Coupling constants for a degenerate Fermi gas confined to a quasi one-dimensional harmonic trap

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Abstract. A theory for the coupling constants between spin polarized degenerate fermionic atoms is developed for the case of atoms confined to a quasi one-dimensional harmonic trap. Exploiting the presence of the Fermi edge and the large number of fermions it is shown that the resulting effective one-dimensional interaction can be parameterized by three sets of interaction coefficients, two for forward scattering and one for backward scattering. In the case of identical fermions, backscattering dominates because the contact part of the effective one-dimensional interaction must be subtracted. Analytic expressions for the interaction coefficients for the effective contact interaction, which is relevant for the inter-component interactions, and for the "p-wave" interaction appearing in next order, are given. As an example, we calculate and discuss in detail the effective coupling constants for the marginally long ranged dipole–dipole interaction.

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Short title: Coupling constants for fermions in one dimension

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1. Introduction

The achievement of Bose-Einstein condensation in dilute ultracold gases [1] also renewed the theoretical interest in fermionic many body systems [2, 3, 4, 5], e.g. their superfluid properties [3, 4, 6, 7, 8, 9]. Experimental successes in obtaining degeneracy in Fermi vapours [1, 2, 3] further stimulated the interest in confined ultracold Fermi gases. Recently, successful production of large samples of fermionic lithium ($10^6 - 10^7$ atoms) with long lifetimes (10^2 sec.) was reported [16, 17].

The present situation in this field is characterized by intensive search for a verification of the presence of a superfluid BCS-like phase. Strong attractive interaction between the fermions is needed for a transition temperature not too far below the temperature of Fermi degeneracy. It is not only the absence of s-wave scattering between identical fermions which causes problems, even the intrinsic interaction between non-identical fermions is generally too weak. Strongly interacting degenerate Fermi gases now can be produced either by mixing bosons with fermions [18, 19] or by magnetically inducing a Feshbach resonance in a two-species mixture of fermions [20, 21, 22, 23].

Here we study the interaction of both identical and non-identical fermions in a one-dimensional harmonic trap where no superfluid transition can occur at finite temperatures. The interactions considered in the present paper are limited to scattering in a single channel when energy is exchanged between the scattering particles and the harmonic trap. Moreover, a large interaction energy would violate the criterion for a one-dimensional regime as is discussed in Section 2.

Using microtrap technology [24, 25, 26, 27, 28], it will become possible in the near future to produce a neutral ultracold quasi one-dimensional Fermi gas in a highly elongated trap.

The confinement of a trapped ultracold gas can be realized by a harmonic external potential. Theoretical work on interacting fermions in a one-dimensional harmonic trap has been published in [30, 31]. It is based on concepts of Luttinger liquid theory [32] (for reviews see [33, 34, 35]). Detailed knowledge of the coupling constants will be necessary for understanding the ground state properties of this novel confined one-dimensional Fermi gas. The coupling constants will decide which type of fluctuations are dominant, density wave, component wave, or pairing fluctuations.

Solvable models require specific forms of the interaction coefficients, especially the exclusion of momentum non-conserving collision events in the harmonic trap (which evidently are present here). We study the coefficients of the pair interaction in order to find out to what extent physical interactions have at least approximately the behaviour required by the exactly solvable models.

We construct an effective one-dimensional potential from a physical interaction potential using the simplest conceivable realization of the projection from three dimensions to
one dimension. We set up the effective interaction operator by taking into account only interaction coefficients of the transverse ground state and study the effects of the one-dimensional geometry on the form of the interaction coefficients.

In many cases, identical spin polarized fermions experience only a weak residual interaction because s-wave scattering is forbidden. This restriction does not hold for a two-component system of spin polarized fermions where significant interactions between the components are possible. For instance, the dipole–dipole interaction can become relevant, especially in the case of polar molecules.

Postulating the existence of an effective local one-dimensional potential with a Fourier transform, we demonstrate that three main scattering processes dominate the interaction coefficients in one dimension. They are identified as two forms of forward and one form of backward scattering. They determine the coupling constants in theories of the interacting one-dimensional degenerate Fermi gas.

