Decomposition Techniques for Bilinear Saddle Point Problems and Variational Inequalities with Affine Monotone Operators on Domains Given by Linear Minimization Oracles

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Abstract

The majority of First Order methods for large-scale convex-concave saddle point problems and variational inequalities with monotone operators are proximal algorithms which at every iteration need to minimize over problem’s domain $X$ the sum of a linear form and a strongly convex function. To make such an algorithm practical, $X$ should be proximal-friendly – admit a strongly convex function with easy to minimize linear perturbations. As a byproduct, $X$ admits a computationally cheap Linear Minimization Oracle (LMO) capable to minimize over $X$ linear forms. There are, however, important situations where a cheap LMO indeed is available, but $X$ is not proximal-friendly, which motivates search for algorithms based solely on LMO. For smooth convex minimization, there exists a classical LMO-based algorithm – Conditional Gradient. In contrast, known to us LMO-based techniques for other problems with convex structure (nonsmooth convex minimization, convex-concave saddle point problems, even as simple as bilinear ones, and variational inequalities with monotone operators, even as simple as affine) are quite recent and utilize common approach based on Fenchel-type representations of the associated objectives/vector fields. The goal of this paper is to develop an alternative (and seemingly much simpler) decomposition LMO-based techniques for bilinear saddle point problems and for variational inequalities with affine monotone operators.

1 Introduction

This paper is a follow-up to our paper [17] and, same as its predecessor, is motivated by the desire to develop first order algorithms for solving convex-concave saddle point problem (or variational inequality with monotone operator) on a convex domain $X$ represented by Linear Minimization Oracle (LMO) capable to minimize over $X$, at a reasonably low cost, any linear function. “LMO-representability” of a convex domain $X$ is an essentially weaker assumption than “proximal friendliness” of $X$ (possibility to minimize over $X$, at a reasonably low cost, any linear perturbation of a properly selected strongly convex function) underlying the vast majority of known first order algorithms. There are important applications giving rise to LMO-represented domains which are not proximal friendly, most notably

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nuclear norm balls arising in low rank matrix recovery and in Semidefinite optimization; here LMO reduces to approximating the leading pair of singular vectors of a matrix, while all known proximal algorithms require much costly computationally full singular value decomposition,

• total variation balls arising in image reconstruction; here LMO reduces to solving a specific flow problem [14], while a proximal algorithm needs to solve a much more computationally demanding linearly constrained convex quadratic program,

• some combinatorial polytopes.

The needs of these applications inspire the current burst of activity in developing LMO-based optimization techniques. In its major part, this activity was focused on smooth (or Lasso-type smooth regularized) Convex Minimization over LMO-represented domains, where the classical Conditional Gradient algorithm of Frank & Wolfe [7] and its modifications are applicable (see, e.g., [5, 6, 8, 9, 13, 14, 15, 16, 21] and references therein). LMO-based techniques for large-scale Nonsmooth Convex Minimization (NCM), convex-concave Saddle Point problems (SP), even bilinear ones, and Variational Inequalities (VI) with monotone operators, even affine ones, where no classical optimization methods work, have been developed only recently. To the best of our knowledge, the related results reduce to LMO-based techniques for large-scale NCM based on Nesterov’s smoothing [1, 20, 23, 24, 4, 18].

An alternative approach to NCM, based on Fenchel-type representations of convex functions and processing the induced by these representations problems dual to the problem of interest, was developed in [4] and was further extended in [17] to convex-concave SP’s and VI’s with monotone operators. The goal of this paper is to develop an alternative to [17] decomposition-based approach to solving convex-concave SP’s and monotone VI’s on LMO-represented domains. In the nutshell, this approach is extremely simple, and it makes sense to present an informal outline of it, in the SP case, right here.

Given convex compact sets $X_1, X_2, Y_1, Y_2$ in Euclidean spaces, consider a convex-concave saddle point “master” problem

$$
\min_{x_1, x_2} \max_{y_1, y_2} \Phi(x_1, x_2; y_1, y_2)
$$

along with two “induced” problems

$$
(P) \min_{x_1} \max_{y_1} \phi(x_1, y_1) := \min_{x_1} \max_{y_2} \phi(x_1, x_2; y_1, y_2)
$$

$$
(D) \min_{x_2} \max_{y_2} \psi(x_2, y_2) := \min_{x_2} \max_{y_1} \psi(x_1, x_2; y_1, y_2)
$$

It is easily seen that $(P)$ and $(D)$ are convex-concave problems and a good approximate solution to the master problem induces straightforwardly equally good approximate solutions to $(P)$ and to $(D)$. More importantly, it turns out that when solving one of the induced problems, say, $(P)$, by an “intelligent,” in certain precise sense, algorithm, information acquired in course of building an $\epsilon$-solution yields straightforwardly an $\epsilon$-solution to the master problem, and thus yields an $\epsilon$-solution to the other induced problem, in our case, to $(D)$.

Now imagine that we want to solve a convex-concave SP problem which “as is” is too complicated for the standard solution techniques (e.g., problem’s domain is not proximal-friendly, or is of huge dimension). Our proposed course of actions is to make the problem of interest the problem $(D)$ stemming from a master problem built in a way which ensures that the associated problem $(P)$ is amenable to an “intelligent” solution algorithm $B$. After such a master problem is built, we solve $(P)$ within a desired accuracy $\epsilon$ by $B$ and use the acquired information to build an $\epsilon$-solution to the problem of interest.
As we shall see, our decomposition approach can, in principle, handle general convex-concave SP’s and affine VI’s. Our emphasis in this paper is, however, on bilinear SP’s and on VI’s with affine operators – the cases which, on one hand, are of primary importance in numerous applications, and, on the other hand, are the cases where our approach is easy to implement and where this approach seems to be more flexible and much simpler than the machinery of Fenchel-type representations developed in [17] (and in fact even covers this machinery, see section 3.3).

The rest of this paper is organized as follows. In sections 2 and 3 we present our decomposition-based approach to SP problems, resp., VI’s with monotone operators, with emphasis on utilizing the approach to handle bilinear SP’s, resp., affine VI’s, on LMO-represented domains. We illustrate our constructions by applying them to Colonel Blotto type matrix game (section 2.6.3) and Nash Equilibrium with pairwise interactions (section 3.2.2); in both these illustrations, decomposition allows to overcome difficulties coming from potentially huge ambient dimensions of the problems. Proofs missing in the main body of the paper are relegated to Appendix.

2 Decomposition of Convex-Concave Saddle Point Problems

2.1 Situation

In this section, we focus on the situation as follows. Given are

1. convex compact sets $X_i$ in Euclidean spaces $\mathcal{X}_i$ and convex compact sets $Y_i$ in Euclidean spaces $\mathcal{Y}_i$, $i = 1, 2$;

2. convex compact sets $X, Y$ such that

$$X \subset X_1 \times X_2 \subset \mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2, Y \subset Y_1 \times Y_2 \subset \mathcal{Y} := \mathcal{Y}_1 \times \mathcal{Y}_2,$$

such that the projections of $X$ onto $\mathcal{X}_i$ are the sets $X_i$, and projections of $Y$ onto $\mathcal{Y}_i$ are the sets $Y_i$, $i = 1, 2$. For $x_1 \in X_1$, we set $X_2[x_1] = \{x_2 : [x_1; x_2] \in X\} \subset X_2$, and for $y_1 \in Y_1$ we set $Y_2[y_1] = \{y_2 : [y_1; y_2] \in Y\} \subset Y_2$. Similarly,

$$X_1[x_2] = \{x_1 : [x_1; x_2] \in X\}, x_2 \in X_2,$$

and

$$Y_1[y_2] = \{y_2 : [y_1; y_2] \in Y\}, y_2 \in Y_2;$$

3. Lipschitz continuous function

$$\Phi(x = [x_1; x_2]; y = [y_1; y_2]) : X \times Y \to \mathbb{R} \quad (1)$$

which is convex in $x \in X$, and concave in $y \in Y$.

We call the outlined situation a direct product one, when $X = X_1 \times X_2$ and $Y = Y_1 \times Y_2$.

2.2 Induced convex-concave functions

We associate with $\Phi$ primal and dual induced functions:

$$\phi(x_1, y_1) := \min_{x_2 \in X_2[x_1]} \max_{y_2 \in Y_2[y_1]} \Phi(x_1, x_2; y_1, y_2) = \max_{y_2 \in Y_2[y_1]} \min_{x_2 \in X_2[x_1]} \Phi(x_1, x_2; y_1, y_2) : X_1 \times Y_1 \to \mathbb{R},$$

$$\psi(x_2, y_2) := \min_{x_1 \in X_1[x_2]} \max_{y_1 \in Y_1[y_2]} \Phi(x_1, x_2; y_1, y_2) = \max_{y_1 \in Y_1[y_2]} \min_{x_1 \in X_1[x_2]} \Phi(x_1, x_2; y_1, y_2) : X_2 \times Y_2 \to \mathbb{R}.$$

(the equalities are due to the convexity-concavity and continuity of $\Phi$ and convexity and compactness of $X_i[\cdot]$ and $Y_i[\cdot]$).
Recall that a Lipschitz continuous convex-concave function $\theta(u,v) : U \times V \to \mathbb{R}$ with convex compact $U, V$ gives rise to the primal and dual problems

$$\text{Opt}(P[\theta, U, V]) = \min_{u \in U} \left[ \bar{\theta}(u) := \max_{v \in V} \theta(u,v) \right]$$

$$\text{Opt}(D[\theta, U, V]) = \max_{v \in V} \left[ \bar{\theta}(v) := \min_{u \in U} \theta(u,v) \right]$$

with equal optimal values:

$$\text{SadVal}(\theta, U, V) := \text{Opt}(P[\theta, U, V]) = \text{Opt}(D[\theta, U, V]),$$

same as gives rise to saddle point residual

$$\epsilon_{\text{sad}}([u; v] | \theta, U, V) = \bar{\theta}(u) - \bar{\theta}(v) = [\bar{\theta}(u) - \text{Opt}(P[\theta, U, V])] + [\text{Opt}(D[\theta, U, V]) - \bar{\theta}(v)].$$

**Lemma 1.** $\phi$ and $\psi$ are convex-concave on their domains, are lower (upper) semicontinuous in their “convex” (“concave”) arguments, and are Lipschitz continuous in the direct product case. Besides this, it holds

$$\text{SadVal}(\phi, X_1, Y_1) = \text{SadVal}(\Phi, X, Y) = \text{SadVal}(\psi, X_2, Y_2), \quad (2)$$

and whenever $\bar{x} = [\bar{x}_1; \bar{x}_2] \in X$ and $\bar{y} = [\bar{y}_1; \bar{y}_2] \in Y$, one has

$$\epsilon_{\text{sad}}([\bar{x}_1; \bar{y}_1] | \phi, X_1, Y_1) \leq \epsilon_{\text{sad}}([\bar{x}; \bar{y}] | \Phi, X, Y), \quad \epsilon_{\text{sad}}([\bar{x}_2; \bar{y}_2] | \psi, X_2, Y_2) \leq \epsilon_{\text{sad}}([\bar{x}; \bar{y}] | \Phi, X, Y). \quad (3)$$

**The strategy** for solving SP problems we intend to develop is as follows:

1. We represent the SP problem of interest as the dual SP problem

$$\min_{x_2 \in X_2} \max_{y_2 \in Y_2} \psi(x_2, y_2) \quad (D)$$

induced by master SP problem

$$\min_{[x_1; x_2] \in X} \max_{[y_1; y_2] \in Y} \Phi(x_1, x_2; y_1, y_2) \quad (M)$$

The master SP problem is built in such a way that the associated primal SP problem

$$\min_{x_1 \in X_1} \max_{y_1 \in Y_1} \phi(x_1, y_1) \quad (P)$$

admits First Order oracle and can be solved by a traditional First Order method (e.g., a proximal one).

2. We solve $(P)$ to a desired accuracy by First Order algorithm producing accuracy certificates [19] and use these certificates to recover approximate solution of required accuracy to the problem of interest.

We shall see that the outlined strategy (originating from [3]\(^1\)) can be easily implemented when the problem of interest is a bilinear SP on the direct product of two LMO-represented domains.

\(^1\)in hindsight, a special case of this strategy was used in [10, 11].
2.3 Regular sub- and supergradients

Implementing the outlined strategy requires some “agreement” between the first order information of the master and the induced SP’s, and this is the issue we address now.

