}^{r} 1/n$, $H(0) = 0$, and $N$ is the residue at a pole of the $\zeta$–function defined by,

$$Z_d(s) \equiv \sum_m \frac{1}{\lambda_m^s},$$

(8)

where the $\lambda_m$ are the eigenvalues (with repeats) of the linear operator, $L$, (3). That is,

$$Z_d(s + r) \to \frac{N_r(d)}{s} + R_r(d) \quad {\text{as}} \ s \to 0.$$  

(9)

The $N$s are simply related to the heat–kernel coefficients and are, therefore, locally computable. $N$ and $R$ will depend on the parameters in the eigenvalues, $\lambda_m$.

I note the important fact that there are no cubic, or higher powers of $\alpha^2$ in (7).

Expression (7) can be implemented easily on a machine, but it does not directly yield the answer in its simplest factorised form without further manipulation. The symmetrised sums appearing in (7),

$$\Sigma_1(k) = \sum_{j=1}^{k} \alpha_j^{2r},$$

$$\Sigma_2(k) = \sum_{i<j=1}^{k} \alpha_i^{2r} \alpha_j^{2t},$$

allow some combinatorial rearrangements to be made at an earlier stage.

For short I now write $M_k(D_1, \ldots, D_k) = M(1, \ldots, k)$ and also define $M(i, j) \equiv M_2(D_i, D_j)$ with $i < j$. The aim is to write everything in terms of the two operator anomaly, $M(i, j)$. I spell out the steps.

From (7), defining handy quantities $A$ and $B$,

$$M(i, j) = \frac{1}{4} \sum_{r=1}^{u} A(r, d)(\alpha_i^{2r} + \alpha_j^{2r}) - \frac{1}{4} \sum_{r=1}^{u} \sum_{t=1}^{u-r} B(r, t, d) \alpha_i^{2r} \alpha_j^{2t}.$$  

(10)

\(^3\) If the manifold is closed only even dimensions are relevant.
Therefore,

\[
\frac{2}{k} \sum_{i<j=1}^{k} M(i, j) = \frac{1}{2k} \sum_{r=1}^{u} A(r, d) \sum_{i<j=1}^{k} (\alpha_i^{2r} + \alpha_j^{2r})
- \frac{1}{2k} \sum_{r=1}^{u} \sum_{t=1}^{u-r} B(r, t, d) \sum_{i<j=1}^{k} \alpha_i^{2r} \alpha_j^{2t}
= \frac{k-1}{2k} \sum_{r=1}^{u} A(r, d) \sum_{j}^{k} \alpha_j^{2r}
- \frac{1}{2k} \sum_{r=1}^{u} \sum_{t=1}^{u-r} B(r, t, d) \sum_{i<j=1}^{k} \alpha_i^{2r} \alpha_j^{2t}
= M(1, 2, \ldots, k),
\]

which is the desired relation. As a check, since \(M(i, j)\) vanishes if \(\alpha_i^2 = \alpha_j^2\) so do all the anomalies, as required.

If there are repeats amongst the \(D_i\), say \(D_i\) appears \(g_i\) times, then it is easy to see that the relation is modified to

\[
M(1, 2, \ldots, k) = \frac{1}{g} \sum_{i<j=1}^{k} (g_i + g_j) M(i, j)
\]

where \(g\) is the total degeneracy, \(g = \sum_i g_i\). Similar formulae can be found relating \(M_k\) to \(M_n\) for any \(n, 1 < n < k\).

Equation (12) is a particular example of a general result of Castillo-Garate, Friedman and M˘antoiu, [7], having the same basic definitions but with \(D_i\) quite general operators. The proof is an inductive one, and uses general properties of the Wodzicki expression for the defect.

In the approach of the present paper, the possibility of such a decomposition arises from the absence of cubic, and higher, terms in the explicit form of the defect.