Recently, a two-component Fermi gas has been realized with different hyperfine states of $^{40}$K. For simplicity, however, we assume the same longitudinal trapping frequency for both components.

The paper is organized as follows. Section 2 develops the theory for the interaction coefficients. It includes a discussion how the one-dimensional regime may be reached. In Section 3 we evaluate the interaction coefficients for an arbitrary interaction of finite range in terms of the zeroth and second moments of the effective one-dimensional interaction potential. In Section 4 we present the results for the marginally long-ranged dipole-dipole interaction. In both cases we assess the relevance of our findings for one- and two-component Fermi vapours in one dimension.

2. Theory

Starting from the case of identical atoms in the same hyperfine state, the pair interaction operator $\hat{V}_3$ for a many particle system in a trap is

$$\hat{V}_3 = \frac{1}{2} \sum V(m, p; q, n) \hat{c}_m \hat{c}_q \hat{c}_p \hat{c}_n,$$

with interaction coefficients

$$V(m, p; q, n) = \int d^3x d^3x' \psi_m^*(x) \psi_p^*(x') V_3(x - x') \psi_q(x) \psi_n(x').$$

$V_3(x - x')$ denotes the two-particle interaction potential while the quantum numbers $m_x, m_y, m_z$ etc., abbreviated by $\mathbf{m}$, characterize the single particle bound state in the potential of the trap.

In our case,

$$V(\mathbf{x}) = \frac{1}{2}m_\perp \omega_x^2 z^2 + \frac{1}{2}m_\perp \omega_y^2 (x^2 + y^2)$$
is the potential of the highly elongated (trap frequencies $\omega_\perp \gg \omega_\ell$) axially symmetric harmonic trap. The atomic mass is denoted by $m_A$. The non-interacting system has single particle levels

$$\hbar \omega_n = \hbar \omega_\ell \left( n_z + \frac{1}{2} \right) + \hbar \omega_\perp (n_x + n_y + 1), \quad n_x, n_y, n_z = 0, 1, \ldots$$

and wave functions which are products of the wave functions of the one-dimensional harmonic oscillator, e.g., for the coordinate $z$ in the elongated direction of the trap ($n = n_z$)

$$\psi_n(z) = \left( \frac{\alpha}{2^n n! \pi^{1/2}} \right)^{1/2} \exp(-\alpha^2 z^2/2) H_n(\alpha z).$$

$H_n$ denotes a Hermite polynomial. Intrinsic length scales of the trap are $\ell = \alpha^{-1}$ with $\alpha^2 = m_A \omega_\ell / \hbar$ and $\ell_\perp = \alpha_\perp^{-1}$ with $\alpha_\perp^2 = m_A \omega_\perp / \hbar$, the spatial extensions of the longitudinal and transverse ground states, respectively.

We consider a system of $1 \ll N < \omega_\perp / \omega_\ell$ fermionic atoms. At $T = 0$, they fill the lowest $N$ states well below the first transverse excited state. Note that each single particle level is occupied by at most one spin-polarized fermion. Then the unperturbed Fermi energy is

$$\epsilon_F = \hbar \omega_\ell \left( N - \frac{1}{2} \right).$$

The transverse ground state energy $\hbar \omega_\perp$ is taken from now on as zero of energy.

Neglect of all terms with transverse quantum numbers not equal to zero defines our quasi one-dimensional model. The validity of this approximation is discussed below.

