A FAREWELL TO UNIMODULARITY

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To the memory of Daniel TESTARD

Abstract

We interpret the unimodularity condition in almost commutative geometries as central extensions of spin lifts. In Connes’ formulation of the standard model this interpretation allows to compute the hypercharges of the fermions.

PACS-92: 11.15 Gauge field theories
MSC-91: 81T13 Yang-Mills and other gauge theories

march 2001

CPT-01/P.4193
hep-th/yymmxxx

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1 Introduction

Connes’ noncommutative geometry \cite{1,2} allows to derive the intricate rules of the Yang-Mills-Higgs model building kit from first principles \cite{3,4,2}: in almost commutative geometries, certain Yang-Mills-Higgs forces appear as pseudo forces associated to gravity, just as in Minkowskian geometry, the magnetic force appears as a pseudo force associated to electricity. The input of the Yang-Mills-Higgs kit, a compact group, three unitary representations and a certain number of coupling constants, is drastically constrained by almost commutative geometry and the simplest possible input is the standard model for leptons. To be more concrete: the following properties of the complete standard model are \textit{ad hoc} in the Yang-Mills-Higgs kit, but they derive from noncommutative geometry:

- Fermions transform according to fundamental or trivial representations under isospin and colour,
- parity violation is explicit, not spontaneous \cite{3},
- strong forces couple vectorially,
- colour is unbroken,
- isospin is broken spontaneously by one doublet of scalars with hypercharge -1/2,
- the gauge group must contain a non-vectorial $U(1)$ \cite{6}.

In the past, quite some effort has been devoted to an understanding of charge quantization, e.g. magnetic monopoles, anomalies, grand unification. In the standard model, charge quantization follows from an even more intriguing property, O’Raifeartaigh’s $\mathbb{Z}_2 \times \mathbb{Z}_3$ reduction \cite{7}. Noncommutative geometry only allows algebra representations for fermions and therefore restricts their $U(1)$ charges to $-1, 0, 1$. However the unimodularity condition, presently the only kill-joy of the geometric formulation of the standard model, always stood in the way to charge quantization. Here we reconcile the unimodularity condition with Connes’ derivation of the Yang-Mills-Higgs kit. For the standard model, this reconciliation implies a strong form of charge quantization.

2 Commutative geometry, gravity and electromagnetism

Let us summarize the punch line of Connes’ geometry. Let $M$, ‘spacetime’, be a compact, Riemannian spin manifold and $\mathcal{A} = C^\infty(M)$ the commutative, associative algebra of differentiable
functions from \(M\) into \(\mathbb{C}\). In a first step, Connes reformulates the geometry of spacetime in algebraic terms without using the commutativity of the algebra. Basic ingredients of his formulation are: the algebra \(\mathcal{A}\), its faithful representation by pointwise multiplication on the Hilbert space \(\mathcal{H} = \mathcal{L}^2(S)\) of square integrable spinors, the self-adjoint Dirac operator \(\partial\) on \(\mathcal{H}\), the charge conjugation or real structure \(J = C\circ\text{complex conjugation}\), and in the four dimensional case the chirality operator \(\chi = \gamma_5\).

In a second step, he replaces the algebra \(C^\infty(M)\) by a noncommutative algebra. If \(M\) instead of describing spacetime was describing phase space then the algebra of observables from quantum mechanics would be an example of this noncommutative algebra represented on the Hilbert space of wave functions.

Connes’ formulation is precise enough to allow repeating Einstein’s derivation of general relativity using only the algebraic data (\(\mathcal{A}, \mathcal{H}, \partial, J, \chi\)) of a four dimensional spacetime. As Einstein’s derivation, Connes’ consists of two strokes, kinematics and dynamics for the gravitational field. Connes’ first stroke is the fluctuating metric \([3][2]\) where he uses essentially the spin lift of algebra automorphisms to the Hilbert space to identify the Dirac operator as gravitational field. The second stroke is the celebrated spectral action \([4]\), that reproduces the Einstein-Hilbert action from the spectrum of the Dirac operator \(\partial\).

In the commutative case, \(\mathcal{A} = C^\infty(M)\), the group of automorphisms is just the group of diffeomorphisms, \(\text{Aut}(\mathcal{A})=\text{Diff}(M)\). We interpret a diffeomorphism \(\varphi\) locally as general coordinate transformation. The receptacle group of automorphisms of the algebra lifted to the Hilbert space is

\[
\text{Aut}_\mathcal{H}(\mathcal{A}) := \{U \in \text{End}(\mathcal{H}), \ UU^* = U^*U = 1, \ UJ = JU, \ U\chi = \chi U, \ i_U \in \text{Aut}(\rho(\mathcal{A}))\}, \quad (1)
\]

with \(i_U(x) := UxU^{-1}\). The first three properties say that a lifted automorphism \(U\) preserves probability, charge conjugation and chirality. The fourth, called covariance property, allows to define the projection \(p : \text{Aut}_\mathcal{H}(\mathcal{A}) \longrightarrow \text{Aut}(\mathcal{A})\) by

\[
p(U) = \rho^{-1}i_U \quad (2)
\]

In our case, a local calculation yields \(\text{Aut}_{C^2(S)}(C^\infty(M)) = \text{Diff}(M) \ltimes MSpin(4)\). We say receptacle because already in six dimensions, \(\text{Aut}_{C^2(S)}(C^\infty(M))\) is larger than \(\text{Diff}(M) \ltimes MSpin(6)\). We still have to construct the lift \(L : \text{Diff}(M) \longrightarrow MSpin(n)\) with \(p(L(\varphi)) = \varphi\). In the coordinates \(\tilde{x}^\mu\), the spin lift \(L(\varphi)\) applied to a spinor \(\psi(\tilde{x}) \in \mathcal{L}^2(S)\) takes the explicit form \([8]\),

\[
(L(\varphi)\psi)(x) = S(\Lambda(\varphi))|_{\varphi^{-1}(x)} \psi(\varphi^{-1}(x)), \quad (3)
\]

