Logarithmic Stability for Coefficients Inverse Problem of Coupled Wave Equations

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Abstract

This paper investigates the identification of two coefficients in a coupled hyperbolic system with an observation on one component of the solution. Based on the Carleman estimate for coupled wave equations a logarithmic type stability result is obtained by measurement data only in a suitably chosen subdomain under the assumption that the coefficients are given in a neighborhood of some subboundary.

Keywords: Logarithmic stability, Identification of coefficients, coupled wave equations, Carleman estimate, Fourier-Bros-Iagolnitzer transform

1 Introduction and main result

Let $T > 0$ and $\Omega \subset \mathbb{R}^n$ be a nonempty bounded domain. Write $n = n(x)$ for the unit outward normal vector of $\partial \Omega$ at $x$. Consider the following coupled hyperbolic system:

$$\begin{cases}
\frac{\partial^2 y_1}{\partial t^2} - \text{div}(a(x)\nabla y_1) + c_{11}(x)y_1 + c_{12}(x)y_2 = 0 & \text{in } Q \triangleq \Omega \times (0, T), \\
\frac{\partial^2 y_2}{\partial t^2} - \text{div}(a(x)\nabla y_2) + c_{21}(x)y_1 + c_{22}(x)y_2 = 0 & \text{in } Q, \\
\frac{\partial y_1}{\partial n} = 0, \frac{\partial y_2}{\partial n} = 0 & \text{on } \partial \Omega \times (0, T), \\
(y_1(0), \partial_t y_1(0)) = (y_{10}, y_{11}), (y_2(0), \partial_t y_2(0)) = (y_{20}, y_{21}) & \text{in } \Omega.
\end{cases} \tag{1}$$

It is well known that wave equations are widely used to describe many kinds of waves in the world. In particular, the system (1) is a simplified model for describing the interaction of waves (e.g., [9, 13, 24]).

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Let $\omega_0$ be a nonempty open subset of $\Omega$. In this work, we consider the coefficients inverse problem for system (1), i.e., establish the conditional stability for identifying the coefficients in the zeroth-order terms $(c_{11}, c_{22})$ simultaneously from suitable observation of one component $y_1$ of the solution $y = (y_1, y_2)$ in $\omega_0 \times (0, T)$. More precisely, we consider the following problem:

**Problem (IP).** Can one recover the coefficients $(c_{11}, c_{22})$ from a suitable observation of $y_1$ on $\omega_0 \times (0, T)$?

Throughout this paper, in order to emphasize the dependence of the solution of (1) on the coefficients, we denote by $(y_1(c_{11}, c_{22}), y_2(c_{11}, c_{22}))$ the solution of (1) with fixed coefficients $c_{12}, c_{21}$.

Coefficient inverse problems are important in various real world applications, including the detection and identification of explosives, nondestructive testing and material characterization. For their significant applications, coefficient inverse problems are widely studied for different equations and systems. Generally speaking, “recover” usually refers to the following two issues:

- determining the coefficients uniquely by the measurement;
- giving an algorithm to compute the coefficients efficiently.

A key step to achieve the above two goals is to establish an inequality which is called a stability estimate:

$$|| (c_{11} - \tilde{c}_{11}, c_{22} - \tilde{c}_{22}) || \leq f( || y_1(c_{11}, c_{22}) - y_1(\tilde{c}_{11}, \tilde{c}_{22}) ||_{\omega \times (0, T)} ),$$

(2)

where $f$ is a non-negative continuous function satisfying $f(0) = 0$.

On one hand, it is clear that if (2) holds, then $y_1(c_{11}, c_{22}) = y_1(\tilde{c}_{11}, \tilde{c}_{22})$ in $\omega \times (0, T)$ implies that $(c_{11}, c_{22}) = (\tilde{c}_{11}, \tilde{c}_{22})$. This implies that the measurement of $y_1$ in $\omega \times (0, T)$ can uniquely determine the coefficients $(c_{11}, c_{22})$.

On the other hand, according to [14], one knows that the stability rate described by the function $f$ is a quasi-optimal convergence rate of Tikhonov regularization with a suitable a priori choice of the regularizing parameters according to noise levels in data $y_1|_{\omega_0 \times (0, T)}$.

In general, there are three common types of $f$:

1. $f(\xi) = C \xi$;
2. $f(\xi) = C \xi^\alpha$ for some $\alpha \in (0, 1)$;
3. $f(\xi) = C |\ln \xi|$.

For the first, the second and the third kinds of $f$, the estimate (2) indicates Lipschitz-type stability, Hölder-type stability and logarithmic-type stability, respectively.
As one main methodology for the coefficient inverse problem, we refer to Bukhgeim and Klibanov [10]. See also Bellassoued and Yamamoto [7], Klibanov [25], Klibanov and Timonov [27] for example. The arguments are based on Carleman estimates, which we discuss. There have been many works: Beilina, Cristofol, Li and Yamamoto [2], Bellassoued [3], Bellassoued and Yamamoto [5], Cannarsa, Floridia and Yamamoto [11], Cannarsa, Floridia, Gölgeleyen and Yamamoto [12], Imanuvilov and Yamamoto [22], Klibanov [26], Lü and Zhang [29], Yu, Liu and Yamamoto [31] and the references therein. Here we do not intend a comprehensive list.

Compared with the case of single partial differential equations, there are much fewer works addressing coefficients inverse problems for coupled systems. By the character of the Carleman estimate, the inverse problems for weakly coupling systems, which mean that the terms of the second order are not coupled, can be done very similarly to the case of a single equation if we adopt data of all the components of the solution. However for strongly coupling cases, it is more difficult to establish underlying Carleman estimates and there are very few researches for inverse problems by Carleman estimates. As for inverse problems for the Lamé systems which are strongly coupled, see e.g., Bellassoud, Imanuvilov and Yamamoto [4], Bellassoued and Yamamoto [6, 7], Imanuvilov and Yamamoto [23], for instance.

Our main target is a weakly coupling hyperbolic system (1), and we describe our main achievements for the inverse problem:

- **Data of one component of the solution:**
  As similar works, we can refer to Benabdallah, Cristofol, Gaitan and Yamamoto [8], Alabau-Boussouira, Cannarsa and Yamamoto [1] for example.

- **Data for the inverse problem can be restricted to an arbitrarily fixed subdomain $\omega$:**
  For a single wave equation, see [3], [5]. The argument is based on the Fourier-Bros-Iagolnitzer transform which is a kind of truncated Laplace transform, and applications to coupling systems require non-trivial consideration.

In this paper, we establish a logarithmic-type stability with the measurement on only ONE component of the solution. In order to present the main result, let us introduce some notations and conditions. Throughout this paper, we assume that $a = a(x) \in C^4(\Omega)$ satisfying

$$a > \theta_1 \quad \text{on } \overline{\Omega}, \quad ||a||_{C^4(\overline{\Omega})} \leq M_0, \quad \left| \frac{\nabla a(x) \cdot (x - x_0)}{2a(x)} \right| \leq 1 - \theta_0, \quad x \in \overline{\Omega} \setminus \omega,$$

from some constants $M_0, \theta_0 > 0$ and $0 < \theta_1 \leq 1$, and subdomain $\omega$ of $\Omega$.

