AN ALMOST SURE INVARIANCE PRINCIPLE FOR ADDITIVE FUNCTIONALS OF MARKOV CHAINS

F. RASSOUL-AGHA AND T. SEPPÄLÄinen

Abstract. We prove an invariance principle for a vector-valued additive functional of a Markov chain for almost every starting point with respect to an ergodic equilibrium distribution. The hypothesis is a moment bound on the resolvent.

1. Introduction.

This note extends a result of Maxwell and Woodroofe [5]. Our notation and presentation follow [5] as closely as possible, and some results from there will be repeated without proofs. The work presented here was motivated by applications to random walk in random environment that are reported elsewhere.

After completing this note we learned of the work of Derriennic and Lin on fractional coboundaries of Banach space contractions [2]. The estimates needed for the invariance principles we prove can be then obtained by applying the Derriennic and Lin machinery, and this way one can even improve the moment hypothesis to just having two moments ($p = 2$ below); see [3]. Thus, currently our note offers alternative probabilistic proofs of the results of [3] under the more restrictive moment hypothesis of [5].

Let $(X_n)_{n \geq 0}$ be a stationary ergodic Markov chain defined on a probability space $(\Omega, \mathcal{F}, P)$, with values in a general measurable space $(\mathcal{X}, \mathcal{B})$. Let $Q(x; dy)$ be its transition probability kernel and $\pi$ the stationary marginal distribution of each $X_n$. Write $E$ for the expectation under $P$. $P_x$ denotes the probability measure obtained by conditioning on $X_0 = x$, and $E_x$ is the corresponding expectation. For $p \geq 1$, we will denote by $L^p(\pi)$ the equivalence class of $\mathcal{B}$-measurable functions $g$ with values in $\mathbb{R}^d$ for some $d \geq 1$ and such that

$$\|g\|_p^p = \int |g(x)|^p \pi(dx) < \infty.$$ 

Here, $| \cdot |$ denotes the $\ell^2$-norm on $\mathbb{R}^d$.

Date: November 26, 2004.

2000 Mathematics Subject Classification. Primary 60F17, secondary 60J10.

Key words and phrases. Invariance principle, functional central limit theorem, additive functional of Markov chain, vector-valued martingale.

T. Seppäläinen was partially supported by National Science Foundation grant DMS-0402231.
Now fix $d$ and an $\mathbb{R}^d$-valued function $g \in L^2(\pi)$ with $\int g \, d\pi = 0$. Define $S_0(g) = 0$ and

$$S_{n+1}(g) = \sum_{k=0}^{n} g(X_k) \quad \text{and} \quad \tilde{S}_n(g) = S_n(g) - E_{X_0}(S_n(g)), \quad \text{for } n \geq 0.$$ 

We are concerned with central limit type results for $S_n(g)$ and $\tilde{S}_n(g)$. This question has been investigated from many angles and under different assumptions; see [5] and its references. A widely used method of Kipnis and Varadhan [4] works for reversible chains. Article [5] adapted this approach to a non-reversible setting, and used growth bounds on the resolvent to obtain sufficient conditions for an invariance principle for $S_n(g)$ under $P$, if $p > 2$.

Derriennic and Lin used then their theory of fractional coboundaries [2] to push the result to an invariance principle for $S_n(g)$ under $P_x$, for $\pi$-a.e. $x$, even when $p = 2$; see [3]. Using their method one can also show that the same almost-sure invariance principle holds for $\tilde{S}_n(g)$. We will show how to further the probabilistic technique of [2] to yield both almost-sure invariance principles (for $S_n(g)$ and $\tilde{S}_n(g)$) when $p > 2$.

Invariance principles for additive functionals of Markov chains have many applications. This note is a byproduct of the authors’ recent work on random walks in a random environment [6, 7] where this invariance principle proved useful.

