Realizable spin models and entanglement dynamics in superconducting flux qubit systems

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Realizable spin models are investigated in a two superconducting flux qubit system. It is shown that a specific adjustment of system parameters in the two flux qubit system makes it possible to realize an artificial two-spin system that cannot be found naturally. For the artificial two-spin systems, time evolution of a prepared quantum state is discussed to quantify quantum entanglement dynamics. The concurrence and fidelity as a function of time are shown to reveal a characteristic entanglement dynamics of the artificial spin systems. It is found that the unentangled input state can evolve to be a maximally entangled output state periodically due to the exchange interactions induced by two-qubit flipping tunneling processes while single-qubit flipping tunneling processes plays a role of magnetic fields for the artificial spins.

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Introduction. Superconducting qubit systems as one of promising candidates have been paid much attentions for quantum information processing and computing. The tunable superconducting devices have provided a variety of possibilities to realize quantum spin models that are not findable naturally. Recent experiments have shown that different types of exchange interactions are observable. Particularly, there have been demonstrated an Ising type interaction in two charge qubits [1] and two flux qubits [2,3] and an XY type interaction in two phase qubits [4,5] as well as superconducting single qubits [6,7,8]. Moreover, such realizations of artificial spin systems make possible to observe entangled states of two qubits [1,2,4,5]. Indeed, for the time evolution of states in the experiments of charge [1] and phase qubits [4], a partial entanglement has been observed. An experiment of a capacitively coupled two phase qubits [5] shows that higher fidelity for the entanglement exhibits in an excited level. The higher fidelity is caused by two-qubit tunneling processes [9] between two qubit states, i.e., flipping both qubits. Such a two-qubit tunneling processes contributes exchange interactions between the two artificial spins.

In this paper, we will theoretically investigate a possible realization of quantum spin models in superconducting flux qubit systems by varying a system parameter. Especially, we use a phase coupling by introducing a connecting wire between the two qubit loops (see Fig. 1) because the phase coupling gives more controllable parameters than the inductive coupling with respect for the manipulation of qubit states [11]. It is shown that the Josephson junction in the connecting superconducting wires plays a role of controller in determining exchange interactions between the two qubits. In general, it is found that the two flux qubit system can map into an XYZ quantum spin model in the presence of magnetic fields. We show that specific values of system parameters generate various types of quantum spin models. Further, to address about time evolution of an input state for the two flux qubit system corresponding to quantum spin models, we introduce the concurrence and fidelity as a function of time as a measure of entanglement and evolution of the state. It turns out that an unentangled (entangled) input state evolves to be an entangled (unentangled) state periodically with a characteristic period of time.

Time evolution of quantum states. A system described by two quantum states can be a qubit. The two states can be represented in terms of pseudo-spin language, i.e., two orthogonal states |↑⟩ and |↓⟩. Then, any normalized pure state of two qubit systems can be written as a linear combination in the basis { |↑↑⟩, |↓↓⟩, |↑↓⟩, |↓↑⟩}:

\[ |ψ⟩ = a |↑↑⟩ + b |↑↓⟩ + c |↓↑⟩ + d |↓↓⟩. \] (1)

For two flux qubit systems, a given Hamiltonian \( H \) generates the time evolution of the state through the Schrödinger equation \( i\hbar \partial_t |ψ(t)⟩ = H |ψ(t)⟩ \). If the Hamiltonian \( H \) is independent of time, the time-dependent state is given by \( |ψ(t)⟩ = \exp \left( -\frac{iH}{\hbar} t \right) |ψ(0)⟩ \), where \( |ψ(0)⟩ = |ψ⟩ \) is a given state at the initial time. By virtue of the unitary transformation \( U \) making the Hamiltonian diagonal, at time \( t \), the state is given by

\[ |ψ(t)⟩ = G(t) |ψ(0)⟩, \] (2)

where the propagator is \( G(t) = U \exp \left[ -\frac{iHt}{\hbar} \right] U^† \) for the time evolution of the state.

