Critical metrics of the volume functional on three-dimensional manifolds

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Abstract
In this paper, we prove the three-dimensional CPE conjecture with nonnegative Ricci curvature. Moreover, we establish rigidity theorems for three-dimensional compact, oriented, connected V-static metrics with nonnegative Ricci curvature. Finally, we obtain classification results on three-dimensional vacuum static space and Miao–Tam critical metric with nonnegative Ricci curvature.

KEYWORDS
CPE metric, Miao–Tam critical metric, V-static metric, vacuum static space

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1 | INTRODUCTION

Let \((M^n, g)\) be a compact, oriented, connected Riemannian manifold with dimension \(n\) at least three, \(\mathcal{M}\) be the set of Riemannian metrics on \(M^n\) of unitary volume, and \(C \subset \mathcal{M}\) be the set of Riemannian metrics with constant scalar curvature. Define the total scalar curvature functional \(\mathcal{R} : \mathcal{M} \to \mathbb{R}\) as

\[
\mathcal{R}(g) = \int_{M^n} R_g \, dM_g,
\]

where \(R_g\) is the scalar curvature on \(M^n\). It is well known that the formal \(L^2\)-adjoint of the linearization of the scalar curvature operator \(\mathcal{Q}_g\) at \(g\) is defined as

\[
\mathcal{Q}_g^*(f) := -(\Delta_g f) g + \text{Hess}_g f - f \text{Ric}_g,
\]

where \(f\) is a smooth function on \(M^n\), and \(\Delta_g, \text{Hess}_g,\) and \(\text{Ric}_g\) stand for the Laplacian, the Hessian form, and the Ricci curvature tensor on \(M^n\), respectively. An Euler–Lagrange equation is given by

\[
\mathcal{Q}_g^*(f) = \text{Ric}_g - \frac{R_g}{n} g.
\] (1.1)

As the terminology used in [8, 13, 33], a CPE metric can be defined as follows.

**Definition 1.1.** A CPE metric is a three-tuple \((M^n, g, f)(n \geq 3)\), where \((M^n, g)\) is a compact (without boundary), oriented Riemannian manifold with constant scalar curvature and \(f : M^n \to \mathbb{R}\) is a nonconstant smooth function satisfying Equation (1.1). Such a function \(f\) is called a potential function.
Besse [8] first posed the following conjecture on CPE metrics:

**CPE Conjecture.** A CPE metric is always Einstein.

In the last decades, many researchers tried to solve this conjecture, but only partial results were proved. In 1983, Lafontaine [26] showed the CPE conjecture under a locally conformally flat assumption and $\text{Ker} \mathcal{L}_g(f) \neq 0$. Chang, Hwang, and Yun [13] avoided the condition on $\text{Ker} \mathcal{L}_g(f)$. In 2000, Hwang [22] obtained the conjecture provided $f \geq -1$. In 2013, Qing and Yuan [33] obtained a positive answer for Bach-flat CPE metrics manifolds. For more references on CPE metrics, see [12, 14, 17, 28] and references therein.

When the dimension $n = 3$, Baltazar [2] proved that the CPE conjecture is true for three-dimensional manifolds with nonnegative sectional curvature. In this paper, we are able to prove that the CPE conjecture is true for three-dimensional manifolds with nonnegative Ricci curvature.

One of our main results is the following:

**Theorem 1.2.** Let $(M^3, g, f)$ be a three-dimensional compact, oriented, connected CPE metric with nonnegative Ricci curvature. Then, $M^3$ is isometric to a standard sphere $S^3$.

In this paper, we also pay attention to another kind of critical point equation. As in [4, 16, 23], we say that $g$ is a $V$-static metric if there is a smooth function $f$ on $M^n$ and a constant $\nu$ satisfying the $V$-static equation

$$\mathcal{L}_g^\nu(f) = \nu g. \quad (1.2)$$

On the one hand, when $\nu = 0$ in (1.5), as in [19, 25], we can have the definition of vacuum static space.

**Definition 1.3.** A vacuum static space is a three-tuple $(M^n, g, f)(n \geq 3)$, where $(M^n, g)$ is a compact, oriented, connected Riemannian manifold with a smooth boundary $\partial M$ and $f : M^n \rightarrow \mathbb{R}$ is a smooth function such that $f^{-1}(0) = \partial M$ satisfying the overdetermined-elliptic system

$$\mathcal{L}_g^\nu(f) = 0. \quad (1.3)$$

Such a function $f$ is called a potential function.

