THE COMPACT-OPEN TOPOLOGY ON THE HOMEOMORPHISM GROUP OF A SURFACE WITHOUT BOUNDARY IS MINIMAL.

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Abstract. We show that the homeomorphism group of a surface without boundary does not admit a Hausdorff group topology strictly coarser than the compact-open topology. In combination with known automatic continuity results, this implies that the compact-open topology is the unique Hausdorff separable group topology on the group if the surface is closed or the complement in a closed surface of either a finite set or the union of a finite set and a Cantor set.

0. Introduction

We begin by recalling the following definition:

Definition 0.1. Let $G$ be a group and $t$ a Hausdorff group topology on $G$. We say that $t$ is minimal if $G$ admits no Hausdorff group topology strictly coarser than $t$.

The notion of minimality captures a key feature of compact groups and as such has received a substantial degree of attention in the literature. For the broader context and further questions we refer the reader to the comprehensive survey [3].

Given a topological space $X$, the compact-open topology on the homeomorphism group of $X$ is the topology generated by all subsets of the form $[K, U] = \{ f \in G | f(K) \subseteq U \}$, for $K \subseteq X$ compact and $U \subseteq X$ open in $X$. The following classical result is due to Arens [1].

Fact 0.2. Let $X$ be a Hausdorff, locally compact and locally connected topological space. Then the compact-open topology is a group topology on $\mathcal{H}(X)$.

Given a manifold $X$, we denote by $\mathcal{H}_0(X)$ the subgroup of $\mathcal{H}(X)$ consisting of all those $h \in \mathcal{H}(X)$ for which there exists an isotopy between $h$ and the identity; i.e. some continuous map $H : X \times [0, 1] \to X$ such that $H(\cdot, t) \in \mathcal{H}(X)$ for all $t \in [0, 1]$, $H(x, 0) = h(x)$ and $h(x, 1) = x$. We denote by $\mathcal{H}_{c0}(X) \leq \mathcal{H}_0(X)$ the subgroup of all of those for which there exists some compact set $K \subseteq X$ such that $H(\cdot, t)$ is supported on $K$ for all $t \in [0, 1]$. The term surface will be used to refer to topological 2-manifolds. The purpose of this short note is to provide a very elementary proof of the following result:

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Theorem 0.3. Let $X$ be a surface without boundary and $G$ be a group satisfying $\mathcal{H}_{c_0}(X) \leq G \leq \mathcal{H}(X)$. Then the restriction to $G$ of the compact-open topology on $\mathcal{H}(X)$ is minimal.

It was first shown by Rosendal in [12] that any group homomorphism from the homeomorphism group of a compact surface to a separable topological group is automatically continuous. This was later generalized by Mann to homeomorphism groups of compact manifolds in arbitrary dimension [7] and later in [8] to certain groups of homeomorphisms of non-compact manifolds. Combining these results with Theorem 0.3 one obtains:

Corollary 0.4. Let $X$ be a closed surface and $F \subseteq X$ either a finite set (possibly empty) or the union of a finite set and a Cantor set. Then the compact-open topology is the unique separable Hausdorff group topology on the group $\mathcal{H}(X \setminus F)$.

Note that the compact-open topology was already known to be the unique complete separable group topology on the homeomorphism group of a compact manifold (see [6] and [7]). The automatic continuity result in [8] applies to the group of homeomorphisms of a manifold $X$ preserving a set $F$ as above. However, as discussed there, in dimension 2 said group endowed with the compact-open topology is isomorphic to $\mathcal{H}(X \setminus F)$ as a topological group via the restriction map.

In [2] Chang and Gartside provide a counterexample to the minimality of the compact-open topology for any compact manifold $X$ with non-empty boundary which admits the following alternative description. The restriction homomorphism $\rho : \mathcal{H}(X) \to \mathcal{H}(\text{int}(X))$ is injective and continuous, where $\text{int}(X) = X \setminus \partial X$. Let $t_{c_0}$ denote the preimage by $\rho$ of the compact-open topology on $\mathcal{H}(\text{int}(X))$. The fact that $X$ is compact. In fact, the same argument shows that this is also true if $X$ is not compact for $t_{c_0}$ defined as above. On the other hand, since $\mathcal{H}_{c_0}(\text{int}(X)) \leq \text{im}(\rho)$, Theorem 0.3 applies and thus we get:

Corollary 0.5. For any surface with boundary the topology $t_{c_0}$ on $\mathcal{H}(X)$ is minimal.

Outline of the proof. The proof of 0.3 is structured as follows. We fix once and for all some surface $X$ without boundary, as well as some $G$ with $H \leq G \leq \mathcal{H}(X)$, where $H = \mathcal{H}_{c_0}(X)$. We denote by $t_{c_0}$ the compact-open topology on $G$ and fix some group topology $t$ on $G$ strictly coarser than $t_{c_0}$. Denote by $N_t(1)$ the collection of open neighbourhoods of the identity in $t$. Fix some arbitrary $g_0 \in H \setminus \{1\}$ supported on a disk and assume the existence of $V \in N_t(1)$ with $g_0 \notin V$.

Section 1 establishes some notation and recalls some basic facts many readers will be familiar with. In Section 2 we see, using an auxiliary result from Section 1 that the assumption $t \leq t_{c_0}$ implies that every $V \in N_t(1)$ is very rich, containing a plethora of fix-point stabilizers of embedded graphs. Finally, in Sections 4 and 5 we show this to be in contradiction with the existence of $V$ as in the previous paragraph.

1. Preliminaries

We do not assume $X$ to be compact. However, we will often use the well-known fact that $X$ admits an exhaustion by compact submanifolds, which follows from

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1 That the two topologies are the same can be seen by applying the criterion of equality between $t_{c_0}$ and $(t_{c_0})_F$ in Theorem 5 from [2] to the set-wise stabilizer of $F$ in $\mathcal{H}_0(X)$. 
a standard argument using the smoothability of surfaces and Whitney embedding theorem (see for instance [9]).

If we fix a metric $d$ on $X$ compatible with its topology, $t_{co}$ can be described as the topology of uniform convergence on compact sets. That is, a base of neighbourhoods of the identity for the compact-open topology on $G$ is given by the collection of sets

$$V_{K,\epsilon} = \{ g \in G \mid \forall x \in K \ d(x, gx), d(x, g^{-1}x) < \epsilon \}$$

where $K$ ranges over all compact subsets of $X$ and $\epsilon$ over all positive reals. Sometimes we might only be interested in the set $V_\epsilon := V_{X,\epsilon}$ (potentially not open in $t_{co}$).

For any subset $A \subseteq X$ and $\epsilon > 0$ we let $N_\epsilon(A) := \{ p \in X \mid d(p, A) < \epsilon \}$. We make and will assume that for any $p$ there is $\delta_p$ such that the ball $B(p, \delta_p)$ is connected.

Given a surface $Y$ and a closed set $F \subseteq Y$ we write $\mathcal{H}(Y, F)$ for the subgroup of all homeomorphisms of $Y$ that fixes $F$ point-wise, $\mathcal{H}_0(Y, F)$ for the subgroup of all homeomorphisms fixing $F$ isotopic to the identity by an isotopy that fixes $F$ at any point in time and $\mathcal{H}_{co}(Y, F)$ for the group of all homeomorphisms isotopic to the identity through a compactly supported isotopy fixing $F$ at any point in time.

### Disks and arcs.

By a disk in a surface $Y$ we intend a homeomorphic image in $Y$ of a standard 2-ball. We write $I$ for the interval $[-1, 1]$. By an arc (in $Y$) from a point $p \in Y$ to a point $q \in Y$ we intend an injective continuous map $\alpha : I \to Y$ with $\alpha(-1) = p$ and $\alpha(1) = q$. We refer to $p$ and $q$ as the endpoints of $\alpha$. By a small abuse of notation, we may use the term $\alpha$ to refer to the image of $\alpha$. We will refer to $\alpha((-1, 1))$ as the interior of $\alpha$. Whenever we concatenate a sequence of arcs or we restrict some arc to some interval $I \subseteq \mathbb{I}$ we always assume an order-preserving reparametrization is applied at the end so that $\mathbb{I}$ is again the domain of the resulting map.

It is a consequence of the Jordan curve theorem and Schönflies theorem that for any arc $\alpha \subseteq \text{int}(Y)$ there exists some homeomorphic embedding $\hat{\alpha} : \mathbb{I} \times I \to Y$ so that $\alpha$ is the restriction of $f$ to $\{0\} \times [-\frac{1}{2}, \frac{1}{2}]$ and $\mathbb{I} \times \mathbb{I}$. We will refer to $\hat{\alpha}$ as a rectangular extension of $\alpha$. By a regular path we mean the concatenation of finitely many arcs intersecting only at the endpoints.

We say that two arcs $\alpha, \beta$, are transverse on an open set $U \subseteq Y$ if $\mathbb{I} := U \cap \alpha \cap \beta$ is finite, coincides with $U \cap \alpha(I) \cap \beta(I)$ and $\alpha$ and $\beta$ cross at $q$ for any $q \in \mathbb{I} \mathbb{B}$. We say that two collection of arcs $\mathcal{A}, \mathcal{B}$ are transverse on $U$ if $\mathcal{A}$ is transverse on $U$ to each arc in $\mathcal{B}$. We will not mention $U$ explicitly when $U = X$. Sometimes we will talk about a collection of arcs being transverse to a certain 1-dimensional compact submanifold $Z$, which simply means it is transverse to some decomposition of $Z$ into arcs.

