The Deformation complex is a homotopy invariant of a homotopy algebra

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“Well, mathematician X likes to write formulas...”
Alexander Beilinson

Abstract

To a homotopy algebra one may associate its deformation complex, which is naturally a differential graded Lie algebra. We show that \(\infty\)-quasi-isomorphic homotopy algebras have \(L_\infty\)-quasi-isomorphic deformation complexes by an explicit construction.

1 Introduction

Given two homotopy algebras \(A, B\) of a certain type (e. g. \(L_\infty\) or \(A_\infty\) algebras), we may define their deformation complexes \(\text{Def}(A)\) and \(\text{Def}(B)\), which are differential graded Lie algebras. Suppose that \(A\) and \(B\) are quasi-isomorphic. For example, there may be an \(L_\infty\) or \(A_\infty\) quasi-isomorphism \(A \to B\). It is natural to ask whether in this case the deformation complexes \(\text{Def}(A)\) and \(\text{Def}(B)\) are quasi-isomorphic as \(L_\infty\) algebras, and whether a quasi-isomorphism may be written down in a (sufficiently) functorial way. The answer to the above question is (not surprisingly) yes, as is probably known to the experts. However, the authors were not able to find a proof of this statement in the literature in the desired generality.

The modest purpose of this note is fill in this gap by presenting the construction of an explicit sequence of quasi-isomorphisms connecting \(\text{Def}(A)\) with \(\text{Def}(B)\).

This note is organized as follows. After a brief description of our construction, we recall, in Section 2, the necessary prerequisites about homotopy algebras. Section 3 is the core of this paper. In this section, we formulate the main statement (see Theorem 3.1), describe various auxiliary constructions, and finally prove Theorem 3.1 in Subsection 3.4. Section 4 is devoted to the notion of homotopy algebra and its deformation complex in the setting of dg sheaves on a topological space. In this section, we give a version of Theorem 3.1 (see Corollary 4.10) and describe its application.

1.1 The construction in a nutshell

For the reader who already knows some homotopy algebra, here is what we will do in this note. First, the homotopy algebras of the type we consider are governed by some operad
P. For example, for $A_\infty$ algebras $P = \text{As}_\infty$ and for $L_\infty$ algebras $P = \text{Lie}_\infty$. Providing $P$ algebra structures on $A$ and $B$ is equivalent to providing operad maps $P \to \text{End}(A)$, $P \to \text{End}(B)$ into the endomorphism operads. The deformation complexes $\text{Def}(A)$, $\text{Def}(B)$ are by definition the deformation complexes of the operad maps $\text{Def}(A) = \text{Def}(P \to \text{End}(A))$, $\text{Def}(B) = \text{Def}(P \to \text{End}(B))$.

Similarly, one may define a two-colored operad $\text{Hom}_P$, whose algebras are triples $(A, B, F)$, where $A$ and $B$ are $P$ algebras and $F$ is a homotopy $(\infty)$-morphism between them. Furthermore, given an $\infty$ quasi-isomorphism $A \sim B$, we may build a colored operad map $\text{Hom}_P \to \text{End}(A, B)$ into the colored endomorphism operad. One may build a deformation complex $\text{Def}(\text{Hom}_P \to \text{End}(A, B))$, which is an $L_\infty$ algebra. Furthermore, there are natural maps

$$\text{Def}(A) \leftarrow \text{Def}(\text{Hom}_P \to \text{End}(A, B)) \rightarrow \text{Def}(B)$$

which one may check to be quasi-isomorphisms. Hence this zigzag constitutes desired explicit and natural quasi-isomorphisms of $L_\infty$ algebras.

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2 Preliminaries

The base field $\mathbb{K}$ has characteristic zero. The underlying symmetric monoidal category is the category of unbounded cochain complexes of $\mathbb{K}$-vector spaces. We will use the notation and conventions about labeled planar trees from [5]. In particular, we denote by $\text{Tree}(n)$ the groupoid of $n$-labeled planar trees. As in [5], we denote by $\text{Tree}_2(n)$ the full subcategory of $\text{Tree}(n)$ whose objects are $n$-labeled planar trees with exactly 2 nodal vertices. For a groupoid $G$, the notation $\pi_0(G)$ is reserved for the set of isomorphism classes of objects in $G$.

We say that an $n$-labeled planar tree $t$ is a pitchfork if each leaf of $t$ has height\footnote{Recall that the height of a vertex $v$ is the length of the (unique) path which connects $v$ to the root vertex.} 3. Figure 2.1 shows a pitchfork while figure 2.2 shows a tree that is not a pitchfork.

Fig. 2.1: An example of a pitchfork

Fig. 2.2: This is not a pitchfork

The notation $\text{Tree}_d(n)$ is reserved for the full sub-groupoid of $\text{Tree}(n)$ whose objects are pitchforks.

Let $C$ be a coaugmented dg cooperad satisfying the following technical condition:

**Condition 2.1** The cokernel $C_\circ$ of the coaugmentation carries an ascending exhaustive filtration

$$0 = F^0C_\circ \subset F^1C_\circ \subset F^2C_\circ \subset \ldots$$

(2.1)
which is compatible with the pseudo-cooperad structure on $C_\circ$.

For example, if the dg cooperad $C$ has the properties
\[ C(1) \cong \mathbb{K}, \quad C(0) = 0 \]  
then the filtration “by arity minus one” on $C_\circ$ satisfies the above technical condition.

For a cochain complex $\mathcal{V}$ we denote by
\[ C(\mathcal{V}) := \bigoplus_{n \geq 1} \left( C(n) \otimes \mathcal{V} \otimes^n \right)_{S_n} \]  
the “cofree” $C$-coalgebra co-generated by $\mathcal{V}$.

We denote by
\[ \text{coDer}(C(\mathcal{V})) \]  
the cochain complex of coderivations of the cofree coalgebra $C(\mathcal{V})$ co-generated by $\mathcal{V}$. In other words, $\text{coDer}(C(\mathcal{V}))$ consists of $\mathbb{K}$-linear maps
\[ D : C(\mathcal{V}) \to C(\mathcal{V}) \]  
which are compatible with the $C$-coalgebra structure on $C(\mathcal{V})$ in the following sense:
\[ \Delta_n \circ D = \sum_{i=1}^{n} \left( \text{id}_C \otimes \text{id}^{(i-1)}_{\mathcal{V}} \otimes D \otimes \text{id}^{n-i}_{\mathcal{V}} \right) \circ \Delta_n \]  
where $\Delta_n$ is the comultiplication map
\[ \Delta_n : C(\mathcal{V}) \to \left( C(n) \otimes (C(\mathcal{V}))^{\otimes n} \right)_{S_n} . \]

The $\mathbb{Z}$-graded vector space (2.4) carries a natural differential $\partial$ induced by those on $C$ and $\mathcal{V}$.