Let us now describe the structure of this note. In Section 2 we will present the setting of [5] and prove an $L^q$ bound, with $q > 2$, on a certain martingale. In Section 3 we will state and prove the main theorem of the note. The proof depends on a vector-valued version of a well-known invariance principle for martingales (Theorem 3 of [5]).

2. A USEFUL MARTINGALE.

For a function $h \in L^1(\pi)$ and $\pi$-a.e. $x \in \mathcal{X}$ define

$$Qh(x) = \int h(y)Q(x;dy).$$

$Q$ is a contraction on $L^p(\pi)$ for every $p \geq 1$. For $\varepsilon > 0$ let $h_\varepsilon$ be the solution of

$$(1 + \varepsilon)h_\varepsilon - Qh_\varepsilon = g.$$

In other words,

$$h_\varepsilon = \sum_{k=1}^{\infty} (1 + \varepsilon)^{-k} Q^{k-1} g.$$

Note that $h_\varepsilon \in L^p(\pi)$, if $g \in L^p(\pi)$. On $\mathcal{X}^2$ define the function

$$H_\varepsilon(x_0, x_1) = h_\varepsilon(x_1) - Qh_\varepsilon(x_0).$$
For a given realization of \((X_k)_{k \geq 0}\), let
\[
M_n(\varepsilon) = \sum_{k=0}^{n-1} H\varepsilon(X_k, X_{k+1}) \quad \text{and} \quad R_n(\varepsilon) = Qh\varepsilon(X_0) - Qh\varepsilon(X_n)
\]
so that
\[
S_n(g) = M_n(\varepsilon) + \varepsilon S_n(h\varepsilon) + R_n(\varepsilon).
\]
Finally, let \(\pi_1\) be the distribution of \((X_0, X_1)\) under \(P\); that is
\[
\pi_1(dx_0, dx_1) = Q(x_0; dx_1)\pi(dx_0).
\]
Let us denote the \(L^p\)-norm on \(L^p(\pi_1)\) by \(\|\cdot\|_p\). The following theorem summarizes results of [5].

**Theorem MW.** Assume that \(g \in L^2(\pi)\) and that there exists an \(\alpha \in (0, 1/2)\) such that
\[
\left\| \sum_{k=0}^{n-1} Q^k g \right\|_2 = O(n^\alpha). \tag{2.1}
\]
Then
(a) The limit \(H = \lim_{\varepsilon \to 0^+} H\varepsilon\) exists in \(L^2(\pi_1)\). Moreover, if one defines
\[
M_n = \sum_{k=0}^{n-1} H(X_k, X_{k+1}),
\]
then, for \(\pi\)-almost every \(x\), \((M_n)_{n \geq 1}\) is a \(P\)-square integrable martingale, relative to the filtration \(\{\mathcal{F}_n = \sigma(X_0, \cdots, X_n)\}_{n \geq 0}\).
(b) One has \(\|h\varepsilon\|_2 = O(\varepsilon^{-\alpha})\), and if \(R_n = S_n(g) - M_n = M_n(\varepsilon) - M_n + \varepsilon S_n(h\varepsilon) + R_n(\varepsilon)\), then
\[
E(|R_n|^2) = O(n^{2\alpha}).
\]

**Proof.** The existence of \(H\) follows from Proposition 1 of [5]. The statement about \(M_n\) follows from Theorem 1 therein. The bounds on \(\|h\varepsilon\|_2\) and \(E(|R_n|^2)\) follow from Lemma 1 and Corollary 4 of [5], respectively. \(\square\)

If, moreover, one has an \(L^p\) assumption on \(g\), then one can say more.

**Theorem 1.** Assume that there exists an \(\alpha < 1/2\) for which (2.1) is satisfied. Assume also that there exists a \(p > 2\) such that \(g \in L^p(\pi)\). Then there exists a \(q \in (2, p)\) such that \(H \in L^q(\pi_1)\) and \((M_n)_{n \geq 1}\) is an \(L^q\)-martingale.

