Strategic arrivals to a queue with service rate uncertainty

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Abstract

This paper studies the problem of strategic choice of arrival time to a single-server queue with opening and closing times when there is uncertainty regarding service speed. A Poisson population of customers need to arrive during a specified acceptance period and are served on a first-come first-served basis. We assume there are two types of customers that differ in their beliefs regarding the service time distribution. The inconsistent beliefs may arise from randomness in the server state along with noisy signals that customers observe. Customers choose their arrival-times with the goal of minimizing their expected waiting times. Assuming that customers are aware of the two types of customers with differing beliefs, we characterize the Nash equilibrium arrival profiles for exponentially distributed service times and provide an explicit solution for a fluid approximation of this game. For general service time distributions we provide an algorithm for computing the equilibrium in a discrete time system. Furthermore, we present a dynamic learning model in which customers make joining decisions based on their signals and past experience. The learning model assumes customers do not know all of the system and uncertainty parameters, or are unable to correctly compute the respective posterior expected waiting times. We numerically compare the long-term average outcome of the learning process, which has an inherent estimation bias due to the unknown signal quality, with the equilibrium solution.

1 Introduction

A bottleneck arrival game is typically modeled as a queue that operates during a specified period of time and has to serve a finite population of incoming customers who can choose their arrival time. In the queueing literature this is known as the $\lambda$/$M$/1 model introduced by Glazer and Hassin in [3]. The characteristics of a congested bottleneck queue are often not the same on different days and customers arriving at the bottleneck may not be aware of the current state on a given day. For example, the duration of an airport security check can differ from day to day and location to location depending on many variables such as staffing issues and specific security alerts. Moreover, customers may have different beliefs regarding the speed of service due to past experience and private information obtained from external sources such as the media or friends. Customers can choose when to arrive to the queue with the goal of minimizing their expected waiting times, but the uncertainty regarding the service times implies that customers are heterogeneous when evaluating expected waiting times. We study
the simplest model for this situation by assuming that there are two possible service states, “fast” and “slow”, and two distinct populations of customers each believing that the state is one of the possibilities. We assume that the total number of customers arriving at any given day is a Poisson random variable and analyze both the case of exponential service times and general service times in a discrete setting.

A Nash equilibrium is given by a set of, possibly mixed, arrival strategies for every type of customer belief. For the case of exponential service times we characterize the Nash equilibrium arrival distributions of both customer types along with an explicit solution for a fluid approximation of a large scale system. We further consider a discrete time system with general service times where customers can only be admitted into the queue at specified time slots. If the time slots are very close then this can approximate the continuous time setting, but the model also captures the dynamics of a system with limited admission times (e.g., a day with three admission periods: morning, afternoon and evening). Furthermore, moving away from exponential service times enables the examination of the effect of the variance of the service time distribution. This framework is useful because it is very general, but explicit analysis seems intractable and so we present an algorithmic procedure for the computation of the Nash equilibrium arrival distributions. For the continuous and discrete time settings we show that there are multiple types of equilibrium arrival distributions, including everyone arriving at the opening, one class of customers at the beginning and the other according to a mixed distribution during the working hours, or both arriving according to a mixed distribution.

We further present a random service model with noisy signals that may result in inconsistent customer beliefs due to information asymmetry. Suppose that the server state is assigned randomly every day according to a fixed probability $p \in [0, 1]$ (for slow service) and a signaling mechanism reveals the true state with a high probability $q > \frac{1}{2}$. This model is similar to the setting of the join or balk game studied in [2]. This results in seemingly inconsistent customer beliefs regarding the service time distribution that correspond to the posterior distributions given the signal observed. Therefore, through this service uncertainty model and signal mechanism we have a game that is in fact consistent with an assumption of homogeneous customers from a Bayesian point of view.

In order to compute the optimal arrival strategies the customers must be aware of all of the system parameters, including the characteristics of customers with different beliefs and the quality of the signal they receive. Furthermore, the customers must be able to use the available information to make posterior updates regarding the sizes of both population types, knowing that some received a wrong signal, and their respective service time distributions. However, in some settings customers may not be aware of all of the relevant system parameters and of the quality of the signal and hence they make decisions based solely on their signal and past experience. We therefore introduce a dynamic learning model where customers update their estimators for the expecting waiting time given their signal and actual delay in the system on previous days with the same signal. The learning procedure is a mixture between randomly testing new time slots (exploration) and choosing a time slot that was optimal in the past on average (exploitation) such that the probability of exploration decreases to zero with time.

The equilibrium in the discrete setting is numerically compared to the dynamic learning model. As the learning model does not assume customer knowledge of the system parameters, i.e., the arrival rates, service time distributions, true state probability $p$ and signal quality $q$, there is an inherent estimation bias of the expected waiting time. This is because on a proportion of $1 - q$ of the days the signal is wrong and the delay observations are misclassified.
We show that this bias increases the average waiting times compared to the full information equilibrium, but less so than the equilibrium arising from a game that ignores the signal quality. Alternatively, the dynamic learning model can be viewed as a form of bounded rationality such that customers cannot make the elaborate posterior computations on a daily basis and therefore employ a simple decision rule that is based on experimentation and past experience.

**Main contributions.** We now summarize the main contributions of our work.

- **Heterogeneous customers:** Thus far, the literature on bottleneck queue arrival games has mostly assumed homogeneous customers. This paper is the first to analyze a game with a discrete (non-fluid) population and customers with heterogeneous beliefs regarding the service time distribution.

- **Asymmetric information:** We present a simple model for a bottleneck queue arrival game with service quality uncertainty and provide equilibrium analysis for both the Markovian and non-Markovian settings.

- **Discrete game with large time slots:** For the discrete game we provide a framework to analyze the equilibrium when the acceptance period is divided into big time slots such that multiple jobs can be served within a time slot. This is more general than discrete systems with small time slots that aim to approximate continuous time, and the resulting equilibrium distributions can be quite different.

- **Dynamic learning:** We compare the equilibrium arrival distributions in the discrete game with the outcome of a learning process in which customers update their decisions based on past experiences by estimating the expected waiting time corresponding to every pair of time choice and signal. This enables us to examine the effect of the biased estimation in the learning process as opposed to the full information equilibrium solution.

**Background and related literature.** The research on strategic queueing deals with many aspects of decision making in queueing systems. An introduction and overview of this research can be found in [5] and [4]. The $\text{?}/\text{M}/1$ model of Glazer and Hassin [3] was the first to consider an endogenous arrival process to a queue that is determined by strategic considerations of the arriving customers. The server was assumed to start working in time zero and customers choose their arrival time with the goal of minimizing expected waiting costs. It was assumed that customers can queue before the server commences operations, an assumption which was later removed in [6] that considered a finite acceptance period. The latter established that the equilibrium arrival distribution has an atom at zero, an interval with no arrivals and a continuous distribution on a continuous interval. The socially optimal arrival distribution was numerically analyzed and it was shown that it is optimal to allow arrivals on a discrete set of times. The model was later extended to include tardiness penalties in [11] and [7], order penalties in [14], a loss system in [8], and a network of queues in [9]. The issues of existence and uniqueness of equilibrium were addressed in detail in [11]. In [1] the model was generalized to general service times, i.e., $\text{?}/G/1$, and it was shown that the equilibrium has the same form and uniqueness properties, but is much harder to compute the arrival distribution. A fluid approximation for a game with heterogeneous customers that differ in their cost functions was presented in [12]. The discrete-time version of the $\text{?}/G/1$ game was analysed in [15], which also presented an algorithm to compute the equilibrium. This discrete-time setting can be used as
an approximation for the continuous time model when taking a large number of small time slots. Moreover, [15] presented the dynamic learning model which is extended here to allow multiple customer types. A more detailed review of the literature on arrival time games can be found in Chapter 4.1 of [4].

Observable queues with customers deciding to join or balk based on noisy signals regarding the service speed and value were studied in [16] and [2]. The impact of uncertainty regarding server capacity on staffing and control of multiserver queues was investigated in [10].

**Paper organization.** The remainder of the paper is organized as follows. Section 2 introduces the queueing and game model. In Section 3 the equilibrium in the continuous time game with exponential service times is analyzed, including an explicit fluid approximation. Section 4 defines the discrete-time game and provides a method to compute the equilibrium distributions. In Section 5 a model with server uncertainty and a noisy signaling mechanism is defined which gives rise to a game with inconsistent customer beliefs but can also be analyzed through a dynamic learning approach. Section 6 presents the dynamic learning model and discusses the estimation bias associated with it. Numerical analysis and comparison of the discrete game and dynamic learning model is given in Section 7. Finally, Section 8 concludes the paper with a discussion of the results and potential future research.

## 2 Model and Preliminaries

A population of \( N \sim \text{Poisson}(\lambda) \) customers need to arrive at a queue during an acceptance period \( T \subseteq [0, T] \) such that \( 0 \in T \). The acceptance period may be a continuous interval \( T = [0, T] \) or a discrete collection of times \( T = \{0, t_1, \ldots, t_n\} \). All customers arriving during the acceptance period are served, even if service is completed after the acceptance period. The jobs in the queue are processed FCFS and if two, or more, jobs arrive at the same time then they are randomly ordered.

