ON THE OUTER AUTOMORPHISM GROUPS OF
TRIANGULAR ALTERNATION LIMIT
ALGEBRAS

S. C. Power
Department of Mathematics
University of Lancaster
Lancaster LA1 4YF
England.

ABSTRACT

Let $A$ denote the alternation limit algebra, studied by Hopenwasser and Power, and by Poon, which is the closed direct limit of upper triangular matrix algebras determined by refinement embeddings of multiplicity $r_k$ and standard embeddings of multiplicity $s_k$. It is shown that the quotient of the isometric automorphism group by the approximately inner automorphisms is the abelian group $\mathbb{Z}^d$ where $d$ is the number of primes that are divisors of infinitely many terms of each of the sequences $(r_k)$ and $(s_k)$. This group is also the group of automorphisms of the fundamental relation of $A$. 
1 Introduction

In Hopenwasser and Power [HP] and in Poon [Po] the alternation limit algebras described below were classified. In this note we determine the quotient group $\text{Out}_{\text{isom}} A = \text{Aut}_{\text{isom}} A / I(A)$ for these algebras where $\text{Aut}_{\text{isom}} A$ is the group of isometric algebra automorphisms and $I(A)$ is the normal subgroup of $\text{Aut} A$ of approximately inner automorphisms. An automorphism $\alpha$ is said to be approximately inner if there exists a sequence $(b_k)$ of invertible elements such that $\alpha(a) = \lim_k b_k a b_k^{-1}$ for all $a$ in $A$.

Let $(r_k), (s_k)$ be sequences of positive integers. Write $T(r_k, s_k)$ for the Banach algebra limit of the system

$$C \rightarrow T_{r_1} \rightarrow T_{r_1s_1} \rightarrow T_{r_1s_1r_2} \rightarrow \ldots,$$

where $T_n$ is the algebra of upper triangular $n \times n$ complex matrices and where the embeddings are unital and are alternately of refinement type $(\rho(a) = (a_{ij} 1_t)$, with $1_t$ the $t \times t$ identity and of standard type $(\sigma(a) = a \oplus \ldots \oplus a, t$ times).

**Theorem 1** $\text{Out}_{\text{isom}}(T(r_k, s_k)) = \mathbb{Z}^d$ where $d$ is the number of primes $p$ that are divisors of infinitely many terms of each of the sequences $(r_k)$ and $(s_k)$. (If $d = \infty$ interpret $\mathbb{Z}^d$ as the countably generated free abelian group.)

The proof uses the methods of [HP]. A major step is to characterise the automorphism group of the fundamental relation, or semigroupoid, which is associated with an alternation algebra. This order-topological result is of independent interest and is stated and proved separately below.

Let $r$ and $s$ be the generalised integers $r_1r_2 \ldots$, and $s_1s_2 \ldots$ respectively and suppose that $p$ is a prime satisfying the condition in the statement of the theorem. Then $p^\infty$ divides $r$ and $s$. Thus we can arrange new formal products $r = t_1t_2 \ldots$, $s = u_1u_2 \ldots$, with $t_k = u_k = p$ for all odd $k$. As noted in [HP], because of the commutation of refinement and standard embeddings, we can easily display a commuting zig zag diagram to show that $T(r_k, s_k)$ and $T(t_k, u_k)$ are isometrically isomorphic. However, with the new formal product we can construct one of the generators of $\text{Out}_{\text{isom}} A$. Consider the automorphism $\alpha$ determined
by the following commuting diagram where the matrix algebras are omitted for notational economy.

\[
\begin{array}{cccccccc}
\mathbb{C} & \xrightarrow{\rho_{t_1}} & \xrightarrow{\sigma_{u_1}} & \xrightarrow{\rho_{t_2}} & \xrightarrow{\sigma_{u_2}} & \xrightarrow{\rho_{t_3}} & \xrightarrow{\sigma_{u_3}} & \cdots & T(t_k, u_k) \\
& \sigma_{t_1} & \rho_{t_1} & & \sigma_{t_3} & \rho_{t_3} & & \\
\mathbb{C} & \xrightarrow{\rho_{t_1}} & \xrightarrow{\sigma_{u_1}} & \xrightarrow{\rho_{t_2}} & \xrightarrow{\sigma_{u_2}} & \xrightarrow{\rho_{t_3}} & \xrightarrow{\sigma_{u_3}} & \cdots & T(t_k, u_k) \\
\end{array}
\]

It will be shown below that \( \alpha \) provides a nonzero coset and that the totality of such cosets provides a generating set for the isometric outer automorphism group.