Kobayashi and Obata [25] (for dimension $n = 3$, see [29]) proved that a static metric $g$ is isometric to a warped-product metric of constant scalar curvature near the hypersurface $f^{-1}(c)$ for a regular value $c$ provided $g$ is locally conformally flat. In the paper [18], Fischer and Marsden showed the local surjectivity for the scalar curvature as a map from the space of metrics to the space of functions at a nonstatic metric on a closed manifold and raised the possibility of identifying all compact vacuum static spaces.

Obata [32] showed that the only compact Einstein manifold, which satisfies (1.3), is the standard sphere. Locally conformally flat vacuum static spaces have been completely classified in [24] and [26] independently. When dimension $n = 3$, as Ambrozio state in [1], the classification of locally conformally flat vacuum static spaces is as follows.

**Theorem A** (Kobayashi [24], Lafontaine [26]). Let $(M^3, g, f)$ be a vacuum static space with positive scalar curvature. If $(M^3, g)$ is locally conformally flat, then $(M^3, g, f)$ is covered by a static space that is equivalent to one of the following tuples:

(i) The standard round hemisphere

$$S_+^3, g_{\text{can}}, f = x_4,$$

where $g_{\text{can}}$ is the metric of standard unit round sphere.

(ii) The standard cylinder over $S^2$ with the product metric

$$\left( \left[ 0, \frac{\pi}{\sqrt{3}} \right] \times S^2, g_{\text{prod}} = dt^2 + \frac{1}{3} g_{\text{can}} : f = \frac{1}{\sqrt{3}} \sin(\sqrt{3}t) \right).$$
iii) For some $m \in \left( 0, \frac{1}{3\sqrt{3}} \right)$, the tuple

$$\left( [r_h(m), r_c(m)] \times S^2, g_m = \frac{dr^2}{1 - r^2 - \frac{2m}{r}} + r^2g_{\text{can}}, f_m = \sqrt{1 - r^2 - \frac{2m}{r}} \right)$$

where $r_h(m) < r_c(m)$ are the positive zeroes of $f_m$.

Based on a similar idea from [9, 10], Qing and Yuan [33] obtained a classification result for Bach-flat vacuum static spaces in any dimension. Ambrozio [1] got some classification results for compact simply connected static three-manifolds with positive scalar curvature. Baltazar-Ribeiro [4] gave a classification theorem for compact static three-manifolds with nonnegative sectional curvature. For more information, see [21, 27, 34, 36] and references therein.

On the other hand, when $\kappa \neq 0$ in (1.5), in [30], Miao and Tam have studied the volume functional on the space of constant scalar curvature metrics with a prescribed boundary metric, and derived a sufficient or necessary condition for a metric to be a critical point. As in [23, 31], we can define the Miao–Tam critical metric as follows.

**Definition 1.4.** A Miao–Tam critical metric is a three-tuple $(M^n, g, f)(n \geq 3)$, where $(M^n, g)$ is a compact, oriented, connected Riemannian manifold with a smooth boundary $\partial M$ and $f : M^n \to \mathbb{R}$ is a smooth function such that $f^{-1}(0) = \partial M$ satisfying the overdetermined-elliptic system

$$\mathcal{L}^r g(f) = g. \quad (1.4)$$

Such a function $f$ is called a potential function.

In [31], Miao–Tam studied these critical metrics under Einstein condition. More precisely, they obtained the following result.

**Theorem B** (Miao–Tam [31]). Let $(M^n, g, f)$ be a connected, compact Einstein Miao–Tam critical metric with smooth boundary $\partial M$, then $M^n$ is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{H}^n$, or $\mathbb{S}^n$.

In the same article, they obtained the same result under the assumption that the metric is locally conformally flat instead of Einstein. Based on a work of Cao–Chen [11], Barros–Diogenes–Ribeiro [6] showed that a four-dimensional Bach-flat simply connected, compact Miao–Tam critical metric with boundary isometric to a standard sphere $\mathbb{S}^3$ must be isometric to a geodesic ball in a simply connected space form $\mathbb{R}^4$, $\mathbb{H}^4$, or $\mathbb{S}^4$.

It is important to note that Baltazar–Ribeiro [4] provided a general Bochner-type formula to show that a three-dimensional compact, oriented, connected Miao–Tam critical metric $(M^3, g, f)$ with smooth boundary $\partial M$ and non–negative sectional curvature, with $f$ assumed to be nonnegative, is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^3$ or $\mathbb{S}^3$. We refer to [3, 5, 7, 35] for more related results.

