ABSTRACT To fuse the information in dual-hesitant Pythagorean fuzzy sets (DHPFSs) more effectively, in this paper, some dual hesitant Pythagorean fuzzy Hamy mean (DHPFHM) operators, which can consider the relationships between being fused arguments, are defined and studied. Afterward, the defined aggregation operators are used to multiple attribute decision-making (MADM) with dual-hesitant Pythagorean fuzzy elements (DHPFEs), and the MADM decision-making model is developed. In accordance with the defined operators and the built model, the dual-hesitant Pythagorean fuzzy weighted Hamy mean (DHPFWHM) operator and the dual-hesitant Pythagorean fuzzy weighted dual Hamy mean (DHPFWDHM) operator are applied to deal with green supplier selection in supply chain management, and the availability and superiority of the proposed operators are analyzed by comparing with some existing approaches. The method presented in this paper can effectively solve the MADM problems, which the decision-making information is expressed by DHPFEs and the attributes are interactive.

INDEX TERMS Multiple attribute decision making (MADM), dual hesitant Pythagorean fuzzy sets (DHPFSs), dual hesitant Pythagorean fuzzy weighted Hamy mean (DHPFWHM) operator, dual hesitant Pythagorean fuzzy weighted dual Hamy mean (DHPFWDHM) operator, supply chain management.

I. INTRODUCTION
In real-life decision making environment, it’s difficult for decision makers (DMs) to give evaluate information by using exact real numbers. In order to overcome this limitation, Zadeh [1] developed the fuzzy set (FS) theory which used the membership degree to depict decision making information instead of crisp results. In accordance of the studied of FS, Atanassov [2] further proposed another function which named non-membership degree as a supplementary. Thus, the intuitionistic fuzzy set (IFS) was found, each intuitionistic fuzzy set is characterized by the functions of membership degree and non-membership degree between 0 and 1, and the sum of them are limited to 1. After that, more and more scholars have studied about the IFS and its extensions in many multiple attribute decision making (MADM) problems [3]–[12]. Xu [13] studied the intuitionistic fuzzy weighted average operator, intuitionistic fuzzy ordered weighted average operator and intuitionistic fuzzy hybrid average operator. Xu and Yager [14] proposed some intuitionistic fuzzy weighted geometric (IFWG) aggregation operators. Wei and Wang [15] extended these IFWG operators to interval valued intuitionistic fuzzy environment. Hung and Yang [16] studied the similarity measures of IFS. Narayanamoorthy et al. [17] firstly developed the definition of interval-valued intuitionistic hesitant fuzzy entropy and built an extended VIKOR model for industrial robots selection. Zhang [18] combined the interval-valued intuitionistic fuzzy set and geometric Bonferroni means to define some new aggregation operators and applied them into multiple attribute decision making problems. Based on Shapley index, Zhou et al. [19] studied the Choquet integral correlation coefficient under extended intuitionistic fuzzy environment and applied to MADM. Zhang et al. [20] proposed some normal intuitionistic fuzzy Heronian mean operators based on Hamacher operation laws. Zhai et al. [21] defined the probabilistic interval-valued intuitionistic hesitant fuzzy sets and studied the distance and similarity measures of them. Based on the grey relational methods, the bidirectional projection and hesitant intuitionistic fuzzy linguistic information,
Zang et al. [22] developed a new approach to solve MADM problems. Yu et al. [23] presented some aggregation operators to fuse intuitionistic uncertain 2-tuple linguistic variables and applied them to MADM with heterogeneous relationships among attributes. Yao and Wang [24] defined the hesitant intuitionistic fuzzy entropy and hesitant intuitionistic fuzzy cross-entropy and studied their applications. Yang et al. [25] proposed an extended VIKOR model under linguistic hesitant intuitionistic fuzzy environment.

However, the scope of evaluation information is limited by using intuitionistic fuzzy set, that is, the sum of membership and non-membership must be less or equal to 1. Thus, Pythagorean fuzzy set (PFS) [26,27] has emerged to overcome this limitation. Similar to IFS, the Pythagorean fuzzy set (PFS) is also consisted of the function of membership degree and non-membership degree, and the sum of squares of them is restricted to 1, thus, it’s clear that the PFS is more widespread than the IFS and can express more decision-making information. For instance, the membership is given as 0.6 and the non-membership is given as 0.8, it’s obvious that this problem is only valid for PFS. In other words, all the intuitionistic fuzzy decision-making problems are the special case of Pythagorean fuzzy decision-making problems, which means that the PFS is more efficient to deal with MADM problems. In previous literature, some researching works have studied by a large amount of investigators. Zhang and Xu [28] proposed the Pythagorean fuzzy TOPSIS model to handle the MADM problems. Consider incomplete weight information, Khan et al. [29] studied MADM problems under Pythagorean hesitant fuzzy environment. Peng and Yang [30] primarily developed two Pythagorean fuzzy operations including the division and subtraction operations to better understand PFS. Reformat and Yager [31] dealt with the collaborative-based recommender system with Pythagorean fuzzy information. Based on traditional TOPSIS method, Khan et al. [32] developed an interval-valued Pythagorean TOPSIS method based on Choquet integral and Khan et al. [33] also studied the TOPSIS model under Pythagorean hesitant fuzzy environment. Gou et al. [34] researched some precious properties of continuous Pythagorean fuzzy information. Garg [35] gave the definition of some new Pythagorean fuzzy aggregation operators. Khan et al. [36] proposed some Pythagorean hesitant fuzzy Choquet integral aggregation operators. Wu and Wei [37] presented some Pythagorean fuzzy Hamacher aggregation operators to fuse Pythagorean fuzzy information. Zeng et al. [38] used the Pythagorean fuzzy ordered weighted average weighted average distance (PFOWAWAD) operator to study Pythagorean fuzzy MADM issues. Ren et al. [39] built the Pythagorean fuzzy TODIM model. Wei and Lu [40] developed some new Maclaurin symmetric mean (MSM) [41] operator based on Pythagorean fuzzy environment. Wei [42] developed some Pythagorean fuzzy interaction aggregation operators based on arithmetic and geometric operations. Wei and Lu [43] proposed some Pythagorean fuzzy power aggregation operators. Zeb et al. [44] presented some approaches to handle MADM problems with risk preference under extended Pythagorean fuzzy environment. Wei and Wei [45] defined ten cosine similarity measures under Pythagorean fuzzy environment. Liang et al. [46] investigated some Bonferroni mean operator with Pythagorean fuzzy information. Liang et al. [47] presented Pythagorean fuzzy Bonferroni mean aggregation operators based on geometric averaging (GA) operators. Khan and Abdullah [48] proposed the interval-valued Pythagorean fuzzy GRA method to solve MADM problems with incomplete weight information. Combined the PFSs [26,27] and dual hesitant fuzzy sets (DHFSs) [49,50], Wei and Lu [51] introduced the definition of the dual hesitant Pythagorean fuzzy sets (DHPFSs) and proposed some dual hesitant Pythagorean fuzzy Hamacher aggregation operators. Obviously, the DHPFSs have the advantages of considering the hesitance of DMs and expressing fuzzy information more effectively and reasonably. Khan et al. [52] investigated the Pythagorean hesitant fuzzy sets for group decision making with incomplete weight information. Liang and Xu [53] extended the TOPSIS method for multiple criteria decision making with hesitant Pythagorean fuzzy sets.

However, in practical MADM problems, there do exist some relationships between being fused arguments, it’s obvious that the dual hesitant Pythagorean fuzzy Hamacher aggregation operators defined by Wei and Lu [51] don’t take the relationships between being fused arguments into consideration. Thus, we need to find another more effective method to fuse dual hesitant Pythagorean fuzzy information. To date, the Hamy mean (HM) [54] operator, which can effectively take the interrelationship between arguments into account, has drawn large quantity of scholars’ attention. Based on the intuitionistic fuzzy information, Hamy operator and Dombi operation laws, Li et al. [3] developed some intuitionistic fuzzy Dombi Hamy operators and applied these aggregation operators to the selection of a car supplier. Consider interval-valued intuitionistic fuzzy information, Wu et al. [4] further proposed some interval-valued intuitionistic fuzzy Dombi Hamy operators and discussed the application for evaluating the elderly tourism service quality in tourism destination by using these operators. To overcome the limitation of intuitionistic fuzzy set, Li et al. [55] defined some Pythagorean fuzzy Hamy operators to select most desirable green supplier. Deng et al. [56] combined the 2-tuple linguistic set (2TLS) and Pythagorean fuzzy set (PFS) to give the definition of 2-tuple linguistic Pythagorean fuzzy set (2TLPFS), and then some Hamy operators are presented under 2-tuple linguistic Pythagorean fuzzy environment. Based on linguistic neutrosophic set (LNS), Liu and You [57] proposed some linguistic neutrosophic Hamy operators for MADM problems. Liu et al. [58] defined some intuitionistic uncertain linguistic Hamy mean operators to select an appropriate and effective health-care waste treatment technology.

From above literature review, we can obtain that the dual hesitant Pythagorean fuzzy set is a meaningful tool to...
depict fuzzy and ambiguous information, the Hamy operator can consider the interrelationship between any number of being fused arguments. In this paper, motivated by traditional Hamy mean (HM) operator and the dual hesitant Pythagorean fuzzy set, we shall develop some dual hesitant Pythagorean fuzzy Hamy mean aggregation operators. The main contribution of this manuscript is to introduce some more reasonable aggregation operator for multiple attribute decision making (MADM) problems, and the way to express evaluation information we proposed is more scientific and effective. In addition, consider the decision maker’s risk attitude, we can dynamic adjust to the parameter to derive different decision making results under different decision making environments.

