Quantum Spectrum of Instanton Solitons
in Five Dimensional Noncommutative $U(N)$ Theories

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We explore quantum states of instanton solitons in five dimensional noncommutative Yang-Mills theories. We start with maximally supersymmetric $U(N)$ theory compactified on a circle $S^1$, and derive the low energy dynamics of instanton solitons, or calorons, which is no longer singular. Quantizing the low energy dynamics, we find $N$ physically distinct ground states with a unit Pontryagin number and no electric charge. These states have a natural D-string interpretation. The conclusion remains unchanged as we decompactify $S^1$, as long as we stay in the Coulomb phase by turning on adjoint Higgs expectation values.

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1 Instantons and Monopoles in Noncommutative Yang-Mills

In Yang-Mills theories in five dimensions, solitons carrying Pontryagin charge, or real time instantons, can appear. Unlike their four-dimensional cousins, magnetic monopoles, the low energy dynamics of these solitons are singular due to well-known singularities in instanton moduli spaces. A singularity appears when any scale parameter vanishes, i.e., when an instanton size becomes zero. This prevents a reliable computation of low energy spectra in sectors with nonzero Pontryagin numbers.

Recently, it was realized that a noncommutative version of such Yang-Mills theories arises naturally from open string propagating on D-branes [1, 2, 3]. While we will not go into details, it suffices to observe that the resulting theory can be understood by allowing coordinates to be noncommuting variables. One effect of this is to remove very small distance scales. Therefore, one expects to find that the instanton cannot be smaller than the size set by the noncommutativity. Indeed, recent study of noncommutative Yang-Mills theory indicates that small instanton singularity [4] of the instanton moduli space is resolved once we deform the Yang-Mills theory in this manner [5, 6, 7, 3]. Naturally, this opens a way to regularize dynamics of five-dimensional solitons.

Before delving into low energy dynamics, it is instructive to understand how solitons themselves are deformed. Easiest way to construct such deformed solitons on noncommutative $R^{1+4}$ would be via the ADHM construction. In string theory context, one is considering the D0 dynamics inside $N$ D4-branes. The Higgs vacua of D0 parameterize the instanton moduli space of the $U(N)$ theory, which can be found by solving a series of D-term conditions [4]. These D-term conditions are nothing but the ADHM equation [8].

However, a little bit more intuitive picture is obtained by considering T-dual of this when compactified on $R^{1+3} \times S^1$. Let the radius of $S^1$ be $R$. One can take the decompactification limit $R \to \infty$ at the end, if one wishes. The T-duality maps D4-branes to D3-branes whose worldvolumes are parallel to $R^{1+3}$ and transverse to the dual circle $\tilde{S}^1$ of radius $\tilde{R} = \alpha'/R$. For $U(N)$ gauge theory on $R^{1+3} \times S^1$, one has $N$ such D3 branes. The instanton charge is given by the D-string winding number along $\tilde{S}^1$, so a single instanton corresponds to a singly wound D-string along $\tilde{S}^1$. This D-string is in general broken up at D3’s, and the open D-string segments between adjacent D3’s can move freely along $R^3$ directions. These open D-string segments can be interpreted as fundamental monopoles [1].

As there are $N$ such intervals, there are $N$ fundamental monopoles, corresponding to the roots in the extended Dynkin diagram. These $N$ solitons constitute an instanton on $R^3 \times S^1$, also
known as a caloron \([9, 10, 11, 12]\). The corresponding Nahm data encodes the positions of the D-string segments and gauge fields on them.

Suppose we introduce noncommutativity on \(R^3 \times S^1\), by turning on a uniform NS-NS tensor field on it;

\[
[x_\mu, x_\nu] = i\theta_{\mu\nu}, \quad i = 1, 2, 3, 4. \tag{1}
\]

The antisymmetric 2-tensor \(\theta_{\mu\nu}\) will be assumed to be covariantly constant. The ADHM formalism of the instanton in the noncommutative Yang-Mills theory is developed in Ref. [5, 7], where it was shown that the ADHM equation (or D-term condition) acquires a triplet of constant terms that can be thought of as Fayet-Iliopoulos term of the D0 worldvolume theory in the presence of D4-branes. This triplet of number \(\zeta_a\)'s \((a = 1, 2, 3)\) are related to the anti-self-dual part of \(\theta\) as

\[
\theta(-) = \zeta^a \left( dx^4 \wedge dx^a - \frac{1}{2} \epsilon^{abc} dx^b \wedge dx^c \right). \tag{2}
\]

