Minimal length in quantum gravity and gravitational measurements

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Abstract – The existence of a minimal length is a common prediction of various theories of quantum gravity. This minimal length leads to a modification of the Heisenberg uncertainty principle to a Generalized Uncertainty Principle (GUP). Various studies showed that a GUP modifies the Hawking radiation of black holes. In this paper, we propose a modification of the Schwarzschild metric based on the modified Hawking temperature derived from the GUP. Based on this modified metric, we calculate corrections to the deflection of light, time delay of light, perihelion precession, and gravitational redshift. We compare our results with gravitational measurements to set an upper bound on the GUP parameter.

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Introduction. – Various approaches to quantum gravity (QG) are expected to play a crucial role in revealing some characteristic features of the fundamental quantum theory of gravity. One common feature among most of these approaches, such as string theory and black-hole physics [1–7], is the existence of a minimal observable length, i.e. the Planck length \( l_p \). The existence of a minimal length leads to the modification of the Heisenberg uncertainty principle to a Generalized Uncertainty Principle (GUP). Various studies showed that a GUP modifies the Hawking radiation of black holes. In this paper, we propose a modification of the Schwarzschild metric based on the modified Hawking temperature derived from the GUP. Based on this modified metric, we calculate corrections to the deflection of light, time delay of light, perihelion precession, and gravitational redshift. We compare our results with gravitational measurements to set an upper bound on the GUP parameter.

\[ [x_i, p_j] = i\hbar \left[ \delta_{ij} - \alpha \left( p\delta_{ij} + \frac{p_ip_j}{p} \right) + \alpha^2 \left( p^2 \delta_{ij} + 3p_ip_j \right) \right], \]

where \( \alpha = \alpha_0 l_p/\hbar \), and \( \alpha_0 \) is a dimensionless constant. The upper bounds on the parameter \( \alpha_0 \) have been calculated in [13] and it was proposed that GUP may introduce an intermediate length scale between the Planck scale and the electroweak scale. Recent proposals suggested that these bounds can be measured using quantum optics techniques in [16] and using gravitational wave techniques [17,18] which may be considered as a milestone in quantum gravity phenomenology. In a series of papers, various phenomenological implications of the new model of GUP were investigated [19–24]. A detailed review along the mentioned lines of minimal length theories and quantum gravity phenomenology can be found in [25–27].
Very recently, Scardigli and Casadio [28] proposed a modification of the Schwarzschild metric to reproduce the modified Hawking temperature [29–33] which was derived from the GUP of eq. (1). This modification of the metric takes the form

\[ dr^2 = F(r)dt^2 - \frac{1}{F(r)}dr^2 - r^2d\Omega^2, \quad (3) \]

with

\[ F(r) = 1 - \frac{2GM}{r} + \frac{G^2M^2}{r^2}, \quad (4) \]

and from comparing the Hawking temperature derived from the GUP with the one derived from the modified metric they concluded that \( \beta = -\pi^2 \epsilon^2 M^2 / 4M_p^2 \), where \( M_p \) is the Planck mass. There is a problem in the form of \( F(r) \) in eq. (4), which implies that the horizon is at a different value from \( r_s = 2GM \) contrary to many arguments based on the GUP [2,7,34]. More importantly, it leads to a negative GUP parameter \( \beta \) < 0 (see footnote 1), which is inconsistent with almost all the motivations that led to the GUP [1–7]; the GUP implies a minimal length \( \Delta x \) (see eq. (15) below). We do not assume a modification form for \( F(r) \); rather, we look for the metric that reproduces the modified Hawking temperature (5), we see that

\[ \Delta x \geq h/\beta \]

would be imaginary if \( \beta \) is negative. In this paper, we continue our investigations of the phenomenological implications of GUP that was studied in [13]. We propose a modification to the Schwarzschild metric to reproduce the modified Hawking temperature derived from the GUP in eq. (2). We use a more general form for \( F(r) \) than the one used by Scardigli and Casadio (see eq. (15) below). We do not assume a modification with \( 1/r^2 \)-dependence. Instead, we consider a metric with a general \( 1/r^n \)-dependence. In addition, this form leads to a horizon at the usual value \( r_s = 2GM \), and yields a positive GUP parameter.

