A NOTE ON TRANSFERS FOR NON-STABLE $K_1$-FUNCTORS OF CLASSICAL TYPE

A. Stavrova

Department of Mathematics and Mechanics
St. Petersburg State University
St. Petersburg, Russia
anastasia.stavrova@gmail.com

1. Introduction

Throughout this text, we assume that $k$ is a field of characteristic 0. Let $Sm_k$ be the category of smooth schemes of finite type over $k$. We will consider $Sm_k$ as a site with Nisnevich topology, and for any presheaf $F$ on $Sm_k$, we denote by $F_{Nis}$ its Nisnevich sheafification. Recall that a presheaf $F$ on $Sm_k$ (or its subcategory) is called $\mathbb{A}^1$-invariant, if for any object $X$ the projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism $F(X \times \mathbb{A}^1) \cong F(X)$.

Let $G$ be a reductive group over $k$. Let $P$ be a (proper) parabolic $k$-subgroup of $G$. For any $k$-scheme $X$, we set $E_P(X) = \langle U_P(X), U_{P-}(X) \rangle \subseteq G(X)$, where $U_P$ and $U_{P-}$ are the unipotent radicals of $P$ and any opposite parabolic subgroup $P^-$. The quotient $G(X)/E_P(X) = K_{1,G,P}^G(X) = W_P(X, G)$ is called the non-stable $K_1$-functor associated to $G$, or the Whitehead group of $G$. Both names go back to Bass’ founding paper [B], where the case $G = \text{GL}_n$ was considered.

It is known that the functor $K_{1,G,P}^G$ on affine $k$-schemes is independent of the choice of a parabolic $k$-subgroup $P$ that intersects properly every semisimple normal subgroup of $G$. If every semisimple normal subgroup of $G$ contains $(\mathbb{G}_m)^2$, then $K_{1,G,P}^G$ takes values in the category of groups. Also, it is $\mathbb{A}^1$-invariant on the category of smooth affine $k$-schemes. We refer to [St13] for these and other basic properties of non-stable $K_1$-functors associated to isotropic groups.

In the present paper we study the Nisnevich sheafification $K_{1,Nis}^{G,P}$ of $K_{1,G,P}^G$ on the category $Sm_k$, where $G$ a simply connected semisimple group of type $A_l$ or $D_l$. It is well-known that for such groups $G$ the functor $K_{1,G,P}^G$, and hence $K_{1,Nis}^{G,P}$, is non-trivial on fields. (Note that if $G$ is simply connected and of type $B_l$ or $C_l$, then $K_{1,Nis}^G$ is trivial; see [C].) The above

---

1 The author acknowledges support of the postdoctoral grant 6.50.22.2014 “Structure theory, representation theory and geometry of algebraic groups” of St. Petersburg State University, and the RFBR grants 14-01-31515-mol_a, 13-01-00429-a.
properties of $K_{1}^{G,P}$ imply that $K_{1,Nis}^{G,P} = K_{1,Nis}^{G}$ is group-valued, independent of the choice of $P$, and $A^{1}$-invariant (see Lemmas 3.2 and 3.3 below).

J. Ross [R] recently gave the following definition of oriented weak transfers, modifying the definition of I. Panin and S. Yagunov [PaYa].

**Definition 1.1.** [R, Definition 2.1] Let $\mathcal{A}$ be an additive category, and $F : Sm_{k}^{op} \to \mathcal{A}$. Assume that $F$ is additive in the sense that $F(X_{1} \coprod X_{2}) \cong F(X_{1}) \oplus F(X_{2})$. We say that $F$ has oriented weak transfers if for any $X, Y \in Sm_{k}$, any finite flat generically étale morphism $f : X \to Y$, and any closed embedding of $Y$-schemes $X \hookrightarrow Y \times A^{n}$ with trivial normal bundle, there is a map $f_{\ast} : F(X) \to F(Y)$ satisfying the following properties.

1. The $f_{\ast}$’s are compatible with disjoint unions: if $X = X_{1} \coprod X_{2}$, $f_{i} : X_{i} \to Y$, $i = 1, 2$ are the morphisms induced by $f$, then the following diagram commutes:

   $$
   \begin{array}{ccc}
   F(X) & \xrightarrow{f_{\ast}} & F(Y) \\
   \cong \downarrow & & \downarrow (f_{1\ast}, f_{2\ast}) \\
   F(X_{1}) \oplus F(X_{2}) & \xrightarrow{(f_{1\ast}, f_{2\ast})} & F(Y)
   \end{array}
   $$

2. The $f_{\ast}$’s are compatible with sections $s : Y \to X$ which are isomorphisms onto connected components of $X$. In the notation of the previous property, if $s$ is an isomorphism onto $X_{1}$, then we require $f_{1\ast} = s_{\ast}$ (for any $Y$-embedding $X \hookrightarrow Y \times A^{n}$).

3. For any morphism $g : Y' \to Y$ which is either smooth, or a closed embedding of a principal smooth divisor, the following diagram commutes:

   $$
   \begin{array}{ccc}
   F(X) & \xrightarrow{g_{\ast}} & F(X \times_{Y} Y') \\
   \downarrow f_{\ast} & & \downarrow f'_{\ast} \\
   F(Y) & \xrightarrow{g'_{\ast}} & F(Y')
   \end{array}
   $$

   Here $f' : X \times_{Y} Y' \to Y'$ is the natural morphism induced by $f$, and $g' : X \times_{Y} Y' \to X$ is the projection.

4. $f_{\ast}$ is compatible with the addition of irrelevant summands: if $i : X \hookrightarrow Y \times A^{n}$ is a closed $Y$-embedding with trivial normal bundle, and $(i, 0) : X \hookrightarrow Y \times A^{n} \times A^{1}$ is the embedding induced by $0 \hookrightarrow A^{1}$, then the corresponding transfer maps $F(X) \to F(Y)$ coincide.

If $\mathcal{A}$ is the category of abelian groups $Ab$, then all transfer maps are required to be group homomorphisms.

One says that a presheaf $F : Sm_{k} \to \mathcal{A}$ has weak transfers for affine varieties, if we are given weak transfers as described above whenever $X$ and $Y$ are affine.

Our main result is the following theorem.

**Theorem 1.2.** Let $k$ be a field of characteristic 0, and let $R$ be a regular ring containing $k$. Let $G$ be a simply connected simple algebraic group over $k$ of type $A_{l}$, $l \geq 2$, or $D_{l}$, $l \geq 3$, such that $G$ contains $(\mathbb{G}_{m,k})^{2}$. Then the functor $K_{1,Nis}^{G}(-)$ on the category $Sm_{k}$ takes values in $Ab$, and is an $A^{1}$-invariant functor with oriented weak transfers for affine varieties.

