Domain decomposition schemes for evolutionary equations of first order with not self-adjoint operators

Petr N. Vabishchevich

Submitted to arXiv.org January 12, 2011

Abstract Domain decomposition methods are essential in solving applied problems on parallel computer systems. For boundary value problems for evolutionary equations the implicit schemes are in common use to solve problems at a new time level employing iterative methods of domain decomposition. An alternative approach is based on constructing iteration-free methods based on special schemes of splitting into subdomains. Such regionally-additive schemes are constructed using the general theory of additive operator-difference schemes. There are employed the analogues of classical schemes of alternating direction method, locally one-dimensional schemes, factorization methods, vector and regularized additive schemes. The main results were obtained here for time-dependent problems with self-adjoint elliptic operators of second order.

The paper discusses the Cauchy problem for the first order evolutionary equations with a nonnegative not self-adjoint operator in a finite-dimensional Hilbert space. Based on the partition of unit, we have constructed the operators of decomposition which preserve nonnegativity for the individual operator terms of splitting. Unconditionally stable additive schemes of domain decomposition were constructed using the regularization principle for operator-difference schemes. Vector additive schemes were considered, too. The results of our work are illustrated by a model problem for the two-dimensional parabolic equation.

Keywords Time-dependent problems · Domain decomposition method · Additive schemes · Operator-splitting difference schemes

PACS 02.60.Lj · 02.70.Bf

Mathematics Subject Classification (2000) 65N06 · 65M06

P.N. Vabishchevich
Keldysh Institute of Applied Mathematics, 4 Miusskaya Square, 125047 Moscow, Russia
Tel.: +7-499-9781014
Fax: +7-499-9720737
E-mail: vabishchevich@gmail.com
1 Introduction

Domain decomposition methods are widely used for the numerical solution of boundary value problems for partial differential equations on parallel computers. Stationary problems [11,12,21,25] are the most extensively studied in the theory of domain decomposition methods. Numerical algorithms with and without overlapping of subdomains are used here in the synchronous (sequential) and asynchronous (parallel) organization of computations.

Domain decomposition methods for unsteady problems are based on two approaches [14]. In the first approach for the approximate solution of time-dependent problems we use the standard implicit approximations in time. After that domain decomposition methods are applied to solve the discrete problem at a new time level. For the optimal iterative methods of domain decomposition the number of iterations does not depend on the discretization steps in time and space [34]. The iteration-free domain decomposition algorithms are constructed for unsteady problems in the second approach. In some cases we can confine ourselves to only one iteration of the Schwarz alternating method for solving boundary value problems for the parabolic equation of second order [6,7]. Special schemes of splitting into subdomains (regionally-additive schemes [20,27]) are constructed, too.

Construction and convergence investigation of the regionally-additive schemes is based on the general results of the theory of splitting schemes [10,13,31]. The most interesting for practice is the situation where the problem operator is decomposed into a sum of three or more noncommutative not self-adjoint operators. In the case of such a multi-component splitting the stable additive splitting schemes are constructed on the basis of the concept of summarized approximation. Additively-averaged schemes of summarized approximation are interesting for using on parallel computers. In the class of splitting schemes of full approximation [19] we highlight the vector additive schemes based on the transition from the single original equation to a system of similar equations [1,2,31]. Additive regularized operator-difference schemes are constructed in the most simple way for multi-component splitting [18,23], where stability is achieved via perturbations of operators of the difference scheme.

Peculiarities of domain decomposition schemes result from the selection of splitting operators. To construct the operators of decomposition for boundary value problems for partial differential equations, it is convenient to use the partition of unit for the computational domain [5,8,16,26,28,29,33]. In the domain decomposition method with overlapping a separate subdomain is associated with a function with values lying between zero and one. Domain decomposition methods for unsteady convection-diffusion problems are studied in works [17,20,30]. In the limiting case the width of subdomain overlapping is equal to the discretization step. In this case the regionally-additive schemes are interpreted as the decomposition without overlapping of subdomains but with appropriate boundary conditions of exchange. Domain decomposition methods for unsteady boundary value problems are summarized in the books [14,19]. More recent studies are presented in the work [32]. In this case we use
different constructions for the splitting operators and for operators of the grid problem at a new time level.

In this paper we construct domain decomposition schemes for the first order evolutionary equations with a general nonnegative operator in a finite Hilbert space. Decomposition operators are constructed separately for the self-adjoint and skew-symmetric parts using the partition of unit in the appropriate spaces. Two classes of unconditionally stable regionally-additive regularized schemes are proposed. Vector additive operator-difference schemes of domain decomposition are considered. The paper is organized as follows. In section 2 there is formulated the Cauchy problem for the evolutionary equation of first order and the corresponding a priori estimate of stability is derived. Decomposition operators are constructed in Section 3. Problems with the self-adjoint operator and skew-symmetric one are considered separately. Unconditionally stable regularized additive schemes of domain decomposition are constructed in Section 4, with the additive and multiplicative perturbation of the operator of transition to the new time level. Vector splitting schemes are discussed in Section 5. In section 6 we consider a model boundary value problem for the two-dimensional parabolic equation along with the results of using different domain decomposition schemes. The main results are summarized in Section 7.

2 The Cauchy problem for the first order evolutionary equation

Let \( H \) be a finite-dimensional real Hilbert space of grid functions with the scalar product and norm \((\cdot, \cdot) \| \cdot \|\), respectively. Let a constant (independent of time \( t \)) grid operator \( A \) is nonnegative in \( H \):

\[
A \geq 0, \quad \frac{d}{dt}A = A \frac{d}{dt}
\]

and \( E \) is the identity operator in \( H \). We search the solution of the Cauchy problem

\[
\frac{du}{dt} + Au = f(t), \quad 0 < t \leq T,
\]

\[
u(0) = u_0.
\]

Problem (1)–(3) results from a finite-difference approximation in space for the approximate solution of boundary value problems for partial differential equations. Similar systems of ordinary differential equations arise in using the finite-element method as well as in applying the finite-volume approach. Let us obtain the standard a priori estimate for problem (1)–(3).

Multiply equation (2) by \( u \) scalarly in \( H \). In view of (1) we obtain inequality

\[
\frac{1}{2} \frac{d}{dt} \| u \|^2 \leq (f, u).
\]

Taking into account

\[
(f, u) \leq \| f \| \| u \|,
\]
from (4) we have
\[
\frac{d}{dt} \|u\| \leq \|f\|.
\]
In view of the Gronwall lemma we obtain the desired estimate
\[
\|u\| \leq \|u^0\| + \int_0^t \|f(\theta)\|d\theta, \tag{5}
\]
which expresses the stability of the solution with respect to the initial data and right-hand side.

