Pieri resolutions for classical groups
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Dedicated to Corrado De Concini on the occasion of his 60th birthday.

Abstract
We generalize the constructions of Eisenbud, Floystad, and Weyman for equivariant minimal free resolutions over the general linear group, and we construct equivariant resolutions over the orthogonal and symplectic groups. We also conjecture and provide some partial results for the existence of an equivariant analogue of Boij–Söderberg decompositions for Betti tables, which were proven to exist in the non-equivariant setting by Eisenbud and Schreyer. Many examples are given.

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Introduction.
In this paper we define new families of equivariant free resolutions. These extend the equivariant resolutions defined in [EFW] over the general linear group, which gave the first proof in characteristic 0 of the existence of the “pure” resolutions conjectured by Boij and Söderberg [BS]. The
proof of acyclicity of the pure resolutions in [EFW] was indirect and used the Borel–Weil–Bott theorem. Here we give a more direct proof based on an explicit description of the differentials due to Olver [Olv]. We also extend the constructions of [EFW] to more general resolutions, and give constructions for other groups. We also provide some evidence for an equivariant form of the Boij–Söderberg conjectures that would have striking consequences for Schur functions.

Let us describe the setup more precisely and give an overview of the paper.

We work over a field $K$ of characteristic zero, and let $V$ be an $n$-dimensional vector space over $K$. We set $A = \text{Sym}(V) \cong K[x_1, \ldots, x_n]$ to be a polynomial ring in $n$ variables, and we consider the general linear group $\text{GL}(V)$ as an algebraic group over $K$. In this paper, all modules are assumed to be graded. We consider finitely generated equivariant $A$-modules $M$. This means that one has an (algebraic) action of $\text{GL}(V)$ on $M$ denoted $g.m$ for $g \in \text{GL}(V)$ and $m \in M$, and an action of $A$ on $M$ denoted $p \cdot m$ for $p \in A$ such that the identity $g.(p \cdot m) = (g.p) \cdot (g.m)$ holds, where $g.p$ denotes the canonical action of $\text{GL}(V)$ on Sym$(V)$. Note that since $\text{GL}(V)$ (and other classical groups used below) are linearly reductive, the category of $\text{GL}(V)$-representations is semisimple, so every graded equivariant $A$-module $M$ has a minimal graded equivariant resolution whose terms are direct sums of free modules of type $A \otimes S_{\lambda}V$ where $S_{\lambda}V$ denotes the irreducible representation of $\text{GL}(V)$ of highest weight $\lambda$. The above statements remain true when we replace $\text{GL}(V)$ by $\text{SO}(V)$ or $\text{Sp}(V)$ whenever $V$ has a nondegenerate orthogonal or symplectic form.

Section 2 extends the results of [EFW]. In that paper, pure free resolutions are constructed in characteristic 0 using representation theory. One of the main constructions was the minimal free resolution of the cokernel of a nonzero map of the form

$$\varphi(\alpha, \beta) : A \otimes S_{\beta}V \to A \otimes S_{\alpha}V$$

(0.1)

where $\alpha$ and $\beta$ are partitions satisfying $\alpha_1 < \beta_1$, and $\alpha_i = \beta_i$ for $i > 1$. In Section 2.1, we give a simpler proof of the correctness of the terms of the minimal resolution of coker$\varphi(\alpha, \beta)$. Then in Section 2.3, we extend the construction of minimal free resolutions in Theorem 2.12 by removing the restrictions on $\alpha$ and $\beta$. In particular, we only assume that $S_{\beta}V$ is a subrepresentation of $A \otimes S_{\alpha}V$, so that a nonzero equivariant map of the form (0.1) exists. We call such maps Pieri maps. The decomposition of the modules in the resolution in terms of $\text{GL}(V)$ representations can be described purely combinatorially in terms of partitions. We also present a simple combinatorial algorithm for writing down a free resolution of the cokernel of a map of the form

$$\varphi(\alpha; \beta^1, \ldots, \beta^r) : \bigoplus_{i=1}^{r} A \otimes S_{\beta^i}V \to A \otimes S_{\alpha}V$$

(0.2)

in Theorem 2.7 (under the natural assumption that $S_{\beta^i}V$ is a subrepresentation of $A \otimes S_{\alpha}V$ for $i = 1, \ldots, r$). In general, the resolution we give may not be minimal, see Example 2.21. However, we give an explicit closed form description for the minimal free resolution in Corollary 2.10 for the special case when $\beta^i$ and $\alpha$ differ in only one entry for each $i = 1, \ldots, r$. We call the minimal resolutions of maps of the form (0.2) Pieri resolutions.

The map $\varphi(\alpha, \beta)$ (and hence $\varphi(\alpha; \beta^1, \ldots, \beta^r)$) can be calculated (up to a scalar multiple) in Macaulay 2 using the PieriMaps package (see [Sam]).

**Example 0.3.** Let $n = 3$, $\alpha = (3,1,0)$, $r = 2$, and $\beta^1 = (5,1,0)$ and $\beta^2 = (3,2,0)$. Representing the module $A \otimes S_{\lambda}V$ by the Young diagram of $\lambda$ (our convention for partitions: the diagram of $\lambda$ has $\lambda_i$ boxes in the $i$th column), we get the following resolution:

$$0 \to \begin{array}{c} \end{array} \to \begin{array}{c} \end{array} \oplus \begin{array}{c} \end{array} \to \begin{array}{c} \end{array} \oplus \begin{array}{c} \end{array} \to \begin{array}{c} \end{array} \to M \to 0$$
where \( M = S_{(3,1,0)} V \oplus S_{(4,1,0)} V \oplus S_{(3,1,1)} V \oplus S_{(4,1,1)} V \).

We remark that the techniques of [EFW] are limited to characteristic 0 because of the failure of the Borel–Weil–Bott theorem in positive characteristic, and our techniques are limited to characteristic 0 because semisimplicity of the general linear group fails otherwise, which means that nonzero equivariant maps of the form (0.1) often do not exist.

Also included in Section 2.3 is how equivariant resolutions can be constructed when \( A \) is replaced by the exterior algebra \( B = \bigwedge V \). The resolutions one obtains are infinite in length, but still simple to describe combinatorially.

In Section 3 we generalize the results of [EFW] to other classical groups. When \( G \) is an orthogonal or symplectic group, we have a standard representation \( F \) (a vector space with a symmetric or skew-symmetric nondegenerate bilinear form). The highest weights of irreducible representations of the group \( G \) occurring in the tensor powers on \( F \) still correspond to partitions. The constructions in the case of the general linear group are functorial and hence extend to vector bundles. We construct the analogues of a family of graded equivariant Sym(\( F \))-modules \( M \) which is analogous to the cokernel of Pieri maps in the \( GL(V) \) case by considering Pieri resolutions of homogeneous bundles over certain homogeneous spaces for \( G \). Then we use the geometric technique (see Theorem 3.5) and the Borel–Weil–Bott theorem for the group \( G \) to describe the minimal free resolution of the module \( M \). They are not pure but can still be considered to be the analogues of the complexes from [EFW]. The Lie types \( B_n, C_n, \) and \( D_n \) are treated in separate subsections. The arguments here are more delicate, since the resolutions are constructed as iterated mapping cones, and for example, in the calculations for type \( B \), one must analyze a connecting homomorphism in a long exact sequence to prove that some repeating representations cancel.

Section 4 is concerned with a possible equivariant analogue of the Boij–Söderberg conjectures which were proved in [ES]. If \( M \) is an equivariant module, let \( F_\bullet \) be its equivariant minimal free resolution. We define its equivariant Betti table \( B(M) \) as follows: if \( F_i = \bigoplus_j A(-j) \otimes V_{i,j} \), then \( B(M)_{i,j} \) is the character of \( V_{i,j} \).

The strong version of the conjecture says that given any finite length equivariant module \( M \) with equivariant resolution \( F_\bullet \), there exist representations \( W, W_1, \ldots, W_r \) such that \( W \otimes F_\bullet \) has a filtration with associated graded

\[
\text{gr}(W \otimes F_\bullet) \cong \bigoplus_{i=1}^r W_i \otimes F(\alpha^i, \beta^i)_\bullet,
\]

where \( F(\alpha^i, \beta^i)_\bullet \) is the minimal free resolution of the map \( \varphi(\alpha^i, \beta^i) \).

The weak version of the conjecture replaces the isomorphism of complexes in (0.4) with an equality of equivariant Betti tables. If we remove the adjective “equivariant,” then the weak version of the conjecture is a theorem of Eisenbud and Schreyer [ES, Theorem 0.2]. Furthermore, their result holds over any field, and the finite length condition can be replaced by an arbitrary codimension. The weak form of the conjecture would already be interesting from the point of view of cohomology tables of homogeneous bundles on projective space. In [ES], a bilinear pairing is defined between minimal resolutions over Sym(\( V \)) and vector bundles on the projective space \( P(V) \) which reveals a duality between the two. This bilinear pairing can also be defined in an equivariant way, and one hopes that a similar kind of duality holds in an equivariant sense.

We present some examples of decompositions predicted by the weak version of the conjecture in Section 4. We also provide some partial results in this direction (see Proposition 4.11) and discuss some of the difficulties in trying to extend the proof of Eisenbud and Schreyer to the equivariant setting and in trying to find counterexamples to the existence of such decompositions.
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1 Background.

In Section 1.1, we define our notation for partitions and representations of GL(V) (which is slightly nonstandard). In Section 1.2, we give Olver’s explicit description of the inclusion arising from a Pieri-type tensor product decomposition.

1.1 Partitions and representation theory.

Let α denote a partition, i.e., a sequence α = (α₁, ..., αₙ) with αᵢ ∈ ℤ and α₁ ≥ α₂ ≥ ... ≥ αₙ ≥ 0. We let ℓ(α) denote the length of α, which is defined to be the largest m such that αₘ ≠ 0. We represent α by its Young diagram D(α) with αᵢ boxes in the i-th column. The dual partition α* is defined by setting αᵢ* to be the number of j such that αⱼ ≥ i, or equivalently, the number of boxes in the i-th row of D(α). The notation α ⊆ β means that αₗ ≤ βₗ for all i, or equivalently, D(α) ⊆ D(β), and in this case, β/α = D(β/α) refers to the skew diagram D(β) \ D(α).

For brevity, we will say (β, α) ∈ VS to mean that β ⊇ α and β/α is a vertical strip, i.e., βᵢ ≤ αᵢ₋₁ for all i, or equivalently, that there is at most one box in each row of β/α. Analogously, (β, α) ∈ HS will mean that β/α is a horizontal strip, i.e., β*/α* is a vertical strip. The notation (β, α) ∉ VS shall mean that either α ⊈ β, or that α ⊆ β but β/α is not a vertical strip, and similarly for (β, α) ∉ HS.

The union of two partitions β ∪ β' is defined to have i-th part max(βᵢ, β'ᵢ), so that D(β ∪ β') = D(β) ∪ D(β'). We will also use the notation α < β (lexicographic ordering) to mean that the first nonzero entry of (β₁ − α₁, β₂ − α₂, ...) is positive. Note that < is a total ordering which extends ⊆.

Fix a vector space V with an ordered basis x₁, ..., xₙ. This ordered basis determines a maximal torus and Borel subgroup T ⊂ B ⊂ GL(V). We identify partitions α with dominant weights of GL(V), and let SαV denote the irreducible representation of GL(V) with highest weight α, thought of as a factor module of Symα(V) = Symα₁(V) ⊗ ... ⊗ Symαₙ(V). We identify the elements of SαV with linear combinations of fillings of D(α) using elements of {1, ..., n}, where i is identified with the basis element xᵢ, which are weakly increasing top to bottom along columns modulo certain relations (see [Sam] for more details). A basis is given by semistandard Young tableaux, i.e., those fillings which are strictly increasing from left to right along rows. Under this identification, a highest weight vector is given by the tableau with all boxes in column i labelled with an i. We refer to this tableau as the canonical tableau. By the Weyl character formula, see [Mac I, Appendix A.8], the character of SαV is given by the Schur polynomial

\[ s_α = s_α(x_1, ..., x_n) = \frac{\det(x_i^{α_i+n-1})}{\det(x_j^{n-1})} \]. (1.1)
In Section 3, we will need to know that the definition of $S_\alpha V$ is functorial with respect to $V$ and extends to vector bundles.

Let $R(\text{GL}(V))$ be the Grothendieck ring of the tensor category of finite-dimensional rational representations of $\text{GL}(V)$. By semisimplicity, every representation can be written uniquely as a direct sum of irreducible representations $S_\alpha V \otimes (\Lambda V)^{\otimes r}$ where $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n = 0)$ and $r \in \mathbb{Z}$. Let $\mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ denote the ring of symmetric functions, and define the character $\text{ch}: R(\text{GL}(V)) \to \mathbb{Z}[x_1, \ldots, x_n]^{S_n}[(x_1 \cdots x_n)^{-1}]$, which is a ring isomorphism given by

$$
\text{ch} \left( \bigoplus_\lambda \left( S_\lambda V \otimes (\Lambda V)^{\otimes c_\lambda} \right) \right) = \sum_\lambda c_{\lambda\alpha} \cdot (x_1 \cdots x_n)^{c_\lambda}. \quad (1.2)
$$

### 1.2 Olver’s description of Pieri inclusions.

The following formula is crucial for the existence of our equivariant resolutions.

**Theorem 1.3 (Pieri’s formula).** Let $\alpha$ be a partition, and let $b$ be a positive integer. We have isomorphisms of $\text{GL}(V)$-modules

$$
S_\beta V \otimes S_\alpha V \cong \bigoplus_{(\beta, \alpha) \in VS} S_\beta V, \quad \text{with } |\beta/\alpha| = b
$$

$$
\bigwedge^b V \otimes S_\alpha V \cong \bigoplus_{(\beta, \alpha) \in HS} S_\beta V, \quad \text{with } |\beta/\alpha| = b
$$

**Proof.** See [Mac] (5.16), (5.17) or [Wey], Corollary 2.3.5. Note that in both sources, the convention for Young diagrams is transpose to ours, and that in [Wey], $L_\lambda E$ is an irreducible representation with highest weight $\lambda^*$. □

In particular, we get inclusions $S_\beta V \to S_\alpha V \otimes S_\beta V$, which are well-defined up to a (nonzero) scalar multiple. We call such maps **Pieri inclusions**.

In fact, one can describe this map explicitly with respect to the basis of semistandard Young tableaux. The following description for the case when $b = 1$, i.e., $\beta/\alpha$ is a single box, and we have a map $S_\beta V \to V \otimes S_\alpha V$, comes from [Olv], §6 where they are called polarization maps. The general case is not treated in [Olv], so we provide a self-contained account after describing Olver’s work.

First, we work with more general “shapes.” That is, diagrams $D(\lambda)$ obtained by dropping the requirement that $\lambda_1 \geq \lambda_2 \geq \cdots$. Given a tableau $T$ with underlying shape $\lambda$ and indices $i < j$, set $\tau_{ij}(T)$ to be the sum of all fillings of shapes obtained by removing a box along with its label from the $j$th column (and the boxes below it are shifted up to fill in the hole) and appending it to the end of the $i$th column. There are of course $\lambda_j$ such ways to do so counting multiplicity. If $i = 0$, then we consider the box to be in the “0th column” (this will correspond to the $V$ in $V \otimes S_\alpha V$). Given an increasing sequence $J = (j_1 < j_2 < \cdots < j_r)$, we define $\tau_J = \tau_{j_{r-1}j_r} \circ \cdots \circ \tau_{j_1j_2}$, and define $\#J = r$. The fillings obtained need not be semistandard, but they are well-defined elements of $V \otimes S_\alpha V$ (see Section 1.1).

Now suppose that $\beta/\alpha$ is a single box in the $k$th column. Given our basis $\{x_1, \ldots, x_n\}$ of $V$, the basis elements of $S_\beta V$ are identified with semistandard tableaux of shape $\beta$ with labels $\{1, \ldots, n\}$, and the basis elements of $V \otimes S_\alpha V$ are identified with elements $x_i \otimes T$ where $1 \leq i \leq n$ and $T$ is a semistandard tableau of shape $\alpha$ (the variable can be thought of as the “0th column.”) Let $B_k$
be the set of strictly increasing sequences \( j_1 < j_2 < \cdots < j_r \) (of all lengths \( r \)) such that \( j_1 = 0 \) and \( j_r = k \). For \( J \in B_k \), define the coefficients

\[
c_J = \prod_{i=2}^{#J-1} (\beta_{j_i} - \beta_k + k - j_i)
\]

(1.4)

(the empty product is 1). Then the Pieri inclusion is

\[
\sum_{J \in B_k} (-1)^{#J} c_J.
\]

Essentially, the Pieri inclusion is obtained by summing “all possible ways to remove a box from a semistandard tableau of shape \( \beta \) to get a variable in \( V \), times the remaining filling of \( \alpha \).” Of course, in general, this filling will not be semistandard but we can use the relations in \( \text{Sym}_V \) to write them in terms of a semistandard basis. Details and some examples can be found in [Sam].

In order to get the general case, one first picks a filtration of partitions \( \beta = \alpha^0 \supset \alpha^1 \supset \cdots \supset \alpha^b = \alpha \) where \( b = |\beta/\alpha| \) and each \( \alpha^j/\alpha^{j+1} \) is a single box. Composing the Olver maps, one gets a map

\[
\text{Sym}_\beta V \to V \otimes \text{Sym}_\alpha^1 V \to \cdots \to V^{\otimes b} \otimes \text{Sym}_\alpha V.
\]

To get the desired inclusion, we compose this with \( V^{\otimes b} \otimes \text{Sym}_\alpha V \to \text{Sym}_b V \otimes \text{Sym}_\alpha V \) where the map on the first component is the canonical projection of a tensor power onto a symmetric power, and the second component is the identity map.

The following lemma explains how to extend the above definition of Olver maps to the case that \( |\beta/\alpha| > 1 \). Since it is crucial for our proof of Theorem 2.2 and is not contained in [Olv], we give a proof.

