Perturbative test of exact vacuum expectation values of local fields in affine Toda theories

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Abstract

Vacuum expectation values of local fields for all dual pairs of non-simply laced affine Toda field theories recently proposed are checked against perturbative analysis. The computations based on Feynman diagram expansion are performed upto two-loops. Agreement is obtained.
1 Introduction

The vacuum expectation values (VEV)s of local fields play an important role in quantum field theory (QFT) and statistical mechanics [1, 2]. In QFT defined as perturbed conformal field theory (CFT), they constitute the basic ingredients for multipoint correlation functions, using short-distance expansions [3, 4]. Some times ago, important progress was made in the calculations of the VEVs in two dimensional integrable QFT. In ref. [4], an explicit expression for the VEVs of the exponential field in the sine-Gordon and sinh-Gordon models - $A_{i}^{(1)}$ affine Toda field theory (ATFT) - was proposed. Moreover, it was shown in [5] that this expression can be obtained as the minimal solution of certain “reflection relations” which involve the Liouville “reflection amplitude” [6], where the sinh-Gordon QFT was considered as a perturbed Liouville conformal field theory. Later, this “reflection relations” method was successfully generalized to other models, for which the VEVs were calculated. We refer the reader to refs. [7, 8, 9, 10, 11] for details.

Among the family of known integrable QFTs, it was thus natural to study the case of dual pairs of non-simply laced ATFTs, beyond the simply laced one which had been previously considered [12]. Beside the technical aspect, such VEVs can provide interesting informations as this class of models appears in various physics contexts [13, 14, 15, 16]. For instance, in [12] the following applications were studied: particular correlation functions in a special case of 3-D $U(1)$ or XY model, VEV of the spin field $\sigma$ in the $Z_{n}$-Ising models [17] perturbed by the leading thermal operator, asymptotics of the cylindrically symmetric solutions of the classical Toda equations. More recently, exact off-shell results for coupled minimal models were considered in [18].

ATFTs can be considered as perturbed Toda field theories (TFT)s. In [19] the “reflection amplitudes” for all non-simply laced Toda field theories (TFT)s were proposed as well as the exact relation between the masses of the particles and the parameters in the action associated with ATFTs. On the one hand, reflection amplitudes are the main objects which can be used for studying the UV asymptotics of the ground state energy $E(R)$ (or effective central charge $c_{\text{eff}}(R)$) for the system on the circle of size $R$ [8, 21]. In particular, the comparison of its asymptotics at small $R$ with the same quantity which can be independently calculated from the $S$-matrix data using TBA method [22, 23] can be considered as a non-trivial test for the $S$-matrix amplitudes [19] proposed in [24, 25]. On the other hand, reflection amplitudes were also used to calculate the exact VEVs for all dual pairs of non-simply laced ATFTs [19].

However, in order to support the exact VEVs, various checks are desirable. The main reason is that the “reflection relations” method has no mathematical proof yet. For the two simplest cases (sine-Gordon and Bullough-Dodd), the exact VEVs have been checked non-perturbatively but also perturbatively using the standard perturbation theory [4, 7], the perturbation theory in the radial [26] or angular quantization [27] framework. The purpose of this paper is to check the VEVs conjectures of general affine Toda theories using standard perturbative calculations up to two-loops. In the next section, we recall some basic facts about ATFTs and the axiomatic equations satisfied by the VEVs which lead to the exact solutions given in [19]. Perturbative analysis follows in sect.3 where we carefully compute $\langle \varphi \rangle$ perturbatively up to one-loop and also the VEVs of some composite operators up to two-loops.
2 Exact vacuum expectation values in affine Toda field theories

Let us first recall some known results about ATFTs which are relevant in further analysis. The ATFT with real coupling $b$ corresponding to the affine Lie algebra $\hat{G}$ is generally described by the action in the Euclidean space:

$$A = \int d^2x \left[ \frac{1}{8\pi} (\partial \phi)^2 + \sum_{i=0}^{r} \mu_{e_i} e^{b e_i \phi} \right], \quad (2.1)$$

where $\{e_i\} \in \Phi_s$ ($i = 1, ..., r$) is the set of simple roots of $\hat{G}$ of rank $r$ and $-e_0$ is a maximal root satisfying

$$e_0 + \sum_{i=1}^{r} n_i e_i = 0. \quad (2.2)$$

The fields in (2.1) are normalized such that:

$$\langle \phi_a(x) \phi_b(y) \rangle = -\delta_{ab} \log |x-y|^2. \quad (2.3)$$

For the simply laced case, since all vertex operators in the potential possess the same conformal dimension they all renormalize in the same way. It is then sufficient to introduce one scale parameter $\mu$ in action (2.1). However, for the non-simply laced case (except $BC_r \equiv A_{2r}^{(1)} - r \geq 2$ - affine Lie algebra in which case three different parameters are necessary) we have to introduce two different parameters$; one is associated with the set of standard roots of length $|e_i|^2 = 2$ and is denoted by $\mu_{e_i} = \mu$ whereas the other, denoted by $\mu_{e_i} = \mu'$, is associated with the set of non-standard roots of length $|e_i|^2 = l^2 \neq 2$.

The ATFTs can be considered as perturbed CFTs. Without the term with the zeroth root $e_0$, the action in (2.1) describes a TFT which is conformal. To do it, one introduce a charge to infinity defined by:

$$Q = b\rho + \frac{1}{b} \rho^\vee \quad \text{where} \quad \rho = \frac{1}{2} \sum_{\alpha > 0} \alpha \quad \text{and} \quad \rho^\vee = \frac{1}{2} \sum_{\alpha > 0} \alpha^\vee \quad (2.4)$$

are respectively the Weyl and dual Weyl vector of $G$. The sums in their definitions run over all positive roots $\{\alpha\} \in \Phi_+$, dual roots $\{\alpha^\vee\} \in \Phi_+^\vee$. Then, the stress-energy tensor $T(z)$, where $z = x_1 + i x_2$, $\bar{z} = x_1 - i x_2$ are complex coordinates of $\mathbb{R}^2$,

$$T(z) = -\frac{1}{2} (\partial z \phi)^2 + Q \cdot \partial^2 z \phi \quad (2.5)$$

1Throughout the paper, we denote an untwisted algebra as $\hat{G}$, while $\hat{G}^\vee$ refers to a twisted one. Furthermore, $G$ denotes a finite Lie algebra.

2For the sinh-Gordon model ($A_{1}^{(1)}$ ATFT) $\mu$ is generally called the cosmological constant.