From now on we write $m$ for $m = (0, 0, m_z)$ etc. and use an effective one-dimensional potential $V_{1\text{eff}}$ which results from averaging the physical potential $V_3$ over the transverse ground state

$$V_{1\text{eff}}(z - z') = \int d^2 x \, d^2 x' \, \psi_{10}(x)^2 \psi_{10}(y)^2 \psi_{10}(x')^2 \psi_{10}(y')^2 V_3(x - x').$$

For example, a three-dimensional contact potential $V_3(x) = g \delta^{(3)}(x)$ (which contributes only for bosons or non-identical fermions to $\hat{V}_3$) gives

$$V_{1\text{eff}}(z) = g \frac{\alpha_\perp^2}{2\pi} \delta(z).$$

Instead of computing $V(m, p; q, n)$ from the real space potential equation (7) we use in our calculations below a Fourier version of equation (2) which reads

$$V(m, p; q, n) = \cos(\pi(|q - m| - |n - p|)/2) \left( \frac{2^{|q-m|} |q-m|! |n-p|! \max(m,q)! \max(n,p)!}{\min(m,q)! \min(n,p)!} \right)^{1/2}$$

$$\times \frac{\alpha^2}{2\pi} \int_0^\infty dv \, v^{(|q-m| + |n-p| - 1)/2} e^{-v} \tilde{V}_{1\text{eff}}(k^2) = 2\alpha^2 v) L_{\min(q,m)}^{(|q-m|)}(v) L_{\min(n,p)}^{(|n-p|)}(v),$$
provided the Fourier transform $\tilde{V}_{\text{eff}}(k_\parallel)$ of $V_{\text{eff}}(z)$

$$\tilde{V}_{\text{eff}}(k_\parallel) = \int dz \exp(i k_\parallel z) V_{\text{eff}}(z)$$

exists. The $z$–component of the Fourier variable $k$ is denoted by $k_\parallel$. $L^{(j)}$ denotes a Laguerre polynomial.

We generalize to the case of two components and discuss the conditions under which the one-dimensional model defined above is valid. First of all we exclude thermal excitations of the transverse levels by demanding $k_B T \ll h\omega_\perp$. Consider now $2N$ fermionic atoms in an elongated harmonic trap which are distributed equally into 2 hyperfine states. Atoms of different components interact through a three-dimensional contact potential $g \delta^{(3)}(\mathbf{x} - \mathbf{x}')$. Interaction between atoms of the same component is much weaker and is therefore neglected here. The relation $N < \omega_\perp/\omega_\ell$ guarantees that all the atoms can be accommodated in the non-interacting ground state. We use the mean field estimate $ng$ for the interaction energy per atom denoting the density in three dimensions of each hyperfine component by $n$. Following D. Gangardt and G. Shlyapnikov [39], this energy must be much smaller than the transverse excitation energy $h\omega_\perp$ to be in the one-dimensional regime. Hereby it is assumed that the width of the transverse ground state of the trap greatly exceeds the three-dimensional scattering length $a = mg/4\pi \hbar^2$ for the scattering of fermions in different hyperfine states [40]. $N$ atoms in the transverse ground state of the trap occupy a volume of $\sim 2\pi \alpha^{-1} \alpha_\perp^{-2} \sqrt{2N}$ and the above criterion may be written as

$$a \alpha \sqrt{2N} \ll 1.$$  \(10\)

Note that the aspect ratio $(\omega_\ell/\omega_\perp)^{1/2}$ does not enter the criterion (10) explicitly. It is not only a sufficiently large ratio of $\omega_\perp/\omega_\ell$ but the reduction of $N$ which leads into the one-dimensional regime provided the Fermi level is always below $h\omega_\perp$.

As an example we consider fermionic potassium atoms in 2 different hyperfine states in an elongated harmonic trap with typical parameters: $\omega_\ell/\omega_\perp = 10^{-4}$, $\alpha = 10^5 \text{ cm}^{-1}$, and $a = 8$ nm. The transverse ground state of the trap be half filled by each component. Then the ratio between the energies of interaction and transverse excitation is as small as 0.05. In what follows we assume that the above criterion is always fulfilled.

We continue to discuss the interaction coefficients $V(m, p; q, n)$. From equation (7) the following features may be deduced: If the atoms interact by the marginally long ranged dipole-dipole interaction which has no intrinsic range, the effective one-dimensional potential has a finite range $d_{1D} \sim \ell_\perp$. This is shown in detail in Section 4. We note that a similar calculation may be performed for the Coulomb interaction which has an infinite range and results in $d_{1D} \sim 3\ell_\perp$. If instead the interatomic potential has a finite range $d_{3D}$ as e.g. the van der Waals interaction then we find $d_{1D} \sim d_{3D}$.  


