THE MINIMAL VOLUME OF LOG CANONICAL SURFACES OF GENERAL TYPE WITH POSITIVE GEOMETRIC GENUS

WENFEI LIU

Abstract. We determine the minimal possible volume of a projective log canonical surface of general type with prescribed positive geometric genus. As applications, we provide effective Noether type inequalities for log canonical threefolds and stable surfaces. Also, we obtain a uniform bound on the order of the symplectic automorphism group $\text{Aut}_s(S)$ of smooth projective surfaces $S$ of general type.

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Date: August 24, 2022.
2010 Mathematics Subject Classification. Primary: 14J29; Secondary: 14E30.
Key words and phrases. log surfaces of general type, minimal volume, geography, Noether inequality.
1. Introduction

Let $(X, B)$ be a projective log canonical surface defined over an algebraically closed field $\mathbb{k}$. The volume $\text{vol}(K_X + B)$ measures the asymptotic growth of the pluri-canonical linear system:

$$h^0(X, |m(K_X + B)|) = \frac{\text{vol}(K_X + B)}{2^{m^2} + o(m^2)}.$$

One says that $K_X + B$ is big if $\text{vol}(K_X + B) > 0$; if this is the case, $(X, B)$ is said to be of general type.

Let us take a subset $\mathcal{C} \subset (0, 1]$, and consider the set $\mathcal{S}(\mathcal{C})$ of projective log canonical surfaces $(X, B)$ of general type such that the coefficients of $B$ lie in $\mathcal{C}$. (When $\mathcal{C} = \emptyset$, we understand that $B$ is the zero divisor and $(X, 0)$ is just a projective log canonical surface without boundary.) Set

$$\mathbb{K}^2(\mathcal{C}) := \{\text{vol}(K_X + B) \mid (X, B) \in \mathcal{S}(\mathcal{C})\}.$$

It is a fundamental result of Alexeev [2] that, if $\mathcal{C}$ satisfies the descending chain condition (DCC), then so does $\mathbb{K}^2(\mathcal{C})$.† In particular, for a DCC set $\mathcal{C} \subset (0, 1]$, any subset of $\mathbb{K}^2(\mathcal{C})$ attains the minimum.

It is thus interesting to find the minima of the volumes of naturally appearing classes of log canonical surfaces. For a fixed subset $\mathcal{C} \subset (0, 1]$ and a nonnegative integer $p_g$, we can define the following subsets of $\mathcal{S}(\mathcal{C})$ and $\mathbb{K}^2(\mathcal{C})$ respectively:

$$\mathcal{S}(\mathcal{C}, p_g) := \{(X, B) \in \mathcal{S}(\mathcal{C}) \mid p_g(X, B) = p_g\}$$

where $p_g(X, B) := h^0(X, K_X + [B])$ is the geometric genus of $(X, B)$, and

$$\mathbb{K}^2(\mathcal{C}, p_g) := \{\text{vol}(K_X + B) \mid (X, B) \in \mathcal{S}(\mathcal{C}, p_g)\}.$$

Obviously, we have $\mathcal{S}(\mathcal{C}) = \bigcup_{p_g \geq 0} \mathcal{S}(\mathcal{C}, p_g)$ and $\mathbb{K}^2(\mathcal{C}) = \bigcup_{p_g \geq 0} \mathbb{K}^2(\mathcal{C}, p_g)$.

The aim of this paper is to find the minimum of $\mathbb{K}^2(\mathcal{C}, p_g)$ when $p_g > 0$ and when the coefficient set $\mathcal{C}$ satisfies the DCC. Actually, as it turns out, the minimum of $\mathbb{K}^2(\mathcal{C}, p_g)$ for $p_g > 0$ can be attained under a weaker condition on $\mathcal{C}$ than the DCC:

**Theorem 1.1.** Let $\mathcal{C} \subset (0, 1]$ be a subset such that $\mathcal{C} \cup \{1\}$ attains the minimum, say $c$. Let $p_g$ be a positive integer. Then the following holds.

(i) The set $\mathbb{K}^2(\mathcal{C}, p_g)$ attains its minimum, and we have

$$\min \mathbb{K}^2(\mathcal{C}, p_g) = \min \mathbb{K}^2(\{c\leq 1, p_g\}).$$

(ii) If $p_g \geq 2$, then

$$\min \mathbb{K}^2(\mathcal{C}, p_g) = \begin{cases} (2c - c^2)(p_g - 1) - 2c^2 & \text{if } c < \frac{p_g - 1}{p_g + 1} \\ p_g - 3 + \frac{4}{p_g + 1} & \text{if } c \geq \frac{p_g - 1}{p_g + 1} \end{cases}.$$

(iii) If $p_g = 1$, then

$$\min \mathbb{K}^2(\mathcal{C}, 1) = \begin{cases} \frac{1}{11}c^2, & \text{if } c \leq \frac{7}{13} \\ -\frac{11}{6}c^2 + 2c - \frac{7}{13}, & \text{if } \frac{7}{13} < c \leq \frac{6}{11} \\ \frac{1}{13}, & \text{if } c > \frac{6}{11}. \end{cases}$$

†See [14] for a new treatment.
For a fixed \( p_g > 0 \), \( \min \mathbb{K}^2(C, p_g) \) is a piecewise smooth, continuous function of \( c = \min(C \cup \{1\}) \). This lies in the fact that we can take the same smooth projective surface \( W \) and reduced divisors \( B'_W \) and \( D \) so that, setting \( B_W^{(c)} = B'_W + cD \), we have \( p_g(W, B_W^{(c)}) = p_g \) and \( \text{vol}(K_W + B_W^{(c)}) = \min \mathbb{K}^2(C, p_g) \); see Section 7.

For the following frequently used sets of coefficients

\[
C_0 = \emptyset, C = \{1\}, C_2 = \left\{ 1 - \frac{1}{n} \mid n \in \mathbb{N}_{\geq 2} \right\} \cup \{1\},
\]

we have \( \min(C_0 \cup \{1\}) = \min(C_1 \cup \{1\}) = 1 \), and \( \min(C_2 \cup \{1\}) = \frac{1}{2} \), and Theorem 1.1 (iii) gives

\[
\min \mathbb{K}^2(C_0, 1) = \min \mathbb{K}^2(C_1, 1) = \frac{1}{143}, \quad \min \mathbb{K}^2(C_2, 1) = \frac{1}{168}.
\]

Therefore, if \((X, B) \in S(C_1)\) has \( \text{vol}(K_X + B) < \frac{1}{143} \) then \( p_g(X, B) = 0 \) and, in fact, \( X \) should be a rational surface (Corollary 6.12).

Theorem 1.1 can be reformulated without involving the coefficient set \( C \):

**Theorem 1.2.** Let \( p_g \) be a positive integer and \( c \in (0, 1] \). Let \((X, B)\) be any projective log canonical surface \( p_g(X, B) = p_g \) and \( \min(C_B \cup \{1\}) = c \).

(i) If \( p_g \geq 2 \), then

\[
\text{vol}(K_X + B) \geq \begin{cases} 
(2c - c^2)(p_g - 1) - 2c^2 & \text{if } c \leq \frac{p_g - 1}{p_g + 1} \\
p_g - 3 + \frac{4}{p_g + 1} & \text{if } c \geq \frac{p_g - 1}{p_g + 1}
\end{cases}
\]

(ii) If \( p_g = 1 \), then

\[
\text{vol}(K_X + B) \geq \begin{cases}
\frac{14}{15}c^2, & \text{if } c \leq \frac{7}{13} \\
\frac{14}{15}c^2 + 2c - \frac{7}{13}, & \text{if } \frac{7}{13} < c \leq \frac{6}{13} \\
\frac{1}{13}, & \text{if } c > \frac{6}{13}.
\end{cases}
\]

Moreover, the inequalities are optimal in the sense that the equalities can be realized by some \((X, B) \in S\{c < 1, p_g\}\).

When \( C \subset \{1\} \), Theorem 1.2 recovers the inequality obtained in [28]:

\[
\text{vol}(K_X + B) \geq p_g(X, B) - 3 + \frac{4}{p_g(X, B) + 1}.
\]

Note that (1.3) is trivially true if \( p_g(X, B) = 1 \), since its right hand side is negative in this case. The upshot of Theorem 1.2 is to provide the optimal lower bound also in this case, at the same time allowing the boundary divisors to have any coefficients in \((0, 1]\).

As a consequence of Theorem 1.1 or 1.2, we can bound the log canonical volume from below by a linear function of the geometric genus.

**Theorem 1.3.** For any projective log canonical surface \((X, B)\) of general type, we have

\[
\text{vol}(K_X + B) \geq (2c - c^2)p_g(X, B) - (2c + c^2)
\]

where \( c := \min(C_B \cup \{1\}) \). Moreover, the inequality is optimal in the following sense:

(i) if \( c < 1 \), then the equality can be attained for \((X, B)\) with \( p_g(X, B) \geq \frac{1}{143} \).

(ii) if \( c = 1 \) then the inequality is strict, but there is a sequence of projective log canonical surfaces \( X_i \) of general type such that

\[
\lim_{i \to \infty} \text{vol}(K_{X_i}) - p_g(X_i) - 3 = 0
\]
The paper is organized as follows: In Section 2 we recall some preliminaries on the birational geometry of surfaces, such as log canonical singularities, the Zariski decomposition, and the volume of \( \mathbb{R} \)-divisors. Then we set out to find the minimal possible volume of a projective log canonical surface of general type with prescribed (positive) geometric genus. It can roughly be divided into the following parts:

(i) In Section 3 we describe how to find the minimal volume of higher models over a fixed log surface \((Z, B_Z)\); see Lemma 3.3. This is then explicitly carried out in the case when the log surface contains an extended canonical type curve; see Section 3.2.

(ii) In Section 4, we introduce the semistable decomposition \(B = B^s + B^{ns}\) for a projective log canonical surface \((X, B)\), and study its behavior under blow-ups. We can then divide the set \(S(C, p_g)\) into several subsets \(S(C, p_g; \kappa)\) according to the Kodaira dimension \(\kappa\) of \((\tilde{X}, B_{\tilde{X}})\), where \((\tilde{X}, B_{\tilde{X}}) \to (X, B)\) is the minimal resolution.

(iii) In Section 5 we find out several necessary conditions that a log surface \((X, B) \in S(C, p_g; \kappa)\) achieving the minimal volume should satisfy; see Proposition 5.6. One of them is that the semistable part \(B^s\) and the non-semistable part \(B^{ns}\) are disjoint; this simplifies the discussion considerably.

(iv) In Section 6 we use the necessary conditions found in Section 5 to give lower bounds on the volumes of log surfaces in \(S(C, p_g; \kappa)\) when \(\kappa \geq 0\). These lower bounds turn out to be optimal by the examples provided in Section 7, which in turn depends on the explicit computation in Section 3. This finishes the search for the minimal volume of log surfaces in \(S(C, p_g; \kappa)\) with \(p_g > 0\) and hence also of \(S(C, p_g)\) with \(p_g > 0\).

In the final Section 8, the main results are applied to give explicit Noether type inequalities for log canonical threefolds of general type and stable surfaces, and to bound the symplectic automorphisms of surfaces of general type.

**Notation and Conventions.**

- We work over an algebraically closed field \(k\) of arbitrary characteristic from Section 2 to Section 7, unless otherwise stated. For the applications in Section 8 we restrict to characteristic 0.

- Given a birational projective morphism \(f: X \to Y\) between two normal varieties, we always choose the canonical divisors \(K_X\) and \(K_Y\) in such a way that \(K_Y = f_* K_X\), so if the dimension is two or \(K_Y\) is \(\mathbb{Q}\)-Cartier then \(K_X - f^* K_Y\) is a well-defined \(\mathbb{Q}\)-divisor supported on the exceptional locus \(\text{Exc}(f)\).

- For a coherent sheaf \(\mathcal{F}\) on a normal projective variety \(X\), we denote \(h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F})\) and \(\chi(\mathcal{F}) = \sum_i (-1)^i h^i(X, \mathcal{F})\), the Euler characteristic of \(\mathcal{F}\). The numbers \(q(X) := h^1(X, \mathcal{O}_X)\) and \(p_g(X) := h^0(X, \mathcal{O}_X(K_X))\) are called the **irregularity** and the **geometric genus** of \(X\) respectively. For a Weil divisor \(D\) on \(X\), we usually write \(H^i(X, D)\) instead of \(H^i(X, \mathcal{O}_X(D))\).

- On a smooth projective surface, a \((-n)\)-curve means a smooth rational curve \(C\) with \(C^2 = -n\).

- For a subset \(C \subset \mathbb{R}\) of real numbers and \(a \in \mathbb{R}\), we define the subset \(C_{\leq a} := \{x \in C \mid x \leq a\}\). The subsets \(C_{< a}, C_{\geq a}\) and \(C_{> a}\) are similarly defined.

**Acknowledgement.** The work has been supported by NSFC (No. 11501012 and No. 11971399) and by the Presidential Research Fund of Xiamen University (No. 20720210006). Thanks go to Valery Alexeev, Sönke Rollenske, Stephen Coughlan and Thomas Bauer for helpful discussions which clarify and improve my arguments. The initial results of this paper were obtained during my trips to Universität Bayreuth, Philipps-Universität Marburg and the...
University of Georgia in 2016. The author enjoyed the hospitality and the inspiring academic atmosphere in these institutions. Over the years leading to this version of the paper, I benefited from discussions on related problems with Chen Jiang, Zhi Jiang, Jingjun Han and Meng Chen at Fudan University; I am also grateful to Yong Hu and Tong Zhang for their interest; a stimulating question of Hang Zhao leads to Theorem 8.1.

2. Preliminaries

We recall the notions of birational geometry in dimension two that are needed in this paper; most of them have a generalization to all dimensions ([18, 11]).

2.1. Divisors. For $\mathbb{F} = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$, an $\mathbb{F}$-divisor on a normal surface $X$ is a formal finite $\mathbb{F}$-linear combination $B = \sum_{j \in J} b_j B_j$, where $B_j$ are distinct prime divisors and $J$ is a finite set. We understand that $J = \emptyset$ if $B = 0$ is the zero divisor. Denote by $C_B = \{b_j\}_{j \in J} \setminus \{0\}$ the set of (nonzero) coefficients of $B$, so that $B = 0$ if and only if $C_B = \emptyset$. The support of $B$ is defined to be $\text{supp}(B) := \bigcup_{b_j \neq 0} \text{supp}(B_j)$.

An $\mathbb{F}$-divisor is said to be $\mathbb{F}$-Cartier if it is a finite $\mathbb{F}$-linear combination of Cartier divisors. Two $\mathbb{F}$-divisors $B$ and $B'$ are said to be $\mathbb{F}$-linearly equivalent, denoted $B \sim_{\mathbb{F}} B'$, if $B - B'$ is a finite $\mathbb{F}$-linear combination of principle divisors. When $\mathbb{F} = \mathbb{Z}$, then a $\mathbb{Z}$-divisor (resp. $\mathbb{Z}$-Cartier) is just a usual Weil (resp. Cartier) divisor and $\mathbb{Z}$-linear equivalence is the usual linear equivalence, which is then denoted by $B \sim B'$. A non-zero effective $\mathbb{Z}$-divisor on a normal surface is usually called a curve. Two $\mathbb{F}$-divisors $B$ and $B'$ are said to be numerically equivalent if $B \cdot C = D \cdot C$ for any curve $C$, and it is denoted by $B \equiv B'$.

For a real number $a$, we write

$$B^{=a} = \sum_{b_j = a} b_j B_j, \quad [B] = \sum_{b_j} [b_j] B_j,$$

where $[b_j]$ denotes the integer satisfying the condition $b_j - 1 < [b_j] \leq b_j$; the $\mathbb{R}$-divisors $B^{>a}$, $B^{<a}$, and $[B]$ are similarly defined. For a prime divisor $P$ on $X$, we denote by $\text{mult}_P B$ the coefficient of $P$ in $B$; for a point $p \in X$, the multiplicity of $B$ at $p$ is $\text{mult}_P B := \sum_{j \in J} b_j \text{mult}_P B_j$.

2.2. (Sub-)log surfaces and their singularities. A sub-log surface (resp. log surface) $(X, B)$ consists of a connected normal surface $X$ and an $\mathbb{R}$-divisor $B$ such that the coefficient set $C_B$ is contained in $\mathbb{R}_{\leq 1}$ (resp. $(0, 1]$) and $K_X + B$ is $\mathbb{R}$-Cartier. We call a sub-log surface $(X, B)$ smooth if $X$ is so. A higher model of a sub-log surface $(X, B)$ consists of a triple $(Y, B_Y, f)$, where $(Y, B_Y)$ is a sub-log surface and $\rho: Y \rightarrow X$ is a birational projective morphism such that $\rho_* B_Y = B$. The higher model is called

- effective if $B_Y$ is so;
- crepant if $K_Y + B_Y = \rho^*(K_X + B)$;
- a resolution if $Y$ is smooth;
- the minimal resolution if it is a crepant resolution and the exceptional locus $\text{Exc}(\rho)$ does not contain any $(-1)$-curves;
- a log resolution if it is a resolution and $\rho_* \pi_1 B$ has a simple normal crossing support, where $\rho_* \pi_1 B$ is the strict transform of $B$.

For simplicity, we often omit the birational morphism $\rho$ and say that $(Y, B_Y)$ is a higher model of $(X, B)$.

Let $(Y, B_Y, \rho)$ be a crepant log resolution of a sub-log surface $(X, B)$. For a prime divisor $E$ on $Y$, $a_E(X, B) := 1 - \text{mult}_E B_Y$ is called the log discrepancy of $E$ with respect to $(X, B)$. 

The sub-log surface \((X, B)\) is said to be sub-klt (resp. sub-log canonical) if the coefficients of \(B_Y\) are less than 1 (resp. at most 1); when \(B\) is effective, we drop the prefix "sub" everywhere. We refer to [1, 17] for a classification of log canonical surface singularities. A nonklt center \(Z \subset X\) of \((X, B)\) is the image of a stratum of the simple normal crossing divisor \(B_Y^{-1}\); it is called accessible if \(\text{supp}B_Y^{>0}\) is singular at some point of \(\rho^{-1}(Z)\); see [3, Definition 2.3]. It is easy to see that the accessibility of \(Z\) does not depend on the choice of the crepant log resolution.

2.3. The Iitaka–Kodaira dimension and the volume of divisors.

Definition 2.1. Let \(X\) be a normal projective surface and \(D\) an \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor on \(X\). The Iitaka–Kodaira dimension \(\kappa(D)\) of \(D\) is defined as follows: \(\kappa(D)\) is \(-\infty\) if \(h^0(X, [mD]) = 0\) for every positive integer \(m > 0\); otherwise \(\kappa(D)\) is defined to be the nonnegative integer \(k\) satisfying

\[
0 < \limsup_{m \to \infty} \frac{h^0(X, [mD])}{m^k} < \infty.
\]

It is clear that \(\kappa(D) \in \{-\infty, 0, 1, 2\}\). The \(\mathbb{R}\)-divisor \(D\) is said to be big if \(\kappa(D) = 2\). The volume of \(D\) is defined to be

\[
\text{vol}(D) = \limsup_{m \to \infty} \frac{h^0(X, [mD])}{m^{2/2}}.
\]

Obviously, \(\text{vol}(D) > 0\) if and only if \(\kappa(D) = 2\). We say a projective log canonical surface \((X, B)\) is of general type if \(K_X + B\) is big. The (log canonical) volume of a log surface \((X, B)\) is defined to be \(\text{vol}(K_X + B)\).

We refer to [21] and [15, Section 2.2] for the basic properties of the volume function.

2.4. The Zariski decomposition. The Zariski decomposition, defined as follows, is an essential tool to compute volumes.

Definition 2.2. Let \(D\) be an \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor on a normal projective surface. Then the Zariski decomposition of \(D\) is a decomposition \(D = P + N\) such that

(i) \(P\) is a nef \(\mathbb{R}\)-divisor, that is, \(P\) is \(\mathbb{R}\)-Cartier and \(P \cdot C \geq 0\) for any curve \(C\) on \(X\);

(ii) \(N\) is either zero or a nonzero effective \(\mathbb{R}\)-divisor whose intersection matrix is negative definite;

(iii) \(P \cdot N_i = 0\) for each irreducible component \(N_i\) of \(N\).