There are two types of customers \( \mathcal{C} := \{a, b\} \) that differ in their belief regarding the service time distribution \( X \). Type \( a \) customers are pessimistic and believe the service rate is “slow” and type \( b \) customers are optimistic and believe the service rate is “fast”. We further denote \( \lambda_i := \lambda \alpha_i \), where \( \alpha_i \) is the proportion of customers with belief \( i \in \mathcal{C} \). We assume that the formal distinction between the types is made in terms of expectations:

\[
\chi_a := \mathbb{E}X_a > \mathbb{E}X_b =: \chi_b .
\]

The reasoning for inconsistent beliefs is information asymmetry. In particular, we later introduce a random environment and signal mechanism such that the system state, slow or fast, is random and the customers receive noisy signals regarding the true state.

**Examples:**

1. Exponential service times with a slower service rate for type \( a \), \( \mu_a < \mu_b \).
2. Deterministic service times: \( \chi_a > \chi_b \).
3. Discrete-time memoryless service: geometric service times with success probabilities \( \frac{1}{\chi_a} < \frac{1}{\chi_b} \).

Customers wish to avoid waiting and choose their arrival time to the queue with the goal of minimizing expected waiting time. An arrival strategy for a customer is given by a probability
distribution on $T$ and can be represented by a cdf $F(t)$. We denote the support of an arrival strategy $F$ by $\sigma(F) := \inf\{S \subseteq T \text{ s.t. } \int_{s \in S} F(ds) = 1\}$. We focus our analysis on the symmetric (within types) case where all customers of type $i \in C$ use the same strategy $F_i$.

Let $A(t)$ denote the cumulative arrival process of customers to the system. Given an arrival profile $(F_a, F_b)$, the amount of customers arriving during an interval $(s, t]$ is a Poisson random variable,

$$A(t) - A(s) \sim \text{Poisson}(\lambda_a(F_a(t) - F_a(s)) + \lambda_b(F_b(t) - F_b(s))).$$

The arrival process is identical for all customers regardless of their individual beliefs on the service rate. Let $W_i(t)$ denote the waiting time for a type $i \in C$ customer arriving at $t$. The distribution of the waiting time depends on the specific assumptions made on the service distribution, and we will consider several cases in this paper. Customers wish to arrive at a time $t$ that minimizes their expected waiting time $w_i(t; F_a, F_b) := E[W_i(t)]$. We are interested in stable arrival profiles $(F_a, F_b)$ such that no customer can decrease their expected waiting time.

**Definition 1.** A symmetric (within types) Nash equilibrium is given by a pair of distributions $(F_a, F_b)$ such that for $i \in C$,

$$w_i(t; F_a, F_b) = w_i(s; F_a, F_b), \quad \forall s, t \in \sigma(F_i),$$

and

$$w_i(t; F_a, F_b) \leq w_i(s; F_a, F_b), \quad \forall t \in \sigma(F_i), s \in T.$$

### 3 Exponential service times

This section provides a characterization of the symmetric (within type) Nash equilibrium arrival profiles for the case of exponential service times and a continuous acceptance period $[0, T]$. The equilibrium distribution has the standard form of a possible atom at zero $F_i(0) > 0$ and a continuous density $f_i(t) > 0$ on a subset of $[0, T]$. The equilibrium solution is not explicit and is given by a set of functional differential equations. The main qualitative observation is that in case a mass of optimistic customers arrive at the opening, i.e., at $t = 0$, then all of the pessimistic will arrive at the opening as well. In Section 3.1 we present the explicit equilibrium solution for a fluid model that can be viewed as an approximation of a large scale system with many small customers that have negligible service times.

Suppose that $T = [0, T]$ and $X_i \sim \text{exp}(\mu_i)$ for $i \in C$, such that $\mu_a < \mu_b$. Due to the memoryless property the expected waiting time of a type $i \in C$ customer arriving at time $t$ is

$$w_i(t; F_a, F_b) = \frac{1}{\mu_i}E[Q_i(t)],$$

where $Q_i$ is the queue length process with rate $\mu_i$. Minimizing expected waiting time is therefore equivalent to minimizing expected queue length in this case and from now on we will consider the expected queue length $q_i(t; F_a, F_b) := E[Q_i(t)]$, given by

$$q_i(t; F_a, F_b) = E[A(t-)] + \frac{1}{2} \sum_{j \in C} \lambda_j(F_j(t) - F_j(t-)) - \mu_i E \int_0^t 1(Q_i(u) > 0) \, du$$

$$= \sum_{j \in C} \lambda_j F_j(t-),$$

$$+ \frac{1}{2} \sum_{j \in C} \lambda_j(F_j(t) - F_j(t-)) - \mu_i \int_0^t P(Q_i(u) > 0) \, du .$$

\[1\]
where \( g(t-) := \lim_{s \uparrow t} g(s) \) for any function \( g \). The expected queue length at time zero is the same for all customers, and equals
\[
q_0 := q_a(0; F_a, F_b) = q_b(0; F_a, F_b) = \frac{\lambda_a F_a(0) + \lambda_b F_b(0)}{2}.
\] (2)

From now on we use the short notation \( q_i(t) \) for \( q_i(t; F_a, F_b) \) while keeping in mind that the queueing processes are always driven by a given pair of strategies \( (F_a, F_b) \). By applying standard arguments we can list several properties that an equilibrium profile satisfies. For more details see [3] and [6].

- \( F_i(0) > 0 \) for at least one of \( i \in C \). Otherwise, by (2) arriving at zero yields no waiting time.
- For any \( i \in C \) there exists some \( t_i > 0 \) such that \( F_i(t_i) = F_i(0) \) and
\[
q_i(t) \geq q_0, \quad t \in [0, t_i].
\] (3)
If \( t_i < T \) then for any \( t \in [t_i, T] \cap \sigma(F_i) \),
\[
q_i(t) = q_0\]
and the cdf \( F_i(t) \) is continuous with density \( f_i(t) \geq 0 \) that satisfies the functional differential equation,
\[
\lambda_a f_a(t) + \lambda_b f_b(t) = \mu_i P(Q_i(t) > 0). \] (4)

The differential equations are obtained by taking derivative of the expectation in (1) which is constant in equilibrium.

We next state two lemmas that will be used to characterize the equilibrium arrival profile. The proof of Lemma 1 relies on a coupling argument for the virtual workload which is detailed in Appendix A.

**Lemma 1.** For any arrival profile \( (F_a, F_b) \) such that \( F_i(t) > 0 \) for at least one of \( i \in C \) the expected queue length faced by type a customers is higher than that faced by type b customers: \( q_a(t) > q_b(t) \) for all \( t > 0 \).

**Lemma 2.** For any Nash equilibrium \( (F_a, F_b) \), if \( F_b(0) > 0 \) then \( F_a(0) = 1 \).

**Proof.** If \( F_b(0) > 0 \) then for any \( t \in \sigma(F_i) \setminus \{0\} \) the equilibrium condition (3) states that \( q_b(t) \geq q_0 \) for all \( t > 0 \), and by Lemma 1 \( q_a(t) > q_b(t) \geq q_0 \), hence type a customers will not arrive at any \( t > 0 \). \( \square \)

The previous two lemmas leave us with two following possibilities for Nash equilibrium arrival profiles that are summarized in the following theorem.

**Theorem 3.** If \( (F_a, F_b) \) are Nash equilibrium arrival strategies then only one of the following holds:

1. \( F_a(0) = 1, \ F_b(0) > 0 \) and \( \int_{t_b}^{T} f_b(t) \, dt = 1 - F_b(t) \), where \( f_b(t) \) is the unique solution to (4).
(ii) \( F_a(0) \in (0,1] \), \( F_b(0) = 0 \) and there exist two arrival sets \( \mathcal{T}_a, \mathcal{T}_b \subseteq (0,T] \) such that 
\[
\int_{t \in \mathcal{T}_a} f_a(t) \, dt = 1 = F_a(0) \quad \text{and} \quad \int_{t \in \mathcal{T}_b} f_b(t) \, dt = 1, \quad \text{where } (f_a(t), f_b(t)) \text{ satisfy (4). Note that} \quad T_a = \emptyset \quad \text{if} \quad F_a(0) = 1.
\]

We conjecture that the second type of equilibrium in fact has a much more specific form, namely that all of the pessimistic customers arrive before the optimistic customers start arriving. Formally, \( \mathcal{T}_a = [t_a, t_b) \) and \( \mathcal{T}_b = [t_b, T] \), where \( 0 < t_a < t_b < T \). This conclusion relies on the following conjecture regarding the equilibrium dynamics in the interior of the acceptance period.

