The space of coset partitions of $F_n$ and Herzog-Schönheim conjecture

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Abstract

Let $G$ be a group and $H_1,...,H_s$ be subgroups of $G$ of indices $d_1,...,d_s$ respectively. In 1974, M. Herzog and J. Schönheim conjectured that if \{H,\alpha_i\}_{i=1}^s, \alpha_i \in G, is a coset partition of $G$, then $d_1,...,d_s$ cannot be distinct. We consider the Herzog-Schönheim conjecture for free groups of finite rank. We define $Y$ the space of coset partitions of $F_n$ and show $Y$ is a metric space with interesting properties. In a previous paper, we gave some sufficient conditions on the coset partition of $F_n$ that ensure the conjecture is satisfied. Here, we show that each coset partition of $F_n$, which satisfies one of these conditions, has a neighborhood $U$ in $Y$ such that all the partitions in $U$ satisfy also the conjecture.

1 Introduction

Let $G$ be a group and $H_1,...,H_s$ be subgroups of $G$. If there exist $\alpha_i \in G$ such that $G = \bigcup_{i=1}^s H_i \alpha_i$, and the sets $H_i \alpha_i$, $1 \leq i \leq s$, are pairwise disjoint, then \{H,\alpha_i\}_{i=1}^s is a coset partition of $G$ (or a disjoint cover of $G$). We denote by $d_1,...,d_s$ the indices of $H_1,...,H_s$ respectively. The coset partition \{H,\alpha_i\}_{i=1}^s has multiplicity if $d_i = d_j$ for some $i \neq j$. The Herzog-Schönheim conjecture is true for the group $G$, if any coset partition of $G$ has multiplicity.

In 1974, M. Herzog and J. Schönheim conjectured that if \{H,\alpha_i\}_{i=1}^s, $\alpha_i \in G$, is a coset partition of $G$, then $d_1,...,d_s$ cannot be distinct. In the 1980’s, in a series of papers, M.A. Berger, A. Felzenbaum and A.S. Fraenkel studied the Herzog-Schönheim conjecture [1, 2, 3] and in [4] they proved the conjecture is true for the pyramidal groups, a subclass of the finite solvable groups. Coset partitions of finite groups with additional assumptions on the subgroups of the partition have been extensively studied. We refer to [5, 19, 20, 17] and also to [18]. In [9], the authors very recently proved that the conjecture is true for all groups of order less than 1440.

In [6], we study the Herzog-Schönheim conjecture in free groups of finite rank, and develop a new approach based on the machinery of covering spaces. The fundamental group of the bouquet with $n$ leaves (or the wedge sum of $n$ circles), $X$, is $F_n$, the free group of finite rank $n$. For any subgroup $H$ of $F_n$ of finite index $d$, there exists a $d$-sheeted covering space $(\tilde{X}_H, p)$ with a fixed
basepoint, which is also a combinatorial object. Indeed, the underlying graph
of $\hat{X}_H$ is a directed labelled graph, with $d$ vertices, that can be seen as a finite
complete bi-deterministic automaton; fixing the start and the end state at the
basepoint, it recognises the set of elements in $H$. It is called the Schreier coset
diagram for $F_n$ relative to the subgroup $H$ [16, p.107] or the Schreier automaton
for $F_n$ relative to the subgroup $H$ [15, p.102].

In $\hat{X}_H$, the $d$ vertices (or states) correspond to the $d$ right cosets of $H$, each
element (or transition) $Hg \rightarrow Hga$, $g \in F_n$, a a generator of $F_n$, describes
the right action of $a$ on $Hg$. We call $\hat{X}_H$, the Schreier graph of $H$, where the $d$
vertices $x_0, x_1, ..., x_{d-1}$ are identified with the corresponding $d$ cosets of $H$.
The transition group $T$ of the Schreier automaton for $F_n$ relative to $H$ des-
cribes the action of $F_n$ on the set of the $d$ right cosets of $H$, and is generated
by $n$ permutations. The group $T$ is a subgroup of $S_d$ such that $T \simeq F_n/\langle N_H \rangle$,
where $N_H = \bigcap_{g \in F_n} g^{-1}Hg$ is the normal core of $H$.

