Maximal atmospheric neutrino mixing from texture zeros and quasi-degenerate neutrino masses

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21 April 2011

Abstract

It is well-known that, in the basis where the charged-lepton mass matrix is diagonal, there are seven cases of two texture zeros in Majorana neutrino mass matrices that are compatible with all experimental data. We show that two of these cases, namely $B_3$ and $B_4$ in the classification of Frampton, Glashow and Marfatia, are special in the sense that they automatically lead to near-maximal atmospheric neutrino mixing in the limit of a quasi-degenerate neutrino mass spectrum. This property holds true irrespective of the values of the solar and reactor mixing angles because, for these two cases, in the limit of a quasi-degenerate spectrum, the second and third row of the lepton mixing matrix are, up to signs, approximately complex-conjugate to each other. Moreover, in the same limit the aforementioned cases also develop a maximal CP-violating CKM-type phase, provided the reactor mixing angle is not too small.

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1 Introduction

It is by now well-established that at least two of the neutrino masses $m_j$ ($j = 1, 2, 3$) are non-zero. The same applies to the angles in the lepton mixing matrix $V$. Its parameterization is usually chosen in analogy to the CKM matrix as [1]

$$V = \begin{pmatrix} c_{13}c_{12} & c_{13}s_{12} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} - s_{23}s_{13}s_{12}e^{i\delta} & s_{23}s_{12}c_{13} \\ s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{13}s_{12}e^{i\delta} & s_{23}s_{12}c_{13} \end{pmatrix},$$

with $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$, the $\theta_{ij}$ being angles of the first quadrant. While the angles $\theta_{12}$ and $\theta_{23}$ are approximately $34^\circ$ and $45^\circ$, respectively, the angle $\theta_{13}$ is compatible with zero [2, 3]. All data on lepton mixing are compatible with the tri-bimaximal matrix

$$V_{\text{HPS}} \equiv \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix},$$

which has been put forward by Harrison, Perkins and Scott (HPS) [4] already in 2002.

Equation (2) has lead to the speculation that there is a non-abelian family symmetry behind the scenes enforcing $s_{23}^2 = 1/2$ in particular. This speculation is in accord with the finding of [6] that the only extremal angle which can be obtained by an abelian symmetry is $\theta_{13} = 0^\circ$, i.e., $\theta_{23} = 45^\circ$ cannot be enforced by an abelian symmetry. A favourite non-abelian group in this context is $A_4$ [7]. For recent developments and other favourite groups see the reviews in [8] and references therein, for attempts on systematic studies see [9, 10, 11] (the latter paper refers to abelian symmetries).

However, there is an alternative to non-abelian groups. It is not necessary that, for instance, $\theta_{23} = 45^\circ$ is exactly realized at some energy scale. It suffices that such a relation is fulfilled with reasonable accuracy. This could happen without need for a non-abelian symmetry. In order to pin down what we mean specifically we consider the structure of the mixing matrix $V$. It has two contributions, the unitary matrices $U_\ell$ and $U_\nu$, stemming from the diagonalization of the charged-lepton mass matrix $M_\ell$ and of the neutrino mass matrix $M_\nu$, respectively. Then the matrix

$$U \equiv U_\ell^\dagger U_\nu = e^{i\tilde{\alpha}} V e^{i\tilde{\sigma}}$$

occurs in the charged-current interaction and the lepton mixing matrix $V$ defined above is obtained by removing the diagonal unitary matrices

$$e^{i\tilde{\alpha}} = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}) \quad \text{and} \quad e^{i\tilde{\sigma}} = \text{diag}(e^{i\sigma_1}, e^{i\sigma_2}, e^{i\sigma_3})$$

from $U$. Without loss of generality we will use the convention $e^{i\sigma_3} = 1$ in the following. Suppose that we have a model in which $U_\ell$ and $U_\nu$ are functions of the charged-lepton and neutrino mass ratios, respectively, and that these mass ratios also parameterize the  

