ON SUBVARIETIES OF ABELIAN VARIETIES WITH DEGENERATE GAUSS MAPPING

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ABSTRACT. Let \(X\) be an abelian variety. Then the 'conormal bundle' of an irreducible subvariety \(Y\) of \(X\) defines an irreducible Lagrangian subvariety \(\Lambda_Y\) of the cotangent bundle \(T^*(X) \cong X \times \text{Lie}(X)^*\) of \(X\). Our aim is to show that the projection morphism \(\gamma : \Lambda_Y \to \text{Lie}(X)^*\) is generically finite unless \(Y\) is stable under translation by a nontrivial abelian subvariety.

Introduction. For a \(d\)-dimensional irreducible subvariety \(Y\) of an abelian variety \(X\) of dimension \(g\) the Gauss mapping \(\Gamma\) attaches to a regular point \(y\) of \(Y\) the tangent space \(T_y(Y)\) of \(Y\) at \(y\), considered as a \(d\)-dimensional linear subspace of the tangent space \(T_y(X)\) of \(X\) at \(y\). If the latter is identified with the Lie algebra \(T_0(X)\) by a translation, this defines a point of the Grassmann variety \(\text{Gr}(d, T_0(X))\). For varying \(y\), this defines an algebraic morphism on the subset \(S = Y_{\text{reg}}\) of regular points of \(Y\)

\[ \Gamma : S \to \text{Gr}(d, T_0(X)). \]

Properties of the Gauss mapping have been extensively studied by Ochiai, Kawamata, Bogomolov, Ueno, Noguchi, Mori, Abramovich, Ran and others; see e.g. for an overview [D], [D2]. In particular, Abramovich [A] has shown that \(\Gamma\) fails to be a generically finite morphism if and only if the subvariety \(Y\) is degenerate in the sense that there exists an abelian subvariety \(A\) of \(X\) of dimension > 0 such that \(A + Y = Y\) holds.

Instead of the above mentioned Gauss mapping \(\Gamma\) involving tangent spaces, we consider another type of Gauss mapping that involves cotangent spaces. An irreducible subvariety \(Y\) of the abelian variety \(X\) defines a Lagrangian subvariety \(\Lambda_Y\) of the cotangent bundle \(T^*(X)\) of \(X\), which for smooth \(Y\) is just the conormal bundle of \(Y\) in \(X\). For an abelian variety the cotangent bundle splits \(T^*(X) \cong X \times T_0^*(X)\). Hence the projection onto the second factor \(T_0^*(X)\) (the dual of the Lie algebra) induces an algebraic morphism

\[ \gamma : \Lambda_Y \to T_0^*(X). \]

The main result of this paper (Theorem 1) is the following analog of the result of Abramovich: If the mapping \(\gamma : \Lambda_Y \to T_0(X)^*\) is not dominant, then \(Y\) is degenerate. For \(d = 1\) and \(d = g - 1\) both \(\gamma\) and \(\Gamma\) can be identified. Otherwise, by the obvious identification \(\text{Gr}(d, T_0(X)) \cong \text{Gr}(g - d, T_0(X)^*)\), the Gauss mapping \(\Gamma\) can also be viewed as a morphism \(S \to \text{Gr}(g - d, T_0(X)^*)\), and as such may be naturally extended to a morphism \(\alpha : \Lambda_Y|S \to V\) from the normal bundle \(\Lambda_Y|S\) over \(S\) to the tautological vector bundle \(V\) of degree \(g - d\) over \(\text{Gr}(g - d, T_0(X)^*)\).

Since \(V\) is a subbundle of the trivial vector bundle \(\text{Gr}(g - d, T_0(X)^*) \times T_0(X)^*\),
we dispose over a natural projection $\beta : V \rightarrow T_0(X)^*$ such that $\gamma|_S = \beta \circ \alpha$. So clearly, $\dim(\Gamma(S)) < d$ implies $\dim(\alpha(\Lambda_Y|_S)) < g$. Hence, by dimension reasons, the mapping $\gamma : \Lambda_Y \rightarrow T_0(X)^*$ cannot be dominant in case of $\dim(\Gamma(S)) < d$. Thus, our main result is closely related to the geometry of the classical Gauss mapping and reproves the result of Abramovich for $\Gamma$ (theorem 2). On the other hand, the result of Abramovich for $\Gamma$ does not seem to imply the result for $\gamma$ in a direct way. But, nevertheless, our proof makes significant use of the methods developed in [A] for the study of the Gauss mapping $\Gamma$.

Our interest in the mapping $\gamma$ comes from the fact, that the generic degrees $d(Y) = \deg(\gamma : \Lambda_Y \rightarrow T_0^*(X))$ are strongly related to the singularities of $Y$ and the perverse Euler-Poincare characteristic of $Y$. Indeed, an irreducible subvariety $Y$ of a complex abelian variety naturally defines a characteristic subvariety $Ch_Y$ of the cotangent bundle $T^*(X)$ via the characteristic variety of the $D$-module associated to the middle perverse intersection cohomology sheaf of $Y$ (see below). This characteristic variety is a finite union of irreducible Lagrangian subvarieties $\Gamma(\nu) \subset T^*(X)$, hence each of them is of the form $\Lambda_{\nu}$ for certain irreducible ‘characteristic’ subvarieties $\nu_Y$ of $Y$, with multiplicities $m(\nu_Y)$. As shown in [FK] the Euler-Poincare characteristic

$$\chi_Y = \sum_i (-1)^i \dim(H^i(Y, IC_Y[d]))$$

of the intersection cohomology groups $H^i(Y, IC_Y[d])$ of $Y$ is the sum of the generic degrees $d(\nu_Y)$ of the Lagrangians $\Lambda_{\nu_Y}$

$$\chi_Y = \sum \nu m(\nu_Y)d(\nu_Y).$$

Notice, $Y$ itself is always one of the ‘characteristic’ subvarieties $\nu_Y$. All components $\nu_Y \neq Y$ are contained in the singular locus $Y_{\text{sing}} = Y \setminus S$. If such $\nu_Y \neq Y$ do occur or not, seems to depend on the structure of the singularities of $Y$ in a rather complicated and not well understood way.