3We choose the convention that the length squared of the long roots are four for $C_r^{(1)}$ and two for the other untwisted algebras.
ensures the local conformal invariance of the TFT. The corresponding central charges were calculated in [28]. Defining \( \mathbf{a} = (a_1, ..., a_r) \), the exponential fields
\[
V_{a}(x) = \exp(\mathbf{a} \cdot \varphi)(x)
\] (2.6)
are spinless conformal primary fields with dimensions:
\[
\Delta(a) = \frac{Q^2}{2} - \frac{(a - Q)^2}{2}.
\] (2.7)
By analogy with the Liouville field theory [29, 30, 6] the physical space of states \( \mathcal{H} \) in the TFTs consists of the continuum variety of primary states associated with the exponential fields (2.6) and their conformal descendents with:
\[
\mathbf{a} = i\mathbf{P} + \mathbf{Q} \quad \text{and} \quad \mathbf{P} \in \mathbb{R}^r.
\] (2.8)
Besides the conformal invariance TFTs possess an extended symmetry generated by \( W(\mathcal{G}) \)-algebra [31]. Indeed, for any arbitrary Weyl group element \( \hat{s} \in \mathcal{W} \) the fields \( V_{Q + \hat{s}(a - Q)}(x) \) are reflection images of each other and are related by the linear transformation:
\[
V_{a}(x) = R_{\hat{s}}(a)V_{Q + \hat{s}(a - Q)}(x)
\] (2.9)
where \( R_{\hat{s}}(a) \) is called the “reflection amplitude”, an important object in CFT which defines the two-point functions of the operator \( V_a \). In [19] the following expression for the reflection amplitude \( R_{\hat{s}}(a) \) for non-simply laced TFT was obtained:
\[
R_{\hat{s}}(a) = \frac{A_{\hat{s}i\mathbf{P}}}{A_i\mathbf{P}}
\] (2.10)
where
\[
A_i\mathbf{P} \equiv A(\mathbf{P}) = \prod_{i=1}^{r} [\pi \mu_i, \gamma(e_i^2b^2/2)]^{i\omega_i\mathbf{P}/b} \times \prod_{\alpha > 0} \Gamma(1 - i\mathbf{P} \cdot \alpha \mathbf{b})\Gamma(1 - i\mathbf{P} \cdot \alpha^\vee /b)
\]
with (2.8), the fundamental co-weights \( \omega_i^\vee \) and we denote \( \gamma(x) = \Gamma(x)/\Gamma(1 - x) \) as usual. We accept (2.10) as the proper analytical continuation of the function \( R_{\hat{s}}(a) \) for all \( a \). For \( \hat{s}_i \in \mathcal{W}_s \), the subset of Weyl group elements associated with the simple roots \( e_i \), notice that the ratio \( A(\hat{s}_i\mathbf{P})/A(\mathbf{P}) \) reduce to the reflection amplitude \( S_L(e_i, \mathbf{P}) \) of the Liouville field theory [1] :
\[
\frac{A(\hat{s}_i\mathbf{P})}{A(\mathbf{P})} = S_L(e_i, \mathbf{P})
\]
\[
= [\pi \mu_i, \gamma(e_i^2b^2/2)]^{-i\mathbf{P} \cdot \mathbf{e}_i^\vee /b} \frac{\Gamma(1 + i\mathbf{P} \cdot \mathbf{e}_i \mathbf{b})\Gamma(1 + i\mathbf{P} \cdot \mathbf{e}_i^\vee /b)}{\Gamma(1 - i\mathbf{P} \cdot \mathbf{e}_i \mathbf{b})\Gamma(1 - i\mathbf{P} \cdot \mathbf{e}_i^\vee /b)}.
\] (2.11)
Then, as ATFTs can be realized as CFTs perturbed by some relevant operators \[33\], in the conformal perturbation theory (CPT) approach one can formally rewrite any \(N\)-point correlation functions of local operators \(O_a(x)\) as:

\[
< O_{a_1}(x_1) \ldots O_{a_N}(x_N) >_{\text{ATFT}} = Z^{-1}(\lambda) < O_{a_1}(x_1) \ldots O_{a_N}(x_N) e^{-\lambda \int d^2 x \Phi_{\text{pert}}(x)} >_{\text{TFT}}
\]

where

\[
Z(\lambda) = < e^{-\lambda \int d^2 x \Phi_{\text{pert}}(x)} >_{\text{TFT}},
\]

\(\Phi_{\text{pert}}\) is the perturbing local field, \(\lambda\) is the CPT expansion parameter which characterizes the strength of the perturbation and \(< \ldots >_{\text{TFT}}\) denotes the expectation value in the TFT. Whereas vertex operators (2.6) satisfy reflection relations (2.9) in the CFT, the CPT framework provides similar relations among their expectation values in the perturbed case. In other words, if dots stands for any local insertion one has:

\[
< V_{a}(x)(\ldots) >_{\text{TFT}} = R_{\hat{s}}(a) < V_{Q+(a-Q)}(x)(\ldots) >_{\text{TFT}}.
\] (2.12)

Then, if we define the one-point function \(G(a)\) as the VEV of the vertex operator \(V_a(x)\) for non-simply laced ATFT by:

\[
G(a) = < \exp(a \cdot \varphi)(x) >_{\text{ATFT}}
\] (2.13)

one can formally rewrite this expression \[\] as:

\[
< e^{a \cdot \varphi}(x) >_{\text{ATFT}} = Z^{-1}(\lambda) \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \int d^2 y_1 \ldots d^2 y_n < e^{a \cdot \varphi}(y_1) e^{b \cdot \varphi}(y_1) \ldots e^{b \cdot \varphi}(y_n) >_{\text{TFT}}
\] (2.14)

Indeed, using (2.12) one expects that similar relations hold for \(G(a)\). If this VEV satisfies the system of functional equations associated with \(\mathcal{W}_s\) then it also automatically satisfies more complicated reflection relations. Furthermore, as was shown in previous works \[14, 15\], ATFTs can be understood as different perturbation of TFTs. The simplest case (beyond the sinh-Gordon model) is the Bullough-Dodd model which can be understood alternatively \[7\] as a perturbed Liouville CFT with coupling constant \(b\) or a perturbed Liouville CFT with coupling constant \(-b/2\). Here one can proceed similarly. We denote \(\Phi_s(\mathcal{G})\) as the set of simple roots of the finite Lie algebra \(\mathcal{G}\), \(\eta\) the extra-root associated with the perturbation and \(\{\epsilon_i\}\) an orthogonal basis \((\epsilon_i \cdot \epsilon_j = \delta_{ij})\) in \(\mathbb{R}^r\). Each one of the ATFT Lagrangian representation, denoted \(L_b[\Phi_s(\mathcal{G})]\), associated with \(\Phi_s(\mathcal{G})\) and the

\footnote{At the moment, there is no rigorous proof of this assumption.}

\footnote{In fact, the integrals in (2.14) are highly infrared divergent. By analogy with the situation appearing in the perturbed Liouville QFT \[7\], one can get around this infrared problem by considering a 2D world-sheet \(\Sigma_g\) topologically equivalent to a sphere equipped by a background metric \(g_{\nu\sigma}(x) = \rho(x)\delta_{\nu\sigma}\). Then the terms \(\rho(y_k)\) which appear in the integrals analogous to those in (2.14) provide an efficient infrared cut-off. We report the reader to \[7\] for details.}
coupling constant $b$ can be rewrite in two different ways:

- $\mathcal{L}_b[\Phi_s(B_r^{(1)})] \equiv \mathcal{L}_b[\Phi_s(B_r) \oplus \eta \equiv e_0 = -(\epsilon_1 + \epsilon_2)],$
  $\equiv \mathcal{L}_{-b}[\Phi_s(D_r) \oplus \eta \equiv -\epsilon_r];$

- $\mathcal{L}_b[\Phi_s(C_r^{(1)})] \equiv \mathcal{L}_b[\Phi_s(C_r) \oplus \eta \equiv e_0 = -2\epsilon_1],$
  $\equiv \mathcal{L}_{-b}[\Phi_s(C_r) \oplus \eta \equiv -2\epsilon_r];$

