EQUIVALENCE AND CHARACTERIZATIONS OF LINEAR RANK-METRIC CODES BASED ON INVARIANTS

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1. Introduction

Over the last decades rank-metric codes have become an active research area due to several applications, such as crisscross error correction [55], post-quantum cryptography [21, 17, 47, 22, 32, 1], space-time coding for MIMO systems [20, 7, 31, 34, 53, 51], network coding [63, 15, 42, 62, 16], distributed data storage [61, 52, 8, 39], and digital image watermarking [29]. They can either be defined as sets of matrices of fixed dimensions over some finite field, where the distance of two elements is measured by the rank of their difference, or equivalently as sets of vectors over an extension field, where the distance is measured as the rank of a vector over the base field. In this work we will use the latter. Furthermore, we will denote an underlying finite field by $\mathbb{F}_{q^m}$, where $q$ is a prime power.

A rank-metric code in $\mathbb{F}_{q^m}^n$ is called linear if it forms an $\mathbb{F}_{q^m}$-linear subspace of $\mathbb{F}_{q^m}^n$. Rank-metric codes attaining the Singleton bound, which upper-bounds the code cardinality for given minimum distance, are called maximum rank distance (MRD) codes. Delsarte [14], Gabidulin [18], and Roth [55] independently introduced a prominent class of linear MRD codes for all possible code parameters, which are today called Gabidulin codes.

Driven by applications and fundamental questions, finding new (linear) MRD codes inequivalent to Gabidulin codes has become one of the most actively studied research problems within the field of rank-metric codes in the last years. The topic has been further encouraged by the non-constructive results in [40], which showed that for many parameter ranges there are plenty of non-Gabidulin linear MRD codes. After early works on generalizing Gabidulin codes using different automorphisms [56, 27], Sheekey [58] was the first to find a general construction for different (linear and non-linear) MRD codes, called twisted Gabidulin codes. Otal and Özbudak independently discovered a special case of twisted Gabidulin codes [44]. Starting from Sheekey’s construction, several generalizations have been proposed, e.g. in [58, 51].

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This work is an extension of the conference paper “Invariants and Inequivalence of Linear Rank-Metric Codes” [41], which appeared in Proceedings of IEEE International Symposium on Information Theory 2019.
Remark 9 and in [33, 45, 60, 50]. Moreover, other non-Gabidulin MRD codes have been constructed, see e.g. [24, 11, 12, 13, 35, 5]. For an overview of non-Gabidulin MRD constructions (also non-linear ones), we refer the reader to the survey [59].

A central question in all of the above mentioned works is whether the new codes are actually inequivalent to known codes. A distinguisher for (generalized) Gabidulin codes, which is based on the dimension of the intersection of the code with itself under some field automorphism, was given in [24]. The idea was extended in [23] to distinguish certain twisted Gabidulin codes. For the other constructions mentioned above, some authors proposed methods tailored to their code construction (e.g., [58]), showed the inequivalence only for special cases (e.g., [50]), or did not study the equivalence problem at all (e.g., [19]). The question of inequivalence is thus still open for many cases. Moreover, not all of the above criteria are efficiently computable. It is thus important to derive easily computable criteria to check if two codes are equivalent.

Since any non-linear code cannot be equivalent to a linear code, we will focus on the question if two linear codes are equivalent.

In this work we widely generalize the results of [24, 23] to sums and intersections of the code under arbitrary field automorphisms. We obtain a class of invariants under equivalence, which can be used as an efficiently computable sufficient tool to prove inequivalence of linear rank-metric codes in general. The method is particularly powerful for codes constructed as evaluation codes of skew polynomials, and hence suitable for the majority of proposed linear MRD code constructions in the literature.

The paper is structured as follows. In Section 2 we give some preliminaries on finite fields, linearized polynomials, and rank-metric and MRD codes. We define the intersection and sum sequences, and the corresponding dimension sequences in Section 3, and show that these are invariants for the equivalence class of a rank-metric code. Moreover, we derive some general properties of these sequences. In Section 4 we compute the sequences for various code families, and use the results to show when two codes from different families are not equivalent. Thereafter, in Section 5 we use the sequences to derive the exact number of inequivalent Gabidulin codes and bounds on the number of inequivalent twisted Gabidulin codes, with an exact formula for twisted Gabidulin codes for the case $m = n$. Furthermore, for small code parameters, we present computational results on the number of equivalence classes of generalized twisted Gabidulin codes. In Section 6 we derive new characterization results for Gabidulin codes, based on our sequences. Finally, we conclude this work in Section 7 and present some open questions for further research.

2. Preliminaries

2.1. Finite Fields and Moore Matrices. Let $q$ be a prime power and denote by $\mathbb{F}_q$ the finite field of size $q$. For a positive integer $m$, the extension field $\mathbb{F}_{q^m}$ is a vector space of dimension $m$ over $\mathbb{F}_q$. It is also well-known that the extension $\mathbb{F}_{q^m}/\mathbb{F}_q$ is Galois, with a cyclic Galois group. More precisely the set

$$\text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) := \{ \sigma : \mathbb{F}_{q^m} \to \mathbb{F}_{q^m} \text{ field automorphism} \mid \sigma(a) = a \text{ for every } a \in \mathbb{F}_q \}$$

is a group endowed with the operation of composition, and is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. The elements of $\text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$ are given by the homomorphisms

$$\theta_i : \mathbb{F}_{q^m} \to \mathbb{F}_{q^m},$$

$$a \mapsto a^{q^i},$$

for every $i = 0, 1, \ldots, m-1$, and the generators are given by all the $\theta_i$’s such that $\gcd(i, m) = 1$.

Moreover, the norm with respect to $\mathbb{F}_{q^m}/\mathbb{F}_q$ is the map

$$N_{\mathbb{F}_{q^m}/\mathbb{F}_q} : \mathbb{F}_{q^m} \to \mathbb{F}_q,$$

$$a \mapsto \prod_{i=0}^{m-1} \theta_i(a) = \prod_{i=0}^{m-1} a^{q^i}.$$
We now introduce the *Moore matrix* and state some important properties that we will widely use in this paper. Over finite fields, the Moore matrix is the $q$-analogue of the Vandermonde matrix.

**Definition 2.1.** Let $\mathbb{L}$ be a field and $k, n$ be positive integers. For a vector $v = (v_1, \ldots, v_n) \in \mathbb{L}^n$ and $\tau \in \text{Aut}(\mathbb{L})$, we denote by $M_{k,\tau}(v)$ the *$\tau$-Moore matrix*, which is defined as

$$M_{k,\tau}(v) := \begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ \tau(v_1) & \tau(v_2) & \cdots & \tau(v_n) \\ \vdots & \vdots & \ddots & \vdots \\ \tau^{k-1}(v_1) & \tau^{k-1}(v_2) & \cdots & \tau^{k-1}(v_n) \end{pmatrix} \in \mathbb{L}^{k \times n}.$$

The following results give a relation between the rank of a Moore matrix and the entries of its defining vector. The next theorem generalizes [30, Corollary 2.38] and is a consequence of [28, Corollary 4.13].

**Theorem 2.2.** Let $F \subseteq L$ be a Galois field extension, $\sigma \in \text{Gal}(\mathbb{L}/F)$ and $E = \mathbb{L}^\sigma$ be the fixed field of $\sigma$, i.e., $E = \{\alpha \in \mathbb{L} \mid \sigma(\alpha) = \alpha\}$. For $g \in \mathbb{L}^n$, consider the $\sigma$-Moore matrix $M_{k,\sigma}(g) \in \mathbb{L}^{k \times n}$. Then, $\text{rk}(M_{k,\sigma}(g)) = \min\{k, r\}$, where $r = \dim(g_1, \ldots, g_n)\mathbb{E}$.

**Definition 2.3.** Let $F_{qm}$ be an extension field of $F_q$, and let $g = (g_1, \ldots, g_n) \in F_{qm}^n$. We define the $F_q$-support of $g$ over $F_q$ the $F_q$-subspace

$$\text{supp}_q(g) := \langle g_1, \ldots, g_n \rangle_{F_q}.$$  

Moreover, we define $\text{rk}_q(g) := \dim_q(\text{supp}_q(g))$, and call it the *$q$-rank* (or simply *rank*) of $g$.

Note that the rank of a matrix $M$ is denoted by $\text{rk}(M)$ and is taken over $F_{qm}$. With the notion of $q$-rank and $F_q$-support we can deduce the following corollary from Theorem 2.2 in the case of finite fields.

**Corollary 2.4.** Let $\theta$ be a generator of $\text{Gal}(F_{qm}/F_q)$ and $g \in F_{qm}^n$. Then $\text{rk}(M_{k,\theta}(g)) = \min\{k, \text{rk}_q(g)\}$. In particular, the set $\{g, \theta(g), \ldots, \theta^{k-1}(g)\}$ is linearly independent over $F_{qm}$.

## 2.2. The Skew Group Algebra $F_q[G]$  

In the context of rank-metric codes, an important related object is the ring of *linearized polynomials*, which was first studied by Ør in [43]. Its elements are polynomials in $F_{qm}[x]$ that involve only monomials of the form $x^q^i$, for some non-negative integers $i$. Their importance is due to the fact that, seen as functions corresponding to their evaluation, they are $F_q$-linear maps from $F_{qm}$ to itself. On the other hand, any $F_q$-linear map from $F_{qm}$ to itself can be represented as a $q$-polynomial of degree at most $q^{m-1}$. Let $\mathcal{L}(F_{qm})$ denote the set of $q$-polynomials with coefficients in $F_{qm}$. This set is closed under addition and composition, and together with these two operations, $\mathcal{L}(F_{qm})$ is a non-commutative ring.

When one only cares about the evaluation in $F_{qm}$, one can reduce to studying the set $\mathcal{L}_m(F_{qm}) := \mathcal{L}(F_{qm})/(x^q^m - x)$, since $a^q^m = a$ for every $a \in F_{qm}$, and $(x^q^m - x)$ is a two-sided ideal. In this framework, one can easily verify that

$$\mathcal{L}_m(F_{qm}) \cong F_q^{m \times m}.$$  

However, this is not the end of the story: The ring $\mathcal{L}_m(F_{qm})$ is also isomorphic to the skew group algebra $F_{qm}[G]$, where $G = \text{Gal}(F_{qm}/F_q) = \langle \theta \rangle$, which is a ring endowed with addition and composition. More in detail, the elements $f, g \in F_{qm}[G]$ are of the form $f = \sum_{i=0}^{m-1} f_i \theta^i$, $g = \sum_{i=0}^{m-1} g_i \theta^i$, for some $f_i, g_i \in F_{qm}$. The addition is defined by $f + g = \sum_{i=0}^{m-1} (f_i + g_i) \theta^i$; the composition is defined on monomials by $(f \theta^i) \circ (g \theta^j) = f \theta^i(g \theta^j)$, and then extended by associativity and distributivity. In this framework, we also have that

$$F_{qm}[G] \cong \text{End}_{F_q}(F_{qm}) = \{ \phi : F_{qm} \to F_{qm} \mid \phi \text{ is } F_q\text{-linear} \}.$$
The importance of this point of view is that it can be generalized to fields of any characteristic, provided that the field extension has a cyclic Galois group. This was the key point of the works by Augot, Loidreau and Robert [3, 2, 4]. For a deeper understanding on this topic over finite fields, the interested reader is referred to [67]. A brief summary of this for general fields can be also found in [38, Chapter 4]. This explains why our notation will follow the skew group algebra setting.

2.3. Rank-Metric and MRD Codes. We now explain the basics of linear rank-metric codes and their equivalence maps, and define maximum rank distance (MRD) codes.

**Definition 2.5.** The rank distance between \( u, v \in \mathbb{F}_{q^m}^n \) is defined as

\[
d_{rk}(u, v) := \text{rk}_q(u - v).
\]

A linear (vector) rank-metric code is an \( \mathbb{F}_{q^m} \)-linear subspace \( C \subseteq \mathbb{F}_{q^m}^n \). If \( C \neq \{0\} \) is a linear rank-metric code, then the minimum distance of \( C \) is the integer

\[
d(C) := \min \{d_{rk}(u, v) \mid u, v \in C, \ u \neq v\} = \min \{\text{rk}_q(u) \mid u \in C, \ u \neq 0\}.
\]

It is easy to verify that the map \( d_{rk} : \mathbb{F}_{q^m}^n \times \mathbb{F}_{q^m}^n \rightarrow \mathbb{N} \) defines a metric on \( \mathbb{F}_{q^m}^n \). From now on we will refer to a linear (vector) rank-metric code \( C \subseteq \mathbb{F}_{q^m}^n \) of dimension \( k \) as an \([n, k]_{q^m} \) code. When the minimum distance \( d = d(C) \) is known, we will call it an \([n, k, d]_{q^m} \) code.

Let \( V, W \) be vector spaces over a field \( \mathbb{F}_{q^m} \). Recall that a map \( \varphi : V \rightarrow W \) is called semilinear, if there exists \( \tau \in \text{Aut}(\mathbb{F}_{q^m}) \) such that, for all \( x, y \in V \) and \( \lambda \in \mathbb{F}_{q^m} \), it holds that

1. \( f(x + y) = f(x) + f(y) \).
2. \( f(\lambda x) = \tau(\lambda)f(x) \).

If \( V = W \), then the set of invertible semilinear maps is a group, called general semilinear group and denoted by \( \Gamma L(V) \). Furthermore, \( \Gamma L(V) \cong \text{GL}(V) \times \text{Aut}(\mathbb{F}) \).

**Definition 2.6.** Two rank-metric codes \( C, C' \subseteq \mathbb{F}_{q^m}^n \) are (semilinearly) equivalent if there exists an \( \mathbb{F}_{q^m} \)-semilinear isometry (i.e., distance-preserving mapping) \( \varphi : (\mathbb{F}_{q^m}^n, d_{rk}) \rightarrow (\mathbb{F}_{q^m}^n, d_{rk}) \) such that \( \varphi(C) = C' \). If \( C, C' \subseteq \mathbb{F}_{q^m}^n \) are equivalent rank-metric codes, then we will write \( C \sim C' \).

The semilinear rank isometries on \( \mathbb{F}_{q^m}^n \) are induced by the semilinear isometries on \( \mathbb{F}_{q^m}^{n \times m} \) (see [6, 36, 66]) and are characterized as follows.

**Theorem 2.7.** [6, Corollary 1][36, Proposition 2] The semilinear \( \mathbb{F}_q \)-rank isometries on \( \mathbb{F}_{q^m}^n \) are of the form

\[
(\lambda, A, \tau) \in (\mathbb{F}_{q^m}^\ast \times \text{GL}_n(q)) \rtimes \text{Aut}(\mathbb{F}_{q^m}),
\]

acting on \( \mathbb{F}_{q^m}^n \) via

\[
(v_1, \ldots, v_n)(\lambda, A, \tau) = (\tau(\lambda v_1), \ldots, \tau(\lambda v_n))A.
\]

In particular, if \( C \subseteq \mathbb{F}_{q^m}^n \) is a rank-metric code with minimum rank distance \( d \), then \( C' = \tau(\lambda C)A \) is a rank-metric code with minimum rank distance \( d \).

Observe that we can always reduce to the case \( \lambda = 1 \), because if \( C \) and \( C' \) are \([n, k]_{q^m} \) codes and \( C' = \tau(\lambda C)A \), then by \( \mathbb{F}_{q^m} \)-linearity, we also have \( \tau(\lambda C)A = \tau(\lambda)\tau(C)A = \tau(C)A \). Hence, \( \lambda \) is only relevant when the considered codes are not linear over \( \mathbb{F}_{q^m} \).

Recall that the standard inner-product (or dot product) of \( u, v \in \mathbb{F}_{q^m}^n \) is \( \langle u; v \rangle := \sum_{i=1}^n u_i v_i \). It is well-known that the map \( (u, v) \mapsto \langle u; v \rangle \) defines an \( \mathbb{F}_{q^m} \)-bilinear, symmetric and nondegenerate form on \( \mathbb{F}_{q^m}^n \).

**Definition 2.8.** The dual of a \([n, k]_{q^m} \) (vector) rank-metric code \( C \) is

\[
C^\perp := \{u \in \mathbb{F}_{q^m}^n \mid \langle u; v \rangle = 0 \text{ for all } v \in C\}.
\]

Note that \( C^\perp \) is an \([n, n - k]_{q^m} \) code.

The following result is the rank-metric analogue of the Singleton bound for codes with the Hamming metric.
Definition 2.11. It was then generalized in [54, 27] as Gabidulin codes, was found independently by Delsarte [14], Gabidulin [18], and Roth [55].

Let \( \theta \) be a generator of \( \mathbb{F}_q \). The condition \( n \leq m \) is also necessary. Therefore we assume \( n \leq m \) throughout the paper.

2.4. Known MRD Constructions. The first construction of MRD codes, generally known as Gabidulin codes, was found independently by Delsarte [14], Gabidulin [18], and Roth [55]. It was then generalized in [54, 27].

Definition 2.12. Let \( k, n, m \) be positive integers such that \( 1 \leq k \leq n \leq m \) and let \( \theta \) be a generator of \( G = \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \). We denote by \( G_{k,\theta} \) the \( \mathbb{F}_{q^m} \)-subspace of the skew group algebra \( \mathbb{F}_{q^m}[\theta] = \mathbb{F}_{q^m}[G] \) generated by the first \( k \) powers of \( \theta \), that is

\[
G_{k,\theta} := \left\{ f_0\text{id} + f_1\theta + \ldots + f_{k-1}\theta^{k-1} \mid f_i \in \mathbb{F}_{q^m} \right\}.
\]

Let \( g = (g_1, \ldots, g_n) \in \mathbb{F}_{q^m}^n \) such that \( \text{rk}_q(g) = n \). The \( \theta \)-Gabidulin code \( G_{k,\theta}(g) \) is defined as

\[
G_{k,\theta}(g) := \{(f(g_1), \ldots, f(g_n)) \mid f \in G_{k,\theta}\}.
\]

Proposition 2.12. [18, 27] The \( \theta \)-Gabidulin code \( G_{k,\theta}(g) \) has cardinality \( q^{km} \) and minimum distance \( d = n - k + 1 \), i.e., \( G_{k,\theta}(g) \) is an MRD code.

The following result gives an explicit expression for the dual of a \( \theta \)-Gabidulin code, which is in turn a \( \theta \)-Gabidulin code.

Proposition 2.13. [18, Sections 2 and 4] [27, Subsection IV.C] Let \( C = G_{k,\theta}(g) \) be a \( \theta \)-Gabidulin code. Then

\[
C^\perp = G_{n-k,\theta}(g'),
\]

where \( g' \) is any non-zero vector in the code \( G_{n-k,\theta}(g^{-1}(n-k-1))(g))^\perp \). Moreover, \( \text{rk}_q(g') = n \).

Gabidulin codes are not the only known MRD codes. There are some other families of codes which attain the Singleton-like bound of Theorem 2.9, that have been discovered in the last years. Here we give an overview of some of these families.

