Convergence rate for homogenization of a nonlocal model with oscillating coefficients

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Abstract
This letter deals with homogenization of a nonlocal model with Lévy-type operator of rapidly oscillating coefficients. This nonlocal model describes mean residence time and other escape phenomena for stochastic dynamical systems with non-Gaussian Lévy noise. We derive an effective model with a specific convergence rate. This enables efficient analysis and simulation of escape phenomena under non-Gaussian fluctuations.

Keywords: Convergence rate, mean residence time, escape phenomena, Lévy noise, nonlocal elliptic equations

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1. Introduction

We consider the homogenization of the following nonlocal partial differential equation

\[
\begin{aligned}
\mathcal{A}_\varepsilon u(x) &= f(x), \\
u_\varepsilon(x) &= g(x),
\end{aligned}
\]

arising in the study of escape phenomena of stochastic dynamical systems under Lévy fluctuations\(^1\). Here \(D\) is a bounded domain in \(\mathbb{R}^d\). When \(f = -1\) and \(g = 0\), the solution of this equation is the mean residence time for such a stochastic system in domain \(D\). Specifically, the nonlocal operator depends on a small positive scale parameter \(\varepsilon\) and is defined by \(\mathcal{A}_\varepsilon u = \frac{1}{2}D(\Theta(x, z))\), where the coefficient \(\Theta(x, z) = \Theta(x, z)\) is of period 1 in \((x, z)\) and for a certain positive constant \(\lambda\), \(\frac{1}{\lambda} < \Theta(x, z) < \lambda\).

The nonlocal linear operator \(D\) and its adjoint operator \(D^*\) are defined as follows\(^1\). Given the antisymmetric kernel function \(\gamma(x, z) = (z - x)\frac{1}{|z - x|^{\frac{d + \alpha}{2}}}\) (note that \(\gamma(z, x) = -\gamma(x, z)\)), the nonlocal divergence \(D\) is defined by \(D(\beta)(x) := \int_{\mathbb{R}^d} (\beta(x, z) + \beta(z, x)) \cdot \gamma(x, z) dz, x \in D\). The adjoint operator \(D^*\) of \(D\) is then given by \(D^*(\phi)(x) = -\phi(z) - \phi(x))\gamma(x, z)\) for \(x, z \in D\).

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By the way, if $\Theta \equiv 1$, then we see a relation with the nonlocal Laplace operator $\frac{1}{2}DD^* = -(-\Delta)^{\alpha/2}$. The nonlocal Laplace operator $(-\Delta)^{\alpha/2}$ is defined by

$$(-\Delta)^{\alpha/2}u(x) = \int_{\mathbb{R}^d\setminus\{x\}} \frac{u(z) - u(x)}{|z - x|^{d+\alpha}} dz,$$

where the integral is in the sense of Cauchy principal value, and it is the generator of a symmetric $\alpha$-stable Lévy motion [2].

### 2. Main result

#### 2.1. Function spaces

In this subsection, we define the following two important spaces. One is the space $\mathcal{W}^D$, which is the counterpart of the classical Sobolev space $H^1(D)$. The other is the space $\mathcal{X}^D$, as an analogue of the usual trace space $H^{1/2}(\partial D)$.

We set $\nu(x, y) = \gamma^2(x, z)$. We introduce a quadratic form[3–5] for $u : \mathbb{R}^d \to \mathbb{R}$

$$\mathcal{E}_D(u, u) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^d} (u(x) - u(z))^2 \nu(x, z) dx dz.$$

Now we define the corresponding Sobolev space [7]:

$$\mathcal{W}^D = \{ u : \mathbb{R}^d \to \mathbb{R} \text{ such that } \mathcal{E}_D(u, u) < \infty \}.$$

We also denote

$$\mathcal{W}^D_0 = \{ u \in \mathcal{W}^D; u = 0 \text{ a.e. on } D^c \}.$$

More precisely, $\mathcal{W}^D_0 = H^{\alpha/2}_0(D)$ from [8, Theorem 3.33]. The space $H^{\alpha/2}_0(D)$ is the standard fractional Sobolev space. For $s < 0$, we denote $H^s(D)$ as the dual space of $H^{-s}(D)$. It is clear that $\mathcal{W}^D$ is a Hilbert subspace of $L^2(D)$, with the induced norm $||u||^2_{\mathcal{W}^D} = ||u||^2_{L^2(D)} + \mathcal{E}_D(u, u)$.

We will use the notation $\mathbb{T}^d$ to denote the $d$-dimensional unit torus. The space $H^{\alpha/2}_0(\mathbb{T}^d)$ of 1-periodic functions $u \in H^{\alpha/2}$ such that $\int_{\mathbb{T}^d} u(y) dy = 0$ will be interest in this study. Provided with the norm, $||u||^2_{H^{\alpha/2}_0(\mathbb{T}^d)} = \left( \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|u(y) - u(z)|^2}{|y-z|^{d+\alpha}} dxdy \right)^{1/2}.$

We let

$$\mathcal{E}_D(u, v) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^d} (u(x) - u(z))(v(x) - v(z)) \nu(x, z) dx dz,$$

and

$$a^\alpha(u, v) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus D^d} \Theta^\alpha(x, z)(u(x) - u(z))(v(x) - v(z)) \nu(x, z) dx dz,$$

if the integrals are absolutely convergent, in particular for $u, v \in \mathcal{W}^D$.