Since the commutator of two coderivations is again a coderivation, the cochain complex (2.4) is naturally a dg Lie algebra.

Recall that, since the $C$-coalgebra $C(\mathcal{V})$ is cofree, every coderivation $D : C(\mathcal{V}) \to C(\mathcal{V})$ is uniquely determined by its composition $p_\mathcal{V} \circ D$ with the canonical projection:
\[ p_\mathcal{V} : C(\mathcal{V}) \to \mathcal{V} . \]  

We denote by
\[ \text{coDer}'(C(\mathcal{V})) \]  
the dg Lie subalgebra of coderivations $D \in \text{coDer}(C(\mathcal{V}))$ satisfying the additional technical condition
\[ D \big|_{\mathcal{V}} = 0 . \]  

Due to [5, Proposition 4.2], the map
\[ D \mapsto p_\mathcal{V} \circ D \]  
induces an isomorphism of dg Lie algebras
\[ \text{coDer}'(C(\mathcal{V})) \cong \text{Conv}(C_\circ, \text{End}_\mathcal{V}) , \]  

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where the differential $\partial$ on $\text{Conv}(C, \text{End}_V)$ comes solely from the differential on $C$ and $V$.

Recall that [5, Proposition 5.2] Cobar$(C)$-algebra structures on a cochain complex $V$ are in bijection with degree 1 coderivations $Q \in \text{coDer}'(C(V))$ (2.11) satisfying the Maurer-Cartan equation $\partial Q + \frac{1}{2}[Q, Q] = 0$. (2.12)

Hence, given a Cobar$(C)$-algebra structure on $V$, we may consider the dg Lie algebra (2.10) and the $C$-coalgebra $C(V)$ with the new differentials $\partial + [Q, ]$, (2.13) and $\partial + Q$, (2.14) respectively.

In this text we use the following “pedestrian” definition of homotopy algebras:

**Definition 2.2** Let $C$ be a coaugmented dg cooperad satisfying Condition 2.1. A homotopy algebra of type $C$ is a Cobar$(C)$-coalgebra $V$.

Using the above link between Cobar$(C)$-algebra structures on $V$ and MC elements $Q$ of coDer$(C(V))$, we see that every homotopy algebra $V$ of type $C$ gives us a dg $C$-coalgebra $C(V)$ with the differential $\partial + Q$. This observation motivates our definition of an $\infty$-morphism between homotopy algebras:

**Definition 2.3** Let $A$, $B$ be homotopy algebras of type $C$ and let $Q_A$ (resp. $Q_B$) be the MC element of coDer$(C(A))$ (resp. coDer$(C(B))$) corresponding to the Cobar$(C)$-algebra structure on $A$ (resp. $B$). Then an $\infty$-morphism from $A$ to $B$ is a homomorphism $F : C(A) \rightarrow C(B)$ of the dg $C$-coalgebras $C(A)$ and $C(B)$ with the differentials $\partial + Q_A$ and $\partial + Q_B$, respectively.

A homomorphism of dg $C$-coalgebras $F$ is called an $\infty$ quasi-isomorphism if the composition

$$A \hookrightarrow C(A) \xrightarrow{F} C(B) \xrightarrow{p_B} B$$

is a quasi-isomorphism of cochain complexes.

We say that two homotopy algebras $A$ and $B$ are quasi-isomorphic if there exists a sequence of $\infty$ quasi-isomorphisms connecting $A$ with $B$.

**Definition 2.4** Let $A$ be a homotopy algebra of type $C$ and $Q$ be the corresponding MC element of coDer$(C(A))$. Then the cochain complex $\text{Def}(A) := \text{coDer}'(C(A))$ (2.15) with the differential $\partial + [Q, ]$ is called the deformation complex of the homotopy algebra $A$.
3 The main statement

We observe that the deformation complex (2.15) of a homotopy algebra $\mathcal{A}$ is naturally a dg Lie algebra. We claim that

**Theorem 3.1** Let $C$ be a coaugmented dg cooperad satisfying Condition 2.1. If $\mathcal{A}$ and $\mathcal{B}$ are quasi-isomorphic homotopy algebras of type $C$ then the deformation complex $\text{Def}(\mathcal{A})$ of $\mathcal{A}$ is connected to the deformation $\text{Def}(\mathcal{B})$ of $\mathcal{B}$ via a sequence of $L_\infty$-quasi-isomorphisms.

**Remark 3.2** For $A_\infty$-algebras this statement follows from the result [9] of B. Keller.

It is clearly sufficient to prove this theorem in the case when $\mathcal{A}$ and $\mathcal{B}$ are connected by a single $\infty$ quasi-isomorphism $F : \mathcal{A} \sim \mathcal{B}$.

We will prove the Theorem by constructing an $L_\infty$ algebra $\text{Def}(\mathcal{A} \overset{F}{\sim} \mathcal{B})$, together with quasi-isomorphisms

$$\text{Def}(\mathcal{A}) \leftarrow \text{Def}(\mathcal{A} \overset{F}{\sim} \mathcal{B}) \rightarrow \text{Def}(\mathcal{B}).$$

The next subsections are concerned with the definition of $\text{Def}(\mathcal{A} \overset{F}{\sim} \mathcal{B})$. The proof of Theorem 3.1 is given in Section 3.4 below.

3.1 The auxiliary $L_\infty$-algebra $\text{Cyl}(\mathcal{C}, \mathcal{A}, \mathcal{B})$

Let $\mathcal{A}, \mathcal{B}$ be cochain complexes. We consider the graded vector space

$$\text{Cyl}(\mathcal{C}, \mathcal{A}, \mathcal{B}) := \text{Hom}(\mathcal{C}_0(\mathcal{A}), \mathcal{A}) \oplus \text{sHom}(\mathcal{C}(\mathcal{A}), \mathcal{B}) \oplus \text{Hom}(\mathcal{C}_0(\mathcal{B}), \mathcal{B})$$

with the differential coming from those on $\mathcal{C}, \mathcal{A}$ and $\mathcal{B}$.

We equip the cochain complex $\text{Cyl}(\mathcal{C}, \mathcal{A}, \mathcal{B})$ with an $L_\infty$-structure by declaring that

$$\{s^{-1}P_1, s^{-1}P_2, \ldots, s^{-1}P_n\}_n := \begin{cases} (-1)^{|P_1|+1}s^{-1}[P_1, P_2] & \text{if } n = 2, \\ 0 & \text{otherwise}. \end{cases} \quad (3.2)$$

$$\{s^{-1}R_1, s^{-1}R_2, \ldots, s^{-1}R_n\}_n := \begin{cases} (-1)^{|R_1|+1}s^{-1}[R_1, R_2] & \text{if } n = 2, \\ 0 & \text{otherwise}. \end{cases} \quad (3.3)$$

for $P_i \in \text{Hom}(\mathcal{C}_0(\mathcal{A}), \mathcal{A}) \cong \text{Conv}(\mathcal{C}_0, \text{End}_A)$, and $R_i \in \text{Hom}(\mathcal{C}_0(\mathcal{B}), \mathcal{B}) \cong \text{Conv}(\mathcal{C}_0, \text{End}_B)$, and $[\cdot , \cdot]$ is the Lie bracket on the convolution algebras $\text{Conv}(\mathcal{C}_0, \text{End}_A)$ and $\text{Conv}(\mathcal{C}_0, \text{End}_B)$, respectively.

