On the Performance of Dense Wireless Networks: No Linear Scaling in Practice

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Abstract

We consider the hierarchical cooperation architecture of Ozgur, Leveque and Tse, which is supposed to yield almost linear scaling of the capacity of a dense wireless network with the number of users $n$. Exploiting recent results on the optimality of “treating interference as noise” in Gaussian interference channels, we are able to optimize the achievable average per-link rate and not just its scaling law. Our optimized hierarchical cooperation architecture significantly outperforms the originally proposed scheme, which is yet good enough to achieve the claimed scaling law. On the negative side, we show that even for very large $n$, the rate scaling is far from linear, and the optimal number of stages $t$ is between 2 and 3, instead of $t \to \infty$ as required for almost linear scaling. Combining our results and the fact that, beyond a certain user density, the network capacity is fundamentally limited by Maxwell laws, as shown by Franceschetti, Migliore and Minero, we argue that there is indeed no intermediate regime of linear scaling for dense networks in practice. On the positive side, we show that our optimized hierarchical cooperation scheme outperforms the classical multi-hop routing for a moderately large network size, having a larger and larger gain as network size increases. Thus, hierarchical cooperation with proper optimization is a very promising technique for ad-hoc wireless networks although it does not achieve linear rate scaling for practical network sizes.

Index Terms

Wireless network capacity, Dense wireless networks, Hierarchical cooperation, Scaling laws.
I. Introduction

Although it is extremely hard to characterize exactly the capacity of wireless networks, significant progress has been made towards the understanding of their theoretical limits. In [1], the capacity of multiple multicast wireless network is approximately determined within a constant gap (with respect to signal-to-noise ratio (SNR) and channel coefficients) that scales linearly with the number of nodes in the network. Also, for multiple flows over a single hop, the capacity approximations were obtained in the form of degrees of freedom (DoF), generalized DoF (GDoF), and \( O(1) \)-gap bounds [2]–[4]. However, the case of multiple flows over multiple hops remains widely unsolved in general. Scaling laws provide a useful way to characterize the behavior of the capacity of such networks when the number of nodes becomes large. Initiated by Gupta and Kumar’s seminal work [5], extensive studies in the past decade have made significant progress in the understanding of the scaling laws of such large wireless networks. The well-known “decode and forward” protocol (aka, multi-hop routing) yields a sum capacity that scales as \( \Theta(\sqrt{n}) \) [5]. In [6], Ozgur, Leveque and Tse proposed a cooperative architecture named hierarchical cooperation that combines local communication and long-range cooperative MIMO communication. Applying \( t \) stages of the basic cooperative scheme to a dense network with \( n \) users in a hierarchical architecture, a capacity scaling of \( \Theta(n^{\frac{1}{t+1}}) \) was shown to be achievable. Therefore, for any \( \epsilon > 0 \), a scaling \( \Theta(n^{1-\epsilon}) \) is achievable for sufficiently large \( t \). While the result of [5] holds for both a “physical model”, that considers the actual standard communication-theoretic complex baseband signal at each node receiver, and a “protocol model”, that considers interference as a distance-based link conflict condition, the result of [6] depends critically on modeling the channel coefficient between any two clusters of transmitting and receiving users as a full-rank matrix, due to independent fading coefficients between different antenna pairs. An interesting dichotomy appeared when in [7] Franceschetti, Migliore and Minero showed that the capacity is fundamentally limited to scale as \( \Theta(\sqrt{n}) \). Instead of assuming an independent fading model, they started from Maxwell’s equations and counted the number of independent electromagnetic modes that can flow across a cut separating two regions of the network, and combined this counting argument with a standard information theoretic cut-set bound. This debate was resolved independently in [8] and [9], by showing that both results are correct and they are applicable in different operating regimes of the network. Summarizing, they concluded that linear scaling is achievable if \( n \leq \frac{\sqrt{A}}{\lambda} \) where \( A \) denotes the area of network and \( \lambda \) denotes the wavelength of the operating carrier frequency. Consider for example a network of area \( A = 1 \) km\(^2\), operating at carrier frequency of 3 GHz (\( \lambda = 0.1 \)m). In this case, for \( n \leq 10^4 \) we would expect that the hierarchical cooperation architecture of [6] yields a linear scaling.
of the sum capacity. Such “linear scaling” of the network capacity with the number of users \( n \) is the holy grail of large wireless networks since it yields constant average rate per source-destination pair in the case where sources and destinations are randomly selected such that their distance is \( O(1) \). In turn, this implies that the network, in the linear scaling regime, is “scalable” since the rate per end-to-end communication session does not vanish as the number of users grows.

While scaling law analysis yields nice and clean results, it is hard to tell how a network really performs in terms of rates, since they fail to characterize the constants of the leading term in \( n \) versus the next significant terms. Therefore, there might be significant regimes where the linear scaling does not manifest. The purpose of this paper is twofold. First, we derive an achievable sum-rate (not just a scaling law) for hierarchical cooperation. Second, we optimize the scheme on the basis of the achievable sum-rate, by appropriately choosing the transmit power, reuse factor, and quantization distortion level at the receivers. Based on this optimization, we can provide an answer to the question: “Is linear scaling achievable in practice?”. Unfortunately, the answer to this question is negative. For example, we show that for \( n \leq 10^4 \) (as in the above example) the optimal number of stages \( t \) is less than 3 (see Fig. 5), i.e., the rate scaling is far from linear\(^1\) Our result can be understood as follows. The linear scaling in \([6]\) is obtained by letting first \( n \rightarrow \infty \) to get the scaling law of a single stage of the scheme, and then letting \( t \rightarrow \infty \) to achieve the linear scaling. In contrast, we start from a network density \( n \), and for each such \( n \) we find the optimal number of hierarchical stages \( t \) with respect to sum-rate, essentially capturing the impact of a large but finite network size.

On the positive side, we show that the optimized hierarchical cooperation scheme outperforms the classical multi-hop routing for a moderately large network size, having a larger and larger gain as network size increases. Thus, hierarchical cooperation with proper optimization is a very promising technique for ad-hoc wireless networks although it does not achieve linear rate scaling for practical network sizes.

**System model:** We consider a network deployed over a unit-area squared region and formed by \( n \) nodes placed on a regular grid with minimum distance \( 1/\sqrt{n} \). The grid topology captures the essence of the problem while avoiding some technicalities due to the random placement of nodes. The network consists of \( n \) source-destination pairs, such that each node is both a source and a destination, and pairs are selected at random over the set of \( n \)-permutation \( \pi \) that do not fix any element (i.e., for which

\(^1\)Notice that \( n = 10^4 \) nodes in 1 km\(^2\) is already a very high density of nodes in practical sensor networks and tactical ad-hoc networks, and even increasing the carrier frequency from 3 GHz to 30 GHz, i.e., pushing the upper bound on the network size to \( n \leq 10^5 \), yields an optimal number of stages \( t \) not larger than 4. Hence, our conclusion stands even considering networks that operate in the mm-wave range \([11]–[13]\).
Fig. 1. Grouping of interfering clusters in the TDMA scheme with reuse factor $L = 3$. Each square represents a cluster and the gray squares represent the concurrent transmitting clusters.

$\pi(i) \neq i$ for all $i = 1, \ldots, n)$. We focus on max-min fairness, such that all source-destination pairs wish to communicate at the same rate. The channel coefficient between a transmitter node $k$ and a receive node $\ell$ at distance $r_{\ell k}$ is given by $h_{\ell k} = r_{\ell k}^{-\alpha/2} \exp(j\theta_{\ell k})$, where $\alpha$ denotes the path-loss exponent and $\theta_{\ell k} \sim \text{Unif}(0, 2\pi]$ denotes a random i.i.d. phase. This independent “phase fading” model is the same assumed in [6]. The phase $\theta_{k\ell}$ depends on the distance between the nodes modulo the carrier wavelength and the random phase model above is based on a far-field assumption: the nodes’ separation is at a much higher scale compared to the carrier wavelength, so that the phases can be modeled as completely random and independent of actual positions.

