Quantum Noncanonical Field Theory: Symmetries and Interaction

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The symmetry properties of a proposal to go beyond relativistic quantum field theory based on a modification of the commutation relations of fields are identified. Poincaré invariance in an auxiliary spacetime is found in the Lagrangian version of the path integral formulation. This invariance is contrasted with the idea of Doubly (or Deformed) Special Relativity (DSR). This analysis is then used to go from the free theory of a complex field to an interacting field theory.

INTRODUCTION

The validity of the framework of relativistic quantum field theory (RQFT) at arbitrarily high energies has been put into question in the last ten years by different approaches in the search for a quantum theory of gravity [1]. In this context, the quantum theory of noncanonical fields was proposed in Refs. [2, 3] as an extension of RQFT. The free theory of noncanonical fields is characterized by a modification of the canonical commutation relations when quantizing in a deformed way a classical, relativistic Hamiltonian in the canonical formalism. It is interesting that an explicit quantization of the theory can be made in Fock space [2], leading to a theory of free particles with Lorentz non-invariant dispersion relations.

The idea of noncanonical fields has been implemented up to now at the level of a free field theory. The consistency of such a field theory when including interaction has never been proved. In fact this is an ambitious program that should start by investigating which is the proper criterion one should use to introduce interaction terms in the theory. Since the theory breaks Lorentz invariance, we no longer have special relativity as a symmetry principle helping us to write interaction terms.

However, recently other symmetry principles aside from special relativity have been considered. These symmetry principles preserve an energy scale together with a velocity scale, and are grouped under the name of Doubly (or Deformed) Special Relativity (DSR) [5]. It might be the case that a symmetry principle as or similar to DSR were identified in the theory.

In this paper we explore this problem and try to define a way to go beyond the free level for a theory of noncanonical fields. In order to proceed, we will first identify a symmetry principle replacing usual relativistic invariance in the theory of a free noncanonical scalar field. Let us just remind the main features of this theory (see Ref. [2] for more details). It is defined by the Hamiltonian

$$H = \int d^3x \, \mathcal{H}(x),$$

$$\mathcal{H}(x) = \Pi^\dagger(x)\Pi(x) + \nabla\Phi^\dagger(x)\nabla\Phi(x) + m^2\Phi^\dagger(x)\Phi(x),$$

(1)

together with the commutation relations

$$[\Phi(x), \Phi^\dagger(x')] = \theta \delta^3(x - x'),$$

(2a)
$$[\Pi(x), \Pi^\dagger(x')] = 0,$$

(2b)
$$[\Phi(x), \Pi^\dagger(x')] = i \delta^3(x - x') = [\Phi^\dagger(x), \Pi(x')],$$

(2c)

where $\Phi$ is a complex scalar field and $\Pi$ is the momentum. The previous commutation relations preserve rotational and translational invariance in space, U(1) rigid symmetry in field space, locality, and lead to RQFT at low energies ($\ll 1/\theta$).

The Heisenberg equations are obtained from $\partial_t \Phi(x) = -i[\Phi(x), H]$ and $\partial_t \Pi(x) = -i[\Pi(x), H]$ [6]. One then gets

$$\partial_t^2 \Phi(x) = (\nabla^2 - m^2)\Phi(x) + i[\theta(\nabla^2 - m^2)]\partial_t \Phi(x).$$

(3)

1 What we refer in this paper as noncanonical field has been named as noncommutative field in Refs. [2, 3]. However, the extension of the concept to fermion fields [4] made more suitable the more rigorous name of noncanonical field. This name fits also better with the path integral formulation used in this paper.
Writing $\Phi$ in momentum space

$$\Phi(x, t) = \int \frac{d^3k}{(2\pi)^3} \Phi(k)e^{-ik\cdot x},$$

a solution exists if and only if

$$\frac{k_0^2}{1 + \theta k_0} - k^2 = m^2.$$  \hspace{1cm} (5)

It is then possible to find a representation of the field in a bosonic Fock space:

$$\Phi(x, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k} \cdot \mathbf{x}} \left[ \sqrt{\frac{E_a}{E_b(E_a + E_b)}} a_k e^{-iE_a t} + \sqrt{\frac{E_b}{E_a(E_a + E_b)}} b^\dagger_{-k} e^{iE_b t} \right],$$

where $a_k, a_k^\dagger, b_k, b_k^\dagger$ are two kinds of bosonic annihilation and creation operators $[a_k, a_k^\dagger] = [b_k, b_k^\dagger] = (2\pi)^3\delta^3(k - k')$, and $E_a = E_a(k), E_b = E_b(k)$ are the absolute values of the two solutions for $k_0$ of the quadratic Eq. (5),

$$E_a(k) = \omega_k \left[ \frac{1}{2} \theta \omega_k + \sqrt{1 + \frac{(\theta \omega_k)^2}{4}} \right] = k_0^+, \hspace{1cm} (7)$$

$$E_b(k) = \omega_k \left[ \frac{1}{2} \theta \omega_k + \sqrt{1 + \frac{(\theta \omega_k)^2}{4}} \right] = -k_0^-, \hspace{1cm} (8)$$

with $\omega_k = \sqrt{k^2 + m^2}$. The Hamiltonian and momentum operators can be written (neglecting an infinite constant term) as

$$H = \int \frac{d^3k}{(2\pi)^3} (E_a(k)a_k^\dagger a_k + E_b(k)b_k^\dagger b_k), \hspace{1cm} (9)$$

$$P = \int \frac{d^3k}{(2\pi)^3} k [a_k^\dagger a_k + b_k^\dagger b_k], \hspace{1cm} (10)$$

showing that this is a theory of free particles of two types, with energies $E_a(k)$ and $E_b(k)$, for each momentum $k$.

