Global Existence of Non-cutoff Boltzmann Equation in Critical Weighted Sobolev Space

Dingqun DENG *

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Abstract

This article proves the global existence to the non-cutoff Boltzmann equation for hard potential in the critical weighted Sobolev space on the whole space. We also present a pseudo-differential calculus method for deriving the regularizing estimate on the linearized Boltzmann equation, with respect to velocity \( v \). To this goals, we split the spatially inhomogeneous linearized Boltzmann operator \( B \) as the sum of a dissipative operator and a compact operator. Also we analyze the spectrum structure of the Fourier transform of \( B \) on spatial variable \( x \). These good properties gives the optimal regular estimate for hard potential: the semigroup \( e^{tB} \) is continuous from weighted Sobolev space \( H(a^{-1/2})H^m_x \) to \( H(a^{1/2})H^m_x \) with a sharp large time decay. We thus obtain the unique global solution on the whole space with small initial data. This work develops the application of pseudo-differential calculus, spectrum analysis as well as semigroup theory to Boltzmann equation.

Keywords: Global existence, pseudo-differential calculus, Boltzmann equation without cut-off, regularizing effect, hypoelliptic estimate, spectrum analysis, one-parameter semigroup.

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Contents

1 Introduction

*email: dingqdeng2-c@my.cityu.edu.hk, Department of Mathematics, City University of Hong Kong, ORCID: 0000-0001-9678-314X
### 1 Introduction

In the present paper, we consider the Boltzmann equation in $d$-dimension:

\[ F_t + v \cdot \nabla_x F = Q(F,F), \]  

(1)

where the unknown $F(x,v,t)$ represents the density of particles in phase space, and spatial coordinate $x \in \mathbb{R}^d$ and velocities $v \in \mathbb{R}^d$ with $d \geq 2$. The Boltzmann collision operator $Q(F,G)$ is a bilinear operator defined for sufficiently smooth functions $F,G$ by

\[ Q(F,G)(v) := \int_{\mathbb{R}^d} \int_{S^{d-1}} B(v - v_s, \sigma)(F'_s G' - F_s G) d\sigma dv_s, \]

where $F' = F(x,v',t), G'_s = G(x,v'_s,t), F = F(x,v,t), G_s = G(x,v_s,t)$, and $(v,v_s)$ are the velocities of two gas particles before collision while $(v',v'_s)$ are the velocities after collision satisfying the following conservation laws of momentum and energy,

\[ v + v_s = v' + v'_s, \quad |v|^2 + |v_s|^2 = |v'|^2 + |v'_s|^2. \]

As a consequence, for $\sigma \in S^{d-1}$, the unit sphere in $\mathbb{R}^d$, we have the $\sigma$-representation:

\[ v' = \frac{v + v_s}{2} + \frac{|v - v_s|}{2} \sigma, \quad v'_s = \frac{v + v_s}{2} - \frac{|v - v_s|}{2} \sigma. \]
Also we define the angle $\theta$ in the standard way

$$\cos \theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma,$$

where $\cdot$ denotes the usual inner product in $\mathbb{R}^d$.

**Collision Kernel**  The collision kernel cross section $B$ satisfies

$$B(v - v_*, \sigma) = |v - v_*|^{\gamma} b(\cos \theta),$$

for some $\gamma \in \mathbb{R}$ and function $b$. Without loss of generality, we can assume $B(v - v_*, \sigma)$ is supported on $(v - v_*) \cdot \sigma \geq 0$ which corresponds to $\theta \in [0, \pi/2]$, since $B$ can be replaced by its symmetrized form $B(v - v_*, \sigma) = B(v - v_*, \sigma) + B(v - v_*, -\sigma)$. Moreover, we are going to work on the collision kernel without angular cut-off, which corresponds to the case of inverse power interaction laws between particles. That is,

$$b(\cos \theta) \approx \theta^{-d+1-2s} \quad \text{on } \theta \in (0, \pi/2),$$

and

$$s \in (0, 1), \quad \gamma \in (-d, \infty).$$

For Boltzmann equation without angular cut-off, the condition $\gamma + 2s < 0$ is called soft potential while $\gamma + 2s \geq 0$ is called hard potential. For mathematical theory of Boltzmann equation, one may refer to [6, 7, 19, 44] for more introduction.

**1.1 Preliminary Result**

We will study the Boltzmann equation (1) near the global Maxwellian equilibrium

$$\mu(v) = (2\pi)^{-d/2}e^{-|v|^2/2}.$$ 

So we set $F = \mu + \mu^{1/2}f$ and the Boltzmann equation (1) becomes

$$f_t + v \cdot \nabla_x f = Lf + \mu^{-1/2}Q(\mu^{1/2}f, \mu^{1/2}f),$$

where $L$ is called the linearized Boltzmann operator given by

$$Lf = \mu^{-1/2}Q(\mu, \mu^{1/2}f) + \mu^{-1/2}Q(\mu^{1/2}f, \mu) = L_1f + L_2f,$$

where $L_1, L_2$ are defined in (5)(6). The kernel of $L$ is $\text{Span}\{\psi_i\}_{i=0}^{d+1}$ defined in (100) and we denote the projection from $L^2$ onto Ker$L$ by

$$Pf := \sum_{i=0}^{d+1} \langle f, \psi_i \rangle_{L^2} \psi_i.$$
We would like to apply the symbolic calculus in [10] for our study as the following. One may refer to the appendix as well as [34] for more information about pseudodifferential calculus. Let $\Gamma = |dv|^2 + |d\eta|^2$ be an admissible metric. Define

$$a(v, \eta) := \langle v \rangle^\gamma (1 + |\eta|^2 + |\eta \wedge v|^2 + |v|^2)^s + K_0 \langle v \rangle^{\gamma + 2s}$$

be a $\Gamma$-admissible weight proved in [10], where $K_0 > 0$ is chosen as following and $|\eta \wedge v| = |\eta||v|\sin \theta_0$ with $\theta_0$ be the angle between $\eta, v$. Applying theorem 4.2 in [10] and lemma 2.1 and 2.2 in [23], there exists $K_0 > 0$ such that the Weyl quantization $a^w : H(ac) \to H(c)$ and $(a^{1/2})^w : H(a^{1/2}c) \to H(c)$ are invertible, with $c$ be any $\Gamma$-admissible metric. The weighted Sobolev spaces $H(c)$ is defined by (96). The real symbol $a$ gives the formal self-adjointness of Weyl quantization $a^w$, which is widely applied in our analysis. By the invertibility of $(a^{1/2})^w$, we have equivalence

$$\| (a^{1/2})^w (\cdot) \|_{L^2} \approx \| \cdot \|_{H(a^{1/2})},$$

and hence we will equip $H(a^{1/2})$ with norm $\| (a^{1/2})^w (\cdot) \|_{L^2}$. Also we will denote the weighted Sobolev norms for $l, n \in \mathbb{R}$:

$$\| f \|_{L^2_x H^m_{x,v}} := \| \langle D_x \rangle^m f \|_{L^2_x v},$$

$$\| f \|_{H(a^{1/2})H^m_{x,v}} := \| \langle D_x \rangle^m (a^{1/2})^w f \|_{L^2_x v},$$

where $\langle D_x \rangle^m f = \mathcal{F}^{-1}(y)^m \mathcal{F}_x f, L^2_{x,v} = L^2(\mathbb{R}^{2d}_{x,v})$.

To analyze the linearized Boltzmann operator rigorously, we use the Carleman representation (97) to define $L = L_1 + L_2$:

$$L_1 f = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d, |h| \geq \varepsilon} dh \int_{E_{0,h}} d\alpha \tilde{b}(\alpha, h) 1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma + 1 + 2s}}{|h|^{d + 2s}} \mu^{1/2}(v + \alpha - h) \left( \mu^{1/2}(v + \alpha - h) f(v) - \mu^{1/2}(v + \alpha) f(v) \right),$$

(5)

$$L_2 f = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d, |h| \geq \varepsilon} dh \int_{E_{0,h}} d\alpha \tilde{b}(\alpha, h) 1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma + 1 + 2s}}{|h|^{d + 2s}} \mu^{1/2}(v + \alpha) \left( \mu^{1/2}(v + \alpha) f(v) - \mu^{1/2}(v + \alpha - h) f(v + \alpha - h) \right),$$

(6)

which are well-defined for Schwartz function according to [10, 23]. Here we use the principal value on $h$ in order to assure the integral are well-defined when the two terms in the parentheses is separated into two integral, where change of variable can be applied. By section 3 in [23], $L = L_1 + L_2$ can be regarded as the standard pseudodifferential operator with symbols in $S(a)$. Then by the unique extension of continuous operator, $L$ is a linear continuous operator from $H(ac)$ into $H(c)$ for any $\Gamma-$admissible weight function $c$. Also, lemma 6.12 gives the formal self-adjointness of $L$. To better applying the previous result, we consider weighted Sobolev norm $\| (a^{1/2})^w (\cdot) \|_{L^2}$, triple norm $\| \cdot \|$ in [5] and the norm $\| \cdot \|_{N^{\gamma_0}}$ in [26], where

$$\| f \|^2 := \int B(v - v_s, \sigma) \left( \mu_s (f' - f)^2 + f_s^2 (\mu')^{1/2} - \mu^{1/2} \right) d\sigma dv_s dv,$$
\[|f|_{N^{*,\gamma}}^2 := \|\langle v \rangle^{\gamma/2 + s} f\|_{L^2}^2 + \int \frac{(f' - f)^2}{d(v, v')^{d+2s}} d(v, v') \leq 1,\]

with \(d(v, v') := \sqrt{|v - v'|^2 + \frac{1}{4}(|v|^2 - |v'|^2)^2}\). Then by (2.13)(2.15) in [26], Proposition 2.1 in [5] and Theorem 1.2 in [10], for \(f \in \mathcal{S}, l \in \mathbb{R}\), we have the equivalence of norms:

\[
\|(a^{1/2})^w f\|_{L^2_v}^2 \approx \|f\|_2^2 \approx |f|_{N^{*,\gamma}}^2 \approx (-L f, f)_{L^2_v} + \|\langle v \rangle f\|_{L^2_v},
\]

where the constant depends on \(l\). These norms essentially describe the behavior of Boltzmann collision operator.

### 1.2 Main Result

Our main result is the global existence of Boltzmann equation without angular cutoff for hard potential in the critical Sobolev space. Formal result on spatially inhomogeneous Boltzmann equation were on torus \(\mathbb{T}^d\), cf. [26], or require more regularity on the initial data, cf. [2]. This work provides the global solution on the whole space \(\mathbb{R}^d\) and require minimum assumption on initial data as in the case of Boltzmann with angular cutoff.

The dissipation estimate on \(L\) is necessary for discovering the solution. To extract the dissipation estimate of linearized Boltzmann equation, we define

\[
A := \begin{cases} 
- L + P \langle \langle v \rangle \rangle^{\gamma+2s} P, & \text{if } \gamma + 2s \geq 0, \\
- L + P, & \text{if } \gamma + 2s \leq 0.
\end{cases}
\]

\[K := A + L.\]

Then we have the following dissipation properties, which is valid for both hard and soft potential.

**Theorem 1.1.** The linearized Boltzmann equation can be split as \(L = -A + K\) by its definition (8). There exists \(\nu_0 > 0\) such that for \(f \in \mathcal{S}(\mathbb{R}^d)\),

\[
\text{Re}(A f, f)_{L^2_v} \geq \nu_0 \|(a^{1/2})^w f\|_{L^2_v}^2
\]

Consequently,

\[
\text{Re}(A f, f)_{L^2_v} \geq \nu_0 \|(a^{1/2})^w f\|_{L^2_v}^2.
\]

Additionally, \(A\) and \(K\) can be regarded as Weyl quantization with symbols in \(S(a)\) and \(S(\langle \langle \eta \rangle \rangle^{-k} \langle \eta \rangle^{-l})\) for any \(k, l \geq 0\) respectively. Hence \(K\) is compact on \(L^2_v\) and maps \(L^2_v\) function into \(\mathcal{S}(\mathbb{R}^d)\).

Define

\[B = -v \cdot \nabla_x + L\]
as in (35), then $B$ generates a strongly continuous semigroup $e^{tB}$ on $L^2$. The closure is necessary for generating the semigroup. Then we can discuss the solution $e^{tB}f_0$ to the linearized Boltzmann equation:

\[
\begin{cases}
  f_t = Bf, \\
  f|_{t=0} = f_0.
\end{cases}
\]

This semigroup $e^{tB}$ generated by the linearized Boltzmann operator plays an important role in the perturbation theory of Boltzmann equation and kinetic equation, since the solution to Boltzmann equation can be written into a perturbation form of the solution to its linearized equation, for instance [28, 42]. Our first main result gives an optimal regularizing effect on the linearized Boltzmann equation for hard potential and a large time decay estimate. With this regularizing estimate established, we can apply the energy estimate to obtain a global solution. Our method is building on the whole space $\mathbb{R}^d$, which is essentially different from torus $\mathbb{T}^d$. Thus, the analysis on the spectrum structure on linearized Boltzmann operator is necessary and provides a new approach in the existence theory of Boltzmann equation without angular cutoff.

**Theorem 1.2.** Assume $\gamma + 2s \geq 0$. Fix $f \in \mathcal{S}(\mathbb{R}^{2d}_{x,v})$. Then for $k \geq 2$, $m \in \mathbb{R}$ $p \in [1, 2]$, we have

\[
\|e^{tB}f\|^2_{H(\alpha^{1/2})H^m_x} \lesssim \frac{e^{-2\sigma_y t}}{t^{2k}} \|f\|^2_{H(\alpha^{-1/2})H^m_x} + \frac{1}{(1 + t)^{d/2(2/p - 1)}} \|a^{-1/2}(v, D_v)f\|_{L^2_y(L^p_x)}^2,
\]

where $\sigma_y > 0$ is defined by (74) and the constant is independent of $f$ and $p$.

Note that the space $H(\alpha^{1/2})H^m_x$ is a better space than $H(\alpha^{-1/2})H^m_x$ when $\gamma + 2s \geq 0$, which gives the optimal regular estimate. Also, our theorem gives the large time decay to Cauchy problem of linearized Boltzmann equation.

The smoothing effect of the linearized Boltzmann operator and Boltzmann collision operator for angular non cut-off collision kernel were discussed in many context. At the beginning, entropy production estimate for non cut-off assumption were established, as in [1, 35]. Their result were widely applied in the theory of non-cutoff Boltzmann equation. Later on, many works discover the optimal regular estimate of Boltzmann collision operator in $v$ in different setting. We refer to [3, 10, 27, 39] for the dissipation estimate of collision operator, and [8, 9, 11, 13, 14, 20–22, 31–33] for smoothing effect of the solution to Boltzmann equation in different aspect. With this kind of regular estimate, existence theory were well-discussed in many papers, for instance [2, 26]. These works show that the Boltzmann operator behaves locally like a fractional operator:

\[Q(f, g) \sim (-\Delta_v)^s g + \text{lower order terms}.\]

More precisely, according to the symbolic calculus developed by Alexandre-Hérau-Li [10], the linearized Boltzmann operator behaves essentially as

\[L \sim \langle v \rangle^\gamma (-\Delta_v - |v \wedge \partial_v|^2 + |v|^2)^s + \text{lower order terms}.\]
This diffusion property shows that the spatially homogeneous Boltzmann equation behaves like fractional heat equation, while the spatially inhomogeneous Boltzmann behaves as the generalized Kolmogorov equation. We refer to [14, 31, 33] for Kac equation, the one dimensional model of Boltzmann equation, and [38] for similar kinetic equation. Thus, the Cauchy problem to Boltzmann equation enjoys a smoothing effect at any positive time, which is essential to the existence theory.

**Theorem 1.3.** Suppose $d \geq 3$, $m > \frac{4}{d}$, $\gamma + 2s \geq 0$. There exists $\varepsilon_0 > 0$ so small that if

$$\|f_0\|_X \leq \varepsilon_0,$$

where $X$ is defined as (83), then there exists an unique global weak solution $f$ to Boltzmann equation

$$f_t = Bf + \Gamma(f,f), \quad f|_{t=0} = f_0,$$

satisfying

$$\|f\|_{L^\infty([0,\infty); L^2_x H^m_x)} + \|f\|_{L^2([0,\infty); H^{(a^{1/2})} H^m_x)} \leq C\varepsilon_0,$$

with some constant $C > 0$.

By (83) and (87), the smallness assumption on initial data $f_0$ can be fulfilled if

$$\|f_0\|_{L^2_x H^m_x} + \|(a^{-1/2})^w f_0\|_{L^2_x(L^p_x)}$$

is sufficient small for some $p \in [1, \frac{2d}{d+2})$. Such choice of Sobolev space is similar to the cutoff case, cf. [43]. The space $H^{(a^{1/2})} H^m_x$ is critical due to estimate (9)(81) on $L$ and $\Gamma(\cdot,\cdot)$ respectively. The weak solution $f$ means that

$$\int_0^\infty (f_t, \varphi)_{L^2_{x,v}} dt = \int_0^\infty (f, B^* \varphi)_{L^2_{x,v}} dt + \int_0^\infty (\Gamma(f,f), \varphi)_{L^2_{x,v}} dt,$$

for $\varphi \in \mathcal{D}((0, \infty); \mathcal{S}(\mathbb{R}^d))$, where $B^* = \overline{v \cdot \nabla_x + L}$. For Cauchy problem to Boltzmann equation near the global Maxwellian without angular cut-off, we refer to [26] for the existence theory on torus and [2, 4, 5] on the whole space. We improve the space $H^k_x$ used in [2] and the space $H^{(a^{1/2})} H^m_x$ we introduce in this paper is actually equivalent the norms $\|\cdot\|_{N^{s,\gamma}}$ and $\||\cdot||$ they introduced, thanks to (7). The assumption $\gamma + 2s \geq 0$ is needed for a spectral gap 3.3 and we deduce a polynomial time decay on the semigroup $e^{tB}$. Hence, we can use the norm $X$ (83) to describe the large time behavior of solution, which was also discussed in [18, 28, 29]. But our result builds on the whole space $\mathbb{R}^d$, which is essentially different from torus $\mathbb{T}^d_x$. 

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Organization of the article Our analysis is organized as follows. In Section 2, we provide a proof of the dissipation of $A$ and the solution $g$ to equation $(v \cdot \nabla_x + A)g = f$, a linearized form of Boltzmann equation. The dissipation and existence to this linearized equation gives rigorously the proof of generating the semigroup $e^{tB}$ on $L^2$. Note that the section 2 is valid for both hard and soft potential. In section 3, we give a rigorous argument for obtaining the spectrum structure to linearized Boltzmann operator for hard potential. In section 4, after establishing the spectrum structure, we can discuss the regular estimate of semigroup $e^{tB}$. In section 5, we deduce the global existence of Boltzmann equation on the whole space. The appendix gives some general theory on Boltzmann equation, functional analysis and pseudo-differential calculus.

Notations Throughout this article, we shall use the following notations. $\mathcal{S}(\mathbb{R}^d)$ is the set of Schwartz functions while $\mathcal{D}(0, \infty)$ is the set of smooth functions with compact support in $(0, \infty)$. For any $v \in \mathbb{R}^d$, we denote $\langle v \rangle = (1 + |v|^2)^{1/2}$. The gradient in $v$ is denoted by $\partial_v$. Write $P_0 = P$ be the orthogonal projection from $L^2$ onto Ker$L$ and $P_1 = I - P_0$, with $I$ being the identity map on $L^2$. Denote a complex number $\lambda = \sigma + i\tau$.

The notation $a \approx b$ (resp. $a \geq b$, $a \lesssim b$) for positive real function $a$, $b$ means there exists $C > 0$ not depending on possible free parameters such that $C^{-1}a \leq b \leq Ca$ (resp. $a \geq C^{-1}b$, $a \leq Cb$) on their domain. Re$(a)$ means the real part of complex number $a$. $[a,b] = ab - ba$ is the commutator between operators.

For pseudo-differential calculus, we write $\Gamma_v = |dv|^2 + |d\eta|^2$, $\Gamma_{x,v} = |dx|^2 + |dy|^2 + |dv|^2 + |d\eta|^2$ to be admissible metrics, where $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$ is the space-velocity variable and $(y, \eta) \in \mathbb{R}^d \times \mathbb{R}^d$ is the corresponding variable in dual space (the variable after Fourier transform). Let $m_v$ be $\Gamma_v$-admissible weight functions, $m_{x,v}$ be $\Gamma_{x,v}$-admissible weight functions. We will write $S(m_v) := S(m_v, \Gamma_v)$, $H_v(m_v) := H_v(m_v, \Gamma_v)$ be the weighted Sobolev space on velocity variable $v$ and $H_{x,v}(m_{x,v}) := H_{x,v}(m_{x,v}, \Gamma_{x,v})$ be the weighted Sobolev space on space-velocity variable $(x, v)$.

Also we will use $C_1 := \sup_{f \in L^2} \frac{\|\nu_1 f\|_{L^2}}{\|\nu_1 f\|_{H_0^{\sigma/2}(\mathbb{R}^d)}}$ in section 2 for hard potential case. $\nu_1$ is equal to $\min\{\frac{\nu_0}{\gamma}, \frac{\nu_0}{2C_1}\}$, if $\gamma + 2s \geq 0$ and to 0 if $\gamma + 2s < 0$, with $\nu_0$ defined in (9). The norm of $X_v(Y_x)$ is defined by

$$\|f\|_{X_v(Y_x)} := \|\|f\|_{Y_x}\|_{X_v}.$$

2 Hypoelliptic Estimate to Linearized Boltzmann Operator

In this section, we are trying to solve equation

$$(v \cdot \nabla_x + A)g = f,$$  \hspace{1cm} (11)

where $g$ is unknown and $f \in \mathcal{S}$. Here $A$ is a elliptic-type operator proved in the theorem 1.1. Thus a standard-type argument in solving elliptic equation but in the language
of pseudo-differential calculus will provide the solution and regularity to equation (11). We will also consider its dual equation

$$(2\pi iv \cdot y + A)g = f$$

in $L^2_v$, in order to apply the spectrum analysis to operator $2\pi iv \cdot y + A$. These result allows us to apply the semigroup theory on $L^2_{x,v}$ and $L^2_v$.

### 2.1 Splitting of the linearized operator

**Proof of Theorem 1.1.** 1. We will continue the argument in [10, 23] and notice that the following statement are valid for both hard and soft potential. By section 3 in [23], the linearized Boltzmann operator can be written into

$$L = -b^w + \mathcal{K}^w,$$

where $b \in S(a)$, $\mathcal{K} \in S(|v|^{\gamma+2s})$ are defined as the following. Fix $0 < \delta \leq 1$. Let $\varphi(t)$ be a positive smooth radial function that equal to 1 when $|t| \leq 1/4$ and 0 when $|t| \geq 1$. Let $\varphi_\delta(v) = \varphi(|v|^2/\delta^2)$ and $\tilde{\varphi}_\delta(v) = 1 - \varphi_\delta(v)$. Then $\varphi_\delta(v) = \varphi(|v|^2/\delta^2)$ equal to 0 for $|v| \geq \delta$ and 1 for $|v| \leq \delta/2$. Then for $f \in \mathcal{F}$,

$$b^w f := -\int \int B\varphi_\delta(v' - v)(\mu\cdot f - f) - \int \int B\varphi_\delta(v' - v)\mu' f - \int \int B\varphi_\delta(v' - v)\mu f, dv \cdot d\sigma,$$

$$\mathcal{K}^w f := \int \int B(\mu\cdot (f' - f) - \int \int B\tilde{\varphi}_\delta(v' - v)(\mu\cdot f' - f) - \int \int B\tilde{\varphi}_\delta(v' - v)(\mu\cdot f - f), dv \cdot d\sigma,$$

where we use Carleman representation for writing into the form of pseudo-differential operator, cf. [10, 23]. These operators satisfy

$$\text{Re}(b^w f, f)_{L^2} + C\|\langle v \rangle^{\gamma/2+s} f\|_{L^2}^2 \approx \|a^{1/2} f\|_{L^2}^2, \quad |\langle \mathcal{K}^w f, f \rangle_{L^2}| \leq C\|\langle v \rangle^{\gamma/2+s} f\|_{L^2}^2, \quad (12)$$

By proposition 2.1 in [5], there exists $\nu_0 > 0$ such that

$$\nu_0\|a^{1/2} f\|_{L^2}^2 \leq (-L f, f)_{L^2}. \quad (13)$$
Note that $\langle v \rangle^{\gamma/2+s} \leq a^{1/2}$ implies $\|\langle v \rangle^{\gamma/2+s}f\|_{L^2} \leq C\|a^{1/2}f\|_{L^2}$. By lemma 6.5, we have

$$
\|\langle v \rangle^{\gamma/2+s}(I-P)f\|_{L^2}^2 \leq C(-Lf, f)_{L^2},
\|\langle v \rangle^{\gamma/2+s}f\|_{L^2}^2 \leq C((-L + P\langle v \rangle^{\gamma+2s}Pf, f)_{L^2}.  \tag{14}
$$

In particular, when $\gamma + 2s \leq 0$, we have

$$
\|\langle v \rangle^{\gamma/2+s}f\|_{L^2}^2 \leq C((-L + P)f, f)_{L^2}.
$$

Therefore, by (12)(14), choosing $C > 0$ sufficiently large, we have

$$
\|a^{1/2}/w f\|_{L^2}^2 \leq C(\text{Re}(bw f, f)_{L^2} + \text{Re}(-Kw f, f)_{L^2} + C\|\langle v \rangle^{\gamma/2+s}f\|_{L^2}^2)
\leq C'( (-L + P\langle v \rangle^{\gamma+2s}f, f)_{L^2},
$$

and similarly for soft potential $\gamma + 2s < 0$,

$$
\|a^{1/2}/w f\|_{L^2}^2 \leq C((-L + P)f, f)_{L^2}.
$$

2. Now we can write

$$
A := \begin{cases} 
-L + P\langle v \rangle^{\gamma+2s}P, & \text{if } \gamma + 2s \geq 0, \\
-L + P, & \text{if } \gamma + 2s \leq 0.
\end{cases}
$$

$K := A + L$.