Definition 2.14. Let \( k, n, m, b \) be positive integers such that \( 1 \leq k \leq n \leq m \) and \( 0 \leq b < m \). Let \( \theta \) be a generator of \( G = \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \) and \( \eta \in \mathbb{F}_{q^m} \). We denote by \( H_{k,\theta}^{n,b} \) the \( \mathbb{F}_q \)-subspace of the skew group algebra \( \mathbb{F}_{q^m}[\theta] = \mathbb{F}_{q^m}[G] \) given by

\[
H_{k,\theta}^{n,b} := \left\{ f_0\text{id} + f_1\theta + \ldots + f_{k-1}\theta^{k-1} + \eta\theta^b(f_0)\theta^k \mid f_i \in \mathbb{F}_{q^m} \right\}.
\]

Moreover, let \( g = (g_1, \ldots, g_n) \in \mathbb{F}_{q^m}^n \) such that \( \text{rk}_q(g) = n \) and suppose that \( N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\eta) = (-1)^{km} \). The \( \theta \)-twisted Gabidulin code with parameters \( \eta \) and \( h \) is defined as

\[
H_{k,\theta}^{n,b}(g) := \left\{ (f(g_1), \ldots, f(g_n)) \mid f \in H_{k,\theta}^{n,b} \right\}.
\]

Proposition 2.15. [58] Let \( g \in \mathbb{F}_{q^m}^n \) such that \( \text{rk}_q(g) = n \) and \( \eta \in \mathbb{F}_{q^m} \) such that \( N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\eta) = (-1)^{km} \). The \( \theta \)-twisted Gabidulin code \( H_{k,\theta}^{n,b}(g) \) has cardinality \( q^{km} \) and minimum distance \( d = n - k + 1 \), i.e., \( H_{k,\theta}^{n,b}(g) \) is an MRD code.

Remark 2.16. Observe that in general a \( \theta \)-twisted Gabidulin code is not \( \mathbb{F}_{q^m} \)-linear, but only \( \mathbb{F}_q \)-linear. It is \( \mathbb{F}_{q^m} \)-linear if and only if \( b = 0 \), in which case, we will denote the set \( H_{k,\theta}^{n,0} \) by \( H_{k,\theta}^{n} \), and the corresponding code by \( H_{k,\theta}^{n}(g) \). Therefore, by Proposition 2.15, the code \( H_{k,\theta}^{n}(g) = \langle g + \eta\theta^b(g), \theta(g), \ldots, \theta^{k-1}(g) \rangle_{\mathbb{F}_{q^m}} \) is an \( [n, k, n - k + 1]_{q^m} \) code.
This family of codes was given by Sheekey in [58], and was first introduced only considering the \( q \)-Frobenius automorphism \( \theta(x) = x^q \). In [58, Remark 9] and [33], it was generalized to any generator of \( \text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \). Further generalizations were given in [45], where the codes obtained are only linear over the prime field.

From now on we fix the following notation. Let \( 1 \leq k \leq n \leq m \) be integers. Choose a positive integer \( \ell \in \mathbb{N} \), which we call the number of twists. Let \( h \in \{0, \ldots, k-1\}^\ell \) and \( t \in \{1, \ldots, n-k\}^\ell \cup \{m-n+1, \ldots, m-k\}^\ell \) such that the \( h_i \)'s are distinct and the \( t_i \)'s are distinct. Furthermore, let \( \eta \in (\mathbb{F}_{q^m})^\ell \) and \( \theta \) be a generator of \( G = \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \). We can now define the generalized twisted Gabidulin codes from [50]:

**Definition 2.17.** We denote by \( I_{k,\theta}^{\eta,t,h} \) the \( \mathbb{F}_{q^m} \)-subspace of the skew group algebra \( \mathbb{F}_{q^m}[\theta] = \mathbb{F}_{q^m}[G] \) given by

\[
I_{k,\theta}^{\eta,t,h} := \left\{ f_0\text{id} + f_1\theta + \ldots + f_{k-1}\theta^{k-1} + \sum_{j=1}^{\ell} \eta_j f_{h_j} \theta^{k-1+t_j} \mid f_i \in \mathbb{F}_{q^m} \right\}.
\]

Moreover, let \( g \in \mathbb{F}_{q^m}^n \) with \( \text{rk}_q(g) = n \). The generalized \( \theta \)-twisted Gabidulin code \( I_{k,\theta}^{\eta,t,h}(g) \) is defined as

\[
I_{k,\theta}^{\eta,t,h}(g) := \left\{ (f(g_1), \ldots, f(g_n)) \mid f \in I_{k,\theta}^{\eta,t,h} \right\}.
\]

Note that generalized \( \theta \)-twisted Gabidulin codes are \( \mathbb{F}_{q^m} \)-linear by definition. In particular, the code \( I_{k,\theta}^{\eta,t,h}(g) \) can be written as

\[
\left\{ \left\{ \theta^i(g) + \eta_i \theta^{k-1+t_i} \mid i \in [\ell] \right\} \cup \left\{ \theta^i(g) \mid i \in \{0, \ldots, k-1\} \setminus \{h_1, \ldots, h_\ell\} \right\} \right\}_{\mathbb{F}_{q^m}}.
\]

In general, there is a sufficient MRD condition if the \( g_i \)'s are chosen from a subfield \( \mathbb{F}_{q^m} \subseteq \mathbb{F}_{q^m}^n \) with \( r 2^\ell \mid m \) and a suitable choice of the \( \eta \) [50] (see also [49, Chapter 7] for more details). Note that this gives codes of length \( n \leq 2^{-\ell} m \). It is an open problem whether longer MRD codes exist in \( I_{k,\theta}^{\eta,t,h} \) for arbitrary \( t \) and \( h \). In the special case \( \ell = 1 \), we write \( t = t_1 \in \mathbb{N} \) and \( h = h_1 \in \mathbb{N}_0 \).

Note that these codes have been originally defined only for \( t \in \{1, \ldots, n-k\}^\ell \). This was done in order to assure that the codes have dimension equal to \( k \). Here we relax this condition, since we can still guarantee that the dimension of \( I_{k,\theta}^{\eta,t,h}(g) \) is equal to \( k \) when the \( t_i \)'s belong to \( \{m-n+1, \ldots, m-k\} \), by Corollary 2.4, using \( \theta^{m-n+k} \) instead of \( g \).

In the following we describe two further constructions due to Gabidulin in [19].

**Definition 2.18.** Let \( k, n, m \) be positive integers such that \( 1 \leq k \leq n \leq m \) and \( m - k > k \). Let \( \theta \) be a generator of \( G = \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \) and \( \eta \in \mathbb{F}_{q^m} \). We denote by \( J_{k,\theta}^{\eta,l} \) the \( \mathbb{F}_{q^m} \)-subspace of the skew group algebra \( \mathbb{F}_{q^m}[\theta] = \mathbb{F}_{q^m}[G] \) given by

\[
J_{k,\theta}^{\eta,l} := \left\{ \theta^i + \theta^{l}(\eta) \theta^{k+i} \mid i \in \{0, \ldots, k-1\} \right\}_{\mathbb{F}_{q^m}}.
\]

Moreover, let \( g \in \mathbb{F}_{q^m}^n \) with \( \text{rk}_q(g) = n \). The new \( \theta \)-Gabidulin code of first kind \( J_{k,\theta}^{\eta,l}(g) \) is defined as

\[
J_{k,\theta}^{\eta,l}(g) := \left\{ (f(g_1), \ldots, f(g_n)) \mid f \in J_{k,\theta}^{\eta,l} \right\}.
\]

Observe that the new \( \theta \)-Gabidulin codes of first kind can be seen as a special case of generalized \( \theta \)-twisted Gabidulin codes in the sense of Definition 2.17, with

\[
\ell = k, \quad h_i = i - 1, \quad t_i = i, \quad \text{and} \quad \eta_i = \theta^{i-1}(\eta)
\]

for \( i \in [k] \).
Definition 2.19. Let \( k, n, m \) be positive integers such that \( 1 \leq k \leq n \leq m \) and \( m - k \leq k \). Let \( \theta \) be a generator of \( G = \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \) and \( \eta \in \mathbb{F}_{q^m} \). We denote by \( \mathcal{J}^{n,1}_{k,\theta} \) the \( \mathbb{F}_{q^n} \)-subspace of the skew group algebra \( \mathbb{F}_{q^m}[\theta] = \mathbb{F}_{q^m}[G] \) given by

\[
\mathcal{J}^{n,1}_{k,\theta} := \left\{ \left( \theta^i + \theta^j(\eta)\theta^{k+i} \right) | i \in \{0, \ldots, m-k-1\} \right\} \cup \left\{ \theta^i | m-k \leq i < k \right\}_{\mathbb{F}_{q^n}}.
\]

Moreover, let \( g \in \mathbb{F}_{q^m}^n \) with \( \text{rk}_q(g) = n \). The new \( \theta \)-Gabidulin code of second kind \( \mathcal{J}^{n,1}_{k,\theta}(g) \) is defined as

\[
\mathcal{J}^{n,1}_{k,\theta}(g) := \left\{ (f(g_1), \ldots, f(g_n)) | f \in \mathcal{J}^{n,1}_{k,\theta} \right\}.
\]

Also the new \( \theta \)-Gabidulin codes of second kind can be seen as a special case of generalized \( \theta \)-twisted Gabidulin codes in the sense of Definition 2.17, with

\[
\ell = m-k, \quad h_i = i-1, \quad t_i = i, \quad \text{and} \quad \eta_i = \theta^{i-1}(\eta)
\]

for \( i \in [m-k] \).

Proposition 2.20. [19] Let \( 1 \leq k \leq m \) be integers, \( \theta \) be a generator of \( \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \), \( g \in \mathbb{F}_{q^m}^n \) with \( \text{rk}_q(g) = n \). Suppose, moreover, that \( N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\eta) \neq (-1)^{km} \).

1. If \( m-k > k \), then the new \( \theta \)-Gabidulin code of first kind \( \mathcal{J}^{n,1}_{k,\theta}(g) \) is an \( [n,k]_{q^m} \) MRD code.
2. If \( m-k \leq k \), then the new \( \theta \)-Gabidulin code of second kind \( \mathcal{J}^{n,1}_{k,\theta}(g) \) is an \( [n,k]_{q^m} \) MRD code.

An overview of the code constructions discussed in this subsection can be found in Figure 1.

![Figure 1](image_url)

**Figure 1.** Overview of discussed code constructions. Boxes \( \square \) represent (possibly non-zero) coefficients that can be chosen independently, filled boxes represent coefficients that depend on other coefficients (dependency indicated by arrows). (a) Gabidulin codes, (b) \( \theta \)-twisted Gabidulin codes, (c) generalized \( \theta \)-twisted Gabidulin codes, (d) new Gabidulin (type I) codes, and (e) new Gabidulin (type II) codes (dimension \( k \) is chosen larger in this case due to requirement \( m-k \leq k \)).

3. Invariants

Let \( \theta \in \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \) and, for a code \( C \) and integer \( i \), denote by \( \theta^i(C) \) the code obtained by applying \( \theta^i \) entry-wise to all codewords of \( C \). Sums and intersections of such codes have been considered for several purposes in the literature of rank-metric codes. Overbeck [47]
Furthermore, it is known that the dimension of the code \( \dim C \) for a Gabidulin code is smaller than for the majority of linear codes. The attack has been modified several times to break multiple modifications of the GPT system [25, 46, 26, 10]. Furthermore, it is known that the dimension of the code \( C + \theta(C) + \cdots + \theta^i(C) \) has much smaller dimension for a Gabidulin code than for the majority of linear codes. The attack has been based on the fact that (for small enough \( i \)) the code \( C + \theta(C) + \cdots + \theta^i(C) \) has been used to show that some generalized \( \theta \)-twisted Gabidulin codes are inequivalent to known constructions. Moreover, in [23], Giuzzi and Zullo considered the dimensions of \( C \cap \theta(C) \) and \( C \cap \theta^2(C) \), in order to give a distinguisher for twisted Gabidulin codes. In the following we generalize these invariants.

**Lemma 3.1.** Let \( \sigma_1, \ldots, \sigma_r \) be distinct elements of \( \Gal(\mathbb{F}_{q^m}/\mathbb{F}_q) \) and let \( C_1, C_2 \) be two equivalent \( [n,k]_{q^m} \) codes. Then the following facts hold.

1. The codes \( S_1 := \sigma_1(C_1) + \sigma_2(C_1) + \cdots + \sigma_r(C_1) \) and \( S_2 := \sigma_1(C_2) + \sigma_2(C_2) + \cdots + \sigma_r(C_2) \) are equivalent. In particular, \( \dim S_1 = \dim S_2 \).
2. The codes \( T_1 := \sigma_1(C_1) \cap \sigma_2(C_1) \cap \cdots \cap \sigma_r(C_1) \) and \( T_2 := \sigma_1(C_2) \cap \sigma_2(C_2) \cap \cdots \cap \sigma_r(C_2) \) are equivalent. In particular, \( \dim T_1 = \dim T_2 \).

**Proof.** Since \( C_1 \) and \( C_2 \) are equivalent, there exist \( \tau \in \Aut(\mathbb{F}_{q^m}) \), \( A \in \GL_n(\mathbb{F}_q) \) such that \( C_1 = \tau(C_2)A \). Therefore,

\[
\begin{align*}
S_1 &= \sigma_1(C_1) + \sigma_2(C_1) + \cdots + \sigma_r(C_1) \\
&= \sigma_1(\tau(C_2))\sigma_1(A) + \sigma_2(\tau(C_2))\sigma_2(A) + \cdots + \sigma_r(\tau(C_2))\sigma_r(A) \\
&\overset{(\star)}{=} \tau(\sigma_1(C_2))A + \tau(\sigma_2(C_2))A + \cdots + \tau(\sigma_r(C_2))A \\
&= \tau(\sigma_1(C_2) + \sigma_2(C_2) + \cdots + \sigma_r(C_2))A = \tau(S_2)A,
\end{align*}
\]

and

\[
\begin{align*}
T_1 &= \sigma_1(C_1) \cap \sigma_2(C_1) \cap \cdots \cap \sigma_r(C_1) \\
&= \sigma_1(\tau(C_2))\sigma_1(A) \cap \sigma_2(\tau(C_2))\sigma_2(A) \cap \cdots \cap \sigma_r(\tau(C_2))\sigma_r(A) \\
&\overset{(\star)}{=} \tau(\sigma_1(C_2))A \cap \tau(\sigma_2(C_2))A \cap \cdots \cap \tau(\sigma_r(C_2))A \\
&= \tau(\sigma_1(C_2) \cap \sigma_2(C_2) \cap \cdots \cap \sigma_r(C_2))A = T_2,
\end{align*}
\]

where the equalities \((\star)\) follow from the fact that \( \Aut(\mathbb{F}_{q^m}) \) is a cyclic group, and therefore abelian, \( \Gal(\mathbb{F}_{q^m}/\mathbb{F}_q) \subseteq \Aut(\mathbb{F}_{q^m}) \) and the \( \sigma_i \)'s fix all the elements in \( \mathbb{F}_q \). \( \square \)

Lemma 3.1 implies that if two \([n,k]_{q^m}\) codes \( C_1, C_2 \) have different dimensions of \( S_1 \) and \( S_2 \) (or of \( T_1 \) and \( T_2 \)), then they must be inequivalent. Hence, checking the dimensions of \( S_1 \) and \( S_2 \) (or of \( T_1 \) and \( T_2 \)) for different choices of the \( \sigma_i \)'s gives a sufficient condition for codes to be inequivalent.

In the following, we restrict to the special case of consecutive powers of a fixed \( \sigma \in \Gal(\mathbb{F}_{q^m}/\mathbb{F}_q) \), i.e., \( \sigma_i = \sigma^{i-1} \), since in this case, we have additional interesting properties. Motivated by Lemma 3.1, we introduce the following setting and definitions. Let \( \mathcal{P}_{q^m}(n) \) denote the set of all \( \mathbb{F}_{q^m} \)-subspaces of \( \mathbb{F}_{q^m}^n \). For any automorphism \( \sigma \in \Gal(\mathbb{F}_{q^m}/\mathbb{F}_q) \) and integer \( 0 \leq i \leq n \), we consider the maps

\[
\begin{align*}
S_i^\sigma : \mathcal{P}_{q^m}(n) &\rightarrow \mathcal{P}_{q^m}(n) \\
C &\mapsto \bigoplus_{j=0}^i \sigma^j(C),
\end{align*}
\]

\[
\begin{align*}
T_i^\sigma : \mathcal{P}_{q^m}(n) &\rightarrow \mathcal{P}_{q^m}(n) \\
C &\mapsto \bigcap_{j=0}^i \sigma^j(C),
\end{align*}
\]
and the integers
\[ s_i^\sigma(C) := \dim(S_i^\sigma(C)), \quad t_i^\sigma(C) := \dim(T_i^\sigma(C)), \]
\[ \Delta_i^\sigma(C) := s_{i+1}^\sigma(C) - s_i^\sigma(C), \quad \Delta_i^\sigma(C) := t_i^\sigma(C) - t_{i+1}^\sigma(C). \]

**Definition 3.2.** With the notation above:
1. \( s_i^\sigma(C) \) is called the \( i \)-th \( \sigma \)-sum-dimension of \( C \), and \( \Delta_i^\sigma(C) \) the \( i \)-th \( \sigma \)-sum-increment of \( C \).
2. \( t_i^\sigma(C) \) is called the \( i \)-th \( \sigma \)-intersection-dimension of \( C \), and \( \Lambda_i^\sigma(C) \) the \( i \)-th \( \sigma \)-intersection-decrement of \( C \).

As a consequence of Lemma 3.1, we get that the sequences \( \{s_i^\sigma(C)\} \), \( \{\Delta_i^\sigma(C)\} \), \( \{t_i^\sigma(C)\} \) and \( \{\Lambda_i^\sigma(C)\} \) are invariants of linear rank-metric codes, i.e., they are stable under code equivalence. They can also be efficiently computed, as shown in Theorem 3.7.

A first property that we show is that the maps \( S_i^\sigma \) and \( T_i^\sigma \) are connected by a duality relation:

**Proposition 3.3.** Let \( C \) be an \( [n, k]_{\mathbb{F}_q^m} \) code. Then \( T_i^\sigma(C) \perp S_i^\sigma(C) \). In particular, \( t_i^\sigma(C) = n - s_i^\sigma(C^\perp) \) and \( \Lambda_i^\sigma(C) = \Delta_i^\sigma(C^\perp) \).

**Proof.** Since \( \sigma(C^\perp) = \sigma(C) \), we get
\[ T_i^\sigma(C^\perp) = \left( \bigcap_{j=0}^i \sigma^j(C) \right) \cap \sum_{j=0}^i \sigma^j(C) = S_i^\sigma(C). \]

The equalities \( t_i^\sigma(C) = n - s_i^\sigma(C^\perp) \) and \( \Lambda_i^\sigma(C) = \Delta_i^\sigma(C^\perp) \) immediately follow, using the fact that \( \dim(U^\perp) = n - \dim(U) \), for any \( U \in \mathcal{P}_{\mathbb{F}_q^m}(n) \).

Using the fact that \( S_0^\sigma(C) = T_0^\sigma(C) = C \) we also get the following relations.

**Proposition 3.4.** Let \( C \) be an \( [n, k]_{\mathbb{F}_q^m} \) code. Then:
1. \( t_i^\sigma(C) = 2k - s_i^\sigma(C) \),
2. \( \Delta_0^\sigma(C) = \Lambda_0^\sigma(C) \).

**Proof.**
1. We have \( T_i^\sigma(C) = C \cap \sigma(C) \) and thus \( t_i^\sigma(C) = \dim(C \cap \sigma(C)) = \dim(C) + \dim(\sigma(C)) - s_i^\sigma(C) = 2k - s_i^\sigma(C) \).
2. \( \Delta_0^\sigma(C) = s_1^\sigma(C) - s_0^\sigma(C) = s_1^\sigma(C) - k = k - t_1^\sigma(C) = t_0^\sigma(C) - t_1^\sigma(C) = \Lambda_0^\sigma(C) \).

We now derive more properties of the \( \sigma \)-sum sequence, before doing the analog for the intersection sequence.

