3-Critical Subgraphs of Snarks

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Abstract. In this paper we further our understanding of the structure of class two cubic graphs, or snarks, as they are commonly known. We do this by investigating their 3-critical subgraphs, or as we will call them, minimal conflicting subgraphs. We consider how the minimal conflicting subgraphs of a snark relate to its possible minimal 4-edge-colourings. We fully characterise the relationship between the resistance of a snark and the set of minimal conflicting subgraphs. That is, we show that the resistance of a snark is equal to the minimum number of edges which can be selected from the snark, such that the selection contains at least one edge from each minimal conflicting subgraph. We similarly characterise the relationship between what we call the critical subgraph of a snark and the set of minimal conflicting subgraphs. The critical subgraph being the set of all edges which are conflicting in some minimal colouring of the snark. Further to this, we define groups, or clusters, of minimal conflicting subgraphs. We then highlight some interesting properties and problems relating to clusters of minimal conflicting subgraphs.

1. Introduction

As is well-known, the edge chromatic number of a cubic graph is either three or four. Such graphs are referred to as cubic class one and cubic class two graphs, respectively. Cubic class two graphs are more commonly known as snarks. Snarks have long been of particular interest in graph theory, largely for the fact that many major problems in graph theory are easily solvable for graphs which are not snarks. Tutte’s 5-flow conjecture [10] and the cycle double cover conjecture [9] are major examples of these problems.

Let $G = (V, E)$ be a graph. A $k$-edge-colouring, $f$, of $G$ is a mapping from the set of edges of $G$ to a set of $k$ colours. That is, $f: E \to \{1, \ldots, k\}$. $f$ is a proper $k$-edge-colouring of $G$ if no two adjacent elements in $E$ are mapped to the same colour. By Vizing’s theorem [2, Theorem 6.2], if $G$ is a graph and $f$ is a proper colouring then the smallest possible value of $k$ is $\Delta$ or $\Delta + 1$, where
$\Delta$ is the maximum degree of any vertex in $G$. If the smallest possible value of $k$ is $\Delta$, then we say that $G$ is class one, or $\Delta$-edge-colourable. Otherwise we say that $G$ is class two, or $(\Delta + 1)$-edge-colourable. Given a $k$-edge-colouring $f$, we call the set $f^{-1}(i)$ a colour class, for each $i \in \{1, \ldots, k\}$. A vertex $v$ is conflicting with regard to $f$ if more than one of the edges incident to $v$ are mapped to the same colour.

The resistance of $G$, denoted as $r(G)$, is defined as the $\min\{||f^{-1}(i)|| : f$ is a proper $(\Delta + 1)$-edge-colouring of $G$ and $f^{-1}(i)$ is a colour class$\}$. That is, the minimum number of edges that can be removed from a graph such that the resulting graph is 3-edge-colourable [7]. As it turns out, somewhat counter-intuitively perhaps, the resistance of $G$ equals the vertex resistance of $G$, denoted as $r_v(G)$, which is the minimum number of vertices that needs to be removed from $G$ such that the resultant graph is class one. If given a 3-edge-colouring of $G$ with $r(G)$ conflicting vertices, we can also find a proper 4-edge-colouring of $G$ with $r(G)$ edges being mapped to one particular colour.

Furthermore, the conflicting vertices in the 3-edge-colouring have a one-to-one relationship with the set of edges mapped to the fourth colour in the 4-edge-colouring. This is a result we use implicitly going forward, in that we do not consider 3-edge colourings with conflicting vertices. We consider only proper 3-edge colourings or proper 4-edge-colourings of cubic graphs. It has also been proven that $r(G) = 0$ or $r(G) \geq 2$ for any cubic graph $G$ [3].