**Lemma 1.6.** With the above notation, the composition \( \text{Sym}_\beta V \to \text{Sym}_b V \otimes \text{Sym}_\alpha V \) is nonzero. In fact, up to nonzero scalar multiples, the following diagram

\[
\begin{array}{cccccc}
\text{Sym}_\beta V & \to & V \otimes \text{Sym}_\alpha^1 V & \to & \cdots & \to & V^{\otimes b} \otimes \text{Sym}_\alpha V \\
\psi & & 1 \otimes \psi & & \cdots & & 1 \otimes \psi \\
V \otimes \text{Sym}_\alpha V & \to & S_2 V \otimes \text{Sym}_\alpha^2 V & \to & \cdots & \to & S_b V \otimes \text{Sym}_\alpha^b V \\
p \otimes 1 & & p \otimes 1 & & \cdots & & p \otimes 1 \\
\end{array}
\]

commutes, where \( p : \text{Sym}_i V \otimes V \to \text{Sym}_{i+1} V \) is the usual projection map, and \( \psi \) denotes the Olver map as described above. Furthermore, the map \( \text{Sym}_\beta V \to \text{Sym}_b V \otimes \text{Sym}_\alpha V \) is nonzero for all filtrations \( \beta = \alpha^0 \supset \alpha^1 \supset \cdots \supset \alpha^b = \alpha \).

**Proof.** Let \( V \) have an ordered basis \( x_1, \ldots, x_n \) so that we may identify symmetric powers of \( V \) with monomials in the \( x_i \). Suppose that \( \alpha \) is obtained from \( \beta \) by removing boxes in columns \( c_1 \geq c_2 \geq \cdots \geq c_b \).

We first show that if the filtration is picked such that \( \alpha^{i-1}/\alpha^i \) is a single box in column \( c_i \), then the composition is nonzero. Let \( T_\lambda \) be the canonical tableau in \( \text{Sym}_\lambda V \). The image of \( T_\beta \) under the composition \( \text{Sym}_\beta V \to \text{Sym}_b V \otimes \text{Sym}_\alpha V \) is

\[
\beta_{c_1} \beta_{c_2} \cdots \beta_{c_b} x_{c_1} x_{c_2} \cdots x_{c_b} \otimes T_\alpha + o(T_\alpha)
\]

where \( o(T_\alpha) \) is a sum of tensors whose \( \text{Sym}_\alpha V \) component is a vector with weight lower than \( T_\alpha \). This is clear by induction on \( b \) from Olver’s description: of all the increasing sequences \( J \) in (1.5)
involved in the map $S_{i}V \otimes S_{\alpha}V \rightarrow S_{i}V \otimes \otimes S_{\alpha}V$, only $J = (0 < c_i)$ has the property that $x_{c_1} \cdots x_{c_{n-1}} \otimes \tau(t_{c_{n-1}})$ is written as a linear combination of basis vectors which can contain $x_{c_1}x_{c_2} \cdots x_{c_i} \otimes \tau(\alpha)$ with a nonzero coefficient when mapped to $S_{\alpha}V \otimes \otimes S_{\alpha}V$.

Now we show that the map is nonzero independent of the chosen filtration. We first assume that $b = 2$. Let $i = c_1$ and $j = c_2$. We may assume that $i \neq j$ or there is nothing to show. Then of course $\beta_i > \beta_j$ because $\beta/\alpha$ is a vertical strip. Let $\beta'$ be $\beta$ with a box in column $j$ removed. Let $\varphi_1$ be the first map $S_\beta V \rightarrow V \otimes S_{\beta'}V$, and let $\varphi_2$ be the second map $V \otimes S_{\beta'}V \rightarrow V \otimes V \otimes S_{\alpha}V$, where the map on the first factor of $V$ under $\varphi_2$ is the identity.

There are two basis elements of $V \otimes S_{\beta'}V$ with nonzero coefficient in $\varphi_1(T_{\beta})$ which can map to $x_i x_j \otimes T_{\alpha}$. Namely, the first is $C_1 x_i x_j \otimes T_{\alpha}$, and the second is $C_2 x_i \otimes L$ for some coefficients $C_1, C_2$, where $L$ is the tableau with $\beta_k$'s in the $k$th column if $k \neq i$ and $k \neq j$, contains $\beta_i - 1$'s and 1 $j$ in the $i$th column, and $\beta_j - 1$'s in the $j$th column. Using Olver’s description, the coefficients are

$$C_1 = \beta_j, \quad C_2 = -\frac{\beta_i \beta_j}{\beta_i - \beta_j + j - i}.$$

Now, we also have

$$\varphi_2(T_{\beta'}) = \beta_i x_i T_{\alpha} + o(T_{\alpha}), \quad \varphi_2(L) = x_j T_{\alpha} + o(T_{\alpha}),$$

so putting it all together,

$$\varphi_2(\varphi_1(T_{\beta})) = C_1 x_i x_j \varphi_2(T_{\beta'}) + C_2 x_i \varphi_2(L) + o(T_{\alpha})$$

$$= \beta_i \beta_j x_i x_j T_{\alpha} - \frac{\beta_i \beta_j}{\beta_i - \beta_j + j - i} x_i x_j T_{\alpha} + o(T_{\alpha})$$

$$= \beta_i \beta_j x_i x_j \left(1 - \frac{1}{\beta_i - \beta_j + j - i}\right) T_{\alpha} + o(T_{\alpha}).$$

(1.7)

Finally, we know that $\beta_i - \beta_j + j - i \geq 2$, so the coefficient of $T_{\alpha}$ in the last expression is nonzero. Hence for $r = 2$, both ways of composing Olver maps give nonzero maps, and hence must be scalar multiples of each other.

For the general case $r \geq 2$, note that any permutation of the order of box removals is valid because $\beta/\alpha$ is a vertical strip. Hence, any two permutations of compositions are scalar multiples of each other because the symmetric group is generated by transpositions.

We take a minute to discuss the scalar multiples that appear in the above proof. Let $f_1$ be the composition $S_\beta V \rightarrow S_{\alpha}V \otimes S_{\alpha}V$ obtained by removing the boxes in increasing order of column index $c_1 \leq c_2 \leq \cdots \leq c_b$, and for a permutation $\sigma \in \mathfrak{S}_b$, let $f_\sigma$ be the composition obtained by removing the boxes in the order $c_{\sigma^{-1}(1)}, c_{\sigma^{-1}(2)}, \ldots, c_{\sigma^{-1}(b)}$. If $\beta/\alpha$ has $b_i$ boxes in the $i$th column, then $f_\sigma = f_\tau$ if $\sigma$ and $\tau$ represent the same left coset in $\mathfrak{S}_b / (\mathfrak{S}_{b_1} \times \ldots \times \mathfrak{S}_{b_n})$, where $\mathfrak{S}_{b_1} \times \ldots \times \mathfrak{S}_{b_n}$ is the subgroup of permutations which maps $\{b_1 + \ldots + b_{i-1} + 1, \ldots, b_1 + \ldots + b_i\}$ amongst themselves for $i = 1, \ldots, n$. We have seen that if $s_i$ is the transposition $(i, i + 1)$ and $c_i \neq c_{i+1}$, then

$$f_{s_i} = \left(1 - \frac{1}{\beta_{c_i} - \beta_{c_{i+1}} + c_{i+1} - c_i}\right) f_1.$$

The formula for $f_\sigma$ in terms of $f_1$ is complicated in general, so we content ourselves with this special case.

The above proof also shows that we can replace symmetric powers with exterior powers.

**Corollary 1.8.** Using the notation of this section, replacing the map $V^{\otimes b} \otimes S_{\alpha}V \rightarrow S_b V \otimes S_{\alpha}V$ by $V^{\otimes b} \otimes S_{\alpha}V \rightarrow \wedge^b V \otimes S_{\alpha}V$ gives a nonzero composition $S_\beta V \rightarrow \wedge^b V \otimes S_{\alpha}V$ whenever $\beta/\alpha$ is a horizontal strip.
Proof. It is enough to note that in modifying the proof of Lemma 1.6 to work for \( \wedge V \), the only change is in (1.7), where \( 1 - \frac{1}{\beta_i - \beta_j + j - i} \) is replaced by \( 1 + \frac{1}{\beta_i - \beta_j + j - i} \), which is also nonzero.

2 Equivariant resolutions for the general linear group.

In Section 2.1, we recall the construction of Eisenbud, Fløystad, and Weyman for pure free equivariant resolutions (Theorem 2.2) which resolves a special class of Pieri inclusions. We generalize this construction to the case of an arbitrary Pieri inclusion \( \varphi(\alpha, \beta) \) and to direct sums of Pieri inclusions. In order to describe our resolution we introduce some combinatorial notions in Section 2.2. The actual resolution (and some generalizations) are given in Section 2.3 and examples are given in Section 2.4.

2.1 Pure free resolutions.

In this section, we describe the equivariant pure resolutions of \([EFW]\) and prove their acyclicity using Lemma 1.6.

First we recall the notion of a pure free resolution. Every graded \( A \)-module \( M \) has a minimal graded free resolution \( F_\bullet \to A \to 0 \) which is unique up to isomorphism. The number of minimal generators of degree \( i \) of \( F_j \) is \( \mathbf{B}(M)_{i,j} \), which are the graded Betti numbers of \( M \). The usual convention for representing Betti numbers is via a Betti diagram/table: this is an array of numbers whose \( i \)th column and \( j \)th row contains \( \mathbf{B}(M)_{i,j} \). Thinking of \( K = A/(x_1, \ldots, x_n) \) as a trivial \( A \)-module, we note that \( \mathbf{B}(M)_{i,j} = \dim_K \text{Tor}^A_i(M, K) \). Then \( M \) has a pure free resolution if for each \( i \), \( \mathbf{B}(M)_{i,j} \) is nonzero for at most one value of \( j \), i.e., each syzygy module of a minimal free resolution of \( M \) is generated in a single degree. We also say that \( \mathbf{B}(M)_{i,j} \) is a pure Betti diagram/table. In this case, we define \( d_i \) to be the degree of \( F_i \), and the sequence \( d = (d_i) \) is the degree of \( F_\bullet \).

Let \( \alpha \) and \( \beta \) be partitions such that \( \beta/\alpha \) is a vertical strip of size \( b > 0 \). By choosing a scalar multiple for the Pieri inclusion \( S_\beta V \to S_\alpha V \), we get a uniquely determined (up to a nonzero scalar) equivariant map of \( A \)-modules

\[
\varphi(\alpha, \beta) : A(-b) \otimes S_\beta V \to A \otimes S_\alpha V
\]

of degree 0. (Here \( A(a) \) denotes a grading shift by \( a \).) Our goal is to describe an equivariant minimal free resolution of the \( A \)-modules \( \text{coker}(\varphi(\alpha, \beta)) \). First, we recall the case when the cokernel has finite length. This corresponds to the case when \( \beta/\alpha \) contains boxes only in the first column.

Set \( e_1 = \beta_1 - \alpha_1 \); for \( i > 1 \) set \( e_i = \alpha_{i-1} - \alpha_i + 1 \). Define a sequence \( d = (d_0, \ldots, d_n) \) by \( d_0 = 0 \) and \( d_i = e_1 + \cdots + e_i \) for \( i \geq 1 \), and define some partitions

\[
\alpha(d, i) = (\alpha_1 + e_1, \alpha_2 + e_2, \ldots, \alpha_i + e_i, \alpha_{i+1}, \ldots, \alpha_n)
\]

for \( 1 \leq i \leq n \). Define graded \( A \)-modules \( F(d)_i \) for \( 0 \leq i \leq n \) by

\[
F(d)_0 = A \otimes S_\alpha V \\
F(d)_i = A(-e_1 - \cdots - e_i) \otimes S_{\alpha(d, i)} V, \quad (1 \leq i \leq n).
\]

The natural action of \( \text{GL}(V) \) on \( A = \bigoplus_{i \geq 0} S_i V \) and of \( \text{GL}(V) \) on \( S_{\alpha(d, i)} V \) gives an action of \( \text{GL}(V) \) on \( F(d)_i \). Picking a Pieri inclusion

\[
\psi : S_{\alpha(d, i)} V \to S_{e_i} V \otimes S_{\alpha(d, i-1)} V
\]
and identifying \( S_e V = \text{Sym}^e V \) gives a degree 0 map \( \partial_i : \mathbf{F}(d)_i \to \mathbf{F}(d)_{i-1} \) defined by \( \partial_i(p(x) \otimes v) = p(x) \cdot \psi(v) \) where \( p(x) \in A \) and \( v \in S_{\lambda(d,i)} V \).

Olver’s description of the Pieri inclusion gives the following lemma, which greatly simplifies the proof of Theorem 2.2 compared to the original proof found in [EFW].

**Lemma 2.1.** Pick \( \mu/\nu \in VS \). Given a partition \( \lambda \) such that \( \lambda/\mu \in VS \) and \( \lambda/\nu \in VS \), the composition \( S_\lambda V \to A \otimes S_\mu V \to A \otimes S_\nu V \), where the first map is a Pieri inclusion and the second map is induced by a Pieri inclusion, is nonzero.

**Proof.** Pick a filtration of partitions \( \lambda = \lambda^0 \supset \lambda^1 \supset \cdots \supset \lambda^r = \nu \) such that \( \lambda^r = \mu \) for \( r = |\lambda/\mu| \), and such that \( \lambda^i/\lambda^{i+1} \) is a single box for all \( j \). By Lemma 1.6, \( \partial_i : A \otimes S_\mu V \to A \otimes S_\nu V \) is equal to a composition of nonzero scalar multiples of Olver maps, one for each piece of the filtration \( \lambda^r \supset \lambda^{r-1} \supset \cdots \supset \lambda^s \). Again by Lemma 1.6, the composition of \( \partial_i \) with the composition of Olver maps \( S_\lambda V \to A \otimes S_\mu V \) corresponding to the filtration \( \lambda^0 \supset \lambda^1 \supset \cdots \supset \lambda^r \) is nonzero, which proves the claim.

For the following theorem, we point out that the notation \( (\lambda, \beta) \notin VS \) means that either \( \beta \not\subseteq \lambda \), or that \( \beta \subseteq \lambda \), but \( \lambda/\beta \) is not a vertical strip.

**Theorem 2.2** (Eisenbud–Fløystad–Weyman). With the notation above,

\[
0 \to \mathbf{F}(d)_n \to \cdots \to \mathbf{F}(d)_1 \to \mathbf{F}(d)_0
\]

is a \( GL(V) \)-equivariant minimal graded free resolution of \( M(d) = \text{coker} \partial_1 = \text{coker} \varphi(\alpha, \beta) \), which is pure of degree \( d \). Furthermore, there is an isomorphism (as \( GL(V) \)-representations)

\[
M(d) \cong \bigoplus_{(\lambda, \beta) \notin VS} S_\lambda V.
\]

The proof in [EFW] uses the Borel–Weil–Bott theorem. However, we appeal only to Olver’s description of the Pieri inclusion.

**Proof of Theorem 2.2.** That \( \mathbf{F}(d)_\bullet \) is a complex is obvious: we have picked the partitions \( \alpha(d, i) \) so that for any partition \( \lambda \) such that \( (\lambda, \alpha(d, i + 1)) \notin VS \), we have \( (\lambda, \alpha(d, i - 1)) \notin VS \), and then we use that the \( S_\lambda V \) are irreducible representations of \( GL(V) \). That it is acyclic follows from almost the same reason: if \( (\lambda, \alpha(d, i - 1)) \notin VS \) and \( (\lambda, \alpha(d, i)) \in VS \), then we have \( (\lambda, \alpha(d, i + 1)) \in VS \) by our choices of \( \alpha(d, i) \). However, one needs to know that the image of \( S_\lambda V \) under the map \( \mathbf{F}(d)_{i+1} \to \mathbf{F}(d)_i \) is not zero, and this is the content of Lemma 2.1.

**Example 2.3.** Let \( \alpha = (3, 1, 0, 0) \) and \( \beta = (5, 1, 0, 0) \), so that \( d = (0, 2, 5, 7, 8) \) and \( e = (2, 3, 2, 1) \). Then \( \alpha(d, i) \) is the partition such that \( D(\alpha(d, i)) \) is the subdiagram of the following diagram consisting of boxes with labels \( \leq i \):

\[
\begin{array}{ccc}
0 & 0 & 3 \\
0 & 2 & 3 \\
1 & 2 & \\
1 & & 
\end{array}
\]

and the complex \( \mathbf{F}(d)_\bullet \) looks like (where we use \( \lambda \) as shorthand for \( S_\lambda V \))

\[
0 \to (5, 4, 2, 1) \to (5, 4, 2, 0) \to (5, 4, 0, 0) \to (5, 1, 0, 0) \to (3, 1, 0, 0).
\]
Remark 2.4 (Pure free resolutions over $\bigwedge V$). Let $B = \bigwedge V$ be the exterior algebra of $V$. Given $\alpha$ and $\beta$ as before such that $\beta/\alpha$ contains boxes only in the first column. Define $\alpha(d,1) = \beta$, and $\alpha(d,i)$ for $i > 1$ by

$$\alpha(d,i)_j = \begin{cases} 
\beta_1, & \text{if } j = 1, \\
\alpha_{j-1} + 1, & \text{if } 2 \leq j \leq \min(i,n+1), \\
\alpha_j, & \text{if } i < j \leq n+1, \\
1, & \text{if } n+1 < j \leq i.
\end{cases}$$

Also, set $F'_0 = B \otimes S_\alpha^* V$ and $F'_i = B(-|\alpha(d,i)/\alpha|) \otimes S_{\alpha(d,i)}^* V$ for $i > 1$. Picking Pieri inclusions

$$S_{\alpha(d,i)}^* V \to \bigwedge V \otimes S_{\alpha(d,i-1)}^* V$$

where $e_i = |\alpha(d,i)/\alpha(d,i-1)|$ gives $B$-linear equivariant differentials $F'_i \to F'_{i-1}$. An analogue of Lemma 2.1 using Corollary 1.8 shows that $F'_\bullet$ is a free resolution of the cokernel of $F'_1 \to F'_0$.

2.2 Critical boxes and admissible subsets.

We consider partitions with at most $n$ parts and identify their Young diagrams as subsets of a grid $L$ of boxes with $n$ columns, going infinitely downwards. The boxes in this grid can be thought of as pairs $(j,i)$ with $j = 0,1,\ldots$ and $1 \leq i \leq n$. In this case, $i$ is the column index of $(j,i)$. Recall that the notation $\lambda > \mu$ refers to lexicographic ordering of partitions (see Section 1.1).

Let $\alpha$ and $\beta$ be partitions such that $(\beta,\alpha) \in \mathcal{V}S$. Let $c_1 < c_2 < \cdots < c_m$ denote the indices of the columns of the skew shape $\beta/\alpha$. Define the set of critical boxes as follows

$$C(\alpha,\beta) = \{(\alpha_{j-1} + 1, j) \in L \mid c_1 < j \leq n\}$$

We shall sometimes refer to the critical boxes by their column indices. Given a subset $J \subseteq C(\alpha,\beta)$, we denote by $\beta(J)$ the smallest partition whose Young diagram contains both $\beta$ and $J$. The subsets $J \subseteq C(\alpha,\beta)$ whose column indices are unions of subsets of consecutive integers of the form $\{c_i + 1, c_i + 2, \ldots, j\}$ are admissible. By convention, the empty set is admissible. The set of all admissible subsets is denoted $\text{Ad}(\alpha;\beta)$, and we define

$$\text{Ad}(\alpha;\beta)_i = \{J \in \text{Ad}(\alpha;\beta) \mid \#J + 1 = i\},$$

where $\#\emptyset = 0$. This definition will only be used in Theorem 2.12.