In order to do so, the remainder of this paper is set out as follows. In the next section, we introduce some basic concepts related to Pythagorean fuzzy set (PFS), dual hesitant Pythagorean fuzzy set (DHPFS) and their operational laws. In Section 3, we shall propose some dual hesitant Pythagorean fuzzy Hamy mean aggregation operators such as: the dual hesitant Pythagorean fuzzy Hamy mean (DHPFHM) operator, the dual hesitant Pythagorean fuzzy weighted Hamy mean (DHPFWHM) operator, the dual hesitant Pythagorean fuzzy dual Hamy mean (DHPFDHM) operator and the dual hesitant Pythagorean fuzzy weighted dual Hamy mean (DHPFWDHM) operator. In Section 4, based on DHPFWHM and DHPFDHM operators, we shall propose some models for multiple attribute decision making (MADM) problems with dual hesitant Pythagorean fuzzy information. In Section 5, we present a numerical example for supplier selection in supply chain management with dual hesitant Pythagorean fuzzy information in order to illustrate the method proposed in this paper. Section 6 concludes the paper with some remarks.

II. PRELIMINARIES

A. PYTHAGOREAN FUZZY SET

The fundamental definition of Pythagorean fuzzy sets (PFSs) [26], [27] are briefly introduced in this part. Afterwards, novel score and accuracy functions of Pythagorean fuzzy numbers (PFN) are developed. Furthermore, the comparison laws of PFNs are proposed.

Definition 1 [26], [27]: Let $X$ be a fix set. A Pythagorean fuzzy set (PFS) is an object which can be denoted as

$$ P = \{(x, (\alpha_p (x), \beta_p (x))) | x \in X\} \quad (1) $$

where the function $\alpha_p : X \rightarrow [0,1]$ indicates the degree of membership and the function $\beta_p : X \rightarrow [0,1]$ indicates the degree of non-membership of the element $x \in X$ to $P$, respectively, and, for each $x \in X$, it holds that

$$(\alpha_p (x))^2 + (\beta_p (x))^2 \leq 1. \quad (2)$$

Definition 2 [28]: Assume that $p_1 = (\alpha_1, \beta_1), p_2 = (\alpha_2, \beta_2)$, and $p = (\alpha, \beta)$ be three Pythagorean fuzzy numbers (PFN), then some basic operation laws of them can be expressed as:

$$ p_1 \oplus p_2 = \left(\sqrt{(\alpha_{p_1})^2 + (\alpha_{p_2})^2 - (\alpha_{p_1})^2 (\alpha_{p_2})^2}, \beta_{p_1}, \beta_{p_2}\right) $$

$$ p_1 \otimes p_2 = \left(\alpha_{p_1} \alpha_{p_2}, \sqrt{(\beta_{p_1})^2 + (\beta_{p_2})^2 - (\beta_{p_1})^2 (\beta_{p_2})^2}\right) $$

$$ \lambda p = \left(\sqrt{1 - (1 - \alpha^2) \lambda^2}, \beta^\lambda\right), \lambda > 0 $$

$$ (p)^\lambda = \left(\alpha^\lambda, \sqrt{1 - (1 - \beta^2)^\lambda}\right), \lambda > 0 $$

$$ p^\beta = (\beta, \alpha). $$

Example 1: Assume that $p_1 = (0.6, 0.2), p_2 = (0.4, 0.9)$, and $p = (0.5, 0.7)$ be three Pythagorean fuzzy numbers (PFN), suppose $\lambda = 4$, then according to above operation laws, we can obtain

$$ p_1 \oplus p_2 = \left(\sqrt{(0.6)^2 + (0.4)^2 - (0.6)^2 (0.4)^2}, 0.2 \times 0.9\right) $$

$$ = (0.68, 0.18) $$

$$ p_1 \otimes p_2 = \left(0.6 \times 0.4, \sqrt{(0.2)^2 + (0.9)^2 - (0.2)^2 (0.9)^2}\right) $$

$$ = (0.24, 0.90) $$

$$ 4 \times p = \left(\sqrt{1 - (1 - 0.5^2)^4}, 0.7^4\right) = (0.8268, 0.2401) $$

$$ (p)^4 = \left(0.5^4, \sqrt{1 - (1 - 0.7^2)^4}\right) = (0.0625, 0.9656) $$

$$ p^\beta = (0.7, 0.5). $$

B. DUAL HESITANT PYTHAGOREAN FUZZY SET

In this section, we shall introduce the basic definition of dual hesitant Pythagorean fuzzy set (DHPFS), which is the generalization of Pythagorean fuzzy set (PFS) [26], [27] and dual hesitant fuzzy set (DHFS) [49], [50]. It’s obvious that the DHPFSs consist of two parts, which means, the function of membership hesitancy and the function of non-membership hesitancy, supporting a more exemplary situation.

Definition 3 [51]: Assume that $X$ be a fix set, then a dual hesitant Pythagorean fuzzy set (DHPFS) on $X$ is developed as:

$$ \tilde{P} = \{(x, (h_p (x), g_p (x))) | x \in X\} \quad (3) $$

in which $h_p (x)$ and $g_p (x)$ are two sets of some values in $[0, 1]$, indicating the function of membership degrees and non-membership degrees of the element $x \in X$ to the set $\tilde{P}$ respectively, satisfies the condition:

$$ \alpha^2 + \beta^2 \leq 1 \quad (4) $$

where $\alpha \in h_p (x), \beta \in g_p (x)$, for all $x \in X$. For convenience, the pair $\tilde{p} (x) = (h_{\tilde{p}} (x), g_{\tilde{p}} (x))$ is called a dual hesitant Pythagorean fuzzy element (DHPFE) denoted by
\[ \tilde{p} = (h, g) \], with the conditions: \( \alpha \in h, \beta \in g, 0 \leq \alpha, \beta \leq 1 \), \( 0 \leq \alpha^2 + \beta^2 \leq 1 \).

**Definition 4 [51]:** Let \( \tilde{p} = (h, g) \) be a DHPFE, \( s(\tilde{p}) = \frac{1}{a} \left( 1 + \frac{1}{n^h} \sum_{a \epsilon h} \alpha^2 - \frac{1}{n^g} \sum_{\beta \epsilon g} \beta^2 \right) \) is the score function of \( \tilde{p} \), and \( H(\tilde{p}) = \frac{1}{n^h} \sum_{a \epsilon h} \alpha^2 + \frac{1}{n^g} \sum_{\beta \epsilon g} \beta^2 \) is the accuracy function of \( \tilde{p} \), where \( n^h \) and \( n^g \) are the numbers of the elements in \( h \) and \( g \) respectively, then, Let \( \tilde{p}_i = (h_i, g_i) \) \( (i = 1, 2) \) be any two DHPFEs, we have the following comparison laws:

- If \( s(\tilde{p}_1) > s(\tilde{p}_2) \), then \( \tilde{p}_1 \) is superior to \( \tilde{p}_2 \), denoted by \( \tilde{p}_1 \succ \tilde{p}_2 \);
- If \( s(\tilde{p}_1) = s(\tilde{p}_2) \), then
  1. If \( p(\tilde{p}_1) = p(\tilde{p}_2) \), then \( \tilde{p}_1 \) is equivalent to \( \tilde{p}_2 \), denoted by \( \tilde{p}_1 \sim \tilde{p}_2 \);
  2. If \( p(\tilde{p}_1) > p(\tilde{p}_2) \), then \( \tilde{p}_1 \) is superior to \( \tilde{p}_2 \), denoted by \( \tilde{p}_1 \succ \tilde{p}_2 \).

**Definition 5 [51]:** Assume that \( \tilde{p}_1 = (h_1, g_1), \tilde{p}_2 = (h_2, g_2) \), and \( \tilde{p} = (h, g) \) be three dual hesitant Pythagorean fuzzy elements (DHPFEs), then some basic operation laws of them can be expressed as

\[
\begin{align*}
\tilde{p}^\lambda &= \bigcup_{a \epsilon h, \beta \epsilon g} \left\{ \alpha^\lambda, \sqrt{1 - (1 - \beta^2)^\lambda} \right\}, \lambda > 0 \\
\lambda \tilde{p} &= \bigcup_{a \epsilon h, \beta \epsilon g} \left\{ \sqrt{1 - (1 - \alpha^2)^\lambda}, \beta^\lambda \right\}, \lambda > 0
\end{align*}
\]

\[
\tilde{p}_1 \oplus \tilde{p}_2 = \bigcup_{a_1 \epsilon h_1, a_2 \epsilon h_2, \beta_1 \epsilon g_1, \beta_2 \epsilon g_2} \left\{ \left( \frac{1}{2} \right) \left( a_1^2 + a_2^2 - (a_1 - a_2)^2 \right), \beta_1 \beta_2 \right\}
\]

\[
\tilde{p}_1 \otimes \tilde{p}_2 = \bigcup_{a_1 \epsilon h_1, a_2 \epsilon h_2, \beta_1 \epsilon g_1, \beta_2 \epsilon g_2} \left\{ \left( \frac{1}{2} \right) \left( a_1 a_2 - (a_1 - a_2)^2 \right), \left( \beta_1 \beta_2 - (\beta_1 - \beta_2)^2 \right) \right\}
\]