The symbol of $A$ belongs to $S(a)$ is given by theorem 3.1 and 3.2 in [23]. On the other hand, for the symbol of $K$, we write

$$
P\langle v \rangle^{\gamma+2s}Pf = \sum_{i=0}^{d+1}\langle \langle v \rangle^{\gamma+2s}Pf, \psi_i \rangle_{L^2}\psi_i(v)
\quad = \int \sum_{i=0}^{d+1}\hat{f}(\eta)\mathcal{F}(P\langle \cdot \rangle^{\gamma+2s}\psi_i)(\eta)\psi_i(v)\,d\eta
\quad = \int e^{2\pi iv\cdot\eta}\hat{f}(\eta)\sum_{i=0}^{d+1}e^{-2\pi iv\cdot\eta}\mathcal{F}(P\langle \cdot \rangle^{\gamma+2s}\psi_i)(\eta)\psi_i(v)\,d\eta.
$$

Thus $K$ has symbol $\sum_{i=0}^{d+1}e^{-2\pi iv\cdot\eta}\mathcal{F}(P\langle \cdot \rangle^{\gamma+2s}\psi_i)(\eta)\psi_i(v) \in S(\langle v \rangle^{-k}\langle \eta \rangle^{-l})$, for $k, l > 0$ as a standard pseudo-differential operator, since $\psi_i \in \mathcal{S}$ has exponential decay. Also $\lim_{(v, \eta) \to \infty} \langle v \rangle^{-k} \langle \eta \rangle^{-l}$, by theorem 4.28 in [46], we obtain that $K$ is compact on $L^2$. \qed

### 2.2 Hypoelliptic of $v \cdot \nabla_x + A$

If $\gamma + 2s \geq 0$, we have $\|\cdot\|_{L^2} \leq C\|(a^{1/2}/w)(\cdot)\|_{L^2}$ by lemma 6.5, since $1 \leq a$ and $(a^{1/2}/w)$ is invertible. We thus write

$$
C_1 := \sup_{f \in L^2} \frac{\|\varphi\|_{L^2}}{\|\varphi\|_{H(a^{1/2})}}, \tag{15}
$$

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Theorem 2.1. Write \( L^2 = L^2_{x,v}(\mathbb{R}^{2d}) \), \( H(a^{1/2}) = H_{x,v}(a^{1/2})(\mathbb{R}^{2d}) \), \( \mathcal{S} = \mathcal{S}(\mathbb{R}^{2d}) \) in this theorem. Denote

\[
\| \cdot \|_H := \| \cdot \|_{L^2} + \| \cdot \|_{H(a^{1/2})}, \quad \| \cdot \|_{H^{-1}} := \min\{\| \cdot \|_{L^2}, \| \cdot \|_{H(a^{-1/2})}\}.
\]

Let

\[
\begin{cases}
\text{Re} \zeta > -\frac{\nu_0}{C_1}, & \text{if } \gamma + 2s \geq 0, \\
\text{Re} \zeta > 0, & \text{if } \gamma + 2s < 0.
\end{cases}
\]

Then for any \( f \in \mathcal{S} \), there exists unique \( g \in H(A_k) \) such that

\[
(\zeta I + v \cdot \nabla_x + A)g = f,
\]

where \( A_k = (K_0 + |y|^4 + |v|^2 + |\eta|^2)^k \) with \( K_0 >> 1 \). Hence, the range of \( \zeta I + v \cdot \nabla_x + A \) with domain \( H(a) \cap \mathcal{S}(\langle v \rangle \langle y \rangle) \) is dense in \( L^2 \).

Remark 2.2. The operator \( v \cdot \nabla_x \) can be replaced by its adjoint \(-v \cdot \nabla_x\).

Proof. Write \( F = v \cdot \nabla_x \). Then \( F \) is a pseudo-differential operator with symbol in \( S(\langle v \rangle \langle y \rangle) \), \( \partial_v F \in \mathcal{O}(y) \) and \( (Ff, f)_{L^2} = (f, -Ff)_{L^2} \) for \( f \in \mathcal{S}(\mathbb{R}^{2d}) \).

1. Fix any \( f \in \mathcal{S} \), we are going to find the strong solution of

\[
\zeta g + Fg + Ag = f.
\]

We start with finding the weak solution. For any \( \varphi \in \mathcal{S} \), by (9)(15), we have

\[
\text{Re}(\bar{\zeta} \varphi - F \varphi + A \varphi, \varphi)_{L^2} \geq \nu_0 \| \varphi \|_{H(a^{1/2})}^2 + \text{Re} \| \varphi \|_{L^2}^2 \geq C \| \varphi \|_{H^{-1}}^2.
\]

Then for any \( f \) satisfying \( \| f \|_{H^{-1}} < \infty \),

\[
\| \varphi \|_{H}^2 \leq C \| \bar{\zeta} \varphi - F \varphi + A \varphi \|_{H^{-1}} \| \varphi \|_{H},
\]

\[
|f, \varphi| \leq C \| f \|_{H^{-1}} \| \varphi \|_{H} \leq C \| \bar{\zeta} \varphi - F \varphi + A \varphi \|_{H^{-1}}.
\]

Let \( \text{Im}(\bar{\zeta}I - F + A) := \{(\bar{\zeta}I - F + A) \varphi : \varphi \in \mathcal{S}\} \). The operator \( T_1 : H(a^{-1/2}) \supset \text{Im}(\bar{\zeta}I - F + A) \rightarrow \mathbb{C} \) and \( T_2 : L^2 \supset \text{Im}(\bar{\zeta}I - F + A) \rightarrow \mathbb{C} \) sending \( \psi := \bar{\zeta} \varphi - F \varphi + A \varphi \) to \( (f, \varphi) \) are linear continuous. Hence \( T_1, T_2 \) extend to a linear functional on \( H(a^{-1/2}) \) and \( L^2 \) respectively. Note that from Theorem 2.6.17 in [34], \( (H(a^{1/2}))^* = H(a^{-1/2}) \) and \( (L^2)^* = L^2 \), there exists unique \( u_1 \in H(a^{1/2}) \) and \( u_2 \in L^2 \) such that for \( \psi \in \mathcal{S} \),

\[
(u_1, \psi)_{L^2} = T_1 \psi, \quad (u_2, \psi)_{L^2} = T_2 \psi.
\]

Thus \( g := u_1 = u_2 \in H(a^{1/2}) \cap L^2 \) satisfies that for \( \varphi \in \mathcal{S} \),

\[
(g, \bar{\zeta} \varphi - F \varphi + A \varphi)_{L^2} = (f, \varphi)_{L^2}.
\]

2. We next show that \( g \in H(a) \cap \mathcal{S}(\langle v \rangle \langle y \rangle) \). Let \( k \in \mathbb{R}, \Lambda_k = (K_0 + |y|^4 + |v|^2 + |\eta|^2)^k \) be admissible metric and \( n = \max\{2k, 1, \gamma + 2s, 2s\} \). Then by lemma 6.4, we choose
\( K_0 > 1 \) so large that \( \Lambda_k^w \in Op(\Lambda_k) \) is invertible with \((\Lambda_k^w)^{-1} \in Op(\Lambda_{-k})\). Such choice of \( n \) assures \( \varphi \) is good enough that the following estimates about adjoint are valid by lemma 6.6. Also equation (16) is valid for \( \varphi \in H(\Lambda_n) \) by density. Thus for any \( \varphi \in H(\Lambda_n) \),

\[
\text{Re}(\zeta \varphi + F \varphi + A \varphi, \Lambda^w_k \Lambda^w_k \varphi)_{L^2} \\
= \text{Re}(\Lambda^w_k(\zeta \varphi + F \varphi + A \varphi), \Lambda^w_k \varphi)_{L^2} \\
= \text{Re}((\zeta I + F + A)\Lambda^w_k \varphi, \Lambda^w_k \varphi)_{L^2} + \text{Re}([\Lambda^w_k, F + A] \varphi, \Lambda^w_k \varphi)_{L^2} \\
\geq \frac{1}{C} \|\Lambda^w_k \varphi\|_{L^2}^2 - \|([\Lambda^w_k, F + A] \varphi, \Lambda^w_k \varphi)_{L^2} |.
\]

(17)

Note that \( \langle \eta \rangle \langle v \rangle \lesssim (1 + |v| + |\eta|) + (1 + |v|^2 + |\eta|^2) \lesssim (1 + |v|^2 + |\eta|^2) \). Then by Young’s inequality with \( \frac{1}{3} + \frac{1}{6} = 1 \), we have

\[
\langle \eta \rangle \langle y \rangle \lesssim \langle \eta \rangle^{3/2} + \langle y \rangle^3 \lesssim (1 + |\eta|^{3/2} + |y|^{3/2})^{\frac{4}{3}} \lesssim (1 + |\eta|^2 + |y|^4)^{3/4} \lesssim \Lambda_3^{3/4}.
\]

(18)

To control the commutators, by (95) and the comments therein, we obtain

\[
[\Lambda^w_k, F] \in Op(\langle \eta \rangle \Lambda_{k-1}) \subset Op(\Lambda_{k-\frac{3}{2}}),
\]

\[
[\Lambda^w_k, A] \in Op(a\Lambda_{k-\frac{3}{2}}),
\]

and hence

\[
\|([\Lambda^w_k, F] \varphi, \Lambda^w_k \varphi)_{L^2}\| = \|((\Lambda^w_k)^{-1} [\Lambda^w_k, F] \varphi, \Lambda^w_k \varphi)_{L^2}\| \lesssim \|\Lambda^w_{k-\frac{3}{2}} \varphi\|_{L^2}^2,
\]

\[
\|([\Lambda^w_k, A] \varphi, \Lambda^w_k \varphi)_{L^2}\| = \|((\Lambda^w_k)^{-1} (\Lambda_{\frac{1}{2}}^w)^{-1} [\Lambda^w_k, A] \varphi, \Lambda^w_{\frac{1}{2}} \Lambda_{\frac{1}{2}}^w \varphi)_{L^2}\| \lesssim \|(a^{1/2})^w \Lambda^w_{k-\frac{3}{2}} \varphi\|_{L^2}^2.
\]

Therefore, (17) becomes

\[
\|\Lambda^w_k \varphi\|_{H^\infty}^2 \lesssim \|([\Lambda^w_k, F + A] \varphi, \Lambda^w_k \varphi)_{L^2}\| + \text{Re}(\zeta \varphi + F \varphi + A \varphi, \Lambda^w_k \Lambda^w_k \varphi)_{L^2} \\
\lesssim \|\Lambda^w_{k-\frac{3}{2}} \varphi\|_{L^2}^2 + \|((a^{1/2})^w \Lambda^w_{k-\frac{3}{2}} \varphi\|_{L^2}^2 + \text{Re}(\varphi, (\zeta - F + A)\Lambda^w_k \Lambda^w_k \varphi)_{L^2},
\]

(19)

where the constant depends on \( \zeta \).

Let \( \delta \in (0, 1) \) and \( \Phi_\delta = (K_0 + \delta^2(|y|^4 + |v|^2 + |\eta|^2))^{-n/2} \). Then we choose \( K_0 > 0 \) sufficiently large such that \( \Phi_\delta^w \in S(\Phi_\delta) \) is invertible by lemma 6.4. Choose \( \varphi = \Phi_\delta^w g \in H(\Lambda_n) \), then

\[
\|([\Lambda^w_k, F + A] \varphi, \Lambda^w_k \varphi)_{L^2}\| = \|((\Phi_\delta^w g, (\zeta - F + A)\Lambda^w_k \Lambda^w_k \varphi)_{L^2}|
\]

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\[ \leq \left| (g, [\Phi^w_\delta, -F + A] \Lambda^w_k \Lambda^w_k \varphi)_{L^2} \right| + \left| (g, (\tilde{\zeta} - F + A) \Phi^w_\delta \Lambda^w_k \Lambda^w_k \varphi)_{L^2} \right| . \tag{20} \]

For the commutator \([\Phi^w_\delta, -F + A]\), by (18),

\[ [\Phi^w_\delta, -F] \in \text{Op}(\delta^{1/2} \Phi_\delta), \quad [\Phi^w_\delta, A] \in \text{Op}(\delta a \Phi_\delta), \]

uniformly in \(\delta\) and thus

\[ \left| (g, [\Phi^w_\delta, -F] \Lambda^w_k \Lambda^w_k \varphi)_{L^2} \right| = \left| (\Lambda^w_k \Phi^w_\delta g, (\Lambda^w_k)^{-1} \Phi^w_\delta ^{-1} [\Phi^w_\delta, F] \Lambda^w_k \Lambda^w_k \Phi^w_\delta g)_{L^2} \right| \]

\[ \lesssim \delta^{1/2} \| \Lambda^w_k \Phi^w_\delta g \|_{L^2}^2, \]

\[ \left| (g, [\Phi^w_\delta, A] \Lambda^w_k \Lambda^w_k \varphi)_{L^2} \right| = \left| ((a^{1/2})^w \Lambda^w_k \Phi^w_\delta g, ((a^{1/2})^w)^{-1} (\Lambda^w_k)^{-1} [\Phi^w_\delta, A] \Lambda^w_k \Lambda^w_k \Phi^w_\delta g)_{L^2} \right| \]

\[ \lesssim \delta \| (a^{1/2})^w \Lambda^w_k \Phi^w_\delta g \|_{L^2}^2. \]

Together with (16), (20) becomes

\[ \left| (\varphi, (\tilde{\zeta} - F + A) \Lambda^w_k \Lambda^w_k \varphi)_{L^2} \right| \]

\[ \leq \left| (g, [\Phi^w_\delta, (-F + A)] \Lambda^w_k \Lambda^w_k \varphi)_{L^2} \right| + \left| (g, (\tilde{\zeta} - F + A) \Phi^w_\delta \Lambda^w_k \Lambda^w_k \varphi)_{L^2} \right| \]

\[ \lesssim \delta^{1/2} \| \Lambda^w_k \varphi \|_{L^2}^2 + \delta \| (a^{1/2})^w \Lambda^w_k \varphi \|_{L^2}^2 + \left| (\Lambda^w_k \Phi^w_\delta f, \Lambda^w_k \varphi)_{L^2} \right| \]

\[ \lesssim \delta^{1/2} \| \Lambda^w_k \varphi \|_{L^2}^2 + \delta \| (a^{1/2})^w \Lambda^w_k \varphi \|_{L^2}^2 + C \| \Lambda^w_k \Phi^w_\delta f \|_{H^{-1}}^2 + \kappa \| \Lambda^w_k \varphi \|_{H^{-1}}^2, \]

for \(\kappa > 0\). Note that \(\Phi_\delta \leq 1\). Return to (19), noticing \(\Phi_\delta \in S(1)\) uniformly in \(\delta\) and choosing \(\delta, \kappa\) sufficiently small, we have \(\| \Phi^w_\delta (\cdot) \|_{L^2} \leq C \| \cdot \|_{L^2}\) and

\[ \| (a^{1/2})^w \Lambda^w_k \varphi \|_{H(\Phi^w_\delta)} + \| \Lambda^w_k \varphi \|_{H(\Phi^w_\delta)} \lesssim \| \Lambda^w_{k-\frac{1}{2}} g \|_{L^2} + \| (a^{1/2})^w \Lambda^w_{k-\frac{1}{2}} g \|_{L^2} + \| \Lambda^w_k f \|_{H^{-1}} , \]

whenever the right hand side is well-defined, where the constant is independent of \(\delta\). Recall the definition (96) of Sobolev space \(H(\Phi^w_\delta)\) and \(H(1) = L^2\), we let \(\delta \to 0\) to conclude that

\[ \| \Lambda^w_k \varphi \|_{H} \lesssim \| \Lambda^w_{k-\frac{1}{2}} g \|_{H} + \| \Lambda^w_k f \|_{H^{-1}} \lesssim \cdots \lesssim \| \Lambda^w_{-n} g \|_{H} + \| \Lambda^w_k f \|_{H^{-1}} , \]

if the right hand side is well-defined. Since \(g \in H(a^{1/2}) \subset H(\Lambda_{-n})\), we obtain that \(g \in H(\Lambda_k)\) for any \(k \geq 0\). Hence \(g \in H(a) \cap H(\langle v \rangle \langle y \rangle)\) if we choose \(k\) sufficiently large and the range of \(\zeta I + F + A\) with domain \(H(a) \cap H(\langle v \rangle \langle y \rangle)\) is dense in \(L^2\).

\(\square\)

**Theorem 2.3.** Write \(L^2 = L^2_v(\mathbb{R}^d),\ H(a^{1/2}) = H_v(a^{1/2})(\mathbb{R}^d), \mathcal{S} = \mathcal{S}(\mathbb{R}^d)\) in this theorem. Denote

\[ \| \cdot \|_H := \| \cdot \|_{L^2} + \| \cdot \|_{H(a^{1/2})}, \quad \| \cdot \|_{H^{-1}} := \min\{ \| \cdot \|_{L^2}, \| \cdot \|_{H(a^{-1/2})} \} . \]
Let $y \in \mathbb{R}^d$ be fixed, $K_0 >> 1$, $\Lambda_{k,l} := (K_0 + |v|^2)^{k/2}(K_0 + |\eta|^2)^{l/2}$ and

$$\begin{align*}
\text{Re}\zeta &> -\frac{\nu_0}{C_1}, \quad \text{if } \gamma + 2s \geq 0, \\
\text{Re}\zeta &> 0, \quad \text{if } \gamma + 2s < 0.
\end{align*}$$

Suppose $f \in \mathcal{S}$, then there exists unique solution $g \in \mathcal{S}$ to

$$(\zeta I + 2\pi iv \cdot y + A)g = f.$$  \hfill (21)

Moreover, for $k,l \geq 0$,

$$\|\langle v \rangle^k \langle D_v \rangle^l g\|_H \lesssim \|\langle v \rangle^k \langle D_v \rangle^l f\|_{H^{-1}}.$$  \hfill (22)

**Remark 2.4.** Later, the lemma 3.2 will shows that the estimate (22) actually give the control for $g = (\lambda I - \hat{A}(g))^{-1} f$.

**Proof.** Notice that the constants in the following estimates will depend on $y$. Let $F = 2\pi iv \cdot y$. Then $F$ can be regarded as a pseudo-differential operator with symbol in $S(\langle v \rangle)$. $\partial_v F \in Op(1)$ and $(Ff, f)_{L^2_v} = (f, -Ff)_{L^2_v}$ if $f \in H(\langle v \rangle)$.

1. A similar argument to step one in theorem 2.1, with replacing $L^2_{x,v}$ by $L^2_v$ and $H_{x,v}$ by $H_v$, gives that for any $f$ with $\|f\|_{H^{-1}} < \infty$, there exists unique $g \in H(a^{1/2}) \cap L^2$ such that

$$\langle g, \zeta f - F\varphi + A\varphi \rangle_{L^2} = \langle f, \varphi \rangle_{L^2},$$

for all $\varphi \in \mathcal{S}$.

2. Let $k,l \in \mathbb{R}$, $n = 2k^+ + 2l^+ + 2 + \max\{\gamma + 2s, 2s\}$ and $\Lambda_{k,l} = (K_0 + |v|^2)^{k/2}(K_0 + |\eta|^2)^{l/2}$ be an admissible metric, where $k^+ = \max\{0, k\}$. Then by lemma 6.4, we choose $K_0$ sufficiently large such that $\Lambda_{k,l}^{w}$ is invertible and $(\Lambda_{k,l}^{w})^{-1} \in S(\Lambda_{-k,-l})$. The choice of $n$ assures $\varphi$ below is good enough that the following equations on adjoint are valid by lemma 6.6. Since $\mathcal{S}$ is dense in $H(\Lambda_{n,n})$, the equation (23) is valid for $\varphi \in H(\Lambda_{n,n})$. For such $\varphi$, we have $\zeta \varphi + F\varphi + A\varphi \in H(\Lambda_{k,l})$ and hence

$$\text{Re}(\zeta \varphi + F\varphi + A\varphi, \Lambda_{k,l}^{w}\Lambda_{k,l}^{w}\varphi)_{L^2_v}$$

$$= \text{Re}(\Lambda_{k,l}^{w}(\zeta \varphi + F\varphi + A\varphi), \Lambda_{k,l}^{w}\varphi)_{L^2_v}$$

$$= \text{Re}(\langle \zeta I + F + A \rangle \Lambda_{k,l}^{w}\varphi, \Lambda_{k,l}^{w}\varphi)_{L^2_v} + \text{Re}([\Lambda_{k,l}^{w}, F + A] \varphi, \Lambda_{k,l}^{w}\varphi)_{L^2_v}$$

$$\geq \frac{1}{C} \|\Lambda_{k,l}^{w}\varphi\|_{L^2_v}^2 - \|([\Lambda_{k,l}^{w}, F + A] \varphi, \Lambda_{k,l}^{w}\varphi)_{L^2_v}\|.$$

(24)

For the commutators, we have

$$[\Lambda_{k,l}^{w}, F + A] = \left( \int_0^1 \partial_\theta A_{k,l}^{w} \partial_\theta (F + A) \, d\theta \right)^w - \left( \int_0^1 \partial_\theta A_{k,l}^{w} \partial_\theta (F + A) \, d\theta \right)^w,$$
where we denote $F,$ $A$ to be their corresponding symbols for convenience of notation. By (95),

$$
\int_0^1 \partial_\eta \Lambda_{k,l} \# \partial_\eta (F + A) \, d\theta \in S((1 + a)\Lambda_{k,l-1})
$$

$$
\int_0^1 \partial_{\theta} \Lambda_{k,l} \# \partial_{\theta} (F + A) \, d\theta \in S((1 + a)\Lambda_{k-1,l}).
$$

In particular when $l = 0$, $\partial_\eta \Lambda_{k,l} = 0$ and hence $[\Lambda_{k,0}^w, F + A] \in Op((1 + a)\Lambda_{k-1,0})$ while when $k = 0$, $\partial_\eta \Lambda_{k,l} = 0$ and hence $[\Lambda_{0,l}^w, F + A] \in Op((1 + a)\Lambda_{0,l-1})$. Similar to the estimate on commutators in the second step of theorem 2.1, we have

$$
\left| \left\langle [\Lambda_{k,l}^w, F + A] \varphi, \Lambda_{k,l}^w \varphi \right\rangle_{L^2} \right| \lesssim \left\| \Lambda_{k-\frac{1}{2},l}^w \varphi \right\|_H^2 + \left\| \Lambda_{k,l-\frac{1}{2}}^w \varphi \right\|_H^2,
$$

$$
\left| \left\langle [\Lambda_{k,0}^w, F + A] \varphi, \Lambda_{k,0}^w \varphi \right\rangle_{L^2} \right| \lesssim \left\| \Lambda_{k-\frac{1}{2},0}^w \varphi \right\|_H^2,
$$

$$
\left| \left\langle [\Lambda_{0,l}^w, F + A] \varphi, \Lambda_{0,l}^w \varphi \right\rangle_{L^2} \right| \lesssim \left\| \Lambda_{0,l-\frac{1}{2}}^w \varphi \right\|_H^2,
$$

and hence (24) gives

$$
\left\| \Lambda_{k,l}^w \varphi \right\|_H^2 \lesssim \left\| \Lambda_{k-\frac{1}{2},l}^w \varphi \right\|_H^2 + \left\| \Lambda_{k,l-\frac{1}{2}}^w \varphi \right\|_H^2 + \text{Re}(\left( \zeta + F + A \right) \varphi, \Lambda_{k,l}^w \varphi)_{L^2},
$$

$$
\left\| \Lambda_{k,0}^w \varphi \right\|_H^2 \lesssim \left\| \Lambda_{k-\frac{1}{2},0}^w \varphi \right\|_H^2 + \text{Re}(\left( \zeta + F + A \right) \varphi, \Lambda_{k,0}^w \varphi)_{L^2},
$$

$$
\left\| \Lambda_{0,l}^w \varphi \right\|_H^2 \lesssim \left\| \Lambda_{0,l-\frac{1}{2}}^w \varphi \right\|_H^2 + \text{Re}(\left( \zeta + F + A \right) \varphi, \Lambda_{0,l}^w \varphi)_{L^2},
$$

where the constant depends on $\zeta$ and $y$. Let $\delta \in (0,1)$ and define $\Phi_\delta = (1 + \delta^2(|v|^2 + |\eta|^2))^{-n}$. Choose $\varphi = \Phi_\delta^w g \in H(\Lambda_{a,n})$, then $\Phi_\delta^w \Lambda_{k,l}^w \varphi \in H(\Lambda_{a,n})$ and (23) gives

$$
\left( \zeta + F + A \right) \varphi, \Lambda_{k,l}^w \varphi \right\rangle_{L^2} = (\Phi_\delta^w g, (\zeta - F + A) \Lambda_{k,l}^w \varphi)_{L^2} = (g, \left[ \Phi_\delta^w, -F + A \right] \Lambda_{k,l}^w \varphi)_{L^2} + (f, \Phi_\delta^w \Lambda_{k,l}^w \varphi)_{L^2}.
$$