By a solution of (1) we mean a weak solution, which is defined as every function $u_\varepsilon \in \mathcal{W}^D$ equal to $g$ a.e. on $D^c$ such that for every $\phi \in \mathcal{W}^D_0$, $a^\alpha(u_\varepsilon, \phi) = \int f \phi dx$, this integral is infinite, e.g., if $D$ is bounded and $f \in L^2(D)$. 

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Remark 1. Without loss of generality, we take the coefficient $\Theta'(x, z)$ to be a symmetric function. In fact, we can define the symmetric and anti-symmetric parts of $\Theta'$:

$$\Theta^s(x, z) = \frac{1}{2}(\Theta'(x, z) + \Theta'(z, x)) \quad \text{and} \quad \Theta^a(x, z) = \frac{1}{2}(\Theta'(x, z) - \Theta'(z, x)).$$

By [6, Lemma 3.2], $(\Theta^s(x, z)\mathcal{D}_u, \mathcal{D}_u \phi) = 0$. Then $\mathcal{D}(\Theta^s(x, z)\mathcal{D}_u) = \mathcal{D}(\Theta^a(x, z)\mathcal{D}_u)$.

Let $G'_D(x, y)$ and $P'_D(x, y)$ be the Green function and Poisson kernel of $D$ for $\mathcal{A}$ respectively. At the same time we let $G_D(x, y)$ and $P_D(x, y)$ be the Green function and Poisson kernel for $\mathcal{A}$ with $\Theta' = 1$ on $D$, respectively. In this case $\mathcal{A} u = -2(-\Delta)^{3/2} u$.

For $g : D^c \to \mathbb{R}$, we let $P'_D[g](x) = g(x)$ for $x \in D^c$ and $P'_D[g](x) = \int_{D^c} g(y)P'_D(x, y)dy$ for $x \in D$. Furthermore, $u = P'_D[g]$ is the unique solution of the following homogeneous Dirichlet problem:

$$\begin{align*}
\mathcal{A}' u &= 0, \quad x \in D, \\
|u'|_{D^c} &= g.
\end{align*}$$

Remark 2 (\[4\]). For $f \in H^{-a/2}(D)$ and $g \in X^D$, we have the existence and uniqueness of equations \[1\] and \[2\] in $\mathcal{A}'$.

Next, for $\omega, \xi \in D^c$, we let $\gamma_D(\omega, \xi) = \int_D \int_D \nu(\omega, x)G_D(x, y)\nu(y, \xi)dxdy = \int_D \nu(\xi, x)P_D(x, \omega)dx$. For $g : D^c \to \mathbb{R}$ we introduce a quadratic form $\mathcal{H}_D(g, g) = \frac{1}{2} \int_{D^c} \int_{D^c} (g(y) - g(\xi))^2 \gamma_D(\xi, \omega)d\omega d\xi$.

Then we define as in \[1\] a new space $X^D = \{g : D^c \to \mathbb{R} \text{ such that } \mathcal{H}_D(g, g) < \infty\}$.

If $g \in X^D$ and $x \in D$, we obtain $\int_D g(z)^2 P_D(x, z)dz < \infty$. We fix an arbitrary (reference) point $x_0 \in D$. For $g \in X^D$, we let $|g|_{X^D}^2 = \int_D g(z)^2 P_D(x_0, z)dz$ (we omit $x_0$ from the notation). Then $X^D$ is a Hilbert space with the induced norm $||g||_{X^D}^2 = |g|_{X^D}^2 + \mathcal{H}_D(g, g)$.

2.2. Effective equation and convergence rate

Our main result is that the heterogeneous model \[1\] is approximated by a homogenized effective model \[3\] below, with convergence rate $1 \over 2$. This is stated in the following theorem.

Theorem 1. For scale parameter $\epsilon$ sufficiently small, the solution $u_\epsilon$ of heterogeneous model \[1\] is approximated by the solution $u_0$ of the following homogenized equation

$$\begin{align*}
\begin{cases}
-a_1(-\Delta)^{a/2} u_0 - a_2 \mathcal{F} u_0(x) &= f(x), \quad x \in D, \\
|u_0|_{D^c} &= g(x),
\end{cases}
\end{align*}$$

with coefficients

$$\begin{align*}
a_1 &= \int_{\mathbb{T}^d \times \mathbb{T}^d} \Theta(y, \eta)d\eta d\eta, \\
a_2 &= \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{T}^d} \Theta(y, \eta)\mathcal{D}_x' \mathcal{X}(y) d\eta d\eta, \\
\zeta(u_0)(x) &= \frac{1}{|D|} \int_D (D' u_0)(x, z)dz, \\
\mathcal{F} u_0(x) &= \mathcal{D}_D \zeta(u_0)(x) = \int_D \left[\zeta(u_0)(x) + \zeta(u_0)(z)\right] \gamma(x, z)dz.
\end{align*}$$
where the function $\chi(y)$ is the unique solution of the following variational problem

$$
\begin{cases}
\hat{a}(\chi, \eta) = \int_{\mathbb{R}^d} \Theta(y, \eta) D_x^\nu dy\,d\eta, \\
\chi \in H^{1/2}_0(\mathbb{T}^d).
\end{cases}
$$