Furthermore,

$$\{T, s^{-1}P\}_2(\mathcal{X}, a_1, \ldots, a_n) = \sum_{0 \leq p \leq n} \sum_{\sigma \in \text{Sh}_{p,n-p}} (-1)^{|T|+|P|(|X_{\sigma,i}|+1)}T(\mathcal{X}_{\sigma,i}, P(X_{\sigma,i}'; a_{\sigma(1)}, \ldots, a_{\sigma(p)}); a_{\sigma(p+1)}, \ldots, a_{\sigma(n)}), \quad (3.4)$$

where $T \in \text{Hom}(\mathcal{C}(\mathcal{A}), \mathcal{B}), P \in \text{Hom}(\mathcal{C}_0(\mathcal{A}), \mathcal{A}), \mathcal{X} \in \mathcal{C}_n(\mathcal{A}), X_{\sigma,i}', X_{\sigma,i}''$ are tensor factors in

$$\Delta_{t_\sigma}(\mathcal{X}) = \sum_i X_{\sigma,i}' \otimes X_{\sigma,i}''.$$
$P$ is extended by zero to $\mathcal{A} \subset C(\mathcal{A})$, and $t_\sigma$ is the $n$-labeled planar tree depicted on figure 3.1.

To define yet another collection of non-zero $L_\infty$-brackets, we denote by $\text{Isom}_p(m, r)$ the set of isomorphism classes of pitchforks $t \in \text{Tree}_p(m)$ with $r$ nodal vertices of height 2. For every $z \in \text{Isom}_p(m, r)$ we choose a representative $t_z$ and denote by $X_{z, i}$, the tensor factors in

$$\Delta_{t_z}(X) = \sum_i X_{z, i}^0 \otimes X_{z, i}^1 \otimes \cdots \otimes X_{z, i}^r,$$  \hspace{1cm} (3.5)

where $X \in C(m)$.

Finally, for vectors $T_j \in \text{Hom}(C(\mathcal{A}), \mathcal{B})$ and $R \in \text{Hom}(C_0(\mathcal{B}), \mathcal{B})$ we set

$$\{s^{-1}R, T_1, \ldots, T_r\}_{r+1}(X, a_1, \ldots, a_m) =$$

$$\sum_{\sigma \in S_r} \sum_{z \in \text{Isom}_p(m, r)} \sum_i \pm (-1)^{|\sigma|+1} R(X_{z, i}^0, T_{\sigma(1)}(X_{z, i}^1; a_{\lambda_z(1)}), \ldots, a_{\lambda_z(n_1^z)}),$$

$$T_{\sigma(2)}(X_{z, i}^2; a_{\lambda_z(n_1^z+1)}, \ldots, a_{\lambda_z(n_1^z+n_2^z)}), \ldots, T_{\sigma(r)}(X_{z, i}^r; a_{\lambda_z(m-n_2^z+1)}, \ldots, a_{\lambda_z(m)}),$$  \hspace{1cm} (3.6)

where $n_z^2$ is the number of leaves adjacent to the $(q+1)$-th nodal vertex of $t_z$, $\lambda_z(l)$ is the label of the $l$-th leaf of $t_z$, the map $R$ is extended by zero to $\mathcal{B} \subset C(\mathcal{B})$ and the sign factor $\pm$ comes from the rearrangement of the homogeneous vectors

$$R, T_1, \ldots, T_r, X_{z, i}^0, X_{z, i}^1, \ldots, X_{z, i}^r, a_1, \ldots, a_m$$  \hspace{1cm} (3.7)

from their original positions in (3.7) to their positions in the right hand side of (3.6).

We observe that, due to axioms of a cooperad, the right hand side of (3.6) does not depend on the choice of representatives $t_z \in \text{Tree}_p(m)$.

The remaining $L_\infty$-brackets are either extended in the obvious way by symmetry or declared to be zero.

We claim that

**Claim 3.3** *The operations*

$$\{ , \ldots, \} : S^n(s^{-1}Cyl(C(\mathcal{A}), \mathcal{B})) \to s^{-1}Cyl(C(\mathcal{A}), \mathcal{B}), \hspace{1cm} n \geq 2$$  \hspace{1cm} (3.8)

*defined above have degree 1 and satisfy the desired $L_\infty$ identities:*

$$\partial\{f_1, f_2, \ldots, f_n\} + \sum_{i=1}^{n} (-1)^{|f_i|+\cdots+|f_{i-1}|+1} \{f_1, \ldots, f_{i-1}, \partial(f_i), f_{i+1}, \ldots, f_n\} =$$

$$\sum_{p=2}^{n-1} \sum_{\sigma \in \text{Sh}_p, n-p} \pm \{f_{\sigma(1)}, f_{\sigma(2)}, \ldots, f_{\sigma(p)}\} p, f_{\sigma(p+1)}, \ldots, f_{\sigma(n)}\},$$  \hspace{1cm} (3.9)
where \( f_j \in s^{-1} \text{Cyl}(C, \mathcal{A}, \mathcal{B}) \) and the usual Koszul rule of signs is applied.

Before proving Claim 3.3, we would like to show that

**Claim 3.4** The MC equation for the \( L_\infty \)-algebra \( \text{Cyl}(C, \mathcal{A}, \mathcal{B}) \) is well defined. Moreover, MC elements of the \( L_\infty \)-algebra \( \text{Cyl}(C, \mathcal{A}, \mathcal{B}) \) are triples:

- a Cobar\((C)\)-algebra structure on \( \mathcal{A} \),
- a Cobar\((C)\)-algebra structure on \( \mathcal{B} \), and
- an \( \infty \)-morphism from \( \mathcal{A} \) to \( \mathcal{B} \).