In addition, the integrand in equation (9) contains further functions including Laguerre polynomials. This weighting provides a soft momentum cut-off at $k > 2k_F$ or $v > 4N$, provided we consider the physically relevant case $m \approx n \approx p \approx q \approx N \gg 1$, but $|m - n| \ll N$, $|m - q| \ll N$, etc.. It is here, where the fermionic nature of the quantum gas becomes relevant.

There are 24 coupling coefficients $V(m, p; q, n)$ for any set of four distinct integers. Due to the symmetries $(m \leftrightarrow q)$, $(n \leftrightarrow p)$, and $(m, q \leftrightarrow n, p)$ only three of them can be different.

It is one of the aims of the present work to study how far atoms confined to a one-dimensional harmonic trap and interacting by physical potentials have interaction coefficients $V(m, p; q, n)$ which behave as

$$V(m, p; q, n) \approx V_a \delta_{m-q,n-p} + V_b \delta_{q-m,n-p} + V_c \delta_{m+q,n+p}.$$ (11)

For such an interaction an exact solution of the model was found in previous work [30]. Fermions in a box have exactly the required behaviour as is shown in the appendix. This is a consequence of local translational invariance inside the box. In the one-dimensional harmonic trap, however, the situation is different. Only in a region around the center of the trap, say at $|z| \leq \delta$, the atoms nearly feel no external force. Here the single particle states are superpositions of plane wave states $\exp(ik_n z)$ with $k_n = \pm \alpha \sqrt{2n - 1}$. Because the relevant states are near the Fermi edge, we have $|k_n| \approx k_F$. According to equation (1), incoming states $\{n, q\}$ are transformed into outgoing states $\{p, m\}$ in the collision process. Denoting a state with $k_n \approx -k_F$ by $(-n)$, three types of collision processes may be discriminated

- a-type: $\{n, q\} \to \{p, m\}$,
- b-type: $\{n, -q\} \to \{p, -m\}$,
- c-type: $\{n, -q\} \to \{-p, m\}$.

(12)

If the atoms move in the same direction, only forward scattering ($\Delta p \ll \hbar k_F$, type a) may occur. If the atoms move in opposite directions, forward scattering ($\Delta p \ll \hbar k_F$, type b) as well as backward scattering ($\Delta p \approx 2\hbar k_F$, type c) are possible. The first two cases were considered in the first two papers of [30]. The last one requires an extension of the bosonization method, which is the aim of the third paper [30]. The couplings $V_a$, $V_b$, and $V_c$ are the analogues of the Luttinger model couplings $g_4$, $g_2$, and $g_1$, respectively [33, 34, 35].

Because the quantum numbers $m, p, q, n$ denote energies in units of $\hbar \omega_F$, momentum conservation can only be approximately fulfilled. If e.g. in the c-type collision process $m + q - p - n = 0$ then, linearizing with respect to $m - N$ etc. gives

$$|k_m| + |k_q| - |k_p| - |k_n| = O(N^{-1}),$$ (13)
and one would conclude that the coefficients $V_{a,b,c}$ would dominate more clearly with increasing $N$.

On the other hand, the larger $N$, the larger is the fraction of atoms in the regions $|z| > \delta$ where momentum is not conserved (recall that the number of atoms in the central region $|z| \leq \delta$ is only $\propto \sqrt{N}$) and this would enhance the momentum non-conserving coefficients.

From the discussion below it will turn out that for short range interactions (approximated by a contact potential) coefficients of type $a,b,$ or $c$ dominate over the coefficients with indices which do not fulfill one of the three momentum conservation relations above. Moreover, we will show that backscattering is strongly suppressed for interactions with a range of more than the interparticle separation.

3. General Discussion

Before going to specific interactions, we address some general questions related to the above results.

