Spaces of algebraic maps from real projective spaces into complex projective spaces

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Abstract. We study the homotopy types of spaces of algebraic (rational) maps from real projective spaces into complex projective spaces. We showed in [1] that in this setting the inclusion of the space of rational maps into the space of all continuous maps is a homotopy equivalence. In this paper we prove that the homotopy types of the terms of the natural ‘degree’ filtration approximate closer and closer the homotopy type of the space of continuous maps and obtain bounds that describe the closeness of the approximation in terms of the degree. Moreover, we compute low dimensional homotopy groups of these spaces. These results combined with those of [1] can be formulated as a single statement about $\mathbb{Z}/2$-equivariant homotopy equivalence between these spaces, where the $\mathbb{Z}/2$-action is induced by the complex conjugation. This generalizes a theorem of [7].

1. Introduction.

1.1. Summary of the contents. Let $M$ and $N$ be manifolds with some additional structure, e.g holomorphic, symplectic, real algebraic etc. The relation between the topology of the space of continuous maps preserving this structure and that of the space of all continuous maps has long been an object of study in several areas of topology and geometry. Early examples were provided by Gromov’s h-principle for holomorphic maps [6]. In these cases the manifolds are complex, the structure preserving maps are the holomorphic ones, and the spaces of holomorphic and continuous maps turn out to be homotopy equivalent. However, in many other cases, the space of structure preserving maps approximates, in some sense, the space of all continuous ones and becomes homotopy equivalent to it only after some kind of stabilization. A paradigmatic example of this type was given in a seminal paper of Segal [13], where the space of rational (or holomorphic) maps of a fixed degree from the Riemann sphere to a complex projective space was shown to approximate the space of all continuous maps in homotopy, with the approximation becoming better as the degree increases. Segal’s result was extended to a variety of other target spaces by various authors (e.g. [2]). Although it has been sometimes stated that these phenomena are inherently related to complex or at least symplectic structures, real analogs of Segal’s result were given in [13], [10], [7], [15]. In fact, Segal formulated the complex and real approximation theorems which he had
proved as a single statement involving equivariant equivalence, with respect to complex conjugation (see the remark after Proposition 1.4 of [13]). A similar idea was used in [7], Theorem 3.7. This theorem amounts to two equivariant ones, one of which is equivalent to (a stable version of) Segal’s equivariant one, while the other one is related to the ‘real version’ of Segal’s theorem proved in [10].

All the results mentioned above (except the ones involving Gromov’s h-principle), assume that the domain of the mappings is one dimensional (complex or real). It is natural to try to generalize them to the situation where the domain is higher dimensional. That such generalizations might be possible was first suggested by Segal (see the remark under Proposition 1.3 of [13]). A large step in this direction appeared to have been made when Mostovoy [11] showed that the homotopy types of spaces of holomorphic maps from $\mathbb{C}P^m$ to $\mathbb{C}P^n$ (for $m \leq n$) approximate the homotopy types of the spaces of continuous maps, with the approximation becoming better as the degree increases. Unfortunately Mostovoy’s published argument contains several gaps. A new version of the paper, currently only available from the author, appears to correct all the mistakes, with the main results remaining essentially unchanged. There are two major changes in the proofs. One is that the space $\text{Rat}_f(p,q)$ of $(p,q)$ maps from $\mathbb{C}P^m$ to $\mathbb{C}P^n$ that restrict to a fixed map $f$ on a fixed hyperplane, used in section 2 of the published article is replaced by the space $\text{Rat}_f(p,q)$ of pairs of $n+1$-tuples of polynomials in $m$ variables that produce these maps. In the published version of the article it is assumed that these two spaces are homotopy equivalent, which is clearly not the case. However, they are homotopy equivalent after stabilisation, both being equivalent to $\Omega^{2m}\mathbb{C}P^n$ - the space of continuous maps that restrict to $f$ on a fixed hyperplane. The second important change is the introduction of a new filtration on the simplicial resolution $X^\Delta \subset \mathbb{R}^N \times Y$ of a map $h : X \to Y$ and an embedding $i : X \to \mathbb{R}^N = \mathbb{C}^{N/2}$. This filtration is defined by means of complex skeleta (where the complex $k$-skeleton of a simplex in a complex affine space is the union of all its faces that are contained in complex affine subspaces of dimension at most $k$) and replaces the analogous “real” filtration in the arguments of section 4.

In [1] a variant of Mostovoy’s idea was applied to the case of algebraic maps from $\mathbb{R}P^m$ to $\mathbb{R}P^n$. This leads naturally to the question whether one can generalize the $\mathbb{Z}/2$-equivariant Theorem 3.7 of [7] to an analogous equivariant equivalence of the spaces of algebraic maps and continuous maps between projective spaces, in which the domain is either a real or a complex projective space of dimension $m > 1$, the range is a complex projective space of dimension greater or equal to that of the domain. (Here, the $\mathbb{Z}/2$-action is induced by complex conjugation.)

Note that Theorem 3.7 of [7] has two parts, in the first the domain being complex and in the second real. Clearly, to prove the first part we need a Mostovoy’s complex theorem. We plan to consider this problem in a future paper. Here we concentrate on the second part, concerning equivariant algebraic maps from real projective spaces to complex projective ones. Our main theorem is new, but it uses the main result of [1] and where our arguments are very similar to those used in that paper we omit their details and refer the reader to [1].

In the remainder of this section we introduce our notation and state the main definitions and theorems.
1.2. Notation and Main Results. We first introduce notation which is analogous to the one used in [H], the presence of \( \mathbb{C} \) indicating that the complex case is being considered (i.e. maps take values in \( \mathbb{C}^n \) or polynomials have coefficients in \( \mathbb{C} \)).

Let \( m \) and \( n \) be positive integers such that \( 1 \leq m < 2 \cdot (n + 1) - 1 \). We choose \( e_m = [1 : 0 : \cdots : 0] \in \mathbb{R}P^m \) and \( e_n' = [1 : 0 : \cdots : 0] \in \mathbb{C}^n \) as the base points of \( \mathbb{R}P^m \) and \( \mathbb{C}P^n \), respectively. Let \( \text{Map}^*(\mathbb{R}P^m, \mathbb{C}P^n) \) denote the space consisting of all based maps \( f : (\mathbb{R}P^m, e_m) \to (\mathbb{C}P^n, e_n') \). When \( m \neq 1 \), we denote by \( \text{Map}^*_0(\mathbb{R}P^m, \mathbb{C}P^n) \) the corresponding path component of \( \text{Map}^*(\mathbb{R}P^m, \mathbb{C}P^n) \) for each \( \epsilon \in \mathbb{Z}/2 = \{0, 1\} = \pi_0(\text{Map}^*(\mathbb{R}P^m, \mathbb{C}P^n)) \) [5]. Similarly, let \( \text{Map}(\mathbb{R}P^m, \mathbb{C}P^n) \) denote the space of all free maps \( f : \mathbb{R}P^m \to \mathbb{C}P^n \) and \( \text{Map}(\mathbb{R}P^m, \mathbb{C}P^n) \) the corresponding path component of \( \text{Map}(\mathbb{R}P^m, \mathbb{C}P^n) \).

We shall use the symbols \( z_i \) when we refer to complex valued coordinates or variables or when we refer to complex and real valued ones at the same time while the notation \( x_i \) will be restricted to the purely real case.

A map \( f : \mathbb{R}P^m \to \mathbb{C}P^n \) is a called an algebraic map of the degree \( d \) if it can be represented as a rational map of the form \( f = [f_0 : \cdots : f_n] \) such that \( f_0, \cdots, f_n \in \mathbb{C}[z_0, \cdots, z_m] \) are homogeneous polynomials of the same degree \( d \) with no common real roots except \( 0 \) in \( \mathbb{R}^{m+1} \). We denote by \( \text{Alg}_d(\mathbb{R}P^m, \mathbb{C}P^n) \) (resp. \( \text{Alg}_d^0(\mathbb{R}P^m, \mathbb{C}P^n) \)) the space consisting of all (resp. based) algebraic maps \( f : \mathbb{R}P^m \to \mathbb{C}P^n \) of degree \( d \). It is easy to see that there are inclusions \( \text{Alg}_d(\mathbb{R}P^m, \mathbb{C}P^n) \subseteq \text{Map}(\mathbb{R}P^m, \mathbb{C}P^n) \) and \( \text{Alg}_d^0(\mathbb{R}P^m, \mathbb{C}P^n) \subseteq \text{Map}^*_0(\mathbb{R}P^m, \mathbb{C}P^n) \), where \([d]_2 \in \mathbb{Z}/2 = \{0, 1\}\) denotes the integer \( d \) mod 2.

Let \( A_d(m, n)(\mathbb{C}) \) denote the space consisting of all \((n + 1)\)-tuples \((f_0, \cdots, f_n) \in \mathbb{C}[z_0, \cdots, z_m]^{n+1} \) of homogeneous polynomials of degree \( d \) with coefficients in \( \mathbb{C} \) and without non-trivial common real roots (but possibly with non-trivial common non-real ones).

Let \( A_d^0(m, n) \subset A_d(m, n)(\mathbb{C}) \) be the subspace consisting of \((n + 1)\)-tuples \((f_0, \cdots, f_n) \in A_d(m, n)(\mathbb{C}) \) such that the coefficient of \( z_0^d \) in \( f_0 \) is 1 and 0 in the other \( f_k \)'s \((k \neq 0) \). Then there is a natural surjective projection map

\[
\Psi_d^0 : A_d^0(m, n) \to \text{Alg}_d^0(\mathbb{R}P^m, \mathbb{C}P^n).
\]

If \( d = 2d^* = 0 \) \((\text{mod} \ 2) \) is an even positive integer, we also have a natural projection map

\[
j_d^0 : A_d^0(m, n) \to \text{Map}^*(\mathbb{R}P^m, \mathbb{C}P^n) \cong \text{Map}^*(\mathbb{R}P^m, S^{2n+1})
\]

defined by

\[
j_d^0(f)([x_0 : \cdots : x_m]) = \left( \frac{f_0(x_0, \cdots, x_m)}{(\sum_{k=0}^{m} x_k^2)^{d^*}}, \cdots, \frac{f_n(x_0, \cdots, x_m)}{(\sum_{k=0}^{m} x_k^2)^{d^*}} \right)
\]

for \( f = (f_0, \cdots, f_n) \in A_d^0(m, n) \). Note that the map \( j_d^0 \) is well defined only if \( d \geq 2 \) is an even integer.

