On perfect packings in dense graphs

József Balogh,* Alexandr V. Kostochka† and Andrew Treglown‡

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Abstract

We say that a graph \( G \) has a perfect \( H \)-packing if there exists a set of vertex-disjoint copies of \( H \) which cover all the vertices in \( G \). We consider various problems concerning perfect \( H \)-packings: Given \( n, r, D \in \mathbb{N} \), we characterise the edge density threshold that ensures a perfect \( K_r \)-packing in any graph \( G \) on \( n \) vertices and with minimum degree \( \delta(G) \geq D \). We also give two conjectures concerning degree sequence conditions which force a graph to contain a perfect \( H \)-packing. Other related embedding problems are also considered. Indeed, we give a degree sequence condition which forces a graph to contain a copy of \( K_r \), thereby strengthening the minimum degree version of Turán’s theorem. We also characterise the edge density threshold that ensures a graph \( G \) contains \( k \) vertex-disjoint cycles.

1 Introduction

Given two graphs \( H \) and \( G \), a perfect \( H \)-packing in \( G \) is a collection of vertex-disjoint copies of \( H \) which cover all the vertices in \( G \). Perfect \( H \)-packings are also referred to as \( H \)-factors or perfect \( H \)-tilings. Hell and Kirkpatrick [7] showed that the decision problem whether a graph \( G \) has a perfect \( H \)-packing is NP-complete precisely when \( H \) has a component consisting of at least 3 vertices. So for such graphs \( H \), it is unlikely that there is a complete characterisation of those graphs containing a perfect \( H \)-packing. Thus, there has been significant attention on obtaining sufficient conditions that ensure a graph \( G \) contains a perfect \( H \)-packing.

A seminal result in the area is the Hajnal-Szemerédi theorem [6] which states that a graph \( G \) whose order \( n \) is divisible by \( r \) has a perfect \( K_r \)-packing provided that \( \delta(G) \geq (r-1)n/r \). Kühn and Osthus [10, 11] characterised, up to an additive constant, the minimum degree which ensures a graph \( G \) contains a perfect \( H \)-packing for an arbitrary graph \( H \).

It is easy to see that the minimum degree condition in the Hajnal-Szemerédi theorem cannot be lowered. Of course, this does not mean that one cannot strengthen this result. Ore-type degree conditions consider the sum of the degrees of non-adjacent vertices in a graph. The following Ore-type result of Kierstead and Kostochka [8] implies the Hajnal-Szemerédi theorem.

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*University of Illinois, Urbana-Champaign, USA and University of California, San Diego, USA, jobal@math.uiuc.edu. This author is supported by NSF CAREER Grant DMS-0745185, UIUC Campus Research Board Grant 11067, and OTKA Grant K76099.

†University of Illinois, Urbana-Champaign, USA and Institute of Mathematics, Novosibirsk, Russia, kostochk@math.uiuc.edu. This author is supported in part by NSF grant DMS-0965587 and by grant 09-01-00244-a of the Russian Foundation for Basic Research.

‡University of Birmingham, Birmingham, UK, treglowa@maths.bham.ac.uk
Theorem 1 (Kierstead and Kostochka [8]) Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Suppose that \( G \) is a graph on \( n \) vertices such that for all non-adjacent \( x \neq y \in V(G) \),

\[
d(x) + d(y) \geq 2(1 - 1/r)n - 1.
\]

Then \( G \) contains a perfect \( K_r \)-packing.

Kühn, Osthus and Treglown [12] characterised, asymptotically, the Ore-type degree condition which ensures a graph \( G \) contains a perfect \( H \)-packing for an arbitrary graph \( H \).

1.1 Degree sequence conditions forcing a perfect \( K_r \)-packing

Chvátal [3] gave a condition on the degree sequence of a graph which ensures Hamiltonicity: Suppose that \( G \) is a graph on \( n \) vertices and that the degrees of the graph are \( d_1 \leq \cdots \leq d_n \). If \( n \geq 3 \) and \( d_i \geq i + 1 \) or \( d_{n-i} \geq n - i \) for all \( i < n/2 \) then \( G \) is Hamiltonian. So in the case when \( n \) is even, this degree sequence condition ensures that \( G \) has a perfect \( K_2 \)-packing (i.e. a perfect matching). We propose the following conjecture on the degree sequence of a graph which forces a perfect \( K_r \)-packing.

Conjecture 2 Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Suppose that \( G \) is a graph on \( n \) vertices with degree sequence \( d_1 \leq \cdots \leq d_n \) such that:

\[ (\alpha) \quad d_i \geq (r - 2)n/r + i \text{ for all } i < n/r; \]

\[ (\beta) \quad d_{n/r+1} \geq (r - 1)n/r. \]

Then \( G \) contains a perfect \( K_r \)-packing.

Note that Conjecture 2, if true, is much stronger than the Hajnal-Szemerédi theorem since the degree condition allows for \( n/r \) vertices to have degree less than \((r - 1)n/r\). Further, Proposition 14 in Section 5 shows that the condition on the degree sequence in Conjecture 2 is essentially “best possible”. Chvátal [3] proved Conjecture 2 in the case when \( r = 2 \). We prove the conjecture in the case when \( G \) is additionally \( K_{r+1} \)-free (see Section 6).

It is also of interest to establish degree sequence conditions which force a single copy of \( K_r \) in a graph \( G \). In Section 7 we give such a result, which is a consequence of the following structural theorem.

Theorem 3 Suppose that \( n, r \in \mathbb{N} \) such that \( n \geq r \) and so that \( r \) divides \( n \). Let \( G \) be a \( K_{r+1} \)-free graph on \( n \) vertices whose degree sequence \( d_1 \leq \cdots \leq d_n \) is such that \( d_{n/r} \geq (r - 1)n/r \). Then \( G \subseteq T(n,r) \).

(Here \( T(n,r) \) denotes the complete \( r \)-partite Turán graph on \( n \) vertices; so each vertex class has size \([n/r]\) or \([n/r]\).)

1.2 Perfect packings in dense graphs of low minimum degree

In Section 3 we consider the following natural problem: Let \( n, r \in \mathbb{N} \) such that \( r \) divides \( n \). Given some \( D \in \mathbb{N} \), what edge density condition ensures that any graph \( G \) on \( n \) vertices and of minimum degree \( \delta(G) \geq D \) contains a perfect \( K_r \)-packing? In Section 4.1 we deal with the case when \( r = 2 \). The following result completely answers this question for \( r \geq 3 \).
Theorem 4 Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n$. Given any $D \in \mathbb{N}$ such that $r - 1 \leq D \leq (r - 1)n/r - 1$ define

$$g(n, r, D) := \max \left\{ \left( \frac{n}{2} - \frac{n/r + 1}{2} \right), D(n - D) + \left( \frac{n - 1 - D}{2} \right) + e(T(D, r - 2)) \right\}.$$ 

Suppose that $G$ is a graph on $n$ vertices with $\delta(G) \geq D$ and $e(G) > g(n, r, D)$. Then $G$ contains a perfect $K_r$-packing. Moreover, there exists a graph $G'$ on $n$ vertices with $\delta(G') \geq D$ and $e(G') = g(n, r, D)$ but such that $G'$ does not contain a perfect $K_r$-packing.

Clearly a graph $G$ of minimum degree $\delta(G) < r - 1$ cannot contain a perfect $K_r$-packing. Further, regardless of edge density, every graph $G$ whose order $n$ is divisible by $r$ and with $\delta(G) \geq (r - 1)n/r$ contains a perfect $K_r$-packing. Thus, Theorem 4 considers all values of $D$ where our problem was not solved previously. We prove Theorem 4 in Section 3. In Section 2 we prove the ‘moreover’ part of Theorem 4. That is, we show that the edge density condition in Theorem 4 is best possible for all values of $D$ such that $r - 1 \leq D \leq (r - 1)n/r - 1$.

