Multi Snapshot Sparse Bayesian Learning for DOA Estimation

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Abstract—March 1, 2016 The directions of arrival (DOA) of plane waves are estimated from multi-snapshot sensor array data using Sparse Bayesian Learning (SBL). The prior source amplitudes are assumed independent zero-mean complex Gaussian distributed with hyperparameters the unknown variances (i.e. the source powers). For a complex Gaussian likelihood with hyperparameter the unknown noise variance, the corresponding Gaussian posterior distribution is derived. For a given number of DOAs, the hyperparameters are automatically selected by maximizing the evidence and promote sparse DOA estimates. The SBL scheme for DOA estimation is discussed and evaluated competitively against LASSO ($\ell_1$-regularization), conventional beamforming, and MUSIC.

Index Terms—relevance vector machine, sparse reconstruction, array processing, DOA estimation, compressive beamforming

I. INTRODUCTION

In direction of arrival (DOA) estimation, compressive beamforming, i.e. sparse processing, achieves high-resolution acoustic imaging and reliable DOA estimation even with a single snapshot\[1],\[2],\[3],\[4],\[5],\[6], outperforming traditional methods\[7].

Multiple measurement vector (MMV, or multiple snapshots) compressive beamforming offers several benefits over established high-resolution DOA estimators which utilize the data covariance matrix\[1],\[5],\[8],\[9]: 1) It handles partially coherent arrivals. 2) It can be formulated with any number of snapshots in contrast to eigenvalue based beamformers. 3) Its flexibility in formulation enables extensions to sequential processing, and online algorithms\[3],\[4]. 4) It achieves higher resolution than MUSIC, even in scenarios that favor these classical high-resolution methods\[9].

We solve the MMV problem in the sparse Bayesian learning (SBL) framework\[8] and use the maximum-a-posteriori (MAP) estimate for DOA reconstruction. We assume complex Gaussian distributions with unknown variances (hyperparameters) both for the likelihood and as prior information for the source amplitudes. Hence, the corresponding posterior distribution is also Gaussian. To determine the hyperparameters, we maximize a Type-II likelihood (evidence) for Gaussian distributions with unknown variances (hyperparameters) both for the likelihood and as prior information for the hyperparameter the unknown noise variance, the corresponding Gaussian posterior distribution is derived. For a given number of DOAs, the hyperparameters are automatically selected by maximizing the evidence and promote sparse DOA estimates. The SBL scheme for DOA estimation is discussed and evaluated competitively against LASSO ($\ell_1$-regularization), conventional beamforming, and MUSIC.

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II. ARRAY DATA MODEL AND PROBLEM FORMULATION

Let $X = [x_1, \ldots, x_L] \in \mathbb{C}^{M \times L}$ be the complex source amplitudes, $x_{ml}$ with $m \in [1, \ldots, M]$ and $l \in [1, \ldots, L]$, at $M$ DOAs (e.g. $\vartheta_m = -90^\circ + \frac{2\pi m}{M}$) and $L$ snapshots at frequency $\omega$. We observe narrowband waves on $N$ sensors for $L$ snapshots $Y = [y_1, \ldots, y_L] \in \mathbb{C}^{N \times L}$. A linear regression model relates the array data $Y$ to the source amplitudes $X$, $Y = AX + N$. (1)

The transfer matrix $A = [a_1, \ldots, a_M] \in \mathbb{C}^{N \times M}$ contains the array steering vectors for all hypothetical DOAs as columns, with the $n$th element $e^{-j(n-1)\vartheta_\omega d \sin \vartheta_m}$ (d is the element spacing and c the sound speed). The additive noise $N \in \mathbb{C}^{N \times L}$ is assumed independent across sensors and snapshots, with each element following a complex Gaussian $CN(0, \sigma^2)$.