For \( m \geq 2 \) and \( g \in \text{Alg}_d^0(\mathbb{R}P^{m-1}, \mathbb{C}P^n) \) a fixed algebraic map, we denote by \( \text{Alg}_d^0(m, n; g) \) and \( F_d^0(m, n; g) \) the spaces defined by

\[
\begin{align*}
\text{Alg}_d^0(m, n; g) & = \{ f \in \text{Alg}_d^0(\mathbb{R}P^m, \mathbb{C}P^n) : f|\mathbb{R}P^{m-1} = g \}, \\
F_d^0(m, n; g) & = \{ f \in \text{Map}^*_0(\mathbb{R}P^m, \mathbb{C}P^n) : f|\mathbb{R}P^{m-1} = g \}.
\end{align*}
\]
It is well-known that there is a homotopy equivalence $F_{\mathbb{F}}^\mathbb{C}(m, n; g) \simeq \Omega^m \mathbb{C}P^n$ (12). Let $A_d^G(m, n; g) \subset A_d^C(m, n)$ denote the subspace given by
$$A_d^G(m, n; g) = (\Psi_d^G)^{-1}(\text{Alg}_d^G(m, n; g)).$$
Observe that if an algebraic map $f \in \text{Alg}_d^G(\mathbb{R}P^m, \mathbb{C}P^n)$ can be represented as $f = [f_0 : \cdots : f_n]$, then the same map can also be represented as $f = [\tilde{g}_m f_0 : \cdots : \tilde{g}_m f_n]$, where $\tilde{g}_m = \sum_{k=0}^m z_k^2$. So there is an inclusion $\text{Alg}_d^G(\mathbb{R}P^m, \mathbb{C}P^n) \subset \text{Alg}_d^G+2(\mathbb{R}^{m, n})$ and we can define the stabilization map $s_d : A_d^G(m, n) \to A_d^G+2(m, n)$ by $s_d(f_0, \cdots, f_n) = (\tilde{g}_m f_0, \cdots, \tilde{g}_m f_n)$. It is easy to see that there is a commutative diagram
$$\begin{array}{ccc}
A_d^G(m, n) & \xrightarrow{s_d} & A_d^G+2(m, n) \\
\Psi_d & \downarrow & \Psi_{d+2} \\
\text{Alg}_d^G(\mathbb{R}P^m, \mathbb{C}P^n) & \xrightarrow{\subset} & \text{Alg}_d^G+2(\mathbb{R}P^m, \mathbb{C}P^n)
\end{array}$$
A map $f \in \text{Alg}_d^G(\mathbb{R}P^m, \mathbb{C}P^n)$ is called an algebraic map of minimal degree $d$ if $f \in \text{Alg}_d^G(\mathbb{R}P^m, \mathbb{C}P^n) \setminus \text{Alg}_{d-2}^G(\mathbb{R}P^m, \mathbb{C}P^n)$. It is easy to see that if $g \in \text{Alg}_d^G(\mathbb{R}P^{m-1}, \mathbb{C}P^n)$ is an algebraic map of minimal degree $d$, then the restriction
$$\Psi_d^G|A_d^C(m, n; g) : A_d^C(m, n; g) \xrightarrow{\sim} \text{Alg}_d^C(m, n; g)$$
is a homeomorphism. Let
$$\begin{cases}
i_{d,C} : \text{Alg}_d^G(\mathbb{R}P^m, \mathbb{C}P^n) \xrightarrow{\subset} \text{Map}_{d|d_2}(\mathbb{R}P^m, \mathbb{C}P^n) \\ i_{d,C}^* : \text{Alg}_d^G(m, n; g) \xrightarrow{\rightarrow} F(m, n; g) \simeq \Omega^m \mathbb{C}P^n
\end{cases}$$
denote the inclusions and let
$$i_d = i_{d,C} \circ \Psi_d : A_d^G(m, n) \to \text{Map}_{d|d_2}(\mathbb{R}P^m, \mathbb{C}P^n).$$
be the natural projection. For a connected space $X$, let $F(X, r)$ denote the configuration space of distinct $r$ points in $X$. The symmetric group $S_r$ of $r$ letters acts on $F(X, r)$ freely by permuting coordinates. Let $C_r(X)$ be the configuration space of unordered $r$-distinct points in $X$ given by $C_r(X) = F(X, r)/S_r$. Note that there is a stable homotopy equivalence $\Omega^m S^m \wedge \cdots \wedge S^1 \simeq \bigvee_{r=1}^\infty F(\mathbb{R}^r, r)/\wedge S_r \left(\wedge^r S^l\right)$ (14), and it is known that there is an isomorphism $H_k(F(\mathbb{R}^r, r)/\wedge S_r \left(\wedge^r S^l\right), \mathbb{Z}) \simeq H_{k-rl}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (N-m)})$ for $k, l \geq 1$ (4, 15), where $\wedge^r X = X \wedge \cdots \wedge X$ ($r$ times).

Let $G_d^{m,N}$ denote the abelian group $G_d^{m,N} = \bigoplus_{r=1}^{\lfloor (N-m)/2 \rfloor} \bigoplus_{r=1}^{\lfloor (N-m)/2 \rfloor} H_{k-(N-m)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (N-m)})$, where the meaning of $(\pm \mathbb{Z})^{\otimes (N-m)}$ is the same as in [15].

Let $D_{\mathbb{K}}(d; m, n)$ be the positive integer defined by
$$D_{\mathbb{K}}(d; m, n) = \begin{cases} (n-m)([\frac{d+1}{2}]+1)-1 & \text{if } \mathbb{K} = \mathbb{R}, \\ (2n-m+1)([\frac{d+1}{2}]+1)-1 & \text{if } \mathbb{K} = \mathbb{C}, \end{cases}$$
and $[x]$ is the integer part of a real number $x$. Note that $D_{\mathbb{C}}(d; m, n) = D_{\mathbb{R}}(d; m, 2n+1)$.

First, recall the following 3 results. Here we use the notation of [1].

Theorem 1.1 ([8, 16]). If $n \geq 2$ and then the natural projection $i_d^* : A_d^G(1, n) \to \text{Map}_{d|d_2}(\mathbb{R}P^1, \mathbb{C}P^n) \simeq \Omega S^n$ is a homotopy equivalence up to dimension $D_1(d, n) = (d+1)(n-1)-1$. 

Theorem 1.2 (I). Let $2 \leq m < n$ be integers and let $g \in \text{Alg}^d_R(\mathbb{R}P^{m-1}, \mathbb{R}P^n)$ be an algebraic map of minimal degree $d$.

(i) The inclusion $i'_{d,R} : \text{Alg}^d_R(m,n; g) \to F_d(m,n; g) \simeq \Omega^m S^n$ is a homotopy equivalence through dimension $D_R(d;m,n)$ if $m + 2 \leq n$ and a homology equivalence through dimension $\text{Map}_R^d(d;m,n)$ if $m + 1 = n$.

(ii) For any $k \geq 1$, $H_k(\text{Alg}^d_R(m,n; g), \mathbb{Z})$ contains the subgroup $G^d_{m,n}$ as a direct summand.

Theorem 1.3 (I). If $2 \leq m < n$ and $d \equiv 0 \pmod{2}$ are positive integers,

\[
\begin{align*}
&j'_d : A^d_R(m,n) \to \text{Map}^\ast(\mathbb{R}P^m, S^n) \\
i'_d : A^d_R(m,n) \to \text{Map}^\ast_R(\mathbb{R}P^m, \mathbb{R}P^n)
\end{align*}
\]

are homotopy equivalence through dimension $D_R(d;m,n)$ if $m + 2 \leq n$ and homology equivalences through dimension $D_R(d;m,n)$ if $m + 1 = n$.

Remark. A map $f : X \to Y$ is called a homotopy (resp. a homology equivalence up to dimension $D$ if $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k < D$ and an epimorphism for $k = D$. Similarly, it is called a homotopy (resp. a homology equivalence through dimension $D$ if $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k \leq D$.

In this paper, from now on let $m, n \geq 2$ be positive integers. Our main results are as follows.

Theorem 1.4. Let $2 \leq m \leq 2n$, and let $g \in \text{Alg}^d_R(\mathbb{R}P^{m-1}, \mathbb{C}P^n)$ be an algebraic map of minimal degree $d$.

(i) The inclusion $i'_{d,C} : \text{Alg}^d_C(m,n; g) \to F_d(m,n; g) \simeq \Omega^m S^{2n+1}$ is a homotopy equivalence through dimension $D_C(d;m,n)$ if $m < 2n$ and a homology equivalence through dimension $\text{Map}_C^d(d;m,n)$ if $m = 2n$.

(ii) For any $k \geq 1$, $H_k(\text{Alg}^d_C(m,n; g), \mathbb{Z})$ contains the subgroup $G^d_{m,2n+1}$ as a direct summand.

Theorem 1.5. If $2 \leq m \leq 2n$ and $d \equiv 0 \pmod{2}$ are positive integers,

\[
\begin{align*}
j'_d : A^d_C(m,n) \to \text{Map}^\ast(\mathbb{R}P^m, S^{2n+1}) \\
i'_d : A^d_C(m,n) \to \text{Map}^\ast_R(\mathbb{R}P^m, \mathbb{C}P^n)
\end{align*}
\]

are homotopy equivalences through dimension $D_C(d;m,n)$ if $m < 2n$ and homology equivalences through dimension $D_C(d;m,n)$ if $m = 2n$.

Corollary 1.6. If $2 \leq m \leq 2n$ and $d \equiv 0 \pmod{2}$ are positive integers, the stabilization map $s_d : A^d_C(m,n) \to A^d_C(m,n)$ is a homotopy equivalence through dimension $D_C(d;m,n)$ if $m < 2n$ and a homology equivalence through dimension $D_C(d;m,n)$ if $m = 2n$.

Note that the complex conjugation on $\mathbb{C}$ naturally induces $\mathbb{Z}/2$-actions on the spaces $\text{Alg}^d_C(m,n; g)$ and $A^d_C(m,n)$. In the same way it also induces a $\mathbb{Z}/2$-action on $\mathbb{C}P^n$ and this action extends to actions on the spaces $\text{Map}^\ast(\mathbb{R}P^m, S^{2n+1})$ and $\text{Map}^\ast_R(\mathbb{R}P^m, \mathbb{C}P^n)$, where we identify $S^{2n+1} = \{(w_0, \cdots, w_n) \in \mathbb{C}^{n+1} : \sum_{k=0}^{n} |w_k|^2 = 1\}$ and regard $\mathbb{R}P^m$ as a $\mathbb{Z}/2$-space with the trivial $\mathbb{Z}/2$-action. Since the maps $i'_{d,C}$,
$j_d^C, i_d^C$ are $\mathbb{Z}/2$-equivariant and $(i_{d,\mathcal{C}}')^\mathbb{Z}/2 = i_{d,\mathcal{R}}^\mathbb{R}$, $(j_d^C)^\mathbb{Z}/2 = j_d^\mathbb{R}$, $(i_d^C)^\mathbb{Z}/2 = i_d^\mathbb{R}$, we easily obtain the following result.

**Corollary 1.7.** Let $2 \leq m \leq 2n$ and $d \geq 1$ be positive integers.

(i) The inclusion map $i_{d,\mathcal{C}} : \text{Alg}^C_d(m, n; g) \to \mathcal{F}_d(m, n; g) \simeq \Omega^m S^{2n+1}$ is a $\mathbb{Z}/2$-equivariant homotopy equivalence through dimension $D_d(\mathbb{R}; m, n)$ if $m < 2n$ and a $\mathbb{Z}/2$-equivariant homology equivalence through dimension $D_d(\mathbb{R}; m, n)$ if $m = 2n$.

(ii) If $d \equiv 0 \pmod{2}$, then

\[
\begin{cases}
j_d^C : A_d^C(m, n) \to \text{Map}^*(\mathbb{R}^m, S^{2n+1}) \\
i_d^C : A_d^C(m, n) \to \text{Map}^0(\mathbb{R}^m, \mathbb{C}P^n)
\end{cases}
\]

are $\mathbb{Z}/2$-equivariant homotopy equivalences through dimension $D_d(\mathbb{R}; m, n)$ if $m < 2n$ and $\mathbb{Z}/2$-equivariant homology equivalences through dimension $D_d(\mathbb{R}; m, n)$ if $m = 2n$.

**Remark.** Let $G$ be a finite group and let $f : X \to Y$ be a $G$-equivariant map. Then a map $f : X \to Y$ is called a $G$-equivariant homotopy (resp. homology) equivalence through dimension $D$ if the induced homomorphism $f_*^*: \pi_k(X^H) \xrightarrow{\cong} \pi_k(Y^H)$ (resp. $f_*^*: H_k(X^H, \mathbb{Z}) \xrightarrow{\cong} H_k(Y^H, \mathbb{Z})$) are isomorphisms for any $k \leq D$ and any subgroup $H \leq G$.

