VOLUME SCAVENGING OF NETWORKED DROPLETS

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Abstract. A system of \( N \) spherical-cap fluid droplets protruding from circular openings on a plane is connected through channels. This system is governed by surface tension acting on the droplets and viscous stresses inside the fluid channels. The fluid rheology is given by the Ostwald-de Waele power law, thus permitting shear thinning. The pressure acting on each droplet is caused by capillarity and given in terms of the droplet volume via the Young-Laplace law. Liquid is exchanged along the network of fluid conduits due to an imbalance of the Laplace pressures between the droplets. In this way some droplets gain volume at the expense of others. This mechanism, christened "volume scavenging," leads to interesting dynamics.

Numerical experiments show that an initial droplet configuration is driven to a stable equilibrium exhibiting 1 super-hemispherical droplet and \( N - 1 \) sub-hemispherical ones when the initial droplet volumes are large. The selection of this "winning" droplet depends not only on the channel network and the fluid volume, but also notably on the fluid rheology. The rheology is also observed to drastically change the transition to equilibrium. For smaller droplet volumes the long-term behavior is seen to be more complicated since the types of equilibria differ from those arising for larger volumes. These observations motivate our analytical study of equilibria and their stability for the corresponding nonlinear dynamical system. The identification of equilibria is accomplished by locating the zeros of a mass polynomial, defined through the constant volume/mass constraint. The key tool in our stability analysis is a pressure-volume work functional, related to the total surface area, which serves as a Lyapunov function for the dynamical system. This functional is useful since equilibria are typically not hyperbolic and linearization techniques not available. Equilibria will be shown to be hierarchically organized in terms of size of the pressure-volume work functional. For larger droplet volumes this ordering exhibits one hierarchy of equilibria. Two hierarchies exist when the volumes are smaller. The minimizing equilibria in either case are asymptotically stable.

Key words. Interacting fluid droplets; equilibria; stability and bifurcation; gradient dynamical system of ODEs; nonlinearity; networked array; shear thinning; constrained optimization; bistability; switchable adhesion

AMS subject classifications. 70K50, 37C75, 34C23, 76D45

1. Introduction. We consider a system of \( N \) spherical-cap fluid droplets protruding from uniform, circular orifices on a flat surface. The droplets are assumed connected through straight fluid channels of uniform circular cross section and possibly variable length. The fluid is homogeneous and incompressible. The rheology of the fluid is governed by the Ostwald-de Waele power law, thus, in particular, permitting shear thinning which is often observed in complex (e.g. biological) fluids. The flow dynamics are dominated by surface tension acting on the droplets and viscous forces within the network channels. The pressure acting on each droplet is given by the Young-Laplace relation. Our physics-based model accounts for volume exchange between neighboring spherical-cap droplets owing to the action of capillary pressures due to surface tension. Volume exchange arises by pressure (curvature) differences that drive liquid from one to another droplet along a network of interconnected channels. In this way, certain droplets gain volume at the expense of others. This mechanism, christened "volume scavenging," leads to interesting dynamics, driving an initial droplet configuration to a stable equilibrium. An array of fluid droplets connected to a common reservoir is displayed in Figure 1.

In the Newtonian case a small perturbation of an initial configuration of super-hemispherical droplets of equal size is known to evolve over time into a configuration consisting of one "winning" super-hemispherical droplet with all other droplets being sub-hemispherical of equal size. Since this evolution is driven by a minimization of surface area, volume scavenging appears to be a coarsening process like Ostwald ripening [10].

The mathematical model we study is a generalization of the Newtonian volume scavenging model introduced by van Lengerich, Vogel and Steen [14, 15]. That model was used to explain the grab-and-release mechanism for a switchable capillary adhesion device, described by Vogel and Steen [16], and
was motivated by a study of Eisner and Aneshansley [5] about the tarsal adhesion of the Florida tortoise beetle (*Hemisphaerota cyanea*) to defend against predators. In this work we consider the generalization of the previous model to power-law fluids. This generalization has far-reaching consequences, both for the analytical treatment of the governing equations and for the observed dynamics of solutions. Our analysis, while extending the previous results of van Lengerich et al. [14, 15], will do so without regard of the underlying conduit networks and without use of linearization techniques which are generally not available in the case of the power-law model. We will give a complete classification of possible equilibria and their stability with the dimensionless average droplet volume \( \bar{V} \) serving as a bifurcation parameter. The full range \( \bar{V} > 0 \) will be discussed (only equilibria with \( \bar{V} > 1 \) were known before). One of our central results will be an exhaustive, analytical identification of hierarchies of stationary droplet configurations, classified by the size and number of large droplets versus small ones. Remarkably, in the case of small volumes (\( \bar{V} < 1 \)), equilibria are found to exhibit two distinct hierarchies.

Volume scavenging has been the focal point of several important studies. Let us mention a few: Wente [18] analyzed the two- and three-droplet regime from the vantage point of catastrophe theory. His seminal work made use of an energy formulation to classify the stability of equilibria. Slater and Steen [11] discussed the inviscid case of \( N \) spherical-cap droplets under the symmetry assumption of equivariance with respect to the permutation group \( S_N \). Stability results were also given by van Lengerich et al. [14, 15] for the Newtonian case with constant viscosity and large average volumes \( \bar{V} > 1 \). This was achieved by linearization about hyperbolic equilibria for a conduit network where the linearized equations turned out to be particularly simple. The existence of a network-independent Lyapunov function made it then possible to extend the stability results to general networks. These works are the starting point of our own study. Some of the findings reported there will be included as special cases in this article. Yet far from being narrowly tailored, our analysis provides a template of arguments which is likely to generalize and be of use beyond this specific problem.

We base our governing equations on the dimensionless model introduced in [14]. In this framework the uniform radius \( R \) of each circular opening is rescaled and non-dimensionalized to be 1. The same length scale is used to rescale the radius \( r \) and the height \( h \) of a spherical-cap droplet protruding from an orifice. Droplet volume \( v \) is normalized such that the volume of a hemispherical droplet is 1. Then radius \( r \), height \( h \) and volume \( v \) of the droplet are related by

\[
2r = h + \frac{1}{h} \quad \text{and} \quad v = \frac{1}{4} h(h^2 + 3).
\]

A schematic of the spherical-cap droplet geometry is given in Figure 2. It is also worthwhile to record the
corresponding normalized surface area \( s_A = s_A(h) \) of the droplet in terms of its height \( h \):

\[
s_A = \frac{3}{2} \left( h^2 + 1 \right).
\]

Note that \( v = v(h) \) is invertible for \( h \in \mathbb{R} \) (with \( h \geq 0 \) being of physical importance in our situation) and that, due to the chosen scalings, \( v = 1 \) if and only if \( h = 1 \). Moreover, we have, of course,

\[
|h| = O(|v|^{1/3}) \quad \text{as} \ |v| \to \infty.
\]

The pressure \( p \) acting on the spherical-cap droplet is caused by surface tension and given by the Young-Laplace law (after non-dimensionalization):

\[
p = \frac{2 \pi}{r} = \frac{4h}{h^2 + 1}.
\]

Hence the capillary pressure is the same for spherical-cap droplets of the same radius of curvature \( r \). More precisely, if \( h_0 \) is a solution of the equation \( p(h) = \lambda \) with \( \lambda \in \mathbb{R}_+ \), then by (1), (4), \( h_0^{-1} \) is also a solution. In fact, \( h_0 \) and \( h_0^{-1} \) are the only solutions. They are distinct if and only if \( \lambda \neq 1 \). Consequently, similar to the terminology used in [14, 15], we call a spherical-cap droplet of height \( h \) “large” if \( h > 1 \) \((v > 1)\) and “small” if \( 0 \leq h \leq 1 \) \((0 \leq v \leq 1)\).

Because of the relations between radius \( r \), height \( h \) and volume \( v \) given above, we can write the pressure \( p = p(h) \) in terms of the droplet volume by setting

\[
P(V) = p(h) \quad \text{whenever} \ V = v(h).
\]

In this way, droplet pressure \( P \) is defined as a function of droplet volume \( V \). Similarly, we obtain the surface area of a droplet as a function of its volume via

\[
S(V) = s_A(h) \quad \text{whenever} \ V = v(h).
\]

**1.1. Networks of interaction.** We now consider a network of conduits, each of circular cross section (with uniform radius) and possibly variable length, connecting \( N \) (\( \geq 2 \)) spherical-cap fluid droplets. The network is described by a simple, connected graph whose adjacency relation is given through a weighted adjacency matrix \((c_{i,j})\): For \( 1 \leq i, j \leq N \),

\[
c_{i,j} = c_{j,i} > 0 \quad \text{if there is a channel connecting droplets} \ i \ and \ j \ (i \neq j),
\]

\[
c_{i,j} = 0 \quad \text{in all other cases}.
\]
The size of $c_{i,j}$ may be interpreted as a measure for the (inverse) length of the channel between droplets $i$ and $j$. If the network is uniform (i.e. its conduits are of equal length), we may assume $c_{i,j} \in [0,1]$ after rescaling. Examples of networks (graphs) with $N = 5$ are given in Figure 3. Important networks include the following simple, connected graphs:

- **Complete network**: Each vertex is adjacent to every other vertex.
- **Star network**: Exactly one vertex (star center) is adjacent to every other vertex. No other edges are present.
- **Linear network**: Each vertex is adjacent to no more than two other vertices. Exactly two vertices are adjacent to only one other vertex.

1.2. Rates of exchange. We denote the volume of droplet $j$ at time $t$ by $V_j = V_j(t)$ and the volumetric flow rate from droplet $i$ to droplet $j$ at time $t$ by $q_{i,j} = q_{i,j}(t)$. Then we obtain the change in volume $V_j$ at time $t$ from conservation of mass (volume):

\[
\frac{d}{dt} V_j = \sum_{i=1}^{N} q_{i,j}, \quad 1 \leq j \leq N. \tag{9}
\]

For a power-law fluid the volumetric flow rate $q_{i,j}$ is assumed to be given by the dimensionless closed-form flow rate–pressure change expression for $1 \leq i, j \leq N$

\[
q_{i,j} = c_{i,j} \Delta s P_{i,j} \quad \text{with} \quad \Delta s P_{i,j} = \frac{1}{|P(V_i) - P(V_j)|^s sgn\left(P(V_i) - P(V_j)\right)}. \tag{10}
\]

The power-law parameter $s > 0$ is the reciprocal of the power-law index of the fluid, see [2]. The case $s = 1$ reduces this relationship to laminar pipe flow of a Newtonian liquid. This is the situation studied in [14, 15]. For $s > 1$ the fluid is shear thinning, i.e. the apparent viscosity decreases with increased stress. For $0 < s < 1$ the fluid is shear thickening where the opposite flow behavior is observed. Shear thinning is commonly seen for real fluids (including biological fluids, paints and foods), while shear thickening is rare. Observed values of the power-law index in the case of shear-thinning fluids are listed in [2, 12]. The corresponding values of the power-law parameter $s$ fall roughly in the interval $1 \leq s \leq 5$. In contrast, rheological data for shear-thickening behavior are scarce. In fact, we will observe in the next section that the shear thickening parameter range $0 < s < 1$ is problematic in our model.

