A CHARACTERISTIC CLASS OF Homeo(X)₀-BUNDLES AND AN ABELIAN EXTENSION OF THE HOMEOMORPHISM GROUP

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Abstract. A Homeo(X)₀-bundle is a fiber bundle with fiber X whose structure group reduces to the identity component Homeo(X)₀ of the homeomorphism group of X. We construct a characteristic class of Homeo(X)₀-bundles as a third cohomology class with coefficients in Z. We also investigate the relation between the universal characteristic class of flat fiber bundles and the gauge group extension of the homeomorphism group. Furthermore, under some assumptions, we construct and study the central S¹-extension and the corresponding group two-cocycle of Homeo(X)₀.

1. Introduction and Main Theorem

1.1. Introduction. Let X be a topological space and c ∈ H¹(X; Z) a non-trivial cohomology class of X. Let Homeo(X) denote the homeomorphism group of X with the compact-open topology and G = Homeo(X)₀ its identity component. Let Gδ denote the group G with the discrete topology. A fiber bundle X → E → B is called a G-bundle if the structure group reduces to G. A G-bundle is called flat if the structure group reduces to the discrete group Gδ. In [2], a characteristic class e_c(E) ∈ H²(B; Z) of the G-bundle is defined if c is in the first cohomology group H¹(X; Z). The universal characteristic class e_c is defined as an element in the second cohomology H²(BG; Z) of the classifying space BG with coefficients in Z. Let

(1.1) t*: H*(BG; Z) → H*(BGδ; Z)

be the map induced from the canonical map Gδ → G. Then the cohomology class t* e_c ∈ H²(BGδ; Z) gives the universal characteristic class of flat G-bundles. Since the cohomology H*(BGδ; Z) of the classifying space of the discrete group is isomorphic to the group cohomology H*grp(G; Z), the universal characteristic class t* e_c gives the second group cohomology class in H²grp(G; Z). On the other hand, the second group cohomology H²grp(G; Z) is isomorphic to the equivalence classes of central Z-extensions of G. In [2], Fujitani constructed a central Z-extension of G that corresponds to the characteristic class t* e_c. The central extension is given as a homeomorphism group of the regular Z-covering of X.

* In this paper, we deal with topological spaces whose homeomorphism groups are topological groups with respect to the compact-open topology. For example, the homeomorphism group of locally compact, locally connected topological space is a topological group.
In the present paper, we consider the case when the first cohomology $H^1(X; \mathbb{Z})$ is trivial and the second cohomology $H^2(X; \mathbb{Z})$ is non-trivial. In this case, the characteristic class $e_c$ above cannot be defined since $H^1(X; \mathbb{Z}) = 0$. On the other hand, we can define a characteristic class $e_c(E)$ in $H^3(B; \mathbb{Z})$ by using a second cohomology class $c \in H^2(X; \mathbb{Z})$. More precisely, the class $e_c$ is obtained as the transgression image of the class $c$ by the Serre spectral sequence of the $G$-bundle $X \to E \to B$ (see section 2). Let $e_c \in H^3(BG; \mathbb{Z})$ denote the universal characteristic class. By the canonical map (1.1) and the isomorphism $H^3(BG; \mathbb{Z}) \cong H^3_{\text{grp}}(G; \mathbb{Z})$, we obtain the third group cohomology class $\iota^*e_c$ in $H^3_{\text{grp}}(G; \mathbb{Z})$.

1.2. Main results. In this paper, we regard the circle $S^1$ as the quotient group $\mathbb{R}/\mathbb{Z}$. Take a cohomology class $c \in H^2(X; \mathbb{Z})$. Let $P \to X$ be a principal $S^1$-bundle such that the first Chern class is equal to $c$. Then there is the following exact sequence

\[(1.2) \quad 0 \to \text{Gau}(P) \to A_G(P) \to G \to 1,\]

where the group $\text{Gau}(P)$ is the gauge group of $P$ and $A_G(P)$ is the bundle automorphisms that cover elements in $G$. Since the group $S^1$ is abelian, the gauge group is also abelian. So the exact sequence (1.2) is an abelian extension. In general, an abelian extension $1 \to A \to \Gamma \to G \to 1$ determines a second group cohomology class $e(\Gamma) \in H^2_{\text{grp}}(G; A)$. Thus the abelian extension (1.2) defines a group cohomology class $e(A_G(P))$ in $H^2_{\text{grp}}(G; \text{Gau}(P))$.

Here we obtain two group cohomology classes $e(A_G(P)) \in H^2_{\text{grp}}(G; \text{Gau}(P))$ and the universal characteristic class $\iota^*e_c \in H^3_{\text{grp}}(G; \mathbb{Z})$. The following theorem describes the relation between these two classes.

**Theorem 1.1.** Let $\delta : H^2_{\text{grp}}(G; \text{Gau}(P)) \to H^3_{\text{grp}}(G; \mathbb{Z})$ denote the connecting homomorphism (defined below). Then the following holds;

$\delta e(\text{Aut}(P)) = \iota^*e_c$.

The connecting homomorphism $\delta$ is given as follows. Since the group $S^1$ is abelian, the gauge group $\text{Gau}(P)$ is isomorphic to the group

$C(X, S^1) = \{ f : X \to S^1 : \text{continuous} \}$.

Let $C(X, \mathbb{R})$ denote the group of all continuous maps from $X$ to $\mathbb{R}$. By the assumption $H^1(X; \mathbb{Z}) = 0$ and the isomorphism $H^1(X; \mathbb{Z}) \cong \text{Hom}(\pi_1(X); \mathbb{Z})$, all elements in $C(X, S^1)$ can be lifted to elements in $C(X, \mathbb{R})$. So we have the following exact sequence

\[(1.3) \quad 0 \to \mathbb{Z} \to C(X, \mathbb{R}) \to C(X, S^1) \cong \text{Gau}(P) \to 0.\]

By the exact sequence (1.3) and the identification $\text{Gau}(P) \cong C(X, S^1)$, we have the connecting homomorphism

$\delta : H^2_{\text{grp}}(G; \text{Gau}(P)) \to H^3_{\text{grp}}(G; \mathbb{Z})$.

This is the map in Theorem 1.1.
From here we assume that the cohomology class $c \in H^2(X; \mathbb{Z})$ is equal to zero in $H^2(X; \mathbb{R})$. Let $\delta : H^2_{\text{grp}}(G; S^1) \to H^3_{\text{grp}}(G; \mathbb{Z})$ be the connecting homomorphism with respect to the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$ of coefficients. By the cohomology long exact sequence

$$\cdots \to H^1(X; S^1) \xrightarrow{\delta} H^2(X; \mathbb{Z}) \to H^2(X; \mathbb{R}) \to \cdots,$$

we take a cohomology class $\rho \in H^1(X; S^1)$ such that $\delta(\rho) = c$. Then we can construct a central $S^1$-extension

$$(1.4) \quad 1 \to S^1 \to A_G(P^\delta_\rho) \to G \to 1$$

by using the principal $(S^1)^\delta$-bundle $P^\delta_\rho$ over $X$ (see section 5). This central $S^1$-extension determines a second group cohomology class $e(A_G(P^\delta_\rho))$ in $H^2_{\text{grp}}(G; S^1)$. For this class $e(A_G(P^\delta_\rho))$, we have the following theorem, that is analogous to Theorem 1.1.