For the following see Chapters 2 and 3 in [5].

**Fact 1.1.** Let $Y$ be a surface and $\{\alpha_i\}_{i=1}^k, \{\beta_j\}_{j=1}^r$ two collections of arcs in $Y$ that can only meet $\partial Y$ at an endpoint and such that $\alpha_i \cap \alpha_{i'}, \beta_j \cap \beta_{j'}$ are finite for $1 \leq i < i' \leq k$ and $1 \leq j < j' \leq r$. Then for any neighbourhood $\mathcal{W}$ of the identity

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2 Working in the universal cover one can first extend the arc to a closed curve (using [11] p.164 or some other of the suggestions found at [https://mathoverflow.net/questions/57766/why-are-there-no-wild-arcs-in-the-plane](https://mathoverflow.net/questions/57766/why-are-there-no-wild-arcs-in-the-plane) and then apply Schönflies theorem (see [10]).

3 That is, there is a disk with $p \in D$ and a homeomorphism $h : D \to B = \{(x, y) \in \mathbb{R}^2 \mid |x| + |y| \leq 1\}$ taking $\alpha \cap D, \beta \cap D$ to $\{(x, y) \mid x = 0\} \cap B$ and $\{(x, y) \mid y = 0\} \cap B$ respectively.
in the restriction of the compact-open topology to $\mathcal{H}(Y, \partial Y)$ there exists $\phi \in \mathcal{W}$ such that $\phi \cdot \alpha_i$ is transverse to $\beta_j$ on $\text{int}(Y)$ for all $1 \leq i \leq k, 1 \leq j \leq r$.

Given a compact surface $Y$ by a triangulation $\mathcal{T}$ of $Y$ we mean a homeomorphism between the geometric realization of some finite simplicial complex, which we assume to contain no double edges $Y$. We can think of it as a collection of disks, the triangles of $\mathcal{T}$ and of arcs, the edges of $\mathcal{T}$. For the following see [10].

Fact 1.2. Every compact surface admits a triangulation. Given two compact surfaces $Y, Y'$ with $Y \subseteq \text{int}(Y')$ every triangulation of $Y$ extends to a triangulation of $Y'$.

We will often repeatedly use the following weak version of Alexander’s Lemma (see [3], Chapter 4):

Fact 1.3. If $D$ is a disk, then $\mathcal{H}(D, \partial D) = \mathcal{H}_0(D, \partial D)$. Therefore any homeomorphism of $X$ supported on an embedded disk is in $H$.

Dehn twists. An annulus $A$ in $X$ is the image of some homeomorphic embedding $h : S^1 \to I \to X$. A core curve of $A$ is the image of $\alpha : S^1 \to I \times S^1, \alpha(s) = (0, s)$ by some such $h$ with $\text{im}(h) = A$ and a Dehn twist over $A$ the homeomorphism of $X$ resulting from pushing forward the homeomorphism $(s, t) \mapsto (s, t+s)$ of $S^1 \to I$ onto $A$ via some such $h$ an then extending it by the identity outside of $A$.

Observation 1.4. Let $A$ be an annulus and $\gamma, \delta$ two disjoint arcs in $A$ joining the two boundary components of $A$. Let $\tau$ a Dehn twist over $A$. Then $\gamma \cup \tau^2 \cdot \delta$ is connected and contains some core curve of $A$.

Embedded graphs. By an embedded graph $\Gamma$ we mean a finite tuple of arcs in $X$ such that for distinct $\gamma, \gamma'$ in $\Gamma$ any intersection point of $\gamma$ and $\gamma'$ is an endpoint of both and no two distinct $\gamma, \gamma'$ can have two endpoints in common. For convenience our notation will often treat $\Gamma$ as a mere set.

We let $V(\Gamma)$ be the set of endpoints of $\Gamma$ in $X$. Alternatively, we say that $\Gamma$ is an embedded $\mathbb{Q}$-graph if $V(\Gamma) = \mathbb{Q}$. We write $\bigcup \Gamma = \bigcup_{\gamma \in \Gamma}\gamma$. We will refer to any neighbourhood of $\bigcup \Gamma$ simply as a neighbourhood of $\Gamma$. The group $G$ acts on the collection of embedded graphs by post-composition.

Given embedded graphs $\Gamma = (\gamma_i)_{i=1}^k$ and $\Gamma' = (\gamma'_i)_{i=1}^{k'}$ we write $\Gamma \simeq \Gamma'$ if $k = k'$ and $\gamma'_j$ is an order preserving reparametrization of $\gamma_j$ for $1 \leq j \leq k$.

We denote by $H_{\cap \Gamma}$ the subgroup consisting of all the elements of $H$ fixing $\bigcup \Gamma$ and by $H_{\Gamma}$ the subgroup of $H$ consisting of all $h \in H$ such that $h \cdot \Gamma \simeq \Gamma$.

Given a neighbourhood $U$ of $\Gamma$ and a homeomorphic embedding $h : U \to X$ preserving the arcs of $\Gamma$ and fixing their endpoints we say that $h$ is orientation preserving at $\Gamma$ if for any $p \in \gamma$ there exists some homeomorphic embedding $\tilde{\gamma} : I \times I \to X$ with $\gamma = \tilde{\gamma}|_{\{0\} \times I}$ and some neighbourhood $V$ of $p$ such that $V, h(V) \subseteq \text{im}(\gamma)$ and if we let $A_0 = \tilde{\gamma}([-1,0) \times I)$, $A_1 = \tilde{\gamma}((0,1] \times I)$ then if $C \subseteq A_i$ for a component $C$ of $V \setminus \gamma$ then also $h(C) \subseteq A_i$.

We denote by $H_{\cup \Gamma}^+$ and $H_{\cap \Gamma}^+$ the subgroups of $H_{\cup \Gamma}$ and $H_{\cap \Gamma}^+$ respectively consisting of those elements that are orientation preserving at $\Gamma$.

Extending partial homeomorphisms. The following facts can be seen as a consequence of Schönflies theorem, the Jordan curve theorem and the classification of compact surfaces.
Definition 2.1. Let \( \mathcal{N} = (\{D_q\}_{q \in \mathcal{Q}}, \{D_\gamma\}_{\gamma \in \Gamma}, (\theta_q)_{q \in \mathcal{Q}}, (\theta_\gamma)_{\gamma \in \Gamma}) \) where \( D_q \) and \( D_\gamma \) are embedded disks and \( \theta_q : I \times I \to D_q \mathcal{Q} = D_q \) such that

- \( \{D_q\}_{q \in \mathcal{Q}} \) are pair-wise disjoint and \( \{D_\gamma\}_{\gamma \in \Gamma} \) are pair-wise disjoint
- \( D_q \cap D_\gamma \) is an arc if \( q \) is an endpoint of \( \gamma \) for \( q \in \mathcal{Q} \) and \( \gamma \in \Gamma \) and empty otherwise

Fact 1.5. For any two families of disjoint embedded disks \( \{D_i\}_{i=1}^k, \{D'_i\}_{i=1}^k \) in some connected surface \( Y \) and any collection of homeomorphisms \( h_i : \partial D_i \cong \partial D'_i \), which we assume to be orientation-preserving if \( Y \) is orientable, there exists \( h \in \mathcal{H}_{\mathcal{Q}}(Y) \) taking \( D_i \) to \( D'_i \) and restricting to \( h_i \) on \( \partial D_i \) for \( 1 \leq i \leq k \).

Fact 1.6. Let \( D \) be a disk and \( \{\alpha_i\}_{i=1}^k, \{\alpha'_i\}_{i=1}^k \) two families of pair-wise disjoint arcs between points in \( D \) such that either \( \alpha'_i, \alpha_i \subseteq \bar{D} \) for all \( i \) or the following holds:
- for all \( 1 \leq i \leq k \) the intersection of \( \alpha_i \) with \( \partial D \) consists of one or two endpoints \( \alpha_i \) and the same is true for \( \alpha'_i \)
- if \( \alpha_i(-1) \in \partial D \), then \( \alpha'_i(-1) = \alpha_i(-1) \) and the same is true if we replace \( -1 \) with \( 1 \) and/or exchange the roles of \( \alpha_i \) and \( \alpha'_i \)

Then there exists some \( h \in \mathcal{H}(D, \partial D) \) such that \( h \circ \alpha_i = \alpha'_i \) for all \( i \). In particular, for any arc \( \alpha \) and any embedded disk \( D \) with \( \alpha \subseteq D \) we have that \( D \) is the image of a rectangular extension of \( \alpha \).

In particular, for any arc \( \alpha \) and any embedded disk \( D \) with \( \alpha \subseteq \bar{D} \) we have that \( D \) is the image of a rectangular extension of \( \alpha \).

Fact 1.7. Let \( D \) be a disk and \( \Gamma \) some embedded \( \mathcal{Q} \)-graph such that \( \bigcup \Gamma \) is simply connected and \( \bigcup \Gamma \cap \partial D \) coincides with the set of points in \( \mathcal{Q} \) belonging to a unique \( \gamma \in \Gamma \). Then any homeomorphic embedding \( h \) of \( \bigcup \Gamma \cap \partial D \) into \( D \) that is the identity on \( \partial D \) extends to a homeomorphism of \( D \).