Interactions produced by a one-dimensional contact potential will henceforth be called "s-wave". Such a contact potential produces interaction coefficients, which are fully symmetric in all their arguments. Therefore they do not contribute to the interaction between identical fermions. We demonstrate how a contact contribution is subtracted from the interaction coefficient

$$V(m,p; q, n) = \int dz \int dz' \psi_m(z) \psi_q(z) V(z - z') \psi_p(z') \psi_n(z').$$

(14)

The one-dimensional potential $V(z)$ could, for instance, be an effective one-dimensional potential calculated from a physical potential via equation (7). If $V(z)$ decays faster than the product $\psi_p(z') \psi_n(z')$ oscillates (for $p \approx n \approx N$ the wavelengths are about $\pi/k_F \approx \pi\ell_\perp$) one can expand $\psi_p(z') \psi_n(z')$ at $z' = z$ up to second order and gets

$$V(m,p; q, n) = \langle V \rangle \int dz \psi_m(z) \psi_q(z) \psi_p(z) \psi_n(z)$$

$$+ \frac{1}{2} \langle z^2 V \rangle \int dz \psi_m(z) \psi_q(z) \frac{d^2}{dz^2} [\psi_p(z) \psi_n(z)] + \ldots$$

$$= V^{(s)}(m,p; q, n) + V^{(p)}(m,p; q, n) + \ldots$$

(15)

We call the first term in equation (15), which corresponds to a contact interaction, $V^{(s)}$ ("s-wave") and the second term $V^{(p)}$ ("p-wave") contribution to the interaction coefficient. Higher order "partial waves" can be defined analogously.

A sufficient condition for the validity of the expansion is $2k_F d_{1D} \ll 1$. Here we use the notation $\langle V \rangle = \int dz V(z)$ and assume that the first moment $\langle z V \rangle$ of $V$ vanishes because
$V$ is an even function. The moments of $V$ may be expressed in terms of the Fourier transform $\tilde{V}(k)$ of the potential and its second derivative $\tilde{V}''(k)$ at $k = 0$

$$\langle V \rangle = \int dz \, V(z) = \tilde{V}(k = 0),$$

$$\langle z^2 V \rangle = \int dz \, z^2 \, V(z) = -\tilde{V}''(k = 0),$$

provided the integrals converge. The Fourier variable $k$ conjugate to $z$ in this case corresponds to $k_\parallel$ as explained above.

We note that the expansion equation (15) is equivalent to inserting the approximation

$$V(z) \approx \langle V \rangle \delta(z) + \frac{1}{2} \langle z^2 V \rangle \frac{d^2}{dz^2} \delta(z) + \ldots$$

into equation (14).

In a contact interaction, the momentum transfer is uniform: Both forward scattering ($k \ll k_F$) as well as backward scattering ($k \approx 2k_F$) are equally present in the potential and hence contribute equally to the first term in equation (15). In "p-wave" scattering, however, the second derivative leads to a suppression of the forward contribution. In the Fourier domain, equation (18) reads

$$\tilde{V}(k) = \tilde{V}(k = 0) + \frac{1}{2} \tilde{V}''(k = 0) k^2 + O(k^4),$$

provided $\tilde{V}(k = 0)$ and $\tilde{V}''(k = 0)$ exist. Again, the first term is called "s–wave" and the second "p–wave".

The dominant interaction between different components in a mixture of Fermi gases with short range interaction is the "s–wave" part. It is characterized by interaction coefficients, which are fully symmetric in its arguments.

In this case, inserting only the "s–wave" part $\tilde{V}(k = 0)$ of equation (19) into equation (9), the $v$-integration can be performed exactly, leading to the result:

$$V^{(s)}(m, p; q, n) = \alpha\langle V \rangle \frac{\cos(\pi(p + q - m - n)/2)}{\sqrt{2\pi}} \left( \frac{\Gamma(m + 1)}{\Gamma(p + 1)\Gamma(q + 1)\Gamma(n + 1)} \right)^{1/2} \times \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a_3)\Gamma(m - q + 1)} 3F_2(-q, a_4; a_5; m - q + 1; 1)$$

with the abbreviations $a_1 = (p + q + n - m + 1)/2$, $a_2 = (m + p - q - n + 1)/2$, $a_3 = (p + q - m - n + 1)/2$, $a_4 = (m + n - p - q + 1)/2$, $a_5 = (m - n - p - q + 1)/2$.