The emphasis of our work is on constructing approximations in time for equation (2). Two-level schemes will be considered. Let \( \tau \) be a step of the uniform grid in time and let \( y^n = y(t^n), \quad t^n = n\tau, \quad n = 0, 1, ..., N, \quad N\tau = T. \) Equation (2) is approximated by the two-level scheme with weights
\[
\frac{y^{n+1} - y^n}{\tau} + A(\sigma y^{n+1} + (1 - \sigma)y^n) = \varphi^n, \quad n = 0, 1, ..., N - 1, \tag{6}
\]
where, for example, \( \varphi^n = f(\sigma t^{n+1} + (1 - \sigma)t^n). \) It is supplemented by the initial condition
\[
y^0 = u^0. \tag{7}
\]
Difference scheme (6), (7) has approximation error \( O(\tau^2 + (\sigma - 0.5)\tau). \)

Grid analog of (5) is the estimate at the time level
\[
\|y^{n+1}\| \leq \|y^n\| + \tau\|\varphi^n\|, \quad n = 0, 1, ..., N - 1. \tag{8}
\]
Let us prove the following statement.

**Theorem 1** Difference scheme (6), (7) is unconditionally stable at \( \sigma \geq 0.5, \) and estimate (8) holds for the difference solution.

**Proof** We write (6) in the form
\[
y^{n+1} = Sy^n + \tau(E + \sigma\tau A)^{-1}\varphi^n, \tag{9}
\]
where
\[
S = (E + \sigma\tau A)^{-1}(E - (1 - \sigma)\tau A) \tag{10}
\]
is the operator of transition to the new time level. From (9) we have
\[
\|y^{n+1}\| = \|S\| \|y^n\| + \tau\|(E + \sigma\tau A)^{-1}\varphi^n\|. \tag{11}
\]

For the last term in the right-hand side of (11) in the class of operators (10), under natural conditions \( \sigma \geq 0 \) we have
\[
\|(E + \sigma\tau A)^{-1}\varphi^n\| \leq \|\varphi^n\|.
\]
We show that if \( \sigma \geq 0.5 \) then for the nonnegative operators \( A \) the following estimate holds
\[
\|S\| \leq 1. \tag{12}
\]
In the Hilbert real space $H$ inequality (12) is equivalent \[9\] to fulfilment of the operator inequality
\[SS^* \leq E.\]

In view of (10) this inequality takes the form
\[(E + \sigma \tau A)^{-1}(E - (1 - \sigma)\tau A)(E - (1 - \sigma)\tau A^*)(E + \sigma \tau A)^{-1} \leq E.\]

Multiplying this inequality on the left by $(E + \sigma \tau A)^{-1}$ and on the right by $(E + \sigma \tau A^*)^{-1}$, we obtain
\[(E - (1 - \sigma)\tau A)(E - (1 - \sigma)\tau A^*) \leq (E + \sigma \tau A)(E + \sigma \tau A^*).\]

It follows that
\[\tau(A + A^*) + (\sigma^2 - (1 - \sigma)^2)\tau^2 AA^* \geq 0.\]

This inequality holds for the nonnegative operators $A$ with $\sigma \geq 0$. In view of (12) we have from (11) required estimate (8).

3 Operators of decomposition

To better understand the formal structure of the domain decomposition operators, we give a typical example. We consider a model unsteady convection-diffusion problem with a constant (independent of time, but depending on the points of a computational domain) coefficients of diffusion and convection. Convective transport is written in the so-called (see, for example, \[21\]) symmetric form. Let in a bounded domain $\Omega$ an unknown function $u(x, t)$ satisfies the following equation
\[
\frac{\partial u}{\partial t} + \frac{1}{2} \sum_{\alpha=1}^{m} \left( v_\alpha(x) \frac{\partial u}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha} (v_\alpha(x)u) \right) - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} \left( k(x) \frac{\partial u}{\partial x_\alpha} \right) = f(x, t), \quad x \in \Omega, \quad 0 < t < T, \tag{13}
\]
where $k(x) \geq \kappa > 0$, $x \in \Omega$. Supplement equation (13) with the homogeneous Dirichlet boundary conditions
\[u(x, t) = 0, \quad x \in \partial \Omega, \quad 0 < t < T. \tag{14}\]

In addition, we define the initial condition
\[u(x, 0) = u^0(x), \quad x \in \Omega. \tag{15}\]

Let us consider a set of functions $u(x, t)$ satisfying boundary conditions (14). We write the unsteady convection-diffusion problem in the form of differential-operator equation
\[
\frac{du}{dt} + Au = f(t), \quad 0 < t < T, \quad t > 0 \tag{16}
\]
We consider the Cauchy problem for evolutionary equation (16):

\[ u(0) = u^0. \]  

(17)

Operators of diffusive and convective transport are treated separately, so that in (16)

\[ A = C + D. \]  

(18)

For the diffusion operator we set

\[ D u = - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_{\alpha}} \left( k(x) \frac{\partial u}{\partial x_{\alpha}} \right). \]

On set of functions (13) in \( \mathcal{H} = \mathcal{L}_2(\Omega) \) diffusive transport operator \( D \) is self-adjoint and positive definite:

\[ D = D^* \geq \kappa \delta E, \quad \delta = \delta(\Omega) > 0, \]  

(19)

where \( E \) is the identity operator in \( \mathcal{H} \).

Convective transport operator \( C \) is defined by the expression

\[ C u = \frac{1}{2} \sum_{\alpha=1}^{m} \left( v_{\alpha}(x) \frac{\partial u}{\partial x_{\alpha}} + \frac{\partial}{\partial x_{\alpha}}(v_{\alpha}(x)u) \right). \]

For any \( v_{\alpha}(x) \) the operator \( C \) is skew-symmetric in \( \mathcal{H} \):

\[ C = -C^*. \]  

(20)

Taking into account representation (18), it follows from (19), (20) that \( A > 0 \) in \( \mathcal{H} \).

