Entanglement sharing via qudit channels: Nonmaximally entangled states may be necessary for one-shot optimal singlet fraction and negativity

Rajarshi Pal
Department of Physics, Indian Institute of Technology Madras, Chennai 600036

Somshubro Bandyopadhyay†
Department of Physics and Center for Astroparticle Physics and Space Science, Bose Institute, EN-80, Sector V, Bidhannagar, Kolkata 700091

We consider the problem of establishing entangled states of optimal singlet fraction and negativity between two remote parties for every use of a noisy quantum channel and trace-preserving LOCC under the assumption that the parties do not share prior correlations. We show that for a family of quantum channels in every finite dimension \(d \geq 3\), one-shot optimal singlet fraction and entanglement negativity are attained only with appropriate nonmaximally entangled states. A consequence of our results is that the ordering of entangled states in all finite dimensions may not be preserved under trace-preserving LOCC.

I. INTRODUCTION

In quantum information theory, entangled states [1, 2] shared between remote parties are considered as resources [2] within the paradigm of local operations and classical communication (LOCC) (see for example, [3]). However, any protocol of entanglement sharing requires sending quantum systems over quantum channels along with local processing irrespective of preshared correlations that may be present between the parties [8–11, 17–20]. It may be noted that recent results [19] strongly suggest that protocols with prior correlations may not provide any efficiency advantage over the ones without correlations.

In this paper we consider the basic protocol between two remote parties, Alice and Bob, who do not share any prior correlation. Such a protocol may be described as follows. Alice locally prepares a pure quantum state \(|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d\) and sends half of it to Bob down a \(d\)-dimensional quantum channel \(\Lambda\). In an ideal scenario where the channel is taken to be noiseless, maximally entangled states are easily established this way. For a noisy channel, which is typically the case, Alice and Bob end up with a mixed state \(\rho_{\psi,\Lambda} = (I \otimes \Lambda) \rho_{\psi}\) where \(\rho_{\psi} = |\psi\rangle \langle \psi|\) or an ensemble for many uses of the channel. Thus in a noisy channel scenario the goal is to establish entangled states that are optimal with respect to some well-defined figure of merit.

Entanglement distillation [8–11] provides a solution by converting many copies of \(\rho_{\psi,\Lambda}\) to fewer near-perfect entangled states thereby requiring many uses of the channel and joint measurements.

The present paper considers a one-shot instance of the entanglement sharing problem where the goal is to establish entangled states of maximum singlet fraction and entanglement negativity [24] achievable for every single use of the channel (see for example, [12, 13]). As we will see, the one-shot optimal values of these two quantities are closely related and exhibit similar properties.

The singlet fraction (or maximally entangled fraction) [8–10, 16, 17] of \(\rho_{\psi,\Lambda}\) is given by

\[
F(\rho_{\psi,\Lambda}) = \max_{|\Phi\rangle} \langle \Phi | \rho_{\psi,\Lambda} | \Phi \rangle
\]

where \(|\Phi\rangle\) is a maximally entangled state in \(\mathbb{C}^d \otimes \mathbb{C}^d\). The motivation behind choosing singlet fraction as our figure of merit lies in the fact that singlet fraction is an effective measure of usefulness of the state \(\rho_{\psi,\Lambda}\) for quantum information processing tasks, e.g. quantum teleportation [2], superdense coding [4], quantum key distribution [5], and distributed computation [6], which typically require entangled states of very high \(F\), ideally close to unity. It is useful to note that the yield in a distillation protocol depends on \(F(\rho_{\psi,\Lambda})\); in fact, for the distillation protocols to work the singlet fraction of the mixed states must exceed a certain threshold value [8–10].

