3-FORMS AND ALMOST COMPLEX STRUCTURES ON 6-DIMENSIONAL MANIFOLDS

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Abstract. This article deals with 3-forms on 6-dimensional manifolds, the first dimension where the classification of 3-forms is not trivial. There are three classes of multisymplectic 3-forms there. We study the class which is closely related to almost complex structures.

Let $V$ be a real vector space. Recall that a $k$-form $\omega$ ($k \geq 2$) is called multisymplectic if the homomorphism

$$\iota : V \to \Lambda^{k-1}V^*, \ v \mapsto \iota_v\omega = \omega(v, \ldots)$$

is injective. There is a natural action of the general linear group $G(V)$ on $\Lambda^kV^*$, and also on $\Lambda^k_{ms}V^*$, the subset of the multisymplectic forms. Two multisymplectic forms are called equivalent if they belong to the same orbit of the action. For any form $\omega \in \Lambda^kV^*$ define a subset

$$\Delta(\omega) = \{v \in V; (\iota_v\omega) \wedge (\iota_v\omega) = 0\}.$$

If $\dim V = 6$ and $k = 3$ the subset $\Lambda^3_{ms}V^*$ consists of three orbits. Let $e_1, \ldots, e_6$ be a basis of $V$ and $\alpha_1, \ldots, \alpha_6$ the corresponding dual basis. Representatives of the three orbits can be expressed in the form

1. $\omega_1 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_4 \wedge \alpha_5 \wedge \alpha_6$,
2. $\omega_2 = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6$,
3. $\omega_3 = \alpha_1 \wedge \alpha_4 \wedge \alpha_5 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_3 \wedge \alpha_5 \wedge \alpha_6$.

Multisymplectic forms are called of type 1, resp. of type 2, resp. of type 3 according to which orbit they belong to. There is the following characterisation of the orbits:

1. $\omega$ is of type 1 if and only if $\Delta(\omega) = V^a \cup V^b$, where $V^a$ and $V^b$ are 3-dimensional subspaces satisfying $V^a \cap V^b = \{0\}$.
2. $\omega$ is of type 2 if and only if $\Delta(\omega) = \{0\}$.
3. $\omega$ is of type 3 if and only if $\Delta(\omega)$ is a 3-dimensional subspace.

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The forms $\omega_1$ and $\omega_2$ have equivalent complexifications. From this point of view the forms of type 3 are exceptional. You can find more about these forms in [V].

A multisymplectic $k$-form on a manifold $M$ is a section of $\Lambda^k T^* M$ such that its restriction to the tangent space $T_x M$ is multisymplectic for any $x \in M$, and is of type $i$ in $x \in M$, $i = 1, 2, 3$, if the restriction to $T_x M$ is of type $i$. A multisymplectic form on $M$ can change its type as seen on

$$
\sigma = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge dx_4 \wedge dx_5 + dx_2 \wedge dx_4 \wedge dx_6 + \sin(x_3 + x_4)dx_3 \wedge dx_5 + \sin(x_3 + x_4)dx_4 \wedge dx_5 \wedge dx_6,
$$

a 3-form on $\mathbb{R}^6$. $\sigma$ is of type 3 on the submanifold given by the equation $x_3 + x_4 = k\pi$, $k \in \mathbb{N}$. If $x_3 + x_4 \in (k\pi, (k + 1)\pi)$, $k$ even, then $\sigma$ is of type 1 and if $x_3 + x_4 \in (k\pi, (k + 1)\pi)$, $k$ odd, then $\sigma$ is of type 2. Let us point out that $\sigma$ is closed and invariant under the action of the group $(2\pi\mathbb{Z})^6$ and we can factor $\sigma$ to get a form changing the type on $\mathbb{R}^6/(2\pi\mathbb{Z})^6$, which is the 6-dimensional torus, that is $\sigma$ is closed on a compact manifold.

The goal of this paper is to study the forms of type 2. We denote $\omega = \omega_2$.

### 3-forms on vector spaces

Let $J$ be an automorphism of a 6-dimensional real vector space $V$ satisfying $J^2 = -I$. Further let $V^C = V \oplus iV$ be the complexification of $V$. There is the standard decomposition $V^C = V^{1,0} \oplus V^{0,1}$. Consider a non-zero form $\gamma$ of type $(3, 0)$ on $V^C$ and set

$$
\gamma_0 = \Re \gamma, \quad \gamma_1 = \Im \gamma.
$$

For any $v_1 \in V$ there is $v_2 + iJv_1 \in V^{0,1}$, and consequently $\gamma(i(v_1 + iJv_1), v_2, v_3) = 0$ for any $v_2, v_3 \in V$. This implies $\gamma_0(i(v_1 + iJv_1), v_2, v_3) = 0$ and $\gamma_1(i(v_1 + iJv_1), v_2, v_3) = 0$. Thus

$$
0 = \gamma_0(i(v_1 + iJv_1), v_2, v_3) = \gamma_0(i(v_1, v_2, v_3) - \gamma_0(Jv_1, v_2, v_3).
$$

Similarly we can proceed with $\gamma_1$ and we get

$$
\gamma_0(i(v_1, v_2, v_3) = \gamma_0(Jv_1, v_2, v_3), \quad \gamma_1(i(v_1, v_2, v_3) = \gamma_1(Jv_1, v_2, v_3)
$$

for any $v_1, v_2, v_3 \in V$. Moreover there is

$$
\gamma_0(w_1, w_2, w_3) = \Re(-\gamma(i^2 w_1, w_2, w_3)) = \Re(-i\gamma(iw_1, w_2, w_3))
$$

$$
= \Im(\gamma(iw_1, w_2, w_3)) = \gamma_1(iw_1, w_2, w_3),
$$

for any $w_1, w_2, w_3 \in V^C$ and that is $\gamma_1(w_1, w_2, w_3) = -\gamma_0(iw_1, w_2, w_3)$. Finally,

$$
\gamma_0(Jv_1, v_2, v_3) = \gamma_0(iw_1, v_2, v_3) = \Re(\gamma(iw_1, v_2, v_3)) = \Re(i\gamma(w_1, v_2, v_3))
$$

$$
= \Re(\gamma(w_1, iv_2, v_3)) = \Re(\gamma(v_1, Jv_2, v_3)) = \gamma_0(v_1, Jv_2, v_3).
$$

Along these lines we obtain

$$
\gamma_0(Jv_1, v_2, v_3) = \gamma_0(v_1, Jv_2, v_3) = \gamma_0(v_1, v_2, Jv_3),
$$

$$
\gamma_1(Jv_1, v_2, v_3) = \gamma_1(v_1, Jv_2, v_3) = \gamma_1(v_1, v_2, Jv_3),
$$

that is both forms $\gamma_0$ and $\gamma_1$ are pure with respect to the complex structure $J$. 

