Numerical Solution of Weakly Regular Volterra Integral Equations of the First Kind
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Abstract

The numerical method for solution of the weakly regular scalar Volterra integral equation of the 1st kind is proposed. The kernels of such equations have jump discontinuities on the continuous curves which starts at the origin. The mid-rectangular quadrature rule is employed for the numerical method construction. The accuracy of proposed numerical method is $O(1/N)$.

1 Introduction

This article deals with the following linear weakly regular Volterra integral equation (VIE) of the first kind

$$\int_0^t K(t, s)x(s)ds = f(t), \quad 0 \leq s \leq t \leq T, \quad f(0) = 0,$$

where kernel is defined as follows

$$K(t, s) := \begin{cases} K_1(t, s), & t, s \in m_1, \quad m_i := \{t, s \mid \alpha_{i-1}(t) \leq s < \alpha_i(t)\}, \\ \vdots & \vdots \\ K_n(t, s), & t, s \in m_n, \quad \alpha_0(t) = 0, \quad \alpha_n(t) = t, \ i = 1, n, \end{cases}$$

$\alpha_i(t), f(t) \in C^1_{[0, T]}$, $K_i(t, s)$ have continuous derivatives (w.r.t. $t$) for $t, s \in m_i$, $K_n(t, t) \neq 0$, $\alpha_i(0) = 0, \quad 0 < \alpha_1(t) < \alpha_2(t) < \cdots < \alpha_{n-1}(t) < t, \ \alpha_1(t), \ldots, \alpha_{n-1}(t)$ increase at least in the small neighborhood $0 \leq t \leq \tau, \ 0 < \alpha_1'(0) \leq \cdots \leq \alpha_{n-1}'(0) < 1$.

Such integral equations are in the core of many mathematical models in physics, economics and ecology. The theory of integral models of evolving systems was initiated in the early works of L. Kantorovich, V. Glushkov and R. Solow in the mid-20th Century. Such theory employs the VIEs of the first kind where bounds of the integration interval can be functions of time. It is to be noted that conventional Glushkov integral model of evolving systems is the special case of the VIE (1.1) where all the functions $K_i(t, s)$ are zeros except of $K_n(t, s)$. 
We stress here that the VIE (1.1) is the ill-posed problem. Such weakly regular equations have been introduced in [7]. It is to be noted that solution of the equation (1.1) may contain some arbitrary constants and can be unbounded as \( t \) goes to 0. Indeed, if

\[
K(t, s) = \begin{cases} 
1, & 0 < s < t/2, \\
-1, & t/2 < s < t,
\end{cases}
\]  

(1.2)

\( f(t) = t \), then equation (1.1) has the solution \( x(t) = c - \frac{\ln t}{\ln 2} \), where \( c \) remains free parameter. Numerical solution of the VIE (1.1) based on combinations of the left and right rectangle rules has been discussed by E.V. Markova and D.N. Sidorov in [3].

In this paper for such VIE with jump discontinuous kernels we propose numerical method and discuss the analytical algorithm for construction of the continuous solutions in the following form:

\[
x(t) = \sum_{i=0}^{N} x_i(\ln t)t^i + t^N u(t). 
\]  

(1.3)

Coefficients \( x_i(\ln t) \) are constructed as polynomials on powers of \( \ln t \) and they may depend on certain number of arbitrary constants. \( N \) defines the necessary smoothness of the functions \( K_i(t, s), f(t) \).

Let us make the following notation \( D(t) := \sum_{i=1}^{n-1} |\alpha_i(t)K^{-1}_n(t, t)| |K_i(t, \alpha_i(t)) - K_{i+1}(t, \alpha_i(t))| \) and briefly outline the main results. Here readers may refer to the papers [1, 2].

**Theorem 1.1.** (Sufficient Conditions of Existence & Uniqueness of Local Solution) Let for \( t \in [0, T] \) the following conditions be satisfied: continuous \( K_i(t, s), i = 1, n \), \( \alpha_i(t) \) and \( f(t) \) have continuous derivatives wrt \( t \), \( K_n(t, t) \neq 0 \), \( 0 = \alpha_0(t) < \alpha_1(t) < \cdots < \alpha_{n-1}(t) < \alpha_n(t) = t \) for \( t \in (0, T] \), \( \alpha_i(0) = 0 \), \( f(0) = 0, D(0) < 1 \), then \( \exists \tau > 0 \) such as eq. (1.1) has a unique local solution in \( C_{[0, \tau]} \). Moreover if \( \min_{\tau \leq t \leq T} (t - \alpha_{n-1}(t)) = h > 0 \). Then eq. (1.1) has unique global solution in \( C_{[0, T]} \).

Let us outline the following conditions.