**Conjecture 4.** There is no time \( t > 0 \) such that both types of customers arrive simultaneously; 
\( \sigma(F_a) \cap \sigma(F_b) \cap (0,T] = \emptyset. \)

The reasoning for this conjecture is as follows: If type \( i \in \mathcal{C} \) customers arrive at a positive rate, \( \lambda_i f_i(t) \), then the functional differential equation (4) maintaining the expected queue length \( q_i(t) \) is constant must be satisfied. Suppose that on an interval \( (t_1, t_2) \) at least one of the types \( i \in \mathcal{C} \) arrives with a positive density, i.e., \( f_i(t) > 0 \) for all \( t \in (t_1, t_2) \). Let \( \Lambda_i(t) := \lambda_a f_a(t) + \lambda_b f_b(t) \), then by Lemma 9 of [11] the functional differential equations (4), one equation for each type \( i \in \mathcal{C} \), admits a unique solution \( \Lambda_i() \) on the interval \( (t_1, t_2) \). Furthermore, these unique solutions are monotone increasing with the initial queue length distribution \( Q_i(t_1) \). We have been able to rule out the existence of a pair of distinct initial distributions \( (Q_a(t_1), Q_b(t_1)) \) along with rates \( \mu_a < \mu_b \) such that the two solutions for two different differential equations coincide on a non-empty interval; \( \Lambda_a(t) = \Lambda_b(t) \) for all \( t \in [t_1, t_2] \). Such a situation seems unlikely but due to the elaborate dynamics of the functional differential equations we have been unable to establish a proof of non-existence.

If Conjecture 4 does not hold then there exist multiple equilibria because any combination of \( (fa(t), fb(t)) \) such that \( \lambda_a(t) = \lambda_b(t) = \lambda_a f_a(t) + \lambda_b f_b(t) \) is also an equilibrium. Otherwise, the uniqueness arguments for the single-class case (e.g., see [11]) can be applied directly to the equilibrium arrival rates on the disjoint arrival intervals.

### 3.1 Fluid approximation

We next consider a fluid model that approximates the case where the expected population size is very big and service rates are very fast: \( \lambda_i \to \infty \) and \( \mu_i \to \infty \) for \( i \in \mathcal{C} \), while maintaining \( \frac{\lambda_i}{\mu_i} \to C_i \in (0,\infty) \). See [11] for rigorous justification of this approximation for the bottleneck queue arrival game.

Suppose now that there are two deterministic fluid populations of volumes \( \lambda_a \) and \( \lambda_b \). Type \( i \in \mathcal{C} \) customers believe that the deterministic output rate is \( \mu_i \), where \( \mu_a < \mu_b \). As before, we denote the strategies of type \( i \) customers by the cdf \( F_i \). The fluid dynamics of the queue at \( t \in [0,T] \) as long as the queue is never empty, which is the case in equilibrium, are given by

\[
q_i(t) = \sum_{j \in \mathcal{C}} \lambda_j \int_0^t dF_j(u) - \mu_i t, \quad i \in \mathcal{C}.
\]

Due to the deterministic dynamics of (5) the queue length process for type \( a \) dominates that of type \( b \), as was established for the stochastic model in the previous section, and therefore we obtain the same structure as in Theorem 3. In particular, type \( a \) customers all arrive no later than any type \( b \) customers and there are three possible equilibrium outcomes for type \( b \).
customers: all arrive at \( t = 0 \), i.e., \( F_b(0) = 1 \), or a positive mass arrives continuously on an interval \([t_b, T]\) and \( 0 \leq F_b(0) < 1 \). The following theorem provides the explicit solution and conditions for all cases. We exclude the degenerate case of \( T > \frac{\lambda_a}{\mu_a} + \frac{\lambda_b}{\mu_b} \) for which there are multiple equilibria such that there are no arrivals at zero, \( F_a(0) = F_b(0) = 0 \), both types can arrive on disjoint intervals and the queue is always empty.

**Theorem 5.** If \( T \leq \frac{\lambda_a}{\mu_a} + \frac{\lambda_b}{\mu_b} \) then the unique Nash equilibrium has the following form:

(i) If \( T \leq \frac{\lambda_a + \lambda_b}{2\mu_b} \) then \( F_a(0) = F_b(0) = 1 \) is the equilibrium.

(ii) If \( \frac{\lambda_a + \lambda_b}{2\mu_b} < T \leq \frac{\lambda_a + 2\lambda_b}{2\mu_b} \) then \( F_a(0) = 1, F_b(0) \in (0, 1) \) and \( f_b(t) = \frac{\mu_a}{\lambda_b} \) for \( t \in (t_b, T) \), where \( t_b = \frac{\lambda_a + \lambda_b}{2\mu_b} \) and \( F_b(0) = \frac{2\lambda_b}{\mu_b} \left( \frac{\lambda_a + 2\lambda_b}{2\mu_b} - T \right) \).

(iii) If \( \frac{\lambda_a + 2\lambda_b}{2\mu_b} < T \leq \frac{\lambda_a}{2\mu_a} + \frac{\lambda_b}{\mu_b} \) then \( F_a(0) = 1, F_b(0) = 0 \) and \( f_b(t) = \frac{\mu_a}{\lambda_b} \) for \( t \in (t_b, T) \), where \( t_b = \frac{\lambda_a}{2\mu_a} \).

(iv) If \( \frac{\lambda_a}{2\mu_a} + \frac{\lambda_b}{\mu_b} < T \leq \frac{\lambda_a + \lambda_b}{2\mu_b} \) then \( F_a(0) \in (0, 1), f_a(t) = \frac{\mu_a}{\lambda_a} \) for \( t \in [t_a, t_b) \) where \( t_a = \frac{\lambda_a F_b(0)}{2\mu_a} \) and \( F_a(0) = 2 \left( \frac{\lambda_a + \lambda_b}{\lambda_a \mu_a} - \frac{\mu_a}{\lambda_a} T \right) \). For type \( b \), \( F_b(0) = 0, f_b(t) = \frac{\mu_b}{\lambda_b} \) for \( t \in [t_b, T) \), where \( t_b = T - \frac{\lambda_b}{\mu_b} \).

**Proof.** The queue size ahead of an arbitrary chosen customer arriving at time \( t = 0 \) is \( q_0 = \frac{\lambda_a F_a(0) + \lambda_b F_b(0)}{2} \), regardless of the customer type, and the respective waiting time for a type \( i \) customer is \( \frac{\mu_i}{\mu} \). If there are no arrivals during \((0, t)\) then the waiting queue for a type \( i \) customer is

\[
q_i(u) = 2q_0 - \mu_i u, \quad u \in (0, t),
\]

and then \( q_i(t_i) = q_0 \) is equivalent to \( t_i = \frac{2q_0}{\mu_i} \). As \( \mu_a < \mu_b \), we have that \( t_a > t_b \). Hence, if \( F_b(0) > 0 \) then at \( t_b \) type \( b \) customers will start arriving at a rate satisfying \( \frac{d}{dt} q_b(t) = 0 \), as \( f_a(t) = 0 \) for all \( t \in (t_b, t_a) \). By (5), for any \( t \in [t_b, t_a) \),

\[
0 = \lambda_b f_b(t) - \mu_b \iff f_b(t) = \frac{\mu_b}{\lambda_b}.
\]

Therefore, the queue size for type \( a \) customers at \( t \in [t_b, t_a) \) is

\[
g_a(t) = 2q_0 + \lambda_b \int_{t_b}^{t} f_b(u) du - \mu_a t = 2q_0 + \mu_b (t - t_b) - \mu_a t
\]

\[
= 2q_0 - \mu_b t_b + (\mu_b - \mu_a) t = q_0 + (\mu_b - \mu_a) t,
\]

where the last equality follows from the definition of \( t_b \), i.e., \( q_b(t_b) = q_0 \). As \( \mu_b - \mu_a > 0 \), this implies that \( q_a(t) > q_0 \) and thus \( f_a(t) = 0 \) for all \( t \geq t_b \) such that \( f_b(t) > 0 \). If for some \( t < T \), \( F_b(t) = 1 \) and \( F_b(u) < 1 \) for all \( u < t \), then there exists some \( \epsilon > 0 \) such that \( q_a(t + \epsilon) > q_b(t) = q_0 \). Therefore, \( f_a(u) = 0 \) for all \( u \in [t, t + \epsilon] \), and as \( f_b(u) = 0 \) as well we have that \( q_b(t + \epsilon) < q_b(t) = q_0 \), which contradicts the equilibrium assumption. The conclusion is similar to that of Theorem 3: if \( F_b(t) > 0 \) then all type \( a \) customers will arrive at \( t = 0 \) and \( q_0 = \frac{\lambda_a + \lambda_b F_b(0)}{2} \).

The three equilibrium cases are then verified as follows:
1. The strategies $F_a(0) = F_b(0) = 1$ are an equilibrium if $t_b = \frac{\lambda_a + \lambda_b}{2\mu_b} \geq T$ implying the equilibrium of case (i).