In this paper, we are able to prove the following theorem on three-dimensional $V$-static space with nonnegative Ricci curvature.

**Theorem 1.5.** Let $(M^3, g, f)$ be a three-dimensional compact, oriented, connected $V$-static metric with a smooth boundary $\partial M$ and a nonnegative Ricci curvature. If $f$ and $\kappa$ satisfy one of the following conditions:

(i) $\kappa \geq 0$ and $f > 0$,  
(ii) $\kappa \leq 0$ and $f < 0$,

then $M^3$ is locally conformally flat.

Specifically, when $\kappa = 0$, we can get the following corollary.
Corollary 1.6. Let \((M^3, g, f)\) be a three-dimensional compact, oriented, connected vacuum static space with smooth boundary \(\partial M\), nonnegative Ricci curvature, and positive \(f\). Then, \((M^3, g, f)\) is covered by a static space that is equivalent to one of the following tuples:

(i) The standard round hemisphere

\[
(S^3_+, g_{\text{can}}, f = x_4),
\]

where \(g_{\text{can}}\) is the metric of standard unit round sphere.

(ii) The standard cylinder over \(S^2\) with the product metric

\[
\left(\left[0, \frac{\pi}{\sqrt{3}}\right] \times S^2, g_{\text{prod}} = dt^2 + \frac{1}{3}g_{\text{can}}, f = \frac{1}{\sqrt{3}} \sin(\sqrt{3}t)\right).
\]

Moreover, when \(\kappa > 0\), we are able to establish the following result under the condition of \(f \geq 0\).

Theorem 1.7. Let \((M^3, g, f)\) be a three-dimensional compact, oriented, connected \(V\)-static metric with smooth boundary \(\partial M\) and nonnegative Ricci curvature. If \(\kappa > 0\) and \(f \geq 0\), \(M^3\) is an Einstein manifold.

As a direct corollary, we can prove the following result.

Corollary 1.8. Let \((M^3, g, f)\) be a three-dimensional compact, oriented, connected Miao–Tam critical metric with smooth boundary \(\partial M\), nonnegative Ricci curvature, and nonnegative \(f\). Then, \(M^3\) is isometric to a geodesic ball in a simply connected space form \(\mathbb{R}^3\) or \(S^3\).

Remark 1.9. After we finished the first version of this paper in January 2021 and posted on the Arxiv (2101.05621v1), 2 months later, Allan Freitas contacted us to claim that they proved that a three-dimensional compact Miao–Tam critical metric with smooth boundary \(\partial M\), positive scalar curvature, nonnegative Ricci curvature, and nonnegative \(f\) is isometric to a geodesic ball in \(S^3\) by using a distinct technique.

The organization of the paper is as follows. In Section 2, we unify the form of equation of CPE metric and \(V\)-static metric. Moreover, we state four lemmas as the preparation. In Section 3, we give the proof of our main theorems and corollaries.

2 | PRELIMINARIES

Let \((M^n, g)(n \geq 3)\) be an \(n\)-dimensional compact, orientable Riemannian manifold. In what follows, we adopt, without further comment, the moving frame notation. For any \(p \in M^n\), we choose \(e_1, \ldots, e_n\) as a local orthonormal frame field at \(p\), \(\omega_1, \ldots, \omega_n\) as its dual coframe field, \(g_{ij} = \delta_{ij}\). Here and hereafter, the Einstein convention of summing over the repeated indices will be adopted.

The decomposition of the Riemannian curvature tensor into irreducible components yields

\[
R_{ijkl} = W_{ijkl} + \frac{1}{n-2}(R_{ik} \delta_{jl} - R_{il} \delta_{jk} + R_{jl} \delta_{ik} - R_{jk} \delta_{il}) - \frac{R}{(n-1)(n-2)}(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}),
\]

where \(R_{ijkl}\) are the components of the Riemannian curvature tensor, \(W_{ijkl}\) are the components of the Weyl tensor, \(R_{ij} := \Sigma_{k,l} R_{ikjl} g_{kl}\) are components of Ricci curvature tensor, and \(R := \Sigma_{ij} R_{ij} g_{ij}\) is the scalar curvature of \(M^n\).
Let $\phi = \sum_{i,j} \phi_{i,j} \omega_i \otimes \omega_j$ be a symmetric $(0,2)$-type tensor defined on $M^n$. By letting $\phi_{i,j,k} := \nabla_k \phi_{i,j}$, $\phi_{i,j,k,l} := \nabla_l \nabla_k \phi_{i,j}$, where $\nabla$ is the operator of covariant differentiation on $M^n$, we have the following Ricci identities:

$$\phi_{i,j,k,l} - \phi_{i,j,l,k} = \phi_{m,j} R_{m,i,k,l} + \phi_{i,m} R_{m,j,k,l}. \quad (2.2)$$