with \(x = \varphi(\tilde{x})\), the local Lorentz transformation

\[
\Lambda(\varphi)|_{\tilde{x}} = \sqrt{J^{-1}\tilde{g}J^{-1}}|_{\varphi(\tilde{x})} J|_{\tilde{x}} \sqrt{\tilde{g}^{-1}}|_{\tilde{x}}, \quad (4)
\]
the matrix \( \tilde{g}_{\mu\nu}(\tilde{x}) = \tilde{g}(\partial/\partial \tilde{x}^\mu, \partial/\partial \tilde{x}^\nu) \) of the metric in the coordinates \( \tilde{x} \), the Jacobian of the diffeomorphism \( J(\tilde{x})^{-1} = \partial \tilde{x}^\mu / \partial x^\mu \) and the spin lift
\[
S : SO(4) \longrightarrow Spin(4)
\]
\[
\Lambda = \exp \omega \longmapsto \exp \left[ \frac{1}{4} \omega_{ab} \gamma^a \gamma^b \right].
\] (5)

\( \omega = -\omega^T \in \mathfrak{so}(4) \). Now we can characterize algebraically the spin group in any dimension \( n \) of spacetime \( M \) as image of the automorphism group under the lift:
\[
\mathcal{M}Spin(n) = L(\text{Aut}(C^\infty(M))) \subset \text{Aut}_{L^2(S)}(C^\infty(M)).
\]
The spin lift \( L \) is of course double-valued. In \( n = 4 \) dimensions, this double-valuedness is accessible to quantum mechanical experiments, e.g. neutrons have to be rotated through an angle of 720° before interference patterns repeat [9].

Consider the flat Dirac operator \( \tilde{\partial} \) in inertial coordinates \( \tilde{x}^\mu \),
\[
\tilde{\partial} = i \delta_{\alpha a} \gamma^a \frac{\partial}{\partial \tilde{x}^\mu}.
\] (6)

We have written \( \delta_{\alpha a} \gamma^a \) instead of \( \gamma^\mu \) to stress that the selfadjoint \( \gamma \) matrices are \( \tilde{x} \)-independent.

Let us ‘fluctuate’ it by changing to general (curved) coordinates \( x = \varphi(\tilde{x}) \):
\[
L(\varphi)\tilde{\partial}L(\varphi)^{-1} =: \partial = ie^{-1}_{\alpha a} \gamma^a \left[ \frac{\partial}{\partial x^\mu} + s(\omega_\mu) \right],
\] (7)

where \( e^{-1} = \sqrt{\mathcal{J}\mathcal{J}^T} \) is a symmetric matrix,
\[
s : \mathfrak{so}(4) \longrightarrow Spin(4)
\]
\[
\omega \longmapsto \frac{1}{4} \omega_{ab} \gamma^a \gamma^b
\] (8)
is the Lie algebra isomorphism corresponding to the lift (5) and
\[
\omega_\mu(x) = \Lambda|_{\varphi^{-1}(x)} \frac{\partial}{\partial x^\mu} \Lambda^{-1}|_x.
\] (9)

For the ‘spin connection’ \( \omega \) we recover the well known expression
\[
\omega^{a}_{\beta\mu}(e) = \frac{1}{2} \left[ (\partial_\beta e^a_\mu) - (\partial_\mu e^a_\beta) + e^m_\mu (\partial_\beta e^a_m) e^{-1}_{a \alpha} \right] e^{-1}_{\beta b} \mid_{a \leftrightarrow b}
\] (10)
of the torsionless spin connection in terms of the first derivatives of \( e \). Modulo flatness (encoded in the constraint that \( e^{-1} = \sqrt{\mathcal{J}\mathcal{J}^T} \) is not a general positive matrix) the fluctuation of the flat Dirac operator produces the general one and the kinematics of the gravitational field is the set of positive matrices \( e \) with smooth spacetime dependence or equivalently the set of all torsionless Dirac operators.

The second stroke is the spectral action, a functional on this latter set, that defines the dynamics of the gravitational field. The beauty of Chamseddine & Connes’ approach [4] to general
relativity is that it works precisely because the Dirac operator $\partial$ plays two roles simultaneously, it defines the dynamics of matter and it parameterizes the set of all Riemannian metrics. Their starting point is the simple remark that the spectrum of the Dirac operator is invariant under diffeomorphisms interpreted as general coordinate transformations. From $\partial \chi = -\chi \partial$ we know that the spectrum of $\partial$ is even. We may therefore consider only the spectrum of the positive operator $\partial^2/\Lambda^2$ where we have divided by a fixed arbitrary energy scale to make the spectrum dimensionless. If it was not divergent the trace $\text{tr} \partial^2/\Lambda^2$ would be a general relativistic action functional. To make it convergent, take a differentiable function $f: \mathbb{R}_+ \to \mathbb{R}_+$ of sufficiently fast decrease such that the action

$$S_{CC} := \text{tr} f(\partial^2/\Lambda^2)$$

(11)

converges. It is still a diffeomorphism invariant action. Using the heat kernel expansion it can be computed asymptotically:

$$S_{CC} = \int_M [\Lambda_c - \frac{m_p^2}{16\pi} R + a(5R^2 - 8\text{Ricci}^2 - 7\text{Riemann}^2)] \sqrt{\det g_{\mu\nu}} d^4x + O(\Lambda^{-2}),$$

(12)

where the cosmological constant is $\Lambda_c = \frac{f_0}{4\pi^2} \Lambda^4$, the Planck mass is $m_p^2 = \frac{f_4}{5\pi} \Lambda^2$ and $a = \frac{f_4}{5760\pi^2}$. The Chamseddine-Connes action is universal in the sense that the ‘cut off’ function $f$ only enters through its first three ‘moments’, $f_0 := \int_0^\infty u f(u) du, f_2 := \int_0^\infty f(u) du$ and $f_4 = f(0)$. Thanks to the curvature square terms the Chamseddine-Connes action is positive and has minima. For instance the 4-sphere with a radius of $(11f_4)^{1/2}(90\pi(1 - (1 - 11/15 f_0 f_4 f_2^{-2})^{1/2}))^{-1/2}$ times the Planck length is a ground state. This minimum breaks the diffeomorphism group spontaneously down to the isometry group $SO(5)$. The little group consists of those lifted automorphisms that commute with the Dirac operator $\partial$. Let us anticipate that the spontaneous symmetry breaking of the Higgs mechanism will be a little brother of this gravitational break down. Physically the gravitational symmetry breaking seems to regularize the initial cosmological singularity.