Because we compactified one direction, the ADHM system actually consists of infinite number of mirror images of D0. T-dualizing this picture to D3-D1 system, the D-term conditions become one-dimensional anti-self-dual equations known as Nahm’s equation [12]:

\[
\frac{dT^a}{dt} + \epsilon_{abc}[T^b, T^c] = \zeta^a + \sum_{i=1}^{N} \delta(t - t_i) a_i \sigma^a a_i^\dagger. \tag{3}
\]

The coordinate \(t\) parameterizes the circumference of the dual circle \(\tilde{S}^1\), and \(t = t_i\)'s are positions of D3-branes along \(\tilde{S}^1\). We have chosen the coordinate \(t\) to have dimension of mass, while \(T^a\)'s have dimension of lengths, by inserting appropriate factors of \(\alpha'\), so that \(0 \leq t \leq 2\pi \tilde{R}/\alpha'\). \(T^a\)'s are \(k \times k\) unitary matrices where \(k\) is the Pontryagin number. The first term on the right hand side comes from the noncommutativity, while the second terms reflects the possibility that D-strings are broken up along each D3. Each \(a_i\) is a \(2k\)-dimensional complex vector \(a_i^{A\alpha}\), \(A = 1, \ldots, k\), \(\alpha = 1, 2\), and \(\sigma^a\)'s are the usual Pauli matrices. Conventional Nahm’s equation for caloron is recovered when \(\zeta^a\)'s are taken to zero.

For a single instanton with \(k = 1\) (a self-dual instanton), the numbers \(T^a\) encode positions of D-string segments along \(R^3\) directions and so the interpretation of \(\zeta^a\)'s are quite clear. In the coordinates \(t\) and \(x^a\)'s, all D-string segments are slanted with the slope of \(\tilde{\zeta}\), as in figure 1. This simple picture contains much of physics we need to understand about instantons and monopoles in noncommutative Yang-Mills theories. (For another view of the slanted strings between D-branes, see Ref. [13].)

The D-string segments are magnetically charged with respect to the (unbroken) \(U(1)\) worldvolume gauge fields on \(D3\)'s, and are nothing but non-Abelian magnetic monopoles. Clearly the
single instanton on $\tilde{S}^1 \times R^3$ consists of a collection of D-string segments that completes a singly-wound D-string, and hence a collection of $N$ distinct monopoles. Interestingly enough, the size $\rho$ of the instanton is related to another measure of distance in the multi-monopole picture as
\[
\rho^2 = 2R \left(|\vec{x}_1 - \vec{x}_2| + |\vec{x}_2 - \vec{x}_3| + \cdots + |\vec{x}_{N-1} - \vec{x}_N| + |\vec{x}_N - \vec{x}_1|\right).
\tag{4}
\]
The parameter $\rho$ defined this way coincides with the conventional definition of instanton size in $R^4$ when $\rho$ is much smaller than $R$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{$N$ D3-branes, parallel to $R^{1+3}$, are represented by horizontal lines. They are located at specific points along $\tilde{S}^1$, whose radius is $1/R$. The D-string segments ending on D3-branes are slanted toward the direction given by $\vec{\zeta}$.}
\end{figure}

One immediate difference between the commutative and the noncommutative cases is that the right hand side can vanish only for the former. In particular, the instanton on a single D4-brane is a $U(1)$ instanton, yet its size is not zero; the noncommutativity blows up the small instanton to be of finite size. For general $N$, the smallest possible value the right hand side can take is obtained when D-string is connected at $N - 1$ of $N$ D3-branes. The right hand side is then,
\[
2R \times |2\pi \tilde{R} \vec{\zeta} / \alpha'|,
\tag{5}
\]
where $\tilde{R}$ is the radius of $\tilde{S}^1$. Using the T-dual relationship $\tilde{R}R / \alpha' = 1$, we find that
\[
\rho_{\text{minimum}}^2 = 4\pi |\vec{\zeta}|,
\tag{6}
\]
which persists as we decompactify $S^1$ to go back to $R^{1+4}$.

