NUMERICAL ESTIMATION OF NON-RELATIVISTIC VLASOV N-BODY MODEL USING SEMI LAGRANGIAN SCHEMES

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Abstract

The main intention of this paper was to deliver some of the distinctive features of the Vlasov equation and Landau damping from the perspective of mathematical physics. The main thrust can be reviewed as: Vlasov equation is understood from the origin point of view. The mean field is limit of the classical N-body problem, is depicted in pure mathematical and also statistical guidelines. We also axiomatically concluded that the Vlasov equation is completely justified as one major source that led to numeral of open problems in mathematical physics: either from molecular chaos to the problems of kinetic theory. We have delivered the mathematics of the Vlasov equations: particularly the traditional partial differential equation analysis in the functional spaces. The problem is observed to be converted as a general transport equation and relying on the well-posedness of the equation and preserving the transport structure the Vlasov equation is solved for pivotal analytic solutions and compared with the computational solution obtained using solver codes. The analysis of the Vlasov-Poisson equation and its qualitative properties and are focused on the mathematical aspects. In this paper Landau damping is identified numerically for 1D model of non relativistic Vlasov equation.

Keywords: Vlasov equation, Landau damping, functional spaces, Semi Lagrangian Schemes

I. Introduction

The mathematical model for several particles interacting system leads to the Vlasov equation, which is actually time-reversible. The subtle irreversible nature inherent in the equation was deciphered by Lev Landau and actually surprised the mathematicians of that time with that prediction based on equation. It is with the assumption of Gaussian system with spatial equilibrium conditions to find out the
result of the linearized Vlasov equation, with Vlasov-Poisson system he did arrive to an entirely new concept of Landau damping while attempting to solve Vlasov equation. The interesting physical phenomenon is that while physicists were trying to understand the magnetized and unmagnetized plasma flows leading to electrostatic wave in homogeneous continuum, it has been noticed that similar kind of peculiar damping which can be described by the Vlasov-Poisson system. This is really in contrast to the traditional treatment of Vlasov equation.

The Landau treatment of the equation for general solution is same as the mathematical treatment of plasma oscillations, and consequently lead to the celebrated Landau damping. Landau himself conducted a mathematical study without worrying about either the physical explanation or the essential interaction mechanism, solved the equation with the methods of Laplace and Fourier transforms, and after identifying the singularities that could be in the complex plane, it was concluded that the field loses and exponential swift takes place and further stated that the rate of decay was in fact function of the wave vector.

The meticulous treatments of Landau damping were carried out in the late sixties, it was revived and renewed at the beginning of the nineteenth century, and accelerated by many mathematicians during twentieth century. In majority of works, analytic arguments take part in the explanation of crucial aspects of the equation and nature of asymptotic expansion for the complexities associated to the linearization and quasi-linearization of Vlasov–Poisson equation. In more or less every case it was observed that the exponential decay of amplitude both in semi linearized and linearized Vlasov–Poisson equation.

In the present paper the nonlinear nature of Vlasov equation is directly considered in both theoretical and computational scenarios. Perturbative arguments are considered to prove the existence theorem and 1D model of the problem is considered for the class of computational method of solution. The solutions are behaving asymptotically as \( t \to +\infty \) however the distributions which are spatially homogeneous the nature of landau damping is predictable with the numerical treatment as well.

II. Mathematical description of the problem

Considering one single particle with mass \( m > 0 \), let \( x(t) \in \mathbb{R}^3 \) denote the position of the particle at time \( t \in \mathbb{R} \) and its velocity may be defined as

\[
\dot{v}(t) = \frac{dx(t)}{dt}.
\]

Let us consider the force be acting on the particle at certain time \( t \) be given as \( F(t, x(t), v(t)) \).

The Newton equations for the particle’s trajectory may be given by

\[
\dot{x}(t) = v(t) \quad \text{(1)}
\]

\[
\dot{v}(t) = m^{-1} F(t, x(t), v(t)) \quad \text{(2)}
\]

In the same direction of assumption let us consider large number of such particles ‘N’ now the case is a collection of a large number (\( N \sim 10^{10} \)) of similar particles which gives rise, if the trajectory of each particle satisfies the above equation, where \( F \) is the
total force exerted by the remaining particles. It is assumed that there are no external forces acting on the system. For systems with such a massive number of degrees of freedom the motion of the individual particles is not the scientifically interesting quantity, since it is not possible to measure by experiments nor can we draw any useful conclusions.

Kinetic theory instead provides a more practical description of such large particle system which could be encoded in the one particle distribution function \( f(t, x, v) \).

A small cube C is considered in the present case and in phase space centered \((x, v)\), It may be proposed to think of the quantity \( f(t, x, v) \), volume of C as an approximation for \( \frac{n(t,x,v)}{N} \), where \( n(t,x,v) \) is the number of particles at any time \( t \) whose physical state is represented by a single point in C.