We call \(P\) the positive part of \(D\), and \(N\) the negative part.

Example 2.3. Let \((X, B)\) be a projective log canonical surface of general type. Then there is a birational morphism \(\pi: (X, B) \to (\bar{X}, \bar{B})\), such that \(\bar{B} = \pi_* B\); \((\bar{X}, \bar{B})\) is a projective log canonical surface with ample \(K_X + B\) and \(K_X + B - \pi^*(K_{\bar{X}} + \bar{B})\) is an effective divisor supported on the exceptional locus \(\text{Exc}(\pi)\); see [12]. We call \((\bar{X}, \bar{B})\), together with the birational morphism \(\pi\), the ample model of \((X, B)\). Let \(P := \pi^*(K_{\bar{X}} + \bar{B})\) and \(N := K_X + B - \pi^*(K_{\bar{X}} + \bar{B})\). Then \(K_X + B = P + N\) is the Zariski decomposition of \(K_X + B\).

Recall that an \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor \(D\) on a normal projective surface \(X\) is pseudo-effective if \(D \cdot H \geq 0\) for any ample divisor \(H\).

Theorem 2.4. Let \(D\) be an \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor on a normal projective surface \(X\). Then the following holds.

(i) The \(\mathbb{R}\)-divisor \(D\) admits a Zariski decomposition if and only if \(D\) is pseudo-effective.

(ii) The negative part \(N\) in the Zariski decomposition depends only on the numerical class of \(D\).
(iii) The positive part $P$ in the Zariski decomposition can be characterized as the maximal nef $\mathbb{R}$-divisor such that $D - P$ is effective.

**Proof.** When $X$ is smooth, the theorem is proved for effective $\mathbb{Q}$-divisors $D$ in [29] and then generalized to pseudo-effective $\mathbb{Q}$-divisors by [13]; see also [4, Theorem 14.14]. The proof for the general case is similar; see [24, Theorem 2.2]. □

We omit the proof of the following easy lemma.

**Lemma 2.5.** Let $D$ be a pseudo-effective $\mathbb{R}$-divisor on a normal projective surface $X$, and $D = P + N$ the Zariski decomposition with positive part $P$.

(i) $\text{vol}(D) = \text{vol}(P) = P^2 \geq D^2$, and the last inequality is an equality if and only if $D$ is nef.

(ii) If $D$ is big and if $E$ is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor such that $N - E$ is not effective, then $\text{vol}(D) > \text{vol}(D - E)$.

(iii) If $E$ is a pseudo-effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor such that $D + E$ is big, then $D + tE$ is big for any $t > 0$.

2.5. The volumes of certain $\mathbb{R}$-divisors depending on a parameter. We compute the volume of an $\mathbb{R}$-divisor of the form $D + tC$ on a normal projective surface, where $D$ is nef and $C$ is an irreducible curve.

**Lemma 2.6.** Let $X$ be a normal projective surface. Suppose that $D$ is a nef $\mathbb{R}$-divisor and $C$ an irreducible curve on $X$. Define

$$t_0 = \sup\{t \in \mathbb{R}_{\geq 0} \mid D + tC \text{ is nef} \} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$  

Then the following holds.

(i) $t_0 = \begin{cases} \frac{-C \cdot D}{C^2} & \text{if } C^2 < 0 \\ \infty & \text{if } C^2 \geq 0 \end{cases}$.

(ii) For $t \in \mathbb{R}_{\geq 0}$ the positive part of $D + tC$ in the Zariski decomposition is $P := D + \min\{t, t_0\}C$, and consequently its volume is

$$\text{vol}(D + tC) = P^2 = D^2 + 2\min\{t, t_0\}C \cdot D + (\min\{t, t_0\})^2C^2.$$  

(iii) If $C \cdot D > 0$, then $D + tC$ is big for any $t > 0$.

**Proof.** (i) and (ii). If $C^2 \geq 0$ then $D + tC$ is nef for any $t \geq 0$ and hence $t_0 = \infty$. In this case the positive part of $D + tC$ is $D + tC$ itself. Now suppose that $C^2 < 0$. Then one verifies $(D + tC) \cdot C < 0$ if and only if $t > -\frac{C \cdot D}{C^2}$. It follows readily that $t_0 = -\frac{C \cdot D}{C^2}$ and $D = P + (t - \min\{t_0, t\})C$ is the Zariski decomposition of $D + tC$ with $P$ as the positive part.

(iii) If $C \cdot D > 0$, then $t_0 > 0$, and for $0 < \epsilon \ll t$, we have by (2.1)

$$\text{vol}(D + tC) \geq \text{vol}(D + \epsilon C) = D^2 + \epsilon C \cdot D + \epsilon^2 C^2 > 0$$  

□

**Lemma 2.7.** Let $X$ be a smooth projective surface, and $B$ an $\mathbb{R}$-divisor such that $K_X + B$ is nef. Let $C$ be an irreducible curve such that $B \cdot C = 0$ and $d := K_X \cdot C > 0$. Then the following holds.

(i) If $p_a(C) > 0$, then, for $0 \leq t \leq 1$, we have

$$\text{vol}(K_X + B + tC) = (K_X + B)^2 + t^2(2p_a(C) - 2) + m(2t - t^2).$$  

□
(ii) If \( p_a(C) = 0 \), then, for \( 0 \leq t \leq 1 \), we have
\[
\text{vol}(K_X + B + tC) = \begin{cases} 
(K_X + B)^2 + d(2t - t^2) - 2t^2 & \text{if } t < \frac{d}{4t^2} \\
(K_X + B)^2 + d - 2 + \frac{4}{4t^2} & \text{if } \frac{d}{4t^2} \leq t \leq 1
\end{cases}
\]

**Proof.** For \( 0 \leq t \leq 1 \), we have
\[
(K_X + B + tC) \cdot C = t(K_X + C)' \cdot C + (1 - t)K_X \cdot C = t(2p_a(C) - 2) + d(1 - t).
\]
(i) Assume that \( p_a(C) > 0 \). Then, for any \( 0 \leq t \leq 1 \), we have \((K_X + B + tC)' \cdot C \geq d(1 - t) \geq 0\) by (2.3). Since \( K_X + B \) is nef, so is \( K_X + B + tC \), and hence
\[
\text{vol}(K_X + B + tC) = (K_X + B + tC)^2
= (K_X + B)^2 + 2t(K_X + B) \cdot C + t^2C^2
= (K_X + B)^2 + 2tK_X \cdot C + t^2C^2
= (K_X + B)^2 + t^2(K_X \cdot C + C^2) + (2t - t^2)K_X \cdot C
= (K_X + B)^2 + t^2(2p_a(C) - 2) + (2t - t^2)K_X \cdot C
= (K_X + B)^2 + t^2(2p_a(C) - 2) + d(2t - t^2)
\]
(ii) Now assume that \( p_a(C) = 0 \), so \( C \cong \mathbb{P}^1 \). By the adjunction formula, we have \( C^2 = K_X \cdot C - 2 = -d - 2 < 0 \). By Lemma 2.6, we have
\[
\text{vol}(K_X + B + tC) = (K_X + B)^2 + 2 \min\{t, t_0\}C \cdot (K_X + B) + (\min\{t, t_0\})^2C^2
= (K_X + B)^2 + 2d \min\{t, t_0\} - (d + 2)(\min\{t, t_0\})^2
\]
where \( t_0 = -\frac{(K_X + B) \cdot C}{C^2} = \frac{d}{4t^2} \). Now (2.2) is obtained by spelling out the two cases \( t < t_0 \) and \( t \geq t_0 \) of the above formula. \( \square \)

3. The minimal volume over a fixed log surface

In this section, we figure out the minimal possible volume of effective higher models \((U, B_U)\) of a fixed smooth projective log surface \((Z, B_Z)\).

3.1. A method of finding the minimal volume over a fixed log surface.

**Lemma 3.1.** Let \( \rho : U \to Z \) be a birational morphism between smooth projective surfaces. Let \( B'_Z \) and \( B''_Z \) be two effective divisors on \( Z \) such that \( \rho \) is an isomorphism over a neighborhood of \( \text{supp}(B'_Z) \). Denote \( m = \max_p \{\text{mult}_p B''_Z\} \) with \( p \in Z \) running through the points blown up by \( \rho \). Then
\[
K_U + \rho_s^{-1}(B'_Z + B''_Z) \geq \rho^*\left(K_Z + B'_Z + \frac{1}{m}B''_Z\right).
\]
**Proof.** We write \( \rho : U \to Z \) as the composition \( \rho_1 \circ \rho_2 \circ \cdots \circ \rho_n \) of blow-ups, and let \( E_i \subset U \) be the total transform of the exceptional curve of \( \rho_i \) (\( 1 \leq i \leq n \)). Then the following equalities (respectively inequality) hold:
- \( K_U = \rho^*K_Z + \sum_{1 \leq i \leq n} E_i \).
- \( \rho^{-1}_s B'_Z = \rho^* B'_Z \), since \( \rho \) is an isomorphism over a neighborhood of \( \text{supp}(B'_Z) \).
- \( \rho^{-1}_s B''_Z \geq \rho^* B''_Z - m \sum_{1 \leq i \leq n} E_i \) by the definition of \( m \).
Combining these facts, one obtains
\[ K_U + \rho_\ast^{-1}(B_Z' + B_Z'') = K_U + \frac{1}{m}B_Z'' + \rho_\ast B_Z' \]
\[ \geq \rho_\ast K_Z + \frac{1}{m}B_Z' = \rho_\ast\left(K_Z + B_Z' + \frac{1}{m}B_Z\right). \]

\[ \square \]

**Notation 3.2.** Let \((Z, B_Z)\) be a smooth projective log surface satisfying the following conditions:

- \(K_Z + B_Z\) is big,
- there is a decomposition \(B_Z = B_Z' + B_Z''\) such that \(B_Z'\) and \(B_Z''\) are effective \(\mathbb{R}\)-divisors without common components and \(\kappa(K_Z + B_Z') \geq 0\).

Define \(\mathcal{S}(Z, B_Z; B_Z')\) to be the set of triples \((U, B_U, \rho_U)\), where \((U, B_U)\) is a smooth effective higher model of \((Z, B_Z)\), and \(\rho_U : U \to Z\) is a birational morphism that is an isomorphism over a neighborhood of \(\text{supp}(B_Z')\). Finally, define
\[ \mathcal{K}^2(Z, B_Z; B_Z') := \{ \text{vol}(K_U + B_U) \mid (U, B_U, \rho_U) \in \mathcal{S}(Z, B_Z; B_Z') \} \]

**Lemma 3.3.** Let \((Z, B_Z)\) be as in Notation 3.2 and \((U, B_U, \rho_U) \in \mathcal{S}(Z, B_Z; B_Z')\). Then the following holds.

\(\text{(i)}\) \(K_U + B_U\) is big.

\(\text{(ii)}\) If \(B_U\) is the strict transform of \(B_Z\) and \(\text{mult}_qB_U^q \leq 1\) for any \(q \in U\), where \(B_U'' = \rho_U^{-1}B_Z''\) is the strict transform, then \(\text{vol}(K_U + B_U) = \min \mathcal{K}^2(Z, B_Z; B_Z')\). In particular, if \((U, B_U, \rho_U) \in \mathcal{S}(Z, B_Z; B_Z')\) resolves the singularities of \(B_Z''\), then \(\text{vol}(K_U + B_U) = \min \mathcal{K}^2(Z, B_Z; B_Z')\).

**Proof.** (i) Since \(\kappa(K_Z + B_Z') \geq 0\), there is an effective \(\mathbb{R}\)-divisor \(D\) on \(Z\) such that \(K_Z + B_Z' \sim_{\mathbb{R}} D\), so \(D + B_Z'' \sim_{\mathbb{R}} K_Z + B_Z\) is a big effective divisor. By Lemma 3.1
\[ K_U + B_U \geq K_U + \rho_{U\ast}^{-1}(B_Z' + B_Z'') \geq \rho_{U\ast}\left(K_Z + B_Z' + \frac{1}{m}B_Z''\right) \]
where \(m = \max\{\text{mult}_pB_Z^p\}\). Note that the last \(\mathbb{R}\)-divisor is \(\mathbb{R}\)-linearly equivalent to \(\rho_{U\ast}(D + \frac{1}{m}B_Z'')\), which is big by Lemma 2.5 (iii). It follows that \(K_U + B_U\) is big.

(ii) Let \((V, B_V, \rho_V)\) be any log surface in \(\mathcal{S}(Z, B_Z; B_Z')\). Then the birational map \(\phi = \rho_V^{-1} \circ \rho_U : U \to V\) over \(Z\) is an isomorphism over an open neighborhood of \(\text{supp}(B_Z)\). Let \(\gamma_U : W \to U\) and \(\gamma_V : W \to V\) be a common resolution of \(U\) and \(V\) such that \(\gamma_U\) and \(\gamma_V\) are isomorphisms over a neighborhood of \(\text{supp}(B_Z')\):

\[ \begin{array}{c}
  W \\
  \gamma_U \searrow \nearrow \gamma_V \\
  U \\
  \phi \downarrow \downarrow \nearrow \rho_V \\
  Z \\
\end{array} \]

Let \(B_W\) be the strict transform of \(B_Z\) and \(\rho_W = \rho_U \circ \gamma_U = \rho_V \circ \gamma_V\). Then \((W, B_W, \rho_W) \in \mathcal{S}(Z, B_Z; B_Z')\) and \(\gamma_{U\ast}B_W = B_U\). Since \(\gamma_U\) is an isomorphism over a neighborhood of \(B_U\) and \(\text{mult}_qB_U^q \leq 1\) for any point \(q \in U\), we have by Lemma 3.1
\[ K_W + B_W \geq \gamma_{U\ast}(K_U + B_U). \]
On the other hand, 
\[ \gamma_{U*}(K_W + B_W) = K_U + B_U \text{ and } \gamma_{V*}(K_W + B_W) \leq K_V + B_V. \]

It follows that 
\[ \text{vol}(K_V + B_V) \geq \text{vol}(K_W + B_W) = \text{vol}(K_U + B_U). \]

Since \((V, B_V, \rho_V) \in S(Z, B_Z; B'_Z)\) is arbitrary, we infer that \(\text{vol}(K_U + B_U) = \min \mathbb{K}^2(Z, B_Z; B'_Z). \)

\[
\Box
\]

3.2. Minimal volumes over log surfaces with an extended canonical type curve.

The task of this subsection is to compute \(\min \mathbb{K}^2\left(Z, B^{(c)}_Z; B'_Z\right)\), where \(Z, B^{(c)}_Z\), and \(B'_Z\) satisfy the following conditions:

- \((Z, B'_Z)\) is smooth projective log canonical surface with \(B'_Z\) reduced and \(K_Z + B'_Z \sim_{\mathbb{Q}} 0\).
- \(B^{(c)}_Z = B'_Z + c(C + D)\), where \(c \in (0, 1]\), \(C\) is a connected reduced curve supporting a curve of canonical type, and \(D\) is a \((-2)\)-curve intersecting \(C\) transversally at exactly one point, and \(\text{supp}(B'_Z) \cap \text{supp}(C + D) = \emptyset\).
- There are no \((-1)\)-curves \(G\) such that \((K_Z + B'_Z) \cdot G = 0\).

We recall that a curve \(C = \sum n_i C_i\) on a smooth projective surface \(Z\) is of canonical type if \(K_Z \cdot C_i = C \cdot C_i = 0\) for all \(i\) ([23]). Indecomposable curves of canonical type are classified into types \(I_b (b \geq 0)\), \(II, III, IV, I_5 (b \geq 0)\), \(II^*, III^*, IV^*\) ([5]).

Lemma 3.3 gives the recipe of finding \(\min \mathbb{K}^2\left(Z, B^{(c)}_Z; B'_Z\right)\):

1. Take \((W, B^{(c)}_W, \rho_W) \in S(Z, B^{(c)}_Z; B'_Z)\) such that \(\rho_W: W \to Z\) resolves the singularities of \(C + D\) and \(B^{(c)}_W = B'_W + c(C^w + D^w)\), where \(B'_W, C^w\) and \(D^w\) are the strict transforms of \(B'_Z, C\) and \(D\) respectively. Then \(\min \mathbb{K}^2\left(Z, B^{(c)}_Z; B'_Z\right) = \text{vol}(K_W + B^{(c)}_W)\).

2. To compute \(\text{vol}(K_W + B^{(c)}_W)\), we need to consider the ample model, say \(\gamma_c: (W, B^{(c)}_W) \to (X^{(c)}, B^{(c)}_{X^{(c)}})\). Then we have \(\text{vol}(K_W + B^{(c)}_W) = (K_{X^{(c)}} + B^{(c)}_{X^{(c)}})^2\). The latter is in turn computed by pulling back via the minimal resolution \(\pi_c: (\tilde{X}^{(c)}, B_{\tilde{X}^{(c)}}) \to (X^{(c)}, B^{(c)}_{X^{(c)}})\). Obviously, we have an induced birational morphism \(\tilde{\gamma}_c: W \to \tilde{X}^{(c)}\) such that \(\gamma_c = \pi_c \circ \tilde{\gamma}_c\). It can be readily checked that \(\tilde{\gamma}_c\) is an isomorphism on a neighborhood of \(B'_W\), and that \(\text{Exc}(\gamma_c) \subset \text{Exc}(\rho_W)\). Hence there is a birational morphism \(\rho_c: \tilde{X}^{(c)} \to Z\) such that the following diagram commutes:

\[
\begin{array}{ccc}
W & \xrightarrow{\gamma_c} & \tilde{X}^{(c)} \\
\rho_W \downarrow & & \downarrow \pi_c \\
Z & & \tilde{X}^{(c)}
\end{array}
\]

(3.1)

We explain the relation among the models \(\tilde{X}^{(c)}\) as \(c\) varies. For \(0 < c < c' \leq 1\), we have \(\text{Exc}(\tilde{\gamma}_c) \subset \text{Exc}(\tilde{\gamma}_{c'})\), and hence there is a birational morphism \(\tilde{\gamma}_{c'c}: \tilde{X}^{(c')} \to \tilde{X}^{(c)}\) such that \(\tilde{\gamma}_{c'c} \circ \tilde{\gamma}_c = \tilde{\gamma}_{c'}\). Since \(\tilde{\gamma}_{c'c}\) sits between \(W\) and \(Z\), there are finitely many values \(0 = c_0 < c_1 < \cdots < c_r = 1\) such that for any \(c_i < c \leq c_{i+1}(0 \leq i \leq r - 1)\), \(\tilde{\gamma}_c: W \to \tilde{X}^{(c)}\) stay the same and \(\tilde{X}^{(c)} \to \tilde{X}_{c_i}\) is a nontrivial birational contraction; we call \(c_i\) (\(1 \leq i \leq r - 1\)) the critical values of \(c\). Note that \(X^{(c)} = Z\) for \(0 < c \ll 1\), and \(\tilde{X}_1\) dominate all the other \(X^{(c)}\) for \(0 < c \leq 1\); we will omit the subscript when \(c = 1\) and denote

\[ \tilde{\gamma} := \tilde{\gamma}_1, \rho := \rho_1, \gamma := \gamma_1, X := X^{(1)} , \text{ and } \tilde{X} := \tilde{X}^{(1)}. \]
Before diving into the details of the computation, we introduce some additional notations and convention:

- We call the curves appearing in \( C + D \) and its inverse images on \( X \) and \( W \) the \textit{visible curves}. If the curve \( C + D \) is a normal crossing divisor, then we will present the visible curves on \( Z, X \) and \( W \) by dual graphs; otherwise, the curves are sketched more concretely to indicate the (unique) worse-than-nodal singularity.

- The strict transforms of \( C_i \) and \( D \) on \( W \) (resp. on \( X \)) are denoted by \( C_w^i \) and \( D_w \) (resp. by \( \tilde{C}_i \) and \( \tilde{D} \)). The \( \rho_W \)-exceptional curves on \( W \) are denoted by \( E_i \). If there is only one such curve, then the subscript is omitted. Correspondingly, their image curves on \( X \) are denoted by \( \tilde{E}_i \). These curves are denoted by black bullets in the dual graphs and dashed lines in the sketches of visible curves respectively.