In the $\theta \to 0$ limit, $E_a(k) = E_b(k) = \omega_k$ corresponds to the particle-antiparticle degeneration of a relativistic theory. The vacuum expectation value of the time ordered product of field operators (propagator) of the free theory is given by

$$\langle 0 | T(\Phi(t, x)\Phi^\dagger(t', x')) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} e^{-i(k_0(t-t') + i k(x-x'))} \frac{i(1 + \theta k_0)}{(k_0 - E_a(k) + i\epsilon)(k_0 + E_b(k) - i\epsilon)}. \hspace{1cm} (11)$$

Note that we are breaking the discrete symmetry $\Phi \leftrightarrow \Phi^\dagger$ with the new symplectic structure. The energies appearing in the Hamiltonian Eq. (9) are different functions of momentum and therefore $C$ transformation is no longer a symmetry of the theory. $P$ and $T$ are still good discrete symmetries, and therefore $CPT$ is broken.

According to Ref. [7], Lorentz symmetry must be broken in the theory with $\theta \neq 0$. In fact, the dispersion relation Eq. (6) is not invariant under conventional Lorentz transformations between inertial observers, though it still satisfies rotational symmetry. In the following section we will try to find out whether some kind of symmetry replacing Lorentz invariance can still be defined for the noncanonical theory, which would be helpful in the construction of an interacting theory.
SYMMETRIES OF THE THEORY OF NONCANONICAL FIELDS

The symmetries of a quantum field theory help us to understand its spectrum. They give information about the relation between the energy and momentum of the excitations over the ground state. In a free field theory, transformations leaving the dispersion relation invariant are transformations which connect solutions of the quantum equation of motion among themselves. We will refer to them as symmetries of the dispersion relation.

In RQFT the dispersion relation is Poincaré invariant. However, the dispersion relation Eq. (5) is not invariant under conventional Lorentz transformations. We start by considering whether a deformed transformation could be defined so as to keep this dispersion relation invariant [8]. A positive answer to this question is the central issue in DSR theories. It is standard in DSR to define auxiliary variables [9, 10]

\[ \kappa_0 = \frac{k_0}{\sqrt{1 + \theta k_0}}; \quad \kappa = k \]  

so that in terms of them the dispersion relation gets the form

\[ \kappa_0^2 - \kappa^2 = m^2. \]  

Note that the previous relation between physical and auxiliary momentum is only possible for \( k_0 > -1/\theta \). But, looking at Eq. (5), we can see that, for all the solutions of the equation motion, \( k_0 > -1/\theta \). Then we will consider transformations on functions of \( k_\mu \) restricted to this domain.

The auxiliary variables satisfy the usual, Lorentz invariant, dispersion relation. We can then define the generators of boosts as the usual expressions

\[ N_i = i(\kappa_0 \partial_{\kappa_i} + \kappa_i \partial_{\kappa_0}), \]  

so that in terms of the original variables

\[ N_i = i \left[ \frac{k_0}{\sqrt{1 + \theta k_0}} \partial_{k_i} + \frac{(1 + \theta k_0)^{3/2}}{1 + \theta k_0/2} k_i \partial_{k_0} \right], \]  

which leaves Eq. (5) invariant by construction.

The algebra of the generators of boosts is not modified with respect to the usual Lorentz algebra, as one can easily check by writing the generators in terms of the auxiliary variables, Eq. (14). This is not the case, however, of the algebra of boost generators with the four-momentum, which is different from the usual Poincaré algebra, and reduces to it in the \( \theta \to 0 \) limit,

\[ [N_i, k_0] = i k_i \frac{(1 + \theta k_0)^{3/2}}{1 + \theta k_0/2}, \quad [N_i, k_j] = i \delta_{ij} \frac{k_0}{\sqrt{1 + \theta k_0}}. \]  

Poincaré algebra has been, therefore, deformed to a new closed, non-linear, algebra.

Our theory, however, is a quantum theory of fields. This means that, once we have identified a principle of relativity for the invariance of the dispersion relation of particles, it is necessary to find the corresponding representation of the symmetry in the Fock space.

Every vector in the Fock space can be written as a sum of tensor products of vectors belonging to the single-particle Hilbert space. In our case we have two kinds of single-particle spaces, that is, those corresponding to particles of types \( a \) and \( b \), respectively. We will define a representation of Lorentz transformations in each of these spaces by saying how they act on a basis of vectors. The basis elements are the states with a given momentum \( k \)

\[ |k\rangle_a = \sqrt{2\omega_k a_k^\dagger |0\rangle}; \quad |k\rangle_b = \sqrt{2\omega_k b_k^\dagger |0\rangle}, \]  

for the corresponding spaces of particles \( a \) and \( b \). We are using the same normalization factors of the relativistic theory; in fact the spatial components of the auxiliary four-momentum variables Eq. (12) transforming linearly under Lorentz transformations are just the components of the momentum \( k \).