The norm of a $(0,4)$-type tensor $T$ is defined as

$$|T|^2 = |T_{ijkl}|^2 = T_{ijkl} T_{ijkl}.$$ 

By the second Bianchi identity

$$R_{lmkl,j} + R_{mlkj} + R_{lkmj} = 0,$$

we have

$$R_{i,j,k} - R_{i,k,j} = R_{l,i,k,j,l} \quad (2.3)$$

and

$$R_{i,k,l} = \frac{1}{2} R_{k} \quad (2.4)$$

A Cotton tensor $C_{i,j,k}$ is given by

$$C_{i,j,k} = R_{j,k,i} - R_{i,k,j} - \frac{1}{2(n-1)} (R_{i} \delta_{j,k} - R_{j} \delta_{i,k}). \quad (2.5)$$

From Definition 1.1, we know that a CPE metric $(M^n, g, \tilde{f})$ satisfies

$$-(\Delta \tilde{f}) \delta_{i,j} + \tilde{f}_{i,j} - \tilde{f} R_{i,j} = Ric - \frac{R}{n} \delta_{i,j}, \quad (2.6)$$

where $R$ is a constant. Replacing $\tilde{f}$ by $f - 1$, we can rewrite (2.6) as

$$-(\Delta f) \delta_{i,j} + f_{i,j} - f R_{i,j} = \frac{R}{n} \delta_{i,j}, \quad (2.7)$$

where $\frac{R}{n}$ is a constant. Obviously, (2.7) is a $V$-static equation. Then, we can unify the equation of CPE metric and $V$-static metric into the following form:

$$-(\Delta f) \delta_{i,j} + f_{i,j} - f R_{i,j} = \kappa \delta_{i,j}, \quad (2.8)$$

where $\kappa$ is a constant. By tracing the above formula, we can get

$$\Delta f = \frac{f R + n \kappa}{1-n}. \quad (2.9)$$

Substituting (2.9) into (2.8), we have

$$f R_{i,j} - f_{i,j} - \frac{f R}{n-1} \delta_{i,j} = \frac{\kappa}{n-1} \delta_{i,j}. \quad (2.10)$$

Now, we present four lemmas as the preparation for proving our main theorems.
Lemma 2.1. Let \((M^n, g)(n \geq 3)\) be an \(n\)-dimensional Riemannian manifold and \(f\) be a smooth solution to (2.8). Then, \(g\) has constant scalar curvature.

Proof. As the proof of Theorem 7 in Miao-Tam [30], differentiating (2.8), we get

\[-f_{,k} f_{,k} \delta_{ij} + f_{,ijj} - f_{,j} R_{ij} - f R_{ij,j} = 0.\]  

(2.11)

From Ricci identity and (2.4), we can derive that

\[0 = -f_{,kk} + f_{,j} R_{ij} - \frac{1}{2} f R_j = -\frac{1}{2} f R_j.\]

Thus, scalar curvature \(R\) is constant. Hence, we complete the proof of Lemma 2.1.

Lemma 2.2. Let \((M^n, g)(n \geq 3)\) be an \(n\)-dimensional Riemannian manifold and \(f\) be a smooth solution to (2.8). Then,

\[f C_{kij} = f_{,h} R_{hjik} + R \frac{n-1}{n-2} (f_{,k} \delta_{ij} - f_{,i} \delta_{kj}) - (f_{,k} R_{ij} - f_{,i} R_{kj}).\]  

(2.12)

Remark 2.3. As in [6] and [3], we can derive

\[f C_{kij} = T_{kij} + W_{kij} f_s\]  

(2.13)

from (2.1) and Lemma 2.2, where \(T_{kij}\) is defined as

\[T_{kij} = \frac{n-1}{n-2} (R_{jk} f_{,i} - R_{ij} f_{,k}) - R \frac{n-2}{n-1} (\delta_{jk} f_{,i} - \delta_{ij} f_{,k}) + \frac{1}{n-2} (\delta_{jk} R_{is} f_{,s} - \delta_{ij} R_{ks} f_{,s}).\]

Proof. Differentiating (2.10), we get

\[f R_{ij,k} = f_{,ijk} + f_{,ij} R_{kj} - f_{,k} R_{ij}.\]  

(2.14)

Combining this equation with Ricci identity, we have

\[f C_{kij} = f (R_{ij,k} - R_{kij}) = f_{,ijk} - f_{,kji} + \frac{R}{n-1} (f_{,k} \delta_{ij} - f_{,i} \delta_{kj}) - (f_{,k} R_{ij} - f_{,i} R_{kj}).\]

(2.15)

Hence, we complete the proof of Lemma 2.2.