Example 2.5. Let $n = 8$, $\alpha = (4,4,3,2,1,0,0,0)$, and $\beta = (5,4,3,2,2,1,0,0)$. In the following picture, $\alpha$ is the set of white boxes, $\beta/\alpha$ is the set of framed boxes, and the critical boxes are marked with the symbol $\times$. The black boxes are holes and are not part of the diagram.

```
\begin{verbatim}
+----------------+
|   ×         × |
|  ×         × |
|  ×         × |
+----------------+
\end{verbatim}
```

From this, we can see that the column indices of the admissible subsets are arbitrary unions of the subsets $\{2\}$, $\{2,3\}$, $\{2,3,4\}$, $\{2,3,4,5\}$, $\{6\}$, $\{7\}$, and $\{7,8\}$.
We will need the combinatorics of admissible subsets in one other setting. Suppose we are given partitions \( \alpha \) and \( \beta^1 > \cdots > \beta^r \) such that \( (\beta^i, \alpha) \in \text{VS} \) for all \( j \), and \( \beta^i \not\subseteq \beta^k \) for \( i \neq k \). Suppose also that \( \beta^i/\alpha \) only contains boxes in a single column \( c_j \) for \( j = 1, \ldots, r \). By our assumptions, \( c_1 < \cdots < c_r \) and \( r \leq n \).

From before, we have already defined the sets \( \text{Ad}(\alpha; \beta^j) \) for \( j = 1, \ldots, r \). Set \( S = \{(I, (J_i)_{i \in I}) \mid I \subseteq \{1, \ldots, r \}, J_i \in \text{Ad}(\alpha; \beta^i)\} \). Note that the \( J_i \) may be empty. For an element \( J = (I, (J_i)_{i \in I}) \in S \), we define

\[
\beta(J) = \bigcup_{i \in I} \beta^i(J_i), \quad s(J) = \sum_{i \in I} (#J_i + 1).
\]

We will define the sets \( \overline{\text{Ad}}(\alpha; \beta)_i \) and \( \text{Ad}(\alpha; \beta)_i \) by induction on \( i \). First, we have \( \overline{\text{Ad}}(\alpha; \beta)_1 = \text{Ad}(\alpha; \beta)_1 = \{(1, \emptyset), \ldots, (\{r\}, \emptyset)\} \). In general, let \( \overline{\text{Ad}}(\alpha; \beta)_i \) be the set of \( J = (I, (J_i)_{i \in I}) \in S \) such that \( #I = i \) and such that there does not exist \( J' \in \overline{\text{Ad}}(\alpha; \beta)_{i'} \) with \( i' < i \) and \( \beta(J') = \beta(J) \). We will refer to this last condition as the irredundance condition. In our setting, this is equivalent to asking that \( \beta(J)/\alpha \) have exactly \( s(J) \) columns. Finally, let \( \text{Ad}(\alpha; \beta)_i \) denote the admissible sets \( J \in \overline{\text{Ad}}(\alpha; \beta)_i \) such that \( \beta(J) \) is a minimal partition (with respect to inclusion) of the set \( \{\beta(J') \mid J' \in \overline{\text{Ad}}(\alpha; \beta)_i\} \).

We claim that if \( J, J' \in \overline{\text{Ad}}(\alpha; \beta)_i \), then \( \beta(J) \neq \beta(J') \). To see this, let \( c_{i_1} \) be the first column index of \( \beta(J)/\alpha \) which is nonempty, and let \( J_{i_1} = \{c_{i_1}, c_{i_1}+1, \ldots, c_{i_1}+k_1\} \) be the longest consecutive sequence of indices such that the bottom box of column \( c_{i_1} + j \) in \( \beta(J) \) is in row \( \alpha c_{i_1} + j - 1 + 1 \) for \( j = 1, \ldots, k_1 \). Define \( c_{i_2} \) to be the next column index of \( \beta(J)/\alpha \), and define \( J_{i_2} \) similarly, etc. We set \( J = (\{i_1, \ldots, i_t\}, (J_{i_1}, \ldots, J_{i_t})) \). Our procedure minimizes \( t \), which we need to do since we are assuming that \( J \in \overline{\text{Ad}}(\alpha; \beta) \). Furthermore, the choice of indices \( \{i_1, \ldots, i_t\} \) uniquely determines the corresponding partitions \( \beta^1, \ldots, \beta^t \) for which \( \beta^k/\alpha \) is a single column in column index \( i_k \). Namely, we need to take \( i_k = j_k \) for all \( k \) by our assumptions on \( \alpha \) and \( \beta^1, \ldots, \beta^r \), so the claim follows.

This definition will be used in Corollary 2.10 and in the proof of Theorem 2.12.

### 2.3 Pieri resolutions for the general linear group.

Let \( \alpha \) and \( \beta^1 > \cdots > \beta^r \) be partitions such that \( (\beta^i, \alpha) \in \text{VS} \) for \( i = 1, \ldots, r \) such that \( \beta^i \not\subseteq \beta^j \) if \( i \neq j \). In this section we are concerned with the minimal free resolution of the cokernel of

\[
\bigoplus_{i=1}^r A(-|\beta^i/\alpha|) \otimes S_{\beta^i}V \to A \otimes S_{\alpha}V, \tag{2.6}
\]

where the maps are induced by Pieri inclusions. The maps of this form give presentations of arbitrary equivariant factors of the free module \( A \otimes S_{\alpha}V \). Let us briefly explain our choice of assumptions on \( \alpha \) and the \( \beta^i \). The assumption on the ordering of the \( \beta^i \) is of course harmless as \( > \) is a total order, and the assumption that \( (\beta^i, \alpha) \in \text{VS} \) is necessary to ensure that a nonzero map of the form \( A(-|\beta^i/\alpha|) \otimes S_{\beta^i}V \to A \otimes S_{\alpha}V \) exists. The assumption \( \beta^i \not\subseteq \beta^j \) is made to eliminate nonminimality: if \( \beta^i \subseteq \beta^j \), then the image of \( A(-|\beta^i/\alpha|) \otimes S_{\beta^i}V \) will be contained in the image of \( A(-|\beta^j/\alpha|) \otimes S_{\beta^j}V \).

We will denote this minimal resolution by \( \text{F}(\alpha; \beta)_\bullet : = \text{F}(\alpha; \beta^1, \ldots, \beta^r)_\bullet \), and call it a Pieri resolution even though we do not yet have the precise knowledge of its terms.

We first give an inductive procedure for building a free resolution using just the knowledge of the structure of Pieri resolutions in the case \( r = 1 \). We use this inductive procedure in Corollary 2.10 to give an explicit description in the case where each \( \beta^i/\alpha \) is a single column. Finally we use this special case to describe explicitly the Pieri resolutions when \( r = 1 \).
Theorem 2.7. Let \( \alpha \) and \( \beta^1 > \cdots > \beta^r \) be partitions such that \( (\beta^i, \alpha) \in \text{VS} \) for \( i = 1, \ldots, r \) and \( \beta^i \not\subseteq \beta^j \) for \( i \neq j \). An equivariant free graded resolution \( F'(\alpha; \beta)_* \) of

\[
M = \text{coker} \left( \bigoplus_{i=1}^r A (-|\beta^i/\alpha|) \otimes S_{\beta^i} V \to A \otimes S_\alpha V \right)
\]

can be expressed as an iterated mapping cone of Pieri resolutions coming from the case \( r = 1 \). The length of \( F'(\alpha; \beta)_* \) is \( \leq n + 1 - c \), where \( c \) denotes the index of the first column of \( \beta^1/\alpha \).

Proof. We do a double induction, first on \( n - c \), and secondly on \( r \). The base case \( n = c \) implies that \( r = 1 \), in which case there is nothing to do.

So suppose \( n > c \) and \( r > 1 \). We first note that

\[
M = \bigoplus_{(\lambda, \alpha) \in \text{VS}} S_\lambda V
\]

(as \( \text{GL}(V) \)-representations) by Lemma 2.1. Define

\[
N = \text{coker} \left( \bigoplus_{i=2}^r \beta^i \to \alpha \right) = \bigoplus_{(\lambda, \alpha) \in \text{VS}} S_\lambda V.
\]  \hspace{1cm} (2.8)

Both \( N \) and \( M \) are generated over \( A \) by \( S_\alpha V \). Choosing an inclusion \( S_\alpha V \to M \), we get a surjection \( N \to M \); let \( N' \) be the kernel of this map. Then \( N' \) is the direct sum of representations \( S_\lambda V \) corresponding to \( \lambda \) such that \( (\lambda, \alpha) \in \text{VS} \), \( (\lambda, \beta^1) \in \text{VS} \), and \( (\lambda, \beta^i) \not\in \text{VS} \) for \( i = 2, \ldots, r \). We describe all \( \lambda \) such that \( (\lambda, \beta^1) \in \text{VS} \) which do not appear in \( N' \).

If \( (\lambda, \beta^1) \in \text{VS} \) and \( (\lambda, \alpha) \not\in \text{VS} \), then \( (\lambda, \beta^1(j)) \in \text{VS} \) for some \( j \in \text{Ad}(\alpha, \beta^1) \). If \( (\lambda, \beta^1) \in \text{VS} \) and \( (\lambda, \beta^i) \in \text{VS} \) for some \( i \geq 2 \), then in particular \( \lambda \supseteq \beta^1 \cup \beta^i \), so \( (\lambda, \beta^1 \cup \beta^i) \in \text{VS} \). These are the only possibilities, and one can write

\[
N' = \bigoplus_{(\lambda, \beta^1) \in \text{VS}} S_\lambda V = \text{coker} \left( \bigoplus_{i=2}^r (\beta^1 \cup \beta^i) \oplus \bigoplus_{\{j\} \in \text{Ad}(\alpha, \beta^1)} \beta^1(j) \to \beta^1 \right).
\]

In fact, in the above presentation, we only need to take those partitions of

\[
\{\beta^1 \cup \beta^2, \ldots, \beta^1 \cup \beta^r\} \cup \bigcup_{\{j\} \in \text{Ad}(\alpha, \beta^1)} \beta^1(j)
\]

which are minimal with respect to inclusion. So let

\[
N' = \text{coker} \left( \bigoplus (\beta^1 \cup \beta^i) \oplus \bigoplus \beta^1(j) \to \beta^1 \right)
\]  \hspace{1cm} (2.9)

be such a minimal presentation.

For \( N' \), the number of relations is \( r - 1 \), so the Pieri resolution has been constructed by induction on \( r \). For \( N' \), the first column index of any \( (\beta^1 \cup \beta^i)/\beta^1 \) or any \( \beta^1(j)/\beta^1 \) is strictly bigger than

12
c because $\beta^1$ is largest in lexicographic order, and by definition of critical boxes. Hence the Pieri resolution of $N'$ has been constructed by induction on $n-c$. Let $(P_*,d)$ and $(P'_*,d')$ be the associated Pieri resolutions for the presentation \((2.8)\) of $N$ and presentation \((2.9)\) of $N'$, respectively. Extend the inclusion $f: N' \subseteq N$ to an equivariant map of resolutions $f_*: P'_* \to P_*$. and let $F_*$ be the mapping cone of $f_*$. We have $F_0 = P_0 = \alpha$ and $F_i = P'_i \oplus P_i$ for $i > 0$. The differentials of $F_*$ are

$$
\begin{bmatrix}
-d' & 0 \\
-f_i & d
\end{bmatrix} : P'_i \oplus P_{i+1} \to P'_{i-1} \oplus P_i,
$$

using the convention that $P'_{-1} = 0$, so it is clear that they are GL($V$)-equivariant. The Pieri resolution of $M$ is a direct summand of $F_*$. Writing $P'_*[-1]$ to mean $P'_i[-1] = P'_{i-1}$, we have a short exact sequence of chain complexes

$$0 \to P_* \to F_* \to P'_*[-1] \to 0$$

whose long exact sequence of homology shows that $F_*$ is acyclic.

The claim about the length of the resolution follows by induction, which gives the easily checked fact that $P'_*$ has length $\leq n - c$. 

While general Pieri resolutions may have complicated inductive constructions which involve many cancellations, there are some special cases when the explicit description can be written down.

**Corollary 2.10.** Let $\alpha$ and $\beta^1 > \cdots > \beta^r$ be partitions with at most $n$ columns such that $\beta^i/\alpha$ only contains boxes in the $c_i$th column. Using the definitions from Section 2.2 define

$$
\begin{align*}
F(\alpha;\beta)_0 &= A \otimes S_{\alpha} V, \\
F(\alpha;\beta)_i &= \bigoplus_{J \in \text{Ad}(\alpha;\beta)_i} A(\beta(\alpha;J)/\alpha) \otimes S_{\beta(J)} V \quad (1 \leq i \leq n + 1 - c_1).
\end{align*}
$$

Then there exist differentials such that $F(\alpha;\beta)_*$ is a minimal free graded resolution of

$$M = \text{coker} \left( \bigoplus_{i=1}^r A(\beta^i/\alpha) \otimes S_{\beta^i} V \to A \otimes S_{\alpha} V \right).$$

**Proof.** We can use the inductive procedure of Theorem 2.7. First, this description agrees with that of Theorem 2.2 in the case that $r = 1$ and $c_1 = 1$, which is clear from the definition of admissible subsets. The case of general $c_1$ can also be deduced from Theorem 2.2. In particular, suppose that $\beta/\alpha$ consists of $m$ boxes in the second column. Let $\bar{\alpha} = (\alpha_2 + m - 1, \alpha_2, \alpha_3, \ldots, \alpha_n)$. Then $F(\alpha;\beta)_i = F(\bar{\alpha};\alpha)_i + 1$ for $i \geq 0$. In other words, the resolution is obtained by removing the first term of the resolution of Theorem 2.2. We leave it to the reader to formulate the easy generalization when “second column” is replaced by “$c_1$th column.”

Using the notation from the proof of Theorem 2.2, $N$ is generated by $\alpha$ with relations $\beta^2, \ldots, \beta^r$, so satisfies the inductive hypothesis. Also, $N'$ is generated by $\beta^1$ with relations $\gamma^1 = \beta^1(c_1 + 1), \gamma^2 = \beta^1 \cup \beta^2, \ldots, \gamma^r = \beta^1 \cup \beta^r$. If $c_2 > c_1 + 1$, then these are minimal relations. Otherwise, if $c_2 = c_1 + 1$, we throw out $\beta^1(c_1 + 1)$ since it contains $\beta^1 \cup \beta^2$. For notation, let $\beta^- = \{\beta^2, \ldots, \beta^r\}$ so that we can write $\text{Ad}(\alpha;\beta^-)$ in place of $\text{Ad}(\alpha;\beta^2, \ldots, \beta^r)$, etc.

By induction, the minimal free resolution of $N'$ is described by the admissible sets $\text{Ad}(\beta^1;\gamma)$. The partitions appearing in the $k$th term of the resolution of $M$ obtained from the mapping cone construction are the partitions of the form $\beta^-(J)$ or $\gamma(J')$ for $J \in \text{Ad}(\alpha;\beta^-)_k$ or $J' \in \text{Ad}(\beta^1;\gamma)_{k-1}$. So we only need to show that this set is equal to $\{\beta(J) \mid J \in \text{Ad}(\alpha;\beta)_k\}$.
Supposing that this has been done, no two partitions which appear in $N$ or $N'$ can be the same because the partitions $\lambda$ of $N'$ have the property that $\lambda_{c_1} > \alpha_{c_1}$, while those of $N$ do not. Hence no cancellations occur, and the mapping cone of an equivariant chain map between the Pieri resolutions of $N'$ and $N$ lifting the inclusion $N' \subseteq N$ will be the Pieri resolution of $M$. This finishes the construction.

To finish the proof, we establish the promised bijection. First we construct a bijection

$$\varphi: \overline{\text{Ad}}(\alpha; \beta)_{k} \xrightarrow{\cong} \overline{\text{Ad}}(\alpha; \beta^-)_{k} \bigsqcup \overline{\text{Ad}}(\beta^1; \gamma)_{k-1}$$

for $k \geq 1$ such that $\beta(J) = \beta^-(\varphi(J))$ or $\beta(J) = \gamma(\varphi(J))$ depending on which set $\varphi(J)$ lives in. If $c_2 = c_1 + 1$ so that $\gamma^2, \ldots, \gamma^r$ are minimal relations for $N'$, then we use subsets of $\{2, \ldots, r\}$ to index admissible subsets of $\text{Ad}(\beta^1; \gamma)$ and $\overline{\text{Ad}}(\beta^1; \gamma)$ for consistency. Pick $J \in \overline{\text{Ad}}(\alpha; \beta)$ and write $J = (I, (J_i)_{i \in I})$. If $1 \notin I$, then $J \in \overline{\text{Ad}}(\alpha; \beta^-)_i$, and we set $\varphi(J)$ to be this copy of itself.

If $1 \in I$ and $J_1 \neq \emptyset$, then we set $\varphi(J) = (I, (\varphi(J_i))_{i \in I})$ where $\varphi(J_1) = J \setminus \{c_1 + 1\}$. If $1 \in I$ and $J_1 = \emptyset$, we set $\varphi(J) = (I \setminus \{1\}, (\varphi(J_i))_{i \in I \setminus \{1\}})$. In both cases, we set $\varphi(J)_i = J_i$ for $i > 1$, so that $\varphi(J) \in \overline{\text{Ad}}(\beta^1; \gamma)$.

To check that $\varphi$ is well-defined, we need to know that given $\varphi(J)$ and $\varphi(J')$ with $s(\varphi(J')) < s(\varphi(J'))$ and $\beta^-(\varphi(J)) = \beta^-(\varphi(J'))$, we have $\beta(J) = \beta(J')$, and we also need to show a similar statement with $\beta^-$ replaced by $\gamma$. Both of these statements are immediate.

Also $\varphi$ is injective by definition. To see that $\varphi$ is surjective, we have to check that given $J = (I, (J_i)_{i \in I})$ and $J' = (I', (J'_i)_{i \in I'})$ with $\# I < \# I'$ and $\beta(J) = \beta(J')$, we have equality of the partitions associated with $\varphi(J)$ and $\varphi(J')$. If both $I$ and $I'$ contain 1 or if neither $I$ nor $I'$ contains 1, then this is clear. If only one of $I$ and $I'$ contains 1, we cannot have $\beta(J) = \beta(J')$ by our assumption that only $\beta^1/\alpha$ contains boxes in the $c_1$-th column.

