Ingappability Index for Quantum Many-Body Systems

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(Dated: January 7, 2022)

We propose an index IG which characterizes the degree of ingappability, namely the difficulty to induce a unique ground state with a nonvanishing excitation gap, in the presence of a symmetry G. IG represents the dimension of the subspace of ambient uniquely-gapped in the entire G-invariant “theory space”. The celebrated Lieb-Schultz-Mattis theorem corresponds, in our formulation, to the case IG = 0 (completely ingappable) for the symmetry G including the lattice translation symmetry. We illustrate the usefulness of the index by discussing the phase diagram of spin-1/2 antiferromagnets in various dimensions, which do not necessarily have the translation symmetry.

Introduction.— Quantum critical phenomena have been a central subject of physics. Most generic quantum many-body systems are expected to have either (spontaneously) symmetry-breaking ground states or a unique ground state below a non-vanishing gap; a fine-tuning of parameters would be required to reach a quantum critical point. Interestingly, however, in quantum many-body systems, the concept of “ingappability” has been developed in the context of Lieb-Schultz-Mattis (LSM) theorem [1] and its generalizations [2–5]: translationally invariant systems under certain symmetry conditions are ingappable, namely must have either a gapless excitation spectrum above ground state(s) or a nontrivial degeneracy of ground states. In such systems, gapped phases with a unique ground state (which we shall call “uniquely gapped” phases for short) are excluded from phase diagrams, while gapless critical phases acquire enhanced stability.

LSM-type theorems only tell whether the system can have a uniquely gapped ground state, under a spatial symmetry such as the translation symmetry. In “gapless” systems which are not constrained by the LSM-type theorems, gapless phases are expected to be less stable and may even disappear from phase diagrams. Nevertheless, a large number of critical phases have been observed numerically and experimentally [6–8]. The existence of those gapless phases suggests a weaker notion of the ingappability.

In this Letter, we propose an integer index IG to characterize the degree of ingappability of a non-uniquely-gapped (NUG) Hamiltonian respecting a symmetry G, where NUG means having gapless low-energy excitations or degenerate ground states. For each NUG Hamiltonian with a symmetry G denoted by a point in a parameter space P, we define its IG as the (co-)dimension of contiguous NUG phases as illustrated in FIG. 1, where IG can be understood as the number of gapping directions [9]. However, IG depends on the chosen parameter space P with a particular set of coupling constants, so it is meaningful to introduce the complete, infinite-dimensional parameter space whose coordinate axes exhaust all G-symmetric coupling constants [10], and we denote IG on it as IG. Thus experimental phase diagrams with a finite number of parameters are its various sections with IG ≤ IG [11] and a lower IG implies being more ingappable. The original LSM-type ingappability corresponds to IG = 0 representing the absence of uniquely-gapped phases. The index IG provides more refined constraints on quantum phase diagrams, and give finer measures of stability of critical phases/points than the LSM-type theorem that only indicates IG = 0 or not.

Our notion of ingappabilities indicated by IG is related to the co-dimension of topologically protected gapless defects/boundaries in gapped phases of free fermions [12, 13] and that of gapless points in the phase diagram of effective field theories [14–18]. Here, as concrete examples, we will consider quantum spin systems on d-dimensional lattices that may not admit a simple description in terms

![FIG. 1. Illustration of IG's on a three-dimensional parameter space: co-dimension IG of NUG Hamiltonians (denoted by dots) is equal to the number of orthogonal vectors (dashed arrows) spanning the subspace of ambient uniquely-gapped phase. Here the curve, surface and solid cube represent the NUG phase to which a NUG Hamiltonian belongs while blank regions are uniquely gapped phases.](image-url)
of non-interacting particles or a known field theory. We claim the following theorem as our main result:

**Inapplicability of spin-1/2 antiferromagnets:** There exists a quantum phase of Hamiltonians with \( I_G \leq d - k \) in spin-1/2 antiferromagnetic systems on cubic lattices in \( d \) dimensions and \( G = G_{\text{onsite}} \times (\mathbb{Z})^k \) where \( G_{\text{onsite}} \) is an onsite symmetry that is one of i) \( SO(3) \) spin-rotation symmetry, ii) dihedral symmetry of \( \pi \)-spin-rotations \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), or iii) a time reversal symmetry \( \mathbb{Z}_2^T \). \((\mathbb{Z})^k\) denotes the translational symmetry along \( k \) lattice direction(s).