The generalized hypergeometric function $3F_2$ reduces here to a finite series [11].
The coefficients $V^{(s)}(m, p; q, n)$ have arguments $m, p, q, n \approx N$ while the differences of the arguments are much smaller than $N$ as was explained above. In this case, the relative variation of the arguments of the functions which determine $V^{(s)}(m, p; q, n)$ according to equation (20) is small with respect to a variation of $N$ but of order unity with respect to variations of the differences of the arguments. Numerical examination of equation (20) shows that $V^{(s)}(m, p; q, n)$ behaves indeed in this way.

Following equation (11), we use the notation $V_a$ for $V(m, p; q, n)$ with $m + p = q + n$. $V_b$ and $V_c$ are defined analogously. Specifically for the coefficients $V_a$, $V_b$, and $V_c$, we find from equation (20)

$$V^{(s)}_{a,b,c}(m, p; q, n \approx N) \propto N^{-0.3}.$$  

For the model interaction in the appendix (see equation (A.6) below) we have instead $V_{a,b,c} \propto N^{-1/2}$ (note $L \approx L_F$) in accordance with an argument given in [31]. In contrast, the momentum non-conserving coefficients are almost independent of $N$. Concerning the neglect of all background coefficients, as in the Tomonaga-Luttinger model with harmonic confinement, we conclude from the present work that this model may be applicable to non-identical fermions if the number of atoms in the trap is limited [29].

In figure 1, we show corresponding results. We have chosen the form of a histogram since the coefficients $V(m, p; q, n)$ with an odd sum of arguments vanish. The “p-wave” interaction coefficients are obtained by inserting only the “p–wave” part $\tilde{V}''(k = 0) k^2$ of equation (19) into equation (9). They can also be given in closed form:

$$V^{(p)}(m, p; q, n) = -\alpha^3 \langle z^2 V \rangle \cos(\pi(p + q - m - n)/2) \left( \frac{\Gamma(m + 1)}{\Gamma(p + 1)\Gamma(q + 1)\Gamma(n + 1)} \right)^{1/2}$$

$$\times \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(b_3)\Gamma(m - q + 1)} \, 3F_2(-q, b_4, b_2; b_5, m - q + 1; 1)$$

with the abbreviations $b_1 = (p + q + n - m - 1)/2$, $b_2 = (m + p - q - n + 3)/2$, $b_3 = (p + q - m - n - 1)/2$, $b_4 = (m + n - p - q + 3)/2$, $b_5 = (m - n - p - q + 3)/2$.

This formula is relevant for identical fermions with a short range interaction in a quasi one-dimensional harmonic trap.

Corresponding results are shown in figure 2. Note that units are different from figure 1. Forward scattering is almost completely suppressed and the backward scattering interaction coefficients $V_c$ become dominant. The $N$–dependence of $V^{(p)}_c$ is different from that of $V^{(s)}$. We find

$$V^{(p)}_c \propto N^{0.5},$$

which agrees for $L \approx L_F$ completely with equation (A.11) below in the limit $k_F d \ll 1$, even if our interaction model in the appendix is extremely crude. The dominance of
$V_c$ over $V_a$ and $V_b$ persists for all $N$. The momentum non-conserving coefficients also increase with $N$ but slightly less than $V_c$.

As was discussed in Section 2, the Fermi algebra removes any fully symmetric part of the interaction coefficients $V(m, p; q, n)$. Inspection of figures 1 and 2 shows that not only the two forward scattering peaks disappear after this procedure but also almost all momentum non-conserving coefficients. In principle, we considered an arbitrary finite ranged potential $V(z)$ here but the approximation (18) and (19) require an interaction of short range to be valid. Note also that figures 3 and 4 below show that the same feature is present in the case of the marginally long ranged dipole-dipole interaction.

We conclude that theories like [30] which exactly solve the model find strongest support by our results if they take into account backscattering between identical fermions.