Note that the two endpoints of a D-string segment is located at different points along $R^3$. Each endpoint is perceived as a magnetic charge with respect to an unbroken $U(1)$ associated with the D3-brane on which the endpoint is located, so this means among other things that the classical solution associated with the segment is not traceless, and so cannot be thought of as an $SU(N)$ configuration. It is necessarily a $U(N)$ configuration. When $N = 1$, the D-string has one connected component but its two endpoints are located at two different points on the D3, in particular. A $U(1)$ instanton on a noncommutative $R^3 \times S^1$ is a magnetic dipole [5]. (See Ref. [14] for the explicit construction of the dipole term for $U(2)$ BPS monopoles.)

2 Moduli Space of a Single Instanton on $R^3 \times S^1$

The moduli space of a unit $U(N)$ periodic instanton has been derived previously in the context of ordinary Yang-Mills theory on $R^3 \times S^1$. With a generic Wilson line, the instanton actually consists of $N$ distinct monopoles, the sum of whose magnetic charges vanish. The moduli space of distinct monopoles are well-understood, and one finds the following hyperKaehler metric for the $4N$ dimensional moduli space [4].

$$g_{\text{total}} = \frac{4\pi^2 R}{e^2} \left( M_{ij} d\vec{x}_i \cdot d\vec{x}_j + (M^{-1})_{ij} (d\xi_i + \vec{v}_{ik} \cdot d\vec{x}_k)(d\xi_j + \vec{v}_{jm} \cdot d\vec{x}_m) \right). \quad (7)$$

$\vec{x}_i$ are $N$ three-vectors, while $\xi_i$ are periodic in $2\pi$. The symmetric matrix $M_{ij}$ depends on differences, $\vec{r}_{ij} = \vec{x}_i - \vec{x}_j$, only, and has the form,

$$
\begin{pmatrix}
\mu_1 + 1/r_{N1} + 1/r_{12} & -1/r_{12} & 0 & \cdots & 0 & -1/r_{N1} \\
-1/r_{12} & \mu_2 + 1/r_{12} + 1/r_{23} & -1/r_{23} & \cdots & 0 & 0 \\
0 & -1/r_{23} & \mu_3 + 1/r_{23} + 1/r_{34} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\end{pmatrix}
\quad (8)
$$

The quantities $\mu_i = 2\epsilon_i / R \geq 0$, with $\sum \epsilon_i = 1$, parameterize the Wilson line, or alternatively the relative positions of D3-branes along the dual circle $\tilde{S}^1$. The vector potentials $\vec{v}_{ij}$ are related to the scalar potentials in $M_{ij}$ by,

$$\vec{V}_i M_{jk} = \vec{V}_i \times \vec{v}_{jk}, \quad (9)$$

which is necessary for the metric to be hyperKaehler.

One way to derive this metric is to consider dynamics of well-separated $N$ monopoles, as in Ref. [9, 15]. The above metric captures the long-range electromagnetic and scalar interactions precisely. In cases of distinct monopoles, no short-distance correction arises, and this simple-minded
derivation gives the right answer. Alternatively, one could start with the Nahm data of a caloron, and derive the corresponding metric in the space of the Nahm data. The above form of the metric was first conjectured by the authors, and subsequently proven by Kraan using Nahm data \cite{16}, where the moduli space is obtained by a finite hyperKaehler reduction of some flat quaternionic space.

The latter method can be easily generalized to noncommutative cases, since the only modification is via the FI constants $\zeta^a$’s. For a single instanton, all that happens is that the moment maps are shifted by an amount proportional to $\zeta^a$’s. Despite the complicated nature of noncommutative Yang-Mills theories, thus, the modification of the instanton moduli space is rather simple. With this in mind, let us consider how noncommutativity may alter the moduli space metric.

Since the long distance behavior of noncommutative theories should be identical to that of the commutative version, we expect the former method to be also informative for well-separated monopoles. So let us concentrate on long-distance interaction between D-string segments. The $i^{th}$ D-string segments appears on $R^3$ as a pair of particles, separated by a fixed vector, $(t_{i+1} - t_i)\vec{\zeta}$. They are charged, respectively, positively with respect to the $i^{th}$ and negatively with respect to the $(i+1)^{th}$ unbroken $U(1)$ on D3-branes; the net electric charge is $(0, \ldots, 0, 1, -1, 0, \ldots, 0)$. From this, we can see easily that the long range interaction is modified only through the correction to $M_{ij}$ (and thus to $\vec{v}_{ij}$) that shifts the harmonic functions $1/|\vec{x}_i - \vec{x}_{i+1}|$ to,

$$
\frac{1}{|\vec{x}_i - \vec{x}_{i+1}|} \rightarrow \frac{1}{|(\vec{x}_i + (t_{i+1} - t_i)\vec{\zeta}/2) - (\vec{x}_{i+1} - (t_{i+2} - t_{i+1})\vec{\zeta}/2)|}.
$$

This harmonic function encodes all long-range interactions via massless fields on the $(i + 1)^{th}$ D3-branes.