Here

$$
\left[ \Phi_\delta^w, -F + A \right] \in Op(\delta(1 + a)\Phi_\delta),
$$

uniformly in $\delta$ and hence similar to theorem 2.1, we have

$$
\left| \left\langle g, [\Phi_\delta^w, -F + A] \Lambda_{k,l}^w \varphi \right\rangle_{L^2} \right| \lesssim \delta \left\| \Lambda_{k,l}^w \Phi_\delta^w g \right\|_H^2.
$$

Thus

$$
\left| \left( \zeta + F + A \right) \varphi, \Lambda_{k,l}^w \varphi \right\rangle_{L^2} \lesssim \delta \left\| \Lambda_{k,l}^w \Phi_\delta^w g \right\|_H^2 + \left| \left\langle \Lambda_{k,l}^w \Phi_\delta^w f, \Lambda_{k,l}^w \Phi_\delta^w g \right\rangle_{L^2} \right| \lesssim 2\delta \left\| \Lambda_{k,l}^w \Phi_\delta^w g \right\|_H^2 + C_\delta \left\| \Lambda_{k,l}^w \Phi_\delta^w f \right\|_{H^{-1}}^2.
$$

Substitute this into (25), by picking $\delta$ sufficiently small and note that $\Phi_\delta \in S(1)$ uniformly in $\delta$, we have $\left\| \Phi_\delta^w (\cdot) \right\|_{L^2} \leq C \cdot \left\| \cdot \right\|_{L^2}$ and

$$
\left\| (a^{1/2})^w \Lambda_{k,l}^w g \right\|_{H(\Phi_\delta^w)} + \left\| \Lambda_{k,l}^w g \right\|_{H(\Phi_\delta^w)} \lesssim \left\| \Lambda_{k-\frac{1}{2},l}^w g \right\|_H + \left\| \Lambda_{k,l-\frac{1}{2}}^w g \right\|_H + \left\| \Lambda_{k,l}^w f \right\|_{H^{-1}},
$$

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whenever the right hand side is well defined, where the constant is independent of $\delta$. Recall the definition (96) and let $\delta \to 0$, we obtain
\[
\|A_{k,l} w g\|_H \lesssim \|A_{k-\frac{1}{2},l} w g\|_H + \|A_{k-\frac{1}{2},l} f\|_{H^{-1}},
\]
whenever the right hand side is well-defined. Similarly, substituting (28) into (26)(27) and letting $\delta \to 0$, we have
\[
\|A_{k,0} w g\|_H \lesssim \|A_{k-\frac{1}{2},0} w g\|_H + \|A_{k,0} f\|_{H^{-1}},
\]
\[
\|A_{0,l} w g\|_H \lesssim \|A_{0,-\frac{1}{2},l} w g\|_H + \|A_{0,0} f\|_{H^{-1}}.
\]
3. Let $k,l \in \mathbb{R}$, and recall $\|g\|_H < \infty$. Note that $\|A_{k,0} w g\|_{L^2} \lesssim \|A_{k_2,0} g\|_{L^2}$ for $k_1 \leq k_2$. Then (31) yields
\[
\|A_{k,0} w g\|_H \lesssim \|A_{k-\frac{1}{2},0} w g\|_H + \|A_{k,0} f\|_{H^{-1}} \lesssim \cdots \lesssim \|A_{-n,0} w g\|_H + \|A_{k,0} f\|_{H^{-1}}.
\]
Similarly (32) gives
\[
\|A_{0,l} w g\|_H \lesssim \|A_{0,-n} w g\|_H + \|A_{0,l} f\|_{H^{-1}}.
\]
Finally, substitute these two estimates into (30),
\[
\|A_{k,l} w g\|_H \lesssim \|A_{k-\frac{1}{2},l} w g\|_H + \|A_{k,-\frac{1}{2},l} g\|_H + \|A_{k,l} f\|_{H^{-1}}
\]
\[
\lesssim \cdots \lesssim \|A_{0,l} w g\|_H + \|A_{k,0} w g\|_H + \|A_{k,l} f\|_{H^{-1}}
\]
\[
\lesssim \|g\|_H + \|A_{k,l} f\|_{H^{-1}},
\]
for $f$ satisfying $\|A_{k,l} f\|_{H^{-1}} < \infty$. Therefore if $g \in \mathcal{S}$, then $g \in H(A_{k,l})$ for any $k,l \geq 0$ and by Sobolev embedding $g \in \mathcal{S}$ is a strong solution to (21). Taking inner product in (21) with $g$, we obtain
\[
\text{Re}((\zeta I + 2\pi i v \cdot y + A)g, g)_{L^2} = (f, g)_{L^2},
\]
\[
\|g\|_H^2 \lesssim \text{Re} \zeta \|g\|_{L^2}^2 + \nu_0 \|g\|_{H^{a_1/2}}^2 \lesssim \|f\|_{H^{-1}} \|g\|_H.
\]
Thus $\|g\|_H \lesssim \|f\|_{H^{-1}}$. Substitute this into (33), we obtain (22), since $\langle \psi \rangle^k \langle D \psi \rangle^l \langle \cdot \rangle_{L^2}$ is equivalent to $\|A_{k,l}^\psi(\cdot)\|_{L^2}$.

### 3 Spectrum Structure

With the tools in section 2, we can give the rigorous proof of generating strongly continuous semigroup as well as the spectrum structure to $\hat{B}(y)$ in this section. We will denote $L^2 = L^2(\mathbb{R}^d)$ in this section.

Recall that $C_1 := \sup_{f \in L^2} \|\varphi\|_{L^2}/\|\varphi\|_{H^{a_1/2}}$. Let $\zeta \in \mathbb{C}$ satisfies $\text{Re} \zeta > -2\nu_1$ with
\[
\nu_1 = \begin{cases}
\min\{\frac{\nu_0}{2}, \frac{\nu_0}{2C_1}\}, & \text{if } \gamma + 2s \geq 0, \\
0, & \text{if } \gamma + 2s < 0.
\end{cases}
\]
3.1 Generating strongly continuous semigroup

By theorem 2.3 and (9), the operator \(-\zeta - 2\pi iy \cdot v - A\) with domain \(D(-\zeta - 2\pi iy \cdot v - A) := \mathcal{S}\) on Banach space \(L^2(\mathbb{R}^d)\) is densely defined, dissipative and has dense range in \(L^2_v\). Thus by theorem 6.8, its closure \(-\zeta - 2\pi iy \cdot v - A\) generates a contraction semigroup. Since \(K\) is compact on \(L^2\), by theorem 6.10, \(-\zeta - 2\pi iy \cdot v + L\) generates a strongly continuous semigroup on \(L^2(\mathbb{R}^d)\).

Similarly, theorem 2.1 and (9) show that the operator \(-\zeta I - v \cdot \nabla_x - A\) with domain \(H(a) \cap H(\langle v\rangle)\) on \(L^2(\mathbb{R}^d_{x,v})\) is densely defined, dissipative and has dense range. Thus its closure \(-\zeta I - v \cdot \nabla_x - A\) generates a contraction semigroup and \(-\zeta I - v \cdot \nabla_x + L\) generates a strongly continuous semigroup on \(L^2(\mathbb{R}^d_{x,v})\).

Recall the definition 6.1 of the closure, and that \(K\) is continuous, we can define

\[
\hat{A}(y) := -2\pi iy \cdot v - A, \\
\hat{B}(y) := -2\pi iy \cdot v + L = -2\pi iy \cdot v - A + K, \\
B := -v \cdot \nabla_x + L,
\]

with \(D(\hat{B}(y)) := D(\hat{A}(y))\). Then \(\hat{A}(y)\), (resp. \(\hat{B}(y)\), \(B\)) generates strongly continuous semigroup on \(L^2(\mathbb{R}^d)\), (resp. \(L^2_v\), \(L^2_{x,v}\)). By Hille-Yoshida theorem, for \(\Re \lambda > -2\nu_1\), \((\lambda I - \hat{A}(y))^{-1} : L^2_v \to L^2_v\) is linear continuous and there exists \(C > 0\) such that for \(\Re \lambda > C\), \((\lambda I - \hat{B}(y))^{-1} : L^2_v \to L^2_v\) and \((\lambda I - B)^{-1} : L^2_{x,v} \to L^2_{x,v}\) are linear continuous.

By (9), we have for \(f, g \in \mathcal{S}\),

\[
||f||_{H(a^{1/2})}^2 + ||f||_{L^2}^2 \lesssim \Re((I + 2\pi iy \cdot v + A)f, f)_{L^2}, \\
((-2\pi iy \cdot v - A)f, g)_{L^2} = (f, (2\pi iy \cdot v - A)g)_{L^2},
\]

where \(I\) is the identity mapping. Applying lemma 3.1 below to \(I + 2\pi iy \cdot v + A\), we have

\[
\hat{A}(y) = \hat{A}(-y)^*, \\
\hat{B}(y) = \hat{B}(-y)^*,
\]

(36)

since \(I\) and \(K\) are self-adjoint bounded operator on \(L^2\).

**Lemma 3.1.** Let \((A, D(A))\), \((B, D(B))\) be two densely defined linear operator on complex Hilbert space \((H, \| \cdot \|)\) such that \(D(A) = D(B)\), \(\Im(A) = \Im(A) = H\). Let \(C > 0\) and suppose for \(f, g \in D(A)\),

\[
\Re(Af, f) \geq \frac{1}{C}||f||^2, \\
(Af, g) = (f, Bg).
\]

Then \(A = B^*\) and \(A^{-1} = A^{-1}\) is continuous on \(H\). Also for \(f \in D(A)\),

\[
\Re(Af, f) \geq \frac{1}{C}||f||^2.
\]

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Proof. From the assumption, we have for $f \in D(A)$ that

$$\|f\| \leq C\|Af\|, \quad \|f\| \leq C\|Bf\|.$$ 

Thus $A, B$ are injective and $A^{-1}, B^{-1}$ are densely defined operator and for $f = A\phi \in D(A^{-1}) = \text{Im}(A)$, $g = B\psi \in D(B^{-1}) = \text{Im}(B)$, with $\phi, \psi \in D(A) = D(B)$, we have

$$\|A^{-1}f\| \leq C\|f\|, \quad \|B^{-1}g\|_{L^2} \leq C\|g\|,$$

$$(A^{-1}f, g) = (\phi, B\psi) = (A\phi, \psi) = (f, B^{-1}g).$$

In particular,

$$\text{Re}(A^{-1}f, f) = \text{Re}(\phi, B\phi) \geq \frac{1}{C}\|\phi\|^2 = \frac{1}{C}\|A^{-1}f\|^2.$$ 

Thus, $A^{-1}, B^{-1}$ are closable and $\overline{A^{-1}}, \overline{B^{-1}}$ are linear bounded operator on $H$. Indeed, for $f \in L^2$, there exists $f_n \in D(A)$ such that $f_n \to f$ in $H$ as $n \to \infty$ and hence $A^{-1}f_n$ is Cauchy in $H$. So $(f_n, A^{-1}f_n) \in G(A)$ converges and $f \in D(A^{-1})$. Thus $\overline{A^{-1}}$ is an closed operator defined on the whole space $H$, and hence continuous on $H$. $\overline{B^{-1}}$ is similar. Thus, for $f, g \in H$, by density argument,

$$(\overline{A^{-1}}f, g) = (f, \overline{B^{-1}}g) = (\overline{B^{-1}*}f, g) = ((B^*)^{-1}f, g),$$

$$(\overline{A^{-1}}f, f) \geq \frac{1}{C}\|\overline{A^{-1}}f\|^2.$$ 

That is $\overline{A^{-1}} = (B^*)^{-1}$ is injective on $H$. On the other hand, $\overline{(A^{-1})^{-1}} = \overline{A}$, since the graph

$$G((\overline{A^{-1}})^{-1}) = \{(g, (\overline{A^{-1}})^{-1}g) : g \in D((\overline{A^{-1}})^{-1}) = \text{Im}(\overline{A^{-1}})\}$$

$$= \{(\overline{A^{-1}}f, f) : f \in L^2\}$$

$$= \{((A^{-1}f, f) : f \in D(A^{-1}) = \text{Im}(A)\}$$

$$= \{(g, Ag) : g \in D(A)\}$$

$$= G(\overline{A}).$$

Thus $A$ is closable and $\overline{A} = (\overline{A^{-1}})^{-1} = B^*$. Also for $f \in D(\overline{A}) = \text{Im}(\overline{A^{-1}})$, there exists $g \in D(\overline{A}^{-1}) = H$ such that $f = \overline{A}^{-1}g$ and hence

$$\text{Re}(\overline{Af}, f) = \text{Re}(g, \overline{A}^{-1}g) \geq \frac{1}{C}\|\overline{A}^{-1}g\|^2 = \frac{1}{C}\|f\|^2.$$

By the definition of closure, $\hat{A}(y)$ will be a dissipative operator as well. Then we have the following basic boundedness on $(\lambda I - \hat{A}(y))^{-1}$. This lemma will be used in the later argument readily. 

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Lemma 3.2. (1) \(D(\hat{A}(y)) \subset H(a^{1/2})\) and
\[
\|f\|_{H(a^{1/2})}^2 \leq \frac{1}{\nu_0} \text{Re}(-\hat{A}(y)f, f)_{L^2_y}
\]
(2) Let \(\text{Re}\lambda \geq -\nu_1, f \in H(a^{-1/2}) \cap L^2\), then \((\lambda I - \hat{A}(y))^{-1} f \in H(a^{1/2}) \cap L^2\) and
\[
\| (\lambda I - \hat{A}(y))^{-1} f \|_{H(a^{1/2})} \leq \frac{2}{\nu_0} \| f \|_{H(a^{-1/2})}, \tag{37}
\]
\[
\| (\lambda I - \hat{A}(y))^{-1} f \|_{L^2} \leq \frac{2}{\nu_1} \| f \|_{L^2}. \tag{38}
\]
Note that the constants are independent of \(\lambda\). In particular, if \(f \in \mathcal{S}\), then \((\lambda I - \hat{A}(y))^{-1} f \in \mathcal{S}\).

Proof. 1. By the non-positiveness (9) of \(A\), for \(g \in \mathcal{S}\), we have
\[
\text{Re}(-2\pi iy \cdot v g - Ag, g)_{L^2_y} \leq -\nu_0 \| g \|_{H(a^{1/2})}^2.
\]
Fix \(f \in D(\hat{A}(y))\). Note that the graph \(G(\hat{A}(y)) = \overline{G(-2\pi iy \cdot v - A)}\), there exists \(f_n \in D(-2\pi iy \cdot vf - A)\) such that \(f_n \to f\) and \(\hat{A}(y)f_n \to \hat{A}(y)f\) in \(L^2\). Thus \(\{f_n\}_{L^2_y}\), \(\{\hat{A}(y)f_n\}_{L^2_y}\) are bounded set and hence
\[
\|f_n\|_{H(a^{1/2})}^2 \leq \frac{\text{Re}(-\hat{A}(y)f_n, f_n)_{L^2_y}}{\nu_0}
\]
is bounded. By Banach-Alaoglu theorem, \(f_n\) is weakly* compact in \(H(a^{1/2})\). That is, there exists a sub-sequence \(\{f_{n_k}\} \subset \{f_n\}\) and \(g \in H(a^{1/2})\) such that for \(\varphi \in \mathcal{S}\),
\[
(f_{n_k}, \varphi)_{H(a^{1/2})} \to (g, \varphi)_{H(a^{1/2})} = (g, (a^{1/2})w(a^{1/2})u \varphi)_{L^2_y}, \quad \text{as} \; n_k \to \infty.
\]
For \(\psi \in \mathcal{S}\), choose \(\varphi = ((a^{1/2})w)^{-1}(a^{1/2})w)^{-1} \psi \in \mathcal{S}\), then
\[
(f_{n_k}, \psi)_{L^2} \to (g, \psi)_{L^2}, \quad \text{as} \; n_k \to \infty.
\]
On the other hand, \(f_n \to f\) in \(L^2\). Thus \(f = g \in H(a^{1/2})\) and
\[
\nu_0 \| f \|_{H(a^{1/2})}^2 \leq \nu_0 \liminf_{n_k \to \infty} \| f_{n_k} \|_{H(a^{1/2})}^2 \leq \liminf_{n_k \to \infty} \text{Re}(-\hat{A}(y)f_{n_k}, f_{n_k})_{L^2_y} \leq \text{Re}(-\hat{A}(y)f, f)_{L^2_y}.
\]
Therefore, \(f \in H(a^{1/2})\) and so \(D(\hat{A}(y)) \subset H(a^{1/2})\). Now we let \(f \in L^2 \cap H(a^{-1/2})\), \(\varphi := (\lambda I - \hat{A}(y))^{-1} f\), then
\[
\nu_0 \| \varphi \|_{H(a^{1/2})}^2 + \text{Re} \| \varphi \|_{L^2}^2 \leq \text{Re}(\lambda - \hat{A}(y))\varphi, \varphi)_{L^2_y}
\]
\[
\leq |(f, \varphi)_{L^2}| \\
\leq \|f\|_{H(a^{1/2})} \|\varphi\|_{H(a^{1/2})}, \\
\|\varphi\|_{H(a^{1/2})} \leq \frac{2}{\nu_0} \|f\|_{H(a^{1/2})},
\]

by our choice (34) of \(\nu_1\) and \(\text{Re}\lambda \geq -\nu_1\).

2. By the first step of theorem 2.3, for \(\text{Re}\lambda \geq -\nu_1\), there exists unique \(g \in H(a^{1/2}) \cap L^2\) such that for \(\varphi \in \mathcal{S}\),

\[
(g, (\lambda - 2\pi i y \cdot v + A)\varphi)_{L^2_v} = (f, \varphi)_{L^2_v}.
\]

Notice \((\lambda - \hat{A}(y))(\lambda - \hat{A}(y))^{-1}f = f\) and use (36), we have

\[
((\lambda - \hat{A}(y))^{-1}f, (\lambda - 2\pi i y \cdot v + A)\varphi)_{L^2_v} = (f, \varphi)_{L^2_v}.
\]

For \(\psi \in \mathcal{S}\), again by theorem 2.3, there exists \(\varphi \in \mathcal{S}\) such that \((\lambda - 2\pi i y \cdot v + A)\varphi = \psi\).

Combine the above two identity, we have for \(\psi \in \mathcal{S}\),

\[
((\lambda - \hat{A}(y))^{-1}f - g, \psi)_{L^2_v} = 0, \\
(\lambda - \hat{A}(y))^{-1}f = g \in H(a^{1/2}) \cap L^2.
\]

In particular when \(f \in \mathcal{S}\), by theorem 2.1, we have \(g \in \mathcal{S}\). Also, since \(\frac{3\alpha}{2} + \hat{A}(y)\) generates a strongly continuous semigroup, by Hille-Yoshida theorem, we have

\[
\|(\lambda I - \frac{3\nu_1}{2} - \hat{A}(y))^{-1}f\|_{L^2} \leq \frac{1}{\text{Re}\lambda} \|f\|_{L^2}, \text{ for Re}\lambda > 0,
\]

\[
\|(\lambda I - \hat{A}(y))^{-1}f\|_{L^2} \leq \frac{1}{\text{Re}\lambda + \frac{3\nu_1}{2}} \|f\|_{L^2}, \text{ for Re}\lambda > -\frac{3\nu_1}{2}.
\]

This proves (38).

\[\square\]

### 3.2 Spectrum structure of \(\hat{B}(y)\)

We next analyze the spectrum structure of \(\hat{B}(y)\) for hard potential \(\gamma + 2s \geq 0\). This theorem yields the essential spectrum of \(\hat{B}(y)\) lies in \(\{\lambda : \text{Re}\lambda \leq -\nu_1\}\), while the complementary set contains only eigenvalues of \(\hat{B}(y)\) which lies in \(\{\lambda \in \mathbb{C} : -\nu_1 + \delta < \text{Re}\lambda \leq 0\}\). That is, \(\hat{B}(y)\) has spectral gap. Also the corresponding eigenfunctions are Schwartz function.

**Theorem 3.3.** Write \(\lambda = \sigma + i\tau\). Suppose \(\gamma + 2s \geq 0\).

1. For any \(y \in \mathbb{R}^d\),

\[
\sigma(\hat{B}(y)) \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda \leq 0\}.
\]
(2). There exists $\tau_1, y_1 > 0$ such that for $y \in \mathbb{R}^d$,

$$\sigma(\hat{B}(y)) \cap \{ \lambda \in \mathbb{C} : -\nu_1 < \text{Re}\lambda \leq 0 \} \subset \{ \lambda \in \mathbb{C} : |\text{Im}\lambda| \leq \tau_1 \};$$

and for $|y| \geq y_1$,

$$\sigma(\hat{B}(y)) \cap \{ \lambda \in \mathbb{C} : -\nu_1 < \text{Re}\lambda \leq 0 \} = \emptyset.$$

(3). For any $\delta > 0$, the set

$$\sigma(\hat{B}(y)) \cap \{ \lambda \in \mathbb{C} : -\nu_1 + \delta < \text{Re}\lambda \leq 0 \}$$

consists of finitely many discrete eigenvalues of finite type without accumulation point.

If $\lambda \in \sigma(\hat{B}(y)) \cap \{ \lambda : -\nu_1 < \text{Re}\lambda \leq 0 \}$ is an eigenvalue and $f \in D(\hat{B}(y))$ is the corresponding eigen-function, then $f \in \mathcal{H}$. Also

$$\sigma(\hat{B}(y)) \cap \{ \text{Re}\lambda = 0 \} = \begin{cases} \emptyset, & \text{if } y \neq 0, \\ \{0\}, & \text{if } y = 0. \end{cases}$$

In particular, if $y = 0$, then $\text{Ker}\hat{L} = \text{Ker}L$.

(4). For any $y_2$, there exists $\sigma_1 \in (0, \nu_1)$ such that for $|y| \geq y_2$,

$$\sigma(\hat{B}(y)) \cap \{ \lambda \in \mathbb{C} : -2\sigma_1 \leq \text{Re}\lambda \leq 0 \} = \emptyset.$$

Proof. 1. For $\text{Re}\lambda > 0$, $\zeta > -\nu_1$, by Hille-Yoshida theorem, $\lambda + \zeta - \hat{A}(y)$ is invertible on $L^2$ and hence

$$\sigma(\hat{A}(y)) \subset \{ \lambda \in \mathbb{C} : \text{Re}\lambda \leq -\nu_1 \}.$$

Since $K$ is compact on $L^2$, $\hat{B}(y) = \hat{A}(y) + K$, corollary XVII.4.4 in [25] gives that essential spectrum of $\hat{B}(y)$ is contained in the essential spectrum of $\hat{A}(y)$:

$$\sigma_{\text{ess}}(\hat{B}(y)) \subset \sigma(\hat{A}(y)) \subset \{ \lambda \in \mathbb{C} : \text{Re}\lambda \leq -\nu_1 \},$$

while $\sigma(\hat{B}(y)) \cap \{ \lambda \in \mathbb{C} : \text{Re}\lambda = -\nu_1 \}$ consists of eigenvalues of finite type of $\hat{B}(y)$ with possible accumulation point on $\{ \text{Re} = -\nu_1 \}$.

2. Lemma 3.5 shows that there exists $\tau_1 \equiv y_1 > 0$ such that for $|\text{Im}\lambda| + |y| \geq \tau_1$,

$$\|(\lambda I - \hat{A}(y))^{-1}K\|_{L^2(L^2)} \leq 1/2$$

and thus $(I - (\lambda I - \hat{A}(y))^{-1}K)^{-1}$ exists with

$$\|(I - (\lambda I - \hat{A}(y))^{-1}K)^{-1}\|_{L^2(L^2)} \leq 2.$$ (41)

For $\delta > 0$, $y \in \mathbb{R}^d$, since $\rho(\hat{A}(y)) \supset \{ \lambda \in \mathbb{C} : \text{Re}\lambda > -\nu_1 \}$ and

$$(\lambda I - \hat{B}(y))^{-1} = (I - (\lambda I - \hat{A}(y))^{-1}K)^{-1}(\lambda I - \hat{A}(y))^{-1},$$
we have $\sigma(\hat{B}(y)) \cap \{\lambda : -\nu_1 + \delta \leq \text{Re}\lambda \leq 0\} \subset \{\lambda : |\text{Im}\lambda| \leq \tau_1\}$, as a bounded set, consists of discrete eigenvalues without accumulation point and hence the number of such eigenvalues is finite. On the other hand, for $|y| \geq y_1$, $\text{Re}\lambda \geq -\nu_1 + \delta$, we have that $I - (\lambda I - \hat{A}(y))^{-1}K$ and $\lambda I - \hat{A}(y)$ are always invertible and hence $\sigma(\hat{B}(y)) \cap \{\lambda : -\nu_1 + \delta \leq \text{Re}\lambda \leq 0\} = \emptyset$.