2 Proof of Theorem 1

Let \( X \), or \( X(r_k, s_k) \), be the Cantor space

\[ X = \prod_{k=-\infty}^{-1}\{1, \ldots, s_{-k}\} \times \prod_{k=1}^{\infty}\{1, \ldots, r_k\}, \]

where we have fixed the sequences \( (r_k) \) and \( (s_k) \). Define the equivalence relation \( \tilde{R} \) on \( X \) to consist of the pairs \( (x, y) \) of points \( x = (x_k), y = (y_k) \) in \( X \) with \( x_k = y_k \) for all large enough and small enough \( k \). \( \tilde{R} \) carries a natural locally compact Hausdorff topology (giving it the structure of an approximately finite groupoid). Write \( R \), or \( R(r_k, s_k) \), for the antisymmetric topologised subrelation of \( \tilde{R} \) consisting of pairs \( (x, y) \) in \( R \) for which \( x \) precedes \( y \) in the lexicographic order. Thus \( (x, y) \in R \) if and only if \( (x, y) \in \tilde{R} \) and, either \( x = y \), or for the smallest \( k \) for which \( x_k \neq y_k \) we have \( x_k < y_k \).

An automorphism of \( R(r_k, s_k) \) is a binary relation isomorphism (implemented by a bijection \( \alpha \) of the underlying space \( X \)), which is a homeomorphism for the (relative groupoid) topology of \( R(r_k, s_k) \). Necessarily \( \alpha \) is a homeomorphism of \( X \).

**Theorem 2** The group of automorphisms of the topological binary relation \( R(r_k, s_k) \)
is $\mathbb{Z}^d$ where $d$ is the number of primes which divide infinitely many terms of each of the sequences $(r_k)$ and $(s_k)$.

**Proof:** Let $\overline{O(x)}$ denote the closure of the $R$-orbit of the point $x$ in $X$. Here $O(x) = \{ y : (y, x) \in R \}$. Recall from [HP] that the pair of points $x, x^+$ is called a gap pair if $x^+ \not\in \overline{O(x)}$ and

$$\overline{O(x^+)} = \overline{O(x)} \cup \{ x \}.$$  

Furthermore $x, x^+$ is a gap pair if and only if

1) there exists $n$ such that $x_m = 1$ for all $m \leq n$,
2) there exists $p$ such that $x_q = r_q$ for all $q \geq p$.

Also if $p$ is the smallest integer for which 2) holds (with $r_p = s_{-p}$ if $p$ is negative), then $x^+$ is given by

$$(x^+_j) = \begin{cases} x_j & \text{if } j < p - 1 \\ x_{p-1} + 1 & \text{if } j = p - 1 \\ 1 & \text{if } j \geq p \end{cases}$$

The usefulness of this for our purpose is that an automorphism $\alpha$ of $R$ necessarily maps gap pairs to gap pairs and so the coordinate description of these pairs leads ultimately to a coordinate description of $\alpha$.

Let $\alpha$ be an automorphism of $R$. Consider the (left) gap point $x_\ast = (\ldots, 1, 1, \hat{1}, r_1, r_2, \ldots)$ where $\hat{1}$ indicates the coordinate position for $s_1$. Then $\alpha(x_\ast)$ is necessarily a (left) gap point, thus

$$\alpha(x_\ast) = (\ldots, 1, 1, z_{-t+1}, z_{-t}, \ldots, z_{t-1}, r_t, r_{t+1} \ldots)$$

for some positive integer $t$. We have

$$\overline{O(x_\ast)} = \{ x = (\ldots, 1, \hat{1}, x_1, x_2, \ldots) : x_k \leq r_k \text{ for all } k \},$$

$$\overline{O(\alpha(x_\ast))} = \{ y = (\ldots, 1, w', y_t, y_{t+1}, \ldots) \},$$