If $f$ is a proper $k$-edge-colouring of a graph $G$ and $|f^{-1}(i)| = r(G)$ for some $i \in \{1, \ldots, k\}$, then we call $f$ a minimal colouring [8]. For cubic graphs, we will use colour sets $\{1, 2, 3\}$ and $\{0, 1, 2, 3\}$ for class one and class two graphs, respectively. We will assume $|f^{-1}(0)| = r(G)$ for a minimal colouring $f$ of $G$. Given a minimal colouring $f$ of $G$, if $f(e) = 0$ for some edge $e \in G$ then we call $e$ a conflicting edge with regard to $f$. Now, let $H \subseteq G$ and let $f_H$ be a proper colouring of $H$. A proper colouring of $G$, $f_G$, with $f_G(e) = f_H(e)$ for all $e \in H$ is called an extension of $f_H$. If $f_G$ is such that the number of conflicting edges in $G - H$ is minimal given $f_H$, then we call $f_G$ a minimal extension of $f_H$.

In this paper, we further understand the complexity of these graphs by extending the definition of conflicting zones introduced in [3], although we opt for the term conflicting subgraphs. We define minimal conflicting subgraphs, as well as consequent concepts such as the buffer subgraph, which is the maximum subgraph containing no edges in or adjacent to any minimal conflicting subgraph; the critical subgraph, which is the subgraph containing all edges which are conflicting in some minimal colouring of the graph; and clusters of minimal conflicting subgraphs which are essentially overlapping minimal conflicting subgraphs (formal definitions to follow). We then prove the following insight about cubic class two graphs. That for any collection of edges $R$ in a cubic graph $G$, such that $R$ contains an edge from every minimal conflicting subgraph in $G$, $G - R$ is 3-edge-colourable. Furthermore, for such an $R$ with minimal possible order, the resistance of $G$ is $|R|$. We are able to then characterise the resistance, as well as the critical subgraph of a graph, in terms of the set of minimal conflicting subgraphs. Finally, we discuss further problems of consideration.
2. Minimal Conflicting Subgraphs

A conflicting subgraph of a cubic graph $G$ is defined as a subgraph $H$ of $G$ which does not admit a proper 3-edge-colouring. That is essentially, a subgraph which itself is not 3-edge-colourable. This idea was introduced in [3], where it was called a conflicting zone. With a view to further understand what makes a cubic graph class two, we extend this idea by defining minimal conflicting subgraphs. The essential idea being to isolate from the graph that which is non 3-edge-colourable.

**Definition 1.** Let $G$ be a subcubic graph and let $M$ be a conflicting subgraph of $G$. If for any $e \in E(M)$ we have that $M - e$ is not a conflicting subgraph, then we call $M$ a minimal conflicting subgraph of $G$. Let

$$M_G = \bigcup \{M \mid M \text{ is a minimal conflicting subgraph of } G\}.$$  

We call $M_G$ the maximal conflicting subgraph of $G$. Let

$$C_G = \{e \in E(G) \mid e \notin M_G \text{ and } e \text{ is adjacent to some } e' \text{ in } M_G\}.$$  

We call $C_G$ the conflict-cut set of $G$. Let

$$B_G = \{e \in E(G) \mid e \notin M_G \cup C_G\}.$$  

We call $B_G$ the buffer subgraph of $G$.

A subcubic graph $G$ is called 3-critical if it has a chromatic index 4 and $G - e$ has chromatic index 3 for every $e \in G$. It is easy to see that this definition coincides with our definition of minimal conflicting subgraphs, in that a minimal conflicting subgraph can be thought of as a 3-critical subgraph. If the 3-critical subgraphs represent only that which is essentially non 3-edge-colourable, then the buffer subgraph represents that which is essentially redundant in contributing to the non colourability of the cubic graph. We list some properties of 3-critical subgraphs, or minimal conflicting subgraphs, of subcubic graphs. First, we present some known properties of 3-critical graphs in general, after which we prove some more pertinent properties for our purposes regarding minimal conflicting subgraphs.

**Proposition 1.** Let $M$ be a 3-critical graph. The following statements are true.

(i) $r(M) = 1$ and every edge $e \in M$ is conflicting in some minimal colouring of $M$.
(ii) $M$ is strictly subcubic.
(iii) $M$ is bridgeless.
(iv) Every vertex in $M$ has degree two or three.
(v) Every vertex in $M$ has at least two neighbours of degree three.