Let $\{H_i\alpha_i\}_{i=1}^s$ be a coset partition of $F_n$, $n \geq 2$, with $H_i < F_n$ of index
d_i > 1, $\alpha_i \in F_n$, $1 \leq i \leq s$. Let $\hat{X}_i$ be the Schreier graph of $H_i$, with transition
group $T_i$, $1 \leq i \leq s$. In [6], we give some sufficient conditions on the transition
groups of the Schreier graphs $\hat{X}_i$, $1 \leq i \leq s$, that ensure the coset partition
$\{H_i\alpha_i\}_{i=1}^s$ has multiplicity. We state the following Theorems from [6] that we
need for the paper:

**Theorem A.** [6, Theorem 1] Let $F_n$ be the free group on $n \geq 2$ generators.
Let $\{H_i\alpha_i\}_{i=1}^s$ be a coset partition of $F_n$ with $H_i < F_n$ of index $d_i$, $\alpha_i \in F_n$,
$1 \leq i \leq s$, and $1 < d_1 \leq ... \leq d_s$. Let $\hat{X}_i$ denote the Schreier graph of
$H_i$, with transition group $T_i$, $1 \leq i \leq s$. If there exists a $d_s$-cycle in $T_s$, then
the index $d_s$ appears in the partition at least $p$ times, where $p$ is the smallest
prime dividing $d_s$.

The transition group $T_s$ is a subgroup of the symmetric group $S_{d_s}$, generated
by $n \geq 2$ permutations. Dixon proved that the probability that a ran-
dom pair of elements of $S_n$ generate $S_n$ approaches $3/4$ as $n \to \infty$, and the
probability that they generate $A_n$ approaches $1/4$ [7]. As $d_s \to \infty$, the probability
that $T_s$ is the symmetric group $S_{d_s}$ approaches $3/4$. So, asymptoti-
cally, the probability that there exists a $d_s$-cycle in $T_s$ is greater than $3/4$.
If $T_s$ is cyclic, there exists a $d_s$-cycle in $T_s$, since $d_s$ divides the order of $T_s$.
That is, Theorem A is satisfied with very high probability and the conjecture is
“asymptotically satisfied with probability greater than $3/4$” for free groups
of finite rank.

Theorem B provides a list of conditions on a coset partition that ensure
multiplicity. Let $w \in F_n$. We denote by $a_{s_i}(w)$ the minimal natural num-
ber, $1 \leq a_{s_i}(w) \leq d_i$, such that $w^{a_{s_i}(w)}$ is a loop at the vertex $H_i\alpha_i$ in $\hat{X}_i$. Let
$o_{\text{max}}(w) = \max\{a_{s_i}(w) | 1 \leq i \leq s\}$ and $k = \max\{o_{\text{max}}(v) \mid v \in F_n\}$, $k$ is the
maximal length of a cycle in $\bigcup_{i=1}^s T_i$. Let $p$ denote the smallest prime dividing
$k$. We show there exists $u \in F_n$ such that $o_{\text{max}}(u) = k$ and $\# \geq 2$, where
\# = \{1 \leq i \leq s \mid o_{\alpha_i}(u) = k\}. Using this notation, we show the following result, which implies under the assumption \(k > d_1\), that there is a finite number of cases not covered by Theorem B.

**Theorem B.** [6, Theorem 2] Let \(F_n\) be the free group on \(n \geq 2\) generators. Let \(\{H_i\alpha_i\}_{i=1}^s\) be a coset partition of \(F_n\) with \(H_i < F_n\) of index \(d_i\), \(\alpha_i \in F_n\), \(1 \leq i \leq s\), and \(1 < d_1 \leq \ldots \leq d_s\). Let \(r\) be an integer, \(2 \leq r \leq s - 1\). If \(k\), \(p\) and \#, as defined above, satisfy one of the following conditions:

(i) \(k > d_{s-2}\).