**A.** Exists polynomial \( K_i^M(t, s) = \sum_{\nu+\mu=0}^{M} K_{\nu+\mu} t^\nu s^\mu, i = 1, n \), \( f^M(t) = \sum_{\nu=1}^{M} f_\nu t^\nu \), \( \alpha_i^M(t) = \sum_{\nu=1}^{M} \alpha_{i\nu} t^\nu, i = 1, n - 1 \), where \( 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_{n-1} < 1 \) such as for \( t \rightarrow +0, s \rightarrow +0 \) the following estimates hold: \( |K_i(t, s) - K_i^M(t, s)| = O((t + s)^{M+1}), i = 1, n \), \( f(t) - f^M(t) = O(t^{M+1}) \), \( |\alpha_i(t) - \alpha_i^M(t)| = O(t^{M+1}), i = 1, n - 1 \).

**B.** For fixed \( q \in (0, 1) \), \( \exists \tau \in (0, T], 0 < \varepsilon < 1 : \max_{t \in [0, \tau]} \varepsilon^M D(t) \leq q < 1 \).
C. Exists $N^*$ such as \( \lim_{t \to 0} \left( \frac{\int_0^t K(t,s)\hat{x}(s)ds - f(t)}{t^{N^*}} \right) = 0. \)

**Theorem 1.2. (Regularization)** Let $\hat{x}(t)$ be the known function such as the condition C is true for $N^* \geq M$. Then eq. (1) has the solution $x(t) = \hat{x}(t) + t^{N^*}u(t)$, where $u(t) \in C[0,T]$ is unique and can be constructed by means of the successive approximations method from the equation $\int_0^t K(t,s)s^N u(s)ds = g(t)$, where $g(t) := -\int_0^t K(t,s)\hat{x}(s)ds + f(t)$.

**Theorem 1.3. (Relaxed Sufficient Condition for Existence & Uniqueness)** Let conditions B and C be satisfied, and $B(j) := K_n(0,0) + \sum_{i=1}^{n-1} (\alpha_i'(0))^{1+i}(K_i(0,0) - K_{i+1}(0,0)) \neq 0$ for $j \in \mathbb{N} \cup \{0\}$. Then eq. (1.1) has unique solution $x(t) = x^M(t) + t^{N^*}u(t)$ in $C[0,T]$, $M \geq N$. Moreover, for $t \to +0$ polynomial $\hat{x}(t) \equiv x^M(t) = \sum_{i=0}^{M} x_i t^i$ is an $M$th order asymptotic approximation of such solution.

The paper is organized as follows. In section 2 we propose the numerical method for solution of the VIE (1.1). In section 3 we demonstrate the efficiency of proposed numerical method on synthetic data. As footnote, we outline the final conclusions in section 4.

## 2 Numerical method

In this section we propose the generic numerical method for weakly singular Volterra integral equations (1.1) based on the mid-rectangular quadrature rule. The accuracy of proposed numerical method is $O(1/N)$. Section 3 illustrates concepts and results of proposed numerical method on synthetic data.

For numerical solution of the equation (1.1) on the interval $[0,T]$ we introduce the following mesh (the mesh can be non-uniform)

$$0 = t_0 < t_1 < t_2 < \ldots < t_N = T, \quad h = \max_{i=1,N}(t_i - t_{i-1}) = O(N^{-1}). \quad (2.1)$$

Let us search for the approximate solution of the equation (1.1) as follows

$$x_N(t) = \sum_{i=1}^{N} x_i \delta_i(t), \quad t \in (0,T), \quad \delta_i(t) = \begin{cases} 1, & \text{for } t \in \Delta_i = (t_{i-1}, t_i]; \\ 0, & \text{for } t \notin \Delta_i \end{cases} \quad (2.2)$$

with coefficients $x_i$, $i = 1, N$ are under determination.
In order to find \( x_0 = x(0) \) we differentiate both sides of the equation (1.1) wrt \( t \):

\[
\begin{align*}
  f'(t) &= \sum_{i=1}^{n} \left( \int_{\alpha_{i-1}(t)}^{\alpha_i(t)} \frac{\partial K_i(t, s)}{\partial t} x(s)ds + \alpha_i'(t)K_i(t, \alpha_i(t))x(\alpha_i(t)) - \\
  &\quad - \alpha_{i-1}'(t)K_i(t, \alpha_{i-1}(t))x(\alpha_{i-1}(t)) \right).
\end{align*}
\]

Therefore

\[
x_0 = \frac{f'(0)}{\sum_{i=1}^{n} K_i(0, 0) [\alpha_i'(0) - \alpha_{i-1}'(0)]}.
\]

(2.3)

Here we assume that conditions of the Theorem 1.1 are satisfied and \( \sum_{i=1}^{n} K_i(0, 0) [\alpha_i'(0) - \alpha_{i-1}'(0)] \neq 0 \).