**Theorem 1.2.** Let $\delta : H^2_{\text{grp}}(G; S^1) \to H^3_{\text{grp}}(G; \mathbb{Z})$ be the connecting homomorphism with respect to the exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$ of coefficients. Then the following holds;

$$\delta e(A_G(P^\delta_\rho)) = \iota^* e_c.$$

Moreover, we construct a group two-cocycle $\mathfrak{G}_{x,a} \in C^2_{\text{grp}}(G; S^1)$ that represents the class $e(A_G(P^\delta_\rho))$ (for definition, see (5.5) in section 5).

**2. Construction of the characteristic class**

Let $X$ be a connected topological space satisfying $H^1(X; \mathbb{Z}) = 0$. Take a non-zero cohomology class $c$ in $H^2(X; \mathbb{Z})$. Let $X \to E \to B$ denote a $G$-bundle with connected base space $B$. Let us consider the Serre spectral sequence $E_r^{p,q}$ of the bundle $X \to E \to B$. Since the structure group $G$ is connected, the local system $H^*(X; \mathbb{Z})$ is trivial. By the assumption $H^1(X; \mathbb{Z}) = 0$, we have

$$E_2^{0,2} = E_2^{0,2} = H^0(B; H^2(X; \mathbb{Z})) = H^2(X; \mathbb{Z})$$

and

$$E_3^{3,0} = E_2^{3,0} = H^3(B; H^0(X; \mathbb{Z})) = H^3(B; \mathbb{Z}).$$

By the transgression map

$$d_3^{0,2} : H^2(X; \mathbb{Z}) = E_2^{0,2} \to E_3^{3,0} = H^3(B; \mathbb{Z}),$$

we obtain the cohomology class $d_3^{0,2} c$ in $H^3(B; \mathbb{Z})$.

**Definition 2.1.** For a $G$-bundle $X \to E \to B$ above, we put

$$e_c(E) = -d_3^{0,2} c \in H^3(B; \mathbb{Z}).$$

By the naturality of the Serre spectral sequence, the cohomology class $d_3^{0,2} c$ has also the naturality with respect to bundle maps. Thus the class $e_c(E)$ gives rise to a characteristic class of $G$-bundles. Let $e_c \in H^3(BG; \mathbb{Z})$ denote the universal characteristic class of $G$-bundles. By the canonical map $\iota^* : H^3(BG; \mathbb{Z}) \to H^3(BG^\delta; \mathbb{Z})$, we obtain the universal characteristic class $\iota^* e_c$ of flat $G$-bundles.
In section 4, we give examples of $G$-bundle that the characteristic class $e_c$ and $\nu^*e_c$ are non-zero.

Remark 2.2. Let $\text{Homeo}(X, c)$ denote the group of homeomorphisms that preserve the cohomology class $c$. Let $X \to E \to B$ be a fiber bundle with the structure group $\text{Homeo}(X, c)$ and $E^{p,q}_r$ the Serre spectral sequence of the bundle. Then we have

$$E^{0,2}_3 = E^0_2 = H^0(B; H^2(X; \mathbb{Z})) = H^2(X; \mathbb{Z})^{\pi_1(B)},$$

where $H^2(X; \mathbb{Z})^{\pi_1(B)}$ is the $\pi_1(B)$-invariant cohomology classes. Since the cohomology class $c$ is in $E^{0,2}_3 = H^2(X; \mathbb{Z})^{\pi_1(B)}$, a characteristic class of $\text{Homeo}(X, c)$-bundles can be defined in the same way.

3. Preliminaries

3.1. Group cohomology and Hochschild-Serre spectral sequence. Let $G$ be a group and $A$ a right $G$-module. The group $p$-cochain $c : G^p \to A$ is a function from $p$-fold product $G^p$ to $A$. Let $C^p(G; A)$ denote the set of all group $p$-cochains. The coboundary map $\delta : C^p_G(G; A) \to C^{p+1}_G(G; A)$ is defined by

$$\delta c(g_1, \ldots, g_{p+1}) = c(g_2, \ldots, g_{p+1}) + \sum_{i=1}^{p} (-1)^i c(g_1, \ldots, g_i g_{i+1}, \ldots, g_{p+1})$$

$$+ (-1)^{p+1} c(g_1, \ldots, g_p) : g_{p+1}$$

for $p > 0$ and $\delta = 0$ for $p = 0$. The cohomology of the cochain complex $(C^*_{\text{grp}}(G; A), \delta)$ is called the group cohomology of $G$ with coefficients in $A$ and denoted by $H^*_{\text{grp}}(G; A)$.

For a group $G$, the group cohomology $H^*_G(G; A)$ is isomorphic to the singular cohomology $H^*(BG^\delta; A)$ of the classifying space $BG^\delta$, where $G^\delta$ denote the group $G$ with discrete topology and $A$ is the local system on $BG^\delta$ (see [1]).

For an exact sequence $1 \to K \to \Gamma \to G \to 1$ of groups, there is the following spectral sequence, which is called the Hochschild-Serre spectral sequence.

Theorem 3.1. [1] There exists the spectral sequence with $E^{p,q}_2 \cong H^p_{\text{grp}}(G; H^q_{\text{grp}}(K; A))$ which converges to $H^*_{\text{grp}}(\Gamma; A)$.

Note that the group $\Gamma$ acts on $H^q_{\text{grp}}(K; A)$ by conjugation and this action induces the $G$-action. By this action, we consider $H^q_{\text{grp}}(K; A)$ as a right $G$-module.

3.2. Abelian extension and second group cohomology. An exact sequence $1 \to A \xrightarrow{\iota} \Gamma \xrightarrow{p} G \to 1$ of groups is called an abelian $A$-extension of $G$ if the group $A$ is abelian. The group $\Gamma$ acts on $A$ by conjugation, that is, for $\gamma \in \Gamma$ and $a \in A$, we put $\gamma \cdot a = \gamma^{-1} a \gamma$. This action induces the right $G$-action on $A$. So we consider the abelian group $A$ as a right $G$-module. It is known that the second group cohomology $H^2_{\text{grp}}(G; A)$ is isomorphic to the equivalence classes of abelian $A$-extensions of $G$ (see [1]). For an abelian $A$-extension $\Gamma$, the corresponding cohomology class $e(\Gamma)$ is defined as follows. Take a section $s : G \to \Gamma$ of the projection $p : \Gamma \to G$. For
any \( g, h \in G \), the value \( s(g)s(h)s(gh)^{-1} \) is in \( i(A) \cong A \). Thus we obtain a group
two-cochain \( c \in C^2_{\text{grp}}(G; A) \) by putting
\[
\tag{3.1}
\]
\[ c(g, h) = s(g)s(h)s(gh)^{-1}. \]
It can be seen that the cochain \( c \) is a cocycle, and its cohomology class \([c]\) does not depend on the section. We put \( e(\Gamma) = [c] \). This cohomology class \( e(\Gamma) \) is the class
that corresponds to the abelian \( A \)-extension \( \Gamma \).