Lemma 2.4. Let \((M^n, g)(n \geq 3)\) be an \(n\)-dimensional Riemannian manifold and \(f\) be a smooth solution to (2.8). Then,

\[f R_{ij} C_{kij,k} = f R_{ij} R_{jk} R_{kij} - f R_{ij} R_{ns} R_{kijh} - \frac{1}{2} \langle V f, V |Ric| \rangle^2 \]

\[-2R_{ij} f_{,k} C_{kij} + \frac{n \alpha}{n-1} |Ric|^2.\]  

(2.15)
Proof. According to Lemma 2.2, we have
\[
fr_{ij}k = r_{ij}(f_{kij})_k - r_{ij}f_{jk}c_{kij}
\]
\[
= r_{ij}
\left[
 f_{j}r_{hjk} + \frac{r}{n-1}(f_{j} \delta_{ij} - f_{i} \delta_{kj}) - (f_{j} r_{ij} - f_{j} r_{kj})
\right]_k
- r_{ij}f_{jk}c_{kij}
\]
\[
= r_{ij}f_{jk}r_{hjk} + r_{ij}f_{jk}r_{hjk,k} + \frac{r^2}{n-1} \Delta f - \frac{r}{n-1} f_{ij} r_{ij}
- \Delta f |Ric|^2 - r_{ij}f_{jk}r_{ij,k} + r_{ij}f_{jk}r_{kj} - r_{ij}f_{jk}c_{kij}.
\]

From (2.3), (2.5), and the fact R is a constant, we know that \(R_{hjk,k} = C_{jhi}\). Substituting this equation, (2.9), and (2.10) into the equation above, we have
\[
fr_{ij}k = r_{ij}(fr_{hk} - f_{i} \frac{r}{n-1} \delta_{hk} - \frac{r}{n-1} \delta_{i} f_{hk})r_{hjk} + r_{ij}f_{hk}c_{jhi}
+ \frac{r^2}{n-1} f + \frac{r}{n-1} (f_{ij})_k
- \frac{r}{n-1} \delta_{ij})r_{ij}
- \frac{r}{n-1} |Ric|^2 - r_{ij}f_{jk}r_{ij,k} + r_{ij}f_{jk}r_{ij} - \frac{r}{n-1} f_{ik} c_{kij}.
\]

That is,
\[
fr_{ij}k = fr_{ij}r_{jk}r_{ki} - fr_{ij}r_{hk}r_{ik} - \frac{1}{2} (\nabla f, \nabla |Ric|^2)
- 2r_{ij}f_{jk}c_{kij} + \frac{n r}{n-1} |Ric|^2 - \frac{r^2}{n-1}.
\]

Thus, we complete the proof of Lemma 2.4. \(\Box\)

Lemma 2.5. Let \((M^3, g)\) be a three-dimensional Riemannian manifold with nonnegative Ricci curvature. Then,
\[
6r_{ij}r_{hj}r_{hi} - 5r |Ric|^2 + r^3 \geq 0.
\]

Proof. For any fixed point \(p \in M\), we choose an orthonormal basis \(\{e_i\}_{i=1}^3\) in \(T_p M\) such that
\[
r_{ij} = \rho \delta_{ij}, \quad r = \rho_1 + \rho_2 + \rho_3.
\]
Without loss of generality, we can assume \(0 \leq \rho_1 \leq \rho_2 \leq \rho_3\).

Similar to the proof of Lemma 2.2 in [20], we can compute that
\[
6r_{ij}r_{hj}r_{hi} - 5r |Ric|^2 + r^3 = 6 \sum_{i=1}^3 \rho_i^3 - 5 \sum_{i=1}^3 \rho_i \sum_{j=1}^3 \rho_j^2 + (\sum_{i=1}^3 \rho_i)^3
\]
\[
= 9 \sum_{i=1}^3 \rho_i^3 - 6 \sum_{i=1}^3 \rho_i \sum_{j=1}^3 \rho_j^2 + (\sum_{i=1}^3 \rho_i)^3
- 3 \sum_{i=1}^3 \left(3 \rho_i^3 - \rho_i \sum_{j=1}^3 \rho_j^2\right)
\]
\[
\begin{align*}
\sum_{i,j,k=1}^{3} \rho_i(\rho_i - \rho_j)(\rho_i - \rho_k) - \sum_{i,j=1}^{3} \rho_i(\rho_i - \rho_j)^2
\end{align*}
\]

