Complexity of the Boundary-Guarding Art Gallery Problem

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Abstract

We resolve the complexity of the boundary-guarding variant of the art gallery problem, showing that it is $\exists \mathbb{R}$-complete, meaning that it is equivalent under polynomial time reductions to deciding whether a polynomial system of equations has a real solution. Introduced by Victor Klee in 1973, the art gallery problem concerns finding configurations of guards which together can see every point inside of an art gallery shaped like a simple polygon. The original version of this problem has previously been shown to $\exists \mathbb{R}$-hard, but until now the complexity of the variant where guards only need to guard the walls of the art gallery was an open problem.

Our results can also be used to provide a simpler proof of the $\exists \mathbb{R}$-hardness of the standard art gallery problem. In particular, we show how the algebraic constraints describing a polynomial system of equations can somewhat naturally occur in an art gallery setting.

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1 Introduction

1.1 Art gallery problem

The original form of the art gallery problem presented by Victor Klee (see O’Rourke [9]) is as follows:

We say that a closed polygon $P$ can be guarded by $n$ guards if there is a set of $n$ points (called the guards) in $P$ such that every point in $P$ is visible to some guard, that is the line segment between the point and guard is contained in the polygon. The problem asks, for a given polygon $P$ (which we refer to as the art gallery), to find the smallest $n$ such that $P$ can be guarded by $n$ guards.

The vertices of $P$ are usually restricted to rational or integral coordinates, but even so an optimal configuration might require guards with irrational coordinates (see Abrahamsen, Adamaszek and Miltzow [1] for specific examples of polygons where this is the case). For this reason, we don’t expect algorithms to actually output the guarding configurations, only to determine how many guards are necessary.

The art gallery problem (and any variant thereof) can be phrased as a decision problem: “can gallery $P$ be guarded with at most $k$ guards?” The complexity of this problem is the subject of this paper. Approximation algorithms can also be studied, see for example Bonnet and Miltzow [4].

1.2 The complexity class $\exists^R$

The decision problem ETR asks whether a sentence of form:

$$\exists X_1 \ldots \exists X_n \Phi(X_1, \ldots, X_n)$$

is true, where $\Phi$ is a formula in the $X_i$ involving addition, subtraction, multiplication, constants 0 and 1, and strict or non-strict inequalities. The complexity class $\exists^R$ consists of problems which can be reduced to ETR in polynomial time. A number of interesting problems have been shown to be complete for this class, including for example the 2-dimensional packing problem [3] and the problem of deciding whether there exists a point configuration with a given order type [8][10].

It is straightforward to show that NP $\subseteq \exists^R$, and it is also known, though considerably more difficult to prove, that $\exists^R \subseteq \text{PSPACE}$ (see Canny [5]). Both inclusions are conjectured to be strict.

1.3 Art gallery variants

There are several variants of this problem. We will be interested in ones involving restrictions on the placement of guards and of the region that must be guarded. Table 1 lists these variants as well as monikers we use to refer to them.

| Guards anywhere inside $P$ | Guard interior of $P$ | Guard only the boundary of $P$ |
|---------------------------|----------------------|-----------------------------|
| Guards on the boundary of $P$ | AG (Art Gallery) | BG (Boundary Guarding) |
|                           | GOB (Guards on Boundary) | BB (Boundary-Boundary) |

Table 1: Variants of the art gallery problem
Lee and Lin [6] showed that all of these variants are NP-hard (the result is stated for the AG and GOB variants, but their constructions also work for the BG and BB variants respectively). More recently, Abrahamsen, Adamaszek, and Miltzow [2] showed that the AG and BOG variants are \( \exists \mathbb{R} \) complete. It is straightforward to extend their proof of membership in \( \exists \mathbb{R} \) to any of these variants, but the \( \exists \mathbb{R} \)-hardness question remained open for the BG and BB variants. We will show that the BG variant is also \( \exists \mathbb{R} \)-hard:

**Theorem 1.1.** The BG variant of the art gallery problem is \( \exists \mathbb{R} \)-complete.

## 2 The problem ETR-INV\(^{rev}\)

The proof of Theorem 1.1 is by reduction of the problem ETR-INV\(^{rev}\) to the BG variant of the art gallery problem.

**Definition 2.1.** (ETR-INV\(^{rev}\)) In the problem ETR-INV\(^{rev}\), we are given a set of real variables \( \{x_1, \ldots, x_n\} \) and a set of inequalities of the form:

\[
    x = 1, \quad xy \geq 1, \quad x\left(\frac{5}{2} - y\right) \leq 1, \quad x + y \leq z, \quad x + y \geq z,
\]

for \( x, y, z \in \{x_1, \ldots, x_n\} \). The goal is to decide whether there is an assignment of the \( x_i \) satisfying these inequalities with each \( x_i \in [\frac{1}{2}, 2] \).

