ON THE SOLUTIONS OF THE DIOPHANTINE EQUATION

\[(x - d)^2 + x^2 + (x + d)^2 = y^n\] FOR D A PRIME POWER

ANGELOS KOUTSIANAS

ABSTRACT. We study the Diophantine equation \((x - d)^2 + x^2 + (x + d)^2 = y^n\) when \(n \geq 3\) and \(d = p^k, p\) a prime, using the characterization of primitive divisors on Lehmer sequences.

1. Introduction

The question when a sum of consecutive powers is a perfect power has a long and rich history. In 1875 Lucas [Luc75] asks for the solutions of equation

\[(1) \quad 1^2 + 2^2 + \cdots + x^2 = y^2\]

and it is until Watson [Wat18] gives a satisfied solution of the problem. In 1956, Schäffer [Sch56] generalises Lucas question and studies the equation

\[(2) \quad 1^k + 2^k + \cdots + x^k = y^n\]

where he proves that (2) has only finitely many solutions except from a finite number of pairs \((k, n)\) which he determines. In [BGP04] the authors complete solve (2) for \(k \leq 11\) while in [Pin07] the case \(n\) is even and \(k \leq 170\) odd is studied.

The last few years many mathematicians have focused on the most general equation

\[(3) \quad x^k + (x + d)^k + \cdots + (x + (r - 1)d)^k = y^n, \quad x, y, d, r, k, n \in \mathbb{Z}, n \geq 2.\]

and many specific cases have been studied. For example, the case \(k = 3, d = 1\) and \(r \leq 50\) is considered in [BPS16] and for \(k = 2, d = 1\) and \(r \leq 10\) in [Pat17]. Moreover, Bai and Zhang [ZB13] solve (3) for \(k = 2, d = 1\) and \(r = x + 1\) and in [Soy17] and [BPSS18] the authors are able to bound \(n\) when the last term of the sum is \(\ell\)-times the first one in terms of \(\ell\) and \(k\) under some conditions for \(\ell\). Finally, Patel and Siksek [PS17] prove that (3) has no solutions for \(k\) being even for almost all \(r \geq 2\).

Among the many different cases of (3) the case of three powers

\[(4) \quad (x - d)^2 + x^2 + (x + d)^2 = y^n\]

has attracted a lot of attention. Equation (4) has been studied for small values of \(k\) and \(d = 1\) in [BPS16] and [Zha14] and for \(d > 1\) in [Zha17] and [AGP17]. In this paper we consider the case \(k = 2\) of (4) and we study the equation

\[(5) \quad (x - d)^2 + x^2 + (x + d)^2 = y^n\]

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A very natural and interesting question is to study equation (5) for an infinitely family of d. Very recently Zhang [Zha17] proved that (4) has no solutions for \( k = 4 \) and \( n \geq 11 \) when \( d \) lies in a suitable infinitely family. In this paper we solve (5) when \( d \) is a prime power. A solution of (5) is called primitive if \((x, y) = 1\). Moreover, a solution is called non-trivial if \( x \neq 0 \).

**Theorem 1.1.** Let \( n \geq 3 \) be an integer. The non-trivial primitive solutions of (5) where \( d = p^b \) with \( b \geq 0 \) and \( p \leq 5000 \) are the ones in Table 1.

We have used the mathematical software package Sage [Dev17] for the computations in this paper. The code can be found at https://sites.google.com/site/angeloskoutsianas/research/code.

| \( p \) | \((|x|, y, b, n)\) |
|---|---|
| 7 | \((3, 5, 1, 3)\) |
| 79 | \((63, 29, 1, 3)\) |
| 223 | \((345, 77, 1, 3)\) |
| 439 | \((987, 149, 1, 3)\) |
| 727 | \((2133, 245, 1, 3)\) |
| 1087 | \((3927, 365, 1, 3)\) |
| 3109 | \((627, 29, 1, 5)\) |
| 3967 | \((27657, 1325, 1, 3)\) |
| 4759 | \((36363, 1589, 1, 3)\) |

**Table 1.** Non-trivial primitive solutions \((|x|, y, b, n)\).

2. **Lucas–Lehmer Sequences**

The characterization of primitive divisors of Lucas–Lehmer sequences in [BHV01] is used to prove Theorem 1.1. We have to recall the main definitions and terminology about Lehmer sequences and we recommend [BHV01] for a more detailed exposition.