- $\mathcal{L}_b[\Phi_s(F_4^{(1)})] \equiv \mathcal{L}_b[\Phi_s(F_4) \oplus \eta \equiv e_0 = -\sqrt{2}\epsilon_1],$
  $\equiv \mathcal{L}_{-b}[\Phi_s(B_4) \oplus \eta \equiv -\sqrt{2}/3\epsilon_1];$

- $\mathcal{L}_b[\Phi_s(G_2^{(1)})] \equiv \mathcal{L}_b[\Phi_s(G_2) \oplus \eta \equiv e_0 = -\sqrt{2}\epsilon_1],$
  $\equiv \mathcal{L}_{-b}[\Phi_s(A_2) \oplus \eta \equiv -\sqrt{2}/3\epsilon_1].$

where the different sets of simple roots are reported in Appendix B. Using (2.12), it implies that the VEV (2.13) must satisfy simultaneously two irreducible systems of functional equations corresponding to two different sets $\mathcal{W}_s$. It results that $G(\alpha)$ obeys the functional equations

$$G(\tau \alpha) = S_L(e_j, P)G(\tau(Q + \delta_j(\alpha - Q)))$$
for all $\delta_j \in \mathcal{W}_s \quad (2.15)$

where

- $B_r^{(1)}$: $(\tau)_{ij} = \delta_{ij}$ for $G \equiv B_r$ and $(\tau)_{ij} = -\delta_{i, r+1-j}$ for $G \equiv D_r$;
- $C_r^{(1)}$: $(\tau)_{ij} = \delta_{ij}$ and $(\tau)_{ij} = -\delta_{i, r+1-j}$ for $G \equiv C_r$;
- $F_4^{(1)}$: $(\tau)_{ij} = \delta_{ij}$ for $G \equiv F_4$ and $(\tau)_{ij} = \delta_{ij}(\delta_{2j} + \delta_{3j} + \delta_{4j} - \delta_{ij})$ for $G \equiv B_4$;
- $G_2^{(1)}$: $(\tau)_{ij} = \delta_{ij}$ for $G \equiv G_2$ and $(\tau)_{ij} = -\delta_{i, 3-j}$ for $G \equiv A_2$

with coupling constant $b$. Notice that by simply looking at the Dynkin diagram symmetry of $B_r^{(1)}$ and $C_r^{(1)}$ - see figure 1 - one can also differently deduce:

$$G(a_1, a_2, ..., a_{r-1}, a_r) = G(-a_1, a_2, ..., a_{r-1}, a_r)$$
for $B_r^{(1)}; \quad (2.16)$

$$G(a_1, a_2, ..., a_{r-1}, a_r) = G(-a_r, -a_{r-1}, ..., -a_2, -a_1)$$
for $C_r^{(1)}$.

The reflection relations (2.13) (or, equivalently the relations (2.16) for $B_r^{(1)}$ and $C_r^{(1)}$) constituted the starting point in deriving the expectation values $G(\alpha)$. Following previous works, we also assumed that $G(\alpha)$ is a meromorphic function of $\alpha$. 
Furthermore, for real coupling constant $b$, the spectrum for any dual pair of non-simply laced ATFT consists of $r$ particles with the masses $M_a$ ($a = 1, ..., r$) expressed in terms of the mass parameter $m$. These spectra are reported in appendix A. The exact relation between the parameters of the action $\mu$ and $\mu'$ and the masses associated with the spectrum of the physical particles was obtained in [19] using the Bethe ansatz method (see for example [35, 36]). We report the reader to [19] for details. By replacing these mass-$\mu$ relations in the “minimal” solution of the functional equations (2.15), the following exact expression for the VEVs (2.13) was proposed [19] :

$$G(a) = \left[ \frac{k(G) \kappa(G)}{\maxU} \right]^{-a^2} \left[ \frac{\mu \gamma(1 + b^2)}{\mu' \gamma(1 + b^2)} \right] \left[ \frac{-\pi \mu \gamma(1 + b^2)^{1/2}}{-\pi \mu' \gamma(1 + b^2)^{1/2}} \right] \frac{d \cdot a \cdot b}{nb}$$

$$\times \exp \int_0^\infty \frac{dt}{t} \left( a^2 e^{-2t} - F(a, t) \right)$$

(2.17)

with

$$F(a, t) = \sum_{\alpha > 0} \left[ \frac{\sinh(a \cdot Q \cdot t) \sinh((a \cdot Q + H(1 + b^2)/2)\sinh((b^2/2^{1/2} + 1)\sinh(H(1 + b^2)t))}{\sinh(t) \sinh(b^2/2 - t) \sinh(H(1 + b^2)t)} \right]$$

where we denote $a_\alpha = a \cdot \alpha$ and

$$d = \frac{\rho \cdot h' - \rho \cdot h}{1 - l^2/2}.$$

The expressions $k(G)$ and $\kappa(G)$ can be found in [19]. Here, it is convenient to introduce the “deformed” Coxeter number [24, 25] :

$$H = h(1 - B) + h' B \quad \text{with} \quad B = \frac{b^2}{1 + b^2}$$

(2.18)

where $h$ (resp. $h'$) is the Coxeter (resp. dual Coxeter) number of $G$ (resp. $G'$). The integral in (2.17) is convergent iff :

$$\alpha \cdot Q - H(b + 1/b) < \Re(e(\alpha \cdot a) < \alpha \cdot Q \quad \text{for all} \quad \alpha \in \Phi_+$$

(2.19)

Notice that the prefactor which was given in ref. [19] was presented in a slightly different, but equivalent, form.
and is defined through analytic continuation outside this domain. Particular case of (2.17) corresponds to the simply laced one for which the result is in perfect agreement with [12].

Similarly, it is straightforward to obtain the VEVs of an ATFT based on a twisted affine Lie algebra \( \hat{\mathcal{G}}^\vee \). The reflection amplitudes corresponding to the TFT, i.e. the conformal part were easily obtained from (2.10) by using the duality relation for the parameters \( \mu_{e_i} \) and \( \mu_{e_i}^\vee \) associated with the dual pairs of ATFTs [19]:

\[
\pi \mu_{e_i} \gamma \left( \frac{b^2 e_i^2}{2} \right) = \left[ \pi \mu_{e_i}^\vee \gamma \left( \frac{e_i^2}{2b^2} \right) \right]^{b^2 e_i^2/2} \tag{2.20}
\]

and the change \( b \to 1/b \). Each one of the Lagrangian associated with \( \hat{\mathcal{G}}^\vee \) can be written in two different ways. In any case, the resulting system of functional equations which has to be satisfied by the VEV is nothing else than the dual of (2.15). To express the corresponding solution in terms of the mass of the physical particles, the mass-\( \mu \)-relations in the twisted case [19] are used. Finally, the result for the VEV \( G(a) \) for all twisted affine Lie algebras is obtained from (2.17) with the change \( b \to 1/b \).

It is similarly straightforward to study the \( BC_r \equiv A_{2r}^{(2)} \) (self-dual) remaining case which was considered in [37].