\[
= \sum_{i,j,k=1}^{3} \rho_i(\rho_i - \rho_j)(\rho_i - \rho_k)
\]

\[
= \sum_{i,j,k} \rho_i(\rho_i - \rho_j)(\rho_i - \rho_k)
\]

\[
= 2[\rho_1(\rho_1 - \rho_2)(\rho_1 - \rho_3) + \rho_2(\rho_2 - \rho_1)(\rho_2 - \rho_3) + \rho_3(\rho_3 - \rho_1)(\rho_3 - \rho_2)]
\]

\[
= 2[\rho_1(\rho_1 - \rho_2)(\rho_1 - \rho_3) + \rho_2(\rho_2 - \rho_1)(\rho_2 - \rho_3) + \rho_3(\rho_3 - \rho_1)(\rho_3 - \rho_2)]
\]

\[
= 0,
\]

where \(i, j, k \neq \) means that \(i, j, k\) are distinct from each other. Thus, we complete the proof of Lemma 2.5.

3 PROOF OF MAIN THEOREMS

In this section, we prove the main theorems of our paper. Let \((M^n, g)(n \geq 3)\) be an \(n\)-dimensional compact, oriented, connected Riemannian manifold and \(f\) be a smooth solution to (2.8). From a direct computation and the definition of \(C_{ijk}\), we get

\[
\text{div}
\left(f^2 \nabla |\text{Ric}|^2\right) = 2f \langle \nabla f, \nabla |\text{Ric}|^2 \rangle + f^2 \Delta |\text{Ric}|^2
\]

\[
= 2f \langle \nabla f, \nabla |\text{Ric}|^2 \rangle + 2f^2 |\nabla \text{Ric}|^2 + 2f^2 R_{ij} R_{jk,kk}
\]

\[
= 2f \langle \nabla f, \nabla |\text{Ric}|^2 \rangle + 2f^2 |\nabla \text{Ric}|^2 + 2f^2 R_{ij} (C_{klj} + R_{kj,i})_k
\]

\[
= 2f \langle \nabla f, \nabla |\text{Ric}|^2 \rangle + 2f^2 |\nabla \text{Ric}|^2 + 2f^2 R_{ij} C_{klj,k}
\]

\[
+ 2f^2 R_{ij} R_{jk,ik}.
\]

Combining equation above with Lemma 2.4 and Ricci identity, we have

\[
\text{div}
\left(f^2 \nabla |\text{Ric}|^2\right) = 4f^2 R_{ij} R_{jk} R_{kl} - 4f^2 R_{ij} R_{nk} R_{jk,ih} - 8f R_{ij} f \delta_{ij} C_{klj}
\]

\[
+ \frac{4nf}{n-1} |\text{Ric}|^2 - 4f^2 R_{ij} C_{kij,k} + 2f^2 |\nabla \text{Ric}|^2
\]

\[
+ 2f^2 R_{ij} C_{kij,k} + 2f^2 R_{ij} R_{jk,ik}
\]

\[
= 6f^2 R_{ij} R_{jk} R_{kl} - 6f^2 R_{ij} R_{nk} R_{jk,ih} - 8f R_{ij} f \delta_{ij} C_{klj}
\]

\[
+ \frac{4nf}{n-1} |\text{Ric}|^2 - 2f^2 R_{ij} C_{kij,k} + 2f^2 |\nabla \text{Ric}|^2 + 2f^2 R_{ij} R_{jk,ik}.
\]

From

\[
(f^2 R_{ij} C_{kij})_k = 2f \delta_{jk} R_{ij} C_{kij} + f^2 R_{ij} C_{kij} + f^2 R_{ij} C_{kij,k}
\]
and the fact that $R$ is constant, it follows that
\[
div\left(f^2 \nabla |\text{Ric}|^2\right) = 6f^2 R_{ij} R_{jk} R_{ki} - 6f^2 R_{ij} R_{hk} R_{ik, jh} - 4f R_{ij} f_k C_{kij} + \frac{4n \pi f}{n - 1} |\text{Ric}|^2 + 2f^2 |\nabla \text{Ric}|^2
\]
\[
= 6f^2 R_{ij} R_{jk} R_{ki} - 6f^2 R_{ij} R_{hk} R_{ik, jh} - 4f R_{ij} f_k C_{kij} + \frac{4n \pi f}{n - 1} |\text{Ric}|^2 + 2f^2 |\nabla \text{Ric}|^2 + 2f^2 |\text{Ricc}||^2 - 2(f^2 R_{ij} C_{kij})_k + 2f^2 |\text{Ricc}||^2.
\]