**Paper organization:** In Section II, we derive an achievable sum-rate for the basic cooperative transmission scheme. We also optimized such rate by appropriately choosing the transmit power, reuse factor, and quantization distortion level of the scheme. This result is extended into a hierarchical cooperation architecture in Section III by considering a modification of the hierarchical structure of [6], where we employ more efficiently the TDMA scheme during the local communication phases. In Section IV, we show that the optimized hierarchical cooperation outperforms multi-hop routing, even for a moderately large network size. Some concluding remarks are provided in Section V.

II. COOPERATIVE TRANSMISSION SCHEME

In this section, we optimize the cooperative transmission scheme proposed in [6] with respect to the achievable sum-rate. We let $R_c(\alpha)$ denote the common message coding rate for all users, expressed in bits per codeword symbol. The scheme delivers $n$ messages using a certain number of time-slots, each
Fig. 2. Achievable sum-rates of the cooperative transmission scheme when path-loss exponent $\alpha = 3$ and network size $n = 10^4$. of which corresponds to the duration of a codeword. Hence, the network sum throughput is given by

$$R_{\text{sum}}(n, \alpha) = R_c(\alpha)T(n, \alpha)$$

where

$$T(n, \alpha) = n/(\text{required number of time slots})$$

is the number of delivered source-destination messages per time-slot ratio (referred to as packet throughput in the following).

The network is divided into $n/M$ clusters, of $M$ nodes each. The cooperative transmission scheme consists of 3-phase:

1) A “local” communication phase is used to form cooperative clusters of transmitters. In this phase, each source distributes $M$ distinct sub-packets of its message to the $M$ neighboring nodes in the same cluster. One transmission is active per each cluster, in a round robin fashion, and clusters are active simultaneously in order to achieve some spatial spectrum reuse. The inter-cluster interference
Fig. 3. Achievable sum-rates of the cooperative transmission scheme when path-loss exponent $\alpha = 7$ and network size $n = 10^4$.

is controlled by the reuse factor $L^2$.

2) A “global” cooperative MIMO transmission phase is used to deliver messages across different clusters. In this phase, one cluster at a time is active, and when a cluster is active it operates as a single $M$-antenna MIMO transmitter, sending $M$ independently encoded data streams to a destination cluster. Each node in the cooperative receiving cluster stores its own received signal.

3) A “local” communication phase is used during which all receivers in each cluster share their own received and quantized signals in order to allow each destination in the cluster to decode its intended message on the basis of the (quantized) $M$-dimensional observation. Quantization and binning (or random hashing of the quantization bits onto channel codewords) is used in this phase, which is a special case of the general Quantize reMap and Forward (QMF) scheme for wireless relay networks [1]. Each destination performs joint typical decoding to obtain its own desired message based on the quantized signals (or bin indices).

$^2$All clusters have one transmission opportunity every $L^2$ time slots (see Fig. 1).
The parameters we need to optimize in the cooperative transmission scheme are the cluster size $M$, the node transmit power $\text{SNR}$, and reuse factor $L$. Regarding the transmit power, it is assumed that $\text{SNR}$ can be chosen arbitrarily with a uniform bound $\text{SNR} \leq \text{SNR}_{\text{max}}$ where the latter is a fixed arbitrarily large constant that does not scale with $n$. As the result of such optimization, we have:

**Theorem 1:** For network size $n$ and path-loss exponent $\alpha$, the cooperative transmission scheme achieves the sum-rate of

$$R_{\text{sum}}(n, \alpha) = \log \left( 1 + \frac{\text{SNR}}{1 + P_I} \right) \frac{\sqrt{n}}{2\sqrt{2L(\text{SNR})}}$$

where

$$\text{SNR} = 2^{2(3+\alpha/\ln 2)}$$ (1)

$$L(\text{SNR}) = \left\lceil \sqrt{\text{SNR}}^{1/\alpha} + 1 \right\rceil$$ (2)

$$P_I = \sum_{i=1}^{\sqrt{n}} 8i\text{SNR}(L(\text{SNR}) - 1)^{-\alpha}.$$ (3)

**Proof:** See Sections II-A, II-B, and II-C.

Theorem 1 implies that all sources can reliably transmit their messages at rate $R_c(\alpha) \approx \log \sqrt{\text{SNR}}$ over the $2\sqrt{2L(\text{SNR})}\sqrt{n}$ time slots. Notice that despite the fact we let $\text{SNR}_{\text{max}}$ to be an arbitrarily large constant, the optimal $\text{SNR}$ depends only on the pathloss $\alpha$ and it is generally not too large. This is because there is a tension between the transmit power of each local link and the reuse factor necessary to keep inter-cluster interference under control. The optimal transmit power is determined in Section II-C (Theorem 1), as a result of this tradeoff. For comparison, notice that in the original scheme of [6] the reuse factor is fixed to 3. Figs. 2 and 3 show that our optimized scheme provides a substantial gain over the scheme of [6] (conventional scheme).

**Remark 1:** By considering only the dominant interfering terms at each receiver, the total interference power is well approximated by:

$$P_I = 8\text{SNR}(L(\text{SNR}) - 1)^{-\alpha}.$$ (4)

With the approximation $L(\text{SNR}) \approx \sqrt{\text{SNR}}^{1/\alpha} + 1$, the achievable sum-rate in Theorem 1 can be simplified as

$$R_{\text{sum}}(n, \alpha) \approx \frac{\alpha \sqrt{n}}{(2\sqrt{2}\ln 2)^{2^{3/\alpha+1/\ln 2}}}.$$ (5)

From [5], we observe that the sum-rate grows almost linearly with the path-loss exponent $\alpha$. Thus, the optimized cooperative transmission scheme can provide a higher rate when the system operates at high
frequencies (e.g., mm-waves) due to the fact that at those frequencies the path-loss exponent becomes large [13], [14].

A. Local Communication

In phases 1 and 3 of the scheme there is no intra-cluster interference since a single transmitter-receiver pair is active in each cluster. However, each receiver suffers from inter-cluster interference from the transmitters in the other clusters. Hence, we are in the presence of a $n/M$ user (symmetric) interference channel. For such channel, the advanced coding schemes have been recently proposed such as message splitting and successive interference cancellation [15], [16], interference alignment [17], and structured coding [18]. However, they require extensive channel state information at transmitters as well as receivers and long delay. On the other hand, it has been recently recognized [19] that there exists a regime of the interference channel gains for which treating interference as noise (TIN) is information-theoretically optimal (within a constant gap). TIN is most attractive in practice since it requires standard “Gaussian” codes and minimum distance decoders. Hence, we shall operate our network in the TIN-optimal regime. For convenience, we recall here the main result of [19]:

Theorem 2 ([19]): For $n$-user interference channel, if the following condition is satisfied, then treating interference as noise (TIN) can achieve the whole capacity region to within a constant gap of $\log(3n)$:

$$\text{SNR}_i \geq \max_{j \neq i} \text{INR}_{ij} \cdot \max_{k \neq i} \text{INR}_{ki}, \forall i = 1, \ldots, n,$$

where $\text{SNR}_i = \frac{P|h_{ii}|^2}{N_0}$ and $\text{INR}_{ij} = \frac{P|h_{ij}|^2}{N_0}$ denote the signal-to-noise ratio of user $i$ and the interference-to-noise ratio of source $j$ at destination $i$, respectively.