By definition of the derivations of the Hochschild-Serre spectral sequence, the
cohomology class \( e(\Gamma) \) can be described as follows (see, for example, [9]).

Lemma 3.2. Let \( 1 \to A \to \Gamma \xrightarrow{\pi} G \to 1 \) be an abelian \( A \)-extension of \( G \) and
\( E_{p,q}^r \) the Hochschild-Serre spectral sequence of the abelian extension. Then the
corresponding cohomology class \( e(\Gamma) \) is equal to the negative of \( d_2^{0,1}(\text{id}_A) \), where
\[ d_2^{0,1} : H^1_{\text{grp}}(A; A)^G = E_2^{0,1} \to E_2^{2,0} = H^2_{\text{grp}}(G; A) \]
is the derivation of the spectral sequence and \( H^1_{\text{grp}}(A; A)^G \) is the \( G \)-equivariant homomorphisms.

3.3. Cohomology of the gauge group. Since the exact sequence
\[ 0 \to \mathbb{Z} \to C(X, \mathbb{R}) \xrightarrow{\pi} C(X, S^1) \cong \text{Gau}(P) \to 0 \]
is a Serre fibration and the space \( C(X, \mathbb{R}) \) is contractible, the gauge group \( \text{Gau}(P) \) is
the \((\mathbb{Z}, 1)\)-type Eilenberg-MacLane space \( K(\mathbb{Z}, 1) \). So the classifying space \( B \text{Gau}(\mathbb{Z}) \)
is the \((\mathbb{Z}, 2)\)-type Eilenberg-MacLane space and therefore we have
\[ H^1(B \text{Gau}(\mathbb{Z}); \mathbb{Z}) = 0, \quad H^2(B \text{Gau}(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}. \]

Let \( i : \mathbb{R} \to C(X, \mathbb{R}) \) and \( j : S^1 \to \text{Gau}(P) = C(X, S^1) \) be inclusions, and \( \text{ev}^R_x : C(X, \mathbb{R}) \to \mathbb{R} \) and \( \text{ev}_x : C(X, S^1) \to S^1 \) the evaluation maps at \( x \in X \). Then we have the following commuting diagram of fibrations
\[
\begin{array}{ccc}
B\mathbb{Z} & \to & B\mathbb{R} \\
\downarrow & & \downarrow i \\
B\mathbb{Z} & \to & BC(X, \mathbb{R}) & \to & B \text{Gau}(P) \\
\downarrow & & \downarrow \text{ev}^R_x & & \downarrow \text{ev}_x \\
B\mathbb{Z} & \to & BR & \to & BS^1,
\end{array}
\]
where we also use the symbols \( i, j, \text{ev}^R_x, \) and \( \text{ev}_x \) for induced maps between the
classifying spaces due to the abuse of the notation. Since the composition \( j \) and \( \text{ev}_x \)
is equal to the identity, the composition \( \text{ev}^*_x : H^*(BS^1; \mathbb{Z}) \to H^*(B \text{Gau}(P); \mathbb{Z}) \) and
\[ j^* : H^*(B \text{Gau}(P); \mathbb{Z}) \to H^*(BS^1; \mathbb{Z}) \]
is also equal to the identity. Thus, the first Chern class of \( B\mathbb{Z} \to BC(X, \mathbb{R}) \to B \text{Gau}(P) \) is equal to \( \text{ev}^*_x c_1 \) and non-zero, where
\( c_1 \in H^2(BS^1; \mathbb{Z}) \) is the universal first Chern class.

Next we consider the first group cohomology \( H^1_{\text{grp}}(\text{Gau}(P); \mathbb{Z}) = \text{Hom}(\text{Gau}(P); \mathbb{Z}) \).
For a homomorphism \( \psi : \text{Gau}(P) \to \mathbb{Z} \), the composition \( \psi \circ \pi : C(X, \mathbb{R}) \to \mathbb{Z} \) is
also a homomorphism. For any element \( f \in C(X, \mathbb{R}) \), the map \( g = f/2 \in C(X, \mathbb{R}) \)
satisfies $f = g + g$. Thus any homomorphism $\phi : C(X, \mathbb{R}) \to \mathbb{Z}$ must be trivial. Since $\pi$ is surjective, we have $\psi = 0$ and therefore $H^1_{\text{grp}}(\text{Gau}(P); \mathbb{Z}) = 0$.

4. Proofs

For a cohomology class $c \in H^2(X; \mathbb{Z})$, let $\text{Homeo}(X, c)$ denote the group of $c$-preserving homeomorphisms.

Lemma 4.1. Let $P \to X$ be a principal $S^1$-bundle with the first Chern class $c \in H^2(X; \mathbb{Z})$. Let $\text{Aut}(P)$ denote the bundle automorphisms of $P$. Then the canonical projection $p : \text{Aut}(P) \to \text{Homeo}(X)$ gives the surjection $\text{Aut}(P) \to \text{Homeo}(X, c)$. In particular, we have the following exact sequence

$$0 \to \text{Gau}(P) \to \text{Aut}(P) \to \text{Homeo}(X, c) \to 1.$$

Remark 4.2. The exact sequence above gives rise to a Serre fibration with respect to the compact-open topology (see [8]).

Proof of lemma 4.1. For a bundle automorphism $F : P \to P$, the homeomorphism $p(F) : X \to X$ preserves the class $c$ by the naturality of the Chern class. Thus the homeomorphism $p(F)$ is in $\text{Homeo}(X, c)$. For a homeomorphism $f \in \text{Homeo}(X, c)$, let $f^*P \to X$ denote the pullback bundle of $P$ by $f$. Since the first Chern class of $f^*P \to X$ is equal to $c$ and the first Chern class completely determines the isomorphic class of principal $S^1$-bundles, the bundles $f^*P \to X$ and $P \to X$ are isomorphic. Thus we have the following diagram

$$\begin{array}{ccc}
P & \cong & f^*P \\
\downarrow & & \downarrow \\
X & \rightarrow & X \\
\downarrow & & \downarrow \\
f : X & \to & X \\
\end{array}$$

and this gives a bundle automorphism of $P$ that covers the homeomorphism $f$. □

Let $A_G(P)$ denote the group of bundle automorphisms that cover elements in $G$. Then, by lemma 4.1 we have the following exact sequence

$$0 \to \text{Gau}(P) \to A_G(P) \to G \to 1. \quad (4.1)$$

Let $E'_{r,p,q}$ and $E''_{r,p,q}$ denote the Hochschild-Serre spectral sequence of the exact sequence (4.1) with coefficients in $\text{Gau}(P)$ and $\mathbb{Z}$ respectively. Then we have