**Example 2:** Assume that \( p_1 = \{0.1, 0.6, 0.4\}, p_2 = \{0.5, 0.2, 0.3\} \), and \( p = \{0.7, 0.6, 0.8\} \) be three Pythagorean fuzzy numbers (PFNs), suppose \( \lambda = 2 \), then according to above operation laws, we can obtain

\[
\begin{align*}
\tilde{p}^2 &= \bigcup_{a \epsilon h, \beta \epsilon g} \left\{ \left( 0.2 \right)^2, \sqrt{1 - (1 - \beta^2)^2}, \sqrt{1 - (1 - 0.8^2)^2} \right\} \\
&= \{0.490, 0.768, 0.933\}
\end{align*}
\]

\[
\begin{align*}
2\tilde{p} &= \bigcup_{a \epsilon h, \beta \epsilon g} \left\{ \left( 1 - (1 - \beta^2)^2 \right), 0.62, 0.82 \right\} \\
&= \{0.860, 0.360, 0.640\}
\end{align*}
\]

\[
\begin{align*}
\tilde{p}_1 \oplus \tilde{p}_2 &= \bigcup_{a_1 \epsilon h_1, a_2 \epsilon h_2, \beta_1 \epsilon g_1, \beta_2 \epsilon g_2} \left\{ \left( \frac{1}{2} \right) \left( 0.1^2 + 0.5^2 - 0.1^2 \times 0.5^2 \right), \left( \frac{1}{2} \right) \left( 0.2 \times 0.4 \times 0.3 \right) \right\} \\
&= \{0.507, 0.721, 0.080, 0.120\}
\end{align*}
\]

\[
\begin{align*}
\tilde{p}_1 \otimes \tilde{p}_2 &= \bigcup_{a_1 \epsilon h_1, a_2 \epsilon h_2, \beta_1 \epsilon g_1, \beta_2 \epsilon g_2} \left\{ \left( \frac{1}{2} \right) \left( 0.1^2 \times 0.5^2, 0.6 \times 0.5 \right) \right\} \\
&= \{0.050, 0.300, 0.440, 0.485\}
\end{align*}
\]

**C. THE HAMY MEAN OPERATOR**

**Definition 6 [59]:** The Hamy mean (HM) operator is defined as follows:

\[
\text{HM}(k)(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \frac{1}{\sum_{i=1}^{k} \tilde{p}_i} \left( \prod_{j=1}^{k} \tilde{p}_j \right)^\frac{1}{k}
\]

where \( k \) is a parameter and \( k = 1, 2, \ldots, n, i_1, i_2, \ldots, i_k \) are integer values taken from the set \( \{1, 2, \ldots, n\} \) of \( k \) integer values, \( C_n^k \) denotes the binomial coefficient and \( C_n^k = \frac{n!}{k!(n-k)!} \).

**III. DUAL HESITANT PYTHAGOREAN FUZZY HAMY MEAN OPERATORS**

In the following, we shall propose some dual hesitant Pythagorean fuzzy Hamy mean (DHPFHM) operator based on the DHPFEs and Hamy mean (HM) operations. In addition, some precious properties, such as idempotency, boundedness and monotonicity, are discussed.

**A. THE DHPFHM AGGREGATION OPERATOR**

**Definition 7:** Let \( \tilde{p}_j = (h_j, g_j) (j = 1, 2, \ldots, n) \) be a group of DHPFEs, and then we define the dual hesitant Pythagorean fuzzy Hamy (DHPFHM) operator as follows:

\[
\text{DHPFHM}(k)(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)
\]

\[
= \bigoplus_{1 \leq i_1 < \cdots < k \leq n} \left( \bigotimes_{j=1}^{k} \tilde{p}_{i_j} \right)^\frac{1}{k}
\]

\[
= \bigotimes_{j=1}^{k} \tilde{p}_{i_j} \bigotimes_{j=1}^{k} \tilde{p}_{i_j} \bigotimes_{j=1}^{k} \tilde{p}_{i_j}
\]

\[
\text{ According to the operation laws of the DHPFEs described in definition 5, we can obtain the Theorem 1.}
\]

**Theorem 1:** Let \( \tilde{p}_j = (h_j, g_j) (j = 1, 2, \ldots, n) \) be a group of DHPFEs, then their fused results by utilizing the DHPFHM operator is also a DHPFE, and (7), as shown at the top of the next page, holds.

\[
\begin{align*}
\text{Proof: Based on Definition 5, we can derive:}
\end{align*}
\]

Thus,

\[
\begin{align*}
\Theta \tilde{p}_j &= \bigcup_{a_j \epsilon h_j, \beta_j \epsilon g_j} \left\{ \left( \frac{k}{j=1} \alpha_j \right)^\frac{1}{k}, \left( \frac{k}{j=1} (1 - \beta_j^2) \right)^\frac{1}{k} \right\}
\end{align*}
\]

\[
= \bigcup_{a_j \epsilon h_j, \beta_j \epsilon g_j} \left\{ \left( \frac{k}{j=1} \alpha_j \right)^\frac{1}{k}, \left( \frac{k}{j=1} (1 - \beta_j^2) \right)^\frac{1}{k} \right\}
\]
Therefore,

\[
\cup_{\alpha \in \mathcal{H}, \beta \in \mathcal{G}} \left\{ \prod_{1 \leq i < \ldots < k \leq n} \left[ 1 - \prod_{j=1}^{k-1} \left( 1 - \alpha_j \right) \right] \prod_{1 \leq i < \ldots < k \leq n} \left[ 1 - \prod_{j=1}^{k-1} \left( 1 - \beta_j \right) \right] \right\} = \left\{ \prod_{1 \leq i < \ldots < k \leq n} \left[ 1 - \prod_{j=1}^{k-1} \left( 1 - \beta_j \right) \right] \right\}
\]

(10)

Furthermore, see (11) at the top of this page.

Thus, we have finished the proof.

Example 3: Assume that \( \tilde{p}_1 = \{0.7, 0.8\}, \{0.4\} \), \( \tilde{p}_2 = \{0.3\}, \{0.6, 0.7\} \), \( \tilde{p}_3 = \{0.1, 0.3\}, \{0.4, 0.6\} \) and \( \tilde{p}_4 = \{0.5\}, \{0.5\} \) be four DHPFEs, suppose \( k = 2 \), then according to the DHPFM operator, we can obtain the fused results as follows. For the membership degree function \( \alpha \), the fused results are shown in (I), at the top of the next page.

Similarly, we can obtain

\[
\beta_2 = \text{DHPFM}^{(2)}(0.4, 0.6, 0.6, 0.5) = 0.5307
\]

\[
\beta_3 = \text{DHPFM}^{(2)}(0.4, 0.7, 0.4, 0.5) = 0.5101
\]

\[
\beta_4 = \text{DHPFM}^{(2)}(0.4, 0.7, 0.6, 0.5) = 0.5602,
\]

So we can get \( \beta = \{0.4798, 0.5307, 0.5101, 0.5602\} \).

Therefore,

\[
\text{DHPFM}^{(2)}\left( \tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4 \right) = \left\{ \{0.3888, 0.4446, 0.4115, 0.4674\}, \{0.4798, 0.5307, 0.5101, 0.5602\} \right\}.
\]

It can be easily proved that the DHPFM operator satisfies the following properties.

Property 1 (Idempotency): If all \( \tilde{p}_j = (h_j, g_j) (j = 1, 2, \ldots, n) \) are equal, i.e. \( \tilde{p}_j = \tilde{p} \) for all \( j \), then

\[
\text{DHPFM}^{(k)}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \tilde{p}
\]

(12)

Proof: For \( \tilde{p}_j = \tilde{p} \) for all \( j \), that means \( \alpha_j = \alpha \) and \( \beta_j = \beta \) for all \( j \), thus we can obtain (13), as shown at the bottom of the next page.

Thus, property 1 is maintained.

Property 2 (Monotonicity): Let \( \tilde{p}_j = (h_j, g_j) \) and \( \tilde{p}_j' = (h_{j}', g_{j}') \), \( j = 1, 2, \ldots, n \), be two set of DHPFEs, if \( h_j \leq h_{j}', g_j \geq g_{j}' \) for all \( j \), then

\[
\text{DHPFM}^{(k)}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \preceq \text{DHPFM}^{(k)}(\tilde{p}_1', \tilde{p}_2', \ldots, \tilde{p}_n')
\]

(14)
\[ \alpha_1 = \text{DHPFHM}^{(2)} (0.7, 0.3, 0.1, 0.5) = \sqrt{1 - \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left(1 - \left(\prod_{j=1}^{k} \alpha_j\right)^{\frac{1}{k}}\right)} = 0.3888 \]

\[ \beta_1 = \text{DHPFHM}^{(2)} (0.4, 0.6, 0.4, 0.5) = \sqrt{1 - \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left(1 - \left(\prod_{j=1}^{k} \beta_j^{\frac{1}{k}}\right)\right)} = 0.4798 \]

**Proof:** For \( h_j \leq h_j' \) for all \( j \), that means \( \alpha_j \leq \alpha_j' \) for all \( j \), thus we can obtain

\[ \left(\prod_{j=1}^{k} \alpha_j\right)^{\frac{1}{k}} \leq \left(\prod_{j=1}^{k} \alpha_j'\right)^{\frac{1}{k}} \leq \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left(1 - \left(\prod_{j=1}^{k} \alpha_j\right)^{\frac{1}{k}}\right) = 0.3888 \]

Then

\[ 1 - \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left(1 - \left(\prod_{j=1}^{k} \alpha_j\right)^{\frac{1}{k}}\right) \leq 1 - \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left(1 - \alpha_j^{\frac{1}{k}}\right) = 0.4798 \]

DHPFHM\(^{(k)}\) \((\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)\)

\[ = \frac{\bigoplus_{1 \leq i_1 < \ldots < i_k \leq n} \left(\hat{p}_{i_1} \otimes \cdots \otimes \hat{p}_{i_k}\right)^{\frac{1}{k}}}{C_n^k} = \bigcup_{\alpha, \beta \in \mathcal{E}} \left\{ \left[1 - \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left(1 - \left(\prod_{j=1}^{k} \alpha_j\right)^{\frac{1}{k}}\right)\right]^{\frac{1}{C_n^k}} \right\} \bigcup_{\alpha, \beta \in \mathcal{E}} \left\{ \left[1 - \left(\prod_{j=1}^{k} \beta_j^{\frac{1}{k}}\right)\right]^{\frac{1}{C_n^k}} \right\} = \bigcup_{\alpha, \beta \in \mathcal{E}} \{\{\alpha\}, \{\beta\}\} = \hat{p} \]
DHPFWHM$_{w}^{(k)}$ ($\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n$) = \frac{\bigoplus_{1 \leq i < j \leq n} \left( \prod_{j=1}^{k} (\tilde{p}_{ij})^{w_j} \right)^{\frac{1}{k}}}{C_n^k}

