ANALYSIS OF A BATCH ARRIVAL RETRIAL QUEUE WITH IMPATIENT CUSTOMERS SUBJECT TO THE SERVER DISASTERS

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Abstract. We consider an $M^X/G/1$ retrial queue with impatient customers subject to disastrous failures at which times all customers in the system are lost. When the server finishes serving a customer and finds the orbit empty, the server becomes dormant until $N$ or more customers accumulate. If a coming batch of customers finds the server idle, one of the arriving customers begins his service immediately and the rest join the orbit to repeat their request later. Otherwise, if the server is dormant or busy or down, all customers of the coming batch enter the orbit. When the server is under repair, customers in the orbit can become impatient after waiting a random amount of time and leave the system. By using the characteristic method for partial differential equations, the steady-state distributions of the server state and the number of customers in the orbit are obtained along with various performance measures. In addition, the reliability of the system is analyzed detailed. Finally, an application to telecommunication networks is provided and the effects of various parameters on the system performance are demonstrated numerically.

1. Introduction. Recently there have been significant contributions to queuing systems with repeated attempts which are characterized by the feature that arriving customers who find all servers busy or down or on vacation will join the retrial group/orbit to try their luck again some later time. For example, Yang et al. [28] considered an $M/G/1$ retrial queue where customers are impatient when the server is busy. Phung-Duc [19] analyzed $M/M/c/K$ retrial queues where customers are impatient when the servers are busy or idle. Retrial queues have been widely used to model the many practical situations in telephone switching systems, telecommunication networks and computer systems. Other applications include stacked aircraft waiting to land, queues of retrial shoppers who may leave along waiting line hoping to return later when the line may be shorter. For detailed overviews of

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the main results and methods, interested readers may refer to the survey papers by Yang and Templeton [29], Falin [9] and the book by Falin and Templeton [10]. For more recent references see the bibliographical overviews in Artalejo [1, 2].

The presence of disasters in queueing systems was first introduced by Jain and Sigman [11]. Queueing systems with disasters can be used to analyze computer networks with virus infections and breakdowns due to a reset order. Disasters attendance in a queueing system can also be considered as a type of clearing mechanism which removes all workload in the system whenever it occurs to the busy server. That is, the arrival of a disaster not only destroys all the unfinished work but also causes the server breakdown. In recent years, a variety of industrial applications have created interest in the modelling of reliability using queueing theory. Queueing systems with a repairable service station have been studied by many authors. Retrial queueing systems with servers failures and repairs were introduced by Aissani [3] and Kulkarni and Choi [12]. Wang et al. [24] studied a repairable $M/G/1$ retrial queueing model from the viewpoint of reliability for the first time, both the queueing indices and reliability characteristics are obtained. Later, Wang et al. [25] dealt with a repairable $M/G/1$ retrial queue with disasters. Recently, Wu and Lian [26] gave a detailed analysis on $M/G/1$ retrial queue with priority and unreliable server under Bernoulli vacation schedule. Mytalas and Zazanis [18] considered an $M^X/G/1$ queue with disasters and repairs under a multiple adapted vacation. Chang et al. [5] analyzed an unreliable-server retrial queue with customer’s feedback and impatience. Chang and Wang [6] investigated unreliable $M/M/1/1$ retrial queues with set-up time.

Queueing systems with impatient customers have drawn significant attention in recent years. Queueing systems with impatient customers are occurring in many practical situations. Applications include real-time telecommunication systems in which data received after a hard deadline is useless, telephone networks in which subscribers give up due to impatience before the requested connection is established and data-base management systems in which queries of users are lost when not answered within a certain time. Furthermore, impatience of customers is becoming a significant factor in qualitatively evaluating and comparing various stochastic models. In the queueing literature, one can find an extensive body of research addressing impatience phenomena observed in single or multi-server systems serving a single class of customers (see Altman and Yechiali [4], Yechiali [30], Perel and Yechiali [20]). Recently, Li et al. [13] studied a supermarket model with impatient customers. However, to the best of our knowledge, no work has been done on the $M^X/G/1$ retrial queueing system with impatient customers.

The first study of the batch arrival queue with $N$-policy was by Lee and Srinivasan [14]. They presented a procedure to determine the optimal threshold $N$ under a linear cost structure. Later, Lee et al. [15] considered this model extensively through different techniques. They discussed the stochastic decomposition property. Further, Lee et al. [16] discussed an $M^X/G/1$ queueing system with $N$-policy and single vacation. Moreover, Choudhury et al. [7] studied an $M^X/G/1$ queue system with an additional second phase of optional service and unreliable server, which consists of a breakdown period and a delay period under $N$-policy. Recently, Liu et al. [17] gave a detailed heavy-traffic asymptotics analysis on a priority polling system with $N$-policy. However, as far as we know, the batch arrival retrial queueing system with $N$-policy has not been discussed.

Our study is motivated by the performance analysis of wireless cellular mobile
networks. There are often retrial, disasters, impatience, feedback and vacation phenomena in wireless cellular mobile networks. Therefore, our model is very suitable for the modelling of wireless cellular mobile networks. The main purpose of the paper is to investigate the impact of retrial, disasters, impatience, feedback and vacation phenomena on queueing performance measures and reliability indexes. There are at least three highlights in our paper. The first one is that we use the characteristic method for partial differential equations to derive the stationary distribution of the system. The second feature is that we realize an extensive analysis of the system from both the queueing and reliability points of view. The third is that we give an application to telecommunication networks. The rest of the paper is organized as follows. The model description and some definitions and conventions are given in Section 2. The stability condition is derived in Section 3. The steady-state distribution of the server state and the orbit size is discussed in Section 4 along with some performance measures. Section 5 relates to the reliability results obtained for this model. An application for the model under discussion and some numerical examples are shown in Section 6. Finally, Section 7 concludes the paper.

2. Model description. We consider an $M^X/G/1$ retrial queueing system. Assume that customers arrive to a single server system according to a compound Poisson process, with rate $\lambda$. The arrival size random variable is denoted by $X$ with probability mass function $c_n = P\{X = n\}, n \geq 1$, probability generating function $c(z) = \sum_{n=1}^{\infty} c_n z^n$ and first two moments $C_1$ and $C_2$, respectively. There is no waiting space and therefore if a coming batch of customers finds the server idle, one of the arriving customers starts his service immediately and the rest join a retrial group (called orbit) to seek service again and again until they find the server idle. Otherwise, if the server is found busy or down or on vacation, all customers of the coming batch enter the orbit. Assume that the retrial times for any repeated customer are exponentially distributed with rate $\delta$. After the customer is served completely, he will decide either to return to the front of the server again for another service with probability $p$ ($p < 1$), or to leave the system forever with probability $\bar{p} = 1 - p$. All customers have i.i.d service time distribution given by

$$B(x) = 1 - \exp\left\{-\int_{0}^{x} \mu(t) dt\right\},$$

where $\mu(x) = \frac{B'(x)}{1 - B(x)}$ is the failure rate function and the first two moments are denoted by $B_1$ and $B_2$. When the server finishes serving a customer and finds the orbit empty, the server enters the $N$-policy vacation (the dormant period). If the server finds $N$ or more customers in the orbit, it enters the idle period to wait for a customer in the orbit or for a new arriving customer. Moreover, there is a Poisson stream of catastrophes with rate $\alpha$. A catastrophe occurs only when the server is busy, and it has no effect on the system when the server is dormant, or idle, or failed. When a catastrophe occurs, all customers in the system are deleted immediately and the server is breakdown. When the server is breakdown, it is sent to repair immediately. After repair the server is as good as new. As soon as the repair of the server is completed, the server is idle if there is at least one customer in the orbit. Otherwise, the server enters the dormant period. The repair time
distribution is given by
\[ R(x) = 1 - \exp \left\{ - \int_{0}^{x} \gamma(t) dt \right\}, \]
where \( \gamma(x) = \frac{R'(x)}{R(x)} \) is the failure rate function and the first two moments are denoted by \( R_1 \) and \( R_2 \). Assume that the customers in the orbit are not informed the repair time. Consequently, when the system is undergoing repair, customers in the orbit will become impatient. The impatient times are exponentially distributed with parameter \( \theta \). Upon completion of a repair, the server will start to serve customers in the orbit and all of these impatient customers will become normal customers.

