A refined version of Grothendieck’s anabelian conjecture for hyperbolic curves over finite fields

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Abstract. In this paper we prove a refined version of a theorem by Tamagawa and Mochizuki on isomorphisms between (tame) arithmetic fundamental groups of hyperbolic curves over finite fields, where one “ignores” the information provided by a “small” set of primes.

§0. Introduction
§1. Review of the local theory
§2. Large and small sets of primes relative to a hyperbolic curve over a finite field
§3. Review of Mochizuki’s cuspidalization theory of proper hyperbolic curves over finite fields
§4. Isomorphisms between geometrically pro-$\Sigma$ arithmetic fundamental groups
§5. On the fundamental theorem of projective geometry

§0. Introduction. Let $k$ be a finite field of characteristic $p > 0$ and $U$ a hyperbolic curve over $k$. Namely, $U = X \setminus S$, where $X$ is a proper, smooth, geometrically connected curve of genus $g$ over $k$ and $S \subset X$ is a divisor which is finite étale of degree $r$ over $k$, such that $2 - 2g - r < 0$. We have the following commutative diagram of profinite groups:

\[
1 \to \pi_1(U \times_k \bar{k}, \ast) \to \pi_1(U, \ast) \to G_k \to 1
\]

\[
1 \to \pi_1^t(U \times_k \bar{k}, \ast) \to \pi_1^t(U, \ast) \to G_k \to 1
\]

in which both rows are exact and all vertical arrows are surjective (and bijective for $r = 0$). Here, $G_k$ is the absolute Galois group $\text{Gal}(\bar{k}/k)$, $\ast$ means a suitable geometric point, and $\pi_1$ (resp. $\pi_1^t$) stands for the étale (resp. tame) fundamental group. The following result is fundamental in the anabelian geometry of hyperbolic curves over finite fields.

Theorem A (Tamagawa, Mochizuki). Let $U, V$ be hyperbolic curves over finite fields $k_U, k_V$, respectively. Let

\[
\alpha : \pi_1(U, \ast) \sim \to \pi_1(V, \ast)
\]

be an isomorphism of profinite groups. Then $\alpha$ arises from a uniquely determined commutative diagram of schemes:

\[
\begin{array}{ccc}
\tilde{U} & \sim \to & \tilde{V} \\
\downarrow & & \downarrow \\
U & \sim \to & V
\end{array}
\]
in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale universal coverings determined by the profinite groups \( \pi_1(U, \ast), \pi_1(V, \ast) \), respectively.

Theorem A was proved by Tamagawa (cf. [Tamagawa], Theorem (4.3)) in the affine case (together with the variant where \( \pi_1 \) is replaced by \( \pi_1' \)), and by Mochizuki (cf. [Mochizuki1], Theorem 3.2) in the proper case. It implies, in particular, that one can embed a suitable category of hyperbolic curves over finite fields into the category of profinite groups. It is essential in the anabelian philosophy of Grothendieck, as was formulated in [Grothendieck], to be able to determine the image of this functor. Recall that the full structure of the profinite group \( \pi_1(U \times_k \bar{k}, \ast) \) is unknown (for any single example of \( U \) which is hyperbolic). Hence, a fortiori, the structure of \( \pi_1(U, \ast) \) is unknown. (Even if we replace the fundamental groups \( \pi_1(U \times_k \bar{k}, \ast), \pi_1(U, \ast) \) by the tame fundamental groups \( \pi_1^t(U \times_k \bar{k}, \ast), \pi_1^t(U, \ast) \), respectively, the situation is just the same.) Thus, the problem of determining the image of the above functor seems to be quite difficult, at least for the moment. In this paper we investigate the following question:

**Question 0.1.** Is it possible to prove any result analogous to the above Theorem A where \( \pi_1(U, \ast) \) is replaced by some (continuous) quotient of \( \pi_1(U, \ast) \) whose structure is better understood?

Let \( \text{Primes} \) be the set of all prime numbers. Let \( \Sigma = \Sigma_X \subset \text{Primes} \) be a set of prime numbers containing at least one prime number different from the characteristic \( p \). Let \( C \) be the full class of finite groups whose cardinality is divisible only by primes in \( \Sigma \). Let \( \Delta_U \defeq \pi_1^t(U \times_k \bar{k}, \ast)_{\Sigma} \) be the maximal pro-\( C \) quotient of \( \pi_1^t(U \times_k \bar{k}, \ast) \). Here, if \( \Sigma \) does not contain \( p \), the structure of \( \Delta_U \) is well understood: \( \Delta_U \) is isomorphic to the pro-\( \Sigma \) completion of a certain well-known finitely generated discrete group (i.e., either a free group or a surface group). Let \( \Pi_U \defeq \pi_1(U, \ast) / \ker(\pi_1^t(U \times_k \bar{k}, \ast) \to \pi_1^t(U \times_k \bar{k}, \ast)_{\Sigma}) \) be the corresponding quotient of \( \pi_1(U, \ast) \). We shall refer to \( \Pi_U \) as the maximal geometrically pro-\( \Sigma \) quotient of the tame fundamental group \( \pi_1^t(U, \ast) \) or, in short, the geometrically pro-\( \Sigma \) tame fundamental group of \( U \). (When \( \Sigma \) does not contain \( p \), we may and shall refer to it as the maximal geometrically pro-\( \Sigma \) quotient of the fundamental group \( \pi_1(U, \ast) \) or, in short, the geometrically pro-\( \Sigma \) fundamental group of \( U \).)

**Question 0.2.** Is it possible to prove any result analogous to the above Theorem A where \( \pi_1(U, \ast) \) is replaced by \( \Pi_U \), for some non-empty set of prime numbers \( \Sigma \) containing at least one prime number different from the characteristic \( p \)?

The first set \( \Sigma \) to consider is the set \( \Sigma \defeq \text{Primes} \setminus \{ \text{characteristic} = p \} \). In this case we shall refer to \( \Pi_U \) as the maximal geometrically prime-to-characteristic quotient of the fundamental group \( \pi_1(U, \ast) \). We have the following result:

**Theorem B** (A Prime-to-\( p \) Version of Grothendieck’s Anabelian Conjecture for Hyperbolic Curves over Finite Fields). Let \( U, V \) be hyperbolic curves over finite fields \( k_U, k_V \), respectively. Let \( \Sigma_U \defeq \text{Primes} \setminus \{ \text{char}(k_U) \} \), \( \Sigma_V \defeq \text{Primes} \setminus \{ \text{char}(k_V) \} \), and write \( \Pi_U, \Pi_V \) for the geometrically pro-\( \Sigma_U \) étale fundamental group of \( U \), and the geometrically pro-\( \Sigma_V \) étale fundamental group of \( V \), respectively. Let

\[ \alpha : \Pi_U \xrightarrow{\sim} \Pi_V \]
be an isomorphism of profinite groups. Then \( \alpha \) arises from a uniquely determined commutative diagram of schemes:

\[
\begin{array}{ccc}
\tilde{U} & \sim & \tilde{V} \\
\downarrow & & \downarrow \\
U & \sim & V
\end{array}
\]

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups \( \Pi_U, \Pi_V \), respectively.

Theorem B was proved by Saïdi and Tamagawa (cf. [Saïdi-Tamagawa1], Corollary 3.10). Our main result in this paper is the following refined version of the above Theorems A and B (cf. Theorem 4.22).

**Theorem C (A Refined Version of the Grothendieck Anabelian Conjecture for Proper Hyperbolic Curves over Finite Fields).** Let \( X, Y \) be proper hyperbolic curves over finite fields \( k_X, k_Y \) of characteristic \( p_X, p_Y \), respectively. Let \( \Sigma_X, \Sigma_Y \subset \text{Primes} \) be sets of prime numbers and set \( \Sigma'_X \overset{\text{def}}{=} \text{Primes} \setminus \Sigma_X, \Sigma'_Y \overset{\text{def}}{=} \text{Primes} \setminus \Sigma_Y \). Assume that neither the \( \Sigma'_X \)-adic representation \( \rho_{\Sigma'_X} : G_{k_X} \to \prod_{l \in \Sigma'_X} \text{GL}(T_l(J_X)) \) nor the \( \Sigma'_Y \)-adic representation \( \rho_{\Sigma'_Y} : G_{k_Y} \to \prod_{l \in \Sigma'_Y} \text{GL}(T_l(J_Y)) \), arising from the Jacobian varieties \( J_X, J_Y \) of \( X, Y \), respectively, is injective. Write \( \Pi_X, \Pi_Y \) for the geometrically pro-\( \Sigma_X \) étale fundamental group of \( X \) and the geometrically pro-\( \Sigma_Y \) étale fundamental group of \( Y \), respectively. Let

\[
\alpha : \Pi_X \sim \Pi_Y
\]

be an isomorphism of profinite groups. Then \( \alpha \) arises from a uniquely determined commutative diagram of schemes:

\[
\begin{array}{ccc}
\tilde{X} & \sim & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \sim & Y
\end{array}
\]

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups \( \Pi_X, \Pi_Y \), respectively.

Note that the extra assumptions on \( \Sigma_X \) and \( \Sigma_Y \) in Theorem C are satisfied if \( \Sigma'_X, \Sigma'_Y \) are finite. We show that sets of primes \( \Sigma_X \) and \( \Sigma_Y \) satisfying the conditions in Theorem C must be of (natural) density \( \neq 0 \), while given any \( \epsilon > 0 \) there exist sets of primes \( \Sigma_X \) and \( \Sigma_Y \) of (natural) density \( < \epsilon \) satisfying the conditions in Theorem C (cf. Remark 2.8).

Theorem C above implies a “similar” version for affine hyperbolic curves (cf. Theorem 4.23).

**Theorem D (A Refined Version of the Grothendieck Anabelian Conjecture for (Not Necessarily Proper) Hyperbolic Curves over Finite Fields).** Let \( U, V \) be (not necessarily proper) hyperbolic curves over finite fields \( k_U, k_V \) of characteristic \( p_U, p_V \), respectively. Let \( \Sigma_U, \Sigma_V \subset \text{Primes} \), be sets of
prime numbers, and set \( \Sigma_U' \triangleq \text{Primes} \setminus \Sigma_U, \Sigma_V' \triangleq \text{Primes} \setminus \Sigma_V \). Write \( \Pi_U, \Pi_V \) for the geometrically pro-\( \Sigma_U \) tame fundamental group of \( U \) and the geometrically pro-\( \Sigma_V \) tame fundamental group of \( V \), respectively. Let

\[
\alpha : \Pi_U \sim \Pi_V
\]

be an isomorphism of profinite groups. Assume that there exist open subgroups \( \Pi_{U'} \subset \Pi_U, \Pi_{V'} \subset \Pi_V \), which correspond to each other via \( \alpha \), i.e., \( \Pi_{V'} = \alpha(\Pi_{U'}) \), corresponding to \( \epsilon \)-étale coverings \( U' \to U, V' \to V \), such that the smooth compactifications \( X' \) of \( U' \) and \( Y' \) of \( V' \) are hyperbolic, and that neither the \( \Sigma_U \)-adic representation \( \rho_{\Sigma_U} : G_{k_U} \to \prod_{l \in \Sigma_U} \text{GL}(T_l(J_{X'})) \) nor the \( \Sigma_V \)-adic representation \( \rho_{\Sigma_V} : G_{k_V} \to \prod_{l \in \Sigma_V} \text{GL}(T_l(J_{Y'})) \), arising from the Jacobian varieties \( J_{X'}, J_{Y'} \) of \( X', Y' \), respectively, is injective. (Here, \( k_{U'}, k_{V'} \) denote the fields of constants of \( U', V' \), respectively.) Then \( \alpha \) arises from a uniquely determined commutative diagram of schemes:

\[
\begin{array}{ccc}
U & \sim & V \\
\downarrow & & \downarrow \\
\tilde{U} & \sim & \tilde{V}
\end{array}
\]

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups \( \Pi_U, \Pi_V \), respectively.

In what follows we explain the steps/ideas of the proof of Theorem C. Starting from an isomorphism

\[
\alpha : \Pi_X \sim \Pi_Y
\]

between profinite groups, one can first, using well-known results on the group-theoretic characterization of decomposition groups in arithmetic fundamental groups as in [Tamagawa] (the so-called local theory), establish a set-theoretic bijection

\[
\phi : \text{X}^{\text{cl}} \setminus E_X \sim \text{Y}^{\text{cl}} \setminus E_Y
\]

between the set of closed points of \( X, Y \), outside some “exceptional sets” \( E_X \subset X^{\text{cl}} \) and \( E_Y \subset Y^{\text{cl}} \), respectively, such that \( \alpha(D_x) = D_{\phi(x)} \) for \( x \in X^{\text{cl}} \setminus E_X \) where \( D_x, D_{\phi(x)} \) denote the decomposition group of \( x, \phi(x) \) in \( \Pi_X, \Pi_Y \), respectively (which are only defined up to conjugation). It is not difficult to prove \( p \triangleq p_X = p_Y \) and \( \Sigma \triangleq \Sigma_X = \Sigma_Y \). As a technical step in the proof we resort to a specific auxiliary prime number \( l \) and consider the \( \mathbb{Z}_l \)-extensions \( k_X^l, k_Y^l \) of \( k_X, k_Y \), respectively. Let \( X^l \triangleq X \times_{k_X} k_X^l \), and \( Y^l \triangleq Y \times_{k_Y} k_Y^l \). Write \( E_{X^l} \triangleq E_X \times_{k_X} k_X^l \) (resp. \( E_{Y^l} \triangleq E_Y \times_{k_Y} k_Y^l \), \( \mathcal{O}_{E_{X^l}}, \mathcal{O}_{E_{Y^l}} \) for the rings of rational functions on \( X^l \), \( Y^l \) whose poles are disjoint from \( E_{X^l}, E_{Y^l} \), respectively, and \( \mathcal{O}_{E_{X^l}^\times}, \mathcal{O}_{E_{Y^l}^\times} \) the multiplicative groups of \( \mathcal{O}_{E_{X^l}}, \mathcal{O}_{E_{Y^l}} \), respectively. We have a natural set-theoretic bijection \( \phi^l : (X^l)^{\text{cl}} \setminus E_{X^l} \sim (Y^l)^{\text{cl}} \setminus E_{Y^l} \). Next, certain finite index subgroups \( H^\times_{X^l}, H^\times_{Y^l} \) of \( \mathcal{O}_{E_{X^l}}^\times, \mathcal{O}_{E_{Y^l}}^\times \) are naturally associated with \( \alpha \) via Kummer theory, such that \( \alpha : \Pi_X \sim \Pi_Y \) induces a commutative diagram:

\[
\begin{array}{ccc}
H^\times_{X^l}/((k_X^l)^{\times}\{\Sigma'\}) & \sim \sim \sim & H^\times_{Y^l}/((k_Y^l)^{\times}\{\Sigma'\}) \\
\downarrow & & \downarrow \\
H^\times_{X^l}/(k_X^l)^{\times} & \left\sim \sim \sim \right \downarrow \phi^l \sim \sim \sim H^\times_{Y^l}/(k_Y^l)^{\times}
\end{array}
\]
in which the vertical arrows are the natural surjective homomorphisms and the horizontal arrows are natural isomorphisms induced by $\alpha$, where $\Sigma' \overset{\text{def}}{=} \Primes \setminus \Sigma$, and $(k_y^l)^\times \{ \Sigma' \}$ (resp. $(k_Y^l)^\times \{ \Sigma' \}$) is the $\Sigma'$-primary part of the multiplicative group $(k^l_Y)^\times$ (resp. $(k_Y^l)^\times$).

The isomorphism $\bar{\rho} : H^\times_{Y,l} / (k_Y^l)^\times \overset{\cong}{\to} H^\times_{X,l} / (k_X^l)^\times$ between subgroups of groups of principal divisors supported outside exceptional sets has the property that it preserves the valuations of functions, with respect to the set-theoretic bijection $\phi : (X^l)^{\cl} \times E_{X,l} \overset{\sim}{\to} (Y^l)^{\cl} \times E_{Y,l}$. We think of elements of $O^\times_{E_{X,l}} / ((k_X^l)^\times \{ \Sigma' \})$ and $O^\times_{E_{Y,l}} / ((k_Y^l)^\times \{ \Sigma' \})$ as "pseudo-functions", i.e., classes of rational functions with divisor supported outside exceptional sets, modulo $\Sigma'$-primary constants. In particular, given a pseudo-function $f' \in O^\times_{E_{X,l}} / ((k_X^l)^\times \{ \Sigma' \})$ (resp. $g' \in O^\times_{E_{Y,l}} / ((k_Y^l)^\times \{ \Sigma' \})$), and a closed point $x \in X^l \times E_X$ (resp. $y \in Y^l \times E_Y$) it makes sense to consider the $\Sigma$-value $f'(x) \in (k(x)^\times)_{\Sigma}$ (resp. $g'(y) \in (k(y)^\times)_{\Sigma}$) of $f'$ (resp. $g'$) (cf. Lemma 4.5 and the discussion before it). Here, $(k(x)^\times)_{\Sigma}$, $(k(y)^\times)_{\Sigma}$ denote the maximal $\Sigma$-primary quotient of the multiplicative group of residues of the residue fields $k(x)$, $k(y)$, respectively. Then the isomorphism $\rho' : H^\times_{X,l} / (k_X^l)^\times \{ \Sigma' \} \to H^\times_{Y,l} / (k_Y^l)^\times \{ \Sigma' \}$ has the property that it preserves the $\Sigma$-value of the pseudo-functions with respect to the set-theoretic bijection $\phi : (X^l)^{\cl} \times E_{X,l} \overset{\sim}{\to} (Y^l)^{\cl} \times E_{Y,l}$. Let $R_{X,l} = \langle H^\times_{X,l} \rangle$, $R_{Y,l} = \langle H^\times_{Y,l} \rangle$ denote the abelian subgroups of $O_{E_{X,l}}, O_{E_{Y,l}}$ generated by $H^\times_{X,l}, H^\times_{Y,l}$, respectively. In fact, $R_{X,l}, R_{Y,l}$ are subalgebras of $O_{E_{X,l}}, O_{E_{Y,l}}$ over $k_X^l, k_Y^l$, respectively, having the same fields of fractions, and $O_{E_{X,l}}, O_{E_{Y,l}}$ are the normalizations of $R_{X,l}, R_{Y,l}$, respectively. We think of the multiplicative groups $H^\times_{X,l} / (k_X^l)^\times, H^\times_{Y,l} / (k_Y^l)^\times$ as subsets of the projective spaces $\mathbb{P}(R_{X,l}), \mathbb{P}(R_{Y,l})$ associated to the infinite-dimensional $k_X^l$-vector space $R_{X,l}$, $k_Y^l$-vector space $R_{Y,l}$, respectively. Using again, in an essential way, the fact that the set $\Sigma$ satisfies the assumptions in Theorem C we show that the isomorphism $\bar{\rho} : H^\times_{Y,l} / (k_Y^l)^\times \to H^\times_{X,l} / (k_X^l)^\times$ viewed as a bijection between subsets of the projective spaces $\mathbb{P}(R_{Y,l})$ and $\mathbb{P}(R_{X,l})$ preserves "partial" collineations in the following sense: given a line $\ell \subset \mathbb{P}(R_{Y,l})$ such that $\ell \cap (H^\times_{Y,l} / (k_Y^l)^\times) \neq \emptyset$ then there exists a unique line $\ell' \subset \mathbb{P}(R_{X,l})$ such that $\ell' \cap (H^\times_{X,l} / (k_X^l)^\times) \neq \emptyset$ and $\bar{\rho}(\ell \cap (H^\times_{Y,l} / (k_Y^l)^\times)) = \ell' \cap (H^\times_{X,l} / (k_X^l)^\times)$. If $H^\times_{X,l} / (k_X^l)^\times = O^\times_{E_{X,l}} / (k_X^l)^\times, H^\times_{Y,l} / (k_Y^l)^\times = O^\times_{E_{Y,l}} / (k_Y^l)^\times$, and $E_X = E_Y = \emptyset$, then $\bar{\rho} : K^\times_{X,l} / (k_X^l)^\times \to K^\times_{Y,l} / (k_Y^l)^\times$ is a bijection between points of the projective spaces $\mathbb{P}(K_{X,l})$ and $\mathbb{P}(K_{Y,l})$, which preserves collineations, where $K_{X,l}$ (resp. $K_{Y,l}$) is the function field of $X^l$ (resp. $Y^l$). Thus, by the fundamental theorem of projective geometry, it arises from a unique semi-linear isomorphism $(K_{X,l}, +) \sim (K_{Y,l}, +)$. Unfortunately, at this stage we are even not able to prove that the exceptional sets $E_X$ and $E_Y$ are finite. This causes a very serious difficulty. To overcome this difficulty, we prove, in §5, a refined version of the fundamental theorem of projective geometry, which may be of interest independently of the topic of this paper (cf. Theorem 5.7), and which applies well in our situation in order to recover the ring structures of $O_{E_{X,l}}, O_{E_{Y,l}}$, respectively. More precisely, given a commutative field $k$ we define the notion of an admissible set $S$ of subsets of $\mathbb{P}^1(k)$ (cf. Definition 5.4) (roughly speaking these are sets consisting of "small" subsets of $\mathbb{P}^1(k)$). For a subset $U \subset \mathbb{P}(V)$ of a projective space $\mathbb{P}(V)$ associated to a $k$-vector space $V$, we define the notion of being $S$-ample where $S$ as above is admissible (cf. Definition 5.6). Roughly speaking, being $S$-ample means that $U$ is "sufficiently
large” in some sense (cf. loc. cit.). Let \( \mathbb{L}(V) \) be the set of lines in \( \mathbb{P}(V) \), and \( \mathbb{L}(V)_U \triangleq \{ \ell \in \mathbb{L}(V) \mid \ell \cap U \neq \emptyset \} \). Our main result is the following (cf. Theorem 5.7).

**Theorem E (A Refined Version of the Fundamental Theorem of Projective Geometry).** Let \( V_i \) be a \( k_i \)-vector space for \( i = 1, 2 \). Assume that \( \dim_{k_i}(V_i) \geq 3 \) for \( i = 1, 2 \). Let \( U_i \) be a subset of \( \mathbb{P}(V_i) \) for \( i = 1, 2 \), and assume that \( U_i \) is \( S_i \)-ample for some admissible set \( S_i \) of subsets of \( \mathbb{P}^1(k_i) \) for \( i = 1, 2 \). Let \( \sigma : U_1 \rightarrow U_2 \) and \( \tau : \mathbb{L}(V_1)_{U_1} \rightarrow \mathbb{L}(V_2)_{U_2} \) be bijections such that for each \( \ell \in \mathbb{L}(V_1)_{U_1} \), one has \( \tau(\ell)_{U_2} = \sigma(\ell)_{U_1} \). Then, each such \( (\sigma, \tau) : (U_1, \mathbb{L}(V_1)_{U_1}) \rightarrow (U_2, \mathbb{L}(V_2)_{U_2}) \) uniquely extends to a collineation \( (\tilde{\sigma}, \tilde{\tau}) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \rightarrow (\mathbb{P}(V_2), \mathbb{L}(V_2)) \). Thus, \( \tilde{\sigma} \) is a bijection between projective spaces which preserves collineation. In particular, there exists an isomorphism \( \mu : k_1 \rightarrow k_2 \), and a \( \mu \)-semi-linear isomorphism of abelian groups \( \lambda : (V_1, +) \rightarrow (V_2, +) \) that induces \( (\sigma, \tau) : (U_1, \mathbb{L}(V_1)_{U_1}) \rightarrow (U_2, \mathbb{L}(V_2)_{U_2}) \). Moreover, such an isomorphism \( (\mu, \lambda) \) is unique up to scalar multiplication.

Theorem E applies well in our case. More precisely, applying Theorem E to the above situation we deduce that there exists a unique isomorphism \( \tilde{\rho} : \mathbb{P}(R_{Y^l}) \rightarrow \mathbb{P}(R_{X^l}) \) which extends the bijection \( \tilde{\rho} : H^X_{Y^l}/(k_Y^l)^{\times} \rightarrow H^Y_{X^l}/(k_X^l)^{\times} \) and \( \tilde{\rho} \) preserves collineation. In particular, the bijection \( \tilde{\rho} \) arises from a \( \psi_0 \)-isomorphism

\[
\psi : (R_{Y^l}, +) \rightarrow (R_{X^l}, +),
\]

where \( \psi_0 : k_Y^l \rightarrow k_X^l \) is a field isomorphism. Namely, \( \psi \) is an isomorphism of abelian groups which is semilinear with respect to \( \psi_0 \) in the sense that \( \psi(ax) = \psi_0(a)\psi(x) \) for \( a \in k_Y^l \) and \( x \in R_{Y^l} \). Further, \( \psi_0 \) is uniquely determined and \( \psi \) is uniquely determined up to scalar multiplication. Moreover, if we normalize the isomorphism \( \psi : (R_{Y^l}, +) \rightarrow (R_{X^l}, +) \), by the condition \( \psi(1) = 1 \), it becomes a ring isomorphism such that the diagram

\[
\begin{array}{ccc}
R_{X^l} & \xrightarrow{\psi} & R_{Y^l} \\
\uparrow & & \uparrow \\
k_X^l & \xrightarrow{\psi_0} & k_Y^l
\end{array}
\]

commutes. Further, \( \psi \) induces a natural commutative diagram

\[
\begin{array}{ccc}
X^l & \xrightarrow{\psi} & Y^l \\
\downarrow & & \downarrow \\
X & \xrightarrow{\psi} & Y
\end{array}
\]

where the horizontal maps are scheme isomorphisms and the vertical maps are natural morphisms. By passing to open subgroups of \( \Pi_X \) and \( \Pi_Y \) which correspond to each other via \( \alpha \), one constructs the desired scheme isomorphism \( \tilde{X} \rightarrow \tilde{Y} \) which is compatible with the isomorphism \( \psi : X \rightarrow Y \). Here, one has to overcome the difficulty that the assumptions on the set \( \Sigma \) in Theorem C are not preserved by passing to open subgroups: even if the representation \( \rho_{\Sigma', X} : G_{k_X} \rightarrow \prod_{i \in \Sigma'} GL(T_{ij}(J_X)) \) is not injective and \( \Pi_X \subset \Pi_X' \) is an open subgroup, the representation \( \rho_{\Sigma', X'} : \)
$G_{k_{X'}} \to \prod_{l \in \Sigma'} \text{GL}(T_l(J_{X'}))$ might be injective. We overcome this problem by introducing certain (weaker but more technical) conditions which are preserved by passing to open subgroups.

In §1, we review the main results of the local theory mainly from [Saïdi-Tamagawa1], and how various invariants of the curve $X$ can be recovered group-theoretically from $\Pi_X$. In §2, we define and discuss the notion of large set of primes relative to a hyperbolic curve over a finite field. In §3, we review the main results of Mochizuki’s theory of cuspidalization of étale fundamental groups of proper hyperbolic curves, which plays an essential role in this paper. In §4, we prove our main results: Theorem C and Theorem D. In §5, we prove the refined version of the fundamental theorem of projective geometry: Theorem E.

Remark 0.3. (i) A function field version of the main results of the present paper is given in [Saïdi-Tamagawa3]. (See also [Saïdi-Tamagawa1] and [Saïdi-Tamagawa2] for the special case $\Sigma = \text{Primes} \setminus \{p\}$.)

(ii) At the moment of writing this paper, we do not know (even in the function field case) if pro-$l$ versions of the above results hold, namely if the above Theorems C and D hold (under a certain Frobenius-preserving assumption) in the case where $\Sigma = \{l\}$ consists of a single prime $l$ which is different from $p$.

§1. Review of the local theory. In this section we briefly review the main results in [Saïdi-Tamagawa1], §1 concerning the local theory in arithmetic fundamental groups of hyperbolic curves over finite fields. Let $X$ be a proper, smooth, geometrically connected curve over a finite field $k = k_X$ of characteristic $p = p_X > 0$. Write $K = K_X$ for the function field of $X$.

Let $S$ be a (possibly empty) finite set of closed points of $X$, and set $U = X \setminus S$. We assume that $U$ is hyperbolic.

Fix a separable closure $K^\text{sep} = K_X^\text{sep}$ of $K$, and write $\overline{k} = \overline{k_X}$ for the algebraic closure of $k$ in $K^\text{sep}$. Write

$$G_K \overset{\text{def}}{=} \text{Gal}(K^\text{sep}/K),$$
$$G_k \overset{\text{def}}{=} \text{Gal}(\overline{k}/k)$$

for the absolute Galois groups of $K$ and $k$, respectively.

The tame fundamental group $\pi_1^t(U)$ with respect to the base point defined by $K^\text{sep}$ (where “tame” is with respect to the complement of $U$ in $X$) can be naturally identified with a quotient of $G_K$. Write $\text{Gal}(K_U^t/K)$ for this quotient. (In case $S = \emptyset$, we also write $K_U^t$ for $K_U^t$.) It is easy to see that $K_U^t$ contains $K\overline{k}$.

Let $\Sigma = \Sigma_X$ be a set of prime numbers that contains at least one prime number different from $p$. Write

$$\Sigma^t \overset{\text{def}}{=} \Sigma \setminus \{p\}.$$ 

Thus, $\Sigma^t \neq \emptyset$ by our assumption. Denote by $\hat{\Sigma}^t$ the maximal pro-$\Sigma^t$ quotient of $\hat{\mathbb{Z}}$. Set $\Sigma' = \Sigma_X' = \text{Primes} \setminus \Sigma_X$. We say that $\Sigma$ is cofinite if $\#(\Sigma') < \infty$.

We define $\overline{K}_U$ to be the maximal pro-$\Sigma$ subextension of $K\overline{k}$ in $K_U^t$. Now, set

$$\Pi_U = \text{Gal}(\overline{K}_U/K),$$

which is a quotient of $\pi_1^t(U) = \text{Gal}(K_U^t/K)$. This fits into the exact sequence

$$1 \to \Delta_U \to \Pi_U \overset{\text{pr}_U}{\to} G_k \to 1.$$
Here, $\Delta_U$ is the maximal pro-$\Sigma$ quotient of $\pi_1(\overline{U})$, where, for a $k$-scheme $Z$, we set $\overline{Z} \overset{\text{def}}{=} Z \times_k \overline{k}$.

Define $\tilde{X}_U$ to be the integral closure of $X$ in $\tilde{K}_U$. Define $\tilde{U}$ to be the integral closure of $U$ in $\tilde{K}_U$, which can be naturally identified with the inverse image (as an open subscheme) of $U$ in $\tilde{X}_U$. Define $\tilde{S}_U$ to be the inverse image (as a set) of $S$ in $\tilde{X}_U$.

For a scheme $Z$, write $Z^{\text{cl}}$ for the set of closed points of $Z$. Then we have

$$X^{\text{cl}} = U^{\text{cl}} \coprod S,$$

$$(\tilde{X}_U)^{\text{cl}} = \tilde{U}^{\text{cl}} \coprod \tilde{S}_U.$$  

Moreover, $(\tilde{X}_U)^{\text{cl}}$ admits a natural action of $\Pi_U$, and the corresponding quotient can be naturally identified with $X^{\text{cl}}$.

For each $\tilde{x} \in (\tilde{X}_U)^{\text{cl}}$, we define the decomposition group $D_{\tilde{x}} \subset \Pi_U$ (respectively, the inertia group $I_{\tilde{x}} \subset D_{\tilde{x}}$) to be the stabilizer at $\tilde{x}$ of the natural action of $\Pi_U$ on $(\tilde{X}_U)^{\text{cl}}$ (respectively, the kernel of the natural action of $D_{\tilde{x}}$ on $k(\tilde{x}) = \overline{k(x)} = \overline{k}$, where $x$ is the image of $\tilde{x}$ in $X^{\text{cl}}$). These groups fit into the following commutative diagram in which both rows are exact:

$$1 \to I_{\tilde{x}} \to D_{\tilde{x}} \to G_{k(x)} \to 1$$

$$\cap \quad \cap \quad \cap$$

$$1 \to \Delta_U \to \Pi_U \to G_{k} \to 1$$

Moreover, $I_{\tilde{x}} = \{1\}$ (respectively, $I_{\tilde{x}}$ is (non-canonically) isomorphic to $\hat{Z}^{\Sigma^\dagger}$), if $\tilde{x} \in \tilde{U}^{\text{cl}}$ (respectively, $\tilde{x} \in \tilde{S}_U$). Since $I_{\tilde{x}}$ is normal in $D_{\tilde{x}}$, $D_{\tilde{x}}$ acts on $I_{\tilde{x}}$ by conjugation. Since $I_{\tilde{x}}$ is abelian, this action factors through $D_{\tilde{x}} \to G_{k(x)}$ and induces a natural action of $G_{k(x)}$ on $I_{\tilde{x}}$.