3. A special case, the sphere and hemisphere.

Explicit expressions for the defect can be obtained when the singularity structure of the \(\zeta\)-function \(Z_d\) is known (or equivalently the heat–kernel coefficients). Such is the case for spheres discussed already in [6,8,5]. The motivation for considering products of operators on the sphere was Branson’s expression for the higher derivative conformal operator (GJMS operator). In that case the parameters, \(\alpha_i, \alpha_j\),
I find it convenient, and a little more flexible, to consider the sphere as the union of two hemispheres. To be more precise, the spectrum on the whole sphere is the union of the Neumann and Dirichlet spectra on a hemisphere. Then I have access to the hemisphere values as well.

For this geometry, the eigenvalues of $L$ take the linear form,

$$\lambda_m = a + m.\omega$$

(13)

where, for the hemisphere, $\omega = 1_d$.\(^4\) Neumann conditions arise when $a = (d - 1)/2$, and Dirichlet when $(d + 1)/2$. It is often sufficient to give the expressions for just one of these conditions as the other is the same up to a sign.

$Z_d$ is then a Barnes $\zeta$–function with explicit pole structure. The residues, $N$ in (9), are easily calculated generalised Bernoulli polynomials.

The formula (7) has been evaluated by brute force for various $d$ and $k$. I will not write out the expressions but rather will appeal to the sum structure (11) which organises them in a more symmetrical way. Then I need give only $M(i, j)$ for each dimension. For display purposes, I set $x = \alpha^2_i$, $y = \alpha^2_j$ and write $M(x, y)$ for $M(i, j)$.

Every $M(x, y)$ has the factor $(x - y)^2$ so I put

$$M(x, y) = (x - y)^2 P(d)$$

(14)

and list $P(d)$, a symmetric polynomial in $x$ and $y$. I have put back the dimension. The Neumann hemisphere values are,

\[
\begin{align*}
P_N(3) &= 0, & P_N(4) &= \frac{1}{48}, & P_N(5) &= \frac{1}{96}, \\
P_N(6) &= \frac{1}{1920} (2 (x + y) - 5), & P_N(7) &= \frac{1}{1920} ((x + y) - 5), \\
P_N(8) &= \frac{1}{1935360} (44 (x^2 + y^2) + 56 x y - 420 (x + y) + 777), \\
P_N(9) &= \frac{1}{967680} (11 (x^2 + y^2) + 14 x y - 168 (x + y) + 588).
\end{align*}
\]

(15)

The Dirichlet values, $P_D(d)$ are the same as these for even dimensions, and opposite in sign for odd.

\(^4\) Other choices for $\omega$ can give other divisions of the sphere. I will not consider these. For integer $\omega$ the spectrum is strictly Zoll.
For comparison, I also give the corresponding spin–half quantities using the same notation. The non–zero values are,

\[ P_{1/2}(4) = \frac{1}{48}, \quad P_{1/2}(6) = \frac{1}{960} (x + y - 5) \]
\[ P_{1/2}(8) = \frac{1}{483840} \left( 11(x^2 + y^2) + 14xy - 168(x + y) + 588 \right), \]

and I note the holographic–like relation

\[ P_N(d) - P_D(d) = P_{1/2}(d - 1), \quad \forall d. \]  

(17)

I should note here that I am using ‘mixed’ (local) boundary conditions for the spinor field. These are conformally covariant. Both types (analogous to N and D) give the same hemisphere eigenvalue set which is as in (13) with \( a = d/2 \). (See the Appendix and [9].)

4. Linear factorisation, again

For the operator (2) the factors occur in ‘conjugate’ pairs, \( \pm \alpha_i \). This means that all calculations involve \( \alpha_i^2 \) from the start, which is somewhat restrictive. In this section I discuss the slightly more flexible product (\( w \) is a constant introduced for expository convenience),

\[ \prod_{i=1}^{2k} (L + w - \alpha_i), \]  

(18)

where, to preserve my previous notation, \( k \) can be an integer or a half–integer. \( L \) can, in fact, be a fairly general operator, not necessarily as in (3). All I require is that the corresponding \( \zeta \)–function, \( Z(s) \),

\[ Z(s, w) \equiv \sum_m \frac{1}{(\lambda_m + w)^s}, \]  

(19)

has reasonable properties such as a finite number of single poles (only) and that it is analytic around \( s = 0 \). In particular I will require the behaviour around the points \( s = r, \ r \in \mathbb{Z}_+ \), which, for a general operator \( L \), may, or may not, be poles. I assume therefore that,

\[ Z(s + r, w) \sim \frac{N(r, w)}{s} + R(r, w), \quad s \to 0 \text{ and } r \in \mathbb{Z}_+, \]  

(20)
where $N(r, w)$ might be zero. The $N$ and $R$ depend on the dimension, $d$, and on the parameters determining the eigenvalues. I do not show these.

