THE GENERIC GRADIENT-LIKE STRUCTURE OF CERTAIN ASYMPTOTICALLY AUTONOMOUS SEMILINEAR PARABOLIC EQUATIONS

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Abstract. We consider asymptotically autonomous semilinear parabolic equations
\[ u_t + Au = f(t, u). \]
Suppose that \( f(t, .) \to f^{\pm} \) as \( t \to \pm \infty \), where the semiflows induced by
(*)
\[ u_t + Au = f^{\pm}(u) \]
are gradient-like. Under certain assumptions, it is shown that generically with respect to a perturbation \( g \) with \( g(t) \to 0 \) as \( |t| \to \infty \), every solution of
\[ u_t + Au = f(t, u) + g(t) \]
is a connection between equilibria \( e^{\pm} \) of (*) with \( m(e^-) \geq m(e^+) \). Moreover, if the Morse indices satisfy \( m(e^-) = m(e^+) \), then \( u \) is isolated by linearization.

1. Introduction

Let \( \Omega \subset \mathbb{R}^m \), \( m \geq 1 \) be a bounded domain with smooth boundary. As an illustrative example for the abstract result in the following section, consider the following problem

(1.1) \[ \partial_t u - \Delta u = f(t, x, u(t, x), \nabla u(t, x)) \]
\[ u(t, x) = 0 \quad x \in \partial \Omega \]
\[ u(t, x) = u_0(x) \quad x \in \Omega \]
Suppose that \( f \) is sufficiently regular and \( f(t, x, u, v) \to f^{\pm}(x, u) \) as \( t \to \pm \infty \) uniformly on compact subsets. Note that the limit nonlinearities \( f^{\pm} \) are independent of the gradient \( \nabla u \). The limit problems

(1.2) \[ \partial_t u - \Delta u = f^{\pm}(x, u(t, x)) \]
\[ u(t, x) = 0 \quad x \in \partial \Omega \]
\[ u(t, x) = u_0(x) \quad x \in \Omega \]
define local gradient-like semiflows on an appropriate Banach space \( \tilde{X} \). It is well-known that for generic \( f^{\pm} \), every equilibrium of (1.2) is hyperbolic. Hence, a solution \( u : \mathbb{R} \to \tilde{X} \) is either an equilibrium solution or a heteroclinic connection. It has been proved [1] that for a generic \( f \) the semiflow induced by

\[ \partial_t u - \Delta u = f(x, u(t, x), 0) \]
\[ u(t, x) = 0 \quad x \in \partial \Omega \]
\[ u(t, x) = u_0(x) \quad x \in \Omega \]
is Morse-Smale. For the above equation, the Morse-Smale property means the following.

1. Every bounded subset of \( \tilde{X} \) contains only finitely many equilibria.
2. Given a pair \((e^-, e^+)\) of equilibria, the stable manifold \(W^s(e^+)\) and the unstable manifold \(W^u(e^-)\) intersect transversally.

An easy consequence of property (2) is stated below.

(2') A connection between \(e^-\) and \(e^+\) can only exist if the respective Morse-indices satisfy \(m(e^+) < m(e^-)\).

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The aim of this paper is to investigate if and how property (1) and (2') can be generalized to semilinear parabolic equations which are asymptotically autonomous, for example (1.1). Roughly speaking, the general situation is as follows: Equilibria in the autonomous case correspond to connections between two equilibria having the same Morse-index, and every bounded set contains only finitely many such connections. Furthermore, a connection between equilibria \(e^-\) and \(e^+\) can only exist if \(m(e^+) \leq m(e^-)\).

The proof of our results is similar to the relevant parts of [1], applying an abstract transversality theorem to a suitable differential operator. As a result, we know that for a dense subset of possible perturbations, 0 is a regular value of this operator. Using the framework of [1], namely the characterization of transversality in terms of the existence of exponential dichotomies on halflines [1 Corollary 4.b.4], we could try to prove that an appropriate generalization of (2) (see [2, 3]) holds with respect to a perturbation for which the abstract differential operator has 0 as a regular value. Following the approach of [1], we would have to assume that the evolution operator defined by the linearized equation at a heteroclinic solution is injective [1 Lemma 4.a.12]. (1) and (2') can be proved to hold for a generic perturbation without the injectivity assumption. For this reason, (2) is replaced by (2').

We will now apply Theorem 2.4 to the concrete problem (1.1). Let \(p > m \geq 1\), \(X := L^p(\Omega)\), which is reflexive, and define an operator

\[
A : W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \to L^p(\Omega)
\]

\[
Au := -\Delta u.
\]

\(A\) is a positive sectorial operator and has compact resolvent. As usual, define the fractional power space \(X^\alpha\) as the range of \(A^{-\alpha}\) equipped with the norm \(\|x\|_\alpha := \|A^\alpha x\|_X\). For \(\alpha < 1\) sufficiently large, the space \(X^\alpha\) is continuously imbedded in \(C^1(\overline{\Omega})\) (see for instance [9, Lemma 37.8]). Hence, \(f\) gives rise to a Nemitskii operator \(\hat{f} : \mathbb{R} \times X^\alpha \to X\), where

\[
\hat{f}(t,u)(x) := f(t,x,u(x),\nabla u(x)).
\]

Suppose that for some \(\delta > 0\)

1. \(f(t,.) \to f^\pm\) uniformly on sets of the form \(\Omega \times B_\eta(0) \times B_\eta(0) \subset \Omega \times \mathbb{R} \times \mathbb{R}^m\), where \(\eta > 0\) and \(f^\pm : \Omega \times \mathbb{R} \to \mathbb{R}\) is continuously differentiable in its second variable with \(\partial_x f^\pm(x,u)\) being continuous,
2. \(f(t,x,\ldots)\) is \(C^\infty\), and
3. each partial derivative of \(f(t,x,\ldots)\) is continuous in \(x\) and Hölder-continuous in \(t\) with Hölder-exponent \(\delta\) uniformly on sets of the form \(\mathbb{R} \times \Omega \times B_\eta(0) \times B_\eta(0) \subset \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^m\), \(\eta > 0\).

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1. non-trivial, that is, except for constant solutions
Let $C_{0}^{0,\delta}(\mathbb{R} \times \bar{\Omega})$ denote the set of all in $t$ H"{o}lder-continuous (with exponent $\delta > 0$) functions $g : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$ with $g(t,x) \rightarrow 0$ as $t \rightarrow \pm \infty$ uniformly on $\Omega$. $C_{0}^{0,\delta}(\mathbb{R} \times \bar{\Omega})$ is endowed with the norm
\[
\|g\| := \sup_{(t,x) \in \mathbb{R} \times \bar{\Omega}} |g(t,x)| + \sup_{(t,x) \neq (t',x) \in \mathbb{R} \times \Omega} \frac{|g(t,x) - g(t',x)|}{|t - t'|^{\delta}}.
\]

**Theorem 1.1.** In addition to the hypotheses above, assume that every equilibrium of the equations
\[
u_{t} + Au = \dot{f}(t, u)
\]
is hyperbolic.

Then there is a residual subset $Y \subset C_{0}^{0,\delta}(\mathbb{R} \times \bar{\Omega})$ such that for all $g \in Y$ and for every bounded solution of $u : \mathbb{R} \rightarrow W^{2,p}(\Omega)$ of
\[
u_{t} + Au = \dot{f}(t, u) + \tilde{g}(t),
\]
it holds that:

1. There are equilibria $e^\pm$ of
\[
u_{t} + Au = \dot{f}(u)
\]
such that $u(t) \rightarrow e^\pm$ in $C(\bar{\Omega})$ as $t \rightarrow \pm \infty$.

2. $m(e^-) \geq m(e^+)$ and $m(e^-) = m(e^+)$ only if
\[
v_{t} + Av = D\tilde{f}(t, u(t))v
\]
does not have a non-trivial ($L^p(\Omega)$-) bounded solution.

Since there are continuous imbeddings $X^1 \subset C(\bar{\Omega}, \mathbb{R}) \subset X^0$, the above theorem follows immediately from Corollary 2.5.

2. **Abstract formulation of the result**

Let $X$ and $Y$ be normed spaces and $X_0 \subset X$ be open. $\mathcal{L}(X,Y)$ is the space of all continuous linear operators $X \rightarrow Y$ endowed with the usual operator norm. The open ball with radius $\varepsilon$ and center $x$ in $X$ is denoted by $B_{\varepsilon}(x)$ and the closed ball with the same radius and center by $B_{\varepsilon}[x]$.

