On Renormalons and Landau Poles in Gauge Field Theories

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Abstract

It is shown that the commonly accepted relationship between the Landau singularity in the running coupling constant of QED or QCD and the renormalon singularities in the Borel sums of perturbation theory expansions is only a particular feature of the restriction of the perturbative $\beta$–function to the one loop level.
1. The success of the Standard Model in describing particle physics at present energies relies heavily on the perturbation theory approach as applied to the underlying gauge field theories: \( SU(3) \times SU(2)_L \times U(1) \). Yet, it is known since the early work of Dyson \[1\] that perturbation theory itself suffers from ambiguities which originate in the fact that physical quantities are not analytic in the coupling constants which define the expansion parameters. This is reflected in a characteristic \( k! \)-growth pattern in the coefficients having the same sign of the large order terms in the perturbation series \[\dagger\]. In theories like QED this growth originates in the large momentum integration region of virtual photons dressed with vacuum polarization corrections and leads to singularities in the associated Borel plane; the so-called UV–renormalons \[4–8\]. In QCD, it is the low momentum integration region of virtual gluons dressed with running couplings which leads to non integrable singularities in the Borel plane the so–called IR–renormalons \[3, 6, 13\].

The study of renormalon properties in gauge theories is at present an active field of research. The rôle of IR–renormalons in the operator product expansion of two point functions and their relationship with non–perturbative inverse power corrections has been extensively discussed in the literature \[3, 14\]. This and further discussions which originated after ref. \[17\] appeared, has led to a new point of view \[18\] concerning renormalons in QCD which focuses on the possibility that their systematic study in a given hadronic process might suggest generic non–perturbative effects of a universal nature. This applies to the case of IR–renormalons as well as to the much less explored rôle of UV–renormalons \[19, 20\] in QCD.

Practically all work on renormalons in the literature is restricted to the effect of the first coefficient \( \beta_1 \) of the \( \beta \)–function. There is a good reason for that: the position of the singularities for positive Borel variable is governed by \( \beta_1 \); and it is the sign of \( \beta_1 \) which fully dictates their fate as UV–like \( (\beta_1 > 0) \) versus IR–like singularities \( (\beta_1 < 0) \). It is also often stated that these singularities are due to the Landau pole in the running coupling constant. The purpose of this note is to discuss some interesting properties which appear when the first two terms of the \( \beta \)–function are fully taken into account, and to show how they clash with this common belief that renormalon singularities are due to the Landau pole in the running coupling constant.

2. We shall first review the usual analysis of renormalon effects in the precise case of the anomalous magnetic moment of the electron \( a_e \) in QED, the same example that Lautrup considered in ref. \[4\]. The relevant Feynman diagrams are those generated by the one–renormalon chain in Fig. 1, and their contribution to \( a_e \) is given by the integral

\[
a_e = \frac{1}{\pi} \int_0^1 dx (1 - x) \alpha_{\text{eff}} \left( \frac{x^2}{1 - x}, \alpha \right),
\]

where \( x \) is the Feynman parameter which combines the propagators in the loop vertex of Fig. 1, and \( \alpha_{\text{eff}} \) the QED effective charge. For large values of the euclidean momentum \( k^2 \) \( (k^2/m^2 = \frac{x^2}{1 - x} \) in our case, with \( m \) the electron mass) the effective charge obeys the renormalization group equation:

\[
\left( m \frac{\partial}{\partial m} + \beta(\alpha)\alpha \frac{\partial}{\partial \alpha} \right) \alpha_{\text{eff}}^{(\infty)} \left( \frac{k^2}{m^2}, \alpha \right) = 0,
\]

\[\dagger\]For a comprehensive review of the subject and a collection of early articles see ref. \[2\]
where the infinity superscript refers to this asymptotic limit for $k^2$.

Equation (1) is an exact integral representation of the contribution to the electron anomaly from the infinite class of diagrams which Fig. 1 represents. In particular, if one takes $\alpha_{\text{eff}} = \alpha \simeq 1/137$ one of course re-obtains the value $a_e = \alpha/2\pi$. When the number of vacuum polarization bubbles is large, the integral is then dominated by the $x \to 1$ region; i.e. large values of the virtual euclidean loop momentum, and becomes equivalent to the integral

$$a_e \simeq \frac{1}{\pi} \int_{m^2}^{\infty} \frac{dk^2}{k^2} \left( \frac{m^2}{k^2} \right)^2 \alpha_{\text{eff}}^{(\infty)} \left( \frac{k^2}{m^2}, \alpha \right).$$  \hspace{1cm} (3)