My basic example is the linear function (13) and $Z$ is then the Barnes function, already used. See the section 6.

I define the multiplicative anomaly, or rather defect, for the ‘linear factorisation’, (18) by (this is just (1) with a different notation),

$$\delta(k, w; \alpha) \equiv \log \det \prod_{i=1}^{2k} (L + w - \alpha_i) - \sum_{i=1}^{2k} \log \det (L + w - \alpha_i).$$

(21)

$\alpha$ is a $2k$–vector, specifying the operator (18).

In the next section I give calculational details of the case of two factors, which is enough to motivate the general formulae,

$$\delta(k, w; \alpha) = \frac{2k - 1}{2k} \sum_{r=1}^{u} \frac{H(r - 1) N(r, w)}{r} \left( \sum_{j=1}^{2k} \alpha_r^j \right)$$

$$+ \frac{1}{2k} \sum_{r=1}^{u} \sum_{t=1}^{u-r} \frac{N(r + t, w)}{r t} \left( \sum_{i<j=1}^{2k} \alpha_r^i \alpha_t^j \right)$$

(22)

which correctly vanishes for one factor, $k = 1/2$.

The upper limit $u$ equals $[\mu]$ where $\mu$ is the order of the eigenvalue set, $\lambda_m$. Most usually, $\mu = d$ for linear operators and $d/2$ for second order ones. The sums over $r$ and $t$ in (22) restrict to the pole set in $Z(s)$.

Just as in the previous discussion, it is possible to express the general product defect in terms of the two–factor defect, either in the same manner as before, see (11), or by rearranging the sums in (22). Then, not surprisingly,

$$\delta(k, w; \alpha) = \frac{1}{k} \sum_{i<j=1}^{2k} \delta(1, w; \alpha_i, \alpha_j).$$

(23)

In the next section I expand on the two factor case and apply it to the (hemi)–sphere in the following section.
5. Two factors.

For two factors the $\zeta$–function is,

$$\zeta(s, w) \equiv \sum_m \frac{1}{(\lambda_m + w - \alpha)^s(\lambda_m + w - \alpha')^s}.$$

The method, as used in [6,5], is to binomially expand each bracket and do the resulting sum over $m$ using (19). This gives,

$$\zeta(s, w) = \sum_{r=0}^{\infty} \sum_{r'=0}^{\infty} \alpha^r \alpha'^{r'} \frac{s(s+1) \ldots (s+r-1) s(s-1) \ldots (s+r'-1)}{r! r'!} Z(2s+r+r', w).$$

For the determinant of the product (defined by $\zeta$–functions), I require $\zeta'(0, w)$.

To expose the powers of $s$, it is convenient to split off the $r = 0$ and $r' = 0$ terms. Written out,

$$\zeta(s, w) = Z(2s, w) + \sum_{r=1}^{\infty} (\alpha^r + \alpha'^{r'}) \frac{s(s+1) \ldots (s+r-1)}{r!} Z(2s+r, w)$$

$$+ \sum_{r, r'=1}^{\infty} \alpha^r \alpha'^{r'} \frac{s(s+1) \ldots (s+r-1) s(s+1) \ldots (s+r'-1)}{r! r'} Z(2s+r+r', w)$$

$$\sim Z(2s, w) + \sum_{r=1}^{\infty} (\alpha^r + \alpha'^{r'}) \left( \frac{s(s+1) \ldots (s+r-1)}{r!} \left( \frac{N(r, w)}{2} + sR(r, w) \right) + \sum_{r, r'=1}^{\infty} \alpha^r \alpha'^{r'} \frac{s(s+1) \ldots (s+r-1) s(s+1) \ldots (s+r'-1)}{r! r'} \right.$$ 