**Proof.** Let \( U \) be a degree 1 element in \( \text{Cyl}(C, \mathcal{A}, \mathcal{B}) \).

We observe that the components of \( \{s^{-1} U, s^{-1} U, \ldots, s^{-1} U\}_n \) in \( \text{Hom}(C_{o}(\mathcal{A}), \mathcal{A}) \) and \( \text{Hom}(C_{o}(\mathcal{B}), \mathcal{B}) \) are zero for all \( n \geq 3 \). Furthermore, for every \( (X, a_1, \ldots, a_k) \in C(\mathcal{A}) \)

\[
\{s^{-1} U, s^{-1} U, \ldots, s^{-1} U\}_n(X, a_1, \ldots, a_k) = 0 \quad \forall \ n \geq k + 1.
\]

Therefore the infinite sum

\[
[\partial, U] + \sum_{n=2}^{\infty} \frac{1}{n!} \{s^{-1} U, s^{-1} U, \ldots, s^{-1} U\}_n
\]

makes sense for every degree 1 element \( U \) in \( \text{Cyl}(C, \mathcal{A}, \mathcal{B}) \) and we can talk about MC elements of \( \text{Cyl}(C, \mathcal{A}, \mathcal{B}) \).

To prove the second statement, we split the degree 1 element \( U \in \text{Cyl}(C, \mathcal{A}, \mathcal{B}) \) into a sum

\[
U = Q_A + s U_F + Q_B,
\]

where \( Q_A \in \text{Conv}(C_0, \text{End}_{\mathcal{A}}) \), \( Q_B \in \text{Conv}(C_0, \text{End}_{\mathcal{B}}) \), and \( U_F \in \text{Hom}(C(\mathcal{A}), \mathcal{B}) \).

Then the MC equation for \( U \) is equivalent to the following three equations:

\[
\partial Q_A + \frac{1}{2} [Q_A, Q_A] = 0 \quad \text{in} \quad \text{Conv}(C_0, \text{End}_{\mathcal{A}}),
\]

(3.11)

\[
\partial Q_B + \frac{1}{2} [Q_B, Q_B] = 0 \quad \text{in} \quad \text{Conv}(C_0, \text{End}_{\mathcal{B}}),
\]

(3.12)

and

\[
[\partial, U_F] + \{U_F, s^{-1} Q_A\}_2 + \sum_{r=1}^{\infty} \frac{1}{r!} \{s^{-1} Q_B, U_F, U_F, \ldots, U_F\}_{r+1} = 0.
\]

(3.13)

Equations (3.11) and (3.12) imply that \( Q_A \) (resp. \( Q_B \)) gives us a Cobar\((C)\)-algebra structure on \( \mathcal{A} \) (resp. \( \mathcal{B} \)). Furthermore, equation (3.13) means that \( U_F \) is an \( \infty \)-morphism from \( \mathcal{A} \) to \( \mathcal{B} \). \( \square \)
3.1.1 Proof of Claim 3.3

The most involved identity on $L_\infty$-brackets defined above is

$$\{\{s^{-1} R_1, s^{-1} R_2\}_2, T_1, \ldots, T_n\}_{n+1}^+ + \sum_{1 \leq p \leq n-1} \pm \{s^{-1} R_1, \{s^{-1} R_2, T_{\sigma(1)}, \ldots, T_{\sigma(p)}\}_{p+1}, T_{\sigma(p+1)}, \ldots, T_{\sigma(n)}\}_{n-p+2}^+ + \sum_{1 \leq p \leq n-1} \pm \{s^{-1} R_2, \{s^{-1} R_1, T_{\sigma(1)}, \ldots, T_{\sigma(p)}\}_{p+1}, T_{\sigma(p+1)}, \ldots, T_{\sigma(n)}\}_{n-p+2} = 0. \quad (3.14)$$

This identity is a consequence of a combinatorial fact about certain isomorphism classes in the groupoid $\text{Tree}(n)$. To formulate this fact, we recall that the set of isomorphism classes of $r$-labeled planar trees with two nodal vertices are in bijection with the set of shuffles

$$\bigsqcup_{p=0}^{r} \text{Sh}_{p,r-p}. \quad (3.15)$$

This bijection assigns to a shuffle $\sigma \in \text{Sh}_{p,r-p}$ the $r$-labeled planar tree $t_\sigma$ shown on figure 3.2.

![Fig. 3.2: Here $\sigma$ is a $(p, r-p)$-shuffle](image)

Next, we observe that $\pi_0(\text{Tree}_n(n))$ is in bijection with the set

$$\bigsqcup_{r \geq 1} \mathcal{S}_n, \quad (3.16)$$

where

$$\mathcal{S}_n = \bigsqcup_{1 \leq q_1 < q_2 < \cdots < q_{r-1} < q_r = n} \{\tau \in \text{Sh}_{q_1, q_2-1, \ldots, n-q_{r-1}} \mid \tau(1) < \tau(q_1+1) < \tau(q_2+1) < \cdots < \tau(q_{r-1}+1)\}. \quad (3.17)$$

This bijection assigns to a shuffle $\tau$ in the set (3.16) the isomorphism class of the pitchfork $t^{\tau}_r$ depicted on figure 3.3.

Note that, in the degenerate cases $r = 1$ and $r = n$, $\mathcal{S}_n$ is the one-element set consisting of the identity permutation $\text{id} \in S_n$. The corresponding pitchforks are shown on figures 3.4 and 3.5, respectively.

For every permutation $\tau \in \mathcal{S}_n$ and a shuffle $\sigma \in \text{Sh}_{p,r-p}$ we can form the following $n$-labeled planar tree

$$t^{\tau}_r \bullet_1 t_\sigma, \quad (3.18)$$

\[2\] It is obvious that, for every $\tau \in \mathcal{S}_n$, $\tau(1) = 1$. 

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where $t \otimes_j t'$ denotes the insertion of the tree $t'$ into the $j$-th nodal vertex of the tree $t$ (see Section 2.2 in [5]).

It is clear that, for distinct pairs $(\tau, \sigma) \in \mathcal{SF}_{n,r} \times \text{Sh}_{p,r-p}$, we get mutually non-isomorphic labeled planar trees.

Let $\tau' \in \mathcal{SF}_{n,r'}$ and $m_i$ be the number of edges which terminate at the $(i+1)$-th nodal vertex of $t' \otimes_j t''$. For every $\tau'' \in \mathcal{SF}_{m_i,r''}$, we may form the $n$-labeled planar tree

$$t' \otimes_j t'' . \quad (3.19)$$

It is clear that, for distinct triples $(\tau', i, \tau'') \in \mathcal{SF}_{n,r'} \times \{1, 2, \ldots, r'\} \times \mathcal{SF}_{m_i,r''}$, the corresponding labeled planar trees (3.19) are mutually non-isomorphic. Furthermore, every tree of the form (3.19) is isomorphic to exactly one tree of the form (3.18) and vice versa. This is precisely the combinatorial fact that is need to prove that identity (3.14) holds.