We assume that all stochastic processes involved in the system are independent of each other.

Throughout the rest of the paper, we denote by \( \bar{F}(x) = 1 - F(x) \) the tail of distribution function \( F(x) \). We also denote \( F^*(s) = \int_{0}^{+\infty} e^{-sx} dF(x), \) \( \bar{F}(s) = \int_{0}^{+\infty} e^{-sx} \bar{F}(x) dx = \frac{1 - F^*(s)}{s} \).

3. Stability condition. In this section, we carry out the necessary and sufficient condition for the system to be stable. The stochastic behaviour of this queueing system can be described by the Markov process \( \{X(t), t \geq 0\} = \{H(t), N(t), \zeta(t), t \geq 0\} \), where
\[
H(t) = \begin{cases} 
0, & \text{if the server is dormant at time } t, \\
1, & \text{if the server is idle at time } t, \\
2, & \text{if the server is busy at time } t, \\
3, & \text{if the server is under repair at time } t,
\end{cases}
\]
and \( N(t) \) corresponds to the number of customers in the orbit at time \( t \). If \( H(t) = 2 \), \( \zeta(t) \) represents the elapsed service time of the customer being in service at time \( t \).

If \( H(t) = 3 \), \( \zeta(t) \) denotes the elapsed repair time at time \( t \).

**Theorem 3.1.** In the case of \( \alpha = 0 \), the system is stable if and only if \( \lambda B_1 C_1 < \bar{p} \); in the case of \( \alpha > 0 \), the system is stable without further conditions.

**Proof.** In the case of \( \alpha = 0 \), we let \( \{t_n; n \in \mathbb{N}\} \) be the sequence of epochs of the end of the service completion times at which the server is idle. Then the sequence \( \{Y_n = N(t_{n+1})\} \) forms a Markov chain which is embedded in our queueing system on the state space \( \mathbb{N} \). It is readily to see that \( \{Y_n, n \in \mathbb{N}\} \) is irreducible and aperiodic.

To prove the sufficient condition, it is very convenient to use criteria based on the mean drift \( \chi_j = E[f(Y_{n+1}) - f(Y_n)|Y_n = j] \). In our case, we consider the function \( f(j) = j \). Then we get
\[ \chi_j = \frac{\lambda B_1 C_1}{\bar{p}} - \frac{j\delta}{\lambda + j\delta}, \quad j \geq 0. \]

Obviously, if \( \frac{\lambda B_1 C_1}{\bar{p}} < 1 \), then we have \( |\chi_j| < \infty \) for all \( j \) and \( \lim_{j \to \infty} \chi_j < 0 \).

Therefore, applying Foster’s criterion (see Pakes [21]), we can guarantee that the embedded Markov chain \( \{Y_n, n \geq 0\} \) is ergodic.

Besides, since \( Y_{n+1} - Y_n \geq -1 \), for chain \( \{Y_n, n \in \mathbb{N}\} \) the mean down drift is bounded below. Thus, the necessity can immediately be verified from Kaplan’s condition (see Sennot et al. [23]), namely \( \chi_j < \infty \) for all \( j \geq 0 \) and there exists \( j_0 \in \mathbb{N} \) such that \( \chi_j \geq 0 \) for \( j \geq j_0 \).

As the arrival stream is a Poisson process, Burke’s theorem (see Cooper [8])
and PASTA property (see Wolff [27]) establish the existence of the steady-state probabilities of \( \{X(t), t \geq 0\} \) if and only if \( \frac{RB}{\bar{C}} < 1 \).

In the case of \( \alpha > 0 \), each time a catastrophe occurring, all customers in the system will be deleted. Therefore, the process \( \{X(t), t \geq 0\} \) is regenerative at epochs of catastrophes occurring. Thus, the regeneration cycle is the time interval between two successive catastrophes. As the catastrophes occur with positive rate \( \alpha \), we can derive that the expected length of a regeneration cycle is finite, which ensures the stability of the system.

\[ \Box \]

4. Analysis of the steady-state probabilities. In this section, we study the steady-state distribution for the system.

For the process \( \{X(t), t \geq 0\} \), we define the probabilities

\[
V_n(t) = P\{H(t) = 0, N(t) = n\}, \quad t \geq 0, 0 \leq n \leq N - 1,
\]

\[
I_n(t) = P\{H(t) = 1, N(t) = n\}, \quad t \geq 0, n \geq 1
\]

and the probability densities \( B_n(x, t) \) and \( R_n(x, t) \) as follows

\[
B_n(x, t)dx = P\{H(t) = 2, N(t) = n, x < \zeta(t) \leq x + dx\}, \quad t \geq 0, x \geq 0, n \geq 0,
\]

\[
R_n(x, t)dx = P\{H(t) = 3, N(t) = n, x < \zeta(t) \leq x + dx\}, \quad t \geq 0, x \geq 0, n \geq 0.
\]

We assume that the system is in steady state, so that we can define the limiting probabilities

\[
V_n = \lim_{t \to \infty} V_n(t), \quad 0 \leq n \leq N - 1,
\]

\[
I_n = \lim_{t \to \infty} I_n(t), \quad n \geq 1
\]

and the limiting probability densities

\[
B_n(x) = \lim_{t \to \infty} B_n(x, t), \quad x \geq 0, n \geq 0,
\]

\[
R_n(x) = \lim_{t \to \infty} R_n(x, t), \quad x \geq 0, n \geq 0.
\]