On the other hand, the first $N-1$ such shifts can be absorbed in the definition of the Cartesian coordinate $\vec{x}_i - \vec{x}_{i+1}$’s. By doing so, the only vestige of this deformation survives in $|\vec{x}_N - \vec{x}_1|$. The modified metric is then identical to the above except that a scalar potential in $M$ is modified in the following manner:

$$
\frac{1}{|\vec{r}_{N1}|} \rightarrow \frac{1}{|\vec{r}_{N1} - 2\pi\vec{\zeta}/R|} = \frac{1}{|\vec{x}_N - \vec{x}_1 - 2\pi\vec{R}\vec{\zeta}/\alpha'|},
$$

while the rest remains unchanged

$$
\frac{1}{|\vec{r}_{i,i+1}|} \rightarrow \frac{1}{|\vec{r}_{i,i+1}|} = \frac{1}{|\vec{x}_i - \vec{x}_{i+1}|}, \quad i = 1, \ldots, N - 1.
$$

The vector potentials in $\vec{u}_{ij}$ are modified according to the relationship in Eq. (9).
One may separate out the noninteracting center-of-mass degrees of freedom by introducing new coordinates,

\[ \vec{x}_{cm} \equiv \frac{\sum \mu_i \vec{x}_i}{\sum \mu_i}, \]  
\[ \vec{r}_1 \equiv \vec{r}_{12}, \]  
\[ \vdots \]  
\[ \vec{r}_{N-1} \equiv \vec{r}_{N-1,N}, \]

and similar redefinitions for canonical conjugate momenta of \( \xi_i \)'s,

\[ \frac{\partial}{\partial \xi_{cm}} \equiv \left( \frac{\sum \mu_i \partial}{\sum \mu_i} \right) / \sum \mu_i, \]  
\[ \frac{\partial}{\partial \psi_1} \equiv \frac{\partial}{\partial \xi_1} - \frac{\partial}{\partial \xi_2}, \]  
\[ \vdots \]  
\[ \frac{\partial}{\partial \psi_{N-1}} \equiv \frac{\partial}{\partial \xi_{N-1}} - \frac{\partial}{\partial \xi_N}. \]

In the new coordinate system, \( \vec{x}_{cm} \) and \( \xi_{cm} \) decouple from the rest. The \( 4(N-1) \) dimensional interacting part of the moduli space is given by the metric,

\[ g = \frac{4\pi^2 R}{e^2} \left( C_{AB} d\vec{r}_A \cdot d\vec{r}_B + (C^{-1})_{AB}(d\psi_A + \vec{\omega}_{AC} \cdot d\vec{r}_C)(d\psi_B + \vec{\omega}_{BD} \cdot d\vec{r}_D) \right). \]  

\( \psi_A \) are periodic in \( 4\pi \), and the symmetric matrix \( C_{AB} \) has the form,

\[ C_{AB} = \left( \mu_{AB} + \frac{\delta_{AB}}{|\vec{r}_A|} + \frac{1}{|\sum_{A=1}^{N-1} \vec{r}_A - 2\pi \vec{\zeta}/R|} \right). \]  

The last term is common to all components. The vector potentials \( \vec{\omega}_{AB} \) are again related to \( C_{AB} \) by

\[ \vec{\nabla}_D C_{AB} = \vec{\nabla}_D \times \vec{\omega}_{AB}. \]

The “reduced mass matrix” \( \mu_{AB} \) is defined by the formula,

\[ \sum_i \mu_i d\vec{x}_i^2 = (\sum_i \mu_i) d\vec{x}_{cm}^2 + \sum_{A=1}^{N-1} \sum_{B=1}^{N-1} \mu_{AB} d\vec{r}_A \cdot d\vec{r}_B. \]  

If and only if \( \vec{\zeta}/R \) is nonzero, the relative moduli space is smooth. Otherwise, there exists a singularity at origin, \( \vec{r}_A = 0, A = 1, \ldots, N-1 \).