We assume $M \gg N$ and thus (1) is underdetermined. In the presence of few stationary sources, the source vector $x_l$ is $K$-sparse with $K \ll M$. We define the $l$th active set $M_l = \{m \in \mathbb{N} | x_{ml} \neq 0\} = \{m_1, m_2, \ldots, m_K\}$, (2) and assume $M_l = M$ is constant across snapshots l. Also, we define $A_M \in \mathbb{C}^{N \times K}$ which contains only the $K$ “active” columns of $A$. The $\| \cdot \|_p$ denotes the vector $p$-norm and $\| \cdot \|_F$ the matrix Frobenius norm.

III. BAYESIAN FORMULATION

Using Bayesian inference to solve the linear problem (1) involves determining the posterior distribution of the complex source amplitudes $X$ from the likelihood and a prior model.

A. Likelihood

Assuming the additive noise (1) complex Gaussian the data likelihood, i.e., the conditional probability density function (pdf) for the single-frequency observations $Y$ given the sources $X$, is complex Gaussian with noise variance $\sigma^2$.

$$p(Y|X; \sigma^2) = \frac{\exp \left(-\frac{1}{2\sigma^2} \|Y - AX\|^2_F\right)}{(\pi\sigma^2)^{NL}}.$$ (3)
### B. Prior

We assume that the complex source amplitudes $x_{ml}$ are independent both across snapshots and across DOAs and follow a zero-mean complex Gaussian distribution with DOA-dependent variance $\gamma_m \in \gamma = [\gamma_1, \ldots, \gamma_M]^T$.

\[
p_{m}(x_{ml}; \gamma_m) = \begin{cases} 
\frac{\delta(x_{ml})}{\pi \sigma_{x_{ml}}^2}, & \text{for } \gamma_m = 0, \\
\frac{1}{\pi \sigma_{x_{ml}}^2 e^{-x_{ml}^2/\gamma_m}}, & \text{for } \gamma_m > 0
\end{cases}
\]

(4)

\[
p(X; \gamma) = \prod_{l=1}^{L} \prod_{m=1}^{M} p_{m}(x_{ml}; \gamma_m) = \prod_{l=1}^{L} \mathcal{CN}(0, \Gamma),
\]

(5)
i.e., the source vector $x_l$ at each snapshot $l \in [1, \ldots, L]$ has a multivariate Gaussian distribution with potentially singular covariance matrix,

\[
\Gamma = \text{diag}(\gamma) = E \left[ x_l x_l^H \right],
\]

(6)
as $\text{rank}(\Gamma) = \text{card}(\mathcal{M}) = K \leq M$. Note that the diagonal elements of $\Gamma$, i.e., the hyperparameters $\gamma \geq 0$, represent source powers. When the variance $\gamma_m = 0$, then $x_{ml} = 0$ with probability 1. The sparsity of the model is thus controlled with the hyperparameters $\gamma$.

### C. Posterior

Given the likelihood for the array observations $Y$ (3) and the prior (5), the posterior pdf for the source amplitudes $X$ can be found using Bayes rule conditioned on $\gamma$, $\sigma^2$,

\[
p(X|Y; \gamma, \sigma^2) \propto p(Y|X; \sigma^2)p(X; \gamma).
\]

(7)
The denominator $p(Y; \gamma, \sigma^2)$ is the evidence term, i.e., the marginal distribution for the data, which for a given $\gamma, \sigma^2$ is a normalization factor and is neglected at first,

\[
p(X|Y; \gamma, \sigma^2) \propto p(Y|X; \sigma^2)p(X; \gamma) \propto e^{-\text{tr}(X^H Y \Sigma_y^{-1} Y)} \propto \mathcal{CN}(\mu_X, \Sigma_x).
\]

(8)

As both $p(Y|X; \sigma^2)$ in (3) and $p(X; \gamma)$ in (5) are Gaussians, their product (8) is Gaussian with posterior mean $\mu_X$ and covariance $\Sigma_x$,

\[
\mu_X = E(X|Y; \gamma, \sigma^2) = \Gamma A^H \Sigma_y^{-1} Y,
\]