1. Lemma. The real 3-forms $\gamma_0|V$ and $\gamma_1|V$ (on $V$) are multisymplectic.

Proof. Let us assume that $v_1 \in V$ is a vector such that for any vectors $v_2, v_3 \in V$ ($\gamma_0|V)(v_1, v_2, v_3) = 0$ or equivalently $\gamma_0(v_1, v_2, v_3) = 0$. There are uniquely determined vectors $u_1, u_2, u_3 \in V^{1,0}$ such that

$$v_1 = u_1 + \bar{u}_1, \quad v_2 = u_2 + \bar{u}_2, \quad v_3 = u_3 + \bar{u}_3.$$ 

Then

$$0 = \gamma_0(v_1, v_2, v_3) = \text{Re}(\gamma(u_1 + \bar{u}_1, u_2 + \bar{u}_2, u_3 + \bar{u}_3)) = \text{Re}(\gamma(u_1, u_2, u_3)) = \gamma_0(u_1, u_2, u_3)$$

(for a fixed $u_1$, and arbitrary $u_2, u_3 \in V^{1,0}$). Because $i\omega \in V^{1,0}$, we find that

$$\gamma_0(i\omega, u_2, u_3) = \gamma_0(u_1, i\omega, u_3) = 0.$$

Moreover $\gamma_1(w, w', w'') = -\gamma_0(i\omega, w', w'')$ for any $w, w', w'' \in V^C$, and we get

$$\gamma_1(u_1, u_2, u_3) = -\gamma_0(i\omega, u_2, u_3) = 0$$

for arbitrary $u_2, u_3 \in V^{1,0}$. Thus

$$\gamma(u_1, u_2, u_3) = \gamma_0(u_1, u_2, u_3) + i\gamma_1(u_1, u_2, u_3) = 0$$

for arbitrary $u_2, u_3 \in V^{1,0}$.

Because $\gamma$ is a non-zero complex 3-form on the complex 3-dimensional vector space $V^{1,0}$, we find that $u_1 = 0$, and consequently $v_1 = 0$. This proves that the real 3-form $\gamma_0|V$ is multisymplectic. We find that the real 3-form $\gamma_1|V$ is also multisymplectic likewise.

2. Lemma. The forms $\gamma_0|V$ and $\gamma_1|V$ satisfy $\Delta(\gamma_0|V) = \{0\}$ and $\Delta(\gamma_1|V) = \{0\}$.

Proof. The complex 3-form $\gamma$ is decomposable, and therefore $\gamma \wedge \gamma = 0$. This implies that for any $w \in V^C$ $(i_\omega \gamma) \wedge (i_\omega \gamma) = 0$. Similarly for any $w \in V^C$ $(i_\omega \bar{\gamma}) \wedge (i_\omega \bar{\gamma}) = 0$.

Obviously $\gamma_0 = (1/2)(\gamma + \bar{\gamma})$. Let $v \in V$ be such that $(i_\omega \gamma_0) \wedge (i_\omega \gamma_0) = 0$. Then

$$0 = (i_\omega \gamma_0) \wedge (i_\omega \gamma_0) = \frac{1}{4}(i_\omega \gamma + i_\omega \bar{\gamma}) \wedge (i_\omega \gamma + i_\omega \bar{\gamma}) = \frac{1}{2}(i_\omega \gamma) \wedge (i_\omega \bar{\gamma}).$$

But $i_\omega \gamma$ is a form of type $(2, 0)$ and $i_\omega \bar{\gamma}$ a form of type $(0, 2)$. Consequently the last wedge product vanishes if and only if either $i_\omega \gamma = 0$ or $i_\omega \bar{\gamma} = 0$. By virtue of the preceding lemma this implies that $v = 0$.

Lemma 2 shows that the both forms $\gamma_0|V$ and $\gamma_1|V$ are of type 2. As a final result of the above considerations we get the following result.

3. Corollary. Let $\gamma$ be a 3-form on $V^C$ of the type $(3, 0)$. Then the real 3-forms $(\text{Re} \gamma)|V$ and $(\text{Im} \gamma)|V$ on $V$ are multisymplectic and of type 2.

Let $\omega$ be a 3-form on $V$ such that $\Delta(\omega) = \{0\}$. This means that for any $v \in V$, $v \neq 0$ there is $(i_v \omega) \wedge (i_v \omega) \neq 0$. This implies that rank $i_v \omega \geq 4$. On the other hand obviously rank $i_v \omega \leq 4$. Consequently, for any $v \neq 0$ rank $i_v \omega = 4$. Thus
the kernel \( K(t_v \omega) \) of the 2-form \( t_v \omega \) has dimension 2. Moreover \( v \in K(t_v \omega) \). Now we fix a non-zero 6-form on \( \theta \) on \( V \). For any \( v \in V \) there exists a unique vector \( Q(v) \in V \) such that
\[
(t_v \omega) \land \omega = t_Q(v) \theta.
\]
The mapping \( Q : V \to V \) is obviously a homomorphism. If \( v \neq 0 \) then \((t_v \omega) \land \omega \neq 0\), and \( Q \) is an automorphism. It is also obvious that if \( v \neq 0 \), then the vectors \( v \) and \( Q(v) \) are linearly independent (apply \( t_v \) to the last equality). We evaluate \( t_Q(v) \) on the last equality and we get
\[
(t_Q(v) t_v \omega) \land \omega + (t_v \omega) \land (t_Q(v) \omega) = 0
\]
\[
-(t_v t_Q(v) \omega) \land \omega + (t_v \omega) \land (t_Q(v) \omega) = 0
\]
\[
-t_v([t_Q(v) \omega] \land \omega] + 2(t_v \omega) \land (t_Q(v) \omega) = 0
\]
Now, apply \( t_v \) to the last equality:
\[
(t_v \omega) \land (t_v t_Q(v) \omega) = 0.
\]
If the 1-form \( t_v t_Q(v) \omega \) were not the zero one then it would exist a 1-form \( \sigma \) such that \( t_v \omega = \sigma \land t_v t_Q(v) \omega \), and we would get
\[
(t_v \omega) \land (t_v \omega) = \sigma \land t_v t_Q(v) \omega \land \sigma \land t_v t_Q(v) \omega = 0,
\]
which is a contradiction. Thus we have proved the following lemma.