Given $\bar{x}_1 \in X_1, \bar{y}_1 \in Y_1$, let $\bar{x}_2 \in X_2[\bar{x}_1]$ and $\bar{y}_2 \in Y_2[\bar{y}_1]$ form a saddle point of the function $\Phi(\bar{x}_1, x_2; \bar{y}_1, y_2)$ (min in $x_2 \in X_2[\bar{x}_1]$, max in $y_2 \in Y_2[\bar{y}_1]$). In this situation we say that $(\bar{x} = [\bar{x}_1; \bar{x}_2], \bar{y} = [\bar{y}_1; \bar{y}_2])$ belongs to the saddle point frontier of $\Phi$, and we denote this frontier by $\mathcal{S}$. Let now $\bar{z} = (\bar{x} = [\bar{x}_1; \bar{x}_2], \bar{y} = [\bar{y}_1; \bar{y}_2]) \in \mathcal{S}$, so that the function $\Phi(\bar{x}_1, x_2; \bar{y}_1, \bar{y}_2)$ attains its minimum over $x_2 \in X_2[\bar{x}_1]$ at $\bar{x}_2$, and the function $\Phi(\bar{x}_1, \bar{x}_2; \bar{y}_1, y_2)$ attains its maximum over $y_2 \in Y_2[\bar{y}_1]$ at $\bar{y}_2$. Consider a subgradient $G$ of $\Phi(\cdot; \bar{y}_1, \bar{y}_2)$ taken at $\bar{x}$ along $X$: $G \in \partial_x \Phi(\bar{x}; \bar{y})$. We say that $G$ is a regular subgradient of $\Phi$ at $\bar{z}$, if for some $g \in E_1$ it holds

$$\forall x = [x_1; x_2] \in X : \langle G, x - \bar{x} \rangle \geq \langle g, x_1 - \bar{x}_1 \rangle;$$

every $g$ satisfying this relation is called compatible with $G$. Similarly, we say that a supergradient $H$ of $\Phi(\cdot; \cdot)$, taken at $\bar{y}$ along $Y$ is a regular supergradient of $\Phi$ at $\bar{z}$, if for some $h \in F_1$ it holds

$$\forall y = [y_1; y_2] \in Y : \langle H, y - \bar{y} \rangle \leq \langle h, y_1 - \bar{y}_1 \rangle;$$

and every $h$ satisfying this relation will be called compatible with $H$.

**Remark 1.** Let $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$, meaning that we are in the direct product case. If $\Phi(x; \bar{y})$ is differentiable in $x$ at $x = \bar{x}$, the partial gradient $\nabla_s \Phi(\bar{x}; \bar{y})$ is a regular subgradient of $\Phi$ at $(\bar{x}, \bar{y})$, and $\nabla_x \Phi(\bar{x}; \bar{y})$ is compatible with this subgradient:

$$\forall x = [x_1; x_2] \in X_1 \times X_2 :$$

$$\langle \nabla_s \Phi(\bar{x}; \bar{y}), x - \bar{x} \rangle = \langle \nabla_{x_1} \Phi(\bar{x}; \bar{y}), x_1 - \bar{x}_1 \rangle + \langle \nabla_{x_2} \Phi(\bar{x}; \bar{y}), x_2 - \bar{x}_2 \rangle \geq \langle \nabla_x \Phi(\bar{x}; \bar{y}), x_1 - \bar{x}_1 \rangle.$$

Similarly, if $\Phi(\bar{x}; y)$ is differentiable in $y$ at $y = \bar{y}$, then the partial gradient $\nabla_y \Phi(\bar{x}; \bar{y})$ is a regular supergradient of $\Phi$ at $(\bar{x}, \bar{y})$, and $\nabla_y \Phi(\bar{x}; \bar{y})$ is compatible with this supergradient.

**Lemma 2.** In the situation of section 2.1, let $\bar{z} = (\bar{x} = [\bar{x}_1; \bar{x}_2], \bar{y} = [\bar{y}_1; \bar{y}_2]) \in \mathcal{S}$, let $G$ be a regular subgradient of $\Phi$ at $\bar{z}$ and let $g$ be compatible with $G$. Let also $H$ be a regular supergradient of $\Phi$ at $\bar{z}$, and $h$ be compatible with $H$. Then $g$ is a subgradient in $x_1$, taken at $(\bar{x}_1, \bar{y}_1)$ along $X_1$, of the induced function $\phi$, and $h$ is a supergradient in $y_1$, taken at $(\bar{x}_1, \bar{y}_1)$ along $Y_1$, of the induced function $\phi$:

\[
\begin{align*}
(a) \quad & \phi(x_1, \bar{y}_1) \geq \phi(\bar{x}_1; \bar{y}_1) + \langle g, x_1 - \bar{x}_1 \rangle, \\
(b) \quad & \phi(\bar{x}_1, y_1) \leq \phi(\bar{x}_1; \bar{y}_1) + \langle h, y_1 - \bar{y}_1 \rangle.
\end{align*}
\]

for all $x_1 \in X_1, y_1 \in Y_1$.

**Regular sub- and supergradient fields of induced functions.** In the sequel, we say that $\phi'_{x_1}(x_1, y_1), \phi'_{y_1}(x_1, y_1)$ are regular sub- and supergradient fields of $\phi$, if for every $(x_1, y_1) \in X_1 \times Y_1$ and properly selected $\bar{x}_2, \bar{y}_2$ such that the point $\bar{z} = (\bar{x} = [x_1; \bar{x}_2], \bar{y} = [y_1; \bar{y}_2])$ is on the SP frontier of $\Phi$, $\phi'_{x_1}(x_1, y_1), \phi'_{y_1}(x_1, y_1)$ are the sub- and supergradients of $\phi$ induced, via Lemma 2, by regular sub- and supergradients of $\Phi$ at $\bar{z}$. Invoking Remark 1, we arrive at the following observation:

**Remark 2.** Let $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$, meaning that we are in the direct product case. If $\Phi$ is differentiable in $x$ and in $y$, then regular sub- and supergradient fields of $\phi$ can be built as follows: given $(x_1, y_1) \in X_1 \times Y_1$, we find $\bar{x}_2, \bar{y}_2$ such that the point $\bar{z} = (\bar{x} = [x_1; \bar{x}_2], \bar{y} = [y_1; \bar{y}_2])$ is on the SP frontier of $\Phi$, and set

\[
\phi'_{x_1}(x_1, y_1) = \nabla_{x_1} \Phi(x_1, \bar{x}_2; y_1, \bar{y}_2), \quad \phi'_{y_1}(x_1, y_1) = \nabla_{y_1} \Phi(x_1, \bar{x}_2; y_1, \bar{y}_2).
\]
2.3.1 Existence of regular sub- and supergradients

The notion of regular subgradient deals with \( \Phi \) as a function of \([x_1; x_2] \in X\) only, the \( y \)-argument being fixed, so that the existence/description questions related to regular subgradient deal in fact with a Lipschitz continuous convex function on \( X \). And of course the questions about existence/description of regular supergradients reduce straightforwardly to existence/description of regular subgradients (by swapping the roles of \( x \)'s and \( y \)'s and passing from \( \Phi \) to \(-\Phi\)). Thus, as far as existence and description of regular sub- and supergradients is concerned, it suffices to consider the situation where

- \( \Psi(x_1, x_2) \) is a Lipschitz continuous convex function on \( X \),
- \( \bar{x}_1 \in X_1 \) and \( \bar{x}_2 \in X_2[\bar{x}_1] \) is a minimizer of \( \Psi(\bar{x}_1, x_2) \) over \( x_2 \in X_2[\bar{x}_1] \).

What we need to understand, is when a subgradient \( G \) of \( \Psi \) taken at \( \bar{x} = [\bar{x}_1; \bar{x}_2] \) along \( X \) and some \( g \) satisfy the relation

\[
\langle G, [x_1; x_2] - \bar{x} \rangle \geq \langle g, x_1 - \bar{x}_1 \rangle \forall x = [x_1; x_2] \in X
\]

and what can be said about the corresponding \( g \)'s. The answer is as follows:

**Lemma 3.** With \( \Psi, \bar{x}_1, \bar{x}_2 \) as above, \( G \in \partial \Psi(\bar{x}) \) satisfies (5) if and only if the following two properties hold:

(i) \( G \) is a “certifying” subgradient of \( \Psi \) at \( \bar{x} \), meaning that \( \langle G, [0; x_2 - \bar{x}_2] \rangle \geq 0 \forall x_2 \in X_2[\bar{x}_1] \) (the latter relation indeed certifies that \( \bar{x}_2 \) is a minimizer of \( \Psi(\bar{x}_1, x_2) \) over \( x_2 \in X_2[\bar{x}_1] \));

(ii) \( g \) is a subgradient, taken at \( \bar{x}_1 \) along \( X_1 \), of the convex function

\[
\chi_G(x_1) = \min_{x_2 \in X_2[\bar{x}_1]} \langle G, [x_1; x_2] \rangle
\]

It is easily seen that with \( \Psi, \bar{x} = [\bar{x}_1; \bar{x}_2] \) as in Lemma 3 (i.e., \( \Psi \) is convex and Lipschitz continuous on \( X \), \( \bar{x}_1 \in X_1 \), and \( \bar{x}_2 \in X_2[\bar{x}_1] \)) minimizes \( \Psi(\bar{x}_1, x_2) \) over \( x_2 \in X_2[\bar{x}_1] \) a certifying subgradient \( G \) always exists; when \( \Psi \) is differentiable at \( \bar{x} \), one can take \( G = \nabla_x \Psi(\bar{x}) \). The function \( \chi_G(\cdot) \), however, not necessary admits a subgradient at \( \bar{x}_1 \); when it does admit it, every \( g \in \partial \chi_G(\bar{x}_1) \), satisfies (5). In particular,

1. **[Direct Product case]** When \( X = X_1 \times X_2 \), representing a certifying subgradient \( G \) of \( \Psi \), taken at \([\bar{x}_1; \bar{x}_2] \in \text{Argmin}_{x_2 \in X_2} \Psi(\bar{x}_1, x_2)\), as \([g; h] \), we have

\[
\langle h, x_2 - \bar{x}_2 \rangle \geq 0 ~ \forall x_2 \in X_2,
\]

whence \( \chi_G(x_1) = \langle g, x_1 \rangle + \langle h, \bar{x}_2 \rangle \), and thus \( g \) is a subgradient of \( \chi_G \) at \( \bar{x}_1 \). In particular, in the direct product case and when \( \Psi \) is differentiable at \( \bar{x} \), (5) is met by \( G = \nabla \Psi(\bar{x}) \), \( g = \nabla_{x_1} \Psi(\bar{x}) \);

2. **[Polyhedral case]** When \( X \) is a polyhedral set, for every certifying subgradient \( G \) of \( \Psi \) the function \( \chi_G \) is polyhedrally representable with domain \( X_1 \) and as such has a subgradient at every point from \( X_1 \);

3. **[Interior case]** When \( \bar{x}_1 \) is a point from the relative interior of \( X_1 \), \( \chi_G \) definitely has a subgradient at \( \bar{x}_1 \).

2.4 Main Result, Saddle Point case

2.4.1 Preliminaries: execution protocols, accuracy certificates, residuals

We start with outlining some simple concepts originating from [19]. Let \( W \) be a convex compact set in Euclidean space \( W \), and \( M(w) : W \rightarrow W \) be a vector field on \( W \). A *t-step execution protocol*
associated with $M, W$ is a collection $\mathcal{I}_t = \{w_i \in W, M(w_i) : 1 \leq i \leq t\}$. A t-step accuracy certificate is a t-dimensional probabilistic (i.e., with nonnegative entries summing up to 1) vector $\lambda$. Augmenting a t-step accuracy protocol $\mathcal{I}_t$ by t-step accuracy certificate $\lambda$ gives rise to two entities:

approximate solution: $w^t = w^t(\mathcal{I}_t, \lambda) := \sum_{i=1}^{t} \lambda_i w_i \in W$;  
residual: $\text{Res}(\mathcal{I}_t, \lambda|W) = \max_{w \in W} \sum_{i=1}^{t} \lambda_i \langle M(w_i), w_i - w \rangle$. (6)

When $W = U \times V$, where $U$ is a closed convex subset of Euclidean space $\mathcal{U}$ and $V$ is a closed convex subset of Euclidean space $\mathcal{V}$, and $M$ is vector field induced by convex-concave function $\theta(u, v) : U \times V \rightarrow \mathbb{R}$, that is,

$M(u, v) = [M_u(u, v); M_v(u, v)] : U \times V \rightarrow \mathcal{U} \times \mathcal{V}$ with $F_u(u, v) \in \partial_u \theta(u, v)$, $F_v(u, v) \in \partial_v [-\theta(u, v)]$

(such a field always is monotone), an execution protocol associated with $(M, W)$ will be called also protocol associated with $\theta, U, V$, or protocol associated with the saddle point problem

$$\min_{u \in U} \max_{v \in V} \theta(u, v).$$

The importance of these notions in our context stems from the following simple observation [19]:

**Proposition 1.** Let $U, V$ be nonempty convex compact domains in Euclidean spaces $\mathcal{U}, \mathcal{V}$, $\theta(u, v) : U \times V \rightarrow \mathbb{R}$ be a convex-concave function, and $M$ be induced monotone vector field: $M(u, v) = [M_u(u, v); M_v(u, v)] : U \times V \rightarrow \mathcal{U} \times \mathcal{V}$ with $M_u(u, v) \in \partial_u \theta(u, v)$, $M_v(u, v) \in \partial_v [-\theta(u, v)]$. For a t-step execution protocol $\mathcal{I}_t = \{w_i = [u_i; v_i] \in W : \sum_{i=1}^{t} M_i = [M_u(u_i, v_i); M_v(u_i, v_i)] : 1 \leq i \leq t\}$ associated with $\theta, U, V$, and t-step accuracy certificate $\lambda$, it holds

$$\epsilon_{sad}(w^t(\mathcal{I}_t, \lambda)|\theta, U, V) \leq \text{Res}(\mathcal{I}_t, \lambda|U \times V).$$ (7)

Indeed, for $[u; v] \in U \times V$ we have

$$\text{Res}(\mathcal{I}_t, \lambda|U \times V) \geq \sum_{i=1}^{t} \lambda_i \langle \lambda_i, w_i - [u; v]\rangle = \sum_{i=1}^{t} \lambda_i \left[\langle M_u(u_i, v_i), u_i - u \rangle - \langle M_v(u_i, v_i), v_i - v \rangle\right] \geq \sum_{i=1}^{t} \lambda_i \left[\theta(u_i, v_i) - \theta(u, v_i)\right] \geq \theta(u^t, v) - \theta(u, v^t),$$

where the inequalities are due to the origin of $M$ and convexity-concavity of $\theta$. The resulting inequality holds true for all $[u; v] \in U \times V$, and (7) follows. 