2. Next consider $F_a(0) = 1$ and $F_b(0) \in (0, 1)$, then $f_b(t) = \frac{\mu_a}{\lambda_a}$ for $t \in [t_b, T]$, where $t_b = \frac{\lambda_a + \lambda_b F_b(0)}{2\mu_b}$. The probability $F_b(0)$ is obtained by solving $F_b(0) + (T - t_b) \frac{\mu_a}{\lambda_a} = 1$, yielding $F_b(0) = \frac{2\lambda_b}{\mu_b} \left( \frac{\lambda_a + 2\lambda_b}{2\mu_b} - T \right)$. Observe that $F_b(0) < 1$ is equivalent to $T > \frac{\lambda_a + \lambda_b}{2\mu_b}$ and $F_b(0) > 0$ is equivalent to $T < \frac{\lambda_a + 2\lambda_b}{2\mu_b}$, yielding case (ii).

3. If $F_a(0) < 1$ then $F_b(0) = 0$, $f_a(t) = \frac{\mu_a}{\lambda_a}$ for $t \in [t_a, t_b)$ where $t_a = \frac{\lambda_a F_a(0)}{2\mu_a}$, $F_b(0) = 0$ and $f_b(t) = \frac{\mu_a}{\lambda_a}$ for $t \in [t_b, T]$. In particular, solving $F_a(0) + (t_b - t_a) \frac{\mu_a}{\lambda_a} = 1$ and $(T - t_b) \frac{\mu_a}{\lambda_a} = 1$ yields $F_a(0) = 2 \left( \frac{\lambda_a \mu_b + \lambda_b \mu_a}{\lambda_a \mu_b} - \frac{\mu_a}{\lambda_a} T \right)$ and $t_b = T - \frac{\lambda_b}{\mu_b}$. We therefore have that $F_a(0) < 1$ is equivalent to $T > \frac{\lambda_a}{2\mu_a} + \frac{\lambda_b}{\mu_b}$, yielding cases (iii) and (iv). As expected, $F_a(0) > 0$ is equivalent to $T \leq \frac{\lambda_a}{\mu_a} + \frac{\lambda_b}{\mu_b}$.

In Figure 1 we illustrate all possible equilibrium outcomes for fixed parameters $T = 1$, $\lambda_a = 1$, $\lambda_b = 2$, $\mu_a = 1$ and varying values of $\mu_b$. When $\mu_b$ is low then all customers arrive at $t = 0$ as seen in Figure 1a. As $\mu_b$ grows the optimistic (type $b$) customers become more optimistic regarding service speed and will spread out more throughout the acceptance period as seen in Figure 1b and Figure 1c. Finally, if $\mu_b$ is very high all of the optimistic customers arrive towards the end of the period which results in pessimistic (type $a$) customers also mixing between $t = 0$ and a continuous interval within the acceptance period.

![Figure 1: Equilibrium arrival distributions (solid red for type $a$ and dashed blue for type $b$) for varying service rate $\mu_b$ of the optimistic type $b$ customers. The other parameters are fixed: $T = 1, \lambda_a = 1, \lambda_b = 2, \mu_a = 1$.](image-url)
4 Discrete-time game

In some systems the acceptance period may be divided into a discrete number of slots and arrivals are only allowed in the designated slots. Furthermore, in [6] it was shown that limiting the number of possible arrival time slots may lower the expected waiting time for all customers in equilibrium. For the discrete system it is easier to compute the equilibrium arrival distribution for general service times. Moreover, this system can also be used to approximate a continuous time system by considering a very large number of small slots. Formally, the acceptance period is split into $T + 1$ equidistant time slots of length $\tau \in \mathbb{N}$:

$$T := \{0, \tau, 2\tau, \ldots, T\tau\}.$$ 

Customers can now choose to arrive in any $t \in T$, and if multiple customers arrive at the same time instant then they are uniformly randomly ordered. For convenience we modify the choice set to the slot numbers, i.e., $T := \{0, 1, 2, \ldots, T\}$ so that choosing $t \in T$ implies the arrival time $t\tau$.

We further assume that the service times are integer valued random variables for either of the types. For $i \in C = \{a, b\}$, let $x_i := (x_i(k); k \geq 1)$ denote the service time distribution with type $i$, where its mean is denoted by $\chi_i := \sum_{k \geq 1} kx_i(k)$. A mixed strategy is now a discrete probability distribution $(p_{i,t}; t \in T)$. A symmetric Nash equilibrium is given by a pair of distributions $p_a$ and $p_b$ with respective cdf's ($F_a, F_b$) satisfying Definition 1.

The discrete model provides flexibility to deal with limited acceptance slots and general service times. Furthermore, the discrete setting enables comparison of the equilibrium outcomes with those of a dynamic learning model that will be introduced in Section 6. The downside of the discrete model is that it is much harder to analyze the Nash equilibrium than the continuous time model with exponential service times of the previous section. Nevertheless, we provide an algorithm for computing equilibrium arrival distributions and present numerical results. We further show that symmetric Nash equilibrium arrival distributions exist for every instance of the game.

We first compute the mean unfinished workload for each customer’s type ($a$ and $b$) given an arrival distribution relying on the more detailed analysis of [15]. This is then used to construct a sufficient and necessary condition so that $(p_a, p_b)$ is Nash equilibrium. Finally, we give an algorithm for computing the equilibrium.

In the following analysis all random variables and expectations are given for a pair of arrival distributions $(p_a, p_b)$, but we omit this from the notation for the sake of brevity. For $i \in C$ and $t \in T$, let $V_{i,t-}$ denote the unfinished workload in system immediately before slot $t$ with type $i$. The probability distribution of $V_{i,t-}$ is denoted by $v_{i,t} = (v_{i,t}(k); k \geq 0)$, i.e., $v_{i,t}(k) = P(V_{i,t-} = k), k \geq 0, i \in C, t \in T$. Since there is no arriving customer before slot 0, i.e., we have $v_{i,0}(k) = 1(l(k = 0))$, where $1(l)$ is the indicator function of an event in the parenthesis.

The total work arriving in slot $t$ is

$$H_{i,t} = \sum_{k=1}^{N_t} X_{i,k},$$

where $N_t$ is a Poisson distributed random variable with mean $\lambda_{a}p_{a,t} + \lambda_{b}p_{b,t}$, and $X_{i,k}$’s are i.i.d. random variables with distribution $x_i$. Observe that $H_{i,t}$ follows a compound Poisson distribution, and denote its probability distribution by $h_{i,t} := (h_{i,t}(k); k \geq 0)$, which is recursively computed (see e.g., [17]). The mean unfinished workload with customer’s type $i \in C$ is computed as follows.
Lemma 6. For \( i \in \mathcal{C} \) and \( t \in \mathcal{T} \), we have
\[
E[V_{i,t-}] = \sum_{u=0}^{\tau-1} \left( (\lambda_a p_{a,u} + \lambda_b p_{b,u}) \chi_i - \tau + \sum_{k=0}^{\tau-1} (\tau - k)v_{i,u} * h_{i,u}(k) \right),
\]
where \( v_{i,u} * h_{i,u}(k) : = \sum_{\ell=0}^{k} v_{i,u}(\ell)h_{i,u}(k - \ell) \), and the probability distribution of the unfinished workload \( (v_{i,t}(k); k \geq 0) \) is recursively calculated by
\[
v_{i,t}(k) = \begin{cases} 
\sum_{\ell=0}^{k} v_{i,t-1} * h_{i,t-1}(\ell), & k = 0 \\
v_{i,t-1} * h_{i,t-1}(\tau + k), & k \geq 1.
\end{cases}
\]
\( \)\( \)\( \)\( \)\( \)

Proof. Since
\[
V_{i,t-} = (V_{i,(t-1)-} + H_{i,t-1} - \tau)^+ , \quad i \in \mathcal{C}, \ t \in \mathcal{T},
\]
then
\[
E[V_{i,t-}] = E[(V_{i,(t-1)-} + H_{i,t-1} - \tau)1_{V_{i,(t-1)-} + H_{i,t-1} \geq \tau}]
\]
\[
= E[V_{i,(t-1)-} + H_{i,t-1} - \tau] + E[(\tau - (V_{i,(t-1)-} + H_{i,t-1}))1_{V_{i,(t-1)-} + H_{i,t-1} \leq \tau - 1}].
\]
Since \( V_{i,(t-1)-} \) and \( H_{i,t-1} \) are independent, the second term in the last equation is given by
\[
E[(\tau - (V_{i,(t-1)-} + H_{i,t-1}))1_{V_{i,(t-1)-} + H_{i,t-1} \leq \tau - 1}] = \sum_{k=0}^{\tau-1} (\tau - k)v_{i,t-1} * h_{i,t-1}(k).
\]
By combining these equations, we obtain (6). Equation (7) immediately follows by (8).

By Definition 1, the arrival-time distributions in equilibrium are given as follows.