**Proof.** First choose a positive \(q < (3 - 2\alpha)p/(1 - 2\alpha + p)\). One can check that since \(2\alpha < 1\) and \(p > 2\), we have \(q \in (2, p)\). Using Hölder’s inequality, we have
\[
\|H_\delta - H_\varepsilon\|_q^q \leq \|H_\delta - H_\varepsilon\|_p^\alpha \|H_\delta - H_\varepsilon\|_2^b,
\]
where \( a = p(q - 2)/(p - 2) < q \) and \( b = q - a \). Next, observe that
\[
\|h_\varepsilon\|_p \leq \sum_{n \geq 1} (1 + \varepsilon)^{-n} \|g\|_p = \|g\|_{p, \varepsilon^{-1}}.
\]
Thus, one has
\[
W(0) = 0,
\]
and, by Lemma 2 of [5],
\[
\] = \max_{\{1 \leq k \leq n\}} \left\{ |||B_k||| \right\}
\]
by [5]. Let \( x \) denote the distribution of Brownian motion with diffusion matrix \( \Gamma \) on the space \( D \) of Borel probability measures on \( [0, 1] \). Let \( \Delta \) denote the Prohorov metric on the space of Borel probability measures on \( D \).

Remark 1. Note that [5] uses \( \|\cdot\|_1 \) for the \( L^2 \)-norm under \( \pi_1 \), while we use \( \|\cdot\|_2 \).

3. The almost sure invariance principle.

First some notation. We write \( A^T \) for the transpose of a vector or matrix \( A \). An element of \( \mathbb{R}^d \) is regarded as a \( d \times 1 \) matrix, or column vector. Define
\[
\mathbb{B}_n(t) = n^{-1/2} S_{[nt]}(g) \quad \text{and} \quad \widetilde{\mathbb{B}}_n(t) = n^{-1/2} \widetilde{S}_{[nt]}(g), \quad \text{for } t \in [0, 1].
\]
Here, \( [x] = \max\{k \in \mathbb{Z} : k \leq x\} \). Let \( D_{\mathbb{R}^d}([0, 1]) \) denote the space of right continuous functions on \( [0, 1] \) taking values in \( \mathbb{R}^d \) and having left limits. This space is endowed with the usual Skorohod topology \( \mathcal{I} \). Let \( \Delta \) denote the Prohorov metric on the space of Borel probability measures on \( D_{\mathbb{R}^d}([0, 1]) \).

For a given symmetric, non-negative definite \( d \times d \) matrix \( \Gamma \), a Brownian motion with diffusion matrix \( \Gamma \) is the \( \mathbb{R}^d \)-valued process \( \{W(t) : 0 \leq t \leq 1\} \) such that \( W(0) = 0 \), \( W \) has continuous paths, independent increments, and for \( s < t \) the \( d \)-vector \( W(t) - W(s) \) has Gaussian distribution with mean zero and covariance matrix \( (t-s)\Gamma \). If the rank of \( \Gamma \) is \( m \), one can produce such a process by finding a \( d \times m \) matrix \( \Lambda \) such that \( \Gamma = \Lambda \Lambda^T \), and by defining \( W(t) = \Lambda B(t) \) where \( B \) is an \( m \)-dimensional standard Brownian motion.

Let \( \Phi_T \) denote the distribution of Brownian motion with diffusion matrix \( \Gamma \) on the space \( D_{\mathbb{R}^d}([0, 1]) \). For \( x \in \mathcal{X} \) let \( \Psi_n(x) \), respectively \( \widetilde{\Psi}_n(x) \), be the distribution of \( \mathbb{B}_n \), respectively \( \widetilde{\mathbb{B}}_n \), on the Borel sets of \( D_{\mathbb{R}^d}([0, 1]) \) under the measure \( P_x \); that is, conditioned on \( X_0 = x \).