(ii) \(k > d_{s-3}\), \(p \geq 3\).

(iii) \(k > d_{s-3}\), \(p = 2\), and \(# = 2\) or \(# \geq 4\).

(iv) \(k > d_{s-r}\) and \(p \geq r\), or \(# = p\), or \(# \geq r + 1\).

Then the coset partition \(\{H_i\alpha_i\}_{i=1}^s\) has multiplicity.

Inspired by [14], in which the author defines the space of left orders of a left-orderable group and show it is a compact and totally disconnected metric space, we define \(Y\) to be the space of coset partitions of \(F_n\) (under some equivalence relation) and show \(Y\) is a metric space. In our case, the metric defined induces the discrete topology.

**Theorem 1.** Let \(F_n\) be the free group on \(n \geq 2\) generators. Let \(Y\) be the space of coset partitions of \(F_n\) (under some equivalence relation). Then \(Y\) is a metric space with a metric \(\rho\) and \(Y\) is (topologically) discrete.

We extend the results from [6] and show that for each coset partition of \(F_n\), which satisfies one of the conditions in Theorems A or B, there exists a neighborhood \(U\) in \(Y\) such that all the coset partitions in \(U\) have multiplicity.

**Theorem 2.** Let \(F_n\) be the free group on \(n \geq 2\) generators. Let \(Y\) be the space of coset partitions of \(F_n\) (under some equivalence relation) with metric \(\rho\). Let \(P_0 = \{H_i\alpha_i\}_{i=1}^s\) be in \(Y\), with \(1 < d_1 \leq \ldots \leq d_s\).

(i) If \(P_0\) satisfies the condition of Theorem A, then every \(P \in Y\) with \(\rho(P, P_0) < \frac{1}{2}\) satisfies the same condition and hence has multiplicity.

(ii) If \(P_0\) satisfies (i) or (ii) of Theorem B, with some \(2 \leq r \leq s - 1\), then every \(P \in Y\) with \(\rho(P, P_0) < 2^{-(r+1)}\) satisfies the same condition and hence has multiplicity.

(iii) If \(P_0\) satisfies (iii) or (iv) of Theorem B, with some \(2 \leq r \leq s - 1\), then every \(P \in Y\) with \(\rho(P, P_0) < 2^{-(r+1)}\) has multiplicity.

The paper is organized as follows. In the first section, we introduce the space of coset partitions of \(F_n\), an action of \(F_n\) on it, a metric and prove Theorem 1. We also give another proof of [6, Theorem 3] using the action defined. In Section 2, we prove Theorem 2.
2 The space of coset partitions of $F_n$

2.1 Action of $F_n$ on the space of its coset partitions

Let $F_n$ be the free group on $n \geq 2$ generators. We define $Y'$ to be the space of coset partitions of $F_n$ (only with subgroups of finite index). For each subgroup $H$ of $F_n$ of finite index $d > 1$, there exists a partition of $F_n$ by the $d$ cosets of $H$. Generally, if $P \in Y'$, then $P = \{H_i\alpha_i\}_{i=1}^{\lambda}$, a coset partition of $F_n$ with $H_i < F_n$ of index $d_i$, $\alpha_i \in F_n$, $1 \leq i \leq s$, and $1 < d_1 \leq \ldots \leq d_s$.

To get some intuition on $Y'$, it is worth recalling that the subgroup growth of $F_n$ is exponential. There exists a natural right action of $F_n$ on $Y'$. Indeed, if $w \in F_n$, then $P \cdot w = P'$, with $P' = \{H_i\alpha_i w\}_{i=1}^{\lambda}$ in $Y'$.