It is also worthwhile to point out at this point that our physics-based model of volume scavenging is related to an ODE system modeling aggregating populations studied by Lizana and Padrón [8] (in the special case of a uniform linear network with $s = 1$). This is remarkable because some of our findings here potentially also apply to such models in population dynamics.

2. Motivation and Numerical Results. The flow is modeled by the system

\[
\frac{d}{dt} V_j = \sum_{i=1}^{N} c_{i,j} \Delta s P_{i,j}, \quad 1 \leq j \leq N \tag{11}
\]

for the unknown droplet volumes $V_j = V_j(t)$ together with the initial volume configuration

\[
V_j(0) = v_j, \quad 1 \leq j \leq N. \tag{12}
\]

The initial value problem (11) and (12) has a local–in time solution for every $s > 0$ (and every choice of initial data). Solutions are unique for every $s \geq 1$. In the case $0 < s < 1$, solutions might lose uniqueness at points where the right-hand side of (11) is not Lipschitz continuous. Indeed, the occurrence of non-unique solutions is readily confirmed when $0 < s < 1$ and $N = 2$.

Non-uniqueness is clearly a nonphysical artifact of our model. Instead of excluding the case $0 < s < 1$ outright, we carry it along as a mathematical curiosity and address some resultant challenges of analytical interest.

Let us give an elementary characterization of the equilibria of system (11). The connectedness of the conduit network immediately implies:
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PROPOSITION 2.1. For \( V^* = (V_1^*, ..., V_N^*) \in \mathbb{R}^N \) to be an equilibrium of the system (11), it is necessary and sufficient that \( P(V_i^*) = P(V_j^*) \), 1 \( \leq \) \( i, j \leq N \).

For \( 0 < s < 1 \), Lipschitz continuity of the right-hand side of (11) is lost at equilibria.

The model equations given by (11) are based on conservation of volume (mass). Indeed, we obtain immediately from the symmetry of the matrix \( (c_{i,j}) \):

PROPOSITION 2.2. Any solution \( \mathbf{V} = \mathbf{V}(t) = (V_1(t), ..., V_N(t)) \) of the system (11) for \( 0 \leq t \leq T \) satisfies \( \sum_{j=1}^{N} V_j(t) = \sum_{j=1}^{N} V_j(0) \).

Hence the average droplet volume

\[
\mathcal{V} = \frac{1}{N} \sum_{j=1}^{N} V_j(t)
\]

is an invariant of the system. Let us explicitly state the underlying (physically relevant)

Standing Assumption: \( \mathcal{V} > 0 \)

and define:

DEFINITION 2.3. For given \( \mathcal{V} \), let

\[
\mathcal{V}(\mathcal{V}) = \left\{ \mathbf{v} = (v_1, ..., v_N) \in \mathbb{R}^N \mid \frac{1}{N} \sum_{j=1}^{N} v_j = \mathcal{V} \right\}.
\]

By Proposition 2.2, the set \( \mathcal{V}(\mathcal{V}) \) is forward invariant in the sense that any solution \( \mathbf{V} = \mathbf{V}(t) \) of (11) and (12) for \( 0 \leq t \leq T \) with initial data in \( \mathcal{V}(\mathcal{V}) \) takes values in \( \mathcal{V}(\mathcal{V}) \) for \( 0 \leq t \leq T \).

Having introduced the relevant parameters \( N, \mathcal{V} \) and \( s \), we now report some numerical experiments to document their impact on the flow dynamics. First we consider solutions of the system (11) with \( 1 < \mathcal{V} \leq 2.5 \) for a uniform linear network with \( 3 \leq N \leq 25 \) for various values of \( s \geq 1 \). Specifically, we take \( c_{i,i+1} = 1 = c_{i+1,i}, 1 \leq i \leq N - 1 \), and \( c_{i,j} = 0 \) in all other cases. As in [14] we choose as initial data a small perturbation of the equilibrium \( (\mathcal{V}, ..., \mathcal{V}) \) consisting of droplets of equal size:

\[
v_j = \mathcal{V} + 10^{-3} \frac{2 j - (N+1)}{N-1}, \quad 1 \leq j \leq N.
\]

The long-term behavior of solutions is computed by using standard numerical integrators. We start with a value of \( \mathcal{V} = 1.01 \) and increase it successively to \( \mathcal{V} = 2.5 \) in increments of 0.01. Similarly to the Newtonian situation in [14], volume scavenging for \( s > 1 \) leads to a depletion of volumes for all but one droplet that emerges as “winner.” In fact, \( N - 1 \) small droplets remain with the winning droplet being large. As we will see later (and was shown in [14] for the case \( s = 1 \)), this configuration corresponds to a stable equilibrium of the system. The winners of volume scavenging are displayed in Figure 4 for \( s = 1.0, s = 1.1, s = 1.2 \) and \( s = 1.3 \). Winning droplets with larger index \( j, 1 \leq j \leq N \), are depicted by darker grays. Figure 4a reproduces the Newtonian “change-of-winner” result of [14] with more details.

The physical mechanism behind volume scavenging, independent of the particular fluid model, was made plausible in [11] in the case \( N = 2 \): When two droplets of equal volumes are large, adding a small volume to one droplet removes the same amount from the other one (mass conservation). While the pressure in the first droplet decreases, it increases in the second droplet, pushing even more volume toward the first droplet. Hence this configuration is unstable. Likewise, when both droplets are small of equal volumes less than 1, the configuration is stable. A similar behavior is observed in soap films [3] and is reminiscent of the “two-balloon experiment” [4, 17]. When \( N > 2 \), the situation is more complicated as
we will demonstrate later in this work. Animations of volume scavenging can be found online at https://www.dropbox.com/s/0t4yk5jkk5uxd0/VAllLog.mpg?dl=0 with a legend given at https://www.dropbox.com/s/8teepdybd2j3op/Legend.pdf?dl=0.

In our computation we have noticed a change in the dynamics with respect to the power-law parameter $s$: The transient times to equilibrium increase by several orders of magnitude as $s$ increases. This phenomenon occurs as the apparent viscosity (friction) of the fluid decreases with increasing shear rates. Hence the longer transients are surprising, at least at first sight, since, naively, we anticipate a shear-thinning fluid ($s > 1$) to become “thinner” than a Newtonian one ($s = 1$). However, on second thought, one realizes that the apparent viscosity of shear-thinning fluids actually increases when shear rates become smaller. In volume scavenging an increase in the apparent viscosity occurs when pressure drops are small. Hence in low-pressure drop regions a shear-thinning fluid is expected to flow slower than a Newtonian one. To rationalize the observed transient behavior further, we refer to the similarity with the “toy problem” $x' = -x^s$, $x(0) = x_0 > 0$. The solution $x(t)$ decays to the equilibrium 0 for every $s \geq 1$. The decay is exponential when $s = 1$. For $s > 1$, however, the decay is only algebraic like $t^{-1/(s-1)}$.

Figure 4 shows that even small variations in $s$ lead to notable changes in the winning droplet configuration. This fact is particularly visible when $N \geq 15$. It appears that the basins of attraction of the
corresponding stable equilibria not only vary drastically with $N$ and $\overline{V}$ (as indicated in [14]), but also with $s$. The exact mechanisms underlying the selection of winning droplets remain largely unknown, although some explanations to this end are given in [14] when $N = 3$ and $s = 1$.

![Diagram](image1.png)

**Fig. 5.** Positive solutions $h$ of Equation (17) in dependence on $\overline{V}$

Our second concern is to highlight possible issues with the occurrence of equilibria in dependence on $\overline{V} > 0$. We concentrate on the case $N = 3$. In light of Proposition 2.1, any equilibrium of the system (11) must consist of $n$ ($0 \leq n \leq 3$) large droplets of height $h > 1$ and $3 - n$ small droplets of height $h^{-1}$ such that the capillary pressure in all droplets is equal. By mass conservation, for a given average droplet volume $\overline{V} > 0$ we have $n v(h) + (3 - n) v(h^{-1}) = 3 \overline{V}$, or equivalently,

$$n h^6 + 3 n h^4 - 12 \overline{V} h^3 + 3 (3 - n) h^2 + 3 - n = 0. \tag{17}$$

Figure 5 shows the positive solutions $h$ of this equation for varying values of $\overline{V}$ with $n \in \{0, 1, 2, 3\}$. We focus on solutions near $h = 1$, $\overline{V} = 1$ (intersection point of dashed lines). For given $n$, the solutions of interest are those with $h > 1$. As expected, in Figures 5a and 5d we encounter equilibria consisting of droplets of the same size. They are the only equilibria which arise for $0 < \overline{V} < 1$ with $n = 0$ (three small droplets of height $h^{-1}$ and zero large droplets) and for $\overline{V} > 1$ with $n = 3$ (three large droplets of height $h > 1$ and zero small droplets). As seen in Figure 5c, equilibria with exactly two large droplets occur for $\overline{V} > 1$, but not for $0 < \overline{V} < 1$. The most interesting case is $n = 1$. Here, Figure 5b confirms that for $\overline{V} > 1$, we have equilibria consisting of exactly one large droplet (and two small ones). If, however, $\overline{V}$ falls in the interval (0.957, 1), two types of equilibria consisting of exactly one large and two small droplets exist. For smaller values of $\overline{V}$, there are no equilibria with $n = 1$. We illustrate all possible equilibria (up to permutation of the droplets) with one large and two small droplets for $\overline{V} = 0.98$ by volume percentages of the total droplet volume $3 \overline{V} = 2.94$ in Figure 6. Here, the two possible heights of the large spherical-cap droplet are $h_L = 1.348$ and $h_I = 1.047$ with the height of the two small droplets given by the reciprocals
$h_S = h^{-1}_L$ and $h_s = h^{-1}_I$, respectively. The corresponding volumes are denoted by $V_L = v(h_L)$, $V_S = v(h_S)$, $V_I = v(h_I)$ and $V_s = v(h_s)$. The only other possible equilibrium consists of three equal, small droplets of volume $\bar{V}$. We will confirm that for $\bar{V} > 1$ only one type of stable equilibrium is present, as seen in the case $s = 1$ [14] and in the inviscid case [11]. If $\bar{V}$ is restricted to an interval of the form $l_0 < \bar{V} < 1$, two types of stable equilibria will be shown to arise side-by-side. The limiting value $l_0 > 0$ will turn out to depend on $N$. In the situation discussed above ($N = 3$) $l_0 = 0.957$, and the stable equilibria will be the ones described by Figures 6a and 6c. If, however, $0 < \bar{V} < l_0$, the only possible equilibrium will be the one consisting of droplets of equal size. This equilibrium is stable.