To quantify entanglement for the time evolution of the state, we introduce the concurrence as the overlap between the state and the spin flipped state at a given time \( t \):

\[ C(ψ(t)) = \text{Tr} \left( σ_y |ψ(t)⟩⟨ψ(t)| \right), \] (3)

where the spin flipped state is given by \( |ψ'(t)⟩ = \sigma_z |ψ(t)⟩ \) with the pauli matrix \( σ_z \). The concurrence ranges from zero (unentangled state) to one (a maximally entangled state). To help understanding the entanglement dynamics, one can define the overlap between the states at the initial time (input state) and at a given time \( t \) (output state) as the fidelity:

\[ F(t) = \text{Tr} \left( |ψ(t)⟩⟨ψ(0)| \right). \] (4)
The Josephson energy of the junctions is given by the sum of the charging and Josephson energies: 

\[ H(T) = \sum_j \left( \frac{C}{2} \sin^2 \frac{\phi_j}{2} + 2E_J \sin^2 \frac{\phi_j}{2} \right), \]

where \( E_J \) is the Josephson energy associated with the geometry of the system is so small that the inductive energy is negligible. The Hamiltonian describing the model is given by the sum of the charging and Josephson energies:

\[ H(\{\phi_i, \phi_j, \phi_m\}) = H_C(\{\phi_i, \phi_j\}) + H_J(\{\phi_i, \phi_j\}), \]

where the phases across the Josephson junctions are \( \phi_i \) and their time derivatives are \( \dot{\phi}_i \). The charging energy of Josephson junctions in the two qubit loops and the connecting wire is given by

\[ H_C = \frac{1}{2} \left( \frac{\Phi_0}{2\pi} \right)^2 \sum_{j \neq k} \sum_{a \in \{a,b\}} C_{jk}^a \phi_j^a \phi_k^a \sin^2 \frac{\phi_j^a}{2} + C^c \phi_j^c \phi_j^c \]

where \( C^a(C^c) \) are the capacitance of the Josephson junctions in the qubit (connecting) loops. \( \Phi_0 = h/2e \) is the unit flux quantum. The Josephson energy of the junctions is given by

\[ H_J = \sum_{j \neq k} \sum_{a \in \{a,b\}} 2E_{Jj}^a \sin^2 \frac{\phi_j^a}{2} + 2E_{Jj}^c \sin^2 \frac{\phi_j^c}{2}, \]

where \( E_{Jj}^a \) are the Josephson energy of junctions in the qubit and connecting loops. For flux qubits, the charging energy is much smaller than the Josephson energy. The number of Cooper pairs \( n \) and the phase \( \phi \) are non-commuting variables, i.e., \([\phi, n] = i\), such that the canonical momentum \( p_\phi \) can be introduced as \( \frac{p_\phi}{\phi} = n = \frac{q}{2\pi} \) with the charge from the Josephson relation \( q = C(\Phi_0)/2\pi\phi \).

In the low energy limit, generally the Hamiltonian of superconducting flux qubit systems can be written in terms of the circulating currents in each qubit loop. For two flux qubit systems, following Ref. [2,13], one can write the two-qubit matrix Hamiltonian in terms of qubit energy levels, single-qubit tunnelings, and two-qubit tunnelings:

\[ H = \begin{pmatrix} E_{\uparrow\uparrow} & -t_1 & -t_2 & -t_2^c \\ -t_1 & E_{\downarrow\downarrow} & -t_1 & -t_1^c \\ -t_2 & -t_1 & E_{\uparrow\downarrow} & -t_1 \\ -t_2^c & -t_1^c & -t_1 & E_{\downarrow\uparrow} \end{pmatrix}, \]

where \( E_{\uparrow\uparrow} \) and \( E_{\downarrow\downarrow} \) are the energies for the two qubit states. The two qubit states correspond to the local minima \( \{\phi_i^0, \phi_m^0\} \) of the Josephson energy \( H_J(\{\phi_i\}) \) with \( E_{\uparrow\downarrow}^0 = E_{\downarrow\uparrow}^0 = \phi_i^0 \) and \( E_{\downarrow\downarrow}^0 = \phi_i^0 \).

By varying the amplitude of \( E_J \), the two flux qubit system can be mapped into a quantum two-spin model. The state of each qubit loop is in a superposed state of which \( |\downarrow\rangle \) and \( |\uparrow\rangle \) represent the diamagnetic and paramagnetic current states, respectively. This schematic of the system show the state \( |\uparrow\downarrow\rangle \) that is one of possible four states. Here, \( \bigotimes \) and \( \otimes \) denote the direction of the magnetic fields, \( f_{i(2)} = \Phi_{i(2)}/\Phi_0 \), in the qubit loops. \( E_{Ji}, E_J, E_J^c \) are the Josephson coupling energies of the Josephson junctions in the qubit loops and the superconducting connecting wire, and \( \varphi \)'s are phase differences across the Josephson junctions.

