A sharpened energy-Strichartz inequality for the wave equation

Giuseppe Negro

Centro de Análise Matemática, Geometria e Sistemas Dinâmicos (CAMGSD), Instituto Superior Técnico, Lisbon, Portugal

Correspondence
Giuseppe Negro, Centro de Análise Matemática, Geometria e Sistemas Dinâmicos (CAMGSD), Instituto Superior Técnico, Avenida Rovisco Pais, 1049-001 Lisbon, Portugal.
Email: giuseppe.negro@tecnico.ulisboa.pt

Abstract
We consider the sharp Strichartz estimate for the wave equation on \( \mathbb{R}^{1+5} \) in the energy space, due to Bez and Rogers. We show that it can be refined by adding a term proportional to the distance from the set of maximizers, in the spirit of the classical sharpened Sobolev estimate of Bianchi and Egnell.

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1 | INTRODUCTION

Let \( \mathcal{H} \) denote the energy space for the wave equation; precisely, \( \mathcal{H} \) is the real Hilbert space obtained as the completion of the Schwartz space with scalar product

\[
\langle f | g \rangle_{\mathcal{H}} := \int_{\mathbb{R}^5} \nabla f_0 \cdot \nabla g_0 \, dx + \int_{\mathbb{R}^5} f_1 g_1 \, dx,
\]

where \( f = (f_0, f_1),\ g = (g_0, g_1) \). Bez and Rogers [2] proved the following sharp inequality: for all \( u : \mathbb{R}^{1+5} \to \mathbb{R} \) satisfying \( u_{tt} = \Delta u \), letting \( u(0) = (u(0), u_t(0)) \), it holds that

\[
\|u\|^2_{L^4(\mathbb{R}^{1+5})} \leq \frac{1}{8\pi} \|u(0)\|^2_{\mathcal{H}}.
\]

Moreover, there is equality in (1) if and only if

\[
u(0) \in M := \left\{ c\Gamma_{\mathbb{R}} f_\star \mid c \in \mathbb{R}, \ \alpha \in S^1 \times \mathbb{R}^7 \times SO(5) \right\},
\]
where \( f_* := (4(1 + |\cdot|^2)^{-2}, 0) \), and \( \Gamma_\alpha \) is a certain representation of the natural symmetry group of (1); we will define this operator in the following section. The distance from \( M \) is

\[
\text{dist}(f, M) := \inf \{ \| f - c\Gamma_\alpha f_* \|_H | c \in \mathbb{R}, \alpha \in S^1 \times \mathbb{R}^7 \times SO(5) \}.
\]

In this note, we will prove that (1) can be sharpened, by adding a term that is proportional to such distance.

**Theorem 1.** There exists an absolute constant \( C > 0 \) such that, for all \( u : \mathbb{R}^{1+5} \to \mathbb{R} \) satisfying \( u_{tt} = \Delta u \) and \( u(0) \in H \), it holds that

\[
C \text{dist}(u(0), M)^2 \leq \frac{1}{8\pi} \| u(0) \|^2_H - \| u \|^2_{L^4(\mathbb{R}^{1+5})} \leq \frac{1}{8\pi} \text{dist}(u(0), M)^2. \tag{2}
\]

This theorem is analogous to [7, Theorem 1.1], which is a sharpening of the conformal Strichartz estimate of Foschi [5], based on the computation of a spectral gap using the Penrose conformal compactification of the Minkowski spacetime. In applying the same approach here, we will face a fundamental new difficulty, related to the lack of conformal invariance of (1). The computation of the spectral gap is more involved: both the relevant quadratic forms and the scalar product of \( H \) are not diagonal in their spherical harmonics expansions. This requires the introduction of several new ingredients. We find it remarkable that a method based on conformal transformations also works in the present nonconformally invariant case.

The upper bound in Theorem 1 follows from a general, abstract argument. The lower bound, on the other hand, is much more delicate and it will be obtained by the method of Bianchi–Egnell [3]. More precisely, we will obtain a local version of Theorem 1, with an effective constant for the lower bound. This local result will then be made global via the profile decomposition of Bahouri and Gérard [1], following the same steps as in the aforementioned [7]. We refer to [4, 7] for further background and references.

Throughout all of this note, we only consider real-valued solutions to the wave equation. This is a natural assumption; for more details, see [7, Remark 3.2].

In the next section, we will prove Theorem 1. The necessary computations with spherical harmonics are collected in the Appendix.

## 2 PROOF OF THEOREM 1

The right-hand inequality in (2) is an immediate consequence of [7, Proposition 2.1]. We will thus focus on the following theorem, a local version of Theorem 1. Once this is proved, the global version will be a consequence of the profile decomposition of Bahouri and Gérard [1], applying verbatim the argument in [7, section 5]. We omit the details.

**Theorem 2.** For all \( u : \mathbb{R}^{1+5} \to \mathbb{R} \) satisfying \( u_{tt} = \Delta u \) and \( \text{dist}(u(0), M) < \| u(0) \|_H \),

\[
\frac{36}{85} \frac{1}{8\pi} \text{dist}(u(0), M)^2 + O(\text{dist}(u(0), M)^3) \leq \frac{1}{8\pi} \| u(0) \|^2_H - \| u \|^2_{L^4(\mathbb{R}^{1+5})}. \tag{3}
\]
Before we can begin with the proof, we give the precise definition of the symmetry operators \( \Gamma_\alpha \):

\[
\Gamma_\alpha f := R(t_0 \sqrt{-\Delta} + \theta) \left[ e^{3\sigma} f_0(\cdot + x_0) \right] e^{5\sigma} f_1(\cdot + x_0) .
\]

Here, the matrix-valued operator \( R \) is given by

\[
R(\cdot) := \begin{bmatrix}
\cos(\cdot) & \sin(\cdot) \\
-\sqrt{-\Delta} \sin(\cdot) & \cos(\cdot)
\end{bmatrix},
\]

and the parameter \( \alpha \) is

\[
\alpha = (t_0, \theta, \sigma, x_0, A), \quad t_0 \in \mathbb{R}, \theta \in S^1, \sigma \in \mathbb{R}, x_0 \in \mathbb{R}^5, A \in SO(5).
\]

**Remark 3.** This definition is analogous to [7, eq. 8], which concerns Foschi’s sharp conformal Strichartz estimate, mentioned in the introduction. However, here there are no Lorentz boosts. This is a first manifestation of the lack of full invariance of (1) under conformal transformations of Minkowski spacetime.