= \bigcup_{\alpha_j \in h_j, \beta_j \in g_j} \left\{ \left\{ \left( \prod_{j=1}^{k} \alpha_j^{w_j} \right)^{\frac{1}{k}} \right\}, \left\{ 1 - \left( \prod_{j=1}^{k} (1 - \beta_j^2)^{w_j} \right)^{\frac{1}{k}} \right\} \right\} \left( 1 \leq i < j \leq n \right) \left( \prod_{j=1}^{k} (\tilde{p}_{ij})^{w_j} \right)^{\frac{1}{k}} \right\}

(19)

\left( \prod_{j=1}^{k} \alpha_j^{w_j} \right)^{\frac{1}{k}} = \bigcup_{\alpha_j \in h_j, \beta_j \in g_j} \left\{ \left\{ (\prod_{j=1}^{k} \alpha_j^{w_j})^{\frac{1}{k}} \right\}, \left\{ 1 - \left( \prod_{j=1}^{k} (1 - \beta_j^2)^{w_j} \right)^{\frac{1}{k}} \right\} \right\}

(22)

That means $\alpha_{DHPFWHM}^{(k)} \leq \alpha'_{DHPFWHM}^{(k)}$, similarly, we can obtain $\beta_{DHPFWHM}^{(k)} \geq \beta'_{DHPFWHM}^{(k)}$, thus DHPFWHM$^{(k)}$ ($\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n$) $\leq$ DHPFWHM$^{(k)}$ ($\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n$) is proved.

**Property 3 (Boundedness):** Let $\tilde{p}_j = (h_j, g_j)$ ($j = 1, 2, \ldots, n$) be a collection of DHPFEs, and let

\[
\tilde{p}^+ = \bigcup_{\alpha_j \in h_j, \beta_j \in g_j} \left\{ \max_i (\alpha_j), \min_j (\beta_j) \right\},
\]

\[
\tilde{p}^- = \bigcup_{\alpha_j \in h_j, \beta_j \in g_j} \left\{ \min_i (\alpha_j), \max_j (\beta_j) \right\}
\]

Then

\[
\tilde{p}^- \leq \text{DHPFWHM}^{(k)} (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \tilde{p}^+
\]

(17)

**Proof:**

From property 1, we can obtain

\[
\text{DHPFWHM}^{(k)} (\tilde{p}^+, \tilde{p}^+, \ldots, \tilde{p}^+) = \tilde{p}^+
\]

\[
\text{DHPFWHM}^{(k)} (\tilde{p}^-, \tilde{p}^-, \ldots, \tilde{p}^-) = \tilde{p}^-
\]

From property 2, we can obtain

\[
\tilde{p}^- \leq \text{DHPFWHM}^{(k)} (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \tilde{p}^+
\]

**B. THE DHPFWHM AGGREGATION OPERATOR**

According to Definition 7, we can obtain that the DHPFWHM operator doesn’t take the importance of being fused arguments into account. However, in many practical MADM problems, we should consider the weights of attribute. To overcome the limitation of DHPFWHM operator, we shall propose the dual hesitant Pythagorean fuzzy weighted Hamy mean (DHPFWHM) operator as follows:

**Definition 8:** Assume that $\tilde{p}_j = (h_j, g_j)$ ($j = 1, 2, \ldots, n$) be a group of DHPFEs, and then we define the dual hesitant Pythagorean fuzzy weighted Hamy mean (DHPFWHM) operator as follows:

\[
\text{DHPFWHM}_w^{(k)} (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)
\]

\[
= \frac{\bigoplus_{1 \leq i < j \leq n} \left( \prod_{j=1}^{k} (\tilde{p}_{ij})^{w_j} \right)^{\frac{1}{k}}}{C_n^k}
\]

(18)

According to the operation laws of the DHPFEs described in definition 5, we can obtain the Theorem 2.

**Theorem 2:** Assume that $\tilde{p}_j = (h_j, g_j)$ ($j = 1, 2, \ldots, n$) be a collection of DHPFEs with weighting vector be $w = (w_1, w_2, \ldots, w_n)^T$ which satisfies $w_j > 0, i = 1, 2, \ldots, n$ and $\sum_{j=1}^{n} w_j = 1$, then their fused results by utilizing the DHPFWHM operator is also a DHPFE, and (19), as shown at the top of this page, holds.

**Proof:** Based on Definition 5, we can obtain:

\[
\left( \tilde{p}_j \right)^{w_j} = \bigcup_{\alpha_j \in h_j, \beta_j \in g_j} \left\{ \left\{ (\prod_{j=1}^{k} \alpha_j^{w_j})^{\frac{1}{k}} \right\}, \left\{ 1 - \left( \prod_{j=1}^{k} (1 - \beta_j^2)^{w_j} \right)^{\frac{1}{k}} \right\} \right\}
\]

(20)

Thus,

\[
\bigoplus_{j=1}^{k} (\tilde{p}_j)^{w_j} = \bigcup_{\alpha_j \in h_j, \beta_j \in g_j} \left\{ \left\{ (\prod_{j=1}^{k} \alpha_j^{w_j})^{\frac{1}{k}} \right\}, \left\{ 1 - \left( \prod_{j=1}^{k} (1 - \beta_j^2)^{w_j} \right)^{\frac{1}{k}} \right\} \right\}
\]

(21)

Therefore, (22), as shown at the top of this page, holds. Thereafter, (23), as shown at the top of the next page, holds. Therefore, (24), as shown at the top of the next page, holds. Thus, we have finished the proof.

**Example 4:** Assume that $\tilde{p}_1 = \{[0.7, 0.8], [0.4]\}$, $\tilde{p}_2 = \{[0.3], [0.6, 0.7]\}$, $\tilde{p}_3 = \{[0.1, 0.3], [0.4, 0.6]\}$ and $\tilde{p}_4 = \{[0.5], [0.5]\}$ be four DHPFEs, suppose $k = 2$ and $w_j = (0.3, 0.2, 0.1, 0.4)$, then according to the DHPFWHM operator, we can obtain the fused results as follows. For the membership degree function $\alpha$, the fused results are shown in (III) at the top of the next page.

Similarly, we can obtain

\[
\alpha_2 = \text{DHPFWHM}_w^{(2)} (0.7, 0.3, 0.3, 0.5) = 0.8351
\]

\[
\alpha_3 = \text{DHPFWHM}_w^{(2)} (0.8, 0.3, 0.1, 0.5) = 0.8210
\]
\[
\Theta \bigoplus_{1 \leq i_1 < \ldots < i_k \leq n} \left( \prod_{j=1}^{k} (\tilde{p}_{i_j})^{w_{i_j}} \right)^{1 \over t}
\]
\[
= \bigcup_{\alpha_j \in h_{i_j}, \beta_j \in g_{i_j}} \left\{ \left\{ 1 - \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} \alpha_j^{w_{i_j}} \right)^{1 \over t} \right) \right\} \times \left\{ 1 - \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} (1 - \beta_j^{w_{i_j}}) \right)^{1 \over t} \right) \right\} \right\}
\]

DHPFWHM\(^{(k)}\) \((\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)\)
\[
= \Theta \bigoplus_{1 \leq i_1 < \ldots < i_k \leq n} \left( \prod_{j=1}^{k} (\tilde{p}_{i_j})^{w_{i_j}} \right)^{1 \over t}
\]
\[
= \bigcup_{\alpha_j \in h_{i_j}, \beta_j \in g_{i_j}} \left\{ \left\{ 1 - \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} \alpha_j^{w_{i_j}} \right)^{1 \over t} \right) \right\} \times \left\{ 1 - \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} (1 - \beta_j^{w_{i_j}}) \right)^{1 \over t} \right) \right\} \right\}
\]

\[
\alpha_1 = \text{DHPFWHM}_{w}^{(2)} (0.7, 0.3, 0.1, 0.5) = \sqrt{1 - \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} \alpha_j^{w_{i_j}} \right)^{1 \over t} \right)}
\]
\[
= 1 - \left( \left( 1 - (0.7^{0.3} \times 0.3^{0.2})^{1 \over t} \right) \times \left( 1 - (0.7^{0.3} \times 0.1^{0.1})^{1 \over t} \right) \times \left( 1 - (0.7^{0.3} \times 0.5^{0.4})^{1 \over t} \right) \right)^{1 \over c_4}
\]
\[
= 0.8107
\]

\[
\alpha_4 = \text{DHPFWHM}_{w}^{(2)} (0.8, 0.3, 0.3, 0.5) = 0.8459,
\]

So we can get \( \alpha = \{0.8107, 0.8351, 0.8210, 0.8459\} \).

For the non-membership degree function \( \beta \), the fused results are shown in (IV) at the top of the next page.

Similarly, we can obtain
\[
\beta_2 = \text{DHPFWHM}_{w}^{(2)} (0.4, 0.6, 0.6, 0.5) = 0.2660
\]
\[
\beta_3 = \text{DHPFWHM}_{w}^{(2)} (0.4, 0.7, 0.4, 0.5) = 0.2677
\]
\[
\beta_4 = \text{DHPFWHM}_{w}^{(2)} (0.4, 0.7, 0.6, 0.5) = 0.2828,
\]

So we can get \( \beta = \{0.2504, 0.2660, 0.2677, 0.2828\} \).