If \( F(T) = 1 \), the output quantum state is the same with the initial input state at \( t = T \), i.e., unentangled (entangled) initial state returns to unentangled (entangled) state. Then, for time evolution of quantum states, entanglement dynamics can be understood from the concurrence and fidelity.
form from Eq. (10):

\[
H = \sum_{j=1}^{N} \sum_{\sigma=\uparrow,\downarrow} B_j^\sigma \sigma_j^\sigma + \sum_{\alpha \neq \beta} J_{\alpha \beta} S_\alpha^\uparrow S_\beta^\downarrow,
\]

where \(B_j^\sigma = -t_1\), \(B_j^\downarrow = 0\), \(B_j^\uparrow = (E_{\uparrow \uparrow} + E_{\uparrow \downarrow} - E_{\downarrow \uparrow} - E_{\downarrow \downarrow})/4\), \(B_j^\downarrow = (E_{\uparrow \uparrow} - E_{\uparrow \downarrow} + E_{\downarrow \uparrow} - E_{\downarrow \downarrow})/4\), \(J_{\alpha \beta} = (t_2^\alpha + t_2^\beta)/2\), \(J_{\alpha \beta} = (t_2^\alpha - t_2^\beta)/2\), and \(J_{\alpha \beta} = (E_{\uparrow \uparrow} - E_{\uparrow \downarrow} + E_{\downarrow \uparrow} - E_{\downarrow \downarrow})/4\). The single qubit transitions play the role of a transverse magnetic field while the energy difference of two-qubit levels correspond to the applied magnetic field parallel to the z-direction of spins. Note that the x- and y-components of the exchange interaction are determined by the two-qubit tunnelings and the z-component of the interaction is the energy difference between the parallel spin state and the anti-parallel spin state. Consequently, Eq. (11) shows that an XYZ quantum spin model with magnetic fields can be realizable in any two flux qubit system [13]. We will discuss a specific realization of a quantum spin model with adjusted system parameters and entanglement dynamics for an input state.

Realizable artificial spin systems. Case I. For \(E_{\uparrow \uparrow} = 0.0 E_j\) and \(E_{\downarrow \downarrow} = 0.7 E_j\), a two-spin Hamiltonian can be constructed by the relations of \(E_{\uparrow \uparrow} = E_{\uparrow \downarrow} = E_{\downarrow \uparrow} = E_{\downarrow \downarrow}\) and \(t_2^\alpha = t_2^\beta = t_2\). The numerical values of the macroscopic quantum tunnelings are obtained as \(t_1 = 0.0075 E_j\) and \(t_2^{(0)} = 0.00024 E_j\). From Eq. (11) the two flux qubit system is described by the corresponding spin Hamiltonian:

\[
H = J S_1^\uparrow S_2^\downarrow + B (S_1^\uparrow + S_2^\downarrow),
\]

where \(B = -t_1\) and \(J = -t_2\). Note that the entanglement dynamics of this spin system is determined only by the single- and two-qubit tunneling amplitudes.

For the spin system, the concurrence is given by

\[
C(t) = \left[ C_0 + C_1 \cos 4Jt \right]^{1/2},
\]

where \(C_0 = [(a + d)^2 - (b + c)^2] / 4 + [(a - d)^2 - (b - c)^2] / 4\) and \(C_1 = -[(a + d)^2 - (b + c)^2] [(a - d)^2 - (b - c)^2] / 2\). It should be noticed that the concurrence does not depend on the magnetic field \(B = -t_1\), i.e., the single-qubit tunneling. The concurrence is an oscillating function with respect of the exchange interaction \(J = -t_2\) with the characteristic period of time \(T = \pi/2J\). This shows that any entangled state cannot be generated by applying the magnetic field in this quantum spin system. At \(t = (m + 1/4) \pi / 2J\) with an integer \(m\), the concurrence reaches its maximum value \(C(t) = 2 |a d - b c|\), i.e., a maximally entangled state. At \(t = \frac{1}{2} J \cos^{-1} \frac{1}{C_1}\), the entanglement disappears, i.e., the input state involves to be unentangled.

