“Bare” Effective Mass in Finite Sized $\nu = 1/2$ Systems

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Abstract

In this note, we discuss the effective mass of quasiparticles in finite sized $\nu = 1/2$ systems in the lowest Landau level, given a natural notion we have of the Fermi surface in these finite sized systems. The effective mass is related to the difference between instantaneous density-density response functions of the ground state and excited states of the system; we do a finite size calculation for this difference for $\nu = 1/2$ systems with 7, 8 and 9 fermions, and conjecture how the difference must look in the thermodynamic limit.
The composite fermion ideas of Jain [1] lead us to believe that we should see a metallic (compressible) state for fermions in the half-filled Landau level. Halperin, Lee and Read [2] and others investigated the details of the physics of such a system analytically, based around a mean-field Chern-Simons Landau-Ginzburg theory. Recent numerical work has also shown [3] that there is a very natural notion of the Fermi surface for $\nu = 1/2$ systems in the lowest Landau level. The basic entity in this picture is what is called a "cluster" wavefunction.

First pick $N$ (= number of fermions) momenta $\{\vec{k}_i\}_{i=1}^N$ such that they form a compact configuration in momentum space. The momenta will have to be chosen consistent with the boundary conditions imposed on the system. In the thermodynamic limit, these boundary conditions will not be important and we will be able to choose this cluster to be circular, but for finite system sizes, the clusters can only be chosen to be approximately circular. Then for any choice of $\{\vec{k}_i\}_{i=1}^N$ there is a $\nu = 1/2$ LLL wavefunction defined as follows.

The wavefunctions are built out of the $\nu = 1/2$ Laughlin state (which is bosonic because it is an even denominator state). In the standard holomorphic coordinates, this has the form

$$\Psi_L(\{z_i\}) = \prod_{i<j}(z_i - z_j)^2 \exp\left(-\frac{1}{4}|z_i|^2\right)$$  (1)

We will be doing finite sized calculations on the torus, and for this it is convenient to equivalently define the bosonic Laughlin state as the ground state of bosons at $\nu = 1/2$ interacting with a hard-core repulsive potential. Then in terms of this bosonic Laughlin state, we can define a fermionic state for the cluster $\{\vec{k}_i\}_{i=1}^N$ as

$$\Psi(\{\vec{k}_i\}_{i=1}^N) = \det_{ij} \exp\left(i\vec{k}_i \cdot \vec{R}_j \right) |\Psi_L(\nu = 1/2)\rangle$$  (2)

Note that this wavefunction has the correct statistics (fermionic) and filling factor. It has been shown elsewhere [3] that for any given set of periodic boundary conditions, the choice of $\{\vec{k}_i\}_{i=1}^N$ that gives the "most circular" compact configuration possible yields a wavefunction that is almost exactly the ground state for fermions interacting with coulomb interactions [3]. For example, the overlap between two arbitrary states is no more than a few percent, while the fermionic slater determinant state (equation 2) for the "most circular" cluster has
overlap better than 99 percent with the actual ground state for the system with coulomb
interactions (this is true for related short-range interactions as well). (We pick a natural
notion of ”most circular” by choosing the set \( \{ \vec{k}_i \}_{i=1}^N \) such that the quantity \( \sum_{i \neq j} (\vec{k}_i - \vec{k}_j)^2 \)
is minimized, or equivalently, such that \( \sum_i (\vec{k}_i - \vec{k}_{av})^2 \) is minimized). We should also point
out that these cluster wavefunctions are *completely independent* of the interaction between
the fermions.

Given this result, and the definition for cluster wavefunctions (equation \( \text{(2)} \)), there is a
very natural definition of the effective mass of ”quasiparticles” in this system.

Let us denote the collection of momenta for the relaxed ”most circular” cluster by \( \{ \vec{k}_0^i \}_{i=1}^N \)
and let \( \{ \vec{k}_i \}_{i=1}^N \) depict an excited cluster (see figure 1 for an example when \( N = 9 \)).

