A RAMSEY SPACE OF INFINITE POLYHEDRA
AND THE RANDOM POLYHEDRON

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ABSTRACT. In this paper we introduce a new topological Ramsey space whose elements are infinite ordered polyhedra. Then, we show as an application that the set of finite polyhedra satisfies two types of Ramsey property: one, when viewed as a category over \( \mathbb{N} \); the other, when considered as a class of finite structures. The (ordered) random polyhedron is the Fraïssé limit of the class of finite ordered polyhedra; we prove that its group of automorphisms is extremely amenable. Finally, we present a countably infinite family of topological Ramsey subspaces; each one determines a class of finite ordered structures which turns out to be a Ramsey class. One of these subspaces is Ellentuck’s space; another one is associated to the class of finite ordered graphs whoseFraïssé limit is the random graph. The Fraïssé limits of these classes are not pairwise isomorphic as countable structures and none of them is isomorphic to the random polyhedron.

INTRODUCTION

A polyhedron is a geometric object built up through a finite or countable number of suitable amalgamations of convex hulls of finite sets; polyhedra are generated in this way by simplexes. Simplicial morphisms are locally linear maps that preserve vertices. An ordered polyhedron is a polyhedron for which we have imposed a linear order on the set of its vertices. As we only consider order-preserving morphisms, ordered polyedra are rigid, i.e., admit no non-trivial automorphisms; this is an easy consequence of the well order principle. In this paper we define a new topological Ramsey space (see \cite{22}) whose elements are essentially infinite ordered polyhedra.

The prototypical topological Ramsey space is due to Ellentuck \cite{4}. It is the set \( \mathbb{N}[\infty] \) of infinite subsets of \( \mathbb{N} \), equipped with the exponential topology, whose basic sets are of the form:

\[
[a, A] = \{ B \in \mathbb{N}[\infty] : a \subset B \& B \subseteq A \},
\]

where \( a \) is a finite subset of \( \mathbb{N} \) and \( A \in \mathbb{N}[\infty] \). Here, \( a \subset B \) means that \( a \) is an initial segment of \( B \). Recall that a set \( \mathcal{X} \subseteq \mathbb{N}[\infty] \) is Ramsey if for every neighborhood \( [a, A] \neq \emptyset \) there exists \( B \in [a, A] \) such that \( [a, B] \subseteq \mathcal{X} \) or \( [a, B] \cap \mathcal{X} = \emptyset \). And it is Ramsey null if for every neighborhood \( [a, A] \) there exists \( B \in [a, A] \) such that \( [a, B] \cap \mathcal{X} = \emptyset \). The main result in \cite{4} states the following:

**Theorem 0.1.** (Ellentuck \cite{4}, 1974) A set \( \mathcal{X} \subseteq \mathbb{N}[\infty] \) is Ramsey if and only if it has the Baire property with respect to the exponential topology. And \( \mathcal{X} \) is Ramsey null if and only if it is meager with respect to the exponential topology.

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The theory of topological Ramsey spaces is developed in [3, 22] following the work of Ellentuck’s. In § 1 we will present the fundamental concepts of that theory. In § 2 we will define our new topological Ramsey space $\mathcal{P}$ and in the rest of the paper we will use $\mathcal{P}$ to study the Ramsey property for the finite ordered polyhedra, both as a category over $\mathbb{N}$ and as class of finite ordered structures. A description of the Fraïssé limit of the class of finite ordered polyhedra is given, we call it the (ordered) random polyhedron. It is proven that the automorphism group of this limit is extremely amenable. In § 3 we present a countable family $\{\mathcal{P}_k\}_{k>0}$ of topological Ramsey subspaces of $\mathcal{P}$. Each $\mathcal{P}_k$ determines a class $\mathcal{AP}_k$ of finite ordered structures which turns out to be a Ramsey class. The automorphism group of its Fraïssé limit is therefore extremely amenable. For instance, $\mathcal{P}_1$ coincides with Ellentuck’s space (see the definition below). The corresponding Ramsey class is off course the class of finite ordered sets whose Fraïssé limit is $\mathcal{Q}$. Each $\mathcal{P}_k$ determines a class $\mathcal{AP}_k$ of finite ordered structures which turns out to be a Ramsey class. The automorphism group of its Fraïssé limit is therefore extremely amenable. In § 4 we will present the fundamental concepts of that theory. This work is a step towards fully understanding these relationships.

Notation. Given a countable set $A$, we will adopt the following notation throughout the paper. If $X \subseteq A$ then $|X|$ denotes the cardinality of $X$:

- $A^{[k]} = \{X \subseteq A : |X| = k\}$, for every $k \in \mathbb{N}$.
- $A^{[\leq k]} = \{X \subseteq A : |X| \leq k\}$, for every $k \in \mathbb{N}$.
- $A^{[< \infty]} = \{X \subseteq A : |X| < \infty\}$.
- $A^{[\infty]} = \{X \subseteq A : |X| = \infty\}$.

1. Ramsey Spaces

The definitions and results throughout this section are taken from [22]. A previous presentation can be found in [3]. Consider a triplet of the form $(\mathcal{R}, \leq, r)$, where $\mathcal{R}$ is a set, $\leq$ is a quasi order on $\mathcal{R}$ and $r : \mathbb{N} \times \mathcal{R} \rightarrow \mathcal{AR}$ is a function with range $\mathcal{AR}$. For every $n \in \mathbb{N}$ and every $A \in \mathcal{R}$, let us write $r_n(A) := r(n, A)$ and $\mathcal{AR}_n := \{r_n(A) : A \in \mathcal{R}\}$. We say that $r_n(A)$ is the $n$th approximation of $A$. In order to capture the combinatorial structure required to ensure the provability of an Ellentuck type Theorem, some assumptions on $(\mathcal{R}, \leq, r)$ will be imposed. The first is the following:

(A.1)

(A.1.1) For any $A \in \mathcal{R}$, $r_0(A) = \emptyset$.

(A.1.2) For any $A, B \in \mathcal{R}$, if $A \neq B$ then $(\exists n) (r_n(A) \neq r_n(B))$.

(A.1.3) If $r_n(A) = r_m(B)$ then $n = m$ and $(\forall i < n) (r_i(A) = r_i(B))$. 