Let’s make the notation \( f_k := f(t_k), \ k = 1, \ldots, N \). In order to define the coefficient \( x_1 \) we rewrite the equation in \( t = t_1 \):

\[
\sum_{i=1}^{n} \int_{\alpha_{i-1}(t_1)}^{\alpha_i(t_1)} K_i(t_1, s)x(s)ds = f_1.
\]

(2.4)

It is to be noted that the lengths of all the segments of integration \( \alpha_i(t_1) - \alpha_{i-1}(t_1) \) in (2.4) are less or equal to \( h \) and an approximate solution is \( x_1 \) then application of the mid-rectangular quadrature rule yields

\[
x_1 = \frac{f_1}{\sum_{i=1}^{n} (\alpha_i(t_1) - \alpha_{i-1}(t_1))K_i(t_1, \frac{\alpha_i(t_1)+\alpha_{i-1}(t_1)}{2})}.
\]

(2.5)

The mesh point of the mesh (2.1) which coincide with \( \alpha_i(t_j) \) we denote as \( v_{ij} \), i.e. \( \alpha_i(t_j) \in \Delta_{v_{ij}} \). Obviously \( v_{ij} < j \) for \( i = 0, n-1, j = 1, N \). It is to be noted that \( \alpha_i(t_j) \) are not always coincide with any mesh point. Here \( v_{ij} \) is used as index of the segment \( \Delta_{v_{ij}} \), such as \( \alpha_i(t_j) \in \Delta_{v_{ij}} \) (or its right-hand side).

Let us now assume the coefficients \( x_0, x_1, \ldots, x_{k-1} \) be known. Equation (1.1) defined in \( t = t_k \) as

\[
\sum_{i=1}^{n} \int_{\alpha_{i-1}(t_k)}^{\alpha_i(t_k)} K_i(t_k, s)x(s)ds = f_k,
\]

we can rewrite as follows: \( I_1(t_k) + I_2(t_k) + \cdots + I_n(t_k) = f_k \), where

\[
I_1(t_k) := \sum_{j=1}^{v_{i,j-1}} \int_{t_{j-1}}^{t_j} K_1(t_k, s)x(s)ds + \int_{t_{i,j-1}}^{t_{i,j}} K_1(t_k, s)x(s)ds,
\]

\[
I_2(t_k) := \sum_{j=1}^{v_{i,j-1}} \int_{t_{j-1}}^{t_j} K_2(t_k, s)x(s)ds + \int_{t_{i,j-1}}^{t_{i,j}} K_2(t_k, s)x(s)ds,
\]

\[
\vdots
\]

\[
I_n(t_k) := \sum_{j=1}^{v_{i,j-1}} \int_{t_{j-1}}^{t_j} K_n(t_k, s)x(s)ds + \int_{t_{i,j-1}}^{t_{i,j}} K_n(t_k, s)x(s)ds.
\]
\[ I_n(t_k) := \int_{\alpha_{n-1}(t_k)}^{t_{v_{n-1,k}}} K_n(t_k, s)x(s) \, ds + \sum_{j=v_{n-1,k}+1}^{k} \int_{t_j}^{t_{v_{n-1,k}+1}} K_n(t_k, s)x(s) \, ds. \]

1. If \( v_{p-1,k} \neq v_{p,k} \), \( p = 2, \ldots, n-1 \) then
\[
I_p(t_k) := \int_{\alpha_{p-1}(t_k)}^{t_{v_{p-1,k}}} K_p(t_k, s)x(s) \, ds + \sum_{j=v_{p-1,k}+1}^{v_{p,k}-1} \int_{t_j}^{t_{v_{p-1,k}+1}} K_p(t_k, s)x(s) \, ds + \]
\[ + \int_{t_{v_{p,k}-1}}^{\alpha_p(t_k)} K_p(t_k, s)x(s) \, ds. \]

2. If \( v_{p-1,k} = v_{p,k} \) then
\[
I_p(t_k) := \int_{\alpha_{p-1}(t_k)}^{\alpha_p(t_k)} K_p(t_k, s)x(s) \, ds.
\]

**Remark 1.** The number of terms in each line of the last formula depends on an array \( v_{ij} \), defined using the input data: functions \( \alpha_i(t) \), \( i = 1, n - 1 \), and fixed (for specific \( N \)) mesh.

Each integral term we approximate using the mid-rectangular quadrature rule, e.g.
\[
\int_{t_{v_{p,k}-1}}^{\alpha_p(t_k)} K_p(t_k, s)x(s) \, ds \approx (\alpha_p(t_k) - t_{v_{p,k}-1}) K_p \left( t_k, \frac{\alpha_p(t_k) + t_{v_{p,k}-1}}{2} \right) x_N \left( \frac{\alpha_p(t_k) + t_{v_{p,k}-1}}{2} \right).
\]

Moreover, on those intervals where the desired function has been already determined, we select \( x_N(t) \) (i.e. \( t \leq t_{k-1} \)).