Notice that the expectation values (2.17) can be used to derive the bulk free energy of the ATFT:

\[
f_{\hat{\mathcal{G}}} = - \lim_{V \to \infty} \frac{1}{V} \ln Z, \tag{2.21}
\]

where \( V \) is the volume of the 2D space and \( Z \) is the singular part of the partition function associated with action (2.1). For specific values \( a \in b\{e_i\} \), with \( \{e_i\} \in \Phi_s \) (\( i = 1, \ldots, r \)) or \( e_0 \), the integral in (2.17) can be evaluated explicitly. Using the exact mass-\( \mu \)-relations and the obvious relations:

\[
\partial_\mu f(\mu) = \sum_{\{i\}} < e^{b e_i \cdot \varphi} > \quad \text{or} \quad \partial_{\mu'} f(\mu') = \sum_{\{i'\}} < e^{b e_{i'} \cdot \varphi} > \tag{2.22}
\]

where \( \{i\} \) and \( \{i'\} \) denotes respectively the whole set of long and short roots, the following bulk free energy was obtained [19]:

\[
f_{\hat{\mathcal{G}}} = \frac{m^2 \sin(\pi/H)}{8 \sin(\pi B/H) \sin(\pi(1-B)/H)}, \quad \hat{\mathcal{G}} = B_{r}^{(1)} \text{ and } C_{r}^{(1)},
\]

\[
f_{\hat{\mathcal{G}}} = \frac{m^2 \cos(\pi(1/3 - 1/H))}{16 \cos(\pi/6) \sin(\pi B/H) \sin(\pi(1-B)/H)}, \quad \hat{\mathcal{G}} = G_{2}^{(1)} \text{ and } F_{4}^{(1)},
\]

and similarly with the change \( B \to (1-B) \) for \( (B_{r}^{(1)})^\vee \), \( (C_{r}^{(1)})^\vee \), \( (G_{2}^{(1)})^\vee \), and \( (F_{4}^{(1)})^\vee \). In particular, these results were in perfect agreement with the values obtained using the Bethe ansatz approach [13].

### 3 Perturbative checks

To support the result (2.17) of [19] beyond the non-perturbative check (provided by the bulk free energy calculation), we present here a perturbative check. We expand the
vacuum expectation value (2.17) in power series in $b$ and compare each coefficient with the one obtained from standard Feynman perturbation theory associated with (2.1). In the first part of this section, we consider the VEV of the field $\langle \varphi \rangle$ which is given by:

$$\langle \varphi \rangle = \frac{\delta}{\delta a} G(a)|_{a=0}. \quad (3.1)$$

Since the result renders the same conclusion for all ATFTs, we choose $D^{(1)}$ series as illustrative examples and omit the details for other simply laced cases ($A^{(1)}_r$ case is trivial as seen shortly). It also provides a useful step to the calculation of $B^{(1)}$ series which is obtained from $D^{(1)}$ through folding procedure. Finally we present the result of an exceptional algebra $G^{(1)}_2$.

In a second part, as an additional check we also consider the “fully connected” composite operator expectation value of $\langle \varphi^a \varphi^b \rangle$ defined by

$$\langle \varphi^a \varphi^b \rangle \equiv \langle \varphi^a \rangle \langle \varphi^b \rangle - \frac{1}{2} \frac{\delta^2 \ln G(a)}{\delta a^2 \delta a^b}|_{a=0} \quad (3.2)$$

Since these quantities are quite complicated to calculate perturbatively, we will content ourselves with considering only some simple combinations of them up to two loops for $B^{(1)}_3$, $C^{(1)}_2$ and $G^{(1)}_2$ cases.

### 3.1 Perturbative checks of $\langle \varphi \rangle$

Using (2.17) and (3.1) one finds the result:

$$\langle \varphi \rangle = \frac{d}{H b} \ln \left[ \frac{\mu^2 (1 + b^2)}{\mu' \mu (1 + b^2)} \right] + B \frac{d}{H b} (L^2/2 - 1) \ln \left[ - \pi \mu' \gamma (1 + b^2) \right] + \frac{b}{2} \int_0^\infty dt \sum_{\alpha > 0} \left[ \frac{\alpha \sinh ((2\alpha \cdot Qb - H(1 + b^2))t) \sinh((b^2|\alpha|^2 + 1)t)}{\sinh(t) \sinh(H(1 + b^2)t) \sinh(b^2|\alpha|^2 + 1)t)} \right]. \quad (3.3)$$

To proceed further, we expand $\langle \varphi \rangle$ order by order in $b$ and write the result as:

$$\langle \varphi \rangle = \frac{1}{b} K + b \mathcal{L} + O(b^2). \quad (3.4)$$

For the simply laced case, this expression is drastically simplified:

$$K = - \sum_{\alpha > 0} \alpha \ln \gamma \left( \frac{\alpha \cdot \rho}{h} \right) \quad (3.5)$$

$$\mathcal{L} = - \int_0^\infty dt \cot(t) \frac{1}{\sinh(bt)} \left\{ \sum_{\alpha > 0} \alpha \sinh((h - 2\alpha \cdot \rho)t) \right\}. \quad \mathcal{L}$$

Let us introduce the component notation: $K_i = e_i \cdot K$ and $\mathcal{L}_i = e_i \cdot \mathcal{L}$.

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7 However, notice that $\langle \varphi \rangle = 0$ identically for $A^{(1)}_r$ series since $\mu' = \mu$, $l^2 = 2$ and $\sum_{\alpha > 0} \sinh((2\alpha \cdot Q - hQ)t) = 0$ \[43\].
For the $D(1)_{r}$ series, the non-perturbative results are given as

\begin{align}
K_1 &= K_{r-1} = K_r = -(1 - \frac{4}{h}) \ln 2; \\
\mathcal{L}_1 &= \mathcal{L}_{r-1} = \mathcal{L}_r = \frac{1}{2h} \left( \xi(1 - \frac{1}{h}) + \xi(1 + \frac{1}{h}) - \xi(1 + \frac{k-1}{h}) - \xi(1 + \frac{k}{h}) \right)
\end{align}

where $h = 2r - 2$ and for $k = 2, 3, \ldots, r - 2$ we have

\begin{align}
K_k &= \ln 2 - (1 - \frac{4}{h}) \ln 2; \\
\mathcal{L}_k &= \frac{1}{2h} \left( - \xi(1 + \frac{k-1}{h}) - \xi(1 + \frac{k}{h}) - \xi(1 + \frac{k+1}{h}) - \xi(1 + \frac{k+2}{h}) \right) \\
&\quad + \xi(1 - \frac{k-2}{h}) + 2\xi(1 - \frac{k-1}{h}) + \xi(1 - \frac{k}{h})
\end{align}

where we define $\xi(x) = \Psi(x) + \Psi(1 - x)$ in terms of the di-gamma function, $\Psi(x) = d\ln \Gamma(x)/dx$.

Perturbative analysis of the action (2.1) begins with shifting $\varphi \rightarrow \varphi_{cl} + \varphi$ such that $\varphi_{cl}$ satisfies the minimum of the ATFT potential. This classical solution reproduces exactly the leading term in (3.4):

\begin{equation}
\varphi_{cl} \cdot e_i = K_i.
\end{equation}

This identity provides the amusing relations among the $\gamma(x)$-functions, when $x$ is related to Lie algebra quantity, which is observed for general case in ref. [12, 19].