While we derived this metric from the asymptotic interactions between the D-string segments, we have many reasons to believe that we actually found the exact metric. First of all, known moduli spaces of distinct monopoles are always such that long distance dynamics determine the metric everywhere. Furthermore, the shift of moment maps one would have considered in Nahm data approaches is exactly the shift of coordinate \( \vec{x}_i \)'s. In the following section, we will proceed to solve for low energy dynamics of a single instanton with this moduli space.
3 Low Energy Dynamics with or without Higgs Expectation

The instanton soliton breaks half of the supersymmetry present in Yang-Mills theory, and its low energy dynamics is given by the sigma model with four complex supercharges \([17]\):

\[
\mathcal{L} = \frac{1}{2} \left( g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu + ig_{\mu\nu} \bar{\eta}^\mu \gamma^0 D t \bar{\eta}^\nu + \frac{1}{6} R_{\mu\nu\rho\sigma} \bar{\eta}^\mu \eta^\rho \bar{\eta}^\sigma \eta^\nu \right)
\]  

(25)

where we introduced a two-component real fermionic coordinates \(\eta^\mu\) for each \(\mu\). Actually, this dynamics does not take into account possible Higgs expectation values. We are already working in broken Coulomb vacua, due to the Wilson line along \(S^1\), so have no reason to exclude adjoint Higgs expectations.

Maximally supersymmetric Yang-Mills theory in five dimension is written in terms of the vector multiplet, which consists of a vector field, five scalar fields and a pair of Dirac fields in five dimensions. Pictorially the five scalar fields encodes the fluctuation of D4- (or D3-) branes along the five Euclidean directions transverse to \(R^{1+3} \times S^1\) (or to \(R^{1+3} \times \tilde{S}^1\)). In the presence of small expectation value of a single Higgs field, the low energy dynamics is modified by a potential term. For instance, if one of the scalars gets a vev, the Lagrangian is corrected to \([18]\),

\[
\mathcal{L} = \frac{1}{2} \left( g_{\mu\nu} \dot{z}^\mu \dot{z}^\nu + g_{\mu\nu} \bar{\eta}^\mu \gamma^0 D t \bar{\eta}^\nu + \frac{1}{6} R_{\mu\nu\rho\sigma} \bar{\eta}^\mu \eta^\rho \bar{\eta}^\sigma \eta^\nu - g^{\mu\nu} G_\mu G_\nu - D_\mu G_\nu \bar{\eta}^\mu \gamma^5 \eta^\nu \right),
\]

(26)

where the Killing vector field \(G\) is defined through,

\[
G = \sum a_i \frac{\partial}{\partial \xi_i},
\]

(27)

with eigenvalue \(a_i\)'s of the adjoint Higgs expectation in a suitable normalization. Canonical quantization conditions enable one to translate the supercharges to geometrical operators on the moduli space,

\[
Q \rightarrow d - i_G
\]

(28)

and similarly for its conjugate. The wavefunction is now represented by differential forms on the moduli space. Here, \(i_G\) denote the contraction of the wavefunction/differential form with \(G\). The SUSY algebra has the central charge,

\[
Z \equiv i \mathcal{L}_G,
\]

(29)

which measures the electric part of the energy, and the Hamiltonian,

\[
H = \frac{1}{2} \left( QQ^\dagger + Q^\dagger Q \right),
\]

(30)

is bounded below by the absolute value of the central charge \(|Z|\). In particular, the BPS bound states without any electric charge should be annihilated by all supercharges.
4 Counting Bound States with No Electric Charge

Let us start with a single instanton of the $U(2)$ theory. This is the simplest case with the nontrivial moduli space. The relative moduli space is a 4-dimensional hyperKähler space with double Taub-NUT centers.

$$g = U(\vec{r}) d\vec{r}^2 + U(\vec{r})^{-1}(d\psi + \vec{\omega} \cdot d\vec{r})^2,$$

(31)

where

$$U(\vec{r}) = \left( \mu + \frac{1}{|\vec{r}|} + \frac{1}{|\vec{r} - 2\pi \vec{\zeta}/R|} \right),$$

(32)

and

$$\vec{\nabla} \times \vec{\omega} = \vec{\nabla} U.$$

(33)

First, suppose that there is no scalar Higgs expectations, so that $G \equiv 0$. Then the problem of finding ground states reduces to that of finding normalizable harmonic forms on the double-centered Taub-NUT. The solution to such a problem is actually well-known for arbitrary number of the Taub-NUT centers; For each Taub-NUT centers, there exists precisely one associated (anti-self-dual) harmonic 2-form [19]. The harmonic forms can be written generally as,

$$d(f/U) \wedge (d\psi + \vec{\omega} \cdot d\vec{r}) - \frac{1}{2} U \epsilon_{ijk} \partial_i (f/U) dr_j \wedge dr_k,$$