2.4.2 Main Result

**Proposition 2.** In the situation and notation of sections 2.1 – 2.3, let $\phi$ be the primal convex-concave function induced by $\Phi$, and let

$\mathcal{I}_t = \{[x_{1,i}; y_{1,i}] \in X_1 \times Y_1, [\alpha_i := \phi_1(x_{1,i}; y_{1,i}); \beta_i := -\phi_1'(x_{1,i}; y_{1,i})] : 1 \leq i \leq t\}$

be an execution protocol associated with $\phi, X_1, Y_1$, where $\phi_1$, $\phi_1'$ are regular sub- and supergradient fields associated with $\Phi, \phi$. Due to the origin of $\phi$, $\phi_1$, $\phi_1'$, there exist $x_{2,i} \in X_2[x_{1,i}]$, $G_i \in \mathcal{X}$, $y_{2,i} \in Y_2[y_{1,i}]$, and $H_i \in \mathcal{Y}$ such that

(a) $G_i \in \partial_2 \Phi(x_i := [x_{1,i}; x_{2,i}], y_i := [y_{1,i}; y_{2,i}]),$
(b) $H_i \in \partial_y [-\Phi(x_i := [x_{1,i}; x_{2,i}], y_i := [y_{1,i}; y_{2,i}])],$
(c) $\langle G_i, x - [x_{1,i}; x_{2,i}] \rangle \geq \langle \phi_1'(x_{1,i}; y_{1,i}), x_{1,i} - x_{1,i} \rangle \forall x \in [x_{1,i}; x_{2,i}] \in X,$
(d) $\langle H_i, y - [y_{1,i}; y_{2,i}] \rangle \geq \langle -\phi_1'(x_{1,i}; y_{1,i}), y_{1,i} - y_{1,i} \rangle \forall y \in [y_{1,i}; y_{2,i}] \in Y.$ (8)
implying that
\[ J_t = \{ z_i = [x_i; x_{2,i}]; y_i = [y_{1,i}; y_{2,i}], F_i = [G_i; H_i] : 1 \leq i \leq t \} \]
is an execution protocol associated with \( \Phi, X, Y \). For every accuracy certificate \( \lambda \) it holds
\[ \text{Res}(J_t, \lambda | X \times Y) \leq \text{Res}(I_t, \lambda | X_1 \times Y_1). \] (9)
As a result, given an accuracy certificate \( \lambda \) and setting
\[ [x^t; y^t] = \left[ \sum_{i=1}^{t} \lambda_i [x_{1,i}; x_{2,i}] ; \left[ y_{1,i}; y_{2,i} \right] \right], \]
we ensure that
\[ \epsilon_{\text{sad}}([x^t; y^t]|\Phi, X, Y) \leq \text{Res}(I_t, \lambda | X_1 \times Y_1), \] (10)
whence also, by Lemma 1,
\[ \epsilon_{\text{sad}}([x^t_1; y^t_1]|\phi, X_1, Y_1) \leq \text{Res}(I_t, \lambda | X_1 \times Y_1), \]
\[ \epsilon_{\text{sad}}([x^t_2; y^t_2]|\psi, X_2, Y_2) \leq \text{Res}(I_t, \lambda | X_1 \times Y_1), \] (11)
where \( \psi \) is the dual function induced by \( \Phi \).

**Proof.** Let \( z := [u_1; u_2]; [v_1; v_2] \in X \times Y \). Then
\[ \sum_{i=1}^{t} \lambda_i \langle F_i, z_i - z \rangle = \sum_{i=1}^{t} \lambda_i \left[ \langle G_i, [x_{1,i}; x_{2,i}] - [u_1; u_2] \rangle + \langle H_i, [y_{1,i}; y_{2,i}] - [v_1; v_2] \rangle \right] \]
\[ \leq \langle \phi'_{x_1}(x_{1,i}, y_{1,i}), x_{1,i} - u_1 \rangle \text{ by (8.c)} \leq \langle \phi'_{y_1}(x_{1,i}, y_{1,i}), y_{1,i} - v_1 \rangle \text{ by (8.d)} \]
\[ \leq \sum_{i=1}^{t} \lambda_i \left[ \langle \alpha_i, x_{1,i} - u_1 \rangle + \langle \beta_i, y_{1,i} - v_1 \rangle \right] \leq \text{Res}(I_t, \lambda | X_1 \times Y_1), \]
and (9) follows. \( \square \)

### 2.5 Application: Solving bilinear Saddle Point problems on domains represented by Linear Minimization Oracles

#### 2.5.1 Situation
Let \( W \) be a nonempty convex compact set in \( \mathbb{R}^N \), \( Z \) be a nonempty convex compact set in \( \mathbb{R}^M \), and let \( \psi : W \times Z \to \mathbb{R} \) be bilinear convex-concave function:
\[ \psi(w, z) = \langle w, p \rangle + \langle z, q \rangle + \langle z, Sw \rangle. \] (12)
Our goal is to solve the convex-concave SP problem
\[ \min_{w \in W} \max_{z \in Z} \psi(w, z) \] (13)
given by \( \psi, W, Z \).
2.5.2 Simple observation

We intend to show that $\psi$ can be represented (in fact, in many ways) as the dual function induced by a bilinear convex-concave function $\Phi$; this is the key element of the outlined in section 2.2 strategy for solving (13).

In the situation described in section 2.5.1, let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be convex compact sets, and let $D \in \mathbb{R}^{m \times N}$, $A \in \mathbb{R}^{n \times M}$, $R \in \mathbb{R}^{m \times n}$. Consider bilinear (and thus convex-concave) function

$$
\Phi(u, w; v, z) = \langle w, p + D^T v \rangle + \langle z, q + A^T u \rangle - \langle v, Ru \rangle : [U \times W] \times [V \times Z] \to \mathbb{R} \quad (14)
$$

(the “convex” argument is $(u, w)$, the “concave” one is $(v, z)$). Assume that a pair of functions

$$
\bar{u}(w, z) : W \times Z \to U,
$$

$$
\bar{v}(w, z) : W \times Z \to V \quad (15)
$$

satisfies

$$
\forall (w, z) \in W \times Z : Dw = R\bar{u}(w, z)
$$

$$
\forall (w, z) \in W \times Z : Az = R^T \bar{v}(w, z) \quad (16)
$$

Denoting $\bar{u} = \bar{u}(w, z)$, $\bar{v} = \bar{v}(w, z)$, we have

$$
(a) \quad \langle w, D^T \bar{v} \rangle = \langle Dw, \bar{v} \rangle = \langle R\bar{u}, \bar{v} \rangle
$$

$$
(b) \quad \langle z, A^T \bar{u} \rangle = \langle Az, \bar{u} \rangle = \langle \bar{u}, R^T \bar{v} \rangle = \langle R\bar{u}, \bar{v} \rangle. \quad (17)
$$

Thus,

$$
\nabla_u \Phi(\bar{u}, w; \bar{v}, z) = Az - R^T \bar{v} = 0
$$

$$
\nabla_v \Phi(\bar{u}, w; \bar{v}, z) = Dw - R\bar{u} = 0
$$

whence

$$
\bar{\psi}(w, z) := \min_{w \in U} \max_{v \in V} \Phi(u, w; v, z) = \Phi(\bar{u}(w, z), w; \bar{v}(w, z), z)
$$

$$
= \langle w, p \rangle + \langle z, q \rangle + \langle Dw, \bar{v}(w, z) \rangle \quad \text{[by (17)]}
$$

We have proved

Lemma 4. In the case of (15), (16), assuming that

$$
\langle Dw, \bar{v}(w, z) \rangle = \langle z, Sw \rangle \quad \forall (w \in W, z \in Z), \quad (18)
$$

$\psi$ is the dual convex-concave function induced by $\Phi$ and the domains $U \times W$, $V \times Z$.

Note that there are easy ways to ensure (16) and (18).

**Example 1.** Here $m = M$, $n = N$, and $D = A^T = R = S$. Assuming $U \supset W$, $V \supset Z$ and setting $\bar{u}(w, z) = w$, $\bar{v}(w, z) = z$, we ensure (15), (16) and (18).

**Example 2.** Let $S = A^T D$ with $A \in \mathbb{R}^{K \times M}$, $D \in \mathbb{R}^{K \times N}$. Setting $m = n = K$, $R = I_K$, $\bar{u}(w, z) = Dw$, $\bar{v}(w, z) = Az$ and assuming that $U \supset DW$, $V \supset AZ$, we again ensure (15), (16) and (18).
2.5.3 Implications

Assume that (15), (16) and (18) take place. Renaming the variables according to \( x_1 \equiv u, \ y_1 \equiv v, \ x_2 \equiv w, \ y_2 \equiv z \) and setting \( X_1 = U, \ X_2 = W, \ Y_1 = V, \ Y_2 = Z, \ X = X_1 \times X_2 = U \times W, \ Y = Y_1 \times Y_2 = V \times Z \), we find ourselves in the direct product case of the situation of section 2.1, and Lemma 4 says that the bilinear SP problem of interest (12), (13) is the dual SP problem associated with the bilinear master SP problem

\[
\min_{u;w} \max_{v;z} \left[ \Phi(u, w; v, z) = \langle w, p + D^T v \rangle + \langle z, q + A^T u \rangle - \langle Ru, v \rangle \right]
\]  

(19)

Since \( \Phi \) is linear in \([w; z]\), the primal SP problem associated with (19) is

\[
\min_{u \in U} \max_{v \in V} \left[ \phi(u, v) = \min_{w \in W} \langle w, p + D^T v \rangle + \max_{z \in Z} \langle v, q + A^T u \rangle - \langle Ru, v \rangle \right].
\]

Assuming that \( W, Z \) allow for cheap Linear Minimization Oracles and defining \( w_*(\cdot), z_*(\cdot) \) according to

\[
w_*(\xi) \in \text{Argmin}_{w} \langle w, \xi \rangle, \ z_*(\eta) \in \text{Argmin}_{z} \langle z, \eta \rangle,
\]

we have

\[
\phi(u, v) = \langle w_*(p + D^T v), p + D^T v \rangle + \langle z_*(-q - A^T u), q + A^T u \rangle - \langle Ru, v \rangle,
\]

\[
\phi'_a(u, v) := A_z(\phi(u, v)), \quad \phi'_v(u, v) := D w_*(p + D^T v) - Ru \in -\partial v[-\phi(u, v)],
\]

(20)

that is, first order information on the primal SP problem

\[
\min_{u \in U} \max_{v \in V} \phi(u, v)
\]

(21)

is available. Note that since we are in the direct product case, \( \phi'_a \) and \( \phi'_v \) are regular sub- and supergradient fields associated with \( \Phi \), \( \phi \).

Now let \( \mathcal{I}_t = \{[u_i; v_i] \in U \times V; [\gamma_i := \phi'_a(u_i, v_i); \delta_i := -\phi'_v(u_i, v_i)] : 1 \leq i \leq t \} \) be an execution protocol generated by a First Order algorithm as applied to the primal SP problem (21), and let

\[
w_i = w_*(p + D^T v_i), \quad z_i = z_*(-q - A^T u_i),
\]

\[
\alpha_i = \nabla_{u;w} \Phi(u_i, w_i; v_i, z_i) = p + D^T v_i,
\]

\[
\beta_i = -\nabla_{z;v} \Phi(u_i, w_i; v_i, z_i) = -q - A^T u_i,
\]

so that

\[
\mathcal{J}_t = \left\{ [u_i; v_i]; [\gamma_i; \alpha_i]; [\delta_i; \beta_i] : 1 \leq i \leq t \right\}
\]

is an execution protocol associated with the SP problem (19). By Proposition 2, for any accuracy certificate \( \lambda \) it holds

\[
\text{Res}(\mathcal{J}_t, \lambda|U \times W \times V \times Z) \leq \text{Res}(\mathcal{I}_t, \lambda|U \times V)
\]

(22)

whence, setting

\[
[[u^t; w^t]; [v^t; z^t]] = \sum_{i=1}^{t} \lambda_i[[u_i; w_i]; [v_i; z_i]]
\]

(23)
and invoking Proposition 1 with $\Phi$ in the role of $\theta$,
\[
\epsilon_{\text{sad}}(\|[u^t; w^t] - [v^t; z^t]\|_\Phi, X_1 \times X_2, Y_1 \times Y_2) \leq \text{Res}(I_t, \lambda |U \times V) \tag{24}
\]
whence, by Lemma 1,
\[
\epsilon_{\text{sad}}([w^t; z^t] | \psi, W, Z) \leq \text{Res}(I_t, \lambda |U \times V). \tag{25}
\]
We have arrived at the following

**Proposition 3.** In the situation of section 2.5.1, let (15), (16) and (18) take place. Then applying to the primal SP problem (21) First Order algorithm $B$ with accuracy certificates, $t$-step execution protocol $I_t$ and accuracy certificate $\lambda_t$ generated by $B$ yield straightforwardly a feasible solution to the SP problem of interest (12) – (13) of the $\epsilon_{\text{sad}}$-inaccuracy $\leq \text{Res}(I_t, \lambda |U \times V)$.