4. Dipole–Dipole Interaction

The real space interaction between two parallel magnetic dipoles $\mu$ is

$$V_{3DD}(x) = \frac{\mu_0}{4\pi} \left[ \frac{\mu^2}{r^3} - 3 \frac{(x \cdot \mu)^2}{r^5} - \frac{8\pi}{3} \mu^2 \delta^{(3)}(x) \right]. \quad (24)$$

The Fourier transform becomes [30]

$$\tilde{V}_{3DD}(k) = \mu_0 \left( \frac{(\mu \cdot k)^2}{k^2} - \mu^2 \right). \quad (25)$$

These functions are singular at $x = 0$ and $k = 0$, respectively.

With trivial redefinitions, all formulae can be taken over to the case of molecules interacting via permanent electric dipoles.

The Fourier version of equation (7) reads

$$\tilde{V}_{1\text{eff}}(k_{\parallel}) = \int \frac{d^2k_{\perp}}{(2\pi)^2} \exp \left( -\frac{k_{\perp}^2}{2\alpha_{\perp}^2} \right) \tilde{V}_3(k), \quad (26)$$

where $k_{\parallel}, k_{\perp}$ are the components of the wave vector $k$ with respect to the axis of the trap.

A straightforward application of equation (26) gives the effective one-dimensional dipole-dipole potential

$$\tilde{V}_{1\text{eff DD}}(k_{\parallel}) = -\frac{\mu_0 \mu^2 \alpha_{\perp}^2}{2\pi} \left\{ \frac{3 \cos^2 \Phi - 1}{2} \left[ 1 + \frac{\kappa^2}{2} e^{\kappa^2/2} \text{Ei}(-\kappa^2/2) \right] + \sin^2 \Phi \right\}. \quad (27)$$
with $\kappa = k_{\parallel}/\alpha_{\perp}$. $\text{Ei}$ denotes the exponential integral and $\Phi$ is the angle between the dipoles and the axis of the trap.

The effective one-dimensional potential is non-analytic at $k = 0$, but has the finite value

$$\tilde{V}_{\text{eff DD}}(k_{\parallel} = 0) = -\frac{\mu_0\mu^2\alpha_{\perp}^2}{4\pi}(1 + \cos^2 \Phi).$$

(28)

For identical fermions, this value must be subtracted from equation (27), as was explained above.

It is possible to calculate exactly the real space version of the effective one-dimensional dipole–dipole interaction:

$$V_{\text{eff DD}}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk_{\parallel} e^{-ik_{\parallel}z} \tilde{V}_{\text{eff DD}}(k_{\parallel})$$

$$= -\frac{\mu_0\mu^2\alpha_{\perp}^3}{8\sqrt{2\pi}} \left\{ (3\cos^2 \Phi - 1) \left[ (1 + \alpha_{\perp}^2 z^2) \exp \left( \alpha_{\perp}^2 z^2 / 2 \right) \text{erfc} \left( \alpha_{\perp} |z| / \sqrt{2} \right) \right. \right.$$

$$\left. + \frac{3}{8\sqrt{2\pi}} \alpha_{\perp} \sin^2 \Phi \delta(z) \right\}.$$

For $|\Phi| < (>) \arccos 1/3$, the potential is attractive (repulsive). Note that in a one-dimensional Fermi gas quantum fluctuations prevent the classical collapse of this configuration until a threshold attractive interaction strength is reached which according to [30] is $|V_{a,b,c}| = O(\hbar\omega_{\ell})$. The range $d_{1D}$ of this effective one-dimensional potential inside the trap follows from $d_{1D}\alpha_{\perp} = 1$, which gives $k_Fd_{1D} = \sqrt{2}$ for a completely filled trap. The real space potential, which has a cusp at $z = 0$, is shown in the inset of figure 4 for the case of dipoles oriented parallel to the axis of the trap ($\Phi = 0$).

