CONTROL OF ORDINARY DIFFERENTIAL EQUATIONS USING BAGARELLO’S OPERATOR APPROACH: CASE OF FORCED HARMONIC OSCILLATOR SYSTEMS

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ABSTRACT. This work deals with a study of an optimal control of a system of nonlinear differential equations by the Bagarello’s operator approach recently introduced in (Int. Jour. of Theoretical Physics, 43, issue 12 (2004), p. 2371 - 2394). The control problem is reduced, by the Pontryagin’s maximum principle, to a system of ordinary differential equations with unknown state and adjoint variables. The solution of such a system is described by an unbounded self-adjoint and densely defined Hamiltonian operator $H$. Some relevant applications are discussed.

1. Introduction

Various methods are used to solve systems of ordinary differential equations (ODEs) in mathematics and applied sciences. The most popular ones refer to factorization [15], linearization [17], perturbation [14, 20], closure approximation [1, 2], discretization [23], Adomian methods [1, 3, 8, 9, 16] to cite those. Recently, in an interesting paper [7], Bagarello adapted ideas coming from quantum mechanics to develop a non-commutative strategy for the analysis of some systems of ODEs. He provided the solution of such systems described by an unbounded self-adjoint and densely defined Hamiltonian operator $H$ of the system and discussed the role of this Hamiltonian in the analysis of the integrals of motion in the system.

The need might occur to exert, in an appropriate way, an influence on systems of ODEs governing phenomena in natural sciences (for instance physics and biology), in social sciences (e.g. economics and politics), in technology and so on, and what results in controlling these systems. Such optimal control problems can be solved by a so-called Pontryagin’s method.

One of the most important systems encountered in the literature is certainly the system of harmonic oscillators, well spread as a basic tool in physics [12, 19]. This is motivated by their role in many applications in various fields of physics and technology. Besides, the harmonic oscillators are also used as a preliminary tool to approach the understanding of complex systems. For instance, quantum
mechanics as well as optimal control were first illustrated using systems of harmonic oscillators. A typical controlled harmonic oscillator is a pendulum made of a string and a ball moving in a vertical plane and submitted to a force (i.e. the control) whose role is to lead the system to rest in a minimum time.

Recently, Andresen et al [4] dealt with the control of an oscillator via the variation of its frequency and gave a good summary of the underlying control theory. Dasanayake [11] and Van Dooren [25] among others, (see references therein), looked for numerical solutions to optimal control problems and made use of pseudospectral methods and Chebyshev series, respectively.

In the present work, and for the first time to the best of our knowledge of the literature, we extend the noncommutative approach developed by Bagarello to the control of systems of nonlinear ordinary differential equations.

The paper is organized as follows : In section 2, we present the theoretical framework describing the optimal control problem and derive the state and adjoint systems of ordinary differential equations by using the Pontryagin’s maximum principle [13, 21]. Then we apply the Bagarello’s operator method to solve such systems. In section 3, some relevant harmonic oscillator controlled systems are investigated and discussed as illustration.

## 2. Theoretical Framework

We consider the following optimal control problem [6, 24]:

\[
\dot{x} = f(x, u), \quad x(0) = x_0 \tag{2.1}
\]

\[
x(T) = x_1 \tag{2.2}
\]

\[
u \in V \subset E^r \tag{2.3}
\]

\[
\min_{u \in V} \int_0^T f(x, u) dt, \tag{2.4}
\]

where \(x, x_0, x_1 \in E^n\); \(u \in V \subset E^r\) is the control (\(V\) the control set); \(T\) denotes the moment and \(E\) is the Euclidean space. All the functions occurring in the formulation of the problem are assumed to be differentiable with respect to \((x, u)\).

In this problem, the minimum is sought in a class of measurable and bounded functions \(u\) defined on \([0, T]\). For instance, \(V\) can be represented in the form

\[
V = \{u \in E^r : h_i(u) \leq 0, i \in I_1, h_i(u) = 0, i \in I_2\}
\]

where the \(h_i\) are functions expressing the constraints on the controls with \(I_1\) and \(I_2\) denoting finite disjoint index sets.

Let us introduce the Hamilton-Pontryagin’s function

\[
\eta(x, u, \psi) = \psi^* f(x, u) - \psi^0 f(x, u)
\]

where \(f = (f_1, \ldots, f_n) \in E^n\), \(\psi = (\psi_1, \ldots, \psi_n) \in E^n\) the adjoint variable depending on \(t\), \(*\) denotes transposition and \(\psi^0\) a number.
By virtue of the Pontryagin’s maximum principle [13, 21], it is necessary to have the existence of adjoint functions that allows us to attach our control system of ordinary differential equations to our functional objective. The characterization of an optimal control $u^*$ gives a representation in terms of the state and adjoint functions.

Given an optimal control $u^*$ and the corresponding system (2.1), there exists adjoint variable $\psi$ satisfying the following equations

$$\frac{d\psi}{dt} = -\frac{\partial \eta}{\partial x}(x,u^*(x,\psi),\psi).$$

Then we consider the Cauchy problem

$$\dot{x} = f(x,u^*(x,\psi)), \quad x(0) = x_0$$
$$\dot{\psi} = -\frac{\partial \eta}{\partial x}(x,u^*(x,\psi),\psi), \quad \psi(0) = \psi_0$$

with the unknowns $x, \psi$.