Of course we would also like to understand the cases $d \equiv 1 \pmod{2}$. The homotopy type of $\text{Alg}^*_d(\mathbb{R}^m, \mathbb{C}P^n)$ appears hard to investigate in general. However, for $d = 1$, $\Psi^1_1 : A_1^C(m, n) \xrightarrow{\cong} \text{Alg}^*_1(\mathbb{R}^m, \mathbb{C}P^n)$ is a homeomorphism and we can prove the following results.

**Theorem 1.8.**

(i) If $2 \leq m < 2n$, the inclusion

$$i_{1, \mathcal{C}} : \text{Alg}^*_1(\mathbb{R}^m, \mathbb{C}P^n) \to \text{Map}^*_1(\mathbb{R}^m, \mathbb{C}P^n)$$

is a homotopy equivalence up to dimension $D_1(1; m, n) = 4n - 2m + 1$.

(ii) If $m = 2n + 4$, the inclusion $i_{1, \mathcal{C}}$ induces an isomorphism

$$\pi_1(\text{Alg}^*_1(\mathbb{R}^{2n}, \mathbb{C}P^n)) \xrightarrow{\cong} \pi_1(\text{Map}^*_1(\mathbb{R}^{2n}, \mathbb{C}P^n)) \cong \mathbb{Z}/2.$$

**Corollary 1.9.**

(i) If $2 \leq m < 2n$ and $d \equiv 0 \pmod{2}$ are positive integers, the space $\text{Alg}^*_d(\mathbb{R}^m, \mathbb{C}P^n)$ is $(2n - m)$-connected and

$$\pi_{2n-m+1}(\text{Alg}^*_d(\mathbb{R}^m, \mathbb{C}P^n)) \cong \begin{cases} \mathbb{Z} & \text{if } m \equiv 1 \pmod{2}, \\ \mathbb{Z}/2 & \text{if } m \equiv 0 \pmod{2}. \end{cases}$$

(ii) If $2 \leq m \leq 2n$ and $\epsilon \in \{0, 1\}$, the two spaces $\text{Alg}^*_d(\mathbb{R}^m, \mathbb{C}P^n)$ and $\text{Map}^*_d(\mathbb{R}^m, \mathbb{C}P^n)$ are $(2n - m)$-connected, and

$$\pi_{2n-m+1}(\text{Alg}^*_d(\mathbb{R}^m, \mathbb{C}P^n)) \cong \pi_{2n-m+1}(\text{Map}^*_d(\mathbb{R}^m, \mathbb{C}P^n)) \cong \begin{cases} \mathbb{Z} & \text{if } m \equiv 1 \pmod{2}, \\ \mathbb{Z}/2 & \text{if } m \equiv 0 \pmod{2}. \end{cases}$$

**Remark.** We conjecture that $\pi_1(\text{Alg}^*_d(\mathbb{R}^m, \mathbb{C}P^n)) = \mathbb{Z}/2$ if $m = 2n \geq 4$ and $d \equiv 0 \pmod{2}$, but at this time we cannot prove this.

This paper is organized as follows. In section 2, we consider the space of algebraic maps $\mathbb{R}^m \to \mathbb{C}P^n$ and recall the stable Theorem obtained in [I]. In
can show that the statement and the proof in it are valid also for spaces of based maps, and we may be true.

Conjecture 2.1. The map \( \Psi^C_d : A^C_d(m, n) \rightarrow \Alg^*_d(\R^m, \C^n) \) is a homotopy equivalence.

Although we cannot prove this conjecture, we show in section 6 that it is true if \( d \rightarrow \infty \) through even integers.

We always have \( \Alg^*_d(\R^m, \C^n) \subset \Alg^*_{d+2}(\R^m, \C^n) \) and \( \Alg_d(\R^m, \C^n) \subset \Alg_{d+2}(\R^m, \C^n) \), because \( \{ f_0 : f_1 : \cdots : f_n \} = \{ \hat{g}_m f_0 : \hat{g}_m f_1 : \cdots : \hat{g}_m f_n \} \).

**Definition 2.2.** For \( \epsilon \in \{ 0, 1 \} \), define subspaces \( \Alg^*_\epsilon(m, n) \subset \Map^*_\epsilon(\R^m, \C^n) \) and \( \Alg_\epsilon(m, n) \subset \Map_\epsilon(\R^m, \C^n) \) by

\[
\begin{align*}
\Alg^*_\epsilon(m, n) &= \bigcup_{k=1}^{\infty} \Alg^*_{\epsilon+2k}(\R^m, \C^n), \\
\Alg_\epsilon(m, n) &= \bigcup_{k=1}^{\infty} \Alg_{\epsilon+2k}(\R^m, \C^n).
\end{align*}
\]

**Theorem 2.3.** If \( 1 \leq m \leq 2n \) and \( \epsilon = 0 \) or \( 1 \), the inclusion maps

\[
\begin{align*}
i : \Alg^*_\epsilon(m, n) &\rightarrow \Map^*_\epsilon(\R^m, \C^n) \\
j : \Alg_\epsilon(m, n) &\rightarrow \Map_\epsilon(\R^m, \C^n)
\end{align*}
\]

are homotopy equivalences.

**Proof.** It follows from [1, Theorem 2.3] that \( j \) is a homotopy equivalence. The statement and the proof in it are valid also for spaces of based maps, and we can show that \( i \) is also a homotopy equivalence.
3. Spectral sequences of the Vassiliev type.

From now on, we assume $2 \leq m \leq 2n$ and let $g \in \text{Alg}^*_d(\mathbb{RP}^{m-1}, \mathbb{CP}^n)$ be a fixed algebraic map of minimal degree $d$, such that $g = [g_0 : \cdots : g_n]$ with $(g_0, \cdots, g_n) \in A_d(m-1, n)$. Note that $(g_0, \cdots, g_n)$ is uniquely determined by $g$ (because of the minimal degree condition).

Let $\mathcal{H}_d = \mathbb{C}[z_0, \cdots, z_m]$ denote the subspace consisting of all homogeneous polynomials of degree $d$. For $\epsilon \in \{0, 1\}$, let $\mathcal{H}_d^\epsilon \subset \mathcal{H}_d$ be the subspace consisting of all homogeneous polynomials $f \in \mathcal{H}_d$ such that the coefficient of $(z_0)^d$ of $f$ is $\epsilon$.

Since $A_d^\epsilon(m, n)$ is the space consisting of all $(n+1)$-tuples $(f_0, \cdots, f_n) \in A_d(m, n)(\mathbb{C})$ such that the coefficient of $z_0^d$ in $f_0$ is $1$ and those of other $f_k$’s are all zero, $A_d^\epsilon(m, n) \subset \mathcal{H}_d^\epsilon \times (\mathcal{H}_d^\epsilon)^n$. Note that $\mathcal{H}_d^\epsilon \times (\mathcal{H}_d^\epsilon)^n$ is an affine space of real dimension $N_d = 2(n+1)\binom{m+d-1}{m}$.

Next, we set $B_k = \{g_k + z_m h : h \in \mathcal{H}_{d-1}\}$ $(k = 0, 1, \cdots, n)$ and define the subspace $A_d^\ast \subset \mathcal{H}_d^\epsilon \times (\mathcal{H}_d^\epsilon)^n$ by $A_d^\ast = B_0 \times B_1 \times \cdots \times B_n$. Note that $A_d^\ast$ is an affine space of real dimension $N_d^\ast = 2(n+1)\binom{m+d-1}{m}$.

**Definition 3.1.** Let $A_d^\epsilon(m, n; g) \subset A_d^\ast$ be the subspace $A_d^\epsilon(m, n; g) = A_d^\ast \cap A_d^\epsilon(m, n)$. Let $\Sigma_d \subset A_d^\ast$ denote the discriminant of $A_d^\epsilon(m, n; g)$ in $A_d^\ast$ defined by $\Sigma_d = A_d^\ast \setminus A_d^\epsilon(m, n; g)$.

Since $g \in \text{Alg}^*_d(\mathbb{RP}^{m-1}, \mathbb{RP}^n)$ has minimal degree $d$, clearly the restriction

$$\Psi_d^\epsilon|_{A_d^\epsilon(m, n; g)} : A_d^\epsilon(m, n; g) \xrightarrow{\cong} \text{Alg}^*_d(m, n; g)$$

is a homeomorphism.

**Lemma 3.2.** (i) If $(f_0, \cdots, f_n) \in \Sigma_d$ and $x = (x_0, \cdots, x_m) \in \mathbb{R}^{m+1}$ is a non-trivial common root of $f_0, \cdots, f_n$, then $x_m \neq 0$.

(ii) $A_d^\epsilon(m, n; g)$ and $A_d^\ast(m, n)$ are simply connected if $m < 2n$.

**Proof.** The proof is completely analogous to that of [1], Lemma 4.1. \hfill $\square$

**Definition 3.3.** (i) For a finite set $x = \{x_1, \cdots, x_l\} \subset \mathbb{R}^N$, let $\sigma(x)$ denote the convex hull spanned by $x$. Note that $\sigma(x)$ is an $(l-1)$-dimensional simplex if and only if vectors $\{x_k - x_1\}_{k=2}^l$ are linearly independent. In particular, it is in general position if $x_1, \cdots, x_l$ are linearly independent over $\mathbb{R}$.

(ii) Let $h : X \to Y$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in Y$, and let $i : X \to \mathbb{R}^n$ be an embedding. Let $X^\Delta$ and $h^\Delta : X^\Delta \to Y$ denote the space and the map defined by

$$X^\Delta = \{(y, w) \in Y \times \mathbb{R}^N : w \in \sigma(i(h^{-1}(y)))\} \subset Y \times \mathbb{R}^N, \ h^\Delta(y, w) = y.$$

The pair $(X^\Delta, h^\Delta)$ is called a simplicial resolution of $(h, i)$. In particular, $(X^\Delta, h^\Delta)$ is called a non-degenerate simplicial resolution if for each $y \in Y$ and any $k$ points of $i(h^{-1}(y))$ span $(k - 1)$-dimensional simplex of $\mathbb{R}^N$.

(iii) For each $k \geq 0$, let $X^\Delta_k \subset X^\Delta$ be the subspace given by

$$X_k^\Delta = \{(y, \omega) \in X^\Delta : \omega \in \sigma(u), u = \{u_1, \cdots, u_l\} \subset i(h^{-1}(y)), l \leq k\}.$$
We make identification $X = X^A$ by identifying the point $x \in X$ with the pair $(h(x), i(x)) \in X^I_A$, and we note that there is an increasing filtration

$$\emptyset = X^A_0 \subset X = X^A_1 \subset X^A_2 \subset \cdots \subset X^A_{k+1} \subset \cdots \subset \bigcup_{k=0}^{\infty} X^A_k = X^A.$$  

**Lemma 3.4** ([11], [15]). Let $h : X \to Y$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in Y$, and let $i : X \to \mathbb{R}^N$ be an embedding.

(i) If $X$ and $Y$ are closed semi-algebraic spaces and the two maps $h$, $i$ are polynomial maps, then $h^A : X^A \xrightarrow{\sim} Y$ is a homotopy equivalence.

(ii) There is an embedding $j : X \to \mathbb{R}^M$ such that the associated simplicial resolution $(\hat{X}^A, \hat{h}^A)$ of $(h, j)$ is non-degenerate, and the space $\hat{X}^A$ is uniquely determined up to homeomorphism. Moreover, there is a filtration preserving homotopy equivalence $q^A : \hat{X}^A \xrightarrow{\sim} X^A$ such that $q^A|X = \text{id}_X$. 

**Remark.** Even when $h$ is not finite to one, it is still possible to define its simplicial resolution and associated non-degenerate one. We omit the details of this construction and refer the reader to [11].