**Remark 4.** In the aforementioned [2, Corollary 1.2], Bez and Rogers actually considered \((0, (1 + | \cdot |^2)^{-3})\) instead of \(f_*\). Both belong to the orbit \(M\); letting \(\alpha = (0, \pi/2, 0, 0, 0)\), we have that \(-2^{-2} \Gamma_\alpha f_* = (0, (1 + | \cdot |^2)^{-3})\). Thus, the choice of which maximizer to consider is inessential.

These operators \( \Gamma_\alpha \) preserve both sides of the Strichartz inequality (1). Precisely, we introduce the notation

\[
Sf(t, x) := \cos(t \sqrt{-\Delta}) f_0(x) + \frac{\sin(t \sqrt{-\Delta})}{\sqrt{-\Delta}} f_1(x),
\]

for the solution operator to the wave equation \( u_{tt} = \Delta u \) with initial data \( u(0) = f \in \mathcal{H} \), and we have that (see [7, eq. 9])

\[
\| \Gamma_\alpha f \|_{\mathcal{H}} = \| f \|_{\mathcal{H}}, \quad \| S \Gamma_\alpha f \|_{L^4(\mathbb{R}^{1+5})} = \| S f \|_{L^4(\mathbb{R}^{1+5})}.
\]

With this newly introduced notation, we can denote the right-hand side of the sought inequality (3), which is known as the *deficit functional* of (1), as follows:

\[
\Phi(f) := \frac{1}{8\pi} \| f \|_{\mathcal{H}}^2 - \| S f \|_{L^4(\mathbb{R}^{1+5})}^2.
\]

We can now begin the proof of Theorem 2. This will occupy the rest of the section.

For \( f \in \mathcal{H} \setminus \{0\} \), the invariance of \( \Phi \) under the operators \( \Gamma_\alpha \) yields \( \Phi(f) = \Phi(cf_* + f_\perp) \), where \( c \neq 0 \) and

\[
\| f_\perp \|_{\mathcal{H}} = \text{dist}(f, M), \quad \text{and} \quad f_\perp \perp T f_* M;
\]
see the proof of [7, Proposition 5.3]. Here ⊥ denotes orthogonality with respect to $\langle \cdot | \cdot \rangle _{\mathcal{H}}$, and the tangent space is given by

$$T_{f_*} M := \text{span}\{ f_*, \nabla_\alpha \Gamma_\alpha f_\star | _{\alpha = 0} \},$$

where $\nabla_\alpha$ is the list of derivatives with respect to all parameters (4). We will give a precise description of such tangent space below.

Obviously, $\Phi(f_\star) = 0$. Moreover, $f_\star$ is a critical point of $\Phi$, being indeed a global minimizer. The expansion of $\Phi$ to second order thus reads

$$\Phi(f) = \Phi(cf_\star + f_\perp) = Q(f_\perp, f_\perp) + O(\|f_\perp\|_{\mathcal{H}}^2),$$

where $Q$ is a quadratic form, whose precise expression we will give in the forthcoming (10). We are thus reduced to proving the following coercivity;

$$Q(f_\perp, f_\perp) \geq \frac{36}{85} \frac{1}{8\pi} \|f_\perp\|_{\mathcal{H}}^2, \text{ for all } f_\perp \not\equiv T_{f_\star} M.$$

As mentioned in the introduction, the scalar product $\langle \cdot | \cdot \rangle _{\mathcal{H}}$ and the associated orthogonality relation $\perp$ are cumbersome to work with; see the forthcoming Remark A.2 in the Appendix. The following lemma shows that we are free to modify $\perp$, leveraging on the fact that $f_\star$ is a maximizer for (1), hence a minimizer for the deficit functional $\Phi$.

**Lemma 5.** Let $\langle \cdot | \cdot \rangle _{\mathcal{H}}$ be a scalar product on $\mathcal{H}$, and denote by $\perp$ the corresponding orthogonality relation. If there is a $C_\star > 0$ such that $Q(g, g) \geq C_\star \|g\|_{\mathcal{H}}^2$ for all $g \perp T_{f_\star} M$, then $Q(f_\perp, f_\perp) \geq C_\star \|f_\perp\|_{\mathcal{H}}^2$ for all $f_\perp \not\equiv T_{f_\star} M$.

**Proof.** Let $f_\perp \not\equiv T_{f_\star} M$ and decompose it as $f_\perp = g + h$, where $g \perp T_{f_\star} M$ and $h \in T_{f_\star} M$. Note that $Q(h, h) = 0$, as $\partial^2 \phi (c f_\star + \varepsilon h)|_{\varepsilon = 0}$ vanishes in that case, as we are differentiating along some curve associated to some symmetry of (1), on which $\Phi$ is constant.

As $f_\star$ is a global minimizer of $\Phi$, in particular $Q$ is a positive-semidefinite quadratic form. We can thus apply the Cauchy–Schwarz inequality to obtain that

$$|Q(g, h)| \leq Q(g, g)Q(h, h) = 0.$$

We conclude that

$$Q(f_\perp, f_\perp) = Q(g, g) \geq C_\star \|g\|_{\mathcal{H}}^2 = C_\star \|f_\perp\|_{\mathcal{H}}^2 + C_\star \|h\|_{\mathcal{H}}^2 \geq C_\star \|f_\perp\|_{\mathcal{H}}^2. \quad \Box$$

From now on we will be working on the sphere $\mathbb{S}^5 \subset \mathbb{R}^6$, whose points we denote by $X = (X_0, \vec{X})$, with $\vec{X} = (X_1, ..., X_5)$; thus, $\sum_0^5 X_j^2 = 1$. We let $d\sigma$ denote the standard hypersurface measure on $\mathbb{S}^5$.