Domain decomposition schemes will be constructed via the partition of unit for the computational domain \( \Omega \). Let domain \( \Omega \) consists of \( p \) separate subdomains

\[ \Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_p. \]

Individual subdomains can overlap. With separate subdomain \( \Omega_\alpha, \alpha = 1, 2, \ldots, p \) we associate function \( \eta_{\alpha}(x), \alpha = 1, 2, \ldots, p \) such that

\[ \eta_{\alpha}(x) = \begin{cases} > 0, \ x \in \Omega_\alpha, \\ 0, \ x \notin \Omega_\alpha \end{cases}, \quad \alpha = 1, 2, \ldots, p, \]  

(21)

where

\[ \sum_{\alpha=1}^{p} \eta_{\alpha}(x) = 1, \quad x \in \Omega. \]  

(22)

In view of (21), (22) we obtain from (18) the following representation

\[ A = \sum_{\alpha=1}^{p} A_{\alpha}, \quad A_{\alpha} = C_{\alpha} + D_{\alpha}, \quad \alpha = 1, 2, \ldots, p, \]  

(23)
where

\[ D_\alpha u = - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} \left( k(x) \eta_\alpha(x) \frac{\partial u}{\partial x_\alpha} \right), \]

\[ C_\alpha u = \frac{1}{2} \sum_{\alpha=1}^{m} \left( v_\alpha(x) \eta_\alpha(x) \frac{\partial u}{\partial x_\alpha} + \frac{\partial}{\partial x_\alpha} (v_\alpha(x) \eta_\alpha(x) u) \right). \]

Similarly (19), (20), we have

\[ D_\alpha = D_\alpha^*, \quad C_\alpha = -C_\alpha^*, \quad \alpha = 1, 2, ..., p. \]  \hspace{1cm} (24)

In view of (24) in splitting (23) the following property holds

\[ A_\alpha \geq 0, \quad \alpha = 1, 2, ..., p, \]  \hspace{1cm} (25)

and the self-adjoint part of operator \( A \) is split into the sum of nonnegative self-adjoint operators, whereas the skew-symmetric one – into the sum of skew-symmetric operators.

It is convenient to represent diffusive transport operator \( D \) as follows

\[ D = G^* G, \quad G = k^{1/2} \text{grad}, \quad G^* = -\text{div} k^{1/2}, \]  \hspace{1cm} (26)

with \( G : H \rightarrow \tilde{H} \), where \( \tilde{H} = (L_2(\Omega))^p \) is the corresponding Hilbert space of vector functions. From this structure, for \( D_\alpha, \alpha = 1, 2, ..., p \) we obtain

\[ D_\alpha = G^* \eta_\alpha G, \quad \alpha = 1, 2, ..., p. \]  \hspace{1cm} (27)

Similarly, \( C_\alpha, \alpha = 1, 2, ..., p \) can be represented as

\[ C_\alpha = \frac{1}{2} (\eta_\alpha C + C \eta_\alpha), \quad \alpha = 1, 2, ..., p. \]  \hspace{1cm} (28)

Representations (27), (28) for operators of diffusive and convective transport demonstrate us clearly the structure of operators in individual subdomains in the splitting based on (21), (22) and allow to verify fulfillment of (24). A similar consideration can be given for the operator of problem (2), (3).

Let us divide the operator \( A \) into the self-adjoint and skew-symmetric parts:

\[ A = C + D, \quad C = \frac{1}{2} (A + A^*), \quad D = \frac{1}{2} (A - A^*). \]  \hspace{1cm} (29)

For the nonnegative operator \( D \) the following representation holds

\[ D = G^* G, \]  \hspace{1cm} (30)

where \( G : H \rightarrow \tilde{H} \). For the partition of unit of the computational domain we consider the corresponding additive representation of unit operators \( E \) and \( \tilde{E} \) in spaces \( H \) and \( \tilde{H} \), respectively. Let

\[ \sum_{\alpha=1}^{p} \chi_\alpha = E, \quad \chi_\alpha \geq 0, \quad \alpha = 1, 2, ..., p, \]  \hspace{1cm} (31)
\[ \sum_{\alpha=1}^{p} \tilde{\chi}_\alpha = \tilde{E}, \quad \tilde{\chi}_\alpha \geq 0, \quad \alpha = 1, 2, ..., p. \] (32)

By analogy with (23)–(25), we use the splitting
\[ A = \sum_{\alpha=1}^{p} A_\alpha, \quad A_\alpha \geq 0, \quad \alpha = 1, 2, ..., p, \] (33)

where
\[ A_\alpha = C_\alpha + D_\alpha, \quad D_\alpha = D_\alpha^* \geq 0, \quad C_\alpha = -C_\alpha^*, \quad \alpha = 1, 2, ..., p. \] (34)

On the basis of (32) for the terms of the self-adjoint part of the operator \( A \) we set
\[ D_\alpha = G^* \tilde{\chi}_\alpha G, \quad \alpha = 1, 2, ..., p. \] (35)

Decomposition of the skew-symmetric part is based on (31):
\[ C_\alpha = \frac{1}{2} (\chi_\alpha C + C \chi_\alpha), \quad \alpha = 1, 2, ..., p. \] (36)

Such an additive representation is a discrete analog of (27), (28) and is interpreted as the corresponding version of the domain decomposition.

4 Regularized domain decomposition schemes

For the approximate solution of the Cauchy problem for equation (2), (3) under condition (33) we apply different splitting schemes. Transition to the new time level is based on solving \( p \) separate subproblems with individual operators \( A_\alpha \), \( \alpha = 1, 2, ..., p \). Taking into account the structure of the operators (see (34)–(36)) we can say that these splitting schemes are regionally-additive and iteration-free.

Currently, the principle of regularization of difference schemes is considered as the basic methodological principle for improving difference schemes [13]. The construction of unconditionally stable additive difference schemes [19] via the principle of regularization will be implemented in the following way.

1. For the initial problem there is constructed some simple difference scheme (the producing difference scheme). This scheme does not satisfy the necessary properties. For example, in constructing additive schemes the producing scheme is not splitting one, or it is conditionally stable or even absolutely unstable.
2. The difference scheme is written in the form for which the stability conditions are known.
3. Quality of the scheme (its stability) is improved via perturbations of operators of the difference scheme with preserving possibility of its computational implementation as an additive scheme.
Concerning to problem (2), (3) it is natural to choose as the producing scheme the following simple explicit scheme

$$\frac{y^{n+1} - y^n}{\tau} + Ay^n = \varphi^n, \quad n = 0, 1, ..., N - 1,$$

which is supplemented by initial conditions (7). Stability of scheme (37) is provided (see the proof of Theorem 1) by fulfillment of inequality

$$A + A^* - \tau AA^* \geq 0.$$  

Inequality (38) with $D > 0$ imposes appropriate restrictions on the time step, i.e. scheme (29), (37) is conditionally stable. Note also that if $D = 0$ than scheme (29), (37) is absolutely unstable. Taking into account splitting (33), we refer this scheme to the class of additive schemes.

To construct additive schemes, we can take more general scheme (6), (7) as a producing one. It is unconditionally stable at $\sigma \geq 0$. In this case the perturbation of scheme operators is oriented only to receive the additive schemes preserving the property of unconditional stability.

Regularization of a difference scheme in order to improve the stability restriction (construction of a splitting scheme) can be performed via some perturbation of the operator $A$. The second possibility is related to perturbation of the operator at the difference derivative in time (for our scheme (ref 37) it is operator $E$). In constructing additive schemes, it is convenient to operate with the transition operator $S$, rewriting producing scheme (37) as follows

$$y^{n+1} = Sy^n + \tau \varphi^n, \quad n = 0, 1, ..., N - 1.$$  

For (37) we have

$$S = E - \tau A.$$  

The regularized scheme is based on perturbation of the operator $S$ and has the following form

$$y^{n+1} = \tilde{S}y^n + \tau \varphi^n, \quad n = 0, 1, ..., N - 1.$$  

Let us consider general restrictions for $\tilde{S}$.