Then the energy of the relaxed configuration is (let \( H \) be the Hamiltonian for fermions
with coulomb interaction, for example)

\[
E_0 = \langle \Psi(\{\vec{k}_0^i\}) | H | \Psi(\{\vec{k}_0^i\}) \rangle \tag{3}
\]

which to a very good approximation is just the ground state energy for \( \nu = 1/2 \) fermions
with coulomb interaction. The energy of the excited cluster is

\[
E = \langle \Psi(\{\vec{k}_i\}) | H | \Psi(\{\vec{k}_i\}) \rangle \tag{4}
\]

In this setting, since the \( |\Psi(\{\vec{k}_i\})\rangle \) are purely functions of the \( \{\vec{k}_i\} \), so are the energies:
\( E = E(\{\vec{k}_i\}) \). The situation here is very similar to Hartree-Fock theory, except here the
Hamiltonian does not contain a contribution from the kinetic energy (which gets quenched
in the LLL). In the absence of the kinetic energy, the energy functional \( E(\{\vec{k}_i\}) \) is translation
invariant; in addition, the energy ends up being a functional of the occupation numbers \( n_k \)
of the configuration, just as in the case of Hartree-Fock (in Hartree-Fock theory, however,
the energy is quadratic in these occupation numbers while in this case, the energy is *not* a
quadratic functional of the occupation numbers).

For simplicity, let us pick excited clusters that consist of a single quasiparticle-quasihole
pair, or in other words, only one of the \( \vec{k}_i \) differ from the \( \vec{k}_0^i \) (See figure [II]). Label the particles
so that \( \vec{k}_i = \vec{k}_0^i \) for \( i = 2, ..., n \). In the thermodynamic limit, if \( \delta \vec{k} \equiv \vec{k}_1 - \vec{k}_0^1 \) is small, we have
\[ \delta E \equiv E - E_0 = \frac{2 \vec{k} \cdot \vec{k}_F}{m^*} \] (5)

and (for \( \nu = 1/2 \)) \( k_F = l_0^{-1} \) where \( l_0 \) is the magnetic length. The definition yields what can be called the “bare effective mass” of the composite fermion. For finite sized systems, however, we need another operational definition of the Fermi momentum. Define \( \vec{k}_{av} = \frac{1}{N} \sum_{i=1}^{N} \vec{k}_i \). Then we can compute

\[ \vec{k}_F = \left( \frac{\vec{k}_1 + \vec{k}_0}{2} \right) - \vec{k}_{av} \] (6)

For finite-sized systems, this definition of the Fermi momentum yields a vector of magnitude approximately equal to 1. The reason we have to use a definition of this kind is that the clusters of momenta for the system sizes we examine are not rotationally invariant.

We can now reexpress the energy difference \( \delta E \) in terms of the static structure factors of the two cluster wavefunctions. First, we notice that in the lowest Landau level, we have

\[ H = \sum_q V(q) \rho(q) \rho(-q) \] (7)

where \( \rho(q) = \sum_{i=1}^{N} \exp(-i\vec{q} \cdot \vec{R}_i) \).

Thus

\[ E = \langle \Psi(\{\vec{k}_i\}) | \sum_q V(q) \rho(q) \rho(-q) | \Psi(\{\vec{k}_i\}) \rangle \]
\[ = \sum_q V(q) \langle \Psi(\{\vec{k}_i\}) | \rho(q) \rho(-q) | \Psi(\{\vec{k}_i\}) \rangle \]
\[ = \sum_q V(q) S[\Psi(\{\vec{k}_i\})](q) \] (8)

where \( S[\Psi(\{\vec{k}_i\})](q) \) is the instantaneous structure factor for the state \( \Psi(\{\vec{k}_i\}) \). Hence

\[ \delta E = \sum_q V(q) (S[\Psi(\{\vec{k}_i\})](q) - S[\Psi(\{\vec{k}_0\})](q)) \] (9)