**Definition 6** (Penrose transform). We identify $f = (f_0, f_1) \in \mathcal{H}$ with the pair $(F_0, F_1)$ of real functions on $\mathbb{S}^5$ via the formulae

$$f_0(x) = (1 + X_0)^2 F_0(X), \quad f_1(x) = (1 + X_0)^3 F_1(X), \quad \text{where } x = \frac{\vec{X}}{1 + X_0}. \quad (6)$$
Remark 7. The map \((X_0, \vec{X}) \mapsto x = \frac{\vec{X}}{1 + X_0}\) is the stereographic projection of \(\mathbb{S}^5 \setminus \{(-1, \vec{0})\}\) onto \(\mathbb{R}^5\). The identification (6) implies (see, for example, [7, eqs. 19–20])

\[
Sf(t, r\omega) = \cos \left( T \sqrt{4 - \Delta_{\mathbb{S}^5}} \right) F_0(X) + \frac{\sin(T \sqrt{4 - \Delta_{\mathbb{S}^5}})}{\sqrt{4 - \Delta_{\mathbb{S}^5}}} F_1(X),
\]

where \(X = (\cos R, \sin R\omega)\), for \(r \geq 0\) and \(\omega \in \mathbb{S}^4\). The variables \((T, X) \in [-\pi, \pi] \times \mathbb{S}^5\) are related to \((t, r\omega) \in \mathbb{R}^{1+5}\) via the formulae \(T = \arctan(t + r) + \arctan(t - r)\), \(R = \arctan(t + r) - \arctan(t - r)\), which, however, we will not need in the sequel.

Under (6), \(f_+\) corresponds to the pair of constant functions \(F_{+0} = 1, F_{+1} = 0\), while \(Tf_+M\) corresponds to the following space of polynomials in \((X_0, X_1, ..., X_5)\):

\[
Tf_+M \equiv \left\{ \begin{bmatrix} (1 + X_0)^2 \left( \sum_{j=0}^5 a_j X_j + a_6 \right) \\ (1 + X_0)^3 (b_0 X_0 + b_1) \end{bmatrix} : a_j, b_j \in \mathbb{R} \right\}.
\]

This can be seen by redoing verbatim the computations in [7, section 3].

Remark 8. Note that (7) is strictly smaller than the analogous tangent space [7, eq. 34] in the conformal case. This is because the present case has less symmetries, as we saw.

We now introduce the system of spherical harmonics on \(\mathbb{S}^5\) that we will use. This fine description of spherical harmonics was not needed in the aforementioned [7].

Definition 9. Let \{\(Y_{\ell, m}\)\} denote a complete orthonormal system of \(L^2(\mathbb{S}^5)\), fixed once and for all according to the following prescription. The indices range on

\[
\ell \in \mathbb{N}_{\geq 0}, \quad m \in N(\ell) := \{m = (m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 : \ell \geq m_1 \geq m_2 \geq m_3 \geq |m_4|\},
\]

and each \(Y_{\ell, m}\) has the form

\[
Y_{\ell, m}(X_0, \vec{X}) = P_{m_1}(X_0) Y_{m}^{\mathbb{S}^4}(\vec{X} / |\vec{X}|),
\]

with \(P_{m_1}\) denoting the 6-dimensional normalized associated Legendre function (see the Appendix), while \(\{Y_{m}^{\mathbb{S}^4}\}_{m \in N(\ell)}\) denotes a orthonormal system of real spherical harmonics in \(L^2(\mathbb{S}^4)\) of degree \(m_1\). For each \(F \in L^2(\mathbb{S}^5)\), let \(\hat{F}(\ell, m) := \int_{\mathbb{S}^5} FY_{\ell, m} d\sigma\).

With these definitions, we can characterize (7) as

\[
f \in Tf_+M \iff \begin{cases} \hat{F}_0(\ell, m) = 0, & \ell \geq 2, \\ \hat{F}_1(\ell, m) = 0, & \ell \geq 2 \text{ or } \ell = 1, m \neq 0, \end{cases}
\]

† In particular, each \(Y_{\ell, m}\) is a spherical harmonic on \(\mathbb{S}^5\) of degree \(\ell\) and such that \(\| Y_{\ell, m} \|_{L^2(\mathbb{S}^5)} = 1\); see [6, Lemma 1, p. 55].
which suggests the introduction of the following alternative orthogonality relation, toward the application of Lemma 5:

\[
\begin{cases}
\hat{G}_0(\ell, m) = 0, \\
\hat{G}_1(\ell, 0) = 0,
\end{cases}
\text{for } \ell = 0, 1, m \in \mathbb{N}(\ell).
\]  \hspace{1cm} (8)

Here, \((F_0, F_1)\) and \((G_0, G_1)\) denote the Penrose transforms (6) of \(f\) and \(g\), respectively. Note that \(\overline{\perp}\) is different from the standard orthogonality \(\perp\) of \(\mathcal{H}\); see the Appendix.

Remark 10. In the conformal case of [7], there is no need to introduce such alternative orthogonality relations. Indeed, the natural scalar product considered there is diagonalized by the spherical harmonics, after the Penrose transform; see [7, eq. 24].