$C_{B}^{k}(X_0, Y)$ denotes the space of all $k$-times continuously differentiable mappings $X_0 \rightarrow Y$ with bounded derivatives up to order $k$. The spaces are endowed with the usual norm
\[
\|y\| := \sup_{x \in X_0} \max\{\|y(x)\|, \ldots, \|D^{k}y(x)\|\}
\]
The space $C_{B}^{k,\delta}(X_0, Y)$ is the subspace of $C_{B}^{k}(X_0, Y)$ consisting of all functions in $C_{B}^{k}(X_0, Y)$ whose $k$-order derivative is Hölder-continuous with exponent $\delta > 0$. In the case $\delta = 0$, we simply set $C_{B}^{k,0}(X_0, Y) := C_{B}^{k}(X_0, Y)$. On $C_{B}^{k,\delta}(X_0, Y)$, we consider the norm
\[
\|y\| := \|y\|_{C_{B}^{k}(X_0, Y)} + \sup_{x,x' \in X_0, x \neq x'} \frac{\|D^{k}y(x) - D^{k}y(x')\|}{\|x - x'\|^\delta}.
\]
Let $\eta > 0$ and $i_{\eta} : B_{\eta}(0) \cap X_0 \rightarrow X_0$ the inclusion mapping. Let $C_{B}^{k,\delta}(X_0, Y)$ denote the set of all functions $f : X_0 \rightarrow Y$ such that $f \circ i_{\eta} \in C_{B}^{k,\delta}(X_0 \cap B_{\eta}(0), Y)$
for all \( \eta > 0 \). We also write \( C_b^k(X_0, Y) := C_b^k,0(X_0, Y) \) and \( C_b(X_0, Y) := C_b^0(X_0, Y) \) for short. These spaces are equipped with an invariant metric

\[
d(f, f') := d(f - f', 0) := \sum_{n \in \mathbb{N}} 2^{-n} \frac{\| (f - f') \circ i_n \|}{1 + \| (f - f') \circ i_n \|}.
\]

This metric induces the respective topology of uniform convergence on bounded sets, that is, \( f_n \to f \) in \( C_b^k(X_0, Y) \) (resp. \( C_b^{k, \delta}(X_0, Y) \)) if \( f_n \circ i_\varepsilon \to f \circ i_\varepsilon \) in \( C_b^k(X_0 \cap B_\varepsilon(0), Y) \) for every \( \varepsilon > 0 \). \( C_B^{k, \delta}(\mathbb{R}, Y) \) denotes the closed subspace of \( C_B^k(\mathbb{R}, Y) \) containing all functions \( x \) with \( x(t) \to 0 \) as \( |t| \to \infty \).

**Definition 2.1.** A family \( T(t, s) \) defined for real numbers \( t \geq s \) of continuous linear operators is called a *linear evolution operator* if \( T(r, t)T(t, s) = T(r, s) \) for all \( r \geq t \geq s \).

**Definition 2.2.** We say that an evolution operator \( T(t, s) \) on a normed space \( X \) admits an *exponential dichotomy* on an interval \( J \) if there are constants \( \gamma, M > 0 \) and a family \( (P(t))_{t \in J} \) in \( \mathcal{L}(X, X) \) such that:

1. \( T(t, s)P(s) = P(t)T(t, s) \) for \( t \geq s \).
2. The restriction \( T(t, s) : \mathcal{R}(P(s)) \to \mathcal{R}(P(t)) \) is an isomorphism. Its inverse is denoted by \( T(s, t) \), where \( s < t \).
3. \( \| T(t, s)(I - P(s))\|_{\mathcal{L}(X, X)} \leq Me^{-\gamma(t-s)} \) for \( t \geq s \).
4. \( \| T(t, s)P(s)\|_{\mathcal{L}(X, X)} \leq Me^{\gamma(t-s)} \) for \( t < s \).

We also refer to the the family of projections as an exponential dichotomy.

**Definition 2.3.** Let \( \pi \) be a semiflow on a normed space \( X \). We say that \( \pi \) is *simple gradient-like* if:

1. Every equilibrium \( e \) of \( \pi \) is isolated\(^2\).
2. For every bounded solution \( u : \mathbb{R} \to X \), one has \( u(t) \to e^- \) as \( t \to -\infty \) and \( u(t) \to e^+ \) as \( t \to \infty \).
3. There is a partial order \( \prec \) on the set \( E \) of all equilibria such that \( e^+ \prec e^- \) whenever \( u \) satisfies (b).
4. If \( u \) is given by (b) and \( e^- = e^+ \), then \( u \equiv e \).

Unless otherwise stated, let \( X \) be a reflexive Banach space and \( A \) a positive sectorial operator defined on subspace \( X^1 \subset X \). \( X^\alpha := \mathcal{R}(A^{-\alpha}) \) denotes the \( \alpha \)-th fractional power space with the norm \( \| x \|_\alpha := \| A^\alpha x \| \). We will assume that the operator \( A \) has compact resolvent.

Fix some \( \delta \in [0, 1] \), and let \( f \in C_b^{1, \delta}(\mathbb{R} \times X^\alpha, X) \) be asymptotically autonomous, that is, there are \( f^\pm \in C_b^{1, \delta}(X^\alpha, X) \) such that \( f(t, .) \to f^\pm \) in \( C_b^{1, \delta}(X^\alpha, X) \) as \( t \to \pm \infty \). We consider solutions of

\[
(2.1) \quad u_t + Au = f(t, u)
\]

and its limit equations

\[
(2.2) \quad u_t + Au = f^\pm(u).
\]

The above equations define evolution operators (respectively semiflows in the autonomous case) on \( X^\alpha \).

\(^2\)The term *simple* refers to this hypothesis.
By an equilibrium $e$ of \eqref{2.2}, we mean a point $e \in X^\alpha$ such that $u : \mathbb{R} \to X^\alpha$, $t \mapsto e$, solves \eqref{2.2}. We say that an equilibrium $e$ is hyperbolic if the linearized equation

$$u_t + Au = Df^\pm(e)u$$

admits an exponential dichotomy $(P(t))_{t \in \mathbb{R}}$. The Morse-index of $e$ is the dimension of the exponential dichotomy, respectively the dimension of the range of its associated projection i.e., $m(e) := \dim \mathcal{R}P(t)$, where $t \in \mathbb{R}$ can be chosen arbitrarily.

**Theorem 2.4.** Assume that:

(a) Every equilibrium $e$ of \eqref{2.2} is hyperbolic.

(b) $f \in C^4(\mathbb{R} \times X^\alpha, X)$

(c) $f(t, \cdot) \to f^\pm$ in $C^1_b(X^\alpha, X)$ as $t \to \pm \infty$.

(d) $f(t, \cdot)$ is $C^\infty$ for each $t \in \mathbb{R}$, $t \mapsto D^k f(t, \cdot)$ Hölder-continuous with Hölder-exponent $\delta$ uniformly on sets of the form $\mathbb{R} \times B_{\eta}(0) \subset \mathbb{R} \times X^\alpha$, and $\|D^k f(t, x)\| \leq C(k, \|x\|_\alpha)$ for all $k \in \mathbb{N} \cup \{0\}$ and all $(t, x) \in \mathbb{R} \times X^\alpha$.

(e) The semiflows induced by \eqref{2.2} are gradient-like.

Let $\beta \in [0, 1]$, and let $C_{B,0}^{\alpha, \beta}(\mathbb{R}, X^\beta)$ denote the complete subspace of all $x \in C_{B}^{\alpha, \beta}(\mathbb{R}, X^\beta)$ with $\|x(t)\|_\alpha \to 0$ as $|t| \to \infty$. Then, for a generic $g \in C_{B,0}^{\alpha, \beta}(\mathbb{R}, X^\beta)$, every bounded solution $u : \mathbb{R} \to X^\alpha$ of

\begin{equation}
\tag{2.3}
u_t + Au = f(t, u) + g(t)
\end{equation}

satisfies:

1. There are equilibria $e^-, e^+$ of the respective limit equation \eqref{2.2} such that $\|u(t) - e^-\|_\alpha \to 0$ as $t \to -\infty$ and $\|u(t) - e^+\|_\alpha \to 0$ as $t \to \infty$.