In the following sections, we review the derivation of the modified Hawking temperature from the GUP, and find the relation between the GUP parameter \( \alpha \) and the metric. Then, we use this metric to find corrections to the general relativistic results of the deflection of light, time delay of light, perihelion precession, and gravitational redshift. We compare our results with experiment to set upper bounds on the GUP parameter \( \alpha_0 \).

**Hawking temperature from GUP.** – The Hawking temperature of a black hole takes the well-known form [36]

\[ T_H = \frac{1}{8\pi GM}, \quad (5) \]

where, from now on, we use natural units, in which \( c = 1 \), \( \hbar = 1 \), \( G = 6.708 \times 10^{-39} \text{GeV}^{-2} \) and \( l_p = \sqrt{G} = 8.19 \times 10^{-33} \text{GeV}^{-1} \).

The GUP modifies the Hawking temperature; that modification was derived in several papers for different forms of the GUP [29–33,37–39]. We will follow the derivation in [37–39]. We start by rewriting eq. (2) as [37]

\[ \Delta x \Delta p \geq \frac{1}{2} \left[ 1 - \alpha \Delta p + 4\epsilon^2 (\Delta p)^2 \right]. \quad (6) \]

Solving eq. (6) for \( \Delta p \) and expanding to second order in \( \alpha \), we get a momentum uncertainty of

\[ \Delta p \geq \frac{1}{2\Delta x} - \frac{1}{4\Delta x^2} + \alpha^2 \frac{5}{8\Delta x^3}. \quad (7) \]

According to [29,31,40], a photon is used to ascertain the position of a quantum particle of energy \( E \) and according to the argument in [41] which demonstrated that the uncertainty principle \( \Delta p \geq 1/\Delta x \) can be written as a lower bound \( E \geq 1/\Delta x \). The uncertainty principle \( \Delta p \Delta x \geq 1/2 \) leads to an energy uncertainty of \( \Delta E \geq 1/2\Delta x \). Similarly, from eq. (7) we get

\[ \Delta E \geq \frac{1}{2\Delta x} - \alpha \frac{1}{4\Delta x^2} + \alpha^2 \frac{5}{8\Delta x^3}. \quad (8) \]

The energy uncertainty can be viewed as the energy of the emitted photon from the black hole, and thus as its characteristic temperature \( T = \Delta E \). Taking the uncertainty in position to be proportional to the Schwarzschild radius \( \Delta x = \mu_r s = 2\mu GM \) gives the temperature

\[ T \simeq \frac{1}{4\mu GM} - \alpha \frac{1}{16\mu^2 G^2 M^2} + \alpha^2 \frac{5}{64\mu^3 G^3 M^3}. \quad (9) \]

Comparing this equation with the standard Hawking temperature (5), we see that \( \mu = 2\pi \) and the temperature from the GUP is

\[ T \simeq \frac{1}{8\pi GM} \left( 1 - \alpha \frac{1}{8\pi GM} + 5 \left( \frac{\alpha}{8\pi GM} \right)^2 \right). \quad (10) \]

**Modified Schwarzschild metric.** – The standard Hawking temperature can be derived from the metric using the surface gravity \( \kappa \)

\[ T_H = \frac{\kappa}{2\pi} \quad (11) \]

where \( \kappa \) is related to the metric by ([42], p. 246)

\[ \kappa = \lim_{\epsilon \to r_s} \sqrt{ -g^{rr} g^{\alpha \alpha} g_{r,\alpha} \epsilon^2}. \quad (12) \]

For the Schwarzschild metric, eq. (3), the surface gravity is simply half the derivative of \( F(r) \) at the Schwarzschild radius

\[ \kappa = \frac{1}{2} F'(r_s). \quad (13) \]

Using \( F(r) = 1 - 2GM/r \), we get the standard Hawking temperature

\[ T_H = \frac{1}{4\pi} F'(r_s) = \frac{1}{8\pi GM}. \quad (14) \]

We can follow the same argument backwards; start from the modified temperature (10) and look for the metric that reproduces it. We assume the metric takes the same form as eq. (3) but \( F(r) \) is modified to

\[ F(r) = \left( 1 - \frac{2GM}{r} \right) \left( 1 + \eta \left( \frac{2GM}{r} \right)^n \right), \quad (15) \]