**Definition 3.5.** Let $Z_d \subset \Sigma_d \times \mathbb{R}^m$ denote the tautological normalization of $\Sigma_d$ consisting of all pairs $(f, x) = (((f_0, \cdots, f_n), x(0) = x(m-1)) \in \Sigma_d \times \mathbb{R}^m$ such that the polynomials $f_0, \cdots, f_n$ have a non-trivial common real root $(x, 1) = (x_0, \cdots, x_m-1, 1)$. Projection on the first factor gives a surjective map $\pi_d^* : Z_d \to \Sigma_d$.

Let $\phi_d : A_d^* \cong \mathbb{N}^d$ be any fixed homeomorphism, and let $H_d$ be the set consisting of all monomials $\varphi_I = z^{i_0} z_1^{i_1} \cdots z_m^{i_m}$ of degree $d$ ($I = (i_0, i_1, \cdots, i_m) \in \mathbb{Z}_{\geq 0}^{m+1}$, $|I| = \sum_{k=0}^{m} i_k = d$). Next, we define the Veronese embedding, which will play a key role in our argument. Let $\psi_d^* : \mathbb{R}^m \to \mathbb{R}^{M_d}$ be the map given by $\psi_d^*(x_0, \cdots, x_{m-1}) = (\varphi_I(x_0, \cdots, x_{m-1}, 1))$, where $M_d := (d+m)$. Now define the embedding $\Phi_d^* : Z_d \to \mathbb{R}^{N_d + M_d}$ by

$$\Phi_d^*((f_0, \cdots, f_n), x) = (\phi_d^*(f_0, \cdots, f_n), \psi_d^*(x)).$$

**Definition 3.6.** Let $(\mathcal{Z}^A(d), \mathcal{\pi}_d^A : \mathcal{Z}^A(d) \to \Sigma_d)$ and $Z^A(d) \to \Sigma_d$) denote the simplicial resolution of $(\pi_d^A, \Phi_d^*)$ and the corresponding non-degenerate simplicial resolution with the natural increasing filtrations

$$\mathcal{Z}^A(d)_0 = \emptyset \subset \mathcal{Z}^A(d)_1 \subset \mathcal{Z}^A(d)_2 \subset \cdots \subset \mathcal{Z}^A(d) = \bigcup_{k=0}^{\infty} \mathcal{Z}^A(d)_k,$$

$$\hat{Z}^A(d)_0 = \emptyset \subset \hat{Z}^A(d)_1 \subset \hat{Z}^A(d)_2 \subset \cdots \subset \hat{Z}^A(d) = \bigcup_{k=0}^{\infty} \hat{Z}^A(d)_k.$$

By Lemma 3.4 the map $\mathcal{\pi}_d^A : \mathcal{Z}^A(d) \xrightarrow{\sim} \Sigma_d$ is a homotopy equivalence. It is easy to see that it extends to a homotopy equivalence $\mathcal{\pi}_d^A : \mathcal{Z}^A(d)_+ \xrightarrow{\sim} \Sigma_d +$, where $X_+$ denotes the one-point compactification of a locally compact space $X$.

Since $\mathcal{Z}^A(d)_{r+} / \mathcal{Z}^A(d)_{r-1} \cong (\mathcal{Z}^A(d)_r \setminus \mathcal{Z}^A(d)_{r-1})_+$, we have the Vassiliev type spectral sequence

$$\{ E_t^{r,s}(d), d_t : E_t^{r,s}(d) \to E_t^{r+t,s+1-t}(d) \} \Rightarrow H^{r+s}_c(\Sigma_d, \mathbb{Z}),$$
Lemma 3.7. (i) If \( r \geq 3 \),
\[
H_k(A^c_d(m, n; g, Z)) \cong H^c_d(N^{d-k-1}_d, 1 \leq k \leq N^d_d - 2).
\]
Using (3.2) and reindexing we obtain a spectral sequence
\[
\{ \tilde{E}^r_{s,t}(d), d^r : \tilde{E}^r_{s,t}(d) \to \tilde{E}^r_{s+r,t+s-t-1}(d) \} \Rightarrow H_{s-r}(A^c_d(m, n; g, Z)).
\]
if \( s - r \leq N^d_d - 2 \), where \( \tilde{E}^1_{r,s}(d) = H^c_d(N^{d+r-s-1}_d, \mathbb{Z}) \) and span an \((r - 1)\)-dimensional simplex in \( \mathbb{R}^{M_d} \).

Lemma 3.8. If \( 1 \leq r \leq \left\lfloor \frac{d+1}{2} \right\rfloor \), \( \mathbb{Z}^{\Delta}(d)_r \setminus \mathbb{Z}^{\Delta}(d)_{r-1} \) is homeomorphic to the total space of a real vector bundle \( \mathbb{E}_d \mathbb{R}^{M_d} \) over \( C_r(\mathbb{R}^m) \) with rank \( \mathbb{E}^r_{d,s} = N^d_d - (2n + 1) r - 1 \).

Proof. The proof is completely analogous to that of \([1], \text{Lemma } 4.3\].

Lemma 3.9. All non-zero entries of \( \tilde{E}^1_{r,s}(d) \) are situated in the range \( s \geq r(2n + 2 - m) \) if \( 1 \leq r \leq d^m \), and \( \tilde{E}^1_{r,s}(d) = 0 \) if \( r > d^m \).

Proof. The proof is completely analogous to that of \([1], \text{Lemma } 4.4\].

Lemma 4.5. \]

Lemma 4.10. If \( 1 \leq r \leq \left\lfloor \frac{d+1}{2} \right\rfloor \), there is a natural isomorphism
\[
\tilde{E}^1_{r,s}(d) \cong H_s(\mathbb{R}^{2n+2}, \mathbb{Z}) \otimes \mathbb{Z}(\mathbb{R}^{2n+1 - m}, \mathbb{Z}).
\]

Proof. By using the Thom isomorphism and Poincaré duality, we obtain the desired isomorphism.

Now we recall the spectral sequence constructed by V. Vassiliev \([15]\). From now on, we will assume that \( m \leq 2n \), and we identify Map\((S^m, S^{2n+1})\) with the space Map\((S^m, \mathbb{R}^{2n+2} \setminus \{0_{2n+2}\})\). We also choose a map \( \varphi : S^m \to \mathbb{R}^{2n+2} \setminus \{0_{2n+2}\} \) and fix it. Observe that Map\((S^m, \mathbb{R}^{2n+2})\) is a linear space and consider the complements \( \mathfrak{A}_m = \text{Map}(S^m, \mathbb{R}^{2n+2}) \setminus \text{Map}(S^m, S^{2n+1}) \) and \( \mathfrak{A}_m = \text{Map}^*(S^m, \mathbb{R}^{2n+2}) \setminus \text{Map}^*(S^m, S^{2n+1}) \).

Note that \( \mathfrak{A}_m \) consists of all continuous maps \( f : S^m \to \mathbb{R}^{2n+2} \) passing through \( 0_{2n+2} \). We will denote by \( \Theta^k_\varphi \subset \text{Map}(S^m, \mathbb{R}^{2n+2}) \) the subspace consisting of all maps \( f \) of the forms \( f = \varphi + p \), where \( p \) is the restriction to \( S^m \) of a polynomial map \( \mathbb{R}^{m+1} \to \mathbb{R}^{2n+2} \) of degree \( \leq k \). Let \( \Theta^k \subset \Theta^k_\varphi \) denote the subspace consisting of all \( f \in \Theta^k_\varphi \) passing through \( 0_{n+1} \). In \([15], \text{page } 111\) Vassiliev uses the space \( \Theta^k \) as a finite dimensional approximation of \( \mathfrak{A}_m \).

Let \( \Theta^k \) denote the subspace of \( \Theta^k \) consisting of all maps \( f \in \Theta^k \) which preserve the base points. By a variation of the preceding argument, Vassiliev also shows that \( \Theta^k \) can be used as a finite dimensional approximation of \( \mathfrak{A}_m \), \([15], \text{page } 112\].

Let \( \mathcal{X}_k \subset \Theta^k \times \mathbb{R}^{m+1} \) denote the subspace consisting of all pairs \((f, \alpha) \in \Theta^k \times \mathbb{R}^{m+1} \) such that \( f(\alpha) = 0_{2n+2} \), and let \( p_k : \mathcal{X}_k \to \Theta^k \) be the projection onto
the first factor. Then, by making use of simplicial resolutions of the surjective maps \( \{ p_k : k \geq 1 \} \), one can construct an associated geometric resolution \( \{ \mathcal{X}_m^n \} \) of \( \mathcal{X}_m^n \), whose cohomology approximates the homology of \( \text{Map}^* (S^m, S^{2n+1}) = \Omega^m S^{2n+1} \) to any desired dimension. From the natural filtration on the approximating space \( F_1 \subset F_2 \subset F_3 \subset \cdots \subset \bigcup_{k=1}^{\infty} F_k = \{ \mathcal{X}_m^n \} \), we obtain the associated spectral sequence:

\[
E_{r,s}^1 : E_{r+s,t-1}^{1} \Rightarrow H_{s-r}(\Omega^m S^{2n+1}, \mathbb{Z}).
\]

The following result follows easily from [15, Theorem 2 (page 112) and (32) (page 114)].

**Lemma 3.11** ([15]). Let \( 2 \leq m \leq 2n \) be integers and let \( X \) be a finite m-dimensional simplicial complex with a fixed base point \( x_0 \in X \).

(i) \( E_{r,s}^1 = H_{r-(2n-m+2)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z}))^{\otimes (2n-m+1)} \) if \( r \geq 1 \), and \( E_{r,s}^1 = 0 \) if \( r < 0 \) or \( s < 0 \) or \( s < (2n-m+2)r \).

(ii) For any \( t \geq 1 \), \( d^t = 0 : E_{r,s}^t \rightarrow E_{r+t,s+t-1}^t \) for all \( (r, s) \), and \( E_{r,s}^1 = E_{s,r}^\infty \).

Moreover, for any \( k \geq 1 \), the extension problem for the graded group

\[
Gr(H_k(\Omega^m S^{2n+1}, \mathbb{Z})) = \bigoplus_{r=1}^\infty E_{r+r+k}^\infty = \bigoplus_{r=1}^\infty E_{r,r+k}^1
\]

is trivial and there is an isomorphism

\[
H_k(\Omega^m S^{2n+1}, \mathbb{Z}) \cong \bigoplus_{r=1}^\infty H_{k-(2n-m+1)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (2n-m+1)}).
\]

**Definition 3.12.** We identify \( \Omega^m S^{2n+1} = \Omega^m (\mathbb{C}^{n+1} \setminus \{0\}) \) and define the map \( j_d^m : A_d^m(m, n; g) \rightarrow \Omega^m S^{2n+1} \) by

\[
j_d^m(f_0, \ldots, f_n)(x_0, \ldots, x_m) = (f_0(x_0, \ldots, x_m), \ldots, f_n(x_0, \ldots, x_m))
\]

for \((f_0, \ldots, f_n, (x_0, \ldots, x_m)) \in A_d^m(m, n; g) \times S^m \).

Now, by applying the spectral sequence \([3.4]\), we prove the following result, which plays a key role in the proof of Theorem 1.4.

**Theorem 3.13.** Let \( m, n \geq 2 \) be positive integers such that \( 2 \leq m \leq 2n \), and let \( g \in \text{Alg}_d(\mathbb{R}P^{m-1}, \mathbb{C}P^n) \) be an algebraic map of minimal degree \( d \).

(i) The map \( j_d^m : A_d^m(m, n; g) \rightarrow \Omega^m S^{2n+1} \) is a homotopy equivalence through dimension \( D_C(d; m, n) \) if \( m < 2n \) and a homology equivalence through dimension \( D_C(d; m, n) \) if \( m = 2n \).

(ii) For any \( k \geq 1 \), \( H_k(A_d^m(m, n; g), \mathbb{Z}) \) contains the subgroup

\[
G_m^d,2n+1 = \bigoplus_{r=1}^{\frac{d+1}{2}} H_{k-(2n-m+1)r}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes (2n-m+1)}),
\]

as a direct summand.