By the method of supplementary variables, we easily obtain the system of equilibrium equations:

\[
\frac{dB_n(x)}{dx} = -(\lambda + \alpha + \mu(x))B_n(x) + (1 - \delta_{n,0})\lambda \sum_{k=1}^{n} c_k B_{n-k}(x), \quad n \geq 0, \quad (1)
\]

\[
\frac{dR_n(x)}{dx} = -(\lambda + n\theta + \gamma(x))R_n(x) + (1 - \delta_{n,0})\lambda \sum_{k=1}^{n} c_k R_{n-k}(x)+(n+1)\theta R_{n+1}(x), \quad n \geq 0, \quad (2)
\]

\[
\lambda V_n = \delta_{n,0}\bar{p} \int_0^\infty B_0(x)\mu(x)dx + \delta_{n,0} \int_0^\infty R_0(x)\gamma(x)dx + (1 - \delta_{n,0})\lambda \sum_{k=1}^{n} c_k V_{n-k}, \quad 0 \leq n \leq N, \quad (3)
\]

\[
(\lambda + n\delta) I_n = \bar{p} \int_0^\infty B_n(x)\mu(x)dx + \int_0^\infty R_n(x)\gamma(x)dx, \quad 1 \leq n \leq N, \quad (4)
\]

\[
(\lambda + n\delta) I_n = \bar{p} \int_0^\infty B_n(x)\mu(x)dx + \int_0^\infty R_n(x)\gamma(x)dx + \lambda \sum_{k=0}^{N-1} c_{n-k} V_k, \quad n \geq N. \quad (5)
\]
The steady-state boundary conditions are

\[ B_n(0) = (n + 1)\delta I_{n+1} + \lambda(1 - \delta_{n,0}) \sum_{k=1}^{n} c_k I_{n-k+1} + p \int_{0}^{\infty} B_n(x) \mu(x) dx, \quad n \geq 0, \]

\[ R_n(0) = \delta_{n,0} \alpha \sum_{k=0}^{\infty} \int_{0}^{\infty} B_k(x) dx, \quad n \geq 0, \]

where \( \delta_{n,m} \) denotes Kronecker’s function, and the normalization condition is

\[ \sum_{n=0}^{N-1} V_n + \sum_{n=1}^{\infty} I_n + \sum_{n=0}^{\infty} \int_{0}^{\infty} B_n(x) dx + \sum_{n=0}^{\infty} \int_{0}^{\infty} R_n(x) dx = 1. \]

In order to solve the system Eqs (1)-(7), let us define the following probability generating functions for \(|z| \leq 1\):

\[ I(z) = \sum_{n=1}^{\infty} I_n z^n, \quad B(x, z) = \sum_{n=0}^{\infty} B_n(x) z^n, \]

\[ V(z) = \sum_{n=0}^{\infty} V_n(z) z^n, \quad R(x, z) = \sum_{n=0}^{\infty} R_n(x) z^n. \]

Then the following theorem gives the solution of the system.

**Theorem 4.1.** The stationary distribution of the process \( \{X(t), t \geq 0\} \) has the following generating functions:

**Case 1:** \( \alpha > 0 \)

\[ I(z) = \begin{cases} \int_{\omega}^{1} \beta(x) \exp \left\{ - \int_{\omega}^{z} \alpha(u) du \right\} dx, & z \neq \omega, \\
\frac{\Phi}{1 + \Phi + \Psi[1 + \alpha \varphi(1)]} \sum_{n=0}^{\infty} z^n \pi_n, & z = \omega, 
\end{cases} \]

\[ V(z) = \frac{\Phi}{1 + \Phi + \Psi[1 + \alpha \varphi(1)]} \sum_{n=0}^{\infty} \frac{z^n \pi_n}{\pi_n}, \]

\[ B(x, z) = \frac{\delta \beta(z) + \left[ \lambda \frac{c(z)}{z^2} - \delta \alpha(z) \right] I(z)}{1 - pB^*(\alpha + b(z))} \exp\{-(\alpha + b(z))x\} \hat{B}(x), \]

\[ R(x, z) = \frac{\alpha \Psi}{1 + \Phi + \Psi[1 + \alpha \varphi(1)]} \exp \left\{ \frac{1}{\delta} \int_{1-(1-z)e^{-sx}}^{z} \frac{b(u) + \gamma (x + \ln(\frac{1-u}{u}))}{1 - u} du \right\}, \]

where

\[ b(z) = \lambda(1 - c(z)), \quad \alpha(z) = \frac{\lambda}{\delta} \cdot \left[ \frac{\bar{c}(z)}{z^2} + p \right] B^*(\alpha + b(z)) - 1 \]

\[ \beta(z) = \left\{ V(z) b(z) - \frac{\alpha \Psi \xi(z)}{1 + \Phi + \Psi[1 + \alpha \varphi(1)]} \int_{\omega}^{1} \frac{1 - pB^*(\alpha + b(z))}{\delta(\bar{p} + zp)B^*(\alpha + b(z)) - z} \right\}. \]
\[\xi(z) = \int_0^\infty \exp \left\{ \frac{1}{\theta} \int_0^z b(u) + \gamma \left( x + \frac{\ln(1-u)}{\theta} \right) \frac{du}{1-u} \right\} \gamma(x)dx,\]

\[\Phi = \frac{W}{WA - B} \left[ \sum_{n=0}^{N-1} \pi_n \right], \quad \Psi = \frac{1}{WA - B} \cdot \frac{\tilde{B}(\alpha)}{1 - pB^\ast(\alpha)},\]

\[W = \frac{\alpha \tilde{B}(\alpha)}{1 - pB^\ast(\alpha)} \int_0^\infty \frac{\xi(x)[1 - pB^\ast(\alpha + b(x))]}{\delta((\bar{p} + zp)B^\ast(\alpha + b(x)) - x)} \exp \left\{ \int_0^x \alpha(u)du \right\} dx,\]

\[A = \int_0^1 \frac{b(x)[1 - pB^\ast(\alpha + b(x))]}{\delta((\bar{p} + zp)B^\ast(\alpha + b(x)) - x)} \left[ \sum_{n=0}^{N-1} z^n \pi_n \right] \exp \left\{ - \int_x^1 \alpha(u)du \right\} dx,\]

\[B = \frac{\alpha \tilde{B}(\alpha)}{1 - pB^\ast(\alpha)} \int_0^1 \frac{\xi(x)[1 - pB^\ast(\alpha + b(x))]}{\delta((\bar{p} + zp)B^\ast(\alpha + b(x)) - x)} \exp \left\{ - \int_x^1 \alpha(u)du \right\} dx,\]

\[\varphi(1) = \lim_{z \to 0} \int_0^\infty \exp \left\{ \frac{1}{\theta} \int_0^\nu b(u) + \gamma \left( x + \frac{\ln(1-u)}{\theta} \right) \frac{du}{1-u} \right\} dx,\]

and \(\pi_n, n = 0, 1, \ldots, N - 1,\) is the probability that the number of customers in the orbit visits \(n\) during a dormant period, which satisfies the following recursive equation:

\[\begin{cases}
\pi_n = \sum_{k=1}^n c_k \pi_{n-k}, & n = 0, 1, \ldots, N - 1, \\
\pi_0 = 1 & \end{cases}\]

and \(\omega\) is the unique root of \(z\) of the equation \((\bar{p} + zp)B^\ast(\alpha + b(z)) = z\) on the interval \([0, 1].\)

Case 2: \(\alpha = 0\)

\[I(z) = \int_0^z \beta_0(x) \exp \left\{ - \int_x^z \alpha_0(u)du \right\} dx,\]

\[V(z) = \frac{\Phi_0}{1 + \Phi_0} \cdot \left( 1 - \frac{\lambda c_1 B_1}{\delta} \right) \cdot \left[ \sum_{n=0}^{N-1} z^n \pi_n \right],\]

\[B(x, z) = \frac{\delta \beta_0(z) + \left[ \lambda c_1(z) \delta - \lambda \alpha_0(z) \right] I(z)}{1 - pB^\ast(b(z))} \exp \{- (b(z))x \} B(x),\]

where

\[\alpha_0(z) = \lambda \delta \left[ \frac{\bar{p}c_1(z)}{\bar{p} + zp} B^\ast(b(z)) - 1 \right], \quad \beta_0(z) = \frac{V(z)b(z)[1 - pB^\ast(b(z))]}{\delta((\bar{p} + zp)B^\ast(b(z)) - z)},\]

\[\Phi_0 = \frac{1}{\int_0^1 \frac{b(x)[1 - pB^\ast(b(x))]}{\delta((\bar{p} + zp)B^\ast(b(x)) - z)} \left[ \sum_{n=0}^{N-1} z^n \pi_n \right] \exp \left\{ - \int_z^1 \alpha_0(u)du \right\} dz.\]
Proof. See the Appendix.