Our construction of oriented weak transfers is inspired by the construction of the norm homomorphism for $R$-equivalence class groups of classical groups due to V. Chernousov and...
A. Merkurjev [ChM1]. In particular, their norm homomorphisms are precisely the oriented weak transfer maps for finite extensions of fields.

We would like to mention that, for isotropic groups $G$ of type $A_l$, the results of B. Kahn and M. Levine [KLe] seem to imply the existence of Voevodsky’s transfers for $K^G_{1,Nis}$ (although this is not stated explicitly). Our proof is much more elementary.

It is shown in [R], that an $A^1$-invariant presheaf on $Sm_k$ with oriented weak transfers has,

essentially, all the same properties as an $A^1$-invariant presheaf with Voevodsky’s transfers.

In particular, the results of [R] produce the following corollaries.

Corollary 1.3. Under the hypothesis of Theorem 1.2, for any $n ≥ 0$, $H^n_{Nis}(-, K^G_{1,Nis})$ is an $A^1$-invariant presheaf on $Sm_k$ with oriented weak transfers.

Proof. This follows from Theorem 1.2 and [R, Corollary 6.12].

Corollary 1.4. Under the hypothesis of Theorem 1.2, for any any smooth $k$-scheme $X$, one has

$$K^G_{1,Nis}(X) ≅ \prod_{i=1}^{k} K^G_{1,Nis}(k(X_i)),$$

where $X_i$, $1 ≤ i ≤ n$, are the connected components of $X$.

Proof. By [R] Theorem 6.14] combined with Theorem 1.2 there is a Gersten-type exact sequence of abelian groups

$$1 → K^G_{1,Nis}(X) → \bigoplus_{x ∈ X^{(0)}} i_x^*(K^G_{1,Nis}) → \bigoplus_{x ∈ X^{(1)}} i_x^*((K^G_{1,Nis})_{−1}) → \ldots$$

Here dy definition $(K^G_{1,Nis})_{−1}(X) = \text{coker}(K^G_{1,Nis}(A^1_X) → K^G_{1,Nis}(G_{m,X}))$. By Lemma 3.3 below we have $K^G_{1,Nis}(A^1_X) ≅ K^G_{1,Nis}(G_{m,X})$, that is, $K^G_{1,Nis})_{−1}$ is trivial. □

The author is indebted to Ivan Panin for pointing out that norm maps for algebraic tori can be defined not only for field extensions, but also for finite flat morphisms of affine schemes.

2. APPLICATIONS IN $A^1$-HOMOTOPY THEORY

For any presheaf on $Sm_k$, we denote by $\text{Sing}_{A^1}(F)$ the simplicial presheaf $U → F(Δ^* × U)$, where $Δ^*$ is the standard cosimplicial object made of affine spaces.

We denote by $\pi_i$ presheaves of homotopy groups of simplicial presheaves, and by $\pi_i^{A^1}$ presheaves of $A^1$-homotopy groups. Their Nisnevich sheafifications are denoted by $\pi_i,Nis$ and $\pi_i^{A^1,Nis}$ respectively.

For any simplicial sheaf of groups $G\dot$, we consider the simplicial sheaf $EG\dot$ with $EG_n = (G_n)^{n+1}$, and the classifying space $BG\dot = EG\dot / G\dot$ as in [MoV]. Note that we have

$$\pi_{n,Nis}(BG\dot) ≅ \pi_{n+1,Nis}(G\dot) \quad (2.1)$$

(see [MoV p. 123]).
Recall that a Nisnevich sheaf of groups $G$ on $Sm_k$ is called strongly $\mathbb{A}^1$-invariant, if for any $X \in Sm_k$ and $i = 0, 1$, the map 

$$H^i_{Nis}(X, G) \to H^i_{Nis}(X \times \mathbb{A}^1, G),$$

induced by the projection $\mathbb{A}^1 \times X \to X$, is bijective. If $G$ is a sheaf of abelian groups, then it is called strictly $\mathbb{A}^1$-invariant, if for any $X \in Sm_k$ and any $i \geq 0$ the map 

$$H^i_{Nis}(X, G) \to H^i_{Nis}(X \times \mathbb{A}^1, G),$$

is bijective.

We prove the following

**Theorem 2.1.** Let $G$ be a simply connected simple algebraic group over $k$ of classical type $A_l$, $l \geq 2$, or $D_l$, $l \geq 3$. Assume that $G$ has isotropic rank $\geq 2$, that is, contains $(\mathbb{G}_m, k)^2$. Then

$$\pi^{h^1}_{0,Nis}(G, e) \cong K^G_{1,Nis}.$$ 

In particular, $\pi^{h^1}_{0,Nis}(G, e)$ is strictly $\mathbb{A}^1$-invariant.

The same statement was previously established in the literature for $G = GL_n$, $n \geq 2$, and for split semisimple groups $G$ of arbitrary type (see [Mo1, p. 87], [Mo2, Theorem 8.1], [W2, Theorem 5.3]). The first key step of the proof is to show that, if $G$ is split simply connected, then $\pi^{h^1}_{0,Nis}(G)$ coincides with the non-stable $K_1$-functor (=Whitehead group) associated to $G$ on smooth affine schemes. This readily implies that $\pi^{h^1}_{0,Nis}(G)$ is actually trivial. Using this result, one concludes that for non-simply connected groups, $\pi^{h^1}_{0,Nis}(G)$ is a quotient of a torus, and thus a Nisnevich sheaf of abelian groups with transfers in the sense of Voevodsky. In particular, it is strongly $\mathbb{A}^1$-invariant. See [Mo1, p. 87].

The same argument actually applies to all isotropic semisimple groups $G$ over $k$ for which their simply connected cover $G^{\text{sc}}$ has trivial non-stable $K_1^G$-functor on all field extensions of $k$. This includes all $k$-rational isotropic groups (in particular, split groups and groups of classical types $B_l, C_l$) and many isotropic groups of exceptional types, see [G].

On the other hand, it is known that for some simply connected isotropic groups of classical types $A_l$ and $D_l$ the non-stable $K_1$-functor is non-trivial on fields, and does not allow any obvious description [G].

Theorem 2.1 has the following corollaries, which are heavily based on [MoV, Mo2].

**Corollary 2.2.** Let $G$ be as in Theorem 2.1. Let $G - X \to Y$ be a (homotopy) principal fibration with structure group $G$. Then it is an $\mathbb{A}^1$-homotopy $G$-principal fibration.

**Proof.** This is [Mo2, Theorem 6.50].

**Corollary 2.3.** Let $G$ be as in Theorem 2.1. Then the classifying space $B\text{Sing}_{\bullet}^\mathbb{A}^1 G$ is $\mathbb{A}^1$-local.