4. Lemma. For any \( v \in V \) there is \( t_Q(v) t_v \omega = 0, \) i. e. \( Q(v) \in K(t_v \omega) \).

This lemma shows that if \( v \neq 0 \), then \( K(t_v \omega) = [v, Q(v)] \). Applying \( t_Q(v) \) to the equality \((t_v \omega) \land \omega = t_Q(v) \theta \) and using the last lemma we obtain easily the following result.

5. Lemma. For any \( v \in V \) there is \((t_v \omega) \land (t_Q(v) \omega) = 0\).

Lemma 4 shows that \( v \in K(t_Q(v) \omega) \). Because \( v \) and \( Q(v) \) are linearly independent, we can see that
\[
K(t_Q(v) \omega) = [v, Q(v)] = K(t_v \omega).
\]
If \( v \neq 0 \), then \( Q^2(v) \in K(t_Q(v) \omega) \), and consequently there are \( a(v), b(v) \in \mathbb{R} \) such that
\[
Q^2(v) = a(v)v + b(v)Q(v).
\]
For any \( v \in V \)
\[
(t_Q(v) \omega) \land \omega = t_{Q^2(v)} \theta.
\]
Let us assume that \( v \neq 0 \). Then
\[
(t_Q(v) \omega) \land \omega = a(v)t_v \theta + b(v)t_{Q(v)} \theta,
\]
and applying \( t_v \) we obtain \( b(v)t_v t_Q(v) \theta = 0 \), which shows that \( b(v) = 0 \) for any \( v \neq 0 \). Consequently, \( Q^2(v) = a(v)v \) for any \( v \neq 0 \).
6. Lemma. Let \( A : V \to V \) be an automorphism, and \( a : V \backslash \{0\} \to \mathbb{R} \) a function such that
\[
A(v) = a(v)v \quad \text{for any} \ v \neq 0.
\]
Then the function \( a \) is constant.

Proof. The condition on \( A \) means that every vector \( v \) of \( V \) is an eigenvector of \( A \) with the eigenvalue \( a(v) \). But the eigenvalues of two different vectors have to be the same otherwise their sum would not be an eigenvector.

Applying Lemma 6 on \( Q^2 \) we get \( Q^2 = aI \). If \( a > 0 \), then \( V = V^+ \oplus V^- \), and
\[
Qv = \sqrt{a}v \quad \text{for} \ v \in V^+, \quad Qv = -\sqrt{a}v \quad \text{for} \ v \in V^-.
\]
At least one of the subspaces \( V^+ \) and \( V^- \) is non-trivial. Let us assume for example that \( V^+ \neq \{0\} \). Then there is \( v \in V^+, \ v \neq 0, \) and \( Qv = \sqrt{a}v \), which is a contradiction because the vectors \( v \) and \( Qv \) are linearly independent. This proves that \( a < 0 \). We can now see that the automorphisms
\[
J_+ = \frac{1}{\sqrt{-a}}Q \quad \text{and} \quad J_- = -\frac{1}{\sqrt{-a}}Q
\]
satisfy \( J^2_+ = -I \) and \( J^2_- = -I \), i.e. they define complex structures on \( V \) and \( J_- = -J_+ \). Setting
\[
\theta_+ = \sqrt{-a}\theta, \quad \theta_- = -\sqrt{-a}\theta
\]
we get
\[
(\iota_v \omega) \wedge \omega = \iota_{J_+v} \theta_+, \quad (\iota_v \omega) \wedge \omega = \iota_{J_-v} \theta_-.
\]

In the sequel we shall denote \( J = J_+ \). The same results which are valid for \( J_+ \) hold also for \( J_- \).

7. Lemma. There exists a unique (up to the the sign) complex structure \( J \) on \( V \) such that the form \( \omega \) satisfies the relation
\[
\omega(Jv_1, v_2, v_3) = \omega(v_1, Jv_2, v_3) = \omega(v_1, v_2, Jv_3) \quad \text{for any} \ v_1, v_2, v_3 \in V.
\]
We recall that such a form \( \omega \) is usually called pure with respect to \( J \).

Proof. We shall prove first that the complex structure \( J \) defined above satisfies the relation. By virtue of Lemma 4 for any \( v, v' \in V \) \( \omega(v, Jv, v') = 0 \). Therefore we get
\[
0 = \omega(v_1 + v_2, J(v_1 + v_2), v_3) = \omega(v_1, Jv_2, v_3) + \omega(v_2, Jv_1, v_3) = -\omega(Jv_1, v_2, v_3) + \omega(v_1, Jv_2, v_3),
\]
which gives
\[
\omega(Jv_1, v_2, v_3) = \omega(v_1, Jv_2, v_3).
\]
Obviously, the opposite complex structure \( -J \) satisfies the same relation. We prove that there is no other complex structure with the same property. Let \( \tilde{J} \) be a complex structure on \( V \) satisfying the above relation. We set \( A = \tilde{J}J^{-1} \). Then we get
\[
\omega(v_1, Av_2, Av_3) = \omega(v_1, \tilde{J}Jv_2, \tilde{J}Jv_3) = \omega(v_1, Jv_2, Jv_3) = -\omega(v_1, Jv_2, Jv_3) = -\omega(v_1, v_2, J^2v_3) = \omega(v_1, v_2, J^2v_3).
\]
Any automorphism $A$ satisfying this identity is $±I$. Really, the identity means that $A$ is an automorphism of the 2-form $τ_ω$. Consequently, $A$ preserves the kernel $K(τ_ω) = [v, Jv]$. On the other hand it is obvious that any subspace of the form $[v, Jv]$ is the kernel of $τ_ω$. Considering $V$ as a complex vector space with the complex structure $J$, we can say that every 1-dimensional complex subspace is the kernel of the 2-form $τ_ω$ for some $v ∈ V, v ≠ 0$, and consequently is invariant under the automorphism $A$. Similarly as in Lemma 6 we conclude, that $A = λI$, $λ ∈ C$.

If we write $λ = λ_0 + iλ_1$, then $A = λ_0I + λ_1J$ and

\[
ω(v_1, v_2, v_3) = ω(v_1, Av_2, Av_3) = ω(v_1, ω_0v_2 + λ_1Jv_2, ω_0v_3 + λ_1Jv_3)
= λ_0^2ω(v_1, v_2, v_3) + λ_0λ_1ω(v_1, v_2, Jv_3) + λ_0λ_1ω(v_1, Jv_2, v_3) + λ_1^2ω(v_1, Jv_2, Jv_3)
\]

\[
(λ_0^2 - λ_1^2 - 1)ω(v_1, v_2, v_3) + 2λ_0λ_1ω(v_1, v_2, Jv_3) = 0
\]

We shall use this last equation together with another one obtained by writing $Jv_3$ instead of $v_3$. In this way we get the system

\[
(λ_0^2 - λ_1^2 - 1)ω(v_1, v_2, v_3) + 2λ_0λ_1ω(v_1, v_2, Jv_3) = 0
-2λ_0λ_1ω(v_1, v_2, v_3) + (λ_0^2 - λ_1^2 - 1)ω(v_1, v_2, Jv_3) = 0.
\]

Because it has a non-trivial solution there must be

\[
\begin{vmatrix}
λ_0^2 - λ_1^2 - 1 & 2λ_0λ_1 \\
-2λ_0λ_1 & λ_0^2 - λ_1^2 - 1
\end{vmatrix} = 0.
\]

It is easy to verify that the solution of the last equation is $λ_0 = ±1$ and $λ_1 = 0$. This finishes the proof.