Let \( \alpha, \beta \) be two algebraic integers such that \( (\alpha + \beta)^2 \) and \( \alpha \beta \) are non-zero coprime rational integers and \( \alpha/\beta \) is not a root of unity. Then the Lehmer sequence associated to the Lehmer pair \((\alpha, \beta)\) is

\[
\tilde{u}_n = \tilde{u}_n(\alpha, \beta) = \begin{cases} 
\frac{\alpha^n - \beta^n}{\alpha - \beta}, & n \text{ odd,} \\
\frac{\alpha^{n/2} - \beta^{n/2}}{\alpha^{1/2} - \beta^{1/2}}, & n \text{ even.}
\end{cases}
\]

**Definition 2.1.** Let \((\alpha, \beta)\) be a Lehmer pair. A prime number \( p \) is called primitive divisor of \( \tilde{u}_n(\alpha, \beta) \) if \( p \) divides \( \tilde{u}_n \) but does not divide \( (\alpha^2 - \beta^2)^2 \cdot \tilde{u}_1 \cdots \tilde{u}_{n-1} \).

In case \( \tilde{u}_n \) has no primitive divisors then the pair \((\alpha, \beta)\) is called \( n \)-defective Lehmer pair. We say that an integer \( n \) is totally non-defective if no Lehmer pair is \( n \)-defective.

**Theorem 2.1** ([BHV01]). Every integer \( n > 30 \) is totally non-defective, and for all prime \( n > 7 \).

**Definition 2.2.** Two Lehmer pairs \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) are equivalent if \( \alpha_1/\beta_1 = \beta_1/\beta_2 \in \{\pm 1, \pm \sqrt{-1}\} \).

For the integers \( 1 \leq n \leq 30 \) the \( n \)-defective Lehmer pairs are completely described (up to equivalence) in [Vou95] (see [BHV01, Theorem C]) and [BHV01, Theorem 1.3].
3. A Lehmer sequence from primitive solutions of (5)

We suppose \( n \geq 3 \). We can rewrite (5) as

\[
3x^2 + 2d^2 = y^n
\]

Let \((x, y)\) be a non–trivial primitive solution of (5). This also implies that \( x, y, d \) are pairwise coprime. We rewrite equation (7) as

\[
(3x)^2 + 6d^2 = 3y^n,
\]

Let \( K = \mathbb{Q}(\sqrt{-6}) \) and write \( \mathcal{O}_K = \mathbb{Z}[\sqrt{-6}] \) for its ring of integers. This has class group isomorphic to \((\mathbb{Z}/2\mathbb{Z})\). We factorise the left-hand side of equation (8) as

\[
(3x + d\sqrt{-6})(3x - d\sqrt{-6}) = 3y^n.
\]

It follows that

\[
(3x + d\sqrt{-6})\mathcal{O}_K = p_3 \cdot 3^n
\]

where \( p_3 \) is the unique prime of \( \mathcal{O}_K \) above 3 and 3 is an ideal of \( \mathcal{O}_K \). We have that \( p_3 \) is not principal.

**Lemma 3.1.** There are no non–trivial primitive solutions of (5) for \( n \) even.

**Proof.** This is an immediate consequence of (9) and the fact that \( p_3 \) is not principal ideal. \( \square \)

Let assume that \( n \) is an odd prime. Since \( p_3 \) is not principal, 3 is not either, and because \( p_3^2 = 3 \) we have,

\[
(3x + d\sqrt{-6})\mathcal{O}_K = p_3^{1-n} \cdot (p_33)^n = (\sqrt{3})^n.
\]

It follows that \( p_33 \) is a principal ideal. Write \( p_33 = (\gamma)\mathcal{O}_K \) where \( \gamma = u' + v'\sqrt{-6} \in \mathcal{O}_K \) with \( u', v' \in \mathbb{Z} \). We can easily prove that \( 3 | u' \). Let \( u' = 3u \) and \( v' = v \). Then we have \( \gamma = 3u + v\sqrt{-6} \). After possibly changing the sign of \( \gamma \) we obtain,

\[
3x + d\sqrt{-6} = \frac{\gamma^n}{3^{(n-1)/2}}.
\]

Subtracting the conjugate equation from this equation, we obtain

\[
\frac{\gamma^n}{3^{(n-1)/2}} - \frac{\gamma^n}{3^{(n-1)/2}} = 2d\sqrt{-6},
\]

or equivalently,

\[
\frac{\gamma^n}{3^{n/2}} - \frac{\gamma^n}{3^{n/2}} = 2d\sqrt{-2}.
\]