A sketch of the proof of this corollary, provided that $\pi^{h^1}_{0,Nis}(G)$ is strictly $\mathbb{A}^1$-invariant, is available, for example, in [W2]. However, since it is rather involved from a logical point of view, we reproduce it below with explicit references.

We note that pointed simplicial sheaves $(BG, e)$ and $(B\text{Sing}_{\bullet}^\mathbb{A}^1 G, e)$ are $\mathbb{A}^1$-weakly equivalent. This follows, for example, from [Mo2, Lemma 6.48], since $G$ and $\text{Sing}_{\bullet}^\mathbb{A}^1 G$ are $\mathbb{A}^1$-weakly equivalent by [MoV, Corollary 3.8 on p. 89].
The proofs in this section rely, in particular, on several results of Morel contained in [Mo2].

**Theorem 2.4.** [Mo2, Theorem 5.46] Let $M$ be a Nisnevich sheaf of abelian groups on $Sm_k$. Then $M$ is strongly $\mathbb{A}^1$-invariant if and only if $M$ is strictly $\mathbb{A}^1$-invariant.

**Theorem 2.5.** [Mo2, Corollary 6.2] For any pointed space $B$, its $\mathbb{A}^1$-fundamental sheaf of groups $\pi_{1,Nis}(B)$ is strongly $\mathbb{A}^1$-invariant, and for any $n \geq 2$ the sheaf of abelian groups $\pi_{n,Nis}(B)$ is strictly $\mathbb{A}^1$-invariant.

**Theorem 2.6.** [Mo2, Corollary 6.3] For any pointed simplicially connected space $B$, the following conditions are equivalent:
1) The space $B$ is $\mathbb{A}^1$-local.
2) The sheaf of groups $\pi_{1,Nis}(B)$ is strongly $\mathbb{A}^1$-invariant and for any integer $n \geq 2$, the $n$-th simplicial homotopy sheaf of groups $\pi_{n,Nis}(B)$ is strictly $\mathbb{A}^1$-invariant.

**Theorem 2.7.** [Mo2, Corollary A.12] Let $G$ be a simplicial presheaf of groups on $Sm_k$ that has the affine Brown-Gersten property for the Nisnevich topology and the affine $\mathbb{A}^1$-invariance property in the sense of [Mo2, Def. A.7]. Then its associated sheaf in the Nisnevich topology $G_{Nis}$ is $\mathbb{A}^1$-local, and its simplicially fibrant replacement $Ex(G_{Nis})$ satisfies the following: for any smooth $k$-algebra $A$ the map

$$G(A) \to Ex(G_{Nis})(A)$$

is a weak equivalence.

**Theorem 2.8.** [Mo2, W1, VW] Let $G$ be an isotropic reductive group over $k$ such that every semisimple normal subgroup of $G$ contains $(\mathbb{G}_m,k)^2$. Then $\text{Sing}_{\mathbb{A}^1}(G)$ is $\mathbb{A}^1$-local, and

$$\pi_i(\text{Sing}_{\mathbb{A}^1}(G)(A)) \cong (\pi_{\mathbb{A}^1}(G))(A)$$

for any smooth $k$-algebra $A$ and any $i \geq 0$.

**Proof.** We follow [Mo2, Proof of Theorem 8.1 assuming Theorem 8.2, p. 201]. One notes that $\text{Sing}_{\mathbb{A}^1}(G)$ has the affine $\mathbb{A}^1$-invariance property (considered in [Mo2, Definition A.7]) by [W, Lemma 3.5].

One also proves that $\text{Sing}_{\mathbb{A}^1}(G)$ has the affine B.G. property in the Nisnevich topology exactly as it is proved for $G = \text{GL}_r$ ($r \geq 3$) in [Mo2, Theorem 9.21]. The only difference is that we substitute the elementary subgroup functor $E(-)$ of $G(-)$ instead of $E_r(-) \subseteq \text{GL}_r(-)$, and refer to [St13, Corollary 3.5] instead of [Vo, Lemma 2.4 (i)], and to [St13, Theorem 1.3] instead of [Vo, Theorem 3.3].

Thus, $\text{Sing}_{\mathbb{A}^1}(G)$ has the affine B.G. property in the Nisnevich topology and the affine $\mathbb{A}^1$-invariance property, so by Theorem 2.7 $\text{Sing}_{\mathbb{A}^1}(G)$ is $\mathbb{A}^1$-local. Then its simplicially fibrant replacement $Ex(\text{Sing}_{\mathbb{A}^1}(G))$ is also $\mathbb{A}^1$-local (e.g. by [MoV, Prop. 3.19 on p. 93]), and hence $\mathbb{A}^1$-fibrant by [MoV, Prop. 2.28 on p. 80]. By [MoV, Corollary 3.8 on p. 89] $G$ and $\text{Sing}_{\mathbb{A}^1}(G)$ are $\mathbb{A}^1$-weakly equivalent. In particular, the presheaves $\pi_i(\text{Sing}_{\mathbb{A}^1}(G))(-)$ and $\pi_{\mathbb{A}^1}(\text{Sing}_{\mathbb{A}^1}(G))(-)$ are isomorphic. It remains to apply the last statement of Theorem 2.7.

**Proof of Theorem 2.7** By Theorem 2.8 we have

$$\pi_0(\text{Sing}_{\mathbb{A}^1}(G)(A),e) \cong (\pi_{\mathbb{A}^1}(G,e))(A)$$
for any smooth $k$-algebra $A$. By definition, $\pi_0(\text{Sing}^{A^1}(G)(A), e)$ is the first Karoubi-Villamayor $K$-theory $KV_1^G(A)$. By [St13 Lemma 3.3]

$$KV_1^G(A) \cong K_1^G(A).$$

One readily sees that there is a natural morphism of presheaves $K_1^G \to \pi_0^{A_1}(G, e)$. Indeed, $\pi_0^{A_1}(G, e)$ is $A_1$-invariant. On the other hand, for any $k$-scheme $X$, any parabolic $k$-subgroup $P$ of $G$, and any $g \in E_P(X)$, there is $\tilde{g} \in E_P(A^1_k)$ such that $\tilde{g}|_0 = 1$ and $\tilde{g}|_1 = g$ (e.g. by [SGA3 Exp. XXVI Cor. 2.5]). Together, this implies that the natural morphism $G \to \pi_0^{A_1}(G, e)$ factors through $K_1^G$. Therefore, $\pi_{0,Nis}(G, e) \cong K_1^G$ as Nisnevich sheaves on $Sm_k$, since they coincide on stalks.