Remark 1. In the special case: no disasters, no N-policy and no feedback, our model becomes the classical $M^X/G/1$ retrial queue. We put $\alpha = 0$, $p = 0$ and $V(z) = 0$ in the main results and obtain

$$I(z) = (1 - \lambda C_1 B_1) \exp \left\{ -\frac{\lambda}{\delta} \int z \frac{c(u)}{u - B^*(b(u))} \, du \right\},$$

$$B(x, z) = \left[ \frac{\lambda c(z)}{z} - \delta \alpha(z) \right] I(z) \exp \{-b(z)x\} B(x),$$

where $I(z)$ is redefined as $I(z) := \sum_{n=0}^{\infty} z^n I_n$. We note that these results are consistent with known results of classical $M^X/G/1$ retrial queue presented in Falin and Templeton [10].

Throughout the rest of this section, we assume $\alpha > 0$.

Next, we are interested in investigating some queueing performance measures of our system.

Corollary 1. Under the steady-state condition, the marginal probability generating functions of the server’s state and orbit size distribution are given by

$$B(z) = \frac{\delta \beta(z) + \left[ \frac{\lambda c(z)}{z} - \delta \alpha(z) \right] I(z)}{1 - pB^*(\alpha + b(z))} B(\alpha + b(z)),$$

$$R(z) = \frac{\alpha \Psi}{1 + \Phi + \Psi[1 + \alpha \varphi(1)]} \int_0^\infty \exp \left\{ \frac{1}{\theta} \int_{\nu}^z b(u) + \gamma \left( x + \frac{\ln(1-u)}{\theta} \right) \, du \right\} dx,$$

$I(z)$ and $V(z)$ are given by the above theorem.

Corollary 2. If the system is in steady state, then

(i) the probability that the server is idle is

$$P_I = I(1) = \frac{1}{1 + \Phi + \Psi[1 + \alpha \varphi(1)]}.$$

(ii) the probability that the server is busy serving a customer is

$$P_B = B(1) = \frac{\Psi}{1 + \Phi + \Psi[1 + \alpha \varphi(1)]}.$$

(iii) the probability that the server is dormant is

$$P_V = V(1) = \frac{\Phi}{1 + \Phi + \Psi[1 + \alpha \varphi(1)]}.$$

(iv) the probability that the server is under repair is

$$P_R = R(1) = \frac{\Psi \alpha \varphi(1)}{1 + \Phi + \Psi[1 + \alpha \varphi(1)]}.$$
Corollary 3. (1) The probability generating function of the number of customers in the orbit is given by

\[
P_o(z) = I(z) + V(z) + 2\mathcal{B}(z) + 2\mathcal{R}(z)
\]

\[
= I(z) + \Phi \sum_{n=0}^{N-1} z^n \pi_n \left[ 1 + \Phi + \Psi[1 + \alpha\varphi(1)] \right] + \delta\beta(z) \left[ \frac{\lambda(c(z) - \delta\alpha(z)) I(z)}{1 - pB^*(\alpha + b(z))} \right] B(\alpha + b(z))
\]

\[
+ \alpha\Psi \sum_{n=0}^{N-1} z^n \pi_n \left[ 1 + \Phi + \Psi[1 + \alpha\varphi(1)] \right] \int_0^\infty \exp \left\{ \frac{1}{\theta} \int_u^z b(u) + \gamma \left( x + \frac{\ln(1 - u)}{\theta} \right) \right\} du \right\} dx.
\]

(2) The probability generating function of the number of customers in the system is given by

\[
P_s(z) = I(z) + V(z) + z\mathcal{B}(z) + 2\mathcal{R}(z)
\]

\[
= I(z) + \Phi \sum_{n=0}^{N-1} z^n \pi_n \left[ 1 + \Phi + \Psi[1 + \alpha\varphi(1)] \right] + \delta\beta(z) \left[ \frac{\lambda(c(z) - \delta\alpha(z)) I(z)}{1 - pB^*(\alpha + b(z))} \right] B(\alpha + b(z))
\]

\[
+ \alpha\Psi \sum_{n=0}^{N-1} z^n \pi_n \left[ 1 + \Phi + \Psi[1 + \alpha\varphi(1)] \right] \int_0^\infty \exp \left\{ \frac{1}{\theta} \int_u^z b(u) + \gamma \left( x + \frac{\ln(1 - u)}{\theta} \right) \right\} du \right\} dx.
\]

Corollary 4. (1) The mean number of customers in the orbit is

\[
L_o = P_o'(1) = I'(1) + V'(1) + 2\mathcal{B}'(1) + 2\mathcal{R}'(1).
\]

(2) The mean number of customers in the system is

\[
L_s = P_s'(1) = I'(1) + V'(1) + \mathcal{B}'(1) + 2\mathcal{B}'(1) + 2\mathcal{R}'(1).
\]

Theorem 4.2. The probability generating function of the stationary orbit size distribution at a departure epoch is given by

\[
\kappa(z) = \frac{(WA - B)[1 + \Phi + \Psi[1 + \alpha\varphi(1)][1 - pB^*(\alpha)]B^*(\alpha + b(z))] \times \left\{ \delta\beta(z) + \left[ \frac{\lambda(c(z)) - \delta\alpha(z)}{\lambda(c(z))} \right] I(z) \right\}}{B^*(\alpha)[1 - pB^*(\alpha + b(z))]}.
\]

Proof. Following the argument of PASTA (see Wolff [27]), we state that a departing customer will see \( j \) customers in the orbit just after a departure if and only if there were \( j + 1 \) customers in the system (or \( j \) customers in the orbit) just before the departure. Denote \( \{\kappa_j; j \in \mathbb{Z}^+\} \) as the probability that there are \( j \) customers in
the orbit at a departure epoch. Then for \( j \in \mathbb{Z}^+ \), we have
\[
\kappa_j = K_0 \bar{p} \int_0^\infty B_j(x) \mu(x) dx,
\]
where \( K_0 \) is the normalizing constant.

Multiplying both sides of (10) by \( z^j \) and taking summation over \( j \in \mathbb{Z}^+ \) and utilizing (37), we get
\[
\kappa(z) = K_0 \bar{p} \left\{ \delta \beta(z) + \left[ \frac{\lambda c(z)}{z} - \delta \alpha(z) \right] I(z) \right\} \frac{B^*(\alpha + b(z))}{1 - pB^*(\alpha + b(z))}, \tag{11}
\]
Utilizing normalizing condition \( \kappa(1) = 1 \), we obtain
\[
K_0 = \frac{(WA - B)\{1 + \Phi + \Psi[1 + \alpha \varphi(1)]\}[1 - pB^*(\alpha)]}{\bar{p}B^*(\alpha)}. \tag{12}
\]
Inserting (12) into (11) gives (9).

