Entanglement, weak values, and the precise inference of joint measurement outcomes for non-commuting observable pairs

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The problem of inferring the outcome of a simultaneous measurement of two non-commuting observables is addressed. We show that for certain pairs with dense spectra, precise inferences of the measurement outcomes are possible in pre- and postselected ensembles, and if the selections involve entangled states with some other system. We show that the problem is related to the problem of assigning weak values to a continuous family of operators, and give explicit examples where this problem is solvable. Some foundational implications are briefly discussed.

A joint measurement of two non-commuting observables can be understood as the simultaneous coupling of the measured system to two independent instruments, each of which is a good probe of either observable when coupled individually [1]. Independently of whatever meaning— if any—one may choose to attach to the instrument readings in this context, no state preparation can be realized so as to systematically ensure a definite outcome in any subsequent measurement of this type [2]. Instead, the mutual back-action of the instruments ensures an uncertainty relation for the joint outcomes with a lower bound that is twice that given by the standard uncertainty principle [3].

Fully accepting these facts, consider a scenario in the same spirit of the so called “King’s problem” [4–8]: Alice sends a particle to Bob, which he then subjects to a joint measurement of a non-commuting observable pair—say \( \hat{x} \) and \( \hat{p} \); finally, he returns the particle to Alice. Suppose Bob chooses not to reveal his readings. The question is: are there conditions under which the readings can still be inferred, precisely, by Alice?

In this let us show that this question can be answered in the affirmative for certain operator pairs, and if she makes appropriate measurements both before and after the particle has passed through Bob’s instruments. Moreover, she will need entanglement resources in order to achieve the task, not unlike other quantum inference problems where entanglement is required for optimality (e.g., [9, 10]). Therefore, we believe the results point to an interesting connection between quantum uncertainty, non-locality and ultimately, classicality. These questions will be addressed briefly at the conclusion. For the moment, we begin with the formal statement of the problem:

**Joint Measurement Inference Problem:** Let \( (\hat{A}_1, \hat{A}_2) \) be a pair of observables of a system with no common eigenstates, and \( \hat{U}(q_1, q_2) \) a unitary operator of the form

\[
\hat{U}(q_1, q_2) = \exp[i(\hat{A}_1 q_1 + \hat{A}_2 q_2)]
\]

(\( h = 1 \)). When \((q_1, q_2)\) are operator-valued, \( \hat{U}(\hat{q}_1, \hat{q}_2) \) is the unitary evolution operator describing an impulsive simultaneous interaction of the system with two separate instruments \((I_1, I_2)\) described by canonical pairs \((\hat{q}_1, \hat{\pi}_1)\) and \((\hat{q}_2, \hat{\pi}_2)\). Given this interaction, the problem is to find a realizable set of conditions not involving the instruments such that for an arbitrary initial state \( \hat{\rho} \) of \((I_1, I_2)\), the conditional post-interaction density matrix \( \hat{\rho}' \) of the instruments, is related to \( \hat{\rho} \) through a quantum operation with a unitary normalized Kraus operator

\[
\hat{\rho}' = \hat{F} \hat{\rho} \hat{F}^\dagger, \quad \hat{F} = \exp \left[ i(\alpha_1 \hat{q}_1 + \alpha_2 \hat{q}_2) \right], \quad (2)
\]

for some \( \hat{\rho} \)-independent pair of real numbers \((\alpha_1, \alpha_2)\). If the problem has a solution, then the conditional probability distribution of the “pointer variables” \((\hat{\pi}_1, \hat{\pi}_2)\) satisfies \( P(\pi_1, \pi_2|\hat{\rho}') = P(\pi_1 - \alpha_1, \pi_2 - \alpha_2|\hat{\rho}) \), and the conditional outcomes can be ascertained to within the uncertainties in \( (\pi_1, \pi_2) \). Thus, the outcomes \((\alpha_1, \alpha_2)\) can be unequivocally ascertained for any single trial in the “sharp” limit \( (\Delta \pi_1, \Delta \pi_2) \to 0 \).