In order to ensure that the interference channel induced by the local communication phases operates in the regime for which TIN is (near) optimal, we can choose appropriately the reuse factor $L$. We first determine the transmit power $P$ according to the cluster area $A$ and the path-loss exponent $\alpha$ as $P = \text{SNR}_i A^\alpha/2$. This choice makes that $\text{SNR}_i = \text{SNR}$ for all $i$. Also, let $\text{INR}$ denote the strongest interference power, i.e., $\text{INR} = \max_{j \neq i} \text{INR}_{ij} = \max_{k \neq i} \text{INR}_{ki}$ where the last equality is due to the symmetric structure of network. Considering the TDMA structure, we obtain that $\text{INR} = (L - 1)^{-\alpha} \text{SNR}$. In our symmetric model, the optimality condition of TIN in Theorem 2 is satisfied if $\text{INR} \leq \sqrt{\text{SNR}}$. We can find $L$ to meet the above condition as

$$L(\text{SNR}) = \left\lceil \sqrt{\text{SNR}}^{1/\alpha} + 1 \right\rceil.$$
Fig. 4. Distributed MIMO channel with finite backhaul capacity of rate $R_0$ where $R_0$ is determined by local communication rate.

Then, the local communication rate of $R^{(1)}(\text{SNR}) = \log \left( 1 + \frac{\text{SNR}}{1 + P_l} \right)$ is achievable by TIN, where the inter-cluster total interference is upper bounded by $P_l$ in (3). Reliable local communication is ensured by letting:

$$R_c(\alpha) \leq R^{(1)}(\text{SNR}). \quad (7)$$

B. Long-Range MIMO Communication

By concatenating phases 2 and 3 of the cooperative scheme, we obtain an equivalent distributed MIMO channel with finite backhaul capacity of rate $R_0$ as shown in Fig. 4 where the $M$ transmit (resp., $M$ receiver) antennas correspond to the $M$ nodes in the source cluster (resp., destination cluster). This model has been extensively studied in [10], [20]. In particular, in [20] we showed that the capacity of this channel is almost achieved by either QMF or Compute and Forward (CoF) [21], depending on the channel coefficients and on the value of $R_0$. Specifically, CoF can outperform QMF if the number of strong interferers at each receiver is relatively small with respect to the total number of nodes in the network (i.e., \textit{sparse network}) and the backhaul capacity is small (i.e., $R_0 \leq 5$). In this regime, the impact of the non-integer penalty of CoF is not severe, while QMF suffers from the quantization noise due to the small backhaul capacity (see [20] for details). In other regimes, QMF shows better performance than CoF because it can fully exploit the MIMO beamforming gain. Since our model can be considered as a \textit{dense network} (i.e., each receiver suffers from $n - 1$ non-negligible interfering nodes), here we choose to employ QMF for the long-range MIMO transmission.
For the MIMO transmission (i.e., phase 2), the transmit power is given by
\[ P_{\text{MIMO}} = \frac{\text{SNR}'}{M} A^0/2 \]
where SNR' can be arbitrary chosen with \( \text{SNR}' \leq \text{SNR}_{\text{max}} \) as in the local communication. Using distance-dependent power control in order to eliminate the effect of the path-loss between the transmit and receiving clusters, the channel matrix of the distributed MIMO channel is \( H \in \mathbb{C}^{M \times M} \), with \((k, \ell)\)-element given by \( \exp(j \theta_{k\ell}) \), with \( \theta_{k\ell} \sim \text{Unif}(0, 2\pi) \). Let \( N_0 \) denote the variance of the additive noise plus inter-cluster interference. As in [6], the local communication of phase 3 can be expanded over \( Q \) time slots for some integer \( Q \), in order to obtain more flexibility in the quantization rate of the underlying QMF scheme. This yields the backhaul capacity of the “equivalent” model as \( R_0 = QR^{(1)}(\text{SNR}) \). An optimal \( Q \) will be chosen later on.

The computation of the rate achievable by QMF for the distributed MIMO channel with finite backhaul capacity is generally difficult since it involves a complicated combinatorial optimization [10]. So far, a closed-form expression was only available for the symmetric Wyner model [10]. In this paper, we derive a closed-form expression of the achievable rate for our model, exploiting the fact that, for large \( n \), the problem becomes symmetric although the matrix \( H \) is “full” and not tri-diagonal as in the Wyner model. Our result is based on asymptotic Random Matrix Theory and the submodular structure of the rate expression:

**Theorem 3:** For a distributed MIMO channel with backhaul capacity equal to \( R_0 \) and random i.i.d. channel coefficients with zero mean and unit variance, QMF achieves the symmetric rate of
\[
R_{\text{QMF}}(R_0, N_0, \text{SNR}) = \min \left\{ R_0 - \log \left( 1 + \frac{N_0}{\sigma_q^2} \right), C \left( \frac{\text{SNR}}{N_0 + \sigma_q^2} \right) \right\}
\]  \hspace{1cm} (8)
for some quantization level \( \sigma_q^2 \geq 0 \), where
\[
C(x) \doteq 2 \log \left( \frac{1 + \sqrt{1 + 4x}}{2} \right) - \frac{\log e}{4x} \left( \sqrt{1 + 4x} - 1 \right)^2. \hspace{1cm} (9)
\]

**Proof:** See Appendix A.

Since the achievable rate in (8) is the minimum of two terms, where the first is an increasing function of \( \sigma_q^2 \) and the second is a decreasing function of \( \sigma_q^2 \), the optimal value of \( \sigma_q^2 \) is attained by solving
\[
R_0 - \log \left( 1 + \frac{N_0}{\sigma_q^2} \right) = C \left( \frac{\text{SNR}}{N_0 + \sigma_q^2} \right),
\]

\(^3\)Inter-cluster interference is zero in a single layer of the hierarchical cooperation, but is non-zero when multiple stages are considered in the hierarchical cooperation architecture. Therefore, it is denoted here by \( N_0 \) not necessarily equal to the normalized noise level 1, for the sake of generality.
Letting
\[ f(\sigma_q^2) \overset{\Delta}{=} R_0 - \log \left( 1 + \frac{N_0}{\sigma_q^2} \right) - \mathcal{C} \left( \frac{\text{SNR}}{N_0 + \sigma_q^2} \right), \]
we can find
\[ \sigma_{q,\min}^2 = \frac{N_0}{2R_0 - 1} \quad \text{and} \quad \sigma_{q,\max}^2 = \frac{N_0 + \text{SNR}}{2R_0 - 1}, \]
such that \( f(\sigma_{q,\min}^2) \leq 0 \) and \( f(\sigma_{q,\max}^2) \geq 0 \). This is because \( \sigma_{q,\min}^2 \) makes the first term of the minimum in (8) zero and \( \sigma_{q,\max}^2 \) is the quantization level of Quantize and Forward (QF) which makes the second term to attain the minimum in (8). Using bisection method, we can quickly find an optimal quantization level \( \sigma_{q,\text{opt}}^2 \). This will be used in this paper to plot the achievable rates of QMF.

Putting together the MIMO rate constraint of Theorem 3 with the rate achievable in phase 1 (7), we find
\[ R_c(\alpha) = \min \{ R^{(1)}(\text{SNR}), R_{\text{QMF}}(QR^{(1)}(\text{SNR}), 1, \text{SNR'})) \}. \]
Since there is no inter-cluster interference (i.e., \( N_0 = 1 \)) in the MIMO communication phase 2, we can find some finite value \( \text{SNR}' \) with \( Q = 1 \) such that
\[ R^{(1)}(\text{SNR}) \leq R_{\text{QMF}}(R^{(1)}(\text{SNR}), 1, \text{SNR'}). \]
Then, we have that \( R_c(\alpha) = R^{(1)}(\text{SNR}) \), where the optimal SNR will be determined in the next section. In fact, we do not have to compute an exact explicit expression for the achievable rate of QMF in this section but the QMF rate will be used in Section III for the hierarchical cooperation architecture, when we shall consider multiple stages of the 3-phase cooperative scheme.