Ahmed Farag Ali et al.
where $\eta$ is a constant $\ll 1$, and $n$ is an integer $\geq 0$. Differentiating $F(r)$ at $r = 2GM$ we get the temperature
\[ T = \frac{1}{4\pi} F'(r_s) = \frac{1 + \eta}{8\pi GM}, \] (16)
which must equal the temperature in eq. (10)
\[ \frac{1 + \eta}{8\pi GM} = \frac{1}{8\pi GM} \left(1 - \frac{\alpha}{8\pi GM} + 5 \left(\frac{\alpha}{8\pi GM}\right)^2\right). \] (17)
Solving for $\eta$
\[ \eta = -\frac{\alpha}{8\pi GM} + 5 \left(\frac{\alpha}{8\pi GM}\right)^2, \] (18)
and to first order in $\alpha$ the metric is modified by the function
\[ F(r) = \left(1 - \frac{2GM}{r}\right) \left(1 - \frac{l_p\alpha_0}{4\pi \frac{(2GM)^{n-1}}{r^n}}\right), \] (19)
where we used $\alpha = l_p\alpha_0$.

A couple of comments are in order here. First, it is easily seen from eq. (19) that if one selects the dimensionless constant $\alpha_0$ to be positive (and thus $\eta$ is negative), then the metric in eq. (19) will have two horizons as is the case for the modified metric in [28]. However, if one selects the dimensionless constant $\alpha_0$ to be negative (and thus $\eta$ is positive), in this case the metric in eq. (19) will have only one horizon, i.e., $r_s = 2GM$. Second, the value of $n$ is still undetermined. In the next section, we will determine the value of $n$ from the modified Newton’s law.

**Modified Newton’s law.** – In this section, we calculate the modified Newton’s law from the modified metric, and we will follow the derivation of gravitational acceleration in [43], pp. 3–32. For a mass falling radially from rest at $r_0$, $d\tau^2 = F(r_0)dt^2$; thus, its energy is
\[ E = F(r) \frac{dt}{d\tau} = F(r_0). \] (20)
Substituting $dr$ from the previous equation in the metric, eq. (3) with $d\Omega = 0$, and solving for $dr/d\tau$
\[ \frac{dr}{d\tau} = F(r) \sqrt{1 - \frac{F(r)}{F(r_0)}}. \] (21)
The proper time and length experienced by a static observer on a spherical shell of radius $r$ is given by
\[ dt_{sh} = F(r)dt, \quad dr_{sh} = \frac{dr}{F(r)}. \] (22)
Thus,
\[ \frac{dr_{sh}}{dt_{sh}} = \sqrt{1 - \frac{F(r)}{F(r_0)}}. \] (23)
Differentiating with respect to $t_{sh}$ and substituting $r = r_0$ we get the acceleration
\[ g = \frac{d^2r_{sh}}{dt_{sh}^2} = -\frac{1}{2\sqrt{F(r)}} F'(r_0), \] (24)
where $F'(r_0)$ is the derivative of $F(r)$ with respect to $r$ evaluated at $r_0$. Substituting the modified function $F(r)$ and expanding to first order in $\alpha_0$, we get
\[ g = \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1/2} \times \left[1 - \frac{(2GM)^{n-2}}{4\pi n^2} \left((1 + 2\alpha GM - nr)l_p\alpha_0\right)\right], \] (25)
which reduces to the standard relativistic result when $\alpha = 0$, and to the Newtonian result after neglecting the relativistic factor $1/\sqrt{1 - 2GM/r}$.

Thus, the modified Schwarzschild metric leads to the modified Newton’s law
\[ F_N = \frac{GMm}{r^2} \left[1 - \frac{(2GM)^{n-2}}{4\pi n^2} \left((1 + 2n)GM - nr\right)l_p\alpha_0\right]. \] (26)
It should be pointed out here that we have neglected the relativistic factor since we would like to keep the GUP correction terms and the other terms to be up to second order in $GM/r$.