Therefore, DHPFWHM\(^{(2)}\) \((\tilde{p}_1, \tilde{p}_2, \tilde{p}_3, \tilde{p}_4)\)
\[
= \left\{ \{0.8107, 0.8351, 0.8210, 0.8459\} \cup \{0.2504, 0.2660, 0.2677, 0.2828\} \right\}.
\]

It can be easily proved that the DHPFWHM operator satisfies the following properties.

**Property 4 (Monotonicity):** Let \( \tilde{p}_j = (h_j, g_j) \) and \( \tilde{p}_j' = (h_j', g_j'), j = 1, 2, \ldots, n, \) be two set of DHPFEs, if \( h_j \leq h_j', g_j \geq g_j' \) for all \( j \), then
\[
\text{DHPFWHM}_{w}^{(k)} (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \text{DHPFWHM}_{w}^{(k)} (\tilde{p}_1', \tilde{p}_2', \ldots, \tilde{p}_n')
\]
(25)

The proof is similar to DHPFH operator, so it’s omitted here.

**Property 5 (Boundedness):** Let \( \tilde{p}_j = (h_j, g_j) (j = 1, 2, \ldots, n) \) be a collection of DHPFEs, and let
\[
\tilde{p}^+ = \bigcup_{\alpha_j \in h_j, \beta_j \in g_j} \left\{ \{\max_i (\alpha_j)\}, \{\min_i (\beta_j)\} \right\}
\]
\[
\tilde{p}^- = \bigcup_{\alpha_j \in h_j, \beta_j \in g_j} \left\{ \{\min_i (\alpha_j)\}, \{\max_i (\beta_j)\} \right\}
\]
Then
\[
\tilde{p}^- \leq \text{DHPFWHM}_{w}^{(k)} (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \tilde{p}^+
\]
(26)
\[
\beta_1 = \text{DHPFWHM}^{(2)}_w(0.4, 0.6, 0.4, 0.5) = \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} \left( 1 - \beta_j^{i_j} \right)^{w_j} \right) \right) \right)^{\frac{1}{C_n^k}}
\]

\[
= \left( \left( \left( 1 - \left( (1 - 0.4^2)^{0.3} \times (1 - 0.6^2)^{0.2} \right)^{\frac{1}{2}} \times (1 - \left( (1 - 0.4^2)^{0.3} \times (1 - 0.6^2)^{0.2} \right)^{\frac{1}{2}}) \right) \right) \right)^{\frac{1}{C_n^k}} = 0.2504
\]

\[
\text{DHPFWHM}^{(k)}_w(\tilde{p}^+, \tilde{p}^+, \ldots, \tilde{p}^+)
\]

\[
= \bigoplus_{1 \leq i_1 < \ldots < i_k \leq n} \left( \frac{k}{C_n^k} \left( \max \tilde{p}_{i_j} \right)^{w_j} \right)^{\frac{1}{2}} \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} \left( \max \alpha_j \right)^{w_j} \right) \right) \right)^{\frac{1}{C_n^k}}
\]

\[
= \frac{1}{C_n^k} \left\{ \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} \min \alpha_j \right)^{w_j} \right) \right) \right\}^{\frac{1}{C_n^k}}
\]

\[
\text{DHPFWHM}^{(k)}_w(\tilde{p}^-, \tilde{p}^-, \ldots, \tilde{p}^-)
\]

\[
= \bigoplus_{1 \leq i_1 < \ldots < i_k \leq n} \left( \frac{k}{C_n^k} \left( \min \tilde{p}_{i_j} \right)^{w_j} \right)^{\frac{1}{2}} \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} \min \alpha_j \right)^{w_j} \right) \right)^{\frac{1}{C_n^k}}
\]

\[
= \frac{1}{C_n^k} \left\{ \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} \max \alpha_j \right)^{w_j} \right) \right) \right\}^{\frac{1}{C_n^k}}
\]

**Proof:** Based on theorem 2, we can obtain (27) and (28), as shown at the top of this page.

From property 4, we can derive

\[
\tilde{p}^- \leq \text{DHPFWHM}^{(k)}_w(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \tilde{p}^+
\]

**C. THE DHPFDHM AGGREGATION OPERATOR**

In the following, based on the dual operation laws, Wu et al. [54] extended the Hamy mean (HM) operator to the dual Hamy mean (DHM) operator which can be depicted as follows.

\[
\text{Definition 9 [54]: The DHM operator is defined as follows:}
\]

\[
\text{DHM}^{(k)}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)
\]

\[
= \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( \frac{1}{k} \left( \sum_{j=1}^{k} \tilde{p}_{i_j} \right) \right) \right)^{\frac{1}{C_n^k}}
\]

where \(k\) is a parameter and \(k = 1, 2, \ldots, n\), \(i_1, i_2, \ldots, i_k\) are \(k\) integer values taken from the set \(\{1, 2, \ldots, n\}\) of \(k\) integer values, \(C_n^k\) denotes the binomial coefficient and \(C_n^k = \frac{n!}{k!(n-k)!}\).

In this section, we shall study the dual Hamy mean (DHM) operator with dual hesitant Pythagorean fuzzy information.
According to Definition 5, we give the definition of the dual hesitant Pythagorean fuzzy dual Hamy mean (DHPFDHM) operator as follows.

**Definition 10:** Assume that \( \tilde{p}_j = (h_j, g_j) \) \((j = 1, 2, \ldots, n)\) be a collection of DHPFEs, and then the dual hesitant Pythagorean fuzzy dual Hamy mean (DHPFDHM) operator can be defined as:

\[
\text{DHPFDHM}^{(k)}(\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) = \left( \bigotimes_{1 \leq i_1 < \ldots < i_k \leq n} \left( \frac{1}{k} \bigoplus_{j=1}^{k} \tilde{p}_{i_j} \right) \right)^{\frac{1}{c_k}}
\]

(31)

Thus,

\[
\frac{1}{k} \left( \bigoplus_{j=1}^{k} \tilde{p}_{i_j} \right) = \bigcup_{i_j \in e_j, \beta_j \in g_j} \left\{ \left\{ \left( 1 - \prod_{j=1}^{k} \left( 1 - \alpha_{i_j}^2 \right)^{\frac{1}{2}} \right) \right\}, \left\{ \prod_{j=1}^{k} \beta_j \right\} \right\}
\]

(33)

Therefore, (35), as shown at the top of this page, holds. Furthermore, (36), as shown at the top of this page, holds. Thus, we have finished the proof.

**Example 5:** Assume that \( \tilde{p}_1 = \{(0.7, 0.8), (0.4)\}, \tilde{p}_2 = \{(0.3), (0.6, 0.7)\}, \tilde{p}_3 = \{(0.1, 0.3), (0.4, 0.6)\} \) and \( \tilde{p}_4 = \{(0.5), (0.5)\} \) be four DHPFEs, suppose \( k = 2 \), then according to the DHPFDHM operator, we can obtain the fused results as follows. For the membership degree \( \alpha \), the fused results are shown in (V) at the top of the next page.

Similarly, we can obtain

\[
\alpha_2 = \text{DHPFDHM}^{(2)}(0.7, 0.3, 0.3, 0.5) = 0.4662
\]

\[
\alpha_3 = \text{DHPFDHM}^{(2)}(0.8, 0.3, 0.1, 0.5) = 0.4646
\]

\[
\alpha_4 = \text{DHPFDHM}^{(2)}(0.8, 0.3, 0.3, 0.5) = 0.4991,
\]

So we can get \( \alpha = \{0.4327, 0.4662, 0.4646, 0.4991\} \).
\[ \alpha_1 = \text{DHPFDHM}^{(2)}(0.7, 0.3, 0.1, 0.5) = \left( \frac{1}{C^3} \right) \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \prod_{j=1}^k \left( 1 - a_j^2 \right)^{\frac{1}{2}} \right) \right) \]

\[ = \left( \frac{1}{C^3} \right) \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( 1 - (0.7)^{\frac{1}{2}} \right) \times \left( 1 - (0.3)^{\frac{1}{2}} \right) \times \left( 1 - (0.1)^{\frac{1}{2}} \right) \right) \right) \]

\[ = 0.4327 \] (V)

\[ \beta_1 = \text{DHPFDHM}^{(2)}(0.4, 0.6, 0.4, 0.5) = \left( \frac{1}{C^3} \right) \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^k \beta_j \right)^{\frac{1}{2}} \right) \right) \]

\[ = \left( \frac{1}{C^3} \right) \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( 1 - (0.4 \times 0.6)^{\frac{1}{2}} \right) \times \left( 1 - (0.4 \times 0.4)^{\frac{1}{2}} \right) \times \left( 1 - (0.4 \times 0.5)^{\frac{1}{2}} \right) \right) \right) \]

\[ = 0.4739 \] (VI)

For the non-membership degree function \( \beta \), the fused results are shown in (VI) at the top of this page.

Similarly, we can obtain

\[ \beta_2 = \text{DHPFDHM}^{(2)}(0.4, 0.6, 0.6, 0.5) = 0.5247 \]

\[ \beta_3 = \text{DHPFDHM}^{(2)}(0.4, 0.7, 0.4, 0.5) = 0.4978 \]

\[ \beta_4 = \text{DHPFDHM}^{(2)}(0.4, 0.7, 0.6, 0.5) = 0.5497 \]

So we can get \( \beta = \{0.4739, 0.5247, 0.4978, 0.5497\} \).

Therefore,

\[ \text{DHPFDHM}^{(2)}(\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{\beta}_4) = \{0.4327, 0.4662, 0.4646, 0.4991\} \cup \{0.4739, 0.5247, 0.4978, 0.5497\} \]

It can be easily proved that the DHPFDHM operator satisfies the following properties. The proof is similar to DHPFH operator, so it’s omitted here.