$$E'_{2,0,1} = H^0_{\text{grp}}(G; H^1_{\text{grp}}(\text{Gau}(P); \text{Gau}(P))) = H^1_{\text{grp}}(\text{Gau}(P); \text{Gau}(P))^G,$$

where the group $H^1_{\text{grp}}(\text{Gau}(P); \text{Gau}(P))^G$ is the $G$-equivariant self-homomorphism on $\text{Gau}(P)$. Note that the $G$-action on $C(X, S^1) = \text{Gau}(P)$ induced from the $A_G(P)$-action is given by the pullback, that is, $\lambda \cdot g = g^*\lambda = \lambda \circ g$ for $\lambda \in C(X, S^1)$ and $g \in G$. Since the first group cohomology $H^1_{\text{grp}}(\text{Gau}(P); \mathbb{Z})$ is trivial, we have

$$E''_{3,0,2} = E'_{2,0,2} = H^0_{\text{grp}}(G; H^2_{\text{grp}}(\text{Gau}(P); \mathbb{Z})) = H^2_{\text{grp}}(\text{Gau}(P); \mathbb{Z})^G,$$

where the group $H^2_{\text{grp}}(\text{Gau}(P); \mathbb{Z})^G$ is the $G$-invariant cohomology classes.
Lemma 4.3. The connecting homomorphism

\[ \delta : H^1_{\text{grp}}(\text{Gau}(P); \text{Gau}(P)) \to H^2_{\text{grp}}(\text{Gau}(P); \mathbb{Z}) \]

with respect to the exact sequence (1.3) of coefficients induces the map

\[ \delta : H^1_{\text{grp}}(\text{Gau}(P); \text{Gau}(P))^G \to H^2_{\text{grp}}(\text{Gau}(P); \mathbb{Z})^G. \]

Proof. Let \([\phi]\) be an element in \(H^1_{\text{grp}}(\text{Gau}(P); \text{Gau}(P))^G\). Note that the cocycle \(\phi\) is a \(G\)-equivariant homomorphism, that is, the following holds

\[ \phi(\lambda \cdot g) = \phi(\lambda) \cdot g \]

for any \(\lambda \in \text{Gau}(P)\) and \(g \in G\). For a point \(x\) in \(X\), let \(s_x(\lambda) : X \to \mathbb{R}\) denote the lift of \(\lambda : X \to S^1\) satisfying \(s_x(\lambda)(x) \in [0,1)\). This map \(s_x\) defines a section \(s_x : \text{Gau}(P) = C(X, S^1) \to C(X, \mathbb{R})\). Then a cocycle \(c_x \in C^2_{\text{grp}}(\text{Gau}(P); \mathbb{Z})\) of \(\delta[\phi] \in H^2_{\text{grp}}(\text{Gau}(P); \mathbb{Z})\) is given by

\[
c_x(\lambda_1, \lambda_2) = \delta(s_x \circ \phi)(\lambda_1, \lambda_2) = s_x(\phi(\lambda_2)) - s_x(\phi(\lambda_1 \lambda_2)) + s_x(\phi(\lambda_1))
\]

for \(\lambda_1, \lambda_2 \in \text{Gau}(P)\). For \(g \in G\), we put \(y = g(x)\). Since \(s_x(\lambda \cdot g) = (s_y(\lambda)) \circ g\) for any \(\lambda \in \text{Gau}(P)\), we have

\[
c_x \cdot g(\lambda_1, \lambda_2) = c_x(\lambda_1, g, \lambda_2 \cdot g) = s_x(\phi(\lambda_2 \cdot g)) - s_x(\phi((\lambda_1 \lambda_2) \cdot g)) + s_x(\phi(\lambda_1 \cdot g)) = s_x(\phi(\lambda_2) \cdot g) - s_x(\phi(\lambda_1 \lambda_2) \cdot g) + s_x(\phi(\lambda_1) \cdot g) = (s_y(\phi(\lambda_2)) - s_y(\phi(\lambda_1 \lambda_2)) + s_y(\phi(\lambda_1))) \circ g = s_y(\phi(\lambda_2)) - s_y(\phi(\lambda_1 \lambda_2)) + s_y(\phi(\lambda_1)) = c_y(\lambda_1, \lambda_2).
\]

Since the image \(\delta[\phi]\) of the connecting homomorphism is independent of the choice of the section \(s_x\), we have

\[
(\delta[\phi]) \cdot g = [c_x \cdot g] = [c_y] = \delta[\phi].
\]

Thus the cohomology class \(\delta[\phi]\) is in \(H^2_{\text{grp}}(\text{Gau}(P); \mathbb{Z})^G\). \(\square\)

Lemma 4.4. The following diagram

\[
\begin{array}{ccc}
H^1_{\text{grp}}(\text{Gau}(P); \text{Gau}(P))^G = E'_{2,0,1} & \xrightarrow{d_{2,0,1}} & H^2_{\text{grp}}(G; \text{Gau}(P)) \\
\downarrow \delta & & \downarrow \delta \\
H^2_{\text{grp}}(\text{Gau}(P); \mathbb{Z})^G = E''_{3,0,2} & \xrightarrow{-d'_{3,0,2}} & H^3_{\text{grp}}(G; \mathbb{Z})
\end{array}
\]

commutes, where the maps \(\delta\) are the connecting homomorphisms and \(d'_{2,0,1}\) and \(d''_{3,0,2}\) are the derivations of the Hochschild-Serre spectral sequences \(E''_{r,p,q}\) and \(E''_{r,p,q}\) above.
Proof. Let \( \phi \) be an element in \( H_1^{gr}(\text{Gau}(P); \text{Gau}(P))^G \) and \( s : G \to \text{A}_G(P) \) a section such that \( s(\text{id}) = \text{id} \). At first, we give a cocycle of the class \( \delta d_{2,1}^{0,1}[\phi] \). Let us define a cochain \( \phi_s : \text{A}_G(P) \to \text{Gau}(P) \) by putting

\[
\phi_s(F) = \phi(F \circ (s(p(F)))^{-1}) \in \text{Gau}(P),
\]

where \( p : \text{A}_G(P) \to G \) is the projection. We can see that the cochain \( \phi_s \) restricts to \( \phi \) on \( \text{Gau}(P) \) and the coboundary \( \delta \phi_s \) defines a cocycle in \( C^2_{\text{grp}}(G; \text{Gau}(P)) \). Thus, by definition of the derivation of Hochschild-Serre spectral sequence, the cocycle of \( d_{2,1}^{0,1}[\phi] \) is given by \( \delta \phi_s \in C^2_{\text{grp}}(G; \text{Gau}(P)) \). Thus, the cocycle of \( \delta d_{2,1}^{0,1}[\phi] \) is given by

\[
\delta(s_x \circ (\delta \phi_s)) \in C^3_{\text{grp}}(G; \mathbb{Z}),
\]

where \( s_x : \text{Gau}(P) \to \mathbb{C}(x, \mathbb{R}) \) is the section defined in Lemma 4.3.