We discuss solutions to the inference problem involving complete pre- and postselection measurements, in which case the Kraus operator is \( \hat{F} \propto (\psi_f | \hat{U}(\hat{q}_1, \hat{q}_2) | \psi_i) \) with pure initial and final states \( | \psi_i \rangle \) and \( | \psi_f \rangle \). Recalling the definition of the weak value of a quantum mechanical observable [11, 12], \( A_w = \langle \psi_f | \hat{A} | \psi_i \rangle / \langle \psi_f | \psi_i \rangle \), the inference problem is then equivalent to finding a pair \((| \psi_i \rangle, | \psi_f \rangle)\) such that weak values \( w^{(\alpha_1 q_1 + \alpha_2 q_2)} \) can be assigned to all elements of the continuous set of operators \( \{ \hat{U}(q_1, q_2) | (q_1, q_2) \in \mathbb{R}^2 \} \). Alice’s task should then be to perform pre- and postselection measurements, yielding initial and final states for which this assignment is possible.

Let us then consider the case of the canonical variable pair \((\hat{x}, \hat{p})\) for a particle in one dimension. A natural first guess at a solution would seem to be a pre-and postselected ensemble defined by initial and final eigenstates of \( \hat{p} \) and \( \hat{x} \) respectively. However, applying the Baker-Hausdorff lemma to the exponential in \( \langle | p \rangle e^{i(\hat{q}_1 \hat{p} + \hat{p} \hat{q}_2)} | x \rangle \), we find the Kraus operator \( \hat{F} = e^{i\hat{q}_1 \hat{p} + i\hat{p} \hat{q}_2} e^{i\hat{p} \hat{q}_1 / 2} \), differing from the desired form by the additional term \( e^{i\hat{p} \hat{q}_1 / 2} \). This term represents the back-action between the two
measurements, as it generates the canonical transformation \((\hat{\delta}x_1, \hat{\delta}x_2) = 1/4(\hat{q}_2, \hat{q}_1)\); in the absence of correlations in the preparation of the instruments, this back-action term can be shown to enforce the uncertainty relations in the final outcomes \((\pi'_1, \pi'_2)\)

\[
\Delta\pi'_1\Delta\pi'_2 \geq \Delta\pi_1\Delta\pi_2 + \frac{1}{16\Delta\pi_1\Delta\pi_2} \geq 1/2. \tag{3}
\]

The conditional uncertainty bounds now coincide with standard uncertainty relations, suggesting that even with additional information from postselection, the standard limit is still unbreachable due to inevitable back-action.

Note, however, that we have only looked at complete measurements on the system. So suppose instead that the selections involve measurements of general observables of the system and an ancillary canonical system, with canonical variables \((\hat{x}_a, \hat{p}_a)\). In particular, consider the conjugate pairs \((\hat{x}_a, \hat{p}_a)\) defined by

\[
\hat{x}_a = (\hat{x} + \hat{x}_a)/2, \quad \hat{p}_a = \hat{p} + \hat{p}_a, \tag{4a}
\]

\[
\hat{x}_a = \hat{x} - \hat{x}_a, \quad \hat{p}_a = (\hat{p} - \hat{p}_a)/2, \tag{4b}
\]

so that \(\hat{x} = \hat{x}_a + \hat{x}_a/2\) and \(\hat{p} = \hat{p}_a/2 + \hat{p}_a\). The linear combination \(\hat{x}_a \hat{q} + \hat{p}_a \hat{q} / 2\) can then be expressed as the sum of the two commuting combinations: \((\hat{x}_a \hat{q} + \hat{p}_a \hat{q})/2\) and \(\hat{x}_a \hat{q} + \hat{p}_a \hat{q}\), each involving commuting operators; thus, we can perform the factorization

\[
e^{i(\hat{x}_a \hat{q} + \hat{p}_a \hat{q})/2} = e^{i(\hat{x}_a \hat{q} + \hat{p}_a \hat{q})} e^{i(\hat{x}_a \hat{q} + \hat{p}_a \hat{q})/2}. \tag{5}
\]

Taking eigenstates of the two terms in the factorization, for instance an initial eigenstate of \((\hat{x}_a, \hat{p}_a)\) and a final eigenstate of \((\hat{x}_a, \hat{p}_a)\), we obtain the unitary normalized Kraus operator

\[
\hat{F} = \frac{\langle x, p_+ | e^{i(\hat{x}_a \hat{q} + \hat{p}_a \hat{q})/2} | x, p_+ \rangle}{\langle x, p_+ | x, p_+ \rangle} = e^{i(\hat{x}_a \hat{q} + \hat{p}_a \hat{q})/2}, \tag{6}
\]

with \(x = x_a + x / 2\) and \(p = p + p_+/2\). With the absence of the back-action term, the ensemble with \(\psi_f = |x, p_+\rangle\) and \(\psi_i = |x, p_+\rangle\) solves the inference problem for \((\hat{x}, \hat{p})\).