Setting $y = \begin{pmatrix} x \\ \psi \end{pmatrix}$, $g(y) = \left( \begin{array}{c}
\frac{f(x,u^*(x,\psi))}{\psi(0)} \\
\frac{\partial \eta}{\partial x}(x,u^*(x,\psi),\psi)
\end{array} \right)$, $g(0) = \begin{pmatrix} x_0 \\ \psi_0 \end{pmatrix}$, we can write the Cauchy problem in the form

$$\dot{y} = g(y), \quad y(0) = y_0 = \begin{pmatrix} x_0 \\ \psi_0 \end{pmatrix},$$

equivalently

$$\begin{align*}
\dot{y}_1 &= g_1(y_1, y_2, \ldots, y_{2n}) \\
\dot{y}_2 &= g_2(y_1, y_2, \ldots, y_{2n}) \\
&\vdots \\
\dot{y}_{2n} &= g_{2n}(y_1, y_2, \ldots, y_{2n}) \\
y_j(0) &= y_j^0, \quad j = 1, \ldots, 2n.
\end{align*}$$

We suppose that the functions $g_j$ above are such that the solution of the Cauchy problem (2.9) exists and is unique.

Then we solve this system by Bagarello’s approach ([7], and references therein), based on a suggestion coming from quantum mechanics. Indeed given a quantum mechanical system $\mathcal{S}$ and the related set of observables $\mathcal{O}_S$ that is the set of all the self-adjoint bounded (or more often unbounded) operators related to $\mathcal{S}$, the evolution of any observable $Y \in \mathcal{O}_S$ satisfies the Heisenberg equation of motion (HOEM) [7]

$$\frac{d}{dt} Y(t) = i[H,Y(t)].$$

Here $[A,B] = AB - BA$ is the commutator between $A, B \in \mathcal{O}_S$; $H$ is assumed to be a densely defined self-adjoint Hamiltonian operator of the system acting on...
some Hilbert space $\mathcal{H}$, given by [7]

$$H(\vec{g}_0) = \frac{1}{2} \sum_{j=1}^{2n} \left\{ p_j g_j(y_1^0, y_2^0, \ldots, y_{2n}^0) + g_j(y_1^0, y_2^0, \ldots, y_{2n}^0) p_j \right\}$$

(2.11)

where $\vec{g}_0 = \left( g_1(\vec{Y}^0), g_2(\vec{Y}^0), \ldots, g_{2n}(\vec{Y}^0) \right)$, $\vec{Y}^0 = (y_1^0, y_2^0, \ldots, y_{2n}^0)$; the initial position $y_j^0$ is considered as an operator acting on the Hilbert space $\mathcal{H}$, $p_j$ another operator acting on $\mathcal{H}$, canonical conjugate momentum associated to each $y_j^0$ such that

$$[y_j^0, p_k] = i\delta_{jk}1, \quad j, k = 1, \ldots, 2n$$

(2.12)

$$[y_j^0, y_k^0] = [p_j, p_k] = 0, \quad j, k = 1, \ldots, 2n.$$  

(2.13)

Standard results in quantum mechanics show that for any differentiable functions $\varphi(y_1^0, y_2^0, \ldots, y_{2n}^0)$, $\hat{\varphi} = (p_1, p_2, \ldots, p_{2n})$, we have

$$[p_j, \varphi(y_1^0, y_2^0, \ldots, y_{2n}^0)] = -i\frac{\partial \varphi}{\partial y_j^0}, \quad j = 1, \ldots, 2n$$

(2.14)

$$[y_j^0, \hat{\varphi} = (p_1, p_2, \ldots, p_{2n})] = i\frac{\partial \hat{\varphi}}{\partial p_j}, \quad j = 1, \ldots, 2n.$$  

(2.15)

**Theorem 1.** [7] If the functions $g_j$ are holomorphic, a formal solution of the HOEM (2.9) is

$$Y(t) = e^{iHt} Y^0 e^{-iHt}.$$  

(2.16)

where $Y^0$ is the initial value of $Y(t)$ and $H$ does not depend explicitly on time.

Furthermore if $H$ is bounded, we get

$$Y(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} [H, Y^0]_k$$

(2.17)

where $[A, B]_k$ is the multiple commutator defined recursively as:

$$[A, B]_0 = B; \quad [A, B]_k = [A, [A, B]_{k-1}].$$

The concepts of integral of motion and extended integral of motion of the system (2.9) are defined as [7]

**Definition 1.** Any holomorphic function $I$ depending on the variables $y_j$, such that

$$I(y_1(t), y_2(t), \ldots, y_{2n}(t)) = I_0 \quad \forall t, \quad I_0 = \text{const.},$$

is called an Integral of motion (IoM) of system (2.9).

**Definition 2.** We call extended integral of motion (EIoM) of the system (2.9) any holomorphic function $J$ depending on the $y_j, p_j$ such that

$$J(y_1(t), y_2(t), \ldots, y_{2n}(t), p_1(t), p_2(t), \ldots, p_{2n}(t)) = J_0 \quad \forall t, J_0 = \text{const.},$$

where $y_j(t) = e^{itH} y_j^0 e^{-itH}$ and $p_j(t) = e^{itH} p_j e^{-itH}, j = 1, \ldots, 2n.$
One of the main advantages of the Bagarello’s strategy is to provide at hand a good approximation of the solution of the SODE. The $N$–th order approximation $(N \in \mathbb{N})$ is

$$
Y_N(t) = \sum_{k=0}^{N} \frac{(it)^k}{k!} [H, Y^0]_k.
$$

(2.18)

Using the integral of motion, we can estimate the approximation’s error of the unknown exact solution of the derived Cauchy’s problem. In this case, one can proceed as follows: let $Y_N(t)$ be the approximated solution of $Y(t)$. We may thus compute and evaluate the error

$$
\Delta_N(t) = I(Y_N(t)) - I(Y^0).
$$

In the next section, in application of our theory, we look for solution to time-optimal control problems.