2. $m(e^+) \leq m(e^-)$ and $m(e^-) = m(e^+)$ only if the linear equation

\begin{equation}
\tag{2.4}
v_t + Av = Df(t, u(t))v
\end{equation}

does not have a non-trivial bounded solution $v : \mathbb{R} \to X^\alpha$.

Note that (2) is equivalent to the existence of an exponential dichotomy for \eqref{2.4} (cf. the proof of Lemma 3.3).

**Proof.**

(1) Since the limit equations \eqref{2.2} are gradient-like, this is a consequence of Lemma 3.2.

(2) This follows from Theorem 4.3 together with Lemma 5.6.

**Corollary 2.5.** Let $E$ be a normed space such that $X^1 \subset E \subset X^0$, the inclusions being continuous.

Moreover, assume the hypotheses of Theorem 2.4. Then the conclusions of Theorem 2.4 hold for a generic $g \in C_{B,0}^{\alpha, \beta}(\mathbb{R}, E)$.

**Proof.** Let $\Phi$ be defined as in Section 4 preceding Theorem 4.3. Let $Y$ denote the set of all $g \in C_{B,0}^{\alpha, \beta}(\mathbb{R}, X)$ such that $0$ is a regular value of $\Phi(\cdot, g)$. It follows from Theorem 4.3 that $Y = \bigcap_{n \in \mathbb{N}} Y_n$, where each $Y_n$ is open and dense in $C_{B,0}^{\alpha, \beta}(\mathbb{R}, X)$. A second application of Theorem 4.3 proves that $Y \cap C_{B,0}^{\alpha, \beta}(\mathbb{R}, X^1)$ is dense in $C_{B,0}^{\alpha, \beta}(\mathbb{R}, X)$.

\footnote{i.e., there is a residual subset of $C_{B}^{\alpha, \beta}(\mathbb{R}, X^\beta)$ such that all $g$ in this subset have the stated property}
By the continuity of the inclusions, each of the sets \( Y_n \cap C_{B,0}^{0,\delta}(\mathbb{R}, E) \) is open in \( C_{B,0}^{0,\delta}(\mathbb{R}, E) \). Moreover, \( Y \cap C_{B,0}^{0,\delta}(\mathbb{R}, X^1) \) is a dense subset of each \( Y_n \cap C_{B,0}(\mathbb{R}, E) \), which proves that \( \bigcap_{n \in \mathbb{N}} (Y_n \cap C_{B,0}(\mathbb{R}, E)) = Y \cap C_{B,0}(\mathbb{R}, E) \) is residual. \( \square \)

3. A skew-product semiflow and convergence of solutions

Let \( Y \subset C_0(\mathbb{R} \times X^\alpha, X) \) denote the subspace, that is, equipped with a metric of convergence uniformly on bounded sets, of all functions \( f : \mathbb{R} \times X^\alpha \rightarrow X \) such that:

1. \( f(t, \cdot) \in C_b^1(X^\alpha, X) \) for all \( t \in \mathbb{R} \)
2. \( t \mapsto Df(t, \cdot) \) is a Hölder-continuous function \( \mathbb{R} \rightarrow C_b^1(X^\alpha, X) \)

The above assumptions are rather strong, but we do not strive for maximum generality here. It is easy to prove

**Lemma 3.1.** For every \( f \in Y \), the translation \( t \mapsto f(t + s, x) \), \( \mathbb{R} \rightarrow Y \) is continuous.

Let \( Y_0 \subset Y \) be a compact subspace of \( Y \) which is invariant with respect to translations. We consider solutions of the semilinear parabolic equation

\[
(3.1) \quad \dot{u} + Au = y(t, u).
\]

These induce a skew-product semiflow \( \pi := \pi_{Y_0} \) on \( Y_0 \times X^\alpha \), where we set \( (y, x) \pi t := (y^t, u(t)) \) if there exits a solution \( u : [0, t] \rightarrow X^\alpha \) of (3.1) with \( u(0) = x \). It follows from [2] Theorem 47.5 that \( \pi \) is continuous.

Now suppose that \( y^t \rightarrow y^- \) as \( t \rightarrow -\infty \) and \( y^t \rightarrow y^+ \) as \( t \rightarrow \infty \), where \( y^-, y^+ \in Y \) are autonomous. It is easily seen that the set \( Y_0 := cl_Y \{ y^t : t \in \mathbb{R} \} = \{ y^t : t \in \mathbb{R} \} \cup \{ y^-, y^+ \} \) is compact. Moreover for \( y^+ \) (resp. \( y^- \)), (3.1) defines a semiflow on \( X^\alpha \), which is denoted by \( \chi_{y^+} \) (resp. \( \chi_{y^-} \)).

It is easy to see that the two lemmas still hold true in a more general setting, replacing the boundedness in \( X^\alpha \) by an asymptotic convergence assumption, admissibility \([7]\) for example.

**Lemma 3.2.** Assume that \( \chi_{y^+} \) (resp. \( \chi_{y^-} \)) is simple gradient-like, and let \( u : \mathbb{R} \rightarrow X^\alpha \) be a bounded solution of (3.1). Then, \( u(t) \) converges to an equilibrium of \( \chi_{y^+} \) (resp. \( \chi_{y^-} \)) as \( t \rightarrow \infty \) (resp. \( t \rightarrow -\infty \)).

In the following proof, we use as before infix notation for the semiflows, i.e. given an arbitrary semiflow \( \pi \) on a metric space \( X \), we write \( x \pi t \) instead of \( \pi(t, x) \). A solution of \( \pi \) or with respect to \( \pi \) is a continuous mapping \( u : I \rightarrow X \) such that \( I \subset \mathbb{R} \) is an interval and \( u(t) = u(t_0) \pi (t - t_0) \) whenever \( [t_0, t] \subset I \). Given \( N \subset X \), \( \text{Inv}_\omega(N) \) denotes the negatively invariant subset of \( N \), i.e. \( x \in \text{Inv}_\omega(N) \) iff there exists a solution \( u : [-\infty, 0] \rightarrow N \) with \( u(0) = x \).

**Proof.** We consider only the case \( t \rightarrow \infty \) because \( t \rightarrow -\infty \) can be treated analogously. Suppose to the contrary that \( N \subset X^\alpha \) is bounded and \( u : \mathbb{R} \rightarrow Y_0 \times X \) is a solution with \( \omega(u) \neq \{ (y^+, e_0) \} \), where \( e_0 \) denotes a minimal equilibrium in \( \{ x : (y^+, x) \in \omega(u) \} \). The minimality refers to the partial order \( \prec \) introduced in Definition 2.3.

Let \( E \subset \{ y^+ \} \times X^\alpha \) denote the set of all equilibria in \( \omega(u) \). Pick an \( \varepsilon > 0 \) such that \( B_\varepsilon(y^+, e_0) \cap E = \{ (y^+, e_0) \} \) and a sequence \( t_n \rightarrow \infty \) with \( u(t_n) \rightarrow (y^+, e_0) \).

There are \( s_n \geq t_n \) such that \( d(u(s_n), (y^+, e_0)) = \varepsilon \) and \( u([t_n, s_n]) \subset B_\varepsilon((y^+, e_0)) \).
We claim that $|t_n - s_n| \to \infty$ as $n \to \infty$. Otherwise, we may assume without loss of generality that $r_n := |t_n - s_n| \to r_0$. The continuity of the semiflow implies that $\partial B_e[(y^+,e_0)] \ni u(s_n) \to (y^+,e_0)\pi r_0$, which is a contradiction. Choosing a subsequence $(s'_n)_n$ of $(s_n)_n$, we can assume that $u(s'_n) \to (y^+,x_0) \in \partial B_e^f[(y^+,e_0)] \cap \operatorname{Inv}^{-1}(B_e^f[(y^+,e_0)])$. Since $\chi_{y^+}$ is simple gradient-like, one has $(y^+,x_0)\pi t \to (y^+,e)$ as $t \to \infty$ for some $e \in E$, in contradiction to the minimality of $e_0$. □

4. SURJECTIVITY

The main result of this section is Theorem 4.3, applying an abstract transversality theorem. One of the key steps towards its proof is Theorem 4.4, stating the surjectivity of certain linear operators. The main ingredient for the proof of Theorem 4.4 is Lemma 4.6, which relies on a geometric idea that can be sketched as follows. Let $u : \mathbb{R} \to X$ be a heteroclinic solutions that is, a solution connecting hyperbolic equilibria $e^-$ and $e^+$. The hyperbolicity of the equilibria implies the existence of exponential dichotomies on intervals of the form $]-\infty,\tau]$ and $]\tau,\infty[$ provided $\tau$ is large enough. The linear equation respectively its solution operators determines a connection between these dichotomies respectively their associated invariant spaces. Perturbing this connection is the idea behind Lemma 4.6.