$$\times s \left( \frac{N(r+r', w)}{2} + sR(r+r', w) \right).$$

Therefore

$$\zeta'(0, w) = 2Z'(0, w) + \sum_{r=1}^{[\mu]} \frac{\alpha^r + \alpha'^{r'}}{r} \left( \frac{1}{2} H(r-1) N(r, w) + R(r, w) \right)$$

$$+ \sum_{r=[\mu]+1}^{\infty} \frac{\alpha^r + \alpha'^{r'}}{r} Z(r, w) + \frac{1}{2} \sum_{r, r'=1}^{\infty} \frac{\alpha^r \alpha'^{r'}}{r r'} N(r+r', w).$$

(24)

I do not wish to enlarge on the generality of the operator $L$ and only say that if $\mu$ is the order of the set of eigenvalues, $\lambda_m$, $r$ is less than, or equal to, $[\mu]$. For the eigenvalues (13), for example, $\mu = d$. $N(r, w)$ is non–zero only if $r \leq [\mu].$
I now convert the infinite series into a finite one in the following, rather round-
about way. Introduce the ‘heat–kernel’, \( K(\tau, w) \), of the operator \( L + w \), explicitly

\[
K(\tau, w) = \sum_{m} e^{-(\lambda_m + w)\tau},
\]

and use the Mellin representation of the \( \zeta \)–functions to write the infinite sum in
(24) as, using an intermediate, calculational regularization,

\[
\sum_{r=d+1}^{\infty} \frac{\alpha^r}{r \Gamma(r)} \int_{0}^{\infty} d\tau \tau^{r-1} K(\tau, w) \]

\[
= \lim_{s \to 0} \int_{0}^{\infty} d\tau \left( \exp(\alpha\tau) - \sum_{r=0}^{d} \frac{(\alpha\tau)^r}{r!} \right) \tau^{s-1} K(\tau, w) \quad (25)
\]

\[
= \lim_{s \to 0} \left( \Gamma(s)Z(s, w - \alpha) - \sum_{r=0}^{d} \frac{\alpha^r}{r!} \Gamma(s + r)Z(s + r, w) \right).
\]

Since the total quantity in (25) is finite, the individual pole terms that arise in
the \( s \to 0 \) limit must cancel yielding the identity between heat–kernel coefficients,

\[
Z(0, w - \alpha) - Z(0, w) = \sum_{r=1}^{[\mu]} \frac{\alpha^r}{r} N(r, w). \quad (26)
\]

The finite remainder is the required result and equals,

\[
Z'(0, w - \alpha) - Z'(0, w) - \sum_{r=1}^{[\mu]} \frac{\alpha^r}{r} R(r, w)
\]

\[
- \gamma(Z(0, w - \alpha) - Z(0, w)) - \sum_{r=1}^{[\mu]} \frac{\alpha^r}{r} \psi(r)N(r, w),
\]

which, in view of (26), yields

\[
\sum_{r=[\mu]+1}^{\infty} \frac{\alpha^r}{r} Z(r, w))
\]

\[
= Z'(0, w - \alpha) - Z'(0, w) - \sum_{r=1}^{[\mu]} \frac{\alpha^r}{r} R(r, w) - \sum_{r=1}^{[\mu]} \frac{\alpha^r}{r} (\psi(r) + \gamma)N(r, w) \quad (27)
\]

\[
= Z'(0, w - \alpha) - Z'(0, w) - \sum_{r=1}^{[\mu]} \frac{\alpha^r}{r} (R(r, w) + H(r - 1)N(r, w)),
\]
where $\gamma$ is the Euler constant.