The fidelity of this quantum spin system is given by

\[
F(t) = \left[ F_0 + \sum_{\sigma=\uparrow,\downarrow} F_1^\sigma \cos 2(B + \sigma J)t + F_2 \cos 4Bt \right]^{1/2},
\]

where \(F_0 = 1 - [(a + d)^2 + (b + c)^2] [(a - d)^2 + (b - c)^2] / 2 - [(a + d)^2 - (b + c)^2] / 2 - (a + d)^2 - (b + c)^2] / 4\), \(F_1^\uparrow = (a + b + c + d)^2 [(a - d)^2 + (b - c)^2] / 4\), \(F_1^\downarrow = (a - b - c + d)^2 [(a - d)^2 + (b - c)^2] / 4\), \(F_2 = [(a + d)^2 - (b + c)^2] / 8\). For zero magnetic field, the quantum state evolves in time due to the exchange interaction. The fidelity has twice longer period of time than the concurrence.

Let us study entanglement when the initial input states are not in an entangled state. We choose the case of \(c = d = 0\), i.e., \(|\Phi(0)\rangle = a |\uparrow\uparrow\rangle + b |\uparrow\downarrow\rangle\), in which the initial state is a product state. It is clearly shown that the exchange interaction makes the artificial spins entangled in the concurrence \(C(t) = (2a^2 - 1) \sin 2Jt\). The period of the concurrence is \(T = \pi / J\).

For \(a = d = b = c\), the initial state can be written in the Bell basis \(|\Psi^\pm, \Phi^\mp\rangle\) with \(|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow \downarrow \rangle \pm |\downarrow \uparrow \rangle\rangle\) and \(|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow \downarrow \rangle \pm |\downarrow \uparrow \rangle\rangle\). The state is given by \(|\phi(t)\rangle = \sqrt{2} (a |\Phi^+\rangle + b |\Phi^-\rangle)\). Initially the input state is an entangled state quantified by the concurrence \(C(0) = 2(a^2 - b^2)\). For the time evolution of the state, the concurrence is not a function of time, i.e., \(C(t) = 2(a^2 - b^2)\). However, the fidelity is a function of time. It does not depend on the exchange interaction, i.e., \(F(t) = 2a^2 - 1\). This can be understood as follows. The initial state can be rewritten in the eigen basis: \(|\phi(t)\rangle = (a + b) |\psi_0\rangle + (a - b) |\psi_3\rangle\), where \(|\psi_0\rangle\) and \(|\psi_3\rangle\) are the ground state and the third excited state, respectively. At a given time \(t\), then, \(|\phi(t)\rangle = (a + b) e^{iE_0 t} |\psi_0\rangle + (a - b) e^{iE_3 t} |\psi_3\rangle\), where the energies for the ground state and he third excited state are \(E_0 = J = 2B\) and \(E_3 = J + 2B\). As a result, for the constant concurrence, the fidelity is oscillating in time.

Case II. For \(E_{\uparrow \uparrow} = 0.6 E_j\) and \(E_{\downarrow \downarrow} = 0.7 E_j\), one find the relations: \(E_{\uparrow \uparrow} = E_{\downarrow \downarrow} = E_{\downarrow \uparrow} = E_{\uparrow \downarrow}\) and \(t_2^{(0)} = 0.0 = 0\). The corresponding two spin system can be written as

\[
H = J (S_1^\uparrow S_2^\downarrow - S_1^\downarrow S_2^\uparrow) + J_{\alpha \beta} S_\alpha^\uparrow S_\beta^\downarrow,
\]

where \(J = -t_2^\uparrow / 2\) and \(J_{\alpha \beta} = (E_{\uparrow \uparrow} - E_{\downarrow \downarrow}) / 2\). The numerical values of the two-qubit tunneling amplitude and the energy difference between the two states are given by \(t_2^\uparrow = 0.00024 E_j\) and \(J_{\alpha \beta} = -0.425045 E_j\). The Hamiltonian describe an anisotropic spin exchange interaction belonging in the class of the XYZ spin model. Interestingly, the x- and z-components of the interaction are anti-ferromagnetic because \(J < 0\) and \(J_{\alpha \beta} < 0\) while the y-component is ferromagnetic. To our knowledge, it is unlikely to find a class of spin Hamiltonian naturally.

