Torus Equivariant $D$-modules and Hypergeometric Systems

Christine Berkesch Zamaere, Laura Felicia Matusevich, and Uli Walther

Abstract. We formalize, at the level of $D$-modules, the notion that $A$-hypergeometric systems are equivariant versions of the classical hypergeometric equations. For this purpose, we construct a functor $\Pi_A^B$ on a suitable category of torus equivariant $D$-modules and show that it preserves key properties, such as holonomicity, regularity, and reducibility of monodromy representation. We also examine its effect on solutions, characteristic varieties, and singular loci. When applied to certain binomial $D$-modules, $\Pi_A^B$ produces saturations of the classical hypergeometric differential equations, a fact that sheds new light on the $D$-module theoretic properties of these classical systems.

In memory of Mikael Passare.

Introduction

Hypergeometric systems of Horn type were introduced in the late nineteenth century as multivariate generalizations of the Gauss hypergeometric equation. While their series solutions have been studied extensively, few of these works answer $D$-module theoretic questions. Indeed, Horn systems depend on parameters whose variation impacts their properties as $D$-modules, and stratifying their parameter spaces accordingly is difficult.

Binomial $D$-modules are generalizations of the $A$-hypergeometric systems of Gelfand, Graev, Kapranov, and Zelevinsky [GGZ87, GKZ89, DMM10b]. They are torus equivariant, the parameters acting as characters. As it turns out, homogenization with respect to the torus action provides an isomorphism between the holomorphic solutions of a classical Horn system and its equivariant binomial counterpart; however, the $D$-modules themselves are related in a more subtle way.

In this article, we introduce an invariantizing functor $\Pi_A^B$ that, in particular, links Horn systems to certain binomial $D$-modules and induces the above homogenization process on solutions. This involves a general procedure for realizing as $D$-modules on $(\mathbb{C}^\ast)^{n-d}$ the modules of invariants of $(\mathbb{C}^\ast)^d$-equivariant $D$-modules on $(\mathbb{C}^\ast)^n$. The construction of $\Pi_A^B$ and the study of its properties occupy Part I, while Part II is concerned with the application of $\Pi_A^B$ to binomial $D$-modules.

Outline for Part I. In §1 we provide background on $D$-modules and define the torus actions we will consider, while we introduce important categories of equivariant $D$-modules in §2. We work with the torus invariants functor for $D$-modules on $X = (\mathbb{C}^\ast)^n$ in §3. Over $\bar{X} = \mathbb{C}^n$ in §4, we define and state initial properties of the functor $\Pi_A^B$, while §5 provides more refined information.

Outline for Part II. In §6 we discuss binomial $D_{\bar{X}}$-modules, and we apply $\Pi_A^B$ to them in §7. This yields explicit characterizations of $D$-module properties for saturated Horn systems in terms of the parameter sets; in §§8-9, we consider other variants of Horn systems, see Definition 0.3. We conclude by explaining the relationship between the image under $\Pi_A^B$ of an $A$-hypergeometric system and the Horn–Kapranov uniformization of discriminantal varieties in §10.

2010 Mathematics Subject Classification. Primary: 14L30, 33C70; Secondary: 13N10, 14M25, 32C38.
CBZ was partially supported by NSF Grants DMS 1303083, OISE 0964985, and DMS 0901123.
LFM was partially supported by NSF Grants DMS 0703866, DMS 1001763, and a Sloan Research Fellowship.
UW was partially supported by NSF Grant DMS 0901123.
Construction of the functor. Throughout, on any \( \mathbb{C} \)-scheme, we mean by “differential operators” the sheaf of \( \mathbb{C} \)-linear differential operators on the corresponding structure sheaf. Since we consider only products of affine spaces and tori as underlying manifolds, which are all \( D \)-affine, we may restrict our attention to global sections.

If \( T := (\mathbb{C}^*)^d \) acts on \( \tilde{X} := \mathbb{C}^n \), then the action extends naturally to the differential operators on \( \tilde{X} \). In order to produce \( D_{\mathbb{C}^n-a} \)-modules from \( T \)-equivariant \( D_X \)-modules, one could try GIT type constructions. However, the GIT quotient of \( \tilde{X} \) by \( T \) is singular and would not fit our desired applications for Horn systems.

Instead, we restrict \( T \)-equivariant \( D_X \)-modules to \( X := \tilde{X} \setminus \text{Var}(x_1 \cdots x_n) \), the open torus of \( \tilde{X} \), and consider the family of toric maps from the quotient of \( X \) by \( T \) to the open torus \( Z = \mathbb{C}^{m} \) in an affine space \( Z = \mathbb{C}^{m} \), where \( m := n - d \). This family is parametrized by invertible matrices \( \tilde{A} \) that extend \( A \) and the Gale duals \( B \) of \( A \). Theorem [3.6] states that, over \( X \), taking \( T \)-invariants and endowing the resulting module with a \( D_Z \)-module structure, a functor denoted \( \Delta_B \), behaves well with respect to a number of \( D \)-module theoretic properties.

Returning to \( \tilde{X} \), we produce a functor \( \Pi_B^A \) that sends \( T \)-equivariant \( D_X \)-modules to \( D_Z \)-modules. With moderate restrictions on its source category, \( \Pi_B^A \) preserves \( L \)-holonomicity, regular holonomicity, and reducibility of monodromy representation, as shown in Theorem [4.2]. Under an additional irreducibility assumption in [15] we obtain an explicit presentation for \( \Pi_B^A \) and describe its impact on solutions, characteristic varieties, and singular loci.

Binomial \( D \)-modules. In Part II, we consider an important class of equivariant \( D_X \)-modules, whose images under the functor \( \Pi_B^A \) are \( D_Z \)-modules of hypergeometric type. We show that there is an equivalence between the (regular) holonomicity of a binomial \( D_X \)-module \( M \) and \( \Pi_B^A(M) \).

Definition 0.1. For a \( d \times n \) integer matrix \( A = (a_{ij}) \in \mathbb{Z}^{d \times n} \), let \( I \) be an \( A \)-graded binomial \( \mathbb{C}[\partial_x] \)-ideal, meaning that \( I \) is generated by elements of the form \( \partial_x^u - \lambda \partial_v^v \). In this definition, \( \lambda = 0 \) is allowed; in other words, monomials are admissible generators in a binomial ideal. A binomial \( D_X \)-module depends on a choice of some \( \beta \in \mathbb{C}^d \) and has the form

\[
D_X/(I + \langle E - \beta \rangle) := D_X/(D_X \cdot I + \langle E - \beta \rangle).
\]

Here \( E_i := \sum_{j=1}^n a_{ij}x_j \partial_{x_i} \) for \( 1 \leq i \leq d \), and \( E - \beta \) is the sequence \( E_1 - \beta_1, \ldots, E_d - \beta_d \) of Euler operators of \( A \), see Remark [1.6].

Example 0.2. For special kinds of binomial ideals, we obtain special kinds of binomial \( D_X \)-modules. The toric ideal

\[
I_A := \langle \partial_x^u - \partial_v^v \mid Au = Av \rangle \subseteq \mathbb{C}[\partial_x]
\]

gives rise to a binomial \( D_X \)-module called an \( A \)-hyypergeometric \( D_X \)-module. The left \( D_X \)-ideal

\[
H_A(\beta) := D_X \cdot I_A + \langle E - \beta \rangle
\]

is called an \( A \)-hypergeometric system, or GKZ-system.

If \( B \) is a Gale dual of \( A \) (see Convention [3.2]), then the lattice basis ideal associated to \( B \) is

\[
I(B) := \langle \partial_w^w - \partial_x^x \mid w = w_+ - w_- \text{ is a column of } B \rangle \subseteq \mathbb{C}[\partial_x]
\]

The lattice basis binomial \( D_X \)-module is \( D_X/(I(B) + \langle E - \beta \rangle) \).

\( \Box \)
Horn hypergeometric systems. The explicit description of $\Pi_B^A$ in Corollary 5.8 yields a method to show that saturated Horn systems arise essentially by applying $\Pi_B^A$ to lattice basis binomial $D$-modules. Corollaries 7.2 and 7.3 characterize holonomicity, regularity, and reducibility of monodromy representation of saturated Horn systems. To explain the use of the word saturated, we now provide three ways of viewing Horn systems.

Definition 0.3. Let $B$ be an $n \times m$ integer matrix of full rank $m$ with rows $B_1, \ldots, B_n$. Let $\kappa \in \mathbb{C}^n$. With $\bar{Z} = \mathbb{C}^m$, let $\eta := [z_1 \partial_{z_1}, \ldots, z_m \partial_{z_m}]$ and construct the following elements of $D_{\bar{Z}}$:

$$q_k := \prod_{b_{ik} > 0} \prod_{\ell = 0}^{b_{ik} - 1} (B_i \cdot \eta + \kappa_i - \ell) \quad \text{and} \quad p_k := \prod_{b_{ik} < 0} \prod_{\ell = 0}^{-|b_{ik}| - 1} (B_i \cdot \eta + \kappa_i - \ell).$$

(1) The Horn hypergeometric system associated to $B$ and $\kappa$ is the left $D_{\bar{Z}}$-ideal $\text{Horn}(B, \kappa) := D_{\bar{Z}} \cdot <q_k - z_k p_k | k = 1, \ldots, m> \subseteq D_{\bar{Z}}$. (0.1)

(2) The saturated Horn hypergeometric system associated to $B$ and $\kappa$ is the left $D_{\bar{Z}}$-ideal $\text{sHorn}(B, \kappa) := D_{\bar{Z}} \cdot <q_k - z_k p_k | k = 1, \ldots, m> \cap D_{\bar{Z}} = D_{\bar{Z}} \cdot \text{Horn}(B, \kappa) \cap D_{\bar{Z}} \subseteq D_{\bar{Z}}$. (0.2)

(3) Assume that $B$ has an $m \times m$ submatrix which is diagonal with strictly positive entries, and assume that the corresponding entries of $\kappa$ are all zero. The normalized Horn hypergeometric system associated to $B$ and $\kappa$ is the left $D_{\bar{Z}}$-ideal $\text{nHorn}(B, \kappa) := D_{\bar{Z}} \cdot \left\{ \frac{1}{z_k} q_k - p_k | k = 1, \ldots, m \right\} \subseteq D_{\bar{Z}}$. (0.3)

Example 0.4. Hypergeometric series are expressions of the form

$$\sum_{k_1, \ldots, k_m = 0}^{\infty} \Omega_{k_1, \ldots, k_m} \frac{z_{k_1} \cdots z_{k_m}}{k_1! \cdots k_m!},$$

where $\Omega_{k_1, \ldots, k_m}$ is a ratio of products of ascending (or descending) factorials of integer linear combinations of the indices $k_1, \ldots, k_m$, translated by certain fixed parameters. They have been extensively studied and include the (generalized) Gauss hypergeometric function(s), Appell functions, Horn series in two variables, Lauricella series, and Kampé de Feriét functions, among others.

All such series are solutions of systems of differential equations of the form $\text{Horn}(B, \kappa)$, where the rows of $B$ are determined by the factorials that appear in the series. Having $k_1! \cdots k_m!$ in the denominator of the series coefficients implies that $B$ has an $m \times m$ identity submatrix, and the corresponding parameters $\kappa_j$ are zero. In particular, all of the above named hypergeometric series are solutions of normalized Horn systems, as in Definition 0.3.(3).

Once $B$ and $\kappa$ are fixed, the holomorphic solutions of the three types of Horn systems in Definition 0.3 coincide; however, the modules themselves are truly different. As illustrated in Example 9.1, it can happen that $D_{\bar{Z}}/\text{sHorn}(B, \kappa)$ is holonomic while $D_{\bar{Z}}/\text{Horn}(B, \kappa)$ is not.

For previous results about Horn $D$-modules, we are aware of only [Sad02, DMS05]. We show in Corollary 5.10 that saturated Horn systems are essentially captured by the image of $\Pi_B^A$. This leads to precise results about saturated Horn systems in Theorems 7.1 and 6.7, in addition to our general results on images of binomial $D$-modules. We use alternate approaches in 8 and 9 to obtain results about normalized and usual Horn systems. In particular, we show that the holonomicity of $D_{\bar{Z}}/\text{Horn}(B, \kappa)$ and $D_{\bar{X}}/(I(B) + \langle E - \beta \rangle)$ is not equivalent; however, Theorem 9.4 provides a sufficient condition on $\kappa$ to ensure the holonomicity of $D_{\bar{Z}}/\text{Horn}(B, \kappa)$. 
**Acknowledgements.** We are grateful to Frits Beukers, Alicia Dickenstein, Brent Doran, Anton Leykin, Ezra Miller, Christopher O’Neill, Mikael Passare, and Bernd Sturmfels, who have generously shared their insight and expertise with us while we worked on this project. Part of this work was carried out at the Institut Mittag-Leffler program on Algebraic Geometry with a view towards Applications and the MSRI program on Commutative Algebra. We thank the program organizers and participants for exciting and inspiring research atmospheres.

**Part I: Torus invariants and \( D \)-modules**

1. **\( D \)-modules and torus actions**

   Fix integers \( n > m > 0 \) and set \( d := n - m \). After reviewing some background on \( D \)-modules and their solutions, we describe the action of the algebraic \( d \)-torus \( T := (\mathbb{C}^*)^d \) on \( \mathbb{C}^n \) and show how this action extends to regular functions, differential operators, and holomorphic germs.

1.1. **\( D \)-modules.** Let \( \bar{X} := \mathbb{C}^n \) with coordinates \( x := (x_1, \ldots, x_n) \). The Weyl algebra \( D_{\bar{X}} \) is the ring of differential operators on \( \bar{X} \), which is a quotient of the free associative \( \mathbb{C} \)-algebra generated by \( x_1, \ldots, x_n, \partial_{x_1}, \ldots, \partial_{x_n} \) by the two-sided ideal

\[
\langle x_i x_j = x_j x_i, \partial_{x_i} \partial_{x_j} = \partial_{x_j} \partial_{x_i}, \partial_{x_i} x_j = x_j \partial_{x_i} + \delta_{ij} \mid i, j \in \{1, \ldots, n\} \rangle,
\]

where \( \delta_{ij} \) is the Kronecker \( \delta \)-function. Let \( \bar{Z} := \mathbb{C}^m \) with coordinates \( z := (z_1, \ldots, z_m) \), and define its Weyl algebra \( D_{\bar{Z}} \) analogously, with \( \partial_{z_i} \) denoting the operator for differentiation with respect to \( z_i \). Throughout this article, let

\[
\theta_i := x_i \partial_{x_i}, \quad \eta_i := z_i \partial_{z_i}, \quad \theta := [\theta_1 \theta_2 \cdots \theta_n], \quad \text{and} \quad \eta := [\eta_1 \eta_2 \cdots \eta_m].
\]

Other relevant rings are the Laurent Weyl algebras \( D_X \) and \( D_Z \), defined as the rings of linear partial differential operators with Laurent polynomial coefficients, that is,

\[
D_{\bar{X}} = \mathbb{C}[x^\pm] \otimes_{\mathbb{C}[x]} D_{\bar{X}} \quad \text{and} \quad D_Z = \mathbb{C}[z^\pm] \otimes_{\mathbb{C}[z]} D_Z,
\]

where \( \mathbb{C}[x^\pm] \) and \( \mathbb{C}[z^\pm] \) denote denote the rings of differential operators on

\[
X := \bar{X} \setminus \text{Var} (x_1 \cdots x_n) = (\mathbb{C}^*)^n \quad \text{and} \quad Z := \bar{Z} \setminus \text{Var} (z_1 \cdots z_m) = (\mathbb{C}^*)^m.
\]

We say that \( L = (L_x, L_{\partial_x}) \in \mathbb{Q}^{2n} \) is a weight vector on \( D_{\bar{X}} \) if \( L_x + L_{\partial_x} \geq 0 \) coordinate-wise. Writing \( 1_n := (1, \ldots, 1) \in \mathbb{Z}^n \), we assume throughout that \( L_x + L_{\partial_x} = c \cdot 1_n \) for some \( c \in \mathbb{Q}_{>0} \).

A weight vector \( L \) defines an increasing filtration \( L \) on \( D_{\bar{X}} \) by

\[
L^k D_{\bar{X}} := \mathbb{C} \cdot \{ x^u \partial_x^v \mid L \cdot (u,v) \leq k \} \quad \text{for} \ k \in \mathbb{Q}.
\]

Set \( L^<k D_{\bar{X}} := \bigcup_{l<k} L^l D_{\bar{X}} \). By our convention, the associated graded ring \( \text{gr}^L D_{\bar{X}} \) is isomorphic to the coordinate ring of \( T^* \bar{X} \cong \mathbb{C}^{2n} \).

If \( P \in L^k D_{\bar{X}} \setminus L^<k D_{\bar{X}} \), then \( P \) is said to have \( L \)-order \( k \). For any \( P \) in \( L^k D_{\bar{X}} \setminus L^<k D_{\bar{X}} \), we denote its symbol by \( \text{in}_L(P) := P + L^<k D \in \text{gr}^L D = L^k D / L^<k D \subseteq \text{gr}^L D \).

The weight filtration induced by \( F = (F, F_{\partial_x}) := (0_n, 1_n) \in \mathbb{Q}^{2n} \) is called the order filtration on \( D_{\bar{X}} \). Its \( k \)th filtered piece \( F^k D_{\bar{X}} \) consists of all operators of order at most \( k \). The associated graded ring of \( D_{\bar{X}} \) with respect to the order filtration is denoted by \( \text{gr}^F D_{\bar{X}} \). Denoting \( \text{in}_F(\partial_{x_i}) = \xi_i \) and abusing language by writing \( x_i \) for \( \text{in}_F(x_i) \), we have \( \text{gr}^F D_{\bar{X}} \cong \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n] \). Similarly, \( \text{gr}^F D_Z \cong \mathbb{C}[z_1, \ldots, z_m, \zeta_1, \ldots, \zeta_m] \), where \( \zeta_i \) is the symbol of \( \partial_{z_i} \).
Let $M$ be a finitely generated left $D_X$-module. A weight vector $L$ on $D_X$ induces a filtration on (a presentation of) $M$. When $L_x + L_{0_y} > 0$, the $L$-characteristic variety of $M$ is $\text{supp} (\text{gr}^L(M)) \subseteq T^* X \cong \mathbb{C}^{2n}$. We denote this set by $\text{Char}^L(M)$; it is well-defined (see [SW08] for references), Zariski closed, and agrees with the zero set of the ideal $\text{ann}(\text{gr}^L(M))$. We say that $M$ is $L$-holonomic if its $L$-characteristic variety has dimension $n$.

The $F$-characteristic variety of $M$ is called its characteristic variety, denoted $\text{Char}(M)$. We say that $M$ is holonomic if its characteristic variety has dimension $n$. Bernstein’s inequality says that the characteristic variety of a left $D_X$-ideal has dimension at least $n$ [Bern72]. Smith [Smi01] refined this result, showing that every component of $\text{Char}^L(M)$ has dimension at least $n$.

The rank of the module $M$ is $\text{rank}(M) := \dim_{\mathbb{C}(x)}(\mathbb{C}(x) \otimes_{\mathbb{C}[x]} M)$. If $M$ is holonomic, then $\text{rank}(M)$ is finite, see [SST00], Proposition 1.4.9.

The projection of $\text{Char}(M) \setminus \text{Var}(\xi_1, \ldots, \xi_n)$ onto the $x$-coordinates is called the singular locus of $M$, denoted $\text{Sing}(M)$. Any $p \in \text{Sing}(M)$ is called a singular point of $M$.

Let $Y$ be a complex manifold of dimension $n$ with a point $p \in Y$. The holonomic left $D_Y$-module $M$ is said to be regular at $p$ if for any curve $C \subseteq Y$ passing through $p$, with smooth locus $\iota : C_s \hookrightarrow Y$, there is an open neighborhood $U$ of $p$ in $Y$ such that all the sheaves $L\iota^*(M|_U)$ are connections on $C_s$ with regular singular ODEs on a smooth compactification $\overline{C}$ of $C_s$. (These sheaves are zero outside the range $-n + 1 \leq k \leq 0$.) The module $M$ is regular holonomic if it is regular holonomic at every point $p \in \overline{X}$.

The notion of regularity is equivalent to requiring that the natural restriction map from formal to analytic solutions of $M$ be an isomorphism in the derived category. As such, it generalizes the classical definition of regular (Fuchsian) singularities for an ordinary differential equation. Note that, since we consider a compactification of $C$, regular holonomicity on $Y$ includes information about the behavior of (derived) solutions at infinity.

Regular holonomicity is a crucial property in the theory of $D$-modules. Bounded complexes of regular holonomic modules provide the input for the de Rham functor, which appears in the Riemann–Hilbert correspondence. A larger category cannot be used, as regularity is needed so that the DeRham functor is fully faithful. Moreover, regular holonomic modules are particularly suitable for computations. For example, the Frobenius algorithm for solving Fuchsian ODEs has been successfully generalized to regular holonomic systems in [SST00] §2.5.

The categories of holonomic and regular holonomic $D_Y$-modules are Abelian and closed under the formation of extensions, and they form full subcategories of the category of $D$-modules. Both holonomicity and regularity are preserved by direct and inverse images along morphisms of smooth varieties. See [BGK+87] for more details.

A holonomic left $D_Y$-module $M$ is said to have irreducible monodromy representation if the module $M(y) := \mathbb{C}(Y) \otimes_{\mathbb{C}[Y]} M$ is an irreducible module over $D(y) := \mathbb{C}(Y) \otimes_{\mathbb{C}[Y]} D_Y$. Here, $\mathbb{C}[Y]$ and $\mathbb{C}(Y)$ are the rings of regular and rational functions on $Y$, respectively. If $M$ does not have irreducible monodromy representation, then its solution space has a nontrivial proper subspace that is monodromy invariant, since the fundamental group of $Y \setminus \text{Sing}(M)$ is finitely presented.

### 1.2. Solutions of $D$-modules.

Let $\mathcal{O}_{p,X}^{an}$ be the space of germs of holomorphic functions at $p \in X$, and use $\bullet$ for the action of $D_X$ on $\mathcal{O}_{p,X}^{an}$. Given a left $D_X$-ideal $J$, the solution space $\text{Sol}_p(D_X/J)$
Further, there is an open set \([\mathcal{SST}_{00}, \text{Theorems 2.5.1 and 2.5.14}]\) in the direction of \(w\) regular holonomic. If \(w\) the direction of \(w\) simultaneously converge at each \(p\) and equal to \(\text{rank}(D_{\bar{X}}/J)\).

**Theorem 1.1** (Kashiwara, see \([\mathcal{SST}_{00}, \text{Theorem 1.4.19}]\)). Let \(J\) be a left \(D_{\bar{X}}\)-ideal such that \(D_{\bar{X}}/J\) is holonomic, and let \(p\) be a nonsingular point of \(D_{\bar{X}}/J\). Then \(\dim_{\mathbb{C}}(\text{Sol}_p(D_{\bar{X}}/J))\) is finite, independent of \(p\), and equal to \(\text{rank}(D_{\bar{X}}/J)\). \(\square\)

Given \(w \in \mathbb{R}^n, (-w, w) \in \mathbb{R}^{2n}\) induces a filtration on \(D_{\bar{X}}\). Note that \(\text{gr}^{(-w, w)}D_{\bar{X}} \cong D_{\bar{X}}\), which is in contrast to the case in \([1,1]\) for a weight vector \(L \in \mathbb{R}^{2n}\); however, for \(P \in D_{\bar{X}}\), we may analogously define \(\text{in}^{(-w, w)}(P) \in D_{\bar{X}}\). If \(J \subseteq D_{\bar{X}}\) is a left \(D_{\bar{X}}\)-ideal, then set \(\text{gr}^{(-w, w)}(J) := \langle \text{in}^{(-w, w)}(P) \mid P \in J \rangle\), which by abuse of notation, we view as a left \(D_{\bar{X}}\)-ideal.

**Definition 1.2.** Let \(J \subseteq D_{\bar{X}}\) be a left \(D_{\bar{X}}\)-ideal. We say that \(w \in \mathbb{R}^n\) is a generic weight vector for \(D_{\bar{X}}/J\) if there exists an \((n\)-dimensional\) open rational polyhedral cone \(\Sigma \subseteq \mathbb{R}^n_{>0}\) with trivial lineality space such that \(w \in \Sigma\) and for all \(w' \in \Sigma\), \(\text{gr}^{(-w', w')}(J) = \text{gr}^{(-w', w')}(J)\).