We shall now consider the vector space $V$ together with a complex structure $J$, and a 3-form $ω$ on $V$ which is pure with respect to this complex structure. Firstly we define a real 3-form $γ_0$ on $V^C$. We set

\[
γ_0(v_1, v_2, v_3) = ω(v_1, v_2, v_3),
γ_0(iv_1, v_2, v_3) = ω(Jv_1, v_2, v_3),
γ_0(iv_1, iv_2, v_3) = ω(Jv_1, Jv_2, v_3),
γ_0(iv_1, iv_2, iv_3) = ω(Jv_1, Jv_2, Jv_3),
\]

for $v_1, v_2, v_3 ∈ V$. Then $γ_0$ extends uniquely to a real 3-form on $V^C$. We can find easily that

\[
γ_0(iw_1, w_2, w_3) = γ_0(w_1, iw_2, w_3) = γ_0(w_1, w_2, iw_3)
\]

for any $w_1, w_2, w_3 ∈ V^C$. Further, we set

\[
γ_1(w_1, w_2, w_3) = -γ_0(iw_1, w_2, w_3) \quad \text{for} \quad w_1, w_2, w_3 ∈ V^C.
\]

It is obvious that $γ_1$ is a real 3-form satisfying

\[
γ_1(iw_1, w_2, w_3) = γ_1(w_1, iw_2, w_3) = γ_1(w_1, w_2, iw_3)
\]

for any $w_1, w_2, w_3 ∈ V^C$. Now we define

\[
γ(w_1, w_2, w_3) = γ_0(w_1, w_2, w_3) + iγ_1(w_1, w_2, w_3) \quad \text{for} \quad w_1, w_2, w_3 ∈ V^C.
\]
It is obvious that \( \gamma \) is skew symmetric and 3-linear over \( \mathbb{R} \) and has complex values. Moreover
\[
\gamma(iw_1, w_2, w_3) = \gamma_0(iw_1, w_2, w_3) + i\gamma_1(iw_1, w_2, w_3)
\]
\[
= -\gamma_1(w_1, w_2, w_3) - i\gamma_0(i^2w_1, w_2, w_3) = -\gamma_1(w_1, w_2, w_3) + i\gamma_0(w_1, w_2, w_3) =
\]
\[
i[\gamma_0(w_1, w_2, w_3) + i\gamma_1(w_1, w_2, w_3)] = i\gamma(w_1, w_2, w_3),
\]
which proves that \( \gamma \) is a complex 3-form on \( V^C \). Now we prove that \( \gamma \) is a form of type \((3, 0)\). Obviously, it suffices to prove that for \( v_1 + iJv_1 \in V_0^{0,1} \) and \( v_2, v_3 \in V \) there is \( \gamma(v_1 + iJv_1, v_2, v_3) = 0 \). Really,
\[
\gamma(v_1 + iJv_1, v_2, v_3) = \gamma(v_1, v_2, v_3) + i\gamma(Jv_1, v_2, v_3)
\]
\[
= \gamma_0(v_1, v_2, v_3) + i\gamma_1(v_1, v_2, v_3) + i\gamma_0(Jv_1, v_2, v_3) - \gamma_1(Jv_1, v_2, v_3)
\]
\[
= \gamma_0(v_1, v_2, v_3) - i\gamma_0(iv_1, v_2, v_3) + i\gamma_0(Jv_1, v_2, v_3) + \gamma_0(iJv_1, v_2, v_3)).
\]
Now \( \gamma_0(iv_1, v_2, v_3) = \omega(J^2v_1, v_2, v_3) = -\omega(v_1, v_2, v_3) = -\gamma_0(v_1, v_2, v_3) \) and the real part of the last expression is zero, further \( \gamma_0(Jv_1, v_2, v_3) = \omega(Jv_1, v_2, v_3) = \gamma_0(iv_1, v_2, v_3) \) and the complex part of the expression is zero as well. Now we get easily the following proposition.

**8. Proposition.** Let \( \omega \) be a real 3-form on \( V \) satisfying \( \Delta(\omega) = \{0\} \), and let \( J \) be a complex structure on \( V \) (one of the two) such that
\[
\omega(Jv_1, v_2, v_3) = \omega(v_1, Jv_2, v_3) = \omega(v_1, v_2, Jv_3).
\]
Then there exists on \( V^C \) a unique complex 3-form \( \gamma \) of type \((3, 0)\) such that
\[
\omega = (\text{Re} \gamma)|V.
\]

**Remark.** The complex structure \( J \) on \( V \) can be introduced also by means of the Hitchin’s invariant \( \lambda \), as in \([H]\). Forms of type 2 form an open subset \( U \) in \( \Lambda^3V^* \). Hitchin has shown that this manifold also carries an almost complex structure, which is integrable. Hitchin uses the following way to introduce an almost complex structure on \( U \). \( U \subset \Lambda^3V^* \) can be seen as a symplectic manifold (let \( \theta \) be a fixed element in \( \Lambda^6V^* \); one defines the symplectic form \( \Theta \) on \( \Lambda^3V^* \) by the equation \( \omega_1 \wedge \omega_2 = \Theta(\omega_1, \omega_2) \theta \). Then the derivative of the Hamiltonian vector field corresponding to the function \( \sqrt{\lambda(\omega)} \) on \( U \) gives an integrable almost complex structure on \( U \). That was for the Hitchin’s construction.