For the spin Hamiltonian, the concurrence is given by

\[
C(t) = \left[ C_0 + \sum_{\sigma=\uparrow,\downarrow} C_1^\sigma \cos 4(J + \sigma J_{\alpha \beta})t + C_2 \cos 8Jt \right]^{1/2},
\]

where \(C_0 = [(a + d)^4 + (a - d)^4] / 4 + 4b^2 c^2\), \(C_1^\uparrow = -2(a + d)^2 b c\), \(C_1^\downarrow = 2(a - d)^2 b c\), and \(C_2 = -(a^2 - d^2)^2 / 2\). The fidelity is given by

\[
F(t) = \left[ F_0 + \sum_{\sigma=\uparrow,\downarrow} F_1^\sigma \cos 2(J + \sigma J_{\alpha \beta})t + F_2 \cos 4Jt \right]^{1/2},
\]

where \(F_0 = 1 - 2(a^2 + d^2) (b^2 + c^2) - (a^2 - d^2)^2 / 2\), \(F_1^\uparrow = (a + d)^2 (b^2 + c^2)\), \(F_1^\downarrow = (a - d)^2 (b^2 + c^2)\), and \(F_2 = (a^2 - d^2)^2 / 2\).
shows that the concurrence and fidelity have a similar dynamic property. However, the fidelity has twice longer period than the concurrence.

Compared to the Case I, for the initial product state $|Ψ(0)| = a|↑↑⟩ + b|↓↓⟩$, the concurrence becomes $C(t) = a^2 \sin 4Jt$ with the period $T = \pi/2J$. The input state $|Ψ(t)| = \sqrt{2}(a|Φ^+⟩ + b|Ψ^+⟩)$ evolves and its concurrence is oscillating with $C(t) = 2\sqrt{a^4 + b^4 - 2a^2b^2 \cos 4Jt}$. Also, the fidelity is given by $F(t) = \sqrt{A_0^2 + A_1^2 \cos 4Jt}$, where $A_0 = 1 - 8a^2b^2$ and $A_1 = 8a^2b^2$. Thus, the Case I and II show a different characteristic entanglement dynamics depending on the realizable two artificial spin models in the flux qubit systems.

Case III. For $E'_J = 0.05E_J$ and $E_{11} = 0.7E_J$. The two-qubit energies have the relations: $E^*_{11} = E_{11}$, and $E^*_{1} = E_{1}$. The two-qubit tunneling becomes $r'^2_2 = 0$. We find another realization of a two spin system:

$$H = J(S^+_1 S^+_2 - S^-_1 S^-_2) + J'_2 S^+_1 S^+_2 + J_2 S^+_1 S^-_2 + B (S^+_1 + S^+_2),$$

where $B = -t_1$, $J = -r'^2_2/2$, and $J_2 = (E_{11} - E^*_{11})/2$. For the system parameters of the superconducting flux qubits, the single- and two-qubit tunneling amplitudes are $t_1 = 0.0024E_J$ and $r'^2_2 = 0.00024E_J$ and the energy difference is $J_2 = -0.05E_J$. The expressions of the concurrence and fidelity are too lengthy to display.

In Fig. 2, we plot the concurrences and fidelities as a function of time $t$ and the initial state parameter $θ$ to give the comparison of entanglement dynamics between the three different spin models for the same initial state $|Ψ(0)| = \cos 2πθ|↑↑⟩ + \sin 2πθ|↓↓⟩$. Explicitly, the different values of system parameters controlling the two flux qubits are given in the captions of the figures. For the time evolution of the initial state, it is shown that unentangled (entangled) state can become entangled (unentangled) state even though the specification of the superconducting devices are different each other.

**Summary.** A two superconducting flux qubit system has been considered to investigate a possible realization of quantum spin models. Three different artificial spin models were demonstrated by varying controllable system parameters. The realizable spin models in the flux qubit system are not likely to find naturally. We discussed the entanglement dynamics of the artificial spin models in the specific parameter values of the two superconducting flux qubit system. It was found that the input unentangled (entangled) state can become maximally entangled (unentangled) state irrespective of the specifications of the superconducting devices. Such a maximally entangled state should be observable experimentally.

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