As a bonus, this algebraic derivation of gravity achieves the unification with electromagnetism by a straightforward central extension of the lift $L$. Since particles and antiparticles have opposite charge, we have to separate them before turning on the electromagnetic field. Technically this is done by doubling the Hilbert space,

$$\mathcal{A} = \mathcal{C}^\infty(M) \ni a, \quad \mathcal{H}_t = \mathcal{L}^2(S) \otimes \mathbb{C}^2 \ni \psi_t = \begin{pmatrix} \psi \\ \psi^c \end{pmatrix}, \quad \rho_t(a) = \begin{pmatrix} a_{14} & 0 \\ 0 & \bar{a}_{14} \end{pmatrix},$$

$$\mathcal{D}_t = \begin{pmatrix} \partial & 0 \\ 0 & \bar{\partial} \end{pmatrix}, \quad J_t = C \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ c c, \quad \chi_t = \gamma_5 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\quad$$

(13)

(14)

We emphasize that $\psi^c$ is not a new degree of freedom but we will make $\psi^c$ the antiparticle of $\psi$ at the end by imposing $J_t \psi_t = \psi_t$. This disentangling of particles and antiparticles is Dirac’s
reinterpretation of the antiparticles as holes. Now \( \text{Aut}_{\mathcal{H}_t}(\mathcal{C}^\infty(M)) = \text{Diff}(M) \ltimes M(\text{Spin}(4) \times U(1)) \) and we wish to extend the lift \( L \) to include the \( U(1) \) gauge transformations. At this point we anticipate that we aim at noncommutativity: for a noncommutative algebra \( \mathcal{A} \) there is a strong link between its group of automorphisms \( \text{Aut}(\mathcal{A}) \) and its group of unitaries

\[
U(\mathcal{A}) := \{ u \in \mathcal{A}, \; uu^* = u^*u = 1 \}.
\]

In our present commutative example \( \mathcal{C}^\infty(M) \), this link is not visible, but its group of unitaries consists precisely of \( U(1) \) gauge transformations, \( U(\mathcal{C}^\infty(M)) = M(\text{U}(1)) \). Furthermore there is a natural class of centrally extended lifts \( L \) on \( \mathcal{A} = \mathcal{C}^\infty(M) \):

\[
L = (L, \ell) : \text{Aut}(\mathcal{A}) \ltimes U(\mathcal{A}) \longrightarrow M(\text{Spin}(4) \times U(1)) \subset \text{Aut}_{\mathcal{H}_t}(\mathcal{A})
\]

\[
(\varphi, u) \longmapsto L(\varphi, u) = (L(\varphi), \ell(u))
\]

with

\[
\ell(u) = \rho_t(u^{q/2})J_t\rho_t(u^{q/2})J_t^{-1}, \quad q \in 2\mathbb{Z} \text{ or } q \in \mathbb{Q}.
\]

We may allow rational charges \( q \) if we do admit spin representations, i.e. multi-valued representations. Let us again fluctuate the flat Dirac operator \( \tilde{D}_t \) and compute the spectral action of the fluctuated Dirac operator:

\[
L(\varphi, u)\tilde{D}_t L(\varphi, u)^{-1} = \begin{pmatrix} \varphi & 0 \\ 0 & C\varphi C^{-1} \end{pmatrix}.
\]

As before, a straightforward calculation yields the covariant derivative:

\[
\varphi = ie^{-1}\mu_a\gamma^a[\partial_\mu + s(\omega_\mu) - qA_\mu], \quad A_\mu = u\partial/\partial x^\mu u^{-1},
\]

identifying indeed the power \( q \) in the central extension \( \text{[10]} \) as electric charge. The second stroke unifies gravity with electrodynamics:

\[
S_{\text{CC}} = \text{tr} f(\tilde{\varphi}^2 / \Lambda^2)
\]

\[
= \int_M [\Lambda_c - \frac{m_e^2}{16\pi}R + a(5R^2 - 8\text{Ricci}^2 - 7\text{Riemann}^2) + \frac{1}{4g^2}F_{\mu\nu}^*F^{\mu\nu}] \sqrt{\text{det} g_{\mu\nu}}d^4x + O(\Lambda^{-2}),
\]

where the electric coupling constant is \( g^2 = \frac{6\pi^2}{f_4} \). Although the algebra of functions on spacetime is commutative, its group of automorphisms, the diffeomorphism group, and its spin lift, the local Lorentz group, are nonAbelian. Consequently general relativity has nonlinear field equations. On the other hand, the group of unitaries remains Abelian and Maxwell’s equations are linear. Therefore the electric charge \( q \) and the electric coupling constant \( g = \epsilon_0^{-1/2} \) only appear as products \( qg \) and by means of a finite renormalization of the coupling constant, we may put \( q = 1 \) for the electron.
3 Lifts in finite, noncommutative geometries and their central extensions

The algebra $\mathcal{A}$ is a real, associative involution algebra with unit, that admits a faithful $^*$ representation $\rho$. In finite dimensions, a simple such algebra is a real, complex or quaternion matrix algebra, $\mathcal{A} = M_n(\mathbb{R})$, $M_n(\mathbb{C})$ or $M_n(\mathbb{H})$, represented irreducibly on the Hilbert space $\mathcal{H} = \mathbb{R}^n$, $\mathbb{C}^n$ or $\mathbb{C}^{2n}$. In the first and third case, the representations are the fundamental ones, $\rho(a) = a$, $a \in \mathcal{A}$, while $M_n(\mathbb{C})$ has two non-equivalent irreducible representations on $\mathbb{C}^n$, the fundamental one, $\rho(a) = a$ and its complex conjugate $\rho(a) = \bar{a}$. In the general case we have sums of simple algebras and sums of irreducible representations. To simplify notations, we concentrate on complex matrix algebras $M_n(\mathbb{C})$ in this section. Indeed the others, $M_n(\mathbb{R})$ and $M_n(\mathbb{H})$, do not have central unitaries close to the identity. In the following it will be important to separate the commutative and noncommutative parts of the algebra:

$$\mathcal{A} = \mathbb{C}^M \oplus \bigoplus_{k=1}^{N} M_{n_k}(\mathbb{C}) \ni a = (b_1, \ldots, b_M, c_1, \ldots, c_N), \quad n_k \geq 2. \tag{20}$$

