GAUGE CONDITIONS FOR THE
CONSTRANED-WZNW–TODA REDUCTIONS

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Abstract

There is a constrained-WZNW–Toda theory for any simple Lie algebra equipped
with an integral gradation. It is explained how the different approaches to these
dynamical systems are related by gauge transformations. Combining Gauss decompos-
tions in relevent gauges, we unify formulae already derived, and explicitly determine
the holomorphic expansion of the conformally reduced WZNW solutions — whose
restriction gives the solutions of the Toda equations. The same takes place also for
semi-integral gradation. Most of our conclusions are also applicable to the affine Toda
theories.

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1 Introduction

Two-dimensional conformal Toda systems have been invented and explicitly integrated on the basis of the group-algebraic approach more than ten years ago in papers\(^1\) \cite{1} and \cite{2}. They are in general associated with any integral gradation of a simple Lie algebra. We shall deal with the general case, that is, not only with the principal gradation that leads to the so-called finite non-periodic Toda systems, but also with their non-abelian versions\(^4\) introduced in refs.\cite{2}. In what follows, and for the sake of brevity, we call them all Toda-type systems. Some time after refs.\cite{1}, \cite{2}, a number of investigations appeared where these systems arose in the context of two dimensional gravity, of various aspects of W-algebras and W-geometries, of constrained WZNW-models, of “topological and anti-topological fusion”, and so on, – see \cite{4}-\cite{10} and references therein. At present, however, the link between these papers is not easy to make, although they basically deal with the same objects and notions. Different notations have often been used, and, more importantly, equivalent formulae may be connected only when one sees that for the various problems it was convenient to use different gauges.

This disaccordance is quite unsuitable to make further progress. This is true, not only for concrete applications, but also for the possibility (and desirability) of extending the ideas to multi-dimensional systems following a path opened, e.g. in refs.\cite{10}, \cite{12}. Accordingly, the present paper is partly a unifying review of the various gauge conditions that have been used and of the relationship between them. Our contribution is, to some extent, a methodological one proposed for a wide audience of people interested in various aspects of integrable systems and related subjects. Systematically combining Gauss decompositions in these different gauges will nevertheless lead us to novel results, such as the explicit formula for the holomorphic decomposition of the WZNW solution.

2 The basic framework

We consider two-dimensional space with coordinates \(z^+, z^-\). In this discussion, the unifying feature is the zero-curvature condition – also called Maurer-Cartan, Lax, and so on:

\[
\left[ \partial_+ + A_+, \partial_- + A_- \right] \equiv \partial_+ A_- - \partial_- A_+ + [A_+, A_-]_\pm = 0. \tag{2.1}
\]

As usual, \(\partial_\pm\) stands for \(\partial/\partial z^\pm\). \(A_\pm\) takes values in a simple finite-dimensional Lie algebra \(\mathcal{G}\). One considers a \(Z\)-gradation \(\mathcal{G} = \bigoplus_{m \in \mathbb{Z}} \mathcal{G}_m \equiv \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+\).

\(^3\)In what follows, and when we use the results obtained by the authors of refs \cite{1}, \cite{2}, in the framework of the algebraic approach, we give reference to their review book \cite{3} (of course, if the result is described there) rather that to the original papers \cite{1}, \cite{2}.

\(^4\)Here we deal with systems that are different from those proposed by A. M. Polyakov at the end of the seventies, and also called by him “nonabelian Toda theories”.

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and, associated with this choice, we have the modified Gauss decompositions, such that any regular element \( U \) of the corresponding Lie group \( G \) may be written under the form:

\[
U = U_{(+)}U_{(0)}U_{(-)}, \quad U_{(\pm)} \in G_{\pm}, \quad U_{(0)} \in G_0.
\]  

\( G_{\pm} \) are the nilpotent algebras generated by \( G_{\pm} \), and \( G_0 \) is the subgroup generated by \( G_0 \). Note that \( G_0 \) may be a non-abelian algebra – just then the Toda theory is called non-abelian. In Eq.2.2 we introduced a notation which we will use throughout the article: group elements that belong to \( G_{\pm} \) or \( G_m \) are given the same subscripts in parenthesis. We shall also use a similar notational rule for elements of \( G_{\pm} \) or \( G_m \)