**Remark 1.1.** The above assumption on $a$ is for Lemma 2.1. More precisely, it is used to establish suitable Carleman estimate for (1), which is the key tool to prove Lemma 2.1. It is a kind of pseudoconvex condition and has already been used by several authors (e.g. [21, 25]).
Let \( \tilde{\omega} \subset \Omega \) be such that \( \omega \subset \tilde{\omega} \) and \( \text{dist}(\partial \tilde{\omega} \setminus \partial \Omega, \partial \omega \setminus \partial \Omega) > 0 \). Let us now define the admissible set of unknown coefficients. Fix constants \( M_1 > 0, \varpi_1, \varpi_2 \in W^{1,\infty}(\tilde{\omega}) \) and let \( A = A(T, \omega, M_1, \varpi_1, \varpi_2) \) be the set of pairs of real valued functions \((c_{11}, c_{22})\) such that

\[
A = \left\{ (c_{11}, c_{22}) \in W^{1,\infty}(\Omega)^2 : ||c_{jj}||_{W^{1,\infty}(\Omega)} \leq M_1, \ c_{jj} = \varpi_j \text{ for } j = 1, 2 \right\}. \tag{3}
\]

For \( s > \frac{3}{2} \), set

\[
X^s(\Omega) = \left\{ u \in H^s(\Omega) : \frac{\partial u}{\partial n} = 0 \right\}.
\]

By the classical well-posedness result for wave equations, similarly to [21, Lemma 2.1], for any \((y_{10}, y_{11}), (y_{20}, y_{21}) \in X^3(\Omega) \times X^2(\Omega)\), the equation (1) has a unique solution

\[
(y_1, y_2) \in \left[ C([0,T]; H^3(\Omega)) \times C^1([0,T]; H^2(\Omega)) \times C^2([0,T]; H^1(\Omega)) \right]^2
\]

satisfying that

\[
\|(y_1, y_2)||_{C([0,T]; H^3(\Omega)) \cap C^1([0,T]; H^2(\Omega)) \cap C^2([0,T]; H^1(\Omega))}^2 \leq C(M_1)(||(y_{10}, y_{11})||_{H^3(\Omega) \cap H^2(\Omega)} + ||(y_{20}, y_{21})||_{H^3(\Omega) \cap H^2(\Omega)}).
\]

**Remark 1.2.** The admissible set \( A \) defined by (6), poses constraints on unknown coefficients:

- **A priori bounds for \((c_{11}, c_{22})\):** This is reasonable because in a physical model, one usually have some rough estimate on the coefficients.

- **We assume to know the values of \((c_{11}(x), c_{22}(x))\), \(x \in \tilde{\omega}\):** This can be interpreted by that one can directly know physical properties near the boundary.

According to the classical well-posedness of wave equations(e.g., [19]), we know that there are plenty of solutions such that \( A \) is nonempty.

Next, we give the condition for \( c_{12} \) and \( c_{21} \):

\[
\{c_{12}, c_{21}\} \subset W^{2,\infty}(\Omega) \text{ and there is a constant } c_0 > 0 \text{ such that}
\]

\[
c_{21} \geq c_0 \text{ or } -c_{21} \geq c_0 \text{ in } \omega_0. \tag{5}
\]

**Remark 1.3.** Condition (5) means that \( y_1 \) can effect \( y_2 \) adequately. Without (5), one cannot obtain information of \( y_2 \) from \( y_1 \) and the observation on \( y_1 \) is not enough to determine the coefficients \((c_{11}, c_{22})\).

Now we are ready to state the main result of this paper.
Theorem 1.1. There exists $T_0 > 0$ such that for all $T > T_0$, we have that
\[
\| (c_{11} - \tilde{c}_{11}, c_{22} - \tilde{c}_{22}) \|_{L^2(\Omega)} \\
\leq C \left( \ln \| \partial_t^j (y_1(c_{11}, c_{22}) - y_1(\tilde{c}_{11}, \tilde{c}_{22})) \|_{L^2(\omega \times (0, T))} \right)^{-1} C(M_1) \\
+ \| \partial_t^j (y_1(c_{11}, c_{22}) - y_1(\tilde{c}_{11}, \tilde{c}_{22})) \|_{L^2(\omega \times (0, T))}
\]
for all $(c_{11}, c_{22}), (\tilde{c}_{11}, \tilde{c}_{22}) \in \mathcal{A}$, where $C = C(T) > 0$ is a constant.

From the proof of Theorem 1.1, one can see that it can be generalized to a system coupled by more than two wave equations by data of reduced numbers of components of data, but in this paper, we do not pursue the full technical generality for presenting the key in a simple way. Following the method in [2], We can discuss similar inverse problems of determining all the coefficients but we need to choose suitable initial values and repeat taking data. Furthermore we can establish a stability estimate in determining other combinations such as $(a, c_{12})$ of two coefficients among $a, c_{11}, c_{12}, c_{21}, c_{22}$ by a single measurement of $y_1$ in $\omega \times (0, T)$, but we do not discuss here. Moreover, the elliptic operator $\text{div} (a \nabla)$ can be generalized to a more general one as $\sum_{j,k=1}^n \partial_{x_j} (a^{jk} \partial_{x_k})$ for suitable $\{a^{jk}\}_{1 \leq j,k \leq n}$. Indeed, by [17, Theorems 4.2 and 4.3], we can prove a similar result for Lemma 2.1. Then the rest of the proof is similar.

The rest of this paper is organized as follows. Section 3 is devoted to presenting some auxiliary result, i.e., a Hölder type stability estimate for the problem we consider with measurement of one component in an open subset of the domain satisfying some geometry conditions, as well as the introduction for the Fourier-Bros-Iagliolntzer transform. Then in Section 3, we give the proof of Theorem 1.1.

2 Preliminaries

In this section, we give some preliminary results. We first recall a Lipschitz type estimate.

Let $\omega_1 \subset \Omega$ be such that $\omega \subset \omega_1 \subset \tilde{\omega}$, dist($\partial \omega_1 \setminus \partial \Omega, \partial \omega \setminus \partial \Omega$) > 0 and dist($\partial \omega_1 \setminus \partial \Omega, \partial \tilde{\omega} \setminus \partial \Omega$) > 0. Let $O_j, j = 1, 2, 3$ be subset of $\omega$ such that $O_1 \subset \subset O_2 \subset \subset O_3 \subset \subset \omega$, $\partial \omega \subset \partial O_3$. Let $\rho \in C^\infty(\mathbb{R}^n)$ be such that $\rho = 1$ in $(\Omega \setminus \omega_1) \cup (\omega \setminus O_3)$ and $\rho = 0$ in $O_2$.

Lemma 2.1. For all $T > 0$ satisfying
\[
T > \sup_{x \in \Omega} |x - x_0| \text{ for some } x_0 \notin \Omega \setminus \omega,
\]
there exists a constant $C > 0$ such that for all $(c_{11}, c_{22}), (\tilde{c}_{11}, \tilde{c}_{22}) \in \mathcal{A},$
\[
\| (c_{11} - \tilde{c}_{11}, c_{22} - \tilde{c}_{22}) \|_{L^2(\Omega)} \\
\leq C \sum_{j=1}^2 \sum_{k=1}^2 \| \partial_{t}^j (y_j(c_{11}, c_{22}) - y_j(\tilde{c}_{11}, \tilde{c}_{22})) \|_{L^2((0, T) \times (\omega \setminus O_2))}.
\]
Lemma 2.1 can be obtained directly by following the proof of Theorem 1.1 in [21] step by step. We omit it here.

Next, we give a brief introduction for Fourier-Bros-Iagolnitzer transform (shortened to F.B.I. transform), which is a generalization of the Fourier transform in this subsection. A detailed introduction to this can be found in [15]. As an important application to a hyperbolic equation, see Robbiano [30], and here we modify the arguments in [30]. Let

\[ F(z) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} e^{iz\tau} e^{-\tau^2} d\tau. \]

Then

\[ F(z) = \frac{\sqrt{\pi}}{2\pi} e^{\frac{1}{4}(|\text{Im}z|^2 - |\text{Re}z|^2)} e^{-\frac{1}{2}(\text{Im}z\text{Re}z)}. \]

For every \( \lambda \geq 1 \), define

\[ F_\lambda(z) \triangleq \lambda F(\lambda z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{iz\tau} e^{-(\lambda z)^2} d\tau. \]

It can be easily seen that

\[ |F_\lambda(z)| = \frac{\sqrt{\pi}}{2\pi} \lambda e^{\frac{\lambda^2}{4}(|\text{Im}z|^2 - |\text{Re}z|^2)}. \]

Let \( s,l_0 \in \mathbb{R} \) and recall that \( i = \sqrt{-1} \). The F.B.I. transformation \( F_\lambda \) for \( f \in \mathcal{S}(\mathbb{R}^{n+1}) \) is defined as follows:

\[ (F_\lambda f)(x,s) \triangleq \int_{\mathbb{R}} F_\lambda(l_0 + is - l)\Phi(l)f(x,l)dl. \] (9)