Since the case of \( k = d \) is reduced to the LSM-type theorems [19–27], we will focus on \( k < d \). To prove the statement, we will first consider the other extreme case \( k = 0 \) or \( G = G_{\text{onsite}} \) where no lattice translation symmetry is required, and take the translations into consideration later. In the following parts, we obtain the above statement for all realistic dimensions \( d = 1, 2, 3 \), and leave \( d \geq 4 \) as conjecture.

**NUG phases with \( I_{G_{\text{onsite}}} \leq 1 \) on spin-1/2 chains.**—To show the existence of such a NUG phase, we consider a typical Heisenberg antiferromagnetic (HAF) spin-1/2 chain with dimerization strength \( \delta \in [-1, 1] \):

\[
H_{\text{HAF}}^{d=1}(\delta) = \sum_{j} [1 + (-1)^j \delta] \vec{S}_j \cdot \vec{S}_{j+1},
\]

which is gapless if \( \delta = 0 \) while uniquely-gapped for other \( \delta \neq 0 \). Here we fix a sublattice (odd, even) structure so that \( \delta \) and \( -\delta \) are inequivalent parameters. We will see that the existence of this \( I_G^P = 1 \) gapless point on the special one-dimensional phase diagram \( \delta \in [-1, 1] \) actually implies \( I_{G_{\text{onsite}}} = 1 \) when we include all arbitrary \( G_{\text{onsite}} \)-symmetric interaction parameters into the parameter space, as we show in the following proof by contradiction.

Let us assume that, upon including some other \( G_{\text{onsite}} \)-symmetric interaction say \( J' \) as in FIG. 2 (a), \( I \geq 2 \) of that gapless point increases to 2. Then it is possible to find a connected path of uniquely-gapped Hamiltonians from \( H_{\text{HAF}}^{d=1}(\delta = +1) \) to \( H_{\text{HAF}}^{d=1}(\delta = -1) \) on the enlarged parameter space. Specifically, this path is an adiabatic transformation between those two Hamiltonians, along which the gap does not close. However, finding such a path should be impossible since \( H_{\text{HAF}}^{d=1}(\delta = +1) \) and \( H_{\text{HAF}}^{d=1}(\delta = -1) \) belong to distinct \( G_{\text{onsite}} \)-SPT phases classified by \( \mathbb{Z}_2 \) [19, 28] realized by spin-1/2's [29], i.e., the two topologically distinct phases under the onsite symmetry

\[
G_{\text{onsite}} = SO(3), \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ or } \mathbb{Z}_2^T.
\]

(Indeed, with an open boundary at the first site, there is a single undimerized spin 1/2 for \( H_{\text{HAF}}^{d=1}(\delta = +1) \) which signals the nontrivial \( G_{\text{onsite}} \)-SPT phase, while the absence of spin-1/2 boundary state for \( H_{\text{HAF}}^{d=1}(\delta = -1) \) implies the trivial \( G_{\text{onsite}} \)-SPT phase.) Then the presumption is not true. The above argument further implies that this gapless point will keep extending to higher-dimensional NUG phases as we include more and more parameters into the parameter space. Any point in this extending NUG phase has \( I_{G_{\text{onsite}}} \leq 1 \) to ensure the non-existence of the any adiabatic path connecting \( H_{\text{HAF}}^{d=1}(\delta = +1) \) and \( H_{\text{HAF}}^{d=1}(\delta = -1) \), so we call it a phase with \( I_{G_{\text{onsite}}} \leq 1 \) for short.

**General construction of the topological-invariant.**—

Now we generalize the method we have used to determine NUG phases with \( I_{G_{\text{onsite}}} \leq 1 \) to cases with higher \( I_{G_{\text{onsite}}} \) s. Let us assume that we can find a \((m - 1)\)-dimensional sphere \( S^{m-1} \) [30] in a selected \( m \)-dimensional parameter space, such that each Hamiltonian on this sphere is uniquely gapped. If the sphere is non-contractible, that is, if it cannot be adiabatically deformed/shrunken to a point without sweeping any NUG Hamiltonian even after we enlarge the dimensionality of the parameter space by introducing arbitrary more \( G_{\text{onsite}} \)-symmetric interaction parameters, then there must be at least one NUG Hamiltonian inside the sphere \( S^{m-1} \). Each time we include one more arbitrary interaction parameter (i.e., one more axis in the parameter space), the NUG point/phase must extend to a phase of one dimension higher without termination; otherwise the sphere would be contractible in the enlarged parameter space, contradicting the assumption. Thus any point in this NUG phase must satisfy \( I_{G_{\text{onsite}}} \leq m \). Specifically, the earlier proof on the spin chain corresponds to \( m = 1 \), making use of the non-contractible \( S^0 \simeq \{H_{\text{HAF}}^{d=1}(\delta = +1), H_{\text{HAF}}^{d=1}(\delta = -1)\} \).