This solution ensemble reveals another surprising result when we consider, with otherwise the same conditions, the joint measurement of all four canonical variables of the system plus the ancilla with four different instruments. The corresponding Kraus operator then involves the quantity \(\langle x, p_+ | e^{i(\hat{x}_a \hat{q} + \hat{p}_a \hat{q})/2} | x, p_+ \rangle\). Disentangling the exponential, we obtain

\[
\hat{F} = e^{i(\hat{x}_a \hat{q} + \hat{p}_a \hat{q} + \hat{x}_a \hat{q} + \hat{p}_a \hat{q})} e^{i(\hat{x}_a \hat{q} + \hat{p}_a \hat{q})/2}, \tag{7}
\]

where \(x_a = x_a - x / 2\) and \(p_a = -(p + p_+/2)\). It is therefore the crossed pairs of commuting observables \((\hat{x}_a, \hat{p}_a)\) and \((\hat{p}_a, \hat{x}_a)\), that generate a back-action term and hence uncertainty relations as those of Eq. \(E\) for initially uncorrelated instruments. These results, as the results of our earlier failed attempt, are consistent with what could be conjectured to be a new type of uncertainty relation for a composite system with two canonical degrees of freedom, allowing for any pre- and postselection described by states of the composite system only. In the absence of any information about correlations in the preparation of the measuring instruments, it appears as if the uncertainties in any possible inference of the readings \((\pi'_x, \pi'_p, \pi'_{x_2}, \pi'_{p_2})\) from a joint measurement of all canonical variables are only constrained by the relation

\[
\Delta\pi'_x \Delta\pi'_p \Delta\pi'_{x_2} \Delta\pi'_{p_2} \geq 1/4. \tag{8}
\]

It is as if entanglement allowed us to arbitrarily deform an uncertainty volume in the “inference phase-space”, provided the total volume of 1/4 is preserved.

Prior to further speculation, we turn our attention to the assignment of weak values associated with the general inference problem. The assignment problem can be cast in terms of a complex linear map

\[
\hat{W} : \hat{A} \rightarrow \text{Tr}_\mathcal{E}(\hat{W} \hat{A}) \quad \hat{W} = \text{Tr}_\mathcal{E} \left( \frac{|\psi_i\rangle \langle \psi_f|}{\langle \psi_f | \psi_i \rangle} \right), \tag{9}
\]

where \(\hat{W}\), henceforth termed the weak value operator, is realized by some pair \(|\psi_i\rangle, |\psi_f\rangle\) of non-orthogonal states in the total Hilbert space of the system and an ancilla. The general question is then to determine for a given set of pairs \((\hat{A}^{(i)}, \alpha^{(i)})\), an \(\hat{W}\) such that \(\hat{W} : \hat{A}^{(i)} \rightarrow \alpha^{(i)}, \forall i\).

Consider the finite-dimensional case first. For the space of linear operators acting on a Hilbert space \(\mathcal{H}_a\) of dimension \(d\), we introduce a basis \(\{E_1, \ldots, E_d\}\), orthonormal with respect to the standard hermitian inner product: i.e., \(\langle E_i | E_j \rangle = \delta_{ij}\) where \(\langle \hat{A} | \hat{B} \rangle \equiv \text{Tr}(\hat{A}^\dagger \hat{B})\). An arbitrary observable \(\hat{A}\) of the system can then be represented as a vector \(a \in \mathbb{C}^d\) through \(\hat{A} = \sum d \cdot E_i = a \cdot \mathbf{E}\) with \(a_i = \langle E_i | \hat{A} \rangle\), and where the dot product is euclidean \((a \cdot b = \sum a_i b_i)\). We will make an exception for the weak value operator \(\hat{W}\), which we choose to expand in terms of the hermitian conjugate basis: \(\hat{W} = w \cdot \mathbf{E}^\dagger\), in which case \(w_a\) is the weak value of \(E_a\), and the weak value of \(\hat{A}\) can be written as \(a \cdot \hat{w}\). We reserve the notation \(I\) for the vector representing the identity operator, where \(I_i = \text{Tr}(E_i)^*\). One constraint on \(w\) is then \(\hat{I} \cdot w = 1\).