Therefore $\varphi$ has the desired properties. Now we show that $\varphi$ restricts to a bijection

$$\varphi: \text{Ad}(\alpha; \beta)_k \xrightarrow{\cong} \text{Ad}(\alpha; \beta^-)_k \bigsqcup \text{Ad}(\beta^1; \gamma)_{k-1}.$$ 

First, if $\beta^-(\varphi(J)) \subseteq \beta^-(\varphi(J'))$, then we have $\beta(J) \subseteq \beta(J')$ and a similar statement holds when we replace $\beta^-$ with $\gamma$. This implies that $\varphi$ is well-defined.

Now we establish surjectivity of $\varphi$. First pick $\varphi(J) \in \overline{\text{Ad}}(\alpha; \beta^-)_k$, we have to show that $J \in \overline{\text{Ad}}(\alpha; \beta)_k$. Suppose not, so that there exists $J' \in \overline{\text{Ad}}(\alpha; \beta)_k$ with $\beta(J') \nsubseteq \beta(J)$. Since $\beta(J)/\alpha$ contains no boxes in the $c_1$-th column, the same is true for $\beta(J')/\alpha$, so we know that $\varphi(J') \in \overline{\text{Ad}}(\alpha; \beta^-)_k$, and that $\beta^-(\varphi(J')) \nsubseteq \beta^-(\varphi(J))$, a contradiction.

Now pick $\varphi(J) \in \overline{\text{Ad}}(\beta^1; \gamma)_{k-1}$, we have to show that $J \in \overline{\text{Ad}}(\alpha; \beta)_k$. Again, suppose not so that there exists $J' \in \overline{\text{Ad}}(\alpha; \beta)_k$ such that $\beta(J') \nsubseteq \beta(J)$. Write $J = (I, (J_i)_{i \in I})$ and $J' = (I', (J'_i)_{i \in I'})$. We know that $1 \in I$. If $1 \notin I'$, then $\varphi(J') \in \overline{\text{Ad}}(\beta^1; \gamma)$, and we get a contradiction as before. So suppose that $1 \notin I'$. By the irredundancy condition in the definition of $\overline{\text{Ad}}$, we have that $s(J)$ is the number of columns of $\beta(J)/\alpha$, and similarly for $J'$. Hence $s(J) > s(J')$, which is a contradiction. We conclude that $J \in \overline{\text{Ad}}(\alpha; \beta)_k$, so $\varphi$ is surjective. Injectivity of $\varphi$ is immediate from injectivity of $\varphi$, so we have the desired bijection.$\square$

Now we specialize to the case that $r = 1$, and write $\beta = \beta^1$. Define equivariant $A$-modules $F(\alpha; \beta)_i$ as follows:

$$F(\alpha; \beta)_0 = A \otimes S_\alpha V,$$

$$F(\alpha; \beta)_i = \bigoplus_{J \in \text{Ad}(\alpha; \beta)_i} A(-|\beta(J)/\alpha|) \otimes S_{\beta(J)} V, \quad (0 < i \leq n).$$
Theorem 2.12 will show that one can choose equivariant differentials so that \( F(\alpha; \beta) \) becomes a minimal free resolution. Note that \( F(\alpha; \beta)_1 = A(-|\beta/\alpha|) \otimes S_\beta V \), and that these modules agree with those defined in Section 2.1 in the case that \( \beta/\alpha \) consists of just boxes in the first column.

Remark 2.11. The definition of \( F(\alpha; \beta) \) is natural from the following point of view. The idea is to consider which representations appear in the kernel of the map

\[
A(-|\beta/\alpha|) \otimes S_\beta V \to A \otimes S_\alpha V,
\]

i.e., those representations of highest weight \( \lambda \) where \( (\lambda, \alpha) \notin VS \), and \( (\lambda, \beta) \in VS \). Then we find a minimal generating set of such representations, and then surject onto them using other representations. This explains the \( F(\alpha; \beta)_2 \) term, and we continue in this way; the language of critical boxes and admissible sets is a convenient way to describe such minimal partitions.

Theorem 2.12. Let \( \alpha \) and \( \beta \) be partitions with at most \( n \) columns such that \( \beta \supseteq \alpha \) and \( \beta/\alpha \) is a vertical strip. Then there exist equivariant differentials \( F(\alpha; \beta)_{i+1} \to F(\alpha; \beta)_i \) making \( F(\alpha; \beta) \) a minimal free graded resolution of \( M = \text{coker}(\varphi(\alpha, \beta)) \). Furthermore, the length of \( F(\alpha; \beta) \) is \( n + 1 - c \), where \( c \) denotes the index of the first column of \( \beta/\alpha \).

Proof. First note that \( M \) is the direct sum (as \( GL(V) \) representations) of \( S_\lambda V \) over all \( \lambda \) such that \( (\lambda, \alpha) \in VS \) and \( (\lambda, \beta) \notin VS \). This follows from Lemma 2.1. Write \( \beta = \alpha + a_{i_1}e_{i_1} + \cdots + a_{i_b}e_{i_b} \) to mean that \( \beta \) is obtained from \( \alpha \) by adding \( a_{i_j} \) boxes in column \( i_j \). We proceed by induction on \( b \).

The case \( b = 1 \) follows from Corollary 2.7.

Now assume \( b > 1 \). Set \( \alpha' = \alpha + a_{i_1}e_{i_1} + \cdots + a_{i_{b-1}}e_{i_{b-1}} \) and define

\[
N = \bigoplus_{(\lambda, \alpha') \in VS} S_\lambda V, \quad \bigoplus_{(\lambda, \beta) \notin VS} S_\beta V, \quad \bigoplus_{(\lambda, \alpha'(i_j + 1)) \notin VS, (1 \leq j \leq b-1)} S_\lambda V,
\]

which is an \( A \)-submodule of \( M \). For clarity, we will abuse notation and abbreviate \( A(-|\beta(J)/\alpha|) \otimes S_{\beta(J)} V \) by simply \( \beta(J) \). The minimal free resolution \( Q_* \) of \( N \) has been constructed in Corollary 2.7. Namely, set \( \gamma_j = \alpha'(i_j + 1) \) for \( 1 \leq j \leq b - 1 \) and \( \gamma_b = \beta \). We have \( \gamma_j \subset \gamma_{j+1} \) if and only if \( i_j = i_{j-1} + 1 \), and there are no other inclusions. Let \( \gamma = \{\gamma^1, \ldots, \gamma^{b-2}, \gamma^b\} \) in the first case, and \( \gamma = \{\gamma^1, \ldots, \gamma^b\} \) in the second case. Then \( Q_k = \bigoplus_{J \in \text{Ad}(\alpha'; \gamma)_k} \bigoplus_{J \in \text{Ad}(\alpha'; \gamma)_k} N' \), where we are adopting the shorthand above.

The quotient \( N' = M/N \) is the direct sum (as a representation of \( GL(V) \); \( N' \) is not an \( A \)-submodule of \( M \) in general)

\[
N' = \bigoplus_{(\lambda, \alpha) \in VS} S_\lambda V, \quad \bigoplus_{(\lambda, \alpha') \notin VS} S_\lambda V,
\]

which is also the cokernel of a Pieri inclusion \( A(-|\alpha'/\alpha|) \otimes S_{\alpha'} V \to A \otimes S_\alpha V \). The minimal free resolution \( Q_*' \) of \( N' \) is

\[
0 \to \bigoplus_{J \in \text{Ad}(\alpha'; \alpha)_n} \alpha'(J) \to \cdots \to \bigoplus_{J \in \text{Ad}(\alpha'; \alpha)_2} \alpha'(J) \to \alpha' \to \alpha,
\]

which follows by induction since \( \alpha'/\alpha \) has \( b - 1 \) columns. Using the short exact sequence

\[
0 \to N \to M \to N' \to 0,
\]
we can construct a $\text{GL}(V)$-equivariant resolution $\tilde{P}_\bullet$ of $M$ whose terms are coordinatewise direct sums of the resolutions of $N$ and $N'$. So the partitions generating the module $\tilde{P}_k$ are

$$\{\alpha'(J) \mid J \in \text{Ad}(\alpha; \alpha')_k\} \bigsqcup \{\gamma(J) \mid J \in \text{Ad}(\alpha'; \gamma)_k\}.$$  

However, $\tilde{P}_\bullet$ is not a minimal resolution of $M$. In particular, any partition that appears as a generator of a module in the minimal resolution of $M$ (and which is not $\alpha$) must contain $\beta$. The terms generated by partitions which do not contain $\beta$ form a subcomplex of $\tilde{P}_\bullet$. After a change of basis, we can show that the positive degree terms of this subcomplex (so excluding the degree 0 term generated by $\alpha$) is also a subcomplex. Let $P_\bullet$ be the quotient of $\tilde{P}_\bullet$ by this subcomplex. Then $P_\bullet$ contains as generators only $\alpha$ in degree 0 and those generators of $\tilde{P}_\bullet$ which contain $\beta$. We claim that $P_\bullet$ is the minimal resolution of $M$.

To show this, it is enough to show that the terms of $P_\bullet$ agree with the terms of $F(\alpha; \beta)_\bullet$, since the generating partitions of the latter are all distinct. So we will be finished if we can establish a bijection

$$\psi: \text{Ad}(\alpha; \beta)_k \xrightarrow{\cong} \{J \in \text{Ad}(\alpha; \alpha')_k \mid \alpha'(J) \supseteq \beta\} \bigsqcup \{J \in \text{Ad}(\alpha'; \gamma)_k \mid \gamma(J) \supseteq \beta\}$$

for all $k$. So pick $J \in \text{Ad}(\alpha; \beta)_k$. We can uniquely write $J = \{i_{j_1} + 1, \ldots, i_{j_1} + k_1\} \cup \cdots \cup \{i_{j_s} + 1, \ldots, i_{j_s} + k_s\}$ so that $i_r + 1 \notin [i_{j_s} + 1, i_{j_s} + k_s]$ unless $r = s$. Then $k_1 + \cdots + k_t = k - 1$. If $i_b \in J$, then $J \in \text{Ad}(\alpha; \alpha')_k$ and $\alpha'(J) \supseteq \beta$, so we define $\psi(J)$ to be this copy of $J$.

Otherwise, if $i_b \notin J$, we first define $I = \{j_1, \ldots, j_t, b\}$ and $\psi(J)_b = \{i_{j_s} + 2, \ldots, i_{j_s} + k_s\}$ for $s = 1, \ldots, t - 1$, and do the same for $s = t$ if $b \neq j_t$. If $b \neq j_s$, set $\psi(J)_b = \emptyset$, and if $b = j_t$, set $\psi(J)_b = \{i_{j_s} + 1, \ldots, i_{j_s} + k_s\}$. Now set $\psi(J) = (I, (\psi(J))_{b \in I})$, which we claim is an element of $\text{Ad}(\alpha'; \gamma)_k$. To see that it is an element of $\tilde{\text{Ad}}(\alpha'; \gamma)_k$, it is enough to show that $\gamma(\psi(J))/\alpha'$ has exactly $k$ columns. This is true because the column indices of $\gamma(\psi(J))/\alpha'$ are precisely $J \cup \{i_b\}$. Now since $\gamma(\psi(J)) = \beta(J)$, if $J' \in \tilde{\text{Ad}}(\alpha'; \gamma)_k$ existed so that $\gamma(\psi(J')) \supseteq \gamma(\psi(J))$, one could use $J'$ to find a corresponding $J'' \in \tilde{\text{Ad}}(\alpha; \beta)$ so that $\beta(J'') \supseteq \beta(J)$. The argument is similar to the one found in the proof of Corollary 2.7, so we omit the details.

The above establishes that $\psi$ is well-defined, and it is injective by construction. To finish, we show that $\psi$ is also surjective. First pick $J \in \text{Ad}(\alpha; \alpha')_k$ such that $\alpha'(J) \supseteq \beta$. Then $i_b \in J$ and $J$ can also be considered an element of $\text{Ad}(\alpha; \beta)_k$, so we’re done in this case. Now pick $J \in \text{Ad}(\alpha'; \gamma)_k$ and write $J = (I, (J_i)_{i \in I})$. By the irredundancy condition in the definition of $\text{Ad}$, it follows that $J' = \{s \mid s \in I\} \cup \bigcup_{i \in I} J_i$ is a set of size $k$. So $J' \setminus \{i_b\} \in \text{Ad}(\alpha; \beta)_k$ and $\psi(J' \setminus \{i_b\}) = J$.

Finally, by definition the length of $F(\alpha; \beta)_\bullet$ is the size $s(J)$ of the largest admissible set $J$. Every admissible set is a subset of $\{c + 1, c + 2, \ldots, n\}$, and this set is admissible, so the second statement of the theorem follows.

There is a natural family of resolutions $F(\alpha; \beta)_\bullet$, which are pure.

**Corollary 2.13.** If $\beta/\alpha$ contains boxes only in the $i$th column and the $n$th column for some $i$, then the complex $F(\alpha; \beta)_\bullet$ is pure.

**Remark 2.14** (Pieri resolutions over $\wedge V$). Following Remark 2.4, it is not hard to see how to construct Pieri resolutions over the exterior algebra $B$. The only thing that changes is that columns and rows are swapped in the notion of critical boxes, and the resulting resolution will have infinite length and is eventually linear.

**Remark 2.15.** In principle, one can resolve a general finitely generated equivariant module $M$. Pick any cyclic $A$-submodule $N$ of $M$. Then $M/N$ has less generators than $M$, and resolutions for
$N$ and $M/N$ can be combined to give a resolution of $M$. We do not attempt to resolve cokernels of maps of the form

$$\bigoplus_{i=1}^r A \otimes S_{\beta_i} V \rightarrow \bigoplus_{j=1}^g A \otimes S_{\alpha_j} V$$

simply because the cokernel is not uniquely defined by the domain and codomain. It is not even enough to specify which maps $A \otimes S_{\beta_i} V \rightarrow A \otimes S_{\alpha_j} V$ are nonzero since for $r > 1$ and $g > 1$, the scalar multiples of the Olver maps used becomes important.

2.4 Examples.

Remark 2.16. Given an irreducible representation $S_{\alpha} V$, we can think of it as a trivial $A$-module, i.e., its annihilator is the maximal ideal $(x_1, \ldots, x_n)$. Then a minimal presentation of $S_{\alpha} V$ is obtained as follows. Let $1 = i_1 < \cdots < i_r \leq n$ denote the indices such that $\alpha_{i_j} < \alpha_{i_{j-1}}$ for $j = 2, \ldots, r$. Let $\beta^j = (\alpha_1, \ldots, \alpha_{i_j} + 1, \ldots, \alpha_n)$, so that

$$S_{\alpha} V = \text{coker} \left( \bigoplus_{i=1}^r A(-1) \otimes S_{\beta_i} V \rightarrow A \otimes S_{\alpha} V \right),$$

and one can write down its minimal resolution using Corollary 2.10.

In particular, if $\alpha = (0, \ldots, 0)$, then $S_{\alpha} V = K$, and the Pieri resolution in this case is the Koszul complex. In general, this resolution is $S_{\alpha} V$ tensored with the Koszul complex on all of the variables $x_1, \ldots, x_n$.

Example 2.17. Let $n = 4$, $\alpha = (5, 3, 1, 0)$, and $\beta = (6, 4, 1, 0)$. The corresponding Young diagram is

```
     "\x"  
    "1\x  
   "1\x  
```

The critical boxes are marked with $\times$, and our Pieri resolution is

$$0 \rightarrow (6, 6, 4, 2) \rightarrow (6, 6, 4, 0) \oplus (6, 4, 4, 2) \rightarrow (6, 6, 1, 0) \oplus (6, 4, 4, 0) \rightarrow (6, 4, 1, 0) \rightarrow (5, 3, 1, 0).$$

Example 2.18. Here is an example of a pure resolution of new type: $n = 4$, $\alpha = (5, 3, 1, 0)$, $\beta = (6, 3, 1, 1)$. The corresponding Young diagram is

```
     1
    "\x  
   "\x  
```

The critical boxes are marked with $\times$. The Pieri resolution $F(\alpha; \beta)$, we get is

$$0 \rightarrow (6, 6, 4, 2) \rightarrow (6, 6, 4, 1) \rightarrow (6, 6, 1, 1) \rightarrow (6, 3, 1, 1) \rightarrow (5, 3, 1, 0).$$

The length of the complex is 4, but the module it resolves has infinite length (the cokernel contains all highest weight modules for partitions $(5 + d, 3, 1, 0)$ for $d \geq 0$). Thus this module is not a maximal Cohen–Macaulay module.
Example 2.19. To illustrate Corollary \[2.10\] let \( \alpha = (4,3,1,0) \), \( \beta^1 = (6,3,1,0) \), and \( \beta^2 = (4,3,3,0) \). Then the column indices of the admissible subsets of \( C(\alpha, \beta^1) \) and \( C(\alpha, \beta^2) \) are, respectively, \( \{2\}, \{2,3\}, \{2,3,4\} \) and \( \{4\} \). In homological degree 2, the candidates for syzygy generators are \( \{\beta^1(2), \beta^1 \cup \beta^2, \beta^2(4)\} \), in degree 3, they are \( \{\beta^1(2,3), \beta^1(2) \cup \beta^2, \beta^1 \cup \beta^2(4)\} \), and in degree 4, they are \( \{\beta^1(2,3,4), \beta^1(2,3) \cup \beta^2, \beta^1(2) \cup \beta^2(4)\} \).

Since \( \beta^1(2,3) \cup \beta^2 = (6,5,4,0) = \beta^1(2,3) \), we remove it from our list of candidates in degree 4. Finally, we pick only those candidates which are minimal in their homological degree with respect to inclusion. The resolution is then

\[
0 \to (6,5,3,2) \to (6,5,3,0) \to (6,3,1,0) \to (4,3,1,0).
\]

Here we are stacking partitions as shorthand for the direct sum of the free \( A \)-modules generated by the corresponding representations.

Example 2.20. Here is an example to show that the representations that appear in a Pieri resolution may have nontrivial multiplicities. Let \( n = 3 \), \( \alpha = (3,1,0) \), \( \beta^1 = (4,3,0) \), \( \beta^2 = (3,3,1) \), and \( \beta^3 = (4,2,1) \). The Pieri resolution is

\[
0 \to (4,4,1) \to (4,4,0) \to (4,3,1) \to (4,2,1) \to (3,1,0).
\]

Again, we use the same shorthand from Example 2.19. Note that the partition \( (4,3,1) \) appears twice as a generator. This is reasonable: the representation \( (4,3,1) \) appears only once in homological degree 0, but appears 3 times in homological degree 1. Also note that this resolution is pure (though not of a Cohen–Macaulay module).