Lemma 7. A pair of arrival-time distributions \( (p_a, p_b) \) is a symmetric (within types) Nash equilibrium if and only if there exist positive numbers \( \bar{w}_i \) \( i \in \mathcal{C} \) such that
\[
p_{i,t} = \frac{1}{\lambda_i} \left( \frac{2}{\lambda_i} (\bar{w}_i - E[V_{i,t-}]) - \lambda_{-i}p_{-i,t} \right)^+, \quad i \in \mathcal{C}, \ t \in \mathcal{T},
\]
where \( -i \) denotes the counterpart of \( i \), i.e., if \( i = a \), then \( -i = b \), and vice versa.

Proof. For \( i \in \mathcal{C} \) and \( t \in \mathcal{T} \), let \( w_{i,t} \) denote the expected waiting time of a tagged type \( i \) arrival customer if she/he chooses slot \( t \). From Definition 1, \( (p_a, p_b) \) is an equilibrium if there exist positive numbers \( \bar{w}_i \) \( i \in \mathcal{C} \) such that
\[
w_{i,t} = \bar{w}_i, \quad t \in \sigma(p_i),
\]
\[
w_{i,t} \geq \bar{w}_i, \quad t \notin \sigma(p_i).
\]
Since
\[
w_{i,t} = E[V_{i,t-}] + \frac{\lambda_a p_{a,t} + \lambda_b p_{b,t}}{2} \chi_i, \quad i \in \mathcal{C}, \ t \in \mathcal{T},
\]
then equations (10) and (11) are rewritten by
\[
p_{i,t} = \frac{1}{\lambda_i} \left( \frac{2}{\lambda_i} (\bar{w}_i - E[V_{i,t-}]) - \lambda_{-i}p_{-i,t} \right), \quad t \in \sigma(p_i),
\]
and
\[
p_{i,t} = 0 \geq \frac{1}{\lambda_i} \left( \frac{2}{\lambda_i} (\bar{w}_i - E[V_{i,t-}]) - \lambda_{-i}p_{-i,t} \right), \quad t \notin \sigma(p_i),
\]
respectively, which imply (9).
4.1 Algorithm for computing equilibrium distributions

Lemma 7 provides a recursive relation in the form of Equation (9) that allows the iterative computation of the probability to arrive in every time slot \( t \in \mathcal{T} \). For every type \( i \in C \) this recursion relies on a given distribution for the other type \( j \neq i \). We leverage this structure to derive a best-response type algorithm that fixes the distribution of one type \( i \) in each iteration and then computes the distribution of the other type \( j \), and then repeating this procedure for the \( j \) given the computed distribution for \( i \). A point of convergence is clearly a Nash equilibrium, although we do not argue that the algorithm is guaranteed to converge. In practice, this procedure finds equilibrium points very efficiently.

For a fixed \( i \in C \), assume that the arrival-time distribution of type \(-i\) customers is given by \( p_{-i} \). Then the best response of type \( i \) customers is computed by a discretized (with accuracy parameter \( \epsilon > 0 \)) search procedure that we call Algorithm 1, which is similar to the methods applied in [6] and [15]. Algorithm 1 is detailed in Appendix B, and here we treat it as a function \( \text{Alg.1}(p_{-i}, \lambda, x_i, \epsilon) \rightarrow p_i \) that returns a probability distribution \( p_i \) that satisfies (9) for any distribution of the other type \( p_{-i} \), system parameters \( \lambda := (\lambda_a, \lambda_b) \), \( x_i \), and accuracy parameter \( \epsilon > 0 \).

Note that best-response here refers to all customers of type \( i \) using the same distribution, and not to an individual best response for an individual customer of type \( i \). This is then used iteratively in Algorithm 2 to find equilibrium arrival distributions, i.e., to find positive numbers \((w_a, w_b)\) and the corresponding arrival-time distributions \((p_a, p_b)\) jointly satisfying (9), which was shown to be equivalent to the Definition 1 of a Nash equilibrium in Lemma 7.

We next give an iterated best-response algorithm to obtain a pair of equilibrium arrival-time distributions. To this end, denote the set of all probability distributions on \( \mathcal{T} \) by \( \mathcal{P} \), i.e., \( \mathcal{P} = \{ p \in (\mathbb{R}_+)^{T+1}; \sum_{u=0}^{T} p_u = 1 \} \). Similarly, for \( t \in \mathcal{T} \), let \( \mathcal{P}_t = \{ p \in (\mathbb{R}_+)^{T+1}; \sum_{u=t}^{T} p_u = 1 \} \). A pair of equilibrium arrival-time distributions is computed as follows.

\textbf{Algorithm 2: Iterated best response}

\begin{enumerate}
    \item \textbf{Input:} \( \lambda := (\lambda_a, \lambda_b) \), \( x := (x_a, x_b) \), \( \epsilon > 0 \), \( \delta > 0 \)
    \item \textbf{Output:} \( (p^e_a, p^e_b) \) (equilibrium)
    \end{enumerate}

\begin{verbatim}
init \( p_0^a := (1, 0, \ldots, 0) \), \( p_0^b := (1, 0, \ldots, 0) \)
init \( k := 0 \), \( \Delta := \delta \)
while \( \Delta \geq \delta \) do
    \begin{enumerate}
        \item set \( p^{k+1}_a := \text{Alg.1}(p^k_a, \lambda, x_a, \epsilon) \), \( p^{k+1}_b := \text{Alg.1}(p^k_a, \lambda, x_b, \epsilon) \)
        \item set \( \Delta := \max\{||p^{k+1}_a - p^k_a||, ||p^{k+1}_b - p^k_b||\} \)
        \item set \( k := k + 1 \)
    \end{enumerate}
end while
set \( (p^e_a, p^e_b) := (p^k_a, p^k_b) \)
return \( (p^e_a, p^e_b) \)
\end{verbatim}

4.2 Existence of equilibrium arrival distributions

In [15] it was shown that an equilibrium exists in the single type game. Furthermore, it was conjectured that the equilibrium is unique, and this conjecture was strengthened by numerical analysis. The existence result can be extended to the two-type game by applying Kakutani’s fixed point theorem. The issue of uniqueness is an open problem even for the single-type game,
but there is reason to believe the conjecture holds in that case. In the multi-class setting it is less clear if the solution should be unique although no examples of multiple equilibria have been found.

**Proposition 8.** A symmetric (within type) pair of equilibrium arrival distributions \((p_{e}^{a}, p_{e}^{b})\) exists for any game parameters.

**Proof.** For type \(i \in \mathcal{C}\) customers, given any arrival distribution \(p_{-i}\) of customers with the other type, denote the outcome of Algorithm 1 by \(BR_{i}(p_{-i}) : [0, 1]^{T+1} \rightarrow [0, 1]^{T+1}\). The algorithm always returns a valid distribution due to the existence result of [15] for the single-type game. Let

\[
BR(p_1, p_2) := (BR_1(p_2), BR_2(p_1)) : [0, 1]^{T+1} \times [0, 1]^{T+1} \rightarrow [0, 1]^{T+1} \times [0, 1]^{T+1},
\]

denote the two dimensional mapping of the pair of distributions \((p_1, p_2)\) to their respective \(BR_i\) sets. By applying Kakutani’s fixed point theorem (e.g. Lemma 20.1 of [13]) we can conclude that \(BR\) has a fixed point \((p_1, p_2) = BR(p_1, p_2)\). In particular:

(i) \([0, 1]^{T+1}\) is a compact and closed set,

(ii) \(BR(p_1, p_2)\) is a single point in \([0, 1]^{T+1} \times [0, 1]^{T+1}\), hence it is trivially a convex set for any \((p_1, p_2)\),

(iii) and \(BR\) has a closed graph (because solutions of (9) are continuous with respect to all coordinates of \(p_i\)).

\[\square\]

5 Signal mechanism

Inconsistent beliefs regarding the service speed may be due to information asymmetry, even if the customers are otherwise homogeneous. On a given day, the server operates in mode \(a\) with probability \(p\), otherwise, it operates in mode \(b\) with probability \(1 - p\). Let \(M\) denote the service mode, i.e., its probability distribution is given by

\[
P(M = a) = p, \quad P(M = b) = 1 - p.
\]

The service time in mode \(i \in \mathcal{C} = \{a, b\}\) is denoted by a random variable \(X_i\) with mean \(\chi_i\). As before, we assume that \(\chi_a > \chi_b\), i.e., the service time in mode \(a\) is slower than one in mode \(b\). Customers receive independent signals \(S \in \{0, 1\}\) about the server mode with 1 indicating the true state. The probability of a correct signal is given by \(P(S = 1) = q\), where \(q \in (\frac{1}{2}, 1]\).