We will study our physics-based model based on analytical techniques that are independent of the underlying network and of the power-law parameter $s$. A key tool for our study of the system (11) will be provided by the following energy functional:

\[
\mathcal{W}(V) = \frac{1}{N} \sum_{j=1}^{N} \int_{0}^{V_j} P(V) \, dV \quad \text{for } V = (V_1, \ldots, V_N) \in \mathbb{R}^N.
\]

$\mathcal{W}$ is a minor modification of the equivalent functional given in [14]. It is a measure of the pressure-volume work due to the capillary pressure acting on each droplet.

Since for any $V_0 \in \mathbb{R}$ and $h_0 = v^{-1}(V_0)$

\[
\int_{0}^{V_0} P(V) \, dV = \int_{0}^{h_0} p(h) v'(h) \, dh = 3 \int_{0}^{h_0} h \, dh = \frac{3}{2} h_0^2,
\]

by Equation (2), $\mathcal{W}$ is related to the total surface area $A(V) = \sum_{j=1}^{N} S(V_j)$ through

\[
\mathcal{W}(V) = \frac{1}{N} A(V) - \frac{3}{2}.
\]

**Proposition 2.4.** $\mathcal{W}(V) \geq 0$ for all $V \in \mathbb{R}^N$, and $\mathcal{W}(V) \to -\infty$ if and only if $\|V\| \to \infty$.

**Lemma 2.5.** Suppose $V = V(t) = (V_1(t), \ldots, V_N(t))$ is any solution of the system (11) on an open interval $I \subset \mathbb{R}$. Then

\[
\frac{d}{dt} \mathcal{W}(V(t)) = -\frac{1}{2N} \sum_{i,j=1}^{N} c_{i,j} \left| P(V_i(t)) - P(V_j(t)) \right|^{p+1} \quad \text{for all } t \in I.
\]

In particular, $\frac{d}{dt} \mathcal{W}(V(t)) \leq 0$ for all $t \in I$, and $\frac{d}{dt} \mathcal{W}(V(t_0)) = 0$ for some $t_0 \in I$ if and only if $V(t_0)$ is an equilibrium of the system (11).
Proof. By direct calculation we obtain
\begin{equation}
\frac{d}{dt} w(V) = \frac{1}{2N} \sum_{j=1}^{N} P(V_j) V_j' + \frac{1}{2N} \sum_{j=1}^{N} P(V_j) V_i' = \frac{1}{2N} \sum_{i,j=1}^{N} c_{i,j} \left( P(V_j) \Delta_i P_{i,j} + P(V_i) \Delta_j P_{j,i} \right).
\end{equation}

This implies the claim.

\[ \therefore \]

Hence solutions stay bounded on “any” time interval of existence starting at \( t = 0 \). Consequently, we obtain a global existence and uniqueness result.

**Proposition 2.6.** Solutions of the initial value problem (11) and (12) exist for all time \( t \geq 0 \). They are unique if \( s \geq 1 \).

### 3. Forward Invariant Sets
We have seen in Proposition 2.2 that, by conservation of mass, the set \( \mathbb{V}(\bar{V}) \) is forward invariant. In this section we study two more, physically relevant sets.

**Definition 3.1.** For given \( \bar{V} \) and \( k \in \mathbb{N} \) with \( 1 \leq k \leq N \), define the sets \( \mathbb{V}^k(\bar{V}) = \mathbb{V}(\bar{V}) \cap \{ 0,\ldots,N \}^N \) and
\[ \mathbb{V}^k(\bar{V}) = \left\{ \mathbf{v} = (v_1,\ldots,v_N) \in \mathbb{V}(\bar{V}) \mid v_k \leq 1 \right\}. \]

Because of Propositions 2.1 and 2.2 we readily have:

**Proposition 3.2.** Let \( \mathbf{V}^* = (V_1^*,\ldots,V_N^*) \) be an equilibrium of the system (11) with \( \bar{V} = \frac{1}{N} \sum_{j=1}^{N} V_j^* > 0 \). Then \( \mathbf{V}^* \) is contained in \( \mathbb{V}^k(\bar{V}) \). In fact, \( V_j^* > 0 \) for all \( 1 \leq j \leq N \).

Next we will show that the set \( \mathbb{V}^k(\bar{V}) \) is forward invariant when \( s \geq 1 \). To obtain this result, we choose \( \epsilon > 0 \) and let
\[ c^\epsilon_{i,j} = \begin{cases} c_{i,j} & \text{if } c_{i,j} > 0, \\ \epsilon & \text{if } c_{i,j} = 0 \text{ and } i \neq j, \\ 0 & \text{otherwise.} \end{cases} \]

**Theorem 3.3.** Suppose that \( s \geq 1 \), or that \( s > 0 \) and the network is complete. Then any solution \( \mathbf{V} = \mathbf{V}(t) \) of the system (11) with \( \mathbf{V}(0) \in \mathbb{V}^k(\bar{V}) \) satisfies \( \mathbf{V}(t) \in \mathbb{V}^k(\bar{V}) \) for all \( t \geq 0 \).

**Proof.** Let \( \mathbf{V} = \mathbf{V}(t) = (V_1(t),\ldots,V_N(t)) \) be a solution originating in \( \mathbb{V}^k(\bar{V}) \), and let us assume for the moment that the network is complete. It suffices to show that \( V_j(t) \geq 0 \) for all \( t \geq 0 \) and \( 1 \leq j \leq N \). The map \( V \rightarrow P(V) \) is positive for \( V > 0 \), increasing on \( (0,1) \) and decreasing on \( (1,\infty) \). It satisfies \( P(0) = \lim_{V \rightarrow -\infty} P(V) = 0 \). Suppose now that \( t_0 \geq 0 \) is a fixed time and \( j \in \{ 1,\ldots,N \} \) an index such that \( 0 = V_j(t_0) \leq V_j(t_0) < V_i(t_0) \), \( 1 \leq i \leq N \). Since \( \bar{V} > 0 \), there is \( k \in \{ 1,\ldots,N \} \) such that \( V_k(t_0) \geq 0 \). Consequently, \( 0 = P(V_j(t_0)) \leq P(V_i(t_0)) \leq P(V_k(t_0)) \), \( 1 \leq i \leq N \). Hence we have from (10) at \( t_0 \) that \( \Delta_i P_{k,j} > 0 \) and \( \Delta_j P_{i,j} \geq 0 \) for \( 1 \leq i \leq N \). Completeness of the network gives then \( \frac{d}{dt} V_j(t_0) > 0 \). However, this result implies that \( V_j(t) \geq 0 \) for \( t \geq 0 \), \( 1 \leq j \leq N \). Together with Proposition 2.2, this proves the claim for the case of a complete network with any \( s > 0 \).

Let us now turn to the case \( s \geq 1 \). For \( \epsilon > 0 \), denote the solution of the system (11) with initial data \( \mathbf{V}(0) = \mathbf{v} \) and with adjacency matrix \( \left( c^\epsilon_{i,j} \right) \), given in (24), by \( \mathbf{V}^\epsilon = \mathbf{V}(t) = (V_1^\epsilon(t),\ldots,V_N^\epsilon(t)) \). The corresponding network is complete. Since our argument above applies and proves that \( V_j^\epsilon(t) \geq 0 \) for \( t \geq 0 \), \( 1 \leq j \leq N \). As we pass to the limit \( \epsilon \rightarrow 0^+ \), continuous dependence of solutions on the parameter \( \epsilon \) shows that the pointwise limit \( \mathbf{V}(t) = \lim_{\epsilon \rightarrow 0^+} \mathbf{V}^\epsilon(t) \) exists for every \( t \geq 0 \) and solves the initial value problem (11) and (12) with initial data \( \mathbf{V}(0) = \mathbf{v} \) and with the original adjacency matrix \( \left( c_{i,j} \right) \). Hence \( V_j(t) \geq 0 \) for \( t \geq 0 \), \( 1 \leq j \leq N \). \( \square \)
Finally let us turn to a physically important observation:

**Theorem 3.4.** Suppose that \( s \geq 1 \) and \( k \in \{1, \ldots, N\} \). Then any solution \( V = V(t) \) of the system (11) with \( V(0) \in V^+_s(\bar{V}) \) satisfies \( V(t) \in V^+_s(\bar{V}) \) for all \( t \geq 0 \).

Hence for \( s \geq 1 \), a small droplet will remain small.

**Proof.** Suppose first that the network is complete. Given \( k \), we may assume that there exists \( t_0 \geq 0 \) and \( j \in \{1, \ldots, N\} \) such that \( V_j(t_0) = 1 \) and \( V_j(t_0) \neq 1 \). For otherwise, we would either have \( V_k(t) < 1 \) for all \( t \geq 0 \), or \( V_i(t) = 1, 1 \leq i \leq N \), whenever \( V_k(t) = 1 \). The latter situation implies that \( V_i(t) = 1, 1 \leq i \leq N \), for all \( t \geq 0 \) by uniqueness of solutions. In either case, the claim would follow. Since for every \( V > 0 \) with \( V \neq 1 \), we have \( P(V) < P(1) = 2 \), we obtain that at time \( t_0 \), \( \Delta_i P_{jk} < 0 \) and \( \Delta_j P_{ik} \leq 0 \) for all \( 1 \leq i \leq N \).

Consequently, by completeness of the network, \( \frac{d}{dt} V_k(t_0) < 0 \). Hence as soon as \( V_k \) reaches the value 1, it must decrease again. In conclusion, \( V_k(t) \leq 1 \) for all \( t \geq 0 \) and \( k \in \{1, \ldots, N\} \).

If the network is not complete, we proceed as before and introduce a complete network with adjacency matrix \( \{c_{ij}\} \). The claim follows as \( c \to 0 \). \( \Box \)

For a complete network it is sufficient to replace the condition \( s \geq 1 \) by the requirement that the initial value problem (11) and (12) with initial data \( V_i(0) = 1, 1 \leq i \leq N \), only permit the solution \( V_i(t) = 1, 1 \leq i \leq N, t \geq 0 \). This is indeed satisfied for \( 0 < s < 1 \) in the case \( N = 2 \), as can be seen by direct calculation.

### 4. Equilibria and Semiflows.

Let us now return to the study of equilibria of the system (11) in \( V^+_s(\bar{V}) \) for fixed \( \bar{V} > 0 \). In light of Proposition 2.1, we consider the question whether an equilibrium can consist of \( n \) large droplets (i.e. droplets of height \( h > 1 \)) and \( N - n \) small droplets (i.e. droplets of height \( h^{-1} \leq 1 \)).

Noting the volume-height relation (1), we can thus cast mass conservation, given by Proposition 2.2, in the form

\[
\alpha v(h) + (1 - \frac{\alpha}{N}) v(h^{-1}) = \frac{n}{N} \left( h^{3} + 3h^{-1} \right) + \left( 1 - \frac{n}{N} \right) \left( h^{3} + 3h^{-1} \right) = \bar{V},
\]

(25)

This equation has a simple symmetry:

(26)

\[ h > 0 \text{ solves (25) with } \alpha = \beta \text{ if and only if } h^{-1} \text{ solves (25) with } \alpha = 1 - \beta. \]

It is convenient to rewrite Equation (25) as \( \mathcal{P}_d(h) = 0 \) where the mass polynomial \( \mathcal{P}_d = \mathcal{P}_d(h; \bar{V}) \) is given by

(27)

\[ \mathcal{P}_d(h) \equiv \alpha h^6 + 3 \alpha h^4 - 4 \bar{V} h^3 + 3(1 - \alpha) h^2 + 1 - \alpha. \]

Let us now address the uniform equilibrium.