The study of singularities of theta divisors has a long tradition after the work of Andreotti and Mayer [AM]. As an application of theorem 1 we show that the theta divisor $Y$ of the Jacobian $X$ of a generic curve, although highly singular by Riemann’s theorem, only admits $Y_{\nu} = Y$ as singular component (theorem 3). For the theta divisor $Y$ of a principally polarized abelian variety $X$, the structure map $p_Y : \Lambda_Y \rightarrow Y$ is a birational morphism, but there are examples [Kr] where there appear other $Y_{\nu}$ than $Y$ defining components of $Ch(Y)$. The above relation between the invariants $\chi_Y$ and $d(\nu_Y)$ is also interesting from the point of view given in [KrW], where it is shown that $\chi_Y$ can be interpreted as the dimension of an irreducible representation $\omega_Y$ of a reductive group $G(Y)$, both canonically attached to $Y$. In the case of theta divisors, the groups $G(Y)$ define an interesting stratification of the moduli space of principally polarized abelian varieties. In this context, we refer to related results on $d(Y)$ from [J], [GM], [SS].

Concerning the definition of $Ch(Y)$, recall that any $D$-module $K$ on a smooth complex algebraic variety $X$ has an associated characteristic variety $Ch(K)$ that
by definition is a subvariety of the cotangent bundle $T^*(X)$ of $X$. If the $D$-module $K$ is holonomic, all the irreducible components of the characteristic variety $Ch(K)$ are subvarieties $\Lambda_{Y_\nu} \subset T^*(X)$ that are defined by the conormal bundles of certain irreducible subvarieties $Y_\nu$ of $X$. See [KS], [G]. Any middle perverse sheaf $P$ on $X$ defines a holonomic $D$-module $K$ via the Riemann-Hilbert correspondence. Applied for the perverse intersection cohomology sheaf $P = IC_Y[d]$ of $Y$, the results mentioned above for $Ch(Y)$ are a special case of the following formula: The sum of the degrees of the mappings $\gamma: \Lambda_{Y_\nu} \to T^*_0(X)$ defined by the components of the characteristic variety $Ch(K)$, with multiplicities, is the Euler-Poincare characteristic $\chi(P)$ of the perverse sheaf $P$ defined by the solutions of the holonomic $D$-module $K$ via the Riemann-Hilbert correspondence [FK]. Notice, $\chi(P) = 0$ implies that $P$ is translation invariant by an abelian subvariety $A$ of $X$ of dimension $> 0$. For simple abelian varieties $X$ a short proof for this can be found in [KrW]. In general, all known proofs of this result are more complicated. We wonder whether theorem 1 can be helpful to find a short argument as in [KrW] also in the non-simple case.

The paper is organized as follows: In §1 we formulate the results and prove theorem 2 and 3. In §2 we review some results from [A]. Then, in the remaining part of the paper the proof of theorem 1 is given via induction on $g = \dim(X)$. In §3 we make some preparations in order to deal with non-simple abelian varieties. Using an argument from [R], we give in §4 the proof in the case of simple abelian varieties $X$ (lemma 3) formulated in a way that is suitable for the induction argument. We remark, if $X$ is simple and in addition $Y$ is smooth, this is reminiscent of a result of Hartshorne [H, §4]. Indeed, then regular 1-forms on $\Lambda_Y$ descend to $Y$ because $\Lambda_Y \to Y$ defines an ample vector bundle unless $Y$ is degenerate. In the remaining part of §4, by iterated use of lemma 3, we reduce the proof of theorem 1 to the codegenerate case (corollary 3) that finally is treated in §5.

§1 Notations and preliminary remarks. Let $X$ be an abelian variety over an algebraically closed field $k$ of dimension $g$. For a closed irreducible subvariety $Y$ of dimension $d$ in $X$ we let $S$ denote a Zariski dense open subset of its regular locus $Y_{reg}$. For the conormal bundle $p_S: T^*_0(X) \to S$ let $\Lambda_Y$ denote the closure of $T^*_0(X)$ in the cotangent bundle $T^*(X)$ of $X$. The cotangent bundle is trivial for an abelian variety, and with respect to the corresponding decomposition

$$T^*(X) \cong X \times T^*_0(X)$$

the structure morphism of the bundle $p_X: T^*(X) \to X$ is given by the projection on the first factor. This defines a structure morphism $p_Y: \Lambda_Y \to Y$ by the upper horizontal arrows of the diagram

$$\Lambda_Y \xleftarrow{i_Y} T^*(X) \xrightarrow{p_X} X,$$

$$\Lambda_S \xrightarrow{p_S} S$$

if we take into account that the image of $p_X \circ i_Y$ is contained in $Y$, since $Y$ is the Zariski closure of $S$ and $\Lambda_Y$ is the Zariski closure of $\Lambda_S$. Notice, $\dim(\Lambda_Y) = g$
and \( \dim(\Lambda_Y \setminus \Lambda_S) < g \). For \( y \in S \) let \( \Lambda_{S,y} \) denote the conormal space \( N^*_y(Y) \) of \( Y \) in \( T^*_y(X) \) at \( y \), i.e. the fiber \( p^*_S(y) \). By a translation, \( T^*_y(X) \) will always be identified with its image in \( T^*_0(X) = -y + T^*_y(X) \), so in this sense \( \Lambda_{S,y} \subset T^*_0(X) \).

The Gauss mapping \( \gamma \). The projection \( T^*(X) \cong X \times T^*_0(X) \to T^*_0(X) \) on the second factor restricted to \( \Lambda_Y \subset T^*(X) \) induces the Gauss mapping
\[
\gamma: \Lambda_Y \to T^*_0(X).
\]

We write \( \lambda = (y, \tau) \in \Lambda_Y \), where \( y = p_Y(\lambda) \in Y \) and \( \tau \in \Lambda_{Y,y} \subset T^*_0(X) \). Then the image under the Gauss mapping is the second component \( \gamma(\lambda) = \tau \in T^*_0(X) \). For \( y \in S \), the conormal vector \( \tau \in T^*_0(X) \), resp. in \( T^*_Y(Y, y) \subset T^*_0(X) \) after translation, annihilates the tangent space \( T_y(Y) \) of \( Y \) at \( y \).

The case \( Y = X \) is exceptional, since in this case the image \( \gamma(\Lambda_Y) \) of the Gauss mapping is contained in \( \{0\} \). For \( Y \neq X \), we can remove both the closure of the zero section in \( \Lambda_Y \) and the zero section in \( T^*(X) \) to obtain a proper morphism between the associated projective conormal bundles
\[
\mathbb{P}\gamma: \mathbb{P}\Lambda_Y \to \mathbb{P}(T^*_0(X)).
\]

Obviously for \( Y \neq X \), the morphism \( \mathbb{P}\gamma \) is dominant if and only if \( \gamma \) is dominant. This allows to ignore the trivial vector \( \tau = 0 \in T^*_0(X) \) in subsequent arguments. If the morphism \( \gamma \) is not dominant, the image of \( \mathbb{P}(\Lambda_Y) \) is a closed subvariety of \( \mathbb{P}(T^*_0(X)) \), hence contained in a hypersurface that is defined as the zero locus of some nontrivial homogeneous polynomial \( F \) on \( T^*_0(X) \).