Now we proceed according to the canonical type curve that \( C \) supports.

**Case: \( C \) supports a curve of canonical type \( I_0 \).** In this case \( C \) is a smooth elliptic curve with \( C^2 = 0 \) and \( D \) is a \((-2)\)-curve intersecting \( C \) transversally at a smooth point. The dual graphs of visible curves on \( W \xrightarrow{\rho_W} Z \) are as follows:

There is exactly one critical value \( c = \frac{2}{3} \). If \( \frac{2}{3} < c \leq 1 \), then \( \tilde{X}^{(c)} = \tilde{X} \). In this case,

\[
\text{vol}(K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}}) = \left( E + cC^w + \frac{1}{3}D_w \right)^2 = -c^2 + 2c - \frac{2}{3}
\]

If \( 0 < c \leq \frac{2}{3} \), then \( \tilde{X}^{(c)} = Z \), \( K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} cC + \frac{2}{3}D \) and hence

\[
\text{vol}(K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}}) = \left( cC + \frac{c}{2}D \right)^2 = \frac{c^2}{2}
\]

In conclusion, we obtain in this case

\[
(3.2) \quad \min K^2 \left( Z, B^c_Z ; B^\prime_Z \right) = \text{vol}(K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}}) = \begin{cases} \frac{4}{3}c^2 & \text{if } c \leq \frac{2}{3} \\ -c^2 + 2c - \frac{2}{3} & \text{if } c > \frac{2}{3} \end{cases}
\]

**Case: \( C \) supports a curve of canonical type \( I_b \) with \( b \geq 1 \).** In this case, \( C = \sum_{i=1}^{b} C_i \) is a cycle of \((-2)\)-curves if \( b \geq 2 \), and \( D \) is \((-2)\)-curve intersecting one component, say \( C_1 \), of \( C \). The dual graph of visible curves on \( W \xrightarrow{\gamma} \tilde{X} \xrightarrow{\rho} Z \) are as follows:

If \( b = 1 \), then \( C \) is a nodal rational curve with \( C^2 = 0 \), and the above dual graph degenerate to
There is exactly one critical value \( c = \frac{1}{2} \). For \( \frac{1}{2} < c \leq 1 \), we have \( \tilde{X}^{(c)} = \tilde{X} \) and

\[
K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} a\tilde{C}_1 + \frac{1}{2} \sum_{2 \leq i \leq b} \tilde{C}_i + \frac{a}{2} \tilde{D} + \sum_{i \in \{1,b\}} \tilde{E}_i
\]

where \( a = \min\{\frac{4}{7}, c\} \). For \( 0 < c \leq \frac{1}{2} \), we have \( \tilde{X}^{(c)} = Z \) and

\[
K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} cC + \frac{c}{2}D
\]

Now it is straightforward to compute

\[
(3.3) \quad \min \mathbb{K}^2 \left(Z, B^{(c)}_Z, B'_Z\right) = \text{vol}(K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}}) = \begin{cases}
\frac{4}{7}c^2 & \text{if } c \leq \frac{1}{7} \\
\frac{7}{2}c^2 + 4c - 1 & \text{if } \frac{1}{7} < c \leq \frac{4}{7}
\end{cases} \quad \text{if } c > \frac{4}{7}
\]

**Case: C supports a curve of canonical type II.** In this case, \( C \) is a cuspidal rational curve with \( C^2 = 0 \) and \( D \) is a \((-2)\)-curve intersecting \( C \) transversally at a smooth point.

The sketches of visible curves on \( W \xrightarrow{\gamma} \tilde{X} \xrightarrow{\beta} Z \) are as follows:

There is exactly one critical value \( c = \frac{1}{7} \). For \( \frac{1}{7} < c \leq 1 \), we have \( \tilde{X}^{(c)} = \tilde{X} \) and

\[
K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} a\tilde{C}_1 + \frac{a}{2} \tilde{D} + \tilde{E}_2
\]

where \( a = \min\{c, \frac{1}{7}\} \). For \( 0 < c \leq \frac{1}{7} \), we have \( \tilde{X}^{(c)} = Z \) and

\[
K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} aC + \frac{a}{2}D
\]

The formula for \( \min \mathbb{K}^2 \left(Z, B^{(c)}_Z, B'_Z\right) = \text{vol}(K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}}) \) is the same with the case of \( I_b, b \geq 1 \); see (3.3).

**Case: C supports a curve of canonical type III.** In this case, \( C = C_1 + C_2 \) consists of two \((-2)\)-curves intersecting at one point with multiplicity two, and \( D \) is a \((-2)\)-curve intersecting \( C \) transversally at a smooth point, say, of \( C_1 \). The sketches of visible curves on \( W \xrightarrow{\gamma} \tilde{X} \xrightarrow{\beta} Z \) are as follows:

There is exactly one critical value \( c = \frac{1}{7} \). For \( \frac{1}{7} < c \leq 1 \), we have \( \tilde{X}^{(c)} = \tilde{X} \) and

\[
K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} a\tilde{C}_1 + \frac{a+1}{3} \tilde{C}_2 + \frac{a}{2} \tilde{D} + \tilde{E}_2
\]

where \( a = \min\{c, \frac{8}{17}\} \). For \( 0 < c \leq \frac{1}{7} \), we have \( \tilde{X}^{(c)} = Z \) and

\[
K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} cC + \frac{c}{2}D
\]
The formula for the minimal volume is as follows:

\[
\min \|K^2 \left( Z, B_Z^{(c)}; B'_Z \right) = \text{vol}(K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}}) = \begin{cases} 
\frac{1}{2}c^2 & \text{if } c \leq \frac{1}{3} \\
-\frac{13}{6}c^2 + \frac{5}{3}c - \frac{4}{3} & \text{if } \frac{1}{3} < c \leq \frac{8}{13} \\
\frac{2}{13} & \text{if } c > \frac{8}{13}
\end{cases}
\]

**Case: C supports a curve of canonical type IV.** In this case, \( C = C_1 + C_2 + C_3 \) consists of three \((-2)\)-curves intersecting at one point, and \( D \) is a \((-2)\)-curve intersecting, say \( C_1 \), transversally at a smooth point. The sketches of visible curves on \( W \xrightarrow{\tilde{\gamma}} \tilde{X} \xrightarrow{\rho} Z \) are as follows:

There is exactly one critical value \( c = \frac{1}{3} \). For \( \frac{1}{3} < c \leq 1 \), we have \( \tilde{X}^{(c)} = \tilde{X} \) and

\[
K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} a\tilde{C}_1 + \frac{1}{3}(\tilde{C}_2 + \tilde{C}_3) + \frac{a}{2} \tilde{D} + \tilde{E}_2
\]

where \( a = \min\{c, \frac{2}{3}\} \). For \( 0 < c \leq \frac{1}{3} \), we have \( \tilde{X}^{(c)} = Z \) and

\[
K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} c\tilde{C}_0^w + \frac{1}{3} \sum_{0 \leq i \leq 4} E_i
\]

The formula for the minimal volume is as follows:

\[
\min \|K^2 \left( Z, B_Z^{(c)}; B'_Z \right) = \text{vol}(K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}}) = \begin{cases} 
\frac{1}{2}c^2 & \text{if } c \leq \frac{1}{3} \\
-\frac{5}{2}c^2 + 2c - \frac{1}{3} & \text{if } \frac{1}{3} < c \leq \frac{2}{5} \\
\frac{1}{19} & \text{if } c > \frac{2}{5}
\end{cases}
\]

**Case: C supports a curve of canonical type I_0, and D is attached to the central component of C.** In this case, the dual graphs of visible curves on \( W \xrightarrow{\rho_W} Z \) are as follows:

There is exactly one critical value \( c = \frac{2}{3} \). For \( \frac{2}{3} < c \leq 1 \), we have \( \tilde{X}^{(c)} = W \) and

\[
K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} a\tilde{C}_0^w + \frac{1}{3} \sum_{1 \leq i \leq 4} C_i^w + \frac{1}{3} D^w + \sum_{0 \leq i \leq 4} E_i
\]
where \( a = \min\{\frac{2}{3}, c\} \). For \( 0 < c \leq \frac{2}{3} \), we have \( \hat{X}(c) = Z \) and

\[
K_{\hat{X}(c)} + B_{\hat{X}(c)} \sim_R cC_0 + \frac{c}{2} \left( \sum_{1 \leq i \leq 4} C_i + D \right)
\]

The formula for the minimal volume is

\[
(3.6) \quad \min K^2 \left( Z, B_Z; B'_Z \right) = \text{vol}(K_{\hat{X}(c)} + B_{\hat{X}(c)}) = \begin{cases} \frac{1}{5}c^2 & \text{if } c \leq \frac{2}{3} \\ -7c^2 + 10c - \frac{40}{3} & \text{if } \frac{2}{3} < c \leq \frac{5}{7} \\ \frac{5}{21} & \text{if } c > \frac{5}{7} \end{cases}
\]

Case: \( C \) supports a curve of canonical type \( I_0^* \), and \( D \) is attached to an end component of \( C \). The dual graphs of the visible curves on \( W \xrightarrow{\gamma} \hat{X} \xrightarrow{\rho} Z \) are as follows:

There is exactly one critical value \( c = \frac{3}{5} \). If \( \frac{3}{5} < c \leq 1 \), we have \( \hat{X}(c) = \hat{X} \) and

\[
K_{\hat{X}(c)} + B_{\hat{X}(c)} \sim_R a\hat{C}_0 + \frac{2}{5}\hat{C}_1 + \frac{a}{2} \sum_{2 \leq i \leq 4} \hat{C}_i + \frac{1}{5}\hat{D} + \hat{E}_1
\]

where \( a = \min\{\frac{2}{3}, c\} \). If \( 0 < c \leq \frac{3}{5} \), we have \( \hat{X}(c) = Z \) and

\[
K_{\hat{X}(c)} + B_{\hat{X}(c)} \sim_R cC_0 + \frac{2c}{3}C_1 + \frac{c}{2} \sum_{2 \leq i \leq 4} C_i + \frac{c}{3}D
\]

The formula for the minimal volume is

\[
(3.7) \quad \min K^2 \left( Z, B_Z; B'_Z \right) = \text{vol}(K_{\hat{X}(c)} + B_{\hat{X}(c)}) = \begin{cases} \frac{1}{5}c^2 & \text{if } c \leq \frac{3}{5} \\ -\frac{2}{5}c^2 + 2c - \frac{3}{8} & \text{if } \frac{3}{5} < c \leq \frac{7}{9} \\ \frac{1}{15} & \text{if } c > \frac{7}{9} \end{cases}
\]
Case: $C$ supports a curve of canonical type $I^*$, $b \geq 1$, and $D$ is attached to an end component of $C$. The dual graphs of the visible curves on $W \xrightarrow{\tilde{\gamma}} \tilde{X} \xrightarrow{\rho} Z$ are as follows:

There is exactly one critical value $c = \frac{1}{2}$. If $\frac{1}{2} < c \leq 1$, we have $\tilde{X}^{(c)} = \tilde{X}$ and

$$K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} \frac{2a}{3} \tilde{C}_1 + a \tilde{C}_2 + \frac{1}{2} \sum_{3 \leq i \leq b+2} \tilde{C}_i + \frac{1}{4} \tilde{C}_{b+3} + \frac{a}{2} \tilde{C}_{b+4} + \frac{1}{4} \tilde{C}_{b+5} + \frac{a}{3} \tilde{D} + \tilde{E}_3$$

where $a = \min \{ \frac{6}{11}, c \}$. If $0 < c \leq \frac{1}{2}$, we have $\tilde{X}^{(c)} = Z$ and

$$K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} \frac{2c}{3} \tilde{C}_1 + c \sum_{2 \leq i \leq b+2} \tilde{C}_i + \frac{c}{2} \sum_{b+3 \leq i \leq b+5} \tilde{C}_i + \frac{c}{3} \tilde{D}$$

The formula for the volume is

$$\min_{KZ} K^2 \left( Z, B_Z^{(c)}; B_Z \right) = \text{vol}(K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}}) = \begin{cases} \frac{1}{6} c^2 & \text{if } 0 < c \leq \frac{1}{2} \\ \frac{11}{6} c^2 + 2c - \frac{1}{2} & \text{if } \frac{1}{2} < c \leq \frac{6}{11} \\ \frac{1}{11} & \text{if } c > \frac{6}{11} \end{cases}$$

Case: $C$ supports a curve of canonical type $II^*$ and $D$ is attached to the end component of the longest chain of $C$. The dual graphs of the visible curves on $W \xrightarrow{\tilde{\gamma}} \tilde{X} \xrightarrow{\rho} Z$ are as follows:
There is exactly one critical value $c = \frac{7}{13}$. If $\frac{7}{13} < c \leq 1$, we have $\tilde{X}^{(c)} = \tilde{X}$ and

$$K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} \tilde{E}_a + \frac{1}{13} \left( \tilde{D} + \sum_{1 \leq i \leq 5} (i + 1) \tilde{C}_i \right) + \frac{a}{6} \left( 6 \tilde{C}_6 + 4 \tilde{C}_7 + 2 \tilde{C}_8 + 3 \tilde{C}_9 \right),$$

where $a = \min \{ \frac{6}{11}, c \}$. If $0 < c \leq \frac{7}{13}$, we have $\tilde{X}^{(c)} = Z$ and

$$K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} c C_6 + \frac{c}{4} \left( D + \sum_{1 \leq i \leq 5} (i + 1) C_i \right) + \frac{c}{6} \left( 4 \tilde{C}_7 + 2 \tilde{C}_8 + 3 \tilde{C}_9 \right).$$

The formula for the minimal volume is

$$(3.9) \quad \min K^2 \left( Z, B^{(c)}_Z; B'_Z \right) = \text{vol}(K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}}) = \begin{cases} \frac{1}{12} c^2 & \text{if } c \leq \frac{7}{13} \\ \frac{11}{144} c^2 + 2 c - \frac{7}{13} & \text{if } \frac{7}{13} < c \leq \frac{6}{11} \\ \frac{1}{144} c^2 & \text{if } c > \frac{6}{11} \end{cases}$$

**Case:** *C supports a curve of canonical type III*. The dual graphs of the visible curves on $W \xrightarrow{\gamma} \tilde{X} \xrightarrow{\rho} Z$ are as follows:

There is exactly one critical value $c = \frac{5}{9}$. If $\frac{5}{9} < c \leq 1$, then $\tilde{X}^{(c)} = \tilde{X}$ and

$$K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} \tilde{E}_4 + \frac{1}{9} \left( \tilde{D} + \sum_{1 \leq i \leq 3} (i + 1) \tilde{C}_i \right) + \frac{a}{4} \left( \sum_{4 \leq i \leq 7} (8 - i) \tilde{C}_i + 2 \tilde{C}_8 \right),$$

where $a = \min \{ \frac{4}{9}, c \}$. If $0 < c \leq \frac{5}{9}$, then $\tilde{X}^{(c)} = Z$ and

$$K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}} \sim_{\mathbb{R}} \frac{c}{5} \left( D + \sum_{1 \leq i \leq 3} (i + 1) C_i \right) + \frac{c}{4} \left( \sum_{4 \leq i \leq 7} (8 - i) \tilde{C}_i + 2 \tilde{C}_8 \right),$$

The formula for the minimal volume is

$$(3.10) \quad \min K^2 \left( Z, B^{(c)}_Z; B'_Z \right) = \text{vol}(K_{\tilde{X}^{(c)}} + B_{\tilde{X}^{(c)}}) = \begin{cases} \frac{1}{30} c^2 & \text{if } c \leq \frac{5}{9} \\ \frac{7}{9} c^2 + 2 c - \frac{5}{9} & \text{if } \frac{5}{9} < c \leq 1 \\ \frac{1}{63} & \text{if } c > \frac{4}{7} \end{cases}$$
Case: *C supports a curve of canonical type IV*. The dual graphs of the visible curves on $W \xrightarrow{\gamma} \hat{X} \xrightarrow{\rho} Z$ are as follows:

In this case the critical value is $c = \frac{4}{7}$. If $\frac{4}{7} < c \leq 1$, then $\tilde{X}(c) = \hat{X}$, and we have

$$K_{\tilde{X}(c)} + B_{\tilde{X}(c)} = \tilde{E}_3 + \frac{1}{7}(\tilde{D} + 2\tilde{C}_1 + 3\tilde{C}_2) + \frac{a}{3} \left( \sum_{3 \leq i \leq 5} (6 - i)\tilde{C}_i + 2\tilde{C}_6 + \tilde{C}_7 \right)$$

where $a = \min\{c, \frac{4}{7}\}$. If $c \leq \frac{4}{7}$, then $\hat{X} = Z$, and we have

$$K_{\hat{X}(c)} + B_{\hat{X}(c)} = \frac{c}{4}(D + 2C_1 + 3C_2) + \frac{c}{3} \left( \sum_{3 \leq i \leq 5} (6 - i)C_i + 2C_6 + C_7 \right)$$

The formula for the minimal volume is

$$\text{(3.11)} \quad \min \mathbb{K}^2 \left( Z, B_Z^{(c)}; B_Z' \right) = \text{vol}(K_{\tilde{X}(c)} + B_{\tilde{X}(c)}) = \begin{cases} \frac{1}{12}c^2 & \text{if } c \leq \frac{4}{7} \\ -\frac{5}{2}c^2 + 2c - \frac{4}{5} & \text{if } \frac{4}{7} < c \leq \frac{7}{8} \\ \frac{1}{35} & \text{if } c > \frac{7}{8} \end{cases}$$

**Remark 3.4.** The above cases does not exhaust the possibilities how the (−2)-curve $D$ is attached to the curve $C$. But the other choices of attachment would result in configurations of (−2)-curves, say $G'$, strictly containing one of the configurations considered above, say $G$. Then the resulting minimal volume for $G'$ would be larger than that for $G$.

4. **The semistable part of the boundary divisor**

In this subsection, we study the part of the boundary divisor $B$ that contributes to the geometric genus of a smooth projective log canonical surface $(X, B)$; see Definition 4.4 and Lemma 4.5.

**Definition 4.1.** A reduced curve $B$ on a smooth projective surface $X$ is called *semistable* if it has at most nodes as singularities and each of its smooth rational components intersects the other components of $B$ in more than one point.

**Lemma 4.2.** Let $B$ be a nodal reduced curve on a smooth projective surface $X$. Then $B$ is semistable if and only if $(K_X + B) \cdot B_i \geq 0$ for any smooth rational component $B_i$ of $B$. 


Proof. Let $B_i$ be a smooth rational component of $B$. Then
\[(K_X + B) \cdot B_i = (K_X + B_i) \cdot B_i + (B - B_i) \cdot B_i = -2 + (B - B_i) \cdot B_i\]
Thus $B_i$ intersects the other components of $B$ in more than one point if and only if $(K_X + B) \cdot B_i \geq 0$.

Lemma 4.3. Let $(X, B)$ be a smooth projective log canonical surface. Then there is a (possibly empty) maximal semistable reduced subcurve $B^s$ of $[B]$, in the sense that any semistable subcurve $B'$ of $[B]$ is contained in $B^s$.

Proof. Since $(X, [B])$ is log canonical, it is well-known that $[B]$ is a nodal curve ([17, Theorem 2.31]). Suppose that $B'$ is a semistable subcurve of $[B]$. Any irreducible component $B_i$ of $B$ with $(K_X + B) \cdot B_i < 0$ cannot be an irreducible component of $B'$, since otherwise we have by Lemma 4.2
\[(K_X + B) \cdot B_i \geq (K_X + B') \cdot B_i \geq 0.\]
Starting with $B^{(0)} = [B]$, we construct inductively $B^{(j+1)} = B^{(j)} - B_i$, where $B_j$ is an irreducible component of $B^{(j)}$ such that $(K_X + B^{(j)}) \cdot B_j < 0$. We stop at a subcurve $B^s$ of $B$ such that $(K_X + B^s) \cdot B_i \geq 0$ for any irreducible component $B_i$ of $B^s$. Then $B^s$ is semistable by Lemma 4.2, and, by the above argument, we have $B' \subset B^s$. Thus $B^s$ is the desired maximal semistable subcurve of $[B]$. □

Definition 4.4. Let $(X, B)$ be a projective log canonical surface and $\pi: (\tilde{X}, B_{\tilde{X}}) \rightarrow (X, B)$ the minimal resolution. Let $B^s_{\tilde{X}}$ and $B^ns_{\tilde{X}}$ be as in Lemma 4.3. Then $B^s := \pi_* B^s_{\tilde{X}}$ and $B^{ns} := \pi_* B^{ns}_{\tilde{X}}$ are called the semistable part and the non-semistable part of $B$ respectively, and $B = B^s + B^{ns}$ is called the semistable decomposition of $B$.