3. Applications

3.1. Problem 1. In classical mechanics, we study the forced harmonic oscillator. The corresponding optimal control problem has the form [18]

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\omega^2 x_1 + u(t) \\
x_1(0) &= x_1^0; \quad x_2(0) = x_2^0; \\
x_1(T) &= 0; \quad x_2(T) = 0; \\
-1 &\leq u \leq 1 \\
T &\rightarrow \inf
\end{align*}
$$

(3.1)

Here $t$ and $T$ denote the time, $x = (x_1, x_2) \in E^2$, $u$ is the control and $\omega$ the oscillation frequency.

In this problem, the infimum is sought in a class of controls $u(t), t \geq 0$. We study a control $u$ that moves the point $x_0$ to the point $x(T) = 0$ in accordance with the corresponding solution of (3.1) during the time $T$. Then we solve the boundary value problem derived from the Pontryagin’s maximum principle.

Let us consider the Hamilton-Pontryagin’s function

$$
\eta(x, u, \psi) = -1 + \psi_1 x_2 + \psi_2 (-\omega^2 x_1 + u)
$$

(3.2)

where $\psi = (\psi_1, \psi_2) \in E^2$.

According to Pontryagin’s maximum principle, we obtain the control $u$ in the form

$$
 u(t) = \begin{cases} 
 1 & \text{if } \psi_2(t) > 0 \\
 -1 & \text{if } \psi_2(t) < 0
\end{cases}
$$

Now we solve the Cauchy problem derived from the Pontryagin’s maximum principle using the technique developed by Bagarello.
In the case \( u(t) = 1 \), we can write

\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\omega^2 x_1 + 1 \\
\dot{\psi}_1 &= \omega^2 \psi_2 \\
\dot{\psi}_2 &= -\psi_1
\end{aligned}
\]

\( (3.3) \)

\[ x_1(0) = x_1^0; \quad x_2(0) = x_2^0; \quad \psi_1(0) = \psi_1^0; \quad \psi_2(0) = \psi_2^0; \]

equivalently

\[
\begin{aligned}
\dot{\tilde{x}}_1 &= x_2 \\
\dot{\tilde{x}}_2 &= -\omega^2 \tilde{x}_1 \\
\dot{\psi}_1 &= \omega^2 \psi_2 \\
\dot{\psi}_2 &= -\psi_1
\end{aligned}
\]

\( (3.4) \)

where \( \tilde{x}_1 = x_1 - \frac{1}{\omega^2} \).

In the case \( u(t) = -1 \) we get

\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\omega^2 x_1 - 1 \\
\dot{\psi}_1 &= \omega^2 \psi_2 \\
\dot{\psi}_2 &= -\psi_1
\end{aligned}
\]

\( (3.5) \)

equivalently

\[
\begin{aligned}
\dot{\hat{x}}_1 &= x_2 \\
\dot{\hat{x}}_2 &= -\omega^2 \hat{x}_1 \\
\dot{\psi}_1 &= \omega^2 \psi_2 \\
\dot{\psi}_2 &= -\psi_1
\end{aligned}
\]

\( (3.6) \)

with the previous initial conditions and where \( \hat{x}_1 = x_1 + \frac{1}{\omega^2} \).

Equations (3.4) and (3.6) reduce to the equivalent system

\[
\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -\omega^2 y_1 \\
\dot{y}_3 &= \omega^2 y_4 \\
\dot{y}_4 &= -y_3
\end{aligned}
\]

\( (3.7) \)

where \( y_1 = \hat{x}_1 \) or \( y_1 = \tilde{x}_1, y_2 = x_2, y_3 = \psi_1, y_4 = \psi_2 \). Using the property of the commutator

\[ g_j p_j = -[p_j, g_j] + p_j g_j, \quad j = 1, \ldots, 4, \]

we compute the Hamiltonian of the previous system in the form

\[ H = p_1 y_2^0 - \omega^2 p_2 y_1^0 + \omega^2 p_3 y_4^0 - p_4 y_3^0. \]
Expecting to develop the solution of (3.7) as the infinite series below
\[
\begin{pmatrix}
y_1(t) \\
y_2(t) \\
y_3(t) \\
y_4(t)
\end{pmatrix} = \begin{pmatrix}
y_0^{\prime} \\
y_0^{\prime} \\
y_0^{\prime} \\
y_0^{\prime}
\end{pmatrix} + it \begin{bmatrix}
y_0^{\prime} \\
y_0^{\prime} \\
y_0^{\prime} \\
y_0^{\prime}
\end{bmatrix} + \frac{(it)^2}{2!} \begin{bmatrix}
y_0^{\prime} \\
y_0^{\prime} \\
y_0^{\prime} \\
y_0^{\prime}
\end{bmatrix} + \frac{(it)^3}{3!} \begin{bmatrix}
y_0^{\prime} \\
y_0^{\prime} \\
y_0^{\prime} \\
y_0^{\prime}
\end{bmatrix} + \ldots
\]