Abrahamsen et al [2] proved the \( \exists \mathbb{R} \)-hardness of the AG and GOB variants using a similar problem called ETR-INV, which essentially differs from ETR-INV\(^{rev}\) only by having a \( xy \leq 1 \) constraint instead of one of the form \( x\left(\frac{5}{2} - y\right) \leq 1 \):

**Definition 2.2.** (Abrahamsen, Adamaszek, Miltzow [2]) (ETR-INV) In the problem ETR-INV, we are given a set of real variables \( \{x_1, \ldots, x_n\} \) and a set of equations of the form:

\[
    x = 1, \quad xy = 1, \quad x + y = z,
\]

for \( x, y, z \in \{x_1, \ldots, x_n\} \). The goal is to decide whether there is a solution to the system of equations with each \( x_i \in [\frac{1}{2}, 2] \).

**Theorem 2.3.** (Abrahamsen, Adamaszek, Miltzow [2]) The problem ETR-INV is \( \exists \mathbb{R} \)-complete.

Geometrically, the constraint \( xy \leq 1 \) seems to be difficult to construct in any variant of the art gallery problem. The inversion gadget in [2] effectively computes \( x\left(\frac{5}{2} - y\right) = 1 \) and uses other gadgets to reverse the second variable. This, however, is not strictly necessary, since the proof of the \( \exists \mathbb{R} \)-hardness of ETR-INV can be easily modified to show the same result for ETR-INV\(^{rev}\).

### 2.1 \( \exists \mathbb{R} \)-hardness of ETR-INV\(^{rev}\)

The proof of the \( \exists \mathbb{R} \)-hardness of ETR-INV in [2] has the interesting property that every time the inversion constraint is used, at least one of the input variables is known \textit{a priori} to be in an interval \( [a, b] \) of length less than \( \frac{1}{2} \). If \( x \) is such a variable, then it is possible with the addition constraints to create an auxiliary variable \( V \) satisfying \( V = \frac{5}{2} - x \). This allows the full inversion constraint can be constructed. The construction of \( V \) follows.

First, construct a variable equal to \( \frac{1}{2} \):

\[
    V_1 = 1
\]
\[ V_2 + V_2 = V_1 \]

Next, let \( V_3 = x + a \) where \( a = 0 \) or \( \frac{1}{2} \), so that \( V_3 \in [1, 2] \) for any value of \( x \). Now:

\[ V_4 + V_4 = V_3 \quad \left( V_4 = \frac{1}{2}x + \frac{1}{2}a \in \left[ \frac{1}{2}, 1 \right] \right) \]

\[ V_5 + V_5 = V_2 \quad \left( V_5 = \frac{1}{4} \right) \]

\[ V_5 + V_1 = V_6 \quad \left( V_6 = \frac{5}{4} \right) \]

\[ V_1 + V_2 = V_7 \quad \left( V_7 = \frac{3}{2} \right) \]

\[ V_8 + V_4 = V_6 \text{ or } V_7 \quad \left( V_7 + \frac{1}{2}x + \frac{1}{2}a = \frac{5}{4} + \frac{1}{2}a \right) \]

\[ V + V = V_7 \quad \left( V + x = \frac{5}{2} \right) \]

Thus, the problem ETR-INV\(^{rev}\) can be shown to be \( \exists R \)-hard as in the proof in [2].

Since the publication of [2], Miltzow and Schmiermann [7] have shown that, subject to some minor technical conditions, a continuous constraint satisfaction problem with an addition constraint, a convex constraint, and a concave constraint is \( \exists R \). This would allow us to use constraints other than exact inversion constraints, but we didn’t find that this could simplify our argument.

3 Art gallery construction

3.1 Notation

Here \( AB \) refers to a line segment, \( \overrightarrow{AB} \) is the line containing that segment, and \( |AB| \) is the length of that segment.

3.2 Wedges and variable gadgets

The first step of the construction is to prove some results that allow us to restrict the possible guarding configurations.

**Definition 3.1.** (Wedge) A *wedge* is any pair of adjacent line segments on the boundary of the art gallery with an internal angle between them less than \( \pi \). The *critical region* of a wedge is the set of points which are visible to the vertex of the wedge.
Lemma 3.2. In a guarding configuration, the critical region of each wedge must contain a guard.