While one may suppose that maximizing \(F(\rho_{\psi,\Lambda})\) given by Eq. (1.1) over all transmitted states \(|\psi\rangle\) will yield the desired result, such a supposition may be unfounded. This is because singlet fraction of a state can increase under local trace-preserving operations (TP-LOCC) [16, 21, 22] which strongly suggests that in a one-shot protocol local post-processing may be required to attain the optimal value. Taking this into account, let \(\rho^L_{\psi,\Lambda} = L(\rho_{\psi,\Lambda})\) denote the density matrix under the action of some...
TP-LOCC operation \( L \) on \( \rho_{\psi,\Lambda} \). Then, for a fixed transmitted state \( |\psi\rangle \), the maximum achievable singlet fraction is defined as [16]

\[
F^* (\rho_{\psi,\Lambda}) = \max_{L \in \text{TP-LOCC}} F (\rho_{\psi,\Lambda}^L),
\]

where the maximization is over all TP-LOCC \( L \). Note that, unlike \( F \) which can increase under TP-LOCC, \( F^* \) is a LOCC monotone [16]. It is important to note that the action of optimal TP-LOCC, say, \( L^* \) on \( \rho_{\psi,\Lambda} \) results in a density matrix, say \( \rho_{\psi,\Lambda}^L = L^*(\rho_{\psi,\Lambda}) \). Thus, we can write

\[
F^* (\rho_{\psi,\Lambda}) = F (\rho_{\psi,\Lambda}^L).
\]

The one-shot optimal singlet fraction for the channel \( \Lambda \) is defined as [13]

\[
F (\Lambda) = \max_{|\psi\rangle} F^* (\rho_{\psi,\Lambda}),
\]

where the maximum is taken over all pure state transmissions. Let us now suppose that \( |\psi_{\text{opt}}\rangle \) is a pure entangled state such that (I.4) holds; then,

\[
F (\Lambda) = F^* (\rho_{\psi,\Lambda}^\text{opt}) = F (\rho_{\psi,\Lambda}^\text{opt}^L).
\]

The one-shot optimal singlet fraction is related to optimal negativity in the following way. For any two-qudit density matrix \( \sigma \) the following inequality holds [23]:

\[
F^* (\sigma) \leq \frac{1 + 2N (\sigma)}{d}
\]

where \( N (\sigma) \) denotes the negativity [24] of the state \( \sigma \). Now, substituting \( \sigma \) by \( \rho_{\psi,\Lambda} \) in the above inequality and maximizing over all transmitted states \( |\psi\rangle \) leads to an upper bound on \( F (\Lambda) \):

\[
F (\Lambda) \leq \frac{1 + 2N (\rho_{\psi,\Lambda}^\text{opt})}{d} \leq \frac{1 + 2N (\Lambda)}{d},
\]

where \( N (\Lambda) = \max_{|\psi\rangle} N (\rho_{\psi,\Lambda}) \) is the optimal negativity.

Thus given a quantum channel \( \Lambda \), the task is to find \( F (\Lambda) \) and \( N (\Lambda) \) and the protocols to achieve these optimal values. Note that, it is quite possible that the optimal values may be attained by sending different pure states. However, the question that deserves utmost importance is whether the optimal states are maximally entangled like noiseless channels.

To the best of our knowledge, the problem concerning one-shot optimal singlet fraction has been completely solved only in the qubit case [13]. In particular, for any qubit channel (which is not entanglement breaking), it was shown that \( |\psi_{\text{opt}}\rangle \), satisfying (I.5), is maximally entangled if and only if the channel is unital, and for any non-unital qubit channel \( |\psi_{\text{opt}}\rangle \) is necessarily nonmaximally entangled (for the specific case of amplitude damping channel; see [12]). Further, it was shown that for any qubit channel \( \Lambda_{\text{qubit}} \), \( F (\Lambda_{\text{qubit}}) \) can be exactly computed and is given by [13]

\[
F (\Lambda_{\text{qubit}}) = \frac{1 + 2N (\rho_{\Phi^+,\Lambda_{\text{qubit}}})}{2}
\]

where \( |\Phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \).