Mathematically \( f(t, x, v) \) is a probability measure density; to find a particle at time \( t \) in the region \( U \times V \)

\[
\int_U \int_V f(t, x, v) dv dx
\]

The function \( f \) has to clearly specify the nature of variation of function when it is greater than or equal to zero. And the same may be represented as

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t, x, v) dv dx
\]

The 6N ordinary differential equations (ODEs) for the particle system are replaced by a single partial differential equation (PDE) on the distribution function \( f \).

Assuming function \( f \) as smooth and considering each individual particle trajectory follows a distribution function which remains constant the PDE may be justified.

\[
\frac{d}{dt} f(t, x(t), v(t)) = 0
\]

The \( x(t), v(t) \) is the solution of the defined equation , if function \( f \) remains regular and hence leads to the basic Vlasov equation from the following PDE on function \( f \).

\[
\frac{\partial}{\partial t} f + v \cdot \nabla_x f + \frac{f(t,x,v)}{m} \cdot \nabla_v f = 0
\]

The central idea for the present problem is to obtain one such solution for the equation which can be necessary and sufficient to prove that it initiated Landau damping in all the non linear cases.

Mathematical description and setup for the numerical scheme, as we have the 1D mathematical model is based on the ritualistic treatment of the Landau damping original equation.

\[
D(\omega, k) = 1 - \sum_{\omega_0} \frac{\omega_0^2 p_{\omega}^2}{k^2} \int \frac{\partial f_0}{\partial \omega} \frac{du}{u^2 - \frac{\omega_0^2}{k^2}}
\]

The equation asymptotically reaches singularity at a point, \( u - \omega / k \) implementing the residue theorem for the condition that has been considered as

\[
\frac{1}{x-x_0} \rightarrow \frac{1}{x-x_0-i_0} = P \frac{1}{x-x_0} + i\pi \delta(x-x_0)
\]
And the integral can be rewritten as given below

\[ D(\omega, k) = 1 - \sum_\alpha \frac{\omega^2 p_\alpha}{k^2} P \int \frac{\partial f_{\alpha_0}}{u - \frac{k^2}{k}} \, du - i\pi \sum_\alpha \frac{\omega^2 p_\alpha}{k^2} \frac{\partial F_{\alpha_0}}{\partial u} \bigg|_{u=\frac{\omega}{k}} \]  

(9)

The proof of backward characteristic can be proved to have the consistency.

**Lemma:** Let \( F \in C^1(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3) \in \text{sup} \{ F \mid (t, x, v) \in (T_0, T) \times \mathbb{R}^3 \times \mathbb{R}^3 \} < \infty \) for all values of \( T > 0 \) then \( Z(s, t, \cdot) : \mathbb{R}^6 \rightarrow \mathbb{R}^6 \) is a diffeomorphism with inverse \( Z^{-1}(s, t, \cdot) : Z(t, s, \cdot) \) for all and for all \( 0 \leq s \leq t \) fixed.

**Proof:** The assertion about the regularity of \( Z \) follows from standard ODEs theory.

The fact that the inverse of \( Z(s, t, \cdot) \) is given by \( Z^{-1}(s, t, \cdot) = Z(t, s, \cdot) \) i.e. \( Z(r; t; Z(t, t, z)) = Z(t; s, z) \).

at time \( r = t \), both members of the above equation is equal to \( z \), so the claim

\[ \text{det} [\nabla_x Z(s; t, z)] = 1 \]

For all values of \( s \) in the range of \([0, t]\) and Hence that is equivalent to

\[ \frac{d}{ds} \| \text{det} [\nabla_x Z(s; t, z)] \| = 0 \]  \( \text{at} 0 \leq s \leq t \)  

(11)

From the equation (11) and \( Z(t; t, Z(t, s, z)) = 1 \) implies that \( \text{det}[\nabla_x Z(s; t, z)] = 1 \)

Therefore using Matrix calculus

\[ \frac{d}{dt} (\text{det} A) = \text{det} A \sum_{i,j} A^{-1}_{ij} \frac{d}{dt} A_{ij} \]  

(12)

the solution \( Z(s; t; z) \) of the system \( \dot{z} = G(t, z) \) will satisfy

\[ \frac{d}{ds} \| \text{det}[\nabla_x Z(s; t, z)] \| = \nabla_x G(s, Z(s; t, z)) \text{det} \nabla_x Z(s; t, z) \]  

(13)

**Computational setup:**

Before taking the computational step it is also necessary to whether the solution of the characteristic system satisfies the backward characteristic.

Given \( x, v \in \mathbb{R}^3 \) and \( 0 \leq s \leq t \), we denote by space \( X(s; t; x; v) \) and \( V(s; t; x; v) \), the solution of the characteristic system equation (1) that satisfies

\[ x = X(t; t; x, v) \text{ and } v = V(t; t; x, v) \]  

(10)

The backward characteristic can be proved with a lemma

With suitable assumptions implemented damping on the velocity profile in the classical plasma physics that can suitable at all scales. As such generally few of the scales are not interested, for example the scales smaller than Debye length and Jeans length.