(34)

where $f = f(\vec{r})$ with certain harmonic functions in $R^3$ spanned by $\vec{r}$. Two regular and normalizable harmonic forms on the moduli space are obtained by setting by setting

$$f = \frac{1}{|\vec{r}|} + \frac{1}{|\vec{r} - 2\pi \vec{\zeta}/R|} \rightarrow \Omega_1$$

(35)

$$f = \frac{1}{|\vec{r}|} - \frac{1}{|\vec{r} - 2\pi \vec{\zeta}/R|} \rightarrow \Omega_2$$

(36)

are regular and normalizable. Call them $\Omega_1$ and $\Omega_2$, respectively. Note that the third obvious solution with $f = 1$ is actually a constant multiple of $\Omega_1$. Thus, when the Higgs expectation (beyond the Wilson line along $S^1$) is absent, we find exactly two bound states with the unit Pontryagin number.

Both $\Omega_1$ and $\Omega_2$ consist of two lumps localized at $\vec{r} = 0$ and at $\vec{r} = 2\pi \vec{\zeta}/R$. There is an illuminating classical picture of these two lumps. The position $\vec{r} = 0$ translates to the statement that the two D-string segments are glued along one of the D3-branes. Similarly $\vec{r} = 2\pi \vec{\zeta}/R$ represents D-string segments glued along the other D3. Thus, the bound states at zero energy are represented as linear combinations of two such glued D-strings.

In more general vacua with one or more Higgs expectation values, these glued D-strings are classical bound states at $\vec{r} = 0$ or $\vec{r} = 2\pi \vec{\zeta}/R$ with net binding energy. For this reason, we expect
that these two states get deformed but remain as quantum bound states, when the potential term due to adjoint Higgs expectations is turned on.

More generally, this suggests that there are \( N \) independent bound states with unit Pontryagin number in the noncommutative \( U(N) \) theory on \( S^1 \times R^3 \). Quantum mechanically, one should find \( N \) normalizable differential forms, satisfying a SUSY condition on the \( 4(N - 1) \) dimensional moduli space. This needs a further work. Here we will be content with finding the classical ground states of the potential when we turn on one additional Higgs field.

Natural candidates for classical ground states are D-string segments glued at all except one D3-brane. Such classical states are represented by points on the moduli space, where \( N - 1 \) of the following \( N \) vectors vanish,

\[
\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_{N-1}, \vec{r}_N + 2\pi \vec{\zeta}/R.
\]

We introduced a new notation

\[
\vec{r}_N \equiv - \sum_{A=1}^{N-1} \vec{r}_A.
\]

Does the bosonic potential vanish at such points? For generic vev of a single Higgs, the nontrivial part of the bosonic potential has the form \(^{18}\),

\[
V = \frac{1}{2} a^A a^B (C^{-1})_{AB}
\]

This potential will vanish where all eigenvalues of \( C \) diverge. Clearly, the point where

\[
\vec{r}_1 = \vec{r}_2 = \cdots = \vec{r}_{N-1} = 0
\]

is one such ground state; All diagonal elements of \( C \) diverges while all off-diagonal elements are finite. On the other hand, one can choose a slightly different coordinates \( \vec{r}'_A \) such that

\[
\vec{r}'_1 = \vec{r}_N + 2\pi \vec{\zeta}/R,
\]

\[
\vec{r}'_2 = \vec{r}_1,
\]

\[
\vdots
\]

\[
\vec{r}'_{N-1} = \vec{r}_{N-2},
\]

which implies

\[
\vec{r}'_N + 2\pi \vec{\zeta}/R \equiv - \sum_A \vec{r}'_A + 2\pi \vec{\zeta}/R = \vec{r}_{N-1}
\]

When accompanied by a related transformation for the angular part, this redefinition leaves the form of the metric invariant. In the new coordinates,

\[
\vec{r}'_1 = \vec{r}'_2 = \cdots = \vec{r}'_{N-1} = 0,
\]
is clearly a zero of the potential $V$. Repeating the exercise, we can see that zeros of $V$ occur where any $N-1$ of $\vec{r}_1, \ldots, \vec{r}_{N-1}, \vec{r}_N + 2\pi\vec{\zeta}/R$ vanish. There are exactly $N$ such points. We surmise that, in the noncommutative $U(N)$ theory on $R^{1+3} \times S^1$, there are exactly $N$ BPS supermultiplets of states with the unit Pontryagin number.