In order to diagnose the non-contractibility, we will find and assign a topological invariant \( Q \) to each \( S^{m-1} \) with the following property: if two \( S^{m-1} \)'s can be deformed to each other in some (maybe enlarged) parameter space without passing through any NUG Hamiltonian, then their assigned \( Q \)'s are equal [see the two loops with \( Q_2 = Q_1 \) in FIG. 2 (b)]. Therefore, such a topological invariant serves as a non-contractibility detector; if a \( S^{m-1} \) does not have the same \( Q \) as a contractible sphere, it must be non-contractible.

Now we will explicitly describe the assignment of the
topological invariant \( Q \) for \( m = 2 \). Namely, here we consider a closed loop \( S^{m-1} = S^1 \) in the parameter space, along which the Hamiltonian has a unique ground state with a gap. We first decompose the lattice coordinate into one vertical (\( V \)) direction and horizontal (\( H \)) directions: \( \vec{r} = (r_V, r_H) \) with lengths \( L_V \) and \( L_{H,1}, \ldots, L_{H,d-1} \), separately, under a periodic boundary condition. Let us temporarily consider \( d = 2 \), i.e., the vector \( r_H \) is a single number \( r_H \). Each loop of uniquely-gapped Hamiltonians can be denoted by \( H[\vec{J}(\tau)] \), where \( \vec{J} \) is a compact notation of all \( G_{\text{onsite}} \)-symmetric interaction parameters and \( \tau \) is the loop parameter that can be freely parameterized such that \( \tau \in [0, L_H] \) with \( H[\vec{J}(\tau = 0)] = H[\vec{J}(\tau = L_H)] \); a concrete example is given in Eqs. (4,5). Then we construct a new \( r_H \)-dependent Hamiltonian \( H(\vec{r}) \)

\[
H(\vec{r}) \equiv H[\vec{J}(r_H)],
\]

by simply replacing the loop parameter \( \tau \) by the horizontal coordinate \( r_H \). We expect that \( H(\vec{r}) \) is also uniquely gapped because we can parametrize the loop \( H[\vec{J}(\tau)] \) to vary slowly enough \( |\partial_\tau \vec{J} | \ll \Delta_r \) so that the nonzero gap \( \Delta_r \) of the Hamiltonians \( H[\vec{J}(\tau)] \) along the loop still holds in \( H(\vec{r}) \) that is spatially adiabatically deformed. We view the \( r_H \)-dependent \( H(\vec{r}) \) as a (quasi-)one-dimensional system along the vertical direction \( x_V \), which is formally an anisotropic thermodynamic limit \( L_V \gg L_H \gg 1 \). Since \( H(\vec{r}) \) also respects \( G_{\text{onsite}} \), it belongs to either the non-trivial one-dimensional \( G_{\text{onsite}} \)-SPT phase or the trivial one. Let us assign \( Q = -1 \) to the original loop \( H[\vec{J}(\tau)] \) if the corresponding \( H(\vec{r}) \) is in the nontrivial \( G_{\text{onsite}} \)-SPT phase while \( Q = +1 \) to the loop \( H[\vec{J}(\tau)] \) if \( H(\vec{r}) \) is in the trivial SPT phase.

Indeed, \( Q \) constructed above qualifies as a topological invariant; if two loops \( H_0[\vec{J}(\tau)] \) and \( H_1[\vec{J}(\tau)] \) are assigned by distinct \( Q_0 \neq Q_1 \), they cannot be deformed continuously to each other. Suppose that we can find a series of loops \( H_s[\vec{J}(\tau)] \) with \( s \in [0, 1] \) deforming \( H_0[\vec{J}(\tau)] \) to \( H_1[\vec{J}(\tau)] \). Each intermediate loop \( H_s[\vec{J}(\tau)] \) gives a \( r_H \)-dependent Hamiltonian \( H_s(\vec{r}) \) by Eq. (3). Then \( H_s(\vec{r}) \) is a path (parametrized by \( s \)) of uniquely-gapped Hamiltonians connecting \( H_0(\vec{r}) \) and \( H_1(\vec{r}) \) that belong to two distinct SPT phases \( (Q_0 \neq Q_1) \), which is impossible by definition of SPT phases.