Note also that when the constructions from Examples 1, 2 are used, there is a significant freedom in selecting the domain $U \times V$ of the primal problem (we only require $U$, $V$ to be convex compact sets “large enough” to ensure the inclusions mentioned in Examples), so that there is no difficulty to enforce $U$, $V$ to be proximal friendly. As a result, we can take as $B$ a proximal First Order method, for example, Non-Euclidean Restricted Memory Level algorithm with certificates (cf. [4]) or Mirror Descent (cf. [17]). The efficiency estimates of these algorithms as given in [4, 17] imply that the resulting procedure for solving the SP of interest (12) – (13) admits non-asymptotic $O(1/\sqrt{t})$ rate of convergence, with explicitly computable factors hidden in $O(\cdot)$. The resulting complexity bound is completely similar to the one achievable with the machinery of Fenchel-type representations [4, 17].

We are about to consider a special case where the $O(1/\sqrt{t})$ complexity admits a significant improvement.

### 2.6 Matrix Game case

Let $S \in \mathbb{R}^{M \times N}$ admit representation
\[
S = A^T D
\]
with $A \in \mathbb{R}^{K \times M}$ and $D \in \mathbb{R}^{K \times N}$. Let also $W = \Delta_N = \{ w \in \mathbb{R}^N_+ : \sum_i w_i = 1 \}$, $Z = \Delta_M$. Our goal is to solve matrix game
\[
\min_{w \in W} \max_{z \in Z} [\psi(w, z) = \langle z, Sw \rangle = \langle Az, Dw \rangle]. \tag{26}
\]
Let $U$, $V$ be convex compact sets such that
\[
V \supset AZ, \quad U \supset DW, \tag{27}
\]
and let us set
\[
\Phi(u, w; v, z) = \langle u, Az \rangle + \langle v, Dw \rangle - \langle u, v \rangle
\]
\[
\bar{u} := \bar{u}(w, z) = Dw
\]
\[
\bar{v} := \bar{v}(w, z) = Az
\]
implying that
\[
\nabla_u \Phi(\bar{u}, w; \bar{v}, z) = Az - \bar{v} = 0,
\]
\[
\nabla_v \Phi(\bar{u}, w; \bar{v}, z) = Dw - \bar{u} = 0,
\]
\[
\Phi(\bar{u}, w; \bar{v}, z) = \langle \bar{u}, Az \rangle + \langle \bar{v}, Dw \rangle - \langle \bar{u}, \bar{v} \rangle = \langle Dw, Az \rangle + \langle Az, Dw \rangle - \langle Dw, Az \rangle
\]
\[
= \langle z, A^T Dw \rangle = \psi(w, z).
\]
It is immediately seen that the function $\psi$ from (26) is nothing but the dual convex-concave function associated with $\Phi$ (cf. Example 2), while the primal function is

\[ \phi(u, v) = \text{Max}(A^T u) + \text{Min}(D^T v) - \langle u, v \rangle; \]  

(28)

here $\text{Min}(p)$ and $\text{Max}(p)$ stand for the smallest and the largest entries in vector $p$. Applying the strategy outlined in section 2.2, we can solve the problem of interest (26) applying to the primal SP problem

\[ \min_{u \in U} \max_{v \in V} \left[ \phi(u, v) = \text{Min}(D^T v) + \text{Max}(A^T u) - \langle u, v \rangle \right] \]  

(29)

an algorithm with accuracy certificates and using the machinery outlined in previous sections to convert the resulting execution protocols and certificates into approximate solutions to the problem of interest (26).

We intend to consider a special case when the outlined approach allows to reduce a huge, but simple, matrix game (26) to a small SP problem (29) – so small that it can be solved to high accuracy by a cutting plane method (e.g., the Ellipsoid algorithm). This is the case when the matrices $A, D$ in (26) are simple.

2.6.1 Simple matrices

Given a $K \times L$ matrix $B$, we call $B$ simple if, given $x \in \mathbb{R}^K$, it is easy to identify the columns $\overline{B}[x]$, $\underline{B}[x]$ of $B$ making the maximal, resp. the minimal, inner product with $x$.

When matrices $A, D$ in (26) are simple, the first order information for the cost function $\phi$ in the primal SP problem (29) is easy to get. Besides, all we need from the convex compact sets $U, V$ participating in (29) is to be large enough to ensure that $U \supset DW$ and $V \supset AZ$, which allows to make $U$ and $V$ simple, e.g., Euclidean balls. Finally, when the design dimension $2K$ of (29) is small, we have at our disposal a multitude of linearly converging, with the converging ratio depending solely on $K$, methods for solving (29), including the Ellipsoid algorithm with certificates presented in [19]. We are about to demonstrate that the outlined situation indeed takes place in some meaningful applications.

2.6.2 Example: Knapsack generated matrices

2 Assume that we are given knapsack data, namely,

- positive integer horizon $m$,
- nonnegative integer bounds $\overline{p}_s$, $1 \leq s \leq m$,
- positive integer costs $h_s$, $1 \leq s \leq m$, and positive integer budget $H$, and
- output functions $f_s(\cdot) : \{0, 1, \ldots, \overline{p}_s\} \rightarrow \mathbb{R}^{r_s}$, $1 \leq s \leq m$.

Given the outlined data, consider the set $\mathcal{P}$ of all integer vectors $p = [p_1; \ldots; p_m] \in \mathbb{R}^m$ satisfying the following restrictions:

\[ 0 \leq p_s \leq \overline{p}_s, 1 \leq s \leq m \quad \text{[range restriction]} \]

\[ \sum_{s=1}^{m} h_s p_s \leq H \quad \text{[budget restriction]} \]

and the matrix $B$ of the size $K \times \text{Card}(\mathcal{P})$, $K = \sum_{s=1}^{m} r_s$, defined as follows: the columns of $B$ are indexed by vectors $p = [p_1; \ldots; p_m] \in \mathcal{P}$, and the column indexed by $p$ is the vector

\[ B_p = [f_1(p_1); \ldots; f_m(p_m)]. \]

\[ \text{The construction to follow can be easily extended from “knapsack generated” matrices to more general “Dynamic Programming generated” ones, see section A.4 in Appendix.} \]
Note that assuming $m$, $\overline{p}_s$, $r_s$ moderate, matrix $B$ is simple – given $x \in \mathbb{R}^K$, it is easy to find $\overline{B}[x]$ and $B[x]$ by Dynamic Programming.

Indeed, to identify $\overline{B}[x]$, $x = [x_1; \ldots; x_m] \in \mathbb{R}^{r_1} \times \ldots \times \mathbb{R}^{r_m}$ (identification of $B[x]$ is completely similar), it suffices to run for $s = m, m-1, \ldots 1$ the backward Bellman recurrence

\[
\begin{align*}
U_s(h) &= \max_{r \in \mathbb{Z}} \{U_{s+1}(h - h_s r) + \langle f_s(r), x_s \rangle : 0 \leq r \leq \overline{p}_s, 0 \leq h - h_s r \} \\
A_s(h) &\in \mathop{\text{Argmax}}_{r \in \mathbb{Z}} \{U_{s+1}(h - h_s r) + \langle f_s(r), x_s \rangle : 0 \leq r \leq \overline{p}_s, 0 \leq h - h_s r \}\}
\end{align*}
\]

with $U_{m+1}(\cdot) = 0$, and then to recover one by one the entries $p_s$ in the index $p \in \mathcal{P}$ of $\overline{B}[x]$ from the forward Bellman recurrence

\[
\begin{align*}
H_1 &= H, p_1 = A_1(H_1) \\
H_{s+1} &= H_s - h_s p_s, p_{s+1} = A_{s+1}(H_{s+1}), 1 \leq s < m.
\end{align*}
\]

2.6.3 Illustration: Attacker vs. Defender.

The “covering story” we intend to consider is as follows. Attacker and Defender are preparing for a conflict to take place on $m$ battlefields. A pure strategy of Attacker is a vector $a = [a_1; \ldots; a_m]$, where nonnegative integer $a_s$, $1 \leq s \leq m$, is the number of attacking units to be created and deployed at battlefield $s$; the only restrictions on $a$, aside of nonnegativity and integrality, are the bounds $a_s \leq \overline{a}_s$, $1 \leq s \leq m$, and the budget constraint $\sum_{s=1}^{m} h_s a_s \leq H_A$ with positive integer $h_s A$ and $H_A$. Similarly, a pure strategy of Defender is a vector $d = [d_1; \ldots; d_m]$, where nonnegative integer $d_s$ is the number of defending units to be created and deployed at battlefield $s$, and the only restrictions on $d$, aside of nonnegativity and integrality, are the bounds $d_s \leq \overline{d}_s$, $1 \leq s \leq m$, and the budget constraint $\sum_{s=1}^{m} h_s D d_s \leq H_D$ with positive integer $h_s D$ and $H_D$. The total loss of Defender (the total gain of Attacker), the pure strategies of the players being $a$ and $d$, is

\[
S_{a,d} = \sum_{s=1}^{m} |\Omega_s|^a_{s,d},
\]

with given $(\overline{p}_s + 1) \times (\overline{d}_s + 1)$ matrices $\Omega^s$. Our goal is to solve in mixed strategies the matrix game where Defender seeks to minimize his total loss, and Attacker seeks to maximize it.

Denoting by $\mathcal{A}$ and $\mathcal{D}$ the sets of pure strategies of Attacker, resp., Defender, representing

\[
\Omega^s = \sum_{i=1}^{r_s} f^i_s [g^i_s]^T, f^i_s = [f^1_s; \ldots; f^{r_s}_s], g^i_s = [g^1_s; \ldots; g^{r_s}_s], r_s = \text{Rank}(\Omega^s),
\]

and setting

\[
K = \sum_{s=1}^{m} r_s,
\]

\[
A_a = \{[f_{a1}^1; f_{a1}^2; \ldots; f_{a1}^{r_1}]; [f_{a2}^1; f_{a2}^2; \ldots; f_{a2}^{r_2}]; \ldots; [f_{am}^1; f_{am}^2; \ldots; f_{am}^{r_m}] \} \in \mathbb{R}^K, a \in \mathcal{A},
\]

\[
D_d = \{[g_{d1}^1; g_{d1}^2; \ldots; g_{d1}^{r_1}]; [g_{d2}^1; g_{d2}^2; \ldots; g_{d2}^{r_2}]; \ldots; [g_{dm}^1; g_{dm}^2; \ldots; g_{dm}^{r_m}] \} \in \mathbb{R}^K, d \in \mathcal{D},
\]

we end up with $K \times M$, $M = \text{Card}(\mathcal{A})$, knapsack-generated matrix $A$ with columns $A_a$, $a \in \mathcal{A}$, and $K \times N$, $N = \text{Card}(\mathcal{D})$, knapsack-generated matrix $D$ with columns $D_d$, $d \in \mathcal{D}$, such that

\[
S := [S_{a,d}]_{a \in \mathcal{A}, d \in \mathcal{D}} = A^T D.
\]

This story is a variation of what is called “Colonel Blotto Game” in Game Theory, see, e.g., [2, 22] and references therein.
As a result, solving the Attacker vs. Defender game in mixed strategies reduces to solving SP problem (26) with knapsack-generated (and thus simple) matrices $A$, $D$ and thus can be reduced to convex-concave SP (29) on the product of two $K$-dimensional convex compact sets. Note that in the situation in question the design dimension $2K$ of (29) will, typically, be rather small (few tens or at most few hundreds), while the design dimensions $M$, $N$ of the matrix game of interest (26) can be huge.

**Numerical illustration.** With the data (quite reasonable in terms of the “Attacker vs. Defender” game)

$$m = 8, \ h_sA = h_sD = 1, 1 \leq s \leq m, \ H_A = H_D = 64 = \overline{d}_s = \overline{\alpha}_s, \ 1 \leq s \leq m$$

and rank 1 matrices $\Omega_s$, $1 \leq s \leq m$, the design dimensions of the problem of interest (26) are as huge as

$$\dim w = \dim z = 97,082,021,465$$

while the sizes of problem (29) are just

$$\dim u = \dim v = 8,$$

and thus (29) can be easily solved to high accuracy by the Ellipsoid method. In the numerical experiment we are about to report, the outlined approach allowed to solve (26) within $\epsilon_{sat}$-inaccuracy as small as 5.0e-9 in just 1537 steps of the Ellipsoid algorithm (110.0 sec on a medium quality laptop). This performance is quite promising, especially when taking into account huge – nearly $10^{11}$ – sizes of the matrix game of interest (26).