**Proposition 4.1.** For every \( \bar{V} > 0 \), there is an equilibrium consisting of \( N \) droplets of equal height (and volume).

The uniform equilibrium arises for \( \bar{V} > 1 \) with \( n = N \) and for \( 0 < \bar{V} \leq 1 \) with \( n = 0 \).

Before we continue with our analysis, let us briefly adapt an approach taken by Slater and Steen [11] in their study of \( N \) inviscid spherical droplets with \( S_N \) symmetry. To this end, we note that for constant \( \bar{V} \), equilibria can be identified with each other if they share the same number of large droplets of the same height (or, equivalently, the same number of small droplets of the same height). Because of the symmetry expressed in (26) we may assume that \( \alpha = \frac{n}{N} \leq \frac{N/2}{N} \). Using ideas proposed in [13], we now introduce the quantity

(28)

\[ \theta(h) = \theta(h) \equiv v(h) - v\left( \frac{1}{h} \right), \]

defined for all positive \( h \) such that \( \mathcal{P}_d(h) = 0 \) for some \( \bar{V} > 0 \) and some \( \alpha = \frac{n}{N}, 1 \leq n \leq \lfloor N/2 \rfloor \). The function \( \theta \) is clearly invertible on its domain. The case \( \theta > 0 \) arises exactly when there exists \( h > 1 \) such
that $P_\alpha(h) = 0$. Hence in this situation we have an equilibrium with exactly $n = \alpha N$ large droplets of height $h > 1$. The negative case $\theta < 0$ corresponds to exactly $n = \alpha N$ small droplets of height $0 < h < 1$ as is apparent from the symmetry (26). $\theta = 0$ occurs exactly when $h = 1$. Hence the corresponding equilibrium is uniform and $V = 1$. It will become evident from (25) and (26) and the later developments that, in this way, $\theta$ is defined for all $\theta > 0$. To include all uniform equilibria, we extend the definition of $\theta$ by setting $\theta = 0$ if there exists $h > 0$ such that $P_\alpha(h) = 0$ for some $V > 0$. Since $P_\alpha(h) = -4V h^3 + 3h^2 + 1$, we have exactly one positive zero $h$ for every choice of $V$. The zero $h$ is in $(0, 1)$ if $V > 1$, it is 1 if $V = 1$, and it lies in $(1, \infty)$ if $0 < V < 1$. Therefore, as we appeal again to (26), we conclude that $\theta = 0$ corresponds either to the uniform equilibrium with $N$ large droplets of height $h^{-1}$ if $V > 1$, or to the uniform equilibrium with $N$ small droplets of height $h^{-1}$ if $0 < V \leq 1$. Hence after this extended definition of $\theta$, $\theta = 0$ represents an equilibrium in the $(V, \theta)$-plane for every $V > 0$. Having such a constant equilibrium for all parameter values is commonly assumed in bifurcation theory [6].

Now let $a = h + h^{-1}$ and $b = h - h^{-1}$. Then $a^2 - b^2 = 4$. Since (28) becomes

$$\theta = \frac{1}{4} b (b^2 + 6),$$

we have

$$b = R(\theta) \equiv \frac{-2^{2/3} + 2^{1/3} \left(\theta + \sqrt{2 + \theta^2}\right)^{2/3}}{\left(\theta + \sqrt{2 + \theta^2}\right)^{1/3}} \quad \text{and} \quad a = A(\theta) \equiv (4 + R^2(\theta)^2)^{1/2}.$$

Equation (25) can now be written in the form $V = \frac{1}{8} a^3 + \frac{3a - 1}{8} b (b^2 + 6)$. Hence we recover a single equation relating $\theta$ and $V$:

$$V = \frac{1}{8} A(\theta)^3 + \left(\alpha - \frac{1}{2}\right) \theta.$$

Equation (31) is - up to a rescaling - the same as in [11]. This might seem curious since the work [11] is concerned with inviscid spherical-cap droplets subject to $S_N$ symmetry. On second thought, equilibria in [11] are determined by the same volume constraint together with the Young-Laplace relation as here.
Figure 7 displays the generic situation: curves of equilibria with \( n \) large droplets determined by Equation (31) plus the uniform equilibrium curve \( \theta = 0 \) (labeled \( n = 0 \) for \( V \leq 1 \) and \( n = N \) for \( V > 1 \)).

Equation (31) allows an interesting first insight in the location of equilibria in dependence on \( V \). To obtain more detailed information about the location and nature of equilibria, we proceed by studying the zeros of the mass polynomial \( P_\alpha \) directly. As we will see, this approach will prove useful: Firstly, we gather details about the size of equilibria which have an immediate bearing on their stability. Secondly, we can easily identify the turning points of the equilibrium curves in Figure 7. Thirdly, our results lay the foundation of our study of hierarchies among the equilibria.

From here onwards it will be convenient to allow any real \( \alpha \) with \( 0 \leq \alpha \leq 1 \) in the definition of \( P_\alpha \), keeping in mind that \( \alpha = \frac{n}{N} \) describes the situation of \( n \) large and \( N - n \) small droplets. Moreover, to find non-uniform equilibria (i.e. equilibria consisting of both large and small droplets), it suffices to assume that \( 1 \leq n \leq N - 1 \) and to study the zeros of \( P_\alpha \) for \( 0 < \alpha < 1 \).

It is tacitly understood in all later developments that zeros are counted according to their multiplicity.

For later use let us introduce the limiting function

\[ L(\alpha) = \alpha^{3/4} (1 - \alpha)^{1/4} + \alpha^{1/4} (1 - \alpha)^{3/4}. \]

\( L \) is strictly increasing on \((0, \frac{1}{2})\) and strictly decreasing on \((\frac{1}{2}, 1)\). It has the maximum 1 at \( \alpha = \frac{1}{2} \). The function is graphed in Figure 8.

The key result on zeros of the mass polynomial \( P_\alpha \) is the following:

**Theorem 4.2.** Let \( V > 0 \). Then \( P_\alpha \), \( 0 < \alpha < 1 \), has either no positive zeros or exactly two. It has exactly two positive zeros \( h_1 = h_1(\alpha) \) and \( h_2 = h_2(\alpha) \) with \( h_1 \leq h_2 \) if and only if

\[ V \geq L(\alpha). \]

In this case, the zeros \( h_1 \) and \( h_2 \) are such that either

\[ h_1 = \left( \frac{1}{\alpha} - 1 \right)^{1/4} = h_2 \quad \text{if} \quad V = L(\alpha), \quad \text{or} \quad h_1 < \left( \frac{1}{\alpha} - 1 \right)^{1/4} < h_2 \quad \text{if} \quad V > L(\alpha). \]

**Proof.** Consider the polynomial \( q_\alpha \), defined by

\[ q_\alpha(h) = \alpha \left( h^4 + 2 \left( \frac{1}{\alpha} - 1 \right)^{1/4} h^3 + 3 \left( \frac{1}{\alpha} - 1 \right)^{1/2} h^2 + 2 \left( \frac{1}{\alpha} - 1 \right)^{1/4} h + \left( \frac{1}{\alpha} - 1 \right)^{1/2} \right). \]

Evidently, \( q_\alpha(h) > 0 \) for \( h > 0 \). Then for \( h > 0 \), we obtain that

\[ P_\alpha(h) \geq q_\alpha(h) \left( h - \left( \frac{1}{\alpha} - 1 \right)^{1/4} \right)^2 \quad \text{if} \quad V \geq L(\alpha), \]

respectively. The claim follows now immediately.

Condition (33) in Theorem 4.2 is both necessary and sufficient for positive zeros of the polynomial \( P_\alpha \).

This surprising result hinges on the factorizability of \( P_\alpha \), a polynomial of degree 6, when \( V = L(\alpha) \). The symmetry (26) hints at this result. Next we record some additional properties of the mass polynomial \( P_\alpha \).
PROPOSITION 4.3. For every $V > 0$ and $0 < \alpha < 1$,

$$\mathcal{P}_\alpha(h) \to \infty \text{ as } h \to \infty, \quad \mathcal{P}_\alpha(0) = 1 - \alpha, \quad \mathcal{P}_\alpha(1) = 4 \left(1 - \sqrt[4]{V}\right),$$

(37) \hspace{1cm} (38)

With this information we obtain:

THEOREM 4.4. Let $V > 1$. For every $0 < \alpha < 1$, $\mathcal{P}_\alpha$ has exactly one zero $l_1 = l_1(\alpha)$ larger than 1 and

$$l_1 > \left(\frac{1}{\alpha} - 1\right)^{1/4}.$$  

(39)

Note that condition (33) is vacuously satisfied. The existence and uniqueness of $l_1 > 1$ is a consequence of (37) when $V > 1$. The estimate (39) follows from (38) since for $V > 1$, $0 < \alpha < 1$, we have $\mathcal{P}_\alpha \left(\frac{1}{\alpha} - 1\right)^{1/4} < 0$. Of course, (39) is useful only if $0 < \alpha < \frac{1}{2}$.

THEOREM 4.5. Let $V = 1$.

(a) If $0 < \alpha < \frac{1}{2}$, $\mathcal{P}_\alpha$ has exactly one zero $l_1 = l_1(\alpha)$ larger than 1 and

$$l_1 > \left(\frac{1}{\alpha} - 1\right)^{1/4}.$$  

(40)

(b) If $\frac{1}{2} \leq \alpha < 1$, $\mathcal{P}_\alpha$ has no zero larger than 1.

Proof. Since $\mathcal{P}_\alpha(1) = 0$ and $\mathcal{P}_\alpha'(1) = 6(2\alpha - 1)$, it is clear in light of (37) that $\mathcal{P}_\alpha$ has a zero larger than 1 if and only if $0 < \alpha < \frac{1}{2}$. If so, this is the only zero larger than 1 since $\mathcal{P}_\alpha$ cannot have more than two positive zeros. The estimate (40) follows again from (38) since $\mathcal{P}_\alpha \left(\frac{1}{\alpha} - 1\right)^{1/4} < 0$ if $V = 1$ and $0 < \alpha < \frac{1}{2}$. \hfill $\Box$

THEOREM 4.6. Let $0 < \sqrt{V} < 1$.