where $y_k \leq r_k$ for all $k \geq t$ and where $w'$ is any word of length $2t - 2$ which precedes (or is equal to) the word $w = z_{-t+1}, z_{-t}, \ldots, z_{t-1}$ in the lexicographic order. Restating this, we
have natural homeomorphisms

\[ \overline{O}(x_*) \approx \Pi_{k=1}^\infty \{1, \ldots, r_k\} \]

\[ \overline{O}(\alpha(x_*)) \approx \{1, \ldots n\} \times \Pi_{k=t}^\infty \{1, \ldots, r_t\} \]

where \( n \) is the number of words \( w' \). Moreover, these identifying homeomorphisms induce isomorphisms between the restrictions \( R|\overline{O}(x_*) \) and \( R|\overline{O}(\alpha(x_*)) \) and the unilateral relations \( R_1 \) and \( R_2 \), respectively, where \( R_1 = R(r_k, u_k) \), with \( u_k = 1 \) for all \( k \), and \( R_2 = R(r'_k, u_k) \), with \( u_k \) as before, \( r'_1 = n \), and \( r'_k = r_{k+t-2} \) for \( k = 2, 3, \ldots \). Since \( \alpha \) induces an isomorphism between the restrictions, we obtain an induced isomorphism \( \beta \) between \( R_1 \) and \( R_2 \). It is well-known that this means that \( r = r' \) where \( r = r_1 r_2 \ldots \) and \( r' = r'_1 r'_2 \ldots \) are generalised integers. (See [P2] for example). Thus we obtain the necessary condition that the integer \( n \) is a divisor of the generalised integer \( r \).

We shall now improve on this necessary condition.

The isomorphism between \( R|\overline{O}(x_*) \) and \( R|\overline{O}(\alpha(x_*)) \) is given explicitly by

\[ \alpha : (\ldots 1, \hat{1}, x_1, x_2, \ldots) \rightarrow (\ldots 1, w', y_t, y_{t+1}, \ldots) \]

where

\[ \frac{||w'|| - 1}{n} + \sum_{k=1}^\infty \frac{(y_{t+k-1} - 1)}{nm_{t+k-1}} m_{t-1} = \sum_{k=1}^\infty \frac{x_k - 1}{m_k}, \tag{1} \]

where \( ||w'|| \) is the cardinality of the set of points in the order interval from the \((2t-2)\)-tuple \((1,1,\ldots,1)\) to \( w' \), and where \( m_k = r_1 r_2 \ldots r_k \) for \( k = 1, 2, \ldots \). The identity (1) follows from the fact that there are unique canonical \( R \)-invariant probability measures on \( \overline{O}(x_*) \) and on \( \overline{O}(\alpha(x_*)) \) and the quantities in (1) are the measures of the subsets \( \overline{O}(\alpha(x)) \) and \( \overline{O}(x) \) respectively.

To verify these facts one must recall how the topology of a topological binary relation is defined. In the case of \( R_1 = R|\overline{O}(x_*) \) fix two words

\[ (x_1, x_2, \ldots, x_t) \leq (x'_1, x'_2, \ldots, x'_t) \]
in lexicographic order. Then the set $E$ of pairs

$$( (x_1, x_2, \ldots, x_\ell, z_{\ell+1}, z_{\ell+2}, \ldots), (x'_1, x'_2, \ldots, x'_\ell, z_{\ell+1}, z_{\ell+2}, \ldots) )$$

is, by definition, a basic open and closed subset for the topology. Notice that for this set, the left and right coordinate projection maps, $\pi_\ell : E \to \mathcal{O}(x_*)$, $\pi_r : \mathcal{O}(x_*) \to \mathcal{O}(x_*)$, are injective. In the language of groupoids, $E$ is a $G$-set. If $\lambda$ is a Borel measure such that $\lambda(\pi_\ell(E)) = \lambda(\pi_r(E))$ for all closed and open $G$-sets $E$, then $\lambda$ is said to be $R$-invariant. It is easy to see that this requirement forces $\lambda$ to be the product measure $\lambda_1 \times \lambda_2 \times \ldots$ where $\lambda_k$ is the uniformly distributed probability measure on $\{1, \ldots, r_k\}$. (One can also bear in mind that $R$-invariant measures are also $\tilde{R}$-invariant, where $\tilde{R}$ is the topological equivalence relation (i.e. groupoid) generated by $R$, and that the $\tilde{R}$-invariant measures correspond to traces on the $C^*$-algebra of $\tilde{R}$. In our context $C^*(\tilde{R})$ is UHF, and the $R$-invariant measure corresponds to the unique trace.)