Point pushing maps. Given an arc \( \alpha \) from a point \( p \in X \) to a point \( q \in X \) and \( \epsilon > 0 \) we denote by \( \mathcal{P}_\epsilon(\alpha) \) the collection of all homeomorphisms that take \( p \) to \( q \) and are supported in some embedded disk \( D \) with \( \alpha \subseteq D \subseteq N_\epsilon(\alpha) \). The existence of rectangular extensions and Fact 1.3 implies that \( \mathcal{P}_\epsilon(\alpha) \neq \emptyset \) and by Fact 1.6 \( \mathcal{P}_\epsilon(\alpha) \subseteq H \). If \( \alpha \) is a regular path and \( U \) an open set containing \( \text{im}(\alpha) \) we let \( \mathcal{P}_\epsilon(\alpha) \) be the collection of all products of the form \( f_k f_{k-1} \ldots f_1 \), where \( f_i \in \mathcal{P}_\epsilon(\alpha_i) \) and \( \alpha = \alpha_1 \ast \alpha_2 \ldots \alpha_k \) for some decomposition of \( \alpha \) into arcs.

Bigons. The following follows from a similar argument to that in the proof of 1.7 in [H].

Fact 1.8. Let \( Y \) be a compact surface with boundary and let \( \{\gamma_i\}_{i=1}^k, \{\gamma'_i\}_{i=1}^k \) be two families of pair-wise disjoint arcs in \( Y \) between boundary points with \( \gamma_i(1), \gamma'_i(1) \subseteq \text{int}(Y) \). Assume that for each \( 1 \leq i \leq k \) and \( 1 \leq j \leq r \) the arcs \( \gamma_i \) and \( \gamma'_j \) are transverse on \( \text{int}(Y) \) and some representative of the homotopy class of \( \gamma'_j \) is disjoint from \( \gamma_i \) on \( \text{int}(Y) \). Then there is an innermost bigon bounded by \( \gamma \) and \( \gamma' \): an embedded disk \( D \) in \( Y \) whose boundary is the union of a subarc of some \( \gamma_i \) and a subarc of some \( \gamma'_j \) and whose interior is disjoint from all \( \gamma_i \) and \( \gamma'_j \).

2. Extending partial homeomorphisms around graphs

Definition 2.1. Let \( \Gamma \) be an embedded \( \mathcal{Q} \)-graph. By a nice system of disks around \( \Gamma \) we mean a tuple \( \mathcal{N} = (\{D_q\}_{q \in \mathcal{Q}}, \{D_\gamma\}_{\gamma \in \Gamma}, (\theta_q)_{q \in \mathcal{Q}}, (\theta_\gamma)_{\gamma \in \Gamma}) \) where \( D_q \) and \( D_\gamma \) are embedded disks and \( \theta_q : I \times I \to D_q \mathcal{Q} = D_q \) such that
- \( \{D_q\}_{q \in \mathcal{Q}} \) are pair-wise disjoint and \( \{D_\gamma\}_{\gamma \in \Gamma} \) are pair-wise disjoint
- \( D_q \cap D_\gamma \) is an arc if \( q \) is an endpoint of \( \gamma \) for \( q \in \mathcal{Q} \) and \( \gamma \in \Gamma \) and empty otherwise
• \( \theta_\gamma \) takes \( \{0\} \times I \) to \( \gamma \cap D_\gamma \) and \( I \times \{-1\} \) and \( I \times \{1\} \) to \( D_\gamma(-1) \cap D_\gamma \) and \( D_\gamma(1) \cap D_\gamma \) respectively.

and moreover for any arc \( \gamma \in \Gamma \) from \( q \) to \( q' \) we have that

• \( \gamma \) can be divided into three consecutive subarcs: \( \gamma = \gamma_q * \gamma' * \gamma_{q'} \) with \( \gamma_q \subseteq D_q \), \( \gamma' \subseteq D_\gamma \) and \( \gamma_{q'} \subseteq D_{q'} \)

• \( \gamma \) intersects \( \partial D_q \) only at the common endpoint of \( \gamma_q \) and \( \gamma' \), which lies in the interior of \( D_q \cap D_\gamma \), and similarly for \( q' \)

We write \( \bigcup \mathcal{N} = \bigcup_{q \in \mathcal{Q}} D_q \cup \bigcup_{\gamma \in \Gamma} D_\gamma \). It is easy to see that \( \bigcup \mathcal{N} \) is a neighbourhood of \( \bigcup \Gamma \). We will refer to any set of this form as a nice neighbourhood of \( \Gamma \).

**Definition 2.2.** In particular, in the situation above each \( D_q \) intersects each arc \( \gamma \in \Gamma \) in either the empty set or a single arc from \( q \) to \( \partial D_q \) touching \( \partial D_q \) in a single point. We will henceforth refer to any disk satisfying this condition for some \( q \in V(\Gamma) \) as intersecting \( \Gamma \) in a star.

**Remark 3.** Given two collection of disks as in the definition, a collection of homeomorphisms \( \{\theta_\gamma\}_{\gamma \in \Gamma} \) always exists.

The following follows from the Jordan curve theorem and Schönflies theorem by standard arguments:

**Fact 2.4.** Let \( \Gamma \) be an embedded \( \mathcal{Q} \)-graph. Then for any \( \epsilon > 0 \) there is a nice system of disks \( \mathcal{N} = \{\{D_q\}_{q \in \mathcal{Q}}, \{D_\gamma\}_{\gamma \in \Gamma}, \{\theta_\gamma\}_{\gamma \in \Gamma}\} \) such that \( \bigcup \mathcal{N} \subseteq \mathcal{N}_\epsilon(\bigcup \Gamma) \) and \( \text{diam}(D_q) < \epsilon \) for all \( q \in \mathcal{Q} \).

The Lemma below is probably known, but we were unable to find a suitable reference.

**Lemma 2.5.** Given any embedded graph \( \Gamma \) and \( \epsilon > 0 \) there exists some \( \delta = \delta(\epsilon, \Gamma) > 0 \) such that for any closed neighbourhood \( U \) of \( \Gamma \) there is some closed neighbourhood \( N \subseteq U \) and with the property that any homeomorphic embedding \( h : N \to X \) which is the identity on \( \bigcup \Gamma \), orientation preserving at \( \Gamma \) and satisfies \( d(\mathcal{p}, h(\mathcal{p})) < \delta \) for all \( \mathcal{p} \in N \) extends to some \( h \in V_c \cap H \). Moreover, if there exists some union \( C \) of connected components of \( X \setminus \bigcup \Gamma \) such that \( h \) restricts to the identity on \( C \cap N \), then we may assume that \( h \) is the identity on \( C \).

**Proof.** Pick some nice system of disks \( \mathcal{N} = \{\{D_q\}_{q \in \mathcal{Q}}, \{D_\gamma\}_{\gamma \in \Gamma}, \{\theta_\gamma\}_{\gamma \in \Gamma}\} \) around \( \Gamma \) with \( \text{diam}(D_q) < \frac{\epsilon}{2} \) for all \( q \in \mathcal{Q} \) and some \( \eta > 0 \) such that \( \text{diam}(\theta_\gamma(J_1 \times J_2)) < \frac{\epsilon}{2} \) for any \( \gamma \in \Gamma \), \( J_1, J_2 \subseteq I \), \( \text{diam}(J_1), \text{diam}(J_2) \leq 2\eta \).

Pick \( t_0 = -1 < t_1 < \ldots < t_m < t_{m+1} = 1 \) such that \( |t_i - t_{i+1}| \leq (\eta, 2\eta) \) for \( 0 \leq i \leq m \). For any \( 1 \leq i \leq m \) let \( J_i = \left[\frac{t_i + t_{i+1}}{2}, \frac{t_i + t_{i+1}}{2}\right] \). Let also \( Q^\gamma_{t_i} = \theta_\gamma(J_0 \times J_i) \) and \( Q^-_{\gamma t_i} = \theta_\gamma([-\eta, 0] \times J_i) \).

Using compactness we can find some \( \delta \in (0, \frac{\eta}{2}) \) such that

1. for any \( \gamma \in \Gamma \) any arc in \( X \setminus \bigcup \Gamma \) from a point in \( \theta_\gamma([-1, 0] \times I) \) to a point \( \theta_\gamma((0, 1] \times I) \) has diameter at least \( 3\delta \)
2. \( d(\text{im}(\theta_\gamma), \text{im}(\theta_{\gamma'})) \geq \delta \) for distinct \( \gamma, \gamma' \in \Gamma \)
3. \( d(\text{im}(\theta_\gamma), D_q) \geq \delta \) for \( \gamma \in \Gamma \) and \( q \in \mathcal{Q} \) such that \( q \) is not an endpoint of \( \gamma \)
4. for any \( \gamma \in \Gamma \) and \( (s, s'), (t, t') \in I^2 \) with \( |s - t| + |s' - t'| \geq \frac{\eta}{2} \) we have \( d(\theta_\gamma(t, t'), \theta_\gamma(s, s')) \geq 2\delta \)

Note that this allows for the possibility that the disk in question intersects \( \Gamma \) in just one or two arcs.
(5) \( d(D_{\gamma(-1)} \cup D_{\gamma(1)}, \theta_\gamma(I \times \left[-1 + \frac{p}{2}, 1 - \frac{p}{2}\right])) \geq 2\delta \)

It is easy to find some nice neighbourhood \( N \) of \( \Gamma \) contained in \( U \) such that:

(a) \( d(N, \partial(N)) \geq \delta \)

(b) for all \( \gamma \in \Gamma \) we have \( \theta_\gamma^{-1}(N \cap D_\gamma) = [\xi, \xi] \times I \) for some \( \xi > 0 \) such that

\[
\text{diam}(\theta_\gamma([-\xi, \xi] \times \{t\})) < \delta \quad \text{for all } t \in I
\]

Assume now that \( h : N \to X \) is a homeomorphic embedding restricting to the identity on \( \Gamma \), orientation preserving at \( \Gamma \) and such that \( d(p, h(p)) < \delta \) for all \( p \in N \).