Example 2.21. Here is an example to show that the mapping cone construction of Theorem \[2.7\] need not return a minimal resolution. Let \( n = 4 \), \( \alpha = (4,2,1,0) \), \( \beta^1 = (5,3,1,0) \) and \( \beta^2 = (5,2,2,0) \). Working out the induction, we get the following resolution

\[
0 \to (5,5,3,2) \to (5,5,3,0) \to (5,5,2,0) \to (5,5,1,0) \to (5,3,1,0) \to (4,2,1,0).
\]

The actual calculation can be done in Macaulay 2 using the package PieriMaps [Sam], and one gets the following graded Betti table:
In particular, the differentials between the pair of representations $(5,2,2)$ and the pair of representations $(5,2,0)$ are isomorphisms. The minimal resolution of the cokernel is

$$0 \to (5,5,3,2) \to (5,5,3,3) \to (5,5,1,0) \to (5,2,2,2) \to (5,3,1,0) \to (5,2,2,0) \to (4,2,1,0).$$

**Remark 2.22.** For the sequence $e = (e_0, \ldots, e_n)$ (with $e_0 = 0$) we can produce $e_1$ pure complexes with shifts $e$ (corresponding to choices of $i$ boxes in the first column and $e_1 - i$ boxes in the last column, for $1 \leq i \leq e_1$). For example, take $n = 4$, $e = (0, 3, 4, 2, 1)$. There are three pure complexes in our family, corresponding to the Young diagrams:

3 Equivariant resolutions for other classical groups.

In this section we generalize the resolutions from [EFW] to other classical groups in the following way. We use the vector representation $F$ of an orthogonal or a symplectic group. Denote by $V_{\lambda}$ the irreducible representation of the corresponding classical group whose highest weight is $\lambda$. The highest weights in each case correspond to partitions with some restrictions (see Table 1 below). Consider the polynomial ring $A = \text{Sym}(F)$. For a pair of representations $V_{\mu}, V_{\lambda}$ such that $V_{\mu} \subset V_{\lambda} \otimes \text{Sym}^i F$ we can ask again about the resolution of the cokernel $M$ of the Pieri map $V_{\mu} \otimes A(-i) \to V_{\lambda} \otimes A$. Using the Pieri resolutions discussed in the previous sections and some sheaf cohomology, we construct resolutions for a certain quotient $N$ of this cokernel in the case that $|\mu| = |\lambda| + i$ (see Remark 3.1). The comments at the end of Section 3.3 explain the relationship between the resolutions of $M$ and $N$. More generally, we can also study direct sums of Pieri maps. We consider the cases of odd orthogonal, symplectic, and even orthogonal groups separately.

This section is independent of Section 4, so the reader not comfortable with the representation theory of classical groups can skip this section without any loss of continuity.

### 3.1 Notation.

Let $F$ be a $(2n+\tau)$-dimensional vector space over $K$ (where $\tau \in \{0, 1\}$) and let $\omega$ be a nondegenerate symplectic or symmetric bilinear form with signature $(n, n + \tau)$ on $F$. Let $G$ be the subgroup of $\text{SL}(F)$ which preserves $\omega$. In order to be precise let us just list the cases:

1. Case $B_n$: We have $\tau = 1$, $\omega$ is symmetric, $G = \text{SO}(F) \cong \text{SO}(2n + 1)$.
2. Case $C_n$: We have $\tau = 0$, $\omega$ is skew-symmetric, $G = \text{Sp}(F) \cong \text{Sp}(2n)$.
3. Case $D_n$: We have $\tau = 0$, $\omega$ is symmetric, $G = \text{SO}(F) \cong \text{SO}(2n)$.

We identify the weight lattice of $G$ with $\mathbb{Z}^n = \mathbb{Z}<\varepsilon_1, \ldots, \varepsilon_n>$ equipped with the standard dot product. The identification is crucial for our applications of Theorem 3.2 so we make this more
precise. In all three cases, we can find a basis $e_1, \ldots, e_{2n+\tau}$ for $F$ such that $1 = \omega(e_i, e_{2n+\tau+1-i}) = \pm \omega(e_{2n+\tau+1-i}, e_i)$ (the sign depending on whether $\omega$ is symmetric or skew-symmetric) for $i = 1, \ldots, n + \tau$, and all other pairings are 0. Representing elements of $G$ as matrices, we take our maximal torus $T$ to be the subgroup of diagonal matrices, and our Borel subgroup $B$ to be the subgroup of upper triangular matrices, so that we have a set of simple roots for $G$. We identify $(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with the character $\lambda: T \to K^*$ given by

$$\text{diag}(d_1, \ldots, d_{2n+\tau}) \mapsto \prod_{i=1}^{n} d_i^\lambda_i$$

To be completely explicit, we list the simple roots and the conditions for a weight to be dominant under this identification in Table 1. We choose this particular identification to be compatible with the identification of weights for $GL_n$. In particular, whenever we have an $n$-tuple $\lambda$, the statement that $\lambda$ is dominant for $GL_n$ will mean that $\lambda$ is a weakly decreasing sequence, and the statement that $\lambda$ is dominant for any other classical group $G$ will mean that it satisfies the appropriate condition according to Table 1.

| Simple roots | $SO(2n+1)$ | $Sp(2n)$ | $SO(2n)$ |
|--------------|-------------|----------|----------|
| $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3,$ | $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3,$ | $\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3,$ |
| $\ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n$ | $\ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n$ | $\ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n$ |
| Dominant weights | $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ | $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ | $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|$ |
| $\rho$ | $(\frac{n+1}{2}, \frac{n+1}{2}, \ldots, \frac{n+1}{2})$ | $(n, n-1, \ldots, 2, 1)$ | $(n-1, n-2, \ldots, 1, 0)$ |

For $\lambda$ a dominant weight of $G$, the notation $V_\lambda$ denotes an irreducible representation of $G$ with highest weight $\lambda$. For $\lambda = (k, 0, \ldots, 0)$, we just write $V_k$. In the case of the symplectic group, $V_k = \text{Sym}^k F$, but this is false in the orthogonal case: in fact, $V_k$ is the kernel of the contraction map $\text{Sym}^k F \to \text{Sym}^{k-2} F$ given by $x_1 \cdots x_k \mapsto \sum_{i < j} \omega(x_i, x_j) x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_k$.

Remark 3.1. To give the reader a sense of why resolving equivariant modules over orthogonal and symplectic groups might be more difficult than the case of the general linear group, we recall the corresponding analogues of Pieri’s rule.

Let $G$ be an orthogonal or symplectic group of rank $n$. The Newell–Littlewood rule (see [Kin, §4]) gives, under the assumption that $n \geq \ell(\lambda) + \ell(\mu)$, that $V_\lambda \otimes V_\mu = \bigoplus N^{\nu}_{\lambda,\mu} V_\nu$ where $N^{\nu}_{\lambda,\mu} = \sum_{\alpha,\beta,\gamma} c^{\lambda}_{\alpha,\beta} c^{\mu}_{\alpha,\gamma} c^{\nu}_{\beta,\gamma}$ and $c^{\ast,\ast}_{\ast,\ast}$ denotes the usual Littlewood–Richardson coefficient for the general linear group.

Now specialize to the case that $\mu = (k)$, so that $c^{\lambda}_{\alpha,\gamma} \neq 0$ in the above sum only if $\alpha$ and $\gamma$ are vertical strips of sizes $l$ and $k - l$ for some $l \leq k$. Hence, if $\lambda$ is a partition such that $\lambda_n = 0$, then $V_\lambda \otimes V_k \cong \bigoplus \mu V_{\mu}^{\otimes n_\mu}$ where the sum is over all $\mu$ which can be obtained from $\lambda$ by first removing a vertical strip of size $l \leq k$, and then adding a horizontal strip of size $k - l$ to the result, and $n_\mu$ is the number of different ways to obtain $\mu$ via this process.

The above formulas can still be interpreted when $\lambda_n \neq 0$ or $n < \ell(\lambda) + \ell(\mu)$. In this case, one needs to use certain modification rules to rewrite $V_\nu$ where $\ell(\nu) > n$ as $V_\eta$ where $\ell(\eta) \leq n$. See [Kin, §3] and [KT, §§2.4–2.5] for more details.

Hence it is not clear how to give a combinatorial description of the terms in an equivariant resolution since in general, the same representation may appear as a generator for several different syzygy modules. Furthermore, the Pieri rule for orthogonal and symplectic groups is not multiplicity free.
Let $X' = \{ V \in \text{Gr}(n, F) \mid \omega|_V = 0 \}$ be the Grassmannian of Lagrangian subspaces of $F$. Let $x_0 \in X'$ be the point representing the subspace $\langle e_1, \ldots, e_n \rangle$, and let $X$ be the connected component of $X'$ which contains $x_0$. Then $X$ is a homogeneous space for $G$, and we can identify $X$ with $G/P$ where $P$ is the parabolic subgroup which stabilizes $x_0$. On $X$, we have the tautological subbundle $\mathcal{R}$ of the trivial bundle $F \times X$ defined by $\mathcal{R} = \{(x, W) \mid x \in W\}$, and also the tautological quotient bundle $\mathcal{Q} = (F \times X)/\mathcal{R}$. Let $\mathcal{Q}^\vee$ denote the orthogonal complement of $\mathcal{R}$ with respect to $\omega$, and define $\mathcal{Q} = (F \times X)/\mathcal{R}^\vee$. Since we have perfect pairings $\omega: \mathcal{R} \times \mathcal{Q} \to K \times X$ and $\omega: \mathcal{R}^\vee \times \mathcal{Q}^\vee \to K \times X$, the form on $F$ gives identifications $\mathcal{Q}^* = \mathcal{R}$ and $\mathcal{Q}^\vee = \mathcal{R}^\vee$. When $F$ is even-dimensional, $\mathcal{R} = \mathcal{R}^\vee$, and $\mathcal{Q} = \mathcal{Q}^\vee$.

We shall not make a distinction between vector bundles and locally free sheaves, so that $\mathcal{R}, \mathcal{Q}, \ldots$ will be a sheaf when we want to calculate its cohomology, and will be a vector bundle when we need to work with its total space.

There is an equivalence between homogeneous bundles over $X = G/P$ and rational representations of $P$ defined by sending a homogeneous bundle $\mathcal{E}$ to its fiber over the point $x_0$, which represents the coset $P$. In the other direction, given a rational $P$-module $U$, we define a homogeneous bundle

$$G \times^P U = G \times U/\{(g, u) \sim (gp, p^{-1}u) \mid p \in P\}$$

with $G$ acting on the first factor by multiplication on the left, and the structure map $G \times^P U \to G/P$ is given by $(g, u) \mapsto gP$. Under this equivalence, $\mathcal{R}$ is associated with an isotropic subspace $R \subset F$, and $\mathcal{Q}$ to the quotient $Q = F/R^\vee$. Hence, $S_\lambda Q$ is a homogeneous bundle over $X$, and we will need to know something about its cohomology.

Let $W$ be the Weyl group of $G$, and let $\ell(\sigma)$ denote the length of $\sigma \in W$. We define a dotted action of $W$ on the weights of $G$ by $\sigma^*(\lambda) = \sigma(\lambda + \rho) - \rho$, where $\rho$ is given in Table I.

**Theorem 3.2.** With the notation above, one of two mutually exclusive cases occurs:

1. There exists $\sigma \in W$ such that $\sigma^*(\alpha) = \alpha$. In this case, $H^i(G/P; S_\alpha Q) = 0$ for all $i$.

2. There exists a unique $\sigma \in W$ such that $\beta = \sigma^*(\alpha)$ is a dominant weight for $G$. Then

$$H^i(G/P; S_\alpha Q) = \begin{cases} V^*_\beta & \text{if } i = \ell(\sigma), \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** This theorem is a special case of the Borel–Weil–Bott theorem for the homogeneous space $G/P$. See [Jan, §II.5] for a proof of the Borel–Weil–Bott theorem, and see [Wey, §4.3] for more details regarding our special case. It should be mentioned that in [Wey, §4.3] there is an error: the $V_\alpha$ should be $V_\alpha^*$ in Theorem 4.3.1, and similarly for the other results in that section. 

**Remark 3.3.** In types $B_n$ and $C_n$, all representations are self-dual, so that $V^*_\beta \cong V_\beta$. The same is true for type $D_n$ when $n$ is even. However, when $n$ is odd, the dual of a representation of type $D_n$ with highest weight $(\beta_1, \ldots, \beta_{n-1}, \beta_n)$ has highest weight $(\beta_1, \ldots, \beta_{n-1}, -\beta_n)$.

### 3.2 General setup.

Now define $B = \text{Sym}(Q)$. Let $\alpha$ and $\beta^1, \ldots, \beta^r$ be partitions with at most $n$ parts such that $\beta^i/\alpha$ is a vertical strip for $i = 1, \ldots, r$. Pieri’s rule extends to the setting of vector bundles, so the sheaf $S_\alpha Q \otimes B$ contains $S_{\beta^i} Q$ as a direct summand with multiplicity one, and hence we have a unique (up to nonzero scalar) $\text{GL}(F)$-equivariant morphism of sheaves

$$\bigoplus_{i=1}^r B \otimes S_{\beta^i} Q \to B \otimes S_\alpha Q.$$
We can resolve the cokernel of this map via the relative version of the minimal Pieri resolution, $F_* := F(\alpha; \beta)_*$ defined in Section 2 (see the remarks before Theorem 2.7), obtained by substituting $Q$ for $V$.

By Theorem 3.2, the sheaves that appear in $F_*$ do not have any higher cohomology, so taking sections, we get a $G$-equivariant acyclic complex

$$0 \to H^0(X; F_n) \to \cdots \to \bigoplus_{i=1}^{r} H^0(X; B \otimes S_{\beta^i} Q) \to H^0(X; B \otimes S_{\alpha} Q) \to M(\alpha; \beta) \to 0, \quad (3.4)$$

where $M(\alpha; \beta)$ is by definition the cokernel of the map preceding it. Letting $p: F \times X \to X$ denote the projection onto the second factor, one has $p_*(\mathcal{O}_\mathcal{R}) = B$ by [Wey, Proposition 5.1.1(b)], so

$$H^0(X; B \otimes S_{\lambda} Q) \cong H^0(F \times X; \mathcal{O}_\mathcal{R} \otimes p^*(S_{\lambda} Q))$$

by the projection formula. Define $A = \text{Sym}(F)$ to be the symmetric algebra of $F$. Since $\mathcal{O}_\mathcal{R}$ is a quotient of $\mathcal{O}_{F \times X}$, the above groups inherit the structure of graded $A$-modules. If we set the generator of $M(\alpha; \beta)$ to have degree 0, then the module $H^0(X; B \otimes S_{\lambda} Q)$ in (3.3) needs to be shifted by degree $-|\lambda/\alpha|$ in order for the differentials to be degree 0. However, these terms are not in general free $A$-modules. So first we find free resolutions for each $H^0(X; B \otimes S_{\lambda} Q)$ -module by taking an iterated mapping cone.

The maps in (3.4) will induce maps between these resolutions, and we can put these resolutions together to get a free resolution of $M(\alpha; \beta)$ as an $A$-module by taking an iterated mapping cone.

Motivated by this, we call modules of the form $H^0(X; B \otimes S_{\lambda} Q)$ geometric modules, and let $G(\lambda)_*$ denote their minimal free resolutions over $A$. In order to calculate $G(\lambda)_*$, we will need the following result.

**Theorem 3.5.** Let $Y$ be a projective variety over $K$ and $V$ be a vector bundle over $Y$. Let $E = K^N \times Y$ denote the trivial vector bundle of rank $N$ over $Y$, and let $S \subset E$ be a subbundle with quotient bundle $T = E/S$. Letting $A$ be the coordinate ring of $K^N$, there exists a complex $F(V)_*$ of free $A$-modules with minimal differentials of degree 0 whose terms are given by

$$F(V)_i = \bigoplus_{j \geq 0} H^j(Y; \bigwedge^{i+j}(T^*) \otimes \mathcal{O}_Y V) \otimes_K A(-i - j) \quad (i \in \mathbb{Z}),$$

and whose homology groups are concentrated in degrees $i \leq 0$, given by

$$H_i(F(V)_*) \cong H^{-i}(Y; \text{Sym}(S^*) \otimes V).$$

**Proof.** See [Wey, Theorem 5.1.2].

For our application, we take $Y = X$, $V = S_{\lambda} Q$, $N = \dim F$, and $S = \mathcal{R}$. By Theorem 3.2, the complex $F(V)_*$ is exact in degrees $i \neq 0$, so we get a free resolution of $H^0(X; B \otimes S_{\lambda} Q)$ over $A$. In fact, this resolution is $G$-equivariant (see [Wey, Theorem 5.4.1]). So we have reduced the problem to calculating the cohomology groups $H^i(X; \bigwedge^{i+j}(\mathcal{R}^*) \otimes S_{\lambda} Q)$. We will do this individually for the special orthogonal and symplectic groups in the following subsections.

Now we can state the main result of this section.

**Theorem 3.6.** The iterated mapping cone of the complexes $G(\lambda)_*$ resolving the terms in (3.4) is a minimal resolution of $M(\alpha; \beta)$.

In order to prove this, we shall make use of the following technical lemma.
Lemma 3.7. Suppose $F_\bullet \to M \to 0$ is a minimal acyclic complex of graded $A$-modules (i.e., the differentials have positive degree). Let $F_\bullet^i \to F_i$ be a minimal free resolution over $A$ with differentials denoted $\partial^i$ for each $i$, so that we have induced differentials $d: F_i^i \to F_i^{i-1}$. Suppose further that each $F_\bullet^i$ has a filtration of subcomplexes $0 = F_\bullet^i[-1] \subseteq F_\bullet^i[0] \subseteq F_\bullet^i[1] \subseteq \cdots \subseteq F_\bullet^i$ such that:

(a) Each homogeneous component of $F_\bullet^i$ either intersects the $j$th graded part of the filtration in zero, or is entirely contained in it.

(b) For a homogeneous element $x$, let $\text{grade}(x)$ be the number $g$ such that $x \in F_\bullet^i[g] \setminus F_\bullet^i[g-1]$. Then whenever $x \in F_\bullet^i$ is homogeneous such that $\partial^i(x) \neq 0$, then $\text{deg}(\partial^i(x)) - \text{deg}(x) = \text{grade}(x) - \text{grade}(\partial^i(x)) + 1$.