**Proposition 3.5.** Let \( C \subseteq \mathbb{F}_q^n \) be an \( [n, k]_{\mathbb{F}_q^m} \) code. Then:
1. \( k = s_0^\sigma(C) \leq s_1^\sigma(C) \leq \ldots \leq s_{n-k}^\sigma(C) \leq n \).
2. \( S_i^\sigma \circ S_j^\sigma = S_{i+j}^\sigma \).
3. \( s_i^\sigma(C) = s_{i+1}^\sigma(C) \) if and only if \( S_i^\sigma(C) \) has a basis of elements in \( \mathbb{F}_q \).
4. If \( s_i^\sigma(C) = s_{i+1}^\sigma(C) \) then \( s_{i+j}^\sigma(C) = s_j^\sigma(C) \) for all \( j \geq 0 \).
5. \( s_{n-k}^\sigma(C) = s_{n-k+j}^\sigma(C) \) for all \( j \geq 0 \).
6. \( k \geq \Delta_0^\sigma(C) \geq \Delta_1^\sigma(C) \geq \ldots \geq \Delta_{n-k}^\sigma(C) = 0 \).
7. \( s_i^\sigma(C) = k + \sum_{j=0}^{i-1} \Delta_j^\sigma(C) \).

**Proof.**
1. By definition we have \( s_0^\sigma(C) = k \). The rest follows from \( S_i^\sigma(C) \subseteq S_{i+1}^\sigma(C) \subseteq \mathbb{F}_q^n \).
2. It holds that \( S_i^\sigma(S_j^\sigma(C)) = \sum_{\ell=0}^i \sigma^\ell(S_j^\sigma(C)) = \sum_{\ell=0}^i \sum_{r=0}^j \sigma^{\ell+r}(C) = \sum_{h=0}^{i+j} \sigma^h(C) = S_{i+j}^\sigma(C) \).
3. Suppose \( s_i^\sigma(C) = s_{i+1}^\sigma(C) \), then \( S_i^\sigma(C) = S_{i+1}^\sigma(C) \), and by part 2, we get \( S_i^\sigma(S_j^\sigma(C)) = S_j^\sigma(C) \). This is true if and only if \( \sigma(S_i^\sigma(C)) = S_j^\sigma(C) \), and we can conclude using [24, Lemma 4.5].
4. The equality $s_i^q(C) = s_{i+1}^q(C)$ implies that $\sigma(S_i^q(C)) = S_i^q(C)$, and therefore, $S_{i+j}^q(C) = \sigma^j(S_i^q(C)) = S_i^q(C)$ for all $j \geq 0$.
5. Let $r^q(C) = \min \{ i \mid s_i^q(C) = s_{i+1}^q(C) \}$. If $r^q(C) \leq n - k$, then by part 4 we can conclude. Suppose by contradiction that $r^q(C) > n - k$. Then we get a chain $k = s_0^q(C) < s_1^q(C) < \ldots < s_{n-k}^q(C) < s_{n-k+1}^q(C)$. This implies that $s_i^q(C) \geq k + i$, and in particular $s_{n-k}^q(C) \geq k + n - k + 1 = n + 1$, but this is impossible since $S_{n-k}^q \subseteq \mathbb{F}_q^n$.
6. First we prove that $\Delta_n^q(C) \leq k$. We have $S_i^q(C) = C + \sigma(C)$ and thus $s_i^q(C) = \dim(C + \sigma(C)) \leq \dim(C) + \dim(\sigma(C)) = s_0^q(C) + k$. Furthermore, we have $\Delta_{n-k} = 0$ by part 5.

Now we prove that $\Delta_i \geq \Delta_{i+1}$. Suppose $\Delta_i = s_{i+1}^q(C) - s_i^q(C) = r$. Then $\dim(S_i^q(C) + \sigma(S_i^q(C))) = \dim(S_i^q(C)) + r$. This implies that $\sigma(S_i^q(C)) = W + U$, where $W \subseteq S_i^q(C)$, $U \cap S_i^q(C) = \{0\}$ and $\dim U = r$. Hence, $S_{i+1}^q(C) = S_i^q(S_i^q(C)) = S_{i+1}^q(C) = S_i^q(C) + U + \sigma(S_i^q(C)) + \sigma(U)$. However, $U \subseteq \sigma(S_i^q(C))$, and therefore $S_{i+2}^q(C) = S_i^q(C) + \sigma(S_i^q(C)) + \sigma(U) = S_{i+1}^q(C) + \sigma(U)$. Since $\dim \sigma(U) = \dim(U) = r$, we get $\Delta_{i+1} = \dim(S_{i+2}^q(C)) - \dim(S_{i+1}^q(C)) \leq r = \Delta_i$.
7. $\sum_{j=0}^{i-1} \Delta_j^q(C) = \sum_{j=0}^{i-1} (s_{j+1}^q(C) - s_j^q(C)) = s_i^q(C) - s_0^q(C) = \sigma_i(C) - k$.

The following results are analogous results for the $\sigma$-intersection sequences.

**Proposition 3.6.** Let $C \subseteq \mathbb{F}_q^n$ be an $[n,k]_q$ code. Then:
1. $k = t_0^q(C) \geq t_1^q(C) \geq \ldots \geq t_k^q(C) \geq 0$.
2. $T^q_\iota \circ T^q_\iota = T^q_{\iota+j}$.
3. $t_\iota^q(C) = t_{\iota+1}^q(C)$ if and only if $T^q_\iota(C)$ has a basis of elements in $\mathbb{F}_q$.
4. If $t_\iota^q(C) = t_{\iota+1}^q(C)$ then $t_{\iota+j}^q(C) = t_\iota^q(C)$ for all $j \geq 0$.
5. $t_k^q(C) = t_{k+1}^q(C)$ for all $j \geq 0$.
6. $k \geq \Delta_0^q(C) \geq \Delta_1^q(C) \geq \ldots \geq \Delta_k^q(C) = 0$.
7. $t_i^q(C) = k - \sum_{j=0}^{i-1} \Delta_j^q(C)$.

**Proof.**
1. This is clear, since $T^q_{\iota+1} \subseteq T^q_\iota \subseteq \mathbb{F}_q^n$.
2. We have $T^q_\iota(T^q_\iota(C)) = \bigcap_{\iota=0}^{i} \sigma^\iota(T^q_\iota(C)) = \bigcap_{\iota=0}^{i} \sigma^\iota = \sigma^{i+j} = \bigcap_{\iota=0}^{i+j} \sigma^\iota = T^q_{\iota+j}(C)$.
3. By Proposition 3.3, we have that $t_\iota^q(C) = t_{\iota+1}^q(C)$ if and only if $s_\iota^q(C) = s_{\iota+1}^q(C)$, which in turn is equivalent to $C^\perp$ having a basis of elements in $\mathbb{F}_q$, by Proposition 3.5. This is in turn equivalent to $C$ having a parity check matrix with entries from $\mathbb{F}_q$, which happens if and only if there exists a basis of elements in $\mathbb{F}_q$ for the code itself.

4.-7. In an analogous way, the remaining statements follow from Proposition 3.5 and the duality result of Proposition 3.3.

The following theorem shows that the invariants presented in this section can be computed efficiently for any given code.

**Theorem 3.7.** Let $C$ be an $[n,k]_q$ code, $\sigma, \sigma_1, \ldots, \sigma_r \in \text{Gal}(\mathbb{F}_q^n/\mathbb{F}_q)$ be automorphisms.
1. The dimension of $\sigma_1(C) + \sigma_2(C) + \ldots + \sigma_r(C)$ can be computed in
   \[ O\left(\min\{n^{\omega-1}r_k, n^{\omega-1}k^{\omega-1}\}\right) \]
   operations over $\mathbb{F}_q^n$, where $2 \leq \omega \leq 3$ is the matrix multiplication exponent.
2. The dimension of $\sigma_1(C) \cap \sigma_2(C) \cap \ldots \cap \sigma_r(C)$ can be computed in
   \[ O\left(\min\{n^{\omega-1}r(n-k), n^{\omega-1}(n-k)^{\omega-1}\}\right) \]
   operations over $\mathbb{F}_q^n$.
3. The sequences $\{s_i^q(C)\}_{i=0}^\infty$, $\{\Delta_i^q(C)\}_{i=0}^\infty$, $\{t_i^q(C)\}_{i=0}^\infty$, $\{\Lambda_i^q(C)\}_{i=0}^\infty$ can be computed in
   \[ O(n^{\omega-1}k(n-k)) \]
operations over $\mathbb{F}_{q^m}$.

**Proof.** Let $G \in \mathbb{F}_{q^m}^{k \times n}$ be a generator and $H \in \mathbb{F}_{q^m}^{(n-k) \times n}$ be a parity-check matrix of the code $C$. Without loss of generality we assume that we know (or can efficiently compute) one of them. The other matrix can be determined from the known one in $O(\max\{n-k, k\}^{w-1}n)$ field operations using Gaussian elimination (for the complexity, see e.g., [64, Theorem 2.10]).

1. The code $\sigma_1(C) + \sigma_2(C) + \ldots + \sigma_r(C)$ is generated by the rows of the matrix

$$
\begin{pmatrix}
\sigma_1(G) \\
\sigma_2(G) \\
\vdots \\
\sigma_r(G)
\end{pmatrix} \in \mathbb{F}_{q^m}^{kr \times n},
$$

so its dimension can be computed by determining the rank of the matrix. Since the rank of an $a \times b$ matrix can be computed in $O(\min\{a^{w-1}b, ab^{w-1}\})$, the claimed complexity follows.

2. Analogously, the dimension of $\sigma_1(C) \cap \sigma_2(C) \cap \ldots \cap \sigma_r(C)$ can be computed by determining the rank of $(\sigma_1(H)^\top, \sigma_2(H)^\top, \ldots, \sigma_r(H)^\top)^\top \in \mathbb{F}_{q^m}^{(n-k)r \times n}$ which is a parity-check matrix of the intersection code.

3. It suffices to show that we can compute $s^\sigma_1(C), \ldots, s^\sigma_{n-k}(C)$ efficiently since

- by Proposition 3.5, the sequence $s^\sigma_i(C)$ converges after at most $n - k$ steps (i.e., $s^\sigma_{n-k+j}(C) = s^\sigma_{n-k}(C)$ for all $j \geq 0$),
- by Proposition 3.3, the sequence $\{t^\sigma_i(C)\}$ can be computed from the $s^\sigma_i(C^\perp)$ sequence of the dual code (which converges after at most $k$ steps), and
- the difference sequences follow by $O(\max\{k, n-k\})$ subtractions.

We can compute these $n-k$ values $s^\sigma_i(C)$ by first determining the column rank profile (i.e., the row indices of leading ones in a column echelon form) of the matrix

$$
\begin{pmatrix}
G \\
\sigma(G) \\
\vdots \\
\sigma^{n-k}(G)
\end{pmatrix} \in \mathbb{F}_{q^m}^{k(n-k+1) \times n},
$$

which can be done in $O(n^{w-1}k(n-k))$ operations [64, Thm. 2.10]. We get $s^\sigma_i(C)$ by counting the elements of the column rank profile that are contained in the first $(i+1)k$ rows. We save a factor $n-k$ compared to naively computing each $s^\sigma_i(C)$ individually as in part 1.

\[\square\]

4. **The Sum and Intersection Sequences for Known MRD Constructions**

In this section we are going to study the properties of the sequences introduced above for Gabidulin, twisted Gabidulin and some generalized twisted Gabidulin codes. Since the intersection sequence is fully determined by the sum sequence, we exemplarily determine the intersection sequence for Gabidulin and (narrow-sense) twisted Gabidulin codes, but not for generalized twisted Gabidulin codes.

For simplicity we will represent the field automorphism $\sigma$ as a power of a generator $\theta$ of the Galois group $\text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$.

4.1. **Gabidulin Codes.** In general a $\theta$-Gabidulin code $C$ can also be a $\bar{\theta}$-Gabidulin code for another generator $\bar{\theta}$ of $\text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$. We will prove in Theorem 5.5 that this cannot happen for many $\bar{\theta}$s. However, it is straightforward to see that a $\theta$-Gabidulin code is always also a $\theta^{-1}$-Gabidulin code.

**Lemma 4.1.** Let $\theta$ be a generator of the Galois group $\text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$ and $g \in \mathbb{F}_{q^m}^n$ such that $\text{rk}_q(g) = n$. Then $G_{k, \bar{\theta}}(g) = G_{k, \theta^{-1}}(\theta^{k-1}(g))$. 


Proof. We have
\[ G_{k,\theta}(g) = \langle g, \theta(g), \ldots, \theta^{k-1}(g) \rangle = \langle \theta^{k-1}(g), \theta^{-1}(\theta^{k-1}(g)), \ldots, \theta^{-(k-1)}(\theta^{k-1}(g)) \rangle = G_{k,\theta^{-1}}(\theta^{k-1}(g)). \]
\[ \square \]

The following result gives the \( \theta^r \)-sequences for \( \theta \)-Gabidulin codes. Note that the computation for the special case in which \( r = 1 \) and \( \theta \) is the \( q \)-Frobenius automorphism was already derived by Overbeck in [47].

**Proposition 4.2.** Let \( C := \mathcal{G}_{k,\theta}(g) \) be a \( \theta \)-Gabidulin code and \( i, r \in \mathbb{N} \) such that \( 0 \leq r < m \).

1. For \( 0 \leq r \leq k \) we have \( \mathcal{S}^\theta_i(C) = \mathcal{S}^\theta_{i+r}(g) \) and for \( m - k \leq r \leq m \) we have \( \mathcal{S}^\theta_i(C) = \mathcal{G}_{k+i(m-r),\theta^{-1}}(\theta^{k-1}(g)) \).

   In particular, we have \( s^\theta_1(C) = \min\{e,n\} \), where
   \[ e = \begin{cases} 
   k + ir & \text{if } 0 \leq r \leq k \\
   k + i(m-r) & \text{if } m-k \leq r \leq m.
   \end{cases} \]

2. If \( k < r \leq n-k \) or \( m-n+k \leq r < m-k \) we have \( s^\theta_1(C) = 2k \).

3. If \( r > k \) and \( r > n-k \), then \( s^\theta_1(C) \geq k + n - r \). If \( r < m-k \) and \( r < m-n+k \), then \( s^\theta_1(C) \geq k + n - m + r \).

Proof. 1. If \( 0 \leq r \leq k \), then \( \mathcal{S}^\theta_i(C) = \langle g, \theta(g), \ldots, \theta^{k-1}(g), \theta^{k}(g), \ldots, \theta^{k+ir-1}(g) \rangle = \mathcal{G}_{k+i(r-m),\theta}(g) \). If \( m-k \leq r \leq m-1 \), we can write \( \theta^r = (\theta^{-1})^{m-r} \), with \( 0 \leq m-r \leq k \), and thus get \( \mathcal{S}^\theta_i(C) = \mathcal{S}^{(\theta^{-1})^{m-r}}_i(\mathcal{G}_{k,\theta^{-1}}(\theta^{k-1}(g))) = \mathcal{G}_{k+i(m-r),\theta^{-1}}(\theta^{k-1}(g)) \), by Lemma 4.1. The computation of \( s^\theta_1(C) \) follows from Corollary 2.4.

2. If \( k < r \leq n-k \), then \( \mathcal{S}^\theta_i(C) = \langle g, \theta(g), \ldots, \theta^{k-1}(g), \theta^{r}(g), \ldots, \theta^{r+k-1}(g) \rangle \), which has dimension \( 2k \), by Corollary 2.4. If \( m-n+k \leq r < m-k \) the statement follows with Lemma 4.1.

3. If \( r > k \) and \( r > n-k \), then \( \mathcal{S}^\theta_i(C) \supseteq \langle g, \theta(g), \ldots, \theta^{k-1}(g), \theta^{r}(g), \ldots, \theta^{n-1}(g) \rangle \), which has dimension \( n + k - r \), by Corollary 2.4. If \( r < m-k \) and \( r < m-n+k \) the statement follows with Lemma 4.1.

\[ \square \]

In particular, for \( m = n \geq 2k \), we have
\[ s^\theta_1(C) = \begin{cases} 
   k + r & \text{if } 0 \leq r \leq k \\
   2k & \text{if } k+1 \leq r \leq n-k-1 \\
   k+(n-r) & \text{if } n-k \leq r \leq n-1.
   \end{cases} \]

We present example sequences and the corresponding statements from Proposition 4.2 in Appendix A.

**Proposition 4.3.** Let \( C := \mathcal{G}_{k,\theta}(g) \) be a \( \theta \)-Gabidulin code and \( i, r \in \mathbb{N} \) such that \( 0 \leq r < m \).

1. For \( 0 \leq r \leq k \) we have \( \mathcal{T}^\theta_i(C) = \mathcal{T}^\theta_{i+r}(g) \) and for \( m-k \leq r < m \) we have \( \mathcal{T}^\theta_i(C) = \mathcal{G}_{k-i(m-r),\theta^{-1}}(\theta^{k-i(m-r)-1}(g)) \).\( ^2 \) In particular, we have \( t^\theta_1(C) = \max\{e,0\} \), where
   \[ e = \begin{cases} 
   k - ir & \text{if } 0 \leq r \leq k \\
   k - i(m-r) & \text{if } m-k \leq r \leq m.
   \end{cases} \]

2. If \( k < r \leq n-k \) or \( m-n+k \leq r < m-k \) we have \( t^\theta_1(C) = 0 \).

\[^1\text{Although not properly defined, we assume that } G_{s,\theta}(g) = \mathbb{F}_m^s \text{ for } s \geq n.\]

\[^2\text{Although not properly defined, we assume that } G_{s,\theta}(g) = \{0\} \text{ for } s \leq 0.\]
3. If \( r > k \) and \( r > n - k \), then \( \ell_{i}^{\theta^{r}}(C) \leq k - n + r \). If \( r < m - k \) and \( r < m - n + k \), then \( \ell_{i}^{\theta^{r}}(C) \leq k - n + m - r \).

**Proof.**
1. It is enough to prove it for \( i = 1 \), then the claim follows by induction, since \( T_{i+1}^{\theta^{r}} = T_{i}^{\theta^{r}} \circ T_{i}^{\theta^{r}} \) by part 2 of Proposition 3.6. For \( 0 \leq r \leq k \) we have \( T_{i}^{\theta^{r}}(C) = (g, \ldots, \theta^{k-1}(g)) \cap (\theta^{r}(g), \ldots, \theta^{k+r-1}(g)) \supseteq (\theta^{r}(g), \ldots, \theta^{k-1}(g)) \). The equality follows by part 1 of Proposition 3.4 and part 1 of Proposition 4.2. If \( m - k \leq r \leq m \) we write \( \theta^{r} = (\theta^{-1})^{m-r} \) and use Lemma 4.1.

2.–3. They follow from part 1 of Proposition 3.4 and parts 2–3 of Proposition 4.2.

In particular, for \( m = n \geq 2k \), we have

\[
\ell_{i}^{\theta^{r}}(C) = \begin{cases} 
  k - r & \text{if } 0 \leq r \leq k \\
  0 & \text{if } k + 1 \leq r \leq n - k - 1 \\
  k - (n - r) & \text{if } n - k \leq r \leq n - 1 
\end{cases}
\]

4.2. **Twisted Gabidulin Codes.** In this subsection we analyze the family of \([n, k]_{q^{m}} \theta\)-twisted Gabidulin codes, i.e., those which are linear over \( \mathbb{F}_{q^{m}} \). The following result is a straightforward computation, analogous to Lemma 4.1.