To preserve the first order approximation, which has generating scheme (39), (40), we subordinate the selection of $\tilde{S}$ to the condition

$$\tilde{S} = E - \tau A + O(\tau^2).$$  

Stability of scheme (41) in the sense that estimate (38) holds, is provided by the inequality

$$\|\tilde{S}\| \leq 1.$$  

In addition, the regularized scheme must be additive, i.e. the transition to the new time level is implemented via solving the individual subproblems for operators $A_\alpha$, $\alpha = 1, 2, ..., p$ in decomposition (33).
The first class of regularized splitting schemes is based on the following additive representation for the transition operator of the producing scheme

\[ S = \frac{1}{p} \sum_{\alpha=1}^{p} S_\alpha, \quad S_\alpha = E - prA_\alpha, \quad \alpha = 1, 2, \ldots, p. \]

The similar additive representation we also use for the transition operator of the regularized scheme

\[ \tilde{S} = \frac{1}{p} \sum_{\alpha=1}^{p} \tilde{S}_\alpha, \quad \alpha = 1, 2, \ldots, p. \]  \hspace{1cm} (44)

Individual terms \( \tilde{S}_\alpha, \quad \alpha = 1, 2, \ldots, p \) are constructed via perturbations of operators \( A_\alpha, \quad \alpha = 1, 2, \ldots, p \). By analogy with (10) we set

\[ \tilde{S}_\alpha = \left( E + \sigma prA_\alpha \right)^{-1} \left( E - (1 - \sigma)prA_\alpha \right), \quad \alpha = 1, 2, \ldots, p. \]  \hspace{1cm} (45)

If \( \sigma \geq 0.5 \) (see proof of Theorem 1) we have

\[ \| \tilde{S}_\alpha \| \leq 1, \quad \alpha = 1, 2, \ldots, p. \]

In view of (44) it provides fulfillment of stability conditions (43).

Using the representation

\[ \tilde{S}_\alpha = E - pr(E + \sigma prA_\alpha)^{-1}A_\alpha, \quad \alpha = 1, 2, \ldots, p \]

we can rewrite regularized additive scheme (41), (44), (45) as follows

\[ \frac{y^{n+1} - y^n}{\tau} + \sum_{\alpha=1}^{p} (E + \sigma prA_\alpha)^{-1}A_\alpha y^n = \varphi^n, \quad n = 0, 1, \ldots, N - 1. \]  \hspace{1cm} (46)

The comparison with producing scheme (33), (37) shows that the regularization is provided by perturbation of the operator \( A \). Our consideration results in the following statement.

**Theorem 2** Additive difference scheme (7), (41), (44), (45) is unconditionally stable at \( \sigma \geq 0.5 \), and for the numerical solution estimate of stability (8) with respect to the initial data and right-hand side holds.

Numerical implementation of scheme (7), (46) can be conducted as follows. Assume

\[ y^{n+1} = \frac{1}{p} \sum_{\alpha=1}^{p} y^{n+1}_\alpha, \quad \varphi^n = \sum_{\alpha=1}^{p} \varphi^n_\alpha. \]

In this case we obtain

\[ \frac{y^{n+1}_\alpha - y^n}{p\tau} + (E + \sigma prA_\alpha)^{-1}A_\alpha y^n = \varphi^n_\alpha, \quad \alpha = 1, 2, \ldots, p. \]  \hspace{1cm} (47)
for the individual components of the approximate solution at the new time level $y_{\alpha}^{n+1}$, $\alpha = 1, 2, ..., p$. Scheme (47) can be rewritten as follows

$$\frac{y_{\alpha}^{n+1} - y_n}{p\tau} + A_\alpha y^n (\sigma y_{\alpha}^{n+1} + (1 - \sigma)y^n) = (E + \sigma p\tau A_\alpha) \varphi_{\alpha}^{n}.$$ 

In this form we can interpret scheme (47) as a variant of the additively-averaged scheme of component-wise splitting [19].

The second class of regularized splitting schemes is based on using not additive (see (44)) but multiplicative representation of the transition operator:

$$\tilde{S} = \prod_{\alpha=1}^{p} S_\alpha, \quad \alpha = 1, 2, ..., p. \quad (48)$$

Taking into account (42), we have

$$S = \prod_{\alpha=1}^{p} S_\alpha + O(\tau^2), \quad S_\alpha = E - \tau A_\alpha, \quad \alpha = 1, 2, ..., p.$$ 

Similarly (45), we set

$$\tilde{S}_\alpha = (E + \sigma \tau A_\alpha)^{-1}(E - (1 - \sigma)\tau A_\alpha), \quad \alpha = 1, 2, ..., p. \quad (49)$$

Under the standard restrictions $\sigma \geq 0.5$ regularized scheme [11], [18], [19] is stable.

**Theorem 3** Additive difference scheme (7), (41), (48), (49) is unconditionally stable at $\sigma \geq 0.5$, and estimate (8) of stability with respect to the initial data and right-hand side is valid for the difference solution.

We present now some possible implementation of the constructed regularized scheme. Let us introduce auxiliary quantities $y^{n+\alpha/p}$, $\alpha = 1, 2, ..., p$ and taking into account (41), (48), we define them from the equations

$$y^{n+\alpha/p} = \tilde{S}_\alpha y^{n+(\alpha-1)/p}, \quad \alpha = 1, 2, ..., p-1,$$

$$y^{n+1} = \tilde{S}_p y^{n+(p-1)/p} + \tau \varphi_n. \quad (50)$$

Similar to (47) we obtain from (50)

$$\frac{y^{n+\alpha/p} - y^{n+(\alpha-1)/p}}{\tau} + (E + \sigma \tau A_\alpha)^{-1} A_\alpha y^{n+(\alpha-1)/p} = \varphi_{\alpha}^{n}, \quad (51)$$

where

$$\varphi_{\alpha}^{n} = \begin{cases} 0, & \alpha = 1, 2, ..., p-1, \\ \varphi^n, & \alpha = p. \end{cases}$$

Rewrite scheme (51) as follows

$$\frac{y^{n+\alpha/p} - y^{n+(\alpha-1)/p}}{\tau} + A_\alpha (\sigma y^{n+\alpha/p} + (1 - \sigma)y^{n+(\alpha-1)/p}) = \tilde{\varphi}_{\alpha}^{n}, \quad (52)$$
where
\[ \tilde{\varphi}_n^\alpha = (E + \sigma \tau A_\alpha)\varphi_n^\alpha, \quad \alpha = 1, 2, ..., p. \]

Scheme (52) is a special variant of the standard component-wise splitting scheme [10, 13, 34]. But unlike these schemes of summarized approximation we constructed here the regularized schemes of full approximation. Regularized schemes (41), (44), (45), constructed using additive representation (44) for the transition operator, are more suitable for parallel computations in compare with regularized schemes (41), (48), (49) which are based on multiplicative representation (48).