A.1 allows us to identify each $A \in \mathcal{R}$ with the sequence $(r_n(A))_n$ of its approximations. In this way, if $\mathcal{AR}$ has the discrete topology, $\mathcal{R}$ can be identified with a subspace of the (metric) space $\mathcal{AR}^\mathbb{N}$ (with the product topology) of all the sequences of elements of $\mathcal{AR}$, see for instance [1, 9]. We will say that $\mathcal{R}$ is **metrically closed** if it is a closed subspace of $\mathcal{AR}^\mathbb{N}$. The basic open sets generating the metric topology on $\mathcal{R}$ inherited from the product topology of $\mathcal{AR}^\mathbb{N}$ are of the form:

(2) \[ [a] = \{ B \in \mathcal{R} : (\exists n)(a = r_n(B)) \} \]

where $a \in \mathcal{AR}$. For each $a \in \mathcal{AR}$ let us define the **length** of $a$, as the unique integer $|a| = n$ such that $a = r_n(A)$ for some $A \in \mathcal{R}$. The **Ellentuck type neighborhoods** are of the form:

(3) \[ [a, A] = \{ B \in \mathcal{R} : (\exists n)(a = r_n(B)) \text{ and } (B \leq A) \} \]

where $a \in \mathcal{AR}$ and $A \in \mathcal{R}$. Let $\mathcal{AR}(A) = \{ a \in \mathcal{AR} : [a, A] \neq \emptyset \}$. We also write $[n, A] := [r_n(A), A]$. Given a neighborhood $[a, A]$ and $n \geq |a|$, let $r_n[a, A]$ be the image of $[a, A]$ by the function $r_n$, i.e., the set $\{ b \in \mathcal{AR} : \exists B \in [a, A] \text{ such that } b = r_n(B) \}$.

1.1. **Ramsey sets.** A set $X \subseteq \mathcal{R}$ is **Ramsey** if for every neighborhood $[a, A] \neq \emptyset$ there exists $B \in [a, A]$ such that $[a, B] \subseteq X$ or $[a, B] \cap X = \emptyset$. A set $X \subseteq \mathcal{R}$ is **Ramsey null** if for every neighborhood $[a, A]$ there exists $B \in [a, A]$ such that $[a, B] \cap X = \emptyset$.

1.2. **Topological Ramsey spaces.** We say that $(\mathcal{R}, \leq, r)$ is a **topological Ramsey space** iff subsets of $\mathcal{R}$ with the Baire property are Ramsey and meager subsets of $\mathcal{R}$ are Ramsey null.

Given $a, b \in \mathcal{AR}$, write

(4) \[ a \sqsubseteq b \text{ iff } (\exists A \in \mathcal{R}) (\exists m, n \in \mathbb{N}) m \leq n, a = r_m(A) \text{ and } b = r_n(A). \]

By A.1, $\sqsubseteq$ can be proven to be a partial order on $\mathcal{AR}$.

**A.2 (Finitization)** There is a quasi order $\leq_{fin}$ on $\mathcal{AR}$ such that:

(i) $A \leq B$ iff $(\forall n) (\exists m) (r_n(A) \leq_{fin} r_m(B)).$

(ii) $\{ b \in \mathcal{AR} : b \leq_{fin} a \}$ is finite, for every $a \in \mathcal{AR}$.

(iii) If $a \leq_{fin} b$ and $c \sqsubseteq a$ then there is $d \sqsupseteq b$ such that $c \leq_{fin} d$.

Given $A \in \mathcal{R}$ and $a \in \mathcal{AR}(A)$, we define the **depth of $a$ in $A$** as

(5) \[ \text{depth}_A(a) := \min\{ n : a \leq_{fin} r_n(A) \}. \]

**Lemma 1.1.** Given $A \in \mathcal{R}$ and $a \in \mathcal{AR}(A)$, $|a| \leq \text{depth}_A(a)$.

**Proof** By axioms A.1.3 and A.2.1. □

**A.3 (Amalgamation)** Given $a$ and $A$ with $\text{depth}_A(a) = n$, the following holds:

(i) $(\forall B \in [n, A]) ([a, B] \neq \emptyset)$.

(ii) $(\forall B \in [a, A]) (\exists A' \in [n, A]) ([a, A'] \subseteq [a, B])$.

**A.4 (Pigeonhole Principle)** Given $a$ and $A$ with $\text{depth}_A(a) = n$, for every $O \subseteq \mathcal{AR}_{|a|+1}$ there is $B \in [n, A]$ such that $r_{|a|+1}[a, B] \subseteq O$ or $r_{|a|+1}[a, B] \subseteq O^c$. 
Theorem 1.2 (See [3, 22]). [Abstract Ellentuck Theorem] Any $(\mathcal{R}, \leq, r)$ with $\mathcal{R}$ metrically closed and satisfying (A.1)-(A.4) is a topological Ramsey space.

Notation: The following is taken from [13]. For metrically closed and satisfying (A.1)-(A.4) is a topological Ramsey space.

In the rest of this section we shall prove the following:

Theorem 1.3. Let $A \in \mathcal{R}$ and $k, n, r \in \mathbb{N}$ be given. Then, there exists $m \in \mathbb{N}$ such that for every coloring $c : \mathcal{AR}_k^m(A) \to r$, there exists $b \in \mathcal{AR}_k^m(A)$ such that $c$ is constant in $\mathcal{AR}_k^m(A, b)$.

For instance, consider the triplet $\mathcal{E} = (\mathbb{N}[\mathbb{N}], \subseteq, i)$, where $\subseteq$ is the inclusion relation and $i : \mathbb{N} \times \mathbb{N}[\mathbb{N}] \to \mathbb{N}[\mathbb{N}]$, is the approximation function:

$$i(n, A) = \text{ the first } n \text{ elements of } A$$

In this case, the instance of the abstract Ellentuck theorem is Theorem 0.1 and the instance of Theorem 1.3 if the finite version of Ramsey’s theorem [21].

From now on, we will refer to $\mathcal{E} = (\mathbb{N}[\mathbb{N}], \subseteq, i)$ as Ellentuck’s space.

2. The topological Ramsey space $\mathcal{P}$

In this § we will consider pairs $(x, S_x)$ satisfying the following:

1. $x \subseteq \mathbb{N}$,
2. $S_x \subseteq x^{<\omega}$ is hereditary, i.e., $u \subseteq v \& v \in S_x \Rightarrow u \in S_x$, and
3. $\bigcup S_x = \bigcup \{u : u \in S_x\} = x$.

Given a pair $(x, S_x)$ satisfying the conditions above and $y \subseteq x$, we let $S_x | y = \{u \cap y : u \in S_x\}$. Given two pairs $(x, S_x)$, $(y, S_y)$ satisfying the conditions (1), (2) and (3); we write

$$\langle y, S_y \rangle \leq \langle x, S_x \rangle \iff y \subseteq x \& S_y \subseteq S_x.$$ 

For instance, if $y \subset x$ then $(y, S_x | y) \leq (x, S_x)$. In particular, if $n \in \mathbb{N}$ and $|x| \geq n$ let $x | n$ be the set of the first $n$ elements of $x$ and $S_x | n = S_x \mid (x | n)$. The pair

$$r_n(x, S_x) = (x | n, S_x | n)$$

is the $n$th approximation of $(x, S_x)$. Notice that $r_n(x, S_x) \leq (x, S_x)$ for all $n \leq |x|$.