Remark 2: We observe that, by choosing \( \text{SNR}' = \text{SNR} \), the achievable coding rate is about \( R_c = R^{(1)}(\text{SNR}) - 0.5 \) when using the optimal quantization level. This has about 0.5 bits improvement compared with noise-power level quantization in \([1]\).

\[ \diamond \]

C. Achievable sum-rate

In order to derive an achievable sum-rate, we will compute the packet throughput \( T(n, \alpha) \). As anticipated before, in the cooperative scheme each source transmits \( M \) distinct sub-packets of its message to the intended destination. To transmit overall \( nM \) sub-packets (in the whole network), phase 1 requires \( (L(\text{SNR})M)^2 \) time slots, phase 2 requires \( n \) time slots, and phase 3 requires \( Q(L(\text{SNR})M)^2 \) time slots. Hence, we have
\[ T(n, \alpha) = \frac{Mn}{(Q + 1)(L(\text{SNR})M)^2 + n}. \]

\[ ^4 \text{QF is a simplified version of QMF without using binning (see [20] for details).} \]
Since the coding rate $R^{(1)}(\text{SNR})$ is independent of $M$, we can find the optimal cluster size $M$ by treating $M$ as a continuous variable and solving $\frac{dT(n,\alpha)}{dM} = 0$. This yields

$$M = \frac{\sqrt{n}}{L(\text{SNR})\sqrt{1+Q}}.$$  

Then, the packet throughput is obtained as

$$T(n,\alpha) = \frac{\sqrt{n}}{2L(\text{SNR})\sqrt{1+Q}}$$

and, accordingly, the achievable sum-rate is given by

$$R_{\text{sum}}(n,\alpha) = R^{(1)}(\text{SNR})\frac{\sqrt{n}}{2L(\text{SNR})\sqrt{1+Q}}.$$  

Next, we will optimize the transmit power $\text{SNR}$ to maximize the above sum-rate. To make the problem tractable, we use the approximations $L(\text{SNR}) \approx \sqrt{\text{SNR}}^{1/\alpha}$ and $R^{(1)}(\text{SNR}) \approx \log(\sqrt{\text{SNR}/8})$. Then, the sum-rate is approximated by

$$\hat{R}_{\text{sum}}(n,\alpha) = \log\left(\frac{\sqrt{\text{SNR}}}{8}\right)\frac{\sqrt{n}}{2\sqrt{2}\sqrt{\text{SNR}}^{1/\alpha}}$$

where $Q = 1$ is chosen because of the argument given in Section II-B. Differentiating and solving $\frac{d\hat{R}_{\text{sum}}(n,\alpha)}{d\text{SNR}} = 0$, we find that the optimal transmit power is given by

$$\text{SNR} = 2^{2(3+\alpha/\ln 2)}.$$  

This concludes the proof of Theorem 1.

### III. Optimizing the Hierarchical Cooperation Architecture

*Hierarchical cooperation* was proposed in [6] by employing the cooperative transmission scheme of Section II as a building block, that can be applied for local communication of a higher stage, i.e., at a larger space scale in the network. In this scheme, we use the symmetric coding rate $R_c(\alpha)$ regardless of the number of hierarchical stages $t$. Based on Section II we choose

$$P_t = \text{SNR}A_i^{\alpha/2}$$

$$L = \left[\sqrt{\text{SNR}}^{1/\alpha} + 1\right]$$

$$\text{SNR} = 2^{2(3+\alpha/\ln 2)}$$

for stages $i = 1, \ldots, t$, where $A_i$ denotes the cluster area of stage $i$. Notice that these choices guarantee that, regardless of the hierarchical stage $i$, the received power of inter-cluster interference is upper bounded by $P_t$ given in (3). For the MIMO communication phase, we choose transmit power

$$P_{\text{MIMO},i} = \frac{\text{SNR}}{M}A_i^{\alpha/2}$$
which also makes the interference power to be not larger than $P_I$. The main result of this section is:

**Theorem 4:** For network size $n$ and path-loss exponent $\alpha$, the hierarchical cooperation scheme with $t \geq 2$ stages achieves the sum-rate of

$$R_{\text{sum}}^{(t)}(n, \alpha) = R_c(\alpha) \frac{n^{t+1}}{(1 + t)L^{t+1} \sqrt{3}}$$

where $L = \left[2^{(3+\alpha/\ln 2)/(\alpha + 1)} + 1 \right]$ and $R_c(\alpha)$ is determined in Section III-A.

**Proof:** See Sections III-A and III-B.

Fig. 6 shows the message coding rate $R_c(\alpha)$ for a range of the path-loss exponent $\alpha$ relevant in practice. Even for $t = 1$ (single stage), the sum-rate in Theorem 4 does not reduce to the previous result of Theorem 1 since in this case we can choose a higher coding rate than $R_c(\alpha, 1)$, because there is no inter-cluster interference in the MIMO communication phase. Instead, the rate $R_c(\alpha)$ is chosen here in order to meet the most stringent constraint for reliable communication with an arbitrary number of stages $t$, i.e., including the case where the MIMO communication phase of stages $i < t$ suffers from inter-cluster interference. From Theorem 4 we observe that a linear scaling can be achieved as $t \to \infty$. 

Fig. 5. Optimal number of hierarchical stages $t$ as a function of network size $n$, when path-loss exponent $\alpha = 7$. 

![Optimal number of hierarchical stages](image)
Fig. 6. Achievable coding rates as function of path-loss exponent $\alpha$.

when the network size $n$ grows faster than the constant term $(1+t)L \frac{2t}{t+1} \sqrt{3}$. However, for a finite network size, this term in the denominator of the packet throughput cannot be neglected since it also grows with $t$. Namely, adding more stages does not necessarily improve the achievable sum-rate. Thus, for given $n$, we can find an optimal number of hierarchical stages to maximize the sum-rate. In order to make the problem manageable, we relax the integer constraint on $t$ and find the optimal $t$ as solution of

$$
\frac{dR_{\text{sum}}^{(n)}}{dt} = 0.
$$

This gives the equation in $t$ as

$$(t + 1)^2 \ln \sqrt{3} + (t + 1) - \ln(n/L) = 0$$

which yields

$$
t_{\text{opt}} = -1 + \frac{-1 + \sqrt{1 + 2 \ln(n/L) \ln 3}}{\ln 3}.
$$

(10)

This shows the following negative result: even for $n$ as large as $10^6$, the optimal number of hierarchical stages is not larger than 3 (see Fig. 5). Hence, for networks of reasonable size, the linear scaling law is a “myth”, even without considering the physical propagation limitations analyzed in [7]. Fig. 7 plots the achievable sum-rate of hierarchical cooperation with an optimal number of stages. The conventional
scheme is the one presented in [6] and the enhanced scheme is the one presented in this paper with sum-rate given by Theorem 4. In our scheme, we have modified the TDMA phases in order to reduce the transmission overhead and further improve the packet throughput (see Section III-B for details). We observe that the enhanced scheme provides a considerable gain over the conventional scheme, having a larger gap as $n$ increases. Nevertheless, the network throughput is clearly sub-linear even for the range of unreasonably large $n$ shown in the figure.