\[1\] However, recently, it has been argued that tri-bimaximal mixing might nevertheless have an accidental origin [5].
deviations of $U_\ell$ and $U_\nu$ from a diagonal form. In $U_\ell$ these ratios are $m_e/m_\mu$, $m_e/m_\tau$ and $m_\mu/m_\tau$. Since the mass hierarchy in the charged-lepton sector is rather strong, $U_\ell$ is approximately a diagonal matrix of phase factors, with the possible exception of the occurrence of $m_\mu/m_\tau$; if this ratio appears in a square root in analogy to the famous formula $\sin \theta_c \simeq \sqrt{m_d/m_s}$ for the Cabibbo angle [12], with quark masses $m_d$ and $m_s$, then $\sqrt{m_\mu/m_\tau} \simeq 0.24$ is even larger than $\sin \theta_c$. The simplest way to avoid such a deviation of $U_\ell$ from a diagonal matrix is to have a model which, through its symmetries, enforces a diagonal $M_\ell$. Switching to $U_\nu$, we point out that up to now the type of neutrino mass spectrum is completely unknown [1]. A particularly exciting possibility would be a quasi-degenerate mass spectrum in which case the neutrino mass ratios could be close to one such that effectively $U_\nu$ is independent of the masses and could look like a matrix of pure numbers, potentially disturbed by phase factors. Thus, with $U_\ell$ sufficiently close to a diagonal matrix and a quasi-degenerate neutrino mass spectrum it might be possible to simulate a mixing matrix $V$ consisting of “pure numbers,” leading for instance to an atmospheric neutrino mixing angle $\theta_{23}$ which is in practice indistinguishable from 45\(^\circ\).

The advantage is that such a scenario could be achieved with texture zeros and that texture zeros in mass matrices may always be explained by abelian symmetries [13], at the expense of an extended scalar sector in renormalizable models\(^2\).

Let us summarize the assumptions of this paper:

- $U_\ell$ is sufficiently close to a diagonal matrix such that in good approximation it does not contribute to $V$.
- The neutrino mass spectrum is quasi-degenerate.
- The symmetry groups we have in mind are abelian, i.e., we deal with texture zeros.

In the following we will show that these assumptions can indeed lead to a realization of maximal atmospheric neutrino mixing, in the framework which consists of Majorana neutrino mass matrices with two texture zeros and a diagonal mass matrix $M_\ell$; two of the viable cases of neutrino mass matrices classified in [15] exhibit precisely the desired features.

In section 2 we review the viable textures presented in [15] and point out models in which they can be realized. Then, in section 3 we discuss the phenomenology of the cases B\(_3\) and B\(_4\) of [15] in the light of the philosophy specified above. The remaining cases are discussed in section 4. We summarize our findings in section 5.

### 2 The framework

Assuming the neutrinos to be Majorana particles, the neutrino mass term is given by

$$\mathcal{L}_{\nu \text{mass}} = \frac{1}{2} \nu_L^T C^{-1} M_{\nu} \nu_L + H.c.,$$

\(^2\)We emphasize that our approach is different from that of [11] where the Froggatt–Nielsen mechanism [14] is used and, therefore, order-of-magnitude relations among the elements of mass matrices are assumed.
Table 1: The viable cases in the framework of two zeros in the Majorana neutrino mass matrix $\mathcal{M}_\nu$ and a diagonal charged-lepton mass matrix $M_\ell$ \cite{15}.

with a symmetric mass matrix $\mathcal{M}_\nu$. In the basis where the charged-lepton mass matrix is diagonal, there are seven possibilities for an $\mathcal{M}_\nu$ with two texture zeros which are compatible with all available neutrino data, as was shown in \cite{15}. These seven viable cases are listed in table 1. The phenomenology of those seven mass matrices has been discussed in \cite{15,16,17}. Moreover, case C has also been investigated in \cite{18}.

There are several ways to construct models where the cases of table 1 together with a diagonal charged-lepton mass matrix are realized by symmetries. Five of the seven mass matrices, but not $B_1$ and $B_2$, have various embeddings in the seesaw mechanism \cite{19}, by placing zeros in the Majorana mass matrix $M_R$ of the right-handed neutrino singlets $\nu_R$ and in the Dirac mass matrix $M_D$ connecting the $\nu_R$ with the $\nu_L$ \cite{20}. With the methods described in \cite{13}, one can then construct models where the zeros in the various mass matrices, including the six off-diagonal zeros in $M_\ell$, are enforced by abelian symmetries.

Four of the seven textures of table 1, namely $A_1$, $A_2$, $B_3$, $B_4$, have a realization in the seesaw mechanism with a diagonal $M_D$ \cite{20,21}: by suitably placing two texture zeros in $M_R$ or, equivalently, in $M_\nu^{-1}$, these four textures can be obtained. Actually, now we are dealing with 14 texture zeros, namely six in $M_\ell$ and $M_D$ each and two in $M_R$. In order to construct models for these four cases, one Higgs doublet is sufficient, but one needs two scalar gauge singlets in order to implement the desired form of $M_R$ \cite{21}.