In [12, 13], specific examples were given which showed that the ordering of entangled states may change under one-sided local action of a qubit channel and the maximum output entanglement may not be achieved for an input maximally entangled state (shown for a system of four qubits having configuration \((\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2)\)). A more systematic way supporting these observations can be found in [12, 13, 19]. For example, in [13] it was pointed out that for qubit channels, the maximum achievable negativity may not be achieved by sending a maximally entangled state: Using (I.3) and (I.7) we see that

\[
N (\rho_{\Phi^+,\Lambda_{\text{qubit}}}) \leq N (\rho_{\psi_{\text{opt}},\Lambda_{\text{qubit}}}) \leq N (\Lambda_{\text{qubit}})
\]

Since \( |\psi_{\text{opt}}\rangle \) is nonmaximally entangled for non-unital channels, the inequality implies that nonmaximally entangled states also lead to maximum achievable entanglement negativity; for an amplitude damping channel the inequality (I.9) is strict [13]. The question for other nonunital channels, however, remains open.

In this paper we extend our previous studies [12, 13] to higher dimensional quantum channels. In particular, we wish to know whether we can find quantum channels in all higher dimensions \( d \geq 3 \) with properties similar to non-unital qubit channels. The main results of this paper are the following.
We present a family of quantum channels \( \Omega \) in every finite dimension \( d \geq 3 \) for which we prove that \( |\psi_{\text{opt}}\rangle \) is nonmaximally entangled. Although we are not able to provide an expression for this optimal state, nonetheless, we obtain a nonmaximally entangled state \( |\psi'\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \) satisfying the inequality:

\[
F(\Omega) = F^*(\rho_{\psi_{\text{opt}},\Omega}) \geq F(\rho_{\psi',\Omega}) > F^*(\rho_{\Phi,\Omega})
\]

where \( |\psi'\rangle \) is the eigenvector corresponding to the largest eigenvalue of the density matrix \( \rho_{\Phi,\Omega} \) with \( \Omega \) being the dual map (see the next section for the definition) and \( |\Phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \) being any maximally entangled state. Note that the first inequality gives us a lower bound on the one-shot optimal singlet fraction and shows that suitable nonmaximally entangled states are better than maximally entangled states. Also note that, since \( F^* \) is a LOCC monotone, the above inequality together with the identity (13) provides a constructive way to demonstrate that ordering of \( F^* \) in general is not preserved under TP-LOCC in all finite dimensions.

Optimal negativity is attained only by appropriate nonmaximally entangled states. Using (I.6), (I.5) and (I.12) it is easy to see that

\[
\mathcal{N}(\Omega) \geq \mathcal{N}(\rho_{\psi_{\text{opt}},\Omega}) > \mathcal{N}(\rho_{\Phi,\Omega})
\]

where \( |\psi_{\text{opt}}\rangle \) is nonmaximally entangled. Thus, in all finite dimensions \( d \geq 3 \) we are able to show by explicit construction that the maximum output entanglement, as measured by negativity, is not always achieved using a maximally entangled input state. This, significantly improves upon the previously known examples.

We also make the following observation. We find that in higher dimensions an expression analogous to (13) does not hold in general. This follows from inequality (see the proof of (10)):

\[
F(\Omega) > \frac{1 + 2\mathcal{N}(\rho_{\Phi,\Omega})}{d}
\]

where \( \mathcal{N}(\rho_{\Phi,\Omega}) \) is the negativity of the density matrix \( \rho_{\Phi,\Omega} \). One may argue that there is no convincing reason why one should have expected the generalization to hold in the first place; however, the exact formula obtained in (13) prompted us to think such a generalization, if it holds, would give us a computable formula for one-shot optimal singlet fraction in all finite dimensions. Unfortunately, our optimism turned out to be misplaced.