Lemma 4.5. Let $(X, B)$ be a smooth projective log canonical surface. Let $B = B^s + B^{ns}$ be decomposition into the the semistable part and the non-semistable part. Then the following holds.

(i) The irreducible components of $[B^{ns}]$ are all smooth rational curves; each connected component of $[B^{ns}]$ intersects $B^s$ in at most one point and its dual graph is a tree.
(ii) $p_g(X, B) = p_g(X, B^s)$.

Proof. (i) By the maximality of $B^s$, any irreducible component subcurve $B_i$ of $[B]$ with $p_a(B_i) > 0$ is contained in $B^s$, so the irreducible components of $[B^{ns}]$ are smooth rational curves.

Suppose that there is a cycle of smooth rational curves $B' \subset [B]$. Then $(K_X + B') \cdot B_i = 0$ and thus $B'$ is semistable by Lemma 4.2. By the maximality of $B^s$, we have $B' \subset B^s$. It follows that the dual graph of each connected component of $[B^{ns}]$ is a tree.

Suppose on the contrary that there is a connected component $B''$ of $[B^{ns}]$ such that $B'' \cdot B^s = 2$. Then one can find a chain of irreducible components $B_i$ of $B''$ with $1 \leq i \leq r$ such that $B_i \cdot B_{i+1} = 1$ for $1 \leq i \leq r - 1$, and $B_1 \cdot B^s$ and $B_r \cdot B^s$ are both positive. But then $B'' + \sum_{1 \leq i \leq r} B_i$ is semistable, contradicting the maximality of $B^s$.

(ii) By definition, we have $p_g(X, B) = p_g(X, [B])$. If $B^s = [B]$ then there is nothing to prove. So we assume that $[B] - B^s > 0$. Let $E$ be an irreducible component of $[B] - B^s$, which is necessarily a smooth rational curve by (i).

Consider the exact sequence of cohomology groups
\[0 \rightarrow H^0(X, K_X + [B] - E) \rightarrow H^0(X, K_X + [B]) \rightarrow H^0(E, (K_X + [B])|_E)\]
Since $\deg(K_X + [B])|_E = (K_X + [B]) \cdot E = -2 + ([B] - E) \cdot E < 0,$
we have $H^0(E, (K_X + |B|)|_E) = 0$ and hence
\[ H^0(X, K_X + |B| - E) \cong H^0(X, K_X + |B|). \]
Since $|B| - E$ and $|B|$ have the same semistable part, namely $B^a$, we finish the proof by induction on the number of irreducible components of $|B| - B^a$. \hfill \Box

Now we describe the behavior of the semistable decomposition of the boundary divisor under blow-ups.

**Lemma 4.6.** Let $(X, B_X)$ be a smooth projective log canonical surface, and $$ \rho: U = X_n \xrightarrow{\rho_n} X_{n-1} \xrightarrow{\rho_{n-1}} \cdots \xrightarrow{\rho_1} X_1 \xrightarrow{\rho_0} X_0 = X $$ is a sequence of blow-ups. For each $1 \leq i \leq n$, let $B_{X_i}$ be the $\mathbb{R}$-divisor on $X_i$ such that $(X_i, B_{X_i})$ is a crepant higher model of $(X, B_Z)$. Let
\[ B_{X_i}^s = B_{X_i}^s + B_{X_i}^{ns} \]
be the decomposition of $B_{X_i}^s$ into the semistable part and the non-semistable part. Then the following holds.

(i) $\text{supp}(B_U^s) \cap \text{supp}(B_{X_i}^s) = \emptyset$.

(ii) $\rho_*, B_U^s = B_X^s$, $\rho_*, B_{X_i}^{ns} = B_{X_i}^{ns}$.

(iii) For $1 \leq i \leq n$, let $p_i \in X_{i-1}$ be the point blown up by $\rho_i$, and $E_i \subset X_i$ the exceptional $(-1)$-curve over $p_i$. Let $\Lambda := \{i \mid 0 \leq i \leq n, p_i \notin \text{supp}(B_{X_{i-1}}^s)\}$, and $E_i$ is the total transform of $E_i \subset X_i$ on $U$. Then
\[ K_U + B_U^s = \rho^*(K_X + B_X^s) + \sum_{i \in \Lambda} E_i. \]

(iv) $\text{supp}(B_U^s) \cap \text{supp}(B_{X_i}^s) \neq \emptyset$ if and only if $\text{supp}(B_X^s) \cap \text{supp}(B_{X_i}^{ns}) \neq \emptyset$.

**Proof.** We show by induction on $i$ that the following statements (a) - (d) hold, from which the lemma follow immediately by induction:

(a) $\text{supp}(B_{X_i}^s) \cap \text{supp}(B_{X_i}^{ns}) = \emptyset$.

(b) $\rho_*, B_{X_i}^s = B_{X_{i-1}}^s$, $\rho_*, B_{X_i}^{ns} = B_{X_{i-1}}^{ns}$, $\rho_*, B_{X_i}^s = B_{X_{i-1}}^s$.

(c) $K_{X_i} + B_{X_i} = \begin{cases} \rho_*(K_{X_{i-1}} + B_{X_{i-1}}^s) & \text{if } p_i \in B_{X_{i-1}}^s, \\ \rho_*(K_{X_{i-1}} + B_{X_{i-1}}^{ns}) + E_i & \text{if } p_i \notin B_{X_{i-1}}^s, \end{cases}$

(d) $\text{supp}(B_{X_i}^s) \cap \text{supp}(B_{X_i}^{ns}) \neq \emptyset$ if and only if $\text{supp}(B_{X_{i-1}}^s) \cap \text{supp}(B_{X_{i-1}}^{ns}) \neq \emptyset$.

Note that $\text{supp}(B_{X_i}^s) \cap \text{supp}(B_{X_i}^{ns}) = \emptyset$ is trivially true, since $(X_0, B_{X_0})$ is log canonical and hence $B_{X_0}^{ns} = 0$. For $i \geq 1$, we can assume by induction that $\text{supp}(B_{X_{i-1}}^s) \cap \text{supp}(B_{X_{i-1}}^{ns}) = \emptyset$.

We distinguish two cases depending on whether the blown-up point $p_i \in X_{i-1}$ lies on $\text{supp}(B_{X_{i-1}}^{ns})$ or not.

**Case 1.** Suppose that $p_i \in \text{supp}(B_{X_{i-1}}^{ns})$. Then $p_i \notin \text{supp}(B_{X_{i-1}}^s)$ by the induction hypothesis. Now we can check directly that
\[ 1 \geq \text{mult}_{E_i} B_{X_i} = \text{mult}_{p_i} B_{X_{i-1}} - 1 = \text{mult}_{p_i} B_{X_{i-1}}^s + \text{mult}_{p_i} B_{X_{i-1}}^{ns} - 1 \geq 0, \]
where the first inequality is because $(X_i, B_{X_i})$ is sub-log canonical. If $p_i \in \text{supp}(B_{X_{i-1}}^s)$ is a node, then $\text{mult}_{p_i} B_{X_{i-1}}^s = 0$, $E_i$ intersects the strict transform $\rho_{i-1}^* B_{X_{i-1}}$ at two points, and
\[ B_{X_i}^s = \rho_{i-1}^* B_{X_{i-1}}^s + E_i = \rho^* B_{X_{i-1}}^s - E_i \]
If \( p_i \in \text{supp}(B^s_{X,-1}) \) is a smooth point, then the connected components of \( E_i + [\rho^{-1}_s B^*_{X,-1}] \) are again trees of smooth rational curves (cf. Lemma 4.5), each intersecting \( \rho^{-1}_s B^*_{X,-1} \) at \textit{less than} two points. It follows that \( E_i \) is not a component of \( B^s_{X,-1} \), and we have

\[
(4.3) \quad B^s_{X,-1} = \rho^{-1}_s B^*_{X,-1} = \rho^* B^s_{X,-1} - E_i
\]

**Case 2.** Suppose that \( p_i \notin \text{supp}(B^s_{X,-1}) \). One checks as before that \( E_i \) is not a component of \( B^s_{X,-1} \), and

\[
(4.4) \quad B^s_{X,-1} = \rho^{-1}_s B^*_{X,-1} = \rho^* B^s_{X,-1}.
\]

In both cases, we have

\[
(4.5) \quad \text{supp}(B^s_{X,-1}) \cap \text{supp}(B^{<0}_{X,-1}) \subset \rho^{-1} \left( \text{supp}(B^s_{X,-1}) \cap \text{supp}(B^{<0}_{X,-1}) \right) = \emptyset
\]

where the equality is because of the induction hypothesis. Thus \( \text{supp}(B^s_{X,-1}) \cap \text{supp}(B^{<0}_{X,-1}) = \emptyset \), which is (a)\(_i\).

Also we have \( \rho^*_s B^s_{X,-1} = B^s_{X,-1} \). It follows directly from the definition that \( \rho^*_s B^{>0}_{X,-1} = B^{>0}_{X,-1} \) and \( \rho^*_s B^{<0}_{X,-1} = B^{<0}_{X,-1} \), and hence

\[
\rho^*_s B^{<0}_{X,-1} = \rho^*_s B^{>0}_{X,-1} - \rho^*_s B^s_{X,-1} = B^{<0}_{X,-1} - B^{>0}_{X,-1} = B^{ns}_{X,-1}.
\]

This proves (b)\(_i\).

Combining (4.2), (4.3), and (4.4) with the fact that \( K_{X,-1} = \rho^*_s K_{X,-1} + E_i \), one obtains (c)\(_i\):

\[
K_{X,-1} + B^s_{X,-1} = \begin{cases} \rho^*(K_{X,-1} + B^s_{X,-1}) & \text{if } p_i \in B^s_{X,-1}, \\ \rho^*(K_{X,-1} + B^s_{X,-1}) + E_i & \text{if } p_i \notin B^s_{X,-1}. \end{cases}
\]

Now we prove (d)\(_i\): Suppose that there is a point \( p \in \text{supp}(B^{<0}_{X,-1}) \cap \text{supp}(B^{ns}_{X,-1}) \). Then \( p \notin \text{supp}(B^{<0}_{X,-1}) \) by (a)\(_i\), and \( p \) is not a node of \( B^{<0}_{X,-1} \). If \( p = p_i \) is blown up by \( \rho_i \), then

\[
\text{mult}_E B_{X,-1} = \text{mult}_p B^s_{X,-1} + \text{mult}_p B^{ns}_{X,-1} - 1 = \text{mult}_p B^{ns}_{X,-1} > 0.
\]

Hence \( \text{supp}(B^{ns}_{X,-1}) = \rho^*_s \text{supp}(B^{ns}_{X,-1}) \) and \( B^s_{X,-1} = \rho^{-1}_s B^{*}_{X,-1} \), which have nonempty intersection. For the other implication, suppose on the contrary that there is a point \( q \in \text{supp}(B^s_{X,-1}) \cap \text{supp}(B^{ns}_{X,-1}) \) while \( \text{supp}(B^s_{X,-1}) \cap \text{supp}(B^{ns}_{X,-1}) = \emptyset \). Then the connected component \( G \) of \( \text{supp}(B^s_{X,-1}) \) containing \( q \) is necessarily contracted by \( \rho_i \). This can only happen if \( G = E_i \), \( p_i = \rho(q) \notin \text{supp}(B^{ns}_{X,-1}) \), and \( p_i \in \text{supp}(B^s_{X,-1}) \) is smooth point. But then, as the computation in Case 1 above shows, \( \text{mult}_E B_{X,-1} = 0 \) and hence \( p \notin \text{supp}(B^{ns}_{X,-1}) \), which is a contradiction.

\[ \square \]

**Remark 4.7.** By Lemma 4.6, given a projective log canonical surface \((X, B_X)\) and an effective higher model \((U, B_U, \rho)\), we have \( \rho_s(B^s_U) = B^s \) and \( \rho_s(B^{ns}_U) = B^{ns} \). Thus the system of \( \mathbb{R} \)-divisors \( \{ B^s_U \}_U \) (resp. \( \{ B^{ns}_U \}_U \)) form a so-called b-\( \mathbb{R} \)-divisor.

**Corollary 4.8.** Let \((X, B_X)\) be a smooth projective log canonical surface, and \( \rho: (U, B_U) \to (X, B_X) \) a crepant resolution. Let \( B^0_U \) and \( B^s_X \) be semistable parts of \( B^0_U \) and \( B_X \) respectively. Then, for any \( m \in \mathbb{Z}_{\geq 0} \), we have \( \dim H^0(U, m(K_U + B^0_U)) = \dim H^0(X, m(K_X + B^s_X)) \). In particular, \( p_9(Y, B^0_U) = p_9(X, B^s_X) \) and \( \kappa(K_U + B^0_U) = \kappa(K_X + B^s_X) \).

**Proof.** By Lemma 4.6 (ii) and (iii), we have \( \rho_s(K_U + B^0_U) = K_X + B^s_X \), and \( K_U + B^0_U - \rho^*(K_X + B^s_X) \) is effective \( \mathbb{R} \)-divisor supported on \( \text{Exc}(\rho) \). Thus there is a natural map \( \rho^*: H^0(X, m(K_X + B^s_X)) \to H^0(U, m(K_U + B^0_U)) \) which is easily seen to be an isomorphism for any positive integer \( m \).

\[ \square \]
5. Necessary conditions for realizing the minimal volumes

In this section we pin down several necessary conditions a projective log canonical surface with prescribed geometric genus should satisfy, in order that it achieves the minimal possible volume.

5.1. Decreasing the volume inside $S(C, p_g; \kappa)$. In this subsection we present several conditions, which forbid a log surface $(X, B) \in S(C, p_g; \kappa)$ to realize the minimal volume. This is shown by constructing another log surface $(X', B') \in S(C, p_g; \kappa)$ with smaller volume.

We need to set up the notation before giving the exact statements.

Notation 5.1. For a projective log canonical surface $(X, B)$, we consider the following diagram:

$$\begin{align*}
\pi & \quad \rho \\
(\tilde{X}, B_{\tilde{X}}) & \quad (X, B) & \quad (Z, B_Z)
\end{align*}$$

where $\pi: (\tilde{X}, B_{\tilde{X}}) \to (X, B)$ is the minimal resolution, and $\rho: \tilde{X} \to Z$ is a minimal model program of the divisor $K_{\tilde{X}} + B_{\tilde{X}}^s$, contracting successively the $(-1)$-curves that are disjoint from (the images of) $B_{\tilde{X}}^s$, the semistable part of $B_{\tilde{X}}$. Set $B_Z = \rho_* B_{\tilde{X}}$ and $B_Z' = \rho_* B_{\tilde{X}}^s$. Then the birational morphism $\rho$ is an isomorphism over a neighborhood of $\text{supp}(B_Z')$, but $(Z, B_Z)$ may have worse than log canonical singularities at $\text{supp}(B_Z) \setminus \text{supp}(B_Z')$. Therefore, we have $(\tilde{X}, B_{\tilde{X}}, \rho) \in S(Z, B_Z; B_Z')$, as in Notation 3.2. The following relations are clear by Lemma 4.5

$$(5.2) \quad p_g(X, B) = p_g(\tilde{X}, B_{\tilde{X}}) = p_g(Z, B_Z) \quad \text{and} \quad \kappa(K_{\tilde{X}} + B_{\tilde{X}}^s) = \kappa(K_Z + B_Z').$$

Notation 5.2. Let $C \subset (0, 1]$ be a subset, $p_g$ a non-negative integer and $\kappa \in \{-\infty, 0, 1, 2\}$. Recall from the introduction that $S(C)$ consists of projective log canonical surfaces $(X, B)$ of general type with $C_B \subset C$, and $S(C, p_g)$ consists of those $(X, B) \in S(C)$ with $p_g(X, B) = p_g$. Now for $\kappa \in \{-\infty, 0, 1, 2\}$, define

$$S(C, p_g; \kappa) = \{(X, B) \in S(C, p_g) \mid \kappa(K_{\tilde{X}} + B_{\tilde{X}}^s) = \kappa\},$$

where $(\tilde{X}, B_{\tilde{X}}) \to (X, B)$ is the minimal resolution. Correspondingly, we denote the set of volumes of log surfaces from $S(C, p_g; \kappa)$ by

$$\mathbb{K}^2(C, p_g; \kappa) := \{\text{vol}(K_X + B) \mid (X, B) \in S(C, p_g; \kappa)\}.$$ 

Remark 5.3. We have two easy observations:

(i) $S(C, p_g; \kappa) = \emptyset$ if $p_g > 0$ and if $\kappa = -\infty$.

(ii) If $C' \subset C \subset (0, 1]$ then $S(C', p_g; \kappa) \subset S(C, p_g; \kappa)$.

Now we can give the first criteria for non-minimal volumes.

Lemma 5.4. Let $C \subset (0, 1]$ be a subset, $p_g$ a non-negative integer, and $\kappa \in \{-\infty, 0, 1, 2\}$. Suppose that $(X, B) \in S(C, p_g; \kappa)$ has ample $K_X + B$ and let $\pi: (\tilde{X}, B_{\tilde{X}}) \to (X, B)$ be the minimal resolution. If one of the following conditions holds:

(a) $\text{supp}(B_{\tilde{X}}^s) \cap \text{supp}(B_{\tilde{X}}^a) \neq \emptyset$,

(b) $\text{supp}(B_{\tilde{X}}^a)$ contains a nonklt center of $(X, B)$,

then there is another projective log canonical surface $(X', B') \in S(C_B, p_g; \kappa)$ such that $\text{vol}(K_{X'} + B') < \text{vol}(K_X + B)$. 

Proof. The construction of the log surface \((X', B')\) runs along the same line as [3, Theorem 3.3], but here one needs to check additionally that the newly constructed \((X', B')\) has the same \(p_g\) and \(\kappa\) as \((X, B)\).

Let \(\alpha: (U, B_U) \to (X, B)\) be an effective crepant resolution. By possibly blowing up points on \([B_U]\) further, we may assume that \(\text{supp}(B_U)\) has normal crossing in a neighborhood of \([B_U]\). Since \(\pi: \tilde{X} \to X\) is the minimal resolution, there is a birational morphism \(\tilde{\alpha}: U \to \tilde{X}\) such that \(\pi \circ \tilde{\alpha} = \alpha\). By Lemma 4.6, the condition (a) (resp. (b)) still holds with \((\tilde{X}, B_{\tilde{X}})\) replaced by \((U, B_U)\); moreover, under the condition (b), one can require that \([B_U^\alpha] \neq 0\) by blowing up \(U\) further, if needed. In both cases, there is an irreducible component \(B_{U,2}\) of \([B_U]\), intersecting some other irreducible component \(B_{U,1}\) that is contained in \(\text{supp}(B_U^\alpha)\). In particular, \(B_{U,2}\) is an accessible non-klt center of \((U, B_U)\).

Take a point \(p \in B_{U,2} \cap B_{U,1}\). Blow up \(p \in U\) and then its pre-images on the strict transforms of \(B_{U,2}\) on the blown-up surfaces. Let \(\beta: (W, B_W) \to (U, B_U)\) be the resulting crepant effective resolution after \(n\) such blow-ups, shown by dual graphs as follows:

![Diagram](image)

where the \(E_i\) \((1 \leq i \leq n)\) denote the exceptional curves of \(\beta\), and the numbers in the brackets above the vertices are the self-intersection numbers of the corresponding curves.