Let \( L = \mathbb{Q}(\sqrt{-6}, \sqrt{3}) = \mathbb{Q}(\sqrt{-2}, \sqrt{3}) \). Write \( \mathcal{O}_L \) for the ring of integers of \( L \) and let

\[
\alpha = \frac{\gamma}{\sqrt{3}} \quad \text{and} \quad \beta = \frac{\gamma}{\sqrt{3}}.
\]

**Lemma 3.2.** Let \( \alpha, \beta \) be as above. Then, \( \alpha \) and \( \beta \) are algebraic integers. Moreover, \((\alpha + \beta)^2\) and \(\alpha \beta\) are non–zero coprime rational integers and \(\alpha/\beta\) is not a unit.
Proof. Let \( \gamma = 3u + v\sqrt{-6} \) be as above with \( u, v \in \mathbb{Z} \). Then
\[
(\alpha + \beta)^2 = 12u^2.
\]
So, \((\alpha + \beta)^2\) is a rational integer. If \((\alpha + \beta)^2 = 0\) then we have \( u = 0 \). However, from (12) and the fact that \( n \) is odd we understand that this can not happen. Clearly, \( \alpha\beta = \frac{\gamma}{3} \) is a non–zero rational integer.

We have to check that \((\alpha + \beta)^2\) and \(\alpha\beta\) are coprime. Suppose they are not coprime. Then there exist a prime \( q \) of \( \mathcal{O}_L \) dividing both. Then \( q \) divides \( \alpha, \beta \) and from equations (10) and (12) we understand that \( q \) divides \((y)\mathcal{O}_L\) and \((2d\sqrt{-2})\mathcal{O}_L\) which is a contradiction to the fact that \((x, y)\) is a non–trivial primitive solution, equivalent \( \gcd(y, r) = 1 \).

Finally, we need to show that \(\alpha/\beta = \frac{\gamma}{\bar{\gamma}} \in \mathcal{O}_K\) is not a root of unity. Since the only roots of unity in \( K \) are \( \pm 1 \) we conclude \( \gamma = \pm \bar{\gamma} \). Then, either \( v = 0 \) or \( u = 0 \) which both can not be hold because of (12).

\( \square \)

From Lemma 3.2 we have that the pair \((\alpha, \beta)\) is a Lehmer pair and we denote by \( \tilde{u}_k \) the associate Lehmer sequence. Substituting into equation (12), we see that
\[
(\alpha - \beta) \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right) = d.
\]
Hence, we get:
\[
\frac{\alpha^n - \beta^n}{\alpha - \beta} = d/v = d'.
\]
We understand that \( v \mid d \). We define
\[
f_n(x, y) = \sum_{i=0}^{(n-1)/2} \binom{n}{2i+1} (-2)^i 3^{n-1-i} x^{n-1-2i} y^{2i}.
\]
After an elementary calculation we have
\[
\tilde{u}_n = f_n(u, v)
\]

4. Proof of Theorem 1.1

We use the Lehmer sequence of Section 2 to prove Theorem 1.1.

**proof of Theorem 1.1** Let \((x, y)\) be a primitive solution of (7) for \( n \) an odd prime and \( d = p^b \) with \( p \neq 2, 3 \). Let \( K = \mathbb{Q}(\sqrt{-6}) \). We recall from Section 2 that there exists \( \gamma = 3u + v\sqrt{-6} \in \mathcal{O}_K \) with \( u, v \in \mathbb{Z} \) such that
\[
3x + p^b\sqrt{-6} = \frac{\gamma^n}{3(n-1)/2},
\]
and the elements
\[
\alpha = \frac{\gamma}{\sqrt{3}} \quad \text{and} \quad \beta = \frac{\bar{\gamma}}{\sqrt{3}}
\]
define Lehmer sequence \( \tilde{u}_k \). It holds that
\[
\tilde{u}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = p^b/v = p^{b'}.
\]
We consider the following cases:

**Case** \( b = 0 \): The equation \( 3x^2 + 2 = y^n \) is a special case of the following lemma due to Nagell [Nag55].
Lemma 4.1. Let $D \geq 3$ be an odd number. Then the equation

\begin{equation}
2 + Dx^2 = y^n, \quad n > 2
\end{equation}

has no integer solutions $(x, y, n)$ with $n \nmid h(-2D)$ where $h(-2D)$ is the class number of $\mathbb{Q}(\sqrt{-2D})$.

In our case we have $D = 3$ and $h(-6) = 2$, so in general there are no solutions of \ref{eq:22} for $b = 0$.