On the rest of the intervals an unknown value \( x_k \) appears in the last terms. We explicitly define it and proceed in the loop for \( k \). The number of these terms is determined from the initial data \( v_{ij} \) analysis. The accuracy of the numerical method is \( O(\frac{1}{N}) \).

### 3 Numerical illustrations

Let us consider three examples. In all cases the uniform mesh is used.

**Example 3.1.**
\[
\int_{0}^{t/3} (1 + t - s)x(s) \, ds - \int_{t/3}^{t} x(s) \, ds = \frac{t^4}{108} - \frac{25t^3}{81}, \ t \in [0, 2],
\]
Table 1: Errors for the 1st Example.

| $h$  | $\varepsilon$          |
|------|------------------------|
| 1/32 | 0.13034091293670258    |
| 1/64 | 0.07804538180930365    |
| 1/128| 0.03989003750757547    |
| 1/256| 0.01975354947865071    |
| 1/512| 0.010027923872257816   |
| 1/1024| 0.005083865773485741  |
| 1/2048| 0.002569318297446435   |
| 1/4096| 0.001288983987251413   |
| 1/8192| 0.0006500302042695694  |

exact solution is $\bar{x}(t) = t^2$. Tab. 3.1 demonstrates the errors $\varepsilon = \max_{0 \leq i \leq n} |\bar{x}(t_i) - x^h(t_i)|$ for various steps $h$.

Example 3.2.

$$\int_{0}^{\frac{1}{2}} (1 + t - s)x(s) \, ds - \int_{\frac{4t}{9}}^{\frac{2t}{9}} x(s) \, ds - 2 \int_{\frac{2t}{9}}^{\frac{4t}{9}} x(s) \, ds + \int_{\frac{4t}{9}}^{t} x(s) \, ds = \frac{11 t^4}{26244} + \frac{547 t^3}{2187}, \; t \in [0, 2],$$

exact solution is $\bar{x}(t) = t^2$. Tab. 3.2 demonstrates the errors $\varepsilon$ for various steps $h$.

Example 3.3.

$$2 \int_{0}^{\frac{\sin \frac{t}{2}}{\sin \frac{t}{3}}} x(s) \, ds - \int_{\frac{2 \sin \frac{t}{3}}{\sin \frac{t}{2}}}^{\frac{2 \sin \frac{t}{3}}{\sin \frac{t}{2}}} x(s) \, ds + \int_{\frac{2 \sin \frac{t}{3}}{\sin \frac{t}{2}}}^{t} x(s) \, ds = \frac{t^3}{3} + \sin \frac{t}{2} - \frac{16}{3} \sin \frac{t}{3} \sin \frac{t}{3}, \; t \in \left[0, \frac{3\pi}{2}\right],$$

exact solution is $\bar{x}(t) = t^2$. Tab. 3.3 demonstrates the errors $\varepsilon$ for various steps $h$.

4 Conclusion

In this article we addressed the novel class of weakly regular linear Volterra integral equations of the first kind first introduced in [7]. We outlined the main results for this class of equation previously derived. The main contribution of this paper is a generic numerical method designed
Table 2: Errors for the 2nd Example.

| $h$   | $\varepsilon$       |
|-------|---------------------|
| 1/32  | 0.13718808476353672 |
| 1/64  | 0.07408554651043886 |
| 1/128 | 0.04531351578371812 |
| 1/256 | 0.022111520501482573|
| 1/512 | 0.011079518173630731|
| 1/1024| 0.005492567505257284|
| 1/2048| 0.0027453216364392574|
| 1/4096| 0.0014125244842944085|
| 1/8192| 0.00077170109943836  |

for solution of such weakly regular equation. The numerical method employes the mid-point quadrature rule and enjoy the the $O(1/N)$ order of accuracy. The illustrative examples demonstrate the efficiency of proposed method. As footnote let us outline that proposed approach enable construction of the 2nd order accurate numerical method. This improvement will be done in our further works.

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Table 3: Errors for the 3rd Example.

| $h$   | $\varepsilon$       |
|-------|----------------------|
| 1/32  | 1.2810138805937967   |
| 1/64  | 0.7064105257311724   |
| 1/128 | 0.3172969521937503   |
| 1/256 | 0.16990268475221626  |
| 1/512 | 0.1178708722029413   |
| 1/1024| 0.07940422358498633  |
| 1/2048| 0.06518995509284764  |
| 1/4096| 0.06004828109245386  |
| 1/8192| 0.046102790104048275 |

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