### 3 From Saddle Point problems to Variational Inequalities with Monotone Operators

In what follows, we extend the decomposition approach (developed so far for convex-concave SP problems) to Variational Inequalities (VI’s) with monotone operators, with the primary goal to handle VI’s with affine monotone operators on LMO-represented domains.

#### 3.1 Decomposition of Variational Inequalities with monotone operators

**3.1.1 Situation**

Let $\mathcal{X}$, $\mathcal{H}$ be Euclidean spaces, $\Theta \subset \mathcal{X} \times \mathcal{H}$ be convex compact set, $\Xi$ be the projection of $\Theta$ onto $\mathcal{X}$, and $H$ be the projection of $\Theta$ onto $\mathcal{H}$. Given $\xi \in \Xi$, $\eta \in H$, we set

$$H_\xi = \{ \eta : [\xi; \eta] \in \Theta \}, \ \Xi_\eta = \{ \xi \in \Xi : [\xi; \eta] \in \Theta \}.$$ 

We denote a point from $\mathcal{X} \times \mathcal{H}$ as $\theta = [\xi; \eta]$ with $\xi \in \mathcal{X}$, $\eta \in \mathcal{H}$. Let, further,

$$\Phi(\xi, \eta) = [\Phi_\xi(\xi, \eta); \Phi_\eta(\xi, \eta)] : \Theta \to \mathcal{X} \times \mathcal{H}$$

be a continuous monotone vector field.

---

4 for implementation details, see section A.5
3.1.2 Induced vector fields

Let $\xi \in \Xi$, and let $\eta = \eta(\xi)$ be a somehow selected, as a function of $\xi \in \Xi$, strong solution to the VI given by $(H_\xi, \Phi_\eta(\xi, \eta))$, that is,

$$\eta(\xi) \in H_\xi \& \langle \Phi_\eta(\xi, \eta(\xi)), \eta - \eta(\xi) \rangle \geq 0 \ \forall \eta \in H_\xi. \quad (30)$$

Let us call $\Phi$ (more precisely, the pair $(\Phi, \eta(\cdot))$) $\eta$-regular, if for every $\xi \in \Xi$, there exists $\Psi = \Psi(\xi) \in \mathcal{X}$ such that

$$\langle \Psi(\xi), \xi' - \xi \rangle \leq \langle \Phi(\xi, \eta(\xi)), [\xi'; \eta'] - [\xi; \eta(\xi)] \rangle \ \forall [\xi'; \eta'] \in \Theta. \quad (31)$$

Similarly, let $\xi(\eta)$ be a somehow selected, as a function of $\eta \in H$, strong solution to the VI given by $(\Xi_\eta, \Phi_\eta(\xi, \eta))$, that is,

$$\xi(\eta) \in \Xi_\eta \& \langle \Phi_\eta(\xi(\eta), \eta), \xi - \xi(\eta) \rangle \geq 0 \ \forall \xi \in \Xi_\eta. \quad (32)$$

Let us call $(\Phi, \xi(\cdot))$ $\xi$-regular, if for every $\eta \in H$ there exists $\Gamma = \Gamma(\eta) \in \mathcal{H}$ such that

$$\langle \Gamma(\eta), \eta' - \eta \rangle \leq \langle \Phi(\xi(\eta), \eta), [\xi'; \eta'] - [\xi(\eta); \eta] \rangle \ \forall [\xi'; \eta'] \in \Theta. \quad (33)$$

When $(\Phi, \eta)$ is $\eta$-regular, we refer to the above $\Psi(\cdot)$ as to a primal vector field induced by $\Phi^5$, and when $(\Phi, \xi)$ is $\xi$-regular, we refer to the above $\Gamma(\cdot)$ as to a dual vector field induced by $\Phi$.

**Example: Direct product case.** This is the case where $\Theta = \Xi \times H$. In this situation, setting $\Psi(\xi) = \Phi_\xi(\xi, \eta(\xi))$, we have for $[\xi'; \eta'] \in \Theta$:

$$\langle \Phi(\xi, \eta(\xi)), [\xi'; \eta'] - [\xi; \eta(\xi)] \rangle = \langle \Phi_\xi(\xi, \eta(\xi)), \xi' - \xi \rangle + \langle \Phi_\eta(\xi, \eta(\xi)), \eta' - \eta(\xi) \rangle \geq \langle \Psi(\xi), \xi' - \xi \rangle,$$

that is, $(\Phi, \eta(\cdot))$ is $\eta$-regular, with $\Psi(\xi) = \Phi_\xi(\xi, \eta(\xi))$. Setting $\Gamma(\eta) = \Phi_\eta(\xi(\eta), \eta)$, we get by similar argument

$$\langle \Phi(\xi(\eta)), [\xi'; \eta'] - [\xi(\eta); \eta] \rangle \geq \langle \Gamma(\eta), \eta' - \eta \rangle, \ \eta, \eta' \in H,$$

that is, $(\Phi, \xi(\cdot))$ is $\xi$-regular, with $\Gamma(\eta) = \Phi_\eta(\xi(\eta), \eta)$.

3.1.3 Main Result, Variational Inequality case

3.1.4 Preliminaries

Recall that the (Minty’s) variational inequality $VI(M, W)$ associated with a convex compact subset $W$ of Euclidean space $\mathcal{W}$ and a vector field $M : W \rightarrow \mathcal{W}$ is

$$\text{find } w \in W : \langle M(w'), w' - w \rangle \geq 0 \ \forall w' \in W \quad \text{VI}(M, W)$$

$w$ satisfying the latter condition is called a weak solution to the VI. A natural measure of inaccuracy for an approximate solution $w \in W$ to $VI(M, W)$ is the dual gap function

$$\epsilon_{VI}(w|M, W) = \sup_{w' \in W} \langle M(w'), w - w' \rangle;$$

weak solutions to the VI are exactly the points of $W$ where this (clearly nonnegative everywhere on $W$) function is zero.

In the sequel we utilize the following simple fact originating from [19]:

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5“a primal” instead of “the primal” reflects the fact that $\Psi$ is not uniquely defined by $\Phi$ – it is defined by $\Phi$ and $\eta$ and by how the values of $\Psi$ are selected when (31) does not specify these values uniquely.
Proposition 4. Let $M$ be monotone on $W$, let $\mathcal{I}_t = \{w_i \in W, M(w_i) : 1 \leq i \leq t\}$ be a $t$-step execution protocol associated with $(M, W)$, $\lambda$ be a $t$-step accuracy certificate, and $w^t = \sum_{i=1}^{t} \lambda_i w_i$ be the associated approximate solution. Then
\[ \epsilon_{\mathcal{V}I}(w^t|M, W) \leq \text{Res}(\mathcal{I}_t, \lambda|W). \]
Indeed, we have
\[
\text{Res}(\mathcal{I}_t, \lambda|W) = \sup_{w^t \in W} \left[ \sum_{i=1}^{t} \lambda_i \langle M(w_i), w - w^t \rangle \right]
\geq \sup_{w^t \in W} \left[ \sum_{i=1}^{t} \lambda_i \langle M(w'), w_i - w^t \rangle \right] \quad \text{[since $M$ is monotone]}
= \sup_{w^t \in W} \langle M(w'), w^t - w' \rangle = \epsilon_{\mathcal{V}I}(w^t|M, W).
\]

3.1.5 Main result

Proposition 5. In the situation of section 3.1.1, let $(\Phi, \eta(\cdot))$ be $\eta$-regular. Then
(i) Primal vector field $\Psi(\xi)$ induced by $(\Phi, \eta(\cdot))$ is monotone on $\Xi$. Moreover, whenever $\mathcal{I}_t = \{\xi_i \in \Xi, \Psi(\xi_i) : 1 \leq i \leq t\}$ and $\mathcal{J}_t = \{\theta_i := [\xi_i; \eta(\xi_i)], \Phi(\theta_i) : 1 \leq i \leq t\}$ and $\lambda$ is a $t$-step accuracy certificate, it holds
\[ \epsilon_{\mathcal{V}I}(\sum_{i=1}^{t} \lambda_i \theta_i|\Phi, \Theta) \leq \text{Res}(\mathcal{J}_t, \lambda|\Theta) \leq \text{Res}(\mathcal{I}_t, \lambda|\Xi). \quad (34) \]
(ii) Let $(\Phi, \xi)$ be $\xi$-regular, and let $\Gamma$ be the induced dual vector field. Whenever $\hat{\theta} = [\hat{\xi}; \hat{\eta}] \in \Theta$, we have
\[ \epsilon_{\mathcal{V}I}(\hat{\eta}|\Gamma, H) \leq \epsilon_{\mathcal{V}I}(\hat{\theta}|\Phi, \Theta). \quad (35) \]

3.2 Implications

In the situation of section 3.1.1, assume that for properly selected $\eta(\cdot), \xi(\cdot), (\Phi, \eta(\cdot))$ is $\eta$-regular, and $(\Phi, \xi(\cdot))$ is $\xi$-regular, induced primal and dual vector fields being $\Psi$ and $\Gamma$. In order to solve the dual VI $VI(\Gamma, H)$, we can apply to the primal VI $VI(\Psi, \Xi)$ an algorithm with accuracy certificates; by Proposition 5.i, resulting $t$-step execution protocol $\mathcal{I}_t = \{\xi_i, \Psi(\xi_i) : 1 \leq i \leq t\}$ and accuracy certificate $\lambda$ generate an execution protocol $\mathcal{J}_t = \{\theta_i := [\xi_i; \eta(\xi_i)], \Phi(\theta_i) : 1 \leq i \leq t\}$ such that
\[ \text{Res}(\mathcal{J}_t, \lambda|\Theta) \leq \text{Res}(\mathcal{I}_t, \lambda|\Xi), \]
whence, by Proposition 4, for the approximate solution
\[ \theta^t = [\xi^t, \eta^t] := \sum_{i=1}^{t} \lambda_i \theta_i = \sum_{i=1}^{t} \lambda_i [\xi_i; \eta(\xi_i)] \]
it holds
\[ \epsilon_{\mathcal{V}I}(\theta^t|\Phi, \Theta) \leq \text{Res}(\mathcal{I}_t, \lambda|\Xi). \]
Invoking Proposition 5.ii, we conclude that $\eta^t$ is a feasible solution to the dual VI $VI(\Gamma, H)$, and
\[ \epsilon_{\mathcal{V}I}(\eta^t|\Gamma, H) \leq \text{Res}(\mathcal{I}_t, \lambda|\Xi). \quad (36) \]
We are about to present two examples well suited for the just outlined approach.
3.2.1 Solving affine monotone VI on LMO-represented domain

Let $H$ be a convex compact set in $\mathcal{H} = \mathbb{R}^N$, and let $H$ be equipped with an LMO. Assume that we want to solve the VI $VI(F,H)$, where

$$F(\eta) = S\eta + s$$

is an affine monotone operator (so that $S + S^T \succeq 0$). Let us set $\mathcal{X} = \mathcal{H}$, select $\Xi$ as a proximal-friendly convex compact set containing $H$, and set $\Theta = \Xi \times H$,

$$\Phi(\xi, \eta) = \left[\begin{array}{cc} S^T & -S^T \\ S & S \end{array}\right] \left[\begin{array}{c} \xi \\ \eta \end{array}\right] + \left[\begin{array}{c} 0 \\ s \end{array}\right].$$

We have

$$S + S^T = \left[\begin{array}{c} S + S^T \\ S \end{array}\right] \succeq 0,$$

so that $\Phi$ is an affine monotone operator with

$$\Phi_\xi(\xi, \eta) = S^T\xi - S^T\eta$$

$$\Phi_\eta(\xi, \eta) = S\xi + s$$

Setting $\xi(\eta) = \eta$, we ensure that $\xi(\eta) \in \Xi$ when $\eta \in H$ and $\Phi_\xi(\xi(\eta), \eta) = 0$, implying (32). Since we are in the direct product case, we can set $\Gamma(\eta) = \Phi_\eta(\xi(\eta), \eta) = S\eta + s = F(\eta)$; thus, $VI(\Gamma, H)$ is our initial VI of interest. On the other hand, setting

$$\eta(\xi) \in \text{Argmin}_{\eta \in H} \langle S\xi + s, \eta \rangle,$$

we ensure (30). Since we are in the direct product case, we can set

$$\Psi(\xi) = \Phi_\xi(\xi, \eta(\xi)) = S^T[\xi - \eta(\xi)];$$

note that the values of $\Psi$ can be straightforwardly computed via calls to the LMO representing $H$. We can now solve $VI(\Psi, \Xi)$ by a proximal algorithm $B$ with accuracy certificates and recover, as explained above, approximate solution to the VI of interest $VI(F, H)$. With the Non-Euclidean Restricted Memory Level method with certificates [4] or Mirror Descent with certificates (see, e.g., [17]), the approach results in non-asymptotical $O(1/\sqrt{t})$-converging algorithm for solving the VI of interest, with explicitly computable factors hidden in $O(\cdot)$. This complexity bound, completely similar to the one obtained in [17], seems to be the best known under the circumstances.