Condition (3) above implies that \( h(N) \subseteq \bigcup N \).

It suffices to show that there exists some \( g \in \mathcal{V}_\gamma \cap H^+_{\Gamma} \) such that \( g \circ h \) extends to some \( g' \in \mathcal{V}_\gamma \cap H^+_{\Gamma} \), since then \( g^{-1}g' \in \mathcal{V}_\gamma \cap H \) will be the extension we need.

To begin with, notice that we may assume without loss of generality that \( h \circ \partial N \) is transverse to the boundary of the \( Q^+_{\gamma, i} \) and \( Q^-_{\gamma, i} \). For any \( \gamma \in \Gamma \) and \( 1 \leq i \leq m \) condition (4) above implies that

\[
h(\theta_\gamma([0, 1] \times \{t_i\})), h(\theta_\gamma([-1, 0] \times \{t_i\})) \subseteq B(\theta_\gamma(0, t_i), 2\delta),
\]

which together with the fact that \( h \) is orientation preserving at \( \Gamma \) implies that \( h(\theta_\gamma([-1, 0] \times \{t_i\})) \subseteq Q^+_{\gamma, i} \) with only one endpoint on \( \partial Q^+_{\gamma, i} \) and similarly for \( \theta_\gamma([-1, 0] \times \{t_i\}) \) and \( Q^-_{\gamma, i} \). We refer to the image of \( [0, \eta] \times \{\frac{t_i + t_{i+1}}{2}\} \) and \( [0, \eta] \times \{t_{i+1} - t_i\} \) by \( \gamma \) as the two vertical sides of \( Q^+_{\gamma, i} \) and similarly for \( Q^-_{\gamma, i} \).

**Claim 2.6.** There is only one subarc of \( h \circ \partial N \) joining the two vertical sides of \( Q^+_{\gamma, i} \) (resp. \( Q^-_{\gamma, i} \)) of \( \theta_\gamma(\{0, \eta\} \times \{1, t_i\}) \).

**Proof.** Conditions (1) and (5) and the fact that \( h \) is orientation preserving at \( \Gamma \) imply that no point in \( \theta_\gamma([-1, 0] \times I) \) can be sent to \( Q^+_{\gamma, i} \) by \( h \). Together with conditions (2) and (3) this implies that no point outside of \( D_{\gamma(-1)} \cup \theta_\gamma([-1, 0] \times I) \cup D_{\gamma(1)} \) can be sent to \( Q^+_{\gamma, i} \) by \( h \) and a similar statement holds for \( Q^-_{\gamma, i} \).

On the other hand, there cannot exist \( p \in D_{\gamma(-1)} \cup \theta_\gamma([-1, 0] \times I) \cup D_{\gamma(1)} \) such that \( h(p) \in \theta_\gamma([-1, 0] \times \{\frac{t_i + t_{i+1}}{2}\}) \). Since by conditions (1) and (5) we have \( d(A, B) \geq \delta \), and the same holds if we exchange the role of the two endpoints of \( \gamma \) and/or of + and −. The moreover part is clear and the result follows.

This implies the existence of arcs \( \alpha^+_{\gamma, i} \subseteq Q^+_{\gamma, i} \) and \( \alpha^-_{\gamma, i} \subseteq Q^-_{\gamma, i} \) from \( \theta_\gamma(\eta, t_i) \) to \( \theta_\gamma(\xi, t_i) \) and from \( \theta_\gamma(-\eta, t_i) \) to \( \theta_\gamma(-\xi, t_i) \) respectively of which only the endpoints \( \theta_\gamma(\xi, t_i) \) and \( \theta_\gamma(-\xi, t_i) \) respectively belong to \( h(N) \). Let \( \beta^+_{\gamma, i} \) be the concatenation of \( \alpha^+_{\gamma, i} \) and the arc with image \( h(\theta_\gamma([0, \xi] \times \{i\})) \) and define \( \beta^-_{\gamma, i} \) in a similar way.

Fact (\ref{fact:homeo}) implies the existence of homeomorphisms \( g^+_{\gamma, i} \in H \) supported on \( Q^+_{\gamma, i} \) and \( g^-_{\gamma, i} \in H \) supported on \( Q^-_{\gamma, i} \) such that \( g^+_{\gamma, i}(\beta^+_{\gamma, i}) = \theta_\gamma([-\eta, 0] \times \{t_i\}) \) and \( g^-_{\gamma, i}(\beta^-_{\gamma, i}) = \theta_\gamma([-\eta, 0] \times \{t_i\}) \). Moreover, we may assume that \( g^+_{\gamma, i} \circ h \) is the identity on \( \theta_\gamma([0, \eta] \times \{t_i\}) \) and similarly for \( g^-_{\gamma, i} \circ h \). Since the \( Q^\pm_{\gamma, i} \) have disjoint interiors and diameter at most \( \leq \frac{\epsilon}{2} \), it follows that the product \( g := \prod_{\gamma \in \Gamma} g^+_{\gamma, i}g^-_{\gamma, i} \) is in \( \mathcal{V}_\gamma \cap H \).

For all \( q \in \mathbb{Q} \) let \( D_q^\gamma \) is the union of \( D_q^\gamma \) and all the sets of the form \( \theta_\gamma(I \times [-1, t]) \) for \( \gamma \in \Gamma \) with \( \gamma(-1) = q \) and \( \theta_\gamma(I \times [t, 1]) \) for \( \gamma \in \Gamma \) with \( \gamma(1) = q \). Notice that \( \text{diam}(D_q^\gamma) \leq \frac{\epsilon}{2} \), since \( \text{diam}(D_q^\gamma) \leq \frac{\epsilon}{6} \) and

\[
\text{diam}(\theta_\gamma([-\eta, \eta] \times [-1, t_0])) \leq \frac{\epsilon}{6},
\]

by the choice of \( \eta \) and similarly for \( \theta_\gamma(I \times [t, 1]) \).
Since \( g \) is supported on \( \bigcup \mathcal{N} \), the image of \( N \) by the rectified map \( g \circ h \) is still contained in \( \bigcup \mathcal{N} \). Additionally, \( g \circ h(N \cap E) = g \circ h(N) \cap E \) whenever \( E \) is either:

- \( D_q' \) for some \( q \in \mathcal{Q} \)
- \( \theta_\gamma([0, \eta] \times [t_i, t_{i+1}]) \) or \( \theta_\gamma([-\eta, 0] \times [t_i, t_{i+1}]) \) for \( \gamma \in \Gamma \) and \( 1 \leq i \leq m - 1 \)

and \( g \circ h \) restricts to the identity on the arc \( \partial E \cap N \). It follows from Fact 1.6 that \( g \circ h \) extends to some \( g' \in H \) which is the product of elements supported on sets \( E \) as above. Since \( \text{diam}(E) \leq \frac{1}{2} \) in both cases, it follows that \( g' \in \mathcal{V}_\frac{1}{2} \). This concludes the proof. The moreover part is clear.

\[ \square \]

3. Neighbourhoods of the identity contain fix-point stabilizers of embedded graphs

We begin with the following observation:

**Observation 3.1.** There do not exist disks \( D, D' \) with \( D \subseteq D' \) and \( \mathcal{V} \in \mathcal{N}_i(1) \) such that \( g \cdot D \subseteq D' \) for all \( g \in \mathcal{V} \).

**Proof.** Indeed, given such \( D, D' \), any \( p \in X \) and any \( \epsilon > 0 \) small enough applying Fact 1.3 two times yields some \( h \in H \) such that \( p \in h \cdot D \subseteq h \cdot D' \subseteq B(p, \epsilon) \). Then \( g \cdot D_0 \subseteq B(p, \epsilon) \) for any \( g \in \mathcal{V}_{\frac{1}{h}} \), where \( D_0 = h \cdot D \). It follows easily from this using the definition of compactness that the conjugates of \( \mathcal{V} \) by the action of \( H \) generate a system of neighbourhoods of \( t_\infty \) at the identity.

\[ \square \]

Lemma 3.3 below can be seen as a consequence of the theory of pseudo-Anosov mapping classes on compact surfaces. For the sake of self-containment and with the potential for higher dimension generalizations in view we provide a more elementary proof.