One-loop perturbative calculation is conveniently done using the classical mass eigenstate representation [22]. $D(1)_{r}$ series representation is given by:

\begin{align}
e_1 &= (-l_1^1, -l_1^2, \ldots, -l_r^{r-2}, 1, 0) \\
e_k &= (l_{k-1}^1, l_{k-1}^2, \ldots, l_k^{r-2} - l_k, 0, 0) \quad \text{for} \quad k = 2, \ldots, r - 2
\end{align}

where $l_k = \frac{2}{\sqrt{h}} \sin \frac{2\theta k}{\pi}$.

The next-to leading order term, i.e. the field expectation value to the one-loop order, $\langle \varphi \rangle$ is given by tadpole diagrams which in general needs to be appropriately regularized. The perturbative result is, however, finite for the mass eigenstate representation and does not depend on the regularization scheme for $D(1)_{r}$ series (and in general for simply laced cases).

To distinguish from the component notation, $\varphi_j$, which is obtained from the proposed VEV, the perturbative mass eigenstate component is denoted as $\Phi^c$. If the values are correct, then the relation between these two quantities should be $\varphi_j = \sum_c \Phi^c e_j^c$ where $e_j^c$ is the $c$-th component of the mass eigenstate representation $e_j$.

$\Phi^c$ vanishes when $c = r - 1$, $r$ and $c = \text{odd} \leq r - 2$,

\begin{equation}
\langle \Phi^c \rangle_0 = 0;
\end{equation}
and otherwise

\[ < \Phi^c >_b = \frac{1}{2} \begin{bmatrix} \frac{c}{2} & \frac{h-c}{2} & r & r-1 \\ c & c & c & c \end{bmatrix} \]

\[ = - b \sqrt{h Z_c} \left( Z_c^2 \ln(4 Z_c^2) - Z_{h+c}^2 \ln(4 Z_{h+c}^2) \right) , \tag{3.11} \]

where \( Z_a = \sin \frac{a\pi}{h} \). The divergent terms cancel each other and the total contribution remains finite.

With the help of various relations of the di-gamma function and trigonometric function one can prove that \( \mathcal{L} \)'s in (3.6) and (3.7) coincide with the ones in (3.11). Considering this as a non-trivial check, one can view this as a useful identity between di-gamma functions and trigonometric functions,

\[ b \mathcal{L}_i = \sum_{c=\text{even}}^{r-2} < \Phi^c >_b e_i^c . \tag{3.12} \]

For example, we have for \( i = 1 \),

\[ \xi\left( \frac{1}{h} \right) + \xi\left( \frac{2}{h} \right) - \xi\left( \frac{1}{2} + \frac{1}{h} \right) - \xi\left( \frac{1}{2} \right) \]

\[ = \sum_{c=\text{even}}^{r-2} 8 \cos\left( \frac{c\pi}{h} \right) \left[ \sin^2\left( \frac{c\pi}{2h} \right) \ln(4 \sin^2\left( \frac{c\pi}{2h} \right)) - \cos^2\left( \frac{c\pi}{2h} \right) \ln(4 \cos^2\left( \frac{c\pi}{2h} \right)) \right] . \tag{3.13} \]

For the non-simply laced case, the situation becomes more involved. By expanding (3.3), one finds the following coefficients:

\[ \mathcal{K} = \frac{d}{h} \ln\left( \frac{2\mu}{l^2 \mu'} \right) - \sum_{\alpha > 0} \alpha^\vee \ln \gamma \left( \frac{\alpha \cdot \rho^\vee}{h} \right) , \]

\[ \mathcal{L} = \frac{d}{h} \left\{ \left( \frac{l^2}{2} - 1 \right) [2\gamma_E + \ln(\pi \mu b^2)] + (h - h'^\vee) \ln\left( \frac{2\mu}{l^2 \mu'} \right) \right\} \]

\[ - \int_0^\infty dt \frac{1}{\sinh(ht)} \left\{ \coth(t) \sum_{\alpha > 0} \alpha \sinh((h - 2\alpha \cdot \rho^\vee)t) \right. \]

\[ - \frac{2}{h} \sum_{\alpha > 0} \alpha^\vee \alpha \cdot (h \rho - h'^\vee \rho^\vee) \cosh((h - 2\alpha \cdot \rho^\vee)t) \right\} , \]

where \( \gamma_E = 0.5772... \) is the Euler’s number.
The explicit value for $B_r^{(1)}$ series takes the form:

$$K_1 = \frac{2}{h} \ln \left( \frac{2\mu'}{\mu} \right) - \ln 2 ; \quad (3.14)$$

$$K_k = \frac{2}{h} \ln \left( \frac{2\mu'}{\mu} \right), \quad k = 2, 3, \cdots, r - 1 ;$$

$$K_r = -(1 - \frac{2}{h}) \ln \left( \frac{2\mu'}{\mu} \right) + \ln 2 ;$$

and

$$L_1 = \frac{1}{h} J + I_1 + \Delta I_1 ; \quad (3.15)$$

$$L_r = \left( \frac{1}{h} - \frac{1}{2} \right) J + I_r + \Delta I_r ;$$

$$L_k = \frac{1}{h} J + I_k + \Delta I_k, \quad k = 2, 3, \cdots, r - 1,$$ 

where

$$J = \left( 2\gamma E + \ln(\pi \mu b^2) + \frac{2}{h} \ln \left( \frac{\mu'}{2\mu} \right) \right) ; \quad (3.16)$$

and

$$I_1 = \frac{1}{2h} \left\{ \xi \left( \frac{1}{h} \right) + \xi \left( \frac{2}{h} \right) - \xi \left( \frac{1}{2} + \frac{1}{h} \right) - \xi \left( \frac{1}{2} \right) \right\} ; \quad (3.17)$$

$$I_r = \frac{1}{2h} \left\{ 2 \xi \left( \frac{1}{h} \right) - \xi \left( \frac{2}{h} \right) - \xi \left( \frac{1}{2} \right) \right\} ;$$

$$I_k = \frac{1}{2h} \left\{ \xi \left( \frac{2k - 2}{h} \right) + 2 \xi \left( \frac{2k - 1}{h} \right) + \xi \left( \frac{2k}{h} \right) - \xi \left( \frac{k}{h} \right) - \xi \left( \frac{k - 1}{h} \right) - \xi \left( \frac{k}{2} + \frac{k - 1}{h} \right) - \xi \left( \frac{1}{2} + \frac{k}{h} \right) \right\} .$$

Note that $I_k$’s ($k = 1, \ldots, r - 1$) are identical to $L_k$’s in (3.7) for $D_{r+1}^{(1)}$ series. $\Delta I$’s are given by:

$$\Delta I_1 = \Delta I_r = \frac{1}{2h^2} \left\{ -2 \xi \left( \frac{1}{h} \right) + 4 \xi \left( \frac{2}{h} \right) - 2 \xi \left( \frac{1}{2} + \frac{1}{h} \right) \right\} ; \quad (3.18)$$

$$\Delta I_k = \frac{1}{2h^2} \left\{ (4 - 4k) \xi \left( \frac{2k - 2}{h} \right) + 4k \xi \left( \frac{2k}{h} \right) - 2k \xi \left( \frac{k}{h} \right) 
+ (2k - 2) \xi \left( \frac{k - 1}{h} \right) + (2k - 2) \xi \left( \frac{1}{2} + \frac{k - 1}{h} \right) - 2k \xi \left( \frac{1}{2} + \frac{k}{h} \right) \right\} ,$$

and turn out to be identical to each other:

$$\Delta I_1 = \Delta I_k = \Delta I_r = \frac{4}{h^2} \ln 2 . \quad (3.19)$$
As noted for the simply laced case, $\mathcal{K}$ is identified with the classical value $\varphi_{cl}$. For $B_r^{(1)}$ series,

\[ b e_i \cdot \varphi_{cl} = \ln \left( \frac{\mu n_i}{\mu e_i} \right) - \frac{1}{h} \sum_{j=0}^{r} n_j \ln \left( \frac{\mu n_i}{\mu e_j} \right), \] (3.20)

which agrees with $\mathcal{K}$ in (3.14).