3.2.2 Solving skew-symmetric VI on LMO-represented domain

Let $H$ be an LMO-represented convex compact domain in $\mathcal{H} = \mathbb{R}^N$, and assume that we want to solve $VI(F,H)$, where

$$F(\eta) = 2Q^TP\eta + f : \mathcal{H} \to \mathcal{H}$$

with $K \times N$ matrices $P, Q$ such that the matrix $Q^TP$ is skew-symmetric:

$$Q^TP + P^TQ = 0.$$  \hspace{1cm} (37)
Let $X = \mathbb{R}^K \times \mathbb{R}^K$, let $\Xi_1, \Xi_2$ be two convex compact sets in $\mathbb{R}^K$ such that
\[ QH \subset \Xi_1, -PH \subset \Xi_2. \tag{38} \]
Let us set $\Xi = \Xi_1 \times \Xi_2$, and let
\[
\Phi(\xi = [\xi_1; \xi_2], \eta) = \begin{bmatrix} I_K & P \\ -I_K & Q \\ -P^T & -Q^T \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \eta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f \end{bmatrix}.
\]
Note that $\Phi$ is monotone and affine. Setting $\xi(\eta) = [Q\eta; -P\eta]$ and invoking (38), we ensure (32); since we are in the direct product case, we can take, as the dual induced vector field,
\[
\Gamma(\eta) = \Phi(\xi(\eta), \eta) = -P^T(Q\eta) - Q^T(-P\eta) + f = [Q^TP - P^TQ]\eta + f = 2Q^TP\eta + f = F(\eta),
\]
so that the dual VI VI$(\Gamma, H)$ is our VI of interest.

On the other hand, setting
\[
\bar{\Xi}(\eta) = [Q\eta; -P\eta]
\]
and invoking (38), we ensure (30); since we are in the direct product case, we can take, as the dual induced vector field,
\[
\bar{\eta}(\xi = [\xi_1; \xi_2]) \in \text{Argmin} \langle f - P^T\xi_1 - Q^T\xi_2, \eta \rangle,
\]
we ensure (30). Since we are in the direct product case, we can define primal vector field as
\[
\Psi(\xi = [\xi_1; \xi_2]) = \Phi(\xi([\xi_1; \xi_2], \bar{\eta}([\xi_1; \xi_2]))) = \begin{bmatrix} \xi_2 + P\bar{\eta}(\xi) \\ -\xi_1 + Q\bar{\eta}(\xi) \end{bmatrix}.
\]
Note that LMO for $H$ allows to compute the values of $\Psi$, and that $\Xi$ can be selected to be proximal-friendly. We can now solve VI$(\Psi, \Xi)$ by a proximal algorithm $B$ with accuracy certificates and recover, as explained above, approximate solution to the VI of interest VI$(F, H)$. When the design dimension $\dim \Xi$ of the primal VI is small, other choices of $B$, like the Ellipsoid algorithm, are possible, and in this case we can end up with linearly converging, with the converging ratio depending solely on $\dim \Xi$, algorithm for solving the VI of interest. We are about to give a related example, which can be considered as multi-player version of the “Attacker vs. Defender” game.

**Example: Nash Equilibrium with pairwise interactions.** Consider the situation as follows: there are
- $L \geq 2$ players, $\ell$-th of them selecting a mixed strategy $w_\ell$ from probabilistic simplex $\Delta_{N_\ell}$ of dimension $N_\ell$,
- encoding matrices $D_\ell$ of sizes $m_\ell \times N_\ell$, and loss matrices $M^{\ell\ell'}$ of sizes $m_\ell \times m_{\ell'}$ such that
\[
M^{\ell\ell} = 0, M^{\ell\ell'} = -[M^{\ell\ell'}]^T, 1 \leq \ell, \ell' \leq L.
\]
• The loss of $\ell$-th player depends on mixed strategies of the players according to
\[
\mathcal{L}_\ell(\eta := [w_1; \ldots; w_L]) = \sum_{\ell' = 1}^{L} w_{\ell'}^T E^{\ell\ell'} w_{\ell'}, \quad E^{\ell\ell'} = D^{T}_\ell M^{\ell\ell'} D_{\ell'} + \langle g_\ell, \eta \rangle.
\]

In other words, every pair of distinct players $\ell, \ell'$ are playing matrix game with matrix $M^{\ell\ell'}$, and the loss of player $\ell$, up to a linear in $[w_1; \ldots; w_L]$ function, is the sum, over the pairwise games he is playing, of his losses in these games, the “coupling constraints” being expressed by the requirement that every player uses the same mixed strategy in all pairwise games he is playing.

We have described convex Nash Equilibrium problem, meaning that for every $\ell$, $\mathcal{L}_\ell(w_1, \ldots, w_L)$ is convex (in fact, linear) in $w_\ell$, is jointly concave (in fact, linear) in $w := (w_1, \ldots, w_{\ell-1}, w_{\ell+1}, \ldots, w_L)$, and $\sum_{\ell=1}^{L} \mathcal{L}_\ell(\eta)$ is the linear function $g, \eta \mapsto g = \sum_\ell g_\ell$, and thus is convex. It is known (see, e.g., [19]) that Nash Equilibria in convex Nash problem are exactly the weak solutions to the VI given by monotone operator
\[
F(\eta := [w_1; \ldots; w_L]) = [\nabla w_1 \mathcal{L}_1(\eta); \ldots; \nabla w_L \mathcal{L}_L(\eta)]
\]
on the domain
\[
H = \Delta_{N_1} \times \cdots \times \Delta_{N_L}.
\]

Let us set
\[
Q = \frac{1}{2} \begin{bmatrix} D_1 & D_2 & \cdots & D_L \\ \end{bmatrix}, \quad P = \begin{bmatrix} M^{1,1} D_1 & M^{1,2} D_2 & \cdots & M^{1,L} D_L \\
M^{2,1} D_1 & M^{2,2} D_2 & \cdots & M^{2,L} D_L \\
\vdots & \vdots & \ddots & \vdots \\
M^{L,1} D_1 & M^{L,2} D_2 & \cdots & M^{L,L} D_L 
\end{bmatrix}.
\]

Then
\[
Q^T P = \frac{1}{2} \begin{bmatrix} D^T_1 M^{1,1} D_1 & D^T_1 M^{1,2} D_2 & \cdots & D^T_1 M^{1,L} D_L \\
D^T_2 M^{2,1} D_1 & D^T_2 M^{2,2} D_2 & \cdots & D^T_2 M^{2,L} D_L \\
\vdots & \vdots & \ddots & \vdots \\
D^T_L M^{L,1} D_1 & D^T_L M^{L,2} D_2 & \cdots & D^T_L M^{L,L} D_L 
\end{bmatrix}
\]
so that $Q^T P$ is skew-symmetric due to $M^{\ell\ell'} = -[M^{\ell'\ell}]^T$. Besides this, we clearly have
\[
F(\eta := [w_1; \ldots; w_L]) = 2Q^T P \eta + f, \quad f = [\nabla w_1 \langle g_1, [w_1; \ldots; w_L] \rangle; \ldots; \nabla w_L \langle g_L, [w_1; \ldots; w_L] \rangle].
\]

Observe that if $D_1, \ldots, D_L$ are simple, so are $Q$ and $P$.

Indeed, for $Q$ this is evident: to find the column of $Q$ which makes the largest inner product with $x = [x_1; \ldots; x_L]$, dim $x_\ell = m_\ell$, it suffices to find, for every $\ell$, the column of $D_\ell$ which makes the maximal inner product with $x_\ell$, and then to select the maximal of the resulting $L$ inner products and the corresponding to this maximum column of $Q$. To maximize the inner product of the same $x$ with columns of $P$, note that
\[
x^T P = [\sum_{\ell=1}^{L} x^T_\ell M^{\ell,1} D_1, \ldots, \sum_{\ell=1}^{L} x^T_\ell M^{\ell,L} D_L],
\]
so that to maximize the inner product of $x$ and the columns of $P$ means to find, for every $\ell$, the column of $D_\ell$ which makes the maximal inner product with $y_\ell$, and then to select the maximal of the resulting $L$ inner products and the corresponding to this maximum column of $P$.  

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We see that if $D_\ell$ are simple, we can use the approach from section 3.2.2 to approximate the solution to the VI generated by $F$ on $H$. Note that in the case in question the dual gap function $\epsilon_{VI}(\eta|F,H)$ admits a transparent interpretation in terms of the Nash Equilibrium problem we are solving: for $\eta = [w_1;\ldots;w_L] \in H$, we have

$$\epsilon_{VI}(\eta|F,H) \geq \epsilon_{Nash}(\eta) := \sum_{\ell=1}^L \left[ \mathcal{L}_\ell(\eta) - \min_{w_\ell \in \Delta N_\ell} \mathcal{L}_\ell(w_1,\ldots,w_{\ell-1},w'_\ell,w_{\ell+1},\ldots,w_L) \right], \tag{39}$$

and the right hand side here is the sum, over the players, of the (nonnegative) incentives for a player $\ell$ to deviate from his strategy $w_\ell$ to another mixed strategy when all other players stick to their strategies as given by $\eta$. Thus, small $\epsilon_{VI}([w_1;\ldots;w_L],\cdot)$ means small incentives for the players to deviate from mixed strategies $w_\ell$.

Verification of (39) is immediate: denoting $f_\ell = \nabla_{w_\ell}(g_\ell,w)$, by definition of $\epsilon_{VI}$ we have for every $\eta' = [w'_1;\ldots;w'_L] \in H$:

$$\epsilon_{VI}(\eta|F,H) \geq \langle F(\eta'),\eta - \eta' \rangle = \sum_\ell \langle \nabla_{w_\ell} \mathcal{L}_\ell(\eta'),w_\ell - w'_\ell \rangle$$

$$= \sum_\ell \langle f_\ell,w_\ell - w'_\ell \rangle + \sum_{\ell,e} \langle D^T_\ell M^e \ell^T D_e w'_\ell,w_\ell - w'_\ell \rangle$$

$$= \sum_\ell \langle \nabla_{w_\ell} \mathcal{L}_\ell(\eta),w_\ell - w'_\ell \rangle = \sum_\ell \langle \mathcal{L}_\ell(\eta) - \mathcal{L}_\ell(w_1,\ldots,w_{\ell-1},w'_\ell,w_{\ell+1},\ldots,w_L) \rangle$$

and (39) follows.

### 3.3 Relation to [17]

Here we demonstrate that the decomposition approach to solving VI’s with monotone operators on LMO-represented domains cover the approach, based on Fenchel-type representations, developed in [17]. Specifically, let $H$ be a compact convex set in Euclidean space $\mathcal{H} = \mathbb{R}^N$, $G(\cdot)$ be a monotone vector field on $H$, and $\eta \mapsto Ax + a$ be an affine mapping from $\mathcal{H}$ to Euclidean space $\mathcal{X} = \mathbb{R}^M$. Given a convex compact set $\Xi \subset \mathcal{X}$, let us set

$$\Theta = \Xi \times H, \ 0(\xi,\eta) := \left[ 0(\xi),\eta \right] := Ax + a; \Phi(\xi,\eta) := G(\eta) - A^T \xi : \Theta \to \mathcal{X} \times \mathcal{H}, \tag{40}$$

so that $\Phi$ clearly is a monotone vector field on $\Theta$. Assume that $\mathcal{T}(\xi) : \Xi \to H$ is a somehow selected strong solution to VI$(\Phi(\xi,\cdot),H)$:

$$\forall \xi \in \Xi : \mathcal{T}(\xi) \in H \ & \ \langle G(\mathcal{T}(\xi)) - A^T \xi,\eta - \mathcal{T}(\xi) \rangle \geq 0 \ \forall \eta \in H; \tag{41}$$

(c.f. (30)); note that required $\mathcal{T}(\xi)$ definitely exists, provided that $G(\cdot)$ is continuous and monotone. Let us also define $\mathcal{T}(\eta)$ as a selection of the point-to-set mapping $\eta \mapsto \text{Argmin} \langle A\eta + a,\xi \rangle$, so that

$$\forall \eta \in H : \mathcal{T}(\eta) \in \Xi \ & \ \langle A\eta + a,\xi - \mathcal{T}(\eta) \rangle \geq 0 \ \forall \xi \in \Xi \tag{42}$$

\[ \text{and} \]
Observe that with the just defined $\Xi$, $H$, $\Theta$, $\Phi$, $\eta(\cdot)$, $\xi(\cdot)$ we are in the direct product case of the situation described in section 3.1.1. Since we are in the direct product case, $(\Phi, \eta(\cdot))$ is $\eta$-regular, and we can take, as the induced primal vector field associated with $(\Phi, \eta(\cdot))$, the vector field

$$\Psi(\xi) = A\eta(\xi) + a = \Phi_\xi(\xi, \eta(\xi)) : \Xi \to \mathcal{X}, \quad (43)$$

and as the induced dual vector field, the field

$$\Gamma(\eta) = G(\eta) - A^T\bar{\xi}(\eta) = \Phi_\eta(\bar{\xi}(\eta), \eta) : H \to \mathcal{X}.$$  

Note that in terms of [17], relations (43) and (41), modulo notation, form what in the reference is called a Fenchel-type representation (F.-t.r.) of vector field $\Psi : \Xi \to \mathcal{X}$, the data of the representation being $\mathcal{H}$, $A$, $a$, $\eta(\cdot)$, $G(\cdot)$, $H$. On a closer inspection, every F.-t.r. of a given monotone vector field $\Psi : \Xi \to \mathcal{X}$ can be obtained in this fashion from some setup of the form (40).