Proof. Straightforward.

Definition 3.3. (Guard regions and guard segments) We will designate certain regions inside the art gallery called guard regions, which are each formed by the intersection of the critical regions of some number of wedges. The critical regions of wedges corresponding to different guard regions must not intersect. A guard region shaped like a line segment is called a guard segment.

Lemma 3.4. If we designate $n$ guard regions, then any guarding configuration has at least $n$ guards, and a guarding configuration with exactly $n$ guards has 1 guard in each guard region.

Proof. Straightforward.

Having one guard in each guard region is a necessary but not sufficient condition for a configuration to be a guarding configuration. The gallery we construct will have some number of guard regions, with additional constraints on the positions of the guards which can be satisfied if and only if a given ETR-INV$^{rev}$ problem has a solution.
3.3 Nooks and continuous constraints

Definition 3.5. (Nook) A nook consists of a line segment on the boundary of the art gallery, called the nook segment, and geometry around it to restrict the visibility of that segment. The critical region of a nook is the region of points which can see some part of the nook segment.

![Figure 3: A nook with nook segment s and critical region R. This nook has a wedge on one of the side walls; it will occasionally be necessary to intersect nooks and wedges in this way.]

A guarding configuration will have some non-zero number of guards in the critical region of a nook, which together must guard the nook segment. In general, no guard needs to see all of the nook segment, instead it can be guarded by a collaboration of several guards. This is used to enforce continuous constraints between guards.

![Figure 4: An art gallery which requires 2 guards. No guard can see the entirety of the nook segment on the left while also being able to see the tips of the wedges on the right, so an optimal solution has two guards which collaborate to guard the nook segment. The allowed positions of \( g_2 \) depend continuously on \( g_1 \).]

In our construction, the critical region of each constraint nook will intersect exactly 2 guard regions.

3.4 Copying

In order to use a variable in multiple constraints, we need to a way to force guards on two different guard segments to have the same relative position.
Lemma 3.6. Suppose segments \(AB\) and \(CD\) are such that \(\overrightarrow{AB}\) and \(\overrightarrow{CD}\) are parallel, and suppose \(\overrightarrow{AC}\) and \(\overrightarrow{BD}\) intersect at a point \(P\), as in Figure 5. If a line through \(P\) intersects \(AB\) at a point \(X\) and intersects \(CD\) at a point \(Y\), then \(\frac{|AX|}{|AB|} = \frac{|CY|}{|CD|}\).

Proof. Triangles \(APB\) and \(CPD\) are similar, so \(\frac{|AP|}{|AB|} = \frac{|CP|}{|CD|}\). Also, the triangles \(AXP\) and \(CYP\) are similar, so \(\frac{|AX|}{|AP|} = \frac{|CY|}{|CP|}\). Multiplying, we obtain \(\frac{|AX|}{|AB|} = \frac{|CY|}{|CD|}\). \(\square\)

Figure 5: By Lemma 3.6 we have \(\frac{|AX|}{|AB|} = \frac{|CY|}{|CD|}\).

This allows us to create nooks which copy between two segments.

Definition 3.7. (Copy nook) A copy nook is a nook whose critical region intersects exactly two guard segments and no other guard regions. Further, the two guard segments should be parallel to the nook segment.

Figure 6: A copy nook. By Lemma 3.6 the position of guard \(g_2\) must be above that of guard \(g_1\) in order to guard the nook.

With these nooks, we can start to arrange the art gallery.

3.5 Art Gallery Setup

The rough setup for the art gallery is shown in Figure 7. The variables \(x_1, \ldots, x_n\) are represented by a bank of guard segments, called the variable segments. The \(y\)-coordinate of each of these guards represents the value of a variable in \([1/2, 2]\), with larger \(y\)-coordinates corresponding to larger values
of the variable. For each time that a variable appears in a constraint, a pair of copy nooks force the guard on one of the input segments to have a position corresponding to the same value in $\left[\frac{1}{2}, 2\right]$. The constraint gadgets then enforce constraints on the input segments.

Figure 7: Diagram of the art gallery construction. The copy nooks would be far enough away that it is impractical to fit them on this diagram.

### 3.6 Specification of the copy nooks

This section is concerned with showing that it is indeed possible to arrange all the copy nooks in such a way that none of them interfere with each other. The critical region of each copy nook can be made as almost as narrow as the convex hull of the segments being copied, and the nook itself can be made arbitrarily small and distant, so it shouldn’t be surprising that it is possible to arrange copy nooks in this way. However, working out the exact details requires some tedious calculation. Figure 8 shows the parameters describing the copy nooks that we will use.
Figure 8: Parameters and measurements for the copy nooks. The nook on the left forces the guard on the variable segment to have a position less than or equal to the corresponding position of the guard on the input segment. The nook on the right enforces the corresponding $\geq$ constraint.