Remark 3: From Fig. 6, we observe that QF almost achieves the upper bound for our model. Thus, QF is better choice in practice, significantly reducing the decoding complexity at destination, since QMF requires joint typical decoding based on the collected bin indices while QF just performs “classical” MIMO decoding based on the quantized received signals (see comment in Section II-B).

A. Achievable coding rate

From Section II, we have the rate-constraint of

$$R_c(\alpha) \leq R^{(1)}_t \triangleq \log \left( 1 + \frac{\text{SNR}}{1 + P_t} \right)$$
for reliable local communication at the bottom stage (i.e., stage 1). Concatenating phases 2 and 3 of stage 1, we can produce an equivalent distributed MIMO channel with backhaul capacity of $QR^{(1)}$ (see Section II-B). Then, the coding rate should satisfy the constraint

$$R_c(\alpha) \leq R_{QMF}(QR^{(1)}, N_0 = P_I + 1, \text{SNR}) \triangleq R^{(2)}.$$  

Lemma 1 below yields that $R^{(2)} \leq R^{(1)}$ for any positive integer $Q \geq 1$. Since $R^{(2)}$ is the local communication rate of stage 2, we can produce a *degraded* distributed MIMO channel with backhaul capacity $QR^{(2)} \leq QR^{(1)}$, resulting in the rate-constraint

$$R_c(\alpha) \leq R_{QMF}(QR^{(2)}, N_0 = P_I + 1, \text{SNR}) \triangleq R^{(3)}.$$  

Due to the smaller backhaul capacity, we have that $R^{(3)} \leq R^{(2)}$. Repeating the above procedure, we obtain that

$$R^{(t+1)} \triangleq R_{QMF}(QR^{(t)}, P_I + 1, \text{SNR}) \leq R^{(t)}$$  

implying that $\{R^{(t)}\}$ is monotonically non-increasing. Hence, there exists a limit

$$\lim_{t \to \infty} R^{(t)} = R^*(\alpha, Q)$$
where such limit depends on $\alpha$ and $Q$. All rate-constraints are satisfied by choosing $R_c(\alpha, Q) = R^*(\alpha, Q)$.

One might have a concern that this choice is not a good one for small $t$. However, Fig. 8 shows that $R^{(t)}$ quickly converges to its positive limit for $Q \geq 2$. Also, we observe that $Q = 2$ is the best choice since it almost achieves the upper bound $R^{(1)}$, while it requires only two time slots in order to deliver the quantization bits to the receivers in phase 3 of each stage. Therefore, we choose the $Q = 2$ and $R_c(\alpha) = R^*(\alpha, 2)$ in the following, for any $t$. The corresponding coding rates are plotted in Fig. 6 as a function of $\alpha$.

**Lemma 1:** For any $Q \geq 1$, the achievable rate of MIMO transmission is upper-bounded by the local communication rate of bottom stage (i.e., stage 1):

$$R_{QMF}(QR^{(1)}, 1 + P_I, \text{SNR}) \leq R^{(1)}.$$  

**Proof:** Taking $Q \to \infty$, we have a naive upper bound:

$$R_{QMF}(QR^{(0)}, P_I + 1, \text{SNR}) \leq R_{QMF}(\infty, P_I + 1, \text{SNR}).$$

Letting $\text{SNR}_{\text{eff}} = \frac{\text{SNR}}{1 + P_I}$, the proof follows from the limit upper bound:

$$R_{QMF}(\infty, P_I + 1, \text{SNR}) = \lim_{M \to \infty} \frac{1}{M} \log \det \left( I + \frac{\text{SNR}_{\text{eff}}}{M} HH^H \right)$$

$$\leq a \lim_{M \to \infty} \frac{1}{M} \log \left( \text{tr} \left( I + \frac{\text{SNR}_{\text{eff}}}{M} HH^H \right) \right)^M$$

$$= b \log(1 + \text{SNR}_{\text{eff}}) = R^{(1)}$$

where (a) is from the inequality of arithmetic and geometric means, i.e., $\log \det(A) \leq M \log \left( \frac{\text{tr}(A)}{M} \right)$ and (b) is due to the fact that $\lim_{M \to \infty} \frac{1}{M} \text{tr}(HH^H) = 1$ by Law of Large Numbers.

**B. Achievable sum-rate**

Focusing on the packet throughput, we first review the work in [6] and then improve it by efficiently using the TDMA scheme during the local communication phases.

**Conventional approach in [6]:** The operation of stage 1 is equivalent to the cooperative transmissions and thus, from Section II-C, the packet throughput is computed as

$$T^{(1)}(n, \alpha) = \frac{\sqrt{n}}{2L\sqrt{1 + Q}}.$$  

(11)

Also, stage 2 employs stage 1 as its local communication (see Fig. 9). Then, the required number of time slots is $(LM)^2/T^{(1)}(M, \alpha)$ for phase 1, $n$ for phase 2, and $Q(LM)^2/T^{(1)}(M, \alpha)$ for phase 3.
The resulting packet throughput is given by

$$T^{(2)}(n, \alpha) = \frac{nM_1}{(1+Q)(LM_1)^2/T^{(1)}(M_1, \alpha) + n}.$$ 

The optimal cluster size $M_1$ is obtained by solving $\frac{dT^{(2)}(n, \alpha)}{dM_1} = 0$, which yields the

$$M_1 = \frac{n^2}{L^2(1+Q)}.$$ 

Then, we obtain the

$$T^{(2)}(n, \alpha) = \frac{n^3}{3(L\sqrt{1+Q})^2}.$$ 

Generalizing to $t$ stages, we obtain the achievable sum-rate of the scheme in [6] as

$$R_{\text{sum}}^{(t)}(n, \alpha) = R_c(\alpha) \frac{n^{\frac{t}{t+1}}}{(t+1)(L\sqrt{1+Q})^t}.$$ 

**Enhanced approach:** We will improve the penalty term associated with TDMA from $L^t$ to $L^{\frac{2t}{t+1}}$. This provides a non-trivial gain especially when $t$ is large, since the former exponentially increases with $t$ while the latter is upper bounded by $L^2$. We explain our approach based on a 2-stage hierarchical cooperation architecture (see Fig. 10) and then extend the result to general $t$. First, we want to emphasize that TDMA scheme is used so that the received power of interference is less than a certain level for all transmissions. It can be noticed that this requirement is satisfied for the transmissions of phases 11 (or phase 13) (i.e., stage 1, phases 1 and 3) without using the TDMA scheme of phase 21 (stage 2, phase 1) since local communications have already included the TDMA operation. However, the TDMA scheme of phase 21 is required for the long-range MIMO communication of phase 12. Based on this observation, we present an alternative approach to efficiently use the TDMA scheme (see Fig. 10): All clusters in phase 21 (or phase 23) are always active (spatial reuse 1); In phase 12, each cluster has a turn to perform
the MIMO transmissions every $L^2$ time slots, which is equivalent to apply the TDMA scheme of phase 21 (or phase 23). In short, this approach applies the TDMA scheme only once to every phase. Then, we can recompute the required number of time slots as follows. Since TDMA scheme is used for all phases in stage 1, it requires $(LM_1)^2 + L^2n + Q(LM_1)^2$ time slots. With the optimal cluster size $M_1 = \sqrt{M_2 \sqrt{1+Q}}$, we can compute the packet throughput for local communication as

$$TL^{(1)}(M_2) = \frac{\sqrt{M_2}}{2L^2\sqrt{1+Q}}$$

yielding the achievable sum-rate:

$$R^{(2)}_{\text{sum}}(n, \alpha) = R_c(\alpha) \frac{nM_2}{(1+Q)M_2^2/TL^{(1)}(M_2) + n}$$

where TDMA is not used in this stage, as shown in Fig. 10. With the optimal cluster size $M_2 = \frac{n^{\frac{2}{3}}}{L^{\frac{2}{3}}(1+Q)}$, we have

$$R^{(2)}_{\text{sum}}(n, \alpha) = R_c(\alpha) \frac{n^{\frac{2}{3}}}{3L^{\frac{2}{3}}(1+Q)}.$$