Next, we give a cocycle of the class \( d_{2,0,2}^{0,2}[\phi] \). A cocycle of \( \delta[\phi] \) is given by the coboundary \( c_x = \delta(s_x \circ \phi) \in C^2_{\text{grp}}(\text{Gau}(P); \mathbb{Z}) \). Put

\[
c'_x = \delta(s_x \circ \phi_x) - p^*(s_x \circ (\delta \phi_s)),
\]

then it can be seen that the cochain \( c'_x \) is in \( C^2_{\text{grp}}(\text{A}_G(P); \mathbb{Z}) \) and the restriction of \( c'_x \) on \( \text{Gau}(P) \) is equal to \( c_x \). Moreover, since the coboundary \( \delta c'_x \) is equal to \( -p^*\delta(s_x \circ (\delta \phi_s)) \), the cocycle of \( d_3^{0,2}[\phi] \) is given by

\[
-\delta(s_x \circ (\delta \phi_s)) \in C^3_{\text{grp}}(G; \mathbb{Z}).
\]

By (4.3) and (4.4), we have \( \delta d_2^{0,1}[\phi] = -d_3^{0,2}[\phi] \) and the lemma follows. \( \square \)

Lemma 4.5. Let \( X \to E \to BG \) be the universal \( G \)-bundle and \( B \text{Gau}(P) \to B\text{A}_G(P) \to BG \) the fibration that corresponds to the exact sequence (1.2). Then there is the following commuting diagram

\[
\begin{array}{ccc}
X & \longrightarrow & E \\
\downarrow f & & \downarrow \phi \\
B \text{Gau}(P) & \longrightarrow & B\text{A}_G(P)
\end{array}
\]

where the map \( f : B \text{Gau}(P) \) is the composition of the classifying map \( X \to BS^1 \) and the map \( BS^1 \to B \text{Gau}(P) \) induced from the inclusion \( S^1 \to \text{Gau}(P) \).

Proof. Let \( P_G = P \times_{\text{Gau}(P)} \text{Gau}(P) \to X \) denote the associated bundle. Then the bundle

\[
E = E \text{G} \times \text{A}_G(P) \to BG
\]

gives one of a model of the universal \( G \)-bundle with fiber \( X \). Since \( E \text{G} \times P_G \to E \) is a principal \( \text{A}_G(P) \)-bundle, there is a bundle map to the universal principal \( \text{A}_G(P) \)-bundle

\[
E \text{G} \times P_G \longrightarrow E \text{A}_G(P) \quad \psi
\]

\[
E \longrightarrow B\text{A}_G(P) \quad \psi
\]
Define the map $\Phi : EG \times P_G \to EG \times E\text{Aut}(P)$ by $\Phi(a, p) = (a, \Psi(a, p))$. Then the map $\Phi$ gives a bundle map from $EG \times P_G \to E$ to $EG \times E\text{Aut}(P) \to EG \times \text{Aut}(P)$ $E\text{Aut}(P) = B\text{Aut}(P)$. Let $\phi : E \to B\text{Aut}(P)$ denote the classifying map that is covered by $\Phi$. Then it can be seen that the map $\phi$ covers the identity on $BG$. Moreover, the restriction $f : X \to B\text{Gau}(P)$ to the fiber gives rise to the classifying map of the bundle $P_G \to M$.

**Proof of Theorem 1.1.** Take a commuting diagram (4.5). Let us consider the Serre map of the bundle $f$.

Moreover, the restriction $f : X \to B\text{Gau}(P)$ to the fiber gives rise to the classifying map of the bundle $P_G \to M$.

Let $\Phi : X \to BG$ be the classifying map that is.

Define the map $\Phi : EG \times P_G \to EG \times E\text{Aut}(P)$ by $\Phi(a, p) = (a, \Psi(a, p))$. Then the map $\Phi$ gives a bundle map from $EG \times P_G \to E$ to $EG \times E\text{Aut}(P) \to EG \times \text{Aut}(P)$ $E\text{Aut}(P) = B\text{Aut}(P)$. Let $\phi : E \to B\text{Aut}(P)$ denote the classifying map that is covered by $\Phi$. Then it can be seen that the map $\phi$ covers the identity on $BG$. Moreover, the restriction $f : X \to B\text{Gau}(P)$ to the fiber gives rise to the classifying map of the bundle $P_G \to M$.

**Proof of Theorem 1.1.** Take a commuting diagram (4.5). Let us consider the Serre spectral sequences $E^r_{p,q}$ and $E''''_{p,q}$ of the fibrations $X \to E \to BG$ and $B\text{Gau}(P) \to B\text{Aut}(P) \to BG$ respectively. Since $H^1(B\text{Gau}(P); \mathbb{Z}) = 0$, we have $E^r_{3,0,2} = E''''_{3,0,2} = H^2(B\text{Gau}(P); \mathbb{Z})$ and $E''''_{3,3,0} = E''''_{3,3,0} = H^3(BG; \mathbb{Z})$. By the naturality of the Serre spectral sequence, we have the commuting diagram

$$
\begin{array}{ccc}
H^2(B\text{Gau}(P); \mathbb{Z}) & \xrightarrow{d''''_{3,0,2}} & H^3(BG; \mathbb{Z}) \\
| & & | \\
f^* & & f^*
\end{array}
$$

Then we have $f^* \text{ev}^*_c = c \in H^2(X; \mathbb{Z})$ since $f : X \to B\text{Gau}(P)$ is the composition of the classifying map $X \to BS^1$ and the map $j : BS^1 \to B\text{Gau}(P)$ (see section 3.3). Thus, the universal characteristic class $e_c \in H^3(BG; \mathbb{Z})$ is equal to $-d''''_{3,0,2} \text{ev}^*_c$. Let $E''''_{p,q}$ denote the Serre spectral sequence of the fibration $B\text{Gau}(P) \to B\text{Aut}(P) \to BG$ (or, equivalently, the Hochschild-Serre spectral sequence of $1 \to \text{Gau}(P) \to \text{Aut}(P) \to G \to 1$). By the commuting diagram

$$
\begin{array}{ccc}
B\text{Gau}(P) & \xrightarrow{\delta} & B\text{Aut}(P) \\
| & & | \\
B\text{Gau}(P) & \xrightarrow{\delta} & B\text{Aut}(P) \xrightarrow{\delta} BG
\end{array}
$$

and the naturality, we have the commuting diagram of cohomologies

$$
\begin{array}{ccc}
H^2(B\text{Gau}(P); \mathbb{Z}) & \xrightarrow{d''''_{3,0,2}} & H^3(BG; \mathbb{Z}) \\
| & & | \\
\text{ev}^* & & \text{ev}^*
\end{array}
$$

here we identify the group cohomology and the singular cohomology of classifying space of discrete groups. Then we have

$$
\text{ev}^* e_c = -\text{ev}^* d''''_{3,0,2} \text{ev}^*_c = -d''''_{3,0,2} \text{ev}^*_c.
$$

On the other hand, by the commuting diagram (4.2) and lemma 3.2 we have

$$
\delta \epsilon(A\text{G}(P)) = -\delta d_{2,0,2}^c (\text{id}_{G\text{Gau}(P)}) = d''''_{3,0,2} \delta (\text{id}_{G\text{Gau}(P)}).
$$
Since the class $\iota^* \text{ev}_x^* c_1$ is equal to $-\delta(\text{id}_{\text{Gau}(P)})$ (see [3, Lemma 2.4]), we have
\[ \iota^* c_1 = \delta(e(A_G(P))) \]
and the theorem follows. \qed

5. Central $S^1$-extension of $G$

In this section, we assume that the topological space $X$ admits the universal covering space $\tilde{X}$. We fix a non-zero cohomology class $c \in H^2(X; \mathbb{Z})$ such that it is equal to zero in $H^2(X; \mathbb{R})$.