According to (2.13) and the fact that $W_{ijkl} = 0$ when $n = 3$, we have
\[
f C_{kij} = T_{kij},
\]
then from the definition of $T_{kij}$, we can get
\[
f |C|^2 = T_{kij} C_{kij} = -4C_{kij} R_{ij} f, k.
\] (3.2)

Besides, from (2.1), we know that
\[
R_{ij} R_{jk} R_{ki} - R_{ij} R_{hk} R_{ik, jh} = 3R_{ij} R_{jk} R_{ki} - \frac{5R}{2} |\text{Ric}|^2 + \frac{R^3}{2}.
\] (3.3)

Thus, we can conclude that when $n = 3$,
\[
div\left(f^2 \nabla |\text{Ric}|^2\right) = 3f^2 (6R_{ij} R_{jk} R_{ki} - 5R |\text{Ric}|^2 + R^3) + 6 \pi f |\nabla \text{Ric}|^2 + 2f^2 |C|^2
\]
\[
-2(f^2 R_{ij} C_{kij})_k + 2f^2 |\nabla \text{Ric}|^2.
\] (3.4)

To prove Theorem 1.2, we need the following results by Hwang [22] and Cheng [15].

**Proposition 3.1** [22]. Let $(M^n, g, f)$ be a CPE metric with $f$ nonconstant. Then, the set $\{x \in M^n : f(x) = -1\}$ has measure zero.

**Proposition 3.2** [15]. Let $M^n$ be a compact connected oriented locally conformally flat $n$-dimensional Riemannian manifold with constant scalar curvature. If the Ricci curvature of $M^n$ is nonnegative, then $M^n$ is isometric to a space form or a Riemannian product $\mathbb{S}^{n-1}(c) \times \mathbb{S}^1$, where $c$ is the sectional curvature of $\mathbb{S}^{n-1}$.

**Proof of Theorem 1.2.** Let $(M^3, g, \tilde{f})$ be a three-dimensional compact, oriented, connected CPE metric with nonnegative Ricci curvature. Then, $\tilde{f}$ is a nonconstant solution to (2.6). Hence, $f = 1 + \tilde{f}$ satisfies (2.8) with $\gamma = -\frac{R}{3}$. From Lemma 2.5 and (3.4), we have
\[
div\left((1 + \tilde{f})^2 \nabla |\text{Ric}|^2\right) \geq -2R(1 + \tilde{f}) |\nabla \text{Ric}|^2 + 2(1 + \tilde{f})^2 |C|^2
\]
\[
-2((1 + \tilde{f})^2 R_{ij} C_{kij})_k + 2(1 + \tilde{f})^2 |\nabla \text{Ric}|^2.
\]

According to a direct computation on (2.6), we can see that $(1 + \tilde{f}) |\text{Ric}|^2 = \tilde{f}_{ij} \hat{R}_{ij}$, and from (2.4) and the fact that $M^3$ has no boundary, it is easy to know that $\int_{M^3} (1 + \tilde{f}) |\text{Ric}|^2 dM_g = 0$. Integrating the inequality above, we can derive that
\[
0 \geq \int_{M^3} 2(1 + \tilde{f})^2 |C|^2 dM_g + 2 \int_{M^3} (1 + \tilde{f})^2 |\nabla \text{Ric}|^2 dM_g.
\]
From Proposition 3.1, we can see that $M^3$ is a locally conformally flat manifold and $\nabla \text{Ric} = 0$. 

Taking trace of (2.6), we have

$$\Delta \tilde{f} = -\frac{R}{2} \tilde{f}. \quad (3.5)$$

Multiplying both sides of this equation by $\tilde{f}$ and integrating on $M^3$, we get

$$-\int_{M^3} |\nabla f|^2 dM_g = \int_{M^3} (\Delta \tilde{f}) \cdot f dM_g = -\frac{R}{2} \int_{M^3} f^2 dM_g.$$

According to the fact that $\tilde{f}$ is not a constant, we can conclude that the scalar curvature $R$ is a positive constant.

By using Proposition 3.2, we can conclude that $M^3$ is isometric to a standard sphere $S^3$ or a Riemannian product $S^2(c) \times S^1$, where $c$ is the sectional curvature of $S^2$.