Similarly, generalizing to a $t$-stage hierarchical scheme, we obtain:

$$R^{(t)}_{\text{sum}}(n, \alpha) = R_c(\alpha) \frac{nM_t}{(1+Q)M_t^2/TL^{(t-1)}(M_t) + n}$$

\[(a) \Rightarrow R_c(\alpha) \frac{nM_t}{n + tL^2(\sqrt{1+Q})^{t+1}M_t^{\frac{t+2}{t+1}}}\]

\[(b) \Rightarrow R_c(\alpha) \frac{n^{\frac{t}{t+1}}}{(1+t)L^{\frac{2t}{t+1}}(\sqrt{1+Q})^t}\]

where (a) is from Lemma 2 below and (b) is from the optimal cluster size

$$M_t = \left(\frac{n}{L^2(\sqrt{1+Q})^{t+1}}\right)^{\frac{1}{t+1}}.$$

**Lemma 2**: The packet throughput of local communication of stage-$t$ is given by

$$TL^{(t)}(n) = \frac{n^{\frac{t}{t+1}}}{(t+1)L^2(\sqrt{1+Q})^t}.$$

**Proof**: The result is provided by induction. By (11), the result holds for $t = 1$. Assuming that it holds for $t-1$, we show that it also holds for $t$. We have:

$$TL^{(t)}(n) = \frac{nM}{(1+Q)M^2/TL^{(t-1)}(M) + L^2n}$$

\[= \frac{nM}{tL^2(1+Q)^{\frac{t-1}{t}}M^{\frac{t+1}{t}} + L^2n}\]
where the second equality is from the hypothesis assumption on $T(t-1)(M)$. We can maximize it by choosing the cluster size $M$ as solution of

$$\frac{dTL(t)(n)}{dM} = 0.$$  \hspace{1cm} (14)

This yields the optimal cluster size such as

$$M = \frac{n^{\frac{1}{t+1}}}{(1 + Q)^{\frac{1}{2}}}. \hspace{1cm} (15)$$

By plugging (15) into (13), we can get:

$$TL(t)(n) = \frac{n^{\frac{1}{t+1}}}{(t + 1)L^2(\sqrt{1 + Q})^t}. \hspace{1cm} (16)$$

This completes the proof.

\hspace{1cm} \blacksquare

IV. COMPARISON WITH MULTI-HOP ROUTING

In this section, we compare the sum-rate performance of our optimized hierarchical cooperation architecture with that of conventional multi-hop routing (i.e., multi-hop decode and forward strategy). In the multi-hop protocol, the packets of a source-destination pair are communicated by successive point-to-point transmissions between relaying nodes. In [22], the performance of this scheme has been analyzed in terms of scaling law. Also, it was shown that the cluster size $M = 1$ maximizes the sum-rate by minimizing the relaying burden. Following this result, we assume $M = 1$ and derive an achievable sum-rate of multi-hop routing for the grid network considered in this paper. Recall that the source-destination pairs are selected at random over the set of $n$-permutation $\pi$ that do not fix any element. As in [22], we assume that the communication between each source-destination pair is relayed by the following simple routing scheme: first proceeding horizontally and then vertically (see Fig. 11).
Fig. 11. Multi-hop routing over the grid network. Red arrow-lines represent a routing path from the source $s$ to the destination $d$.

done for local communication in Section II, the distance-dependent power control is applied and the interference is controlled by the reuse factor $L$, chosen to enforce the optimality condition of TIN as $L(SNR) = \left\lceil \sqrt{SNR}^{1/\alpha} + 1 \right\rceil$. The corresponding achievable coding rate for reliable hop transmission is given by

$$R_c(SNR) = \log \left( 1 + \frac{SNR}{1 + P_I} \right)$$

(17)

where $P_I$ is in \[3\].

Next, we need to compute the packet throughput of multi-hop routing. This can be done by computing the average relaying traffic (i.e., the required number of time slots to deliver all messages). The notation of Fig. 11 will be used in the following. Focusing on the center node $r$, the relaying traffic is generated either by the source nodes located in the same horizontal slab or the destination nodes located in the same vertical slab as $r$. Let $S = \{s_1, \ldots, s_{|S|}\}$ denote the set of source nodes located in the right side of $r$ (see Fig. 11). Also, let $d_i$ denote the corresponding destination node for $i = 1, \ldots, |S|$. First, we compute the average relaying traffic generated by the source nodes in $S$. This traffic is generated only when $d_i$ is located in the right half-space, i.e., $d_i \in D$ in Fig. 11. Then, we have

$$\mathbb{E} \left[ \sum_{i=1}^{|S|} 1_{\{d_i \in D\}} \right] = \sum_{i=1}^{|S|} \mathbb{P}(d_i \in D)$$

(18)

$$= |S| \frac{1}{2} = \frac{\sqrt{n}}{4}$$

(19)
where \( 1_{\{E\}} \) denotes the indicator function of an event \( E \) and \( \mathbb{P}(\cdot) \) is the probability measure induced by the random source-destination assignment. In the above, we use the fact that \( \mathbb{P}(d_i \in \mathcal{D}) = \frac{1}{2} \) for \( i = 1, \ldots, |\mathcal{S}| \) since for any given source node, its destination can be located in the right half-space with probability \( \frac{1}{2} \). With the exactly same argument, the average traffic generated by the source nodes located in the left side of \( r \) is also equal to \( \sqrt{\pi} \). Then, the overall average traffic generated by the source nodes located in the same horizontal slab as \( r \) is equal to \( \sqrt{\pi} \). Also, the same computation can be applied to the average traffic generated by the destination nodes located in the same vertical slab as \( r \), which is equal to \( \sqrt{\pi} \) (i.e., \( \sqrt{\pi} \) generated from the destination nodes located in the upper side of \( r \) and \( \sqrt{\pi} \) generated from the destination nodes located in the lower side of \( r \)). Then, the overall average relaying traffic is equal to \( \sqrt{\pi} \). For other relay nodes, we can easily see that the average traffic is not larger than \( \sqrt{n} \). Including the impact of TDMA, the achievable sum-rate of multi-hop routing is given by

\[
R_{\text{sum}}(n, \alpha) = \log \left( 1 + \frac{\text{SNR}}{1 + P_I} \right) \frac{\sqrt{n}}{L^2(\text{SNR})}
\]

where \( L(\text{SNR}) = \left\lfloor \sqrt{\text{SNR}}^{1/\alpha} + 1 \right\rfloor \). Finally, we can find an optimal transmit power by differentiating and solving \( \frac{dR_{\text{sum}}}{d\text{SNR}} = 0 \), yielding the optimal transmit power

\[
\text{SNR} = 2^{2\left(3+\alpha/(2\ln 2)\right)}
\]

which is very close to the value found in Theorem 1 for the single stage of the hierarchical cooperation architecture.