Let $\nu(x)$ denote the right hand quantity of (1). Then the coordinates for $\alpha(x)$ are calculated from the identity (1), bearing in mind that the ambiguity arising from the equality $\nu(x) = \nu(x^+)$, for a gap pair $x, x^+$, is resolved by the known correspondence of left and right gap points.

Note that if $x$ is in $\overline{\mathcal{O}(x_*)}$, and $\alpha(x) = y = (y_k)$, and $\|w\| = 1$ (so that $y_{-t+1}, y_{-t}, \ldots, y_t$ are all equal to 1), then, by (1),

$$\nu(\alpha(x)) = \sum_{k=1}^{\infty} \frac{y_k - 1}{m_k} = \sum_{k=1}^{\infty} \frac{y_{t+k-1} - 1}{m_{t+k-1}} = \frac{n\nu(x)}{m_{t-1}}.$$ 

We have obtained the identity $\nu(\alpha(x)) = c\nu(x)$, with $c = n/m_{t-1}$, for all points $x$ in $\overline{\mathcal{O}(x_*)}$ for which $\nu(x)$ is small. In fact, because of the $R$-invariance of the measures on $\overline{\mathcal{O}(x_*)}$ and $\overline{\mathcal{O}(\alpha(x_*))}$, which we shall call $\lambda_1$ and $\lambda_2$ respectively, it follows that $\nu(\alpha(x)) = c\nu(x)$ for all points $x$ for which $\alpha(x) \in \overline{\mathcal{O}(x_*)}$. To be more precise about this, consider the left gap points

$$g = (\ldots, \hat{1}, 1, \ldots, 1, r_{\ell+1}, \ldots),$$

$$x = (\ldots, \hat{1}, w, r_{\ell}, r_{\ell+1}, \ldots),$$

$$x' = (\ldots, \hat{1}, w, r_{\ell} - 1, r_{\ell+1}, \ldots),$$
where \( w \) is some word \( w_1, w_2, \ldots, w_{T-1} \). Note that the set

\[
E = \{((\ldots, 1, \hat{w}, r_{\ell}, z_{\ell+1}, z_{\ell+2}, \ldots), (\ldots, 1, \ldots, 1, z_{\ell+1}, z_{\ell+2}, \ldots)) : z_j \leq r_j \}
\]

has \( \pi_\ell(E) = \overline{O(x)} \setminus \overline{O(x')} \) and \( \pi_r(E) = \overline{O(g)} \), and so \( \nu(g) = \nu(x) - \nu(x') \). Since \( \alpha \) preserves orbits and \( G \)-sets we also deduce that

\[
\nu(\alpha(g)) = \lambda_1(\overline{O(\alpha(g))}) = \lambda_1(\pi_r((\alpha \times \alpha)(E)))
\]

\[
= \lambda_1(\pi_\ell((\alpha \times \alpha)(E))) = \lambda_1(\overline{O(\alpha(x))} \setminus \overline{O(\alpha(x'))})
\]

\[
= \nu(\alpha(x)) - \nu(\alpha(x')).
\]

Thus, if we choose \( \ell \) large, so that we know that \( \nu(\alpha(g)) = \nu(\alpha(x)) \), we deduce that

\[
\nu(\alpha(x)) - \nu(\alpha(x')) = \nu(\alpha(g)) = \nu(x) - \nu(x'),
\]

from which it follows that \( \nu(\alpha(x)) = \nu(x) \) for general points \( x \) with \( \alpha(x) \) in \( \overline{O(x')} \).