For \(f = (f_0, f_1) \in \mathcal{H}\), a Taylor expansion to second order of the deficit functional (5) shows that the quadratic form \(Q(f, f)\) equals

\[
\frac{1}{4\pi} \left[ 2\frac{\langle f_\ast | f \rangle_{\mathcal{H}}}{\| f_\ast \|_{\mathcal{H}}^2} + \frac{\| f_\ast \|_{\mathcal{H}}^2}{\| f \|_{\mathcal{H}}^2} - 3 \int_{\mathbb{R}^{1+5}} (Sf_\ast)^2(Sf)^2 \right].
\]  \hspace{1cm} (9)

We already observed that \(\Phi(f_\ast) = 0 = \partial_\varepsilon \Phi(f_\ast + \varepsilon f)|_{\varepsilon = 0}\). This yields

\[
\| Sf_\ast \|_{L^4(\mathbb{R}^{1+5})}^2 = \frac{1}{8\pi} \| f_\ast \|_{\mathcal{H}}^2, \quad \int_{\mathbb{R}^{1+5}} (Sf_\ast)^3Sf = \frac{\| Sf_\ast \|_{L^4(\mathbb{R}^{1+5})}^4}{\| f_\ast \|_{\mathcal{H}}^2} \langle f_\ast | f \rangle_{\mathcal{H}},
\]

which we can insert into (9) to obtain the simpler expression

\[
Q(f, f) = \frac{16\pi}{\| f_\ast \|_{\mathcal{H}}^2} \left[ \frac{1}{(8\pi)^2} \left( 2\langle f_\ast | f \rangle_{\mathcal{H}}^2 + \| f_\ast \|_{\mathcal{H}}^2 \| f \|_{\mathcal{H}}^2 \right) - 3 \int_{\mathbb{R}^{1+5}} (Sf_\ast)^2(Sf)^2 \right].
\]  \hspace{1cm} (10)

It is easy to see that \(Q(f, f) = Q((f_0, 0), (f_0, 0)) + Q((0, f_1), (0, f_1))\) (see [7, eq. 50]). The computations in the Subsection A.1 of the Appendix show that, for all \(f \overline{\perp} T_{f_\ast} M\),

\[
Q((f_0, 0), (f_0, 0)) = \frac{1}{4\pi} \left[ \sum_{\ell=2}^\infty \sum_{m \in \mathbb{N}(\ell)} \alpha_{\ell, m} \hat{F}_0(\ell, m)^2 + \beta_{\ell, m} \hat{F}_0(\ell + 1, m) \hat{F}_0(\ell, m) \right],
\]  \hspace{1cm} (11)

while

\[
Q((0, f_1), (0, f_1)) = \frac{1}{4\pi} \left[ \sum_{m \in \mathbb{N}(1), m_1 = 1} \frac{\hat{F}_1(1, m)^2}{9} \right. \\
\left. + \sum_{\ell=2}^\infty \sum_{m \in \mathbb{N}(\ell)} \alpha_{\ell, m} \frac{\hat{F}_1(\ell, m)^2}{(\ell + 2)^2} + \beta_{\ell, m} \frac{\hat{F}_1(\ell, m) \hat{F}_1(\ell + 1, m)}{(\ell + 2)(\ell + 3)} \right],
\]  \hspace{1cm} (12)
where the coefficients are

\[
\alpha_{\ell,m} = \frac{\ell^4 + 8\ell^3 + 11\ell^2 - 20\ell - 12 + 6m_1^2 + 18m_1}{(\ell + 1)(\ell + 3)}, \quad \beta_{\ell,m} = (\ell - 1)(\ell + 6) \sqrt{\frac{(\ell + 1 - m_1)(\ell + 4 + m_1)}{(\ell + 2)(\ell + 3)}}.
\]

These formulae show that \( Q \) has a kind of tridiagonal structure; for example, neglecting all summands with \( m \neq 0 \), we can formally write (11) as

\[
\begin{bmatrix}
\hat{F}_0(2,0) & \hat{F}_0(3,0) & \hat{F}_0(4,0) & \cdots \\
\frac{1}{2}\beta_{2,0} & \alpha_{2,0} & 0 & 0 & \hat{F}_0(2,0) \\
0 & \frac{1}{2}\beta_{3,0} & \alpha_{3,0} & 0 & \hat{F}_0(3,0) \\
0 & 0 & \frac{1}{2}\beta_{4,0} & \alpha_{4,0} & \hat{F}_0(4,0) \\
& & & & \ddots \\
& & & & \ddots \\
& & & & \ddots \\
& & & & \ddots \\
& & & & \ddots \\
\end{bmatrix}
\]

Remark 11. The analogous quadratic form [7, eq. 52] for the conformal case is diagonal.

To exploit such tridiagonal structure, we introduce the following criterion.

**Lemma 12 (Diagonal dominance).** Let \( L \in \mathbb{N}_{\geq 0} \) and let

\[
\{ a_{\ell,m}, b_{\ell,m} : \ell \in \mathbb{N}_{\geq L}, \ m \in \mathbb{N}(\ell) \}
\]

be real sequences satisfying

\[
\begin{align*}
|a_{\ell,m}| & \geq \frac{1}{2} |b_{\ell,m}|, \\
|a_{\ell,m}| & \geq \frac{1}{2} (|b_{\ell,m}| + |b_{\ell-1,m}|), \quad \ell > L,
\end{align*}
\]

Then the quadratic functional \( T \), defined by

\[
T(F) = \sum_{\ell=L}^{\infty} \sum_{m \in \mathbb{N}(\ell)} a_{\ell,m} \hat{F}(\ell, m)^2 + b_{\ell,m} \hat{F}(\ell, m) \hat{F}(\ell + 1, m),
\]

satisfies \( T(F) \geq 0 \) for all \( F \in L^2(S^5) \).

**Proof.** With the convention that \( b_{\ell,m} = 0 \) if \( \ell < L \) or \( \ell < m_1 \), we can bound \( T(F) \) from below by

\[
T(F) \geq \sum_{\ell \geq L, \ m \in \mathbb{N}(\ell)} \frac{|b_{\ell,m}|}{2} \hat{F}(\ell, m)^2 + \frac{|b_{\ell-1,m}|}{2} \hat{F}(\ell, m)^2 + b_{\ell,m} \hat{F}(\ell, m) \hat{F}(\ell + 1, m)
\]

\[
\geq \sum_{\ell \geq L, \ m \in \mathbb{N}(\ell)} \frac{1}{2} |b_{\ell,m}| (\hat{F}(\ell, m) + \text{sign}(b_{\ell,m}) \hat{F}(\ell + 1, m))^2 \geq 0.
\]

We can finally apply this lemma to obtain the desired lower bound. Recall that the relation \( \mathcal{I} \) has been defined in (8).
Proposition 13. For all $f$ $\overline{\perp} T f^* M$,

$$Q(f, f) \geq \frac{36}{85} \frac{1}{8\pi} \|f\|_H^2.$$ 

Once Proposition 13 is proved, Lemma 5 will imply the same lower bound with the standard orthogonality $\perp$ instead of $\overline{\perp}$, thus establishing the required coercivity of $Q$ and completing the proof of Theorem 2, hence of Theorem 1.

