Equivalence of physical and SRB measures in random dynamical systems

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Abstract
We give a geometric proof, offering a new and quite different perspective on an earlier result of Ledrappier and Young on random transformations (Ledrappier and Young 1988 Probab. Theory Relat. Fields 80 217–40). We show that under mild conditions, sample measures of random diffeomorphisms are SRB measures. As sample measures are the limits of forward images of stationary measures, they can be thought of as the analog of physical measures for deterministic systems. Our results thus show the equivalence of physical and SRB measures in the random setting, a hoped-for scenario that is not always true for deterministic maps.

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In this paper, we prove for random dynamical systems a result one would have liked to have for deterministic systems (referring to systems defined by maps or flows) except that for deterministic systems, such a result is likely not true without some additional hypotheses.

Ideal picture for deterministic systems
To motivate our result, consider first a deterministic system on $\mathbb{R}^d$ (or on a finite dimensional manifold) with an attractor. An ‘ideal picture’—which we do not claim to be mathematically proven or even necessarily true but which physicists often take for granted—might be as follows: Lebesgue measure in the basin, transported forward by the map or flow, converges

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to an invariant measure on the attractor. This measure, called a physical measure in [5], is the
natural invariant measure from an observational point of view. For systems with some hyper-
bolicity, it is also an SRB measure, characterized by having smooth conditional measures on
unstable manifolds; see, e.g. [5, 19].

The equivalence of physical and SRB measures can be justified heuristically as follows:
as mass is transported forward by a system with hyperbolicity, it is compressed along stable
directions and spread out along unstable directions, eventually aligning itself with unstable
manifolds. Reasoning geometrically as we have done, it follows that the limiting distribution
will have the SRB property. This indeed was how Ruelle first constructed SRB measures for
Axiom attractors in [17].

Reality is a little more complex outside of the Axiom A category, however: first, there is no
 guarantee that the pushed forward measures will converge. Second, Newhouse’s phenomenon
of infinitely many sinks [14] implies that for maps that are not uniformly hyperbolic, accu-
mulation points of the pushed forward measures can fracture into many ergodic components,
some of which can be Dirac measures supported on sinks. Another example to keep in mind
is the figure-eight attractor [15]. This is a rather extreme example, but it points to the fact that
without adequate control, a sequence of measures that seemingly aligns itself with unstable
manifolds need not converge to an SRB measure.

Random dynamical systems

By a random dynamical system (RDS) in this paper, we refer to the composition of i.i.d.
sequences of random diffeomorphisms. RDS are used to model dynamical systems with a
stochastic component or experiencing small random fluctuations. Solutions of stochastic
differential equations (SDE) are known to have representations as stochastic flows of
diffeomorphisms, the time-$t$-maps of which are compositions of i.i.d. sequences of random
diffeomorphisms; see, e.g. [1, 8].

In the world of RDS, it is quite natural for the stationary measure to have a density, so let
us for the moment confuse the stationary measure with Lebesgue measure. Also, ergodicity is
achieved easily in such RDS, and with ergodicity, one does not have to be concerned with
the fracturing of the limit measure. Under these assumptions, all of which are quite mild for RDS,
we prove that the reasoning in the ‘ideal picture’ above is valid.

Main result (informal version). Consider an ergodic RDS \( \{f^n_\omega\} \) , the stationary measure \( \mu \)
of which has a density. Assume the system has a positive Lyapunov exponent. Then for almost
every sample path \( \omega \), \( (f^n_\theta - I)_\omega \mu \) converges as \( n \to \infty \) to a random SRB measure \( \mu_{\omega} \). Here \( \theta^n \)
is time shift on the sequence of random maps.

A precise formulation is given in section 1. Under the conditions above, we have also an
entropy equality, which asserts that pathwise entropy is equal to the sum of positive Lyapunov
exponents. That follows easily once we have the SRB property, by a proof identical to that for
deterministic systems.

The results above are not new. They were first proved by Ledrappier and Young [10] and
subsequently extended to random endomorphisms by Liu, Qian and Zhang [13]; see also the
more recent book [16] of Qian, Xie and Zhu. In these earlier proofs, the authors showed that
the RDS satisfies an entropy equality, from which they deduced the SRB property of \( \mu_{\omega} \) by
appealing to another theorem. This last result, which provides the crucial link to random SRB
measures, is not elementary, especially when zero Lyapunov exponents are present; see [9]
for a complete proof in the nonrandom case. We mention also the recent result [3] of Brown
and Rodriguez-Hertz for random surface diffeomorphisms, proved under an assumption of
randomness for \( E^s \).
The proof presented here is new and different, and we think it has the following merits: one, it is conceptually more transparent and confirms the intuition behind the ‘ideal picture’ discussed above. Two, it highlights clearly the differences between deterministic and random dynamical systems; and three, our proof is more generalizable as we will show in forthcoming papers. For example, the proof of the entropy formula in [10] involves conditional densities on the stable foliation, ruling out immediately direct generalizations to semiflows defined by dissipative PDEs, for which stable manifolds are always infinite dimensional.

Finally, one of our motivations for presenting a more accessible proof is that there has been some renewed interest in random dynamical systems, and in the idea of random SRB measures in particular. We mention two recent applications in which these ideas have appeared: one is the reliability of biological and engineered systems (see, e.g. [11]) and the other is in climate science (see, e.g. [4]).

1. Setting and statement of results

We begin with the definition of a random dynamical system, abbreviated as RDS. Let $\Omega$ be a Polish space, and let $P$ be a Borel probability measure on $\Omega$. Let $M$ be a compact Riemannian manifold, and consider a Borel measurable mapping $\omega \mapsto f_\omega$ from $\Omega \to \text{Diff}^2(M)$, the space of $C^2$ diffeomorphisms from $M$ onto itself equipped with the $C^2$-metric. An RDS consists of compositions of sequences of maps from $\{f_\omega, \omega \in \Omega\}$ chosen i.i.d. with law $P$. For $\omega = (\omega_n)_{n \in \mathbb{Z}} \in \Omega^\mathbb{Z}$ and $n \in \mathbb{Z}$, we write

$$f^n_\omega = \begin{cases} f_{\omega_n} \circ \cdots \circ f_{\omega_1} & n > 0 \\ \text{Id} & n = 0 \\ f_{\omega_{-(n-1)}}^{-1} \circ \cdots \circ f_{\omega_0}^{-1} & n < 0. \end{cases}$$

One considers also one-sided compositions $f^n_\omega^+$ for $\omega^+ \in \Omega^{\mathbb{Z}^+} := \prod_{n > 0} \Omega$ and $n > 0$.

There are several ways to view an RDS. One is as a Markov chain $(X_n)$ on $M$ defined by fixing an initial condition $X_0 \in M$ and setting $X_{n+1} = f_{\omega_n}(X_n)$. Equivalently, we define the transition probabilities of the chain by

$$P(E \mid x) = P\{\omega : f_\omega x \in E\}$$

for $x \in M$ and Borel sets $E \subset M$. A Borel probability measure $\mu$ on $M$ is said to be stationary if for all Borel sets $E \subset M$,

$$\mu(E) = \int P(E \mid x) \mu(dx).$$

Another viewpoint is to represent an RDS as a measure-preserving skew product map. Here it is important to distinguish between the two-sided and one-sided cases. Let $\theta : \Omega^\mathbb{Z} \to \Omega^\mathbb{Z}$ be the leftward shift preserving the probability $P = P^\mathbb{Z}$ on $\Omega^\mathbb{Z}$, and let $\theta^+ : \Omega^{\mathbb{Z}^+} \to \Omega^{\mathbb{Z}^+}$ be the corresponding shift preserving $P^+ = P^{\mathbb{Z}^+}$. Then the skew product maps corresponding to the RDS above are given by

$$\tau : M \times \Omega^\mathbb{Z} \to M \times \Omega^\mathbb{Z} \quad \text{with} \quad \tau(x, \omega) = (f_\omega x, \theta \omega)$$

and

$$\tau^+ : M \times \Omega^{\mathbb{Z}^+} \to M \times \Omega^{\mathbb{Z}^+} \quad \text{with} \quad \tau^+(x, \omega^+) = (f_\omega x, \theta^+ \omega^+),$$

and lemma 1 identifies the relevant invariant measures of $\tau$ and $\tau^+$.
Lemma 1.

(a) A Borel probability measure \( \mu \) on \( M \) is a stationary measure of the Markov chain \( (X_n) \) if and only if \( \mu \times P^+ \) is an invariant measure of \( \tau : M \times \Omega^Z \rightarrow M \times \Omega^Z \).

(b) Given \( \mu \) as above, there is a unique \( \tau \)-invariant probability measure \( \mu^* \) on \( M \times \Omega^Z \) that projects onto \( \mu \times P^+ \).

The next lemma gives more information on the disintegration of \( \mu^* \) on \( M \)-fibers, i.e. the family of probability measures \( \{ \mu_\omega, \omega \in \Omega^Z \} \) on \( M \) with the property that for all continuous \( \varphi : M \times \Omega^Z \rightarrow \mathbb{R} \), we have

\[
\int \varphi(x, \omega) d\mu^*(x, \omega) = \int \left( \int \varphi(x, \omega) d\mu_\omega(x) \right) dP(\omega).
\]

Lemma 2.

(a) The measures \( \mu_\omega \) are invariant in the sense that for each \( \omega = (\omega_n) \in \Omega^Z \),

\[
(f_\omega)_* \mu_\omega = \mu_{\theta \omega}.
\]

(b) For \( P \)-a.e. \( \omega \in \Omega^Z \),

\[
(f_{\omega, n}^n)_* \mu \rightarrow \mu_\omega \text{ weakly as } n \rightarrow \infty.
\]

It follows that \( \mu_\omega \) depends only on \( \omega_n \) for \( n \leq 0 \).

Lemmas 1 and 2 are standard; see, e.g. chapter 1 of [1] for details.

Lemma 2(b) tells us that the \( \mu_\omega \), which are called sample measures, are in fact the conditional distributions of \( \mu \) given the history of the dynamical system, \( \omega^- = (\omega_n)_{n \leq 0} \). Intuitively, they represent what we see at time 0 given that the transformations \( f_{\omega, n}, n \leq 0 \), have occurred.

Given an RDS together with a stationary measure \( \mu \), certain properties of deterministic systems \((f, m)\), where \( f \) is a single diffeomorphism and \( m \) an invariant measure, extend in a straightforward way to the RDS via their skew product representations. We assume throughout that

\[
\int \log^+ \|f_\omega\|_{C^2} dP(\omega), \quad \int \log^+ \|f_\omega^{-1}\|_{C^2} dP(\omega) < \infty.
\]

These conditions are satisfied by the time-one maps of a large class of SDEs [6]. Under these assumptions, the following are known: for one-sided skew products, Lyapunov exponents of \( f_{\omega, n}^n \) are defined \( \mu^* \)-a.e. for \( P^+ \)-a.e. \( \omega^+ = (\omega_n)_{n > 0} \), as are stable manifolds corresponding to negative Lyapunov exponents. For the two-sided skew-product, Lyapunov exponents of \( f_{\omega, n}^n \) are defined \( \mu^* \)-a.e. as are stable and unstable manifolds. Lyapunov exponents are nonrandom. Another nonrandom quantity of the RDS is pathwise entropy, which we denote by \( h_\mu(\{f_{\omega, n}^n\}) \). See [1, 7] for more information.

As a direct generalization of the idea of SRB measures in the deterministic case, we have the following:

Definition 3. Let \( \{f_\omega\} \) and \( \mu \) be given. We say the \( \mu_\omega \) are random SRB measures if

1. \( f_{\omega, n}^n \) has a positive Lyapunov exponent \( \mu^* \)-a.e.
2. For \( P \)-a.e. \( \omega \), the sample measure \( \mu_\omega \) has absolutely continuous conditional measures on unstable manifolds.

The main result of this paper can now be stated formally as follows:
Main theorem. Let \( \{ f_\omega \} \) be a RDS satisfying (1), and let \( \mu \) be an ergodic stationary measure. We assume that

1. \( \mu \ll \text{Leb} \) with a continuous density.
2. \( \{ f^n_\omega \} \) has a positive Lyapunov exponent \((\mu \times P^+)-\text{a.e.}\).

Then the \( \mu_\omega \) are random SRB measures.

Corollary. Let \( \{ f_\omega \} \) be as in the main theorem. Then the entropy formula

\[
h_\mu(\{ f^n_\omega \}) = \sum_{\lambda_i > 0} m_i \lambda_i
\]

holds. Here \( h_\mu(\{ f^n_\omega \}) \) is pathwise entropy, and \( \lambda_1 > \lambda_2 > \cdots > \lambda_d \) denote the Lyapunov exponents of \( \{ f^n_\omega \} \) with multiplicities \( m_i, 1 \leq i \leq d \).

As noted in the Introduction, the results above were first proved in [10]. They were subsequently extended to random endomorphisms in [13], and to compositions that are not necessarily i.i.d. in [16]. In all of these papers, the result in the corollary is first proved, and the result in the main theorem is deduced from that by appealing to the RDS version of the entropy formula characterization for SRB measures. Here we prove these results in the opposite order: we give a direct proof of the SRB property of \( \mu_\omega \). Once that is proved, the corollary follows immediately by a proof identical to that in the deterministic case.

Our proof of the main theorem will proceed as follows. For \( P \)-a.e. \( \omega \), we consider \( (f^n_\theta - n)_\omega := \mu^n_\omega \), which we know converges to \( \mu_\omega \) as \( n \to \infty \) by lemma 2(b). It suffices to show that \( \mu_\omega \) has smooth conditional probabilities on unstable manifolds, and we will prove that by showing that the geometric argument in the ‘ideal picture’ in section 1 can, in fact, be made rigorous for RDS.

One of the technical novelties of this paper is our analysis of orbits with finite pasts. For RDS, this is both important and natural, for the set of ‘typical’ points changes with knowledge of the past: with zero knowledge of the past, \( \mu \)-a.e. \( x \) is ‘typical’; starting from time \( -n \), typicality as seen at time 0 is with respect to \( \mu^n_\omega \), and as \( n \to \infty \), this measure becomes \( \mu_\omega \).

The following notation will be used throughout:

- On \( M: T_xM \) is the tangent space at \( x \), \( \| \cdot \| \) is the norm on \( T_xM \), \( d(\cdot, \cdot) \) is the distance on \( M \) inherited from the Riemannian metric, and \( B(x, r) = \{ y \in M: d(x, y) < r \} \).
- If \( E \subset T_xM \) is a subspace, then \( E(r) = \{ v \in T_x: \| v \| \leq r \} \).
- On \( \mathbb{R}^d, d \geq 1 \), norms are denoted \( \| \cdot \| \), and balls centered at the origin by \( B(\cdot) \); see section 2.1 for detail.

2. Preliminaries and main proposition

In this section, we consider exclusively the two-sided skew product

\[
\tau: M \times \Omega^\mathbb{Z} \to M \times \Omega^\mathbb{Z} \quad \text{given by} \quad \tau(x, \omega) = (f_\omega x, \theta_\omega \omega)
\]

with invariant probability measure \( \mu^* \). Sections 2.1–2.4 contain some preliminary facts that will be used later on. In section 2.5, we formulate the main proposition (proposition 12) and explain why it implies the main theorem. The proof of proposition 12 will occupy the rest of this paper.
2.1. Two-sided charts for random maps (mostly review)

Assuming the existence of a strictly positive Lyapunov exponent, we first record some properties enjoyed by two-sided Lyapunov charts at $\mu^*$-a.e. $(x, \omega)$ for the skew-product map $\tau$. Details of chart construction will be omitted as the results are entirely analogous to those for deterministic maps, and such charts have been used before for RDS (see, e.g. [1], chapter 4 for more detail). We will include only those properties that are relevant for subsequent discussion.

**Proposition 4 (Linear picture).** There exist $\lambda_0 > 0$ and a $\tau$-invariant Borel measurable subset $\Gamma \subset M \times \Omega^Z$ with $\mu^*(\Gamma) = 1$ such that on $\Gamma$ there is a measurable splitting

$$(x, \omega) \mapsto E^x_{\omega} = T_xM$$

with respect to which the following hold for each $(x, \omega)$:

(a) $\lim_{n \to \infty} \frac{1}{n} \log \|d\phi^{-n}_{(x, \omega)} \| = -\lambda_0$ ;

(b) $\lim_{n \to \infty} \frac{1}{n} \log \|d\phi^{-n}_{(x, \omega)} \| < 0$ ; and

(c) $\lim_{n \to \infty} \frac{1}{n} \log \|d\phi^{-n}_{(x, \omega)} \| = 0$.

Here, $\pi_{(x, \omega),u}$ denotes the projection onto $E^u_{(x, \omega)}$ along $E^u_{(x, \omega)}$.