(10)
\[
\Sigma_x = E((x_l - \mu_x)(x_l - \mu_x)^H|Y; \gamma, \sigma^2) = \left( \frac{1}{\sigma^2} A^H A + \Gamma^{-1} \right)^{-1} = \Gamma - \Gamma A^H \Sigma_y^{-1} \Gamma A,
\]

(11)
where the array data covariance $\Sigma_y$ and its inverse are derived from (1) and using the matrix inversion lemma

\[
\Sigma_y = E(y_l y_l^H) = \sigma^2 I_N + A \Gamma A^H,
\]

(12)
\[
\Sigma_y^{-1} = -\sigma^2 I_N - \sigma^2 A \left( \frac{1}{\sigma^2} A^H A + \Gamma^{-1} \right)^{-1} A^H \sigma^2 - 2 I_N - \sigma^2 A \Sigma_x A^H \sigma^2 - 2.
\]

(13)
If $\gamma$ and $\sigma^2$ are known then the MAP estimate is the posterior mean,

\[
\hat{X}_{\text{MAP}} = \mu_X = \Gamma A^H \Sigma_y^{-1} Y.
\]

(14)
The diagonal elements of $\Gamma$ control the row-sparsity of $\hat{X}_{\text{MAP}}$ as for $\gamma_m = 0$ the corresponding $m$th row of $\hat{X}_{\text{MAP}}$ becomes $0^T$. Thus, the active set $\mathcal{M}$ is equivalently defined by

\[\mathcal{M} = \{ m \in \mathbb{N} | \gamma_m > 0 \}.\]

(15)

### D. Evidence

The hyperparameters $\gamma, \sigma^2$ in (10–13) are estimated by a type-II maximum likelihood, i.e., by maximizing the evidence which was treated as constant in (8). The evidence is the product of the likelihood (3) and the prior (5) integrated over the complex source amplitudes $X$,

\[
p(Y; \gamma, \sigma^2) = \int_{\mathbb{C}^M} p(Y|X; \sigma^2)p(X; \gamma) dX = \frac{e^{-\text{tr}(Y^H \Sigma_y^{-1} Y)}}{(\pi^N \text{det} \Sigma_y)^L},
\]

(16)

where $dX = \prod_{l=1}^{L} \prod_{m=1}^{M} \text{Re}(dX_{ml}) \text{Im}(dX_{ml})$, and $\Sigma_y$ is the data covariance (12). The $L$-snapshot marginal log-likelihood becomes

\[
\log p(Y; \gamma, \sigma^2) \propto -\text{tr}(Y^H \Sigma_y^{-1} Y) - L \log \text{det} \Sigma_y
\]

\[\propto -\text{tr}(\Sigma_y^{-1} \Sigma_y) - \log \text{det} \Sigma_y.
\]

(17)

where we define the data sample covariance matrix,

\[
S_y = YY^H / L.
\]

(18)

Note that (17) does not involve the inverse of $S_y$ hence it works well even for few snapshots (small L).

The hyperparameter estimates $\gamma, \sigma^2$ are obtained by maximizing the evidence,

\[
(\gamma, \sigma^2) = \arg \max_{\gamma \geq 0, \sigma^2 > 0} \log p(Y; \gamma, \sigma^2).
\]

(19)
The maximization is carried out iteratively using derivatives of the evidence for $\gamma$ (see Sec. III-E) as well as conventional noise estimates (see Sec. III-F) as explained in Sec. III-G.

### E. Source power estimation (hyperparameters $\gamma$)

We impose the diagonal structure $\Gamma = \text{diag}(\gamma)$, in agreement with (5), and form derivatives of (17) with respect to the diagonal elements $\gamma_m$, cf. [20]. Using

\[
\frac{\partial \Sigma_y^{-1}}{\partial \gamma_m} = -\Sigma_y^{-1} \frac{\partial \Sigma_y}{\partial \gamma_m} \Sigma_y^{-1} = -\Sigma_y^{-1} a_m A^H \Sigma_y^{-1},
\]