**NUG phases with \( I_{G_{\text{onsite}}} \leq 2 \) in spin-1/2’s on the square lattice.**—Now let us apply the general construction discussed above to the concrete system of spin-1/2’s on the square lattice. We first consider a potentially non-contractible loop in the parameter space by considering the dimerized Heisenberg antiferromagnet

\[
H_{\text{dHAF}}(\delta_H, \delta_V) = \sum_{\vec{r}} \left\{ (1 + (-1)^{r_H} \delta_H) \vec{S}_{\vec{r} + \hat{x}_V} \cdot \vec{S}_{\vec{r}} + (1 + (-1)^{r_V} \delta_V) \vec{S}_{\vec{r} + \hat{x}_H} \cdot \vec{S}_{\vec{r}} \right\}, \tag{4}
\]

where we have introduced the dimerization strengths \( |\delta_H, \delta_V| \leq 1 \) along both directions to span a two-dimensional parameter space as shown in FIG. 3. The Hamiltonians near the four sides \( \delta_H = \pm 1 \) and \( \delta_V = \pm 1 \) of the square in the parameter space are in decoupled ladder or decoupled four-spin-paquette phases, which are all only-gapped [31], while antiferromagnetic long range orders occur deep inside the square [32]. Then we consider the loop parametrized by \( \tau \in [0, L_H] \) winding along this square boundary for \( W \) times:

\[
\theta(\tau = L_H) = \theta(\tau = 0) + 2\pi W, \tag{5}
\]

where \( \theta(\tau) \equiv \text{Arg}[\delta_H(\tau) + \sqrt{-1} \delta_V(\tau)] \) in FIG. 3.

Then we assign a topological invariant \( Q_W = \pm 1 \) to this loop depending on whether its corresponding \( r_H \)-dependent Hamiltonian \( H(\vec{r}) \equiv H[\vec{J}(r_H)] \) is \( G_{\text{onsite}} \)-SPT trivial or not. To do so, we apply the bulk-edge correspondence and make an open boundary perpendicular to the vertical direction: cutting all the bonds connecting \( r_V = L_V \) and \( r_V = 1 \), and count the total number of undimerized spin 1/2’s along \( r_V = 1 \) in FIG. 4 as follows.

Such undimerized spins can take place where \( \delta_H(\vec{r}_H) \) changes sign, i.e., \( \theta(\vec{r}_H) = \pm \pi/2 \mod 2\pi \), so that they cannot be dimerized by neighboring spins along the horizontal direction. Thus, we only need to focus around the \( r_H \) where \( \theta(\vec{r}_H) = \pm \pi/2 - \epsilon \) changes to \( \theta(\vec{r}_H) = \pm \pi/2 + \epsilon \) with \( \epsilon \approx 0^+ \). However, when \( \theta(\vec{r}_H) \approx -\pi/2 \), we have \( \delta_V \approx -1 \) so the boundary spins there are readily dimerized along the vertical direction as in FIG. 4 (b). On the other hand at \( \theta(\vec{r}_H) \approx +\pi/2 \), the undimerized spins are decoupled from the bulk spins. We show a special paradigm of \( \theta(\vec{r}_H) = \pi/2 + \epsilon \) changing to \( \theta(\vec{r}_H) = \pi/2 - \epsilon \) in FIG. 4 (a) where the boundary hosts 3 undimerized spin 1/2’s. In general, the number of these spin 1/2’s is always odd, since that sign-changing point is exactly the interface between two distinct \( G_{\text{onsite}} \)-SPT chains along the boundary; this interface must host an odd number of spin 1/2’s, independent of lattice details [33, 34]. Thus the total number of undimerized boundary spin 1/2’s is

![FIG. 3. The definition of \( \theta(\tau) \) and the featureless phases along the loop within which gapless points take place.](image-url)
the winding number \( W \mod 2 \), which means

\[
Q_W = \exp(i \pi W) = \exp \left( i \pi \int_{r_H} \frac{d\theta(r_H)}{2\pi} \right), \tag{6}
\]

because the bulk-edge correspondence of \( G_{\text{ onsite}} \)-SPT phases implies that the nontrivial (trivial) phase hosts an odd (even) number of spin 1/2’s on the boundary, respectively [35]. Equation (6) means that the loop with \( W = 1 \) is non-contractible to a point \((W = 0)\), thereby detecting a NUG phase with \( I_{G_{\text{ onsite}}} \leq 2 \) extended without termination from a gapless point in FIG. 3 as more interaction parameters are included. The loop is similar to a Floquet system [36–41] but spatially periodic here. The following discussion of multi-variable extension does not have Floquet analogs.