**Proof.** These are known properties of 3-critical graphs and we omit the proofs. □

**Proposition 2.** Let $G$ be a bridgeless cubic graph. The following statements are true.
(i) The distance between any two disjoint minimal conflicting subgraphs of $G$ is at least one.

(ii) Every conflicting subgraph in $G$ contains a minimal conflicting subgraph.

Proof. (i) This follows on directly from Proposition 1 (iv).

(ii) Let $M$ be a conflicting subgraph of $G$. Choose an edge $e \in M$. We check $e$ by considering $r(M - \{e\})$. If $r(M - \{e\}) \neq 0$ then remove $e$ from $M$. If $r(M - \{e\}) = 0$ then leave $M$ as is and mark $e$ as checked. Continue checking edges in $M$ until every edge is checked. Once every edge is checked, $M$ is then a minimal conflicting subgraph. □

We begin our investigation into these structures. We consider their existence relative to conflicting edges in minimal colourings. Note that although our primary interest is in cubic graphs, some results are applicable to subcubic graphs as well and are stated as such.

**Proposition 3.** Let $G$ be a subcubic class two graph and let $f$ be a minimal colouring of $G$. For each conflicting edge $e$ with regard to $f$, there exists at least one minimal conflicting subgraph which contains $e$ and also contains no other conflicting edge with regard to $f$.

Proof. Let $f$ be a minimal colouring of $G$ and let $R = \{e_1, \ldots, e_r\}$ be the set of conflicting edges with regard to $f$. For each $i \in \{1, \ldots, r\}$ let $M_i = \{e_i\}$ and conduct the following process. Choose an edge $e$ not contained in $M_i \cup R$ which is adjacent to some edge in $M_i$. Add edge $e$ to $M_i$. While $r(M_i) = 0$, we keep adding such edges. Since $r(G - (R - \{e_i\}))$ must equal 1, we know that eventually we will have $r(M_i) = 1$. If $r(M_i) = 1$ then $M_i$ is a conflicting subgraph which contains no other conflicting edge with regard to $f$ besides $e_i$. By Proposition 2 (ii), $M_i$ contains a minimal conflicting subgraph. Since $r(M_i - e_i) = 0$, this minimal conflicting subgraph must contain $e_i$. This completes the proof. □

From Proposition 3, it is clear that the resistance of a cubic class two graph is less than or equal to the number of distinct minimal conflicting subgraphs contained in the graph. We may be inclined to think that the number of minimal conflicting subgraphs is in some way upper bounded by resistance, however, this is not the case. The flower snarks and Loupekine snarks represent counter examples to this idea. Each of the graphs in these classes have resistance 2. However, the order of the graphs can be arbitrarily large. Furthermore, the number of possible single vertices which can be removed from the said graphs to leave behind a minimal conflicting subgraph is also arbitrarily large. Thus the number of minimal conflicting subgraphs is not bounded by resistance.

While no such upper bound exists, there does exist an essential relationship between resistance and minimal conflicting subgraphs. This relationship, as is proven in the following theorem, provides much insight into possible conflicting edges and minimal colourings of snarks. First, we present an important definition.
Definition 2. Let $G$ be a subcubic class two graph with minimal conflicting subgraphs $M_1, \ldots, M_r$. A representative conflicting subset of $G$ is a set of distinct edges $R = e_1, \ldots, e_s \subset E(G)$ such that $R \cap M_i \neq \emptyset$ for each $i$.

We note that for a graph $G$ there may exist representative conflicting subsets of varying order.

Theorem 1. Let $G$ be a subcubic class two graph. Then

$$r(G) = \min\{|R| : R \text{ is a representative conflicting subset of } G\}$$

Proof. Let $\mathcal{M} = \{M_1, \ldots, M_m\}$ be the set of all minimal conflicting subgraphs in $G$ and let $R$ be a representative conflicting subset of $G$. Note that no $M_i$ in $\mathcal{M}$ is a subgraph of $G - R$. Assume now that $G - R$ is not 3-edge-colourable. Then $G - R$ contains some minimal conflicting zone $M'$ by Proposition 2 (ii). But $M'$ is also contained in $G$, which is a contradiction since $M'$ is not contained in $\mathcal{M}$. Therefore, $G - R$ is 3-edge-colourable.