There is another way of introducing the (Hitchin’s) almost complex structure on \( U \). Given a 3-form \( \omega \in U \) we choose the complex structure \( J_\omega \) on \( V \) (one of the two), whose existence is guaranteed by the lemma 7. Then we define endomorphisms \( A_{J_\omega} \) and \( D_{J_\omega} \) of \( \Lambda^kV^* \) by
\[
(A_{J_\omega}\Omega)(v_1, \ldots, v_k) = \Omega(J_\omega v_1, \ldots, J_\omega v_k),
\]
\[
(D_{J_\omega}\Omega)(v_1, \ldots, v_k) = \sum_{i=1}^k \Omega(v_1, \ldots, v_{i-1}, J_\omega v_i, v_{i+1}, \ldots, v_k).
\]
Then \( A_{J_\omega} \) is an automorphism of \( \Lambda^V^* \) and \( D_{J_\omega} \) is a derivation of \( \Lambda^V^* \). If \( k = 3 \) then the automorphism \( -\frac{1}{2}(A_{J_\omega} + D_{J_\omega}) \) of \( \Lambda^3V^* (= T_U) \) gives a complex structure on \( U \) and coincides with the Hitchin’s one.
3-forms on manifolds

We use facts from the previous section to obtain some global results on 3-forms on 6-dimensional manifolds. We shall denote by $X, Y, Z$ the real vector fields on a (real) manifold $M$ and by $V, W$ the complex vector fields on $M$. $\mathcal{X}(M)$ stands for the set of all (real) vector fields on $M$, $\mathcal{X}^c(M)$ means all the complex vector fields on $M$.

A 3-form $\omega$ on $M$ is called the form of type 2 if for every $x \in M$ there is $\Delta(\omega_x) = \{0\}$. Let $\omega$ be a form of type 2 on $M$ and let $U \subset M$ be an open orientable submanifold. Then there exists an everywhere nonzero differentiable 6-form on $U$.

In each $T_x M$, $x \in U$ construct $J_-$ and $J_+$ as in Lemma 7. The construction is evidently smooth on $U$. Thus

9. Lemma. Let $\omega$ be a form of type 2 on $M$ and let $U \subset M$ be an orientable open submanifold. Then there exist two differentiable almost complex structures $J_+$ and $J_-$ on $U$ such that

(i) $J_+ + J_- = 0$,
(ii) $\omega(J_+ X_1, X_2, X_3) = \omega(X_1, J_+ X_2, X_3) = \omega(X_1, X_2, J_+ X_3)$,
(iii) $\omega(J_- X_1, X_2, X_3) = \omega(X_1, J_- X_2, X_3) = \omega(X_1, X_2, J_- X_3)$,

for any vector fields $X_1, X_2, X_3$.

At each point $x \in M$ consider a 1-dimensional subspace of the space $T^1 T^1 x(M)$ of tensors of type $(1, 1)$ at $x$ generated by the tensors $J_+ x$ and $J_- x$. The above considerations show that it is a 1-dimensional subbundle $\mathcal{J} \subset T^1 T^1(M)$.

10. Lemma. The 1-dimensional vector bundles $\mathcal{J}$ and $\Lambda^6 T^* (M)$ are isomorphic.

Proof. Let us choose a riemannian metric $g_0$ on $TM$. If $x \in M$ and $v, v' \in T_x M$ we define a riemannian metric $g$ by the formula

$$g(v, v') = g_0(v, v') + g_0(J_+ v, J_+ v') = g_0(v, v') + g_0(J_- v, J_- v').$$

It is obvious that for any $v, v' \in T_x M$ we have

$$g(J_+ v, J_+ v') = g(v, v'), \quad g(J_- v, J_- v') = g(v, v').$$

We now define

$$\sigma_+(v, v') = g(J_+ v, v'), \quad \sigma_-(v, v') = g(J_- v, v').$$

It is easy to verify that $\sigma_+$ and $\sigma_-$ are nonzero 2-forms on $T_x M$ satisfying $\sigma_+ + \sigma_- = 0$.

We define an isomorphism $h: \mathcal{J} \to \Lambda^6 T^* M$. Let $x \in M$ and let $A \in \mathcal{J}_x$. We can write

$$A = a J_+, \quad A = -a J_-.$$

We set

$$hA = a \sigma_+ \wedge \sigma_+ \wedge \sigma_+ = -a \sigma_- \wedge \sigma_- \wedge \sigma_-.$$
11. Corollary. There exist two almost complex structures $J_+$ and $J_-$ on $M$ such that

(i) $J_+ + J_- = 0$.
(ii) $\omega(J_+ X_1, X_2, X_3) = \omega(X_1, J_+ X_2, X_3) = \omega(X_1, X_2, J_+ X_3)$,
(iii) $\omega(J_- X_1, X_2, X_3) = \omega(X_1, J_- X_2, X_3) = \omega(X_1, X_2, J_- X_3)$,

for any vector fields $X_1, X_2, X_3$ if and only if the manifold $M$ is orientable.

Hence the assertions in the rest of the article can be simplified correspondingly if $M$ is an orientable manifold.

12. Lemma. Let $J$ be an almost complex structure on $M$ such that for any vector fields $X_1, X_2, X_3 \in \mathcal{X}(M)$ there is

$$\omega(JX_1, X_2, X_3) = \omega(X_1, JX_2, X_3) = \omega(X_1, X_2, JX_3).$$

If $\nabla$ is a linear connection on $M$ such that $\nabla \omega = 0$, then also $\nabla J = 0$.

Proof. Let $Y \in \mathcal{X}(M)$, and let us consider the covariant derivative $\nabla_Y$. We get

$$0 = (\nabla_Y \omega)(JX_1, X_2, X_3) = Y(\omega(JX_1, X_2, X_3) - \omega(\nabla_Y J X_1, X_2, X_3))$$
$$- \omega(J \nabla_Y X_1, X_2, X_3) - \omega(JX_1, \nabla_Y X_2, X_3) - \omega(JX_1, X_2, \nabla_Y X_3),$$

$$0 = (\nabla_Y \omega)(X_1, JX_2, X_3) = Y(\omega(JX_1, X_2, X_3) - \omega(\nabla_Y X_1, JX_2, X_3))$$
$$- \omega(X_1, \nabla_Y J X_2, X_3) - \omega(X_1, J \nabla_Y X_2, X_3) - \omega(X_1, JX_2, \nabla_Y X_3).$$

Because the above expressions are equal we find easily that

$$\omega((\nabla_Y J)X_1, X_2, X_3) = \omega(X_1, (\nabla_Y J)X_2, X_3).$$

We denote $A = \nabla_Y J$. Extending in the obvious way the above equality, we get

$$\omega(A X_1, X_2, X_3) = \omega(X_1, A X_2, X_3) = \omega(X_1, X_2, A X_3).$$

Moreover $J^2 = -I$, and applying $\nabla_Y$ to this equality, we get

$$AJ + JA = 0.$$

We know that $K(\iota_X \omega) = [X, JX]$. Furthermore

$$\omega(X, AX, X') = \omega(X, X, AX') = 0, \quad \omega(X, AJX, X') = \omega(X, JX, AX') = 0,$$

which shows that $A$ preserves the distribution $[X, JX]$. By the very same arguments as in Lemma 7 we can see that $A = \lambda_0 I + \lambda_1 J$. Consequently

$$(\lambda_0 I + \lambda_1 J)J + J(\lambda_0 I + \lambda_1 J) = 0$$
$$-2\lambda_1 I + 2\lambda_0 J = 0,$$

which implies $\lambda_0 = \lambda_1 = 0$. Thus $\nabla_Y J = A = 0$.