We can similarly extend this identity to points in the set

\[
X_0 = \{(y_k) \in X : \exists k_0 \text{ such that } y_k = 1 \text{ for all } k \leq k_0 \}
\]

and the extension of \( \nu \) given by

\[
\nu(y) = \sum_{k=1}^{\infty} (y_{-k} - 1)s_0s_1 \ldots s_{k-1} + \sum_{k=1}^{\infty} \frac{y_k - 1}{m_k}
\]

for \( y \) in \( X_0 \), where \( s_0 = 1 \). The range of \( \nu \) on the gap points of \( X_0 \) is the additive cone of rationals of the form \( \ell/m_k \) for some \( k = 1, 2, \ldots \) and some natural number \( \ell \). The identity \( \nu(\alpha(x)) = \nu(x) \) for \( x \) in \( X_0 \) shows that multiplication by \( c \) is a bijection of the cone. From this we obtain the necessary condition that \( c \) has the form

\[
c = p_1^{a_1} \ldots p_d^{a_d}
\]

where \( a_i \in \mathbb{Z}, 1 \leq i \leq d \), and where \( p_1, \ldots p_d \) are primes which divide infinitely many terms of the sequence \( (r_k) \).

We now improve further on this condition by considering the fact that \( \alpha \) is a homeomorphism of \( X \) and is determined by its restriction to \( X_0 \).
Suppose, by way of contradiction, that \( a_1 \neq 0 \) and that \( p_1 \) does not divide infinitely many terms of the sequence \( (s_k) \). Note that \( c \) only depends on \( \alpha \), thus, replacing \( \alpha \) by its inverse if necessary, we may assume that \( a_1 > 0 \). By relabelling we may also assume that \( p_1 \) divides no terms of the sequence. Without loss of generality assume that \( s_1 > 1 \) and consider the proper clopen subset \( E \) of points \( y = (y_k) \) in \( X \) with \( y_{-1} = 1 \). We show that \( \alpha(E) \) is dense, which is the desired contradiction. Observe first that the range of \( \nu \) on \( E \cap X_0 \) is the union of the intervals \( [ks_1, ks_1 + 1] \) for \( k = 0, 1, 2, \ldots \). Pick \( x \) in \( X_0 \) arbitrarily, pick \( j \) large, and consider the countable set

\[
F_j(x) = \{ x' \in X_0 : x' = (x'_k) \text{ and } x'_k = x_k \text{ for all } k \geq -j \}.
\]

The range of \( \nu \) on \( F_j(x) \) is an arithmetic progression of period \( s_1 s_2 \ldots s_j \). In view of the identity \( \nu(\alpha(y)) = cv(y) \), the range of \( \nu \) on \( \alpha(E) \cap X_0 \) is the union of the intervals \([cks_1, cks_1 + c]\), which is an arithmetic progression of intervals of period \( cs_1 \). It follows from our hypothesis on \( p_1 \) that one of these intervals contains a point in \( \nu(F_j(x)) \), and so \( \alpha(E) \) meets \( F_j(x) \). Since the intersection of the sets \( F_1(x), F_2(x), \ldots \) is the singleton \( x \), it follows that \( x \) lies in the closure of \( \alpha(E) \). Since \( X_0 \) is dense it follows that \( \alpha(E) \) is dense as desired.

We have now shown that if \( \alpha \) is an automorphism of \( R = R(r_k, s_k) \), then \( \nu(\alpha(x)) = cv(x) \) for all \( x \) in \( X_0 \) where \( c \) has the form \( c = p_1^{a_1} p_2^{a_2} \ldots p_d^{a_d} \) where \( a_1, \ldots, a_d \) are integers and where \( p_1, \ldots, p_d \) are primes which divide infinitely many terms of \( (r_k) \) and of \( (s_k) \). It is also clear from the above that for each such \( c \) there is at most one automorphism \( \alpha \) satisfying the identity \( \nu(\alpha(x)) = cv(x) \). It follows that the map

\[
\alpha \to (a_1, \ldots, a_d)
\]

is an injective group homomorphism from \( \text{Aut}R \) to \( \mathbb{Z}^d \). (\( d \) may be infinite.) It remains to show that this map is surjective. One way to do this is to start with \( c \) of the required form above and to show that the bijection of \( X_0 \) induced by multiplication by \( c \) (that is, the bijection \( \alpha \) satisfying \( \nu(\alpha(x)) = cv(x) \)) does extend to an order preserving homeomorphism of \( X \) which defines an automorphism of \( R \). Another way, which we now follow, is to make the connection between \( R(r_k, s_k) \) and \( T(r_k, s_k) \), and to determine generators of \( \text{Aut}R \) in terms of commuting diagrams, as we indicated after the statement of Theorem 1.
Consider the diagram