Note that because the ratio of the length of the nook segment to the length of the input or variable segment is $1 : k - 1$, the points at the openings of the nooks occur $\frac{1}{k}$th of the way between the nook segment and a guard segment.

The segments in each bank occur at regular intervals with a spacing of a distance $d_0$. The two grids should be horizontally aligned, as in Figure 9, but no input segment should have a guard segment directly below it. We will need $n$ variable segments, and let $i$ be the number of spaces needed for input segments, and let $j$ be the index of the furthest input segment from the first variable segment.
Ideally, the constraint gadgets should only depend on these parameters up to a change of scale. This can be accomplished if $d_0$ is a fixed multiple of $k - 1$. We will use $d_0 = 9(k - 1)$. As we will see, this gives sufficient space between segments for the copy nooks and constraint gadgets to work.

All lines between nook segments and input segments should have slope between $-\frac{1}{2}$ and $-1$. The steepest of these lines in Figure 8 has slope:

$$-\frac{(m - h)k}{m-h-1}d = -(h + 2)\frac{(m - h)k}{(m - h - 1)d},$$

and the shallowest of these lines has slope:

$$-\frac{(m - h - 1)k}{m-h}d = -h\frac{(m - h - 1)k}{(m - h)d}$$

The values of $d$ range from $((j - i - n)d_0$ to $jd_0$. We have set $d_0 = 9(k - 1)$, so as long as $k \geq 9$, we have that $8k \leq d_0 \leq 9k$. The value of $m$ will be chosen in such that a way that $m \geq 2h$, so if $h \geq 10$:

$$\frac{m - h}{m - h - 1} \leq \frac{10}{9}$$

So we have:

$$(h + 2)\frac{(m - h)k}{(m - h - 1)d} \leq \frac{10(h + 2)k}{9(j - i - m)d_0} \leq \frac{5(h + 2)}{36(j - i - m)} \leq 1$$

$$h\frac{(m - h - 1)k}{(m - h)d} \geq \frac{9hk}{10jd_0} \geq \frac{h}{10j} \geq \frac{1}{2}$$

Rearranging:

$$h + 2 \leq \frac{36}{5}(j - i - m) \text{ and } h \geq 5j,$$

so it is sufficient to have $h + 2 \leq 6(j - i - n)$ and $h \geq 5j$. So if $h = 5j$ and $j \geq 2 + 6(i + n)$, then this is satisfied, and the slopes of lines between nook segments and guard segments will be between $-1$ and $-\frac{1}{2}$ for any $k \geq 9$ and $m \geq 2h$. 

Figure 9: The variable and input segments.
Next, we need to choose \( m \) and \( k \) so that none of the possible copy nooks interfere with each other. By the construction of the segment banks, \( d \) can take values \( qd_0 \) for \( q < j \). First, choose \( h \) to be an odd integer, say \( h = 2\ell + 1 \). Now, for some integer \( p \), set \( m = (\ell + 2)h + ph(h + 2) \), so \( m + 1 = (\ell + 2)(2\ell + 1) + 1 + ph(h + 2) = (\ell + 1)(h + 2) + ph(h + 2) \). This means that the vertical offsets of the nook segments are integer multiples of \( d \), in particular:

\[
\frac{m}{h} = (\ell + 2 + p(h + 2)), \quad \frac{m + 1}{h + 2} = (\ell + 1 + ph)
\]

The value of \( d \) is a multiple of \( d_0 \) between 1 and \( j \), and the variable segments have coordinates between 0 and \( nd_0 \). So no two possible copy nooks have the same \( x \)-coordinate so long as the set:

\[
\{(\ell + 2)q + (h + 2)pq + r, (\ell + 1)q + hpq + r : 1 \leq q < j, 0 \leq r < n\}
\]

contains \( 2n(j - 1) \) distinct values. Because \( h \) is odd, \( hq_1 = (h + 2)q_2 \) only when \( q_1 = (h + 2) \) and \( q_2 = h \), so since \( j < h \), \( hpq_1 \) and \( (h + 2)pq_2 \) always differ by at least \( p \) for any allowed values of \( q_1 \) and \( q_2 \). So if we choose \( p > (\ell + 2)j + n \), then each copy nook has a nook segment at a different \( x \)-coordinate.