The statement of the previous lemma can be in a way reversed, and we get
13. Proposition. Let $\omega$ be a real 3-form on a 6-dimensional differentiable manifold $M$ satisfying $\Delta(\omega_x) = \{0\}$ for any $x \in M$. Let $J$ be an almost complex structure on $M$ such that for any vector fields $X_1, X_2, X_3 \in \mathcal{X}(M)$ there is

$$\omega(JX_1, X_2, X_3) = \omega(X_1, JX_2, X_3) = \omega(X_1, X_2, JX_3).$$

Then there exists a symmetric connection $\tilde{\nabla}$ on $M$ such that $\tilde{\nabla}\omega = 0$ if and only if the following conditions are satisfied

(i) $d\omega = 0$,

(ii) the almost complex structure $J$ is integrable.

Proof. First, we prove that the integrability of the structure $J$ and the fact that $\omega$ is closed implies the existence of a symmetric connection with respect to which $\omega$ is parallel.

For any connection $\nabla$ on $M$ we shall denote by the same symbol its complexification. Namely, we set

$$\nabla_{X_0+iX_1}(Y_0+iY_1) = (\nabla_{X_0}Y_0 - \nabla_{X_1}Y_1) + i(\nabla_{X_0}Y_1 + \nabla_{X_1}Y_0).$$

Let us assume that there exists a symmetric connection $\tilde{\nabla}$ such that $\tilde{\nabla}J = 0$. We shall consider a 3-form $\gamma$ of type $(3, 0)$ such that $(\text{Re}\, \gamma)|TM = \omega$. Our next aim is to try to find a symmetric connection

$$\nabla_V W = \tilde{\nabla}_V W + Q(V, W)$$

satisfying $\nabla_V \gamma = 0$. Obviously, the connection $\nabla$ is symmetric if and only if $Q(V, W) = Q(W, V)$.

Moreover, $\nabla_V \gamma = 0$ hints that $\nabla J = 0$.

$$0 = (\nabla_V J)W = \nabla_V(JW) - J\nabla_V W = \tilde{\nabla}_V(JW) + Q(V, JW) - J\tilde{\nabla}_V W - JQ(V, W),$$

which shows that we should require

$$Q(JV, W) = Q(V, JW) = JQ(V, W).$$

Because $\tilde{\nabla}J = 0$, we can immediately see that for any $V \in \mathcal{X}(M)$ the covariant derivative $\tilde{\nabla}_V \gamma$ is again a form of type $(3, 0)$. Consequently there exists a uniquely determined complex 1-form $\rho$ such that

$$\tilde{\nabla}_V \gamma = \rho(V)\gamma.$$

Then

$$\nabla_V \gamma(W_1, W_2, W_3) = V(\gamma(W_1, W_2, W_3)) - \gamma(\nabla_V W_1, W_2, W_3) - \gamma(W_1, \nabla_V W_2, W_3) - \gamma(W_1, W_2, \nabla_V W_3)$$

$$= V(\gamma(W_1, W_2, W_3)) - \gamma(\nabla_V W_1, W_2, W_3) - \gamma(W_1, \nabla_V W_2, W_3) - \gamma(W_1, W_2, \nabla_V W_3)$$

$$- \gamma(Q(V, W_1), W_2, W_3) - \gamma(W_1, Q(V, W_2), W_3) - \gamma(W_1, W_2, Q(V, W_3))$$

$$= \rho(V)\gamma(W_1, W_2, W_3)$$

$$- \gamma(Q(V, W_1), W_2, W_3) - \gamma(W_1, Q(V, W_2), W_3) - \gamma(W_1, W_2, Q(V, W_3)).$$

In other words $\nabla_V \gamma = 0$ if and only if

$$\rho(V)\gamma(W_1, W_2, W_3) = \gamma(Q(V, W_1), W_2, W_3) + \gamma(W_1, Q(V, W_2), W_3) + \gamma(W_1, W_2, Q(V, W_3)).$$
Sublemma. If $d\gamma = 0$, then $\rho$ is a form of type $(1,0)$.

Proof. Let $V_1 \in T^{0,1}(M)$. Because $\nabla$ is symmetric $d\gamma = \mathcal{A}^\circ(\nabla\gamma)$, where $\mathcal{A}$ denotes the alternation. We obtain

$$0 = -4!(d\gamma)(V_1, V_2, V_3, V_4) = \sum_\pi \text{sign}(\pi)(\nabla_{V_{\pi_1}}\gamma)(V_{\pi_2}, V_{\pi_3}, V_{\pi_4})$$

$$+ \sum_\tau \text{sign}(\tau)(\nabla_{V_{\tau_1}}\gamma)(V_{\tau_2}, V_{\tau_3}, V_{\tau_4}) = 3!(\nabla_{V_1}\gamma)(V_2, V_3, V_4) = 3!\rho(V_1)\gamma(V_2, V_3, V_4).$$

The first sum is taken over all permutations $\pi$ satisfying $\pi 1 > 1$, and the second one is taken over all permutations of the set $\{2, 3, 4\}$. The first sum obviously vanishes, and $\rho(V_1) = 0$. This finishes the proof.

We set now

$$Q(V, W) = \frac{1}{8}[\rho(V)W - \rho(JV)JW + \rho(W)V - \rho(JW)JV].$$

It is easy to see that $Q(JV, W) = Q(V, JW) = JQ(V, W)$. For $V, W_1, W_2, W_3 \in T^{1,0}(M)$ we can compute

$$8\gamma(Q(V, W_1), W_2, W_3)$$

$$= \gamma(\rho(V)W_1 - \rho(JV)JW_1 + \rho(W_1)V - \rho(JW_1)JW_1, W_2, W_3)$$

$$= \gamma(2\rho(V)W_1 + 2\rho(W_1)V, W_2, W_3) = 2\rho(V)\gamma(W_1, W_2, W_3) + 2\rho(W_1)\gamma(V, W_2, W_3),$$

where we used for $V \in T^{1,0}(M)$ that $\rho(JV) = i\rho(V)$ and $\gamma(JV, V', V'') = i\gamma(V', V'', V')$, since $\gamma$ is of type $(3,0)$ and $\rho$ of type $(1,0)$.