Let $G$ be a profinite group. Then, define $\text{Sub}(G)$ (respectively, $\text{OSub}(G)$) to be the set of closed (respectively, open) subgroups of $G$.

By conjugation, $G$ acts on $\text{Sub}(G)$. More generally, let $H$ and $K$ be closed subgroups of $G$ such that $K$ normalizes $H$. Then, by conjugation, $K$ acts on $\text{Sub}(H)$. We denote by $\text{Sub}(H)_K$ the quotient $\text{Sub}(H)/K$ by this action. In particular, $\text{Sub}(G)_G$ is the set of conjugacy classes of closed subgroups of $G$.

For any closed subgroups $H, K$ of $G$ with $K \subset H$, we have a natural inclusion $\text{Sub}(K) \subset \text{Sub}(H)$, as well as a natural map $\text{Sub}(H) \to \text{Sub}(K)$, $J \mapsto J \cap K$. By using this latter natural map, we define

$$\overline{\text{Sub}}(G) \overset{\text{def}}{=} \lim_{\text{projlim}} \text{Sub}(H).$$

Observe that $\overline{\text{Sub}}(G)$ can be identified with the set of commensurate classes of closed subgroups of $G$. (Closed subgroups $J_1$ and $J_2$ of $G$ are called commensurate (to each other), if $J_1 \cap J_2$ is open both in $J_1$ and in $J_2$.)

With these notations, we obtain natural maps

$$D = D[U] : (\tilde{X}_U)^{\text{cl}} \to \text{Sub}(\Pi_U), \tilde{x} \mapsto D_{\tilde{x}},$$

...
\[ I = I[U] : (\tilde{X}_U)_{\text{cl}} \to \text{Sub}(\Delta_U) \subset \text{Sub}(\Pi_U), \tilde{x} \mapsto I_{\tilde{x}}, \]

which fit into the commutative diagram

\[
\begin{array}{ccc}
(\tilde{X}_U)_{\text{cl}} & \xrightarrow{D} & \text{Sub}(\Pi_U) \\
\| & \downarrow & \\
(\tilde{X}_U)_{\text{cl}} & \xrightarrow{I} & \text{Sub}(\Delta_U)
\end{array}
\]

where the vertical arrow stands for the natural map \( \text{Sub}(\Pi_U) \to \text{Sub}(\Delta_U), \ J \mapsto J \cap \Delta_U \). By composition with the natural map \( \text{Sub}(\Pi_U) \to \text{Sub}(\Pi_U) \), \( D, I \) yield

\[
D = D[U] : (\tilde{X}_U)_{\text{cl}} \to \text{Sub}(\Pi_U),
\]

\[
I = I[U] : (\tilde{X}_U)_{\text{cl}} \to \text{Sub}(\Delta_U) \subset \text{Sub}(\Pi_U).
\]

Note that, unlike the case of \( D, I \), the maps \( \overline{D}, \overline{I} \) are essentially unchanged if we replace \( U \) by any covering corresponding to an open subgroup of \( \Pi_U \).

Since the maps \( D, I \) are \( \Pi_U \)-equivariant, they induce natural maps

\[
D_{\Pi_U} = D[U]_{\Pi_U} : X_{\text{cl}} \to \text{Sub}(\Pi_U)_{\Pi_U},
\]

\[
I_{\Pi_U} = I[U]_{\Pi_U} : X_{\text{cl}} \to \text{Sub}(\Delta_U)_{\Pi_U} \subset \text{Sub}(\Pi_U)_{\Pi_U},
\]

respectively.

**Definition 1.1.** Let \( f : A \to B \) be a map of sets.

(i) We define \( \mu_f : B \to \mathbb{Z} \cup \{ \infty \} \) by \( \mu_f(b) = \sharp(f^{-1}(b)) \). (Thus, \( f \) is injective (respectively, surjective) if \( \mu_f(b) \leq 1 \) (respectively, \( \mu_f(b) \geq 1 \)) for any \( b \in B \). We also have \( f(A) = \{ b \in B \mid \mu_f(b) \geq 1 \} \).)

(ii) We say that \( f \) is quasi-finite, if \( \mu_f(b) < \infty \) for any \( b \in B \).

(iii) We say that an element \( a \) of \( A \) is an exceptional element of \( f \) (in \( A \)), if \( \mu_f(f(a)) > 1 \). We refer to the set of exceptional elements as the exceptional set.

(iv) We say that a pair \( (a_1, a_2) \) of elements of \( A \) is an exceptional pair of \( f \) (in \( A \)), if \( a_1 \neq a_2 \) and \( f(a_1) = f(a_2) \) hold.

(v) We say that \( f \) is almost injective (in the strong sense), if the exceptional set of \( f \) is finite. (Observe that almost injectivity implies quasi-finiteness.)

**Definition 1.2.** Denote by \( E_{\tilde{U}} \) the exceptional set of \( \overline{D} \) in \((\tilde{X}_U)_{\text{cl}}\).

**Proposition 1.3.** Let \( \overline{\rho} \) denote the natural morphism \( \tilde{X}_U \to \overline{X} \). Then, for each \( \overline{x} \in X_{\text{cl}} \), \( \overline{D}_{\overline{\rho}^{-1}(\overline{x})} \) is injective.

**Proof.** Cf. [Saïdi-Tamagawa1], Proposition 1.8(iii). \( \square \)

**Definition 1.4.** We define \( E_U \) to be the image of \( E_{\tilde{U}} \) in \( X_{\text{cl}} \). (This can be identified with \( E_G/\Pi_U \).

Next, we shall explain how various invariants and structures of \( U \) can be recovered group-theoretically (or \( \varphi \)-group-theoretically) from \( \Pi_U \), in the following sense.
Definition 1.5. (i) We say that \( \Pi = (\Pi, \Delta, \varphi_{\Pi}) \) is a \( \varphi \)-(profinite) group, if \( \Pi \) is a profinite group, \( \Delta \) is a closed normal subgroup of \( \Pi \) and \( \varphi_{\Pi} \) is an element of \( \Pi/\Delta \).

(ii) An isomorphism from a \( \varphi \)-group \( \Pi = (\Pi, \Delta, \varphi_{\Pi}) \) to another \( \varphi \)-group \( \Pi' = (\Pi', \Delta', \varphi_{\Pi'}) \) is an isomorphism \( \Pi \cong \Pi' \) as profinite groups that induces \( \Delta \cong \Delta' \), hence also \( \Pi/\Delta \cong \Pi'/\Delta' \), such that the last isomorphism sends \( \varphi_{\Pi} \) to \( \varphi_{\Pi'} \).

From now on, we regard \( \Pi_U \) as a \( \varphi \)-group by \( \Pi_U = (\Pi_U, \Delta_U, \varphi_k) \), where \( \varphi_k \) stands for the \( \sharp(k) \)-th power Frobenius element in \( G_k = \Pi_U/\Delta_U \). We shall say that an isomorphism \( \alpha : \Pi_U \cong \Pi'_U \) as profinite groups is Frobenius-preserving, if \( \alpha \) is an isomorphism as \( \varphi \)-groups.

Definition 1.6. (i) Given an invariant \( F(U) \) (e.g., a number, a set of numbers, etc.) that depends on the isomorphism class (as a scheme) of a hyperbolic curve \( U \) over a finite field, we say that \( F(U) \) can be recovered group-theoretically (respectively, \( \varphi \)-group-theoretically) from \( \Pi_U \), if any isomorphism (respectively, any Frobenius-preserving isomorphism) \( \Pi_U \cong \Pi_V \) implies \( F(U) = F(V) \) for two such curves \( U, V \). (ii) Given an additional structure \( \mathcal{F}(U) \) (e.g., a family of subgroups, quotients, elements, etc.) on the profinite group \( \Pi_U \) that depends functorially on a hyperbolic curve \( U \) over a finite field (in the sense that, for any isomorphism (as schemes) between two such curves \( U, V \), any isomorphism \( \Pi_U \cong \Pi_V \) induced by this isomorphism \( U \cong V \) (which is unique up to composition with inner automorphisms) preserves the structures \( \mathcal{F}(U) \) and \( \mathcal{F}(V) \), we say that \( \mathcal{F}(U) \) can be recovered group-theoretically (respectively, \( \varphi \)-group-theoretically) from \( \Pi_U \), if any isomorphism (respectively, any Frobenius-preserving isomorphism) \( \Pi_U \cong \Pi_V \) between two such curves \( U, V \) preserves the structures \( \mathcal{F}(U) \) and \( \mathcal{F}(V) \).

Proposition 1.7. The following invariants and structures can be recovered group-theoretically from \( \Pi_U \):

(i) The subgroup \( \Delta_U \) of \( \Pi_U \), hence the quotient \( G_k = \Pi_U/\Delta_U \).

(ii) The subsets \( \Sigma \) and \( \Sigma^\dagger \) of \( \mathfrak{Primes} \).

Proof. Cf. [Saïdi-Tamagawa1], Proposition 1.15(i)(ii). □

Finally, we shall explain that the set of decomposition groups in \( \Pi_U \) can be recovered group-theoretically from \( \Pi_U \). First, we shall treat decomposition groups at points of \( \tilde{S}_U \).

Theorem 1.8. (i) The set of inertia groups at points of \( \tilde{S}_U \) (i.e., the image of the injective map \( I|_{\tilde{S}_U} : \tilde{S}_U \to \text{Sub}(\Delta_U) \subset \text{Sub}(\Pi_U) \)) can be recovered \( \varphi \)-group-theoretically from \( \Pi_U \).

(ii) The set of decomposition groups at points of \( \tilde{S}_U \) (i.e., the image of the injective map \( D|_{\tilde{S}_U} : \tilde{S}_U \to \text{Sub}(\Pi_U) \)) can be recovered \( \varphi \)-group-theoretically from \( \Pi_U \).

Proof. Cf. [Saïdi-Tamagawa1], Theorem 1.18. □

Next, we shall consider decomposition groups at points of \( \tilde{U}^{\text{cl}} \). This is done along the lines of [Tamagawa], §2, but slightly more subtle than the case of [Tamagawa], due to the existence of the exceptional set \( E_{\tilde{U}} \).

Theorem 1.9. The following hold.

(i) The set of decomposition groups at points of \( \tilde{U}^{\text{cl}} \) (respectively, \( \tilde{U}^{\text{cl}} \setminus E_{\tilde{U}} \), respectively, \( E_{\tilde{U}} \)) (i.e., the image of the map \( D|_{\tilde{U}^{\text{cl}}} : \tilde{U}^{\text{cl}} \to \text{Sub}(\Pi_U) \) (respectively,
Definition 2.2. We say that the pair \((P, X)\) has property \(P_\Sigma\) if the following holds:

\[ P_{\Sigma}(f, g): \exists a, b \in k^\times \{\Sigma'\}, \text{ such that } f = a + bg. \]

\[ Q_{F,\Sigma}(f, g): \forall x \in X^{\cl} \setminus X(F)^{\cl}, \exists a_x, b_x \in k(x)^\times \{\Sigma'\}, \text{ such that } f(x) = a_x + b_x g(x). \]

\[ \overline{P}_{\Sigma}(f, g): \exists a, b \in \bar{k}^\times \{\Sigma'\}, \text{ such that } f = a + bg. \]

\[ \overline{Q}_{F,\Sigma}(f, g): \forall x \in X^{\cl} \setminus X(F)^{\cl}, \exists a_x, b_x \in \bar{k}^\times \{\Sigma'\}, \text{ such that } f(x) = a_x + b_x g(x). \]

Here the symbol \(\forall'\) means “for all but finitely many”.

Proof. Cf. [Saidi-Tamagawa], Theorem 1.24 and Corollary 1.25. \(\square\)

Section 2. Large and small sets of primes relative to a hyperbolic curve over a finite field. Throughout this section, let \(\Sigma \subset \text{Primes}\) be a set of prime numbers, and set \(\Sigma' = \text{Primes} \setminus \Sigma\). Let \(k\) be a finite field of characteristic \(p > 0\) and set \(\Sigma^\uparrow = \Sigma \setminus \{p\}\). Write

\[ \hat{\mathbb{Z}}^\uparrow_{\Sigma} \defeq \prod_{\ell \in \Sigma^\uparrow} \mathbb{Z}_\ell. \]

For a prime number \(\ell \in \text{Primes} \setminus \{p\}\) let

\[ \chi_\ell : G_k \to \mathbb{Z}_\ell^\times \]

be the \(\ell\)-adic cyclotomic character, and define the \(\Sigma\)-part of the cyclotomic character by:

\[ \chi_\Sigma \defeq (\chi_\ell)_{\ell \in \Sigma^\uparrow} : G_k \to (\hat{\mathbb{Z}}^\uparrow_{\Sigma})^\times = \prod_{\ell \in \Sigma^\uparrow} \mathbb{Z}_\ell^\times. \]

Thus, we have

\[ k_{\ker(\chi_\Sigma)} = k^\Sigma \defeq k(\zeta_\ell \mid \ell \in \Sigma^\uparrow, j \in \mathbb{Z}_{\geq 0}). \]

For a prime number \(\ell \in \text{Primes}\), let \(G_{k,\ell} \subset G_k\) be the pro-\(l\)-Sylow subgroup of \(G_k\). (Recall that \(G_k \simeq \hat{\mathbb{Z}}\) and \(G_{k,\ell} \simeq \mathbb{Z}_\ell\).) Next, we recall the notion of \(k\)-small and \(k\)-large set of primes. (Cf. [Saidi-Tamagawa3], §3 for more details).

Definition 2.1. \((k\)-Small/\(k\)-Large Set of Primes\) (i) We say that the set \(\Sigma\) is \(k\)-small if the \(\Sigma\)-part \(\chi_\Sigma\) of the cyclotomic character is not injective.

(ii) We say that the set \(\Sigma\) is \(k\)-large if the set \(\Sigma'\) is \(k\)-small.

Note that a \(k\)-large set of primes is not \(k\)-small, by [Saidi-Tamagawa3], Proposition 3.3.

The following results are slight generalizations of results in [Saidi-Tamagawa3], §3. Let \(X\) be a proper, smooth, geometrically connected curve over \(k\), \(f, g : X \to \mathbb{P}^1_k\) nonconstant \(k\)-morphisms, and \(F\) a proper subfield of \(k\) containing \(k\). Write \(X(F)^{\cl} \subset X^{\cl}\) for the image of \(X(F)\) in \(X^{\cl}\).

Definition 2.2. We say that the pair \((f, g)\) has property \(P_\Sigma\) (respectively, \(Q_{F,\Sigma}\), \(P_\Sigma\) and \(\overline{P}_{F,\Sigma}\)) if the following holds:

\[ P_\Sigma(f, g): \exists a, b \in k^\times \{\Sigma'\}, \text{ such that } f = a + bg. \]

\[ Q_{F,\Sigma}(f, g): \forall x \in X^{\cl} \setminus X(F)^{\cl}, \exists a_x, b_x \in k(x)^\times \{\Sigma'\}, \text{ such that } f(x) = a_x + b_x g(x). \]

\[ \overline{P}_{\Sigma}(f, g): \exists a, b \in \bar{k}^\times \{\Sigma'\}, \text{ such that } f = a + bg. \]

\[ \overline{Q}_{F,\Sigma}(f, g): \forall x \in X^{\cl} \setminus X(F)^{\cl}, \exists a_x, b_x \in \bar{k}^\times \{\Sigma'\}, \text{ such that } f(x) = a_x + b_x g(x). \]
Proposition 2.3. (i) We have the following implications:

\[
P_{\Sigma}(f, g) \iff \overline{P}_{\Sigma}(f, g) \\
Q_{F, \Sigma}(f, g) \implies \overline{Q}_{F, \Sigma}(f, g)
\]

(ii) Assume that \(\Sigma\) is \(k\)-large. Then we have the following implication:

\[
\overline{Q}_{F, \Sigma}(f, g) \implies \overline{P}_{\Sigma}(f, g).
\]

Proof. (i) The equivalence in the first row is given in [Saïdi-Tamagawa3], Definition/Proposition 3.5. The remaining implications are immediate.

(ii) Similar to the proof of [Saïdi-Tamagawa3], Proposition 3.11. Indeed, as in the proof of loc. cit., if property \(P_{\Sigma}(f, g)\) does not hold one deduces that there exists a non-empty open subscheme \(V \subset X\) such that for every \(x \in V^{\text{cl}} \setminus X(F)^{\text{cl}}\), one has \(k(x) \subset K\) where \(K/k\) is a subextension of \(\bar{k}/k\) such that \(\bar{k}/K\) is infinite. In particular, for every \(x \in V^{\text{cl}}\), one has \(k(x) \subset K\) or \(k(x) \subset F\). This is not possible:

Let \(\phi : V \to \mathbb{A}^1_k\) be a finite \(k\)-morphism, \(a \in \bar{k} \setminus K \cup F\), and \(x \in \phi^{-1}(a) \subset V^{\text{cl}}\), then \(k(a) \subset k(x) \subset K \cup F\), which is absurd. □

Definition/Proposition 2.4. For a pair \((f, g)\) as in the above discussion, a positive integer \(m\), and a set of prime numbers \(\Sigma \subset \text{Primes}\). We define the following properties:

\[
P_{\Sigma}^{(m)}(f, g) : \exists a, c \in k^\times \{\Sigma'\}, \text{ such that } f = a(1 + cg)^m.
\]

\[
\overline{P}_{\Sigma}^{(m)}(f, g) : \exists a, c \in \bar{k}^\times \{\Sigma'\}, \text{ such that } f = a(1 + cg)^m.
\]

\[
\overline{Q}_{F, \Sigma}^{(m)}(f, g) : \forall x \in X^{\text{cl}} \setminus X(F)^{\text{cl}}, \exists a_x, c_x \in \bar{k}^\times \{\Sigma'\}, \text{ such that } f(x) = a_x(1 + c_x g(x))^m.
\]

Then:

(i) The implications

\[
P_{\Sigma}^{(m)}(f, g) \iff \overline{P}_{\Sigma}^{(m)}(f, g) \implies \overline{Q}_{F, \Sigma}^{(m)}(f, g)
\]

hold.

(ii) If \(\Sigma\) is \(k\)-large, then the implication

\[
\overline{Q}_{F, \Sigma}^{(m)}(f, g) \implies \overline{P}_{\Sigma}^{(m)}(f, g)
\]

holds.

Proof. Similar to the proof of Proposition 2.3. □

The following is the first application of the \(k\)-largetness property to the (geometrically pro-\(\Sigma\), tame) fundamental groups of hyperbolic curves over \(k\).

Proposition 2.5. Let \(U\) be a hyperbolic curve over \(k\), \(X\) the smooth compactification of \(U\), \(g\) the genus of \(X\), \(r\) the cardinality of \(X^\text{cl}_k \setminus U^\text{cl}_k\), and \(\Pi_U\) the geometrically pro-\(\Sigma\) tame fundamental group of \(U\) (cf. §1). Assume that \(\Sigma\) is \(k\)-large. Then the following invariants and structures can be recovered group-theoretically from \(\Pi_U\) (cf. Definition 1.6 for the meaning of being recovered group-theoretically).

(i) The prime number \(p\).

(ii) The \(\varphi\)-th power Frobenius element \(\varphi_k \in G_k\).
(iii) The cardinality $q \overset{\text{def}}{=} \#(k)$ (or, equivalently, the isomorphism class of the finite field $k$).

Proof. First, consider the natural character

$$\rho^\text{det} : G_k \to \text{Aut}(\bigwedge^{\max}_\Sigma (\Delta^a)^{\Sigma_1}) = \left(\hat{\mathbb{Z}}^{\Sigma_1}\right)^\times,$$

which can be group-theoretically recovered, by Proposition 1.7(i)(ii). As in [Tamagawa], Proposition 3.4 and its proof, $\rho^\text{det}$ coincides with $\lambda \cdot (\chi^a)$, where $a = g$ (resp. $a = g + r - 1$) for $r = 0$ (resp. $r > 0$), and $\lambda$ is a certain character with values in $\{\pm 1\}$. (Note that $\lambda = 1$ when $r = 0$.) It follows from the hyperbolicity assumption $2 - 2g - r < 0$ that $a > 0$. In particular, we have $(\rho^\text{det})^2 = (\chi^a)^2$.

(i) For each $N \in \mathbb{Z}_{>0}$, let $k_N/k$ denote the unique finite subextension of $\bar{k}/k$ of degree $N$. Then $G_{k_N} = (G_k)^N \subset G_k$ can be recovered group-theoretically. Consider the coinvariant quotient $\hat{\mathbb{Z}}^{\Sigma_1}_{(\rho^\text{det})^2(G_{k_N})}$ and define $w_{X,N}$ to be its cardinality, which is a group-theoretic invariant. As $(\rho^\text{det})^2 = (\chi^a)^2$, this invariant is computed as:

$$w_{X,N} = (q^{2aN} - 1)\Sigma = \frac{\left(q^{2aN} - 1\right)\Sigma}{\left(q^{2aN} - 1\right)\Sigma'},$$

where, for a positive integer $n$, $n = n_\Sigma n_{\Sigma'}$ stands for the unique decomposition where every prime divisor of $n_\Sigma$ (resp. $n_{\Sigma'}$) belongs to $\Sigma$ (resp. $\Sigma'$). Set

$$M_0 \overset{\text{def}}{=} \inf \mathcal{M}, \quad \mathcal{M} \overset{\text{def}}{=} \{M \in \mathbb{R}_{>0} \mid \exists C > 0, \forall N \in \mathbb{Z}_{>0}, w_{X,N} \leq CM^N\},$$

which is also group-theoretic.

We claim that $M_0 = q^{2a}$. Indeed, since $w_{X,N} \leq q^{2aN} - 1 \leq q^{2aN}$, we have $q^{2a} \in \mathcal{M}$. On the other hand, set $F_0 \overset{\text{def}}{=} \bar{k}/\ker((\rho^\text{det})^2)$. As $\Sigma$ is $k$-large, $F_0$ is a proper subfield of $\bar{k}$. Take a prime $l$ so that $k(l) \overset{\text{def}}{=} F_0 \cap k^l$ is finite, where $k^l$ denotes the unique $\mathbb{Z}_l$-extension of $k$ (cf. Proposition 2.13 below). Write $[k(l) : k] = l^{n_0}$. Then for each $n \geq 0$, we have

$$w_{X,l^n} = (q^{2al^n} - 1)\Sigma = \frac{\left(q^{2al^n} - 1\right)\Sigma}{\left(q^{2al^n} - 1\right)\Sigma'} \geq \frac{\left(q^{2al^n} - 1\right)\Sigma}{\left(q^{2al^n} - 1\right)\Sigma'} \geq \frac{\left(q^{2al^n} - 1\right)\Sigma}{\left(q^{2al^n} - 1\right)\Sigma'} \geq \frac{q^{2al^n}}{q^{2al^n}}.$$

Thus, if $(0 <)M < q^{2a}$, we have $M \notin \mathcal{M}$. The claim now follows.

Now, $p$ can be recovered as the unique prime divisor of $M_0 = q^{2a}$.

(ii) Similar to [Tamagawa], Proposition 3.4(i)(ii).

(iii) Similar to [Tamagawa], Proposition 3.4(iii). \hfill \Box

The notion of $k$-small/$k$-large set of primes can be naturally generalized as follows, by replacing the multiplicative group $\mathbb{G}_{m,k}$ by an abelian variety. Let $A$ be an abelian variety over $k$, and $T(A) = \prod_{l \in \mathbb{Z}_{\times}} T_l(A)$ the (full) Tate module of $A \times_k \bar{k}$. Let $T_\Sigma(A) = \prod_{l \in \Sigma} T_l(A)$ be the maximal pro-$\Sigma$ quotient of $T(A)$. Recall that one has a natural Galois representation $\rho_{A,\Sigma} : G_k \to \text{Aut}(T_\Sigma(A))$.

**Definition 2.6.** (A-Small/A-Large Set of Primes) Let $A$ be an abelian variety over $k$.

(i) We say that the set $\Sigma$ is $A$-small if the Galois representation $\rho_{A,\Sigma} : G_k \to \text{Aut}(T_\Sigma(A))$ is not injective.

(ii) We say that the set $\Sigma$ is $A$-large if the set $\Sigma'$ is $A$-small.
Lemma 2.7. Let $A$ be an abelian variety over $k$ of dimension $g > 0$ and let $\Sigma \subset \text{Primes}$ be a set of prime numbers. If $\Sigma$ is $A$-large, then $\Sigma$ is $k$-large.

Proof. It is well-known that the $2g$-th exterior power of the representation $\rho_{A,\Sigma}$ coincides with the $g$-th power of $\chi_\Sigma$. Hence, an open subgroup of $\text{Ker}(\rho_{A,\Sigma})$ (of index $|g|$) is contained in $\text{Ker}(\chi_\Sigma)$, from which the assertion follows. □

Remark 2.8. Let $A$ be an abelian variety over $k$ of dimension $g > 0$ and let $\Sigma \subset \text{Primes}$ be a set of prime numbers.

(i) By Lemma 2.7 and [Saïdi-Tamagawa3], Remark 3.4.1, if a set of primes $\Sigma \subset \text{Primes}$ is $A$-large, then $\Sigma$ is of (natural) density $\neq 1$.

(ii) On the other hand, for any given $\epsilon > 0$, there exists a set of primes $\Sigma \subset \text{Primes}$ such that $\Sigma$ is $A$-large and that the (natural) density of $\Sigma$ is $< \epsilon$. Indeed, take a prime number $r \neq p$ satisfying $\frac{2g(2g+1)}{2r} < r - 1$. Let $\Sigma \overset{\text{def}}{=} \bigcup_{k=1}^{2g} \{ l \in \text{Primes} \mid l^k \equiv 1 \pmod{r} \} \cup \{ r, p \}$. Observe that the condition $l^k \equiv 1 \pmod{r}$ is equivalent to saying that $l \mod{r} \in \mu_k(\mathbb{F}_r)$, hence the (natural) density of $\Sigma$ is $\leq \frac{\sum_{k=1}^{2g} k}{r} \leq \frac{2g(2g+1)}{2(r-1)} < \epsilon$. We claim that $\Sigma' \overset{\text{def}}{=} \text{Primes}\setminus \Sigma$ is $A$-small. Indeed, $l \in \Sigma'$ implies that $l \neq r, p$ and $r$ does not divide $\sharp \text{GL}_{2g}(\mathbb{F}_l) = (l^{2g} - 1)(l^{2g-1} - 1) \cdots (l^2 - 1)(l-1)\frac{l^{2g(2g-1)}}{2}$. Consider the following homomorphism $G_k \to \text{Aut}(T_{\Sigma'}(A)) \to \text{Aut}(T_l(A)) \to \text{Aut}(A[l])$ where the far left map is the natural $\Sigma'$-adic representation and the right maps are the natural ones. Note that the kernel of the natural surjective map $\text{Aut}(T_l(A)) \to \text{Aut}(A[l])$ is pro-$l$. In particular, the image of the $r$-Sylow subgroup of $G_k$ in $\text{Aut}(A[l])$ is trivial, hence the image of the $r$-Sylow subgroup of $G_k$ in $\text{Aut}(T_{\Sigma'}(A))$ is trivial and $\Sigma'$ is $A$-small.

Next, let $X$ be a proper, smooth, and geometrically connected hyperbolic curve over the finite field $k$. We apply the notations in §1 to $U = X$. In particular, $\Delta_X$ and $\Pi_X$ denote the maximal pro-$\Sigma$ quotient of the geometric fundamental group $\pi_1(X \times_k \bar{k})$ and the maximal geometrically pro-$\Sigma$ quotient of the arithmetic fundamental group $\pi_1(X)$, respectively. For the definition of the exceptional set $E_X \subset X^{\text{cl}}$, see Definition 1.4. Further, let $J_X$ denote the Jacobian variety of $X$.

Definition 2.9. (i) We denote by $F_X$ the compositum of $k(x)$ in $\bar{k}$ for all $x \in E_X$. (Note that $F_X$ depends on $\Sigma$, as so does $E_X$.)

(ii) Let $F$ be a proper subfield of $\bar{k}$ containing $k$: $k \subset F \subset \bar{k}$. We say that $X$ is almost $\Sigma$-separated with respect to $F$ if $E_X \subset X(F)^{\text{cl}}$ or, equivalently, if $F_X \subset F$.

(iii) We say that $X$ is almost $\Sigma$-separated if it is almost $\Sigma$-separated with respect to some proper subfield of $\bar{k}$ containing $k$ or, equivalently, if $F_X \subset \bar{k}$.

Let $k, k'$ be finite fields and $X, X'$ proper, smooth, geometrically connected curves over $k, k'$, respectively. Let $f : X' \to X$ be a finite, generically étale morphism (as schemes), which induces a finite separable extension $k'(X')/k(X)$ of function fields and a finite extension $k'/k$ of constant fields. (In particular, we may identify $\bar{k'} = k$.) Let $L'/k(X)$ denote the Galois closure of $k'(X')/k(X)$.

Definition 2.10. (i) We say that $f$ is a $\Sigma$-covering if the cardinality of the finite group $\text{Gal}(L'/k(X)) \bar{k}$ is divisible only by primes in $\Sigma$.

(ii) We say that $f$ is tame-Galois if $k'(X')/k(X)$ is a Galois extension (i.e., $L' = k'(X')$) and is at most tamely ramified everywhere on $X$. 
Proposition 2.11. Assume that \( f : X' \to X \) is a \( \Sigma \)-covering.

(i) Assume that \( f \) is étale. Then we have \( E_{X'} = f^{-1}(E_X) \). Further, \( X \) is almost \( \Sigma \)-separated if and only if so is \( X' \). More precisely, if \( X' \) is almost \( \Sigma \)-separated with respect to \( F' \), then \( X \) is almost \( \Sigma \)-separated with respect to \( F' \); and, if \( X \) is almost \( \Sigma \)-separated with respect to \( F \), then \( X' \) is almost \( \Sigma \)-separated with respect to some finite extension of \( Fk' \).

(ii) Assume that \( f \) is tame-Galois. Then we have \( E_{X'} \subset f^{-1}(E_X) \cup S' \), where \( S' \subset (X')^{\text{cl}} \) is the non-étale locus of \( f \). Further, if \( X \) is almost \( \Sigma \)-separated, then so is \( X' \). More precisely, if \( X \) is almost \( \Sigma \)-separated with respect to \( F \), then \( X' \) is almost \( \Sigma \)-separated with respect to some finite extension of \( Fk' \).

Proof. (i) When \( f \) is an étale \( \Sigma \)-covering, we have the following commutative diagram:

\[
\begin{array}{ccc}
(\hat{X}')^{\text{cl}} & \xrightarrow{\overline{T}} & \text{Sub}(\Pi_{X'}) \\
\parallel & & \parallel \\
(\hat{X})^{\text{cl}} & \xrightarrow{\overline{T}} & \text{Sub}(\Pi_X)
\end{array}
\]

from which we get \( E_{X'} = f^{-1}(E_X) \). It is clear that, if \( X' \) is almost \( \Sigma \)-separated with respect to \( F' \), then \( X \) is almost \( \Sigma \)-separated with respect to \( F' \). Next, assume that \( X \) is almost \( \Sigma \)-separated with respect to \( F \). Define \( F' \) to be the finite extension of \( F \) corresponding to the open subgroup \( (G_F)^d) \subset G_F \) (of index \( d \) ), where \( d \) is the degree of \( f \). Then \( F' \supset Fk' \) (as \( |Fk' : F| = [k' : k] | [k'(X') : k(X)] = d \) ), and \( X' \) is almost \( \Sigma \)-separated with respect to \( F' \).

