Escaping Spurious Local Minima of Low-Rank Matrix Factorization Through Convex Lifting

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Abstract

This work proposes a rapid global solver for nonconvex low-rank matrix factorization (MF) problems that we name MF-Global. Through convex lifting steps, our method efficiently escapes saddle points and spurious local minima ubiquitous in noisy real-world data, and is guaranteed to always converge to the global optima. Moreover, the proposed approach adaptively adjusts the rank for the factorization and provably identifies the optimal rank for MF automatically in the course of optimization through tools of manifold identification, and thus it also spends significantly less time on parameter tuning than existing MF methods, which require an exhaustive search for this optimal rank. On the other hand, when compared to methods for solving the lifted convex form only, MF-Global leads to significantly faster convergence and much shorter running time. Experiments on real-world large-scale recommendation system problems confirm that MF-Global can indeed effectively escapes spurious local solutions at which existing MF approaches stuck, and is magnitudes faster than state-of-the-art algorithms for the lifted convex form.

1 Introduction

The matrix factorization (MF) problem that recovers a low-rank matrix $X$ from noisy and partially observed data by minimizing a loss with respect to its factorization $(W, H)$ such that $X = WH^\top$ finds thriving interest in various applications, such as recommendation systems [42], phase retrieval [11], and the Internet of things [39]. As one of the most popular and simple nonconvex problems in machine learning and the simplest special case of neural networks (with only one hidden layer and the activation function being the identity map), lately it also receives growing theoretical interests in studying both its optimization landscape [19, 48, 20, 32, 33, 17] and whether simple methods for it can make $(W, H)$ efficiently converge to a local optimum without being stuck at strict saddle

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points [34, 13, 35, 56]. In particular, it has been shown in recent works like [19, 20, 48, 13, 56] that for a quadratic or linear loss, either under some stringent and non-verifiable assumptions on the global solutions as well as the observed data entries, or provided that a high enough rank is given, methods for MF can find the global optima with high probability. However, these theoretical results either need to overestimate the rank to a certain degree, or require that the entries observed are noiseless and decided by a uniform random sampling. The former means that one needs to start from an unnecessarily high rank, in which case the practical efficiency of matrix factorization approaches is by and large eliminated, making such theoretically guaranteed approaches infeasible for large-scale problems in real-world applications because the increased rank will lead to prohibitively high cost per iteration. The latter requirement is also impractical for two reasons. First, the uniform sampling assumption clearly violates the real scenario in recommendation systems such that entries recommended to users (and thus those of which we might get an observation) are actively selected by mechanisms of an existing recommendation system. Second, measurement errors and data corruption are omnipresent in real-world applications, so the noiseless assumption can hardly be satisfied.

Recently, it has also been shown by [53] that there indeed exists a family of MF problems whose number of spurious local minima grows exponentially with the problem dimension, which validates that those theoretical guarantees above could fail when the problem to solve fails some of the assumptions in those theoretical analyses. In practice, it is unlikely that any method considers the nonconvex MF formulation only could escape all such spurious local minima to reach the global optima, even if such a method avoids all strict saddle points. Moreover, those theoretical analyses all focus on very specific problem cases to utilize properties that cannot be extended to general problems, and thus a practically efficient and general approach for finding global minima of low-rank matrix factorization problems is still in an urgent need. We hence propose in this work an efficient and practical algorithmic framework for MF to escape from such stationary points that are not global optima by utilizing convex lifting steps. Our framework is a flexible one that can be combined with any algorithm for MF to easily provide it with an efficient escape from spurious stationary points and hence global convergence to global optima, and we name this method MF-Global.

Alternative to the nonconvex MF problem that minimizes \((W, H)\), low-rank matrix completion (MC) that directly minimizes their product \(X\) can lead to a convex formulation, so convergence to the global optima is guaranteed [10, 9, 41, 40], and such a convex formulation forms the cornerstone of the lifting steps in our framework. However, such convergence guarantee for global optimality comes with much higher computational and spatial costs. In particular, when the matrix dimension is high, it is impractical to explicitly compute and store \(X\), and thus in implementation, one still needs to form some factorization of \(X\). To make such factorization of a reasonable size, we must then enforce all iterates to be of low-rank, and thus the nuclear norm regularization is needed. However, accurate computation of the nuclear norm requires a partial singular value decomposition (SVD) of the positive singular values, whose exact computation from scratch could be expensive. Although the state of the art for convex MC [49, 23, 55] works around this difficulty by using either the Lanczos or the power method to compute approximate SVDs in a lower cost through matrix-vector products, these approximate SVDs are still rather expensive, and the practical efficiency of such approaches for MC is hardly comparable to that of methods for MF. In contrast, our framework utilizes MF methods as much as possible, and only resorts to a convex lifting step when we need to escape a spurious stationary point, so the encounters with (approximate) SVDs is reduced to minimal. We also leverage on recent advancements in numerical linear algebra to incorporate a limited-memory Krylov subspace method [36] to make our computation of approximate SVDs much more efficient than existing MC approaches. The resulting MF-Global is therefore magnitudes faster than the state of the art for MC in our numerical experiments with real-world large-scale recommendation system data.

The main goal of this work is to propose a highly efficient framework for MF problems that absorbs the advantages of both the nonconvex but simpler MF form and the convex MC form. Like methods for the convex MC, our framework possesses guaranteed convergence to the global optima, while it fully utilizes the simpler form of nonconvex MF without the expensive nuclear norm to obtain better parallelism and a much lower per iteration cost. Partly inspired by [54] that accelerates certain SDP problems by polynomial optimization, our MF-Global alternates between solving the smooth MF problem and a convex lifting, or escape, step that optimizes the convex MC problem with only one iteration to escape from spurious stationary points in the nonconvex formulation and to decide the rank of the matrices in the next MF round.
Another advantage of MF-Global over those for MF is the ability to find the optimal rank without parameter search. Existing methods for MF need to pre-specify the rank of $W$ and $H$ as a parameter in the optimization problem, and the optimal value for $k$ is found through an exhaustive search such that multiple MF problems with different ranks are solved, and thus efficiency of MF methods over MC approaches is not as significant as it would appear due to this additional parameter search time. On the other hand, through tools of manifold identification \[29, 22, 31\], our method is guaranteed to quickly and automatically fixate the rank at the MF steps to that of the global optimum to which the iterates converge, and thus MF-Global is empirically much more efficient than using a MF solver only through saving the expensive cost on parameter search.

**Contributions.** The main contributions of this work is four-fold.

1. **Global optimality guarantee with no assumption:** We propose MF-Global, a universally safeguarded approach for MF that can effectively escape all spurious stationary points and hence possesses convergence guarantees to the global optima, without any requirement on the data or the MF solver.

2. **Minimizing the need for expensive SVDs:** Through the steps that focus on the easier-to-optimize MF form, MF-Global requires significantly fewer times of computing partial SVDs than MC approaches, and thus achieves better efficiency and parallelism.

3. **Provable rank identification:** We show that after a finite number of iterations, all iterates of MF-Global automatically have the same rank as the global optimum that they converge to. By knowing the optimal rank in advance, MF-Global conducts approximate SVDs more efficiently than existing MC approaches because computation of the singular vectors/values beyond the optimal rank is avoided. MF-Global also totally removes the parameter search time needed in MF approaches for tuning the rank.

4. **Superior empirical performance:** Numerical results show that on real-world recommendation system data whose dimension is as high as a million by hundreds of thousands, MF-Global is magnitudes faster than existing methods for MC, and is also superior than the state of the art for MF by escaping spurious stationary points at which the MF approach gets stuck at.

**Organization.** This paper is organized as follows. In Section 2 we describe the problem formulation and details of MF-Global. Theoretical results are then stated in Section 3 with their proofs in the appendix. Section 4 reviews existing works and contrasts our approach with them. Experiments in Section 5 provides empirical evidence of the superior efficiency of the proposed MF-Global over existing MF and MC solvers. Finally, Section 6 concludes this work. Experiments and implementation details, additional experiments, and proofs are in the appendices.