Similarly we can compute $\gamma(W_1, Q(V, W_2), W_3)$ and $\gamma(W_1, W_2, Q(V, W_3))$. Without a loss of generality we can assume that the vector fields $W_1, W_2, W_3$ are linearly independent (over $\mathbb{C}$). Then we can find uniquely determined complex functions $f_1, f_2, f_3$ such that

$$V = f_1W_1 + f_2W_2 + f_3W_3.$$

Then we get

$$\rho(W_1)\gamma(V, W_2, W_3) + \rho(W_2)\gamma(W_1, V, W_3) + \rho(W_3)\gamma(W_1, W_2, V)$$

$$= f_1\rho(W_1)\gamma(W_2, W_3) + f_2\rho(W_2)\gamma(W_1, W_3) + f_3\rho(W_3)\gamma(W_1, W_2, W_3)$$

$$= \rho(f_1W_1 + f_2W_2 + f_3W_3)\gamma(W_1, W_2, W_3) = \rho(V)\gamma(W_1, W_2, W_3).$$

Finally we obtain

$$\gamma(Q(V, W_1), W_2, W_3) + \gamma(W_1, Q(V, W_2), W_3) + \gamma(W_1, W_2, Q(V, W_3))$$

$$= \rho(V)\gamma(W_1, W_2, W_3).$$

which proves $\nabla_V\gamma = 0$.

Let us continue in the main stream of the proof. We shall now use the complex connection $\nabla$. For $X, Y \in TM$ we shall denote $\nabla^0_X Y = \text{Re} \nabla_X Y$ and $\nabla^1_X Y =$
In $\nabla_X Y$. This means that we have $\nabla_X Y = \nabla_X^0 Y + i\nabla_X^1 Y$. For a real function $f$ on $M$ we have

$$\nabla_X (fY) = \nabla_X^0 (fY) + i\nabla_X^1 (fX),$$

$$\nabla_X (fY) = (Xf)Y + f\nabla_X Y = [(Xf)Y + f\nabla_X^0 Y] + i(f\nabla_X^1 Y),$$

which implies

$$\nabla_X^0 (fY) = (Xf)Y + f\nabla_X^0 Y, \quad \nabla_X^1 (fY) = f\nabla_X^1 Y.$$

This shows that $\nabla^0$ is a real connection while $\nabla^1$ is a real tensor field of type $(1, 2)$. We have also

$$0 = \nabla_X Y - \nabla_Y X - [X, Y] = \nabla_X^0 Y + i\nabla_X^1 Y - \nabla_Y^0 X - i\nabla_Y^1 X - [X, Y] =$$

$$= [\nabla_X^0 Y - \nabla_Y^0 X - [X, Y]] + i[\nabla_X^1 Y - \nabla_Y^1 X],$$

which shows that

$$\nabla_X^0 Y - \nabla_Y^0 X - [X, Y] = 0, \quad \nabla_X^1 Y - \nabla_Y^1 X = 0.$$

These equations show that the connection $\nabla^0$ is symmetric, and that the tensor $\nabla^1$ is also symmetric. Moreover, we have

$$\nabla_X (JY) = \nabla_X^0 (JY) + i\nabla_X^1 (JY),$$

$$\nabla_X (JY) = J\nabla_X Y = J\nabla_X^0 Y + iJ\nabla_X^1 Y,$$

which gives

$$\nabla_X^0 J = 0, \quad \nabla_X^1 (JY) = J\nabla_X^1 Y.$$

For the real vectors $X, Y_1, Y_2, Y_3 \in TM$ we can compute

$$0 = (\nabla_X \gamma)(Y_1, Y_2, Y_3) = X(\gamma(Y_1, Y_2, Y_3)) -$$

$$- \gamma(\nabla_X Y_1, Y_2, Y_3) - \gamma(Y_1, \nabla_X Y_2, Y_3) - \gamma(Y_1, Y_2, \nabla_X Y_3) =$$

$$= X(\gamma(Y_1, Y_2, Y_3)) -$$

$$- \gamma(\nabla_X^0 Y_1, Y_2, Y_3) - \gamma(\nabla_X^0 Y_2, Y_3) - \gamma(Y_1, \nabla_X^0 Y_2, Y_3) - \gamma(Y_1, Y_2, \nabla_X^0 Y_3) -$$

$$- i[\gamma(\nabla_X^1 Y_1, Y_2, Y_3)] + \gamma(Y_1, \nabla_X^1 Y_2, Y_3) - \gamma(Y_1, Y_2, \nabla_X^1 Y_3) =$$

$$= [X(\gamma_0(Y_1, Y_2, Y_3)) - \gamma_0(\nabla_X^0 Y_1, Y_2, Y_3) - \gamma_0(Y_1, \nabla_X^0 Y_2, Y_3) - \gamma_0(Y_1, Y_2, \nabla_X^0 Y_3) +$$

$$+ \gamma_1(\nabla_X^1 Y_1, Y_2, Y_3) + \gamma_1(Y_1, \nabla_X^1 Y_2, Y_3) + \gamma_1(Y_1, Y_2, \nabla_X^1 Y_3)] +$$

$$+ i[X(\gamma_1(Y_1, Y_2, Y_3)) - \gamma_1(\nabla_X^0 Y_1, Y_2, Y_3) - \gamma_1(Y_1, \nabla_X^0 Y_2, Y_3) - \gamma_1(Y_1, Y_2, \nabla_X^0 Y_3) -$$

$$- \gamma_0(\nabla_X^1 Y_1, Y_2, Y_3) - \gamma_0(Y_1, \nabla_X^1 Y_2, Y_3) - \gamma_0(Y_1, Y_2, \nabla_X^1 Y_3)].$$

This shows that the real part (as well as the complex one, which gives in fact the same identity) is zero. Using the relations between $\gamma_0$ and $\gamma_1$ we get

$$0 = X(\gamma_0(Y_1, Y_2, Y_3)) - \gamma_0(\nabla_X^0 Y_1, Y_2, Y_3) - \gamma_0(Y_1, \nabla_X^0 Y_2, Y_3) - \gamma_0(Y_1, Y_2, \nabla_X^0 Y_3) -$$
\[ -\gamma_0(J\nabla_X Y_1, Y_2, Y_3) - \gamma_0(Y_1, J\nabla_X^1 Y_2, Y_3) - \gamma_0(Y_1, Y_2, J\nabla_X^1 Y_3) = \]
\[ = X(\gamma_0(Y_1, Y_2, Y_3)) - \gamma_0(\nabla_X^0 Y_1 + J\nabla_X^1 Y_2, Y_3) \]
\[ - \gamma_0(Y_1, \nabla_X^0 Y_2 + J\nabla_X^1 Y_2, Y_3) - \gamma_0(Y_1, Y_2, \nabla_X^0 Y_3 + J\nabla_X^1 Y_3). \]

We define now
\[ \tilde{\nabla}_X Y = \nabla_X^0 Y + J\nabla_X^1 Y. \]
It is easy to verify that \( \tilde{\nabla} \) is a real connection. Moreover, the previous equation shows that
\[ \tilde{\nabla}\gamma_0 = 0. \]

Furthermore, it is very easy to see that the connection \( \tilde{\nabla} \) is symmetric.