Lemma 2.1. The natural right action of $F_n$ on $Y'$ is faithful

Proof. Let $w \in F_n$. Then $P \cdot w = P$ for every $P \in Y'$ if and only if $w$ belongs to the intersection of all the subgroups of finite index of $F_n$. As $F_n$ is residually finite [12, p.158], the intersection of all the subgroups of finite index of $F_n$ is trivial, so $w = 1$, that is the action is faithful.

Let $P = \{H_i\alpha_i\}_{i=1}^{\lambda}$ in $Y'$ and let $\tilde{X}_i$ be the Schreier graph of $H_i$, $1 \leq i \leq s$ (as defined in the introduction). Let $w \in F_n$. We denote by $o_{\alpha_i}(w)$ the minimal natural number, $1 \leq o_{\alpha_i}(w) \leq d_i$, such that $w^{o_{\alpha_i}(w)}$ is a loop at the vertex $H_i\alpha_i$ in $\tilde{X}_i$ or equivalently $w^{o_{\alpha_i}(w)} \in \alpha_i^{-1}H_i\alpha_i$ [6, Section 4.1].

Lemma 2.2. Let $P = \{H_i\alpha_i\}_{i=1}^{\lambda}$ in $Y'$. Then $|\text{Orb}_{F_n}(P)| \leq d_1 \ldots d_s$, where $\text{Orb}_{F_n}(P)$ denotes the orbit of $P$ under the action of $F_n$. Furthermore, for $w \in F_n$, $|\text{Orb}_w(P)| = \text{lcm}(o_{\alpha_1}(w), \ldots, o_{\alpha_s}(w))$, where $\text{Orb}_w(P)$ denotes the orbit of $P$ under the action of $\langle w \rangle$.

Proof. From the definition of the action of $F_n$ on $P$, $F_n$ permutes between the cosets of $H_1$, between the cosets of $H_2$ and so on. So, $|\text{Orb}_{F_n}(P)| \leq d_1 \ldots d_s$. The size of $\text{Orb}_w(P)$ is equal to the minimal natural number such that $P \cdot w^k = P$, that is $k = \text{lcm}(o_{\alpha_1}(w), \ldots, o_{\alpha_s}(w))$. Indeed, $P \cdot w^k = P$, if and only if $w^k \in \bigwedge_{i=1}^{\lambda} \alpha_i^{-1}H_i\alpha_i$, that is if and only if $\text{lcm}(o_{\alpha_1}(w), \ldots, o_{\alpha_s}(w))$ divides $k$ [6, Lemma 4.10]. As $k = \text{lcm}(o_{\alpha_1}(w), \ldots, o_{\alpha_s}(w))$ is minimal such that $P \cdot w^k = P$, $|\text{Orb}_w(P)| = \text{lcm}(o_{\alpha_1}(w), \ldots, o_{\alpha_s}(w))$.

In [6, Theorem 3], we give a condition on a partition $P$ that ensures the same subgroup appears at least twice in $P$. We state a shortened version of the result and give another proof using the action of $F_n$ on $Y'$.

Theorem C. [6, Theorem 3] Let $P = \{H_i\alpha_i\}_{i=1}^{\lambda}$ in $Y'$. If there exist $1 \leq j, k \leq s$ such that $\bigcap_{i=1}^{\lambda} \alpha_i^{-1}H_j\alpha_i \subseteq \bigcap_{i \neq j, k} \alpha_i^{-1}H_i\alpha_i$. Then $H_j = H_k$. 