Below we formulate a system of adapted charts for the two-sided dynamics. Let $R^u = \text{dim } E^u, R^c = \text{dim } E^c$ (recall that since $\mu$ is an ergodic stationary measure, hence $(\tau, \mu^*)$ is ergodic, we have that $\text{dim } E^u_{(x, \omega)}$ is constant along $\Gamma$). For $w = u + v \in R^u \times R^c$, we define $w = \max \{|u|, |v| \}$ where $|u|$ and $|v|$ are Euclidean norms on $R^u$ and $R^c$ respectively. For $r > 0$, we let $B^u(r) = \{v \in R^u : |v| \leq r \}$, and write $B(r) = B^u(r) + B^c(r)$.

**Proposition 5 (Nonlinear picture).** Fix $\delta_0, \delta_1, \delta_2 > 0$ with $\delta_0, \delta_2 \ll \lambda_0$ and $\delta_1$ sufficiently small, and let $\lambda = \lambda_0 - \delta_0$. Shrinking $\Gamma$ by a set of $\mu^*$-measure 0 (and continuing to call it $\Gamma$), there are defined on $\Gamma$

(i) a Borel measurable family of invertible linear maps

$L_{(x, \omega)} : R^u \times R^c \to T_xM$,

with $L_{(x, \omega)} R^u = E^u_{(x, \omega)}$ and $L_{(x, \omega)} R^c = E^c_{(x, \omega)}$, and

(ii) a measurable function $\phi : \Gamma \to [1, \infty)$ satisfying $e^{-\delta_0} \leq \frac{\phi}{\text{dim } E^u} \leq e^{\delta_0}$, with respect to which the following hold. Let the chart at $(x, \omega)$ be given by

$\Phi_{(x, \omega)} : B(\delta_0 l(x, \omega)^{-1}) \to M$ with $\Phi_{(x, \omega)} = \exp \circ L_{(x, \omega)}$,

and define the connecting maps between charts to be

$\tilde{f}_{(x, \omega)} = \Phi^{-1}_{(x, \omega)} \circ f \circ \Phi_{(x, \omega)} : B(\delta_1 l(x, \omega)^{-1}) \to R^u \times R^c$.

Then (a) for any $y, y' \in \Phi_{(x, \omega)} B(\delta_0 l(x, \omega)^{-1})$, we have

$\|d(y, y') \| \leq \|\Phi^{-1}_{(x, \omega)} y - \Phi^{-1}_{(x, \omega)} y'\| \leq l(x, \omega) d(y, y')$;

and (b) $\tilde{f}_{(x, \omega)}$ satisfies

(b1) $|(\tilde{f}_{(x, \omega)})_u u| \geq e^{\bar{\lambda}} |u|$ for $u \in R^u$ and $|(\tilde{f}_{(x, \omega)})_v v| \leq e^{\delta_0} |v|$ for $v \in R^c$;

(b2) $\text{Lip}(\tilde{f}_{(x, \omega)}) = (\tilde{f}_{(x, \omega)})_u u(\delta x) \leq \delta$ for all $\delta \in (0, \delta_1)$; and

(b3) $\text{Lip}(\tilde{f}_{(x, \omega)}) \leq l(x, \omega)$.
A difference in proposition 5 from the single diffeomorphism case is that in (b2) and (b3) above, we need to take into consideration the possibly unbounded sequence of $C^2$ norms $\|f^+_n\|_{C^2}, n \in \mathbb{Z}$ We account for this by taking $l \geq l_0$, where 

$$l_0(\omega) := \sup_{n \in \mathbb{R}} \left( e^{-|n|/5} \max \{ \|f_n\|_{C^2}, \|f_n^{-1}\|_{C^2} \} \right)$$

is finite $\mathbb{P}$-almost surely by our integrability condition (1).

As in the deterministic case, we also have the notion of uniformity sets, i.e. sets of the form $\Gamma_{k} := \{ l \leq l_0 \} \text{ for fixed } l_0 \geq 1$.

2.2. Continuity of $E^u$ and $E^s$

For a single diffeomorphism, the continuity of $E^u$ and $E^s$ on uniformity sets is well known. We formulate and prove here the RDS versions that will be needed later on. Form $\omega \in \Omega^2$, we write $\omega = (\omega^-, \omega^+) \in \Omega^2 \times \Omega^2$, where $\Omega^2 = \prod_{n \leq 0} \Omega$ and $\Omega^2 = \prod_{n \geq 0} \Omega$.

**Proposition 6 (Continuity of $E^u$ and $E^s$).** Let $l_0 \geq 1$ be fixed. Then

(a) for fixed $\omega^+ \in \Omega^2_+, (x, \omega) \mapsto E^u_{(x, \omega)}$ is continuous among $(x, \omega) \in \{ l \leq l_0 \} \cap \{ \omega^+ = \omega^+ \}$.

(b) for fixed $\omega^- \in \Omega^2_-$, $(x, \omega) \mapsto E^s_{(x, \omega)}$ is continuous among $(x, \omega) \in \{ l \leq l_0 \} \cap \{ \omega^+ = \omega^- \}$.

Specifically, for any $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon, l_0)$ and $\eta = \eta(\varepsilon, l_0)$ such that if $(x, \omega), (y, \omega') \in \{ l \leq l_0 \}$ are such that $\omega_i = \omega'_{i-1}$ for all $0 \leq i \leq n_0 - 1$, then

$$d(x, y) < \eta \implies d_{\mathbb{H}}(E^u_{(x, \omega)}, E^u_{(y, \omega')}) < \varepsilon.$$ 

Here, for $E \subset T_M, E' \subset T_M$, we have written $d_{\mathbb{H}}(E, E')$ for the Hausdorff distance between the unit balls of $E$ and $E'$. Since all considerations are local, we will assume, via the use of charts, that we are working in Euclidean space where there is a canonical identification of tangent spaces. Part (a) is standard; $E^u$ depends only on the future $\omega^+ = (\omega_t)_{t \geq 1}$. Later on we will need the ‘finite past’ version of Part (b), which says that the dependence of $E^u$ on the far past, i.e. on $(\omega_{-i})_{i \geq n}$ for large $n$, is weak, and we give a proof of it.

**Proof of (b).** Let $x, y \in M$ be nearby points and $\omega, \omega' \in \Omega^2$ be such that $(x, \omega), (y, \omega') \in \{ l \leq l_0 \}$. Assume that $\omega_i = \omega'_{i-1}$ for all $0 \leq i \leq n_0 - 1$ for some $n_0 \in \mathbb{N}$ to be specified. Crucially, in the argument below we work exclusively with the maps $f^{-i}_\omega$, $0 \leq i \leq n_0$. For ease of notation, let us write $f^{-i} := f^{-1}_{\omega_{-i-1}} \circ \cdots \circ f^{-1}_{\omega_n}$, $x_{-i} = f^{-i}x, y_{-i} = f^{-i}y$, and $d f^{-i}_\omega = (df^{-1}_\omega)^{x_{-i}}$. Let $\nu \in E^u_{(x, \omega)}$ be a unit vector and write $\nu = \tilde{\nu} + \nu'$ according to the splitting $E^u_{(x, \omega)} \oplus E^s_{(y, \omega')}$ of $T_x M$. It suffices to bound $\|\nu'\|$ for $\varepsilon$ small. To begin, we estimate

$$\|df^{-n_0}_{\omega'}\nu\| \leq \|df^{-n_0}_{\omega'}\nu\| + \|df^{-n_0}_{\omega} - df^{-n_0}_{\omega'}\|.$$

The first term is bounded by $l_0 e^{-n_0(\lambda - l_0)}$ by proposition 5. For the second term, we bound

$$\|df^{-n_0}_{\omega} - df^{-n_0}_{\omega'}\| \leq \sum_{j=0}^{n_0-1} \|df^{-1}_{x_{-(j+1)}} \circ \cdots \circ df^{-1}_{x_{-(j-1)}} \circ (df^{-1}_{x_{-j}} - df^{-1}_{y_{-j}}) \circ df^{-1}_{y_{-j}} \circ \cdots \circ df^{-1}_{y_{0}}\|

\leq \sum_{j=0}^{n_0-1} \prod_{m=0}^{n_0-j/2} \|df^{-1}_{x_{-m}}\| \cdot \|d f^{-1}_{x_{-j}}\| \cdot d(x_{-j}, y_{-j})

\leq n_0 l_0^2 e^{(2\lambda + n_0)\delta} d(x, y) = C_0 d(x, y).$$
Here, for $\omega \in \Omega$ we write $\|df_x^{-1}\|, \|df_x^{-1}\|$ for uniform norms over $M$ and have used repeatedly the bound $\|df_x^{-1}\|, \|df_x^{-1}\| \leq e^{\delta_1}l_1(\omega) \leq e^{\delta_1}l_0$. We now compute a lower bound on $\|df_x^{-m}\|$:
\[
\|df_x^{-m}\| \geq \|df_x^{-m}\| - \|df_x^{-m}\| = \|df_x^{-m}\| - \|df_x^{-m}\| \geq \|df_x^{-m}\| - \|df_x^{-m}\|,
\]
having used the estimate $\|\dot{\omega}\| \leq \|\alpha\| \leq l_0$. Collecting,
\[
\|\dot{\Gamma}\| \leq (l_0^2 + l_0^2e^{-n_0(\lambda - \delta_0)}) + l_0^2e^{-n_0(\lambda - \delta_0)}C_\eta d(x, y).
\]
Fix $n_0 = n_0(\epsilon, l_0)$ large enough that the first term is $< \epsilon/2$. Now, choose $\eta = \eta(\epsilon, l_0, n_0)$ so that the second term is $< \epsilon/2$ when $d(x, y) < \eta$.

2.3. Graph transforms and unstable manifolds

We begin by recalling the definition of local unstable manifolds.

Proposition 7 (Unstable manifold theorem). Let $\Gamma$ be as in proposition 5, and let $\delta > 0$ be sufficiently small. Then there is a unique family of measurably-varying maps $\{g_{(x, \omega)} : B^1(\delta l(x, \omega)^{-1}) \to \mathbb{R}^c\}_{(x, \omega) \in \Gamma}$ and a constant $C > 0$ such that
\[
g_{(x, \omega)}(0) = 0 \quad \text{and} \quad \tilde{f}_{(x, \omega)}(\text{graph } g_{(x, \omega)}) \supset \text{graph } g_{\tau(x, \omega)}
\]
for every $(x, \omega) \in \Gamma$. Moreover,

1. $g_{(x, \omega)}$ is $C^{1+\text{Lip}}$, and $(dg_{(x, \omega)})_0 = 0$;
2. $\text{Lip}(g_{(x, \omega)}) \leq 1/10, \text{Lip}(dg_{(x, \omega)}) \leq C_l(x, \omega)$; and
3. if $z_1, z_2 \in \tilde{f}_{(x, \omega)}^{-1}(\text{graph } g_{\tau(x, \omega)})$, then
\[
\|\tilde{f}_{(x, \omega)}^{-1}z_1 - \tilde{f}_{(x, \omega)}^{-1}z_2\| \geq (e^\delta - \delta)|z_1 - z_2|.
\]

We write $W_{u_{(x, \omega), \delta}} = \Phi_{(x, \omega)}(\text{graph } g_{(x, \omega)})$, where $g_{(x, \omega)}$ is as above. The sets $W_{u_{(x, \omega), \delta}}$ are the local unstable manifolds at $(x, \omega)$; The global unstable manifold
\[
W_u = \bigcup_{n \geq 0} W_{u_{(x, \omega), \delta}} ^n,
\]
is an immersed submanifold of $M$.

Since proposition 7 is well-known, we omit its full proof. We do, however, note that it can be proved by graph transform techniques, some details of which we recall here for later use. For a Lipschitz continuous map $g : B^1(\delta l(x, \omega)^{-1}) \to \mathbb{R}^c$, we define the graph transform $\tilde{T}_{(x, \omega)}g$ of $g$, when it exists, to be the mapping $\tilde{T}_{(x, \omega)}g : B^1(\delta l(\tau(x, \omega))^{-1}) \to \mathbb{R}^c$ for which
\[
\tilde{f}_{(x, \omega)}^{-1}\text{graph } g \supset \text{graph } \tilde{T}_{(x, \omega)}g.
\]

The following lemma summarizes what we will need about $\tilde{T}_{(x, \omega)}$.

Lemma 8. Let $\delta > 0$ be sufficiently small.

(a) Let $g : B^1(\delta l(x, \omega)^{-1}) \to \mathbb{R}^c$ be such that

(1) $g$ is $C^{1+\text{Lip}}$ with Lip($g$) $\leq 1/10$ and

(ii) there exists $z \in \text{graph } g$ such that: $z \in B(\frac{1}{2}l(x, \omega)^{-1}) \cap \tilde{f}_{(x, \omega)}^{-1}B(\frac{1}{2}l(\tau(x, \omega))^{-1})$. Then $\tilde{T}_{(x, \omega)}g : B^1(\delta l(\tau(x, \omega))^{-1}) \to \mathbb{R}^c$ exists, with $\text{graph } \tilde{T}_{(x, \omega)}g \subset B(l(\tau(x, \omega))^{-1})$.

Moreover, $\tilde{T}_{(x, \omega)}g$ is $C^{1+\text{Lip}}$ and satisfies Lip($\tilde{T}_{(x, \omega)}g$) $\leq 1/10$. 

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(b) Let $g^1, g^2$ be as in (a), with (ii) replaced by $g'(0) = 0$. Then $T_{(x,\omega)}g^1(0) = 0$, and $|T_{(x,\omega)}g^1 - T_{(x,\omega)}g^2|' \leq c|g^1 - g^2|'$, where $|h|' := \sup_{u \in B'(d((x,\omega)^{-1}),\delta)} \frac{|h(u)|}{|u|}$ and $c \in (0,1)$ is a constant independent of $(x, \omega)$.

Lemma 8 is standard and its proof is omitted.

Next, we recall the following distortion estimate along unstable leaves. As is well-known, the quality of such distortion estimates is a function only of the uniformity estimates at the end of the trajectory, as we describe below.

**Lemma 9.** Let $\delta > 0$ be sufficiently small. Then for any $l_0 > 1$, there exists $D = D(l_0) > 0$ for which the following holds. Let $(x, \omega) \in \Gamma$ be such that $l(x, \omega) \leq l_0$, and let $p_1, p_2 \in W_{(x,\omega),\delta}^n$. For arbitrary $n \geq 1$, write $W = W_{r^{-1}}^n$. Then,

$$\log \frac{\det(d_{\omega^{-1}} f^n_{x,\omega})|(f^n_{x,\omega} p_1)|}{\det(d_{\omega^{-1}} f^n_{x,\omega})|(f^n_{x,\omega} p_2)|} \leq D(l_0)d(p_1, p_2).$$

As before, to control the possible unboundedness of the sequence $\|f^n_{x,\omega}\|_{C^2}$, we incorporated $l_1$ into the definition of $l$ as in section 2.1. Details are left to the reader.

### 2.4. Stacks of unstable leaves

All $\mu_{\omega}$-typical points have $W^u$-leaves passing through them, so $\mu_{\omega}$ itself can be thought of as being supported on a union of $W^u$-leaves. At issue is whether the conditional measures of $\mu_{\omega}$ on these leaves are in the Lebesgue measure class. One way to articulate these ideas geometrically is to group nearby $W^u$-leaves into a stack. We introduce here some language that will be useful later on.

**Switching axes.** Let $x, y \in M$ be nearby points, and let $TzM = E_x \oplus F_x, TM = E_y \oplus F_y$ be such that $dM(E_x, E_y), dM(F_x, F_y) \ll 1$. For $z = x, y$, write $\pi_z : TM \to E_z$ for the projection parallel to $F_z$. Given a mapping $\phi_z : \text{Dom}(\phi_z) \to F_z$ defined on a set $\text{Dom}(\phi_z) \subset E_z$, we write $\phi'_z : \text{Dom}(\phi'_z) \subset E_z \to F_z$ for the mapping, if it can be uniquely defined, such that

$$\exp_x \text{graph } \phi'_z = \exp_y \text{graph } \phi_z.$$

Below we give a condition to guarantee the well-definedness of $\phi'_z$. $\text{Lip}(\cdot)$ refers to Lipschitz constants with respect to the norms $\norm{\cdot}$.

**Lemma 10.** Given $L > 1, \rho > 0$, there exist $\epsilon_1 = \epsilon_1(L, \rho) \ll \rho$ and $\epsilon_2 = \epsilon_2(L)$ such that the following holds. Assume that

(i) $\norm{\pi_x}, \norm{\pi_y} \leq L$;

(ii) $d(x,y) < \epsilon_1$;

(iii) $\phi_z : E_z(2\rho) \to F_z(\frac{1}{2} \rho)$ is a Lipschitz mapping with $\text{Lip}(\phi_z) \leq 1/10$.

Then $\phi'_z$ exists, is defined on $E_z(\rho)$ with $\phi'_z(E_z(\rho)) \subset F_z(\rho)$, and has $\text{Lip}(\phi'_z) \leq 2 \text{Lip}(\phi_z)$. Moreover, $\exp_x \text{graph } \phi'_z |_{E_z(\rho)} \subset \exp_x \text{graph } \phi'_z |_{E_z(\rho)}$. 