we compute the multiple commutator
\[
\begin{pmatrix}
[H, y_0^{\prime}]_{2p} \\
[H, y_0^{\prime}]_{2p} \\
[H, y_0^{\prime}]_{2p} \\
[H, y_0^{\prime}]_{2p}
\end{pmatrix} = \begin{pmatrix}
\omega^{2p} y_0^{\prime} \\
\omega^{2p} y_0^{\prime} \\
\omega^{2p} y_0^{\prime} \\
\omega^{2p} y_0^{\prime}
\end{pmatrix}, \quad \begin{pmatrix}
[H, y_0^{\prime}]_{2p+1} \\
[H, y_0^{\prime}]_{2p+1} \\
[H, y_0^{\prime}]_{2p+1} \\
[H, y_0^{\prime}]_{2p+1}
\end{pmatrix} = \begin{pmatrix}
-\imath \omega^{2p} y_0^{\prime} \\
\imath \omega^{2p} y_0^{\prime} \\
\imath \omega^{2p} y_0^{\prime} \\
\imath \omega^{2p} y_0^{\prime}
\end{pmatrix}
\]
and obtain
\[
\begin{pmatrix}
y_1(t) \\
y_2(t) \\
y_3(t) \\
y_4(t)
\end{pmatrix} = \begin{pmatrix}
y_0^{\prime} \cos(\omega t) + \frac{x_0^{\prime}}{\omega} \sin(\omega t) \\
y_0^{\prime} \cos(\omega t) - \frac{y_0^{\prime}}{\omega} \sin(\omega t) \\
y_0^{\prime} \cos(\omega t) + \frac{y_0^{\prime}}{\omega} \sin(\omega t) \\
y_0^{\prime} \cos(\omega t) - \frac{y_0^{\prime}}{\omega} \sin(\omega t)
\end{pmatrix}
\]

In order to determine the time \( T \) such that
\[
\begin{cases}
x_1(T) = 0 \\
x_2(T) = 0
\end{cases}
\]

and taking into account \( y_1 = \tilde{x}_1 = x_1 - \frac{1}{\omega^2}, y_2 = x_2 \), we can write
\[
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix} = \begin{pmatrix}
(x_0^{\prime} - \frac{1}{\omega^2}) \cos(\omega t) + \frac{x_0^{\prime}}{\omega} \sin(\omega t) + \frac{1}{\omega^2} \\
x_0^{\prime} \cos(\omega t) - (x_0^{\prime} - \frac{1}{\omega^2}) \omega \sin(\omega t)
\end{pmatrix}
\]
and
\[
\cos(\omega T) = \frac{\omega}{x_2} (x_1^{\prime} - \frac{1}{\omega^2}) \sin(\omega T); \quad \sin(\omega T) = \frac{-x_2^{\prime} / \omega}{\omega^2 (x_1^{\prime} - \frac{1}{\omega^2})^2 + (x_2^{\prime})^2}.
\]
Thus
\[
\tan \omega T = \frac{x_2^{\prime}}{\omega (x_1^{\prime} - \frac{1}{\omega^2})} \quad \text{or} \quad \omega T = \arctan \frac{x_2^{\prime}}{\omega (x_1^{\prime} - \frac{1}{\omega^2})};
\]
\[
\tilde{T} = \frac{1}{\omega} \arctan \frac{x_2^{\prime}}{\omega (x_1^{\prime} - \frac{1}{\omega^2})}, \quad x_2^{\prime} \left( x_1^{\prime} - \frac{1}{\omega^2} \right) > 0
\]

Analogously, as \( y_1 = \tilde{x}_1 = x_1 + \frac{1}{\omega^2}, y_2 = x_2 \),
\[
\begin{pmatrix}
x_1(t) \\
x_2(t)
\end{pmatrix} = \begin{pmatrix}
(x_1^{\prime} + \frac{1}{\omega^2}) \cos(\omega t) + \frac{x_0^{\prime}}{\omega} \sin(\omega t) - \frac{1}{\omega^2} \\
x_0^{\prime} \cos(\omega t) - (x_1^{\prime} + \frac{1}{\omega^2}) \omega \sin(\omega t)
\end{pmatrix}
\]
and the relations
\[
\cos(\omega T) = \frac{\omega}{x_2^0} \left( x_1^0 + \frac{1}{\omega^2} \right) \sin(\omega T); \quad \sin(\omega T) = \frac{x_2^0}{\omega^2(x_1^0 + \frac{1}{\omega^2})^2 + (x_2^0)^2}
\]

imply
\[
\tan \omega T = \frac{x_2^0}{\omega(x_1^0 + \frac{1}{\omega^2})} \quad \text{or} \quad \omega T = \arctan \frac{x_2^0}{\omega(x_1^0 + \frac{1}{\omega^2})};
\]
\[
\hat{T} = \frac{1}{\omega} \arctan \frac{x_2^0}{\omega(x_1^0 + \frac{1}{\omega^2})}, \quad x_2^0 \left( x_1^0 + \frac{1}{\omega^2} \right) > 0.
\]

The optimal time solution to problem (3.1) is given considering the two following cases: if \( x_2^0 > 0 \), then \( \hat{T} \) is the optimal time and if \( x_2^0 < 0 \), then \( \tilde{T} \) is the optimal time.

3.2. Problem 2. Considering a pendulum with large oscillations the corresponding optimal control problem has the form [26]

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\beta x_2 - \sin x_1 + u(t) \\
x_1(0) &= x_1^0, \quad x_2(0) = x_2^0, \\
x_1(T) &= 0; \quad x_2(T) = 0; \\
-1 &\leq u \leq 1 \\
T &\rightarrow \inf
\end{align*}
\]

where \( u(t) = \frac{1}{m} F(t) \), \( 0 \leq t \leq T \) is the control submitted to the following constraint:
\[
u \in V = \left\{ u \in E^1 : |u| \leq 1 \right\}; \quad (3.10)
\]
x_1^0, x_2^0 are given positive constants, m is the mass, F the force and \( \beta > 0 \) a constant.