An equitable $k$-colouring of a graph $G$ is a proper $k$-colouring of $G$ such that any two colour classes differ in size by at most one. Let $n, r \in \mathbb{N}$ such that $r$ divides $n$. Notice that a graph $G$ on $n$ vertices has a perfect $K_r$-packing if and only if the complement $\overline{G}$ of $G$ has an equitable $n/r$-colouring. So, for example, the Hajnal-Szemerédi theorem can be stated in terms of equitable colourings. Here we also find it more convenient to work with equitable colourings. Indeed, rather than prove Theorem 1 directly, Kierstead and Kostochka proved the equivalent statement for equitable colourings. Here we also find it more convenient to work with equitable colourings. Thus, instead of proving Theorem 4 directly we prove the following equivalent result.

Theorem 5 Let $n, r \in \mathbb{N}$ such that $r \geq 3$ and $r$ divides $n$. Given any $D \in \mathbb{N}$ such that $n/r \leq D \leq n - r$ define

$$f(n, r, D) := \min \left\{ \left( \frac{n/r + 1}{2} \right), D + e(T(n - D - 1, r - 2)) \right\}.$$ 

Suppose that $G$ is a graph on $n$ vertices with $\Delta(G) \leq D$ and $e(G) < f(n, r, D)$. Then $G$ has an equitable $n/r$-colouring. Moreover, there exists a graph $G'$ on $n$ vertices with $\Delta(G') \leq D$ and $e(G') = f(n, r, D)$ but such that $G'$ does not have an equitable $n/r$-colouring.

(Note that here $\overline{T(n, r)}$ denotes the complement of the Turán graph $T(n, r)$.)

1.3 Vertex-disjoint cycles in dense graphs

Given $k \in \mathbb{N}$, Corrádi and Hajnal [5] proved that every graph $G$ on $n \geq 3k$ vertices and of minimum degree $\delta(G) \geq 2k$ contains at least $k$ vertex-disjoint cycles. So when $n = 3k$, the Corrádi-Hajnal theorem is precisely the Hajnal-Szemerédi theorem in the case when $r = 3$. Recently, Allen, Böttcher, Hladký and Piguet (see [1]) characterised the density threshold that ensures a sufficiently large $n$-vertex graph $G$ contains at least $k$ vertex-disjoint triangles where $0 \leq k \leq n/3$. As an application of Theorem 4 we characterise the density threshold that ensures an $n$-vertex graph $G$ contains at least $k$ vertex-disjoint cycles where $n \geq 7k/2$. 

3
Theorem 6 Let \( n, k \in \mathbb{N} \) such that \( n \geq 7k/2 \). Suppose that \( G \) is a graph on \( n \) vertices so that
\[
e(G) > (2k - 1)(n - k).
\]
Then \( G \) contains \( k \) vertex-disjoint cycles. Moreover, there exists a graph \( G' \) on \( n \) vertices with \( e(G') = (2k - 1)(n - k) \) such that \( G' \) does not contain \( k \) vertex-disjoint cycles.

We prove Theorem 6 in Section 4.2. Notice that \( G' := K_n - E(K_{n-2k+1}) \) does not contain \( k \) vertex-disjoint cycles and \( e(G') = (2k - 1)(n - k) \).

2 The extremal examples for Theorems 4 and 5

In this section we will give the extremal examples for Theorem 5. Since Theorems 4 and 5 are equivalent, the complements of the extremal graphs for Theorem 5 are the extremal graphs for Theorem 4.

Proposition 7 Suppose that \( n, r \in \mathbb{N} \) such that \( r \geq 3 \) and \( r \) divides \( n \). Then there exists a graph \( G_1 \) on \( n \) vertices such that \( \Delta(G_1) = n/r \),
\[
e(G_1) = \binom{n/r + 1}{2},
\]
but such that \( G_1 \) does not have an equitable \( n/r \)-colouring.

Proof. Let \( G_1 \) denote the disjoint union of a clique \( V \) on \( n/r + 1 \) vertices and an independent set \( W \) of \( (1 - 1/r)n - 1 \) vertices. So every independent set in \( G_1 \) contains at most one vertex from \( V \). But since \(|V| = n/r + 1\), \( G_1 \) does not have an equitable \( n/r \)-colouring. Further, \( \Delta(G_1) = n/r \) and \( e(G_1) = \binom{n/r + 1}{2} \). \( \square \)

Proposition 8 Suppose that \( n, r \in \mathbb{N} \) such that \( r \geq 3 \) and \( n = kr \) for some \( k \geq 2 \). Further, let \( D \in \mathbb{N} \) such that \( n/(r-1) \leq D \leq n - r \). Then there exists a graph \( G_2 \) on \( n \) vertices such that \( \Delta(G_2) = D \),
\[
e(G_2) = D + e(T(n-D-1,r-2)),
\]
but such that \( G_2 \) does not have an equitable \( n/r \)-colouring.

Proof. Let \( G_2 \) denote the disjoint union of a copy \( K \) of \( K_{1,D} \) and a copy of \( T(n-D-1,r-2) \). So \(|G| = n\). Let \( v \) denote the vertex of degree \( D \) in \( K \). The largest independent set in \( G_2 \) that contains \( v \) is of size \( r - 1 \). Thus, \( G_2 \) does not have an equitable \( n/r \)-colouring. Further, \( e(G_2) = D + e(T(n-D-1,r-2)) \).

Since \( n/(r-1) \leq D \) we have that \( n - 1 \leq (r-1)D \). Thus,
\[
\left\lfloor \frac{n-D-1}{r-2} \right\rfloor - 1 \leq \frac{n-D-1}{r-2} 
\]
This implies that \( \Delta(G_2) = D \). \( \square \)
Clearly Propositions 7 and 8 show that one cannot lower the edge density condition in Theorem 5 in the case when \( n/(r-1) \leq D \leq n-r \). The following result, together with Proposition 7, shows that Theorem 5 is best possible in the case when \( n/r \leq D \leq n/(r-1) \).

**Proposition 9** Let \( n, r \in \mathbb{N} \) such that \( r \geq 3 \) and \( r \) divides \( n \geq 2r \). Suppose that \( D \in \mathbb{N} \) such that \( n/r \leq D \leq n/(r-1) \). Then

\[
f(n, r, D) = \left( \frac{n/r + 1}{2} \right).
\]

The following simple consequence of Turán’s theorem will be used in the proof of Theorem 5.

**Fact 10** Let \( n, r \in \mathbb{N} \) such that \( r \leq n \). Then

\[
e(T(n, r)) \leq \left( 1 - \frac{1}{r} \right) \frac{n^2}{2} \quad \text{and thus} \quad e(T(n, r)) \geq \frac{n^2}{2r} - \frac{n}{2}.
\]

We will also require the following easy result.

**Lemma 11** Let \( n, r \in \mathbb{N} \) such that \( r \geq 4 \) and \( r \) divides \( n \geq 3r \). Suppose that \( D \in \mathbb{N} \) such that \( n/r \leq D < (n + r)/(r-1) \). Then

\[
f(n, r, D) = \left( \frac{n/r + 1}{2} \right).
\]

**3 Proof of Theorem 5**

**3.1 Preliminaries**

Suppose for a contradiction that the result is false. Let \( G \) be a counterexample with the fewest vertices. That is, \( n = |V(G)| = rk \) for some \( k \in \mathbb{N} \), \( \Delta(G) \leq D \) for some \( D \in \mathbb{N} \) such that \( n/r \leq D \leq n-r \), \( e(G) < f(n, r, D) \) and \( G \) has no equitable \( n/r \)-colouring. By the Hajnal-Szemerédi theorem, \( \Delta(G) \geq n/r \). Notice that given fixed \( n \) and \( r \), \( f(n, r, D) \) is non-increasing with respect to \( D \). Thus, we may assume that \( \Delta(G) = D \).