**NUG phases with \( I_{G_{\text{ onsite}}} \leq 3 \) on spin-1/2 cubic lattices.**—Our construction of the bound \( I_{G_{\text{ onsite}}} \leq m \) based on a topological invariant on \( S^{m-1} \) in the parameter space can be extended to \( m > 2 \). Let us illustrate the construction for \( m = 3 \), with respect to the spin-1/2’s on the \( d = 3 \) cubic lattice. We consider the following typical Hamiltonian: \((J_V, J_{H;1,2} > 0)\)

\[
H^{t=4}_{\text{H AB}}(\vec{S}_{r}, \delta_{V}) = \sum_{r} \left\{ J_V [1 + (-1)^{r^y} \delta_{V}] \vec{S}_{r+\hat{x}_V} \cdot \vec{S}_{r} \right. \\
\left. + \sum_{n=1}^{2} J_{H;n} [1 + (-1)^{r^{H;n}} \delta_{H;n}] \vec{S}_{r+\hat{x}_{H;n}} \cdot \vec{S}_{r} \right\}, \tag{7}
\]

where, in addition to dimerization strengths along each direction, the antiferromagnetic exchange couplings \( J_V \) and \( J_{H;1,2} \) are also necessary control parameters in the following construction of a non-contractible sphere.

We start from the surfaces of the cube in the parameter space, topologically equivalent to a sphere, consisting of six faces defined by either one of the following conditions: \( \delta_{V} = \pm 1 \), \( \delta_{H;1} = \pm 1 \), and \( \delta_{H;2} = \pm 1 \). The Hamiltonians on twelve edges of these faces belong to the phase of decoupled four-leg spin tube, which are uniquely-gapped [42]. Thus, we simply set \( J_V = J_{H;1,2} = 1 \) on the twelve edges. By contrast, as we approach the center of each face where the Hamiltonian is reduced to a set of decoupled bilayer antiferromagnets, antiferromagnetic long-range order emerges if we keep \( J_{V,H;1,2} = 1 \) [43]. To avoid having the long-range order, we gradually increase \( J_V \) from 1 to, say, 3 on the face \( \delta_{V} = 1 \) (and similarly on the face \( \delta_{V} = -1 \)) when approaching from an edge to the face center so that the Hamiltonians on this entire face are uniquely-gapped. This is possible because there is no phase transition even when \( J_V \) is increased from 1 to infinity near the edges [42]. Similarly, we increase \( J_{H;j} \) from edges to the center on each face \( \delta_{H;j} = \pm 1 \). The resultant parameter surface, denoted as \( C \), of this parameter cube [44] is topologically equivalent to a sphere \( S^2 \), which is shown to be non-contractible below.

Following the general construction, we assign the topological invariant \( Q \) to this sphere through a \( r_H \)-dependent Hamiltonian \( \vec{H}(\vec{r}) = \vec{H}(r_{H;1}, r_{H;2}) \) as follows. Since we have a periodic boundary condition along the horizontal directions, \( \vec{H}(r_{H;1}, r_{H;2}) \) can be seen as a mapping from the lattice torus \( T^2 = [0, L_{H;1}] \times [0, L_{H;2}] \) to the parameter-cube surface \( C \cong S^2 \) constructed above. Then we take \( \vec{H}(r_{H;1}, r_{H;2}) \) as a typical mapping which wraps the “sphere” \( C \) by \( T^2 \) once as in FIG. 5. Next, we determine its topological-invariant assignment \( Q \) by studying which one-dimensional \( G_{\text{ onsite}} \)-SPT phase that \( \vec{H}(\vec{r}) \) belongs to when viewed as a one-dimensional system along the vertical unit vector \( \hat{x}_V \). Again, we make an open boundary at \( r_V = 1 \) perpendicular to the vertical direction \( \hat{x}_V \) and count undimerized spins there. Each loop in FIG. 5 is a two-dimensional analog of a one-dimensional boundary in FIG. 4, and from similar consideration we find that undimerized spins can only take place on each loop where both \( \delta_{H;1} \) and \( \delta_{H;2} \) change sign and \( \delta_{V} > 0 \). Such a situation occurs exactly once for \( \vec{H}(r_{H;1}, r_{H;2}) \) circled in FIG. 5, and therefore odd number(s) of undimerized spins are dangling at the certain position on the horizontal boundary, which implies that \( Q = -1 \). In