Since \( \dim(\Lambda_Y) = g \), for \( Y \neq X \) the following assertions are equivalent:

(a) \( \dim(\gamma(\Lambda)) < g \)

(b) The proper morphism \( \mathbb{P}\gamma: \mathbb{P}\Lambda_S \to \mathbb{P}(T^*_0(X)) \) is not dominant.

(c) The image \( \gamma(\Lambda_S) \) of the Gauss mapping is contained in the zero locus of a nontrivial homogeneous polynomial \( F \) on \( T^*_0(X) \).

(d) \( \gamma: \Lambda_Y \to T^*_0(X) \) is not dominant.

(e) \( \gamma: \Lambda_Y \to T^*_0(X) \) is not generically finite.

(f) For any point \( y \) of general position in \( S \) there exists a curve \( C \) in \( S \) containing \( y \), which is contracted by \( \gamma \).

Similarly as for irreducible \( Y \), one defines the Gauss mapping \( \gamma \) for reducible closed varieties \( Y \). In this case the Gauss mapping is dominant if and only if the Gauss mapping for one of its irreducible components is dominant.

An irreducible variety \( Y \) in \( X \) will be called degenerate, if there exists an abelian subvariety \( A \subset X \) of positive dimension with the property \( A + Y = Y \). (This notion differs from the one used in [R]). Our main result is

**Theorem 1.** For a closed irreducible subvariety \( Y \) of \( X \) the following assertions are equivalent

(a) The Gauss mapping \( \gamma: \Lambda_Y \to T^*_0(X) \) is not dominant.

(b) \( Y \) is degenerate.
The Gaß mapping $\Gamma$. If we assign to each point $y \in S$ the tangent space $T_y(Y)$ in $T_y(X)$, this gives rise to the Gaß mapping $\Gamma$, now with image in the Graßmann variety

$$\Gamma : S \rightarrow Gr(d, T_0(X)).$$

Here again the tangent space $T_y(Y)$ is considered as a subspace of $T_0(X)$, using translation by $y \in X$.

The following reproves theorem 4 of Abravich [A]; see also [R], chapter II.

**Theorem 2.** For a closed irreducible subvariety $Y$ of $X$ the following holds: If the Gaß mapping $\Gamma : S \rightarrow Gr(d, T_0(X))$ is not generically finite, then the Gaß mapping $\gamma : \Lambda_Y \rightarrow T_0^*(X)$ is not dominant and hence $Y$ is degenerate.

**Proof.** If $\Gamma$ is not generically finite, for any point $y$ of $S$ in general position there exists an algebraic curve $C$ containing $y$, which is contracted under $\Gamma$. Then for $y' \in C$ we have $T_y(Y) = T_{y'}(Y)$ in $T_0(X)$, hence $\Lambda_{S,y} = N^*_y Y = N^*_{y'} Y = \Lambda_{S,y'}$. Since then $\gamma(y, v) = \gamma(y', v)$ holds for all $v \in \Lambda_{S,y} \subset T_0^*(X)$, the curve $C \times \{v\} \subset \Lambda_S$ is contracted by $\gamma$, for all $v \in N^*_y(Y) \subset T_0^*(X)$. So there exist points in general position contracted by $\gamma$, hence $\gamma : \Lambda_S \rightarrow T_0^*(X)$ can not be dominant by dimension reasons. Therefore $Y$ is degenerate by theorem 1. \[\square\]

**Theorem 3.** For the theta divisor $Y$ of the Jacobian $X$ of a generic regular projective complex curve $C$ the characteristic variety $Ch(Y)$ is irreducible.

**Proof.** We can assume that $C$ is not hyperelliptic. Since $\dim(Y) = g - 1$, we need not distinguish between $\Gamma$ and $\gamma$. So $d(Y) = \binom{2g-2}{g-1}$ is the generic degree of $Y$ for the classical Gaß mapping [GH, p. 360]. Also the perverse Euler-Poincare characteristic $\chi_Y$ of the theta divisor is $\binom{2g-2}{g-1}$; see [W, p.273]. Hence $d(Y_v) = 0$ for every irreducible component $\Lambda_{Y_v} \neq \Lambda_Y$ of $Ch(Y)$, as follows from the formula of [FK]. So all $Y_v \neq Y$ are degenerate by theorem 1. If $Ch(Y)$ were not irreducible, therefore some $Y_v \neq Y$ and hence also $Y$ would contain $A + A$ for some $y \in Y$ and some abelian subvariety $A \subset X$ of dimension $> 0$. Since the Jacobian of a generic curve of genus $g$ is an irreducible abelian variety [CG], this would imply $A = X$ which is not possible. \[\square\]

**Concerning the proof of Theorem 1.** To show (a) $\implies$ (b) will cover the rest of this paper. The converse is trivial: Suppose $Y$ is degenerate and $A + Y = Y$ holds. For an abelian variety $A \subset X$ of $\dim(A) > 0$, let $\tilde{Y}$ denote the image of $Y$ in the quotient $B = X/A$. Notice, $A + Y = Y$ implies $T_y(A) \subset T_y(Y)$ and hence $\Lambda_{Y,y} \subset T_0^*(B)$. Therefore $\gamma(\Lambda_S) = \tilde{\gamma}(\Lambda_{\tilde{Y}}) \subset T_0^*(B)$, for the corresponding Gaß mapping $\tilde{\gamma}$ of $\tilde{Y} \subset B$. Since $\dim(\gamma(\Lambda_S)) = \dim(\tilde{\gamma}(\Lambda_{\tilde{Y}})) \leq \dim(B) < \dim(X)$, the morphism $\gamma : \Lambda_Y \rightarrow T_0^*(X)$ is not dominant.

We prove the assertion (a) $\implies$ (b) of theorem 1 by induction on the dimension $d$ of $Y$. The case $d = 0$ is trivial. So, let us fix some $d > 0$. Suppose theorem 1 is already proven for irreducible subvarieties $Y'$ of dimension $\dim(Y') < d$ of an arbitrary abelian variety $X'$. This assumption will be maintained during the proof.
almost until the end of the paper. Furthermore, it is easy to see that for the proof we may assume that \( Y \) generates \( X \), i.e.
\[
\langle Y \rangle = X.
\]
Under these assumptions, we then show that the assertion of theorem 1 also holds for varieties \( Y \) of dimension \( d \). Before we proceed, let us recall from [A] the following

§2 Characterization of degenerate subvarieties. For reduced and irreducible subvarieties \( Y \) of an abelian variety \( X \) define
\[
Z(Y) = \{ y \in Y \mid \exists X' \subset X, X' \text{ closed subgroup of } \dim(X') > 0, y + X' \subset Y \}.
\]
Then, according to loc. cit. the following holds

Proposition 1. If \( Y \) is Zariski closed in \( X \), then \( Z(Y) \) is Zariski closed in \( Y \).