Moreover, if $f \in C^\infty(D)$ and $\chi \in L^\infty(\mathbb{T}^d)$, then $u_\epsilon$ has the following asymptotic expansion

$$
u_\epsilon = u_0 - \epsilon H \frac{1}{|D|} \int_D (D^2 u_0)(x, z) dz \, \chi(\frac{x}{\epsilon}) + R(\epsilon),$$

and there exists a constant $C$ (independent of $\epsilon$) such that the remainder is estimated as

$$
\|R(\epsilon)\|_{V^0} = \left\| u_\epsilon - (u_0 - \epsilon \frac{H}{|D|} \int_D (D^2 u_0)(x, z) dz \, \chi(\frac{x}{\epsilon})) \right\|_{V^0} \leq C \epsilon^{1/2}.
$$

This says that $u_\epsilon \to u_0$, in Sobolev space $V^0$, with convergence rate $\frac{\epsilon}{2}$.

In order to prove this theorem, we recall some lemmas. The next result is due to [7].

**Lemma 1.** Let $D \subset \mathbb{R}^d$ be bounded, open and Lipschitz, $|\partial D| = 0$.

- If $g \in \mathcal{X}^D$, then $P_{D}^\nu [g] \in \mathcal{V}^D$ and $\mathcal{E}_D(P_{D}^\nu [g], P_{D}^\nu [g]) = \mathcal{H}_D(g, g)$.
- If $u \in \mathcal{V}^D$, then $g = u|_{\partial D} \in \mathcal{X}^D$ and $\mathcal{E}_D(u, u) \geq \mathcal{H}_D(g, g)$.

Let $X = (X_t)_{t \geq 0}$ be a Lévy process with $(0, \nu, 0)$ as the Lévy triplet on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We introduce the time of the first exit of $X$ from $D$, $\tau_D = \tau_D(X) = \inf \{t \geq 0 : X_t \notin D \}$.

**Lemma 2.** The assumptions are the same as in Lemma 1. Then for every $g \in \mathcal{X}^D$, there exists a positive constant $C(D)$ such that $\|P_{D}^\nu [g]\|_{V^0} \leq C(D)\|g\|_{\chi^0}$.

**Proof.** We write $U \subset \subset D$ if $U$ is an open set, its closure $\bar{U}$ is bounded, and $\bar{U} \subset D$. Let $\tilde{u}^\nu$ be the unique solution of the homogeneous Dirichlet problem [1] and $\mathbb{E}^\nu$ be the expectation for $X_t$ start at $x \in D$. We have $\mathbb{E}^\nu \tilde{u}^\nu(x, \tau_D) = \int_D \tilde{u}^\nu(y) P_{D}^\nu (x, y) dy$. Then we obtain

$$
\|P_{D}^\nu [g]\|_{L^2(D)} \leq \int_D \int_D g(y) P_{D}^\nu (x, y) dy \, dx \leq \int_D g(y)^2 P_{D}^\nu (x, y) dy \, dx,
$$
due to the fact that $\int_D P_{D}^\nu (x, y) dy = 1$, for $x \in D$. That is to say $\|P_{D}^\nu [g]\|_{L^2(D)} \leq \mathbb{E}^\nu \tilde{u}^\nu (x, \tau_D)$.

Note that $\mathbb{E}^\nu \tilde{u}^\nu (x, \tau_D)$ is a closed martingale [2, Remark 4.4] and for $x \in U$, $x \rightarrow \int_{U^c} g(y)^2 P_{U}^\nu (x, y) dy$ satisfies the Harnack inequality [3]. Thus

$$
\|P_{D}^\nu [g]\|_{L^2(U)}^2 \leq \lim_{x \in U \subset \subset D} \int_U \int_{U^c} g(y)^2 P_{U}^\nu (x, y) dy \, dx \leq \lim_{x \in U \subset \subset D} \int_U \int_{U^c} g(y)^2 P_{U}^\nu (x, y) dy \, dx = C(D)\|g\|_{\mathcal{V}^D}^2.
$$

By Lemma 1 we obtain

$$
\|P_{D}^\nu [g]\|_{L^2(D)}^2 = \|P_{D}^\nu [g]\|_{L^2(U)}^2 + \mathcal{E}_D(P_{D}^\nu [g], P_{D}^\nu [g]) \leq C(D)\|g\|_{\mathcal{V}^D}^2 + \mathcal{H}_D(g, g) \leq C(D)\|g\|_{\chi^0}^2.
$$

Hence Lemma 2 follows. \qed
Next, we obtain a natural estimate concerning the solution $u_\varepsilon$ for the original heterogeneous equation (1).