The insertion in the integrand above of the one loop solution to the renormalization group equation in (2) with $\beta(\alpha) = [\beta_1 = \frac{2}{3}\pi; \beta_2 = \frac{1}{2}]$; i.e., the insertion of the expression

$$\frac{\alpha}{\alpha_{\text{eff}}^{(\infty)} \left( \frac{k^2}{m^2}, \alpha \right)} = 1 - \frac{\beta_1}{2\pi} \alpha \log \frac{k^2}{m^2}$$  \hspace{1cm} (4)

in (3), and the change of variables:

$$Bz^2 = 1 - \frac{\alpha}{\alpha_{\text{eff}}^{(\infty)} \left( \frac{k^2}{m^2}, \alpha \right)}, \quad \text{where} \quad B \equiv \frac{\beta_1}{2\pi},$$  \hspace{1cm} (5)

leads then to the Borel sum

$$a_e = \frac{1}{\pi} \int_0^{\infty} dz \ e^{-\frac{z}{2-Bz}} \frac{1}{2-Bz}. $$  \hspace{1cm} (6)

The singularity (UV renormalon) at $z = 2/B$ reminds us that we only know the function for $z < 2/B$, just as we only know $\alpha_{\text{eff}}^{(\infty)} \left( \frac{k^2}{m^2}, \alpha(m) \right)$ in Eq. (4) up to a momentum $k^2 = \Lambda_{L}^2 = \exp \left( \frac{1}{\beta_1} \right)$. At this momentum (the Landau scale) the effective charge becomes infinity. The association of the UV-renormalon ambiguity with the Landau pole comes from this observation.

Notice that in Eq. (6) the leading UV renormalon at $z = 1/B$ does not contribute, so that the ambiguity starts at $z \geq 2/B$. The reason for it is the factor $1 - x$ in the numerator in eq. (4), which appears because of the electron helicity conservation of the electromagnetic interaction, and leads to 2–powers of the factor $m^2/k^2$ in the euclidean momentum version of the same integral in (3). In the language of UV–effective operators of Parisi, this corresponds to the tree level insertion of the operator $(m^3/\Lambda_{L}^4) \bar{\psi}(x)\sigma_{\mu\nu}\psi(x)F^{\mu\nu}(x)$.

3. Let us now study the same $a_e$ observable but in the presence of the first two terms of the QED $\beta$–function:

$$\beta(\alpha) = [\beta_1 = \frac{2}{3}\pi; \beta_2 = \frac{1}{2}] \left( \frac{\alpha}{\pi} \right)^2.$$  \hspace{1cm} (7)

In this equation we have no loss of generality since any $\beta$–function can be brought into this form by a (perturbative) coupling constant redefinition.

To simplify the notation we shall denote

$$\alpha(k) \equiv \alpha_{\text{eff}}^{(\infty)} \left( \frac{k^2}{m^2}, \alpha \right) \quad \text{and} \quad \delta \equiv \frac{\beta_2}{\beta_1^2}.$$  \hspace{1cm} (8)
The general solution of eq. (2) is then
\[
\frac{\alpha(m)}{\alpha(k)} = 1 - B\alpha(m) \log \frac{k^2}{m^2} + \delta B\alpha(m) \left[ \log \frac{\alpha(m)}{\alpha(k)} + \log \frac{1 + \delta B\alpha(k)}{1 + \delta B\alpha(m)} \right], \tag{9}
\]
where \(\alpha(m)\) is the boundary condition at \(k^2 = m^2\). The shape of these solutions in the plane \(\log \frac{k^2}{m^2}, \alpha(k)\) is shown in Fig. 2. In full generality, and depending on whether the boundary condition is chosen to be \(\alpha(m) \geq 0; -\frac{1}{\delta B} \leq \alpha(m) \leq 0;\) or \(\alpha(m) \leq -\frac{1}{\delta B}\) the solutions lie along the curves shown in the regions I, II, or III of Fig. 2. These regions correspond to the analogous ones in the \(\beta\)-function of Fig. 3 where the arrows show the flow of the effective charge as \(k^2 \to \infty\). The three regions are separated by the fixed points at \(\alpha^* = 0\) and \(\alpha^{**} = -\frac{1}{\delta B}\).