Proposition 2. If \( Y \) is closed in \( X \) and \( Z(Y) = Y \) holds, \( Y \) is degenerate.

In loc. cit. this is stated in the more general context of semiabelian varieties.

Lemma 1. Suppose \( U \) is a Zariski open dense subvariety of \( Y \), and suppose \( Y \) is closed in \( X \). Then \( Z(U) = U \) implies \( Z(Y) = Y \).

Proof. Indeed \( Z(U) \subset Z(Y) \) by definition, hence \( Y = \overline{U} = \overline{Z(U)} = \overline{Z(Y)} \). Since \( Z(Y) \) is Zariski closed in \( Y \) by proposition 1, we get \( Z = Z(Y) \).

Remark 1. Keeping lemma 1 and proposition 2 in mind, we may replace \( Y \) by some Zariski dense open subset \( U \) of the nonsingular locus \( S = Y_{\text{reg}} \) of \( Y \). For simplicity, we then often tacitly write \( U = S \) by abuse of notation.

Remark 2. Suppose \( Y \), or a Zariski open dense subset \( U \) of \( Y \), somehow is written as the union of (not necessarily finitely many) subvarieties \( F \). Then \( Z(F) = F \) for all these \( F \) implies \( Z(U) = U \), hence \( Z(Y) = Y \).

§3 Exact sequences of abelian varieties. 1) Let \( X' \subset X \) be a nontrivial abelian subvariety of dimension \( < g \). The image of \( Y \subset X \) under the quotient mapping \( q: X \to \tilde{X} = X/X' \) will be considered as a closed irreducible subvariety \( \tilde{Y} \) of \( \tilde{X} \), endowed with the reduced subscheme structure
\[
\begin{array}{c}
0 \rightarrow X' \xrightarrow{i} X \xrightarrow{q} \tilde{X} \rightarrow 0.
\end{array}
\]

Our assumption \( \langle Y \rangle = X \) implies \( \langle \tilde{Y} \rangle = \tilde{X} \). Hence, \( \dim(\tilde{Y}) > 0 \) and the fibers \( F_y \) of the morphism \( q: Y \to \tilde{Y} \) have dimension
\[
\dim(F_y) < d = \dim(Y).
\]
For \( \tilde{y} \in \tilde{Y} \), there exists \( y \in Y \) so that \( F_y = q^{-1}(\tilde{y}) \subset y + X' \).
2) For the proof of theorem 1 we may replace \( X \) by a finite etale covering and \( Y \) by its inverse image. This allows to assume that \( X \) splits (non-canonically) into a direct product

\[
X = X' \times \tilde{X}.
\]

Therefore we may tacitly assume that some splitting of the exact sequence exists, and has been chosen. Then

\[
T^*(X) = T^*(X') \times T^*(\tilde{X}) \quad \text{and} \quad F_{\tilde{y}} = Y \cap (\tilde{y} + X').
\]

3) For (regular) points \( y_1, y_2 \) in \( Y \) with \( q(y_1) = q(y_2) \), the fibers \( \Lambda_{Y,y_1} = N_{y_1}^* (Y) \) and \( \Lambda_{Y,y_2} = N_{y_2}^* (Y) \) usually do not coincide. Let \( i : X' \to X \) be the inclusion or more generally any of its translates \( i(x') = y + x' \). Then we claim

**Lemma 2.** There exists a Zariski dense open subset \( U \) of the set of regular points of \( \tilde{S} \) of \( \tilde{Y} \), such that for regular points \( y \) of \( Y \) in \( q^{-1}(U) \) there exists a canonical exact sequence of vector spaces

\[
0 \longrightarrow \Lambda_{S,\tilde{y}} \longrightarrow \Lambda_{S,y} \xrightarrow{T^*(i)} \Lambda_{F_{\tilde{y}},y} \longrightarrow 0.
\]

So for fixed \( \tilde{y} = q(y) \) in \( \tilde{Y} \) with fiber \( F_{\tilde{y}} \) in \( X' \), the variation of the conormal spaces \( \Lambda_{S,y} \) for \( y \in F_{\tilde{y}} \) is controlled by the variation of the conormal spaces \( \Lambda_{F_{\tilde{y}},y} \).

Notice, for a subvariety \( Y' \) of a translate of \( X' \), we can define \( \Lambda_{Y'} \) in \( T^*(X) \), and also \( \Lambda_{Y'} \) in \( T^*(X') \). The prime index will indicate that the ambient space is a translate of \( X' \).

**Proof of lemma 2.** Consider \( 0 \to T_y(F_{\tilde{y}}) \to T_y(Y) \to T_{\tilde{y}}(\tilde{Y}) \to 0 \). This exact sequence of tangent spaces, at a point \( y \) where \( q \) is a smooth morphism locally, maps to \( 0 \to T(X') \to T(X) \to T(\tilde{X}) \to 0 \). Hence, by the snake lemma we get \( 0 \to N_y'(F_{\tilde{y}}) \to N_y(Y) \to N_{\tilde{y}}(\tilde{Y}) \to 0 \). The exact sequence in our assertion is the dual sequence. This easily shows the claim. \( \square \)

Remark: There is also an exact sequence \( 0 \to \Lambda_{S,y} \to \Lambda_{F_{\tilde{y}},y} \to T_{\tilde{y}}(\tilde{S}) \to 0 \).

4) Since \( \Lambda_S = \bigcup_{y \in S} \Lambda_{S,y} \), the image \( \gamma(\Lambda_Y) \subset T^*_0(X) \) of the Gauss mapping is the Zariski closure of the union

\[
\gamma(\Lambda_S) = \bigcup_{y \in S} \gamma(\Lambda_{S,y}).
\]

Here, of course, \( S \) could be replaced by any Zariski dense open subset. We conclude that the image of \( \gamma(\Lambda_Y) \) under the linear mapping

\[
T^*(i) : T^*_0(X) \to T^*_0(X')
\]

is the Zariski closure of

\[
T^*(i)(\bigcup_{y \in S} \gamma(\Lambda_{S,y})) = \bigcup_{y \in S} T^*(i)(\gamma(\Lambda_{S,y})) = \bigcup_{y \in S} \gamma'(\Lambda_{F_{\tilde{y}},y})
\]
where \( \gamma' : \Lambda'_{F_y,y} \to T^*_0(X') \) denotes the Gauß mapping for \( X' \), instead of \( X \).