We consider the following (Banach) spaces:

$$
\mathcal{X} := C^0_B(\mathbb{R}, X) \cap C^{1,\delta}_B(\mathbb{R}, X^1)
$$

$$
\mathcal{Y} := C^0_{B,0}(\mathbb{R}, X^\beta) := \{ y \in C^0_B(\mathbb{R}, X^\beta) : y(t) \to 0 \text{ as } t \to \pm \infty \} \quad 0 \leq \beta \leq 1
$$

$$
\mathcal{Z} := C^0_B(\mathbb{R}, X).
$$

Here, we choose $\|x\|_{\mathcal{X}} := \|x\|_{C^1_B(\mathbb{R}, X)} + \|x\|_{C^0_B(\mathbb{R}, X^1)}$

A function $f : \mathbb{R} \times X^\alpha \to X^0$ gives rise to a Nemitskii operator $\hat{f}$ defined by

$$
\hat{f}(u)(t) := f(t, u(t)).
$$

Lemma 4.1. Under the hypotheses (b), (c) and (d) of Theorem 2.4, $\hat{f}$ maps bounded Hölder-continuous functions to bounded Hölder-continuous functions, that is, $\hat{f}(C^{0,\delta}_B(\mathbb{R}, X^\alpha)) \subset C^{0,\delta}_B(\mathbb{R}, X^0)$

Lemma 4.2. Under the hypotheses (b) and (d) of Theorem 2.4, the mapping $\hat{f} : C^0_B(\mathbb{R}, X^\alpha) \to C^0_B(\mathbb{R}, X)$ as defined above is $C^\infty$.

Proof. Suppose that $u, u', v \in C^{0,\delta}_B(\mathbb{R}, X^\alpha)$ satisfy $\|u\|, \|u'\| \leq M$, and set $B(t) := D_x f(t, u(t))$.

By the assumptions on $D_x f$ and $D_x^2 f$, there are constants $C_1 := C_1(M)$ and $C_2 := C_2(M)$ such that for arbitrary $v \in C^{0,\delta}_B(\mathbb{R}, X^\alpha)$ and $t, s \in \mathbb{R}^+$

$$
\|B(t)v(t)\|_0 \leq C_1 \|v\|_{C^0_B(\mathbb{R}, X^\alpha)}
$$

$$
\|B(t+s)\nu(t+s) - B(t)\nu(t)\|_0 \leq C_2 s^\delta \|\nu\|_{C^0_B(\mathbb{R}, X^\alpha)} + C_1 s^\delta \|\nu\|_{C^{0,\delta}_B(\mathbb{R}, X^\alpha)}.
$$
Now, set $B'(t, y) := D_x f(t, u(t) + y) - D_x f(t, u(t))$. We have

\[ B'(t, y) = \int_0^1 D^2_x f(t, u(t) + \lambda y) y d\lambda \]

\[ B'(t, y_1) - B'(t, y_2) = \int_0^1 \left[ D^2_x f(t, x + y_1) - D^2_x f(t, x) \right] y_1 d\lambda \]

By the assumptions on $D^k_x f$, there are constants $C_3 := C_3(M)$ and $C_4 := C_4(M)$ such that for all $y, y_1, y_2 \in B_M(0) \subset X^\alpha$ and all $z, z_1, z_2 \in X^\alpha$

\[ \|B'(t, y)\|_0 \leq C_3 \|y\|_\alpha \]

\[ \|B'(t, y_1) - B'(t, y_2)\|_0 \leq C_4 \left( \|y_1\|_\alpha + \|y_2\|_\alpha \right) \]

It follows that $[D\tilde{f}(u)]u(t) := Df(t, u(t))u(t)$ satisfies

\[ (4.1) \quad \left\| \hat{D}\tilde{f}(u + u') - D\tilde{f}(u) \right\|_{\mathcal{L}(C^0_{B,\delta}(R, X^\alpha), C^0_{B,\delta}(R, X))} \leq C_5(M) \|u'\|_{C^0_{B,\delta}(R, X^\alpha)} \]

In particular, one has $D\tilde{f} \in C_b(C^0_{B,\delta}(R, X^\alpha), \mathcal{L}(C^0_{B,\delta}(R, X^\alpha), C^0_{B,\delta}(R, X)))$.

Moreover,

\[ f(t, x + y) = f(t, x) + Df(t, x)y + \int_0^1 (Df(t, x + \lambda y) - Df(t, x)) y d\lambda, \]

so

\[ \hat{f}(u + u') - \hat{f}(u) - D\tilde{f}(u)u' = \int_0^1 \left( D\tilde{f}(u + \lambda u') - D\tilde{f}(u) \right) u' d\lambda, \]

and by (4.1),

\[ \left\| \hat{f}(u + u') - \hat{f}(u) - D\tilde{f}(u)u' \right\| \leq C_5(M) \|u'\|^2, \]

which proves that $\hat{f}$ is continuously differentiable and $D\hat{f}$ as defined above is indeed the derivative. The higher derivatives can be treated analogously.

Define $\Phi := \Phi f : X \times Y \to Z$ by

\[ \Phi(u, g)(s) := u_4(s) + Au(s) - f(s, u(s)) - g(s). \]

$\Phi$ is continuous by the choice of $X, Y,$ and $Z$.

Recall that a subset of a topological space is nowhere dense if the interior of its closure is empty. A countable union of nowhere dense sets is called meager and the complement of a meager set residual. The following theorem is the main result of this section.

**Theorem 4.3.** Under the hypotheses of Theorem 2.3, the set of all $y \in Y$ such that $0$ is a regular value of $\Phi(., y)$ is residual (in $Y$).

\[ D_x \Phi(x_0, y) : X \to Z \] is surjective whenever $\Phi(x_0, y) = 0$
In order to prove Theorem 4.4, we need to check the premises of the following theorem, which is a simplified version of [12, Theorem 2.1] (see also [14, Theorem 5.4]).

**Theorem 4.4.** Let $X, Y, Z$ be open subsets of Banach spaces, $r$ a positive integer, and $\Phi : X \times Y \to Z$ a $C^r$ map. Assume that the following hypotheses are satisfied:

1. For each $(x, y) \in \Phi^{-1}([0])$, $D_x \Phi(x, y) : X \to Z$ is a Fredholm operator of index less than $r$.
2. For each $(x, y) \in \Phi^{-1}([0])$, $D \Phi(x, y) : X \times Y \to Z$ is surjective.
3. The projection $p : (x, y) \mapsto y : \Phi^{-1}([0]) \to Y$ is $\sigma$-proper, that is, there is a countable system of subsets $V_n \subset \Phi^{-1}([0])$ such that $\bigcup_{n \in \mathbb{N}} V_n = \Phi^{-1}([0])$ and for each $n \in \mathbb{N}$ the restriction $p_n : V_n \cap \Phi^{-1}([0]) \to Y$ of $p$ is proper.

Then the set of all $y \in Y$ such that $0$ is a regular value of $\Phi(., y)$ is residual in $Y$.

Using Lemma 4.2, it is easy to see that $\Phi$ is $C^\infty$. In particular, we have

$$D \Phi(u_0, v_0)(u, v) = u_t + Au - D \hat{f}(u_0)u - v.$$

Now, suppose that $\Phi(u_0, v_0) = 0$, that is, $u_0$ is a solution of

$$u_t + Au = \hat{f}(u) + v_0.$$

Under the assumptions of Theorem 4.3, it follows from Lemma 3.2 that $u(t)$ converges to a (hyperbolic) equilibrium $e^\pm$ of the respective limit equation as $t \to \pm \infty$.

**Proof of Theorem 4.3.** Initially, define

$$\mathcal{X}_n := \{x \in \mathcal{X} : \|x(t)\|_\alpha < n \text{ for all } t \in \mathbb{R}\} \quad n \in \mathbb{N}.$$

It is clear that $\mathcal{X} = \bigcup_{n \in \mathbb{N}} \mathcal{X}_n$.

Since each equilibrium of (2.2) is hyperbolic, there are only finitely many equilibria $e$ with $\|e\|_\alpha \leq n$. Hence, there is an $m \in \mathbb{N}$ such that $m(e) \leq m$ whenever $e$ is an equilibrium of (2.2) with $\|e\|_\alpha \leq n$.