**Lemma 4.4.** Let \( \theta \) be a generator of the Galois group \( \text{Gal}(\mathbb{F}_{q^{m}}/\mathbb{F}_{q}) \) and \( g \in \mathbb{F}_{q^{m}}^{n} \) such that \( \text{rk}_{\eta}(g) \), and \( \eta \in \mathbb{F}_{q^{m}}^{*} \). Then \( \mathcal{H}_{k, \theta}^{\eta}(g) = \mathcal{H}_{k, \theta^{-1}}^{\eta^{-1}}(\theta^{k}(g)) \).

**Proof.** We have

\[
\mathcal{H}_{k, \theta}^{\eta}(g) = (g + \eta \theta^{k}(g), \theta(g), \ldots, \theta^{k-1}(g)) = (\eta^{-1} \theta^{-k}(\theta^{k}(g)) + \theta^{k}(g), \theta^{-1}(\theta^{k}(g)), \theta^{-2}(\theta^{k}(g)), \ldots, \theta^{-k+1}(\theta^{k}(g))) = \mathcal{H}_{k, \theta^{-1}}^{\eta^{-1}}(\theta^{k}(g)).
\]

**Proposition 4.5.** Let \( g \in \mathbb{F}_{q^{m}}^{n} \) with \( \text{rk}_{\eta}(g) = n \), and \( C := \mathcal{H}_{k, \theta}^{\theta}(g) \) be a \( \theta \)-twisted Gabidulin code, where \( \eta \in \mathbb{F}_{q^{m}}^{*} \), and \( i, r \in \mathbb{N} \) such that \( 0 \leq r < m \).

1. For \( 1 \leq r \leq k - 1 \) we have \( \mathcal{S}_{i}^{\theta^{r}}(C) = \mathcal{G}_{k+i+r, \theta}(g) \) and for \( m - k - 1 \leq r < m \) we have \( \mathcal{S}_{i}^{\theta^{r}}(C) = \mathcal{G}_{k+i+(m-r)+1, \theta^{-1}}(\theta^{k}(g)) \).

3. Although not properly defined, we assume that \( \mathcal{H}_{i, \theta}^{\eta}(g) = \mathbb{F}_{q^{m}}^{n} \) for \( s \geq n \).
2. If \( k \leq r \leq n - k \), then \( \mathcal{S}^\theta_r(C) = \langle g + \eta \theta^k(g), \theta(g), \ldots, \theta^{k-1}(g), \theta^r(g) + \theta^r(\eta) \theta^{r+k}(g), \theta^{r+1}(g), \ldots, \theta^{r+k-1}(g) \rangle \) which has dimension \( 2k \) by Corollary 2.4.

   If \( m - n + k \leq r < m - k \) the claim follows with Lemma 4.4, writing \( \theta^r = (\theta^{-1})^{m-r} \).

3. If \( r \geq k \) and \( k + r > n - 1 \), then

\[
\mathcal{S}^\theta_r(C) \supseteq \langle g + \eta \theta^k(g), \theta(g), \ldots, \theta^{k-1}(g), \theta^r(g) + \theta^r(\eta) \theta^{r+k}(g), \theta^{r+1}(g), \ldots, \theta^{n-1}(g) \rangle,
\]

which has dimension \( k + n - r \), by Corollary 2.4.

   If \( r \leq m - k \) and \( r < m - n + k + 1 \) the claim follows again with Lemma 4.4.

\[\square\]

In particular, for \( m = n \geq 2k \), we have

\[
s^\theta_1(C) = \begin{cases} 
k + r + 1 & \text{if } 0 \leq r < k \\2k & \text{if } k \leq r \leq n - k \\k + (n - r) + 1 & \text{if } n - k < r \leq n - 1 \end{cases}.
\]

We present example sequences and the corresponding statements from Proposition 4.5 in Appendix A.

**Proposition 4.6.** Let \( g \in \mathbb{F}^n_q \) with \( \text{rk}_q(g) = n \), and \( C := \mathcal{H}_{k,\theta}^0(g) \) be a \( \theta \)-twisted Gabidulin code, where \( \eta \in \mathbb{F}^n_q \), and \( i, r \in \mathbb{N} \) such that \( 0 \leq r < m \).

1. For \( 1 \leq r \leq k - 1 \) we have \( \mathcal{T}^\theta_i(C) = \mathcal{T}^\theta_{i(r)}(C) = \mathcal{G}_{k-i(r-1)(\theta^{r+1}(g))} \) and for \( m - k + 1 \leq r < m - 1 \) we have \( \mathcal{T}^\theta_i(C) = \mathcal{G}_{k-i(m-r-1)(\theta^{k-i(m-r)-1}(g))} \). In particular, for any \( i > 0 \), we have \( \mathcal{t}^\theta_i(C) = \max\{e, 0\} \), where

\[
e = \begin{cases} 
k - ir - 1 & \text{if } 1 \leq r \leq k - 1 \\
k - i(m - r) - 1 & \text{if } m - k + 1 \leq r \leq m - 1 \end{cases}.
\]

2. If \( k \leq r \leq n - k \) or \( m - n + k \leq r < m - k \) we have \( \mathcal{t}^\theta_1(C) = 0 \).

3. If \( r \geq k \) and \( r > n - k \), then \( \mathcal{t}^\theta_1(C) \leq k - n + r \). If \( r \leq m - k \) and \( r < m - n + k \), then \( \mathcal{t}^\theta_1(C) \leq k - n + m - r \).

**Proof.**

1. For \( i = 1 \) the claim follows from part 1 of Proposition 3.4 and part 1 of Proposition 4.5. That \( \mathcal{T}^\theta_i(C) \) is equal to some Gabidulin code can be shown analogously to the proof of Proposition 4.3. For larger \( i \) the claim follows by induction, since \( \mathcal{T}^\theta_{i+1} = \mathcal{T}^\theta_i \circ \mathcal{T}^\theta_i \) by part 2 of Proposition 3.6, together with part 1 of Proposition 4.3.

2–3. They follow from part 1 of Proposition 3.4 and parts 2–3 of Proposition 4.5.

\[\square\]

In particular, for \( m = n \geq 2k \), we have

\[
\mathcal{t}^\theta_i(C) = \begin{cases} 
k - r - 1 & \text{if } 0 \leq r < k \\0 & \text{if } k \leq r \leq n - k \\k - (n - r) - 1 & \text{if } n - k < r \leq n - 1 \end{cases}.
\]

The previous results imply:

**Corollary 4.7.** Let \( 1 < k < n - 1 \), \( g, h \in \mathbb{F}^n_q \) with \( \text{rk}_q(g) = \text{rk}_q(h) = n \) and let \( \theta, \bar{\theta} \) be generators of \( \text{Gal}(\mathbb{F}^n_q/\mathbb{F}_q) \), i.e., there exists \( 1 \leq r < m \) with \( \bar{\theta} = \theta^r \). If

- \( r < n - 1 \), or
- \( r > m - n + 1 \), or
- \( k < 2n - m - 1 \), or
- \( k > m - n + 1 \),

then \( \mathcal{H}^\theta_{k,\theta}(g) \) is not equivalent to \( \mathcal{G}_{k,\bar{\theta}}(h) \).

\[\footnote{Although not properly defined, we assume that \( \mathcal{G}_{s,\bar{\theta}}(g) = \{0\} \) for \( s \leq 0 \).} \]
Proof. By Proposition 4.2 we have \( s_1^r(\mathcal{G}_{k,\theta}(h)) = k + 1 \). Hence, if \( s_1^r(\mathcal{H}_{k,\theta}(g)) = s_1^{r'}(\mathcal{H}_{k,\theta}(g)) \neq k + 1 \) then the two codes are inequivalent by Lemma 3.1.

If \( r \in \{1, \ldots, \max(k - 1, n - k)\} \cup \{\min(m - k + 1, m - n + k), \ldots, m - 1\} \) we have that \( s_1^{r'}(\mathcal{H}_{k,\theta}(g)) > k + 1 \) by parts 1 and 2 of Proposition 4.5. If \( r \in \{\max(k, n - k + 1), \ldots, n - 3\} \cup \{m - n + 3, \ldots, \min(m - k, m - n + k - 1)\} \) we have that \( s_1^{r'}(\mathcal{H}_{k,\theta}(g)) > k + 1 \) by part 3 of Proposition 4.5. This proves the statement for the first two conditions.

Assume now that \( n - 2 < r < m - n + 2 \). This implies that \( \max(k, n - k) < r < \min(m - k, m - n + k) \). Part 3 of Proposition 4.5 implies that

\[
s_1^{r'} \geq k + n - r > 2n - m
\]

and also that

\[
s_1^{r'} \geq k + n - m + r > 2k + n - m.
\]

Thus \( s_1^{r'} \) is strictly greater than \( k + 1 \) with the conditions of parts 3 and 4 above.

This implies the following general result on the equivalence of Gabidulin and twisted Gabidulin codes in dependance on \( m \) and \( n \). The special case \( m = n \) of was already proved in [58] and [33].

Corollary 4.8. Let \( 1 < k < n - 1 \). If \( m < 2n - 2 \) (in particular \( m = n \)), then twisted Gabidulin codes of length \( n \) and dimension \( k \) over \( \mathbb{F}_{q^m} \) are never equivalent to a Gabidulin code.

Proof. For \( m < 2n - 2 \) we get that \( n - 1 \geq m - n + 2 \), hence either the first or the second condition of Corollary 4.7 is fulfilled.

4.3. Generalized Twisted Gabidulin Codes. In this subsection, we analyze some special instances of generalized twisted Gabidulin codes with \( \ell = 1 \) twists. We derive (parts of) the sequences \( s_1^r \) for three cases (recall that \( t = 1, h = 0 \) are narrow-sense twisted Gabidulin codes discussed in the previous section):

- \( h = 0 \) and arbitrary \( 1 < t \leq n - k \) for any evaluation point vector \( g \) (Proposition 4.11),
- \( h = k - 1 \) and arbitrary \( m - n + 1 \leq t < m - k \) for any \( g \) (Corollary 4.14),
- and almost all \( t, h \) for any \( g \) spanning a subfield of \( \mathbb{F}_{q^m} \) (Proposition 4.16).

This enables us to give several classes of generalized twisted Gabidulin codes that are inequivalent to any Gabidulin or (narrow-sense) twisted Gabidulin code (cf. Corollaries 4.12, 4.13, 4.17). The techniques can be carried over to more classes of twisted Gabidulin codes. In order to demonstrate the suitability of the approach without becoming too technical, we concentrate on the cases mentioned above.

For the sake of better readability we refrain from determining the intersections sequences in this subsection.

The following result is again analogous to Lemmas 4.1 and 4.4.

Lemma 4.9. Let \( \theta \) be a generator of the Galois group \( \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \), \( g \in \mathbb{F}_{q^m}^n \) with \( r_0(g) = n \) and \( \eta \in \mathbb{F}_{q^m}^* \). Then \( \mathcal{I}_{k,\theta}^{\eta, t,h}(g) = \mathcal{I}_{k,\theta}^{\eta, m-(k+t-1),k-h-1}(g^{k-1}(g)) \).

Proof. We have the following chain of equalities

\[
\mathcal{I}_{k,\theta}^{\eta, m-(k+t-1),k-h-1}(g^{k-1}(g)) \\
= \left\{ \theta^i(g^{k-1}(g)) \right\}_{i \in \{0, \ldots, k-1\} \setminus \{k-h-1\}} \cup \{\eta \theta^{-(k-1+m-(k+t-1))}(g^{k-1}(g)) + \theta^{-(k-h-1)}(g^{k-1}(g))\} \\
= \left\{ \theta^i(g) \right\}_{i \in \{0, \ldots, k-1\} \setminus \{h\}} \cup \{\eta \theta^{k+t-1}(g) + \theta^h(g)\} \\
= \mathcal{I}_{k,\theta}^{\eta, t,h}(g).
\]
Note that for $h = 0, t = 1$ this gives a different equality from the one in Lemma 4.4. Both results together give the following identities.

**Corollary 4.10.**

$$\mathcal{I}_{k,\theta}^{n,1.0}(g) = \mathcal{I}_{k,\theta}^{n,m-k,k-1}(g^{k-1}(g)) = \mathcal{I}_{k,\theta}^{n-1.0,m-k,k-1}(\theta^k(g)) = \mathcal{I}_{k,\theta}^{n-1,m-k,k-1}(\theta^{2k-1}(g))$$

We now determine the first elements of the $\theta^r$-sum sequence for generalized twisted Gabidulin codes for the case $h = 0$ and $1 < t \leq n - k$.

**Proposition 4.11.** Let $g \in \mathbb{F}_q^n$ with $\text{rk}_q(g) = n$, and $C := \mathcal{I}_{k,\theta}^{n,1.0}(g)$ be a generalized $\theta$-twisted Gabidulin code with $h = 0$ and $1 < t \leq n - k$.

1. If $1 \leq r \leq k - 1$, then $s_{\theta}^r(C) \geq \min\{n, k + ir\}$ and if $t + ir \leq n - k$ then
   $$s_{\theta}^r(C) = \min\{k + ir + \min\left\{i, \left\lceil \frac{r}{m-r} \right\rceil \right\}, n\}.$$
   If $m - k + 1 \leq r \leq m - 1$, then $s_{\theta}^r(C) \geq \min\{n, k + i(m - r)\}$ and if $t + i(m - r) \leq n - k$ then
   $$s_{\theta}^r(C) = \min\left\{k + i(m - r) + \min\left\{i, \left\lceil \frac{r}{m-r} \right\rceil \right\}, n\right\}.$$

2. If $k \leq r \leq n - k - 1$ or $m - n + k + 1 \leq r \leq m - k$ we have $s_{\theta}^r(C) = 2k$.

3. If $r \geq k$ and $r \geq n - k$, then $s_{\theta}^r(C) \geq k + n - r - 1$. If $r \leq m - k$ and $r \leq m - n + k$, then $s_{\theta}^r(C) \geq k + n - m + r - 1$.

**Proof.**

1. If $1 \leq r \leq k - 1$ we have
   $$S_{\theta}^{r}(C) = \langle g + \eta \theta^{k+t-1}(g), \theta(g), \theta^2(g), \ldots, \theta^{k-1}(g) \rangle + \langle \theta^r(g) + \theta^r(\eta)\theta^{k+i(t-1)}(g), \theta^{r+1}(g), \ldots, \theta^{r+k-1}(g) \rangle + \langle \theta^{ir}(g) + \theta^{ir}(\eta)\theta^{k+i(r-1)}(g), \theta^{ir+1}(g), \ldots, \theta^{ir+k-1}(g) \rangle = \langle g + \eta \theta^{k+t-1}(g), \theta(g), \theta^2(g), \ldots, \theta^{k+n-1}(g) \rangle + \langle \theta^{k+t+r-1}(g), \theta^{k+t+r+2r-1}(g), \ldots, \theta^{k+n+r}(g) \rangle.$$

   The first summand contains $\min\{n, k + ir\}$ linearly independent elements and implies $s_{\theta}^r(C) \geq \min\{n, k + ir\}$. If $t + ir \leq n - k$, all involved powers of $\theta$ are smaller than $n$.

   In this case, the number of elements from the second summand that are not contained in the first summand is $\min\{i, \left\lceil \frac{r}{m-r} \right\rceil \}$.

   If $m - k + 1 \leq r \leq m - 1$, then analogously
   $$S_{\theta}^{r}(C) = \langle \theta^{ir}(g) + \theta^{ir}(\eta)\theta^{k+i(r-1)}(g), \theta^{ir+1}(g), \ldots, \theta^{ir+k-1}(g) \rangle + \langle \theta^{k+t-1}(g), \ldots, \theta^{k+t+i(r-1)-1}(g), \rangle = \langle g, \theta^{k+n-1}(g), \theta^{k+n-1(m-r)}(g), \theta^{k+n-1(m-r)+1}(g), \ldots, \theta^{k+n-1(m-r)+1}(g), \rangle$$
   which implies the second statement.

2. If $k \leq r \leq n - k - 1$ we have
   $$S_{\theta}^{r}(C) = \langle \theta(g), \theta^{k-1}(g), \theta^{k+1}(g), \ldots, \theta^{k+(r-1)}(g), g + \eta \theta^{k+t-1}(g), \theta^r(g) + \eta \theta^{k+t+r-1}(g) \rangle$$
   which has dimension $2k$. Similarly we get only linearly independent elements if $m - n + k + 1 \leq r \leq m - k$.

3. If $r \geq \max(k, n - k)$, then
   $$S_{\theta}^{r}(C) = \langle g + \eta \theta^{k+t-1}(g), \theta(g), \ldots, \theta^{k-1}(g), \theta^{r+1}(g), \ldots, \eta \theta^{a-1}(g) \rangle$$
   which has dimension $k + n - r - 1$, by Corollary 2.4. Similarly the claim follows for $r \leq \min(m - k, m - n + k)$. □
Corollary 4.12. Let $1 < k < n - 1$, $g, h \in \mathbb{F}_{q^m}^n$ with $rk_q(g) = rk_q(h) = n$ and let $\theta, \bar{\theta}$ be generators of $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$, i.e., there exists $1 \leq r < m$ with $\theta = \theta^r$. Moreover let $t \in \{2, \ldots, n - k\}$.

1. If
   - $1 < r < n - 2$, or
   - $m - n + 2 < r < m - 1$, or
   - $r \in \{1, m - 1\}$ and $t < n - k$, or
   - $k < \min(2n - m - 2, n - t)$, or
   - $m - n + 2 < k < n - t$,
   then $T_{k,\bar{\theta}}^{n,t,0}(g)$ is not equivalent to $G_{k,\theta}(h)$.

2. If
   - $1 < r < n - 3$ and $k > 2$, or
   - $m - n + 3 < r < m - 1$ and $k > 2$, or
   - $r \in \{1, m - 1\}$ and $t < n - k - 1$, or
   - $k = 2, t < n - 3$ and $r \in \{1, \ldots, [(n - 2)/2]\} \cup \{[m - (n - 2)/2, \ldots, m - 1]\}$,
   then $T_{k,\bar{\theta}}^{n,t,0}(g)$ is not equivalent to $H_{k,\bar{\theta}}^{q,0}(h)$.

Proof.

1. We use again the fact that, by Proposition 4.2 we have $s_1^{\bar{\theta}}(G_{k,\theta}(h)) = k + 1$. Hence, if $s_1^{\bar{\theta}}(T_{k,\bar{\theta}}^{n,t,0}(g)) = s_1^{\bar{\theta}}(I_{k,\bar{\theta}}^{n,t,0}(g)) \neq k + 1$ then the two codes are inequivalent by Lemma 3.1.

   If $1 < r < k$, then by part 1 of Proposition 4.11 we have
   
   $s_1^{\theta^r}(I_{k,\theta}^{n,t,0}(g)) \geq \min\{n, k + r\} > k + 1$.

   If $k \leq r < n - k$, then by part 2 of Proposition 4.11 we have
   
   $s_1^{\theta^r}(I_{k,\theta}^{n,t,0}(g)) = 2k > k + 1$.

   If $\max(k, n - k) \leq r < n - 2$, then by part 3 of Proposition 4.11 we have
   
   $s_1^{\theta^r}(I_{k,\theta}^{n,t,0}(g)) \geq k + n - r - 1 > k + 1$.

   Analogously the statement follows if $m - n + 2 < r < m - 1$ or $r \in \{1, m - 1\}$ and $t < n - k$.