Indeed, after replacing \( S \) by some Zariski open dense subset (also denoted \( S \) by abuse of notation) there exists a commutative diagram

\[
\begin{array}{ccc}
\Lambda_S \otimes & T^*(i) & \Lambda_{F_y,y} \\
\downarrow & & \downarrow \\
T^*(X) & T^*(X') & \gamma' \\
\downarrow & & \downarrow \\
T^*_0(X) & T^*_0(X') &
\end{array}
\]

Therefore

\[
T^*(i) (\gamma(\Lambda_Y)) = \bigcup_{y \in S} \gamma'(\Lambda'_{F_y,y}) .
\]

5) Now assume that the Gauß mapping

\[
\gamma : \Lambda_Y \to T^*_0(X)
\]

is not dominant; in addition we assume \( Y \neq X \). Then using 2), there exists a homogenous polynomial \( F \neq 0 \) on \( T^*_0(X) = T^*_0(X') \oplus T^*_0(\tilde{X}) \) whose zero locus contains the image of the Gauß mapping \( \gamma \). Suppose

- \( \tau' \neq 0 \) in \( T^*_0(X') \) is some fixed vector of general position in \( T^*_0(X') \).
- \( \tau' \) is contained in the subvectorspace \( N_{F_y} \) of \( T^*_0(X') \) for \( y \in U \subset Y \) in some fixed Zariski dense open subset \( U \) of \( S \).

Then, by lemma 2, there exists \( \tilde{\tau} \) in \( T^*_0(\tilde{X}) \) such that the vector \( \tau = (\tau', \tilde{\tau}) \) in \( T^*_0(X') \oplus T^*_0(\tilde{X}) = T^*_0(X) \) is contained in the linear subspace \( \Lambda_S \otimes \) of \( T^*_0(X) \), and such that furthermore

\[
(\tau', \tilde{\tau}) + \Lambda_{\tilde{S}, \tilde{y}} \subset \Lambda_S \otimes \subset \gamma(\Lambda_S) \subset T^*_0(X) .
\]

So, the polynomial \( F \) vanishes on all the vectors \( (\tau', \tilde{\tau}) + \Lambda_{\tilde{S}, \tilde{y}} \).

Notice, \( \Lambda_{\tilde{S}, \tilde{y}} = \{0\} \times W \) holds for some linear subspace \( W \) of \( V = T^*(\tilde{X}) \). For \( v \in V \) there exists an expansion of \( F(\tau', v) \)

\[
F(\tau', v) = F_{m,\tau'}(v) + F_{m-1,\tau'}(v) + \ldots + F_0,\tau'(v) ,
\]

where the \( F_{\nu,\tau'}(v) \) are homogenous polynomials of degree \( \nu \) on \( V \). We may suppose \( \tilde{F} := F_{m,\tau'} \neq 0 \), since otherwise \( F \) would not depend on \( v \in V \); and since \( \tau' \) in \( T^*(X') \) is of general position by our assumptions, this would give as a contradiction \( F = 0 \). For any \( v \in V \) and any fixed vector \( \tilde{\tau} \in V \) we have (symbolically)

\[
\tilde{F}(v) = \lim_{t \to \infty} t^{-m} \cdot F(\tau', \tilde{\tau} + t \cdot v) .
\]

Since \( F(\tau', \tilde{\tau} + W) = 0 \) vanishes, we get: For every \( v \in W \subset V \) and any \( t \in k^{*} \) also \( t^{-m} \cdot F(\tau', \tilde{\tau} + t \cdot v) = 0 \) vanishes. We summarize this as follows:

- \( \tilde{F} \neq 0 \) on \( V = T^*_0(\tilde{X}) \)
- \( \tilde{F} = 0 \) on \( W = \Lambda_{\tilde{S}, \tilde{y}} \subset T^*_0(\tilde{X}) \).
The polynomial \( \tilde{F} \) does not depend on the particular point \( y \in U \). It only depends on the fixed decomposition \( T_0^*(X) = T_0^*(X') \oplus T_0^*(\tilde{X}) \) and on the point \( \tau' \) in \( T_0^*(X') \), where the latter is in general position by assumption. We obtain

**Corollary 1.** Suppose \( Y \) is closed and irreducible in \( X \) such that the Gauß mapping \( \gamma : \Lambda Y \to T_0^*(X) \) is not dominant. For \( 0 \to X' \to X \to \tilde{X} \to 0 \) given with \( 0 < \dim(X') < g \), together with \( \tau' \) in \( \gamma'(\Lambda_{F_y} \gamma) \subset T^*(i) (\gamma(\Lambda_S)) \) of sufficiently general position in \( T_0^*(X') \), the Gauß mapping

\[
\tilde{\gamma} : \Lambda Y \to T_0^*(\tilde{X})
\]

is not dominant for all \( \tilde{y} \) in a Zariski dense open subset of \( \tilde{S} \subset \tilde{Y} \).

**Proof.** We may assume \( Y \neq X \), since otherwise \( \tilde{Y} = \tilde{X} \) and the assertion would be trivial. So, for \( \tilde{y} \in \tilde{S} \), we know that \( (\tau', \tilde{\tau}) + \Lambda_{\tilde{S}_\tilde{y}} \subset \gamma(\Lambda_S) \) for some \( \tilde{\tau} \), as shown in 5) above. For \( \tilde{y} \in \Lambda_{\tilde{S}_\tilde{y}} \) therefore \( \tilde{F}(\tilde{y}) = 0 \) holds for a fixed nontrivial polynomial \( \tilde{F} \) on \( T_0^*(\tilde{X}) \), not depending on \( \tilde{y} \). Hence \( \tilde{\gamma} \) can not be dominant. \( \square \)

**Proposition 3.** For \( Y \) closed and irreducible in \( X \) suppose

\[
\gamma : \Lambda Y \to T_0^*(X)
\]

is not dominant. Furthermore, given \( 0 \to X' \to X \to \tilde{X} \to 0 \) for an abelian subvariety \( 0 \neq X' \subset X \), suppose

\[
\tilde{\gamma} : \Lambda Y \to T_0^*(\tilde{X})
\]

is dominant. Then, for all \( \tilde{y} \) in a Zariski dense open subset of \( \tilde{S} \subset \tilde{Y} \), none of the Gauß mappings

\[
\gamma' : \Lambda_{F_{\tilde{y}}} \to T_0^*(X')
\]

is dominant (and the same for the irreducible components \( Y' \) of these fibers \( F_{\tilde{y}} \)).