Case $v = p^b$: Then we have that the Lehmer sequence $\tilde{a}_n$ is $n$–defective, because $p \mid (\alpha^2 - \beta^2)$, and so by [BV01] $n \leq 30$. From Lemma 4.4 we understand that there are no solutions for $n \leq 30$.

Case $1 < v < p^b$: From the definition of $f_n$ we can see that this can not happen unless $n = p$ and $v = p^{b-1}$. So, we have to solve the equation

\begin{equation}
\Delta(u) = \left( \frac{p}{2i+1} \right) 2^{2i+1} \cdot 3^\frac{p-1}{2} i^{-2i} u^{p-1-2i} (p')^{2i} - \left( \frac{p}{2i+3} \right) 2^{2i+1} 3^\frac{p-1}{2} i^{-2i} u^{p-1-2i} (p')^{2i} \Rightarrow
\end{equation}

\begin{equation}
\Delta_i(u) = 2^3 3^\frac{p-1}{2} i^{-2i} u^{p-1-2i} (p')^{2i} \left( 3 \left( \frac{p}{2i+1} \right) u^2 - 2 \left( \frac{p}{2i+3} \right) p^2 \right)
\end{equation}

Let choose a $u_0$ such that $\Delta_i(u) \geq 0$ for any $|u| \geq u_0$ for every $i$. Since the constant term of $f_p(u, p^{b-1})$ is $2^{2p+1} (p^{b-1})^{p-1} > p$ we have that $f_p(u, p^{b-1}) > p$ for $|u| \geq u_0$. Thus \ref{eq:22} has no solutions. From the above the constant $u_0$ is easily computable. \hfill \square

Case $v = 1$: For this case we need the following lemma.

Lemma 4.3. Let

\[
B = \begin{cases} 
    p - 1 & \text{if } \left( \frac{-6}{p} \right) = 1 \\
    p + 1 & \text{if } \left( \frac{-6}{p} \right) = -1.
\end{cases}
\]

Let

\[
B_p := \max (7, B).
\]

Then $n \leq B_p$. 
Proof. Recall that the exponent $n$ is an odd prime. Suppose $n > 7$. By the theorem of Bilu, Hanrot and Voutier, $\tilde{u}_n = (\alpha^n - \beta^n)/(\alpha - \beta) = p^b$ is divisible by $p$ while $p$ divides neither $(\alpha^2 - \beta^2)^2 - 96u^2v^2$ nor the terms $\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{n-1}$. Note that $p$ does not divide $6v$. Let $p$ be a prime of $K = \mathbb{Q}(\sqrt{-6})$ above $p$. As $(\alpha + \beta)^2$ and $\alpha\beta$ are coprime integers, and as $\alpha, \beta$ satisfy (20), we see that $\gamma, \overline{\gamma}$ are not divisible by $p$. We claim the multiplicative order of the reduction of $\gamma/\overline{\gamma}$ modulo $\mathbb{F}_p$ divides $B_p$.

If $-6$ is a square modulo $p$, then $\mathbb{F}_p = \mathbb{F}_p$ and so the multiplicative order divides $p - 1 = B_p$. Otherwise, $\mathbb{F}_p = \mathbb{F}_p^*$. However, $\gamma/\overline{\gamma}$ has norm 1, and the elements of norm 1 in $\mathbb{F}_p^*$ form a subgroup of order $p + 1 = B_p$. Thus in either case

$$ (\gamma/\overline{\gamma})^{B_p} \equiv 1 \pmod{p} $$

This implies that $p \mid \tilde{u}_n$. As $p$ is primitive divisor of $\tilde{u}_n$ we see that $n \leq B_p$, proving the lemma. □

From Lemma 4.3 we know that $n \leq b_p$. For the values of $n \leq B_p$ we have to solve the equation

$$ p^b = f_n(u, 1) \tag{24} $$

For the rest when we write $f_n(u)$ we mean $f_n(u, 1)$.