**Lemma 3.** Let $f$ in $H^{-\alpha/2}(D)$, $g$ in $X^0_D$ and $u_\varepsilon$ be the unique solution of the original heterogeneous equation (1). Then there exist two positive constants $C_1$, $C_2$ such that

$$
||u_\varepsilon||_{X^0_D} \leq C_1||f||_{H^{-\alpha/2}(D)} + C_2||g||_{X^0_D}.
$$

**Proof.** From Lemma 2, we can obtain the following conclusion. For every $g \in X^0_D$ there exists $G \in X^0_D$ and a linear operator $\rho$ such that $\rho(G) = G|_{D^c} = g$ and $||G||_{X^0_D} \leq C_1||f||_{H^{-\alpha/2}(D)} + C_2||g||_{X^0_D}$. For every $v \in V^0_D$, we have

$$
||\langle A^\varepsilon G, v \rangle \| = \frac{1}{2}||D(\Theta^\varepsilon D^\varepsilon G), v\| \leq \lambda C ||D^\varepsilon G||_{L^2(D \times \mathbb{R}^d)} ||D^\varepsilon v||_{L^2(D \times \mathbb{R}^d)} \leq 4\lambda C ||G||_{X^0_D} ||v||_{X^0_D}.
$$

Then we obtain $D(\Theta^\varepsilon D^\varepsilon) \in (V^0_D)^*$. That is to say, for $x \in D$, we have $f - D(\Theta^\varepsilon D^\varepsilon) \in H^{-\alpha/2}$. Recall that

$$
\alpha'(u, v) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times D} \Theta^\varepsilon(x, z)(u(x) - u(z))(v(x) - v(z)) \rho \ v(x, z) dx dz,
$$

for every $v \in V^0_D$. We can find a unique $v_\varepsilon \in V^0_D$ such that $\alpha'(v_\varepsilon, v) = \langle f - \frac{1}{2}D(\Theta^\varepsilon D^\varepsilon)G, v \rangle_{V^0_D}$. Due to the Poincaré inequality \[4\], there exists a constant $C \geq 1$, for every $u \in V^0_D$,

$$
||u||^2_{L^2(D)} \leq C \int_{\mathbb{R}^d \times \mathbb{R}^d \times D} (u(x) - u(z))^2 \rho v(x, z) dx dz.
$$

We thus obtain $||u||_{V^0_D} \leq (2C + 1)||D^\varepsilon u||_{L^2(D \times \mathbb{R}^d)}$. In other words, the space $V^0_D$ can be equipped by the norm $||D^\varepsilon u||_{L^2(D \times \mathbb{R}^d)}$. Then $\alpha'(v, v) \geq C ||v||^2_{V^0_D}$ and $|$|$\alpha'(u, v)\| |\leq C ||u||_{V^0_D} ||v||_{V^0_D}$. From the Lax-Milgram theorem,

$$
\frac{1}{\lambda} ||v_\varepsilon||^2_{V^0_D} \leq \alpha'(v_\varepsilon, v_\varepsilon) = \langle f - \frac{1}{2}D(\Theta^\varepsilon D^\varepsilon)G, v_\varepsilon \rangle_{V^0_D} \leq \frac{1}{2}D(\Theta^\varepsilon D^\varepsilon)\|G\|_{H^{-\alpha/2}(D)} ||v_\varepsilon||_{V^0_D}.
$$

We can see that

$$
||v_\varepsilon||_{V^0_D} \leq \lambda \|f - \frac{1}{2}D(\Theta^\varepsilon D^\varepsilon)G\|_{H^{-\alpha/2}(D)}.
$$

Set $u_\varepsilon = v_\varepsilon + G$. By the linearity of $\rho$, we have $\rho(u_\varepsilon) = \rho(G) = g$. Furthermore,

$$
\alpha'(u_\varepsilon, v) = \alpha'(v_\varepsilon, v) + \alpha'(G, v) = (f, v),
$$

which means that $u_\varepsilon$ is the unique solution of the original heterogeneous equation (1). Then

$$
||u_\varepsilon||_{X^0_D} \leq ||u_\varepsilon - G||_{X^0_D} + ||G||_{X^0_D} \leq C ||v_\varepsilon||_{V^0_D} + \sqrt{C(D)}||G||_{X^0_D}.
$$

On the other hand,

$$
< D(\Theta^\varepsilon D^\varepsilon G), v \rangle_{V^0_D} = \int_{\mathbb{R}^d \times \mathbb{R}^d \times D} \Theta^\varepsilon(x, z)D^\varepsilon Gdz \rho dx \leq \lambda C ||D^\varepsilon G||_{L^2(D \times \mathbb{R}^d)} ||D^\varepsilon v||_{L^2(D \times \mathbb{R}^d)} \leq C ||G||_{X^0_D} ||D^\varepsilon v||_{L^2(D \times \mathbb{R}^d)} \leq C ||G||_{X^0_D} ||v||_{V^0_D}.
$$

That implies $||D(\Theta^\varepsilon D^\varepsilon)G|| \leq C||G||_{X^0_D}$. Hence $||u_\varepsilon||_{X^0_D} \leq C_1||f||_{H^{-\alpha/2}(D)} + C_2||g||_{X^0_D}$. This completes the proof. \[\square\]
2.3. Proof of Theorem 1

We are now ready to prove our main result in Theorem 1.