The physical region is clearly the one in I. For a given initial condition \(\alpha(m) > 0\), the running coupling \(\alpha(k)\) grows and hits a Landau pole at a finite value of \(k^2\) that we call \(\Lambda^2_L\), where \(\alpha(\Lambda_L) \to \infty\). With an initial condition fixed in region I, there is no solution to the renormalization group equation when \(k^2 \geq \Lambda^2_L\). Yet the integral over the euclidean momentum in eq. (3) generated by perturbation theory runs up to \(\infty\). It has been recently shown by Grunberg \[21\] that in the presence of the first two terms in the power expansion of the \(\beta\)-function it is still possible to find an exact change of variables which maps \(k^2\) integrals in the euclidean space, like the one in (3), to integrals in the \(z\)-variable of the Borel plane. The change of variables in question is
\[
B^\frac{z}{2} = \frac{1 - \frac{\alpha(m)}{\alpha(k)}}{1 + \delta B\alpha(m)}, \tag{10}
\]
and leads to the Borel sum
\[
a_e = \frac{1}{2\pi} \int_0^\infty dz \, e^{-\frac{z}{2}} \frac{e^{-z\delta B}}{(1 - B^\frac{z}{2})^{1+\delta}}. \tag{11}
\]
However this is only a formal expression. With the initial condition \(\alpha(m) > 0\) fixed, the change of variables in (10) is only well defined in the region where \(m^2 \leq k^2 < \Lambda^2_L\), which corresponds to \(0 \leq B^{\frac{z}{2}} < (1 + \delta B\alpha(m))^{-1} < 1\). Let us however inspect what happens when one goes beyond this physically allowed region i.e., when \((1 + \delta B\alpha(m))^{-1} \leq B^{\frac{z}{2}} < \infty\). This is equivalent to taking a certain arbitrary \(\alpha(k)\) in the non–physical region III, where \((1 + \delta B\alpha(m))^{-1} < B^{\frac{z}{2}} \leq 1\), and then in the region II, where \(1 \leq B^{\frac{z}{2}} < \infty\). With this proviso, eq. (11) has a definite (but arbitrary) meaning. From this point of view, the singularity at \(z = 2/B\) acts as a reminder of this arbitrariness. For example, one can define an “extended” \(\alpha(k)\) valid at all \(k^2\)-values, \(m^2 \leq k^2 \leq \infty\), such that in the physical region \(m^2 \leq k^2 < \Lambda^2_L\) it coincides with the physical \(\alpha(k)\) and that it satisfies the perturbative renormalization group equation in (3), except at a finite number of singularities. This extended \(\alpha(k)\) has the following form:
\[
\frac{\alpha(m)}{\alpha(k)} = 1 - B\alpha(m) \log \frac{k^2}{m^2} + \delta B\alpha(m) \log \left| \frac{\alpha(m) (1 + \delta B\alpha(k))}{\alpha(k) (1 + \delta B\alpha(m))} \right|. \tag{12}
\]
This expression leads to the trajectory with the arrows displayed on Fig. 2, where the arrows correspond to the flow as one integrates over \(z\) going first from region I to region III, and

\[\text{Footnote: Notice the absolute value in the argument of the logarithm, in contrast to eq. (3).}\]
then to region II. In so doing, one sees that one crosses three singularities: one in which 
\( \alpha(k) \) goes to infinity—the conventionally called Landau singularity—and two in which 
\( \log{\frac{k^2}{m^2}} \) goes to infinity—at the fixed points of \( \beta(\alpha) \): \( \alpha^* = 0 \) and \( \alpha^{**} = -\frac{1}{\beta^2} \). With this “extended”
definition of \( \alpha(k) \), the change of variables in (10) leads to the Borel sum

\[
a_e = \frac{1}{2\pi} \int_0^\infty \frac{dz e^{-\frac{z}{\alpha}}}{(1 - \frac{Bz}{2}) |(1 - \frac{Bz}{2})|^\delta}. \tag{13}
\]

The two-loop Borel integrals (11) and (13) still have the singularity at \( z = 2/B \), as in the one-loop case. This may lead one to believe that, exactly as in the one-loop case, this singularity is caused by the corresponding Landau singularity in euclidean momentum. However this is not so. To see this explicitly let us expand eq. (9), or eq. (12), for \( \alpha(k) \to +\infty \), i.e., near the Landau pole:

\[
B^2 \log \frac{k^2}{m^2} = 1 - \delta B^2 \log \left( 1 + \frac{1}{\delta B^2} \right) - \frac{\alpha}{2\delta B} \frac{1}{\alpha^2(k)} + \mathcal{O} \left( \frac{1}{\alpha^3(k)} \right). \tag{14}
\]