Furthermore, there is an $\varepsilon = \varepsilon(n) > 0$ such that $\|e - e'\|_\alpha > 2\varepsilon$ for every pair $(e, e')$ of equilibria with $\|e\|_\alpha \leq n$ and $\|e'\|_\alpha \leq n$. Define

$$\mathcal{X}_{n,m} := \{x \in \mathcal{X}_n : x(t) \in \bigcup_e B_\varepsilon[e] \text{ for } |t| \geq m\},$$

where the union is taken over all equilibria $e$ with $\|e\|_\alpha \leq n$.

Let $(u_0, y_0) \in \mathcal{X}_n \times \mathcal{Y}$ be a solution of $\Phi(u_0, y_0) = 0$. By our assumptions and [12, Lemma 4.4.11], assumption (CH) in Lemma 3.4 is satisfied. Hence, it follows from Theorem 4.3 and Lemma 4.4.4 that for every solution $u_0 \in \mathcal{X}_n$, $D_x \Phi(u_0, y_0) : \mathcal{X} \to \mathcal{Z}$ is a Fredholm operator and its (Fredholm) index is bounded by $m$. Furthermore, (4.2)

$$L(u, v) := u_t + Au - D_n f(t, u_0)u + v$$

defines a surjective operator $\mathcal{X} \times W \to \mathcal{Z}$, where $W = \text{span}\{w_1, \ldots, w_m\}$ and $w_1, \ldots, w_m \in \mathcal{Y}$ have compact support.

In order to apply Theorem 4.3, we need to show that the map $(x, y) \mapsto y : \Phi^{-1}([0]) \to \mathcal{Y}$ is $\sigma$-proper, that is, there is a family $(V_n)_n$ with $\Phi^{-1}([0]) = \bigcup_{n \in \mathbb{N}} V_n$ such that for each $n \in \mathbb{N}$ the map

$$(x, y) \mapsto y : V_n \to \mathcal{Y}$$

is proper.
Let \((x, y) \in \Phi^{-1}(\{0\})\) with \(x \in \mathcal{X}_n\). Since \(y(t) \to 0\) as \(t \to \pm \infty\), \(x\) converges to an equilibrium as \(t \to \pm \infty\) (Lemma 5.1). Hence,
\[
\Phi^{-1}(\{0\}) = \bigcup_{(n,m) \in \mathbb{N} \times \mathbb{N}} \left(\Phi^{-1}(\{0\}) \cap (\mathcal{X}_{n,m} \times \mathcal{Y})\right).
\]

Let \((x_n, y_n)\) be a sequence in \(V_{n,m}\) with \(y_n \to y_0\) in \(\mathcal{Y}\). Using the compactness of the evolution operators on \(\mathcal{X}^\alpha\) defined by
\[
u_t + Au = f(t, u) + y \quad y \in \mathcal{Y},
\]
it follows that there is a solution \(x_0 : \mathbb{R} \to \mathcal{X}^\alpha\) and a subsequence \((x_n')_n\) such that \(x'_n \to x_0\) uniformly on bounded sets. Suppose that the convergence is not uniform with respect to \(t \in \mathbb{R}\). In this case, there are a subsequence \((x_n'')_n\), a sequence \((t_n)_n\) and an \(\eta > 0\) such that \(\|x''_n(t_n) - x_0(t_n)\| \geq \eta\) for all \(n \in \mathbb{N}\). Moreover, we can assume without loss of generality that \(t_n \to \infty\) or \(t_n \to -\infty\).

By the choice of \(V_{n,m}\), there are equilibria \(e^\pm\) with \(x''_n(t) \in B_{\varepsilon}(e^\pm)\) for all \(t\) with \(|t| \geq m\). Hence, one has \(x_0(t) \in B_{\varepsilon}[e^\pm]\) for \(|t| \geq m\). Using assumption (c) of Theorem 2.4 and \[9\] Theorem 4.5, it follows that there is a solution \(u : \mathbb{R} \to B_{\varepsilon}[e]\) (either \(e = e^+\) or \(e = e^-\)) of one of the limit equations such that \(\|u(0) - e\|_\alpha \geq \eta > 0\). We can assume without loss of generality that \(B_{\varepsilon}[e]\) is an isolating neighborhood for \(e\), which means that \(u \equiv e\). This is an obvious contradiction, so
\[
\sup_{t \in \mathbb{R}} \|x_n(t) - x_0(t)\|_\alpha \to 0 \quad \text{as} \quad n \to \infty.
\]

By \[1\] Lemma 4.6.1, one has \(x_n \to x_0 \in \mathcal{X}\), which proves that the map defined by (4.1) is proper.

Now, it follows from Theorem 4.4 that there is a residual subset \(\mathcal{Y}_n \subset \mathcal{Y}\) such that for every \(\mathcal{Y} \subset \mathcal{Y}_n\), 0 is a regular value of \(\Phi(\cdot, y) : \mathcal{X}_n \to \mathcal{Z}\).

This completes the proof since a countable intersection of residual sets is residual. 

\[\square\]

**Lemma 4.5.** For every \(F \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)\) with \(\det F > 0\), there is an \(\hat{F} \in C^\infty([0, 1], \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n))\) such that \(\hat{F}(0) = \text{id}, \hat{F}(1) = F\), and \(\det F(t) > 0\) for all \(t \in [0, 1]\).

The proof is omitted.

**Lemma 4.6.** Suppose that:

(CH) \(B \in C^{\alpha, \delta}(\mathcal{L}(\mathcal{X}^\alpha, \mathcal{X}))\) with \(B(t) \to B^+\) as \(t \to \infty\) and \(B(t) \to B^-\) as \(t \to -\infty\). There further exists an \(m^+ \in \mathbb{N}\) (resp. \(m^-\)) such that the evolution operator defined by solutions of
\[
u_t + Au = B^+ u \quad \text{resp.} \quad B^- u
\]

admits an exponential dichotomy \(P\) defined for \(t \in \mathbb{R}^+\) (resp. \(t \in \mathbb{R}^-\)) with \(|t|\) large and \(\dim \mathcal{R}(P) = m^+\) (resp. \(m^-\)).

If \(m^- = m^+ = m\), then there exist \(t_1 \leq t_2\) and an \(R \in C^\infty([t_1, t_2], \mathcal{L}(\mathcal{X}^\alpha, \mathcal{X}))\) such that there does not exist a bounded non-trivial (mild) solution of
\[
u_t + Au = B(t) u + \begin{cases} R(t) u & t \in [t_1, t_2] \\ 0 & \text{otherwise.} \end{cases}
\]

Lemma 4.7. Suppose that \( A \) is a positive sectorial operator having compact resolvent. Let \( X_1 \subset X^1 = \mathcal{D}(A) \) be an arbitrary finite-dimensional subspace. Then, there are a closed subspace \( X_2 \subset X \) and \( B' \in \mathcal{L}(X, X) \) such that \( X = X_1 \oplus X_2 \), \((A - B')x = 0\) for all \( x \in X_1 \), and \((A - B')x \in X_2\) for all \( x \in X_2 \cap \mathcal{D}(A) \).

Proof. The claim is trivial for \( X_1 = \{0\} \), so we will assume that \( X_1 \neq \{0\} \).

Let \( P \in \mathcal{L}(X, X_1) \) denote an otherwise arbitrary projection, and let \( R(\mu, A) \in \mathcal{L}(X, X) \) denote the resolvent of \( A + \mu I \). We have \([3\) Theorem 5.2 in Chapter 2\]

\[
\| R(\mu, A) \| \leq \frac{M}{|\mu|},
\]

so every real \( \mu > 0 \) sufficiently large is in the resolvent set of

\[
A + \mu I - AP = (A + \mu I)(I - R(\mu, A)AP).
\]

Moreover, the resolvent \( R'(\mu) \) of \((4.6)\) is compact, and \( \frac{1}{\mu} \) is an eigenvalue of \( R'(\mu). \)

Let \( X = X'_1 \oplus X'_2 \) be the associated decomposition of \( X \), where \( X'_1 \supset X_1 \) is the generalized eigenspace associated with \( \frac{1}{\mu} \) and \( X'_2 \) is \( R'(\mu) \) invariant.