For this problem the Hamilton-Pontryagin’s function is
\[
\eta(x, u, \psi) = -1 + \psi_1 x_2 + \psi_2(-\beta x_2 - \sin x_1 + u) \quad (3.11)
\]

where \( \psi = (\psi_1, \psi_2) \in E^2 \).

By virtue of the Pontryagin’s maximum principle the control is given by
\[
u(t) = \begin{cases} 
1 & \text{if } \psi_2(t) > 0 \\
-1 & \text{if } \psi_2(t) < 0
\end{cases}
\]

and the Cauchy problem derived from the Pontryagin’s maximum principle is
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\beta x_2 - \sin x_1 + u(t) \\
\dot{\psi}_1 &= \psi_2 \cos x_1 \\
\dot{\psi}_2 &= -\psi_1 + \beta \psi_2
\end{align*}
\]

with the initial conditions
\[
x_1(0) = x_1^0, x_1^0 > 0; \quad x_2(0) = x_2^0, x_2^0 > 0; \quad \psi_1(0) = \psi_1^0; \quad \psi_2(0) = \psi_2^0;
\]
equivalent to

\[
\begin{aligned}
\dot{y}_1 &= y_2 \\
y_2 &= -\beta y_2 - \sin y_1 + u(t) \\
y_3 &= y_4 \cos y_1 \\
y_4 &= -y_3 + \beta y_4
\end{aligned}
\] (3.13)

\[
y_1(0) = y_1^0; \quad y_2(0) = y_2^0; \quad y_3(0) = y_3^0; \quad y_4(0) = y_4^0
\]

where \(y_1 = x_1, y_2 = x_2, y_3 = \psi_1, y_4 = \psi_2\).

The Hamiltonian of the system is

\[
H = p_1 y_2^0 + p_2 \left[ -\beta y_2^0 - \sin y_1^0 + u(t) \right] - i\beta + p_3 y_4^0 \cos y_1^0 + p_4 (-y_3^0 + \beta y_4^0). \quad (3.14)
\]

The solution in the (2.13) becomes here

\[
Y_2(t) = \left( \begin{array}{c} y_1^0 \\ y_2^0 \\ y_3^0 \\ y_4^0 \end{array} \right) + it \left[ H, \left( \begin{array}{c} y_1^0 \\ y_2^0 \\ y_3^0 \\ y_4^0 \end{array} \right) \right] + \frac{(it)^2}{2!} \left[ H, \left[ H, \left( \begin{array}{c} y_1^0 \\ y_2^0 \\ y_3^0 \\ y_4^0 \end{array} \right) \right] \right]
\]

where the commutators are given by

\[
\begin{pmatrix}
[H, y_1^0] \\
[H, y_2^0] \\
[H, y_3^0] \\
[H, y_4^0]
\end{pmatrix} = \begin{pmatrix}
-iy_2^0 \\
-iy_4^0 \cos y_1^0 \\
-i(-y_3^0 + \beta y_4^0)
\end{pmatrix};
\]

\[
\begin{pmatrix}
[H, y_1^0] \\
[H, y_2^0] \\
[H, y_3^0] \\
[H, y_4^0]
\end{pmatrix} = \begin{pmatrix}
(-i)^2 \left[ -\beta y_2^0 - \sin y_1^0 + u(t) \right] \\
-\beta \left[ -\beta y_2^0 - \sin y_1^0 + u(t) \right] - \cos(y_1^0) y_2^0 \\
(-i)^2 \left[ -y_2^0 y_4^0 \sin(y_1^0) + (y_3^0 - \beta y_4^0) \cos(y_1^0) \right] \\
(-i)^2 \left[ -y_2^0 \cos(y_1^0) + \beta(-y_3^0 + \beta y_4^0) \right]
\end{pmatrix}.
\]

Finally we get

- For \(u(t) = -1\),

\[
\begin{pmatrix}
\dot{y}_{1,2}(t) \\
\dot{y}_{2,2}(t)
\end{pmatrix} = \begin{pmatrix}
y_1^0 + ty_2^0 + \frac{\ell^2}{2}(-\beta y_2^0 - \sin y_1^0 - 1) \\
y_2^0 + ty_2^0 + \frac{\ell^2}{2}(-\beta y_2^0 - \sin y_1^0 - 1) - y_2^0 \cos y_1^0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\dot{x}_{1,2}(t) \\
\dot{x}_{2,2}(t)
\end{pmatrix} = \begin{pmatrix}
x_1^0 + tx_2^0 + \frac{\ell^2}{2}(-\beta x_2^0 - \sin x_1^0 - 1) \\
x_2^0 + tx_2^0 + \frac{\ell^2}{2}(-\beta x_2^0 - \sin x_1^0 - 1) - x_2^0 \cos x_1^0
\end{pmatrix}
\]