To estimate the value of $n$, we compare our result with phenomenologically well motivated approaches that modify Newton’s law of gravity at short distance such as Randall-Sundrum II model [44] which implies a modification of Newton’s law on a brane [45] as follows:
\[ F_{RS} = \begin{cases} \frac{GMm}{r^2} \left(1 + \frac{4\Lambda_R}{3\pi r^2}\right), & r < \Lambda_R, \\ \frac{GMm}{r^2} \left(1 + \frac{2\Lambda_R}{3\pi r^2}\right), & r \gg \Lambda_R, \end{cases} \] (27)
where $\Lambda_R$ is a characteristic length scale.

When $n = 2$ in the modified Newton’s law of eq. (26) we get to first order in $1/r$
\[ F_N = \frac{GMm}{r^2} \left(1 + \frac{l_p\alpha_0}{2\pi r}\right), \] (28)
which clearly agrees with the Randall-Sundrum II result. We conclude that the most likely value for $n$ is 2 and thereon we set $n = 2$. Thus, the function $F(R)$ in the modified metric takes the form
\[ F(r) = \left(1 - \frac{2GM}{r}\right) \left(1 - \frac{l_p\alpha_0 GM}{2\pi \frac{1}{r^2}}\right). \] (29)
A couple of comments are in order here. First, we obtain these corrections in the framework of semiclassical gravity approach. Thus, we keep the LHS of the Einstein equations as it is and we assign the QG corrections to the RHS of the Einstein equations. Therefore, using Mathematica and employing the metric element given by eq. (29), the RHS of Einstein equations is non-zero, as expected. In addition, the RHS of Einstein Equations has only diagonal terms and all of them are perturbation terms of the first order in the perturbative parameter which reads
\[ \epsilon = \frac{l_p\alpha_0}{2\pi}. \]
The non-zero diagonal terms that appear in the RHS of Einstein equations are of the form

\[ G^t_t = G^r_r = \pm \epsilon \frac{GM(4GM - r)}{r^5}, \]

\[ G^\phi_\phi = G_\phi^\phi = -\epsilon \frac{GM(6GM - r)}{r^5}. \]

Therefore, these GUP corrections can be treated as first order perturbation terms around the vacuum solution, i.e., Schwarzschild solution, and the proposed solution can be considered a solution of the Einstein equations in a perturbative sense.

Second, it is evident that the specific expression for the metric element \( F(r) \) implies the existence of another horizon at

\[ r = \sqrt{\ell_p \alpha_0 GM/2\pi}, \]  

\[ (30) \]

which will be an inner horizon. Thus, the spacetime contains an inner and outer horizon, as well as a timelike singularity and the conformal structure should be the same as that of a Reissner-Nordstrom black hole. This implies the existence of a Cauchy horizon which in turn leads to mass inflation.

**Deflection of light.** — When light approaches a massive body, such as the Sun, it gets deflected from a straight line by an angle given by ([46], p. 189)

\[ \Delta \phi = 2 \int_{r_0}^{\infty} \frac{1}{r F(r)} \left( \frac{r^2 F(r_0)}{r_0^2} - 1 \right)^{-1/2} \, dr - \pi, \]  

\[ (31) \]

where \( r_0 \) is the distance of closest approach to the Sun. In general relativity the deflection angle is given by ([46], p. 190)

\[ \Delta \phi_{GR} \simeq \frac{4GM}{r_0}. \]  

\[ (32) \]

To find the deflection angle predicted by the modified metric, we need to use \( F(r) \) from eq. (29) in eq. (31). To simplify the calculations, we make the transformation \( \alpha = r_0/r \) in eq. (31)

\[ \Delta \phi = 2 \int_0^1 \frac{1}{\sqrt{F \left( \frac{r_0}{\alpha} \right)}} \left( \frac{F(r_0)}{F \left( \frac{r_0}{\alpha} \right)} - u^2 \right)^{-1/2} \, du - \pi. \]  

\[ (33) \]