Finally, let \( Q \in \mathcal{L}(X, X'_1) \) denote the projection with kernel \( X'_2 \). The operator

\[
A - (AP + A(I - P)Q)_{B'}
\]

vanishes on \( X'_1 \). Let \( C \) satisfy the relation \( X'_1 = X_1 \oplus C \), and set \( X_2 := C \oplus X'_2 \). \( \square \)

Proof of Lemma 4.6. Let the evolution operator \( T(t, s) \) be defined by

\[
u + Au = B(t)u,
\]

and consider the bundles

\[
U := \{(s, x) \in \mathbb{R} \times X^\alpha : \text{there exists a solution } u : \mathbb{R}^- \to X \text{ with } u(s) = x
\]

and \( \sup_{t \in \mathbb{R}^-} \| u(t) \|_\alpha < \infty \}

\[
S := \{(s, x) \in \mathbb{R} \times X^\alpha : \sup_{t \in \mathbb{R}^+} \| T(t, s)x \|_\alpha < \infty \}.
\]

\( U \) and \( S \) are positively invariant, that is, \( (s, x) \in U \) (resp. \( S \)) implies \( (t, T(t, s)x) \in U \) (resp. \( S \)) for all \( t \geq s \).

It is well-known that, for small \( t \in \mathbb{R} \) (resp. large \( t \in \mathbb{R} \)), \( \dim U(t) = m \) and \( \text{codim} S(t) = m \) (see for instance \([R\) Lemma 4.a.11\)). Choose \( t_1 < t_2 \) such that \( \dim U(t) = m \) for all \( t \leq t_1 \) and \( \text{codim} S(t) = m \) for all \( t \geq t_2 \).

Let \( X = S(t_2) \oplus C_S \), \( X_1 := U(t_1) + C_S \), and \( X = X_1 \oplus X_2 \) with \( X_2 \subset S(t_2) \). For \( t \geq s \geq t_2 \), the evolution operator \( T(t, s) \) induces an isomorphism \( X/S (s) \to X/S(t) \), so \( X = T(t, t_2)C_S \oplus S(t + t_2) \) for every \( t \in \mathbb{R}^+ \). By standard regularity results and choosing \( t_2 \) larger if necessary, we can thus assume without loss of generality that \( C_S \subset X^1 \) so that \( X_1 = U(t_1) + C_S \subset X^1 \).

Let \( F : X_1 \to X_1 \) be a linear endomorphism with \( \det F > 0 \) which takes \( U(t_1) \) to \( C_S \), let \( \hat{F} \) be given by Lemma 4.5 and set \( G(t_1 + \xi(t_2 - t_1)) := \hat{F}(\xi) \) for \( \xi \in [0, 1] \).

Let \( B' \) be defined by Lemma 4.7 and let \( X = X_1 \oplus \hat{X}_2 \) with an \((A - B')\)-invariant complement \( \hat{X}_2 \). \( \hat{P} \in \mathcal{L}(X, X_1) \) denotes the projection along \( \hat{X}_2 \). Consider the semigroup \( S(t) \) defined by

\[
\dot{u} + Au = B'u.
\]
We can now define the modified evolution operator $\tilde{T}(t,s)$ by

$$
\tilde{T}(t,s)(x) := \begin{cases}
G(t)G(s)^{-1}x & x \in X_1 \text{ and } [s,t] \subset [t_1,t_2] \\
S(t-s)x & x \in \tilde{X}_2 \text{ and } [s,t] \subset [t_1,t_2] \\
T(t,s)x & [s,t] \cap [t_1,t_2] = \emptyset.
\end{cases}
$$

One has $\tilde{T}(t_2,t_1)x = F(x)$ for all $x \in U(t_1)$, so $\tilde{T}(t_2,t_1)U(t_1) \subset C_\delta$, which proves that there does not exist a full bounded solution of $\tilde{T}$.

Assume that $u$ is a solution of $\tilde{T}$ defined for $t \in [a,b] \subset [t_1,t_2]$. We have

$$
\tilde{P}u_t = (-A + B')\tilde{P}u + G_t(t)G(t)^{-1}\tilde{P}u
$$

(4.7)

$$
(1 - \tilde{P})u_t = (-A + B')(1 - \tilde{P})u,
$$

(4.8)

where the term $(-A + B')\tilde{P}u$ has been added deliberately. Consequently, every solution of $\tilde{T}(t,s)$ is also a solution of (4.5), where

$$
R(t) := B' + G_t(t)G(t)^{-1}\tilde{P} - B(t)
$$

is obtained by comparing the sum of (4.7) and (4.8) with (4.5). □

**Lemma 4.8.** Let $B \in L^\infty(\mathbb{R}, \mathcal{L}(X^\alpha, X))$ with $B(t) \to B^+$ as $t \to \infty$ and $B(t) \to B^-$ as $t \to -\infty$.

Assume there exists an $m \in \mathbb{N}$ such that each of the evolution operators defined by solutions of

$$
u_t + Au = B^+_t \text{ (resp. } B^-_t)$$

admits an exponential dichotomy $P$ defined for $t \in \mathbb{R}^+$ (resp. $t \in \mathbb{R}^-$) with $|t|$ large and $\dim \mathcal{R}(P) = m$.

Moreover, suppose that the only bounded mild solution $u : \mathbb{R} \to X^\alpha$ of

$$
u_t + Au = B(t)u
$$

(4.9)\)

is $u \equiv 0$.

Then, for every $h \in L^\infty(\mathbb{R}, X)$, there is a unique mild solution $u_0 \in C_B(\mathbb{R}, X^\alpha)$ of

$$
u_t + Au = B(t)u + h
$$

(4.10)

Proof. It follows from [9] Theorem 4.3 that (4.10) generates a skew-product semiflow on a suitable phase space $W \times X^\alpha$, where $W := \overline{\{B(t) : t \in \mathbb{R}\}}$, $p$ is a sufficiently large integer, and the closure is taken in $L^p_{\text{loc}}(\mathbb{R}, \mathcal{L}(X^\alpha, X))$. Note that $W = \{\tilde{B}^-, \tilde{B}^+\} \cup \{B(t) : t \in \mathbb{R}\}$, where $\tilde{B}^\pm(t) \equiv B^\pm$.

Now [8] Theorem C implies that the evolution operator $T(t,s)$ defined by mild solutions of (4.5) admits an exponential dichotomy. Our claim follows using the same formula as [4] Theorem 7.6.3 (see also [11] Lemma 4.a.7 and [11] Lemma 4.a.8)). □

**Theorem 4.9.** Suppose that (CH) holds, and let $m := \max\{m^-, m^+\}$. Then there are $w_1, \ldots, w_m \in \mathcal{Y}$ having compact support such that the operator $\tilde{L} : \mathcal{X} + \text{span}\{w_1, \ldots, w_m\} \to \mathcal{Z}$

$$
\tilde{L}(u,w) := L(u,w) = u_t + Au - B(t)u - w
$$

is surjective.
Proof. Consider the spaces
\[ X' := \mathbb{R}^{m^+} \times X \]
\[ (X')^\alpha := \mathbb{R}^{m^+} \times X^\alpha \]
and
\[ X' := C^{1,\delta}_B(\mathbb{R}, X') \cap C^{0,\delta}_B(\mathbb{R}, (X')^1) \]
\[ Y' := C^{0,\delta}_B(\mathbb{R}, \mathbb{R}^{m^+}) \times Y \]
\[ Z' := C^{0,\delta}_B(\mathbb{R}, X'). \]

We define an operator \( L' : X' \times Y' \to Z' \), where
\[ L'((x, u), (w', w)) := (x_t + \mu \arctan(t)x - w', u_t + Au - B(t)u - w) \]
and \( \mu = 1 \) if \( m^- < m^+ \) and \( \mu = -1 \) otherwise.

In both cases and for both limit equations i.e., \( t \to \pm \infty, (0, 0) \) is an equilibrium having Morse index \( m \). It follows easily that \( L \) is surjective if we prove that \( L' \) is surjective.

For the sake of simplicity, we will henceforth assume that \( m^- = m^+ \).