- For \(u(t) = 1\),

\[
\begin{pmatrix}
\dot{y}_{1,2}(t) \\
\dot{y}_{2,2}(t)
\end{pmatrix} = \begin{pmatrix}
y_1^0 + ty_2^0 + \frac{\ell^2}{2}(-\beta y_2^0 - \sin y_1^0 + 1) \\
y_2^0 + ty_2^0 + \frac{\ell^2}{2}(-\beta y_2^0 - \sin y_1^0 + 1) - y_2^0 \cos y_1^0
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\dot{x}_{1,2}(t) \\
\dot{x}_{2,2}(t)
\end{pmatrix} = \begin{pmatrix}
x_1^0 + tx_2^0 + \frac{\ell^2}{2}(-\beta x_2^0 - \sin x_1^0 + 1) \\
x_2^0 + tx_2^0 + \frac{\ell^2}{2}(-\beta x_2^0 - \sin x_1^0 + 1) - x_2^0 \cos x_1^0
\end{pmatrix}
\]
Taking into account the relations
\[
\begin{align*}
\begin{cases}
\tilde{x}_{1,2}(T) = 0 \\
\tilde{x}_{2,2}(T) = 0
\end{cases}
\quad \text{and} \quad
\begin{cases}
\hat{x}_{1,2}(T) = 0 \\
\hat{x}_{2,2}(T) = 0
\end{cases}
\end{align*}
\]
we obtain
\[
(i) \quad \tilde{T} = \frac{(-\beta x_2^0 - \sin x_1^0 - 1)(x_2^0 + \beta x_1^0) + x_1^0 x_2^0 \cos x_1^0}{(-\beta x_2^0 - \sin x_1^0 - 1)(\sin x_1^0 + 1) - (x_2^0)^2 \cos x_1^0}
\]
and \(a_2 \left( (a_1 c_2 - a_2 c_1)^2 + (a_2 b_1 - a_1 b_2)(b_1 c_2 - b_2 c_1) \right) = 0\)

with \(a_1, b_1, c_1\) and \(a_2, b_2, c_2\) being respectively the coefficients of the quadratic polynomials \(\tilde{x}_{1,2}(t)\) and \(\tilde{x}_{2,2}(t)\).

\[
(ii) \quad \hat{T} = \frac{(-\beta x_2^0 - \sin x_1^0 + 1)(x_2^0 + \beta x_1^0) + x_1^0 x_2^0 \cos x_1^0}{(-\beta x_2^0 - \sin x_1^0 + 1)(\sin x_1^0 + 1) - (x_2^0)^2 \cos x_1^0}
\]
and \(A_2 \left[ (A_1 C_2 - A_2 C_1)^2 + (A_2 B_1 - A_1 B_2)(B_1 C_2 - B_2 C_1) \right] = 0\)

with \(A_1, B_1, C_1\) and \(A_2, B_2, C_2\) being respectively the coefficients of the quadratic polynomials \(\hat{x}_{1,2}(t)\) and \(\hat{x}_{2,2}(t)\).

If
\[
\frac{(-\beta x_2^0 - \sin x_1^0 - 1)(x_2^0 + \beta x_1^0) + x_1^0 x_2^0 \cos x_1^0}{(-\beta x_2^0 - \sin x_1^0 - 1)(\sin x_1^0 + 1) - (x_2^0)^2 \cos x_1^0}
< \frac{(-\beta x_2^0 - \sin x_1^0 + 1)(x_2^0 + \beta x_1^0) + x_1^0 x_2^0 \cos x_1^0}{(-\beta x_2^0 - \sin x_1^0 + 1)(\sin x_1^0 + 1) - (x_2^0)^2 \cos x_1^0},
\]
then \(\tilde{T}\) is the solution of the problem. Otherwise \(\hat{T}\) is the solution.

Let us estimate the error using the following integral of motion
\[
I(x_1, x_2) = x_2 + \beta x_1 + \int_0^t [\sin x_1(\tau) - u(\tau)] d\tau = x_2^0 + \beta x_1^0.
\]

Then
\[
(i) \quad \text{for} \ u(t) = -1,
\]
\[
\tilde{\Delta}_2(t) = I(\tilde{x}_{1,2}, \tilde{x}_{2,2}) - I(x_1^0, x_2^0) = t(-\sin x_1^0 - 1) - \frac{t^2}{2} x_2^0 \cos x_1^0
\]
\[
+ \int_0^t \left\{ \sin \left[ x_1^0 + \tau x_2^0 + \frac{t^2}{2} (-\beta x_2^0 - \sin x_1^0 - 1) \right] + 1 \right\} d\tau;
\]
\[
|\tilde{\Delta}_2(t)| \leq \frac{t^2}{2} x_2^0 + 4t;
\]

leading to \(|\tilde{\Delta}_2(t)| < 1\) for \(t \in \left[ 0, -\frac{4 + \sqrt{16 + 2x_2^0}}{x_2^0} \right] \).
(ii) for \( u(t) = 1 \),
\[
\hat{\Delta}_2(t) = I(\hat{x}_{1,2}, \hat{x}_{2,2}) - I(x_1^0, x_2^0) = t(-\sin x_1^0 + 1) - \frac{t^2}{2} x_2^0 \cos x_1^0
+ \int_0^t \left\{ \sin \left[ x_1^0 + \tau x_2^0 + \frac{\tau^2}{2} (-\beta x_2^0 - \sin x_1^0 + 1) \right] - 1 \right\} d\tau;
\]
\[
\left| \hat{\Delta}_2(t) \right| \leq \frac{t^2}{2} x_2^0 + 4t;
\]
yielding \( \left| \hat{\Delta}_2(t) \right| < 1 \) for \( t \in \left[ 0, -4 + \sqrt{16 + 2 x_2^0} \right] \).