Of course the zero-curvature condition Eq.2.1 is invariant under the general gauge transformation

\[
A_{\pm} \rightarrow A_{\pm}^{\text{h}} = h^{-1}A_{\pm} h + h^{-1}\partial_{\pm} h,
\]

where \( h \) is any element of \( G \). A basic requirement of the Toda dynamics is that the above vector potentials satisfy the condition \( A_{\pm} \in G_0 \oplus G_{\pm} \) in some gauges. These gauges will be called triangular. This triangular gauge condition is left invariant by the gauge transformations

\[
A_{\pm} \rightarrow A_{\pm}^{h(0)} = h_{(0)}^{-1}A_{\pm} h_{(0)} + h_{(0)}^{-1}\partial_{\pm} h_{(0)}, \quad \text{with } h_{(0)} \in G_0.
\]

We shall call them \( G_0 \)-gauge transformations, in the present paper. The abelian and non-abelian Toda theories have abelian and non-abelian \( G_0 \)-gauge groups respectively. Obviously the solution of the zero-curvature condition is

\[
A_{\pm} = g^{-1}\partial_{\pm} g,
\]

with \( g \in G \). Thus under the gauge transformation Eq.2.3

\[
g \rightarrow gh.
\]

Following ref.\[3\], let us recall that the Toda equation may be written as\[5\]

\[
\partial_{\pm}(\hat{g}_{(0)}\partial_{\mp}\hat{g}_{(0)}) = [X_{(-1)}, \hat{g}_{(0)}^{-1}X_{(1)}\hat{g}_{(0)}],
\]

where \( X_{(\pm1)} \) are fixed elements of \( G_{\pm1} \) such that the whole \( G_{\pm1} \) may be recovered by adjoint action of \( G_0 \), and where the dynamical fields are lumped into \( \hat{g}_{(0)}(z_+, z_-) \in G_0 \). We shall see how this is connected with Eq.2.1 in different gauges. Let us mention some well known examples. The elements \( X_{(\pm1)} \) could be the generators of a \( sl(2, \mathbb{C}) \)-subalgebra of \( G \), and the corresponding \( \mathbb{Z} \)-gradation be performed by

\[\text{Of course there are equivalent decompositions such as } U = U'_{(-)}U''_{(0)}U'_{(+)} \text{ where the elements of } G_{\pm} \text{ and } G_0 \text{ are in different orders.}
\]

\[\text{Note also that the equation } \partial_{\pm}(\hat{g}_{(0)}^{-1}\partial_{\mp}\hat{g}_{(0)}) = [X_{-j}, \hat{g}_{(0)}^{-1}X_{i}\hat{g}_{(0)}], \quad 1 \leq i, j \leq n, \text{ recently proposed in } [13] \text{ as a natural generalization of the so-called special geometry, represents, at least formally, a multi-dimensional analogue of Eq.2.7. This generalized equation may probably be considered as an appropriate reduction of the 2n-dimensional WZNW-model } [14], \text{ just with the same reasonings as in } [14] \text{ for the usual 2D case.}\]
the Cartan element $H = [X_{(1)}, X_{(-1)}]$ of this $sl(2, \mathbb{C})$, i.e., $[H, \mathcal{G}_m] = m \mathcal{G}_m$. In particular, for the principal (canonical) gradation of a simple Lie algebra $\mathcal{G}$ (with the abelian subalgebra $\mathcal{G}_0$ and $X_{(\pm 1)}$ decomposable over the Chevalley generators of the positive and negative simple roots, respectively), equation (2.7) reduces to the abelian Toda system

$$\partial_+ \partial_- x_i = \exp \rho_i, \quad 1 \leq i \leq r \equiv \text{rank } \mathcal{G}; \quad \rho_i = \sum_{j=1}^{r} k_{ij} x_j,$$

(2.8)

where $x_i(z_+, z_-)$ are the Toda fields, $k$ is the Cartan matrix of $\mathcal{G}$. Recall here the defining relations for the Cartan ($h_i$) and Chevalley ($X_{\pm i}$) generators of $\mathcal{G}$:

$$[h_i, h_j] = 0, \quad [h_i, X_{\pm j}] = \pm k_{ji} X_{\pm j}, \quad [X_i, X_{-j}] = \delta_{ij} h_i; \quad 1 \leq i, j \leq r.$$

(2.9)

Of course, the simple Lie algebras $\mathcal{G}$, in general, could be also supplied with non-principal gradations, in particular those associated with non-principal $sl(2, \mathbb{C})$ subalgebras of $\mathcal{G}$. For such gradations the subalgebra $\mathcal{G}_0$ is not abelian.