3. Note that for $f \in \mathcal{S}$, we have $\text{Re}(f, (-2\pi iy \cdot v + L)f)_{L^2} \leq 0$. Thus by definition of closure,

$$\{\text{Re}(f, \hat{B}(y)f)_{L^2} : f \in D(\hat{B}(y))\} \subset \{\text{Re}(f, (-2\pi iy \cdot v + L)f)_{L^2} : f \in \mathcal{S}\} \subset (-\infty, 0].$$

Let $\lambda \in \sigma(\hat{B}(y))$ satisfies $\text{Re}\lambda > -\nu_1$, then $\lambda$ is an eigenvalue to $\hat{B}(y)$. Suppose $f \in D(\hat{B}(y))$ is the corresponding eigen-function, then

$$\hat{B}(y)f = \lambda f,$$

$$(\hat{B}(y)f, f)_{L^2} = \lambda \|f\|_{L^2}^2.$$ 

Taking the real part, we have $\text{Re}\lambda \leq 0$.

For this eigen-function $f \in D(\hat{B}(y)) \subset H(a^{1/2}) \cap L^2$, we have

$$\hat{A}(y)f = Kf - \lambda f.$$ 

By (36), we have for $\varphi \in \mathcal{S}$ that

$$(\lambda f - \hat{A}(y)f, \varphi)_{L^2} = (f, (\lambda - 2\pi i y \cdot v + A)\varphi)_{L^2} = (Kf, \varphi)_{L^2}$$

Thus $f$ is a weak solution of $(\lambda + 2\pi iy \cdot v + A)f = Kf$, i.e. (23). Notice that from theorem 1.1, $K \in S((\nu)^{-n}(\eta)^{-n})$ is a good pseudo-differential operator. Apply the estimate (22) to $f$, for $k, l \geq 0$, we have

$$\|a^{1/2}w\langle \nu\rangle^k\langle Dv\rangle^l f\|_{L^2} \lesssim \|a^{-1/2}w\langle \nu\rangle^k\langle Dv\rangle^l Kf\|_{L^2} \lesssim \|f\|_{L^2},$$

whenever the right hand side is well-defined. Thus $f \in H((\langle \nu\rangle^k\langle \eta\rangle^l))$ for any $k, l \in \mathbb{R}$ and hence belongs to $\mathcal{S}$ by Sobolev embedding theorem. Now $f \in \mathcal{S}$ is smooth enough that the closure in $\hat{A}(y)$ can be canceled and the eigen-equation becomes

$$(-2\pi iy \cdot v + L)f = \lambda f,$$

$$(Lf, f)_{L^2} = \text{Re}\lambda \|f\|_{L^2}^2.$$ 

If $\text{Re}\lambda = 0$, then $Lf = 0$ and $-2\pi iy \cdot vf = i\text{Im}\lambda f$, which implies $y = \lambda = 0$. Also $\overline{L}f = 0$ yields $f \in \mathcal{S}$, hence $\text{Ker}\overline{L} \in \text{Ker}L$.

4. It suffices to prove (4) when $y_2 \in (0, y_1)$. We claim that there exists $0 < \sigma_1 < \nu_1$ such that for $|y| \in [y_2, y_1]$, $\sigma(\hat{B}(y)) \cap \{\lambda : -2\sigma_1 \leq \text{Re}\lambda \leq 0\} = \emptyset$. We prove this by contradiction. Suppose this fails, then for $n \in \mathbb{N}$, there exists eigenvalues

$$\lambda_n \in \sigma(\hat{B}(y)), \quad y_n \in \mathbb{R}^d,$$
with \(-\frac{1}{n} \leq \text{Re}\lambda \leq 0, |y_n| \in [y_2, y_1]\). Since \(\sigma(\hat{B}(y)) \subset \{\lambda : |\text{Im}\lambda| \leq \tau_1\}\), we find that \(\{\lambda_n\}\) is a bounded sequence. Let \(f_n \in D(\hat{B}(y))\) be the corresponding eigen-function to \(\lambda_n\) such that \(\|f_n\|_{L^2} = 1\). Then by statement (3), we have \(f_n \in \mathcal{S}\) and hence

\[
\begin{align*}
(2\pi i y_n \cdot v + L)f_n &= \lambda_n f_n, \\
\text{Re}\lambda_n &= (L f_n, f_n)_{L^2} \leq -\nu_0 \|P_1 f_n\|_{L^2}^2,
\end{align*}
\]

by (13) for some \(\nu_0 > 0\). Thus \(\lim_{n \to \infty} \|P_1 f_n\|_{L^2} = 0\) and \(\lim_{n \to \infty} \|P_0 f_n\|_{L^2} = 1\). Since \(\text{Ker}L\) is a finite dimensional space, \(P_0 f_n\) converges in \(L^2\). Thus up to a sub-sequence, we obtain

\[
y_n \to y_0, \lambda \to i\lambda_0, f_n \to f_0, \text{ as } n \to \infty,
\]

for some \(y_0 \in [y_2, y_1]\), \(\lambda_0 \in \mathbb{R}\), \(f_0 \in \text{Ker}L\). Also for \(\varphi \in \mathcal{S}\), \((L f_n - L f_0, \varphi)_{L^2} = (f_n - f_0, L\varphi)_{L^2} \to 0\). Up to a sub-sequence, we have \(\lim_{n \to \infty} L f_n = L f_0\) almost everywhere. Thus taking limit in (42),

\[
(2\pi i y_0 \cdot v + L)f_0 = 2\pi i y_0 \cdot v f_0 = i\lambda_0 f_0.
\]

Hence \(\lambda_0 = y_0 = 0\), which contradicts to \(y_0 \in [y_2, y_1]\). Therefore, there exists \(\sigma_1 > 0\) such that for \(|y| \in [y_2, y_1]\),

\[
\sigma(\hat{B}(y)) \cap \{\lambda : -2\sigma_1 \leq \text{Re}\lambda \leq 0\} = \emptyset.
\]

Together with (40), we prove (4). \(\square\)

The existence of eigenvalues and eigenfunctions to \(\hat{B}(y)\) with expansions and derivatives have been well studied in different contexts, cf. [36, 45]. But since our definition (5)(6) on \(L\) and \(\hat{A}(y), \hat{B}(y)\) are different from the above works, we provide a slightly different proof in order to make our argument self-contained.

**Theorem 3.4.** Assume \(\gamma + 2s \geq 0\).

1. There exists \(y_0 > 0\), \(\sigma_0 \in (0, \nu_1 / 2)\) and \(\lambda_j(|y|) \in C^\infty([0, y_0])\) such that for \(|y| \leq y_0\),

\[
\begin{align*}
\sigma(\hat{B}(y)) \cap \{\lambda : \text{Re}\lambda \geq -\nu_1\} &= \{\lambda_j(|y|)\}_{j=0}^{d+1}, \\
\rho(\hat{B}(y)) \supset \{\lambda : -2\sigma_0 \leq \text{Re}\lambda \leq -\frac{\sigma_0}{2}\} \cup \{\lambda : \text{Re}\lambda \geq -2\sigma_0, |\lambda| \geq \frac{\sigma_0}{2}\}.
\end{align*}
\]

The eigenvalues \(\lambda_j(y)\) and corresponding eigenfunctions \(\varphi_j(y)\) have the asymptotic expansions:

\[
\begin{align*}
\lambda_j(y) &= -2\pi i \eta_{0,j}|y| + \eta_{1,j}|y|^2 + O(|y|^3), \quad (|y| \to 0), \\
\varphi_j(y, v) &= \varphi_{0,j} + |y|\varphi_{1,j}(y),
\end{align*}
\]

with \(\eta_{0,j} \in \mathbb{R}\) selected from lemma 6.14 and \(\eta_{1,j} < 0, \varphi_{0,j}, \varphi_{1,j}(y) \in \mathcal{S}\).

2. Denote the eigen-projection to the eigenvalue \(\lambda_j(y)\) by \(P_j(y)\). Then \(P_j\) is of finite rank and there exists \(C > 0\) such that for \(|y| \leq y_0\),

\[
\|P_j(y)f\|_{H^{(a - 1/2)}} \leq C\|f\|_{H^{(a)}}.
\]

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Proof. Let $\varphi \in D(\hat{B}(y))$ be the eigenfunction corresponding to eigenvalue $\lambda \in \sigma(\hat{B}(y))$ with $\Re \lambda \in (-\nu_1, 0]$, then $\varphi \in \mathcal{H}$ by theorem 3.3 (3). Hence the eigenvalue problem $\hat{B}(y)\varphi = \lambda \varphi$ becomes

$$(-2\pi iy \cdot v + L)\varphi = \lambda \varphi,$$

(44)

where $\lambda = \lambda(y)$ is the eigenvalue and $\varphi = \varphi(y)$ is the eigenfunction.

1. We claim that $\lambda(y)$ depends only $|y|$. Indeed, let $R$ be an orthogonal matrix acting on $v$, i.e. $R$ is a rotation. Then by lemma 6.12,

$$\lambda(y)R\varphi = -2\pi iy \cdot Rv \varphi(Rv) + RL\varphi = -2\pi iR^\perp y \cdot v\varphi(Rv) + LR\varphi.$$

Thus $\lambda(y)$ is an eigenvalue of both $\hat{B}(R^\perp y)$ and $\hat{B}(y)$. So $\lambda$ depends only on $|y|$ and we will write $\lambda(y) = \lambda(|y|)$ for convenience. Now we pick a orthogonal matrix $R$ such that $R^\perp y = (|y|, 0, \cdots, 0) \in \mathbb{R}^d$, then $R(\hat{B}(y))\varphi = R(\lambda \varphi)$ becomes

$$(2\pi i|y|v_1 + L)R\varphi(y) = \lambda(|y|)R\varphi(y),$$

and hence $R\varphi(y)$ depends only of $|y|$. $R\varphi(y)$ is the eigenfunction of $2\pi i|y|v_1 + L$.

2. Denote $r = |y|$. Notice that we assume $r \in \mathbb{R}$ in the following, though $|y| \geq 0$. For $-\nu_1 < \Re \lambda \leq 0$, next we will solve eigen-equation:

$$(2\pi irv_1 + L)\varphi = \lambda(r)\varphi.$$  

(45)

We apply the Macro-micro decomposition, i.e. do the projection $P_0$ and $P_1$, we obtain

$$\begin{cases}
P_0(-2\pi irv_1(P_0\varphi + P_1\varphi)) = \lambda P_0\varphi,
P_1(-2\pi irv_1\varphi) + LP_1\varphi = \lambda P_1\varphi.
\end{cases}$$  

(46)

The second equation yields

$$(-\lambda - P_1(2\pi irv_1) + L)P_1\varphi = P_1(2\pi irv_1 P_0\varphi).$$  

(47)

3. For $\Re \lambda > -\nu_1$, denote

$$S := -\lambda P_1 - P_1(2\pi irv_1)P_1 + L,$$

with domain $D(S) := \mathcal{H} \cap (\Ker L)^\perp$. Regard $(\Ker L)^\perp$ equipped with $L^2$ norm as the whole space in this step, then $D(S)$ is densely defined and hence closable. We claim that $(S)^{-1}$ exists on $(\Ker L)^\perp$ with the closure taken in $(\Ker L)^\perp$. Indeed, recall (36) that

$$2\pi irv_1 - A^L = (-2\pi irv_1 - A)^{*-L^2},$$

where the closure and adjoint are taken on $L^2$ with domain $\mathcal{H}$, denoted by $(\cdot)^{L^2}$ and $(\cdot)^{*-L^2}$ respectively while the usual notation $(\cdot), (\cdot)^*$ are taken on $(\Ker L)^\perp$. By lemma
6.8, it suffices to show that $S$ and its adjoint are dissipative on $(\text{Ker}L)^\perp$. However, it’s hard to prove $S$ has dense range on $(\text{Ker}L)^\perp$ and hence we can’t use the former technique. For $f \in \mathcal{S}$, $\text{Re} \lambda > -\nu_1 \geq -\nu_0$, by (13),

$$\text{Re}(Sf, f)_{L^2} \leq (-\nu_1 - \text{Re} \lambda) \|P_1 f\|_{L^2}^2 \leq 0. \quad (48)$$

Hence the closure $\overline{S}$ of $S$ on $(\text{Ker}L)^\perp$ and closure $\overline{S}^{L^2}$ on $L^2$ satisfies

$$\text{Re}(\overline{S} f, f)_{L^2} \leq (-\nu_1 - \text{Re} \lambda) \|P_1 f\|_{L^2}^2 \leq 0, \text{ for } f \in D(\overline{S}),$$

$$\text{Re}(\overline{S}^{L^2} f, f)_{L^2} \leq 0, \text{ for } f \in D(\overline{S}^{L^2}). \quad (49)$$

Notice that

$$S = -2\pi i\nu_1 - A - \lambda P_1 + (2\pi i\nu_1)P_0 + P_0(2\pi i\nu_1)P_1 + K, \quad (50)$$

we can find the domain of $S^*$, the adjoint taken on $(\text{Ker}L)^\perp$:

$$D(S^*) = \{g \in (\text{Ker}L)^\perp : f \mapsto (S f, g)_{L^2} \text{ is continuous on } \mathcal{S} \cap (\text{Ker}L)^\perp\}$$

$$\subset \{g \in L^2 : f \mapsto (S f, g)_{L^2} \text{ is continuous on } \mathcal{S} \cap L^2\}$$

$$= D((-2\pi i\nu_1 - A)^{\ast L^2}).$$

Then by definition 6.2, for $f \in D(S^*)$,

$$\text{Re}(S^* f, f)_{L^2} = \text{Re}((-2\pi i\nu_1 - A)^{\ast L^2} f, f)_{L^2}$$

$$+ \text{Re}(-\lambda P_1 + (2\pi i\nu_1)P_0 + P_0(2\pi i\nu_1)P_1 + K)^{\ast L^2} f, f)_{L^2}$$

$$= \text{Re}(f, (2\pi i\nu_1 - A)^{\ast L^2} f)_{L^2}$$

$$+ \text{Re}(f, (-\lambda P_1 + P_0(2\pi i\nu_1)P_1 + L) f)_{L^2}$$

$$\leq 0,$$

where the last inequality is similar to (49). Thus $S$ and its adjoint are dissipative on Banach space $(\text{Ker}L)^\perp$. By theorem 6.8, $\overline{S}$ generates a contraction semigroup on $(\text{Ker}L)^\perp$. Hence, by 6.9, $(-\lambda + \eta)P_1 - P_1(2\pi i\nu_1)P_1 + \overline{L}$ exists on $(\text{Ker}L)^\perp$, for $\text{Re} \eta > 0$, $\text{Re} \lambda > -\nu_1$. Thus $\overline{S}$ is invertible on $(\text{Ker}L)^\perp$. Also by (48), for $f \in (\text{Ker}L)^\perp$,

$$\|\overline{S}^{-1} f\|_{L^2} \leq \frac{1}{\nu_1 + \text{Re} \lambda} \|P_1 f\|_{L^2}. \quad (51)$$

One can easily apply the spectrum theory to get that $\overline{S}^{-1}$ is smooth with respect to $\lambda$. In the following, we will regard $\overline{S}$ and $(\overline{S})^{-1}$ as operators on $(\text{Ker}L)^\perp$. 

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4. In this step, we claim that for \( f, g \in \mathcal{S}, \) \(((\overline{S})^{-1}f, g)_{L^2}\) is smooth with respect to \( r \), where we write \( S_r \) to show the dependence on \( r \). Indeed, for \( g \in \mathcal{S} \cap (\text{Ker} L)^\perp \), let \( f := (\overline{S})^{-1}g \in (\text{Ker} L)^\perp \), then by (50),

\[
\lambda + 2\pi irv_1 + Af = Kf + \lambda P_0 f + (2\pi irv_1)P_0 f + P_0(2\pi irv_1)P_1f - g.
\]

Applying the argument of step 4 in theorem 3.3, we have \( f \in \mathcal{S} \) and for \( k, l \geq 0, \)

\[
\|v\|^k(D_v)^l f\|_{L^2} \lesssim \|f\|_{L^2} + \|v\|^k(D_v)^l g\|_{L^2}.
\]

(52)

Thus \((\overline{S})^{-1}\) maps \( \mathcal{S} \) into \( \mathcal{S} \). Also \( P_1 = I - P_0 \) maps \( \mathcal{S} \) into \( \mathcal{S} \). For \( r_1, r_2 \in \mathbb{R}, \)

\[
((\overline{S}_{r_1})^{-1}f - (\overline{S}_{r_2})^{-1}f, g)_{L^2} = 2\pi i((\overline{S}_{r_1})^{-1}(P_1(r_1 - r_2)v_1P_1)(\overline{S}_{r_2})^{-1}f, g)_{L^2},
\]

\[
\left|((\overline{S}_{r_1})^{-1}f - (\overline{S}_{r_2})^{-1}f, g)_{L^2}\right| \leq \frac{2\pi|v_1 - r_2|}{v_1 + Re\lambda}\|(P_1(v_1)P_1)\left(\overline{S}_{r_2}\right)^{-1}f\|_{L^2}\|g\|_{L^2}
\]

\[
\rightarrow 0, \quad \text{as} \quad r_1 \rightarrow r_2.
\]

Hence \( \partial_r((\overline{S})^{-1}f, g)_{L^2} = ((\overline{S}_{r_1})^{-1}(P_12\pi iv_1)(\overline{S}_{r_2})^{-1}f, g)_{L^2} \). Noticing that

\[
(\overline{S}_{r_1})^{-1}(P_1v_1)(\overline{S}_{r_1})^{-1} - (\overline{S}_{r_2})^{-1}(P_1v_1)(\overline{S}_{r_2})^{-1}
\]

\[
= (\overline{S}_{r_1})^{-1}(P_1v_1)(\overline{S}_{r_1})^{-1} - (\overline{S}_{r_2})^{-1}(P_1v_1)(\overline{S}_{r_2})^{-1}
\]

\[
+ (\overline{S}_{r_2})^{-1}(P_1v_1)(\overline{S}_{r_2})^{-1} - (\overline{S}_{r_2})^{-1}(P_1v_1)(\overline{S}_{r_2})^{-1},
\]

we can apply induction to find \((\overline{S})^{-1}f, g)_{L^2}\) is smooth with respect to \( r \) and

\[
\partial_r^n((\overline{S})^{-1}f, g)_{L^2} = \left(((\overline{S}_{r_1})^{-1}(P_12\pi iv_1))^n(\overline{S}_{r_2})^{-1}f, g\right)_{L^2}.
\]

5. Now we can return to (47). Notice that \( 2\pi irv_1P_0\varphi \in \mathcal{S} \) and \( P_1 = I - P_0 \) maps \( \mathcal{S} \) into \( \mathcal{S} \), we have

\[
P_1\varphi = \overline{S}^{-1}P_1(2\pi irv_1P_0\varphi) \in \mathcal{S}.
\]

(53)

Substitute this into (46),

\[
P_0\left(-2\pi irv_1(P_0\varphi + \overline{S}^{-1}P_1(2\pi irv_1P_0\varphi))\right) - \lambda P_0\varphi = 0.
\]

Denote \( T := P_0(-2\pi iv_1\overline{S}^{-1}P_1(2\pi iv_1P_0)) \), then \((Tf, g)_{L^2}\) is smooth respect to \( r \), and

\[
rP_0(-2\pi iv_1P_0\varphi) + r^2TP_0\varphi - \lambda P_0\varphi = 0.
\]

(54)

By lemma 6.14 in appendix, the operator \( P_0(-2\pi iv_1P_0) \) has eigenvalues \( \eta_{0,j} \) and eigenfunctions \( \psi_{0,j} \). We expand

\[
P_0\varphi = \sum_{j=0}^{d+1} C_j \psi_{0,j}.
\]

(55)

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Then

\[-2\pi ir \sum_{j=0}^{d+1} C_j \eta_{0,j} \psi_{0,j} + r^2 \sum_{j=0}^{d+1} C_j T \psi_{0,j} - \sum_{j=0}^{d+1} C_j \lambda \psi_{0,j} = 0.\]

Taking inner product with \( \{\psi_{0,k}\}_{k=0}^{d+1} \), we obtain

\[-2\pi i r \left( \begin{array}{c} C_0 \eta_{0,0} \\ \vdots \\ C_{d+1} \eta_{0,d+1} \end{array} \right) + r^2 \left( \begin{array}{c} \sum_{j=0}^{d+1} C_j (T \psi_{0,j}, \varphi_{0,0})_{L^2} \\ \vdots \\ \sum_{j=0}^{d+1} C_j (T \psi_{0,j}, \varphi_{0,d+1})_{L^2} \end{array} \right) - \left( \begin{array}{c} C_0 \lambda \\ \vdots \\ C_{d+1} \lambda \end{array} \right) = 0,

\]

where \( (T \psi_{0,j}, \psi_{0,k})_{L^2} \) is the matrix with \( T_{jk} := (T \psi_{0,j}, \psi_{0,k})_{L^2} \) being the element of its \( k \)th row, \( j \)th column. Notice that \( (T \psi_{0,j}, \psi_{0,k})_{L^2} = 4\pi \langle S^{-1} P_1 (v_1 \psi_{0,j}), P_1 (v_1 \psi_{0,k}) \rangle_{L^2} \) and \( \psi_{0,j} = v_j \mu^{1/2} \) \( (j = 2, \ldots, d) \). The reflection

\[ v \mapsto (v_1, \ldots, -v_j, \ldots, v_d) \quad (j = 2, \ldots, d) \]

and rotation

\[ R_{ij} := (\ldots, v_i, \ldots, v_j, \ldots) \mapsto (\ldots, v_j, \ldots, v_i, \ldots) \quad (i, j = 2, \ldots, d) \]

commutes with \( S^{-1}, v_1, P_0, P_1 \). Thus by reflection, for \( j = 2, \ldots, d \), \( k = 0, \ldots, d + 1 \), \( k \neq j \), we have \( v_1 S^{-1} P_1 v_1 \psi_{0,j} \) is odd about \( v_j \) while \( \psi_{0,k} \) is even about \( v_j \) and hence, \( T_{jk} = 0 \). By rotation, \( T_{22} = T_{33} = \cdots = T_{dd} = 4\pi \langle S^{-1} P_1 (v_1 \psi_{0,j}), P_1 (v_1 \psi_{0,k}) \rangle_{L^2} < 0 \) and are independent of \( r \).

6. In this step, we will solve the eigenvalues \( \lambda \) and eigenvector \( (C_0, \ldots, C_{d+1}) \) of (56) by implicit function theorem. The eigen-equation (56) has a non-trivial solution \( (C_0, \ldots, C_{d+1}) \) if and only if the corresponding matrix is of 0 determinant. That is

\[
0 = \left| \lambda I_{d+2} + 2\pi i r \left( \begin{array}{c} \eta_{0,0} \\ \vdots \\ \eta_{0,d+1} \end{array} \right) - r^2 \left( (T \psi_{0,j}, \psi_{0,k})_{L^2} \right)_{k,j=0}^{d+1} \right| = \left| \lambda_3 + 2\pi i r \left( \begin{array}{c} \eta_{0,0} \\ \eta_{0,1} \\ \eta_{0,d+1} \end{array} \right) - r^2 (T_{jk})_{k,j=0,1,d+1}^{d+1} \right| = \prod_{j=2}^{d} (\lambda - r^2 T_{jj}).
\]

We can easily get \( d - 1 \) solutions:

\[
\lambda_j(r) = r^2 T_{jj},
\]

(58)
with $T_{jj} < 0$. Choose the corresponding eigenvectors of (56) to be $\xi_2, \ldots, \xi_d$, the standard unit vector in $\mathbb{R}^{d+2}$. Then by (55), we can choose

$$P_0\varphi_j = \psi_{0,j} = \psi_j, \quad (j = 2, \ldots, d).$$

(59)

In order to find the other 3 solutions, we write $\lambda = (2\pi ir)\eta, \zeta := (C_0, C_1, C_{d+1})$ and

$$f(r, \eta) := \eta I_3 + \begin{pmatrix} \eta_{0,0} & \eta_{0,1} & \eta_{0,d+1} \\ \eta_{0,0} & \eta_{0,1} & \eta_{0,d+1} \\ \eta_{0,0} & \eta_{0,1} & \eta_{0,d+1} \end{pmatrix} - \frac{r}{2\pi i} (T_{jk})_{k,j=0,1,d+1}$$

Suppose $r \in [-1, 1], |\eta| \leq \frac{\nu_1}{4\pi}$, then $\text{Re}\lambda > -\nu_1$. Since the above eigenvectors are unit vector $\xi_j (j = 2, \ldots, d)$, we obtain that the eigen-equation (57) is equivalent to

$$f(r, \eta)\zeta^T = 0,$$

since $r = 0$ means $\lambda = 0$ from (56). To apply the implicit function theorem 6.13, we consider $\eta = \eta^R + i\eta^I$ and

$$g(r, \eta^R, \eta^I) := (\text{Re}(\det f(r, \eta)), \text{Im}(\det f(r, \eta))).$$

Then $g : (-1, 1) \times (-\frac{\nu_1}{4\pi}, \frac{\nu_1}{4\pi})^2 \to \mathbb{R}^2$ is smooth and

$$g|_{r=0, \eta=-\eta_{0,j}} = 0,$$

$$\det(\partial_{\eta^R, \eta^I} g)|_{r=0, \eta=-\eta_{0,j}} = \prod_{k=0,1,d+1,k\neq j} (\eta_{0,k} - \eta_{0,j})^2 \neq 0,$$

for $j = 0, 1, d + 1$. Hence, implicit function theorem yields that there exists $y_0 > 0$ and $\eta_j(r) \in C^\infty([-y_0, y_0]; \mathbb{C})$ such that for $j = 0, 1, d + 1$,

$$\eta_j(0) = -\eta_{0,j}, \quad \det f(r, \eta_j(r)) = 0, \quad \text{for } r \in [-y_0, y_0].$$

(60)

Thus,

$$\lambda_j(r) = 2\pi i r \eta_j(r) \in C^\infty([-y_0, y_0]; \mathbb{C}), \quad j = 0, 1, d + 1,$$

(61)

is the other 3 eigenvalues.