3.3. Problem 3. Let us examine the optimal control problem with the state equation being the Van der Pol equation \cite{5, 10}
\[
\ddot{x} + \varepsilon \dot{x}(x^2 - 1) + x = u(t) \quad (3.15)
\]
such that the control \( u(t) \in [\alpha, \beta] \). The problem has the form
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \varepsilon x_2(1 - x_1^2) + u(t) \\
x_1(0) &= x_1^0, \quad x_2(0) = x_2^0; \\
x_1(T) &= 0; \quad x_2(T) = 0; \\
\alpha &\leq u \leq \beta \\
T &\rightarrow \inf
\end{align*}
\] (3.16)
where \( \varepsilon, \alpha, \beta \) are real constants.

The Hamilton-Pontryagin’s function is
\[
\eta(x, u, \psi) = -1 + \psi_1 x_2 + \psi_2 \left[ -x_1 + \varepsilon x_2(1 - x_1^2) + u(t) \right]. \quad (3.17)
\]

According to the Pontryagin’s maximum principle the supremum of the function \( \eta \) depending on \( x_1, x_2, \psi_1, \psi_2, u \) with respect to \( u \) is reached when the control takes the following form
\[
u(t) = \begin{cases} 
\beta & \text{if } \psi_2(t) > 0 \\
\alpha & \text{if } \psi_2(t) < 0.
\end{cases}
\]

Two cases are examined

(i) Taking \( u(t) = \alpha \), we get the following Cauchy problem
\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \varepsilon x_2(1 - x_1^2) + \alpha \\
\psi_1 &= (1 + 2\varepsilon x_1 x_2) \psi_2 \\
\psi_2 &= -[\psi_1 + \varepsilon (1 - x_1^2) \psi_2] \\
x_1(0) &= x_1^0; \quad x_2(0) = x_2^0; \quad \psi_1(0) = \psi_1^0; \quad \psi_2(0) = \psi_2^0;
\end{align*}
\] (3.18)
which, according to (2.9), reduces to

\[
\begin{align*}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= -y_1 + \varepsilon y_2 (1 - y_1^2) + \alpha \\
\dot{y}_3 &= (1 + 2\varepsilon y_1 y_2) y_4 \\
\dot{y}_4 &= -[y_3 + \varepsilon (1 - y_1^2) y_4] \\
y_1(0) &= y_1^0; \quad y_2(0) = y_2^0; \quad y_3(0) = y_3^0; \quad y_4(0) = y_4^0.
\end{align*}
\]  

(3.19)

The Hamiltonian of the system is computed as follows

\[
H = p_1 y_2^0 + p_2 \{ -y_1^0 + \varepsilon y_2^0 [1 - (y_1^0)^2] + \alpha \} \\
+ p_3 (1 + 2\varepsilon y_1^0 y_2^0) y_4^0 - p_4 \{ y_3^0 + \varepsilon [1 - (y_1^0)^2] y_4^0 \}.
\]  

(3.20)

The first order approximation solutions are given by

\[
\begin{bmatrix}
H, y_1^0 \\
H, y_2^0 \\
H, y_3^0 \\
H, y_4^0
\end{bmatrix} = \begin{bmatrix}
-i \{-y_1^0 + \varepsilon y_2^0 [1 - (y_1^0)^2] + \alpha \} \\
- i \{(1 + 2\varepsilon y_1^0 y_2^0) y_4^0 \} \\
i \{y_3^0 + \varepsilon [1 - (y_1^0)^2] y_4^0 \}
\end{bmatrix};
\]

\[
\begin{bmatrix}
\dot{y}_1 (t) \\
\dot{y}_2 (t)
\end{bmatrix} = \begin{bmatrix}
y_1^0 + t y_2^0 \\
y_2^0 + t \{-y_1^0 + \alpha y_2^0 [1 - (y_1^0)^2] \}
\end{bmatrix}.
\]

In the initial variables

\[
\begin{bmatrix}
\dot{x}_{1.1} (t) \\
\dot{x}_{2.1} (t)
\end{bmatrix} = \begin{bmatrix}
x_1^0 + t x_2^0 \\
x_2^0 + t \{-x_1^0 + \alpha x_2^0 [1 - (x_1^0)^2] \}
\end{bmatrix}.
\]

(ii) For \( u(t) = \beta \), the corresponding derived Cauchy problem is given by

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \varepsilon x_2 (1 - x_1^2) + \beta \\
\dot{\psi}_1 &= (1 + 2\varepsilon x_1 x_2) \psi_2 \\
\dot{\psi}_2 &= -[\psi_1 + \varepsilon (1 - x_1^2) \psi_2] \\
x_1(0) &= x_1^0; \quad x_2(0) = x_2^0; \quad \psi_1(0) = \psi_1^0; \quad \psi_2(0) = \psi_2^0.
\end{align*}
\]  

(3.21)

This SODE differs from (3.19) only by the term \( \beta \) replacing \( \alpha \). Then replacing mutatis mutandis \( \alpha \) by \( \beta \) the SODE remains the same as in (3.19). Then the first order approximation solutions are given in the form

\[
\begin{bmatrix}
\dot{y}_{1.1} (t) \\
\dot{y}_{2.1} (t)
\end{bmatrix} = \begin{bmatrix}
y_1^0 + t y_2^0 \\
y_2^0 + t \{-y_1^0 + \beta y_2^0 [1 - (y_1^0)^2] \}
\end{bmatrix}
\]

or equivalently in terms of the original variables,

\[
\begin{bmatrix}
\dot{x}_{1.1} (t) \\
\dot{x}_{2.1} (t)
\end{bmatrix} = \begin{bmatrix}
x_1^0 + t x_2^0 \\
x_2^0 + t \{-x_1^0 + \beta x_2^0 [1 - (x_1^0)^2] \}
\end{bmatrix}.
\]