The Toda field equations Eq.2.7 follow from the zero-curvature condition if one requires that there exist triangular gauges where

$$A_{\pm} \in \mathcal{G}_0 \oplus \mathcal{G}_{\pm 1}.$$

(2.10)

We shall characterize these gauges as being restricted-triangular. Of course, since these conditions are not invariant under the general transformations Eq.2.3, we may only require that there exist gauges such that the vector potentials take certain restricted forms. It is clearly not true that any vector potential may be gauge-transformed to become restricted-triangular. Thus the restricted-triangularity condition is not a matter of gauge choice. In fact, it determines the dynamics. Other hypothesis may be contradictory, that is, only give trivial dynamics (such as $A_{\pm} \in \mathcal{G}_{\pm m} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{\pm 1}$); or lead to higher Toda flows (such as $A_{\pm} \in \mathcal{G}_0 \oplus \mathcal{G}_{\pm m}$, with $m > 1$). The equations associated with the last possibility also are integrable [3]. The holomorphic expansions in different gauges could be studied, following the method described here, but we shall not do so, at present.

### 3 Relevant gauges

In this section we discuss three different gauges which are relevant for this or that problems under consideration, namely “general triangular”, “Toda”, and “WZNW” gauges.

#### 3.1 General triangular gauges

To begin with, we discuss properties that hold in any triangular gauge, without assuming that the restriction condition Eq.2.11 holds. We distinguish these gauges...
by a superscript \((t)\). This is essentially a review of some results from \[3\]. Two useful forms for the Gauss decomposition of the corresponding \(g^{(t)}\) of Eq. 2.5 are as follows

\[
g^{(t)} = M_{(-)} N_{(+)} g_{(0)} - = M_{(+)} N_{(-)} g_{(0)} +,
\]

(3.1)

Under the \(G_0\)-gauge transformation Eq. 2.4, \(M_{(\pm)}, N_{(\pm)}\) are clearly invariant. On the other hand, \(g^{(t)}_{(0)} \rightarrow g^{(t)}_{(0)} h_{(0)}\), and we introduce the \(G_0\)-gauge-invariant

\[
g_{(0)} \equiv g_{(0)} + \frac{1}{g_{(0)}}
\]

(3.2)

Next, it is easily seen that the triangular gradation properties are satisfied iff \(M_{(\pm)}\) are holomorphic functions, that is

\[
\partial \mp M_{(\pm)} = 0,
\]

(3.3)

and one deduces from Eq. 2.5 that

\[
A^{(t)}_{(+)} = g_{(0)}^{-1} N_{(+)}^{-1} \partial_{(+)} N_{(+)} g_{(0)} - + g_{(0)}^{-1} \partial_{(-)} g_{(0)} - ,
\]

\[
A^{(t)}_{(-)} = g_{(0)}^{-1} N_{(-)}^{-1} \partial_{(-)} N_{(-)} g_{(0)} + + g_{(0)}^{-1} \partial_{(+)} g_{(0)} + .
\]

(3.4)

According to Eq. 3.3, one may define two holomorphic functions \(L_{(\pm)}(z_{\pm}) \in G_{\pm}\) such that

\[
dM_{(\pm)}/dz_{\pm} = L_{(\pm)} M_{(\pm)}.
\]

(3.5)

On the other hand, it follows from Eqs. 3.1 and 3.2 that

\[
K \equiv M_{(+)}^{-1} M_{(-)} = N_{(-)} g_{(0)} N_{(+)}^{-1}.
\]

(3.6)

The differential equations 3.7 give

\[
K^{-1} \partial_{(-)} K = L_{(-)}, \quad \partial_{(+)} K K^{-1} = -L_{(+)}.
\]

(3.7)

Substituting in these equations the element \(K\) expressed via the second part of the formula (3.6), we arrive at

\[
N_{(-)}^{-1} \partial_{(-)} N_{(-)} = g_{(0)} L_{(-)} g_{(0)}^{-1}, \quad N_{(+)}^{-1} \partial_{(+)} N_{(+)} = g_{(0)}^{-1} L_{(+)} g_{(0)}.
\]

(3.8)

It finally follows from Eqs. 3.2, 3.4, and 3.8 that, in any triangular gauge, \(^7\)

\[
A^{(t)}_{(+)} = g_{(0)}^{-1} L_{(+)} g_{(0)} + + g_{(0)}^{-1} \partial_{(-)} g_{(0)} - ,
\]

\[
A^{(t)}_{(-)} = g_{(0)}^{-1} L_{(-)} g_{(0)} - + g_{(0)}^{-1} \partial_{(+)} g_{(0)} + .
\]

(3.9)