![FIG. 4](image-url) (a) An odd number of undimerized spins on the boundary \( r_V = 1 \) where \( \theta(r_H) \) changes from \( \pi/2 - \epsilon \) to \( \pi/2 + \epsilon \) around \( r_H = x_0 \). (b) Boundary spins are dimerized into bulk where \( \theta(r_H) \approx -\pi/2 \).

![FIG. 5](image-url) \( \vec{H}(r_{H;1}, r_{H;2}) \) wraps the torus \( T^2 \) around \( C \cong S^2 \). The position where undimerized boundary spins appear is indicated by the circle “o”.

![FIG. 5](image-url)
general, we can wrap the torus $T^2$ around the cube $\mathcal{C}$ repeatedly for $W$ times by $\mathcal{H}^{(W)}(r_{H1}, r_{H2})$, and thus the topological-invariant assignment for this general wrapping, a higher-dimensional analog of Eq. (6), is

$$Q_W = \exp(i\pi W) = \exp \left( i \pi \int_{\mathcal{F}_H} \frac{d\Omega_{2}(\vec{r}_H)}{4\pi} \right), \quad (8)$$

where $d\Omega_2$ is the differential solid angle of the “sphere” $\mathcal{C}$ spanned by the vector $(\delta_{H1}, \delta_{H2}, \delta_{H3})$. Specifically, it means that the cube surface $\mathcal{C}$, which is of $W = 1$, cannot be contracted continuously to a point ($W = 0$) due to their different $Q_W$’s. Thus its non-contractibility signals an extending NUG phase with $I_{G_{\text{ onsite}}} \leq 3$, as announced.

**NUG phases $I_G \leq d - k$ with $k$ translations.**—So far we have shown the ingappability index in realistic dimensions $d = 1, 2, 3$ without translations imposed [45]. When $G = G_{\text{ onsite}} \times \mathbb{Z}^k$ is imposed on the $d$-dimensional Hamiltonian with $\mathbb{Z}^k$ translations along directions $(k < d)$, we can construct a non-contractable $S^{d-k-1}$ in the parameter space by stacking identical $(d-k)$-dimensional $\mathcal{H}(\vec{r})$ such as Eq. (3) when $d-k=2$, along those $k$ directions. (When $d = k+1$, it is the weak $G$ SPT [46].) It proves our statement of the existence of NUG phases with $I_G \leq d - k$.

**Critical point and symmetry-protected gapless phases.**—When the NUG phase is critical, it is described in terms of a Renormalization-Group (RG) fixed point or a scale-invariant field theory. The ingappability index $I_G$ strongly restricts its nature: A critical point in the NUG phase with a finite $I_G$ can have at most $I_G$ relevant (or marginally relevant) operators which gap out the system to yield a unique ground state.

In particular, when $d = 1$ and $k = 0$ with $G = SO(3)$, the conformal field theory describing the critical point, the SU(2) level-1 Wess-Zumino-Witten model, indeed possesses only $I_G = d - k = 1$ relevant operator [47]. This observation should be extended to systems only with discrete onsite symmetries [20]. On the other hand, there are many open questions about the field theory describing the critical point in higher dimensions. The present result gives a powerful constraint on the possible field theory, and an insight into the role of translational symmetries on the stability of critical phases.

Additionally, $I_G$ can be used to classify gapless critical phases, which are inaccessible by classification theory of conventional gapped topological phases due to the absence of gaps. Gapless critical phases are protected by symmetry $G$ in that a lower symmetry generically results in a larger $I_G$, and trivial uniquely gapped phases correspond to $I_G = \infty$. Moreover, Eqs. (6) and (8) do not exclude $W, W \in \mathbb{Z}_{\text{even}}$ doubly winding/wrapping around the loop/cube to be contractible. Thus even number and $I_G$ form a finer symmetry-protected classification of the gapless phases generalizing previous proposals by LSM ingappabilities [48–50].