Write $\mathcal{AP}$ to denote the set of all pairs $(x, S_x)$ such that $|x| < \omega$; and $\mathcal{P}$ to denote its complement, whose elements will be written $(A, S_A), (B, S_B), \ldots$ in capital letters. There is a well defined surjective function

$$\mathcal{P} \times \mathbb{N} \rightarrow \mathcal{AP} \quad r((A, S_A), n) = r_n(A, S_A)$$

In the rest of this section we shall prove the following:

Theorem 2.1. $(\mathcal{P}, \leq, r)$ is a topological Ramsey space.

The proof of Theorem 2.1 will be divided into several lemmas, showing that $(\mathcal{P}, \leq, r)$ satisfies the conditions of the abstract Ellentuck theorem.
Lemma 2.2. $\langle P, \leq, r \rangle$ satisfies axiom A.1

1. For every $(A, S_A) \in P$, $r_0(A, S_A) = \emptyset$.
2. If $(A, S_A) \neq (B, S_B)$ then there exists $n$ such that $r_n(A, S_A) \neq r_n(B, S_B)$.
3. If $r_n(A, S_A) = r_m(B, S_B)$ then $n = m$ and for every $i < n$, $r_i(A, S_A) = r_i(B, S_B)$.

[Proof] Straightforward. \hfill \square

Hence each element of $P$ can be identified with the sequence of its approximations. Next we consider $P$ as a subset of the product space $AP^N$, regarding $AP$ as a discrete space.

Lemma 2.3. $P$ is a closed subset of $AP^N$.

[Proof] By Lemma 2.2, the injection $P \to AP$ given by $\varphi(A, S_A) = (r_0(A, S_A), r_1(A, S_A), \ldots)$ is continuous. Let us show that $\varphi(P)$ is closed. Given a closure point $\alpha = \{ (a^j, S_{a^j}) \} \in \varphi(P) \subset AP^N$ and a sequence $\{(A^k, S_{A^k})\}_{k \in N}$ in $P$, if $\{(A^k, S_{A^k})\}_{k \in N}$ converges to $\alpha$ then

$$(\forall n \in \mathbb{N}) \ (\exists k_n \in \mathbb{N}) \ k \geq k_n \Rightarrow (\forall j \leq n) \ r_j(A^k, S_{A^k}) = (a^j, S_{a^j})$$

Taking a strictly increasing sequence $k_n < k_{n+1}$ for all $n \in \mathbb{N}$ we get $n > m \Rightarrow r_m(A^{k_n}, S_{A^{k_n}}) = (a^{k_m}, S_{a^{k_m}})$

Define $A = \bigcup_{n \in \mathbb{N}} a^{k_n}$ and $S_A = \bigcup_{n \in \mathbb{N}} S_{a^{k_n}}$. Then $(A, S_A) \in P$ and $\varphi(A, S_A) = \alpha$ by construction. \hfill \square

Let us define the preorder $\leq_{fin}$ and the partial order $\sqsubseteq$ on $AP$ as follows:

(12) $$(a, S_a) \leq_{fin} (b, S_b) \iff (a, S_a) \leq (b, S_b) \& \max(a) = \max(b)$$

(13) $$\quad (a, S_a) \sqsubseteq (b, S_b) \iff a \sqsubseteq b \& (a, S_a) \leq (b, S_b)$$

Here we are using the same symbol $\sqsubseteq$ to indicate that the set $a$ is an initial segment of the set $b$. The following two lemmas are straightforward; we leave the details to the reader.

Lemma 2.4. $\langle P, \leq, r \rangle$ satisfies axiom A.2

1. If $(A, S_A) \leq (B, S_B)$ then $\forall n \exists m$, $r_n(A, S_A) \leq_{fin} r_m(B, S_B)$.
2. For every $(a, S_a) \in AP$ the set $\{(b, S_b) : (b, S_b) \leq_{fin} (a, S_a)\}$ is finite.
3. If $(a, S_a) \leq_{fin} (b, S_b)$ and $(c, S_c) \sqsubseteq (a, S_a)$ then there is $(d, S_d) \sqsubseteq (b, S_b)$ such that $(c, S_c) \leq_{fin} (d, S_d)$.

Before stating Lemma 2.5 below, let us adapt from §1.2 the definition of basic open sets, for the Ellentuck-like topology of $P$. These will be sets $[(a, S_a); (A, S_A)]$, with $(a, S_a) \in AP$ and $(A, S_A) \in P$, such that $(B, S_B) \in [(a, S_a); (A, S_A)]$ if and only if

(14) $$(B, S_B) \leq (A, S_A) \& (\exists n) \ r_n(B, S_B) = (a, S_a).$$

In particular,
Remark 2.7. □

result follows from Lemmas 2.2, 2.3, 2.4, 2.5, 2.6.

Proof of Theorem 2.1

Now we can prove that \((P, \leq, r)\) satisfies axiom A.3

Let \(n = \operatorname{depth}_{(B, S_B)}(a, S_a)\).

1. If \((a, S_a) \in \{a; (B, S_B)\}\) then \([a, S_a) : (A, S_A) \neq \emptyset\).

2. For every \((A, S_A) \in \{(a, S_a); (B, S_B)\}\) there exists \((A', S_{A'}) \in \{a; (B, S_B)\}\) such that \(-\emptyset \neq [(a, S_a); (A', S_{A'})] \subseteq [(a, S_a); (A, S_A)]\).

For every natural number \(n\), let

\[ \mathcal{AP}_n := \{(a, S_a) \in \mathcal{AP} : |a| = n\}. \]

If \((a, S_a) \in \mathcal{AP}_n\), then we say that the length of \((a, S_a)\) is \(n\) or simply write \(|(a, S_a)| = n\).

Also, as in the general setting, for every natural number \(n\) write

\[ r_n[(a, S_a); (A, S_A)] = \{r_n(B, S_B) : (B, S_B) \in [(a, S_a); (A, S_A)]\}. \]

Finally, we prove the following

Lemma 2.6. Pigeonhole principle A.4 for \((P, \leq, r)\):

Let \(n = \operatorname{depth}_{(B, S_B)}(a, S_a)\), \(k = |(a, S_a)|\) and \(c : \mathcal{AP}_{k+1} \to \{0, 1\}\) be any partition. There exists \((A, S_A) \in \{a; (B, S_B)\}\) such that \(c\) is constant in \(r_{k+1}[(a, S_a); (A, S_A)]\).

