Local algebras with radical cubic zero are PCM-free

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Abstract
An artin algebra is said to be PCM-free if every finitely generated Gorenstein projective module with a projective submodule is projective. In this paper, we show that artin local algebras with radical cubic zero are PCM-free.

1 Introduction
Throughout this paper, $R$ is a commutative artin ring, $\Lambda$ is an artin algebra, that is an $R$-algebra which is a finitely generated $R$-module and $\text{mod}\Lambda$ the category of finitely generated left $\Lambda$-modules. We recall from [AM] a complex $P^\ast$ of projective $\Lambda$-modules is totally acyclic if it is acyclic and for each projective $\Lambda$-module $Q$ the Hom complex $\text{Hom}_\Lambda(P^\ast, Q)$ is acyclic. A $\Lambda$-module $M$ is called a (finitely generated) Gorenstein projective module ([EJ]) provided that there is a totally acyclic complex $P^\ast$ such that the zeroth cocycle $Z^0P^\ast$ is isomorphic to $M$. In this case, the complex $P^\ast$ is called a complete resolution of $M$. In the literature, Gorenstein projective modules are also called modules of G-dimension zero ([AuB]).

We denote $G(\Lambda)$ as the full subcategory of $\text{mod}\Lambda$ consisting of all Gorenstein projective modules, and the full subcategory of $\text{mod}\Lambda$ being composed of all projective module by $P(\Lambda)$. It is well known that a projective module is Gorenstein projective. That is, $P(\Lambda) \subset G(\Lambda)$. An artin algebra is called CM-free provided that $G(\Lambda) = P(\Lambda)$, that is, its all finitely generated Gorenstein projective modules are projective. An extreme case has been considered by Xiaowu Chen in [Ch]: a connected artin ring with radical square zero is either CM-free or self-injective.

In this paper, another extreme case is considered. We recall from [T] that a Gorenstein projective module is said to be $P$-Gorenstein projective if it contains a nonzero-projective submodule. An artin algebra is PCM-free if every finitely generated $P$-Gorenstein projective

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module is projective. Note that a CM-free algebra is PCM-free. However, the converse is not true in general.

For an algebra Λ, denote by \( J \) its Jacobson radical. The algebra Λ is said to be with radical cubic zero provided that \( J^3 = 0 \). Let us remark that there exists an upper triangular matrix algebra which is not CM-free is such a Gorenstein algebra that its Jacobson radical satisfies the radical cubic zero and square nonzero; see the remarks after [Ch]. Here we would like to address a problem: for an algebra with the radical cubic zero, when are finitely generated Gorenstein projective modules projective?

The aim of the paper is to show that for an artin local algebra with radical cubic zero the study of its finitely generated \( \mathcal{P} \)-Gorenstein projective modules always belongs to the one extreme case as follows.

**Theorem 1** Let \((\Lambda, S)\) be an artin local algebra with radical cubic zero. Then \( \Lambda \) is PCM-free.

We draw an immediate consequence of Theorem 1. Recall that a local algebra \( \Lambda \) is \( n \)-Gorenstein provided that G-dimension of the simple module is less than or equal to \( n \). A local algebra \( \Lambda \) is \( n \)-regular if projective dimension of the simple module is less than or equal to \( n \) (see [C]).

**Corollary 1** Let \((\Lambda, S)\) be an artin local algebra with radical cubic zero. If \( \Lambda \) is 1-Gorenstein, then it is either 1-regular or self-injective.

**Proof.** It suffices to notice the following fact: If G-dimension of \( S \) is less than or equal to 1, then \( S \) admits a surjective Gorenstein projective precover \( \psi : G \to S \) where \( K = \text{Ker}\psi \) satisfies \( K \) is projective and \( G \) is indecomposable. That is, there exists an exact sequence \( 0 \to K \to G \to S \to 0 \). If \( K \neq 0 \), by the theorem 1, we know that \( G \) is projective. So projective dimension of \( S \) is less than or equal to 1. That is, \( \Lambda \) is 1-regular. If \( K = 0 \), \( S \) is Gorenstein projective. So, \( \Lambda \) is self-injective.

In the next section, we start by recalling the definition of irreducible morphisms, give several preliminary involving properties of Gorenstein projective modules, and finally prove the above theorem.