\(^7\)For simplicity of notation, \(\partial_{\pm}\) only act on the first following function, unless parenthesis indicate otherwise.
3.2 Toda gauges

For the Toda dynamics, there exist restricted-triangular gauges where the more restrictive condition Eq. 2.10 holds. These gauges will be characterized by a superscript (rt). If one is in such a gauge, and using the formulae just written, it is easy to see that, \( N_{(\pm)} \in G_{\pm 1} \), \( M_{(\pm)} \in G_{\pm 1} \), and \( L_{(\pm)} \in G_{\pm 1} \). In accordance with our general convention, we may write them as \( N_{(\pm 1)} \), \( M_{(\pm 1)} \), \( L_{(\pm 1)} \). According to ref. [3], \( L_{(\pm 1)} \) may be obtained from fixed elements \( X_{(\pm 1)} \in G_{\pm 1} \) by adjoint action of \( G_0 \). Thus we may write

\[
L_{(\pm 1)} = y_{(0)\pm}X_{(\pm 1)}y_{(0)\pm}^{-1},
\]

where, according to Eqs. 3.3, 3.5, \( y_{(0)\pm} \) are holomorphic: \( \partial_{\pm}y_{(0)\pm} = 0 \). Therefore

\[
A_{+}^{(T)} = \left( y_{(0)+}^{-1}g_{(0)+}^{-1} \right) X_{(1)} \left( y_{(0)+}^{-1}g_{(0)+}^{-1} \right) + g_{(0)-}^{-1}\partial_{+}g_{(0)-},
\]

\[
A_{-}^{(T)} = \left( y_{(0)-}^{-1}g_{(0)-}^{-1} \right) X_{(-1)} \left( y_{(0)-}^{-1}g_{(0)-}^{-1} \right) + g_{(0)+}^{-1}\partial_{-}g_{(0)+}.
\]

The precise form Eq. 2.7 of the Toda equation comes out directly from the zero-curvature condition with specific conditions introduced in ref. [3]. There are two associated gauge choices which we call Toda gauges, and distinguish with a superscript \( T \). They may be characterized by the fact that the gradation-zero part in one of the connections, say \( A_{+}^{(T)} \) is set equal to zero, while the gradation-one component of the other \( A_{-}^{(T)} \) is chosen to be constant. Next we verify that, starting from any restricted-triangular gauge, there exists a \( G_0 \)-gauge transformation

\[
A_{+}^{(T)} \equiv h_{(0)}^{-1}(A_{+}^{(rt)} + \partial_{+})h_{(0)}, \quad \text{i.e., } g^{(T)} \equiv g^{(rt)}h_{(0)},
\]

such that

\[
A_{(0)+}^{(T)} = \left( g_{(0)-}^{-1}h_{(0)}^{-1} \right) \partial_{+} \left( g_{(0)-}h_{(0)} \right) = 0,
\]

\[
A_{(-1)-}^{(T)} = \left( y_{(0)-}^{-1}g_{(0)-}^{-1} \right) X_{(-1)} \left( y_{(0)-}^{-1}g_{(0)-}^{-1} \right) + g_{(0)+}^{-1}\partial_{-}g_{(0)+} = \text{constant}.
\]

In the last equations we made use of Eqs. 3.11. Indeed, it is enough to choose

\[
h_{(0)} = g_{(0)-}^{-1}y_{(0)-}.
\]

Next, it is easily seen that the Toda equations 2.7 come out from the flatness condition Eq. 2.1 in this gauge, if we identify

\[
\hat{g}_{(0)} = y_{(0)+}^{-1}g_{(0)+}h_{(0)},
\]

and one gets

\[
A_{+}^{(T)} = \hat{g}_{(0)+}^{-1}X_{(1)}\hat{g}_{(0)}, \quad A_{-}^{(T)} = \hat{g}_{(0)+}^{-1}\partial_{-}\hat{g}_{(0)} + X_{(-1)}.
\]
Finally, one may rewrite $h_{(0)}$, $g^{(T)}$, and $\hat{g}_0$ in terms of the original quantities introduced above for an arbitrary restricted triangular gauge, obtaining,

$$
\begin{align*}
    h_{(0)} &= (y_{(0)} - g_{(0)})^{-1} = g_{(0)}^{-1} + y_{(0)} + \hat{g}_0; \\
    g^{(T)} &= g^{(T)}_0 - y_{(0)}^{-1}; \\
    \hat{g}_0 &= y_{(0)}^{-1} + g_{(0)} + g_{(0)}^{-1} y_{(0)}^{-1} = y_{(0)}^{-1} + g_{(0)} y_{(0)}^{-1}.
\end{align*}
$$

(3.17) (3.18) (3.19)

Note that the functions (3.16) for the case of the abelian Toda system (2.8) are reduced to the form

$$
\begin{align*}
    A^{(T)}_+ &= \sum_j e^{\rho_j} X_j, \\
    A^{(T)}_- &= -\sum_j \partial_+ x_j h_j + \sum_j X_{-j}.
\end{align*}
$$

(3.20)

where the sum runs over the step operators associated with simple roots.