Proof. Let

\[ X = \{m \in B : m > \max(a)\} \text{ and there exists } u \in S_a \text{ such that } u \cup \{m\} \in S_B \}

For \(i \in \{0, 1\}\), let

\[ X_i = \{m \in X : c((a \cup \{m\}, S_B \mid a \cup \{m\})) = i\}. \]

By the classical pigeon hole principle, there is \(i_0 \in \{0, 1\}\) such that \(|X_{i_0}| = \infty\). So let

\[ A = (B \mid n) \cup X_{i_0} \text{ and } S_A = S_B \mid A \]

Then \((A, S_A) \in \{a; (B, S_B)\}\) is as required.

□

Now we can prove that \((P, \leq, r)\) is a topological Ramsey space:

[ Proof of Theorem A.4.1 ] In virtue of the abstract Ellentuck theorem, the required result follows from Lemmas 2.2, 2.3, 2.4, 2.5, 2.6. □

Remark 2.7. (Ellentuck’s space as a subspace of \(P\)) Notice that we can identify each \(A \in \mathbb{N}[\infty]\) with the pair \((A, A[\leq 1])\). In this way, we can view Ellentuck’s space \(E\) as a (closed) subspace of \(P\).
Now we give an alternative proof to the well known fact that $\mathcal{E}$ is a Ramsey space. Recall the approximation function $i : \mathbb{N} \times \mathbb{N}[\leq 1] \to \mathbb{N}[<\infty]$, given by

$$i(n, A) = \text{the first n elements of A.}$$

**Corollary 2.8.** (Ellentuck [4], 1974) $\mathcal{E} = (\mathbb{N}[\infty], \subseteq, i)$ is a topological Ramsey space.

**Proof** Fix $\mathcal{X} \subseteq \mathbb{N}[\infty]$ with the Baire property with respect to the exponential topology of $\mathcal{E}$. Since $\mathcal{E}$ as a closed subspace of $\mathcal{P}$, it is easy to show that the set

$$\mathcal{X}' = \{(A, A[\leq 1]) : A \in \mathcal{X}\} \subseteq \mathcal{P}$$

has the Baire property with respect to the Ellentuck-like topology of $\mathcal{P}$. Given a nonempty neighborhood $[a, A]$ in $\mathcal{E}$, let $S_a = a[\leq 1]$ and $S_A = A[\leq 1]$. Then, consider the neighborhood $[(a, S_a), (A, S_A)]$ in $\mathcal{P}$. Applying Theorem 2.1 we obtain $(B, S_B) \in [(a, S_a); (A, S_A)]$ such that $[(a, S_a); (B, S_B)] \subseteq \mathcal{X}$ or $[(a, S_a); (B, S_B)] \cap \mathcal{X} = \emptyset$.

Notice that, by necessity, $S_B = B[\leq 1]$. Hence, $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$.

If $\mathcal{X}$ is meager with respect to the exponential topology of $\mathcal{E}$ then the same argument works but in addition the case $[a, B] \subseteq \mathcal{X}$ will never happen, by the meagerness of $\mathcal{X}$. This completes the proof. $\square$

We will see later in Theorem 4.1 that we can understand $\mathcal{AP}$ as a category (over $\mathbb{N}$) with suitable embeddings, and that it is in fact a Ramsey category in the sense of [5]. We will also see that it is a Ramsey class of finite structures in the sense of [10, 13, 19] (for instance). But before that, we will give a geometric interpretation of the Ramsey space $(\mathcal{P}, \leq, r)$.

3. THE RAMSEY SPACE $\mathcal{P}$ GEOMETRICALLY INTERPRETED

3.1. Simplexes. Let $V = \{v_0, \ldots, v_n\} \subset \mathbb{R}^d$ be an affinely independent finite subset of some euclidean space, which we call the ambient space. The $n$-simplex generated by $V$, written $\langle V \rangle$, is the convex hull generated by the points of $V$; i.e. the set of convex combinations,

$$\sum_{i=0}^{n} a_i v_i, \quad a_0, \ldots, a_n \in [0, \infty); \quad a_0 + \cdots + a_n = 1$$

A vertex of $\langle V \rangle$ is a point in $V$. The integer $n$ is the affine dimension of $\langle V \rangle$. Thus $\dim(\langle V \rangle) + 1 = |V|$. A linear morphism between simplexes $\langle W \rangle \xrightarrow{f} \langle V \rangle$ is the restriction of a linear function $f$ between the respective ambient spaces, such that $f(\langle W \rangle) \subseteq \langle V \rangle$. An embedding is a vertex-preserving linear injective morphism, i.e. $f(W) \subseteq V$. For $0 \leq m \leq n$, a $m$-face (or $m$-subsimplex) of $\langle V \rangle$ is the image $\rho(f) \subseteq \langle V \rangle$ of an $m$-simplex $\langle W \rangle$ through an embedding $\langle W \rangle \xrightarrow{f} \langle V \rangle$. Notice

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1 This can be deduced from two facts: (a) Since $\mathcal{E}$ is closed in $\mathcal{P}$, every meager subset of $\mathcal{E}$ is still meager in $\mathcal{P}$; and (b) Subsets of $\mathcal{P}$ with the Baire property constitute a $\sigma$-algebra.
that the faces of \( V \) can be obtained from the subsets of \( V \). By convention we will consider \( \emptyset \) as the empty face with affine dimension 0. Given \( n \in \mathbb{N} \), we write
\[
\Delta^n = \langle \{ e_0, \ldots, e_n \} \rangle \subset \mathbb{R}^{n+1}
\]
for the \textbf{standard} \( n \)-simplex generated by the canonical basis.