5.1. Construction of the central $S^1$-extension. By the cohomology long exact sequence
\[ \cdots \rightarrow H^1(X; \mathbb{R}) \rightarrow H^1(X; S^1) \rightarrow H^2(X; \mathbb{Z}) \rightarrow H^2(X; \mathbb{R}) \rightarrow \cdots, \]
there is an element $\rho \in H^1(X; S^1)$ such that $\delta(\rho) = c$. By the isomorphism $H^1(X; S^1) \cong \text{Hom}(\pi_1(X); S^1)$, we regard the class $\rho$ as a homomorphism $\rho : \pi_1(X) \rightarrow S^1$. Let $P^\delta_\rho \rightarrow X$ be a principal $S^1$-bundle with the holonomy homomorphism $\rho$, that is, we put $P^\delta_\rho = \tilde{X} \times_{\rho} S^1\delta$.

**Lemma 5.1.** Let $\text{Aut}(P^\delta_\rho)$ denote the bundle automorphisms of $P^\delta_\rho$ and $\text{Homeo}(X, \rho)$ the group of $\rho$-preserving homeomorphisms. Then the canonical projection $\pi : \text{Aut}(P^\delta_\rho) \rightarrow \text{Homeo}(X)$ gives the surjection $\text{Aut}(P^\delta_\rho) \rightarrow \text{Homeo}(X, \rho)$ and its kernel is isomorphic to $S^1$, that is, there is the following exact sequence of groups
\[ 1 \rightarrow S^1 \rightarrow \text{Aut}(P^\delta_\rho) \rightarrow \text{Homeo}(X, \rho) \rightarrow 1. \]

**Proof.** At first, we show that the image $\pi(\text{Aut}(P^\delta_\rho))$ is contained in $\text{Homeo}(X, \rho)$. Take a bundle automorphism $F \in \text{Aut}(P^\delta_\rho)$ that covers a homeomorphism $f : X \rightarrow X$. Let $\gamma : [0, 1] \rightarrow X$ be a loop and $\tilde{\gamma} : [0, 1] \rightarrow P^\delta_\rho$ a lift of $\gamma$. Then $F\tilde{\gamma} : [0, 1] \rightarrow P^\delta_\rho$ is a lift of the loop $f\gamma$. Since $\rho : \pi_1(X) \rightarrow S^1\delta$ is the holonomy of $P^\delta$, the value $\rho(f\gamma) \in S^1\delta$ is given by
\[ F(\tilde{\gamma}(1)) = F(\tilde{\gamma}(0)) \cdot \rho(f\gamma). \]
Since the left-hand side of (5.2) is equal to $F(\tilde{\gamma}(0) \cdot \rho(\gamma)) = F(\tilde{\gamma}(0)) \cdot \rho(\gamma)$, we have $f^* \rho(\gamma) = \rho(f\gamma) = \rho(\gamma)$. Thus the homeomorphism $f$ preserves the holonomy $\rho$ and we have $\pi(\text{Aut}(P^\delta_\rho)) \subset \text{Homeo}(X, \rho)$.

Next we show that the map $\pi : \text{Aut}(P^\delta_\rho) \rightarrow \text{Homeo}(X, \rho)$ is surjective. For a homeomorphism $f_\gamma : X \rightarrow X$ that preserves $\rho$, we take a lift $\tilde{f} : \tilde{X} \rightarrow \tilde{X}$. Then the map $\tilde{X} \times S^1\delta \rightarrow \tilde{X} \times S^1\delta$ defined by $(\tilde{x}, u) \rightarrow (\tilde{f}(\tilde{x}), u)$ induces a bundle automorphism $F : P^\delta_\rho \rightarrow P^\delta_\rho$ that covers $f$. Thus the map $\pi : \text{Aut}(P^\delta_\rho) \rightarrow \text{Homeo}(X, \rho)$ is surjective.

The kernel of $\pi : \text{Aut}(P^\delta_\rho) \rightarrow \text{Homeo}(X, \rho)$ is the gauge group $\text{Gau}(P^\delta_\rho)$ and this is isomorphic to $C(X, S^1\delta) \cong S^1\delta$ since the fiber of $P^\delta_\rho$ is the discrete group $S^1\delta$. \qed
By restricting the above exact sequence (5.1) to the subgroup \(G\) of \(\text{Homeo}(X, \rho)\), we have the abelian \(S^1\)-extension

\[
1 \longrightarrow S^1 \longrightarrow A_G(P^\delta_\rho) \longrightarrow G \longrightarrow 1.
\]

Moreover, the group \(S^1\) is in the center of \(A_G(P^\delta_\rho)\), this is a central \(S^1\)-extension of \(G\). Thus we obtain the corresponding group cohomology class \(e(A_G(P^\delta_\rho)) \in H^2_{\text{grp}}(G; S^1)\).

Let \(P_\rho \to X\) be a principal \(S^1\)-bundle defined by \(P_\rho = \tilde{X} \times_\rho S^1\). Then we have the following commuting diagram of groups

\[
\begin{array}{cccccc}
1 & \longrightarrow & S^1 & \longrightarrow & A_G(P^\delta_\rho) & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow j & & \downarrow k & & \downarrow & \\
1 & \longrightarrow & \text{Gau}(P_\rho) & \longrightarrow & A_G(P^\delta_\rho) & \longrightarrow & G & \longrightarrow & 1,
\end{array}
\]

where \(j : S^1 \to \text{Gau}(P_\rho)\) is the inclusion. Recall that \(e(A_G(P^\delta_\rho)) \in H^2_{\text{grp}}(G; S^1)\) and \(e(A_G(P_\rho)) \in H^2_{\text{grp}}(G; \text{Gau}(P_\rho))\) denote the cohomology classes that correspond to the central extension \(A_G(P^\delta_\rho)\) and the abelian extension \(A_G(P_\rho)\) respectively. By the commuting diagram above, we have the following proposition.