A Lorentz transformation \( \Lambda \) is represented in the Fock space by a unitary operator \( U_\Lambda \), according to

\[ |k\rangle_a = U_\Lambda |k\rangle_a; \quad |k\rangle_b = U_\Lambda |k\rangle_b \]  

where $k'$ is the momentum of the state which results of applying a standard Lorentz transformation on a state with momentum $k$ and energy $\omega_k$ in special relativity. Then Lorentz transformations in the Fock space are just those of RQFT

$$U_\Lambda a_k^\dagger U_\Lambda^\dagger = \sqrt{\frac{\omega_{k'}}{\omega_k}} a_{k'}^\dagger,$$

$$U_\Lambda b_k^\dagger U_\Lambda^\dagger = \sqrt{\frac{\omega_{k'}}{\omega_k}} b_{k'}^\dagger.$$  

Any state in the Fock space can be written as a linear combination of states which result from acting with a product of $a^\dagger, b^\dagger$ operators on the vacuum. Then Eqs. (19) and (20) define the transformation of any state in the Fock space. Once we have identified a unitary representation of the Lorentz transformations in the Fock space we can work out the transformations of the Hamiltonian Eq. (9) and the momentum operator Eq. (10).

Under Lorentz transformations,

$$U_\Lambda H U_\Lambda^\dagger = \int \frac{d^3k'}{(2\pi)^3} \left[ E_a(k) a_k^\dagger a_{k'} + E_b(k) b_k^\dagger b_{k'} \right],$$

$$U_\Lambda P U_\Lambda^\dagger = \int \frac{d^3k'}{(2\pi)^3} \kappa \left[ a_k^\dagger a_{k'} + b_k^\dagger b_{k'} \right].$$

There is no way to write the transformed operators as a function of the Hamiltonian and the momentum operator so that there is no closed deformed Poincaré algebra including the Hamiltonian and the momentum as generators. This can be seen from the following argument. The effect of the transformation of the Hamiltonian on the Fock space is to replace the coefficients $E_a(k')$, $E_b(k')$ of $a_k^\dagger a_{k'}$, $b_k^\dagger b_{k'}$ by $E_a(k)$, $E_b(k)$. Similarly the effect on the momentum operator is to replace the coefficient $k'$ of $a_k^\dagger a_{k'}$, $b_k^\dagger b_{k'}$ by $k$. Although there is a nonlinear relation between the coefficients in the original and transformed operators, the linear dependence on $a^\dagger a$, $b^\dagger b$ prevents us to identify a nonlinear closed algebra.

However, if we project on the one particle sector the transformation laws of the Hamiltonian and the momentum operator we recover the nonlinear deformation of the Poincaré algebra Eq. (10). The energy-momentum relations for the particles of the free noncanonical theory can then be understood as a consequence of nonlinear deformations of the Poincaré algebra depending on the noncommutativity length scale $\theta$.

The role of the ultraviolet (UV) parameter $\theta^{-1}$, as well as the deformation of relativistic invariance, can be compared with the typical situation in DSR theories. Eq. (5) shows that when $|k| \to \infty$, then $E_b \to \theta^{-1}$. This and the fact that $E_b(k)$ is a monotonically increasing function means that $\theta^{-1}$ is the supremum of the energy of particle b. Since Eq. (6) is invariant under the deformed boosts generated by $N_i$ in Eq. (15), every inertial observer will agree on this supreme value for $E_b$.

This fact gives a direct physical meaning to the parameter $\theta$ in the theory of noncanonical fields, in analogy to the similar role played by the Planck mass, $M_P$, in DSR theories. However, there is not an observer-independent energy scale associated with particle a. Therefore, we have a DSR 3 (i.e, an observer-independent energy cutoff) realization in the one-particle $b'$ sector and a smoothly modified special relativity (i.e. without any cutoff in energy or momentum) in the one-particle $a'$ sector [11]. The same conclusions can be obtained from the expression for the auxiliary variables. One has a one to one mapping Eq. (12) between $-\infty < \kappa_0 < \infty$ and $-1/\theta < k_0 < \infty$.

There is no extension of the deformed Poincaré symmetry of the single-particle sector to the multiparticle sectors of the Fock space. In this sense we can say that although we can find a parallelism between the introduction of an UV scale through noncommutativity of field operators and the attempts to make the relativity principle compatible with a new invariant energy scale (DSR) in the one particle sector, the quantum theory of noncanonical fields is not a quantum field theory realization of DSR.

Another way to arrive to the same conclusion is by contrasting the trivial additive composition law of energy and momentum in the free theory of noncanonical fields with the necessity for considering a nontrivial composition law in DSR, owing to the nonlinear representation in momentum space of Lorentz transformations.