**Lemma 3.2.** Suppose that \( \alpha_1, \alpha_2, \ldots, \alpha_m \) are pair-wise transverse regular paths in \( X \) and \( U_1, \ldots, U_m \) open sets such that \( \alpha_1(1) \in U_1 \). Denote by \( \mathcal{I} \) the collection of self-intersection points of the \( \alpha_i \) and of intersection points between different \( \alpha_i \). Then for any \( \mu > 0 \) there exists some \( f_1 \in H \) such that if we let \( f = f_m f_{m-1} \ldots f_1 \) then \( \alpha_1 \setminus N_\mu(\mathcal{I}) \subseteq \alpha \cdot U_1 \) for \( 1 \leq l \leq m \).

**Proof.** Consider first the case \( m = 1 \). Let \( \alpha = \alpha_1 \), \( U_1 = U \) and \( p = \alpha(-1) \). Let \( \alpha = \alpha_1 \ast \ldots \ast \alpha_k \) be a decomposition of \( \alpha \) into arcs not containing any point of \( \mathcal{I} \) in their interior and write \( \alpha^i = \alpha^1 \ast \ldots \ast \alpha^i \). Let \( \vec{\alpha} = \alpha \setminus N_\mu(\mathcal{I}) \) and pick some \( \nu > 0 \) be smaller than \( \frac{\mu}{2} \) and \( d(\vec{\alpha}^i, \vec{\alpha}^j) \) for \( 1 \leq i < j \leq m \).

We choose \( g_i \in \mathcal{P}_\nu(\alpha^i) \) by induction so that if we write \( \vec{g}_i = g_i g_{i-1} \ldots g_1 (\vec{g}_0 = 1) \) then for all \( 0 \leq i \leq m \) the set \( \vec{g}_i \cdot U \) contains some path \( \beta^i \) from \( \vec{B}_\nu(p) \) to \( \vec{\alpha}^i(1) \) such that \( \vec{\alpha} \setminus N_{2\nu}(\mathcal{I}) \subseteq \beta^i \) and \( \alpha^i \subseteq \beta^i \) if \( i \geq 1 \).

In the base case we simply take as \( \beta^0 \) some non-trivial arc in \( U \) ending in \( \alpha(-1) \) such that \( \beta^0 \ast \alpha^0 \) is still an arc. Suppose now that \( i \geq 1 \) and the result has been shown for \( i - 1 \). Let \( \vec{\alpha}^i \) be a rectangular extension of \( \alpha^i \) with \( im(\vec{\alpha}^i) \subseteq N_\nu(\alpha^i) \) and let \( D = im(\vec{\alpha}^i) \). We may assume that \( D \cap \beta^{i-1} \) is a single arc \( \gamma \) from \( \partial D \) to \( \alpha^i(-1) \) (use a rectangular extension of an arc \( \gamma' \ast \beta \), where \( \gamma' \) is a subarc of \( \beta^{i-1} \)). By Fact 1.6 there exist some \( g_i \) supported in \( D \) such that \( g_i \cdot \gamma = \gamma \ast \alpha^i \). If \( 1 \leq i \leq m - 1 \), the construction ensures that \( \alpha^i \setminus N_{2\nu}(\mathcal{I}) \subseteq \vec{g}_{i+1} \cdot U \), from it which it follows that \( \vec{g}_m \cdot U \supseteq (g_{i+1} \cdot U) \bigcup_{j=i+2}^m N_\nu(\alpha^j) \supseteq \alpha^i \setminus N_{2\nu}(\mathcal{I}) \).
For the general case one can proceed similarly. Take \( \nu = \min\{\frac{1}{2} \mu, d(\tilde{\alpha}_t, \tilde{\alpha}_u)\}_{v \neq l} \), \( \tilde{\alpha}_t = \alpha_l \setminus N_{\mu}(I) \). We may assume that \( U_i \cap \bigcup_{v \neq l} N_{\nu}(\alpha_l) = \emptyset \). For \( 1 \leq l \leq m \) let \( T_l \) be the set of self-intersection points of \( \alpha_l \) and pick some \( f_l \in T_{\nu}(\alpha_l) \) such that \( \alpha_l \setminus N_{\mu}(T_l) \subseteq f_l \cdot U_l \). Let \( \tilde{f}_l = f_l f_{l-1} \cdots f_1 \). Then \( \tilde{f}_l(U_l) = f_l(U_l) \) and

\[

f \cdot U_l \supseteq (f_l \cdot U_l) \setminus \bigcup_{\nu \neq l} N_{\nu}(\alpha_l) \supseteq \alpha_l \setminus N_{\mu}(I).

\]

\[ \square \]

**Lemma 3.3.** Let \( D, E_1, \ldots, E_k \) be disks in \( X \) such that \( E_i \) is not contained in \( D \) for any \( 1 \leq i \leq k \). Let also \( K \subseteq X \) be a compact subset and \( \epsilon \) a positive real. Then there exists \( h, h' \in H \) fixing \( D \) such that for any \( 1 \leq i, j \leq k \) and any connected component \( C \) of the complement of \( h \cdot E_i \cup h' \cdot E_j \) either:

- \( K \cap C = \emptyset \)
- \( \text{diam}(C) < \epsilon \)
- \( C \subseteq N_\epsilon(D) \)

**Proof.** For \( 1 \leq i \leq k \) choose some \( p_i \in E_i \setminus D \). Pick some compact submanifold \( Y \) such that \( N_\epsilon(D), N_\epsilon(K) \subseteq Y \) and let \( T \) a triangulation of \( Y \). We can choose \( T \) so that each triangle has diameter at most \( \frac{\epsilon}{2} \) and \( \{p_i\}_{i=1}^k \cap T_D = \emptyset \). Let \( T_D \) be the collection of triangles disjoint from \( D \) and \( T_D \) be the collection of all their vertices. Given adjacent \( u, v \in T_D \) we denote by \( [u, v] \) the corresponding triangle side, an arc in \( X \).

Let \( \eta = \min\{\frac{\epsilon}{2}, \frac{1}{2} d([v, v'], [v, v] \cap T_D, v \neq v')\} \). For each \( v \in T_D \) choose disks \( F_v, F_v' \) such that \( v \in F_v, F_v \subseteq F_v' \) and \( F_v' \subseteq B(v, \eta) \) and let \( A_v \) be the annulus \( F_v' \setminus F_v \).

For \( 1 \leq i \leq k \) it is easy to find some regular path \( \alpha_i \) in \( Y \setminus D \) starting at \( p_i \) and such that for each triangle \( T \) in \( T \) that does not intersect \( D \) and each side \([u, v]\) of \( T \) there exists some arc \( \alpha_i^{[u, v]} \subseteq \alpha_i \cap T \) from a point \( u' \in F_u \) to a point \( v' \in F_v \).

We may also make the choice in such a way that the \( \alpha_i \) are pair-wise transverse and that if we denote by \( T \) the set consisting of all self-intersection points of \( \alpha_i \) for some \( i \) and of intersection points of \( \alpha_i \) and \( \alpha_j \) for different \( i, j \), then \( T \) is contained in \( \bigcup_{v \in T_D} F_v \) and disjoint from all the \( \alpha_i^{[u, v]} \).

Lemma 3.2 applied to the collection of paths \( \{\alpha_i\}_{i=1}^k \) with a suitably small constant \( \mu \) provides some \( f_i \in H \) fixing \( D \) such that for each \( 1 \leq i \leq k \) and each edge \([u, v]\) in a triangle \( T \) in \( T_D \) we have \( \alpha_i^{[u, v]} \subseteq f_i \cdot E_i \).

For each \( v \in T_D \) let \( \tau_v \in H \) be a Dehn twist over the annulus \( A_v \). Let \( \tau = \prod_{v \in T_D} \tau_v^2 \) and \( h' = \tau h \). Then by virtue of Observation 1.4 for each \( 1 \leq i, j \leq k \) the set \( E_{i,j} := h \cdot E_i \cup h' E_j \) contains:

- a core curve \( \beta_v \) of each of the annuli \( A_v, v \in T_D \)
- for each edge \([u, v']\) of some triangle \( T \) in \( T_D \) some path from \( \beta_v \) to \( \beta_{v'} \) in \( T \)

It easily follows that every connected component in the complement of \( E_{i,j} \) is either contained in \( N_\epsilon(D) \) or it has diameter less than \( \epsilon \) or else it is contained in \( N_\epsilon(Y^c) \) and is thus disjoint from \( K \). \[ \square \]

**Corollary 3.4.** Let \( D, E_1, \ldots, E_k \) be disks in \( X \). Assume that \( E_i \not\subseteq D \) for \( 1 \leq i \leq k \) and there exists \( \mathcal{V} \subseteq N_\epsilon(1) \) such that for all \( g \in \mathcal{V} \) there is \( 1 \leq i \leq k \) with \( g \cdot E_i \cap D = \emptyset \). Then for any \( \epsilon > 0 \) and any compact set \( K \subseteq X \) there is \( \mathcal{V}_D^{K, \epsilon} \subseteq N_\epsilon(1) \) such that for any \( g \in \mathcal{V}_D^{K, \epsilon} \) either:
such that $T$ transverse to the edges of $\Gamma$.

Suppose that $\ell_1$ (1) there exists $g \in \mathcal{V}$ such that $g \cdot D \cap E_i \neq \emptyset$ for all $1 \leq i \leq k$.

Proof. Suppose that $\mathcal{V} \in \mathcal{N}_1(1)$ fails to satisfy the property. Up to making $D$ smaller, we may assume that $E_i \subseteq D$ for all $1 \leq i \leq k$.