Its group of unitaries is

$$U(\mathcal{A}) = U(1)^M \times \bigtimes_{k=1}^{N} U(n_k) \ni u = (v_1, \ldots, v_M, w_1, \ldots, w_N) \tag{21}$$

and its group of central unitaries

$$U^c(\mathcal{A}) := U(\mathcal{A}) \cap \text{center}(\mathcal{A}) = U(1)^{N+M} \ni u_c = (v_{c1}, \ldots, v_{cM}, w_{c1}1_{n_1}, \ldots, w_{cN}1_{n_N}). \tag{22}$$

The component of the automorphisms group $\text{Aut}(\mathcal{A})$, that is connected to the identity, is the group of inner automorphisms, $\text{Aut}(\mathcal{A})^e = \text{In}(\mathcal{A})$. There are additional, discrete automorphisms, the complex conjugation and, if there are identical summands in $\mathcal{A}$, their permutations. These discrete automorphisms do not concern us here. An inner automorphism is of the form $i_u(a) = uau^{-1}$ for some unitary $u \in U(\mathcal{A})$. Multiplying $u$ with a central unitary $u_c$ of course does not affect the inner automorphism $i_{ucu} = i_u$. Note that this ambiguity distinguishes between ‘harmless’ central unitaries $v_{c1}, \ldots, v_{cM}$ and the others, $w_{c1}, \ldots, w_{cN}$, in the sense that

$$\text{In}(\mathcal{A}) = U^n(\mathcal{A})/U^{nc}(\mathcal{A}), \tag{23}$$

where we have defined the group of noncommutative unitaries

$$U^n(\mathcal{A}) := \bigtimes_{k=1}^{N} U(n_k) \ni w \tag{24}$$
and $U^{nc}(\mathcal{A}) := U^n(\mathcal{A}) \cap U^c(\mathcal{A}) \ni w_c$. The map

\[
i : U^n(\mathcal{A}) \longrightarrow \text{In}(\mathcal{A})
\]

\[(1, w) \mapsto i_w \quad \text{(25)}\]

has kernel $\text{Ker} \ i = U^{nc}(\mathcal{A})$.

The lift of an inner automorphism to the Hilbert space has a simple closed form \[3\], $L = \hat{L} \circ i^{-1}$ with

\[
\hat{L}(w) = \rho(1, w)J\rho(1, w)^{-1}. \quad \text{(26)}
\]

It satisfies $p(\hat{L}(w)) = i(w)$. If the kernel of $i$ is contained in the kernel of $\hat{L}$ then the lift is well defined, as e.g. for $\mathcal{A} = \mathbb{H}$, $U^{nc}(\mathbb{H}) = \mathbb{Z}_2$.

\[
\begin{array}{c}
\text{Aut}_\mathcal{H}(\mathcal{A}) \\
p \uparrow \quad L \downarrow \hat{L} \\
\text{In}(\mathcal{A}) \leftarrow \leftarrow U^n(\mathcal{A}) \xrightarrow{\det} U^{nc}(\mathcal{A})
\end{array}
\quad \text{(27)}
\]

For more complicated real or quaternionic algebras, $U^{nc}(\mathcal{A})$ is finite and the lift $L$ is multi-valued with a finite number of values. For noncommutative, complex algebras, their continuous family of central unitaries can not be eliminated except for very special representations and we face a continuous infinity of values. The solution of this problem follows an old strategy: ‘If you can’t beat them, adjoin them’. Who is them? The harmful central unitaries $w_c \in U^{nc}(\mathcal{A})$ and adjoining means central extending. The central extension \[\text{II}\], generalizes naturally from the algebra $\mathbb{C}$ to our present setting:

\[
\ell(w_c) = p\left(\prod_{j_1=1}^{N} (w_{cj_1})^{q_{j_1}}, \ldots, \prod_{j_M=1}^{N} (w_{cj_M})^{q_{jM}}, \prod_{j_{M+1}=1}^{N} (w_{cj_{M+1}})^{q_{M+1,j_{M+1}}1_{n_{1}}}, \ldots, \prod_{j_{M+N}=1}^{N} (w_{cj_{M+N}})^{q_{M+N,j_{M+N}}1_{n_{N}}}\right)J\rho(\ldots)J^{-1} \quad \text{(28)}
\]

with the $(M+N) \times N$ matrix of charges $q_{kj}$. The extension satisfies indeed $p(\ell(w_c)) = 1 \in \text{In}(\mathcal{A})$ for all $w_c \in U^{nc}(\mathcal{A})$.

Having adjoined the harmful, continuous central unitaries, we may now streamline our notations and write the group of inner automorphisms as

\[
\text{In}(\mathcal{A}) = \left(\bigotimes_{k=1}^{N} SU(n_k)\right) / \Gamma \ni [w_\varphi] = [(w_{\varphi 1}, \ldots, w_{\varphi N})] \mod \gamma. \quad \text{(29)}
\]
Γ is the discrete group
\[ \Gamma = \prod_{k=1}^{N} \mathbb{Z}_{n_k} \ni (z_1^{n_1}, \ldots, z_N^{1_{n_N}}), \quad z_k = \exp[-m_k 2\pi i / n_k], \quad m_k = 0, \ldots, n_k - 1. \]

The quotient is factor by factor. This way to write inner automorphisms is convenient for complex matrices, but not available for real and quaternionic matrices. Equation (30) remains the general characterization of inner automorphisms.