We first show that \( k \geq 4 \). Indeed, if \( n = 2r \) then \( f(n, r, D) \leq \left( \frac{3}{2} \right) = 3 \). But it is easy to see that every graph \( G_1 \) on \( 2r \) vertices and with \( e(G_1) \leq 2 \) has an equitable 2-colouring. If \( n = 3r \) then \( f(n, r, D) \leq \left( \frac{4}{3} \right) = 6 \). Consider any graph \( G_1 \) on \( 3r \) vertices with \( e(G_1) \leq 5 \) and \( 3 \leq \Delta(G_1) \leq 5 \). Let \( x \) denote the vertex in \( G_1 \) where \( d_{G_1}(x) = \Delta(G_1) \). Since \( 3 \leq d_{G_1}(x) \leq 5 \), \( x \) lies in an independent set \( I \) in \( G_1 \) of size \( r \). But then \( G_1 - I \) contains \( 2r \) vertices and at most \( 2 \) edges. So \( G_1 - I \) has an equitable 2-colouring and hence \( G_1 \) has an equitable 3-colouring.

Let \( v \in V(G) \) such that \( d_G(v) = D \). Set \( G^* \) := \( G - (N_G(v) \cup \{v\}) \). Since \( f(n, r, D) \leq D + e(T(n-D-1, r-2)) \) we have that \( e(G^*) \leq e(T(n-D-1, r-2)) \). Thus, by Turán’s theorem, \( G^* \) contains an independent set of size \( r-1 \). Hence, \( v \) lies in an independent set in \( G \) of size \( r \). Amongst all such independent sets of size \( r \) that contain \( v \), choose a set \( I = \{x_1, \ldots, x_{r-1}\} \) such that \( d_{G_1}(x_1) + \cdots + d_{G_1}(x_{r-1}) \) is maximised.

Set \( G' := G - I \), \( n' := |V(G')| = n-r \) and \( D' := \Delta(G') \leq D \). Notice that \( D' \geq n'/r \). (Indeed, if not, then by the Hajnal-Szemerédi theorem \( G' \) contains an equitable \( n'/r \)-colouring. Thus, as \( I \) is an independent set in \( G \) this gives us an equitable \( n/r \)-colouring of \( G \), a contradiction.) Furthermore, \( D' \leq n' - r \). If not then

\[
e(G) \geq D + D' \geq 2(n' - r + 1) = 2n - 4r + 2.
\]
Thus, we have that
\[ e(G) < f(n, r, D) \leq f(n, r, n - 2r + 1) \leq (n - 2r + 1) + e(\overline{T}(2r - 2, r - 2)) \leq (n - 2r + 1) + (r + 3) = n - r + 4. \]
Therefore, \( 2n - 4r + 2 < n - r + 4 \) and so \( n < 3r + 2 \) a contradiction since \( n = kr \geq 4r \).

Since \( n' / r \leq D' \leq n' - r \), if \( e(G') < f(n', r, D') \) then the minimality of \( G \) implies that \( G' \) has an equitable \( n'/r \)-colouring. This then implies that \( G \) has an equitable \( n/r \)-colouring, a contradiction.

Thus,
\[ e(G') \geq f(n', r, D'). \tag{1} \]

We now split our argument into three cases.

3.2 **Case 1:** \( f(n', r, D') = \binom{n'/r + 1}{2} \).

By (1), \( e(G') \geq \binom{n'/r + 1}{2} = \binom{n/r}{2} \). Since \( d_G(v) = D \geq n/r \),
\[ e(G) \geq \frac{n}{r} + \binom{n/r}{2} = \binom{n/r + 1}{2} \geq f(n, r, D), \]
a contradiction, as desired.

3.3 **Case 2:** \( D' \leq D - 1 \) and \( f(n', r, D') = D' + e(\overline{T}(n' - D' - 1, r - 2)) \).

The following claim will be useful.

**Claim 12** \( D' < \frac{r - 1}{2r - 3}n - \frac{(r^2 - r + 1)}{2r - 3} \).

**Proof.** Note that
\[ D + D' + e(\overline{T}(n' - D' - 1, r - 2)) \leq e(G) < f(n, r, D) \leq D + e(\overline{T}(n - D - 1, r - 2)). \tag{2} \]
Since \( D' \leq D - 1 \), clearly \( e(\overline{T}(n' - D, r - 2)) \leq e(\overline{T}(n' - D' - 1, r - 2)) \). Thus, (2) implies that
\[ D' + e(\overline{T}(n' - D, r - 2)) < e(\overline{T}(n - D - 1, r - 2)). \tag{3} \]
One can obtain \( \overline{T}(n - D - 1, r - 2) \) from \( \overline{T}(n' - D, r - 2) \) by adding \( r - 1 \) vertices and at most
\[ (n' - D) + \frac{n - D - 2}{r - 2} \]
edges. \tag{4}
Hence (3) and (4) give
\[ D' < n' - D + \frac{n - D - 2}{r - 2}. \]
Rearranging, and using that \( D' \leq D - 1 \) and \( n' = n - r \) we get that
\[ \left( 2 + \frac{1}{r - 2} \right) D' < \left( 1 + \frac{1}{r - 2} \right) n - \frac{(r^2 - r + 1)}{r - 2}. \]
Thus,
\[ D' < \frac{r - 1}{2r - 3}n - \frac{(r^2 - r + 1)}{2r - 3}, \]
as desired. \( \square \)
Since we are in Case 2 we have that
\[ D' + e(\mathcal{T}(n - r - D' - 1, r - 2)) \leq \binom{n'/r + 1}{2} = \binom{n/r}{2}. \] (5)

Notice that for fixed \( n \) and \( r \), \( D' + e(\mathcal{T}(n - r - D' - 1, r - 2)) \) is non-increasing as \( D' \) increases. Hence, \( (5) \) and Claim 12 imply that
\[ D'' + e(\mathcal{T}(n - r - D'' - 1, r - 2)) \leq \frac{n^2}{2r^2} - \frac{n}{2r} \] (6)
where \( D'' := \lfloor (n - 1)n/(2r - 3) - (r^2 - r + 1)/(2r - 3) \rfloor \). Note that
\[ n - r - \frac{r - 1}{2r - 3} - \frac{(r^2 - r + 1)}{2r - 3} = \frac{r - 2}{2r - 3} + \frac{4 - r^2}{2r - 3}. \]

So Fact 10 and (6) imply that
\[
\left( \frac{r - 1}{2r - 3} - \frac{(r^2 - r + 1)}{2r - 3} - \frac{(2r - 4)}{2r - 3} \right) + \frac{1}{2(r - 2)} \left( \frac{r - 2}{2r - 3} + \frac{4 - r^2}{2r - 3} \right)^2 - \frac{1}{2} \left( \frac{r - 2}{2r - 3} + \frac{4 - r^2}{2r - 3} \right) \leq \frac{n^2}{2r^2} - \frac{n}{2r}.
\]

Next we will move all terms from the previous equation to the left hand side and simplify. The coefficient of \( n^2 \) is
\[ \frac{r - 2}{2(2r - 3)^2} - \frac{1}{2r^2} = \frac{r^3 - 6r^2 + 12r - 9}{2r^2(2r - 3)^2}. \] (7)