5. Reliability analysis. In this section, we discusses some reliability indexes of the queueing system under study.

Let \( A(t) \) be the pointwise availability of the server at time \( t \), that is, the probability that the server is either serving a customer or the server is dormant or the server is idle. So that the steady state availability of the server is \( A = \lim_{t \to \infty} A(t) \).

**Corollary 5.** The steady state availability of the server is given by
\[
A = 1 - P_R = \frac{1 + \Phi + \Psi}{1 + \Phi + \Psi[1 + \alpha \varphi(1)]}.
\]

**Corollary 6.** The steady state failure frequency of the server is given by
\[
W_f = \alpha P_B = \frac{\alpha \Psi}{1 + \Phi + \Psi[1 + \alpha \varphi(1)]}.
\]

We suppose that the system is empty at time \( t = 0 \). Denote by \( \tau \) the time to the first failure of the server, then the reliability function of the server is
\[
\varsigma(t) = P(\tau > t).
\]

In order to find the reliability of the server, we construct a new queueing system where the failure states of the server are assumed to be absorbent states. In this new queueing system, we use the same notations as in the previous section, and then we can deduce the following differential equations:
\[
\frac{\partial B_n(x,t)}{\partial x} + \frac{\partial B_n(x,t)}{\partial t} = -(\lambda + \alpha + \mu(x))B_n(x,t) + (1 - \delta_{n,0})\lambda \sum_{k=1}^n c_k B_{n-k}(x,t),
\]
\( n \geq 0, \) \( \tag{13} \)

\[
\frac{dV_n(t)}{dt} = -\lambda V_n(t) + \delta_{n,0} \bar{p} \int_0^\infty B_0(x,t) \mu(x) dx + (1 - \delta_{n,0})\lambda \sum_{k=1}^n c_k V_{n-k}(t),
\]
\( 0 \leq n < N, \) \( \tag{14} \)

\[
\frac{dI_n(t)}{dt} = -(\lambda + n\delta)I_n(t) + \bar{p} \int_0^\infty B_n(x,t) \mu(x) dx, \quad 1 \leq n < N, \tag{15}
\]
the boundary conditions:

\[ B_n(0, t) = (n + 1)\delta I_{n+1}(t) + \lambda(1 - \delta_{n,0})\sum_{k=1}^{N} c_k I_{n-k+1}(t) + p\int_0^\infty B_n(x, t)\mu(x)dx, \quad n \geq 0, \]

and the initial conditions:

\[ V_n(0) = \delta_{n,0}, \quad I_n(0) = 0, \quad B_n(x, 0) = 0. \]

By taking Laplace transforms of these equations, we obtain

\[
\frac{\partial \tilde{B}_n(x, s)}{\partial x} = -(s + \lambda + \alpha + \mu(x))\tilde{B}_n(x, s) + (1 - \delta_{n,0})\lambda\sum_{k=1}^{n} c_k \tilde{B}_{n-k}(x, s), \quad n \geq 0,
\]

\[
(s + \lambda)\tilde{V}_n(s) = \delta_{n,0} + \delta_{n,0}\tilde{p}\int_0^\infty \tilde{B}_0(x, s)\mu(x)dx + (1 - \delta_{n,0})\lambda\sum_{k=1}^{n} c_k \tilde{V}_{n-k}(s), \quad 0 \leq n < N,
\]

\[
(s + \lambda + n\delta)\tilde{I}_n(s) = \tilde{p}\int_0^\infty \tilde{B}_n(x, s)\mu(x)dx, \quad 1 \leq n < N,
\]

\[
(s + \lambda + n\delta)\tilde{I}_n(s) = \tilde{p}\int_0^\infty \tilde{B}_n(x, s)\mu(x)dx + \lambda\sum_{k=0}^{N-1} c_k \tilde{V}_k(s), \quad n \geq N,
\]

Define the following generating functions

\[
\tilde{I}(z, s) = \sum_{n=0}^\infty \tilde{I}_n(s)z^n, \quad \tilde{B}(z, x, s) = \sum_{n=0}^\infty \tilde{B}_n(x, s)z^n, \quad \tilde{V}(z, s) = \sum_{n=0}^{N-1} \tilde{V}_n(s)z^n.
\]

Multiplying (18) by \( z^n \) and summing over \( n \), then proceeding in the usual manner, we get

\[
\tilde{B}(s, x, z) = \frac{\delta \tilde{I}(s, z) + \lambda \tilde{I}(s, z)\frac{c(z)}{z}}{1 - pB^*(s + \alpha + b(z))}\exp\{-(s + \alpha + b(z))x\}B(x).
\]

Multiplying (20) and (21) by \( z^n \) and summing over \( n \) and utilizing (19), we obtain

\[
\delta((\tilde{p} + zp)B^*(s + \alpha + b(z)) - z)\tilde{I}(s, z) - (s + \lambda)\tilde{I}(s, z) = [s\tilde{V}_0(s) + \tilde{V}(s, z)\beta(z) - 1][1 - pB^*(s + \alpha + b(z))].
\]

In the same manner as Section 4, we have

\[
\tilde{I}(s, z) = \int_0^z \beta(s, x)\exp\left\{-\int_x^z \alpha(s, u)du\right\}dx, \quad z \neq \infty,
\]
Substituting (30) into (29) yields
\[ \tilde{I}(s, \varpi) = \frac{[s\tilde{V}_0(s) + \tilde{V}(s, z)b(z) - 1][1 - pB^*(s + \alpha + b(z))]}{\lambda p c(z) + (s + \lambda)p} B^*(s + \alpha + b(z)) - (s + \lambda), \quad z = \varpi, \] (26)

where
\[ \alpha(s, z) = \frac{\lambda p c(z) + (s + \lambda)p}{\delta((\bar{p} + zp)B^*(s + \alpha + b(z)) - z)}, \] (27)
\[ \beta(s, z) = \frac{s\tilde{V}_0(s) + \tilde{V}(s, z)b(z) - 1][1 - pB^*(s + \alpha + b(z))]}{\delta((\bar{p} + zp)B^*(s + \alpha + b(z)) - z)}, \] (28)

and \( \varpi \) is the root of the equation \((\bar{p} + zp)B^*(s + \alpha + b(z)) = z \) inside \( |z| = 1 \), \( \text{Re}(s) > 0 \).

Further, by mathematical induction and from (19), we can easily prove that \( \tilde{V}_n(s), n = 1, \ldots, N - 1, \) satisfy the following relationship
\[ \tilde{V}_n(s) = \frac{\lambda}{s + \lambda} \tilde{V}_0(s)\Pi_n, \quad n = 1, \ldots, N - 1, \]
where \( \Pi_n \) satisfy the following recursive equation
\[ \begin{cases}
\Pi_n &= c_n + \frac{\lambda}{s + \lambda} \sum_{k=1}^{n-1} c_k \Pi_{n-k}, \quad n = 2, 3, \ldots, N - 1 \\
\Pi_1 &= c_1.
\end{cases} \]

So that
\[ \tilde{V}(s, z) = \left\{ \frac{\lambda}{s + \lambda} \sum_{n=1}^{N-1} z^n \Pi_n + 1 \right\} \tilde{V}_0(s). \] (29)

As the arrival stream is a Poisson process, we have
\[ \tilde{V}_0(s) = \int_0^\infty e^{-sx} e^{-\lambda x} dx = \frac{1}{s + \lambda}. \] (30)