**Remark 4:** In this section, we compare the two schemes for a network of size \( n \leq 10^5 \), for which the optimal number of hierarchical stages \( t_{\text{opt}} \) is small (\( t_{\text{opt}} \leq 3 \) for \( n \leq 10^5 \) from (10)). When \( t \leq 3 \), it turns out that \( Q = 1 \) can also guarantee a positive coding rate as shown in Fig. 8. Hence, the optimal \( Q \) can be obtained by either \( Q = 1 \) or \( Q = 2 \), depending on the actual network size \( n \) and path-loss exponent \( \alpha \). Taking this into account, we calculate the achievable rate of hierarchical cooperation by maximizing over \( t \) and \( Q \), as

\[
R_{\text{sum}}(n, \alpha) = \max_{t=1, \ldots, \max; Q=1,2} R^{(t)} \frac{n^{\alpha+1}}{(1 + t)L^{2t} \left(\sqrt{1 + Q}\right)^t}
\]

where the coding rate \( R^{(t)} \) is defined in Section III-A as a function of \( \alpha \) and \( Q \). Here, we employ a different coding rate depending on the number of hierarchical stages \( t \), instead of using its limit as in Section III since for \( Q = 1 \), \( R^{(t)} \) is far from its limit for large \( t \). For the interesting network sizes (i.e., \( n \leq 10^5 \)), the optimal number of hierarchical stages is not large (\( t_{\text{opt}} \leq 3 \) for \( n \leq 10^5 \) from (10)) and hence, the optimization problem in (22) can be quickly solved by limiting search up to \( t_{\text{max}} = 4 \). Namely, we only need to compare the 8 pairs of \( (t, Q) \) in order to find the optimal values.
Fig. 12. Performance comparison of hierarchical cooperation and multi-hop routing when path-loss exponent $\alpha = 7$.

Fig. 12 plots the achievable sum-rates of optimized hierarchical cooperation in (22) and multi-hop routing in (20). We observe that for $n \leq 10^5$, the optimal $Q$ of hierarchical cooperation is equal to 1, i.e., the phase 3 does not need to use time-expansion and transmit the quantization bits on multiple time slots. This is because for small network sizes (assuming that $n = 10^5$ can be considered as “small”), the optimal number of hierarchical stages is also small (i.e., not larger than 3) and hence the coding rate is large enough with $Q = 1$ (as shown in Fig. 8), which minimizes the number of time slots. When $Q = 1$, we can see that QF cannot achieve the performance of QMF since the “backhaul” capacity of phase 3 (without expansion) is not large enough. Therefore, we reach the interesting conclusion that binning is actually required in order to achieve good sum-throughput. Also, we observe that the optimized hierarchical cooperation architecture provides a higher sum-rate than multi-hop routing, where the relative gain increases with the network size.

It is also interesting to assess whether the optimized hierarchical cooperation architecture is able to provide attractive performances for future wireless networks. Consider again the example made in Section I which may be representative of a university campus. The network has area $\mathcal{A} = 1\text{km}^2$ and
\( n = 3 \times 10^4 \) nodes (e.g., students, faculty, and employees). Also, we assume that wireless network operates in the mm-wave range, with carrier frequency 38 GHz and bandwidth 800 MHz, according to [23]. From Fig. 12, the optimized hierarchical cooperation can achieve the sum-rate of 170 bits/s/Hz. Then, the achievable rate per user can be computed as \( \frac{170 \times 800 \times 10^6}{3 \times 10^4} \approx 4.53 \text{ Mbps} \) while, in the case of multi-hop routing, the achievable rate per user would be \( \approx 2.4 \text{ Mbps} \).

V. CONCLUDING REMARKS

In this work we have considered the optimization of the actual achievable rate of the hierarchical cooperation architecture for dense wireless networks originally proposed in [6]. For the local communication phases, we focused on simple Gaussian codes, single-user decoding, and treating interference as noise (TIN). However, we built on the recent results on the approximate optimality of TIN [19] in order to optimize the spatial reuse factor and the transmit power. It turns out that even though we assume an arbitrarily large per-node power constraint (as long as it is a fixed constant that does not scale with \( n \)), then the optimal transmit power is a constant value that depends only on the path-loss exponent \( \alpha \). For the global MIMO communication phase, we considered the optimization of the QMF approach of [1], observing that the combination of phase 2 and 3 of the MIMO cooperative scheme is formally analogous to the well-investigated MIMO MAC channel with central processing and backhaul links of finite capacity. For such model, we have found new closed-form expressions for the achievable rate in the case of large number of nodes and random i.i.d. channel coefficients, extending the formulas provided for the symmetric Wyner model in [10]. Finally, we optimized further the achievable sum capacity by considering a variation of the original hierarchical scheme in [6], where we combine the TDMA phases of the hierarchical stages for better overall spectral efficiency.

The result of our optimization yields the performance of the hierarchical cooperation architecture in terms of actual achievable rates. We believe that these rates cannot be easily beaten for this type of network model, as well as any network with random independent placements of the nodes and random assignment of the source-destination pairs, for schemes that do not make “non-physical assumptions” on the communication channel model (i.e., consider actual signal and noise power, and not artificial collision-based interference models such as the “protocol model”) and do not assume unreasonable knowledge of the network global state (i.e., this rules out interference alignment schemes based on the knowledge of the network state with infinite precision). Furthermore, since the scheme considered here involves only “Gaussian” single-user coding, the analysis of this paper is suitable to be extended to more refined finite-length analysis [24], where the tradeoff between coding length and block error probability can be
also investigated, thus illuminating also issues about the latency of such networks. This interesting aspect, however, is out of the scope of the present paper and it is deferred to future work.

The main conclusions of our work are summarized as follows. First, we notice a negative result: despite the scheme of [6] is supposed to yield (almost) linear scaling of the sum capacity with the network size, we found that even for unreasonably large networks (up to $n = 10^7$) the optimal number of hierarchical stages is not larger than 4, which means that the effective scaling is significantly worse than linear. On the other hand, combining our results with the regime of $n$ in which the physical propagation limits of Maxwell’s laws (as illuminated by [7]–[9]) do not kick in, we see that linear scaling cannot be practically achieved. In contrast, we also have some positive results. Thanks to the analysis provided in this paper, we can actually compare the sum rate (not just the scaling law) achievable by the hierarchical cooperation architecture with that achievable by conventional multi-hop routing. To the best of our knowledge, this is the first time that such quantitative comparison is made. Hence, we were able to clarify the long-standing question of whether hierarchical cooperation can yield throughput advantages over conventional multi-hop routing in terms of actual user rates. In practically relevant network scenarios, representative of a possible next-generation of wireless devices capable of device-to-device communication, we found that an order of 100% (a factor of 2) can be expected. There is also a more subtle aspects that we have not fully investigated here, which is worthwhile to be pointed out: we found that for practical network sizes the optimal number of hierarchical stages $t$ is small. This means that an optimized hierarchical cooperation scheme delivers the messages to their destination in a small number of hops (including both local and global MIMO hops). Instead, multi-hop routing yields on average $\Theta(\sqrt{n})$ number of hops to deliver messages. This may yield a significant difference in latency and in coding rate when considering the effect of finite block length and non-zero block error rate [24].

In conclusion, with proper optimization, the hierarchical cooperation architecture appears to be a feasible and attractive approach to dense infrastructure-less wireless networks.

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APPENDIX A

ACHIEVABLE RATE OF DISTRIBUTED MIMO CHANNEL WITH FINITE BACKHAUL CAPACITY

In this section, we derive a closed-form expression of achievable rate of QMF for the distributed MIMO channel with finite backhaul capacity of rate $R_0$ (see Fig. 4). Let $H \in \mathbb{C}^{M \times M}$ denote the channel matrix,
with $k$-th row $h_k$, for $k = 1, \ldots, M$, and i.i.d. elements with zero mean and unit variance. Also, we assume that the transmit power of each user is equal to $\frac{\text{SNR}}{M}$ (i.e., the sum-power is fixed as SNR) and we let $N_0$ denote the total received interference plus noise power. We focus on the symmetric user rate.