Let $|R|$ be minimal. Since $G - R$ is 3-edge-colourable, we know that $r(G) \leq |R|$. Assume that $r(G) < |R|$ and let $f$ be a minimal colouring of $G$. Let $R'$ be the conflicting edges with regard to $f$. By Proposition 3, every element in $R'$ is contained in some minimal conflicting subgraph of $G$. If $R'$ is not a representative conflicting subset then there exists some minimal conflicting subgraph $M' \subset G$ which contains no conflicting edges with regard to $f$. In which case, we have a minimal conflicting subgraph of $G$ which is properly coloured by $f$ using just three colours, a contradiction. If $R'$ is a representative conflicting subset, then the minimality of $|R|$ is contradicted since $|R'| = r(G) < |R|$. Therefore, $r(G) = |R|$.

A cubic graph $G$ may have resistance $r(G)$, but given that information there is no way of knowing which combination of $r(G)$ edges may be removed from $G$ to render colourability. Theorem 1 is significant in that it informs us exactly which combinations of $r(G)$ edges are sufficient for this purpose. The requisite is that we can identify the minimal conflicting subgraphs of $G$. Another way of understanding the result is that we can choose a minimal colouring, relative to conflicting edges, by simply selecting a combination of edges from each minimal conflicting subgraph, as long as this is done minimally.

Furthermore, with Theorem 1 we further note that if there exists some edge $e$ which is contained in exactly one minimal conflicting subgraph $M$, but $M$ has non-empty intersection with some other minimal conflicting subgraph $M'$, then $e$ may not be conflicting in any minimal colouring of $G$. Another way of saying this is, if $e \in M$ where $M$ is a minimal conflicting subgraph of $G$, then it is not necessarily the case that $r(G - e) = r(G) - 1$. Equivalently, we could say that there does not necessarily exist some minimal colouring of $H \subset G$ which can be extended to a minimal colouring of $G$. It is possible to have a minimal colouring of a subgraph $H \subset G$ with say, $r_1$ conflicting edges, such that a minimal extension has $r_2$ further conflicting edges, but $r_1 + r_2 > r(G)$. What is also clear is that if every minimal conflicting subgraph of $G$ is disjoint, then $r(G) = |\mathcal{M}|$, where $\mathcal{M}$ is the set of all minimal conflicting subgraphs in $G$. Consequent to this discussion, we define the following.
Definition 3. Let $G$ be a subcubic graph. Let

$$K_G = \{e \in G \mid f(e) = 0 \text{ for some minimal colouring } f \text{ of } G\}.$$ 

We call $K_G$ the critical subgraph of $G$.

As we did with resistance, we are also able to explicitly characterise the critical subgraph in terms of the minimal conflicting subgraphs.

Theorem 2. Let $G$ be a subcubic class two graph. Then

$$K_G = \bigcup \{R : R \text{ is a representative conflicting subset of } G \text{ of minimal order}\}.$$ 

Proof. Let $R$ be a representative conflicting subset of $G$ with minimal order. Then the edges in $R$ are the conflicting edges of some minimal colouring of $G$. Therefore, the edges in $R$ are all critical.

Let $e$ be a critical edge of $G$. Then it is a conflicting edge in some minimal colouring $f$ of $G$. Let $R$ be the set of all conflicting edges in $G$ with regard to $f$. Since $G - R$ is colourable, $R$ is a representative conflicting subset of $G$. Since $f$ is minimal, $R$ must have a minimal order. Therefore, $e$ is contained in the union of all representative conflicting subsets of $G$ with a minimal order. \[\square\]

It is clear that $K_G \subseteq M_G$. We present an example where $K_G \subset M_G = G$, an example where $K_G = M_G \subset G$, and an example where $K_G = M_G = G$.