(ii) Set \( G \stackrel{\text{def}}{=} \text{Gal}(k'(X')/k(X)) \) and \( \Delta_G \stackrel{\text{def}}{=} \text{Gal}(k'(X')\overline{k}/k(X)\overline{k}) \). Let \( \hat{X} \to X \) (resp. \( \hat{X}' \to X' \)) be the proétale covering corresponding to \( \Pi_X \) (resp. \( \Pi_{X'} \)). Note that \( \hat{X}' \to X \) is a profinite Galois covering with group \( \Pi_{X',G} \) which sits naturally in the following exact sequence \( 1 \to \Pi_{X'} \to \Pi_{X',G} \to \Pi_X \) \( 1 \to \Delta_{X',G} \to \Pi_{X',G} \to \Pi_X \) \( 1 \to \Delta_X \to \Delta_{X',G} \to \Delta_G \to 1 \). Note that if we view \( X \) as an orbicurve, being the stack-theoretic quotient of \( X' \) by the action of the finite group \( G \), then \( \Pi_{X',G} \) is nothing but the geometrically pro-\( \Sigma \) étale fundamental group of the orbicurve \( X \).

Now, let \( x_1' \in E_{X'} \subset (X')^{\text{cl}} \). Then there exists \( \tilde{x}'_1, \tilde{x}'_2 \in (\hat{X})^{\text{cl}} \), \( \tilde{x}'_1 \neq \tilde{x}'_2 \), such that \( \tilde{x}'_1 \) is above \( x_1' \) and that \( D_{\tilde{x}'_1}, D_{\tilde{x}'_2} \) are commensurate in \( \Pi_{X'} \). Let \( \tilde{x}_1, \tilde{x}_2 \in \hat{X}^{\text{cl}} \), \( x_1, x_2 \in (X')^{\text{cl}} \) and \( x_1, x_2 \in \hat{X}^{\text{cl}} \), the images of \( \tilde{x}'_1, \tilde{x}'_2 \), respectively. Then \( D_{\tilde{x}_1}, D_{\tilde{x}_2} \subset \Pi_X \) are commensurate to each other; hence either \( \tilde{x}_1 = \tilde{x}_2 \) or \( \tilde{x}_1, \tilde{x}_2 \in E_X \). In the latter case, we have \( x_1' \in f^{-1}(E_X) \), as desired. In the former case, in particular, the images of \( \tilde{x}'_1, \tilde{x}'_2 \) in \( X^{\text{cl}} \) are equal, hence there exists \( \sigma \in \Delta_{X',G} \) such that \( \sigma \cdot \tilde{x}'_1 = \tilde{x}'_2 \). Write \( Z \stackrel{\text{def}}{=} D_{\tilde{x}'_1} \cap D_{\tilde{x}'_2} \subset \Pi_{X'} \subset \Pi_{X',G} \) for simplicity. First, we follow the proof of [Saidi-Tamagawa1], Lemma 1.7 in order to deduce that \( \sigma \) is torsion. More precisely, let \( Z_0 \stackrel{\text{def}}{=} Z \cap \sigma Z \sigma^{-1} \). Then as in loc. cit. we deduce that \( \sigma \) commutes with any element of \( Z_0 \), i.e., \( Z_0 \subset \overline{Z}_{\Pi_{X',G}}(\langle \sigma \rangle) \). (Here, given a profinite group \( G \) and a closed subgroup \( H \), we write \( Z_G(H) \) for the centralizer of \( H \) in \( G \).) Moreover, arguing by contradiction, suppose \( \sigma \neq 1 \) and let \( \overline{N} \) be a sufficiently small open characteristic subgroup of \( \Delta_{X',G} \) such that \( \sigma \notin \overline{N} \) and set \( \overline{H} \stackrel{\text{def}}{=} \langle \overline{N}, \sigma \rangle \subset \Delta_{X',G} \). Then, as in loc. cit. \( Z_0 \) normalizes \( \overline{H} \) and the image of \( \sigma \) in \( \overline{H}^{\text{ab}} \) is nontrivial and fixed by the action of \( Z_0 \). By observing the Frobenius
weights in the action of \( \mathbb{Z}_l \) we deduce that \( \langle \sigma \rangle \cap \Delta_{X'} = \{1\} \) and \( \sigma \) has finite order.

If \( \sigma = 1 \), then \( \tilde{x}'_1 = \tilde{x}'_2 \) and we are done. Suppose that \( \sigma \neq 1 \). Then it follows from [Mochizuki2], Lemma 4.1(iii) that there exists a unique closed point \( \tilde{y}' \in X' \) such that \( \langle \sigma \rangle \subset I_{\tilde{y}'} \), where \( I_{\tilde{y}'} \) is the inertia subgroup at \( \tilde{y}' \) (necessarily finite). We claim that \( Z_{I_{X',G}}(\langle \sigma \rangle) \) is commensurate to the decomposition group \( D_{\tilde{y}'} \) at \( \tilde{y}' \). Indeed, first there exists an open subgroup \( D_{\tilde{y}'}^0 \subset D_{\tilde{y}'} \) such that \( D_{\tilde{y}'}^0 \subset Z_{I_{X',G}}(\langle \sigma \rangle) \), as follows easily from the fact that \( D_{\tilde{y}'} \) acts by inner conjugation on its normal subgroup \( I_{\tilde{y}'} \) and the group of automorphisms of \( I_{\tilde{y}'} \) is finite. Second, we have a natural exact sequence \( 1 \rightarrow Z_{X',G}(\langle \sigma \rangle) \rightarrow Z_{I_{X',G}}(\langle \sigma \rangle) \rightarrow G_k \). Now, our second claim is that \( Z_{X',G}(\langle \sigma \rangle) \) is finite. Indeed, after possibly replacing the orbicurve \( X \) by a suitable étale cover, corresponding to an open subgroup of \( I_{X',G} \), we can assume that \( \langle \sigma \rangle = I_{\tilde{y}'} \). The assertion then follows from the fact that \( I_{\tilde{y}'} \) is normally terminal in \( \Delta_{X',G} \) (cf. [Mochizuki2], Lemma 4.1(i)). This implies that the subgroups \( Z_0 \) and \( D_{\tilde{y}'}^0 \) of \( Z_{I_{X',G}}(\langle \sigma \rangle) \) are commensurate (cf. above exact sequence), hence \( D_{\tilde{x}'_1} \) and \( D_{\tilde{y}'} \) are commensurate in \( \Pi_{X',G} \), since \( Z_0 \) is open in \( D_{\tilde{x}'_1} \) and \( D_{\tilde{y}'} \) is open in \( D_{\tilde{y}'}^0 \). In particular, \( D_{\tilde{x}'_1} \) and \( D_{\tilde{y}'} \) are commensurate in \( \Pi_X \), where \( \tilde{y} \) is the image of \( \tilde{y}' \) in \( X^{cl} \). Hence, either \( \tilde{x}'_1 = \tilde{y} \) or \( \tilde{x}'_1 \in E_X \). In the former case, we have \( x_1 = y \), hence \( x_1 \in S' \), as desired. In the latter case, we have \( x_1 \in E_X \), hence \( x_1' \in f^{-1}(E_{X'}) \), as desired. \( \square \)

The following application of the \( J_X \)-largeness property is crucial in later sections.

**Proposition 2.12.** If \( \Sigma \) is \( J_X \)-large, then \( X \) is almost \( \Sigma \)-separated. More precisely, then \( X \) is almost \( \Sigma \)-separated with respect to some finite extension \( (\neq k) \) of \( \overline{k}^{Ker(\rho_{J_X,S'})} \).

**Proof.** Let \( F_0 \triangleq \overline{k}^{Ker(\rho_{J_X,S'})} \). Then \( k \subset F_0 \subset \overline{k} \). Let \( F \) denote the finite extension of \( F_0 \) corresponding to the open subgroup \( (G_{F_0})^e \subset G_{F_0} \) (of index \( \leq e \)), where \( e = 2 \) (resp. \( e = 1 \)) if \( X \) is hyperelliptic (resp. otherwise). Now, we claim that the field \( F \) satisfies the above property.

Consider the morphism \( \delta : X \times X \rightarrow J_X, (P,Q) \mapsto cl(P - Q) \). Then \( \delta|_{X \times X \setminus \{X\}} \) is quasi-finite with geometric fibers of cardinality \( \leq e \), where \( \iota : X \rightarrow X \times X \) is the diagonal map (cf. [Saïdi-Tamagawa1], Claim 1.9(i) and its proof). Thus, we have a quasi-finite map \( X(\bar{k}) \times X(\bar{k}) \setminus \iota(X)(\bar{k}) \rightarrow J_X(\bar{k}) \) with fibers of cardinality \( \leq e \).

Let \( \tilde{x}, \tilde{x}' \in \tilde{X}^{cl} \) be an exceptional pair of the map \( \overline{D} \) and \( \tilde{x}, \tilde{x}' \in \tilde{X}^{cl} = X(\bar{k}) \) the images of \( \tilde{x}, \tilde{x}' \), respectively. By Proposition 1.3, we have \( \tilde{x} \neq \tilde{x}' \). Then the image of \( (\tilde{x}, \tilde{x}') \in X(\bar{k}) \times X(\bar{k}) \setminus \iota(X)(\bar{k}) \) via the composed map \( X(\bar{k}) \times X(\bar{k}) \setminus \iota(X)(\bar{k}) \xrightarrow{\delta} J_X(\bar{k}) \rightarrow J_X(\bar{k})/\{J_X(\bar{k})\} = \{\bar{1}\} \) is trivial (cf. [Saïdi-Tamagawa1], proof of Proposition 1.8(iii)). Thus, the image of \( (\tilde{x}, \tilde{x}') \) in \( J_X(\bar{k}) \) lies in the subgroup \( J_X(\bar{k}) \{\Sigma'\} \) which is contained in \( J_X(F_0) \) by the choice of \( F_0 \). It follows from this that \( (\tilde{x}, \tilde{x}') \in X(F) \times X(F) \) since the above map \( \delta|_{X \times X \setminus \{X\}} \) is quasi-finite with geometric fibers of cardinality \( \leq e \). \( \square \)

**Proposition 2.13.** (i) If \( X \) is almost \( \Sigma \)-separated, then there exists a prime number \( l \), such that, for every finite extension \( k'/k \), and every finite extension \( F/F_X \), the field \( k' \cap F \) is finite, where \( k' \) denotes the unique \( \mathbb{Z}_l \)-extension of \( k \). (We shall refer to such a prime number \( l \) as being \( (X,\Sigma) \)-admissible.) In particular, the set \( E_X \cap X(k')^{cl} \) is finite.

(ii) If, moreover, \( \Sigma \) is \( J_X \)-large (resp. \( \Sigma \) is \( J_X \)-large and \( J_X \) has positive \( p \)-rank), then the prime number \( l \) in (i) can be chosen to be in \( \Sigma \cup \{p\} \) (resp. \( \Sigma \)).
Proof. (i) As $1 \neq G_{F_X} \subset G_k = \hat{\mathbb{Z}}$, there exists an $l \in \mathcal{P}_{\text{Primes}}$ such that the image of $G_{F_X}$ in $\mathbb{Z}_l$ is nontrivial, or, equivalently, open in $\mathbb{Z}_l$. Such an $l$ satisfies the desired property.

(ii) By Proposition 2.12, there exists an open subgroup $1 \neq H \subset \text{Ker}(\rho_{J_X,\Sigma'})$ that is contained in $G_{F_X}$. So, as in the proof of (i), take any $l \in \mathcal{P}_{\text{Primes}}$ such that the image of $H$ under the projection $G_k = \hat{\mathbb{Z}} \to \mathbb{Z}_l$ is nontrivial. Then the assertions in (i) hold for this $l$.

Now, assume that $\Sigma$ is $J_X$-large (resp. $\Sigma$ is $J_X$-large and $J_X$ has positive $p$-rank), and suppose $l \in \Sigma' \setminus \{p\}$ (resp. $l \in \Sigma'$). Then the image of the $l$-adic representation $\rho_{J_X,\{l\}} : G_k \to \text{GL}(T_l(J_X))$ is infinite and almost pro-$l$, hence the image of $\text{Ker}(\rho_{J_X,\Sigma'}) \subset \text{Ker}(\rho_{J_X,\{l\}})$ in $\mathbb{Z}_l$ is trivial. Thus, we must have $l \in \Sigma \cup \{p\}$ (resp. $l \in \Sigma$) automatically. \hfill $\square$

§3. Review of Mochizuki’s cuspidalization theory of proper hyperbolic curves over finite fields. In this §3, we review the main results of Mochizuki’s theory of cuspidalizations of arithmetic fundamental groups of proper hyperbolic curves over finite fields, developed in [Mochizuki1], which plays an important role in this paper. We maintain the notations of §1 and further assume $X = U$. (Thus, the finite set $S$ in §1 is empty, and, in this §3, we save the symbol $S$ for another finite set of closed points of $X$.) Accordingly, $X$ is a proper hyperbolic curve over a finite field $k = k_X$.

Recall that $\Delta_X$ stands for the maximal pro-$\Sigma$ quotient of $\pi_1(X)$, that $\Pi_X$ stands for $\pi_1(X)/\text{Ker}(\pi_1(X) \to \Delta_X)$, and that they fit into the following exact sequence:

$$1 \to \Delta_X \to \Pi_X \xrightarrow{\text{pr}_X} G_k \to 1.$$ 

Similarly, if we write $X \times X \overset{\text{def}}{=} X \times_k X$, then we obtain (by considering the maximal pro-$\Sigma$ quotient $\Delta_{X \times X}$ of $\pi_1(X \times X)$) an exact sequence:

$$1 \to \Delta_{X \times X} \to \Pi_{X \times X} \to G_k \to 1,$$

where $\Pi_{X \times X}$ (respectively, $\Delta_{X \times X}$) may be identified with $\Pi_X \times_{G_k} \Pi_X$ (respectively, $\Delta_X \times \Delta_X$).

Definition 3.1. (cf. [Mochizuki1], Definition 1.1(i).) Let $H$ be a profinite group equipped with a homomorphism $H \to \Pi_X$. Then we shall refer to the kernel $I_H$ of $H \to \Pi_X$ as the cuspidal subgroup of $H$ (relative to $H \to \Pi_X$). We shall refer to an inner automorphism of $H$ by an element of $I_H$ as a cuspidally inner automorphism. We shall say that $H$ is cuspidally abelian (respectively, cuspidally pro-$\Sigma^*$, where $\Sigma^*$ is a set of prime numbers) (relative to $H \to \Pi_X$) if $I_H$ is abelian (respectively, if $I_H$ is a pro-$\Sigma^*$ group). If $H$ is cuspidally abelian, then observe that $H/I_H$ acts naturally (by conjugation) on $I_H$. We shall say that $H$ is cuspidally central (relative to $H \to \Pi_X$) if this action of $H/I_H$ on $I_H$ is trivial. Also, we shall use the same terminology for $H \to \Pi_X$ when $\Pi_X$ is replaced by $\Delta_X$, $\Pi_{X \times X}$, or $\Delta_{X \times X}$.

For a finite subset $S \subset X^{\text{cl}}$ write $U_S \overset{\text{def}}{=} X \setminus S$. Let $\Delta_{U_S}$ be the maximal cuspidally (relative to the natural map to $\Delta_X$) pro-$\Sigma^+$ quotient of the maximal pro-$\Sigma$ quotient of the tame fundamental group of $\overline{U_S}$ (where “tame” is with respect to the complement of $U_S$ in $X$), and let $\Pi_{U_S}$ be the corresponding quotient $\pi_1(U_S)/\text{Ker}(\pi_1(U_S) \to \Delta_{U_S})$ of $\pi_1(U_S)$. Thus, we have an exact sequence:

$$1 \to \Delta_{U_S} \to \Pi_{U_S} \to G_k \to 1,$$
which fits into the following commutative diagram:

\[
\begin{array}{c}
1 \\ \downarrow \\
\Delta U \rightarrow \\
\downarrow \\
1 \\ \downarrow \\
\Delta X \rightarrow \\
\end{array}
\]

\[
\begin{array}{c}
\Pi U \rightarrow \\
\Pi X \rightarrow \\
G \rightarrow \\
\end{array}
\]

Further, let \(\iota : X \rightarrow X \times X\) be the diagonal morphism, and write

\[
U_{X \times X} \overset{\text{def}}{=} X \times X \setminus \iota(X).
\]

We shall denote by \(\Delta_{U_{X \times X}}\) the maximal cuspidally (relative to the natural map to \(\Delta_{X \times X}\)) pro-\(\Sigma\) quotient of the maximal pro-\(\Sigma\) quotient of the tame fundamental group of \((U_{X \times X})_k\) (where “tame” is with respect to the divisor \(\iota(X) \subseteq X \times X\)), and by \(\Pi_{U_{X \times X}}\) the corresponding quotient \(\pi_1(U_{X \times X})/\ker(\pi_1(U_{X \times X}) \onto \Delta_{U_{X \times X}})\) of \(\pi_1(U_{X \times X})\). Thus, we have an exact sequence:

\[1 \rightarrow \Delta_{U_{X \times X}} \rightarrow \Pi_{U_{X \times X}} \rightarrow G_k \rightarrow 1\]

which fits into the following commutative diagram:

\[
\begin{array}{c}
1 \\ \downarrow \\
\Delta_{U_{X \times X}} \rightarrow \\
\downarrow \\
\Pi_{U_{X \times X}} \rightarrow \\
\downarrow \\
\Pi_X \rightarrow \\
G_k \rightarrow \\
\end{array}
\]

Finally, set

\[
M_X \overset{\text{def}}{=} \text{Hom}_{\hat{\mathbb{Z}}^{\Sigma}}(H^2(\Delta_X, \hat{\mathbb{Z}}^{\Sigma}), \hat{\mathbb{Z}}^{\Sigma}).
\]

Thus, \(M_X\) is a free \(\hat{\mathbb{Z}}^{\Sigma}\)-module of rank 1, and \(M_X\) is isomorphic to \(\hat{\mathbb{Z}}^{\Sigma}(1)\) as a \(G_k\)-module (where the “(1)” denotes a “Tate twist”, i.e., \(G_k\) acts on \(\hat{\mathbb{Z}}^{\Sigma}(1)\) via the cyclotomic character) (cf. [Mochizuki1], the discussion following Proposition 1.1).

For the rest of this \(\S\), let \(X, Y\) be proper, hyperbolic curves over finite fields \(k_X, k_Y\) of characteristic \(p_X, p_Y\), respectively. Let \(\Sigma_X\) (respectively, \(\Sigma_Y\)) be a set of prime numbers that contains at least one prime number different from \(p_X\) (respectively, \(p_Y\)). Write \(\Delta_X\) (respectively, \(\Delta_Y\)) for the maximal pro-\(\Sigma\) quotient of \(\pi_1(X)\) (respectively, the maximal pro-\(\Sigma\) quotient of \(\pi_1(Y)\)), and \(\Pi_X\) (respectively, \(\Pi_Y\)) for the quotient \(\pi_1(X)/\ker(\pi_1(X) \onto \Delta_X)\) of \(\pi_1(X)\) (respectively, the quotient \(\pi_1(Y)/\ker(\pi_1(Y) \onto \Delta_Y)\) of \(\pi_1(Y)\)).

Let

\[
\alpha : \Pi_X \overset{\sim}{\rightarrow} \Pi_Y
\]

be an isomorphism of profinite groups.

The following is one of the main results of Mochizuki’s theory (cf. [Mochizuki1], Theorem 1.1(iii)).
Theorem 3.2. (Reconstruction of Maximal Cuspidally Abelian Extensions) Let \( \iota_X : X \to X \times X \) (respectively, \( \iota_Y : Y \to Y \times Y \)) be the diagonal morphism, and write \( U_{X \times X} \overset{\text{def}}{=} X \times X \setminus \iota(X) \) (respectively, \( U_{Y \times Y} \overset{\text{def}}{=} Y \times Y \setminus \iota(Y) \)). Denote by \( \Pi_{U_{X \times X}} \to \Pi_{U_{X \times X}}^{c-ab}, \Pi_{U_{Y \times Y}} \to \Pi_{U_{Y \times Y}}^{c-ab} \) the maximal cuspidally (relative to the natural surjections \( \Pi_{U_{X \times X}} \to \Pi_{X \times X}, \Pi_{U_{Y \times Y}} \to \Pi_{Y \times Y} \), respectively) abelian quotients. Then there is a commutative diagram:

\[
\begin{array}{ccc}
\Pi_{U_{X \times X}}^{c-ab} & \overset{\alpha^{c-ab}}{\longrightarrow} & \Pi_{U_{Y \times Y}}^{c-ab} \\
\downarrow & & \downarrow \\
\Pi_{X \times X} & \overset{\alpha \times \alpha}{\longrightarrow} & \Pi_{Y \times Y}
\end{array}
\]

where \( \alpha^{c-ab} \) is an isomorphism which is well-defined up to cuspidally inner automorphism (i.e., an inner automorphism of \( \Pi_{U_{Y \times Y}}^{c-ab} \) by an element of the cuspidal subgroup \( \text{Ker}(\Pi_{U_{Y \times Y}}^{c-ab} \to \Pi_{Y \times Y}) \)). Moreover, the correspondence

\[ \alpha \mapsto \alpha^{c-ab} \]

is functorial (up to cuspidally inner automorphism) with respect to \( \alpha \).

Proof. See [Mochizuki1], Theorem 1.1(iii). (See also [Saïdi-Tamagawa1], Theorem 2.2.) \( \square \)

Let \( \tilde{x} \in \tilde{X}^{cl} \) and \( x \) the image of \( \tilde{x} \) in \( X \). In this and next \( \S\S \), we sometimes refer to the decomposition group \( D_x \) as the decomposition group of \( \Pi_X \) at \( x \), and denote it simply by \( D_x \). Thus, \( D_x \) is well-defined only up to conjugation by an element of \( \Pi_X \).

For the rest of this \( \S \), we shall assume that \( \alpha \) is Frobenius-preserving (cf. Definition 1.5). (Note that this assumption is automatically satisfied in the case where \( \Sigma_X \) is \( k_X \)-large and \( \Sigma_Y \) is \( k_Y \)-large, cf. Proposition 2.5(ii).) Thus, by Theorem 1.9, one deduces naturally from \( \alpha \) a bijection

\[ \phi : X^{cl} \setminus E_X \sim Y^{cl} \setminus E_Y \]

such that

\[ \alpha(D_x) = D_{\phi(x)} \]

holds (up to conjugation) for any \( x \in X^{cl} \setminus E_X \). Here, considering the images of \( D_x \) and \( D_{\phi(x)} \) in \( G_{k_X} = \Pi_X / \Delta_X \sim \Pi_Y / \Delta_Y = G_{k_Y} \) (cf. Proposition 1.7(i)), we obtain \([k(x) : k_X] = [k(\phi(x)) : k_Y] \), hence \( \sharp(k(x)) = \sharp(k(\phi(x))) \) by Proposition 2.5(iii).

As an important consequence of Theorem 3.2 we deduce the following:

Corollary 3.3. With the above assumptions, let \( S \subset X^{cl} \setminus E_X, T \subset Y^{cl} \setminus E_Y \) be finite subsets that correspond to each other via \( \phi \). Then, \( \alpha, \alpha^{c-ab} \) induce isomorphisms (well-defined up to cuspidally inner automorphisms, i.e., inner automorphisms by elements of \( \text{Ker}(\Pi_{U_{Y \times Y}}^{c-ab} \to \Pi_{Y \times Y}) \))

\[ \alpha_{S,T}^{c-ab} : \Pi_{U_S}^{c-ab} \sim \Pi_{V_T}^{c-ab} \]

lying over \( \alpha \), where \( U_S \overset{\text{def}}{=} X \setminus S, V_T \overset{\text{def}}{=} Y \setminus T \), and \( \Pi_{U_S} \to \Pi_{U_S}^{c-ab}, \Pi_{V_T} \to \Pi_{V_T}^{c-ab} \), are the maximal cuspidally abelian quotients (relative to the maps \( \Pi_{U_S} \to \Pi_X \), \( \Pi_{V_T} \to \Pi_Y \)).
These isomorphisms are functorial with respect to $\alpha$, $S$, $T$, as well as with respect to passing to connected finite étale coverings of $X$, $Y$, which arise from open subgroups of $\Pi_X$, $\Pi_Y$, in the following sense: Let $\xi : X' \to X$ (respectively, $\eta : Y' \to Y$) be a finite étale covering which arises from the open subgroup $\Pi_{X'} \subset \Pi_X$ (respectively, $\Pi_{Y'} \subset \Pi_Y$), such that $\alpha(\Pi_{X'}) = \Pi_{Y'}$; set $U'_{S'} \overset{\text{def}}{=} X' \setminus S'$, $V'_T \overset{\text{def}}{=} Y' \setminus T'$, $S' \overset{\text{def}}{=} \xi^{-1}(S)$, $T' \overset{\text{def}}{=} \eta^{-1}(T)$; and denote by $\alpha'$ the isomorphism $\Pi_{X'} \to \Pi_{Y'}$, induced by $\alpha$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
\Pi_{U'_{S'}} & \overset{(\alpha')_{c-ab}}{\longrightarrow} & \Pi_{V'_{T'}} \\
\downarrow & & \downarrow \\
\Pi_{U_S} & \overset{\alpha_{S,T}}{\longrightarrow} & \Pi_{V_T}
\end{array}
$$

where the vertical arrows are the natural maps.

**Proof.** The proof of [Mochizuki1], Theorem 2.1(i) (where $E_X = E_Y = \emptyset$ is assumed) works as it is. See also [Saïdi-Tamagawa1], Corollary 2.3. $\square$

Next, let

$$1 \to M_X \to \mathcal{D} \to \Pi_{X \times X} \to 1$$

be a fundamental extension, i.e., an extension whose corresponding extension class in $H^2_{\text{et}}(X \times X, M_X)$ (via the natural identification $H^2(\Pi_{X \times X}, M_X) \overset{\sim}{\longrightarrow} H^2_{\text{et}}(X \times X, M_X)$ (cf. [Mochizuki1], Proposition 1.1)) coincides with the first (étale) Chern class of the diagonal $\iota(X)$ (cf. [Mochizuki1], Proposition 1.5). Let $x, y \in X(k)$ and write $D_x, D_y \subset \Pi_X$ for the associated decomposition groups (which are well-defined up to conjugation). Set

$$D_x \overset{\text{def}}{=} \mathcal{D}|_{D_x \times G_k \Pi_X}, \quad D_{x,y} \overset{\text{def}}{=} \mathcal{D}|_{D_x \times G_k \Pi_y}.
$$

Thus, $D_x$ (respectively, $D_{x,y}$) is an extension of $\Pi_X$ (respectively, $G_k$) by $M_X$. Similarly, if $D = \sum_i m_i x_i$, $E = \sum_j n_j y_j$ are divisors on $X$ supported on $k$-rational points, then set

$$D_D \overset{\text{def}}{=} \sum_i m_i D_{x_i}, \quad D_D, E \overset{\text{def}}{=} \sum_{i,j} m_i n_j D_{x_i, y_j},
$$

where the sums are to be understood as sums of extensions of $\Pi_X$, $G_k$, respectively, by $M_X$, i.e., the sums are induced by the additive structure of $M_X$.

For a finite subset $S \subset X(k)$, we shall write

$$D_S \overset{\text{def}}{=} \prod_{x \in S} D_x
$$

where the product is to be understood as a fiber product over $\Pi_X$. Thus, $D_S$ is an extension of $\Pi_X$ by a product of copies of $M_X$ indexed by the points of $S$. We shall refer to $D_S$ as the $S$-cuspidalization of $\Pi_X$. Observe that if $T \subset X(k)$ is a finite subset containing $S$, then we obtain a natural projection morphism $\mathcal{D}_T \to D_S$. More generally, for a finite subset $S \subset X^{\text{cl}}$ which does not necessarily consist of $k$-rational points, one can still construct the object $D_S$ by passing to a finite extension $k_S$ of $k$ over which the points of $S$ are rational, performing the above construction over $k_S$, and then descending to $k$. (See [Mochizuki1], Remark 5 for more details.)
Proposition 3.4. (Maximal Geometrically Cuspidally Central Quotients)

(i) For $S \subset X^{cl}$ a finite subset, the $S$-cuspidalization $D_S$ of $\Pi_X$ may be identified with the quotient $\Pi_{U_S} \twoheadrightarrow \Pi^{c-cn}_{U_S} \overset{\text{def}}{=} \Pi_{U_S} / \text{Ker}(\Delta_{U_S} \twoheadrightarrow \Delta^{c-cn}_{U_S})$ of $\Pi_{U_S}$, where $\Delta^{c-cn}_{U_S}$ is the maximal cuspidally central quotient of $\Delta_{U_S}$ relative to the natural map $\Delta_{U_S} \twoheadrightarrow \Delta_X$.

(ii) The fundamental extension $D$ may be identified with the quotient $\Pi_{U_{XX}} \twoheadrightarrow \Pi^{c-cn}_{U_{XX}} \overset{\text{def}}{=} \Pi_{U_{XX}} / \text{Ker}(\Delta_{U_{XX}} \twoheadrightarrow \Delta^{c-cn}_{U_{XX}})$ of $\Pi_{U_{XX}}$, where $\Delta^{c-cn}_{U_{XX}}$ is the maximal cuspidally central quotient of $\Delta_{U_{XX}}$ relative to the natural map $\Delta_{U_{XX}} \twoheadrightarrow \Delta_{X^{XX}}$.

Proof. See [Mochizuki1], Proposition 1.6(iii)(iv). (Precisely speaking, Proposition 1.6(iii) loc. cit. only treats the special case where $S \subset X(k)$ holds. However, the proof for the general case is easily reduced to this special case by passing to a finite extension of $k$. cf. Remark 5, loc. cit.) See also [Saïdi-Tamagawa1], Proposition 2.4. □

Remark 3.5. Let $D$ (respectively, $\mathcal{E}$) be a fundamental extension of $X$ (respectively, $Y$). The isomorphism $\alpha : \Pi_X \sim \Pi_Y$ induces an isomorphism:

$$D \sim \mathcal{E}$$

up to cyclotomically inner automorphisms (i.e., inner automorphisms by elements of $M_X, M_Y$) and the actions of $(k_X^{\times})^{\Sigma_X^1}, (k_Y^{\times})^{\Sigma_Y^1}$, where $(k_X^{\times})^{\Sigma_X^1}$ (respectively, $(k_Y^{\times})^{\Sigma_Y^1}$) is the maximal $\Sigma_X^1$- (respectively, $\Sigma_Y^1$-) quotient of $k_X^{\times}$ (respectively, $k_Y^{\times}$) (cf. [Mochizuki1], Proposition 1.4(ii)). Moreover, let $S \subset X^{cl} \backslash E_X$ and $T \subset Y^{cl} \backslash E_Y$ be as in Corollary 3.3 and write $D_S$ (respectively, $\mathcal{E}_T$) for the $S$-cuspidalization of $\Pi_X$ (respectively, the $T$-cuspidalization of $\Pi_Y$). Then the isomorphism $D \sim \mathcal{E}$ induces an isomorphism

$$D_S \sim \mathcal{E}_T$$

lying over $\alpha$.

On the other hand, let $\Pi_{U_S} \twoheadrightarrow \Pi^{c-cn}_{U_S}$ and $\Pi_{V_T} \twoheadrightarrow \Pi^{c-cn}_{V_T}$ be the maximal geometrically cuspidally central quotients (here, $U_S \overset{\text{def}}{=} X \backslash S$, $V_T \overset{\text{def}}{=} Y \backslash T$) (cf. Proposition 3.4). Note that the isomorphism $\alpha^{\text{c-ab}}_{S,T} : \Pi^{c-ab}_{U_S} \sim \Pi^{c-ab}_{V_T}$ in Corollary 3.3 naturally induces an isomorphism

$$\Pi^{c-cn}_{U_S} \sim \Pi^{c-cn}_{V_T}$$

lying over $\alpha$, which is well-defined up to cuspidally inner automorphism. Now, by Proposition 3.4(i), $\Pi^{c-cn}_{U_S}$ (respectively, $\Pi^{c-cn}_{V_T}$) may be identified with $D_S$ (respectively, $\mathcal{E}_T$). Thus, we deduce another isomorphism

$$D_S \sim \mathcal{E}_T$$

lying over $\alpha$.