## 2 Algorithmic framework

Given the problem dimension $m, n > 0$, we consider the following matrix factorization problem:

$$
\min_{W \in \mathbb{R}^{m \times k}, H \in \mathbb{R}^{n \times k}} \left( F(W, H) := f(WH^\top) + \frac{\lambda}{2} \left( \|W\|_F^2 + \|H\|_F^2 \right) \right),
$$

(MF)

where $\lambda > 0$ and $k \in \mathbb{N}$ are pre-specified parameters and $\|\cdot\|_F$ is the Frobenius norm. We assume without loss of generality $m \leq n$, which can easily be achieved by a matrix transpose. On the other hand, the convex lifting formulation we consider is also known as the matrix completion problem.

$$
\min_{X \in \mathbb{R}^{m \times n}} F(X) := f(X) + \lambda\|X\|_*,
$$

(MC)

where $\|\cdot\|_*$ is the nuclear norm that sums up the singular values of the given matrix. For (MF) and (MC) we assume the loss $f$ is convex and has $L$-Lipschitz-continuous gradient for some $L > 0$, and although $X$ is too large to be stored explicitly, we can afford either forming $\nabla f(X)$ or computing $\nabla f(X)v$ for any given $v$ efficiently. Therefore, both terms in (MF) are smooth, so many efficient and highly parallelizable algorithms can be applied to solve (MF) with a low per-iteration cost, but the nonconvex nature of $F(W, H)$ makes it hard to find the global optima of (MF). On the other hand, (MC) is more costly to minimize because even evaluating the objective value requires a round of expensive SVD, which is often the bottleneck of methods for (MC).

The standard approach for solving (MC) is the proximal gradient (PG) method, whose exact form in the case of (MC) with a given stepsize $\alpha_t > 0$ is as follows.

$$
X_{t+1} = \text{prox}_{\alpha_t \lambda \|\cdot\|_*} \left( X_t - \alpha_t \nabla f(X_t) \right)
$$

$$
= \arg\min_Y \left( Q_{X_t}^\alpha(Y) := \langle \nabla f(X_t), Y - X_t \rangle + (2\alpha_t)^{-1}\|Y - X_t\|_F^2 + \lambda\|Y\|_* \right),
$$

(1)
where $(A, B) = \text{trace}(A^T B)$ is the inner product of matrices $A, B$, and given any function $g$, prox$_g$ is its proximal operator: prox$_g(x) := \arg \min_y \{\|x - y\|^2/2 + g(y)\}$. When $g = \beta \| \cdot \|_*$ for $\beta > 0$, it is known that for any $\hat{Z}_t$ with rank($\hat{Z}_t$) = $k_t$ and SVD $\hat{Z}_t = U_t \text{diag}(\Sigma_t) V_t^T$, where $\hat{\Sigma}_t \in \mathbb{R}^{k_t}$, $\mathbb{R}^{k_t} \geq 0$ is the nonnegative orthonormal of $\mathbb{R}^{k_t}$, and $U_t \in \mathbb{R}^{m \times k_t}, V_t \in \mathbb{R}^{n \times k_t}$ are orthonormal (matrices with orthonormal columns), the proximal operation has a closed-form solution \[ \text{prox}_{\beta \| \cdot \|_*}(\hat{Z}_t) = U_t \text{diag}(\Sigma_t^+) V_t^T, \quad \Sigma_t := \left[ \Sigma_t - \beta e \right]_+, \tag{2} \]

where $e := (1, \ldots, 1)$, and $[\cdot]_+$ is the projection onto $\mathbb{R}^{k_t}_{\geq 0}$. Computing the exact SVD of $X_t - \alpha_t \nabla f(X_t)$ needed in \ref{eq:prox} is of cost $O(m^2 n)$, which can be prohibitively large when we are not even able to afford to explicitly form $X$. Therefore, we will conduct inexact PG steps whose SVD part is approximate but the truncation in \ref{eq:prox} using the approximate SVD is exact.

Our method MF-Global is an iterative process such that at the $t$th iteration with the current iterate $X_t$, it conducts the following two operations in order.

1. **MF phase:** Solve \ref{eq:main} using a descent method to a certain stopping tolerance, starting from the point $(\hat{W}_t, \hat{H}_t)$ obtained from the SVD of $X_t$ computed in the inexact PG step, to get $(\hat{W}_t, \hat{H}_t)$ satisfying a certain stopping criterion. We call such an intermediate iterate $\hat{X}_t := \hat{W}_t \hat{H}_t^T$.

2. **Convex lifting step:** Conduct an inexact PG step at $\hat{X}_t$ with error measure $\epsilon_t > 0$ to obtain the next iterate $X_{t+1}$ as follows.

\[
Z_t := \hat{X}_t - \alpha_t \nabla f(\hat{X}_t), \quad X_{t+1} \in \text{prox}_{\alpha_t \lambda \| \cdot \|_2}(Z_t) := \{Y \mid \min_{g \in \partial Q_{\alpha_t X_t}^*} \|g\| \leq \epsilon_t\}. \tag{3}
\]

In the convex lifting step, standard PG approaches for \ref{eq:main} fixes $\alpha_t \equiv \alpha \in (0, 2/L)$. However, it is known that the convergence speed of standard PG is slow, thus we follow \cite{Borwein} for the stepsize with linesearch to accelerate the practical convergence of the convex lifting step. Given $\alpha_{max} \geq \alpha_{min} > 0$, we first compute

\[
\alpha_t^{BB} := \max \left\{ \alpha_{min}, \min \left\{ \alpha_{max}, \frac{\langle X_t - X_{t-1}, \nabla f(X_t) - \nabla f(X_{t-1}) \rangle}{\|X_t - X_{t-1}\|^2} \right\} \right\}. \tag{4}
\]

But different from \cite{Borwein}, our backtracking linesearch does not search for an $\alpha_t$ that gives sufficient descent. Instead, given $\beta, \delta \in (0, 1)$, we find the smallest nonnegative integer $i$ such that

\[
\alpha_t = \alpha_t^{BB} \beta^i, \quad f(X_{t+i}) \leq f(X_t) + \langle \nabla f(X_t), X_{t+i} - X_t \rangle + \delta \alpha_t^{-1} \|X_{t+i} - X_t\|^2. \tag{5}
\]

From the $L$-Lipschitz continuity of $\nabla f$, \ref{eq:prox} is satisfied for any $\alpha_t \leq 2\delta/L$, so the backtracking procedure will terminate in finite steps.

For the proximal part in \ref{eq:prox}, we use the following procedure to generate an iterate satisfying \ref{eq:prox} First, we compute a rank-$k_t$ approximate eigendecomposition of $Z_t Z_t^T$ to get

\[
Z_t Z_t^T \approx \hat{U}_t \text{diag}(\hat{\Sigma}_t^i \odot \hat{\Sigma}_t) \hat{U}_t^T \tag{6}
\]

for $\hat{\Sigma}_t \in \mathbb{R}^{k_t}$ and $\hat{U}_t$ orthonormal, where $\odot$ denotes entry-wise multiplication of two vectors. We then conduct an exact SVD on $\hat{U}_t^T Z_t \in \mathbb{R}^{k_t \times n}$ to obtain $\hat{U}_t^T Z_t = \hat{U}_t \text{diag}(\hat{\Sigma}_t^i V_t^T$ with cost $O(k_t^3 + n k_t^2)$, which is much cheaper than SVD for $Z_t$ when $k_t \ll m$. The approximate SVD is then obtained by

\[
Z_t \approx \hat{Z}_t := \hat{U}_t \text{diag}(\hat{\Sigma}_t^i) V_t^T, \quad U_t := \hat{U}_t \hat{U}_t. \tag{7}
\]

Clearly, $U_t$ and $V_t$ are both orthonormal, so this is actually the exact SVD of $\hat{Z}_t = \hat{U}_t \hat{U}_t^T Z_t$. The inexact PG step is then finished by applying \ref{eq:prox} to $\hat{Z}_t$ in \ref{eq:prox}. Note that we only store $(\hat{U}_t, \hat{\Sigma}_t^i, V_t)$ without explicitly forming $X_{t+i}$ throughout the process. Here we slightly abuse the notation to let $\hat{\Sigma}_t$ denote only the coordinates with nonzero values, and let $U_t$ and $V_t$ contain only the corresponding columns. This procedure is summarized in Algorithm 1.