**Proof.** If \( \gamma' : \Lambda_{F_{\tilde{y}}} \to T_0^*(X') \) is dominant for some \( \tilde{y} \in \tilde{S} \subset \tilde{Y} \) in general position, there exists a conormal vector \( \tau' \neq 0 \) of general position in \( T_0^*(X') \) such that

\[
\tau' \in \gamma'(\Lambda_{F_{\tilde{y}}})
\]

holds for some point \( y \) on the fiber \( F_{\tilde{y}} \subset Y \), i.e. \( q(y) = \tilde{y} \). Hence by corollary 1

\[
\tilde{\gamma} : \Lambda Y \to T_0^*(\tilde{X})
\]

is not dominant, which contradicts our assumptions. \( \square \)

By induction theorem 1 holds for varieties \( Y' \) of dimension \( < d \). Thus in the situation of proposition 3, we can apply theorem 1 to the irreducible components \( Y' \) of the fibers \( F_{\tilde{y}} \) in \( Y \). They have dimension \( \dim(Y') \leq \dim(Y) - \dim(\tilde{Y}) < d \) for generic \( \tilde{y} \) in \( \tilde{Y} \), since \( \tilde{Y} \) has dimension \( > 0 \) by our assumption \( \langle Y' \rangle = X \). So, in the situation of the induction step of the proof for theorem 1, after renaming \( X' \) by \( A \) the last proposition implies

**Proposition 4.** Suppose \( Y \) is irreducible of dimension \( d = \dim(Y) \), closed in \( X \) and also generates \( X \). Suppose there exists an abelian subvariety \( A \subset X \) such that

- \( \gamma : \Lambda Y \to T_0^*(X) \) is not dominant.
- \( \tilde{\gamma} : \Lambda Y \to T_0^*(\tilde{X}) \) is dominant, for the image \( \tilde{Y} \) of \( Y \) in \( \tilde{X} = X/A \).
Then $Y$ is degenerate.

**Proof.** The assumptions on the Gauß mappings imply $A \neq 0$ and $\tilde{Y} \neq \tilde{X}$. By the other assumptions $X \neq 0$. Hence $\dim(\tilde{Y}) > 0$ by $\langle \tilde{Y} \rangle = \tilde{X}$. Then by proposition 3 all the Gauß mappings $\gamma': \Lambda_Y' \to T^*_0(X')$ for the irreducible components $Y'$ of the fibers $F_{\tilde{y}}$ (for $\tilde{y}$ in general position) are not dominant. By the general induction assumption underlying the proof of theorem 1, we conclude that for all these $\tilde{y}$ the components $Y'$ are degenerate. Hence $Z(Y') = Y'$ holds, and therefore $Z(U) = U$ holds for some Zariski dense open subset $U$ of $Y$ by remark 2. Thus $Y$ is degenerate by proposition 1 and 2. □

In the situation of the induction step for theorem 1, proposition 3 also implies

**Corollary 2.** Suppose $Y$ is irreducible and closed in $X$ of dimension $d$ and generates $X$. Suppose $\tilde{Y}$ is not degenerate and $\gamma: \Lambda_Y \to T^*_0(X)$ is not dominant. Then for any abelian subvariety $X' \subset X$ with quotient $\tilde{X} := X/X'$ and image $\tilde{Y}$ of $Y$ in $\tilde{X}$, the Gauß mapping $\tilde{\gamma}: \Lambda_{\tilde{Y}} \to T^*_0(\tilde{X})$ is not dominant.

**Proof.** We can assume $\dim(X') > 0$, so that proposition 4 can be applied. □

§4 Fibers of the Gauß mapping. Suppose $Y \subseteq X$ is irreducible and closed, and suppose the Gauß mapping

$$\gamma : \Lambda_Y \to T^*_0(X)$$

is not dominant. In this section we furthermore assume $\langle Y \rangle = X$ and $Y \neq X$. In this setting we now use the following argument of [R], or [KrW]: First observe that under these assumptions all nonempty fibers of the Gauß mapping $\gamma$

$$Z_\tau = \gamma^{-1}(\tau) \subset \Lambda_Y$$

have dimension

$$\dim(Z_\tau) \geq 1.$$ 

This follows from the upper semicontinuity of fiber dimensions, since it holds for generic points $\tau$ in $\gamma(\Lambda_Y)$. Notice, the image $Z_\tau = p_Y(Z_\tau)$ in $Y$

$$Z_\tau \hookrightarrow \Lambda_Y \xrightarrow{\gamma} T^*_0(X)$$

has the property that $p_Y: Z_\tau \to Y_\tau$ is a set theoretic bijection, since over any $y \in Y_\tau$ the points $z \in Z_\tau$ are uniquely determined by the condition $\gamma(z) = \tau$. Indeed, set theoretically,

$$Z_\tau = Y_\tau \times \{\tau\} \subset X \times T^*_0(X) = T^*(X).$$

Now assume $\tau \neq 0$. Then $\tau$ defines a nontrivial linear form

$$\tau : T_0(X) \to k.$$
whose kernel contains all tangent vectors in $T_y(Y)$, considered as vectors in $T_0(X)$ via a translation by $y \in X$. In particular, all tangent vectors in $T_y(Y_τ)$ at regular points $y$ of $Y_τ$ are contained in the kernel of the linear form $τ$.

So, for given $τ ≠ 0$, let us fix some point $y = y_τ \in Y_τ$. Then the translate $Y_τ - y_τ$ contains zero, and the abelian subvariety $X'$ of $X$ generated by $Y_τ - y_τ$ is a nontrivial abelian subvariety $X' \subsetneq X$. Indeed, $dim(Y_τ) ≥ 1$ implies $X' ≠ 0$ and $τ(T_0(X')) = 0$ implies $X' ≠ X$. Furthermore by construction of $X'$

$$Z_τ \subset y_τ + X'$$

In this situation, a priori, the abelian variety $X' = X'(τ, y_τ)$ may depend on the choice of $τ \in γ(Λ)$ and also on the choice of $y = y_τ$ in $Y_τ$, so that

$$Y \tau \downarrow \downarrow ↘ ↙ → X \downarrow \downarrow ⊂ T_0(X) \ni τ \quad \tilde{F}_{y_τ} = Y \cap (y_τ + X'(τ, y_τ)) \downarrow \downarrow ↘ ↙ → X$$

and

$$Y = \bigcup_{τ ∈ γ(Λ_Y)} Y_τ .$$

**Rigidity Property.** *For all $τ$ in a Zariski open dense subset of $γ(Λ_Y)$ and all $y_τ$ in a Zariski open dense subset of $Y_τ$, the abelian variety $X' = X'(τ, y_τ)$ does not depend on the choice of $τ$ and $y_τ$.***

*Proof.* There exist only countably many abelian subvarieties $X'$ in $X$, and $X' = X'(τ, y_τ)$ depends algebraically on $τ$ and $y_τ$. Replace $k$ by an uncountable extension field.