By Theorem 1.2 the functor $K_{1,Nis}$ takes values in $Ab$, is $A_1$-invariant and has oriented weak transfers. By Corollary 1.3 the functor $\mathcal{H}^n_{Nis}(-, (K^G_{1,Nis})_n)$ is $A_1$-invariant for any $n \geq 0$. This completes the proof of the fact that $\pi_{0,Nis}(\text{Sing}^{A^1}(G), e)$ is strictly $A_1$-invariant.

Proof of Corollary 2.3 We follow the sketch of the proof given in [W2].

First, note that $\pi_0(B \text{Sing}^{A^1}(G), e)$ classifies Nisnevich $\text{Sing}^{A^1}(G)$-torsors; see, for example, [JL Corollary 10 and Remark 5]. Then $B \text{Sing}^{A^1}(G)$ is simplicially connected, i.e. $\pi_{0,Nis}(B \text{Sing}^{A^1}(G), e)$ is trivial. Hence by Theorem 2.6 it is enough to check that the sheaf of groups $\pi_{1,Nis}(B)$ is strongly $A_1$-invariant and for any integer $n \geq 2$, the sheaf of groups $\pi_{n,Nis}(B)$ is strictly $A_1$-invariant.

By (2.1), we have

$$\pi_{n,Nis}(\text{Sing}^{A^1}(G)) \cong \pi_{n+1,Nis}(B \text{Sing}^{A^1}(G)).$$

The presheaf $\text{Sing}^{A^1}(G)$ is $A_1$-local by Theorem 2.8. Hence, by Theorem 2.5 $\pi_{1,Nis}(\text{Sing}^{A^1}(G))$ is strongly $A_1$-invariant, and $\pi_{n,Nis}(\text{Sing}^{A^1}(G))$ is strictly $A_1$-invariant for $n \geq 2$. By (2.1) it means that $\pi_{2,Nis}(B \text{Sing}^{A^1}(G))$ is strongly $A_1$-invariant, and $\pi_{n,Nis}(B \text{Sing}^{A^1}(G))$ is strictly $A_1$-invariant for $n \geq 3$.

Since the sheaf $\pi_{2,Nis}(B \text{Sing}^{A^1}(G))$ is abelian and strongly $A_1$-invariant, it is strictly $A_1$-invariant by Theorem 2.6.

It remains to show that $\pi_{1,Nis}(B \text{Sing}^{A^1}(G)) \cong \pi_{0,Nis}(\text{Sing}^{A^1}(G))$ is strongly $A_1$-invariant. By Theorem 2.8 we have

$$\pi_{0,Nis}(\text{Sing}^{A^1}(G), e) \cong \pi_{0,Nis}(G, e).$$

By Theorem 2.6 the sheaf $\pi_{0,Nis}(G, e)$ is strictly $A_1$-invariant. Therefore, $\pi_{1,Nis}(B \text{Sing}^{A^1}(G))$ is strictly $A_1$-invariant. 

$\square$

3. Nisnevich sheafification of a non-stable $K_1$-functor

In order to prove Theorem 1.2 we need to consider non-stable $K_1$-functors in a more general situation than in the introduction. Let $R$ be a commutative ring with 1, and let $G$ be a reductive scheme over $R$ in the sense of [SGA3].

Definition 3.1. A parabolic subgroup $P$ in $G$ is called strictly proper, if it intersects properly every normal semisimple subgroup scheme of $G$. 

Assume that $G$ has a parabolic $R$-subgroup $P$ that is strictly proper. Since the base Spec $R$ is affine, $P$ has a Levi $R$-subgroup $L$, and there is a unique opposite parabolic $R$-subgroup $P^-$ of $G$ such that $P \cap P^- = L$ \cite[Exp. XXVI, Cor. 2.3 and Th. 4.3.2]{SGA3}. For any $R$-scheme $X$, we set

$$E_P(X) = \langle U_{P^+}(X), U_{P^-}(X) \rangle \subseteq G(X),$$

where $U_P$ and $U_{P^-}$ are the unipotent radicals of $P$ and $P^-$. Since any two Levi $R$-subgroups of $P$ are conjugate by an element of $U_P(R)$ by \cite[Exp. XXVI, Cor. 1.8]{SGA3}, the group $E_P(X)$ is indeed independent of the choice of $L$ and $P^-$.

Again, the quotient

$$G(X)/E_P(X) = K^{G,P}_1(X) = W_P(X,G)$$

is again called the non-stable $K_1$-functor associated to $G$, or the Whitehead group of $G$. It is a functor on the category of $R$-schemes.

If $R$ is a semilocal ring, or if every semisimple normal subgroup of $G$ contains $(\mathbb{G}_m, k)^2$ locally in Zariski topology on Spec $R$, then for any $R$-algebra $A$, the group $E_P(A) = E(A)$ is independent of the choice of a strictly proper parabolic $R$-subgroup $P$ \cite[Exp. XXVI, Cor. 2.1]{SGA3}, see \cite[Theorem 2.1]{St13}. In this case $K^{G,P}_1(A)$ is a group, since a conjugate of a strictly proper parabolic subgroup is a strictly proper parabolic subgroup.

**Lemma 3.2.** Let $R$ be a smooth $k$-algebra, where $k$ is a field. Let $G$ be a reductive group scheme over $R$ having a strictly proper parabolic $R$-subgroup. Assume that every semisimple normal subgroup of $G$ contains $(\mathbb{G}_m, k)^2$ locally in Zariski topology on Spec $R$. Then the functor $K^{G,P}_{1,Nis} = K^{G,P}_{1,Nis}$ on the category of smooth $R$-schemes is independent of the choice of a strictly proper parabolic $R$-subgroup $P$ of $G$, and takes values in the category of groups.

**Proof.** Let $P$ be a strictly proper parabolic $R$-subgroup of $G$. Since the presheaf $E_P$ is a subgroup presheaf of $G$, and the sheafification functor (on presheaves of sets) is exact, its Nisnevich sheafification $E_{P,Nis}$ is a subgroup presheaf of the Nisnevich sheaf $G$. If $Q$ is another strictly proper parabolic $R$-subgroup of $G$, then $E_{P,Nis} = E_{Q,Nis}$ as sheaves on the category of smooth $R$-schemes, since they are both subsheaves of $G$ and coincide on stalks. In particular, $E_{P,Nis}$ is a subsheaf of normal subgroups of the sheaf $G$ on the category of all smooth $R$-schemes.