The Linear type electron plasma waves of plasma can be obtained by solving the linearized Vlasov equation together with the Poisson equation, which was first treated by Vlasov.

The dispersion relation is given by

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\[ \frac{\partial f}{\partial t} + v \cdot \nabla_x f + \frac{F}{m} \nabla_v f = 0 \quad (14) \]

Where the density of electron is represented as \( f(t, x, v) > 0 \), in space (x= position, v=velocity) \( m \) is the mass of an electron.

\[ F = F(t, x) \] is given a notation as the mean-field of electrostatic force

\[ F = -eE \quad (15) \]

\[ E = \nabla^{-1} \left( 4\pi \times \rho \right) \quad (16) \]

Where

\[ e > 0 \] is the absolute electron charge,

\[ E = E(t, x) \] is the electric field, and

\[ \rho = \rho(t, x) \] is the density of charges

\[ \rho = \left[ (\rho_i) - e \int f \, dv \right] \quad (17) \]

Where \( \rho_i \) is the density of charges due to ions.

III. Results and Discussion

The damping can be determined from the slope of the equilibrium distribution curve at any of the phase velocities of the mode. Depending on the nature of the slope it can be confirmed that whether the Landau damping is of growth mode or leading to the instability. If the particle velocity is marginally more than the density of the distribution it is more likely to gain the energy from the wave field and leads to Landau damping Fig. 1, on the other hand if the mean field energy is less than the density distribution the particles lose energy to the wave which leads to Landau growth Fig. 2, as shown below. This has been successfully predicted numerically and supported with the discrete analytic data plotted on the graphs as shown in the Fig. 1 and Fig. 2.

![Fig. 1 Vlasov Solution Converging](image-url)
Fig.2 Vlasov Solution leading to damping

References

I. Biskamp. D, “Magnetohydrodynamic Turbulence”, doi:10.1017/cbo9780511535222, 2003.

II. Blaisdell, G. A,“Review of "Implicit Large Eddy Simulation: Computing Turbulent Fluid Dynamics””, AIAA Journal, 46(12), 3168-3170. doi:10.2514/1.40931, 2008.

III. Braun. W.&Hepp. K, “The Vlasov dynamics and its fluctuations in the 1/N limit of interacting classical particles”, Communications in Mathematical Physics, 56(2), 101-113. doi:10.1007/bf01611497, 1977.

IV. Dafermos. C. M & Feireisl. E, Handbook of differential equations: evolutionary equations Boston: Elsevier, 2004.

V. Degond. P,“Spectral theory of the linearized Vlasov-Poisson equation”, Transactions of the American Mathematical Society, 294(2), 435-435. doi:10.1090/s0002-9947-1986-0825714-8, 1986.

VI. Derfler. H & Simonen, T. C,”Landau Waves: An Experimental Fact”, Physical Review Letters, 17(4), 172-175. doi:10.1103/physrevlett.17.172, 1966.

VII. Elskens. Y, “Irreversible behaviors in Vlasov equation and many-body Hamiltonian dynamics: Landau damping, chaos and granularity”, Topics in Kinetic Theory, 89-108. doi:10.1090/fic/046/03, 2005.
VIII. Glassey. R & Schaeffer. J, “On time decay rates in landau damping”, Communications in Partial Differential Equations, 20(3-4), 647-676. doi:10.1080/03605309508821107, 1995.

IX. Glassey. R & Schaeffer. J, “Time decay for solutions to the linearized Vlasov equation”, Transport Theory and Statistical Physics, 23(4), 411-453. doi:10.1080/00411459408203873, 1994.

X. Horst. E, “On the asymptotic growth of the solutions of the vlasov-poisson system”, Mathematical Methods in the Applied Sciences, 16(2), 75-85. doi:10.1002/mma.1670160202, 1993.

XI. Horst. E. & Neunzert. H, “On the classical solutions of the initial value problem for the unmodified non-linear Vlasov equation I general theory”, Mathematical Methods in the Applied Sciences, 3(1), 229-248. doi:10.1002/mma.1670030117, 1981.

XII. Ng. K. Y & Landau damping. doi:10.2172/1002000, 2010.

XIII. Pfaffelmoser. K, “Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data”, Journal of Differential Equations, 95(2), 281-303. doi:10.1016/0022-0396(92)90033-j, 1992.

XIV. Robert. R, Statistical Mechanics and Hydrodynamical Turbulence. Proceedings of the International Congress of Mathematicians, 1523-1531. doi:10.1007/978-3-0348-9078-6_85, 1995.

XV. Ryutov. D. D, “Landau damping: half a century with the great discovery”, Plasma Physics and Controlled Fusion, 41(3A). doi:10.1088/0741-3335/41/3a/001, 1999.

XVI. Vlasov. A. A, “The Vibrational Properties of an Electron Gas”, Soviet Physics Uspekhi, 10(6), 721-733. doi:10.1070/pu1968v010n06abeh003709, 1968.