For the approximate eigendecomposition, as mentioned in Section 1, we cannot directly form $\hat{X}_t$ or $Z_t$ to conduct exact eigendecomposition needed in \ref{eq:prox}. Fortunately, given the factorized smaller matrices $W_t$ and $H_t, \hat{X}_t v$ and $Z_t v$ for any $v$ can be computed efficiently. The state of the art for \ref{eq:main} hence utilizes this property to use the power method to compute an approximate
Algorithm 1: \((U_t, \Sigma_t, V_t) = \text{Inexact Proximal Gradient} (\lambda, \alpha_{t}^{BB}, \beta, \hat{X}_t, R_t)\)

\[\text{for } i = 0, 1, \ldots \text{ do}\
\quad \alpha_t \leftarrow \alpha_{t}^{BB}, \text{ and compute an orthonormal matrix } \hat{U}_t \in \mathbb{R}^{m \times k} \text{ from a top-}k_t \text{ approximate eigendecomposition of } Z_t Z_t^\top \text{ as in } (6) \text{ for } Z_t \text{ defined in } (3) \text{ with the current } \alpha_t, \text{ using } R_t \text{ as the initial point for } \hat{U}_t \text{ and let } k_t := \text{rank}(R_t).\
\quad \text{Compute the exact SVD of } \hat{U}_t^\top Z_t \text{ to get } \hat{U}_t, \hat{\Sigma}_t, V_t, \text{ and then get } \Sigma_t \text{ from } \hat{\Sigma}_t \text{ as in } (2).\
\text{if } (5) \text{ holds then Terminate}\
\text{end for}\]

Algorithm 2: Algorithm Framework of MF-Global for MF

Input : \(\lambda > 0, \alpha_{\max} \geq \alpha_{\min} > 0, \text{ initial rank } k, \text{ nonnegative sequences } \{\epsilon_t\} \text{ with } \epsilon_t \to 0\)

\[\text{Initialize } \hat{W}_0 \in \mathbb{R}^{m \times k}, \hat{H}_0 \in \mathbb{R}^{n \times k}.
\text{for } t = 0, \ldots \text{ do}\
\quad \text{Compute } (\hat{W}_t, \hat{H}_t) \text{ as an approximate solution to } (\text{MF}) \text{ with rank } k \text{ (starting from } (\hat{W}_t, \hat{H}_t)) \text{ satisfying } F(\hat{W}_t, \hat{H}_t) \leq F(X_t).
\quad \text{Compute } \alpha_{t}^{BB} \text{ by } (4) \text{ and obtain } \hat{U}_t, \hat{\Sigma}_t, V_t \text{ for the SVD of } X_{t+1} \text{ by Algorithm 1 that satisfies } (3) \text{ with } \hat{X}_t = \hat{W}_t \hat{H}_t^\top \text{ and } R_t \text{ initialized using } \hat{W}_t, \hat{W}_t \text{ with small random perturbations.}
\quad k \leftarrow \text{rank}(\text{diag}(\Sigma_t^\top)), \hat{W}_{t+1} \leftarrow U_t \text{diag}(\sqrt{\Sigma_t}), H_{t+1} \leftarrow V_t \text{diag}(\sqrt{\Sigma_t}^\top).
\text{end for}\

\[
\text{eigendecomposition of } Z_t Z_t^\top. \text{ Given an initial point } R_t, \text{ the power method computes } \hat{U}_t^{(0)} = \text{orth}(R_t) \text{ and } \hat{U}_t^{(i)} = \text{orth}(Z_t Z_t^\top \hat{U}_t^{(i-1)}\top) \text{ for } i > 0 \text{ where for any matrix } A, \text{ orth}(A) \text{ is its orthogonalization, and then takes } \hat{U}_t = \hat{U}_t^{(T_t)} \text{ for some } T_t. \text{ An inefficient part of the power method is that } \hat{U}_t^{(j)} \text{ for } j < i - 1 \text{ is not used in obtaining } \hat{U}_t^{(i)}. \text{ Thus, we adopt the limited-memory Krylov subspace method of } [35] \text{ instead to form the best approximation } \hat{U}_t^{(i)} \text{ using both } \text{orth}(Z_t Z_t^\top \hat{U}_t^{(i-1)}\top) \text{ and } \{\hat{U}_t^{(i-1)}, \ldots, \hat{U}_t^{(i-M)}\} \text{ for some } M > 0 \text{ to improve the slow convergence of the power method. When compared with the power method, this approach takes slightly more storage and computation per iteration, but lead to much improved convergence speed and thus greatly reduces the time cost of the approximate eigendecomposition. For initializing } R_t, \text{ we use all columns of } \hat{W}_t \text{ and } \hat{W}_t, \text{ and add in few columns truncated in } (2) \text{ in previous iterations with random perturbation when no truncation happens in } (2) \text{ and the rank of orth}(R_t) \text{ stops increasing, so we can ensure that eventually the approximate eigendecomposition approaches the exact one and } \epsilon_t \to 0 \text{ in } (3).\]

The MF stage requires an initial point \((\hat{W}_t, \hat{H}_t)\) from \(X_t\), and we know that for any \(\hat{W}_t \hat{H}_t^\top = X_t\), the value of \(f\) remains the same, so the key is the part of the norms. The following lemma justifies our usage of \((\text{MC})\) as the convex lifting for \((\text{MF})\) and provides the optimal \((\hat{W}_t, \hat{H}_t)\). The first part of it is directly from [42].

**Lemma 2.1.** Given any \(X \in \mathbb{R}^{m \times n}\), we have

\[
\|X\|_* = \min_{W,H:WH^\top = X} \frac{1}{2} \left( \|W\|_F^2 + \|H\|_F^2 \right).
\]

Moreover, if the SVD of \(X = USV^\top = \sum_{i=1}^k \sigma_i u_i v_i^\top\), where \(S = \text{diag}(\sigma_1, \ldots, \sigma_k)\) and \(\sigma_i > 0\) are the singular values, \(k = \text{rank}(X)\), \(U = [u_1, \ldots, u_k] \in \mathbb{R}^{m \times k}\) and \(V = [v_1, \ldots, v_k] \in \mathbb{R}^{n \times k}\) are both orthonormal, the minima of (8) are exactly those

\[
\hat{W} := [\sqrt{\sigma_1} u_{\tau(1)}, \ldots, \sqrt{\sigma_k} u_{\tau(k)}], \quad \hat{H} := [\sqrt{\sigma_1} v_{\tau(1)}, \ldots, \sqrt{\sigma_k} v_{\tau(k)}],
\]

where \(\tau\) is any random permutation of \(\{1, \ldots, k\}\). Therefore, for any global optimum \((W^*, H^*)\) of \((\text{MF})\), \(X^* = W^* (H^*)^\top\) is also a global optimum to \((\text{MC})\) provided that there is a global optimum \(\hat{X}\) of \((\text{MC})\) with \(\text{rank}(\hat{X}) \leq k\), and for any optimal solution \(X^*\) of \((\text{MC})\) with SVD \(X^* = U^* S^* (V^*)^\top\), \(W^* = U^* (S^*)^{1/2}\) and \(H^* = V^* (S^*)^{1/2}\), form a global solution of \((\text{MC})\) for any \(k \geq \text{rank}(X^*)\).
3 Analysis

we first discuss the global convergence of MF-Global associated with \( \{ \epsilon_i \} \) in (3). Rank identification of Algorithm 2 is then proven under a nondegeneracy condition, which shows that for any subsequence \( \{ \tilde{X}_t \} \) of the iterates that converge to a solution \( X^* \), \( \text{rank}(\tilde{X}_t) = \text{rank}(X^*) \) for all \( i \) large enough, so the optimal rank for MF can be automatically decided on the fly.