The question of whether \(w\) has other constraints in \(\mathbb{C}^d\) besides \(\hat{I} \cdot w = 1\) brings us to the significance of entanglement. Suppose that one could write \(\hat{W}\) in the form \(\hat{W} = \frac{\langle \chi_f | \chi_i \rangle}{\langle \chi_f | \chi_i \rangle^2}\) for normalized \(|\chi_i\rangle\) and \(|\chi_f\rangle\) in the system Hilbert space. It is then easy to see that \(\hat{W}\) satisfies the relations \(\hat{W}^2 = \hat{W}, \quad \hat{W} \hat{W}^\dagger = \frac{|\langle \chi_f | \chi_i \rangle|^2}{|\langle \chi_f | \chi_i \rangle|^2}\), and \(\hat{W}^\dagger \hat{W} = \frac{|\langle \chi_f | \chi_i \rangle|^2}{|\langle \chi_f | \chi_i \rangle|^2}\). These conditions lead to non-trivial constraints involving the components of the weak vector \(w\), the simplest of which are:

\[
\gamma_{ij}^* w_i w_j = 1 \quad \text{and} \quad w \cdot w^* = |\langle \chi_f | \chi_i \rangle|^2 \geq 1, \tag{10}
\]

where \(\gamma_{ij} = \text{Tr}(E_i E_j)\). On the other hand, define two
vectors in a Hilbert space $\mathcal{H}_s \times \mathcal{H}_a$, with $\dim(H_s) \geq d$

$$|\Phi\rangle = \sum_{i=1}^{n} |i\rangle_s |i\rangle_a, \quad \text{and} \quad |z\rangle = z \cdot \hat{E}^\dagger |\Phi\rangle. \quad (11)$$

Using these as initial and final states for a pre-and post-election and tracing over the ancilla, we obtain a realization of $\hat{W}$ with $w = z/(z \cdot I)$, where no constraints are required for $z$ other than $z \cdot I \neq 0$. Thus we see that with entanglement, it is possible to realize any complex vector $w$ that solves the associated linear-algebraic problem of satisfying $a^{(i)} = a^{(i)} \cdot w$ for a vector set $\{a^{(i)}\}$, provided the problem is solvable and consistent with $I \cdot w = 1$.

Turning then to the linear algebra problem associated with $W : U(q_1, q_2) \rightarrow e^{i(\alpha_1 q_1 + \alpha_2 q_2)}$, we now show that the problem has no solution if $\hat{B}_\theta \equiv A_1 \cos \theta + A_2 \sin \theta$ has a discrete spectrum with $\theta$ in any finite subinterval of $[0, \pi]$, as is always the case in the finite-dimensional case. First note by uniqueness of the exponential expansion, that we must have $W : \hat{B}_\theta \rightarrow \beta^k_\theta$ where $\beta^k_\theta = (\alpha_1 \cos \theta + \alpha_2 \sin \theta)$ for all integer powers $k$; on the other hand, by the Cayley-Hamilton theorem, we also know that $\hat{B}_\theta$ annihilates its characteristic polynomial $p_\theta(z) = \det(z I - \hat{B}_\theta)$, and hence, by linearity of weak values, we conclude that $p_\theta(\beta^k_\theta) = 0$ and hence that $\beta^k_\theta$ must be one of the eigenvalues $b^{(i)}_\theta$ of $\hat{B}_\theta$. But setting $\theta = 0$ or $\pi/2$ we see that $(\alpha_1, \alpha_2)$ must be a pair of eigenvalues $(a_1, a_2)$ of $(A_1, A_2)$. The conjunction of conditions can be visualized as the intersection of the curve $\beta^k_\theta = a_1 \cos \theta + a_2 \sin \theta$ with all zeroes of $p_\theta(z)$ plotted as a function of $\theta$. If $\hat{B}_\theta$ has a discrete spectrum in some interval, then either the intersection occurs at discrete values of $\theta$, in which case the solution fails, or else there must exist a root behaving like $b^{(i)}_\theta = a_1 \cos \theta + a_2 \sin \theta$ in that interval. But this can only occur if $A_1$ and $A_2$ have a common eigenstate, in contradiction with our stated assumptions. Consequently, solutions to the inference problem can only be found in the infinite dimensional case and only for operators $A_1$ and $A_2$ such that the combination $\hat{B}_\theta$ has a dense spectrum in some band around $a_1 \cos \theta + a_2 \sin \theta$ for some pair of eigenvalues $(a_1, a_2)$ of $(A_1, A_2)$.