(c) The induced maps $d: F_\bullet^i \to F_\bullet^{i-1}$ satisfy the inequality $\text{deg}(d(x)) - \text{deg}(x) \geq \text{grade}(x) - \text{grade}(D(x)) + 1$ whenever $x$ is homogeneous.

Then the iterated mapping cone of the $F_i$ forms a minimal resolution of $M$.

Proof. The differentials in the mapping cone have the form $D: F_k^i \to F_{k-1+i}^{i-j}$. It is enough to prove that each such map is either 0, or has positive degree. More specifically, we will show by double induction on $j$ and $k$ that

$$\text{deg}(D(x)) - \text{deg}(x) \geq \text{grade}(x) - \text{grade}(D(x)) + 1 \quad (3.8)$$

whenever $D(x) \neq 0$.

The case $j = 0$ is the content of (b). Now suppose $j > 0$. In general, if a nonzero differential $D: F_k^i \to F_{k-1+i}^{i-j}$ exists, then it was induced from a diagram

$$
\begin{array}{c}
\text{F}_k^{i-j} \\
\partial^{i-j} \downarrow \\
\text{F}_{k-1+j}^{i-j}
\end{array}
\xrightarrow{d_1} \begin{array}{c}
\text{F}_{k-1+j}^{i-\ell} \\
\text{D}
\end{array} \xrightarrow{d_2} \begin{array}{c}
\text{F}_k^i
\end{array}
$$

for some $\ell < j$. If $j > 1$ and $k \geq 0$, then $\ell > 0$, so each of $d_1$, $d_2$, and $\partial^{i-j}$, satisfies (3.8) by induction on $j$. If $j = 1$ and $k > 0$, then we also use induction on $k$. For the case $j = 1$ and $k = 0$, (3.8) is the content of (c). Now we have for $x \in F_k^i$

$$
\begin{align*}
\text{deg}(D(x)) - \text{deg}(x) + (\text{deg}(\partial^{i-j}D(x)) - \text{deg}(D(x))) \\
= (\text{deg}(d_2(x)) - \text{deg}(x)) + (\text{deg}(d_1d_2(x)) - \text{deg}(d_2(x))) \\
\geq (\text{grade}(x) - \text{grade}(d_2(x)) + 1) + (\text{grade}(d_2(x)) - \text{grade}(d_1d_2(x)) + 1).
\end{align*}
$$

By (b), we have that $\text{deg}(\partial^{i-j}D(x)) - \text{deg}(D(x)) = \text{grade}(D(x)) - \text{grade}(\partial^{i-j}D(x)) + 1$, and using (a), we have $\text{grade}(d_1d_2(x)) = \text{grade}(\partial^{i-j}D(x))$. So we conclude that

$$
\text{deg}(D(x)) - \text{deg}(x) \geq \text{grade}(x) - \text{grade}(D(x)) + 1,
$$

as desired.

To apply this lemma to our situation, we let $G(\lambda)_\bullet[0]$ be the subcomplex of $G(\lambda)_\bullet$ consisting of the $H^0$ terms, and $G(\lambda)_\bullet[1] = G(\lambda)_\bullet$. We will show in each case that $H^i$ terms are 0 for $i > 1$. The fact that the $H^0$ terms form a subcomplex follows from minimality of the complex in Theorem 3.5. Also, (a) and (c) follow from the grading given by Theorem 3.5 so in each case we will only need to verify that (b) holds. For $x$ homogeneous coming from an $H^1$ term, we have to show that $\text{deg}(d(x)) - \text{deg}(x) \geq 2$ if $d(x)$ lies in an $H^0$ term to verify (c). In all other cases, we only need to show that $\text{deg}(d(x)) - \text{deg}(x) \geq 1$ to verify (c).
3.3 Type $B_n$: Odd orthogonal groups.

**Theorem 3.9.** Let $G$ be of type $B_n$. If $\lambda_n > 0$, then

$$H^0(X; S_{\lambda} \mathcal{Q} \otimes \bigwedge^i \mathcal{R}^\vee) = \bigoplus_{\mu \subseteq \lambda, \text{HS}} V_{\mu},$$

(3.10)

and all higher cohomology vanishes. If $\lambda_n = 0$, then

$$H^0(X; S_{\lambda} \mathcal{Q} \otimes \bigwedge^i \mathcal{R}^\vee) = \bigoplus_{\mu \subseteq \lambda, \text{HS}} V_{\mu},$$

(3.11)

and all cohomology vanishes.

**Proof.** First we calculate the cohomology groups of $S_{\lambda} \mathcal{Q} \otimes \bigwedge^i \mathcal{R}$. We use that $\mathcal{R} = \mathcal{Q}^*$ and hence $igwedge^i \mathcal{R} = \bigwedge^{n-i} \mathcal{Q} \otimes (\bigwedge^n \mathcal{Q})^{-1}$. So by Pieri’s formula, $S_{\lambda} \mathcal{Q} \otimes \bigwedge^i \mathcal{R} = \bigoplus_{\mu} S_{\mu} \mathcal{Q} \otimes (\bigwedge^n \mathcal{Q})^{-1}$ where the sum is over all $\mu$ such that $|\mu/\lambda| = n-i$ and $\mu_j - \lambda_j \leq 1$ for all $j$. The highest weight of the representation $S_{\mu} \mathcal{Q} \otimes (\bigwedge^n \mathcal{Q})^{-1}$ is $(\mu_1 - 1, \ldots, \mu_n - 1)$. Since this is a dominant weight for $G$ if and only if $\mu_n - 1 \geq 0$, we conclude from Theorem 3.2 that

$$H^0(X; S_{\lambda} \mathcal{Q} \otimes \bigwedge^i \mathcal{R}) = \bigoplus_{\mu \subseteq \lambda, \text{HS}} V_{\mu},$$

(3.13)

Now suppose that $\mu_n = 0$ (which can only happen if $\lambda_n = 0$). Let $w \in W$ be the reflection given by the simple root $\varepsilon_n$, i.e., it acts on weights by changing the sign of the last coordinate. In this case, $\rho = (\frac{2n-1}{2}, \frac{2n-3}{2}, \ldots, \frac{1}{2})$. So $w^*(\mu_1 - 1, \ldots, \mu_{n-1} - 1, -1) = (\mu_1 - 1, \ldots, \mu_{n-1} - 1, 0)$. If $\mu_{n-1} \geq 1$, this weight is dominant. Otherwise, if $\mu_{n-1} = 0$, let $w_1 \in W$ be the reflection given by the simple root $\varepsilon_{n-1} - \varepsilon_n$, which permutes the last two coordinates. Then $w_1^*$ fixes $(\mu_1 - 1, \ldots, -1, 0)$, so there is no cohomology in this case. Hence we conclude that

$$H^1(X; S_{\lambda} \mathcal{Q} \otimes \bigwedge^i \mathcal{R}) = \bigoplus_{\mu \subseteq \lambda, \text{HS}} V_{\mu},$$

(3.14)

and all other cohomology vanishes.

There is a nonsplit exact sequence

$$0 \to \mathcal{R} \to \mathcal{R}^\vee \to \mathcal{O}_X \to 0,$$

where the trivialization of the cokernel comes from the fact that it has rank 1 and the corresponding $P$-module has the zero weight. This sequence gives rise to a short exact sequence

$$0 \to \bigwedge^i \mathcal{R} \to \bigwedge^i \mathcal{R}^\vee \to \bigwedge^{i-1} \mathcal{R} \to 0.$$
Now we tensor with $S_{\lambda}Q$ and take the long exact sequence of cohomology to get

$$
0 \rightarrow H^0(X; S_{\lambda}Q \otimes \bigwedge^i \mathcal{R}) \rightarrow H^0(X; S_{\lambda}Q \otimes \bigwedge^i \mathcal{R}^\vee) \rightarrow H^0(X; S_{\lambda}Q \otimes \bigwedge^{i-1} \mathcal{R}) \xrightarrow{\delta} \\
H^1(X; S_{\lambda}Q \otimes \bigwedge^i \mathcal{R}) \rightarrow H^1(X; S_{\lambda}Q \otimes \bigwedge^i \mathcal{R}^\vee) \rightarrow H^1(X; S_{\lambda}Q \otimes \bigwedge^{i-1} \mathcal{R}) \rightarrow 0.
$$

By (3.14), the higher cohomology vanishes if $\lambda_n \geq 1$, and we conclude (3.10) by semisimplicity of $G$. For $\lambda_n = 0$, (3.11) and (3.12) follow from (3.13) and (3.14) if we can show that $\delta$ is an isomorphism, and this is the content of Lemma 3.15. \hfill \Box

**Lemma 3.15.** If $\lambda_n = 0$ and $i \geq 1$, then $\delta$ is an isomorphism.

**Proof.** Consider the short exact sequence

$$
0 \rightarrow S_{\lambda}Q \otimes \bigwedge^i \mathcal{R} \rightarrow S_{\lambda}Q \otimes \bigwedge^i \mathcal{R}^\vee \rightarrow S_{\lambda}Q \otimes \bigwedge^{i-1} \mathcal{R} \rightarrow 0. \tag{3.16}
$$

Let $\mu \subseteq \lambda$ be a partition such that $(\lambda, \mu) \in HS$ and $|\lambda/\mu| = i - 1$, so that $V_\mu$ is a subrepresentation of $H^0(X; S_{\lambda}Q \otimes \bigwedge^{i-1} \mathcal{R})$ and of $H^1(X; S_{\lambda}Q \otimes \bigwedge^i \mathcal{R})$. Inside of $S_{\lambda}Q \otimes \bigwedge^i \mathcal{R}$ is a subbundle isomorphic to $S_{\mu-\varepsilon_n}Q$ (since $\lambda_n = 0$), and inside of $S_{\lambda}Q \otimes \bigwedge^{i-1} \mathcal{R}$ is a subbundle isomorphic to $S_\mu Q$. Let $E$ be the subbundle of $S_{\lambda}Q \otimes \bigwedge^i \mathcal{R}^\vee$ which is the extension of these two subbundles:

$$
0 \rightarrow S_{\mu-\varepsilon_n}Q \rightarrow E \rightarrow S_\mu Q \rightarrow 0.
$$

In order to show that $\delta$ is an isomorphism between the two copies of $V_\mu$, it is enough to show that $E$ is a nontrivial extension. To show this, we will use the equivalence between homogeneous bundles and rational $P$-modules, and show that the corresponding short exact sequence on the fiber over the $P$-fixed point of $X$ is a non-split extension.

Let $R$, $R^\vee$, and $Q$ denote the fibers of $\mathcal{R}$, $\mathcal{R}^\vee$, and $Q$, respectively, over the $P$-fixed point of $X = G/P$. Recall that we have a basis $e_1, \ldots, e_{2n+1}$ for $F$ such that $e_i$ and $e_{2n+2-i}$ are dual basis vectors. Then

$$
R = \langle e_1, \ldots, e_n \rangle, \quad R^\vee = \langle e_1, \ldots, e_n, e_{n+1} \rangle, \quad Q = \langle e_1^*, \ldots, e_n^* \rangle = \langle e_{n+1}, \ldots, e_{2n+2} \rangle.
$$

Also, let $G_0$ be the subgroup of $P$ which leaves $\langle e_{n+1}, \ldots, e_{2n+1} \rangle$ invariant. By definition, $P$ leaves $R$ invariant, so $G_0 \cong \text{GL}(R)$, and hence is linearly reductive. Thus any rational $P$-module is completely reducible as a $G_0$-module, and we will refer to such a decomposition as the associated graded module. To keep track of the $P$-action, we should write the fiber of (3.16) as

$$
0 \rightarrow S_{\lambda}Q \otimes \bigwedge^i R \rightarrow S_{\lambda}Q \otimes \bigwedge^i R^\vee \rightarrow S_{\lambda}Q \otimes \bigwedge^{i-1} R \otimes R^\vee/R \rightarrow 0.
$$

Let $g \in P$ be defined by $g(e_{n+1}) = e_n + e_{n+1}$, $g(e_{n+2}) = -\frac{1}{2}e_n - e_{n+1} + e_{n+2}$ and $g(e_i) = e_i$ for $i \not\in \{n+1, n+2\}$. (We are only interested in the value of $g(e_{n+1})$, the value of $g(e_{n+2})$ given is needed to ensure that $g$ preserves the symplectic form.) Then $S_{\mu}Q$ is a summand of $S_{\lambda}Q \otimes \bigwedge^{i-1} R \otimes R^\vee/R$, and hence is a summand of the associated graded of $S_{\lambda}Q \otimes \bigwedge^i R^\vee$. We claim that $g$ does not fix $S_\mu Q$, which will show that $E$ is a nontrivial extension of $S_\mu Q$ and $S_{\mu-\varepsilon_n}Q$.

Consider an Olver map (see Corollary 1.8)

$$
S_{\mu}Q \otimes \bigwedge^n Q \rightarrow S_{\lambda}Q \otimes \bigwedge^{n-i+1} Q.
$$

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Since $\mu_n = 0$, the canonical tableau $T$ inside of $S_\mu Q \otimes \wedge^n Q$ contains a single $e_n^*$ in the $n$th column, and its image is a linear combination over possible ways to remove $n - i + 1$ boxes from $T$. Since $\lambda_n = 0$, some of these summands will be of the form $T_\lambda \otimes e_i^* \wedge e_n^*$ where $e_i^* \in \wedge^{n-i} Q$. Hence when we tensor this Olver map with $\wedge^n R \otimes R^\vee/R$, we can contract this $e_n^*$ with a copy of $e_n$ for such summands. The main point is that under the isomorphism

$$S_\lambda Q \otimes \bigwedge^{n-i+1} Q \otimes \bigwedge^n R \otimes R^\vee/R \cong S_\lambda Q \otimes \bigwedge^i R \otimes R^\vee/R,$$

the image of the canonical tableau $T$ is a linear combination $T = \sum_j e_j T_j \otimes e_{I(j)} \otimes e_{n+1}$ where for at least one value of $j$ with $c_j \neq 0$, we have $e_{I(j)} \wedge e_{n+1} \neq 0$. But the action of $g$ on $T$ is given by replacing the $e_{n+1}$ factor by $e_n + e_{n+1}$, so $gT$ does not lie in $S_\mu Q$, but in the span of both $S_\mu Q$ and $S_{\mu - e_n} Q$. So we have established that $E$ is a nontrivial extension.

Thus, $E$ cannot have a $V_\mu$-isotypic component in its global sections. This follows, for example, from the identification of cohomology of homogeneous bundles with an induction functor (see [Jan, Proposition I.5.12]) and Frobenius reciprocity (see [Jan, Proposition I.3.4]) since the extension of $S_\mu Q$ and $S_{\mu - e_n} Q$ has no $P$-submodule isomorphic to $S_\mu Q$.

We now show that the hypothesis of Lemma 3.7(c) is satisfied. Suppose that $\lambda(i)$ and $\lambda(i+1)$ are partitions appearing in the $i$th and $(i+1)$st degrees of the Pieri resolution $F(\alpha; \beta)_*$ such that the differential

$$H^0(X; S_{\lambda(i+1)} Q \otimes \bigwedge^j R^\vee) \to H^0(X; S_{\lambda(i)} Q \otimes \bigwedge^j R^\vee)$$

is not minimal. The first term has homogeneous degree $-|\lambda(i+1)/\alpha| - j$ and the second has homogeneous degree $-|\lambda(i)/\alpha| - j$, so in order for this happen, we would need that $|\lambda(i)| = |\lambda(i+1)|$. But in this case, this differential is induced by a degree 0 map of the form $S_{\lambda(i+1)} Q \otimes B \to S_{\lambda(i)} Q \otimes B$ which is zero unless $\lambda(i+1) = \lambda(i)$. But in the latter case, the Pieri resolution we started with is not minimal.

If we have a horizontal map from an $H^1$ term to an $H^0$ term (and hence $\lambda(i)_n = \lambda(i+1)_n = 0$), then it must have degree at least 2: if it had smaller degree, then $|\lambda(i+1)| - |\lambda(i)| \leq 1$, but then (3.11) and (3.12) show that the map must be 0. Hence the hypothesis of Lemma 3.7(c) holds.

More generally, suppose that $N$ is the cokernel of an equivariant map of the form

$$\bigoplus_{i=1}^r A(-|\beta^i/\alpha|) \otimes V_{\beta_i} \to A \otimes V_\alpha$$

where each of $\beta^1, \ldots, \beta^r$ is obtained by adding vertical strips to $\alpha$. Let $M$ be the sections of the cokernel of the map of sheaves $\bigoplus_{i=1}^r S_{\beta_i} Q \otimes B \to S_\alpha Q \otimes B$. From the description above, we have a surjection $N \to M \to 0$. Letting $N_0$ be the kernel, we know that $N_0$ is generated by all partitions obtained by removing a box from $\alpha$. If there was only one such partition, we can resolve $N_0$ by induction on the size of $\alpha$. Otherwise, call these partitions $\alpha'_1, \ldots, \alpha'_s$, and add the relations $\alpha'_{i+1}, \ldots, \alpha'_s$ to $N$ to get a new module $L_i$. Then we have a short exact sequence

$$0 \to L'_i \to L_i \to L_{i-1} \to 0$$

where $L'_i$ is generated by $\alpha'_i$. So we can get a resolution for $L_i$ via resolutions of both $L'_i$ and $L_{i-1}$. Note that $L_0 = M$ and $L_s = N$, so we have described a sequence of quotients

$$N = L_s \to L_{s-1} \rightarrow \cdots \rightarrow L_1 \rightarrow L_0 = M$$

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for obtaining a resolution of $N$. Unfortunately, it will usually be far from minimal, and one must know something about which cancellations will occur. These remarks also apply to the cases of the symplectic group and the even orthogonal group.

### 3.4 Type $C_n$: Symplectic groups.

**Theorem 3.17.** Let $G$ be a group of type $C_n$. Then

$$H^0(X; S_\lambda Q \otimes \bigwedge^i R^\vee) = \bigoplus_{\substack{\mu \subseteq \lambda \leq \lambda \cap \rho \equiv i \\mid \mu/\lambda \in HS \\wedge (\lambda, \mu) \in HS \vee \mu \geq \lambda \\wedge \mu_{n-1} \geq |\mu_n - 1| \\wedge |\mu/\lambda| = n - i \\wedge (\mu, \lambda) \in HS}} V_\mu,$$

(3.18)

and all higher cohomology vanishes.

**Proof.** The dominant weights are the same for the symplectic and odd orthogonal groups, so using (3.13) and the fact that $R^\vee = R$ gives (3.18).