3.2. \textbf{Simplicial sets and rigid polyhedra}. A geometric polyhedron is the amalgamated sum of a countable collection of standard simplexes through a countable family of embeddings; a simplicial set is the prescription of these embeddings, so it is essentially a recipe for gluing simplexes. Simplicial sets always are induced by the category of finite ordinals; see [11, pp.171-174]. Here there is another approach: A \textbf{finite} simplicial set is a set \( S \subset \mathbb{N}[<\infty] \) such that
\[
S = S_x \times S_x \text{ for some } (x, S_x) \in \mathcal{AP}
\]
The elements of \( x = \cup S \) correspond to the vertices in a geometric realization of \( S \). For instance, since \( |\cup S| < \infty \), define
\[
V_S = \{ e_j : j \in \cup S \} \subset \mathbb{R}^{\max(|\cup S|)+1}
\]
as the corresponding subset of the canonical basis of \( \mathbb{R}^{\max(|\cup S|)+1} \). We generate from \( S \) a polyhedron \( K \) by taking the amalgamated sum of the family of simplexes \( \{ \langle V_u \rangle : u \text{ is maximal in } (S, \subset) \} \) glued by the (linear) embeddings
\[
\langle V_u \rangle \leftrightarrow \langle V_u \cap v \rangle \leftrightarrow \langle V_v \rangle
\]
induced by the inclusions \( V_u \cap v \subset V_u \) and \( V_u \cap v \subset V_v \). Notice that the geometric object \( K \) does not depend on \( V_S \) but rather on \( (S, \subset) \). As an example, one could have decided to embed \( K \) in \( \mathbb{R}^{\cup S} \) instead of \( \mathbb{R}^{\max(|\cup S|)+1} \), by taking any bijection \( \cup S \leftrightarrow |\cup S| \); this leads to a polyhedron which is isomorphic to \( K \). For instance, if \( S_x = 2^x \) is the power set of \( x \) then \( K = \Delta^{2^x} \subset \mathbb{R}^{2^x+1} \) is a standard simplex, which is off course isomorphic to \( \langle x \rangle \), the simplex generated by \( x \).

Conversely, given a geometric polyhedron \( K \) with a finite set \( V_K \) of vertices in some ambient space \( \mathbb{R}^n \), we get a simplicial set by letting
\[
S_K = \{ V \subset V_K : \langle V \rangle \subset K \}
\]
In other words, \( V \in S_K \) iff \( \langle V \rangle \) is a face of \( K \). Notice that, by injecting \( V_K \) into the canonical basis of \( \mathbb{R}^n \) if necessary, \( (V_K, S_K) \) is isomorphic to some \( (x, S_x) \in \mathcal{AP} \) and hence \( S_K \) is essentially a simplicial set. From the above discussion, it should now be clear that each simplicial set \( S = S_x \) coming from some pair \( (x, S_x) \in \mathcal{AP} \) represents, geometrically, the set of faces of some polyhedron \( K \) with finite dimension.

A \textbf{finite ordered polyhedron} is a finite geometric polyhedron for which we have prefixed a linear order on the set of its vertices. It corresponds to a pair \( (x, S_x) \in \mathcal{AP} \). An \textbf{embedding} \( (x, S_x) \xrightarrow{f} (y, S_y) \) is an injective function \( x \xrightarrow{f} y \) such that \( u \in S_x \Rightarrow f(u) \in S_y \). A \textbf{rigid embedding} is an embedding \( (x, S_x) \xrightarrow{f} (y, S_y) \) such that \( f \) is order-preserving: \( i < j \Rightarrow f(i) < f(j) \).

The following will be useful in the sequel:
Lemma 3.1. Each finite ordered polyhedron can be rigidly embedded in some simplex.

[Proof] If \((x, S_x) \in AP\) is a finite polyhedron let \(n = \max(x)\) be the maximum of \(x\). Then the inclusions \(x \subseteq n, S_x \subseteq 2^n\) induce a rigid embedding

\[
(x, S_x) \rightarrow (n, 2^n)
\]

Geometrically, this is just the inclusion of the polyhedron \(K\) determined by \((x, S_x)\) in the standard simplex \(\Delta^n \subseteq \mathbb{R}^{n+1}\).

\(\square\)

3.3. The Ramsey space \(P\) geometrically interpreted. Let \(\{e_n : n \in \mathbb{N}\}\) be the canonical basis of the real separable Hilbert space \(l^2\). By the correspondence

\(\forall u \in S\)

each \((A, S_A) \in P\) can be considered as polyhedron \(K \subseteq l^2\) with an infinite set of vertices \(V_K\) for which we have fixed a linear order \((V_K, <)\) induced by the usual linear order on \(\mathbb{N}\). We call \(P\) the Ramsey space of infinite countable ordered polyhedra. There is also a notion of rigid embeddings in this case.

4. Categorical Ramsey properties for polyhedra

4.1. Ramsey Categories. In this \(\S\) we follow [5]. For any category \(D\), write \(\text{Obj}(D)\) for the collection of its objects. If \(d, d' \in \text{Obj}(D)\), write \((d, d')_D\) for the collection of subobjects of \(d\) which are isomorphic to \(d'\). Consider a category \(D\) satisfying the following properties:

1. \(\text{Obj}(D) = \mathbb{N}\), the set of non-negative integers.
2. For each pair of non-negative integers \(m \leq n\) there is an integer \(r_{m,n}\) such that \(\binom{m}{n}_D\) is a finite set with \(r_{m,n}\) elements. In particular, \(r_{n,0} = 1\).
3. All morphisms of \(D\) are embeddings \([6]\).

\(D\) is said to be Ramsey iff in addition it satisfies

4. The Ramsey property: Given \(m, n, r \in \mathbb{N}\) there is \(N_{(m,n,r)} \in \mathbb{N}\) such that for all \(N \geq N_{(m,n,r)}\) and for every \(r\)-coloring \(\binom{N}{n}_D \rightarrow r\) there exists \(m' \in \binom{N}{m}_D\) such that \(\binom{m'}{n}_D\) is monochromatic. \([4]\)

4.2. Finite polyhedra as Ramsey Category. In virtue of Theorem [2.4] the following instance of Theorem [3.3] holds. This is the version of finite Ramsey’s theorem corresponding to the Ramsey space \(P\) (Recall the notation at the end of \(\S\) [1.2]):

Theorem 4.1. Let \((A, S_A) \in P\) and \(k, n, r \in \mathbb{N}\) be given. Then, there exists \(m \in \mathbb{N}\) such that for every coloring \(c : AP^m_k((A, S_A)) \rightarrow r\), there exists \((b, S_b) \in AP((A, S_A))^m_k\) such that \(c\) is constant in \(AP^m_k((A, S_A), (b, S_b))\).

\(\text{For finite sets this means that all morphisms of } \mathcal{D} \text{ are injective functions.}\)

\(\text{We will assume the usual identification } r = \{0, \ldots, r-1\}, \text{ for every integer } r > 0 \text{ and } 0 = \emptyset.\)
Thanks to Theorem 4.1 it is now easy to prove that the category of finite polyhedra is Ramsey. Notice that we do not need to check the conditions of sufficiency stated in the main Theorem \cite[p.418]{5} but simply translate Theorem 4.1 into its geometric/categorical version.

**Theorem 4.2. [Categorical Ramsey property for polyhedra]** The category \( \mathcal{P} \) of finite ordered polyhedra and rigid embeddings is Ramsey.

**[Proof]** Fix \( k, n, r \in \mathbb{N} \) and apply Theorem 4.1 to \( A = \mathbb{N} \) and \( S_A = \mathbb{N}^{<\infty} \).