**Definition 1.3.** Let \(J \subseteq D_{\bar{X}}\) be a left \(D_{\bar{X}}\)-ideal, and assume that \(w \in \mathbb{R}^n\) is a generic weight vector for \(D_{\bar{X}}/J\). Write \(\log(x) := (\log(x_1), \ldots, \log(x_n))\). A formal solution \(\phi\) of \(D_{\bar{X}}/J\) is called a basic Nilsson solution of \(D_{\bar{X}}/J\) in the direction of \(w\) if it has the form \(\phi(x) = \sum_{u \in C} x^{w+u}p_u(\log(x))\), for some vector \(v \in \mathbb{C}^n\), such that the following conditions are satisfied:

1. \(C\) is contained in \(\Sigma^* \cap \mathbb{Z}^n\), where \(\Sigma^*\) is as in Definition 1.2. Here, \(\Sigma^*\) is the dual cone of \(\Sigma\), which consists of the vectors \(u \in \mathbb{R}^n\) with \(u \cdot w' \geq 0\) for all \(w' \in \Sigma\).
2. The \(p_u\) are polynomials, and there exists \(k \in \mathbb{Z}\) such that \(\deg(p_u) \leq k\) for all \(u \in C\).
3. \(p_0 \neq 0\).

The set \(\text{supp}(\phi) := \{u \in C \mid p_u \neq 0\}\) is called the support of \(\phi\). The \(\mathbb{C}\)-span of the basic Nilsson solutions of \(D_{\bar{X}}/J\) in the direction of \(w\) is called the space of formal Nilsson solutions of \(D_{\bar{X}}/J\) in the direction of \(w\), denoted \(\mathcal{N}_w(D_{\bar{X}}/J)\).

**Theorem 1.4** (\([\mathcal{SST}_{00}, \text{Theorems 2.5.1 and 2.5.14}]\)). Let \(J\) be a left \(D_{\bar{X}}\)-ideal such that \(D_{\bar{X}}/J\) is regular holonomic. If \(w \in \mathbb{R}^n\) is a generic weight vector for \(D_{\bar{X}}/J\), then
\[\dim_{\mathbb{C}}(\mathcal{N}_w(D_{\bar{X}}/J)) = \text{rank}(D_{\bar{X}}/J)\].

Further, there is an open set \(U \subseteq \bar{X} \setminus \text{Sing}(D_{\bar{X}}/J)\) such that the basic Nilsson solutions of \(D_{\bar{X}}/J\) in the direction of \(w\) simultaneously converge at each \(p \in U\) and form a basis for \(\text{Sol}_p(D_{\bar{X}}/J)\). \(\square\)

### 1.3. Torus actions in the Weyl algebra.

**Convention 1.5.** Fix a \(d \times n\) integer matrix \(A\) of rank \(d\) whose columns span \(\mathbb{Z}^d\) as a lattice. We denote the columns of \(A\) by \(a_1, \ldots, a_n\). Throughout this article we assume that \(A\) is pointed; in other words, there exists \(h \in \mathbb{R}^d\) such that \(h \cdot a_i > 0\) for all \(i = 1, \ldots, n\). \(\square\)

We consider the action of the torus \(T = (\mathbb{C}^*)^d\) on \(\bar{X}\) given by
\[(t_1, \ldots, t_d) \circ (x_1, \ldots, x_n) := (t^{a_1}x_1, \ldots, t^{a_n}x_n)\].
We let \( \mathcal{T} \) denote the torus orbit of \( 1_n \) in \( \bar{X} \). By our assumptions on \( A \), the map \( T \to \mathcal{T} \) given by \( t \mapsto t \circ 1_n \) is an isomorphism of varieties. This torus action induces an action of \( T \) on \( D_{\bar{X}} \) via
\[
t \circ x_i := t^{a_i} x_i, \quad t \circ \partial_{x_i} := t^{-a_i} \partial_{x_i}, \quad \text{for } 1 \leq i \leq n \text{ and } t = (t_1, \ldots, t_d) \in T.
\] (1.1)
An element \( P \in D_{\bar{X}} \) is torus homogeneous of weight \( a \in \mathbb{Z}^d \) if \( t \circ P = t^a P \) for all \( t \in T \). A left \( D_{\bar{X}} \)-ideal is torus equivariant if and only if it is generated by torus homogeneous elements.

The torus action on \( D_{\bar{X}} \) imposes a multigrading, called the \( A \)-grading, given by
\[
\deg(x_i) := a_i \quad \text{and} \quad \deg(\partial_{x_i}) := -a_i.
\]
The grading group is \( \mathbb{Z} \cdot A = \mathbb{Z}^d \), the Abelian group generated by the columns of \( A \). An element \( P = \sum \lambda_{u,v} x^u \partial^v \in D_{\bar{X}} \) has degree \( a \in \mathbb{Z} \cdot A = \mathbb{Z}^d \) if and only if \( Au - Av = a \) whenever \( \lambda_{u,v} \neq 0 \).

The term \( A \)-graded is used to emphasize that the torus action is defined by \( A \).

The torus action also passes to germs of functions on \( \bar{X} \). For \( t \in T \) and \( x \in \bar{X} \), denote \( t \circ x := (t^{a_1} x_1, \ldots, t^{a_n} x_n) \). A function that satisfies \( \varphi(t \circ x) = t^{\beta} \varphi(x_1, \ldots, x_n) \) for all \( t \) in an open subset of \( T \) has degree \( \beta \in \mathbb{C}^d \). Note that any \( \beta \in \mathbb{C}^d \) may occur as a weight of a function. The subspace of \( \mathcal{O}_{p,\bar{X}}^{an} \) consisting of \( A \)-graded germs of degree \( \beta \) is denoted \( \left[ \mathcal{O}_{p,\bar{X}}^{an} \right]_\beta \).

**Remark 1.6.** The space \( \left[ \mathcal{O}_{p,\bar{X}}^{an} \right]_\beta \) can be viewed as the set of \( \beta \)-eigenvectors for the Euler operators \( E \) of \( A \) from Definition 0.1. As \( \beta \) defines a character of \( T \), the action of \( T \) on \( X \) (really, on \( T^*X \)) can be viewed as an action of its Lie algebra. Loosely, the operators in \( E \) are the (Fourier transforms of the) pushforwards of generators of the Lie algebra of the torus. Thus, the solutions of the Euler operators \( E - \beta \) are precisely the functions which are infinitesimally torus homogeneous of weight \( \beta \). In other words,
\[
\text{Hom}(D_{\bar{X}}/(E-\beta), \mathcal{O}_{p,\bar{X}}^{an}) \cong \left[ \mathcal{O}_{p,\bar{X}}^{an} \right]_\beta.
\]

**Remark 1.7.** If \( J \) is an \( A \)-graded left \( D_{\bar{X}} \)-ideal, then any series solution of \( D_{\bar{X}}/J \) may be decomposed as a sum of \( A \)-homogeneous series, each of which is also a solution of \( D_{\bar{X}}/J \). In particular, for a generic weight vector \( \mathcal{W} = (D_{\bar{X}}/J) \) of Definition 1.3 is spanned by basic Nilsson series \( \phi(x) = \sum_{u \in \mathbb{C}} x^{e+u}p_u(\log(x)) \) whose support is contained in \( \Sigma \cap \ker_{\mathbb{C}}(A) \). Further, if \( \langle E - \beta \rangle \subseteq J \), then \( Av = \beta \).

2. Categories of \( A \)-graded \( D \)-modules

In this section, \( Y \) is either \( X \) or \( \bar{X} \), and \( \mathcal{D}\text{-mods}(Y) \) is the category of finitely generated \( D_{\bar{Y}} \)-modules. We introduce certain subcategories of \( \mathcal{D}\text{-mods}(Y) \) that will be used in this article, and prove some of their basic properties.

**Definition 2.1.** Let \( \mathcal{A}\text{-mods}(Y) \) denote the category of finitely generated \( A \)-graded \( D_{\bar{Y}} \)-modules with \( A \)-graded morphisms of \( A \)-degree zero. We let \( \mathcal{A}\text{-hol}(Y) \) denote the full subcategory of \( \mathcal{A}\text{-mods}(Y) \) given by \( A \)-graded holonomic \( D_{\bar{Y}} \)-modules.

**Remark 2.2.** Recall that \( \mathcal{T} = T \circ 1_n \cong T \). The ring of differential operators on \( \mathcal{T} \) is the subgroup \( D_{\mathcal{T}} \) of \( D_{\bar{X}}/(\langle x^u - 1 \mid u \in \mathbb{Z}^n, Au = 0 \rangle \cdot D_{\bar{X}}) \) generated by the monomials \( x^u \) with \( u \in \mathbb{Z}^n \) and the operators \( E_1, \ldots, E_d \). This is indeed a ring and is isomorphic to \( D_T \) via \( x^u \mapsto t^{Au} \) for \( u \in \mathbb{Z}^n \) and \( E_i \mapsto t^{i} \partial_t \) for \( i = 1, \ldots, d \). Note that \( [D_{\mathcal{T}}]^T = \mathbb{C}[E_1, \ldots, E_d] := \mathbb{C}[E] \) is (isomorphic to) a polynomial ring in \( d \) variables generated by the \( T \)-equivariant vector fields on \( X \). The containment \( \bar{D}_{\bar{X}} \subseteq D_{\bar{X}} \) is such that we may endow \( D_{\bar{X}} \) with a \( [D_{\mathcal{T}}]^T \)-action using the \( E_i \). We shall use this lifted action in the sequel without further mention.
Definition 2.3. We denote by $\mathcal{T}$-$\text{hol}(Y)$ the smallest full subcategory of $A$-$\text{mod}(Y)$ that contains each module $M$ with the property that, for each homogeneous element $\gamma \in M$, the $D_\mathcal{T}$-module $D_\mathcal{T}/(D_\mathcal{T} \cdot \text{ann}_{D_\mathcal{T}}(\gamma))$ is holonomic.

Note that there is a natural functor from $\mathcal{T}$-$\text{hol}(X)$ to $\mathcal{T}$-$\text{hol}(X)$ given by restriction to $X$. Also, note that the categories $A$-$\text{mod}(Y)$, $A$-$\text{hol}(Y)$, and $\mathcal{T}$-$\text{hol}(Y)$ are all Abelian, and $A$-$\text{mod}(Y)$ and $A$-$\text{hol}(Y)$ are clearly closed under extensions.

Lemma 2.4. The category $\mathcal{T}$-$\text{hol}(Y)$ is closed under extensions.

Proof. Let $0 \to M \to N \to P \to 0$ be a short exact sequence in $A$-$\text{mod}(Y)$ with $M$ and $P$ in $\mathcal{T}$-$\text{hol}(Y)$. For any homogeneous $\gamma \in N$, let $J = D_\mathcal{T} \cdot \text{ann}_{D_\mathcal{T}}(\gamma + M)$, the annihilator of $\gamma + M \in P$, be generated by $P_1, \ldots, P_k \in [D_\mathcal{T}]^T$. Then $P_1 \cdot \gamma, \ldots, P_k \cdot \gamma$ all lie in $M$ and are hence annihilated by ideals $I_1, \ldots, I_k$ with generators in $[D_\mathcal{T}]^T$. Therefore

$$D_\mathcal{T} \cdot \text{ann}_{D_\mathcal{T}}(\gamma) \supseteq \left(\bigcap_j I_j \cdot P_1\right) + \cdots + \left(\bigcap_j I_j \cdot P_k\right) \supseteq \left(\bigcap_j I_j\right) \cdot J.$$ 

Note that if $J_1$ and $J_2$ are left $D_\mathcal{T}$-ideals such that each $D_\mathcal{T}/J_i$ is holonomic, then $D_\mathcal{T}/(J_1 \cap J_2)$ is also holonomic because there is an exact sequence

$$0 \to \frac{D_\mathcal{T}}{J_1 \cap J_2} \to \frac{D_\mathcal{T}}{J_1} \oplus \frac{D_\mathcal{T}}{J_2} \to \frac{D_\mathcal{T}}{J_1 + J_2} \to 0.$$ 

Since each $D_\mathcal{T}/J_i$ is holonomic by assumption, so is $D_\mathcal{T}/\bigcap_j I_j$. Now if $J_1, J_2$ are such that $D_\mathcal{T}/J_i$ are holonomic for each $i$, then $D_\mathcal{T}/(J_1 J_2)$ is also holonomic. Indeed, the exact sequence

$$0 \to \frac{J_2}{J_1 J_2} \to \frac{D_\mathcal{T}}{J_1 J_2} \to \frac{D_\mathcal{T}}{J_2} \to 0$$

shows that it suffices to prove holonomicity for $J_2/(J_1 J_2)$, which is the quotient of the holonomic module $(D_\mathcal{T}/J_1)^k$ under the map induced by any surjection $D_\mathcal{T}^k \twoheadrightarrow J_2$. Thus $D_\mathcal{T}/(\bigcap_j I_j \cdot J)$ is holonomic, so the same is true for $D_\mathcal{T}/D_\mathcal{T} \cdot \text{ann}_{D_\mathcal{T}}(\gamma)$, as desired. \hfill $\square$

Theorem 2.5. The category $A$-$\text{hol}(Y)$ is a subcategory of $\mathcal{T}$-$\text{hol}(Y)$.

Proof. We begin by proving the result when the matrix $A$ consists of the top $d$ rows of an $n \times n$ identity matrix and $Y = \bar{X}$. In this case, the torus $\mathcal{T}$ is simply $\text{Var}(x_{d+1} - 1, \ldots, x_n - 1) \subseteq X$. Thus $D_\mathcal{T}$ is the $\mathbb{C}[x_1^\pm, \ldots, x_d^\pm]$-algebra generated by $\theta_1, \ldots, \theta_d$, while the invariant ring $[D_\mathcal{T}]^T$ is $\mathbb{C}[\theta_1, \ldots, \theta_d]$. If $M$ is a holonomic $A$-graded $D_\bar{X}$-module and $\gamma \in M$ is a fixed homogeneous element, then the annihilator of $\gamma$ is a holonomic $A$-graded ideal, say $I := \text{ann}_{D_\bar{X}}(\gamma)$. As such, the $b$-function $b_{\gamma,i}(s)$ for restriction to $x_i = 0$, $1 \leq i \leq d$, exists (i.e., is nonzero) and satisfies

$$b_{\gamma,i}(\theta_i) \in I + V_i^{-1}D_\bar{X}. \quad (2.1)$$

Here $V_i^*D_\bar{X}$ denotes the Kashiwara–Malgrange filtration on $D_\bar{X}$, which is given by

$$V_i^*D_\bar{X} := \text{Span}_\mathbb{C}\{x^u \partial^v \in D_\bar{X} \mid v_i - u_i \leq k\}.$$ 

(A discussion of $b$-functions for restriction can be found in [SST00, \S 5.1-5.2].) Taking the $A$-degree zero part of any equation expressing the containment $\text{(2.1)}$ shows that $b_{\gamma,i}(\theta_i) \in I$ because no element of $V_i^{-1}D_\bar{X}$ has $A$-degree 0. (Recall that $A$ consists of the top $d$ rows of an $n \times n$ identity matrix.) In other words, for all $i$, the $D_\bar{X}$-annihilator of $\gamma$ contains a polynomial in $E_i$. Now since $\mathbb{C}[\theta_i]/(b_{\gamma,i}(\theta_i))$ is Artinian, so is the quotient $\mathbb{C}[\theta_1, \ldots, \theta_d]/\text{ann}_{D_\mathcal{T}}(\gamma)$ of $\mathbb{C}[\theta_1, \ldots, \theta_d]/\sum_i(\gamma_i)$. Theorem 2.7 below completes this special case.
We next observe that the case $Y = X$ follows from the case $Y = \bar{X}$ (for any fixed $A$). Indeed, if $M$ is holonomic and $A$-graded on $X$, then the pushforward of $M$ to $\bar{X}$ (which is just $M$, viewed as a $D_{\bar{X}}$-module) has the same properties and is therefore in $\mathcal{F}_{\text{hol}}(\bar{X}) \supseteq \mathcal{F}_{\text{hol}}(X)$.

We next consider $Y = X$ but general $A$. The theory of elementary divisors ascertains the existence of matrices $P \in \text{GL}(d, \mathbb{Z})$ and $Q \in \text{GL}(n, \mathbb{Z})$ such that $PAQ$ is the top $d$ rows of a diagonal $n \times n$ matrix. Substituting $PA$ for $A$ is just a change of coordinates on the grading group and the Euler operators. On the other hand, replacing $A$ by $AQ$ corresponds to the change on the grading group and the Euler operators induced by the monomial change of coordinates $x \mapsto x^\ell$ on $X$. This change is $T$-invariant on the category of $D_X$-modules, and thus also on $A_{\text{hol}}(X)$ and $\mathcal{F}_{\text{hol}}(X)$. Now $\mathbb{Z}A = \mathbb{Z}^d$ implies that $PAQ$ can be chosen to equal the first $d$ rows of an $n \times n$ identity matrix, which settles the case that $A$ is arbitrary and $Y = X$.

Finally, we reduce the case $Y = \bar{X}$ to the case $Y = X$. To that end, use induction on $n$. Let $i: X \hookrightarrow \bar{X}$ be the natural embedding and fix a module $M$ in $A_{\text{hol}}(\bar{X})$. For $n = 0$, the result is trivial. For $n = 1$, consider the exact sequence of local cohomology

$$0 \to H^0_{x_1}(M) \to M \to i_* i^* M \to H^1_{x_1}(M) \to 0.$$ 

The outer terms are supported in $x_1 = 0$. By Kashiwara equivalence, the case $n = 0$, and since $\mathcal{F}_{\text{hol}}(\bar{X})$ is closed under extensions, the proposition holds for $M$ precisely if it holds for $i_* i^* M$. But we already proved it holds for $i^* M$, so the case $n = 1$ is proven.

The general case is slightly more involved, but it suffices to indicate the case $n = 2$. Given $i_1: X_1 = \bar{X} \setminus \text{Var}(x_1) \hookrightarrow \bar{X}$ and $i_2: X_2 = X_1 \setminus \text{Var}(x_2) \hookrightarrow X_1$, there are two exact sequences:

$$0 \to H^0_{x_1}(M) \to M \to i_1_* i_1^* M \to H^1_{x_1}(M) \to 0 \quad \text{and} \quad 0 \to H^0_{x_2}(i_1_* i_1^* M) \to i_2_* i_2^* i_1^* M \to i_2_* i_2^* i_1^* i_1^* M \to H^1_{x_2}(i_1_* i_1^* M) \to 0.$$

By Kashiwara equivalence and the case $n = 1$, the outer terms in both sequences satisfy the proposition. As $\mathcal{F}_{\text{hol}}(\bar{X})$ is closed under extensions, the proposition holds for $M$ if and only if it holds for $i_1_* i_1^* M$, and that happens if and only if it holds for $i_2_* i_2^* i_1^* i_1^* M$. However, the latter module is the pushforward to $\bar{X}$ of the restriction of $M$ to $X$, so the case $n = 2$ follows from the case $Y = X$ above.

**Example 2.6.** The module $D_{\bar{X}}/(D_{\bar{X}} \cdot E)$ is in $\mathcal{F}_{\text{hol}}(\bar{X})$ but does not belong to $A_{\text{hol}}(\bar{X})$. 

The following result can be used to give alternative descriptions for the objects of $\mathcal{F}_{\text{hol}}(Y)$, where $Y$ is still $X$ or $\bar{X}$. The equivalence of the second and last items in Theorem 2.7 was proven in [SST00, Proposition 2.3.6] for cyclic modules, using different methods.

**Theorem 2.7.** The following are equivalent for a finitely generated $A$-graded $D_{\mathcal{F}}$-module $M$:

1. $M$ is regular holonomic.
2. $M$ is holonomic.
3. $M$ is $L$-holonomic for all $L = (L_x, L_\partial_x)$ on $X$ with $L_x + L_\partial_x = c \cdot 1_n$ for some $c \in \mathbb{Q}_{>0}$.
4. $\mathbb{C}[E]/\text{ann}_{\mathbb{C}[E]}(\gamma)$ is Artinian for all $\gamma \in M$.

**Proof.** The isomorphism between $D_{\mathcal{F}}$ and $D_{\mathcal{T}}$ in Remark 2.2 induces an equivalence of categories between $A\text{-mod}_{\mathcal{F}}(\mathcal{F})$ and the category of finitely generated left $D_{\mathcal{T}}$-modules that are equivariant with respect to the action induced by $\mathcal{T}$ acting on itself by multiplication. Under this equivalence of categories, (regular) holonomic modules correspond to (regular) holonomic modules. Moreover,
the filtration on $D_\mathcal{F}$ induced by a weight vector $L = (L_x, L_{\partial_x})$ with $L_x + L_{\partial_x} = c \cdot 1_n$ for some $c > 0$ corresponds, under our isomorphism, to the filtration induced on $D_T$ by the weight vector $((AL_x)^t, c \cdot 1_d - (AL_x)^t)$. Given these considerations, it is enough to prove the statements in the context of $D_T$-modules.

First note that the module $D_T/((t_i \partial_i - \beta_i \mid i = 1, \ldots, d) \cdot D_T)$ is regular holonomic and $L'$-holonomic for any $L' = (L'_t, L'_{\partial_t}) \in \mathbb{C}^{2d}$ such that $L'_t + L'_{\partial_t} = c \cdot 1_d$ for some constant $c > 0$. Regularity follows from the fact that these are Fuchsian equations in disjoint variables, while $L'$-holonomicity is clear. This implies that if $I$ is an equivariant left $D_T$-ideal whose degree zero part $I_0$ is an Artinian ideal in the polynomial ring $\mathbb{C}[t_1 \partial_{t_1}, \ldots, t_d \partial_{t_d}]$, then the module $D_T/I$ is regular holonomic, and $L'$-holonomic, since the categories involved are Abelian and closed under extensions. This shows that (4) implies (1), (2), and (3) in the cyclic case.

For the reverse implications, still in the cyclic case, suppose that $I$ is an equivariant left $D_T$-ideal. If $\mathbb{C}[t \partial_t]/I_0 = \mathbb{C}[t \partial_t]/\text{ann}_{\mathbb{C}[t \partial_t]}(1)$ is not Artinian, then $D_T/I$ has infinite rank, and therefore cannot be $L'$-holonomic or regular holonomic. The previous sentence uses the fact that $L'$-holonomic modules are holonomic [SST00, Theorem 1.4.12].

Finally, we consider the general case. Let $M$ be a finitely generated $A$-graded equivariant $D_\mathcal{F}$-module. Since the categories involved are closed under extensions, it is not hard to show that $M$ is holonomic ($L$-holonomic, regular holonomic) if and only if it has a finite composition chain such that each composition factor is cyclic and holonomic ($L$-holonomic, regular holonomic). Note that the maximal length of such a composition chain depends only on $M$ (and is called the holonomic length of $M$).

Let $\gamma \in M$, and let $N$ be the $\mathcal{F}$-submodule of $M$ generated by $\gamma$. The $\mathbb{C}[E]$-annihilator of $\gamma$ as an element of $M$ is equal to the $\mathbb{C}[E]$-annihilator of $\gamma$ as an element of $N$. Since $N$ is cyclic, we have already shown that $N$ satisfies (1), (2), or (3) precisely if it satisfies (4). But $M/N$ has smaller holonomic length than $M$, so the result follows by induction.

\section{Torus invariants of $D_X$-modules}

In this section, we consider the exact functor $[-]^T : M \mapsto M_0$ on $A$-modules $(X)$, which selects the $A$-degree zero part (that is, the torus invariant part) of a module $M$. The natural target of this functor is the category of $[D_X]^T$-modules. Viewing $X$ as the product $(X/\mathcal{F}) \times \mathcal{F}$, $M_0$ inherits an action by $D_{X/\mathcal{F}}$. Any étale morphism $X/\mathcal{F} \to Z$ induces (by $D$-affinity) an action of $D_Z$ on $D_{X/\mathcal{F}}$, and hence on the category of $D_{X/\mathcal{F}}$-modules. This section explains how we may consider the category of $D_Z$-modules to be the target of the invariant functor $[-]^T$.