Approximate assignments of the form $e^{i(\alpha_1 q_1 + \alpha_2 q_2 + a(q^r))}$ for some power $s$ can nevertheless be constructed in the finite-dimensional case. A generic form for the leading correction is obtained when $\hat{B}_\theta$ has a minimal polynomial $m_\theta(z)$ of degree $s$ (for all $\theta$), and assuming the linear independence of all symmetrized operators $S_{m_\theta}$, generated by the expansion $(\hat{A}_1 t + \hat{A}_2)^k = \sum_{l=0}^{k} \frac{k!}{l!(k-l)!} S_{l,k-l}$ for all $k < s$. Then one can assign $W : (\hat{A}_1 q_1 + \hat{A}_2 q_2)^k \rightarrow (\alpha_1 q_1 + \alpha_2 q_2)^k$ for all $k < s$ with arbitrary $(\alpha_1, \alpha_2)$. The leading order term in the exponent is then found to be $-i\psi [m_\theta(z)/s!$, evaluated at $z = (\alpha_1 \cos \theta + \alpha_2 \sin \theta)$ (An example is a pair of orthogonal spin directions, in a spin $j$ representation, in which case $m_\theta(z) = p_\theta(z)$ and $s = 2j + 1$).

Proceeding with the continuous variable case for a one-dimensional canonical system, we generalize previous results for two standard basis sets in the Weyl representation (see e.g. [13] [14]): the Heisenberg $\hat{T}_\zeta$ basis and its reciprocal $\hat{\Delta}_\eta$ basis, where $\zeta$ and $\eta$ are two-component dense indices valued on the standard symplectic plane of dimension two. Throughout, we denote the canonical observable pair as $\hat{n} = (\hat{x}, \hat{p})$ and use the symplectic product notation $\zeta \wedge \eta \equiv (\zeta_1 \eta_2 - \zeta_2 \eta_1)$. The Heisenberg basis consists of the translation operators $\hat{T}_\zeta = e^{i\eta \wedge \zeta}$ satisfying $\hat{T}_\zeta^\dagger \hat{n} \hat{T}_\zeta = \hat{n} + \zeta$, while the $\hat{\Delta}_\eta$ operators form with the $\hat{T}_\zeta$ a Fourier-transform pair

$$\hat{n}_\eta = \int_\zeta e^{i\zeta \wedge \eta} \hat{T}_\zeta, \quad \hat{T}_\zeta = \int_\eta e^{i\eta \wedge \zeta} \hat{n}_\eta. \quad (12)$$

The integration measure is defined as $\int_\zeta \equiv (2\pi)^{-1} \int d^2 \zeta$, etc., with the $2\pi$ replaced by $h = 2\pi\hbar$ when using units; note the orthonormality of both bases with respect to this measure, i.e., $(\zeta|\eta') = 2\pi\delta(\eta - \eta')$, etc. A summary of useful algebraic relations is

$$\hat{T}_\zeta \hat{T}_{\zeta'} = e^{i\phi_1} \hat{T}_{\zeta + \zeta'}, \quad \hat{n}_\eta \hat{n}_{\eta'} = 4e^{i\phi_2} \hat{T}_{(\eta + \eta')}^{(2)}, \quad (13)$$

$$\hat{T}_\zeta \hat{n}_\eta = e^{i\phi_3} \hat{n}_{\eta + \zeta/2}, \quad \hat{n}_\eta \hat{T}_\zeta = e^{i\phi_4} \hat{n}_{\eta - \zeta/2}, \quad (14)$$