Let \(Y\) denote the belief of an arbitrary customer, then it is given by

\[
Y = MS + (a + b - M)(1 - S).
\]

The marginal probability for a customer to receive a type \(i \in \mathcal{C}\) signal is

\[
P(Y = i) = \begin{cases} 
pq + (1 - p)(1 - q), & i = a, \\
p(1 - q) + (1 - p)q, & i = b. \end{cases}
\]
Let $\alpha_{ji} := P(Y_h = j|Y_k = i)$ denote the conditional probability that any other customer $h$ has a belief $Y_h = j$ given that customer $k$ received signal $i$ for $(i, j) \in \mathcal{C}^2$. If the true service mode is $M$ and customer $k$ has a belief $Y_k = i$ after receiving a signal then

$$\frac{P(\{MS_h + (a + b - M)(1 - S_h) = j\} \cap \{MS_k + (a + b - M)(1 - S_k) = i\})}{P(MS_k + (a + b - M)(1 - S_k) = i)},$$

and therefore,

$$\alpha_{j1} = \begin{cases} \frac{pq^2 + (1-p)(1-q)^2}{pq(1-p)(1-q)}, & j = a, \\ \frac{pq(1-q) + (1-p)(1-q)q}{pq(1-p)(1-q)}, & j = b, \end{cases} \quad (14)$$

and

$$\alpha_{j2} = \begin{cases} \frac{p(1-q)q + (1-p)(1-q)q}{p(1-q)(1-p)q}, & j = a, \\ \frac{p(1-q)^2 + (1-p)q^2}{p(1-q)(1-p)q}, & j = b. \end{cases} \quad (15)$$

The total number of customers is a Poisson random variable with mean $\lambda$. Therefore, for a customer with belief $Y = i \in \mathcal{C}$ the number of customers with the same belief is Poisson with mean $\lambda \alpha_{ii}$ and the number of customers with the opposite belief $j \neq i$, is $\lambda \alpha_{ij}$.

The posterior expected service time of a customer with belief $Y = i$ is

$$\zeta_i := E[X_M|Y = i] = \chi_a P(M = a|Y = i) + \chi_b P(M = b|Y = i),$$

where $\eta_{ij} := P(M = j|Y = i)$ $(i, j \in \mathcal{C})$ is given by

$$(\eta_{aa}, \eta_{ba}) := \frac{(pq, (1-p)(1-q))}{pq + (1-p)(1-q)} , \quad (\eta_{ab}, \eta_{bb}) := \frac{(p(1-q), (1-p)q)}{p(1-q) + (1-p)q} .$$

Therefore we have

$$\zeta_i = \begin{cases} \frac{\chi_a p + \chi_b (1-p)(1-q)}{p(1-p)(1-q)}, & i = a, \\ \frac{\chi_a p(1-q) + \chi_b (1-p)q}{p(1-q) + (1-p)q}, & i = b. \end{cases} \quad (16)$$

We assumed here that customers can fully compute all of the above conditional distributions, and explicitly that they know $p$ and $q$. If the only information available to them is that $q > \frac{1}{2}$ then $\zeta_i = \chi_i$ is a reasonable assumption. However, there needs to be additional information or some prior belief in order to determine $\lambda_a$ and $\lambda_b$. For example, in this context the Markovian model of Section 3 can be interpreted in the following ways: (1) customers believe their signal due to not knowing the values of $p$ or $q$ and have some (true or not) belief regarding the population sizes $\lambda_1$ and $\lambda_2$, or (2) customers are not fully rational, i.e., a bounded rationality assumption that they use the posterior mean $\lambda_{\alpha_{ij}}$ for the population sizes but do not use the posterior distribution of service times and just believe $\zeta_i = \chi_i$. In this sense the analysis of Section 4 is more general because one can assume that $X_i$ and $\lambda_i$ already correspond to the posterior distributions for any $i \in \mathcal{C}$. In the dynamic model of Section 6 a customer may have a different belief every day so we will refer specifically to beliefs on a given day and not to customer types.
6 Dynamic customer learning

We now present a dynamic model where customers need to choose the arrival time every day to a system with server uncertainty and noisy signals. Every day \( d = 1, 2, \ldots \) the system is as described in Section 5: the server is slow, i.e., with mean service time \( \chi_a \), with probability \( p \in [0, 1] \) and fast, i.e., with mean service time \( \chi_b \), with probability \( 1 - p \). Customers receive independent signals \( S_d \in \{0, 1\} \) about the system state with 1 indicating the true state. The probability of a correct signal is \( q \in \left( \frac{1}{2}, 1 \right] \). The true state is \( M \in C \) and the belief of an arbitrary, denoted by \( Y_d \), satisfies

\[
Y_d = MS_d + (a + b - M)(1 - S_d) .
\]

The server state and signals on any day are independent of each other and of the states and signals in all other days. We assume customers do not know \( p \) or \( q \) but they do know that \( q > \frac{1}{2} \), and so the signal is informative. The acceptance period is the discrete grid \( T = \{0, \tau, \ldots, T\tau\} \) as in Section 4.

We assume that there is a finite large pool of \( N \) potential customers that join independently at any given day with a probability \( \delta_N \) such that \( \delta_N \rightarrow 0 \) as \( N \rightarrow \infty \) and \( N\delta_N = \lambda \). The total arrivals in a given day is therefore \( A \sim \text{Bin}(N, \delta_N) \) and can be approximated by Poisson(\( \lambda \)) for large \( N \). This formulation enables giving customers identities which is important for the learning framework as each customer has their own personal history of waiting time observations.

On day \( d \geq 1 \) a customer observes information \( Y_d = y \), \( y \in C \), and chooses a time slot \( t_d = t \in T \) according to a decision rule that will be elaborated on later, and records the waiting time \( W_{y,t}^{(d)} \). Let \( \tilde{W} \in \mathbb{R}^{2 \times (T+1)} \) denote the matrix of average waiting times such that \( \tilde{W}_{y,t}^{(d)} \) is the average waiting time experience on days (up to day \( d \)) when he joined the system, received information was \( y \) and the chosen time slot was \( t \). The daily decisions involve randomization between past experience and random experimentation. We denote the experimentation probability as a function of trials as

\[
\theta(x) = e^{c_1 - e^{-c_2 x}} , \tag{17}
\]

where \( c_1, c_2 > 0 \). A customer with belief \( y \) on the \( d \)th day that joined and held belief \( y \) chooses their arrival time uniformly from \( T \) with probability \( \theta(d) \) and the time slot with minimal average experienced waiting time (on previous days with belief \( y \)) with probability \( 1 - \theta(d) \). Therefore the decision rule can be written as follows,

\[
t_y^{(d)} = \begin{cases} \text{Uniform}(T), & \text{w.p. } \theta(d) , \\
\arg \min \tilde{W}_{y,t}^{(d)}, & \text{w.p. } 1 - \theta(d) . \end{cases} \tag{18}
\]

By (17) and (18) we have that as \( d \) grows the probability of uniform exploration diminishes and the minimal average waiting time is chosen with an arbitrarily high probability. Consider a tagged customer and their experienced average waiting time \( \tilde{W}_{y,t}^{(d)} \) on days with belief \( y \in C \) such that the specific slot \( t \) has been sampled. Clearly, as the number of days grows all
customers will join and receive both types of signals infinitely often. The limiting proportion of the number of times slot $t \in \mathcal{T}_\infty$ is chosen is given by

$$p_t := \lim_{D \to \infty} \frac{1}{D} \sum_{d=1}^{D} 1(t_{y}^{(d)} = t). \quad (19)$$

There is an inherent bias in the average waiting time estimation because the decision rule does not involve the system parameters ($\lambda, x_a, x_b$) or the state probabilities ($p, q$). In particular, a proportion of $q$ days are wrongly classified as the signal received is wrong. In Section 8 we will discuss the potential of investigating smarter estimation schemes.

7 Numerical analysis

We first show several examples of the equilibrium arrival distributions for different service time distributions for the discrete model of Section 4. The distributions are computed using Algorithm 2. We further compare the above to the long-term arrival distributions in the respective dynamic learning model (with the same system parameters) obtained by simulation. In all of the examples we assume that the unit of time is a single minute and that the length of the acceptance period is $(T + 1)\tau = 60$ minutes.

Numerical examples for the Nash equilibrium distributions (Section 4)

In all examples the mean number of arriving customers with beliefs $a$ and $b$ are given by $\lambda := (\lambda_a, \lambda_b) = (5, 5)$; the pair of the service time distributions with beliefs $a$ and $b$ are denoted by $x := (x_a, x_b)$ with mean $({\chi}_a, {\chi}_b) = (4, 2)$. The service time distributions $(x_a, x_b)$ considered are deterministic, geometric, and a mixture of two geometric distributions. The number of slots in the acceptance period are 2, 6, and 20 slots, i.e., $\tau = 30$, $\tau = 10$ and $\tau = 3$, respectively (see Table 1). In Case (III), the parameters of the service time distributions were chosen so that their coefficient of variations (CV’s, for short) are twice of that of Case (II).

| Table 1: List of numerical examples |
|---|---|---|
| | (1) 30[min] × 2 | (2) 10[min] × 6 | (3) 3[min] × 20 |
| (I) Deterministic distribution | (I-1) | (I-2) | (I-3) |
| (II) Geometric distribution | (II-1) | (II-2) | (II-3) |
| (III) Mix. of geometric distributions | (III-1) | (III-2) | (III-3) |

In each case, the equilibrium arrival-time distributions are computed by

$$(p^e_a, p^e_b) := \text{Alg.} 2(\lambda, x_a, x_b, \epsilon, \delta), \quad (20)$$

where $\epsilon = \delta := 10^{-5}$. Figure 2 shows the results of Case (I), i.e., the service time distribution of each belief of customers follows the deterministic distribution. Figures 3 and 4 show the
Results of Cases (II) and (III), respectively.