**Proof.** Consider the spectral sequence \([3.4]\). First, note that, by Lemma 3.11 there is a filtration preserving homotopy equivalence \( \tilde{\Delta} : \tilde{\mathcal{Z}}(d) \cong \mathcal{Z}(d) \). Note also that the image of the map \( j_d^m \) lies in a space of polynomial mappings, which approximates the space of continuous mappings \( S^m \rightarrow S^{2n+1} \). Since \( \tilde{\mathcal{Z}}(d) \) is non-degenerate, the map \( j_d^m \) naturally extends to a filtration preserving map \( \tilde{\Delta} : \tilde{\mathcal{Z}}(d) \rightarrow \{ \mathcal{X}_m^n \} \) between resolutions. Thus the filtration preserving maps

\[
\mathcal{Z}(d) \xrightarrow{\tilde{\Delta}} \tilde{\mathcal{Z}}(d) \xrightarrow{\tilde{\Delta}} \{ \mathcal{X}_m^n \}
\]
induce a homomorphism of spectral sequences \( \{ \hat{\theta}^t_{r,s} : \hat{E}^{t}_{r,s}(d) \to E^{t}_{r,s} \} \), where \( \{ E^{t}_{r,s} : d^t \} \Rightarrow H_{s-r}(\Omega^m S^n, \Z) \).

Observe that, by Lemma 3.10(ii) of Lemma 3.11 and the naturality of Thom isomorphism, for \( r \leq \lfloor \frac{d+1}{2} \rfloor \) there is a commutative diagram

\[
\begin{array}{c}
\hat{E}^{1}_{r,s}(d) \xrightarrow{T} H_{s-r}(2n-m+2)(C_{r}(\mathbb{R}^m), (\pm \Z)^{(2n-m+1)}) \\
\hat{\theta}^{1}_{r,s} \downarrow \quad \|
\end{array}
\]

(3.5) \[ E^{1}_{r,s} \xrightarrow{T} H_{s-r}(2n-m+2)(C_{r}(\mathbb{R}^m), (\pm \Z)^{(2n-m+1)}) \]

Hence, if \( r \leq \lfloor \frac{d+1}{2} \rfloor \), \( \hat{\theta}^{1}_{r,s} : \hat{E}^{1}_{r,s}(d) \xrightarrow{\cong} E^{1}_{r,s} \) and thus so is \( \hat{\theta}^{\infty}_{r,s} : \hat{E}^{\infty}_{r,s}(d) \xrightarrow{\cong} E^{\infty}_{r,s} \).

Next, we will compute the number

\[ D_{\text{min}} = \min\{N \mid N \geq s-r, \ s \geq (2n+2-m)r, \ 1 \leq r < \lfloor \frac{d+1}{2} \rfloor + 1 \}. \]

It is easy to see that \( D_{\text{min}} \) is the largest integer \( N \) which satisfies the inequality \((n+1-m)r > r+N \) for \( r = \lfloor \frac{d+1}{2} \rfloor + 1 \), hence

\[ (3.6) \quad D_{\text{min}} = (2n-m+1)(\lfloor \frac{d+1}{2} \rfloor + 1) - 1 = D_C(d; m, n). \]

We note that, for dimensional reasons, \( \hat{\theta}^{\infty}_{r,s} : \hat{E}^{\infty}_{r,s}(d) \xrightarrow{\cong} E^{\infty}_{r,s} \) is always an isomorphism if \( r \leq \lfloor \frac{d+1}{2} \rfloor \) and \( s-r \leq D_C(d; m, n) \).

On the other hand, from Lemma 3.9 it easily follows that \( \hat{E}^{1}_{r,s}(d) = E^{1}_{r,s} = 0 \) if \( s-r \leq D_C(d; m, n) \) and \( r > \lfloor \frac{d+1}{2} \rfloor \). Hence, we have:

\[ (3.6.1) \quad \text{If } s \leq r+D_C(d; m, n), \text{ then } \hat{\theta}^{\infty}_{r,s} : \hat{E}^{\infty}_{r,s}(d) \xrightarrow{\cong} E^{\infty}_{r,s} \text{ is always an isomorphism.} \]

Hence, by using the comparison Theorem of spectral sequences, we have that \( j'_{d} \) is a homology equivalence through dimension \( D_C(d; m, n) \). Since \( A^C_d(m, n; g) \) and \( \hat{\Omega}^m S^{2n+1} \) are simply connected if \( m < 2n \), \( j'_d \) is a homotopy equivalence through dimension \( D_C(d; m, n) \) if \( m < 2n \). Hence, (i) is proved.

It remains to show (ii). Since \( d^t = 0 \) for any \( t \geq 1 \), from the equality \( d^t \circ \hat{\theta}^{t}_{r,s} = \hat{\theta}^{t}_{r+t,s+t-1} \circ d^t \) and some diagram chasing, we obtain \( \hat{E}^{1}_{r,s}(d) = \hat{E}^{\infty}_{r,s}(d) \) for all \( r \leq \lfloor \frac{d+1}{2} \rfloor \). Moreover, since the extension problem for the graded group

\[ \text{Gr}(H_k(\Omega^m S^{2n+1}, \Z)) = \bigoplus_{r=1}^{\infty} E^{\infty}_{r,k+r} = \bigoplus_{r=1}^{\infty} E^{1}_{r,k+r} \]

is trivial, by using (3.6.1) we can prove that the associated graded group

\[ \text{Gr}(H_k(A^C_d(m, n; g), \Z)) = \bigoplus_{r=1}^{\infty} \hat{E}^{\infty}_{r,k+r}(d) = \bigoplus_{r=1}^{\infty} \hat{E}^{1}_{r,k+r}(d) \]

is also trivial until the \( \lfloor \frac{d+1}{2} \rfloor \)-th term of the filtration. Hence, \( H_k(A^C_d(m, n; g), \Z) \) contains the subgroup

\[ \bigoplus_{r=1}^{\lfloor \frac{d+1}{2} \rfloor} \hat{E}^{1}_{r,k+r}(d) = \bigoplus_{r=1}^{\lfloor \frac{d+1}{2} \rfloor} \hat{E}^{\infty}_{r,k+r}(d) \cong \bigoplus_{r=1}^{\lfloor \frac{d+1}{2} \rfloor} H_{k-r}(2n-m+2)(C_{r}(\mathbb{R}^m), (\pm \Z)^{(2n-m+1)}) \]

as a direct summand, which proves the assertion (ii). \( \square \)
Corollary 3.14. Let \( m, n \geq 2 \) be positive integers such that \( 2 \leq m \leq 2n \), let \( g \in \text{Alg}^*_d(\mathbb{R}P^{m-1}, \mathbb{C}P^n) \) be an algebraic map of minimal degree \( d \), and let \( \mathbb{F} = \mathbb{Z}/p \) (\( p: \text{prime} \)) or \( \mathbb{F} = \mathbb{Q} \). Then the map \( j'_d: A^C_d(m, n; g) \to \Omega^m S^{2n+1} \) induces an isomorphism on the homology group \( H_k(\ , \mathbb{F}) \) for any \( 1 \leq k \leq D_C(d; m, n) \).

Proof. In the proof of Theorem 3.13 replace the homology groups \( H_k(\ , \mathbb{Z}) \) and \( H_k(\ , (\pm \mathbb{Z})^{\otimes k}) \) by \( H_k(\ , \mathbb{F}) \) and \( H_k(\ , (\pm \mathbb{F})^{\otimes k}) \) and use the same argument. □

Let \( \gamma_m: S^m \to \mathbb{R}P^m \) and \( \gamma^C_m: S^{2n+1} \to \mathbb{C}P^n \) denote the usual double covering and the Hopf fibration map, respectively. Let \( \gamma^#_m: \text{Map}^*(\mathbb{R}P^m, \mathbb{C}P^n) \to \Omega^m \mathbb{C}P^n \) be given by \( \gamma^#_m(h) = h \circ \gamma_m \). It is easy to verify that the following diagram

\[
\begin{array}{cccc}
A^C_d(m, n; g) & \rightarrow & \text{Alg}^C_d(m, n; g) & \rightarrow & F_d(m, n; g) \\
j'_d |_{\Omega^m S^{2n+1}} & \cong & \Omega^m \mathbb{C}P^n & \cong & \text{Map}^*_d(\mathbb{R}P^m, \mathbb{C}P^n) \\
\end{array}
\]

(3.7)

is commutative, where \( i': F_d(m, n; g) \hookrightarrow \text{Map}^*_d(\mathbb{R}P^m, \mathbb{C}P^n) \) and \( \Psi'_d \) denote the inclusion and the restriction \( \Psi'_d = \Psi^C_d|A^C_d(m, n; g) \), respectively.

Lemma 3.15. If \( 2 \leq m < 2n \) and \( g \in \text{Alg}^*_d(\mathbb{R}P^{m-1}, \mathbb{R}P^n) \) a fixed map of minimal degree \( d \), then the map \( \gamma^#_m \circ i': F_d(m, n; g) \to \Omega^m \mathbb{C}P^n \) is a homotopy equivalence through dimension \( D_C(d; m, n) \).

Proof. Since there is a homotopy equivalence \( F_d(m, n; g) \simeq \Omega^m \mathbb{C}P^n \), the two spaces \( F_d(m, n; g) \) and \( \Omega^m \mathbb{C}P^n \) are simple. So it suffices to show that the map \( \gamma^#_m \circ i' \) is a homotopy equivalence through dimension \( D_C(d; m, n) \).

Let \( \mathbb{F} = \mathbb{Z}/p \) (\( p: \text{prime} \)) or \( \mathbb{F} = \mathbb{Q} \), and consider the induced homomorphism \( (\gamma^#_m \circ i')_*: H_k(\gamma^#_m \circ i', \mathbb{F}) \to H_k(\Omega^m \mathbb{C}P^n, \mathbb{F}) \). Since \( \Omega^m \gamma^C_m \) is a homotopy equivalence by Corollary 3.14 and the commutativity of the diagram (3.7), \( (\gamma^#_m \circ i')_* \) is an epimorphism for any \( 1 \leq k \leq D_C(d; m, n) \).

However, since there is a homotopy equivalence \( F_d(m, n; g) \simeq \Omega^m \mathbb{C}P^n \), we have \( \dim_F H_k(F_d(m, n; g), \mathbb{F}) = \dim_F H_k(\Omega^m \mathbb{C}P^n, \mathbb{F}) < \infty \) for any \( k \). Hence, \( H_k(\gamma^#_m \circ i', \mathbb{F}) \) is an isomorphism for any \( 1 \leq k \leq D_C(d; m, n) \). By the Universal Coefficient Theorem \( \gamma^#_m \circ i' \) induces an isomorphism on \( H_k(\ , \mathbb{F}) \) for any \( 1 \leq k \leq D_C(d; m, n) \).

Proof of Theorem 1.4. Since \( \Psi'_d: A^C_d(m, n; g) \xrightarrow{\cong} \text{Alg}^C_d(m, n; g) \) is a homomorphism, the assertion (ii) follows from Theorem 3.13. Because \( \gamma^#_m \circ i' \) a homotopy equivalence through dimension \( D(d; m, n) \), by using the diagram (3.7) and Theorem 3.13 we easily obtain the assertion (i). □

4. The space \( A^C_d(m, n) \).

In this section we shall consider the unstable problem for the space \( A^C_d(m, n) \), where \( d = 2d^* \geq 2 \) is even.