Proof of Proposition 13. We observed that $Q(f, f) = Q((f_0, 0), (f_0, 0)) + Q((0, f_1), (0, f_1))$. We consider the term $Q((f_0, 0), (f_0, 0))$ first. Defining the quadratic functional

$$T : \{ F_0(\ell, m) = 0, \text{ for } \ell = 0, 1, m \in \mathbb{N}(\ell) \} \rightarrow \mathbb{R},$$

$$T(f_0) := Q((f_0, 0), (f_0, 0)) - \frac{36}{85} \frac{1}{8\pi} \|(f_0, 0)\|_H^2,$$

it will suffice to show that $T$ satisfies the conditions of Lemma 12; notice that the orthogonality $(f_0, 0) \overline{\perp} T f^* M$ is encoded in the domain of $T$. We perform the change of variable

$$\hat{F}_0(\ell, m) = \frac{\hat{H}(\ell, m)}{\sqrt{(\ell + 1)(\ell + 3)}},$$

so that, by (11) and by the formula (A.4) in the Appendix, we have

$$T(H) = \sum_{\ell=2}^{\infty} \sum_{m \in \mathbb{N}(\ell)} a_{\ell, m} \hat{H}(\ell, m)^2 + b_{\ell, m} \hat{H}(\ell, m) \hat{H}(\ell + 1, m),$$

where

$$a_{\ell, m} = \frac{1}{4\pi} \frac{\ell^4 + 8\ell^3 + 11\ell^2 - 20\ell - 12 + 6m_1^2 + 18m_1}{(\ell + 1)^2(\ell + 3)^2} - \frac{36}{85} \frac{1}{8\pi} \frac{(\ell + 2)^2}{(\ell + 1)(\ell + 3)},$$

$$b_{\ell, m} = \sqrt{\frac{(\ell + 1 - m_1)(\ell + 4 + m_1)}{(\ell + 1)(\ell + 4)}} \left( \frac{1}{4\pi} \frac{(\ell - 1)(\ell + 6)}{(\ell + 2)(\ell + 3)} - \frac{36}{85} \frac{1}{8\pi} \right).$$

Notice that $b_{\ell, 0}$ is a rational function: the change of variable (14) was chosen to obtain this. Note also that, for all $\ell \geq 2$, $a_{\ell, m} \geq a_{\ell, 0}$ and $b_{\ell, m} \leq b_{\ell, 0}$. Therefore,

$$a_{2,m} - \frac{1}{2} b_{2,m} \geq a_{2,0} - \frac{1}{2} b_{2,0} = 0,$$

while, for $\ell > 2$,

$$a_{\ell, m} - \frac{1}{2} (b_{\ell, m} + b_{\ell-1,m}) \geq a_{\ell,0} - \frac{1}{2} (b_{\ell,0} + b_{\ell-1,0})$$

$$= \frac{1}{4\pi(\ell + 1)(\ell + 3)} \left( \frac{\ell^2 + 4\ell + 15}{(\ell + 1)(\ell + 3)} - \frac{18}{85} \right) > 0.$$
So, the conditions (13) of Lemma 12 are satisfied, and we can conclude that

\[ Q((f_0, 0), (f_0, 0)) \geq \frac{36}{85} \frac{1}{8\pi} \| (f_0, 0) \|^2_{H^1}, \quad \text{for all } (f_0, 0) \perp T_{f_0} M. \]

To prove the analogous inequality for \( Q((0, g_1), (0, g_1)) \), we let

\[ T : \{ \hat{F}_1(\ell, 0) = 0, \text{ for } \ell = 0, \ell = 1 \} \rightarrow \mathbb{R} \]

\[ T(f_1) := Q((0, f_1), (0, f_1)) - \frac{36}{85} \frac{1}{8\pi} \| (0, f_1) \|^2_{H^1}. \]

We perform the change of variable

\[ \hat{F}_1(\ell, m) = \frac{\hat{H}(\ell, m)(\ell + 2)}{\sqrt{(\ell + 1)(\ell + 3)}}, \]

so that, by (12) and by the formula (A.3) in the Appendix,

\[ T(H) = \sum_{m \in \mathbb{N}(1), m_1 = 1} \tilde{a}_{1,m} \hat{H}(1, m)^2 + \tilde{b}_{1,m} \hat{H}(1, m) \hat{H}(2, m) + \sum_{\ell = 2}^{\infty} \sum_{m \in \mathbb{N}(\ell)} a_{\ell,m} \hat{H}(\ell, m)^2 + b_{\ell,m} \hat{H}(\ell, m) \hat{H}(\ell + 1, m), \]

where \( \tilde{a}_{1,m} = \frac{93}{2720}, \tilde{b}_{1,m} = -\frac{36}{85} \frac{1}{8\pi} \sqrt{\frac{3}{5}} \), while \( a_{\ell,m} \) and \( b_{\ell,m} \) equal (15) for \( \ell \geq 2 \). For \( \ell = 1, 2 \) and \( m_1 = 1 \) we have that

\[ a_{1,m} - \frac{1}{2} |\tilde{b}_{1,m}| = \frac{1}{\pi} \left( \frac{93}{2720} - \frac{36}{85} \frac{1}{8\pi} \sqrt{\frac{3}{5}} \right) > \frac{1}{100} > 0, \]

\[ a_{2,m} - \frac{1}{2} (b_{2,m} + |\tilde{b}_{1,m}|) = \frac{1}{\pi} \left( \frac{64}{1275} - \frac{2\sqrt{7}}{255} - \frac{9\sqrt{15}}{1700} \right) > \frac{2}{1000} > 0. \]