### 2 Proof of the Theorem

In this section we present the proofs of Theorem 1. Let \( \Lambda \) be an artin algebra. Recall that for each \( \Lambda \)-module \( M \), its syzygy module \( \Omega(M) \) is defined to be the kernel of its projective...
cover \( P \to M \). Recall that in a short exact sequence \( 0 \to M_1 \to P \xrightarrow{p} M \to 0 \) with \( P \) projective, we have \( M_1 \cong \Omega(M) \oplus Q \) for a projective module \( Q \); moreover, \( Q = 0 \) if and only if \( p \) is a projective cover. Here we state several properties of Gorenstein projective modules for later use. For the proofs, we refer to [Ch, Lemma 2.1, 2.2].

**Lemma 1** Let \( M \) be a Gorenstein projective \( \Lambda \)-module which is indecomposable and non-projective. Then \( \Omega(M) \) is also an indecomposable non-projective Gorenstein projective \( \Lambda \)-module.

Recall that \( J \) denotes the Jacobson radical of \( \Lambda \).

**Lemma 2** Let \( M \) be a Gorenstein projective \( \Lambda \)-module without projective direct summands. Assume that \( J^n = 0 \) for \( n \geq 2 \). Then \( J^{n-1}M = 0 \) and \( J^{n-2}M \) is semi-simple.

Now we introduce the notion of an irreducible morphism and state its property for later use. For the proof of its property, we refer to [AuR1, Proposition 4.1].

**Definition 1** A morphism \( f : B \to C \) in the category \( \text{mod}\Lambda \) is said to be Irreducible if (a) \( f \) is not a splittable proper epimorphism or a splittable proper monomorphism; (b) given a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{g} & \bar{C} \\
\downarrow{h} & & \downarrow{f} \\
B & \xrightarrow{f} & C
\end{array}
\]

either \( h \) is splittable epimorphic or \( g \) is splittable monomorphic.

**Lemma 3** Let \( M \) be an indecomposable \( G \)-projective module. The following conditions are equivalent:

1. There exists an irreducible monomorphism \( M \to Q \) for a projective \( Q \).
2. \( M \) is isomorphic to a summand of \( JQ \) for a projective \( Q \).

Let \( P_1 \xrightarrow{f} P_0 \to M \to 0 \) be a minimal projective resolution of a module \( M \) in \( \text{mod}\Lambda \). We call \( \text{Coker} \ f^* \) the Auslander transpose of \( M \), and denote it by \( \text{Tr} M([\text{AuR}]) \). Consider the stable category \( \text{mod}\Lambda \) of \( \text{mod}\Lambda \), by the [AuR, Proposition 2.2], we have the following lemma.

**Lemma 4** Let \( M, N \) be in \( \text{mod}\Lambda \). Then there exists an isomorphism \( \text{Tor}_1^\Lambda(\text{Tr} M, N) \cong \text{Hom}_\Lambda(M, N) \).
Let $D(\cdot)$ be the nature functor $\text{Hom}_R(\cdot, E(R/r))$ where $r$ is Jacobson radical of $R$ and $E(R/r)$ is the injective envelope. We say that an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\text{mod}\Lambda$ is almost split if $f$ is left almost split and $g$ is right almost split (see [AuR]). It is well known that there exists a unique almost split sequence for a non-injective indecomposable $\Lambda$-module $M$ and the form is $0 \to M \to N \to \text{Tr} DM \to 0$. Now the time has come when we can prove our theorem [1].

**The proof of Theorem** [1] Let $PG(\Lambda)$ be the subcategory of $\text{mod}\Lambda$ composed of all finitely generated $P$-Gorenstein projective modules. Assume that $\Lambda$ is not PCM-free. Take a $M$ in $PG(\Lambda)$ to be indecomposable and non-projective. Set $Q$ to be the nonzero-projective submodule $M$. Considering the monomorphism $0 \to Q \to M$, there is the monomorphism $0 \to JQ \to JM$. Note that $J^3 = 0$. By the lemma 2 we have $JM$ is semisimple and then $JQ$ is semisimple. Let $S$ be the simple submodule of $JQ$. Then there exists an indecomposable projective module $P$ such that the simple module $S$ is isomorphic to the summand of $JP$. It follows immediately from the lemma 3 that there exists an irreducible morphism $f : S \to P.$ Take a short exact sequence $0 \to S \to E \to \text{Tr} DS \to 0$ such that it is the almost split sequence of $S.$ Then there exist the morphisms $h : E \to P$ and $g : \text{Tr} DS \to P/S$ such that the following diagram