All of the seven cases of table 1 can be realized as models in scalar-triplet extensions of the Standard Model \cite{13}, i.e., in the type II seesaw mechanism \cite{22} without any right-handed neutrino singlets.

\footnote{There are three more viable cases of texture zeros in $M_\nu^{-1}$ which do not correspond to texture zeros in $M_\nu$. \cite{21}.}
3 Cases B$_3$ and B$_4$

In this section we discuss the cases B$_3$ and B$_4$ which correspond to the Majorana mass matrices

\[
\begin{align*}
B_3 & : \mathcal{M}_\nu \sim \begin{pmatrix} \times & 0 & \times \\ 0 & 0 & \times \\ \times & \times & \times \end{pmatrix}, \\
B_4 & : \mathcal{M}_\nu \sim \begin{pmatrix} \times & \times & 0 \\ \times & \times & \times \\ 0 & \times & 0 \end{pmatrix}.
\end{align*}
\]

The symbol $\times$ denotes non-zero matrix elements. The equations which follow from these cases have the form

\[
\sum_{j=1}^{3} V_{\alpha j} V_{\alpha j} \mu_j = \sum_{j=1}^{3} V_{\alpha j} V_{\beta j} \mu_j = 0 \quad \text{with} \quad \mu_j \equiv m_j e^{2i\sigma_j}
\]

and $\alpha \neq \beta$, where B$_3$ is given by $(\alpha, \beta) = (\mu, \epsilon)$ and B$_4$ by $(\alpha, \beta) = (\tau, \epsilon)$.

Equation (7) can be considered from a linear-algebra perspective. Defining line vectors

\[
z_{\alpha} = (V_{\alpha j}), \quad z_{\beta} = (V_{\beta j})
\]

of $V$, equation (7) tells us that, because of the unitarity of $V$, the line vector

\[
(V_{\alpha 1} \mu_1^*, V_{\alpha 2} \mu_2^*, V_{\alpha 3} \mu_3^*)
\]

is orthogonal to both, $z_{\alpha}$ and $z_{\beta}$. Therefore, this vector must be proportional to the line vector $z_{\gamma}$ with $\gamma \neq \alpha, \beta$, and we obtain

\[
z_{\gamma} = (V_{\gamma j}) = \frac{e^{i\phi}}{N_{\alpha}} \left( V_{\alpha j} \mu_j^* \right) \quad \text{with} \quad N_{\alpha}^2 = \sum_{k=1}^{3} |V_{\alpha k}|^2 m_k^2.
\]

We use equation (10) for the discussion of the physical consequences of cases B$_3$ and B$_4$. We begin with case B$_3$ where $\gamma = \tau$. Defining $\epsilon = s_{13} e^{i\delta}$, $t_{12} = s_{12}/c_{12}$ and $t_{23} = s_{23}/c_{23}$, from equations (1) and (10) we find the following relations:

\[
B_3 : \quad \frac{\mu_1}{m_3} = \frac{V_{\mu 3} V_{\tau 3}^*}{V_{\mu 1} V_{\tau 1}^*} = -\frac{t_{12} t_{23} - \epsilon^*}{t_{12} + t_{23}\epsilon} t_{23}, \quad \frac{\mu_2}{m_3} = \frac{V_{\mu 3} V_{\tau 3}^*}{V_{\mu 2} V_{\tau 3}^*} = -\frac{t_{23} + t_{12} \epsilon^*}{1 - t_{12} t_{23} \epsilon} t_{23}.
\]

Alternatively, one can use the procedure of [16] to arrive at these expressions.