Indeed, the terms in the expression

$$\pm \{s^{-1} R_1, s^{-1} R_2\}_2, T_1, \ldots, T_n\}_{n+1}$$

involve trees of the form (3.18) and the terms in the expressions

$$\sum_{1 \leq p \leq n-1} \pm \{s^{-1} R_1, s^{-1} R_2, T_{\sigma(1)}, \ldots, T_{\sigma(p)}\}_{p+1}, T_{\sigma(p+1)}, \ldots, T_{\sigma(n)}\}_{n-p+2}$$

and

$$\sum_{1 \leq p \leq n-1} \pm \{s^{-1} R_2, s^{-1} R_1, T_{\sigma(1)}, \ldots, T_{\sigma(p)}\}_{p+1}, T_{\sigma(p+1)}, \ldots, T_{\sigma(n)}\}_{n-p+2}$$

involve trees of the form (3.19).

Thus it only remains to check that the sign factors match.
The remaining identities on $L_\infty$-brackets are simpler and we leave their verification to the reader.

Claim 3.3 is proved. □

3.2 The $L_\infty$-algebra $\text{Cyl}(C, \mathcal{A}, \mathcal{B})^{sF_1}$ and its MC elements

Let

$$F_1 : \mathcal{A} \to \mathcal{B}$$

be a map of cochain complexes.

We may view $sF_1$ as a degree 1 element in $\text{Cyl}(C, \mathcal{A}, \mathcal{B})$:

$$sF_1 \in s\text{Hom}(\mathcal{A}, \mathcal{B}) \subset s\text{Hom}(C(\mathcal{A}), \mathcal{B}) \subset \text{Cyl}(C, \mathcal{A}, \mathcal{B}).$$

Since $F_1$ is compatible with the differentials on $\mathcal{A}$ and $\mathcal{B}$, $sF_1$ is obviously a MC element of $\text{Cyl}(C, \mathcal{A}, \mathcal{B})$ and, in view of Claim 3.4, $sF_1$ corresponds to the triple:

- the trivial Cobar($C$)-algebra structure on $\mathcal{A}$,
- the trivial Cobar($C$)-algebra structure on $\mathcal{B}$, and
- a strict$^3$ $\infty$-morphism $F_1$ from $\mathcal{A}$ to $\mathcal{B}$.

Let $Q_1, Q_2, \ldots, Q_m$ be vectors in $\text{Cyl}(C, \mathcal{A}, \mathcal{B})$. We recall that the components of

$$\{F_1, F_1, \ldots, F_1, s^{-1}Q_1, s^{-1}Q_2, \ldots, s^{-1}Q_m\}_{n+m}$$

in $\text{Hom}(C_\circ(\mathcal{A}), \mathcal{A})$ and $\text{Hom}(C_\circ(\mathcal{B}), \mathcal{B})$ are zero if $n + m > 2$. Furthermore, for every $(X, a_1, \ldots, a_k) \in C(\mathcal{A})$

$$\{F_1, F_1, \ldots, F_1, s^{-1}Q_1, s^{-1}Q_2, \ldots, s^{-1}Q_m\}_{n+m}(X, a_1, \ldots, a_k) = 0 \in \mathcal{B}$$

provided $n + m \geq k + 2$.

Therefore we may twist (see [7, Remark 3.11.]) the $L_\infty$ algebra on $\text{Cyl}(C, \mathcal{A}, \mathcal{B})$ by the MC element $sF_1$. We denote by

$$\text{Cyl}(C, \mathcal{A}, \mathcal{B})^{sF_1}$$

the $L_\infty$-algebra obtained in this way.

It is not hard to see that$^4$

$$\text{Cyl}_\circ(C, \mathcal{A}, \mathcal{B})^{sF_1} := \text{Hom}(C_\circ(\mathcal{A}), \mathcal{A}) \oplus s\text{Hom}(C_\circ(\mathcal{A}), \mathcal{B}) \oplus \text{Hom}(C_\circ(\mathcal{B}), \mathcal{B})$$

(3.22)

is an $L_\infty$-subalgebra of $\text{Cyl}(C, \mathcal{A}, \mathcal{B})^{sF_1}$. Furthermore, Claim 3.4 implies that

Claim 3.5 MC elements of the $L_\infty$-algebra (3.22) are triples:

- $A$ Cobar($C$)-algebra structure on $\mathcal{A}$,
- $A$ Cobar($C$)-algebra structure on $\mathcal{B}$,
- $A$ strict $\infty$-morphism $F_1$ from $\mathcal{A}$ to $\mathcal{B}$.

$^3$ i.e. an $\infty$-morphism $F : \mathcal{A} \rightsquigarrow \mathcal{B}$ whose all higher structure maps are zero.

$^4$ In $\text{Cyl}_\circ(C, \mathcal{A}, \mathcal{B})^{sF_1}$, we have $s\text{Hom}(C_\circ(\mathcal{A}), \mathcal{B})$ instead of $s\text{Hom}(C(\mathcal{A}), \mathcal{B})$. 

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• an $\infty$-morphism $F : A \sim B$ for which the composition

$$A \hookrightarrow C(A) \xrightarrow{F} C(B) \xrightarrow{p_B} B$$

coincides with $F_1$. 

\[\square\]

**Remark 3.6** Using the ascending filtration (2.1) on the pseudo-operad $C_o$, we equip the $L_\infty$-algebra $\text{Cyl}(C, A, B)$ with the complete descending filtrations:

$$\text{Cyl}(C, A, B) = \mathcal{F}_0 \text{Cyl}(C, A, B) \supset \mathcal{F}_1 \text{Cyl}(C, A, B) \supset \mathcal{F}_2 \text{Cyl}(C, A, B) \supset \ldots ,$$  

(3.23)

where (for $m \geq 1$)

$$\mathcal{F}_m \text{Cyl}(C, A, B) := \{ Q' \oplus F \oplus Q'' \in \text{Hom}(C_o(A), A) \oplus \text{sHom}(C(A), B) \oplus \text{Hom}(C_o(B), B) \}$$  

(3.24)

$$Q'(X, a_1, \ldots, a_k) = 0, \quad F(X, a_1, \ldots, a_k) = 0, \quad Q''(X, b_1, \ldots, b_k) = 0 \quad \forall X \in \mathcal{F}^{m-1}C(k) \}.$$

The same formulas define a complete descending filtration on the $L_\infty$-algebras $\text{Cyl}(C, A, B)^{sF_1}$ and $\text{Cyl}_o(C, A, B)^{sF_1}$.

We observe that

$$\text{Cyl}_o(C, A, B)^{sF_1} = \mathcal{F}_1 \text{Cyl}_o(C, A, B)^{sF_1}$$  

(3.25)

and hence $\text{Cyl}_o(C, A, B)^{sF_1}$ is pro-nilpotent. Later, we will use this advantage of $\text{Cyl}_o(C, A, B)^{sF_1}$ over $\text{Cyl}(C, A, B)^{sF_1}$.