PATH INTEGRAL FORMULATION OF THE FREE THEORY

In order to apply our understanding from the point of view of symmetries of the free noncanonical field theory to the introduction of interactions it is convenient to use the path integral formulation of field theory. In this section
we translate the standard derivation \[12\] of the path integral approach of a quantum theory to the free noncanonical field theory, which reduces to finding a representation of the scalar propagator

\[
\langle 0 | T \left( \Phi(x_A) \Phi^\dagger(x_B) \right) | 0 \rangle
\]  

(23)
as a path integral.

First we introduce the basis of eigenstates \(|\pi(x); t\rangle\)

\[
\Pi(t, x)|\pi(x); t\rangle = \pi(x)|\pi(x); t\rangle
\]

(24)
where \(\Pi(t, x)\) are the Heisenberg-picture operators

\[
\Pi(t, x) = e^{iHt} \Pi(x) e^{-iHt}.
\]

(25)
This allows to express the scalar propagator as

\[
\int \left[ \prod_x d\pi'(x) \right] \left[ \prod_x d\pi(x) \right] \langle 0 | \pi'(x); t_f \rangle \langle \pi'(x); t_f | T \left( \Phi(x_A) \Phi^\dagger(x_B) \right) | \pi(x); t_i \rangle \langle \pi(x); t_i | 0 \rangle.
\]

(26)
Owing to the noncommutativity of fields there are no eigenstates of the complex scalar field operator. But it is possible to introduce the linear combinations

\[
\Phi_c(t, x) = \Phi(t, x) - \frac{i\theta}{2} \Pi(t, x)
\]

(27)
and the basis of eigenstates \(|\phi_c(x); t\rangle\)

\[
\Phi_c(t, x)|\phi_c(x); t\rangle = \phi_c(x)|\phi_c(x); t\rangle.
\]

(28)
Note that \(\Phi_c\) and \(\Pi\) form a canonical conjugate pair of variables.

Then one can repeat step by step the standard derivation of the path integral formulation \[12\]. First one can use the completeness conditions

\[
\int \left[ \prod_x d\pi(x) \right] |\pi(x); t\rangle \langle \pi(x); t| = 1
\]

(29)
\[
\int \left[ \prod_x d\phi_c(x) \right] |\phi_c(x); t\rangle \langle \phi_c(x); t| = 1
\]

(30)
for all \(t\) with \(t_i < t < t_f\) to write the matrix elements of the time ordered product of two field operators between eigenstates of \(\Pi(t_i, x)\) and \(\Pi(t_f, x)\) as a path integral\[2\]

\[
\langle \pi'(x); t_f | T \left( \Phi(x_A) \Phi^\dagger(x_B) \right) | \pi(x); t_i \rangle \propto \int_{\pi(t_i, x)=\pi(x)}^{\pi(t_f, x)=\pi(x)} \left[ \prod_x d\pi(t, x) \right] \left[ \prod_x d\phi(t, x) \right] \phi(x_A) \phi^*(x_B) \exp \left[ -i \int_{t_i}^{t_f} dt \left\{ \int d^3x \left( \phi(t, x) \phi(t, x) \frac{i\theta}{2} - \frac{i\theta}{2} \right) + c.c. \right. \right]
\]

(31)
where we have used the new integration variable \(\phi(x) \equiv \phi_c(x) + i\frac{\theta}{2} \pi(x)\) instead of \(\phi_c(x)\) \[13\].

Next one can calculate, from the Fock space representation of the free noncanonical theory discussed in the Introduction section, the projection of the vacuum on the eigenstates \(|\pi(x)\rangle\)

\[
\langle \pi(x)|0\rangle = N \exp \left( -\frac{1}{2} \int d^3y d^3z \Delta(y, z) \pi^*(y) \pi(z) \right),
\]

(32)
\[2\] The arbitrariness in the normalization of the integration measure leaves an undetermined proportionality factor in all equations involving a path integral.
where $N$ is a normalization constant and
\[
\Delta(y, z) = \int \frac{d^3k}{(2\pi)^3} \left( \frac{E_a(k) + E_b(k)}{\omega_k^2} \right) e^{ik(y-z)}.
\]

Using the relation
\[
f(\infty) + f(-\infty) = \lim_{\epsilon \to 0^+} \epsilon \int_{-\infty}^{\infty} dt f(t) e^{-\epsilon|t|}
\]

one has
\[
\lim_{t_j \to -\infty} \langle 0| \pi'(x); t_j \rangle \langle \pi(x); t_i | 0 \rangle = |N|^2 \exp \left( -\frac{1}{2} \epsilon \int d^3y d^3z \int_{-\infty}^{\infty} dt \Delta(y, z) \pi^*(t, y) \pi(t, z) e^{-\epsilon|t|} \right).
\]