For all other values of \( \ell \) and \( m \), the assumptions of Lemma 12 have already been verified; see (16) for the \( \ell = 2, m_1 = 2 \) case (recall that, by convention, \( b_{1,m} = 0 \) if \( m_1 > 1 \)), and (17) for all the other cases. We conclude that

\[ Q((0, f_1), (0, f_1)) \geq \frac{36}{85} \frac{1}{8\pi} \| (0, f_1) \|^2_{H^1}, \quad \text{for all } (0, f_1) \perp T_{f_1} M, \]

which completes the proof.

\[ \square \]

Remark 14. The same proof shows that \( C = \frac{36}{85} \frac{1}{8\pi} \) is the largest constant such that the quadratic form \( Q(f, f) - C\| f \|^2_{H^1} \) is diagonally dominant in the sense of Lemma 12. This is the reason why the constant \( \frac{36}{85} \frac{1}{8\pi} \) appears in Theorem 2.
APPENDIX A: COMPUTATIONS WITH SPHERICAL HARMONICS

In this Appendix, we compute expressions for the scalar product \((f | g)_H\) and the quadratic form \(Q(f, f)\) in terms of the Penrose transforms \((F_0, F_1)\) and \((G_0, G_1)\) of \(f\) and \(g\), respectively (see Definition 6).

Following [6, p. 54], we introduce the normalized associated Legendre functions of degree \(\ell \in \mathbb{N}_{\geq 0}\), order \(m \in \{0, 1, \ldots, \ell\}\) and dimension 6 to be the functions\(^1\)

\[
P^m_{\ell}(6; t) = \mathcal{N}_{\ell, m}(1 - t^2)^{\frac{m}{2}} P_{\ell - m}(2m + 6; t), \quad t \in [-1, 1],
\]

where \(P_{\ell - m}(2m + 6; \cdot)\) is the Legendre polynomial of degree \(\ell - m\) in dimension \(2m + 6\). The normalization constant (recall \(|S^n| = 2\pi^{\frac{n+1}{2}} / \Gamma(\frac{n+1}{2})\))

\[
\mathcal{N}_{\ell, m} = \sqrt{\frac{(2\ell + 4)(\ell + m + 3)!}{(\ell - m)!(2m + 4)!}} \frac{|S^{2m+4}|}{|S^{2m+5}|}
\]

is chosen to ensure the orthonormality

\[
\int_{-1}^{1} P^m_{\ell}(6; t) P^m_{\ell'}(6; t)(1 - t^2)^{\frac{3}{2}} dt = \delta_{\ell, \ell'}, \quad (A.1)
\]

We adopt the convention that \(P^m_{\ell}(6; \cdot) = 0\) if \(m > \ell\).

Recall from Definition 9 that

\[
Y_{\ell, m}(X_0, \vec{X}) = P^m_{\ell}(6; X_0) Y^{S^4}_{m} \left( \frac{\vec{X}}{|\vec{X}|} \right),
\]

where \(\{Y^{S^4}_{m}\}\) is a fixed orthonormal system of spherical harmonics on \(S^4\) of degree \(m_1\); here

\[
\ell \in \mathbb{N}_{\geq 0}, \quad m \in \mathbb{N}(\ell) := \{m = (m_1, m_2, m_3, m_4) \in \mathbb{Z}^4 : \ell \geq m_1 \geq m_2 \geq m_3 \geq |m_4|\}.
\]

Note that \(\mathbb{N}(\ell) \subset \mathbb{N}(\ell + 1)\). We now introduce the following coefficient, defined for \(\ell, m_1 \in \mathbb{Z}\):

\[
C_{5}(\ell, m_1) = \begin{cases} 
\frac{1}{2} \sqrt{\frac{(\ell-m_1+1)(\ell+m_1+4)}{(\ell+2)(\ell+3)}}, & 0 \leq m_1 \leq \ell, \\
0, & \text{otherwise}.
\end{cases}
\]

This appears in the next lemma.

**Lemma A.1.** For all \(\ell, \ell' \in \mathbb{N}_{\geq 0}\) and all \(m \in \mathbb{N}(\ell), m' \in \mathbb{N}(\ell')\),

\[
\int_{S^5} X_0 Y_{\ell, m}(X) Y_{\ell', m'}(X) d\sigma = \begin{cases} 
C_{5}(\min(\ell, \ell'), m_1), & |\ell' - \ell| = 1 \text{ and } m = m', \\
0, & \text{otherwise}.
\end{cases}
\]

\(^1\) In terms of Gegenbauer polynomials, given via the generating function \(\sum_{\ell=0}^{\infty} C^{(\ell)}(t)e^{-\ell} = (1 - 2rt + r^2)^{-\nu}\), it holds that \(P^m_{\ell}(6; t) = \mathcal{N}_{\ell, m}(1 - t^2)^{\frac{m}{2}} C_{\ell-m}(t) / C_{\ell-m}(1)\).
Proof. Letting $X_0 = \cos R$, we have that $d\sigma = (\sin R)^4 dR \, d\sigma_{S^4}$; thus,

$$
\int_{S^5} X_0 Y_{\ell,m} Y_{\ell',m'} \, d\sigma = \int_{S^4} Y_{m}^{S^4} Y_{m'}^{S^4} \int_0^\pi \cos(R) P_{\ell}^{m_1}(6; \cos R) P_{\ell'}^{m_1}(6; \cos R) (\sin R)^4 dR
$$

$$
= \delta_{m,m'} \int_{-1}^1 X_0 P_{\ell}^{m_1}(6;X_0) P_{\ell'}^{m_1}(6;X_0)(1 - X_0^2)^{\frac{3}{2}} \, dX_0.
$$