5 Vector schemes of domain decomposition

Difference schemes for unsteady problems can often be treated as appropriate iterative methods for the approximate solution of stationary problems. Great opportunities in this direction provide the vector additive schemes [1, 31].

Instead of the single unknown \( u(t) \) we consider \( p \) unknowns \( u_\alpha, \alpha = 1, 2, ..., p \), which are determined from the system

\[ \frac{du_\alpha}{dt} + \sum_{\beta=1}^{p} A_{\beta} u_\beta = f(t), \quad \alpha = 1, 2, ..., p, \quad 0 < t \leq T. \]  

(53)

The system of equations (53) is supplemented with the initial conditions

\[ u_\alpha(0) = u_0^\alpha, \quad \alpha = 1, 2, ..., p, \]  

(54)

which follow from (2). Obviously, each function is a solution of problem (2), (3), (33). The approximate solution of (2), (3), (33) will be constructed on the basis of one or another difference scheme for vector problem (53), (54).

To solve problem (53), (54), we use the following two-level scheme

\[ \frac{y_{\alpha}^{n+1} - y_{\alpha}^{n}}{\tau} + \sum_{\beta=1}^{\alpha} A_{\beta} y_{\beta}^{n+1} + \sum_{\beta=\alpha+1}^{p} A_{\beta} y_{\beta}^{n} = \varphi^n, \]

\[ \alpha = 1, 2, ..., p, \quad n = 0, 1, ..., N - 1. \]  

(55)

For this difference scheme we use the initial conditions

\[ y_\alpha(0) = u_0^\alpha, \quad \alpha = 1, 2, ..., p. \]  

(56)

Numerical implementation of this scheme is based on the successive inversion of operators \( E + \tau A_\alpha, \alpha = 1, 2, ..., p. \)

**Theorem 4** Vector additive difference scheme (53), (55), (56) is unconditionally stable, and for the components of the difference solution the following stability estimate with respect to the initial data and right-hand side

\[ \|y_{\alpha}^{n+1}\| \leq \|y_{\alpha}^{n}\| + \tau\|\varphi^0 - Au^0\| + \tau \sum_{k=1}^{n} \|\varphi^k - \varphi^{k-1}\|, \]  

(57)
\( \alpha = 1, 2, ..., p, \ n = 0, 1, ..., N - 1, \) \hspace{1cm} (57)

is valid.

**Proof** To study vector scheme (55), (56), it is convenient to use the approach from the work [22]. Subtracting the \( \alpha \)-th equation for \( y_{\alpha+1}^{n} \) from the \( \alpha + 1 \)-th equations of system (55) for \( y_{\alpha+1}^{n+1} \), we get

\[
(E + \tau A_{\alpha+1}) \frac{y_{\alpha+1}^{n+1} - y_{\alpha+1}^{n}}{\tau} = \frac{y_{\alpha+1}^{n} - y_{\alpha}^{n}}{\tau}, \quad \alpha = 1, 2, ..., p - 1. \hspace{1cm} (58)
\]

Similarly, considering the equations for \( y_{1}^{n+1} \) and \( y_{p}^{n} \), we obtain

\[
(E + \tau A_{1}) \frac{y_{1}^{n+1} - y_{1}^{n}}{\tau} = \frac{y_{p}^{n} - y_{p-1}^{n}}{\tau} + \frac{\varphi^{n} - \varphi^{n-1}}{\tau}. \hspace{1cm} (59)
\]

Taking into account nonnegativity of operators \( A_{\alpha} \), \( \alpha = 1, 2, ..., p \), from (58) we obtain

\[
\| \frac{y_{\alpha+1}^{n+1} - y_{\alpha+1}^{n}}{\tau} \| \leq \| \frac{y_{\alpha+1}^{n} - y_{\alpha}^{n}}{\tau} \|, \quad \alpha = 1, 2, ..., p - 1. \hspace{1cm} (60)
\]

Similarly, from (59) we have

\[
\| \frac{y_{1}^{n+1} - y_{1}^{n}}{\tau} \| \leq \| \frac{y_{p}^{n} - y_{p-1}^{n}}{\tau} \| + \left\| \frac{\varphi^{n} - \varphi^{n-1}}{\tau} \right\|. \hspace{1cm} (61)
\]

From (60), (61), we derive at each time level the following estimate

\[
\| \frac{y_{\alpha+1}^{n+1} - y_{\alpha}^{n}}{\tau} \| \leq \left\| \frac{y_{\alpha+1}^{n} - y_{\alpha}^{n}}{\tau} \right\| + \left\| \frac{\varphi^{n} - \varphi^{n-1}}{\tau} \right\|, \quad \alpha = 1, 2, ..., p, \quad n = 1, 2, ..., N - 1. \hspace{1cm} (62)
\]

From (62) we get

\[
\| \frac{y_{\alpha+1}^{n+1} - y_{\alpha}^{n}}{\tau} \| \leq \left\| \frac{y_{\alpha+1}^{n} - y_{\alpha}^{n}}{\tau} \right\| + \sum_{k=1}^{n} \frac{\varphi^{k} - \varphi^{k-1}}{\tau}, \quad \alpha = 1, 2, ..., p, \quad n = 1, 2, ..., N - 1. \hspace{1cm} (63)
\]

From (55) with \( \alpha = 1 \), taking into account splitting (33) and initial conditions (56), we obtain

\[
\| \frac{y_{1}^{1} - y_{1}^{0}}{\tau} \| \leq \| \varphi^{0} - Au^{0} \|. \hspace{1cm} (64)
\]

In view of (60) we can rewrite inequality (63) as follows

\[
\| \frac{y_{\alpha+1}^{n+1} - y_{\alpha}^{n}}{\tau} \| \leq \| \varphi^{0} - Au^{0} \| + \sum_{k=1}^{n} \frac{\varphi^{k} - \varphi^{k-1}}{\tau}, \hspace{1cm} (65)
\]
\[ \alpha = 1, 2, \ldots, p, \quad n = 1, 2, \ldots, N - 1. \]  

Taking into account the obvious inequality

\[ \|y_n^{n+1}\| \leq \|y_n^n\| + \tau \left\| \frac{y_n^{n+1} - y_n^n}{\tau} \right\|, \quad \alpha = 1, 2, \ldots, p, \]

we obtain from (64) required estimate (57).