Using Lemma 3.3 take $h \in \mathcal{G}$ such that $\text{supp}(g_0) \subseteq h \cdot D =: D'$, let $\mathcal{V}' = \mathcal{V}_{h^{-1}}$ and pick some compact set $L \subset X$ and $\epsilon > 0$ such that $\mathcal{V}_{L,\epsilon} \subseteq \mathcal{V}'$ and $\mathcal{N}_1(D')$ is contained in a disk $D''$. Let also $\mathcal{V}_1 = \mathcal{V}_{h^{-1}} \in \mathcal{N}_1(1)$ be such that $g_0 \notin \mathcal{V}_1$. Consider the intersection $\mathcal{V}_0 := \mathcal{V}_1 \cap \mathcal{V}_{L,\epsilon}$, where $\mathcal{V}_{L,\epsilon}$ is given by Corollary 3.4 applied to $\mathcal{V}'$ and $D'$, $h \cdot E_i$. For any given $g \in \mathcal{V}_0$ at least one of the following possibilities holds:

- $g \cdot D' \subseteq D''$
- $\text{diam}(g \cdot D') < \epsilon$
- $g \cdot D' \cap L = \emptyset$

If $g \in \mathcal{V}_0$ satisfies the second or third possibility, then $g_0^{-1} \in \mathcal{V}_{L,\epsilon} \subseteq \mathcal{V}_1$ and thus $g_0 \cdot \mathcal{V}_1$, contradicting the choice of $\mathcal{V}_1$. Hence the first alternative must always hold, contrary to Observation 3.1.

Lemma 3.5. Let $D, E_1, \ldots, E_k$ be disks in $X$ and $\mathcal{V} \in \mathcal{N}_1(1)$. Then for any $\mathcal{V} \in \mathcal{N}_1(1)$ there exists $g \in \mathcal{V}$ such that $g \cdot D \cap E_i \neq \emptyset$ for all $1 \leq i \leq k$.

Proof. Suppose that $\mathcal{V} \in \mathcal{N}_1(1)$ fails to satisfy the property. Up to making $D$ smaller, we may assume that $E_i \subseteq D$ for all $1 \leq i \leq k$.

Using Lemma 3.3 take $h \in \mathcal{G}$ such that $\text{supp}(g_0) \subseteq h \cdot D =: D'$, let $\mathcal{V}' = \mathcal{V}_{h^{-1}}$ and pick some compact set $L \subset X$ and $\epsilon > 0$ such that $\mathcal{V}_{L,\epsilon} \subseteq \mathcal{V}'$ and $\mathcal{N}_1(D')$ is contained in a disk $D''$. Let also $\mathcal{V}_1 = \mathcal{V}_{h^{-1}} \in \mathcal{N}_1(1)$ be such that $g_0 \notin \mathcal{V}_1$. Consider the intersection $\mathcal{V}_0 := \mathcal{V}_1 \cap \mathcal{V}_{L,\epsilon}$, where $\mathcal{V}_{L,\epsilon}$ is given by Corollary 3.4 applied to $\mathcal{V}'$ and $D'$, $h \cdot E_i$. For any given $g \in \mathcal{V}_0$ at least one of the following possibilities holds:

- $g \cdot D' \subseteq D''$
- $\text{diam}(g \cdot D') < \epsilon$
- $g \cdot D' \cap L = \emptyset$

If $g \in \mathcal{V}_0$ satisfies the second or third possibility, then $g_0^{-1} \in \mathcal{V}_{L,\epsilon} \subseteq \mathcal{V}_1$ and thus $g_0 \cdot \mathcal{V}_1$, contradicting the choice of $\mathcal{V}_1$. Hence the first alternative must always hold, contrary to Observation 3.1.

Lemma 3.6. For any $\mathcal{V} \in \mathcal{N}_1(1)$ and any compact set $L \subset X$ there exists some embedded graph $\Gamma$ such that $H^+_1 \subseteq \mathcal{V}$ and $L$ is contained in the closure of one connected component $\mathcal{V}_0$ of $X \setminus \bigcup \Gamma$.

Proof. Take $\mathcal{V}_0 = \mathcal{V}_0^{-1} \in \mathcal{N}_1(1)$ with $\mathcal{V}_0 \subseteq \mathcal{V}$. Let $K$ a compact set and $\epsilon > 0$ be such that $\mathcal{V}_{K,\epsilon} \subseteq \mathcal{V}_0$. Now, pick some disk $D$ with $\text{diam}(D) < \epsilon$ and some triangulation $\mathcal{T}$ of a compact submanifold $Y$ of $X$ containing $\mathcal{V}_0(K \cup L)$ in which each triangle has diameter strictly less than $\epsilon$. Applying Lemma 3.5 we can find some $g' \in \mathcal{V}_0$ such that $g' \cdot D \cap T \neq \emptyset$ for each triangle $T$ in $\mathcal{T}$. Let $Y'$ be a compact submanifold such that $g \cdot D \subseteq \text{int}(Y')$, equipped with a triangulation $\mathcal{T}'$ that restricts to $\mathcal{T}$ on $Y$. We may also assume that $g' \cdot \partial D$ is transverse to all the internal edges of $\mathcal{T}'$, so that for any triangle $T \in \mathcal{T}'$ if it intersects $\mathcal{T}$ if and only if it intersects $\mathcal{T}$. Let $F$ be the union of all triangles of $\mathcal{T}'$ intersecting $g \cdot D$ and $\mathcal{V}$ the collection of vertices of $\mathcal{T}'$ that lie in $F$.

Claim 3.7. There is $\phi \in \mathcal{V}_{K,\epsilon}$ such that $\phi g \cdot D$ intersects exactly the same triangles of $\mathcal{T}'$ as $g \cdot D$ and $\mathcal{V} \subseteq \phi g \cdot D$.

Proof. Indeed, we can easily find a family of disjoint disks $\{D_u\}_{u \in \mathcal{V}}$ contained in $\mathcal{T}$ and $\{\phi_u\}_{u \in \mathcal{V}} \subseteq H$, where $\phi_u$ is supported on $D_u$, $u \in \mathcal{V}$, $g \cdot D$ and $D_u$ is either disjoint from $K$ or is contained in some small neighbourhood of some triangle $T \in \mathcal{T}$ and has diameter less than $\epsilon$. We then let $\phi = \prod_{u \in \mathcal{V}} \phi_u$.

Using Fact 11 we can slightly perturb $g := \phi g$ within $\mathcal{V}_0$ so that $g \cdot \partial D$ is transverse to the edges of $\mathcal{T}'$, while preserving the conclusion of Claim 3.7.

We construct an embedded graph $\Gamma$ in $X$ as follows. For each triangle $T$ in $\mathcal{T}'$ such that $T \cap g \cdot D \neq \emptyset$ and every connected component $C$ of $\mathcal{T} \setminus g \cdot D$ we pick
a vertex \( q_c \in C \cap \hat{C} \) and given two components \( C, C' \) in adjacent triangles \( T, T' \) satisfying \( C \cap C' \neq \emptyset \) add an arc between \( v_c \) and \( v_{c'} \) inside \( T \cup T' \) intersecting \( C \cap C' \) in a single point. The choice can clearly be made in such a way that two of the resulting arcs can only intersect at a common endpoint. We can assume that \( T \) contains at least 3-triangles so that no two of the resulting edges can have the same pair of endpoints.

Let \( U_0 \) be the connected component of \( X \setminus \bigcup \Gamma \) containing \( g \cdot D \).

**Claim 3.8.** \( K \cup L \subseteq U_0 \)

**Proof.** The inclusion \( N_e(K \cup L) \subseteq Y \) implies that for any triangle \( T \) in \( T \) with \( (K \cup L) \cap T \neq \emptyset \) the triangle \( T \) and any triangle in \( T \) adjacent to it must intersect \( g \cdot D \) (be contained in \( F \)) and thus that all the vertices of \( T \) must belong to \( Y \), since they are all in \( \text{int}(Y) \). It follows that for each component \( C \) of \( T \setminus \bigcup g \cdot D \) and every edge \( \gamma \in \Gamma \) with \( v_c \) as an endpoint and crossing some side \( \gamma \) of \( \Gamma \) there is some subarc \( \gamma' \subseteq \gamma \) intersecting \( g \cdot D \) only in \( \gamma'(-1) \) and intersecting \( \bigcup \Gamma \) only in \( \gamma(1) \in \gamma \cap \sigma \). It follows that any component \( V \) of \( T \setminus \bigcup \Gamma \) intersects \( g \cdot D \) and therefore that \( K \cup L \subseteq U_0 \).

**Claim 3.9.** For any neighbourhood \( N \) of \( \bigcup \Gamma \) there exists some \( f \in V_{K,e} \) such that \( f \cdot (U_0 \setminus N) \subseteq g \cdot D \).

**Proof.** It is easy to see that \( U_0 \setminus g \cdot D \) is the union of all the connected components of \( T \setminus (\bigcup \Gamma \cup g \cdot D) \) bordering \( g \cdot D \) as \( T \) ranges among all the triangles of \( T' \) that intersect \( g \cdot D \) non-trivially but are not contained in \( g \cdot D \). We construct \( f \) as a homeomorphism preserving each of the triangles in \( T' \) intersecting \( g \cdot D \) and fixing their complement in \( X \). We can first define \( f \) on the 1-skeleton of \( T' \), fixing \( u, v \) and preserving \( [u, v] \cap \bigcup \Gamma \) for every edge \( [u, v] \) and mapping \( [u, v] \setminus N \) into \( [u, v] \cap g \cdot D \) and then use [1.6] to find an extension to the interior of the triangles with the same property.