On the other hand, $K^{G,P}_{1,Nis}$ is also the Nisnevich sheafification of the quotient $G/E_{P,Nis}$. Therefore, $K^{G,P}_{1,Nis} = K^{G,P}_{1,Nis}$ is group-valued and independent of the choice of $P$. \hfill $\square$

Let $k$ be a field of characteristic $0$, and let $R$ be a regular ring containing $k$. Let $G$ be a reductive group over $k$, such that every semisimple normal subgroup of $G$ contains $(\mathbb{G}_m, k)^2$. By \cite[Theorem 1.3]{St13}, in this case the natural inclusion $R \to R[t]$ induces an isomorphism

$$K^G_1(R) \cong K^G_1(R[t]).$$

(3.1)

If $R$ be a local regular $k$-algebra, and $K$ is the field of fractions of $R$, then the natural map

$$K^G_1(R) \to K^G_1(K)$$

is injective by \cite[Theorem 1.4]{St13}. We extend these properties to $K^{G}_{1,Nis}$.

Note that Theorem 2.8 together with \cite[Theorem 4.18]{Choud} implies the $\mathbb{A}^1$-invariance of $K^{G}_{1,Nis}$ right away, however, we decided to include an independent proof, since it is very easy.
Lemma 3.3. Let $k$ be a field of characteristic 0, and let $G$ be a reductive group over $k$, such that every semisimple normal subgroup of $G$ contains $(\mathbb{G}_m,k)^2$.

(i) For any smooth $k$-scheme $X$, the natural map

$$K^G_{1,Nis}(X) \to \prod_{i=1}^k K^G_{1,Nis}(k(X_i))$$

is injective, where $X_i$, $1 \leq i \leq n$, are the connected components of $X$.

(ii) if $G$ is simply connected semisimple, then the functor $K^G_{1,Nis}$ on $Sm_k$ is $\mathbb{A}^1$-invariant and $\mathbb{G}_m$-invariant, that is, $K^G_{1,Nis}(\mathbb{G}_m,X) \cong K^G_{1,Nis}(X) \cong K^G_{1,Nis}(\mathbb{A}^1_X)$ for any $X \in Sm_k$.

Proof. By [ST13, Theorem 1.4], (i) holds for $X = \text{Spec} R$, where $R$ is a henselian local $k$-algebra. In general, consider the presheaf $F$ on $Sm_k$ given by

$$F(X) = \prod_{i=1}^k K^G_{1,Nis}(k(X_i)).$$

It is easy to see that $F$ is a Nisnevich sheaf. Then the natural morphism of functors $K^G_{1,Nis} \to F$ is injective, since it is injective on stalks.

In order to prove (ii), we first prove that $K^G_{1,Nis}(R) \cong K^G_{1,Nis}(R[t])$ for any essentially smooth $k$-domain $R$. First, assume that $R = K$ is a field extension of $k$. By (i), the homomorphism

$$f : K^G_{1,Nis}(K[t]) \to K^G_{1,Nis}(K(t))$$

is injective. On the other hand, $f$ is surjective, since the natural homomorphism

$$g : K^G_{1,Nis}(K) = K^G_1(K) \to K^G_1(K(t)) = K^G_{1,Nis}(K(t))$$

is an isomorphism by [Th Théorème 5.8 and Théorème 7.2]. This implies that $f$ is an isomorphism. Therefore, the map

$$f^{-1} \circ g : K^G_{1,Nis}(K) \to K^G_{1,Nis}(K[t])$$

is an isomorphism.

Now assume that $R$ is an essentially smooth $k$-domain, and let $K$ be its field of fractions. By (i), the natural map

$$K^G_{1,Nis}(R[t]) \to K^G_{1,Nis}(K(t))$$

is injective. Then the map $K^G_{1,Nis}(R[t]) \to K^G_{1,Nis}(K[t])$ is also injective. Then the commutative diagram

$$\begin{array}{ccc}
K^G_{1,Nis}(R[t]) & \xrightarrow{f \circ 0} & K^G_{1,Nis}(R) \\
\downarrow & & \downarrow \\
K^G_{1,Nis}(K[t]) & \xrightarrow{f \circ 0} & K^G_{1,Nis}(K)
\end{array}$$

implies that the map $K^G_{1,Nis}(R[t]) \xrightarrow{f \circ 0} K^G_{1,Nis}(R)$ is injective, and hence an isomorphism.

To finish the proof of (ii), consider an open cover $X = \bigcup_i U_i$, where $U_i$ are smooth connected affine $k$-schemes. Then $\mathbb{A}^1_X = \bigcup_i \mathbb{A}^1_{U_i}$ is also an open cover. Assume that

$$x \in \ker(K^G_{1,Nis}(\mathbb{A}^1_X) \xrightarrow{f_0} K^G_{1,Nis}(X)),$$
where \( f_0 \) is the "restriction to 0" homomorphism. Then \( \left. x \right|_{U_i} = 1 \) for any \( U_i \), since by the above

\[
K_{1,Nis}^G(\mathbb{A}^1_{U_i}) \cong K_{1,Nis}^G(U_i).
\]

Since \( K_{1,Nis}^G \) is a Zariski sheaf, this implies that \( x = 1 \).

The proof of \( \mathbb{G}_m\)-invariance is the same as the proof of \( \mathbb{A}^1\)-invariance, except that we show that the "restriction to 1" homomorphism \( K_{1,Nis}^G(\mathbb{G}_m, X) \to K_{1,Nis}^G(X) \) is injective. \( \square \)

4. Proof of Theorem 1.2

We begin the proof of Theorem 1.2. From now on, all regular rings that we consider are assumed to be essentially smooth over \( k \). Note that we have established in Lemma 3.3 that \( K_{1,Nis}^G \) is \( \mathbb{A}^1\)-invariant, i.e. homotopy invariant.

We recall the following construction of V. Chernousov and A. Merkurjev. As shown in [ChM1] p. 189–190 (see also [ChM2] for more details), for any simply connected semisimple algebraic group \( G \) over \( k \) of classical type \( A_l, B_l, C_l, D_l \) there exists a rational reductive \( k \)-group \( H \) and a \( k \)-torus \( T \) such that \( G \) fits into an exact sequence of algebraic group homomorphisms

\[
1 \to G \to H \xrightarrow{\mu} T \to 1.
\]

Using this diagram, one studies the group \( H(K)/RG(K) \), where \( K \) is a field extension of \( k \) and \( RG(K) \) is the group of \( R \)-trivial elements in \( G(K) \). Note that, since \( H \) is rational, one has \( RH(K) = e \).

By [ChM1] Lemma 1.2 the sequence (4.2) implies that the group of \( R \)-equivalence classes \( G(K)/RG(K) \) is abelian. By [G] Théorème 7.2 one has \( RG(K) = E_P(K) \) and \( G(K)/RG(K) = K_1^G(K) \) for any proper parabolic subgroup \( P \) of \( G \), therefore, \( K_1^G(K) = K_{1,Nis}^G(K) \) is abelian.