The role of the components $A_+$ and $A_-$ in the Toda gauge may be, of course, reversed by letting

$$
\begin{align*}
    \tilde{A}^{(T)}_+ &= \hat{g}_0 A^{(T)}_+ \omega_+^{-1} + \hat{g}_0 \partial_+ \omega_+^{-1}, \\
    \tilde{A}^{(T)}_- &= \hat{g}_0 \partial_- \omega_+^{-1} + X_{(1)}, \\
    \tilde{A}^{(T)}_- &= \hat{g}_0 X_{(-1)} \omega_+^{-1}.
\end{align*}
$$

(3.21) (3.22)

### 3.3 W-gauges

In accordance with [5]-[8], the WZNW dynamics is best seen by going to other gauges which we characterize with a superscript ($W^\pm$). They are gauges of the axial type where one component, that is, $A^{(W^\pm)}_\pm$ vanishes. In this subsection, we rederive results of refs.[5]-[8], for completeness. The change of gauge is most easily achieved starting from the T-gauge. Let us write, for the ($W^+$) gauge,

$$
A^{(W^+)}_\pm = \omega_+ A^{(T)}_\pm \omega_+^{-1} + \omega_+ \partial_\pm \omega_+^{-1},
$$

(3.23)

It follows from Eq.3.16 that, if

$$
\omega_+^{-1} \partial_+ \omega_+ = \hat{g}_0^{-1} X_{(1)} \hat{g}_0,
$$

(3.24)

then

$$
A^{(W^+)}_+ = 0, \\
A^{(W^+)}_- \equiv -J = \omega_+ (X_{(-1)} + \hat{g}_0^{-1} \partial_- \hat{g}_0 + \partial_-) \omega_+^{-1}.
$$

(3.25)

It is clear from Eq.3.24 that $\omega_+ \in G_+$ as the notation anticipated. Thus the W-gauges are not triangular. On the other hand, it is easy to see that $J$ satisfies

$$
\partial_+ J = 0, \\
J_{(-)} = -X_{(-1)}.
$$

(3.26)

The first equation follows from the zero-curvature condition Eq.2.1, since $A^{(W^+)}_+ = 0$. The second may be verified from the explicit expression just given (following the general convention, $J_{(-)}$ is the negative-gradation component of $J$).
The other W gauge is treated similarly, starting from the other Toda gauge potentials \( \tilde{A}^{(T)}_\pm \):

\[
A^{(W-)}_\pm = \omega^{-1}_\pm \tilde{A}^{(T)}_\pm \omega_\pm + \omega^{-1}_\pm \partial_\pm \omega_\pm, \quad \text{that is,} \quad g^{(W-)}_\pm = \tilde{g}^{T}_\pm \omega_\pm; \tag{3.27}
\]

\[
\omega_\pm \partial_\pm \omega_\pm^{-1} = \hat{g}(0) X_\mp \hat{g}^{-1}(0); \tag{3.28}
\]

\[
A^{(W-)}_\pm = 0, \quad A^{(W-)}_+ \equiv -\bar{J} = \omega^{-1}_\mp (X_\mp + \hat{g}(0) \partial_\mp \hat{g}^{-1}(0) + \partial_\mp) \omega^{-1}_\mp; \tag{3.29}
\]

\[
\bar{J}_+ = X_\mp, \quad \partial_- \bar{J} = 0. \tag{3.30}
\]

In accordance with [8]-[11], the group element \( \omega \) which parametrises the solution of the constrained WZNW equation is defined as follows:

\[
\omega \equiv \omega_+ \omega_0 \omega_-; \quad \text{where} \quad \omega_0 = \hat{g}^{-1}(0). \tag{3.31}
\]

Indeed, by explicit computation one verifies that the currents \( J \) and \( \bar{J} \) introduced above are the corresponding WZNW currents, that is,

\[
J \equiv \partial_- \omega \omega^{-1}, \quad \bar{J} \equiv -\omega^{-1} \partial_+ \omega. \tag{3.32}
\]

Therefore, Eqs. 3.26 and 3.31 show that \( \omega \) is a solution of the corresponding conformally reduced WZNW model.