The coefficient of \( n \) is
\[ \frac{r - 1}{2r - 3} - \frac{(r - 2)}{2(2r - 3)} + \frac{1}{2r} + \frac{(4 - r^2)}{(2r - 3)^2} = \frac{r^2 - 4r + 9}{2r(2r - 3)^2}. \] (8)

The constant term is
\[ -\frac{(r^2 + r - 3)}{2r - 3} + \frac{(r^2 - 4)^2}{2(2r - 2)(2r - 3)^2} + \frac{(r^2 - 4)}{2(2r - 3)} = \frac{-r^4 + 3r^3 + 4r^2 - 26r + 28}{2(r - 2)(2r - 3)^2}. \] (9)

Since \( n \geq 4r \), (7)–(9) imply that
\[ \frac{8(r^3 - 6r^2 + 12r - 9)}{(2r - 3)^2} + \frac{2(r^2 - 4r + 9)}{(2r - 3)^2} + \frac{-r^4 + 3r^3 + 4r^2 - 26r + 28}{2(r - 2)(2r - 3)^2} \leq 0. \] (10)

Multiplying (10) by \( 2(r - 2)(2r - 3)^2 \) we get
\[ 15r^4 - 121r^3 + 364r^2 - 486r + 244 \leq 0. \]

This yields a contradiction, since it is easy to check that
\[ 15r^4 - 121r^3 + 364r^2 - 486r + 244 > 0 \]
for all \( r \in \mathbb{N} \) such that \( r \geq 3 \).
3.4 Case 3: $D' = D$ and $f(n', r, D') = D' + e(\overline{T}(n' - D' - 1, r - 2))$.

By (1) we have that

$$e(G') \geq f(n', r, D') = D' + e(\overline{T}(n' - D' - 1, r - 2)).$$

(11)

Consider any vertex $x \in V(G')$ such that $d_{G'}(x) = D'$. Since $\Delta(G) = D$, $x$ is not adjacent to any vertex in $I = \{v, x_1, \ldots, x_r\}$. Further, $I$ was chosen such that $d_G(x_1) + \cdots + d_G(x_{r-1})$ is maximised. Thus, $d_G(x_1) = \cdots = d_G(x_{r-1}) = D$. Together with (11) this implies that

$$e(G) \geq (r + 1)D + e(\overline{T}(n' - D - 1, r - 2)).$$

(12)

Since $e(G) < f(n, r, D) \leq D + e(\overline{T}(n - D - 1, r - 2))$, (12) implies that

$$rd + e(\overline{T}(n' - D - 1, r - 2)) < e(\overline{T}(n - D - 1, r - 2)).$$

(13)

One can obtain $\overline{T}(n - D - 1, r - 2)$ from $\overline{T}(n' - D - 1, r - 2)$ by adding $r$ vertices and at most

$$(n' - D - 1) + \frac{2(n - D - 3)}{r - 2} + 1 \text{ edges.}$$

(14)

Thus, (13) and (14) imply that

$$rd < n - r - D + \frac{2(n - D - 3)}{r - 2}$$

and so

$$\left(r + 1 + \frac{2}{r - 2}\right)D < \left(1 + \frac{2}{r - 2}\right)n + \frac{(-r^2 + 2r - 6)}{r - 2} < \left(1 + \frac{2}{r - 2}\right)n.$$  

(15)

If $r = 3$ then (15) implies that

$$D < \frac{n}{2}.$$  

Since $f(n', 3, D) = \min\{(n'/2+1)/(n' - D - 1)\}$ it is easy to see that if $f(n', 3, D) = D + (n' - D - 1)/(n' - 2)$ then $D \geq 2n'/3 + 1 = 2n/3 - 1$. Thus, $2n/3 - 1 \leq D < n/2$, a contradiction since $n \geq 4r = 12$.

If $r \geq 4$ then (15) implies that

$$D < \frac{n}{r - 1} = \frac{n'}{r - 1} + \frac{r}{r - 1}.$$  

Since $n' \geq 3r$, Lemma 11 implies that $f(n', r, D') = (n'/r + 1)$ and so we are in Case 1, which we have already dealt with.

4 Perfect matchings and cycles in dense graphs

4.1 Perfect matchings in dense graphs

In this section we establish the density threshold that ensures every graph $G$ on an even number $n$ of vertices and of minimum degree $\delta(G) \geq d$ contains a perfect matching. Note that we only
consider values of \( d \) such that \( 1 \leq d < n/2 \), since if \( \delta(G) \geq n/2 \) then \( G \) has a perfect matching, regardless of the edge density.

For a positive even \( n \) and an integer \( 0 \leq d < n/2 \), let \( A, B \) and \( C \) be disjoint sets with \( |A| = d+1 \), \( |B| = d \), \( |C| = n - 2d - 1 \). Let \( H = H(n, d) \) be the graph with the vertex set \( A \cup B \cup C \) such that \( H[B \cup C] = K_{n-d-1} \), and each vertex in \( A \) is adjacent to each vertex in \( B \) and to no vertex in \( C \). So \( H \) does not contain a perfect matching. Let

\[
h(n, d) := |E(H(n, d))| = \left( \frac{n-d-1}{2} \right) + d(d+1).
\]

Note that for a fixed even \( n \), \( h(n, d) \) decreases with \( d \) in the interval \([0, n/3 - 5/6]\) and increases with \( d \) in \([n/3 - 5/6, 0.5n - 1]\).

**Proposition 13** For an even positive \( n \) and integer \( 1 \leq d < n/2 \), let \( f(2, n, d) \) denote the maximum integer \( c \) such that some \( n \)-vertex graph with minimum degree at least \( d \) and at least \( c \) edges has no perfect matching. Then

\[
f(2, n, d) = \max \{ h(n, d), h(n, 0.5n - 1) \}.
\]

**Proof.** The examples of \( H(n, d) \) show that \( f(2, n, d) \geq \max \{ h(n, d), h(n, 0.5n - 1) \} \). If \( G \) is an \( n \)-vertex graph with \( \delta(G) \geq n/2 \), then \( G \) has a perfect matching. Thus, it is enough to prove that if an \( n \)-vertex graph \( G \) with \( d \leq \delta(G) < n/2 \) has no perfect matching, then

\[
e(G) \leq h(n, d') \text{ for some } d \leq d' < 0.5n.
\]
4.2 Proof of Theorem 6

Suppose for a contradiction that the result is false. Then there is a graph $G$ on $n \geq 7k/2$ vertices with

$$e(G) > (2k - 1)(n - k)$$

(19)

but such that $G$ does not contain $k$ vertex-disjoint cycles.

Let $v_1 \in V(G)$ such that $d_G(v_1) = \delta(G)$. If $\delta(G) \geq 2k$ then the Corrádi-Hajnal theorem implies that $G$ contains $k$ vertex-disjoint cycles, a contradiction. So $d_G(v_1) \leq 2k - 1$. Let $v_2 \in V(G - v_1)$ such that $d_{G-v_1}(v_2) = \delta(G - v_1)$. Again we may assume that $d_{G-v_1}(v_2) \leq 2k - 1$. Repeating this argument we obtain distinct vertices $v_1, \ldots, v_{n-3k}$ so that $G' := G - \{v_1, \ldots, v_{n-3k}\}$ is a graph on $3k$ vertices with $\delta(G') \leq 2k - 1$. The choice of $v_1, \ldots, v_{n-3k}$ and (19) implies that

$$e(G') > (2k - 1)(n - k) - (2k - 1)(n - 3k) = 2k(2k - 1).$$

(20)

If $k = 1$ this implies that $|G'| = 3$ and $e(G') > 2$, a contradiction. When $k = 2$ we have that $|G'| = 6$ and $e(G') > 12$. But then $G'$ contains two vertex-disjoint triangles, a contradiction. Thus, $k \geq 3$.