   The fourth and fifth conditions imply that $t < n - k$. If the first three conditions are not fulfilled then $n - 2 \leq r \leq m - n + 2$, which we assume now. This implies that $\max(k, n - k) \leq r \leq \min(m - k, m - n + k)$. Part 3 of Proposition 4.11 implies that
   
   $s_1^{\theta^r}(I_{k,\theta}^{n,t,0}(g)) \geq k + n - r - 1 \geq 2n - m - 1$

   and also that
   
   $s_1^{\theta^r}(I_{k,\theta}^{n,t,0}(g)) \geq k + n - m + r - 1 \geq 2k + n - m - 1$.

   Thus $s_1^{\theta^r}(I_{k,\theta}^{n,t,0}(g))$ is strictly greater than $k + 1$ with the last two conditions above.

2. Here we use the fact that, by Proposition 4.5 we have $s_1^{\bar{\theta}}(H_{k,\bar{\theta}}^{q,0}(h)) = k + 2$ and $s_2^{\bar{\theta}}(H_{k,\bar{\theta}}^{q,0}(h)) = k + 3$. Hence, if $s_1^{\bar{\theta}}(T_{k,\bar{\theta}}^{n,t,0}(g)) = s_1^{\theta^r}(I_{k,\theta}^{n,t,0}(g)) \neq k + 2$ or $s_2^{\bar{\theta}}(I_{k,\bar{\theta}}^{n,t,0}(g)) = s_2^{\theta^r}(I_{k,\theta}^{n,t,0}(g)) \neq k + 3$ then the two codes are inequivalent by Lemma 3.1.

   If $1 < r < k$, then by part 1 of Proposition 4.11 we have
   
   $s_2^{\theta^r}(I_{k,\theta}^{n,t,0}(g)) \geq \min\{n, k + 2r\} > k + 3$.

   If $k \leq r < n - k$, then by part 2 of Proposition 4.11 we have
   
   $s_1^{\theta^r}(I_{k,\theta}^{n,t,0}(g)) = 2k$.
which is greater than \(k+2\) if \(k>2\). If \(\max(k,n-k)\leq r<n-3\), then by part 3 of Proposition 4.11 we have
\[
s_1^\theta(\mathcal{I}_{k,\theta}^{n,k-1}(g)) \geq k+n-r-1 > k+2.
\]
Analogously the statement follows if \(m-n+3 < r < m-2\) or \(r \in \{1,m-1\}\) and \(t < n-k-1\).
If \(k=2\) and \(t < n-3\) we get
\[
s_2^\theta(\mathcal{I}_{k,\theta}^{n,k-1}(g)) = 6 > k+3
\]
with the given conditions.

\[\square\]

**Corollary 4.13.**
1. Let \(1 < k < n-t < n-1\). If \(m < 2n-4\) then generalized twisted Gabidulin codes of length \(n\) and dimension \(k\) over \(\mathbb{F}_{q^m}\) with hook \(h=0\) and twist \(t>1\) are never equivalent to a Gabidulin code.
2. Let \(2 < k < n-t-1 < n-2\). If \(m < 2n-6\) then generalized twisted Gabidulin codes of length \(n\) and dimension \(k\) over \(\mathbb{F}_{q^m}\) with hook \(h=0\) and twist \(t>1\) are never equivalent to a twisted Gabidulin code.

**Proof.**
1. For \(m < 2n-4\) we get that \(n-2 > m-n+2\), hence one of the first three conditions of part 1 of Corollary 4.12 is fulfilled.
2. Similarly, for \(m < 2n-6\) one of the first three conditions of part 2 of Corollary 4.12 is fulfilled.

**Corollary 4.14.** Let \(g \in \mathbb{F}_{q^m}^n\) with \(rk_q(g) = n\), and \(C := \mathcal{I}_{k,\theta}^{n,k-1}(g)\) be a generalized \(\theta\)-twisted Gabidulin code with \(h=k-1\) and \(m-n+1 \leq t < m-k\).

1. If \(1 \leq r \leq k-1\), then \(s_1^\theta(C) \geq \min\{n, k+i\}\) and if \(t+ir \leq n-k\) then
\[
s_1^\theta(C) = \min\{k + ir + \min(i, \left\lfloor \frac{r}{2} \right\rfloor), n\}.
\]
If \(m-k+1 \leq r \leq m-1\), then \(s_1^\theta(C) \geq \min\{n, k+i(m-r)\}\) and if \(t+i(m-r) \leq n-k\) then
\[
s_1^\theta(C) = \min\{k + i(m-r) + \min(i, \left\lceil \frac{m-(k+1-t)}{m-r} \right\rceil), n\}.
\]
2. If \(k \leq r \leq n-k-1\) or \(m-n+k+1 \leq r \leq m-k\) we have \(s_1^\theta(C) = 2k\).
3. If \(r \geq k\) and \(r \leq n-k\), then \(s_1^\theta(C) \geq k+n-r-1\). If \(r \leq m-k\) and \(r \leq m-n+k\), then \(s_1^\theta(C) \geq k+n-m+r+1\).

**Corollary 4.15.**
1. Let \(1 < k < n-t < 2n-m\). If \(m < 2n-4\) then generalized twisted Gabidulin codes of length \(n\) and dimension \(k\) over \(\mathbb{F}_{q^m}\) with hook \(h=k-1\) and twist \(t\) are never equivalent to a Gabidulin code.
2. Let \(2 < k < n-t-1 < 2n-m-1\). If \(m < 2n-6\) then generalized twisted Gabidulin codes of length \(n\) and dimension \(k\) over \(\mathbb{F}_{q^m}\) with hook \(h=k-1\) and twist \(t\) are never equivalent to a twisted Gabidulin code.

As a third class of generalized twisted Gabidulin codes, we study the family of codes in which the evaluation points \(g_i\) span a subfield of \(\mathbb{F}_{q^m}\). This is a relevant case since it includes the codes of maximal length (for a given field extension \(m\)) for which the codes can be guaranteed to be MRD by the sufficient condition in [50]. We exclude the smallest and largest two choices of both \(t\) and \(h\) for the sake of an easier proof, but the result might as well hold for these cases.
Proposition 4.16. Let $g \in \mathbb{F}_q^n$ with $\text{rk}_q(g) = n$, where $\mathbb{F}_q^n \subseteq \mathbb{F}_q^m$ is a subfield (i.e., the $g_i$ span a subfield of $\mathbb{F}_q^m$), and $C := \mathcal{I}_{k,\theta}^{t,h}(g)$ be a generalized $\theta$-twisted Gabidulin code with $\eta \in \mathbb{F}_q^m$, $2 < t < n - k - 1$ and $1 < h < k - 2$ (note that this implies $4 < k < n - 4$). Then, for $0 < r < m$ with $\gcd(r,m) = 1$, we have

$$s_1^\theta(C) \geq k + 3.$$  

Proof. Since the $g_i$ span a subfield $\mathbb{F}_q^n$ of $\mathbb{F}_q^m$, we have $\theta^{n+r}(g) = \theta^r(g)$ for all $r$. Due to this periodicity, it suffices to show the claim for $r = 1, \ldots, \left\lfloor \frac{n - 1}{2} \right\rfloor$ since $\gcd(m,n) = 1$ implies $r \not\equiv 0 \mod n$ (due to $n \mid m$), we can always write $\theta^r(g) = \theta^{\text{mod } n}(g)$, and for $(r \mod n) > n/2$, we can express the code with respect to $g' = \theta^{(r \mod n)}(g)$ instead of $g$. Recall also that by Corollary 2.4, $\theta^0(g), \ldots, \theta^{n-1}(g)$ are linearly independent. The set $S_1^\theta(C)$ contains

$$\{\theta^i(g)\}_{i \in \{0, \ldots, k-1\} \setminus \{h\}} + \{\theta^j(g)\}_{j \in \{\max(r,k), \ldots, \min(n,k+r-1)\} \setminus \{h+r\}} + (\eta \theta^{k-1+t}(g) + \theta^0(g), \theta^r(\eta) \theta^{k-1+t+r}(g) + \theta^{h+r}(g))$$

which contains $k + 3$ linearly independent codewords, as we show now. Note that the first summand above always has dimension $k - 1$, thus we need to find four extra linearly independent elements. We distinguish three cases:

- **Case $r \geq 4$:** Due to $4 \leq r \leq n/2$ and $n - k \geq 5$, we get:
  - If $h + r \in \{k, \ldots, n-1\}$, then there are three distinct elements, say $i_1, i_2, i_3$, in $\{k, \ldots, n-1\} \setminus \{h + r\}$, for which $\theta^i_j(g) \in \theta^r(C) \subseteq S_1^\theta(C)$ for $j = 1, 2, 3$. The fourth linearly independent element is $\theta^{h+r}(g) + \theta^r(\eta) \theta^{k-1+t+r}(g)$.
  - If $h + r \not\in \{k, \ldots, n-1\}$, then there are four distinct integers $i_1, \ldots, i_4 \in \{k, \ldots, n-1\}$ with $\theta^{i_j}(g) \in \theta^r(C) \subseteq S_1^\theta(C)$.

- **Case $r = 3$:** For each $i \in \{0, \ldots, k + 2\}$, we have that $\theta^i(g) \in S_1^\theta(C)$ if $i \not\in \{h, h + 3\}, \theta^3(g) + \eta \theta^{k-1+t}(g) \in S_1^\theta(C)$, and $\theta^{h+r}(g) + \theta^r(\eta) \theta^{k-1+t+r}(g) \in S_1^\theta(C)$. Since $\theta^{k-1+t}(g) \neq \theta^{h+r}(g)$ and $\theta^{k-1+t+r}(g) \neq \theta^{h+r}(g)$, there are at least $k + 3$ linearly independent codewords in $S_1^\theta(C)$.

- **Case $r \in \{1, 2\}$:** Due to $2 \leq h \leq k - 3$, we have $r \leq h$ and $h + r \leq k - 1$, hence the codeword $\theta^i(g)$ is contained in $C$ or $\theta^r(C)$ for any $i = 0, \ldots, k - 1 + r$. Furthermore, $k - 1 + r < k - 1 + t < k - 1 + t + r < n$, so the codewords $\eta \theta^{k-1+t}(g) + \theta^0(g)$ and $\theta^r(\eta) \theta^{k-1+t+r}(g) + \theta^{h+r}(g)$ are linearly independent from $g, \theta(g), \ldots, \theta^{k-1+r}(g)$. Thus, there are at least $k + 3$ linearly independent elements in $S_1^\theta(C)$. □

This implies:

Corollary 4.17. Let $g \in \mathbb{F}_q^n$ with $\text{rk}_q(g) = n$, where $\mathbb{F}_q^n \subseteq \mathbb{F}_q^m$ is a subfield (i.e., $n$ divides $m$ and the $g_i$'s span a subfield of $\mathbb{F}_q^m$), and $C := \mathcal{I}_{k,\theta}^{t,h}(g)$ be a generalized twisted Gabidulin code with $\eta \in \mathbb{F}_q^m$, $2 < t < n - k - 1$ and $1 < h < k - 2$.

Then, $C$ is not equivalent to a $\theta$-Gabidulin code or a $\theta$-twisted Gabidulin code in the narrow sense ($t = 1, h = 0$) for any generator $\theta'$ of $\text{Gal}(\mathbb{F}_q^m/\mathbb{F}_q)$.

Proof. Let $\theta' \in \text{Gal}(\mathbb{F}_q^m/\mathbb{F}_q)$ be a generator of the Galois group. Then, there is an $r \in \{1, \ldots, m-1\}$ with $\theta' = \theta^r$, where $\gcd(r,m) = 1$. By Proposition 4.16, we have $s_1^\theta(C) \geq k + 3$. On the other hand, a $\theta'$-Gabidulin code $C'$ fulfills $s_1^\theta(C') = k + 1$ and any (narrow-sense) $\theta'$-twisted Gabidulin code $C''$ gives $s_1^\theta(C'') = k + 2$. Hence, the codes cannot be equivalent. □

5. Number of Inequivalent Known Codes

In this section we use the results from the previous section to determine upper and lower bounds on the number of inequivalent Gabidulin and twisted Gabidulin codes. For generalized twisted Gabidulin codes, we compute lower bounds on the number of equivalence classes for some small parameters.
5.1. Gabidulin Codes. In the following, $Gr(k, F_{q^m}^n)$ denotes the $k$-dimensional Grassmannian of $F_{q^m}^n$, that is the set of all the $k$-dimensional $F_{q^m}$-subspaces of $F_{q^m}^n$. Moreover, let $Gab_q(k, n, m, \theta)$ be the set of all $[n, k]_{q^m}$ $\theta$-Gabidulin codes, i.e.,

$$Gab_q(k, n, m, \theta) := \{ U \in Gr(k, F_{q^m}^n) \mid U = G_{k, \theta}(g) \text{ for some } g \in F_{q^m}^n \text{ with } \text{rk}_q(g) = n \}.$$ 

Furthermore, define $F := \{ \theta \in \text{Gal}(F_{q^m}/F_q) \mid \theta \text{ is a generator of } \text{Gal}(F_{q^m}/F_q) \}$, and

$$Gab_q(k, n, m, \theta) := \{ U \in Gr(k, F_{q^m}^n) \mid U \text{ is a } \theta\text{-Gabidulin code for some } \theta \in F \} = \bigcup_{\theta \in F} Gab_q(k, n, m, \theta).$$

For a fixed $\theta$, Berger provided the following result about the equality of two Gabidulin codes.

**Theorem 5.1.** [6, Theorem 2] Let $u, v \in F_{q^m}^n$ be two vectors such that $\text{rk}_q(u) = \text{rk}_q(v) = n$. Then, for any generator $\theta$ of $\text{Gal}(F_{q^m}/F_q)$, $G_{k, \theta}(u) = G_{k, \theta}(v)$ if and only if $u = \lambda v$ for some $\lambda \in F_{q^m}^*$.

This leads to the following results.

**Corollary 5.2.** Let $k, n, m$ be integers such that $2 \leq k \leq n - 2$ and $n \leq m$, and let $\theta$ be a generator of $\text{Gal}(F_{q^m}/F_q)$. Then,

1. $$|Gab_q(k, n, m, \theta)| = \prod_{i=1}^{n-1} (q^m - q^i),$$

and hence

2. $$|Gab_q(k, n, m)| \leq \frac{\phi(m)}{2} \prod_{i=1}^{n-1} (q^m - q^i).$$

**Proof.** 1. This follows from Theorem 5.1.

2. This directly follows from part 1 and Lemma 4.1.

However, one can expect that some of those Gabidulin codes will be equivalent to each other. Then the natural question is “how many inequivalent Gabidulin codes are there?”. Recently, Schmidt and Zhou provided a lower bound on this number.

**Theorem 5.3.** [57, Theorem 1.2] For any $k, n, m$ integers such that $1 \leq k \leq n - 1$ and $2 \leq n \leq m - 2$, the number of inequivalent Gabidulin codes of dimension $k$ in $F_{q^m}^n$ is at least

$$\frac{1}{m}\left[\frac{m}{n}\right]\frac{q-1}{q^m - 1}.$$

In order to determine the number of equivalent Gabidulin codes, for any such code $C$, we need to give an estimate of the number of automorphisms $\theta$ for which $C$ is a $\theta$-Gabidulin code. Note that the notion of $\theta$-Gabidulin codes requires $\theta$ to be a generator of the Galois group. Since $\text{Gal}(F_{q^m}/F_q) \cong (\mathbb{Z}/m\mathbb{Z})^*$ and the set of generators is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^*$, we introduce the following notation. For a code $C$ we define the set

$$A_C := \{ \theta \text{ generator of } \text{Gal}(F_{q^m}/F_q) \mid C \text{ is a } \theta\text{-Gabidulin code} \}.$$

If we fix a generator $\theta$ of $\text{Gal}(F_{q^m}/F_q)$, the set $A_C$ corresponds to the set

$$A_{C, \theta} := \{ r \in (\mathbb{Z}/m\mathbb{Z})^* \mid C \text{ is a } \theta^r\text{-Gabidulin code} \}.$$

We now state an auxiliary result.

**Lemma 5.4.** Let $3 \leq k \leq n-3$ be integers and $\theta$ be a generator of $\text{Gal}(F_{q^m}/F_q)$. Let moreover $C$ be a $\theta$-Gabidulin code.
Theorem 5.5. Let \( C, \theta \) be integers with \( 2 < k < n - 2 \) and \( n \leq m \). Then

\[
\left| \text{Gab}_q(k, n, m) \right| \geq \frac{\phi(m)}{2m} \prod_{i=1}^{n-1} \left( q^{m} - q^i \right).
\]

Proof. First we provide an upper bound on the cardinality of the set \( C, \theta \), where \( C \) is an \( \theta \)-Gabidulin code, using a double counting argument. Consider the number

\[
\sum_{b \in \mathbb{Z}/m\mathbb{Z}} \left| C, \theta \cap \{b, b+1, \ldots, b+n-2\} \right|.
\]

On one hand, by Lemma 5.4, we have that it is upper-bounded by \( 2m \). On the other hand, every \( r \in A, \theta \) is counted exactly \( n - 1 \) times. Therefore we get

\[
(n-1)|A, \theta| = \sum_{b \in \mathbb{Z}/m\mathbb{Z}} \left| C, \theta \cap \{b, b+1, \ldots, b+n-2\} \right| \leq 2m,
\]

from which we deduce that \( |A, \theta| \leq \frac{2m}{n-1} \). At this point, combining this upper bound with Corollary 5.2, we get the desired lower bound, since every \( \theta \)-Gabidulin code \( C \) is counted at most \( |A, \theta| \) times.  \( \square \)
Moreover, for $1 \leq k < n \leq m$ we define $N_q(k, n, m)$ as the number of inequivalent $[n, k]_{q^m}$ Gabidulin codes, i.e.

$$N_q(k, n, m) := |\text{Gab}_q(k, n, m)/\sim|.$$

**Theorem 5.6.** Let $\mathbb{F}_q$ be a finite field of characteristic $p$ and $2 < k < n - 2$ be integers.

1. If $m = n$, then

$$N_q(k, m, m) = \frac{\phi(m)}{2}.$$

2. If $m > n$, then

$$\frac{(n - 1)\phi(m)}{2m^2[\mathbb{F}_q : \mathbb{F}_p]} \prod_{i=2}^{n} \frac{q^{m-i+1} - 1}{q^i - 1} \leq N_q(k, m, n) \leq \frac{\phi(m)}{2} \prod_{i=2}^{n} \frac{q^{m-i+1} - 1}{q^i - 1}.$$

**Proof.**

1. Let $\theta_1, \theta_2$ be two generators of $\text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$, and $g, h \in \mathbb{F}_{q^m}^*$ be two vectors such that $r_{kq}(g) = r_{kq}(h) = m$. Consider two Gabidulin codes $C = \mathcal{G}_{k, \theta_1}(g)$ and $C' = \mathcal{G}_{k, \theta_2}(h)$. First we show that if $\theta_1 = \theta_2$ then $C \sim C'$. Since $n = m$ then $\text{supp}(g) = \text{supp}(h)$. Therefore, there exists $A \in \text{GL}_m(q)$ such that $gA = h$. This implies that $C \cdot A = C'$ and therefore, $C \sim C'$. Now, suppose that $\theta_1 = \theta_2^{-1}$. Then, by the previous argument, we can assume $g = h$. Since by Lemma 4.1, $\mathcal{G}_{k, \theta_1}^{-1}(h) = \mathcal{G}_{k, \theta_2}(\theta_1^{-k}(h))$, we obtain again $C \sim C'$. Finally, if $\theta_1 \notin \{\theta_2, \theta_2^{-1}\}$, then $\theta_1 = \theta_2^r$, with $r \notin \{1, -1\}$, and by Proposition 4.2, we have $s_{\theta_1}^{s_1}(C') = k + 1$ and $s_{\theta_1}^{s_2}(C) \geq k + 2$. By Lemma 3.1 we deduce that they cannot be equivalent. Since there are exactly $\phi(m)$ generators for $\text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$, we conclude.