By the construction of \(\beta\), one sees easily that \(B_W = \beta^{-1}_* B_U + a \sum_{1 \leq i \leq n} E_i\), where \(a \in (0, 1]\) is the coefficient of \(B_{U,1}\) in \(B_U\). Decreasing the coefficients of the exceptional divisors \(E_i\) in \(B_W\), we set

\[
B_W' = B_W - a \sum_{1 \leq i \leq n} \frac{i}{n} E_i = \beta^{-1}_* B_U + a \sum_{1 \leq i \leq n-1} \left(1 - \frac{i}{n}\right) E_i.
\]

Since \(K_W + B_W = \beta^* \alpha^*(K_X + B)\) is big and nef, and \(B_W' < B_W\), we have by Lemma 2.5

\[
\text{vol}(K_W + B_W') < \text{vol}(K_W + B_W) = \text{vol}(K_X + B)
\]

Taking \(n\) large enough, for example, \(n > \frac{\alpha^2}{\text{vol}(K_X + B)}\), we have

\[
(K_W + B_W')^2 = \left( K_W + B_W - a \sum_{1 \leq i \leq n} \frac{i}{n} E_i \right)^2
\]

\[
= (K_W + B_W)^2 + a^2 \left( \sum_{1 \leq i \leq n} \frac{i}{n} E_i \right)^2
\]

\[
= \text{vol}(K_X + B) - \frac{\alpha^2}{n} > 0
\]

Together with the fact that \(\beta_*(K_W + B_W') = K_U + B_U\) is big, (5.5) implies that \(K_W + B_W'\) is big for \(n > \frac{\alpha^2}{\text{vol}(K_X + B)}\).

Let \(\pi': (W', B_W') \to (X', B')\) be the contraction onto the ample model. By (5.4), we have

\[
0 < \text{vol}(K_{X'} + B') = \text{vol}(K_W + B_W') < \text{vol}(K_X + B).
\]

Next we check that \((X', B') \in S(C, p_g; \kappa)\).

Write \(\gamma = \alpha \circ \beta: W \to X\). As one checks above, \((K_W + B_W') \cdot E_i = 0\) holds for \(1 \leq i \leq n-1\), while for any \(\gamma\)-exceptional curve \(F\) other than the \(E_i\), \(1 \leq i \leq n\), we have \((K_W + B_W') \cdot F \leq 0\).

It follows that all \(\gamma\)-exceptional curves except for \(E_n\) are contracted to points on the ample
model \((X', B')\). Note also that the coefficient of \(E_n\) in \(B'_n\) is 0. Therefore, the coefficients of \(B'\) are among those of \(B\).

The semistable parts of \(B_W\) and \(B_W'\) are the same, which is denoted by \(B_W\). It follows from Lemma 4.5 and Corollary 4.8 that

\[ p_g(X', B') = p_g(W, B'_W) = p_g(W, B'_W) = p_g(U, B'_U) = p_g \]

and

\[ \kappa(K_W + B'_W) = \kappa(K_U + B'_U) = \kappa. \]

In conclusion, we have constructed under one of the conditions (a) and (b) a projective log canonical surface \((X', B')\) in \(S(\mathcal{C}_B, p_g; \kappa)\) and it has a smaller volume than \((X, B)\). \(\square\)

**Lemma 5.5.** Let \(C \subset (0, 1]\) be a subset such that \(\min(C \cup \{1\})\) attains the minimum, say \(c\). Then, for \(p_g \in \mathbb{Z}_{\geq 0}\) and \(\kappa \in \{0, 1, 2\}\), the following holds.

(i) Suppose that \((X, B)\) in \(S(\mathcal{C}, p_g; \kappa)\) has ample \(K_X + B\), and let \(\pi: (\hat{X}, B_{\hat{X}}) \to (X, B)\) be the minimal resolution. If one of the following conditions (a) and (b) holds:

(a) \(\max \mathcal{C}_{B_X} > c\);

(b) there is an irreducible component \(B_0 = B'_0\) of \(B_{\hat{X}}\), say, with coefficient \(b_0 > 0\), such that \(K_{\hat{X}} + B_{\hat{X}} - b_0B_0\) is still big;

then there is another log surface \((X', B')\) in \(S(\{c, 1\} \cap \mathcal{C}, p_g; \kappa)\) such that \(K_{X'} + B'\) is ample and \(\text{vol}(K_{X'} + B') < \text{vol}(K_X + B_X)\):

(ii) The volume set \(\mathbb{K}^2(\mathcal{C}, p_g; \kappa)\) attains the minimum, and we have \(\min \mathbb{K}^2(\mathcal{C}, p_g; \kappa) = \min \mathbb{K}^2(\{c, 1\} \cap \mathcal{C}, p_g; \kappa)\).

**Proof.** (i) Under either condition, we will define a boundary divisor \(\hat{B}\) strictly smaller than \(B_{\hat{X}}\) and then take \((X', B')\) to be ample model of \((\hat{X}, \hat{B})\).

We first deal with the condition (a). Write \(B_{\hat{X}} = \sum_{j \in J} b_j\hat{B}_j\) as the sum of distinct prime divisors \(\hat{B}_j\), and let \(J_{\text{ns}} \subset J\) be the index set for the components \(\hat{B}_j\) contained in \(B_{\hat{X}}^\text{ns}\). Now define a new boundary divisor on \(\hat{X}\) as follows:

\[ \hat{B} := B_{\hat{X}}^\text{ns} + \sum_{j \in J_{\text{ns}}} \min\{b_j, c\} \hat{B}_j. \]

Since \(\kappa(K_{\hat{X}} + B_{\hat{X}}^\text{ns}) \geq 0\), \(K_{\hat{X}} + \hat{B}\) is still big by Lemma 2.5 (iii). Let \(\pi': (\hat{X}, \hat{B}) \to (X', B')\) be the ample model. Under the condition (a), \(K_{\hat{X}} + \hat{B}\) is strictly smaller than the big and nef \(K_{\hat{X}} + B_{\hat{X}}\), and thus \(\text{vol}(K_{X'} + B') = \text{vol}(K_{\hat{X}} + \hat{B}) < \text{vol}(K_{\hat{X}} + B_{\hat{X}}) = \text{vol}(K_X + B_X)\) by Lemma 2.5 (ii). By construction we have \(\mathcal{C}_{\hat{B}} \subset (\mathcal{C}_{B_{\hat{X}}})_{<c} \cup \{c, 1\}\). The components \(\hat{B}_j, j \in J\) contracted by \(\pi\: \hat{X} \to X\) are necessarily contracted by \(\pi'\), since \((K_X + \hat{B}) \cdot \hat{B}_j \leq (K_{\hat{X}} + B_{\hat{X}}) \cdot \hat{B}_j = 0\). Now the components \(\hat{B}_j\) in \(B_{\hat{X}}^\text{ns}\) with \(b_j < c\) are contracted by \(\pi\). Therefore, \(\mathcal{C}_{B'} \subset \mathcal{C}_B \cap \{c, 1\}\). Since the semistable part of \(\hat{B}\) and \(B_{\hat{X}}\) are the same, namely \(B_{\hat{X}}^\text{ns}\), we infer that \((X', B') \in S(\{c, 1\} \cap \mathcal{C}, p_g; \kappa)\).

Under the condition (b), we define \(\hat{B} = B_{\hat{X}} - b_0B_0\). The arguments are similar to (and easier than) those under condition (a).

(ii) we have \(\inf \mathbb{K}^2(\mathcal{C}, p_g; \kappa) \geq \inf \mathbb{K}^2(\{c, 1\} \cap \mathcal{C}, p_g; \kappa)\) by (i). On the other hand, the inclusion \(\mathbb{K}^2(\{c, 1\} \cap \mathcal{C}, p_g; \kappa) \subset \mathbb{K}^2(\mathcal{C}, p_g; \kappa)\) implies that \(\inf \mathbb{K}^2(\mathcal{C}, p_g; \kappa) \leq \inf \mathbb{K}^2(\{c, 1\} \cap \mathcal{C}, p_g; \kappa)\). Therefore, \(\inf \mathbb{K}^2(\mathcal{C}, p_g; \kappa) = \inf \mathbb{K}^2(\{c, 1\} \cap \mathcal{C}, p_g; \kappa)\). By [2], the latter attains the minimum. It follows that \(\mathbb{K}^2(\mathcal{C}, p_g; \kappa)\) also attains the same minimum. \(\square\)
5.2. Necessary conditions for realizing the minimal volumes. Based on the criteria for non-minimal volumes in Section 5.1, we can now formulate several necessary conditions for realizing the minimal volumes inside $S(C,p_g;\kappa)$ with $\kappa \geq 0$. A notable one is the separation of the semistable part of the boundary divisor from the non-semistable part. This enables us, in searching for the minimal volume, to operate as if we were always on a surface with non-negative Kodaira dimension.

**Proposition 5.6.** Let $C \subset \{0,1\}$ be a subset, possibly empty, such that $C \cup \{1\}$ attains the minimum, say $c$. Let $p_g \in \mathbb{Z}_{\geq 0}, \kappa \in \{0,1,2\}$ be two nonnegative integers. Suppose that $(X,B) \in S(C,p_g;\kappa)$ has ample $K_X + B$ and $\text{vol}(K_X + B) = \min \mathbb{R}^2(C,p_g;\kappa)$, and let $\pi: (\tilde{X},\tilde{B}) \to (X,B)$ be the minimal resolution. Then the following holds.

(i) $(\tilde{X},\tilde{B})$ has klt singularities in a neighborhood of $\text{supp}(B^p_X) \subset \tilde{X}$. In particular, $\text{supp}(B^p_{\tilde{X}})$ and $\text{supp}(B^p_{\tilde{X}})$ are disjoint.

(ii) There is an inclusion $\mathcal{C}_{B^p_{\tilde{X}}} \subset (0,c] \subset C$.

(iii) If $\tilde{B}_0$ is irreducible component of $B^p_{\tilde{X}}$, say, with coefficient $b_0 > 0$, then $K_{\tilde{X}} + B_{\tilde{X}} - b_0\tilde{B}_0$ is not big.

(iv) There is an inclusion $\mathcal{C}_{B} \subset \{c,1\} \cap C$.

**Proof.** (i) Suppose on the contrary that $(\tilde{X},\tilde{B})$ is not klt along $\text{supp}(B^p_{\tilde{X}})$. Then one of the conditions (a) and (b) of Lemma 5.4 holds, and this would violate the minimality of $\text{vol}(K_X + B)$.

(ii) we have $\mathcal{C}_{B^p_{\tilde{X}}} \subset (0,1)$ by (i) and $\mathcal{C}_{B^p_{\tilde{X}}} \subset (0,c]$ by Lemma 5.5 (ia). (ii) is proved by combining these two inclusions.

(iii) is a direct consequence of Lemma 5.5 (ib).

(iv) We have $\mathcal{C}_{B} \subset ((\mathcal{C}_{B_{\tilde{X}}})_{\leq} \cup \{1\}) \cap C \subset \{c,1\} \cap C$, where the first inclusion is by (ii), and the second inclusion is because $c = \min(\mathcal{C} \cup \{1\})$. \hfill $\Box$

**Corollary 5.7.** Keep the notation and assumptions of Proposition 5.6. Let $\rho: (\tilde{X},B^p_{\tilde{X}}) \to (Z,B^p_Z)$ be the $(K_{\tilde{X}} + B^p_{\tilde{X}})$-minimal model as in (5.1), with

$$B^p_Z = \rho_*B^p_{\tilde{X}}, B^\prime_Z = \rho_\ast B^p_{\tilde{X}}, \text{ and } B_Z = \rho_*B_{\tilde{X}}$$

Then the following holds.

(i) $(Z,B^\prime_Z)$ has log canonical singularities, and $\text{supp}(B^p_Z) \cap \text{supp}(B^\prime_Z) = \emptyset$.

(ii) There is an inclusion $\mathcal{C}_{B^p_Z} \subset (0,c] \subset C$.

(iii) If $B_{Z,0}$ is an irreducible component of $B^p_Z$, say, with coefficient $b_0 > 0$, then $K_{Z} + B_{Z} - b_0B_{Z,0}$ is not big.

(iv) $B^p_X$ is the strict transform of $B^p_Z$.

**Proof.** (i) is clear from the corresponding property of $(\tilde{X},B^p_{\tilde{X}} + B^p_{\tilde{X}})$ as in Proposition 5.6 and from the fact that $\rho: \tilde{X} \to Z$ is an isomorphism over a neighborhood of $B^\prime_Z$.

(ii) follows from Proposition 5.6 (ii), since $C_{B^p_Z} \subset C_{B^p_{\tilde{X}}}$.

(iii) follows from Proposition 5.6 (iii), since the bigness of $K_Z + B_Z - b_0B_{Z,0}$ would imply that of $K_{\tilde{X}} + B_{\tilde{X}} - b_0\tilde{B}_0$ by Lemma 3.3, where $\tilde{B}_0 \subset \tilde{X}$ is the strict transform of $B_{Z,0}$.

(iv) As noticed in Notation 5.1, we have $(\tilde{X},B_{\tilde{X}},\rho) \in S(Z,B^p_Z;B^\prime_Z)$. By Lemma 3.3, $K_{\tilde{X}} + B^p_{\tilde{X}} + \rho^{-1}_*B^\prime_Z$ is big. By Proposition 5.6, we infer that $B^p_{\tilde{X}} = \rho^{-1}_*B^\prime_Z$. \hfill $\Box$
6. Lower bounds of $\mathcal{K}^2(\mathcal{C}, p_g)$

In this section we fix a (possibly empty) coefficient set $\mathcal{C} \subset (0, 1]$ such that $\mathcal{C} \cup \{1\}$ attains the minimum, say $c$, and a nonnegative integer $p_g$. Our aim is to uniformly bound $\text{vol}(K_X + B)$ for $(X, B) \in \mathcal{S}(\mathcal{C}, p_g)$ from below. Note that, by taking the ample model of $(X, B)$, we can assume that $K_X + B$ is ample. If $p_g > 0$, the set $\mathcal{S}^2(\mathcal{C}, p_g)$ can be decomposed as follows (see Notation 5.2):

$$\mathcal{S}^2(\mathcal{C}, p_g) = \bigcup_{0 \leq \kappa \leq 2} \mathcal{S}^2(\mathcal{C}, p_g; \kappa).$$

Thus it suffices to give, separately for $0 \leq \kappa \leq 2$, the lower bounds of $\text{vol}(K_X + B)$ for $(X, B) \in \mathcal{S}^2(\mathcal{C}, p_g; \kappa)$. Most of the lower bounds found turn out to be optimal, as the examples in Section 7 show.

6.1. The lower bound of $\mathcal{K}^2(\mathcal{C}, p_g; 2)$.

**Proposition 6.1.** For any $(X, B) \in \mathcal{S}(\mathcal{C}, p_g; 2)$, we have $\text{vol}(K_X + B) \geq \max\{1, p_g - 2\}$.

**Proof.** Suppose that $(X, B) \in \mathcal{S}(\mathcal{C}, p_g; 2)$ has ample $K_X + B$ and $\text{vol}(K_X + B) = \min \mathcal{K}^2(\mathcal{C}, p_g; 2)$. It suffices to show the inequality of the lemma for this log surface.

Note that $(X, B)$ and its minimal resolution $\pi: (\tilde{X}, B_{\tilde{X}}) \to (X, B)$ are subject to the restrictions given by Proposition 5.6. Since $(X, B) \in \mathcal{S}(\mathcal{C}, p_g; 2)$, we have $\kappa(K_{\tilde{X}} + B_{\tilde{X}}^a) = 2$, where $B_{\tilde{X}}^a$ is the semistable part of $B_{\tilde{X}}$. It follows from Proposition 5.6 (iii) that $B_{\tilde{X}}^a = B_{\tilde{X}}$. Then $B = \pi_* B_{\tilde{X}}$ is reduced and $K_X + B$ is Cartier. Thus $\text{vol}(K_X + B) = (K_X + B)^2 \geq 1$ holds.

Next we show that $\text{vol}(K_X + B) \geq p_g - 2$. For this purpose, we may assume that $p_g \geq 3$. Let $\phi: X \to \mathbb{P}^n$ with $n = p_g - 1$ be the canonical map defined by the linear system $|K_X + B|$. Let $|M|$ be the movable part of $|K_X + B|$, so $M$ is nef and $K_X + B = M + N$, where $N$ is the fixed part of $|K_X + B|$. If $\phi(X)$ is a surface, then we have

$$\text{vol}(K_X + B) = (K_X + B)^2 \geq M^2 \geq \deg \phi(X) \geq p_g - 2,$$

where the last inequality is by [10, Proposition 0]. If $\phi(X)$ is a curve, then we have $M \equiv mF$, where $F$ is a general element of the pencil, and $K_X + B = M + N = mF + N$. Also $m = \deg \phi(X) \geq p_g - 1$ by [10, Proposition 0]. Thus we have

$$\text{vol}(K_X + B) = (K_X + B)^2 = (mF + N) \cdot (mF + N) \geq m(K_X + B) \cdot F \geq p_g - 1.$$

In conclusion, we have shown that $\text{vol}(K_X + B) \geq \max\{1, p_g - 2\}$. \hfill \square

6.2. The lower bound of $\mathcal{K}^2(\mathcal{C}, p_g; 1)$.

**Proposition 6.2.** For any $(X, B) \in \mathcal{S}(\mathcal{C}, p_g; 1)$, the following holds.

(i) If $p_g \geq 2$, then

$$\text{vol}(K_X + B) \geq \begin{cases} (2c - c^2)(p_g - 1) - 2c^2 & \text{if } c < \frac{p_g - 1}{p_g + 1} \\ p_g - 3 + \frac{4}{p_g + 1} & \text{if } c \geq \frac{p_g - 1}{p_g + 1} \end{cases}$$

(ii) If $p_g \leq 1$, then

$$\text{vol}(K_X + B) \geq \begin{cases} 2c - 3c^2 & \text{if } c \leq \frac{1}{3} \\ \frac{4}{9} & \text{if } c > \frac{1}{3} \end{cases}$$
Proof. Step 0. Suppose that \((X, B) \in \mathcal{S}(C, p_0; 1)\) has ample \(K_X + B\) and \(\text{vol}(K_X + B) = \min K^2(C, p_0; 1)\). It suffices to show the inequalities of the proposition for this log surface. Then \((X, B)\) and the minimal resolution \(\pi: (\tilde{X}, B_{\tilde{X}}) \to (X, B)\) are subject to the restrictions of Proposition 5.6. Since \((X, B) \in \mathcal{S}(C, p_0; 1)\), we have \(\kappa(K_{\tilde{X}} + B_{\tilde{X}}) = 1\). For \(l\) sufficiently large and divisible, the linear system \(|l(K_{\tilde{X}} + B_{\tilde{X}})|\) defines a fibration \(\tilde{f}: \tilde{X} \to C\). For a general fiber \(\tilde{F}\) of \(\tilde{f}\), we have \((K_{\tilde{X}} + B_{\tilde{X}}) \cdot \tilde{F} = 0\) and \(g(\tilde{F}) \leq 1\).

Since the divisor \(K_{\tilde{X}} + B_{\tilde{X}}\) is big, there is a component \(\tilde{B}_0\) of \(B_{\tilde{X}}\), horizontal with respect to the fibration \(f\). Let \(b > 0\) be the coefficient of \(\tilde{B}_0\) in \(B_{\tilde{X}}\). Then the log canonical divisor \(K_{\tilde{X}} + B_{\tilde{X}} + b\tilde{B}_0\) is big by Lemma 2.6. Due to the minimality of \(\text{vol}(K_X + B)\) we infer from Proposition 5.6 that

- \(B_{\tilde{X}} = b\tilde{B}_0\) with \(b \in (0, c_{\tilde{B}_0})\), and
- \(\tilde{B}_0 \cap \text{supp}(B_{\tilde{X}}) = \emptyset\), thus \(K_{\tilde{X}} \cdot \tilde{B}_0 = (K_{\tilde{X}} + B_{\tilde{X}}) \cdot \tilde{B}_0 > 0\).