The final result for the propagator is
\[
\langle 0| T (\Phi(x_A)\Phi^+(x_B)) | 0 \rangle \propto \int \prod_{t, x} \frac{d\pi(t, x)}{2\pi} \prod_{t, x} d\phi(t, x) \phi(x_A)\phi^*(x_B) \
\quad \exp \left[ -i \int_{-\infty}^{\infty} dt \left\{ \int d^3x \left( \phi(t, x) \partial_t \pi^*(t, x) - \frac{i\theta}{2} \pi(t, x) \partial_t \pi^*(t, x) + c.c. \right) + H[\pi(t), \phi(t)] \right\} \right.
\quad \left. -\frac{1}{2} \epsilon \int d^3y d^3z \Delta(y, z) \pi^*(t, y) \pi(t, z) e^{-\epsilon|t|} \right].
\]

In order to get the Lagrangian\(^3\) version of the path integral formulation one has to integrate over $\pi$. Since the argument of the exponential is quadratic in $\pi$ the integral will be proportional to the integrand evaluated at the stationary point of its argument. Then one has
\[
\langle 0| T (\Phi(x_A)\Phi^+(x_B)) | 0 \rangle \propto \int \prod_{t, x} \frac{d\phi(t, x)}{2\pi} \phi(x_A)\phi^*(x_B) e^{iS[\phi]}
\]

with\(^4\)
\[
S[\phi] = \int dt d^3x \phi^*(t, x) \left[ -\frac{\partial^2}{1 + i\theta \partial_t - i\epsilon} + \nabla^2 - m^2 \right] \phi(t, x).
\]

Because we are expressing the action as a functional of noncanonical variables, inverses of differential operators appear in the action. A more explicit way of writing the action is
\[
S[\phi] = \int dt dt' d^3x \phi^*(t, x) \left[ -D(t-t')\partial_t^2 + \delta(t-t')(\nabla^2 - m^2) \right] \phi(t', x),
\]

where
\[
D(t-t') = \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{(1 + \theta\omega - i\epsilon)};
\]

in this expression, the invariance under time translations is manifest. The invariance under time translations is in agreement with the conservation of energy that can be deduced from the explicit time independence of the Hamiltonian $H$. At this point we can introduce the generating functional of Green functions
\[
Z[j] \propto \int \prod_{t, x} \frac{d\phi(t, x)}{2\pi} e^{iS[\phi] + i \int d^4x (j^*(x)\phi(x) + j(x)\phi^*(x))},
\]

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\(^3\) Strictly speaking we should refer to this as a generalized Lagrangian formulation since we are not using canonical field variables.

\(^4\) We have assumed field configurations such that total derivatives in the generalized Lagrangian density can be ignored.
where once more the argument in the exponential is quadratic in $\phi$. Then $Z[j]$ is an exponential with an argument quadratic in $j$. From this result one can calculate the two-point Green function from the generating functional $Z[j]$

$$G^{(2)}(t_A, x_A; t_B, x_B) = -\frac{1}{Z[j]} \left. \frac{\delta^2 Z[j]}{\delta j^*(t_A, x_A) \delta j(t_B, x_B)} \right|_{j=0}$$

$$= \int \frac{d^3k}{(2\pi)^4} e^{-ik_0(t_A-t_B)+ik(x_A-x_B)} \frac{i(1+\theta k_0)}{(k_0 - E_a(k) + i\epsilon)(k_0 + E_b(k) - i\epsilon)}$$

and one can check that it coincides with the result for the vacuum expectation value of the time ordered product of field operators in the operator formalism Eq. (11).

The generating functional $Z[j]$ in Eq. (41) provides a formalism for the free noncanonical theory as an integral in a complex field configuration space.

### Map to the relativistic free field theory

The effect of the noncommutativity of fields in the path integral formulation can be seen through the $\theta$-dependence of the action Eq. (38). If we introduce

$$\phi(k_0, x) = \int dt e^{ik_0t} \phi(t, x)$$

then the action can be written as

$$S[\phi] = \int \frac{dk_0}{2\pi} \int d^3x \phi^*(k_0, x) \left[ \frac{k_0^2}{1 + \theta k_0 - i\epsilon} + \nabla^2 - m^2 \right] \phi(k_0, x).$$

At this point it is convenient to split the integration on $k_0$ into two pieces

$$S[\phi] = S^\theta[\phi^\theta] + \bar{S}^\theta[\bar{\phi}^\theta]$$

with

$$\phi^\theta(t, x) = \int_{-1/\theta}^{1/\theta} dk_0 e^{-ik_0t} \phi(k_0, x),$$

$$\bar{\phi}^\theta(t, x) = \int_{-\infty}^{-1/\theta} \frac{dk_0}{2\pi} e^{-ik_0t} \phi(k_0, x).$$

In the first piece ($k_0 > -1/\theta$) one can make a change of variables

$$\kappa_0 = \frac{k_0}{\sqrt{1 + \theta k_0}}$$

and introduce a new field variable

$$\phi_r(\kappa_0, x) = \sqrt{\frac{d\kappa_0}{dk_0}} \phi(k_0, x),$$

$$\phi_r(\tau, x) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} e^{-i\kappa_0 \tau} \phi_r(\kappa_0, x),$$

which leads to

$$S^\theta[\phi^\theta] = S_r[\phi_r],$$

where $S_r$ is the action of the relativistic theory of the free complex scalar field $\phi_r$

$$S_r[\phi_r] = \int_{-\infty}^{\infty} d\tau d^3x \phi_r^*(\tau, x) \left[ -\partial^2_r + \nabla^2 - m^2 + i\epsilon \right] \phi_r(\tau, x).$$
The relation Eq. (51) is the translation to the path integral formulation of the possibility to find a nonlinear change of variables which brings the dispersion relation of the noncanonical theory into the relativistic dispersion relation. This nonlinear change of variables was the starting point to identify a deformed Poincaré invariance in the one particle sectors including the Hamiltonian as one of the generators.