Here is our main theorem.
**Theorem 2.** Assume there are \( p > 2 \) and \( \alpha < 1/2 \) for which \( g \in L^p(\pi) \) and \( E(|R_n^2|) = O(n^{2\alpha}) \). Then

\[
\lim_{n \to \infty} \Delta(\Phi_D, \Psi_n(x)) = 0 \quad \text{for} \quad \pi\text{-a.e.} \; x,
\]

where \( D = E(M_1 M_1^T) = \int HH^T d\pi_1 \).

**Remark 2.** The above result improves Theorem 2 of [5] which stated that

\[
\lim_{n \to \infty} \int \Delta(\Phi_D, \Psi_n(x)) \pi(dx) = 0.
\]

**Remark 3.** Due to Theorem MW, (2.1) guarantees the bound on \( E(|R_n^2|) \) in Theorem 2.

**Proof.** The proof is essentially done in [5]. We explain below how to apply Borel-Cantelli’s Lemma to strengthen their result to an almost sure statement.

Let \( M_n^*(t) = n^{-1/2} M_{[nt]} \). We have

\[
\sup_{0 \leq t \leq 1} |\mathbb{B}_n(t) - M_n^*(t)| \leq n^{-1/2} \max_{k \leq n} |R_k|.
\]

Therefore to conclude the proof we need to show two things:

for \( \pi\)-almost every \( x \), under the probability measure \( P_x \) the processes

\[
M_n^*
\]

converge weakly to a Brownian motion with diffusion matrix \( \mathcal{D} \), (3.1)

and

\[
n^{-1/2} \max_{k \leq n} |R_k| \to 0 \quad \text{in} \; P_x\text{-probability, for} \; \pi\text{-a.e.} \; x.
\]

(3.2)

Statement (3.1) follows from the martingale invariance principle stated as Theorem 3 in [6]. The limits needed as hypotheses for that theorem follow from ergodicity and the square-integrability of \( H \). We leave this check to the reader.

To prove (3.2), let \( n_j = j^r \) for a large enough integer \( r \). Fix \( 0 < \gamma < 1 \), and let \( m_j = [n_j^{1-\gamma}] \), \( \ell_j = [n_j^\gamma] \). Here \( [x] = \min \{ n \in \mathbb{Z} : x \leq n \} \). Since \( R_n = S_n(g) - M_n \), one can write

\[
n_j^{-1/2} \max_{1 \leq i \leq n_j} |R_i| \leq n_j^{-1/2} \max_{0 \leq k \leq m_j} |R_{k\ell_j}|
\]

\[
+ n_j^{-1/2} \max_{0 \leq k < m_j} \max_{k \ell_j \leq i \leq (k+1)\ell_j} |M_i - M_{k\ell_j}|
\]

\[
+ n_j^{-1/2} \max_{0 \leq k < m_j} \max_{k \ell_j \leq i \leq (k+1)\ell_j} |S_i(g) - S_{k\ell_j}(g)|.
\]

Recalling that \( E(|R_n|^2) = O(n^{2\alpha}) \) with \( \alpha < 1/2 \), one can apply Corollary 3 of [5] to get that for any \( \delta > 0 \)

\[
P(\max_{0 \leq k \leq m_j} |R_{k\ell_j}| \geq \delta \sqrt{n_j}) = O(\ell_j^{2\alpha} m_j^\beta / n_j) = O(j^{-r(1-2\gamma\alpha-(1-\gamma)\beta)}),
\]

where \( \ell_j = [n_j^\gamma] \) and \( m_j = [n_j^{1-\gamma}] \).
for any $\beta > 1$. Choosing $\beta$ close enough to 1 and $r$ large enough, the above becomes summable. Borel-Cantelli’s Lemma implies then that the first term on the right-hand-side of (3.3) converges to 0, $P$-a.s.

The second martingale term on the right-hand side of (3.3) tends to 0 in $P_x$-probability for $\pi$-a.e. $x$, by the functional central limit theorem for $L^2$-martingales; see Theorem 3 of [6], for example. So it all boils down to showing that the last term in (3.3) goes to 0 $P$-a.s.