For our purposes, the important cases are those for which the composition $X \to X/\mathcal{F} \to Z$ is a monomial map. These situations are parameterized by two pieces of data: the possible splittings of $X$ and the Gale duals of $A$. We encode this information in two matrices, $\tilde{A}$ and $B$, and denote our invariants functor $[-]^T$ by $\Delta_{\tilde{A}, B}^T$, as defined in \cite{3.4}. We construct this functor in Definition \ref{3.6} and then show in Theorem \ref{3.6} that, upon restricting its source to $\mathcal{F}$-$\text{hol}(X)$, it behaves well with respect to several $D$-module theoretic properties.

\textbf{Notation 3.1.} If $G$ is an integer $p \times q$ matrix then we denote by $\mu_G$ the monomial morphism from $(\mathbb{C}^*)^p$ to $(\mathbb{C}^*)^q$ that sends $v \in (\mathbb{C}^*)^p$ to $v^G = (v^{g_1}, \ldots, v^{g_q})$, where $g_1, \ldots, g_q$ are the columns of $G$. We also denote the corresponding morphism on the structure sheaves by $\mu_G$. For example, $\mu_A : T \to X$ is the map with image $\mathcal{F}$. \hfill $\diamond$
Recall that \( m = n - d \) and \( \mathbb{Z}A = \mathbb{Z}^d \). The splittings of \( X \) that factor through monomial maps from \( X \) to \( (X/\mathcal{T}) \times T \) are in bijection with those matrices \( \tilde{A} \in \text{GL}(n, \mathbb{Z}) \) whose top \( d \) rows agree with \( A \). We denote the bottom \( n - d \) rows of such \( \tilde{A} \) by \( \tilde{A}^\perp \). We let \( C^\perp \) and \( C \) respectively denote the \( n \times d \) and \( n \times m \) matrices that form the left and right parts of the inverse matrix \( \tilde{C} \) to \( \tilde{A} \). Note that \( \mu_C : X \to X/\mathcal{T} \) comes from a splitting of \( X \).

**Convention 3.2.** An \( n \times m \) integer matrix \( B \) is a Gale dual of \( A \) if the columns of \( B \) span \( \ker Q(A) \). For the remainder of this article, fix a Gale dual \( B \) of \( A \) and a matrix \( \tilde{A} \in \text{GL}(n, \mathbb{Z}) \) whose top \( d \) rows agree with \( A \). Since \( \tilde{A}\tilde{C} \) is the identity, \( AC = 0 \). Thus the equation \( AB = 0 \) implies the existence of a full rank integer \( m \times m \) matrix \( K \) such that \( B = CK \), inducing \( \mu_K : X/\mathcal{T} \to Z \) via \( y \mapsto y^K \) as in Notation 3.1. Call the composition \( \mu_K \circ \mu_C : X \to X/\mathcal{T} \to Z \) the split Gale morphism attached to \((\tilde{A}, B)\).

**Notation 3.3.** Suppose \((C^*)^q\) has a monomial action on \((C^*)^p\) given by a matrix \( H \), and let \( \mathcal{D} := (C^*)^q \circ 1_p \subseteq (C^*)^p \). Let \( \tilde{H} \in \text{GL}(p, \mathbb{Z}) \) be such that its top \( q \) rows agree with \( H \), and denote its bottom \( p - q \) rows by \( H^\perp \). Let \( \tilde{G} = (\tilde{H})^{-1} \), and let \( G \) denote the matrix given by the final \( p - q \) columns of \( \tilde{G} \). Using \( v \) and \( y \) for the respective coordinates of \((C^*)^p \) and \((C^*)^{p-q} \cong (C^*)^p / \mathcal{D} \), set

\[
\chi_i := v_i \partial_{v_i}, \quad \chi := [\chi_1 \chi_2 \cdots \chi_p], \quad \lambda_i := y_i \partial_{y_i}, \quad \text{and} \quad \lambda := [\lambda_1 \lambda_2 \cdots \lambda_{p-q}].
\]

In accordance with Notation 3.1, the matrix \( G \) induces a quotient map \( \mu_G : (C^*)^p \to (C^*)^{p-q} \cong (C^*)^p / \mathcal{D} \), and via \( \mu_G \), \( y \) acts as \( vG \) and \( \lambda \) acts as \( H^\perp \chi \), both of which lie in \([D(C^*)^p]^{(C^*)^q}\). From this we obtain an action of \( D(C^*)^{p-q} \cong \mathbb{C}[y^{\pm 1}]\langle \lambda \rangle \) on \([D(C^*)^p]^{(C^*)^q}\). We denote by \( \Upsilon_G \) the process of endowing a \([D(C^*)^p]^{(C^*)^q}\)-module with a \( D(C^*)^p\)-module structure via \( \mu_G \).

We will apply Notation 3.3 in two cases. The first is for \( \tilde{A} \) and \( \mu_C : X \to X/\mathcal{T} \), and the second is for \( K^{-1} \) and \( \mu_K : X/\mathcal{T} \to Z \), where \( X/\mathcal{T} \) has a trivial torus action. Note that the functors \( \Upsilon_{\tilde{C}}^{-1} \) and \( \Upsilon_K^{-1} \) are related to the maps in the split Gale morphism \( X \to X/\mathcal{T} \to Z \) attached to \((\tilde{A}, B)\).

**Definition 3.4.** Using Convention 3.2 and Notation 3.3, define the functor

\[
\Delta_B^\tilde{A} := \Upsilon_K^{-1} \circ \Upsilon_{\tilde{C}}^{-1} \circ [-]^T : M \mapsto M_0,
\]

which sends a finitely generated \( D_X \)-module \( M \) to the \([D_X]^T\)-module \( M_0 \), viewed as \( D_Z \)-module.

**Example 3.5.** Suppose that \( \tilde{A} = \tilde{C} \) is an identity matrix and \( B = C \). In this situation, we may write \( X = T \times Z \), and \( T \) acts on \( X \) by \( t \circ (s, z) = (ts, z) \). Even in this case, it becomes apparent why, if we desire nice output, we must restrict \( \Delta_B^{\tilde{A}} \) to \( \mathcal{T}\text{-}\text{hol}(X) \). To wit, the torus invariants of \( D_{T \times Z} \) are generated by \( D_Z \) and \( \mathbb{C}[t_1 \partial_{t_1}, \ldots, t_d \partial_{t_d}] \). In particular, while \([D_{T \times Z}]^T = D_Z[t\partial_t] \) is a \( D_Z \)-module, it is not finitely generated. Fortunately, if \( M = \bigoplus_{\alpha \in \mathbb{Z}A} M_\alpha = \bigoplus_{\alpha \in \mathbb{Z}A} M_0 \cdot t^\alpha \) is in \( A\text{-}\text{mods}(T \times Z) \), then \([M]^T = M_0 \) is a finitely generated \( D_Z[t\partial_t] \)-module. In order for \( M_0 \) to be a finitely generated \( D_Z \)-module, more assumptions are needed to govern the impact of \( t\partial_t \); the holonomicity requirement in the definition of \( \mathcal{T}\text{-}\text{hol}(X) \) provides precisely that.

We are now prepared to state the main result of this section, which concerns the restriction of \( \Delta_B^{\tilde{A}} \) to \( \mathcal{T}\text{-}\text{hol}(X) \) (see Definition 2.3). This restriction guarantees that the output of the functor is a finitely generated \( D_Z \)-module and compatible with a number of \( D \)-module theoretic properties. Recall from [2] that we have the following inclusions of categories:

\[
A\text{-}\text{hol}(X) \subseteq \mathcal{T}\text{-}\text{hol}(X) \subseteq A\text{-}\text{mods}(X) \subseteq D\text{-}\text{mods}(X).
\]
**Theorem 3.6.** If $M$ is in $\mathcal{F}\text{-hol}(X)$, then $\Delta^\mathfrak{A}_B(M)$ is a finitely generated $D_Z$-module, so that we have a restricted functor

$$
\Delta^\mathfrak{A}_B: \mathcal{F}\text{-hol}(X) \rightarrow \mathfrak{D}\text{-mod}(Z).
$$

Further, the following statements hold for $M$ in $\mathcal{F}\text{-hol}(X)$.

1. Let $L := (L_x, L_{\partial_x}) \in \mathbb{Q}^{2n}$ be a weight vector with $L_x + L_{\partial_x} = c \cdot 1_n$ for some $c > 0$. Let $L' := (L_xB, c \cdot 1_m - L_x B) \in \mathbb{Q}^{2n}$. Then $M$ is $L$-holonomic if and only if $\Delta^\mathfrak{A}_B(M)$ is $L'$-holonomic.

2. A module $M$ in $A\text{-hol}(X)$ is regular holonomic if and only if $\Delta^\mathfrak{A}_B(M)$ is regular holonomic.

3. If $M$ has reducible monodromy representation, then so does $\Delta^\mathfrak{A}_B(M)$. If the columns of $B$ span $\ker Z(A)$ as a lattice, then the converse also holds.

Note that if $L = F$ induces the order filtration on $D_X$, then $L'$ induces the order filtration on $D_Z$. Thus, as a special case of Theorem 3.6(1), if $M$ is in $\mathcal{F}\text{-hol}(X)$, then $M$ is holonomic if and only if $\Delta^\mathfrak{A}_B(M)$ is holonomic.

**Proof of Theorem 3.6** This proof occupies the remainder of this section and will be accomplished through a series of reductions.

**Lemma 3.7.** If Theorem 3.6 holds when $K$ is the identity matrix, then it holds in general.

**Proof.** We wish to show that, for general $K$, Theorem 3.6 holds for the functor $\Upsilon^K_{\mathfrak{A}}$. Up to a coordinate change on the base and range, $\Upsilon^K_{\mathfrak{A}}$ corresponds to the covering map induced by a diagonal matrix. In this case, $D_{X/\mathcal{F}}$ is a free $D_Z$-module of rank $|\det(K)| = |\ker Z(A) : \mathbb{Z}B|$, and in particular, $\Upsilon^K_{\mathfrak{A}}$ preserves finiteness, as desired.

Suppose now $L = (L_y, L_{\partial_y})$ is a weight vector on $X/\mathcal{F}$ with $L_x + L_{\partial_x} = c \cdot 1_m$ for some $c > 0$. On $Z$, consider the weight $L' = (L'_y, L'_{\partial_y})$ given by $L'_y = L_y K$ and $L'_{\partial_y} = c \cdot 1_m - L_y K$. This is compatible with $z = y^K$, and the weights of $\lambda$ and $\eta$ are $c$. Hence to the $L$-filtered $D_{X/\mathcal{F}}$-module $M$ corresponds the $L'$-filtered $D_Z$-module $\Upsilon^K_{\mathfrak{A}}(M)$ (which as an underlying filtered set is exactly $M$). Hence, (1) of Theorem 3.6 holds for $\Upsilon^K_{\mathfrak{A}}$.

Part (2) follows from the fact that $\mu_K$ (see Notation 3.1) is an algebraic coordinate change that induces (locally on $X/\mathcal{F}$) a diffeomorphism. Exponential solution growth along the germ of an embedded curve on $X/\mathcal{F}$ is then equivalent to such growth along its image in $Z$.

Here only the first half of (3) applies, and this follows because $\Delta^\mathfrak{A}_B$, and thus $\Upsilon^K_{\mathfrak{A}}$, is exact. \[\square\]

We continue with the proof of Theorem 3.6, assuming now that $K$ is the identity matrix. In parallel to the proof of Theorem 2.5, we next prove the theorem in the case of the simplest diagonal torus action. In this situation, we write $X = T \times Z$, where $T$ acts on $X$ by scaling the first factor, namely $t \circ (s, z) = (ts, z)$, so the matrix $A$ consists of the first $d$ rows of the $n \times n$ identity matrix $\tilde{A}$.

**Lemma 3.8.** If $\tilde{A} = \tilde{C}$ and $K$ are identity matrices, then Theorem 3.6 holds.

**Proof.** Since $\tilde{A}$ is the identity map in this lemma, we suppress writing $\Upsilon^\mathfrak{A}_{\tilde{C}}$ in the remainder, so that $[M]^T$ represents $\Delta^\mathfrak{A}_B(M)$. With a slight abuse of notation, write $T = \mathcal{F}$ and $X = T \times Z$.

Recall that $\mathcal{F}\text{-hol}(T \times Z)$, $A\text{-hol}(T \times Z)$, $L$-holonomic $A$-graded modules on $T \times Z$ with $A$-degree zero morphisms, and regular holonomic $A$-graded modules on $T \times Z$ with $A$-degree zero
morphisms are all Abelian categories that are closed under extensions. Thus, in order to show \((1)\) and \((2)\) of Theorem 3.6 in this case, it is enough by Theorem 2.7 to consider a module \(N\) for which there exists a \(\beta \in \mathbb{C}^{*}\) such that for each nonzero homogeneous element \(\gamma \in N\), \(\text{ann}_{C[E]}(\gamma) = \langle t\partial_{t} - \beta - \deg(\gamma) \rangle\). Then \(D_{Z} \cdot \gamma = D_{Z}[E] \cdot \gamma = [D_{T \times Z}]^{T} \cdot \gamma\) for all \(\gamma \in N\). In other words, each \(t_{i}\partial_{t_{i}}\) acts on \(N\) as a multiplication by some scalar \(\beta_{i}\).

Since \([N]^{T} = N_{0}\) is a finitely generated \([D_{T \times Z}]^{T}\)-module and \([D_{T \times Z}]^{T} = D_{Z}[t\partial_{t}]\), we see that \([N]^{T}\) is a finitely generated \(D_{Z}\)-module. To continue, note that since \(N = \bigoplus_{\alpha \in \mathbb{Z}A} N_{0} \cdot t^{\alpha}\) and \(D_{T} = \bigoplus_{\alpha \in \mathbb{Z}A} \mathbb{C}[t\partial_{t}] \cdot t^{\alpha}\), we have that, as \(D_{T \times Z}\)-modules,

\[
N = \frac{DT}{\langle t\partial_{t} - \beta \rangle} \otimes_{\mathbb{C}} [N]^{T},
\]

with \(D_{T \times Z} = D_{T} \otimes_{\mathbb{C}} D_{Z}\), so that \(D_{T}\) is acting on \(D_{T}/(D_{T} \cdot \langle t\partial_{t} - \beta \rangle)\) and \(D_{Z}\) is acting on \([N]^{T}\).

For \((1)\) and \((2)\), consider a weight vector \(L = (L_{t}, L_{z}, L_{\partial_{t}}, L_{\partial_{z}})\), where \((L_{t}, L_{z}) = c \cdot 1_{n}\) for some rational number \(c > 0\). Set \(L_{T} := (L_{t}, L_{\partial_{t}})\) and \(L_{Z} := (L_{z}, L_{\partial_{z}})\), and note that the \(L'\) in the theorem is just \(L_{Z}\). Thus \([\text{gr}^{L}(D_{T \times Z})]^{T} = \text{gr}^{L_{T}}([D_{T}]^{T}) \otimes_{\mathbb{C}} \text{gr}^{L_{Z}}(D_{Z})\), and \((3.2)\) implies that

\[
\text{gr}^{L}(N) = \text{gr}^{L} \left( \frac{DT}{\langle t\partial_{t} - \beta \rangle} \otimes_{\mathbb{C}} [N]^{T} \right) = \text{gr}^{L_{T}} \left( \frac{DT}{\langle t\partial_{t} - \beta \rangle} \right) \otimes_{\mathbb{C}} \text{gr}^{L_{Z}}([N]^{T}).
\]

The module

\[
H(\beta) := D_{T}/\langle t\partial_{t} - \beta \rangle.
\]

is a regular connection on \(T\). It follows from \((3.3)\) that \(N\) is \(L\)-holonomic if and only if \([N]^{T}\) is \(L'\)-holonomic. Thus, \((1)\) and \((2)\) hold for \(N\), from which the general case of the lemma follows.

We now consider \((3)\), still assuming that for each homogeneous element \(\gamma \in N\), \(\text{ann}_{C[E]}(\gamma) = \langle t\partial_{t} - \beta - \deg(\gamma) \rangle\). Set

\[
N(t, z) := \mathbb{C}(t, z) \otimes_{\mathbb{C}[t, z]} N \quad \text{and} \quad D_{T \times Z}(t, z) := \mathbb{C}(t, z) \otimes_{\mathbb{C}[t, z]} D_{T \times Z}.
\]

From \((3.2)\), we see that

\[
N(t, z) = \left( \left( \mathbb{C}(t) \otimes_{\mathbb{C}[t]} H(\beta) \right) \otimes_{\mathbb{C}} \left( \mathbb{C}(z) \otimes_{\mathbb{C}[z]} [N]^{T} \right) \right) \otimes_{\mathbb{C}(t) \otimes_{\math{C}} C(t, z)} C(t, z).
\]

Since \(\mathbb{C}(t) \otimes_{\mathbb{C}[t]} H(\beta)\) is an irreducible \((\mathbb{C}(t) \otimes_{\mathbb{C}[t]} D_{T})\)-module, irreducibility of \(N(t, z)\) is equivalent to that of \(\mathbb{C}(z) \otimes_{\mathbb{C}[z]} [N]^{T}\), and \((3)\) holds for \(N\).

For the general case of \((3)\), suppose now that \(M\) is in \(\mathcal{F}\)-holo\(X\) such that for some nonzero \(\gamma \in M\), \(\mathbb{C}[t\partial_{t}]/\text{ann}_{\mathbb{C}[t\partial_{t}]}(\gamma)\) is Artinian, but \(\text{ann}_{\mathbb{C}[t\partial_{t}]}(\gamma)\) is not a maximal ideal. Then \(M\) has a filtration by \(A\)-graded modules

\[
0 = M^{(0)} \subsetneq M^{(1)} \subsetneq \cdots \subsetneq M^{(r)} = M,
\]

where \(r > 1\) and the \(\mathbb{C}[t\partial_{t}]\)-annihilators of nonzero homogeneous elements \(\gamma^{(i)}\) in the successive quotients \(M^{(i)}/M^{(i-1)}\) have the form \(\langle t\partial_{t} - \beta^{(i)} - \deg(\gamma^{(i)}) \rangle\) for suitable \(\beta^{(i)}\). Since \(M\) is finitely generated, one may choose such a finite filtration that works simultaneously for all homogeneous elements of \(M\). In particular, \(M\) does not have irreducible monodromy representation, since \(\mathbb{C}(t, z) \otimes_{\mathbb{C}[t, z]} M^{(1)}\) provides a nonzero nontrivial submodule of \(\mathbb{C}(t, z) \otimes_{\mathbb{C}[t, z]} M\). At the same time, applying the exact functor \(\mathbb{C}(z) \otimes_{\mathbb{C}[z]} [-]^{T}\) to \((3.5)\) also shows that \([M]^{T}\) has reducible monodromy representation, completing the proof of \((3)\), and thus of Lemma 3.8. \(\square\)
Continuing with the proof of Theorem 3.6 by Lemma 3.7 we are left to consider other choices for $A$ in Lemma 3.8 with $K$ still equal to the identity matrix, so that $Z = X/\mathcal{T} \cong (\mathbb{C}^*)^m$ and $D_Z$ acts on $D_X$ via $C$. Let $X' := (\mathbb{C}^*)^d \times (\mathbb{C}^*)^m$ and consider the change of coordinates $\mu_A: X' \to X$. The action of $T$ on $X'$ given by identifying $T$ with $(\mathbb{C}^*)^d$ agrees with the action of $T$ on $X$ through $\mu_A$. At the same time, $\mu_A$ identifies $X'/T$ with $X/\mathcal{T}$, so since $K$ is the identity matrix, $Z = X/\mathcal{T}$.

If $M$ is in $\mathcal{T}$-$\text{hol}(X)$, then $\Delta_B^A(M) = \Upsilon_{A'}([\mu_A^*(M)]^T)$, with the $D_{(\mathbb{C}^*)^m}$-action coming from the decomposition $X' = (\mathbb{C}^*)^d \times (\mathbb{C}^*)^m$. This implies parts (2) and (3) of Theorem 3.6. For part (1), note that the monomial map $\mu_A$ identifies the filtration on $D_X$ induced by $L$ with the filtration on $D_{X'}$ induced by $L \cdot (\bar{A})^{-1} = \bar{L} \cdot \bar{C} = (LC^\perp, LC)$. Thus $M$ is $L$-holonomic on $X$ if and only if $\mu_A^*(M)$ is $LC$-holonomic on $X'$. Now by Lemma 3.8, $\mu_A^*(M)$ is $(LC^\perp, LC)$-holonomic on $X'$ if and only if $\Upsilon_{A'C'}([\mu_A^*(M)]^T)$ is $LC$-holonomic on $Z = (\mathbb{C}^*)^m$, completing the proof. \qed

We conclude this section with an example to illustrate how a module $M$ with irreducible monodromy representation could have a reducible image under $\Delta_B^A$.

**Example 3.9.** Consider the case that $\bar{A}$ is the $2 \times 2$ identity matrix, $B = [0 \ 2]^t$, $K = [2]$, with $M = D_X/D_X \cdot \langle x_1 \partial_{x_1}, \partial_{x_2} - 1 \rangle = D_X \cdot \exp(x_2)$. Here,

$$\Delta_B^A(M) = \frac{D_Z \oplus D_Z \cdot z^{1/2}}{D_Z \cdot \langle (2z \partial_z, -1 \cdot z^{1/2}), (-z, (2z \partial_z - 1) \cdot z^{1/2}) \rangle}$$

is isomorphic to $D_Z/D_Z \cdot \langle 4z \partial_z^2 + 2\partial_z - 1 \rangle$, whose solution space is spanned by $\exp(\sqrt{z})$ and $\exp(-\sqrt{z})$. The fundamental group $\mathbb{Z}$ of the regular locus of $\Delta_B^A(M)$ acts on the solution space by switching the two distinguished generators. In particular, $f = \exp(\sqrt{z}) + \exp(-\sqrt{z})$ generates a monodromy invariant subspace. As $f$ is holomorphic on $z \neq 0, \infty$, it satisfies the first order equation $(\partial_z \cdot \frac{1}{f}) \bullet f = 0$. \qed

4. Torus invariants and $D_X$-modules

Let $i: X \hookrightarrow \bar{X}$ and $j: Z \hookrightarrow \bar{Z}$ be the natural inclusions. In Convention 3.2, we fixed a splitting of $X$, encoded by an $n \times n$ matrix $\bar{A}$, and a Gale dual $B$ of $A$. In Theorem 3.6, we examined the functor $\Delta_B^A: \mathcal{T}$-$\text{hol}(X) \to \mathcal{D}$-$\text{mod}(Z)$, which is given by taking torus invariants and adjusting module structure (see Definition 3.4).

**Definition 4.1.** The main subject of study in this article is the functor $\Pi_B^A$ on $\mathcal{T}$-$\text{hol}(\bar{X})$ given by

$$\Pi_B^A := j_+ \circ \Delta_B^A \circ i^*,$$

where $j_+$ is the $D$-module direct image and $i^*$ is the $D$-module inverse image.

In this section, we extend our results about $\Delta_B^A$ from §3 to statements for $\Pi_B^A$. In §5 we provide further consequences when we further restrict the source category of $\Pi_B^A$.

Note that functor $\Pi_B^A: \mathcal{T}$-$\text{hol}(\bar{X}) \to \mathcal{D}$-$\text{mod}(\bar{Z})$ is exact. Indeed, since $i: X \to \bar{X}$ is faithfully flat, $i^*$ is exact. Note also that $i^*$ sends $\mathcal{T}$-$\text{hol}(\bar{X})$ to $\mathcal{T}$-$\text{hol}(X)$, the restricted source category for $\Delta_B^A$. Since $\Delta_B^A$ essentially takes the $A$-degree 0 part of a module, it is exact. We thus obtain the result from the exactness of $j_+$. 
**Theorem 4.2.** If $M$ is in $\mathcal{T}_{\text{hol}}(\bar{X})$, then $\Pi_B^\text{A}(M)$ is a finitely generated $D_Z$-module. In particular, $\Pi_B^\text{A}: \mathcal{T}_{\text{hol}}(\bar{X}) \to \mathcal{D}_{\text{mods}}(Z)$, and the following statements hold.

1. Let $L := (L_x, L_\partial_x) \in \mathbb{Q}^{2n}$ be a weight vector with $L_x + L_\partial_x = c \cdot 1_n$ for some $c > 0$. Let $L' := (L_x, B, c \cdot 1_m - L_x B) \in \mathbb{Q}^{2m}$. If $M$ is $L$-holonomic, then $\Pi_B^\text{A}(M)$ is $L'$-holonomic.
2. If $M$ in $A_{\text{hol}}(\bar{X})$ is regular holonomic, then $\Pi_B^\text{A}(M)$ is regular holonomic.
3. If $M$ has reducible monodromy representation, then so does $\Pi_B^\text{A}(M)$. If the columns of $B$ span $\ker_Z(\bar{A})$ as a lattice, then the converse also holds.