In order to find the corresponding eigenvectors, it’s equivalent to solve

$$f(r, \eta_j(r)) \begin{pmatrix} C_0 \\ C_1 \\ C_{d+1} \end{pmatrix} = 0,$$

where the characteristic polynomial $\det f(r, \eta)$ has three distinct solution $\eta_j(r)$ when $r$ is sufficiently small. Hence each eigenvalue $\eta_j$ correspond to distinct eigenvector $\zeta := (C_0, C_1, C_{d+1})$. Denote $f^{(i)}(r, \eta)$ to be the matrix $f(r, \eta)$ with the $i$th row and
ith column removed, \( f^{(j)}(r, \eta) \) to be the \( i \)th column of \( f(r, \eta) \) with the \( i \)th component removed, \( \zeta' \) to be \( \zeta \) with the \( i \)th component removed. Then \( f^{(j)}(0, \eta_j) \) is invertible by our choice of \( \eta_j(r) \) from (101) and hence \( f^{(j)}(r, \eta_j(r)) \) is invertible whenever \( r \) is sufficiently small. So the \( j \)th row in \( f(r, \eta_j(r)) \) can be eliminated by Gaussian elimination and we can choose the \( j \)th component of \( \zeta_j(r) \) to be 1 and determine the rest of \( \zeta_j(r) \) by \( \zeta^j(r) := -(f^{(j)}(r, \eta_j(r)))^{-1}f^{(j)}(r, \eta_j(r)) \). Then \( \zeta^j(0) = 0 \) and \( \zeta_j(r) \) is smooth with respect to \( r \in [-y_0, y_0] \). We then select

\[
P_0 \varphi_j(r) = \zeta_j(r) \cdot (\psi_{0,0}, \psi_{0,1}, \psi_{0,d+1}) =: \psi_{0,j} + rC_{0,j}(r)\varphi_{0,j},
\]

for \( j = 0, 1, d + 1 \). Then \( P_0 \varphi_j(r) \in \mathcal{S} \) is a linear combination of \( \psi_{0,0}, \psi_{0,1}, \psi_{0,d+1} \) and \( C_{1,j}(r) \in C^\infty([-y_0, y_0]) \).

Write \( \lambda_j(r) = -2\pi ir\eta_{0,j} + r^2\eta_{1,j} + O(r^3) \ (j = 0, 1, d + 1) \) and substitute eigenvalue and eigenfunction \( \lambda_j(r) \), \( P_0 \varphi_j(r) \) into (54), we obtain

\[
P_0(-2\pi iv_1 P_0 \varphi_j) + rTP_0 \varphi_j - (-2\pi i\eta_{0,j} + r\eta_{1,j} + O(r^2))P_0 \varphi_j = 0,
\]
as \( r \to 0 \) and hence considering order \( O(r) \),

\[
\eta_{1,j} = \frac{(TP_0 \varphi_j, P_0 \varphi_j)_{L^2}}{\|P_0 \varphi_j\|_{L^2}^2} < 0.
\]

7. At last we can select \( P_1 \varphi_j(r) \) by (53) and statement (2) follows from (52)(53)(62). Then by step one, the rotation \( R^T \varphi_j \) gives the eigenfunctions of \( \hat{B}(y) \).

Recall the choice (58)(61) of \( \lambda_j \). If \( |y| \leq y_0 \) and \( \lambda \) is any eigenvalues of \( \hat{B}(y) \) with \( \text{Re}\lambda \geq -2\sigma_0 \), then \( \lambda \) is the root of (57) and must be \( \lambda_j(r) \) for some \( j = 0, \ldots, d + 1 \). Fix \( 0 < \sigma_0 < \nu_1/2 \) and choose \( y_0 > 0 \) sufficiently small such that for \( |r| \leq y_0, j = 0, \ldots, d + 1 \),

\[
|\lambda_j(r)| < \frac{\sigma_0}{2}.
\]

Then for \( |y| \leq y_0 \),

\[
\sigma(\hat{B}(y)) \subset \left\{ \lambda : -\frac{\sigma_0}{2} < \text{Re}\lambda \leq 0, |\text{Im}\lambda| < \frac{\sigma_0}{2} \right\},
\]

\[
\rho(\hat{B}(y)) \supset \left\{ \lambda : -2\sigma_0 \leq \text{Re}\lambda \leq -\frac{\sigma_0}{2} \right\} \cup \left\{ \lambda : \text{Re}\lambda \geq -2\sigma_0, |\lambda| \geq \frac{\sigma_0}{2} \right\}.
\]

The following lemma provides the decay of \( \|\lambda I - \hat{A}(y)^{-1}K\|_{\mathcal{S}(L^2)} \) when \( |y| \) or \( |\tau| \) is large, which will gives the invertibility of \( \lambda I - \hat{B}(y) \) when \( |y| \) or \( |\tau| \) is large.

**Lemma 3.5.** Let \( |\tau| \geq 1 \), \( \text{Re}\lambda \geq -\nu_1 \). Then

\[
\|\lambda I - \hat{A}(y)^{-1}K\|_{\mathcal{S}(L^2)} \to 0, \text{ as } |y| + |\tau| \to \infty.
\]
Proof. Let \( f \in \mathcal{S} \) and denote \( \varphi = (\lambda I - \hat{A}(y))^{-1}Kf \), \( \Phi(v) = e^{-\pi|v|^2} \), \( \Phi_\varepsilon(v) = e^{-d\varepsilon^2|v|^2} \) with \( \varepsilon \in (0, 1) \). Then \( \Phi = \Phi_1 \).

\[
\|\varphi\|_{L^2} \leq \|\Phi_\varepsilon \ast \varphi - \varphi\|_{L^2} + \|\Phi_\varepsilon \ast \varphi\|_{L^2}.
\]

(64)

For the first term, we have

\[
\|\Phi_\varepsilon \ast \varphi - \varphi\|_{L^2} = \|(\hat{\Phi}(\varepsilon y) - 1)\hat{\varphi}\|_{L^2}
\]

\[
= \left( \int |(e^{-\pi|\varepsilon y|^2} - 1)|^2 dv \right)^{1/2}
\]

\[
\leq \left( \int_{|y| \geq \frac{1}{\sqrt{s}}} 4\varepsilon^2 |\langle \eta \rangle \langle y \rangle \phi \rangle^2 dv + \int_{|y| \leq \frac{1}{\sqrt{s}}} |1 - e^{-\pi\varepsilon^2}|^2 dv \right)^{1/2}
\]

\[
\lesssim (\varepsilon s/2 + (1 - e^{-\pi\varepsilon^2})) \|\langle \eta \rangle \langle y \rangle \phi \|_{L^2}
\]

\[
\lesssim (\varepsilon s/2 + (1 - e^{-\pi\varepsilon^2})) \|f\|_{L^2},
\]

(65)

where the last inequality following from lemma 3.2 and the boundedness of \( K \). For the second term, we would like to apply the calculation similar to lemma 4.2 in [45]. Let \( \sigma_2 > 1 \), depending on \( \tau \) and \( |y| \), to be chosen later. Since \( Kf \in \mathcal{S} \), we have \( \varphi \in \mathcal{S} \) and so

\[
(\lambda + 2\pi i y + A)\varphi(v) = Kf,
\]

and hence

\[
\varphi = \frac{\sigma_2(v)^{-1} \varphi - \sigma \varphi + Kf - A\varphi}{\sigma_2(v)^{-1} + i\tau + 2\pi i y}.
\]

Noticing \( \sigma_2 > 1 \), we have

\[
|\Phi_\varepsilon \ast \varphi| = \left| \int \Phi_\varepsilon(v-u)\varphi(u) du \right|
\]

\[
\leq \int \Phi_\varepsilon(v-u) \frac{|\varphi| + |\sigma(u)\varphi| + \langle u \rangle |Kf(u)|}{(1 + (\tau + 2\pi i y)^2 u^2/\sigma_2^2)^{1/2}} du
\]

\[
+ \left| \int \Phi_\varepsilon(v-u) \frac{A\varphi(u)}{\sigma_2(u)^{-1} + i\tau + 2\pi i y} du \right|
\]

(66)

\[= I_1 + I_2.\]

For \( I_1 \), by Hölder inequality,

\[
|I_1|^2 = \left| \int \Phi_\varepsilon(v-u) \frac{|\varphi| + |\sigma(u)\varphi| + \langle u \rangle |Kf|}{(1 + (\tau + 2\pi i y)^2 u^2/\sigma_2^2)^{1/2}} du \right|^2
\]

\[
\leq \int \Phi_\varepsilon(v-u) \left( |\varphi| + |\sigma(u)\varphi| + \langle u \rangle |Kf| \right)^2 du \times \int \frac{\Phi_\varepsilon(v-u)}{1 + (\tau + 2\pi i y)^2 u^2/\sigma_2^2} du.
\]

(67)
We will use the decomposition: $u = \tilde{u}\frac{y}{|y|} + u'$ with $\tilde{u} = \frac{w}{|y|}$ and $u' = u - \tilde{u}\frac{y}{|y|}$. Then $y \perp u'$ and

$$
\int \frac{\Phi_\varepsilon(v-u)}{1 + (\tau + 2\pi u \cdot y)^2 \sigma_2^2} du \leq \int \int_{\mathbb{R}^d-1} \frac{e^{-d - e^{-\frac{|\tau - \tilde{\tau}|^2}{4\sigma_2^2}}}}{1 + (\tau + 2\pi \tilde{u}|y|)^2 \tilde{u}^2 \sigma_2^2} du' d\tilde{u}
$$

$$
\leq \int \frac{1}{1 + (\tau + 2\pi \tilde{u}|y|)^2 \tilde{u}^2 \sigma_2^2} d\tilde{u} =: J.
$$

It's easy to get

$$
J \leq \varepsilon^{-1} \int \frac{1}{1 + (\tau + 2\pi \tilde{u}|y|)^2 \sigma_2^2} d\tilde{u} \lesssim \frac{\sigma_2}{\varepsilon|y|}.
$$

On the other hand, notice $|\tilde{u}| \leq \frac{|\tau|}{4\pi|y|}$ implies $|\tau + 2\pi \tilde{u}|y| \geq |\tau| - |2\pi \tilde{u}|y| \geq \frac{|\tau|}{2}$.

$$
J = \frac{1}{\varepsilon} \int_{|\tilde{u}| \leq \frac{|\tau|}{4\pi|y|}} \frac{1}{1 + (\tau + 2\pi \tilde{u}|y|)^2 \tilde{u}^2 \sigma_2^2} d\tilde{u} + \frac{1}{\varepsilon} \int_{|\tilde{u}| \geq \frac{|\tau|}{4\pi|y|}} \frac{1}{1 + (\tau + 2\pi \tilde{u}|y|)^2 \tilde{u}^2 \sigma_2^2} d\tilde{u}
$$

$$
\leq \frac{1}{\varepsilon} \int_{|\tilde{u}| \leq \frac{|\tau|}{4\pi|y|}} \frac{1}{1 + (\tau + 2\pi \tilde{u}|y|)^2 \sigma_2^2} d\tilde{u} + \frac{1}{\varepsilon} \int_{|\tilde{u}| \geq \frac{|\tau|}{4\pi|y|}} \frac{1}{1 + (\tau + 2\pi \tilde{u}|y|)^2 \tilde{u}^2 \sigma_2^2} d\tilde{u}
$$

$$
\lesssim \frac{\sigma_2}{\varepsilon|\tau|}.
$$

Thus combining the above two estimates,

$$
J \lesssim \frac{\sigma_2}{\varepsilon(|\tau|^2 + |y|^2)^{1/2}}.
$$

Recall (67) and applying (22), we have

$$
\|I_1\|^2_{L^2_y} \lesssim J \times \int \int \Phi_\varepsilon(v-u)(|\varphi| + |\sigma(u)\varphi| + \langle u \rangle |Kf|)^2 dudv
$$

$$
\lesssim J \times \|\varphi| + |\sigma(u)\varphi| + \langle u \rangle |Kf|\|^2_{L^2_u}
$$

$$
\lesssim J \times \|\langle u \rangle |Kf|\|^2_{L^2_u}
$$

$$
\lesssim \frac{\sigma_2}{\varepsilon(|\tau|^2 + |y|^2)^{1/2}} \|f\|^2_{L^2_y},
$$

(70)

since $K \in S(\langle v \rangle^{-1})$, where the constant may depend of $\sigma$, the real part of $\lambda$. For part $I_2$,

$$
I_2 = \left| \frac{\langle u \rangle^{-2}\Phi_\varepsilon(v-u)}{\sigma_2(u)^{-1} + i\tau + 2\pi iu \cdot y} \langle u \rangle^2 A\varphi(u) \right|_{L^2_y}
$$

$$
= \left| \langle v-u \rangle^d \langle D_u \rangle^s \left( \frac{\langle u \rangle^{-2}\Phi_\varepsilon(v-u)}{\sigma_2(u)^{-1} + i\tau + 2\pi iu \cdot y}, \langle u \rangle^{-d}\langle D_u \rangle^{-s} \langle u \rangle^2 A\varphi(u) \right) \right|_{L^2_y}
$$

31
We analyze the first part: Note $\sigma_2 > 1$, we have

\[
\left\| \frac{(v-u)^d}{\sigma_2(u)^{-1} + i\tau + 2\pi i u \cdot y} \right\|_{L^2_\sigma} \leq \left\| \frac{(v-u)^d}{\sigma_2(u)^{-1} + i\tau + 2\pi i u \cdot y} \right\|_{L^2_\sigma} + \left\| \frac{(v-u)^{-d}}{\sigma_2(u)^{-1} + i\tau + 2\pi i u \cdot y} \right\|_{L^2_\sigma} + \left\| \frac{(v-u)^{-2d}}{\sigma_2(u)^{-1} + i\tau + 2\pi i u \cdot y} \right\|_{L^2_\sigma}
\]

\[
\leq \left( \int \left| \frac{\sigma_2 e^{-d-1} \exp\left(-\frac{\pi}{2} \frac{v-u}{\epsilon} \right)^2}{\sigma_2 + (i\tau + 2\pi i u \cdot y)(u)^2} \right|^2 \right)^{1/2} + \left( \int \left| \frac{|y| \epsilon^{-d} \exp\left(-\frac{\pi}{2} \frac{v-u}{\epsilon} \right)^2}{\sigma_2 + (i\tau + 2\pi i u \cdot y)(u)^2} \right|^2 \right)^{1/2}
\]

\[
=: J_1.
\]

For the first integral, we apply the estimate (68) (69), while for the second integral, we make a rough estimate by canceling $(i\tau + 2\pi i u \cdot y)(u)^2$. Thus

\[
|J_1|^2 \lesssim \int \frac{\sigma_2 e^{-d-2} \Phi_\tau(v-u)}{\sigma_2 + (\tau + 2\pi u \cdot y)^2}(u)^2 \, du + \int \frac{|y|^2 \epsilon^{-d} \Phi_\tau(v-u)}{\sigma_2^4} \, du
\]

\[
\lesssim \epsilon^{-d-2} J + \epsilon^{-d} \sigma_2^{-4} |y|^2
\]

\[
\lesssim \frac{\sigma_2}{\epsilon^{d+3}(|\tau|^2 + |y|^2)^{1/2}} + \frac{|y|^2}{\epsilon^d \sigma_2^4}.
\]

Thus similar to $I_1$, we have

\[
\left\| I_2 \right\|_{L^2_\sigma}^2 \lesssim |J_1|^2 \times \left\| (v-u)^{-d} \langle D_u \rangle^{-s} \langle u \rangle^2 A \varphi(u) \right\|_{L^2_\sigma}^2
\]

\[
\lesssim |J_1|^2 \times \left\| \langle D_u \rangle^{-s} \langle u \rangle^2 A \varphi(u) \right\|_{L^2_\sigma}^2
\]

\[
\lesssim |J_1|^2 \times \left\| (a^{1/2}) \langle w \rangle^{2+\gamma/2+s} \varphi \right\|_{L^2_\sigma}^2
\]

\[
\lesssim |J_1|^2 \times \left\| \langle u \rangle^{2+\gamma/2+s} K f \right\|_{L^2_\sigma}^2
\]

\[
\lesssim \left( \frac{\sigma_2}{\epsilon^{d+3}(|\tau|^2 + |y|^2)^{1/2}} + \frac{|y|^2}{\epsilon^d \sigma_2^4} \right) \left\| f \right\|_{L^2_\sigma}^2,
\]

32
where we apply (22) and $A \in S(a) \subset S((\eta)^{2s}(v)^{\gamma+2s})$, $K \in S((u)^{-2-\gamma/2-s})$. Together with (66) (70), we have

$$
\| \Phi_{\varepsilon} \ast f \|_{L^2}^2 \lesssim \left( \frac{\sigma_2}{\varepsilon (|\tau|^2 + |y|^2)^{1/2}} + \frac{\sigma_2}{\varepsilon^{d+3} (|\tau|^2 + |y|^2)^{1/2}} + \frac{|y|^2}{\varepsilon^d \sigma_2^2} \right) \| f \|_{L^2}^2
$$

$$
\lesssim \left( \frac{\sigma_2}{\varepsilon^{d+3} (|\tau|^2 + |y|^2)^{1/2}} + \frac{|y|^2}{\varepsilon^d \sigma_2^2} \right) \| f \|_{L^2}^2,
$$

for $\varepsilon \in (0, 1)$. Choose $\delta \in (0, 1/(6d + 32))$ small and

$$
\sigma_2 := (1 + |\tau|^2 + |y|^2)^{1/(d+4)\delta}, \quad \varepsilon := (1 + |\tau|^2 + |y|^2)^{-\delta}
$$

Then, since $|\tau| \geq 1$,

$$
\| \Phi_{\varepsilon} \ast f \|_{L^2}^2 \lesssim \left( \frac{1}{(1 + |\tau|^2 + |y|^2)^{\delta}} + \frac{1}{(1 + |\tau|^2 + |y|^2)^{1-(3d+16)\delta}} \right) \| f \|_{L^2}^2.
$$

Together with (65),

$$
\| \varphi \|_{L^2} \lesssim \frac{1}{(1 + |\tau|^2 + |y|^2)^{\delta/2}} \| f \|_{L^2}.
$$

(72)

The next theorem shows the uniformly boundedness of $(\lambda I - \widehat{B}(y))^{-1}$ and $(I - (\lambda I - \widehat{A}(y))^{-1} K)^{-1}$.

**Theorem 3.6.** Take $y_0, \sigma_0 > 0$ from theorem 3.4 and with this $y_0$, we choose $\sigma_1$ from the statement (4) in theorem 3.3. Let $\sigma_y$ to be equal to $\sigma_0$ when $|y| \leq y_0$, and equal to $\sigma_1$ when $|y| \geq y_0$. Such choice assures that $(\sigma_y + i\tau)I - \widehat{B}(y)$ is invertible. Then

$$
\sup_{y \in \mathbb{R}^d, \text{Re} \lambda = -\sigma_y} \| (I - (\lambda I - \widehat{A}(y))^{-1} K)^{-1} \|_{\mathcal{L}(L^2)} < \infty,
$$

$$
\sup_{y \in \mathbb{R}^d, \text{Re} \lambda = -\sigma_y} \| (I - \widehat{B}(y))^{-1} \|_{\mathcal{L}(L^2)} < \infty.
$$

(73)

**Proof.** 1. Fix $\lambda \in \rho(\widehat{A}(y)) \cap \rho(\widehat{B}(y))$. Since $\widehat{B}(y) = \widehat{A}(y) + K$, we have

$$
\lambda I - \widehat{B}(y) = \lambda I - \widehat{A}(y) - K = (\lambda I - \widehat{A}(y))(I - (\lambda I - \widehat{A}(y))^{-1} K),
$$

$$
I = (\lambda I - \widehat{A}(y))^{-1} (\lambda I - \widehat{B}(y)) + (\lambda - \widehat{A}(y))^{-1} K.
$$

Thus $I - (\lambda I - \widehat{A}(y))^{-1} K$ is invertible on $L^2$ and

$$
(\lambda I - \widehat{B}(y))^{-1} = (I - (\lambda I - \widehat{A}(y))^{-1} K)^{-1} (\lambda I - \widehat{A}(y))^{-1}
$$

$$
= (\lambda I - \widehat{A}(y))^{-1} + (\lambda - \widehat{A}(y))^{-1} K (\lambda I - \widehat{B}(y))^{-1}.
$$

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Hence
\[(λI - \hat{A}(y))^{-1} = (λ - \hat{A}(y))^{-1} + (λ - \hat{A}(y))^{-1}K(I - (λI - \hat{A}(y))^{-1}K)^{-1}(λ - \hat{A}(y))^{-1}.\]

Next we prove the continuity of \((I - (λI - \hat{A}(y))^{-1}K)^{-1}\). For any \(y_1, y_2 ∈ \mathbb{R}^d\), \(\text{Re} λ_1, \text{Re} λ_2 ≥ 2σ_1\) with \(λ_1 ∈ \rho(\hat{A}(y_1)) \cap \rho(\hat{B}(y_1)), λ_2 ∈ \rho(\hat{A}(y_2)) \cap \rho(\hat{A}(y_2))\), we have
\[
(I - (λ_1I - \hat{A}(y_1))^{-1}K)^{-1} - (I - (λ_2I - \hat{A}(y_2))^{-1}K)^{-1}
= (I - (λ_1I - \hat{A}(y_1))^{-1}K)^{-1}((λ_1I - \hat{A}(y_1))^{-1} - (λ_2I - \hat{A}(y_2))^{-1})K
(I - (λ_2I - \hat{A}(y_2))^{-1}K)^{-1}
\]

We first deal with the middle term. Let \(f ∈ \mathcal{S}\), then \(Kf ∈ \mathcal{S}\), \((λ_2I - \hat{A}(y_2))^{-1}Kf ∈ \mathcal{S}\) from theorem 2.3. Thus
\[
((λ_1I - \hat{A}(y_1))^{-1} - (λ_2I - \hat{A}(y_2))^{-1})Kf
= (λ_1I - \hat{A}(y_1))^{-1}(λ_2I - \hat{A}(y_2) - λ_1I + \hat{A}(y_1))(λ_2I - \hat{A}(y_2))^{-1}Kf
= (λ_1I - \hat{A}(y_1))^{-1}(λ_2I - λ_1I + 2πi(y_2 - y_1) \cdot v)(λ_2I - \hat{A}(y_2))^{-1}Kf.
\]

By (22),
\[
\|(λ_1I - \hat{A}(y_1))^{-1} - (λ_2I - \hat{A}(y_2))^{-1})Kf\|_{L^2_x}
\leq C\|(λ_2 - λ_1 + 2πi(y_2 - y_1) \cdot v))\|\,(λ_2I - \hat{A}(y_2))^{-1}Kf\|_{L^2_x}
\leq C\|v\|\,(λ_2I - \hat{A}(y_2))^{-1}Kf\|_{L^2_x}
\leq C\|v\|\,(λ_2I - \hat{A}(y_2))^{-1}Kf\|_{L^2_x}
\leq C\|(λ_2 - λ_1 + |y_2 - y_1|)\|f\|_{L^2_x}.
\]

Thus \((λI - \hat{A}(y))^{-1}\) is continuous with respect to \((λ, y)\). Since \(((λ_1I - \hat{A}(y_1))^{-1} - (λ_2I - \hat{A}(y_2))^{-1})K\) is bounded on \(L^2_x\), the above estimate is also valid for \(f ∈ L^2\) by density argument. Fix \(λ_2, y_2\) and set \((λ_1, y_1)\) sufficiently close to \((λ_2, y_2)\) such that
\[
\|(λ_1I - \hat{A}(y_1))^{-1} - (λ_2I - \hat{A}(y_2))^{-1})K\|_{\mathcal{L}(L^2)} \leq \frac{1}{2\|(λ_2I - \hat{A}(y_2))^{-1}\|_{\mathcal{L}(L^2)}}.
\]
Therefore, applying lemma 6.3 to \((I - (λI - \hat{A}(y_1))^{-1}K)^{-1}\), we have
\[
||(λI - \hat{A}(y))^{-1} - (I - (λI - \hat{A}(y))^{-1}K)^{-1}\|_{\mathcal{L}(L^2)}
\leq \frac{1}{2\|(λ_2I - \hat{A}(y_2))^{-1}\|_{\mathcal{L}(L^2)}}\|(λI - \hat{A}(y))^{-1} - (λ_2I - \hat{A}(y_2))^{-1}K\|_{\mathcal{L}(L^2)}\|(I - (λ_2I - \hat{A}(y_2))^{-1}K)^{-1}\|_{\mathcal{L}(L^2)}
\leq C\|(λI - \hat{A}(y))^{-1} - (λ_2I - \hat{A}(y_2))^{-1}K\|_{\mathcal{L}(L^2)}\|(I - (λ_2I - \hat{A}(y_2))^{-1}K)^{-1}\|_{\mathcal{L}(L^2)}.
\]

In conclusion, \((λI - \hat{A}(y))^{-1}\), \((I - (λI - \hat{A}(y))^{-1}K)^{-1}\) and hence \((λI - \hat{B}(y))^{-1}\), as operators on \(L^2\), are continuous with respect to \(λ ∈ \sigma(\hat{B}(y)) \cap \sigma(\hat{A}(y))\) and \(y ∈ \mathbb{R}^d\).