The first two examples are specific graphs, while the third is the interesting general case of a hypo-Hamiltonian snark.

Example 1. The subcubic graph $G$ depicted below consists of four identical minimal conflicting subgraphs, $M_1, M_2, M_3$ and $M_4$. Thus $M_G = G$. $M_1 \cap M_2$, $M_2 \cap M_3$ and $M_3 \cap M_4$ are represented by the thicker edges. We have $r(G) = 2$ and $K_G = (M_1 \cap M_2) \cup (M_3 \cap M_4) \subseteq M_G = G$. The sets of two edges, one each from $(M_1 \cap M_2)$ and $(M_3 \cap M_4)$, are the only representative conflicting subsets of minimal order. Thus, even though $K_G = (M_1 \cap M_2) \cup (M_3 \cap M_4)$ is itself 3-edge-colourable, any minimal colouring of $G$ must contain a conflicting edge in each of $(M_1 \cap M_2)$ and $(M_3 \cap M_4)$.

Example 2. The snark $G$ depicted below consists of three identical non-overlapping minimal conflicting subgraphs $M_1, M_2$ and $M_3$. $M_1 \cup M_2 \cup M_3$ is represented by the thicker edges. Any set of three edges, one each from $M_1, M_2$ and
$M_3$, is a representative conflicting subset. Therefore $r(G) = 3$ and $K_G = M_1 \cup M_2 \cup M_3 = M_G \subset G$.

**Example 3.** The graph $G$ depicted below is a hypo-Hamiltonian snark. Let $e$ be any given edge in $G$. Then $e$ is contained in a Hamiltonian cycle of $G - v$ where $v$ is a vertex distance 1 from $e$. In the diagram, $e$ is conflicting in a minimal 4-edge-colouring of $G$ with two conflicting edges. Therefore, $K_G = G$. The 3-coloured chordal edges and the alternatively 1-2 coloured edges in the Hamiltonian cycle are not depicted in the diagram. Since hypo-Hamiltonian snarks are bicritical [8], we note that this implies that every minimal conflicting subgraph of a hypo-Hamiltonian snark contains all but one vertex.

3. Further Considerations

3.1. Clusters

We have noticed that it is typically the case in smaller snarks that minimal conflicting subgraphs have non-empty intersections. To facilitate further brief discussion, it serves to formally define groups of minimal conflicting subgraphs in terms of non-empty intersections, as well as distinguish between different types of these groups.
**Definition 4.** Let $G$ be a subcubic class two graph. Let $\mathcal{M} = \{M_1, \ldots, M_m\}$ be a collection of minimal conflicting subgraphs of $G$.

(i) If for every $i \in \{1, \ldots, m\}$ with $i \neq j$ there exists some $j \in \{1, \ldots, m\}$ such that $M_i \cap M_j \neq \emptyset$, and $M \cap M_i = \emptyset$ for any other minimal conflicting subgraph $M \notin \mathcal{M}$, then we call $\mathcal{M}$ a cluster of minimal conflicting subgraphs.

(ii) If $\mathcal{M}$ is a cluster and $\bigcap M_i \neq \emptyset$ then we call $\mathcal{M}$ a dense cluster.

(iii) If $\mathcal{M}$ is a cluster and is not dense then it is a sparse cluster.

(iv) If $\mathcal{M}$ is sparse cluster such that for every $i, j \in \{1, \ldots, m\}$ we have that $M_i \cap M_j \neq \emptyset$, then it is a densely sparse cluster.

We prove and discuss some immediate results on these structures. Our investigations suggest that it serves to consider strictly subcubic clusters and cubic clusters separately.

**Proposition 4.** The following statements are true.

(i) There exists no cubic dense cluster.

(ii) Let $G$ be a bridgeless cubic graph. If $M_G = G$ then $G$ consists entirely of one sparse cluster of minimal conflicting zones.