**Conclusions.**—In this work, we have proposed an ingappability index $I_G$ as a measure of the stability of NUG phases. This gives a strong constraint on the relevant operators when the NUG phase is critical, and can be used to classify symmetry-protected critical phases. The LSM-type theorems corresponds to a very special case $I_G = 0$ which requires the lattice translation symmetry. As a concrete application, we demonstrated that $d$-dimensional antiferromagnetic spin-1/2 systems with $k$-dimensional lattice translational symmetries have $I_G \leq d - k$. Although we have focused on $d - k \leq 3$, we conjecture that the generalized theorem can be extended to arbitrary $d$ and $k$ by a similar construction of non-contractible hyperspheres replacing the solid angle in Eq. (8) by $d\Omega_{d-k-1}$ spanned by $(d-k)$-tuple dimerization strengths. Moreover, we expect that the result can be generalized to other systems, such as SU($N$) “spin” models realizable in cold-atomic systems [49–58].

This work was supported in part by MEXT/JSPS KAKENHI Grant Nos. JP17H06462, JP19H01808 and JP19K03680, and JST CREST Grant No. JP-MJCR19T2.

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[1] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. 16, 407 (1961).
[2] I. Affleck and E. H. Lieb, Lett. Math. Phys. 12, 57 (1986).
[3] M. Oshikawa, M. Yamanaka, and I. Affleck, Phys. Rev. Lett. 78, 1984 (1997).
[4] M. Oshikawa, Phys. Rev. Lett. 84, 1535 (2000).
[5] M. B. Hastings, Phys. Rev. B 69, 104431 (2004).
[6] F. C. Alcaraz and M. J. Martíns, J. Phys. A: Math. Theor. 22, L865 (1989).
[7] M. Führinger, S. Rachel, R. Thomale, M. Greiter, and P. Schmitteckert, Ann. Phys. (Berlin) 17, 922 (2008).
[8] Y. Matsumoto, S. Nakatsuji, K. Kuga, Y. Karaki, N. Horie, Y. Shimura, T. Sakakibara, A. H. Nevidomskyy, and P. Coleman, Science 331, 316 (2011).
[9] Here and after, “gapping” means gapping the NUG Hamiltonian to be uniquely gapped.
[10] Another situation, where considering such a complete parameter space is necessary and natural, is the renormalization-group transformations during which infinitely-many interactions are generated in general.
[11] As more and more interaction couplings are introduced into the parameter space, the gapping directions of the certain NUG Hamiltonian cannot decrease.
[12] J. C. Teo and C. L. Kane, Phys. Rev. B 82, 115120 (2010).
[13] S. Ryu, A. P. Schnyer, A. Furusaki, and A. W. W. Ludwig, New Journal of Physics 12, 065010 (2010).
[14] C. Cordova, D. Freed, H. T. Lam, and N. Seiberg, SciPost Physics 8, 1 (2020).
[15] C. Cordova, D. S. Freed, H. T. Lam, and N. Seiberg, SciPost Physics Proceedings 8 (2020).
[16] A. Kapustin and L. Spodyneiko, Phys. Rev. B 101, 235130 (2020).
[17] A. Kapustin and L. Spodyneiko, arXiv preprint arXiv:2003.09519 (2020).
[18] P.-S. Hsin, A. Kapustin, and R. Thorngren, Phys. Rev. B 102, 245113 (2020).
[19] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 83, 035107 (2011).
[20] Y. Fuji, Phys. Rev. B 93, 104425 (2016).
[21] H. Watanabe, H. C. Po, A. Vishwanath, and M. Zaletel, Proc. Natl. Acad. Sci. USA 112, 14551 (2015).
[22] M. Cheng, M. Zaletel, M. Barkeshli, A. Vishwanath, and P. Bonderson, Phys. Rev. X 6, 041068 (2016).
[23] H. C. Po, H. Watanabe, C.-M. Jian, and M. P. Zaletel, Phys. Rev. Lett. 119, 127202 (2017).