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The proof is straightforward and is left to the reader (for more detail, see section 5 of [2]).
For $l_0 > 1$, let
\[ \Gamma_{l_0, \omega} = \{ x \in M : (x, \omega) \in \Gamma \text{ and } l(x, \omega) \leq l_0 \}, \]
and let $\overline{A}$ denote the closure of the set $A$.

Lemma 11. Let $\delta > 0$ be as in proposition 7, and let $l_0 > 1$ be fixed. For all $r = r(l_0, \delta)$ and $\epsilon = \epsilon(l_0, \delta, r)$ sufficiently small (in particular, $\epsilon \ll r$), the following holds. Fix $\omega \in \Omega^2$ and $x_0 \in \Gamma_{l_0, \omega}$. View $x_0$ as a reference point, and write $E^{u/\text{cs}}_u(r) = E^{u/\text{cs}}_{(x_0, \omega)}(r)$ and $E_u(r) = E_u^0(r) + E_u^1(r)$. Then

(a) for each $x \in \overline{B(x_0, \epsilon)} \cap \Gamma_{l_0, \omega}$, there is a $C^{1+\text{Lip}}$ map $\gamma_x : E^u_0(r) \to E^u_1(r)$ with $\text{Lip}(\gamma_x) \leq 1$ such that the connected component of $W^u_{(x_0, \omega), \delta} \cap \exp_{x_0}(E_u(r))$ containing $x$ coincides with $\exp_{x_0}(\text{graph } \gamma_x)$;

(b) the assignment $x \mapsto \gamma_x$ varies continuously in the uniform norm on $C(E^u_0(r), E^u_1(r))$ as $x$ varies in $B(x_0, \epsilon) \cap \Gamma_{l_0, \omega}$.

Proof. For $x \in \Gamma_{l_0, \omega}$, define $\check{g}_{(x, \omega)} = L_{(x, \omega)} \circ g_{(x, \omega)} \circ L_{(x, \omega)}^{-1}$, where $g_{(x, \omega)}$ is as in lemma 7 and $L_{(x, \omega)}$ is as in proposition 5, so that $\check{g}_{(x, \omega)}$ is a graph map from $E^u_{(x, \omega)}$ to $E^u_{(x, \omega)}$ in $T_xM$.

For (a), we use lemma 10 to change the axes of $\check{g}_{(x, \omega)}$ from $E^u_{(x, \omega)}$ to $E^u_{(x, \omega)}$; (i) in lemma 10 is satisfied with $L = l_0$, and (ii) follows from the continuity of $E^u_{\text{cs}}$ subspaces through points of $\Gamma_{l_0, \omega}$ (proposition 6). To arrange for (iii), observe that our control on $\text{Lip}(\check{g}_{(x, \omega)})$ is only in the adapted norm $| \cdot |$, not the Riemannian metric $\| \cdot \|$ on $T_xM$; generally we have only the very poor bound $\text{Lip}(\check{g}_{(x, \omega)}) \leq l_0 \text{Lip}(g_{(x, \omega)})$. This is remedied by truncating the domain of $\check{g}_{(x, \omega)}$ to $E^u_{(x, \omega)}(2r)$, where $r > 0$ is chosen sufficiently small so that (i) $\check{g}_{(x, \omega)}$ is defined on $E^u_{(x, \omega)}(2r)$, and (ii) $\text{Lip}(\check{g}_{(x, \omega)}|_{E^u_{(x, \omega)}(2r)}) \leq 1/10$. For the latter, we take advantage of the fact that $\text{Lip}(\check{g}_{(x, \omega)}|_{E^u_{(x, \omega)}(2r)}) \leq l_0$ and our control on $\text{Lip}(\check{g}_{(x, \omega)}|_{E^u_{(x, \omega)}(2r)})$. A simple computation implies this can be arranged by taking $r = \min\{20C \delta^{-1}, 1, \frac{1}{2}\delta^{-2}\}$, with $C, \delta$ as in proposition 7.

(b) follows from the continuity of $E^u_{\text{cs}}$ subspaces (proposition 6) and the contraction estimate for the graph transform (lemma 8). Details are left to the reader; a similar argument is carried out in the proof of proposition 26 in section 5 of this paper; see also lemma 5.5 in [2].

We refer to
\[ \mathcal{S}_{\omega} := \bigcup_{x \in B(x_0, \epsilon) \cap \Gamma_{l_0, \omega}} \xi(x) \quad \text{where} \quad \xi(x) := \exp_{x_0}(\text{graph } \gamma_x) \]
as a stack of unstable leaves through $\overline{B(x_0, \epsilon)} \cap \Gamma_{l_0, \omega}$.

2.5. Main proposition and proof of main theorem

For $\omega \in \Omega^2$ let us write $\mu^\omega_u = (f^n_{\omega - \omega})_\ast \mu$, recalling that $\mu^\omega_u \rightharpoonup \mu_\omega$ weakly with $\mathbb{P}$-probability 1 by lemma 2. Our plan is to fix $\omega$, and track the orbits of a small fraction of $\mu$-typical points from time $-n$ to time 0 with the aid of Lyapunov charts. We will show that their images at time 0 are increasingly aligned with local unstable manifolds, and that as $n \to \infty$, the weak limits of this sequence of small ‘pieces’ of $\mu^\omega_u$ possess the SRB property. This is summarized in the following main proposition of this paper. The notation is as in section 2.2.
Proposition 12 (Main proposition). For all sufficiently large $l_0 > 1$, there is a positive $P$-measure set of $\omega$ and a small constant $c > 0$ for which the following hold. On each $\Gamma_{l_0, \omega}$, there is a stack $S_\omega$ of local unstable manifolds with the following properties:

(a) A fraction $\geq c$ of $\mu_\omega$ is supported on $S_\omega$, i.e. $\mu_\omega = \nu_1 + \nu_2$ where $\nu_1, \nu_2$ are both positive measures, and $\nu_1$ is supported on $S_\omega$ with $\nu_1(S_\omega) \geq c$.

(b) Let $\Xi$ be the quotient set of $S_\omega$ into unstable leaves. Then the conditional probabilities of $\nu_1$ on elements of $\Xi$ are absolutely continuous with respect to Lebesgue measure, with densities uniformly bounded above and below.

The proof of proposition 12 will occupy the rest of this paper. We first complete the proof of the main theorem assuming this result.

Let $f : M \to$ be a (single) diffeomorphism preserving a probability measure $\sigma$ with at least one positive Lyapunov exponent $\sigma$-a.e.. We say that a measurable partition $\eta$ of $M$ is subordinate to unstable manifolds if for $\sigma$-a.e. $x$, $\eta(x)$, the element of $\eta$ containing $x$, is a relatively compact subset of $\mathcal{W}^u(x)$ and contains an open neighborhood of $x$ in $\mathcal{W}^u(x)$. To say that $\sigma$ is an SRB measure is equivalent to saying that its conditional measures on the elements of $\eta$ are absolutely continuous with respect to the Riemannian measures on unstable manifolds (see, e.g. [9] for details).

These ideas extend readily to RDS. We say a partition $\eta$ of $M \times \Omega^2$ is subordinate to unstable manifolds if for $\mu^*$-a.e. $(x, \omega)$, $\eta(x, \omega)$ is a relatively compact subset of the global unstable manifold $\mathcal{W}^u(x)$ and contains an open neighborhood of $x$ in $\mathcal{W}^u(x)$. The definition of random SRB measures in definition 3 is equivalent to $\mu^*$ having absolutely continuous conditional measures on elements of $\eta$.

Proof of main theorem assuming proposition 12. Let $\eta$ be a partition of $M \times \Omega^2$ subordinate to unstable manifolds, and let $\mu^*_\tau$ be the quotient measure of $\mu^*$ on $(M \times \Omega^2)/\eta$. For $\sigma$-a.e. element $\alpha$ of $\eta$, let $m_\alpha$ denote the Riemannian measure on $\alpha$. We define a (possibly sigma-finite) measure $\nu$ on $M \times \Omega^2$ by letting

$$\nu(A) = \int m_\alpha(A \cap \alpha) \, d\mu^*_\tau(\alpha)$$

for every Borel set $A \subset M \times \Omega^2$, and decompose $\mu^*$ into an absolutely continuous part $\mu^*_a$ and a singular part $\mu^*_s$ with respect to $\nu$. Since $\mu^*_a$ is preserved by $\tau$, and $(\tau, \mu^*)$ is ergodic, we have either $\mu^*_a(M \times \Omega^2) = 1$, in which case $\mu^*$ is SRB, or $\mu^*_a(M \times \Omega^2) = 0$.

Assume, to derive a contradiction, that $\mu^* = \mu^*_s$. By definition, there exists a Borel set $A \subset M \times \Omega^2$ with $\nu(A^c) = 0$ and $\mu^*_s(A) = 0$. In particular, $m_\alpha(A^c \cap \alpha) = 0$ and $\mu^*_s(A \cap \alpha) = 0$ for $\mu^*_s$-almost every $\alpha \in \eta$. We conclude that for such $\alpha$, $\mu^*_s$ and $m_\alpha$ are mutually singular. This contradicts proposition 12, which implies that for a $\mu^*_\tau$-positive measure set of $\alpha \in \eta$, we have that $\mu^*_s$ has a nontrivial absolutely continuous component.

We conclude that $\mu^*_a(M \times \Omega^2) > 0$, hence $\mu^* = \mu^*_a$ and the proof is complete.

3. New chart systems and iterated graph transforms

As explained in the Introduction, our plan is to realize $\mu^*_a$ as the limit of $(f^n_{\omega^+})_\mu$ as $n \to \infty$. In this section, we begin to prepare for this pushing-forward process, with the following simplifications: (i) we will start from time 0 rather than time $-n$, i.e. we will consider $(f^n_{\omega^+})_\mu, n = 1, 2, \ldots$, for some $\omega^+ \in \Omega^2$; (ii) we will consider pushing forward $\mu$ near one
where $\omega, \Theta \subset \Omega^Z$, and $\Theta^+$ denotes its projection to $M \times \Omega^Z$. We would like to have charts defined at $(\mu \times \mathbb{P}^+)$-a.e. $(x, \omega^+)$, so we can push forward small pieces of graphs transversal to $E^{\omega^+}$ in the chart at $x$. Adapted charts for one-sided RDS have been constructed before (see, e.g. chapter 3 of [12]), but the authors’ knowledge, there are no existing constructions in the literature that are suitable for our purposes; see the discussion following proposition 15.

The construction we present here proceeds roughly as follows: since $\mu$-a.e. $x \in M$ is generic with respect to $\mu_x$ for some $\omega \in \Omega^Z$, one may associate to $(\mu \times \mathbb{P}^+)$-a.e. $(x, \omega^+)$ a choice of $\omega \in \Omega^Z$ that (i) agrees with $\omega_x^+$ on its $\Omega^Z$-coordinate and for which (ii) $(x, \omega^+) \in \Gamma$ where $\Gamma$ is as in proposition 5. We may then equip $(x, \omega^+)$ with the chart at $(x, \omega)$ from proposition 5. This is essentially how we will proceed, but first we need to take care of measurability issues. Given a Borel measurable set $\Theta \subset M \times \Omega^Z$, let $\Theta^+$ denote its projection to $M \times \Omega^Z$.

**Lemma 13.** Given any Borel set $\Theta \subset M \times \Omega^Z$ that is a countable union of compact subsets, there is a Borel measurable function
$$\hat{\omega}^- : \Theta^+ \to \Omega^Z$$
with the property that for any $(x, \omega^+) \in \Theta$, we have
$$(x, \hat{\omega}(x, \omega^+)) \in \Theta, \quad \text{where} \quad \hat{\omega}(x, \omega^+) := (\hat{\omega}^-(x, \omega^+), \omega^+) \in \Omega^Z.$$

Lemma 13 is a direct application of the measurable selection criterion in lemma 14. We will explain in the appendix how lemma 14 can be deduced from a well known result.

**Lemma 14.** Let $X, Y$ be Polish spaces. Let $G \subset X \times Y$ be a compact subset and set $G_X \subset G$ be the projection of $G$ onto $X$. Then, there exists a Borel measurable mapping $\psi : G_X \to Y$ with the property that for any $x \in G_X$, we have $(x, \psi(x)) \in G$.

We apply lemma 13 to $\Theta = \Gamma$ where $\Gamma$ is as in proposition 5, noting that (perhaps diminishing $\Gamma$ by a $\mu^*$-null set), $\Gamma$ can be represented as a countable union of compact sets by the inner regularity of $\mu^*$. We then obtain $\Gamma^+ \subset M \times \Omega^Z$ and $\hat{\omega}^- : \Gamma^+ \to \Omega^Z$, i.e. to each $(x, \omega^+) \in \Gamma^+$, we associate in a measurable way a ‘past’ $\hat{\omega}^-(x, \omega^+)$ so that $(x, \hat{\omega}) \in \Gamma$. Recall that $E^\omega_{(x, \omega^+)}$, which depends only on future iterates, is well defined but without a past there is no intrinsic notion of $E^\omega_{(x, \omega^+)}$. We now define $\hat{E}^\omega_{(x, \omega^+)} := \hat{E}^\omega_{(x, \hat{\omega})}$, and let $\hat{\Phi}^{(x, \omega^+)} := \hat{\Phi}^{(x, \hat{\omega})}$ be a chart at $(x, \omega^+)$. Recall $\hat{\Phi}^{(x, \omega^+)}$, along the $\tau^+$-orbit of $(x, \omega^+)$, with $\hat{\phi}^{(x, \omega^+)} = \hat{\Phi}^{(x, \omega^+)}$.

**Proposition 15.** Let $(x, \omega^+) \in \Gamma^+$, and let $(x, \hat{\omega}) \in \Gamma$ be given by lemma 13. Then for each $k = 0, 1, 2, \ldots$, this induces an $(\tau^+)^k(x, \omega^+)$ the splitting
$$T_{\hat{\mu}^+} : M = \hat{E}^\rho_{(x, \omega^+)} \oplus E^{\omega^+}_{(x, \omega^+)}$$
where
$$\hat{E}^\rho_{(x, \omega^+)} := E^\rho_{(x, \hat{\omega})}, \quad E^{\omega^+}_{(x, \omega^+)} := E^{\omega^+}_{(x, \hat{\omega})}.$$
We also define for each \( k \)
\[
\tilde{\Phi}_k^{(k)} = l(\tau^k(x, \bar{\omega})) \\
\text{where } l \text{ is as in proposition 5, and define at } (\tau^+)^k(x, \omega) \text{ a chart given by}
\]
\[
\tilde{\Phi}_k^{(k)} = \Phi^{\tau^k(x, \bar{\omega})} = \exp_{\tilde{\omega}^k}^{x} \circ L_{(x, \omega)^+}^{(k)}; \quad \tilde{L}_{(x, \omega)^+}^{(k)} := L_{\tau^k(x, \bar{\omega})}
\]
where \( \Phi(\cdot), L(\cdot) \) are as in proposition 5. Then for each \( k \),
\[
(x, \omega^+)^+ \rightarrow \tilde{E}^{(k)}_{(x, \omega^+)^+}, \quad \tilde{\Phi}_k^{(k)}_{(x, \omega^+)^+}, \quad \tilde{L}_k^{(k)}_{(x, \omega^+)^+}
\]
are measurable functions, and the properties of the maps \( \tilde{E}^{(k)}_{(x, \omega^+)^+}, \tilde{\Phi}_k^{(k)}_{(x, \omega^+)^+} \) etc. even though these objects are attached to the point \( (\tau^+)^k(x, \omega^+) \).

With regard to differences with existing constructions of one-sided charts (as used in, e.g. chapter III of [12]), previously constructed charts are only guaranteed to have size at least \( C^{-1}e^{-n\delta} \); at time \( n \) where \( C \) depends on the initial point. In contrast, the construction in proposition 15 has the property that the chart size at time \( n \) is \( \sim (\tilde{E}^{(n)}_{(x, \omega^+)^+})^{-1} = l(\tau^n(x, \bar{\omega}))^{-1} \). For \( (\mu \times \mathbb{P}^+) \)-typical \( (x, \omega^+) \), these chart sizes are guaranteed to rise above some minimum size for infinitely many \( n \), a property crucial for our constructions in sections 4–6.

In the rest of this section, the selection function \( \hat{\omega} : \Gamma^+ \rightarrow \Omega^\omega \) given by lemma 13 is fixed, and the chart system in use will be
\[
\{ \hat{\Phi}_k^{(k)}_{(x, \omega^+)^+}, k = 0, 1, 2, \ldots \mid (x, \omega^+) \in \Gamma^+ \}.
\]

Uniformity sets
For \( l_0 > 1 \) and \( k \geq 0 \), we let
\[
\Gamma^+_0 = \{ (x, \omega^+) \in \Gamma^+ \mid \tilde{L}_0^{(k)}(x, \omega^+) \leq l_0 \}.
\]

These are clearly versions of the uniformity sets described in section 2.2.