**Step 1:** First, we will derive the homogenized equation for \( \varepsilon \) sufficiently small. For a function \( \nu(x, y) \), we define

\[
(D_y^* \nu)(x, z, y) = -(\nu(z, y) - \nu(x, y))y(x, z)
\]

and

\[
(D_x D_y^* \nu)(x, y) = 2 \int_{\mathbb{R}^d} -\nu(z, y) - \nu(x, y))y^2(x, z)\,dz \\
= -2(-\Delta)^{\eta/2} \nu(x, y).
\]

Denote \( \eta = \frac{1}{\varepsilon} \) a variable on the period: \( \eta \in \mathbb{T}^d \). We look for a formal asymptotic expansion:

\[
u_\varepsilon = \nu_0(x, \frac{x}{\varepsilon}) + \varepsilon \frac{1}{\varepsilon} \nu_1(x, \frac{x}{\varepsilon}) + o(\frac{1}{\varepsilon})
\]

with \( \nu_i(x, y) \), for \( i = 1, 2 \), such that

\[
\begin{cases}
\nu_i(x, y) \text{ is defined for } x \in D \text{ and } y \in \mathbb{T}^d, \\
\nu_i(\cdot, y) \text{ is } 1\text{-periodic.}
\end{cases}
\]

For every function \( h \), we denote \( h^\varepsilon(x) = h(\frac{x}{\varepsilon}) \). Thus

\[
D^* \nu_\varepsilon = \mathcal{A}_1 \nu_0 + \varepsilon \frac{1}{\varepsilon} (\mathcal{A}_0 \nu_0)^\varepsilon + \varepsilon \frac{1}{\varepsilon} \mathcal{A}_1 \nu_1 + (\mathcal{A}_0 \nu_1)^\varepsilon + o(\varepsilon)
\]

where

\[
\mathcal{A}_0 \nu(x, y) := (D_y^* \nu)(x, y, \eta), \mathcal{A}_1 \nu(x, y) := (D_y^* \nu)(x, z, y).
\]

Then we have \( \mathcal{A}_0 \nu_0 = 0 \). Furthermore, \( \nu_0(x, y) = \nu_0(x) \).

Let \( C_{per}(\mathbb{T}^d) \) be the subspace of \( C(\mathbb{R}^d) \) of 1-periodic functions. For every \( \nu \in \mathcal{M}(D, C_{per}(\mathbb{T}^d)) \) (\( \mathcal{M}(D) \) is the space of functions in \( C^{\infty} \) with compact support), we denote \( \nu_\varepsilon = \nu(x, \frac{x}{\varepsilon}) \). We conclude

\[
(f, \nu_\varepsilon) = \frac{1}{2} (\Theta^*(\mathcal{A}_1 u_0 + (\mathcal{A}_0 u_1)^\varepsilon, D^* \nu_\varepsilon) = \frac{1}{2} (\Theta^*(\mathcal{A}_1 u_0 + (\mathcal{A}_0 u_1)^\varepsilon, D^* \nu_\varepsilon)_{L^2(D \times D)} + \int_{D \times D^\varepsilon} \Theta^*(u_0(x) - g(z))\nu_\varepsilon(x)\gamma^\varepsilon(x, z)\,dz
da \xrightarrow{\varepsilon \to 0} I_1 + I_2.
\]

By [15, Lemma 2.34], for every \( \nu \in \mathcal{M}(D, C_{per}(\mathbb{T}^d)) \), \( I_1, I_2 \) converges, as \( \varepsilon \) goes to 0. Now take \( \nu_\varepsilon(x) = \nu_0(x) + \varepsilon \frac{1}{\varepsilon} \nu_1(x, \frac{x}{\varepsilon}) \), where \( \nu_0(x) \in L^2(D) \) and \( \nu_1 \in L^2(D, C_{per}(\mathbb{T}^d)) \). As \( \varepsilon \) goes to 0, we have

\[
I_1 \xrightarrow{\varepsilon \to 0} \frac{1}{2} \int_{T^d \times T^d} \int_{D \times D} \Theta(y, \eta)(D^* u_0 + D^* u_1)(D^* \nu_0 + D^* \nu_1)\,dz
da dy d\eta. \tag{5}
\]

On one hand, let \( \nu_0 = 0 \), we have \( I_2 \to 0 \) and

\[
\int_{D \times D} (\Theta(y, \eta)D^* u_1, D^* \nu_1)_{L^2(T^d \times T^d)}\,dz = - \int_{D \times D} (\Theta(y, \eta)D^* u_0, D^* \nu_1)_{L^2(T^d \times T^d)}\,dz. \tag{6}
\]
For all \( w, v \in H^{\alpha/2}_\#(\mathbb{T}^d) \), we introduce two quadratic forms: \( \hat{a}(w, v) = \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{T}^d} \Theta(y, \eta) D_y^\alpha w D_y^\alpha v dyd\eta \).