where $\phi_1 = \frac{1}{2} \zeta \wedge \phi, \phi_2 = 2\eta \wedge \eta'$ and $\phi_3 = \eta \wedge \zeta$. For the expansion of an observable $\hat{A}$ in either basis

$$\hat{A} = \int_\zeta a(\zeta) \hat{T}_\zeta = \int_\eta \tilde{a}(\eta) \hat{n}_\eta, \quad (15)$$

the expansion functions $a(\zeta)$, and $\tilde{a}(\eta)$ are respectively known as the Weyl transform and the Weyl symbol of $\hat{A}$ and form a Fourier transform pair; for the identity operator, $I(\zeta) = 2\pi\delta(\zeta)$ and $I(\eta) = 1$. The exception again is the weak value operator $\hat{W}$, which is expanded in the Hermitian conjugate of the Heisenberg basis $\hat{W} = \int_\zeta w(\zeta) \hat{T}_\zeta$, in which case the Weyl transform of $\hat{W}(\eta)$ corresponds to the function $w(-\zeta)$. With these conventions, $W : \hat{A} \rightarrow \int_\zeta a(\zeta) w(\zeta) = \int_\eta \tilde{a}(\eta) \hat{w}(\eta)$.

In the infinite-dimensional case, the weak value assignment associated with the inference problem involves integral equations with solutions obtained by inverting potentially complicated kernels. For the moment, it will suffice to show a generic solution for a reduced set of observable pairs, leaving open the question of the general set of pairs for which the problem is solvable. We consider pairs of the form $(\hat{p}, f(\hat{x}))$ for functions $f(x)$ satisfying conditions determined by the generic solution. Letting $\hat{U}(t_1, t_2) = \exp[i(f(\hat{x})t_2 - \hat{p}t_1)]$, the problem is then to find $w_{\psi\phi}(\zeta)$ such that for all $(t_1, t_2)$,

$$e^{i(\phi t_2 - \kappa t_1)} = \int_\zeta K(t_1|\zeta) w_{\psi\phi}(\zeta), \quad (16)$$

where the kernel $K(t_1|\zeta) = \Tr \left[ \hat{U}(t_1, t_2) \hat{T}_\zeta^\dagger \right]$ can be shown
to take the form
\[ K(t|\zeta) = \delta(\zeta_1 - t_1) \int_{-\infty}^{\infty} du e^{i[g(u|\zeta_2) t_2 - \zeta u}], \]
with \( g(u|\zeta_1) = \frac{1}{2} \int_{-1}^{1} ds f(u - s\zeta_1/2) \). If \( f(x) \) allows the exchange \( \int d\zeta_2 \int du \leftrightarrow \int du \int d\zeta_2 \) (see 15), then it is easily verified that \( W_{\zeta,\phi}(\zeta) \) must be the Fourier transform with respect to the \( \zeta_2 \) variable of a \( \phi \)-function at some real root \( u_\phi(\zeta_1) \) of the equation \( g(u|\zeta_1) = \phi \). Thus,
\[ W_{\zeta,\phi}(\zeta) = e^{i[u_\phi(\zeta_1) t_2 - \zeta u]}, \]
The function \( f \) must therefore be such that \( g(u|\zeta_1) = \phi \) admits a real branch for all \( \zeta_1 \) for given \( \phi \). These conditions admit solutions for polynomial \( f(x) \), but of odd degree only. We have yet to find solutions for bounded \( f(x) \) (one side or both).