To evaluate the latter integral, we first assume without loss of generality $\ell' \geq \ell$. From the recursion relation for the Legendre polynomials \cite[Lemma 3, p. 39]{6}, we obtain

\begin{equation}
0 = a_{m_1}^{m_1} P_{\ell}^{m_1}(6;X_0) - b_{m_1}^{m_1} X_0 P_{\ell-1}^{m_1}(6;X_0) + c_{m_1}^{m_1} P_{\ell-2}^{m_1}(6;X_0),
\end{equation}

with

\begin{align*}
& a_{m_1}^{m_1} = \sqrt{\frac{(\ell-m_1)(\ell+m_1+3)}{(2\ell+4)(\ell+m_1+2)}}, \quad b_{m_1}^{m_1} = \sqrt{\frac{2\ell+2}{\ell+m_1+2}}, \quad c_{m_1}^{m_1} = \sqrt{\frac{\ell-m_1-1}{2\ell}}.
\end{align*}

Multiplying (A.2) by $P_{\ell'-1}^{m_1}(6;X_0)(1 - X_0^2)^{\frac{3}{2}}$ and then integrating, we infer from (A.1) that, as $\ell' \geq \ell$,

$$
\int_{-1}^1 P_{\ell}^{m_1}(6;X_0) P_{\ell'-1}^{m_1}(6;X_0)(1 - X_0^2)^{\frac{3}{2}} \, dX_0 = \frac{a_{m_1}^{m_1}}{b_{m_1}^{m_1}} \delta_{\ell',\ell-1} = c_5(\ell-1, m_1) \delta_{\ell',\ell-1}.
$$

This completes the proof. \hfill \Box

To compute a convenient expression for $\langle f \mid g \rangle_{H^s}$, where as usual $f = (f_0, f_1)$ and $g = (g_0, g_1)$, we start by rewriting it in terms of the fractional Laplacian, as follows:

$$
\langle f \mid g \rangle_{H^s} = \int_{\mathbb{R}^5} \sqrt{-\Delta f_0} \sqrt{-\Delta g_0} \, dx + \int_{\mathbb{R}^5} f_1 g_1 \, dx.
$$

We identify $f$ to $(F_0, F_1)$ and $g$ to $(G_0, G_1)$ via the Penrose transform of Definition 6, and we recall the fractional Laplacian formula (see, e.g., \cite[Lemma A.3]{8})

$$
\sqrt{-\Delta f_0}(x) = (1 + X_0)^3 \sqrt{4 - \Delta_{S^5}}(F_0)(X),
$$

where $x$ and $X = (X_0, \vec{X})$ are related by the stereographic projection $x = \vec{X}/(1 + X_0)$, as in Definition 6. Obviously, the same formula holds for for $g_0$ and $G_0$.

Recalling the Jacobian $dx = (1 + X_0)^{-5}d\sigma$, we have that

$$
\langle f \mid g \rangle_{H^s} = \int_{S^5} \sqrt{4 - \Delta_{S^5}} F_0 \sqrt{4 - \Delta_{S^5}} G_0 (1 + X_0) \, d\sigma + \int_{S^5} F_1 G_1(1 + X_0) \, d\sigma.
$$

We now use Lemma A.1 to compute

$$
\int_{S^5} F_1 G_1(1 + X_0) \, d\sigma = \sum_{\ell=0}^{\infty} \sum_{m \in \mathbb{N}(\ell)} \tilde{F}_1(\ell,m) \tilde{G}_1(\ell,m) + C_5(\ell, m_1) \left( \hat{F}_1(\ell,m) \hat{G}_1(\ell + 1,m) + \tilde{F}_1(\ell + 1,m) \tilde{G}_1(\ell,m) \right).
$$
As $-\Delta_{S^5} Y_{\ell, m} = \ell(\ell + 4) Y_{\ell, m}$, we have that $\sqrt{4 - \Delta_{S^5}} Y_{\ell, m} = (\ell + 2) Y_{\ell, m}$. Thus, 
\[
\int_{S^5} \sqrt{4 - \Delta_{S^5}} F_0 \sqrt{4 - \Delta_{S^5}} G_0 (1 + X_0) \, d\sigma \text{ is equal to }
\]
\[
\sum_{\ell \geq 0} \sum_{m \in \mathbb{N}(\ell)} (\ell + 2)^2 \hat{F}_0(\ell, m) \hat{G}_0(\ell, m) + C_5(\ell, m_1)(\ell + 2)(\ell + 3)(\hat{F}_0(\ell, m) \hat{G}_0(\ell + 1, m) + \hat{F}_0(\ell + 1, m) \hat{G}_0(\ell, m)). \tag{A.4}
\]

**Remark A.2.** These formulae show that $\langle \cdot | \cdot \rangle_H$ is not diagonal in $\hat{F}_0(\ell, m)$ and $\hat{F}_1(\ell, m)$. This is the reason why the orthogonality $\mathbf{f} \perp \mathcal{T} \mathbf{f}^* \mathbf{M}$ is difficult to characterize in terms of these coefficients.

As an application, we now compute $\|\mathbf{f}^*\|_H^2$. The Penrose transform of $\mathbf{f}^*$ is the pair of constant functions $(F^*_0, F^*_1) = (1, 0)$, so $\hat{F}^*_0(0, 0) = \sqrt{|S^5|}$ and $\hat{F}^*_0(\ell, m) = 0$ for all $\ell > 0$. We conclude from (A.4) that
\[
\|\mathbf{f}^*\|_H^2 = 4|S^5| = 4\pi^3.
\]

**A.1 | The quadratic form**

In this subsection, we evaluate an expression for the quadratic form $Q(\mathbf{f}, \mathbf{f})$; recall (10). We need to compute it for $\mathbf{f}^* \mathcal{T}^* \mathbf{f}^* \mathbf{M}$, that is,
\[
\hat{F}_0(\ell, m) = 0, \quad \hat{F}_1(\ell, 0) = 0, \quad \text{for } \ell = 0, 1 \text{ and } m \in \mathbb{N}(\ell), \tag{A.5}
\]

which, by (A.4), immediately imply $\langle \mathbf{f}^* | \mathbf{f} \rangle_H = 0$. Thus,
\[
Q((f_0, 0), (f_0, 0)) = \frac{1}{4\pi} \| (f_0, 0) \|_H^2 - \frac{12}{\pi^2} \int_{\mathbb{R}^{1+5}} (S \mathbf{f}^*)^2 (S(f_0, 0))^2,
\]
\[
Q((0, f_1), (0, f_1)) = \frac{1}{4\pi} \| (0, f_1) \|_H^2 - \frac{12}{\pi^2} \int_{\mathbb{R}^{1+5}} (S \mathbf{f}^*)^2 (S(0, f_1))^2. \tag{A.6}
\]