The lift \( L(w_\varphi) = (\hat{L} \circ i^{-1})(w_\varphi) \) is multi-valued with, depending on the representation, up to \( |\Gamma| = \prod_{j=1}^{N} n_j \) values. More precisely the multi-valuedness of \( L \) is indexed by the elements of the kernel of the projection \( p \) restricted to the image \( L(\text{In}(A)) \). Depending on the choice of the charge matrix \( q \), the central extension \( \ell \) may reduce this multi-valuedness. Extending harmless central unitaries is useless for any reduction. With the multi-valued group homomorphism

\[
(h_\varphi, h_c) : U^n(A) \rightarrow \text{In}(A) \times U^{nc}(A) \quad (w_j) \rightarrow ((w_{\varphi j}, w_{c j})) = ((w_j(\det w_j)^{-1/n_j}, (\det w_j)^{1/n_j})),
\]

we can write the two lifts \( L \) and \( \ell \) together in closed form \( \mathbb{L} : U^n(A) \rightarrow \text{Aut}_H(A) \):

\[
\mathbb{L}(w) = L(h_\varphi(w)) \ell(h_c(w)) = \rho \left( \prod_{j_1=1}^{N} (\det w_{j_1})^{\tilde{q}_{j_1}}, \ldots, \prod_{j_M=1}^{N} (\det w_{j_M})^{\tilde{q}_{Mj_M}}, w_1 \prod_{j_{M+1}=1}^{N} (\det w_{j_{M+1}})^{\tilde{q}_{M+1j_{M+1}}}, \ldots, w_N \prod_{j_{N+M}=1}^{N} (\det w_{j_{N+M}})^{\tilde{q}_{N+Mj_{N+M}}} \right) \times J\rho(...).J^{-1}.
\]

We have set

\[
\tilde{q} := \left( q - \begin{pmatrix} 0_{M \times N} \\ 1_{N \times N} \end{pmatrix} \right) \begin{pmatrix} n_1 \\ \vdots \\ n_N \end{pmatrix}^{-1}.
\]

Due to the phase ambiguities in the roots of the determinants, the extended lift \( \mathbb{L} \) is multi-valued in general. It is single-valued if the matrix \( \tilde{q} \) has integer entries, e.g. \( q = \begin{pmatrix} 0 \\ 1_N \end{pmatrix} \), then \( \tilde{q} = 0 \) and \( \mathbb{L}(w) = \hat{L}(w) \). On the other hand \( q = 0 \) gives \( \mathbb{L}(w) = \hat{L}(i^{-1}(h_\varphi(w))) \), not always well defined as already noted. Unlike the extension \( \ell \) of general relativity, equation (16), and unlike the map \( i \), the extended lift \( \mathbb{L} \) is not necessarily even. We do impose this symmetry...
\( \mathbb{L}(u) = \mathbb{L}(-u) \) which translates into conditions on the charges, conditions that depend on the details of the representation \( \rho \).

The lift \( \mathbb{L} \) is the 'representation up to a phase' that we have studied earlier in the case of the standard model \([10]\). Let us note that \( \mathbb{L} \) is not the most general lift. We could have added the harmless central unitaries, and, if the representation \( \rho \) is reducible, we could have chosen different charge matrices in different irreducible components.

## 4 The standard model

The internal algebra \( \mathbb{A} \) is chosen as to reproduce \( SU(2) \times U(1) \times SU(3) \) as subgroup of \( U(\mathbb{A}) \),

\[
\mathbb{A} = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \ni (a, b, c).
\]  

(34)

The internal Hilbert space is copied from the Particle Physics Booklet \([11]\),

\[
\mathcal{H}_L = (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C}^2 \otimes \mathbb{C}^N \otimes \mathbb{C}), \\
\mathcal{H}_R = (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}^3) \oplus (\mathbb{C} \otimes \mathbb{C}^N \otimes \mathbb{C}).
\]  

(35)\hspace{1cm} (36)

In each summand, the first factor denotes weak isospin doublets or singlets, the second denotes \( N \) generations, \( N = 3 \), and the third denotes colour triplets or singlets. Let us choose the following basis of the internal Hilbert space, counting fermions and antifermions independently as explained in section 2, \( \mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R \oplus \mathcal{H}_L^c \oplus \mathcal{H}_R^c = \mathbb{C}^{90} \):

\[
\begin{pmatrix}
(u_d)_L, (c_s)_L, (t_b)_L, (\nu_e)_L, (\nu_\mu)_L, (\nu_\tau)_L; \\
u_R, c_R, t_R, e_R, \mu_R, \tau_R;
\end{pmatrix}, \\
\begin{pmatrix}
(u_d)_L, (c_s)_L, (t_b)_L, (\nu_e)_L, (\nu_\mu)_L, (\nu_\tau)_L; \\
u_R^c, c_R^c, t_R^c, e_R^c, \mu_R^c, \tau_R^c.
\end{pmatrix}
\]

This is the current eigenstate basis, the representation \( \rho \) acting on \( \mathcal{H} \) by

\[
\rho(a, b, c) := \begin{pmatrix}
\rho_L & 0 & 0 & 0 \\
0 & \rho_R & 0 & 0 \\
0 & 0 & \tilde{\rho}_L & 0 \\
0 & 0 & 0 & \tilde{\rho}_R
\end{pmatrix}
\]  

(37)

with

\[
\rho_L(a) := \begin{pmatrix}
a \otimes 1_N \otimes 1_3 & 0 \\
0 & a \otimes 1_N
\end{pmatrix}, \quad \rho_R(b) := \begin{pmatrix}
b_1N \otimes 1_3 & 0 & 0 \\
0 & \tilde{b}_1N \otimes 1_3 & 0 \\
0 & 0 & \tilde{b}_1N
\end{pmatrix},
\]  

(38)
\[ \rho_L^c(b, c) := \left( \begin{array}{cc} 1_2 \otimes 1_N \otimes c & 0 \\ 0 & 0 \end{array} \right), \quad \rho_R^c(b, c) := \left( \begin{array}{ccc} 1_N \otimes c & 0 & 0 \\ 0 & 1_N \otimes c & 0 \\ 0 & 0 & \bar{b}_1 \otimes 1_N \end{array} \right). \tag{39} \]