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Proof. From the assumption, there exists \( w \in \bigcap_{i \neq j, k} \alpha_i^{-1} H_i \alpha_i \), \( w \notin \bigcap_{i = 1}^{i=s} \alpha_i^{-1} H_i \alpha_i \). Then \( P \cdot w = \{ H_i \alpha_i w \}_{i=1}^{i=s} \) gives \( F_n = \bigcup_{i \neq j, k} H_i \alpha_i \cup H_j \alpha_j w \cup H_k \alpha_k w \). So, \( H_j \alpha_j w \cup H_k \alpha_k w = H_j \alpha_j \cup H_k \alpha_k \), with \( H_i \alpha_i w \neq H_j \alpha_j \) and \( H_k \alpha_k w \neq H_k \alpha_k \). As \( H_j \alpha_j w \cap H_k \alpha_k \alpha_j = 0 \) and \( H_k \alpha_k w \cap H_k \alpha_k = \emptyset \), \( H_j \alpha_j w \subseteq H_k \alpha_k \) and \( H_k \alpha_k w \subseteq H_j \alpha_j \). From \( H_j \alpha_j w \cup H_k \alpha_k w = H_j \alpha_j \cup H_k \alpha_k \) again, we have \( H_j \alpha_j w = H_k \alpha_k \) and \( H_k \alpha_k w = H_k \alpha_k \), that is \( H_k \alpha_k w \alpha_j^{-1} = H_j \) a subgroup, so \( H_k \alpha_k w \alpha_j^{-1} = H_k \), that is \( H_k = H_j \). Further, we recover \( \alpha_j(w) = o_k(w) = 2 \) and \( w^2 \in \bigcap_{i=1}^{i=s} \alpha_i^{-1} H_i \alpha_i \), as in the proof of [6, Theorem 3]. \( \square \)

We recall that for each subgroup \( H \) of index \( d \) in \( F_n \) (or in any group), there is a transitive action of the group on the set of right cosets of \( H \), that is given two cosets \( H \alpha \) and \( H \beta \) of \( H \), there exists \( w \) such that \( H \alpha \cdot w = H \beta \). So, the following question arises:

**Question 2.3.** Let \( P = \{ H_i \alpha_i \}_{i=1}^{i=s} \) and \( P' = \{ H_i \beta_i \}_{i=1}^{i=t} \) in \( Y' \). Does there necessarily exist \( w \in F_n \) such that \( P' = P \cdot w \)?

### 2.2 Topology in the space of coset partitions of \( F_n \)

We refer to [10] for more details. Let \( Y' \) be the space of coset partitions of \( F_n \). Given \( P = \{ H_i \alpha_i \}_{i=1}^{i=s} \) in \( Y' \), with \( d_s \geq ... \geq d_1 > 1 \), we identify \( P \) with the \( s \)-tuple \( (H_s, ..., H_1) \) and we consider \( H_s \) at the first place, \( H_{s-1} \) at the second place and so on. Let \( P' \in Y' \), \( P' = \{ K_i \beta_i \}_{i=1}^{i=t} \). We define a function \( d : Y' \times Y' \to \mathbb{R} : \)

\[
d(P, P') = \begin{cases} 
2^{-k} & \text{if } k \text{ is the first place at which } K_i \neq H_i \\
0 & \text{if } t = s; \ H_i = K_i, \forall 1 \leq i \leq s 
\end{cases}
\]

The function \( d : Y' \times Y' \to \mathbb{R} \cup \{ \infty \} \) is a semi-metric if for all \( P, P', P'' \in Y' \), \( d \) satisfies \( d(P, P') = d(P', P) \) (symmetry) and \( d(P, P'') \leq d(P, P') + d(P', P'') \) (triangle inequality). A standard argument shows:

**Lemma 2.4.** The function \( d \) is a semi-metric.

**Proof.** Let \( P, P', P'' \in Y' \). Clearly, \( d(P, P') = d(P', P) \). Assume \( d(P, P') = 2^{-k} \), \( d(P', P'') = 2^{-\ell} \), and \( d(P, P'') = 2^{-m} \). If \( k > 1 \) or \( \ell > 1 \), then \( m = \min \{k, \ell\} \) and \( d(P, P'') = 2^{-\min(k, \ell)} \leq 2^{-k} + 2^{-\ell} \). If \( k = \ell = 1 \), then \( m \geq 1 \) and \( d(P, P'') = 2^{-m} < 1 \). \( \square \)