To simplify the integral, we expand the integrand to first order in \( \alpha_0 \) and to second order in \( 1/r_0 \)

\[ \Delta \phi = \Delta \phi_{GR} + \int_0^1 \frac{u^2 + 1}{2\pi r_0 \sqrt{1 - u^2}} \frac{GM}{r_0} \ell_p \alpha_0 \, du \]  

\[ (34) \]

which evaluates in terms of the gamma function to

\[ \Delta \phi = \Delta \phi_{GR} + \frac{\Gamma \left( \frac{3}{2} \right)}{2\sqrt{\pi}} \frac{GM}{r_0^3} \ell_p \alpha_0. \]  

\[ (35) \]

The best accuracy of measuring the deflection of light by the Sun is from measuring the deflection of radio waves from distant quasars using the Very Long Baseline Array (VLBA) [47], which achieved an accuracy of \( 3 \times 10^{-4} \); thus,

\[ \frac{\Delta \phi}{\Delta \phi_{GR}} < 3 \times 10^{-4}. \]  

\[ (36) \]

Assuming that light grazes the surface of the Sun \( r_0 \simeq R_G \approx 6.96 \times 10^8 \text{m} = 3.53 \times 10^{24} \text{GeV}^{-1} \) and \( M = M_\odot = 1.99 \times 10^{30} \text{kg} = 1.116 \times 10^{57} \text{GeV} \), we get an upper bound on \( \alpha_0 \) of

\[ \alpha_0 < 1.4 \times 10^{41}. \]  

\[ (37) \]

This bound is larger than the bound set by the electroweak scale \( 10^{17} \) but not incompatible with it. However, studying the effects of the GUP, which is model independent, on gravitational phenomena might prove useful in understanding the effects of quantum gravity in that regime.

**Time delay of light.** — In general relativity, the time taken by light to travel from \( r = r_1 \) to \( r = r_2 \) passing by a massive body, such as the Sun, is slightly longer than what is expected in flat spacetime, and the time of the round trip is given by

\[ T = 2 \left( t(r_1, r_0) + t(r_2, r_0) \right), \]  

\[ (38) \]

with ([46], p. 202)

\[ t(r, r_0) = \int_{r_0}^{r} \frac{1}{F(r)} \left( 1 - \frac{F(r)}{F(r_0)} \right)^{-1/2} \, dr, \]  

\[ (39) \]

and \( r_0 \) is the distance of closest approach to the Sun. General relativity predicts a time of travel ([46], p. 203)

\[ T_{GR} = 2\sqrt{r_1^2 - r_0^2} + 2\sqrt{r_2^2 - r_0^2} + 4GM \left( 1 + \ln \left( \frac{4\pi r_2 r_0}{r_0} \right) \right). \]  

\[ (40) \]

Using the modified function \( F(r) \), we apply the transformation \( u \equiv r_0/r \) in eq. (39)

\[ t(r_1, r_0) = \int_{r_0}^{1} \frac{r_0}{u^2 F \left( \frac{r_0}{u} \right)} \left( 1 - u^2 \frac{F \left( \frac{r_0}{u} \right)}{F(r_0)} \right)^{-1/2} \, du. \]  

\[ (41) \]

Expanding the integrand to first order in \( \alpha_0 \) and \( M/r_0 \) we get

\[ \Delta T = \Delta T_{GR} + \left[ \int_{r_0}^{1} \frac{du}{\sqrt{1 - u^2}} + \int_{r_0}^{1} \frac{du}{\sqrt{1 - u^2}} \right] \frac{3GM \ell_p \alpha_0}{2\pi r_0} \]  

\[ \Delta T_{GR} + \left[ \pi - \sin^{-1} \frac{r_0}{r_1} - \sin^{-1} \frac{r_0}{r_2} \right] \frac{3GM \ell_p \alpha_0}{2\pi r_0}. \]  

\[ (42) \]

The best accuracy of measuring the delay was obtained from the delay in the travel time of radio waves from Earth to the Cassini spacecraft [48] when it was at a geocentric distance of \( 8.43 \text{AU} = 6.39 \times 10^{27} \text{GeV}^{-1} \), and the closest distance of the photons to the Sun
was \( r_0 = 1.6R_\odot = 5.64 \times 10^{24}\text{ GeV}^{-1} \). The experiment achieved an accuracy of \( 2.3 \times 10^{-5} \), which means that
\[
\frac{\delta \Delta T}{\Delta T_{GR}} < 2.3 \times 10^{-5}.
\] (43)

setting an upper bound on \( \alpha_0 \) of
\[
\alpha_0 < 5.8 \times 10^{40}
\] (44)

which is slightly less than the bound from the deflection of light but still compatible with the bound set by electroweak scale.