Furthermore, if $\mu_n = 0$, then $S_\mu Q \otimes (\bigwedge^n Q)^{-1}$ has no cohomology. To see this, let $w \in W$ be the reflection given by the simple root $2\varepsilon_n$, which changes the sign of the last coordinate. Then $w^* \cdot \lambda$ fixes $(\mu_1 - 1, \ldots, \mu_n - 1)$ since $\rho = (n, n - 1, \ldots, 2, 1)$.

The application of Lemma 3.7 follows as in the previous section.

### 3.5 Type $D_n$: Even orthogonal groups.

**Theorem 3.19.** Let $G$ be a group of type $D_n$. Then

$$H^0(X; S_\lambda Q \otimes \bigwedge^i R^\vee) = \bigoplus_{\substack{\mu \subseteq \lambda \leq \lambda \cap \rho \equiv i-2 \\mid \mu/\lambda \in HS \\wedge (\lambda, \mu) \in HS}} V_\mu,$$

(3.20)

If $\lambda_{n-1} = \lambda_n = 0$ and $i \geq 2$, we also have

$$H^1(X; S_\lambda Q \otimes \bigwedge^i R^\vee) = \bigoplus_{\substack{\mu \subseteq \lambda \leq \lambda \cap \rho \equiv i-2 \\mid \mu/\lambda \in HS \\wedge (\lambda, \mu) \in HS}} V_\mu,$$

(3.21)

and all other cohomology vanishes.

Note that by Remark 3.3, we only need to refer to the dual in (3.20).

**Proof.** As before, $S_\lambda Q \otimes \bigwedge^i R^\vee = \bigoplus_{\mu} S_\mu Q \otimes (\bigwedge^n Q)^{-1}$ where the sum is over all $\mu$ such that $|\mu/\lambda| = n - i$ and $\mu_j - \lambda_j \leq 1$ for all $j$. The highest weight of the representation $S_\mu Q \otimes (\bigwedge^n Q)^{-1}$ is $(\mu_1 - 1, \ldots, \mu_n - 1)$. This is a dominant weight if and only if $\mu_{n-1} - 1 \geq |\mu_n - 1|$, so we conclude (3.20) from Theorem 3.2. Note that the condition $\mu_{n-1} - 1 \geq |\mu_n - 1|$ happens in exactly two cases: if $\mu_n \geq 1$, or if $\mu_n = 0$ and $\mu_{n-1} \geq 2$. So we need to study the cases when $\mu_n = 0$ and $\mu_{n-1} \in \{0, 1\}$. Of course, this forces $\lambda_n = 0$.

Let $w \in W$ be the reflection given by the simple root $\varepsilon_{n-1} + \varepsilon_n$. This acts on weights by

$$w(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) = (\alpha_1, \ldots, \alpha_{n-2}, -\alpha_n, -\alpha_{n-1}).$$
Also, we have $\rho = (n-1, n-2, \ldots, 1, 0)$. In the case that $\mu_{n-1} = 1$, we see that $w^*(\mu_1 - 1, \ldots, 0, -1) = (\mu_1 - 1, \ldots, 0, -1)$, so there is no cohomology. On the other hand, if $\mu_{n-1} = 0$ (which can only happen if $i \geq 2$ and $\lambda_{n-1} = 0$), then we have $w^*(\mu_1 - 1, \ldots, \mu_{n-2} - 1, -1, -1) = (\mu_1 - 1, \ldots, \mu_{n-2} - 1, 0, 0)$. There are two possibilities: either $\mu_{n-2} \geq 1$ or $\mu_{n-2} = 0$. In the first case, the resulting weight is dominant. In the second case, let $w_1 \in W$ be the reflection given by the simple root $\varepsilon_{n-2} - \varepsilon_{n-1}$, which permutes the $(n-2)_{st}$ and $(n-1)_{st}$ coordinates. Then $w_1^*(\mu_1 - 1, \ldots, -1, 0, 0) = (\mu_1 - 1, \ldots, -1, 0, 0)$, so there is no cohomology. We conclude (3.21) and all higher cohomology of $S_{\varepsilon} Q \otimes \Lambda^i R^j$ vanishes.

Now we apply Lemma 3.7. There are only two possibilities for a horizontal differential to be non-minimal: the first is when both terms are $H^0$ terms or both are $H^1$ terms (and these are ruled out just as in the odd orthogonal case), and the second is when one term is an $H^0$ term and the other is an $H^1$. In this case, then the differential has degree $\leq 1$ only if the partitions from (3.1) that they resolve differ by a single box or are the same. Then the situation can be ruled out because the partitions which appear in the $H^0$ and $H^1$ have different sizes by the descriptions given in (3.20) and (3.21).

4 Toward equivariant Boij–Söderberg cones.

To give some context, we review the non-equivariant version of Boij–Söderberg cones in Section 4.1 and present the Boij–Söderberg algorithm for writing Betti tables as linear combinations of pure Betti tables. In Section 4.2, we formulate a conjectural equivariant analogue of Boij–Söderberg decompositions and provide some partial results. Finally, in Section 4.3, we present some examples of these decompositions, and show that the equivariant analogue of the Boij–Söderberg algorithm does not hold.

4.1 Boij–Söderberg cones in general.

Let $A = K[x_1, \ldots, x_n]$ as usual, and pick $c \leq n$. The Boij–Söderberg cone, denoted $\Delta$, is the cone generated by all Betti tables corresponding to Cohen–Macaulay modules of codimension $c$ with pure free resolutions. Given two degree sequences $d = (d_0, \ldots, d_c)$ and $d' = (d'_0, \ldots, d'_c)$, we say that $d \leq d'$ if $d_i \leq d'_i$ for $i = 0, \ldots, n$. Let $\Pi$ denote the poset of all degree sequences. Any maximal chain $C$ in $\Pi$ forms a simplicial cone inside of $\Delta$ by taking the cone generated by the pure Betti tables coming from the degree sequences of $C$. By (4.2), these simplicial cones are well-defined. In fact, the union of all such $C$ forms a simplicial fan $\mathcal{F}$ (see [BS, Proposition 2.9]) which we call the Boij–Söderberg fan. Recall from Section 2.1 that $B(M)$ denotes the graded Betti table of a module $M$. The following was conjectured by Boij and Söderberg:

Theorem 4.1 (Eisenbud–Schreyer). Let $K$ be a field of arbitrary characteristic. Let $M$ be a finitely generated Cohen–Macaulay graded $A$-module of codimension $c$. Then $B(M)$ can be written as a positive rational linear combination of Betti tables of pure Cohen–Macaulay modules of codimension $c$. This linear combination is unique if we require that the degree sequences of these pure free diagrams form a chain in $\Pi$.

Proof. See [ES] §7.

Hence the cone $\Delta$ contains the Betti tables of all finitely generated Cohen–Macaulay graded $A$-modules of codimension $c$. There is a simple algorithm for producing the linear combination whose idea originally appeared in [BS] §2.3. First, define the impurity $i(\beta)$ of a Betti diagram $\beta$
to be the number of its nonzero entries minus the number of nonzero columns. So a pure diagram has impurity 0. Given a Betti diagram \( B \), define \( d_i = \min\{j \mid B_{i,j} \neq 0\} \) for \( i = 0, \ldots, c \). This is the top degree sequence of \( B \). The bottom degree sequence can be defined by replacing \( \min \) with \( \max \). Let \( D \) be a pure Betti diagram of degree sequence \( d = (d_0, \ldots, d_c) \): the Herzog–Kühl equations \([HK, \text{Theorem 1}]\) state that if \( B(M) \) is a pure Betti table of degree \( d \), then one has

\[
B(M)_{i,d_i} = (-1)^{i+1} q \prod_{j \in \{0,i\}} \frac{d_j - d_0}{d_j - d_i}
\]

(4.2)

for some positive rational number \( q \) if and only if \( M \) is Cohen–Macaulay. Pick this rational multiple to be minimal with respect to the property of \( D \) having integral entries (this might not be the Betti diagram of a module), and let \( r \) be the largest rational number such that \( B' = B - rD \) has non-negative entries. Then from Theorem \([4.1] \) \( i(B') < i(B) \), so we repeat the process, which must terminate after finitely many steps.

Note that our choice of \( D \) at each stage ensures that the degree sequences of the pure diagrams used in the resulting linear combination which expresses \( B \) will form a chain in \( \Pi \). Example \([4.21]\) will show that the equivariant analogue of this algorithm does not hold.

4.2 An equivariant conjecture.

In this section, we study the Betti tables of equivariant Cohen–Macaulay modules. The support of an equivariant module is also equivariant, hence must either be all of \( \text{Spec} A \), or \( \text{Spec} A \) minus the homogeneous maximal ideal \((x_1, \ldots, x_n)\). We conclude that a Cohen–Macaulay equivariant \( A \)-module is either free or has finite length.

Given an equivariant graded free resolution of a finite length (codimension \( n \)) Cohen–Macaulay \( A \)-module \( M \), we wish to categorify the expression of \( B(M) \) as a linear combination of pure free diagrams. Without loss of generality, we will henceforth assume that \( M \) is a polynomial representation. In particular, by clearing denominators and taking high enough multiples, we can replace the pure free diagrams by the equivariant pure free diagrams of Section \([2.1]\) and the integer coefficients \( r \) will be replaced by Schur positive symmetric functions (the characters of some \( GL(V) \)-representation).

To get an equivariant Betti diagram out of an equivariant module \( M \), we let \( B(M)_{i,j} \) be the character of the minimal generating representations in degree \( j \) of the \( i \)th syzygy module of an equivariant graded minimal free resolution of \( M \), as described in the introduction.

The equivariant Betti diagram follows the usual convention of Betti diagrams, namely that the \( i \)th column and \( j \)th row contains \( B(M)_{i,j-i} \). We will say that \( B(M) \) is pure if each column contains at most one nonzero entry, just as in the non-equivariant setting.

In order to make things precise, first define \( SQ = SQ(n) \) to be the quotient field of the ring of symmetric functions \( Z[x_1, \ldots, x_n]^{S_n} \). Every element of \( SQ \) can be written as \( A/B \) where \( A \) and \( B \) are symmetric functions. Supposing that one can write \( A = \sum \lambda a_{\lambda} s_{\lambda} \) and \( B = \sum \lambda b_{\lambda} s_{\lambda} \) such that \( a_{\lambda} \geq 0 \) and \( b_{\lambda} \geq 0 \) for all \( \lambda \), we say that \( A/B \) is Schur positive, and also write \( A/B \succeq_s 0 \) and \( A/B \in SQ_{\geq 0} \). Note that the product of two Schur positive fractions is still Schur positive. This notion of Schur positive fractions seems to be the correct replacement for positive rational numbers in the equivariant setting.

The equivariant graded Betti tables live in the \( SQ \)-vector space \( SB = \bigoplus_{-\infty}^{\infty} SQ^{n+1} \), where we think of the elements of this vector space as tables with \( n + 1 \) columns and infinitely many rows. If \( d \leq \overline{d} \) are two degree sequences, let \( SB_{d,\overline{d}} \) be the finite dimensional subspace of \( SB \) consisting of those tables whose nonzero entries lie within \([d, \overline{d}]\). We are interested in the “cone” \( C \) whose generators are the pure equivariant resolutions of Section \([2.1]\). By a cone with generators, we mean
the set of finite linear combinations of the generators using Schur positive coefficients. While $C$ is not a cone in the usual sense, $C$ is convex when we consider $SB$ as a $Q$-vector space. Given a set of elements $S \subseteq SB$, we write $SQ \geq 0S$ to denote the set of finite Schur positive linear combinations of elements of $S$.

**Problem 4.3** (Weak version). Let $K$ be a field of characteristic 0. Let $M$ be an equivariant Cohen–Macaulay graded $A$-module of finite length. Is it true that $B(M)$ can be written as a Schur positive linear combination of Betti tables of pure Cohen–Macaulay modules of finite length?

It is not too hard to answer Problem 4.3 affirmatively in the case when the impurity is concentrated in one column (see Proposition 4.11). If Problem 4.3 can be answered affirmatively in the general case, then we can further ask if the equalities hold on the level of complexes.

**Problem 4.4** (Strong version). Let $M$ be a finite length equivariant $A$-module with equivariant minimal free resolution $F_\bullet$. Does there exist degree sequences $d^1, \ldots, d^r$ and representations $W, W_1, \ldots, W_r$ such that $W \otimes F_\bullet$ has a filtration of subcomplexes whose associated graded is isomorphic to

$$
\bigoplus_{i=1}^r W_i \otimes F(d^i)_\bullet,
$$

where the $F(d^i)_\bullet$ is the pure resolution of degree $d^i$ described in Section 2.1 and the isomorphism is of equivariant complexes? Can we write $W \otimes F_\bullet$ as a direct sum of subcomplexes instead of having to pass to an associated graded?

Problem 4.4 is false in the non-equivariant case as the next example shows.

**Example 4.5.** Let $A = K[x, y]$ and let $M = A/(x, y^2)$. Then $M$ is Cohen–Macaulay and has Betti table $\left( \begin{array}{ccc} 1 & 1 & - \\ - & 1 & 1 \end{array} \right)$. For any positive integer $n$, we cannot find a filtration $0 \to N \to M^{\oplus n} \to M^{\oplus n}/N \to 0$ of $M^{\oplus n}$ such that each piece has a pure free resolution because $x$ would annihilate both $N$ and $M^{\oplus n}/N$, which means that the middle entry of the first row of their Betti tables must be nonzero.

We indicate some facts which may be of use in trying to answer Problem 4.3 affirmatively. However, we first point out a fact which makes finding a counterexample particularly difficult.

**Proposition 4.6.** If $A$ is any weight positive (positive in the monomial symmetric function basis) symmetric function, then there exists a Schur polynomial $s_\lambda$ such that $As_\lambda$ is Schur positive.

**Proof.** Given two Schur polynomials $s_\lambda$ and $s_\mu$, let $W_1, \ldots, W_N$ be the weights of $\mu$. Then for each $\lambda + W_i$, there either exists a nonidentity $\sigma \in \mathfrak{S}_n$ such that $\sigma^*(\lambda + W_i) = \lambda + W_i$, or there exists a unique $\sigma_i$ such that $\sigma^*(\lambda + W_i) = \lambda^i$ is a dominant weight (a partition). Recall that $\sigma^*(\lambda + W_i)$ is defined to be $\sigma(\lambda + W_i + \rho) - \rho$ where $\rho = (n - 1, n - 2, \ldots, 1, 0)$. In the second case, we say that $\lambda + W_i$ is nondegenerate, and we claim that $s_\lambda s_\mu = \sum_i (-1)^{\ell(\sigma_i)} s_{\lambda^i}$, the sum over $i$ such that $\lambda + W_i$ is nondegenerate. First, define $a_{ij} = \det(x_j^{\gamma+n-i})_{i,j=1}^n$ for all $\gamma \in \mathbb{N}^n$. For a weight $W$, set $a_W = m^{-1}_W \sum_{\tau \in \mathfrak{S}_W} x^\tau(W)$ where $x^W = x_1^{W_1} \cdots x_n^{W_n}$ and $m_W$ is the integer needed so that the
coefficients in \( o_W \) are 1. Hence we have \( s_\mu = \sum_W c_W o_W \) for some coefficients \( c_W \). The equation
\[
s_\lambda o_W = \frac{a_{\lambda+\rho} o_W}{a_\rho} \quad \text{(by (1.1))}
\]
\[
= a_\rho^{-1} m_W^{-1} \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} x^{\sigma(\lambda+\rho)} \sum_{\tau \in \mathfrak{S}_n} x^{\tau(W)}
\]
\[
= a_\rho^{-1} m_W^{-1} \sum_{\sigma, \tau' \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} x^{\sigma(\lambda+\rho+\tau'(W))} \quad \text{(setting } \tau' = \sigma^{-1}\tau)\]
\[
= a_\rho^{-1} m_W^{-1} \sum_{\tau \in \mathfrak{S}_n} (-1)^{\ell(\sigma)} a_{\sigma, \lambda+\tau(W)}
\]
is valid for arbitrary choices of \( \sigma_\tau \in \mathfrak{S}_n \). We have that \( \lambda + \tau(W) \) is degenerate (say corresponding to the permutation \( \sigma_\tau \)) if and only if the determinant \( a_{\sigma, \lambda+\tau(W)} \) is 0 since this corresponds to the matrix having repeated rows. Hence only nondegenerate weights contribute to the sum \( s_\lambda \sum_W c_W o_W \), and in the nondegenerate case, one has \( a_{\sigma, \lambda+\tau(W)}/a_\rho = s_{\sigma, \lambda+\tau(W)} \) assuming that \( \sigma_\tau \) has been chosen so that the subscript of \( s \) is a partition. This proves the claim.

In our case, we can choose \( \lambda \) such that for every weight \( W \) appearing in \( A \), we have that \( \lambda + W \) is dominant, and hence if \( A = \sum_i x^{W_i} \), then \( A s_\lambda = \sum_i s_{\lambda+W_i} \). \( \square \)

In particular, if \( A/B = A'/B' \) and \( A' \) and \( B' \) are Schur positive symmetric functions, then it is not necessarily the case that \( A \) and \( B \) are both Schur positive nor that both \( -A \) and \( -B \) are both Schur positive, as the next example shows.

**Example 4.7.** Let \( n = 2 \). Then \( s_4 - s_{3,1} \) is not Schur positive as a symmetric function, but it is equal to \( s_3(s_4 - s_{3,1})/s_3 = s_7/s_3 \), which is Schur positive in our sense. In this case, \( s_4 - s_{3,1} = x_1^4 + x_2^4 \) in the monomial symmetric function basis.

Furthermore, Proposition 4.6 is not a necessary condition.

**Example 4.8.** Let \( n = 2 \). Then \( s_4 - s_{3,1} - s_{2,2} = x_1^4 - x_1^2 x_2^2 + x_2^4 \) is not positive in the monomial symmetric function basis, but the identity
\[
\frac{s_3^2(s_4 - s_{3,1} - s_{2,2})}{s_3^3} = \frac{s_{19} + 2s_{18,1} + 2s_{17,2} + 2s_{16,3} + 3s_{15,4} + 4s_{14,5} + 5s_{13,6} + 6s_{12,7} + 7s_{11,8} + 8s_{10,9}}{s_3^3}
\]
holds, so \( x_1^4 - x_1^2 x_2^2 + x_2^4 \in \mathcal{S}Q_\geq 0 \).

However, there do exist simple necessary conditions. For one thing, a symmetric function which is Schur positive in our sense must be positive when we do the substitution \( x_1 = x_2 = \cdots = 1 \). Less trivially, if we partially order the weights of a symmetric function by dominance order (\( \lambda \) is said to dominate \( \mu \) if \( \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \) for all \( i \)), those monomials with maximal weights must have a positive coefficient.