**Proof.** Since $\mathcal{T} \subseteq X \subseteq \bar{X}$, Theorem 2.7 ensures that, with input from $\mathcal{T}_{\text{hol}}(\bar{X})$, $i^*$ returns objects in $\mathcal{T}_{\text{hol}}(X)$. Therefore by Theorem 3.6 and the fact that the direct image functor $j_+$ will preserve finite generation, $\Pi_B^\text{A}(M)$ is a finitely generated $D_Z$-module when $M$ is in $\mathcal{T}_{\text{hol}}(\bar{X})$. The properties considered in (1), (2), and (3) are compatible with the inverse and direct image functors in the directions of the implications stated, so the proof reduces to Theorem 3.6. □

The reverse implications in the first two items of Theorem 4.2 do not hold, because the restriction $i^*$ of a module which is not (regular, $L$-) holonomic might be (regular, $L$-) holonomic. In Theorem 7.1, we show that these converses do hold for binomial $D_X$-modules.

We conclude this section with a description of the characteristic varieties and singular loci of $\Pi_B^\text{A}(M)$ in terms of those of $\Delta_B^\text{A}(M)$. In §5.3 we will obtain more refined descriptions of these objects under additional assumptions. An application of the following Proposition 4.3 to the singular locus of the image under $\Pi_B^\text{A}$ of the $A$-hypergeometric system $H_A(\beta)$ can be found in §10.

**Proposition 4.3.** Let $M$ be a nonzero regular holonomic module in $\mathcal{T}_{\text{hol}}(\bar{X})$, and recall that $i: X \hookrightarrow \bar{X}$ is the natural inclusion. Then the characteristic variety of $\Pi_B^\text{A}(M)$ is the union of the characteristic variety of $\Delta_B^\text{A}(D_X \otimes_{D_X} M)$ with the coordinate hyperplane conormals in $\bar{Z}$:

\[
\text{Char}(\Pi_B^\text{A}(M)) = \text{Char}(\Delta_B^\text{A}(D_X \otimes_{D_X} M)) \cup \text{Var}(z_1\zeta_1, \ldots, z_m\zeta_m) \subseteq T^*\bar{Z}.
\]

Further, the singular locus of $\Pi_B^\text{A}(M)$ is the union of the singular locus of $\Delta_B^\text{A}(D_X \otimes_{D_X} M)$ with the coordinate hyperplanes in $\bar{Z}$:

\[
\text{Sing}(\Pi_B^\text{A}(M)) = \text{Sing}(\Delta_B^\text{A}(D_X \otimes_{D_X} M)) \cup \text{Var}(z_1z_2\cdots z_m) \subseteq \bar{Z}.
\]

**Proof.** Note that $D_X \otimes_{D_X} M = i^*M$. Since $\Pi_B^\text{A}(M)$ is the direct image of $\Delta_B^\text{A}(i^*M)$ along the open embedding $j: Z \hookrightarrow \bar{Z}$, [Gin86, Theorem 3.2] implies that the characteristic variety of $\Pi_B^\text{A}(M)$ is the union of the characteristic variety of $\Delta_B^\text{A}(i^*M)$ with the set of coordinate hyperplane conormals in $\bar{Z}$. In addition, saturation and projection commute with taking associated graded objects, so the final statement holds. □

5. Torus invariants for a fixed torus character

In this section, let $Y$ be $X$ or $\bar{X}$. We now restrict our attention to certain subcategories of $\mathcal{T}_{\text{hol}}(Y)$ called $\mathcal{T}_{\text{irred}}(Y)$ and $\mathcal{T}_{\text{irred}}(Y, \beta)$, over which we are able to obtain more detailed information about the functors $\Delta_B^\text{A}$ and $\Pi_B^\text{A}$, still following Convention 3.2. We provide explicit expressions for these functors, show how they affect solutions, and, under certain assumptions on $B$, describe
some of their geometric properties. In Corollary 5.10 we explain the relationship between lattice basis binomial \( D_X \)-modules and saturated Horn \( D_Z \)-modules.

**Definition 5.1.** Let \( \mathcal{T} \)-irre\( \mathcal{D}(Y) \) denote the subcategory of \( \mathcal{T} \)-hol\( (Y) \) consisting of modules \( M \) such that for each nonzero homogeneous element \( \gamma \in M \), \( \text{ann}_{\mathbb{C}[E]}(\gamma) \) is a maximal ideal in \( \mathbb{C}[E] \).

To provide a \( D \)-module theoretic understanding of the definition of \( \mathcal{T} \)-hol\( (Y) \), we note the parallel to Theorem 2.7. Recall that each \( A \)-graded \( D \)-module \( M \) has a filtration

\[
0 = M^{(0)} \subset M^{(1)} \subset \cdots \subset M^{(r)} = M,
\]

such that for each \( i \) and each nonzero homogeneous element \( \gamma \) in \( M^{(i)}/M^{(i-1)} \) has the form \( \langle E - \beta^{(i)} - \text{deg}(\gamma) \rangle \) for some \( \beta^{(i)} \). Thus, an \( A \)-graded \( D \)-module \( M \) is irreducible if and only if there is such a filtration with \( r = 1 \).

In the sequel, we consider subcategories of \( \mathcal{T} \)-irre\( \mathcal{D}(Y) \) given by fixing \( \beta \). To this end, let \( \mathcal{T} \)-irre\( \mathcal{D}(Y, \beta) \) denote the subcategory of \( \mathcal{T} \)-irre\( \mathcal{D}(Y) \) given by objects \( M \) such that for each nonzero homogeneous element \( \gamma \in M \), \( \text{ann}_{\mathbb{C}[E]}(M) = \langle E - \beta - \text{deg}(\gamma) \rangle \). Note that when \( \gamma \in D_Y \) is homogeneous, \( \gamma(E - \beta) = (E - \beta - \text{deg}(\gamma)) \gamma \). Thus, when \( M = D_Y/J \) is a cyclic element in \( \mathcal{T} \)-irre\( \mathcal{D}(Y) \), it lies in \( \mathcal{T} \)-irre\( \mathcal{D}(Y, \beta) \) precisely when \( \langle E - \beta \rangle \) is contained in \( J \).

**Example 5.2.** If \( I \subset \mathbb{C}[\partial_x] \) is an \( A \)-graded ideal, then \( D_X/(I + \langle E - \beta \rangle) \) is in \( \mathcal{T} \)-irre\( \mathcal{D}(X, \beta) \).

### 5.1. Explicit expressions for the functors \( \Delta^1_B \) and \( \Pi^1_B \)

We provide explicit computations of \( \Delta^1_B \) and \( \Pi^1_B \) on \( \mathcal{T} \)-irre\( \mathcal{D}(X, \beta) \). We begin by first relating \( [D_X]^T/(E - \beta) \) and \( D_Z \) via \( D_X/\mathcal{T} \).

**Convention 5.3.** For each \( \beta \in \mathbb{C}^d \), fix a vector \( \kappa(\beta) = \kappa \in \mathbb{C}^n \) such that \( A \kappa = \beta \). In addition, recall that the \( m \times m \) matrix \( K \) from Convention 3.2 has full rank, and therefore its Smith normal form is an integer diagonal matrix with nonzero diagonal entries called the elementary divisors of \( K \). Let \( \varkappa := (\varkappa_1, \ldots, \varkappa_m) \) denote the elementary divisors of \( K \).

**Proposition 5.4.** Let \( B \) be a Gale dual of \( A \) whose columns span \( \ker_Z(A) \) as a lattice, so that \( [D_X]^T \) is \( \mathbb{C} \)-spanned by monomials \( x^u \cdot \theta^\mu \), where \( v \in \mathbb{Z}^m \) and \( u \in \mathbb{N}^n \). Denote the rows of \( B \) by \( B_1, \ldots, B_n \). For \( v \in \mathbb{Z}^m \) and \( u \in \mathbb{N}^n \), define

\[
\delta_{B, \kappa}(x^u \cdot \theta^\mu) := z^v \prod_{i=1}^n (B_i \cdot \eta + \kappa_i)^{u_i}. \tag{5.1}
\]

This extends linearly to a surjective homomorphism of \( \mathbb{C} \)-algebras

\[
\delta_{B, \kappa} : [D_X]^T \longrightarrow D_Z,
\]

whose kernel is the (two-sided) \( [D_X]^T \)-ideal generated by the sequence \( E - \beta \).

**Corollary 5.5.** Let \( B \) be any Gale dual of \( A \), and let \( \kappa \) and \( \varkappa \) be as in Convention 5.3 and set \( \varepsilon_C := \sum_{i=1}^m c_i \). Then there is an isomorphism, denoted \( \Delta_{B, \kappa} \), given by the composition

\[
\delta_{C, \kappa + \varepsilon_C} : [D_X]^T/\langle E - \beta \rangle \sim \longrightarrow D_Z
\]

and the isomorphism induced by \( \mu_K \):

\[
D_Z \cong \bigoplus_{0 \leq k < \varkappa} D_Z \cdot z^{k/\varkappa}, \tag{5.2}
\]

where the comparison \( k < \varkappa \) is component-wise and \( k/\varkappa \) is the vector \( (k_1/\varkappa_1, \ldots, k_m/\varkappa_m) \).
Proof. The map $\delta_{C,\kappa+\varepsilon_C}$ is an isomorphism by Proposition 5.4, so it is enough to understand (5.2). First note that by identifying $y_i$ with $z_i^{1/\kappa_i}$,

$$\bigoplus_{0 \leq k < \kappa} D_Z \cdot z^{k/\kappa} \cong \frac{D_Z[y_1, \ldots, y_m]}{\langle y_i^{\kappa_i} - z_i \mid i = 1, \ldots, m \rangle}.$$ 

By Convention 5.3, there are matrices $P, Q \in \text{GL}(m, \mathbb{Z})$ such that $PKQ$ is the diagonal matrix whose diagonal entries are the components of $\kappa$. The maps $\mu_P$ and $\mu_Q$ both induce isomorphisms of $D_Z$; for the matrix $P$, whose columns are denoted by $p_1, \ldots, p_m$, this isomorphism is given by $z_i \mapsto z^{p_i}$ and $\eta_i \mapsto \sum_{j=1}^m p_{ij} \eta_j$, where $P^{-1} = [p_{ij}]$.

Since $P$ and $Q$ induce isomorphisms, we may assume that $K$ is diagonal with diagonal entries $\kappa_1, \ldots, \kappa_m$. In this case, the ring homomorphism $D_Z \rightarrow D_Z[y_1, \ldots, y_m]/\langle y_i^{\kappa_i} - z_i \mid i = 1, \ldots, m \rangle$ by $z_i \mapsto y_i$ and $\eta_i \mapsto \kappa_i \eta_i$ is clearly an isomorphism. \hfill $\square$

Proof of Proposition 5.4 If $Bv = Bv'$, then $v = v'$ because $B$ has full rank. Moreover, since the columns of $B$ span $\ker Z(A)$ as a lattice, the elements $x^{Bv} \theta^u$ form a basis of $[D_X]^T$ as a $\mathbb{C}$-vector space. Thus $\delta_{B,\kappa}$ is well-defined.

For $\delta_{B,\kappa}$ to be a ring homomorphism, it is enough to show that $\delta_{B,\kappa}(\theta_i^k x^{Bv}) = \delta_{B,\kappa}(\theta_i^k) \delta_{B,\kappa}(x^{Bv})$. This follows from two key identities that hold for any $1 \leq i \leq n$ and $k \in \mathbb{Z}$:

$$\theta_i^k x^{Bv} = x^{Bv} (\theta_i + B_i \cdot v)^k \quad \text{and} \quad [B_i \cdot \eta + \kappa_i]^k z^v = z^v [B_i \cdot v + B_i \cdot \eta + \kappa_i]^k.$$ 

For surjectivity, observe that $z_i^{\pm 1} = \delta_{B,\kappa}(x^{\pm B_{1i}})$. We thus need to show that $\eta_1, \ldots, \eta_m$ belong to the image of $\delta_{B,\kappa}$. For notational convenience, assume that the first $m$ rows of $B$ are linearly independent. Call $N$ the corresponding submatrix of $B$. Let $N_i^{-1}$ denote the $i$th row of $N^{-1}$. Then $\delta_{B,\kappa}(N_i^{-1} \cdot [\theta_1 - \kappa_1, \ldots, \theta_m - \kappa_m]) = \eta_i$.

Now consider $F \in \ker(\delta_{B,\kappa})$. Since $F \in [D_X]^T$, we can write $F = \sum_i x^{Bv_i} p_i(\theta)$, where the $u_i$ are distinct, the $p_i$ are polynomials in $n$ variables, and the sum is finite. Then $\delta_{B,\kappa}(F) = \sum_i z^{u_i} p_i(B \eta + \kappa) = 0$. Since the zero operator annihilates every monomial $z^\mu$ with $\mu \in \mathbb{Z}^m$, $0 = \delta_{B,\kappa}(F) \cdot z^\mu = \sum_i z^{u_i} p_i(B \mu + \kappa) z^\mu$, and, since the $v_i$ are distinct, $p_i(B \mu + \kappa) = 0$ for all $\mu \in \mathbb{Z}^m$ and for all $i$. Hence each $p_i$ vanishes on the Zariski closure of $\kappa + \ker Z(A)$, so by the Nullstellensatz, every $p_i(\theta)$ is an element of $\mathbb{C}[\theta] \cdot (E - \beta)$. It follows that $F$ belongs to the $[D_X]^T$-ideal generated by $E - \beta$, as desired. \hfill $\square$

Example 5.6. Consider the matrices

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -3 & -2 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 \\ 0 & -3 \\ 3 & 0 \\ -2 & 1 \end{bmatrix}, \quad \text{and} \quad K = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}.$$ 

From Corollary 5.5 using $y_2^3 = 1/(z_1^2 z_2)$, we obtain the isomorphism

$$[D_X]^T / \langle E - \beta \rangle \cong D_X / \mathcal{J} \cong D_Z \oplus D_Z \cdot \left(\frac{1}{z_1^2 z_2}\right)^{1/3} \oplus D_Z \cdot \left(\frac{1}{z_1^2 z_2}\right)^{2/3}.$$ 

For a module $M$ in $\mathcal{T}$-irred$(X, \beta)$ or $\mathcal{T}$-irred$(\tilde{X}, \beta)$, the isomorphism $\tilde{\delta}_{B,\kappa}$ can be used to explicitly compute $\Delta_{\tilde{B}}(M)$ or $\Pi_{\tilde{B}}(M)$, respectively.
Corollary 5.7. Suppose that \( \phi: (D_X)^p \to (D_X)^q \) is a presentation matrix for a module \( M \) in \( \mathcal{I}-\text{irred}(X, \beta) \), and let \( \kappa \) and \( \varkappa \) be as in Convention 5.3. Then the \( D_X \)-module \( \Delta_B(M) \) is presented up to isomorphism by

\[
\tilde{\delta}_{B,\kappa} \left( \frac{[D_X]^T}{\langle E - \beta \rangle} \otimes [D_X]^T \Delta_B(\phi) \right) : (D_Z)^p \longrightarrow \left( \bigoplus_{0 \leq k < \varkappa} D_Z \cdot z^{k/\varkappa} \right)^q.
\]

In particular, if \( J + \langle E - \beta \rangle \) is a torus equivariant left \( D_X \)-ideal, then there is an isomorphism

\[
\Delta_B \left( \frac{D_X}{J + \langle E - \beta \rangle} \right) \cong \bigoplus_{0 \leq k < \varkappa} D_Z \cdot z^{k/\varkappa} / \delta_{B,\kappa}(J).
\]

**Proof.** If \( K \) is not invertible over \( \mathbb{Z} \) and \( M \) is irreducible, change coordinates so that \( K \) is diagonal with diagonal entries \( \varkappa \). Then write (in the new coordinates) \( M = (\bigoplus_{\varepsilon} D_X / \mathcal{I} \cdot \varepsilon) / I \). Applying Corollary 5.5 and sorting by (rational) powers of \( z \), \( M = (\bigoplus_{0 \leq k < \varkappa} \bigoplus_{\varepsilon} D_Z \cdot \varepsilon \cdot z^{k/\varkappa}) / I \), where \( I \) is being expanded in terms of the \( D_Z \cdot z^{k/\varkappa} \). The result now follows from Corollary 5.5.

**Corollary 5.8.** Suppose that \( \phi: (D_X)^p \to (D_X)^q \) is a presentation matrix for a module \( M \) in \( \mathcal{I}-\text{irred}(X, \beta) \), and let \( \kappa \) and \( \varkappa \) be as in Convention 5.3. Then the \( D_X \)-module \( \Pi_B(M) \) is isomorphic to the quotient of \( (D_Z)^q \) by

\[
\left( \bigoplus_{0 \leq k < \varkappa} D_Z \cdot z^{k/\varkappa} \right)^q \cap \text{image} \left( \tilde{\delta}_{B,\kappa} \left( \frac{[D_X]^T}{\langle E - \beta \rangle} \otimes [D_X]^T D_X \cdot \Delta_B(\phi) \right) \right).
\]

In particular, if \( I + \langle E - \beta \rangle \) is a torus equivariant left \( D_X \)-ideal, then there is an isomorphism

\[
\Pi_B \left( \frac{D_X}{I + \langle E - \beta \rangle} \right) \cong \bigoplus_{0 \leq k < \varkappa} D_Z \cdot z^{k/\varkappa} / \delta_{B,\kappa}(D_X / I) \cap \left( \bigoplus_{0 \leq k < \varkappa} D_Z \cdot z^{k/\varkappa} \right).
\]

Note that if we change our choice of \( \kappa \) in Convention 5.3, we obtain different isomorphisms in Proposition 5.4 and Corollary 5.5. Thus, different choices of \( \kappa \) will produce (isomorphic) presentations of \( \Delta_B(M) \) (respectively, \( \Pi_B(M) \)) if \( M \) is an element of \( \mathcal{I}-\text{irred}(X, \beta) \) (respectively, \( \mathcal{I}-\text{irred}(\overline{X}, \beta) \)). Likewise, different choices of Gale duals \( B \) will also produce different isomorphisms, and therefore different presentations.

**Example 5.9** (Example 5.6, first variant). We use Corollary 5.8 to apply \( \Pi_B(M) \) to the \( A \)-hypergeometric \( D_X \)-module associated to \( A \) and \( \beta \) (see Example 0.2). Let \( C_1, \ldots, C_4 \) denote the rows of \( C \). Under \( \tilde{\delta}_{C,\kappa+i\varepsilon} \), the generators of \( I_A = \langle \partial^2_{x_3} - \partial_{x_2} \partial_{x_4}, \partial_{x_3} \partial^3_{x_4} - \partial_{x_1} \partial_{x_4}, \partial^2_{x_2} - \partial_{x_1} \partial_{x_3} \rangle \) map respectively to

\[
(C_3 \cdot \lambda + \kappa_3 - 2)(C_3 \cdot \lambda + \kappa_3 - 3) - y_2(C_2 \cdot \lambda + \kappa_2 + 1)(C_4 \cdot \lambda + \kappa_4 + 1),
\]

\[
(C_2 \cdot \lambda + \kappa_2 + 1)(C_3 \cdot \lambda + \kappa_3 - 2) - \frac{y_2}{y_1}(C_1 \cdot \lambda + \kappa_1)(C_4 \cdot \lambda + \kappa_4 + 1), \quad \text{and}
\]

\[
(C_2 \cdot \lambda + \kappa_2 + 1)(C_2 \cdot \lambda + \kappa_2) - \frac{y_2^2}{y_1}(C_1 \cdot \lambda + \kappa_1)(C_3 \cdot \lambda + \kappa_3 - 2).
\]
Thus, under $\delta_{B,\kappa}$, the binomial generators of $I_A$ map to

$$(B_3 \cdot \eta + \kappa_3)(B_3 \cdot \eta + \kappa_3 - 1) - \left( \frac{1}{z_1^2 z_2} \right)^{\frac{1}{3}} (B_2 \cdot \eta + \kappa_2)(B_4 \cdot \eta + \kappa_4),$$

and

$$(B_2 \cdot \eta + \kappa_2)(B_3 \cdot \eta + \kappa_3) - \left( \frac{z_1}{z_2} \right)^{\frac{1}{3}} (B_1 \cdot \eta + \kappa_1)(B_4 \cdot \eta + \kappa_4),$$

and

$$(B_2 \cdot \eta + \kappa_2)(B_2 \cdot \eta + \kappa_3 - 1) - \left( \frac{1}{z_1 z_2^2} \right)^{\frac{1}{3}} (B_1 \cdot \eta + \kappa_1)(B_3 \cdot \eta + \kappa_3).$$

Going modulo these equations in $D_{\bar{Z}} \oplus D_{\bar{Z}} \cdot \left( \frac{1}{z_1^2 z_2} \right)^{\frac{1}{3}} \oplus D_{\bar{Z}} \cdot \left( \frac{1}{z_1 z_2^2} \right)^{\frac{1}{3}}$, we obtain $\Pi_B^\Delta(H_A(\beta))$ by Corollary 5.8.

With Corollary 5.8 in hand, we now have the tools to relate lattice basis binomial $D_X$-modules and saturated Horn $D_Z$-modules via $\Pi_B^\Delta$ (see Example 0.2 and Definition 0.3(2)). This result will be further exploited in Part II.

**Corollary 5.10.** If $\kappa$ is as in Convention 5.3 then

$$\Pi_B^\Delta \left( \frac{D_X}{I(B) + \langle E - \beta \rangle} \right) \cong \bigoplus_{0 \leq k < \kappa} D_{\bar{Z}} \cdot \text{sHorn}(B, \kappa + Ck) \cdot z^{k/\kappa}.$$

**Proof.** If the columns of $B$ span $\ker_{\bar{Z}}(A)$ as a lattice, then the statement follows immediately from the definitions of the systems, via Corollary 5.8. For an arbitrary choice of Gale dual $B$, we again apply Corollary 5.8. Since passage of $z^{k/\kappa}$ through the equations of $\text{sHorn}(B, \kappa)$ results in a shift by $Ck$, the summands separate to yield the desired isomorphism. □

**Example 5.11** (Example 5.9 continued). Since (5.3) does not lie in a single $D_Z$-summand of $[D_X]^T / \langle E - \beta \rangle$, we see immediately that $\Pi_B^\Delta(D_X / (I_A + \langle E - \beta \rangle))$ does not possess the nice decomposition that arises in the lattice basis situation. After multiplying (5.4) and (5.5) by $z_1^{-1}$, it is also clear that neither of these elements lie in a single $D_Z$-summand of $[D_X]^T / \langle E - \beta \rangle$. □

**Example 5.12** (Example 5.6, second variant). We now use Corollary 5.8 to compute $\Pi_B^\Delta(D_X / (I(B) + \langle E - \beta \rangle))$. In this example, $I(B) = \langle \partial_3 - \partial_{x_1} \partial_{x_4}, \partial_3^2, \partial_{x_4} - \partial_2^3 \rangle$. Since $\varepsilon_C = [1, 1, -5, 3]^T$, under $\delta_{B,\kappa}$, the binomials generating $I(B)$ map respectively to

$$(C_3 \cdot \lambda + \kappa_3 - 5)(C_3 \cdot \lambda + \kappa_3 - 6)(C_3 \cdot \lambda + \kappa_3 - 7)$$

and

$$-g_1^{-1}(C_1 \cdot \lambda + \kappa_1 + 1)(C_4 \cdot \lambda + \kappa_4 + 3)(C_4 \cdot \lambda + \kappa_4 + 2)$$

and

$$(C_1 \cdot \lambda + \kappa_1 + 1)(C_1 \cdot \lambda + \kappa_1)(C_4 \cdot \lambda + \kappa_4 + 3)$$

and

$$-g_2^{-1}(C_2 \cdot \lambda + \kappa_2 + 1)(C_2 \cdot \lambda + \kappa_2)(C_2 \cdot \lambda + \kappa_2 - 1).$$

(5.6)
Then under $\tilde{\mathcal{S}}_{B,\kappa}$, the expressions of (5.6) map to the generators of $s\text{Horn}(B, \kappa)$, while $y_2$ times the expressions of (5.6) map respectively to

$$[(B_3 \cdot \eta + \kappa_3 - 2)(B_3 \cdot \eta + \kappa_3 - 3)(B_3 \cdot \eta + \kappa_3 - 4)$$

$$- z_1(B_1 \cdot \eta + \kappa_1)(B_4 \cdot \eta + \kappa_4 + 1)(B_4 \cdot \eta + \kappa_4)] \cdot \left( \frac{1}{z_1^2 z_2} \right)^{\frac{3}{2}}$$

and

$$[(B_1 \cdot \eta + \kappa_1)(B_1 \cdot \eta + \kappa_1 - 1)(B_4 \cdot \eta + \kappa_4 + 1)$$

$$- z_2(B_2 \cdot \eta + \kappa_2 + 1)(B_2 \cdot \eta + \kappa_2)(B_2 \cdot \eta + \kappa_2 - 1)] \cdot \left( \frac{1}{z_1^2 z_2} \right)^{\frac{3}{2}}. \tag{5.7}$$

Similarly, we can compute $y_2^2$ times the expressions of (5.6). Together, these show that the image of the lattice basis binomial $D_\bar{X}$-module $D_{\bar{X}}/(I(B) + (E - \beta))$ under $\Pi_B^\delta$ is isomorphic to

$$\frac{D_Z}{s\text{Horn}(B, \kappa)} \oplus \frac{D_Z}{s\text{Horn}(B, \kappa + c_2)} \cdot \left( \frac{1}{z_1^2 z_2} \right)^{\frac{3}{2}} \oplus \frac{D_Z}{s\text{Horn}(B, \kappa + 2c_2)} \cdot \left( \frac{1}{z_1^2 z_2} \right)^{\frac{3}{2}}.$$

This agrees with Corollary 5.10. \hfill \Box

### 5.2. Solution spaces

We now consider the effect of $\Pi_B^\delta$ on solutions of objects in $\mathcal{T}$-$\text{irred}(\bar{X}, \beta)$. Using $b_1, \ldots, b_m$ to denote the columns of a Gale dual $B$ of $A$, the map $\mu_B : X \to Z$ given by $x \mapsto x^B := (x_1^{b_1}, \ldots, x_1^{b_m})$ is onto, since $\mathbb{C}$ is algebraically closed. If $p \in X$, this surjection induces a map $\mu_B^p : \mathcal{O}^{an}_{Z^p,\eta} \to \mathcal{O}^{an}_{Z,\eta}$ via $g(z) \mapsto g(x^B)$. If the columns of $B$ span $\ker_{\bar{Z}}(A)$ as a lattice, then $\mu_B^p$ is an isomorphism. Otherwise, it will be a $[\ker_{\bar{Z}}(A) : ZB]$-to-1 covering map. Let $s_1, \ldots, s_{[\ker_{\bar{Z}}(A) : ZB]}$ denote the distinct sections of this map. Together, these can be used to analyze the behavior of solution spaces under $\Pi_B^\delta$.