Observations from numerical examples:

1. Belief \( a \) customers (Belief \( b \) customers) tend to arrive in the first half (the latter half) of the acceptance period. This is similar to the equilibrium results for the Markovian model of Section 3.

2. As the number of slots in the acceptance period increases, the mean waiting time (MWT, for short) of each belief of customers decreases. Note that the MWT’s are computed based on the beliefs of customers (see e.g., Formula (12)).

3. There are time slots \( t > 0 \) in which both customers with beliefs \( a \) and \( b \) arrive with positive probabilities, see slot 4 in Case (I-2) and slot 12 in Case (I-3).

4. Figures 2, 3, and 4 show that as the CV’s of the service time distributions increase, the MWT’s of both belief \( a \) and \( b \) customers increase.

Figure 2: Case (I): Deterministic distribution, where the equilibrium arrival distributions are computed by (20)

Figure 3: Case (II): Geometric distribution, where CV’s of beliefs \( a \) and \( b \) service time distributions are \((0.87, 0.71)\), and the equilibrium arrival distributions are computed by (20)
Numerical examples for the dynamic learning model (Section 6)

Next, we consider several examples for the dynamic learning model under the following parameter settings: the mean number of population is given by $\lambda = 10$; the server uncertainty and the strength of the signal are given by $p = 0.5$ and $q = 0.9$, respectively; the service time distributions in modes $a$ and $b$ are given by $x_a$ and $x_b$ with mean $\chi_a = 4$ and $\chi_b = 2$, respectively. With these parameter settings, the mean number of arrival customers with beliefs $a$ and $b$ are given by $\lambda_a := \lambda(pq + (1 - p)(1 - q)) = 5$ and $\lambda_b := \lambda(p(1 - q) + (1 - p)q) = 5$, respectively, and are thus equal to the arrival rates in previous the equilibrium examples. As in the Nash equilibrium examples (see Table 1), we consider the same three service time distributions and the three cases for the number of slots in the acceptance period. Given the above parameter settings, the results of this subsection are displayed in Figures 5, 6, and 7. The average arrival distributions are obtained by simulations such that for each belief ($a$ and $b$), every customer arrives at the system 1,000 times on average.

By comparing the results of this subsection (see Figures 5, 6, 7) with ones in the last subsection (see Figures 2, 3, 4), we can see the outcome is quite different in all examples. In Figures 5, 6, and 7, the shapes of the arrival distributions obtained by the dynamic learning model tend to spread out over the acceptance period. On the other hand, in Figures 2, 3, and 4, the equilibrium arrival distributions of beliefs $a$ and $b$ customers tend to lie in the first and the last half of the acceptance period, respectively. This further results in differences in the expected waiting times.
The equilibrium computation for the above examples does not take into account the information structure given by the probabilities $p$ and $q$. A more appropriate comparison is therefore with the game that includes the posterior update of expectations as detailed in Sec-
tion 5. For an arrival customer with belief $i$ ($i \in C$) in the dynamic learning model, the posterior arrival rates of the other customers with beliefs $a$ and $b$ are given by (14) and (15),

$$\nu_i := \lambda(\alpha_a i, \alpha_b i), \quad i \in C.$$  

Next consider the service time distribution of an arrival customer with belief $i$ ($i \in C$). Similar to (16), let $z_i$ ($i \in C$) denote the service time with mean $\zeta$, then

$$z := (z_a, z_b) = (\eta_{aa} x_a + \eta_{ba} x_b, \eta_{ab} x_a + \eta_{bb} x_b).$$

Those observations are summarized as follows: in the view point of belief $a$ (belief $b$) arrival customers, the arrival rates to the system is estimated by $\nu_a$ ($\nu_b$), and the service time random variables of beliefs $a$ and $b$ customers are distributed as $z$.

We conclude that the equilibrium arrival distributions, denoted by $(\hat{p}_a^e, \hat{p}_b^e)$, are computed by

$$\hat{p}_a^e := \text{Alg.2}(\nu_a, z, \epsilon, \delta), \quad \hat{p}_b^e := \text{Alg.2}(\nu_b, z, \epsilon, \delta).$$

(21)

The results are displayed in Figures 8–16. In what follows, we compare the preceding numerical results of the dynamic learning model (see Figures 5, 6, 7) with the equilibrium arrival distributions $(\hat{p}_a^e, \hat{p}_b^e)$ computed by (21). Under the parameter settings in this subsection (i.e., $\lambda = 10$, $p = 0.5$, $q = 0.9$, $\chi_a = 4.0$, and $\chi_b = 2.0$), we have $\nu_a = (8.2, 1.8)$, $\nu_b = (1.8, 8.2)$, and $(\zeta_a, \zeta_b) = (3.8, 2.2)$.

The results are displayed in Figures 8–16, and we can see that the resulting arrival distributions and mean waiting times are closer to the outcome of the learning model. However, the distribution resulting from the learning dynamics is more spread out than the equilibrium distribution. In particular, the optimistic type $b$ customers consistently arrive at $t = 0$ with positive probabilities even when the equilibrium prediction is that they only start arriving later. On the other hand, pessimistic type $a$ customers arrive at the last slots with positive probability even though they would not do so in equilibrium. These discrepancies are due to the estimation bias resulting from ignoring the system parameters and information structure.

![Figure 8: Case (I-1): Deterministic distribution, where the equilibrium arrival distributions are computed by (21)](image-url)
Figure 9: Case (I-2): Deterministic distribution, where the equilibrium arrival distributions are computed by (21)

Figure 10: Case (I-3): Deterministic distribution, where the equilibrium arrival distributions are computed by (21)

Figure 11: Case (II-1): Geometric distribution, where CV’s of the service time distributions in modes a and b are (0.87, 0.71), and the equilibrium arrival distributions are computed by (21)
Figure 12: Case (II-2): Geometric distribution, where CV’s of the service time distributions in modes a and b are (0.87, 0.71), and the equilibrium arrival distributions are computed by (21)

Figure 13: Case (II-3): Geometric distribution, where CV’s of the service time distributions in modes a and b are (0.87, 0.71), and the equilibrium arrival distributions are computed by (21)

Figure 14: Case (III-1): Mixture of geometric distributions, where CV’s of the service time distributions in modes a and b are (1.74, 1.42), and the equilibrium arrival distributions are computed by (21)
Figure 15: Case (III-2): Mixture of geometric distributions, where CV’s of the service time distributions in modes $a$ and $b$ are $(1.74, 1.42)$, and the equilibrium arrival distributions are computed by (21)

Figure 16: Case (III-3): Mixture of geometric distributions, where CV’s of the service time distributions in modes $a$ and $b$ are $(1.74, 1.42)$, and the equilibrium arrival distributions are computed by (21)
8 Conclusion

This paper has introduced a general framework to analyze the arrival distributions to a bottleneck queue with multiple customer types that differ in their belief regarding the service time distribution. Two models were presented. First, a game where rational customers that know all of the system parameters, including the mechanism driving uncertainty, with the solution given by a Nash equilibrium. Second, a dynamic learning model in which customers adapt their decisions based on past experience. Constructive procedures have been provided for the computation of the arrival distribution in both cases.

Our framework can be extended in several natural directions. From an economic point of view, more elaborate cost functions can be considered, e.g. tardiness or order penalties and rewards for service. The customers may have heterogeneous features that may be due to individual preferences (e.g. waiting costs) or partial information (e.g. random value of service). Such games may also combine multiple $n > 2$ types of customers or beliefs. From a queueing perspective, different system configurations may be considered, e.g. a multi-server system with uncertainty regarding the number of servers. Both our equilibrium and learning analysis can be extended to the above settings.

The learning model of Section 6 can be seen as a simple first attempt at formulating interesting learning dynamics for the arrival time problem to a bottleneck queue. There are several issues that are perhaps of interest to explore further. Notably, accurate asymptotic analysis of the average arrival distribution. This can be combined with further investigation of the bias brought on by the average waiting time estimation, and can it be improved by decision rules that are still reasonably simple. For example, we assumed that the information quality is unknown, i.e. customers just know that $q > \frac{1}{2}$. If we further assume that customer observe service times, then $q$ can be estimated as follows. Denote by $X_i^d$ the service time observed on day $d$ when the belief was $y \in C$, and let $ar{X}_i$ be the respective average service time on days (up to day $d$) that the service belief was $y$. By the law of large numbers, $\bar{X}^{(d)}_a \xrightarrow{p} q \chi_a + (1 - q) \chi_b$, and this yields an asymptotically unbiased estimator

$$
\hat{q}^{(d)}_a = \frac{\bar{X}^{(d)}_a - \chi_b}{\chi_a - \chi_b}.
$$

An interesting question is then if the above can be used to make a simpler decision role that relies on multiple estimators for the system parameters, i.e. arrival and service rates as well as the information strength. Of course, such a learning process again assumes customers have the ability and resources to make more elaborate computations for the decision making, but it is interesting to examine what can be achieved with slightly more sophisticated estimation and decision rules.