\( \square \)

## 5. Ramsey classes

In this section we present some basic concepts on Ramsey classes, Fraïssé theory and extremely amenability of automorphism groups. For more details and in-depth treatment of these subjects see \cite{2, 11, 12, 13}.

### 5.1. Structures

A **signature** \( L = \langle (R_i)_{i \in I}, (F_j)_{j \in J} \rangle \) is a couple of two countable sets, we say that \( (R_i)_{i \in I} \) is a set of **relation symbols** while \( (F_j)_{j \in J} \) is a set of **function symbols**. Each relation (resp. function) symbol \( R_i \) (resp. \( F_j \)) has an associated integer number \( n(i) > 0 \) (resp. \( m(j) \geq 0 \)) called its **arity**.

Each signature \( L \) has an associated countable first order language which contains the symbols of \( L \), a countable set of variable symbols (say \( x_1, x_2, \ldots \) etc.); the usual connective symbols (\( \land, \lor, \neg, \to \)), quantifiers (\( \forall, \exists \)) and a binary symbol for the equality (\( = \)). With little abuse of notation we will still refer to this language with the letter \( L \); it might also have constant symbols (function symbols with arity 0) though we will not insist on them.

Given a language \( L \), a **\( L \)-structure** \( \mathcal{A} = \langle (R_i^\mathcal{A})_{i \in I}, (F_j^\mathcal{A})_{j \in J} \rangle \) is a triple constituted as follows:

- A non empty set \( A \neq \emptyset \) called the **universe** of the structure;
- a set of relations \( (R_i^A)_{i \in I} \) where \( R_i^A \subseteq A^{n(i)} \) for each \( i \in I \); and
- a set of functions \( (F_j^A)_{j \in J} \) where \( F_j^A : A^{m(j)} \to A \) for each \( j \in J \).

We say that \( \mathcal{A} \) is a **relational structure** if \( L \) has no function symbols.

A **morphism** of \( L \)-structures \( \mathcal{A} \to \mathcal{B} \) is a map \( A \to B \) between the respective universe sets, such that for each relation symbol \( R_i \) and each function symbol \( F_j \); it satisfies

- \( (a_1, \ldots, a_{n(i)}) \in R_i^A \) iff \( (\pi(a_1), \ldots, \pi(a_{n(i)})) \in R_i^B \) for all \( a_1, \ldots, a_{n(i)} \in A \).
- \( \pi(F_j^A(a_1, \ldots, a_{m(j)})) = F_j^B(\pi(a_1), \ldots, \pi(a_{m(j)})) \) for all \( a_1, \ldots, a_{m(j)} \in A \).

When \( \pi \) is bijective we say that it is an **isomorphism** and \( \mathcal{A}, \mathcal{B} \) are **isomorphic** structures, or just \( A \cong B \). When \( \pi \) is injective we say that it is an **embedding**. In particular, we say that \( \mathcal{A} \) is a **substructure** of \( \mathcal{B} \), and write \( \mathcal{A} \leq \mathcal{B} \) whenever \( A \subseteq B \) and the inclusion map \( A \to \mathcal{B} \) is an embedding. In that case

- \( R_i^A = R_i^B \cap A^{n(i)} \) for each \( i \in I \).
- \( F_j^A = F_j^B \mid A^{m(j)} \) for each \( j \in J \).

Finally, an **automorphism** of an \( L \)-structure \( \mathcal{A} \) is an isomorphism of \( \mathcal{A} \) on itself; we write \( \text{Aut}(\mathcal{A}) \) for the group of automorphisms of \( \mathcal{A} \).
5.2. **Substructures.** Let \( X \subset B \) be a subset of the universe of some \( L \)-structure \( B \). The **substructure** of \( B \) **generated** by \( X \) is the smallest substructure of \( B \) containing \( X \). Denote such a substructure by \( B_X \). Since the intersection of substructures is a substructure, we have

\[
B_X = \bigcap \{ A \leq B : X \text{ is a subset of the universe of } A \}
\]

We say that \( X \) **spans** \( B_X \). A structure \( B \) is **finitely generated** iff it is spanned by a finite subset. A structure \( B \) is **locally finite** iff all finitely generated substructures of \( B \) are finite.

**Remark 5.1.** Relational structures are locally finite.

5.3. **Age of a structure.** The **age** of a \( L \)-structure \( A \) is the class \( \text{Age}(A) \) of all finite \( L \)-structures which are isomorphic to some substructure of \( A \).

5.4. **Ultrahomogeneous structures.** A locally finite structure \( F \) is **ultrahomogeneous** iff each isomorphism between any two finite substructures of \( F \) can be extended to some automorphism of \( F \). A **Fraïssé structure** is an (infinite) countable ultrahomogeneous structure.

The following results (Proposition 5.2 and Theorems 5.3, 5.4) provide a characterization (Theorem 5.4) of all classes of finite structure which are the age of a Fraïssé structure. See for instance [2, 12] for more details.

**Proposition 5.2.** A locally finite \( L \)-structure \( A \) is ultrahomogeneous iff the following holds: If \( B, C \in \text{Age}(A) \) and \( B \leq C \) then each embedding \( B \to A \) can be extended to an embedding \( C \to A \). \( \square \)

**Theorem 5.3** (Fraïssé). Any two (infinite) countable ultrahomogeneous \( L \)-structures having the same age are isomorphic. \( \square \)

**Theorem 5.4.** A non empty class of finite \( L \)-structures \( C \) is the age of a Fraïssé structure iff it satisfies:

1. \( C \) is closed under isomorphisms: If \( A \in C \) and \( A \cong B \) then \( B \in C \).
2. \( C \) is hereditary: If \( A \in C \) and \( B \leq A \) then \( B \in C \).
3. \( C \) contains structures with arbitrarily high finite cardinality.
4. Joint embedding property: If \( A, B \in C \) then there is \( D \in C \) such that \( A \leq D \) and \( B \leq D \).
5. Amalgamation property: Given \( A, B_1, B_2 \in C \) and embeddings \( A \xrightarrow{f_1} B_i \), \( i \in \{1, 2\} \), there is \( D \in C \) and embeddings \( B_i \xrightarrow{g_i} D \) such that \( g_1 \circ f_1 = g_2 \circ f_2 \).