Assume now that $\Xi$ is LMO-representable, and we have at our disposal $G$-oracle which, given on input $\eta \in H$, returns $G(\eta)$. This oracle combines with LMO for $\Xi$ to induce a procedure which, given on input $\eta \in H$, returns $\Gamma(\eta)$. As a result, we can apply the decomposition machinery presented in sections 3.1, 3.2 to reduce solving VI($\Psi$, $\Xi$) to processing VI($\Gamma$, $H$) by an algorithm with accuracy certificates. It can be easily seen by inspection that this reduction recovers constructions and results presented in [17, sections 1 – 4]. The bottom line is that the developed in sections 3.1, 3.2 decomposition-based approach to solving VI’s with monotone operators on LMO-represented domains essentially covers the developed in [17] approach based on Fenchel-type representations of monotone vector fields.

References

[1] Argyriou, A., Signoretto, M., and Suykens, J. A. K. “Hybrid Conditional Gradient-Smoothing Algorithms with Applications to Sparse and Low Rank Regularization”, in Regularization, Optimization, Kernels, and Support Vector Machines, J. A. K. Suykens, M. Signoretto, and A. Argyriou editors, (2014), 53–82.

[2] Bellman, R. “On Colonel Blotto and analogous games” – SIAM Review 11 (1969), 66-68.

[3] Cox, B., “Applications of accuracy certificates for problems with convex structure” (2011) – Ph.D. Thesis, Georgia Institute of Technology

https://smartech.gatech.edu/jspui/bitstream/1853/39489/1/cox_bruce_a_201105_phd.pdf

[4] Cox, B., Juditsky, A., Nemirovski, A. (2013), “Dual subgradient algorithms for large-scale nonsmooth learning problems” – Mathematical Programming Series B (2013), Online First, DOI 10.1007/s10107-013-0725-1. E-print: arXiv:1302.2349.

[5] Demyanov, V., Rubinov, A. Approximate Methods in Optimization Problems – Elsevier, Amsterdam 1970.

[6] Dunn, J. C., Harshbarger, S. “Conditional gradient algorithms with open loop step size rules” – Journal of Mathematical Analysis and Applications 62:2 (1978), 432–444.

6 “covers” instead of “is equivalent” stems from the fact that the scope of decomposition is not restricted to the setups of the form (40)).
[7] Frank, M., Wolfe, P. “An algorithm for quadratic programming” – *Naval Res. Logist. Q.* 3:1-2 (1956), 95–110.

[8] Freund, R., Grigas, P. “New Analysis and Results for the Conditional Gradient Method” – *Mathematical Programming* (2014), http://dx.doi.org/10.1007/s10107-014-0841-6

[9] Garber, D., Hazan, E. “Faster Rates for the Frank-Wolfe Method over Strongly-Convex Sets” – to appear 32nd International Conference on Machine Learning (ICML 2015), arXiv:1406.1305 (2014).

[10] Gol’shtein, E.G. “Direct-Dual Block Method of Linear Programming” – *Avtomat. i Telemekh.* 1996 No. 11, 3–9 (in Russian; English translation: *Automation and Remote Control* 57:11 (1996), 1531–1536.

[11] Gol’shtein, E.G., Sokolov, N.A., Gol’shtein E.G., Sokolov N.A. "A decomposition algorithm for solving multicommodity production-and-transportation problem” – *Ekonomika i Matematicheskie Metody,* 33:1 (1997), 112-128.

[12] Grant, M., Boyd, S. CVX: Matlab software for disciplined convex programming, version 2.0 beta. http://cvxr.com/cvx, September 2013.

[13] Harchaoui, Z., Douze, M., Paulin, M., Dudik, M., Malick, J. “Large-scale image classification with trace-norm regularization” – In CVPR, 2012.

[14] Harchaoui, Z., Juditsky, A., Nemirovski, A. “Conditional Gradient Algorithms for Norm-Regularized Smooth Convex Optimization” – *Mathematical Programming.* DOI 10.1007/s10107-014-0778-9 Online first, April 18, 2014.

[15] Jaggi, M. “Revisiting Frank-Wolfe: Projection-free sparse convex optimization” – In ICML, 2013.

[16] Jaggi, M., Sulovský, M. “A simple algorithm for nuclear norm regularized problems” – In ICML, 2010.

[17] Juditsky, A., Nemirovski, A. “Solving Variational Inequalities with Monotone Operators on Domains Given by Linear Minimization Oracles” – *Mathematical Programming,* Online First, March 22, 2015
http://www.springerlink.com/openurl.asp?genre=article&id=doi:10.1007/s10107-015-0876-3

[18] Lan, G., Zhou, Y. (2014) “Conditional Gradient Sliding for Convex Optimization” – submitted to *SIAM Journal on Optimization.* http://www.ise.ufl.edu/glan/files/2015/09/CGS08-31.pdf

[19] Nemirovski, A., Onn, S., Rothblum, U., “Accuracy certificates for computational problems with convex structure” – *Mathematics of Operations Research,* 35 (2010), 52-78.

[20] Pierucci, F., Harchaoui, Z., Malick, J. “A smoothing approach for composite conditional gradient with nonsmooth loss”, http://hal.inria.fr/hal-01096630/document (2014).

[21] Pshenichnyi, B.N., Danilin, Y.M. *Numerical Methods in Extremal Problems.* Mir Publishers, Moscow 1978.

[22] Robertson, B. “The Colonel Blotto game” – *Economic Theory* 29:1 (2006), 1–24.
Appendix

A.1 Proof of Lemma 1

It suffices to prove the \( \phi \)-related statements. Lipschitz continuity of \( \phi \) in the direct product case is evident. Further, the function \( \theta(x_1, x_2; y_1) = \max_{y_2 \in Y_2[y_1]} \Phi(x_1, x_2; y_1, y_2) \) is convex and Lipschitz continuous in \( x = [x_1; x_2] \in X \) for every \( y_1 \in Y_1 \), whence \( \phi(x_1, y_1) = \min_{x_2 \in X_2[x_1]} \theta(x_1, x_2; y_1) \) is convex and lower semicontinuous in \( x_1 \in X_1 \) (note that \( X \) is compact). On the other hand, \( \phi(x_1, y_1) = \max_{y_2 \in Y_2[y_1]} \min_{x_2 \in X_2[x_1]} \Phi(x_1, x_2; y_1, y_2) = \max_{y_2 \in Y_2[y_1]} \chi(x_1; y_1, y_2) := \min_{x_2 \in X_2[x_1]} \Phi(x_1, x_2; y_1, y_2) \), so that \( \chi(x_1; y_1, y_2) \) is concave and Lipschitz continuous in \( y = [y_1; y_2] \in Y \) for every \( x_1 \in X_1 \), whence

\[
\phi(x_1, y_1) = \max_{y_2 \in Y_2[y_1]} \chi(x_1; y_1, y_2)
\]

is concave and upper semicontinuous in \( y_1 \in Y_1 \) (note that \( Y \) is compact).

Next, we have

\[
\begin{align*}
\text{SadVal}(\phi, X_1, X_2) &= \inf_{x_1 \in X_1} \left[ \sup_{y_1 \in Y_1} \left[ \sup_{y_2:[y_1;y_2] \in Y} \left[ \inf_{x_2:[x_1;x_2] \in X} \Phi(x_1, x_2; y_1, y_2) \right] \right] \right] \\
&= \inf_{x_1 \in X_1} \left[ \sup_{y_1 \in Y_1} \inf_{[y_1;y_2] \in Y} \sup_{x_2:[x_1; x_2] \in X} \Phi(x_1, x_2; y_1, y_2) \right] \\
&= \inf_{x_1 \in X_1} \left[ \sup_{x_2:[x_1;x_2] \in X} \inf_{[y_1;y_2] \in Y} \sup_{y_1 \in Y_1} \sup_{x_2:[x_1; x_2] \in X} \Phi(x_1, x_2; y_1, y_2) \right] \quad \text{[by Sion-Kakutani Theorem]} \\
&= \inf_{x_1 \in X_1} \sup_{x_2:[x_1;x_2] \in X} \Phi(x_1, x_2; y_1, y_2) = \text{SadVal}(\Phi, X, Y),
\end{align*}
\]

as required in (2). Finally, let \( \bar{x} = [\bar{x}_1; \bar{x}_2] \in X \) and \( \bar{y} = [\bar{y}_1; \bar{y}_2] \in Y \). We have

\[
\Phi(\bar{x}_1) - \text{SadVal}(\phi, X_1, Y_1) = \Phi(\bar{x}_1) - \text{SadVal}(\Phi, X, Y) \quad \text{[by (2)]} \\
= \sup_{y_1 \in Y_1} \phi(\bar{x}_1, y_1) - \text{SadVal}(\Phi, X, Y) \\
= \sup_{y_1 \in Y_1} \sup_{y_2:[y_1;y_2] \in Y} \inf_{x_2:[x_1; x_2] \in X} \Phi(\bar{x}_1, x_2; y_1, y_2) - \text{SadVal}(\Phi, X, Y) \\
= \sup_{[y_1;y_2] \in Y} \inf_{x_2:[x_1; x_2] \in X} \Phi(\bar{x}_1, x_2; y_1, y_2) - \text{SadVal}(\Phi, X, Y) \\
= \inf_{x_2:[x_1;x_2] \in X} \sup_{y= [y_1;y_2] \in Y} \Phi(\bar{x}_1, x_2; y) - \text{SadVal}(\Phi, X, Y) \\
\leq \sup_{y= [y_1;y_2] \in Y} \Phi(\bar{x}_1, \bar{x}_2; y) - \text{SadVal}(\Phi, X, Y) \\
= \Phi(\bar{x}) - \text{SadVal}(\Phi, X, Y)
\]
We conclude that

$$\text{SadVal}(\phi, X_1, Y_1) - \phi(\bar{y}_1) = \text{SadVal}(\Phi, X, Y) - \phi(\bar{y}_1) \quad [\text{by (2)}]$$

$$= \text{SadVal}(\Phi, X, Y) - \inf_{x_1 \in X_1} \phi(x_1, \bar{y}_1)$$

$$= \text{SadVal}(\Phi, X, Y) - \inf_{x_1 \in X_1} \left[ \inf_{x_2 : [x_1, x_2] \in X} \sup_{y_1, y_2 : [\bar{y}_1, \bar{y}_2] \in Y} \Phi(x_1, x_2; \bar{y}_1, y_2) \right]$$

$$= \text{SadVal}(\Phi, X_1, Y) - \inf_{x_1 \in X_1} \sup_{y_1, y_2 : [\bar{y}_1, \bar{y}_2] \in Y} \Phi(x; \bar{y}_1, y_2)$$

$$\leq \text{SadVal}(\Phi, X, Y) - \inf_{x_1 : [x_1, x_2] \in X} \Phi(x; \bar{y}_1, \bar{y}_2)$$

$$= \text{SadVal}(\Phi, X, Y) - \Phi(\bar{y}).$$

We conclude that

$$e_{\text{sad}}([\bar{x}_1; \bar{y}_1]|\phi, X_1, Y_1) = [\bar{\phi}(\bar{x}_1) - \text{SadVal}(\phi, X_1, Y_1)] + [\text{SadVal}(\phi, X_1, Y_1) - \phi(\bar{y}_1)]$$

$$\leq [\Phi(\bar{x}) - \text{SadVal}(\Phi, X, Y)] + [\text{SadVal}(\Phi, X, Y) - \Phi(\bar{y})] = e_{\text{sad}}([\bar{x}; \bar{y}]|\Phi, X, Y),$$

as claimed in (3). \qed

A.2 Proof of Lemma 2

For \(x_1 \in X_1\) we have

$$\phi(x_1; \bar{y}_1) = \min_{x_2 : [x_1, x_2] \in X} \max_{y_2 : [\bar{y}_1, y_2] \in Y} \Phi(x_1, x_2; \bar{y}_1, y_2) \geq \min_{x_2 : [x_1, x_2] \in X} \Phi(x_1, x_2; \bar{y}_1, \bar{y}_2)$$

$$\geq \min_{x_2 : [x_1, x_2] \in X} \Phi(\bar{x}; \bar{y}) + (G, [x_1; x_2] - [\bar{x}_1; \bar{x}_2]) \quad [\text{since } \Phi(x; \bar{y}) \text{ is convex and } G \in \partial_x \Phi(\bar{x}; \bar{y})]$$

$$\geq \phi(\bar{x}_1; \bar{y}_1) + \langle g, x_1 - \bar{x}_1 \rangle \quad [\text{by definition of } g, G],$$

as claimed in (a). “Symmetric” reasoning justifies (b). \qed

A.3 Proof of Lemma 3

Assume that (5) holds true. Then \(G\) clearly is certifying, implying that

$$\chi_G(\bar{x}_1) = \langle G, [\bar{x}_1; \bar{x}_2] \rangle,$$

and therefore (5) reads

$$\langle G, [x_1; x_2] \rangle \geq \chi_G(\bar{x}_1) + \langle g, x_1 - \bar{x}_1 \rangle \quad \forall x = [x_1; x_2] \in X,$$

where, taking minimum in the left hand side over \(x_2 \in X_2[x_1],\)

$$\chi_G(x_1) \geq \chi_G(\bar{x}_1) + \langle g, x_1 - \bar{x}_1 \rangle \quad \forall x_1 \in X_1,$$

as claimed in (ii).