3 Proof of Theorem 1.1

Let \( \chi \in C^\infty(\omega) \) satisfying that \( 0 \leq \chi \leq 1 \) and

\[ \chi(x) = \begin{cases} 1, & \text{if } x \in \omega \setminus O_2, \\ 0, & \text{if } x \in O_1. \end{cases} \] (10)

Put

\[ w_j = (y_j(c_{11}, c_{22}) - \tilde{y}_j(\tilde{c}_{11}, \tilde{c}_{22})), \quad g_j = (\tilde{c}_{jj} - c_{jj})\chi \tilde{y}_j. \]

Let \((U_1, U_2) = (\chi \partial_tw_1, \chi \partial_tw_2)\) and \((V_1, V_2) = (\chi \partial_t^2w_1, \chi \partial_t^2w_2)\), respectively. By the assumption of \( A \) in (3), there exists \( C(M_0, M_1) > 0 \) such that

\[ ||(U_1, U_2)||_{L^2(-T,T;H^2(\omega))^2}^2 + ||(\partial_t U_1, \partial_t U_2)||_{L^2(-T,T;H^3_0(\omega))^2}^2 \leq C(M_0, M_1)^2. \] (11)
Then by fundamental calculation, we have that
\[
\begin{align*}
\partial_t^2 U_1 - \text{div}(a(x) \nabla U_1) + c_{11} U_1 + c_{12} U_2 &= -(\nabla a \cdot \nabla \chi) U_1 - a[\Delta, \chi] U_1 \quad \text{in } \omega \times (-T, T), \\
\partial_t^2 U_2 - \text{div}(a(x) \nabla U_2) + c_{21} U_1 + c_{22} U_2 &= -(\nabla a \cdot \nabla \chi) U_2 - a[\Delta, \chi] U_2 \quad \text{in } \omega \times (-T, T), \\
U_1 &= U_2 = 0 \quad \text{in } \partial \omega \times (-T, T), \quad (12) \\
(U_1(0), \partial_t U_1(0)) &= (0, \partial_t^2 g_1(x, 0)) \quad \text{in } \omega, \\
(U_2(0), \partial_t U_2(0)) &= (0, \partial_t^2 g_2(x, 0)) \quad \text{in } \omega.
\end{align*}
\]

and
\[
\begin{align*}
\partial_t^2 V_1 - \text{div}(a(x) \nabla V_1) + c_{11} V_1 + c_{12} V_2 &= -(\nabla a \cdot \nabla \chi) v_1 - a[\Delta, \chi] v_1 \quad \text{in } \omega \times (-T, T), \\
\partial_t^2 V_2 - \text{div}(a(x) \nabla V_2) + c_{21} V_1 + c_{22} V_2 &= -(\nabla a \cdot \nabla \chi) v_2 - a[\Delta, \chi] v_2 \quad \text{in } \omega \times (-T, T), \\
V_1 &= V_2 = 0 \quad \text{on } \partial \omega \times (-T, T), \quad (13) \\
(V_1(0), \partial_t V_1(0)) &= (\partial_t^2 g_1(x, 0), \partial_t^3 g_1(x, 0)) \quad \text{in } \omega, \\
(V_2(0), \partial_t V_2(0)) &= (\partial_t^2 g_2(x, 0), \partial_t^3 g_2(x, 0)) \quad \text{in } \omega.
\end{align*}
\]

Let \( \Phi \in C_0^\infty(\mathbb{R}) \) satisfying the following conditions:
\[
\begin{align*}
\Phi &\in C_0^\infty\left(\left[ -\frac{L}{2}, \frac{L}{2} \right]; [0, 1] \right), \\
\Phi &= 1 \text{ on } \left[ -\frac{L}{4}, \frac{L}{4} \right], \\
|\Phi'| &\leq \frac{1}{4L}, \quad |\Phi''| \leq \frac{1}{4L},
\end{align*}
\]

where \( L > 0 \) will be chosen later.

Take
\[
K = \left[ -\frac{L}{2}, -\frac{L}{4} \right] \cup \left[ \frac{L}{4}, \frac{L}{2} \right], \quad K_0 = \left[ -\frac{L}{8}, \frac{L}{8} \right].
\]

and let \( l_0 \in K_0 \) in (9).

Let \( U_j^F(x, s) \) and \( V_j^F(x, s) \) be the F.B.I. transform of \( U_j \) and \( V_j \), respectively, \( j = 1, 2 \), i.e.,
\[
U_j^F(x, s) = \int_{\mathbb{R}} F_\lambda(l_0 + is - l) \Phi(l) U_j(x, l)dl
\]

and
\[
V_j^F(x, s) = \int_{\mathbb{R}} F_\lambda(l_0 + is - l) \Phi(l) V_j(x, l)dl.
\]
Since
\[ \partial_s^2 U^F_j(x, s) = -\partial_s \int_{\mathbb{R}} i \partial_l F_\lambda(l_0 + is - l) \Phi(l) U_j(x, l) dl \]
\[ = i \partial_s \int_{\mathbb{R}} F_\lambda(l_0 + is - l) (\Phi'(l) U_j(x, l) + \Phi(l) \partial_l U_j(x, l)) dl \]
\[ = - \int_{\mathbb{R}} F_\lambda(l_0 + is - l) (\Phi''(l) U_j(x, l) + 2\Phi'(l) \partial_l U_j(x, l) + \Phi(l) \partial_l^2 U_j(x, l)) dl, \]
we have that
\[
\begin{align*}
\partial_s^2 U^F_1(x, s) &+ \text{div}(a(x) \nabla U^F_1) - c_{11} U_1^F - c_{12} U_2^F = G_1 + H_1 \quad \text{in } \omega \times \mathbb{R}, \\
\partial_s U^F_2 + \text{div}(a(x) \nabla U^F_2) - c_{21} U_1^F - c_{22} U_2^F = G_2 + H_2 \quad \text{in } \omega \times \mathbb{R}, \\
\frac{\partial U^F_1}{\partial \nu} = \frac{\partial U^F_2}{\partial \nu} = 0 \quad \text{on } \partial \omega \times \mathbb{R},
\end{align*}
\]
where
\[ G_j(x, s) = \int_{\mathbb{R}} F_\lambda(l_0 + is - l) \Phi(l) (\nabla a \cdot \nabla \chi) u_j + a[\Delta, \chi] u_j) dl. \]
\[ H_j(x, s) = - \int_{\mathbb{R}} F_\lambda(l_0 + is - l) (\Phi''(l) U_j(x, l) + 2\Phi'(l) \partial_l U_j(x, l)) dl. \]

Recall that \( \omega_0 \) is an arbitrary fixed nonempty subset of \( \omega \) such that \( \overline{\omega_0} \subset \omega \). Let \( \tilde{\omega}_0 \subset \subset \omega_0 \) be a nonempty open subset. By [18, Lemma 1.1], we know that there exists a function \( \hat{\psi} \in C^2(\overline{\omega}) \) such that
\[
\begin{align*}
\hat{\psi}(x) > 0, &\quad \forall x \in \omega, \\
\hat{\psi}(x) = 0, &\quad \forall x \in \partial \omega, \\
|\nabla \hat{\psi}(x)| > 0, &\quad \forall x \in \omega \setminus \tilde{\omega}_0.
\end{align*}
\]
We can conclude from (15) that there exist two constants \( \beta_1, \beta_2 > 0 \), where \( \beta_2 > (\beta_1 + ||\hat{\psi}||_{L^\infty(\omega)})/2 \), and \( \omega_2 \subset \subset \omega \) such that
\[ \hat{\psi}(x) \leq \beta_1, \quad \forall x \in O_2 \]
and that
\[ \hat{\psi}(x) \geq \beta_2, \quad \forall x \in \omega_0. \]
It follows from the last condition in (15) that the maximum value of \( \hat{\psi} \) can only be attained in \( \omega_0 \), i.e., there exists a point \( \hat{x} \in \omega_0 \) such that
\[ \hat{\psi}(\hat{x}) = \max_{x \in \omega} \hat{\psi}(x). \]
Let
\[ \theta = e^\ell, \quad \ell = \zeta \phi, \quad \phi = e^{\mu \psi}, \quad \psi = \psi(x, s) \equiv \frac{\hat{\psi}(x)}{M|\hat{\psi}|_{L^\infty(\omega)}} + b^2 - s^2. \tag{19} \]

Here \(1 < b_0 < b \leq 2\), and
\[ \frac{1 - \frac{\beta_2}{||\psi||_{L^\infty(\omega)}}}{b_0^2 - 1} < M < \frac{1 - \frac{\beta_1}{||\psi||_{L^\infty(\omega)}}}{b_0^2 - 1}, \]
where \(\lambda\) and \(\mu\) are parameters, \(x \in \omega\) and \(s \in (-b, b)\).