For analyzing the worst-case convergence, we define \( \Omega^* \) as the solution set to (MC) and the optimal objective as \( F^* \). The following result establishes the global convergence of MF-Global.

**Theorem 3.1.** Consider (MC) with \( f \) lower-bounded. Then \( \Omega^* \) is compact and \( F \) is coercive. Moreover, let \( \text{dist}(X, \Omega^*):= \min_{X \in \Omega^*} ||X - X^*|| \), if \( \nabla f \) is Lipschitz continuous and \( \sum i^2 < \infty \) in Algorithm 2 then for any initialization of \( \tilde{W}_0, \tilde{H}_0 \), we always have \( \text{dist}(X_t, \Omega^*) \to 0 \), there is at least one limit point of \( \{ X_t \} \), any such limit point is a global solution to (MC) and \( F(X_t) \to F^* \).

Although our global convergence is guaranteed by the inexact PG step, existing analyses for inexact PG that utilizes the geometry of the iterates like those in [15, 47, 24, 21] are not applicable, due to that the additional MF phase could move the iterates arbitrarily in the level set. Therefore, another contribution of this work is developing new proof techniques for obtaining global convergence guarantee for general nonmonotone inexact PG combined with other optimization steps.

Inspired by the proof techniques of [26], under the premise of convergence, we show by the idea of manifold identification from partial smoothness [29, 22, 31] of the nuclear norm [16] that \( \text{rank}(X_t) \) will eventually become fixed even in the presence of the inexactness from the approximate SVD.

**Theorem 3.2.** Consider a sequence of iterates \( \{ X_t \} \) generated by Algorithm 2 from some starting point \( X_0 \) with \( \epsilon_t \to 0 \) in (3). Then the following hold.

1. For a subsequence \( \{ \tilde{X}_{i_t} \} \), such that \( \tilde{X}_{i_t} \to X^* \) for some \( X^* \in \Omega^* \), \( X_{i_t+1} \to X^* \) as well.
2. For the same subsequence in the previous item, if \( X^* \) satisfies the nondegeneracy condition

\[
0 \in \text{relative interior } (\partial F (X^*)),
\]

then there is \( i_0 \geq 0 \) such that \( \text{rank}(X_{i_t+1}) = \text{rank}(X^*) \) for all \( i \geq i_0 \).

Theorem 3.2 implies that when the full iterate sequence converge to a point \( X^* \), we get \( k_t:= \text{rank}(X_t) = \text{rank}(X^*) \) for all \( t \) large enough, and thus through the help of Lemma 2.1 the optimal rank for MF is automatically found. This also shows that the convex lifting step can effectively make the iterates escape from saddle points and spurious local minima resulted from an insufficient rank.

4 Related works

**Methods for (MF).** State-of-the-art practically efficient solvers for (MF) include (semi-asynchronous) parallel stochastic gradient (SG) [14] and parallel block-coordinate descent (BCD) [57, 51] that fully exploit modern multicore environments. The SG approach has good performance in terms of the prediction criterion when suitable parameters are used, but as an optimization approach it is not as efficient as the BCD approach. Moreover, it requires more overall running time for tuning additional parameters. We thus use polyMF-SS of [51] that conducts exact line search for the update step obtained by BCD as our solver for (MF) and also as the baseline for comparison, because it is shown to outperform [57]. Another popular approach for (MF) is Riemannian optimization, including [50] [37] [8] [59] [13], that explicitly enforces a fixed-rank constraint on \( X \) through retractions. The main drawback of the Riemannian approaches is, just like other methods for (MF) that they have no convergence guarantee to a local or global optimum. In addition, the retractions could be more expensive, so it is also often observed in practice that the Riemannian approaches are empirically slower than methods directly working on \( (W, H) \).

**Methods for (MC).** The convex problem (MC) falls in the category of regularized optimization, and many efficient algorithms in the optimization literature are available. However, most methods for regularized optimization focus on the scenario that the proximal operation can be conducted efficiently but obtaining information of the smooth term is the major computational bottleneck, which is apparently not the case for (MC). Practical methods specifically designed for (MC) all take into serious account the expensive SVD involved in the nuclear norm [49, 23, 55], and they
all focused on the most popular setting that $f$ is a quadratic term. In this case, high-order methods like proximal (quasi-)Newton is useless because the subproblem has the same form as the original problem itself. Therefore, these methods all consider first-order methods, including inexact PG and inexact accelerated PG (APG) [38] [5] [6]. [49] used the Lanczos method to conduct approximate SVD, and [23] proposed to apply the power method for approximate SVD and to use the rank $k$ decided by their inexact PG to conduct another convex optimization step with respect to a subproblem of dimension $k \times k$ after each PG step. The power method in [23] effectively uses the current iterate as warmstart and is much more efficient than the Lanczos approach in [49], but the additional convex optimization step turns out to be time-consuming and hard to parallelize. To improve the efficiency of [49] and [23], Yao et al. [55] combined the two approaches to conduct inexact APG using the power method, and significantly improved the running time upon both, but the part on the power method becomes the new bottleneck of [55]. Although the major computation of matrix-matrix products at each iteration of the power method is embarrassingly parallel, they are memory-heavy but computation-light operations, so its parallelism is confined by the memory bandwidth and thus quite low. Different from [23] [55], for approximate SVDs we use the limited-memory Krylov subspace method [36] that also utilizes the warmstart from the current iterate, but has faster convergence and better parallelism than the power method because it balances computation-heavy and memory-heavy operations better, so our PG step can also be conducted more efficiently.

More recently, motivated by PG’s ability of manifold identification, [2] proposed to alternate between an exact PG step and a Riemannian (truncated) Newton step on the currently identified manifold for general regularized optimization, and applied this method to a toy problem of [MC] in their experiments. Different from ours, their usage of manifold identification is for showing that their method could obtain superlinear convergence, but their algorithm is not feasible for large-scale problems considered in this work because they considered exact PG only and require the explicit computation of $X_t$.

Convergence of Inexact Methods The global convergence of our method is achieved through the safeguard of inexact PG steps, but we do not require the inexact PG step to always decrease the objective value. This feature combines with the MF phase makes the analysis difficult. Existing analyses for inexact PG [15] [47] [24] [21] utilize telescope sums of inequalities in the form of $\|X_{t+1} - X^*\|^2 \leq \text{error term} + \|X_t - X^*\|^2 - \alpha (F(X_t) - F(X^*))$ for any $X^* \in \Omega$ and some $\alpha > 0$ to prove convergence and rates. Therefore, those approaches cannot allow for alternating between inexact PG and other steps because that will nullify the technique of telescope sums. On the other hand, analyses compatible with other steps like those in [46] [7] [27] require strict decreasing of the objective in the inexact PG step, which imposes an additional burden.

The work of [54] that applies inexact PG to the SDP relaxation of polynomial optimization problems is probably the closest to ours in that they allow an alternative step and nonmonotone inexact PG. However, their alternative step is only accepted when it decreases the objective by an absolute amount of some $\epsilon > 0$, which makes the alternative steps eventually always rejected when the objective converges to the global optimum, but we do not have such restrictions. One possible criticism for our analysis is the lack of a rate. We emphasize that complexity analysis is easily attainable if additional assumptions are imposed, but those worst-case rates anyway will be pessimistic and faraway from the fast empirical convergence we observed, and we thus prefer the current result with novel techniques.