II. RESULTS

A quantum channel \( \Lambda \) is a trace preserving completely positive map characterized by a set of Kraus operators \( \{A_i\} \) satisfying \( \sum A_i^*A_i = I \) (see for example, [18]). The dual map \( \hat{\Lambda} \), described in terms of the Kraus operators \( \{A_i^*\} \), is the adjoint map with respect to the Hilbert-Schmidt inner product. We say that a channel \( \Lambda \) is unital if its action preserves the identity: \( \Lambda(I) = I \), and nonunital if it does not, i.e., \( \Lambda(I) \neq I \). Moreover, the dual map \( \hat{\Lambda} \) is trace-preserving, and hence a channel, iff \( \Lambda \) is unital. The one-sided action of a \( d \)-dimensional map \( \hat{\Psi} \in \{\Lambda, \hat{\Lambda}\} \) on a pure state \( |\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \) gives rise to a mixed state which can be conveniently expressed as:

\[
\rho_{\psi,\hat{\Psi}} = (I \otimes \hat{\Psi}) \rho_{\psi} = \sum_i (I \otimes K_i) \rho_{\psi} (I \otimes K_i^\dagger)
\]

where the Kraus operators \( \{K_i\} \) describe the channel \( \hat{\Psi} \) and \( \rho_{\psi} = |\psi\rangle \langle \psi| \) is the density matrix corresponding to the pure state \( |\psi\rangle \). We now give two useful lemmas which are applicable to any quantum channel \( \Lambda \). The first lemma was proved in [14].

**Lemma 1.** For a \( d \)-dimensional quantum channel \( \Lambda \), \( F^*(\rho_{\Phi,\Lambda}) = F^*(\rho_{\Phi^+,\Lambda}) \) where \( |\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle \) and \( |\Phi\rangle \) is any maximally entangled state in \( \mathbb{C}^d \otimes \mathbb{C}^d \).

The proof is simple. Since every maximally entangled state \( |\Phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \) can be written as \( |\Phi\rangle = (U \otimes V) |\Phi^+\rangle \) for some \( U, V \in SU(d) \), using the identity \( (I \otimes V) |\Phi^+\rangle = (V^T \otimes I) |\Phi^+\rangle \) we can write \( |\Phi\rangle = (W \otimes I) |\Phi^+\rangle \) where \( W = UV^T \) is also a unitary operator. Because the channel \( \Lambda \) acts only on the second qudit, we have \( \rho_{\Phi,\Lambda} = (W \otimes I) \rho_{\Phi^+,\Lambda} (W^\dagger \otimes I) \). Thus the density matrices \( \rho_{\Phi,\Lambda} \) and \( \rho_{\Phi^+,\Lambda} \) are connected by a local unitary operator acting on the first system. Because the first system never interacts with the channel, this local unitary can always be absorbed in the post-transmission optimal TP-LOCC associated with the state transformations (defined earlier) \( \rho_{\Phi,\Lambda} \rightarrow \rho_{\Phi,\Lambda} \) and \( \rho_{\Phi^+,\Lambda} \rightarrow \rho_{\Phi^+,\Lambda} \). Therefore, \( F^*(\rho_{\Phi,\Lambda}) = F^*(\rho_{\Phi^+,\Lambda}) \).
Lemma 2. For a \(d\)-dimensional quantum channel \(\Lambda\), \(F(\Lambda) \geq \lambda_{\max}(\rho_{\Phi+\Lambda})\) where \(|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii\rangle\) and \(\lambda_{\max}(\rho_{\Phi+\Lambda})\) is the largest eigenvalue of the density matrix \(\rho_{\Phi+\Lambda}\).

Proof. The proof is along the same lines as in the qubit case [13]. We begin by noting that for any \(|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d\),

\[
F(\Lambda) \geq \max_{\psi} F(\rho_{\psi,\Lambda}) = \max_{\psi} (\langle \Phi | \rho_{\psi,\Lambda} | \Phi \rangle) \tag{II.1}
\]
where \(|\Phi\rangle\) is maximally entangled. Using the relations \(|\Phi\rangle = (U \otimes V) |\Phi^+\rangle\) for some \(U, V \in SU(d)\) and \((I \otimes V) |\Phi^+\rangle = (V^T \otimes I) |\Phi^+\rangle\), it is straightforward to show that

\[
F(\rho_{\psi,\Lambda}) = \left( \langle \psi | \rho_{\Phi^+,\Lambda} | \psi \rangle \right) \tag{II.2}
\]
where \(\hat{\Lambda}\) is the dual channel. From (II.1) and (II.2) we therefore get

\[
F(\Lambda) \geq \lambda_{\max}(\rho_{\Phi^+,\Lambda}) = \lambda_{\max}(\rho_{\Phi^+,\Lambda})
\]
where we have used \(\lambda_{\max}(\rho_{\Phi^+,\Lambda}) = \lambda_{\max}(\rho_{\Phi^+,\Lambda})\) proved in [13] for any \(d\) dimensional channel \(\Lambda\).