Substituting (30) into (29) yields
\[ \tilde{V}(s, z) = \frac{\lambda}{(s + \lambda)^2} \sum_{n=1}^{N-1} z^n \Pi_n + \frac{1}{s + \lambda}. \] (31)

Inserting (30) and (31) into (28) gives
\[ \beta(s, z) = \frac{\left\{ \frac{s}{s + \lambda} + \left[ \frac{\lambda}{(s + \lambda)^2} \sum_{n=1}^{N-1} z^n \Pi_n + \frac{1}{s + \lambda} \right] b(z) - 1 \right\} [1 - pB^*(s + \alpha + b(z))]}{\delta((\bar{p} + zp)B^*(s + \alpha + b(z)) - z)}. \]

From (23) and (24), we get
\[ \tilde{B}(s, z) = \int_0^\infty \tilde{B}(s, x, z) dx 
= \left\{ \delta \beta(s, z) + \left[ \frac{\lambda}{z} - \delta \alpha(s, z) \right] \tilde{I}(s, z) \right\} \frac{\tilde{B}(s + \alpha + b(z))}{1 - pB^*(s + \alpha + b(z))}, \] (32)

So we summarize our results in the following theorem.
Theorem 5.1. The Laplace transform of $\zeta(t)$ is given by

$$\tilde{\zeta}(s) = \int_{\infty}^{1} \beta(s, x) \exp \left\{ - \int_{x}^{1} \alpha(s, u) du \right\} dx + \frac{\lambda}{(s + \lambda)^2} \sum_{n=1}^{N-1} \Pi_n + \frac{1}{s + \lambda}$$

$$+ \left\{ \delta \beta(s, 1) + [\lambda - \delta \alpha(s, 1)] \int_{\infty}^{1} \beta(s, x) \exp \left\{ - \int_{x}^{1} \alpha(s, u) du \right\} dx \right\} \frac{\tilde{B}(s + \alpha)}{1 - pB^*(s + \alpha)}.$$ (33)

where $\varpi$ is the root of the equation $(\bar{p} + zp)B^*(s + \alpha + b(z)) = z$ inside $|z| = 1$, $\text{Re}(s) > 0$.

Proof. From (26), (31) and (32), we obtain

$$\tilde{I}(s, 1) = \int_{\infty}^{1} \beta(s, x) \exp \left\{ - \int_{x}^{1} \alpha(s, u) du \right\} dx,$$

$$\tilde{V}(s, 1) = \frac{\lambda}{(s + \lambda)^2} \sum_{n=1}^{N-1} \Pi_n + \frac{1}{s + \lambda},$$

$$\tilde{B}(s, 1) = \left\{ \delta \beta(s, 1) + [\lambda - \delta \alpha(s, 1)] \tilde{I}(s, 1) \right\} \frac{\tilde{B}(s + \alpha)}{1 - pB^*(s + \alpha)}.$$

Hence we have

$$\tilde{\zeta}(s) = \tilde{I}(s, 1) + \tilde{V}(s, 1) + \tilde{B}(s, 1).$$

By direct calculation we can obtain (33). $\square$

From Theorem 5.1 we have

Corollary 7. The mean time to the first failure (MTTFF) of the server is given by

$$MTTFF = \tilde{\zeta}(0) =$$

$$\int_{\infty}^{1} \beta(0, x) \exp \left\{ - \int_{x}^{1} \alpha(0, u) du \right\} dx + \frac{1}{\lambda} \sum_{n=1}^{N-1} \Pi_n + \frac{1}{\lambda}$$

$$+ \left\{ \delta \beta(0, 1) + [\lambda - \delta \alpha(0, 1)] \int_{\infty}^{1} \beta(0, x) \exp \left\{ - \int_{x}^{1} \alpha(0, u) du \right\} dx \right\} \frac{\tilde{B}(\alpha)}{1 - pB^*(\alpha)}.$$ (34)

where $\varpi$ is the root of the equation $(\bar{p} + zp)B^*(\alpha + b(z)) = z$ on the interval $[0,1]$.

Proof. From (33) and the following equation

$$MTTFF = \int_{0}^{+\infty} \zeta(t) dt = \tilde{\zeta}(s)|_{s=0},$$

we obtain (34). $\square$
6. Application to cellular mobile networks. In this section, we give an application for the model under discussion and some numerical results as follows. Retrial phenomenon naturally arises in various systems such as call centers, cellular mobile networks and computer communication networks. For a comprehensive survey on theory and applications of retrial queues in these systems the reader is referred to the survey paper by Phung-Duc [22]. Cellular mobile networks are governed by numerous base stations having a finite number of channels, each of them with a specific influence zone which is called cell. Here, we consider a single cell of a cellular mobile network that consists of one base station and one channel providing exponential distributed service. We assume that the calls arrive at the channel according to a compound Poisson process with parameter $\lambda$. A originating call is blocked if it finds the channel busy. Any blocked originating call goes to a buffer named orbit and retries for service after an exponentially distributed time with parameter $\delta$. Furthermore, the channel is unreliable when it is serving a call. Sometimes the channel gets attacked by a virus or the server is breakdown and the calls in the system (including the call in the channel and the calls in the buffer) are lost forever. We assume that the virus arrival rate is $\alpha$. When the server fails it is sent to repair immediately and the calls in the orbit may be getting impatient. The impatient times are exponentially distributed with parameter $\theta$. Then the channel can be modeled as a batch arrival single server retrial queue with impatient customers subject to the server disasters.

For the purpose of a numerical illustration, we assume that all distribution functions involved are exponential, i.e. $B(x)$ and $R(x)$ are exponential distribution functions with rates $1/B_1$ and $1/R_1$, respectively. Here we choose the following arbitrary values: $1/B_1 = 5$, $1/R_1 = 0.5$, $\lambda = 1$ and $c(z) = \frac{x}{2-x}$. Of course, in all the below cases, the parametric values are chosen under the stability condition. Numerical results are presented in Figures 1-4.

In Figures 1-4, the mean number of customers in the system $L_s$ is plotted against the parameters $p$, $\delta$, $\theta$ and $\alpha$, respectively. In each of the pictures, we have presented three curves which correspond to $N = 2$, 5 and 10. Figure 1 displays the dependence
Figure 2. The mean system size versus $\delta$ with $(\alpha, p, \theta) = (0.1, 0.1, 0.7)$.

Figure 3. The mean system size versus $\theta$ with $(\alpha, p, \delta) = (0.1, 0.1, 3)$.

of the mean number of customers in the system $L_s$ on the feedback probability $p$ when the other parameters are fixed. As is to be expected, the mean number of customers in the system increases monotonously as the feedback probability $p$ increases. Moreover, we find that when the feedback probability $p$ gets larger, the mean number of customers in the system gets infinite. That is because the system becomes unstable. From Figure 2, we observe that the mean number of customers in the system $L_s$ decreases monotonously as the value $\delta$ increases. It should be pointed out that when $\delta$ is large, the curves are almost parallel to $x$-axis. That is to say, the retrial parameter $\delta$ hardly affects the mean number of customers in the system when the retrial rate is large. This confirms the asymptotic behavior under high rate of retrials. In Figure 3, we observe that the mean number of customers in the
system decreases monotonously as the value \( \theta \) increases, which agrees with intuitive expectations. In Figure 4, it is easy to see that the mean number of customers in the system increases monotonously as the value \( \alpha \) increases. It seems to be not in conformity with the reality. However, this phenomenon can be explained taking into account the fact that the catastrophe arrivals not only remove all the customer in the system but also cause the server breakdown.