Definition 4.1. Define \( \psi_{m,n}: A^C_d(m, 2n+1) \to A^C_d(m, n) \) by

\[
\psi_{m,n}(f_0, \cdots, f_{2n+1}) = (f_0 + \sqrt{-1}f_1, f_2 + \sqrt{-1}f_3, \cdots, f_{2n} + \sqrt{-1}f_{2n+1}).
\]

It is easy to see that
Lemma 4.2. $\psi_{m,n} : A^2_B(m, 2n + 1) \to A^2_B(m, n)$ is a homeomorphism. \hfill $\square$

Lemma 4.3. (i) Map$^*(\mathbb{R}P^m, S^{2n+1})$ is $(2n - m)$-connected.

(ii) $\pi_{2n-m+1}(\text{Map}^*(\mathbb{R}P^m, S^{2n+1})) \cong \begin{cases} \mathbb{Z} & \text{if } m \equiv 1 \pmod{2}, \\ \mathbb{Z}/2 & \text{if } m \equiv 0 \pmod{2}. \end{cases}$

Proof. (i) We argue by induction on $m$. For $m = 1$ the result follows from the homotopy equivalence $\text{Map}^*(\mathbb{R}P^1, S^{2n+1}) \simeq \Omega S^{2n+1}$. Suppose that the space $\text{Map}^*(\mathbb{R}P^{m-1}, S^{2n+1})$ is $(2n - m + 1)$-connected for some $m \geq 2$. Since $\Omega^m S^{2n+1}$ is $(2n-m)$-connected, from the restriction fibration sequence $\Omega^m S^{2n+1} \to \text{Map}^*(\mathbb{R}P^m, S^{2n+1}) \to \text{Map}^*(\mathbb{R}P^{m-1}, S^{2n+1})$, we deduce that $\text{Map}^*(\mathbb{R}P^m, S^{2n+1})$ is $(2n-m)$-connected. Hence, (i) has been proved.

(ii) First, consider the case $m = 1$. Since $\text{Map}^*(\mathbb{R}P^1, S^{2n+1}) \simeq \Omega S^{2n+1}$, (ii) clearly holds for $m = 1$. Next, consider the case $m = 2$. If we consider the fibration sequence $\text{Map}^*(\mathbb{R}P^2, S^{2n+1}) \to \Omega S^{2n+1} \to \Omega S^{2n+1}$ induced from the cofibration sequence $S^2 \to S^1 \to \mathbb{R}P^2$, an easy computation shows that $\pi_{2n-1}(\text{Map}^*(\mathbb{R}P^2, S^{2n+1})) \cong \mathbb{Z}/2$. Hence, (ii) holds for $m = 2$, too.

Now we assume that $m \geq 3$ and consider the fibration sequence

$$\text{Map}^*(\mathbb{R}P^m/\mathbb{R}P^{m-2}, S^{2n+1}) \to \text{Map}^*(\mathbb{R}P^m, S^{2n+1}) \xrightarrow{\text{res}} \text{Map}^*(\mathbb{R}P^{m-2}, S^{2n+1}).$$

Since $\text{Map}^*(\mathbb{R}P^{m-2}, S^{2n+1})$ is $(2n - m + 2)$-connected, there is an isomorphism $\pi_{2n-m+1}(\text{Map}^*(\mathbb{R}P^m, S^{2n+1})) \cong \pi_{2n-m+1}(\text{Map}^*(\mathbb{R}P^m/\mathbb{R}P^{m-2}, S^{2n+1}))$. Thus it remains to show the following:

$$\pi_{2n-m+1}(\text{Map}^*(\mathbb{R}P^m/\mathbb{R}P^{m-2}, S^{2n+1})) \cong \begin{cases} \mathbb{Z} & \text{if } m \equiv 1 \pmod{2}, \\ \mathbb{Z}/2 & \text{if } m \equiv 0 \pmod{2}. \end{cases}$$

If $m \equiv 1 \pmod{2}$, $\mathbb{R}P^m/\mathbb{R}P^{m-2} = S^m \vee S^m$ and there is an isomorphism

$$\pi_{2n-m+1}(\text{Map}^*(\mathbb{R}P^m/\mathbb{R}P^{m-2}, S^{2n+1})) \cong \pi_{2n-m+1}(\Omega^m S^{2n+1} \times \Omega^m S^{2n+1}) \cong \mathbb{Z}.$$

Hence (4.1) holds for $m \equiv 1 \pmod{2}$. Finally suppose that $m \equiv 0 \pmod{2}$. Then, because $\mathbb{R}P^m/\mathbb{R}P^{m-2} = S^m \vee 2 e_m$, there is a fibration sequence

$$\text{Map}^*(\mathbb{R}P^m/\mathbb{R}P^{m-2}, S^{2n+1}) \to \Omega^{m-1} S^{2n+1} \to \Omega^{m-1} S^{2n+1}.$$

From the homotopy exact sequence induced by the above sequence, we deduce that $\pi_{2n-m+1}(\text{Map}^*(\mathbb{R}P^m/\mathbb{R}P^{m-2}, S^{2n+1})) \cong \mathbb{Z}/2$. \hfill $\square$

Definition 4.4. (i) Let $\gamma_n : S^n \to \mathbb{R}P^n$ and $\gamma^C_n : S^{2n+1} \to \mathbb{C}P^n$ denote the usual double covering and the Hopf fibering as before. Define the following two maps

$$\begin{cases} 
\gamma_n : \text{Map}^*(\mathbb{R}P^m, S^n) \to \text{Map}^*(\mathbb{R}P^m, \mathbb{R}P^n) \\
\gamma^C_n : \text{Map}^*(\mathbb{R}P^m, S^{2n+1}) \to \text{Map}^*(\mathbb{R}P^m, \mathbb{C}P^n)
\end{cases}$$

by $\gamma_n(h) = \gamma_n \circ h$ and $\gamma^C_n(h') = \gamma^C_n \circ h'$. Since $\text{Map}^*(\mathbb{R}P^m, S^n)$ and $\text{Map}^*(\mathbb{R}P^m, S^{2n+1})$ contain the subspace of constant maps, the images of $\gamma_n$ and that of $\gamma^C_n$ are contained in $\text{Map}^*_0(\mathbb{R}P^m, \mathbb{R}P^n)$ and $\text{Map}^*_0(\mathbb{R}P^m, \mathbb{C}P^n)$, respectively. Thus we obtain two maps

$$\begin{cases} 
\gamma_n : \text{Map}^*(\mathbb{R}P^m, S^n) \to \text{Map}^*_0(\mathbb{R}P^m, \mathbb{R}P^n), \\
\gamma^C_n : \text{Map}^*(\mathbb{R}P^m, S^{2n+1}) \to \text{Map}^*_0(\mathbb{R}P^m, \mathbb{C}P^n).
\end{cases}$$
(ii) Let $\mu_n : \mathbb{R}P^{2n+1} \to \mathbb{C}P^n$ denote the usual projection given by
$$\mu_n([x_0 : x_1 : \cdots : x_{2n+1}] = [x_0 + \sqrt{-1}x_1 : \cdots : x_{2n} + \sqrt{-1}x_{2n+1}],$$
and define the map $\mu_n : \text{Map}_0^{\mathbb{R}P^m, \mathbb{R}P^{2n+1}} \to \text{Map}_0^{\mathbb{R}P^m, \mathbb{C}P^n}$ by $\mu_n(h) = \mu_n \circ h$.

**Lemma 4.5.** (i) If $1 \leq m < n$, $\gamma_n : \text{Map}^*(\mathbb{R}P^m, S^n) \xrightarrow{\sim} \text{Map}_0^*(\mathbb{R}P^m, \mathbb{R}P^n)$ is a homotopy equivalence.

(ii) If $2 \leq m \leq 2n$, $\gamma_n : \text{Map}^*(\mathbb{R}P^m, S^{2n+1}) \xrightarrow{\sim} \text{Map}_0^*(\mathbb{R}P^m, \mathbb{C}P^n)$ is a homotopy equivalence.

**Remark.** Since $\pi_1(\text{Map}^*(\mathbb{R}P^1, S^{2n+1})) = 0$ and $\pi_1(\text{Map}_0^*(\mathbb{R}P^1, \mathbb{C}P^n)) = \mathbb{Z}$, (ii) of Theorem 4.5 does not hold for $m = 1$.

**Proof.** Since (i) follows from [1, Lemma 4.17], it remains to prove (ii). We prove it by induction on $m$. First, assume that $m = 2$, and consider the following commutative diagram of restriction fibration sequences
\[
\begin{array}{c}
\Omega S^{2n+1} \xrightarrow{r} \Omega S^{2n+1} \\
\downarrow \cong \quad \downarrow \gamma_n \quad \downarrow \Omega n_n \\
\Omega^2 \mathbb{C}P^n \xrightarrow{r} \text{Map}_0^*(\mathbb{R}P^2, \mathbb{C}P^n)
\end{array}
\]

Since $\Omega^2\gamma_n$ is a homotopy equivalence and $\Omega\gamma_n : \pi_k(\Omega S^{2n+1}) \xrightarrow{\sim} \pi_k(\Omega \mathbb{C}P^n)$ is an isomorphism for any $k \geq 2$, by the Five Lemma the induced homomorphism $\gamma'_n : \pi_k(\text{Map}^*(\mathbb{R}P^2, S^{2n+1})) \xrightarrow{\sim} \pi_k(\text{Map}_0^*(\mathbb{R}P^2, \mathbb{C}P^n))$ is an isomorphism for any $k \geq 2$. On the other hand, because the homotopy exact sequence of the lower row of (4.2) is
\[
0 \to \pi_1(\text{Map}_0^*(\mathbb{R}P^2, \mathbb{C}P^n)) \to \mathbb{Z} = \pi_1(\Omega \mathbb{C}P^n) \xrightarrow{\partial} \pi_0(\Omega^2 \mathbb{C}P^n) = \mathbb{Z} \to 0,
\]
we see that $\pi_1(\text{Map}_0^*(\mathbb{R}P^2, \mathbb{C}P^n)) = 0$. Thus, since $\text{Map}^*(\mathbb{R}P^2, S^{2n+1})$ is $(2n-2)$-connected (by Lemma 4.3), the induced homomorphism $\Omega\gamma'_n : \pi_1(\Omega S^{2n+1}) \xrightarrow{\sim} \pi_1(\Omega \mathbb{C}P^n)$ is an isomorphism. Hence, $\gamma'_n : \gamma_n$ is an isomorphism for any $k \geq 1$ and the assertion is true for $m = 2$.

Now suppose that $\gamma'_n : \text{Map}^*(\mathbb{R}P^m, S^{2n+1}) \xrightarrow{\sim} \text{Map}_0^*(\mathbb{R}P^m, \mathbb{C}P^n)$ is a homotopy equivalence for some $m \geq 3$, and consider the following commutative diagram of restriction fibration sequences
\[
\begin{array}{c}
\Omega^m S^{2n+1} \xrightarrow{r} \text{Map}^*(\mathbb{R}P^m, S^{2n+1}) \\
\downarrow \cong \quad \downarrow \gamma_n \quad \downarrow \Omega n_n \\
\Omega^m \mathbb{C}P^n \xrightarrow{r} \text{Map}_0^*(\mathbb{R}P^m, \mathbb{C}P^n)
\end{array}
\]

Since $\Omega^m\gamma_n$ and $\gamma'_n$ are homotopy equivalences, by the Five Lemma, we see that $\gamma_n : \text{Map}^*(\mathbb{R}P^m, S^{2n+1}) \xrightarrow{\sim} \text{Map}_0^*(\mathbb{R}P^m, \mathbb{C}P^n)$ is also a homotopy equivalence.