Step 1. We pass to the \((K_{\tilde{X}} + B_{\tilde{X}})\)-minimal model \(\rho: (\tilde{X}, B_{\tilde{X}}) \to (Z, B_Z)\), as in Notation 5.1. Then \(\kappa(K_Z + B_Z) = \kappa(K_{\tilde{X}} + B_{\tilde{X}}) = 1\), and \(|l(K_Z + B_Z)|\) is base point free, defining a fibration \(f: Z \to C\) so that \(\tilde{f} = f \circ \rho\). The general fiber \(F\) of \(f\) is isomorphic to that of \(\tilde{f}\). Note also that \(\tilde{B}_0\) is not contracted by \(\rho: \tilde{X} \to Z\), since \(\rho\) only contracts curves vertical with respect to \(f\); denote \(B_{Z,0} = \rho_* \tilde{B}_0\).

Step 2. In this step, we decompose \(\rho: \tilde{X} \to Z\) into a sequence of simple blow-ups, say

- \(\rho: \tilde{X} = Z_n \xrightarrow{\rho_n} Z_{n-1} \xrightarrow{\rho_{n-1}} \cdots \xrightarrow{\rho_2} Z_1 \xrightarrow{\rho_1} Z_0 = Z\),

and compute \(\text{vol}(K_X + B)\) in terms of these blow-ups. For each \(1 \leq i \leq n\), let \(p_i \in Z_{i-1}\) be the point blown up by \(\rho_i\) and \(E_i \subset Z_i\) the \(\rho_i\)-exceptional \((-1)\)-curve over \(p_i\). Let \(E_i \subset \tilde{X}\) be the total transform of \(E_i\). Using the projection formula, we have for \(1 \leq i, j \leq n\)

\[E_i \cdot E_j = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}\]

and \(E_i \cdot \rho^* D = 0\) for any \(\mathbb{R}\)-divisor \(D\) on \(Z\). Let \(B_{Z_i}, B'_{Z_i}, B''_{Z_i} = b B_{Z_{i-1},0}\) be the pushforward of the \(\mathbb{R}\)-divisors \(B_{\tilde{X}}, B'_{\tilde{X}}, B''_{\tilde{X}} = b \tilde{B}_0\) to \(Z_i\), respectively, and set \(\mu_i := \text{mult}_{\rho} B_{Z_{i-1},0}\). Then

\[K_{\tilde{X}} = \rho^* K_Z + \sum_{1 \leq i \leq n} E_i, \quad B_{\tilde{X}}'' = \rho^* B_{Z} + B''_{\tilde{X}} = \rho_* B_Z'' = \rho^* B_Z'' - \sum_{1 \leq i \leq n} b_i E_i,\]

where the last equality is by Corollary 5.7. Combining the above equalities, we obtain

\[K_{\tilde{X}} + B_{\tilde{X}} = \rho^* (K_Z + B_Z) - \sum_{1 \leq i \leq n} (b \mu_i - 1) E_i.\]

Setting \(d = K_Z \cdot B_{Z,0}\), we have

Claim 6.3. \(\text{vol}(K_X + B) = (2b - b^2) d + b^2 \left(2 p_0(\tilde{B}_0) - 2\right) + \sum_{1 \leq i \leq n} \left[(b - b^2) \mu_i + (b \mu_i - 1)\right].\)

Proof of Claim 6.3. we have

\[(K_{\tilde{X}} + B_{\tilde{X}}) \cdot \tilde{B}_0 = K_{\tilde{X}} \cdot \tilde{B}_0 = \left(\rho^* K_Z + \sum_{1 \leq i \leq n} E_j \right) \cdot \left(\rho^* B_{Z,0} - \sum_{1 \leq i \leq n} \mu_i E_j \right)\]

\[(6.1) = K_Z \cdot B_{Z,0} + \sum_{1 \leq i \leq n} \mu_i = d + \sum_{i} \mu_i\]
where the first equality is because $B_X^s$ and $\tilde{B}_0$ do not intersect. By the adjunction formula,

\begin{equation}
\tilde{B}_0^2 = 2p_a(\tilde{B}_0) - 2 - K\cdot \tilde{B}_0 = 2p_a(\tilde{B}_0) - 2 - \left(d + \sum_i \mu_i\right).
\end{equation}

Note also that $K\cdot B_X^s = \rho^*(K_Z + B'_Z) + \sum_{1 \leq i \leq n} \mathcal{E}_j$ and hence

\begin{equation}
(K\cdot B_X^s)^2 = \left(\rho^*(K_Z + B'_Z) + \sum_{1 \leq i \leq n} \mathcal{E}_j\right)^2 = (K_Z + B'_Z)^2 - n = -n.
\end{equation}

Combining (6.1), (6.2) and (6.3), we obtain

$$\text{vol}(K_X + B) = (K\cdot B_X^s)^2 = (K\cdot B_X^s + b\tilde{B}_0)^2$$

$$= \left(K\cdot B_X^s + 2b(K\cdot B_X^s)\cdot \tilde{B}_0 + b^2\tilde{B}_0^2\right)$$

$$= -n + 2b\left(d + \sum_{1 \leq i \leq n} \mu_i\right) + b^2\left(2p_a(\tilde{B}_0) - 2 - d - \sum_{1 \leq i \leq n} \mu_i\right)$$

$$= (2b - b^2)d + b^2(2p_a(\tilde{B}_0) - 2) + \sum_{1 \leq i \leq n} [(b - b^2)\mu_i + (b\mu_i - 1)].$$

\hfill \Box

**Step 3.** In this step, we get rid of the $\mu_i$’s and $d$ in Claim 6.3 by observing the following lower bounds:

**Claim 6.4.** For each $1 \leq i \leq n$, we have $b\mu_i > 1$.

*Proof of Claim 6.4.* In fact, $b\mu_i - 1 = (K\cdot B_X^s)\cdot \mathcal{E}_i = (K_X + B)\cdot \pi_*\mathcal{E}_i > 0$, where the last inequality is because $\mathcal{E}_i$ is not contracted by $\pi$ and $K_X + B$ is ample. \hfill \Box

**Claim 6.5.** We have $d \geq \max\{1, p_g - 1\}$

*Proof of Claim 6.5.* It is clear that $d = (K_Z + B'_Z)\cdot B_{Z,0} \geq 1$. In case $p_g \geq 2$, $K_Z + B'_Z - (p_g - 1)F$ is numerically equivalent to some effective vertical curve, and hence $d = (K_Z + B'_Z)\cdot B_{Z,0} \geq (p_g - 1)F\cdot B_{Z,0} \geq p_g - 1$. \hfill \Box

In view of Claim 6.3, the following Claim 6.6 follows from Claims 6.4 and 6.5, together with the fact that $b - b^2 > 0$.

**Claim 6.6.** We have $\text{vol}(K\cdot B_X^s) \geq (2b - b^2)\max\{1, p_g - 1\} + b^2(2p_a(\tilde{B}_0) - 2)$ with equality holds if and only if $d = \max\{1, p_g - 1\}$, and $n = 0$, that is, $\rho: \tilde{X} \to Z$ is an isomorphism.

If $p_a(\tilde{B}_0) > 0$ then $b = c < 1$, and hence $\text{vol}(K_X + B) \geq (2c - c^2)\max\{1, p_g - 1\}$ by Claim 6.6. In particular, in this case $\text{vol}(K_X + B)$ is strictly larger than the lower bounds given in Proposition 6.2.

**Step 4.** From now on, we can assume that $p_a(\tilde{B}) = 0$. Under this assumption, we have

\begin{equation}
b = \min\left\{c, \frac{\tilde{B}_0^2 + 2}{\tilde{B}_0^2}\right\} = \min\left\{c, \frac{m + \sum_i \mu_i}{2 + \sum_i \mu_i}\right\}
\end{equation}
and $b = \frac{d + \sum \mu_i}{2 + d + \sum \mu_i}$ if and only if $\tilde{B}_0$ is contracted by $\pi: \tilde{X} \to X$. By Claim 6.5, we have $d \geq \max\{1, p_g - 1\}$ and hence

\begin{equation}
\frac{d + \sum \mu_i}{2 + d + \sum \mu_i} \geq \frac{d}{2 + d} \geq \frac{\max\{1, p_g - 1\}}{\max\{1, p_g - 1\} + 2}
\end{equation}

with equality if and only if $n = 0$ and $d = \max\{1, p_g - 1\}$.

(i) Suppose that $p_g \geq 2$. In this case, we have $\max\{1, p_g - 1\} = p_g - 1$. We divide the discussion according to the value of $c$.

(i.1) If $c < \frac{p_g - 1}{p_g + 1}$ then by (6.5)

\[ \frac{d + \sum \mu_i}{2 + d + \sum \mu_i} \geq \frac{p_g - 1}{p_g + 1} > c. \]

It follows from (6.4) that $b = c$, and by Claim 6.6 we have

\begin{equation}
\vol(K_X + B) \geq (2c - c^2)d - 2c^2 \geq (2c - c^2)\max\{1, p_g - 1\} - 2c^2
\end{equation}

with equalities if and only if $n = 0$ and $d = \max\{1, p_g - 1\}$.

(i.2) If $c \geq \frac{p_g - 1}{p_g + 1}$ then by (6.4) and (6.5) we have

\[ b = \min\left\{ c, \frac{d + \sum \mu_i}{2 + d + \sum \mu_i} \right\} \geq \min\left\{ \frac{p_g - 1}{p_g + 1}, \frac{d + \sum \mu_i}{2 + d + \sum \mu_i} \right\} = \frac{p_g - 1}{p_g + 1}\]

with equalities if and only if $n = 0$ and $d = p_g - 1$. By Claim 6.6 we have

\[ \vol(K_X + B) \geq (2b - b^2)m - 2b^2 \geq (2b - b^2)(p_g - 1) - 2b^2 \geq p_g - 3 + \frac{4}{p_g + 1}\]

with equalities if and only if $n = 0$, $b = \frac{p_g - 1}{p_g + 1}$, and $d = p_g - 1$.

(ii) Suppose that $p_g \leq 1$. Again, we divide the discussion according to the value of $c$.

(ii.1) If $c < \frac{1}{3}$, then

\[ \frac{d + \sum \mu_i}{2 + d + \sum \mu_i} \geq \frac{d}{2 + d} \geq \frac{1}{3} > c. \]

It follows from (6.4) that $b = c$, and by Claim 6.6 we have

\begin{equation}
\vol(K_X + B) \geq (2c - c^2)d - 2c^2 \geq (2c - c^2) - 2c^2 = 2c - 3c^2
\end{equation}

where equalities hold if and only if $n = 0$ and $d = 1$.

(ii.2) If $c \geq \frac{1}{3}$, then by (6.4) and (6.5) we have

\[ b \geq \min\left\{ c, \frac{d + \sum \mu_i}{2 + d + \sum \mu_i} \right\} \geq \min\left\{ \frac{1}{3}, \frac{d + \sum \mu_i}{2 + d + \sum \mu_i} \right\} = \frac{1}{3}\]

with equalities if and only if $n = 0$ and $d = 1$. By Claim 6.6 we have

\[ \vol(K_X + B) \geq (2b - b^2)d - 2b^2 \geq (2b - b^2) - 2b^2 \geq \frac{1}{3}\]

with equalities if and only if $n = 0$, $d = 1$, and $b = \frac{1}{3}$.

Obviously, the lower bounds obtained in Step 4 (when $p_a(\tilde{B}_0) = 0$) is smaller than the one obtained in Step 3 (when $p_a(\tilde{B}_0) > 0$). The proof of Proposition 6.2 is now complete. \(\square\)

**Corollary 6.7.** For any projective log canonical surface of general type $(X, B)$, we have

\begin{equation}
\vol(K_X + B) \geq (2c - c^2)p_g(X, B) - (2c + c^2)
\end{equation}

where $c := \min(C_B \cup \{1\})$. 
Proof. If \( p_g := p_g(X, B) \geq 2 \) then \((X, B) \in S(C, p_g; \kappa)\) with \( \kappa \in \{1, 2\} \). One checks directly that the following strict inequality holds

\[
(2c - c^2)(p_g - 1) - 2c^2 < p_g - 3 + \frac{4}{p_g + 1} \tag{6.9}
\]

Thus the lemma holds in this case by Propositions 6.1 and 6.2. If \( p_g \leq 1 \), then the right hand side of (6.8) is negative, so the inequality also holds. \( \square \)

6.3. The lower bound of \( \kappa^2(C, p_g; 0) \).

**Proposition 6.8.** For any \((X, B) \in S(C, p_g; 0)\), we have

\[
\text{vol}(K_X + B) \geq \begin{cases} 
\frac{1}{14} c^2 & \text{if } c \leq \frac{7}{11} \\
\frac{11}{6} c^2 - 2c - \frac{7}{11} & \text{if } \frac{7}{11} < c \leq \frac{6}{11} \\
\frac{6}{11} c & \text{if } c > \frac{6}{11}
\end{cases}
\tag{6.10}
\]

**Proof.** **Step 0.** Suppose that \((X, B) \in S(C, p_g; 0)\) has ample \( K_X + B \) and \( \text{vol}(K_X + B) = \min \kappa^2(C, p_g; 0) \). It suffices to show the inequalities of the proposition for this log surface. Then \((X, B)\) and the minimal resolution \( \pi: (\tilde{X}, \tilde{B}) \rightarrow (X, B) \) are subject to the restrictions of Proposition 5.6. In particular, we have \( B^\text{ns}_{\tilde{X}} = (B^\text{ns}_X)_{<c} \). Also, observe that \((B^\text{ns}_X)_{<c}\) is necessarily contracted by \( \pi: \tilde{X} \rightarrow X \). Therefore,

\[
\text{vol}(K_X + B) = \text{vol}(K_{\tilde{X}} + B_{\tilde{X}}) = \text{vol}(K_{\tilde{X}} + B^c_{\tilde{X}} + B^\text{ns}_{\tilde{X}}) = \text{vol}(K_{\tilde{X}} + B^c_{\tilde{X}} + c[B^\text{nt}_{\tilde{X}}]).
\]

**Step 1.** In this step, we pass to the \((K_{\tilde{X}} + B^c_{\tilde{X}})\)-minimal model \( \rho: (\tilde{X}, B^c_{\tilde{X}}) \rightarrow (Z, B^c_Z) \) as in Notation 5.1. Then \( Z, B^c_Z := \rho_* B^c_{\tilde{X}} \) and \( B^c_Z + B''_Z = \rho_* (K_{\tilde{X}} + B_{\tilde{X}}) \) is big. Using Notation 3.2, we have \((X, B^c_Z + c[B^\text{nt}_Z]) \in S(Z, B^c_Z + B''_Z)\) and \( \text{vol}(K_X + B) \in \kappa^2(Z, B^c_Z + B''_Z) \), where \( B^c_Z = B^c_Z + c[B''_Z] \). Hence it suffices to bound the set \( \kappa^2(Z, B^c_Z + B''_Z) \) from below.

**Step 2.** In this step we prove that \( \text{supp}(B''_Z) \) is connected. Otherwise, we can write \( B''_Z = D_1 + D_2 \) with \( D_1 \) and \( D_2 \) two effective \( \mathbb{R} \)-divisors with disjoint supports. Let \( P_1 \leq D_1 \) and \( P_2 \leq D_2 \) be the positive parts. Then \( P = P_1 + P_2 \) is the positive part of \( B''_Z \). Since \( P_1, P_2 = 0 \) and \( B''_Z \) is big, we have \( 0 < P^2 = P_1^2 + P_2^2 \). So either \( P_1^2 > 0 \) or \( P_2^2 > 0 \). It follows that either \( D_1 \) or \( D_2 \) is big, contradicting Proposition 5.6 (iii).

**Step 2.** In this step, we assume that \( p_a([B''_Z]) \geq 2 \). By the adjunction formula \( [B''_Z]^2 \geq 2 \).

**Claim 6.9.** There is no smooth rational component \( D \) of \( [B''_Z] \) such that \( D \cdot ([B''_Z] - D) = 1 \).

**Proof of Claim 6.9.** Otherwise \( D^2 = -2 \) and \( ([B''_Z] - D)^2 = [B''_Z]^2 > 0 \), so \( [B''_Z] - D \) is big, contradicting Proposition 5.6 (iii). \( \square \)

Let \( \mu \) be the maximal multiplicity of \( [B''_Z] \) at a point. By Lemma 3.1, we have

\[
K_{\tilde{X}} + B^c_{\tilde{X}} + c[B^\text{nt}_{\tilde{X}}] \geq \rho^*(K_Z + B_Z) + \frac{1}{\mu} \rho^*(c[B''_Z]) \equiv \frac{1}{\mu} \rho^*[B''_Z], \tag{6.11}
\]

and hence by (6.10)

\[
\text{vol}(K_X + B) = \text{vol}(K_{\tilde{X}} + B^c_{\tilde{X}} + c[B^\text{nt}_{\tilde{X}}]) \geq \frac{1}{\mu^2} \text{vol}([B''_Z]). \tag{6.12}
\]
If \( \mu \leq 3 \), then by (6.12) we have
\[
\text{vol}(K_X + B) \geq \frac{1}{9} \text{vol}(\lceil B'_Z \rceil) \geq \frac{1}{9} \lceil B'_Z \rceil^2 \geq \frac{2}{9},
\]
where the second inequality is because of Lemma 2.5 (i).

Now suppose that \( \mu \geq 4 \). Let \( p \in \lceil B'_Z \rceil \) be a point of multiplicity \( \mu \) and \( \tilde{Z} \to Z \) the blow-up of \( p \). Let \( \mathcal{B}'_Z \) be the strict transform of \( B'_Z \). Then \( \mathcal{B}'_Z \) has at most \( \mu \) connected components, and we have
\[
p_a([B''_Z]) = p_a([\mathcal{B}'_Z]) + \frac{\mu(\mu - 1)}{2} \geq 1 - \mu + \frac{\mu(\mu - 1)}{2} = \frac{(\mu - 1)(\mu - 2)}{2}.
\]
It follows that
\[
\text{vol}([B''_Z]) \geq \lceil B'_Z \rceil^2 = K_Z \lfloor B'_Z \rceil + \lceil B'_Z \rceil^2 = 2p_a([B''_Z]) - 2 \geq \mu(\mu - 3),
\]
where the first inequality is by Lemma 2.5 (i), so by (6.12) we have
\[
\text{vol}(K_X + B) \geq \frac{1}{9} \text{vol}([B'_Z]) \geq 1 - \frac{3}{\mu} \geq 1 - \frac{1}{4}.
\]

**Step 3.** In this step we assume that \( p_a([B''_Z]) \leq 1 \). Then every connected subcurve of \([B''_Z]\) has arithmetic genus at most 1; each smooth rational component of \([B''_Z]\), if present, has zero intersection number of \( K_Z \) and hence should be a \((-2)\)-curve. It is then not hard to see that, the assumption \( p_a([B''_Z]) \leq 1 \) together with Proposition 5.6 implies that \([B''_Z] = C + D\) where \( C \) supports a curve of canonical type and \( D \) is a \((-2)\)-curve such that \( C \cdot D = 1 \). By the explicit computation done in Section 3.2, the minimal possible volume occurs when \( C \) supports a curve of type II*, and it serves as the uniform lower bound of Proposition 6.8. \( \square \)

Combining Propositions 6.1, 6.2, and 6.8, we obtain

**Theorem 6.10.** Let \( \mathcal{C} \subset (0, 1]\) be a subset such that \( \mathcal{C} \cup \{1\} \) attains the minimum, say \( c \), and let \( p_g \) be a positive integer. Then the following holds for any \((X, B) \in \mathcal{S}(\mathcal{C}, p_g)\).

(i) If \( p_g \geq 2 \), then
\[
\text{vol}(K_X + B) \geq \begin{cases} (2c - c^2)(p_g - 1) - 2c^2 & \text{if } c < \frac{p_g - 1}{p_g + 1} \\ p_g - 3 + \frac{4}{p_g + 1} & \text{if } c \geq \frac{p_g - 1}{p_g + 1} \end{cases}
\]

(ii) If \( p_g = 1 \), then
\[
\text{vol}(K_X + B) \geq \begin{cases} \frac{1}{13}c^2 & \text{if } c \leq \frac{7}{13} \\ -\frac{11}{6}c^2 + 2c - \frac{7}{13} & \text{if } \frac{7}{13} < c \leq \frac{6}{13} \\ \frac{1}{13}c & \text{if } c > \frac{6}{13} \end{cases}
\]

In Section 7 we will show that Theorem 6.10 is sharp in that the equalities can be achieved for any given \( p_g > 0 \).