Observe that since \( E^\omega \) subspaces depend only on the future, they have no dependence on the measurable selection made at time 0. As in proposition 6(a), it follows that \( E_{(x, \omega^+)^+} = E_{(x, \omega^+)^+}^{(k)} \) varies continuously across points of \( (\tau^+)^k(\Gamma^+_0) \). The situation for \( \tilde{E}^{\tau^+} \) is different, and the following observations are crucial:

**Remark 16.**

(a) We claim that the subspaces \( \tilde{E}^{\tau^+}_{(x, \omega^+)^+} \) and \( E_{(x, \omega^+)^+}^{\tau^+} \) are uniformly separated when \( (x, \omega^+) \), \( (x', \omega^+) \in \Gamma^+_0 \) are sufficiently close. While we do not have that \( \tilde{E}^{\tau^+}_{(x, \omega^+)^+} \) and \( E_{(x, \omega^+)^+}^{\tau^+} \) are close, we have \( \| \tilde{L}_0^{(k)}_{(x, \omega^+)} \| \leq l_0 \) where \( \tilde{L}_0^{(k)}_{(x, \omega^+)} : T_xM \rightarrow E_{(x, \omega^+)^+}^{\tau^+} \) is the projection parallel to \( E_{(x, \omega^+)^+}^{\tau^+} \). This together with the continuity of \( E^\omega \) as discussed above implies uniform separation, as claimed.
(b) In light of proposition 6(b), the dependence of $\tilde{E}^{u(k)}$ on the measurable selection becomes weaker and weaker as $k$ is increased; that is, although $\tilde{E}^{u(k)}$ does not vary continuously, nearby $\tilde{E}^{u(k)}$ subspaces become very well aligned for $k \gg 1$.

3.2. Transforms of graphs (with possibly large slopes)

Graph transforms were discussed in section 2.3; what is new here is that we have to consider graphs with possibly large though uniformly bounded slopes, the reason being the observation in remark 16(a). This subsection gives a priori bounds for a single step of the graph transform. Let $(x, \omega^+)$ ∈ $\Gamma^+$ be fixed; we will omit mention of $(x, \omega^+)$ in the remainder of section 3, writing $\tilde{\Phi}(k) = \tilde{\Phi}(x_{\omega^+})$, $\tilde{f}(k) = \tilde{f}(x_{\omega^+})$, $\tilde{l}(k) = \tilde{l}(x_{\omega^+})$ etc. For $k \geq 0$ and $\delta \in (0, 1)$, let

$$g : B^g(\delta(\tilde{l}(k))^{-1}) \to \mathbb{R}^e$$

be a mapping. The graph transform $T^{(k)}g = T^{(k)}(x_{\omega^+})g$, if it is defined, is a mapping

$$T^{(k)}g : B^{(\rho'(\tilde{l}(k+1))^{-1})} \to \mathbb{R}^e$$

with $\tilde{f}(k)$ (graph $g$) ⊆ graph $T^{(k)}g$

for some $\rho' \in (0, 1)$. We write $K_0 = \text{Lip}(g)$ for the Lipschitz constant of the initial graph $g$. The following lemma does not distinguish between large and small $K_0$.

**Lemma 17.** For any $K_0 > 0$, there exist constants $r_1 = r_1(\lambda, \delta_0, K_0) \in (0, 1)$, $C_1 = C_1(\lambda, \delta_0, K_0)$, $C_2 = C_2(\lambda, \delta_0, K_0)$ such that the following holds when $\delta < r_1$. Let $k \geq 0$, $\rho \in (0, K_0^{-1})$, and let

$$g : B^g(\rho(\tilde{l}(k))^{-1}) \to \mathbb{R}^e$$

be a $C^1+\text{Lip}$ map for which (i) $g(0) = 0$, (ii) graph $g \subset B(\delta(\tilde{l}(k))^{-1})$, and (iii) $\text{Lip}(g) \leq K_0$.

Then, the graph transform

$$T^{(k)}g : B^{(\rho'(\tilde{l}(k+1))^{-1})} \to \mathbb{R}^e$$

with $\rho' = \min\{\rho e^{\lambda/2}, 1\}$

is defined, and is a $C^1+\text{Lip}$ map for which (i') $T^{(k)}g(0) = 0$, (ii') graph $T^{(k)}g \subset B(\delta(\tilde{l}(k))^{-1})$, (iii') $\text{Lip}(T^{(k)}g) \leq K_0 e^{-\lambda/2}$, and (iv')

$$||dT^{(k)}g||_{0} \leq e^{-\lambda/2}||dg||_{0},$$

$$\text{Lip}(dT^{(k)}g) \leq C_1 \tilde{l}(k) + C_2 \text{Lip}(dg).$$

**Proof.** For short, let us write $\tilde{g} = T^{(k)}g$, $\tilde{f} = \tilde{f}(k)$, $\tilde{l} = \tilde{l}(k)$.

Let

$$C = C(K_0) = \{u + v : u \in \mathbb{R}^e, v \in \mathbb{R}^e, \text{ and } |v| \leq K_0 |u|\}.$$ 

Observe that $\tilde{d}_0$ maps $C$ strictly into its interior. Let $r_1 = r_1(\lambda, \delta_0, K_0) > 0$ be small enough that for $\delta \in (0, r_1)$,

$$\tilde{d}_0 C \subset C(K_0 e^{-\lambda/2}) \text{ for all } z \in B(\delta \tilde{l}^{-1}).$$

More precisely, if $w = u + v \in C \subset \mathbb{R}^e \oplus \mathbb{R}^e$ and $\tilde{d}_0(\tilde{g})(w) = u' + v' \in \mathbb{R}^e \oplus \mathbb{R}^e$, then

$$|u'| \geq e^{\lambda}|u| - \delta \max\{|u|, |v|\} \geq (e^{\lambda} - \delta \max\{1, K_0\})|u|,$$

and

$$|v'| \leq e^{\delta}|v| + \delta \max\{|u|, |v|\} \leq e^{\delta}|v| + \delta \max\{1, K_0\}|u|,$$

and $u' + v' \in C$ for $\delta \in (0, r_1)$.
Let now \( \rho, \delta, \) and let \( g \) be as in the hypothesis of lemma 17. We let 
\[
\phi : B^\rho(\rho \delta^{-1}) \to \mathbb{R}^n
\]
be given by 
\[
\phi(u) = \pi^n \circ f (u, g(u)),
\]
and define 
\[
\hat{g} = T^{(k)} g = \pi^n \circ f \circ (\text{Id} \times g) \circ \phi^{-1}
\]
where \( \text{Id} \) refers to the identity map restricted to \( B^\rho(\rho \delta^{-1}) \). As the existence of \( \hat{g} \) and its first derivative properties follow largely from standard arguments involving the invariant cones condition above, we leave them to the reader, providing below only the bound for \( \text{Lip}(\hat{g}) \).

Let \( \hat{u}_1, \hat{u}_2 \in B^\rho(\rho \delta^{-1}) \) with \( u_i = \phi^{-1}(\hat{u}_i), i = 1, 2 \). Then,
\[
|\hat{d}g_{u_1} - \hat{d}g_{u_2}| \leq |\pi^{\text{cx}} \circ (\hat{d}f_{(u_1, g(u_1))} - \hat{d}f_{(u_2, g(u_2))})| \cdot |\text{Id} + dg_{u_1}| \cdot |\text{d}\phi^{-1}_{u_1}|
\]
\[
+ |\pi^{\text{cx}} \circ \hat{d}f_{(u_2, g(u_2))}(dg_{u_1} - dg_{u_2})| \cdot |\text{d}\phi^{-1}_{u_2}|
\]
\[
+ |\pi^{\text{cx}} \circ \hat{d}f_{(u_2, g(u_2))}(\text{Id} + dg_{u_2})| \cdot |\text{d}\phi^{-1}_{u_1} - \text{d}\phi^{-1}_{u_2}| .
\]

(2)

We bound the last term of (2) as follows:

- First we have \( |\text{d}\phi^{-1}_{u_1} - \text{d}\phi^{-1}_{u_2}| \leq e^{-\lambda/2} \) for all \( \hat{u} \in B^\rho(\rho \delta^{-1}) \). Using the fact that \( \hat{d}f_{\hat{u}} = \hat{d}f_0 + (\hat{d}f_{\hat{u}} - \hat{d}f_0) \) and \( |\hat{d}f_{\hat{u}} - \hat{d}f_0| \leq \hat{l}(\hat{u}) \leq \delta \) for \( z \in B^\rho(\rho \delta^{-1}) \) by proposition 5, we have

\[
|\text{d}\phi_{u_1} - \text{d}\phi_{u_2}| \leq |\hat{d}f_{(u_1, g(u_1))} \circ (\text{Id} + dg_{u_1}) - \hat{d}f_{(u_2, g(u_2))} \circ (\text{Id} + dg_{u_2})|
\]
\[
\leq |\hat{d}f_{(u_1, g(u_1))} - \hat{d}f_{(u_2, g(u_2))}| \cdot |\text{Id} + dg_{u_1}| + |\hat{d}f_{(u_2, g(u_2))}(\text{Id} + dg_{u_1}) - (\text{Id} + dg_{u_2})|
\]
\[
\leq \hat{l} \max\{1, K_0\} |u_1 - u_2| \cdot \max\{1, K_0\} + (e^\delta + \delta) \text{Lip}(dg)|u_1 - u_2|
\]
\[
\leq e^{-\lambda/2} \left( \max\{1, K_0\}^2 \hat{l} + (e^\delta + \delta) \text{Lip}(dg) \right) |\hat{u}_1 - \hat{u}_2|.
\]

- Since for \( w \in \mathbb{R}^n \) with \( |w| = 1 \), we have

\[
|\pi^{\text{cx}} \circ \hat{d}f_{(u_2, g(u_2))}(w + dg_{u_2}w)| \leq e^\delta K_0 + \delta \max\{1, K_0\},
\]

this is an upper bound for \( |\pi^{\text{cx}} \circ \hat{d}f_{(u_2, g(u_2))}(\text{Id} + dg_{u_2})| \).

Finally, plugging these back into (2), we obtain
\[
|\hat{d}g_{u_1} - \hat{d}g_{u_2}| \leq (C_1 \hat{l} + C_2 \text{Lip}(dg)) \cdot |\hat{u}_1 - \hat{u}_2|,
\]
where
\[
C_1 = e^{-\lambda} \max\{1, K_0\}^2 + (e^\delta K_0 + \delta \max\{1, K_0\}) \max\{1, K_0\}^2 e^{-\lambda/2},
\]
\[
C_2 = e^{-\lambda}(e^\delta + \delta) + (e^\delta K_0 + \delta \max\{1, K_0\})(e^\delta + \delta) e^{-\lambda/2}.
\]
Lemma 17 provides us with the following information: in general, \( C_2 > 1 \), which is not useful for controlling the growth of \( \text{Lip}(dT^{(k)}g) \) as we iterate the graph transform. However, when \( K_0 \) is small enough depending mostly on \( \lambda \) (also \( \delta_0 \) and \( \delta \)), then \( C_2(K_0) < 1 \). We fix \( \hat{B}_0 \) small enough that \( C_2(K_0)e^{\hat{z}_2} < 1 \), and write \( C_i = C_i(K_0), i = 1, 2 \). Furthermore, we let \( \hat{r}_1 := \hat{r}_1(K_0) \) be small enough that on \( B(\hat{r}_1(\hat{1}^{(1)}))^{-1} \), the cones \( C_i(K_0) \) are invariant under \( df_C \).

### 3.3. Iteration of graph transforms

We now consider iterated graph transforms along the orbit of \( (x, \omega^+) \in \Gamma^+ \), introducing first the following notation: given \( g \) and a sequence of numbers \( d_0, d_1, \ldots \), we say

\[
g_{k+1} = T^{(k)} \circ \cdots \circ T^{(0)}g, \quad k = 0, 1, 2, \ldots,
\]

are the graph transforms of \( g \) on \( B(d_k(\hat{1}^{(k)}))^{-1} \) if

\[
\text{graph } g_0 = \text{graph } g \cap B(d_0(\hat{1}^{(0)}))^{-1},
\]

and for each \( k \geq 0 \), we let

\[
\text{graph } g_{k+1} = \hat{f}^{(k)}(\text{graph } g_k) \cap B(d_{k+1}(\hat{1}^{(k+1)}))^{-1},
\]

assuming the graph transforms above are well defined. In this definition, we allow the domain of definition of \( g_k \) to be a proper subset of \( B^p(d_k(\hat{1}^{(k)}))^{-1} \) containing \( 0 \) (but when we write \( h : U \rightarrow V \), it will be implied that \( h \) is defined on all of \( U \)).

Let \( K_0 \) and \( \hat{r}_1 \) be as in section 3.2.

**Proposition 18.** Given \( K_0, \lambda_0, \delta_0 \), there exist \( C \geq 1 \) (independent of \( K_0 \)), \( m_0 = m_0(K_0) \) and \( r_0 = r_0(K_0, m_0) > 0 \) for which the following hold. Let \( r_0 \leq \hat{r}_0 \). Then there exists \( m_1 \in \mathbb{Z}^+ \) depending on \( \text{Lip}(dg) \) in addition to the constants above with the following properties. Let

\[
g : B^p(r_0(\hat{1}^{(0)}))^{-1} \rightarrow \mathbb{R}^x
\]

be a \( C^{1+\lambda_0} \) map with (i) \( g(0) = 0 \) and (ii) \( \text{Lip}(g) \leq K_0 \). We let \( g_k \) be the graph transforms of \( g \) with \( d_k = r_0 \) for \( k \leq m_0 \) and \( d_k = \hat{r}_1 \) for \( k > m_0 \). Then for all \( k \geq m_0 + m_1 \),

\[
g_k : B^p(\hat{r}_1(\hat{1}^{(1)}))^{-1} \rightarrow \mathbb{R}^x
\]

is defined and satisfies

\[
\text{Lip}(g_k) \leq K_0, \quad |(dg_k)_0| \leq e^{-k\lambda/2}|(dg)_0|, \quad \text{and } \text{Lip}(dg_k) \leq C\hat{l}^{(k)}.
\]

**Proof.** We assume \( K_0 > \hat{K}_0 \) (omitting the first part of the proof if \( K_0 \leq \hat{K}_0 \)). Let \( m_0 \) be such that \( K_0 e^{-m_0\lambda/2} < \hat{K}_0 \). Fix \( r_0 > 0 \) sufficiently small so that for each \( 0 \leq k \leq m_0 - 1 \), we have for \( z \in B(r_0(\hat{1}^{(k)}))^{-1} \) that \( (df^{\hat{k}})_C(K_0e^{-k\lambda/2}) \leq C(K_0e^{-(k+1)\lambda/2}) \) (notation as in the proof of lemma 17). By the estimates in the proof of lemma 17, the choice of \( r_0 \) depends on \( m_0, \hat{K}_0 \).

With \( r_0 \leq \hat{r}_0 \) now fixed and \( \{g_k\} \) the graph transform sequence as defined in the statement, we obtain from a simple induction that \( g_{m_0} \) is defined on \( B^p(r_0(\hat{1}^{(m_0)}))^{-1} \) and \( \text{Lip}(g_{m_0}) \leq K_0 e^{-m_0\lambda/2} < \hat{K}_0 \). Since \( \hat{K}_0 \)-cones are preserved on charts of size \( B(\hat{r}_1(\hat{1}^{(1)}))^{-1} \), \( \text{Lip}(g_k) \leq \hat{K}_0 \) will hold for \( k \geq m_0 \). Moreover, one easily checks (see (iv) in lemma 17) that \((dg_k)_0 \leq e^{-k\lambda/2}|(dg)_0|\) holds for all \( k \).

Next, we grow \( g_k \) so that its graph stretches all the way across the chart, letting \( m_1 = m_1(r_0, \hat{r}_1) \) be such that \( g_{m_0+m_1} \) is defined on all of \( B^p(\hat{r}_1(\hat{1}^{(m_0+m_1)}))^{-1} \).
It remains to bound \( \text{Lip}(\text{dg}_l) \). Let \( a = \text{Lip}(\text{dg}_m) \). Though \( \text{Lip}(\text{dg}_r) \) may have grown during the first \( m_0 \) iterates, \( a \) is determined by \( \text{Lip}(\text{dg}), K_0, \lambda \) and \( m_0 \) (lemma 17). Applying lemma 17 again repeatedly from step \( m_0 \) on, we obtain

\[
\text{Lip}(\text{dg}_{m_0+i}) \leq C_1 \left( \ell(m_0+i) \right) + C_2 \left( \ell(m_0+i-1) \right) + C_3 \left( \ell(m_0+i-2) \right) + \cdots + C_4 a.
\]

Let \( C = 2C_1 \sum (C_2 e^{C_3})^i \), and choose \( m_l \geq m_0 \) large enough that \( C_1 \cdot C_2^m a < \frac{1}{2} C \). The desired properties are achieved for \( k \geq m_0 + m_1 \).