Using equation [5], we get

\[ \frac{1}{m^*} = \sum_q V(q) \left( \frac{S[\Psi(\{\vec{k}_i\})](q) - S[\Psi(\{\vec{k}_0\})](q)}{2 \delta \vec{k} \cdot \vec{k}_F} \right) \] (10)
We reiterate that the quantity in the large brackets in equation \[10\] depends only on the cluster wavefunctions, which are strictly independent of the interaction; all the interaction dependence of the effective mass in this calculation follows rather simply from equation \[10\] once we have calculated the relevant (interaction-independent) static structure factors. Clearly, \(1/m^*\) is a linear functional of \(V(q)\). In the thermodynamic limit, we should get the same answer for \(1/m^*\) regardless of what particle-hole pair we create, just so long as the pair is created close to the Fermi surface. However, in finite sized systems, we cannot get arbitrarily close to the Fermi surface (defined by our relaxed cluster). Moreover, the relaxed cluster is not really circular, so depending on what the choice of particle-hole pair we make, we will not necessarily get the same result for \(1/m^*\) each time. In figures 2, 3, 4 we have shown the results of the calculation of \(S(Ψ(\{\vec{k}_i\}))(q) - S(Ψ(\{\vec{k}_0\}))(q)\) for 7, 8 and 9 fermions at \(ν = 1/2\).

For each system size, we can vary the geometry (in other words, the angle between the sides of the box and the ratio of their lengths). Doing so allows us to probe a larger set of momenta than any single geometry will allow us to do, and in the process we hope to get a better handle on the physics in the thermodynamic limit. In the process of changing the geometry, the shape of the ”relaxed” cluster \(\{\vec{k}_0^i\}\) will also change. In fact, if we change the geometry enough, we will encounter a phase transition to another region where the most relaxed cluster has a different set of momenta. In collecting the data for the figures 2, 3, 4, we picked only those geometries whose most relaxed cluster looked reasonably circular. In addition, in order not to have to deal with excitations transverse to the Fermi surface, we threw out the data from all excited clusters if the angle between \(δ\vec{k}\) and \(\vec{k}_F\) exceeded 60 degrees. The choices made above are somewhat arbitrary, but are reflective of the usual definition of the effective mass in the thermodynamic limit.

The fact that we have finite sized systems produces the sort of spread in the data in figures 4, 3, 4. Essentially, all the different calculations for \(1/m^*\) yield slightly different answers, but the qualitative features in the results for \(S(Ψ(\{\vec{k}_i\}))(q) - S(Ψ(\{\vec{k}_0\}))(q)\) all follow the same pattern: the plot has a positive peak for some value of \(q\), and then for a larger value of \(q\), there is a negative peak. We can provide a heuristic argument for why the data looks
this way, and moreover, suggest what it must look lie in the thermodynamic limit.

Figure 5 shows that the presence of a particle lying just outside the Fermi surface (and a hole just inside) will suppress the matrix elements of \( \rho(q)\rho(-q) \) for \( q \) a little greater than \( 2k_F \) and enhance it for \( q \) a little less than \( 2k_F \). This is in the thermodynamic limit, but for our finite systems, \( k_F \) is actually of the same order as \( |\delta k| \) (see figure 4). As a result we see a positive peak at \( q \) substantially less than \( 2k_F \) and a negative peak for \( q \) somewhere around \( 2k_F \). It seems clear that if we had the ability to examine large enough systems, the data from all the different clusters ought to collapse onto a single curve, and moreover, the curve should consist of a very strong positive peak just under \( 2k_F \) and a very strong negative peak just over \( 2k_F \).

We can get an inkling for what this curve looks like by averaging out the data for the different clusters for a given system size, and this is shown in figure 6.