We emphasize that the above stability estimates (57) are received for each individual component \( y_n^{n+1}, \alpha = 1, 2, \ldots, p \). Each of them or their linear combination

\[ y_n^{n+1} = \sum_{\alpha=1}^{p} c_{\alpha} y_n^{n+1}, \quad c_{\alpha} = \text{const} \geq 0, \quad \alpha = 1, 2, \ldots, p \]

can be treated as an approximate solution of our problem (2), (3), (33) at time moment \( t = t_n^{n+1} \).

6 Model problem

To illustrate possibilities of the constructed here domain decomposition schemes, let us consider the simplest boundary value problem for the parabolic equation. We consider the problem in a rectangle

\[ \Omega = \{ \mathbf{x} \mid \mathbf{x} = (x_1, x_2), \ 0 < x_\alpha < l_\alpha, \ \alpha = 1, 2 \}. \]

In \( \Omega \) the following boundary value problem

\[ \frac{\partial u}{\partial t} = \sum_{\alpha=1}^{2} \frac{\partial^2 u}{\partial x_\alpha^2}, \quad \mathbf{x} \in \Omega, \quad 0 < t < T, \]  

\[ u(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial \Omega, \quad 0 < t < T, \]  

\[ u(\mathbf{x}, 0) = u^0(\mathbf{x}), \quad \mathbf{x} \in \Omega \]  

is solved.

The approximate solution is searched at the nodes of a uniform rectangular grid in \( \Omega \):

\[ \bar{\omega} = \{ \mathbf{x} \mid \mathbf{x} = (x_1, x_2), \ 0 < x_\alpha < l_\alpha, \ i_\alpha = 0, 1, \ldots, N_\alpha, \ N_\alpha h_\alpha = l_\alpha \} \]

and let \( \omega \) be the set of internal nodes (\( \bar{\omega} = \omega \cup \partial \omega \)). For grid functions \( y(\mathbf{x}) = 0, \ \mathbf{x} \in \partial \omega \) we define the Hilbert space \( H = L_2(\omega) \) with the scalar product and norm

\[ (y, w) \equiv \sum_{\mathbf{x} \in \omega} y(\mathbf{x})w(\mathbf{x})h_1h_2, \quad ||y|| \equiv (y, y)^{1/2}. \]
Approximating problem (65), (66) in space, we obtain the differential-difference equation

\[
\frac{dy}{dt} + Ay = 0, \quad x \in \omega, \quad 0 < t < T,
\]  

(68)

where

\[
Ay = -\frac{1}{h_1^2}(y(x_1 + h_1, x_2) - 2y(x_1, x_2) - y(x_1 - h_1, x_2))
\]

\[
- \frac{1}{h_2^2}(y(x_1, x_2 + h_2) - 2y(x_1, x_2) - y(x_1, x_2 - h_2)), \quad x \in \omega.
\]  

(69)

In the space \(H\) the operator \(A\) is self-adjoint and positive definite [13,15]:

\[
A = A^* \geq (\delta_1 + \delta_2)E, \quad \delta_\alpha = \frac{4}{h_\alpha^2} \sin^2 \frac{\pi h_\alpha}{2l_\alpha}, \quad \alpha = 1, 2.
\]  

(70)

Taking into account (67), we supplement equation (69) with the initial condition

\[
y(x, 0) = u^0(x), \quad x \in \omega.
\]  

(71)

For simplicity, operators of domain decomposition in the investigated problem (68)–(71) will be constructed without the explicit definition of operators \(G\) and \(\tilde{G}\) as well as space \(\tilde{H}\), focusing on decomposition (21), (22). We set

\[
A_\alpha y = -\frac{1}{h_1^2}\eta_\alpha(x_1 + 0.5h_1, x_2)(y(x_1 + h_1, x_2) - y(x_1, x_2))
\]

\[
+ \frac{1}{h_1^2}\eta_\alpha(x_1 - 0.5h_1, x_2)(y(x_1, x_2) - y(x_1 - h_1, x_2))
\]

\[
- \frac{1}{h_2^2}\eta_\alpha(x_1, x_2 + 0.5h_2)(y(x_1, x_2 + h_2) - y(x_1, x_2))
\]

\[
+ \frac{1}{h_2^2}\eta_\alpha(x_1, x_2 - 0.5h_2)(y(x_1, x_2) - y(x_1, x_2 - h_2)), \quad \alpha = 1, 2, ..., p.
\]  

(72)

In view of (21), (22) we have

\[
A = \sum_{\alpha=1}^{p} A_\alpha, \quad A_\alpha = A_\alpha^*, \quad \alpha = 1, 2, ..., p.
\]  

(73)

Thus, we consider the class of additive schemes (33).

Numerical calculations for problem (65)–(67) are performed in the unit square \((l_1 = l_2 = 1)\) where the solution has the form

\[
u(x, t) = \sin(n_1 \pi x_1) \sin(n_2 \pi x_2) \exp(-\pi^2(n_1^2 + n_2^2)t)
\]  

(74)

for a natural \(n_1\) and \(n_2\). For this solution we set the corresponding initial conditions (67). Decomposition is performed with respect to one of two variables into four subdomains (see Fig. 1) with overlapping.Disconnected subdomains can be considered as some single subdomain and the decomposition in Fig. 1
Fig. 1 Domain decomposition

can be treated as the decomposition into two subdomains described via two functions: \( \eta_\alpha = \eta_\alpha(x_1), \alpha = 1, 2. \)

For problems of type (65)–(67) two cases of domain decomposition are highlighted: decomposition with and without overlapping of subdomains. Methods without overlapping of subdomains are connected with the explicit formulation of certain conditions at the common boundaries. In our case a special problem at the interfaces is not formulated, but for algorithms without overlapping we can derive the corresponding exchange boundary conditions.

For the domain decomposition methods the fundamental issue is exchange of calculation data between different subdomains. Standard explicit schemes can be used. In this case the domain decomposition can be associated with separate subsets of grid nodes: \( \omega_\alpha, \alpha = 1, 2, \) where \( \omega = \omega_1 \cup \omega_2. \) In the case of (65)–(67) (the seven point stencil in space), the transition to the new time level via the explicit scheme for finding the approximate solution on grid \( \omega_\alpha, \alpha = 1, 2 \) is performed using the solution values at nodes adjacent to the interface. We need to transfer the data of size \( \sim \partial \omega_\alpha, \alpha = 1, 2. \) For the approximate solution of problem (68)–(71) we can consider two possibilities with minimum overlapping of subdomains. The first employs the domain decomposition with interfaces at integer nodes — the boundary nodes belong to several subdomains (two in our case of decomposition with respect to one variable). The second possibility is realized when the boundary of subdomains passes through the half-integer nodes of the corresponding variable.