Methods for escaping saddle points. There has recently been a thriving interest in studying smooth optimization methods that can escape strict saddle points with at least one negative eigenvalue in the Hessian [28] [25] [44] [12] [45] [1]. However, these methods are unable to deal with degenerate saddle points where the smallest eigenvalue of the Hessian is exactly zero, and nor could they deal with spurious local minima that might be arbitrarily far away from the global ones. On the other hand, our method can handle all such difficult cases appearing in [MF] by resorting to convex lifting, and Lemma 2.1 together with Theorem 3.1 show that our method indeed will converge to the global optima, so it is more desirable for the matrix factorization case. Moreover, these methods for escaping strict saddles are mainly of theoretical interest but their empirical performance is not very impressive, while our method is designed for practical large-scale usage and greatly outperforms state-of-the-art methods for [MC] and [MF] by a large margin on real-world data.

Absence of spurious stationary points for [MF]. To cope with the possible spurious local minima and degenerate saddle points of matrix factorization, there are also growing interest in analyzing its optimization landscape. Most such works consider the quadratic loss defined below, where given an
index set $\Omega$ of observed entries, $(P_{\Omega}(X))_{i,j}$ returns $X_{i,j}$ when $(i,j) \in \Omega$ and 0 otherwise.

$$f(X) = \|P_{\Omega}(A-X)\|_F^2. \quad (11)$$

[19, 48, 20] confined their analyses to [11] and discussed only the ideal case in which each $(i,j)$ belongs to $\Omega$ with a fixed probability $p \in (0, 1]$ and the observations are noiseless, and showed the absence of spurious local minima, but these assumptions generally fail in practice. In particular, for applications like recommendation systems, elements of $\Omega$ are already biased selections by an existing system and will never obey the independent random assumption, and real-world data always contain noisy observations and measurement corruption. [13, 56] focused on [11] with $\Omega = \{1, \ldots, m\} \times \{1, \ldots, n\}$ only, and their techniques are not generalizable to other problems. Moreover, these two works utilized only gradient descent, which is barely used (if any) in large-scale real applications. Different from these works, we do not have any assumption on the underlying data, as it has been shown recently in [53] that some problem classes of matrix factorization indeed contain numerous spurious local minima, yet our method can still find the global optima in the presence of non-strict saddle points and spurious local optima with ultra-high practical efficiency.

5 Experiments

We conduct three sets of experiments to evaluate our algorithm. In particular, we examine the rank of $X_t$ and compare MF-Global with solvers for (MF) and for (MC). We focus on the case of (11) for its popularity and use four public large-scale recommendation system data sets and a server with eight cores. All experiments utilize all available cores for all algorithms. Detailed data statistics and experiment settings are in Appendix A.3. To compare different methods, we consider two criteria, one is the relative objective value from the optimization perspective, and the other is the relative RMSE from the task-oriented angle. Their technical definitions are in Appendix A.3. Here we only mention that the smaller is the better for both, while their values at an optimum should both be 0.

5.1 Stabilization of the rank

We first demonstrate numerically that MF-Global is able to identify the optimal rank quickly. We check the rank of $X_t$ in Algorithm 2 over time and the results are in Figure 1, where we use solid lines and dash lines to respectively show the relative objective value and the rank of $X_t$. Moreover, the gray line represents the rank at the optimum. We can see that the rank of $X_t$ increases quickly at first, and then soon stabilizes at the rank of the optimal solution in all cases.

![Figure 1: Rank and relative objective of the iterates of MF-Global over running time.](image)

5.2 Comparison with solving (MF) only

We next compare our method with one that solves (MF) only. The purpose of this experiment is to show that solving (MF) can indeed get stuck at spurious stationary points, while MF-Global can efficiently escape from such points. Note that using the relative objective alone is sufficient for this comparison. To solve (MF), we directly use the solver in our MF phase, namely, polyMF-SS of [51], with their original random initialization to avoid starting from the origin, which is a known saddle point. We favor their method to directly assign $k$ in (MF) as the final rank of an optimal solution, but we emphasize that in real-world applications, finding this $k$ will require additional efforts in parameter search. Comparisons of polyMF-SS and MF-Global are shown in Figure 2.

We observe that in both iterations and running time, polyMF-SS is fast at first, but then its convergence quickly slows down, suggesting that likely the iterates are attracted to a saddle point or a spurious
local minimum that is strictly worse than the global optima in terms of objective function value. On the other hand, \textbf{MF-Global} could be slightly slower at the early stage sometimes, but it maintains a steady convergence towards the global optimum throughout and always outperforms \text{polyMF-SS} eventually. Another observation is that although we did not prove a convergence rate for Algorithm 2 in the numerical experiments, the observed convergence speed is nearly linear.

Overall speaking, \textbf{MF-Global} is as efficient as running a solver for \text{[MF]} alone, while it provides additional advantages like rank identification and guaranteed convergence to global optima.

![Comparison between polyMF-SS and MF-Global using the relative objective. Top row: iterations. Bottom row: time (seconds).](image)

### 5.3 Comparison with methods for \text{(MC)}

Now that it is clear \textbf{MF-Global} is advantageous over solvers for \text{[MF]} alone, we proceed to compare it with the state-of-the-art solvers for \text{(MC)}. In particular, we compare it with the following:

- \text{Active-ALT} [23]: This method alternates between conducting inexact PG and solving a lower-dimensional convex subproblem. In the approximate SVD, they use the power method with warmstart input from the output of the previous iteration plus random columns as a safeguard.
- \text{AIS-Impute} [55]: An inexact APG method that also uses the power method for approximate SVD. They use the combination of the outputs of the previous iteration and the iteration preceding it to warmstart the power method.

The inexact APG method in [49] is not included because the underlying APG part is the same as that of AIS-Impute, but their approximate SVD using Lanczos is less efficient, as shown by [55].

The results are shown in ![Figure 3](image) Clearly, \textbf{MF-Global} outperforms the other two for \text{(MC)} significantly on both criteria. Particularly, our results exemplify that for reaching a satisfactory RMSE, \textbf{MF-Global} is magnitudes faster than state-of-the-art solvers for \text{(MC)}.

![Comparison between MF-Global and methods for \text{(MC)} in running time (seconds). Top row: relative objective. Bottom row: relative RMSE (time in log scale).](image)
6 Conclusions

In this work, we propose an efficient algorithm for finding global optima of the nonconvex matrix factorization problem. Our method conducts aperiodic convex lifting steps to effectively escape spurious stationary points of the nonconvex problems, and exhibits extremely fast convergence in numerical experiments. The proposed approach pushes forward the state of the art by being magnitudes faster than existing methods, especially on large-scale problems. We hope this work will inspire more research on safeguarding fast nonconvex optimization approaches through convex lifting as well as accelerating solvers for convex optimization through nonconvex steps. Based on the results of this work, we will release an open-source package for the matrix factorization problem soon.

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A Implementation and Experiment Details

This section supplies details in our implementation and the experiments not described in the main paper.

A.1 Implementation details

We now provide implementation details of Algorithm 2.

Initialization. For initializing, it is known that the origin is a saddle point for (MF) with (11). Therefore, we start from a point near the origin with a pre-specified rank $k_0$ for $\tilde{W}_0$ and $\tilde{H}_0$.

Rank of $R_t$ for the Krylov subspace method. As mentioned in the main paper, we use the columns of $\tilde{W}_t$ and $W_t$ plus some small random perturbation of columns truncated in previous iterations. The truncation step in (2) attempts to decrease the rank of $X_t$, while the combination of $\tilde{W}_t$ and $W_t$ gives $R_t$ a rank that tends to be higher than that of $X_t$. In particular, if columns of $\tilde{W}_t$ and $W_t$ are linearly independent, then the rank of $R_t$ is twice that of $X_t$. Thus in the fastest case, the rank of $X_t$ doubles at each iteration. Adding back the previously truncated columns to $R_t$ then ensures that the rank of $R_t$ is indeed strictly larger than that of $X_t$ even if the linear independence condition fails. When the rank of $X_t$ stabilizes, we eventually maintain the rank of $R_t$ to be rank($X_t$) + 1 to prevent unnecessary computation.