2. By part 2 of Corollary 5.2, we have that there are at most $\frac{\phi(m)}{2} \prod_{i=1}^{n-1}(q^m - q^i)$ many Gabidulin codes. Moreover, consider the action

$$\text{GL}_m(q) \times \text{Gab}_q(k, n, m) \rightarrow \text{Gab}_q(k, n, m) \quad (A, \mathcal{G}_{k, \theta}(g)) \mapsto \mathcal{G}_{k, \theta}(gA).$$

We have that, by Theorem 5.1, $\mathcal{G}_{k, \theta}(g) \cdot A = \mathcal{G}_{k, \theta}(g')$, if and only if $gA = \lambda g_0$, for some $\lambda \in \mathbb{F}_{q^m}^*$. Since $g = (g_1, \ldots, g_n)$ is such that the $g_i$’s are $\mathbb{F}_q$-linearly independent, it easily follows that $gA = \lambda g_0$ if and only if $A = \lambda I_n$. Therefore, the action defined in (1) induces a free action of $\text{GL}_m(q)/\mathbb{F}_q^*$ with the same orbits. Since this action is free, and every orbit is contained in an equivalence class, we have that

$$N_q(k, m, n) \leq |\text{Gab}_q(k, n, m)| \frac{|\mathbb{F}_q^*|}{|\text{GL}_m(q)|} = \frac{\phi(m)}{2} \prod_{i=1}^{n-1} \frac{q^{m-i} - 1}{q^{i+1} - 1}.$$

We now prove the lower bound. By Theorem 5.5, we have at least $\frac{\phi(m)}{2m^2} \prod_{i=1}^{n-1}(q^m - q^i)$ many distinct Gabidulin codes. Considering again the action in (1), we get that the number of orbits under that action, is at least

$$\frac{\phi(m)}{2m^2} \prod_{i=1}^{n-1} \frac{q^{m-i} - 1}{q^{i+1} - 1}.$$

If we now consider the equivalence classes of Gabidulin codes, it remains to study the action of the subgroup $\text{Aut}(\mathbb{F}_{q^m})$, which has cardinality exactly $m[\mathbb{F}_q : \mathbb{F}_p]$. Therefore, an equivalence class can be union of at most $m[\mathbb{F}_q : \mathbb{F}_p]$ orbits of the action (1), which leads to the desired result.

\[ \blacksquare \]

The first part of Theorem 5.6 already appeared in [41, Theorem 1]. It provides the exact number of inequivalent Gabidulin codes in the case $n = m$. Moreover, for the general case $n < m$, the second part of Theorem 5.6 gives both an upper and a lower bound on this number. It is important to observe that, whenever $(n - 1)\phi(m) > 2m[\mathbb{F}_q : \mathbb{F}_p]$, the lower bound improves the one given in Theorem 5.3, due to Schmidt and Zhou [57].
5.2. Twisted Gabidulin Codes. Let $F$ be the set of generators of $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$. We denote by $T\text{Gab}_q(k,n,m,\theta)$ the set of all $[n, k]_{q^m}$ $\theta$-twisted Gabidulin codes, and by $T\text{Gab}_q(k,n,m)$ the set of all $[n, k]_{q^m}$ twisted Gabidulin codes i.e.

$$T\text{Gab}_q(k,n,m,\theta) := \{U \in \text{Gr}(k, \mathbb{F}_{q^n}^m) \mid U = \mathcal{H}_k^\theta(g) \text{ for some } \eta \in \mathbb{F}_{q^n}^* \text{ and } g \in \mathbb{F}_{q^n}^m \}
$$

with $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\eta) \neq (-1)^{km}$ and $rk_q(g) = n$,

$$T\text{Gab}_q(k,n,m) := \{U \in \text{Gr}(k, \mathbb{F}_{q^n}^m) \mid U \text{ is a } \theta\text{-twisted Gabidulin code for some } \theta \in F \} = \bigcup_{\theta \in F} T\text{Gab}_q(k,n,m,\theta).$$

As for Gabidulin codes, the dual of a $\theta$-twisted Gabidulin code with $b = 0$ is another $\theta$-twisted Gabidulin code. For the case $n = m$ this duality result based on the Delsarte bilinear form was shown in [58, 33]. One can show a similar duality result for the general case $n \leq m$, whose proof can be also found in [38, Chapter 7].

**Theorem 5.7.** [38, Theorem 7.21] Let $k, n, m$ be positive integers such that $2 \leq k \leq n-2$ and $n \leq m$. Let $g \in \mathbb{F}_{q^m}^m$ with $rk_q(g) = n$, $\eta \in \mathbb{F}_{q^m}^*$ with $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\eta) \neq (-1)^{km}$ and consider the $\theta$-twisted Gabidulin code $C := \mathcal{H}_k^\theta(g)$. Then, for any non-zero $g' \in \mathcal{G}_{n-1,\theta}(\theta^{-(n-k-1)}(g))'$, we have $C^\perp = \mathcal{H}_k^{\theta\eta}(g')$, where

$$\eta' = (-1)^n \eta \frac{\theta^{k-n+1}(D) \theta^{k-n}((\theta^{n-k}(g')); g))}{\theta^{k-n}(D)} (\theta^{n-k}(g'); g)$$

and $D := \det(M_{n,\theta}(g))$. Moreover $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\eta') = (-1)^{nm}N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\eta)$.

As for $\theta$-Gabidulin codes, one can find the exact number of $\theta$-twisted Gabidulin codes for a given generator $\theta$ of $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$, as we show in the following.

**Theorem 5.8.** Let $u, v \in \mathbb{F}_{q^n}^*$ be two vectors such that $rk_q(u) = rk_q(v) = n$. Then, for any generator $\theta$ of $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ and $\eta, \eta' \in \mathbb{F}_{q^m}^*$, $\mathcal{H}_k^\theta(u) = \mathcal{H}_k^{\eta\theta}(v)$ if and only if there exists $\lambda \in \mathbb{F}_{q^m}^*$ such that $u = \lambda v$ and $\eta' = \frac{\eta \theta^k(\lambda)}{\lambda}$.

**Proof.** We divide the proof in three cases.

Case $3 \leq k \leq \frac{n}{2}$: Suppose that $C := \mathcal{H}_k^\theta(u) = \mathcal{H}_k^{\eta\theta}(v)$. Then $\theta(u)$ can be written as

$$\theta(u) = \sum_{i=1}^{k-1} \lambda_i \theta^i(v) + \lambda_k (v + \eta' \theta^k(v)),$$

for some $\lambda_i \in \mathbb{F}_{q^m}$ not all zeros. Let $r := \max\{i \in [k] \mid \lambda_i \neq 0\}$. If $r = k$, then we would have $\theta(\theta(u)) = \theta(\lambda_k (v + \eta' \theta^k(v))) + \theta(\sum_{i=1}^{k-1} \lambda_i \theta^i(v)) \in C$, but this is not possible, since we would have $C = \langle v + \eta' \theta^k(v), \theta(v), \ldots, \theta^{k-1}(v) \rangle C + \langle v, \theta(v), \ldots, \theta^{k-1}(v) \rangle C + \langle \lambda_k \theta^k(v) \rangle C$, for some $\mu \in \mathbb{F}_{q^m}$, which has dimension $k+1$ by Corollary 2.4. Then $r < k$. Suppose that $r > 1$, then $0 < k - r < k$ and $\theta^{k-r}(\theta(u)) \in C$, and we obtain

$$\theta^{k-r}(\theta(u)) = \sum_{i=1}^{r} \theta^{k-r}(\lambda_i) \theta^{k-r+i}(v) \in C.$$

Also in this case, we obtain $C = \langle v, \theta(v), \ldots, \theta^{k-1}(v), \theta^k(v) \rangle$, which has dimension $k+1$ by Corollary 2.4. Hence the only possibility is $r = 1$, i.e. $\theta(u) = \lambda_1 \theta(v)$, or equivalently, $u = \lambda v$ for some $\lambda \in \mathbb{F}_{q^m}^*$. It remains to study the conditions on $\eta$ and $\eta'$. At this point we have $C = \langle v + \eta' \theta^k(v), \theta(v), \theta^{k-1}(v) \rangle C$. We have $C = \langle \lambda v + \eta \theta^k(\lambda) \theta^k(v) + \sum_{i=1}^{k-1} \mu_i \theta^i(v), \mu \theta^k(v), v, \theta(v), \ldots, \theta^{k-1}(v) \rangle$, therefore,

$$v + \eta' \theta^k(v) = \mu \lambda v + \mu \eta \theta^k(\lambda) \theta^k(v) + \sum_{i=1}^{k-1} \mu_i \theta^i(v),$$

for some $\mu, \mu_i \in \mathbb{F}_{q^m}$, which we can rewrite as

$$(1 - \mu \lambda) v - \sum_{i=1}^{k-1} \mu_i \theta^i(v) + (\eta' - \eta \theta^k(\lambda)) \theta^k(v) = 0.$$
By Corollary 2.4, \( v, \theta(v), \ldots, \theta^k(v) \) are linearly independent (since obviously \( k < n \)), hence \( \mu_i = 0 \) for every \( i \in [k-1] \), \( \mu = \lambda^{-1} \) and \( \eta' = \frac{v}{\lambda} \). 

**Case \( k = 2 \):** Suppose \( C = \langle u + \eta \theta^2(u), \theta(u) \rangle = \langle v + \eta' \theta^2(v), \theta(v) \rangle \). Then \( \theta(u) \) can be written as linear combination of \( v + \eta' \theta^2(v), \theta(v) \), i.e. \( \theta(u) = \lambda_1 \theta(v) + \lambda_2 v + \lambda_2 \eta' \theta^2(v) \). Then one can write

\[
\begin{align*}
u + \eta \theta^2(u) = & \theta^{-1}(\lambda_2)\theta^{-1}(v) + \theta^{-1}(\lambda_1)v + (\theta(\lambda_2) + \theta^{-1}(\lambda_2)\theta^{-1}(\eta'))\theta(v) + \\
& \theta(\lambda_1) \theta^2(v) + \theta(\lambda_2) \theta(\eta') \theta^2(v).
\end{align*}
\]

By Corollary 2.4 we deduce that \( \lambda_2 = 0 \), and therefore, \( \theta(u) = \lambda_1 \theta(v) \), or equivalently, \( u = \lambda v \) for some \( \lambda \in \mathbb{F}_{q^m}^* \). The relation between \( \eta \) and \( \eta' \) is derived in the same way as done in the proof of the case \( 3 \leq k \leq \frac{n}{2} \).

**Case \( k > \frac{n}{2} \):** It follows by the duality result in Theorem 5.7, and the cases \( k = 2 \) and \( 3 \leq k \leq \frac{n}{2} \).

**Corollary 5.9.** Let \( k, n, m \) be integers such that \( 2 \leq k \leq n - 2 \) and \( n \leq m \), and let \( \theta \) be a generator of \( \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \). Then,

\[
|\text{TGab}_\theta(k, n, m, \theta)| = \left(1 - \frac{1}{q-1}\right) \prod_{i=0}^{n-1} \left(q^m - q^i\right).
\]

**Proof.** We have exactly \( \prod_{i=0}^{n-1} (q^m - q^i) \) many choices for the vector \( g \) and \( (q^m - 1) - \frac{q^m - 1}{q-1} \) choices for the element \( \eta \) with norm different from \( (-1)^{km} \). By Theorem 5.8, the total number has to be divided by the number of non-zero multiples of \( g \), which is \( q^m - 1 \). \( \square \)

**Corollary 5.10.** Let \( k, n, m \) be integers such that \( 2 \leq k \leq n - 2 \) and \( n \leq m \). Then,

\[
|\text{TGab}_\theta(k, n, m)| \leq \frac{\phi(m)}{2} \left(1 - \frac{1}{q-1}\right) \prod_{i=0}^{n-1} \left(q^m - q^i\right).
\]

**Proof.** It directly follows from Corollary 5.9 and Lemma 4.4. \( \square \)

For a prime power \( q \), and two integers \( k, m \) we consider the set \( X_q(m, k) := \{ \alpha \in \mathbb{F}_{q^m} \mid N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha) \neq (-1)^{km}\} \), and the left group action

\[
\text{Aut}(\mathbb{F}_{q^m}) \times X_q(m, k) \rightarrow X_q(m, k)
\]

\[
(\tau, \alpha) \mapsto \tau(\alpha).
\]

Observe that the one above is well-defined. Indeed, if \( \alpha \in X_q(m, k) \), and \( \tau \in \text{Aut}(\mathbb{F}_{q^m}) \), then

\[
N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\tau(\alpha)) = \prod_{i=0}^{m-1} \theta^i(\tau(\alpha)) = \prod_{i=0}^{m-1} \tau(\theta^i(\alpha)) = \tau(N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha)).
\]

Since \( (-1)^{km} \) belongs to the prime field, and therefore is fixed by any automorphism \( \tau \in \text{Aut}(\mathbb{F}_{q^m}) \), we have that also \( \tau(N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\alpha)) \neq (-1)^{km} \).

**Theorem 5.11.** [41, Theorem 2] Let \( \mathbb{F}_q \) be a finite field of characteristic \( p \) and \( 2 < k < n - 2 \) be integers. Denote by \( X_q(m, k) \) the cardinality of the set of orbits of this group action. If \( m = n \), then the number of inequivalent twisted Gabidulin codes is

\[
X_q(m, k) \frac{\phi(m)}{2}.
\]

**Proof.** The proof is similar to the one of part 1 of Theorem 5.6. Let \( \theta_1, \theta_2 \) be two generators of \( \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \), and \( g, h \in \mathbb{F}_{q^m}^* \) be two vectors such that \( \text{rk}_g(h) = n \), and let \( \eta, \eta' \in \mathbb{F}_{q^m} \) with norm not equal to \( (-1)^{km} \). Consider two twisted Gabidulin codes \( C = \mathcal{H}_{k, \theta_1}^q(g) \) and \( C' = \mathcal{H}_{k, \theta_2}^q(h) \). Suppose \( \theta_1 \notin \{ \theta_2, \theta_2^{-1} \} \), then \( \theta_1 = \theta_2^r \), with \( r \notin \{1, -1\} \), and by Proposition 4.5, we have \( s_1^{\theta_1}(C') = k + 2 \) and \( s_1^{\theta_1}(C) \geq k + 3 \). By Lemma 3.1 we conclude that they cannot
be equivalent. Now, recall that $C \sim C'$ if and only if there exist $\tau \in \text{Aut}(\mathbb{F}_q^m)$ and $A \in \text{GL}_n(q)$ such that $C' = \tau(C)A$. When $C = H_{k, \theta, 2}^\tau(g)$ we get $\tau(C)A = H_{k, \theta, 2}^\tau(\tau(g))$. Assume $\eta' = \tau(\eta)$ for some $\tau \in \text{Aut}(\mathbb{F}_q^m)$. If $\theta_2 = \theta_1$ then, $H_{k, \theta, 2}^\tau(g) = \text{supp}(\tau(g)) = \text{supp}(h)$. This implies that there exists $A \in \text{GL}_m(q)$ such that $\tau(g)A = h$ and $\tau(C)A = C'$. Hence, for every $\theta_2$ generator of $\text{Gal}(\mathbb{F}_q^m/\mathbb{F}_q)$, and for every representative $\eta$ in an orbit of the action defined in (2), we have exactly one equivalent class of twisted Gabidulin codes. Moreover, observe that $H_{k, \theta, 2}^\tau(g) = H_{k, \theta, 2}^{\tau^{-1}}(g)$, by Lemma 4.4. This shows that $C$ and $C'$ are equivalent if and only if $\theta_2 = \theta_1$ and $\eta = \tau(\eta')$ for some $\tau \in \text{Aut}(\mathbb{F}_q^m)$ or $\theta_1 = \theta_2^{-1}$ and $\eta^{-1} = \tau(\eta')$ for some $\tau \in \text{Aut}(\mathbb{F}_q^m)$. By counting, we get exactly $\phi(m) \chi_q(m, k)$ inequivalent twisted Gabidulin codes.

5.3. Computational Results on the Number of Generalized Twisted Gabidulin Codes. Due to the huge variation of parameters, studying the exact number of generalized $\theta$-twisted Gabidulin codes with the same techniques as in the previous two subsections would become extremely technical. To nevertheless give an idea on the suitability of the invariants for distinguishing these codes, we present computational results on the number of inequivalent generalized $\theta$-twisted Gabidulin codes with one twist ($\ell = 1$) here.

We fix the following parameters

- $n \geq 6$
- $m = 2n$
- $2 \leq k \leq n - 2$
- $g \in \mathbb{F}_q^m$ (chosen at random such that $rk_q(g) = n$)
- $\eta \in \mathbb{F}_q^m \setminus \mathbb{F}_q$ (chosen at random).

Note that $\mathbb{F}_q^m$ is a subfield of $\mathbb{F}_q^n$ and the choice of $m$ gives the smallest $m$ such that the sufficient MRD condition in [50] can be applied. Also, $g$ and $\eta$ are chosen to satisfy this MRD condition.

For each such choice of fixed parameters, we consider the parameter set

$$\text{param}(n, m) := \{(\theta, t, h) : \theta \in \text{Gal}(\mathbb{F}_q^m/\mathbb{F}_q) \text{ generator}, 1 \leq t \leq n - k, 0 \leq h < k\} \sim_p,$$

where we say that two parameter triples are equivalent, $(\theta_1, t_1, h_1) \sim_p (\theta_2, t_2, h_2)$, if $\theta_1 = \theta_2^{-1}$, $t_1 = n - k + 1 - t_2$, and $h_1 = k - 1 - h_2$. The choice of equivalence in parameters, $\sim_p$, is due to the symmetry of generalized $\theta$-twisted Gabidulin codes described in Lemma 4.9.

We consider the set of codes

$$gT\text{Gab}_q(k, n, m, g, \eta) := \left\{T^\eta_{k, t, h}(g) : (\theta, t, h) \in \text{param}(n, m)\right\}$$

and count the number of equivalence classes $|gT\text{Gab}_q(k, n, m, g, \eta)| \sim |$ among these codes.

Table 1 presents lower and upper bounds on $|gT\text{Gab}_q(k, n, m, g, \eta)| \sim |$, where each table entry contains three bounds $LB1(n, k)$, $LB2(n, k)$, and $UB(n, k)$ that are computed as follows:

- $LB1(n, k)$: Lower bound on $|gT\text{Gab}_q(k, n, m, g, \eta)| \sim |$ obtained by computing the sequences $\{s_i\}(C)_{i=1}^{k-1}$ and $\{g_i\}(C)_{i=1}^k$ (consecutive sums/intersections) for all codes $C$ in $gT\text{Gab}_q(k, n, m, g, \eta)$, where $\sigma$ ranges through all elements of the Galois group $\text{Gal}(\mathbb{F}_q^m/\mathbb{F}_q)$.
- $LB2(n, k)$: Lower bound on $|gT\text{Gab}_q(k, n, m, g, \eta)| \sim |$ obtained by computing the dimensions of the sums $\sigma_1(C) + \sigma_2(C) + \sigma_3(C)$ and intersections $\sigma_1(C) \cap \sigma_2(C) \cap \sigma_3(C)$ for 100 random choices of pairwise distinct automorphisms $\sigma_1, \sigma_2, \sigma_3 \in \text{Gal}(\mathbb{F}_q^m/\mathbb{F}_q)$.
- $UB(n, k)$: Upper bound on $|gT\text{Gab}_q(k, n, m, g, \eta)| \sim |$ given by $|\text{param}(n, m)|$.