Suppose that a module $M$ in $\mathcal{T}$-$\text{irred}(\bar{X}, \beta)$ is such that at a sufficiently generic (nonsingular) point $p \in X$, $\text{Sol}_p(M)$ has a basis of basic Nilsson solutions in the direction of a generic weight vector $w \in \mathbb{R}^n$. (This occurs when $M$ is regular holonomic, but regularity is not a necessary condition, see Theorems 1.4 and 6.5.) Then $\text{Sol}_p(M)$ is spanned by vectors of the form $\phi = (\phi_1, \ldots, \phi_r)$, where the $\phi_i$ are Nilsson solutions that converge at $p$. Further, by Remark 1.7, any solution $\phi$ of $M$ can be written $\phi = x^\kappa f(x^L)$, where $L$ is a collection of $m$ vectors that $\mathbb{Z}$-span $\ker_{\bar{Z}}(A)$.

**Theorem 5.13.** Let $M$ be a module in $\mathcal{T}$-$\text{irred}(\bar{X}, \beta)$ and let $p \in X$ be a generic nonsingular point of $M$ so that $\mu_B(p) = p^B \in Z$ is a nonsingular point of $\Pi_B^\delta(M)$. Choose $\kappa$ so that $A\kappa = \beta$ and $\kappa C = 0$. If $\text{Sol}_p(M)$ has a basis of basic Nilsson solutions in the direction of a generic weight vector, then $\text{Sol}_p(\Pi_B^\delta(M))$ is isomorphic to the sum over $i \in \{1, \ldots, [\ker_{\bar{Z}}(A) : ZB]\}$ of the images of the maps

$$\text{Sol}_p(M) \longrightarrow \text{Sol}_p(\Pi_B^\delta(M)) \quad \text{given by} \quad x^\kappa f(x^L) \mapsto f(s_i(z)).$$

**Corollary 5.14.** If $M$ is a regular holonomic module in $\mathcal{T}$-$\text{irred}(\bar{X}, \beta)$, then the rank of $\Pi_B^\delta(M)$ is equal to $[\ker_{\bar{Z}}(A) : ZB] \cdot \text{rank}(M)$. Moreover, if the columns of $B$ span $\ker_{\bar{Z}}(A)$ as a lattice (so that $[\ker_{\bar{Z}}(A) : ZB] = |\det(K)| = 1$) and $p \in X$ is a generic nonsingular point of $M$, then there is an isomorphism $\text{Sol}_p(M) \cong \text{Sol}_p(\Pi_B^\delta(M))$. \hfill \Box

**Proof of Theorem 5.13.** Since $p \in X$ is nonsingular for $M$, restriction provides an isomorphism between the solutions of $M$ at $p$ and those of $i^* M = D_X \otimes_{\bar{X}} M$ at $p$. By the genericity of $p$,
\[ p^B \in Z \text{ will be a nonsingular point of } \Delta_{B}^{\hat{A}}(i^*M), \text{ and moreover, the solutions of } \Delta_{B}^{\hat{A}}(i^*M) \text{ at } p^B \text{ can be extended to solutions of } \Pi_{B}^{\hat{A}}(M) \text{ at } p^B. \text{ This means that we may assume that } M \text{ is a } D_X\text{-module and work with the functor } \Delta_{B}^{\hat{A}} \text{ instead of } \Pi_{B}^{\hat{A}}. \text{ Further, as solutions of } M \text{ are represented by vectors of functions, for simplicity, it is enough to consider case that } M \text{ is cyclic.} \\

Consider first the case that the columns of } B \text{ do not } Z\text{-span ker}_Z(A). \text{ Then } \mu_K: X/\mathcal{T} \to Z \text{ induces a } |\det(K)|\text{-fold cover of } Z, \text{ which induces an isomorphism between the solutions of } \Delta_{B}^{\hat{A}}(M) \text{ at } p^B \text{ and the sum over all sections of } \mu_K \text{ of the image of the solution space of } \Delta_{B}^{\hat{A}}(M) = \Upsilon^{\hat{A}}_C([M])^T. \text{ Thus, we have reduced the proof of the theorem to the case that the columns of } B \text{ span ker}_Z(A) \text{ as a lattice, so that } \mu_B \text{ is an isomorphism. We assume this case for the remainder of the proof; in particular, without loss of generality, we may assume that } B = C = \mathcal{L}. \\

Consider now the case that } X = T' \times Z, \text{ where } T' = (\mathbb{C}^*)^d, \text{ and } T' \times Z \text{ has a torus action given by } t \circ (s, z) = \left(t_1^{a_1} s_1, \ldots, t_1^{a_d} s_d, z_1, \ldots, z_n\right). \text{ Let } E' \text{ denote the Euler operators on } T' \times Z, \text{ viewed as elements in } D_{T'}. \text{ If } N \text{ is a module in } \mathcal{T}\text{-irred}(T' \times Z, \beta), \text{ then by the same argument used to obtain (5.2), } N \text{ can be expressed as a module over } D_{T' \times Z} = D_{T'} \otimes_C D_Z \text{ as} \\

\[ N = \frac{D_{T'}}{\langle E' - \beta \rangle} \otimes_C \Upsilon^{\hat{A}}_C([N])^T = \frac{D_{T'}}{\langle E' - \beta \rangle} \otimes_C \Delta_{C}^{\hat{A}}(N). \] (5.8) \\

Suppose that } (t_0, z_0) \in T' \times Z \text{ is a generic nonsingular point of } N. \text{ From (5.8), we see that} \\

\[ \text{Sol}_p(N) = \text{Hom}_{D_{T' \times Z}}(N, \mathcal{O}_{T' \times Z}^{\text{an}}) \cong \text{Hom}_{D_{T'}}\left(\frac{D_{T'}}{\langle E' - \beta \rangle}, \mathcal{O}_{T', t_0}^{\text{an}}\right) \otimes_C \text{Hom}_{D_Z}\left(\Delta_{C}^{\hat{A}}(N), \mathcal{O}_{Z, z_0}^{\text{an}}\right). \] (5.9) \\

For the isomorphism (5.9), use the fact that } \text{Hom}_{D_{T'}}\left(\frac{D_{T'}}{\langle E' - \beta \rangle}, \mathcal{O}_{T', t_0}^{\text{an}}\right) \text{ is spanned by the homomorphism } 1 \mapsto \prod_{i=1}^d t_i^{a_i/a_i}. \text{ In particular, it is a one-dimensional } C\text{-vector space.} \\

We now return to the general case that } M \text{ is in } \mathcal{T}\text{-irred}(X, \beta). \text{ Let } \phi: X \to T' \times Z \text{ be a } T\text{-equivariant change of coordinates, whose existence follows from the existence of a Smith normal form for } \hat{A}. \text{ Let } C \text{ and } C' \text{ denote the appropriate } C\text{-matrices for } X \text{ and } T' \times Z, \text{ respectively. With } (t_0, z_0) := \phi(p), \text{ since } \kappa C = 0, \text{ we have the following isomorphisms:} \\

\[ \text{Sol}_p(M) \cong \text{Sol}_{\phi(p)}(\phi^* M) \cong \text{Sol}_{t_0}(\phi_+ M)|_{T'} \otimes_C \text{Sol}_{z_0}(\Delta_{C'}^{\hat{A}}(\phi_+ M)) \] (using an argument similar to (5.8)) \\

\[ \cong \text{Sol}_{t_0}(D_{T'}/\langle E' - \beta \rangle) \otimes_C \text{Sol}_{z_0}(\Delta_{C'}^{\hat{A}}(\phi_+ M)) \] (by applying (5.9) to } \phi_+(M)) \\

\[ \cong \text{Sol}_{(t_0, \ldots, t_0)}(D_{\mathcal{T}}/\langle E - \beta \rangle) \otimes_C \text{Sol}_{z_0}(\Delta_{C}^{\hat{A}}(M)) \]

\[ \cong \text{Sol}_{p, \mu}(\Delta_{C}^{\hat{A}}(M)). \]

The final equality follows since } \text{Sol}_{(t_0, \ldots, t_0)}(D_{\mathcal{T}}/\langle E - \beta \rangle) \text{ is spanned by the function } x^\kappa. \text{ In fact, it is spanned by any function } x^v \text{ such that } Av = \beta, \text{ but observe that if } Au = 0, \text{ then } x^u \equiv 1 \text{ on } \mathcal{T}. \\

Finally, note that two different versions of } [-]^T \text{ are used above; however, they are compatible via the isomorphism } \phi \text{ because } \mu_B = \pi_2 \circ \phi: X \to T' \times Z \to Z. \]

**Example 5.15.** The assumption that the module } M \text{ belongs to } \mathcal{T}\text{-irred}(X, \beta) \text{ is crucial for Theorem 5.13. The key fact here is that } \langle E - \beta \rangle \text{ is a maximal ideal in } C[E].
Consider instead the case that $d = m = 1$ and $X = T \times Z$, where $T$ acts by scaling on the first factor. The torus equivariant module $M := D_{T \times Z} / D_{T \times Z} \cdot ((t \partial_t)^3, t \partial_t - \partial_z)$ is holonomic, and its solution space at a nonsingular point $(t_0, z_0)$ is

$$\text{Sol}_{(t_0, z_0)}(M) = \text{Span}_\mathbb{C}\{1, z + \log(t), \log^2(t) + 2z \log(t)\}.$$ 

In this case, since $\text{ann}_{\mathbb{C}[E]}(M) = ((t \partial_t)^3)$, the operator $t \partial_t$ does not act as a constant on $M$, and therefore, the operator $t \partial_t - \partial_z$ cannot be written modulo $\text{ann}_{\mathbb{C}[E]}(M)$ as an element of $D_Z$. This is the reason that the solutions of $M$ are not as well-behaved as in Theorem 5.13.

**Remark 5.16.** Note that in the proof of Theorem 5.13, $\mu_K$ is used to describe the behavior of solutions when the columns of $B$ do not $\mathbb{Z}$-span $\ker_Z(A)$. Combining this argument with the direct sum decomposition in Corollary 5.10 reveals that for sufficiently generic $p \in X$, there is an isomorphism of solution spaces for lattice basis binomial and Horn hypergeometric $D$-modules:

$$\text{Sol}_p \left( \frac{D_X}{I(B) + \langle E - \kappa \rangle} \right) \cong \text{Sol}_p \left( \frac{D_Z}{\mathfrak{s}\text{Horn}(B, \kappa)} \right).$$

### 5.3. Characteristic varieties and singular loci

The homomorphism $\delta_{B, \kappa}$ can be used to explain the image of the $L$-characteristic variety of $M$ under $\Delta^\kappa_B$ when $M$ is in $\mathcal{T}$-irred$(X, \beta)$.

**Proposition 5.17.** Let $M$ be a module in $\mathcal{T}$-irred$(X, \beta)$, let $L := (L_x, L_{\partial_x}) \in \mathbb{Q}^{2n}$ be a weight vector with $L_x + L_{\partial_x} = c \cdot 1_n$ for some $c > 0$, and set $L' := (L_x, c \cdot 1_m - L_x B) \in \mathbb{Q}^{2m}$. If $B$ has columns that $\mathbb{Z}$-span $\ker_Z(A)$, then the $L'$-characteristic variety and singular locus of $\Delta^\kappa_B(M)$ are geometric quotients of the $L$-characteristic variety and singular locus of $M$, respectively.

**Proof.** Note first that taking $L$-associated graded objects commutes with taking invariants. Also, taking invariants produces a $[D_X]^L$-module $[M]^L$. Moreover, by the definition of $\mathcal{T}$-irred$(X, \beta)$ and Proposition 5.4, we see that $[M]^L$ is already naturally a $D_Z$-module that is isomorphic to $\Delta^\kappa_B(M)$. We now have the result, since taking torus invariants induces categorical quotients, which in this case separates orbits.

Combining Propositions 4.3 and 5.17, we can compute the characteristic variety and singular locus of $\Pi^\kappa_B(M)$ in terms of those for $M$ under certain assumptions.

**Corollary 5.18.** Let $M$ be a nonzero regular holonomic module in $\mathcal{T}$-irred$(X, \beta)$. If $B$ has columns that $\mathbb{Z}$-span $\ker_Z(A)$, then $\text{Char}(\Pi^\kappa_B(M))$ and $\text{Sing}(\Pi^\kappa_B(M))$ are the unions of the geometric quotient of the $\text{Char}(D_X \otimes D_X M)$ and $\text{Sing}(D_X \otimes D_X M)$ with, respectively, the coordinate hyperplane conormals in $D_X$ and the coordinate hyperplanes in $D_X$.

### Part II: Binomial $D$-modules and hypergeometric systems

#### 6. Binomial $D$-modules

We introduced binomial $D$-modules in Definition 0.1 and Example 0.2. In this section, we summarize results from [DMM10b, CF12, BMW13] regarding the holonomicity and regularity of binomial $D$-modules. We also generalize results from [DMM12, SW12] to provide criteria for regular holonomicity and reducibility of monodromy representation for binomial $D$-modules.
6.1. Overview. The holonomicity of a binomial $D$-module is controlled by the primary decomposition of the underlying binomial ideal. Primary decomposition of binomial ideals has several special features. Eisenbud and Sturmfels have shown that the associated primes of a binomial ideal are binomial ideals themselves and that the primary decomposition of a binomial ideal can be chosen to be binomial \[\text{[ES96]}.\] A more detailed study of the primary components of a binomial ideal, geared towards applications to binomial $D$-modules, appears in \[DMM10a].

Binomial ideals can have two types of primary components, \textit{toral} and \textit{Andean}, according to how they behave with respect to the inherited torus action. We recall from \[\text{[ES96]}\] that if $I \subseteq \mathbb{C}[\partial_x]$ is a binomial ideal, then every associated prime of $I$ is of the form $I' + \langle \partial_{x_i} \mid i \notin \sigma \rangle$, where $\sigma \subseteq \{1, \ldots, n\}$, $I'$ is generated by elements of $\mathbb{C}[\partial_{\sigma}] := \mathbb{C}[\partial_{x_i} \mid i \in \sigma]$, and the intersection $I' \cap \mathbb{C}[\partial_{\sigma}]$ is a toric ideal after rescaling the variables in $\mathbb{C}[\partial_{\sigma}]$.

**Definition 6.1.** Let $I$ be an $A$-graded binomial ideal in $\mathbb{C}[\partial_x]$, and let $p = I' + \langle \partial_{x_i} \mid i \notin \sigma \rangle$ be an associated prime of $I$ as above, where we have rescaled the variables so that $I' \cap \mathbb{C}[\partial_{\sigma}]$ is a toric ideal. Denote by $A_{\sigma}$ the submatrix of $A$ consisting of the columns in $\sigma$. If $I' \cap \mathbb{C}[\partial_{\sigma}] = I_{A_{\sigma}}$, then $p$ is called a \textit{toral} associated prime of $I$, and the corresponding primary component $p$ is also called toral. An associated prime (or primary component) that is not toral is called \textit{Andean}.

It is the Andean components of a binomial ideal that cause failure of holonomicity for the corresponding binomial $D_X$-module. To make this precise, let $V$ be an $A$-graded $\mathbb{C}[\partial_x]$-module. The set of quasidegrees of $V$ is $qdeg(V) := \{ \alpha \in \mathbb{Z}^d \mid \text{V}_\alpha \neq 0 \}$, where the closure is taken with respect to the Zariski topology in $\mathbb{C}^d$. If $I$ is an $A$-graded binomial $\mathbb{C}[\partial_x]$-ideal, then the set $\bigcup qdeg(\mathbb{C}[\partial_x]/\mathcal{C})$, where the union runs over the Andean components $\mathcal{C}$ of $I$, is called the \textit{Andean arrangement} of $I$.

The following theorem collects results about binomial $D_X$-modules. Except for items (iv) and (v), which are from \[\text{[BMW13]}\] and \[\text{[CF12]}\] respectively, all of these facts are proved in \[\text{[DMM10b]}\].

**Theorem 6.2 (\text{[DMM10b]} \text{[CF12]} \text{[BMW13]}).** Let $I \subseteq \mathbb{C}[\partial_x]$ be an $A$-graded binomial ideal, and consider the binomial $D_X$-module $M = D_X/(I + \langle E - \beta \rangle)$.

(i) The module $M$ is holonomic if and only if $\beta$ does not lie in the Andean arrangement of $I$.

(ii) The module $M$ is holonomic if and only if its holonomic rank is countable.

(iii) The module $M$ is holonomic if and only if its holonomic rank is finite.

(iv) The module $M$ is holonomic if and only if $M$ is $L$-holonomic for all $L = (L_x, L_{\partial_x}) \in \mathbb{Q}^{2n}$ such that $L_x + L_{\partial_x} = c \cdot 1_n$ for some constant $c > 0$. Moreover, if $M$ fails to be $L$-holonomic for some $L$, then $\text{Char}^L(M)$ has a component in $T^*X$ of dimension more than $n$.

(v) Assume that $\beta$ does not lie on the Andean arrangement of $I$. Consider the set

$$\{ \mathcal{C} \text{ toral primary component of } I \mid \beta \in qdeg(\mathcal{C}) \}. \quad (6.1)$$

Then $M$ is regular holonomic if and only if each ideal in (6.1) is standard $\mathbb{Z}$-graded.

(vi) A lattice basis binomial $D_X$-module associated to a matrix $B$ is regular holonomic if and only if it is holonomic and the ideal $I(B)$ is standard $\mathbb{Z}$-graded.

(vii) The holonomic rank of $M$ is minimal as a function of $\beta$ if and only if $\beta$ does not belong to the union of the Andean arrangement of $I$ with $\text{qdeg} \left( \bigoplus_{i \in d} H^i_m(\mathbb{C}[\partial_x]/I_{\text{toral}}) \right)$, where $H^i_m(-)$ denotes local cohomology with support $m = \langle \partial_{x_1}, \ldots, \partial_{x_n} \rangle$ and $I_{\text{toral}}$ is the intersection of the toral primary components of $I$.

(viii) Suppose that the Andean arrangement of $I$ is not all of $\mathbb{C}^d$. For each toral associated prime of $I$, let $A_\sigma$ be the matrix from Definition 6.1, and denote by $\text{vol}(A_\sigma)$ the lattice volume of the
polytope $\text{conv}(0, A_\sigma)$ in the lattice $\mathbb{Z}A_\sigma$. The minimal rank attained by $M$ is the sum over the toral primary components of $I$ of $\text{mult}(I, \sigma) \cdot \text{vol}(A_\sigma)$, where $\text{mult}(I, \sigma)$ is the multiplicity of the associated prime $I_{A_\sigma} + \langle \partial_{x_j} \mid j \notin \sigma \rangle$ in $I$.

(ix) Define a module $P_1$ by the exact sequence

$$0 \rightarrow \mathbb{C}[\partial_{x_0}] / I \rightarrow \bigoplus_{\mathcal{C}} \mathbb{C}[\partial_{x_0}] / \mathcal{C} \rightarrow P_1 \rightarrow 0$$

where the direct sum is over the primary components $\mathcal{C}$ of $I$. If $\beta$ lies outside the union of $q\text{deg}(P_1)$ with the Andean arrangement of $I$, then $M$ is isomorphic to the direct sum over the modules $D_X / (\mathcal{C} + \langle E - \beta \rangle)$, where $\mathcal{C}$ lies in (6.1). $\square$

The Andean arrangement and all other quasidegree sets in Theorem 6.2 are unions of finitely many integer translates of subspaces of the form $\mathbb{C}A_\sigma$, where $\sigma \subseteq \{1, \ldots, n\}$. If the Andean arrangement of $I$ is not all of $\mathbb{C}^d$, then the other quasidegree sets are also proper subsets of $\mathbb{C}^d$.

The holonomicity of an $A$-hypergeometric system was first shown in [GGZ87, Ado94], and information on the rank of these systems can be found in [MMW05, Berk11].

6.2. Regular holonicity and Nilsson series. We now discuss Nilsson solutions of binomial $D$-modules, generalizing statements in [DMM12] for $A$-hypergeometric systems. We will make use of a homogenization operation. In this direction, set

$$\rho(A) := \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & & & \\ \vdots & & A \\ 0 & & & & & & & & & & & \\ \end{bmatrix}.$$ 

Given an $A$-graded binomial ideal $I \subseteq \mathbb{C}[\partial_{x_0}]$, let $\rho(I)$ denote the homogenization of $I$ with respect to an additional variable $\partial_{x_0}$. Here, $\partial_{x_0}$ corresponds to a variable $x_0$ giving rise to coordinates $(x_0, x_1, \ldots, x_n)$ on $\hat{X} := \mathbb{C}^{n+1}$. Note that $\rho(I)$ is $\rho(A)$-graded. For a fixed $\beta_0 \in \mathbb{C}$ and $\beta \in \mathbb{C}^d$, write $E_\rho - (\beta_0, \beta)$ for the sequence of $d + 1$ Euler operators associated to $\rho(A)$ and the vector $(\beta_0, \beta)$. We define the homogenization of the binomial $D$-module $M = D_{\hat{X}} / (I + \langle E - \beta \rangle)$ to be

$$\rho(M, \beta_0) := \frac{D_{\hat{X}}}{\rho(I) + \langle E_\rho - (\beta_0, \beta) \rangle}.$$ 

A vector $w \in \mathbb{R}^n$ is a generic binomial weight vector for a binomial $D$-module $M = D_{\hat{X}} / (I + \langle E - \beta \rangle)$, if it satisfies the conditions from Definition 1.2 and also, for all $w' \in \Sigma$, we have $\text{gr}^w(I) = \text{gr}^{w'}(I)$. We say that $w \in \mathbb{R}^n$ is a perturbation of $w_0 \in \mathbb{R}_{>0}^n$ if there exists an open cone $\Sigma$ as in the previous definition, with $w \in \Sigma$ such that $w_0$ lies in the closure of $\Sigma$. Our proofs also make use of the homogenization of a basic Nilsson solution $\phi = \sum_{u \in \mathcal{C}} x^{u+u_0}p_u(\log(x))$ of $M$ in the direction of a binomial weight vector $w \in \mathbb{R}^n$ for $M$. From the definition of $\rho(\phi)$ in [DMM12, §3], it follows as in [DMM12, Proposition 3.17] that if $|u| \geq 0$ for all $u \in \mathcal{C}$, then $\rho(\phi)$ is a basic Nilsson solution of $\rho(M, \beta_0)$ in the direction of $(0, w)$.