The variant of domain decomposition with boundaries through integer nodes is shown in Fig. 2. Assume that the decomposition is carried out in
Domain decomposition schemes with not self-adjoint operators

spatial variable $x_1$, i.e. $\theta = x_1$. The boundary of subdomains here passes through the node $\theta_i$. Thus, for this decomposition operators (72) take the form

$$A_1 y = \frac{1}{h_1}(y(x_1, x_2) - y(x_1 - h_1, x_2))$$

$$- \frac{1}{2h_2}(y(x_1, x_2 + h_2) - 2y(x_1, x_2) - y(x_1, x_2 - h_2)),$$

$$A_2 y = -\frac{1}{h_1}(y(x_1 + h_1, x_2) - y(x_1, x_2))$$

$$- \frac{1}{2h_2}(y(x_1, x_2 + h_2) - 2y(x_1, x_2) - y(x_1, x_2 - h_2)), \quad x_1 = \theta_i.$$

This decomposition can be associated with using the Neumann boundary conditions as the exchange boundary conditions. The relationship between the individual subdomains is minimal and requires exchange of data at $\theta = \theta_i$. This case can be identified by the operators of decomposition (72) as follows:

$$R(\tilde{\chi}_\alpha) = [0, 1], \quad \alpha = 1, 2, \ldots, p. \quad (75)$$

The values of $\eta_\alpha(x_1 \pm 0.5h_1, x_2), \eta_\alpha(x_1, x_2 \pm 0.5h_1), \alpha = 1, 2$ for (72), (74) is equal to 0 or 1.

The second possibility, which is associated with decomposition through half-integer nodes, is depicted in Fig. 3. In this case instead of (75) we have

$$R(\tilde{\chi}_\alpha) = [0, 1/2, 1], \quad \alpha = 1, 2, \ldots, p. \quad (76)$$
At node \( \theta = \theta_i \) we use the difference approximation with the flux reduced by half. For the decomposition in the variable \( x_1 \) the operators of decomposition \( A_2 \) seem like this

\[
A_1 y = \frac{1}{2h_1^2} (y(x_1, x_2) - y(x_1 - h_1, x_2)) \\
- \frac{1}{4h_2^2} (y(x_1, x_2 + h_2) - 2y(x_1, x_2) - y(x_1, x_2 - h_2)),
\]

\[
A_2 y = -\frac{1}{h_1^2} (y(x_1 + h_1, x_2) - y(x_1, x_2)) + \frac{1}{2h_1^2} (y(x_1, x_2) - y(x_1 - h_1, x_2)) \\
- \frac{3}{4h_2^2} (y(x_1, x_2 + h_2) - 2y(x_1, x_2) - y(x_1, x_2 - h_2)), \quad x_1 = \theta_i.
\]

For calculations in the domain \( \Omega_1 \) (see Fig. 3) we employ adjacent to the interface data from the domain \( \Omega_2 \) — at node \( \theta = \theta_i \). Thus, for this domain decomposition exchanges are minimal and coincide with the exchanges in the explicit scheme.

\[
\begin{array}{c}
\eta_1(\theta) \\
\eta_2(\theta)
\end{array}
\]

\[
\begin{array}{cccc}
\Omega_1 & & & \Omega_2 \\
\theta_{i-1/2} & \theta_i & \theta_{i+1/2} & \theta_{i+1}
\end{array}
\]

**Fig. 4** Decomposition through integer nodes with the width of overlapping 3h

The considered variants of decomposition \( \eta_1, \eta_2 \) correspond to the minimum overlapping of subdomains. At the discrete level the width of overlapping is governed by the mesh size \( (h \text{ and } 2h, \text{ respectively}) \). Similar variants can be constructed for a higher overlapping of subdomains. For the decomposition presented in Fig. 4 we have

\[
R(\tilde{\chi}_\alpha) = [0, 1/3, 2/3, 1], \quad \alpha = 1, 2, ..., p. \quad (77)
\]

Obviously, in this case we have a greater volume of data exchange, but at the same time the transition from one domain to another is much smoother. The latter allows us to expect a higher accuracy of the approximate solution.

Consider now the results of approximate solving problem \( (65) - (67) \), which has the exact solution \( (73) \). Let \( n_1 = 2, n_2 = 1, T = 0.01 \) and the grid is square \( N_1 = N_2 \). Calculations have been performed using regularized schemes with the additive (scheme \( (7), (41), (45), (45) \) and multiplicative (scheme \( (7), (41), (48), (49) \)) perturbation of the transition operator at \( \sigma = 1 \), as well as vector additive scheme \( (33), (55), (56) \). The results are compared with the difference
solution obtained via implicit scheme (1), (6), (7) at $\sigma = 1$. The error of the approximate solution was estimated through value $\varepsilon(t^n) = \|y^n(x) - u(x, t^n)\|$ at a particular time step.

**Fig. 5** Error at $N_1 = N_2 = 32$ and $N = 10$

Considering decomposition (75) (the width of overlapping is $h$) with the space grid $N_1 = N_2 = 32$ and time grid $N = 10$ ($\tau = 0.001$), we can compare the error norms of the difference solution obtained using different schemes (see Fig. 5). Figure 6–8 shows the local error at the final time moment. The error is localized in the area of overlapping and it is much lower for the vector scheme of decomposition in compare with the additive and multiplicative variants of regularized additive schemes.

In contrast to the implicit scheme, the error of the approximate solution grows with increasing the space grid for domain decomposition schemes (Fig. 9). In this case the width of overlapping is reduced by half.

The dependence of results on the width of overlapping is shown in Fig. 10. It is easy to see that decomposition (77) demonstrates the approximate solution of essentially higher accuracy in compare with decomposition (75) (compare Fig. 5 with Fig. 10).

**7 Conclusions**

1. In this paper we have constructed the operators of domain decomposition for solving evolutionary problems. Splitting of the general not self-adjoint nonnegative finite-dimensional operator is performed separately for its self-
2. There are constructed unconditionally stable regularized additive schemes for the Cauchy problem for evolutionary equations of first order based on splitting of the problem operator into the sum of not self-adjoint nonnegative operators. Regularization is based on the principle of regularization for

adjoint and skew-symmetric parts. This preserves the property of nonnegativity for operator terms associated with individual subdomains.

Fig. 6 Error of scheme (1), (2), (28), (39)

Fig. 7 Error of scheme (1), (2), (25), (26)
operator-difference schemes with perturbation of the transition operator for the explicit scheme. Both additive and multiplicative splittings are considered. It was highlighted the relationship of such regularized schemes with the additive schemes of summarized approximation: additively-averaged schemes as well as standard component-wise splitting ones.
3. Vector additive schemes of full approximation are selected among the splitting schemes for evolutionary equations. They are based on the transition to a system of similar problems with a special component-wise organization for searching the approximate solution at the new time level.