By Lemma 3.3 this implies that \( K_{1,Nis}^G \) takes values in \( Ab \).

The sequence (4.2) also implies the following. If

\[
1 \to G' \to H' \xrightarrow{\mu'} T' \to 1.
\]

is another short exact sequence of \( K \)-groups, where \( H' \) is reductive and \( T' \) is a torus, and if \( \beta : T \to T' \) is a group homomorphism such that

\[
\beta(\mu(H(F))) \subseteq \mu'(H'(F))
\]

for any field extension \( F/K \), then by [ChM1] Lemma 3.1 there exists an open dense \( K \)-subscheme \( U \subseteq H \) and a morphism \( \eta : U \to H' \) such that \( \mu' \circ \eta = \beta \circ \mu_{|U} \) and \( \eta(1_H) = 1_{H'} \). By [ChM1] Lemma 3.2 the induced map \( \tilde{\eta}(K) : U(K) \to H'(K)/RG'(K) \) extends uniquely to a homomorphism

\[
\tilde{\beta} : H(K) \to H'(K)/RG'(K)
\]

such that \( \tilde{\beta}(RG(K)) = 1 \). Actually, \( \tilde{\beta} \) is given by the following formula [ChM1] Proposition 1.4: for any \( g \in H(K) \) and any \( g_1, g_2 \in U(K) \) such that \( g = g_1g_2 \) set

\[
\tilde{\beta}(g) = \tilde{\eta}(g_1)\tilde{\eta}(g_2).
\]
One shows that \( \tilde{\beta} \) is correctly defined, independent of the choice of \((U, \eta)\) and functorial in \( K \) \cite[p. 183]{ChM1}. Since \( E_P(K) = RG(K) \), and \( \mu' \circ \eta = \beta \circ \mu|_U \), we see that \( \tilde{\beta} \) induces a homomorphism

\[
\tilde{\beta} : K_1^G(K) \to K_1^{G'}(K).
\]

We are going to generalize this construction to regular local \( k \)-rings. We will need the following easy lemma.

**Lemma 4.1.** Let \( A \) be a semilocal ring such that all residue fields of \( A \) are infinite. Let \( X \) be an affine group scheme over \( A \). Assume that \( X \) is rational, i.e. contains a dense open \( A \)-subscheme \( V \) which is isomorphic to an open subscheme of \( \mathbb{A}^n_A \) for some \( n \geq 1 \). Then

(i) \( X(A) \) is dense in \( X \).

(ii) \( X(A) = V(A) \cap (A - \eta) \).

(iii) if \( A \) is local, then for any ideal \( I \) of \( A \) the natural homomorphism \( X(A) \to X(A/I) \) is surjective.

**Proof.** Straightforward. \( \square \)

**Lemma 4.2.** In the above notation, let \( R \) be a regular local domain containing \( k \) such that \( K \) is its field of fractions. Assume that \( G, H, T \) are defined over \( R \) and \( \beta : T \to T_R' \) is a homomorphism of \( R \)-groups, and the condition (4.3) holds for all field extensions \( F \) of \( R \) only. Then \( \tilde{\beta} \) restricts to a correctly defined homomorphism

\[
\tilde{\beta} : K_1^G(R) \to K_1^{G'}(R).
\]

Moreover, \( \tilde{\beta} \) is functorial with respect to any morphism of regular local \( k \)-rings \( R \to R' \).

**Proof.** Repeating the beginning of the proof of \cite[Lemma 3.1]{ChM1} with \( R \) instead of \( k \) (note that \( R \) contains an infinite field and \( H \) is rational, hence \( H(R) \) is dense in \( H \) by Lemma 4.1), one obtains an open \( R \)-subscheme \( U \subseteq H \) and an \( R \)-morphism \( \eta : U \to H_R' \) such that \( \mu' \circ \eta = \beta \circ \mu|_U \). We explain how to secure that \( 1_H \in U(R) \) and \( \eta(1_H) = 1_{H'} \). There is an \( R \)-point \( g \in U(R) \subseteq H(R) \); set \( g' = \eta(g) \). Consider \( U_1 = g^{-1}U \) and \( \eta_1 : U_1 \to H_R' \), \( \eta_1(g^{-1}x) = g'^{-1}(g_1x) \). Then \( 1_H = g^{-1}g \in U_1(R) \), and \( \eta_1(1_H) = 1_{H'} \). Moreover, we have

\[
\begin{align*}
\mu'(\eta_1(g^{-1}x)) &= \mu'(g'^{-1}(g_1x)) = \mu'(g)^{-1}\mu'(g_1x) \\
&= \mu'(\eta(g))^{-1}\mu'(\eta(x)) = \beta(\mu(g))^{-1}\beta(\mu(x)) = \beta(\mu(g_1x)),
\end{align*}
\]

as required. Thus, we can replace \( U \) by \( U_1 \) and \( \eta \) by \( \eta_1 \).

The formula (4.4) together with Lemma 4.1 implies that \( \tilde{\beta} : H(K) \to H'(K)/RG'(K) \) restricts to

\[
\tilde{\beta} : H(R) \to H'(R)/(RG'(K) \cap H'(R)).
\]

Now we show that

\[
\tilde{\beta}(G(R)) \subseteq G'(R)/(RG'(K) \cap G'(R)).
\]

Assume that for \( g_1, g_2 \in U(R) \) we have \( g_1g_2 \in G(R) \). Then

\[
\mu'(\eta(g_1)\eta(g_2)) = \mu'(\eta(g_1))\mu'(\eta(g_2)) = \beta(\mu(g_1))\beta(\mu(g_2)) = \beta(\mu(g_1g_2)) = 1.
\]

Therefore, \( \eta(g_1)\eta(g_2) \in G'(R) \). This implies that \( \tilde{\beta}(G(R)) \subseteq G'(R)/(RG'(K) \cap G'(R)) \).
Since \( K_i^G(R) \) injects into \( K_i^{G'}(K) \) and \( E_P(K) = R G'(K) \) for any proper parabolic subgroup \( P' \) of \( G' \), we have \( R G'(K) \cap G(R) = E_P(R) \). Therefore, \( \hat{\beta} \) induces a correctly defined homomorphism

\[
\hat{\beta} : K_i^G(R) = G(R)/E_P(R) \to G'(R)/E_P(R) = K_i^{G'}(R).
\]

The formula (4.4) also implies that \( \hat{\beta} \) is functorial with respect to any morphism of regular local \( k \)-rings \( R \to R' \).