### 3.3 What if $F_1$ is a quasi-isomorphism?

Starting with a chain map (3.20) we define two maps of cochain complexes:

$$P \mapsto f(P) = F_1 \circ P : \text{Hom}(C_o(A), A) \to \text{Hom}(C_o(A), B)$$  

(3.26)

$$R \mapsto \tilde{f}(R) = R \circ C_o(F_1) : \text{Hom}(C_o(B), B) \to \text{Hom}(C_o(A), B)$$  

(3.27)

and observe that, as the cochain complex, $\text{Cyl}_o(C, A, B)^{sF_1}$ is precisely the cochain complex $\text{Cyl}(f, \tilde{f})$ defined in (A.2), (A.3) of Appendix A.

Hence, using Lemma A.1, we deduce the following statement:

**Proposition 3.7** If the chain map $F_1 : A \to B$ induces an isomorphism on the level of cohomology then so do the following canonical projections:

$$\pi_A : \text{Cyl}_o(C, A, B)^{sF_1} \to \text{Hom}(C_o(A), A),$$  

(3.28)

$$\pi_B : \text{Cyl}_o(C, A, B)^{sF_1} \to \text{Hom}(C_o(B), B).$$  

(3.29)

The maps $\pi_A$ and $\pi_B$ are strict homomorphisms of $L_\infty$-algebras.

Proof. Since we work over a field of characteristic zero, the functors $\text{Hom}$, $\otimes$, as well as the functors of taking (co)invariants with respect to actions of symmetric groups preserve quasi-isomorphisms. Therefore the maps (3.26) and (3.27) are quasi-isomorphisms of cochain complexes.

Thus the first statement follows directly from Lemma A.1.

The second statement is an obvious consequence of the definition of $L_\infty$-brackets on $\text{Cyl}_o(C, A, B)^{sF_1}$. \[\square\]
3.4 Proof of Theorem 3.1

We will now give a proof of Theorem 3.1. Let $A$ and $B$ be homotopy algebras of type $C$. As said above, we may assume, without loss of generality, that $A$ and $B$ are connected by a single $\infty$ quasi-isomorphism:

$$F : A \rightsquigarrow B.$$  \hfill (3.30)

We denote by $\alpha_{\text{Cyl}}$ the MC element of $\text{Cyl}_0(C, A, B)^{sF_1}$ which corresponds to the triple

- the homotopy algebra structure on $A$,
- the homotopy algebra structure on $B$, and
- the $\infty$-morphism $F$.

Due to (3.25), we may twist (see [7, Remark 3.11.]) the $L_\infty$-algebra $\text{Cyl}_0(C, A, B)^{sF_1}$ by the MC element $\alpha_{\text{Cyl}}$. We denote by

$$\text{Def}(A \xrightarrow{F} B)$$  \hfill (3.31)

the $L_\infty$-algebra which is obtained from $\text{Cyl}_0(C, A, B)^{sF_1}$ via twisting by the MC element $\alpha_{\text{Cyl}}$.

We also denote by $Q_A$ (resp. $Q_B$) the MC element of $\text{Conv}(C, \text{End}_A)$ (resp. $\text{Conv}(C, \text{End}_B)$) corresponding to the homotopy algebra structure on $A$ (resp. $B$) and recall that $\text{Def}(A)$ (resp. $\text{Def}(B)$) is obtained from $\text{Conv}(C, \text{End}_A)$ (resp. $\text{Conv}(C, \text{End}_B)$) via twisting by the MC element $Q_A$ (resp. $Q_B$).

It is easy to see that

$$\pi_A(\alpha_{\text{Cyl}}) = Q_A, \quad \pi_B(\alpha_{\text{Cyl}}) = Q_B.$$  \hfill (3.32)

Since $\pi_A$ (3.28) and $\pi_B$ (3.29) are strict $L_\infty$-morphisms, they do not change under twisting by MC elements. Thus, we conclude that, the same maps $\pi_A$ and $\pi_B$ give us (strict) $L_\infty$-morphisms

$$\pi_A : \text{Def}(A \xrightarrow{F} B) \rightarrow \text{Def}(A),$$

$$\pi_B : \text{Def}(A \xrightarrow{F} B) \rightarrow \text{Def}(B).$$  \hfill (3.33)

According to [7, Proposition 6.2], twisting preserves quasi-isomorphisms. Thus, due to Proposition 3.7, the two arrows in (3.33) are (strict) $L_\infty$ quasi-isomorphisms, as desired. Theorem 3.1 is proven. \qed

4 Sheaves of homotopy algebras

For a topological space $X$ we consider the category $\text{dgSh}_X$ of dg sheaves (i.e. sheaves of unbounded cochain complexes of $\mathbb{K}$-vector spaces). We recall that $\text{dgSh}_X$ is a symmetric monoidal category for which the monoidal product is the tensor product followed by sheafification.

Given coaugmented dg cooperad $C$ (satisfying condition (2.1)) one may give the following naive definition of a homotopy algebra of type $C$ in the category $\text{dgSh}_X$:
Definition 4.1 (Naive!) We say that a dg sheaf $\mathcal{A}$ on $X$ carries a structure of a homotopy algebra of type $C$ if $\mathcal{A}$ is an algebra over the dg operad $\text{Cobar}(C)$.

One can equivalently define a homotopy algebra of type $C$ by considering coderivations of the cofree $C$-coalgebra (in the category $\text{dgSh}_X$)

$$C(\mathcal{A}) := \bigoplus_n \left( C(n) \otimes \mathcal{A}^\otimes n \right)_{S_n}. \tag{4.1}$$

In other words, a homotopy algebra of type $C$ on $\mathcal{A}$ is a degree 1 coderivation $Q$ of $C(\mathcal{A})$ satisfying the MC equation and the additional condition

$$Q \big|_{\mathcal{A}} = 0.$$

Given such a coderivation $Q$, it is natural to consider the $C$-coalgebra (4.1) with the new differential

$$\partial + Q \tag{4.2}$$

where $\partial$ comes from the differentials on $C$ and $\mathcal{A}$.

This observation motivates the following naive definition of $\infty$-morphism of homotopy algebra in $\text{dgSh}_X$:

Definition 4.2 (Naive!) Let $\mathcal{A}$ and $\mathcal{B}$ be homotopy algebras of type $C$ in $\text{dgSh}_X$ and let $Q_\mathcal{A}$ and $Q_\mathcal{B}$ be the corresponding coderivations of $C(\mathcal{A})$ and $C(\mathcal{B})$ respectively. An $\infty$-morphism $F : \mathcal{A} \to \mathcal{B}$ is a map of sheaves

$$F : C(\mathcal{A}) \to C(\mathcal{B})$$

which is compatible with the $C$-coalgebra structure and the differentials $\partial + Q_\mathcal{A}$, $\partial + Q_\mathcal{B}$.