(20)
\[
\frac{\partial \log \text{det} \Sigma_y}{\partial \gamma_m} = \text{tr}(\Sigma_y^{-1} \frac{\partial \Sigma_y}{\partial \gamma_m}) = a_m^H \Sigma_y^{-1} a_m,
\]

(21)
the derivative of (17) is

\[
\frac{\partial \log p(Y; \gamma, \sigma^2)}{\partial \gamma_m} = \frac{1}{\gamma_m \text{tr} \Sigma_y^{-1} a_m^H \Sigma_y^{-1} a_m}
\]

(22)
where $\mu_m = \gamma_m a_m^H \Sigma_y^{-1} Y$ is the $m$th row of $\mu_X$ in (10). Assuming $\mu_m$ given (from previous iterations or initialization) and forcing (22) to zero gives the $\gamma_m$ update (SBL1):

\[
\gamma_m = \frac{1}{\text{tr} \Sigma_y^{-1} a_m^H \Sigma_y^{-1} a_m}
\]

(SBL1)
Given: $\mathbf{A} \in \mathbb{C}^{N \times M}$, $\mathbf{Y} \in \mathbb{C}^{N \times K}$, $K = 3$

Initialize, here: $\sigma_0^2 = 0.1$, $\gamma_0 = 1$, $\epsilon_{\text{min}} = 0.001$, $j_{\text{max}} = 500$

1. Initialize $j = 0$, $\sigma^2 = \sigma_0^2$, $\gamma = \gamma_0$
2. while ($\epsilon > \epsilon_{\text{min}}$) and ($j < j_{\text{max}}$)
3. \quad $j = j + 1$, $\gamma_{\text{old}} = \gamma_{\text{new}}$, $\Gamma = \text{diag}(\gamma_{\text{new}})$
4. $\Sigma_y = \sigma^2 I_N + \mathbf{A}_M \Gamma \mathbf{A}_M^H$
5. $\mu_m = \gamma_m a_m^H \Sigma_y^{-1} \gamma_{\text{new}}$
6. $\gamma_{\text{new}} = \left\{ \begin{array}{ll}
\frac{1}{L} \| \mu_m \|^2 / \sqrt{a_m^H \mu_m a_m} & (\text{SBL}) \\
\frac{1}{L} \| \mu_m \|^2 + (\Sigma_x)_{m,m} & (\text{M-SBL})
\end{array} \right.$
7. $\mathbf{M} = \{ m \in \mathbb{N} | K \text{ largest peaks in } \gamma \} = \{ m_1 \ldots m_K \}$
8. $\mathbf{A}_M = (a_{m_1}, \ldots, a_{m_K})$
9. $\langle \sigma^2 \rangle_{\text{new}} = \frac{1}{N-K} \text{tr} \left( (I_N - \mathbf{A}_M \mathbf{A}_M^+) \Sigma_y \right)$
10. $\epsilon = \| \gamma_{\text{new}} - \gamma_{\text{old}} \|_1 / \| \gamma_{\text{old}} \|_1$
11. end

Output: $\mathcal{M}$, $\gamma_{\text{new}}$, $\langle \sigma^2 \rangle_{\text{new}}$

**TABLE I**

| Algorithm | Exhaust | SBL | M-SBL |
|-----------|---------|-----|-------|
| RMSE (°)  | c)      |     |       |

**SBL ALGORITHM:** In line 6 choose SBL, SBL1 or M-SBL.

When the sample data covariance $\Sigma_y$ is positive definite (i.e. usually when $L \geq 2N$) we can replace $\Sigma_y^{-1}$ in (SBL1) with $\hat{\Sigma}_y^{-1}$ [see (24)]

$$\gamma_{\text{new}} = \frac{1}{\sqrt{L}} \| \mu_m \|^2 / \sqrt{a_m^H \hat{\Sigma}_y^{-1} a_m}.$$  (SBL)

The SBL estimate tends to converge faster as the denominator does not change during iterations.