Using (27) in (24) gives my final answer for two factors,

\[
\zeta'(0, w) = 2Z'(0, w) + Z'(0, w - \alpha) - Z'(0, w) + Z'(0, w - \alpha') - Z'(0, w)
\]

\[
+ \sum_{r=1}^{[\mu]} \frac{\alpha^r + \alpha'^r}{r} \left( \frac{1}{2} H(r - 1) N(r, w) + R(r, w) \right)
\]

\[
- \sum_{r=1}^{[\mu]} \frac{\alpha^r + \alpha'^r}{r} (R(r, w) + H(r - 1) N(r, w)) + \frac{1}{2} \sum_{r, r' = 1}^{\infty} \frac{\alpha^r \alpha'^{r'}}{r r'} N(r + r', w)
\]

\[
= Z'(0, w - \alpha) + Z'(0, w - \alpha')
\]

\[
- \frac{1}{2} \sum_{r=1}^{[\mu]} \frac{\alpha^r + \alpha'^r}{r} (H(r - 1) N(r, w)) + \frac{1}{2} \sum_{r, r' = 1}^{\infty} \frac{\alpha^r \alpha'^{r'}}{r r'} N(r + r', w).
\]

(28)

In the special case of $\alpha' = -\alpha$

\[
\zeta'(0, w) = Z'(0, w - \alpha) + Z'(0, w + \alpha)
\]

\[
- \sum_{r=1}^{[\mu]} \frac{(1 + (-1)^r)\alpha^r}{2r} H(r - 1) N(r, w) + \frac{1}{2} \sum_{r, r' = 1}^{\infty} (-1)^{r'} \frac{\alpha^{r + r'}}{r r'} N(r + r', w)
\]

\[
= Z'(0, w - \alpha) + Z'(0, w + \alpha)
\]

(29)

\[
- \sum_{\rho=1}^{[\mu/2]} \frac{\alpha^{2\rho}}{2\rho} H(2\rho - 1) N(2\rho, w) + \frac{1}{2} \sum_{r, r' = 1}^{\infty} \frac{(-1)^r + (-1)^{r'}}{2} \frac{\alpha^{r + r'}}{r r'} N(r + r', w).
\]

The last term can be reworked so as to combine with the preceding one. I give
the details for those who might want to see the nuts and bolts. Setting $r + r' = 2\rho$,

$$
\frac{1}{2} \sum_\begin{array}{c} r, r' = 1 \\ r + r' \leq [\mu] \end{array} \infty (-1)^r + (-1)^{r'} \frac{\alpha^{r + r'}}{rr'} N(r + r', w)
$$

$$
= \sum_{\rho = 1}^{[\mu/2]} \frac{\alpha^{2\rho}}{2} N(2\rho, w) \sum_{r' = 1}^{2\rho - 1} (-1)^{r'} \frac{1}{(2\rho - r') r'}
$$

$$
= \sum_{\rho = 1}^{[\mu/2]} \frac{\alpha^{2\rho}}{4\rho} N(2\rho, w) \sum_{r' = 1}^{2\rho - 1} (-1)^{r'} \left( \frac{1}{r'} + \frac{1}{2\rho - r'} \right)
$$

$$
= \sum_{\rho = 1}^{[\mu/2]} \frac{\alpha^{2\rho}}{2\rho} N(2\rho, w) \sum_{r' = 1}^{2\rho - 1} (-1)^{r'} \frac{1}{r'}
$$

$$
= \frac{1}{2} \sum_{\rho = 1}^{[\mu/2]} \frac{\alpha^{2\rho}}{2\rho} N(2\rho, w) (H(\rho - 1) - 2H_O(\rho - 1)),
$$

where $H_O(r) \equiv \sum_{k=0}^{r} 1/(2k + 1)$.

Then, combining this with the penultimate term in (29), one encounters the simplification,

$$
-H(2\rho - 1) + \frac{1}{2} H(\rho - 1) - H_O(\rho - 1) = -2H_O(\rho - 1)
$$

and the total quantity, from (29), is, finally,

$$
\zeta'(0, w) = Z'(0, w - \alpha) + Z'(0, w + \alpha) - \sum_{\rho = 1}^{[\mu]} \frac{\alpha^{2\rho}}{\rho} H_O(\rho - 1) N(2\rho, w),
$$

(30)

where the (spin–zero) logdet defect, $\delta_0(d, \alpha)$, is the (negative of) the final term. This is the expression given in [6] derived more simply there by expanding in $\alpha^2$ from the start. The equality is gratifying.