(a) Suppose $0 < \alpha < \frac{1}{2}$ and $\sqrt{V} \geq L(\alpha)$. Then $\mathcal{P}_\alpha$ has exactly two zeros $l_1 = l_1(\alpha)$ and $l_2 = l_2(\alpha)$ and

$$1 < l_1 \leq \left(\frac{1}{\alpha} - 1\right)^{1/4} \leq l_2. \quad \text{The two zeros } l_1 \text{ and } l_2 \text{ are such that either}$$

$$l_1 = \left(\frac{1}{\alpha} - 1\right)^{1/4} = l_2 \quad \text{if } \sqrt{V} = L(\alpha), \text{ or}$$

$$l_1 < \left(\frac{1}{\alpha} - 1\right)^{1/4} < l_2 \quad \text{if } \sqrt{V} > L(\alpha).$$

(b) If $\frac{1}{2} \leq \alpha < 1$ or $\sqrt{V} < L(\alpha)$, $\mathcal{P}_\alpha$ has no zero larger than 1.

Proof. By Theorem 4.2, $\mathcal{P}_\alpha$, $0 < \alpha < 1$, has either no positive zeros or exactly two. The latter occurs if and only if (33) holds true. If so, since $\mathcal{P}_\alpha(0) > 0$, $\mathcal{P}_\alpha(1) > 0$ and $\mathcal{P}_\alpha \left(\frac{1}{\alpha} - 1\right)^{1/4} \leq 0$, Proposition 4.3 implies that either the number $\left(\frac{1}{\alpha} - 1\right)^{1/4}$ and the positive zeros of $\mathcal{P}_\alpha$ lie in the interval $(0, 1)$, or they all lie in the interval $(1, \infty)$. Now $\left(\frac{1}{\alpha} - 1\right)^{1/4}$ lies in $(1, \infty)$ if and only if $0 < \alpha < \frac{1}{2}$. The remaining claims follow from Theorem 4.2. \hfill $\Box$

The shaded area in Figure 8 displays permissible pairs $(\alpha, \sqrt{V})$ to which part (a) of Theorem 4.6 applies.

All possible non-uniform equilibria are now completely described by Theorems 4.4 to 4.6 when we note that each such equilibrium is – up to permutation – given by the number of its large droplets $n$ of height $h > 1$ and the number of its small droplets $N - n$ of height $h^{-1}$. We note, in particular, that, for $0 < \sqrt{V} \leq 1$, the condition $0 < \alpha < \frac{1}{2}$ translates to $N \geq 3$. Hence non-uniform equilibria with $0 < \sqrt{V} \leq 1$
require \( N \geq 3 \). For \( \overline{V} > 1 \), every equilibrium contains at least one large droplet. If \( N \geq 2 \) and \( 0 < \overline{V} < L(\frac{1}{N}) \), then only the uniform equilibrium consisting of \( N \) droplets of equal size is possible.

Finally, we end this section with some important remarks about the structure of the dynamical system, given by (11) and (12): In the case \( s \geq 1 \), Proposition 2.6 together with Proposition 2.2 shows that the initial value problem (11) and (12) defines a semiflow on \( \mathbb{V}(\overline{V}) \). By Proposition 2.1 and Lemma 2.5, this semiflow is a gradient dynamical system with Lyapunov function \( \mathcal{W} \), see [7]. When \( 0 < s < 1 \), the situation is more delicate. Here, Proposition 2.1 and Lemma 2.5 imply that the initial value problem (11) and (12) defines a generalized semiflow (in the sense of Ball) on \( \mathbb{V}(\overline{V}) \) with \( \mathcal{W} \) serving as Lyapunov function, see [1]. We can now conclude [1, 7]:

**Proposition 4.7.** The (generalized) semiflow on \( \mathbb{V}(\overline{V}) \), given by the initial value problem (11) and (12) with \( \mathbf{V}(0) \in \mathbb{V}(\overline{V}) \), has a global attractor which consists of all equilibria. Specifically, every semi-trajectory \( \mathbf{V} = \mathbf{V}(t) \) of the system (11) with \( \mathbf{V}(0) \in \mathbb{V}(\overline{V}) \) converges to an equilibrium of (11) in \( \mathbb{V}(\overline{V}) \) as \( t \to \infty \).

5. **Stability of Equilibria.** The stability of equilibria was discussed for the Newtonian case \( s = 1 \) with \( \overline{V} > 1 \) in [14]. The arguments there were based on the observation that the Lyapunov function \( \mathcal{W} \) is independent of the particular network of fluid channels. The authors exploited this observation by working with a network of their choice (star network) and combining local information about equilibria (obtained by linearization) with the Lyapunov Stability Theorem. Instead of pursuing an approach as in [14], we will give an argument which is independent of any particular network (except the number \( N \)), and is also independent of the power-law parameter \( s > 0 \) since the energy functional \( \mathcal{W} \) is as well. Our discussion contains the stability findings in [14] as a special case. At the core of our approach lies the following simple equivalence:

**Proposition 5.1.** A point \( \mathbf{V}^* \in \mathbb{V}(\overline{V}) \) is a stable equilibrium of the system (11) if and only if it is a local minimizer of the Lyapunov function \( \mathcal{W} \) on \( \mathbb{V}(\overline{V}) \).

Note that a local minimizer of \( \mathcal{W} \) is actually a strict local minimizer by Lemma 2.5. Also, in light of Proposition 3.2 we may replace \( \mathbb{V}(\overline{V}) \) by \( \mathbb{V}(\overline{V}) \) here.

Consider the Lagrangian \( \mathcal{L} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \), given for \( \mathbf{V} = (V_1, \ldots, V_N) \) by

\[
\mathcal{L}(\mathbf{V}, \lambda) = \mathcal{W}(\mathbf{V}) - \lambda F(\mathbf{V}) \quad \text{with} \quad F(\mathbf{V}) \equiv \frac{1}{N} \sum_{k=1}^N V_k - \overline{V}.
\]

If \( \mathcal{W} \) assumes a local minimum on \( \mathbb{V}(\overline{V}) \) at \( \mathbf{V}^* = (V_1^*, \ldots, V_N^*) \), then there exists \( \lambda^* \in \mathbb{R} \) such that \( (\mathbf{V}^*, \lambda^*) \) is a local minimizer of \( \mathcal{L} \) on \( \mathbb{R}^N \times \mathbb{R} \). For \( \mathcal{L} \) to have a local minimum on \( \mathbb{R}^N \times \mathbb{R} \) at \( (\mathbf{V}^*, \lambda^*) \), it is necessary that

\[
D_V \mathcal{L}(\mathbf{V}^*, \lambda^*) = 0, \quad F(\mathbf{V}^*) = 0, \quad \text{and} \quad \phi D^2_V \mathcal{L}(\mathbf{V}^*, \lambda^*) \phi^T \geq 0 \quad \text{for all} \ \phi \in \mathbb{R}^N \ 	ext{with} \ (DF(\mathbf{V}^*)) \phi^T = 0.
\]

Here, \( DF \) denotes the total derivative (row vector) of \( F \), while \( D_V \mathcal{L} \) and \( D^2_V \mathcal{L} \) denote the total derivative and the Hessian of \( \mathcal{L} \) with regard to \( V \), respectively. Clearly, in our situation we have \( D^2_V \mathcal{L} = D^2 \mathcal{W} \), the Hessian of \( \mathcal{W} \). Conversely, if \( (\mathbf{V}^*, \lambda^*) \in \mathbb{V}(\overline{V}) \times \mathbb{R} \) is such that the condition (44) holds and such that

\[
\phi D^2_V \mathcal{L}(\mathbf{V}^*, \lambda^*) \phi^T > 0 \quad \text{for all nonzero} \ \phi \in \mathbb{R}^N \ 	ext{with} \ (DF(\mathbf{V}^*)) \phi^T = 0,
\]

\( \mathcal{W} \) assumes a strict local minimum on \( \mathbb{V}(\overline{V}) \) at \( \mathbf{V}^* \). Conditions (44)–(46), are classical, see e.g. [9]. Equation (44) just states the requirement that \( \mathbf{V}^* \) be an equilibrium of (11) in \( \mathbb{V}(\overline{V}) \).

If \( \mathbf{V}^* = (V_1^*, \ldots, V_N^*) \in \mathbb{V}(\overline{V}) \) is an equilibrium of (11), \( \mathbf{V}^* \) consists of \( n \) entries for large droplets of volume \( V_L > 1 \) (and height \( h_L > 1 \)) and \( N - n \) entries for small droplets of corresponding volume \( 0 < V_S \leq \)}
1 (and height $h_S$). Note that $V_S = 1$ immediately requires $n = 0$ and $\bar{V} = 1$. Let us define

$$\delta_S \equiv \begin{cases} P'(V_S) & \text{if } 0 \leq n \leq N - 1, \\ 0 & \text{if } n = N \end{cases} \quad \text{and} \quad \delta_L \equiv \begin{cases} P'(V_L) & \text{if } 1 \leq n \leq N, \\ 0 & \text{if } n = 0. \end{cases}$$

Then the necessary condition (45) for a local minimizer can be cast in the form

$$\delta_L \left( \sum_{k=1}^n \phi_k^2 \right) + \delta_S \left( \sum_{k=n+1}^N \phi_k^2 \right) \geq 0 \quad \text{for all } (\phi_1, \ldots, \phi_N) \text{ with } \sum_{j=1}^N \phi_j = 0.$$  

The sufficient condition (46) for a strict local minimizer reduces to

$$\delta_L \left( \sum_{k=1}^n \phi_k^2 \right) + \delta_S \left( \sum_{k=n+1}^N \phi_k^2 \right) > 0 \quad \text{for all nonzero } (\phi_1, \ldots, \phi_N) \text{ with } \sum_{j=1}^N \phi_j = 0.$$  

Next we have for $V = v(h)$,

$$P'(V) = \frac{16 (1 - h^2)}{3 (h^2 + 1)^3}.$$  

Hence $\delta_L < 0$ if $1 \leq n \leq N;$ $\delta_S > 0$ if $1 \leq n \leq N - 1,$ or if $n = 0$ and $0 < V < 1$; and

$$\delta_S = -h^4_L \delta_L \quad \text{if } 1 \leq n \leq N - 1.$$  

Consequently, we have immediately from (48) that $V^*$ is unstable if there exist $i, j$, $1 \leq i < j \leq N$ such that $V^*_i = V^*_j = V_L > 1$. Hence we have shown:

**Theorem 5.2.** An equilibrium of the system (11) in $\mathcal{V}(\bar{V})$ is unstable if it contains two or more large droplets.

For $\bar{V} > 1$, any equilibrium $V^*$ in $\mathcal{V}(\bar{V})$ must contain at least one entry $V^*_i > 1$. Moreover, since $W$ assumes a global minimum on $\mathcal{V}(\bar{V})$, the minimizer must be an equilibrium of the system (11) by (44). Hence we conclude:

**Corollary 5.3.** An equilibrium of the system (11) in $\mathcal{V}(\bar{V})$ with $\bar{V} > 1$ is stable if and only if it contains exactly one large droplet.

Theorem 5.2 is reminiscent of related results in [18] for two and three spherical-cap droplets and in [11] for $N$ coupled inviscid droplet oscillators with $S_N$ symmetry, while Corollary 5.3 was proved in [14] for $s = 1$, exploiting the hyperbolicity of equilibria in the Newtonian regime.