## 4 Explicit solution of the WZNW model

As noted in refs. [5] - [8], the Toda field \( \hat{g}(0) \) is recovered from \( \omega \) by using the fact that Eq. 3.31 is a Gauss decomposition, so that, when we compute matrix elements between states \( |\lambda_j\rangle \) which are annihilated by \( G_+ \), we get \( \langle \lambda_j | \omega | \lambda_j \rangle = \langle \lambda_j | \hat{g}(0) | \lambda_j \rangle \). Of course \( \omega \) contains more informations. It is the purpose of the present section to establish its full connection with the group-elements which appeared in the triangular gauges of subsections 3.1 and 3.2. It follows from Eqs. 3.24 and 3.28 that

\[
\omega^{-1}_+ \partial_+ \omega_+ = \left( g_0 y_{0-} \right)^{-1} L_1 g_0 y_0 = y^{-1}_0 N^{-1}_1 \partial_+ N_1 y_0 - ;
\]

\[
\omega^{-1}_- \partial_- \omega_- = \left( g_0 y_{0+} \right)^{-1} L_- g_0^{-1} y_0 = y^{-1}_0 N^{-1}_- \partial_- N_- y_0 + .
\]

Since \( \omega_{\pm} \in G_{\pm} \), a more convenient form of these formulae is

\[
\omega^{-1}_+ \partial_+ \omega_+ = \left[ y_{0-} N_1 y^{-1}_0 \right]^{-1} \partial_+ \left[ y_{0-} N_1 y^{-1}_0 \right];
\]

and

\[
\omega^{-1}_- \partial_- \omega_- = \left[ y_{0+} N^{-1}_- y_{0+} \right] \partial_- \left[ y_{0+} N^{-1}_- y_{0+} \right]^{-1}.
\]

where we used the fact that \( \partial_\mp y_0 = 0 \). Thereof,

\[
\omega_+ = q_+ (z_-) y_0 N_1 y^{-1}_0, \quad \omega_- = y^{-1}_0 N^{-1}_- y_0 + q_-(z_+), \quad \text{Eq. 4.1}
\]
where \( q_+(-) \) and \( q_-(+) \) are arbitrary elements of \( G_+ \) and \( G_- \), respectively. Consequently,

\[
\omega = q_+(z-) y_{(0)-} N_{(1)} \tilde{g}_{(0)}^{-1} N_{(-1)}^{-1} y_{(0)+} q_-(z_+),
\]

\[
= q_+(z-) y_{(0)}^- M_{(-1)}^{-1} M_{(1)} y_{(0)+} q_-(z_+). \tag{4.2}
\]

Here \( q_+(z_-) \) and \( q_-(z_+) \) are arbitrary elements of \( G_+ \) and \( G_- \), respectively. Since \( y_{(0)\pm} \) and \( M_{(\pm1)} \) are only functions of \( z_{\pm} \) respectively, this shows that \( \omega^{-1} \) has the holomorphic decomposition

\[
\omega^{-1} = g_L(z_+) g_R(z_-), \tag{4.3}
\]

\[
g_L(z_+) \equiv q_{-(+)}^{-1}(z_+) h_{(0)+}^{-1}(z_+) M_{(1)}^{-1}(z_+), \quad g_R(z_-) \equiv M_{(-1)}(z_-) h_{(0)-}^{-1}(z_-) q_{+(+)}^{-1}(z_-). \tag{4.4}
\]

This of course agrees with the fact that \( \omega \) is a solution of the WZNW equations \( \partial_+ J = \partial_- \bar{J} = 0 \). As is well known, \( g_R \) and \( g_L \) are solutions of the equations \( dg_R/dz^- = -g_R J, \ dg_L/dz^+ = J g_L \). These holomorphic relations have been called generalised Bargmann, or generalised Frobenius, or Drinfeld-Sokolov equations. The arbitrary functions \( q_{(\pm)}^{-1}(z_{\mp}) \) reflect the usual gauge invariance of the WZNW solution, and of the holomorphic equations just recalled.