A metric is a semi-metric with the additional requirement that \( d(P, P') = 0 \) implies \( P = P' \). Identifying points with zero distance in a semi-metric \( d \) is an equivalence relation that leads to a metric \( \tilde{d} \). The function \( \tilde{d} \) then is a metric in \( Y'/\equiv \), with \( P \equiv P' \) if and only if \( d(P, P') = 0 \). If the answer to Question 2.3 is positive, then \( Y'/\equiv \) is the same as the quotient of \( Y' \) by the action of \( F_n \). We denote \( Y'/\equiv \) by \( Y \) and \( \tilde{d} \) by \( \rho \).
We denote by $B_r(P_0) = \{ P \in Y \mid \rho(P, P_0) < 2^{-r}\}$, the open ball of radius $2^{-r}$ centered at $P_0$. A set $U \subset Y$ is open if and only if for every point $P \in U$, there exists $\epsilon > 0$ such that $B_\epsilon(P) \subset U$. A space $Y$ is **totally disconnected** if every two distinct points of $Y$ are contained in two disjoint open sets covering the space. A point $P$ in a metric space $Y$ is an isolated point of $Y$ if there exists a real number $\epsilon > 0$ such that $B_\epsilon(P) = \{ P \}$. If all the points in $Y$ are isolated, then $Y$ is discrete. The space $Y$ is (topologically) discrete if $Y$ is discrete as a topological space, that is the metric may be different from the discrete metric.

**Theorem 2.5.** The metric space $Y$ is (topologically) discrete.

*Proof. We show that all the points in $Y$ are isolated. Let $P = \{ H_i \alpha_i \}_{i=1}^s$ in $Y$, with $d_s \geq ... \geq d_1 > 1$. Then for $\epsilon < 2^{-(s+1)}$, $B_\epsilon(P) = \{ P \}. \square$

This implies that $Y$ is Hausdorff, bounded and totally disconnected, facts that could be easily proved directly using $\rho$. A metric space $X$ is uniformly discrete, if there exists $\epsilon > 0$ such that for any $x, x' \in X$, $x \neq x'$, $\rho(x, x') > \epsilon$. The space $Y$ is not uniformly discrete.

**Remark 2.6.** Given an arbitrary group $G$, one can define in the same way the space $Y$, the metric $\rho$ and obtain the same topological properties. The action of $G$ on $Y$ can also be defined in the same way, but it is not necessarily faithful anymore.

### 3 Extension of Theorem B: Proof of Theorem 2

**Proof of Theorem 2.** Let $P_0 = \{ H_i \alpha_i \}_{i=1}^s$, with $1 < d_{H_1} \leq ... \leq d_{H_s}$. Let $P \in Y$, $P = \{ K_i \beta_i \}_{i=1}^t$, with $1 < d_{K_1} \leq ... \leq d_{K_t}$.

(i) If $\rho(P, P_0) < \frac{1}{2}$, then $K_i = H_s$. So, if there exists a $d_s$-cycle in $T_{H_s}$, the index $d_s$ appears in $P_0$ and in $P$ at least $p$ times, where $p$ is the least prime dividing $d_s$. Note that this implies necessarily $\rho(P, P_0) \leq 2^{-p-1}$.

(ii), (iii) If $\rho(P, P_0) < 2^{-(s+1)}$, $2 \leq r \leq s - 1$, then $K_i = H_s$, $K_{t-1} = H_{s-1}$, ..., $K_{t-r} = H_{s-r}$. If $k$, the maximal length of a cycle in $\bigcup_{i=1}^s T_{H_i}$, satisfies $k > d_{H_{s-r}}$, then $k > d_{K_{t-r}}$ also. Furthermore, $k$ occurs in $\bigcup_{i=1}^s T_{H_i}$ and also in $\bigcup_{j=t-r}^{j=t} T_{K_j}$, since $\bigcup_{i=s-r}^{i=s} T_{H_i} \bigcup_{j=t-r}^{j=t} T_{K_j}$. If $P_0$ satisfies condition (i) or (ii) of Theorem B, then $P$ satisfies the same condition and hence has multiplicity.