The terms of \( \mathcal{O} \left( \frac{1}{\alpha(k)} \right) \) cancel, which means that the Landau pole of \( \alpha(k) \) at

\[
k^2 = \Lambda^2_L = m^2 \exp \left\{ \frac{1}{B^2} - \delta \log \left( 1 + \frac{1}{\delta B^2} \right) \right\}, \tag{15}
\]

goes like

\[
\alpha(k) = \frac{\pi}{\beta^2} \frac{1}{\log{\Lambda^2_L - \log{k^2}}}^{1/2}; \tag{16}
\]

i.e., it is a squared root singularity and therefore integrable up to the Landau scale \( \Lambda^2_L \) where \( Bz/2 = (1 + \delta B^2(m))^{-1} < 1 \). From that point onwards the integration in \( z \) can be done using the extended \( \alpha(k) \) in eq. (12), following the arrows on Fig. 2 from region I to region III until \( \alpha = \alpha^{**} \) where one hits the Borel singularity at \( z = z_n \). This is to be contrasted with the one-loop behaviour in eq. (4) where

\[
\alpha(k) = \frac{1}{B} \log{\frac{\Lambda^2_L}{\log{\Lambda^2_L - \log{k^2}}}}, \tag{17}
\]

and the singularity is not integrable. The euclidean origin of the \( z = 2/B \) singularity in the Borel sum at the two-loop level is the \( \log{\frac{k^2}{m^2}} \) singularity corresponding to \( \alpha^{**} = -\frac{1}{\beta^2} \); not the Landau pole! Of course in the limit \( \beta^2 \to 0 \) one recovers the one-loop situation in which \( \alpha^{**} \to \infty \), and the Landau pole.

4. The basic features we have discussed in the case of the anomalous magnetic moment of the electron in QED are rather generic and they appear as well in the case of IR–renormalon calculus in QCD where, typically, one encounters euclidean integrals like

\[
R_n(Q^2) = \int_0^{Q^2} \frac{dk^2}{k^2} \left( \frac{k^2}{Q^2} \right)^n \alpha_s \left( \frac{k}{Q}, \alpha_s(Q) \right), \tag{18}
\]

\( ^5 \)Although one can invoke \( 1/N_F \) arguments to effect the limit \( \beta_2 \to 0 \) in QED, there is no analogue in the case of QCD.
with \( n > 0 \), so that the integral is infrared convergent. The scale \( Q^2 \) in these integrals corresponds to a sufficiently large choice of euclidean momentum at which the QCD running coupling \( \alpha_s(Q) \) is reasonably small. Here the ambiguity problem appears because of the integration over virtual euclidean momentum in the infrared region \( 0 \leq k^2 \leq Q^2 \), where the extrapolation of the perturbative \( \alpha_s(Q) \) coupling is not well defined. It is often stated in the literature that the reason for the ambiguity is the existence of the Landau pole in the region of integration. We shall see that, as already shown in the case of \( a_e \) in QED, this is only correct if one rather arbitrarily restricts the QCD beta function to its first term.

The discussion runs parallel to the one in the previous section, except that now the running coupling \( \alpha_s \left( \frac{k}{Q}, \alpha_s(Q) \right) \equiv \alpha_s(k) \) obeys the QCD renormalization group equation

\[
\frac{d\alpha_s(k)}{d \log \frac{k^2}{Q^2}} = -b_0 \alpha_s^2(k) - b_1 \alpha_s^3(k),
\]

where we are using the notation

\[
b_0 \equiv -\frac{1}{2\pi} \beta_1^\text{QCD} = \frac{1}{12\pi} (11N_c - 2n_f),
\]

\[
b_1 \equiv \frac{1}{2\pi^2} \beta_2^\text{QCD} = -\frac{1}{8\pi^2} \left\{ \frac{17}{3}N_c^2 - \frac{N_f^2 - 1}{2N_c - 5} - \frac{5}{3}N_c n_f \right\}.
\]

with \( b_0, b_1 > 0 \) in the physically interesting case \( N_c = n_f = 3 \). We can consider eq. (19) to be a generic case since any other \( \beta \)-function can be brought into this form by an appropriate coupling constant redefinition. Of course this redefinition will entail in general an infinite power series. The solutions of eq. (19) have the pattern shown in Fig. 4, with region I corresponding to the physical region with the boundary condition \( \alpha_s(Q) \geq 0 \). The equivalent change of variables to the one in eq. (10) is

\[
z = \frac{1 - \alpha_s(Q)}{\alpha_s(k)}; \quad z_n = \frac{n}{b_0};
\]