The next proposition gives practical criteria for a projective log canonical surface \((X, B)\) to lie in \( \mathcal{S}(\mathcal{C}, p_g; \kappa) \) with \( \kappa \geq 0 \).

**Proposition 6.11.** Let \((X, B)\) be a smooth projective log canonical surface of general type, and let \( B = B^* + B^\text{as} \) be the semistable decomposition. Suppose that one of the following holds:

(i) \( q(X) > 0 \) and either \( \text{supp}(B^\text{as}) \) is empty or it consists of rational curves.

(ii) \( q(X) = 0 \) and \( B^* \neq 0 \).

Then \( \kappa(K_X + B^*) \geq 0 \). Moreover, in case (ii), we have \( p_g(X, B) > 0 \).
Proof. (i) Suppose that \( q(X) > 0 \) and \( \text{supp}(B^{\text{ns}}) \) consists of rational curves. Suppose on the contrary that \( \kappa(K_X + B^e) = -\infty \). Then \( X \) is birational to a ruled surface and the Albanese map \( \alpha : X \to A \) is a \( \mathbb{P}^1 \)-fibration. Any minimal model program on \( X \) contracts only rational curves, which are necessarily contained in fibers of \( \alpha \); in other words, it is over \( A \). Since \( \kappa(K_X + B^e) = -\infty \), we have \( (K_X + B^e) \cdot F < 0 \) for a general fiber \( F \) of \( \alpha \). On the other hand, since \( B^{\text{ns}} \) consists of rational curves, it is contracted by \( \alpha \) and thus \( B^{\text{ns}} \cdot F = 0 \). Due to the bigness of \( K_X + B^e + B^{\text{ns}} \), we have \( (K_X + B^e) \cdot F = (K_X + B^e + B^{\text{ns}}) \cdot F > 0 \), which is a contradiction.

(ii) Now suppose that \( q(X) = 0 \) and \( B^e \neq 0 \). Then \( h^0(B^e, K_{X'}) > 0 \) because every connected component of \( B^e \) has positive arithmetic genus. A portion of the long exact sequence associated to the short exact sequence \( 0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(K_X + B^e) \to \mathcal{O}_{K_{X'}}(K_{X'}) \to 0 \) reads

\[
H^0(X, K_X + B^e) \to H^0(B^e, K_{X'}) \to H^1(X, K_X)
\]

Since \( q(X) = 0 \), we have \( H^1(X, K_X) = 0 \) and thus the first map of (6.13) is surjective. It follows that \( H^0(X, K_X + B^e) \neq 0 \) and hence \( \kappa(K_X + B^e) \geq 0 \).

Corollary 6.12. Suppose that \( (X, B) \in S(\{1\}, p_g) \) has \( \text{vol}(K_X + B) < \frac{1}{143} \). Then \( \tilde{X} \) is a rational surface and \( B^e_{\tilde{X}} = 0 \), where \( \pi : (\tilde{X}, B_{\tilde{X}}) \to (X, B) \) is the minimal resolution.

Proof. Since \( \text{vol}(K_X + B) < \frac{1}{143} \), we have \( \kappa(K_{\tilde{X}} + B^e_{\tilde{X}}) = -\infty \) by Theorem 1.1 applied to the case \( \mathcal{C} = \{1\} \). Note that the non-rational components of \( B_{\tilde{X}} \) are necessarily contained in the semistable part \( B^e_{\tilde{X}} \), so \( B^e_{\tilde{X}} \) consists of rational curves. By Proposition 6.11 (i), applied to \( (\tilde{X}, B_{\tilde{X}}) \), we have \( q(\tilde{X}) = 0 \) and \( B^e_{\tilde{X}} = 0 \). It follows that \( \tilde{X} \) is a rational surface.

7. Finding the minimal volumes

Given a subset \( \mathcal{C} \subset \{0, 1\} \) such that \( \min(\mathcal{C} \cup \{1\}) = c \), a positive integer \( p_g \), and \( \kappa \in \{0, 1, 2\} \), we have obtained low bounds of \( \mathbb{K}^2(\mathcal{C}, p_g; \kappa) \) in Section 6. We construct in this section projective log surfaces \( (X, B) \in S(C, p_g; \kappa) \) realizing the low bounds in most cases. In particular, those with \( \text{vol}(K_X + B) = \min \mathbb{K}^2(\mathcal{C}, p_g; \kappa) \) are presented when \( p_g > 0 \).

7.1. Log surfaces in \( S(\{1\}, p_g; 2) \) achieving the minimal volume. The following theorem of Sakai gives a characterization of the equality case of Proposition 6.1. Note that the proof of [25, Theorem 6.7] works in all characteristic.

Theorem 7.1 ([25], Theorem 6.7). If \( (X, B) \in S(\{1\}, p_g; 2) \) has ample \( K_X + B \) and \( \text{vol}(K_X + B) = p_g - 2 \), then the minimal resolution \( (\tilde{X}, B_{\tilde{X}}) \) of \( (X, B) \) is one of the following:

(i) \( \mathbb{P}^2 \), nodal quartic curve,
(ii) \( \mathbb{P}^2 \), nodal quintic curve,
(iii) \( \tilde{X} = \mathbb{F}_e, B_{\tilde{X}} \) a nodal curve in \( |3\Gamma_0 + (2e + k + 2)F| \) with \( k \geq 1, e \geq 0 \),
(iv) \( \tilde{X} = \mathbb{F}_e, B_{\tilde{X}} \) a nodal curve in \( |3\Gamma_0 + (2e + 2)F| \) with \( e > 0 \),

where \( \mathbb{F}_e = \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-e)) \to \mathbb{P}^1 \) denotes a Hirzebruch surface with fiber \( F \) and a section \( \Gamma_0 \) such that \( \Gamma_0^2 = -e \).

Corollary 7.2. \( \min \mathbb{K}^2(\mathcal{C}, p_g; 2) = \max\{1, p_g - 2\} \) if and only if either \( 0 \leq p_g \leq 2 \) or \( 1 \in \mathcal{C} \).

Proof. One can find smooth projective surfaces of general type with \( 0 \leq p_g \leq 2 \) and \( K_X^2 = 1 \) (see [6, 7]). If \( p_g \geq 3 \), Theorem 7.1 provides examples \( (X, B) \in S(\{1\}, p_g; 2) \) attaining the lower bound of Proposition 6.1, that is, \( \text{vol}(K_X + B) = p_g - 2 \); the same theorem shows that the inequality of Proposition 6.1 is strict if \( 1 \notin \mathcal{C} \).
7.2. Log surfaces in $S(\mathcal{C}, p_g; 1)$ achieving the minimal volume. In this subsection, we construct for each $p_g > 0$ a log surface $(X, B) \in S(\mathcal{C}, p_g; 1)$ realizing the equality of Proposition 6.2. By the proof of Proposition 6.2, we need to construct a smooth projective log canonical surface $(Z, B')$ such that

(i) $B'$ is a (reduced) semistable curve such that $\kappa(K - B') = 1$ and $K + B'$ is nef,

(ii) $B$ is a smooth rational curve with $K + D = \max\{p_g - 1, 1\}$, and

(iii) $\text{supp}(B') \cap \text{supp}(D) = \emptyset$.

Then, setting $B'' = B' + cD$, $(X, B)$ can be taken as the ample model of $(Z, B'')$.

Example 7.3. Let $p_g \geq 2$ be an integer. Let $h: S \to \mathbb{P}^1$ be a relatively minimal rational elliptic surface with a section $\Gamma$. Then by the canonical bundle formula, we can take $K_S = -F$, where $F$ is a fiber of $f$. Set $B' = \sum_{1 \leq i \leq p_g} F_i$, where $F_1, \ldots, F_{p_g}$ are the general fibers of $h$. Then $K_S + B' \sim (p_g - 1)F$.

Let $p_i$ be the intersection point of $\Gamma$ and $F_i$ for $1 \leq i \leq p_g$, and let $p: Z \to S$ be the blow-up of $\{p_1\} \leq i \leq p_g$. Let $B'$ and $D$ be the strict transforms of $B'$ and $\Gamma$ on $Z$, respectively, and set $B'' = B' + cD$. Then $B'$ is the semistable part of $B''$, $K + B' \sim (p_g - 1)f*t$ for $t \in \mathbb{P}^1$, and $K + B''$ is big for $c = \min(K \cup \{1\})$. Thus $p_g(Z, B''_g) = p_g$.

Let $\pi: (S', B'(c)) \mapsto (X(c), B'(c))$ be the contraction to the ample model. Then $B''$ is contracted by $\pi$, and $D$ is contracted by $\pi$ if and only if $c \geq \frac{p_g - 1}{p_g + 1}$. It follows that $B'' = c\pi_* D$ and hence $(X(c), B'(c)) \in S(\{c\} < 1, p_g; 1)$. The volume is exactly the lower bound of Proposition 6.2:

$$\text{vol} \left( K_{X(c)} + B'(c) \right) = \text{vol} \left( K + B'' \right) = \begin{cases} (2c - c^2)(p_g - 1) - 2c^2 & \text{if } c < \frac{p_g - 1}{p_g + 1} \\ p_g - 3 + \frac{4}{p_g + 1} & \text{if } c \geq \frac{p_g - 1}{p_g + 1} \end{cases}$$

Example 7.4. Let $p_g \geq 2$ be an integer. Take $(Z, B''_g) = (\mathbb{P}_e, \Gamma + e\Gamma_0)$, where $e = p_g - 1$, $\mathbb{P}_e$ is the Hirzebruch surface with a section $\Gamma_0$ such that $\Gamma_0 = -e$, and $\Gamma \in |2(\Gamma_0 + eF)|$ is a general element. Then the ample model $(X(c), B'(c))$ lies in $S(\{c\} < 1, p_g; 1)$ and $\text{vol}(K_{X(c)} + B'(c))$ achieves the lower bound of Proposition 6.2. In this example, we have $(B''(c))^c \neq 0$.

Example 7.5. Let $S$ be a (classical) Enriques surface and $h: S \to \mathbb{P}^1$ an elliptic fibration. Then $h$ has two double fibers, say $2F_1$ and $2F_2$, and a double section $\Gamma$; see [8, Theorem 5.7.2], we have $K_S \sim F_2 - F_1$. Note that the double fibers are necessarily of type $2h_i, i \in \{1, 2\}$, and hence the reduced parts $F_i$ have at most nodes as singularities, and the curves $\Gamma$ and $F_i$ intersect transversally at one point, say $p_i$. Let $p: Z \to S$ be the blow-up of $p_i$. Then $B''_i$ and $D$ be the strict transforms of $F_i$ and $\Gamma$ respectively. Then $K_S + B''_i = p^*(K_S + F_i) \sim \rho^* F_2$ has Iitaka–Kodaira dimension 1. Moreover, we have

$$B''_i \cap D = \emptyset, \quad K_S \cdot D = 1, \quad p_g(Z, B''_i) = 1.$$

Let $B''_i = B''_i + cD$ and $\pi: (Z, B''_i) \mapsto (X(c), B'(c))$ the contraction to the ample model. Then, as in Example 7.8, we have $(X(c), B'(c)) \in S(\{c\} < 1, 1; 1)$. One computes easily that

$$\text{vol} \left( K_{X(c)} + B'(c) \right) = \text{vol} \left( K + B'' \right) = \begin{cases} 2c - 3c^2 & \text{if } c \leq \frac{1}{4} \\ \frac{1}{4} & \text{if } c > \frac{1}{4} \end{cases}$$

As a consequence of Proposition 6.2 and the above examples, we obtain

Theorem 7.6. Let $\mathcal{C} \subset (0, 1]$ be a subset such that $\mathcal{C} \subset (0, 1]$ attains the minimum, say $c$. Then the following holds.

For $p_g \geq 2$, we have

$$\min K^2(C, p_g; 1) = \min K^2(\{c\} < 1, p_g; 1) = \begin{cases} (2c - c^2)(p_g - 1) - 2c^2 & \text{if } c < \frac{p_g - 1}{p_g + 1} \\ p_g - 3 + \frac{4}{p_g + 1} & \text{if } c \geq \frac{p_g - 1}{p_g + 1} \end{cases}$$

For $p_g = 1$, we have

$$\min K^2(C, 1; 1) = \min K^2(\{c\} < 1, p_g; 1) = \begin{cases} 2c - 3c^2 & \text{if } c < \frac{1}{3} \\ \frac{1}{4} & \text{if } c \geq \frac{1}{3} \end{cases}$$

**Corollary 7.7.** The inequality of Corollary 6.8 is optimal in the following sense:

(i) If $c < 1$, then the equality can be attained for $(X, B)$ with $p_g(X, B) \geq \frac{1}{1+c}.$

(ii) If $c = 1$, then the inequality is strict, but there is a sequence of projective log canonical surfaces $X_i$ of general type such that

$$\lim_{i \to \infty} \frac{\text{Vol}(K_{X_i}) - p_g(X_i) + 3}{X_i} = 0$$

7.3. **Log surfaces in $S(C, p_g; 0)$ achieving the minimal volume.** In this subsection, we construct for a log surface $(X, B) \in S(C, 1; 0)$ realizing the equality of Proposition 6.8. If the characteristic of the base field $k$ is 2, the same can be done when $p_g = 0.$

The point is to exhibit a smooth projective log canonical surface $(Z, B'_Z)$ together with a simple normal crossing curve $C + D$ such that

(i) $B'_Z$ is a (reduced) semistable curve such that $K_Z + B'_Z \sim_{Q} 0.$

(ii) $C = \sum_{1 \leq i \leq 9} C_i$ supports a curve of canonical type $\Pi^*$ and $D$ is a $(-2)$-curve intersecting $C$ transversally at exactly one point, so that the dual graphs is

```
   D  C1  C2  C3  C4  C5  C6  C7  C8  C9
```

(iii) $\text{supp}(B'_Z) \cap \text{supp}(C + D) = \emptyset$

Let $\rho : W \to Z$ be the blow up of all nodes of $C + D$ and $B'_W^{(c)} = \rho^{-1}B'_Z + c\rho^{-1}(C + D).$ Then the ample model $(X'^{(c)}, B'^{(c)})$ of $(W, B'^{(c)})$ lies in $S(\{c\}, p_g; 0)$ and $\text{Vol}(K_X + B) = \min K^2(\{c\}, p_g; 0),$ where $p_g = p_g(Z, B'_Z) \in \{0, 1\};$ see Section 3.2. In Examples 7.8, 7.9 and 7.10, the coefficient set of $B'^{(c)}$ is as follows, depending on the value of $c$:

$$C_{B'^{(c)}} = \begin{cases} \{c\} & \text{if } c < \frac{7}{13} \\ \emptyset & \text{if } c \geq \frac{7}{13} \end{cases}$$

**Example 7.8.** Let $Z$ be a K3 surface, $C + D$ the reduced simple normal crossing divisor, consisting of $(-2)$-curves with dual graph as above (see [27] for the existence of such a K3 surface, at least in characteristic zero). It suffices to take $B'_Z = 0,$ and we have $p_g(Z, B'_Z) = p_g(Z) = 1$ in this case.

**Example 7.9.** Let $A$ be a nodal cubic curve on $\mathbb{P}^2$ with at most nodes as singularities and a line $L$ intersecting $A$ transversally at 3 points. We blow up the intersection points $A \cap L$ and their infinitely near points on the strict transforms of $A$ successively to arrive at the following configuration of curves
where $A_Z$ and $L_Z$ are the strict transforms of $A$ and $L$ respectively, the white bullets denote $(-2)$-curves, and the black bullets denote $(-1)$-curves. We take $B'_Z = A_Z$, $C + D$ the sum of all the $(-2)$-curves visible in the dual graph, and we have $p_g(Z, B'_Z) = 1$ in this case.

**Example 7.10.** Suppose that char $k = 2$. Let $Z$ be a classical/supersingular Enriques surface of type $\tilde{E}_8$ ([26]), so there is a configuration $C + D$ of $(-2)$-curves as above. We take $B'_Z = 0$, and we have

$$p_g(Z, B'_Z) = p_g(Z) = \begin{cases} 0 & \text{if } Z \text{ is classical} \\ 1 & \text{if } Z \text{ is supersingular} \end{cases}$$

By Proposition 6.8 and the above examples, we obtain

**Theorem 7.11.** Let $C \subset (0, 1]$ be a subset such that $C \subset (0, 1]$ attains the minimum, say $c$. Then

$$\min K^2(C, 1; 0) = \min K^2(\{c\}, 1; 0) = \begin{cases} \frac{1}{127}c^2 & \text{if } c \leq \frac{7}{13} \\ -\frac{11}{8}c^2 + 2c - \frac{7}{13} & \text{if } \frac{7}{13} < c \leq \frac{6}{11} \\ \frac{1}{143} & \text{if } c > \frac{6}{11} \end{cases}$$

We are going to address the sharpness of the inequality in Proposition 6.8 in the case $p_g = 0$. First, a simple lemma on the algebraic fundamental group.

**Lemma 7.12.** Let $Z$ be a smooth projective surface and $C$ a big connected curve on $Z$. If the algebraic fundamental group $\pi_1^{alg}(C)$ is trivial, then so is $\pi_1^{alg}(S)$.

**Proof.** Suppose on the contrary that there is an étale cover $f: \tilde{Z} \to Z$ of degree $d > 1$. Then $f^*C = \bigcup_{1 \leq i \leq d} \tilde{C}_i$, where the $\tilde{C}_i$’s are pairwise disjoint and $\tilde{C}_i \cong C$. Since each $C_i$ is big, this contradicts the Hodge index theorem. $\square$

**Proposition 7.13.** Let $C \subset (0, 1]$ be a subset such that $C \cup \{1\}$ attains the minimum, say $c$. Let $(X, B) \in S(C, p_g; 0)$ be such that $K_X + B$ is ample and $\text{vol}(K_X + B) = \min K^2(C, 1; 0)$. Then the minimal resolution $\pi: (\tilde{X}, B_{\tilde{X}}) \to (X, B)$ has the following characterization:

(i) $\tilde{X}$ is birationally a $K3$ surface, or a rational surface, or in case char $k = 2$, $\tilde{X}$ is a classical/supersingular Enriques surface. In particular, the algebraic fundamental group $\pi_1^{alg}(\tilde{X})$ is trivial and if $k \cong \mathbb{C}$ then the topological fundamental group $\pi_1(\tilde{X})$ is trivial.

(ii) The semistable part $B^s_X$ and the non-semistable part $B^{ns}_X$ of $B_X$ are disjoint.

(iii) if $c \leq \frac{7}{13}$ then $B^{ns}_X$ a simple normal crossing divisor consisting of $(-2)$-curves, and its dual graph is

(iv) if $c > \frac{7}{13}$ then $B^{ns}_X$ is obtained by blowing up the configuration of curves in (ii), so there is exactly one $(-1)$-curve, say $E$, not intersecting $B^s_X$, and the dual graph of $B^{ns}_X + E$ is
where the two white bullets adjacent to $E$ represent $(-3)$-curves and all the other white bullets are $(-2)$-curves.

(v) If $c \geq \frac{7}{11}$ then $B = 0$, and if $c < \frac{7}{11}$ then $B = cB_0$ where $B_0$ is the image of the $(-3)$-curve to the right of $E$ in the dual graph of (ii).

Proof. The statements (ii)-(iv) follow from the proof of Proposition 6.8 and the computation of (3.9) in Section 3.2. (v) can be seen by finding the Zariski decomposition on $K_{\tilde{X}} + B_{\tilde{X}} + c[B^{ns}_{\tilde{X}}]$.