Wipf and Rao (18): Eq.(18) followed the EM approach to estimate the M-SBL:

$$\gamma_{\text{new}} = \frac{1}{L} \| \mu_m \|^2 + (\Sigma_x)_{m,m}.$$  (M-SBL)

The sequence of parameter estimates in the EM iteration has been proven to converge [21]. However, the convergence is only guaranteed towards a local optimum of the marginal log-likelihood (17). As shown in Sec. IV all the update rules (SBL1)-(M-SBL) converge provided $|\partial \gamma_{\text{new}} / \partial \gamma_m| < 1$.

**F. Noise variance estimation (hyperparameter $\sigma^2$)**

Obtaining a good noise variance estimate is important for fast convergence of the SBL method, as it controls the sharpness of the peaks. For a given set of active DOAs $\mathcal{M}$, stochastic maximum likelihood [14], [16] provides an asymptotically efficient estimate of $\sigma^2$.

Let $\mathbf{\Gamma}_M = \text{diag}(\gamma_{\text{new}})$ be the covariance matrix of the $K$ active sources obtained above with corresponding active steering matrix $\mathbf{A}_M$ which maximizes (17). The corresponding data covariance matrix is

$$\Sigma_y = \sigma^2 I_N + \mathbf{A}_M \mathbf{\Gamma}_M \mathbf{A}_M^H,$$  (23)

where $I_N$ is the identity matrix of order $N$. The data covariance models (12) and (23) are identical. At the optimal solution ($\mathbf{\Gamma}_M, \sigma^2$), Jaffer’s necessary condition ([17];Eq.(6)) must be satisfied

$$\mathbf{A}_M^H (\Sigma_y - \Sigma_y) \mathbf{A}_M = 0.$$  (24)

Substituting (23) into (24) gives

$$\mathbf{A}_M^H (\Sigma_y - \Sigma_y) \mathbf{A}_M = \mathbf{A}_M^H \mathbf{A}_M \mathbf{\Gamma}_M \mathbf{A}_M^H \mathbf{A}_M.$$  (25)

Multiplying (25) from right and left with the pseudo inverse $\mathbf{\hat{A}}_M = (\mathbf{\hat{A}}_M^H \mathbf{A}_M)^{-1} \mathbf{\hat{A}}_M^H$ and $\mathbf{\hat{A}}_M^H$ respectively and subtracting $\Sigma_y$ from both sides yields [16]

$$\langle \sigma^2 \rangle_{\text{new}} = \frac{1}{N-K} \text{tr} \left( (I_N - \mathbf{A}_M \mathbf{\hat{A}}_M^H) \Sigma_y \right).$$  (26)

This estimate requires $K < N$ and will underestimate the noise for small $L$.

Several estimators for the noise $\sigma^2$ are proposed based on EM [8], [12], [13], [22], [23]. Empirically, neither of these converge well in our application. For a comparative illustration in Sec. IV we use the iterative noise $\sigma^2$ EM estimate in [23],

$$\langle \sigma^2 \rangle_{\text{new}} = \frac{1}{L} \| (\mathbf{Y} - \mathbf{A}_M \mathbf{\mu}_X) \|_F^2 + (\langle \sigma^2 \rangle_{\text{old}}^2 M{\sum_{i=1}^{M(M+1)}}) / N.$$  (27)

**G. SBL Algorithm**

Given the observed $\mathbf{Y}$, we iteratively update $\mathbf{\mu}_X$ (10) and $\mathbf{\Sigma}_y$ (12) by using the current $\gamma$. Either SBL, SBL1, or M-SBL can update $\gamma_m$ for $m = 1, \ldots, M$ and then (26) is used to estimate $\sigma^2$. The algorithm is summarized in Table I.