7. Conclusion. In this paper we have analyzed a single server retrial queue with impatient customers and \( N \)-policy vacation subject to the server breakdowns and repairs. Essentially, a breakdown is represented by a catastrophe occurring on the server which removes all the customers in the system. We have studied the system with the help of embedded Markov chains. The necessary and sufficient condition for the system stability is discussed by using Lyapunov functions. The joint distribution of the server’s state, the number of customers in the orbit in steady-state is derived by applying the characteristic method for partial differential equations. Analytical expressions for various performance measures of interest and the reliability of the server are investigated. An application for the model under discussion is provided and numerical examples have carried out to observe the influence of the main parametric values.

Appendix. In this appendix we are going to prove Theorem 4.1.

Proof. Proceeding in the usual manner with (1), we get

\[
B(x, z) = B(0, z) \exp\{-(\alpha + b(z))x\} \bar{B}(x).
\]  

(35)

Multiplying (6) by \( z^n \), summing up from \( n = 0 \) to \( \infty \) and using (35), we have

\[
B(0, z) = \frac{\delta I'(z) + \lambda I(z) \frac{c(z)}{z}}{1 - pB^*(\alpha + b(z))}.
\]

(36)
Substituting (36) into (35) yields
\[ B(x, z) = \frac{\delta I'(z) + \lambda I(z) e^{cz}}{1 - pB^*(\alpha + b(z))} \exp\{-(\alpha + b(z))x\} \bar{B}(x). \] (37)

Multiplying (2) by \( z^n \) and summing up from \( n = 0 \) to \( \infty \), we obtain
\[ \frac{\partial R(x, z)}{\partial x} - \theta (1 - z) \frac{\partial R(x, z)}{\partial z} = -(b(z) + \gamma(x)) R(x, z). \] (38)

Thus the set (2) has been reduced to a partial differential equation which we proceed to solve. Let \( F(x, z) = \ln R(x, z) \), \( f(z) = F(0, z) = \ln R(0, z) \). As a result, (38) is reduced to
\[ \frac{\partial F(x, z)}{\partial x} - \theta (1 - z) \frac{\partial F(x, z)}{\partial z} = -(b(z) + \gamma(x)) \] (39)
with the initial condition \( F(0, z) = f(z) \).

To solve (39), we consider the associated ordinary differential equations
\[ \frac{dx}{1} = \frac{dz}{-\theta (1 - z)} = \frac{dF(x, z)}{(b(z) + \gamma(x))}. \] (40)

The general solution of (40) is \( \psi_2 = G(\psi_1) \), where \( G(\cdot) \) is an arbitrary function and \( \psi_1, \psi_2 \) are two independent solutions of the first and second equations of (40). The first equation of (40) gives
\[ \psi_1 = x - \ln(1 - z) \theta \]
and the second equation of (40) yields
\[ \psi_2 = F(x, z) - \frac{1}{\theta} \int_0^z b(u) + \gamma \left( \psi_1 + \frac{\ln(1-u)}{\theta} \right) du. \]

Thus \( \psi_2 = G(\psi_1) \) provides
\[ F(x, z) - \frac{1}{\theta} \int_0^z b(u) + \gamma \left( \psi_1 + \frac{\ln(1-u)}{\theta} \right) du = G \left( x - \frac{\ln(1 - z)}{\theta} \right). \] (41)

Putting \( x = 0 \) in (41) and using the initial condition \( F(0, z) = f(z) \), we obtain
\[ f(z) - \frac{1}{\theta} \int_0^z b(u) + \gamma \left( \frac{\ln(1-u)}{\theta} - \frac{\ln(1-z)}{\theta} \right) du = G \left( -\frac{\ln(1 - z)}{\theta} \right). \]

Hence, we have
\[ G(z) = f(1 - e^{-\theta z}) - \frac{1}{\theta} \int_0^{1-e^{-\theta z}} b(u) + \gamma \left( \frac{\ln((1-u)e^{\theta z})}{\theta} \right) du. \] (42)

Eliminating the arbitrary function \( G(\cdot) \) from (41) and (42), we get
\[ F(x, z) - \frac{1}{\theta} \int_0^z b(u) + \gamma \left( x + \frac{\ln(1-z)}{\theta} \right) du = f(\nu), \]
Therefore, the solution of equation (38) is
\[
R(x, z) = R(0, \nu) \exp \left\{ \frac{1}{\theta} \int_{\nu}^{z} b(u) + \gamma \left( x + \frac{\ln(1 + u)}{\nu} \right) \frac{1}{1 - u} \, du \right\}. \quad (43)
\]

From (7), we obtain
\[
R(0, z) = R_0(0) = \alpha B(0, 1) \tilde{B}(\alpha) = [\delta I'(1) + \lambda I(1)] \frac{\alpha \tilde{B}(\alpha)}{1 - \rho^* \tilde{\alpha}}. \quad (44)
\]

Substituting (44) into (43) yields
\[
R(x, z) = [\delta I'(1) + \lambda I(1)] \frac{\alpha \tilde{B}(\alpha)}{1 - \rho^* \tilde{\alpha}} \exp \left\{ \frac{1}{\theta} \int_{\nu}^{z} b(u) + \gamma \left( x + \frac{\ln(1 + u)}{\nu} \right) \frac{1}{1 - u} \, du \right\}. \quad (45)
\]

Multiplying (4) and (5) by \( z^n \) and then taking summation over all possible values of \( n \geq 1 \) and utilizing (3) by noting that \( \lambda \sum_{n=0}^{N-1} \frac{z^n}{z} \sum_{k=0}^{N-1} V_k c_{n-k} = \lambda V_0 - V(z)b(z) \), we get
\[
\lambda I(z) + \delta z I'(z) = \tilde{p}B(0, z) B^*(\alpha + b(z)) + R(0, \nu)\xi(z) - V(z)b(z). \quad (46)
\]

Substituting (36) and (44) into (46) yields
\[
\delta \{ (\tilde{p} + z\rho) B^*(\alpha + b(z)) - z \} I'(z) + \lambda \left\{ \left[ \frac{\rho c(z)}{z} + p \right] B^*(\alpha + b(z)) - 1 \right\} I(z) \nonumber \\
= \left\{ V(z)b(z) - [\delta I'(1) + \lambda I(1)] \frac{\alpha \tilde{B}(\alpha)\xi(z)}{1 - \rho^* \tilde{\alpha}} \right\} [1 - \rho^* \tilde{\alpha} (\alpha + b(z))]. \quad (47)
\]

Now, we consider the coefficient \( f(z) = (\tilde{p} + z\rho) B^*(\alpha + b(z)) - z \).

It is easy to see that

(i) \( f(0) = \tilde{p} B^*(\alpha + \lambda) > 0 \),

(ii) \( f(1) = B^*(\alpha) - 1 \leq 0 \),

(iii) \( f'(z) = (2p\lambda c'(z) + (\tilde{p} + z\rho)\lambda c''(z)) B^*(\alpha + b(z)) + (\tilde{p} + z\rho)[\lambda c'(z)]^2 B^{**}(\alpha + b(z)) \geq 0 \).

Therefore \( f(z) \) has exactly one root \( \omega \) in the interval \([0, 1]\). Next, we analyze the solution as two cases.