The inverse implication can be proved easily.

Let us use the standard definition of integrability of a \( k \)-form \( \omega \) on \( M \), that is every \( x \in M \) has a neighbourhood \( N \) such that \( \omega \) has the constant expresion in \( dx^1, x^2 \) being suitable coordinate functions on \( N \).

14. Corollary. Let \( \omega \) be a real 3-form on a 6-dimensional differentiable manifold \( M \) satisfying \( \Delta(\omega_x) = \{0\} \) for any \( x \in M \). Let \( J \) be an almost complex structure on \( M \) such that for any vector fields \( X_1, X_2, X_3 \in \mathcal{X}(M) \) there is
\[ \omega(JX_1, X_2, X_3) = \omega(X_1, JX_2, X_3) = \omega(X_1, X_2, JX_3). \]
Then \( \omega \) is integrable if and only if there exists a symmetric connection \( \nabla \) preserving \( \omega \), that is \( \nabla \omega = 0 \).

**Proof.** Let \( \nabla \) be a symmetric connection such that \( \nabla \omega = 0 \). Then according to the previous proposition \( d\omega = 0 \) and \( J \) is integrable. Then we construct the complex form \( \gamma \) on \( T^c M \) of type \((3, 0)\) such that \( \text{Re} \gamma | T_x M = \omega \), for any \( x \in M \) (point by point, according to Proposition 8). Moreover if \( \omega \) is closed than so is \( \gamma \). That is \( \gamma = f \cdot dz^1 \wedge dz^2 \wedge dz^3 \), where \( z^1, z^2, \) and \( z^3 \) are (complex) coordinate functions on \( M \), \( dz^1, dz^2, dz^3 \) are a basis of \( \Lambda^{1,0} M \) and \( f \) a function on \( M \). Further
\[ 0 = d\gamma = \partial \gamma + \overline{\partial} \gamma = \partial f \cdot dz^1 \wedge dz^2 \wedge dz^3 + \overline{\partial} f \cdot dz^1 \wedge dz^2 \wedge dz^3. \]
Evidently \( \partial \gamma = 0 \), which means \( \overline{\partial} f = 0 \) and \( f \) is holomorphic. Now we exploit a standard trick. There exists a holomorphic function \( F(z^1, \bar{z}^2, \bar{z}^3) \) such that \( \frac{\partial F}{\partial \bar{z}^3} = f \). We introduce new complex coordinates \( \bar{z}^1 = F(z^1, z^2, z^3), \bar{z}^2 = z^2 \), and \( \bar{z}^3 = z^3 \). Then \( \gamma = fdz^1 \wedge dz^2 \wedge dz^3 = d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 \). Now write \( \bar{z}^1 = x^1 + ix^4, \bar{z}^2 = x^2 + ix^5, \) and \( \bar{z}^3 = x^3 + ix^6 \) for real coordinate functions \( x^1, x^2, x^3, x^4, x^5, \) and \( x^6 \) on \( M \). There is
\[ \text{Re} \gamma = \text{Re}(d(x^1 + ix^4) \wedge d(x^2 + ix^5) \wedge d(x^3 + ix^6)) \]
\[ = dx^1 \wedge dx^2 \wedge dx^3 - dx^1 \wedge dx^5 \wedge dx^6 + dx^2 \wedge dx^4 \wedge dx^6 - dx^3 \wedge dx^4 \wedge dx^5. \]
And \( \omega = (\text{Re} \gamma)|TM \) is an integrable on \( M \).

Conversely, if \( \omega \) is integrable, then for any \( x \in M \) there is a basis \( dx_1, \ldots, dx_6 \) of \( T^* N \) in some neighbourhood \( N \subset M \) of \( x \) such that \( \omega \) has constant expression in all \( T_x M, x \in N \). Then the flat connection \( \nabla \) given by the coordinate system \( x_1, \ldots, x_6 \) is symmetric and \( \nabla \omega = 0 \) on \( N \). We use the partition of the unity and extend \( \nabla \) over the whole \( M \).

We can reformulate the Proposition 13 as "The Darboux theorem for type 2 forms".
15. **Corollary.** Let $\omega$ be a real 3-form on a 6-dimensional differentiable manifold $M$ satisfying $\Delta(\omega_x) = \{0\}$ for any $x \in M$. Let $J$ be an almost complex structure on $M$ such that for any vector fields $X_1, X_2, X_3 \in \mathfrak{X}(M)$ there is

$$\omega(JX_1, X_2, X_3) = \omega(X_1, JX_2, X_3) = \omega(X_1, X_2, JX_3).$$

Then $\omega$ is integrable if and only if the following conditions are satisfied

(i) $d\omega = 0$,

(ii) the almost complex structure $J$ is integrable.

16. **Observation.** There is an interesting relation between structures given by a form of type 2 on 6-dimensional vector spaces and $G_2$-structures on 7-dimensional ones ($G_2$ being the exceptional Lie group, the group of automorphisms of the algebra of Cayley numbers and also the group of automorphism of the 3-form given below), i.e. structures given by a form of the type

$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 + \alpha_1 \wedge \alpha_4 \wedge \alpha_5 - \alpha_1 \wedge \alpha_6 \wedge \alpha_7 + \alpha_2 \wedge \alpha_4 \wedge \alpha_6 + \alpha_2 \wedge \alpha_5 \wedge \alpha_7 +$$

$$+ \alpha_3 \wedge \alpha_4 \wedge \alpha_7 - \alpha_3 \wedge \alpha_5 \wedge \alpha_6,$$

where $\alpha_1, \ldots, \alpha_7$ are the basis of the vector space $V$. If we restrict form of this type to any 6-dimensional subspace of $V$ we get a form of type 2.

$G_2$ structures are well studied and there is known a lot of examples of $G_2$ structures.

Thus any $G_2$ structure on a 7-dimensional manifold gives a structure of type 2 on any 6-dimensional submanifold. Thus we get a vast variety of examples. See for example [J].

**References**

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