[24] Y. Ogata and H. Tasaki, Commun. Math. Phys. 372, 951 (2019).
[25] Y. Ogata, Y. Tachikawa, and H. Tasaki, Commun. Math. Phys. 385, 79 (2021).
[26] D. V. Else and R. Thorngren, Phys. Rev. B 101, 224437 (2020).
[27] Y. Yao and M. Oshikawa, Phys. Rev. Lett. 126, 217201 (2021).
[28] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, Phys. Rev. B 85, 075125 (2012).
[29] M. Nakamura and S. Todo, Phys. Rev. Lett. 89, 077204 (2002) and references therein.
[30] Throughout this Letter, sphere is used in the topological sense; no particular shape or location in the parameter space is specified.
[31] G. Y. Chitov, B. W. Ramakko, and M. Azzouz, Phys. Rev. B 77, 224433 (2008).
[32] N. Katoh and M. Imada, J. Phys. Soc. Jpn. 62, 3728 (1993).
[33] M. Hagiwara, K. Katsumata, J. Affleck, B. I. Halperin, and J. P. Renard, Phys. Rev. Lett. 65, 3181 (1990).
[34] S. H. Glarum, S. Geschwind, K. M. Lee, M. L. Kaplan, and J. Michel, Phys. Rev. Lett. 67, 1614 (1991).
[35] More strictly speaking, a nontrivial (trivial) one-dimensional $G_{\text{on-site}}$-SPT phase hosts a (non-)projective representation at the boundary.
[36] D. V. Else and C. Nayak, Phys. Rev. B 93, 201103 (2016).
[37] C. W. von Keyserlingk and S. L. Sondhi, Phys. Rev. B 93, 245145 (2016).
[38] A. C. Potter, T. Morimoto, and A. Vishwanath, Phys. Rev. X 6, 041001 (2016).
[39] R. Roy and F. Harper, Phys. Rev. B 94, 125105 (2016).
[40] R. Roy and F. Harper, Phys. Rev. B 95, 195128 (2017).
[41] K. Shiozaki, arXiv preprint arXiv:2110.10665 (2021).
[42] Y. Fuji, Phys. Rev. B 93, 104425 (2016).
[43] T. Matsuda and K. Hida, J. Phys. Soc. Jpn. 59, 2223 (1990).
[44] The parameter setting in the interior of the cube can be done rather arbitrarily.
[45] For example, with any coupling that respects $G_{\text{on-site}}$ but breaks translation symmetry, the inequality $T_{G_{\text{on-site}}} \leq d - k$ of the NUG phase still holds.
[46] L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. 98, 106803 (2007).
[47] In terms of the SU(2) level-1 Wess-Zumino-Witten model, the local relevant perturbation is of the form of $tr(g)$ with $g$ a SU(2)-matrix-valued field. The next leading SO(3) invariant operator is $tr(g)^2$, which can be marginally relevant. However, when it induces a gap, the ground states are doubly degenerate.
[48] S. C. Furuya and M. Oshikawa, Phys. Rev. Lett. 118, 021601 (2017).
[49] Y. Yao, C.-T. Hsieh, and M. Oshikawa, Phys. Rev. Lett. 123, 180201 (2019).
[50] Y. Yao and M. Oshikawa, Phys. Rev. X 10, 031008 (2020).
[51] C. Wu, J.-P. Hu, and S.-C. Zhang, Phys. Rev. Lett. 91, 186402 (2003).
[52] C. Honerkamp and W. Hofstetter, Phys. Rev. Lett. 92, 170403 (2004).
[53] M. A. Cazalilla, A. Ho, and M. Ueda, New J. Phys. 11 (2009).
[54] A. V. Gorshkov, M. Hermele, V. Gurarie, C. Xu, P. S. Julienne, J. Ye, P. Zoller, E. Demler, M. D. Lukin, and A. Rey, Nature physics 6, 289 (2010).
[55] S. Taie, R. Yamazaki, S. Sugawa, and Y. Takahashi, Nature Physics 8, 825 (2012).
[56] G. Pagano, M. Mancini, G. Cappellini, P. Lombardi, F. Schäfer, H. Hu, X.-J. Liu, J. Catani, C. Sias, M. Inguscio, and L. Fallani, Nature Physics 10, 198 (2014).
[57] F. Scaccia, C. Hofrichter, M. Höfer, P. De Groot, I. Bloch, and S. Fölling, Nature Physics 10, 779 (2014).
[58] X. Zhang, M. Bishof, S. Bromley, C. Kraus, M. Sfornova, P. Zoller, A. M. Rey, and J. Ye, science 345, 1467 (2014).