From equation (6), we have

\[ |D| \int_D \hat{a}(u_1, v) dx = - \int_D (\int_D \mathcal{D}^* u_0(x, z) dz) \hat{a}(\chi, v) dx, \]

where \( \chi(y) \) is the unique solution of the following variational problem

\[
\begin{cases}
\hat{a}(\chi, v) = \int_{\mathbb{T}^d \times \mathbb{T}^d} \Theta(y, \eta) D_y^\alpha v dyd\eta, \\
\chi \in H^{\alpha/2}_\#(\mathbb{T}^d),
\end{cases}
\]

for all \( v \in H^{\alpha/2}_\#(\mathbb{T}^d) \). So

\[ u_1(x, y) = -\frac{1}{|D|} \int_D (D^* u_0)(x, z) dz \cdot \chi(y) \in L^2(\mathbb{R}^d, H^{\alpha/2}_\#(\mathbb{T}^d)). \]

Moreover, let \( v_1 = 0 \), we conclude that

\[ I_2 \rightarrow \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{T}^d} \Theta(y, \eta) dyd\eta \int_{D \times D^c} (u_0(x) - g(z))v_0(x)v(x, z)dxdz, \]

as \( \varepsilon \) goes to 0. Substituting the representation of \( u_1 \) in (8) into the equation (5), we have

\[
\begin{cases}
-a_1 (-\Delta)^{\alpha/2} u_0 - a_2 \mathcal{F} u_0(x) = f(x), & x \in D, \\
u_0 |_{D^c} = g(x).
\end{cases}
\]

Here

\[
a_1 = \int_{\mathbb{T}^d \times \mathbb{T}^d} \Theta(y, \eta) dyd\eta, \quad a_2 = \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{T}^d} \Theta(y, \eta) D_y^\alpha \chi dyd\eta,
\]

\[
\zeta(u_0)(x) = \frac{1}{|D|} \int_D (D^* u_0)(x, z) dz, \quad \mathcal{F} u_0(x) = \mathcal{D}_{D^c} \zeta(u_0)(x) = \int_D [\zeta(u_0)(x) + \zeta(u_0)(z)] \gamma(x, z) dz.
\]

**Step 2:** In this step, we use the letter \( C \) for a constant independent of \( \varepsilon \). We will prove the convergence rate to be \( \frac{1}{2} \), in the Sobolev space \( \mathcal{V}^\alpha \).

Setting

\[ Z_\varepsilon(x) = u_\varepsilon(x) - (u_0 + \varepsilon \frac{1}{\varepsilon} u_1(x, \frac{x}{\varepsilon})), \]

we have

\[
\begin{align*}
\mathcal{A}_\varepsilon Z_\varepsilon &= -\varepsilon \frac{i\alpha}{2} \mathcal{D}(\Theta D_y^\alpha u_1)^\varepsilon := \frac{1}{2} \varepsilon \frac{i\alpha}{2} F_\varepsilon(x) \quad \text{in} \ D, \\
Z_\varepsilon|_{D^c} &= -\varepsilon \frac{i\alpha}{2} u_1(x, \frac{x}{\varepsilon}) := \varepsilon \frac{i\alpha}{2} K_\varepsilon(x) := \varepsilon \frac{i\alpha}{2} K(x, \frac{x}{\varepsilon}) \quad \text{on} \ D^c.
\end{align*}
\]

We can easily check that \( F_\varepsilon \in H^{-\alpha/2}(D) \). Let us now look at the function \( K_\varepsilon \). We prove the following estimate: \( \|K_\varepsilon\|_{\chi^0} \leq C \varepsilon^{-\alpha/2} \).

For a large enough constant \( N \), we set \( M_\varepsilon = \{ x, |x| > N \} \). Introduce the function \( m_\varepsilon \) defined as follows:

\[
\begin{align*}
m_\varepsilon &= 1 \quad \text{if} \quad \text{dist}(x, \partial D) \leq \varepsilon, x \in D \text{ or } x \in D^c/M_\varepsilon \\
m_\varepsilon &= 0 \quad \text{if} \quad \text{dist}(x, \partial D) \geq 2\varepsilon, x \in D \text{ or } x \in M_\varepsilon \\
\|\nabla m_\varepsilon\|_{L^\infty(\mathbb{R}^d)} &\leq \varepsilon^{-1} C,
\end{align*}
\]
Moreover, we set \( m^\varepsilon \in C^\infty(\mathbb{R}^d/\partial M) \), and the derivative of the function \( m^\varepsilon \) at \( x \in D : \text{dist}(x, \partial D) = 2\varepsilon \) is 0. Set \( \psi_\varepsilon = m^\varepsilon K_\varepsilon \). The support of \( \psi_\varepsilon \) in the domain \( D \) is a neighbourhood of thickness \( 2\varepsilon \) which we denote by \( U^\varepsilon \).

First of all, we prove the estimates \( \| \psi_\varepsilon \|_{L^2(U^\varepsilon)} \leq C\varepsilon^{-\alpha/2} \).

Clearly, from the definition of \( m_\varepsilon \) and the regularity properties of \( u_0 \), we have \( \| \psi_\varepsilon \|_{L^2(U^\varepsilon)} \leq C \).