2. By (41) (38), there exists \(τ_1 > 0\) such that for \(y ∈ \mathbb{R}^d\), \(||(λI - \hat{A}(y))^{-1}f||_{\mathcal{L}(L^2)}\) and \(||(I - (λI - \hat{A}(y))^{-1}K)^{-1}\|_{\mathcal{L}(L^2)}||\) are uniformly bounded on \(\{λ ∈ \mathbb{C} : |\text{Im} λ| ≥ τ_1\}\).
For $|\tau|$ large, we can use this uniformly bounded estimate, while for $|\tau|$ small, we can apply the continuity in step one.

Choose $y_0, \sigma_0$ in theorem 3.4 and apply this $y_0$ to theorem 3.3 (4) to find $\sigma_1 \in (0, \nu_1)$ such that

\[
\sigma(\hat{B}(y)) \cap \{ \lambda \in \mathbb{C} : -2\sigma_0 \leq \text{Re}\lambda \leq 0 \} = \{ \lambda_j \}_{j=0}^{d+1}, \text{ for } |y| \leq y_0.
\]

\[
\sigma(\hat{B}(y)) \cap \{ \lambda \in \mathbb{C} : -2\sigma_1 \leq \text{Re}\lambda \leq 0 \} = \emptyset, \text{ for } |y| \geq y_0.
\]

Then theorem 3.4 gives that $\rho(\hat{B}(y)) \supset \{ \lambda : \text{Re}\lambda = \sigma_y \}$ for $y \in \mathbb{R}^d$ where $(\lambda I - \hat{B}(y))^{-1}$ is continuous and we derive the bound (73).

\[\square\]

4 Regularizing estimate for linearized equation

Assume $\gamma + 2s \geq 0$ in this section, then $\| \cdot \|_{L^2} \lesssim \| \cdot \|_{H^{(a/2)}}$. We are concerned with the proof to our main theorem 1.2: the regularizing estimate of semigroup $e^{tB}$.

**Theorem 4.1.** Let $f \in \mathcal{S}$, $y \in \mathbb{R}^d$, choose $\sigma_y$ as (74). Then for $\tau \in \mathbb{R}$, we have $\lambda := \sigma_y + i\tau \in \rho(\hat{B}(y)) \cap \rho(\hat{A}(y))$ as in theorem 3.6. There exists $C > 0$ independent of $y$ such that for $f \in \mathcal{S}$,

\[
\|(\lambda I - \hat{B}(y))^{-1}f\|_{H^{(a/2)}} \leq C\|f\|_{H^{(a-1/2)}}.
\]

**Proof.** Write $\varphi = (\lambda I - \hat{B}(y))^{-1}f$. Then

\[
(\lambda I - \hat{A}(y))\varphi = f + K\varphi.
\]

A similar argument to step 4 in theorem 3.3 gives $\varphi \in \mathcal{S}$ and hence taking inner product with $\varphi$, we have

\[
\|\varphi\|^2_{H^{(a/2)}} \lesssim \text{Re}((\lambda I - \hat{A}(y))\varphi, \varphi)_{L^2}
\]

\[
= \text{Re}(f + K\varphi, \varphi)_{L^2}
\]

\[
\lesssim \|f\|_{H^{(a-1/2)}}\|\varphi\|_{H^{(a/2)}} + \|\varphi\|^2_{L^2}.
\]

Recall that $(\lambda I - \hat{B}(y))^{-1} = (I - (\lambda I - \hat{A}(y))^{-1}K)(\lambda I - \hat{A}(y))^{-1}$ and notice the two inverse operator on the right are uniformly bounded in $L^2$ from theorem 3.6, we have

\[
\|\varphi\|_{H^{(a/2)}} \lesssim \|f\|_{H^{(a-1/2)}} + \|(\lambda I - \hat{B}(y))^{-1}f\|_{L^2}
\]

\[
\lesssim \|f\|_{H^{(a-1/2)}} + \|(\lambda I - \hat{A}(y))^{-1}f\|_{L^2}
\]

\[
\lesssim \|f\|_{H^{(a-1/2)}},
\]

where the last inequality follows from (38). Also these constants are independent of $y$. \[\square\]
With the smoothing effect of \((\lambda I - \hat{B}(y))^{-1}\), we can prove our main theorem 1.2.

**Proof of Theorem 1.2.** 1. Write \(\lambda = \sigma + i\tau\). Take \(y_0, \sigma_0 > 0\) from theorem 3.4 and with this \(y_0\), we choose \(\sigma_1\) from the statement (4) in theorem 3.3. Define

\[
\sigma_y = \begin{cases} 
\sigma_0, & \text{if } |y| \leq y_0, \\
\sigma_1, & \text{if } |y| \geq y_0.
\end{cases}
\]

Then for \(|y| \leq y_0\),

\[
\rho(\hat{B}(y)) \supset \{ \lambda : -2\sigma_y \leq \text{Re}\lambda \leq -\sigma_y \} \cup \{ \lambda : \text{Re}\lambda \geq -2\sigma_y, |\text{Im}\lambda| \geq \sigma_y \},
\]

and for \(|y| \geq y_0\),

\[
\rho(\hat{B}(y)) \supset \{ \lambda \in \mathbb{C} : -2\sigma_y < \text{Re}\lambda \}.
\]

2. For \(f \in \mathcal{S}\), by corollary III.5.15 in [24], there exists \(\sigma_2 > 0\) such that

\[
e^{t\hat{B}(y)} f = \frac{1}{2\pi i} \lim_{n \to \infty} \int_{\sigma_2 - in}^{\sigma_2 + in} e^{\lambda t}(\lambda I - \hat{B}(y))^{-1} f d\lambda,
\]

where the limit is taken in \(L^2\). Since \(e^{\lambda t}(\lambda I - \hat{B}(y))^{-1} f\) is analytic with respect to \(\lambda \in \rho(\hat{B}(y))\), the Cauchy theorem on holomorphic function and theorem XV.2.2 in [25] yield that

\[
\int_{\sigma_2 - in}^{\sigma_2 + in} e^{\lambda t}(\lambda I - \hat{B}(y))^{-1} f d\lambda
\]

\[
= \left( \int_{-\sigma_2 + in}^{\sigma_2 + in} + \int_{-\sigma_2 - in}^{-\sigma_2 + in} + \int_{-\sigma_2 - in}^{-\sigma_2 - in} \right) e^{\lambda t}(\lambda I - \hat{B}(y))^{-1} f d\lambda + \sum_{|y| \leq y_0} e^{\lambda t} f
\]

where \(n > \sigma_y\) and \(P_j = \int_{\Gamma_j} (\lambda I - \hat{B}(y))^{-1} d\lambda\), with \(\Gamma_j\) being the smooth boundary of neighborhood that contains only eigenvalue \(\lambda_j(|y|)\) and separate from other eigenvalues.

3. For the first term, noticing that \((\lambda I - \hat{B}(y))^{-1} = (I - (\lambda I - \hat{A}(y))^{-1})^{-1}(\lambda I - \hat{A}(y))^{-1}\), we apply (73)(72) to get

\[
\lim_{n \to \infty} \left\| \int_{-\sigma_2 - in}^{\sigma_2 + in} e^{\lambda t}(\lambda I - \hat{B}(y))^{-1} f d\lambda \right\|_{L^2}
\]

\[
\leq e^{\sigma_2 t} (\sigma_2 + \sigma_y) \lim_{n \to \infty - \sigma_1 \leq \text{Re}\lambda \leq \sigma_2} \sup ||(\lambda I - \hat{A}(y))^{-1} f||_{L^2} d\lambda
\]

\[
\leq C e^{\sigma_2 t} \lim_{n \to \infty} \sup \frac{1}{(1 + |n|^2 + |y|^2)^{\delta/2}} ||f||_{L^2}
\]

\[
= 0,
\]

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for some $\delta > 0$, since $f \in \mathcal{S}$. Similarly, the second term satisfies

$$
\lim_{n \to \infty} \int_{-\sigma - in}^{\sigma + in} e^{\lambda t} (\lambda I - \hat{B}(y))^{-1} f \, d\lambda = 0, \quad \text{in } L^2.
$$

(76)

4. Now we investigate the third integral. Notice that $\partial_x^k((\lambda I - \hat{B}(y))^{-1} f) = k!(-i)^{k}(\lambda I - \hat{B}(y))^{-k-1}f$, for $\lambda \in \rho(\hat{B}(y))$. Using integration by parts, we have, for $k \geq 2$,

$$
\int_{-\sigma - in}^{\sigma + in} e^{\lambda t} (\lambda I - \hat{B}(y))^{-1} f \, d\lambda
= \sum_{j=1}^{k} \frac{e^{(\sigma + i\tau)t} (j - 1)!}{i t^j} (\lambda I - \hat{B}(y))^{-j} f \bigg|_{\tau = n}^{\tau = m} + k! \int_{-\sigma - in}^{\sigma + in} e^{\lambda t} (\lambda I - \hat{B}(y))^{-k-1} d\lambda.
$$

By theorem 3.6 and (72), we have

$$
(\lambda I - \hat{B}(y))^{-j} f = \left((\lambda I - \hat{A}(y))^{-1} K\right)^{-1} (\lambda I - \hat{A}(y))^{-1} f,
$$

$$
\| (\lambda I - \hat{B}(y))^{-j} f \|_{L^2} \lesssim \| (\lambda I - \hat{A}(y))^{-1} f \|_{L^2} \to 0, \quad \text{as } \tau \to \infty.
$$

Thus,

$$
\lim_{n \to \infty} \int_{-\sigma - in}^{\sigma + in} e^{\lambda t} (\lambda I - \hat{B}(y))^{-1} f \, d\lambda = \lim_{n \to \infty} \frac{k!}{t^k} \int_{-\sigma - in}^{\sigma + in} e^{\lambda t} (\lambda I - \hat{B}(y))^{-k-1} d\lambda,
$$

with the limit taken in $L^2$. For $k \geq 2, g \in L^2$ and $l, m \geq 0$, we have

$$
\left| \left( \int_{-\sigma - in}^{\sigma + in} e^{\lambda t} (\lambda I - \hat{B}(y))^{-k-1} f \, d\lambda, g \right)_{L^2} \right|
\leq e^{-\sigma t} \int_{-\sigma - in}^{\sigma + in} \left| (\lambda I - \hat{B}(y))^{-k} f, (\lambda I - \hat{B}(y)^* )^{-1} g \right|_{L^2} d\lambda
\leq e^{-\sigma t} \int_{-\sigma - in}^{\sigma + in} \| (\lambda I - \hat{B}(y))^{-k} f \|_{H^{(a^{1/2})}} \| (\lambda I - \hat{B}(y)^* )^{-1} g \|_{H^{(a^{-1/2})}} d\lambda
\leq e^{-\sigma t} \left( \int_{-\sigma - in}^{\sigma + in} \| (\lambda I - \hat{B}(y))^{-k} f \|_{H^{(a^{1/2})}}^2 d\lambda \right)^{1/2}
\times \left( \int_{-\sigma - in}^{\sigma + in} \| (\lambda I - \hat{B}(y)^* )^{-1} g \|_{H^{(a^{-1/2})}}^2 d\lambda \right)^{1/2}.
$$

(77)

Denote $\varphi = (\lambda I - \hat{B}(y))^{-1} f$. By theorem 4.1, for $\beta \in \mathbb{R}, k \geq 2$, we have

$$
\int_{-\sigma - in}^{\sigma + in} \| (\lambda I - \hat{B}(y))^{-k} f \|_{H^{(a^{1/2})}}^2 d\lambda
$$

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\[ \lesssim \int_{-\sigma_y}^{-\sigma_y+i\nu} \| (\lambda I - \hat{B}(y))^{-1} f \|^2_{H(a^{-1/2})} d\lambda \]
\[ \lesssim \int_{-\sigma_y}^{i\nu} \lim_{\varepsilon \to 0} \left( \| \varphi - \Phi_\varepsilon \ast \varphi \|_{H(a^{-1/2})} + \| \Phi_\varepsilon \ast \varphi \|_{H(a^{-1/2})} \right)^2 d\tau. \quad (78) \]

where \( \hat{\Phi}(\eta) := \langle \eta \rangle^{-d-1} \) and \( \Phi_\varepsilon := \varepsilon^{-d} \hat{\Phi}(\frac{\varepsilon}{\varepsilon}). \) That is \( \Phi \ast \varphi = \mathcal{F}^{-1}(\hat{\Phi} \mathcal{F} \varphi) = \hat{\Phi}(\varepsilon \nabla_v) \varphi. \)

On one hand,
\[ \| \varphi - \Phi_\varepsilon \ast \varphi \|_{H(a^{-1/2})} \lesssim \| \varphi - \Phi_\varepsilon \ast \varphi \|_{L^2} \to 0, \quad \text{as} \ \varepsilon \to 0. \]

On the other hand,
\[ \| \Phi_\varepsilon \ast \varphi \|_{H(a^{-1/2})} \lesssim \| (a^{-1/2})^w \hat{\Phi}(\varepsilon \nabla_v) \varphi \|_{L^2}. \]

Write
\[ \varphi = \frac{-A\varphi + K\varphi + f}{-\sigma_y + i\tau + 2\pi i y \cdot v}. \quad (79) \]

Then
\[ \| (a^{-1/2})^w \hat{\Phi}(\varepsilon \nabla_v) \varphi \|_{L^2} = \| (a^{-1/2})^w \hat{\Phi}(\varepsilon \nabla_v) \left( \frac{-A\varphi + K\varphi + f}{-\sigma_y + i\tau + 2\pi i y \cdot v} \right) \|_{L^2}. \]

Since \( \sigma_y > 0, \) we have \((-\sigma_y + i\tau + 2\pi i y \cdot v)^{-1} \in S(\sigma_y^2 + (\tau + 2\pi y \cdot v)^2)^{-1/2} \subseteq S(1), \)
\(-\sigma_y + i\tau + 2\pi iy \cdot v \in S((\sigma_y^2 + (\tau + 2\pi y \cdot v)^2)^{1/2}) \) uniformly in \( y. \) Denoting \( \Psi(v) = -\sigma_y + i\tau + 2\pi i y \cdot v, \) we have
\[ \| (a^{-1/2})^w \hat{\Phi}(\varepsilon \nabla_v) \left( \frac{-A\varphi + K\varphi + f}{-\sigma_y + i\tau + 2\pi i y \cdot v} \right) \|_{L^2} \]
\[ = \| \Psi^{-1}(a^{-1/2})^w \hat{\Phi}(\varepsilon \nabla_v) \Psi^{-1}(a^{-1/2})^w (-A\varphi + K\varphi + f) \|_{L^2} \]
\[ \lesssim \left\| \hat{\Phi}(\varepsilon \nabla_v)(a^{-1/2})^w (-A\varphi + K\varphi + f) \right\|_{\lambda} \mid_{L^2_v}. \]

Therefore (78) becomes
\[ \int_{-\sigma_y}^{-\sigma_y+i\nu} \| (\lambda I - \hat{B}(y))^{-k} f \|^2_{H(a^{1/2})} d\lambda \]
\[ \lesssim \int_{-\sigma_y}^{i\nu} \lim_{\varepsilon \to 0} \| \Phi_\varepsilon \ast \varphi \|^2_{H(a^{-1/2})} d\tau \]
\[ \lesssim \int_{-\sigma_y}^{i\nu} \lim_{\varepsilon \to 0} \left\| \hat{\Phi}(\varepsilon \nabla_v)(a^{-1/2})^w (-A\varphi + K\varphi + f) \right\|^2_{L^2_v} d\tau \]

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\[
\liminf_{\varepsilon \to 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \frac{|\Phi(\varepsilon \nabla_v)(a^{-1/2})w(-A\varphi + K\varphi + f)(v)|^2}{\sigma^2_y + (\tau + 2\pi y \cdot v)^2} \, d\tau \, dv \\
\lesssim \liminf_{\varepsilon \to 0} \int_{\mathbb{R}} \left| \frac{\Phi_{\varepsilon} * ((a^{-1/2})w(-A\varphi_1 + K\varphi_1 + f))(v)}{\sigma^2_y + \tau^2} \right|^2 \, d\tau \\
\lesssim \int_{\mathbb{R}} \left( \frac{|(a^{-1/2})w(-A\varphi_1 + K\varphi_1 + f)|^2}{\sigma^2_y + \tau^2} \right) \, d\tau \\
\lesssim \|f\|_{L^2(H)}^2 
\]

by Fatou’s lemma, dominated convergence theorem and theorem 4.1, where \( A \in Op(a) \) and the constant is independent of \( y \) and \( \varphi_1 := (\sigma + i(\tau - 2\pi v \cdot y) - \hat{B}(y))^{-1}f \).

Similarly, \( \hat{B}(y)^* = \hat{B}(-y) \) in the second integral of (77) satisfies the same estimate. Therefore,

\[
\left| \left( \int_{-\sigma_\gamma}^{-\sigma_\gamma + \infty} e^{\lambda t} (\lambda I - \hat{B}(y))^{-k} f \, d\lambda, g \right) \right| \lesssim e^{-\sigma_\gamma t} \|f\|_{L^2(H)} \|g\|_{L^2(H)} \\
\left\| \left( \int_{-\sigma_\gamma}^{-\sigma_\gamma + \infty} e^{\lambda t} (\lambda I - \hat{B}(y))^{-1} f \, d\lambda \right) \right\|_{L^2(H)} \lesssim \frac{e^{-\sigma_\gamma t k^2}}{t^k} \|f\|_{L^2(H)}
\]

for \( f \in \mathcal{S} \) and hence for \( f \in L^2(H) \) by density.

5. Since when \(|y|\) is fixed, \( \lambda_j(|y|) \) is isolated eigenvalue of \( \hat{B}(y) \), by theorem XV.2.2. in [25], we have for \(|y| \leq y_0\),

\[
\text{Res}\{e^{\lambda t} (\lambda I - \hat{B}(y))^{-1} f : \lambda = \lambda_j(y)\} = e^{\lambda_j(y) t} P_j f,
\]

where \( P_j \) is the projection from \( L^2 \) into the eigenspace corresponding \( \lambda_j(y) \). Recall the behavior of \( \lambda_j(|y|) \) and \( P_j \) in theorem 3.4, we can choose \( y_0 \) so small that \( \text{Re} \lambda_j(y) \leq -C_0|y|^2 \), for any \(|y| \leq y_0\) and some \( C_0 > 0 \). Thus, substituting the estimate in step three and four into (75), we have

\[
\|e^{tB} f\|_{H^{a/2}(H)} \lesssim \frac{e^{-\sigma_\gamma t k^2}}{t^k} \|f\|_{L^2(H)} + 1_{|y| \leq y_0} \sum_{j=0}^{d+1} |e^{\lambda_j(y) t}| \|P_j f\|_{H^{a/2}(H)} \\
\lesssim \frac{e^{-\sigma_\gamma t C_k}}{t^k} \|f\|_{L^2(H)} + 1_{|y| \leq y_0} e^{-C_0|y|^2 t} \|f\|_{H^{a/2}(H)}.
\]

Since \( e^{tB}, e^{t\hat{B}(y)} \) generate strongly continuous semigroup on \( L^2_{x,v}, L^2_v \) respectively, \( B = \mathcal{F}_x^{-1} \hat{B}(y) \mathcal{F}_x \) on \( L^2 \), we have, by (98),

\[
e^{tB} = \mathcal{F}_x^{-1} e^{t\hat{B}(y)} \mathcal{F}_x.
\]

Then by Hausdorff-Young’s inequality and Hölder’s inequality, for \( f \in \mathcal{S}(\mathbb{R}^{2d}), p \in [1, 2], q \in [1, \infty) \) satisfying \( \frac{1}{p} + \frac{1}{2q} = 1 \), we have that

\[
\|e^{tB} f\|_{H^{a/2}(H)^2}^2 \]
\[
\begin{align*}
&= \int \langle y \rangle^{2m} \int |(a^{1/2})^w e^{i\tilde{B}(y)} \mathcal{F}_x f|^2 \, dxdy \\
&\leq \int \langle y \rangle^{2m} \left( e^{-2\sigma_1^2 C_k} \frac{1}{t^{2k}} + 1_{|y| \leq y_0} e^{-2C_0|y|^2t} \right) \int |(a^{1/2})^w \mathcal{F}_x f|^2 \, dxdy \\
&\leq \frac{e^{-2\sigma_1^2 C_k}}{t^{2k}} \|f\|_{H_x(a^{1/2})}^2 \\
&\quad + \int \left( \int 1_{|y| \leq y_0} e^{-2C_0|y|^2t} \frac{a}{2} \right)^{q-1} \left( \int |\mathcal{F}_x (a^{1/2})^w f|^{2q} \, dy \right)^{\frac{1}{q}} \, dv \\
&\leq \frac{e^{-2\sigma_1^2 C_k}}{t^{2k}} \|f\|_{H_x(a^{1/2})}^2 + \frac{C_p}{(1+t)^{(d/2)(p-1)}} \|a^{1/2}f\|_{L_v^p(L^p_v)}^2,
\end{align*}
\]
with constant \(C_p\) uniformly bounded on \(p \in [1, 2]\). \(\square\)

5 Global Existence for Hard Potential

In this section, we will discuss the global existence to Boltzmann equation (10) for hard potential \(\gamma + 2s \geq 0\). Here we introduce a norm \(X\) similar to [28] and deduce an energy estimate on this space. Also we will apply the estimate on \(\Gamma\) from [26].

5.1 Estimate on nonlinear term \(\Gamma\)

By Theorem 2.1 in [26] and (7), for \(f, g, h \in \mathcal{S}, \, n \geq 0,\)
\[
\|\langle \Gamma(f, g, h) \rangle_{L^p_x}\| \leq C \|f\|_{L^p_x} \|g\|_{L^p_x} \|h\|_{L^p_x},
\]
(80)

Note that \(\mathcal{F}_x \Gamma(f, g) = \int \Gamma(\tilde{f}(x - y), \tilde{g}(y)) \, dy\), where \(\tilde{f} = \mathcal{F}_x f\). Hence, for \(m \in \mathbb{R},\)
\[
\begin{align*}
&\left| \left( \langle D_x \rangle^m (a^{1/2})^w \Gamma(f, g), \langle D_x \rangle^m (a^{1/2})^w h \right)_{L^p_x} \right| \\
&= \left| \langle y \rangle^m (a^{1/2})^w \int \Gamma(\tilde{f}(y - z), \tilde{g}(z)) \, dz, \langle y \rangle^m (a^{1/2})^w \tilde{h}(y) \right)_{L^p_x} \\
&\leq C_m \int \int \|\tilde{f}(y - z)\|_{L^p_x} \|\tilde{g}(z)\|_{L^p_x} \|\langle y \rangle^m (a^{1/2})^w \tilde{h}(y)\|_{L^p_x} \, dxdy \\
&\quad + C_m \int \int \|\tilde{f}(y - z)\|_{L^p_x} \|\tilde{g}(z)\|_{L^p_x} \|\langle y \rangle^m (a^{1/2})^w \tilde{h}(y)\|_{L^p_x} \, dxdy \\
&\leq C_m \|\langle D_x \rangle^m f\|_{L^p_x} \|\langle (a^{1/2})^w \tilde{g}(y)\|_{L^p_x} \|\langle D_x \rangle^m (a^{1/2})^w \tilde{h} \|_{L^p_x} + C_m \|\tilde{f}(y)\|_{L^p_x} \|\tilde{g}(y)\|_{L^p_x} \|\langle D_x \rangle^m (a^{1/2})^w h\|_{L^p_x}.
\end{align*}
\]
by Hölder’s inequality and Fubini’s theorem since \( f, g \in \mathcal{S} \). In particular, when \( m > \frac{d}{2} \), we have \( \| \hat{g} \|_{L_x^1} \leq \| y \|^{-m} \| \hat{g} \|_{L_y^2} \leq C \| \langle D_x \rangle^m g \|_{L_x^2} \) and hence,

\[
\left| \langle D_x \rangle^m (a^{1/2})^w \Gamma(f, g), \langle D_x \rangle^m (a^{1/2})^w h \right|_{L_x^2, v} \\
\leq C_m \| \langle D_x \rangle^m f \|_{L_x^2, v} \| \langle D_x \rangle^m (a^{1/2})^w g \|_{L_x^2, v} \| \langle D_x \rangle^m (a^{1/2})^w h \|_{L_x^2, v},
\]

\[
\left\| \langle D_x \rangle^m (a^{1/2})^w \Gamma(f, g) \right\|_{L_x^2, v} \leq C_m \| \langle D_x \rangle^m f \|_{L_x^2, v} \| \langle D_x \rangle^m (a^{1/2})^w g \|_{L_x^2, v}.
\]

On the other hand, applying Hölder’s inequality on \( x \) in (80),

\[
\left( (a^{1/2})^w \Gamma(f, g), (a^{1/2})^w h \right)_{L_x^2, v} \leq C \| f \|_{L_x^2, v} \| (a^{1/2})^w g \|_{L_x^2, v} \sup_{x \in \mathbb{R}^d} \| (a^{1/2})^w h \|_{L_x^2, v},
\]

\[
\left\| (a^{1/2})^w \Gamma(f, g) \right\|_{L_x^1} \leq C \| f \|_{L_x^2, v} \| (a^{1/2})^w g \|_{L_x^2, v}.
\]

Thus \( \Gamma(f, g) \), initially defined on \( \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \), can uniquely extend to a bilinear continuous operator on \( L_x^2 H_x^m \times H(a^{1/2}) H_x^m \) and \( L_x^2 \times H(a^{1/2}) L_x^2 \).