In the limit $\vec{\zeta} = 0$, the moduli space becomes singular. In the Coulomb phase, at least part of $N$ states seems to survive. This can be seen explicitly for the $U(2)$ case. In this limit, the two Taub-NUT centers coalesce into one, and the relative moduli space becomes an $Z_2$ orbifold of the single-center Taub-NUT. In the process, $\Omega_2$ vanishes by itself (or disappear into the orbifold point), while $\Omega_1$ remains finite and well-defined. This state $\Omega_1$ is similar to the threshold bound states in monopole dynamics [20]. It is not clear how many actually survive the limit for general $N$, but it seems not farfetched to expect at least one of them does. The turning on Higgs expectation instead of or in conjunction with the Wilson line should not decrease the number of states, and we expect to have at least one pure instanton state in the Coulomb phase of the $U(N)$ theory.

5 Decomplexification

One can take the decompactification limit by sending $R \to \infty$. To reach a sensible moduli space metric in such a limit, we need to rescale the relative coordinates $\vec{r}_A$ by

$$\vec{y}_A = R \vec{r}_A, \quad A = 1, \ldots, N - 1,$$

and thus

$$\vec{y}_N \equiv - \sum_{A=1}^{N-1} \vec{y}_A = R \vec{r}_N,$$

also. Upon such a rescaling the constant piece $\mu_{AB}$ gets multiplied by $1/R$ and can be ignored. The remaining pieces are written as,

$$\frac{8\pi^2}{e^2} \left( \sum_A \left( \frac{1}{y_A} d\vec{y}_A^2 + y_A (D\psi_A)^2 \right) + \frac{(\sum_A d\vec{y}_A)^2}{|\vec{y}_N + 2\pi\vec{\zeta}|} - \frac{(\sum_A y_A D\psi_A)^2}{|\vec{y}_N + 2\pi\vec{\zeta}| + \sum_A |\vec{y}_A|} \right),$$

where

$$D\psi_A \equiv d\psi_A + \vec{\omega}_{AB}(\vec{r}_C) \cdot d\vec{r}_B = d\psi_A + \vec{\omega}_{AB}(\vec{y}_C) \cdot d\vec{y}_B,$$

remains unchanged under the rescaling. All sums are for $A = 1, \ldots, N - 1$. The resulting metric is called the Calabi metric [21].

While the details of the dynamics have changed upon $R \to \infty$, the ground state structure of the bosonic potential, which becomes confining, did not. The bosonic potential still vanishes at $N$
different points, where $N - 1$ of $N$ vectors, $\vec{y}_1$, $\vec{y}_2$, ..., $\vec{y}_{N-1}$, $\vec{y}_N + 2\pi \vec{C}$, vanish. Thus, as long as the theory is in the Coulomb phase, there must be $N$ independent supermultiplets of states with a unit Pontryagin number.

If one approaches the symmetric phase where the $U(N)$ gauge symmetry is restored, the low energy effective potential disappears, and some of the $N$ quantum states might disappear. In the case of $U(2)$, it can be seen explicitly that $\Omega_1$ becomes nonnormalizable while $\Omega_2$ remains normalizable. It is unclear what physics is responsible for such disappearance of some instanton states, or whether this low energy phenomenon is meaningful at all in the full Yang-Mills theory context.

For the case electrically charged case with the potential, there is a recent work in the decompactified limit \cite{22}.

6 Summary

We explored the low energy dynamics of five-dimensional $U(N)$ Yang-Mills theory in the noncommutative setting. One may consider the noncommutativity as a convenient short-distance regulator that allows us to discuss the quantum states of instanton solitons. We computed the moduli space metric of a single $U(N)$ instanton, which is smoothed out thanks to the noncommutativity, and wrote down the supersymmetric low energy dynamics explicitly.

In the Coulomb phase, where the symmetry is broken to $U(1)^N$ by a Wilson line or adjoint Higgs expectation values, there are exactly $N$ supermultiplets of states with the unit Pontryagin numbers and no electric charge. States with electric charges in addition to the Pontryagin number can be studied with the given low energy dynamics, which we have not attempt here.

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