Parallelization. The parallelization of the Krylov subspace method is purely that of the matrix operations, and it is directly achieved by the inherent parallelization of matrix operations in MATLAB, while we do no additional explicit steps to improve the parallelism. For the MF step, the algorithm is implemented in C/C++ and parallelization is achieved through OpenMP by working simultaneously on multiple elements of one row of $W_t$ and one row of $H_t$.

Computation of (5). The key difficulty in computing the right-hand side of (5) is the norm term when $X_t$ are not formed explicitly. This is achieved through the observation that

$$\|X_{t+1} - X_t\|^2 = \langle X_{t+1} - X_t, X_{t+1} - X_t \rangle = \langle X_{t+1}, X_{t+1} \rangle - 2\langle X_{t+1}, X_t \rangle + \langle X_t, X_t \rangle,$$

and the inner products can be computed efficiently through $U_t, \Sigma_t, V_t$. 

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A.2 Selection of \( R_t \)

For the approximate eigendecomposition in [6], the key to the efficiency and asymptotic convergence to the exact eigendecomposition of both the power method and the limited-memory Krylov space method is to have a proper input \( R_t \) for warmstarting. It is well-known that the optimality condition for [MC] is

\[
X = \text{prox}_{\alpha \|\cdot\|_1} (X - \alpha \nabla f(X))
\]

(12)

for any \( \alpha > 0 \). Therefore, let \( Z := X - \alpha \nabla f(X) \), for \([2](12)\) indicates that the singular vectors of \( X \) are exactly those of \( Z \) corresponding to the singular values not truncated to 0. Hence, for a sequence \( \{X_t\} \) that approaches the solution set of [MC] we can hope that the singular vectors of \( X_t \) will also eventually stabilize. Through this intuition, we take the singular vectors of \( X \) so that we know all nonzero singular values are found. Therefore, we will add in one column from (7) and \( W_t \) from the output of the MF phase to form

\[
\hat{R}_t := \text{orth} ([U_t, W_t])
\]

(13)

to construct the warmstart input to the approximate eigendecomposition in obtaining \( X_{t+1} \) from \( Z_t \).

To further guarantee that \( \epsilon_t \rightarrow 0 \) in \([3]\) we need to ensure that the rank of \( R_t \) is sufficiently large, and that \( R_t \) will approach the singular vectors corresponding to the singular values not truncated out. Ideally, we hope that the output of our approximate eigendecomposition will be exactly all the eigenvalues or singular values that are retained nonzero, plus the largest one that is truncated out in [2] so that we know all nonzero singular values are found. Therefore, we will add in one column with randomness to \( \hat{R}_t \) to form \( R_t \) whenever the rank of \( X_t \) and \( X_{t-1} \) are the same, and there is at most one singular truncated out in the inexact PG step at the \( (t-1) \)th iteration. Utilizing this idea, we retain the singular vector \( u_t \) that corresponds to the largest truncated singular value in the latest iteration that such a truncation took place, and compute its projection \( u_t \) to \( \ker (R_t) \), where \( \ker (A) \) denotes the null space of \( A \) for any matrix \( A \). (It is possible and acceptable that \( u_t = 0 \).) The warmstart input is then formed by

\[
R_t := \text{orth} ([R_t, u_t + \xi_t]) \quad \xi_t \in \ker ([R_t, u_t]^\top), \quad \|\xi_t\| \leq \psi_t,
\]

(14)

where \( \xi_t \) is a random vector and \( \{\psi_t\} \) is a sequence such that \( \psi_t \downarrow 0 \).

A.3 Experiment details

Environment. Experiments on the first three datasets are conducted on an Amazon AWS EC2 c6i.4xlarge instance with an 8-core Intel Xeon Ice Lake processor and 32GB memory, while for the largest yahoo-music that consumes more memory, an m6i.4xlarge instance with the same processor and 64GB memory is used. All algorithms are implemented in MATLAB and C/C++, and our experiments are run in MATLAB 2022b.

Datasets. The datasets used are listed in Table 1. They are obtained through the following links.

- movielens100k: [https://www.kaggle.com/prajitdatta/movielens-100k-dataset](https://www.kaggle.com/prajitdatta/movielens-100k-dataset) For the training/test split, we used the split from ua.
- movielens10m: [https://www.kaggle.com/smritisingh1997/movielens-10m-dataset](https://www.kaggle.com/smritisingh1997/movielens-10m-dataset) For the training/test split, we used the split from ra.
- Netflix: [https://www.kaggle.com/netflix-inc/netflix-prize-data](https://www.kaggle.com/netflix-inc/netflix-prize-data)
- Yahoo-music: the R2 one at [https://webscope.sandbox.yahoo.com/catalog.php?datatype=r](https://webscope.sandbox.yahoo.com/catalog.php?datatype=r)

The only preprocessing we did was to transpose the data matrices when necessary to conform to our assumption of \( m \leq n \). For all data sets, we use their original training/test split. For \( \lambda \), we follow the values provided in [23] obtained through cross-validation, while the final \( k \) is the rank of a global optimum, obtained by running our algorithm with the given \( \lambda \) till the objective cannot be further improved.
| Data set     | m     | n     | $|\Omega|$   | $|\Omega_{\text{test}}|$ | $\lambda$ | final $k$ |
|-------------|-------|-------|-------------|-----------------|--------|----------|
| movielens100k | 943   | 1682  | 90570       | 9430            | 15     | 68       |
| movielens10m  | 65133 | 71567 | 9301274     | 698780          | 100    | 50       |
| netflix       | 17770 | 2649429 | 99072112  | 1408395         | 300    | 68       |
| yahoo-music   | 624961| 1000990| 252800275  | 4003960         | 10000  | 52       |

**Evaluation criteria.** To evaluate the algorithms, we first run MF-Global till the objective cannot be further improved, and take the obtained output $X^*$ as the reference point for the optimal solution. The first criterion we consider is the relative objective \[ \frac{F(X) - F^*}{F^*} \] for any point $X$, where $F^* := F(X^*)$ is the optimal objective value. The second measure is the relative root mean squared error (RMSE), which is computed as \[
\frac{\text{RMSE}(X) - \text{RMSE}(X^*)}{\text{RMSE}(0) - \text{RMSE}(X^*)},
\]
where given the test set $\Omega_{\text{test}}$, the RMSE at $X$ is computed by
\[
\text{RMSE}(X) := \sqrt{\frac{\| P_{\Omega_{\text{test}}} (X - A) \|_F^2}{|\Omega_{\text{test}}|}}.
\]

**Settings of our method.** In all experiments, we use the following simple setting for our method. We start from $k_0 = 8$ in initializing Algorithm 2 to fully utilize all cores available for better parallelism from the beginning. For simplicity in computing the speedup, this is also the setting in the speedup experiment when we actually have fewer cores available so that the speedup is indeed measured under the same amount of computation. For switching between the MF phase and the convex lifting phase, we run three epochs of polyMF-SS at each MF phase, where one epoch is counted as one sweep over the training data. In our setting for (5), we use $\beta = 0.5$ and $\delta = 0.99$ throughout.

**Comparison with polyMF-SS.** In the comparison with polyMF-SS, we also examined the behavior in terms of iterations. For polyMF-SS, one iteration is counted as one sweep over the training data (or one epoch), while for our method, we count either one convex lifting step or one iteration of polyMF-SS as one iteration.

**B Proofs**

**B.1 Proof of Lemma 2.1**

The first result of Lemma 2.1 is well-known, and here we provide a proof for completeness.

**Proof.** We first prove $[8]$ Denoted by $v^*$ the optimal value of the right-hand side of [8]. By construction, we see that $(\hat{W}, \hat{H})$ is a feasible point of the minimization problem. It follows that
\[
v^* \leq \frac{1}{2} \left( \| \hat{W} \|_F^2 + \| \hat{H} \|_F^2 \right) = \| X \|_*.
\]

On the other hand, for any $(W, H)$ satisfying $X = WH^T$, it holds that
\[
\| X \|_* = \text{tr}(S) = \text{tr} \left( U^T WH^T V \right) \leq \left\| U^T W \right\|_F \left\| V^T H \right\|_F \leq \frac{1}{2} \left( \| W \|_F^2 + \| H \|_F^2 \right) \text{.} \quad (15)
\]

Therefore, we obtain $\| X \|_* \leq v^*$. Then, [8] holds true.

We next show that all optimal solutions for the minimization problem in [8] take the form as [9]. Given any $(W, H)$ satisfying $X = WH^T$ and $\| X \|_* = \frac{1}{2} \left( \| W \|_F^2 + \| H \|_F^2 \right)$, by inequality (15) we
conclude that $W^TU$ and $H^TV$ are linearly dependent and it holds that $\|W^TU\|_F = \|H^TV\|_F$. As a consequence, we see that $W^TU = H^TV$. Moreover, it follows that

$$S = U^TXV = U^TH^TV = U^THW^TU, \quad S = V^TX^TU = V^THH^TV.$$  

Since $S$ is a diagonal matrix, we see that $(W, H)$ has the form in (9) as desired.

Last, we prove that [MF] and [MC] are equivalent under the rank condition. Let $(W^*, H^*)$ be an optimal solution of [MF] and denote $X^* := W^*(H^*)^\top$ with rank$(X^*) = k$. Suppose that problem [MC] has an optimal solution $\tilde{X}$ satisfying rank$(\tilde{X}) \leq k$. Then, on the one hand, it holds that $F(X) \leq F(X^*)$. On the other hand,

$$F(\tilde{X}) = f(\tilde{X}) + \lambda\|X\|_*$$

$$= f(\tilde{X}) + \min_{WW^T = X} \frac{1}{2}\left(\|W\|_F^2 + \|H\|_F^2\right)$$

$$= \min_{WW^T = X} f(WH^\top) + \frac{1}{2}\left(\|W\|_F^2 + \|H\|_F^2\right),$$

which implies that $F(\tilde{X}) \geq F(X^*)$. Therefore, $F(\tilde{X}) = F(X^*)$. Thus, the proof is completed. \(\square\)

### B.2 Proof of Theorem 3.1

**Proof.** The coerciveness of $F$ follows directly from the fact that the nuclear norm is coercive and that $f$ is lower-bounded. This implies that the level sets of $F$ are bounded, and thus so is $\Omega^*$.

From the assumption on $\epsilon_t^2$, we have

$$\infty > C^2 := \sum_{t=0}^{\infty} \epsilon_t^2. \tag{16}$$

From (3) we have that there exists $g_t \in \mathbb{R}^{m \times n}$ such that $\|g_t\| \leq \epsilon_t$ and

$$g_t \in \nabla f(\tilde{X}_t) + \frac{1}{\alpha_t}(X_{t+1} - \tilde{X}_t) + \partial(\lambda\|X_{t+1}\|_*) . \tag{17}$$

From the termination criterion of backtracking (5) and the convexity of $\|\cdot\|_*$, we have the following two inequalities:

$$f(X_{t+1}) \leq f(\tilde{X}_t) + \langle \nabla f(\tilde{X}_t), X_{t+1} - \tilde{X}_t \rangle + \frac{\delta}{\alpha_t}\|X_{t+1} - \tilde{X}_t\|^2, \tag{18}$$

$$\lambda\|X_{t+1}\|_* \leq -\langle \xi, \tilde{X}_t - X_{t+1} \rangle + \lambda\|\tilde{X}_t\|_* \quad \forall \xi \in \partial(\lambda\|X_{t+1}\|_*) . \tag{19}$$

By combining (17), (18) and (19) we get the following inequality:

$$F(X_{t+1}) \leq F(\tilde{X}_t) + g_t, X_{t+1} - \tilde{X}_t - \frac{1 - \delta}{\alpha_t}\|X_{t+1} - \tilde{X}_t\|^2$$

$$\leq F(\tilde{X}_t) + \|g_t\|\|X_{t+1} - \tilde{X}_t\| - \frac{1 - \delta}{\alpha_{\text{max}}}\|X_{t+1} - \tilde{X}_t\|^2$$

$$\leq F(\tilde{X}_t) + \epsilon_t\|X_{t+1} - \tilde{X}_t\| - \Gamma\|X_{t+1} - \tilde{X}_t\|^2$$

$$\leq F(\tilde{X}_t) + \epsilon_t\|X_{t+1} - \tilde{X}_t\| - \Gamma\|X_{t+1} - \tilde{X}_t\|^2 , \tag{20}$$

where $\Gamma := (1 - \delta)/\alpha_{\text{max}}$, and the last inequality is from that the MF phase does not increase the objective value. (20) implies that

$$\Gamma\|X_{t+1} - \tilde{X}_t\|^2 \leq F(\tilde{X}_t) - F(X_{t+1}) + \epsilon_t\|X_{t+1} - \tilde{X}_t\|. \tag{21}$$
By summing (21) from $t = 0$ to $t = k$, we have that
\[
\Gamma \sum_{t=0}^{k} \|X_{t+1} - \tilde{X}_t\|^2 \leq F(X_0) - F(X_{k+1}) + \sum_{t=0}^{k} \epsilon_t \|X_{t+1} - \tilde{X}_t\|
\]
\[
\leq F(X_0) - F^* + \sum_{t=0}^{k} \epsilon_t \|X_{t+1} - \tilde{X}_t\|
\]
where (23) is from the Cauchy-Schwarz inequality. By applying the quadratic formula to (24), we obtain that
\[
\sqrt{\sum_{t=0}^{k} \|X_{t+1} - \tilde{X}_t\|^2} \leq \frac{C + \sqrt{C^2 + 4\Gamma (F(X_0) - F^*)}}{2\Gamma}.
\]
This implies that
\[
\sum_{t=0}^{k} \epsilon_t \|X_{t+1} - \tilde{X}_t\| \leq \frac{\sqrt{\sum_{t=0}^{k} \|X_{t+1} - \tilde{X}_t\|^2}}{2}\frac{C + \sqrt{C^2 + 4\Gamma (F(X_0) - F^*)}}{2\Gamma}.
\]
Combining (26) and (22) we have that \{F(X_t)\} is an upper-bounded sequence. Then from the coerciveness of $F$, the sequence \{X_t\} is bounded, and thus it has at least one limit point.

Now, let us assume that $\text{dist}(X_t, \Omega^*)$ does not approach zero for contradiction. Then there exists $\sigma > 0$ and a subsequence $\{X_{t_k}\}_k$ such that $\text{dist}(X_{t_k}, \Omega) > \sigma > 0$ for any $k$. Since $\{X_{t_k}\}$ is also a bounded sequence, we have that there exists a subsequence $\{X_{t_k'}\} \subseteq \{X_{t_k}\}$ such that
\[
X_{t_k'} \rightarrow \tilde{X} \notin \Omega^*.
\]
From (17) we have that
\[
g_t + \nabla f(X_{t+1}) - \nabla f(\tilde{X}_t) \in \partial F(X_{t+1}).
\]
From that $\sum_{t=0}^{\infty} \epsilon_t^2 < \infty$, we have $\epsilon_t \to 0$ and hence $g_t \to 0$. From (25) we have that $X_{t+1} - \tilde{X}_t \to 0$. This together with the Lipschitz continuity of $\nabla f$ and the boundedness of $\alpha_t$ implies that $\nabla f(X_{t+1}) - \nabla f(\tilde{X}_t) \to 0$ and $\alpha_t^{-1}(X_{t+1} - \tilde{X}_t) \to 0$. These results together imply that
\[
g_t + \nabla f(X_{t+1}) - \nabla f(\tilde{X}_t) - \frac{1}{\alpha_t}(X_{t+1} - \tilde{X}_t) \to 0.
\]
From (27), (29) and the outer semi-continuity of $\partial F$ (see Proposition 8.7) and Proposition 20.37 in (30) we have that
\[
0 \in \partial F(\tilde{X}^*).
\]
From the convexity of $F$, we get $\tilde{X}^* \in \Omega^*$, contradicting (27) Therefore, we conclude $\text{dist}(X_t, \Omega^*) \to 0$. This also implies that any limit point of $\{X_t\}$ must lie in $\Omega^*$.