The analysis of equation (11) proceeds in the following way. We define

\[
\rho_j = \left( \frac{m_j}{m_3} \right)^2 \quad (j = 1, 2),
\]

take the absolute values of the two expressions in equation (11) and eliminate

\[
\zeta \equiv 2 t_{12} t_{23} s_{13} \cos \delta = \frac{t_{12}^2 t_{23}^2 + s_{13}^2 - \rho_1 (t_{12}^2 + t_{23}^2 s_{13}^2) / t_{23}^2}{1 + \rho_1 / t_{23}^2}.
\]

Then we end up with a cubic equation for $t_{23}^2$:

\[
t_{23}^6 + t_{23}^4 \left[ s_{13}^2 + c_{13}^2 \left( c_{12}^2 \rho_1 + s_{12}^2 \rho_2 \right) \right] - t_{23}^2 \left[ s_{13}^2 \rho_1 \rho_2 + c_{13}^2 \left( s_{12}^2 \rho_1 + c_{12}^2 \rho_2 \right) \right] - \rho_1 \rho_2 = 0.
\]
Inspection of this equation shows that it has a unique positive root. Thus, given the neutrino masses $m_1$, $m_2$, $m_3$ and the mixing angles $\theta_{12}$ and $\theta_{13}$, equation (14) determines $\theta_{23}$. Using equations (11) and (13), we adopt the following philosophy:

\[
\text{input: } m_{1,2,3}, \theta_{12}, \theta_{13} \Rightarrow \text{predictions: } \theta_{23}, \delta, 2\sigma_{1,2}.
\] (15)

Since the Majorana phases $2\sigma_{1,2}$ are not directly accessible to experimental scrutiny, we will later consider instead the observable $m_{\beta\beta}$, the effective mass in neutrinoless double-beta decay.

An approximate solution of equation (14) for a quasi-degenerate neutrino mass spectrum is given by

\[
t_{23}^2 \simeq 1 - \frac{1}{2} \frac{\Delta m_{31}^2}{m_1^2} (1 + s_{13}^2) \quad \text{or} \quad s_{23}^2 \simeq \frac{1}{2} \frac{1}{8} \frac{\Delta m_{31}^2}{m_1^2} (1 + s_{13}^2),
\] (16)

where corrections of order $(\Delta m_{31}^2/m_1^2)^2$ and $\Delta m_{21}^2/m_1^2$ have been neglected. The latter term is small because we know from experiment that $\Delta m_{21}^2/|\Delta m_{31}^2| \sim 1/30$ [2, 3]. We observe that the leading correction to $t_{23}^2$ is independent of $s_{13}^2$.

Case B$_4$ is treated analogously. We obtain

\[
B_4: \quad \frac{\mu_1}{m_3} = V^*_{\mu 3} V_{\tau 3} = -\frac{t_{12} + t_{23} \epsilon^*}{t_{12} t_{23} - \epsilon} \frac{1}{t_{23}}, \quad \frac{\mu_2}{m_3} = \frac{V^*_{\mu 2} V_{\tau 3}}{V^*_{\mu 3} V_{\tau 2}} = -\frac{1}{t_{23}} + \frac{t_{12} t_{23} \epsilon^*}{t_{23} + t_{12} \epsilon} \frac{1}{t_{23}}.
\] (17)

Comparison with equation (11) shows that in the present case the cubic equation for $t_{23}^2$ is obtained from equation (14) by the replacement $\rho_1 \rightarrow 1/\rho_1$, $\rho_2 \rightarrow 1/\rho_2$. It is then easy to show that the atmospheric mixing angles in the cases B$_3$ and B$_4$ are related by

\[
t_{23}^2|_{B_4} = \left( t_{23}^2|_{B_3} \right)^{-1} \quad \text{or} \quad s_{23}^2|_{B_4} = 1 - s_{23}^2|_{B_3}.
\] (18)

Accordingly, the curves for B$_4$ in figure 1 are obtained from those of B$_3$ by reflection at the dashed line.

In figures 1 and 2 we have plotted $s_{23}^2$ and $\cos \delta$ versus $m_1$, respectively, for cases B$_3$ and B$_4$. For definiteness, for the solar and reactor mixing angles and the mass-squared differences we have used the best-fit values listed in [3]: $s_{12}^2 = 0.316$, $\Delta m_{21}^2 = 7.64 \times 10^{-5}$ eV$^2$, which are the same values for both normal and inverted spectrum, and $s_{13}^2 = 0.017$, $\Delta m_{31}^2 = 2.45 \times 10^{-3}$ eV$^2$ for the normal and $s_{13}^2 = 0.020$, $\Delta m_{31}^2 = -2.34 \times 10^{-3}$ eV$^2$ for the inverted spectrum. The two figures illustrate nicely that in all four instances (cases B$_3$ and B$_4$ and both spectra) in the limit $m_1 \rightarrow \infty$ we find $s_{23}^2 \rightarrow 1/2$ and $\cos \delta \rightarrow 0$.