Proceeding with the realization of solution ensembles, it proves instructive to revisit the new measurement problem from a constructive viewpoint. From the definition of \( W(\zeta) \) or equivalently from the above results for \( f(x) = x \), we have \( W : \hat{T}_0 \rightarrow W(\zeta) = e^{i\eta_1 \zeta} \) for some \( \eta_1 = (x, p) \). When the problem is stated in the reciprocal basis, this amounts to finding conditions for which \( W(\eta) = 2\pi \delta(\eta - \eta_1) \), or equivalently, \( W = \hat{\Delta}_{\eta_1} \). We can now work in analogy with the discrete case by introducing a maximally entangled state \( |\Phi_0\rangle = \int dx |x\rangle_\eta |x\rangle_\eta \), and two derived basis sets for the combined Hilbert space:
\[ |\Phi_0\rangle \equiv \hat{T}_{\eta_1} |\Phi_0\rangle \quad \text{and} \quad |\Psi_0\rangle \equiv \hat{\Delta}_{\eta_1} |\Phi_0\rangle, \]
with \( \langle \Psi_0 \Phi_0 \rangle = e^{i\eta_1 \zeta} \). Tracing out the ancilla from the outer product \( |\Phi_{\zeta_1}\rangle \langle \Psi_{\eta_1}| \) for two specific states and normalizing, we find that
\[ \hat{W} = e^{i\eta_1 \zeta} \hat{T}_{\eta_1} \hat{\Delta}_{\eta_1} \hat{\Delta}_{\eta_1 + \zeta_1/2}, \]
which is the desired result with \( \eta_1 = \eta f + \zeta_1/2 \). One can verify that \( |\Phi_{\zeta_1}\rangle, |\Psi_{\eta_1}\rangle \) are indeed \( |x\rangle_{\eta f}, |x\rangle_\eta \), with \( \zeta = (x_1, p_1), \eta = (x_2, p_2) \).

One noteworthy aspect of this solution is the delta-function Weyl symbol for \( \hat{W} \). The ensemble therefore has the remarkable property that the weak value of any observable is the respective Weyl symbol evaluated at \( \eta \). Thus, when observables with classical Weyl symbols are weakly (hence, unsharply) probed, the conditional effects on the instruments from this ensemble will be indistinguishable from those of a classical system with definite phase-space localization. (see also 17).

Any semblance of standard textbook classicality proves illusory, of course, in light of the full repertoire of non-local observables that could be probed weakly. Adapting equation (7) to the present language by introducing the time inversion operation \( \hat{T}^\tau \equiv (\zeta_1, -\zeta_2) \), we find for the composite system the \( \hat{W}^{(x, a)}(\zeta_1, \zeta_1') = e^{i(\eta_1 \zeta + \eta_0 \zeta_1') \zeta_1} e^{i(\eta_1 \zeta + \eta_0 \zeta_1') \zeta_1'} \) where for the ancilla, \( \eta_0 = (x_0, p_0) = (\eta f - \zeta_1/2) \). Note that time inversion arises naturally from \( \hat{r}^{(a)}(\eta f) = \hat{r}^{(a)}(\eta f) \). For test functions, the corresponding Weyl symbol is then \( \hat{w}^{(x, a)}(\eta f, \eta f') = e^{-i(\eta f + \eta f') \zeta_1} \delta(\eta f - \eta f') \). Thus, while joint weak measurements yield Weyl symbols for system and ancilla observables \( \hat{A} \) and \( \hat{B} \) respectively, weak measurements of \( \hat{A} \hat{B} \) will differ from the product of the Weyl symbols by terms involving “crossed” derivatives to all orders (equivalently, the weak value of \( \hat{A} \hat{B} \) determines correlations in the outcomes of the joint weak measurement of \( \hat{A} \) and \( \hat{B} \)). The signature of entanglement is therefore a joint Weyl symbol that is essentially a non-local object in the combined “phase-space”.

We conclude with some remarks on interpretational issues. First, it is hard to ignore the fact that for those conditions solving the canonical inference problem, the conditionally sharp outcomes \( (x, p) \) are in consistent correspondence with numbers representing physical properties of the system in other contexts. To wit: we find algebraic correspondence with the initial and final state labels through the transformations of Eq. (4a), and at the level of weak values, with phase-space functions evaluated at \( (x, p) \). These correspondences seem to indicate a certain inner consistency of the quantum framework which, under special circumstances, allows for a statistically unambiguous operational assignment of values to both canonical variables. Finally, it has been suggested earlier that it is not one, but rather a pair of vector states (e.g., our initial and final states) that provide the complete description of a quantum system at a given instant. The present results provide a good indication of the extent to which the descriptive power of the two-vector framework is enhanced by the property of entanglement. It appears that through entanglement, the framework becomes flexible enough to incorporate, among many other possibilities, the classical description of a single system, at least at the kinematic level. We believe this realization raises interesting questions, particularly regrading our perception of classicality in the macroscopic domain.

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