To evaluate the latter integrals, we first note that, by [7, Corollary 3.7],
\[
\int_{\mathbb{R}^{1+5}} (S \mathbf{f}^*)^2 (S \mathbf{f})^2 =
\]
\[
\frac{1}{2} \int_{S^1 \times S^5} \left[ \cos(2T) \left( \begin{array}{c} \cos \left( 2\sqrt{4 - \Delta_{S^5}} \right) F_0 + \sin \left( \frac{2\sqrt{4 - \Delta_{S^5}}}{4 - \Delta_{S^5}} \right) F_1 \end{array} \right) \right]^2 (\cos T + X_0)^2 \, dT \, d\sigma;
\]

moreover, by Lemma A.1 (recall the convention $Y_{\ell - 1, m} = 0$ if $\ell - 1 < 0$ or $\ell - 1 < m_1$),
\[
(\cos T + X_0) Y_{\ell, m} = \cos(T) Y_{\ell, m} + C_5(\ell - 1, m_1) Y_{\ell - 1, m} + C_5(\ell, m_1) Y_{\ell + 1, m}.
\]
Let $A_\ell(T) := \cos(2T) \cos((\ell + 2)T)$. We have that
\[
\frac{12}{\pi^2} \int_{\mathbb{R}^{1+5}} (Sf_*)_2^2(S(f,0))^2 = \frac{6}{\pi^2} \int_{S_1 \times S_5} \left[ \sum_{\ell,m} A_\ell(T) \hat{F}_0(\ell,m) (\cos T + X_0) Y_{\ell,m}(X) \right]^2 \, dT \, d\sigma
\]
\[
= \frac{6}{\pi^2} \int_{S_1 \times S_5} \left[ \sum_{\ell,m} (A_\ell(T) \cos(T) \hat{F}_0(\ell,m) + A_{\ell-1}(T) C_5(\ell - 1,m_1) \hat{F}_0(\ell - 1,m) \right.
\]
\[
+ A_{\ell+1}(T) C_5(\ell,m) \hat{F}_0(\ell + 1,m) \right] \, dT \, d\sigma
\]
\[
= \frac{3}{8\pi^2} \sum_{\ell,m} \int_{-\pi}^{\pi} \left[ (\cos((\ell - 1)T) + \cos((\ell + 3)T)) (\hat{F}_0(\ell,m) + 2C_5(\ell - 1,m_1) \hat{F}_0(\ell - 1,m)) \right.
\]
\[
+ (\cos((\ell + 1)T) + \cos((\ell + 5)T)) (\hat{F}_0(\ell,m) + 2C_5(\ell,m_1) \hat{F}_0(\ell + 1,m)) \right]^2 \, dT.
\]

By the orthogonality (A.5), the sum runs on $\ell \geq 1$, so the four cosines in the latter integral are orthogonal on $[-\pi, \pi]$. Using this we evaluate the integral, and rearrange terms, to conclude that
\[
\frac{12}{\pi^2} \int_{\mathbb{R}^{1+5}} (Sf_*)_2^2(S(f,0))^2 \text{ equals}
\]
\[
\frac{3}{8\pi^2} \sum_{\ell \geq 2} \sum_{m \in \mathbb{N}(\ell)} \left[ (4 + 8C_5(\ell,m_1))^2 + 16C_5(\ell,m_1)\hat{F}_0(\ell,m)\hat{F}_0(\ell + 1,m) \right.
\]
\[
\left. + 3 \sum_{\ell \geq 2} \sum_{m \in \mathbb{N}(\ell)} \left[ \frac{2\ell^2 + 8\ell - m_1^2 - 3m_1 + 4}{2(\ell + 1)(\ell + 3)} \hat{G}_1(\ell,m)^2 + 2C_5(\ell,m_1)\hat{G}_1(\ell,m)\hat{G}_1(\ell + 1,m) \right] \right].
\]

Inserting this, and (A.4), into (A.6) yields the formula (11) of the main text.

To compute the other term, it is convenient to let
\[
\hat{G}_1(\ell,m) := \frac{\hat{F}_1(\ell,m)}{\ell + 2}.
\]

Arguing as before, we see that
\[
\frac{6}{\pi^2} \int_{S_1 \times S_5} \left[ \sum_{\ell,m} \cos(2T) \sin((\ell + 2)T) \hat{G}_1(\ell,m)(\cos T + X_0) Y_{\ell,m}(X) \right]^2 \, dT \, d\sigma
\]
\[
= \frac{3}{8\pi^2} \sum_{\ell \geq 2} \sum_{m \in \mathbb{N}(\ell)} \left[ 2\ell^2 + 8\ell - m_1^2 - 3m_1 + 4 \right. \hat{G}_1(\ell,m)^2 + 2C_5(\ell,m_1)\hat{G}_1(\ell,m)\hat{G}_1(\ell + 1,m) \right]
\]
\[
+ \sum_{0 \neq m \in \mathbb{N}(1)} \frac{1}{2} \hat{G}_1(1,m)^2 + 2C_5(1,1)\hat{G}_1(1,m)\hat{G}_1(2,m).
\]
This is very similar to the right-hand side of (A.7), however, it has extra summands in $\ell' = 1$ and $0 \neq m \in \mathbb{N}(1)$, due to the orthogonality (A.5); indeed, notice that the term $\hat{F}_1(1, m) = (\ell' + 2)\hat{G}_1(\ell', m)$ needs not vanish. Inserting this, together with the formula (A.3), into (A.6) finally yields the formula (12) of the main text.

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**ORCID**

Giuseppe Negro [https://orcid.org/0000-0002-4913-7863](https://orcid.org/0000-0002-4913-7863)

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