For the remainder of section 3 we fix \( K_0 > 0 \), \( r_0 < r_0(K_0, m_0(K_0)) \) and assume proposition 18 has been applied to a fixed \( C^{1+\text{Lip}} \) graphing function \( g : B_r(r_0(\hat{f}^0)^{-1}) \rightarrow \mathbb{R}^2 \), \( \text{Lip}(g) \leq K_0 \), obtaining the graph transform sequence \( \{ g_k \} \), with all notation (e.g. \( m_0, m_1 \)) as in the conclusions of proposition 18.

First, we give a distortion estimate in this setting.

**Lemma 19.** Write \( a_0 = \text{Lip}(\text{dg}) \). Then for any \( k \geq m_0 + m_1 \), there exists a constant \( D = D(K_0, a_0, r_0, \hat{f}^0, \hat{f}^k) \) with the following property. Write \( \gamma_j = \hat{\Phi}^{(j)}(\text{graph } g_j) \) for \( 0 \leq j \leq k \), and let \( p_1, p_2 \in \gamma_j \). Then,

\[
\left| \frac{\log \det(f_{\omega}^k + T\gamma_0)(f_{\omega}^k)^{-1}p_1}{\log \det(f_{\omega}^k + T\gamma_0)(f_{\omega}^k)^{-1}p_2} \right| \leq D.
\]

Note that \( D \) does not depend on \( k \) except through the value of \( \hat{f}^k \).

**Proof.** For \( i = 1, 2 \), write \( p_i^j = p_i \) and \( p_i^0 = (f_{\omega}^1)^{-1}p_i^k \). For \( 0 \leq j \leq k \) set \( p_i^j = f_{\omega}^j p_i^0 \in \gamma_j \). We decompose

\[
\left| \frac{\log \det(f_{\omega}^j + T\gamma_0)(p_i^0)}{\log \det(f_{\omega}^j + T\gamma_0)(p_i^0)} \right| \leq \left| \frac{\log \det(f_{\omega}^{m_0+m_1} + T\gamma_0)(p_i^0)}{\log \det(f_{\omega}^{m_0+m_1} + T\gamma_0)(p_i^0)} \right| + \left| \frac{\log \det(f_{\omega}^{k-(m_0+m_1)} + T\gamma_0)(p_i^0)}{\log \det(f_{\omega}^{k-(m_0+m_1)} + T\gamma_0)(p_i^0)} \right|.
\]

The first RHS term is the sum of \( m_0 + m_1 \) terms, each of which is bounded from above in terms of \( \| df_{\omega}^j \|, \| df_{\omega}^{-1} \| \), \( 1 \leq j \leq m_0 + m_1 \); these in turn are controlled by the value \( l_1(\hat{\omega}) \leq \ell(\hat{f}^0) \) of the function \( l_1 \) as in section 2.1, (recall \( \hat{\omega} = \hat{\omega}(x, \omega^+) \) as in section 3.1). By these considerations, this term is bounded \( \leq D_1 \), where \( D_1 = D_1(K_0, a_0, r_0, \hat{f}^0) \) (noting that \( m_0, m_1 \) depend on \( K_0, a_0, r_0 \)).

For the second RHS term, observe that the graphing functions \( g_j, j \geq m_0 + m_1 \), satisfy \( \text{Lip}(g_j) \leq 1/10 \) and \( \text{Lip}(\text{dg}_j) \leq C \ell(\hat{f}^k) \). A distortion estimate analogous to that in lemma 9 applies to bound this term \( \leq D_2(\ell(\hat{f}) \cdot d(\hat{\Phi}, p_1, p_2)) \).

The proof is complete on setting \( D = D_1 + D_2 \).

The next lemma gives sufficient conditions for switching of axes (lemma 10) in the present context. Let \( g_k : \text{Dom}(\hat{g}_k) \subset E^{\hat{f}} \rightarrow E^{\hat{f}_{\lambda}} \) be given by \( \hat{g}_k = \hat{L}_{\lambda}^{(k)} \circ g_k \circ \hat{L}_{\lambda}^{(k)} \) of \( \hat{L}_{\lambda}^{(k)} \).

**Lemma 20.** For any \( \hat{l} > 1 \) there exists \( r_3 = r_3(\hat{l}) \) with the following properties. Let \( k \geq m_0 + m_1 \) and let \( r_1 = r_3(\hat{f}^0) \). Then

(a) \( \text{Dom}(\hat{g}_k) \) contains \( E^{\hat{f}_{\lambda}}(r_1) \); and

(b) \( \| (\text{dg}_k) \| \leq (20 \hat{f}^0)^{-1} \) holds, then we have \( \text{Lip}(\hat{g}_k) \leq 1/10 \) on \( E^{\hat{f}}(r_1) \).

Since \( \| (\text{dg}_k) \| \leq K_0 e^{-k/2} \) (proposition 18) and \( \hat{f}^k \leq e^{k/2} \hat{f}^0 \) (proposition 5), the condition in (b) is satisfied for all large enough \( k \) depending on \( \hat{f}^0 \) and \( K_0 \).
Proof. Item (a) is guaranteed when \( r_{1} \left( \hat{f} \right) \) is taken \( \leq \left( \hat{r}_{1} \hat{f} \right)^{-1} \). For (b), for \( r > 0 \) we estimate \( \text{Lip}(\hat{g}_k|\hat{E}^{u}\cap\Theta) \) as follows. Let \( \hat{u} \in \hat{E}^{u}(k) (r) \), \( \hat{u} = \hat{L}^{(k)}(\omega^{*})u, u \in \mathbb{R}^{n} \), and estimate

\[
\| (d\hat{g}_k)_{0} \| \leq (\| (d\hat{g}_k)_{0} \| + \text{Lip}(d\hat{g}_k) \| \hat{u} \|) \lesssim (\hat{L}^{(k)}(\omega^{*})0) + (\hat{L}^{(k)}(\omega^{*})2\text{Lip}(d\hat{g}_k) \| \hat{u} \|) \lesssim \hat{L}^{(k)}(\omega^{*})0 + C(\hat{L}^{(k)})^{3} \cdot r,
\]

where \( C \) is as in the end of section 3.2. Taking \( r_{2} \left( \hat{f} \right) \leq \left( 20\hat{C}^{3} \right)^{-1} \) ensures the second RHS term is \( \leq 1/20 \) while the first term is \( \leq 1/20 \) when the condition in (b) is met. \( \square \)

In the rest of the paper, \( \mathcal{K}_0 \) is fixed, as is \( \delta \in (0, \hat{r}_{1}(\mathcal{K}_0)) \) sufficiently small for the purposes of proposition 7 and lemmas 8 and 9.

4. Setup for the rest of the proof

For \( \omega \in \Omega^{\mathbb{Z}} \), we will realize \( \mu^{n}_{\mathcal{W}} \) as the weak limit as \( n \to \infty \) of \( \mu^{n}_{\mathcal{W}} = (f_{\mathcal{T}^{n}_{\mathcal{W}}})_{\mathcal{W}} \mu \), obtained by pushing \( \mu \) forward from time \(-n\) to 0. But we will not push forward all of \( \mu \), only a small bit of it, as that is all that is needed to show \( \mu^{*} \) has the SRB property; see section 2.5. As a matter of fact, we will push forward a very localized bit of \( \mu \) (located on a ‘source set’), and consider only the part that arrives in a localized region (the ‘target set’), both suitably chosen. This section describes and justifies the main ingredients of this setup; details including order of choice of constants are given in section 5.1.

4.1. Uniformity sets of \( \mu^{n}_{\mathcal{W}} \)-typical points

Let \( l_0 > 1 \) be fixed implicitly throughout. We fix a compact subset \( \Theta_0 \subset \Gamma_{l_0} \), and for now fix \( n \in \mathbb{Z}^{+} \). We define \( \Theta_n := \Theta_0 \cap \tau^{-n} \Theta_0 \), so that \( \Theta_n \) consists of points \( (x, \omega) \) that are good in the sense of being in a uniformity set both at time 0 and at time \( n \). As in section 3.1, a measurable selection (lemma 13) \( \hat{w}^{n}_{\omega} : \Theta_n \to \Omega^{\mathbb{Z}} \) enables us to systematically assign ‘pasts’ to points in \( \Theta_n^{\omega} \), a positive \( (\mu \times \mathbb{P}^{\omega}) \)-measure set.

Now since we are interested in \( \mu^{n}_{\mathcal{W}} = (f_{\mathcal{T}^{n}_{\mathcal{W}}})_{\mathcal{W}} \mu \), we want to consider orbits starting from time \(-n\) and not from 0. For \( \omega \in \Omega^{\mathbb{Z}} \), we write \( x_{-n} = f_{-n}^{\omega}x \), and define

\[
M^{n}_{\omega} = \{ x \in M : (x_{-n}, (\theta^{-n} \omega)^{\uparrow}) \in \Theta_n^{\omega} \}.
\]

Then \( M^{n}_{\omega} \) is a subset of \( \mu^{n}_{\mathcal{W}} \)-typical points.

The ideas from section 3.1 carry over in a straightforward way, though the notation gets more cumbersome. Let \( x \in M^{n}_{\omega} \). For \( k = 0, 1, \ldots, n \), let \( x_{-k} = f_{-k}^{\omega}x \). As before,

\[
E^{u}_{(x_{-k}, (\theta^{-k} \omega)^{\uparrow})} = E^{u}_{(x_{-k} \theta^{-k} \omega)}
\]

are well defined, as \( E^{\omega} \)-subspaces depend only on the future. To define \( E^{\omega} \), for brevity let us write \( \hat{w} = (\hat{w}^{n}_{\omega} (x_{n}, (\theta^{-n} \omega)^{\uparrow}), (\theta^{-n} \omega)^{\uparrow}) \). We define

\[
E^{u}_{(x_{-n}, (\theta^{-n} \omega)^{\uparrow})} = E^{u}_{(x_{-n}, \theta^{-n} \omega)}
\]

and for \( k = 0, 1, \ldots, n - 1 \), we define the \( E^{\omega} \)-subspace at \( (x_{-k}, (\theta^{-k} \omega)^{\uparrow}) \) as

\[
E^{u}_{(x_{-k}, (\theta^{-k} \omega)^{\uparrow})} = E^{u}_{(x_{-k} \theta^{-k} \omega)}.
\]
Chart maps \( \Phi^{(a-k)}_{x,\gamma(\theta-\omega)+} \) connecting maps \( \tilde{\pi}^{(a-k)}_{x,\gamma(\theta-\omega)+} \) and the \( l \)-function \( \tilde{\pi}^{(a-k)}_{x,\gamma(\theta-\omega)+} \) are all defined as before. Observe that by our choice of \( \Theta_n \), we have that
\[
\tilde{\pi}^{(0)}_{x,\gamma(\theta-\omega)+}, \tilde{\pi}^{(n)}_{x,\gamma(\theta-\omega)+} \leq l_0.
\]

We finish by recording the following observation. Let \( M_n = \{ x \in \mathcal{M} : (x, \omega) \in \Theta_0 \} \); that is, \( M_n \) is a uniformity set for the two-sided dynamics restricted to the fiber \( \mathcal{M} \times \{ \omega \} \). Since \( \mu_n \rightarrow \mu \) weakly, one should expect that as \( n \rightarrow \infty \), the uniformity set \( M_n \) of \( \mu_n \)-typical points should converge to \( M_n \) in some sense. Below this is made precise.

**Lemma 21.** For any \( \delta > 0 \), there exists \( N_0 = N_0(\delta) \in \mathbb{N} \) such that for any \( n \geq N_0 \), we have that \( M_n \subset \mathcal{N}_\delta(M_n) \). In particular,
\[
\lim_{n \rightarrow \infty} \sup_{x \in M_n} \text{dist}(x, M_n) = 0.
\]

**Proof.** By standard compactness arguments, it suffices to prove that for any sequence \( \{ x^a \} \subset \mathcal{M} \) converging to a point \( x \in \mathcal{M} \) for which \( x^a \in M_n \) for all \( n \), we have that \( x \in M_n \). For each \( n \geq 1 \) write \( x^n_a = \int_{\mathcal{M}} x^a \), and let \( \tilde{\omega}_n \) be defined by \( \tilde{\omega}_n = \theta^n(\omega)(x^n_a, (\theta^{-n}\omega)^+) \). Observe that \( \tilde{\omega}_n \rightarrow \tilde{\omega} \) as \( n \rightarrow \infty \), and that \( (x^n_a, \tilde{\omega}_n) \in \tau^n \Theta_0 \subset \Theta_0 \) for all \( n \geq 0 \) by our measurable selection construction. Since \( (x^n_a, \tilde{\omega}_n) \) converges to \( (x, \omega) \), and \( \Theta_0 \) is compact, we obtain that \( (x, \omega) \in \Theta_0 \), i.e. \( x \in M_n \). \( \square \)

### 4.2. Accumulating \( \mu_n \)-mass

Let \( \beta_0 > 0 \) be a very small number. We fix \( l_0 > 1 \) sufficiently large so that \( \mu \Gamma_{l_0} \geq 1 - \beta_0/3 \). Fix a compact set \( \Theta_0 \subset \Gamma_{l_0} \) with \( \mu \Theta_0 \geq 1 - \beta_0/2 \) and for each \( n \geq 0 \) let \( \Theta_n = \Theta_0 \cap \tau^{-n} \Theta_0 \), so that \( \mu^n(\Theta_n) \geq 1 - \beta_0 \). For each \( n \), fix a measurable selection \( \tilde{\omega}_n : \Theta_n \rightarrow \Omega^+ \) as in lemma 13. Finally, \( M_n \) is as defined in section 4.1.

Below, we determine a set of \( \omega \) for which \( M_n \) has sufficiently large \( \mu_n \)-mass for an infinite sequence of \( n \).

**Lemma 22.** Let \( \epsilon > 1 \). For each \( n \geq 1 \) define
\[
\mathcal{G}^{(n)} = \{ \omega \in \Omega^+ : \mu_n(M_n) \geq 1 - \epsilon \cdot \beta_0 \},
\]
and set \( \mathcal{G} = \limsup_{n \rightarrow \infty} \mathcal{G}^{(n)} = \bigcap_{n \geq 1} \bigcup_{n \geq N} \mathcal{G}^{(n)} \). Then, we have
\[
P(\mathcal{G}) \geq \frac{\epsilon - 1}{\epsilon}.
\]

**Proof.** We claim that
\[
\theta^{-n} \mathcal{G}^{(n)} = \{ \omega \in \Omega^+ : \mu \{ y \in \mathcal{M} : (y, \omega^+) \in \Theta_n \} \geq 1 - \epsilon \beta_0 \}.
\]
(3)

Assuming this for the moment, observe that \( \theta^{-n} \mathcal{G}^{(n)} \) depends only on the \( \Omega^+ \)-coordinate of \( \omega \), hence
\[
P^+(\theta^{-n} \mathcal{G}^{(n)}) = P(\theta^{-n} \mathcal{G}^{(n)}) = P(\mathcal{G}^{(n)})
\]
by the P-invariance of the shift \( \theta \). We now estimate:
\[ 1 - \beta_0 \leq (\mu \times \mathbb{P}^+)(\Theta_+^\circ) \]

\[ = \left( \int_{(\theta^{-n}g(M))^\circ} + \int_{\Omega + \chi_{\theta^{-n}g(M)^\circ}} \right) \mu\{x \in M : (x, \omega^+) \in \Theta_+^\circ\} \, d\mathbb{P}^+ (\omega^+) \]

\[ \leq \mathbb{P}^+((\theta^{-n}g(M)^\circ) + (1 - c\beta_0)(1 - \mathbb{P}^+((\theta^{-n}g(M)^\circ))). \]

Rearranging, we obtain \( \frac{\lambda_1}{\lambda_2} \leq \mathbb{P}^+((\theta^{-n}g(M)^\circ) = \mathbb{P}(G(n)), \) hence \( \mathbb{P}(G) \geq \frac{\epsilon}{\lambda_2} \).

It remains to check (3). Observe that \( \theta^E \in G(n) \) holds iff \( \mu(E, \omega^+) \geq 1 - c\beta_0 \). We evaluate

\[ M_{\omega^+}^n = \{ x \in M : (f_{\omega^+}^n(x, \omega^+) \in \Theta_+^\circ \} = \{ x \in M : ((f_{\omega^+}^n)^{-1}, \omega^+) \in \Theta_+^\circ \} \]

\[ = f_{\omega^+}^n(y \in M : (y, \omega^+) \in \Theta_+^\circ). \]

Since \( \mu(E, \omega^+) = (f_{\omega^+}^n, \mu), \) equation (3) follows immediately. \( \square \)

Our next step is to coordinate for each \( \omega \in G \) for a positive amount of \( \mu \)-mass to come from a small, fixed region (the ‘source set’) and to land in a small, fixed region (the ‘target set’) under \( f_{\omega^+}^n \) for some infinite sequence \( \{n_i\} \).