Consider the case when $\delta(G') \geq k - 1 \geq 2$. It is easy to check that $g(3k, 3, k-1) = \left(\frac{3k}{2}\right) - \left(\frac{k+1}{2}\right) = 2k(2k - 1)$. Since $G'$ does not contain a perfect $K_3$-packing, Theorem 4 implies that

$$e(G') \leq 2k(2k - 1),$$

a contradiction to (20), as desired.

Now consider the case when $s := \delta(G') \leq k - 2$. For $2 \leq s \leq k - 2$, $g(3k, 3, s) = \left(\frac{3k}{2}\right) - \left(\frac{s}{2}\right) - (3k - 1 - s)$. Since $G'$ does not contain a perfect $K_3$-packing, Theorem 4 implies that

$$e(G') \leq \left(\frac{3k}{2}\right) - \left(\frac{s}{2}\right) - (3k - 1 - s).$$

(21)

If $s = 0, 1$ then it is easy to see that (21) also holds. (In this case, $\left(\frac{s}{2}\right) := s(s-1)/2 = 0$.)

If $k$ is even then, since $\delta(G') = s$, $v_{n-3k}$ must have at most $s + 1$ neighbours in $V(G')$, $v_{n-3k-1}$ has at most $s + 2$ neighbours in $V(G') \cup \{v_{n-3k}\}$ and so on until $v_{n-7k/2+1}$ has at most $s + k/2$ neighbours in $V(G') \cup \{v_{n-3k}, \ldots, v_{n-7k/2+2}\}$. Hence, (21) implies that

$$e(G) \leq \left(\frac{3k}{2}\right) - \left(\frac{s}{2}\right) - (3k - 1 - s) + (s + 1) + \cdots + (s + k/2) + (n - 7k/2)(2k - 1).$$

Comparing with (19), after rearranging and simplifying we get

$$\frac{5k}{2}(2k - 1) < \frac{3k(3k - 1)}{2} - \frac{s(s-1)}{2} - 3k + 1 + s + \frac{s k}{2} + \frac{k^2}{8} + \frac{k}{4}.$$

This implies that

$$\frac{3k^2}{4} + \frac{7k}{2} < -s^2 + s(3 + k) + 2.$$  

(22)

Note that $-s^2 + s(3 + k) + 2$ is maximised when $s = (3 + k)/2$. So (22) implies that

$$\frac{3k^2}{4} + \frac{7k}{2} < \frac{(3 + k)^2}{4} + \frac{(3 + k)^2}{2} + 2,$$

and therefore

$$2k^2 + 8k < 17,$$

a contradiction as $k \geq 3$. The case when $k$ is odd is similar.
5 The extremal examples for Conjecture 2

**Proposition 14** Suppose that \(n, r, k \in \mathbb{N}\) such that \(r \geq 2\) divides \(n\) and \(1 \leq k < n/r\). Then there exists a graph \(G\) on \(n\) vertices whose degree sequence \(d_1 \leq \cdots \leq d_n\) satisfies

- \(d_i = (r - 2)n/r + k - 1\) for all \(1 \leq i \leq k\);
- \(d_i = (r - 1)n/r\) for all \(k + 1 \leq i \leq (r - 2)n/r + k\);
- \(d_i = n - k - 1\) for all \((r - 2)n/r + k + 1 \leq i \leq n - k + 1\);
- \(d_i = n - 1\) for all \(n - k + 2 \leq i \leq n\),

but such that \(G\) does not contain a perfect \(K_r\)-packing.

**Proof.** Let \(G'\) denote the complete \((r - 2)\)-partite graph whose vertex classes \(V_1, \ldots, V_{r-2}\) each have size \(n/r\). Obtain \(G\) from \(G'\) by adding the following vertices and edges: Add a set \(V_{r-1}\) of \(2n/r - 2k + 1\) vertices to \(G'\), a set \(V_r\) of \(k\) vertices and a set \(V_0\) of \(k\) vertices. Add all edges from \(V_0 \cup V_{r-1} \cup V_r\) to \(V_1 \cup \cdots \cup V_{r-2}\). Further, add all edges with both endpoints in \(V_{r-1} \cup V_r\). Add all possible edges between \(V_0\) and \(V_r\).

So \(V_0\) is an independent set, and there are no edges between \(V_0\) and \(V_{r-1}\). This implies that any copy of \(K_r\) in \(G\) containing a vertex from \(V_0\) must also contain at least one vertex from \(V_r\). But since \(|V_0| > |V_r|\) this implies that \(G\) does not contain a perfect \(K_r\)-packing. Furthermore, \(G\) has our desired degree sequence. \(\square\)

Notice that the graphs \(G\) considered in Proposition 14 satisfy (\(\beta\)) from Conjecture 2 and only fail to satisfy (\(\alpha\)) in the case when \(i = k\) (and in this case \(d_k = (r - 2)n/r + k - 1\)).

Let \(n, r \in \mathbb{N}\) such that \(r\) divides \(n\). Denote by \(T^*(n, r)\) the complete \(r\)-partite graph on \(n\) vertices with \(r - 2\) vertex classes of size \(n/r\), one vertex class of size \(n/r - 1\) and one vertex class of size \(n/r + 1\). Then \(T^*(n, r)\) does not contain a perfect \(K_r\)-packing. Furthermore, \(T^*(n, r)\) satisfies (\(\alpha\)) but condition (\(\beta\)) fails; we have that \(d_{n/r + 1} = (r - 1)n/r - 1\) here. Thus, together \(T^*(n, r)\) and Proposition 14 show that, if true, Conjecture 2 is essentially best possible.

6 Some special cases of Conjecture 2

The following is a simple consequence of Chvátal’s theorem.

**Theorem 15 (Chvátal [3])** Suppose that \(G\) is a graph on \(n \geq 2\) vertices and the degrees of the graph are \(d_1 \leq \cdots \leq d_n\). If

\[
d_i \geq i \quad \text{or} \quad d_{n-i+1} \geq n-i \quad \text{for all} \quad 1 \leq i \leq n/2
\]

then \(G\) contains a Hamilton path.

It is easy to see that Theorem 15 implies Conjecture 2 in the case when \(r = 2\). We now give a simple proof of Conjecture 2 in the case when \(G\) is \(K_{r+1}\)-free.

**Theorem 16** Let \(n, r \in \mathbb{N}\) such that \(r \geq 2\) divides \(n\). Suppose that \(G\) is a graph on \(n\) vertices with degree sequence \(d_1 \leq \cdots \leq d_n\) such that:

\[
\text{\textit{\(d_i \geq i\)} \quad \text{or} \quad \text{\(d_{n-i+1} \geq n-i\)} \quad \text{for all} \quad 1 \leq i \leq n/2
}
\]
Thus assume that further suppose that no vertex $x \in V(G)$ of degree less than $(r - 1)n/r$ lies in a copy of $K_{r+1}$. Then $G$ contains a perfect $K_r$-packing.