By (4.33) in [28], there exists a constant \(C = C(\mu) > 0\) and \(\zeta_0 = \zeta_0(\mu)\) so that for all \(\zeta \geq \zeta_0\), the solution \((U_1^F, U_2^F) \in H^1((-b, b) \times \omega)\) to (14) satisfies that
\[
\zeta \mu^2 \int_{-b_0}^{b_0} \int_{\omega} \theta^2 \phi(|\nabla U_1^F|^2 + |\partial_s U_1^F|^2 + \zeta^2 \mu^2 \phi^2 |U_1^F|^2 + |\nabla U_2^F|^2 + |\partial_s U_2^F|^2 + \zeta^2 \mu^2 \phi^2 |U_2^F|^2) \, dx \, ds \\
\leq C \left[ \int_{-b}^{b} \int_{\omega} \theta^2 \phi(|G_1(x, s) + H_1(x, s)|^2 + |G_2(x, s) + H_2(x, s)|^2) \, dx \, ds \\
+ \zeta \mu^2 \int_{-b}^{b} \int_{\omega} \theta^2 \phi(|\nabla U_1^F|^2 + |\partial_s U_1^F|^2 + |\nabla U_2^F|^2 + |\partial_s U_2^F|^2) \, dx \, ds \right] \\
+ \zeta^3 \mu^4 \int_{-b}^{b} \int_{\omega} \theta^2 \phi^3(|U_1^F|^2 + |U_2^F|^2) \, dx \, ds \right] \\
+ C \int_{(-b_0, b_0) \cup (b_0, b)} \int_{\omega} \theta^2 |(\partial_s U_1^F|^2 + |U_1^F|^2 + |\partial_s U_2^F|^2 + |U_2^F|^2) \, dx \, ds.
\]

Let us get rid of the terms of \(U_2^F\) in the second and third integrals in the right hand side of (20).

Let \(\omega_{0,j} (j = 1, 2)\) satisfy that \(\omega_0 \subset \omega_{0,1} \subset \omega_{0,2} \subset \omega_0\). We choose cutoff functions \(\eta_j \in C^\infty(\overline{\omega}; [0, 1])\) \((j = 1, 2, 3)\) satisfying
\[
\begin{align*}
\eta_j(x) &= 1, \quad \forall x \in \omega_{0,j-1}, \\
0 &< \eta_j \leq 1, \quad \forall x \in \omega_{0,j}, \\
\eta_j(x) &= 0, \quad \forall x \in \omega \setminus \omega_{0,j}. \tag{21}
\end{align*}
\]

It is easy to see that
\[
\theta^2 \phi \eta_1^2 U_2^F \left[ \partial_s^2 U_2^F + \text{div}(a(x) \nabla U_2^F) \right] \\
= \partial_s \left( \theta^2 \phi \eta_1^2 U_2^F \partial_s U_2^F \right) - \theta^2 \phi \eta_1^2 |\partial_s U_2^F|^2 - \left( \theta^2 \phi \eta_1^2 \right)_s U_2^F \partial_s U_2^F + \text{div} \left( \theta^2 \phi \eta_1^2 a U_2^F \nabla U_2^F \right) \\
- \theta^2 \phi \eta_1^2 a |\nabla U_2^F|^2 - a \nabla \left( \theta^2 \phi \eta_1^2 \right) U_2^F \nabla U_2^F. \tag{22}
\]
Integrating (22) on \((-b, b) \times \omega\) and noting that \(U^F_2(-b) = U^F_2(b) = 0\) in \(\omega\), by (14) and (19), we see that there exists \(\zeta_1 > 0\) such that for all \(\zeta \geq \zeta_1\),

\[
\int_{-b}^{b} \int_{\omega_0}^{b} \theta^2 \phi (|\nabla U^F_2|^2 + |\partial_s U^F_2|^2) \, dx \, ds
\]

\[
\leq C \left[ \int_{-b}^{b} \int_{\omega} \theta^2 [G_2(x, s) + H_2(x, s)]^2 \, dx \, ds + \zeta^2 \int_{-b}^{b} \int_{\omega_{0,1}} \theta^2 \phi^3 |U^F_2|^2 \, dx \, ds \right].
\]  

(23)

Now we estimate \(\int_{-b}^{b} \int_{\omega_{0,1}} \theta^2 \phi^3 |U^F_2|^2 \, dx \, ds\). It is easy to see that

\[
\theta^2 \phi^3 \eta^2_2 U^F_2 \left[ \partial_s^2 U^F_1 + \text{div} \left( a \nabla U^F_1 \right) \right]
\]

\[
= \theta^2 \phi^3 \eta^2_2 U^F_1 \left[ \partial_s^2 U^F_2 + \text{div} \left( a \nabla U^F_2 \right) \right] + \partial_s \left[ \theta^2 \phi^3 \eta^2_2 \left( U^F_2 \partial_s U^F_1 - \partial_s U^F_2 \partial_s U^F_1 \right) \right]
\]

\[
- \partial_s \left( \theta^2 \phi^3 \eta^2_2 \right) U^F_2 \partial_s U^F_1 + \partial_s \left[ \partial_s \left( \theta^2 \phi^3 \eta^2_2 \right) U^F_2 U^F_1 \right] - \partial_s \left[ \partial_s \left( \theta^2 \phi^3 \eta^2_2 \right) U^F_1 \right] U^F_2
\]

\[
+ \text{div} \left[ \theta^2 \phi^3 \eta^2_2 \left( U^F_2 U^F_1 - U^F_1 U^F_2 \right) \right] - a \nabla \left( \theta^2 \phi^3 \eta^2_2 \right) U^F_2 \nabla U^F_1
\]

\[
+ \text{div} \left[ a \nabla \left( \theta^2 \phi^3 \eta^2_2 \right) U^F_2 U^F_1 \right] - \text{div} \left[ a \nabla \left( \theta^2 \phi^3 \eta^2_2 \right) U^F_1 U^F_2 \right].
\]  

(24)

Integrating (24) on \((-b, b) \times \omega\) and noting that \(U^F_1(-b) = U^F_1(b) = 0\) in \(\omega\), by (14), we find that

\[
\int_{-b}^{b} \int_{\omega_{0,1}} \theta^2 \phi^3 |U^F_2|^2 \, dx \, ds
\]

\[
\leq C \left[ \int_{-b}^{b} \int_{\omega} \theta^2 \left( |G_1(x, s) + H_1(x, s)|^2 + |G_2(x, s) + H_2(x, s)|^2 \right) \, dx \, ds \right]
\]

\[
+ \zeta^2 \mu^2 \int_{-b}^{b} \int_{\omega_{0,2}} \theta^2 \phi^5 (|\nabla U^F_1|^2 + |\nabla U^F_2|^2) \, dx \, ds.
\]  

(25)

Similar to (23), we can obtain that

\[
\int_{-b}^{b} \int_{\omega_{0,2}} \theta^2 \phi^5 (|\nabla U^F_1|^2 + |\nabla U^F_2|^2) \, dx \, ds
\]

\[
\leq C \left[ \int_{-b}^{b} \int_{\omega} \theta^2 [G_1(x, s) + H_1(x, s)]^2 \, dx \, ds + \zeta^2 \mu^2 \int_{-b}^{b} \int_{\omega_{0}} \theta^2 \phi^7 |U^F_2|^2 \, dx \, ds \right].
\]  