The convergence rate $\epsilon$ measures the relative improvement of the estimated total source power,

$$\epsilon = \| \gamma_{\text{new}} - \gamma_{\text{old}} \|_1 / \| \gamma_{\text{old}} \|_1.$$  (28)

The algorithm stops when $\epsilon \leq \epsilon_{\text{min}}$ and the output is the active set $\mathcal{M}$ (15) from which all relevant source parameter estimates are computed.
b) SBL

SBL1

LASSO

0
10
20
300
Snapshots
Fig. 2. Convergence at SNR=0 dB with L=50. a) γ at iteration 1, 10 , 200 for SBL. Convergence of (b) SBL and (c and d) M-SBL for 100 Monte Carlo simulations. Convergence is shown for γ (left) and σ^2/(σ^2) (right). In (b-c) the noise estimate (σ^2)^new is based on (26) and in d) (27).

IV. EXAMPLE

For multiple sources with well separated DOAs and similar magnitudes, conventional beamforming (CBF) and LASSO/SBL methods provide similar DOA estimates. They differ, however, in their behavior whenever two sources are closely spaced. Thus, we examine 3 sources at DOAs [−3, 2, 75]° with magnitudes [12, 22, 20] dB[9].

We consider an array with N=20 elements and half wavelength intersensor spacing. The DOAs are assumed to be on an angular grid [−90:0:5:90]°, M=361, and L=50 snapshots are observed. The noise is modeled as id complex Gaussian, though robustness to array imperfections [24] and extreme noise distributions [25] can be important. The single-snapshot array signal-to-noise ratio (SNR) is SNR = 10 \log_{10} E \{ ||A x_t||^2 \} / E \{ ||n_t||^2 \}. Then, for L snapshots the noise power σ^2 is

σ^2 = E[||N||^2_{x_t}/L/N = 10^{-SNR/10} E||A X||^2_{x}/L/N. (29)

The estimated (σ^2) new (26) deviates from (σ^2) (29) randomly.

Figure 1 compares DOA estimation methods for the simulation. The LASSO solution is found considering multiple snapshots [9] and programmed in CVX[26]. SBL and M-SBL are calculated using the pseudocode on Table I. CBF suffers from low-resolution and the effect of sidelobes in contrast to sparsity based methods as shown in the power spectra in Fig. 1a.

At array SNR=0 dB the histogram in Fig. 1b shows that CBF poorly locates the neighboring DOAs at broadside. SBL and M-SBL localize the sources well. The root mean squared error (RMSE) in Fig. 1c shows that CBF has low resolution as the main lobe is too broad (see Fig. 1a) and MUSIC performs well for SNR > 5 dB. For this case we include exhaustive search, which defines a lower performance bound and requires 361!/(3!358!)=7.8·10^6 evaluations. LASSO and the SBL methods perform better than MUSIC and offer similar accuracy to the exhaustive search.

We compare the convergence of SBL and M-SBL at array SNR=0 dB (Fig. 2). The spatial spectrum (Fig. 2a) shows how the estimate γ improves with SBL iterations from initially locating only the main peak to locating also the weaker sources. SBL exhibits faster convergence than M-SBL to γ min = −60 dB where the algorithm stops (Figs. 2b versus 2c). M-SBL underestimates σ^2 significantly when using (27) (Fig. 2d).

The average number of iterations for SBL and SBL1 decreases with SNR but increases for M-SBL (Fig. 3a). For SBL and SBL1 the CPU time (Macbook Pro 2014) is nearly constant with number of snapshots (Fig. 3b). The number of estimated parameters (γ, σ^2) is independent on the number of snapshots, but increasing the number of snapshots improves the estimation accuracy (lower RMSE). Contrarily, for LASSO the number of degrees of freedom in X increases as do CPU time with number of snapshots increases.

V. CONCLUSIONS

A sparse Bayesian learning (SBL) algorithm is derived for high-resolution DOA estimation from multi-snapshot complex-valued array data. The algorithm uses evidence maximization based on derivatives to estimate the source powers and the noise variance. The method uses the estimated source power at each potential DOA as a proxy for an active DOA promoting sparse reconstruction.

Simulations indicate that the proposed SBL algorithm is a factor of 2 faster than established EM approaches at the same estimation accuracy. Increasing the number of snapshots improves the estimation accuracy while the computational effort is nearly independent of snapshots.

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