By the rigidity property we can assume that $X'$ is a fixed nontrivial proper abelian subvariety attached to $Y \subset X$, so that for all $y$ in a Zariski dense open subset $U \subset Y$

$$F_y = Y \cap (\tilde{y} + X') = Y \cap (y_τ + X'(τ, y_τ))$$

contains $Y_τ$ and hence is of positive dimension $dim(F_y) ≥ 1$; and the image $\tilde{Y}$ of $Y$ in $\tilde{X} = X/X'$ is irreducible of dimension

$$dim(\tilde{Y}) < dim(Y) .$$

To summarize, this shows

**Lemma 3.** *For irreducible closed $Y \neq X$ with $⟨Y⟩ = X$ and non-dominant Gauß mapping $γ : Λ_Y \rightarrow T_0(X)$ there exits an abelian subvariety $0 \neq X' \subsetneq X$ such that $dim(\tilde{Y}) < dim(Y)$ holds for the image $\tilde{Y}$ of $Y$ in $\tilde{X} = X/X'$, with the fibers of the Gauß mapping $γ$ contained in translates of $X'$.*

Indeed $⟨Y⟩ = X$ can be assumed without restriction of generality.

In the situation of lemma 3, we now assume that $Y$ is not degenerate with a non-dominant Gauß mapping $γ$, and let us also assume that theorem 1 holds for
varieties of dimension $< \dim(Y)$. Then proposition 4 can be applied; it shows that the induced Gauß mapping
\[ \tilde{\gamma}: \tilde{Y} \to T^*_0(\tilde{X}) \]

is not dominant. Then, $\langle \tilde{Y} \rangle = \tilde{X}$ is inherited from $\langle Y \rangle = X$. So suppose
\[ \tilde{Y} \neq \tilde{X}. \]

Then, if $\tilde{Y} \neq \tilde{X}$, we can apply once again lemma 3, now for the pair $(\tilde{Y}, \tilde{X})$, to construct an exact sequence
\[ 0 \to \tilde{X}' \to \tilde{X} \to \tilde{X} \to 0 \]
such that
- $\langle \tilde{\tilde{Y}} \rangle = \tilde{\tilde{X}}$
- $\tilde{\tilde{\gamma}}: \tilde{\tilde{Y}} \to T^*_0(\tilde{\tilde{X}})$ is not dominant.

Obviously, this construction can be iterated and terminates after finitely many steps since
\[ \cdots < \dim(\tilde{\tilde{Y}}) < \dim(\tilde{Y}) < \dim(Y), \]
thus provides an abelian subvariety $A \subseteq X$ containing $X'$, so that the image of $Y$ in $B = X/A$ is equal to $B$.

A closed irreducible variety $Y$ in $X$ will be called codegenerate (with respect to $A$), if there exists an abelian subvariety $A \neq X$ in $X$ such that the image of $Y$ in $B = X/A$ is equal to $B$.

Using this notion in the context of the induction step for the proof of theorem 1, we have now shown in this situation that for $Y$ closed irreducible of dimension $\dim(Y) = d$ the following corollary holds.

**Corollary 3.** Suppose the Gauß mapping
\[ \gamma: \Lambda_Y \to T^*_0(X) \]
is not dominant, then either
- $(a)$ $Y$ is degenerate, or
- $(b)$ $Y$ is codegenerate with respect to an abelian subvariety $A$, so that the fibers of the Gauß mapping $\gamma$ are contained in translates of $A$.

§5 **The proof of theorem 1 in the codegenerate case.** To complete the proof of the induction step (hence the proof of theorem 1) for $Y$ of dimension $d$ with non-dominant Gauß mapping, it remains to consider the codegenerate case $\tilde{Y} = \tilde{B}$ of the last corollary 3. Of course, without restriction of generality we can assume in addition that $Y$ is not degenerate. That $Y$ is not degenerate implies (by the induction assumption of theorem 1 and proposition 1 and 2): For a Zariski dense open subset $U$ of $\tilde{Y} = B$, the Gauß mappings of the fibers $F_{\tilde{y}}$, $\tilde{y} \in U$ of the projection $q: Y \to \tilde{Y} = B$ are nondegenerate. Indeed, if this non-degeneracy holds for a single fiber $F_{\tilde{y}}$ where $\tilde{y}$ is supposed to be in general position, it holds for all fibers $F_{\tilde{y}}$ with $\tilde{y}$ in a Zariski dense open subset $U$ of $\tilde{Y}$ by a specialization argument.

Assuming these conditions all together, we claim: $Y$ is degenerate. This gives a contradiction which implies the induction step for the proof of theorem 1.
Recall that in the last section we found an exact sequence

\[ 0 \rightarrow A \xrightarrow{i} X \xrightarrow{q} B \rightarrow 0 \]

with \( B = \tilde{Y} \) such that the fibers \( Z_\tau \) of the Gauss mapping \( \gamma : \Lambda Y \rightarrow T^*_0(X) \) map bijectively to varieties \( Y_\tau \subset Y \) that are contained in the fibers 

\[ F_\tilde{y} = Y \cap (\tilde{y} + A) \subset Y \]

of the projection \( q : Y \rightarrow \tilde{Y} = B \). Here, without restriction of generality, we assume that \( B \) splits, so that \( B \) can be considered as a subvariety of \( X \) complementary to \( A \). Since \( Y = \bigcup_{\tilde{y} \in B} F_\tilde{y} \), the variety \( Y \) is degenerate if all the \( F_\tilde{y} \) are degenerate for all \( \tilde{y} \) in some Zariski open dense subset of \( \tilde{Y} \) (using proposition 1 and 2). Therefore, if \( Y \) were not degenerate, by the induction assumption of theorem 1 we conclude that the Gauss mapping

\[ \gamma_A : \Lambda_{F_\tilde{y}} \rightarrow T^*_0(A) \]

is dominant for all points \( \tilde{y} \) of general position in \( \tilde{Y} \).