Moreover, we have
\[
\mathcal{D}^* \psi_\varepsilon = \mathcal{D}^* (m_\varepsilon K_\varepsilon) = (m_\varepsilon K_\varepsilon)(x) - (m_\varepsilon K_\varepsilon)(z) \gamma(y) = m_\varepsilon(x) \left( (\mathcal{D}^* K)(x, z, \frac{x}{\varepsilon}) + \varepsilon^{-\frac{1}{2}} (\mathcal{D}^* K)^1_{y}(z, \frac{x}{\varepsilon}, \frac{z}{\varepsilon}) \right) + (\mathcal{D}^* m_\varepsilon)(x) (z) \cdot K_\varepsilon(z). \]

That is to say [\( \mathcal{D}^* \psi_\varepsilon \leq C \int_{U^\varepsilon} (\mathcal{D}^* \psi_\varepsilon)^2 dzd\mathbf{x} \leq \int_{U^\varepsilon} (m_\varepsilon)^2 (x) (\mathcal{D}^* K)^2 (x, z, \frac{x}{\varepsilon}) dzd\mathbf{x} \]
\[
+ \varepsilon^{-1-a} \int_{U^\varepsilon} (m_\varepsilon)^2 (x) \left( (\mathcal{D}^* K)^1_{y}(z, \frac{x}{\varepsilon}, \frac{z}{\varepsilon}) \right)^2 dzd\mathbf{x} \]
\[
+ \int_{U^\varepsilon} (\mathcal{D}^* m_\varepsilon)^2 (x) \cdot (K_\varepsilon)^2 (z) dz = J_1 + J_2 + J_3. \]

We set \( K(x, \frac{z}{\varepsilon}) = h(x) \chi(\frac{z}{\varepsilon}) \), where \( h(x) = \frac{1}{|D|} \int_D (\mathcal{D}^* u_0)(x, z) dz \). Clearly, from the regularity properties of \( u_0, \chi \), we have
\[
J_1 = \int_{U^\varepsilon} (m_\varepsilon)^2 (x) (\mathcal{D}^* h)^2 (x, z, \frac{x}{\varepsilon}) dzd\mathbf{x} \leq C \int_{U^\varepsilon} (\mathcal{D}^* h)^2 (x, z) dzd\mathbf{x} \leq C, \]
\[
J_2 = \varepsilon^{-1-a} \int_{U^\varepsilon} (m_\varepsilon)^2 (x) \left( (\mathcal{D}^* \chi)^1_y \right)^2 h^2 (z) dzd\mathbf{x} \leq C \int_{U^\varepsilon} (m_\varepsilon)^2 (x) h^2 (z) dzd\mathbf{x}, \]
\[
+ \varepsilon^{-1-a} \int_{U^\varepsilon} (m_\varepsilon)^2 (x) \left( (\mathcal{D}^* \chi)^1_y \right)^2 h^2 (z) dzd\mathbf{x} \leq C \int_{U^\varepsilon} (m_\varepsilon)^2 (x) h^2 (z) dzd\mathbf{x}, \]
\[
+ \varepsilon^{-1-a} \int_{U^\varepsilon} (m_\varepsilon)^2 (x) \left( (\mathcal{D}^* \chi)^1_y \right)^2 (x, z) dzd\mathbf{x} \leq C \varepsilon^{-1-a} \int_{U^\varepsilon} (\mathcal{D}^* u_0)^2 dzd\mathbf{x}, \]
\[
J_3 = \int_{U^\varepsilon} (\mathcal{D}^* m_\varepsilon)^2 (x, z) \cdot (K_\varepsilon)^2 (z) dzd\mathbf{x} = \int_{U^\varepsilon} \frac{(m_\varepsilon^\varepsilon)(x)(x-z)^2}{|x-z|^{d+a}} dz = C \varepsilon^{-2} \int_{U^\varepsilon} (\mathcal{D}^* u_0)^2 dzd\mathbf{x}, \]
\[
J_3 + C \int_{U^\varepsilon} (\mathcal{D}^* u_0)^2 dzd\mathbf{x} \leq C \varepsilon^{-2} \int_{U^\varepsilon} (\mathcal{D}^* u_0)^2 dzd\mathbf{x}. \]

That is to say \( \mathcal{E}_{U^\varepsilon}(\psi_\varepsilon, \psi_\varepsilon) \leq J_1 + J_2 + J_3 \leq C \varepsilon^{-1-a} \int_{U^\varepsilon} (\mathcal{D}^* u_0)^2 dzd\mathbf{x}. \)

We can use a result from Chapter 1, Lemma 1.5, which states that there exists positive constants \( C \), independent of \( \varepsilon \), such that \( \|u_0\|_{L^2(D)} \leq C \varepsilon^\gamma \|u_0\|_{H^1(D)} \). We can conclude
\[
\int_{U^\varepsilon} (\mathcal{D}^* u_0)^2 dzd\mathbf{x} = C \varepsilon \|u_0\|^2_{H^1(D)}. \]