At this point we can explain why only isospin doublets and singlets and colour triplets and singlets are allowed in the fermionic representation: all other irreducible group representations cannot be extended to algebra representation. While the tensor product of two group representations is again a group representation, the tensor product of two algebra representations is not an algebra representation. The apparent asymmetry between particles and antiparticles – the former are subject to weak, the latter to strong interactions – disappears after application of the lift \( \mathbb{L} \) with

\[ J = \left( \begin{array}{cc} 0 & 1_{15N} \\ 1_{15N} & 0 \end{array} \right) \circ \text{complex conjugation.} \tag{40} \]

For the sake of completeness, we record the chirality as matrix

\[ \chi = \left( \begin{array}{cccc} -1_{8N} & 0 & 0 & 0 \\ 0 & 1_{7N} & 0 & 0 \\ 0 & 0 & -1_{8N} & 0 \\ 0 & 0 & 0 & 1_{7N} \end{array} \right). \tag{41} \]

The internal Dirac operator

\[ \mathcal{D} = \begin{pmatrix} 0 & \mathcal{M} & 0 & 0 \\ \mathcal{M}^* & 0 & 0 & 0 \\ 0 & 0 & \mathcal{M} & 0 \\ 0 & \mathcal{M}^* & 0 & 0 \end{pmatrix} \tag{42} \]

contains the fermionic mass matrix of the standard model,

\[ \mathcal{M} = \begin{pmatrix} \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \otimes M_u \otimes 1_3 + \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \otimes M_d \otimes 1_3 & 0 \\ 0 & \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \otimes M_e \end{pmatrix}, \tag{43} \]

with

\[ M_u := \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & m_t \end{pmatrix}, \quad M_d := C_{KM} \begin{pmatrix} m_d & 0 & 0 \\ 0 & m_s & 0 \\ 0 & 0 & m_b \end{pmatrix}, \tag{44} \]

\[ M_e := \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}. \tag{45} \]

From the booklet we know that all indicated fermion masses are different from each other and that the Cabibbo-Kobayashi-Maskawa matrix \( C_{KM} \) is non-degenerate in the sense that no quark is simultaneously mass and weak current eigenstate.
In the commutative setting, $A = C^\infty(M)$, $H = L^2(S)$, the algebraic formulation of the fact that the Dirac operator is a first order differential operator reads $[[\partial, \rho(a)], J\rho(\bar{a})J^{-1}] = 0$ for all $a, \bar{a} \in A$. In Connes’ noncommutative geometry this property becomes an axiom, ‘the first order axiom’, $[[D, \rho(a)], J\rho(\bar{a})J^{-1}] = 0$. In the example of the standard model, this axiom entails the existence of a gauge group that commutes with the electro-weak interactions and with the fermionic mass matrix and whose fermion representation is vectorial [12]. One of the important features of Connes’ coding of geometry via ‘spectral triples’ $(A, H, D, J, \chi)$ is that they can be tensorized. In the case of two Riemannian manifolds, this tensor product describes the direct product. This tensor product also generalizes the one, that Connes used to unify electromagnetism with gravity, equations (13,14). The tensor product of a Riemannian geometry $M$ and a zero dimensional one, i.e. with finite dimensional algebra and Hilbert space like the internal space of the standard model, has as Dirac operator $D_t = \partial \otimes 1 + \gamma_5 \otimes D$, the free, massive Dirac operator. Its fluctuations with $\mathbb{L}$ produce the minimal couplings to gravity and to the non-Abelian gauge bosons, and the Yukawa couplings to the Higgs boson which in the example of the standard model comes out to be an isospin doublet, colour singlet with hypercharge $-1/2$. The spectral action $S_{CC}$ then yields [4], in addition to the gravitational action, the entire bosonic action of the standard model including the Higgs sector with its spontaneous symmetry breaking. The constraints for the coupling constants, $g_2^2 = g_3^2 = 3\lambda$ occur because the Yang-Mills actions and the $\lambda|\Phi|^4$ term stem from the same heat kernel coefficient $f_{4a_4}$.

Let us go back to the standard model as a Yang-Mills-Higgs model and suppose that god has given the fermionic representation content of isospin and colour. The hypercharges can then be chosen arbitrarily, five rational numbers, $y_1, \ldots, y_5$. $y_1$ is the hypercharge of the left-handed quarks, $y_2$ of the left-handed leptons, $y_3$ of the right-handed up-quarks, $y_4$ the hypercharge of the right-handed down quarks and $y_5$ of the right-handed leptons. The Lorentz force prohibits massless particles with non-vanishing electric charge. Therefore the hypercharge $y_2$ of the purely left-handed neutrinos must be different from zero and by a finite renormalization we can set $y_2 = -1/2$. If we want left- and right-handed particles to have the same electric charge, then we must impose the three conditions

$$y_3 = \frac{1}{2} + y_1 = y_1 - y_2, \quad y_4 = -\frac{1}{2} + y_1 = y_1 + y_2, \quad y_5 = -\frac{1}{2} + y_2 = 2y_2.$$ (46)

The hypercharges are then completely fixed by putting $y_1 = 1/6$ which amounts to choose the electric charge of the quarks. Let us summarize nature’s choice of the fermionic hypercharges,

$$y_1 = \frac{1}{6}, \quad 6y_1 = 1 \mod 2 \quad \text{and} \quad 1 \mod 3,$$
$$y_2 = -\frac{1}{2}, \quad 6y_2 = 1 \mod 2 \quad \text{and} \quad 0 \mod 3,$$
$$y_3 = \frac{2}{3}, \quad 6y_3 = 0 \mod 2 \quad \text{and} \quad 1 \mod 3,$$

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\begin{align*}
y_4 &= -\frac{1}{3}, \quad 6y_4 = 0 \mod 2 \quad \text{and} \quad 1 \mod 3, \\
y_5 &= -1, \quad 6y_5 = 0 \mod 2 \quad \text{and} \quad 0 \mod 3.
\end{align*}

(47)