Fix some \( \tau' \neq 0 \) in \( T^*_0(A) \) in general position; fix some \( \tilde{y} \), now with \( \tilde{y} \in U \), so that \( \gamma_A : \Lambda_{F_\tilde{y}} \rightarrow T^*_0(A) \) is dominant. Since \( \gamma_A \) is dominant and since \( \tau \) has general position in \( T^*_0(A) \), there exists

\[ y \in F_\tilde{y} \subset \tilde{y} + A \text{ so that } \tau' \text{ is contained in } N^*_y(F_\tilde{y}) = \Lambda_{F_\tilde{y},y} . \]

Since \( \tilde{S} \) is Zariski dense in \( \tilde{Y} = B \), we get in \( T^*(B) = B \times T^*_0(B) \)

\[ \Lambda_{\tilde{S},\tilde{y}} = \Lambda_{B,\tilde{y}} = \{\tilde{y}\} \times \{0\} . \]

By lemma 2 we therefore obtain

**Lemma 4.** For \( \tilde{y} \) in a suitably chosen Zariski open dense subset \( U \) of \( \tilde{Y} = B \), our assumptions imply that \( \tau' \) (chosen in general position) is in \( N^*_y(F_\tilde{y}) \) such that

\[ \Lambda_{S,y} \cong \Lambda_{F_\tilde{y},y}' . \]

In other words: \( \tau' \in T^*_0(A) \) can be uniquely lifted to a point in \( \Lambda_{S,y} \), once the corresponding base point \( y \in F_\tilde{y} \) over \( \tilde{y} \in U \) has been specified.

For the conormal bundle in \( T^*(A) \) we now write \( \Lambda_{F,\tilde{y}}^A \) instead of \( \Lambda_{F,\tilde{y}} \). Notice that \( \gamma_A : \Lambda_{F,\tilde{y}}^A \rightarrow T^*_0(A) \) is dominant and hence generically finite, for all \( \tilde{y} \) a Zariski open subset \( U \) of \( \tilde{Y} \). We therefore obtain

**Lemma 5.** For fixed \( \tau' \neq 0 \) with general position in \( T^*_0(A) \) and fixed \( \tilde{y} \) with general position in \( B \), there exist only finitely many points \( y \in Y \) mapping to \( \tilde{y} \) (i.e. \( y \in F_\tilde{y} \)) for which \( \tau' \) is contained in the conormal bundle \( \Lambda_{F_\tilde{y},y} \) of \( F_\tilde{y} \) at \( y \).

Combining lemma 4 and 5 we get
Lemma 6. We have the following diagram

\[
\begin{array}{c}
\text{dense open subset of } \gamma(\Lambda_S) \\
\downarrow \downarrow \downarrow \downarrow \\
\bigcup_{y \in U} \gamma'(\Lambda_{F_y}) \\
\downarrow \\
T^*(i) \circ \gamma \circ \Lambda_S \\
\end{array}
\]

\[
\begin{array}{c}
T_0^*(X) \\
\downarrow \\
T_0^*(A) \\
\end{array}
\]

and the image under the lower horizontal morphism of each \( \gamma'(\Lambda_{F_y}) \) for \( \tilde{y} \in U \) is Zariski dense in \( T_0^*(A) \). Furthermore, for \( \tau' \) of sufficiently general position in \( T_0^*(A) \) the points \( \gamma(\lambda) \in T_0^*(X) \) in \( \gamma(\Lambda_S) \), which map under \( T^*(i) : T_0^*(X) \to T_0^*(A) \) to the point \( \tau' \), correspond to points \( \lambda \) in \( \Lambda_S \)

\[
\lambda = (y, \tau) \in \Lambda_S \subseteq X \times T_0^*(X)
\]

for which the base point

\[
\tilde{y} = q(y) \in B = X/A
\]

of \( y \) can be arbitrarily prescribed within a dense open subset of \( B \). Once this base point \( \tilde{y} \in B \) is fixed, there exists at least one, but at most finitely many choices for the point \( y \in X \) such that

\[
\tilde{y} = q(y) \text{ and } \lambda = (y, \tau) = (y, \tau', \tilde{\tau}) \in \Lambda_S
\]

holds for some \( \tilde{\tau} \) in \( T_0^*(B) \).

Hence there exists a Zariski dense open subset \( V \subset \Lambda_S \), such that on \( V \) the mapping \( \varphi = (q \circ p_S, T^*(i) \circ \gamma) \) defines a generically finite and hence dominant morphism

\[
\Lambda_S \supset V \xrightarrow{\varphi} B \times T_0^*(A)
\]

\[
\lambda = (y, \tau) \mapsto (\tilde{y}, \tau').
\]

For a point \( \tau \in \gamma(\Lambda_S) \) in general position consider the fiber \( Z_\tau \subset \Lambda_S \). Now \( V \cap Z_\tau \) is a Zariski dense open subset of \( Z_\tau \) by the choice of \( \tau \). Furthermore, for all points \( \lambda \) in \( V \cap Z_\tau \) by definition \( \lambda = (y, \tau) \) holds; and the fixed \( \tau \) maps to a fixed \( \tau' \) of general position in \( T^*(A) \). In other words: The second component \( T^*(i) \circ \gamma \) of the morphism \( \varphi \) is constant on \( Z_\tau \). Since \( \varphi \) is finite on \( V \), therefore the first component

\[
q \circ p_S : V \cap Z_\tau \to B
\]

of the morphism \( \varphi \) is a finite morphism on \( V \cap Z_\tau \), and

\[
V \cap Z_\tau \ni (y, \tau) \mapsto \tilde{y} = q(y)
\]

Since \( X' \subset A \), using the notations of the argument preceding corollary 3, we already know from the beginning of this section that

\[
Z_\tau \subset y_\tau + X' \subset y_\tau + A = \tilde{y}_\tau + A
\]

Here without restriction of generality we assumed \( X = A \times B \), and \( \tilde{y}_\tau \in B \) is the image of \( y_\tau \) under the projection \( q : X \to B = X/A \). Recall that \( y_\tau \) was some
fixed chosen point in $Y_\tau$ and only depends on $\tau$. Hence the inclusion $Z_\tau \subset \tilde{y}_\tau + A$ implies

$$q \circ p_S(V \cap Z_\tau) = \tilde{y}_\tau.$$ 

Since $q \circ p_S$ is finite on $V \cap Z_\tau$, the intersection $V \cap Z_\tau$ therefore contains only finitely many point. But this contradicts $\dim(V \cap Z_\tau) = \dim(Z_\tau) \geq 1$ and finishes the proof of the induction step for theorem 1. □

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