Case 1: \( \alpha > 0 \).

In this case, we see that \( 0 < \omega < 1 \). If \( 0 \leq z < \omega \), then \( f(z) \) is strictly positive. Hence the general solution of (47) on the interval \( 0 \leq z < \omega \) is
\[
I(z) = \exp \left\{ - \int_{0}^{z} \alpha(x) \, dx \right\} \left[ I(0) + \int_{0}^{z} \beta(x) \exp \left\{ \int_{0}^{x} \alpha(u) \, du \right\} \, dx \right], \quad 0 \leq z < \omega,
\]
\[
\beta(z) = \left\{ V(z)b(z) - [\delta I'(1) + \lambda I(1)] \frac{\alpha \tilde{B}(\alpha)\xi(z)}{1 - \rho^* \tilde{\alpha}} \right\} [1 - \rho^* \tilde{\alpha} (\alpha + b(z)) - z]. \quad (48)
\]
We note that when $x \to \omega$, the function $\alpha(x) \to -\infty$. Then the integral $\int_0^\omega \alpha(x)dx$ is divergent and so the function $\exp\{-\int_0^z \alpha(x)dx\} \to +\infty$ as $z \to \omega$. On the other hand, $I(z) < \infty$.

So that

$$I(0) = -\int_0^\omega \beta(x) \exp\left\{\int_0^x \alpha(u)du\right\}dx.$$  \hspace{2cm} (49)

Since $I(0) = 0$, (48) is reduced to

$$I(z) = \int_z^\omega \beta(x) \exp\left\{-\int_x^z \alpha(u)du\right\}dx, \quad 0 \leq z < \omega.$$  \hspace{2cm} (50)

Next, we consider the case $\omega < z \leq 1$. When $\omega < z \leq 1$ the function $f(z)$ is strictly negative and so

$$I(z) = \exp\left\{-\int_1^z \alpha(x)dx\right\} \left[I(1) + \int_1^z \beta(x) \exp\left\{\int_1^x \alpha(u)du\right\}dx\right], \quad \omega < z \leq 1.$$  \hspace{2cm} (51)

In the same manner as above, we can determine the constant $I(1)$:

$$I(1) = \int_1^\omega \beta(x) \exp\left\{-\int_x^1 \alpha(u)du\right\}dx,$$  \hspace{2cm} (52)

and therefore (51) is reduced to

$$I(z) = \int_z^\omega \beta(x) \exp\left\{-\int_x^z \alpha(u)du\right\}dx, \quad \omega < z \leq 1.$$  \hspace{2cm} (53)

With the help of (49), (50) and (53) can be written as the joint formula

$$I(z) = \int_z^\omega \beta(x) \exp\left\{-\int_x^z \alpha(u)du\right\}dx, \quad z \neq \omega.$$  \hspace{2cm} (54)

For $z = \omega$, we have directly from (47) that

$$I(\omega) = \frac{V(\omega)b(\omega) - \left[\delta I'(1) + \lambda I(1)\right] \frac{\alpha^\omega_0(\omega) + \lambda \alpha^\omega_0(\omega)}{1 - pB^\omega(\alpha + b(\omega))}}{\lambda \left(\frac{\alpha^\omega_0(\omega) + \lambda \alpha^\omega_0(\omega)}{1 - pB^\omega(\alpha + b(\omega)) - 1}\right)}.$$  \hspace{2cm} (55)

Further, by mathematical induction, we can easily get that $V_n, n = 0, 1, \cdots, N - 1$ satisfy the following relationship

$$V_n = V_0 \pi_n, \quad n = 0, 1, \cdots, N - 1.$$  \hspace{2cm} (56)

So that

$$V(z) = V_0 \left[\sum_{n=0}^{N-1} z^n \pi_n\right].$$  \hspace{2cm} (57)

From (49), we get

$$V_0 = [\delta I'(1) + \lambda I(1)]W.$$  \hspace{2cm} (58)

From (52), we have

$$\delta I'(1) + \lambda I(1) = \frac{V_0 A - I(1)}{B}.$$  \hspace{2cm} (59)

Combining (55) and (56), we obtain

$$V_0 = \frac{I(1)W}{WA - B},$$  \hspace{2cm} (60)

$$\delta I'(1) + \lambda I(1) = \frac{I(1)}{WA - B}.$$  \hspace{2cm} (61)
Substituting (57) into (54), (58) into (37) and (45), and taking \( z = 1 \), we arrive at
\[
V(1) = I(1)\Phi,
\]
\[
\mathfrak{B}(1) = \int_0^{\infty} B(x, 1)dx = I(1)\Psi,
\]
\[
\mathfrak{R}(1) = \int_0^{\infty} R(x, 1)dx = I(1)\Psi\alpha\varphi(1).
\]
From the normalization condition (8), we obtain
\[
I(1) + V(1) + \mathfrak{B}(1) + \mathfrak{R}(1) = 1.
\]
Finally, substituting (59)-(61) into (62), we can find the unknown constant \( I(1) \):
\[
I(1) = \frac{1}{1 + \Phi + \Psi[1 + \alpha\varphi(1)]}.
\]

Case 2: \( \alpha = 0 \).

In this special case, it is easy to see that \( f(z) \) has exactly one root \( \omega = 1 \) on the interval \([0,1]\). Then \( f(z) \) is strictly positive on the interval \([0,1)\).

From (47), we get
\[
I'(z) + \alpha_0(z)I(z) = \beta_0(z).
\]
Note that
\[
\lim_{z \to 1-0} \alpha_0(z) = \frac{\lambda}{\delta} \cdot \frac{\bar{p}(C_1 - 1) + \lambda C_1 B_1}{\lambda C_1 B_1 - \bar{p}} < \infty,
\]
\[
\lim_{z \to 1-0} \beta_0(z) = V_0 \left[ \sum_{n=0}^{N-1} \pi_n \right] \frac{\bar{p}}{\delta} \cdot \frac{\lambda C_1}{\bar{p} - \lambda C_1 B_1} < \infty.
\]
Thus (63) can be considered for \( z \in [0,1] \).

Now solving the first-order differential equation (63), we get
\[
I(z) = \int_0^z \beta_0(x) \exp \left\{ -\int_x^z \alpha_0(u)du \right\} dx, \quad 0 \leq z \leq 1.
\]
Taking \( z = 1 \) in (64) yields
\[
V(1) = I(1)\Phi_0,
\]
\[
\mathfrak{B}(1) = \int_0^{\infty} B(x, 1)dx = \frac{B_1}{\bar{p}} \left\{ \delta\beta_0(1) + [\lambda - \delta\alpha_0(1)]I(1) \right\}.
\]
From the normalization condition (8), we can find the unknown constant \( I(1) \):
\[
I(1) = \frac{\bar{p} - \delta\beta_0(1)B_1}{\bar{p} + \bar{p}\Phi_0 + [\lambda - \delta\alpha_0(1)]B_1} = \frac{\bar{p} - \lambda C_1 B_1}{(1 + \Phi_0)\bar{p}}.
\]
From (65) and (66), we obtain \( I(1) + V(1) = 1 - \frac{\lambda C_1 B_1}{\bar{p}} \) as the probability that the server is idle or in a dormant period and \( \mathfrak{B}(1) = \frac{\lambda C_1 B_1}{\bar{p}} \) as the probability that the server is busy.

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