**Lemma 4.3.** In the above notation, assume that \( K \) is a field of fractions of a (not necessarily local) regular \( k \)-domain \( A \), \( G, H, T \) are defined over \( A \), and \( \beta \) is an \( A \)-morphism, satisfying condition (4.3) for all field extensions of \( A \). Let

\[
i : K_{1,\text{Nis}}^G(A) \to K_{1,\text{Nis}}^{G'}(K)
\]

be the natural homomorphism. Then \( \beta(i(K_{1,\text{Nis}}^G(A))) \subseteq K_{1,\text{Nis}}^{G'}(K) \) inside \( K_{1,\text{Nis}}^{G'}(K) = K_i^{G'}(K) \). Moreover,

\[
\hat{\beta} \circ i : K_{1,\text{Nis}}^G(A) \to K_{1,\text{Nis}}^{G'}(A)
\]

is functorial with respect to any homomorphism of \( k \)-domains \( A \to A' \).

**Proof.** We prove the claim by induction on dimension of \( A \). If \( \dim A = 0 \), then \( A \) is a field and the statement is trivial. In general, note that by Lemma 3.3

\[
K_{1,\text{Nis}}^{G'}(A) = \bigcap_{m \in \text{max}(A)} K_{1,\text{Nis}}^{G'}(A_m) \subseteq K_{1,\text{Nis}}^{G'}(K).
\]

Therefore, in order to prove the claim, we can assume that \( A = A_m = R \) is a regular local \( k \)-domain with a maximal ideal \( m \).

Fix \( x \in K_{1,\text{Nis}}^G(R) \), and set \( y = \hat{\beta}(i(x)) \in K_{1,\text{Nis}}^{G'}(K) \). Let \( \phi^h : R \to R^h \) be the henselization of \( R \). By Lemma 1.2 there is a correctly defined map

\[
\hat{\beta} : K_i^G(R^h) = K_{1,\text{Nis}}^{G'}(R^h) \to K_i^{G'}(R^h) = K_{1,\text{Nis}}^{G'}(R^h).
\]

Set \( y^h = \hat{\beta}(\phi^h(x)) \in K_i^{G'}(R^h) \). The functoriality of \( \hat{\beta} \) for local rings established in Lemma 1.2 together with the sheaf property of \( K_{1,\text{Nis}}^{G'}(K) \) implies that \( y \in K_i^G(K) \) and \( y^h \in K_i^{G'}(R^h) \) are mapped to the same element in \( K_{1,\text{Nis}}^{G'}(K \otimes_R R^h) \). Let \( \phi' : R \to R' \) be an étale local ring homomorphism, such that \( \phi^h \) factors through \( R' \), \( y^h \) lifts to \( y' \in K_{1,\text{Nis}}^{G'}(R') \), and \( y, y' \) are still mapped to the same element in \( K_{1,\text{Nis}}^{G'}(K \otimes_R R') \).

Let \( f_1, \ldots, f_k \in m \) be such that

\[
\text{Spec}(R) \setminus m = \bigcup_{i=1}^k \text{Spec}(R_{f_i}).
\]

Note that since \( R \) is local, \( \dim(R_{f_i}) < \dim(R) \) for any \( i \), and thus \( R_{f_i} \)'s satisfy the claim of the lemma. In particular, \( y \in K_{1,\text{Nis}}^{G'}(R_{f_i}) \subseteq K_{1,\text{Nis}}^{G'}(K) \) for any \( i \). Let \( L_i \) be the fraction field of \( R_{f_i} \otimes_R R' \). Since \( R' \) is a flat \( R \)-module, the map \( R_{f_i} \otimes_R R' \to K \otimes_R R' \) is injective, and, clearly, the map \( R_{f_i} \otimes_R R' \to L_i \) factors through it. Since by Lemma 3.3 the map

\[
K_{1,\text{Nis}}^{G'}(R_{f_i} \otimes_R R') \to K_{1,\text{Nis}}^{G'}(L_i)
\]

is injective, the map

\[
K_{1,\text{Nis}}^{G'}(R_{f_i} \otimes_R R') \to K_{1,\text{Nis}}^{G'}(K \otimes_R R')
\]
is also injective. It follows that \( y, y' \) are mapped to the same element in \( K'_{1,Nis}(R_f \otimes_R R') \) for every \( i \).

Since \( \text{Spec}(R_f) \), \( 1 \leq i \leq k \), and \( \text{Spec}(R') \) together form a Nisnevich cover of \( \text{Spec}(R) \), and the elements \( y \) and \( y' \) are mapped to the same element in each \( K'_{1,Nis}(R_f \otimes_R R') \), we conclude that \( y \in K'_{1,Nis}(R) \).

The functoriality of \( \hat{\beta} \circ i \) with respect to a \( k \)-homomorphism \( A \to A' \) follows immediately from Lemma 4.2 applied to all henselizations of maximal localizations of \( A \) and \( A' \). □

Consider two regular \( k \)-algebras \( R \) and \( S \) and a finite flat ring homomorphism \( f : R \to S \). One can define a natural "norm" homomorphism

\[
N_{S/R} : T(S) \to T(R)
\]

extending the usual norm in the field case, see [Pa, §2]. For any such \( R \) and \( S \) consider the \( S \)-torus \( T' = R_{S/R}(T_S) \). One has \( T'(R) = T(S) \), and the norm map defines a homomorphism of \( R \)-groups

\[
N_{S/R} : T' \to T_R.
\]

Set \( H' = R_{S/R}(H) \) and \( G' = R_{S/R}(G) \). Note that \( H' \) is rational, since \( H \) is, and \( G' \) is simply connected semisimple group such that every semisimple normal subgroup of \( G' \) contains \( G_{m, R}^2 \). We have a short exact sequence of \( R \)-groups

\[
1 \to G' \to H' \to T' \to 1,
\]

and an \( R \)-group morphism \( N_{S/R} : T' \to T_R \).

Observe that \( N_{S/R} \) satisfies condition (4.3) for any field \( K \) over \( R \). Indeed, \( S \otimes_R K \) is a product of finite field extensions of \( K \), and hence one can apply Merkurjev’s norm principle [M Theorem 3.9].

**Lemma 4.4.** In the above setting, one has \( K'_{1,Nis}(R) = K'_{1,Nis}(S) \).

**Proof.** For any \( R \)-algebra \( A \) we have \( K'_{1,Nis}(A) = K'_{1,Nis}(A \otimes_R S) \). Hence there is a morphism of presheaves on the category of smooth \( R \)-algebras \( \phi : K'_{1,Nis}(-) \to K'_{1,Nis}(- \otimes_R S) \). The latter presheaf is a Nisnevich sheaf, hence the morphism factors through a morphism

\[
\phi : K'_{1,Nis}(-) \to K'_{1,Nis}(- \otimes_R S).
\]

Assume that \( A \) is a henselian local ring. Since \( A \otimes_R S \) is finite over \( A \), we conclude that \( A \otimes_R S \) is a finite product of henselian local rings. Then \( K'_{1,Nis}(A \otimes_R S) = K'_{1,Nis}(A \otimes_R S) \).