\[
\begin{array}{ccccccccc}
\mathbb{C} & \overset{\rho_1}{\longrightarrow} & M_{r_1} & \overset{\sigma_1}{\longrightarrow} & M_{s_1} \otimes M_{r_1} & \overset{\rho_2}{\longrightarrow} & M_{s_1} \otimes M_{r_1} \otimes M_{r_2} & \overset{\sigma_2}{\longrightarrow} & \cdots & B \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\mathbb{C} & \overset{\rho_1}{\longrightarrow} & T_{r_1} & \overset{\sigma_1}{\longrightarrow} & T_{s_1} \odot T_{r_1} & \overset{\rho_2}{\longrightarrow} & T_{s_1} \odot T_{r_1} \otimes T_{r_2} & \overset{\sigma_2}{\longrightarrow} & \cdots & A
\end{array}
\]

The vertical maps are inclusions, where \( T_{s_1} \odot T_{r_1} \), for example, is realised in terms of the lexicographic order on the indices \((i,j,k)\) of the minimal projections \( e_{ii} \otimes e_{jj} \otimes e_{kk} \) in \( M_{s_1} \otimes M_{r_1} \otimes M_{r_2} \). (For more detail concerning this discussion, read the introduction of [HP].) The maximal ideal space of the diagonal C*-algebra \( A \cap A^* \) is naturally identified with the space \( X \). Indeed, \( x = (x_k) \) in \( X \) corresponds to the point in the intersection of the Gelfand supports of the projections

\[ e(x, N) = e_{x_{-N-N}} \otimes \cdots \otimes e_{x_{-1-1}} \otimes e_{x_1,1} \otimes \cdots \otimes e_{x_N,N} \]

for \( N = 1, 2, \ldots \). Furthermore, \((x, y)\) belongs to \( R = R(r_k, s_k) \) if and only if for all large \( N \) there is a matrix unit in the appropriate upper triangular matrix algebra with initial projection \( e(y, N) \) and final projection \( e(x, N) \). (In fact \( R \) is the fundamental relation of the limit algebra \( A \).)

Suppose now that \( r_k = s_k = p \) for all odd \( k \) and let \( \alpha \) be the isometric automorphism of \( T(r_k, s_k) \) determined by the diagram given in the introduction. Let \( \alpha \) also denote the induced automorphism of \( R \). We prove that \( \nu(\alpha(x)) = p^{-1} \nu(x) \), completing the proof of the theorem.

Let us calculate \( \alpha(e(x, N)) \), where \( N \) is even, \( x = (\ldots, 1, \hat{1}, 2, 1, \ldots) \), and where we abuse notation somewhat and write \( e(x, N) \) for the image of \( e(x, N) \) in the limit algebra. Let \( d(N) = s_N \ldots s_1 r_1 \ldots r_N \), and let \( e(x, N) \) occupy position \( a(N) \) in the lexicographic ordering of the \( d(N) \) matrix units. Consider the following part of the diagram defining \( \alpha \).

\[
\begin{array}{cc}
T_{s_N \ldots r_N} & \overset{\rho_p}{\longrightarrow} & T_{s_N \ldots r_{N+1}} \overset{i}{\longrightarrow} A \\
\sigma_p & \downarrow & \\
T_{s_N \ldots r_{N+1}} & \overset{i}{\longrightarrow} A
\end{array}
\]
Then
\[ \rho_p(e(x, N)) = \sum_{k=1}^{p} e(x, N) \otimes e_{kk}. \]
On the other hand, \( \sigma_p(e(x, N)) \) is the summation of the diagonal matrix units in positions \( a(N), a(N) + d(N), \ldots, a(N) + (p - 1)d(N) \) in the lexicographic order. Let these projections correspond to the matrix unit tensors with subscripts \( z^{(i)} = (z^{(i)}_{-N}, \ldots, z^{(i)}_{N+1}) \) for \( 1 \leq i \leq p \), and denote the projections themselves by \( f_1, \ldots, f_p \), respectively. It follows (from the partial diagram above) that the homeomorphism \( \alpha : X \to X \) maps the support of \( e(x, N) \) onto the union of the supports of \( f_1, \ldots, f_p \). Denote these supports by \( E(x, N), F_1, \ldots, F_p \) respectively. Since \( X_0 \) is invariant for \( \alpha \),
\[ \alpha(E(x, N) \cap X_0) = \bigcup_{k=1}^{p} F_k \cap X_0. \]
Notice that \( x \) is the unique point in \( E(x, N) \cap X_0 \) with the property that if \( y \in E(x, N) \cap X_0 \) and \( (x, y) \in \tilde{R} \) then \( (x, y) \in R \). The point in the union of \( F_1 \cap X_0, \ldots, F_p \cap X_0 \) with this minimum property is the point
\[ u = (\ldots 1 1 z^{(1)}_{-N}, \ldots, z^{(1)}_{N+1}, 1, 1, \ldots) \]
and so \( \alpha(x) = u \). Finally one can verify that \( \nu(x) = p^{-1} \) and \( \nu(u) = p^{-2} \), as desired. \( \square \)