Alternatively, it is possible to identify a Poincaré group of symmetries of the action (i.e.: a group of transformations which leave the action invariant). To see this one has to consider a transformation on the complex field leaving invariant the component $\phi ^\theta$ and translating the Poincaré transformation of the relativistic complex scalar field $\phi _r$ to the related component $\bar{\phi }^\theta$. The action $S[\phi ]$ is obviously invariant under such transformations because it is a sum of a contribution $S^\theta [\phi ^\theta]$, which is invariant due to the relation Eq. (51) and the Poincaré invariance of the relativistic action, and a contribution $S^\theta [\bar{\phi }^\theta]$ depending only on the field component which is invariant under the transformation. One of the generators of these transformations corresponds to the translation in the time argument $\tau$ of the variable $\phi _r$, which has nothing to do with a translation in the time argument $t$ of the field $\phi$ generated by the Hamiltonian of the free noncanonical field theory. The effect of the group of symmetries of the action on the variable $\phi _r$ is

$$\phi _r ^\prime (\tau ^\prime , x ^\prime) = \phi _r (\tau , x) ,$$

where $(\tau ^\prime , x ^\prime)$ is the ordinary Poincaré transformation of the Minkowskian four vector $(\tau , x)$. Therefore the spacetime realization of the identified Poincaré group of symmetries of the action requires the use of the auxiliary spacetime formed by the elements $\{ (\tau , x) \}$. We are identifying a Poincaré invariance of the path integral formulation by considering a non-conventional representation of Poincaré transformations on the space of configurations of the noncanonical field variables. It may appear bizarre to identify a Poincaré symmetry which is not a symmetry of the physical spacetime. In order to interpret this symmetry as a spacetime symmetry an auxiliary time variable $\tau$ must be introduced. This group of symmetries also leaves the equal-time commutation relations [2] invariant. If these transformations could be interpreted as changes of reference frame, then the commutation relations would not select a preferred reference frame. This mathematical trick will prove to be useful in order to study the renormalization of the theory.

At the level of the generating functional of Green functions $Z[j]$ one can introduce a similar decomposition of the sources

$$j(t, x) = j^\theta (t, x) + \bar{j}^\theta (t, x)$$

with

$$\bar{j}^\theta (t, x) = \int _{-\infty} ^{-1/\theta} \frac{dk_0}{2\pi} e^{-ika_t} j(k_0, x) ,$$

$$j^\theta (t, x) = \int _{-1/\theta} ^{\infty} \frac{dk_0}{2\pi} e^{-ika_t} j(k_0, x) .$$

One gets

$$Z[j] = Z_r[j_r] Z^\theta [\bar{j}^\theta] ,$$

where $Z_r[j_r]$ is the generating functional of Green functions of the relativistic free theory of a complex field with

$$j_r(\tau , x) = \int dt K^\theta (\tau , t) j^\theta (t, x)$$

and

$$K^\theta (\tau , t) = \int _{-\infty} ^{\infty} \frac{dk_0}{2\pi} e^{i(k_0 - \tau \kappa_0)} \sqrt{\frac{d\kappa_0}{dk_0}} ,$$

with

$$k_0 = \kappa_0 \left[ \sqrt{1 + \left( \frac{\theta \kappa_0}{2} \right)^2} + \frac{\theta \kappa_0}{2} \right] .$$

Eq. (57) gives the explicit relation between the physical time $t$ and the auxiliary time $\tau$ through the correspondence Eq. (58) between functions defined in the physical spacetime and functions defined in the auxiliary spacetime.
Then one has

$$\left(-i\right) \frac{\delta Z[j]}{\delta j(t,x)} = Z^0[j^0] \int dt' \int_{-1/\theta}^{\infty} \frac{dk_0}{2\pi} e^{-ik_0(t-t')} \int dr K_\theta(\tau, t') \left(-i\right) \frac{\delta Z[r]}{\delta j_r(\tau, x)} + Z_r[j_r] \int dt' \int_{-\infty}^{-1/\theta} \frac{dk_0}{2\pi} e^{-ik_0(t-t')} \left(-i\right) \frac{\delta Z^0[j^0]}{\delta j^0(t', x)} \int_{-1/\theta}^{\infty} \frac{dk_0}{2\pi} e^{-ik_0(t-t')} \left(-i\right) \frac{\delta Z[j]}{\delta j(t, x)}$$

(61)

The splitting of the generating functional in Eq. (57) just corresponds at the level of the separation of the contributions of modes with $k_0 > -1/\theta$ and those with $k_0 < -1/\theta$.