Beyond the classical result, however, renormalization should be carefully incorporated unlike in the simply laced case. The classical mass eigenstate representation of $B_r^{(1)}$ is obtained by folding the one of $D_r^{(1)}$ (3.9),

\[ e_1 = (-l_1^1, -l_1^2, \ldots, -l_1^{r-1}, 1) \]
\[ e_k = (l_{k-1}^1 - l_k^1, l_{k-1}^2 - l_k^2, \ldots, l_{k-1}^{r-1} - l_k^{r-1}, 0) \text{ for } k = 2, \ldots, r - 1 \]
\[ e_r = (l_{r-1}^1, l_{r-1}^2, \ldots, l_{r-1}^{r-1}, 0) \]

from which we obtain the one-loop contribution $\langle \varphi \rangle_b$:

\[ \langle \Phi^r \rangle_b = 0 ;\]
\[ \langle \Phi^c \rangle_b = \frac{b}{4 \sqrt{h} Z_c^2} g_r Z_{2c} \text{ when } c = \text{ odd } \leq r - 1 ; \]
\[ \langle \Phi^c \rangle_b = \frac{b}{4 \sqrt{h} Z_c^2} \left\{ 4 Z_c \left( g_r^2 Z_c^2 - g_r \frac{m_g}{m_{\text{phys}}} Z_{2c}^2 \right) + g_r Z_{2c} \right\} \text{ when } c = \text{ even } \leq r - 1 , \] (3.22)

where $Z_a = \sin(a \pi / h)$ as is given in (3.11). $g_a$ is the Euclidean integration of the tad-pole diagram,

\[ g_a = \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 + m_a^2}, \] (3.23)

where $m_a$ is the physical mass equivalent to $M_a$ in Appendix A up to this order of $b^2$. Its explicit value is given by $m_a = 2m_0 \sin(\frac{\pi a}{r})$ for $a = 1, 2, \ldots, r - 1$ and $m_r = m_0$ with $m_0^2 = 2^{2+\frac{r}{2}} (\pi \mu b^2) (\mu' / \mu)^{\frac{r}{2}}$.

Here, to evaluate the one-loop diagram we are using the normal ordering with respect to the free field theory. In this scheme $g_a$ is given by

\[ g_a = \frac{1}{4\pi} \left( \ln \left( \frac{m_a}{2} \right)^2 + 2\gamma_E \right) = \left[ J + \ln \left( \frac{m_a}{m_0} \right)^2 + \frac{2}{h} \ln 2 \right]. \] (3.24)

Then, using the identity

\[ \left( \sum_{c=\text{odd}}^{r-1} - \sum_{c=\text{even}}^{r-1} \right) \csc^2 \left( \frac{c\pi}{h} \right) \sin \left( \frac{2c\pi}{h} \right) \sin \left( \frac{2k\pi}{h} \right) = \frac{k}{2h}, \quad k = 1, \ldots, r - 1 , \] (3.25)
we find that the $J$ parts of (3.22) agree exactly with those of (3.15), i.e.

$$bL_i|_J\text{-part} = \sum_c e_i^c < \Phi^c >_b|_J\text{-part}.$$  

Furthermore, since the term $\ln(m_a/m_0)^2$ in (3.24) reproduces $I_k$’s which are the same as $L_k$’s in (3.7) of $D_r^{(1)}$ series for $k = 1, \ldots, r - 1$, the agreement (3.12) in $D_r^{(1)}$ case immediately implies

$$bL_i|_I\text{-part} = \sum_c e_i^c < \Phi^c >_b|_I\text{-part}.$$  

Finally, $\Delta I_k$ terms come from the last term $\frac{2}{n}\ln 2$ in (3.24). This establishes the exact agreement between the perturbative and nonperturbative results for the $B_r^{(1)}$ case.

For the exceptional algebra $G_2^{(1)}$, we have

$$K_1 = \frac{1}{2} \ln(\frac{\mu'_{3}}{3\mu}) + 2 \ln \gamma(\frac{1}{6}) - 4 \ln \gamma(\frac{1}{3});$$

$$K_2 = -\frac{1}{2} \ln(\frac{\mu'_{3}}{3\mu}) - \ln \gamma(\frac{1}{6}) + 2 \ln \gamma(\frac{1}{3});$$

$$L_1 = \frac{1}{6} \left[ 4\gamma_E + \ln((\pi \mu b^2)(\frac{\pi \mu' b^2}{3})) \right] + \frac{1}{2} \ln 3 + \frac{2}{9} \ln 2;$$

$$L_2 = -\frac{1}{3} \left[ 2\gamma_E + \frac{1}{2} \ln((\pi \mu b^2)(\frac{\pi \mu' b^2}{3})) \right] - \frac{2}{9} \ln 2 - \frac{1}{4} \ln 3.$$  

On the other hand, the corresponding one-loop diagram is given by:

$$< \Phi^1 >_b = \frac{1}{2} \left[ \begin{array}{c} 1 \\ \gamma \end{array} \right]$$  

and

$$< \Phi^2 >_b = \frac{1}{2} \left[ \begin{array}{c} 2 \\ \gamma \\ 2 \\ \gamma \end{array} \right] + \left[ \begin{array}{c} 1 \\ \gamma \\ 1 \\ \gamma \\ 1 \end{array} \right].$$

After explicit calculations as in the previous case, we find $< \Phi^2 >_b = bL_2$ and $< \Phi^1 >_b = \frac{1}{2}L_1 + L_2$ which completes the perturbative check for $G_2^{(1)}$.  

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3.2 Perturbative checks of the composite operators $<\varphi_a \varphi_b>$

From the expression (2.17) and using (3.2) we have the VEV of composite operator,

$$G^{ab} \equiv <\varphi^a \varphi^b>$$

$$= -\delta^{ab} \sum_{i=1}^n \frac{n_i}{H(1+b^2)} \ln(-\pi \mu_t \gamma(1+b^2 e^2/2)) + \int_0^\infty \frac{dt}{t} [\delta^{ab} e^{-2t} - \mathcal{F}^{ab}]$$

where

$$\mathcal{F}^{ab} = b^2 t^2 \sum_{\alpha > 0} \alpha^a \alpha^b \frac{\sinh((1+b^2 \alpha^2/2)t)}{\sinh(t)} \sinh(b^2 \alpha^2 t/2) \sinh((1+b^2) H t)$$

These are in general rather complicated quantities to calculate perturbatively due to various divergences to be taken care of up to some finite part. For some combinations such as the relative value of the composite operator $G^{aa} - G^{rr}$ ($a = 1, \cdots, r-1$), however, the propagators are renormalized with an over-all renormalization constant and therefore, most of the complications due to the renormalization scheme disappears. Therefore, such quantities provides an additional independent check of the non-perturbative result in a simple way. Since the perturbative calculation is done in the classical mass eigenstate representation, in this section we will use the mass eigenstate representation for $\alpha$ in (3.29).