If $P_0$ satisfies condition (iii) or (iv) of Theorem B, then $P_0$ has multiplicity, with $d_{H_{s-i}} = d_{H_{s-j}}$ for some $i \neq j$, $0 \leq i, j \leq r$. So, $d_{K_{t-i}} = d_{K_{t-j}}$ also, that is $P$ has multiplicity. \square

To conclude, we ask the following natural question: does there exist a metric on the space of coset partitions $Y'$ that induces a non-discrete topology and yet can give rise to a result of the form of Theorem 2?
References

[1] M.A. Berger, A. Felzenbaum, A.S. Fraenkel, *Improvements to two results concerning systems of residue sets*, Ars Combin. 20 (1985), 69-82.

[2] M.A. Berger, A. Felzenbaum, A.S. Fraenkel, *The Herzog-Schönheim conjecture for finite nilpotent groups*, Canad. Math. Bull. 29 (1986), 329-333.

[3] M.A. Berger, A. Felzenbaum, A.S. Fraenkel, *Lattice paralleloptopes and disjoint covering systems*, Discrete Math. 65 (1987), 23-44.

[4] M.A. Berger, A. Felzenbaum, A.S. Fraenkel, *Remark on the multiplicity of a partition of a group into cosets*, Fund. Math. 128 (1987), 139-144.

[5] M.A. Brodie, R.F. Chamberlain, L.C Kappe, *Finite coverings by normal subgroups*, Proc. Amer. Math. Soc. 104 (1988), 669-674.

[6] F. Chouraqui, *The Herzog-Schönheim conjecture for finitely generated groups*, ArXiv 1803.08301.

[7] J. D. Dixon, *The probability of generating the symmetric group*, Math. Z. 110 (1969), 199-205.

[8] M. Herzog, J. Schönheim, *Research problem no. 9*, Canad. Math. Bull., 17 (1974), 150.

[9] L. Margolis, O. Schnabel, *The Herzog-Schönheim conjecture for small groups and harmonic subgroups*, ArXiv 1803.03569.

[10] J. Munkres, *Topology*, Pearson Modern Classics for Advanced Mathematics Series, 2000.

[11] Š. Porubský, J. Schönheim, *Covering systems of Paul Erdős. Past, present and future. Paul Erdős and his mathematics*, Júnos Bolyai Math. Soc., 11 (2002), 581-627.

[12] D.J.S. Robinson, *A Course in the Theory of Groups*, Graduate Texts in Mathematics 80, Springer-Verlag, Berlin, Heidelberg, New York (1980).

[13] J.J. Rotman, *An Introduction to Algebraic Topology*, Graduate Texts in Mathematics 119, Springer-Verlag, Berlin, Heidelberg, New York (1988).

[14] A. Sikora, *Topology on the spaces of orderings of groups*, Bull. London Math. Soc. 36 (2004), 519-526.

[15] C.C. Sims, *Computation with finitely presented groups*, Encyclopedia of Mathematics and its Applications 48, Cambridge University Press (1994).

[16] J. Stillwell, *Classical Topology and Combinatorial Group Theory*, Graduate Texts in Mathematics 72, Springer-Verlag, Berlin, Heidelberg, New York (1980).
[17] Z.W. Sun, *Finite covers of groups by cosets or subgroups*, Internat. J. Math. 17 (2006), n.9, 1047-1064.

[18] Z.W. Sun, *Classified publications on covering systems*, http://math.nju.edu.cn/zwsun/Cref.pdf.

[19] M.J. Tomkinson, *Groups covered by abelian subgroups*, London Math. Soc. Lecture Note Ser. 121, Cambridge Univ. Press (1986).

[20] M.J. Tomkinson, *Groups covered by finitely many cosets or subgroups*, Comm. Algebra 15(1987), 845-859.

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