(iii) There exists a strictly subcubic cluster with $n$ minimal conflicting subgraphs for each $n \geq 1$.

**Proof.**

(i) Every dense cluster has a representative conflicting subset of order 1. No cubic graph can have resistance 1. Therefore, no dense cluster can be cubic.

(v) Since the distance between any two clusters must be at least one by Proposition 2, that one edge cannot be contained in any minimal conflicting subgraph. Thus $M_G = G$ implies that $G$ consists entirely of one cluster of minimal conflicting subgraphs. By (i), $G$ cannot be dense, and is therefore sparse.

(iii) Consider Example 1, but with $n$ minimal conflicting subgraphs $M_1, \ldots, M_n$. As in Example 1, let $M_i$ intersect with $M_{i+1}$ for $i \in \{1, \ldots, n-1\}$. The result is a strictly subcubic cluster with $n$ minimal conflicting subgraphs.

Now, any cluster with two minimal conflicting subgraphs is trivially dense, and we have seen that we can easily find such clusters. In Proposition 4 (iii), the clusters are however sparse for $n \geq 3$. The question of whether there exists dense clusters with three or more minimal conflicting subgraphs remains.

**Problem 1.** For which $n \geq 3$ does there exist a dense cluster with $n$ minimal conflicting subgraphs?

Recall Example 3, that in a hypo-Hamiltonian snark, the removal of any vertex leaves behind a minimal conflicting subgraph containing all the remaining vertices. Thus any two minimal conflicting subgraphs intersect, and
there is no single vertex which is present in every minimal conflicting subgraph. Hypo-Hamiltonian snarks are therefore densely sparse clusters. From our investigations, we suspect that the only cubic densely sparse clusters are those which are similar to Example 3. That is, possibly hypo-Hamiltonian, and consequently where the removal of any one vertex leaves behind a minimal conflicting subgraph. In such cases as well, resistance is necessarily 2. Recall as well that every edge in a hypo-Hamiltonian snark is critical. Thus, we formulate the following conjecture.

**Conjecture 1.** Let \( G \) be a bridgeless cubic graph. Then \( K_G = G \) if and only if \( r(G) = 2 \) and \( G \) is a densely sparse cluster.

### 3.2. Snark reduction

Although triviality in snarks is not well-defined, snarks have generally been considered to contain more triviality if they are easily reducible to smaller snarks by some well-defined reduction (many variations of snark reductions have been considered by previous authors, (see for example \([6,8]\)). However, there exist many large snarks with arbitrary resistance which can be easily reduced to smaller snarks with less resistance. Reduction of resistance cannot be considered trivial, since some structural complexity is contributing to that resistance being present. Thus we propose instead that snarks which can be reduced in size, without reducing resistance, should be considered more trivial. Given Theorem 1, these would typically be snarks with a large buffer subgraph. Thus we may perhaps formally define a snark to contain no triviality if its buffer subgraph is empty. The notion of a maximal conflicting subgraph and buffer subgraph, therefore, opens up a new avenue of consideration regarding snark reductions. That is, reducing the snark to contain only the essentially uncolourable, by removing the buffer subgraph.

This new notion of reducibility relates interestingly to a problem of oddness and resistance in snarks. Recall that the oddness of a graph \( G \) is the minimum number of odd components in a 2-factor of \( G \), denoted as \( \omega(G) \) \([4]\). In \([1]\) we showed that the ratio of oddness to resistance can be arbitrarily large. We did this by constructing a class of graphs with increasing oddness, but constant resistance. Interestingly, each of the graphs in the class defined contains exactly three disjoint minimal conflicting subgraphs (thus resistance is 3 in each graph). The increase in oddness, whilst keeping resistance constant, is as a result of adding particular subgraphs to the buffer subgraph. This leads to an interesting reformulation of the disproved conjecture which was posed in \([3]\), which states that \( \omega(G) \leq 2r(G) \) for any snark \( G \).

**Conjecture 2.** Let \( G \) be a snark with an empty buffer subgraph. Then \( \omega(G) \leq 2r(G) \).

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