**Proposition 6.3.** Let $I \subseteq \mathbb{C}[\partial_{x_0}]$ be an $A$-graded binomial ideal, and set $M = D_{\hat{X}} / (I + \langle E - \beta \rangle)$. Suppose that $\beta_0 \in \mathbb{C}$ is generic, in that, if $J$ is the intersection of the $\mathcal{C}$ in (6.1), then $(\beta_0, \beta)$ is not in the Andean arrangement of $P_{\rho(I)}$ from Theorem 6.2 (ix). If $\beta$ is not in the Andean arrangement of $I$, then the binomial $D$-module $\rho(M, \beta_0)$ is regular holonomic.
Proof. Since $\beta$ is not in the Andean arrangement of $I$, we may assume that all primary components of $I$ are toral. Since $\beta_0$ is generic and $\rho$ commutes with taking primary decompositions, $\rho(I)$ also has only toral primary components. Further, by Theorem 6.2.(ix), $\rho(M, \beta_0)$ is isomorphic to $\bigoplus D_X / (\rho(C) + \langle E_\rho - (\beta_0, \beta) \rangle)$, where the direct sum is over the toral primary components $C$ of $I$. Since each ideal $\rho(C)$ is $\mathbb{Z}$-graded, the result now follows from Theorem 6.2.(v). □

We now generalize [Berk10, Theorem 7.3], which was first stated for $A$-hypergeometric systems.

Proposition 6.4. Let $I \subseteq \mathbb{C}[\partial_x]$ be an $A$-graded binomial ideal, and set $M = D_X / (I + \langle E - \beta \rangle)$. If $\beta$ does not lie in the Andean arrangement of $I$ and $\beta_0 \in \mathbb{C}$ is generic as in Proposition 6.3, then

$$\text{rank}(M) = \text{rank}(\rho(M, \beta_0)).$$

Proof. We may assume that $I$ is equal to the intersection of the $C$ in (6.1). By Theorem 6.2(i) and the genericity of $\beta_0$, $M$ and $\rho(M, \beta_0)$ are holonomic and thus of finite rank. Given the collection $\{C\}$ of toral primary components of $I$, we may extend $0 \rightarrow \mathbb{C}[\partial_x] / I \rightarrow \bigoplus \mathbb{C}[\partial_x] / C$ to a primary resolution of $\mathbb{C}[\partial_x] / I$, see [BM09, §4]. To compute the rank of $M$, we then apply Euler–Koszul homology to this resolution and follow the resulting spectral sequence, as in the proof of [BM09, Theorem 4.5]. That argument and the fact that the theorem has been proven for $A$-hypergeometric systems in [Berk10, Theorem 7.3] imply that this procedure and its associated numerics are compatible with $\rho$, and this compatibility yields the desired result. □

With the definitions of homogenization and Propositions 6.3 and 6.4 in hand, the necessary ingredients are in place to apply the original proof of [DMM12, Theorem 6.4] to arbitrary holonomic binomial $D$-modules. This result, appearing now in Theorem 6.5, provides a method to compute the rank of these modules via certain Nilsson solutions. Note also that the statement runs in parallel to Theorem 1.4, without requiring regular holonomicity.

Theorem 6.5. Let $I \subseteq \mathbb{C}[\partial_x]$ be an $A$-graded binomial ideal, and set $M = D_X / (I + \langle E - \beta \rangle)$. Assume that $\beta$ does not lie in the Andean arrangement of $I$. If $w \in \mathbb{R}^n$ is a generic binomial weight vector of $M$ that is a perturbation of $1_n$, then

$$\dim_{\mathbb{C}}(\mathcal{N}_w(M)) = \text{rank}(M).$$

Further, there exists an open set $U \subseteq X$ such that the basic Nilsson solutions of $M$ in the direction of $w$ simultaneously converge at each $p \in U$, and as such, they form a basis for $\text{Sol}_w(M)$. □

Finally, we generalize [DMM12, Proposition 7.4] in Theorem 6.6. Combining this result with Theorem 1.4, we obtain a new criterion for the regular holonomicity of binomial $D$-modules. In particular, instead of considering a family of derived solutions, regularity of a binomial $D$-module can be tested through a single collection of Nilsson solutions.

Theorem 6.6. Let $I \subseteq \mathbb{C}[\partial_x]$ be an $A$-graded binomial ideal such that $M = D_X / (I + \langle E - \beta \rangle)$ is not regular holonomic. If $\beta$ does not lie in the Andean arrangement of $I$, then there exists a generic binomial weight vector $w \in \mathbb{R}^n$ of $M$ such that $\dim_{\mathbb{C}}(\mathcal{N}_w(M)) < \text{rank}(M)$.

Proof. The original proof of [DMM12, Proposition 7.4] can be applied, since $\rho$ is commutes with taking primary decompositions. Note that Propositions 6.3 and 6.4 provide the binomial generalizations of the originally toric results needed in this proof. □
6.3. Irreducible monodromy representation. We now characterize when a binomial $D$-module has reducible monodromy representation. To achieve this, we extend [SW12, Theorems 3.1 and 3.2] (see also [Beu11a, Sai11]) on $A$-hypergeometric systems to arbitrary binomial $D$-modules.

Given $\sigma \subseteq \{1, \ldots, n\}$, let $A_\sigma$ denote the submatrix of $A$ given by the columns of $A$ that lie in $\sigma$, and let $I_{A_\sigma} \subseteq \mathbb{C}[\partial_{z_i} \mid i \not\in \sigma]$ denote the toric ideal given by $A_\sigma$ in the variables corresponding to $\sigma$.

A face $G$ of $A$, denoted $G \preceq A$, is a subset $G$ of the column set of $A$ such that there is a linear functional $\phi_G : \mathbb{Z}A \to \mathbb{Z}$ that vanishes on $G$ and is positive on any element of $A \setminus G$. If $G \preceq A$, then the parameter $\beta \in \mathbb{C}^d$ is $G$-resonant if $\beta \in \mathbb{Z}A + \mathbb{C}G$. If $\beta$ is $H$-resonant for all faces $H$ properly containing $G$, but not for $G$ itself, then $G$ is called a resonance center for $\beta$. It is said that $A$ is a pyramid over a face $G$ if $\text{vol}_{\mathbb{Z}G}(G) = \text{vol}_{\mathbb{Z}A}(A)$, where volume is a normalized (or simplicial) volume computed with respect to the given ambient lattices.

Theorem 6.7. Let $I$ be an $A$-graded binomial ideal in $\mathbb{C}[\partial_x]$ and suppose that $\beta \in \mathbb{C}^d$ does not lie in the Andean arrangement of $I$. The binomial $D_X$-module $D_X/(I + \langle E - \beta \rangle)$ has irreducible monodromy representation if and only if the following conditions hold:

1. For some $\sigma \subseteq \{1, \ldots, n\}$, the intersection of the toral components $\mathcal{C}$ of $I$ for which $\beta \in \text{qdeg}(\mathcal{C})$ equals $I_{A_\sigma} + \langle \partial_{z_i} \mid i \not\in \sigma \rangle$.
2. For any $\sigma$ as in (1), there is a unique resonance center $G \preceq A_\sigma$ of $\beta$, and $A_\sigma$ is a pyramid over $G$.

Proof. We may assume that $I$ is equal to the intersection of the toral components $\mathcal{C}$ of $I$ for which $\beta \in \text{qdeg}(\mathcal{C})$. If $I$ is not prime, pick an associated prime of $I$. After possibly rescaling the variables, since $\beta$ does not lie in the Andean arrangement of $I$, such an associated prime is of the form $I_{A_\sigma} + \langle \partial_{z_i} \mid i \not\in \sigma \rangle$ for some $\sigma \subseteq \{1, \ldots, n\}$. Let $N = D_X/(I_{A_\sigma} + \langle \partial_{z_i} \mid i \not\in \sigma \rangle + \langle E - \beta \rangle)$. Then $N$ is a submodule of $M$ such that $\mathbb{C}(x) \otimes_{\mathbb{C}[x]} N$ is a nonzero proper submodule of $\mathbb{C}(x) \otimes_{\mathbb{C}[x]} M$. Thus $M$ always has reducible monodromy representation in this case.

We may now assume that $I$ is prime. Then since $\beta$ does not lie in the Andean arrangement of $I$, there exists a subset $\sigma \subseteq \{1, \ldots, n\}$ such that $I = I_{A_\sigma} + \langle \partial_{z_i} \mid i \not\in \sigma \rangle$, after possibly rescaling the variables. Since the pyramid condition implies the uniqueness of a resonance center $G$ of $\beta$ by [SW12, Lemma 2.9], we have reduced the proof of the statement for binomial $D$-modules to the $A$-hypergeometric setting, which was already proven in [SW12, Theorems 3.1 and 3.2].

7. Torus invariants and binomial $D_X$-modules

In this section, we strengthen our results on the transfer of (regular) holonomicity through $\Pi_B^A$ from Theorem 4.2, still following Convention 3.2, in the case that the input is a binomial $D_X$-module. Remark 7.3 provides an explicit description of the characteristic variety of $\Pi_B^A$ applied to a regular holonomic binomial $D_X$-module.

Theorem 7.1. Let $I \subseteq \mathbb{C}[\partial_x]$ be an $A$-graded binomial ideal. Then the following hold for the binomial $D_X$-module $M = D_X/(I + \langle E - \beta \rangle)$.

1. Let $L := (L_x, L_{\partial_x}) \in \mathbb{Q}^{2m}$ be a weight vector with $L_x + L_{\partial_x} = c \cdot 1_n$ for some $c > 0$. Let $L' := (L_x B, c \cdot 1_m - L_x B) \in \mathbb{Q}^{2m}$. Then the binomial $D_X$-module $M$ is $L$-holonomic if and only if $\Pi_B^A(M)$ is $L'$-holonomic.
2. If $M$ is holonomic, then the rank of $\Pi_B^A(M)$ is equal to $|\ker_{\mathbb{Z}}(A) : \mathbb{Z}B| \cdot \text{rank}(M)$.
3. The module $M$ is regular holonomic if and only if $\Pi_B^A(M)$ is regular holonomic.
Corollary 7.2. Let \( I \subseteq \mathbb{C}[[\partial_z]] \) be an \( A \)-graded binomial ideal. The statements of Theorem 6.2 hold if we replace the binomial \( D_X \)-module \( M = D_X/(I + \langle E - \beta \rangle) \) by \( \Pi_B^A(M) \). Minor modifications are needed; for clarity, we explain the changes in four items.

\((\text{iv})\) The weight vector \( L \) must be replaced with \( L' := (L_B, c \cdot 1_m - L_B) \in \mathbb{Q}^{2m} \). In the second statement, \( T^*X \) and \( n \) are respectively replaced by \( T^*Z \) and \( m \).

\((\text{vi})\) The saturated Horn \( D_Z/sHorn(B, \kappa) \) is (regular) holonomic if and only if the binomial \( D_X \)-module \( D_X/(I(B) + \langle E - A\kappa \rangle) \) is (regular) holonomic. In particular, it follows that \( D_Z/sHorn(B, \kappa) \) is regular holonomic if and only if it is holonomic and the rows of the matrix \( B \) sum to \( 0_m \).

\((\text{viii})\) The minimal rank value above must be scaled by \([\ker_Z(A) : \mathbb{Z}B]\).

\((\text{ix})\) If \( \beta \) does not lie in the union of \( q\deg(P_i) \) with the Andean arrangement of \( I \), then \( \Pi_B^A(M) \) is isomorphic to the direct sum over the toral primary components \( \mathcal{C} \) of \( I \) of the modules \( \Pi_B^A(D_X/(\mathcal{C} + \langle E - \beta \rangle)) \), where \( \mathcal{C} \) lies in \( (6.1) \).

Proof. Only \((\text{vi})\) does not follow immediately from Theorem 7.1. For this, we must also use the decomposition of \( \Pi_B^A(D_X/(I(B) + \langle E - A\kappa \rangle)) \) from Corollary 5.10. Note that each summand in this decomposition is actually isomorphic to the first summand, \( D_Z/sHorn(B, \kappa) \). The parameter shifts \( \kappa + Ck \) in the remaining summands are a red herring, thanks to the graded shifts induced by multiplication by \( z^{k/\kappa} \).

Proof of Theorem 7.1. The forward implication of \((1)\) is Theorem 4.2.(1). For the converse, we first define a map \( \psi_{B, \kappa} : D_Z \to [D_X]^T/(E - \beta) \) that is essentially an inverse to \( \delta_{B, \kappa} \) from \((5.1)\), but without the assumption that the columns of \( B \) span \( \ker_Z(A) \) as a lattice. For notational convenience, assume that the first \( m \) rows of \( B \) are linearly independent. Call \( N \) the corresponding submatrix of \( B \). Let \( N_i^{-1} \) denote the \( i \)th row of \( N^{-1} \). To define \( \psi_{B, \kappa} \), set

\[
\psi_{B, \kappa}(z^u) := x^{Bu} + \langle E - \beta \rangle \quad \text{and} \quad \psi_{B, \kappa}(\eta_i) := N_i^{-1} \circ \theta_1 - \kappa_1, \ldots, \theta_m - \kappa_m + \langle E - \beta \rangle.
\]

By the proof of Corollary 5.5, \( \psi_{B, \kappa} \) is injective. Note also that \( \psi_{B, \kappa} \) respects the \( L' \)- and \( L \)-filtrations on its source and target, respectively. Therefore \( \text{gr}^{L'}(\psi_{B, \kappa}) : \text{gr}^{L'}(D_Z) \to \text{gr}^{L}(D_X/(E - \beta)) \) induces the a dominant map from \( \text{Char}^{L}(D_X \otimes_{D_X} M) \) to \( \text{Char}^{L'}(\Delta_B^A(M)) \), which we denote by \( \Psi_{B, \kappa} \). In addition, since each fiber of \( \Psi_{B, \kappa} \) is a \( T \)-orbit, it is of dimension \( d \). It now follows that

\[
\dim \text{Char}^{L'}(\Delta_B^A(M)) = \dim \text{Char}^{L}(D_X \otimes_{D_X} M) - d. \tag{7.1}
\]

Now suppose that \( M \) is not \( L \)-holonomic. Then in the proof of Theorem 6.2, we show that there is a component of \( \text{Char}^{L}(D_X \otimes_{D_X} M) \) of dimension greater than \( n \). Since \( \text{Char}^{L'}(\Delta_B^A(M)) \) is contained in \( \text{Char}^{L'}(\Pi_B^A(M)) \), we have by \((7.1)\) that

\[
\dim \text{Char}^{L'}(\Pi_B^A(M)) > n - d = m.
\]

Therefore \( \Pi_B^A(M) \) is not \( L' \)-holonomic, completing the proof of \((1)\).

To prove \((2)\), since \( M \) is holonomic, by Theorem 6.5 there is a generic binomial weight vector \( w \in \mathbb{R}^n \) such that at a nonsingular point \( p \), the basic Nilsson solutions of \( M \) in the direction of \( w \) span \( \text{Sol}_p(M) \). Thus Theorem 5.13 implies \((2)\).

The forward implication of \((3)\) is Theorem 4.2.(2), so it remains to establish the converse in the binomial setting. Suppose that \( M \) is not regular holonomic. Then by Theorem 6.6, there exists a
generic weight vector \( w \in \mathbb{R}^n \) for \( M \) such that \( \dim_{\mathbb{C}} \mathcal{N}_w(M) < \text{rank}(M) \). Thus by Theorem 5.13 applied to these Nilsson solutions, we see that, with \( \ell = wB \),

\[
\dim_{\mathbb{C}} \mathcal{N}_\ell(\Pi_B^A(M)) = [\ker Z(A) : \mathbb{Z}B] \cdot \dim_{\mathbb{C}} \mathcal{N}_w(M).
\]

By slight perturbation of \( w \), if necessary, \( \ell \) will be a generic weight vector for \( \Pi_B^A(M) \). But then by \( \Box \), we see that \( \dim_{\mathbb{C}} \mathcal{N}_\ell(\Pi_B^A(M)) < [\ker Z(A) : \mathbb{Z}B] \cdot \text{rank}(M) = \text{rank}(\Pi_B^A(M)) \). Thus Theorem 1.4 implies that \( \Pi_B^A(M) \) cannot be regular holonomic.

**Remark 7.3.** Let \( I \) be an \( A \)-graded binomial \( \mathbb{C}[\partial_x] \)-ideal, and assume that \( \beta \in \mathbb{C}^d \) lies outside of the Andean arrangement of \( I \), so that \( M = D_B(I + \langle E - \beta \rangle) \) is holonomic. Theorem 4.3 in \[CF12\] states that \( \text{Char} (M) \) is equal to the union of the characteristic varieties of the binomial \( D_B \)-modules corresponding to associated primes of \( I \) whose components belong to (6.1). Since prime binomial ideals are isomorphic to toric ideals, it suffices to compute the characteristic varieties of \( A \)-hypergeometric systems, which is done explicitly by Schulze and Walther in \[SW08\]. Therefore, combining \[CF12\] \[SW08\] with Proposition 5.17 yields a description of the characteristic variety of \( \Delta_B^\partial(D_B(I + \langle E - \beta \rangle)) \). If \( M \) is regular holonomic, then Proposition 4.3 provides an explicit description of \( \text{Char}(\Pi_B^A(M)) \).

**Example 7.4.** If \( A = [1 2 3] \) and \( B = \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 0 & 1 \end{bmatrix} \), then when \( \kappa = [0, 0, 0] \),

\[
\text{Char} \left( \frac{D_X}{I(B) + \langle E - Ak \rangle} \right) = \text{Var}(\langle \xi_1, \xi_2, \xi_3 \rangle) \cup \text{Var}(\langle \xi_1, \xi_2, x_3 \rangle)
\]

by computation in \texttt{Macaulay2} \[M2\]. Thus by Proposition 4.3,

\[
\text{Char} \left( \frac{D_Z}{\text{shorn}(B, \kappa)} \right) = \text{Var}(\langle z_1 \xi_1, z_2 \xi_2 \rangle).
\]

On the other hand, \texttt{Macaulay2} \[M2\] reveals that

\[
\text{Char} \left( \frac{D_Z}{\text{Hor}(B, \kappa)} \right) = \text{Var}(\langle z_1 \xi_1, z_2 \xi_2 \rangle) \cup \text{Var}(\langle z_1, 4z_2 - 1 \rangle),
\]

so the component \( \text{Var}(\langle z_1, 4z_2 - 1 \rangle) \) of \( \text{Char}(D_Z/\text{Hor}(B, \kappa)) \) cannot be obtained from those of \( D_X/I(B) + \langle E - A\kappa \rangle \). In particular, the holonomicity of \( D_X/I(B) + \langle E - A\kappa \rangle \) does not by itself guarantee the holonomicity of \( D_Z/\text{Hor}(B, \kappa) \). However, we show in Theorem 9.4 that this will be the case under strong conditions on \( \beta \). In this example, this assumption is equivalent to lying outside of the Andean arrangement of \( A \). Example 9.1 will further address the subtleties of the holonomicity of (non-saturated) Horn \( D \)-modules.

We conclude this section by addressing the irreducibility of monodromy representation of Horn \( D_Z \)-modules. Note first that Theorem 4.2 and Theorem 6.7 together provide a test for the reducibility of monodromy representation for the image of a binomial \( D_X \)-module under \( \Pi_B^A \).

**Corollary 7.5.** If \( \beta = A\kappa \) does not lie in the Andean arrangement of \( I(B) \), then the Horn \( D_Z \)-modules \( D_Z/\text{Hor}(B, \kappa), D_Z/\text{nHorn}(B, \kappa), \) and \( D_Z/\text{shorn}(B, \kappa) \) all have irreducible monodromy representation if and only if (1) and (2) in Theorem 6.7 hold for \( I = I(B) \).
**8. On normalized Horn systems**

We now concentrate on a certain subclass of normalized Horn systems, whose members arise from matrices $B$ that contain an $m \times m$ identity submatrix. For convenience in the notation, we assume that this identity submatrix is formed by the first $m$ rows of $B$. Note that the most widely studied classical hypergeometric systems (e.g., Gauss $\binom{2}{q}F_{1}$, its generalizations $\binom{p}{q}F_{p}$, the Appell and Horn systems in two variables, the four Lauricella families of systems in $m$ variables, etc.) satisfy this hypothesis (see Example 0.4). Thus the following results apply to all of these classical systems.

The approach of this section generalizes and makes more precise an idea of Beukers; namely, classical Horn systems can be obtained from their corresponding $A$-hypergeometric systems by setting certain variables equal to one (see [Beu11b, §§11-13]).

**Theorem 8.1.** Suppose that the top $m$ rows of $B$ form an identity matrix and $\kappa_1 = \cdots = \kappa_m = 0$. Let $r$ denote the inclusion $r: Z \hookrightarrow X$ given by $(z_1, \ldots, z_m) \mapsto (z_1, \ldots, z_m, 1, \ldots, 1)$. If $r^*$ is a restriction of $D_X$-modules, then there is an equality

$$
\frac{D_Z}{nHorn(B, \kappa)} = r^* \left( \frac{D_X}{I(B) + \langle E - A\kappa \rangle} \right).
$$

**Corollary 8.2.** Under the hypotheses of Theorem 8.1, the (regular) holonomicity of the modules $D_Z/nHorn(B, \kappa)$ and $D_X/(I(B) + \langle E - A\kappa \rangle)$ are equivalent. In particular, Corollary 7.2.(vi) holds when $sHorn(B, \kappa)$ is replaced by $nHorn(B, \kappa)$.

**Proof.** If $D_X/(I(B) + \langle E - A\kappa \rangle)$ is (regular) holonomic, then so is $D_Z/nHorn(B, \kappa)$ by Theorem 8.1 since restrictions preserve (regular) holonomicity. For the converse, if $D_X/(I(B) + \langle E - A\kappa \rangle)$ is not (regular) holonomic, then neither is $D_Z/sHorn(B, \kappa)$ by Corollary 7.2.(vi). Since $nHorn(B, \kappa) \subseteq sHorn(B, \kappa)$, and the category of (regular) holonomic $D_Z$-modules is closed under quotients of $D_X$-modules, $D_Z/nHorn(B, \kappa)$ also fails to be (regular) holonomic.

By [SST00, §5.2], the restriction $r^*$ of a cyclic $D_X$-module $D_X/J$ is given by

$$
r^* \left( \frac{D_X}{J} \right) = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\langle x_{m+1}, \ldots, x_n \rangle} \otimes_{\mathbb{C}[x]} \frac{D_X}{J}.
$$

(8.1)

Note that the restriction of a cyclic $D_X$-module is not necessarily cyclic. To establish Theorem 8.1, our first task is to show that the restriction module $r^*(D_X/(I(B) + \langle E - A\kappa \rangle))$ is indeed cyclic. To do this, we compute the $b$-function for the restriction (see [SST00, §§5.1-5.2]).

**Lemma 8.3.** If the matrix formed by the top $m$ rows of $B$ has rank $m$, then the $b$-function of $I(B) + \langle E - A\kappa \rangle$ for restriction to $\Var(x_{m+1}, \ldots, x_n)$ divides $s$.

**Proof.** Begin with the change of variables $x_j \mapsto x_j + 1$ for $m + 1 \leq j \leq m$, and let $J$ denote the $D_X$-ideal obtained from $I(B) + \langle E - A\kappa \rangle$ via this change of variables. As usual, set $\beta = A\kappa$. We now wish to compute the $b$-function of $J$ for restriction to $\Var(x_{m+1}, \ldots, x_n)$.