Recall that the fundamental relation \( R(A) \) of a canonical triangular subalgebra \( A \) of an AF C*–algebra \( B \) is the topological binary relation on the Gelfand space \( M(A \cap A^*) \) induced by the partial isometries of \( A \) which normalise \( A \cap A^* \). (See [P2].) In [HP] we identified \( R(A) \), for \( A = T(r_k, s_k) \), with \( R(r_k, s_k) \). (This identification is also effected in the proof above by virtue of the fact that a matrix unit system determines \( R(A) \).) Let \( \beta \) be an isometric automorphism of \( A \). Then \( \beta \) induces an automorphism of \( R(A) \) (because \( \beta(A \cap A^*) = A \cap A^* \) and \( \beta \) maps the normaliser onto itself). Thus \( \beta \) determines an automorphism of \( R(r_k, s_k) \) and so by the last theorem there is an isometric automorphism \( \alpha \) of \( A \) such that \( \gamma = \alpha^{-1} \circ \beta \) induces the trivial automorphism of \( R(r_k, s_k) \). This means that \( \gamma \) is an isometric automorphism with \( \gamma \) equal to the identity map on \( A \cap A^* \).

**Lemma** Let \( \gamma \) be an automorphism of \( T(r_k, s_k) \) which is the identity on the diagonal subalgebra (and which is not necessarily isometric). Then \( \gamma \) is approximately inner.
Proof: Let \( A = T(r_k, s_k) \) and let \( A_1 \to A_2 \to \ldots \) be the direct system defining \( A \). The hypothesis is that \( \gamma(c) = c \) for all \( c \) in \( C = A \cap A^* \). This ensures that \( \gamma(\tilde{A}_n) = \tilde{A}_n \) where \( \tilde{A}_n \) is the subalgebra generated by \( A_n \) and \( A \). To see this, recall from Lemma 1.2 of [P1] that there are contractive maps \( P_n : A \to \tilde{A}_n \) which are defined in terms of limits of sums of compressions by projections in \( C \), and so, for \( a \) in \( \tilde{A}_n \), \( \gamma(a) = \gamma(P_n(a)) = P_n(\gamma(a)) \).

The restriction automorphism \( \gamma|_{\tilde{A}_n} \) is necessarily inner. Indeed identify \( \tilde{A}_n \) with \( T_r \otimes D \), for appropriate \( r \), where \( D \) is an abelian approximately finite C*-algebra and let \( u_i \in D, 1 \leq i \leq r - 1 \), be the invertible elements such that \( \gamma(e_{i,i+1}) = e_{i,i+1} \otimes u_i \). Also set \( u_0 = 1 \). Then it follows that \( \gamma(a) = u^{-1}au \), where

\[
  u = \sum_{i=1}^{r} e_{i,i} \otimes u_0 u_1 \ldots u_{r-1}
\]

Furthermore, since \( \gamma(e_{1,r}) = e_{1,r} \otimes u_0 u_1 \ldots u_{r-1} \), it follows that \( \|u\| \leq \|\gamma\| \). Similarly \( \|u^{-1}\| \leq \|\gamma^{-1}\| \). The inner automorphisms \( Adu^{-1} \), for varying \( n \), thus form a uniformly bounded sequence which converge pointwise on each \( A_n \), and so determine an approximately inner automorphism. \( \square \)