4. The numerical solution of the boundary value problem for the parabolic equation in a rectangle was conducted. The calculations allow to compare various schemes of domain decomposition and to show the accuracy dependence of the approximate solution on the width of overlapping. The vector additive scheme of domain decomposition demonstrates the best results in terms of accuracy.

References

1. Abrashin, V.: A variant of the method of variable directions for the solution of multidimensional problems of mathematical-physics. I. Differ. Equations 26(2), 243–250 (1990)
2. Abrashin, V., Vabishchevich, P.: Vector Additive Schemes for Second-Order Evolution Equations. Differential Equations 34(12), 1673–1681 (1998)
3. Cai, X.C.: Additive Schwarz algorithms for parabolic convection-diffusion equations. Numer. Math. 60(1), 41–61 (1991)
4. Cai, X.C.: Multiplicative Schwarz methods for parabolic problems. SIAM J. Sci Comput. 15(3), 587–603 (1994)
5. Dryja, M.: Substructuring methods for parabolic problems. In: R. Glowinski, Y.A. Kuznetsov, G.A. Meurant, J. Périaux, O. Widlund (eds.) Fourth International Symposium on Domain Decomposition Methods for Partial Differential Equations. SIAM, Philadelphia, PA (1991)
6. Kuznetsov, Y.: New algorithms for approximate realization of implicit difference schemes. Sov. J. Numer. Anal. Math. Model. 3(2), 99–114 (1988)
7. Kuznetsov, Y.: Overlapping domain decomposition methods for FE-problems with elliptic singular perturbed operators. Fourth international symposium on domain decomposition methods for partial differential equations, Proc. Symp., Moscow/Russ. 1990, 223-241 (1991) (1991)
8. Laevsky, Y.: Domain decomposition methods for the solution of two-dimensional parabolic equations. In: Variational-difference methods in problems of numerical analysis, 2, pp. 112–128. Comp. Cent. Sib. Branch, USSR Acad. Sci., Novosibirsk (1987). In Russian
9. Lax, P.D.: Linear algebra and its applications. 2nd edition. Pure and Applied Mathematics. A Wiley-Interscience Series of Texts, Monographs &amp; Tracts. New York, NY: Wiley, xvi, 376 p. (2007)
10. Marchuk, G.: Splitting and alternating direction methods. In: P.G. Ciarlet, J.L. Lions (eds.) Handbook of Numerical Analysis, Vol. I, pp. 197–462. North-Holland (1990)
11. Mathew, T.: Domain decomposition methods for the numerical solution of partial differential equations. Lecture Notes in Computational Science and Engineering 61. Berlin: Springer, xiii, 764 p. (2008)
12. Quarteroni, A., Valli, A.: Domain decomposition methods for partial differential equations. Numerical Mathematics and Scientific Computation. Oxford: Clarendon Press, xv, 360 p. (1999)
13. Samarskii, A.: The theory of difference schemes. Pure and Applied Mathematics, Marcel Dekker. 240. New York, NY: Marcel Dekker. 786 p. (2001)
14. Samarskii, A., Matus, P., Vabishchevich, P.: Difference schemes with operator factors. Mathematics and its Applications (Dordrecht). 546. Dordrecht: Kluwer Academic Publishers, x, 384 p. (2002)
15. Samarskii, A., Nikolaev, E.: Numerical methods for grid equations. Birkhäuser (1989)
16. Samarskii, A., Vabishchevich, P.: Numerical methods for grid equations. Birkhäuser (1989)
17. Samarskii, A., Vabishchevich, P.: Vector additive schemes of domain decomposition for parabolic problems. Differ. Equations 31(9), 1522–1528 (1995)
18. Samarskii, A., Vabishchevich, P.: Regularized additive full approximation schemes. Doklady. Mathematics 57(1), 83–86 (1998)
19. Samarskii, A., Vabishchevich, P.: Factorized finite-difference schemes for the domain decomposition in convection-diffusion problems. Differ. Equations 34(7), 972–979 (1997)
20. Samarskii, A., Vabishchevich, P.: Domain decomposition methods for parabolic problems. In: C.H. Lai, P. Bjorstad, M. Gross, O. Widlund (eds.) Eleventh International Conference on Domain Decomposition Methods, pp. 341–347. DDM.org (1999)
21. Samarskii, A., Vabishchevich, P.: Parallel domain decomposition algorithms for time-dependent problems of mathematical physics. In: Advances in Numerical Methods and Applications, pp. 293–299. World Scientific (1994)
22. Samarskii, A.A., Vabishchevich, P.N.: Regularized difference schemes for evolutionary second order equations. Math. Models and Methods in Applied Sciences 2(3), 295–315 (1992)
23. Smith, B.: Domain decomposition. Parallel multilevel methods for elliptic partial differential equations. Cambridge: Cambridge University Press, xii, 224 p. (1996)
24. Toselli, A., Widlund, O.: Domain decomposition methods – algorithms and theory. Springer Series in Computational Mathematics 34. Berlin: Springer, xv, 450 p. (2005)
25. Vabishchevich, P.: Difference schemes with domain decomposition for solving nonstationary problems. U.S.S.R. Comput. Math. Math. Phys. 29(6), 155–160 (1989)
26. Vabishchevich, P.: Regional-additive difference schemes for nonstationary problems of mathematical physics. Mosc. Univ. Comput. Math. Cybern. (3), 69–72 (1989)
27. Vabishchevich, P.: Parallel domain decomposition algorithms for time-dependent problems of mathematical physics. In: Advances in Numerical Methods and Applications, pp. 293–299. World Scientific (1994)
28. Vabishchevich, P.: Regionally additive difference schemes with a stabilizing correction for parabolic problems. Comput. Math. Math. Phys. 34(12), 1573–1581 (1994)
30. Vabishchevich, P.: Finite-difference domain decomposition schemes for nonstationary convection-diffusion problems. Differ. Equations 32(7), 929–933 (1996)
31. Vabishchevich, P.: Vector additive difference schemes for first-order evolutionary equations. Computational mathematics and mathematical physics 36(3), 317–322 (1996)
32. Vabishchevich, P.: Domain decomposition methods with overlapping subdomains for the time-dependent problems of mathematical physics. Comput. Methods Appl. Math. 8(4), 393–405 (2008)
33. Vabishchevich, P., Verakhovskij, V.: Difference schemes for component-wise splitting-decomposition of a domain. Mosc. Univ. Comput. Math. Cybern. 1994(3), 7–11 (1994)
34. Yanenko, N.: The method of fractional steps. The solution of problems of mathematical physics in several variables. Berlin-Heidelberg-New York: Springer Verlag, VIII, 160 p. with 15 fig. (1971)