**Proof.** We prove the theorem by induction on $n$. In the case when $n = r$ then $d_{n/r+1} = d_2 \geq (r - 1)r/r = r - 1$. This implies that every vertex in $G$ has degree $r - 1$. Hence $G = K_r$ as desired. So suppose that $n > r$ and the result holds for smaller values of $n$. Let $x_1 \in V(G)$ such that $d_G(x_1) = d_1 \geq (r - 2)n/r + 1$. If $d_G(x_1) \geq (r - 1)n/r$ then $\delta(G) \geq (r - 1)n/r$. Thus $G$ contains a perfect $K_r$-packing by the Hajnal-Szemerédi theorem. So we may assume that $(r - 2)n/r + 1 \leq d_G(x_1) < (r - 1)n/r$. In particular, $x_1$ does not lie in a copy of $K_{r+1}$. We first find a copy of $K_r$ containing $x_1$. If $r = 2$, $x_1$ has a neighbour and so we have our desired copy of $K_2$. So assume that $r \geq 3$. Certainly $N_G(x_1)$ contains a vertex $x_2$ such that $d_G(x_2) \geq (r - 1)n/r$. Thus, $[N_G(x_1) \cap N_G(x_2)] \geq (r - 3)n/r + 1 > 0$. So if $r = 3$ we obtain our desired copy of $K_r$. Otherwise, we can find a vertex $x_3 \in N_G(x_1) \cap N_G(x_2)$ such that $d_G(x_3) \geq (r - 1)n/r$. We can repeat this argument until we have obtained vertices $x_1, \ldots, x_r$ that together form a copy $K'_r$ of $K_r$.

Let $G' := G - V(K'_r)$ and set $n' := n - r = |V(G')|$. Since $G$ does not contain a copy of $K_{r+1}$ containing $x_1$, every vertex $x \in V(G') \setminus V(K'_r)$ sends at most $r - 1$ edges to $K'_r$ in $G$. Thus, $d_{G'}(x) \geq d_G(x) - (r - 1)$ for all $x \in V(G')$. So if $d_G(x) \geq (r - 1)n/r$ then $d_{G'}(x) \geq (r - 1)n/r - (r - 1) = (r - 1)n'/r$ for all $x \in V(G')$. If a vertex $y \in V(G')$ does not lie in a copy of $K_{r+1}$ in $G$ then clearly $y$ does not lie in a copy of $K_{r+1}$ in $G'$. This means that no vertex $y \in V(G')$ of degree less than $(r - 1)n'/r$ lies in a copy of $K_{r+1}$.

Let $d'_1 \leq \cdots \leq d'_{n'}$ denote the degree sequence of $G'$. It is easy to check that $d'_i \geq (r - 2)n'/r + i$ for all $i < n'/r$ and that $d'_{n'/r+1} \geq (r - 1)n'/r$. Indeed, since $x_1 \in V(K'_r)$ where $d_G(x_1) = d_1$, we have that $d'_i \geq d_{i+1} - (r - 1)$ for all $1 \leq i \leq n'$. Thus, for all $1 \leq i < n'/r = n/r - 1$, $d'_i \geq d_{i+1} - (r - 1) \geq (r - 2)n/r + (i + 1) - (r - 1) = (r - 2)n'/r + i$. Similarly, $d'_{n'/r+1} = d'_{n/r} \geq d_{n/r+1} - (r - 1) \geq (r - 1)n/r - (r - 1) = (r - 1)n'/r$. Hence, by induction $G'$ contains a perfect $K'_r$-packing. Together with $K'_r$ this gives us our desired perfect $K_r$-packing in $G$. 

7 Degree sequences forcing a copy of $K_r$ in a graph

**Proof of Theorem 3.** Consider any $x_1 \in V(G)$ such that $d_G(x_1) \geq (r - 1)n/r$. Since $d_{n/r} \geq (r - 1)n/r$ we can greedily select vertices $x_2, \ldots, x_{r-1}$ such that

- $x_1, \ldots, x_{r-1}$ induce a copy of $K_{r-1}$ in $G$;
- $d_G(x_i) \geq (r - 1)n/r$ for all $1 \leq i \leq r - 1$.

Note that since $G$ is $K_{r+1}$-free, $\cap_{i=1}^{r-1} N_G(x_i)$ is an independent set. The choice of $x_1, \ldots, x_{r-1}$ implies that $|\cap_{i=1}^{r-1} N_G(x_i)| \geq n/r$. Let $V_1$ denote a subset of $\cap_{i=1}^{r-1} N_G(x_i)$ of size $n/r$. Thus $V_1$ contains a vertex $x_1^1$ of degree at least $(r - 1)n/r$.

As before we can find vertices $x_2^1, \ldots, x_{r-1}^1$ such that

- $x_1^1, \ldots, x_{r-1}^1$ induce a copy of $K_{r-1}$ in $G$;
- $d_G(x_i^1) \geq (r - 1)n/r$ for all $1 \leq i \leq r - 1$. 

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So \( \cap_{i=1}^{r-1} N_G(x_i^1) \) is an independent set of size at least \( n/r \). Let \( V_2 \) denote a subset of \( \cap_{i=1}^{r-1} N_G(x_i^1) \) of size \( n/r \). Note that \( N_G(x_1^1) \cap V_1 = \emptyset \) since \( x_1^1 \in V_1 \) and \( V_1 \) is an independent set. Thus as \( V_2 \subseteq N_G(x_1^1) \), \( V_1 \cap V_2 = \emptyset \).

Our aim is to find disjoint sets \( V_1, \ldots, V_r \subseteq V(G) \) of size \( n/r \) and vertices \( x_1^1, \ldots, x_r^1, x_1^2, \ldots, x_r^2, \ldots, x_1^{r-1}, \ldots, x_r^{r-1} \) with the following properties:

- \( G[V_j] \) is an independent set for all \( 1 \le j \le r \);
- Given any \( 1 \le j \le r - 1 \), \( x_k^j \in V_k \) for each \( 1 \le k \le j \);
- \( d_G(x_k^j) \ge (r-1)n/r \) for all \( 1 \le j \le r - 1 \) and \( 1 \le k \le r - 1 \);
- \( x_1^j, \ldots, x_r^j \) induce a copy of \( K_{r-1} \) in \( G \) for all \( 1 \le j \le r - 1 \).

Clearly finding such a partition \( V_1, \ldots, V_r \) of \( V(G) \) implies that \( G \subseteq T(n, r) \).

Suppose that for some \( 1 < j < r \) we have defined sets \( V_1, \ldots, V_j \) and vertices \( x_1^1, \ldots, x_j^1, \ldots, x_j^{j-1}, \ldots, x_r^{r-1} \) with our desired properties. Since \( d_{n/r} \ge (r-1)n/r \) and \( V_1, \ldots, V_j \) are independent sets of size \( n/r \) we can choose vertices \( x_1^j, \ldots, x_j^j \) such that for all \( 1 \le k \le j \),

- \( x_k^j \in V_k \) and \( d_G(x_k^j) \ge (r-1)n/r \).

This degree condition, together with the fact that \( x_1^1, \ldots, x_j^j \) lie in different vertex classes, implies that these vertices form a copy of \( K_j \) in \( G \). We now greedily select further vertices \( x_{j+1}^j, \ldots, x_{r-1}^j \) such that

- \( x_1^j, \ldots, x_{r-1}^j \) induce a copy of \( K_{r-1} \) in \( G \);
- \( d_G(x_k^j) \ge (r-1)n/r \) for all \( j+1 \le k \le r - 1 \).

So \( \cap_{i=1}^{r-1} N_G(x_i^j) \) is an independent set of size at least \( n/r \). Let \( V_{j+1} \) denote a subset of \( \cap_{i=1}^{r-1} N_G(x_i^j) \) of size \( n/r \). Note that, for each \( 1 \le k \le j \), \( N_G(x_k^j) \cap V_k = \emptyset \) since \( x_k^j \in V_k \) and \( V_k \) is an independent set. Thus as \( V_{j+1} \subseteq N_G(x_k^j) \) for each \( 1 \le k \le j \), \( V_{j+1} \) is disjoint from \( V_1 \cup \cdots \cup V_j \).