Take an arbitrary \( h \in H^+_{\bigcup \Gamma} \) and let \( h_0 \in H^+_{\bigcup \Gamma} \) the map that agrees with \( h \) on \( U_0 \) and is the identity outside of \( V_0 \). Let \( \delta = \delta(\Gamma, e) \) be the constant provided by Lemma 2.5. Continuity of \( h_0 \) implies the existence of a neighbourhood \( V \) of \( \Gamma \) such that \( d(p, h \cdot p) < \delta \) for all \( p \in V \). Lemma 2.5 then provides some \( h_1 \in V_\epsilon \cap H \subseteq V_0 \) agreeing with \( h_0 \) on \( N \cup (X \setminus U_0) \), where \( N \subseteq V \) is some neighbourhood of \( \Gamma \).

Then Claim 3.9 provides some \( f \in V_e \cap H^+_{\bigcup \Gamma} \) such that \( f \cdot (U_0 \setminus N) \subseteq g \cdot D \) so that

\[
\text{supp}(h_1^{-1}h_0) \subseteq \text{supp}(g^{-1}f \cdot (U_0 \setminus N) \subseteq D)
\]

It follows that \( h_0 \in h_1 V_{g^{-1}}^{2} \subseteq V_0^{8} \). On the other hand, \( h_1^{-1}h \in H \) is in \( V_0 \), since its support is disjoint from \( K \), and thus \( h \in V_0 \subseteq V \).

**Corollary 3.10.** For any \( V \in N_e(1) \) and any open \( U \subseteq X \) there exists some \( Q \)-embedded graph \( \Gamma \) with \( Q \subseteq U \) such that \( H^+_{\bigcup \Gamma} \subseteq V \). Moreover, we can assume \( \Gamma \) to be transverse to any given finite collection of arcs.

**Proof.** We first observe that for any finite set \( F \) of points, any ball \( B = B(p, \delta) \) and any \( W \in N_e(1) \) there exists some \( g \in W \) such that \( g \cdot F \subseteq B \). Indeed, take \( W_0 = W_{0}^{-1} \in N_e(1) \) with \( W_{0}^{2} \subseteq W \) and by Lemma 3.6 some embedded graph \( \Delta \) such that \( H^+_{\bigcup \Delta} \subseteq W_0 \) and \( \bigcup F \subseteq V \) for some connected component \( V \) of \( X \setminus \bigcup \Delta \). Since \( t \in t_0 \) there exists some \( g_0 \in W_0 \) such that \( g_0 \cdot F \subseteq U \) and then some \( g_1 \in H^+_{\bigcup \Gamma} \) such that \( g_1 \cdot (g_0 \cdot F) \subseteq B \) by Fact 1.6.
Proof. The first alternative must take place, but this contradicts Corollary 3.10.

Claim 3.12. For any $\mathcal{V} \in \mathcal{N}_i(1)$ there exists some embedded graph $\Gamma$ with $H^{+}_{\Gamma} \subseteq \mathcal{V}$ which is transverse to the $\beta_i$ and satisfies the second alternative above.

Proof. Suppose not. Notice that by Fact 1.15 there exists some $f \in H$ preserving $D$ and mapping $im(\beta_1)$ homeomorphically onto $im(\beta_{i+1})$ (cyclically). If we let $\mathcal{V}_0 = \mathcal{V} \cap \mathcal{V}^j$, then for any $\Gamma$ transverse to the $\beta_i$ such that $H^{+}_{\Gamma} \subseteq \mathcal{V}_0$ necessarily the first alternative must take place, but this contradicts Corollary 3.10.

Now, given $\alpha$ and $\epsilon > 0$ as in the premise, choose some embedded disk $\mathcal{E}$ with $\alpha \subseteq \hat{\mathcal{E}} \subseteq \mathcal{E} \subseteq \mathcal{N}_i(\alpha)$, $\mathcal{V}_0 = \mathcal{V}_0^{-1} \in \mathcal{N}_i(1)$ with $\mathcal{V}_0^{\delta} \subseteq \mathcal{V}$ and $\delta > 0$ such that

- $B(p,\delta), B(q,\delta)$ are connected
- $B(p,\delta) \cup B(q,\delta) \subseteq \mathcal{E}$
- $\mathcal{V}_{2\delta} \subseteq \mathcal{V}_0$
- $d(p, q) > 3\delta$

Now choose an embedded disk $\mathcal{D} \subseteq \hat{\mathcal{E}}$ bounded by $\beta_i^{-1}$ as above so that, $p \in \beta_1(\hat{\mathcal{D}})$, $q \in \beta_3(\hat{\mathcal{D}})$ and $diam(\beta_1), diam(\beta_3) < \delta$.

If $\Gamma$ satisfies $H^{+}_{\Gamma} \subseteq \mathcal{V}_0$ and is as in the second alternative, then for $\eta > 0$ small enough and some arc $\kappa$ in $\mathcal{D}$ from some $p' \in \beta_1(\hat{\mathcal{D}})$ to $q' \in \beta_3(\hat{\mathcal{D}})$ we have $N_{\eta}(im(\kappa)) \subseteq \mathcal{E} \setminus \mathcal{D}$. On the other hand, there is $h$ supported in the disjoint union $B(p,\delta) \cap B(q,\delta)$ such that $h \cdot (p, q) = (p', q')$. Then $h \in \mathcal{V}_{2\delta}$ and

$$\mathcal{P}_y(\kappa)^h \subseteq (H^{+}_{\Gamma})^h \subseteq \mathcal{V}_0^{\delta} \subseteq \mathcal{V},$$

while on the other hand $\emptyset \neq \mathcal{P}_y(\kappa)^h \subseteq \mathcal{P}_*(\alpha)$.

\[\square\]

4. Untangling embedded graphs

Lemma 4.1. Let $\mathcal{Q}$ be a finite set of points, $\Gamma = (\gamma_1, \ldots, \gamma_k)$ a $\mathcal{Q}$-embedded graph, $Y$ a compact subsurface of $X$ such that $\Gamma$ is transverse to $\partial Y$ and let $g \in H$ supported on $Y$ be such that $g_{\partial Y} \in \mathcal{H}_0(\partial Y)$ and $\Gamma$ and $\Gamma' := g \cdot \Gamma$ are transverse on $int(Y)$.

Then for any $\mathcal{V} \in \mathcal{N}_i(1)$ there exists $h_0 \in H^{+}_{\Gamma}$ and $h_1 \in \mathcal{V}$ supported on $Y$ such that $h_0 h_1 \cdot \Gamma \simeq \Gamma'$.

Proof. We prove the result by induction on $N := |(\bigcup \Gamma) \cap (\bigcup \Gamma') \cap int(Y)| < \infty$.

Pick some $\mathcal{V}_0 = \mathcal{V}_0^{-1} \in \mathcal{N}_i(1)$ such that $\mathcal{V}_0^{\delta} \subseteq \mathcal{V}$.
Let $\mathcal{A}$ be the collection of maximal subarcs of some $\gamma_i$ contained in $Y$. Notice that any two distinct $\alpha, \alpha' \in \mathcal{A}$ satisfy $\alpha \cap \alpha' = \emptyset$. If $N = 0$ then for all $\alpha \in \mathcal{A}$ we have that $\alpha \cup \alpha'$ is the boundary of some disk $D \subseteq Y$ with $D \cap (\bigcup \Gamma \cup \bigcup \Gamma') = \emptyset$.

Then for any $\epsilon > 0$ one can easily find some $h_1 \in \mathcal{V}_\epsilon \cap H$ supported on $Y$ such that $h_1 \cdot \alpha$ lies inside the closed bigon bounded by $\alpha$ and $\alpha'$ for all $\alpha \in \mathcal{A}$ and then some $h_0 \in H^+_{\mathcal{A}}$, such that $h_0 h_1 \cdot \Gamma \approx \Gamma'$.

Assume now that $N > 0$. Then by Claim 4.2 there are $1 \leq i, j \leq k$ and a bigon $B$ in $Y$ whose interior is disjoint from $\Gamma \cup \Gamma'$ and which is bounded by the union of a subarc of $\gamma_i$ and a subarc of $\gamma'_j$ meeting only at the endpoints.

It is easy to find a disk $D$ with $B \subseteq D$ intersecting $\Gamma$ in a single subarc $\alpha_i \subset \gamma_i$ and $\Gamma'$ in a single subarc $\alpha'_j \subset \gamma'_j$ and containing no other points from $\bigcup \Gamma \cap \bigcup \Gamma'$ other than the two intersection points in $\partial B$. Then Fact 1.4 yields some $f$ supported on $D$ such that $f^{-1} \cdot \alpha'_j$ and $\alpha_i$ do not cross.

We can then apply the induction hypothesis to $f^{-1} \cdot \gamma$ and $\gamma_0$. We obtain $h_0' \in H^+_{\Gamma}$ and $h_1' \in \mathcal{V}_0$ supported on $Y$ such that $f^{-1} \cdot \Gamma = h_0' h_1' \cdot \Gamma$.