An important disadvantage of the above naive definitions is that they do not admit an analogue of the homotopy transfer theorem [10, Theorem 10.3.2]. For this reason we propose “more mature” definitions based on the use of the Thom-Sullivan normalization [1], [12, Appendix A].

Let $\mathfrak{U}$ be a covering of $X$ and $\mathcal{A}$ be a dg sheaf on $X$. The associated cosimplicial set $\mathfrak{U}^\Delta(\mathcal{A})$ is naturally a cosimplicial cochain complex. So, applying the Thom-Sullivan functor $N^\text{TS}$ to $\mathfrak{U}^\Delta(\mathcal{A})$, we get a cochain complex

$$N^\text{TS} \mathfrak{U}(\mathcal{A}) \tag{4.3}$$

which computes the Cech hyper-cohomology of $\mathcal{A}$ with respect to the cover $\mathfrak{U}$.

Let us assume, for simplicity, that there exists an acyclic covering $\mathfrak{U}$ for $\mathcal{A}$. In particular, $\check{H}_\mathfrak{U}(\mathcal{A}) \cong H(\mathcal{A})$ agrees with the sheaf cohomology of $\mathcal{A}$.

Then, we have the following definition:

Definition 4.3 A homotopy algebra structure of type $C$ on a dg sheaf $\mathcal{A}$ is $\text{Cobar}(C)$-algebra structure on the cochain complex (4.3).

Remark 4.4 Since, the Thom-Sullivan normalization $N^\text{TS}$ is a symmetric monoidal functor from cosimplicial cochain complexes into cochain complexes, a homotopy algebra structure on $\mathcal{A}$ in the sense of naive Definition 4.1 is a homotopy algebra structure on $\mathcal{A}$ in the sense of Definition 4.3.
Remark 4.5 Let \( \mathcal{U}' \) be another acyclic covering of \( X \) and \( \mathcal{V} \) be a common acyclic refinement of \( \mathcal{U} \) and \( \mathcal{U}' \). Since the functor \( N^{TS} \) preserves quasi-isomorphisms, the cochain complexes \( N^{TS} \mathcal{U}(A) \) and \( N^{TS} \mathcal{U}'(A) \) are connected by the following pair of quasi-isomorphisms:

\[
N^{TS} \mathcal{U}(A) \xrightarrow{\sim} N^{TS} \mathcal{V}(A) \xleftarrow{\sim} N^{TS} \mathcal{U}'(A).
\] (4.4)

Hence, using the usual homotopy transfer theorem [10, Theorem 10.3.2], we conclude that the notion of homotopy algebra structure on a dg sheaf \( A \) is, in some sense, independent on the choice of acyclic covering.

Proceeding further in this fashion, we give the definition of an \( \infty \)-morphism (and \( \infty \) quasi-isomorphism) in the setting of sheaves:

Definition 4.6 Let \( A \) and \( B \) be dg sheaves on \( X \) equipped with structures of homotopy algebras of type \( C \). An \( \infty \)-morphism \( F \) from \( A \) to \( B \) is an \( \infty \)-morphism

\[
F : N^{TS} \mathcal{U}(A) \xrightarrow{\sim} N^{TS} \mathcal{U}(B)
\] (4.5)

of the corresponding homotopy algebras (in the category of cochain complexes) for some acyclic cover \( \mathcal{U} \). If (4.5) is an \( \infty \) quasi-isomorphism then, we say that, \( F \) is an \( \infty \) quasi-isomorphism from \( A \) to \( B \).

Remark 4.7 Again, since the Thom-Sullivan normalization \( N^{TS} \) is a symmetric monoidal functor from cosimplicial cochain complexes into cochain complexes, an \( \infty \)-morphism in the sense of naive Definition 4.2 gives us an \( \infty \)-morphism in the of Definition 4.6.

4.1 The deformation complex in the setting of sheaves

Let \( X \) be a topological space and \( A \) be a dg sheaf on \( X \). Let us assume that \( \mathcal{U} \) is an acyclic (for \( A \)) cover of \( X \) and \( A \) carries a homotopy algebra of type \( C \) defined in terms of this cover \( \mathcal{U} \).

Definition 4.8 The deformation complex \textit{of the sheaf of homotopy algebras} \( A \) is

\[
\text{Def}(A) := \text{Def}(N^{TS} \mathcal{U}(A)).
\]

Remark 4.9 The above definition of the deformation complex is independent on the choice of the acyclic cover in the following sense: Let \( \mathcal{U}' \) be another acyclic cover of \( X \). Since the cochain complexes \( N^{TS} \mathcal{U}(A) \) and \( N^{TS} \mathcal{U}'(A) \) are connected by the pair of quasi-isomorphisms (4.4), Theorem 3.1 and the homotopy transfer theorem imply that the deformation complexes corresponding to different acyclic coverings are connected by a sequence of quasi-isomorphisms of dg Lie algebras.

Theorem 3.1 has the following obvious implication

Corollary 4.10 Let \( A \) and \( B \) be dg sheaves on \( X \) equipped with structures of homotopy algebras of type \( C \). If \( A \) and \( B \) are connected by a sequence of \( \infty \) quasi-isomorphisms then \( \text{Def}(A) \) and \( \text{Def}(B) \) are quasi-isomorphic dg Lie algebras. \( \square \)
4.2 An application of Corollary 4.10

In applications we often deal with honest (versus $\infty$) algebraic structures on sheaves and maps of sheaves which are compatible with these algebraic structures on the nose (not up to homotopy). Here we describe a setting of this kind in which Corollary 4.10 can be applied.

Let $O$ be a dg operad and Cobar($C$) be a resolution of $O$ for which the cooperad $C$ satisfies condition (2.1).

Every dg sheaf of $O$-algebras $\mathcal{A}$ is naturally a sheaf of Cobar($C$)-algebras. Hence, $\mathcal{A}$ carries a structure of homotopy algebra of type $C$ and we define the deformation complex of $\mathcal{A}$ as

$$\text{Def}(\mathcal{A}) := \text{Def}(\text{N}^{\text{TS}}\text{U}(\mathcal{A})).$$

**Theorem 4.11** Let $\mathcal{A}$ and $\mathcal{B}$ be dg sheaves of $O$-algebras on a topological space $X$. If there exists a sequence of quasi-isomorphisms of dg sheaves of $O$-algebras

$$\mathcal{A} \xleftarrow{\sim} \mathcal{A}_1 \xrightarrow{\sim} \mathcal{A}_2 \xleftarrow{\sim} \cdots \xrightarrow{\sim} \mathcal{A}_n \xrightarrow{\sim} \mathcal{B},$$

then the dg Lie algebras $\text{Def}(\mathcal{A})$ and $\text{Def}(\mathcal{B})$ are quasi-isomorphic.