With \( p \) (and so \( m \)) chosen this way, the distance between adjacent copy nooks will be at least \( d_0 \). The copy nooks have a height of \( m + 1 \). The lines between copy nooks and variable or input segments have slope less than \(-\frac{1}{2}\), so allowing a distance of \( 2(m + 1) \) between the copy nooks is sufficient to prevent nooks from interfering or obstructing each other. So choose \( k \) so that \( d_0 = 9(k - 1) > 2(m + 1) \).

This shows that we can construct the copy nooks themselves, but it remains to show that the critical region of each nook doesn’t intersect any guard segments that it isn’t supposed to.

**Lemma 3.8.** If all is as above, then each copy nook only has exactly 1 variable segment and 1 guard segment in its critical region.

**Proof.** For each pair of segments, there are 4 nearby segments which might intersect the critical region of one of their copy nooks. These are shown in Figure [10]

![Figure 10: The critical regions of the copy nooks and nearby segments which they must avoid. All of the lines shown have slope less than \(-\frac{1}{2}\).](image)
The segment highlighted in red is the closest (in this diagram, though also in general) to intersecting the critical region. Figure 11 shows measurements for the lines near this segment.

Figure 11: Closeup near the highlighted segment. The additional red line has slope $-\frac{1}{2}$, and shows the maximum allowed value of $\alpha$.

Since $k \geq 2$, it is sufficient to have $\alpha \leq 3$ in order for the critical region to avoid the highlighted segment. We compute that:

$$\alpha = \frac{(k-1)m-h}{kh - (k-1)m-h - 1}$$

This is $\leq 3$ so long as $\frac{(k-1)m-h}{kh} \leq \frac{3}{2}$. Since $k \geq 2$, $m \leq 8h$ is sufficient for this to hold. We set $m = (\ell + 2)h + ph(h + 2)$, and $p$ will always be at least 4, so this is already sufficient for $m$ to be at least $8h$.

There is a similar constraint on $m$ for each of the other nearby segments, but $m > 8h$ is sufficient in every case.

3.7 Constraint Gadgets

Next, we need to create the constraint gadgets which will actually enforce the constraints. Figure 12 shows how a constraint gadget can be created without interfering with the copy nooks.
Each constraint gadget will have some number (either 2 or 3) of input guard segments, but may also take up additional slots in the bank of input segments. These slots will be left empty.

Three wedges form each of these guard segments; two will be on the top wall of the art gallery, but one must be on the bottom of the gadget. Since the distance from the constraint gadgets to the top wall depends on the number of variables and constraints, the width of these wedges also needs to depend on these parameters. All other parameters of each constraint gadget are fixed up to a choice of scale.

The input segments are tied to the variable segments by the copy gadgets. In order for these gadgets to work, the region $R_1$ should not be obstructed by the walls of the art gallery. Additionally, guards on the variable segments might be able to see anything in the region $R_2$, so the nooks which enforce the constraint should have nook segments which don’t intersect this region. Also, the guard regions for any auxiliary guards used should avoid intersecting $R_2$ so that they don’t interfere with the copy gadgets, and the critical regions of the wedges making up these guard regions should not interfere with the wedges making up the variable gadgets for the input segments. Figure 12 shows examples of constraint nooks and an auxiliary guard region (yellow) which meet these criteria.

Constraints of the form $x = 1$ will not have a dedicated constraint gadget. Instead, these can be created by adding a wedge to a guard segment to turn it into a guard point, as in Figure 13.
The remaining constraints each need a constraint gadget. These constraints are:

\[ xy \geq 1, \quad x \left(\frac{5}{2} - y\right) \leq 1, \quad x + y \leq z, \quad x + y \geq z \]

### 3.7.1 Inversion

The first gadgets we will create are the \( xy \geq 1 \) and \( x \left(\frac{5}{2} - y\right) \leq 1 \) gadgets. Lemma 3.9 shows how this type of constraint can arise geometrically.

**Lemma 3.9.** Start with non-parallel line segments \( AB \) and \( CD \), as in Figure 14, and say that \( \overrightarrow{AD} \) and \( \overrightarrow{BC} \) intersect at a point \( P \). Let \( E \) be the point on \( \overrightarrow{AD} \) such that \( \overrightarrow{PE} \) and \( \overrightarrow{AB} \) are parallel, and let \( F \) be the point on \( \overrightarrow{CD} \) such that \( \overrightarrow{PF} \) and \( \overrightarrow{AB} \) are parallel. Draw a line through \( P \) intersecting \( AB \) at \( X \) and \( CD \) at \( Y \). Then \( \frac{|EA|}{|EB|} = \frac{|FC|}{|FD|} \), and letting \( \alpha^2 = |EA||EB| \) and \( \beta^2 = |FC||FD| \) we have \( \frac{|EX|}{\alpha} \cdot \frac{|FY|}{\beta} = 1 \).