We write \( \psi := n_{i,n} \mu \), which we recall is continuous by hypothesis. With \( \beta_0 \) as before, let us define \( \alpha_0 = \beta_0/\text{Leb}(\mathcal{M}) \), so that

\[ \mu\{\psi \geq \alpha_0\} \geq 1 - \beta_0. \]

**Lemma 23.** For any \( \epsilon > 0 \), there exists a constant \( c = c(\epsilon) > 0 \) such that for any \( \omega \in G \), we have the following. There are points \( \hat{p}_- \in \{ \psi \geq \alpha_0\}, \hat{p} \in M, \) and a sequence \( n_i \to \infty \) for which

\[ \mu_{\omega^+}^n(M_{\omega^+}^n \cap B(\hat{p}, \epsilon) \cap f_{\omega^+}^n \cap B(\hat{p}_-, \epsilon)) \geq c \quad \text{for all } n = n_i. \] (4)

Note that in lemma 23, the points \( \hat{p}, \hat{p}_- \in M \) and the subsequence \( n_i \) all depend on \( \omega \), whereas the constant \( c = c(\epsilon) \) is independent of \( \omega \).

**Proof.** Let \( \omega \in G \). To start, fix a subsequence \( n_i \to \infty \) along which \( \omega \in G(n) \) for all \( n = n_i \).

In pursuit of the ‘source set’ \( B(\hat{p}_-, \epsilon) \) and ‘target set’ \( B(\hat{p}, \epsilon) \), we refine \( (n_i) \) successively several times in the following argument.

Fix an open cover of \( \{ \psi \geq \alpha_0\} \) by balls of radius \( \epsilon \) with centers \( p_j \in \{ \psi \geq \alpha_0\}, 1 \leq j \leq J \). For each \( n = n_i \), we estimate:

\[ c_- := \frac{1}{J} (1 - (1 - \epsilon + 1)\beta_0) \leq \frac{1}{J} \mu(f_{\omega^+}^n \cap \{ \psi \geq \alpha_0\}) \leq \frac{1}{J} \sum_{j=1}^J \mu(B(p_j, \epsilon) \cap f_{\omega^+}^n M_{\omega^+}^n). \]

For each \( i \), there exists \( j = j(i) \) so that \( \mu(B(p_j, \epsilon) \cap f_{\omega^+}^n M_{\omega^+}^n) \geq c_- \). Since there are only finitely many \( j \), by the Pidgeonhole principle we may refine \( (n_i) \) so that

\[ \mu_{\omega^+}^n(B(\hat{p}_-, \epsilon) \cap M_{\omega^+}^n) \geq c_- \]

holds for \( \hat{p}_- = p_j \) for some fixed \( j_- \in \{1, \cdots, J\} \).

Continuing, fix an open cover of \( M \) by balls of radius \( \epsilon \) with centers \( p_j^* \in M, 1 \leq j \leq J' \). For each \( n = n_i \), we estimate:
This means in particular that \( \epsilon \in G \) with Lip, and by remark 16(a) we have that
\[
\mu^n(B(\hat{p}, \epsilon) \cap M^n_w \cap f^n_{\hat{p}, -w} B(\hat{p}, \epsilon)) \geq c
\]
where \( \hat{p} = p^j \) for some fixed \( j \in \{1, \ldots, J'\} \). \( \Box \)

4.3. Disintegration of \( \mu \) in the 'source set' onto \( u \)-graphs

Fix \( \omega \in \mathcal{G} \). We assume in this subsection that \( \epsilon > 0 \) is specified, and lemma 23 has been applied to obtain \( \hat{p} \ldots \hat{p} \in M \), a sequence \( \{n\} \) and \( c = c(\epsilon) > 0 \). For \( n = n_\omega \), define
\[
\Lambda^n = M^n_w \cap B(\hat{p}, \epsilon) \cap f^n_{\hat{p}, -w} B(\hat{p}, \epsilon) \quad \text{and} \quad \Lambda^n = \int^n_{\omega}(\Lambda^n) . \tag{5}
\]
We now specify how \( \mu \) restricted to \( \Lambda^n \) will be decomposed into measures on graphs to be pushed forward.

For \( x \in \Lambda^n \), we have \( \langle x, (\theta^{-\omega})^+ \rangle \in \Gamma_n^+ \). This means in particular that \( \langle x, (\theta^{-\omega})^+ \rangle \) possesses a natural \( E^n_{(\omega \mid \theta^{-\omega})^+} \), and a systematic and measurable assignment of \( E^n_{(\omega \mid \theta^{-\omega})^+} \). We will disintegrate \( \mu \) onto the leaves of a smooth foliation \( \mathcal{F}_\omega \), chosen in such a way that the leaves of \( \mathcal{F}_\omega \), suitably restricted, are graphs from open subsets of \( E^n_{(\omega \mid \theta^{-\omega})^+} \) for each \( x \in \Lambda^n \); we will refer to such graphs as '\( u \)-graphs'.

To define \( \mathcal{F}_\omega \), we fix a reference point \( q^\omega \in \Lambda^n \). As the discussion is entirely local, we can choose a neighborhood of \( q^\omega \) in \( M \) with a subset of \( T_{q^\omega} M \) via \( \exp_{q^\omega} \) and define \( \mathcal{F}_\omega \) to be the collection of \((\dim E^n)\)-dimensional hyperplanes in \( T_{q^\omega} M \) parallel to \( \mathcal{F}^n_{(\omega \mid \theta^{-\omega})^+} \).

**Lemma 24.** For all sufficiently small \( \epsilon > 0 \) depending on \( l_0, \psi = \frac{\pi}{X^\alpha} \) and \( \alpha_0 \), there exist constants \( K_\epsilon = K_\epsilon(l_0), r_\omega = r_\omega(l_0, \epsilon) \) for which the following hold for all \( \omega \in \mathcal{G} \). Assume lemma 23 has been applied. Let \( n = n_\omega \) and let \( \mathcal{F}_\omega \) be as above. Then for every \( x \in \Lambda^n \),

(a) there is a function \( h^-_x \equiv h^-_x(\omega) \),
\[
h^-_x : B^n(r^\omega(\hat{h}^{(0)}_x(\omega)), 1) \to \mathbb{R}^\alpha \quad \text{with} \quad \text{Lip}(h^-_x), \text{Lip}(dh^-_x) < K_\epsilon ,
\]

such that if \( \mathcal{F}^n(x) \) is the leaf of \( \mathcal{F}^n \) containing \( x \), then
\[
\hat{h}^{(0)}_x(\omega) \subset \mathcal{F}^n(x) ;
\]

(b) \( \mathcal{F}^{(0)}_{(\omega \mid \theta^{-\omega})^+} B(r^{(0)}_x(\omega), 1) \subset \{ \psi \geq \alpha_0/2 \} .
\]

**Proof.** First we require \( \epsilon \) to be small enough that \( \mathcal{N}_\epsilon \{ \psi \geq \alpha_0 \} \subset \{ \psi \geq \alpha_0/2 \} \), where \( \mathcal{N}_\epsilon \) denotes the \( \epsilon \)-neighborhood of a set (recall that the density \( \psi \) of \( \mu \) was assumed continuous—see section 2). It follows that \( B(\hat{p} \ldots \hat{p}, 2\epsilon) \subset \{ \psi \geq \alpha_0/2 \} .
\]

To check (a) and (b) for \( n = n_\omega \), observe that by lemma 6, \( x \mapsto E^n_{(\omega \mid \theta^{-\omega})^+} \) varies continuously for \( (x, (\theta^{-\omega})^+) \in \Gamma^n_\omega \), and by remark 16(a) we have \( \| 1_{(\omega \mid \theta^{-\omega})^+} \| \leq l_0 \). This means that by choosing \( \epsilon \) sufficiently small to align nearby \( E^n \)-subspaces, we are assured that there is uniform separation between \( E^n_{(\omega \mid \theta^{-\omega})^+} \), i.e. \( \mathcal{F}^n(x) \), and \( E^n_{(\omega \mid \theta^{-\omega})^+} \) for all
\( x \in \Lambda^a \). This separation provides a constant \( K_\infty \) in (a). That is, if \( h^-_k \) is chosen to satisfy
\[ \Phi_{x, (\theta^- n \omega)^+} \left( \text{graph } h^-_k \right) \subset F^n_\infty (x), \text{ then Lip}(h^-_k), \text{Lip}(dh^-_k) < K_\infty. \]
We shrink \( r_- \) as needed to ensure that \( \Phi_{x, (\theta^- n \omega)^+} \left( \text{graph } h^-_k \right) \subset B(\hat{r}_-, 2\epsilon) \), hence (b) holds.

Now the constants above need to work for all \( n \), and not be chosen one \( n = n_i \) at a time. This
requires that the modulus of continuity of \( x \rightarrow E^*_x(\theta^- n \omega)^+ \) be independent of \( (\theta^- n \omega)^+ \), which
is true because \( \langle x, (\theta^- n \omega)^+ \rangle \) is contained in the compact set \( \Theta^n_+ \subset \Theta^n_0 \).

Returning to the measure \( \mu_\infty \), and continuing to confuse a neighborhood of \( M \) with a subset
of \( T_\infty M \), we have that \( \left( \frac{\epsilon}{2} \text{Leb}_{B_{\hat{r}_-}(\theta^-)} \right) \leq \mu_\infty \), and the conditional densities of \( \frac{\epsilon}{2} \text{Leb}_{B_{\hat{r}_-}(\theta^-)} \)
the leaves of \( F^n_\infty \) are constant functions.

5. Geometry of pushed-forward \( u \)-graphs

We have set up in section 4 a situation that can be described as follows: for each \( \omega \) in a positive
\( P \)-measure set, there is a sequence \( n_i \) such that for each \( n = n_i \), there is a set \( \Lambda^n_\omega \subset M \), and a
collection of \( u \)-graphs associated with \( x \in \Lambda^n_\omega \) that together carry positive \( \mu \)-measure. These
\( u \)-graphs are to be transported forward by \( f^n_\omega \). We consider small pieces of these \( u \)-graphs
that remain inside suitable Lyapunov charts for all \( n \) steps, and refer to the images at the end
as \( W^n \)-leaves. In this section, we will focus on the geometry of the \( W^n \) leaves and the manner to
which they converge to (real) unstable manifolds of the RDS. We will begin by making precise
the order of the various choices of constants and constructions.

5.1. Stacks of \( W^n \)-leaves: details of construction

We now bring together the following three sets of ingredients we have prepared: the setup in
section 4 in which we accumulate certain sets of points with controlled finite pasts (lemmas 23
and 24), graph transforms for ‘slanted’ graphs developed in sections 3.2 and 3.3 (proposition
18), and the switching of axes and consolidation of images onto stacks (lemma 10).

(A) Initial choices. To start, fix a small \( \beta_0 > 0 \) and let \( \beta_0 > 1 \) be such that
\( \mu^* \{ \beta \leq \beta_0 \} \geq 1 - \beta_0/3. \) With \( \Theta_0, \Theta_\infty \subset G \) as in the beginning of section 4.2, let \( G \) be as in
lemma 22 with \( \epsilon = 2 \) so that \( \mathbb{P}(G) \geq 1/2. \) In all that follows, \( \omega \in G \) is fixed. As previously, we
write \( M_\omega = \{ x \in M : (x, \omega, \theta_0) \in \Theta_0 \} \) and, for \( n \geq 0, M^n_\omega = \{ x \in M : (x_{-n}, (\theta^- n \omega)^+ \in \Theta^n_\omega \}. \)

(B) Choices of \( \epsilon_\infty, r_- \), source and target sets, and a lower bound for \( \{ n_i \}. \) Our aim by the end of
part (B) is to have constructed the following objects:

(a) ‘source’ and ‘target’ sets \( B(\hat{r}_-, \epsilon_\infty), B(\hat{r}_-, \epsilon_\infty) \) (as in lemma 23);
(b) a reference point \( x_\infty \in \Lambda_\infty \cap B(\hat{r}_-, \epsilon_\infty) \) and a reference box \( E^*_\infty(r_-) = E^*_\infty(r_\infty), \)
\( E^\infty_{\epsilon_\infty} = E^\infty_{\epsilon_\infty} \) suitable for constructing stacks of (i) \( W^n \)-leaves (as in lemma 11) and (ii)
appropriately truncated, pushed-forward \( u \)-graphs (called \( W^n \)-leaves) through the target
set \( B(\hat{r}_-, \epsilon_\infty) \) (see figure 1).

The main work in constructing (a) and (b) is to identify the parameters \( \epsilon_\infty, r_- \), which we undertake
now, starting with \( r_- \).

For (b)(i), lemma 11 requires that we take \( r_- \), sufficiently small in terms of \( \epsilon_\infty, \delta. \) For (b)(ii),
to each \( x \in \Lambda^n_\omega \) is associated a graph-transform-image (in the sense of section 3) of a \( u \)-graph
at \( x_{-n} := f^{\infty n} x \) (to be made precise in (C) below). The image, what we call a \( W^n \)-leaf, will be
a graph defined on the \( \left( E^\infty_{\epsilon_\infty} \right) \times E^\infty_{\epsilon_\infty} \)-axes in \( T_\infty M \). We seek to switch axes to
a common reference box $E_*(r_*) = E_*^n(r_*) \times E_*^u(r_*)$ centered at a reference point $x_*$ (to be determined). Taking $r_* \leq \frac{1}{2}r_j$ with $r_j = r_j(l_0)$ as in lemma 20 ensures that truncations of $W^u$ leaves will have small-enough $\text{Lip}$ constants for the purposes of lemma 10(iii), provided that the $W^u$-leaves are sufficiently parallel to $E^{u,n}_*(\xi)$ (see end of (C)). This completely fixes the value of $r_*$.  

We now identify two sets of conditions on $\epsilon_*$, to be used in the construction of the ‘source’ and ‘target’ sets. At the source set, lemma 24 imposes two conditions on $\epsilon_*$: one is that it has to be small enough so that $E^u$ is sufficiently well-aligned through points of $\int_{\lambda_*} W^u_{\lambda}$, when restricted to a ball of radius $\epsilon_*$; this is needed to guarantee the separation of $E^u$ and $E^\theta_\cdot$ in the sense of remark 16(a). The other is that the entire $2\epsilon_*$-ball should be contained in $\{\psi \geq \alpha_0/2\}$ (lemma 24(b)).  

At the target set we require $\epsilon_*$ be suitable for constructing the stacks of both $W^u$ and $W^s$ leaves. Both require that $\epsilon_*$ be small enough in terms of $l_0, \delta$ and $r_*$ (lemmas 10 and 11). Additionally, for the $W^u$ stack we need to make the $E^{u,n}_\lambda, E^{u,*}_\lambda$-axes at $x \in \Lambda_*$ line up with the $E^u_{\lambda,\epsilon_*}$ axes at the reference point $x_*$. The $E^u$-axes are aligned by shrinking $\epsilon_*$ (proposition 6(a)); to align the $E^u$ with $E^u_*$ requires shrinking $\epsilon_*$ and taking $\min\{n_i\}$ sufficiently large (proposition 6(b) and remark 16(b)).  

These are our requirements on $\epsilon_*$ and $r_*$. With $\epsilon_*$ determined, we are correctly situated to apply lemma 23 with $\epsilon = \epsilon_*$, fixing once and for all the ‘source’ and ‘target’ regions $B(\hat{p}_-, \epsilon_*)_s$, $B(\hat{p}_*, \epsilon_*)_t$ respectively, and the potentially viable subsequence $n_i$ along which we have the bound $\mu_{\lambda_i}^u(\Lambda^u_n) = \mu(\Lambda^u_n) \geq c_*$ for $n = n_i$; here $c_* := c(\epsilon_*)$ is as in lemma 23 and $\Lambda^u_\epsilon, \Lambda^u_{\epsilon_*}$

Figure 1. On the left-hand side is the foliation $\mathcal{F}_n$ through the ball $B(\hat{p}_-, \epsilon)$ (the ‘source set’). Surrounding each $x \in \Lambda^u_\epsilon$ is the ‘chart box’ $\hat{B}(\epsilon^0_{(\theta_-, \omega)})(r_{(\hat{B}(\epsilon^0_{(\theta_-, \omega)}))^{-1}})$ (parallelogram); the bolded portion of the $\mathcal{F}_n$-leaf through each $x$ is a $u$-graph represented by the graphing function $h_*^-\epsilon$ as in lemma 24. Note that the chart boxes at $x \in \Lambda^u_\epsilon$ are not well-aligned with each other, and the functions $h_*^-\epsilon$ may have large slopes. On the right side (the ‘target set’) is a rectangle of the form $E_*^n(r) \times E_*^u(r)$ centered at a reference point $x_* \in B(\hat{p}, \epsilon)$. For $n$ large enough, $f^n_{\theta_*, \epsilon_*}$ maps portions of the little pieces of $u$-graphs on the left across this box, intersecting it in sets that are graphs of functions from $E_*^n(r)$ to $E_*^u(r)$ as shown.
are the ‘source’ and ‘target’ sets as in (5). Finally, we fix an arbitrary point \( x_* \in M_{\omega} \cap B(\hat{p}, \epsilon_*) \) to be used as reference point; that \( M_{\omega} \cap B(\hat{p}, \epsilon_*) \neq \emptyset \) is guaranteed by lemma 23. This completes the construction of \( E_*(r_*) \).