Now, the above two isomorphisms between $D_S$ and $\mathcal{E}_T$ coincide with each other up to cyclotomically inner automorphisms and the actions of $(k_X^{\times})^{\Sigma_X^1}, (k_Y^{\times})^{\Sigma_Y^1}$.

Another main result of Mochizuki’s theory is the following, which allows us to recover $\varphi$-group-theoretically the maximal cuspidally pro-$l$ extension of $\Pi_X$, in the case where the set of cusps consists of a single rational point.
Theorem 3.6. (Reconstruction of Maximal Cuspidally Pro-l Extensions)

Let \( x_* \in X(k_X) \), \( y_* \in Y(k_Y) \), and set \( S \overset{\text{def}}{=} \{ x_* \} \), \( T \overset{\text{def}}{=} \{ y_* \} \), \( U_S \overset{\text{def}}{=} X \setminus S \), \( V_T \overset{\text{def}}{=} Y \setminus T \). Assume that the Frobenius-preserving isomorphism \( \alpha : \Pi_X \sim \Pi_Y \) maps the decomposition group of \( x_* \) in \( \Pi_X \) (which is well-defined up to conjugation) to the decomposition group of \( y_* \) in \( \Pi_Y \) (which is well-defined up to conjugation).

Then, for each prime \( l \in \Sigma^\dagger \) (thus, \( l \neq p \)), there exists a commutative diagram:

\[
\begin{array}{ccc}
\Pi_U^c \rightarrow & \Pi_Y^c \\
\downarrow & \downarrow \\
\Pi_X & \rightarrow & \Pi_Y \\
\end{array}
\]

in which \( \Pi_U \rightarrow \Pi^c_U \), \( \Pi_V \rightarrow \Pi^c_V \) are the maximal cuspidally pro-\( l \) quotients (relative to the maps \( \Pi_U \rightarrow \Pi_X \), \( \Pi_V \rightarrow \Pi_Y \), respectively), the vertical arrows are the natural surjections, and \( \alpha^c_l \) is an isomorphism well-defined up to composition with a cuspidally inner automorphism (i.e., an inner automorphism by an element of \( \text{Ker}(\Pi^c_V \rightarrow \Pi_Y) \)), which is compatible relative to the natural surjections

\[
\Pi^c_U \rightarrow \Pi^c_{ab,U} \quad \Pi^c_V \rightarrow \Pi^c_{ab,V}
\]

where the subscript “\( \text{c-ab} \)" denotes the maximal cuspidally pro-\( l \) abelian quotient, with the isomorphism

\[
\alpha^c_{S,T} : \Pi^c_{ab,U} \sim \Pi^c_{ab,V}
\]

in Corollary 3.3. Moreover, \( \alpha^c_l \) is compatible, up to cuspidally inner automorphisms, with the decomposition groups of \( x_* \), \( y_* \) in \( \Pi^c_U \), \( \Pi^c_V \). \( \blacksquare \)

Proof. See [Mochizuki1], Theorem 3.1. (See also [Saidi-Tamagawa1], Theorem 2.6.) \( \blacksquare \)

If \( n \) is an integer all of whose prime factors belong to \( \Sigma^\dagger \), then we have the Kummer exact sequence

\[
1 \rightarrow \mu_n \rightarrow G_m \rightarrow G_m \rightarrow 1,
\]

where \( G_m \rightarrow G_m \) is the \( n \)-th power map. We shall identify \( \mu_n \) with \( M_X/nM_X \) according to the identification in [Mochizuki1], the discussion at the beginning of §2.

Consider a subset

\( E \subset X^{cl} \).

(We will set \( E = E_X \) eventually, but \( E \) is arbitrary for the present.) Let \( S \subset X^{cl} \setminus E \) be a finite set. If we consider the above Kummer exact sequence on the étale site of \( U_S \overset{\text{def}}{=} X \setminus S \) and pass to the inverse limit with respect to \( n \), then we obtain a natural homomorphism

\[
\Gamma(U_S, \mathcal{O}_U^X) \rightarrow H^1(\Pi_U, M_X)
\]

(cf. loc. cit.). (Note that here it suffices to consider the group cohomology of \( \Pi_U \) (i.e., as opposed to the étale cohomology of \( U_S \)), since the extraction of \( n \)-th roots
of an element of $\Gamma(U_s, \mathcal{O}^\times_{U_s})$ yields finite étale coverings of $U_s$ that correspond to open subgroups of $\Pi_{U_s}$. The above homomorphism induces a natural injective homomorphism

$$\Gamma(U_s, \mathcal{O}^\times_{U_s})/(k^\times \{\Sigma'\}) \to H^1(\Pi_{U_s}, M_X)$$

where $k^\times \{\Sigma'\}$ stands for the $\Sigma'$-primary part of the multiplicative group $k^\times$ (since the abelian group $\Gamma(U_s, \mathcal{O}^\times_{U_s})/(k^\times \{\Sigma'\})$ is finitely generated and free of $\Sigma'$-primary torsion, hence injects into its pro-$\Sigma$ completion). In particular, by allowing $S$ to vary among all finite subsets of $X^{cl} \setminus E$, we obtain a natural injective homomorphism

$$\mathcal{O}_E^\times/(k^\times \{\Sigma'\}) \to \lim_{S} H^1(\Pi_{U_s}, M_X),$$

where

$$\mathcal{O}_E^\times \overset{\text{def}}{=} \{ f \in K^\times_X | \text{sup}(\text{div}(f)) \cap E = \emptyset \}$$

is the multiplicative group of the units in the ring $\mathcal{O}_E$ of functions on $X$ which are regular at all points of $E$. (Here, $K_X$ denotes the function field of $X$.)

**Proposition 3.7. (Kummer Classes of Functions)** Suppose that $S \subset X^{cl} \setminus E$ is a finite subset. Write

$$\Delta_{U_s} \to \Delta_{U_s}^{c-ab} \to \Delta_{U_s}^{c-cn}$$

for the maximal cuspidally abelian and the maximal cuspidally central quotients, respectively, relative to the map $\Delta_{U_s} \to \Delta_X$, and

$$\Pi_{U_s} \to \Pi_{U_s}^{c-ab} \to \Pi_{U_s}^{c-cn}$$

for the corresponding quotients of $\Pi_{U_s}$ (i.e., $\Pi_{U_s}^{c-ab} \overset{\text{def}}{=} \Pi_{U_s} / \text{Ker}(\Delta_{U_s} \to \Delta_{U_s}^{c-ab})$, $\Pi_{U_s}^{c-cn} \overset{\text{def}}{=} \Pi_{U_s} / \text{Ker}(\Delta_{U_s} \to \Delta_{U_s}^{c-cn})$). If $x \in X^{cl}$, then we shall write

$$D_x[U_S] \subset \Pi_{U_s}$$

for the decomposition group at $x$ in $\Pi_{U_s}$ (which is well-defined up to conjugation), and $I_x[U_S] \overset{\text{def}}{=} D_x[U_S] \cap \Delta_{U_s}$ for the inertia subgroup of $D_x[U_S]$. Thus, when $x \in S$ we have a natural isomorphism of $M_X$ with $I_x[U_S]$ (cf. [Mochizuki1], Proposition 1.6(ii)(iii)). Then:

(i) The natural surjections above induce the following isomorphisms:

$$H^1(\Pi_{U_s}^{c-cn}, M_X) \sim H^1(\Pi_{U_s}^{c-ab}, M_X) \sim H^1(\Pi_{U_s}, M_X).$$

In particular, we obtain the following natural injective homomorphisms:

$$\Gamma(U_s, \mathcal{O}^\times_{U_s})/(k^\times \{\Sigma'\}) \to H^1(\Pi_{U_s}^{c-cn}, M_X) \sim H^1(\Pi_{U_s}^{c-ab}, M_X) \sim H^1(\Pi_{U_s}, M_X),$$

$$\mathcal{O}_E^\times/(k^\times \{\Sigma'\}) \to \lim_{S} H^1(\Pi_{U_s}^{c-cn}, M_X) \sim H^1(\Pi_{U_s}^{c-ab}, M_X) \sim \lim_{S} H^1(\Pi_{U_s}, M_X),$$

where $S$ varies among all finite subsets of $X \setminus E$.

(ii) Restricting cohomology classes of $\Pi_{U_s}$ to the various $I_x[U_S]$ for $x \in S$ yields a natural exact sequence:

$$1 \to (k^\times)^{\Sigma^t} \to H^1(\Pi_{U_s}, M_X) \to \left( \bigoplus_{s \in S} \hat{\mathbb{Z}}^{\Sigma^t} \right)$$
Observe that $\gamma_x$ for the associated Kummer classes. If the natural surjection $\Pi \to G_k$ is defined as a certain fiber product, this section is equivalent to a collection of sections determined by the principal divisors (with support in $S$). A similar statement holds when $\Pi_U$ is replaced by $\Pi^c_{U_S}$ or $\Pi^ab_{U_S}$.

(iii) If $f \in \Gamma(U_S, O^\times_{U_S})$, write $f'$ for its image in $\Gamma(U_S, O^\times_{U_S})/(k^\times \{\Sigma'\})$. Write

$$\kappa_{f'}^c \in H^1(\Pi^c_{U_S}, M_X), \quad \kappa_{f'}^ab \in H^1(\Pi^ab_{U_S}, M_X), \quad \kappa_{f'} \in H^1(\Pi_{U_S}, M_X)$$

for the associated Kummer classes. If $x \in (X^{cl} \setminus E) \setminus S$, then $D_x[U_S]$ maps, via the natural surjection $\Pi_{U_S} \to G_k$, isomorphically onto the open subgroup $G_{k(x)} \subset G_k$ (where $k(x)$ is the residue field of $X$ at $x$). Moreover, the images of the pulled back classes

$$\kappa_{f'}^c|_{D_x[U_S]} = \kappa_{f'}^ab|_{D_x[U_S]} = \kappa_{f'}|_{D_x[U_S]} \in H^1(D_x[U_S], M_X) \simeq H^1(G_{k(x)}, M_X) \simeq (k(x)^\times)^{\Sigma^\dagger}$$

in $(k(x)^\times)^{\Sigma^\dagger}$ are equal to the image in $(k(x)^\times)^{\Sigma^\dagger}$ of the value $f(x) \in k(x)^\times$ of $f$ at $x$.

**Proof.** See [Saïdi-Tamagawa1], Proposition 3.1. (See also [Mochizuki1], Proposition 2.1.) □

**Remark 3.8.** (cf. [Mochizuki1], Remark 12.) In the situation of Proposition 3.7(iii), assume $x \in X(k)$ and $S \subset X(k)$ for simplicity. If we think of the extension $\Pi^c_{U_S}$ of $\Pi_X$ as being given by the extension $\mathcal{D}_S$, where $\mathcal{D}$ is a fundamental extension of $\Pi_X \times X$ (cf. Proposition 3.4(i)), then it follows that the image of $D_x[U_S]$ in $\Pi^c_{U_S}$ may be thought of as the image of $D_x[U_S]$ in $\mathcal{D}_S$. This image of $D_x[U_S]$ in $\mathcal{D}_S$ amounts to a section of $\mathcal{D}_S \to \Pi_X \to G_k$ lying over the section $s_x : G_k \to \Pi_X$ corresponding to the rational point $x$ (which is well-defined up to conjugation). Since $\mathcal{D}_S$ is defined as a certain fiber product, this section is equivalent to a collection of sections (regarded as “cyclotomically outer homomorphisms”, i.e., well-defined up to composition with an inner automorphism of $\mathcal{D}_{y,x}$ by an element of $\text{Ker} (\mathcal{D}_{y,x} \to G_k)$)

$$\gamma_{y,x} : G_k \to \mathcal{D}_{y,x},$$

where $y$ ranges over all points of $S$. Namely, from this point of view, Proposition 3.7(iii) may be regarded as saying that the image in $(k(x)^\times)^{\Sigma^\dagger} = (k^\times)^{\Sigma^\dagger}$ of the value $f(x)$ of the function $f \in \Gamma(U_S, O^\times_{U_S})$ at $x \in X(k)$ may be computed from its Kummer class, as soon as one knows the sections $\gamma_{y,x} : G_k \to \mathcal{D}_{y,x}$ for $y \in S$. Observe that $\gamma_{y,x}$ depends only on $x$, $y$, and not on the choice of $S$. 

24
Definition 3.9. (cf. [Mochizuki1], Definition 2.1.) For \(x, y \in X(k)\) with \(x \neq y\), we shall refer to the above section (regarded as a cyclotomically outer homomorphism)

\[\gamma_{y,x} : G_k \rightarrow \mathcal{D}_{y,x}\]

as the Green’s trivialization of \(\mathcal{D}\) at \((y,x)\). If \(D\) is a divisor on \(X\) supported on \(k\)-rational points \(\neq x\), then multiplication of the various Green’s trivializations for the points in the support of \(D\) yields a section (regarded as a cyclotomically outer homomorphism)

\[\gamma_{D,x} : G_k \rightarrow \mathcal{D}_{D,x}\]

which we shall refer to as the Green’s trivialization of \(\mathcal{D}\) at \((D,x)\).

Definition 3.10. (cf. [Mochizuki1], Definition 2.2.) Let the notations and the assumptions as in Corollary 3.3.

(i) Write \(\mathcal{D}\) (respectively, \(\mathcal{E}\)) for the fundamental extension of \(\Pi_{X \times X}\) (respectively, \(\Pi_{Y \times Y}\)) that arises as the quotient of \(\Pi_{X \times X}^\text{c-ab}\) (respectively, \(\Pi_{Y \times Y}^\text{c-ab}\)) by the kernel of the maximal cuspidally central quotient \(\Delta_{X \times X}^\text{c-ab} \rightarrow \Delta_{Y \times Y}^\text{c-cn}\) (respectively, \(\Delta_{Y \times Y}^\text{c-ab} \rightarrow \Delta_{X \times X}^\text{c-cn}\)) (cf. Proposition 3.4(ii)). The isomorphism \(\alpha^\text{c-ab}\) induces naturally an isomorphism:

\[\alpha^\text{c-cn} : \mathcal{D} \overset{\sim}{\rightarrow} \mathcal{E}\]

We shall say that \(\alpha\) is \((S,T)\)-locally Green-compatible outside exceptional sets if, for every pair of points \((x_1, x_2)\) in \(X(k)\) corresponding via \(\phi\) to a pair of points \((y_1, y_2)\) in \(Y(k)\), such that \(x_1 \in (X^\text{cl} \setminus E_X) \setminus S, y_1 \in (Y^\text{cl} \setminus E_Y) \setminus T, x_2 \in S, y_2 \in T\), the isomorphism

\[\mathcal{D}_{x_1,x_2} \overset{\sim}{\rightarrow} \mathcal{E}_{y_1,y_2}\]

(obtained by restricting \(\alpha^\text{c-cn}\) to the various decomposition groups) is compatible with the Green’s trivializations. We shall say that \(\alpha\) is \((S,T)\)-locally bi-principally Green-compatible outside exceptional sets if, for every point \(x \in X(k) \cap S\) and every principal divisor \(P\) supported on \(k\)-rational points \(\neq x\) contained in \(X^\text{cl} \setminus E_X\) corresponding via \(\phi\) to a pair \((y,Q)\) (so \(y \in Y(k) \cap T\)) with \(Q\) principal, the isomorphism

\[\mathcal{D}_{P,x} \overset{\sim}{\rightarrow} \mathcal{E}_{Q,y}\]

obtained from \(\alpha^\text{c-cn}\) is compatible with the Green’s trivializations.

(ii) We shall say that \(\alpha\) is totally globally Green-compatible (respectively, totally globally bi-principally Green-compatible) outside exceptional sets if, for all pair of connected finite étale coverings \(\xi : X' \rightarrow X, \eta : Y' \rightarrow Y\) that arise from open subgroups \(\Pi_{X'} \subset \Pi_X, \Pi_{Y'} \subset \Pi_Y\), corresponding to each other via \(\alpha\), then for any subset \(S \subset X^\text{cl} \setminus E_X\) that corresponds, via \(\phi\), to \(T \subset Y \setminus E_Y\) the isomorphism

\[\Pi_{X'} \overset{\sim}{\rightarrow} \Pi_{Y'}\]

induced by \(\alpha\) is \((S',T')\)-locally Green-compatible (respectively, \((S',T')\)-locally bi-principally Green-compatible) outside exceptional sets, where \(S' \overset{\text{def}}{=} \xi^{-1}(S) \subset X'^\text{cl}, T' \overset{\text{def}}{=} \eta^{-1}(T) \subset Y'^\text{cl}\) are the inverse images of \(S, T\), respectively.
Remark 3.11. In [Saïdi-Tamagawa1], Definition 3.4, we adopted a slightly different notion of being “\((S,T)\)-locally (or totally globally) principally Green-compatible outside exceptional sets”, where the divisor \(Q\) of \(Y\) appearing in (i) above is not assumed to be principal. Here we adopt the above notion of being “\((S,T)\)-locally (or totally globally) bi-principally Green-compatible outside exceptional sets”, since it is more natural in our settings (although both notions work).

Proposition 3.12. (Total Global Green-Compatibility Outside Exceptional Sets) In the situation of Theorem 3.2, assume further that \(\alpha\) is Frobenius-preserving. Then the isomorphism \(\alpha\) is totally globally, and totally globally bi-principally, Green-compatible outside exceptional sets.

Proof. Similar to the proof of Proposition 3.8 in [Saïdi-Tamagawa1]. (See also [Mochizuki1], Corollary 3.1.) □

§4. Isomorphisms between geometrically pro-\(\Sigma\) arithmetic fundamental groups.

We maintain the notations of §3.

Definition/Remark 4.1. Let \(J = J_X\) be the Jacobian variety of \(X\). Let \(\text{Div}^0_{X \setminus E}\) be the group of degree zero divisors on \(X\) which are supported on points in \(X \setminus E\). Write \(D_{X \setminus E}\) for the kernel of the natural homomorphism \(\text{Div}^0_{X \setminus E} \to J(k)^{\Sigma}\). Here, \(J(k)^{\Sigma}\) stands for the maximal pro-\(\Sigma\) quotient \(J(k)/\{J(k)\{\Sigma'\}\}\) of \(J(k)\), where, for an abelian group \(M\), \(M\{\Sigma'\}\) stands for the subgroup of torsion elements \(a\) of \(M\) such that every prime divisor of the order of \(a\) belongs to \(\Sigma'\). Then \(D_{X \setminus E}\) sits naturally in the following exact sequence:

\[0 \to \text{Pri}_{X \setminus E} \to D_{X \setminus E} \to J(k)\{\Sigma'\} \to 0,\]

where \(\text{Pri}_{X \setminus E} \stackrel{\text{def}}{=} \mathcal{O}_E^\times/k_X^\times\) stands for the group of principal divisors supported in \(X \setminus E\). Further, let \(D_{X \setminus E}\) be the inverse image of \(D_{X \setminus E}\) in \(\lim\inf H^1(\Pi_\text{ab}^{\text{pro-}\Sigma}, M_X)\) (cf. Proposition 3.7(ii)). Then \(D_{X \setminus E}\) sits naturally in the following exact sequence

\[0 \to \mathcal{O}_E^\times/(k_X^\times\{\Sigma'\}) \to D_{X \setminus E} \to J(k)\{\Sigma'\} \to 0.\]

Now, let \(X, Y\) be proper hyperbolic curves over finite fields \(k_X, k_Y\) of characteristic \(p_X, p_Y\), respectively, and define \(K_X, K_Y\) to be the function fields of \(X, Y\), respectively. Let \(\Sigma_X\) (respectively, \(\Sigma_Y\)) be a set of prime numbers that contains at least one prime number different from \(p_X\) (respectively, \(p_Y\)). Write \(\Delta_X\) (respectively, \(\Delta_Y\)) for the maximal pro-\(\Sigma_X\) quotient of \(\pi_1(X)\) (respectively, the maximal pro-\(\Sigma_Y\) quotient of \(\pi_1(Y)\)), and \(\Pi_X\) (respectively, \(\Pi_Y\)) for the quotient \(\pi_1(X)/\text{Ker}(\pi_1(X) \to \Delta_X)\) of \(\pi_1(X)\) (respectively, the quotient \(\pi_1(Y)/\text{Ker}(\pi_1(Y) \to \Delta_Y)\) of \(\pi_1(Y)\)).

Let

\[\alpha : \Pi_X \cong \Pi_Y\]

be an isomorphism of profinite groups and write \(\Sigma \stackrel{\text{def}}{=} \Sigma_X = \Sigma_Y\) (cf. Proposition 1.7(ii)).
Theorem 4.2. (Reconstruction of Pseudo-Functions) Assume that $\alpha$ is Frobenius-preserving. Then:

(i) The bijection $\phi : X^{cl} \setminus E_X \sim Y^{cl} \setminus E_Y$ induced by $\alpha$ (where $E_X$ and $E_Y$ are the exceptional sets), together with the isomorphisms in Corollary 3.3, induce natural bijections $\bar{\rho} : D_{Y} \setminus E_Y \sim D_{X} \setminus E_X$, $\rho' : D_{Y} \setminus E_Y \sim D_{X} \setminus E_X$, which fit into the following commutative diagrams

\[
\begin{array}{cccccccccc}
0 & \longrightarrow & \text{Pri}_{X \setminus E_X} = \mathcal{O}_{E_X}^N / k_X^N & \longrightarrow & D_{X \setminus E_X} & \longrightarrow & J_X(k_X)\{\Sigma'\} & \longrightarrow & 0 \\
& & \downarrow \bar{\rho} & & \uparrow & & & & \\
0 & \longrightarrow & \text{Pri}_{Y \setminus E_Y} = \mathcal{O}_{E_Y}^N / k_Y^N & \longrightarrow & D_{Y \setminus E_Y} & \longrightarrow & J_Y(k_Y)\{\Sigma'\} & \longrightarrow & 0
\end{array}
\]

and

\[
\begin{array}{cccccccccc}
0 & \longrightarrow & \mathcal{O}_{E_X}^N / (k_X^N \{\Sigma'\}) & \longrightarrow & D_{X \setminus E_X} & \longrightarrow & J_X(k_X)\{\Sigma'\} & \longrightarrow & 0 \\
& & \downarrow \rho' & & \uparrow & & & & \\
0 & \longrightarrow & \mathcal{O}_{E_Y}^N / (k_Y^N \{\Sigma'\}) & \longrightarrow & D_{Y \setminus E_Y} & \longrightarrow & J_Y(k_Y)\{\Sigma'\} & \longrightarrow & 0
\end{array}
\]

Moreover, the following diagram commutes

\[
\begin{array}{ccc}
D_{X \setminus E_X} & \xleftarrow{\rho'} & D_{Y \setminus E_Y} \\
\downarrow & & \downarrow \\
D_{X \setminus E_X} & \xleftarrow{\bar{\rho}} & D_{Y \setminus E_Y}
\end{array}
\]

where the vertical maps are the natural ones.

(ii) The bijection $\bar{\rho}$ in (i) induces a natural isomorphism $\bar{\rho} : \overline{H}_Y \sim \overline{H}_X$.

Here, $\overline{H}_X \overset{\text{def}}{=} \text{Ker}(\text{Pri}_{X \setminus E_X} \xrightarrow{\bar{\rho}^{-1}} D_{Y \setminus E_Y} \rightarrow J_Y(k_Y)\{\Sigma'\})$ and $\overline{H}_Y \overset{\text{def}}{=} \text{Ker}(\text{Pri}_{Y \setminus E_Y} \xrightarrow{\bar{\rho}} D_{X \setminus E_X} \rightarrow J_X(k_X)\{\Sigma'\})$ are finite index subgroups of $\mathcal{O}_{E_X}^N / k_X^N$ and $\mathcal{O}_{E_Y}^N / k_Y^N$, respectively. (More precisely, the quotients $(\mathcal{O}_{E_X}^N / k_X^N) / \overline{H}_X$ and $(\mathcal{O}_{E_Y}^N / k_Y^N) / \overline{H}_Y$ are $\Sigma'$-primary finite abelian groups that are embeddable into $J_Y(k_Y)\{\Sigma'\}$ and $J_X(k_X)\{\Sigma'\}$, respectively.) Moreover, let $H'_X$ (resp. $H'_Y$) be the inverse image of $\overline{H}_X$ (resp. $\overline{H}_Y$) in $\mathcal{O}_{E_X}^N / (k_X^N \{\Sigma'\})$ (resp. $\mathcal{O}_{E_Y}^N / (k_Y^N \{\Sigma'\})$). Then the isomorphism $\rho'$ in (i) induces a natural isomorphism $\rho' : H'_X \sim H'_Y$ which fits into the following commutative diagram

\[
\begin{array}{ccc}
H'_X & \xleftarrow{\rho'} & H'_Y \\
\downarrow & & \downarrow \\
\overline{H}_X & \xleftarrow{\bar{\rho}} & \overline{H}_Y
\end{array}
\]

where the vertical maps are the natural surjections.
In particular, $\rho'$ induces a natural isomorphism

$$\tau : (k^\times_Y)^\Sigma = k^\times_Y/(k^\times_Y \{\Sigma'\}) \xrightarrow{\sim} k^\times_X/(k^\times_X \{\Sigma'\}) = (k^\times_X)^\Sigma.$$ (iii) The diagram in (ii) is functorial in $X, Y$, in the following sense: if $\xi : X' \to X$ is a finite étale covering, arising from an open subgroup $\Pi_{X'} \subset \Pi_X$, which corresponds to a finite étale covering $Y' \to Y$ via $\alpha$ (thus, $\Pi_{Y'} \defeq \alpha(\Pi_{X'})$), then $\alpha$ induces natural isomorphisms $\gamma' : H^\prime_Y \xrightarrow{\sim} H^\prime_X$, and $\tau : \Pi_Y \xrightarrow{\sim} \Pi_X$, which fit into the following commutative diagrams

$$\begin{array}{ccc} H^\prime_X & \xleftarrow{\gamma'} & H^\prime_Y \\ \uparrow & & \uparrow \\ H^\prime_X & \xleftarrow{\rho'} & H^\prime_Y \end{array}$$

and

$$\begin{array}{ccc} \Pi^\prime_X & \xleftarrow{\tau} & \Pi^\prime_Y \\ \uparrow & & \uparrow \\ \Pi_X & \xleftarrow{p} & \Pi_Y \end{array}$$

where the vertical maps are natural homomorphisms induced by the natural injective homomorphisms $O^\times_{E_X}/k^\times_X \{\Sigma'\} \hookrightarrow O^\times_{E_X}/k^\times_X \{\Sigma'\}$, $O^\times_{E_Y}/k^\times_Y \{\Sigma'\} \hookrightarrow O^\times_{E_Y}/k^\times_Y \{\Sigma'\}$.

In particular, $\alpha$ induces a natural isomorphism

$$\tau : (k^\times_Y)^\Sigma = k^\times_Y/(k^\times_Y \{\Sigma'\}) \xrightarrow{\sim} k^\times_X/(k^\times_X \{\Sigma'\}) = (k^\times_X)^\Sigma$$

which extends $\tau : (k^\times_X)^\Sigma \xrightarrow{\sim} (k^\times_X)^\Sigma$ in (ii).

**Proof.** Similar to the proof of [Saïdi-Tamagawa1], Theorem 3.6. (See also [Saïdi-Tamagawa3], Lemma 4.5 for a similar statement in the birational case.) Here, the commutativity of the diagrams in (iii) follows basically from the functoriality of Kummer theory, together with the commutativity of the diagram

$$\begin{array}{ccc} M^\prime_X & \xleftarrow{\gamma'} & M^\prime_Y \\ \uparrow & & \uparrow \\ M_X & \xleftarrow{\rho'} & M_Y \end{array}$$

where the horizontal arrows are isomorphisms induced by $\alpha$ (or, more precisely, by $\alpha^{-1}$, via the functoriality of $H^2$) and the vertical arrows are isomorphisms defined geometrically via the identification of $M_X$ and $M^\prime_X$ (resp. $M_Y$ and $M^\prime_Y$) with the Tate module of $\mathbb{G}_m$ over $k_X = k^\prime_X$ (resp. $k^\prime_Y = k_Y$). The commutativity of this last diagram follows from the fact that the above (geometrically defined) isomorphism $M_X \xrightarrow{\sim} M^\prime_X$ (resp. $M_Y \xrightarrow{\sim} M^\prime_Y$) is identified with the composite of the $(X^t_{k_X} : X_{k_X})$-multiplication map $M_X \to M_X$ (resp. the $(Y^t_{k_Y} : Y_{k_Y})$-multiplication map $M_Y \to M_Y$) and the inverse of the natural isomorphism $M^\prime_X \xrightarrow{\sim} (X^t_{k_X} : X_{k_X})M_X \subset M_X$ (resp. $M^\prime_Y \xrightarrow{\sim} (Y^t_{k_Y} : Y_{k_Y})M_Y \subset M_Y$) induced by the inclusion $M^\prime_X \to M_X$ (resp. $M^\prime_Y \to M_Y$) arising from the functoriality of $H^2$, and the fact that $(X^t_{k_X} : X_{k_X}) = (\Delta_X : \Delta_X) = (\Delta_Y : \Delta_Y) = (Y^t_{k_Y} : Y_{k_Y})$. □
Definition 4.3. We say that \( \alpha \) is pseudo-constants-additive if \( \tau : (\bar{k}^\times_X)^\Sigma \sim (\bar{k}^\times_Y)^\Sigma \) in Theorem 4.2(iii) satisfies the following: For \( \eta \in \bar{k}^\times_Y \) and \( \zeta \in \bar{k}^\times_X \), if
\[
1 + \eta \neq 0 \quad \text{and} \quad \tau(\eta') = \zeta',
\]
then there exist \( \alpha, \beta \in \bar{k}^\times_X\{\Sigma'\} \), such that
\[
\alpha + \beta \zeta \neq 0 \quad \text{and} \quad \tau((1 + \eta)') = (\alpha + \beta \zeta)'.
\]
Here, for an element \( \xi \) of \( \bar{k}^\times_X \) (resp. \( \bar{k}^\times_Y \)), \( \xi' \) denotes its image in \( (\bar{k}^\times_X)^\Sigma \) (resp. \( (\bar{k}^\times_Y)^\Sigma \)).

Theorem 4.4. (Pseudo-Constants-Additive Isomorphisms) Assume that \( \Sigma \) is \( k_X \)-large and \( k_Y \)-large, that at least one of \( X \) and \( Y \) is almost \( \Sigma \)-separated, and that \( \alpha \) is pseudo-constants-additive. Then \( \alpha \) arises from a uniquely determined commutative diagram of schemes:
\[
\begin{array}{ccc}
\bar{X} & \sim & \bar{Y} \\
\downarrow & & \downarrow \\
X & \sim & Y
\end{array}
\]
in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings determined by the groups \( \Pi_X, \Pi_Y \).

The rest of this section is devoted to the proof of Theorem 4.4 and deducing corollaries.

First, since \( \Sigma \) is \( k_X \)-large and \( k_Y \)-large, \( \alpha \) is Frobenius-preserving by Proposition 2.5(ii). Thus, we may apply Theorem 1.9, Theorem 3.12 and Theorem 4.2 to \( \alpha \). In particular, \( \alpha \) is totally globally bi-principally Green-compatible outside exceptional sets by Theorem 3.12. Also, we see that both \( X \) and \( Y \) are almost \( \Sigma \)-separated. Indeed, let \( F_X \) (resp. \( F_Y \)) denote the compositum of \( k(x) \) (resp. \( k(y) \)) in \( \bar{k}_X \) (resp. \( \bar{k}_Y \)) for all \( x \in E_X \) (resp. \( y \in E_Y \)). Then it follows from Theorem 1.9(i) that \( G_{F_X} \subset k_X \) corresponds to \( G_{F_Y} \subset k_Y \) via the natural isomorphism \( G_{k_X} \sim G_{k_Y} \) induced by \( \alpha \) (cf. Proposition 1.7(i)). Thus, \( F_X \subseteq k_X \) (i.e., \( G_{F_X} \neq \{1\} \)) is equivalent to \( F_Y \not\subseteq k_Y \) (i.e., \( G_{F_Y} \neq \{1\} \)).