Taking into account the relations

\[
\begin{align*}
\dot{x}_{1.1} (T) &= 0 \\
\dot{x}_{2.1} (T) &= 0 \quad \text{and} \quad \dot{x}_{1.1} (T) = 0 \quad \dot{x}_{2.1} (T) = 0,
\end{align*}
\]
we get the following results

\[ \tilde{T} = \frac{x_0^0}{x_1^0 - \alpha x_2^0 [1 - (x_1^0)^2]} \quad \text{and} \quad \hat{T} = \frac{x_2^0}{x_1^0 - \beta x_2^0 [1 - (x_1^0)^2]}, \]

with the relations

\[ (x_2^0)^2 + (x_1^0)^2 - \alpha x_1^0 x_2^0 [1 - (x_1^0)^2] = 0 \quad \text{or} \quad (x_2^0)^2 + (x_1^0)^2 - \beta x_1^0 x_2^0 [1 - (x_1^0)^2] = 0. \]

The optimal time solution to the problem is

\[ \min(\tilde{T}, \hat{T}) = \begin{cases} 
\tilde{T} & \text{if } \frac{x_0^0}{x_1^0 - \alpha x_2^0 [1 - (x_1^0)^2]} < \frac{x_2^0}{x_1^0 - \beta x_2^0 [1 - (x_1^0)^2]}, \\
\hat{T} & \text{otherwise}. 
\end{cases} \]

An integral of motion of the SODE \((3.16)\) is

\[ J(x_1, x_2) = x_2 + \varepsilon \left( \frac{1}{3} x_1^3 - x_1 \right) + \int_0^t [x_1(\tau) - u(t)] d\tau \tag{3.22} \]

\[ = x_2 + \varepsilon \left( \frac{1}{3} (x_1^0)^3 - x_1^0 \right). \]

Setting

(i) for \(u(t) = \alpha,\)

\[ \tilde{\Delta}_1(t) = J(\tilde{x}_{1,1}, \tilde{x}_{2,1}) - J(x_1^0, x_2^0) = \frac{1}{3} (x_2^0)^3 t^2 + \left[ \frac{1}{2} x_2^0 + \varepsilon \frac{1}{3} x_1^0 (x_2^0)^2 \right] t^2 + \left\{ \alpha \left[ x_2^0 - (x_1^0)^2 x_2^0 \right] + \varepsilon x_2^0 [(x_1^0)^2 - 1] \right\} t, \]

and using the Cardan’s formulas for cubic polynomials, we obtain

\[ |\tilde{\Delta}_1(t)| < 1 \quad \text{for } t \in \left[ 0, \sqrt[3]{-\frac{q}{2} + \sqrt{Q}} - \sqrt[3]{-\frac{q}{2} - \sqrt{Q}} - \frac{A}{3} \right], \]

where

\[ Q = \left( \frac{p}{3} \right)^3 + \left( \frac{q}{2} \right)^2, \quad p = -\frac{1}{3} A^2 + B, \quad q = \frac{2}{27} A^3 - \frac{1}{3} AB - C, \]

\[ A = \frac{3}{2} + \varepsilon |x_2^0|^2, \quad B = 3 |\alpha | x_2^0 [ (x_1^0)^2 x_2^0 - 1] + \varepsilon |x_2^0| [(x_1^0)^2 - 1] |x_2^0|, \]

\[ C = \frac{3}{|x_2^0|^3}. \]
(ii) and for $u(t) = \beta$,

$$\hat{\Delta}_1(t) = \frac{1}{3}(x_0^0)^3 t^3 + \left[\frac{1}{2}x_0^0 + \varepsilon \frac{1}{3}x_1^0(x_2^0)^2\right] t^2$$

$$+ t \left\{ \beta \left[ x_2^0 - (x_1^0)^2 x_2^0 - 1 \right] + \varepsilon x_2^0 \left[ (x_1^0)^2 - 1 \right] \right\}$$

we obtain $|\hat{\Delta}_1(t)| < 1$ for $t \in \left[0, \sqrt{-\frac{q_1}{2}} + \sqrt{Q_1} - \sqrt{-\frac{q_1}{2}} - \sqrt{Q_1} - A_1 \frac{3}{2}\right]$, where

$$Q_1 = \left(\frac{p_1}{3}\right)^2 + \left(\frac{q_1}{2}\right)^2, \quad p_1 = -\frac{1}{3}A_1^2 + B_1, \quad q_1 = \frac{2}{27}A_1^3 - \frac{1}{3}A_1 B_1 - C_1$$

$$A_1 = \frac{3}{2} + \frac{\varepsilon x_1 x_2}{|x_2|^2}; \quad B_1 = \frac{3 |\beta [x_2^0 - (x_1^0)^2 x_2^0 - 1] + \varepsilon x_2^0 [(x_1^0)^2 - 1]|}{|x_2|^3}$$

$$C_1 = \frac{3}{|x_2|^8}.$$

4. Concluding remarks

The non-commutative strategy developed by F. Bagarello for the analysis of ordinary differential equations having already proved its efficiency, we successfully extended the method to problems in optimal control theory. In the present paper, three time-optimal problems relative to forced harmonic oscillator systems have been studied and we gave an explicit solution for the first problem while for the two others we gave approximated solutions and evaluated the approximation error.

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