**Proposition 5.2.** Let \(j_* : H^2_{\text{grp}}(G; S^1) \to H^2_{\text{grp}}(G; \text{Gau}(P_\rho))\) be the map of coefficients change by \(j\). Then we have \(j_*(e(A_G(P^\delta_\rho))) = e(A_G(P_\rho))\) in \(H^2_{\text{grp}}(G; \text{Gau}(P_\rho))\).

**Proof.** Recall that a cocycle \(c\) of \(e(A_G(P^\delta_\rho))\) is given by

\[
c(g, h) = s(g)s(h)s(gh)^{-1},
\]

where \(s : G \to A_G(P^\delta_\rho)\) is a section. Since \(k \circ s : G \to A_G(P_\rho)\) is a section of \(A_G(P_\rho) \to G\), a cocycle \(c'\) of \(e(A_G(P_\rho))\) is given by

\[
c'(g, h) = k(s(g))k(s(h))k(s(gh))^{-1} = k(s(g)s(h)s(gh)^{-1}) = j(c(g, h)).
\]

Thus we have \(j_*(e(A_G(P^\delta_\rho))) = e(A_G(P_\rho))\). \(\square\)

**Proof of Theorem 1.2.** By the commuting diagram of coefficients

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & S^1 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow j & & \downarrow & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & C(X, \mathbb{R}) & \longrightarrow & C(X, S^1) = \text{Gau}(P_\rho) & \longrightarrow & 1,
\end{array}
\]

we have the following commuting diagram of cohomologies

\[
\begin{array}{cccc}
H^2_{\text{grp}}(G; S^1) & \delta & H^3_{\text{grp}}(G; \mathbb{Z}) \\
\downarrow j_* & & \downarrow & \\
H^2_{\text{grp}}(G; \text{Gau}(P_\rho)) & \delta & H^3_{\text{grp}}(G; \mathbb{Z}),
\end{array}
\]
where the maps $\delta$ are the connecting homomorphisms. Together with Theorem 1.4 and proposition 5.2, we obtain

$$\delta e(A_G(P^\delta_\rho)) = \delta j_* e(A_G(P^\delta_\rho)) = \delta e(A_G(P_\rho)) = \epsilon^* \epsilon_\cdot.$$  

\[ \square \]

5.2. **Cocycle description of $e(A_G(P^\delta_\rho))$.** By the similar construction explained in [3], we obtain the group two-cocycle that represents the class $e(A_G(P^\delta_\rho))$. Let $c \in H^2(X; \mathbb{Z})$ and $\rho \in H^1(X; S^1)$ be a cohomology class as above. Let $\alpha \in C^1(X; S^1)$ be a cocycle that represents the cohomology class $\rho$, where $(C^*(X; S^1), d)$ denotes the singular cochain complex of $X$ with coefficients in $S^1$. The singular cochain $C^*(X; S^1)$ and the cohomology $H^*(X; S^1)$ are the right $G$-modules by pullback. Since $g^* \rho = \rho$ for any homeomorphism $g \in G$, the cocycle $g^* \alpha - \alpha$ is a coboundary. Thus there is a cochain $\hat{\kappa}_\alpha (g) \in C^0(X; S^1)$ such that

$$g^* \alpha - \alpha = d \hat{\kappa}_\alpha (g).$$

Then a two-cochain $\Theta_{x,\alpha}$ in $C^2(G; S^1)$ is defined by

$$(5.5) \quad \Theta_{x,\alpha}(g, h) = \int_x^{hx} g^* \alpha - \alpha = \hat{\kappa}_\alpha (g) (hx) - \hat{\kappa}_\alpha (g) (x),$$

where $x \in X$. Here the symbol $\int_x^{hx}$ denotes the pairing of the cocycle $g^* \alpha - \alpha$ and a path from $x$ to $hx$. By the same arguments in [5] Theorem 3.1, we have the following proposition.

**Proposition 5.3.** The two-cochain $\Theta_{x,\alpha} \in C^2_{\text{grp}} (G; S^1)$ is a cocycle and the cohomology class $[\Theta_{x,\alpha}]$ is independent of the choice of the point $x \in X$ and the singular cochain $\alpha$.

Then the following theorem holds.

**Theorem 5.4.** The cohomology class $[\Theta_{x,\alpha}]$ is equal to $e(A_G(P_\rho))$.

For the proof of Theorem 5.4, we show the following lemmas. Let us recall that $p : P^\delta_\rho \to X$ is the principal $S^1$-bundle with the holonomy $\rho$.

**Lemma 5.5.** The pullback $p^* \rho \in H^1(P^\delta_\rho; S^1)$ is equal to zero.

**Proof.** Let $\gamma : [0, 1] \to P^\delta_\rho$ be a loop. Then $\gamma$ is a lift of the loop $p\gamma$ in $X$. Since $\rho : \pi(X) \to S^1$ is the holonomy homomorphism, we have

$$\pi^* \rho (\gamma) = p (p^* \gamma) = \gamma (1) - \gamma (0) = 0$$

and the lemma follows. \[ \square \]

By lemma 5.5, we take a singular cochain $\theta' \in C^0(P^\delta_\rho; S^1)$ satisfying $d \theta' = p^* \alpha$. Take a base point $y \in P^\delta_\rho$ and put $p (y) = x$. For any $u \in S^1$, let $P^\delta_\rho (u)$ denote the connected component of $P^\delta_\rho$ that contains the point $y \cdot u$. Then we put

$$z (u) = \theta' (y) - \theta' (y \cdot u) + u \in S^1.$$
By the straight forward calculation, we have the equality $z(u) = z(u + \rho(\gamma))$ for any $\gamma \in \pi_1(X)$. Thus, the value $z(u)$ defines the continuous function $z : P^\delta_\rho \to S^1$. Note that $z(y \cdot u) = z(u)$ and $z(y) = 0$. We put $\theta = \theta' + z \in C^0(X; Z)$. Since $dz = 0$, we have $d\theta = p^*\alpha$. Let $\tau \in C^1_{\text{grp}}(A_G(P^\delta_\rho); S^1)$ denote a cochain defined by

$$
\tau(\phi) = \int_y^y \theta = \theta(y) - \theta(y).
$$

**Lemma 5.6.** The restriction $\tau|_{S^1} : S^1 \to S^1$ is equal to the identity.

**Proof.** By definition of $\tau$, we have

$$
\tau(u) = \theta(y \cdot u) - \theta(y) = \theta'(y \cdot u) + z(y \cdot u) - \theta'(y) - z(y) = u
$$

for any $u \in S^1$. □

**Lemma 5.7.** The equality

$$
-\delta \tau = \pi^*\mathbf{G}_{x, \alpha} \in C^2_{\text{grp}}(A_G(P^\delta_\rho); S^1)
$$

holds, where $\pi : A_G(P^\delta_\rho) \to G$ is the projection.