With $w = (0_n, 1_d) \in \mathbb{R}^n$, the vector $(-w, w)$ induces a filtration on $D_X$, and the $b$-function we wish to compute is the generator of the ideal $\text{gr}(-w, w)(J) \cap \mathbb{C}[s]$, where $s := \theta_{m+1} + \cdots + \theta_n$. Note...
that, since the submatrix of $B$ formed by its first $m$ rows has rank $m$, the submatrix of $A$ consisting of its last $n - m = d$ columns has rank $d$. Thus there are vectors $\nu^{(m+1)}, \ldots, \nu^{(n)} \in \mathbb{R}^d$ such that $(\nu^{(j)} A)_k = \delta_{jk}$ for $m + 1 \leq k \leq n$. For $m + 1 \leq j \leq n$,

$$\sum_{i=1}^{d} \nu_i^{(j)} E_i - \nu^{(j)} \cdot \beta = \sum_{k=1}^{m} (\nu^{(j)} A)_k \theta_k + \theta_j - \nu^{(j)} \cdot \beta \in \langle E - \beta \rangle.$$ 

Using our change of variables and multiplying by $x_j$ with $m + 1 \leq j \leq n$, we obtain

$$\sum_{k=1}^{m} (\nu^{(j)} A)_k x_j \theta_k + x_j^2 \partial x_j + \theta_j - \nu^{(j)} \cdot \beta x_j \in J.$$ 

Taking initial terms with respect to $(-w, w)$ of this expression, it follows that $\theta_j \in \text{gr}(-w, w)(J)$ for each $m + 1 \leq j \leq n$. Therefore $s = \theta_{m+1} + \cdots + \theta_n \in \text{gr}(-w, w)(J)$, and the result follows. 

\textbf{Proof of Theorem 8.1} By Lemma 8.3 $r^*(D_{\bar{X}}/(I(B) + \langle E - A\kappa \rangle))$ is of the form $D_{x}/J$. In order to find the ideal $J$, we must to perform the intersection

$$(I(B) + \langle E - A\kappa \rangle) \cap R_m, \text{ where } R_m := \mathbb{C}[x_1, \ldots, x_n] \langle \partial x_1, \ldots, \partial x_m \rangle \subseteq D_{\bar{X}}, \quad (8.2)$$

and set $x_{m+1} = \cdots = x_n = 1$. We proceed by systematically producing elements of the intersection (8.2). Using the same argument as in the proof of Lemma 8.3 we see that for $m + 1 \leq j \leq n$, each $\theta_j$ can be expressed as a linear combination of $\theta_1, \ldots, \theta_m$ and the parameters $\kappa$ modulo $D_{\bar{X}} \cdot \langle E - A\kappa \rangle$. By our assumption on $B$, $\theta_j$ can be written explicitly as follows:

$$\theta_j = \kappa_j + \sum_{i=1}^{m} b_j \theta_i \mod D_{\bar{X}} \cdot \langle E - A\kappa \rangle \quad \text{for } m + 1 \leq j \leq n. \quad (8.3)$$

Now if $P \in D_{\bar{X}}$, then there is a monomial $\mu$ in $x_{m+1}, \ldots, x_n$ so that the resulting operator $\mu P$ can be written in terms of $x_1, \ldots, x_n, \partial x_1, \ldots, \partial x_m$ and $\theta_{m+1}, \ldots, \theta_n$. In addition, working modulo $D_{\bar{X}} \cdot \langle E - A\kappa \rangle$ for $j > m$ one can replace $\theta_j$ by the expressions (8.3). Thus $\mu P$ is an element of $R_m$ modulo $R_m \cdot \langle E - A\kappa \rangle$. If this procedure is applied to $E_i - (A\kappa)_i$, the result is zero. We now apply it to one of the generators $\partial_x^{(b_k)} + \partial_x^{(b_k)}$ of $I(B)$, where $b_1, \ldots, b_m$ denote the columns of $B$. An appropriate monomial in this case is $\mu_k = \prod_{j=m+1}^{n} x_j^{|b_{jk}|}$. Then the fact that $b_{kk} = 1$ for $1 \leq k \leq m$ and (8.3) together imply that

$$\mu_k(\partial_x^{(b_k)_+} - \partial_x^{(b_k)_-})$$

$$= (\prod_{b_{jk} < 0} x_j^{-b_{jk}}) \partial x_k \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}} \partial x_j - \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}} \prod_{b_{jk} < 0} x_j^{-b_{jk}} \partial x_j - b_{jk} - \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}} \prod_{b_{jk} < 0} x_j^{-b_{jk}} \partial x_j - b_{jk} - \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}} \prod_{b_{jk} < 0} x_j^{-b_{jk}} \partial x_j - b_{jk} - \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}} \prod_{b_{jk} < 0} x_j^{-b_{jk}} \partial x_j - b_{jk} - \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}} \prod_{b_{jk} < 0} x_j^{-b_{jk}} \partial x_j - b_{jk} - \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}}$$

$$= (\prod_{b_{jk} < 0} x_j^{-b_{jk}}) \partial x_k \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}} \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}} \prod_{b_{jk} < 0} x_j^{-b_{jk}} \partial x_j - b_{jk} - \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}} \prod_{b_{jk} < 0} x_j^{-b_{jk}} \partial x_j - b_{jk} - \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}} \prod_{b_{jk} < 0} x_j^{-b_{jk}} \partial x_j - b_{jk} - \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}} \prod_{b_{jk} < 0} x_j^{-b_{jk}} \partial x_j - b_{jk} - \prod_{j > m, b_{jk} > 0} x_j^{b_{jk}}$$

Note that setting $x_{m+1} = \cdots = x_n = 1$ in (8.4), we obtain the $k$th generator of the normalized Horn system $n\text{Horn}(B, \kappa)$, since $b_{jk} < 0$ implies $j > m$. Now suppose that $P$ is an element of the intersection (8.2). In particular, $P$ belongs to $I(B) + \langle E - A\kappa \rangle$, so there are $P_1, \ldots, P_m, Q_1, \ldots, Q_d \in D_{\bar{X}}$ such that

$$P = \sum_{k=1}^{m} P_k(\partial_x^{(b_k)_+} - \partial_x^{(b_k)_-}) + \sum_{i=1}^{d} Q_i(\partial_x^{E_i} - (A\kappa)_i).$$
If we multiply $P$ on the left by a monomial in $x_{m+1}, \ldots, x_n$ and set $x_{m+1} = \cdots = x_n = 1$, the result is the same as if we set $x_{m+1} = \cdots = x_n = 1$ on $P$ directly. Thus we choose an appropriate monomial $\mu$ such that

$$\mu P = \sum_{k=1}^{m} \tilde{P}_k \mu_k (\partial_x^{(b_k)\downarrow} - \partial_x^{(b_k)\uparrow}) + \mu \sum_{i=1}^{d} Q_i (E_i - (A\kappa)_i)$$

for some operators $\tilde{P}_1, \ldots, \tilde{P}_m$. But then, the result of setting $x_{m+1} = \cdots = x_n = 1$ on $\mu P$ (the same as if this were done to $P$) is a combination of the generators of $\text{nHorn}(B, \kappa)$.

We have shown that $r^* (I(B) + \langle E - A\kappa \rangle)$ contains the generators of $\text{nHorn}(B, \kappa)$. Since the module $D_Z/\text{nHorn}(B, \kappa)$ has a nontrivial solution space, it follows that the $b$-function for restriction of $I(B) + \langle E - A\kappa \rangle$ to $\text{Var}(x_{m+1} - 1, \ldots, x_n - 1)$ is $s$. Thus $r^* (D_Z/\langle I(B) + \langle E - A\kappa \rangle \rangle)$ is indeed a nonzero cyclic $D_Z$-module, and it is equal to $D_Z/\text{nHorn}(B, \kappa)$, completing the proof.

9. Characteristic varieties of Horn $D_Z$-modules

In this section, we study the characteristic variety of the Horn $D_Z$-module $D_Z/\text{Horn}(B, \kappa)$. By Corollary [7.2(vi)], the saturated Horn $D_Z$-module $D_Z/\text{sHorn}(B, \kappa)$ is holonomic if and only if the corresponding lattice basis binomial $D_X$-module is holonomic. This provides a characterization for holonomicity of $D_Z/\text{sHorn}(B, \kappa)$ in terms of the Andean arrangement of $I(B)$, via Theorem [6.2(i)]. While it is clear that if $D_Z/\text{Horn}(B, \kappa)$ is holonomic, then so is $D_Z/\text{sHorn}(B, \kappa)$, the following example shows that the converse is not true.

Example 9.1. For the matrices

$$B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \kappa = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \beta = A\kappa = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

a Macaulay2 [M2] computation shows that the binomial $D_X$-module $D_X/(I(B) + \langle E - \beta \rangle)$ is holonomic; this is the easiest way of verifying the holonomicity of $D_Z/\text{sHorn}(B, \kappa)$, as it avoids computing a saturation. On the other hand, $D_Z/\text{Horn}(B, \kappa)$ is not holonomic, since its characteristic variety has the component $\text{Var}(\langle z_3, z_1 \gamma_1 + z_2 \gamma_2 \rangle) \subseteq T^* Z = C^6$. Since the rank of $D_Z/\text{Horn}(B, \kappa)$ equals that of $D_Z/\text{sHorn}(B, \kappa)$ and $D_X/(I(B) + \langle E - \beta \rangle)$, which in this case is 6, this example provides an instance of a Horn system of finite rank that is not holonomic. On the other hand, we have seen that binomial $D_X$-modules and saturated Horn $D_Z$-modules of finite rank are necessarily holonomic by Theorems [6.2(iii)] and [7.1(i)].

In Theorem [9.4], the main result of this section, we provide a sufficient condition for the holonomicity of a Horn $D_Z$-module by constraining the parameters. The key notions follow.

Let $A$ be as in Convention [1.5] and let $B$ be a Gale dual of $A$. If $\gamma \subseteq \{1, \ldots, m\}$, we denote by $B_{\gamma}$ the matrix whose columns are the columns of $B$ indexed by $\gamma$. Let $A[\gamma]$ be a matrix for which $B_{\gamma}$ is a Gale dual, where we assume that the matrix $A$ is included in $A[\gamma]$ as its first $d$ rows. We compute the Andean arrangement of $I(B_{\gamma})$ using the grading induced by the matrix $A[\gamma]$. 
Definition 9.2. A parameter $\beta \in \mathbb{C}^d$ is toral for $B$ if it does not lie in the Andean arrangement of $I(B)$. We say $\beta \in \mathbb{C}^d$ is completely toral for $B$ if it is toral for $B$ and for every $\gamma \subseteq \{1, \ldots, m\}$, $\beta$ does not lie in the projection of the Andean arrangement of $I(B_\gamma)$ onto the first $d$ coordinates.

Lemma 9.3. The set of completely toral parameters for $B$ is Zariski open in $\mathbb{C}^d$.

Proof. This follows from the fact that Andean arrangements are unions of affine spaces.

Theorem 9.4. If $\beta$ is completely toral for $B$ and $A\kappa = \beta$, then $D_Z/Horn(B, \kappa)$ is holonomic.

Before proving this theorem, we state some consequences. Note that these results generalize and refine previous work of Sadykov [Sad02] and Dickenstein, Matusevich, and Sadykov [DMS05].

Corollary 9.5. If $B$ has a completely toral parameter, then $D_Z/Horn(B, \kappa)$ is holonomic for generic choices of $\kappa$.

Proof. Since the set of completely toral parameters for $B$ is Zariski open in affine $d$-space, the fact that it is nonempty implies that it is dense. Thus the result follows from Theorem 9.4.

Corollary 9.6. If $m = 2$, then $D_Z/Horn(B, \kappa)$ is holonomic if and only if $A\kappa$ is toral for $B$. Equivalently, $D_Z/Horn(B, \kappa)$ is holonomic if and only if $D_X/(I(B) + \langle E - A\kappa \rangle)$ is holonomic.

Proof. Since $m = 2$, the only possible matrices $B_\gamma$ for $\emptyset \subsetneq \gamma \subsetneq \{1, 2\}$ have one column. In this case, the Andean arrangement of $I(B_\gamma)$ is empty, and therefore a parameter is toral for $B$ if and only if it is completely toral for $B$. Since $D_X/(I(B) + \langle E - A\kappa \rangle)$ is holonomic precisely when $A\kappa$ is toral, the result follows from Theorem 9.4.

Example 9.7 (Example 9.1 continued). The condition that $A\kappa$ be completely toral for $B$ is sufficient to guarantee that $D_Z/Horn(B, \kappa)$ is holonomic, but it is not necessary. If we let $B'$ be $B$ in Example 9.1 with third column multiplied by $-1$, then $I(B) = I(B')$. Using the same $A$, it follows that $B$ and $B'$ have the same (empty set of) completely toral parameters. Thus, since $A\kappa = \beta = [2, 0, 0, 0]^t$ is not completely toral for $B$, it is not completely toral for $B'$. However, it is still the case that $D_Z/Horn(B', \kappa)$ is holonomic. To see why this might be the case, note that multiplying the third column of $B$ by $-1$ corresponds to the change of variables $z_3 \mapsto 1/z_3$. Thus the higher dimensional component that prevented the Horn module $D_Z/Horn(B, \kappa)$ from being holonomic does not appear in the characteristic variety of $D_Z/Horn(B', \kappa)$ because it is moved to infinity when $B$ is replaced by $B'$.

To prove Theorem 9.4, we describe the characteristic variety of a Horn module under the assumption that the parameter $\beta$ is completely toral for $B$. To achieve this, for each subset $\gamma \subsetneq \{1, \ldots, m\}$, we show that the intersection of $Char(D_Z/Horn(B, \kappa))$ with $\{(z, \zeta) \in T^*\mathbb{Z} \mid z_i \neq 0 \text{ if } i \in \gamma, \ z_i = 0 \text{ if } i \notin \gamma\}$ is naturally contained in the characteristic variety of a saturated Horn module arising from the matrix $B_\gamma$. The completely toral condition on $\beta$ guarantees that this saturated Horn module is holonomic, thus providing a bound the dimension of $Char(D_Z/Horn(B, \kappa))$. We discuss this approach further after Corollary 9.10.

We first need some facts about the primary decomposition of lattice basis ideals. To begin, we study lattice ideals, which play a fundamental role in this primary decomposition. Our key lattice will be $\mathbb{Z}B$, which is spanned by the columns of a Gale dual $B$ of a matrix $A$. 

...
Let $\mathcal{L} \subseteq \mathbb{Z}^n$ be a lattice. The ideal
\[ I_{\mathcal{L}} := \langle \partial_x^u - \partial_x^v \mid u - v \in \mathcal{L} \rangle \subseteq \mathbb{C}[\partial_x] \]
is called a lattice ideal. Note that the toric ideal $I_A$ is the lattice ideal associated to $\ker_{x}(A)$.

**Proposition 9.8.** The variety $\text{Var}(\langle \text{gr}^F(I_{ZB}), \text{gr}^F(E) \rangle) \subseteq T^*X = (\mathbb{C}^*)^n \times \mathbb{C}^n$ is $T$-equivariant of dimension at most $n$, with generic $T$-orbit of dimension $d$.

**Proof.** Since the generators of the ideal $\langle \text{gr}^F(I_{ZB}), \text{gr}^F(E) \rangle$ are homogeneous with respect to the $T$-action on $\mathbb{C}[x^\pm, \xi]$ given by $t \circ x_i = t^{a_i}x_i$ and $t \circ \xi_i = t^{-a_i}\xi_i$, the given variety is $T$-equivariant. The rest of this argument is based on the fact that the binomial $D_X$-module $D_X/(I_{ZB} + \langle E - \beta \rangle)$ is holonomic for all parameters $\beta$, and its characteristic variety can be described set-theoretically. For the holonomicity statement, we use the primary decomposition of $I_{ZB}$ from [ES96]: if $\varphi$ is the order of the finite group $\ker_{x}(A)/ZB$, then there are $g$ group homomorphisms $\rho_1, \ldots, \rho_g: \ker_{x}(A) \to \mathbb{C}^*$, called partial characters, that extend the trivial character $ZB \to \mathbb{C}^*$, so that
\[ I_{ZB} = I(\rho_1) \cap \cdots \cap I(\rho_g), \quad \text{where} \quad I(\rho_j) = \langle \partial_x^u - \partial_x^v(\rho_j(u - v)) \mid u - v \in \ker_{x}(A) \rangle. \]

Note that $\mathbb{C}[\partial_x]/I(\rho_j)$ is isomorphic to $\mathbb{C}[\partial_x]/I_A$ via $\partial_x^i \mapsto \tilde{\rho}_j(c_i)\partial_x^i$, where $\tilde{\rho}_j$ is any extension of $\rho_j$ to $\mathbb{Z}^n$ and $c_i$ is the $i$th standard basis vector. By Theorem 6.2(i), $D_X/(I_{ZB} + \langle E - \beta \rangle)$ is holonomic for all $\beta$, since all of the primary (actually, prime) components of $I_{ZB}$ are toral.

Now by [CF12, Theorem 4.3], $\text{Char}(D_X/(I_{ZB} + \langle E - \beta \rangle))$ is the union of the characteristic varieties of the $D_X/I(\rho_j) + \langle E - \beta \rangle$. Each of these is isomorphic to $D_X/H_A(\beta) = D_X/(I_A + \langle E - \beta \rangle)$ by appropriately rescaling the variables, so we first describe the characteristic variety of an $A$-hypergeometric $D_X$-module. To do this, we recall notions and results from [SW08]. The polyhedral subcomplex of the face lattice of $\text{conv}(\{0, a_1, \ldots, a_n\})$ consisting of faces not containing the origin is called the $A$-umbrella $\Phi(A)$. (More general umbrellas were introduced by Schulze and Walther in [SW08] in order to study $L$-characteristic varieties of $A$-hypergeometric systems. Here we need only consider the umbrella induced by the order filtration on the Weyl algebra $D_X$.) Note that when the rational row span of $A$ contains $1_n$, $\Phi(A)$ is isomorphic to the face lattice of $\text{conv}(A)$.

**Notation 9.9.** If $\sigma \subseteq \{1, \ldots, n\}$, let $A_\sigma$ denote the matrix consisting of the columns of $A$ indexed by $\sigma$, and let $B^\sigma$ denote the matrix consisting of the rows of $B$ indexed by $\sigma$. For $\tau \subseteq \{1, \ldots, n\}$, let $C_\tau$ be the closure of the conormal space to the $T$-orbit of the point $1_\tau$, where $\langle 1_\tau \rangle_i = 1$ if $i \in \tau$ and $\langle 1_\tau \rangle_i = 0$ if $i \in \tau^c = \{1, \ldots, n\} \setminus \tau$. Then $C_\tau$ has dimension $n$ and has defining ideal equal to the radical of $\langle \xi_{\tau^c}, \text{gr}^F(E_\tau), \text{gr}^F(I_{A_\tau}) \rangle$, where $\xi_{\tau^c} = \{\xi_i \mid i \in \tau^c\}$ and $E_\tau$ is the sequence of Euler operators given by the matrix $A_\tau$. It is shown in [SW08] that $\text{Char}(D_X/H_A(\beta)) = \bigcup_{\tau \in \Phi(A)} C_\tau$, which is equidimensional and set-theoretically independent of $\beta$.

If $\tau \subseteq \{1, \ldots, n\}$, then since $C_\tau$ has dimension $n$, so does the ring
\[ K_\tau := \frac{\mathbb{C}[x, \xi]}{\langle \xi_{\tau^c}, \text{gr}^F(E_\tau), \text{gr}^F(I_{A_\tau}) \rangle}. \]
We claim that $\mathbb{C}[x^\pm] \otimes K_\tau$ is also of dimension $n$. If $\tau$ is not a pyramid, then the row span of $A_\tau$ does not contain basis vectors. Thus a generic point on $C_\tau$ has all $x$-coordinates nonzero, so $\dim(\mathbb{C}[x^\pm] \otimes K_\tau) = \dim(K_\tau) = n$. On the other hand, if $\tau$ is a pyramid over the $i$th column $a_i$ of
by the pyramid assumption and since 
In particular, 
where \( (\mathbb{R}) \) establishes the claim. 

We next claim that for any \( \tau \), the generic \( T \)-orbit in \( C_\tau \) is of the full dimension \( d \). To see this, observe that a generic point of \( C_\tau \) has \( x_i \neq 0 \) for \( i \in \tau \). Moreover, if \( i \notin \tau \), then the variable \( x_i \) is not involved in the definition of the ideal of \( C_\tau \), so a generic point in \( C_\tau \) must be nonzero in the \( x \)-coordinates indexed by \( \tau^c \). Since the orbit of a point with nonzero coordinates in \( \xi_\tau \) and \( x_{\tau^c} \) has dimension \( d \), the claim holds.

The above discussion regarding \( \text{Char}(D_X/H_A(\beta)) \) also applies to the study of the characteristic varieties of \( D_X/(I(\rho_j) + (E - \beta)) \), after a rescaling of the variables.

We now consider the ideal \( \langle \text{gr}^F(I_{ZB}), \text{gr}^F(E) \rangle \subseteq \mathbb{C}[x^\pm, \xi] \). The monomials in \( \xi \) that belong to \( \text{gr}^F(I_{ZB}) \) also belong to \( \text{gr}^F(I_A) \), as \( I_{ZB} \subseteq I_A \). If \( \xi^u \in \text{gr}^F(I_A) \), then there is a \( v \in \mathbb{N}^n \) such that \( \partial_x^u - \partial_x^v \in I_A \) and \( |u| > |v| \); in particular, \( u - v \in \text{ker}_Z(A) \). Since \( g > 0 \) is the order of \( \text{ker}_Z(A)/zb \), \( g(u - v) = gu - gv \in zb \). This implies that the monomial \( (\xi^u)^g \) belongs to \( \text{gr}^F(I_{ZB}) \). We conclude that, up to radical, \( \text{gr}^F(I_{ZB}) \) and \( \text{gr}^F(I_A) \) have the same monomials in \( \xi \).

This means that, in order to compute the dimension of \( \text{Var}(\langle \text{gr}^F(I_{ZB}), \text{gr}^F(E) \rangle) \subseteq T^*X \), we need only consider the components of this variety contained in the sets \( \{ (x, \xi) \in T^*X \mid \xi_i \neq 0, i \in \tau; \xi_i = 0, i \notin \tau \} \) for \( \tau \in \Phi(A) \).

Fixing \( \tau \in \Phi(A) \), consider the ideal \( \langle \text{gr}^F(I_{ZB}), \text{gr}^F(E), \xi_i \mid i \notin \tau \rangle : \xi_\tau^\infty \), where \( \xi_\tau := \prod_{i \in \tau} \xi_i \).

Since \( \tau \in \Phi(A) \), the ideal \( \text{gr}^F(I_{ZB^\tau}) \) is the lattice ideal corresponding to \( B^\tau \) in the variables \( \xi_\tau \) and can be decomposed as an intersection of ideals that are isomorphic to \( \text{gr}^F(I_{A_\tau}) \), up to a rescaling of the variables. Thus the zero set of \( \langle \text{gr}^F(I_{ZB}), \text{gr}^F(E), \xi_i \mid i \notin \tau \rangle : \xi_\tau^\infty \) is the union of \( \text{Var}(\langle \text{gr}^F(I(\rho_{\tau,j})), \text{gr}^F(E_i), \xi_i \mid i \notin \tau \rangle : \xi_\tau^\infty) \), where the \( \rho_{\tau,j} \) are the partial characters appearing in the prime decomposition of \( I_{ZB^\tau} \). Since \( I(\rho_{\tau,j}) \) is obtained from \( I_{A_\tau} \) by rescaling the variables, it follows that \( \text{Var}(\langle \text{gr}^F(I(\rho_{\tau,j})), \text{gr}^F(E_i), \xi_i \mid i \notin \tau \rangle : \xi_\tau^\infty) \) has dimension \( n \) with generic \( T \)-orbit of full dimension \( d \), completing the proof.

**Corollary 9.10.** The ring 
\[
R := \mathbb{C}[z^\pm, \xi] \otimes_{\mathbb{C}[z]} \mathbb{C}[z, \zeta] \left\langle \begin{array}{ll}
-z^{w_+} (B_z \zeta)^{(Bw)_-} - z^{w_-} (B_z \zeta)^{(Bw)_+} & \text{if } w \in \mathbb{Z}^m, |Bw| = 0, \\
(B_z \zeta)^{(Bw)_-} & \text{if } w \in \mathbb{Z}^m, |Bw| < 0
\end{array} \right.
\]
has dimension at most \( m \), where \( z \zeta \) is the \( m \times 1 \) vector with entries \( z_1 \zeta_1, \ldots, z_m \zeta_m \).