**Remark 4.** Note that we have so far used the fact that $g \in L^2(\pi)$. It is only to control the third term in (3.3) that we need a higher moment.

Define, for $\delta > 0$,

$$B'_j = \max_{0 \leq k < n_j} \max_{\ell_j \leq i \leq (k+1)\ell_j} \left| S_i(g) - S_{k\ell_j}(g) \right| \geq \delta \sqrt{n_j}.$$ 

Since $g \in L^p(\pi)$, one can write:

$$P(B'_j) \leq P(\max_{i \leq n_j} |g(X_i)| \geq \delta \sqrt{n_j}/\ell_j)
\leq n_j \pi(|g| \geq \delta \sqrt{n_j}/\ell_j) = O(j^{-r(p/2-1-\gamma p)}).$$

By choosing $\gamma$ small enough and $r$ large enough, one can make sure that $P(B'_j)$ is summable. By Borel-Cantelli’s Lemma, the third term in (3.3) converges to 0, $P$-a.s.

Finally, note that if $n_{j-1} \leq n \leq n_j$, then

$$\max_{k \leq n} \frac{|R_k|}{\sqrt{n}} \leq \left( \frac{j}{j - 1} \right)^{r/2} \max_{k \leq n_j} \frac{|R_k|}{\sqrt{n_j}}$$

and so (3.2) follows. \hfill \Box

**Remark 5.** In the above proof we only needed the martingale term in (3.3) to converge in $P_x$-probability. The $L^q$-bounds of Theorem 1 imply that it actually goes to 0 $P$-a.s., making (3.2) also true $P$-a.s. All this is of course under the assumptions $p > 2$ and (2.1) with $\alpha < 1/2$. In [3] it is shown that the same almost-sure convergence happens even when $p = 2$.

We also have a similar result for $\tilde{S}_n(g)$:

**Theorem 3.** Assume there are $p > 2$ and $\alpha < 1/2$ for which $g \in L^p(\pi)$ and condition (2.1) is satisfied. Then $n^{-1/2} \max_{k \leq n} |E_x(S_k(g))|$ converges to 0 as $n$ goes to infinity for $\pi$-almost every $x$. Consequently, for $\pi$-almost every $x$,

$$\lim_{n \to \infty} n^{-1/2} \max_{k \leq n} |S_k(g) - \tilde{S}_k(g)| = 0$$

and, therefore,

$$\lim_{n \to \infty} \Delta(\Phi_\mathcal{D}, \tilde{\Psi}_n(x)) = 0$$

for $\pi$-a.e. $x$.

The diffusion matrix $\mathcal{D}$ is as defined in Theorem 2.
Before we start the proof, we need to reprove a maximal inequality of [5], this time for a Markov transition operator rather than a shift. For a probability transition kernel \( Q \) and a function \( g \) in its domain, define \( T_n(g, Q) = \sum_{k=0}^{n-1} Q^k g \). We then have the following:

**Proposition 1.** Let \( Q \) be a probability transition kernel with invariant measure \( \pi \).

Let \( g \in L^2(\pi) \) be such that

\[
\int |T_n(g, Q)|^2 d\pi \leq C(g, Q)n,
\]

for some \( C(g, Q) < \infty \) and all \( n \geq 1 \). Then we have

\[
\pi \left( \max_{j \leq n} |T_j(g, Q)| > \lambda \right) \leq \frac{2^{6k}C(g, Q)n^{1+2^{-k}}}{\lambda^2},
\]

for all \( n \geq 1, k \geq 0 \), and \( \lambda > 0 \).