For (i), consider the minimal model $\rho: (\tilde{X}, B_{\tilde{X}}) \to (Z, B_0)$ of $K_{\tilde{X}} + B_{\tilde{X}}$. Then $(Z, B_0)$ is a smooth projective log canonical surface as in the beginning of this subsection. In particular, there is a big connected curve $C + D$ such that $K_Z \cdot (C + D) = 0$. It follows that $\kappa(Z) \leq 0$. Moreover, $\pi_1^{alg}(S)$ is trivial by Lemma 7.12. By the Enriques–Kodaira classification of algebraic surfaces, if $\text{char } \mathbb{C}/\mathbb{Z} \neq 2$, then $\tilde{X}$ is birational to a K3 surface or a rational surface. If $\text{char } \mathbb{C}/\mathbb{Z} = 2$, then $\tilde{X}$ can also be a classical/supersingular Enriques surface, which does not admit any nontrivial étale cover. □

**Corollary 7.14.** Let $C \subset (0, 1]$ be a subset such that $C \subset (0, 1]$ attains the minimum, say $c$. Then

$$\min k^2(C, 0; 0) \geq \begin{cases} \frac{1}{12}c^2 & \text{if } c \leq \frac{7}{11} \\ \frac{11}{13} + 2c - \frac{7}{11} & \text{if } \frac{7}{11} < c \leq \frac{6}{11} \\ \frac{1}{11} & \text{if } c > \frac{6}{11} \end{cases}$$

and equality holds if and only if $\text{char } \mathbb{C}/\mathbb{Z} = 2$.

Proof. The inequality follows from Proposition 6.2. By Example 7.10 the equality holds if $\text{char } \mathbb{C} = 2$. Now suppose that $\text{char } \mathbb{C} \neq 2$ and $(X, B) \in S(C, 0; 0)$ realizes the equality. Then by Proposition 7.13 (i) the minimal resolution $(\tilde{X}, B_{\tilde{X}})$ of $(X, B)$ is birationally a K3 surface or a rational surface. In both cases, the fact that $\kappa(K_{\tilde{X}} + B_{\tilde{X}}) = 0$ implies that $p_g = p_g(K_{\tilde{X}}, B_{\tilde{X}}) = 1$, which is a contradiction to the assumption that $p_g = 0$. In other words, the inequality is strict if $\text{char } \mathbb{C} \neq 2$. □

7.4. **Proofs of the main results.** We conclude this section by giving the proofs of the main results presented in the introduction.

**Proof of Theorem 1.1.** Combine Lemma 5.5 (ii), Theorems 6.10, 7.6, and 7.11. □

**Proof of Theorem 1.2.** This is just a reformulation of Theorem 1.1. □

**Proof of Theorem 1.3.** Combine Corollaries 6.7 and 7.7. □

8. **Applications**

In this section, we work over an algebraically closed field $\mathbb{C}$ of characteristic 0. We apply the main results to several closely related problems.
8.1. **A Noether type inequality for log canonical threefolds.** Given \( c \in (0, 1) \) and a positive integer \( n \), [16] proved a Noether type inequality of the form \( \text{vol}(K_X + B) \geq a_n(c) p_g(X, B) - b_n(c) \) for any \( n \)-dimensional projective log canonical pairs \((X, B)\) such that \( K_X + B \) is big and \( \min(C_B \cup \{1\}) = c \), where \( a_n(c) \) and \( b_n(c) \) are positive constants depending on \( c \). For example, we have the optimal value: \( a_1(c) = b_1(c) = 1 \). By Corollary 6.7, we have the optimal values: \( a_2(c) = 2c - c^2 \) and \( b_2(c) = 2c + c^2 \). Using this, we can also make \( a_3(c) \) and \( b_3(c) \) explicit as follows:

**Theorem 8.1.** Let \((X, B)\) be a projective log canonical threefold such that \( K_X + B \) is big. Denote by \( C_B \) the coefficient set of \( B \) and \( c = \min(C_B \cup \{1\}) \). Then

\[
\text{vol}(K_X + B) \geq a_3(c) \cdot p_g(X, B) - b_3(c)
\]

where \( b_3(c) = c \) and \( a_3(c) \) is specified as follows:

\[
a(c) = \begin{cases} 
\frac{1}{168} c^2, & \text{if } c \leq \frac{7}{13} \\
\frac{1}{24} c^2 + \frac{1}{2} c - \frac{7}{52}, & \text{if } \frac{7}{13} < c \leq \frac{6}{11} \\
\frac{1}{52}, & \text{if } c > \frac{6}{11}
\end{cases}
\]

**Proof.** Let \( v^+_2(\{c, 1\}) \) be the minimal volume of projective log canonical surface of general type \((S, B_S)\) with coefficient in \( \{c, 1\} \) and \( p_g(S, B_S) > 0 \). Then, by Proposition 6.2, we have

\[
v^+_2(\{c, 1\}) = \begin{cases} 
\frac{1}{12} c^2, & \text{if } c \leq \frac{7}{13} \\
\frac{1}{11} c^2 + 2c - \frac{7}{13}, & \text{if } \frac{7}{13} < c \leq \frac{6}{11} \\
\frac{1}{52}, & \text{if } c > \frac{6}{11}
\end{cases}
\]

Now, by [16, Theorem A.2] and its proof, we have

\[
\text{vol}(K_X + B) \geq a_3(c) p_g(X, B) - b_3(c)
\]

where

\[
a_3(c) = \frac{1}{4} \min\{a_2(c), v^+_2(\{c, 1\})\} = \begin{cases} 
\frac{1}{168} c^2, & \text{if } c \leq \frac{7}{13} \\
\frac{1}{24} c^2 + \frac{1}{2} c - \frac{7}{52}, & \text{if } \frac{7}{13} < c \leq \frac{6}{11} \\
\frac{1}{52}, & \text{if } c > \frac{6}{11}
\end{cases}
\]

\[
b_3(c) = \frac{1}{4} \max\{a_2(c) + b_2(c), v^+_2(\{c, 1\})\} = \frac{1}{4} (a_2(c) + b_2(c)) = c
\]

\[\square\]

8.2. **A Noether type inequality for stable surfaces.** A stable surface is a projective surface with semi-log-canonical singularities whose canonical class is ample. They were introduced by [19] to compactify the moduli spaces of canonical surfaces of general type. It is natural and often possible to extend the results for canonical surfaces of general type to stable surfaces. For example, for a stable surface \( X \), the inequality \( K^2_X > p_g(X) - 3 \) holds if \( X \) is either normal or Gorenstein (see [28, 22] and Corollary 6.7). By Example 8.4 given at the end of this subsection, this is not true for general stable surfaces. Instead, we are able to provide a weaker Noether type inequality that holds for all stable surfaces:

**Theorem 8.2.** Let \( X \) be a stable surface. Then \( K^2_X \geq \frac{1}{113} p_g(X) \).

Before giving the proof of Theorem 8.2, we recall how one computes the canonical volume and the geometric genus of a possibly non-normal stable surface in terms of its normalization. So let \( X \) be a non-normal stable surface and \( \nu: \bar{X} \to X \) the normalization. Let \( D \subset \bar{X} \) the conductor divisor, which is a reduced curve on \( \bar{X} \). Then the generically two-to-one map
The normal surface $\bar{X}$ is often not connected. We can write

$$\bar{X} = \bigcup_{1 \leq i \leq n} \bar{X}_i$$

as the (disjoint) union of its irreducible components. Let $\bar{D}_i$ be the part of $\bar{D}$ on $\bar{X}_i$. Then $\bar{X}_i, \bar{D}_i$ are all (connected) projective log canonical surfaces with ample $K_{\bar{X}_i} + \bar{D}_i$, and we have

$$K_{\bar{X}}^2 = (K_X + \bar{D})^2 = \sum_{1 \leq i \leq n} (K_{\bar{X}_i} + \bar{D}_i)^2. \tag{8.1}$$

The computation of the geometric genus $p_g(X)$ is more subtle. First of all, there is a natural inclusion obtained by pulling back the sections of $\mathcal{O}_X(K_X)$ restricted to the Gorenstein locus of $(X, B)$ and then extending to global sections of $\mathcal{O}_X(K_X + \bar{D})$:

$$\nu^* : H^0(X, K_X) \hookrightarrow H^0(\bar{X}, K_{\bar{X}} + \bar{D}).$$

In particular, we have

$$\sum_{1 \leq i \leq n} p_g(\bar{X}_i, \bar{D}_i) = \dim_k H^0(\bar{X}, K_{\bar{X}} + \bar{D}) \geq \dim_k H^0(X, K_X) = p_g(X). \tag{8.2}$$

The image of the map $\nu^*$ consists of sections whose residue at $\bar{D}_\nu$ is $\tau$-anti-invariant ([17, Proposition 5.8]). Let $\text{res}: H^0(\bar{X}, K_{\bar{X}} + \bar{D}) \rightarrow H^0(\bar{D}_\nu, \text{Diff}_\nu(0))$ be the residue map. Then there is a short exact sequence

$$0 \rightarrow H^0(X, K_X) \rightarrow H^0(\bar{X}, K_{\bar{X}}) \rightarrow \text{Im}(\text{res})^- \rightarrow 0,$$

where $\text{Im}(\text{res})^-$ denotes $\tau$-anti-invariant part of $\text{Im}(\text{res})$. It follows that

$$p_g(X) = p_g(\bar{X}) - \dim \text{Im}(\text{res})^-$$

In particular, if $\text{Im}(\text{res}) = 0$, that is, if all of the global sections of $K_{\bar{X}} + \bar{D}$ vanish along $\bar{D}$, then we have

$$p_g(X) = p_g(\bar{X}) = \sum_i p_g(\bar{X}_i). \tag{8.3}$$

Now we can turn to

**Proof of Theorem 8.2.** Let $\bar{X} \rightarrow X$ be the normalization and $\bar{D} \subset \bar{X}$ the conductor divisor. Write $\bar{X} = \bigcup_{1 \leq i \leq n} \bar{X}_i$ as the union of its irreducible components. Applying Theorem 1.1 with $C = \{1\}$, we obtain for each $1 \leq i \leq n$

$$(K_{\bar{X}_i} + \bar{D}_i)^2 \geq \frac{1}{143} p_g(\bar{X}_i, \bar{D}_i)$$

and it follows from (8.1) and (8.2) that

$$K_{\bar{X}}^2 = \sum_{1 \leq i \leq n} (K_{\bar{X}_i} + \bar{D}_i)^2 \geq \sum_{1 \leq i \leq n} \frac{1}{143} p_g(\bar{X}_i, \bar{D}_i) \geq \frac{1}{143} p_g(X).$$

$\square$
Remark 8.3. (i) If the equality in Theorem 8.2 holds, then we see from the proof that 
\[(K_{\tilde{X}_i} + \tilde{D}_i)^2 = \frac{1}{143} p_g(\tilde{X}_i, \tilde{D}_i)\] for each \( 1 \leq i \leq n \), so the conductor divisors \( \tilde{D}_i \) are actually empty by Proposition 7.13 (v). In other words, \( X = \tilde{X} = \tilde{X}_1 \), which is normal, and \( p_g(X) = 1 \).

(ii) Using the same proof, one can show that \((K_X + B)^2 \geq \frac{1}{143} p_g(X, B)\) for any projective semi-log-canonical surface \((X, B)\) with reduced \( B \) and ample \( K_X + B \).

Finally we construct a series of stable surfaces with \( K_X^2 = \frac{25}{84} p_g(X) \) where \( p_g(X) \) can take any positive integer.

Example 8.4. We start with a configuration of a nodal cubic curve \( C \) and three lines \( L_1, L_2, L_3 \) on \( \mathbb{P}^2 \) satisfying the following conditions (see Figure 1):

- each pair of the curves in \( \{L_1, L_2, L_3, C\} \) intersect transversally, and
- there is exactly one triple point on \( L_1 + L_2 + L_3 + C \), which is \( \{g_0\} = L_1 \cap L_2 \cap C \).

![Figure 1](image1.png)

**Figure 1.** Three lines \( L_1, L_2, L_3 \) and one cubic curve \( C \) on \( \mathbb{P}^2 \)

Let \( \rho: \tilde{X} \to \mathbb{P}^2 \) be a composition of blow-ups at the singularities of \( L_1 + L_2 + L_3 + C \) as well as their infinitely near points over \( L_3 \) and \( C \), such that \( \rho^{-1}(L_1 + L_2 + L_3 + C) \) is a simple normal crossing curve with dual graph as in Figure 2:

![Figure 2](image2.png)

**Figure 2.** Dual graph of \( \rho^{-1}(C + L_1 + L_2 + L_3) \subset \tilde{X} \)
• the curves $\tilde{L}_i$ ($1 \leq i \leq 3$) and $\tilde{C}$ are the strict transforms of $L_i$ and $C$ respectively,
• the white bullets without labels denote $(-2)$-curves,
• the black bullets $E_i, F_j$ ($1 \leq i, j \leq 3$) and $G_k$ ($0 \leq k \leq 3$) denote the $\rho$-exceptional $(-1)$-curves over $e_i, f_j, g_k$ respectively.

Note that the $\tilde{L}_i$ are smooth rational curves with $\tilde{L}_1^2 = \tilde{L}_2^2 = -3, \tilde{L}_3^2 = -16$, and $\tilde{C}$ is a smooth elliptic curve with $\tilde{C}^2 = -5$.

Let $\tilde{B} \subset \tilde{X}$ be the reduced subcurve of $\rho^{-1}(C + L_1 + L_2 + L_3)$ consisting of the components that are not $(-1)$-curves (corresponding to the black bullets in the dual graph). Then the semistable part is $\tilde{B}^s = \tilde{C}$ and, as before, we set $\tilde{B}^{ns} := \tilde{B} - \tilde{B}^s$. Since $K_{\tilde{X}} + C \sim 0$, $K_{\tilde{X}} + \tilde{B} \sim K_{\tilde{X}} + \tilde{B} - \rho^*(K_{\tilde{Y}} + C)$ is linearly equivalent to an effective divisor with the same support as $\tilde{B}^{ns} + E_3 + F_3$, which is big.

Let $\pi : (\tilde{X}, \tilde{B}) \to (X, B)$ be the ample model. Then $\pi$ contracts $\tilde{B} - \tilde{L}_1 - \tilde{L}_2$. We denote $\tilde{L}_i = \pi_*L_i$ for $i = 1, 2, \tilde{E}_i = \pi_*E_i$ and $\tilde{F}_j = \pi_*F_j$ for $1 \leq i, j \leq 3$, and $G_k = \pi_*G_k$ for $0 \leq k \leq 3$. Then we have $B = L_1 + L_2$, and $L_1 \equiv L_2 \equiv \mathbb{P}^1$; see Figure 3 for a sketch of the visible curves on $\tilde{X}$, where the bullets denotes the singularities on $\tilde{X}$; $\tilde{e}_i, \tilde{f}_i$ are quotient singularities of type $\frac{1}{2}(1, 1), \tilde{e}_2, \tilde{f}_2$ of type $\frac{1}{3}(1, 2), \tilde{e}_3, \tilde{f}_3$ of type $\frac{1}{6}(1, 6), \tilde{I}_3 := \pi(\tilde{L}_3)$ of type $\frac{1}{60}(1, 1)$, and finally $\tilde{c} = \pi(\tilde{C})$ is a simple elliptic singularity of $\tilde{X}$. Note that $\{\tilde{y}_i\} := G_0 \cap \tilde{L}_i$, $i = 1, 2$, are smooth points of $\tilde{X}$.

The positive part of $K_{\tilde{X}} + \tilde{B}$ is

$$\pi^*(K_{\tilde{X}} + \tilde{B}) = K_{\tilde{X}} + \tilde{C} + \sum b_j \tilde{B}_j + \tilde{L}_1 + \tilde{L}_2 + \frac{7}{8} \tilde{L}_3,$$

where the $\tilde{B}_j$ are the visible $(-2)$-curves and $b_j \in (0, 1)$ are appropriate coefficients. The volume of $K_{\tilde{X}} + \tilde{B}$ is then

$$\text{vol}(K_{\tilde{X}} + \tilde{B}) = \pi^*(K_{\tilde{X}} + \tilde{B})^2 = \pi^*(K_{\tilde{X}} + \tilde{B})(7E_3 + 7F_3 + \tilde{L}_1 + \tilde{L}_2) = \frac{25}{84}.$$ 

One sees easily that $p_g(\tilde{X}, \tilde{B}) = h^0(\tilde{X}, K_{\tilde{X}} + \tilde{C}) = 1$. The different along $B = L_1 + L_2$ is

$$\text{Diff}_{\tilde{B}}(0) = \frac{1}{2}(\tilde{e}_1 + \tilde{f}_1) + \frac{2}{3}(\tilde{e}_2 + \tilde{f}_2) + \frac{6}{7}(\tilde{e}_3 + \tilde{f}_3),$$

and it is clear that $H^0(\tilde{B}, \text{Diff}_{\tilde{B}}(0)) = 0$.

We take $n$ copies $(\tilde{X}_s, \tilde{B}_s) = (L_{s1} + L_{s2})$ of $(\tilde{X}, \tilde{B} = L_1 + L_2)$ and glue them in a cycle along the $\tilde{B}_s$, to obtain a stable surface $X^{(n)}$ (see Figure 4): The boundary component $L_{s2}$ of $(\tilde{X}_s, \tilde{B}_s)$ is identified with the boundary component $\tilde{L}_{s+1,1}$ of $(\tilde{X}_{s+1}, \tilde{B}_{s+1})$, so that the
differents $\text{Diff}_{L^s,2}(0)$ and $\text{Diff}_{L^{s+1},3}(0)$ match, where $1 \leq s \leq n$ is taken modulo $n$: By (8.1)

\[ K_{X^{(n)}}^2 = n(K_X + \bar{B})^2 = \frac{25}{84} n, \quad p_g(X^{(n)}) = n. \]

If $n \geq 5$ then these stable surfaces violate the inequality $K_{X^{(n)}}^2 > p_g(X^{(n)}) - 3$ which was suggested as a working hypothesis in [22].

\textbf{Remark 8.5.} $X^{(1)}$ is obtained from $(\bar{X}, \bar{B})$ by glueing $L_1$ and $\bar{L}_2$, and there is an étale cyclic covering map $X^{(n)} \to X^{(1)}$ of degree $n$. In fact, if $k = \mathbb{C}$ then the fundamental group is $\pi_1(X^{(n)}) \cong \mathbb{Z}$ for each $n \geq 1$.

\textbf{8.3. Bounding symplectic automorphisms of surfaces of general type.}

\textbf{Theorem 8.6.} Let $S$ be a smooth projective surface of general type over an algebraically closed field $k$ of characteristic 0. Let $\text{Aut}_s(S)$ denote the group of symplectic automorphisms of $S$, i.e., those automorphisms inducing the trivial action on $H^0(S, K_S)$. If $p_g(S) \geq 34$, then $|\text{Aut}_s(S)| \leq 12$.

\textbf{Proof.} Let $G = \text{Aut}_s(S)$ and $\pi: S \to X = S/G$ the quotient map. Then there is an effective divisor $B$ with coefficients in $\mathcal{C}_2$ such that $(X, B)$ is klt and $K_X = \pi^*(K_X + B)$. It follows that $\text{vol}(K_S) = |G|\text{vol}(K_X + B)$. Since $G$ induces the trivial action on $H^0(S, K_S)$, we have $p_g(X, B) = p_g(S)$. By Proposition 6.2 for the coefficient set $\mathcal{C}_2$, we have $\text{vol}(K_X + B) \geq \frac{3}{4} p_g - \frac{5}{4}$. Therefore,

\[ \text{vol}(K_S) = |G|\text{vol}(K_X + B) \geq |G| \left( \frac{3}{4} p_g - \frac{5}{4} \right). \]

Combining the Bogomolov–Miyaoka–Yau inequality $\text{vol}(K_S) \leq 9 \chi(O_S) \leq 9(p_g + 1)$ (see [20, Theorem 5.1]), we obtain

\[ |G| \left( \frac{3}{4} p_g - \frac{5}{4} \right) \leq 9(p_g + 1) \]

and it follows that

\[ |G| \leq \frac{36(p_g + 1)}{3p_g - 5} = 12 + \frac{96}{3p_g - 5}. \]

Now one sees easily that $|G| \leq 12$ if $p_g \geq 34$. \hfill \Box

\textbf{Remark 8.7.} (i) In Du’s thesis [9], it was shown that $|\text{Aut}_s(S)| \leq 417$ if $\chi(O_S) \geq 21$.

(ii) Since surfaces of general type with bounded $p_g(S)$ or $\chi(O_S)$ form a bounded family, Theorem 8.6 (and also [9]) implies that there is a universal constant $N$ such that $|\text{Aut}_s(S)| \leq N$ holds for any smooth projective surface of general type.
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School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, P. R. China
Email address: wliu@xmu.edu.cn