Since we want to enforce inversion constraints on variables in the range \([\frac{1}{2}, 2]\), we will use geometry so that \( |EB| = 4|EA| \) (and therefore \( |FD| = 4|FC| \)), so \( \alpha = 2|EA| = \frac{1}{2}|EB| \) and \( \beta = 2|FC| = \frac{1}{2}|FD| \). This means that \( \frac{|EX|}{\alpha} \) and \( \frac{|FY|}{\beta} \) will map the segments \( AB \) and \( CD \) respectively onto \( [\frac{1}{2}, 2] \).

**Figure 14:** By Lemma 3.9 the value of \( |EX||FY| \) is independent of the position of \( X \).

**Proof of Lemma 3.9.** The triangles \( PEX \) and \( PFY \) are similar, so \( \frac{|EX|}{|EP|} = \frac{|FP|}{|FY|} \), so \( |EX||FY| = |FP||EP| \). In particular, when \( X = A \), we have \( |EA||FD| = |FP||EP| \), and when \( X = B \), we have \( |EB||FC| = |FP||EP| \), so \( |EA||FD| = |EB||FC| \) and \( \frac{|EA|}{|EB|} = \frac{|FC|}{|FD|} \).

Now \( |EX||FY| = |FP||EP| = |EA||FD| = |EB||FC| \), so:

\[ |EX||FY| = \sqrt{|FP||EP||EB||FC|} = \sqrt{\alpha^2 \beta^2} = \alpha \beta \]

As long as \( \overrightarrow{AB} \) and \( \overrightarrow{CD} \) are not parallel, Lemma 3.9 will work with any arrangement of the points \( A, B, C \) and \( D \). Importantly for the \( xy \geq 1 \) gadget, it is okay if the segments \( AB \) and \( CD \) intersect.
We do want to enforce inversion constraints between segments which are parallel, and so we will need an additional idea, set out in Lemma 3.10.

**Lemma 3.10.** Suppose line segments $AB$, $CD$, and $EF$ are such that $\overrightarrow{AB}$, $\overrightarrow{CD}$, and $\overrightarrow{EF}$ all intersect at a point $O$, as in Figure 15. Also suppose that the ratios $\frac{|OA|}{|OB|}$ and $\frac{|OC|}{|OD|}$ are the same. Let $P$ be the point where the lines $BE$ and $AF$ intersect, and $Q$ be the point where $DE$ and $CF$ intersect. We obtain a mapping $\varphi$ as follows: draw a line from a point $X$ on $AB$ through $P$, and let $Y$ be the intersection of this line with $EF$. Now draw a line through $Z$ and $Q$. The intersection of this line with $CD$ is $\varphi(X)$.

The map $\varphi$ is linear, that is $\frac{|AX|}{|AB|} = \frac{|C\varphi(X)|}{|CD|}$.

![Figure 15: By Lemma 3.10, $\frac{|AX|}{|AB|} = \frac{|C\varphi(X)|}{|CD|}$. If $|OA| = |OB|$, then $|OX| = |O\varphi(X)|$](image)

Proof. Place the figure in the vector space $\mathbb{R}^2$ with the point $O$ at $(0,0)$. Now the pairs of vectors $\{E,A\}$ and $\{E,C\}$ are each bases for $\mathbb{R}^2$. Let $\theta$ be the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ which takes vector $V$ to $(t,s)$ where $V = tE + sA$, and let $\psi$ be a similar map which writes $V$ as $tE + sC$. Now the linear map $\psi^{-1} \circ \theta$ fixes points on the line containing $E$ and $F$, and sends $A$ to $C$. Also

$$\theta(B) = \frac{|OB|}{|AO|} = \frac{|OC|}{|OD|} = \psi(D)$$

So $\psi^{-1} \circ \theta$ sends $B$ to $D$. Now by linearity, this composition sends $P$ to $Q$, and so sends $X$ to $\varphi(X)$. So $\varphi$ is linear.