The expression (30) is repeated, and generalised, in [10] where some history was attempted. A derivation of the basic cancellations, which avoids using the heat–kernel, as in (25), was also given.
6. Spherical defects again. Some explicit expressions

The defect in (30) is a polynomial in $\alpha^2$. On the sphere some specific calculations were performed in [6,8] and I extend these to the present situation.

For spherical domains, $\lambda_m$ is as in (13) and, as before, $Z(s, w)$ is a Barnes $\zeta$–function. The general form of the defect polynomial is easily found since the residues, $N$, of the Barnes $\zeta$–function are given in terms of generalised Bernoulli polynomials, which are readily found.

Without loss of generality, I can now set $w = 0$ since it only adds to the constant $a$.

Machine evaluation of the last term in (30) yields for the hemisphere spin–zero $N$–defect, $\delta_0(d, \alpha)\big|_N$,

$$
\begin{align*}
    d = 2, & \quad \alpha^2 \\
    d = 3, & \quad \frac{\alpha^2}{2} \\
    d = 4, & \quad \frac{\alpha^4}{9} - \frac{\alpha^2}{24} \\
    d = 5, & \quad \frac{\alpha^4}{18} - \frac{\alpha^2}{12} \\
    d = 6, & \quad \frac{23\alpha^6}{5400} - \frac{\alpha^4}{72} + \frac{3\alpha^2}{640} \\
    d = 7, & \quad \frac{23\alpha^6}{10800} = \frac{\alpha^4}{72} + \frac{\alpha^2}{60} \\
    d = 8, & \quad \frac{11\alpha^8}{132300} - \frac{23\alpha^6}{25920} + \frac{37\alpha^4}{17280} - \frac{5\alpha^2}{7168} \\
    d = 9, & \quad \frac{11\alpha^8}{264600} - \frac{23\alpha^6}{32400} + \frac{7\alpha^4}{2160} - \frac{\alpha^2}{280}
\end{align*}
$$

The $D$–hemisphere values, $\delta_0(d, \alpha)\big|_D$, are the same as the $N$ ones in even dimensions and opposite in sign for odd. Hence, by addition, the defect is zero on odd– dimensional spheres and twice the values in (31) for even. The even values (on the sphere) have been calculated recently by Cognola, Elizalde and Zerbini, [3], using a different method involving the Wodzicki residue. Our overlapping results agree.

Figs. 1 and 2 plot the defect on some odd and even hemispheres. I note that, in each case, the roots approximately coincide, more exactly with increasing dimension when the periodicity tends to unity.
Fig. 1. Scalar Neumann defect, $\delta$, on even hemispheres, dimensions 4, 6, 8, plotted against the spectral parameter, $\alpha$.

Fig. 2. Same as Fig. 1 but for odd hemispheres, dimensions 5, 7, 9.

For variety I also give the spin–half hemisphere polynomial defects, $\delta_{1/2}(d, \alpha)$. The two boundary conditions now give the same values. This means that, since the defect is zero on odd–dimensional closed spaces, the hemisphere values are also zero.
for odd dimensions, as calculation confirms. The even values for $\delta_{1/2}(d, \alpha)$ are, up to spin degeneracy,

\[
d = 2, \quad \alpha^2
\]
\[
d = 4, \quad \frac{\alpha^4}{9} - \frac{\alpha^2}{6}
\]
\[
d = 6, \quad \frac{23 \alpha^6}{5400} - \frac{\alpha^4}{36} + \frac{\alpha^2}{30}
\]
\[
d = 8, \quad \frac{11 \alpha^8}{132300} - \frac{23 \alpha^6}{16200} + \frac{7 \alpha^4}{1080} - \frac{\alpha^2}{140}
\]
\[
d = 10, \quad \frac{563 \alpha^{10}}{571536000} - \frac{11 \alpha^8}{317520} + \frac{299 \alpha^6}{777600} - \frac{41 \alpha^4}{27216} + \frac{\alpha^2}{630}.
\]

(32)

I note the linear counterpart of the relation, (17),

\[
\delta_0(d, \alpha) \big|_N - \delta_0(d, \alpha) \big|_D = \delta_{1/2}(d - 1, \alpha), \quad \forall \, d,
\]

(33)

which connects bulk and boundary quantities. This relation complements one in [11] which identifies the difference in the $N$– and $D$–hemisphere scalar logdets, for even $d$, with the Dirac (squared) logdet on the odd boundary.