**Proof.** It is suffices to prove this theorem in the case when $\mathcal{A}$ and $\mathcal{B}$ are connected by a single quasi-isomorphism

$$f : \mathcal{A} \xrightarrow{\sim} \mathcal{B}$$

(4.6)

of dg sheaves of $O$-algebras.

Since the functor $\text{N}^{\text{TS}}$ preserves quasi-isomorphisms, $f$ induces a quasi-isomorphism

$$f_* : \text{N}^{\text{TS}}\text{U}(\mathcal{A}) \xrightarrow{\sim} \text{N}^{\text{TS}}\text{U}(\mathcal{B})$$

(4.7)

for any acyclic cover $\mathfrak{U}$.

Furthermore, since $\text{N}^{\text{TS}}$ is compatible with the symmetric monoidal structure, the map $f_*$ is compatible with the $O$-algebra structures on $\text{N}^{\text{TS}}\text{U}(\mathcal{A})$ and $\text{N}^{\text{TS}}\text{U}(\mathcal{B})$.

Therefore, $f_*$ may be viewed as an $\infty$ quasi-isomorphism from $\mathcal{A}$ to $\mathcal{B}$.

Thus Corollary 4.10 implies the desired statement. \qed

4.3 A Concluding remark about Definitions 4.3, 4.6, and 4.8

For certain applications, Definitions 4.3, 4.6, and 4.8 may still be naive. One may ask about a possibility to extend the notion of homotopy algebras to the setting of twisted complexes [2], [3], [4], [8]. For some application one may need a universal way of keeping track on “dependencies on covers” by using the notion of hypercover. For other applications one may need a notion of deformation complex which would also govern deformations of $\mathcal{A}$ as a sheaf or possibly as a (higher) stack.

However, for applications considered in [6], the framework of Definitions 4.3, 4.6, and 4.8 is sufficient.

A Cylinder type construction

Given a pair $(f, \tilde{f})$ of maps of cochain complexes

$$V \xrightarrow{f} W \xleftarrow{\tilde{f}} \tilde{V},$$

(A.1)
we form another cochain complex \( Cyl(f, \tilde{f}) \). As a graded vector space
\[
Cyl(f, \tilde{f}) := V \oplus sW \oplus \tilde{V}
\]
and the differential \( \partial^{Cyl} \) is defined by the formula:
\[
\partial^{Cyl}(v + sw + \tilde{v}) := \partial v + s(f(v) - \partial w + \tilde{f}(\tilde{v})) + \partial \tilde{v}.
\]
The equation
\[
\partial^{Cyl} \circ \partial^{Cyl} = 0
\]
is a consequence of \( \partial^2 = 0 \) and the compatibility of \( f \) (resp. \( \tilde{f} \)) with the differentials\(^5\) on \( V \), \( W \), and \( \tilde{V} \).

We have the obvious pair of maps of cochain complexes:
\[
V \xleftarrow{\pi_V} Cyl(f, \tilde{f}) \xrightarrow{\pi_{\tilde{V}}} \tilde{V}
\]
\[
\pi_V(v + sw + \tilde{v}) = v,
\]
\[
\pi_{\tilde{V}}(v + sw + \tilde{v}) = \tilde{v}.
\]

We claim that

**Lemma A.1** If \( f \) and \( \tilde{f} \) are quasi-isomorphisms of cochain complexes, then so are \( \pi_V \) and \( \pi_{\tilde{V}} \).

Proof. Let us prove that \( \pi_V \) is surjective on the level of cohomology.

For this purpose, we observe that for every cocycle \( v \in V \) its image \( f(v) \) in \( W \) is cohomologous to some cocycle of the form \( \tilde{f}(v') \), where \( v' \) is a cocycle in \( \tilde{V} \). The latter follows easily from the fact that \( f \) and \( \tilde{f} \) are quasi-isomorphisms.

In other words, for every degree \( n \) cocycle \( v \in V \) there exists a degree \( n \) cocycle \( v' \in V \) and a degree \( (n - 1) \) vector \( w \in W \) such that
\[
f(v) - \tilde{f}(v') - \partial w = 0.
\]
Hence, \( v + sw - v' \) is a cocycle in \( Cyl(f, \tilde{f}) \) such that \( \pi(v + sw - v') = v \).

Let us now prove that \( \pi_V \) is injective on the level of cohomology.

For this purpose, we observe that the cocycle condition for \( v + sw + v' \in Cyl(f, \tilde{f}) \) is equivalent to the three equations:
\[
\partial v = 0,
\]
\[
\partial v' = 0,
\]
and
\[
f(v) + \tilde{f}(v') - \partial w = 0.
\]

Therefore, for every cocycle \( v + sw + v' \in Cyl(f, \tilde{f}) \), the vectors \( v \) and \( v' \) are cocycles in \( V \) and \( \tilde{V} \), respectively, and the cocycles \( f(v) \) and \( -\tilde{f}(v') \) in \( W \) are cohomologous.

Hence, \( v + sw + v' \in Cyl(f, \tilde{f}) \) is a cocycle and \( v \) is exact then so is \( v' \), i.e. there exist vectors \( v_1 \in V \) and \( v'_1 \in \tilde{V} \) such that
\[
v = \partial v_1, \quad v' = \partial v'_1.
\]

\(^5\)By abuse of notation, we denote by the same letter \( \partial \) the differential on \( V \), \( W \), and \( \tilde{V} \).
Subtracting the coboundary of \( v_1 \oplus s0 \oplus v'_1 \) from \( v \oplus sw \oplus v' \) we get a cocycle in \( \text{Cyl}(f, \tilde{f}) \) of the form
\[
0 \oplus s(w - f(v_1) - \tilde{f}(v'_1)) \oplus 0 \tag{A.11}
\]
Since \( w - f(v_1) - \tilde{f}(v'_1) \) is a cocycle on \( W \) and \( \tilde{f} \) is a quasi-isomorphism, there exists a cocycle \( \tilde{v} \in \tilde{V} \) and a vector \( w_1 \in W \) such that
\[
w - f(v_1) - \tilde{f}(v'_1) - \tilde{f}(\tilde{v}) - \partial(w_1) = 0. \tag{A.12}
\]
Hence the cocycle (A.11) is the coboundary of
\[
0 \oplus (-sw_1) \oplus \tilde{v} \in \text{Cyl}(f, \tilde{f}).
\]
Thus \( \pi_V \) is indeed injective on the level of cohomology.
Switching the roles \( V \leftrightarrow \tilde{V}, f \leftrightarrow \tilde{f}, \) and \( \pi_V \leftrightarrow \pi_{\tilde{V}} \) we also prove the desired statement about \( \pi_{\tilde{V}} \).

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