### 5.2 Estimate on space \( X \)

Let \( \delta > 0 \) be a small constant chosen later. We define inner product

\[
(f, g)_X = \delta(f, g)_{L_x^2 H_x^m} + \int_0^\infty (e^{\tau B} f, e^{\tau B} g)_{L_x^2 H_x^m} d\tau,
\]

(83)

and the corresponding norm \( \| \cdot \|_X^2 := (\cdot, \cdot)_X \).

For \( m \in \mathbb{R} \), assume \( f_0 \in L_x^2 H_x^m \cap D(B) \). By semigroup theory, the solution to

\[
f_t = B f, \quad f|_{t=0} = f_0,
\]

(84)

is \( f = e^{tB} f_0 \in D(B) \). Notice that \( \langle D_x \rangle \) commutes with \( A \), thus by a similar argument in theorem 3.2 (1), we can estimate the closure of \( v \cdot \nabla_x + A \) on \( L_x^2, v \) as following.

\[
\text{Re}(\langle v \cdot \nabla_x + A \rangle f, f)_{L_x^2 H_x^m} \geq \nu_0 \| f \|_{H(a^{1/2}) H_x^m}, \quad \text{for } f \in H(\langle v \rangle, y) \cap H(a), \]

\[
\text{Re}(\langle v \cdot \nabla_x + A \rangle f, f)_{L_x^2 H_x^m} \geq \nu_0 \| f \|_{H(a^{1/2}) H_x^m}, \quad \text{for } f \in D(v \cdot \nabla_x + A) = D(B).
\]

(85)

Thus for \( m \in \mathbb{R} \), \( t_0 > 0 \),

\[
\frac{1}{2} \frac{d}{dt} \| f \|_{L_x^2 H_x^m}^2 = \text{Re}(\langle -v \cdot \nabla_x - A + K \rangle f, f)_{L_x^2 H_x^m} \leq -\nu_0 \| f \|_{H(a^{1/2}) H_x^m}^2 + C \| f \|_{L_x^2 H_x^m}^2,
\]

\[
\frac{1}{2} \sup_{0 \leq t \leq t_0} \| f(t) \|_{L_x^2 H_x^m}^2 + \nu_0 \int_0^{t_0} \| f(t) \|_{H(a^{1/2}) H_x^m}^2 dt \leq \frac{1}{2} \| f_0 \|_{L_x^2 H_x^m}^2 + C \int_0^{t_0} \| f \|_{L_x^2 H_x^m}^2 dt
\]

\[
\leq \frac{1}{2} \| f_0 \|_{L_x^2 H_x^m}^2 + C t_0 \sup_{0 \leq t \leq t_0} \| f \|_{L_x^2 H_x^m}^2,
\]

(81)
if we choose $t_0 = \frac{1}{4v}$. The estimate on the sun dual semigroup $(e^{tB})^\circ$ of $e^{tB}$ satisfies the same estimate, cf. II.2.6 in [24], since $B^* = (-v \cdot \nabla_x + L)^* = v \cdot \nabla_x + \bar{L}$. Therefore

$$
\int_0^{t_0} \| f(t) \|^2_{L^2_{\mathbb{R}^n}} dt = \int_0^{t_0} \| e^{tB} f_0 \|^2_{L^2_{\mathbb{R}^n}} dt
$$

$$
= \int_0^{t_0} \lim_{n \to \infty} |(e^{tB} f_0, g_n)_{L^2_{\mathbb{R}^n}}|^2 dt
$$

$$
\leq \liminf_{n \to \infty} \| f_0 \|^2_{H^{(a-1/2)}_{\mathbb{R}^n}} \int_0^{t_0} \| e^{tB^*} g_n \|^2_{H^{(a-1/2)}_{\mathbb{R}^n}} dt
$$

$$
\leq \frac{1}{2v_0} \liminf_{n \to \infty} \| f_0 \|^2_{H^{(a-1/2)}_{\mathbb{R}^n}} \| g_n \|^2_{L^2_{\mathbb{R}^n}}
$$

$$
\leq \frac{1}{2v_0} \| f_0 \|^2_{H^{(a-1/2)}_{\mathbb{R}^n}}, \quad \text{(86)}
$$

for some sequence $\{g_n\} \subset \mathcal{S}$ with $\| g_n \|_{L^2_{\mathbb{R}^n}} = 1$. For large time $t \geq t_0$, we apply theorem (1.2) to $f = e^{tB} f_0$ with $k = 2$, then

$$
\int_0^{t_0} \| e^{tB} f_0 \|^2_{H^{(a-1/2)}_{\mathbb{R}^n}} dt
$$

$$
\leq \int_0^{t_0} \left( e^{-2\sigma^2 t^4} \right) \left( (a^{-1/2})^w f_0 \right)^2_{L^2_{\mathbb{R}^n}} dt
$$

$$
\leq C \left( (a^{-1/2})^w f_0 \right)^2_{L^2_{\mathbb{R}^n}} + \| (a^{-1/2})^w f_0 \|^2_{L^2_{\mathbb{R}^n}}
$$

for $d \geq 3$ and $\frac{d}{2} (\frac{2}{p} - 1) > 1$. Together with (86), we have

$$
\int_0^{t_0} \| e^{tB} f_0 \|^2_{L^2_{\mathbb{R}^n}} dt \leq C \left( (a^{-1/2})^w f_0 \right)^2_{L^2_{\mathbb{R}^n}} + \| (a^{-1/2})^w f_0 \|^2_{L^2_{\mathbb{R}^n}}. \quad \text{(87)}
$$

### 5.3 Proof of Theorem 1.3

Combining the above two sections, we can prove the global existence to

$$
f_t = B f + \Gamma(f, f), \quad f|_{t=0} = f_0, \quad \text{(88)}
$$

with $f \in X$. For notational convenience during the proof, we define a total norm

$$
\mathbb{G}(f(t)) := \delta \| f(t) \|^2_{L^2_{\mathbb{R}^n}} + \int_0^\infty \| e^{tB} f(t) \|^2_{L^2_{\mathbb{R}^n}} dt + 2\delta v_0 \int_0^t \| f(\tau) \|^2_{H^{(a-1/2)}_{\mathbb{R}^n}} d\tau. \quad \text{(89)}
$$
Theorem 5.1. Let $d \geq 3$, $m > \frac{d}{2}$. There exists $\varepsilon_0 > 0$ such that if
\[
\|f(0)\|_{X}^2 \leq \varepsilon_0^2 \quad \text{and} \quad \sup_{0 \leq t < \infty} \mathcal{G}(g) \leq 2\varepsilon_0^2,
\]
then the solution $f$ to linear equation
\[
f_t = Bf + \Gamma(g, f), \quad f|_{t=0} = f_0,
\]
is well defined and satisfies
\[
\sup_{0 \leq t < \infty} \mathcal{G}(f) \leq 2\varepsilon_0^2.
\]

Proof. A rather standard procedure (adding the vanishing term $\varepsilon (D_{x,v})^M (1 + |v|^2 + |x|^2)^{2M} (D_{x,v})^M f$ for some large $M$ and standard parabolic equation theory with mollified initial data, cf. [37]) gives the existence of solution to (90). So we will focus on the proof of (91). Let $f = f(t)$ be a local solution to (90), then we have
\[
\frac{1}{2} \frac{d}{dt} \|f\|_{X}^2 = \frac{\delta}{2} \frac{d}{dt} \|f\|_{L^2_x H^{m-\frac{d}{2}}_x}^2 + \frac{1}{2} \frac{d}{dt} \int_0^\infty \|e^{\tau B} f\|_{L^2_x H^{m-\frac{d}{2}}_x}^2 d\tau
\]
\[
= \delta \text{Re}(B f, f)_{L^2_x H^m} + \int_0^\infty \text{Re}(e^{\tau B} B f, e^{\tau B} f)_{L^2_x H^m} d\tau
\]
\[
+ \delta \text{Re}(\Gamma(g, f), f)_{L^2_x H^m} + \int_0^\infty \text{Re}(e^{\tau B} \Gamma(g, f), e^{\tau B} f)_{L^2_x H^m} d\tau
\]
\[
=: I_1 + I_2 + I_3 + I_4.
\]
For $I_1$, $I_2$, we use theorem 1.1, $\frac{d}{dt} e^{\tau B} f = e^{\tau B} B f$ and (85) to obtain
\[
I_1 + I_2 \leq -\delta \nu_0 \|f\|_{L^2_x H^{m-\frac{d}{2}}_x}^2 + C \delta \|f\|_{L^2_x H^m}^2 + \int_0^{\infty} \frac{1}{2} \frac{d}{d\tau} \|e^{\tau B} f\|_{L^2_x H^m}^2 d\tau
\]
\[
\leq -\delta \nu_0 \|f\|_{L^2_x H^{m-\frac{d}{2}}_x}^2 + C \delta \|f\|_{L^2_x H^m}^2 - \frac{1}{2} \|f\|_{L^2_x H^m}^2
\]
\[
\leq -\delta \nu_0 \|f\|_{L^2_x H^{m-\frac{d}{2}}_x}^2 - \frac{1}{4} \|f\|_{L^2_x H^m}^2,
\]
where we choose $\delta = \frac{1}{14}$. For $I_3$, $I_4$, we apply (81) to get
\[
I_3 \leq C \delta \|a^{-1/2} w \Gamma(g, f)\|_{L^2_x H^m} \|f\|_{L^{(a+1/2)}_x H^m} \|f\|_{H^{(a+1/2)}_x H^m}
\]
\[
\leq C \delta \|g\|_{L^2_x H^m} \|f\|_{H^{(a+1/2)}_x H^m}.
\]
At last, we apply (81)(82)(87) to $I_4$
\[
I_4 \leq \int_0^{\infty} \|e^{\tau B} \Gamma(g, f)\|_{L^2_x H^m} \|e^{\tau B} f\|_{L^2_x H^m} d\tau
\]
\[
\leq C \left( \int_0^{\infty} \|e^{\tau B} \Gamma(g, f)\|_{L^2_x H^m}^2 d\tau \right)^{1/2} \left( \int_0^{\infty} \|e^{\tau B} f\|_{L^2_x H^m}^2 d\tau \right)^{1/2}
\]
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\[ \leq C \left( \|a^{-1/2} \omega \Gamma(g,f)\|_{L^2_t H^m_x}^2 + \|a^{-1/2} \omega \Gamma(g,f)\|_{L^2_t(L^1)}^2 \right)^{1/2} \left( \int_0^\infty \|e^{\tau B} f\|^2_{L^2_t H^m_x} d\tau \right)^{1/2} \]

\[ \leq C \|g\|_{L^2_t H^m_x} \|f\|_{H(a^{1/2}) H^m_x} \left( \int_0^\infty \|e^{\tau B} f\|^2_{L^2_t H^m_x} d\tau \right)^{1/2} \]

As a conclusion,

\[ \frac{\delta}{dt} \|f\|^2_{L^2_t H^m_x} + \frac{d}{dt} \int_0^\infty \|e^{\tau B} f\|^2_{L^2_t H^m_x} d\tau + 2\delta \|Bf\|^2_{H(a^{1/2}) H^m_x} \]

\[ \leq C \|g\|_{L^2_t H^m_x} \|f\|^2_{H(a^{1/2}) H^m_x} + C \|g\|_{L^2_t H^m_x} \|f\|_{H(a^{1/2}) H^m_x} \left( \int_0^\infty \|e^{\tau B} f\|^2_{L^2_t H^m_x} d\tau \right)^{1/2}. \]

Then taking the integral on \( t \), thanks to (89), we have

\[ \sup_{0 \leq t < \infty} \mathbb{G}(f) \leq \|f(0)\|^2_{H^m_x} + C \sup_{0 \leq t < \infty} \|g\|_{L^2_t H^m_x} \int_0^\infty \|f(t)\|^2_{H(a^{1/2}) H^m_x} dt \]

\[ + C \int_0^\infty \|g(t)\|^2_{H(a^{1/2}) H^m_x} \|f(t)\|_{H(a^{1/2}) H^m_x} dt \sup_{0 \leq t < \infty} \left( \int_0^\infty \|e^{\tau B} f\|^2_{L^2_t H^m_x} d\tau \right)^{1/2} \]

\[ \leq \|f(0)\|^2_{H^m_x} + \frac{C'}{\sqrt{2}} \sup_{0 \leq t < \infty} \mathbb{G}(g)^{1/2} \mathbb{G}(f). \]

since \( \cdot \|_{L^2} \leq C \cdot \|_{H(a^{1/2})} \) for hard potential. Thus if \( \mathbb{G}(g) \leq 2\varepsilon_0^2 \) and \( \|f(0)\|^2_{H^m_x} \leq \varepsilon_0^2 \) with \( \varepsilon_0 \in (0, \frac{1}{2\pi}) \), we have

\[ \sup_{0 \leq t < \infty} \mathbb{G}(f) \leq \|f(0)\|^2_{H^m_x} + C' \sup_{0 \leq t < \infty} \varepsilon_0 \mathbb{G}(f), \]

\[ \sup_{0 \leq t < \infty} \mathbb{G}(f) \leq 2\|f(0)\|^2_{H^m_x} \leq 2\varepsilon_0^2. \]

\[ \square \]

After obtaining the uniform energy estimate, we can apply a standard iteration to prove our global existence result.

**Proof of Theorem 1.3.** 1. Let \( f^0 = 0 \) and \( f^{n+1} (n \geq 0) \) be the solutions to

\[ f_t^{n+1} = B f^{n+1} + \Gamma(f^n, f^{n+1}), \quad f^{n+1}|_{t=0} = f_0. \]  

(92)

Then \( d^n = f^{n+1} - f^n \) \((n \geq 1)\) solves

\[ d_t^n = B d^n + \Gamma(f^n, d^n) + \Gamma(d^{n-1}, f^n), \quad d^{n+1}|_{t=0} = 0, \]

while \( d^0 = f^1 \) satisfies \( \mathbb{G}(d^0) \leq 2\varepsilon_0^2 \). Next we will assume \( \mathbb{G}(d^{n-1}) \leq (2C'\varepsilon_0)^{2n} (n \geq 1) \) for some constant \( C' \) found at (93), then similar to the proof in theorem 5.1, we have

\[ \frac{1}{2} \frac{d}{dt} \|d^n\|^2_{H^m_x} = \frac{\delta}{2} \frac{d}{dt} \|d^n\|^2_{L^2_t H^m_x} + \frac{1}{2} \frac{d}{dt} \int_0^\infty \|e^{\tau B} d^n\|^2_{L^2_t H^m_x} d\tau \]
As in theorem 5.1, we have

\[ I_1 + I_2 \leq -\delta \varphi_0 \|d^n\|^2_{H(a^{1/2})H^p} + C \delta \|d^n\|^2_{L_x^2 L^2_T} + \int_0^\infty \frac{1}{2} \frac{d}{d\tau} \|e^{\gamma B} d^n\|^2_{L_x^2 L^2_T} \, d\tau \]

\[ \leq -\delta \varphi_0 \|d^n\|^2_{H(a^{1/2})H^p} - \frac{1}{4} \|d^n\|^2_{L_x^2 L^2_T}, \]

\[ I_3 \leq C \delta \left( \|f^n\|_{L_x^2 L^2_T} \|d^n\|_{H(a^{1/2})H^p} + \|d^{n-1}\|_{L_x^2 L^2_T} \|f^n\|_{H(a^{1/2})H^p} \right) \|d^n\|_{H(a^{1/2})H^p}, \]

\[ I_4 \leq \int_0^\infty \|e^{\gamma B} (f^n, d^n) + \Gamma(d^{n-1}, f^n))\|_{L_x^2 L^2_T} \|e^{\gamma B} d^n\|_{L_x^2 L^2_T} \, d\tau \]

\[ \leq C \left( \|f^n\|_{L_x^2 L^2_T} \|d^n\|_{H(a^{1/2})H^p} + \|d^{n-1}\|_{L_x^2 L^2_T} \|f^n\|_{H(a^{1/2})H^p} \right) \left( \int_0^\infty \|e^{\gamma B} d^n\|^2_{L_x^2 L^2_T} \, d\tau \right)^{1/2}, \]

where \( \delta = \frac{1}{4C} \) as in theorem 5.1. Thus

\[ \frac{1}{2} \frac{d}{d\tau} \|d^n\|^2_X + \delta \varphi_0 \|d^n\|^2_{H(a^{1/2})H^p} \]

\[ \leq C \delta \left( \|f^n\|_{L_x^2 L^2_T} \|d^n\|_{H(a^{1/2})H^p} + \|d^{n-1}\|_{L_x^2 L^2_T} \|f^n\|_{H(a^{1/2})H^p} \right) \|d^n\|_{H(a^{1/2})H^p} \]

\[ + C \left( \|f^n\|_{L_x^2 L^2_T} \|d^n\|_{H(a^{1/2})H^p} + \|d^{n-1}\|_{L_x^2 L^2_T} \|f^n\|_{H(a^{1/2})H^p} \right) \mathbb{G}(d^n) \]

Taking integral on \( t \) and using the uniform energy bound in theorem 90, we deduce that

\[ \sup_{0 \leq t < \infty} \mathbb{G}(d^n) \]

\[ \leq C' \sup_{0 \leq t < \infty} \|f^n\|_{L_x^2 H^p} \int_0^\infty \|d^n\|^2_{H(a^{1/2})H^p} \, dt \]

\[ + \sup_{0 \leq t < \infty} \|d^{n-1}\|_{L_x^2 H^p} \int_0^\infty \|f^n\|_{H(a^{1/2})H^p} \|d^n\|_{H(a^{1/2})H^p} \, dt \]

\[ + C \int_0^\infty \left( \|f^n\|_{H(a^{1/2})H^p} \|d^n\|_{H(a^{1/2})H^p} + \|d^{n-1}\|_{L_x^2 H^p} \|f^n\|_{H(a^{1/2})H^p} \right) \, dt \mathbb{G}(d^n)^{1/2} \]

\[ \leq C' \sup_{0 \leq t < \infty} \left( \mathbb{G}(f^n)^{1/2} + \mathbb{G}(d^{n-1})^{1/2} \mathbb{G}(f^n)^{1/2} \mathbb{G}(d^n)^{1/2} \right), \quad (93) \]
where \( C' \) is independent of \( n \). Thus, when \( G(d^{n-1}) \leq (2C'\varepsilon_0)^{2n} \), using (91), we have

\[
\sup_{0 \leq t < \infty} G(d^n) \leq C' \left( \sqrt{2}\varepsilon_0 \sup_{0 \leq t < \infty} G(d^n) \right) + (2C'\varepsilon_0)^n \sqrt{2}\varepsilon_0 \sup_{0 \leq t < \infty} G(d^n)^{1/2},
\]

\[
\sup_{0 \leq t < \infty} G(d^n) \leq (2C'\varepsilon_0)^{2(n+1)},
\]

where we choose \( \varepsilon_0 \) such that \( C' \sqrt{2}\varepsilon_0 \leq 1 - \frac{1}{\sqrt{2}} \). Recalling the energy (89) and choose \( \varepsilon_0 \) sufficiently small, the sequence \( \{f^n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( L^\infty([0, \infty); L^2_t H^m_x) \) and \( L^2([0, \infty); H^{(a^1/2)}_t H^m_x) \). Hence its limit \( f \) solves (88) in the weak sense. We then deduce that

\[
\|f\|_{L^\infty([0, \infty); L^2_t H^m_x)} + \|f\|_{L^2([0, \infty); H^{(a^1/2)}_t H^m_x)} \leq C''\varepsilon_0,
\]

(94)

by passing the limit \( n \to \infty \).

2. To prove the uniqueness, we suppose that there exists another solution \( g \) with the same initial data satisfying (94). Then the difference \( f - g \) satisfies

\[
\partial_t(f - g) = B(f - g) + \Gamma(f, f - g) + \Gamma(f - g, g),
\]

in the weak sense. Then by (82)(85), for \( T > 0 \),

\[
\frac{1}{2} \frac{d}{dt} \|f - g\|^2_{L^2_t H^m_x} = \text{Re}(B(f - g), f - g)_{L^2_t H^m_x} + \text{Re}(\Gamma(f, f - g) + \Gamma(f - g, g), f - g)_{L^2_t H^m_x}
\]

\[
\leq -\nu_0 \|f - g\|^2_{H^{(a^1/2)}_t H^m_x} + C\|f - g\|^2_{L^2_t H^m_x} + C\|f\|_{L^2_t H^m_x} \|f - g\|_{H^{(a^1/2)}_t H^m_x} + C\|f - g\|_{L^2_t H^m_x} \|g\|_{H^{(a^1/2)}_t H^m_x} \|f - g\|_{H^{(a^1/2)}_t H^m_x}.
\]

Taking integral on \( t \in [0, T] \),

\[
\sup_{0 \leq t \leq T} \|f - g\|^2_{L^2_t H^m_x} + 2\nu_0 \int_0^T \|f - g\|^2_{H^{(a^1/2)}_t H^m_x} dt 
\]

\[
\leq C \int_0^T \|f - g\|^2_{L^2_t H^m_x} dt + C \sup_{0 \leq t \leq T} \|f\|_{L^2_t H^m_x} \int_0^T \|f - g\|^2_{H^{(a^1/2)}_t H^m_x} dt 
\]

\[
+ C \sup_{0 \leq t \leq T} \|f - g\|_{L^2_t H^m_x} \int_0^T \|g\|_{H^{(a^1/2)}_t H^m_x} \|f - g\|_{H^{(a^1/2)}_t H^m_x} dt 
\]

\[
\leq C \int_0^T \|f - g\|^2_{L^2_t H^m_x} dt + C''\varepsilon_0 \int_0^T \|f - g\|^2_{H^{(a^1/2)}_t H^m_x} dt 
\]

\[
+ C''\varepsilon_0 \sup_{0 \leq t \leq T} \|f - g\|_{L^2_t H^m_x} \left( \int_0^T \|f - g\|^2_{H^{(a^1/2)}_t H^m_x} dt \right)^{1/2},
\]

and when \( \varepsilon_0 > 0 \) is sufficiently small, we can deduce the uniqueness by Gronwall’s inequality.

\( \square \)

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Appendix

**Operator theory** For operator theory, one may refer to [25,40,41].

**Definition 6.1.** Let \((A,D(A))\) be a closable linear operator on Banach space \(X\). Let \(G(A) := \{(x,Ax)|x \in D(A)\}\) and

\[
D(\overline{A}) := \{x \in X : \exists y \in X \text{ s.t. } (x,y) \in G(A)\}.
\]

Denote \(\overline{A}\) maps \(x \in D(\overline{A})\) to the corresponding \(y\). Such \(\overline{A}\) is well-defined and is called the closure of \(A\). Then \(G(\overline{A}) = G(A)\).

**Definition 6.2.** Let \((A,D(A))\) be a linear unbounded densely defined operator from Hilbert space \(H_1\) into Hilbert space \(H_2\) with domain \(D(A)\). Define

\[
D(A^*) = \{y \in H_2 | x \mapsto \langle Ax,y \rangle \text{ is continuous from } H_1 \text{ to } \mathbb{C}\}.
\]

Take \(y \in D(A^*)\), since \(\overline{D(A)} = H_1\), the functional \(F_y(x) := \langle Ax,y \rangle\) has a unique bounded extension on \(H_1\). Hence Riesz representation theorem ensures the existence of a unique \(z \in H_1\) such that \(F_y(x) = \langle x,z \rangle\) for \(x \in H_1\). Define \(A^*y := z\), then

\[
\langle Ax,y \rangle = \langle x,A^*y \rangle.
\]

The operator \(A^*(H_2 \to H_1)\) is linear and is called the adjoint of \(A\). Then \((\overline{A})^* = A^*\) and \((A^*)^{-1} = (A^{-1})^*\) if \(A^{-1}\) exists.

**Lemma 6.3** ([30], Theorem 17.2). Let \(T,S\) be any operator in \(\mathcal{L}(L^2)\) such that \(S\) is invertible and \(\|T\| < \frac{1}{\|S^{-1}\|}\), then

\[
(S - T)^{-1} = \sum_{n=0}^{\infty} (S^{-1}T)^n S^{-1},
\]

and hence \(\|(S - T)^{-1}\| \leq (1 - \|S^{-1}\|\|T\|)\|S^{-1}\|\).