This is exactly what the heuristic described by figure 5 would lead us to expect. The data we obtain can be used with equation 10 to obtain the effective mass of quasiparticles in this theory, and for coulomb interactions we end up with \( \frac{1}{m^*} \sim C\frac{\epsilon^2}{\epsilon} \) where \( C \) is a number in the range 0.2 – 0.4 depending on the geometry and choice of excited cluster. This is in agreement with estimates made by others [2,4]. For arbitrary interactions, we can make some useful statements about the form that \( \frac{S[\Psi(\{\vec{k}_i\})](q)-S[\Psi(\{\vec{k}_0\})](q)}{2\delta \vec{k} \cdot \vec{k}_F} \) must take in the thermodynamic limit. In the lowest Landau level, we can describe the interaction in terms of Haldane pseudopotentials. On account of the symmetry properties of the Laguerre polynomials that these pseudopotentials are defined in terms of, only the odd pseudopotentials matter in the case of fermions. In other words, in the thermodynamic limit, an exact curve (in place of figure 6) multiplied by \( V(q) \) and integrated with respect to \( 2\pi q dq \) should receive contributions only from the odd pseudopotentials. For example, a constant potential cannot make any contribution to the effective mass, since a constant potential is described by the zeroth Laguerre polynomial. We should stress that this is an important requirement for any formula for the effective mass of quasiparticles in the theory, when examined strictly within the lowest Landau level.
We also note that Jain and Kamilla have recently shown how to evaluate the variational energy of composite fermion wavefunctions on the sphere for much larger system sizes (∼ 40-50 CFs). This they achieve by modifying the definition of the CF wavefunction from the original one proposed by Jain. The properties of these modified wavefunctions are apparently very similar to the original CF wavefunction. The $\nu = 1/2$ wavefunctions in the periodic geometry can also be modified along the lines of, potentially allowing the calculation described here to be carried out on much larger systems.

Finally, let us reiterate that within the given framework, there is no way we will observe the kind of effective mass divergence Halperin, Lee and Read calculate (which seems to depend strongly on the type of interaction between the fermions); this is because the “bare” effective mass we calculate is based on the static structure factor of the cluster wavefunction, which is interaction independent. The interaction dependence of the effective mass then follows straightforwardly via equation 10.

This work was supported by NSF DMR–9400362.
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FIGURES

FIG. 1. 9 particle cluster in momentum space, shown here for a slightly oblique torus geometry. For this geometry, the cluster shown on the left is the ”most circular” cluster we can exhibit for 9 particles. The cluster on the right depicts a basic particle-hole excitation of the ”relaxed cluster on the left, and the arrow shows the fermi momentum, as per our operational definition in equation 3.

FIG. 2. 7 fermion data for $\frac{S[\Psi(\{\vec{k}_i\})](q) - S[\Psi(\{\tilde{\vec{k}}_i\})](q)}{2\delta k \cdot \vec{k}_F}$. The relaxed cluster is the same in all cases, though we have taken all possible excited clusters, and plotted the cumulative data for a number of geometries.

FIG. 3. 8 fermion data for $\frac{S[\Psi(\{\tilde{\vec{k}}_i\})](q) - S[\Psi(\{\tilde{\vec{k}}_0\})](q)}{2\delta k \cdot \vec{k}_F}$. Same comments apply as for 7 fermion data.

FIG. 4. 9 fermion data for $\frac{S[\Psi(\{\tilde{\vec{k}}_i\})](q) - S[\Psi(\{\tilde{\vec{k}}_0\})](q)}{2\delta k \cdot \vec{k}_F}$. Same comments apply as for 7 fermion data.

FIG. 5. The particle-hole pair shown suppresses processes contributing to $S(q)$ for the excited cluster relative to the relaxed cluster for $q$ slightly greater than $2k_F$ and enhances processes for $q$ slightly less than $2k_F$.

FIG. 6. 9 fermion data from figure 4 averaged (and convolved with a gaussian to obtain a smoother plot). There is a small bump for small $q$ that is an artifact of the finite size of the system, corresponding to processes along the smallest possible momenta, after which there are no processes until almost twice that $q$ value.
Figure 1:
Figure 2:
Figure 3:
Figure 4:
Figure 5:
Figure 6: