Automorphisms tower of the generalised Thompson groups $G_{n,r}$ and $T_{n,r}$

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Abstract

We show that for the Higman-Thompson groups $G_{n,r}$ and $T_{n,r}$, $\text{Aut}(\text{Aut}(G_{n,r})) = \text{Aut}(G_{n,r})$ and $\text{Aut}(\text{Aut}(T_{n,r})) = \text{Aut}(T_{n,r})$. This extends results of Brin and Guzmán for Thompson’s group $T$, and generalisations of Thompson’s group $F$.

1 Introduction

In the seminal paper [2], Brin characterises the automorphisms of Thompson’s group $F$ and $T$. The follow-up paper [3], analyses automorphisms of generalisations of the groups $F$ and $T$ including the groups $F_n$ and $T_n$. However, the techniques used in analysing the automorphisms of these groups did not extend to analyse automorphisms of the groups $T_{n,r}$, when $r$ is not equal to $n-1$, and $G_{n,r}$ for all valid $n$ and $r$.

The recent papers [1] and [6] address this gap. The paper [1] shows that the group of automorphisms $G_{n,r}$ is a subgroup of the group of rational homeomorphisms of the Cantor space $C_{n,r}$ whilst the paper [6] extends this result to the group $T_{n,r}$.

The group $R_n$ of rational homeomorphisms of Cantor space $C_n$ were introduced in the article [4] of Grigorchuk, Nekreshevych and Suchanski. These consists of homeomorphisms which have finitely many ‘local actions’ and so are homeomorphisms induced by finite state machines called transducers. The group $R_{n,r}$ is the analogue for Cantor space $C_{n,r}$, the disjoint union of $r$-copies of $C_n$. The paper [1] demonstrates that $\text{Aut}(G_{n,r})$ is a subgroup of $R_{n,r}$ consisting of those homeomorphisms $h$, such that $h$ and $h^{-1}$ are induced by ‘synchronizing’ transducers. The synchronizing condition in the previous sentence, is a strong form of the synchronizing condition which appears in the literature surrounding the Černy conjecture and the road colouring problem (see for instance [8] and [9]). In our context, a transducer is synchronizing if there is a natural $k$ such that after processing any given input word of length $k$ in the transducer, the resulting state is independent of where we begin processing this input word.

The article [6] builds upon results of [1] and shows, amongst other things, that $\text{Aut}(T_{n,r})$ is isomorphic to a subgroup of $\text{Aut}(G_{n,r})$.

In this article, making use of results and techniques introduced in [1] and [6], we prove that $\text{Aut}(\text{Aut}(G_{n,r})) = \text{Aut}(G_{n,r})$ and $\text{Aut}(\text{Aut}(T_{n,r})) = \text{Aut}(T_{n,r})$. This extends results of Brin, and Brin and Guzmán concerning automorphisms of $F$, $T$ and some generalisations of $F$. 


Our approach is slightly different between the \( G_{n,r} \) and \( T_{n,r} \) case, however the strategy used in the \( T_{n,r} \) case applies also to \( G_{n,r} \). Our proof in the \( G_{n,r} \) case is very much in the spirit of [1]. For \( T_{n,r} \) we follow a different approach. By Rubin’s Theorem (Theorem 5.7) the group \( \text{Aut}(\text{Aut}(T_{n,r})) \) is isomorphic to its normalizer in the group of homeomorphisms of the circle. The result then follows, since \( T_{n,r} \) is generated by elements which act as the identity on a closed subset with non-empty interior and such elements are preserved by conjugation.

The article is organised as follows. In Section 2, we define and introduce the relevant terms, objects, and machinery that will be required in later sections. In Section 3 we show that a homeomorphism of Cantor space \( \mathcal{C}_{n,r} \) which is an element of the group \( \mathcal{R}_{n,r} \) and induces, by conjugation, an automorphism of \( \text{Aut}(G_{n,r}) \) must in fact be an automorphism of \( G_{n,r} \). In Section 4 we show first of all, using Rubin’s Theorem, that \( \text{Aut}(\text{Aut}(G_{n,r})) \) is isomorphic to the normaliser of \( \text{Aut}(G_{n,r}) \) in the group of homeomorphisms of \( \mathcal{C}_{n,r} \). We then argue that an element of the group of homeomorphisms of \( \mathcal{C}_{n,r} \) which normalisers \( \text{Aut}(G_{n,r}) \), must in fact be an element of \( \mathcal{R}_{n,r} \). The article concludes in Section 5 where we show that \( \text{Aut}(\text{Aut}(T_{n,r})) = \text{Aut}(T_{n,r}) \).

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2 Preliminaries

This section will be devoted to setting up notation and making the relevant definitions that will be used in later sections.

2.1 General notation and conventions

Natural numbers shall include the set 0. For \( j \in \mathbb{N} \) we denote by \( \mathbb{N}_j \) the set of all natural numbers bigger than or equal to \( j \).

For \( n \in \mathbb{N}_2 \), we set \( \mathbb{Z}[1/n] := \{ a/c \mid a, c \in \mathbb{Z} \} \) the set of \( n \)-adic rationals.

Let \( X \) be a set, then \( \text{Sym}(X) \) will denote the group of bijections from \( X \) to \( X \).

Let \( \mathcal{X} \) be a topological space. We denote by \( H(\mathcal{X}) \) the group of self-homeomorphisms of \( \mathcal{X} \). For a subgroup \( G \leq H(\mathcal{X}) \) we denote by \( N_{H(\mathcal{X})}(G) \) the normaliser of \( G \) in \( H(\mathcal{X}) \).

We shall write functions to the right of their arguments. In keeping with this convention, for a group \( G \) and elements \( g, h \in G \), we write \( g^h \) for the product \( h^{-1}gh \).

Throughout this work we shall mainly be concerned with one of the Cantor spaces defined in the next section.

2.2 Words and Cantor space

Set \( X_n := \{ 0, 1, \ldots, n - 1 \} \), and \( \mathfrak{i} := \{ 1, 2, \ldots, \hat{i} \} \). Set \( X_n^* \) to be the set of all finite words (including the empty word) in the alphabet \( X_n \), and set \( X_{n,r}^* := \mathfrak{i} \times X_n^* \). We identify \( X_{n,r}^* \) with the set consisting of the empty word and all finite words over the alphabet \( \mathfrak{i} \sqcup X_n \) which begin with an element of \( \mathfrak{i} \) and contain no other letters from \( \mathfrak{i} \). We shall use \( \epsilon \) for the empty word. Set \( X_n^+ := X_n^* \setminus \{ \epsilon \} \) likewise set \( X_{n,r}^+ := X_{n,r}^* \setminus \{ \epsilon \} \). For \( j \in \mathbb{N} \) we denote by \( X_n^j \) and \( X_{n,r}^j \) the set of all words in \( X_n^* \) and \( X_{n,r}^* \) of length \( j \).
Set $X^\omega_n$ to be the set of all infinite words over the alphabet $X_n$ and set $X^\omega_{n,r} := \bar{r} \times X^\omega_n$. We identify $X^\omega_{n,r}$ with the set of all infinite words over the alphabet $\bar{r} \sqcup X_n$ which begin with a letter in $\bar{r}$ and have no other occurrence of an element of $\bar{r}$.

We may equip $X_n$ and $\bar{r}$ with the discrete topology. Thus under the product topology, the sets $X^\omega_n$ and $X^\omega_{n,r}$ are homeomorphic to Cantor space. We denote by $\mathcal{C}_n$ the space $X^\omega_n$ and $\mathcal{C}_{n,r}$ the space $X^\omega_{n,r}$.

Given a word $\nu \in X^+_n \sqcup X^*_n$ we set $U_\nu := \{\nu \rho \mid \rho \in \mathcal{C}_n\}$, if $\nu = \epsilon$. Depending on the context $U_\epsilon$ represents either $\mathcal{C}_{n,r}$ or $\mathcal{C}_n$, whenever we use this notation, it will be clear which set is meant. The set $\{U_\nu \mid \nu \in X^*_n\}$ and $\{U_\nu \mid \nu \in X^*_n\}$ form a clopen basis for the topology on $\mathcal{C}_{n,r}$ and $\mathcal{C}_n$, respectively.

The set $X^*_n$ is a monoid under concatenation. We also observe that concatenating an elements of $X^+_n$ with an element of $X^*_n$ results in an element of $X^*_n$. We represent this operation by simply writing the elements beside each other: that is for $\Gamma \in X^+_n \sqcup X^*$ and $\gamma \in X^*_n$, we write $\Gamma \gamma$ for the concatenation of $\Gamma$ and $\gamma$. For $V \subset \mathcal{C}_n$ and $U \subset X^*_{n,r} \sqcup X^*_n$ we denote by $UV$ the set $\{\mu \nu \mid \mu \in U \mbox{ and } \nu \in V\}$.

We may order the elements of $X^*_n$ and $X^*_{n,r}$ as follows. Let $X$ be either $X^*_{n,r}$ or $X^*_n$. Given two elements $\nu, \eta \in X$ we write $\nu \leq \eta$ if $\nu$ is a prefix of $\eta$. If $\nu \not\leq \eta$ and $\eta \not\leq \nu$, then we say $\nu$ is incomparable to $\eta$ and write $\nu \perp \eta$ to denote this. Let $\nu, \eta \in X$ such that $\nu \leq \eta$, then we write $\eta - \nu$ for the word $\tau \in X^*_n \sqcup X^*_{n,r}$ such that $\eta = \nu \tau$. Let $U \subset X^*$ and $\nu \in X$ then, we write $u \leq \nu$ if is a prefix of every element of $U$. If $U$ contains only one element $\delta$ then we may also write $u \leq \delta$.

**Definition 2.1.** Let $X^*$ be one of $X^*_{n,r}$ or $Xns$. A subset $\bar{u}$ of $X^*$ is called an antichain (for $X^*$) if $\bar{u}$ consists of pairwise incomparable elements. An antichain $\bar{u}$ for $X^*$ is called complete if for any word $\nu \in X^*$ either there is some element of $\bar{u}$ which is a prefix of $\nu$ or $\nu$ is a prefix of some element of $\bar{u}$.

The natural ordering on the sets $\bar{r}$ and $X_n$ induced from $\mathbb{N}$, means that we may consider the lexicographic ordering $<_{\text{lex}}$ on the sets $X^*_{n,r}$ and $X^*_n$. That is for $\nu, \mu \in X^*_n$ or $\nu, \mu \in X^*_{n,r}$, $\nu <_{\text{lex}} \mu$ if either $\nu$ is a prefix of $\mu$ or there are words $u \in X^*_{n,r} \sqcup X^*_n$, $v, w \in X^*_n$ and $a, b \in \bar{r}$ or $a, b \in X_n$ such that $a$ is less than $b$ in the natural ordering on $\bar{r}$ or $X_n$ and $\nu = uaw$ and $\nu = uwb$.

In this article we assume that all antichains are ordered lexicographically.

### 2.3 Transducers on $\mathcal{C}_n$ or $\mathcal{C}_{n,r}$

We shall mainly be concerned with elements of $H(\mathcal{C}_n)$ or $H(\mathcal{C}_{n,r})$ which are induced by finite transducers. For a more thorough discussion we refer the readers to the articles [2] and [3].

**Definition 2.2.** A transducer is a tuple $T = (X_i, X_o, Q_T, \pi_T, \lambda_T)$ where:

(i) $X_i$ is the input alphabet and $X_o$ is the output alphabet.

(ii) $Q_T$ is the set of states of $T$.

(iii) $\pi_T : X_i \times Q_T \rightarrow Q_T$ is the transition function and,

(iv) $\lambda_T : X_i \times Q_T \rightarrow X^*_i$ is the output function.

If $|Q_T| < \infty$ then we say that $T$ is a finite transducer. If for all $a \in X_i \sqcup \{\epsilon\}$ and any state $q \in Q_T$, $|\lambda_T(a,q)| = |a|$, then $T$ is called synchronous. In the case that $X_i = X_o = X$ then we shall write $T = (X, Q_T, \pi_T, \lambda_T)$. If we fix a state $q \in Q_T$ from which we begin processing inputs
then we say that $T$ is initialised at $q$ and we denote this by $T_q$. For $q \in Q_T$, we call $T_q$ an initial transducer. Given an initial transducer $T_{q_0}$ we shall write $T$ for the underlying transducer with no initialised states.

We inductively extend the domain of the transition and rewrite functions of $T$ to $X_i^* \times Q_T$ by the following rules: for a word $w \in X_i^*$, $a \in X_i$ and any state $q \in Q_T$ we have $\pi_T(wa,q) = \pi_T(a,\pi_T(w,q))$ and $\lambda_T(wa,q) = \lambda_T(w,q)\lambda_T(a,\pi_T(w,q))$. We then extend the domain of $\pi_T$ and $\lambda_T$ to $X_i^* \times Q_T$.

In this paper we shall insist that for $h \in X_i^*$ and any state $q \in Q_T$ we have $\lambda_T(h,q) \in X_o^\omega$. This means that for a state $q \in Q_T$ the initial transducer $T_q$ induces a continuous function $h_T : (h_q$ if it is clear from the context that $q$ is a state of $T)$ from $X_i^\omega$ to $X_o^\omega$.

We now collect some terminology and notation about transducers relevant to this article.

Let $T = (X_i, X_o, Q_T, \pi_T, \lambda_T)$ be a transducer. For $q \in Q_T$ we denote by $\text{im}(q)$ the image of the map $h_q$. For $q \in Q_T$, if $h_q$ is a homeomorphism from $X_i^\omega \rightarrow X_o^\omega$, then we call $q$ a homeomorphism state. Two states $q_1$ and $q_2$ of $T$ are called $\omega$-equivalent if $h_{q_1} = h_{q_2}$.

Let $q$ be a state of $T$, then $q$ is called a state of incomplete response if for some $a \in X_i$ $\lambda_T(a,q)$ is not equal to the greatest common prefix of the set $\{(a\delta)h_q \mid \delta \in X_i^\omega\}$.

If $T$ is an initial transducer with initial state $q_0$, then $q$ is called accessible if there is a word $w \in X_i^*$ such that $\pi_T(w,q_0) = q$. If all the states of $T_{q_0}$ are accessible then $T_{q_0}$ is called accessible.

An initial transducer $T_{q_0}$ is called minimal if $T_{q_0}$ is accessible, has no states of incomplete response and no pair of $\omega$-equivalent states. Given an initial transducer $T_{q_0}$ there is a unique minimal transducer $S_{p_0}$ $\omega$-equivalent to $T_{q_0}$.

Given two transducers $A = (X, Q_A, \pi_A, \lambda_A)$ and $B = (X, Q_B, \pi_B, \lambda_B)$ then the product $A * B = (X, Q_{A \times B}, \pi_{A \times B}, \lambda_{A \times B})$ is the transducer defined as follows. The set of states $Q_{A \times B}$ of $A * B$ is equal to the cartesian product $Q_A \times Q_B$. The transition and output function of $A * B$ are given by the following rules: for states $q \in Q_A$, $p \in Q_B$ and $i \in X$ we have $\pi_{A \times B}(i, (q,p)) = (\pi_A(i,q), \pi_B(\lambda_A(i,q),p))$ and $\lambda_{A \times B}(i, (q,p)) = \lambda_B(\lambda_A(i,q),p)$. For two initial transducers $A_{q_0}$ and $B_{p_0}$, where $A$ and $B$ are the resulting transducers with no initialised states, the product of the initial transducer $A_{q_0} * B_{p_0}$, is the initial transducer $(A * B)_{(q_0, p_0)}$. It is straightforward to see that $h_{(A * B)_{(q_0, p_0)}} = h_{A_{q_0}} \circ h_{B_{p_0}}$.

Now we introduce transducers over $\mathcal{C}_{n,r}$.

**Definition 2.3.** An initial transducer for $\mathcal{C}_{n,r}$ is a tuple $A_{q_0} = (\hat{r}, X_n, R, S, \pi, \lambda, q_0)$ such that:

(i) $R$ is a finite set, and the set $Q$ of states of $A$ is the disjoint union $R \sqcup S$. The state $q_0 \in R$ is the initial state

(ii) $\pi : \hat{r} \times \{q_0\} \sqcup X_n \times Q \rightarrow Q \setminus \{q_0\}$ and $\lambda : \hat{r} \times q_0 \sqcup X_n \times Q \rightarrow X_n^* \sqcup X_n^\omega$

Extending the transition and output functions $\pi_A$ and $\lambda_A$ to the set $\hat{r} \times \{q_0\} \sqcup X_n^\omega \times Q$ in a similar way to before, they also satisfy the following rules:

(1) For a state $r \in R$ and for $i$ in $\hat{r} \sqcup X_n$ such that $\pi(i, r)$ is defined, if $\pi(i, r) \in R$ then $\lambda(i, r) = \epsilon$, otherwise $\lambda(i, r) \in X_n^*$. 

(2) For $x \in X_n$ and $q \in S$, $\lambda(x, q) \in X_n^*$ and $\pi(x, q) \in S$.

(3) For a state $s \in S$ and $\delta \in \mathcal{C}_n$ we have that $\lambda(\delta, s) \in \mathcal{C}_n$.

(4) If there is a word $w \in X_n^+$ and a state $q \in Q$ such that $\pi(w, q) = q$ then $q \in S$. 

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These rules serve the purpose of ensuring that whenever an element of $C_{n,r}$ is is processed through $A_{q_0}$ the output is also in $C_{n,r}$.

Let $A_{q_0}$ be an initial transducer on $C_{n,r}$ as above and let $q$ be a state of $A_{q_0}$. Let $A_q$ denote the initial transducer $A_{q_0}$ where we process inputs from the state $q$. Observe that $A_{q_0}$ induces a continuous function $h_{A_{q_0}}$ (or $h_{q_0}$ if it clear that $q_0$ is the initial state of $A$) from $C_{n,r}$ to itself. Furthermore every non-initial state $q$ of $A_{q_0}$ induces a continuous function $h_q$ from $C_n$ to $C_{n,r}$ if $q \in R$ or from $C_n$ to itself otherwise. Once again we denote by $\text{im}(q)$ the image of $q$ and call $q$ a homeomorphism state if $h_q$ is a homeomorphism from its domain to its range.

We extend, in the natural way, the definition of, synchronous transducers, accessibility, accessible transducers, and states of incomplete response given in the general setting to the specific setting of transducers over $C_{n,r}$.

An initial transducer $A_{q_0}$ for $C_{n,r}$ is called minimal if $A_{q_0}$ is accessible, no states of $A$ are states of incomplete response and for any distinct pair $q_1, q_2$ of states of $A_{q_0}$, $A_{q_1}$ and $A_{q_2}$ are not $\omega$-equivalent. In [1] the authors show, by slight modifications of arguments in [2], that for an initial transducer $A_{q_0}$ for $C_{n,r}$ there is a unique minimal transducer under $\omega$-equivalence.

The product $(A \ast B)(q_0, p_0) = (\hat{r}, X_n, R_A \times R_B \sqcup S_A \times S_B, \pi_{A \ast B}, \lambda_{A \ast B})$ of the initial transducers $A_{q_0}$ and $B_{p_0}$ over $C_{n,r}$ is defined as follows. The set of states $Q_{A \ast B} = R_A \times R_B \sqcup S_A \times S_B$, and the state $(q_0, p_0) \in R_A \times R_B$ is the initial state. The transition and output functions are defined as follows. First for $a \in \hat{r}$ we have $\pi_{A \ast B}(a,(q_0, p_0)) = (\pi_A(a, q_0), \pi_B(\lambda_A(a, q_0), p_0))$, and $\lambda_{A \ast B}(a,(q_0, p_0)) = \lambda_B(\lambda_A(a, q_0), p_0)$. Now for any pair $(q, p) \in Q_{A \ast B}$ and for any $i \in X_n$ we have $\pi_{A \ast B}(i,(q, p)) = (\pi_A(i, q), \pi_B(\lambda_B(i, q), p))$ and $\lambda_{A \ast B}(i,(q, p)) = \lambda_B(\lambda_B(i, q), p)$. Now observe that as $A$ and $B$ satisfy condition [1] to [4] above then so does the product $(A \ast B)(q_0, p_0)$. Furthermore, as before, it is a straightforward observation that $h_{(A \ast B)(q_0, p_0)} = h_{A_{q_0}} \circ h_{B_{p_0}}$.

Let $A_{q_0}$ be a finite, accessible, transducer for $C_n$ or $C_{n,r}$ such that $h_{q_0}$ is a homeomorphism. Then it is a result of [2] that $h_{q_0}^{-1}$ can also be represented by a finite transducer. We shall require some of the details of the inversion algorithm of the article [2].

For each state $q \in Q_A \setminus \{q_0\}$ we define a function $L_q : X_n^* \to X_n^*$ as follows: $(\nu)L_q$ is the greatest common prefix of the set $(U_\nu)h_{q_0}^{-1}$. If $A$ is a transducer for $C_n$ then $L_{q_0}$ is defined as for a non-initial state of $A$; if $A$ is a transducer for $C_{n,r}$ then $L_{q_0} : X_{n,r}^* \to X_{n,r}^*$ maps $\nu \in X_{n,r}^*$ to the greatest common prefix of the set $(U_\nu)h_{q_0}^{-1}$. For $q \in Q$ let $Q[q] := \{(w, q) \mid U_w \subset \text{im}(q), (w)L_q = \epsilon\}$. It is a result in [4] that $Q[q]$ is finite for all $q \in Q_A$.

Set $Q'_A = \{Q[q] \mid q \in Q_A\}$. Let $\pi'_A : \hat{r} \times \{(\epsilon, q_0)\} \sqcup X_n \times Q'_A \to Q'_A \setminus \{(\epsilon, q_0)\}$ be defined by:

(a) for $a \in \hat{r}$, $\pi'_A(a, (\epsilon, q_0)) = (a - \lambda_A((a)L_{q_0}, q_0), \pi_A((a)L_{q_0}, q_0))$,

(b) for $a \in X_n$ and $(w, q) \in Q'_A \setminus \{(\epsilon, q_0)\}$,

\[ \pi'_A(a, (w, q)) = (a - \lambda_A((wa)L_q, q), \pi_A((wa)L_q, q)) \]

If $A_{q_0}$ is a transducer for $C_n$ then $\pi'_A : X_n \times Q'_A \to Q'_A$ is defined as in point (b) for all states of $Q'_A$. It is a result in [4] that the transition function $\pi'_A$ is well-defined. Let $\lambda'_A : \hat{r} \times (\epsilon, q_0) \sqcup X_n \times Q'_A \to X_{n,r}^* \sqcup X_n^*$ be defined by:

(i) for $a \in \hat{r}$, $\lambda'_A(a, (\epsilon, q_0)) = (a)L_{q_0}$,

(ii) for $a \in X_n$ and $(w, q) \in Q'_A \setminus \{(\epsilon, q_0)\}$, $\lambda'_A(a, (w, q)) = (wa)L_{q_0}$.

If $A_{q_0}$ is a transducer over $C_{n,r}$ then $\lambda'_A$ is defined for all states in $Q'_A$ as in point (ii). It is a result of [4] that the transducer $A_{(\epsilon, q_0)}$ with state set $Q'[A]$ and transition and output functions $\pi'_A$ and $\lambda'_A$ satisfies $h_{(\epsilon, q_0)} = h_{q_0}^{-1}$. We call the minimal transducer representing $A_{(\epsilon, q_0)}$ the inverse (transducer) of $A$. 5
We observe that an initial minimal, synchronous transducer for $\mathcal{C}_n$ or $\mathcal{C}_{n,r}$, $T_{q_0}$ if and only if for any state $q \in Q_A$ the map $\lambda_T(\cdot, q) : X_n \to X_n$ is a bijection. In this for all states $q \in Q_T(\epsilon, q)$ is a state of $T(\epsilon, q)$ and $\lambda_A(\cdot, q) : X_n \to X_n$ is the inverse of the map $\lambda_A(\cdot, q)$. Thus, for a minimal, synchronous invertible transducer $T_{q_0}$ we shall say simply that $T$ is invertible.

The set of homeomorphism induced by finite, minimal transducer for $\mathcal{C}_n$ forms a group $\langle [I] \rangle$ $R_n$. Likewise the set of homeomorphism induced by finite, minimal transducers for $\mathcal{C}_{n,r}$ forms a group $\langle [I] \rangle$ $R_{n,r}$.

### 2.4 The subgroups $B_{n,r}$ and $\mathcal{T}B_{n,r}$ of $R_{n,r}$

We now specify some subgroups of $R_{n,r}$ based on a combinatorial property of the transducer inducing the homeomorphisms.

We begin with the combinatorial property of the transducers.

**Definition 2.4.** A transducer (initial or non-initial) $T = (X_I, X_O, Q_T, \pi_T, \lambda_T)$ is said to be **synchronizing at level** $k$ for some natural number $k \in \mathbb{N}$, if there is a map $s : X^k_T \to Q_T$ such that for a word $\Gamma \in X^k_T$ and for any state $q \in Q_T$ we have $\pi_T(\Gamma, q) = (\Gamma)s$. We say that $T$ is **synchronizing** if it is synchronizing at level $k$ for some $k \in \mathbb{N}$.

We will denote by $Core(T)$ the sub-transducer of $T$ induced by the states in the image of a. We call this sub-transducer the **core of $T$**. If $T$ is equal to its core then we say that $T$ **core**. Viewed as a graph $Core(T)$ is a strongly connected transducer.

**Definition 2.5.** If $T$ is an initial transducer $T_{q_0}$ which is invertible, then we say that $T_{q_0}$ is bi-synchronizing if both $T_{q_0}$ and its inverse are synchronizing. Note that when $T$ is synchronous, then we shall say $T$ is bi-synchronizing if $T$ and its inverse are synchronizing.

We say that a transducer $A_{q_0}$ over $\mathcal{C}_{n,r}$ is synchronizing at level $k$ for a natural number $k \in \mathbb{N}$ if given any word $\Gamma$ of length $k$ in $X^k_n$ the active state of $A_{q_0}$ when $\Gamma$ is processed from any non-initial state of $A_{q_0}$ is completely determined by $\Gamma$. We say that $A_{q_0}$ is synchronizing if it is synchronizing at level $k$ for some $k \in \mathbb{N}$. Thus we may also extend the notions of ‘core’ for synchronizing transducers over $\mathcal{C}_{n,r}$. If the minimal transducer representing $A_{(q_0)}$ is also synchronizing then we say that $A_{q_0}$ is bi-synchronizing.

**Definition 2.6.** Let $T_{q_0}$ be an initial synchronizing transducer for $\mathcal{C}_n$ or $\mathcal{C}_{n,r}$, then $T_{q_0}$ is said to have trivial core if $Core(T_{q_0})$ consists of the single state transducer inducing the identity homeomorphism on $\mathcal{C}_n$.

The set $B_{n,r}$ of all homeomorphisms in $R_{n,r}$ which may be represented by a bi-synchronizing transducers forms a subgroup of $R_{n,r}$ (1). The subgroup of $B_{n,r}$ consisting of all elements which can be represented by a bi-synchronizing transducer with trivial core is the Higman-Thompson group $G_{n,r}$. The condition that the core is trivial means that elements of $G_{n,r}$ are homeomorphisms of $\mathcal{C}_{n,r}$ given by prefix replacement maps. That is, given $q \in G_{n,r}$, there are complete antichians $\overline{u} = \{u_0, \ldots, u_l\}$, $\overline{v} = \{v_0, \ldots, v_l\}$ for $X^a_{n,r}$ and a map $\tau \in Sym\{0, 1, \ldots, l\}$ such that for $0 \leq a \leq l$ and $\rho \in \mathcal{C}_n$, $(u_a)^\rho = v_{\tau(a)}\rho$. Let $\overline{T_{n,r}}$ be the subgroup of $G_{n,r}$ consisting of those elements $q$ such that there are complete antichians $\bar{u} = \{u_0, \ldots, u_l\}$, $\bar{v} = \{v_0, \ldots, v_l\}$ for $X^a_{n,r}$ and $b \in \{0, 1, \ldots, l\}$ such that for $0 \leq a \leq l$ and $\rho \in \mathcal{C}_n$, $(u_a)^\rho = v_{(a+b) \mod l} \rho$. We observe that $\overline{T_{n,r}}$ is isomorphic to the Higman-Thompson group $T_{n,r}$.

The following result is proved in (4).

**Theorem 2.7** (Bleak, Maissel, Navas and O). $Aut(G_{n,r}) \cong B_{n,r}$. 

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The group $B_{n,r}$ contains a further subgroup of interest to this article. In order to define this group we require a few more definitions.

Let $S_r$ be the circle $[0, r]$ with the end points identified. Let $\mathcal{N} \subseteq H(S_r)$ be the subgroup of orientation preserving elements which induce bijections from the $\mathbb{Z}[1/n] \cap [0, r]$ to itself.

Let $\sim$ be the equivalence relation on $\mathcal{C}_{n,r}$ given by $\rho \sim \delta$ if either there is a word $\nu \in X_{n,r}^*$ and $a \in X_n \setminus \{0\}$ or $a \in \hat{r} \setminus \hat{0}$ such that $\rho = \nu an - 1n - 1\ldots$ and $\delta = \nu a - 1000\ldots$ or $\rho = 000\ldots$ and $\delta = r - 1n - 1\ldots$.

Let $g \in \mathcal{N}$, and let $x \in [0, r]$. We observe that $x$ has non-unique $n$-ary expansion which is an element of $\mathcal{C}_{n,r}$ precisely when $x \in \mathbb{Z}[1/n] \cap (0, r)$ and in this case the $n$-ary expansions of $x$ take the form $\mu an - 1n - 1\ldots$ and $\gamma = \mu a - 1000\ldots$ for some $\mu \in X_{n,r}^*$ and $a \in X_n \setminus \{0\}$ or $a \in \hat{r} \setminus \hat{0}$. Let $\bar{x}_1, \bar{x}_2 \in \mathcal{C}_{n,r}$ be the $n$-ary expansions of $x$, such that $\bar{x}_1 < \text{lex} \bar{x}_2$ in the lexicographic ordering of $\mathcal{C}_{n,r}$ induced by the natural order on the sets $\hat{r}$ and $X_n$ (set $\bar{x}_1 = \bar{x}_2$ if $x$ has a unique $n$-ary expansion). Observe that for $\nu \in X_{n,r}^*$ the clopen set $U_{\nu}$ corresponds to an interval $[a/n\nu, (a + 1)/n\nu] \subseteq [0, r]$ for some $a \in \mathbb{N}$ and $c \in \mathbb{Z}$. Let $\bar{g} : \mathcal{C}_{n,r} \to \mathcal{C}_{n,r}$ be defined by for $x \in [0, r]$, and $a = 1, 2$, $(\bar{x}_a)\bar{g} = (x)g_a$. As elements of $\mathcal{T}B_{n,r}$ map $\mathbb{Z}[1/n] \cap [0, r]$ bijectively into itself, the map $\bar{g}$ is well-defined. Since every element of $\mathcal{C}_{n,r}$ corresponds to $\bar{x}_a, a = 1, 2$, for some $x \in [0, r]$, and since $g$ is a homeomorphism of $S_r$, one may deduce that $\bar{g}$ is a homeomorphism of $\mathcal{C}_{n,r}$. More specifically the map $\iota : \mathcal{N} \to H(\mathcal{C}_{n,r})$ by $g \mapsto \bar{g}$ is an injective homeomorphism.

Let $\bar{N}$ denote the image of $\mathcal{N}$ under $\iota$. Set $\bar{T}B_{n,r} : \bar{N} \cap B_{n,r}$ and let $\bar{T}B_{n,r}$ be the pre-image of $\bar{T}B_{n,r}$ in $\bar{N}$. Likewise let $T_{n,r}$ be the pre-image of $T_{n,r}$ under $\iota$. The following result is proved in [9]:

**Theorem 2.8 (O).** $\text{Aut}(T_{n,r}) \cong \bar{T}B_{n,r}$.

Although we have been careful in the preliminary sections to distinguish between elements of $\mathcal{R}_{n,r}$ and the transducers inducing the homeomorphisms, for the remainder of the article we shall not emphasise this distinction and identify homeomorphisms. For instance we shall refer to a transducer as an element of $\mathcal{R}_{n,r}$ meaning that the homeomorphism induced by this transducer belongs to $\mathcal{R}_{n,r}$, this should hopefully not result in any ambiguities.

## 3 Rational automorphisms of $\text{Aut}(G_{n,r})$ are bi-synchronizing

In this section we show that any rational homeomorphism which induces an endomorphism of $B_{n,r}$ by conjugation must be in fact be an element of $B_{n,r}$.

**Lemma 3.1.** Let $A_{q_0} \in \mathcal{R}_{n,r}$ be such that $h_{q_0}^{-1} G_{n,r} h_{q_0} \leq B_{n,r}$. Let $B_{(\epsilon, q_0)}$ be the not necessarily minimal inverse of $A_{q_0}$ generated by the inversion algorithm. Suppose there is a state $(\mu, p)$ of $B_{(\epsilon, q_0)}$ and non-empty words $w_1, w_2, w_1, w_2$ such that

(i) $\pi_B(w_i, q_0) = (\mu, p), \lambda_B(w_i, q_0) = w_i, i = 1, 2,$ and

(ii) $w_1$ and $w_2$ are incomparable.

Then for all words $\Delta \in X_{n,r}^*$ such that $U_\Delta \subseteq \text{im}((\mu, p))$ there is a finite set $\mathcal{I}(\Delta)$ of synchronizing words for the strictly accessible states of $A_{q_0}$, such that every element of $U_\Delta$ has a prefix in $\mathcal{I}(\Delta)$.

**Proof.** Notice that it suffices to show that the conclusion of the lemma holds for all minimal words $\Delta \in X_{n,r}^*$ such that $U_\Delta \subseteq \text{im}((u, p))$.

Let $q_1$ and $q_2$ be distinct states accessible from the state $q_0$ by non-empty words $\nu_{q_1}$ and $\nu_{q_2}$ of the same length. Notice that $\nu_{q_1}$ and $\nu_{q_2}$ must be incomparable. Therefore, for each such pair $q_1, q_2$ there is an element $g_{q_1, q_2} \in G_{n,r}$ such that $(U_{w_i})g_{q_1, q_2} = U_{\nu_{q_i}}, i = 1, 2,$ and $g_{q_1, q_2}$ acts...
on the cone $U_{w_i}$ by replacing the prefix $w_i$ with $\nu_{q_i}$. Let $k \in \mathbb{N}$ be minimal such that for any pair $q_1, q_2$ of states of $A_{q_0}$ which are strictly accessible from $q_0$ by a pair of distinct words $\nu_{q_1}, \nu_{q_2}$ of the same length, then the minimal transducer representing $h^{-1}_{\nu_{q_1} q_1, q_0} h_{q_0}$ is synchronizing at level $k$. Notice that $k$ exists since the set of states of $A_{q_0}$ is finite. By choosing a larger value of $k$, we may also assume that $k$ is greater than the smallest integer $l$ such that for any $\Gamma \in \mathcal{X}^l$ $U_{\lambda_B(\Gamma, (\mu, p))} \subset \text{im}(\mu, p)$. Such a $k$ exists since $\text{im}(\mu, p)$ is clopen.

Let $\Xi \in \mathcal{X}^n_{\mu}$ be minimal such that $\Xi \subset \text{im}(\mu, p)$ and fix $\Gamma \in \mathcal{X}^n_{\mu}$ such that $\lambda_B(\Gamma, (\mu, p)) = \Delta$ has prefix $\Xi$. Let $\pi_B(\Gamma, (\mu, p)) = (\phi, s)$ and let $\rho_1, \rho_2, \ldots, \rho_d$ be minimal elements of $\mathcal{X}^n_{\mu}$ such that $\bigcup_{1 \leq a \leq d} U_{\rho_a} = \text{im}(\phi, s)$.

Let $q_1, q_2$ be an arbitrary pair of states which are strictly accessible from $q_0$ by a pair of distinct words $\nu_{q_1}, \nu_{q_2}$ of the same length. Let $C(q_1, q_2)_{p_0}$ be the transducer representing the element $g_{q_1, q_2}$. We observe that for $i = 1, 2$, $\pi_{BC(q_1, q_2)} A(u_i, ((\epsilon, q_0), p_0, q_0)) = ((\mu, p), \text{id}, q_i)$. Since $\text{id}$ induces the identity homeomorphism, we may identify, for $i = 1, 2$ the state $(\mu, p, \text{id}, q_i)$ with the state $(\mu, p, q_i)$ of $B((\epsilon, q_0), p_0)$. Let $t_i = \pi_A(\Delta, q_i)$, then, since the minimal transducer representing the product $h_{\nu_{q_1} q_1, q_0}^{-1} h_{q_0}$ is synchronizing at level $k$, the states $((\phi, s), t_1)$ and $((\phi, s), t_2)$ of $B((\epsilon, q_0), p_0)$ must be $\omega$-equivalent after removing the incomplete response. We observe that $A((\phi, s), t_1)$, the greatest common prefix of $\text{im}(\phi, s, t_1)$, is the greatest common prefix of the set $\{A(\rho_a, t_1) | 1 \leq a \leq d\}$. This is because as $A$ is minimal, the greatest common prefix of $(U_{\rho_a})_A$ is precisely $A(\rho_a, t_1)$ and $\text{im}(\phi, s, t_1) = \bigcup_{1 \leq a \leq d} (U_{\rho_a})_A$.

Let $\xi$ be the greatest common prefix of the set $\bigcup_{1 \leq a \leq d} (U_{\rho_a})_A$. Thus, given a word $\delta_1 \in \mathcal{C}_n$, such that $(\delta_1) B((\phi, s), t_1) = \rho_0 \delta_2$, after removing incomplete response, we have $(\delta_1) B((\phi, s), t_1) = \lambda_A(\rho_a, t_1)(\delta_2) A_\pi(\rho_a, t_1) - \xi_1$. Now since $((\phi, s), t_1)$ is $\omega$-equivalent to $((\phi, s), t_2)$, we must have: $\lambda_A(\rho_a, t_1)(\delta_2) A_\pi(\rho_a, t_1) - \xi_1 = \lambda_A(\rho_a, t_2)(\delta_2) A_\pi(\rho_a, t_2) - \xi_2$. Observe that $\lambda_A(\rho_a, t_1) - \xi_1 = \lambda_A(\rho_a, t_2) - \xi_2$. This is because if, without loss of generality, $\lambda_A(\rho_a, t_1) - \xi_1$ is a prefix of $\lambda_A(\rho_a, t_2) - \xi_2$, then $\lambda_A(\rho_a, t_2) - \xi_2 - \lambda_A(\rho_a, t_1) - \xi_1$ must be a prefix of $\text{im}(\pi_A(\rho_a, t_1) as $U_{\rho_a} \subset \text{im}(\phi, s)$. This contradicts the fact that $A_{q_0}$ is assumed to have no states of incomplete response. Therefore we conclude that $(\delta_2) A_\pi(\rho_a, t_1) = (\delta_2) A_\pi(\rho_a, t_2)$ and, as $U_{\rho_a} \subset \text{im}(\phi, s)$, it follows that $\delta_2$ can be any value in $\mathcal{C}_n$ and so we conclude that $\pi_A(\rho_a, t_1)$ is $\omega$-equivalent to $\pi_A(\rho_a, t_2)$. However, as $A$ is assumed to be minimal, $\pi_A(\rho_a, t_1) = \pi_A(\rho_a, t_2)$. Since $q_1$ and $q_2$ were arbitrary states of $A$ such that they are accessible from $q_0$ by the distinct words of the same length, we conclude that for all such pair of states, $q'_1, q'_2$, $\pi_A(\Delta \rho_a, q'_1) = \pi_A(\Delta \rho_a, q'_2)$ for all $1 \leq a \leq d$.

For $j \in \mathbb{N}_0$ let $Q_A[j]$ be the set of states which are accessible from $q_0$ by all words of length $j$ in $X^*_n$. There is a minimal $m \in \mathbb{N}$ such that $\bigcup_{1 \leq j \leq m} Q_A[j]$ consists of all strictly accessible states of $A_{q_0}$. By the previous paragraph, for any $1 \leq j \leq m$ and $q_1, q_2 \in Q_A[j]$,

$\pi_A(\Delta \rho_a, q'_1) = \pi_A(\Delta \rho_a, q'_2)$

for all $1 \leq a \leq d$. For each $(j, l)$, $1 \leq j, l \leq m$, $j \neq l$, choose a pair $(q_{1,(j,l)}, q_{2,(j,l)})$ of states such that $q_{1,(j,l)}$ is accessible from $q_0$ by a non-empty word $\nu_{q_{1,(j,l)}}$ of length $j$ and $q_{2,(j,l)}$ is accessible from $q_0$ by a non-empty word $\nu_{q_{2,(j,l)}}$ of length $l$ and $\nu_{q_{1,(j,l)}}$ is incomparable to $\nu_{q_{2,(j,l)}}$. Note that this is possible since this corresponds to choosing incomparable words in $X^*_n$ of lengths $j$ and $l$ respectively.

For each such pair $(q_{1,(j,l)}, q_{2,(j,l)})$ let $g_{q_{1,(j,l)}, q_{2,(j,l)}} \in G_{n,r}$ be chosen such that $(U_{w_i}) g_{(q_{1,(j,l)}, q_{2,(j,l)})} = U_{\nu_{q_{1,(j,l)}}}$, $i = 1, 2$, and $g_{q_{1,(j,l)}, q_{2,(j,l)}}$ acts on the cone $U_{\nu_{q_{1,(j,l)}}}$ with $\nu_{q_{1,(j,l)}}$. Let $k' \in \mathbb{N}$ be minimal such that for any pair $q_{1,(j,l)}, q_{2,(j,l)}$ of states of $A_{q_0}$ which are strictly accessible from $q_0$ by incomparable words $\nu_{q_{1,(j,l)}}, \nu_{q_{2,(j,l)}}$ of lengths $j$ and $l$ respectively, then the minimal transducer representing $h^{-1}_{\nu_{q_{1,(j,l)}}, q_{1,(j,l)}} h_{q_0}$ is synchronizing at level $k'$. We may assume that $k'$ is greater than or equal to $k$.

For $\gamma \in X_{n,k}^*$ let $V(\gamma)$ be the minimal subset of $X^*_n$ such that $\bigcup_{p \in V(\gamma)} U_p = \text{im}(\pi_B(\gamma, (\mu, p)))$. Since $\Delta \text{im}(\phi, s) = \bigcup_{1 \leq a \leq d} \Delta U_{\rho_a}$, then we have that, for any $1 \leq a \leq d$,
any element of \( U_{\Delta \rho} \) has a prefix in the set \( \{ \lambda_B(\Gamma, (\mu, p)) \rho \ | \ \gamma \in X_n^{k-k}, \rho \in V(\gamma) \} \). Fix \( \gamma \in X_n^{k-k} \), let \( \zeta \in X_n^* \) be such that \( \lambda_B(\Gamma, (\mu, p)) = \Delta \zeta \), and let \( (\delta', \rho') = \pi_B(\Gamma, (\mu, p)) \). For any element \( \delta \in \im((\delta', \rho')) \) it must be the case that \( \zeta \delta \in U_{\rho} \) for some \( 1 \leq a \leq d \). Let \( V(\gamma) = \{ \rho'_1, \rho'_2, \ldots, \rho'_l \} \). By definition we have \( \bigcup_{1 \leq l \leq \xi} U_{\rho'_l} = \im((\delta', \rho')) \). We must have that \( \zeta \rho'_l \), \( 1 \leq b \leq e \), must have some \( \rho_a \), \( 1 \leq a \leq d \), as a prefix. Thus repeating the arguments above we see that, for \( j \neq l \), \( 1 \leq j, l \leq m \), \( \pi_A(\Delta \zeta \rho'_b, q_{1, j, l}) = \pi_A(\Delta \zeta \rho'_b, q_{2, j, l}) \) for all \( 1 \leq b \leq e \). However since \( \zeta \rho'_l \) is a prefix in the set \( \{ \rho_a \ | \ 1 \leq a \leq d \} \), it must be the case that any pair \( q_1, q_2 \in Q_A[z], \) for \( z \in \{ j, l \} \), satisfies \( \pi_A(\Delta \zeta \rho'_b, q_1) = \pi_A(\Delta \zeta \rho'_b, q_2) \). Thus we see that for any pair \( (q'_1, q'_2) \) of strictly accessible states \( \pi_A(\Delta \zeta \rho'_b, q'_1) = \pi_A(\Delta \zeta \rho'_b, q'_2) \) for all \( 1 \leq b \leq e \). Since \( \gamma \in X_n^{k-k} \) was arbitrary and by the observation that for every \( 1 \leq a \leq d \), every element of \( U_{\Delta \rho} \) has a prefix in the set \( \{ \lambda_B(\Gamma, (\mu, p)) \rho \ | \ \gamma \in X_n^{k-k}, \rho \in V(\gamma) \} \), we conclude that for every \( 1 \leq a \leq d \), there is a finite set \( \mathcal{S}(\Delta \rho) \subset X_n^+ \) such that every element of \( \Delta \rho \) has a prefix in \( \mathcal{S}(\Delta \rho) \) and \( \mathcal{S}(\Delta \rho) \) is a set of synchronizing words for all strictly accessible states of \( A_{\rho} \).

Now for each \( \Gamma \in X_n^k \) such that \( \Xi \) is a prefix of \( \lambda_B(\Gamma, (\mu, p)) \) let \( V(\Gamma) \) be the minimal subset of \( X_n^+ \) such that \( \bigcup_{\rho \in V(\Gamma)} U_{\rho} = \im(\pi_B(\Gamma, (\mu, p))) \). Then the arguments above demonstrates that for each \( \Gamma \in X_n^k \) such that \( \Xi \) is a prefix of \( \lambda_B(\Gamma, (\mu, p)) \) and any \( \rho \in V(\Gamma) \), there is a finite set \( \mathcal{S}(\lambda_B(\Gamma, (\mu, p))) \) consisting of synchronizing words for the strictly accessible states of \( A_{\rho} \) such that every element of \( \lambda_B(\Gamma, (\mu, p)) \) has a prefix in \( \mathcal{S}(\lambda_B(\Gamma, (\mu, p))) \). Observe that since \( U_{\Xi} \subset \im((\mu, p)) \) every element of \( U_{\Xi} \) has a prefix in the set \( \{ \lambda_B(\Gamma, (\mu, p)) \rho \ | \ \Gamma \in X_n^k, \Xi \) is a prefix of \( \lambda_B(\Gamma, (\mu, p)), \rho \in V(\Gamma) \} \). Setting \( \mathcal{S}(\Xi) = \{ \mathcal{S}(\lambda_B(\Gamma, (\mu, p))) \rho \ | \ \Gamma \in X_n^k, \Xi \) is a prefix of \( \lambda_B(\Gamma, (\mu, p)), \rho \in V(\Gamma) \} \) we are done. □

**Lemma 3.2.** Let \( A_{q_0} \in \mathcal{R}_{n,r} \) and let \( B_{(c,q_0)} \) be the not necessarily minimal inverse of \( A_{q_0} \) generated by the inversion algorithm. Then there is a state \( (\mu, p) \) of \( B_{(c,q_0)} \) and non-empty words \( u_1, u_2, w_1, w_2 \) such that

(i) \( \pi_B(u_i, q_0) = (\mu, p), \lambda_B(u_i, q_0) = w_i, i = 1, 2, \) and

(ii) \( w_1 \) and \( w_2 \) are incomparable.

**Proof.** Let \( B_{(c,q_0)} \) be the not necessarily minimal inverse of \( A_{q_0} \) generated by the inversion algorithm. Let \( j \in \mathbb{N} \) be minimal such that for all \( \gamma \in X_n^k, \lambda_A(\gamma, q_0) \neq \epsilon \) and \( n^j \geq |A_{q_0}| \). Thus we may find distinct words \( w_1, w_2 \in X_n^k \) such that \( \pi_A(w_1, q_0) = \pi_A(w_2, q_0) = p \in Q_A \). Since \( p \) is non-trivially accessible, it follows that \( \im(p) \subset \mathcal{C} \), thus, let \( v \in X_n^k \) be a minimal length word such that \( U_v \subset \im(p) \) (note that \( \im(p) \) is clopen). Let \( u_i = \lambda_A(u_i, q_0) \) and observe that \( \lambda_B(u_i, (\epsilon, q_0)) = L_{q_0}(u_i, v) = w_1 L_{p}(v) \) for \( i = 1, 2 \), since \( A_{q_0} \) induces a homeomorphism of \( \mathcal{C}_{n,r} \). Let \( p' = \pi_B(\lambda_B(\rho, v), p) \) and \( p = \mu - \lambda_A(\rho, v), p) \), then it follows that \( \pi_B(w, (\epsilon, q_0)) (\mu, p') \). We observe that since \( w_1 \) and \( w_2 \) are incomparable, then \( w_1 L_{p}(v) \) and \( w_2 L_{p}(v) \) are also incomparable. Thus we have found a state \( (\mu, p') \) and words \( w, v_1, w L_{p}(v), i = 1, 2 \) as required. □

**Theorem 3.3.** Let \( A_{q_0} \in \mathcal{R}_{n,r} \) be such that \( h_{q_0,1} G_{n,r} h_{q_0} \leq B_{n,r} \). Then for each word \( \gamma \) in \( X_n^+ \) there is a unique strictly accessible state \( q_\gamma \) of \( A_{q_0} \) such that \( \pi_A(\gamma, q_\gamma) = q_\gamma \). In particular \( A_{q_0} \) is synchronizing.

**Proof.** Let \( B_{(c,q_0)} \) be the not necessarily minimal inverse of \( A_{q_0} \) generated by the inversion algorithm. Let \( \delta \in X_n^+ \), and \( \gamma_1, \gamma_2 \) be distinct words of equal length. Let \( q'_1 = \pi_A(\gamma_1, q_0) \) and \( q'_2 = \pi_A(\gamma_2, q_0) \). There are states \( q_1 \) and \( q_2 \) accessible from \( q'_1 \) and \( q'_2 \) by non-empty words \( \nu_1 \) and \( \nu_2 \) respectively, and \( i, j \in \mathbb{N} \) such that \( \pi_A(\delta^i, q_1) = q_1 \) and \( \pi_A(\delta^i, q_2) = q_2 \). If \( q_1 \neq q_2 \), then by Lemma 3.2 there is a state \( q \) accessible from \( q_1 \) and \( q_2 \) by the same non-empty word \( \nu \) and \( k \in \mathbb{N} \) such that \( \pi_A(\delta^k, q) = q \). Set \( \Gamma_1 = \gamma_1 \nu \) and \( \Gamma_2 = \gamma_2 \nu \) and observe that \( \Gamma_1 \) and \( \Gamma_2 \) are incomparable.
Let $\rho_a = \lambda_A(\Gamma_a, q_0)$, $a = 1, 2$, and $\nu = \lambda_A(\delta^k, q)$. Let $N \in \mathbb{N}$ be such that $U_{\nu N} \subset \text{im}(q)$ and $L_{\nu}(\nu N) = \xi$ (a prefix of $\delta^N$). Set $\nu^m := \nu^N - \lambda_A(\xi, q)$, where $\nu = \mathcal{P}_2$, and let $p = \pi_A(\xi, q)$.

We observe that $L_{\nu}(\rho_1 \nu N) = \Gamma_1 \xi$ and $L_{\nu}(\rho_2 \nu N) = \Gamma_2 \xi$. Thus we have $\pi_B(\rho_a \nu N, (\epsilon, q_0)) = (\nu^m, p)$, $a = 1, 2$, $\lambda_B(\rho_a \nu N, (\epsilon, q_0)) = \eta_a$. Since $\Gamma_1$ and $\Gamma_2$ are incomparable it must be the case that $\rho_1 \nu N$ and $\rho_2 \nu N$ are also incomparable.

Now as $\Gamma_1 \xi$ is incomparable to $\Gamma_2 \xi$, we see that $\Gamma_a \rho_a \nu N$, $a = 1, 2$, and $(\nu^m, p)$ satisfy the conditions of Lemma 3.1. Moreover, for $\xi \in X^*_n$ such that $\xi \xi = \delta^N$, since $\xi \delta^N$ is in $\text{im}(\nu^m, p)$, it follows that there is an $M \in \mathbb{N}$, and a (possibly non-trivial) rotation $\phi$ of $\delta$ with prefix $\xi$ such that $U_{\phi M} \subset \text{im}(\nu^m, p)$ (since $\text{im}(\nu^m, p)$ is clopen). Now by Lemma 3.4 it follows that there is a $K \in \mathbb{N}$ such that for all $k \in \mathbb{N}_K$ all strictly accessible states of $A_{q_0}$ read $\phi^k$ to the same location. This last statement is only possible precisely when there is a unique state $q_0$ such that $\pi_A(\delta, q_0) = \delta$.

**Corollary 3.4.** Let $A_{q_0} \in \mathcal{R}_{n,r}$ be such that $h_{q_0}^{-1}G_{n,r}h_{q_0} \leq B_{n,r}$ and $h_{q_0}G_{n,r}h_{q_0}^{-1} \leq B_{n,r}$. Then $A_{q_0}$ is bi-synchronizing.

### 4 Endomorphisms of $\text{Aut}(G_{n,r})$ are rational

In this section we show that any rational homeomorphism which induces an endomorphism of $B_{n,r}$ by conjugation must be rational by extending arguments of the paper [1].

Let $X$ be a topological space. A subset $A \subset X$ is called *somewhere dense* if there is an open set $U \subset X$ such that $A \cap U$ is dense in $U$. We require the following crucial result of [1]:

**Theorem 4.1.** Let $(X, G)$ and $(Y, H)$ be space-group pairs. Assume that $X$ is Hausdorff, locally compact, with no isolated points and that for every $x \in X$ and every open neighbourhood $U$ of $x$, the set $\{x \in U \mid g \in G \text{ and } g|_{X-U} = \text{id}|_{X-U}\}$ is somewhere dense. Further assume that the same holds for $(Y, H)$.

Then for a given group isomorphism $\phi : G \to H$, there is a homeomorphism $\varphi : X \to Y$ such that $g \phi = \varphi^{-1} g \varphi$ for every $g \in G$.

The following corollary is immediate and follows in a similar fashion to Corollary 3.2 of [1].

**Corollary 4.2.** $\text{Aut}(B_{n,r}) \cong N_H(\mathcal{C}_{n,r})(B_{n,r})$.

In order to show that the group $N_H(\mathcal{C}_{n,r})(B_{n,r})$ is a subgroup of $\mathcal{R}_{n,r}$ we begin, following [1], by showing first that it is a subgroup of the following group

**Definition 4.3.** Let $\sim$ be the relation on $\mathcal{C}_{n,r}$ defined by $\delta \sim \rho$ if and only if there are elements $\nu, \mu \in X^*_n, r$, and $\gamma \in \mathcal{C}_n \cup \mathcal{C}_{n,r}$ such that $\delta = \nu \gamma$ and $\rho = \mu \gamma$. Let $H_{n,r,\sim}$ be the subgroup of $H(\mathcal{C}_{n,r})$ consisting of all elements that preserve the relation $\sim$.

The result below is a trivial modification to Lemma 3.3 of [1] and is proved in the same way.

**Lemma 4.4.** Let $X$ be a topological space, $\sim$ an equivalence relation on $X$, $H_{\sim}$ the subgroup of $H(X)$ consisting of all elements that preserve $\sim$. Let $G \leq H_{\sim}$ and let $G_0$ be the subgroup of $G$ that fixes each equivalence class of $\sim$, that is, $x \sim xy$ for all $g \in G_0$. Suppose that $G_0$ acts transitively on each equivalence class of $\sim$. Then $\{h \in H(X) \mid h^{-1}Gh \subseteq G\} \subseteq H_{\sim}$.

We require a few definitions.

**Definition 4.5.** For $h \in \mathcal{C}_{n,r}$ and $\nu \in X^*_n, r$ set $(\nu)\theta_h$ to be the maximal common prefix of the set $(U_\nu)h$. For $\nu \in X^*_n, r$, define a map $h_\nu : \mathcal{C}_n \to \mathcal{C}_{n,r}$, $(\nu)\theta_h$ is empty, and $h_\nu : \mathcal{C}_n \to \mathcal{C}_{n,r}$ otherwise, by $(\rho)h_\nu = (\nu)h - (\nu)\theta_h$. Set $h_\epsilon = h$. For $\nu \in X^*_n$, we call $h_\nu$ the local map of $h$ at $\nu$. 

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**Definition 4.6.** Let $V \subset \mathcal{C}_{n,r}$ be a clopen set. There is a minimal antichain $\mathbf{u}$ such that $\cup_{\nu \in \mathbf{u}} U_{\nu} = U$. We call the collection of clopen sets $U(\mathbf{u})$ the decomposition of $V$ and denote it by $\text{Dec}(U)$.

The following definitions appear in Section 6 of [1].

**Definition 4.7.** Let $h \in H(\mathcal{C}_{n,r})$ and $V, W$ be proper clopen subsets of $\mathcal{C}_{n,r}$. We say that $h$ acts on $V$ and $W$ *almost in the same fashion* if there is a $k \in \mathbb{N}$ such that for every pair $U_\nu \in \text{Dec}(V)$ and $U_\eta \in \text{Dec}(W)$, and any $\xi \in X^*_n$ of length at least $k$, $h_{\nu\xi} = h_{\eta\xi}$. If $k = 0$ above, then we say that $h$ acts on $V$ and $W$ in the same fashion.

**Definition 4.8.** Let $h \in H(\mathcal{C}_{n,r})$ and $V, W$ be proper clopen subsets of $\mathcal{C}_{n,r}$. We say that $h$ acts on $V$ and $W$ *almost in the same fashion uniformly* if there is a $k \in \mathbb{N}$ such that for every pair $U_\nu \in \text{Dec}(V)$ and $U_\eta \in \text{Dec}(W)$, and any pair $\xi, \zeta \in X^*_n$ of lengths at least $k$, $h_{\nu\xi} = h_{\eta\zeta}$. If $k = 0$ above, then we say that $h$ acts on $V$ and $W$ in the same fashion uniformly.

**Remark 4.9.** Let $g \in G_{n,r}$ then for any pair $V, W$ of clopen subsets of $\mathcal{C}_{n,r}$, $g$ acts on $V$ and $W$ almost in the same fashion uniformly.

**Lemma 4.10.** Let $h \in B_{n,r}$, then for any pair $V, W$ of clopen subsets of $\mathcal{C}_{n,r}$, $h$ acts on $V$ and $W$ almost in the same fashion.

**Proof.** This is essentially a consequence of the synchronizing property. For if $h \in B_{n,r}$, then $h = h_{q_0}$ for a finite, minimal, bi-synchronizing transducer $A_{q_0} = \langle \hat{r}, X_n, Q_A, \pi_A, \lambda_A \rangle$. Let $k$ be the synchronizing level of $A_{q_0}$.

Let $\nu, \eta \in X^*_{n,r}$ be such that for any word $\nu \in \text{Dec}(V)$ and $U_\eta \in \text{Dec}(W)$, and let $\xi \in X^*_n$ of length at least $k$ be arbitrary. Since $|\xi| \geq k$, we observe that $\pi_A(\nu\xi, q_0) = \pi_A(\eta\xi, q_0)$. Let $\theta h = \lambda_A(\nu\xi, q_0)$. Thus it follows, by definition of local maps, that $h_{\eta\xi} = h_{\pi_A(\eta\xi, q_0)} = h_{\pi_A(\nu\xi, q_0)} = h_{\nu\xi}$. This concludes the proof.

The following lemma is straight-forward, and is Lemma 6.13 of [1].

**Lemma 4.11.** Let $g \in H(\mathcal{C}_{n,r})$, $h \in B_{n,r}$, $\nu, \eta \in X^*_{n,r}$. Suppose that $(U_\nu)g = U_{\nu'}$ and $(U_\eta)g = U_{\eta'}$, $g$ acts on $U_\nu$ and $U_\eta$ in the same fashion, and $h$ acts on $U_{\nu'}$ and $U_{\eta'}$ in the same fashion. Then $gh$ acts on $U_\nu$ and $U_\eta$ in the same fashion.

The following lemma weakens Lemma 6.14 of [1] slightly.

**Lemma 4.12.** Let $g, h \in H(\mathcal{C}_{n,r})$, $\nu, \eta \in X^*_{n,r}$. Suppose that $g$ acts on $U_\nu$ and $U_\eta$ in the same fashion, and $h \in B_{n,r}$. Then $gh$ acts on $U_\nu$ and $U_\eta$ almost in the same fashion.

**Proof.** Since $h \in B_{n,r}$ there is a finite, minimal, bi-synchronizing transducer $A_{q_0} = \langle \hat{r}, X_n, Q_A, \pi_A, \lambda_A \rangle$ such that $h = h_{q_0}$. Let $k$ be the synchronizing level of $A_{q_0}$. Observe that for any $\nu \in X^*_{n,r}$, $h_{\nu q_0} = h_{\pi_A(\nu, q_0)}$. This is because as $A_{q_0}$ is minimal, $(\nu)\theta h = \lambda_A(\nu, q_0)$. Thus since $A_{q_0}$ is synchronizing at level $k$, for any pair, $\nu, \eta \in X^n_{n,r}$ and any word $\xi \in X^n_n$ of length at least $j$, $h_{\nu\xi} = h_{\eta\xi}$.

Let $J \in \mathbb{N}$ be such that for any $\varsigma \in X^J_n$, we have $(\nu\varsigma)\theta h$ and $(\eta\varsigma)\theta h$ are non-empty. Let $J \in \mathbb{N}$ be such that for any word $\zeta \in X^*_n$ of length $j$, $(\zeta)\theta g_{\nu\varsigma} = (\zeta)\theta g_{\nu\varsigma}$. Let $\rho \in \mathcal{C}_n$, $\varsigma \in X^J_n$ and $\zeta \in X^J_n$, then we have $(\nu\varsigma\zeta\rho)g = (\nu\varsigma)\theta h(\zeta)\theta g_{\nu\varsigma}(\rho)g_{\nu\varsigma\zeta}$.
likewise,

\[(\eta \zeta \rho) g = (\eta \zeta) \theta_g (\zeta) \theta_{g_{\nu \eta}} (\rho) g_{\eta \zeta} \cdot \]

Observe that

\[((\nu \zeta) \theta_g (\zeta) \theta_{g_{\nu \zeta}} (\rho) g_{\nu \zeta}) h = ((\nu \zeta) \theta_g (\zeta) \theta_{g_{\nu \zeta}}) \theta_h ((\rho) g_{\nu \zeta}) h_{(\nu \zeta) \theta_g (\zeta) \theta_{g_{\nu \zeta}}} , \]

likewise,

\[((\eta \zeta) \theta_g (\zeta) \theta_{g_{\eta \zeta}} (\rho) g_{\eta \zeta}) h = ((\eta \zeta) \theta_g (\zeta) \theta_{g_{\eta \zeta}}) \theta_h ((\rho) g_{\eta \zeta}) h_{(\eta \zeta) \theta_g (\zeta) \theta_{g_{\eta \zeta}}} . \]

Now by choice of \( j \), since \((\zeta) \theta_{g_{\nu \zeta}} = (\zeta) \theta_{g_{\eta \zeta}} \), and \( g_{\nu \zeta} = g_{\eta \zeta} \), we have \( h_{(\nu \zeta) \theta_g (\zeta) \theta_{g_{\nu \zeta}}} = h_{(\eta \zeta) \theta_g (\zeta) \theta_{g_{\eta \zeta}}} \) and so

\[((\rho) g_{\nu \zeta}) h_{(\nu \zeta) \theta_g (\zeta) \theta_{g_{\nu \zeta}}} = ((\rho) g_{\eta \zeta}) h_{(\eta \zeta) \theta_g (\zeta) \theta_{g_{\eta \zeta}}} . \]

Let \( \varphi \) be the greatest common prefix of the set \((\mathcal{C}_n) g_{\nu \zeta} h_{(\nu \zeta) \theta_g (\zeta) \theta_{g_{\nu \zeta}}} = (\mathcal{C}_n) g_{\eta \zeta} h_{(\eta \zeta) \theta_g (\zeta) \theta_{g_{\eta \zeta}}} . \)

Observe that \((\nu \zeta) \theta_g h \) must be equal to \((\nu \zeta) \theta_g (\zeta) g_{\nu \zeta} \theta_h \varphi \). Likewise, \((\eta \zeta) \theta_g h \) must be equal to \((\eta \zeta) \theta_g (\zeta) g_{\eta \zeta} \theta_h \varphi \).

Therefore we conclude that

\[(\nu \zeta \rho) g h - (\nu \zeta) \theta_g h = (\nu \zeta) \theta_g h - (\eta \zeta) \theta_g h \]

and so \((\rho) g h_{\nu \zeta} = (\rho) g h_{\eta \zeta} \). Since \( \rho \) was arbitrary in \( \mathcal{C}_n \), \( \zeta \) was arbitrary in \( X_{n \rho}^d \) and \( \zeta \) was arbitrary in \( X_{n \rho}^d \) we are done.

The following is a key result of \([\mathbb{I}]\), and will be crucial in arguing that an element of \( H(\mathcal{C}_{n, r}) \) which induces an isomorphism by conjugation from \( \mathcal{B}_{n, r} \) to a subgroup of \( \mathcal{B}_{n, r} \) must in fact be an element of \( \mathcal{R}_{n, r} \).

Proposition 4.13. \([\mathbb{I}]\). Corollary 6.8] If \( h \in H_{n, r, \sim} \) then there exists \( \nu, \eta \in X_{n, r}^+ \), which are incomparable, such that \( U_\nu \cup U_\eta \neq \mathcal{C}_{n, r} \) and \( h_\nu = h_\eta \) i.e. \( h \) acts on \( U_\nu \) and \( U_\eta \) in the same fashion.

The lemma below should be compared with Lemma 6.16 of \([\mathbb{I}]\) and is proved similarly.

Lemma 4.14. Let \( h \in H(\mathcal{C}_{n, r}) \) be such that \( h^{-1} \mathcal{B}_{n, r} h \subseteq \mathcal{B}_{n, r} \). Then for every \( \nu, \eta \in X_{n, r}^+ \) such that \( U_\nu \cup U_\eta \neq \mathcal{C}_{n, r} \), the map \( h \) acts on \( U_\nu \) and \( U_\eta \) almost in the same fashion.