**Proof.** For the pullback $p^*\mathbf{R}_G(g) \in C^0(P^\delta_\rho; S^1)$, we have

$$
d(p^*\mathbf{R}_G(g)) = p^*(d\mathbf{R}_G(g)) = p^*g^*\alpha - p^*\alpha
$$

$$
= \phi^*p^*\alpha - p^*\alpha = p^*d\theta - d\theta = d(\phi^*\theta - \theta),
$$

where $\phi \in A_G(P^\delta_\rho)$ is a lift of $g \in G$. Thus, for $\phi, \psi \in A_G(P^\delta_\rho)$ that covers $g, h \in G$ respectively, we have

$$
\pi^*\mathbf{G}_{x, \alpha}(\phi, \psi) = \mathbf{G}_{p(y), \alpha}(g, h) = \mathbf{R}_G(g)(hp(y)) - \mathbf{R}_G(g)(p(y))
$$

$$
= \mathbf{R}_G(g)(p\psi(y)) - \mathbf{R}_G(g)(p(y))
$$

$$
= p^*\mathbf{R}_G(g)(\psi(y)) - p^*\mathbf{R}_G(g)(y)
$$

$$
= (\phi^*\theta - \theta)(\psi(y)) - (\phi^*\theta - \theta)(y)
$$

$$
= (\theta(\phi(y)) - \theta(y)) - (\theta(\phi(y)) - \theta(y)) - (\theta(\psi(y)) - \theta(y))
$$

$$
= -\delta \tau(\phi, \psi)
$$

and the lemma follows. □

**Proof of theorem 5.4.** Let $E^{p,q}_r$ denote the Hochschild-Serre spectral sequence of $0 \to S^1 \to A_G(P^\delta_\rho) \to G \to 1$ with coefficients in $S^1$. Then there is the derivation

$$
d_2^{0,1} : H^{1,0}_{\text{grp}}(S^1; S^1) = E_2^{0,1} \to E_2^{2,0} = H^2_{\text{grp}}(G; S^1).
$$

By lemma 5.6 and the definition of the derivation of the spectral sequence, we have $d_2^{0,1}(\mathbf{G}_{S^1}) = -[\mathbf{G}_{x, \alpha}]$. On the other hand, by lemma 3.2, we have $d_2^{0,1}(\mathbf{G}_{S^1}) = -e(A_G(P^\delta_\rho))$. Thus we have $[\mathbf{G}_{x, \alpha}] = e(A_G(P^\delta_\rho))$. □
6. Examples

In this section, we give two examples that the universal characteristic classes $e_c$ and $\iota^*e_c$ are non-trivial. The first example is the complex projective space $\mathbb{C}P^n$. The proof of the following theorem is the same as [6, Theorem 1.2], so I will omit it.

**Theorem 6.1.** Let $n$ be a positive integer and $c \in H^2(\mathbb{C}P^n; \mathbb{Z})$ the generator of cohomology. Let $G$ denote the identity component $\text{Homeo}(\mathbb{C}P^n)_0$ of the homeomorphism group. Then the universal characteristic classes $e_c \in H^3(BG; \mathbb{Z})$ and $\iota^*e_c \in H^3(BG^8; \mathbb{Z})$ are non-zero.

The cohomology class $c$ in Theorem 6.1 is non-zero in $H^2(\mathbb{C}P^n; \mathbb{R})$. So this example does not satisfy the assumption in section 5. The following example satisfies the assumption in section 5.

**Theorem 6.2.** Let $c$ be the non-zero element in $H^2(\mathbb{R}P^3; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Let $G$ denote the identity component $\text{Homeo}(\mathbb{R}P^3)_0$ of the homeomorphism group. Then the universal characteristic classes $e_c \in H^3(BG; \mathbb{Z})$ and $\iota^*e_c \in H^3(BG^8; \mathbb{Z})$ are non-zero.

**Proof.** Since $\mathbb{R}P^3$ is homeomorphic to $SO(3)$, the group $SO(3)$ is included in $G$ as the left translations. Thus we have the commuting diagram

$$
\begin{array}{ccc}
SO(3) & \longrightarrow & ESO(3) \\
\downarrow & & \downarrow f \\
G & \longrightarrow & EG \\
\downarrow & & \downarrow f \\
BSO(3) & \longrightarrow & BG.
\end{array}
$$

Recall that the universal $G$-bundle $\mathbb{R}P^3 \to E \to BG$ is given as $E = EG \times_G \mathbb{R}P^3$. For the base point $b \in \mathbb{R}P^3$ that corresponds to the unit $1 \in SO(3)$, we define a map $\phi : ESO(3) \to E$ by putting $\Phi(x) = [F(x), b]$ for $x \in ESO(3)$. Then we have the commuting diagram

$$
\begin{array}{ccc}
SO(3) & \longrightarrow & ESO(3) \\
\downarrow & & \downarrow \phi \\
\mathbb{R}P^3 & \longrightarrow & E \\
\downarrow & & \downarrow f \\
BSO(3) & \longrightarrow & BG.
\end{array}
$$

Let us consider the Serre spectral sequences of the two fibrations in (6.1). By the naturality of the Serre spectral sequence, the pullback $f^*e_c \in H^3(BSO(3); \mathbb{Z})$ of the universal characteristic class $e_c$ is equal to the transgression image of $c \in H^2(\mathbb{R}P^3; \mathbb{Z}) \cong H^2(SO(3); \mathbb{Z})$ with respect to the fibration $SO(3) \to ESO(3) \to BSO(3)$. Since $ESO(3)$ is contractible, the transgression map is injective and thus the class $f^*e_c$ is non-zero. So the class $e_c$ is also non-zero. Next we consider the
following commuting diagram

\[
\begin{array}{c}
H^3(BG; \mathbb{Z}) \xrightarrow{\iota^*} H^3(BG^\delta; \mathbb{Z}) \\
\downarrow f^* \quad \quad \quad \downarrow f^*
\end{array}
\]

\[
H^3(BSO(3); \mathbb{Z}) \xrightarrow{\iota^*} H^3(BSO(3)^\delta; \mathbb{Z}).
\]

Since the map \( \iota^*: H^3(BSO(3); \mathbb{Z}) \rightarrow H^3(BSO(3)^\delta; \mathbb{Z}) \) is injective [7], the class \( \iota^* f^* e_c \in H^3(BSO(3)^\delta; \mathbb{Z}) \) is non-zero. Thus the class \( \iota^* e_c \in H^3(BG^\delta; \mathbb{Z}) \) is also non-zero. \( \square \)

**Remark 6.3.** By Theorem [1.2] and Theorem [6.2], the class \( e(A_\mathcal{G}(P^3)) \) is also non-zero for \( G = \text{Homeo}(\mathbb{R}P^3)_0 \). Thus, this is an example that the cocycle (5.5) is cohomologically non-trivial.

**Remark 6.4.** For \( c \neq 0 \in H^2(\mathbb{R}P; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \) and \( G = \text{Homeo}(\mathbb{R}P^2)_0 \), the universal characteristic class \( e_c \in H^3(BG; \mathbb{Z}) \) is equal to zero.