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Appendices

A Detailed proof of Lemma 1

To prove Lemma 1 we construct a coupling of the virtual waiting time and queueing processes for both types of customers and show that for every sample path type $a$ customers face a longer queue and waiting time than type $b$ customers for any arrival time $t \in T$.

Lemma 9. Let $X_{i,k} \sim \exp(\mu_i)$ denote the job size distribution of the $k$'th arrival when the service distribution is of type $i \in C$ customers. If $\mu_a < \mu_b$ then the virtual waiting time $V_a$ and the queue length $Q_a$ of type $a$ customers is stochastically larger than the virtual waiting time $V_b$ and queue length $Q_b$ of type $b$ customers for any given strategy profile $(F_a, F_b)$:

$$V_b(t) \leq V_a(t), \quad Q_b(t) \leq Q_a(t), \quad \forall t \in [0, T] \, ,$$

which further implies that

$$EV_b(t) \leq EV_a(t), \quad EQ_b(t) \leq EQ_a(t), \quad \forall t \in [0, T] \, .$$

Proof. The virtual waiting time, or workload, for type $i \in C$ customers can be constructed as follows: the input process of work is a non-homogeneous Poisson process defined by the arrival strategies, each job has a size of $X_i$, and work is continuously processed at a rate of one per unit of time. The following arguments are for a sample path constructed of the identical arrival process and coupled sequences of job sizes $\{X_{i,k}\}_{k=1}^{\infty}$ such that $X_{a,k} \geq X_{b,k}$ for all $k \geq 1$.

Specifically, let $U_k \sim \text{Unif}[0, 1]$ and denote

$$X_{ki} = -\frac{1}{\mu_i} \log U_k \, ,$$

then clearly $X_{a,k} = \frac{\mu_b}{\mu_a} X_{b,k} > X_{b,k}$ for all $k \geq 1$. For any strategy profile $(F_a, F_b)$, the virtual waiting time for a type $i \in C$ arriving at $t \in T$ is given by

$$V_i(t) := A_i(t) - t + L_i(t) \, ,$$

where $A_i(t)$ is a non-homogeneous compound Poisson process with cumulative rate $\lambda_a F_a(t) + \lambda_b F_b(t)$ and jump sizes $X_i$, and

$$L_i(t) := \left( -\inf_{s \in [0,t]} \{A_i(s) - s\} - V_0 \right)^+ \, .$$

Observe that $L_i(t)$ is a non-decreasing process that increases only when $V_i(t) = 0$ and that $A_a(0) > 0$ if and only if $A_b(0) > 0$. Suppose that $A_a(0), A_b(0) > 0$ and denote $\tau_i = \inf\{t \geq 0 : V_i(t) = 0\}$, then as $A_a(t) \geq A_b(t)$ we have that

$$V_a(t) = A_a(t) - t \geq A_b(t) - t = V_b(t), \quad t \in [0, \tau_b] \, .$$

Furthermore, as $X_{a,k} > X_{b,k}$ for all jobs, the number of departures from queue $a$ is at most equal to the number of departures from queue $b$ during $[0, \tau_b]$ because both servers are working continuously. The common arrival time of jobs to both systems further implies that

$$Q_a(t) \geq Q_b(t), \quad t \in [0, \tau_b] \, .$$
The process $L_b(t)$ increases after $\tau_b$ as long as $V_b(t) = 0$, but clearly $0 = V_b(t) \leq V_a(t)$ and $0 = Q_b(t) \leq Q_a(t)$ during any such period. If a jump $A_b(t) - A_b(t-)$ occurs at $t < \tau_a$ then $V_a(t)$ has a bigger jump than $V_b(t)$ both and the dominance remains until the next time $s$ such that $V_b(s) = 0$. This continues until $\tau_a$ for which

$$V_a(\tau_a) = V_b(\tau_a) = Q_a(\tau_a) = Q_b(\tau_a) = 0,$$

and both $V_i$ stay at zero until the next arrival time and the same process starts again with a new jump for both processes. The same argument is valid when $A_a(0) = A_b(0) = 0$ and the process starts at the time of the first arrival. We conclude that for every sample path $V_a(t) \geq V_b(t)$ for all $t \in T$ and $V_a(s) > V_b(s)$ for all $S \subset T$ such that $S$ is a non-empty union of positive intervals, i.e., the set $S$ has non-zero measure. Similarly, the same holds for $Q_a(t)$ and $Q_b(t)$.

**B Best response algorithms**

This appendix details Algorithm 1 and a necessary subroutine that we denote Algorithm 1-1. The purpose of the algorithm is to find the symmetric within type best response arrival distribution $p_i$, that satisfies the equilibrium condition (9), when the customers of the other type arrive according to $p_{-i}$. The outline is as follows: (1) finding the first time slot $t \in T$ such that $p_{it} > 0$, (2) a bisection search for the value of $p_{it}$ which uniquely defines all probabilities $p_{iu}$ for $u > t$, (3) stopping the procedure when $\sum_{t=1}^{T} p_{iu} = 1$. Algorithm 1 is designed for step (1) and checking the equilibrium condition of (3), and Algorithm 1-1 performs the bisection search of part (2).

Note that the implementation of the algorithm may be refined to make it more efficient, but the purpose of the description here is to provide a concise description. For example, in line 4 of Algorithm 1-1, we may choose $p_{i,R}^{(R)}$ in a slightly better way, e.g., $p_{i,R}^{(R)} = \min \{1, \text{Equation (13)}\}$, but we’ve chosen $p_{i,R}^{(R)} := 1$ for the simplicity of the presentation.

**Algorithm 1**: Symmetric best response of type $i$ to $p_{-i}$.

**Input**: $p_{-i}, \lambda := (\lambda_a, \lambda_b), x_i, \epsilon > 0$

**Output**: $p_i$

1. init $\theta := 0$
2. while $\theta \leq T$ do
   1. init $p_i := (0, 0, \ldots, 0)$
   2. compute $(w_{i,t}; 0 \leq t \leq \theta)$ by (12)
   3. set $\overline{w}_i := w_{i,\theta}, w_{\min} := \min(w_{i,t}; 0 \leq t \leq \theta - 1)$, where $\min(\phi) := \infty$
   4. if $\overline{w}_i < w_{\min}$ then
      1. compute $(p_{i,t}; \theta \leq t \leq T)$ by Alg.1-1$(p_i, p_{-i}, \lambda, x_i, \theta, \epsilon)$
   5. else
      1. $\theta := \theta + 1$
   6. end if
3. end while
4. return $p_i$
Algorithm 1-1: Subroutine in Algorithm 1 (Bisection search).

Input: $p_i, p_{-i}, \lambda := (\lambda_a, \lambda_b), x_i, \theta, \epsilon > 0$

Output: $p_i, \theta$

init $p_i^{(L)} := (0, 0, ..., 0), p_i^{(M)} := (0, 0, ..., 0), p_i^{(R)} := (0, 0, ..., 0)$

set $p_{i, \theta} := 1, p_{i, \theta} := (p_{i, \theta}^{(R)} + p_{i, \theta}^{(L)})/2$

while true do

for $k = L, M, R$ do

compute $w_{i, \theta}$ by (12) with $p_i := p_i^{(k)}$

init $\overline{w}_i^{(k)} := w_{i, \theta}$

compute $(p_{i,t}^{(k)}, \theta + 1 \leq t \leq T)$ by (9) with $\overline{w}_i := \overline{w}_i^{(k)}, p_i := p_i^{(k)}$

end for

if $1 - \epsilon < ||p_i^{(M)}|| < 1 + \epsilon$ then

set $p_i := p_i^{(M)}, \theta := \infty$

break

else if $||p_i^{(L)}|| > 1$ then

set $\theta := \theta + 1$

break

else if $||p_i^{(M)}|| < 1$ then

set $p_{i, \theta}^{(L)} := p_{i, \theta}^{(M)}, p_{i, \theta}^{(M)} := (p_{i, \theta}^{(L)} + p_{i, \theta}^{(R)})/2$

else if $||p_i^{(M)}|| > 1$ then

set $p_{i, \theta}^{(R)} := p_{i, \theta}^{(M)}, p_{i, \theta}^{(M)} := (p_{i, \theta}^{(L)} + p_{i, \theta}^{(R)})/2$

end if

end while

return $p_i, \theta$

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