Some remarks are at order. First of all, by a numerical comparison it turns out that the approximate formula (16) works quite well. The deviation from the exact value of $s_{23}^2$ is less than 3% at $m_1 = 0.08$ eV and the approximation rapidly improves at larger $m_1$. Secondly, from equations (11) and (17) we read off that cases B$_3$ and B$_4$ do not allow $s_{13} = 0$ because this would lead to $\mu_1 = \mu_2$. However, this observation does not give a strong restriction on $s_{13}$, as we find numerically. Thirdly, the lower bound on

\[\Delta m_{31}^2 = m_3^2 - m_1^2 > 0 \quad \text{whereas} \quad \Delta m_{31}^2 = m_3^2 - m_1^2 \quad \text{can have either sign:} \quad \Delta m_{31}^2 > 0 \quad \text{indicates} \quad \text{the normal order of the neutrino mass spectrum and} \quad \Delta m_{31}^2 < 0 \quad \text{the inverted order.} \]
Figure 1: $s_{23}^2$ as a function of $m_1$. In descending order the full curves refer to case B$_3$ (inverted spectrum), case B$_4$ (normal spectrum), case B$_3$ (normal spectrum), and case B$_4$ (inverted spectrum). The dashed line indicates the value 0.5, i.e., maximal atmospheric mixing. In this plot, for $s_{12}^2$, $s_{13}^2$, $\Delta m_{21}^2$ and $\Delta m_{31}^2$ the best-fit values of [3] have been used.

$s_{13}$ is correlated with a lower bound on $m_1$. The reason is that, in our treatment of cases B$_3$ and B$_4$, $\cos \delta$ is computed via equation (13) after the determination of $s_{23}^2$ by equation (14); then the condition $|\cos \delta| \leq 1$ leads to the lower bound on $m_1$. For the inverted spectrum we obtain numerically that the lower bound $m_1 \gtrsim 0.05 \text{ eV}$ is rather stable for $s_{13}^2 \gtrsim 0.0001$. The normal spectrum allows smaller values of $m_1$, for instance, $m_1 \gtrsim 0.03 \text{ eV}$ at $s_{13}^2 \simeq 0.0001$ and $m_1 \gtrsim 0.01 \text{ eV}$ at $s_{13}^2 \simeq 0.01$. Therefore, cases B$_3$ and B$_4$ do not automatically entail a quasi-degenerate spectrum which would require something like $m_1 > 0.1 \text{ eV}$. In accord with the philosophy of this paper we really have to postulate such a spectrum and only for quasi-degeneracy we obtain an atmospheric mixing angle sufficiently close to $45^\circ$.

Computing an approximation for $\cos \delta$ is a bit laborious. It turns out that, due to the smallness of $s_{13}^2$, it is necessary to expand $\cos \delta$ to second order in both

$$
\Delta_1 = \frac{\Delta m_{31}^2}{m_1^2} \quad \text{and} \quad \Delta_2 = \frac{\Delta m_{21}^2}{m_1^2},
$$

(19)
Figure 2: $\cos \delta$ as a function of $m_1$. For further details see the legend of figure 1.

in order to obtain a reasonable accuracy. The result is

$$\cos \delta \simeq \mp \frac{s_{13} t_{12}}{4} \left\{ \left( 1 - \frac{1}{t_{12}^2} \right) \left( \Delta_1 - \frac{1}{2} \Delta_2^2 \right) + \frac{s_{12}^2 c_{13}^2 - 1}{s_{13}^2} \left( \Delta_2 - \frac{1}{2} \Delta_2^2 \right) \right\}, \quad (20)$$

where the minus and plus signs correspond to $B_3$ and $B_4$, respectively. At $m_1 = 0.16$ eV the approximation (20) deviates from the exact value by less than 1% (5%) assuming a normal (inverted) spectrum. Actually, the sign difference in equation (20) between cases $B_3$ and $B_4$ holds to all orders; with equations (13) and (18) it is easy to show that

$$\cos \delta \big|_{B_4} = - \cos \delta \big|_{B_3}, \quad (21)$$

in perfect agreement with the numerical computation.