By Lemma 4.6 there are \( t_1 \leq t_2 \) and \( R \in C^\infty([t_1, t_2], \mathcal{L}(X^\alpha, X)) \) such that
\[ u_t + Au - B(t)u = \begin{cases} R(t)u & t \in [t_1, t_2] \\ 0 & \text{otherwise} \end{cases} \]

(4.11)
do not have a non-trivial bounded solution. The evolution operator \( T(t, s) \) defined by (4.11) has an exponentially stable subspace of finite codimension for \( t \geq s \geq t_2 \), that is, \( X = \tilde{X}_1 \oplus \tilde{X}_2 \) with \( \text{codim} \tilde{X}_2 = m^+ \) and for some \( M, \delta > 0 \)
\[ \|T(t, s)x\|_\alpha \leq Me^{-\delta(t-s)}\|x\|_\alpha \text{ for } x \in \tilde{X}_2 \text{ and } t \geq t_2. \]

(4.12)

Suppose that \( \tilde{X} := T(t_2, t_1)X^1 \subset X^1, \tilde{X}_2 := \tilde{X} \cap X_2 \) and \( \tilde{X} = \tilde{X}_1 \oplus \tilde{X}_2 \).

**Sublemma 4.10.** For every \( \eta \in \tilde{X}_1 \), there is a \( w \in \mathcal{Y} \) and a solution \( v : [t_1, t_2] \to X^\alpha \) of
\[ v_t + Av = B(t)v + w \]
with \( v(t_1) = 0 \) and \( v(t_2) = \eta \).

Proof. Let \( u : [t_1, t_2] \to X^\alpha \) be a solution of \( T(t, s) \) with \( u(t_2) = \eta \neq 0 \). Note that, by standard regularity results, e.g. [1] Lemma 4.a.6, one has \( u \in C^{1,\delta}(\mathbb{R}, X) \cap C^{0,\delta}([t_1, t_2], X^1) \).

Let \( x : \mathbb{R} \to \mathbb{R} \) be \( C^\infty \) with \( x(t) = 0 \) for \( t \leq t_1 \) and \( x(t) = 1 \) for \( t \geq t_2 \). Setting \( v(t) := u(t) \cdot x(t) \), one has
\[ v_t(s) = u_t(s) \cdot x(t) + u(t) \cdot x_t(s) \]
\[ = (-A + B(s)) u(s) x(s) + u(s) x_t(s) \]
\[ = v(s) \]
and
\[ v(t_1) = x(t_1) \cdot \eta = 0 \]
\[ v(t_2) = x(t_2) \cdot \eta = \eta \]
as claimed. \( \square \)
Let \( \eta_1, \ldots, \eta_n \) be a basis for \( \tilde{X}_1 \), and choose \( w_1, \ldots, w_n \) and \( v_1, \ldots, v_n \) according to Sublemma 4.10. It follows from Lemma 4.11 that for every \( h \in \mathbb{Z} \), there exists a unique mild solution \( u_0 \in C_B(\mathbb{R}, X^\alpha) \) of
\[
(4.13) \quad u_t + Au - B(t)u = R(t)u + h.
\]
Let \( v_1 : [t_1, \infty[ \to X^\alpha \) denote the solution of
\[
v_1 + Av - B(t)v = R(t)u_0 \quad v(t_1) = 0,
\]
and let \( v_1(t_2) = \eta \oplus \eta' \in \tilde{X}_1 \oplus \tilde{X}_2 \).
There is a \( w_0 \in \text{span}\{w_1, \ldots, w_n\} \) such that the solution \( v_2 : [t_1, \infty[ \to X^\alpha \) of
\[
v_1 + Av - B(t)v = -w_0 \quad v(t_1) = 0
\]
satisfies \( v_2(t_2) = \eta \).
It follows that \( v_0 := v_1 - v_2 \) is a solution of
\[
v_t + Av - B(t)v = R(t)u_0 + w_0 \quad v(t_1) = 0
\]
with \( v_0(t_2) \in \tilde{X}_2 \subset X_2 \).
Using (4.12), one concludes that \( \sup_{t \in \mathbb{R}} \| v_0(t) \|_\alpha < \infty \). Furthermore, \( u_0 - v_0 \) is a bounded mild solution of
\[
u_t + Au - B(t)u - w_0 = h,
\]
so by [1, Lemma 4.6], one has \( u_0 - v_0 \in \mathcal{X} \) and thus \( L(u_0 - v_0, w_0) = h \), which completes the proof of Theorem 4.10.

**Lemma 4.11.** Suppose that \( A \) is a sectorial operator having compact resolvent and \( B \) satisfies (CH). Let the operator \( L := L_B \) be defined by
\[
L_B u := u_t + Au - B(t)u
\]
Then \( \dim \mathcal{N}(L_B) \leq m^- \).

**Proof.** This is an immediate consequence of the existence of an exponential dichotomy on an interval \([-\infty, t_0]\) for small \( t_0 \), which follows from [1, Lemma 4.11]. \( \square \)

5. **Adjoint equations**

Throughout this section, suppose that \( X \) is a reflexive Banach space, \( A \) is a positive sectorial operator defined on \( X^1 \subset X \). As usual, we write \( \langle x, x^* \rangle := x^*(x) \). The adjoint operator \( A^* \) with respect to this pairing is a positive sectorial operator on the dual space \( X^* \) [6, Theorem 1.10.6]. Let \( A^{*, \alpha} \) denote the \( \alpha \)-th fractional power of the operator \( A^* \) and \( X^{*, \alpha} \) the \( \alpha \)-th fractional power space defined by \( A^{*, \alpha} \).

For the rest of this section, fix some \( \alpha \in [0, 1] \), and suppose that (CH) holds. Recall that (CH) means in particular that \( B(t) \to B^\pm \) as \( t \to \pm \infty \). We also write \( B(\pm \infty) \) to denote \( B^\pm \).
We will exploit the relationship between
\[
(5.1) \quad u_t + Au = B(t)u
\]
and its adjoint equation, where the adjoint is taken formally with respect to the pairing \( \langle x, y \rangle := \langle x, A^{*, \alpha} y \rangle \) between \( X \) and \( X' := X^{*, \alpha} \). The adjoint equation for (5.1) reads as follows.
\[
(5.2) \quad v_t + A^*v = (B(t)A^{-\alpha})^* A^{*, \alpha}v =: B'(t)v
\]
Lemma 5.1.
\[ \langle x, A^\alpha y \rangle = \langle x, (A^\alpha)^* y \rangle = \langle x, (A^\alpha)^* y \rangle \quad \forall (x,y) \in X^\alpha \times X^{*, \alpha} \]

Proof. Recall that \( A^{*, \alpha} = (A^\alpha)^* \) by definition. We have [3] p. 70]
\[
A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} \, dt,
\]
where the integral is taken in \( \mathcal{L}(X, X) \).
Hence, for \( x \in X \) and \( y \in X^* \), one has
\[
\langle A^{-\alpha} x, y \rangle = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \langle e^{-At} x, y \rangle \, dt
= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \langle x, e^{-A^* t} y \rangle \, dt
= \langle x, (A^*)^{-\alpha} y \rangle.
\]

Lemma 5.2. Let \( B \in \mathcal{L}(X^\alpha, X) \). Then \( B' := (BA^{-\alpha})^* A^{*, \alpha} \in \mathcal{L}(X^{*, \alpha}, X^* ) \) with \( \|B'\| \leq \|B\| \).

Proof. Let \( (x, y) \in X \times X^{*, \alpha} \). We have
\[
|\langle x, B'y \rangle| = |\langle BA^{-\alpha} x, A^{*, \alpha} y \rangle|
\leq \|B\|_{\mathcal{L}(X^\alpha, X)} \|x\|_{X^\alpha} \|A^{*, \alpha} y\|_{X^{*, \alpha}},
\]
which shows that \( \|B'y\|_{X^*} \leq \|B\|_{\mathcal{L}(X^\alpha, X)} \|y\|_{X^{*, \alpha}} \).