Property 6 (Idempotency): If all \( \tilde{\beta}_j = (h_j, g_j) \) \( (j = 1, 2, \ldots, n) \) are equal, i.e. \( \tilde{\beta}_j = \tilde{\beta} \) for all \( j \), then

\[ \text{DHPFDHM}^{(k)}(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n) = \tilde{\beta} \] (37)

Property 7 (Boundedness): Let \( \tilde{\beta}_j = (h_j, g_j) \) \( (j = 1, 2, \ldots, n) \) be a collection of DHPFES, and let

\[ \tilde{\beta}^+ = \bigcup_{h_j, g_j} \{\max_i (\alpha_i), \min_i (\beta_j)\} \]

\[ \tilde{\beta}^- = \bigcup_{h_j, g_j} \{\min_i (\alpha_i), \max_i (\beta_j)\} \]

Then

\[ \tilde{\beta}^- \leq \text{DHPFDHM}^{(k)}(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n) \leq \tilde{\beta}^+ \] (38)

Property 8 (Monotonicity): Let \( \tilde{\beta}_j = (h_j, g_j) \) and \( \tilde{\beta}_j' = (h_j', g_j') \) \( j = 1, 2, \ldots, n \), be two set of DHPFEs, if \( h_j \leq h_j' \), \( g_j \geq g_j' \) for all \( j \), then

\[ \text{DHPFDHM}^{(k)}(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n) \leq \text{DHPFDHM}^{(k)}(\tilde{\beta}_1', \tilde{\beta}_2', \ldots, \tilde{\beta}_n') \] (39)

D. THE DHPFWDM AGGREGATION OPERATOR

According to Definition 10, we can obtain that the DHPFDHM operator doesn’t take the importance of being fused arguments into account. However, in many practical MADM problems, we should consider the weights of attribute. To overcome the limitation of DHPFDHM operator, we shall propose the dual hesitant Pythagorean fuzzy weighted dual Hamy mean (DHPFWDM) operator as follows.

Definition 11: Assume that \( \tilde{\beta}_j = (h_j, g_j) \) \( (j = 1, 2, \ldots, n) \) be a collection of DHPFEs, and then the dual hesitant Pythagorean fuzzy weighted dual Hamy mean (DHPFWDM) operator can be defined as:

\[ \text{DHPFWDM}^{(k)}(\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_n) = \left( \bigotimes_{1 \leq i_1 < \ldots < i_k \leq n} \left( \frac{1}{k} \left( \bigoplus_{j=1}^k w_j \tilde{\beta}_j \right) \right) \right) \] (40)
DHPFWHM_{w}^{(k)}(\hat{p}_{1}, \hat{p}_{2}, \ldots, \hat{p}_{n})
\begin{align*}
= \bigoplus_{1 \leq i_{1} \leq \ldots \leq i_{k} \leq n} \left( \frac{1}{C_{k}^{n}} \right)^{\frac{k}{2}} \\
= \bigcup_{I \in G_{ij}} \left\{ \left[ 1 - \prod_{1 \leq i_{1} \leq \ldots \leq i_{k} \leq n} \left( 1 - \left( 1 - \left( \prod_{j=1}^{k} \alpha_{j} w_{j} \right)^{\frac{1}{2}} \right) \right) \right] \cdot \left( \prod_{j=1}^{k} \left( 1 - \left( \prod_{j=1}^{k} \left( \prod_{j=1}^{k} \beta_{j} w_{j} \right)^{\frac{1}{2}} \right) \right) \right) \right\} \tag{41}
\end{align*}

\begin{align*}
\frac{1}{k} \left( \bigoplus_{j=1}^{k} w_{j} \hat{p}_{j} \right) = \bigcup_{I \in G_{ij}} \left\{ \left[ \sqrt{1 - \prod_{j=1}^{k} \left( 1 - \left( 1 - \left( \prod_{j=1}^{k} \alpha_{j} w_{j} \right)^{\frac{1}{2}} \right) \right) \right] \cdot \left( \prod_{j=1}^{k} \left( 1 - \left( 1 - \left( \prod_{j=1}^{k} \beta_{j} w_{j} \right)^{\frac{1}{2}} \right) \right) \right) \right\} \tag{44}
\end{align*}

\begin{align*}
\bigoplus_{1 \leq i_{1} \leq \ldots \leq i_{k} \leq n} \left( \frac{1}{k} \left( \bigoplus_{j=1}^{k} \hat{p}_{j} \right) \right) \bigotimes_{j=1}^{k} w_{j} \hat{p}_{j} = \bigcup_{I \in G_{ij}} \left\{ \left[ \prod_{1 \leq i_{1} \leq \ldots \leq i_{k} \leq n} \left( 1 - \left( 1 - \left( \prod_{j=1}^{k} \alpha_{j} w_{j} \right)^{\frac{1}{2}} \right) \right) \right] \cdot \left( \prod_{j=1}^{k} \left( 1 - \left( 1 - \left( \prod_{j=1}^{k} \beta_{j} w_{j} \right)^{\frac{1}{2}} \right) \right) \right) \right\} \tag{45}
\end{align*}

According to the operation laws of the DHPFEs described in definition 5, we can obtain Theorem 4.

**Theorem 4:** Assume that \( \hat{p}_{j} = (h_{j}, g_{j}) \) \((j = 1, 2, \ldots, n)\) be a collection of DHPFEs with weighting vector \( w = (w_{1}, w_{2}, \ldots, w_{n})^{T} \) which satisfies \( w_{j} > 0, i = 1, 2, \ldots, n \) and \( \sum_{j=1}^{n} w_{j} = 1 \), then their fused results by utilizing the DHPFWDM operator is also a DHPFE, and (41), as shown at the top of this page, holds.

**Proof:** Based on Definition 5, we can obtain:

\begin{align*}
w_{j} \hat{p}_{j} = \bigcup_{I \in G_{ij}} \left\{ \left[ \sqrt{1 - \left( 1 - \left( \prod_{j=1}^{k} \alpha_{j} w_{j} \right) \right) ^{\frac{1}{2}}} \right] \cdot \left( \prod_{j=1}^{k} \beta_{j} w_{j} \right) \right\} \tag{42}
\end{align*}

Thus,

\begin{align*}
\bigoplus_{j=1}^{k} w_{j} \hat{p}_{j} \bigotimes_{j=1}^{k} \hat{p}_{j} = \bigcup_{I \in G_{ij}} \left\{ \left[ \prod_{1 \leq i_{1} \leq \ldots \leq i_{k} \leq n} \left( 1 - \left( 1 - \left( \prod_{j=1}^{k} \alpha_{j} w_{j} \right) \right) ^{\frac{1}{2}} \right) \right] \cdot \left( \prod_{j=1}^{k} \beta_{j} w_{j} \right) \right\} \tag{43}
\end{align*}

Therefore, (44), as shown at the top of this page, holds. Thereafter, (45), as shown at the top of this page, holds. Therefore, (46), as shown at the top of the next page, holds. Thus, we have finished the proof.

**Example 6:** Assume that \( \hat{p}_{1} = \{0.7, 0.8\}, \{0.4\} \), \( \hat{p}_{2} = \{0.3, 0.6, 0.7\} \), \( \hat{p}_{3} = \{0.1, 0.3\}, \{0.4, 0.6\} \) and \( \hat{p}_{4} = \{0.5\}, \{0.5\} \) be four DHPFEs, suppose \( k = 2 \) and \( w_{j} = (0.3, 0.2, 0.1, 0.4) \), then according to the DHPFWDM operator, we can obtain the fused results as follows. For the membership degree function \( \alpha \), the fused results are shown in (VII) at the top of the next page.

So we can get \( \alpha = (0.2477,0.2574,0.2695,0.2797) \).

For the non-membership degree function \( \beta \), the fused results are shown in (VIII) at the top of the next page.

Therefore,\n
\[ \text{DHPFWDM}_{w}^{(2)}(\hat{p}_{1}, \hat{p}_{2}, \hat{p}_{3}, \hat{p}_{4}) = \{0.2477,0.2574,0.2695,0.2797\}, \{0.8378,0.8495,0.8464,0.8587\} \]
DHPFWDMH\(^{(k)}\) \((\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n)\)

\[
\begin{align*}
&= \left[ \bigotimes_{1 \leq i_1 < \ldots < i_k \leq n} \left( \frac{1}{k} \left( \prod_{j=1}^{k} w_j \tilde{p}_{i_j} \right) \right) \right]^{\frac{1}{c_k^n}} \\
&= \bigcup_{\alpha_i \in h_j, \beta_i \in g_j} \left\{ \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} \left( 1 - \alpha_j^{\ast} \right)^{w_j} \right)^\frac{1}{k} \right) \right) \right\}^{\frac{1}{c_k^n}} \\
&\quad \times \left( \prod_{1 \leq i_1 < \ldots < i_k \leq n} \left( 1 - \left( \prod_{j=1}^{k} \left( 1 - \beta_j^{\ast} \right)^{w_j} \right)^\frac{1}{k} \right) \right) \right\}^{\frac{1}{c_k^n}} \right. \\
&= \left( \prod_{1 \leq i_1 < \ldots < i_n \leq n} \left( 1 - \left( \prod_{j=1}^{k} \left( 1 - \alpha_j^{\ast} \right)^{w_j} \right)^\frac{1}{k} \right) \right) \right\}^{\frac{1}{c_k^n}} \\
&= 0.2477 \\
\end{align*}
\] (VII)

\[
\begin{align*}
\alpha_1 &= \text{DHPFWDMH}^{(2)}_{w} (0.7, 0.3, 0.1, 0.5) = \left( \prod_{1 \leq i_1 < \ldots < i_n \leq n} \left( 1 - \left( \prod_{j=1}^{k} \left( 1 - \alpha_j^{\ast} \right)^{w_j} \right)^\frac{1}{k} \right) \right) \right\}^{\frac{1}{c_k^n}} \\
&= \left( \prod_{1 \leq i_1 < \ldots < i_n \leq n} \left( 1 - \left( \prod_{j=1}^{k} \left( 1 - \alpha_j^{\ast} \right)^{w_j} \right)^\frac{1}{k} \right) \right) \right\}^{\frac{1}{c_k^n}} \\
&= 0.2477 \\
\end{align*}
\] (VII)

\[
\begin{align*}
\beta_1 &= \text{DHPFWDMH}^{(2)}_{w} (0.4, 0.6, 0.4, 0.5) = \left( \prod_{1 \leq i_1 < \ldots < i_n \leq n} \left( 1 - \left( \prod_{j=1}^{k} \left( 1 - \beta_j^{\ast} \right)^{w_j} \right)^\frac{1}{k} \right) \right) \right\}^{\frac{1}{c_k^n}} \\
&= \left( \prod_{1 \leq i_1 < \ldots < i_n \leq n} \left( 1 - \left( \prod_{j=1}^{k} \left( 1 - \beta_j^{\ast} \right)^{w_j} \right)^\frac{1}{k} \right) \right) \right\}^{\frac{1}{c_k^n}} \\
&= 0.8378 \\
\end{align*}
\] (VIII)

It can be easily proved that the DHPFWDMH operator satisfies the following properties. The proof is similar to DHPFWHM operator, so it’s omitted here.