Let us now focus on the case $n = 1$. Since for $(\phi_1, \ldots, \phi_N)$ with $\sum_{j=1}^N \phi_j = 0$

$$\phi_1^2 = \left( \sum_{j=2}^N \phi_j \right)^2 \leq (N - 1) \sum_{j=2}^N \phi_j^2,$$

we can use (49) and (51) to obtain the sufficient condition for a strict local minimizer

$$\delta_L \left( N - 1 - h^4_L \right) > 0.$$  

**Theorem 5.4.** An equilibrium of the system (11) in $\mathcal{V}(\bar{V})$ is stable if it contains exactly one large droplet of height $h > (N - 1)^{1/4}$. 
It follows from Theorems 4.5 and 4.6 with $\alpha = \frac{1}{V}$ that such equilibria arise for $V = 1$ if $N \geq 3$, and for $0 < V < 1$ if $V > L\left(\frac{1}{N}\right)$. For $V > 1$ this result is already contained in Corollary 5.3 by Theorem 4.4.

Next, in the case $n = 1$, let us choose $\phi_1 = 1$ and $\phi_j = -(N-1)^{-1}$, $2 \leq j \leq N$. Then (48) reduces to the following necessary condition for a local minimizer

\begin{equation}
\delta_L \left(N - 1 - h_L^2\right) \geq 0.
\end{equation}

Hence an equilibrium is unstable if it contains a large droplet of height less than $(N-1)^{1/4}$. Theorem 4.6 shows that such equilibria with $n = 1$ occur for $0 < V < 1$ if $V > L\left(\frac{1}{N}\right)$. There the height of the large droplet for this unstable equilibrium was denoted by $h_L(\alpha)$ with $\alpha = \frac{1}{N}$.

The borderline case $n = 1$ and $h_L = (N-1)^{1/4}$ with $0 < V < 1$ is left to be discussed. By Theorem 4.6, this situation arises when $V = L\left(\frac{1}{N}\right)$ with $N \geq 3$. To obtain a stability result in this case, let us argue directly. First

\begin{equation}
P''(V) = \frac{256}{9} \frac{h(h^2 - 2)}{(h^2 + 1)^3} \text{ where } V = \nu(h).
\end{equation}

Next let $\phi = \phi(t) = (\phi_1(t), \ldots, \phi_N(t))$ be a smooth curve. Then

\begin{equation}
\frac{d^2}{dt^2} \mathcal{W}(\phi) = \frac{1}{N} \sum_{k=1}^{N} \left( P'(\phi_k) \left( \phi_k' \right)^2 + P(\phi_k) \phi_k'' \right), \quad \text{and}
\end{equation}

\begin{equation}
\frac{d^3}{dt^3} \mathcal{W}(\phi) = \frac{1}{N} \sum_{k=1}^{N} \left( P''(\phi_k) \left( \phi_k' \right)^3 + 3 P'(\phi_k) \phi_k' \phi_k'' + P(\phi_k) \phi_k''' \right).
\end{equation}

Let $V_L = \nu(h_L)$ and $V_S = \nu(h_S^{-1})$ with $h_L = (N-1)^{1/4}$. For fixed $\phi_0 \in \mathbb{R}$, we set

\begin{equation}
\phi_k(t) = \begin{cases} V_L + (N-1) \phi_0 t & \text{if } k = 1, \\ V_S - \phi_0 t & \text{otherwise}. \end{cases}
\end{equation}

Then $\phi(t) = (\phi_1(t), \ldots, \phi_N(t))$ is a smooth curve with trace in $\mathcal{V}(V)$. Moreover,

\begin{equation}
\frac{d^2}{dt^2} \mathcal{W}(\nu(t)) \bigg|_{t=0} = 0, \quad \text{and}
\end{equation}

\begin{equation}
\frac{d^3}{dt^3} \mathcal{W}(\nu(t)) \bigg|_{t=0} = \frac{256}{9} \frac{(N-1)^{3/4} - 1}{(N-1)^{7/4}} rac{N-1}{N} \phi_0^3.
\end{equation}

Here we have made use of (55). Since $V^* = \phi(0)$ is an equilibrium of the desired form with $\phi_0$ arbitrary, $\mathcal{W}$ does not assume a local minimum at $V^*$. Hence we have found:

**Theorem 5.5.** An equilibrium of the system (11) in $\mathcal{V}(V)$ is unstable if it contains a large droplet of height $1 < h \leq (N-1)^{1/4}$.

Finally, let us turn to the case $n = 0$. The corresponding equilibria are uniform of height $0 < h_S \leq 1$. Hence we have $0 < V \leq 1$. Equation (49) immediately gives the sufficient condition for a strict local minimizer in the form

\begin{equation}
\delta_S > 0.
\end{equation}

Clearly, this condition holds true for the uniform equilibrium when $0 < V < 1$. This condition fails, however, for $V = 1$ since then $\delta_S = 0$. To obtain a stability result for $n = 0$, $V = 1$, we proceed differently.
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First, if \( N = 2 \), \( \mathbf{V}^* = (1, 1) \) is the only possible equilibrium of (11) by Proposition 4.1 and Theorem 4.5. Hence the only choice for a global minimizer of \( \mathcal{W} \) on \( \mathbb{V}(\mathbb{V}) \) is \( \mathbf{V}^* \) which must be stable.

Next assume \( N \geq 3 \). Let \( \mathbf{V}^* = (1, \ldots, 1) \) and consider \( \phi = \phi(t) = \mathbf{V}^* + t (\phi_1, \ldots, \phi_N) \) where \( \phi_1, \phi_2 \in \mathbb{R} \) are fixed, \( \phi_3 = -\left( \phi_1 + \phi_2 \right) \) and \( \phi_k = 0 \) for \( 3 < k \leq N \). Since \( \sum_{k=1}^{N} \phi_k = 0, \phi(t) \) is a curve with trace in \( \mathbb{V}(1) \). From (56) and (57) we obtain

\[
\left. \frac{d^2}{dt^2} \mathcal{W} (\phi(t)) \right|_{t=0} = 0, \quad \text{and} \quad \left. \frac{d^3}{dt^3} \mathcal{W} (\phi(t)) \right|_{t=0} = \frac{1}{N} p''(1) \sum_{k=1}^{N} \phi_k^3 = -\frac{3}{N} p''(1) \phi_1 \phi_2 \left( \phi_1 + \phi_2 \right).
\]

Since \( p''(1) \neq 0 \) by (55) and since the right-hand side of Equation (64) can take positive and negative values for appropriate choices of \( \phi_1 \) and \( \phi_2 \), \( \mathbf{V}^* \) is not a local minimizer of \( \mathcal{W} \) on \( \mathbb{V}(1) \). For \( \mathbb{V} > 1 \) the uniform equilibrium in \( \mathbb{V}(\mathbb{V}) \) is always unstable by Theorem 5.2. Hence we can summarize:

**Theorem 5.6.** The uniform equilibrium of the system (11) in \( \mathbb{V}(\mathbb{V}) \) is stable if and only if either \( 0 < \mathbb{V} < 1 \) and \( N \geq 3 \), or \( 0 < \mathbb{V} \leq 1 \) and \( N = 2 \).

This result implies that for \( N = 2 \) and \( s > 0 \), the initial value problem (11) and (12) with initial data \( V_1(0) = 1 = V_2(0) \) has the unique solution \( V_1(t) = 1 = V_2(t), t \geq 0 \). Similarly, if \( 0 < \mathbb{V} < 1, N \geq 2 \) and \( s > 0 \), the uniform equilibrium is the unique solution of (11) and (12) with initial data \( V_i(0) = \mathbb{V}, 1 \leq i \leq N \). These findings for the case \( N = 2 \) can be confirmed independently by direct arguments.

We have now determined the stability of all equilibria in \( \mathbb{V}(\mathbb{V}) \) and, by Proposition 3.2, in \( \mathbb{V}(\mathbb{V}) \) for every \( \mathbb{V} > 0 \). The stability results are new for the case \( s \neq 1 \). They are also new for the case \( 0 < \mathbb{V} \leq 1 \) when \( s = 1 \).

6. Energy Hierarchies of Equilibria. In the following our objective is to order equilibria by size of the energy functional \( \mathcal{W} \). Let us first motivate the problem by pursuing an approach based on the variable \( \theta \), introduced in (28): If \( \mathbf{V} \) is an equilibrium of (11) consisting of \( n \) droplets of height \( h > 0 \) and \( N - n \) droplets of height \( h^{-1} \), \( 1 \leq n \leq \lfloor \frac{N}{2} \rfloor \), then by (18) and (19) with \( \alpha = \frac{n}{N} \), we obtain

\[
\mathcal{W}(\mathbf{V}) = \frac{3}{2} \left( \alpha h^2 + (1 - \alpha) \frac{1}{h^2} \right).
\]

Using again \( \alpha = h + \frac{1}{2} \) and \( b = h - \frac{1}{4} \), we find \( \mathcal{W}(\mathbf{V}) = \frac{3}{2} (b + \frac{1}{2}) b^2 + \frac{1}{16} (\alpha - \frac{1}{2}) a b \). Consequently, by (30), we have a function in \( \theta \)

\[
w(\theta) = \frac{3}{2} + \frac{3}{4} \mathcal{B}(\theta)^2 + \frac{3}{2} \left( \alpha - \frac{1}{2} \right) \mathcal{A}(\theta) \mathcal{B}(\theta)
\]

such that \( \mathcal{W}(\mathbf{V}) = w(\theta) \) with \( h = \frac{1}{2} (\mathcal{A}(\theta) + \mathcal{B}(\theta)) \). This reduced energy functional is displayed in Figure 9.

Let us consider an example. We choose \( \mathbb{V} = 1.2 \) and determine \( \theta \) and \( w(\theta) \) from Equations (31) and (65) for \( \alpha \in \{ \frac{1}{16}, \frac{1}{8}, \frac{1}{4} \} \). Noting that Equation (31) has both a positive \((\oplus)\) and negative \((\ominus)\) solution for \( \theta \), we display the results in the following tables:

| \( \Theta \) | \( \alpha = \frac{1}{16} \) | \( \alpha = \frac{1}{8} \) | \( \alpha = \frac{1}{4} \) | \( \Theta \) | \( \alpha = \frac{1}{16} \) | \( \alpha = \frac{1}{8} \) | \( \alpha = \frac{1}{4} \) |
|---|---|---|---|---|---|---|---|
| \( w(\theta) \) | 1.427 | 1.646 | 1.814 | \( w(\theta) \) | 1.899 | 1.898 | 1.897 |

For \( \theta > 0 \) we have exactly \( n = \alpha N \) large droplets, while for \( \theta < 0 \) there are exactly \( n = (1 - \alpha) N \) large droplets. Hence the example above suggests (choosing \( N = 16 \) for instance) that for the same average
volume $\bar{V}$, the larger the energy functional $\mathcal{W}$ (or $\omega$), the more large droplets are contained in an equilibrium.