Main results

Let us now consider the \(d\)-dimensional quantum channel \(\Omega\) defined by the Kraus operators \(A_i\) for \(i = 0, \ldots, d - 1\),

\[
A_0 = \text{diag}(1, x_1, x_2, \ldots, x_{d-1}); \quad (A_m)_{ij} = \sqrt{1-x_m^2} \delta_{ij} \delta_{mj} \quad i, j = 0, \ldots, d - 1 \quad \forall \ m = 1, \ldots, d - 1 \tag{II.3}
\]
where \(0 < x_i < 1\) for every \(i\) and \(x_i \neq x_j\) for at least one pair \((i, j)\). That the Kraus operators defined above indeed describe a legitimate quantum channel can be seen as follows. First, it is easy to check that

\[
(A_m^d A_m)_{ik} = (1-x_m^2) \delta_{ik} \delta_{mk} \quad A_m^d A_0 = \text{diag}(1, x_1^2, x_2^2, \ldots, x_{d-1}^2) \tag{II.4}
\]
Clearly the operators \(A_m^d A_0\) are positive and moreover, Eqs. (II.3) lead to

\[
A_0^d A_0 + \sum_{m=1}^{d-1} A_m^d A_m = I_{d \times d}. \tag{II.5}
\]

We now state our result.

Theorem 1. For the \(d\)-dimensional quantum channel \(\Omega\) described above, the following inequalities hold in every finite dimension \(d \geq 3\):

\[
F(\Omega) \geq F(\rho_{\psi',\Omega}) > F^+(\rho_{\Phi,\Omega}) \tag{II.6}
\]
where \(|\psi'\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d\) is a pure state, not maximally entangled, and \(|\Phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d\) is any maximally entangled state.

The inequalities (II.6) are established through the following results.

Lemma 3. For any maximally entangled state \(|\Phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d\),

\[
\lambda_{\max}(\rho_{\Phi^+,\Omega}) > \frac{1 + 2N(\rho_{\Phi^+,\Omega})}{d} \tag{II.7}
\]
for all \(d \geq 3\).

Proof. First we obtain \(\lambda_{\max}(\rho_{\Phi^+,\Omega})\). The action of the Kraus operators given by (II.3) on \(|\Phi^+\rangle\) are given by:

\[
(I \otimes A_0) |\Phi^+\rangle = \frac{1}{\sqrt{d}} \left( |00\rangle + \sum_{i=1}^{d-1} x_i |i\rangle |i\rangle \right) = |\phi_0\rangle, \tag{II.8}
\]
and for $m = 1, \ldots, d - 1$
\[
(I \otimes A_m) |\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle A_m |i\rangle
\]
\[
= \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle \sqrt{1 - x_i^2} \delta_{im} |0\rangle \cdot A_m |i\rangle = \sqrt{1 - x_i^2} \delta_{im} |0\rangle
\]
\[
= \frac{1}{\sqrt{d}} \sqrt{1 - x_i^2} |m\rangle |0\rangle = |\phi_m\rangle.
\]
(II.9)

Thus,
\[
\rho_{\Phi^+, \Omega} = \sum_{m=0}^{d-1} (I \otimes A_m) \rho_{\Phi^+} (I \otimes A_m^\dagger)
\]
\[
= |\phi_0\rangle \langle \phi_0| + \sum_{m=1}^{d-1} |\phi_m\rangle \langle \phi_m|.
\]
(II.10)