Let \( x \in X^{cl} \setminus E_X \) and \( y \overset{def}{=} \phi(x) \in Y^{cl} \setminus E_Y \). Then, as \( \alpha \) preserves the decomposition groups at \( x, y \), respectively, \( \alpha \) induces naturally an isomorphism
\[
\tau_{x,y} : (k(y)^\times)^\Sigma \sim (k(x)^\times)^\Sigma
\]
(cf. Proposition 3.7(iii))), which fits into a commutative diagram
\[
\begin{array}{ccc}
(k(x)^\times)^\Sigma & \overset{\tau_{x,y}}{\sim} & (k(y)^\times)^\Sigma \\
\uparrow & & \uparrow \\
(k_X^\times)^\Sigma & \overset{\tau}{\sim} & (k_Y^\times)^\Sigma
\end{array}
\]
where the vertical maps are the natural homomorphisms.

Next, we shall think of elements of \( \mathcal{O}_{E_X}^\times/(k_X^\times), \mathcal{O}_{E_Y}^\times/(k_Y^\times) \) as principal divisors of rational functions on \( X, Y \), respectively, and denote them \( f, g, \ldots \), where \( f, g \) are rational functions on \( X, Y \), whose supports of divisors are disjoint from \( E_X, E_Y \),
respectively. We will denote the elements of \( \mathcal{O}_E^X/(k_X^\Sigma') \), \( \mathcal{O}_E^Y/(k_Y^\Sigma') \), by \( f', g', \ldots \), where \( f, g \) are rational functions on \( X, Y \), whose supports of divisors are disjoint from \( E_X, E_Y \), respectively, and refer to them as “pseudo-functions”

Consider the commutative diagram (4.1). Let 

\[ \begin{array}{ccc}
    & x & \\
\text{In particular, in terms of divisors, if:} & \longrightarrow & \\
\end{array} \]

For 

\[ \begin{array}{ccc}
    & \longrightarrow & \\
\text{functions in} & \longrightarrow & \\
\end{array} \]

is totally globally bi-principally Green-compatible outside exceptional sets.

**Lemma 4.5. (Recovering the Valuations and the \( \Sigma \)-Values of Pseudo-Functions)**

Consider the commutative diagram (4.1). Let \( x \in X^{cl} \setminus E_X \) and \( y \overset{\text{def}}{=} \phi(x) \in Y^{cl} \setminus E_Y \). Then the following implications hold:

(i) For \( \bar{f} \in \mathcal{H}_Y \) and \( \bar{g} \in \mathcal{H}_X \):

\[ \bar{\rho}(\bar{f}) = \bar{g} \implies v_y(\bar{f}) = v_x(\bar{g}). \]

In particular, in terms of divisors, if:

\[ \bar{f} = y_1 + y_2 + \cdots + y_n - y'_1 - \cdots - y'_n, \]

then:

\[ \bar{g} = x_1 + x_2 + \cdots + x_n - x'_1 - \cdots - x'_n, \]

where \( y_i \overset{\text{def}}{=} \phi(x_i) \) (resp. \( y'_i \overset{\text{def}}{=} \phi(x'_i) \)) for \( i \in \{1, \ldots, n\} \) (resp. \( i' \in \{1, \ldots, n'\} \)). In other words, the isomorphism \( \bar{\rho} : \mathcal{H}_Y \overset{\sim}{\to} \mathcal{H}_X \) preserves the valuations of the classes of functions in \( \mathcal{H}_X, \mathcal{H}_Y \) with respect to the bijection \( \phi : X^{cl} \setminus E_X \overset{\sim}{\to} Y^{cl} \setminus E_Y \) between points. Further, the isomorphism \( \bar{\rho} \) preserves the degrees of the classes of functions in \( \mathcal{H}_X, \mathcal{H}_Y \).

(ii) For \( f' \in H'_Y \) and \( g' \in H'_X \):

\[ v_y(\bar{f}) = 0 \text{ and } \rho(f') = g' \implies v_x(\bar{g}) = 0 \text{ and } \tau_{x,y}(f'(y)) = g'(x) \]

where

\[ y = \phi(x) \text{ and } \tau_{x,y} : (k(y)^\Sigma) \sim (k(x)^\Sigma) \]

is the natural identification above. In other words, the isomorphism \( \rho' : H'_Y \overset{\sim}{\to} H'_X \) preserves the \( \Sigma \)-values of the pseudo-functions in \( H'_X, H'_Y \) with respect to the bijection \( \phi : X^{cl} \setminus E_X \overset{\sim}{\to} Y^{cl} \setminus E_Y \) between points.

**Proof.** Assertion (i) follows from Proposition 3.7(i)(ii), together with the fact that \( [k(x) : k_X] = [k(\phi(x)) : k_Y] \) for each \( x \in X^{cl} \setminus E_X \) (cf. discussion before Corollary 3.3), and assertion (ii) follows from Proposition 3.7(iii), together with the fact that \( \alpha \) is totally globally bi-principally Green-compatible outside exceptional sets. \( \square \)
Lemma 4.6. Let \( x \in X^{cl} \setminus E_X \) and \( y \overset{\text{def}}{=} \phi(x) \in Y^{cl} \setminus E_Y \). The natural identification

\[
\tau_{x,y} : (k(y)^\times) \Sigma \cong (k(x)^\times) \Sigma,
\]

induced by \( \alpha \), satisfies the following property: For \( \eta \in k(y)^\times \) and \( \zeta \in k(x)^\times \), if

\[
1 + \eta \neq 0 \quad \text{and} \quad \tau_{x,y}(\eta') = \zeta',
\]

then there exist \( \alpha, \beta \in k_X^\times \{ \Sigma' \} \), such that

\[
\alpha + \beta \zeta \neq 0 \quad \text{and} \quad \tau_{x,y}((1 + \eta)') = (\alpha + \beta \zeta)'.
\]

Proof. Similar to the proof of [Saâidi-Tamagawa3, Lemma 4.10].

Let \( l \in \text{Primes} \) be a prime number which is both \((X, \Sigma)\)-admissible and \((Y, \Sigma)\)-admissible, i.e., for every finite extension \( k' \) of \( k_X \) (resp. \( k_Y \)), \((k')^l \cap F_X \) (resp. \((k')^l \cap F_Y \)) is finite, where \((k')^l \) is the maximal pro-\( l \) extension of \( k' \), and, in particular, \( E_X \cap X((k')^l)^{cl} \) (resp. \( E_Y \cap Y((k')^l)^{cl} \)) is finite. Let \( X^l, Y^l \) be the normalization of \( X, Y \) in \( K_X k_X^l, K_Y k_Y^l \), respectively. Set \( E_{X^l} \overset{\text{def}}{=} E_X \times_{k_X} k_X^l, E_{Y^l} \overset{\text{def}}{=} E_Y \times_{k_Y} k_Y^l \), and write \( \mathcal{O}_{E_{X^l}}^\times, \mathcal{O}_{E_{Y^l}}^\times \) for the group of multiplicative functions on \( X^l, Y^l \) whose supports of divisors are disjoint from \( E_{X^l}, E_{Y^l} \), respectively. Define \( D_{X^l \setminus E_{X^l}}, D_{X^l \setminus E_{X^l}} \) (resp. \( D_{Y^l \setminus E_{Y^l}}, D_{Y^l \setminus E_{Y^l}} \)) in a similar way as in Definition 4.1. Thus, we have natural exact sequences

\[
0 \to \mathcal{O}_{E_{X^l}}^\times / (k_X^l)^\times \to D_{X^l \setminus E_{X^l}} \to J_X(k_X^l)\{\Sigma'\} \to 0
\]

and

\[
0 \to \mathcal{O}_{E_{X^l}}^\times / (k_X^l)^\times \{\Sigma'\} \to D_{X^l \setminus E_{X^l}} \to J_X(k_X^l)\{\Sigma'\} \to 0
\]

( resp. similar sequences for \( D_{Y^l \setminus E_{Y^l}} \) and \( D_{Y^l \setminus E_{Y^l}} \)).

The isomorphism \( \alpha : \Pi_X \sim \Pi_Y \) induces natural isomorphisms \( \bar{\rho} : D_{Y^l \setminus E_{Y^l}} \sim D_{X^l \setminus E_{X^l}} \), and \( \rho' : D_{Y^l \setminus E_{Y^l}} \sim D_{X^l \setminus E_{X^l}} \) (by passing to finite subextensions of \( k_X^l / k_X \) and \( k_Y^l / k_Y \) corresponding to each other by \( \alpha \), cf. Theorem 4.2(i)), which fit into the following commutative diagram

\[
\begin{array}{ccc}
D_{X^l \setminus E_{X^l}} & \xleftarrow{\rho'} & D_{Y^l \setminus E_{Y^l}} \\
\downarrow & & \downarrow \\
D_{X^l \setminus E_{X^l}} & \xleftarrow{\bar{\rho}} & D_{Y^l \setminus E_{Y^l}}
\end{array}
\]

where the vertical maps are the natural ones. The above isomorphism \( \bar{\rho} \) implies the existence of subgroups \( \overline{\Pi}_{X^l} \) (resp. \( \overline{\Pi}_{Y^l} \)) of \( \mathcal{O}_{E_{X^l}}^\times / (k_X^l)^\times \) (resp. \( \mathcal{O}_{E_{Y^l}}^\times / (k_Y^l)^\times \)), which are functorial in \( X \) and \( Y \), and a natural isomorphism

\[
\bar{\rho} : \overline{\Pi}_{Y^l} \sim \overline{\Pi}_{X^l},
\]

which lies above the isomorphism \( \bar{\rho} : \overline{\Pi}_Y \sim \overline{\Pi}_X \) in Theorem 4.2(ii). (Use similar arguments as those in the proof of Theorem 4.2(ii).) Moreover, let \( H'_{X^l} \)
(resp. \(H'_{X_l}\)) be the inverse image of \(\overline{\mathcal{P}}_{X_l}\) (resp. \(\overline{\mathcal{P}}_{Y_l}\)) in \(O^\times_{E_{X_l}/((k^l_X)^\times\{\Sigma'\})}\) (resp. \(O^\times_{E_{Y_l}/((k^l_Y)^\times\{\Sigma'\})}\)). Then there exists a natural isomorphism \(\rho': H'_{X_l} \cong H'_{X_l}\) (cf. Theorem 4.2(ii)), and \(H'_{X_l}\) (resp. \(H'_{Y_l}\)) is a subgroup of \(O^\times_{E_{X_l}/((k^l_X)^\times\{\Sigma'\})}\) (resp. \(O^\times_{E_{Y_l}/((k^l_Y)^\times\{\Sigma'\})}\)). Further, let

\[
H^\times_{X_l} \overset{\text{def}}{=} \{ f \in O^\times_{E_{X_l}} \mid \tilde{f} \in \overline{\mathcal{P}}_{X_l}\} \quad \text{(resp. } H^\times_{Y_l} \overset{\text{def}}{=} \{ f \in O^\times_{E_{Y_l}} \mid \tilde{f} \in \overline{\mathcal{P}}_{Y_l}\}\).
\]

Then \(H^\times_{X_l}\) (resp. \(H^\times_{Y_l}\)) is a subgroup of \(O^\times_{E_{X_l}}\) (resp. \(O^\times_{E_{Y_l}}\)), and the quotient \(O^\times_{E_{X_l}} / H^\times_{X_l}\) (resp. \(O^\times_{E_{Y_l}} / H^\times_{Y_l}\)) is embeddable into \(J_Y(k^l_Y)\{\Sigma'\}\) (resp. \(J_X(k^l_X)\{\Sigma'\}\)), hence is a \(\Sigma'\)-primary torsion abelian group. (More precisely, let \(r\) be a prime number. Then the \(r\)-primary part of \(O^\times_{E_{X_l}} / H^\times_{X_l}\) (resp. \(O^\times_{E_{Y_l}} / H^\times_{Y_l}\)) is trivial (resp. finite, resp. embeddable into a finite direct sum of \(\mathbb{Q}_r / \mathbb{Z}_r\)), if \(r \in \Sigma\) (resp. \(r \in \Sigma' \setminus \{l\}\), resp. \(r\) is any prime.) We have a commutative diagram

\[
\begin{array}{ccc}
H'_{X_l} & \overset{\rho'}{\leftarrow} & H'_{Y_l} \\
\downarrow & & \downarrow \\
\overline{\mathcal{P}}_{X_l} & \overset{\overline{\rho}}{\leftarrow} & \overline{\mathcal{P}}_{Y_l}
\end{array}
\]

where the vertical maps are the natural surjections (cf. Theorem 4.2(ii)), and lies above the commutative diagram (4.1) in Theorem 4.2(ii).

We state the following results (Remark 4.7, Lemma 4.8, Corollary 4.10 and Lemmas 4.11-4.13) only for \(X\), but similar statements also hold for \(Y\).

**Remark 4.7.** If \(\Sigma\) is \(J_Y\)-large, then \(F_Y\) is contained in a finite extension (\(\neq \bar{k}_Y\)) of \(k^l_Y\) by Proposition 2.12. Thus, any \(l\) such that the image of the pro-\(l\) Sylow subgroup \(G_{k_Y,l}\) of \(G_{k_Y}\) under \(\rho_{J_Y,\Sigma}\) is finite is \((Y, \Sigma)\)-admissible. (Further, by Proposition 2.13(ii), such an \(l\) is automatically in \(\Sigma \cup \{p\}\).) Let us take such an \(l\). Then it follows that \((\rho_{J_Y,\Sigma}(G_{k_Y}) : \rho_{J_Y,\Sigma}(G_{k^l_Y})) < \infty\), hence \(\sharp(J_Y(k^l_Y)\{\Sigma'\}) < \infty\). Thus, in this case, the quotient \(O^\times_{E_{X_l}} / H^\times_{X_l}\) is finite.

**Lemma 4.8.** Let \(f_1, \ldots, f_n \in O_{E_{X_l}}\). Then, for all but finitely many \(c \in k^l_X\), \(f_1 - c, \ldots, f_n - c \in O^\times_{E_{X_l}}\).

**Proof.** We may and shall assume \(n = 1\) and set \(f \overset{\text{def}}{=} f_1\). (The assertion for general \(n\) can be reduced to this special case immediately.) The function \(f \in O_{E_{X_l}}\) descends to a finite extension of \(k_X\). More precisely, there exists a finite subextension \(k_0/k_X\) of \(k^l_X/k_X\), such that \(f \in O_{E_{X_{k_0}}}\), where \(E_{X_{k_0}} \overset{\text{def}}{=} E_X \times_{k_X} k_0\) and \(O_{E_{X_{k_0}}} = O_{E_X} \otimes_{k_X} k_0\). Then \(f\) defines a \(k_0\)-morphism \(f : X_{k_0} \to \mathbb{P}^1_{k_0}\) of degree, say, \(d\). One has \(f^{-1}(\mathbb{P}^1_{k_0}(k^l_{cl})) \subset X(k^l_{cl})\), where \(k'\) is the compositum of finite extensions of \(k^l\) of degree \(\leq d\), which is finite over \(k^l\). On the other hand, \(E_{X_{k_0}} \subset X_{k_0}(F)\), where \(F \overset{\text{def}}{=} F_{X_{k_0}}\). By Proposition 2.13, we see that \(S := f^{-1}(\mathbb{P}^1_{k_0}(k^l_{cl})) \cap E_{X_{k_0}}\) is finite. Now, for each \(c \in k^l \setminus f(S)(k^l)\) (\(\subset \mathbb{P}^1_{k_0}(k^l)\)), one has \(f - c \in O^\times_{E_{X_l}}\), as desired. \(\square\)
Definition 4.9. (i) For a subset $S$ of a field of characteristic $p > 0$, we denote by $\langle S \rangle$ the abelian subgroup (or, equivalently, $\mathbb{F}_p$-vector subspace) generated by $S$.
(ii) $R_{X^i} \overset{\text{def}}{=} \langle H_{X^i}^X \rangle$, $R_{Y^i} \overset{\text{def}}{=} \langle H_{Y^i}^X \rangle$.

Corollary 4.10. (i) $(k_{X^i})^X + \mathcal{O}_{E_{X^i}}^X = \mathcal{O}_{E_{X^i}}^X$.
(ii) $\langle \mathcal{O}_{E_{X^i}}^X \rangle \in \mathcal{O}_{E_{X^i}}^X$. In particular, $\langle \mathcal{O}_{E_{X^i}}^X \rangle$ coincides with the $k_{X^i}$-subalgebra generated by $\mathcal{O}_{E_{X^i}}^X$.

Proof. (i) Clearly $k_{X^i} + \mathcal{O}_{E_{X^i}}^X \subset \mathcal{O}_{E_{X^i}}^X$. To show the opposite, take any $f \in \mathcal{O}_{E_{X^i}}^X$. Then, by Lemma 4.8, there exists $c \in (k_{X^i})^X$ such that $f - c \in \mathcal{O}_{E_{X^i}}^X$. Thus, $f = c + (f - c) \in (k_{X^i})^X + \mathcal{O}_{E_{X^i}}^X$, as desired.
(ii) The first assertion follows directly from (i). The second assertion follows from the first assertion, together with the fact that $\mathcal{O}_{E_{X^i}}$ is a $k_{X^i}$-algebra. □

So far, we have only resorted to the assumptions that $\Sigma$ is $k_X$-large and $k_Y$-large and that at least one of $X$ and $Y$ is almost $\Sigma$-separated. From now, we will resort to the other assumption that $\alpha$ is pseudo-constants-additive.

Lemma 4.11. Let $f \in H_{X^i}^X$, and assume that $1 + f \in \mathcal{O}_{E_{X^i}}^X$. Then $1 + f \in H_{X^i}^X$.

Proof. Replacing $k_X$, $k_Y$ by suitable finite subextensions of $k_{X^i}/k_X$ and $k_{Y^i}/k_Y$ corresponding to each other by $\alpha$, we may assume that $f \in H_{X^i}^X$. Then the proof is similar to [Saidi-Tamagawa3], Lemma 3.16, using Proposition 2.4. More precisely, write $(\rho')^{-1}(f') = g'$. We have $((1 + f')^m \in H_{X^i}^X$ and we may write $(\rho')^{-1}(((1 + f')^m) = h'$ with $h \in H_{X^i}^X$ for some positive integer $m$ divisible only by primes in $\Sigma'$. By evaluating the pseudo-function $h'$ at all points in $Y^{\text{cl}} \setminus Y(F_Y)^{\text{cl}}$, and using Lemma 4.6 and Proposition 2.4, we deduce that $h' = (\rho')^{-1}(((1 + f')^m) = ((1 + c g')^m \in H_{Y^i} \subset \mathcal{D}_{Y^i|E_Y}$ for some $c \in (k_{Y^i})^X \{\Sigma'\}$ (cf. loc. cit. for more details). Hence, $(\rho')^{-1}((1 + f') = (1 + c g') \in \mathcal{D}_{Y^i|E_Y}$ since $\mathcal{D}_{Y^i|E_Y}$ admits no nontrivial $\Sigma'$-primary torsion ($\mathcal{D}_{Y^i|E_Y}$ has no nontrivial torsion, and the kernel of the natural surjective map $\mathcal{D}_{Y^i|E_Y} \to \mathcal{D}_{Y^i|E_Y}$ is $\Sigma$-primary torsion). As $(1 + c g') \in \mathcal{O}_{E_Y^i}/(k_{Y^i})^X \{\Sigma'\}$, we have $(1 + f') \in H_{X^i}^X$ by definition, as desired. □

Lemma 4.12. (i) $R_{X^i}$ coincides with the $k_{X^i}$-subalgebra generated by $H_{X^i}^X$.
(ii) Let $f_1, \ldots, f_n \in R_{X^i}$. Then, for all but finitely many $c \in k_{X^i}$, $f_1 - c, \ldots, f_n - c \in H_{X^i}^X$.
(iii) $(k_{X^i})^X + H_{X^i}^X = R_{X^i}$.
(iv) $R_{X^i} \cap \mathcal{O}_{E_{X^i}}^X = R_{X^i} = H_{X^i}^X$.

Proof. (i) First, note that $k_{X^i} = (k_{X^i})^X \cup \{0\}$ is contained in $R_{X^i} = \langle H_{X^i}^X \rangle$, as $(k_{X^i})^X \subset H_{X^i}^X$. Thus, it suffices to prove that $R_{X^i}$ is stable under multiplication. But this just follows from the fact that $H_{X^i}^X$ (which is a multiplicative subgroup) is stable under multiplication.
(ii) We may and shall assume $n = 1$ and set $f \overset{\text{def}}{=} f_1$. (The assertion for general $n$ can be reduced to this special case immediately.) As $f \in R_{X^i} = \langle H_{X^i}^X \rangle$, $f$ can be written as $f = g_1 + \cdots + g_m$ with $g_1, \ldots, g_m \in H_{X^i}^X$. We shall prove the assertion for $f$ by induction on $m$. The case $m = 0$ (i.e., $f = 0$) is easy: any $c \in k_{X^i} \{0\}$ satisfies the desired property. The case $m = 1$ (i.e., $f = g_1 \in H_{X^i}^X$) follows immediately from Lemma 4.8 and Lemma 4.11. More precisely, by Lemma 4.8 (and the case
m = 0), for all but finitely many \(c \in k_X^l\), one has \(f - c \in \mathcal{O}_E^X_{E_X^l}\) and \(-c \in H_X^X\), hence

\[
f - c = \frac{f - c}{f} \cdot f = \left(\frac{-c}{f} + 1\right) f \in H_X^X
\]

by Lemma 4.11. Now, assume \(m > 1\) and suppose that the assertion holds for \(m - 1\). Then it follows from Lemma 4.8 and the induction hypothesis that, for all but finitely many \(c \in k_X^l\), one has \(f - c \in \mathcal{O}_E^X_{E_X^l}\) and \((g_1 + \cdots + g_{m-1}) - c \in H_X^X\).

Now, as

\[
f - c = \frac{f - c}{g_m} \cdot g_m = \left(\frac{g_1 + \cdots + g_{m-1}}{g_m} - c\right) g_m,
\]

one has \(f - c \in H_X^X\) by Lemma 4.11.

(iii) Clearly \((k_X^l)^{\times} + H_X^X \subset R_X^l\). To show the opposite, take any \(f \in R_X^l\). Then, by (ii), there exists \(c \in (k_X^l)^{\times}\) such that \(f - c \in H_X^X\). Thus, \(f = c + (f - c) \in (k_X^l)^{\times} + H_X^X\), as desired.

(iv) Clearly \(R_X^l \cap \mathcal{O}_E^X_{E_X^l} \supset R_X^l \ni H_X^X\), hence it suffices to prove \(R_X^l \cap \mathcal{O}_E^X_{E_X^l} \subset H_X^X\).

So, take any \(f \in R_X^l \cap \mathcal{O}_E^X_{E_X^l}\). By (iii), there exist \(c \in (k_X^l)^{\times}\) and \(g \in H_X^X\) such that \(f = c + g\). As

\[
f = \frac{f}{c} \cdot c = \left(1 + \frac{g}{c}\right) c,
\]

one has \(f \in H_X^X\) by Lemma 4.11. \(\square\)

**Lemma 4.13.** \(\text{Fr}(\mathcal{O}_E^X_{E_X^l}) = \text{Fr}(R_X^l)\) and \(\mathcal{O}_E^X_{E_X^l}\) is the normalization of \(R_X^l\).

**Proof.** Write \(K_X^l = \text{Fr}(\mathcal{O}_E^X_{E_X^l}) = K_X^l k_X^l\) and \(N_X^l = \text{Fr}(R_X^l)\).

**Step 1.** \(\mathcal{O}_E^X_{E_X^l}\) is the integral closure of \(R_X^l\) in \(K_X^l\).

Indeed, as \(\mathcal{O}_E^X_{E_X^l}\) is the intersection of discrete valuation rings \(\mathcal{O}_{E_X^l,x}(x \in E_X^l)\), \(\mathcal{O}_E^X_{E_X^l}\) is integrally closed. On the other hand, as \(\mathcal{O}_E^X_{E_X^l}/R_X^l\) is torsion, each element of \(\mathcal{O}_E^X_{E_X^l}\) is integral over \(R_X^l\). As \(\mathcal{O}_E^X_{E_X^l} = (\mathcal{O}_E^X_{E_X^l})_l\), \(\mathcal{O}_E^X_{E_X^l}\) is integral over \(R_X^l\), as desired.

**Step 2.** \(\mathcal{O}_E^X_{E_X^l}/(k_X^l)^{\times}\) and \(R_X^l/(k_X^l)^{\times}\) are free \(\mathbb{Z}\)-modules of countably infinite rank.

Indeed, each of these groups is injectively mapped into \(\text{Div}^0_{X^l \setminus E_X^l}\) with torsion cokernel. Now, the assertion follows from the fact that \(\text{Div}^0_{X^l \setminus E_X^l}\) itself is a free \(\mathbb{Z}\)-module of countably infinite rank. (Note that \((X^l)^{\text{cl}} \setminus E_X^l\) is a countably infinite set.)

**Step 3.** \(K_X^l/N_X^l\) is finite.

Indeed, since \(k_X^l \subset N_X^l \subset K_X^l\) and \(K_X^l\) is a (regular) function field of one variable over \(k_X^l\), it suffices to prove that \(N_X^l \ni k_X^l\). But this follows, for example, from Step 1 or Step 2.

**Step 4.** \(K_X^l/N_X^l\) is separable.

Indeed, otherwise, \(N_X^l \subset (K_X^l)^p\), since \(K_X^l/N_X^l\) is a finite extension of (regular) function fields of one variable over a perfect field \(k_X^l\). Then \(R_X^l \subset \mathcal{O}_E^X_{E_X^l} \cap (K_X^l)^p = (\mathcal{O}_E^X_{E_X^l})^p\) (where the last equality follows from the fact that \(\mathcal{O}_E^X_{E_X^l}\) is normal), and \(R_X^l \subset ((\mathcal{O}_E^X_{E_X^l})^p)^{\times} = (\mathcal{O}_E^X_{E_X^l})^p\). Thus, one has \(\mathcal{O}_E^X_{E_X^l}/R_X^l \to \mathcal{O}_E^X_{E_X^l}/(\mathcal{O}_E^X_{E_X^l})^p\).

But this is impossible, since the \(p\)-primary part of the torsion abelian group \(\mathcal{O}_E^X_{E_X^l}/R_X^l\)
is embeddable into a finite direct sum of \( \mathbb{Q}_p/\mathbb{Z}_p \) (hence is isomorphic to a direct sum of a finite number of copies of \( \mathbb{Q}_p/\mathbb{Z}_p \) and a \( p \)-primary finite abelian group), while \( \mathcal{O}_{E_X}^{\times}/(\mathcal{O}_{E_X}^{\times})^p \) is an \( \mathbb{F}_p \)-vector space of countably infinite dimension by Step 2.

Step 5. \( K_X^{\times}/N_X^{\times} \) has finite torsion.

Indeed, the homomorphism \( N_X^{\times} \to K_X^{\times} \) of fields comes from a finite, generically étale \( k_X^{l} \)-morphism \( X^l \to Z \) (where \( Z \) is a proper, smooth, geometrically connected curve over \( k_X^{l} \) with function field \( N_X^{l} \)) of degree \( d = [K_X^{l} : N_X^{l}] \). Then

\[
K_X^{\times}/N_X^{\times} = (K_X^{l}/(k_X^{l})^{\times})/(N_X^{l}/(k_X^{l})^{\times}) = \text{Pri}_X^{l}/\text{Pri}_Z.
\]

Considering the commutative diagram with two rows exact:

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \text{Pri}_Z & \longrightarrow & \text{Div}_Z & \longrightarrow & \text{Pic}_Z & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Pri}_X^{l} & \longrightarrow & \text{Div}_X^{l} & \longrightarrow & \text{Pic}_X^{l} & \longrightarrow & 0
\end{array}
\]

in which the vertical arrows are induced by the pull-back of divisors by the morphism \( X^l \to Z \), one obtains an exact sequence

\[
0 \to \ker(\text{Pic}_Z \to \text{Pic}_X^{l}) \to \text{Pri}_X^{l}/\text{Pri}_Z \to \text{Div}_X^{l}/\text{Div}_Z.
\]

Now, on the one hand, by considering the norm map \( \text{Pic}_X^{l} \to \text{Pic}_Z \), one sees that

\[
\ker(\text{Pic}_Z \to \text{Pic}_X^{l}) \subset \text{Pic}_Z[d] = \text{Pic}_Z^0[d] = J_Z(k_X^{l})[d],
\]

hence that \( \ker(\text{Pic}_Z \to \text{Pic}_X^{l}) \) is finite. On the other hand, by considering the definition of \( \text{Div}_Z \to \text{Div}_X^{l} \), one sees that the torsion of \( \text{Div}_X^{l}/\text{Div}_Z \) (all of which arises from the finitely many ramified points of the generically étale morphism \( X^l \to Z \)) is finite. Thus, the assertion follows.

Step 6. Let \( \tilde{R}_X^{l} \) denote the normalization of \( R_X^{l} \) in \( N_X^{l} \). Then \( (\mathcal{O}_{E_X}^{\times}/\tilde{R}_X^{\times}) < \infty \).

Indeed, as \( \mathcal{O}_{E_X}^{\times}/\tilde{R}_X^{\times} \leftarrow \mathcal{O}_{E_X}^{\times}/R_X^{\times} \), \( \mathcal{O}_{E_X}^{\times}/\tilde{R}_X^{\times} \) is torsion. On the other hand, as \( \mathcal{O}_{E_X}^{\times} \) is integral over \( \tilde{R}_X^{l} \) and \( \tilde{R}_X^{l} \) is integrally closed, one has \( \mathcal{O}_{E_X}^{\times}/\tilde{R}_X^{\times} \hookrightarrow K_X^{\times}/N_X^{\times} \). Now, the assertion follows from Step 5.

Step 7. End of proof: \( \mathcal{O}_{E_X}^{\times} = \tilde{R}_X^{l} \) and \( K_X^{\times} = N_X^{\times} \).

Indeed, by Step 6, we may write \( \mathcal{O}_{E_X}^{\times} = \tilde{R}_X^{l} f_1 \cup \cdots \cup \tilde{R}_X^{l} f_r \) for some \( f_1, \ldots, f_r \in \mathcal{O}_{E_X}^{\times} \). This, together with Corollary 4.10(i), implies that \( \mathcal{O}_{E_X}^{\times} = (\tilde{R}_X^{l} f_1 \cup \cdots \cup \tilde{R}_X^{l} f_r) + k_X^{l} \), and that \( \mathcal{O}_{E_X}^{\times}/k_X^{l} = \tilde{R}_X^{l} f_1 \cup \cdots \cup \tilde{R}_X^{l} f_r \) as a union of \( k_X^{l} \)-vector subspaces (where \( \tilde{R}_X^{l} f_i \) denotes the image of \( \tilde{R}_X^{l} f_i \) in \( \mathcal{O}_{E_X}^{\times}/k_X^{l} \)). As \( k_X^{l} \) is an infinite field, we must have \( \mathcal{O}_{E_X}^{\times}/k_X^{l} = \tilde{R}_X^{l} f_i \) for some \( i \), hence \( \mathcal{O}_{E_X}^{\times} = k_X^{l} + \tilde{R}_X^{l} f_i \).