While this claim appears intuitive and obvious, it is neither: For simplicity take $\bar{V} > 1$ and compare two different equilibria. One has $n_1$ large droplets, the other one $n_2$. If $n_1 < n_2$, each of the $n_1$ large droplets in the first equilibrium is expected to have larger volume (hence larger surface area) than each of the $n_2$ large droplets in the second equilibrium. Yet there are only $n_1$ large droplets in the first equilibrium, while there are $n_2 > n_1$ in the second. A similar picture arises for the small droplets. As the table with negative ($\theta$) solutions for $\theta$ illustrates, the actual differences in the energy functional (or surface area) can be tiny.

We will prove below that the claim above is indeed correct if $\bar{V} > 1$. For $0 < \bar{V} < 1$, the situation is, however, more subtle since for constant $\alpha$, Equation (31) has two distinct positive solutions for $\theta$ and no negative ones as seen in Figure 7.

To obtain more precise information about the ordering of equilibria in terms of $\mathcal{W}$, we put the approach above aside and instead examine the zeros of the mass polynomial $\mathcal{P}_\alpha$, given by (27), in more detail. For $0 < \bar{V} \leq 1$, we define $\alpha^*(\bar{V})$ to be the unique value of $\alpha \in (0, \frac{1}{2})$ such that $\bar{V} = L(\alpha)$. Now consider the map $\alpha \mapsto h_\alpha(\alpha)$, defined on $(0, 1)$ for $\bar{V} > 1$ by Theorem 4.4 and on $\left(0, \alpha^*(\bar{V})\right)$ for $0 < \bar{V} \leq 1$ by Theorems 4.5 and 4.6 with $h_\alpha(\alpha) > \left(\frac{1}{\alpha} - 1\right)^{1/4}$. In each case, $h_\alpha(\alpha)$ is a simple root of the equation $\mathcal{P}_\alpha(h) = 0$. Hence it is elementary to conclude that $\alpha \mapsto h_\alpha(\alpha)$ is a smooth map and $\frac{d}{d\alpha} h_\alpha(\alpha) \neq 0$.

Since $h_\alpha(\alpha) > \left(\frac{1}{\alpha} - 1\right)^{1/4}$, we have $h_\alpha(\alpha) \to \infty$ as $\alpha \to 0^+$. Consequently, $\frac{d}{d\alpha} h_\alpha(\alpha) < 0$. Since $h_\alpha(\alpha)$ is also bounded from below as $\alpha$ approaches the right endpoint of its interval of definition, we obtain:

**PROPOSITION 6.1.** Let $\alpha_0 = 1$ if $\bar{V} > 1$, and $\alpha_0 = \alpha^*(\bar{V})$ if $0 < \bar{V} \leq 1$. Then the map $\alpha \mapsto h_\alpha(\alpha)$ is smooth and strictly decreasing on $(0, \alpha_0)$. Moreover, it extends continuously to $(0, \alpha_0]$ such that

(a) in case $\bar{V} > 1$, $h_\alpha(\alpha_0) > 1$ is the unique positive root of $\mathcal{P}_\alpha(h) = 0$,

(b) in case $\bar{V} = 1$, $h_\alpha(\alpha_0) = 1$, and

(c) in case $0 < \bar{V} < 1$, $h_\alpha(\alpha_0) = \left(\frac{1}{\alpha_0} - 1\right)^{1/4} > 1$.

Next let us investigate the map $\alpha \mapsto h_\alpha(\alpha)$, defined on $\left(0, \alpha^*(\bar{V})\right)$ for $0 < \bar{V} < 1$ by Theorem 4.6 with $1 < h_\alpha(\alpha) < \left(\frac{1}{\alpha} - 1\right)^{1/4}$. Again, $h_\alpha(\alpha)$ is a simple root of the equation $\mathcal{P}_\alpha(h) = 0$, and thus $\alpha \mapsto h_\alpha(\alpha)$ is a smooth map and $\frac{d}{d\alpha} h_\alpha(\alpha) \neq 0$. Now let $h_\alpha(0)$ be the unique positive root of the equation $\mathcal{P}_\alpha(h) = -4\bar{V} h^3 + 3h^2 + 1 = 0$. It follows readily that this definition extends $h\alpha$ to a smooth function on $\left(0, \alpha^*(\bar{V})\right)$. Since $\mathcal{P}_\alpha(1) = -4\bar{V} + 4 > 0$, $\mathcal{P}_\alpha\left(\frac{1}{2\bar{V}}\right) = \frac{1}{4\bar{V}^2} + 1$, and $\lim_{\alpha \to \infty} \mathcal{P}_\alpha(h) = -\infty$, we obtain

$$h_\alpha(0) > \max\left\{1, \frac{1}{2\bar{V}}\right\}.$$ 

This estimate together with the equation $\frac{d}{d\alpha} (\mathcal{P}_\alpha(h_\alpha(\alpha))) = 0$ implies that

$$h_\alpha(0) = \frac{h_\alpha(0)^6 + 3h_\alpha(0)^4 - 3h_\alpha(0)^2 - 1}{6h_\alpha(0)^2 \left[2\bar{V} h_\alpha(0) - 1\right]} > 0.$$ 

Hence by continuity, $\frac{d}{d\alpha} h_\alpha(\alpha) > 0$ for all $\alpha$. Since $h_\alpha$ is increasing and $h_\alpha(\alpha) < \left(\frac{1}{\alpha} - 1\right)^{1/4}$ on $\left(0, \alpha^*(\bar{V})\right)$, the limit $\lim_{\alpha \to \alpha^*(\bar{V})} h_\alpha(\alpha)$ exists. In fact, we obtain:

**PROPOSITION 6.2.** Let $0 < \bar{V} < 1$. Then the map $\alpha \mapsto h_\alpha(\alpha)$ is smooth and strictly increasing on $\left(0, \alpha^*(\bar{V})\right)$. It extends continuously to $\left(0, \alpha^*(\bar{V})\right)$ such that $h_\alpha(0) > 1$ is the unique positive root of $\mathcal{P}_\alpha(h) = 0$ and $h_\alpha\left(\alpha^*(\bar{V})\right) = \left(\frac{1}{\alpha^*(\bar{V})} - 1\right)^{1/4}$. 


Now we define the function $\kappa = \kappa(h)$ for $h > 1$ by

$$\kappa(h) = \frac{4 \sqrt{V} h^3 - 3 h^2 - 1}{h^6 + 3 h^4 - 3 h^2 - 1}.$$  

If $V = 1$, this definition extends smoothly to all $h \geq 1$ with $\kappa(1) = \frac{2}{3}$. The function $\kappa$ is trivially related to the polynomial $\mathcal{P}_a$, given by (27), in the following way:

**Proposition 6.3.** Suppose $h_0 > 1$ is a root of the equation $\mathcal{P}_a(h) = 0$ for some $0 \leq a \leq 1$. Then $\kappa(h_0) = a$.

Finally, we introduce the functional $\mathcal{S} = \mathcal{S}(h)$ for $h > 1$ by

$$\mathcal{S}(h) = \frac{3}{2} \left( h^2 + 1 - \kappa(h) \right) h$$

and

$$\frac{d\mathcal{S}}{dh}(h) = -6 \left( \frac{h^2 - 1}{h^4 + 4 h^2 + 1} \right)^2.$$  

A remark is in order: For integers $N \geq 2$ and $0 \leq n \leq N$, let $h_0 > 1$ be a root of the equation $\mathcal{P}_a(h) = 0$ with $a = \frac{n}{N}$. Then $\mathcal{S}(h_0) = \mathcal{W}(\nu_0)$ where $\nu_0$ is any equilibrium consisting of exactly $n = \alpha N$ large droplets of height $h_0$. Hence $\mathcal{S}$ interpolates the values of $\mathcal{W}$ along the equilibria.

**Theorem 6.4.** Let $\alpha_0 = 1$ if $V > 1$, and $\alpha_0 = \alpha^{\star}(V)$ if $0 < V \leq 1$. Then the maps $\alpha \mapsto \mathcal{S}(h_1(\alpha))$ and $\alpha \mapsto \mathcal{S}(h_1(\alpha))$ are strictly increasing on $(0, \alpha_0)$.

**Proof.** Let $h = h_1$ or $h = h_1$ on $(0, \alpha_0)$. Since $h(\alpha) > 1$ and $\kappa(h(\alpha)) = \alpha > 0$ on $(0, \alpha_0)$, the definition (68) of $\kappa$ shows that $4 \sqrt{V} h(\alpha)^3 - 3 h(\alpha)^2 - 1 > 0$. Consequently, by Propositions 6.1 and 6.2, we have

$$\frac{d}{d\alpha} \left( \frac{4 \sqrt{V} h(\alpha)^3 - 3 h(\alpha)^2 - 1}{h^6 + 3 h^4 - 3 h^2 - 1} \right) h < 0$$

if $h = h_1$, and

$$\frac{d}{d\alpha} \left( \frac{4 \sqrt{V} h(\alpha)^3 - 3 h(\alpha)^2 - 1}{h^6 + 3 h^4 - 3 h^2 - 1} \right) h > 0$$

if $h = h_1$. Hence $\mathcal{S}(\alpha) = \mathcal{W}(h(\alpha))$ is strictly decreasing on $(0, \alpha_0)$ if $h = h_1$, and strictly increasing on $(0, \alpha_0)$ if $h = h_1$. In the first case, we obtain from Proposition 6.1 that $\lim_{\alpha \to \alpha_0^+} \mathcal{S}(\alpha)$ exists and is non-negative. Thus $\mathcal{S}(\alpha) > 0$ on $(0, \alpha_0)$. In the second case, Proposition 6.2 proves that $\lim_{\alpha \to \alpha_0^+} \mathcal{S}(\alpha)$ exists and equals 0. Hence $\mathcal{S}(\alpha) < 0$ on $(0, \alpha_0)$. Finally, the claim follows since $h(\alpha) > 1$ on $(0, \alpha_0)$ and

$$\frac{d}{d\alpha} \left( \mathcal{S}(h(\alpha)) \right) = -6 \left( \frac{h(\alpha)^2 - 1}{h(\alpha)^4 + 4 h(\alpha)^2 + 1} \right)^2 \frac{d}{dh}(h(\alpha)).$$

Now, for fixed $N$, we let $N^\star(V)$ be the largest integer $K$ such that $K \leq a^{\star}(V) N$ if $0 < V < 1$. We also define $N^\star(1)$ to be the largest integer $K$ such that $K < \frac{1}{2} N$. Note that for $0 < V < 1$, the condition $N^\star(V) \geq 1$ is equivalent to $L \left( \frac{1}{N^2} \right) \leq V < 1$. Now we set:

(i) If $V > 1$, let $\beta_N = \mathcal{W}(\nu_n)$, $1 \leq n \leq N$, where $\mathcal{W}$ is an equilibrium of the system (11) with exactly $n$ large droplets.

(ii) If $V = 1$, let $\gamma_N = \mathcal{W}(\nu_n)$, $0 \leq n \leq N^\star(1)$, where $\mathcal{W}$ is an equilibrium of the system (11) with exactly $n$ large droplets.