Here, if $M^3$ is isometric to $S^2(c) \times S^1$, we have

$$(R_{ij})_{3 \times 3} = \begin{pmatrix} c & c & 0 \\ c & c & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3 \times 3},$$

and $R = 2c$. Substituting (3.5) into (2.6), we can deduce that

$$\tilde{f}(c \delta_{ij} - R_{ij}) + \tilde{f}_{,ij} = R_{ij} - \frac{2c}{3} \delta_{ij}.$$

Then, $\tilde{f}_{,11} = \frac{c}{3}, \tilde{f}_{,22} = \frac{c}{3}, \tilde{f}_{,33} = -c \tilde{f} - \frac{2c}{3}$. Thus, for any $q \in S^1$, we can get that $\int_{S^2 \times \{q\}} \Delta S^2 \times \{q\} \tilde{f} d\text{vol} = \int_{S^2 \times \{q\}} \frac{2c}{3} d\text{vol} \neq 0$, which is a contradiction.

Thus, we finish the proof of Theorem 1.2. \qed

Remark 3.3. In the process of proving Theorem 1.2, after getting the result that $M^3$ is a locally conformally flat manifold, we can also deduce that $M^3$ is isometric to a standard sphere $S^3$ by using Corollary 1.3 in [13].

Similarly, we can prove that a three-dimensional $V$-static metric is locally conformally flat under some suitable conditions.

Proof of Theorem 1.5. Let $(M^3, g, f)$ be a three-dimensional compact, oriented, connected $V$-static metric with a smooth boundary $\partial M$ and nonnegative Ricci curvature. Integrating (3.4), we can deduce

$$0 \geq 6\kappa \int_{M^3} f |\nabla \text{Ric}|^2 dM_g + 2 \int_{M^3} f^2 |C|^2 dM_g + 2 \int_{M^3} f^2 |\nabla \text{Ric}|^2 dM_g \quad (3.6)$$

from Lemma 2.5. Then, if $\kappa \geq 0$, $f > 0$, or $\kappa \leq 0$, $f < 0$, $M^3$ is locally conformally flat. Thus, we complete the proof of Theorem 1.5. \qed

When $\kappa = 0$, since for case (iii) in Theorem A, Ricci curvature is always strictly negative, Corollary 1.6 can be directly deduced from Theorem 1.5 and Theorem A.

When $\kappa > 1$, from (3.1), we can prove Theorem 1.7 on the basis of the following calculation on $n$-dimensional compact, oriented, connected $V$-static metric:

$$\text{div}(f \nabla |\text{Ric}|^2) = \frac{1}{f} \text{div}(f^2 \nabla |\text{Ric}|^2) - \langle \nabla f, \nabla |\text{Ric}|^2 \rangle$$

$$= 6f R_{ij} R_{jk} R_{kl} - 6f R_{ij} R_{jk} R_{kh} - 8R_{ij} f_{,k} C_{kij}$$

$$+ \frac{4n\kappa}{n-1} |\text{Ric}|^2 - 2f R_{ij} C_{kij,k} + 2f |\nabla \text{Ric}|^2 - \langle \nabla f, \nabla |\text{Ric}|^2 \rangle.$$
Using Lemma 2.4 again, we have
\[
\text{div}\left(f \nabla |\mathbf{Ric}|^2\right) = 4f R_{ij} R_{kj} - 4f R_{ik} R_{jk} f_{ij} \\
- 4f R_{ij} f_{k} C_{kij} + \frac{2n \kappa}{n-1} |\mathbf{Ric}|^2 + 2f |\nabla \mathbf{Ric}|^2.
\] (3.7)

**Proof of Theorem 1.7.** Let \((M^3, g, f)\) be a three-dimensional compact, oriented, connected \(V\)-static metric with smooth boundary \(\partial M\), nonnegative Ricci curvature, and nonnegative \(f\). As the proof of Theorem 1.5, integrating (3.7), we can derive that
\[
0 \geq \int_{M^3} f(2 |\nabla \mathbf{Ric}|^2 + |C|^2) dM_g + \int_{M^3} 3 \kappa |\mathbf{Ric}|^2 dM_g
\]
from (3.2), (3.3), and Lemma 2.5. Hence, \(M^3\) is an Einstein manifold when \(\kappa > 0\). Thus, we can complete the proof of Theorem 1.7. \(\square\)

When \(\kappa = 1\), from Theorem 1.7 and Theorem B, we can complete the proof of Corollary 1.8.

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