**Corollary 4.6.** If $2 \leq m \leq 2n$, $\text{Map}_0^*(\mathbb{R}P^m, \mathbb{C}P^n)$ is $(2n-m)$-connected and
\[
\pi_{2n-m+1}(\text{Map}_0^*(\mathbb{R}P^m, \mathbb{C}P^n)) \cong \begin{cases} 
\mathbb{Z} & \text{if } m \equiv 1 \pmod{2}, \\
\mathbb{Z}/2 & \text{if } m \equiv 0 \pmod{2}.
\end{cases}
\]

**Proof.** This follows from Lemma 4.3 and Lemma 4.5.
Corollary 4.7. If \( 2 \leq m \leq 2n \), the map \( \mu_{m, \#} : \text{Map}_0^\#(\mathbb{R}P^m, \mathbb{R}P^{2n+1}) \xrightarrow{\cong} \text{Map}_0^\#(\mathbb{R}P^m, \mathbb{C}P^n) \) is a homotopy equivalence.

**Proof.** Consider the commutative diagram

\[
\begin{array}{ccc}
\text{Map}_0^{\#}(\mathbb{R}P^m, S^{2n+1}) & \xrightarrow{\gamma_{2n+1, \#}} & \text{Map}_0^{\#}(\mathbb{R}P^m, \mathbb{C}P^n) \\
\downarrow \cong & & \downarrow \cong \\
\text{Map}_0^{\#}(\mathbb{R}P^m, \mathbb{R}P^{2n+1}) & \xrightarrow{\mu_{m, \#}} & \text{Map}_0^{\#}(\mathbb{R}P^m, \mathbb{C}P^n)
\end{array}
\]

Since \( \gamma_{2n+1, \#} \) and \( \gamma_{m, \#}^C \) are homotopy equivalences by Lemma 4.5, the assertion easily follows from the above commutative diagram. \( \square \)

Now we can prove Theorem 1.5 and Corollary 1.6.

**Proof of Theorem 1.5.** First, it is easy to see the diagram

\[
A^0_d(m, 2n + 1) \xrightarrow{j^0_d} \text{Map}^*(\mathbb{R}P^m, S^{2n+1})
\]

is commutative. Since \( j^0_d \) is a homotopy equivalence through dimension \( D_\mathbb{R}(d; m, 2n + 1) \) if \( m < 2n \) and a homotopy equivalence through dimension \( D_\mathbb{R}(d; m, 2n + 1) \) if \( m = 2n \), by Theorem 1.3 so is the map \( j^C_d \). Because \( D_\mathbb{C}(d; m, n) = D_\mathbb{R}(d; m, 2n + 1) \), we see that the map \( j^C_d \) is a homotopy equivalence through dimension \( D_\mathbb{C}(d; m, n) \) if \( m < 2n \) and a homotopy equivalence through dimension \( D_\mathbb{C}(d; m, n) \) if \( m = 2n \).

It remains to show that the same holds for map \( i^C_d \). However, this follows easily from the facts that \( i^C_d = \gamma_{m, \#}^C \circ j^C_d \) and \( \gamma_{m, \#}^C \) is a homotopy equivalence (by Lemma 4.5). This completes the proof of Theorem 1.5. \( \square \)

**Proof of Corollary 1.6.** Since \( j^C_{d+2} \circ s_d = j^C_d \), the assertion easily follows from Theorem 1.5. \( \square \)

5. The stabilized space \( A^C_{\infty + \epsilon}(m, n) \).

Although we cannot prove Conjecture 2.1, we can prove the following stabilized version.

**Definition 5.1.** For \( \epsilon = 0 \) or 1, let \( A^C_{\infty + \epsilon}(m, n) \) denote the stabilized space \( A^C_{\infty + \epsilon}(m, n) = \lim_{k \to \infty} A^C_{2k + \epsilon}(m, n) \), where the limit is taken over the stabilization maps

\[
A^C_{2k + \epsilon}(m, n) \xrightarrow{s_{2k + \epsilon}} A^C_{2k + \epsilon}(m, n) \xrightarrow{s_{2k + \epsilon + 2}} A^C_{2k + \epsilon}(m, n) \xrightarrow{s_{2k + \epsilon + 2}} \cdots
\]

From the commutative diagram

\[
\begin{array}{ccc}
A^C_{2k + \epsilon}(m, n) & \xrightarrow{s_{2k + \epsilon}} & A^C_{2k + \epsilon}(m, n) & \xrightarrow{s_{2k + \epsilon + 2}} & \cdots \\
\downarrow \Psi^C_{2k + \epsilon} & & \downarrow \Psi^C_{2k + \epsilon + 2} & & \cdots \\
\text{Alg}_{2k + \epsilon}^C(\mathbb{R}P^m, \mathbb{C}P^n) & \xrightarrow{c} & \text{Alg}_{2k + \epsilon + 2}(\mathbb{R}P^m, \mathbb{C}P^n) & \xrightarrow{c} & \cdots
\end{array}
\]

we obtain a stabilized map \( \Psi^C_{\infty + \epsilon} = \lim_{k \to \infty} \Psi^C_{2k + \epsilon} : A^C_{\infty + \epsilon}(m, n) \to \text{Alg}^*_C(m, n) \).
Proposition 5.2. If $2 \leq m \leq 2n$ and $\epsilon = 0$, the map
\[ \Psi^C_{\infty+0} : A^C_{\infty+0}(m, n) \xrightarrow{\sim} \text{Alg}_0^C(m, n) \simeq \text{Map}_0^*(\mathbb{R}P^m, \mathbb{C}P^n) \]
is a homotopy equivalence if $m < 2n$ and a homology equivalence if $m = 2n$.

Proof. The assertion easily follows from Theorem 2.3, Corollary 1.7 and the commutative diagram
\[
\begin{array}{ccc}
A^0_{\infty}(m, n) & \xrightarrow{\lim_{k \to \infty}} & \text{Map}_0^*(\mathbb{R}P^m, \mathbb{C}P^n) \\
\Psi^C_{\infty+0} & | & \\
\text{Alg}_0^C(m, n) & \xrightarrow{i} & \text{Map}_0^*(\mathbb{R}P^m, \mathbb{C}P^n) \quad \square
\end{array}
\]

6. The space $\text{Alg}_1^*(\mathbb{R}P^m, \mathbb{C}P^n)$.

In this section we investigate the homotopy of $A^C_1(m, n) \cong \text{Alg}_1^*(\mathbb{R}P^m, \mathbb{R}P^n)$.

Definition 6.1. For integers $1 \leq m \leq 2n$, let $V_{2n+1,m}$ denote the real Stiefel manifold of all orthogonal $m$-frames in $\mathbb{R}^{2n+1}$ and $(b_1, \ldots, b_m) \in V_{2n+1,m}$ any element such that $b_k = t(b_{k,1}, \ldots, b_{k,2n+1}) \in \mathbb{R}^{2n+1}$ ($1 \leq k \leq m$). Consider the $(n+1)$-tuple of polynomials defined by
\[
(f_0, \ldots, f_n) = (z_0, \ldots, z_m) \left( \begin{array}{ccccccc}
1 & 0 & 0 & \cdots & 0 & 0 \\
\sqrt{-1}b & c_1 & c_2 & \cdots & c_{n-1} & c_n
\end{array} \right),
\]
where $b \in \mathbb{R}^m$ and $c_k \in \mathbb{C}^m$ ($k = 1, 2, \ldots, n$) are given by
\[
\begin{cases}
b_k = t(b_{1,1}, b_{1,2}, b_{1,3}, \ldots, b_{1,m}), \\
c_k = t(b_{2k,1} + \sqrt{-1}b_{2k+1,1}, b_{2k,2} + \sqrt{-1}b_{2k+1,2}, \ldots, b_{2k,m} + \sqrt{-1}b_{2k+1,m})
\end{cases}
\]
Since it is easy to see that $(f_0, \ldots, f_n) \in A^C_1(m, n)$, one can define the map $\varphi_{m,n} : V_{2n+1,m} \to A^C_1(m, n)$ by $\varphi_{m,n}(b_1, \ldots, b_m) = (f_0, \ldots, f_n)$.

Lemma 6.2. If $1 \leq m \leq 2n$, the map $\varphi_{m,n} : V_{2n+1,m} \xrightarrow{\sim} A^C_1(m, n)$ is a homotopy equivalence.

Proof. Let us consider the element $(f_0, \ldots, f_n) \in \mathbb{C}[z_0, \ldots, z_m]^{n+1}$ of the form
\[
(f_0, \ldots, f_n) = (z_0, \ldots, z_m) \left( \begin{array}{ccccccc}
1 & 0 & \cdots & 0 \\
a_{1,0} & a_{1,1} & \cdots & a_{1,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m,0} & a_{m,1} & \cdots & a_{m,n}
\end{array} \right),
\]
where $a_{k,j} = b_{k,j} + \sqrt{-1}c_{k,j}$ ($b_{k,j}, c_{k,j} \in \mathbb{R}$) and we write
\[
\begin{cases}
a = (b_{1,0}, b_{2,0}, \ldots, b_{m,0}) \in \mathbb{R}^m \\
B = (b_{k,j})_{1 \leq k \leq m, 1 \leq j \leq n}, \quad C = (c_{k,j})_{1 \leq k \leq m, 0 \leq j \leq n},
\end{cases}
\]
It is easy to see that the polynomials \( f_0, \ldots, f_n \) have no common real root beside zero if and only if the equation
\[
(z_0, z_1, \ldots, z_m) \left( \begin{array}{cccccc}
1 & 0 & 0 & 0 & \cdots & 0 \\
b_{1,0} & c_{1,0} & b_{1,1} & c_{1,1} & \cdots & b_{1,n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
b_{m,0} & c_{m,0} & b_{m,1} & c_{m,1} & \cdots & b_{m,n} \\
\end{array} \right) = \left( \begin{array}{c}
0 \\
0 \\
\vdots \\
0 \\
\end{array} \right)
\]
has no non-zero solution. So we see that \( (f_0, \ldots, f_n) \in A_1^C(m, n) \) if and only if the \((m \times (2n + 1))\)-matrix \((B, C)\) has rank \(m\). Thus the map \( \Phi : V_{2n+1,m} \times \mathbb{R}^m \to A_1^C(m, n) \) given by \((B, C, a) \mapsto (f_0, \ldots, f_n)\) is clearly a homeomorphism, where \((f_0, \ldots, f_n)\) is defined by (6.1). Since \( \varphi_{m, n} = \Phi|V_{2n+1,m} \times \{0_m\} \), the map \( \varphi_{m, n} \) is a homotopy equivalence.

**Lemma 6.3.** (i) If \( 2 \leq m \leq 2n \), \( \text{Map}^*_1(\mathbb{RP}^m, \mathbb{CP}^n) \) is \((2n - m)\)-connected.

(ii) If \( 1 \leq m \leq 2n \), \( V_{2n+1,m} \) is \((2n - m)\)-connected.