The general formula for the effective mass in neutrinoless double-beta decay (for reviews see for instance [23]) is given by the formula

$$m_{\beta\beta} = |(\mathcal{M}_\nu)_{ee}| = m_3 \left| c_{13}^2 \left( c_{12}^2 \frac{\mu_1}{m_3} + s_{12}^2 \frac{\mu_2}{m_3} \right) + (\epsilon^*)^2 \right|. \quad (22)$$

With $\mu_1$ and $\mu_2$ from equations (11) and (17) we specify it to cases $B_3$ and $B_4$, respectively. Inspection of the same equations reveals a simple procedure to switch from $B_3$ to $B_4$:

$$\frac{\mu_j}{m_3} \big|_{B_4} = \left( \frac{m_3}{\mu_j} \big|_{B_3} \right)^* \quad (j = 1, 2). \quad (23)$$

\footnote{Note that the coefficient of $\Delta_1 \Delta_2$ is zero.}
In $\mu_1$ and $\mu_2$ we need to insert the numerical values obtained for $t_{23}$ and $\delta$. Equation (13) determines $\cos \delta$, therefore, $\sin \delta$ is only determined up to a sign. However, since $\sin \delta \leftrightarrow -\sin \delta$ corresponds to $\epsilon \leftrightarrow \epsilon^*$, this sign has no effect on $m_{\beta\beta}$ because this observable is computed by an absolute value.

The effective mass $m_{\beta\beta}$ applied to the cases $B_3$ and $B_4$ has the property that, if we do not care that $\theta_{23}$ and $\cos \delta$ are actually determined by equations (14) and (13) and simply plug in $\theta_{23} = 45^\circ$ and $\cos \delta = 0$, we obtain the equality $m_{\beta\beta} = m_3$. Since for a quasi-degenerate neutrino mass spectrum $m_1 \simeq m_3$ holds, this demonstrates that we should expect

$$m_{\beta\beta} \simeq m_1$$  \hspace{1cm} (24)

in the limit of quasi-degeneracy. Numerically it turns out that the deviation of $m_{\beta\beta}/m_1$ from one is very small—even at $m_1 = 0.05$ eV, for the inverted spectrum, the ratio $m_{\beta\beta}/m_1$ deviates from one by only $-3.2\%$, at $m_1 = 0.1$ eV the deviation is $-1.5$ per mill. For the normal spectrum this ratio is even closer to one. This renders a plot $m_{\beta\beta}$ vs. $m_1$ superfluous. The smallness of $m_{\beta\beta}/m_1 - 1$ is partially explained by the smallness of $s_{13}^2$ which brings the phases of both $\mu_1$ and $\mu_2$ close to $\pi$ \cite{16}. One can check that choosing a large (and thus unphysical) $s_{13}^2$ there is indeed a substantial deviation of $m_{\beta\beta}/m_1$ from one at the lower end of our range of $m_1$.

### 4 The remaining cases

Here we will show that the remaining five cases of two texture zeros in $M_\nu$ are either such that the assumption of a quasi-degenerate spectrum is incompatible with the data or that they do not conform to the philosophy put forward in this paper.

Cases A1 and A2 are incompatible with quasi-degenerate neutrino masses, as was noticed in \cite{15}. This can be seen in the following way. From $(M_\nu)_{ee} = 0$, assuming a quasi-degenerate spectrum and using equation (1) we readily find

$$s_{13}^2 \gtrsim c_{13}^2 (c_{12}^2 - s_{12}^2) = c_{13}^2 \cos(2\theta_{12}),$$  \hspace{1cm} (25)

in contradiction to our experimental knowledge on $\theta_{13}$ and $\theta_{12}$.

Next we consider cases B1 and B2. Taking into account that one knows from experiment that $s_{13}^2$ is small, in first order in $s_{13}$ for $B_1$ one obtains \cite{15}

$$\frac{\mu_1}{m_3} \simeq -\left[\frac{t_{23}^2 + s_{13} (e^{-i\delta} t_{23} + e^{i\delta} / t_{23})}{t_{12}}\right], \quad \frac{\mu_2}{m_3} \simeq -\left[\frac{t_{23}^2 - s_{13} (e^{-i\delta} t_{23} + e^{i\delta} / t_{23})}{t_{12}}\right].$$  \hspace{1cm} (26)

Now we ask the question if the assumption of a quasi-degenerate mass spectrum compellingly leads to $t_{23} \simeq 1$. The answer is “no” because we could choose $s_{13}/(t_{23} t_{12}) \simeq 1$ in order to achieve quasi-degeneracy; even with the experimentally allowed values for $s_{13}$ and $t_{12}$ we would obtain a rather small $t_{23} \simeq s_{13}/t_{12}$, far from maximal atmospheric neutrino mixing. Case B2 can be discussed analogously.