**Proof.** We will proceed by induction on \( k \). For \( k = 0 \) the lemma follows from Chebyshev’s and Jensen’s inequalities, as well as the invariance of \( \pi \) under \( Q \). Let us assume that the lemma has been proved for some \( k \geq 0 \). We will prove it for \( k+1 \). To this end, choose \( n \geq 1 \) and \( \lambda > 0 \). Let \( m = \lceil \sqrt{n} \rceil \). Then \([n/m] \leq m\), and

\[
\pi \left( \max_{j \leq n} |T_j(g, Q)| > \lambda \right) \leq \pi \left( \max_{i \leq n/m} |T_{im}(g, Q)| > \lambda/2 \right)
\]

\[
+ m \ max_{i \leq n/m} \pi \left( \max_{j \leq m} |T_{j+im}(g, Q) - T_{im}(g, Q)| > \lambda/2 \right)
\]

\[
\leq \pi \left( \max_{i \leq m} |T_i(T_m(g, Q), Q^m)| > \lambda/2 \right)
\]

\[
+ m \ max_{i \leq m} \pi \left( \max_{j \leq m} |T_j(Q^{im}g, Q)| > \lambda/2 \right)
\]

\[
\leq 4 \cdot 2^{6k}C(T_m(g, Q), Q^m)m^{1+2^{-k}} \lambda^2
\]

\[
+ \ max_{i \leq m} 4 \cdot 2^{6k}C(Q^{im}g, Q)m^{2+2^{-k}} \lambda^2.
\]

But one has

\[
\int |T_n(T_m(g, Q), Q^m)|^2 d\pi = \int |T_{mn}(g, Q)|^2 d\pi \leq C(g, Q)mn
\]

and, therefore, \( C(T_m(g, Q), Q^m) \leq C(g, Q)m \). Similarly,

\[
\int |T_n(Q^{im}g, Q)|^2 d\pi = \int |Q^{im}T_n(g, Q)|^2 d\pi \leq \int Q^{im}|T_n(g, Q)|^2 d\pi
\]

\[
= \int |T_n(g, Q)|^2 d\pi \leq C(g, Q)n.
\]
Thus, $C(Q_m g, Q) \leq C(g, Q)$. Above, we have used Jensen’s inequality to bring $Q$ outside the square and then the fact that $\pi$ is invariant under $Q$. Now, we have

$$\pi \left( \max_{j \leq n} |T_n(g, Q)| > \lambda \right) \leq \frac{8 \cdot 2^{6k} C(g, Q) m^{2+2-k}}{\lambda^2}.$$ 

Since $m \leq 2\sqrt{n}$, it follows that

$$\pi \left( \max_{j \leq n} |T_n(g, Q)| > \lambda \right) \leq \frac{2^{6k+6} C(g, Q) n^{1+2-k-1}}{\lambda^2}$$

which is the claim of the lemma, for $k + 1$. □

The following is then immediate:

**Corollary 1.** For any $\beta > 1$ there is a constant $\Gamma$, depending only on $\beta$, for which

$$\pi \left( \max_{j \leq n} |T_j(g, Q)| > \lambda \right) \leq \frac{\Gamma C(g, Q)n^\beta}{\lambda^2}$$

for all $\lambda > 0$ and $n \geq 1$.

We can now prove the theorem.

**Proof of Theorem 3.** Observe that $E_x(S_n(g)) = T_n(g, Q)$. Now, recall that $n_j = j^r$, for an integer $r$ large enough. Also, for $0 < \gamma < 1$ we have $m_j = \lceil n_j^{1-\gamma} \rceil$, $\ell_j = \lceil n_j^{\gamma} \rceil$.

Then,

$$n_j^{-1/2} \max_{i \leq n_j} |E_x(S_i(g))| \leq n_j^{-1/2} \max_{k \leq m_j} |E_x(S_{k\ell_j}(g))|$$

(3.5)

$$+ n_j^{-1/2} \max_{k \leq m_j} \max_{\ell_j \leq \ell \leq (k+1)\ell_j} |E_x(S_i(g)) - E_x(S_{k\ell_j}(g))|.$$ (3.6)