(26)

Combing (20), (23), (25) and (26), we know there exists a constant \(\mu_2 > 0\) such that for all \(\mu \geq \mu_2\), one can find two constants \(C = C(\mu) > 0\) and \(\zeta_2 = \zeta_2(\mu)\) so that for all \(\zeta \geq \zeta_2\), the solution \((U^F_1, U^F_2) \in H^1((-b, b) \times \omega)^2\) to (14) satisfies that

\[
\lambda^2 \phi^2 (|\nabla U^F_1|^2 + |\nabla U^F_2|^2 + \lambda^2 \mu^2 \phi^2 |U^F_1|^2 + |\nabla U^F_1|^2 + |\nabla U^F_2|^2 + \lambda^2 \mu^2 \phi^2 |U^F_2|^2) \, dx \, ds
\]

10
\[
\leq C \left[ \int_{-b}^{b} \int_{\omega} \theta^2 \left| (G_1(x, s) + H_1(x, s))^2 + G_2(x, s) + H_2(x, s) \right|^2 \, dx \, ds \right. \\
+ \int_{-b}^{b} \int_{\omega_0} \lambda^5 \mu^7 \theta^2 \phi^5 |U_1^F|^2 \, dx \, ds \bigg] \\
+ C \int_{(b, -b)} \int_{(b, b)} \int_{\omega} \theta^2 |(\partial_x U_1^F)|^2 + |U_1^F|^2 + |\partial_x U_2^F|^2 + |U_2^F|^2 \, dx \, ds.
\]

(27)

Set \( \phi_j = e^{\mu_j}, j = 1, 2, 3, 4 \), where

\[
\psi_1 = \frac{\beta_1}{M ||\hat{\psi}||_{L^\infty(\omega)}} + b^2, \quad \psi_2 = \frac{\hat{\psi}(\hat{x})}{M ||\hat{\psi}||_{L^\infty(\omega)}} + b^2 = \frac{1}{M} + b^2, \\
\psi_3 = \frac{\beta_2}{M ||\hat{\psi}||_{L^\infty(\omega)}} + b^2 - 1, \quad \psi_4 = \frac{\hat{\psi}(\hat{x})}{M ||\hat{\psi}||_{L^\infty(\omega)}} + b^2 - b_0^2 = \frac{1}{M} + b^2 - b_0^2.
\]

(28)

From the bound for \( M \) we know \( \psi_1 < \psi_4 \). By the property of F.B.I. transformation, we have that

\[
\int_{-b}^{b} \int_{\omega_0} \zeta^5 \mu^7 \theta^2 \phi^5 |U_1^F|^2 \, dx \, ds \\
\leq \zeta^5 \mu^7 \max_{x \in \omega_0, s \in [-b, b]} (\theta^2 \phi^5) \int_{-b}^{b} \int_{\omega_0} |U_1^F(x, s)|^2 \, dx \, ds \\
\leq \zeta^5 \mu^7 \phi_2^5 \int_{-b}^{b} \int_{\omega_0} \left| \int_{\mathbb{R}} \Phi(l_0 + is - l)\Phi(l)U_1(x, l)dl \right|^2 \, dx \, ds \\
\leq \zeta^5 \mu^7 \phi_2^5 \int_{-b}^{b} \int_{\omega_0} \left| \int_{\mathbb{R}} \sqrt{\frac{\pi}{2\pi}} \lambda e^{\frac{x^2}{2}}(s^2-l_0-l^2)\Phi(l)U_1(x, l)dl \right|^2 \, dx \, ds \\
\leq \frac{\lambda^2}{4\pi} \zeta^5 \mu^7 \phi_2^5 \int_{-b}^{b} e^{\frac{x^2}{2}} \sup \Phi \int_{\omega_0} \left| \int_{-b}^{b} U_1(x, l)dl \right|^2 \, dx \\
\leq \frac{\lambda^2 Lb}{2\pi} \zeta^5 \mu^7 \phi_2^5 \int_{-b}^{b} e^{\frac{x^2}{2}} \left| \int_{\omega_0} \left| U_1(x, l) \right|^2 \, dl \right|^2 \, dx.
\]

From the definition of \( H_j \), we see that

\[
\int_{-b}^{b} \int_{\omega} |H_j(x, s)|^2 \, dx \, ds \\
= \int_{-b}^{b} \int_{\omega} \left| - \int_{\mathbb{R}} F_\lambda(l_0 + is - l)(\Phi''(l)U(x, l) + 2\Phi'(l)\partial_l U(x, l))dl \right|^2 \, dx \, ds \\
\leq \int_{-b}^{b} \int_{\omega} \left| \int_{\mathbb{R}} \sqrt{\frac{\pi}{2\pi}} \lambda e^{\frac{x^2}{2}}(s^2-l_0-l^2)(\Phi''(l)U(x, l) + 2\Phi'(l)\partial_l U(x, l))dl \right|^2 \, dx \, ds
\]

(30)
Consequently,
\[
\int_{-b}^{b} \int_{\omega} |G_j(x, s)|^2 dx ds
\]
\[
= \int_{-b}^{b} \int_{\omega} F_\lambda(l_0 + is - l) \Phi(l)((\nabla a \cdot \nabla \chi) U_j + a[\Delta, \chi] U_j) dl \bigg| \bigg|^2 dx ds
\]
\[
\leq \int_{-b}^{b} \int_{\omega} \frac{\lambda^2 e^{\frac{\lambda^2 b^2}{2}}}{2\pi} \int_{-\frac{2}{\lambda} l_0 - l}^{\frac{2}{\lambda} l_0 - l} e^{-\frac{\lambda^2}{2}(s^2 - |l_0 - l|^2)} ((\nabla a \cdot \nabla \chi) U_j + a[\Delta, \chi] U_j) dl \bigg| \bigg|^2 dx ds
\]
\[
\leq \frac{\lambda^2 b L}{2\pi} e^{\frac{\lambda^2 b^2}{2}} \max \{|\nabla \chi|^2, |\Delta \chi|^2\} \int_{-\frac{2}{\lambda} l_0 - l}^{\frac{2}{\lambda} l_0 - l} (|U_j(x, l)|^2 + |\nabla U_j(x, l)|^2) dx dl
\]
\[
\leq \frac{\lambda^2 b M_0^2}{2\pi} e^{\frac{\lambda^2 b^2}{2}} \max \{|\nabla \chi|^2, |\Delta \chi|^2\} \int_{-\frac{2}{\lambda} l_0 - l}^{\frac{2}{\lambda} l_0 - l} (|U_j(x, l)|^2 + |\nabla U_j(x, l)|^2) dx dl.
\]
Thus, we have
\[
\int_{-b}^{b} \int_{\omega} |H_j(x, s)|^2 dx ds
\]
\[
\leq e^{2\lambda^2 b L} \frac{2\lambda^2}{\pi} e^{\frac{\lambda^2 b^2}{2}(b^2 - (\xi^2)^2)} \max\{|\Phi'(l)|^2, |\Phi''(l)|^2\} \int_K \int_K (|U_j(x, l)|^2 + |\partial_l U_j(x, l)|^2) dx dl,
\]
\[
\int_{-b}^{b} \int_{\omega} \theta^2 |G_j(x, s)|^2 dx ds
\]
\[
\leq e^{2\lambda^2 b L} \frac{2\lambda^2}{\pi} e^{\frac{\lambda^2 b^2}{2}(b^2 - (\xi^2)^2)} \max\{|\Phi'(l)|^2, |\Phi''(l)|^2\} \int_K \int_K (|U_j(x, l)|^2 + |\partial_l U_j(x, l)|^2) dx dl,
\]
\[
\int_{-b}^{b} \int_{\omega} \theta^2 |G_j(x, s)|^2 dx ds
\]
\[
\leq e^{2\lambda^2 b M_0^2} \frac{\lambda^2 b^2}{2\pi} e^{\frac{\lambda^2 b^2}{2}} \max\{|\nabla \chi|^2, |\Delta \chi|^2\} \int_{-\frac{2}{\lambda} l_0 - l}^{\frac{2}{\lambda} l_0 - l} (|U_j(x, l)|^2 + |\nabla U_j(x, l)|^2) dx dl.
\]
For simplicity of notations, without loss of generality, we assume that $T = 1$. Substituting (29), (32), and (33) into (27), we obtain that
\[
\zeta^3 \mu^4 \phi^3 e^{2\lambda^2 b M_0^2} \int_{-1}^{1} \int_{\omega \setminus O_2} (|U_1^F|^2 + |U_2^F|^2) dx ds
\]
\[ \leq \zeta^3 \mu^4 \int_{-b_0}^{b_0} \int_\omega \theta^2 \phi^3 \left( |U_1|^2 + |U_2|^2 \right) dx ds \\
\leq \zeta^2 \mu^2 \int_{-b_0}^{b_0} \int_\omega \theta^2 \phi \left( |\nabla U_1|^2 + |\partial_s U_1|^2 + \zeta^2 \mu^2 \phi^2 |U_1|^2 + |\nabla U_2|^2 + |\partial_s U_2|^2 + \zeta^2 \mu^2 \phi^2 |U_2|^2 \right) dx ds \\
\leq C \left[ e^{2\zeta \phi_2} \frac{4\lambda^2 b L}{\pi} e^{\frac{x^2}{2}(b^2 - \frac{4}{\pi})} \right] \max_K \{ |\Phi'(l)|^2, |\Phi(l)|^2 \} \int_\omega \int_K \left( |U_1(x,l)|^2 + |\partial_t U_1(x,l)|^2 + |U_2(x,l)|^2 + |\partial_t U_2(x,l)|^2 \right) dx ds \\
\quad + \left[ e^{2\zeta \phi_1} \frac{\lambda^2 b L M_0^2}{\pi} e^{\frac{x^2}{2} b^2} \right] \left( \max_K \{ |\nabla \chi|^2, |\Delta \chi|^2 \} \right) \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \int_{O_2 \setminus O_1} \left( |u_1(x,l)|^2 + |\nabla u_1(x,l)|^2 \right) dx ds \\
\quad + \left( \frac{\lambda^2 L b}{2\pi} e^{\frac{x^2}{2} b^2} \right) \left( \max_K \{ |\partial_t U_1|^2 + |U_1|^2 + |\partial_t U_2|^2 + |U_2|^2 \} \right) \int_{\omega_0} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} |U_1(x,l)|^2 dx ds \\
+ C e^{2\zeta \phi_4} \int_{(-b_0,0) \cup (b_0,0)} \int_{\omega_0} \left( |\partial_s U_1|^2 + |U_1|^2 + |\partial_s U_2|^2 + |U_2|^2 \right) dx ds. \] 