At this point O’Raifeartaigh [7] remarks that after renormalizing the hypercharges by a factor 6, the isomorphism $U(n) \to [SU(n) \times U(1)]/\mathbb{Z}_n$ induced by the multi-valued homomorphism $h$, equation (31), extents to the fermion representations for isospin, $n = 2$, and colour, $n = 3$. Indeed $\rho(u) = (\det u)^z u, \ u \in U(n), \ z \in \mathbb{Z}$, defines a representation of $U(n)$ and under the isomorphism (31) it induces the fundamental representation of $SU(n)$ with $U(1)$ charge $1 + zn$:

$$(\det u)^z u = u_c^{zn} u_c u_\phi = u_c^{1+zn} u_\phi.$$ (48)

The $U(n)$ representation $\rho(u) = (\det u)^z u = u_c^{zn}$ induces the $SU(n)$ singlet representation with $U(1)$ charge $zn$. $U(1)$ charges are one modulo $n$ for fundamental multiplets, zero modulo $n$ for singlets, precisely nature’s choice (17). In other words nature only represents a quotient of $SU(2) \times U(1) \times SU(3)$ on fermions (and bosons). This faithfully represented quotient is $[SU(2) \times U(1) \times SU(3)]/[\mathbb{Z}_2 \times \mathbb{Z}_3]$. The $\mathbb{Z}_n$s are the centers of the $SU(n)$s but they do act on the $U(1)$ which is not the case of $\Gamma$ in (60) below. O’Raifeartaigh’s reduction is a stronger restriction than charge quantization, the hypercharges $\times 6$ are not only to be integers, they must satisfy the conditions in the second and third columns of equations (17).

If we take the standard model as a noncommutative geometry, isospin and colour of the fermions are given by the geometry. Now what does this geometry tell us about the hypercharges? The most general algebra representation compatible with the first order axiom, involves four ‘charges’ $\tilde{y}_1, \ldots, \tilde{y}_4$. Each $\tilde{y}_i$ can take only 2 values, $-1$ or $+1$,

$$\rho_L(a) := \begin{pmatrix} a \otimes 1_N \otimes 1_3 & 0 \\ 0 & a \otimes 1_N \end{pmatrix}, \quad \rho_R(b) := \begin{pmatrix} b_3 1_N \otimes 1_3 & 0 & 0 \\ 0 & b_4 1_N \otimes 1_3 & 0 \\ 0 & 0 & b_5 1_N \end{pmatrix},$$ (49)

$$\rho^{\epsilon}_L(b, c) := \begin{pmatrix} 1_2 \otimes 1_N \otimes c & 0 \\ 0 & b_2 1_2 \otimes 1_N \end{pmatrix}, \quad \rho^{\epsilon}_R(b, c) := \begin{pmatrix} 1_N \otimes c & 0 & 0 \\ 0 & 1_N \otimes c & 0 \\ 0 & 0 & b_2 1_N \end{pmatrix},$$ (50)

with $b_j := [(1 + \tilde{y}_j)b/2 + (1 - \tilde{y}_j)b/2]$. With the algebra automorphism of $\mathbb{C}$, $b \mapsto \bar{b}$, we can always arrange $\tilde{y}_2 = -1$. We must have $\tilde{y}_3 = \tilde{y}_4$, otherwise the right-handed leptons would be electrically neutral leading to charged neutrinos in conflict with the Lorentz force. On the other hand, we must have $\tilde{y}_3 \neq \tilde{y}_4$, otherwise the bottom and top masses would coincide after spontaneous symmetry breaking. We are back at the representation of equations (38,39), possibly after a permutation of $u$ with $d^c$ and of $d$ with $u^c$. 12
We have the following groups,

\[
U(\mathcal{A}) = SU(2) \times U(1) \times U(3) \ni u = (u_0, v, w),
\]

(51)

\[
U^c(\mathcal{A}) = \mathbb{Z}_2 \times U(1) \times U(1) \ni u_c = (u_{c0}, v_c, w_c1_3),
\]

(52)

\[
U^n(\mathcal{A}) = SU(2) \times U(3) \ni (u_0, w),
\]

(53)

\[
U^{nc}(\mathcal{A}) = \mathbb{Z}_2 \times U(1) \ni (u_{c0}, w_c1_3),
\]

(54)

\[
\text{In}(\mathcal{A}) = [SU(2) \times SU(3)]/\Gamma \ni u_\varphi = (u_{\varphi 0}, w_\varphi),
\]

(55)

\[
\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_3 \ni \gamma = (\exp[-m_02\pi i/2], \exp[-m_22\pi i/3]),
\]

(56)

with \(m_0 = 0,1\) and \(m_2 = 0,1,2\). Let us compute the receptacle of the lifted automorphisms,

\[
\text{Aut}_L(\mathcal{A}) = [U(2)_L \times U(3)_c \times U(N)_{qL} \times U(N)_{\ell L} \times U(N)_{uR} \times U(N)_{dR}]/[U(1) \times U(1)] \times U(N)_{eR}.
\]

(57)

The subscripts indicate on which generation multiplet the \(U(N)s\) act, \(qL\) for the \(N = 3\) left-handed quarks, \(\ell L\) for the left-handed leptons, \(uR\) for the right-handed up-quarks and so forth. The kernel of the projection down to the automorphism group \(\text{Aut}(\mathcal{A})\) is

\[
\ker p = [U(1) \times U(1) \times U(N)_{qL} \times U(N)_{\ell L} \times U(N)_{uR} \times U(N)_{dR}]/[U(1) \times U(1)] \times U(N)_{eR},
\]

(58)

and its restrictions to the images of the lifts are

\[
\ker p \cap \text{L(\text{In}(\mathcal{A}))} = \mathbb{Z}_2 \times \mathbb{Z}_3, \quad \ker p \cap \text{L}(U^n(\mathcal{A})) = \mathbb{Z}_2 \times U(1).
\]

(59)

As a side remark we anticipate that the maximally extended standard model in noncommutative geometry [8], that gauges \(U(N)_{uR}\) and \(U(N)_{dR}\) simultaneously, is not viable: it has massless, physical Higgs scalars and also \(m_u = m_d\) [8].