Repeating this argument we obtain our desired sets \( V_1, \ldots, V_r \subseteq V(G) \) and vertices \( x_1^1, \ldots, x_r^1, x_1^2, \ldots, x_r^2, \ldots, x_1^{r-1}, \ldots, x_r^{r-1} \).

The following consequence of Theorem 3 gives a condition on the degree sequence of a graph \( G \) that forces \( G \) to contain a copy of \( K_{r+1} \).

**Corollary 17** Suppose that \( n, r \in \mathbb{N} \) where \( n \ge r \ge 2 \). Let \( n = mr + s \) where \( m, s \in \mathbb{N} \) such that \( 0 \le s \le r - 1 \). Let \( G \) be a graph on \( n \) vertices whose degree sequence \( d_1 \le \cdots \le d_n \) satisfies the following conditions:

(a) \( d_{m+s} \ge n-m \);

(b) \( d_n \ge n - m + 1 \).

Then \( G \) contains a copy of \( K_{r+1} \).
Further, can be lowered here. Indeed, the Turán graph $T_r$ shows that we cannot have a lower value in the second part of the condition in Question 19. Note that Theorem 15 answers this question in the affirmative when $d_1 = 2$. The following example shows that we cannot omit condition (b). Further, $T^*(n, r)$ does not contain a copy of $K_{r+1}$ but satisfies (b) and only just fails to satisfy (a).

(Recall that the graph $T^*(n, r)$ was defined in Section 5.)

8 Possible extensions of Conjecture 2

If one can prove Conjecture 2, it seems likely it can be used to prove the following conjecture.

Conjecture 18 Suppose $\gamma > 0$ and $H$ is a graph with $\chi(H) = r$. Then there exists an integer $n_0 = n_0(\gamma, H)$ such that the following holds. If $G$ is a graph whose order $n \geq n_0$ is divisible by $|H|$, and whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies

- $d_i \geq (r - 2)n/r + i + \gamma n$ for all $i < n/r$,

then $G$ contains a perfect $H$-packing.

We also suspect that the following ‘Chvátal-type’ degree sequence condition forces a graph to contain a perfect $K_r$-packing.

Question 19 Let $n, r \in \mathbb{N}$ such that $r \geq 2$ divides $n$. Suppose that $G$ is a graph on $n$ vertices with degree sequence $d_1 \leq \cdots \leq d_n$ such that for all $i \leq n/r$:

- $d_i \geq (r - 2)n/r + i$ or $d_{n-i(r-1)+1} \geq n - i$.

Does this condition imply that $G$ contains a perfect $K_r$-packing?

Note that Theorem 15 answers this question in the affirmative when $r = 2$. The following example shows that we cannot have a lower value in the second part of the condition in Question 19.

Proposition 20 Suppose that $n, r, k \in \mathbb{N}$ such that $r \geq 2$ divides $n$ and $1 \leq k \leq n/r$. Then there exists a graph $G$ on $n$ vertices whose degree sequence $d_1 \leq \cdots \leq d_n$ satisfies

- $d_{n-i(r-1)+1} \geq n - i$ for all $i \in [n/r] \setminus \{k\}$;
\[ d_{n-k(r-1)+1} = n - k - 1, \]

but such that \( G \) does not contain a perfect \( K_r \)-packing.

**Proof.** Let \( G \) be the graph on \( n \) vertices with vertex classes \( V_1, V_2 \) and \( V_3 \) of sizes \( k, (r-1)k-1 \) and \( n-rk+1 \) respectively and with the following edges: There are all possible edges between \( V_1 \) and \( V_2 \) and between \( V_2 \) and \( V_3 \). Further add all possible edges in \( V_2 \) and all edges in \( V_3 \). Thus, \( V_1 \) is an independent set and there are no edges between \( V_1 \) and \( V_3 \).

The degree sequence of \( G \) is

\[ \begin{align*}
(r-1)k - 1, \ldots, (r-1)k - 1, & \quad n - k - 1, \ldots, n - k - 1, \\
n - rk + 1 \times & \quad n - \times, \ldots, n - 1, \\
(r-1)k - 1 \times & \quad n - rk + 1 \times \end{align*} \]

Hence \( G \) satisfies our desired degree sequence condition. Every copy \( K'_r \) or \( K_r \) in \( G \) that contains a vertex from \( V_1 \) must contain \( r-1 \) vertices from \( V_2 \). But since \( |V_1|(r-1) > |V_2| \) this implies that \( G \) does not contain a perfect \( K_r \)-packing. \( \square \)

The \( r \)th power of a Hamilton cycle \( C \) is obtained from \( C \) by adding an edge between every pair of vertices of distance at most \( r \) on \( C \). Seymour [13] conjectured the following strengthening of Dirac’s theorem.

**Conjecture 21 (Pósa-Seymour, see [13])** Let \( G \) be a graph on \( n \) vertices. If \( \delta(G) \geq \frac{r}{r+1}n \) then \( G \) contains the \( r \)th power of a Hamilton cycle.

Pósa (see [4]) had earlier proposed the conjecture in the case of the square of a Hamilton cycle (that is, when \( r = 2 \)). Komlós, Sárközy and Szemerédi [9] proved Conjecture 21 for sufficiently large graphs. More recently, Chau, DeBiasio and Kierstead [2] proved Pósa’s conjecture for graphs of order at least \( 2 \times 10^8 \).

In the case when \( r+1 \) divides \( |G| \), a necessary condition for a graph \( G \) to contain the \( r \)th power of a Hamilton cycle is that \( G \) contains a perfect \( K_{r+1} \)-packing. Further, notice that the minimum degree condition in Conjecture 21 is the same as the condition in the Hajnal-Szemerédi theorem with respect to perfect \( K_{r+1} \)-packings. Thus an obvious question is whether the condition in Conjecture 2 forces a graph to contain the \( (r-1) \)th power of a Hamilton cycle. Interestingly though, when \( r = 3 \), this is not the case.

**Proposition 22** Suppose that \( C, n \in \mathbb{N} \) such that \( C \ll n \) and \( 3 \) divides \( n \). Then there exists a graph \( G \) whose degree sequence \( d_1 \leq \cdots \leq d_n \) satisfies

\[ d_i \geq \frac{n}{3} + C + i \quad \text{for all } 1 \leq i \leq \frac{n}{3} \]

but such that \( G \) does not contain the square of a Hamilton cycle.

**Proof.** Choose \( C, K, n \in \mathbb{N} \) so that \( C \ll K \ll n \). Let \( G \) denote the graph on \( n \) vertices consisting of three vertex classes \( V_1 = \{v\}, V_2 \) and \( V_3 \) where \( |V_2| = n/3 + C + 1 \) and \( |V_3| = 2n/3 - C - 2 \) which contains the following edges:

- All edges from \( v \) to \( V_2 \);
- All edges between \( V_2 \) and \( V_3 \) and all possible edges in \( V_3 \);
Figure 1: The example from Proposition 22 in the case when $K = 2$ and $|V_2| = 8$.

- There are $K$ vertex-disjoint stars in $V_2$, each of size $\lceil |V_2|/K \rceil$, $\lfloor |V_2|/K \rfloor$, which cover all of $V_2$ (see Figure 1).

Let $d_1 \leq \cdots \leq d_n$ denote the degree sequence of $G$. There are $n/3 + C - K + 1 \leq n/3 - 2C - 1$ vertices in $V_2$ of degree $2n/3 - C$. Since $C \ll K \ll n$, the remaining $K$ vertices in $V_2$ have degree at least $2n/3 - C - 2 + \lceil |V_2|/K \rceil \geq 2n/3 + C + 1$. Since $d_G(v) = n/3 + C + 1$ and $d_G(x) = n - 2$ for all $x \in V_3$, we have that $d_i \geq \frac{n}{3} + C + i$ for all $1 \leq i \leq \frac{n}{3}$.