Further conditions will be imposed on the lower bound for \( \{n_i\} \).

(C) Pushing forward \( E^n \)-leaves. For each \( x \in \Lambda^n \), we let \( x_{-n} = f_{\omega}^{-n} x \in \Lambda^n \), and push forward the graph of \( h_{x_-}^n \) (here \( x_\cdot \) plays the role of \( r \) in lemma 24) by applying proposition 18. Letting \( K_0 = K_- = K_\cdot(\delta_b) \) and \( r_- = r_\cdot(\delta_b) \) be as in lemma 24, and \( h_0 = h_\cdot(\delta_b) \) be as in proposition 18, we set \( n_0 = \min\{r_0, r_-\} \). Then for all \( n = n_0 + m_1 \) where \( m_0, m_1 \) are as in proposition 18 (depending on \( r_-, K_- \)), the graph transform \( h_n := h^n_{r_\cdot(\delta_b)} = \mathcal{T}(n-1) \circ \cdots \circ \mathcal{T}(0) h_{r_-, x} \) is defined on all of \( B^n(h_0^n(\delta_{n-1}(\theta^-\omega_\cdot)))^{-1} \) and satisfies

\[
\|dh_n\| \leq K \cdot e^{-n\lambda/2}, \quad \text{and} \quad \text{Lip}(dh_n) \leq C \cdot (\xi_n(\theta^-\omega_\cdot))^{-1},
\]

where \( C \) is as in proposition 18. We define

\[
W_n^{\omega} = \Phi(n(\cdot, \theta^-\omega_\cdot)) \text{ graph } h_n
\]

and let \( \hat{h}_n : \text{Dom}(\hat{h}_n) \subset W_n^{\omega} \to E^n_{(r_\cdot(\theta^-\omega_\cdot))} \) be its mapping graph. To perform the switching of axes to \( E^n_{(r_\cdot(\theta^-\omega_\cdot))} \) as discussed in Paragraph (B), we guarantee \( \|dh_n\| \leq \frac{1}{20}(\text{lemma 20}) \) by taking \( \min\{n_i\} \) sufficiently large so that \( \|dh_n\| \leq (20n_0)^{-1} \).

(D) Collecting onto stacks. Let

\[
\Lambda := B(\hat{p}, \epsilon_*) \cap M_{\omega}
\]

where \( \hat{p} \in M \) is as in (B). Applying lemma 11, we let \( \mathcal{S} = \cup_{x \in \Lambda} \xi(x) \) be the stack of \( W_n^{\omega} \)-leaves and \( \Xi \) the partition of \( \mathcal{S} \) into \( \xi(x) \), where \( \xi(x) \), the \( W_n^{\omega} \) leaf through \( x \in \Lambda \), has the form \( \xi(x) = \exp_{x_n} \text{ graph } \gamma_n x \) for some \( \gamma_n : E^n_{(r_\cdot(\theta^-\omega_\cdot))} \to E^n_{(r_\cdot(\theta^-\omega_\cdot))} \) with \( \text{Lip}(\gamma_n) \leq 1 \).

For \( n = n_0 \) sufficiently large, we now define the corresponding stack \( S^n \):

**Lemma 25.** There exists \( N_* \in \mathbb{N} \), depending on all the parameters above, such that the following holds for all \( n = n_0 + N_* \).

(a) For each \( x \in \Lambda^n \), we have that the connected component of \( W_n^{\omega} \cap \exp_{x_n} E_*(r_*) \) containing \( x \) coincides with \( \exp_{x_n} \text{ graph } \gamma_n x \), where \( \gamma_n : E^n_{(r_\cdot(\theta^-\omega_\cdot))} \to E^n_{(r_\cdot(\theta^-\omega_\cdot))} \) is a \( C^{1+\text{Lip}} \) mapping with \( \text{Lip}(\gamma_n) \leq 1 \).

(b) The partition \( \Xi^n \) of \( S^n \) is \( \cup_{x \in \Lambda^n} \xi^n(x) := \exp_{x_n} \text{ graph } \gamma_n x \) is measurable.

**Proof of lemma 25.** For (a) we apply lemma 10 to switch the axes of \( W_n^{\omega} = \exp_{x_n} \text{ graph } h_n \) to the common axes \( E^n_{(r_\cdot(\theta^-\omega_\cdot))} \), \( E^n_{(r_\cdot(\theta^-\omega_\cdot))} \), having already verified conditions (i)–(iii) in lemma 10 in paragraphs (B) and (C).

For (b), measurability of \( \Xi^n \) follows from the fact that \( S^n \) is the union of at-most finitely many sets of the form

\[
\mathcal{S}_x^n := \left( \bigcup_{z \in \Lambda^n \cap f_{\theta^-\omega_\cdot}^{-1} B(z_{-n} \cdot, \epsilon_n)} W_n^{\omega} \right) \cap \exp_{x_n} \left( E^n_{*(r_*)} \right), \quad x \in \Lambda^n,
\]

where \( \epsilon_n = \epsilon_n(x) \) is chosen so that \( z \mapsto \gamma^n_z \) varies continuously over \( z \in \Lambda^n \cap f_{\theta^-\omega_\cdot}^{-1} B(z_{-n} \cdot, \epsilon_n) \).

\[\square\]
5.2. Limiting properties of $W^n$-leaves

Now that we have grouped nearby $W^n$ and truncated $W^n$ leaves into ‘stacks’ $\mathcal{S}$ and $\mathcal{S}^n$, we turn our attention to the limiting properties of $\mathcal{S}^n$ and its relation to $\mathcal{S}$. Recall that for $x \in \Lambda^n$ and $y \in \Lambda$, $\gamma^\omega_n, \gamma^\omega_{\nu} : E^\omega_n(r_x) \to E^\omega_n(r_x)$ are the graphing functions of the leaves of $\mathcal{S}^n$ and $\mathcal{S}$ through $x$ and $y$ respectively.

**Proposition 26.** For any $\epsilon > 0$, there exist $\bar{n}_0 = \bar{n}_0(\epsilon) \geq N_\omega$ and $\bar{\eta} = \bar{\eta}(\epsilon) > 0$ with the following property. For any $n = n_\epsilon \geq \bar{n}_0$ and any $x \in \Lambda^n, y \in \Lambda$ with $d(x, y) < \bar{\eta}$, we have that $\|\gamma^\omega_{\nu} - \gamma^\omega_n\| < \epsilon$ where $\|\cdot\|$ refers to the uniform norm on $C(E^\omega_n(r_x), E^\omega_n(r_x))$.

It follows that $\limsup_{n \to \infty} \mathcal{S}^n \subset \mathcal{S}$. The statement of proposition 26 is all that we need; we do not prove, nor use, the continuity of $\gamma^\omega_{\nu}$. However, it follows from a version of the arguments below that ‘oscillations’ in the Hausdorff distance between nearby $\xi^n$-leaves can be made uniformly, arbitrarily small for all sufficiently large $n$. Indeed, the following is a modification of a standard argument for proving the continuity of actual $W^n$-leaves (see, e.g. section 5 in [2]).

**Proof of proposition 26.** In this proof, we will assume as before a canonical identification of the tangent spaces at $x$ and $y$, which are very close. Also, we will, for simplicity, use the notation of two-sided charts, assuming $(x, \omega^\ell)$ and $(y, \omega^\ell)$ are such that $\omega_i = \omega^\ell_i$ for all $i > -n$ where $n$ is as in the proposition. No relation between $\omega_i$ and $\omega^\ell_i$ for $i \leq -n$ is assumed, as that will depend on the selection lemma.

**Plan of proof.** Let $0 : \mathbb{R}^n \to \mathbb{R}^\ell$ denote the zero function. We consider $0$ as a function in the chart at $\tau^{-k}(y, \omega)$ for some $k \ll \bar{n}_0$ (both $k$ and $\bar{n}_0$ to be determined), and let $0^k_i : B^\omega_i(\delta^2 L_\omega^{-1}) \to \mathbb{R}^\ell$ be given by $0^k_i := \tau^{-1}(\omega, y) \circ \cdots \circ \tau^{-1}(v_x) 0$ (where $\tau^{-1}$ is the graph transform as in section 2.3). Likewise, we consider $0$ as a function in the chart at $\tau^{-k}(x, \omega^\ell)$, and let $0^{\omega^\ell}_i : B^\omega_i(\delta^2 L_\omega^{-1}) \to \mathbb{R}^\ell$ be given by $0^{\omega^\ell}_i := \tau^{-1}(\omega^\ell, y) \circ \cdots \circ \tau^{-1}(v_x) 0$. Leaving it to the reader to check that the switching of axes (lemma 10) can be performed, we obtain two functions $\gamma^\omega_{\nu} \circ \gamma^\omega_{\nu}^{\omega^\ell} : E^\omega_n(r_x) \to E^\omega_n(r_x)$ whose graphs are contained in $\exp_x(\text{graph } 0^k_i)$ and $\exp_x(\text{graph } 0^{\omega^\ell}_i)$ respectively. We will bound $\|\gamma^\omega_{\nu} - \gamma^\omega_i\|$ via the triangle inequality

$$\|\gamma^\omega_{\nu} - \gamma^\omega_i\| \leq \|\gamma^\omega_{\nu} - \gamma^\omega_{\nu}^{\omega^\ell}\| + \|\gamma^\omega_{\nu}^{\omega^\ell} - \gamma^\omega_{\nu}^{\omega^\ell}\| + \|\gamma^\omega_{\nu}^{\omega^\ell} - \gamma^\omega_{\nu}\|. \tag{7}$$

For given $\epsilon > 0$, to prove $\|\gamma^\omega_{\nu} - \gamma^\omega_i\| < \epsilon$ for all $n \geq \bar{n}_0$, we plan to first choose $k = k(\epsilon, \ell_0)$ and then $\bar{n}_0 = \bar{n}_0(k, \epsilon, \ell_0)$.

We isolate below another ‘change-of-chart’ type estimate that will be used several times in the proof of (7). The proof is straightforward and left to the reader.

**Lemma 27.** Let $g_1, g_2 : B^\omega(\delta(\nu, \omega)^{-1}) \to \mathbb{R}^\ell$ be Lipschitz graphing maps in the chart at $(y, \omega)$. For $i = 1, 2$, let $g_i := (L_{\nu, \omega} \circ g_i \circ L_{\nu, \omega}^{-1})|_{E^\omega_n(r_x)}$ and assume that graph $g_i \subset E^\omega_n(2r_x) \times E^\omega_n(\frac{1}{2} r_x)$ with $\text{Lip}(g_i) \leq 1/10$. Let $\gamma_i : E^\omega_n(r_x) \to E^\omega_n$ be such that $\exp_x(\text{graph } \gamma_i) \subset \exp_x(\text{graph } g_i) \cap \exp_x(\text{graph } g_i)$. Then $\text{Lip}(\gamma_i) \leq 1/5$, and

$$\|\gamma_1 - \gamma_2\| \leq \mathcal{C}|g_1 - g_2|, \tag{8}$$

where $\mathcal{C} = \mathcal{C}(\ell_0) > 0$. The same holds when $(y, \omega)$ is replaced by $(x, \omega^\ell)$.

**First and third terms in (7):** we use the contraction estimate in lemma 8 to obtain

$$|0^k_i - g(x, \omega^\ell)| \leq c^k$$

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where $c$ is as in lemma 8 and $g_{(y, w)}$ is the graphing map of the unstable manifold in the chart at $(y, w)$. By lemma 27,
\[
\|\tilde{w}^k - \gamma_y\| \leq C|0_x^k - g_{(y, w)}| \leq C \epsilon^k.
\]
We require $k$ to be large enough that $C \epsilon^k < \epsilon/3$.

The first term on the right side of (7), $\|\gamma_y^0 - \tilde{w}^k\|$, is treated similarly, provided that $n - k$ is large enough that in the chart at $\tau^{-k}(x, w')$, $\text{Lip}(T_{\tau^{-k}(y, w')}) \circ \cdots \circ T_{\tau^{-k}(x, w')}h_c \leq L/10$.

We require $\bar{n}_0 \geq k + m_0 + m_1$ where $m_0, m_1$ are as in proposition 18 and depend on $r_-$ and $K$.

Let $k = k(\epsilon)$ be fixed from here on.

Second term in (7). Given $0 < \tilde{\epsilon} < 1$ to be determined, we claim that for $\tilde{y}$ small enough and $\bar{n}_0$ large enough depending on $l_0, k$, and $\tilde{\epsilon}$, the following hold for $x, y$ with $d(x, y) < \tilde{y}$:

(a) $d(f^{-k}_x x, f^{-k}_y y), \quad d_H(E_{\tau^{-k}(x, w')}, E_{\tau^{-k}(y, w')}) < \tilde{\epsilon}$, and

(b) $\|f^{-k}_x x \in \Phi_{\tau^{-k}(y, w)}B(1, r_1(\tau^{-k}(y, w)^{-1})^2)$ for all $0 \leq i \leq k$.

Item (b) and $d(f^{-k}_x x, f^{-k}_y y) < \tilde{\epsilon}$ follow from the fact that $\text{Lip}(f^{-k}_x) \leq \|df^{-k}_x\| \leq l_0^2 i(k+1)^{1/2}$ and $l(\tau^{-k}(y, w)) \leq l_0^2 \epsilon^k$, and that both bounds depend on $l_0$ and $k$ alone. To control $d_H(E_{\tau^{-k}(x, w')}, E_{\tau^{-k}(y, w')})$, we apply proposition 6 to $\tau^{-k}(y, w), \tau^{-k}(x, w') \in \{l \leq l_0^2 \epsilon^k\}$, and require that $\bar{n}_0 \geq n_0 + k$, where $n_0 = n_0(\bar{\epsilon}, l_0^2 \epsilon^k)$ is as in proposition 6.

Now let $0^i_y : B^c(\delta l(\tau^{-k}(y, w)^{-1})^{-1}) \to \mathbb{R}^c$ be the function whose graph is the component of $(\Phi_{\tau^{-k}(y, w)})^{-1}(\exp^{-1}_{\tau^{-k}(y, w)})$ in $B(\delta l(\tau^{-k}(y, w)^{-1})^{-1})$ containing $(\Phi_{\tau^{-k}(y, w)})^{-1}f^{-k}_x x$.

By choosing $\tilde{\epsilon}$ sufficiently small, we may assume, by item (a) above, that $\text{Lip}(0^i_y) \leq L/10$, $\text{Lip}(d_{0^i_y}) \leq 1$, and $|0 - 0^i_y|$ is as small as we wish. This together with item (b) permits us to apply lemma 8(b) to ensure that the graph transform
\[
0^{i, kn}_y : = T_{\tau^{-k}(y, w)} \circ \cdots \circ T_{\tau^{-i}(y, w)}, 0^i_y : B^c(\delta l(\tau^{-k}(y, w)^{-1})^{-1}) \to \mathbb{R}^c
\]
is well defined. Moreover, with the modulus of continuity of $T_{\tau^{-k}(y, w)}, 1 \leq i \leq k$, depending only on $l_0^2 \epsilon^k$, we may choose $\tilde{\epsilon}$ sufficiently small to guarantee that $|0^{i, kn}_y - 0^i_y| \leq \epsilon/k$ where $\epsilon$ is as in (7) and $C$ is as in lemma 27.

The $\exp_{\tau^{-k}(y, w)}^{-1} \Phi_{\tau^{-k}(y, w)}$-images of the graphs of $0^{i, kn}_y$ and $0^i_y$, when restricted to $E_x(r_\alpha)$ are precisely the graphs of $\tilde{w}^k_x$ and $\tilde{w}^k_y$ respectively. Another application of lemma 27 gives
\[
\|\tilde{w}^k_x - \tilde{w}^k_y\| \leq \epsilon/k.
\]

For each $w \in \mathcal{G}$, the constructions of section 5 are fixed for the remainder of the paper.

6. Proof of SRB property

We now complete the proof of the main proposition (proposition 12).