Proof. Since \( h^{-1} \mathcal{B}_{n, r} h \subseteq \mathcal{B}_{n, r} \) then by Lemma 4.11 \( h \in H_{n, r, \sim} \). By Proposition 4.13 there exists incomparable cones \( \nu, \eta \in X_{n, r}^+ \) such that \( h_\nu = h_\eta \).

Let \( \nu', \eta' \in X_{n, r}^+ \) be incomparable. There is an element \( g \in G_{n, r} \) such that \( g \) acts as a prefix replacement map on the cone \( U_{\nu'} \) replacing the prefix \( \nu' \) with the prefix \( \nu \) and \( g \) acts as a prefix replacement map on the cone \( U_{\eta'} \) replacing the prefix \( \eta' \) with the prefix \( \eta \).

Let \( f = h^{-1} g^{-1} h \in \mathcal{B}_{n, r} \), then observe that \( h = (gh)f \). Now since \( g \) acts on \( U_{\nu'} \) and \( U_{\eta'} \) in the same fashion, \( (U_{\nu'})g = U_\nu \) and \( (U_{\eta'})g = U_\eta \), it follows, since \( h \) acts on \( U_\nu \) and \( U_\eta \) in the same fashion, that \( gh \) acts on \( U_{\nu'} \) and \( U_{\eta'} \) in the same fashion (by Lemma 4.11). Since \( f \in \mathcal{B}_{n, r} \) it follows, by Lemma 4.12 that \( h = ghf \) acts on \( U_{\nu'} \) and \( U_{\eta'} \) almost in the same fashion.

Now suppose \( \nu', \eta' \in X_{n, r}^+ \) are as in the statement of the lemma but \( \nu' \leq \eta \) or \( \eta' \leq \nu' \). Since either \( \nu \) is a prefix of \( \eta \) or vice versa, we may find an element \( \rho \in X_{n, r}^+ \) which is incomparable both to \( \nu \) and \( \eta \). By the previous paragraph we therefore have that \( h \) acts on \( U_\nu \) and \( U_\rho \) almost in the same fashion and \( h \) acts on \( U_\eta \) and \( U_\rho \) almost in the same fashion. However this means that \( h \) acts on \( U_\nu \) and \( U_\eta \) almost in the same fashion.

We can now prove the main result of this section using the following result which may be deduced from the proof of Corollary 6.17 of \([\mathbb{I}]\).
Lemma 4.15. Let \( h \in H(\mathfrak{C}_{n,r}) \) and suppose that for any pair \( \nu, \eta \in X^+_{n,r} \) such that \( U_\nu \cup U_\eta \neq \mathfrak{C}_{n,r} \), the map \( h \) acts on \( U_\nu \) and \( U_\eta \) almost in the same fashion. Then the set of distinct local actions of \( h \) is finite.

Corollary 4.16. Let \( h \in H(\mathfrak{C}_{n,r}) \) be such that \( h^{-1}B_{n,r}h \subseteq B_{n,r} \). Then the set of distinct local actions of \( h \) is finite.

5 Automorphism tower of \( T_{n,r} \)

In this section we show, using a different approach that \( \text{Aut}(\text{Aut}(T_{n,r})) = \text{Aut}(T_{n,r}) \). The approach here also works for \( G_{n,r} \) giving an alternative proof that \( \text{Aut}(\text{Aut}(G_{n,r})) = \text{Aut}(G_{n,r}) \).

In order to prove the main result of this section, we require a few definitions first.

Definition 5.1. Let \( \mathcal{X} \) be a topological space and \( G \leq H(\mathcal{X}) \). Let \( h \in H(\mathcal{X}) \), then \( h \) is said to locally agree with \( G \) if for every point \( x \in \mathcal{X} \) there is an open neighbourhood \( U \) of \( \mathcal{X} \) and an element \( g \in G \) such that \( h|_U = g|_U \). The group \( G \) is said to be full if every element of \( H(\mathcal{X}) \) which locally agrees with \( G \) is in fact an element of \( G \).

Remark 5.2. The group \( T_{n,r} \) is a full group of homeomorphisms of the circle.

Definition 5.3. Let \( G \leq H(S_r) \) and \( D \subset S_r \). Then \( G \) is said to acts \( o-k \)-transitively if for every pair \( x_1, x_2, \ldots, x_k \) and \( y_1, y_2, \ldots, y_k \) of \( k \)-tuples of points in \( D \) such that \( x_1 < x_2 < \ldots < x_k \) and \( y_1 < y_2 < \ldots < y_k \) on some closed interval of \( S_r \) (order induced from the ordering on \( \mathbb{R} \)), then there is a \( g \in G \) such that \( (x_i)g = y_i \) for all \( 1 \leq i \leq k \).

Remark 5.4. The group \( T_{n,r} \) acts \( o-k \)-transitively on the set \( \mathbb{Z}[1/n] \cap [0,r) \). Thus since \( T_{n,r} \leq TB_{n,r} \), \( TB_{n,r} \) acts \( o-k \)-transitively on \( \mathbb{Z}[1/n] \cap [0,r) \).

Definition 5.5. Let \( g \in T_{n,r} \), then \( g \) is said to have small support if there is a proper, open subset \( U \) of the circle \( S_r \) such that \( g|_{S_r \setminus U} \) is the identity map on \( S_r \setminus U \).

The following result is straight-forward.

Lemma 5.6. \( T_{n,r} \) is generated by its elements of small support.

Proof. Let \( g \in T_{n,r} \) be non-trivial. Since \( S_r \) is Hausdorff, there is a proper open set \( U \) such that \( U \cap (U)g = \emptyset \) and \( U \cup (U)g \neq S_r \). Let \( h \) be the homeomorphism of \( S_r \) such that \( h|_{(U)g} = g^{-1} \), \( h|_U = g \) and \( h \) is the identity elsewhere. Note that since \( T_{n,r} \) is full, then \( h \in T_{n,r} \), moreover as \( h \) is the identity map on \( S_r \setminus (U \cup (U)g) \) then \( h \) is an element of small support. Further observe that \( g = (gh)h^{-1} \) is a product of elements of small support.

The following result is again due to McCleary and Rubin (\cite{5}) and is an analogue of Theorem 4.4 for the circle.

Theorem 5.7 (McCleary and Rubin). Let \( G \) be a group acting on the circle \( S_r \) by orientation preserving homeomorphism. Assume that \( G \) acts \( o-3 \)-transitively on a dense subset of \( S_r \). Then for each automorphism \( \alpha \) of \( G \), there is a unique element \( h \) of \( H(S_r) \) such that \( (f)\alpha = h^{-1}fh \) for every \( f \in G \).

As a corollary we have:

Corollary 5.8. \( \text{Aut}(TB_{n,r}) \cong N_{H(S_r)}(TB_{n,r}) \).

We can now prove the main result of this section. We begin with the following lemma.
Lemma 5.9. Let \( g \in TB_{n,r} \) be an element which acts as the identity on a closed subset of \( S_r \) with non-empty interior. Then \( g \in T_{n,r} \).

Proof. Since \( g \in TB_{n,r} \), there is a minimal bi-synchronizing transducer \( A_{q_0} \) such that \( h_{q_0} = \bar{g} = (g) \). Now since \( g \) acts as the identity on a closed subset of \( S_r \), there are integers \( a \in \mathbb{N} \) and \( c \in \mathbb{Z} \) such that \( g \) acts as the identity on the interval \([a/n, (a+1)/n]\). Hence the map \( h_{q_0} \) acts as the identity on a proper clopen subset of \( C_{n,r} \). This is true precisely if Core(\( A_{q_0} \)) is the single state identity transducer, and this is true precisely when \( h_{q_0} \in T_{n,r} \).

Theorem 5.10. Let \( h \in N_{H(S_r)}(TB_{n,r}) \) then \( h \in TB_{n,r} \).

Proof. We begin with the following observation. If for all \( h \in N_{H(S_r)}(TB_{n,r}) \) and any element \( g \in T_{n,r} \) of small support, \( h^{-1}gh \) is an element of small support in \( T_{n,r} \), then \( N_{H(S_r)}(T_{n,r}) = TB_{n,r} \). This is because, as \( T_{n,r} \) is generated by elements of small support, it follows that for all \( h \in N_{H(S_r)}(TB_{n,r}) \) and any \( g \in T_{n,r} \), \( h^{-1}gh \) is an element of \( T_{n,r} \). It therefore suffices to show that for all \( h \in N_{H(S_r)}(TB_{n,r}) \) and any element \( g \in T_{n,r} \) of small support, \( h^{-1}gh \) is an element of small support in \( T_{n,r} \).

Let \( h \in N_{H(S_r)}(TB_{n,r}) \) and \( g \in T_{n,r} \) be an element of small support. Let \( E \subset S_r \) be a closed subset with non-empty interior on which \( g \) acts as the identity. Observe that \( h^{-1}gh \in TB_{n,r} \) acts as the identity on the closed subset with non-empty interior \( (E)h \subset S_r \). Therefore by Lemma 5.9 \( h^{-1}gh \in T_{n,r} \) and is an element of small support. Thus all elements of \( N_{H(S_r)}(T_{n,r}) \) conjugate elements of small support in \( T_{n,r} \) to elements of small support in \( T_{n,r} \). It therefore follows that \( N_{H(S_r)}(T_{n,r}) = TB_{n,r} \).

Corollary 5.11. \( \text{Aut}(\text{Aut}(T_{n,r})) = \text{Aut}(T_{n,r}) \).

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