We could alternatively prove Lemma 3.10 by working an additional dimension. We construct an arrangement of lines in $\mathbb{R}^3$ (see Figure 16) of which Figure 15 is the image under a linear projection, and so that the segments $O'B'$ and $O'D'$ are related by a reflection in $\mathbb{R}^3$. Since the three lines intersect in a point, the pairs of segments $(A'B', E'F')$ and $(C'D', E'F')$ are each coplanar, and so $F'A'$ and $E'B'$ intersect at a point $P'$ in $\mathbb{R}^3$, and similarly the point $Q'$ exists. This means that the map $\varphi$ can be defined on the figure in $\mathbb{R}^3$, and the symmetry of the 3-dimensional geometry descends to the 2-dimensional figure, making $\varphi$ linear.
Unlike in the parallel copying gadgets, the mappings from $AB$ to $EF$ and from $EF$ to $CD$ are in general not linear, only the composition is. This geometry can be combined with the inversion geometry to create a nook which enforces an inversion constraint between two parallel segments. Figure 17 shows how to create an $xy \geq 1$ constraint in this way.

Figure 17: An inversion nook between two parallel segments. The nook attempts to make an angled copy between guard segment $GH$ and the “phantom” segment $CD$, but it ends up hitting segment $AB$ instead.
To create the \( xy \geq 1 \) gadget, we choose geometry as in figure 17 with \(|AB| = |GH|\), \(|FD| = 4|FC|\), and \(|EB| = 4|EA|\). By Lemma 3.9 we have \( \frac{|GX| + |EA|}{2|EA|} = \frac{|FZ|}{2|FC|} \), and so by Lemma 3.9 we have that \( \frac{|GX| + |EA|}{2|EA|} \cdot \frac{EZ}{2|EA|} = \frac{|FZ|}{2|FC|} \cdot \frac{EZ}{2|EA|} = 1 \).

By positioning the two guard segments appropriately, we can create the \( xy \geq 1 \) gadget (Figure 18).

Figure 18: The \( xy \geq 1 \) gadget. The constraint \( xy \geq 1 \) is symmetrical, so it isn’t surprising that the nook will be symmetrical if the nook segment is chosen to lie on a horizontal line.

The \( x \left( \frac{5}{2} - y \right) \leq 1 \) gadget is created in a similar way:
Again \(|AB| = |GH|, |FD| = 4|FC|\), and \(|EB| = 4|EA|\), so \(\frac{|GX|+|EA|}{2|EA|} \cdot \frac{EZ}{2|EA|} = 1\). The segment \(GH\) is now oriented in the reverse direction compared to Figure 17, and the nook is now arranged such that guards see more of the nook segment as \(|AZ|\) or \(|GX|\) increases. Figure 20 shows the full \(x\left(\frac{5}{2} - y\right) \leq 1\) gadget:

Figure 20: The \(x\left(\frac{5}{2} - y\right) \leq 1\) gadget. Unlike with the \(\geq\) inversion gadget, constructing the geometry in a naive way requires solving a quadratic, so the positions of the vertices would potentially be irrational. The solution is to choose a setup as in Figure 19 and then use a linear transform to position the segments appropriately.
3.7.2 Addition

The addition constraint is the one constraint that involves more than two variables. In order to make this work, we use the fact that a single guard has 2 coordinates, so a combination of nooks which only interact with 2 guards each can enforce a constraint which continuously depends on 3 variables.

Figure 21: Guards $g_1$, $g_2$, and $g_3$ see parts of the nook segments of nooks $N_1$, $N_2$, and $N_3$ respectively. In order to see the rest of each of these nooks, the guard $g_4$ needs to be in the intersection of the shaded regions. This places a constraint on the positions of $g_1$, $g_2$, and $g_3$, since there needs to be at least one point in the intersection of all 3 regions. Each $x + y \leq z$ constraint will have a corresponding $x + y \geq z$ constraint, so there will never be more than one point in this intersection.

In order to make this constraint correspond to addition and not some other constraint, we will need a lemma about geometry:

**Lemma 3.11.** Let the line segments $AB$, $CD$, and $EF$ have the same length and lie on the same vertical line, as in Figure 22, and suppose $|CB| = |DE|$. Let points $P$, $Q$, and $R$ lie on a vertical line, with $|QP| = |QR|$. Note that $\overrightarrow{AP}$, $\overrightarrow{DQ}$, and $\overrightarrow{FR}$ intersect a single point, and the same is true of $\overrightarrow{BP}$, $\overrightarrow{CQ}$, and $\overrightarrow{ER}$.

Choose points $X$ and $Y$ on $AB$ and $EF$ respectively. Now $\overrightarrow{XP}$ and $\overrightarrow{YQ}$ intersect at a point $I$. Draw a line through points $I$ and $R$. This intersects $CD$ at a point $Z$.