Therefore, \( \phi \) is an isomorphism on Nisnevich stalks, hence an isomorphism. □

Let \( R = \prod_{i=1}^k R_i \) be the decomposition of \( R \) into a product of domains, and let \( K_i \) be the field of fractions of \( R_i \). Set \( S_i = S \otimes_R R_i \). Then we can apply Lemma 4.3 to \( N_{S_i/R_i} : R_{S_i/R_i}(T) \to T_{R_i} \) for each \( i \). Combining it with Lemma 4.3, we obtain a homomorphism

\[
f_* : K'_{1,Nis}(S) \to K'_{1,Nis}(R),
\]

that is the composition of \( i : K'_{1,Nis}(S) = K'_{1,Nis}(R) \to K'_{1,Nis}(K) \) with the restriction of the homomorphism

\[
\prod_{i=1}^k \hat{N}_{S_i/R_i} : \prod_{i=1}^k K'_{1,Nis}(K_i) \to \prod_{k=1}^n K'_{1,Nis}(K_i).
\]
We claim that the transfer map $f_*$ defined above is compatible with any base change $R \to R'$. This would imply that it satisfies the properties (3) and (4) of [R, Definition 2.1]. Indeed, set $S' = S \otimes_R R'$. The norm map is compatible with base change in the sense that the following diagram commutes (see [Pa, p. 5]):

$$
\begin{array}{ccc}
R_{S'/R'}(T) & \xrightarrow{N_{S'/R'}} & T_{R'} \\
\downarrow & & \downarrow \\
R_{S/R}(T) & \xrightarrow{N_{S/R}} & T_R
\end{array}
$$

By Lemma 4.3, the composition $\prod \hat{N}_{S_i/R_i} \circ i$ is also compatible with $R \to R'$.

Now we check that $f_*$ satisfies property (1) of [R, Definition 2.1]. Assume that $S = S_1 \times S_2$ is a product of two regular $k$-algebras, and let $f_1 : R \to S_1$ and $f_2 : R \to S_2$ be the natural maps. We need to show that $f_* = (f_1*, f_2*)$. The norm map is compatible with the decomposition of $S$, see [Pa, p. 5]. The construction in Lemma 4.3 is also compatible, since $R_{S/R}(T) = R_{S_1/R}(T) \times R_{S_2/R}(T)$.

The property (2) of [R, Definition 2.1] for $f_*$ follows immediately from the fact that the norm map satisfies normalization property of [Pa, p. 5]. The property (5) of [R, Definition 2.1] is trivially true, since $K_{1,Nis}^G$ is $A^1$-invariant. Thus, Theorem 1.2 is proved.

References

[B] H. Bass, *K-theory and stable algebra*, Publ. Math. IHÉS 22 (1964), 5–60.

[ChM1] V. Chernousov, A. S. Merkurjev, *R-equivalence and special unitary groups*, J. Algebra 209 (1998), 175–198.

[ChM2] V. Chernousov, A. S. Merkurjev, *R-equivalence in spinor groups*, J. Amer. Math. Soc. 14 (2001), 509–534.

[Choud] U. Choudhury, *Connectivity of motivic H-spaces*, Algebraic & Geometric Topology 14 (2014) 37–55.

[CTO] J.-L. Colliot-Thélène, M. Ojanguren, *Espaces Principaux Homogènes Localement Triviaux*, Publ. Math. IHÉS 75, no. 2 (1992), 97–122.

[SGA3] M. Demazure, A. Grothendieck, *Schémas en groupes*, Lecture Notes in Mathematics, vol. 151–153, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[G] Ph. Gille, *Le problème de Kneser-Tits*, Sém. Bourbaki 983 (2007), 983-01–983-39.

[J] J.F. Jardine, *On the homotopy groups of algebraic groups*, J. Algebra 81 (1983), 180–201.

[JL] J.F. Jardine, Z. Luo, *Higher principal bundles*, Math. Proc. Camb. Phil. Soc. 140 (2006), 221–243.

[KLe] B. Kahn, M. Levine, *Motives of Azumaya algebras*, Journal of the Inst. of Math. Jussieu 9 (2010), 481–599.

[MVW] C. Mazza, V. Voevodsky, Ch. Weibel, *Lecture notes on motivic cohomology*, volume 2 of Clay Mathematics Monographs. American Mathematical Society, Providence, RI, 2006.

[M] A. S. Merkurjev, *Norm principle for algebraic groups*, St. Petersburg Math. J. 7 (1996), 243–264.

[Mo1] F. Morel, *On the Friedlander-Milnor conjecture for groups of small rank*, Current developments in mathematics, 2010, 45–93. Int. Press, Somerville, MA, 2011.

[Mo2] F. Morel, *A^1-algebraic topology over a field*, Lecture Notes in Mathematics, 2052. Springer, Heidelberg, 2012. x+259 pp.

[MoV] F. Morel, V. Voevodsky, *A^1-homotopy theory of schemes*, Publ. Math. I.H.É.S. 90 (1999), 45–143.

[Pa] I. Panin, On Grothendieck—Serre’s conjecture concerning principal $G$-bundles over reductive group schemes: II, preprint 2013, arXiv:0905.1423v3.

[PaYa] I. Panin, S. Yagunov, Rigidity for orientable functors, J. Pure Appl. Algebra 172 (2002), 49–77.
V. Petrov, A. Stavrova, *Elementary subgroups of isotropic reductive groups*, St. Petersburg Math. J. 20 (2009), 625–644.

J. Ross, *Cohomology of presheaves with oriented weak transfers*, preprint 2014, arXiv:1405.0176.

A. Stavrova, *Homotopy invariance of non-stable $K_1$-functors*, J. K-Theory 13 (2014), 199–248.

M. Wendt, *$\mathbb{A}^1$-homotopy of Chevalley groups*, J. K-Theory 5 (2010), 245–287.

M. Wendt, *Rationally trivial torsors in $\mathbb{A}^1$-homotopy theory*, J. K-Theory 7 (2011), 541–572.

K. Völkel, M. Wendt, *On $\mathbb{A}^1$-fundamental groups of isotropic reductive groups*, 2012 arXiv:1207.2364.

T. Vorst, The general linear group of polynomial rings over regular rings. Comm. Algebra 9(5), 499–509 (1981)