The kernel of \(i\) is \(\mathbb{Z}_2 \times U(1)\) in sharp contrast to the kernel of \(\hat{L}\) which is trivial. The isospin \(SU(2)_L\) and the colour \(SU(3)_c\) are the image of the lift \(\hat{L}\). If \(q \neq 0\), the image of \(\ell\) consists of one \(U(1) \ni w_c = \exp[i\theta]\) contained in the five flavour \(U(N)s\). Its embedding depends on \(q\):

\[
\mathbb{L}(1_2, 1, w_c1_3) = \ell(w_c) = \begin{Diag}
& u_{11} \otimes 1_N \otimes 1_3, u_{21} \otimes 1_N \otimes 1_3, u_{31} \otimes 1_N \otimes 1_3, u_{41} \otimes 1_N \otimes 1_3, u_{51} \otimes 1_N \\
& \bar{u}_{11} \otimes 1_N \otimes 1_3, \bar{u}_{21} \otimes 1_N \otimes 1_3, \bar{u}_{31} \otimes 1_N \otimes 1_3, \bar{u}_{41} \otimes 1_N \otimes 1_3, \bar{u}_{51} \otimes 1_N
\end{Diag}
\]

(60)

with \(u_j = \exp[iy_j\theta]\) and

\[
y_1 = q_2, \quad y_2 = -q_1, \quad y_3 = q_1 + q_2, \quad y_4 = -q_1 + q_2, \quad y_5 = -2q_1.
\]

(61)
Independently of the embedding, we have indeed derived the three conditions (46), that in the Yang-Mills-Higgs version had to be imposed. In other words, in noncommutative geometry the massless electroweak gauge boson necessarily couples vectorially.

Our goal is now to find the minimal extension $\ell$, that renders the extended lift symmetric, $\mathbb{L}(-u_0,-w) = \mathbb{L}(u_0,w)$, and that renders $\mathbb{L}(1_2,w)$ single valued. The first requirement means \{ $\tilde{q}_1 = 1$ and $\tilde{q}_2 = 0$ \} modulo 2, with

$$\left(\begin{array}{c}
\tilde{q}_1 \\
\tilde{q}_2
\end{array}\right) = \frac{1}{3} \left(\begin{array}{c}
q_1 \\
q_2
\end{array} - \left(\begin{array}{c}
0 \\
1
\end{array}\right)\right).$$

The second requirement means that $\tilde{q}$ has integer coefficients.

The first extension, that comes to mind, has $q = 0$, $\tilde{q} = \left(\begin{array}{c}0 \\
-1/3\end{array}\right)$. With respect to the interpretation (29) of the inner automorphisms, one might object that this is not an extension at all. With respect to the generic characterization (23) it certainly is a non-trivial extension. Anyhow it fails both tests. The most general extension, that passes both tests has the form

$$\tilde{q} = \left(\begin{array}{c}2z_1 + 1 \\
2z_2
\end{array}\right), \quad q = \left(\begin{array}{c}6z_1 + 3 \\
6z_2 + 1
\end{array}\right), \quad z_1, z_2 \in \mathbb{Z}.$$  

Consequently $y_2 = -q_1$ cannot vanish, the neutrino comes out electrically neutral in compliance with the Lorentz force. Let us normalize the hypercharges to $y_2 = -1/2$ and compute the last remaining hypercharge $y_1$,

$$y_1 = \frac{q_2}{2q_1} = \frac{1}{6} + \frac{z_2}{1 + 2z_1}. \quad (64)$$

We can change the sign of $y_1$ by permuting $u$ with $d^c$ and $d$ with $u^c$. Therefore it is sufficient to take $z_1 = 0, 1, 2, \ldots$ The minimal such extension, $z_1 = z_2 = 0$, recovers nature’s choice $y_1 = \frac{1}{6}$. Its lift,

$$\mathbb{L}(u_0, w) = \rho(u_0, \det w, w)J\rho(u_0, \det w, w)J^{-1}, \quad (65)$$

is the fermionic representation of the standard model considered as $SU(2) \times U(3)$ Yang-Mills-Higgs model. This lift is double-valued as the gravitational spin lift (3). The double-valuedness of $\mathbb{L}$ comes from the discrete group $\mathbb{Z}_2$ of central unitaries $(\pm 1_2, 1_3) \in \mathbb{Z}_2 \subseteq \Gamma \subseteq U^{nc}(A)$ and cannot be removed by any central extension of the form (28). On the other hand O’Raifeartaigh’s $\mathbb{Z}_2, \pm(1_2, 1_3) \in \mathbb{Z}_2 \subseteq U^{nc}(A)$ is not a subgroup of $\Gamma$. It reflects the symmetry of $\mathbb{L}$.

5 Conclusion

Central extensions of the lift of automorphisms play three roles: in the commutative case they unify gravity and electromagnetism. There, the central unitaries were harmless and the
extensions optional. In general, an extension is mandatory to reduce the multi-valuedness of the lift and to reestablish its symmetry which may be lost when suppressing the harmless unitaries. In the case of the internal space of the standard model, the non-extended lift is at least 6-valued. The minimal extension leading to maximal reduction is the key to the hypercharges of all fermions. The minimal remaining multi-valuedness is double, precisely as for the lift in spacetime. This raises the question if the two double-valuednesses are related and if the internal one is also accessible to experiment.

Still we wonder: Why does the algebra of the standard model need a commutative part $\mathbb{C}$ and produce unitaries which are harmless and only make a virtual appearance? The $\mathbb{C}$ plays two essential roles. It prints the seat tickets, that indicate where in the receptacle $\text{Aut}_H(\mathcal{A})$ of the lifted automorphisms the harmful unitaries are to be seated. Its second role is more physical [6]: Without at least one (non-vectorial) $\mathbb{C}$ in the internal algebra, the symmetry breaking induced by the spectral action gives identical masses to Dirac spinors in the same irreducible multiplet.

As always, it is a pleasure to acknowledge Bruno Iochum’s constructive critique.

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