In such case there is a unique (up to isomorphism, infinite) countable Fraïssé structure \( F \) such that \( \text{Age}(F) = C \); this \( F \) is the **Fraïssé limit** of \( C \) and we write \( F = \text{FLim}(C) \). \( \square \)

5.5. **Fraïssé classes.** A **Fraïssé class** is a class of finite structures \( C \) satisfying the conditions (1)...(5) of Theorem 5.4 above.
5.6. **Colorings of structures.** Given $L$-structures $A, B, C$ we write

- $\text{Emb}(A, B)$ for the set of embeddings from $A$ to $B$ as a substructure; and
- $\left( \begin{array}{c} B \\ A \end{array} \right)$ for the set of substructures of $B$ which are isomorphic to $A$.

Given an integer $r > 0$, if $A \leq B \leq C$ then we write $C \twoheadrightarrow (B)_{A}^{r}$ whenever for each $r$-coloring $c : \left( \begin{array}{c} C \\ A \end{array} \right) \rightarrow r$

of the set $\left( \begin{array}{c} C \\ A \end{array} \right)$, there exists $B' \in \left( \begin{array}{c} C \\ B \end{array} \right)$ such that $\left( \begin{array}{c} B' \\ A \end{array} \right)$ is monochromatic.

**Remark 5.5.** There is an obvious surjective map $\text{Emb}(A, B) \twoheadrightarrow \left( \begin{array}{c} B \\ A \end{array} \right)$. If $A$ is a **rigid structure** (i.e., it does not admit non trivial automorphisms) then $\text{Emb}(A, B)$ and $\left( \begin{array}{c} B \\ A \end{array} \right)$ coincide.

**Remark 5.6.** An **order structure** is a structure in a signature containing the binary symbol $<$, interpreted as a linear order on its universe. If $A, B$ are two order structures in the same signature and $A$ is finite, then $\text{Emb}(A, B)$ and $\left( \begin{array}{c} B \\ A \end{array} \right)$ coincide.

5.7. **Ramsey classes of structures.** A Fraïssé class $C$ has the **Ramsey property** iff, for every integer $r > 1$ and every $A, B \in C$ such that $A \leq B$, there is $C \in C$ such that $C \twoheadrightarrow (B)_{A}^{r}$

See [5, 6, 10, 15, 16, 17, 18, 19] for details and examples.

Let $S_{\infty}$ be the infinite countable symmetric group, i.e., the polish group of bijections from $\mathbb{N}$ to $\mathbb{N}$. Then we have the following.

**Lemma 5.7.** The group $\text{Aut}(F)$ of an infinite countable structure $F$ is a closed subgroup of $S_{\infty}$.

**Remark 5.8.** Hence $\text{Aut}(F)$, with the induced topology as a subspace of $S_{\infty}$, is a polish group. On the other hand, if $A \leq F$ is a finite substructure and $n = |A|$ is the cardinality of its universe, then the action of $S_{\infty}$ on $\mathbb{N}^{[n]}$ given by

$$(\pi, F) \rightarrow \{ \pi(x) : x \in F \},$$

for all $\pi \in S_{\infty}$ and $F \in \mathbb{N}^{[n]}$, naturally induces an action of $\text{Aut}(F)$ on $\left( \begin{array}{c} F \\ A \end{array} \right)$.

Remember that a topological group $G$ is **extremely amenable** or has the **fixed point on compacta property**, if for every continuous action of $G$ on a compact space $X$ there exists $x \in X$ such that for every $g \in G$, $g \cdot x = x$. If $G$ is an extremely amenable group, then its **universal minimal flow** is a singleton. This is a remarkable result in Topological Dynamics. See for instance [10, 8, 7, 13, 19, 24] for more details.
Theorem 5.9. Let $F$ be a Fraïssé structure and $C = \text{Age}(F)$. The polish group $\text{Aut}(F)$ is extremely amenable if and only if $C$ has the Ramsey property and all the structures of $C$ are rigid.

6. The infinite random polyhedron

6.1. Ordered polyhedra as a Ramsey class. It is easy to see that each pair $(x, S_x) \in \mathcal{P} \cup \mathcal{AP}$ is a relational structure whose universe is $x$ and in which $S_x$ is a countable family of relations over $x$. The notions of substructure, homorphism, etc are induced by our geometric interpretation of the Ramsey space $\mathcal{P}$ (see §3). Furthermore, each one of these structures is rigid by construction. In particular, as we showed in Theorem 4.1 the following holds:

Theorem 6.1. $\mathcal{AP}$ is a Ramsey class of rigid finite structures. □

In virtue of of Theorem 5.9, we obtain the following:

Corollary 6.2. Let $\mathcal{P} = \text{FLim}(\mathcal{AP})$, the Fraïssé limit of $\mathcal{AP}$. Then, $\text{Aut}(\mathcal{P})$ with the Polish topology inherited from $S_\infty$ is extremely amenable. □

6.2. The random polyhedron. Consider a countably infinite set $\omega$. We are going to define a family $S_\omega \subseteq \omega^{[<\infty]}$, as follows:

Hold a coin. Even a biased one will be fine, as long as the probability of each side of the coin is not 0. Define a family $T_\omega \subseteq \omega^{[<\infty]}$ probabilistically in the following way: for every $u \in \omega^{[<\infty]}$ such that $|u| > 1$ flip the coin, and say that $u$ is in $T_\omega$ if and only if you get heads. Set

$$S_\omega := \omega[1] \cup \{v : (\exists u \in T_\omega) v \subseteq u\}$$

It is easy to prove that $S_\omega$ is hereditary and $\bigcup S_\omega = \omega$. Hence $(\omega, S_\omega)$ is an infinite polyhedron. We say that $(\omega, S_\omega)$ is an infinite countable random polyhedron.

Lemma 6.3. With probability 1, each finite polyhedron can be embedded in any infinite countable random polyhedron.

[Proof]. Without loss of generality, let us assume that the the probability of getting a head in one flip of the coin is $\frac{1}{2}$. Fix a random polyhedron $(\omega, S_\omega)$, as constructed in §6.2. Since $\omega^{[<\infty]}$ is infinite countable, there is a bijection $\varphi : \omega^{[<\infty]} \rightarrow \mathbb{N}$. Then $S_\omega$ is a random event in the probability space

$$\Omega = \left(2^{\omega^{[<\infty]}}, \equiv 2^{\omega}; \mathbb{B}\left(2^{\omega}\right), P\right)$$

which is just the Cantor space $2^\omega$ with the $\sigma$-algebra $\mathbb{B}\left(2^\omega\right)$ of Borel subsets and the probability measure $P$ on $2^\omega$ induced by the coin-flipping. By our assumptions; given a finite sequence $a \in 2^n$ the probability that the $n$th initial segment of an infinite countable random sequence of flips $z \in 2^\omega$ coincides with $a$ is