It can be seen that for many parameters, we obtain lower and upper bounds that are quite close to each other. For instance, for $[n, k] = [11, 5]$, there are at least 145 and at most 150 equivalence classes. The bounds are also almost tight for $[n, k] = [15, 6]$ ($LB2 = 212, UB = 216$) and $[n, k] = [25, 10]$ ($LB2 = 1489, UB = 1500$).
Table 1. Lower (LB1 and LB2) and Upper (UB) bounds on the number of equivalence classes \( |gT\Gamma\alpha\beta_2(k, n, m, g, \eta)| \sim | \) (cf. (3)) as described in Section 5.3. Each cell is formatted as 

\[ \text{LB1}(n,k) \]
\[ \text{LB2}(n,k) \]
\[ \text{UB}(n,k) \]

| n \( \mod \) k | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|--------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| n \( \mod \) 2 | 4  | 8  | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 | 44 | 48 | 52 | 56 | 60 | 64 | 68 | 72 | 76 | 80 | 84 |

For even length, the best possible ratio (attained for \( k=n/2 \)) in all cases close to, but never above 0.5. This might be due to a further symmetry in the code parameters, besides \( \sim_p \), resulting in equivalent codes, but it needs to be investigates further.

Although the lower bounds on consecutive sums and intersections (UB1) are in general quite good (for odd lengths, they find roughly half of the possible equivalence classes), random sums and intersections (UB2) of codes under automorphisms appear to perform better in distinguishing generalized twisted Gabidulin codes. Our observation is that consecutive sums of two twisted codes always have the same dimensions if the code parameters are related by certain symmetries that the random sums can often distinguish. This should be further investigated in future work.

Another interesting observation is that UB1 is symmetric in \( k \) for a given length (i.e., \( UB1(n,k) = UB1(n,n-k) \) for all \( n,k \)). When looking closer at the raw results, one can observe that the number of equivalence classes for \( [n,k] \) from the consecutive sums \( (s_\eta^1) \) in all checked cases equals the number obtained for \( [n,n-k] \) from the consecutive intersections \( (t_\eta^1) \). This symmetry, together with \( t_\eta^1(C) = n - s_\eta^1(C^\perp) \) (cf. Proposition 3.3), shows that
the bounds obtained from consecutive sums of the \([n, k]\) twisted codes and the duals of the \([n, n-k]\) codes (which are again \([n, k]\) codes) are the same. This indicates, but certainly does not prove, that the duals of the tested twisted Gabidulin codes are equivalent to twisted Gabidulin codes. To the best of our knowledge, nothing is known about the duals of twisted Gabidulin codes, except for Theorem 5.7 (Sheekey’s original twisted Gabidulin codes, i.e., \(t = 1, h = 0\)).

We computed the values of Table 1 using the computer algebra SageMath v8.1 [65]. The computation of the entire table using a non-optimized implementation took less than the equivalent of 45 days on a single thread of an Intel X5650 CPU (launch year 2010). Note, for comparison, that checking the equivalence of only two codes of length 25 using the definition, Definition 2.7, would involve brute-forcing through all full-rank matrices in \(\mathbb{F}_3^{25 \times 25}\), which are more than \(2^{989}\) many. This is far away from what is assumed to be computable on an ordinary computer today.

6. Characterization Results for Gabidulin Codes

In this section we study the \(\sigma\)-sequences, in order to derive characterization results of some families of codes. Unfortunately, \(m\) large enough almost all the rank-metric codes have the same sequence \(\{s^\sigma_i(C)\}\), hence it seems unlikely that asymptotically we can get nice characterizations. This is explained by the following result, due to Coggia and Couvreur.

**Proposition 6.1.** [10, Proposition 2] If \(C\) is an \([n, k]_q^m\) code chosen at random and uniformly among all the possible \([n, k]_q^m\) codes, then for any non-negative integer \(b\) and for a positive integer \(i < k\), we have

\[
\Pr\{s^\sigma_i(C) \leq \min\{n, (i+1)k\} - b\} = \mathcal{O}(q^{-mb}),
\]

for \(m \to +\infty\).

However, this happens only when \(m\) is big. In particular, one can expect that codes which have no maximal dimension (usually) have good algebraic structures. Moreover, restricting to MRD codes and the case \(n = m\) has a different effect. An idea of this different behavior is explained by the following result due to Payne in 1971. The original result is formulated in a completely different way, since it was determined in the framework of hyperovals and linearized o-polynomials. See [9] for more details.

**Theorem 6.2.** [48] Let \(C\) be an \([n, 2]_q^2\) MRD code. Then there exists a generator \(\theta\) of \(\text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\) such that \(s^\theta_i(C) = 2 + i\) for every \(i \in \{0, \ldots, n-2\}\).

In common language, Theorem 6.2 states that all the \([n, 2]_q^2\) MRD codes are Gabidulin codes.

Now we are going to use the sequences for characterizing Gabidulin codes. The following result follows from [23, Lemma 3.5], but we are going to include a proof for completeness, which uses the tools developed in this work. Note that for MRD codes the same result was shown in [24, Proposition 4.6].

**Lemma 6.3.** Let \(0 < k < n \leq m\) be integers, \(C\) be an \([n, k]_q^m\) code, and \(\theta\) be a generator of \(\text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\). If \(s^\theta_1(C) = k + 1\), then there exists \(g \in \mathbb{F}_{q^m}^n\), \(0 \leq t \leq k\) such that

\[
C := C_1 \oplus \mathcal{G}_{t, \theta}(g),
\]

where \(C_1\) is an \([n, k-t]_q^m\) code which has a basis of rank 1 vectors, and \(\text{rk}_q(g) > t\).

**Proof.** Let \(U_1 = \{v \in \mathbb{F}_q^m \mid \text{rk}_q(v) = 1\}\). Then we can write \(C = C_1 \oplus C'\), where \(C_1 = (C \cap U_1)\). Hence \(C' \cap U_1 = \emptyset\). In particular, if \(\dim(C_1) = k - t\), then \(C'\) is an \([n, t, d]_q^m\) code with \(d > 1\). Moreover, \(C + \theta(C) = C_1 \oplus (C' + \theta(C'))\). Therefore, we can assume \(C_1 = \{0\}\), and that the minimum distance of \(C\) is greater than 1, for the rest of the proof.

We proceed by induction on \(k\). For \(k = 1\) it is trivially true. Suppose now that \(k \geq 2\) and that the statement is true for \(k - 1\); we want to prove the lemma for an \([n, k]_q^m\) code \(C\).
By hypothesis, \( s_1^\theta(C) = k + 1 \), hence \( t_1^\theta(C) = k - 1 \), by Proposition 3.4. Consider the code \( D := C \cap \theta(C) = T_1^\theta(C) \). This code has dimension \( k - 1 \). Moreover, \( D \cap U_1 = \emptyset \), and
\[
k - 1 \geq t_1^\theta(C) = t_1^\theta(C) - \Lambda_1^\theta(C) \geq t_0^\theta(C) - \Lambda_0^\theta(C) = k - 2.
\]
However, observe that \( t_1^\theta(D) = t_2^\theta(C) \). Therefore, if \( t_2^\theta(C) = k - 1 \), then \( t_2^\theta(D) = t_2^\theta(D) \), which implies, by part 3 of Proposition 3.6, that \( D \) has a basis of elements in \( \mathbb{F}_q^m \) or, equivalently, that \( D = (D \cap U_1) \). Since \( k - 1 > 0 \), \( D \subseteq C \) and \( C \cap U_1 = \emptyset \), which is a contradiction. Hence, we necessarily have that \( t_1^\theta(D) = t_2^\theta(C) = k - 2 \). Thus, by inductive hypothesis, \( D = G_{k-1}(h) \), for some \( h \in \mathbb{F}_q^m \) with \( \mathrm{rk}_q(h) \geq k \). Moreover, \( \theta^{-1}(h) \in \theta^{-1}(D) = \theta^{-1}(C) \cap C \subseteq C \). Therefore, \( C \supseteq \langle \theta^{-1}(h), \eta, \ldots, \theta^{k-2}(h) \rangle \). Since \( \mathrm{rk}_q(h) \geq k \), by Corollary 2.4 we get \( C = G_{k,\theta}(g) \), where \( g := \theta^{-1}(h) \). We only need to show that \( \mathrm{rk}_q(g) > k \). Suppose \( \mathrm{rk}_q(g) = k \), and let \( \{ f_1, \ldots, f_k \} \) be a basis for \( \text{supp}_q(g) \). Then, there exists \( A \in \text{GL}_n(q) \) such that \( gA = (f_1, \ldots, f_k, 0, \ldots, 0) \). Moreover, the code \( C \cdot A = G_{k,\theta}(gA) \sim C \) and has the same parameters. It is easy to see that this code has generator matrix \( (I_k | 0) \), since the last \( n - k \) entries of the code \( C \cdot A \) are all zeros. This implies that \( C \cdot A \) has (a basis of) codewords of rank 1, which yields a contradiction. Hence, \( \mathrm{rk}_q(g) > k \) and this concludes the proof. \( \square \)

**Remark 6.4.** Observe that in Lemma 6.3, the notation \( G_{k,\theta}(g) \) is used to indicate the code \( \{(f(g_1), \ldots, f(g_n)) \mid f \in G_{k,\theta}\} \), which is not necessarily a \( \theta \)-Gabidulin code, since \( \mathrm{rk}_q(g) \) can be smaller than \( n \). Moreover, we have that \( C_1 = \{0\} \) if and only if the minimum distance of \( C \) is strictly greater than 1.

From Lemma 6.3 we can derive a new criterion for characterizing a Gabidulin code. In order to put all the criteria together, we state a very general characterization theorem which includes also results from [24] and [37].

**Theorem 6.5** (Characterization of \( \theta \)-Gabidulin codes). Let \( C \subseteq \mathbb{F}_q^m \) be a linear code of dimension \( k \) and let \( \theta \) be a generator of \( \text{Gal}(\mathbb{F}_q^m/\mathbb{F}_q) \). The following are equivalent:

1. \( C \) is a \( \theta \)-Gabidulin code of dimension \( k \).
2. \( \Lambda^\theta_k \) is a \( \theta \)-Gabidulin code of dimension \( n - k + 1 \).
3. \( C \) is MRD and \( s_1^\theta(C) = k + 1 \).
4. \( C \) is MRD and \( t_1^\theta(C) = k - 1 \).
5. \( (s_1^\theta(C))_{i=0}^{i=k} = (k, k+1, \ldots, n) \) and \( d(C) > 1 \).
6. \( (t_1^\theta(C))_{i=0}^{i=k} = (k, k-1, \ldots, 0) \) and \( d(C^-) > 1 \).
7. \( s_1^\theta(C) = k + 1 \) and \( s_1^{n-k}(C) = n \) and \( d(C) > 1 \).
8. \( t_1^\theta(C) = k - 1 \) and \( s_1^k(C) = 0 \) and \( d(C^-) > 1 \).
9. \( (\Delta_1^\theta(C))_{i=0}^{i=k} = (1, 1, \ldots, 1, 0) \) and \( d(C) > 1 \).
10. \( (\Delta_1^\theta(C))_{i=0}^{i=k} = (1, 1, \ldots, 1, 0) \) and \( d(C^-) > 1 \).
11. \( \Lambda_0^\theta(C) = \Lambda_{n-k-1}^\theta(C) = 1 \) and \( d(C) > 1 \).
12. \( \Lambda_0^\theta(C) = \Lambda_{n-k-1}^\theta(C) = 1 \) and \( d(C^-) > 1 \).
13. \( C = \text{rowsp}(I_k \mid X) \), where:
   (a) \( \text{rk}(\theta(X) - X) = 1 \),
   (b) the \( q \)-rank of the first row of \( \theta(X) - X \) is \( n - k \),
   (c) the \( q \)-rank of the first column of \( \theta(X) - X \) is \( k \).

**Proof.**

1. \( \iff \) 2. This is well known and can be found e.g. in [18].
1. \( \iff \) 3. This was shown in [24, Theorem 4.8].
1. \( \iff \) 5. That 1. implies 5. was shown in part 1 of Proposition 4.2. It remains to show the other direction. Since \( s_1^\theta(C) = 1 \), then, by Lemma 6.3 and the fact that \( d > 1 \), we have that \( C = G_{k,\theta}(g) \), where \( \mathrm{rk}_q(g) > k \). Moreover, \( S_0^{n-k}(C) = G_{n,\theta}(g) \). By hypothesis, we also have that \( n = s_1^{n-k}(C) = \dim(G_{n,\theta}(g)) \). This implies, by Corollary 2.4, that \( \dim(G_{n,\theta}(g)) = \mathrm{rk}_q(g) \). Therefore, \( C \) is a \( \theta \)-Gabidulin code.
5. \(\iff\) 7. Moreover, if \(s^0_1(C) = k + 1\), and \(s^0_{n-k}(C) = n\), then \(\Delta^0_0(C) = s^0_1(C) - k = 1\), and by part 7 of Proposition 3.5, we have
\[
n = s^0_{n-k}(C) = k + \sum_{i=0}^{n-k-1} \Delta^0_i(C) \leq k + \sum_{i=0}^{n-k-1} \Delta^0_i(C) = k + (n - k).
\]

Therefore, \(\Delta^0_i(C) = 1\), for every \(i = 0, \ldots, n - k - 1\).

5. \(\iff\) 9. The equivalence follows from the definition of \(\Delta^0_i\) and part 7 of Proposition 3.5.

9. \(\iff\) 11. If \(\Delta^0_{n-k-1}(C) = \Delta^0_0(C) = 1\), by Proposition 3.5, we have \(1 = \Delta^0_0(C) \geq \ldots \geq \Delta^0_{n-k-1}(C) = 1\), hence we have all equalities.

4., 6., 8., 10., 12. It is easy to see that 3. \(\iff\) 4., 5. \(\iff\) 6., 7. \(\iff\) 8., 9. \(\iff\) 10. and 11. \(\iff\) 12. by Proposition 3.4.

1. \(\iff\) 13. This was shown in [37, Theorem 11].

\[\square\]

**Remark 6.6.** As explained in Section 3, we can efficiently compute the sum or intersection sequences of any given code. For the characterization result above, however, we need to check if the code (or its dual) has minimum distance greater than one. Although determining the exact minimum rank distance of a code is a computationally heavy task, checking if the minimum distance is one or greater can be done very efficiently by determining the \(F_q\)-subfield subcode of the code. This method of finding rank one codewords is explained in detail in [26].

After this characterization result, we conclude the section by proving that Gabidulin’s new codes of Definitions 2.18 and 2.19 are actually the classical Gabidulin codes, whenever they are MRD.

**Theorem 6.7.** Let \(1 \leq k \leq m\) be integers and \(\theta\) be a generator of \(\text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\). Let, moreover, \(\eta \in \mathbb{F}_{q^m}\) with \(\mathcal{N}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\eta) \neq (-1)^{km}\), and \(g \in \mathbb{F}_{q^m}^n\) be such that \(\text{rk}_q(g) = n\).

1. If \(m - k > k\), then the code \(C := \mathcal{J}^{\eta}_k(g)\) is a \(\theta\)-Gabidulin code.

2. If \(m \leq k\), then the code \(D := \mathcal{J}^{\eta}_k(g)\) is a \(\theta\)-Gabidulin code.

**Proof.**

1. It is immediate to observe that \(C = \mathcal{G}_{k, \theta}(h)\), where \(h = g + \eta \theta^k(g)\). Then, since by Proposition 2.20 \(C\) is MRD, we conclude using part 2 of the characterization result in Theorem 6.5.

2. By Proposition 2.20, we know that \(D\) is MRD. Moreover, if we compute \(\mathcal{S}^\theta_0(D)\) we get that \(\theta(D)\) is generated by
- \(\theta^i(g) + \theta^i(\eta) \theta^{k+i}g\) for \(1 \leq i < m - k\), which are already contained in \(D\),
- \(\theta^i(g)\) for \(m - k < i \leq k\), which includes a new, linearly independent, vector \(\theta^k(g)\), and
- \(\theta^{m-k}(g) + \theta^{m-k}(\eta) \theta^{m}(g) = \theta^{m-k}(g) + \theta^{m-k}(\eta)g\), which is a linear combination of \(\theta^{m-k}(g)\) (which is in \(D\)), \(\theta^k(g)\) (which is in \(\theta(D)\)), and \(g + \theta^k(\eta) \theta^k(g)\) (which is in \(D\)).

Hence, their sum, \(\mathcal{S}^\theta_0(D)\) has dimension \(k + 1\). We conclude using part 2 of the characterization result in Theorem 6.5.

\[\square\]

**Remark 6.8.** At this point one may ask whether a characterization result similar to Theorem 6.5 can be developed for twisted Gabidulin codes, in particular if it is possible to characterize these codes via their \(\sigma\)-sequences. Unfortunately, this seems to be impossible, even for the case \(n = m\), in which we can show that there exists many codes with the same sequence of \(\sigma\)-sums:

Let \(g \in \mathbb{F}_{q^m}^n\) be such that \(\text{rk}_q(g) = n\), \(\theta\) be a generator of \(\text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)\) and \(\eta \in \mathbb{F}_{q^m}^*\) such that \(\mathcal{N}_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\eta) \neq (-1)^{km}\), and consider the \(\theta\)-twisted Gabidulin code \(C := \mathcal{H}_{k, \theta}(g)\). Moreover, in
this setting we can define codes of the form
\[ G_{I,\theta}(g') := \langle \theta_i(g') \mid i \in I \rangle, \]
for some \( I \subseteq \{0, 1, \ldots, n-1\} \), and \( g' \in \mathbb{F}_{q^n}^m \) with \( \text{rk}_q(g') = n \). Observe that Gabidulin codes are a particular case of \( G_{I,\theta}(g') \), with \( I = \{0, \ldots, k-1\} \). It was shown in [12, Theorem 2.2] that this family of codes corresponds to the family of codes having left and right idealizers isomorphic to \( \mathbb{F}_{q^n}^m \). Moreover, in [33], it was proved that the right idealizer of the twisted Gabidulin code \( C \) is isomorphic to \( \mathbb{F}_{q^n}^m \) whenever \( 1 < k < n-1 \) (only in the case \( n = m \)). Therefore, codes of the form \( G_{I,\theta}(g') \) cannot be equivalent to twisted Gabidulin codes, since the right idealizer is invariant under code equivalence. For more details, the interested reader is referred to [12, 33].

Now we consider a set \( I = \{0, \ldots, k\} \setminus \{j\} \), for some \( 1 \leq j \leq k-1 \), a vector \( g' \in \mathbb{F}_{q^n}^m \) with \( \text{rk}_q(h) = n \) and the corresponding code \( G_{I,\theta}(h) \). Using Corollary 2.4, one can easily show that \( s^\theta_i(G_{I,\theta}(g')) = \min\{e, n\} \), where
\[
e = \begin{cases} k & \text{if } i = 0 \\ k + 1 + i & \text{if } i \geq 1. \end{cases}
\]
In particular the sequence of the \( \theta \)-sum dimensions of \( C \) is equal to the one of \( G_{I,\theta}(g') \). It is easy to see that also the sequences of \( \theta \)-intersection dimensions coincide. This shows that a result in the spirit of Theorem 6.5 seems not possible for twisted Gabidulin codes. Moreover, it was proved in [12] that when \( n = 7 \) and \( q \) is odd, or when \( n = 8 \) and \( q \equiv 1 \mod 3 \), the codes \( G_{I,\theta}(g') \) with \( I = \{0, 1, 3\} \) are MRD. Therefore, even with the further assumption to take an MRD code, the characterization of \( \theta \)-twisted Gabidulin codes in terms of their \( \theta \)-sequences seems not feasible. A step in this direction was done in [23], where a characterization of \( \theta \)-Gabidulin codes involving \( t^0_\theta(C), t^1_\theta(C) \), and \( t^2_\theta(C) \) was given. However, the characterization given there required also the existence of a certain element of maximum rank \( n \) with some special properties. An interesting problem would be if one can find better conditions involving the whole sequences \( \{s^\theta_i(C)\} \) and \( \{t^\theta_i(C)\} \), even for different \( \theta \)'s, in order to characterize twisted Gabidulin codes.