For $C_{2}^{(1)}$, the composite operator value is given by :

$$G^{12} = 0 ;$$

$$G^{11} - G^{22} = \int_0^\infty \frac{dt}{t} \frac{b^2 t \sinh(1+b^2 t) (4 \cosh(4+8b^2 t) - 4)}{\sinh(t) \cosh(b^2 t) \sinh(4+6b^2 t)}$$

$$= \ln 2 + b^2(0.79221 \ldots) + \mathcal{O}(b^4) .$$

The corresponding value is confirmed perturbatively: $<\varphi^1 \varphi^2> = 0$ since there is no vertex at all for this case. The other one is given by :

$$<\varphi^1 \varphi^1 - \varphi^2 \varphi^2>$$

$$= \left[ \begin{array}{c} 1 \\ \times \end{array} \right] - \left[ \begin{array}{c} 2 \\ \times \end{array} \right] + \left[ \begin{array}{c} 1 \\ \times \end{array} \right] - \frac{1}{2} \times \left[ \begin{array}{c} 1 \\ \times \end{array} \right] + \mathcal{O}(b^4)$$

$$= \ln 2 + b^2(0.79221 \ldots) + \mathcal{O}(b^4)$$

whose Feynman integration is done in the Appendix C. This agrees with the non-perturbative results (3.30).

For the case $B_3^{(1)}$, the non-perturbative result gives :

$$G^{12} = G^{23} = 0 ;$$

$$G^{11} - G^{33} = -2 \int_0^\infty \frac{dt}{t} (\mathcal{F}^{11} - \mathcal{F}^{33}) = b^2(-0.195326 \ldots) + \mathcal{O}(b^4) ;$$

$$G^{22} - G^{33} = -2 \int_0^\infty \frac{dt}{t} (\mathcal{F}^{22} - \mathcal{F}^{33}) = -\ln 3 + b^2(-0.321552 \ldots) + \mathcal{O}(b^4)$$
where

\[
\mathcal{F}^{11} - \mathcal{F}^{33} = \frac{b^2 t^2}{\sinh(t) \sinh((6 + 5b^2)t)} \times \\
\left\{ \frac{\sinh((1 + b^2)t)}{\sinh(b^2 t)} \left( - \cosh((4 + 3b^2)t) + 2 \cosh(b^2 t) - \cosh((2 + b^2)t) \right) + \frac{\sinh((1 + b^2/2)t)}{2 \sinh(b^2 t/2)} \left( \cosh((4 + 4b^2)t) + \cosh((2 + b^2)t) - 2 \right) \right\} ;
\]

\[
\mathcal{F}^{22} - \mathcal{F}^{33} = \frac{b^2 t^2}{\sinh(t) \sinh((6 + 5b^2)t)} \times \\
\left\{ \frac{\sinh((1 + b^2)t)}{\sinh(b^2 t)} \left( \cosh((4 + 3b^2)t) - \cosh((2 + b^2)t) \right) + \frac{\sinh((1 + b^2/2)t)}{2 \sinh(b^2 t/2)} \left( \cosh((4 + 4b^2)t) + \cosh((2 + b^2)t) - 2 \right) \right\} .
\]

The perturbative calculation gives the result, \(\langle \langle \Phi_1 \Phi_3 \rangle \rangle = \langle \langle \Phi_2 \Phi_3 \rangle \rangle = 0\) since there is no vertex at all in this case. The relative values of the composite operators are:

\[
\langle \langle \Phi_1 \Phi_1 - \Phi_3 \Phi_3 \rangle \rangle = \left[ \begin{array}{c}
1 \\
\frac{2}{2} + \\
\frac{1}{2} \times \\
\frac{3}{3}
\end{array} \right] + \mathcal{O}(b^4)
\]

\[
= b^2(-0.195326\ldots) + \mathcal{O}(b^4) ; \tag{3.33}
\]

\[
\langle \langle \Phi_2 \Phi_2 - \Phi_3 \Phi_3 \rangle \rangle = \left[ \begin{array}{c}
2 \\
\frac{1}{2} \times \\
\frac{3}{3} + \\
\frac{2}{2}
\end{array} \right] + \mathcal{O}(b^4)
\]

\[
= -\ln 3 + b^2(-0.321552\ldots) + \mathcal{O}(b^4) \tag{3.34}
\]

which exactly reproduces (3.32).
Similar check can be done for $G_2^{(1)}$. $G^{12}$ is not vanishing but is given by:

$$G^{12} = -√3 b^2 \sum_{\alpha > 0} \left( \frac{1}{2} \alpha_1 + \alpha_2 \right) \cdot \alpha_1 \cdot \alpha_2 \right) \cdot \alpha_1 \cdot \alpha_2 \right) $$

$$\times \int dt \frac{\sinh((1 + b^2 \alpha^2/2)t) \cosh((1 + b^2)H - 2b\alpha \cdot Q)t}{\sinh(t) \sinh(b^2 \alpha^2 t/2) \sinh((1 + b^2)Ht)}$$

$$= \frac{2}{√3} b^2 (0.0488314 \ldots) + O(b^4) \quad (3.35)$$

whose value is also obtained from the perturbative diagram

$$<< \Phi^1 \Phi^2 >> = \frac{1}{2} \times \begin{array}{c}
1 \\
1 \\
2
\end{array} + O(b^4). \quad (3.36)$$

Finally, the relative value of the composite operators

$$G^{11} - G^{22} = -\int_0^\infty dt \frac{\sinh(t) \sinh((6 + 4b^2)t)}{\sinh(t) \sinh((6 + 4b^2)t)} \times \left( \sinh((1 + b^2)t) \left( \cosh(2t) - \cosh(4t + 2b^2 t) \right) \right)$$

$$+ \sinh(t + \frac{b^2 t}{3}) \left( 2 \cosh(\frac{2b^2 t}{3}) - \cosh(4t + \frac{10b^2 t}{3}) - \cosh(2t + \frac{4b^2 t}{3}) \right) \right)$$

$$= \frac{1}{2} \ln 3 + \frac{2}{3} b^2 (0.183165 \ldots) + O(b^4) \quad (3.37)$$

is reproduced by the following perturbative diagrams,

$$<< \Phi^1 \Phi^1 - \Phi^2 \Phi^2 >> = \begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix} + \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix} + \frac{1}{2} \times \begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix}$$

$$- \begin{bmatrix}
2 \\
2 \\
2
\end{bmatrix} + \frac{1}{2} \times \begin{bmatrix}
2 \\
2 \\
2
\end{bmatrix} + \frac{1}{2} \times \begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix} + O(b^4). \quad (3.38)$$

It is straightforward to generalize the above perturbative calculation to other remaining cases and to confirm the proposed VEV.