From the definition of subdifferential, the convexity of $F$, and (28) we see that for any $X^* \in \Omega^*$,
\[
0 \leq F(X_{t+1}) - F(X^*)
\]
\[
\leq \langle g_t + \nabla f(X_{t+1}) - \nabla f(\tilde{X}_t) - \frac{1}{\alpha_t}(X_{t+1} - \tilde{X}_t), X_{t+1} - X^* \rangle
\]
\[
\leq \|g_t + \nabla f(X_{t+1}) - \nabla f(\tilde{X}_t) - \frac{1}{\alpha_t}(X_{t+1} - \tilde{X}_t)\| \|X_{t+1} - X^*\|.
\]
By the boundedness of $\{X_{t_k}\}$, $\|X_{t+1} - X^*\|$ is upper bounded, while the first norm term in (30) approaches to 0 as shown in (29). We thus conclude from the sandwich lemma that $F(X_t) \to F^*$. @
B.3 Proof of Theorem 3.2

Before proving Theorem 3.2, we will need to define partly smooth functions first. As we consider the nuclear norm only, which is convex, we can use the following simplified definition.

Definition B.1 (Partly smooth [29, 22]). A convex function $\Psi$ is partly smooth at a point $x^*$ relative to a set $\mathcal{M}$ containing $x^*$ if $\partial \Psi(x^*) \neq \emptyset$ and:

1. Around $x^*$, $\mathcal{M}$ is a $C^2$-manifold and $\Psi|_\mathcal{M}$ is $C^2$.
2. The affine span of $\partial \Psi(x)$ is a translate of the normal space to $\mathcal{M}$ at $x^*$.
3. $\partial \Psi$ is continuous at $x^*$ relative to $\mathcal{M}$.

Loosely speaking, this means that $\Psi|_\mathcal{M}$ is smooth at $x^*$, but the value of $\Psi$ changes drastically along directions leaving $\mathcal{M}$ around $x^*$. We will use the partial smoothness of the nuclear norm to prove Theorem 3.2.

Proof of Theorem 3.2

1. Let us denote the exact solution of (1) at the $t_i$-th iteration given $\hat{X}_{t_i}$ as $\hat{X}_{t_i}$. Following the proof of [58, Proposition 1], we have from the optimality of $X^*$, which implies $\text{prox}_{\alpha \lambda \| \cdot \|_\mathcal{M}}(X^* - \alpha \nabla f(X^*)) = X^*$, that

$$0 \leq \left\| \hat{X}_{t_i} - X^* \right\| = \left\| \text{prox}_{\alpha \lambda \| \cdot \|_\mathcal{M}}\left(\hat{X}_{t_i} - \alpha \nabla f(\hat{X}_{t_i}) - \hat{X}_{t_i} + \left(X^* - \text{prox}_{\alpha \lambda \| \cdot \|_\mathcal{M}}(X^* - \alpha \nabla f(X^*))\right)\right) \right\| \leq \left\| \hat{X}_{t_i} - X^* \right\| + \left\| \text{prox}_{\alpha \lambda \| \cdot \|_\mathcal{M}}\left(\hat{X}_{t_i} - \alpha \nabla f(\hat{X}_{t_i})\right) - \text{prox}_{\alpha \lambda \| \cdot \|_\mathcal{M}}(X^* - \alpha \nabla f(X^*)) \right\| \leq \left\| \hat{X}_{t_i} - X^* \right\| + \left\| (\hat{X}_{t_i} - \alpha \nabla f(\hat{X}_{t_i})) - (X^* - \alpha \nabla f(X^*)) \right\| \leq 2 \left\| \hat{X}_{t_i} - X^* \right\| + \alpha \| \nabla f(\hat{X}_{t_i}) - \nabla f(X^*) \| \leq (2 + L\alpha \lambda \max) \left\| \hat{X}_{t_i} - X^* \right\| \to 0, \tag{31}$$

where (31) is from the nonexpansiveness of the proximal operation of any convex function and (32) is from the upper bound in (4) and the assignment of $\alpha_i$ in (5). Therefore, (32) leads to

$$\left\| \hat{X}_{t_i+1} - \hat{X}_{t_i} \right\| \to 0. \tag{33}$$

On the other hand, $Q_{X_{t_i}}^{\alpha_i} (\cdot)$ is $(\alpha_i)^{-1}$-strongly convex for all $i$, so the minimizer is unique and by definition it is $\hat{X}_{t_i+1}$. Moreover, through straightforward computation, we know that such strong convexity of $Q_{X_{t_i}}$ and the condition (3) imply that

$$0 \leq \frac{1}{2\alpha_i} \left\| X_{t_i+1} - \hat{X}_{t_i+1} \right\|^2 \leq Q_{X_{t_i}}^{\alpha_i} (X_{t_i+1}) - Q_{X_{t_i}}^{\alpha_i} (\hat{X}_{t_i+1}) \leq \frac{\alpha_i}{2} \min_{g_{t_i} \in \partial Q_{X_{t_i}}^{\alpha_i} (X_{t_i+1})} \| g_{t_i} \|^2 \leq \frac{\alpha_i}{2 \alpha_i^2} \to 0. \tag{34}$$

The limit is obtained from that $\epsilon_i \to 0$ and that $\alpha_i$ is upper-bounded by $\alpha_{\max}$ from (4). By combining (33) and (34), it is clear that

$$\left\| X_{t_i+1} - X^* \right\| \to 0, \quad \left\| X_{t_i+1} - \hat{X}_{t_i} \right\| \to 0, \tag{35}$$

and the former proves the desired result.
2. It is well-known that $\lambda \|X\|_\ast$ is partly smooth at every $X$ relative to the manifold
\[
M_X := \{ Y \in \mathbb{R}^{m \times n} \mid \text{rank}(Y) = \text{rank}(X) \},
\]
see, for example, [16] or [2, Example 2.4]. Therefore, the result of the second item of Theorem 3.2 is equivalent to $X_{t_i+1} \in M_X^\ast$ for all $i$ large enough. As $\epsilon_i \to 0$, we see that all conditions of [26, Theorem 1] are satisfied, and therefore the conclusion follows.

\[\square\]

C Additional Experiments

We present some additional experimental results in this section.

C.1 Performance in terms of iterations

We present comparison between different methods in terms of iteration count. For AIS-Impute and Active-ALT, we use their original definitions for counting iterations. For MF-Global, one sweep over the training data in the MF phase is counted as one iteration, while one convex lifting step is also counted as one iteration. This result is shown in Figure 4, and although each method might have different per-iteration costs, we observe a trend similar to that in the comparison in running time that MF-Global is much faster than approaches for (MC).

![Image of Figure 4 showing comparison between MF-Global and methods for (MC) in iterations. Top row: relative objective. Bottom row: relative RMSE (iteration count in log scale).]

C.2 Speedup

We examine in Figure 5 the speedup of different methods with respect to the number of cores. The speedup of each method is independently measured by
\[
\text{Speedup}(x) = \frac{\text{Running time of } x \text{ cores}}{\text{Running time of 1 core}}.
\]

We skip the small dataset movielens100k in this experiment because no method achieves observable speedup on this data, as it is too small for the benefits of multicore parallelization to compensate the additional overhead of initializing multithread processes.

We can see that methods for (MC) have barely any speedup at all, so they are less suitable for large-scale problems. On the other hand, although MF-Global sacrificed some speedup when compared with polyMF-SS alone, it can still utilize the multiple cores available in modern computing environments to have speedup increasing with number of cores used.
Figure 5: Speedup comparison of algorithms.