**Property 9 (Boundedness):** Let \( \bar{p}_j = (h_j, g_j) (j = 1, 2, \ldots, n) \) be a collection of DHPFEs, and let

\[
\begin{align*}
\bar{p}^+ &= \bigcup_{\alpha_i \in h_j, \beta_i \in g_j} \{ \max_i (\alpha_j), \min_i (\beta_j) \} \\
\bar{p}^- &= \bigcup_{\alpha_i \in h_j, \beta_i \in g_j} \{ \min_i (\alpha_j), \max_i (\beta_j) \}
\end{align*}
\]

Then

\[
\bar{p}^- \leq \text{DHPFWDMH}^{(k)}_{w} (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \bar{p}^+ \quad (47)
\]

**Property 10 (Monotonicity):** Let \( \bar{p}_j = (h_j, g_j) \) and \( \bar{p}_j' = (h_j', g_j') \), \( j = 1, 2, \ldots, n \), be two set of DHPFEs, if \( h_j \leq h_j' \), \( g_j \geq g_j' \), for all \( j \), then

\[
\text{DHPFWDMH}^{(k)}_{w} (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n) \leq \text{DHPFWDMH}^{(k)}_{w} (\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n) \quad (48)
\]

**IV. AN APPROACH TO MADM WITH DHPFES**

In this section, we shall use the DHPFWHM and DHPFWDMH operators to deal with MADM problems with DHPFES. Suppose there are \( m \) alternatives \( A = \{A_1, A_2, \ldots, A_m\} \), each alternative is characterized by \( n \) attributes \( G = \{G_1, G_2, \ldots, G_n\} \) with weighting vector be \( w_j = \{w_1, w_2, \ldots, w_n\} \). Then the dual hesitant Pythagorean fuzzy matrix can be constructed as \( \tilde{P} = (\tilde{p}_{ij})_{m \times n} \), each

...
Supplier selection in supply chain management is classical MADM problems for supplier selection in supply chain management with dual hesitant Pythagorean fuzzy information.

Step 1: We aggregate the decision information given in matrix $P = (\tilde{p}_{ij})_{m \times n}$ by the DHPFWM operator (49), as shown at the top of this page.

Or the DHPFDHM operator (50), as shown at the top of this page, to obtain the overall fused results $\tilde{p}_i$ ($i = 1, 2, \ldots, m$).

Step 2: Compute the scores values $S (\tilde{p}_i)$ ($i = 1, 2, \ldots, m$) of $\tilde{p}_i$ ($i = 1, 2, \ldots, m$). If there is no difference between any two scores $S (\tilde{p}_i)$ and $S (\tilde{p}_j)$, then we need to compute the accuracy values $H (\tilde{p}_i)$ and $H (\tilde{p}_j)$ of $\tilde{p}_i$ and $\tilde{p}_j$, respectively, and then determine the ordering of all the alternatives $A_i$ and $A_j$ based on the accuracy results $H (\tilde{p}_i)$ and $H (\tilde{p}_j)$.

Step 3: Determine the ordering of all the alternatives $A_i$ ($i = 1, 2, \ldots, m$) and select the best one(s) according to the scores values $S (\tilde{p}_i)$ ($i = 1, 2, \ldots, m$).

Step 4: End.

V. NUMERICAL EXAMPLE AND COMPARATIVE ANALYSIS

A. NUMERICAL EXAMPLE

Supplier selection in supply chain management is classical MADM [62-67]. Thus, in this section we shall present a numerical example for supplier selection in supply chain management with DHPFEs in order to demonstrate the method proposed in this paper. Suppose there is a problem to deal with the supplier selection in supply chain management which is classical MADM problems. There are five prospect suppliers $\eta_i$ ($i = 1, 2, 3, 4, 5$) for four attributes $\delta_j$ ($j = 1, 2, 3, 4$). The four attributes include product quality ($\delta_1$), service ($\delta_2$), delivery ($\delta_3$) and price ($\delta_4$), respectively. In order to avoid influence each other, the decision makers are required to evaluate the five suppliers $\eta_i$ ($i = 1, 2, 3, 4, 5$) under the above four attributes in anonymity and the decision matrix $P = (\tilde{p}_{ij})_{5 \times 4}$ is presented in Table 1, where $\tilde{p}_{ij}$ ($i = 1, 2, 3, 4, 5, j = 1, 2, 3, 4$) are in the form of DHPFEs. (Suppose weighting vector be $w_j = (0.25, 0.34, 0.27, 0.14)$).

In what follows, we can utilize our developed methods to deal with the supplier selection in supply chain management with DHPFEs.

Step 1: We aggregate the DHPFEs given in matrix by utilizing the DHPFWM operator to obtain the overall preference values $\tilde{p}_i$ of the supplier in supply chain management $\eta_i$ ($i = 1, 2, 3, 4, 5$). Take alternative $\eta_1$ for an example (here, we take $k = 2$), we have (IX), as shown at the bottom of the next page.

Step 2: Compute the scores results $s (\tilde{p}_i)$ ($i = 1, 2, 3, 4, 5$) of the overall dual hesitant Pythagorean fuzzy preference values $\tilde{p}_i$ ($i = 1, 2, 3, 4, 5$):

\[
s(\tilde{p}_1) = 0.7797, \quad s(\tilde{p}_2) = 0.7999, \quad s(\tilde{p}_3) = 0.7935
\]

\[
s(\tilde{p}_4) = 0.8313, \quad s(\tilde{p}_5) = 0.7716
\]
Step 3: Determine the ordering of all the suppliers \( \eta_i \) \((i = 1, 2, 3, 4, 5)\) based on the scores values \( s(\tilde{p}_i) \) \((i = 1, 2, 3, 4, 5)\): \( \eta_4 > \eta_5 > \eta_3 > \eta_1 > \eta_2 \), and it’s clear that the most desirable supplier is \( \eta_4 \).

Similarly, if we utilize the DHPFWDHM operator to solve this MADM, the decision making steps can be described as follows.

Step 1': Aggregate all DHPFEs \( \tilde{p}_j \) \((j = 1, 2, 3, 4)\) by using the dual hesitant Pythagorean fuzzy weighted dual Hamy mean (DHPFWHM) operator to derive the overall DHPFEs \( \tilde{p}_i \) \((i = 1, 2, 3, 4, 5)\) of the supplier \( \eta_i \). Take supplier \( \eta_1 \) for an example (here, we take \( k = 2 \)), we have \((X, X)\), as shown at the bottom of this page.

Step 2': Compute the score results \( s(\tilde{p}_i) \) \((i = 1, 2, 3, 4, 5)\) of the overall DHPFEs \( \tilde{p}_i \) \((i = 1, 2, 3, 4, 5)\) of the suppler \( \tilde{p}_i \):

\[
\begin{align*}
\tilde{p}_1 &= \text{DHPFWHM}_w^{(k)}(\tilde{p}_{11}, \tilde{p}_{12}, \tilde{p}_{13}, \tilde{p}_{14}) \\
&= \frac{\bigoplus_{1 \leq i_1 < \cdots < i_k \leq n} \left( \bigotimes_{j=1}^{k} (\tilde{p}_{ij})^{w_j} \right)^{\frac{1}{k}}}{C_n^k} \\
&= \bigcup_{\alpha_j \in \mathbb{H}_j, \beta_j \in \mathbb{H}_j} \left\{ \left[ \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left( 1 - \prod_{j=1}^{k} \alpha_j^{w_j} \right)^{\frac{1}{k}} \right]^{\frac{1}{C_n^k}} \right\} \\
&= \{\{0.4, 0.5\} \{0.7\}, \{0.5, 0.6\} \{0.4, 0.5\}, \{0.3, 0.4\} \{0.8\}, \{0.5, 0.6\}, \{0.6\}\} \\
&= 0.8062, 0.8128, 0.8199, 0.8265, 0.8194, 0.8262, 0.8327, 0.8394, 0.8182, 0.8249, 0.8315, 0.8382, 0.8311, 0.8379, 0.8440, 0.8506, \{0.3494, 0.3646\}\} \quad (IX)
\end{align*}
\]

\[
\begin{align*}
\tilde{p}_1 &= \text{DHPFDHM}_w^{(k)}(\tilde{p}_{11}, \tilde{p}_{12}, \tilde{p}_{13}, \tilde{p}_{14}) \\
&= \left( \bigotimes_{1 \leq i_1 < \cdots < i_k \leq n} \left( \frac{1}{k} \left( \bigoplus_{j=1}^{k} w_j \tilde{p}_{ij} \right) \right)^{\frac{1}{k}} \right) \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left( 1 - \prod_{j=1}^{k} \beta_j^{w_j} \right)^{\frac{1}{k}} \frac{1}{C_n^k} \\
&= \bigcup_{\alpha_j \in \mathbb{H}_j, \beta_j \in \mathbb{H}_j} \left\{ \left[ \prod_{1 \leq i_1 < \cdots < i_k \leq n} \left( 1 - \prod_{j=1}^{k} \alpha_j^{w_j} \right)^{\frac{1}{k}} \right]^{\frac{1}{C_n^k}} \right\} \\
&= \{\{0.4, 0.5\} \{0.7\}, \{0.5, 0.6\} \{0.4, 0.5\}, \{0.3, 0.4\} \{0.8\}, \{0.5, 0.6\}, \{0.6\}\} \\
&= 0.2191, 0.2315, 0.2323, 0.2441, 0.2377, 0.2503, 0.2511, 0.2629, 0.2342, 0.2463, 0.2468, 0.2582, 0.2530, 0.2652, 0.2656, 0.2769, \{0.8895, 0.8997\}\} \quad (X)
\end{align*}
\]
The prominent characteristic of the DHPFWHM and DHPFWDHM operators are crucial in real MADM problems. These features of the DHPFWHM and DHPFWDHM operators are crucial in real MADM problems.