Lemma 5.3. Let \( J \subset \mathbb{R} \) be an open interval, let \( u : J \rightarrow X^\alpha \) be a solution of (5.1) and \( v : -J \rightarrow X^{*, \alpha} \) be a solution of (5.2). Then
\[
(u(t), v(-t)) \equiv C \quad \text{for all } t \in J.
\]

Proof. We consider the function \( h(t) := (u(t), v(-t)) \), which is defined for all \( t \in J \). Note that \( B \) is Hölder-continuous by (CH). Lemma 5.2 implies that \( B' \) is also Hölder-continuous. Therefore, \( u \) and \( v \) are continuously differentiable in \( X \) respectively \( X^* \). One has
\[
h_t(s) = \lim_{h \to 0} \frac{1}{h} ((u(s+h) - u(s), A^{*, \alpha} v(-s-h)) + (A^\alpha u(s), v(-s-h) - v(-s)))
= \langle u_t(s), A^{*, \alpha} v(-s) \rangle + \langle A^\alpha u(s), -v_t(-s) \rangle
= (-Au(s) + B(t)u(s), v(-s)) + \langle A^\alpha u(s), A^* v(-s) - B'(-t)v(-s) \rangle = 0
\]

Lemma 5.4. Let \( J \subset \mathbb{R} \) be an interval and \( P : J \rightarrow \mathcal{L}(X^\alpha, X^\alpha) \) an exponential dichotomy for the evolution operator \( T(t,s) \) on \( X^\alpha \) defined by (5.1).
Then \( P' : -J \rightarrow \mathcal{L}(X^{*, \alpha}, X^{*, \alpha}) \), \( P'(t) := A^{*, \alpha} P(-t)^* A^{*, \alpha} \), is an exponential dichotomy for the evolution operator \( T'(t,s) \) defined by (5.2).

Proof. It is easy to see that \( P' \) is well-defined and continuous (Lemma 5.2). We need to check the assumptions of an exponential dichotomy (Definition 2.2).
Suppose that \( (x,y) \in X^\alpha \times X^{*, \alpha} \) and \([s,t] \subset J\).
(1) From Lemma 5.3 we obtain
\[(x, P'(-s)T'(-s, -t)y) = (T(t, s)P(s)x, y) = (P(t)T(t, s)x, y) = (x, T'(-s, -t)P'(-t)y).\]

Since \(A^\alpha : X^\alpha \to X\) is an isomorphism, it follows that \(P'(-s)T'(-s, -t) = T'(-s, -t)P'(-t)\).

(2) To show that \(T'(-s, -t) : \mathcal{R}(P'(-t)) \to \mathcal{R}(P'(-s))\) is an isomorphism, it is sufficient to show that it is injective. Suppose that \(T'(-s, -t)y = 0\) for some \(y \in \mathcal{R}(P'(-t))\). For \(x \in X^\alpha\), we have
\[0 = (x, T'(-s, -t)y) = (T(t, s)x, P'(-t)y) = (T(t, s)P(s)x, y),\]
so \((x, y) = 0\) for all \(x \in \mathcal{R}(P(t))\). This in turn implies \((x, y) = (x, P'(-t)y) = (P(t)x, y) = 0\) for all \(x \in X^\alpha\), that is, \(y = 0\).

(3) The estimates for \(y \in \mathcal{R}(P'(-t))\) and \(y \in \mathcal{R}(I - P'(-t))\) can be deduced using roughly the same arguments. Hence, we will treat only the case \(y \in \mathcal{R}(P'(-t))\).

Suppose that
\[\|T(t, s)x\|_\alpha \leq M e^{-\gamma(s-t)}\|x\|_\alpha \quad s > t \quad x \in \mathcal{R}(P(s)).\]

We have
\[\langle x, A^{*, \alpha}T'(-s, -t)y \rangle = (x, T'(-s, -t)P'(-t)y) = (P(s)x, T'(-s, -t)P'(-t)y) = (T(t, s)P(s)x, y) \leq CMe^{-\gamma(s-t)}\|x\|_X\|A^{*, \alpha}y\|_{X^*}.\]
Thus, \(\|A^{*, \alpha}T'(-s, -t)y\|_{X^*} \leq CMe^{-\gamma(s-t)}\|A^{*, \alpha}y\|_{X^*}\), where the constant \(C\) is determined by the family \(P(.)\) of projections.

\[\square\]

To sum it up, we have proved that (5.2) satisfies (CH). In comparison to (5.1), the Morse indices \(m^-\) and \(m^+\) are obviously swapped. This is caused by the reversal of the time variable.

Let the spaces \(X, Y, Z\) be defined as in the previous section, and let \(X', Y', Z'\) denote their dual counterparts, that is,
\[X' := C^{1, \delta}_B(\mathbf{R}, X^*) \cap C^{0, \delta}_B(\mathbf{R}, X^{*, 1})\]
\[Z' := C^{0, \delta}_B(\mathbf{R}, X^*).\]
We consider the operators \(L \in \mathcal{L}(X, Z)\) (resp. \(L' \in \mathcal{L}(X', Z')\)) defined by
\[Lu := u_t + Au - B(t)u\]
and
\[L'v := v_t + A*v - B'(t)v.\]

**Lemma 5.5.** If \(\mathcal{R}(L) \supset X\), then \(\mathcal{N}(L') = \{0\}\). Analogously, if \(\mathcal{R}(L') \supset X'\), then \(\mathcal{N}(L) = \{0\}\).
Proof. Assume that $L'v = 0$ for some $v \in X'$ and let $u \in X$ satisfy $u(t) \to 0$ as $|t| \to \infty$. Integration by parts shows that

$$
\int_{-a}^{a} \langle (Lu)(s), A^{\alpha}\varepsilon(-s) \rangle ds
$$

$$
= \int_{-a}^{a} \langle u(s), A^{\alpha}\varepsilon(-s) \rangle + \langle A^{\alpha}u(s), A^{\varepsilon}v(-s) - B'(-s)v(-s) \rangle ds
$$

$$
= (u(a), v(-a)) - (u(-a), v(-a)) + \int_{-a}^{a} \langle A^{\alpha}u(s), v_{t}(-s) + (A^{\varepsilon} - B'(-s))v(-s) \rangle ds.
$$

Consequently for all $u \in X$ with $u(t) \to 0$ as $|t| \to \infty$, one has

$$
(5.3) \quad \int_{-a}^{a} \langle (Lu)(s), A^{\alpha}\varepsilon(-s) \rangle ds \to 0 \text{ as } a \to \infty.
$$

Arguing by contradiction, suppose that $v(t_{0}) \neq 0$ for some $t_{0} \in \mathbb{R}$. Since [6, Theorem 2.6.8] $X^1$ is dense in $X$, there is an $x_{0} \in X^1$ such that $\langle x_{0}, A^{\alpha}\varepsilon(t_{0}) \rangle \neq 0$. Choose $w \in C_{B}^{1,\delta}(\mathbb{R}, X^1)$ such that $w(t_{0}) = x_{0}$ and $w(t) = 0$ for all $t \in \mathbb{R}$ with $|t - t_{0}| \geq \varepsilon$. For small $\varepsilon > 0$, we have

$$
C := \int_{-\infty}^{\infty} \langle w(s), A^{\alpha}\varepsilon(-s) \rangle ds \neq 0.
$$

We further have $w \in X \subset \mathcal{R}(L)$, that is, $w = Lu$ for some $u \in X$. Since $w(t) = 0$ for $|t|$ sufficiently large, it follows from (CH) respectively from the existence of exponential dichotomies at $\infty$ and $-\infty$ that $u(t) \to 0$ as $|t| \to 0$. Hence, one has $C = 0$ by (5.3), which is a contradiction.

Using the Hahn-Banach theorem, the second claim can be treated similarly. \qed

Lemma 5.6. Suppose that $L$ is surjective. Then:

1. $m^{-} \geq m^{+}$;
2. If $m^{-} = m^{+}$, then $L$ is also injective.

Proof. (1) Assume to the contrary that $m^{-} < m^{+}$. Let $P^{-}$ (resp. $P^{+}$) denote the projections associated with the exponential dichotomy at $-\infty$ (resp. $+\infty$), which are given by (CH). Let $(P^{-})'$ and $(P^{+})'$ defined by Lemma 5.4. Note that $\dim \mathcal{R}(P^{-})' = m^{-}$ and $\dim \mathcal{R}(P^{+})' = m^{+}$.

By $T'(t,s)$, we mean the evolution operator on $X^{a,s}$ defined by (5.2). Let $t_{1} < 0 < t_{2}$ so that $(P^{+})'(t_{1})$ and $(P^{-})'(t_{2})$ are defined. Since $m^{-} < m^{+}$, the operator $(P^{-}(t_{2}))'(t_{1}) : \mathcal{R}(P^{+})'(t_{1}) \to \mathcal{R}(P^{-})'(t_{2})$ is not injective. Therefore, there exists a non-trivial bounded solution of [5.2], in contradiction to Lemma 5.5.

(2) $L'$ is injective by Lemma 5.5. We can now apply Lemma 5.8 to $L'$, showing that $L'$ is also surjective. Finally, Lemma 5.5 implies that $L$ is injective as claimed. \qed
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