**Proof.** Let \( b_1, \ldots, b_m \) denote the columns of \( B \). Since \( B \) has rank \( m \), we may assume that the first \( m \) rows of \( B \) form a matrix \( M \) of rank \( m \). Let \( M^{-1} \) denote the \( i \)th row of the matrix \( M^{-1} \in \mathbb{Q}^{m \times m} \).

Consider the ring homomorphism \( R \rightarrow \mathbb{C}[x^\pm, \xi]/\langle \text{gr}^F(I_{ZB}), \text{gr}^F(E) \rangle \) given by \( z_1 \mapsto x^{b_1} \) and \( \zeta_i \mapsto x^{-b_i} M^{-1} \cdot [x_1 \xi_1, \ldots, x_m \xi_m]^t \). Since this homomorphism is injective, it gives rise to a dominant morphism between the corresponding varieties. We conclude that \( \dim(R) \leq n - d = m \) because
this morphism is also constant on $T$-orbits, the dimension of $\text{Var}(\langle \text{gr}^F(I_{ZB}), \text{gr}^F(E) \rangle)$ is $n$, and the generic $T$-orbit (in every component) of this variety has dimension $d$.

Our next step towards a proof of Theorem 9.4 is to find the equations satisfied by the components of $\text{Char}(D_Z/\text{Horn}(B, \kappa))$ that lie in $T^*Z = T^*Z \setminus \text{Var}(z_1 \ldots z_n)$. To do this, we first summarize facts about the primary decomposition of lattice basis ideals from [HS00, DMM10a].

For each associated prime $p$ of $I(B)$, there exist subsets $\sigma \subseteq \{1, \ldots, n\}$ and $\omega \subseteq \{1, \ldots, m\}$ such that $p = \mathbb{C}[\partial_x] : J + \langle \partial_{x_i} | i \notin \sigma \rangle$ and $J \subseteq \mathbb{C}[\partial_{x_i} | i \in \sigma]$ is one of the associated primes of the lattice ideal $I_{ZB_{\sigma,\omega}}$, where $B_{\sigma,\omega}$ is the submatrix of $B$ whose rows are indexed by $\sigma$ and whose columns are indexed by $\omega$. Further, in order for $B_{\sigma,\omega}$ to appear as a toral component in the primary decomposition of $I(B)$, the matrix whose rows are $B_i$ for $i \notin \sigma$ consists of a square, invertible block with columns indexed by $j \notin \omega$, together with a zero block.

If $p$ as above is associated to $I(B)$, then so is $\mathbb{C}[\partial_x] : J' + \langle \partial_{x_i} | i \notin \sigma \rangle$ for every associated prime $J'$ of $I_{ZB_{\sigma,\omega}}$. Recall that the associated primes of a lattice ideal are all isomorphic (by rescaling the variables) to the toric ideal arising from the saturation $(\mathbb{Q}B_{\sigma,\omega}) \cap \mathbb{Z}^n$ of the lattice $\mathbb{Z}B_{\sigma,\omega}$.

If $p$ is a toral associated prime of $I(B)$, then so are the other associated primes arising from the same lattice ideal $I_{ZB_{\sigma,\omega}}$. Here, the $p$-primary component of $I(B)$ is of the form $(I(B) + \mathbb{C}[\partial_x] : J) : \langle \prod_{i \in \sigma} \partial_{x_i} \rangle^\infty + M_{\sigma}$, where $J$ is an associated prime of $I_{ZB_{\sigma,\omega}}$ as above and $M_{\sigma}$ is a monomial ideal generated in the variables $\partial_{x_j}$ for $j \notin \sigma$. The monomial ideal $M_{\sigma}$ depends only on the rows of $B$ indexed by $\{1, \ldots, n\} \setminus \sigma$ and is therefore the same in each primary component involving a fixed associated prime $J$ of $I_{ZB_{\sigma,\omega}}$. Consequently, $I(B) + \mathbb{C}[\partial_x] : I_{ZB_{\sigma,\omega}} + M_{\sigma}$ is contained in the intersection of the primary components of $I(B)$ arising from the lattice ideal $I_{ZB_{\sigma,\omega}}$.

We will also use that if $p$ and $p'$ are two toral associated primes of $I(B)$ arising from the same lattice ideal $I_{ZB_{\sigma,\omega}}$ with primary components $\mathcal{C}$ and $\mathcal{C'}$, then $\text{qdeg}(\mathbb{C}[\partial_x]/\mathcal{C}) = \text{qdeg}(\mathbb{C}[\partial_x]/\mathcal{C'})$.

**Lemma 9.11.** Let $\mathcal{S}$ be a set of pairs $(\sigma, \omega)$, where $\sigma \subseteq \{1, \ldots, n\}$ and $\omega \subseteq \{1, \ldots, m\}$, such that if $J$ is an associated prime of $I_{ZB_{\sigma,\omega}}$, then $J + \langle \partial_{x_i} | i \notin \sigma \rangle$ is a toral associated prime of $I(B)$. Then for each pair $(\sigma, \omega) \in \mathcal{S}$, the ideal $I_{ZB_{\sigma,\omega}} + M_{\sigma}$ is contained in the intersection of the primary components of $I(B)$ arising from the lattice ideal $I_{ZB_{\sigma,\omega}}$. Further, if $J_{\sigma,\omega} := \left\langle \prod_{i \notin \sigma} B_i z_\zeta^{u_i} \prod_{i \in \sigma} \partial_{x_i}^{w_i} \in M_{\sigma} \right\rangle + \left\langle z^w(B_{\sigma,\omega} z_\zeta)^{(Bw)} - z^{-w}(B_{\sigma,\omega} z_\zeta)^{(Bw)} \right\rangle_{\mathbb{Z}^n} \subseteq \mathbb{C}[z^\pm, \zeta]$ is contained in the intersection of the primary components of $I(B)$ arising from the lattice ideal $I_{ZB_{\sigma,\omega}}$, then the ring $R_{\sigma,\omega} := \mathbb{C}[z^\pm, \zeta]/J_{\sigma,\omega}$ has dimension at most $m$. Consequently, the ring $R_{\mathcal{S}} := \mathbb{C}[z^\pm, \zeta]/\prod_{(\sigma,\omega) \in \mathcal{S}} J_{\sigma,\omega}$ also has dimension at most $m$.

**Proof.** It is enough to show that $\dim(R_{\sigma,\omega}) \leq m$. Note that since $\sqrt{M_{\sigma}} = \langle \partial_{x_i} | i \notin \sigma \rangle$, the polynomials $B_i z_\zeta$, for $i \notin \sigma$, belong to the radical of $\langle \prod_{i \notin \sigma} B_i z_\zeta^{u_i} \prod_{i \notin \sigma} \partial_{x_i}^{u_i} \in M_{\sigma} \rangle$. Recall that the matrix with rows $B_i$ for $i \notin \sigma$ consists of a square, invertible block with columns indexed by $j \notin \omega$, as well as a zero block. Thus $\langle \partial_{x_j} z_\zeta | j \in \{1, \ldots, m\} \setminus \omega \rangle$ belongs to the radical of $\langle \prod_{i \notin \sigma} B_i z_\zeta^{u_i} \prod_{i \notin \sigma} \partial_{x_i}^{u_i} \in M_{\sigma} \rangle$, so the statement follows from Corollary 9.10.

**Proof of Theorem 9.4** Since $\beta = A\kappa$ is toral for $B$, the module $D_Z/\text{Horn}(B, \kappa)$ is holonomic by Corollary 7.2(vii) and Theorem 6.2(i). The components of $\text{Char}(D_Z/\text{Horn}(B, \kappa))$ that are not contained in $\text{Var}(\prod_{i=1}^n x_i)$ form $\text{Char}(D_Z/\text{Horn}(B, \kappa))$, and consequently have dimension $m$. 


We now consider the components of \( \operatorname{Char}(D_{\mathcal{Z}}/\operatorname{Horn}(B, \kappa)) \) that are contained in coordinate hyperplanes. To do this, fix \( \gamma \subseteq \{1, \ldots, m\} \), and note that it is enough to show that the components of \( \operatorname{Char}(D_{\mathcal{Z}}/\operatorname{Horn}(B, \kappa)) \) that are contained in \( \operatorname{Var}(\prod_{j \notin \gamma} z_j) \) but not in \( \operatorname{Var}(\prod_{j \notin \gamma'} z_j) \) for any \( \gamma' \) strictly containing \( \gamma \) are of dimension at most \( m \). For simplicity of notation for the remainder of this proof, set \( B := B_{\gamma} \) and \( A := A[\gamma] \).

By [DMM10b, Proposition 6.4], \( \hat{I}(B) + \langle E - \beta \rangle = \hat{I} + \langle E - \beta \rangle \), where \( \hat{I} \) is the intersection of the associated components \( \mathcal{C} \) of \( I(B) \) for which \( \beta \in \text{qdeg}(\mathbb{C}[\partial_z]/\mathcal{C}) \). (Here, quasidegrees are computed with respect to the \( A \)-grading.) Since \( \beta \) is completely toral for \( B \), the components \( \mathcal{C} \) that appear in the intersection \( \hat{I} \) are all toral components of \( I(B) \). By the discussion before Lemma 9.11, \( \hat{I} \) contains a product of ideals of the form \( I_{zB_{\sigma, \omega}} + \hat{M}_{\sigma} \), where \( \sigma \subseteq \{1, \ldots, n\} \), \( \omega \subseteq \gamma \), and \( \hat{M}_{\sigma} \) is a monomial ideal whose radical is \( \langle \partial_{x_i} \mid i \notin \sigma \rangle \) and whose generators involve only those variables. Let \( S \) be the set of pairs \( (\sigma, \omega) \) that appear in this product.

We claim that, with \( S_{\gamma} := \mathbb{C}[z, \zeta, z_{\gamma}]/(z_{\gamma}) \), there is a surjection from \( S_{\gamma} / \prod_{(\sigma, \omega) \in S} J_{\sigma, \omega} \) to \( S_{\gamma}/S_{\gamma} \cdot \operatorname{gr}^F(\operatorname{Horn}(B, \kappa)) \), where \( J_{\sigma, \omega} \subseteq S_{\gamma} \) are ideals with generators as in Lemma 9.11. With this surjection, Lemma 9.11 implies that the components of \( \operatorname{Char}(D_{\mathcal{Z}}/\operatorname{Horn}(B, \kappa)) \) contained in \( \operatorname{Var}(\prod_{j \notin \gamma} z_j) \) but not in \( \operatorname{Var}(\prod_{j \notin \gamma'} z_j) \) for any \( \gamma' \supseteq \gamma \) have dimension at most \( m \), as desired.

To establish this final claim, let \( f := \prod_{(\sigma, \omega) \in S} f_{\sigma, \omega} \), where each \( f_{\sigma, \omega} \) is either of the form:

1. \( x_\sigma^u \partial_{x_{\omega}}^w \), where \( u \in \mathbb{Z}[\omega] \) and \( x_\sigma = \{ x_i \mid i \in \sigma \} \); or
2. \( x_\sigma^u \partial_{x_{\omega}}^w \), where \( \partial_{x_{\omega}} \in \hat{M}_{\sigma} \) and \( x_{\omega} = \{ x_i \mid i \notin \sigma \} \).

Note that \( f \) is invariant under the \( T \)-action, and it belongs to \( D_X \cdot I(B) + \langle E - \beta \rangle \) because

\[
\prod_{(\sigma, \omega) \in S} (I_{zB_{\sigma, \omega}} + \hat{M}_{\sigma}) + \langle E - \beta \rangle \subseteq I(B) + \langle E - \beta \rangle.
\]

In fact, since \( \hat{B} \) is a submatrix of \( B \) and for every \( (\sigma, \omega) \in S \), \( B_{\sigma, \omega} \) consists of \( B_{\sigma, \omega} \) and a block of zeros, it follows that \( f \) belongs to the subring of \( [D_X]^T_B \) generated by \( \theta \) and \( x^B u \), where \( u \in \mathbb{Z}^m \). We denote this subring by \( [D_X]^T_B \) and define a homomorphism \( \delta_{B, \kappa} : [D_X]^T_B \rightarrow D_Z \) by \( x^B u p(\theta) \mapsto z^u \rho(B \eta + \kappa) \). As in Proposition 5.4, this \( \delta_{B, \kappa} \) is surjective, with kernel given by \( \langle E - \beta \rangle \).

Since \( f \in (D_X \cdot I(\hat{B}) + \langle E - \beta \rangle) \cap [D_X]^T_B \), it can be written as a combination of the \( |\gamma| \) binomial generators of \( I(\hat{B}) \) and the sequence \( E - \beta \), all of which belong to \( [D_X]^T_B \). Further, the coefficients in this combination also belong to \( [D_X]^T_B \), since \( g \in [D_X]^T_B \) and \( h \in [D_X]^T \) \( \backslash [D_X]^T_B \) together imply that \( hg \notin [D_X]^T_B \). Applying \( \partial_{\delta_{B, \kappa}} \) to such an expression for \( f \) and clearing denominators, we conclude that there exists a \( \nu \in \mathbb{N}^{|\gamma|} \) such that \( z^\nu \delta_{B, \kappa}(f) \) is a \( D_{\mathcal{Z}} \)-combination of the generators of \( \operatorname{Horn}(B, \kappa) \) indexed by \( \gamma \).

Allowing \( f \) to range through all possibilities allowed by (i) and (ii) above, the images of the set of \( \operatorname{gr}^F(z^\nu \delta_{B, \kappa}(f)) \) in \( S_{\gamma} \) generate precisely the ideal \( S_{\gamma} \cdot \prod_{(\sigma, \omega) \in S} J_{\sigma, \omega} \). This yields the containment

\[
S_{\gamma} \cdot \prod_{(\sigma, \omega) \in S} J_{\sigma, \omega} \subseteq S_{\gamma} \cdot \operatorname{gr}^F(\operatorname{Horn}(B, \kappa)),
\]

which induces the surjection of rings needed to complete the proof. \( \square \)
10. Torus Invariants and the Horn–Kapranov Uniformization

In this section, we consider the singular locus of the image of an \( A \)-hypergeometric system under \( \Pi_A^\Lambda \), still following Convention 3.2. We show that our framework for constructing \( \Pi_A^\Lambda \) can be thought of as a generalization of Kapranov’s ideas in [Kap91].

In this direction, we restrict our attention to the case that the \( \mathbb{Q} \)-rowspan of \( A \) contains \( 1_n \), so that the \( A \)-hypergeometric system \( D_X/H_A(\beta + E) = D_X/H_A(\beta) \) is regular holonomic for all \( \beta \) [Hot98, SW08]. This allows us to apply Proposition 4.3, which provides a precise description of the singular locus \( \text{Sing}(\Pi_A^\Lambda(D_X/H_A(\beta))) \). Namely, it is the union of \( \text{Sing}(\Delta_A^\Lambda(D_X/H_A(\beta))) \) with the coordinate hyperplanes in \( \mathbb{Z} \). The former is a geometric quotient by the torus \( T \) of \( \text{Sing}(D_X/H_A(\beta)) \). We note that in [HT11], Gröbner bases and \( D \)-module theory are used to determine the singular locus of the Lauricella \( F_C \) system, as well as that of its associated binomial \( D \)-module.

The secondary polytope of \( A \) is the Newton polytope of the principal \( A \)-determinant, which is the defining equation for the codimension one part of \( \text{Sing}(D_X/H_A(\beta)) \). Since the principal \( A \)-determinant is \( A \)-graded, its Newton polytope lies in an \( m \)-dimensional subspace of \( \mathbb{C}^n \). Thus, as a direct consequence of Proposition 4.3, we have the following result.

**Corollary 10.1.** The Newton polytope of the defining polynomial of the codimension one part of \( \text{Sing}(\Pi_A^\Lambda(D_X/H_A(\beta))) \) is the secondary polytope of \( A \) viewed in \( \mathbb{R}^m \).

By following essentially the same argument as the one given in the proof in [Kap91] Theorem 2.1.a, we recover the Horn–Kapranov uniformization of the \( A \)-discriminant from our setup for the construction of \( \Pi_A^\Lambda \). Let \([n] := \{1, \ldots, n\}\), and recall the definition of \( C_{[n]} \) from Notation 9.9.

Since we have assumed that the rational row span of \( A \) contains the vector \( 1_n \), the \( A \)-discriminant is the defining polynomial of the projection onto the \( x \)-coordinates of \( C_{[n]} \setminus \text{Var}(\xi_1, \ldots, \xi_n) \), as long as this projection is a hypersurface. Otherwise, the \( A \)-discriminant is defined to be the polynomial \( 1 \). In the hypersurface case, the reduced \( A \)-discriminant is the polynomial \( \nabla A \) given by saturating \( x_1 \cdots x_n \) out of the principal \( A \)-determinant, and thus cuts out the intersection of the \( A \)-discriminant hypersurface with \( (\mathbb{C}^*)^n \). Note that the \( A \)-discriminant is \( A \)-graded, and thus \( \nabla A \) is invariant with respect to the torus action.

Since the algebraic counterpart of projection is elimination, we see that \( \nabla A \), considered as a polynomial in \( 2n \) variables, vanishes on \( C_{[n]} \setminus \text{Var}(\langle x_1 \cdots x_n \rangle \cap \langle \xi_1, \ldots, \xi_n \rangle) \), and in particular on \( C_{[n]} \setminus \text{Var}(x_1 \cdots x_n) \). Write \( \nabla A \) as a finite sum \( \sum_{w \in \mathbb{Z}^m} \lambda_w x^B w \). If \( (\bar{x}, \bar{\xi}) \) belongs to \( C_{[n]} \setminus \text{Var}(x_1 \cdots x_n, \xi_1 \cdots \xi_n) \), then

\[
0 = \nabla A(\bar{x}, \bar{\xi}) = \sum_{w \in \mathbb{Z}^m} \lambda_w \bar{x}^B w = \sum_{w \in \mathbb{Z}^m} \lambda_w \bar{x}^B w \bar{\xi}^B w = \sum_{w \in \mathbb{Z}^m} \lambda_w (\bar{x} \bar{\xi})^B w,
\]

as \( \xi^B w = 1 \) on \( C_{[n]} \setminus \text{Var}(x_1 \cdots x_n, \xi_1 \cdots \xi_n) \). Dehomogenizing, we obtain

\[
\delta_{B,\kappa} \left( \sum_{w \in \mathbb{Z}^m} \lambda_w (x \xi)^B w \right) = \sum_{w \in \mathbb{Z}^m} \lambda_w (B^B z \xi)^B w = \sum_{w \in \mathbb{Z}^m} \lambda_w ((B^B z \xi)^B)^w,
\]

where, for \( x \in X \) and \( b_1, \ldots, b_m \) the columns of \( B \), we write \( x^B := (x^{b_1}, \ldots, x^{b_m}) \), as in §5.
Replacing $z^\nu$ by $(s_1, \ldots, s_m)$, it is now easy to see that the dehomogenized reduced $A$-discriminant is the defining equation for the variety with the desired parametrization:

$$\left( \mathbb{C}^m \setminus \bigcup_{i=1}^n \text{Var}((B s)_i) \right) \ni s \mapsto (B s)^B,$$

known as the Horn–Kapranov uniformization.

**REFERENCES**

[Ado94] Alan Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. 73 (1994), 269–290. 24

[Berk10] Christine Berkesch, *Euler–Koszul methods in algebra and geometry*, PhD thesis, Purdue University, 2010. 25

[Berk11] Christine Berkesch, *The rank of a hypergeometric system*, Compos. Math. 147 (2011), no. 1, 284–318. 24

[BM09] Christine Berkesch and Laura Felicia Matusevich, *A-graded methods for monomial ideals*, J. Algebra, 322 (2009), 2886–2904. 25

[BMW13] Christine Berkesch Zamaere, Laura Felicia Matusevich, and Uli Walther, *Singularities of binomial $D$-modules*, preprint (2013). 22, 23

[Bern72] I. N. Bernštejn, *Analytic continuation of generalized functions with respect to a parameter*, Funkcional. Anal. i Priložen. 6 (1972), no. 4, 26–40. 5

[Beu11a] Frits Beukers, *Irreducibility of $A$-hypergeometric systems*, Indag. Math. (N.S.) 21 (2011), no. 1-2, 30–39. 26

[Beu11b] Frits Beukers, *Notes on $A$-hypergeometric functions*, Séminaires & Congrès, 23 (2011), 25–61. 29

[BGK+87] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers, *Algebraic $D$-modules*, Perspectives in Mathematics, 2. Academic Press Inc., Boston, MA, 1987. 5

[CF12] Francisco-Jesús Castro-Jiménez and María-Cruz Fernández-Fernández, *On irregular binomial $D$-modules*, Math. Z. 272 (2012), no. 3-4, 1321–1337. 22, 23, 28, 33

[DMM12] Alicia Dickenstein, Federico Martínez, and Laura Felicia Matusevich, *Nilsson solutions for irregular hypergeometric systems*, Rev. Mat. Iberoam. 28 (2012), no. 3, 723–758. 22, 24, 25

[DMM10a] Alicia Dickenstein, Laura Felicia Matusevich, and Ezra Miller, *Combinatorics of binomial primary decomposition*, Math Z., 264, no. 4, (2010) 745–763. 23, 35

[DMM10b] Alicia Dickenstein, Laura Felicia Matusevich, and Ezra Miller, *Binomial $D$-modules*, Duke Math. J. 151 no. 3 (2010), 745–763. 1, 22, 23, 36

[DMS05] Alicia Dickenstein, Laura Felicia Matusevich, and Timur Sadykov, *Bivariate hypergeometric $D$-modules*, Advances in Mathematics, 196 (2005), 78–123. 23, 32

[ES96] David Eisenbud and Bernd Sturmfels, *Binomial ideals*, Duke Math. J. 84 (1996), no. 1, 1–45. 23, 33

[GGZ87] I. M. Gel’fand, M. I. Graev, and A. V. Zelevinskiĭ, *Holonomic systems of equations and series of hypergeometric type*, Dokl. Akad. Nauk SSSR 295 (1987), no. 1, 14–19. 1, 24

[GKZ89] I. M. Gel’fand, A. V. Zelevinskiĭ, and M. M. Kapranov, *Hypergeometric functions and toric varieties*, Funktsional. Anal. i Prilozhen. 23 (1989), no. 2, 12–26. Correction in ibid, 27 (1993), no. 4, 91. 1

[Gin86] V. Ginsburg, *Characteristic varieties and vanishing cycles*, Invent. Math. 84 (1986), 327–402. 15

[HT11] Ryohei Hattori and Nobuki Takayama, *The singular locus of Lauricella’s $F_C^*$, 2011, available at arXiv:1110.6675. 37

[HS00] Serkan Hoşten and Jay Shapiro, *Primary decomposition of lattice basis ideals*. (English summary) Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998). J. Symbolic Comput. 29 (2000), no. 4-5, 625–639. 35

[Hot98] Ryoshi Hotta, *Equivariant $D$-modules*, 1998, available at arXiv:math/9805021. 37

[Kap91] M. M. Kapranov, *A characterization of $A$-discriminant hypersurfaces in terms of the logarithmic Gauss map*, Math. Ann. 290 (1991), 277–285. 37

[M2] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at http://www.math.uiuc.edu/Macaulay2/. 28, 31
[MMW05] Laura Felicia Matusevich, Ezra Miller, and Uli Walther, Homological methods for hypergeometric families, J. Amer. Math. Soc. 18 (2005), no. 4, 919–941.

[Sad02] Timur Sadykov, On the Horn system of partial differential equations and series of hypergeometric type, Math. Scand. 91 (2002), no. 1, 127–149.

[Sai11] Mutsumi Saito, Irreducible quotients of A-hypergeometric systems, Compos. Math. 147 (2011), no. 2, 613–632.

[SST00] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama, Gröbner Deformations of Hypergeometric Differential Equations, Springer–Verlag, Berlin, 2000.

[SW08] Mathias Schulze and Uli Walther, Irregularity of hypergeometric systems via slopes along coordinate subspaces, Duke Math. J. 142 (2008), no. 3, 465–509.

[SW12] Mathias Schulze and Uli Walther, Resonance equals reducibility for A-hypergeometric systems, Algebra Number Theory 6 (2012), no. 3, 527–537.

[Smi01] Gregory G. Smith, Irreducible components of characteristic varieties, J. Pure Appl. Algebra 165 (2001), no. 3, 291–306.

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455.

E-mail address: cberkesc@umn.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843.

E-mail address: laura@math.tamu.edu

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907.

E-mail address: walther@math.purdue.edu