Case C is a bit more involved—for details see \cite{18}. In the case of the inverted spectrum, maximal atmospheric neutrino mixing is not compelling. For the normal ordering of the spectrum, using the experimental knowledge on the mass-squared differences and the
mixing angles $\theta_{12}$ and $\theta_{13}$ it follows that $t_{23}$ is extremely close to one and that the spectrum is quasi-degenerate. However, if we do not use the experimental information on $s^2_{13}$, we could assume $c^2_{13}$ being small instead which would then admit $t_{23}$ being smaller than one. This is in contrast to cases B_3 and B_4 where, for quasi-degeneracy, $t_{23}$ is always close to one, independently of the values $s^2_{12}$ and $s^2_{13}$ assume.

5 Conclusions

In this paper we have considered the possibility that neutrinos differ from charged fermions not only in their Majorana nature but also in a quasi-degenerate mass spectrum, in stark contrast to the hierarchical mass spectra of the charged fermions. The appealing aspect of this assumption is that it is already under scrutiny by present experiments, and more experiments will join in the near future [1]. Such experiments search for neutrinoless double-beta decay, whose decay amplitude is proportional to the effective mass $m_{\beta\beta}$, and for a deviation in the shape of the endpoint spectrum of the $\beta$-decay of $^3$H which is, in essence, sensitive to the average of the squares of the neutrino masses if the spectrum is quasi-degenerate. Moreover, that neutrino mass effects in cosmology have not yet been observed puts already a stringent although model-dependent bound on the sum of the masses.

However, the aspect on which we elaborated in this paper was the possibility to obtain near maximal atmospheric neutrino mixing from a quasi-degenerate neutrino mass spectrum. The idea is quite simple: if we have a model with symmetries enforcing a diagonal charged-lepton mass matrix and the atmospheric neutrino mixing angle being a function of the neutrino mass ratios, then in the limit of quasi-degeneracy this mixing angle will become independent of the masses. We have found two instances in the framework of two texture zeros in the Majorana neutrino mass matrix where in this limit atmospheric neutrino mixing becomes maximal, namely the cases B_3 and B_4 of [15]. We have shown that these two textures have the following properties if the neutrino mass spectrum is quasi-degenerate:

1. Using the mass-squared differences as input, the value of $s^2_{23}$ tends to $1/2$ irrespective of the values of $s^2_{12}$ and $s^2_{13}$; therefore, maximal atmospheric neutrino mixing has to be considered a true prediction of the textures B_3 and B_4 in conjunction with quasi-degeneracy.

2. If $s^2_{13}$ is not exceedingly small, then CP violation in lepton mixing becomes maximal too, i.e., $\cos \delta$ tends to zero.

3. Exact vanishing of $s^2_{13}$ is forbidden because this would entail $\Delta m^2_{21} = 0$, however, values as small as $s^2_{13} = 0.0001$ are nevertheless possible.

The results for the cases B_3 and B_4 can be understood in the following way. With the usual phase convention (i) for the mixing matrix, in equation (10) we have $e^{i\phi} = 1$ and, therefore, $V_{\mu j} \simeq -V^*_{r j}$ for $j = 1, 2$ and $V_{\mu 3} \simeq V^*_{\tau 3}$ for a quasi-degenerate spectrum. The signs we obtained here are convention-dependent and have no physical significance. That the exact relation $V_{\mu j} = V^*_{\tau j}$ for $j = 1, 2, 3$ is a viable and predictive restriction of $V$ was
already pointed out in [24], later on in [25] a model was constructed where this relation is enforced by a generalized CP transformation and softly broken lepton numbers, and it was also shown that such a mixing matrix leads to $\theta_{23} = 45^\circ$ and $s_{13}\cos\delta = 0$ at the tree level. While in [25] the symmetry structure is non-abelian and a type of $\mu-\tau$ interchange symmetry (see [26] for some early references, and [27] for a recent paper and references therein), the textures $B_3$ and $B_4$ can be enforced by abelian symmetries and, provided the neutrino mass spectrum is quasi-degenerate, we have the approximate relations $\theta_{23} \simeq 45^\circ$ and $\cos\delta \simeq 0$. Therefore, we have shown that maximal atmospheric neutrino mixing could have an origin completely different from $\mu-\tau$ interchange symmetry, it could simply be a consequence of texture zeros and quasi-degeneracy of the neutrino mass spectrum.

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