We claim that there exists $h''_0 \in H^+_{\Gamma}$, $\hat{h} \in H_{[h_0', h_1']}$, $h''_1 \in \mathcal{V}_0$ supported on $Y$ such that $f h''_0 = h''_0 h''_1 \hat{h}$. A simple calculation then shows that $h_0 := h_0' h''_0 \in H^+_{\Gamma}$ and $h_1 := h''_0 h''_1 \in \mathcal{V}_0 \subseteq \mathcal{V}$ verify the properties we need.

Notice that $f' := f h_0'$ is supported on $D' = (h_0')^{-1} \cdot D$. It brings the arc $\beta''_j := (f h_0')^{-1} \cdot \alpha'_j \subseteq h_1 \cdot \gamma_j \cap D$, which is disjoint from the arc $\beta_i := (h_0')^{-1} \cdot \alpha_i \subseteq \gamma_i \cap D$, to a position in which it intersects the latter in exactly two points, creating a bigon inside $D'$. The proof now reduces to the following:

Claim 4.2. There is $h''_0 \in \mathcal{V}_0^2$ supported in $D'$ bringing, $\beta''_j$ to a curve intersecting $\beta_i$ in exactly two points (bounding a bigon).

Indeed, once such $h''_0$ is given, we first see using Fact 1.4 the existence of $\hat{h} \in H^+_{[h_1', h_1]}$ supported on $D'$ such that

$$(h''_0 \hat{h})^{-1} (\beta_i) \cap \beta''_j = \hat{h}^{-1} (h''_0)^{-1} (\beta_i) \cap \beta''_j = f'^{-1} (\beta_i) \cap \beta''_j$$

and then the existence of $h''_0 \in H^+_{[h_1', h_1]}$ such that $h''_0 h''_1 \hat{h} = f'$ easily follows from the same fact.

To prove the claim pick some disk $E$ in $D'$ which is divided into two smaller disks by an arc of $\beta_i$ and has a diameter small enough so that any homeomorphism supported on $E$ belongs to $\mathcal{V}_0$. Choose points $p \in \beta''_j$, $q \in E$ in the same connected component $C_0$ of $D' \setminus \beta_i$, as well as some arc $\omega$ in $C_0$ from $p$ to $q$.

Let $\epsilon > 0$ be small enough that $N_\epsilon(\omega) \subseteq C_0$. By Lemma 3.11 there exists some $\phi \in \mathcal{P}_\epsilon(\omega) \cap \mathcal{V}_0$. The arc $\phi \cdot \beta''_j$ does not intersect $\beta_i$, but it intersects $E$, so it is easy to see that we can choose some $\theta$ supported on $E$ such that $h''_0 : = \theta \phi \in \mathcal{V}_0^2$ brings $\beta''_j$ into the desired configuration (we may assume $\partial E$ and $\phi \cdot \beta''_j$ are transverse).

Remark 3. We may not have simply chosen $\omega$ to be a point-pushing map along a path crossing $\beta$, and dispense of $\theta$ altogether, since the condition $\phi \in N_\epsilon(\omega)$ is too weak to guarantee multiplying on the left by $\phi$ creates only one new bigon.

\\

Lemma 4.4. For any $\epsilon > 0$ and any embedded graph $\Gamma = (\gamma_j)_{j=1}^k$ we have

$$H^+_{[\Gamma]} \subseteq \mathcal{V}_\epsilon H^+_{\bigcup \Gamma} \mathcal{V}_\epsilon H^+_{\bigcup \Gamma}.$$
Proof. Let $Q = V(\Gamma)$. It suffices to show that for any homeomorphism $h$ of $\cup \Gamma$ preserving the arcs of $\Gamma$ with orientation and fixing their endpoints there is some $g \in \mathcal{V}_h \mathcal{H}_{\Gamma}^+ \mathcal{V}_\Gamma$ such that $g_\Gamma = h$.

Fact 2.4 there exists some family $\{D_q\}_{q \in Q}$ of disks with $q \in \hat{D}_q$ and $\delta \in (0, \frac{\epsilon}{10})$ such that $D_q$ intersects $\Gamma$ in a star, $\text{diam}(D_q) < \frac{\epsilon}{10}$ and $B(q, 5\delta) \subseteq D_q$.

By continuity of $h$ there are subarcs $\gamma_j \subseteq \gamma_j \setminus Q$ such that:

$$\gamma_j \setminus \gamma_j \cup h \cdot (\gamma_j \setminus \gamma_j) \subseteq N_{\delta}(\gamma_j(\{-1, 1\}))$$

By Facts 1.7 and 2.4 there exist some $\phi_j \in H$ supported in some small neighborhood of $\bigcup \gamma_j \setminus \gamma_j$ such that $\phi_j \circ h$ preserves $\gamma_j$ and fixes the endpoints of $\gamma_j$ (thus preserving $\gamma_j$ as well). It is not difficult to see that one can choose $\phi_j \in \mathcal{V}_{\Delta} \phi_j$ with disjoint supports, so that $\phi := \prod_{j=1}^k \phi_j \in \mathcal{V}_{\Delta} \phi_j$.

The following easy consequence of Fact 2.4 is left to the reader.

Claim 4.5. There exists some $g_1 \in \mathcal{V}_{\Delta}$ such that $g_1 \cdot \gamma_j \cap \bigcup \Gamma = \emptyset$ for all $1 \leq j \leq k$.

Consider now the partial homeomorphism $h' = \left( g_1 \circ h \circ g_1^{-1} \right) |_{\bigcup \gamma_j \setminus \gamma_j}$. Notice that the fact that $g_1 \cdot \gamma_j$ admits a rectangular extension implies the existence of some disk $D_j$ disjoint from $\bigcup \Gamma$, with $\partial D_j$ transverse to $g_1 \cdot \gamma_j$ and $D_j \cap g_1 \cdot \gamma_j = g_1 \cdot \gamma_j$.

Together with Fact 1.6 this implies that there is some $g_2 \in H$ which extends $h'$, is orientation preserving at each arc $g_1 \cdot \gamma_j$ and is supported in the complement of $\bigcup \Gamma \cup \bigcup_{j=1}^k g_1 \cdot (\gamma_j \setminus \gamma_j)$. In particular $g_2^{\Delta} \gamma_j$ is the identity on $\bigcup_{j=1}^k (\gamma_j \setminus \gamma_j)$.

On the one hand, $\phi \in \mathcal{V}_{\Delta}$ for all $\eta \in \{0, 1\}$ the component of $\gamma_j \setminus \gamma_j$ in $B(\gamma_j(\eta), \delta)$ is mapped by $\phi$ into $B(\gamma_j(\eta), 5\delta) \subseteq \hat{D}_q$. On the other hand, $\gamma_j$, so the image of said component is disjoint from $\bigcup \gamma_j \setminus \gamma_j$. It follows from Fact 1.7 that there exists some $g_3 \in H$ supported on $\bigcup_{q \in Q} D_q$ fixing $\bigcup_{j=1}^k \gamma_j$ such that $(g_3)_s = \phi_1 s$. Notice that $g_3 \in \mathcal{V}_{\Delta}$. The element $\phi^{-1} g_3 g_1^{-1} g_2 g_1 \in \mathcal{V}_{\Delta} \mathcal{V}_{\Delta} \mathcal{V}_{\Delta} \mathcal{V}_{\Delta} \mathcal{V}_{\Delta} \mathcal{V}_{\Delta} \mathcal{V}_{\Delta} \mathcal{V}_{\Delta} \mathcal{V}_{\Delta} \mathcal{V}_{\Delta}$ agrees with $h$ on the entire $\bigcup \Gamma$ and is orientation preserving at $\Gamma$ so we are done.

5. Concluding the proof

Proof of Theorem 4.3. Pick some $V_0 = V_0^{-1} \in N_1(1)$ such that our fixed element $g_0$ does not belong to $V_0$ and let $D_0$ be a disk on which $g_0$ is supported. By Corollary 3.10 there is some embedded $\mathcal{Q}$-graph $\Gamma$ such that $H_{\Gamma}^{\Delta} \subseteq V_0^0$, $Q \cap D_0 = \emptyset$ and $\Gamma$ is transverse to $\partial D_0$. Using Fact 1.1 we can find some $f \in H$ supported on $D_0$ such that $f g_0 \cdot \Gamma$ and $\Gamma$ are transverse on $\hat{D}_0$. By Lemma 4.3 there are $h_0, h_1 \in \mathcal{V}_{\Delta}$ supported on $D_0$ such that $h_0 h_1 \cdot \Gamma \simeq \Gamma'$.

The element $\psi := h_1^{-1} h_0^{-1} f g_0$ is the identity in a neighborhood of $Q$ and satisfies $\psi \cdot \Gamma \simeq \Gamma$. It follows that $\psi \in \mathcal{H}_{\Gamma}^+$. Finally, note that $H_{\Gamma}^{\Delta} \subseteq V_0^0$ by Lemma 4.3 so that

$$g_0 \in f^{-1} h_0 h_1 H_{\Gamma}^{\Delta} \subseteq V_0 H_{\Gamma}^{\Delta} V_0 H_{\Gamma}^{\Delta} \subseteq V_0^8,$$

a contradiction.

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