(iii) If $0 < V < 1$ and $N^\star(1) \geq 1$, let $\lambda_N = \mathcal{W}(\nu_n)$, $1 \leq n \leq N^\star(1)$, where $\mathcal{W}$ is an equilibrium of the system (11) with exactly $n$ large droplets of height $h \geq \left( \frac{N}{n} - \frac{1}{4} \right)^{\frac{1}{4}}$.

(iv) If $0 < V < 1$, let $\sigma_N = \mathcal{W}(\nu_n)$, $0 \leq n \leq N^\star(1)$, where $\mathcal{W}$ is an equilibrium of the system (11) with exactly $n$ large droplets of height $h \leq \left( \frac{N}{n} - 1 \right)^{\frac{1}{4}}$.

Observe, in particular, that $\beta_N$, $\gamma_0$ and $\sigma_0$ denote the value of the functional $\mathcal{W}$ at the uniform equilibria when $V > 1$, $V = 1$, and $0 < V < 1$, respectively.
Theorem 6.5.
(a) If \( \nabla V > 1 \), then \( \beta_n < \beta_{n+1} \), \( 1 \leq n \leq N-1 \).
(b) If \( \nabla V = 1 \), then \( \gamma_n < \gamma_{n+1} \), \( 1 \leq n \leq N^*(1) - 1 \), and \( \gamma_n < \gamma_0 \), \( 1 \leq n \leq N^*(1) \).
(c) If \( 0 < \nabla V < 1 \), then \( \lambda_n < \lambda_{n+1} \), \( 1 \leq n \leq N^*(\nabla) - 1 \), and \( \sigma_n < \sigma_{n+1} \), \( 0 \leq n \leq N^*(\nabla) - 1 \).

Proof. Let \( a_0 \) be given as in Theorem 6.4 and let \( h = h_L \) or \( h = h_I \) on \( (0, a_0) \). Then with \( V_L = \nu(h(a)) \) and \( V_S = \nu(1/h(a)) \), \( \mathcal{A}(h(a)) = \alpha \int_0^V P(V) dV + (1 - \alpha) \int_{V_S}^V P(V) dV \). Hence setting \( \alpha = \frac{n}{N} \), we obtain

\[
\mathcal{A} \left( h \left( \frac{n}{N} \right) \right) = \begin{cases} 
\beta_n & \text{if } \nabla V > 1, h = h_L, \text{ and } 1 \leq n \leq N-1, \\
\gamma_n & \text{if } \nabla V = 1, h = h_I, \text{ and } 1 \leq n \leq N^*(1), \\
\lambda_n & \text{if } 0 < \nabla V < 1, h = h_L, \text{ and } 1 \leq n \leq N^*(\nabla) - 1, \\
\sigma_n & \text{if } 0 < \nabla V < 1, h = h_I, \text{ and } 1 \leq n \leq N^*(\nabla) - 1. 
\end{cases}
\]

If either \( \nabla V > 1 \) or \( 0 < \nabla V < 1 \), \( h_L \) extends to a continuous function on \( (0, a_0) \) such that \( h_L(a) > 1 \) for \( 0 < a \leq a_0 \). Thus \( \mathcal{A}(h_L(1)) = \beta_N \) if \( \nabla V > 1 \), and \( \mathcal{A}(h_L(1)) = \lambda_{N^*(\nabla)} \) if \( 0 < \nabla V < 1 \), provided that \( N^*(\nabla) \geq 1 \).

If \( \nabla V = 1 \), \( h_I \) extends to a continuous function on \( (0, 1/2) \) such that \( h_I(a) > 1 \) for \( 0 < a < 1/2 \) and \( h_I(1/2) = 1 \). Hence, by (70), the map \( \alpha \rightarrow \mathcal{A}(h_L(a)) \) extends continuously to \( (0, 1/2) \) such that

\[
\mathcal{A} \left( h_L \left( \frac{1}{2} \right) \right) = \mathcal{A}(1) = 2 \int_0^1 P(V) dV + \left( 1 - \frac{2}{3} \right) \int_0^1 P(V) dV = \int_0^1 P(V) dV = \gamma_0.
\]

Finally, if \( 0 < \nabla V < 1 \), \( h_I \) extends to a continuous function on \( (0, a_0) \) such that \( h_I(a) > 1 \) for \( 0 \leq a \leq a_0 \). Hence \( \mathcal{A}(h_I(0)) = \sigma_0 \) and \( \mathcal{A}(h_I(1)) = \sigma_{N^*(\nabla)} \). The ordering of the quantities \( \beta_n, \gamma_n, \lambda_n \) and \( \sigma_n \) follows now immediately from the strict monotonicity of the maps \( \alpha \rightarrow \mathcal{A}(h_L(a)) \) and \( \alpha \rightarrow \mathcal{A}(h_I(a)) \).

In the case \( \nabla V < 1 \) with \( N^*(\nabla) \geq 1 \), the system (11) potentially exhibits two types of stable equilibria: the uniform equilibrium with droplets of height \( h_0 = h_I(0) \) and equilibria with exactly one large droplet of height \( h_N = h_L(1/2) \). It is natural to ask for which stable equilibrium the functional \( \mathcal{W} \) is minimal. We can easily give a partial answer: Since \( k(h) \sim h^{-3} \) as \( h \rightarrow \infty \) and \( h_N = h_L(1/2) \rightarrow \infty \) as \( N \rightarrow \infty \), we obtain from (70) that \( \mathcal{A}(h_N) \rightarrow 0 \) as \( N \rightarrow \infty \), while \( \mathcal{A}(h_0) = \frac{3}{2} h_0^{-2} > 0 \). Hence for sufficiently large \( N \) (with constant \( 0 < \nabla V < 1 \)), the stable equilibria with one large droplet are the global minimizers of \( \mathcal{W} \). If, on the other hand, \( \nabla V = L \left( \frac{1}{N} \right) \), then \( \alpha^*(\nabla) = \frac{1}{N} \) and, by Theorem 4.6, \( h_1 \left( \frac{1}{N} \right) = h_L(1/2) \). By Theorem 5.5, the corresponding equilibrium with exactly one large droplet are unstable. Hence they cannot be minimizers of \( \mathcal{W} \). Indeed, since \( \sigma_0 < \sigma_1 = \lambda_1 \) by Theorem 6.5, \( \mathcal{W} \) takes its global minimum value at the uniform equilibrium. Clearly, there exists \( \epsilon > 0 \) (depending on \( N \)) such that this conclusion remains valid if \( L \left( \frac{1}{N} \right) \leq \nabla V < L \left( \frac{1}{N} \right) + \epsilon \).

The possible occurrence of two types of stable equilibria (“bistability”) for \( \nabla V < 1 \) is an interesting observation which lends support to the switching mechanism of the adhesion device presented by Vogel and Steen in [16]. The device can switch between two stable states: attached and detached. We also note that in the case of bistability droplets can evolve towards the uniform equilibrium. If so, the uniform equilibrium is a local minimizer of the energy functional \( \mathcal{W} \), and yet no coarsening occurs.

7. Concluding Remarks. In the preceding sections we concentrated on the occurrence of equilibria, their stability and their ordering for fixed \( \nabla V > 0 \). It is now instructive to shed light on the equilibria and their stability when \( \nabla V \) varies. We make use of the equilibrium curves obtained via Equation (31) and identify equilibria as stable or unstable. We classify as follows:

- \( L^N, S^N \): uniform equilibria, consisting of \( N \) large (L) or \( N \) small (S) droplets, respectively
- \( L^k S^m \): non-uniform equilibria, consisting of \( k \) large (L) droplets of height \( h > (\frac{2}{N} - 1)^{1/4} \) and \( m = N - k \) small (S) droplets
• \( l^k s^m \): non-uniform equilibria, consisting of \( k \) large (\( l \)) droplets of height \( h \leq (\frac{N}{k} - 1)^{1/4} \) and \( m = N - k \) small (\( s \)) droplets.

Figure 10 displays the situation \( N = 2, 3 \). A generic bifurcation diagram (for odd \( N \)) is shown in Figure 11: Equilibria of the type \( L^k S^m \) and \( l^k s^m \) are both present on equilibrium curves with turning point in the half-plane \( V < 1 \). Such equilibria lie immediately above and below the turning point, respectively. The turning point itself occurs for \( \bar{V} = L \left( \frac{1}{N} \right) \). Up to a rescaling, the same value of \( V \) was given by Slater and Steen [11] in the case \( n = 1 \), confirmed by our general result here. At this turning point a saddle-node
bifurcation occurs with equilibria of the type \( L^1 \mathbb{S}^{N-1} \) stable and all other equilibria on the same solution curve unstable. Our results on the limiting function \( L \) furnish the location of all turning points with \( n \in \{1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor \}. \) Equilibria of the type \( L^k \mathbb{S}^m \) extend all the way to the point \( (\mathbb{V}, \theta) = (1, 0) \) and then switch to the type \( L^m \mathbb{S}^k \). No equilibria of the type \( L^k \mathbb{S}^m \) are present on the equilibrium curve with \( \alpha = \frac{1}{2} \), i.e. for \( N \) even and \( n = \frac{N}{2} \). This curve consists of two solution branches symmetric about the axis \( \theta = 0 \). In the case \( N = 2 \) this situation corresponds to a supercritical pitchfork bifurcation as indicated in Figure 10a. Our results are in agreement with Wente’s work for \( N = 2 \) and 3 [18]. The bistability range \( L \left( \frac{N}{2} \right) < \mathbb{V} < 1 \), i.e. the interval of \( \mathbb{V} \)-values where two different types of stable equilibria can occur, is indicated in Figures 10b and 11.

When \( \mathbb{V} > 1 \), the basins of attraction of stable equilibria are sensitive to changes in the initial data (and model parameters) as seen in Section 2. A similar behavior is expected when \( \mathbb{V} < 1 \). We leave it to future work to study the basins of attraction, especially for \( \mathbb{V} \) in the bistability range.

We have now given a complete characterization of all equilibria and their stability for volume scavenging of Newtonian and power-law fluids. While our stability and bifurcation results bear resemblance with the case of \( N \) inviscid droplets with \( S_N \) symmetry [11], there are notable differences: The inviscid flow in [11] exhibits periodic and quasi-periodic solutions as well as chaotic behavior. In the viscous situation discussed here, the fluid rheology (for any \( s > 0 \)) renders stable equilibria asymptotically stable and restricts \( \omega \)-limit sets to a finite number of equilibria. This viscous behavior for \( \mathbb{V} < 1 \) allows us to rationalize the mechanism for the adhesion device discussed in [16]. It is, however, remarkable that, in turn, our results on the ordering of equilibria in terms of the pressure-volume work functional \( W \) (and hence total surface area) apply to the inviscid case as well. Our description of the turning points of the equilibrium curves as special values of the limiting function \( L \) carry over directly. Most of these findings are new, both for the viscous and inviscid case.

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