We claim that \( f_i \in N_X^{l} \). Indeed, otherwise, \( 1, f_i \in K_X^{\times} \) are linearly independent over \( N_X^{\times} \). Namely, \( N_X^{l} \oplus N_X^{l} f \hookrightarrow K_X^{\times} \), hence, in particular, \( \tilde{R}_X^{l} \oplus \tilde{R}_X^{l} f \hookrightarrow \mathcal{O}_{E_X}^{\times} \). As \( k_X^{l} \subset \tilde{R}_X^{l} \subset \tilde{R}_X^{l} \), one has \( k_X^{l} \oplus \tilde{R}_X^{l} f \subset \tilde{R}_X^{l} \oplus \tilde{R}_X^{l} f \). This implies that \( k_X^{l} + \tilde{R}_X^{l} f \subset \tilde{R}_X^{l} + \tilde{R}_X^{l} f \subset \mathcal{O}_{E_X}^{\times} \), which is absurd.
Now, \( f \in N_{X_1} \cap \mathcal{O}_{X_1}^{X_1} = \tilde{R}_X^X \), where the last equality follows from Step 1, together with the definition of \( \tilde{R}_X^X \). Thus, \( \mathcal{O}_{E_{X_1}} = k_X^f + \tilde{R}_X^X f = k_X^f + \tilde{R}_X^X = \tilde{R}_X^X \), as desired. In particular, \( K_{X_1} = \text{Fr}(\mathcal{O}_{E_{X_1}}) = \text{Fr}(\tilde{R}_X^X) = N_{X_1} \). □

Next, we will denote by \( \mathbb{P}(R_{X_1}) = (R_{X_1} \setminus \{0\})/(k_X^f)^X \) (resp. \( \mathbb{P}(R_{Y_1}) = (R_{Y_1} \setminus \{0\})/(k_Y^f)^X \)) the projective space associated to the infinite-dimensional \( k_X^f \)-vector space \( R_{X_1} \) (resp. the \( k_Y^f \)-vector space \( R_{Y_1} \)) and by \( \mathbb{L}(R_{X_1}) (\subset 2^{\mathbb{P}(R_{X_1})}) \) (resp. \( \mathbb{L}(R_{Y_1}) (\subset 2^{\mathbb{P}(R_{Y_1})}) \)) the set of lines on \( \mathbb{P}(R_{X_1}) \) (resp. \( \mathbb{P}(R_{Y_1}) \)). We view \( \mathcal{U}_{X_1} = \tilde{R}_X^X/(k_X^f)^X \) and \( \mathcal{U}_{Y_1} = R_{Y_1}^X/(k_Y^f)^X \) (cf. Lemma 4.12(iv)) as subsets of the projective spaces \( \mathbb{P}(R_{X_1}) \) and \( \mathbb{P}(R_{Y_1}) \), respectively. Let \( F_X/k_X \) and \( F_Y/k_Y \) be the extensions in Definition 2.9. We define the following sets of subsets of \( \mathbb{P}^1(k_X^f) \) and \( \mathbb{P}^1(k_Y^f) \):

\[
S_{X_1} = \{ S \subset \mathbb{P}^1(k_X^f) \mid \#(S) < \infty \}, \quad S_{Y_1} = \{ S \subset \mathbb{P}^1(k_Y^f) \mid \#(S) < \infty \}.
\]

**Lemma 4.14.** The sets \( S_{X_1} \) and \( S_{Y_1} \) are admissible (cf. Definition 5.4(ii) for the meaning of being admissible).

**Proof.** This follows from the fact that \( k_X^f \) and \( k_Y^f \) are infinite. See Remark 5.5. □

**Lemma 4.15.** The subset \( \mathcal{U}_{X_1} \subset \mathbb{P}(R_{X_1}) \) is \( S_{X_1} \)-ample, and the subset \( \mathcal{U}_{Y_1} \subset \mathbb{P}(R_{Y_1}) \) is \( S_{Y_1} \)-ample (cf. Definition 5.6(iii) for the meaning of being \( S_{X_1} \)-ample and \( S_{Y_1} \)-ample).

**Proof.** We prove that \( \mathcal{U}_{X_1} \) is \( S_{X_1} \)-ample. The proof that \( \mathcal{U}_{Y_1} \) is \( S_{Y_1} \)-ample is similar. We have to prove the following equality \( \mathbb{L}(R_{X_1}) \cap \mathcal{U}_{X_1} = \mathbb{L}(R_{X_1}) \cap S_{X_1} \) (cf. Definition 5.6(iii)(2)). More precisely, let \( \ell \in \mathbb{L}(R_{X_1}) \) be a line in the projective space \( \mathbb{P}(R_{X_1}) \) such that \( \ell \cap \mathcal{U}_{X_1} \neq \emptyset \), then we have to prove that \( \ell \setminus \ell \cap \mathcal{U}_{X_1} \in S_{X_1} \), i.e., \( \#(\ell \setminus \ell \cap \mathcal{U}_{X_1}) < \infty \). We have \( \ell = \mathbb{P}(V) \), where \( V \) is a 2-dimensional \( k_X^f \)-vector subspace of \( R_{X_1} \). As \( \ell \cap \mathcal{U}_{X_1} \neq \emptyset \) we can take \( f \in V \cap R_{X_1}^X \). Further, taking any \( g \in V \setminus k_X^f f \), we may write \( V = \{ a f + b g \mid a, b \in k_X^f \} \). Then \( \ell = \overline{\{ f \}} \cup \{ (h - c) f \mid c \in k_X^f \} \), where \( h \overset{\text{def}}{=} \frac{g}{f} \in R_{X_1}^X \). Now, by Lemma 4.12(ii), \( h - c \in R_{X_1}^X \) (hence \( h - c \in \mathcal{U}_{X_1} \)) for all but finitely many \( c \in k_X^f \), as desired. □

**Lemma 4.16.** The natural isomorphism \( \bar{\rho} : \mathcal{U}_{Y_1} \xrightarrow{\sim} \mathcal{U}_{X_1} \) induces a natural bijection \( \bar{\tau} : \mathbb{L}(R_{Y_1}) \cap \mathcal{U}_{Y_1} \xrightarrow{\sim} \mathbb{L}(R_{X_1}) \cap \mathcal{U}_{X_1} \) with the following property: If \( \ell \in \mathbb{L}(R_{Y_1}) \cap \mathcal{U}_{Y_1} \) then \( \bar{\tau}(\ell) \cap \mathcal{U}_{X_1} = \bar{\rho}(\ell) \cap \mathcal{U}_{Y_1} \).

**Proof.** We will define the map \( \bar{\tau} \). Let \( \ell \in \mathbb{L}(R_{Y_1}) \cap \mathcal{U}_{Y_1} \). Then by Lemma 4.15, \( \ell \setminus \ell \cap \mathcal{U}_{Y_1} \) is finite, hence \( \ell \cap \mathcal{U}_{Y_1} \) is infinite. Take two distinct points \( \overline{f_1} \), \( \overline{g_1} \in \ell \cap \mathcal{U}_{Y_1} \), and take any liftings \( f_1, g_1 \in R_{Y_1}^X \) of \( \overline{f_1}, \overline{g_1} \), respectively. Set \( V_1 = \{ af_1 + bg_1 \mid a, b \in k_Y^f \} \), so that \( V_1 \) is a 2-dimensional \( k_Y^f \)-vector subspace of \( R_{Y_1}^X \) and that \( \ell = \mathbb{P}(V_1) \subset \mathbb{P}(R_{Y_1}) \). Then we have \( \ell = \overline{\{ g_1 \}} \cup \{ (1 + ch_1) f_1 \mid c \in k_Y^f \} \), where \( h_1 \overset{\text{def}}{=} \frac{g_1}{f_1} \in R_{Y_1}^X \), and \( \ell \cap \mathcal{U}_{Y_1} = \overline{\{ g_1 \}} \cup \{ (1 + ch_1) f_1 \mid c \in k_Y^f, (1 + ch_1) \in R_{Y_1}^X \} \). Take any \( c \in (k_Y^f)^X \) such that \( (1 + c h_1) \in R_{Y_1}^X \). Set \( f_2 \overset{\text{def}}{=} \rho'(f_1), g_2 \overset{\text{def}}{=} \rho'(g_1) \), where \( \rho' : H_{Y_1}^X = R_{Y_1}^X / ((k_Y^f)^X \{ S' \}) \to H_{X_1}^X = R_{X_1}^X / ((k_X^f)^X \{ S' \}) \) is the natural isomorphism (cf. discussion before Lemma 4.11), and take any liftings \( f_2, g_2 \in R_{X_1}^X \) of \( f_2', g_2' \), respectively. Set \( h_2 \overset{\text{def}}{=} \frac{g_2}{f_2} \in R_{X_1}^X \),
so that \( h_2' = \frac{\rho_2'}{f_2'} = \rho'(h_1') \). Further set \( V_2 = \{af_2 + bg_2 \mid a, b \in k_X^l \} \). Since \( \bar{\rho} \) is bijective, one has \( \bar{f}_2 \neq \bar{g}_2 \), hence \( V_2 \) is a 2-dimensional \( k_X^l \)-vector subspace of \( R_X^l \). Set \( \bar{\tau}(\ell) \defeq \ell' \defeq \mathbb{P}(V_2) \subset \mathbb{P}(R_X^l) \). As \( \bar{f}_2 \in \ell' \), one has \( \ell' \in \mathbb{L}(R_X^l)_{U_{X^l}} \).

By evaluating the pseudo-function \( \rho'((1 + ch_1)'') \) at all points \( x \in (X^l)^{cl} \) above \( X^{cl} \setminus X(F_X)^{cl} \), and applying Lemma 4.5, Lemma 4.6, and Proposition 2.3 to suitable finite subextensions of \( k_X^l/k_X \) and \( k_Y^l/k_Y \), we deduce that \( \rho'((1 + ch_1)''') = (\alpha + \beta h_2)'' \in R_X^l/(k_X^l, {\Sigma}') \) for some \( \alpha \in (k_X^l, {\Sigma}') \) and \( \beta \in (k_X^l)^{\times} \), hence \( \bar{\rho}(1 + ch_1) = 1 + \gamma h_2 \) for \( \gamma = \frac{\beta}{\alpha} \in (k_X^l)^{\times} \). This, together with \( \bar{\rho}(\bar{f}_1) = \bar{f}_2, \bar{\rho}(\bar{g}_1) = \bar{g}_2 \), implies that \( \bar{\rho}(U_{U_{X^l}}) \subset \bar{\tau}(\ell)_{U_{X^l}} \). Further, \( \bar{\tau} \) is bijective since it has an inverse \( \bar{\tau}' : \mathbb{L}(R_X^l)_{U_{X^l}} \sim \mathbb{L}(R_Y^l)_{U_{X^l}} \) which is naturally deduced from \( \alpha^{-1} : \Pi_Y \sim \Pi_X \).

By using \( \bar{\tau}' \), we also conclude that \( \bar{\rho}(U_{U_{X^l}}) = \ell'_{U_{X^l}} \), as desired. \( \square \)

**Lemma 4.17.** (Recovering the Additive Structure of \( R_X^l \) and \( R_Y^l \)) The following hold.

(i) There exist natural isomorphisms \( \bar{\rho} : \mathbb{P}(R_X^l) \sim \mathbb{P}(R_X^l) \) and \( \bar{\tau} : \mathbb{L}(R_Y^l) \sim \mathbb{L}(R_X^l) \) which extend the isomorphisms \( \rho : U_Y \sim U_X \) and \( \tau : \mathbb{L}(R_Y^l)_{U_{Y^l}} \sim \mathbb{L}(R_X^l)_{U_{X^l}} \), respectively, such that for every \( \ell \in \mathbb{L}(R_Y^l) \), we have \( \bar{\tau}(\ell) = \bar{\rho}(\ell) \) set-theoretically.

(ii) The bijection \( \bar{\rho} \) arises from a \( \psi_0 \)-isomorphism

\[
\psi : (R_Y^l, +) \sim (R_X^l, +),
\]

where \( \psi_0 : k_Y^l \sim k_X^l \) is a field isomorphism. Namely, \( \psi \) is an isomorphism of abelian groups which is compatible with \( \psi_0 \) in the sense that \( \psi(ax) = \psi_0(a)\psi(x) \) for \( a \in k_Y^l \) and \( x \in R_Y^l \). Further, \( \psi_0 \) is uniquely determined and \( \psi \) is uniquely determined up to scalar multiplication.

**Proof.** Assertion (i) follows formally from Lemma 4.16 and Theorem 5.7. Assertion (ii) follows from Theorem 5.7. \( \square \)

**Lemma 4.18.** (Recovering the Ring Structure of \( R_X^l \) and \( R_Y^l \)) The following hold.

(i) If we normalize the isomorphism

\[
\psi : (R_Y^l, +) \sim (R_X^l, +),
\]

in Lemma 4.17 by the condition \( \psi(1) = 1 \) (which is possible as \( \bar{\rho}(\bar{1}) = \bar{1} \)), it becomes a ring isomorphism such that the diagram

\[
\begin{array}{ccc}
R_X^l & \sim & R_Y^l \\
\uparrow & & \uparrow \\
k_X^l & \sim & k_Y^l
\end{array}
\]

commutes.

(ii) \( \psi \) induces a natural commutative diagram

\[
\begin{array}{ccc}
K_X^l & \sim & K_Y^l \\
\uparrow & & \uparrow \\
K_X & \sim & K_Y
\end{array}
\]
where the horizontal maps are field isomorphisms and the vertical maps are natural inclusions. Further, $\psi : K_{Y'} \sim K_{X'}$ is Galois-equivariant with respect to the isomorphism $G_{k_X} \sim G_{k_Y}$ induced by $\alpha : \Pi_X \sim \Pi_Y$ (cf. Proposition 1.7(i)).

(iii) $\psi$ induces a natural commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\psi} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{\psi} & Y
\end{array}
$$

where the horizontal maps are scheme isomorphisms and the vertical maps are natural projections. Further, $\psi : X' \sim Y'$ is Galois-equivariant with respect to the isomorphism $G_{k_X} \sim G_{k_Y}$ induced by $\alpha : \Pi_X \sim \Pi_Y$ (cf. Proposition 1.7(i)).

**Proof.** (i) The proof is similar to the proof of [Saïdi-Tamagawa3], Lemma 4.14. More precisely, let $f \in R_{Y'}^X$, then $\psi \circ \mu_f$ and $\mu_{\psi(f)} \circ \psi$ are $\psi_0$-isomorphisms $(R_{Y'}, +) \sim (R_{X'}, +)$, where $\mu_g$ denotes the $g$-multiplication map. Note that $\psi(f) \in R_{X'}^X$, since $\overline{\psi(f)} = \overline{\rho(f)} \in R_{X'}^X/(k_X)^X \subset K_{X'}^X/(k_X)^X$. The isomorphisms $R_{Y'}^X/(k_Y)^X \sim R_{X'}^X/(k_X)^X$ they induce coincide with each other:

$$
\overline{\psi \circ \mu_f} = \overline{\rho \circ \mu_f} = \mu_{\overline{\rho(f)}} \circ \overline{\rho} = \mu_{\overline{\psi(f)}} \circ \overline{\psi},
$$

where the second equality follows from the multiplicativity of $\overline{\rho}$. Further, we have

$$
\psi \circ \mu_f(1) = \psi(f) = \mu_{\psi(f)}(1) = \mu_{\overline{\psi(f)}} \circ \overline{\psi}(1).
$$

Thus, the equality $\psi \circ \mu_f = \mu_{\psi(f)} \circ \psi$ follows from the uniqueness in Theorem 5.7. This equality means that $\psi(fg) = \psi(f)\psi(g)$ holds for $f \in R_{Y'}^X$ and $g \in R_{Y'}^X$, which, together with the fact that $R_{Y'}^X = \langle R_{Y'}^X \rangle$, implies the multiplicativity of $\psi$.

(ii) By considering fields of fractions, we see that $\psi : R_{Y'}^X \sim R_{X'}^X$ naturally induces $\psi : K_{Y'} \sim K_{X'}$. Since the isomorphism $\overline{\rho} : \mathcal{U}_{Y'} \sim \mathcal{U}_{X'}$ is Galois-equivariant under the compatible actions of $G_{k_X}$ and $G_{k_Y}$ respectively, and since the isomorphism $\psi : R_{Y'}^X \sim R_{X'}^X$, $1 \mapsto 1$ is uniquely determined by $\overline{\rho}$, it follows that $\psi : R_{Y'}^X \sim R_{X'}^X$ is Galois-equivariant under the compatible actions of $G_{k_X}$ and $G_{k_Y}$ respectively, hence so is $\psi : K_{Y'} \sim K_{X'}$. Now, the fact that the field isomorphism $\psi : K_{Y'} \sim K_{X'}$ maps $K_Y$ isomorphically to $K_X$ follows from this.

(iii) This follows immediately from (ii). □

Next, let $X' \rightarrow X$ be a finite étale covering corresponding to an open subgroup $\Pi_{X'} \subset \Pi_X$ and $Y' \rightarrow Y$ the corresponding étale covering via $\alpha$ (i.e., $\Pi_{Y'} \overset{\text{def}}{=} \alpha(\Pi_{X'})$). (We refer to the case where $X' \rightarrow X$ is Galois (or, equivalently, $Y' \rightarrow Y$ is Galois) as a Galois case.) Then $\alpha$ induces, by restriction to $\Pi_{X'}$, an isomorphism

$$
\alpha : \Pi_{X'} \sim \Pi_{Y'}.
$$

To apply the preceding arguments to this isomorphism, we need to show that the assumptions of Theorem 4.4 hold.
Lemma 4.19. (i) $\Sigma$ is $k_{X'}$-large and $k_{Y'}$-large.
(ii) $X'$ and $Y'$ are $\Sigma$-separated. More precisely, $F_{X'}$ and $F_{Y'}$ are finite extensions of $F_X$ and $F_Y$, respectively. In particular, a prime number is $(X, \Sigma)$-admissible (resp. $(Y, \Sigma)$-admissible) if and only if it is $(X', \Sigma)$-admissible (resp. $(Y', \Sigma)$-admissible).
(iii) $\alpha : \Pi_{X'} \cong \Pi_{Y'}$ is pseudo-constants-additive.

Proof. (i) This follows from the fact that $k_{X'}$ and $k_{Y'}$ are finite extensions of $k_X$ and $k_Y$, respectively.
(ii) This follows from Proposition 2.11(i).
(iii) By Theorem 4.2(iii), the isomorphism $\alpha$ isomorphism $\alpha$ in the Galois case, these isomorphisms are Galois-equivariant with respect to the isomorphism $G_{k_{Y'}}$, just coincides with the isomorphism $\tau : (k^\times_{X'})^\Sigma \cong (k^\times_X)^\Sigma$ induced by $\alpha : \Pi_X \cong \Pi_Y$. Thus, the pseudo-constants-additivity of $\alpha : \Pi_{X'} \cong \Pi_{Y'}$ follows from that of $\alpha : \Pi_X \cong \Pi_Y$. □

Now, we may apply Lemma 4.18 to $\alpha : \Pi_{X'} \cong \Pi_{Y'}$ to obtain: (i) a ring isomorphism $\psi' : R_{(Y')} \cong R_{(X')}$ compatible with a field isomorphism $\psi_0 : k_{Y'} \cong k_{X'}$; (ii) a field isomorphism $\psi' : K_{(Y')} \cong K_{(X')}$ compatible with a field isomorphism $\psi : K_{Y'} \cong K_{X'}$ and Galois-equivariant with respect to the isomorphism $G_{k_{X'}} \cong G_{k_{Y'}}$ induced by $\alpha : \Pi_{X'} \cong \Pi_{Y'}$; and (iii) a scheme isomorphism $\psi' : (X') \cong (Y')$ compatible with a scheme isomorphism $\psi : X' \cong Y'$ and Galois-equivariant with respect to the isomorphism $G_{k_{X'}} \cong G_{k_{Y'}}$ induced by $\alpha : \Pi_{X'} \cong \Pi_{Y'}$. Further, in the Galois case, these isomorphisms are Galois-equivariant with respect to the isomorphism $\alpha : \Pi_X \cong \Pi_Y$. Indeed, this Galois-equivariance can be proved just similarly as in the proof of Lemma 4.18(ii).

Lemma 4.20. (i) The following diagram of rings is commutative:

\[
\begin{array}{c}
R_{(X')} & \xleftarrow{\psi'} & R_{(Y')}
\end{array}
\]

In particular, the following diagram of fields is commutative:

\[
\begin{array}{c}
k_{X'} & \xleftarrow{\psi_0} & k_{Y'}
\end{array}
\]

(ii) The following diagram of fields is commutative:

\[
\begin{array}{c}
K_{(X')} & \xleftarrow{\psi'} & K_{(Y')}
\end{array}
\]

In particular, the following diagram of fields is commutative:

\[
\begin{array}{c}
K_{X'} & \xleftarrow{\psi'} & K_{Y'}
\end{array}
\]

\[
\begin{array}{c}
K_X & \xleftarrow{\psi} & K_Y
\end{array}
\]
(iii) The following diagram of schemes is commutative:

\[
\begin{array}{ccc}
(X')^l & \xrightarrow{\psi'} & (Y')^l \\
\downarrow & & \downarrow \\
X^l & \xrightarrow{\psi} & Y^l
\end{array}
\]

In particular, the following diagram of schemes is commutative:

\[
\begin{array}{ccc}
X' & \xrightarrow{\psi'} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\psi} & Y
\end{array}
\]

Proof. (i) To prove the commutativity of the first diagram, we may and shall assume that we are in the Galois case. Let \(\iota_X\) and \(\iota_Y\) denote the natural inclusions \(R_X^l \to R_{(X')}^l\) and \(R_Y^l \to R_{(Y')}^l\), respectively. Applying Theorem 4.2(iii) to various finite covers of \(X\) and \(Y\), we see that the diagram in question induces a commutative diagram

\[
\begin{array}{ccc}
R_{(X')}^l/(k_{X'}^l)^\times & \xleftarrow{\psi'} & R_{(Y')}^l/(k_{Y'}^l)^\times \\
\uparrow & & \uparrow \\
R_X^l/(k_X^l)^\times & \xleftarrow{\iota_X} & R_Y^l/(k_Y^l)^\times
\end{array}
\]

In particular, we have

\[
\psi' \circ \iota_Y (R_{Y^l}^\times) \cdot (k_{X'}^l)^\times = \iota_X \circ \psi (R_{Y^l}^\times) \cdot (k_{X'}^l)^\times.
\]

Consider the \(\Pi_X\)-fixed parts of both sides of this equality. Then, since \(\psi' : R_{(Y')}^l \xrightarrow{\sim} R_{(X')}^l\) is Galois-equivariant with respect to \(\alpha : \Pi_X \xrightarrow{\sim} \Pi_Y\) and since \(\alpha(\Pi_X) = \Pi_Y\), we obtain

\[
\psi' \circ \iota_Y (R_{Y^l}^\times) = \iota_X \circ \psi (R_{Y^l}^\times),
\]

hence

\[
R \overset{\text{def}}{=} \psi' \circ \iota_Y (R_{Y^l}) = \langle \psi' \circ \iota_Y (R_{Y^l}^\times) \rangle = \langle \iota_X \circ \psi (R_{Y^l}^\times) \rangle = \iota_X \circ \psi (R_{Y^l}).
\]

Accordingly, each of \(\psi' \circ \iota_Y\) and \(\iota_X \circ \psi\) induces an isomorphism \(R_{Y^l} \xrightarrow{\sim} R\) that maps 1 to 1. Now, the desired equality \(\psi' \circ \iota_X = \iota_Y \circ \psi\) follows from the uniqueness assertion in Theorem 5.7.

The commutativity of the second diagram follows from that of the first diagram.

(ii) The commutativity of the first diagram follows from that of the first diagram in (i). The commutativity of the second diagram follows from that of the first diagram.

(iii) This follows immediate from (ii). □
Corollary 4.21. The isomorphism $\alpha$ induces a natural isomorphism $\tilde{X} \sim \tilde{Y}$ which fits into a commutative diagram

$$
\begin{array}{ccc}
\tilde{X} & \sim & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \sim & Y
\end{array}
$$

where the vertical maps are the pro-étale coverings corresponding to $\Pi_X$ and $\Pi_Y$, respectively.

Proof. This follows from Lemma 4.20 by passing to open subgroups of $\Pi_X$ and $\Pi_Y$ which correspond to each other via $\alpha$, together with the Galois-equivariance of various isomorphisms in the Galois case. □

This finishes the proof of Theorem 4.4. □

Theorem 4.22. (A Refined Version of the Grothendieck Conjecture for Proper Hyperbolic Curves over Finite Fields) Let $X$, $Y$ be proper hyperbolic curves over finite fields $k_X$, $k_Y$ of characteristic $p_X$, $p_Y$, respectively. Let $\Sigma_X, \Sigma_Y \subset \mathfrak{Primes}$ be sets of prime numbers containing at least one prime number different from $p_X$, $p_Y$, respectively, and set $\Sigma'_X \equiv \mathfrak{Primes} \setminus \Sigma_X, \Sigma'_Y \equiv \mathfrak{Primes} \setminus \Sigma_Y$. Assume that $\Sigma_X$ is $J_X$-large and that $\Sigma_Y$ is $J_Y$-large. Write $\Pi_X, \Pi_Y$ for the geometrically pro-$\Sigma_X$ étale fundamental group of $X$ and the geometrically pro-$\Sigma_Y$ étale fundamental group of $Y$, respectively. Let $\alpha : \Pi_X \sim \Pi_Y$

be an isomorphism of profinite groups. Then $\alpha$ arises from a uniquely determined commutative diagram of schemes:

$$
\begin{array}{ccc}
\tilde{X} & \sim & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \sim & Y
\end{array}
$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the pro-étale coverings corresponding to the groups $\Pi_X, \Pi_Y$, respectively.

Proof. By Proposition 1.7, we have $\Sigma \equiv \Sigma_X = \Sigma_Y$. By Lemma 2.7, $\Sigma$ is $k_X$-large and $k_Y$-large. By Proposition 2.5(i), we have $p \equiv p_X = p_Y$. By Proposition 2.12, $X$ and $Y$ are almost $\Sigma$-separated. Now, by Theorem 4.4, it suffices to prove that $\alpha$ is pseudo-constants-additive.

Let the notations be as in the proof of Theorem 4.4. To prove that $\alpha$ is pseudo-constants-additive, set

$$
\tilde{T}_{X^l} \equiv \{ f \in \mathcal{O}_{E_{X^l}} \mid \text{At least one pole of } f \text{ is } k_X^l \text{-rational.}\},
$$

$$
\tilde{T}^\times_{X^l} \equiv \{ f \in \mathcal{O}^\times_{E_{X^l}} \mid f, f^{-1} \in \tilde{T}_{X^l} \},
$$

and define $T_{X^l}$ to be the intersection of $\tilde{T}_{X^l}$ with $H_{X^l}^\times$. Further, set $\tilde{T}^\times_{X^l} \equiv T_{X^l}/(k_X^l)^\times$. We define $\tilde{T}_{Y^l}$, $\tilde{T}^\times_{Y^l}$ and $\overline{T}^\times_{Y^l}$ similarly. Since the divisors of
functions are preserved under $\tilde{\rho}$ (cf. Lemma 4.5(i)) and since the degrees of extensions of residue fields are preserved under $\phi : X^c \setminus E_X \to Y^c \setminus E_Y$ (cf. discussion before Corollary 3.3), we have $\tilde{\rho}(T_{Y^c}^\infty) = T_{X^c}^\infty$.

Observe that $\text{PGL}_2(k_X^l)$ acts on $K_{X^c} \setminus k_X^l = \mathbb{P}^1(K_{X^c}) \setminus \mathbb{P}^1(k_X^l)$ via linear fractional transformation:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mod (k_X^l)^\times \in \text{PGL}_2(k_X^l), \ h \in K_{X^c} \setminus k_X^l \implies A \cdot h \overset{\text{def}}{=} \frac{ah + b}{ch + d}.$$ 

Claim: For any $h \in K_{X^c} \setminus k_X^l$, there exists $A \in \text{PGL}_2(k_X^l)$ such that $f \overset{\text{def}}{=} A \cdot h$ satisfies $f, f + 1 \in H_{X^c}^\infty$. (In particular, $T_{X^c}^\infty \neq \emptyset$.) A similar statement holds for $Y$.

Indeed, for simplicity, set $P_X \overset{\text{def}}{=} \text{PGL}_2(k_X^l)$ and write $P_X h$ for the $P_X$-orbit of $h$. Take a finite subextension $k_0/k_X$ of $k_X^l/k_X$ such that $h \in K_{X^c}k_0$. Then $h$ can be regarded as a finite $k_0$-morphism $X_{k_0} \to \mathbb{P}^1_{k_0}$. On the one hand, by the Weil estimate, we have $\sharp(X_{k_0}(k_0)_{\text{cl}}) = \infty$. On the other hand, by Proposition 2.13, $\sharp(h((X_{k_0}(k_0)_{\text{cl}}) \cap h(E_X \times_{k_X} k_0))) < \infty$. Thus, there exist infinitely many points $x \in X(k_X^l)$ whose image under $h$ is not contained in $h(E_X \times_{k_X} k_0) \cup \{\infty\}$. Take such an $x$ and set $d \overset{\text{def}}{=} h(x) \in k_X^l = \mathbb{P}^1(k_X^l) \setminus \{\infty\}$. Then the set of zeros of $h - d \in K_{X^c}^l$ intersects trivially with $E_{X^c}$ and includes at least one $k_X^l$-rational point. Thus, $h_1 \overset{\text{def}}{=} \frac{1}{h - d} \in \tilde{T}_{X^c}^\infty \cap P_X h$. Next, it follows from the above argument again (applied to $h_1$ instead of $h$) that for all but finitely many $c \in k_X^l$, the set of zeros of $h_1 - c \in K_{X^c}^l$ intersects trivially with $E_{X^c}$ and includes at least one $k_X^l$-rational point, hence $h_1 - c \in \tilde{T}_{X^c}^\infty \cap P_X h$. Further, by Remark 4.7, $(\mathcal{O}_{E_{X^c}}^\infty : H_{X^c}^\infty) < \infty$, hence there exists an infinite subset $C$ of $k_X^l$ such that for all $c \in C$, $h_1 - c \in \tilde{T}_{X^c}^\infty \cap P_X h$ and $h_1 - c$ belongs to the same coset of $\mathcal{O}_{E_{X^c}}^\infty / H_{X^c}^\infty$. Now, take mutually distinct three elements $a, b, c \in C$. Then

$$f \overset{\text{def}}{=} \frac{b - c}{a - b} \cdot \frac{h_1 - a}{h_1 - c} \in \tilde{T}_{X^c}^\infty \cap H_{X^c}^\infty \cap P_X h \subset T_{X^c}^\infty \cap P_X h$$

and

$$f + 1 \overset{\text{def}}{=} \frac{a - c}{a - b} \cdot \frac{h_1 - b}{h_1 - c} \in \tilde{T}_{X^c}^\infty \cap H_{X^c}^\infty \cap P_X h \subset T_{X^c}^\infty \cap P_X h,$$

as desired.

As the degree of a function depends only on its $P_X$-orbit, the above claim particularly implies that the set of degrees of functions in $K_{X^c} \setminus k_X^l$ coincides with that of $H_{X^c}^\infty$: $\deg(K_{X^c} \setminus k_X^l) = \deg(H_{X^c}^\infty)$, and, similarly, that the set of degrees of functions in $K_{Y^c} \setminus k_Y^l$ coincides with that of $H_{Y^c}^\infty$: $\deg(K_{Y^c} \setminus k_Y^l) = \deg(H_{Y^c}^\infty)$. In particular, it follows from Lemma 4.5(i) that the gonality $\gamma_{X^c}$ of $X^c$ coincides with the gonality $\gamma_{Y^c}$ of $Y^c$.

Now, take any $h \in K_{Y^c} \setminus k_Y^l$ attaining the gonality: $\deg(h) = \gamma_{Y^c}$. Then, by the above claim, there exists $f \in P_Y h$ (where $P_Y \overset{\text{def}}{=} \text{PGL}_2(k_Y^l)$) such that $f, f + 1 \in H_{Y^c}^\infty$. As $f \in P_Y h$, we have $\deg(f) = \deg(h) = \gamma_{Y^c}$.