Then 

\[ \int_{-1}^{1} \int_{\omega \setminus O_2} \left( |U_1|^2 + |U_2|^2 \right) dx ds \]

\[ \leq C \zeta^{-3} \mu^{-4} \phi_3^{-3} e^{-2\zeta \phi_3} \left[ e^{2\zeta \phi_2} \frac{4\lambda^2 b L}{\pi} e^{\frac{x^2}{2}(b^2 - \frac{4}{\pi})} \right] \max_K \{ |\Phi'(l)|^2, |\Phi(l)|^2 \} \]

\[ \times \left( \int_{\omega} \int_K \left( |U_1(x,l)|^2 + |\partial_t U_1(x,l)|^2 + |U_2(x,l)|^2 + |\partial_t U_2(x,l)|^2 \right) dx ds \\
+ \left( \frac{\lambda^2 b L M_0^2}{\pi} e^{\frac{x^2}{2} b^2} \right) \left( \max_K \{ |\nabla \chi|^2, |\Delta \chi|^2 \} \right) \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \int_{O_2 \setminus O_1} \left( |u_1(x,l)|^2 + |\nabla u_1(x,l)|^2 \right) dx ds \\
+ \left( \frac{\lambda^2 L b}{2\pi} e^{\frac{x^2}{2} b^2} \right) \left( \max_K \{ |\partial_t U_1|^2 + |U_1|^2 + |\partial_t U_2|^2 + |U_2|^2 \} \right) \int_{\omega_0} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} |U_1(x,l)|^2 dx ds \\
+ C \zeta^{-3} \mu^{-4} \phi_3^{-3} e^{-2\zeta (\phi_3 - \phi_4)} \int_{(-b_0,0) \cup (b_0,0)} \int_{\omega} \left( |\partial_s U_1|^2 + |U_1|^2 + |\partial_s U_2|^2 + |U_2|^2 \right) dx ds \]

\[ \leq C \left[ \zeta^{-3} \mu^{-4} \phi_3^{-3} \left( e^{2\zeta (\phi_2 - \phi_3)} \frac{4\lambda^2 b L}{\pi} e^{\frac{x^2}{2}(b^2 - \frac{4}{\pi})} \right) \max_K \{ |\Phi'(l)|^2, |\Phi(l)|^2 \} + e^{-2\zeta (\phi_3 - \phi_4)} \right] \\
+ e^{2\zeta (\phi_1 - \phi_3)} \frac{\lambda^2 b L M_0^2}{\pi} e^{\frac{x^2}{2} b^2} \max_K \{ |\nabla \chi|^2, |\Delta \chi|^2 \} C(M_0, M_1)^2 \\
+ \left( \frac{\lambda^2 L b}{2\pi} \phi_3 e^{2\zeta (\phi_2 - \phi_3)} e^{\frac{x^2}{2} b^2} \right) \int_{\omega_0} \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} |U_1(x,l)|^2 dx ds \].
Define
\[ \tau \triangleq 1 - e^{-\mu(\frac{1}{M}(1 - \frac{\rho_2}{\|\omega\|_{L^\infty(\omega)}) + 1 - \rho_2^2))}, \]
then
\[ \phi_3 - \phi_4 = \tau \phi_3 > 0. \]  
(36)

Let \( \zeta = \frac{\lambda^2 b^2}{4\tau \rho_3} \) and \( A > 1 \). By choosing \( L = 8Ab \), we have
\[ \int_{-1}^{1} \int_{\Omega \setminus O_2} \left( |U F|^2 + |U F'_2|^2 \right) dxds \]
\[ \leq C \left[ \zeta^{-3} \mu^{-4} \phi_3^{-3} \left( e^{2\zeta(\phi_2 - \phi_3)} 32A^2b^2 \right) e^{-\frac{\lambda^2 b^2}{2}(A^2-1)} \max_{K} \{|\Phi''(l)|^2, |\Phi'(l)|^2\} + e^{-2\zeta(\phi_3 - \phi_4)} \right. 
\[ + e^{2\zeta(\phi_1 - \phi_3)} \frac{8A^2b^2M_0^2}{\pi} e^{\frac{\lambda^2 b^2}{2}} \max\{|\nabla \chi|^2, |\Delta \chi|^2\} \right] C(M_0, M_1)^2 
\[ + \zeta \mu^{-3} \frac{4A^2b^2 \phi_3^2}{\phi_3} e^{2\zeta(\phi_2 - \phi_3)} e^{\frac{\lambda^2 b^2}{2}} \int_{\Omega} \int_{-4Ab} \left. |U_1(x, l)|^2 dx dl \right] 
\leq C \left[ \left( \frac{4\tau \phi_3}{\lambda^2 b^2} \right)^3 \mu^{-4} \phi_3^{-3} \left( e^{2\frac{\lambda^2 b^2}{4\tau \rho_3}(\phi_2 - \phi_3)} 32A^2b^2 \right) e^{-\frac{\lambda^2 b^2}{2}(A^2-1)} \max_{K} \{|\Phi''(l)|^2, |\Phi'(l)|^2\} \right. 
\[ + e^{-\frac{\lambda^2 b^2}{2}} + e e^{\frac{\lambda^2 b^2}{4\tau \rho_3}(\phi_2 - \phi_3)} \frac{8A^2b^2M_0^2}{\pi} e^{\frac{\lambda^2 b^2}{2}} \max\{|\nabla \chi|^2, |\Delta \chi|^2\} \right] C(M_0, M_1)^2 
\[ + \frac{\lambda^2 b^2}{4\tau \rho_3} \mu^{-3} \frac{4A^2b^2 \phi_3^2}{\phi_3} e^{2\frac{\lambda^2 b^2}{4\tau \rho_3}(\phi_2 - \phi_3)} e^{\frac{\lambda^2 b^2}{2}} \int_{\Omega} \int_{-4Ab} \left. |U_1(x, l)|^2 dx dl \right] \] 
(37)