The elements \( g_{L,R} \) can be written in the form of the modified Gauss decomposition, namely

\[
g_{L,R} = g_{(-),L,R} g_{(0)L,R} g_{(+),L,R}; \tag{4.5}
\]

where

\[
g_{(-)L} = q^{-1}_{(-)}, \quad g_{(0)L} = y_{(0)+}, \quad g_{(+),L} = M_{(1)}^{-1};
\]

\[
g_{(-)R} = M_{(-1)}, \quad g_{(0)R} = y_{(0)-}, \quad g_{(+),R} = q_{(+)}^{-1}. \tag{4.6}
\]

Finally, we write down a few useful relations,

\[
\partial_+ g_{(+),L} \equiv \partial_+ M_{(-1)}^{-1} = -L_{(1)} M_{(1)}^{-1} = -L_{(1)} g_{(+),L} \equiv -F(z_+) g_{(+),L},
\]

and

\[
\partial_+ g_{(+),R} \equiv \partial_+ M_{(-1)} = M_{(-1)} L_{(-1)} = g_{(-),R} L_{(-1)} \equiv -F(z_-) g_{(-),R}. \tag{4.7}
\]

Thereof,

\[
F(z_+) = L_+ = \sum_\alpha \phi_{+\alpha} X_{(1)\alpha} = h_{(0)+} X_{(1)} (h_{(0)+})^{-1} = g_{(0)L}^{-1} X_+ g_{(0)L},
\]

\[
F(z_-) = L_- = \sum_\alpha \phi_{-\alpha} X_{(-1)\alpha} = h_{(0)-} X_- (h_{(0)-})^{-1} = g_{(0)R}^{-1} X_- g_{(0)R}. \tag{4.7}
\]

The explicit form of the solution of conformally reduced WZNW equation is crucial for the geometrical interpretation of the corresponding Toda systems. Indeed, as shown in ref. [10], the solutions of the holomorphic equations \( dg_R/dz^- = -g_R J, \ dg_L/dz^+ = J g_L \), for \( A_n \)-Toda in a certain gauge, coincide with the embedding functions that defines the associated W-holomorphic surface in \( CP^n \). This result may be generalized to any conformal Toda theory [18].
5 Outlook

Besides the W-geometrical aspect just mentioned, the transition between the matrix elements and the connections given in different gauges is of a great importance for the study of various aspects of the Toda-type theories, in particular for a formulation of the boundary value problem in terms of the characteristic integrals for the system (2.7).

One may, of course, also consider the case of half-integral gradations. These can be incorporated into the scheme by extending $A_{\pm}$ so that they take their values in $G_{\frac{1}{2}}$ as well as $G_0$ and $G_{\pm 1}$. If there are no further constraints, the field equations (2.7) then generalize to a set [3, 13]

$$\partial_+(\hat{g}_{(0)}^{-1}\partial_-\hat{g}_{(0)}) = [X_-, \hat{g}_{(0)}^{-1}X_+\hat{g}_{(0)}] + [\hat{\psi}, \hat{g}_{(0)}^{-1}\hat{\psi}\hat{g}_{(0)}],$$

$$\partial_-\psi = [X(1), \hat{g}_{(0)}\hat{\psi}\hat{g}_{(0)}^{-1}], \quad \partial_+\tilde{\psi} = [X(-1), \hat{g}_{(0)}^{-1}\hat{\psi}\hat{g}_{(0)}];$$

and are derivable from the effective action [8]

$$I^{\text{eff}} = I(\hat{g}_{(0)}) - \int \text{tr}(X(1)\hat{g}_{(0)}^{-1}X(-1)\hat{g}_{(0)}) + \int \text{tr}((\partial_+\psi)\hat{g}_{(0)}(\partial_-\tilde{\psi})\hat{g}_{(0)}^{-1})$$

$$+ \int \text{tr}([X(-1), \psi]\partial_+\psi) + \int \text{tr}([X(1), \tilde{\psi}]\partial_-\tilde{\psi}),$$

where $\psi$ and $\tilde{\psi}$ are the grade $\frac{\pm 1}{2}$ fields. On the other hand, one may start from the principle that the constraints be a maximal set of first-class constraints, as in [8]; then half of the grade-$(\frac{1}{2})$ fields $\psi$ must be zero, and the effective action takes the more complicated form

$$I^{\text{eff}} = I(\hat{g}_{(0)}) - \int \text{tr}(X(1)\hat{g}_{(0)}^{-1}X(-1)\hat{g}_{(0)})$$

$$- \int <(\partial_+\eta)(A - BD^{-1}C)(\partial_-\tilde{\eta})> - \int <[X(-1), \partial_+\eta], BD^{-1}\eta >$$

$$+ \int <\tilde{\eta}D^{-1}C, [X(1), \partial_-\tilde{\eta}] > - \int <[X(-1), \eta]D^{-1}[X(1), \tilde{\eta}] >,$$