As $\rho_{\Phi^+, \Omega}$ is already in the diagonal form, it is straightforward to obtain its largest eigenvalue,
\[
\lambda_{\text{max}} (\rho_{\Phi^+, \Omega}) = \frac{1}{d} \left( 1 + \sum_{i=1}^{d-1} x_i^2 \right).
\]
(II.11)

Next, we compute negativity $N (\rho_{\Phi^+, \Omega})$. The partial transposed matrix corresponding to $\rho_{\Phi^+, \Omega}$ is given by
\[
\rho_{\Phi^+, \Omega}^\Gamma = \frac{1}{d} \left[ |00\rangle \langle 00| + \sum_{i=1}^{d-1} x_i (|0i\rangle \langle i0| + |i0\rangle \langle 0i|) + \sum_{i,j=1}^{d-1} x_i x_j |ij\rangle \langle ji| + \sum_{i=1}^{d-1} (1 - x_i^2) |i0\rangle \langle i0| \right]
\]
with easily computed eigenvalues,
\[
\frac{1}{d} (\text{multiplicity } d) ; \pm x_i^2, i = 1, \ldots, d - 1 ; \pm \frac{x_i x_j}{d}, i < j i, j = 1, \ldots, d - 1.
\]
As negativity is defined as the absolute value of the sum of the negative eigenvalues [24], we have
\[
N (\rho_{\Phi^+, \Omega}) = \frac{1}{d} \left( \sum_{i=1}^{d-1} x_i^2 + \sum_{1 \leq i < j \leq d-1} x_i x_j \right).
\]
(II.12)

From (II.11) and (II.12) we see that the inequality (II.7) holds provided:
\[
(d - 2) \sum_{i=1}^{d-1} x_i^2 \geq 2 \sum_{1 \leq i < j \leq d-1} x_i x_j.
\]

Now, $\sum_{1 \leq i < j \leq d-1} (x_i - x_j)^2 > 0$, since for at least one pair $(i, j)$, $x_i \neq x_j$ (as given in the definition of the channel), the above inequality always holds for all $d \geq 3$. This completes the proof.

Let us now note the consequences of the above lemma.
Since $F (\Omega) \geq \lambda_{\text{max}} (\rho_{\Phi^+, \Omega})$ (from Lemma 2), we see that
\[
F (\Omega) \geq \frac{1 + 2 N (\rho_{\Phi^+, \Omega})}{d}.
\]

Thus the generalization of the formula (I.8) that allows us to compute optimal fidelity exactly for qubit channels does not hold in general in higher dimensions.
We now show that $|\psi_{\text{opt}}\rangle$ is a nonmaximally entangled state. From Eq. (16), we have $F^*(\rho_{\Phi+\Omega}) \leq \frac{1+2N(\rho_{\Phi+\Omega})}{d}$. Using this inequality, and (2), and the inequality (II.7) we immediately obtain $F(\Omega) \geq \lambda_{\text{max}}(\rho_{\Phi+\Omega}) > F^*(\rho_{\Phi+\Omega})$. Since $F^*(\rho_{\Phi+\Omega}) = F^*(\rho_{\Phi,\Omega})$ for any maximally entangled state $|\Phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ [Lemma 1], we have

$$F(\Omega) \geq \lambda_{\text{max}}(\rho_{\Phi+\Omega}) > F^*(\rho_{\Phi,\Omega}).$$

(II.13)

Noting that $F(\Omega) = F^*(\rho_{\psi_{\text{opt}},\Omega})$, we get $F^*(\rho_{\psi_{\text{opt}},\Omega}) > F^*(\rho_{\Phi,\Omega})$ for all maximally entangled states $|\Phi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$. We therefore conclude that $|\psi_{\text{opt}}\rangle$ must be nonmaximally entangled. While we are unable to obtain $|\psi_{\text{opt}}\rangle$, the following lemma gives us a possible candidate and allows us to obtain a lower bound on $F(\Omega)$.