Then \( \|\psi_\varepsilon\|_{L^2(U^\varepsilon)}^2 = \|\psi_\varepsilon\|_{L^2(U^\varepsilon)}^2 + \mathcal{E}_{U^\varepsilon}(\psi_\varepsilon, \psi_\varepsilon) \leq C \varepsilon^{-\alpha} \). That is to say
\[
\|\psi_\varepsilon\|_{\mathcal{L}^2(U^\varepsilon)}^2 \leq C \varepsilon^{-\alpha}. \]
Secondly, we will show \( \|K_e\|_{\mathcal{X}^0} = \|\psi_e\|_{\mathcal{X}^0} + C \). Observe now that \( \psi_e = K_e \) on \( D^c/M^c \) and \( \psi_e = 0 \) on \( M^c \). Then
\[
\|K_e\|_{\mathcal{X}^0}^2 = C \int_{D^c/M^c \times M^c} (K_e(x) - K_e(z))^2 \gamma_D(x, z) dz dx + \int_{D^c/M^c \times M^c} (K_e(x) - K_e(z))^2 \gamma_D(x, z) dz dx
\]
\[
+ \int_{M^c \times M^c} (K_e(x) - K_e(z))^2 \gamma_D(x, z) dz dx,
\]
\[
\|\psi_e\|_{\mathcal{X}^0}^2 = C \int_{D^c/M^c \times M^c} (\psi_e(x) - \psi_e(z))^2 \gamma_D(x, z) dz dx + \int_{D^c/M^c \times D^c/M^c} (K_e(x) - K_e(z))^2 \gamma_D(x, z) dz dx.
\]
Recall that [7, Theorem 2.6]
\[
\gamma_D(x, z) \approx \begin{cases} 
\nu(\delta_D(z))\nu(\delta_D(x)) & \text{if } \text{diam}(D) \leq \delta_D(x), \delta_D(z), \\
\nu(\delta_D(z))/V(\delta_D(x)) & \text{if } \delta_D(x) \leq \text{diam}(D) \leq \delta_D(z), \\
\nu(r(x, z))V(\delta_D(z)) & \text{if } \delta_D(x), \delta_D(z) \leq \text{diam}(D),
\end{cases}
\]
here \( \delta_D(x) = \text{dist}(x, \partial D), r(x, y) = \delta_D(x) + |x - y| + \delta_D(x) \) and \( V(r) = Cr^{d/2} \). Then, we have
\[
\int_{M^c \times M^c} (K_e(x) - K_e(z))^2 \gamma_D(x, z) dz dx \leq C \int_{M^c \times M^c} (K_e(x) - K_e(z))^2 (\delta_D(x))^{-\alpha} (\delta_D(z))^{-\alpha} dz dx \leq C,
\]
and
\[
\int_{D^c/M^c \times M^c} (K_e(x) - K_e(z))^2 \gamma_D(x, z) dz dx
\]
\[
= C \int_{D^c/M^c \times \{z : \delta_D(z) \leq \text{diam}(D)\}} \int_{M^c} (K_e(x) - K_e(z))^2 \gamma_D(x, z) dz dx
\]
\[
+ C \int_{D^c/M^c \times \{z : \delta_D(z) \leq \text{diam}(D)\}} \int_{M^c} (K_e(x) - K_e(z))^2 \gamma_D(x, z) dz dx
\]
\[
\leq C.
\]
Then, we get the conclusion
\[
\|K_e\|_{\mathcal{X}^0}^2 = \|\psi_e\|_{\mathcal{X}^0}^2 + C. \tag{12}
\]
Next, we will show \( \|\psi_e\|_{\mathcal{X}^0}^2 \leq \max(C, 1) \|\psi_e\|_{\mathcal{X}^0}^2 + C \). In fact,
\[
\|\psi_e\|_{\mathcal{X}^0}^2 = \|\psi_e\|_{L^2(D)}^2 + C \int_{D \times \mathbb{R}^d} (D^* \psi_e)^2 dx dz = \|\psi_e\|_{L^2(U^c)}^2 + C \int_{U^c \times \mathbb{R}^d} (D^* \psi_e)^2 dx dz
\]
\[
+ C \int_{D \setminus U^c} \int_{D \setminus U^c} (\nu^c(x)^2 K^2(\frac{x}{\nu^c}, \frac{z}{\nu^c})) dy(x, z) dz dx
\]
\[
\leq \max(C, 1) \|\psi_e\|_{\mathcal{X}^0}^2 + C \int_{D \setminus U^c} \int_{D \setminus U^c} (\nu^c(x)^2 K^2(\frac{x}{\nu^c}, \frac{z}{\nu^c})) dy(x, z) dz dx.
\]
From the fact that the derivative of the function \( \nu^c \) at \( x \in D : \text{dist}(x, \partial D) = 2\epsilon \) is 0, we have
\[
\|\psi_e\|_{\mathcal{X}^0}^2 \leq \max(C, 1) \|\psi_e\|_{\mathcal{X}^0}^2 + C. \tag{13}
\]
Finally, we will get the convergence rate. Combining (11), (12), (13) and [7, Corollary 5.1] we conclude

$$\|K_\epsilon\|_{\mathcal{X}^D}^2 = \|\psi_\epsilon\|_{\mathcal{X}^D}^2 + C \leq C \|\psi_\epsilon\|_{\mathcal{V}^D}^2 + C \leq C \|\psi_\epsilon\|_{\mathcal{V}^D}^2 + C \leq C \epsilon^{-\alpha}.$$ 

We thus estimate from Lemma 3

$$\|Z_\epsilon\|_{\mathcal{V}^D} \leq C \epsilon^{\frac{1+\alpha}{2}} \|F_\epsilon\|_{H^{-\alpha/2}(D)} + C \epsilon^{\frac{1+\alpha}{2}} \|K_\epsilon\|_{\mathcal{X}^D} \leq C \epsilon^{\frac{1+\alpha}{2}} + \epsilon^{\frac{1+\alpha}{2}} \epsilon^{-\frac{\alpha}{2}} c_{11} \leq C \epsilon^{\frac{1}{2}}.$$ 

This completes the proof of Theorem 1.

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