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Appendix A

Differently to the simply laced case for which the mass ratios correspond to the classical values, mass ratios for non-simply laced case get quantum corrections [24, 25]. The mass spectrum for the dual cases remains the same with the change $b \rightarrow 1/b$, where the mass spectrum depends only on one parameter $m$:

\[
\begin{align*}
B^{(1)}_r & : M_r = m, \quad M_a = 2m \sin(\pi a/H), \quad a = 1, 2, \ldots, r - 1 \\
C^{(1)}_r & : M_a = 2m \sin(\pi a/H), \quad a = 1, 2, \ldots, r \\
G^{(1)}_2 & : M_1 = m, \quad M_2 = 2m \cos(\pi (1/3 - 1/H)) \\
F^{(1)}_4 & : M_1 = m, \quad M_2 = 2m \cos(\pi (1/3 - 1/H)), \quad M_3 = 2m \cos(\pi (1/6 - 1/H)), \quad M_4 = 2M_2 \cos(\pi / H). \\
\end{align*}
\]

For non-simply laced Lie algebras, the Coxeter and dual Coxeter numbers are:

\[
\begin{align*}
h_{B^{(1)}_r} & = h_{(C^{(1)}_r)} = 2r, \quad h_{(B^{(1)}_r)\nu} = h_{A^{(2)}_{2r-1}} = 2r - 1, \quad h_{(C^{(1)}_r)\nu} = h_{D^{(2)}_{r+1}} = 2(r + 1), \\
h_{F^{(1)}_4} & = 12, \quad h_{(F^{(1)}_4)\nu} = 9, \quad h_{G^{(1)}_2} = 6, \quad h_{(G^{(1)}_2)\nu} = h_{D^{(3)}_4} = 4.
\end{align*}
\]

Appendix B : Notations

\[
\begin{align*}
\Phi_s(A_2) & = \sqrt{2} \varepsilon_1, \quad \sqrt{3/2} \varepsilon_1 - 1/\sqrt{2} \varepsilon_2; \\
\Phi_s(B_r) & = \varepsilon_i - \varepsilon_{i+1} \quad 1 \leq i \leq r - 1, \quad \varepsilon_r; \\
\Phi_s(C_r) & = \varepsilon_i - \varepsilon_{i+1} \quad 1 \leq i \leq r - 1, \quad 2\varepsilon_r; \\
\Phi_s(D_r) & = \varepsilon_i - \varepsilon_{i+1} \quad 1 \leq i \leq r - 1, \quad \varepsilon_r + \varepsilon_{r-1}; \\
\Phi_s(F_4) & = \varepsilon_i - \varepsilon_{i+1} \quad i \in \{2, 3\}, \quad \varepsilon_4, \quad \frac{1}{2} (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4); \\
\Phi_s(G_2) & = \sqrt{2/3} \varepsilon_2, \quad 1/\sqrt{2} \varepsilon_1 - \sqrt{3/2} \varepsilon_2; \\
\end{align*}
\]

and

\[
\begin{align*}
\overline{\Phi}_s(C_r) & = \Phi_s(C_r)|_{\varepsilon_i \leftrightarrow \varepsilon_{i+1}}; \quad \overline{\Phi}_s(D_r) = \Phi_s(D_r)|_{\varepsilon_i \leftrightarrow \varepsilon_{r+1-i}}; \\
\overline{\Phi}_s(A_2) & = \Phi_s(A_2)|_{\varepsilon_1 \leftrightarrow \varepsilon_2}; \quad \overline{\Phi}_s(B_4) = \Phi_s(B_4)|_{\varepsilon_i \leftrightarrow \varepsilon_i, \quad i \in \{2, 3, 4\}}.
\end{align*}
\]

Appendix C : Feynman integrals

The Feynman integration for $C^{(1)}_2$ in (3.31) is presented as the following. The lowest order diagrams (order of $b^0$) are represented as the Feynman integration:

\[
\langle\langle \Phi^1 \Phi^1 - \Phi^2 \Phi^2 \rangle\rangle_{b=0} = -4\pi \int \frac{d^2p_E}{(2\pi)^2} \left( \frac{1}{p_E^2 + M_1^2} - \frac{1}{p_E^2 + M_2^2} \right) = \ln \frac{M_2^2}{M_1^2} = \ln 2, \quad (C.1)
\]

where \( p_E \) is the Euclidean momentum. \( M_i \) is the physical mass and its value at the integration is considered up to this appropriate perturbative order in \( b \). Since the wavefunction and mass renormalization is already done, the next-to leading order diagrams (order of \( b^2 \)) are represented as:

\[
<< \Phi_1^4 \Phi_1^4 - \Phi_2^2 \Phi_2^2 >>_b = (4\pi)^2 \int \frac{d^2p_E}{(2\pi)^2} \left( \frac{32 I_{12}}{(p_E^2 + M_1^2)^2} - \frac{16 I_{11}}{p_E^2 + M_2^2} \right) = 0.79221 \ldots \text{(C.2)}
\]

where

\[
I_{ij} = \int \frac{d^2k_E}{(2\pi)^2} \frac{1}{(k_E^2 + M_i)^2} \left( \frac{1}{(k_E + p_E)^2 + M_j} \right)
\]

\[
= \int_0^1 \frac{dx}{4\pi x(1-x)p_E^2 + (1-x)M_i^2 + xM_j^2} \text{ (C.3)}
\]

The Feynman integrations (3.33) and (3.34) of the next-to leading order for \( B_3^{(1)} \) are given by:

\[
<< \Phi_1 \Phi_2 >\Phi_1 \Phi_2 >>_b = (4\pi)^2 \int \frac{d^2p_E}{(2\pi)^2} \left( \frac{(I_{33} + 2 I_{12})}{(p_E^2 + M_1^2)^2} - \frac{(2 I_{13} + 2 I_{23})}{(p_E^2 + M_3^2)^2} \right) = -0.195326 \ldots
\]

\[
<< \Phi_1^3 \Phi_1^3 >>_b = (4\pi)^2 \int \frac{d^2p_E}{(2\pi)^2} \left( \frac{(I_{33} + I_{11} + 9 I_{22})}{(p_E^2 + M_1^2)^2} - \frac{(2 I_{13} + 2 I_{23})}{(p_E^2 + M_3^2)^2} \right) = -0.321552 \ldots \text{(C.4)}
\]

The Feynman integrations (3.36) and (3.38) of the next-to leading order for \( G_2^{(1)} \) are evaluated respectively as:

\[
<< \Phi_1^2 \Phi_2^2 >>_b = (4\pi)^2 \frac{2}{\sqrt{3}} \int \frac{d^2p_E}{(2\pi)^2} \left( \frac{I_{11}}{(p_E^2 + M_1^2)(p_E^2 + M_2^2)} \right) = \frac{2}{\sqrt{3}} \times 0.0488314 \ldots
\]

\[
<< \Phi_1^4 \Phi_1^4 - \Phi_2^2 \Phi_2^2 >>_b = (4\pi)^2 \int \frac{d^2p_E}{(2\pi)^2} \left( \frac{(2 I_{12} + \frac{4}{3} I_{11})}{(p_E^2 + M_1^2)^2} - \frac{(9 I_{12} + I_{11})}{p_E^2 + M_2^2} \right) = \frac{2}{3} \times 0.183165 \ldots \text{(C.5)}
\]

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