$$
\begin{array}{ccccccc}
0 & \to & S & \to & E & \to & \text{Tr} DS & \to & 0 \\
& & & & h & & g & \\
0 & \to & S & \to & f & \to & P & \to & P/S & \to & 0
\end{array}
$$

commutates. Note that $f$ is irreducible, $h$ is split. This gives rise to the almost split exact sequence of $S$:

$$
0 \to S \to P \oplus B \to \text{Tr} DS \to 0
$$

Hence we have a commutative diagram
This gives us the commutative diagram

$$
\begin{array}{cccccc}
0 & 0 \\
\downarrow & \downarrow \\
B & B \\
\downarrow & \downarrow \\
0 & S & P \oplus B & \text{Tr } DS & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & S & P & P/S & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 \\
\end{array}
$$

That $\sigma$ is irreducible follows from the relationship between irreducible morphisms and almost split sequence.

A): If $\theta = 0$, by the indecomposable module $\text{Tr } DS$, $\text{Tr } DS \cong P/S$. And we have the almost split sequence

$$
\eta : 0 \to S \to P \to \text{Tr } DS \to 0
$$

If $M$ is isomorphic to $S$, by a projective submodule $Q$ of $M$, $M$ is projective. It is contradicted with our assumption. Suppose that $M$ is not isomorphic to $S$. Note that the exact sequence $\eta$ is an almost split sequence, we know that $\text{Hom}_\Lambda(S, M) = 0$. It follows from Lemma 4 that

$$
\text{Tor}_1^\Lambda(\text{Tr } S, M) = 0 \quad \text{and} \quad \text{Ext}_1^\Lambda(\text{Tr } S, DM) \cong D \text{Tor}_1^\Lambda(\text{Tr } S, M) = 0
$$

Let $0 \to M \to P_1 \to M_1 \to 0$ be an exact sequence with a projective module $P_1$. This means that $\text{Ext}_1^\Lambda(\text{Tr } S, DM_1) = 0$. Use the functor $\text{Hom}_\Lambda(-, \Lambda)$ to the almost split sequence $\eta$, by the non-projective module $S$, we obtain the short exact sequence

$$
0 \to DS \to P' \to \text{Tr } S \to 0
$$

with the projective module $P'$. Since $\text{Ext}_1^\Lambda(\text{Tr } S, DM_1) = 0$, we have that

$$
\text{Ext}_1^\Lambda(M_1, S) \cong \text{Ext}_1^\Lambda(DS, DM_1) \cong \text{Ext}_1^\Lambda(\text{Tr } S, DM_1) = 0
$$
Therefore, $M_1$ is projective. It is contradicted with the non-projective module $M$.

**B)**: If $\theta \neq 0$, we have the monomorphism $\sigma : P \to \text{Tr} DS$. Noting the almost split sequence $0 \to S \to E \to \text{Tr} DS \to 0$, we have a commutative diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\sigma} & \text{Tr} DS \\
\downarrow{\alpha} & & \downarrow{0} \\
E & \xrightarrow{\sigma} & 0.
\end{array}
\]

Since $\sigma$ is irreducible, it follows that $\alpha$ is a splittable monomorphism. That is, there exists an epimorphism $\beta : E \to P$ such that $\beta\alpha = 1_P$. Consider the commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{\gamma} & E_1 \oplus P \\
\downarrow{0} & & \downarrow{0} \\
0 & \xrightarrow{\beta} & S \\
\downarrow{\gamma} & & \downarrow{\beta} \\
E & \xrightarrow{\gamma} & E_1 \oplus P \\
\downarrow{\beta} & & \downarrow{\beta} \\
0 & \xrightarrow{\text{Tr} DS} & 0
\end{array}
\]

with $E \cong (P \oplus E_1)$. If $\beta\gamma \neq 0$, it is contradicted with the monomorphism $\sigma$. Hence we have the isomorphisms $E/S \cong (E_1 \oplus P)/S \cong (E_1/S) \oplus P \cong \text{Tr} DS$. This implies that $P = 0$. The result contraries to the assumption of the theorem. This contradiction completes the proof of the theorem.

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