(iii) If \( 1 \leq m \leq 2n \), \( \pi_{2n-m+1}(V_{2n+1,m}) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 1 \pmod{2}, \\ \mathbb{Z}/2 & \text{if } m \equiv 0 \pmod{2}. \end{cases} \)

**Proof.** (i) Consider the restriction fibration sequence
\[(6.2) \quad \Omega^m \mathbb{CP}^n \overset{j}{\to} \text{Map}^*_1(\mathbb{RP}^m, \mathbb{CP}^n) \overset{r}{\to} \text{Map}^*_1(\mathbb{RP}^{m-1}, \mathbb{CP}^n).\]
We prove the assertion (i) by induction on \(m\). First, consider the case \(m = 2\). Since \( \pi_1(\Omega^2 \mathbb{CP}^n) = 0 \) and \( \text{Map}^*_1(\mathbb{RP}^2, \mathbb{CP}^n) = \Omega \mathbb{CP}^n \), taking \(m = 2\) in (6.2) and using the induced exact sequence
\[0 \to \pi_1(\text{Map}^*_1(\mathbb{RP}^2, \mathbb{CP}^n)) \overset{r}{\to} \pi_1(\Omega \mathbb{CP}^n) = \mathbb{Z} \overset{j}{\to} \mathbb{Z} = \pi_0(\Omega^2 \mathbb{CP}^n) \to 0,
\]
we see that \( \pi_1(\text{Map}^*_1(\mathbb{RP}^2, \mathbb{CP}^n)) = 0 \). Since \( \pi_k(\Omega^2 \mathbb{CP}^n) \cong \pi_{k+2}(S^{2n+1}) = 0 \) and \( \pi_k(\Omega \mathbb{CP}^n) \cong \pi_{k+1}(S^{2n+1}) = 0 \) for \(2 \leq k \leq 2n - 2\), applying (6.2) to \(m = 2\) we see that \( \pi_1(\text{Map}^*_1(\mathbb{RP}^2, \mathbb{CP}^n)) = 0 \). Hence, the case \(m = 2\) has been proved. Next, assume that \( \text{Map}^*_1(\mathbb{RP}^{m-1}, \mathbb{CP}^n) \) is \((2n - m + 1)\)-connected for some \(m \geq 3\). Since \(3 \leq m < 2n\), \( \Omega^m \mathbb{CP}^n = \Omega^m S^{2n+1} \) is \((2n - m)\)-connected. Hence, from (6.2) we easily deduce that \( \text{Map}^*_1(\mathbb{RP}^m, \mathbb{CP}^n) \) is \((2n - m)\)-connected. We have proved (i).

(ii) The assertion (ii) can be easily proved by induction on \(m\) by making use of the following fibration sequence:
\[(6.3) \quad S^{2n-m-1} \to V_{2n+1,m} \to V_{2n+1,m-1}.\]
We omit the details.

(iii) First, let \(m \equiv 0 \pmod{2}\). It is known that \(H^{2n-m+2}(V_{2n+1,m}, \mathbb{Z}/p) = 0\) for any odd prime \(p \geq 3\),
\[H^*(V_{2n+1,m}, \mathbb{Z}/2) = E[x_j : 2n - m + 1 \leq j \leq 2n] \quad (|x_j| = j)\]
and that \(S^q(x_{2n-m+1}) = x_{2n-m+2}\). Hence, the \((2n-m+2)\)-skeleton of \(V_{2n+1,m}\) is \(S^{2n-m+1} \cup \partial e^{2n-m+2}\) (up to homotopy equivalence), and we have \(\pi_{2n-m+1}(V_{2n+1,m}) = \mathbb{Z}/2\). If \(m \equiv 1 \pmod{2}\), (6.3) induces the exact sequence
\[\pi_{N+1}(V_{2n+1,m-1}) = \mathbb{Z}/2 \overset{\partial}{\to} \pi_N(S^{2n-m+1}) = \mathbb{Z} \to \pi_N(V_{2n+1,m}) = 0,
\]
where \(N = 2n - m + 1\). Hence, \(\pi_{2n-m+1}(V_{2n+1,m}) \cong \mathbb{Z}\). Thus (iii) has also been proved.
Corollary 6.4. If $1 \leq m \leq 2n$, $\text{Alg}^+_1(\mathbb{RP}^m, \mathbb{CP}^n)$ is $(2n - m)$-connected and

$$\pi_{2n-m+1}(\text{Alg}^+_1(\mathbb{RP}^m, \mathbb{CP}^n)) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 1 \pmod{2}, \\ \mathbb{Z}/2 & \text{if } m \equiv 0 \pmod{2}. \end{cases}$$

Proof. Since there is a homotopy equivalence $V_{2n+1,m} \simeq \text{Alg}^+_1(\mathbb{RP}^m, \mathbb{CP}^n)$, the assertion follows from Lemma 6.3.

Definition 6.5. For $1 \leq m \leq 2n$, define the map $i_m : V_{2n+1,m} \to \text{Map}^+_1(\mathbb{RP}^m, \mathbb{CP}^n)$ by $i_m = i_1^1 \circ \varphi_{m,n}$.

For $2 \leq m \leq 2n$, it is easy to verify that the following diagram is commutative

$$
\begin{array}{ccc}
S^{2n-m+1} & \xrightarrow{j} & V_{2n+1,m} \\
\downarrow{s_m} & & \downarrow{i_m} \\
\Omega^m \mathbb{CP}^n & \xrightarrow{j} & \text{Map}^+_1(\mathbb{RP}^m, \mathbb{CP}^n) \\
\end{array}
$$

(6.4)

where we identify $S^{2n-m+1} \simeq \Omega(2n-2-m)/\Omega(2n+1-m)$ and the two rows are fibration sequences.

Lemma 6.6. If $2 \leq m \leq 2n$, the map $s_m : S^{2n-m+1} \to \Omega^m \mathbb{CP}^n$ is a homotopy equivalence up to dimension $D(1;m,n) = 4n - 2m + 1$.

Proof. By means of a method similar to the one used in the proof of [17, Lemma 3.1] we can show that $s_m : \pi_{2n-m+1}(S^{2n-m+1}) \xrightarrow{\sim} \pi_{2n-m+1}(\Omega^m S^{2n+1}) \cong \pi_{2n-2m+1}(\Omega^m \mathbb{CP}^n)$ is an isomorphism. Thus we can identify $s_m$ with the $m$-fold suspension $E_m : S^{2n-m+1} \to \Omega^m S^{2n+1} \simeq \Omega^m \mathbb{CP}^n$ (up to homotopy equivalence). Hence $s_m$ is a homotopy equivalence up to dimension $4n - 2m + 1$.

Lemma 6.7. If $n \geq 2$, $i_1^*: \pi_k(V_{2n+1,1}) \to \pi_k(\text{Map}^+_1(\mathbb{RP}^1, \mathbb{CP}^n))$ is an isomorphism for any $2 \leq k < 4n - 1$ and an epimorphism for $k = 4n - 1$.

Proof. After identifications (up to homotopy equivalence) $V_{2n+1,1} = S^2$ and $\text{Map}^+_1(\mathbb{RP}^1, \mathbb{CP}^n) = \Omega^1 \mathbb{CP}^n = \Omega S^{2n+1} \times S^1$, the map $i_1$ can be viewed as a map $\tilde{i}_1 : S^2 \to \Omega S^1$. Let $q_1 : \Omega \mathbb{CP}^n = \Omega S^{2n+1} \times S^1 \to \Omega S^{2n+1}$ denote the projection onto the first factor. Note that the composite $q_1 \circ \tilde{i}_1$ can be identified with the map $S^2 \simeq \Omega^1_{(2n+1)}(\mathbb{R}) \to \Omega S^{2n+1}$ given in [8, Corollary 5 (2)]. Hence it follows from [8, Corollary 5] that $q_1 \circ \tilde{i}_1$ is a homotopy equivalence up to dimension $N(1,2n+2) = 4n - 1$. Recalling that $q_{1*} : \pi_l(\Omega \mathbb{CP}^n) \xrightarrow{\sim} \pi_l(\Omega S^{2n+1})$ is an isomorphism for all $l \geq 2$, we see that $\pi_k(\tilde{i}_1)$ is an isomorphism for any $2 \leq k < 4n - 1$ and an epimorphism for $k = 4n - 1$.

Proposition 6.8. If $2 \leq m \leq 2n$, the map $\tilde{i}_m : V_{2n+1,m} \to \text{Map}^+_1(\mathbb{RP}^m, \mathbb{CP}^n)$ is a homotopy equivalence up to dimension $D_C(1;m,n) = 4n - 2m + 1$.

Proof. The proof proceeds by induction on $m$. First, consider the case $m = 2$. Since $\pi_1(V_{2n+1,2}) = \pi_1(\text{Map}^+_1(\mathbb{RP}^2, \mathbb{CP}^n)) = 0$ by Lemma 6.3 the map $\tilde{i}_2$ induces an isomorphism on $\pi_1(\ )$. Hence it suffices to show that $\tilde{i}_2^* : \pi_k(V_{2n+1,2}) \to \pi_k(\text{Map}^+_1(\mathbb{RP}^2, \mathbb{CP}^n))$ is an isomorphism for any $2 \leq k < D_C(1;2,n) = 4n - 1$ and an epimorphism for $k = D_C(1;2,n)$. However, recalling the commutative diagram (6.4) for $m = 2$, and using the Five Lemma, Lemma 6.6 and Lemma 6.7 we see that...
\[ \pi_k(\hat{i}_2) \] is an isomorphism for any \( 2 \leq k < 4n - 1 = D_C(1; 2, n) \) and an epimorphism for \( k = 4n - 1 = D_C(1; 2, n) \). Hence, the case \( m = 2 \) is proved.

Now assume that the map \( \hat{i}_{m-1} \) is a homotopy equivalence up to dimension \( D_C(1; m - 1, n) = 4n - 2m + 3 \) for some \( m \geq 3 \). Since, by Lemma 6.6 \( \hat{s}_m \) is a homotopy equivalence up to dimension \( D_C(1; m, n) = 4n - 2m + 1 \), the Five Lemma and the diagram (6.4) imply that the map \( \hat{i}_m \) is a homotopy equivalence up to dimension \( D_C(1; m, n) \).

**Proof of Theorem 1.8**

(i) Since \( i_1^C = i_{1, C} \circ \Psi_1^C \) and the projection map \( \Psi_1^C : A_1^C(m, n) \xrightarrow{\sim} \text{Alg}_1^C(\mathbb{R}P^m, \mathbb{C}P^n) \) is a homeomorphism, it suffices to show that \( i_1^C \) is a homotopy equivalence up to dimension \( D_C(1; m, n) \). As \( i_m = i_1^C \circ \psi_{m,n} \) and \( \psi_{m,n} \) is a homotopy equivalence, this follows from Proposition 6.8. Hence, (i) has been proved.

(ii) Since \( \hat{s}_{2n^*} : \pi_1(S^1) \to \pi_1(\Omega^{2n}\mathbb{C}P^n) \) is an epimorphism by Lemma 6.0 and \( \pi_1(S^1) \cong \mathbb{Z} \cong \pi_1(\Omega\mathbb{C}P^n) \), \( \hat{s}_{2n} \) induces an isomorphism on \( \pi_1(\cdot) \). Consider the following commutative diagram of the exact sequences induced from (6.4) for \( m = 2n \):

\[
\begin{array}{cccccc}
\pi_2(V_{2n+1,2n-1}) & \xrightarrow{\partial} & \pi_1(S^1) & \longrightarrow & \pi_1(V_{2n+1,2n}) = \mathbb{Z}/2 & \longrightarrow & 0 \\
\hat{i}_{2n-1} \downarrow \cong & & & & & & \\
\pi_2(\text{Map}_{1}^{*}) & \xrightarrow{\partial'} & \pi_1(\Omega\mathbb{C}P^n) & \longrightarrow & \pi_1(\text{Map}_{1}^{*}(\mathbb{R}P^{2n-1}, \mathbb{C}P^n)) & \longrightarrow & 0 \\
\hat{i}_{2n} \downarrow \cong & & & & & & \\
\end{array}
\]

where \( \text{Map}_{1}^{*} = \text{Map}_{1}^{*}(\mathbb{R}P^{2n-1}, \mathbb{C}P^n) \). Since, by Proposition 6.8 the induced homomorphism \( \hat{i}_{2n-1} : \pi_2(V_{2n+1,2n-1}) \xrightarrow{\cong} \pi_2(\text{Map}_{1}^{*}(\mathbb{R}P^{2n-1}, \mathbb{C}P^n)) \) is an isomorphism, \( \hat{i}_{2n} \) induces an isomorphism on \( \pi_1(\cdot) \). 

**Proof of Corollary 1.9** The assertion (i) follows from Lemma 1.8 and Theorem 1.5. The assertion (ii) also easily follows from Corollary 4.0 Lemma 6.2 Lemma 6.3 Proposition 6.8 and Theorem 1.8.

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