6.1. Construction of partitions respecting unstable manifolds

Let $S$ be as in section 5.1, paragraph (D), i.e. $S$ is a stack of local unstable manifolds through points in $\Lambda$, with $\Xi$ denoting the partition into unstable leaves. To capture the conditional
measures on $\Xi$ of any measure $\nu$ supported on $S$, a standard procedure is to construct a sequence of finite partitions $\alpha_1, \alpha_2, \cdots$ of $S$ with the following properties:

(a) The sequence $\{\alpha_m\}$ is increasing, i.e. $\alpha_{m+1} \supseteq \alpha_m$ for all $m \geq 1$.
(b) For each $m$, we have $\alpha_m \subseteq \Xi$, i.e. $\alpha_m$ consists of intact $\xi$-leaves.
(c) $\bigvee_{m=1}^{\infty} \alpha_m \equiv \Xi$, where $\equiv$ denotes equivalence mod 0 with respect to $\nu$.

Then properties of the conditional measures of $\nu$ on $\Xi$ can be deduced from its conditional measures on $\alpha_m$ as $m \to \infty$.

Complicating matters in our setting is that the measure of interest is the limit of a sequence of measures that are not supported on $S$ but on nearby stacks $S^n$ of $W^n$ leaves; see section 5. To accommodate these approximating measures, we will construct partitions similar to the following inductive procedure. Fix an increasing sequence $\{\alpha_m\}$.

Continuing to use notation from section 5, we define $\tilde{\mu}_n = \mu_n|_{\alpha_n}$. On refining the sequence $\{n_i\}$, let us assume that $\tilde{\mu}_n$ converges weakly as $n \to \infty$ to a measure $\tilde{\mu}_\omega$. Note that $\tilde{\mu}_\omega \leq \mu_\omega$, and that $\mu_\omega$ is supported on $\Lambda$ (by lemma 21), with $\tilde{\mu}_\omega(\Lambda) \geq c_\omega > 0$ (lemma 23).

Lemma 28. There is a decreasing sequence of compact subsets $\Delta_1 \supset \Delta_2 \supset \cdots$ of $\Lambda$ with the following properties:

(i) Each $\Delta_m$ is partitioned into disjoint compact subsets $\{\Delta_{m,k}, 1 \leq k \leq K_m\}$ and the $\{\Delta_{m,k}\}$ are nested in the sense that each $\Delta_{m+1,k} \subseteq \Delta_{m,k'}$ for some $k'$.
(ii) The sets $S_{m,k} := \bigcup_{x \in \Delta_{m,k}} \xi(x)$ are compact and pairwise disjoint among $1 \leq k \leq K_m$.
(iii) We have

$$\lim_{m \to \infty} \max_{1 \leq k \leq K_m} \text{diam}^{\nu}(S_{m,k}) = 0.$$

(iv) Defining $\Delta_\infty = \bigcap_{m \geq 1} \Delta_m$, we have $\tilde{\mu}_\omega(\Delta_\infty) \geq \frac{1}{2} c_\omega$.

Proof. For ease of notation, in the following proof, let us suppress the $\omega$ and write $\tilde{\mu} := \tilde{\mu}_\omega$.

Define $\Sigma = \bigcup_{x \in S} E_\omega^x$, which as is easily checked is a transversal to the $\xi$-leaves comprising $S$. Set $\tilde{\Sigma} = \Sigma \cap S$ and let $\pi : S \to \tilde{\Sigma}$ denote the projection along $\xi$-leaves. Project $\tilde{\mu}$ to its transverse measure $\tilde{\mu}^T = \tilde{\mu}_\omega^T$ on $\tilde{\Sigma}$. For each $m \geq 1$, let $Q_m$ be a partition of $\Sigma$ into cubes of side lengths $\approx 1/2^m$ with the following properties:

(1) The sequence $Q_m, m \geq 1$ is increasing, i.e. $Q_{m+1} \supseteq Q_m$ for each $m \geq 1$.
(2) We have $\bigvee_{m=1}^{\infty} Q_m \equiv \varepsilon$, the partition of $\Sigma$ into points $\tilde{\mu}^T$-mod 0.
(3) For each $C \in Q_m$, we have $\tilde{\mu}^T(\partial C) = 0$.

With the $Q_m$ fixed, we define finite collections $Q_m, m \geq 1$, of disjoint compact sets via the following inductive procedure. Fix an increasing sequence $c_1 \leq c_2 \leq \cdots < 1$ for which $\prod_{m=1}^{\infty} c_m = \frac{1}{2}$. To start, for each $C \in Q_1$ fix a compact subset $\tilde{C} \subseteq C \cap \pi(\Lambda)$ for which $\text{dist}(C, \partial C) > 0$ and $\tilde{\mu}(C) \geq c_1 \tilde{\mu}(C)$. We set $Q_1 = \{\tilde{C} : C \in Q_1\}$, so that

$$\tilde{\mu}^T \bigcup_{C \in Q_1} \tilde{C} \geq c_1 \tilde{\mu}^T(\tilde{\Sigma}) \geq c_1 c_\omega.$$
We construct successively $Q_1, Q_2, \ldots$ of disjoint compact subsets with the rule that

(i) for each $C \subseteq Q_m$, we have that $C \subset C'$ for some $C' \subseteq Q_{m-1}$, and
(ii) for each $C' \subseteq Q_{m-1}$, we have $\hat{\mu}^T((\cup_{C \subseteq Q_m, C \subset C'} C)) \geq c_m \hat{\mu}^T(C')$.

With the $\{Q_m\}$ in hand, we now define the array of compact sets $\Delta_{m,k}$ as follows: for each $m \geq 1$ and $1 \leq k \leq K_m := |Q_m|$, we define $\Delta_{m,1, \ldots, \Delta_{m,K_m}}$ to be the collection of sets of the form $\Delta \cap \pi^{-1}(C)$ as $C$ ranges over $Q_m$.

Item (i)–(iv) follow immediately.

What we have done in lemma 28 is to group the unstable leaves in $S$ into finer and finer substacks with a Cantor-like structure transversally, and to do that, we have had to give up on a little bit of $\hat{\mu}_\omega$-measure. Let

$$ S_\infty := \bigcup_{x \in \Delta_\infty} \xi(x). $$

**Corollary 29.** There is a decreasing sequence of open sets $U_1 \supseteq U_2 \supseteq \ldots$ with $\cap_m U_m = S_\infty$ and a sequence of partitions $\beta_m = \{\beta_{m,k}\}$ of $U_m$ into finitely many disjoint open sets, with the properties that

(i) the partitions $\beta_m$ are nested, i.e. each $\beta_{m,k} \subset \beta_{m-1,k'}$ for some $k'$;
(ii) each $\beta_{m,k}$ contains intact leaves of the compact substack $S_{m,k}$, and
(iii) for each $\xi$ in $S_\infty$ if $\beta_{m,k}(\xi)$ is the element of $\beta_m$ containing $\xi$, then $\beta_{m,k}(\xi) \downarrow \xi$ as $m \to \infty$.

Corollary 29 follows easily from lemma 28. The sets $\{\beta_{m,k}\}$ can be chosen quite arbitrarily as long as they have the stated properties.

### 6.2. Pushed forward measures and their conditional densities

Recall that for each $n = n_i$, we have constructed a stack $S^n$ and a partition of $S^n$ into sets $\xi^n(\cdot)$ that are approximate $W^n$-leaves (lemma 25). The next lemma establishes that for each $m$, by taking $n$ large enough, the partition $\beta_m$ will respect a definite fraction of $\xi^n$-leaves. Let $N_\eta(\cdot)$ denote the $\eta$-neighborhood of a set.

**Lemma 30.** For each $m \geq 1$, there exist $\eta_m > 0$ and $N_m \in \mathbb{N}$ with the property that the following hold for all $n = n_i \geq N_m$.

(a) Define

$$ \Lambda_{m,k} := \Lambda_n \cap N_{\eta_m}(\Delta_{m,k}), \quad S_{m,k} := \bigcup_{x \in \Lambda_{m,k}} \xi^n(x). $$

Then $S_{m,k} \subset \beta_{m,k}$.
(b) Letting $\Lambda_m = \bigcup_k \Lambda_{m,k}$, we have

$$ \hat{\mu}_\omega(\Lambda_{m,k}) \geq \frac{2}{3} \hat{\mu}_\omega(\Lambda_{m,k}), \quad \text{hence} \quad \hat{\mu}_\omega(\Lambda_m) \geq \frac{1}{3} \mu_\omega. $$

**Proof.** (a) follows immediately from proposition 26 and (b) from the fact that $\hat{\mu}_\omega$ is assumed to converge to $\hat{\mu}_\omega$. \qed
While we have used $\mu^n_m|_{S^n_m}$ to ensure that our partitions are catching a definite fraction of $\mu^n_m$, we are primarily interested in $\mu^n_m|_{S^n_m}$, where $S^n_m = \cup_k S^n_{m_k}$. We now turn our attention to $\mu^n_m|_{S^n_m}$, focusing on a part of this measure with controlled conditional densities.

Recall from section 4.3 that we disintegrate $\mu$ on the leaves of a foliation $\mathcal{F}^n_n$ to be carried forward by $f^n_n$, and that $\mathcal{F}^n_n$ is defined on a ball $B \subseteq \{ \psi \ge \alpha_0/2 \}$, where $\psi = \frac{d\mu}{d\text{Leb}}$. Define

$$\nu^n_m := \frac{\alpha_0}{2} \text{Leb}|_B, \quad \text{and} \quad \nu^n := \left( f^n_n \right)^* \left( \mu^n_m \right)_{S^n_m}.$$

Since $\nu^n_m \leq \mu$, we have that $\nu^n \leq \mu^n_m$ for all $n$.

Furthermore, let $(\nu^n)_{\xi \in \Xi}$ denote the (normalized) disintegration measures of $\nu^n$ along $\Xi^n_n$ with transversal measure $\nu^n_\xi$ on $S^n_m / \Xi^n_n$. For $\xi \in \Xi^n_n$, let $\phi(\xi)$ denote the leaf of $\mathcal{F}^n_n$ containing $f^n_n$. It is easy to see that $\nu^n_\xi$ is $\left( f^n_n \right)^* \text{Leb} \phi(\xi)_\xi$ normalized.

**Lemma 31.**

(a) For measurable $C \subseteq \Lambda^n$, $\nu^n(C) \geq \frac{\alpha_0}{2\|\psi\|_{\infty}} \cdot \mu^n_m(C)$.

(b) For a.e. $\xi \in \Xi^n_n$, $\nu^n_\xi$ is absolutely continuous with density $\rho^n_\xi : \xi \to (0, \infty)$ satisfying the distortion estimate

$$\left| \log \frac{\rho^n_\xi(p_1)}{\rho^n_\xi(p_2)} \right| \leq D$$

for any $p_1, p_2 \in \xi$. Here $D = D(l_0, K_\omega, r_\omega) > 0$, where $K_\omega, r_\omega$ are as in lemma 24; in particular, $D$ is independent of $\xi$ and $n$.

**Proof.** The estimate in (a) follows from the simple bound $\mu^n_m(C) \leq \|\psi\|_{\infty} \text{Leb}(f^n_n C)$, and Item (b) follows from the distortion estimate in lemma 19 applied to $K_0 = K_\omega, r_0 = r_\omega$.

6.3. Passing to the weak limit as $n \to \infty$ and completing the proof

Let $\nu^n_m := \nu^n|_{S^n_m}$. From lemmas 30(b) and 31(a), we have that for every $m$,

$$\nu^n_m(S^n_m) \geq \frac{\alpha_0}{2\|\psi\|_{\infty}} c_\omega$$

for every $n = n_1 \geq N_m$. We fix such an $n(m)$ for each $m$; clearly $n(m) \to \infty$ as $m \to \infty$. Let $\nu^\star$ be any limit point of the sequence $\nu^n_m$ as $m \to \infty$. Then $\nu^\star$ is supported on $\mathcal{S}_\infty$ with $\nu^\star \leq \mu_\omega$. Moreover, the lower bound (9) passes to $\nu^\star(S_\infty)$. Let $\nu^n_\xi$ denote the conditional measures of $\nu^\star$ on the leaves $\xi \in \Xi$. To complete the proof of proposition 12, it suffices to show that for a.e. $\xi$, the measure $\nu^n_\xi$ is absolutely continuous.

For this, we state below a lemma that will be used to deduce properties of the conditional measures of $\nu^\star$ on leaves of $\Xi$ from those of $\nu^n_m$ on $\Xi^n_n$. First, we need some notation: let $C^\star \subseteq E^\star_n$ be a cube. We let $\text{Leb}(C^\star) = \text{Leb}(C^\star) / \text{Leb}(E^\star_n(r_n))$, and define

$$V_{C^\star} := \exp_{C^\star}(C^n + E^\star_n(r_n)).$$

Recall that $\nu^n_m|_{S^n_m} = \nu^n_m|_{\beta_{\omega}}$ for $n = n(m)$.

**Lemma 32.** There exists $A > 1$ such that for any $C^\star \subseteq E^\star_n(r_n)$, we have, for all large enough $m$ and $n = n(m)$:
\[
\frac{1}{A} \cdot \text{Leb}(C^w) \leq \frac{\nu_n^m(\beta_{m,k} \cap V_C^w)}{\nu_m^m(\beta_{m,k})} \leq A \cdot \text{Leb}(C^w). \tag{10}
\]

It follows that for a.e. \( \xi \in \Xi \), we have \( \pi^* u_{\xi} \ll \text{Leb} \) with \( d(\pi^* u_{\xi}) \in [\frac{1}{A} \cdot A, A] \), where \( \pi^* \) is projection onto \( E^w_\ast(r) \) along \( E^w_\ast \).

**Proof of lemma 32.** Let \( C^w \) be fixed. That (10) holds for each \( \nu_n^m \) follows from the fact that it holds for each \( \nu_n^m \) by lemma 31(b).

To pass these bounds to \( \nu_{\xi}^* \), we let \( m \) and \( k \) be fixed to begin with. Since

\[
\nu_{m'}^m(\beta_{m,k} \cap V_C^w) = \sum_{k': \beta_{m',k'} \subseteq \beta_{m,k}} \nu_{m'}^m(\beta_{m',k'} \cap V_C^w),
\]

and (10) holds for all \( m' \geq m \) and all \( k' \), it follows that for fixed \( \beta_{m,k} \), (10) holds with \( \nu_{m'}^m \) replaced by \( \nu_{m}^m \). Letting \( m' \to \infty \), we obtain that it holds with \( \nu_{\xi}^* \) in the place of \( \nu_{m}^m \).

Continuing to keep \( C^w \) fixed but letting \( m \to \infty \) and running through all \( \beta_{m,k} \in \beta_m \) for each \( m \), we obtain by corollary 29(iii) that for a.e. \( \xi \in \Xi \),

\[
\frac{1}{A} \cdot \text{Leb}(C^w) \leq \pi^* u_{\xi}(C^w) \leq A \cdot \text{Leb}(C^w).
\]

As cubes form a basis for the topology on \( E^w_\ast(r) \), the assertion follows. \( \square \)

**Appendix**

Below, we deduce lemma 14 from the following well-known theorem.

**Theorem A.1 (Kuratowski–Ryll–Nardzewski measurable selection theorem).** Let \((\Xi, M)\) be a measurable space, \( Z \) a Polish space, and let \( F : \Xi \to 2^Z \) be a set-valued mapping such that

- \( F(\xi) \) is closed and nonempty for each \( \xi \in \Xi \),
- for any open \( U \subset Z \), we have \( \{ \xi \in \Xi : F(\xi) \cap U \neq \emptyset \} \in M \).

Then, there exists a measurable map \( f : \Xi \to Z \) for which \( f(\xi) \in F(\xi) \) for all \( \xi \in \Xi \).

For an account of measurable selection theorems, see, e.g. the survey [18].

**Proof of lemma 14.** Let \( X, Y \) be Polish and let \( G \subset X \times Y \) be a compact subset, writing \( G_X \) for the projection of \( G \) onto \( X \). Applying theorem A.1 to \((\Xi, M) = (X, \text{Bor}(X)), Z = Y \) and \( F(x) := \{ y \in Y : (x, y) \in G \} \), it suffices to show that for any open \( U \subset Y \),

\[ V_U := \{ x \in X : F(x) \cap U \neq \emptyset \} \]

is a Borel measurable subset of \( X \).
For this, note that because $Y$ is Polish, we may represent $U$ as the countable union of closed sets $U_i$, so $U = \cup_i U_i$. Moreover, as one easily checks,

$$V_U = \bigcup_i V_{U_i}.$$  

It suffices to show that each $V_{U_i}$ is closed. For this, let $\{x^n\} \subset V_{U_i}$ be a sequence converging to a point $x \in X$. To show $x \in V_{U_i}$, fix for each $n$ an element $y^n \in F(x^n) \cap U_i$. By compactness of $G \cap (X \times U_i)$, it follows that a subsequence of $(x^n, y^n)$ converges to an element $(x^*, y^*)$ of $G \cap (X \times U_i)$. But $x = x^*$, hence $y^* \in F(x)$; since $y^* \in U_i$, it follows that $x \in V_{U_i}$. \hfill $\square$

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