C. COMPARATIVE ANALYSIS

The prominent characteristic of the DHPFWHM and DHPFWDHM operators is that they can consider the interrelationship among any number of DHFNs. Next, we shall compare our developed methods with dual hesitant Pythagorean fuzzy weighted average (DHPFWA) and dual hesitant Pythagorean fuzzy weighted geometric (DHPFWG) operators [51], the comparative analysis results are listed as follows.

According to Table 1 and attribute weights, the fused values by DHPFWA operator are shown in (XI) at the top of the this page.
Then based on the score function of dual hesitant Pythagorean fuzzy elements (DHPFEs), we can obtain the score results of \( \tilde{p}_i \) as:

\[
\begin{align*}
    s(\tilde{p}_1) &= 0.4317, & s(\tilde{p}_2) &= 0.4822, & s(\tilde{p}_3) &= 0.4977 \\
    s(\tilde{p}_4) &= 0.6592, & s(\tilde{p}_5) &= 0.5005
\end{align*}
\]

Rank all the suppliers in supply chain management \( \eta_i \) \((i = 1, 2, 3, 4, 5)\) in accordance with the scores \( s(\tilde{p}_i) \) \((i = 1, 2, 3, 4, 5)\) of the overall DHPFEs \( \tilde{p}_i \) \((i = 1, 2, 3, 4, 5)\): \( \eta_4 > \eta_5 > \eta_3 > \eta_2 > \eta_1 \) and thus the most desirable supplier in supply chain management is \( \eta_4 \).

According to Table 1 and attribute weights, the fused values by DHPFWM operator are shown in (XII) at the top of the previous page.

Then based on the score function of DHPFEs, we can obtain the score results of \( \tilde{p}_i \) as:

\[
\begin{align*}
    s(\tilde{p}_1) &= 0.3851, & s(\tilde{p}_2) &= 0.4099, & s(\tilde{p}_3) &= 0.3584 \\
    s(\tilde{p}_4) &= 0.4939, & s(\tilde{p}_5) &= 0.4410
\end{align*}
\]

Rank all the suppliers in supply chain management \( \eta_i \) \((i = 1, 2, 3, 4, 5)\) in accordance with the scores \( s(\tilde{p}_i) \) \((i = 1, 2, 3, 4, 5)\) of the overall DHPFEs

---

**TABLE 1.** Dual hesitant Pythagorean fuzzy decision matrix.

| \( \delta_1 \) | \( \delta_2 \) | \( \delta_3 \) | \( \delta_4 \) |
|----------------|----------------|----------------|----------------|
| \( \eta_1 \)  | \{0.4,0.5\}, \{0.7\} | \{0.5,0.6\}, \{0.4,0.5\} | \{0.3,0.4\}, \{0.8\} | \{0.5,0.6\}, \{0.6\} |
| \( \eta_2 \)  | \{0.7\}, \{0.5\} | \{0.3,0.5,0.6\}, \{0.5\} | \{0.3\}, \{0.7,0.8,0.9\} | \{0.6\}, \{0.5,0.6\} |
| \( \eta_3 \)  | \{0.6,0.8\}, \{0.3\} | \{0.3\}, \{0.8,0.9\} | \{0.3,0.4,0.5\}, \{0.7\} | \{0.6,0.7,0.8\}, \{0.4\} |
| \( \eta_4 \)  | \{0.8\}, \{0.4\} | \{0.7,0.8,0.9\}, \{0.3\} | \{0.2,0.3\}, \{0.4\} | \{0.2\}, \{0.7,0.8,0.9\} |
| \( \eta_5 \)  | \{0.1,0.2\}, \{0.3\} | \{0.3,0.4,0.5\}, \{0.6\} | \{0.5,0.6\}, \{0.3\} | \{0.3,0.4,0.5\}, \{0.6\} |

---

**TABLE 2.** Ordering by the DHPFWHM operators.

| Parameter | \( s(\eta_1) \) | \( s(\eta_2) \) | \( s(\eta_3) \) | \( s(\eta_4) \) | \( s(\eta_5) \) | Ordering |
|-----------|----------------|----------------|----------------|----------------|----------------|-----------|
| \( k = 1 \) | 0.7972 | 0.8261 | 0.8429 | 0.8595 | 0.7940 | \( \eta_4 \succ \eta_3 \succ \eta_2 \succ \eta_1 \succ \eta_5 \) |
| \( k = 2 \) | 0.7797 | 0.7999 | 0.7935 | 0.8313 | 0.7716 | \( \eta_4 \succ \eta_3 \succ \eta_2 \succ \eta_1 \succ \eta_5 \) |
| \( k = 3 \) | 0.7740 | 0.7878 | 0.7694 | 0.8184 | 0.7619 | \( \eta_4 \succ \eta_3 \succ \eta_2 \succ \eta_1 \succ \eta_5 \) |
| \( k = 4 \) | 0.7712 | 0.7815 | 0.7591 | 0.8121 | 0.7569 | \( \eta_4 \succ \eta_3 \succ \eta_2 \succ \eta_1 \succ \eta_5 \) |

---

**TABLE 3.** Ordering o by the DHPFWHDM operators.

| Parameter | \( s(\eta_1) \) | \( s(\eta_2) \) | \( s(\eta_3) \) | \( s(\eta_4) \) | \( s(\eta_5) \) | Ordering |
|-----------|----------------|----------------|----------------|----------------|----------------|-----------|
| \( k = 1 \) | 0.1182 | 0.1357 | 0.1365 | 0.1551 | 0.1659 | \( \eta_5 \succ \eta_4 \succ \eta_3 \succ \eta_2 \succ \eta_1 \) |
| \( k = 2 \) | 0.1308 | 0.1537 | 0.1569 | 0.2260 | 0.1903 | \( \eta_5 \succ \eta_4 \succ \eta_3 \succ \eta_2 \succ \eta_1 \) |
| \( k = 3 \) | 0.1377 | 0.1600 | 0.1655 | 0.2515 | 0.1973 | \( \eta_5 \succ \eta_4 \succ \eta_3 \succ \eta_2 \succ \eta_1 \) |
| \( k = 4 \) | 0.1421 | 0.1630 | 0.1704 | 0.2598 | 0.2003 | \( \eta_5 \succ \eta_4 \succ \eta_3 \succ \eta_2 \succ \eta_1 \) |
According to Table 4, we can easily obtain that the ordering are slightly different and the best alternative are some, however, our defined operators are mainly characteristic of the advantages that can take the interrelationship between any number of being fused arguments into consideration and consider the human’s hesitance in practical MADM problems, obviously, the DHPFWA and DHPFWG operators defined by Wei and Lu [51] cannot consider the interrelationship between the being fused arguments. In addition, in complicated decision-making environment, the decision maker’s risk attitude is an important factor to think about, our methods can make this come true by altering the parameters \( k \) whereas the DHPFWA and DHPFWG operators presented by Wei and Lu [51] don’t have the ability that dynamic adjust to the parameter according to the decision maker’s risk attitude, so it is difficult to solve the risk multiple attribute decision making in real practice.

VI. CONCLUSION

The DHPFEs have applied the advantages of DHFSs and PFSs. They can flexibly denote decision-making information as well as effectively characterize the reliability of information. Thus, it is meaningful to study MADM problems with DHPFEs. In this paper, based on the Hamy mean (GHM) operator and dual Hamy mean (DHM) operator, we develop some dual hesitant Pythagorean fuzzy Hamy mean aggregation operators: the dual hesitant Pythagorean fuzzy Hamy mean (DHPFGHM) operator, the dual hesitant Pythagorean fuzzy weighted Hamy mean (DHPFWHM) operator, the dual hesitant Pythagorean fuzzy dual Hamy mean (DHPFDHM) operator and the dual hesitant Pythagorean fuzzy weighted dual Hamy mean (DHPFWDHM) operator. Of course, the precious merits of these defined operators are investigated. Moreover, we have adopted DHPFWHM and DHPFWDHM operators to build the decision-making model for MADM problems. In the end, we take a concrete instance for appraising the suppliers selection in supply chain management to demonstrate our defined model and to testify its accuracy and scientific. However, the scope of the evaluation information which is expressed by DHPFEs is still limited, it must satisfies the sum square of maximum membership and maximum non-membership is less or equal to 1. Thus, how to overcome this limitation need to be further studied. In the future, we shall continue studying the MADM problems with the application and extension of the developed operators to other domains [66]–[73].

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