By Parseval’s identity, we get that
\[ \| \Phi U_j \|_{L^2((\omega \setminus O_2) \times (-\frac{1}{4}, \frac{1}{4}))}^2 \leq \int_{-\frac{1}{4}}^{\frac{1}{4}} \int_{\omega \setminus O_2} |\Phi(t)U_j(x, t)|^2 dxdt \]
\[ = \int_{R} \int_{\Omega \setminus O_2} |\Phi(t)U_j(x, t)|^2 dxdt = \frac{1}{2\pi} \int_{R} \int_{\Omega \setminus O_2} |\Phi(l_0)U_j(x, l_0(t))|^2 dxdt \]
\[ \leq \frac{1}{\pi} \int_{R} \int_{\omega} |(1 - F)\Phi(l_0)U_j(x, l_0(t))|^2 dxdt \]
\[ + 2 \int_{R} \int_{\Omega \setminus O_2} |F \ast \Phi(\cdot)U_j(x, \cdot)(l_0)|^2 dxdl_0. \] 
(38)
The first term in the right hand side of (38) reads

$$\frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| (1 - F_\lambda) \Phi(l_0) U_j(x, l_0)(t) \right|^2 dx dt$$

$$= \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| (1 - e^{-(t/\lambda)^2}) \Phi(l_0) U_j(x, l_0)(t) \right|^2 dx dt$$

$$\leq \frac{2}{\pi \lambda^2} \int_{\mathbb{R}} \int_{\mathbb{R}} |t \Phi(l_0) U_j(x, l_0)(t)|^2 dx dt$$

$$\leq \frac{4}{\lambda^2} \int_{\mathbb{R}} \int_{\mathbb{R}} |\Phi'(l_0) u_j(x, l_0) + \Phi(l_0) \partial_{l_0} U_j(x, l_0)|^2 dx dl_0$$

$$\leq \frac{8}{\lambda^2} \left[ \frac{1}{4L} \right] \int_{K_0} \int_{\mathbb{R}} |U_j(x, l_0)|^2 dx dl_0 + \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{\mathbb{R}} |\partial_{l_0} U_j(x, l_0)|^2 dx dl_0 \right]$$

$$\leq \frac{8}{\lambda^2} \left[ \frac{1}{32A\beta} \right] \int_{K_0} \int_{\mathbb{R}} |U_j(x, l_0)|^2 dx dl_0 + \int_{-\frac{4A\beta}{2}}^{\frac{4A\beta}{2}} \int_{\mathbb{R}} |\partial_{l_0} U_j(x, l_0)|^2 dx dl_0 \right].$$

Let

$$U_{j,\lambda}^F(x, l_0) \triangleq U_j^F(x, 0) = \int_{\mathbb{R}} F_\lambda(l_0 - l) \Phi(l) U_j(x, l) dl = F_\lambda * \Phi(\cdot) U_j(x, \cdot)(l_0). \quad (40)$$

By applying the Cauchy integral formula, for $\rho \in (0, 1)$ and by setting $z = \kappa + \rho e^{i\phi}$, we have that

$$U_{j,\lambda}^F(x, \kappa) = \frac{1}{2\pi i} \int_{|z - \kappa| = \rho} \frac{U_{j,\lambda}^F(x, z)}{z - \kappa} dz = \frac{1}{2\pi i} \int_0^{2\pi} U_{j,\lambda}^F(x, \kappa + \rho e^{i\phi}) d\phi$$

$$= \frac{1}{2\pi i} \int_0^1 \int_0^{2\pi} U_{j,\lambda}^F(x, \kappa + \rho e^{i\phi}) d\phi d\rho$$

$$= \frac{1}{2\pi i} \int_{-1}^1 \int_{-\sqrt{1-l_0^2}}^{\sqrt{1-l_0^2}} U_{j,\lambda}^F(x, l_0 + is) J(l_0, s) ds dl_0$$

$$= \frac{1}{2\pi i} \int_{-1}^1 \int_{-\sqrt{1-l_0^2}}^{\sqrt{1-l_0^2}} U_j^F(x, s) ds dl_0.$$ 

Thus,

$$|U_{j,\lambda}^F(x, \kappa)|^2 \leq \frac{1}{\pi^2} \int_{-1}^1 \int_{-1}^1 |U_j^F(x, s)|^2 ds dl_0.$$

(42)
Integrating (42) with respect to $x$ over $\omega$, and with respect to $\kappa$ over $[-\frac{L}{2}, \frac{L}{2}]$ we get that

$$
\int_{-4A}^{4A} \int_{\omega \setminus O_2} |U_j^F(x, \kappa)|^2 \, dx \, d\kappa
$$

$$
\leq \frac{1}{\pi^2} \int_{-4A}^{4A} \int_{-1}^{1} \left( \int_{-1}^{1} \int_{\omega \setminus O_2} |U_j^F(x, s)|^2 \, dx \, ds \right) \, dl_0 \, d\kappa
$$

$$
\leq \frac{16A}{\pi^2} \int_{-1}^{1} \int_{\omega \setminus O_2} |U_j^F(x, s)|^2 \, dx \, ds.
$$

Substituting (39) and (43) into (38) yields

$$
||\Phi U_j||_{L^2((\omega \setminus O_2) \times (-4A, 4A))}^2
$$

$$
\leq \frac{8}{\lambda^2} \left[ \frac{1}{(32A)^2} \int_{K_0} \int_{\omega} |U_j(x, l_0)|^2 \, dx \, dl_0 + \int_{-4A}^{4A} \int_{\omega} |\partial_0 U_j(x, l_0)|^2 \, dx \, dl_0 \right]
$$

$$
+ \frac{16A}{\pi^2} \int_{-1}^{1} \int_{\omega \setminus O_2} |U_j^F(x, s)|^2 \, dx \, ds.
$$

Let $C_2 = \max\{|\nabla \chi|^2, |\Delta \chi|^2\}$ and suppose that $A$ is sufficiently enough such that $\frac{1}{2}(A^2 - 1)\tau + 1 > e^{\frac{\mu}{\mu_M ||\nabla||_{L^\infty(\omega)}} + 1}$. Then

$$
||\langle\Phi U_1, \Phi U_2\rangle||_{L^2((\omega \setminus O_2) \times (-4A, 4A))}^2
$$

$$
\leq \frac{8}{\lambda^2} \sum_{j=1}^{2} \left[ \frac{1}{(32A)^2} \int_{K_0} \int_{\omega} |U_j(x, l_0)|^2 \, dx \, dl_0 + \int_{-4A}^{4A} \int_{\omega} |\partial_0 U_j(x, l_0)|^2 \, dx \, dl_0 \right]
$$

$$
+ \frac{16A}{\pi^2} \int_{-1}^{1} \int_{\omega \setminus O_2} |U_j^F(x, s)|^2 \, dx \, ds
$$

$$
\leq \frac{8}{\lambda^2} \sum_{j=1}^{2} \left[ \frac{1}{(32A)^2} \int_{K_0} \int_{\omega} |U_j(x, l_0)|^2 \, dx \, dl_0 + \int_{-4A}^{4A} \int_{\omega} |\partial_0 U_j(x, l_0)|^2 \, dx \, dl_0 \right]
$$

$$
+ \frac{16A}{\pi^2} \int_{-1}^{1} \int_{\omega \setminus O_2} |U_j^F(x, s)|^2 \, dx \, ds
$$

$$
\leq \frac{8}{\lambda^2} \sum_{j=1}^{2} \left[ \frac{1}{(32A)^2} \int_{K_0} \int_{\omega} |U_j(x, l_0)|^2 \, dx \, dl_0 + \int_{-4A}^{4A} \int_{\omega} |\partial_0 U_j(x, l_0)|^2 \, dx \, dl_0 \right]
$$

$$
+ \frac{16A}{\pi^2} \int_{-1}^{1} \int_{\omega \setminus O_2} |U_j^F(x, s)|^2 \, dx \, ds
$$

$$
\leq \frac{8}{\lambda^2} + \frac{1024A^2\tau^4}{\mu^4 \pi^2 \lambda^5 b^3} \left( \frac{32A^2}{\pi} e^{-(A^2 - 1)\tau_{\phi}^2} \right) + \frac{1}{\lambda^2 b^2}
$$