It is possible to give a simple spectral argument that encompasses all these relations. I outline it in an appendix.

7. The Wodzicki approach

Wodzicki has computed the defect for a $\zeta$–function of the form

\[
\sum_{n=1}^{\infty} \frac{P(n)}{\prod_i (n - a_i)^{s}}
\]

where $P(n)$ is a polynomial. (See Kassel, [12], §6.6.) This is equivalent to the hemisphere case if $P(n)$ is the relevant, Barnes degeneracy. The usual way of dealing with this form is to write the single factor $\zeta$–function (which is a Barnes one) as a sum of Hurwitz functions. The same technique holds for a general $P(n)$ and the defect can be evaluated by the present method. The details might be presented at a later time.

\footnote{This can be understood from the fact that the local spinor boundary conditions are a combination of N and D.}
8. Conclusion

The defect for a product of second order scalar operators compared to a sum of the operators is given in terms of the defect for just two factors. Some examples are given. The defect for a product of ‘linear’ factors is also given which reduces to a known polynomial for two conjugate factors. Applied to hemispheres this yields explicit polynomials whose plots reveal the unexpected feature of (approximately) coincident roots which, at the moment, I cannot explain, except that it involves the infinite dimensional limit. I also briefly give the corresponding spin–half defects which turn out to be related to the scalar defects in one dimension higher.

Added note Sept.2023

I thank Danilo Diaz for communications and pointing out a sign error which has now been corrected. An alternative Shintani-type approach with a relation to a Casimir energy has recently appeared, [13], agreeing with the updated results here.

Appendix A bulk–boundary relation

I now show that the relations (17) and (33) follow from a very basic property of Barnes ζ–functions.

All I need are the eigenvalues on the hemisphere. These are, in \( d \) dimensions,

\[
\lambda^N_m(d) = \frac{d-1}{2} + m \cdot 1_d, \quad \lambda^D_m(d) = \lambda^N_m(d) + 1,
\]

for Neumann and Dirichlet conditions respectively. \( m \) ranges over all the non–negative integers. The spin–half eigenvalues are,

\[
\lambda^{1/2}_m(d) = \lambda^N_m(d) + \frac{1}{2},
\]

up to spin degeneracy, which I ignore. It is easy to see that,

\[
\{\lambda^N_m(d)\} - \{\lambda^D_m(d)\} \equiv \{\lambda^{1/2}_m(d-1)\},
\]

as eigenvalue sets.

Assembling the eigenvalues into Barnes ζ–functions, \( \zeta_d(s, a \mid \omega) \), this statement leads to an example of the general recursion relation, due to Barnes,

\[
\zeta_d(s, a + \omega_1 \mid \omega) - \zeta_d(s, a \mid \omega) = -\zeta_{d-1}(s, a \mid \omega_2, \ldots, \omega_d), \tag{35}
\]
which specialises here to,

$$\zeta^N(s,d) - \zeta^D(s,d) = \zeta^{1/2}(s,d - 1). \quad (36)$$

Thus all quantities derivable from the $\zeta$–function such as the conformal and multiplicative anomalies, logdets, free energies, vacuum energies etc., will obey this bulk–boundary relation in which the bulk quantity is the difference of two boundary conditions. Examples have appeared in the main body of this paper and also in [5,11].

This result carries through to product operators. It also extends to other values of the parameters, $\omega$, for example to $\omega = (q, 1_{d-1})$ which gives a lune of angle $\pi/q$ instead of the hemisphere. The right–hand side of (35) is then independent of $q$, which corresponds to the fact that the boundary of any lune is a full sphere, metrically. This geometry is useful in discussions of Rényi and entanglement entropies.

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