**Lemma 4.** Let $|\psi'\rangle$ be the eigenvector corresponding to the eigenvalue $\lambda_{\text{max}}(\rho_{\Phi+\tilde{\lambda}})$. Then, $\lambda_{\text{max}}(\rho_{\Phi+\Omega}) = F(\rho_{\psi',\Omega})$. Moreover, $|\psi'\rangle$ is not maximally entangled.

**Proof.** From Eq. (II.2) we know that for any pure state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, $F(\rho_{\psi,\Omega}) = \langle \psi | \rho_{\Phi+\tilde{\lambda}} | \psi \rangle$. As $|\psi'\rangle$ is the eigenvector corresponding to the eigenvalue $\lambda_{\text{max}}(\rho_{\Phi+\tilde{\lambda}})$, this means,

$$\lambda_{\text{max}}(\rho_{\Phi+\tilde{\lambda}}) = \langle \psi' | \rho_{\Phi+\tilde{\lambda}} | \psi' \rangle = F(\rho_{\psi',\Omega}).$$

Using the identity $\lambda_{\text{max}}(\rho_{\Phi+\tilde{\lambda}}) = \lambda_{\text{max}}(\rho_{\Phi+\lambda})$ [13] for any quantum channel $\Lambda$, we therefore have

$$\lambda_{\text{max}}(\rho_{\Phi+\lambda}) = F(\rho_{\psi',\Omega}).$$

On the other hand we have already shown that $\lambda_{\text{max}}(\rho_{\Phi+\lambda}) > F^*(\rho_{\Phi,\Omega})$. Therefore, $F(\rho_{\psi',\Omega}) > F^*(\rho_{\Phi,\Omega})$ for any maximally entangled state $|\Phi\rangle$ from which we conclude that $|\psi'\rangle$ is not a maximally entangled state.

Inequalities (II.13) and Lemma 4 conclude the proof of the theorem.

**III. CONCLUSIONS**

For any given $d$-dimensional quantum channel $\Lambda$ with $d \geq 2$, its one-shot optimal singlet fraction $F(\Lambda)$ defines the maximum singlet fraction achievable for entangled states established between two remote observers for every use of the channel. Recall that

$$F(\Lambda) = F^*(\rho_{\psi_{\text{opt}},\Lambda}) = F^*(\rho_{\psi_{\text{opt}},\Lambda}).$$

(III.1)

Thus, $F(\Lambda)$ quantifies how useful a channel $\Lambda$ is either for direct applications for quantum information processing tasks, e.g. teleportation [17] or for entanglement distillation where the yield depends upon the singlet fraction of the noisy states.

For qubit channels $F(\Lambda)$ can be exactly computed and the relevant questions have been satisfactorily answered before [13]. The results, however, point towards two counter-intuitive features. Foremost among them is that $|\psi_{\text{opt}}\rangle$ is nonmaximally entangled if and only if the channel is nonunital. And the next is, for nonunital qubit channels maximum achievable entanglement negativity using a maximally entangled state cannot be more than what is attained by sending $|\psi_{\text{opt}}\rangle$. In fact, for an amplitude damping channel (a nonunital channel) it was further shown that optimal negativity is obtained only by a nonmaximally entangled state.

Motivated by the above results we wanted to understand how well the results and observations made for qubit channels hold in higher dimensions. We presented a family of qudit channels $\Omega$ in all finite dimensions $d \geq 3$ for which we proved properties similar to nonunital qubit channels. In particular, we proved that one-shot optimal singlet fraction and negativity are attained only using appropriate nonmaximally entangled states. However, we also find that a generalized version of the formula that allows us to compute the optimal singlet fraction exactly for qubit channels does not hold in general in higher dimensions.

While a lot of results had been obtained characterizing quantum channels, we believe that much less is understood when it comes to characterizing quantum channels through the notions of optimal singlet fraction and entanglement measures. In higher dimensions almost every interesting question is left open, and probably a good way to address them is to solve the questions for specific channels of interest e.g. a depolarizing channel. Such results can provide us with useful insights. Another paradigm within which we can ask similar questions is entanglement distribution in the presence of preshared correlations.
Part of this work was completed when Rajarshi Pal was a long term visitor at Bose Institute during May 2015- March 2016. S.B. is supported in part by SERB (Science and Engineering Research Board), DST, Govt. of India–Project No. EMR/2015/002373.