The following notations will be frequently used in this section. Let $x$ and $y$ denote the $M$-dimensional transmit and receiver vectors, respectively. Let $S \subseteq [1 : M]$ denote the row index set of $H$. For given $S \subseteq [1 : M]$, $H_S$ represents the channel sub-matrix of the inputs $x$ to the outputs $y_S$.

This model has been extensively studied in [10], [20]. Interested readers should refer to [10], [20] for some discussion and comparison of various relaying schemes. The cut-set upper bound of such channel is given by

$$R_{\text{upper}} = \frac{1}{M} \min_{S \subseteq [1 : M]} |S| R_0 + \log \det \left( I + \frac{\text{SNR} H_S H_S^H}{N_0} \right).$$

This result follows by considering cut-set bound for two cuts: one obtained by separating the destination from the receivers and the other obtained by separating the receivers from the transmitters. The achievable rate of QMF (with given quantization distortion level $\sigma^2_{q,i}$) is derived in [10] as

$$R_{\text{QMF}} = \frac{1}{M} \min_{S \subseteq [1 : M]} \sum_{i \in S} \left( R_0 - \log \left( 1 + \frac{N_0}{\sigma^2_{q,i}} \right) \right) + \log \det \left( I + \text{diag} \left( \frac{\text{SNR}}{N_0 + \sigma^2_{q,i}} \right) H_S H_S^H \right).$$

(23)

Also, Quantize and Forward (QF), a simplified version of QMF that does not include binning after quantization, and directly forwards the quantization bits to the central receiver, achieves the rate of

$$R_{\text{QF}} = \frac{1}{M} \log \det \left( I + \text{diag} \left( \frac{(2R_0 - 1)\text{SNR}}{2R_0 N_0 + \text{SNR} \|h_i\|^2} \right) HH^H \right)$$

which is obtained from (23) by setting the quantization level as

$$\sigma^2_{q,i} = \frac{N_0 + \text{SNR} \|h_i\|^2}{2R_0 - 1} \text{ for } i = 1, \ldots, M.$$  

(24)

Next, we will derive the closed-form expressions of the above rates in order to prove Theorem 3. This will be obtained from asymptotic Random Matrix Theory results, and using the submodular structure of the rate expression. We first provide some lemmas that will be used to prove the theorem.

**Definition 1:** Let $\Omega = [1 : M]$ be a finite ground set. A set function $f : 2^\Omega \rightarrow \mathbb{R}$ is submodular if for every set $A, B \subseteq \Omega$ with $A \subseteq B$ and every $x \notin B$, the following is satisfied:

$$f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B).$$

(25)
Intuitively, submodular functions capture the concept of diminishing returns: as the set becomes larger the benefit of adding a new element to the set will decrease.

Lemma 3: Suppose that a set function $f(S)$ only depends on the size of subset $|S|$, i.e., for any $S_1, S_2 \subseteq [1 : M]$

$$f(S_1) = f(S_2) \text{ if } |S_1| = |S_2|. \tag{26}$$

Define

$$g(\beta) \Delta f(S)$$

where $0 \leq \beta = \frac{|S|}{M} \leq 1$. If $f(S)$ is submodular, then $g(\beta)$ is concave when $M \to \infty$.

Proof: Since $f(S)$ is submodular, the following inequality holds from Definition 1

$$f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B) \tag{27}$$

for any $A \subset B \subset [1 : M]$ and $x \notin B$. From (27) and the assumption of $f(S)$ only depending on the size of subset $|S|$, the following inequality also holds for any $\beta$ and $\beta'$ with $\beta' > \beta$:

$$g(\beta + \Delta) - g(\beta) \geq g(\beta' + \Delta) - g(\beta') \tag{28}$$

where $\Delta = \frac{1}{M}$. Letting $M \to \infty$, the (28) implies that $\dot{g}(\beta) \geq \dot{g}(\beta')$ for any $\beta < \beta'$, i.e., $\dot{g}(\beta)$ is monotonically decreasing function. Therefore, $g(\beta)$ is a concave function.

The following is the main result of this section, which completes the proof of Theorem 3

Lemma 4: In the large system limit (i.e., $M \to \infty$), we have:

$$R_{\text{upper}} = \min \left\{ R_0, C \left( \frac{\text{SNR}}{N_0} \right) \right\}$$

$$R_{\text{QMF}} = \min \left\{ R_0 - \log \left( 1 + \frac{N_0}{\sigma_q^2} \right), C \left( \frac{\text{SNR}}{N_0 + \sigma_q^2} \right) \right\}$$

$$R_{\text{QF}} = C \left( \frac{(2^R_0 - 1)\text{SNR}}{2^R_0 N_0 + \text{SNR}} \right)$$

where $C(\cdot)$ is given by [9].

Proof: We only prove the closed-form expression of cut-set upper bound since the others are
straightforwardly proved along the same lines. Letting $\beta = \frac{|S|}{M}$, we have:

$$R_{\text{upper}} = \lim_{M \to \infty} \min_{S \subseteq [1:M]} \left| S \right| R_0 + \frac{1}{M} \log \det \left( \mathbf{I} + \frac{\text{SNR} \mathbf{H}_S \mathbf{H}_S^H}{\mathbf{N}_0} \frac{1}{M} \right)$$

$$\xrightarrow{(a)} \min_{S \subseteq [1:M]} (1 - \beta) R_0 + C \left( \frac{\text{SNR}}{\mathbf{N}_0}, \beta \right) \quad \text{as } M \to \infty$$

$$= \min_{0 \leq \beta \leq 1} (1 - \beta) R_0 + C \left( \frac{\text{SNR}}{\mathbf{N}_0}, \beta \right)$$

$$\xrightarrow{(b)} \min \left\{ R_0, C \left( \frac{\text{SNR}}{\mathbf{N}_0} \right) \right\}.$$

- The (a) is obtained from asymptotic Random Matrix Theory in [25], given by

$$\frac{1}{M} \log \det \left( \mathbf{I} + \frac{\text{SNR}}{M} \mathbf{H}_S \mathbf{H}_S^H \right) \to C(\text{SNR}, \beta) \quad \text{as } M \to \infty$$

where

$$C(\text{SNR}, \beta) = \frac{\beta \log \left( 1 + \frac{\text{SNR}}{\beta} - \frac{1}{4} \mathcal{F} \left( \frac{\text{SNR}}{\beta}, \beta \right) \right) + \log \left( 1 + \text{SNR} - \frac{1}{4} \mathcal{F} \left( \frac{\text{SNR}}{\beta}, \beta \right) \right)}{4 \text{SNR}} \mathcal{F} \left( \frac{\text{SNR}}{\beta}, \beta \right)$$

$$\mathcal{F}(x, z) = \left( \sqrt{x(1 + \sqrt{z})^2 + 1} - \sqrt{x(1 - \sqrt{z})^2 + 1} \right)^2.$$

- The (b) is due to the fact that $f(S^c) \triangleq \log \det \left( \mathbf{I} + \frac{\text{SNR} \mathbf{H}_S \mathbf{H}_S^H}{\mathbf{N}_0} \right)$ is a submodular [26] and hence, from Lemma 3, the $C \left( \frac{\text{SNR}}{\mathbf{N}_0}, \beta \right)$ is a concave function. Thus, the minimum of (29) is attained at the boundary, i.e., either for $\beta = 0$ or for $\beta = 1$. Also, we use the simple notation as $C(\text{SNR}, 1) = C(\text{SNR}).$

With the same arguments, we can prove the closed-form expressions of achievable rates of QMF and QF.

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