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Similarly, we have

\[
\|\langle \Phi V_1, \Phi V_2 \rangle \|^2_{L^2((\omega \setminus \Omega_2) \times (-\frac{L}{2}, \frac{L}{2}))} \leq \left[ \frac{8}{\lambda^2} + C \frac{1024 A \lambda^3}{\mu^4 \pi^2 \lambda^2 b^5} \left( \frac{32 A \lambda^2 b^2}{\pi} e^{-\left( A^2 - 1 - \frac{\phi_1 - \phi_3}{\tau_3} \right) \frac{1}{2} \lambda^2 b^2} + e^{-\frac{1}{2} \lambda^2 b^2} + C_2 \frac{8 AM^2 \lambda^2 b^2}{\pi} e^{\frac{\lambda^2 b^2}{2} (\frac{\phi_1 - \phi_3}{\tau_3} + 1)} \right) \right] C(M_0, M_1)^2
\]

\[
+ \frac{C_2 A \lambda^3 \lambda^6 b^6}{4\pi^2} \left( \frac{\phi_2}{\phi_3} \right)^5 e^{\left( \frac{\phi_2 - \phi_3}{\tau_3} + 1 \right) \frac{3}{2} \lambda^2 b^2} \|V_1\|^2_{L^2((\omega_0 \setminus (-\frac{L}{2}, \frac{L}{2}))}
\]

and

\[
\| (c_{11} - \tilde{c}_{11}, c_{22} - \tilde{c}_{22}) \|_{L^2(\Omega)} \leq C \sum_{j=1}^{2} \| \partial_{x_j} (w_1, w_2) \|_{L^2((\omega \times (0, T)))}
\]

\[
\leq C \left( \| (U_1, U_2) \|_{L^2((\omega \setminus \Omega_2) \times (0, T))} + \| (V_1, V_2) \|_{L^2((\omega \setminus \Omega_2) \times (0, T))} \right)
\]

\[
\leq C \left( \| (\Phi U_1, \Phi U_2) \|_{L^2((\omega \setminus \Omega_2) \times (0, T))} + \| (\Phi V_1, \Phi V_2) \|_{L^2((\omega \setminus \Omega_2) \times (0, T))} \right)
\]

\[
\leq 2 \left[ \frac{8}{\lambda^2} + C \frac{1024 A \lambda^3}{\mu^4 \pi^2 \lambda^2 b^5} \left( \frac{32 A \lambda^2 b^2}{\pi} e^{-\left( A^2 - 1 - \frac{\phi_1 - \phi_3}{\tau_3} \right) \frac{1}{2} \lambda^2 b^2} + e^{-\frac{1}{2} \lambda^2 b^2} + C_2 \frac{8 AM^2 \lambda^2 b^2}{\pi} e^{\frac{\lambda^2 b^2}{2} (\frac{\phi_1 - \phi_3}{\tau_3} + 1)} \right) \right] C(M_0, M_1)^2
\]

\[
+ \frac{C_2 A \lambda^3 \lambda^6 b^6}{4\pi^2} \left( \frac{\phi_2}{\phi_3} \right)^5 e^{\left( \frac{\phi_2 - \phi_3}{\tau_3} + 1 \right) \frac{3}{2} \lambda^2 b^2} \left( \|U_1\|^2_{L^2((\omega_0 \setminus (-\frac{L}{2}, \frac{L}{2}))} + \|V_1\|^2_{L^2((\omega_0 \setminus (-\frac{L}{2}, \frac{L}{2})))} \right)
\]

\[
\leq 2 \left[ \frac{8}{\lambda^2} + C \frac{1024 A \lambda^3}{\mu^4 \pi^2 \lambda^2 b^5} \left( \frac{32 A \lambda^2 b^2}{\pi} e^{-\left( A^2 - 1 - \frac{\phi_1 - \phi_3}{\tau_3} \right) \frac{1}{2} \lambda^2 b^2} + e^{-\frac{1}{2} \lambda^2 b^2} + C_2 \frac{8 AM^2 \lambda^2 b^2}{\pi} e^{\frac{\lambda^2 b^2}{2} (\frac{\phi_1 - \phi_3}{\tau_3} + 1)} \right) \right] C(M_0, M_1)^2
\]

\[
+ \frac{C_2 A \lambda^3 \lambda^6 b^6}{4\pi^2} \left( \frac{\phi_2}{\phi_3} \right)^5 e^{\left( \frac{\phi_2 - \phi_3}{\tau_3} + 1 \right) \frac{3}{2} \lambda^2 b^2} \sum_{j=1}^{2} \| \partial_{x_j} w_1 \|_{L^2((Q_\omega))}
\]

Let \( \lambda \geq 0 \) be such that

\[
\| (c_{11} - \tilde{c}_{11}, c_{22} - \tilde{c}_{22}) \|_{L^2(\Omega)} \leq \frac{C_3}{\lambda^2} C(M_0, M_1) + e^{C_4 \lambda^2} \sum_{j=1}^{2} \| \partial_{x_j} w_1 \|_{L^2((Q_\omega))},
\]

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where $C_3$ and $C_4$ are two constants independent of $\lambda$. By taking
\[
\lambda = \left( \sum_{j=1}^{2} \frac{\ln ||\partial_t^j w_1||_{L^2(Q_0)}}{C_4} \right)^{\frac{1}{2}},
\]
if $||\partial_t^j w_1||_{L^2(Q_0)}$ is small enough, then
\[
|| (a - \hat{a}, c_{11} - \hat{c}_{11}, c_{22} - \hat{c}_{22}) ||_{L^2(\Omega)} \leq \frac{C_3C_4}{\sum_{j=2}^{3} \ln ||\partial_t^j w_1||_{L^2(Q_0)}} C(M_0, M_1) + \sum_{j=1}^{2} ||\partial_t^j w_1||_{L^2(Q_0)}
\]
\[
\leq C\left( \sum_{j=1}^{2} \frac{\ln ||\partial_t^j w_1||_{L^2(Q_0)}}{||\partial_t^j w_1||_{L^2(Q_0)}} \right)^{-1} C(M_0, M_1) + \sum_{j=1}^{2} ||\partial_t^j w_1||_{L^2(Q_0)} \right) \right)
\]
\[
\leq C\left( \sum_{j=1}^{2} \frac{\ln ||\partial_t^j w_1||_{L^2(Q_0)}}{||\partial_t^j w_1||_{L^2(Q_0)}} \right)^{-1} C(M_0, M_1) + \sum_{j=1}^{2} ||\partial_t^j w_1||_{L^2(Q_0)} \right).
\]

Otherwise, there exists a constant $m > 0$ such that $||\partial_t^j w_1||_{L^2(Q_0)} \geq m$. Thus, by (11) we have
\[
|| (c_{11} - \hat{c}_{11}, c_{22} - \hat{c}_{22}) ||_{L^2(\Omega)} \leq C(M_0, M_1)
\]
\[
= \frac{C(M_0, M_1)}{m} \leq C \sum_{j=1}^{2} ||\partial_t^j w_1||_{L^2(Q_0)} \leq C \sum_{j=1}^{2} ||\partial_t^j w_1||_{L^2(Q_0)}.
\]

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