**Perihelion precession.** – In general relativity, the orbit of a particle around a massive body, such as Mercury around the Sun, precesses in each revolution by an angle given by \([46], \text{p. } 195\)
\[
\Delta \phi = 2\Delta \phi_{GR} = \frac{\pi}{2} \int_{r_-}^{r_+} \frac{\rho(r_-)}{\rho(r_+)} \frac{dr}{\sqrt{F(r)}}
\]
\[
= -2\pi + 2 \int_{r_-}^{r_+} \frac{dr}{\sqrt{F(r)}} \left[ \frac{r^2 - r_+^2}{r_+^2 r_-^2} \left( \frac{F(r_+)}{F(r_-)} - \frac{F(r_-)}{F(r_+)} \right) - \frac{u^2}{r^2} \right]^{-\frac{1}{2}}
\] (45)

where \( r_- \) and \( r_+ \) are the minimum and maximum values of \( r \), respectively. The precession predicted by general relativity is given by \([46], \text{p. } 197\)
\[
\Delta \phi_{GR} \simeq \frac{3\pi GM}{r_+ - r_-}
\] (46)

Using the modified function \( F(r) \), we change the integration variable in eq. (45) to \( u \equiv r_- / r \)
\[
\Delta \phi = -2\pi + 2 \int_{r_- / r}^{1} \frac{du}{\sqrt{F(u)}}
\]
\[
\times \left[ \frac{r^2 - r_+^2}{r_+^2 r_-^2} \left( \frac{F(r_+)}{F(r_-)} - \frac{F(r_-)}{F(r_+)} \right) - \frac{u^2}{r^2} \right]^{-\frac{1}{2}}
\] (47)

We expand the integrand to first order in \( \alpha \) and to first order in \( (1/r_+ + 1/r_-) \) to obtain
\[
\Delta \phi = \Delta \phi_{GR} + \frac{1}{4\pi} \int_{r_- / r}^{1} \frac{\left( r_+^2 - r_-^2 \right) \frac{d\alpha_0}{F(r)}}{\sqrt{F(r)}} du
\]
\[
= \Delta \phi_{GR} + \frac{1}{4} \left( \frac{1}{r_+} + \frac{1}{r_-} \right) \frac{d\alpha_0}{F(r_+)}
\] (48)

The best measurement of the perihelion precession of Mercury is \( 42.980 \pm 0.002 \) as per century \([49,50]\), which amounts to an accuracy of \( 4.6 \times 10^{-5} \). Thus, we have
\[
\frac{\delta \Delta \phi}{\Delta \phi_{GR}} < 4.6 \times 10^{-5}.
\] (49)

Using the perihelion and aphelion of Mercury \( r_- = 4.60 \times 10^{10} \text{ m} = 2.33 \times 10^{20} \text{ GeV}^{-1} \) and \( r_+ = 6.98 \times 10^{10} \text{ m} = 3.54 \times 10^{20} \text{ GeV}^{-1} \), we get an upper bound on \( \alpha_0 \) of
\[
\alpha_0 < 1.6 \times 10^{35}
\] (50)

which is five orders of magnitude less than the bound from the time delay and the deflection of light.

**Gravitational redshift.** – Suppose that light was emitted from radius \( r_1 \) and received at \( r_2 \); by how much will the light be red-shifted? In the metric (3) put \( dr = 0 \) and \( \phi = 0 \), and solve for \( dt \). Since the time measured by a remote observer is the same for the two radii, we get
\[
dt = \frac{dr_1}{\sqrt{F(r_1)}} = \frac{dr_2}{\sqrt{F(r_2)}}
\] (51)

and the relative frequency is
\[
\frac{\omega_2}{\omega_1} = \frac{dr_1}{dr_2} = \sqrt{\frac{F(r_1)}{F(r_2)}}.
\] (52)