In general we do not expect solutions of (24) for big $n$ and we prove that by showing that there are no solutions of the congruence equation

$$ p^b \equiv f_n(u) \pmod{s} \tag{25} $$

for $s = p, p \pm 1$. This elementary criterion works for almost all cases. However, there are pairs $(p, n)$ for which it does not work. For these cases we do the following. We pick an integer $s$ coprime to $p$ and let $t_s$ be the order of $p$ at $\mathbb{Z}_s^*$. We define

$$ W_{s,p}(f_n) := \{p^k \mod{s} : k = 1, \ldots, t_s - 1\} \bigcap \{f_n(i) \mod{s} : i \in [1, s]\} \tag{26} $$

If $W_{s,p}(f_n) = \emptyset$ then we can conclude that $t_s \mid t$. Using many different $s$ and taking lcm of the $t_s$ we can find a number $t$ such that if there exists a solutions of (24) then we have $t \mid b$. Then we check if there exists an integer $l$ such that $p^l \equiv 1 \pmod{l}$ but $f_n(u) \not\equiv 1 \pmod{l}$ for all $u \in \mathbb{Z}$. If that holds then we have proved that (24) has no solutions. In practice, this method works for all $n \geq 5$ apart from $(p, n) = (3109, 5)$.

For $n = 3, 5$ the problem can be reduced to the problem of solving a certain $S$–unit equation\footnote{It can also be reduced to the problem of computing integral points on the elliptic curves $9y^2 - 2 = cx^3$.} Let consider the case $n = 3$. We have that $f_3(u) = 9u^2 - 2$. We define $L = \mathbb{Q}(\sqrt{2})$ and $\epsilon = 1 + \sqrt{2}$ be a generator of the free part of the unit group of $L$. Then for a prime $p$ such that $\left(\frac{2}{p}\right) = 1$ let $w$ be a generator of a prime ideal $p$ in $L$ such that $p \mid p$. For an element $x \in L$ we denote by $\bar{x}$ its conjugate. Then we can prove that

$$ 3x - \sqrt{2} = (-1)^{b_1}e^{b_1}w^b. \tag{27} $$

Taking conjugate and subtracting we have

$$ 1 = (-1)^{b_1}e^{b_1}w^b(-\sqrt{2})^{-3} + (-1)^{b_1}e^{b_1}w^b(-\sqrt{2})^{-3} \tag{28} $$

for $c = 1, p, p^2$. However, for big $p$ it is hard to compute the integral points on the elliptic curve.\footnote{The class number of $L$ is 1.}
For $n = 5$ we have $f_5(u) = 45u^4 - 60u^2 + 4$, so $(15u^2 - 10)^2 - 80 = 5p^b$. Similar to the case $n = 3$ and working over $N = \mathbb{Q}(\sqrt{5})$ for $p \neq 2, 5$ we have

\[(15u^2 - 10) - 4\sqrt{5} = (-1)^{b_0} \epsilon^{b_1} 5^{b_2} w^b\]

where $b_0, b_1, b_2, b \in \mathbb{Z}$, $\epsilon = (1 + \sqrt{5})/2$ and $w$ a generator\(^3\) of a prime $p$ in $N$ above $p$. Taking conjugate and subtracting we end up to the following $S$–unit equation

\[1 = (-1)^{b_0} \epsilon^{b_1} \sqrt{5}^{b_2} w^{b-3} - (-1)^{b_0} \epsilon^{b_1} (-\sqrt{5})^{b_2} w^{b-3}.\]

Using standard and well–known algorithms (see [Sma98], [Sma95], [TDW89], [TDW92]) we can find an upper bound for $b$ in (28) and (30). Since we have the upper bound for $b$ we can compute $u$ looking for integer solutions of (24). In case the upper bound is very big we can use ideas from [Sma99] and [Kou17, Section 6.4] to reduce the number of candidates for $b$.

We have written a Sage script that does all the above computations. Finally, the complete list of primitive solutions of (5) with $d = p^b$ and $p \leq 5000$ are those in table [1].

**Lemma 4.4.** There are no $n$–defective pairs for the Lehmer pair $(\alpha, \beta)$ where $\alpha = \frac{2}{\sqrt{3}}$ and $\beta = \frac{2}{\sqrt{5}}$ for $v = p^b$.

**Proof.** It holds that $\alpha = \frac{2}{\sqrt{3}} = \sqrt{3}u + v\sqrt{-2}$ and $\beta = \sqrt{3}u - v\sqrt{-2}$. So, $(\alpha + \beta)^2 = 12u^2$ and $(\alpha - \beta)^2 = -8v^2$. Then from the definition of equivalence Lehmer pairs we understand that the pair $(\pm 12u^2, \mp 8v^2)$ has to be in tables 2 or 4 in [BHV01] (see also table 2 in [You95]). This never happens for $n > 5$.

Let consider the case $n = 3$. Because $v = p^b$ and $f_3(x, y) = 9x^2 - 2u^2$ the equation $f_3(u, p^b) = 1$ has never an integer solution (taking the equation mod 4). Similarly for $n = 5$. \hfill $\square$

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