For the first term, we can use the above corollary to write

$$\pi \left( n_j^{-1/2} \max_{k \leq m_j} |E_x(S_{k\ell_j}(g))| > \varepsilon \right) = \pi \left( n_j^{-1/2} \max_{k \leq m_j} |T_{k\ell_j}(g, Q)| > \varepsilon \right)$$

$$= \pi \left( \max_{k \leq m_j} |T_k(T_{\ell_j}(g, Q), Q^{\ell_j})| > \varepsilon \sqrt{n_j} \right)$$

$$\leq \frac{\Gamma C(T_{\ell_j}(g, Q), Q^{\ell_j}) m_j^\beta}{\varepsilon^2 n_j}$$

$$\leq \frac{\Gamma C \ell_j^{2\alpha} m_j^\beta}{\varepsilon^2 n_j} = O(j^{-r(1-2\alpha\gamma-(1-\gamma)\beta)})$$

since

$$\int |T_n(T_{\ell_j}(g, Q), Q^{\ell_j})|^2 d\pi = \int |T_{n\ell_j}(g, Q)|^2 d\pi \leq C(n\ell_j)^{2\alpha} \leq C\ell_j^{2\alpha} n,$$
If one chooses $\beta$ small enough and $r$ large enough, then the term on line (3.3) goes to 0, $\pi$-a.s., by Borel-Cantelli’s Lemma. For the term on line (3.6) we have

$$\pi \left( n_j^{-1/2} \max_{k \leq n_j} \max_{k \ell_j \leq i \leq (k+1) \ell_j} |E_x(S_i(g)) - E_x(S_{k\ell_j}(g))| \geq \varepsilon \right)$$

$$\leq \pi \left( \max_{i \leq n_j} |Q^i g| \geq \varepsilon \sqrt{n_j / \ell_j} \right)$$

$$\leq n_j \max_{i \leq n_j} \pi \left( |Q^i g| \geq \varepsilon \sqrt{n_j / \ell_j} \right)$$

$$\leq O\left(j^{-r(p/2-1-\gamma p)}\right),$$

since

$$\int |Q^i g|^p d\pi \leq \int Q^i(|g|^p)d\pi = \int |g|^p d\pi < \infty.$$

Using Borel-Cantelli’s Lemma, we get that the term on line (3.6) also converges to 0, $\pi$-a.s., if one chooses $\gamma$ small enough and $r$ large enough.

Therefore, we have shown that $n_j^{-1/2} \max_{i \leq n_j} |E_x(S_i(g))|$ converges to 0, $\pi$-a.s. The claim of the theorem follows then as in (3.4), by considering $n_j \leq n \leq n_{j+1}$. □

References

[1] Billingsley, P. (1999). *Convergence of probability measures*. 2nd edn. John Wiley & Sons Inc., New York.

[2] Derriennic, Y. and Lin, M. (2001). Fractional Poisson equations and ergodic theorems for fractional coboundaries. *Israel J. Math.* 123 93–130.

[3] Derriennic, Y. and Lin, M. (2003). The central limit theorem for Markov chains started at a point. *Probab. Theory Related Fields* 125 73–76.

[4] Kipnis, C. and Varadhan, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* 104 1–19.

[5] Maxwell, M. and Woodroofe, M. (2000). Central limit theorems for additive functionals of Markov chains. *Ann. Probab.* 28 713–724.

[6] Rassoul-Agha, F. and Seppäläinen, T. (2004). An almost sure invariance principle for random walks in a space-time i.i.d. random environment. *Probab. Theory Related Fields* To appear.

[7] Rassoul-Agha, F. and Seppäläinen, T. (2004). Ballistic random walk in a random environment with a forbidden direction. *Ann. Probab.* Submitted.

Mathematical Biosciences Institute, Ohio State University, Columbus, OH 43210

E-mail address: firas@math.ohio-state.edu

URL: www.math.ohio-state.edu/~firas

Mathematics Department, University of Wisconsin-Madison, Madison, WI 53706

E-mail address: seppalai@math.wisc.edu

URL: www.math.wisc.edu/~seppalai