where the $\eta$'s are the remaining grade-(\pm \frac{1}{2}) fields, $<,>$ denotes the restriction of the Cartan inner-product to these fields and the matrices $\{A, B, C, D\}$ are defined as $\text{Ad}_{g_0} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. In fact, this principle, has been implicitly applied in the case of the integral gradations allowing to rid off $\text{dim } G_0 - \text{dim } G_{\pm 1}$ additional (nonprimary) fields in the stringy non-abelian systems associated with non-principal $sl(2, C)$ subalgebras of $B_r$ [1]. This turned out to be especially important for the study of non-trivial background metrics in the target space, e.g., for the black holes generated by the non-abelian $B_2$-Toda fields, and their $osp(2|4)$ superextension, see [11, 14]. It is interesting to mention related reasonings of the article [15] which argue that the weight-1/2 fermion fields in the superconformal field theory are “auxiliary” ones, and that their role is only to ensure the closure of the corresponding Lie
superalgebra. A similar conclusion takes place for the aforementioned non-abelian
$B_2$ (and $osp(2|4)$) - Toda systems, as well as for their generalizations to the Lie
(super)algebras of higher ranks.

Consider the supersymmetrical extensions\[16\] of the equation (2.7),

$$D_+ \left( \hat{g}(0)^{-1} D_- \hat{g}(0) \right) = \left[ Y_{(-1)}, \hat{g}(0)^{-1} Y(1) \hat{g}(0) \right]_+ ,$$

(5.4)

associated with a classical Lie algebras and superalgebra $G^*$ supplied with a $Z$-
gradation. Here $D_\pm$ are the supercovariant derivatives in $2|2$-superspace; $\hat{g}(0)$ is a
regular element of the Grassmann span of the Lie group $G_0 = \text{Lie} G_0$; $Y_{(\pm1)}$ are
fixed (odd) elements of $G_{\pm1}$. Most of the results given above can be derived for the
system (5.4) as well, with the corresponding modifications. (For the relation with
a supersymmetric extension of the constrained-WZNW-model see also \[17\].) It is
interesting to emphasize that, under the relevant conditions, the component-form of
Eqs.5.4 coincides with Eqs.5.1. This is, of course, only formal, since the fermionic
components entering in Eq.5.4 are Majorana spinors with anticommuting values,
contrary to the functions $\psi$ and $\tilde{\psi}$ in Eq.5.1.

Finishing up this paper, we would like to mention that most of the relations and
conclusions given here are applicable to the periodic Toda-type system associated
with the affine Lie Kac-Moody algebras $\hat{G}$, of course, with the appropriate mod-
ifications. For many interesting cases, these equations, in particular those called
last time also as affine and conformal affine Toda theories, — which are given in
quite general a form in ref.\[3\] following the results of ref.\[19\], together with their
group-algebraic integration scheme — can be rewritten in the form Eq.2.7. Here
the meaning of the elements $X_{(\pm1)}$ is different, however, and the relevant grading
operator $\mathcal{H}, [\mathcal{H}, G_m] = mG_m$, of course does not belong to the corresponding finite-
dimensional Lie algebra $G$. It can be constructed in a degenerate representation of
$\hat{G}$ with the spectral parameter $\lambda$ as $\mathcal{H} = H + c \lambda d/d\lambda$. Then, following the same
reasonings as in ref.\[11\] for the construction of a nontrivial background metric in
the target space, one can describe the black holes associated with the non-canonical
gradation of the affine algebra $B^{(1)}_2$, and the soliton solutions of the model, for ex-
ample. Here, at a formal level, the difference between the corresponding finite and
affine cases arises from the insertion, in the expression for $X_{(\pm1)}$, of the additional
term like $\lambda^{\mp1} X_{\mp M}$, where $M$ is the maximal root of $B_2$. (In the general case a
modification of $X_{(\pm1)}$ is the relevant element of the Heisenberg subalgebra of the
responding affine Lie algebra.) The final equations, of course, do not depend on
$\lambda$, and provide the simplest example of a nontrivial background metric, while the
canonical gradation of the affine Lie algebras leads, as in the finite-dimensional case,
to constant metrics in the target space of the corresponding $\sigma$-model. It seems quite
believable that a quantization procedure for such systems, completely integrable at
the classical level, could follow along the line of the corresponding construction given
in \[3\], and in \[20\] for finite-dimensional systems, with the necessary modification of
such objects as the Casimir operator of the second order, the Whittaker vectors,
etc.
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