**Pseudo-differential calculus** We recall some notation and theorem of pseudo differential calculus. For details, one may refer to Chapter 2 in the book [34], Proposition 1.1 in [16] and [15,17] for details. Set \(\Gamma = |dv|^2 + |dy|^2\), but also note that the following are also valid for general admissible metric. Let \(M\) be an \(\Gamma\)-admissible weight function. That is, \(M : \mathbb{R}^{2d}(0, +\infty)\) satisfies the following conditions:

(a). (slowly varying) there exists \(\delta > 0\) such that for any \(X,Y \in \mathbb{R}^{2d}\), \(|X - Y| \leq \delta\) implies\n
\[
M(X) \approx M(Y);
\]
(b) (temperance) there exists $C > 0$, $N \in \mathbb{R}$, such that for $X, Y \in \mathbb{R}^{2d}$,

$$\frac{M(X)}{M(Y)} \leq C(X - Y)^N.$$  

A direct result is that if $M_1, M_2$ are two $\Gamma$-admissible weight, then so is $M_1 + M_2$ and $M_1M_2$. Consider symbols $a(v, \eta, \xi)$ as a function of $(v, \eta)$ with parameters $\xi$. We say that $a \in S(\Gamma) = S(M, \Gamma)$ uniformly in $\xi$, if for $\alpha, \beta \in \mathbb{N}^d$, $v, \eta \in \mathbb{R}^d$,

$$|\partial^\alpha_v \partial^\beta_\eta a(v, \eta, \xi)| \leq C_{\alpha, \beta} M,$$

with $C_{\alpha, \beta}$ a constant depending only on $\alpha$ and $\beta$, but independent of $\xi$. The space $S(M, \Gamma)$ endowed with the seminorms

$$\|a\|_{k; S(M, \Gamma)} = \max_{0 \leq |\alpha| + |\beta| \leq k} \sup_{(v, \eta) \in \mathbb{R}^{2d}} |M(v, \eta)^{-1} \partial^\alpha_v \partial^\beta_\eta a(v, \eta, \xi)|,$$

becomes a Fréchet space. Sometimes we write $\partial_\eta a \in S(M, \Gamma)$ to mean that $\partial_\eta a \in S(M, \Gamma)$ $(1 \leq j \leq d)$ equipped with the same seminorms. We formally define the pseudo-differential operator by

$$(op_a)u(x) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{2\pi i(x-y) \cdot \eta} a((1-t)x + ty, \xi)u(y) dyd\xi,$$

for $t \in \mathbb{R}$, $f \in \mathcal{S}$. In particular, denote $a(v, D_v) = op_0 a$ to be the standard pseudo-differential operator and $a^w(v, D_v) = op_{1/2} a$ to be the Weyl quantization of symbol $a$. We write $A \in Op(M, \Gamma)$ to represent that $A$ is a Weyl quantization with symbol belongs to class $S(M, \Gamma)$. One important property for Weyl quantization of a real-valued symbol is the self-adjoint on $L^2$ with domain $\mathcal{S}$.

Let $a_1(v, \eta) \in S(M_1, \Gamma), a_2(v, \eta) \in S(M_2, \Gamma)$, then $a_1^wa_2^w = (a_1 \# a_2)^w$, $a_1 \# a_2 \in S(M_1M_2, \Gamma)$ with

$$a_1 \# a_2(v, \eta) = a_1(v, \eta)a_2(v, \eta) + \int_0^1 (\partial_\eta a_1 \# \partial_\eta a_2 - \partial_v a_1 \# \partial_v a_2) d\theta,$$

$$g \# h(Y) := \frac{2^{2d}}{\theta^{2n}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-\frac{4\pi i}{\theta}(X-Y_1)(X-Y_2)}(4\pi i)^{-1}(\sigma \partial_{Y_1} \partial_{Y_2})g(Y_1)h(Y_2) dY_1dY_2,$$

with $Y = (v, \eta)$, $\sigma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. For any non-negative integer $k$, there exists $l, C$ independent of $\theta \in [0, 1]$ such that

$$\|g \# h\|_{k; S(M_1M_2, \Gamma)} \leq C\|g\|_{l; S(M_1, \Gamma)}\|h\|_{l; S(M_2, \Gamma)}.$$  

Thus if $\partial_\eta a_1, \partial_\eta a_2 \in S(M'_1, \Gamma)$ and $\partial_v a_1, \partial_v a_2 \in S(M'_2, \Gamma)$, then $[a_1, a_2] \in S(M'_1M'_2, \Gamma)$, where $[\cdot, \cdot]$ is the commutator defined by $[A, B] := AB - BA$.  

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We can define a Hilbert space $H(M, \Gamma) := \{ u \in \mathcal{S} : \| u \|_{H(M, \Gamma)} < \infty \}$, where
\[
\| u \|_{H(M, \Gamma)} := \int M(Y)^2 \| \varphi_Y u \|_{L^2}^2 |dY|^{1/2} dY < \infty,
\] (96)
and $(\varphi_Y)_{Y \in \mathbb{R}^{dd}}$ is any uniformly confined family of symbols which is a partition of unity. If $a \in S(M)$ is an isomorphism from $H(M')$ to $H(M'M^{-1})$, then $(a^w u, a^w v)$ is an equivalent Hilbertian structure on $H(M)$. Moreover, the space $\mathcal{S}(\mathbb{R}^d)$ is dense in $H(M)$ and $H(1) = L^2$.

Let $a \in S(M, \Gamma)$, then $a^w : H(M_1, \Gamma) \to H(M_1/M, \Gamma)$ is linear continuous, in the sense of unique bounded extension from $\mathcal{S}$ to $H(M_1, \Gamma)$. Also the existence of $b \in S(M_1, \Gamma)$ such that $b\# a = a\# b = 1$ is equivalent to the invertibility of $a^w$ as an operator from $H(MM_1, \Gamma)$ onto $H(M_1, \Gamma)$ for some $\Gamma$-admissible weight function $M_1$.

For the metric $\Gamma = |dv|^2 + |d\eta|^2$, the map $J^t = \exp(2\pi i D_v \cdot D_\eta)$ is an isomorphism of the Fréchet space $S(M, \Gamma)$, with polynomial bounds in the real variable $t$, where $D_v = \partial_v/i$, $D_\eta = \partial_\eta/i$. Moreover, $a(x, D_v) = (J^{-1/2}a)^w$.

Let $m_K(v, \eta)$ be a $\Gamma$-admissible weight function depending on $K$, $c$ be any $\Gamma$-admissible weight. Then lemma 2.1 and 2.3 in [23] can be reformulated as the following.

**Lemma 6.4.** Assume $a_K \in S(m_K), \partial_\eta(a_K) \in S(K^{-\kappa}m_K)$ uniformly in $K$ and $|a_K| \gtrsim m_K$. Then
(1) $a_K^{-1} \in S(m_K^{-1})$, uniformly in $K$, for $K > 1$.
(2) There exists $K_0 > 1$ sufficiently large such that for all $K > K_0$, $a_K^w : H(m_Kc) \to H(c)$ is invertible and its inverse $(a_K^w)^{-1} : H(c) \to H(m_Kc)$ satisfies
\[
(a_K^w)^{-1} = G_{1,K}(a_K^{-1})^w = (a_K^{-1})^w G_{2,K},
\]
where $G_{1,K} \in \mathcal{L}(H(m_Kc)), G_{2,K} \in \mathcal{L}(H(c))$ with operator norm smaller than 2. Also, by the equivalence of invertibility, $(a_K^w)^{-1} \in \text{Op}(m_K^{-1})$.

**Lemma 6.5.** Let $m, c$ be $\Gamma$-admissible weight such that $a \in S(m)$. Assume $a^w : H(mc) \to H(c)$ is invertible. If $b \in S(m)$, then there exists $C > 0$, depending only on the seminorms of symbols to $(a^w)^{-1}$ and $b^w$, such that for $f \in H(mc)$,
\[
\| b(v, D_v) f \|_{H(c)} + \| b^w(v, D_v) f \|_{H(c)} \leq C \| a^w(v, D_v) f \|_{H(c)}.
\]
Consequently, if $a^w : H(m_1) \to L^2 \in \text{Op}(m_1), b^w : H(m_2) \to L^2 \in \text{Op}(m_2)$ are invertible, then for $f \in \mathcal{S}$,
\[
\| b^w a^w f \|_{L^2} \lesssim \| a^w b^w f \|_{L^2},
\]
where the constant depends only on seminorms of symbols to $a^w, b^w, (a^w)^{-1}, (b^w)^{-1}$.

**Lemma 6.6.** Let $a \in S(m)$. Then for $f, g \in \mathcal{S}$,
\[
(a^w f, g)_{L^2} = (f, a^w g)_{L^2}.
\]
Thus by density argument, this identity is also valid for $f, g \in H(m)$.
Carleman representation and cancellation lemma  Now we have a short review of some useful facts in the theory of Boltzmann equation. One may refer to [1, 10] for details. The first one is the so called Carleman representation. For a measurable function $F(v, v_*, v', v'_*)$, if any sides of the following equation is well-defined, then

$$
\int_{\mathbb{R}^d} \int_{S^{d-1}} b(\cos \theta) |v - v_*|^\gamma F(v, v_*, v', v'_*) \, d\sigma dv_*
$$

$$
= \int_{\mathbb{R}_+} \int_{E_0,h} \tilde{b}(\alpha, h) 1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma + 1 + 2s}}{|h|^{d+2s}} F(v, v + \alpha - h, v - h, v + \alpha) \, da dh,
$$

(97)

where $\tilde{b}(\alpha, h)$ is bounded from below and above by positive constants, and $\tilde{b}(\alpha, h) = \tilde{b}(|\alpha|, |h|)$, $E_0,h$ is the hyper-plane orthogonal to $h$ containing the origin. The second is the cancellation lemma. Consider a measurable function $G(|v - v_*|, |v - v'|)$, then for $f \in \mathcal{F}$,

$$
\int_{\mathbb{R}^d} \int_{S^{d-1}} G(|v - v_*|, |v - v'|) b(\cos \theta) (f'_* - f_*) \, d\sigma dv_* = S_{v_*} f(v),
$$

where $S$ is defined by, for $z \in \mathbb{R}^d$,

$$
S(z) = 2\pi \int_0^{\pi/2} b(\cos \theta) \sin \theta \left( G\left( \frac{|z|}{\cos \theta/2}, \frac{|z| \sin \theta/2}{\cos \theta/2} \right) - G(|z|, |z| \sin(\theta/2)) \right) \, d\theta.
$$

Semigroup theory  Here we write some well-known result from semigroup theory. One may refer to [24] for more details.

**Definition 6.7.** An operator $(A, D(A))$ is dissipative if and only if for every $x \in D(A)$ there exists $j(x) \in \{ x' \in X' : \langle x, x' \rangle = \|x\|^2 = \|x'\|^2 \}$ such that

$$
\text{Re}(Ax, j(x)) \leq 0.
$$

**Theorem 6.8.** For a densely defined, dissipative operator $(A, D(A))$ on a Banach space $X$ the following statements are equivalent.

(a) The closure $\overline{A}$ of $A$ generates a contraction semigroup.

(b) $\text{Im}(\lambda I - A)$ is dense in $X$ for some (hence all) $\lambda > 0$.

**Theorem 6.9.** (Hille-Yosida Theorem) For a linear operator $(A, D(A))$ on a Banach space $X$, the following properties are equivalent.

(a) $(A, D(A))$ generates a strongly continuous contraction semigroup.

(b) $(A, D(A))$ is closed, densely defined, and for every $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > 0$ one has $\lambda \in \rho(A)$ and $\| (\lambda I - A)^{-1} \| \leq \frac{1}{\text{Re}\lambda}$.

**Theorem 6.10.** Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on a Banach space $X$ satisfying $\| T(t) \| \leq Me^{\omega t}$ for all $t \geq 0$ and some $\omega \in \mathbb{R}$, $M \geq 1$. If $B \in L(X)$, then $C := A + B$ with $D(C) := D(A)$ generates a strongly continuous semigroup $(S(t))_{t \geq 0}$ satisfying $\| S(t) \| \leq Me^{(\omega + M\|B\|)t}$ for all $t \geq 0$.  

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At last, we state the lemma that was applied in our main theorem.

**Theorem 6.11.** On $L^2_{x,v}$,

$$e^{tB} = \mathcal{F}_x^{-1} e^{t\hat{B}(y)} \mathcal{F}_x. \quad (98)$$

**Proof.** Recall that $B$, $\hat{B}(y)$ generate strongly continuous semigroup $e^{tB}$, $e^{t\hat{B}(y)}$ on $L^2_{x,v}$ and $L^2_v$ respectively. Thus, it suffices to prove (98) on $\mathcal{S}(\mathbb{R}^{2d})$. By the construction of semigroup as Hille-Yoshida theorem II.3.5 in [24], for $n > 0$ sufficiently large, we let

$$B_n : = nB(nI - B)^{-1} = n^2(nI - B)^{-1} - nI,$$

$$\hat{B}_n(y) : = n\hat{B}_n(y)(nI - \hat{B}_n(y))^{-1} = n^2(nI - \hat{B}_n(y))^{-1} - nI.$$ 

Then $B_n$, $\hat{B}_n(y)$ are bounded operator on $L^2_{x,v}$ and $L^2_v$ respectively and for $f \in \mathcal{S}$, $B_nf \to Bf$, $\hat{B}_n(y)f \to \hat{B}(y)f$ as $n \to \infty$. Also,

$$e^{tB_n}f = \lim_{n \to \infty} e^{tB_n}f, \quad \text{for } f \in L^2(\mathbb{R}^{2d}),$$

$$e^{t\hat{B}(y)}f = \lim_{n \to \infty} e^{t\hat{B}_n(y)}f, \quad \text{for } f \in L^2(\mathbb{R}^d).$$

Now fix $f \in \mathcal{S}(\mathbb{R}^{2d})$ and let $\varphi(x,v) = (nI - B)^{-1}f \in L^2_{x,v}$. Then $(nI + v \cdot \nabla_x + A)\varphi = f + K\varphi$. Applying theorem 3.1 to operator $nI + v \cdot \nabla_x + A$, we have $nI + v \cdot \nabla_x + A = nI - v \cdot \nabla_x + A$ on $L^2$. Thus for $\psi \in \mathcal{S}(\mathbb{R}^{2d})$,

$$(\varphi, (n - v \cdot \nabla_x + A)\psi)_{L^2_{x,v}} = (f + K\varphi, \psi)_{L^2_{x,v}}.$$ 

On the other hand, applying theorem 2.1 to $f + K\varphi \in \mathcal{S}(\mathbb{R}^{2d})$, there exists $g \in \cap_k \mathcal{H}((K_0 + |v|^2 + |y|^4 + |\eta|^2)^k)$ ($K_0 >> 1$) such that for $g \in \mathcal{S}(\mathbb{R}^{2d})$,

$$(g, (n - v \cdot \nabla_x + A)\psi)_{L^2_{x,v}} = (f + K\varphi, \psi)_{L^2_{x,v}}.$$ 

We choose $k >> 1$ such that $\mathcal{H}((K_0 + |v|^2 + |y|^4 + |\eta|^2)^k) \subset \mathcal{H}(a) \cap \mathcal{H}(\langle v \rangle \langle y \rangle)$. Then by density of $\mathcal{S}$ in $\mathcal{H}((K_0 + |v|^2 + |y|^4 + |\eta|^2)^k)$, we have for $\psi \in \mathcal{H}((K_0 + |v|^2 + |y|^4 + |\eta|^2)^k)$,

$$(\varphi - g, (n - v \cdot \nabla_x + A)\psi)_{L^2_{x,v}} = 0.$$ 

Applying theorem 2.1 and its remark 2.2 to any $h \in \mathcal{S}(\mathbb{R}^{2d})$, there exists $\psi \in \mathcal{H}((K_0 + |v|^2 + |y|^4 + |\eta|^2)^k)$ such that

$$(n - v \cdot \nabla_x + A)\psi = h.$$ 

Hence for $h \in \mathcal{S}(\mathbb{R}^{2d})$,

$$(\varphi - g, h)_{L^2_{x,v}} = 0.$$ 

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So \( g = (nI - B)^{-1}f \in H((K_0 + |v|^2 + |y|^4 + |\eta|^2)^k) \subset H(a) \cap H(\langle v \rangle \langle y \rangle) \). Therefore, \((nI + v \cdot \nabla_x - L)\varphi = f\), where we can cancel the closure. Taking the Fourier transform \( \mathcal{F} \) acting on spatial variable, \((nI + 2\pi iv \cdot y - L)\mathcal{F}\varphi(y,v) = \mathcal{F}f(y,v)\) and so

\[
\mathcal{F}\varphi(y,v) = (nI - \hat{B}(y))^{-1}\mathcal{F}f,
\]

\[
\varphi(x,v) = (nI - B)^{-1}f = \mathcal{F}^{-1}(nI - \hat{B}(y))^{-1}\mathcal{F}f.
\]

By definition of strongly continuous semigroup generated by bounded operator, we have

\[
e^{tB_n}f = \lim_{N \to \infty} \sum_{j=0}^{N} \frac{(tn^2(nI - B)^{-1} - tn)^j}{j!}f, \quad \text{in } L^2_{x,v},
\]

\[
e^{t\hat{B}_n(y)} = \lim_{N \to \infty} \sum_{j=0}^{N} \frac{(tn^2(nI - \hat{B}_n(y))^{-1} - tn)^j}{j!}f, \quad \text{in } L^2_v.
\]

Hence the above limits are also valid in the sense of almost everywhere. For each \( j \geq 0 \),

\[
\lim_{N \to \infty} \sum_{j=0}^{N} \frac{(tn^2(nI - B)^{-1} - tn)^j}{j!}f = \mathcal{F}^{-1}(tn^2(nI - \hat{B}_n(y))^{-1} - tn)^j \mathcal{F}f,
\]

where the limit is taken in \( L^2_{x,v} \). Noting that for any convergent sequence \( \{f_N\} \subset L^2_{x,v} \), \( \{\mathcal{F}^{-1}f_N\} \) is also a convergent sequence in \( L^2_{x,v} \) and

\[
\| \mathcal{F}^{-1}f_N - \mathcal{F}^{-1}f_N \|_{L^2_{x,v}} \leq \| \mathcal{F}^{-1}f_N - \mathcal{F}^{-1}f_N \|_{L^2_{x,v}} + \| \mathcal{F}^{-1}f_N - \mathcal{F}^{-1}f_N \|_{L^2_{x,v}} \to 0.
\]

Therefore, combining the above estimates,

\[
e^{tB_n}f = \lim_{N \to \infty} \sum_{j=0}^{N} \frac{(tn^2(nI - B)^{-1} - tn)^j}{j!}f = \mathcal{F}^{-1} \lim_{N \to \infty} \sum_{j=0}^{N} \frac{(tn^2(nI - \hat{B}_n(y))^{-1} - tn)^j}{j!} \mathcal{F}f = \mathcal{F}^{-1}e^{t\hat{B}_n(y)} \mathcal{F}f,
\]

where the first limit is taken in \( L^2_{x,v} \), the last equality is viewed as almost everywhere point-wise limit (up to a subsequence of \( N \)).
Preliminary lemma on linearized Boltzmann operator $L$

**Lemma 6.12.** (1). $\text{Ker} L = \{ \mu, v_1 \mu, \cdots, v_d \mu, |v|^2 \mu \}$.

(2). For $f, g \in \mathcal{S}$,

$$ (L_1 f, g)_{L^2} = (f, L_1 g)_{L^2}, \quad (L_2 f, g)_{L^2} = (f, L_2 g)_{L^2}. \quad (99) $$

(3). For any rotation $R$ on $\mathbb{R}^d$, we have $RL_1 = L_1 R$ and $RL_2 = L_2 R$.

**Proof.** 1. Let $f \in \text{Ker} L$, then

$$ \mathcal{T} f = 0, \quad \mathcal{A} f = K f. $$

By theorem 2.3, $f \in \mathcal{S}$ and so the collision invariant identity can be applied to get $\text{Ker} L$ as in the cut-off case, for instance [19].

2. In order to prove (99), we can apply the following change of variable in $(L_1 f, g)$ and $(L_2 f, g)$ respectively:

$$ \begin{align*}
\mu^{1/2}(v + \alpha - h)g(v)\mu^{1/2}(v + \alpha)f(v - h) \\
\mapsto \mu^{1/2}(v + \alpha)g(v + h)\mu^{1/2}(v + \alpha + h)f(v), \quad v \mapsto v + h,
\end{align*} $$

and

$$ \begin{align*}
\mu^{1/2}(v + \alpha - h)g(v)\mu^{1/2}(v - h)f(v + \alpha) \\
\mapsto \mu^{1/2}(v - \alpha - h)g(v)\mu^{1/2}(v - h)f(v - \alpha), \quad \alpha \mapsto -\alpha,
\end{align*} $$

$$ \begin{align*}
\mu^{1/2}(v - h)g(v + \alpha)\mu^{1/2}(v + \alpha - h)f(v), \quad v \mapsto v + \alpha,
\mu^{1/2}(v + \alpha - h)g(v)\mu^{1/2}(v + \alpha - h)f(v) \\
\mapsto \mu^{1/2}(v - h)g(v + \alpha)\mu^{1/2}(v + \alpha - h)f(v), \quad v \mapsto v - \alpha + h,
\end{align*} $$

$$ \begin{align*}
\mu^{1/2}(v + \alpha - h)g(v + \alpha - h)f(v) \\
\mapsto \mu^{1/2}(v + \alpha - h)g(v + \alpha - h)f(v), \quad (h, \alpha) \mapsto (-h, -\alpha).
\end{align*} $$

3. For any rotation $R$, i.e. orthogonal matrix acting on $v$. Noticing $\alpha \perp h$ and $\mu(v)$ depends only on $|v|$, we apply change of variable $(\alpha, h) \mapsto (R\alpha, Rh)$ to get

$$ RL_1 f(v) : = L_1 f(Rv) $$

$$ = \lim_{\varepsilon \to 0} \int_{|h| \geq \varepsilon} \int_{E_{0,h}} \tilde{b}(\alpha, h)1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma + 1 + 2s}}{|h|^{d + 2s}} \mu^{1/2}(Rv + \alpha - h) $$

$$ \quad \left( (\mu^{1/2}(Rv + \alpha)f(Rv - h) - \mu^{1/2}(Rv + \alpha - h)f(Rv) \right) d\alpha d\mu $$

$$ = \lim_{\varepsilon \to 0} \int_{|h| \geq \varepsilon} \int_{E_{0,h}} \tilde{b}(\alpha, h)1_{|\alpha| \geq |h|} \frac{|\alpha + h|^{\gamma + 1 + 2s}}{|h|^{d + 2s}} \mu^{1/2}(v + R^\perp \alpha - R^\perp h) $$

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and hence the matrix representation of $A$ is symmetric. Denote $e_i = (1,0,...,0)$ to be a $d$ dimensional vector, then

$$A_e = P(e_1)P(e_2)\cdots P(e_d)$$

Proof. Denote $A = P(v)P$ in this proof. Since $R_f = \sum_{i=0}^{d+1} (f(e_i), e_i)$, we have $A_f = \sum_{i=0}^{d+1} (f(e_i), e_i)$, to be the eigenvalues and orthonormal eigenfunctions:

$$v_0 = \mu_1, v_i = \psi_i, i = 1, \ldots, d, v_{d+1} = \sqrt{1/(\mu_1^2 - d\mu_2^2)}.$$

Lemma 6.14. The eigenvalue problem in $\ker L$:

$$P(v,P\circ a) = \mu_1$$

Theorem 6.13. Implicit Function Theorem. Let $U \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open subset. Assume $(x_0,y_0) \in U$, $f(x,y) = 0$. Then there exists an open set $V \subset U$ with $(x_0,y_0) \in V$, an open set $W \subset \mathbb{R}^m$ with $y_0 \in W$ and a $C^1$ mapping $g : W \rightarrow \mathbb{R}^n$ such that

$$f(x,y) = 0 \quad \text{for } x \in V \cap (g^{-1}(W))$$

$$\frac{df}{dx}(x_0,y_0) \neq 0.$$
by the properties of Gamma function and noticing that $\mu^{1/2}$ is even about $v_1$ while $v_1 \psi_i$ is odd about $v_1$ if and only if $i \neq 1$. Then

$$|\lambda I - A| = \begin{vmatrix} \lambda & -e_1 & 0 \\ -e_1 & \lambda I_d & -\sqrt{\frac{2}{d}} e_1^T \\ 0 & -\sqrt{\frac{2}{d}} e_1^T & \lambda \end{vmatrix} = \lambda^{d-1} \begin{vmatrix} \lambda & -1 & 0 \\ -1 & \lambda & -\sqrt{\frac{2}{d}} \\ 0 & -\sqrt{\frac{2}{d}} & 0 \end{vmatrix} = \lambda^d (\lambda^2 - \frac{d+2}{d}).$$

Thus $\lambda = -\sqrt{\frac{d+2}{d}}, \sqrt{\frac{d+2}{d}}, 0$ ($d$ multiplicity). For $\lambda = 0$, the corresponding unit eigenvectors are

$$(-\sqrt{\frac{2}{d+2}}, 0, \cdots, 0, \sqrt{\frac{d}{d+2}}), \xi_2, \cdots, \xi_d,$$

where $\xi_j$ are unit vectors in $\mathbb{R}^{d+2}$ whose standard orthonormal base is $(\xi_0, \xi_1, \ldots, \xi_{d+1})$. For $\lambda = \sqrt{\frac{d+2}{d}}, -\sqrt{\frac{d+2}{d}}$, the corresponding unit eigenvectors are

$$\left(\sqrt{\frac{d}{2(d+2)}}, \frac{1}{\sqrt{2}}, 0, \cdots, 0, \sqrt{\frac{1}{d+2}}\right), \left(\sqrt{\frac{d}{2(d+2)}}, -\frac{1}{\sqrt{2}}, 0, \cdots, 0, \sqrt{\frac{1}{d+2}}\right),$$

respectively. Therefore, the eigenvalues and eigenfunctions of $A$ are as in (101).

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