We mention a criterion for determining if a symmetric function is equal to a Schur positive fraction. However, this condition seems difficult to check for large examples. We need some notation. Let \( f \) be a polynomial in \( d \) variables. Write \( f = \sum_I c_I x^I \) and let \( \text{Log}(f) = \{ I \mid c_I \neq 0 \} \subset \mathbb{N}^d \). We define \( \text{conv}(\text{Log}(f)) \) to be the convex hull of this set. Given a face \( F \) of this polytope, let \( f_F = \sum_{I \in F} c_I x^I \).

**Proposition 4.9.** Let \( f \) be a symmetric function in \( d \) variables. Then \( f \) is a Schur positive fraction if and only if \( f_F(r_1, \ldots, r_d) > 0 \) for all positive real numbers \( r_1, \ldots, r_d \) and all faces \( F \) of \( \text{conv}(\text{Log}(f)) \).
Proof. In [Han, V.6], Handelman shows that for any \( f \) (not necessarily symmetric), there exists a polynomial \( g \) with positive coefficients such that \( gf \) has positive coefficients if and only if \( f_F(r_1, \ldots, r_d) > 0 \) for all positive real numbers \( r_1, \ldots, r_d \) for each face \( F \) of \( \text{conv}(\log(f)) \).

Now let \( f \) be a symmetric function which satisfies the hypotheses of the theorem. Then we can find some \( g \) such that \( gf \) has positive coefficients. We also have the identity

\[
    f = \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \sigma(g) f = \frac{1}{d!} \prod_{\sigma \in \mathfrak{S}_d} h \sigma(g)
\]

for some polynomial \( h \). Since \( \prod_{\sigma} \sigma(g) \) is symmetric, we conclude that the same is true for \( f \prod_{\sigma} \sigma(g) \), and hence \( h \) is symmetric. In particular, we expressed \( f \) as a quotient of two monomial positive symmetric functions. Hence we know that \( f \) is a quotient of two Schur positive symmetric functions by Proposition 4.6. \( \square \)

Now we give the setup for an equivariant version of Boij–Söderberg cones.

First, \( C \) lives in a proper subspace of \( SB \). In order to describe this subspace, let \( S_M(d) = \text{ch}(M_d) \) be the \textit{equivariant Hilbert function} of \( M \), and let \( H_M(t) = \sum_{d \geq 0} S_M(d) t^d \in \mathbb{Q}Q_{\geq 0}[t] \) be the \textit{equivariant Hilbert series} of \( M \). Then we have

\[
    H_{A(-j)}(t) = t^j \sum_{d \geq 0} s_d t^d = \frac{t^j}{(1 - x_1 t) \cdots (1 - x_n t)}
\]

as elements of \( \mathbb{Q}Q_{\geq 0}[t] \). We can write an equivariant resolution \( F^* \) for \( M \):

\[
    0 \to \bigoplus_j (A(-j) \otimes B_{n,j}) \to \cdots \to \bigoplus_j (A(-j) \otimes B_{0,j}) \to M \to 0,
\]

where \( B_{i,j} \) is notation for the corresponding representation with that character. Since this resolution has degree 0 maps, and the (equivariant) Hilbert function is an additive function on degree 0 exact sequences, we get

\[
    H_M(t) = \sum_{i=0}^n (-1)^i H_{F_i}(t) = \sum_{i=0}^n \sum_j (-1)^i \text{B}_{i,j} t^j
\]

Knowing that \( M \) is of finite length, \( H_M(t) \) must be a polynomial living in \( \mathbb{Q}Q_{\geq 0}[t] \), and hence the numerator \( \sum_{i=0}^n \sum_j (-1)^i \text{B}_{i,j} t^j \) is divisible by \((1 - x_1 t) \cdots (1 - x_n t)\). In particular, we conclude that

\[
    \sum_{i=0}^n \sum_j (-1)^i \text{B}_{i,j} x_k^{-j} = 0 \quad \text{for } k = 1, \ldots, n, \quad (4.10)
\]

which gives \( n \) linearly independent equations. The equivariant Betti diagrams live in the \( \mathbb{Q}Q \)-subspace defined by these equations. A simple dimension count allows us to conclude the following.

**Proposition 4.11.** \( \text{If } B(M) \text{ is an equivariant Betti table of a finite length module } M \text{ which is pure in all degrees except possibly one, then } B(M) \text{ is a Schur positive linear combination of pure Betti tables.} \)

\[
    \text{Proof. Set } d = \min \{ j \mid B(M)_{i,j} \neq 0 \} \text{ and } \overline{d} = \max \{ j \mid B(M)_{i,j} \neq 0 \}. \text{ By our assumption, } d \text{ and } \overline{d} \text{ agree except in at most one coordinate, call this coordinate } k \text{ if it exists. If it does not, then } M \text{ is pure, and there is nothing to show. Otherwise, } \dim SB_{\frac{d}{\overline{d}}} = n + 1 + \overline{d} - d_k, \text{ and the subspace}
\]

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cut out by the equivariant Herzog–Kühl equations \((4.10)\) has codimension \(n\). Furthermore, we have \(1 + d_k - d_j\) linearly independent equivariant pure Betti tables \(B(j)\) coming from the degree sequences \(d(j)\) for \(d_k \leq j \leq d_k\) which are defined by \(d(j)_i = d_i\) for \(i \neq k\) and \(d(j)_k = j\) otherwise. Hence, \(B(M)\) must be a linear combination \(\sum_j c_j B(j)\) of these Betti tables. By comparing which coefficients are zero or nonzero, we immediately get \(c_j = B(M)_{k,j}/B(j)_{k,j} \in \mathbb{S}_{\mathbb{Q} \geq 0}\). \(\square\)

**Corollary 4.12.** Every pure equivariant Betti table of a finite length module is a Schur positive scalar multiple of a Betti table arising from Theorem 2.2.

One could attempt to mimic the proof of Eisenbud and Schreyer to prove Conjecture 4.3. The main problem seems to be that in the case of the field \(\mathbb{S}_{\mathbb{Q}}\), the boundaries of cones are rather complicated, and so one does not have a nice description of the exterior facets as in the non-equivariant case.

**4.3 Examples.**

We now give some examples which use an equivariant analogue of the Boij–Söderberg algorithm to find a decomposition of Betti tables. The idea of the algorithm is the same as the one presented in Section 4.1 except that one uses coefficients in \(\mathbb{S}_{\mathbb{Q} \geq 0}\) instead of positive rational numbers. The correctness of the algorithm in these cases is a consequence of Proposition 4.11. First, we state two propositions which give some families of identities among Schur polynomials and then give some examples with actual numbers. In Example 4.21 we will give an example showing that this algorithm does not always work.

Pick \(a > b > 0\). Let \(\alpha = (a,b,0)\), \(\beta^1 = (a+1,b,0)\), and \(\beta^2 = (a,b+1,0)\). Then the Pieri resolution of the cokernel of \(\beta^1 \oplus \beta^2 \to \alpha\) is

\[
0 \to (a+1,b+1,b+1) \to (a+1,b+1,0) \oplus (a,b+1,b+1) \to (a+1,b,0) \oplus (a,b+1,0) \to (a,b,0).
\]

**Proposition 4.13.** We have the following equivariant isomorphism of graded Betti tables:

\[
\begin{array}{ccc}
(a,b,0) & (a+1,b,0) & (a+1,b+1,0) \\
\vdots & \vdots & \vdots \\
(b+1,b+1,0) & \oplus & (a,b+1,0) \\
(a+1,b+1,0) & \oplus & (b+1,b+1,b+1) \\
(a,b+1,b+1) & \oplus & (b+1,b,0) \\
\end{array}
\]

where the \(\vdots\) spans \(b-1\) rows of zeroes.
In particular, we deduce the following three identities:

\[(b + 1, b + 1, 0) \otimes (a, b, 0) \cong (a + 1, b + 1, 0) \otimes (b, b, 0) \oplus (a, b + 1, b + 1) \otimes (b, 0, 0) \]
\[(b + 1, b + 1, 0) \otimes ((a + 1, b, 0) \oplus (a, b + 1, 0)) \cong (a + 1, b + 1, 0) \otimes (b + 1, b, 0) \oplus (a, b + 1, b + 1) \otimes (b + 1, 0, 0) \]
\[(b + 1, b + 1, 0) \otimes (a + 1, b + 1, b + 1) \cong (a + 1, b + 1, 0) \otimes (b + 1, b + 1, b + 1) \oplus (a, b + 1, b + 1) \otimes (b + 1, b + 1, 1). \]

Let \( h_k = s_{(k)} \) be the \( k \)th complete homogeneous symmetric function of degree \( k \) in \( n \) variables. We remark that all three identities can be proven from the following lemma by expressing both sides as \( 3 \times 3 \) determinants.

**Lemma 4.14.** The following identity holds

\[ s_\lambda s_\mu = \det(h_{\lambda_i + \mu_{n+1-i} - i+j})_{i,j=1}^n \]

if we interpret \( h_0 = 1 \) and \( h_k = 0 \) for \( k < 0 \).

**Proof.** See [Mac, I.3, Example 8(c)]. \( \square \)

**Example 4.15.** Consider the Pieri resolution with \( n = 3, \alpha = (2, 1, 0), \beta^1 = (3, 1, 0), \) and \( \beta^2 = (2, 2, 0) \), which is

\[ 0 \to \begin{array}{cccc} & & & \end{array} \to \begin{array}{cccc} & & & \end{array} \oplus \begin{array}{cccc} & & & \end{array} \to \begin{array}{cccc} & & & \end{array} \oplus \begin{array}{cccc} & & & \end{array} \to \begin{array}{cccc} & & & \end{array}. \]

The algorithm writes the graded Betti diagram as

\[ \begin{pmatrix} 8 & 21 & 15 & - \\ - & - & 1 & 3 \end{pmatrix} = \frac{5}{2} \begin{pmatrix} 3 & 8 & 6 & - \\ - & - & - & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 2 & - & - \\ - & - & 2 & 1 \end{pmatrix}. \]

Multiplying both sides by 6, we can write this equation as

\[ 6 \left( \begin{array}{cccc} & & & \end{array} \right) = 15 \left( \begin{array}{cccc} & & & \end{array} \right) + \left( \begin{array}{cccc} & & & \end{array} \right), \]

which gives a decomposition of equivariant Betti diagrams:

\[ \begin{array}{cccc} & & & \end{array} \otimes \begin{array}{cccc} & & & \end{array} \oplus \begin{array}{cccc} & & & \end{array} \oplus \begin{array}{cccc} & & & \end{array} = \begin{array}{cccc} & & & \end{array} \otimes \begin{array}{cccc} & & & \end{array} \oplus \begin{array}{cccc} & & & \end{array} \oplus \begin{array}{cccc} & & & \end{array}. \]

We can instead set \( \beta^2 = (a, b, 1) \) and get
Proposition 4.16. Set \( c = a - b + 1 \). We have the following equivariant isomorphism of graded Betti tables:

\[
\begin{array}{cccc}
(a, b, 0) & (a + 1, b, 0) & (a + 1, b, 1) & (a + 1, b, 0) \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\approx
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
(a + 1, a + 1, 0) & (a + 1, a + 1, 1) & (c, c, 0) & (c, c, 1) \\
\end{array}
\]

where the \( \vdots \) spans \( a - b - 1 \) rows of zeroes.

We deduce the following three identities:

\[
(a - b + 1, a - b + 1, 0) \otimes (a, b, 0) \cong (a + 1, b, 1) \otimes (a - b, a - b, 0)
\]

\[
\oplus (a + 1, a + 1, 0) \otimes (a - b, 0, 0)
\]

\[
(a - b + 1, a - b + 1, 0) \otimes ((a + 1, b, 0) \oplus (a, b, 1)) \cong (a + 1, b, 1) \otimes (a - b + 1, a - b, 0)
\]

\[
\oplus (a + 1, a + 1, 0) \otimes (a - b + 1, 0, 0)
\]

\[
(a - b + 1, a - b + 1, 0) \otimes (a + 1, a + 1, 1) \cong (a + 1, b, 1) \otimes (a - b + 1, a - b + 1, a - b + 1)
\]

\[
\oplus (a + 1, a + 1, 0) \otimes (a - b + 1, a - b + 1, 1).
\]

Example 4.17. Consider the Pieri resolution with \( n = 3, \alpha = (2, 1, 0), \beta^1 = (3, 1, 0), \) and \( \beta^2 = (2, 1, 1) \), which is

\[
0 \rightarrow \text{matrix} \rightarrow \text{matrix} \oplus \text{matrix} \rightarrow \text{matrix} \oplus \text{matrix} \rightarrow \text{matrix}
\]

The algorithm writes the graded Betti diagram as

\[
\begin{pmatrix}
8 & 18 & 6 & - \\
- & - & 10 & 6
\end{pmatrix} = \begin{pmatrix}
3 & 8 & 6 & - \\
- & - & 10 & 6
\end{pmatrix} + 5 \begin{pmatrix}
1 & 2 & - & - \\
- & - & 2 & 1
\end{pmatrix}.
\]

Multiplying both sides by 6, we can write this equation as

\[
6 \begin{pmatrix}
8 & 18 & 6 & - \\
- & - & 10 & 6
\end{pmatrix} = 6 \begin{pmatrix}
3 & 8 & 6 & - \\
- & - & 10 & 6
\end{pmatrix} + 10 \begin{pmatrix}
3 & 6 & - & - \\
- & - & 6 & 3
\end{pmatrix},
\]

which gives a decomposition of equivariant Betti diagrams:

\[
\begin{array}{cccc}
\text{matrix} & \oplus & \text{matrix} & \oplus \text{matrix}
\end{array}
\]
Example 4.18. A more complicated Pieri resolution: let \( n = 3, \alpha = (3,1,0), \beta^1 = (4,1,0), \) and \( \beta^2 = (3,3,0). \) Then the Pieri resolution looks like

\[
0 \to \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \to \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \to \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}.
\]

The algorithm writes the graded Betti diagram as

\[
\begin{pmatrix}
15 & 24 & - & - \\
-10 & 24 & - & - \\
- & 3 & 8 & -
\end{pmatrix} = \frac{8}{5} \begin{pmatrix}
8 & 15 & - & - \\
- & - & 10 & - \\
- & - & - & 3
\end{pmatrix} + \frac{8}{5} \begin{pmatrix}
1 & - & - & - \\
- & 5 & 5 & - \\
- & - & - & 1
\end{pmatrix} + 3 \begin{pmatrix}
3 & - & - & - \\
- & - & - & 15 \\
- & - & - & 8
\end{pmatrix}.
\]

The first step of the algorithm subtracts \( \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \) times the first pure diagram on the right hand side from \( \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \) times the Betti diagram on the left hand side, which yields the following table:

| (4,4,0) | (4,3,1) | (4,3,2) | (6,4,0) | (6,3,1) | (5,4,1) | (6,5,0) | (6,4,1) | (5,5,2) | (6,4,3) | (5,5,3) | (5,4,4) |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| (3,3,2) | (5,3,2) | (4,4,2) | (4,3,3) | (5,5,1) | (5,4,2) |
|         |         |         |         |         |         |
|         |         |         |         |         |         |
|         |         |         |         |         |         |

where a collection of partitions in the same entry denotes their direct sum. The associated isomorphism of representations (after simplifications) is

\[
T_0 = T_1 \oplus T_2 \oplus T_3,
\]

where

\[
T_0 = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}, \quad T_1 = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}, \quad T_2 = \left( \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \oplus \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right) \otimes \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \otimes \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}.
\]

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Example 4.19. Here is an elaboration of Example 2.20. While the module resolved is not Cohen–Macaulay, we can throw in an extra relation $\beta^0 = (5, 1, 0)$ to make the cokernel $M$ have finite length. Then the equivariant Betti diagram of $M$ is

\[
\begin{array}{c|c|c|c|c}
(3, 1, 0) & (5, 1, 0) & (4, 3, 0) & (4, 2, 1) & (3, 3, 1) \\
\hline
(5, 3, 0) & (5, 2, 1) & (4, 4, 0) & (4, 2, 2) & (3, 3, 2) \\
2 \cdot (4, 3, 1) & 2 \cdot (4, 3, 1) & (4, 4, 1) & (4, 3, 2) & (4, 3, 2)
\end{array}
\]

and its decomposition is

\[
\bigotimes T = \bigotimes \oplus \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes.
\]

Example 4.20. Now we give a decomposition for a non-Cohen–Macaulay equivariant module. Let $n = 3$, $\alpha = (1, 0, 0)$ and $\beta = (2, 1, 0)$ so that the Pieri resolution is

\[
0 \to \bigotimes \to \bigotimes \oplus \bigotimes \to \bigotimes \to \bigotimes.
\]

The decomposition (after some simplifications) is

\[
\bigotimes T = \bigotimes \oplus \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes.
\]

where $\varnothing$ denotes the empty partition of 0, i.e., $S_3 V = K$ is the trivial representation of $GL(V)$.

Example 4.21. This example shows that the equivariant analogue of the Boij–Söderberg algorithm does not hold. Let $\dim V = 3$ and $A = \text{Sym}(V)$, let $M$ be the cokernel of the Pieri map $A \otimes S_3 V \to$
$A \otimes S_2 V$, and let $N$ be the cokernel of the Pieri map $A \otimes S_{3,1} V \to A \otimes S_{1,1} V$. Then the equivariant Betti table of the $A$-module $M \oplus M \oplus N$ is

\[
T = \begin{array}{ccc}
\begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} \\
\begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} \\
\begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} \\
\begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} \\
\begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} \\
\end{array}
\end{array}
\]

The equivariant analogue of the Boij–Söderberg algorithm fails on this example. The top degree sequence of this diagram is $(0, 1, 3, 5)$, whose corresponding pure Betti diagram is

\[
T' = \begin{array}{ccc}
\begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} \\
\begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} & \begin{array}{c}
\oplus
\end{array} \\
\end{array}
\end{array}
\]

In the non-equivariant case, the algorithm says to subtract the largest rational multiple of $T'$ from $T$ which makes the resulting table have nonnegative entries. In this case, the rational multiple is $4/3$, and corresponds to getting rid of the entry in the first row and second column. The equivariant version would say to replace $4/3$ by $2s_3/s_{3,1}$. Alternatively, we can subtract $2s_3 T'$ from $s_{3,1} T$. The second row and third column of the resulting table contains $-s_6,3 + s_6,2,1 + s_5,4 + s_5,2,2 + s_4,4,1 - s_3,3,3$, which is not in $SQ_{\geq 0}$ because its most dominant weight $(6, 3, 0)$ has a negative coefficient. A similar phenomena occurs when we replace top degree sequence with bottom degree sequence in the Boij–Söderberg algorithm.

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