$$G^{(2)}(t_A, x_A; t_B, x_B) = \int dt'_A dt'_B \int_{-1/\theta}^{\infty} \frac{dk_{AB}}{2\pi} e^{ik_{AB}(t_A-t'_A)} e^{-ik_{AB}(t_B-t'_B)} \int dr A dr B K^\theta(\tau, t'_A) K(\tau, t'_B) \int dt A \int dt B \delta Z^0[j^0] \int_{-1/\theta}^{\infty} \frac{dk_{AB}}{2\pi} e^{ik_{AB}(t_A-t'_A)} e^{-ik_{AB}(t_B-t'_B)} \tilde{G}^\theta(t'_A, x_A; t'_B, x_B)$$

where

$$\tilde{G}^\theta(t_A, x_A; t_B, x_B) = -\left[\frac{\delta Z^0[j]}{\delta j^0(t, x)} \right]_{j^0=0} \int_{-1/\theta}^{\infty} \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{-ik_0(t_A-t_B)+ik(x_A-x_B)} \frac{i(1+\theta k_0)}{(k_0 - E_a(k))(k_0 + E_b(k))}$$

(63)

The expression for the propagator Eq. (62) is just the result of considering a second derivative of the generating functional $Z[j]$ with respect to the sources $j(t, x)$ together with the decomposition of the source as a sum of two independent components $j^\theta, j^0$. Note that Eq. (63) is just the free propagator Eq. (62) restricted to modes with $k_0 < -1/\theta$ where the $i\epsilon$ factors can be omitted owing to the factor $1 + \theta k_0$ in the numerator.

INTERACTION

In order to go beyond the free theory of a complex noncanonical field including interactions, a first attempt could be based on the addition of a term proportional to $[\Phi(x)\Phi(x)]^2$ in the Hamiltonian density Eq. (11). If one goes to the path integral formulation following the same steps of the free theory one finds that the relation with the relativistic theory found at the level of the free theory is lost. A consistent generalization of the standard perturbative analysis of the relativistic theory is problematic. The Poincaré invariance of the free theory is lost and, lacking a characterization in terms of symmetries of the interacting theory, renormalizability will also be lost.

In a second attempt to go beyond the free theory one could take the Lagrangian version of the path integral formulation of the free noncanonical free theory Eq. (18) as a starting point, and then add directly at this level a term proportional to $(\phi^*\phi)^2$. In this case too one looses the relation with the relativistic theory found at the level of the free theory. This is due to the fact that a decomposition of the field into modes with $k_0 > -1/\theta$ and $k_0 < -1/\theta$ does not lead in this case to a splitting of the action (and the generating functional) because the non-quadratic terms couple the two types of modes. Then the Poincaré invariance of the free theory is lost.

In order to consider a theory with interactions maintaining the Poincaré invariance of the free theory it is necessary to keep the splitting of the action Eq. (15) also in the interacting theory. The Poincaré invariance restricts the possible nonquadratic terms in the first contribution to the action through Eq. (15), but not in the second one. A simple way to achieve a perturbatively renormalizable interacting theory is to introduce the interaction only through a term proportional to $(\phi^*\phi)^2$ in $S_r$. One has then a splitting at the level of the generating functional of Green functions, a separation of modes with $k_0 > -1/\theta$ and $k_0 < -1/\theta$, and a simple relation of the Green functions of the noncanonical field theory and those of the relativistic theory. Then the multiplicative renormalizability of the relativistic field theory can be translated to the noncanonical theory.

In fact the relation between the propagator of the noncanonical field theory and the relativistic propagator Eq. (92) of the free theory applies also in the interacting theory. All the contributions from modes with $k_0 < -1/\theta$ can be expressed in terms of the function $\tilde{G}^\theta$. The contribution from modes with $k_0 > -1/\theta$ to a given $n$-point Green function of the quantum noncanonical field theory, $G^{(n)}$, can be expressed in terms of the $m$-point Green functions $G^{(m)}_r$ of
the relativistic theory with interaction (with $m \leq n$). In the case of the four-point Green function one has

$$G^{(4)}(t_A, x_A; t_B, x_B; t_C, x_C; t_D, x_D) =$$

$$\int \frac{dt_A'}{1/\theta} \frac{dt_B'}{1/\theta} \frac{dt_C'}{1/\theta} \frac{dt_D'}{1/\theta} \int_{-\infty}^{\infty} dk_{A0}dk_{B0}dk_{C0}dk_{D0} e^{ik_{A0}(t_A-t_A')}e^{ik_{B0}(t_B-t_B')}e^{-ik_{C0}(t_C-t_C')}e^{-ik_{D0}(t_D-t_D')}$$

$$\int d\tau_B d\tau_C d\tau_D K^\theta_{\tau_B}(t_B, t_B') K^\theta_{\tau_C}(t_C, t_C') K^\theta_{\tau_D}(t_D, t_D') G^{(4)}_{\tau=1}(t_A, x_A; t_B, x_B; t_C, x_C; t_D, x_D)$$