Now assume that (i) and (ii) hold true. By (i), \(\chi_G(\bar{x}_1) = \langle G, [\bar{x}_1; \bar{x}_2] \rangle,\) and by (ii) combined with the definition of \(\chi_G,\)

$$\forall x = [x_1; x_2] \in X : \langle G, [x_1; x_2] \rangle \geq \chi_G(x_1) \geq \chi_G(\bar{x}_1) + \langle g, x_1 - \bar{x}_1 \rangle = \langle G, \bar{x} \rangle + \langle g, x_1 - \bar{x}_1 \rangle,$$

implying (5). \qed
A.4 Dynamic Programming generated simple matrices

Consider the situation as follows. There exists an evolving in time system $S$, with state $\xi_s$ at time $s = 1, 2, \ldots, m$ belonging to a given finite nonempty set $\Xi$. Further, every pair $(\xi, s)$ with $s \in \{1, \ldots, m\}$, $\xi \in \Xi$ is associated with nonempty finite set of actions $A_\xi$, and we set

$$S_s = \{(\xi, a) : \xi \in \Xi, a \in A_\xi\}.$$  

Further, for every $s$, $1 \leq s < m$, a transition mapping $\pi_s(\xi, a) : S_s \to \Xi_{s+1}$ is given. Finally, we are given vector-valued functions (“outputs”) $\chi_s : S_s \to \mathbb{R}^s$.

A trajectory of $S$ is a sequence $\{(\xi_s, a_s) : 1 \leq s \leq m\}$ such that $(\xi_s, a_s) \in S_s$ for $1 \leq s \leq m$ and

$$\xi_{s+1} = \pi_s(\xi_s, a_s), \quad 1 \leq s < m.$$  

The output of a trajectory $\tau = \{(\xi_s, a_s) : 1 \leq s \leq m\}$ is the block-vector

$$\chi(\tau) = [\chi(\xi_1, a_1); \ldots; \chi_m(\xi_m, a_m)].$$

We can associate with $S$ the matrix $D = D[S]$ with $K = r_1 + \ldots + r_m$ rows and with columns indexed by the trajectories of $S$; specifically, the column indexed by a trajectory $\tau$ is $\chi(\tau)$.

For example, knapsack generated matrix $D$ associated with knapsack data from section 2.6.2 is of the form $D[S]$ with system $S$ as follows:

- $\Xi$, $s = 1, \ldots, m$, is the set of nonnegative integers which are $\leq H$;
- $A_\xi$ is the set of nonnegative integers $a$ such that $a \leq \bar{p}_s$ and $\xi - h_s p_\xi \geq 0$;
- the transition mappings are $\pi_s(\xi, a) = \xi - ah_s$;
- the outputs are $\chi_s(\xi, a) = f_s(a), 1 \leq s \leq m$.

In the notation from section 2.6.2, vectors $[p_1; \ldots; p_m] \in \mathcal{P}$ are exactly the sequences of actions $a_1, \ldots, a_m$ stemming from the trajectories of the just defined system $S$.

Observe that matrix $D = D[S]$ is simple, provided the cardinalities of $\Xi$ and $A_\xi$ are reasonable. Indeed, given $x = [x_1; \ldots; x_m] \in \mathbb{R}^n = \mathbb{R}^{r_1} \times \ldots \times \mathbb{R}^{r_m}$, we can identify $\overline{D}[x]$ by Dynamic Programming, running first the backward Bellman recurrence

$$U_s(\xi) = \max_{a \in A_\xi} \{x_s^T \chi_s(\xi, a) + U_{s+1}(\pi_s(\xi, a))\}$$

$$A_s(\xi) = \text{Argmax}_{a \in A_\xi} \{x_s^T \chi_s(\xi, a) + U_{s+1}(\pi_s(\xi, a))\}, \quad \xi \in \Xi, s = m, m-1, \ldots, 1$$

(where $U_{m+1}(\cdot) \equiv 0$), and then identifying the (trajectory indexing the) column of $D$ corresponding to $\overline{D}[x]$ by running the forward Bellman recurrence

$$\xi_1 \in \text{Argmax}_{\xi \in \Xi} U_1(\xi) \Rightarrow a_1 \in A_1(\xi_1) \Rightarrow \ldots \Rightarrow \xi_{s+1} \in A_{s+1}(\xi_{s+1}) \Rightarrow \ldots, s = 1, 2, \ldots, m-1.$$
A.5 Attacker vs. Defender via Ellipsoid algorithm

In our implementation,

1. Relation (38) is ensured by specifying $U, V$ as centered at the origin Euclidean balls of radius $R$, where $R$ is an upper bound on the Euclidean norms of the columns in $D$ and in $A$ (such a bound can be easily obtained from the knapsack data specifying the matrices $D, A$).

2. We processed the monotone vector field associated with the primal SP problem (29), that is, the field

$$F(u, v) = [F_u(u, v) = \overline{A}[u] - v; F_v(u, v) = u - D[v]]$$

by Ellipsoid algorithm with accuracy certificates from [19]. For $\tau = 1, 2, \ldots$, the algorithm generates search points $[u_\tau; v_\tau] \in \mathbb{R}^K \times \mathbb{R}^K$, with $[u_1; v_1] = 0$, along with execution protocols $I^\tau = \{[u_i; v_i], F(u_i, v_i) : i \in I_\tau\}$, where $I_\tau = \{i \leq \tau : [u_i; v_i] \in U \times V\}$, augmented by accuracy certificates $\lambda^\tau = \{\lambda^\tau_i \geq 0 : i \in I_\tau\}$ such that $\sum_{i \in I_\tau} \lambda^\tau_i = 1$. From the results of [19] it follows that for every $\epsilon > 0$ it holds

$$\tau \geq N(\epsilon) := O(1)K^2 \ln \left(\frac{2R + \epsilon}{\epsilon} \right) \Rightarrow \text{Res}(I^\tau, \lambda^\tau|U \times V) \leq \epsilon. \quad (44)$$

3. When computing $F(u_i, v_i)$ (this computation takes place only at productive steps – those with $[u_i; v_i] \in U \times V$), we get, as a byproduct, the columns $A^i = \overline{A}[u_i]$ and $D^i = D[v_i]$ of matrices $A, D$, along with the indexes $a^i, d^i$ of these columns (recall that these indexes are pure strategies of Attacker and Defender and thus, according to the construction of $A, D$, are collections of $m$ nonnegative integers). In our implementation, we stored these columns, same as their indexes and the corresponding search points $[u_i; v_i]$. As is immediately seen, in the case in question the approximate solution $[w^\tau; z^\tau]$ to the SP problem of interest (26) induced by execution protocol $I^\tau$ and accuracy certificate $\lambda^\tau$ is comprised of two sparse vectors

$$w^\tau = \sum_{i \in I_\tau} \lambda^\tau_i \delta_{d^i}^D, \quad z^\tau = \sum_{i \in I_\tau} \lambda^\tau_i \delta_{a^i}^A \quad (45)$$

where $\delta_{d^i}^D$ is the “$d$-th basic orth” in the simplex $\Delta_N$ of probabilistic vectors with entries indexed by pure strategies of Defender, and similarly for $\delta_{a^i}^A$. Thus, we have no difficulties with representing our approximate solutions\(^7\), in spite of their huge ambient dimension.

According to our general theory and (44), the number of steps needed to get an $\epsilon$-solution $[w; z]$ to the problem of interest (i.e., a feasible solution with $\epsilon_{sad}([w; z]; \psi, W, Z) \leq \epsilon$) does not exceed $N(\epsilon)$, with computational effort per step dominated by the necessity to identify $\overline{A}[u_i], D[v_i]$ by Dynamic Programming.

In fact, we used the outlined scheme with two straightforward modifications.

- First, instead of building the accuracy certificates $\lambda^\tau$ according to the rules from [19], we used the best, given execution protocols $I^\tau$, accuracy certificates by solving the convex program

$$\min_{\lambda} \left\{ \text{Res}(I^\tau, \lambda|U \times V) := \max_{y \in U \times V} \sum_{i \in I_\tau} \lambda_i (F(u_i, v_i), [u_i; v_i] - y) : \lambda_i \geq 0, \sum_{i \in I_\tau} \lambda_i = 1 \right\} \quad (46)$$

\(^7\)Note that applying Caratheodory theorem, we could further “compress” the representations of approximate solutions – make these solutions convex combinations of at most $K + 1$ of $\delta_{d^i}^D$’s and $\delta_{a^i}^A$’s.
In our implementation, this problem was solved from time to time, specifically, once per $4K^2$ steps. Note that with $U, V$ being Euclidean balls, (46) is well within the scope of Matlab Convex Programming solver CVX [12].

- Second, given current approximate solution (45) to the problem of interest, we can compute its saddle point inaccuracy exactly instead of upper-bounding it by $\text{Res}(I^T, \lambda^T|U \times V)$. Indeed, it is immediately seen that

$$
\epsilon_{\text{sad}}([w^T; z^T]|\psi, W; Z) = \max(A^T[\sum_{i \in I_r} \lambda_i^r D_i^r]) - \min(D^T[\sum_{i \in I_r} \lambda_i^r A_i^r]).
$$

In our implementation, we performed this computation each time when a new accuracy certificate was computed, and terminated the solution process when the saddle point inaccuracy became less than a given threshold (1.e-4).

### A.6 Proof of Proposition 5

(i): Let $\xi_1, \xi_2 \in \Xi$, and let $\eta_1 = \overline{\eta}(\xi_1)$, $\eta_2 = \overline{\eta}(\xi_2)$. By (31) we have

$$
\langle \Psi(\xi_2), \xi_2 - \xi_1 \rangle \geq \langle \Phi(\xi_2, \eta_2), [\xi_2 - \xi_1; \eta_2 - \eta_1] \rangle
$$

$$
\langle \Psi(\xi_1), \xi_1 - \xi_2 \rangle \geq \langle \Phi(\xi_1, \eta_1), [\xi_1 - \xi_2; \eta_1 - \eta_2] \rangle
$$

Summing inequalities up, we get

$$
\langle \Psi(\xi_2) - \Psi(\xi_1), \xi_2 - \xi_1 \rangle \geq \langle \Phi(\xi_2, \eta_2) - \Phi(\xi_1, \eta_1), [\xi_2 - \xi_1; \eta_2 - \eta_1] \rangle \geq 0,
$$

so that $\Psi$ is monotone.

Further, the first inequality in (34) is due to Proposition 4. To prove the second inequality in (34), let $I_t = \{\xi_i \in \Xi, \Psi(\xi_i) : 1 \leq i \leq t\}$, $J_t = \{\theta_i := [\xi_i; \overline{\eta}(\xi_i)], \Phi(\theta_i) : 1 \leq i \leq t\}$, and let $\lambda$ be $t$-step accuracy certificate. We have

$$\theta = [\xi; \eta] \in \Theta \Rightarrow \sum_{i=1}^t \lambda_i \langle \Phi(\theta_i), \theta_i - \theta \rangle \leq \sum_{i=1}^t \lambda_i \langle \Psi(\xi_i), \xi_i - \xi \rangle \text{ [see (31)]}
$$

$$\leq \text{Res}(I_t, \lambda|\Xi)
$$

$$\Rightarrow \text{Res}(J_t, \lambda|\Theta) = \sup_{\theta = [\xi; \eta] \in \Theta} \sum_{i=1}^t \lambda_i \langle \Phi(\theta_i), \theta_i - \theta \rangle \leq \text{Res}(I_t, \lambda|\Xi).
$$

(i) is proved.

(ii): Let $\eta \in H$. Invoking (33), we have

$$
\langle \Gamma(\eta), \hat{\eta} - \eta \rangle \leq \langle \Phi([\overline{\xi}(\eta); \eta], [\hat{\xi}; \hat{\eta}] - [\overline{\xi}(\eta); \eta]) \rangle \leq \epsilon_{V_1}(\hat{\theta}|\Phi, \Theta),
$$

and (35) follows. \hfill \Box