If all is as above, then $\frac{1}{2}(|AX| + |EY|) = |CZ|$.
Figure 22: Setup for the addition gadget. By Lemma 3.11 we will have that $\frac{1}{2}(|AX| + |EY|) = |CZ|$. 

Proof. A homography of $\mathbb{R}^2$ (or more precisely $\mathbb{RP}^2$) is a transform which sends straight lines to straight lines. We want to find such a map which fixes points $A, B, C, D, E, F, X, Y, Z$ while sending the points $P, Q$ and $R$ to infinity. Additionally, lines through $P$ should be sent to lines with slope $-1$, lines through $Q$ should be sent to lines with slope $0$, and lines through $R$ should be sent to lines with slope $+1$ (see Figure 23).

Figure 23: The transformed geometry. The only parameters of the original geometry which can be recovered after transforming are the lengths of $AB$ and $BC$. 
In the transformed geometry, it is clear by elementary linear algebra that $\frac{1}{2} (|AX| + |EY|) = |CZ|$. A degrees-of-freedom argument is sufficient to show that such a transformation should exist, but for completeness we will give it explicitly. Let $x_0$ be the $x$-coordinate of $A,B,C,...$, $x_1$ be the $x$-coordinate of $P,Q$ and $R$, $y_0$ be the $y$-coordinate of $Q$, and let $a = |QP| = |QR|$, so $P$ has $y$-coordinate $y_0 + a$ and $Q$ has $y$-coordinate $y_0 - a$. Then the transform with the desired properties is given by:

$$
\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} x_0 + a & 0 & -(x_1 + a)x_0 \\ y_0 & x_0 - x_1 & -y_0x_0 \\ 1 & 0 & -x_1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
$$

Writing the map in this form makes it easy to check what happens to lines through $P$, $Q$, and $R$. In particular, for a $3 \times 3$ matrix $A$, if:

$$
A \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}
$$

then the map:

$$
\lambda \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = A \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
$$

sends lines through $(p_x, p_y)$ to lines parallel to $\begin{bmatrix} a \\ b \end{bmatrix}$.

Like all homographies of $\mathbb{R}^2$, the transformation used in the proof of Lemma 3.11 can be obtained geometrically as a projection from a plane, through a point, and onto another plane, as in Figure 24.

![Figure 24](image)

Figure 24: A geometric realization of the transformation from Lemma 3.11. Only rotation and reflection is required from here to obtain the same transformed figure as in the lemma.
Abrahamsen et al [2] use an instance of the same type of geometry for their addition gadget. The verification of their gadget given in that paper could be generalized to give an alternate proof of Lemma 3.11.

### 3.7.3 The $\geq$ addition gadget

![Figure 25: The $x + y \geq z$ addition gadget. From left to right, the input segments represent the variables $x$, $z$, and $y.$](image)

We want the constraint to be $x + y \geq z$, not $\frac{1}{2}(x + y) \geq z$, so the middle nook has to adjust the scale and offset accordingly. Figure 26 shows how this is done.
Figure 26: The positions of the auxiliary guard correspond to values of $x + y$ in the range $[1, 4]$, while the segment should correspond to values in the range $[\frac{1}{2}, 2]$. The middle nook in this gadget is adjusted to compensate for this.

### 3.7.4 The $\leq$ addition gadget

The $x + y \leq z$ addition gadget is very similar to the $x + y \geq z$ one, just with the nooks oriented in the opposite way.
With these gadgets, we can complete the reduction of ETR-INV to the BG variant of the art gallery problem, and hence prove Theorem 1.1.

4 Conclusions

It is interesting to note that our construction, while intended for the BG variant of the art gallery problem, is also sufficient to show the $\exists \mathbb{R}$-hardness of the standard AG variant. Indeed, all the guarding configurations considered are also guarding configurations in the AG variant. This is similar to the construction from [2], which simultaneously showed the $\exists \mathbb{R}$-hardness of the AG and BOG variants. It doesn’t seem to be possible to adapt our construction to the BB variant.

In our construction, each nook enforces a constraint between only two guards. While it is possible to put multiple guard regions in the critical region of one nook, the types of constraints created seem to be unable to depend continuously on more than 2 of the guards at a time. Addition constraints are only possible because the guards themselves have two coordinates, so a single nook can in principal enforce a constraint on as many as 4 variables. In the GOB variant, guards have only one coordinate, but have to cover the entire 2-dimensional interior of the art gallery, so constraints that affect more than 2 guards can be created. In the BB variant, neither of these ideas work, so it seems unlikely that a problem like ETR-INV could be reduced to it in this way. It is very possible that the BB variant is only NP.

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