$$+ \int \frac{dt_A'}{1/\theta} \frac{dt_B'}{1/\theta} \frac{dt_C'}{1/\theta} \frac{dt_D'}{1/\theta} \int_{-\infty}^{\infty} dk_{A0}dk_{C0} \int_{-\infty}^{\infty} dk_{B0}dk_{D0} e^{ik_{A0}(t_A-t_A')}e^{ik_{B0}(t_B-t_B')}e^{-ik_{C0}(t_C-t_C')}e^{-ik_{D0}(t_D-t_D')}$$

$$\int d\tau_A d\tau_C K^\theta_{\tau_A}(t_A, t_A') K^\theta_{\tau_C}(t_C, t_C') G^{(4)}_{\tau=1}(t_B, x_B; t_C, x_C; t_D, x_D) + 3 \text{ permutations}$$

$$+ \int \frac{dt_A'}{1/\theta} \frac{dt_B'}{1/\theta} \frac{dt_C'}{1/\theta} \frac{dt_D'}{1/\theta} \int_{-\infty}^{\infty} dk_{A0}dk_{B0}dk_{C0}dk_{D0} e^{ik_{A0}(t_A-t_A')}e^{ik_{B0}(t_B-t_B')}e^{-ik_{C0}(t_C-t_C')}e^{-ik_{D0}(t_D-t_D')}$$

$$[\tilde G^\theta(t_A', x_B; t_C', x_D)\tilde G^\theta(t_B', x_B; t_D', x_D) + \text{permutation}] . \quad (64)$$

Noticeably for the fully connected Green functions one gets a simpler expression with just the first term on the right hand side of Eq. (64), and modes with $k_0 < -1/\theta$ are irrelevant.

All these relations are a consequence of the correspondence between actions Eqs. 51 and the fact that $\tilde G^\theta$ is an exponential with an argument which is quadratic in the sources (we do not consider non-quadratic terms in the field components with $k_0 < -1/\theta$). The inclusion of interaction terms compatible with renormalizability in the $\tilde S^\theta$ contribution to the action would require a generalization of the standard arguments (power counting, symmetries) to this contribution.

Since we have introduced the interaction directly at the level of the Lagrangian version of the path integral formulation there is no clear way to identify the Hamiltonian of the interacting theory and there is no clear physical interpretation of the theory defined by the Green functions $G^{(n)}(x_1, x_2, ..., x_n)$. In particular it is not possible to relate these objects to expectation values of time ordered products of field operators in the interacting theory. Therefore, it is not clear how to link the Green functions to the amplitude of scattering processes.

**SUMMARY AND DISCUSSION**

The algebraic properties of the solution of the free theory of a noncanonical complex field have been identified. It is a theory of free particles of two types each with a different energy-momentum relation. These relations can be seen as a consequence of nonlinear deformations of the Poincaré algebra in the one particle sectors of the theory.

The derivation of the path integral formulation of quantum field theory can be translated to the case of a free noncanonical field theory and the generalized Lagrangian version of the path integral formulation can be identified. A partial mapping to the relativistic theory makes manifest that the main effect of the noncommutativity is a modification of the time variable identified as the argument of the field variable in the path integral formulation. We have identified a Poincaré invariance of the theory in this formulation. Translations in the (physical) time variable $t$ are symmetry transformations of the free theory which are not included in the Poincaré group of symmetries associated to the mapping to the relativistic theory.

The requirement to keep this mapping with the relativistic theory (and the associated Poincaré invariance) to a theory with interaction leads to a nontrivial way of introducing interactions in the theory of a noncanonical complex field. A general expression for the Green functions in terms of their relativistic counterparts is given, which defines a deformation of the interacting relativistic theory of a complex field parametrized by the noncommutativity length scale $\theta$.

The non-quadratic terms in the action involve products of fields at different times (nonlocal interactions). It is not clear whether it is possible to find an appropriate set of additional variables allowing to prove the equivalence of the path integral formulation with a canonical formalism at the level of the interacting theory. For this reason we can not say at this moment whether the path integral formulation yields a unitary S-matrix and the physical interpretation of the theory defined by the noncanonical deformation of the relativistic Green functions remains an open problem.

This difficulty may be due to the fact that an interpretation as a theory of particles is a property of RQFT that may not necessarily be present in every quantum field theory. In particular, this seems to be the case of the quantum noncanonical field theory. It is interesting that different arguments 14 suggest that the absence of a direct
interpretation in terms of particles might also be a characteristics of an extension of RQFT trying to incorporate gravity effects. At this moment however we do not have a clue of how noncanonical commutation relations could arise as a trace of the gravitational interaction.

Many of the arguments that we have used in the quantum theory of a noncanonical complex field can be applied to other attempts to go beyond relativistic quantum field theory. A discussion along these lines of the quantum field theory formulation of DSR-like theories and its relation with canonical implementations of DSR in position space [15] will be presented elsewhere [16].

We would like to thank Stefano Liberati, Florian Girelli, Mikhail Plyushchay and Lorenzo Sindoni for enlightening discussions. J.I. and D.M. also thank SISSA (Trieste) for hospitality. This work has been partially supported by CICYT (grant FPA2006-02315) and DGIID-DGA (grant 2008-E24/2). J.I. acknowledges a FPU grant and D.M. a FPI grant from MEC.

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