It follows from Lemma 1 and the preceding discussion that

\[
  \text{Aut}_{isom} A/I(A) = \text{Aut}_R(A) = \mathbb{Z}^d.
\]

Remark 1. Suppose that \( \delta \in \text{Aut} A \). Then \( \delta \) determines a scaled group homomorphism \( \delta_\ast : K_0(A) \to K_0(A) \) which preserves the algebraic order on the scale \( \Sigma(A) \) of \( K_0(A) \). Thus, by the main theorem of [P3], (which can also be found in [P4]) there is an isometric algebra automorphism of \( A \), \( \phi \) say, with \( \phi_\ast = \delta_\ast \). In particular \( \psi = \phi^{-1} \circ \delta \) has \( \psi_\ast \) trivial. This means that if \( P : A \to A \cap A^* \) is the diagonal expectation, then \( P(\psi(e)) = e \) for each projection \( e \) in \( A \cap A^* \). Thus to show that \( \text{Aut} A/I(A) = \mathbb{Z}^d \) it remains only to show that such automorphisms \( \psi \) are approximately inner.

Remark 2. There are approximately inner automorphisms of alternation algebras which are not inner. To see this, consider the standard limit algebra \( A = \lim_{\to}(T_{2^n}, \sigma) \).

Let \( \lambda \) be a unimodular complex number and let \( d_n = \lambda e_{1,1} + \lambda^2 e_{2,2} + \ldots + \lambda^{2^n} e_{2^n,2^n} \). Then
\[ d_n a d_n^{-1} = d_m a d_m^{-1} \] if \( a \in T_{2^n} \) and \( m > n \), from which it follows that \( \alpha(a) = \lim_n(d_n a d_n^{-1}) \) is an isometric approximately inner automorphism.

Suppose now that \( \alpha \) is inner, and \( \alpha(a) = g a g^{-1} \) for some invertible \( g \) in \( A \). Since \( \alpha(c) = c \) for all \( c \) in the masa \( C \) it follows that \( g \in C \). In particular \( \|\alpha - \beta\| \leq \frac{1}{4} \) for some inner automorphism \( \beta \) of the form \( \beta(a) = h a h^{-1} \) where, for some large enough \( n \), \( h \in T_{2^n} \cap (T_{2^n})^* \).

However, in \( T_{2^m} \), for large \( m \), the diagonal element \( h \) has matrix entries which are periodic with period \( 2^n \). One can now verify that if \( \lambda \) is chosen so that no power of order \( 2^k \) is unity then for large enough \( m \) there exist matrix units \( e \in T_{2^m} \) such that \( \|\lambda e - heh^{-1}\| > \frac{1}{4} \), a contradiction.

**Remark 3.** Let \((x, y)\) be a point in \( R(C^*(A(r_k, s_k)))\) with \( x = (\ldots, x_{-2}, x_{-1}, x_1, x_2, \ldots), \ y = (\ldots, y_{-2}, y_{-1}, y_1, y_2, \ldots) \). Then, although \( \nu(x) \) and \( \nu(y) \) may be infinite, we may define \( d(x, y) \) as the sum

\[
\sum_{k=1}^\infty (y_{-k} - x_{-k}) s_{0} s_{1} \ldots s_{k-1} + \sum_{k=1}^\infty \frac{y_k - x_k}{r_1 r_2 \ldots r_k}
\]

because only finitely many terms are nonzero. Since \( d(x, y) = d(x, z) + d(z, y) \), and \( (x, y) \in R(r_k, s_k) \) if and only if \( d(x, y) \geq 0 \), it follows that \( d(x, y) \) is a continuous real valued cocycle determining \( A(r_k, s_k) \) as an analytic subalgebra of \( C^*(A(r_k, s_k)) \). See [V], where some special cases are discussed as well as some general aspects of analyticity.

*Added Dec 1992:* Unfortunately the proof of the classification of alternation algebras given in [HP] and [P4] appears to be incomplete. (It is not clear, in [P4], whether \( q \) can be chosen with the desired properties.) However the present paper is independent of [HP] and the arithmetic progression argument above can be adapted, to the case of an isomorphism \( \alpha \) between two alternation algebras, to show that the supernatural numbers for the standard multiplicities are finitely equivalent.

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