Set $g' \overset{\text{def}}{=} \rho'(f') \in H_{X^c}^\infty$, $g_1' \overset{\text{def}}{=} \rho'((f + 1)') \in H_{X^c}^\infty$ and take any lifts $g, g_1 \in H_{X^c}^\infty$ of $g', g_1'$, respectively. Then, by Lemma 4.5(i), $\deg(g) = \deg(f) = \gamma_{Y^c} = \gamma_{X^c}$.
As the pole divisors of $f$ and $f + 1$ coincide, the pole divisors of $g$ and $g_1$ coincide by Lemma 4.5(i). Also, as $f$ admits at least one $k_Y^l$-rational pole, $g$ admits at least one $k_X^l$-rational pole, say, $x$. Now, by considering the leading terms of Laurent expansions of $g$ and $g_1$ at the $k_X^l$-rational pole $x$, we see that there exists $\beta \in (k_X^l)^{\times}$, such that the pole divisor $D_{\beta}$ of $g_1 - \beta g$ is strictly smaller than the pole divisor $D$ of $g$. (That is to say, the divisor $D - D_{\beta}$ is effective and non-zero.) As $\deg(g) = \gamma_X$, this implies that $g_1 - \beta g$ is constant, hence we may write $g_1 = \alpha + \beta g$ with $\alpha \in k_X^l$. By evaluating this equation at $\phi^{-1}(y_1)$, where $y_1 \in (Y^l)^{\text{cl}}$ is a zero of $f$, we see $\alpha' = 1'$, i.e., $\alpha \in (k_X^l)^{\times}\{\Sigma'\}$. Similarly, by evaluating the equation $\frac{\alpha}{g} = \frac{\alpha}{g} + \beta$ at $\phi^{-1}(y_2)$, where $y_2 \in (Y^l)^{\text{cl}}$ is a pole of $g$, we see $\beta' = 1'$, i.e., $\beta \in (k_X^l)^{\times}\{\Sigma'\}$. Now, the proof of the assertion follows just similarly to the last paragraph of the proof of [Saïdi-Tamagawa3], Lemma 4.9. □

As a consequence of Theorem 4.22 we can deduce the following refined version of the Grothendieck conjecture for (not necessarily proper) hyperbolic curves over finite fields.

**Theorem 4.23.** (A Refined Version of the Grothendieck Conjecture for (Not Necessarily Proper) Hyperbolic Curves over Finite Fields) Let $U, V$ be (not necessarily proper) hyperbolic curves over finite fields $k_U, k_V$ of characteristic $p_U, p_V$, respectively. Let $\Sigma_U, \Sigma_V \subset \text{Primes}$ be sets of prime numbers and set $\Sigma'_U \overset{\text{def}}{=} \text{Primes} \setminus \Sigma_U$, $\Sigma'_V \overset{\text{def}}{=} \text{Primes} \setminus \Sigma_V$. Write $\Pi_U, \Pi_V$, for the geometrically pro-$\Sigma_U$ tame fundamental group of $U$ and the geometrically pro-$\Sigma_V$ tame fundamental group of $V$, respectively. Let

$$\alpha : \Pi_U \overset{\sim}{\twoheadrightarrow} \Pi_V$$

be an isomorphism of profinite groups. Assume that there exist open subgroups $\Pi_{U'}, \Pi_{V'} \subset \Pi_U, \Pi_{V'}, \subset \Pi_V$, which correspond to each other via $\alpha$, i.e., $\Pi_{V'} \overset{\text{def}}{=} \alpha(\Pi_{U'})$, corresponding to étale coverings $U' \rightarrow U, V' \rightarrow V$, such that the smooth compactifications $X'$ of $U'$ and $Y'$ of $V'$ are hyperbolic, that $\Sigma_U$ is $J_{X'}$-large and that $\Sigma_V$ is $J_{Y'}$-large. Then $\alpha$ arises from a uniquely determined commutative diagram of schemes:

$$
\begin{array}{ccc}
\tilde{U} & \twoheadrightarrow & \tilde{V} \\
\downarrow & & \downarrow \\
U & \twoheadrightarrow & V
\end{array}
$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups $\Pi_U, \Pi_V$, respectively.

The rest of this section is devoted to the proof of Theorem 4.23. Let $X, Y, X', Y'$ be the smooth compactifications of $U, V, U', V'$, respectively. We consider any open subgroup $\Pi_{U''} \subset \Pi_U$ corresponding to an étale covering $U'' \rightarrow U$ such that $\Pi_{U''}$ is normal in $\Pi_U$, and that $\Pi_{U''}$ is contained in $\Pi_{U'}$. We refer to such a subgroup $\Pi_{U''} \subset \Pi_U$ (resp. a covering $U'' \rightarrow U$) as a nice subgroup of $\Pi_U$ (resp. a nice covering of $U$). Set $\Pi_{V''} \overset{\text{def}}{=} \alpha(\Pi_{U''})$, which corresponds to an étale covering $V'' \rightarrow V$. Let $X'', Y''$ be the smooth compactification of $U'', V''$, respectively. (Note that $X'', Y''$ are hyperbolic, as they dominate $X', Y'$, respectively.) By Theorem 1.8(i), $\alpha : \Pi_{U''} \overset{\sim}{\twoheadrightarrow} \Pi_{V''}$ induces $\alpha : \Pi_{X''} \overset{\sim}{\twoheadrightarrow} \Pi_{Y''}$.
Note that by (the proof of) Theorem 4.22, applied to \( \alpha : \Pi_X \rightarrow \Pi_Y \), induced by \( \alpha : \Pi_U \rightarrow \Pi_V \), we have \( \Sigma \overset{\text{def}}{=} \Sigma_U = \Sigma_V \); \( p \overset{\text{def}}{=} p_U = p_V \); \( \Sigma \) is \( k_X \)-large and \( k_Y \)-large; \( X' \) and \( Y' \) are almost \( \Sigma \)-separated; and \( \alpha : \Pi_X \rightarrow \Pi_Y \) is pseudo-

constants-additive.

**Lemma 4.24.** (i) \( \Sigma \) is \( k_X \)-large and \( k_Y \)-large.

(ii) \( X'' \) and \( Y'' \) are almost \( \Sigma \)-separated.

(iii) \( \alpha : \Pi_X \rightarrow \Pi_Y \) is pseudo-constants-additive.

**Proof.** (i) This follows from the fact that \( k_X \) (resp. \( k_Y \)) is a finite extension of \( k_X \) (resp. \( k_Y \)).

(ii) This follows from Proposition 2.11, together with the fact that \( X'' \rightarrow X' \), \( Y'' \rightarrow Y' \) are tame-Galois.

(iii) This follows from the fact that the diagram

\[
\begin{array}{ccc}
(\bar{k}_X^{(X''})^\Sigma & \overset{\tau}{\leftrightarrow} & (\bar{k}_Y^{(Y'')})^\Sigma \\
\uparrow & & \uparrow \\
(\bar{k}_X^{(X')})^\Sigma & \overset{\tau}{\leftrightarrow} & (\bar{k}_Y^{(Y')})^\Sigma \\
\end{array}
\]

is commutative, where the horizontal arrows are isomorphisms induced by \( \alpha : \Pi_X \rightarrow \Pi_Y \) and \( \alpha : \Pi_X \rightarrow \Pi_Y \), and the vertical arrows are isomorphisms induced (via Kummer theory) by the natural homomorphisms \( \Pi_X \rightarrow \Pi_Y \) and \( \Pi_Y \rightarrow \Pi_Y \), (induced by \( \Pi_U \rightarrow \Pi_U \) and \( \Pi_Y \rightarrow \Pi_Y \), respectively). Here, as in the proof of Theorem 4.2, the commutativity of this diagram follows basically from the functoriality of Kummer theory, together with the commutativity of the diagram

\[
\begin{array}{ccc}
M_X^{} & \leftrightarrow & M_Y^{} \\
\uparrow & & \uparrow \\
M_X' & \leftrightarrow & M_Y' \\
\end{array}
\]

where the horizontal arrows are isomorphisms induced by \( \alpha \) (or, more precisely, by \( \alpha^{-1} \), via the functoriality of \( H^2 \)) and the vertical arrows are isomorphisms defined geometrically via the identification of \( M_X \) and \( M_Y \) (resp. \( M_X' \) and \( M_Y' \)) with the Tate module of \( \mathbb{G}_m \) over \( \bar{k}_X = k_X \) (resp. \( \bar{k}_Y = k_Y \)). The commutativity of this last diagram follows from the fact that the above (geometrically defined) isomorphism \( \bar{M}_X' \cong \bar{M}_X \) (resp. \( \bar{M}_Y' \cong \bar{M}_Y \)) is identified with the composite of the \( (X''_{k_X} : X''_{k_Y}) \)-multiplication map \( M_{X'} \rightarrow M_X \) (resp. the \( (Y''_{k_Y} : Y''_{k_Y}) \)-multiplication map \( M_Y \rightarrow M_Y \)) and the inverse of the natural isomorphism \( \bar{M}_X' \cong (X''_{k_X} : X''_{k_Y})M_{X'} \subset M_X \) (resp. \( \bar{M}_Y' \cong (Y''_{k_Y} : Y''_{k_Y})M_Y \subset M_Y \)) induced by the inclusion \( M_{X''} \rightarrow M_{X'} \) (resp. \( M_{Y''} \rightarrow M_{Y'} \)) arising from the functoriality of \( H^2 \), and the fact that \( (X''_{k_X} : X''_{k_Y}) = (\Delta U' : \Delta U'') = (\Delta V' : \Delta V'') = (Y''_{k_Y} : Y''_{k_Y}) \). \( \square \)

Thus, by Theorem 4.4, we obtain an isomorphism \( X'' \cong Y'' \). Further, this isomorphism is Galois-equivariant with respect to \( \alpha : \Pi_X \rightarrow \Pi_Y \). Indeed, this Galois-equivariance can be proved just similarly as in the proof of Lemma 4.18(ii).

Further, let \( U''_1 \rightarrow U, U''_2 \rightarrow U \) be nice coverings of \( U \) such that \( \Pi_{U''_1} \subset \Pi_{U''_2} \), \( V''_2 \rightarrow V, V''_1 \rightarrow V \) corresponding nice coverings of \( V \) (via \( \alpha : \Pi_U \rightarrow \Pi_V \), and
\(X''_1, X''_2, Y''_1, Y''_2\) the smooth compactifications of \(U''_1, U''_2, V''_1, V''_2\), respectively. Then the following diagram

\[
\begin{array}{ccc}
X''_2 & \longrightarrow & Y''_2 \\
\downarrow & & \downarrow \\
X''_1 & \longrightarrow & Y''_1
\end{array}
\]

is commutative, where the horizontal arrows are (Galois-equivariant) isomorphisms induced by \(\alpha : \Pi X''_2 \xrightarrow{\sim} \Pi Y''_2\) and \(\alpha : \Pi X''_1 \xrightarrow{\sim} \Pi Y''_1\), and the vertical arrows are natural (finite) morphisms induced by finite étale coverings \(U''_2 \rightarrow U''_1\) and \(V''_2 \rightarrow V''_1\). Indeed, the proof of this commutativity is similar to the proof of Lemma 4.20. More precisely, recall the proof of Theorem 4.4. The isomorphism \(X''_i \xrightarrow{\sim} Y''_i\) \((i \in \{1, 2\})\) is induced by the isomorphism \(R_{(Y''_i)} \xrightarrow{\sim} R_{(X''_i)}\) obtained by applying Theorem 5.7 to

\[
\mathbb{P}(R_{(X''_i)}) \subset \mathbb{P}(R_{(Y''_i)})
\]

\[
\left( k^{l}_{(X''_i)} \right)^{\times} \xrightarrow{\sim} \left( k^{l}_{(Y''_i)} \right)^{\times}
\]

where

\[
\langle H^{\times}_{(X''_i)} \rangle = R_{(X''_i)} \subset \mathcal{O}_{E_{(X''_i)}}
\]

\[
\langle H^{\times}_{(Y''_i)} \rangle = R_{(Y''_i)} \subset \mathcal{O}_{E_{(Y''_i)}}
\]

and

\[
H^{\times}_{(X''_i)} = R_{(X''_i)} \subset \mathcal{O}_{E_{(X''_i)}}
\]

\[
H^{\times}_{(Y''_i)} = R_{(Y''_i)} \subset \mathcal{O}_{E_{(Y''_i)}}
\]

But the problem here (which does not occur in the proof of Lemma 4.20) is that in general the natural inclusions \(K_{(X''_i)} \hookrightarrow K_{(X''_2)}\), \(K_{(Y''_i)} \hookrightarrow K_{(Y''_2)}\) induced by the (ramified) coverings \(f : X''_2 \rightarrow X''_1\) and \(g : Y''_2 \rightarrow Y''_1\), respectively, may not induce inclusions \(\mathcal{O}_{E_{(X''_i)}} \hookrightarrow \mathcal{O}_{E_{(X''_2)}}\), \(\mathcal{O}_{E_{(Y''_i)}} \hookrightarrow \mathcal{O}_{E_{(Y''_2)}}\), respectively. This is because it is unclear if \(f^{-1}(E''_{X_2}) = E''_{X_2}, g^{-1}(E''_{Y_1}) = E''_{Y_2}\) hold.

Here, the remedy is to resort to Proposition 2.11(ii) instead of Proposition 2.11(i). So, for each nice covering \(U'' \rightarrow U\) (resp. \(V'' \rightarrow V\)), define \(E_{X''}\) (resp. \(E_{Y''}\)) to be the inverse image of \(E_X \cup (X' \setminus U)\) (resp. \(E_Y \cup (Y' \setminus V)\)) in \(X''\) (resp. \(Y''\)). Then we have \(E_{X''} \subset E_{X'}, E_{Y''} \subset E_{Y'}\). In particular, let \(l\) be a \((X', \Sigma)\)-admissible prime number (then \(l\) is automatically \((Y', \Sigma)\)-admissible). Then \(l\) is \((X'', \Sigma)\)-admissible and \((Y'', \Sigma)\)-admissible for all nice coverings \(U'' \rightarrow U\) and \(V'' \rightarrow V\). Now, replacing \(E_{X''}, E_{Y''}\) by \(E_{X'}, E_{Y'}\) in the various definitions, we obtain

\[
\langle H^{\times}_{(X''_i)} \rangle = R_{(X''_i)} \subset \mathcal{O}_{E_{(X''_i)}}
\]

\[
\langle H^{\times}_{(X''_i)} \rangle = R_{(X''_i)} \subset \mathcal{O}_{E_{(X''_i)}}
\]
and

\[ \langle H_{(Y''')}^t \rangle = \mathcal{R}_{(Y''')} \subset \mathcal{O}_{(Y''')} \]

\[ \mathcal{H}_{(Y''')}^t = \mathcal{R}_{(Y''')}^t \subset \mathcal{O}_{(Y''')}^t \]

As in Lemma 4.13, we have Fr(\( \mathcal{R}_{(X''')}^t \)) = \( K_{(X''')} \) and Fr(\( \mathcal{R}_{(Y''')}^t \)) = \( K_{(Y''')} \). Then we may apply Theorem 5.7 to

\[ \mathbb{P}(\mathcal{R}_{(X''')}^t) \quad \mathbb{P}(\mathcal{R}_{(Y''')}^t) \]

\[ \cup \quad \cup \]

\[ \mathcal{R}_{(X''')}^t/(k_{X'''})^t \sim \mathcal{R}_{(Y''')}^t/(k_{Y'''}^t)^t \]

to obtain \( \mathcal{R}_{(Y''')} \sim \mathcal{R}_{(X''')} \). By the uniqueness assertion of Theorem 5.7, the diagram

\[ \begin{array}{c}
R_{(X''')} \leftarrow \leftarrow R_{(Y''')} \\
\uparrow \quad \uparrow \\
\mathcal{R}_{(X''')} \leftarrow \leftarrow \mathcal{R}_{(Y''')} 
\end{array} \]

commutes, where the vertical arrows are natural inclusions. In particular, \( R_{(Y''')} \sim \mathcal{R}_{(X''')} \) and \( \mathcal{R}_{(Y''')} \sim \mathcal{R}_{(X''')} \) induce the same isomorphisms \( K_{Y''} \sim K_{X''} \) and \( X'' \sim Y'' \).

Now, as in the proof of Lemma 4.20, we can prove that the diagrams

\[ \begin{array}{c}
\mathcal{R}_{(X''')}^t \leftarrow \leftarrow \mathcal{R}_{(Y''')}^t \\
\uparrow \quad \uparrow \\
\mathcal{R}_{(X''')}^t \leftarrow \leftarrow \mathcal{R}_{(Y''')}^t \\
\quad K_{X''}^t \leftarrow \leftarrow K_{Y''}^t \\
\quad \uparrow \quad \uparrow \\
K_{X''}^t \leftarrow \leftarrow K_{Y''}^t 
\end{array} \]

and

\[ \begin{array}{c}
X'' \rightarrow Y''_2 \\
\downarrow \quad \downarrow \\
X''_1 \rightarrow Y''_1 
\end{array} \]

commute, as desired.

Now, passing to nice open subgroups of \( \Pi_U \) and \( \Pi_V \) which correspond to each other via \( \alpha \), we obtain an isomorphism

\[ \tilde{X}_U \sim \tilde{Y}_V \]
which is Galois-equivariant with respect to $\alpha : \Pi_U \to \Pi_V$, where $\tilde{X}_U$, $\tilde{Y}_V$ are the integral closures of $X$, $Y$ in (the function fields of) $\tilde{U}$, $\tilde{V}$, respectively. (Note that in general $\tilde{X}_U$, $\tilde{Y}_V$ do not coincide with $\tilde{X}$, $\tilde{Y}$.) Further, dividing both sides of this isomorphism by the actions of $\Pi_U$ and $\Pi_V$, it follows that the isomorphism $\tilde{X}_U \to \tilde{Y}_V$ fits into a commutative diagram

$$
\begin{array}{ccc}
\tilde{X}_U & \xrightarrow{\sim} & \tilde{Y}_V \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sim} & Y
\end{array}
$$

Finally, removing the ramification loci from this last diagram, we obtain the desired commutative diagram

$$
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\sim} & \tilde{V} \\
\downarrow & & \downarrow \\
U & \xrightarrow{\sim} & V
\end{array}
$$

in which the horizontal arrows are isomorphisms and the vertical arrows are the profinite étale coverings corresponding to the groups $\Pi_U$, $\Pi_V$, respectively.

This finishes the proof of Theorem 4.23. $\square$

§5. On the fundamental theorem of projective geometry. Throughout this section all fields are assumed to be commutative. For a field $k$ and a vector space $V$ over $k$, define $\mathbb{P}(V)$ to be the projective space associated to $V$:

$$
\mathbb{P}(V) \overset{\text{def}}{=} (V \setminus \{0\})/k^\times,
$$

and define $\mathbb{L}(V)$ ($\subset 2^\mathbb{P}(V)$) to be the set of lines on $\mathbb{P}(V)$. Thus, $\mathbb{P}(V)$ (resp. $\mathbb{L}(V)$) is also identified with the set of 1-dimensional (resp. 2-dimensional) $k$-vector subspaces of $V$. For each $x \in V \setminus \{0\}$, denote the point of $\mathbb{P}(V)$ corresponding to $x$ by $\bar{x}$. We say that a set of points in $\mathbb{P}(V)$ is collinear, if there exists a line in $\mathbb{L}(V)$ which contains all of them. We say that a set of lines in $\mathbb{L}(V)$ is concurrent, if there exists a point in $\mathbb{P}(V)$ which is contained in all of them. We set $\mathbb{P}^n(k) \overset{\text{def}}{=} \mathbb{P}(k^n)$ for each $n \geq 0$.

It is well-known and easily proved that the projective space $(\mathbb{P}(V), \mathbb{L}(V))$ satisfies the following proposition.

Proposition 5.1 (cf. [EDM], 343). Let $k$ be a field and $V$ a vector space over $k$.

(i) (Axioms of projective geometry) The following (I)-(III) hold.

(I) If $p, q \in \mathbb{P}(V)$, $p \neq q$, then there exists a unique $\ell \in \mathbb{L}(V)$, such that $p, q \in \ell$. We denote this line $\ell$ by $p \vee q$.

(II) If $p_0, p_1, p_2, q_1, q_2 \in \mathbb{P}(V)$, $p_0, p_1, p_2$ are not collinear, $q_1 \neq q_2$, $p_0, p_1, q_1$ are collinear and $p_0, p_2, q_2$ are collinear, then $p_1 \vee p_2, q_1 \vee q_2$ are concurrent.

(III) If $\ell \in \mathbb{L}(V)$, then $\sharp(\ell) \geq 3$.

(ii) (Desargues' theorem) If $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{P}(V)$, $p_1, p_2, p_3$ are not collinear, $q_1, q_2, q_3$ are not collinear, $p_1 \neq q_1$, $p_2 \neq q_2$ and $p_3 \neq q_3$, then: “$p_1 \vee q_1, p_2 \vee q_2, p_3 \vee q_3$ are concurrent” if and only if “$(p_1 \vee p_2) \cap (q_1 \vee q_2), (p_2 \vee p_3) \cap (q_2 \vee q_3), (p_3 \vee p_1) \cap (q_3 \vee q_1)$, $(p_1 \vee p_2) \cap (q_1 \vee q_2)$ are collinear”. $\square$
Further, such an isomorphism that the information carried by the pair \((k, V)\) is equivalent to that carried by the pair \((\mathbb{P}(V), \mathbb{L}(V))\).

To be more precise, from now on, let \(k_i\) be a field and \(V_i\) a vector space over \(k_i\), for \(i = 1, 2\).

**Definition 5.2.** (i) An (semilinear) isomorphism \((k_1, V_1) \xrightarrow{\sim} (k_2, V_2)\) is a pair \((\mu, \lambda)\), where \(\mu\) is an isomorphism \(k_1 \xrightarrow{\sim} k_2\) of fields and \(\lambda\) is an isomorphism \(V_1 \xrightarrow{\sim} V_2\) of abelian groups, such that, for each \(a \in k_1\) and \(x \in V_1\), one has \(\lambda(ax) = \mu(a)\lambda(x)\).

(In fact, when \(V_i \neq 0\), \(\mu\) is determined uniquely by \(\lambda\), hence we may say that \(\lambda\) is an (a semilinear) isomorphism.)

(ii) A collineation \((\mathbb{P}(V_1), \mathbb{L}(V_1)) \xrightarrow{\sim} (\mathbb{P}(V_2), \mathbb{L}(V_2))\) is a pair \((\sigma, \tau)\), where \(\sigma\) is a bijection \(\mathbb{P}(V_1) \xrightarrow{\sim} \mathbb{P}(V_2)\) and \(\tau\) is a bijection \(\mathbb{L}(V_1) \xrightarrow{\sim} \mathbb{L}(V_2)\), such that, for each \(\ell \in \mathbb{L}(V_1)\), one has \(\tau(\ell) = \sigma(\ell) \overset{\text{def}}{=} \{\sigma(p) \mid p \in \ell\}\). (In fact, \(\tau\) is determined uniquely by \(\sigma\), hence we may say that \(\sigma\) is a collineation.)

**Theorem 5.3 (Fundamental Theorem of Projective Geometry, cf. [Artin]).**

(i) Each isomorphism \((\mu, \lambda) : (k_1, V_1) \xrightarrow{\sim} (k_2, V_2)\) naturally induces a collineation \((\sigma, \tau) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \xrightarrow{\sim} (\mathbb{P}(V_2), \mathbb{L}(V_2))\) by setting \(\sigma(x) \overset{\text{def}}{=} \lambda(x)\) for \(x \in V_1 \setminus \{0\}\).

(ii) Assume that \(\dim_{k_i}(V_i) \geq 3\) for \(i = 1, 2\). Then, for each collineation \((\sigma, \tau) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \xrightarrow{\sim} (\mathbb{P}(V_2), \mathbb{L}(V_2)),\) there exists an isomorphism \((\mu, \lambda) : (k_1, V_1) \xrightarrow{\sim} (k_2, V_2)\) that induces \((\sigma, \tau) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \xrightarrow{\sim} (\mathbb{P}(V_2), \mathbb{L}(V_2))\) (in the sense of (i)). Further, such an isomorphism \((\mu, \lambda)\) is unique up to scalar multiplication. More precisely, if \((\mu, \lambda), (\mu', \lambda')\) are such isomorphisms (that induce the same collineation \((\sigma, \tau)\)), then there exists an (in fact, a unique) element \(a \in k_1^\times\) such that \(\mu' = \mu\) and \(\lambda'(\cdot) = \lambda(a \cdot -)\). \(\square\)

The aim of this section is to give a refined version of the fundamental theorem of projective geometry, where certain “partial” collineations defined over “sufficiently large” subsets of projective spaces are considered. To formulate it precisely, let us first define what are “sufficiently large” subsets of projective spaces.

**Definition 5.4.** Let \(k\) be a field. Let \(\mathcal{S}\) be a set of subsets of \(\mathbb{P}^1(k)\).

(i) We say that \(\mathcal{S}\) is \(\text{PGL}_2\)-stable, if, for any \(S \in \mathcal{S}\) and any \(\sigma \in \text{PGL}_2(k)\), one has \(\sigma(S) \in \mathcal{S}\).

(ii) Let \(m, n\) be integers \(\geq 0\). Then we say that \(\mathcal{S}\) is \((m, n)\)-admissible, if \(\mathcal{S}\) is \(\text{PGL}_2\)-stable and, for any \(0 \leq m' \leq m, 0 \leq n' \leq n\), any \(S_1, \ldots, S_{m'} \in \mathcal{S}\), and any \(p_1, \ldots, p_{n'} \in \mathbb{P}^1(k)\), one has

\[
S_1 \cup \cdots \cup S_{m'} \cup \{p_1, \ldots, p_{n'}\} \subseteq \mathbb{P}^1(k).
\]

(Thus, if \(m_1 \geq m_2 \geq 0, n_1 \geq n_2 \geq 0\), then \(\mathcal{S} : (m_1, n_1)\)-admissible \(\implies\) \(\mathcal{S} : (m_2, n_2)\)-admissible.)

(iii) We say that \(\mathcal{S}\) is admissible, if \(\mathcal{S}\) is \((m, n)\)-admissible for all integers \(m, n \geq 0\).

(iv) Assume that \(\mathcal{S}\) is \(\text{PGL}_2\)-stable. Then, for each 1-dimensional projective space \(\ell\) over \(k\), we set

\[
\mathcal{S}_\ell \overset{\text{def}}{=} \{S \subset \ell \mid \alpha(S) \in \mathcal{S}\ \text{for some} \ \alpha : \ell \xrightarrow{\sim} k \mathbb{P}^1(k)\}.
\]

(By assumption, we see that “for some \(\alpha\)” in this definition may be replaced by “for all \(\alpha\)” and that \(\mathcal{S} = \mathcal{S}_{\mathbb{P}^1(k)}\).
Remark 5.5. We have:
\[ S: (m, n)\text{-admissible} \iff \forall S \in \mathcal{S}, \sharp(\mathbb{P}^1(k)) > m \sharp(S) + n \iff \sharp(k) > m \sharp(S) + n - 1. \]
In particular, if \( S = \{ \emptyset \} \), we have:
\[ S: (m, n)\text{-admissible} \iff \sharp(\mathbb{P}^1(k)) > n \iff \sharp(k) \geq n. \]

Definition 5.6. Let \( k \) be a field, \( V \) a vector space over \( k \), and \( U \) a subset of \( \mathbb{P}(V) \).

(o) For each line \( \ell \in \mathbb{L}(V) \), we write \( \ell_U \overset{\text{def}}{=} \ell \cap U \) and \( \ell_U^c \overset{\text{def}}{=} \ell \cap (\mathbb{P}(V) \setminus U) = \ell \setminus \ell_U \).

(i) We define \( \mathbb{L}(V)_U \subset \mathbb{L}(V) \) by:
\[ \mathbb{L}(V)_U \overset{\text{def}}{=} \{ \ell \in \mathbb{L}(V) \mid \ell_U \neq \emptyset \}. \]

(ii) Let \( S \) be a \( \text{PGL}_2 \)-stable set of subsets of \( \mathbb{P}^1(k) \). Then we define \( \mathbb{L}(V)_{U, S} \subset \mathbb{L}(V) \) by:
\[ \mathbb{L}(V)_{U, S} \overset{\text{def}}{=} \{ \ell \in \mathbb{L}(V) \mid \ell_U^c \in S_\ell \}. \]

(iii) Let \( S \) be a \( \text{PGL}_2 \)-stable set of subsets of \( \mathbb{P}^1(k) \). We say that \( U \) is \( S \)-ample, if the following conditions (1)(2) hold.
(1) \( U \neq \emptyset \).
(2) \( \mathbb{L}(V)_U \subset \mathbb{L}(V)_{U, S} \). Equivalently, for each \( \ell \in \mathbb{L}(V) \), either \( \ell_U = \emptyset \) or \( \ell_U^c \in S_\ell \).

(When \( S \) is \( (1, 0) \)-admissible, one automatically has \( \mathbb{L}(V)_{U, S} \subset \mathbb{L}(V)_U \), and the above condition (2) is then equivalent to: \( \mathbb{L}(V)_U = \mathbb{L}(V)_{U, S} \).)

Now, return to the situation of the fundamental theorem of projective geometry. Namely, let \( k_i \) be a field and \( V_i \) a vector space over \( k_i \), for \( i = 1, 2 \). The main result in this section is the following refinement of the fundamental theorem of projective geometry.

Theorem 5.7. Assume that \( \dim_{k_i}(V_i) \geq 3 \) for \( i = 1, 2 \). Let \( U_i \) be a subset of \( \mathbb{P}(V_i) \) for \( i = 1, 2 \), and assume that \( U_i \) is \( S_i \)-ample for some \((3, 2)\)-admissible set \( S_i \) of subsets of \( \mathbb{P}^1(k_i) \) for \( i = 1, 2 \). Let \( \sigma : U_1 \overset{\sim}{\rightarrow} U_2 \) and \( \tau : \mathbb{L}(V_1)_{U_1} \overset{\sim}{\rightarrow} \mathbb{L}(V_2)_{U_2} \) be bijections such that for each \( \ell \in \mathbb{L}(V_1)_{U_1} \), one has \( \tau(\ell)_{U_2} = \sigma(\ell_{U_1}) \). Then, each such \( (\sigma, \tau) : (U_1, \mathbb{L}(V_1)_{U_1}) \overset{\sim}{\rightarrow} (U_2, \mathbb{L}(V_2)_{U_2}) \) uniquely extends to a collineation \( (\tilde{\sigma}, \tilde{\tau}) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \overset{\sim}{\rightarrow} (\mathbb{P}(V_2), \mathbb{L}(V_2)) \). In particular, there exists an isomorphism \( (\mu, \lambda) : (k_1, V_1) \overset{\sim}{\rightarrow} (k_2, V_2) \) that induces \( (\sigma, \tau) : (U_1, \mathbb{L}(V_1)_{U_1}) \overset{\sim}{\rightarrow} (U_2, \mathbb{L}(V_2)_{U_2}) \), and such an isomorphism \( (\mu, \lambda) \) is unique up to scalar multiplication.

Proof. Step 0. The second assertion follows from the first assertion and Theorem 5.3. So, let us concentrate on the proof of the first assertion.

Step 1. Claim: If \( p \in U_1 \), then \( \tau : \mathbb{L}(V_1)_{U_1} \overset{\sim}{\rightarrow} \mathbb{L}(V_2)_{U_2} \) induces a bijection of subsets \( \mathbb{L}(V_1)_{\{ p \}} \overset{\sim}{\rightarrow} \mathbb{L}(V_2)_{\{ \sigma(p) \}} \).

Indeed, for each \( \ell \in \mathbb{L}(V_1)_{\{ p \}} \), one has
\[ \ell \in \mathbb{L}(V_1)_{\{ p \}} \iff p \in \ell \iff p \in \ell_{U_1} \iff \sigma(p) \in \sigma(\ell_{U_1}) = \tau(\ell)_{U_2} \iff \sigma(p) \in \tau(\ell) \iff \tau(\ell) \in \mathbb{L}(V_2)_{\{ \sigma(p) \}}, \]
as desired.