Substituting \( F(r) \) from (29), and expanding to first order in \( \alpha_0 \)
\[
\frac{\omega_2}{\omega_1} = \sqrt{1 - 2GM/r_1} \left( 1 + \frac{GM}{4\pi} \frac{1}{r_2^2} \left( \frac{1}{r_1^2} - 1 \right) \frac{d\alpha_0}{F(r_1)} \right).
\] (53)

Subtracting one from the previous result we get
\[
\frac{\omega_2 - \omega_1}{\omega_1} = (S - 1) \left( 1 + \frac{S}{4\pi} \frac{GM}{r_2^2} \left( \frac{1}{r_1^2} - 1 \right) \frac{d\alpha_0}{F(r_1)} \right),
\] (54)

where
\[
S \equiv \sqrt{1 - 2GM/r_1} \left( 1 + \frac{GM}{4\pi} \frac{1}{r_2^2} \left( \frac{1}{r_1^2} - 1 \right) \frac{d\alpha_0}{F(r_1)} \right)
\] (55)

The most accurate measurement of gravitational redshift is from Gravity Probe A \([51]\) in 1976. The satellite was at an altitude of \( 10^7 \text{ m} = 5.07 \times 10^{22} \text{ GeV}^{-1} \), and achieved an accuracy of \( 7.0 \times 10^{-5} \), which means that the new term in eq. (54)
\[
\frac{S}{S - 1} \frac{GM}{4\pi} \left( \frac{1}{r_2^2} - 1 \right) \frac{d\alpha_0}{F(r_1)} < 7.0 \times 10^{-5}.
\] (56)

Using the mass and radius of the Earth, \( M_\odot = 5.97 \times 10^{24} \text{ kg} = 3.35 \times 10^{51} \text{ GeV} \), \( r_1 = R_\odot = 6.38 \times 10^6 \text{ m} = 3.23 \times 10^{22} \text{ GeV}^{-1} \), and \( r_2 = r_1 + 10^7 \text{ m} \). The corrections are negative since \( r_2 > r_1 \) and hence the bound is given as follows:
\[
\alpha_0 < 2.5 \times 10^{38}.
\] (57)

This bound is stringent too and compatible with the bound set by the electroweak scale.

It is noteworthy that comparing the version of GUP we employ in our analysis here (see eq. (6)) and the corresponding one used in \([28]\) (see eq. (17) in \([28]\)), one can say that the corresponding GUP parameters, i.e., \( \alpha_0 \) and

20005-p5
\begin{table}[h]
\centering
\begin{tabular}{|l|c|}
\hline
Experiment & Bound on $\alpha_0$ \\
\hline
Deflection of light & $1.4 \times 10^{41}$ \\
Time delay of light & $5.8 \times 10^{40}$ \\
Perihelion precession & $1.6 \times 10^{35}$ \\
Gravitational redshift & $2.5 \times 10^{38}$ \\
\hline
\end{tabular}
\caption{Bounds on the GUP parameter $\alpha_0$ from gravitational tests.}
\end{table}

$\beta$, respectively, are similar, namely $\alpha_0^2 \sim \beta$. Utilizing this similarity, one can conclude that the bounds on the two aforesaid GUP parameters based on the tests for the deflection of light and the perihelion precession are essentially equivalent.

Conclusions. – In this paper, we proposed a modification to the Schwarzschild metric based on the GUP in eq. (2) to reproduce the modified Hawking temperature derived from the GUP. This modification preserves the horizon at $2GM$, and predicts the existence of another horizon which might be the radius of the black-hole singularity. We assumed a modification with a general $1/r^n$-dependence, and determined the value $n = 2$ by comparing the modified Newton’s law derived from the modified metric with phenomenologically well-motivated approaches that modify Newton’s law of gravitation at short distance such as the Randall-Sundrum II model. We computed corrections to the general relativistic results of the deflection of light, time delay of light, perihelion precession, and gravitational redshift. We compared our results with measurements to obtain upper bounds on the GUP parameter $\alpha_0$ (see table 1). The bounds we found in this paper are greater than those reported in previous work from corrections to quantum-mechanical predictions [13-15,52]. However, investigating the implications of the GUP on gravitational phenomena might prove useful for understanding the effects of quantum gravity in that regime. In addition, because the GUP is model independent, this understanding can help to evaluate the results of different theories of quantum gravity.

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