[1] E. Schrödinger, Naturwissenschaften, Die gegenwärtige Situation in der Quantenmechanik. Naturwissenschaften, 23, 807–812, 823–828, 844–849. http://dx.doi.org/10.1007/BF01491891 [2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. 81, 865 (2009).
[3] E. Chitambar, D. Leung, L. Mancinska, M. Ozols, A. Winter, Everything You Always Wanted to Know About LOCC (But Were Afraid to Ask), Commun. Math. Phys. 328, 303 (2014).
[4] C. H. Bennett and S. J. Wiesner. Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states, Phys. Rev. Lett. 69, 2881 (1992).
[5] A. K. Ekert, Quantum cryptography based on Bell’s theorem, Phys. Rev. Lett. 67, 661 (1991).
[6] H. Rohrig, H. Buhrman, Distributed quantum computing, Lecture notes in Computer Science, 2747, 1 (2003).
[7] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett. 70, 1895 (1993).
[8] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. Mixed-state entanglement and quantum error correction, Phys. Rev. A 54, 3824 (1996).
[9] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, Quantum privacy amplification and the security of quantum cryptography over noisy channels, Phys. Rev. Lett. 77, 2818 (1996).
[10] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Purification of noisy entanglement and faithful teleportation via noisy channels, Phys. Rev. Lett. 76, 722 (1996).
[11] M. Horodecki, P. Horodecki, and R. Horodecki. Inseparable two spin- 1 2 density matrices can be distilled to a singlet form. Phys. Rev. Lett. 78, 574 (1997).
[12] S. Bandyopadhyay and A. Ghosh, Optimal fidelity for a quantum channel may be attained by nonmaximally entangled states, Phys. Rev. A 86, 020304(R) (2012).
[13] Rajarshi Pal, Somshubhro Bandyopadhyay, and Sibasish Ghosh, Entanglement sharing through noisy qubit channels: One-shot optimal singlet fraction, Phys. Rev. A 90, 052304 (2014).
[14] Mario Ziman, Vladimir Buzek, Entanglement-induced state ordering under local operations arXiv:quant-ph/0510017.
[15] Mario Ziman, Vladimir Buzek, Entanglement measures: state ordering vs local operations, Quantum Communication and Security (edited by M.Zukowski et al.), pp. 196-204 (IOS Press, 2007).
[16] F. Verstraete and H. Verschelde, Fidelity of mixed states of two qubits, Phys. Rev. A 66, 022307 (2002).
[17] M. Horodecki, P. Horodecki, and R. Horodecki, General teleportation channel, singlet fraction, and quasidistillation, Phys. Rev. A 60, 1888 (1999).
[18] B. Schumacher, Sending entanglement through noisy quantum channels, Phys. Rev. A 54, 2614 (1996).
[19] A. Streltsov, R. Augusiak, M. Demianowicz, and M. Lewenstein, Progress towards a unified approach to entanglement distribution, Phys. Rev. A 92, 012335 (2015).
[20] M. Zuppardo, T. Krisnanda, T. Paterek, S. Bandyopadhyay, A. Banerjee, P. Deb, S. Halder, K. Modi, M. Paternostro, Excessive distribution of quantum entanglement, Phys. Rev. A 93, 012305 (2016).
[21] P. Badziag, M. Horodecki, P. Horodecki, and R. Horodecki, Local environment can enhance fidelity of quantum teleportation. Phys. Rev. A 62, 012311 (2000).
[22] S. Bandyopadhyay, Origin of noisy states whose teleportation fidelity can be enhanced through dissipation, Phys. Rev. A 65, 022302 (2002).
[23] F. Verstraete and H. Verschelde, Optimal teleportation with a mixed state of two qubits, Phys. Rev. Lett. 90, 097901 (2003).
[24] G. Vidal and R. F. Werner, Computable measure of entanglement, Phys. Rev. A 65, 032314 (2002).