It maps the \( k^2 \) integral \( R_n \) onto an integral in the \( z \)-variable of the Borel plane:

\[
R_n(Q^2) = -\frac{1}{n} \int_0^\infty dz e^{-\frac{z}{\alpha_s(Q)}} \frac{e^{-\frac{b_1}{b_0}}}{(1 - \frac{z}{z_n})^{1+\delta_n}}, \quad \delta_n \equiv n \frac{b_1}{b_0}.
\]

For a fixed \( n \), the Borel integral (23) is singular at \( z = z_n \), which according to (22) happens when \( \alpha_s(k) = -\frac{b_0}{b_1} \) and has nothing to do with the Landau pole at which, by definition, \( \alpha_s(k) = \infty \). In fact at the Landau pole, which now occurs when

\[
k^2 = \Lambda^2_L = Q^2 \exp \left\{ -\frac{1}{b_0 \alpha_s(Q)} + \frac{b_1}{b_0^2} \log \left( 1 + \frac{b_0}{b_1 \alpha_s(Q)} \right) \right\},
\]

\( \alpha_s(k) \rightarrow \infty \) as a squared root singularity:

\[
\alpha_s(k) = \frac{1}{(2b_1)^{1/2}} \frac{1}{(\log \Lambda^2_L - \log k^2)^{1/2}},
\]

and it is therefore integrable.
Notice that the ambiguity in eq. (23) can be parameterized by a power–like contribution in $Q^2$. Indeed, making the change of variables

$$\frac{z}{z_n} = 1 + \frac{\alpha(Q)\omega}{z_n(1 + \frac{b_1}{b_0}\alpha(Q))} \quad (26)$$

one finds that the ambiguity in $R_n(Q^2), \delta R_n(Q^2)$, is given by

$$\delta R_n(Q) = \left\{ \frac{z_{n+\delta_n}^{1+\delta_n}}{n} e^{z_{n+\delta_n} b_1} \int_\omega d\omega e^{-\omega} \left( \frac{1}{\alpha(Q)} + \frac{b_1}{b_0} \right) \delta_n e^{-\frac{\omega z_n}{\alpha(Q)}} \right\} \quad (27)$$

$$\delta R_n(Q^2) = \{\text{coefficient}\} \times \left( \frac{\Lambda^2}{Q^2} \right)^n \quad (28)$$

where $\int_\omega d\omega e^{-\omega} \omega^{-(1+\delta_n)}$ denotes an integral with an arbitrary prescription to skip the singularity at $\omega = 0$. Although, as repeatedly emphasized, the Landau singularity is not directly responsible for the singularity at $z = z_n$, the final result of eq. (28) is insensitive to this fact and the form of (28) is the same as in the case of the one-loop $\beta$ function.

Before we conclude, we would like just to mention that an analogous reasoning for $a_e$ in QED leads, mutatis mutandis, to an equation like (28) for the analogous quantity $\delta a_e$ but with $n = -2$ and $Q^2 = m^2$. Therefore it is also true for a two-loop $\beta$ function that the ambiguity $\delta a_e$ can be given by the tree-level insertion of the local operator $(m^3/\Lambda^4) \bar{\psi}(x)\sigma_{\mu\nu}\psi(x)F^{\mu\nu}(x)$, just as in the one-loop case.

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**Note Added:**

The analysis of renormalon effects in the case of a $\beta$ function with a perturbative fixed point has been considered by Grunberg in ref. [21] and also by Yu.L. Dokshitzer and N.G. Uraltsev in [hep-ph/9512407]. They reach similar conclusions to ours. We thank N.G. Uraltsev for drawing his paper to our attention.
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**Figure Captions**

Fig. 1: Feynman diagrams which define the one–renormalon chain contribution to $a_e$ in eq. (1).

Fig. 2: General pattern of the QED renormalization group solution in eq. (9). $\tau$ stands for $\log \frac{k^2}{m^2}$. The regions I, II and III correspond to the choice $\alpha(m) \geq 0; -\frac{1}{B} \leq \alpha(m) \leq 0$; and $\alpha(m) \leq -\frac{1}{B}$ as boundary condition.

Fig. 3: The QED $\beta$–function in perturbation theory at the two–loop level. The arrows show the flow of the effective charge as $k^2 \to \infty$.

Fig. 4: General pattern of the QCD renormalization group solution in eq. (19). $\tau$ stands for $\log \frac{k^2}{Q^2}$. The regions I, II and III correspond to the choice $\alpha_s(Q) \geq 0; -\frac{b_0}{b_1} \leq \alpha_s(Q) \leq 0$; and $\alpha_s(Q) \leq -\frac{b_0}{b_1}$ as boundary condition.
Fig. 3

Fig. 4