7. Conclusion

We showed that the sum and the intersection sequence of a rank-metric code (under an automorphism) are invariants of the equivalence class of the code. This gives an efficiently computable criterion to check if two codes are inequivalent.

We determined many exact and some bounds on the values of these sequences for known maximum rank distance code families, namely Gabidulin, twisted Gabidulin and generalized twisted Gabidulin codes. Based on these results we derived sufficient conditions on the parameters \( n \) (length), \( k \) (dimension) and \( m \) (field extension degree) of the codes, such that (generalized) twisted Gabidulin codes are inequivalent to Gabidulin codes, and such that certain generalized twisted Gabidulin codes are inequivalent to narrow sense twisted Gabidulin codes.

Furthermore, we used the invariants to derive upper and lower bounds on the number of inequivalent classical and (generalized) twisted Gabidulin codes, where for Gabidulin codes with \( m = n \) the bounds coincide and give the exact number of inequivalent Gabidulin codes. Finally, we gave a characterization of Gabidulin codes in terms of these sequences. We used this latter result to show that Gabidulin’s new codes (which correspond to generalized twisted Gabidulin codes with certain parameters) are in fact Gabidulin codes in the original sense. A similar characterization for twisted Gabidulin codes is not possible, since there is a counter example of an inequivalent code construction that has the same sum (or intersection) sequence as a twisted Gabidulin code. The question remains if it is possible to characterize twisted Gabidulin codes by its sum or intersection sequence together with another easily computable criterion.
Besides the main purpose of being an easily computable criterion to verify if a new code construction is inequivalent to other known code constructions, the results of this paper might also be of interest for code-based cryptography. Although not explicitly mentioned, the sum sequence of Gabidulin codes has frequently been used as a distinguisher from random linear codes, for attacking McEliece type of cryptosystems that use Gabidulin codes for the private key. On one hand, it is therefore a promising idea to use non-Gabidulin MRD codes for the design of new cryptosystems. On the other hand, the invariants of the code families treated in this work are all again quite different from the behavior of a random code, which raises the question if similar distinguisher attacks (as for Gabidulin codes) are possible for (generalized) twisted Gabidulin codes.

For the sake of simplicity we have formulated all main results in this paper for codes that are evaluated in a full rank vector \( g \in \mathbb{F}_{q^m}^n \). However, most of the results can easily be carried over to the case where \( g \) does not have full rank. Moreover, we have shown how the invariants can be used to show the inequivalence of certain generalized twisted Gabidulin codes to other code families. This can similarly be done for many more subfamilies of generalized twisted Gabidulin codes, and more generally for any other type of evaluation code based on linearized polynomials.

Lastly, we would like to state the open problem of generalizing Theorem 5.11, which gives the exact number of inequivalent twisted Gabidulin codes in the case \( n = m \), to the case \( n < m \). This requires an estimate on the cardinality of the set of generators \( \theta \) of \( \text{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q) \) for which a given code is \( \theta \)-twisted Gabidulin, which would then imply the number of inequivalent codes by Theorem 5.8.

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APPENDIX A. EXAMPLES

Tables 2 and 3 exemplify, for two different sets of code parameters, Proposition 4.2 (sequences for Gabidulin codes) and Proposition 4.5 (sequences for Twisted Gabidulin codes) from Section 4, and also show the actual sequences $s_i$ for comparison. The sequences were computed using SageMath v8.1 [65].

Table 2: Example illustrating Proposition 4.2 (sequences for Gabidulin codes) and Proposition 4.5 (sequences for Twisted Gabidulin codes).

| $r$ | Code | Case in Prop. 4.2 or Prop. 4.5 | $s_1^q(C), s_2^q(C), \ldots$ |
|-----|------|-------------------------------|--------------------------|
| 1 Gabidulin Actual Sequence | (4, 5, 6, 7, 8, 8, \ldots) |
| 1a: $r \leq k$ | (4, 5, 6, 7, 8, 8, \ldots) | $s_1 \geq -3$ |
| 1b: $r < \min\{m-k, m-n+k\}$ | (5, 6, 7, 8, 8, \ldots) | $s_1 \geq -3$ |
| Tw. Gab. Actual Sequence | (5, 6, 7, 8, 8, \ldots) |
| 1a: $r < k$ | (5, 6, 7, 8, 8, \ldots) | $s_1 \geq -3$ |
| 1b: $r < \min\{m-k+1, m-n+k\}$ | (5, 6, 7, 8, 8, \ldots) | $s_1 \geq -3$ |
| 2 Gabidulin Actual Sequence | (5, 7, 8, 8, \ldots) |
| 1a: $r \leq k$ | (5, 7, 8, 8, \ldots) | $s_1 \geq -2$ |
| 1b: $r < \min\{m-k, m-n+k\}$ | (6, 8, 8, \ldots) | $s_1 \geq -2$ |
| Tw. Gab. Actual Sequence | (6, 8, 8, \ldots) |
| 1a: $r < k$ | (6, 8, 8, \ldots) | $s_1 \geq -2$ |
| 3b: $r < \min\{m-k+1, m-n+k\}$ | (6, 8, 8, \ldots) | $s_1 \geq -2$ |
| Gabidulin | Actual Sequence | (6,8,8,...) |
|-----------|-----------------|-------------|
| 3         | 1a: $r \leq k$  | (6,8,8,...) |
|           | 3b: $r < \min\{m-k, n-k\}$ | $s_1 \geq -1$ |
| Tw. Gab.  | Actual Sequence | (6,8,8,...) |
|           | 2a: $k \leq r \leq n-k$ | $s_1 = 6$ |
|           | 3b: $r < \min\{m-k+1, m+n-k\}$ | $s_1 \geq 0$ |
| 4         | 2a: $k \leq r \leq n-k$ | $s_1 = 6$ |
|           | 3b: $r < \min\{m-k, m+n-k\}$ | $s_1 \geq 0$ |
| Tw. Gab.  | Actual Sequence | (6,8,8,...) |
|           | 2a: $k \leq r \leq n-k$ | $s_1 = 6$ |
|           | 3b: $r < \min\{m-k+1, m+n-k\}$ | $s_1 \geq 0$ |
| 5         | 2a: $k \leq r \leq n-k$ | $s_1 = 6$ |
|           | 3b: $r < \min\{m-k, m+n-k\}$ | $s_1 \geq 0$ |
| Tw. Gab.  | Actual Sequence | (6,8,8,...) |
|           | 3a: $\max\{k, n-k\} < r$ | $s_1 \geq 5$ |
|           | 3b: $r < \min\{m-k, m+n-k\}$ | $s_1 \geq 2$ |
| Tw. Gab.  | Actual Sequence | (6,8,8,...) |
|           | 3a: $\max\{k-1, n-k\} < r$ | $s_1 \geq 5$ |
|           | 3b: $r < \min\{m-k+1, m+n-k\}$ | $s_1 \geq 2$ |
| 7         | 3a: $\max\{k, n-k\} < r$ | $s_1 \geq 4$ |
|           | 3b: $r < \min\{m-k, m+n-k\}$ | $s_1 \geq 3$ |
| Tw. Gab.  | Actual Sequence | (6,8,8,...) |
|           | 3a: $\max\{k-1, n-k\} < r$ | $s_1 \geq 4$ |
|           | 3b: $r < \min\{m-k+1, m+n-k\}$ | $s_1 \geq 3$ |
| 8         | 3a: $\max\{k, n-k\} < r$ | $s_1 \geq 3$ |
|           | 3b: $r < \min\{m-k, m+n-k\}$ | $s_1 \geq 4$ |
| Tw. Gab.  | Actual Sequence | (6,8,8,...) |
|           | 3a: $\max\{k-1, n-k\} < r$ | $s_1 \geq 3$ |
|           | 3b: $r < \min\{m-k+1, m+n-k\}$ | $s_1 \geq 4$ |
| 9         | 3a: $\max\{k, n-k\} < r$ | $s_1 \geq 2$ |
|           | 3b: $r < \min\{m-k, m+n-k\}$ | $s_1 \geq 5$ |
| Tw. Gab.  | Actual Sequence | (6,8,8,...) |
|           | 3a: $\max\{k-1, n-k\} < r$ | $s_1 \geq 2$ |
|           | 3b: $r < \min\{m-k+1, m+n-k\}$ | $s_1 \geq 5$ |
| 10        | 3a: $\max\{k, n-k\} < r$ | $s_1 \geq 1$ |
|           | 2b: $m-n+k \leq r < m-k$ | $s_1 = 6$ |
| Tw. Gab.  | Actual Sequence | (6,8,8,...) |
|           | 3a: $\max\{k-1, n-k\} < r$ | $s_1 \geq 1$ |
|           | 2b: $m-n+k \leq r \leq m-k$ | $s_1 = 6$ |
| 11        | 3a: $\max\{k, n-k\} < r$ | $s_1 \geq 0$ |
|           | 2b: $m-n+k \leq r < m-k$ | $s_1 = 6$ |
| Tw. Gab.  | Actual Sequence | (6,8,8,...) |
|           | 3a: $\max\{k-1, n-k\} < r$ | $s_1 \geq 0$ |
|           | 2b: $m-n+k \leq r \leq m-k$ | $s_1 = 6$ |
| 12        | 3a: $\max\{k, n-k\} < r$ | $s_1 \geq -1$ |
|           | 1b: $m-k \leq r$ | (6,8,8,...) |
| Tw. Gab.  | Actual Sequence | (6,8,8,...) |
|           | 3a: $\max\{k-1, n-k\} < r$ | $s_1 \geq -1$ |
|           | 2b: $m-n+k \leq r \leq m-k$ | $s_1 = 6$ |
| 13        | 3a: $\max\{k, n-k\} < r$ | (5,7,8,8,...) |
### Table 3: Example illustrating Proposition 4.2 (sequences for Gabidulin codes) and Proposition 4.5 (sequences for Twisted Gabidulin codes). Code parameters: $q = 3$, $m = 23$, $n = 20$, $k = 9$. Evaluation points $g = (α^{18291492625}, α^{3015747946}, α^{61931009420}, α^{46672256788}, α^{48458087457}, α^{55265722774}, α^{75032150823}, α^{705986837}, α^{6759919186}, α^{27228306115}, α^{3139590947}, α^{609822449453}, α^{539311149991}, α^{67999350263}, α^{30419168464}, α^{84049490879}, α^{46827933410}, α^{6711480594}, α^{516852126798}, α^{31714555456}) ∈ \mathbb{F}_q^m$, and twist coefficient $η = α^{6706030966} ∈ \mathbb{F}_q^m$, where $α$ is a primitive element of $\mathbb{F}_q$ with $α^23 = 2α^3 + 2α + 2$.

| $r$ | Code | Case in Prop. 4.2 or Prop. 4.5 | $(s_1^r$ $(C), s_2^r$ $(C), ...)$ |
|-----|------|-------------------------------|---------------------------------|
| 1   | Gabidulin | Actual Sequence | (10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 20, ... ) |
|     |       | 1a: $r ≤ k$ | (10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 20, ... ) |
|     |       | 3b: $r < \min\{m - k, m - n + k\}$ | $s_1 ≥ 7$ |
|     | Tw. Gab. | Actual Sequence | (11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 20, ... ) |
|     |       | 1a: $r < k$ | (11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 20, ... ) |
|     |       | 3b: $r < \min\{m - k, m - n + k\}$ | $s_1 ≥ 7$ |
| 2   | Gabidulin | Actual Sequence | (11, 13, 15, 17, 19, 20, 20, ... ) |
|     |       | 1a: $r ≤ k$ | (11, 13, 15, 17, 19, 20, 20, ... ) |
|     |       | 3b: $r < \min\{m - k, m - n + k\}$ | $s_1 ≥ 8$ |
|     | Tw. Gab. | Actual Sequence | (12, 14, 16, 18, 20, 20, ... ) |
|     |       | 1a: $r < k$ | (12, 14, 16, 18, 20, 20, ... ) |
|     |       | 3b: $r < \min\{m - k + 1, m - n + k\}$ | $s_1 ≥ 8$ |
| 3   | Gabidulin | Actual Sequence | (12, 15, 18, 20, 20, ... ) |
|     |       | 1a: $r ≤ k$ | (12, 15, 18, 20, 20, ... ) |
|     |       | 3b: $r < \min\{m - k, m - n + k\}$ | $s_1 ≥ 9$ |
|     | Tw. Gab. | Actual Sequence | (13, 16, 19, 20, 20, ... ) |
|     |       | 1a: $r < k$ | (13, 16, 19, 20, 20, ... ) |
|     |       | 3b: $r < \min\{m - k + 1, m - n + k\}$ | $s_1 ≥ 9$ |
| 4   | Gabidulin | Actual Sequence | (13, 17, 20, 20, ... ) |
|     |       | 1a: $r ≤ k$ | (13, 17, 20, 20, ... ) |
|     |       | 3b: $r < \min\{m - k, m - n + k\}$ | $s_1 ≥ 10$ |
|     | Tw. Gab. | Actual Sequence | (14, 18, 20, 20, ... ) |
|     |       | 1a: $r < k$ | (14, 18, 20, 20, ... ) |
|     |       | 3b: $r < \min\{m - k + 1, m - n + k\}$ | $s_1 ≥ 10$ |
| 5   | Gabidulin | Actual Sequence | (14, 19, 20, 20, ... ) |
|     |       | 1a: $r ≤ k$ | (14, 19, 20, 20, ... ) |
|     |       | 3b: $r < \min\{m - k, m - n + k\}$ | $s_1 ≥ 11$ |
|     | Tw. Gab. | Actual Sequence | (15, 20, 20, ... ) |
|     |       | 1a: $r < k$ | (15, 20, 20, ... ) |
|     |       | 3b: $r < \min\{m - k + 1, m - n + k\}$ | $s_1 ≥ 11$ |
| 6   | Gabidulin | Actual Sequence | (15, 20, 20, ... ) |
|     |       | 1a: $r ≤ k$ | (15, 20, 20, ... ) |
|     |       | 3b: $r < \min\{m - k, m - n + k\}$ | $s_1 ≥ 12$ |
|     | Tw. Gab. | Actual Sequence | (16, 20, 20, ... ) |
|     |       | 1a: $r < k$ | (16, 20, 20, ... ) |
|     |       | 3b: $r < \min\{m - k + 1, m - n + k\}$ | $s_1 ≥ 12$ |
| 7   | Gabidulin | Actual Sequence | (16, 20, 20, ... ) |
|     |       | 1a: $r ≤ k$ | (16, 20, 20, ... ) |
| Gabidulin & Actual Sequence | Tw. Gab | Actual Sequence | 3a: $r < \min\{m - k, m - n + k\}$ | $s_1 \geq 13$ |
|-----------------------------|---------|-----------------|---------------------------------|-------------------|
| 8 Gabidulin & Actual Sequence | Tw. Gab | Actual Sequence | 3a: $r < k$ | $(17, 20, 20, \ldots)$ |
|                           | 3b: $r < \min\{m - k + 1, m - n + k\}$ | $s_1 \geq 13$ |
| 9 Gabidulin & Actual Sequence | Tw. Gab | Actual Sequence | 3a: $r < k$ | $(18, 20, 20, \ldots)$ |
|                           | 3b: $r < \min\{m - k + 1, m - n + k\}$ | $s_1 \geq 14$ |
| 10 Gabidulin & Actual Sequence | Tw. Gab | Actual Sequence | 2a: $k \leq r \leq n - k$ | $s_1 = 18$ |
|                           | 3b: $r < \min\{m - k, m - n + k\}$ | $s_1 \geq 15$ |
| 11 Gabidulin & Actual Sequence | Tw. Gab | Actual Sequence | 2a: $k \leq r \leq n - k$ | $(18, 19, 20, \ldots)$ |
|                           | 3b: $r < \min\{m - k, m - n + k\}$ | $s_1 \geq 17$ |
| 12 Gabidulin & Actual Sequence | Tw. Gab | Actual Sequence | 3a: $\max\{k, n - k\} < r$ | $(18, 19, 20, \ldots)$ |
|                           | 2b: $m - n + k \leq r < m - k$ | $s_1 \geq 17$ |
|                           | 3b: $r < \min\{m - k, m - n + k\}$ | $s_1 \geq 18$ |
| 13 Gabidulin & Actual Sequence | Tw. Gab | Actual Sequence | 3a: $\max\{k, n - k\} < r$ | $(18, 20, \ldots)$ |
|                           | 2b: $m - n + k \leq r < m - k$ | $s_1 \geq 16$ |
|                           | 3b: $r < \min\{m - k, m - n + k\}$ | $s_1 = 18$ |
| 14 Gabidulin & Actual Sequence | Tw. Gab | Actual Sequence | 3a: $\max\{k, n - k\} < r$ | $(18, 20, \ldots)$ |
|                           | 1b: $m - k \leq r$ | $s_1 \geq 15$ |
|                           | 2b: $m - n + k \leq r < m - k$ | $(18, 20, \ldots)$ |
| 15 Gabidulin & Actual Sequence | Tw. Gab | Actual Sequence | 3a: $\max\{k, n - k\} < r$ | $(17, 20, \ldots)$ |
|                           | 1b: $m - k \leq r$ | $s_1 \geq 14$ |
|                           | 2b: $m - n + k \leq r \leq m - k$ | $(18, 20, \ldots)$ |
| 16 Gabidulin & Actual Sequence | Tw. Gab | Actual Sequence | 3a: $\max\{k, n - k\} < r$ | $(16, 20, \ldots)$ |
|                           | 1b: $m - k \leq r$ | $s_1 \geq 13$ |
|                           | 2b: $m - n + k \leq r \leq m - k$ | $(17, 20, \ldots)$ |
| 17 Gabidulin & Actual Sequence | Tw. Gab | Actual Sequence | 3a: $\max\{k, n - k\} < r$ | $(15, 20, \ldots)$ |
|                           | 1b: $m - k \leq r$ | $s_1 \geq 12$ |
| Gabidulin | Actual Sequence | Tw. Gab. | Actual Sequence | Gabidulin | Actual Sequence | Tw. Gab. | Actual Sequence | Gabidulin | Actual Sequence | Tw. Gab. | Actual Sequence |
|-----------|-----------------|----------|-----------------|-----------|-----------------|----------|-----------------|-----------|-----------------|----------|-----------------|
| 18        | (14, 19, 20, 20, ...) | (15, 20, 20, ...) | (13, 17, 20, 20, ...) | (12, 15, 18, 20, 20, ...) | (11, 13, 15, 17, 19, 20, 20, ...) | (10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 20, ...) |
| 3a: $\max\{k-1, n-k\} < r$ | $s_1 \geq 12$ | $s_1 \geq 11$ | $s_1 \geq 10$ | $s_1 \geq 9$ | $s_1 \geq 8$ | $s_1 \geq 7$ |
| 1b: $m - k < r$ | (16, 20, 20, ...) | (14, 19, 20, 20, ...) | (13, 17, 20, 20, ...) | (12, 15, 18, 20, 20, ...) | (11, 13, 15, 17, 19, 20, 20, ...) | (10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 20, ...) |

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