Step 2. Claim: If \( p \in \mathbb{P}(V_1) \), then there exists a unique point \( p' \in \mathbb{P}(V_2) \), such that \( \tau : \mathbb{L}(V_1)_{U_1} \xrightarrow{\sim} \mathbb{L}(V_2)_{U_2} \) induces a bijection of subsets \( \mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1} \xrightarrow{\sim} \mathbb{L}(V_2)_{\{p'\}} \cap \mathbb{L}(V_2)_{U_2} \). If, moreover, \( p \in U_1 \), then \( p' = \sigma(p) \).

Step 2-1. Claim: Two lines \( \ell_1, \ell_2 \in \mathbb{L}(V_1)_{U_1} \) are concurrent, if and only if so are \( \tau(\ell_1), \tau(\ell_2) \in \mathbb{L}(V_2)_{U_2} \).

Indeed, it suffices to prove the ‘only if’ part, since the ‘if’ part is obtained by applying the ‘only if’ part to \( \sigma^{-1} : U_2 \xrightarrow{\sim} U_1 \). Now, the ‘only if’ part is clear if \( \ell_1 = \ell_2 \). So, assume \( \ell_1 \neq \ell_2 \). Then, as \( \ell_1, \ell_2 \) are concurrent, there is a \((\text{unique})\) point \( p \in \mathbb{P}(V) \) such that \( \ell_1 \cap \ell_2 = \{p\} \). If \( p \in U_1 \), then \( \sigma(p) \in \tau(\ell_1), \tau(\ell_2) \) by Step 1, hence \( \tau(\ell_1), \tau(\ell_2) \) are concurrent, as desired. So, we may and shall assume that \( p \notin U_1 \). For each \( i = 1, 2 \), choose \( p_i \in (\ell_i)_{U_1} = \ell_i \setminus ((\ell_i)_i) \neq \varnothing \) (1,0)-admissibility). As \( p_1 \in \ell_1 \setminus \ell_2 \) and \( p_2 \in \ell_2 \setminus \ell_1 \), one has \( p_1 \neq p_2 \). So, set \( m \overset{\text{def}}{=} p_1 \cap p_2 \). As \( p_1 \in U_1 \), one has \( m \in \mathbb{L}(V_1)_{U_1} \). Next, take \( q \in m \cup \{p_1, p_2\} = m \setminus (m \cup \{p_1, p_2\}) \neq \varnothing \) (1,2)-admissibility). Consider the projection (or perspective mapping) \( \alpha : \ell_1 \xrightarrow{\sim} \ell_2 \) with respect to the center \( q \). More precisely, \( \alpha \) is defined by \( \{\alpha(x)\} = \{x \cap q \} \) for each \( x \in \ell_1 \). (In particular, \( \alpha(p_1) = p_2 \) and \( \alpha(p_2) = p_1 \).) Take \( q_1 \in (\ell_1)_{U_1} \cap \alpha^{-1}((\ell_2)_{U_2}) \setminus \{p_1\} \neq \varnothing \) (2,1)-admissibility), and set \( q_2 \overset{\text{def}}{=} \alpha(q_1) \) and \( n = q_1 \cap q_1 = q_1 \cap q_2 \). (Thus, \( q_2 \in ((\ell_2)_{U_1} \cap \alpha((\ell_1)_{U_1})) \setminus \{p_2\} \), and, as \( q_1 \in U_1 \), one has \( n \in \mathbb{L}(V_1)_{U_1} \). Now, one has \( q, p_1, q_1, p_2, q_2 \in U_1, q, p_1, q_1 \) not collinear, \( p_2 = q_2 \), \( q, p_1, p_2 \in m \) and \( q, q_1, q_2 \in n \). Accordingly, one has \( \sigma(q), \alpha(p_1), \alpha(q_1, \sigma(p_2), \sigma(q_2) \in U_2, \sigma(q), \sigma(p_1), \sigma(q_1) \) not collinear, \( \sigma(p_2) \neq \sigma(q_2) \), and, by Step 1, \( \sigma(p_1), \sigma(p_2), \sigma(q) \in \tau(m) \) and \( \sigma(q_1), \sigma(q_2), \sigma(q) \in \tau(n) \). Further, as \( p_1, q_1 \in \ell_1 \) (\( p_1 \neq q_1 \)) and \( p_2, q_2 \in \ell_2 \) (\( p_2 \neq q_2 \)) one has \( \sigma(p_1), \sigma(q_1) \in \tau(\ell_1) \) (with \( \sigma(p_1) \neq \sigma(q_1) \)) and \( \sigma(p_2), \sigma(q_2) \in \tau(\ell_2) \) (with \( \sigma(p_2) \neq \sigma(q_2) \)) by Step 1. Now, by Proposition 5.1(i),(II), this implies that \( \tau(\ell_1) = \sigma(p_1) \cap \sigma(q_1) \) and \( \tau(\ell_2) = \sigma(p_2) \cap \sigma(q_1) \) are concurrent, as desired.

Step 2-2. Claim: Three lines \( \ell_1, \ell_2, \ell_3 \in \mathbb{L}(V_1)_{U_1} \) are concurrent, if and only if so are \( \tau(\ell_1), \tau(\ell_2), \tau(\ell_3) \in \mathbb{L}(V_2)_{U_2} \).

Indeed, it suffices to prove the ‘only if’ part, since the ‘if’ part is obtained by applying the ‘only if’ part to \( \sigma^{-1} : U_2 \xrightarrow{\sim} U_1 \). Now, the ‘only if’ part follows from (the ‘only if’ part of) Step 2-1 if \( \ell_i = \ell_j \) for some \( i \neq j \). So, assume that \( \ell_1, \ell_2, \ell_3 \) are mutually distinct and set \( \ell_1 \cap \ell_2 \cap \ell_3 = \{p\} \). If \( p \in U_1 \), then \( \sigma(p) \in \tau(\ell_1), \tau(\ell_2), \tau(\ell_3) \) by Step 1, hence \( \tau(\ell_1), \tau(\ell_2), \tau(\ell_3) \) are concurrent, as desired. So, we may and shall assume that \( p \notin U_1 \). Take \( p_1 \in (\ell_1)_{U_1} \neq \varnothing \) and \( p_2 \in (\ell_2)_{U_1} \neq \varnothing \). Take \( p_3 \in (\ell_3)_{U_1} \setminus ((p_1 \cup p_2) \cap \ell_3) = \ell_3 \setminus ((\ell_3)_i \cup ((p_1 \cup p_2) \cap \ell_3)) \neq \varnothing \) (1,1)-admissibility). Take \( q_1 \in (\ell_1)_{U_1} \setminus \{p_1\} = \ell_1 \setminus ((\ell_1)_i \cup \{p_1\}) \neq \varnothing \) (1,1)-admissibility). Let \( \alpha : \ell_2 \xrightarrow{\sim} (p_1 \cup p_2) \) be the projection with respect to the center \( q_1 \): \( \{\alpha(x)\} = (x \cap q_1) \cap (p_1 \cup p_2) \). Take \( q_2 \in (\ell_2)_{U_1} \cap \alpha^{-1}((p_1 \cup p_2)_{U_1}) \setminus \{p_2\} = \ell_1 \setminus ((\ell_1)_i \cup \alpha^{-1}((p_1 \cup p_2)_{U_1}) \cup \{p_2\}) \neq \varnothing \) (2,1)-admissibility), and set \( r_12 \overset{\text{def}}{=} \alpha(q_2) \in (p_1 \cup p_2)_{U_1} \setminus \{p_1, p_2\} \). Let \( \beta : \ell_3 \xrightarrow{\sim} (p_2 \cup p_3) \) be the projection with respect to the center \( q_2 \): \( \{\beta(x)\} = (x \cap q_2) \cap (p_2 \cup p_3) \) and \( \gamma : \ell_3 \xrightarrow{\sim} (p_3 \cup p_1) \) the projection with respect to the center \( q_1 \): \( \{\gamma(x)\} = (x \cap q_1) \cap (p_3 \cup p_1) \). Take \( q_3 \in ((\ell_3)_{U_1} \cap \alpha^{-1}((p_2 \cup p_3)_{U_1}) \cap \gamma^{-1}((p_3 \cup p_1)_{U_1}) \setminus \{p_3\} \cup ((q_1 \cup q_2) \cap \ell_3) = \ell_3 \setminus ((\ell_3)_i \cup \alpha^{-1}((p_2 \cup p_3)_{U_1}) \cup \gamma^{-1}((p_3 \cup p_1)_{U_1}) \cup \{p_3\} \cup ((q_1 \cup q_2) \cap \ell_3) \neq \varnothing \) (here, we use the (3,2)-admissibility assumption fully), and set \( r_{32} \overset{\text{def}}{=} \beta(q_3) \in \ell_3 \setminus ((\ell_3)_i \cup \alpha^{-1}((p_2 \cup p_3)_{U_1}) \cup \gamma^{-1}((p_3 \cup p_1)_{U_1}) \cup \{p_3\} \cup ((q_1 \cup q_2) \cap \ell_3) \neq \varnothing \).
(p_2 ∨ p_3)_{U_1} \setminus \{p_2, p_3\} and \ r_{31} \overset{\text{def}}{=} \gamma(q_3) \in (p_3 ∨ p_1)_{U_1} \setminus \{p_3, p_1\}. Now, one has \ p_1, p_2, p_3, q_1, q_2, q_3, r_{12}, r_{23}, r_{31} \in U_1 with \ p_1, p_2, p_3 \ not \ collinear \ and \ q_1, q_2, q_3 \ not \ collinear, \ p_1, q_1 \in \ell_1 \ with \ p_1 \neq q_1, \ p_2, q_2 \in \ell_2 \ with \ p_2 \neq q_2, \ p_3, q_3 \in \ell_3 \ with \ p_3 \neq q_3, \ \{r_{12}\} = (p_1 ∨ p_2) \cap (q_1 ∨ q_2), \ \{r_{23}\} = (p_2 ∨ p_3) \cap (q_2 ∨ q_3), \ \{r_{31}\} = (p_3 ∨ p_1) \cap (q_3 ∨ q_1), \ and \ \ell_1 = p_1 ∨ q_1, \ \ell_2 = p_2 ∨ q_2, \ \ell_3 = p_3 ∨ q_3 \ are \ concurrent. \ Then, \ by \ the \ \{\implies\ \} \ part \ of \ Proposition \ 5.1(ii), \ r_{12}, r_{23}, r_{31} \ are \ collinear. \ Accordingly \ (by \ Step \ 1), \ one \ has \ \sigma(p_1), \ \sigma(p_2), \ \sigma(p_3), \ \sigma(q_1), \ \sigma(q_2), \ \sigma(q_3), \ \sigma(r_{12}), \ \sigma(r_{23}), \ \sigma(r_{31}) \in U_2 \ with \ \sigma(p_1), \ \sigma(p_2), \ \sigma(p_3) \ not \ collinear \ and \ \sigma(q_1), \ \sigma(q_2), \ \sigma(q_3) \ not \ collinear, \ \sigma(p_1), \ \sigma(q_1) \in \tau(\ell_1) \ with \ \sigma(p_1) \neq \sigma(q_1), \ \sigma(p_2), \ \sigma(q_2) \in \tau(\ell_2) \ with \ \sigma(p_2) \neq \sigma(q_2), \ \sigma(p_3), \ \sigma(q_3) \in \tau(\ell_3) \ with \ \sigma(p_3) \neq \sigma(q_3), \ \{\sigma(r_{12})\} = (\sigma(p_1) \lor \sigma(p_2)) \cap (\sigma(q_1) \lor \sigma(q_2)), \ \{\sigma(r_{23})\} = (\sigma(p_2) \lor \sigma(p_3)) \cap (\sigma(q_2) \lor \sigma(q_3)), \ \{\sigma(r_{31})\} = (\sigma(p_3) \lor \sigma(p_1)) \cap (\sigma(q_3) \lor \sigma(q_1)), \ \sigma(r_{12}), \ \sigma(r_{23}), \ \sigma(r_{31}) \ are \ collinear. \ Now, \ by \ (the \ \langle\ \iff\ \rangle \ part \ of \ Proposition \ 5.1(ii), \ \tau(\ell_1) = \sigma(p_1) \lor \sigma(q_1), \ \tau(\ell_2) = \sigma(p_2) \lor \sigma(q_2), \ \tau(\ell_3) = \sigma(p_3) \lor \sigma(q_3) \ are \ concurrent, \ as \ desired.

**Step 2-3. Claim:** If \( p \in \mathbb{P}(V_i) \) for \( i = 1, 2 \), one has

\[
\bigcap_{\ell \in L(V_i) \cap L(V_i)_{U_1}} \ell = \{p\}.
\]

Indeed, we may assume that \( i = 1 \). First, \( \supseteq \) is clear. So, to prove \( \subseteq \), it suffices to show that the left-hand side is of cardinality at most one. Then, since two distinct lines intersect at most one point, it suffices to prove that there are at least two elements \( \ell \in L(V_1) \cap L(V_1)_{U_1} \). Consider the two cases separately: (i) \( p \in U_1 \); and (ii) \( p \not\in U_1 \). In case (i), as \( \text{dim}(\mathbb{P}(V_1)) \geq 2 \), hence there exist \( q, r \in \mathbb{P}(V_1) \) such that \( p, q, r \) are not collinear. Then \( p \lor q \) and \( p \lor r \) are two distinct lines that belong to \( L(V_1) \cap L(V_1)_{U_1} \). In case (ii), as \( U_1 \neq \emptyset \), take \( q \in U_1 \). (Thus, \( q \neq p \).) As \( \text{dim}(\mathbb{P}(V_1)) \geq 2 \), hence there exists \( r \in \mathbb{P}(V_1) \) such that \( p, q, r \) are not collinear. Observe \( q \lor r \in L(V_1)_{U_1} \), and take \( s \in (q \lor r)_{U_1} \setminus \{q, r\} = (q \lor r) \setminus ((q \lor r)_{U_1} \cup \{q, r\}) \neq \emptyset \) \( (1, 2) \)-admissibility). Now, \( p \lor q \) and \( p \lor s \) are two distinct lines that belong to \( L(V_1) \cap L(V_1)_{U_1} \).

**Step 2-4. Claim:** If \( p \in \mathbb{P}(V_1) \),

\[
\bigcap_{\ell \in L(V_1) \cap L(V_1)_{U_1}} \tau(\ell)
\]

is a subset of \( \mathbb{P}(V_2) \) of cardinality one. (Denote it by \( \{p'\} \).)

Indeed, for each pair \( \ell, m \in L(V_1) \cap L(V_1)_{U_1} \) with \( \ell \neq m \), \( \tau(\ell) \cap \tau(m) \) is of cardinality one by Step 2-1. So, set \( \tau(\ell) \cap \tau(m) = \{p'_{\ell, m}\} \). In fact, the point \( p'_{\ell, m} \) does not depend on the pair \( \ell, m \). Indeed, let \( \ell', m' \in L(V_1) \cap L(V_1)_{U_1} \) be any pair with \( \ell' \neq m' \). If \( \sharp \{\ell, m, \ell', m'\} = 2 \), then it is clear that \( p'_{\ell, m} = p'_{\ell', m'} \). If \( \sharp \{\ell, m, \ell', m'\} = 3 \), then it follows from Step 2-2 that \( \tau(\ell), \tau(m), \tau(\ell'), \tau(m') \) are concurrent, hence \( p'_{\ell, m} = p'_{\ell', m'} \). If \( \sharp \{\ell, m, \ell', m'\} = 4 \), then, again by Step 2-2, one has

\[
p'_{\ell, m} = p'_{\ell, m'} = p'_{\ell', m'}.
\]

Now, write \( p' = p'_{\ell, m} \) for some (or, equivalently, all) pair \( \ell, m \in L(V_1) \cap L(V_1)_{U_1} \) with \( \ell \neq m \). Here, note that, as shown in the proof of Step 2-3, \( L(V_1) \cap L(V_1)_{U_1} \)
is of cardinality at least two, hence at least one such pair \( \ell, m \) exists. Then, by definition, one has

\[
\bigcap_{\ell \in \mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_2)_{U_1}} \tau(\ell) = \{p'\},
\]

as desired.

**Step 2-5. End of Step 2.**

Let \( p \in \mathbb{L}(V_1) \) and define \( p' \in \mathbb{L}(V_2) \) as in Step 2-4. Then one has

\[
\tau(\mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}) \subset \mathbb{L}(V_2)_{\{p'\}} \cap \mathbb{L}(V_2)_{U_2}.
\]

Applying this to \( \sigma^{-1} : U_2 \sim U_1 \) and \( p' \in \mathbb{L}(V_2) \), we obtain

\[
\tau^{-1}(\mathbb{L}(V_2)_{\{p'\}} \cap \mathbb{L}(V_2)_{U_2}) \subset \mathbb{L}(V_1)_{\{p''\}} \cap \mathbb{L}(V_1)_{U_1},
\]

for some unique \( p'' \in \mathbb{L}(V_1) \). Combining these containment relations, we conclude

\[
\mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1} \subset \mathbb{L}(V_1)_{\{p''\}} \cap \mathbb{L}(V_1)_{U_1},
\]

which, together with Step 2-3, implies that \( \{p\} \supset \{p''\} \), hence \( p = p'' \), and that

\[
\tau(\mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1}) = \mathbb{L}(V_2)_{\{p'\}} \cap \mathbb{L}(V_2)_{U_2},
\]

as desired.

The uniqueness of \( p' \) is clear by Step 2-3. This uniqueness, together with Step 1, implies \( p' = \sigma(p) \) for \( p \in U_1 \).

**Step 3.** We define \( \tilde{\sigma} : \mathbb{P}(V_1) \rightarrow \mathbb{P}(V_2) \) to be the map that sends \( p \in \mathbb{P}(V_1) \) to \( p' \in \mathbb{P}(V_2) \) defined in Step 2. (Thus, in particular, \( \tilde{\sigma}(p) = \sigma(p) \) if \( p \in U_1 \).)

Claim: \( \tilde{\sigma} \) is a collineation. (More precisely, there exists a (unique) bijection \( \tilde{\tau} : \mathbb{L}(V_1) \sim \mathbb{L}(V_2) \), such that \( (\tilde{\sigma}, \tilde{\tau}) \) is a collineation.)

**Step 3-1. Claim: \( \tilde{\sigma} : \mathbb{P}(V_1) \rightarrow \mathbb{P}(V_2) \) is a bijection.**

Indeed, starting with \( \sigma^{-1} : U_2 \sim U_1 \) instead of \( \sigma \), we obtain \( \tilde{\sigma}^{-1} : \mathbb{P}(V_2) \rightarrow \mathbb{P}(V_1) \), which turns out to be the inverse of \( \tilde{\sigma} \) from the uniqueness assertion of Step 2.

**Step 3-2. Claim: If \( \ell \in \mathbb{L}(V_1)_{U_1} \), then \( \tilde{\sigma}(\ell) = \tau(\ell) = (\in \mathbb{L}(V_2)_{U_2}) \).**

Indeed, let \( \ell \in \mathbb{L}(V_1)_{U_1} \) and \( p \in \ell \). Then, just by the definition of \( \tilde{\sigma} \), we have \( \tilde{\sigma}(p) \in \tau(\ell) \). Namely, we obtain \( \tilde{\sigma}(\ell) \subseteq \tau(\ell) \). Applying this to \( \sigma^{-1} : U_2 \sim U_1 \) and \( \tau(\ell) \in \mathbb{L}(V_2)_{U_2} \) (and noting that \( \tilde{\sigma}^{-1} = \tilde{\sigma}^{-1} \) as shown in Step 3-1), we obtain \( \tilde{\sigma}^{-1}(\tau(\ell)) \subseteq \tau^{-1}(\tau(\ell)) = \ell \), or, equivalently, \( \tau(\ell) \subseteq \tilde{\sigma}(\ell) \). Combining these, we obtain \( \tilde{\sigma}(\ell) = \tau(\ell) \), as desired.

**Step 3-3. Claim: Three points \( p_1, p_2, p_3 \in \mathbb{P}(V_1) \) are collinear, if and only if so are \( \tilde{\sigma}(p_1), \tilde{\sigma}(p_2), \tilde{\sigma}(p_3) \).**

Indeed, it suffices to prove the ‘only if’ part, since the ‘if’ part is obtained by applying the ‘only if’ part to \( \sigma^{-1} : U_2 \sim U_1 \). If \( \sharp\{p_1, p_2, p_3\} \leq 2 \), the assertion is clear. So, we may assume that \( p_1, p_2, p_3 \) are mutually distinct. In particular, the line \( \ell \) containing \( p_1, p_2, p_3 \) (whose existence is ensured by the collinearity of
$p_1, p_2, p_3)$ is unique. Next, if $\ell \in \mathbb{L}(V_1)\setminus U_1$, then the assertion follows immediately from Step 3-2. So, we may assume that $\ell \notin \mathbb{L}(V_1)\setminus U_1$, i.e., $\ell \cap U_1 = \emptyset$.

Take $q_1 \in U_1 \neq \emptyset$. As $\ell \cap U_1 = \emptyset$, we have $q_1 \notin \ell$, hence there exists a unique plane $P \subset \mathbb{P}(V_1)$ containing both $\ell$ and $q_1$. We shall construct various points in $P \cap U_1$. Let $\alpha : (q_1 \cup p_3) \sim (q_1 \cup p_2)$ be the projection with respect to the center $p_1$: $\{\alpha(x)\} = (x \cup p_1) \cap (q_1 \cup p_2)$. Take $q_2 \in (q_1 \cup p_3)\setminus (q_1 \cup p_2)$, then $\{q_2\} \cap (q_1 \cup p_3)\setminus (q_1 \cup p_2) \neq \emptyset ((2,1)\text{-admissibility})$, and set $q_3 \equiv \alpha(q_2) \in (q_1 \cup p_2)\setminus (q_1 \cup p_3)$ for some $\alpha((q_1 \cup p_3)\setminus (q_1 \cup p_2)) \neq \emptyset ((1,1)\text{-admissibility})$. Let $\beta : (r_1 \cup p_3) \sim (r_1 \cup p_2)$ be the projection with respect to the center $q_1$: $\{\beta(x)\} = (x \cup q_1) \cap (r_1 \cup p_2)$. Let $\gamma : (r_1 \cup p_3) \sim (r_1 \cup q_1)$ be the projection with respect to the center $q_2$: $\{\gamma(x)\} = (x \cup q_2) \cap (r_1 \cup q_1)$. Take $r_2 \in (r_1 \cup p_3)\setminus (r_1 \cup p_2) \sim (r_1 \cup q_1)\setminus (r_1 \cup p_2)$, then $\{r_2\} \cap (r_1 \cup p_3)\setminus (r_1 \cup p_2) \neq \emptyset ((3,1)\text{-admissibility})$, and set $r_3 \equiv \beta(r_2) \in (r_1 \cup p_2)\setminus \{r_1\}$.

First, $q_1, q_2, q_3$ are not collinear. Indeed, otherwise, $q_1, p_2, p_3$ must also be collinear, which contradicts the choice of $q_1$. Second, $r_1, r_2, r_3$ are not collinear. Indeed, otherwise, $r_1, p_2, p_3$ must also be collinear, which contradicts the choice of $r_1$. (Here, $r_1 \in U_1$ and $p_3 \notin U_1$.) Third, $q_3 \neq r_1$. Indeed, this follows from the definition of $r_1$. Fourth, $q_2 \neq r_2$. Indeed, otherwise, $q_2 = r_2 \in r_1 \cup p_3$, hence $r_1 \in q_2 \cup p_3 = q_1 \cup p_3$, which contradicts the choice of $r_1$. Fifth, $q_3 \neq r_3$. Indeed, otherwise, $q_3 = r_3$, hence $r_1 \in r_1 \cup p_2 = r_3 \cup p_2 = q_3 \cup p_2 = q_1 \cup p_2$, which contradicts the choice of $r_1$.

Thus, one may apply (the ‘$\iff$’ part of) Proposition 5.1(ii) to $q_1, q_2, q_3, r_1, r_2, r_3$. Then there exists $s \in \mathbb{P}(V)$, such that $s, q_1, r_1$ are collinear, $s, q_2, r_2$ are collinear, and $s, q_3, r_3$ are collinear. By definition, $s = \gamma(r_2) \in U_1$.

By Step 3-2, $\sigma(q_1), \sigma(q_2), \sigma(q_3)$ are not collinear, $\sigma(r_1), \sigma(r_2), \sigma(r_3)$ are not collinear, $\sigma(s), \sigma(q_1), \sigma(r_1)$ are collinear, $\sigma(s), \sigma(q_2), \sigma(r_2)$ are collinear, and $\sigma(s), \sigma(q_3), \sigma(r_3)$ are collinear. Also, as $\sigma$ is a bijection, one has $\sigma(q_1) \neq \sigma(r_1)$, $\sigma(q_2) \neq \sigma(r_2)$ and $\sigma(q_3) \neq \sigma(r_3)$. Thus, applying (the ‘$\implies$’ part of) Proposition 5.1(ii) to $\sigma(q_1), \sigma(q_2), \sigma(q_3), \sigma(r_1), \sigma(r_2), \sigma(r_3)$, we conclude that $\tilde{\sigma}(p_1) \equiv (\sigma(q_2) \vee \sigma(q_3)) \cap (\sigma(r_2) \vee \sigma(r_3))$, $\tilde{\sigma}(p_2) \equiv (\sigma(q_3) \vee \sigma(q_1)) \cap (\sigma(r_3) \vee \sigma(r_1))$, $\tilde{\sigma}(p_3) \equiv (\sigma(q_1) \vee \sigma(q_2)) \cap (\sigma(r_1) \vee \sigma(r_2))$, and $\sigma(p_1)$ are collinear. Indeed, $\sigma(p) = \tilde{\sigma}(p)$ and $\sigma(p) = \tilde{\sigma}(p)$. Thus, applying (the ‘$\iff$’ part of) Proposition 5.1(ii) to $\sigma(q_1), \sigma(q_2), \sigma(q_3), \sigma(r_1), \sigma(r_2), \sigma(r_3)$, we conclude that $\tilde{\sigma}^{-1}(\ell) = (\sigma(p) \vee \sigma(q_3)) \cap (\sigma(r_2) \vee \sigma(r_3))$, $\tilde{\sigma}^{-1}(\ell) = (\sigma(q_3) \vee \sigma(q_1)) \cap (\sigma(r_3) \vee \sigma(r_1))$, $\tilde{\sigma}^{-1}(\ell) = (\sigma(q_1) \vee \sigma(q_2)) \cap (\sigma(r_1) \vee \sigma(r_2))$, and $\sigma(p_1)$ are collinear. Indeed, $\sigma(p) = \tilde{\sigma}^{-1}(\ell)$. Thus, applying (the ‘$\iff$’ part of) Proposition 5.1(ii) to $\sigma(q_1), \sigma(q_2), \sigma(q_3), \sigma(r_1), \sigma(r_2), \sigma(r_3)$, we conclude that $\tilde{\sigma}^{-1}(\ell) = \sigma(p) \cap \sigma(q_3)$.

Step 3-4. Claim: If $\ell \in \mathbb{L}(V_1)$, then $\tilde{\sigma}(\ell) \in \mathbb{L}(V_2)$.

Indeed, for each pair $p, q \in \ell$ with $p \neq q$, set $\ell'_{p,q} \equiv \tilde{\sigma}(p) \vee \tilde{\sigma}(q) \in \mathbb{L}(V_2)$. In fact, the line $\ell'_{p,q}$ does not depend on the pair $p, q$. Indeed, let $p', q' \in \ell$ be any pair with $p' \neq q'$. If $\sharp\{p, q, p', q'\} = 2$, then it is clear that $\ell'_{p,q} = \ell'_{p',q'}$. If $\sharp\{p, q, p', q'\} = 3$, then it follows from Step 3-3 that $\tilde{\sigma}(p), \tilde{\sigma}(q), \tilde{\sigma}(p'), \tilde{\sigma}(q')$ are collinear, hence $\ell'_{p,q} = \ell'_{p',q'}$. If $\sharp\{p, q, p', q'\} = 4$, then, again by Step 3-3, one has $\ell'_{p,q} = \ell'_{p',q'} = \ell'_{p',q'}$.

Now, write $\ell' = \ell'_{p,q}$ for some (or, equivalently, all) pair $p, q \in \ell$ with $p \neq q$. Here, note that, by Proposition 5.1(i)(III), at least one such pair $p, q$ exists. Then, by definition, $\tilde{\sigma}(\ell) \subset \ell'$ or, equivalently, $\sigma(\ell) \subset \tilde{\sigma}^{-1}(\ell')$.

Applying this to $\sigma^{-1} : U_2 \sim U_1$ and $\ell' \in \mathbb{L}(V_2)$, we obtain $\tilde{\sigma}^{-1}(\ell') = \sigma^{-1}(\ell') \subset \ell''$ for some $\ell'' \in \mathbb{L}(V_1)$. Combining these containment relations, we conclude $\ell \subset \ell''$, which implies $\ell = \ell''$ and $\tilde{\sigma}(\ell) = \ell'$, as desired.
Step 3-5. End of Step 3.

By Step 3-4, we may define \( \tilde{\tau} : \mathbb{L}(V_1) \to \mathbb{L}(V_2) \) to be the map that sends \( \ell \in \mathbb{L}(V_1) \) to \( \tilde{\sigma}(\ell) \in \mathbb{L}(V_2) \). (Note that \( \tilde{\tau} : \mathbb{L}(V_1) \to \mathbb{L}(V_2) \) is an extension of \( \tau : \mathbb{L}(V_1)_{U_1} \to \mathbb{L}(V_2)_{U_2} \), by Step 3-2.) Applying this to \( \sigma^{-1} : U_2 \to U_1 \), we may also define \( \tilde{\tau}' : \mathbb{L}(V_2) \to \mathbb{L}(V_1) \) to be the map that sends \( \ell' \in \mathbb{L}(V_2) \) to \( \tilde{\sigma}^{-1}(\ell') = \tilde{\sigma}^{-1}(\ell^\prime) \in \mathbb{L}(V_1) \). By definition, it is immediate to prove that \( \tilde{\tau}' \) is the inverse map of \( \tilde{\tau} \), and, in particular, that \( \tilde{\tau} \) is a bijection. Now, by the very definition of \( \tilde{\tau} \), \((\tilde{\sigma}, \tilde{\tau}) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \to (\mathbb{P}(V_2), \mathbb{L}(V_2)) \) is a collineation, as desired.

Step 4. Uniqueness. If \((\tilde{\sigma}, \tilde{\tau}) : (\mathbb{P}(V_1), \mathbb{L}(V_1)) \to (\mathbb{P}(V_2), \mathbb{L}(V_2)) \) is a collineation, then, for each \( p \in \mathbb{P}(V_1) \), the bijection \( \tilde{\tau} : \mathbb{L}(V_1) \to \mathbb{L}(V_2) \) induces a bijection of subsets \( \mathbb{L}(V_1)_{\{p\}} \to \mathbb{L}(V_2)_{\{\tilde{\sigma}(p)\}} \). So, if \((\tilde{\sigma}, \tilde{\tau}) \) extends \((\sigma, \tau) : (U_1, \mathbb{L}(V_1)_{U_1}) \to (U_2, \mathbb{L}(V_2)_{U_2}) \), then \( \tilde{\tau} \) (or, equivalently, \( \tau \)) induces a bijection of subsets \( \mathbb{L}(V_1)_{\{p\}} \cap \mathbb{L}(V_1)_{U_1} \to \mathbb{L}(V_2)_{\{\tilde{\sigma}(p)\}} \cap \mathbb{L}(V_2)_{U_2} \). Thus, the uniqueness assertion in Theorem 5.7 follows from the uniqueness assertion in Step 2. (Recall the fact that \( \tilde{\tau} \) is determined by \( \tilde{\sigma} \) uniquely.) \( \Box \)

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