A necessary condition for a graph $G$ to contain the square of a Hamilton cycle is that, for every $x \in V(G)$, $G[N(x)]$ contains a path of length 3. Note that $N(v) = V_2$ and $G[V_2]$ does not contain a path of length 3. So $G$ does not contain the square of a Hamilton cycle. □

Notice that we can set $C = o(\sqrt{n})$ in Proposition 22. We finish by raising the following question.

Question 23 What can be said about degree sequence conditions which force a graph to contain the $r$th power of a Hamilton cycle? In particular, can one establish a degree sequence condition that ensures a graph $G$ on $n$ vertices contains the $r$th power of a Hamilton cycle and which allows for “many” vertices of $G$ to have degree “much less” than $rn/(r + 1)$?

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References

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Appendix

Here we give proofs of Proposition 9 and Lemma 11. The following fact will be used in both of these proofs.

**Fact 24** Fix \(n, r \in \mathbb{N}\) such that \(r \geq 3\) and \(r\) divides \(n \geq 2r\). Define

\[
h(x) := x + \frac{(n-x-1)^2}{2(r-2)} - \frac{1}{2}(n-x-1).
\]

Then \(h(x)\) is a decreasing function for \(x \in [0,n/(r-1)]\). Moreover, if \(n \geq 3r\) then \(h(x)\) is a decreasing function for \(x \in [0,(n+r)/(r-1)]\).

**Proof.** Notice that

\[
h'(x) = \frac{3}{2} - \frac{(n-x-1)}{r-2} = \frac{x}{r-2} + \frac{1-n}{r-2} + \frac{3}{2}.
\]
So for \( x \leq n/(r - 1) \),
\[
h'(x) \leq \frac{n}{(r - 1)(r - 2)} + \frac{1 - n}{r - 2} + \frac{3}{2} = -\frac{n}{r - 1} + \frac{1}{r - 2} + \frac{3}{2}.
\]
Note that \( 3(r - 1)/2 + (r - 1)/(r - 2) < n \) since \( n \geq 2r \) and \( r \geq 3 \). Thus,
\[
h'(x) \leq -\frac{n}{r - 1} + \frac{1}{r - 2} + \frac{3}{2} < 0.
\]
If \( x \leq (n + r)/(r - 1) \) then
\[
h'(x) \leq \frac{n + r}{(r - 1)(r - 2)} + \frac{1 - n}{r - 2} + \frac{3}{2} = -\frac{n}{r - 1} + \frac{1}{r - 2} + \frac{r}{(r - 1)(r - 2)} + \frac{3}{2}.
\]
If \( n \geq 3r \) then \( n > 3r/2 + 4 \). So \( n > 3(r - 1)/2 + (2r - 1)/(r - 2) \). Thus,
\[
h'(x) \leq -\frac{n}{r - 1} + \frac{1}{r - 2} + \frac{r}{(r - 1)(r - 2)} + \frac{3}{2} < 0,
\]
as desired. \( \square \)

**Proof of Proposition 9.** We need to show that, for all \( D \in \mathbb{N} \) such that \( n/r \leq D \leq n/(r - 1) \),
\[
\frac{n^2}{2r^2} + \frac{n}{2r} = \left( \frac{n/r + 1}{2} \right) \leq D + e(\mathcal{T}(n - D - 1, r - 2)).
\]
Since \( D \leq n/(r - 1) \), Facts 10 and 24 imply that
\[
D + e(\mathcal{T}(n - D - 1, r - 2)) \geq D + \frac{(n - D - 1)^2}{2(r - 2)} - \frac{(n - D - 1)}{2} \geq \frac{n}{r - 1} + \frac{1}{2(r - 2)} \left[ \frac{(r - 2)}{r - 1} n - 1 \right] - \frac{1}{2} \left[ \frac{(r - 2)}{r - 1} n - 1 \right] \geq \frac{(r - 2)}{2(r - 1)^2} n^2 - \frac{(r - 2)}{2(r - 1)^2} n.
\]
Thus, it suffices to show that
\[
\frac{(r - 2)}{2(r - 1)^2} n - \frac{r - 2}{2(r - 1)} \geq \frac{n}{2r^2} + \frac{1}{2r}. \tag{23}
\]
Notice that
\[
\frac{r - 2}{2(r - 1)^2} - \frac{1}{2r^2} = \frac{(r - 2)^2 - (r - 1)^2}{2r^2(r - 1)^2} = \frac{r^3 - 3r^2 + 2r - 1}{2r^2(r - 1)^2} \tag{24}
\]
and
\[
\frac{r - 2}{2(r - 1)} + \frac{1}{2r} = \frac{r^2 - r - 1}{2r(r - 1)}.
\]
Since \( n \geq 2r \), (23) implies that it suffices to show that
\[
\frac{r^3 - 3r^2 + 2r - 1}{r(r - 1)^2} - \frac{r^2 - r - 1}{2r(r - 1)} \geq 0. \tag{25}
\]
Note that \( r^3 \geq 4r^2 - 4r + 3 \) as \( r \geq 3 \). Thus, \( 2(r^3 - 3r^2 + 2r - 1) \geq (r^2 - r - 1)(r - 1) \). So indeed (25) is satisfied, as desired. \( \square \)
Proof of Lemma 11. We need to show that, for all \( D \in \mathbb{N} \) such that \( n/r \leq D < (n + r)/(r - 1) \),

\[
\frac{n^2}{2r^2} + \frac{n}{2r} = \left(\frac{n/r + 1}{2}\right) \leq D + e(T(n - D - 1, r - 2))
\]

Since \( D < (n + r)/(r - 1) \) we have that \( D \leq n/(r - 1) + 1 \). So Facts 10 and 24 imply that

\[
D + e(T(n - D - 1, r - 2)) \geq D + \frac{(n - D - 1)^2}{2(r - 2)} - \frac{(n - D - 1)}{2} \\
\geq \frac{n}{r - 1} + 1 + \frac{1}{2(r - 2)} \left[ \frac{(r - 2)}{r - 1} n - 2 \right]^2 - \frac{1}{2} \left[ \frac{(r - 2)}{r - 1} n - 2 \right]
\]

\[
\geq \frac{(r - 2)}{2(r - 1)^2} n^2 - \frac{(r - 2)}{2(r - 1)} n - \frac{n}{r - 1}
\]

Thus, it suffices to show that

\[
\frac{(r - 2)}{2(r - 1)^2} n^2 - \frac{(r - 2)}{2(r - 1)} n - \frac{n}{r - 1} \geq \frac{n}{2r^2} + \frac{1}{2r}
\]

(26)

Notice that

\[
\frac{r - 2}{2(r - 1)^2} + \frac{1}{r - 1} + \frac{1}{2r} = \frac{r^2 + r - 1}{2r(r - 1)}.
\]

Since \( n \geq 3r \), (24) and (26) imply that it suffices to show that

\[
\frac{3(r^3 - 3r^2 + 2r - 1)}{2r(r - 1)^2} - \frac{r^2 + r - 1}{2r(r - 1)} \geq 0.
\]

(27)

Note that \( 2r^3 - 9r^2 + 8r - 4 \geq 0 \) as \( r \geq 4 \). Thus, \( 3(r^3 - 3r^2 + 2r - 1) \geq (r^2 + r - 1)(r - 1) \). So indeed (27) is satisfied, as desired. \( \square \)