$$P(z \upharpoonright n = a) = \frac{1}{2^n}$$

which is the measure of the basic open subset $[a]$ in the product topology of $2^\omega$; see equation (2) in §1. Via the Cantor map, $\Omega$ is isomorphic to the unit interval $[0, 1]$ with the Lebesgue measure. The key argument is that the probability of a singleton in $\Omega$ coincides with the Lebesgue measure of a point in $[0, 1]$, so it vanishes.
The bijection $\varphi$ sends the family of faces of our random polyhedron to some infinite sequence $x = \varphi(S_\omega) \in 2^\mathbb{N}$. In order to show the statement of Lemma 6.3, by Lemma 3.1, it is enough to see that any finite simplex can be embedded in $(\omega, S_\omega)$. This is equivalent to say that $(\omega, S_\omega)$ has substructures which are isomorphic to simplexes of arbitrarily large finite dimension; thus for any positive integer $m$ there must be some $n \geq m$ such that $T_\omega$ has some element $u$ with cardinality $|u| = n$. Let us suppose the contrary; then there is some integer $m > 0$ such that $u \in T_\omega \Rightarrow |u| \leq m$. This implies that the sequence $x = \varphi(S_\omega)$ has finitely many 1’s and corresponds, via the Cantor map, to a rational number in $[0, 1]$; which is a contradiction since the probability of $[0, 1] \cap \mathbb{Q}$ is 0.

**Theorem 6.4.** Let $P = \text{FLim}(AP)$, the Fraïssé limit of $AP$. Then $P$ is an infinite ordered polyhedron which is isomorphic to $(\omega, S_\omega)$, as a polyhedron, and to $(\mathbb{Q}, \leq)$, as an ordered set.

**Proof**] By Lemma 6.3 we get $\text{Age}(\omega, S_\omega) = \text{Age}(P) = AP$. □

**Corollary 6.5.** All infinite countable random polyhedra are isomorphic as countable structures.

**Proof**] By Theorem 6.3 □

The above facts allow us to call $(\omega, S_\omega)$ the random polyhedron.

## 7. Topological Ramsey subspaces of $P$

In this section, for every integer $k > 0$, we will define a topological Ramsey space $P_k$. It turns out that each $P_k$ will be a closed subspace of $P$. In particular, $P_1 = E$, Ellentuck’s space, and $P_2$ is a topological Ramsey space whose elements are essentially the countably infinite ordered graphs. The corresponding set of approximations $AP_2$ is the class of finite ordered graphs, which is a Ramsey class whose Fraïssé limit is the ordered random graph. It is well-known that the automorphism group of the ordered random graph is, as in the case of the ordered random polyhedron, extremely amenable (see [10, 19]).

**Remark 7.1.** Though they have the same probability distribution, the random polyhedron and the random graph are not isomorphic as countable structures.

### 7.1. The subspace $P_k$

Given $k > 0$, consider pairs of the form $(A, S_A)$ where:

- $A \in \mathbb{N}^{[\infty]}$.
- $S_A \subseteq A^{[\leq k]}$.
- $\bigcup S_A = A$.
- $S_A$ is hereditary, i.e., $(u \subseteq v \Rightarrow u \in S_A \Rightarrow u \in S_A)$.

Let us define $P_k$ as the collection of all the pairs $(A, S_A)$ as above. Now, if we consider the restrictions to $\mathbb{N} \times P_k$ of the approximation function $r : \mathbb{N} \times P \rightarrow AP$ and the restrictions to $P_k$ and $AP_k$ of the pre-orders $\leq$ and $\leq_{fin}$ defined on $P$ and $AP$, respectively, we can state the following:

**Theorem 7.2.** For every integer $k > 0$, the triplet $(P_k, r, \leq)$ is a topological Ramsey space. In fact, it is a closed subspace of $(P, r, \leq)$. 


Proof] Given $k > 0$, to show that $(\mathcal{P}_k, r, \leq)$ is a topological Ramsey space, proceed as in the proof of Theorem 2.1. To show that is a closed subspace of $(\mathcal{P}, r, \leq)$, proceed as in the proof of Corollary 2.8.

Actually, it is easy to show that given integers $k' > k > 0$, $\mathcal{P}_k$ is a closed subspace of $\mathcal{P}_{k'}$. As mentioned above, $\mathcal{P}_1$ is Ellentuck’s space $\mathcal{E}$. In virtue of Theorem 7.2 we can state the following:

Corollary 7.3. For every $k > 0$, the family of approximations $\mathcal{A}\mathcal{P}_k$ is a Ramsey class of finite ordered structures. Hence, the automorphism group of $\text{Flim}(\mathcal{A}\mathcal{P}_k)$ is extremely amenable.

7.2. The random $k$-polyhedron. Given a countable set $\omega$, for every $k > 0$ probabilistically define a family $T^k_\omega \subseteq \omega[|k|]$, and a family $S^k_\omega \subseteq \omega[\leq|k|]$, proceeding just as in § 6.2. Notice that the corresponding versions of Lemma 6.3 and Corollary 6.5 can be easily proven in this context. It turns out that the resulting pair $(\omega, S^k_\omega)$ is characterized by the following, up to isomorphism: with probability 1, each element of $\mathcal{A}\mathcal{P}_k$ can be embedded in $(\omega, S^k_\omega)$. We then call $(\omega, S^k_\omega)$ the random $k$-polyhedron. The following is similar to Theorem 6.4.

Theorem 7.4. Let $\mathbb{P}_k = \text{Flim}(\mathcal{A}\mathcal{P}_k)$, the Fraïssé limit of $\mathcal{A}\mathcal{P}_k$. Then $\mathbb{P}_k$ is isomorphic to $(\omega, S^k_\omega)$, as countable structure, and to $(\mathbb{Q}, \leq)$, as an ordered set.

So $\mathbb{P}_k$ is the ordered random $k$-polyhedron. Following § 6.2 it is also clear that the random polyhedron contains an isomorphic copy of the random $k$-polyhedron.

7.3. The ordered random graph. The case $k = 2$ is of special notice. Observe that $\mathcal{A}\mathcal{P}_2$ is just the class of finite ordered graphs. Hence, as it is well-known, its Fraïssé limit $\mathbb{P}_2 = \text{Flim}(\mathcal{A}\mathcal{P}_2)$ is the random ordered graph.

8. Final comments

All these examples reveal that, in general, there seems to be a tight relationship between topological Ramsey spaces and Ramsey classes of ordered finite structures, and therefore between topological Ramsey spaces and extremely amenable automorphism groups. This raises several questions. For instance, consider an abstract setting similar to the one presented in § 2.2. Given a Ramsey class $\mathcal{C}$ of ordered structures, what is the precise description of a topological Ramsey space $\mathcal{R}$ (if any), such that $\mathcal{A}\mathcal{R} = \mathcal{C}$?

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