The large distance behaviour is

$$V_{\text{eff DD}}(|z| \gg \alpha_{\perp}^{-1}) \rightarrow -\frac{\mu_0\mu^2}{4\pi} (3\cos^2 \Phi - 1) \frac{1}{|z|^3}.$$

(30)

This is exactly the behaviour of the initial potential equation in three dimensions (24) along the $z$-axis for two parallel dipoles.

We now turn to the interaction coefficients for the dipole–dipole interaction. The analytic procedures outlined in equations (15) to (19) are not applicable to the dipole–dipole interaction. We thus resort to a numerical evaluation and consider the case of dipoles oriented parallel to the $z$-axis: The sequence $V(m = 107, p = 103; q = 105, n)$ of interaction coefficients is displayed in figure 3, as calculated from the full equation
(26), i.e., for the interaction between fermionic atoms in different hyperfine states. Here and in figure 4 a filling factor \( F = N \omega_t/\omega_\perp \approx 1 \) is assumed.

It is seen that the backward scattering coefficient \( V_c \) is suppressed in comparison to \( V_a \approx V_b \) (for large \( N \), \( V_a \rightarrow V_b \) hold due to the long range of the interaction. This is in accordance with the results for the model interaction in the box (equation (20) below, case 2) and \( k_F d = O(1) \).

The \( N \)-dependence of the coefficients \( V_a \), \( V_b \), and \( V_c \) for the full dipole-dipole interaction is

\[
V_a, V_b \propto N^{-0.3}, \quad V_c \propto N^{-0.1}
\]

while the background coefficients which dominate more than in the case of the contact interaction (cf. figure 1) remain almost constant.

In figure 4, we show the interaction coefficients for identical fermions, i.e., for the effective one-dimensional potential \( \tilde{V}_{\text{eff DD}}(k) - \tilde{V}_{\text{eff DD}}(k = 0) \). It can be seen that in this case the approximation

\[
V(m, p; q, n) \approx V_c^{(\text{non-s})} \delta_{m+q,n+p}
\]

is appropriate, i.e., only backward scattering is relevant.

The \( N \)-dependence of \( V_c^{(\text{non-s})} \) is different for different filling factors \( F = N \omega_t/\omega_\perp \). For \( F \approx 1 \) the \( N \)-dependence is estimated as \( V_c^{(\text{non-s})} \propto N^{-0.5} \) while for \( F \ll 1 \) it becomes \( V_c^{(\text{non-s})} \propto N^{0.5} \). Both results agree with equation (23) as well as with equation (A.7) (note that in the first case \( k_F d \approx 1 \) while \( k_F d \ll 1 \) for the second case). The background coefficients grow slightly with \( N \).

From our results, we conclude that neglect of the background coefficients finds justification for the case of identical fermions in traps with a small filling factor after the s-wave part has been subtracted.

Finally, we estimate \( V_a \approx V_b \) for fermions in different hyperfine states, which interact via the magnetic dipole–dipole interaction. For \( m, p, q, n \approx N \) the coefficients \( V_{a,b,c}^{(s)}(m, p; q, n) \) do not exceed \( V(N, N; N, N) \). With dipoles oriented parallel to the trap axis and using the "s-wave" approximation in equation (20) we get

\[
V^{(s)}(N, N, N, N) \approx -1.34 \frac{\mu_0 \mu^2}{4\pi} \frac{m_A}{\hbar^2} \frac{\alpha_\perp}{\sqrt{F}} \hbar \omega_t.
\]

We refer to the planned highly anisotropic trap in [28] and find with \( \omega_\perp = 12\pi \cdot 10^5 \text{s}^{-1} \) for \(^{53}\text{Cr}\):

\[
\left| \frac{V_{a,b,c}^{(s)}}{\hbar \omega_t} \right| < 0.2 \frac{\sqrt{F}}{F}.
\]

Provided \( F \) is small, \( |V_b| \) can become of the order of the unperturbed level splitting \( \hbar \omega_t \), which is required for interactions to be relevant.
5. Discussion and Summary

The prospect of realizing a quasi one-dimensional gas of interacting fermions in a trap will allow a comparison of experiments with emerging theories \cite{30, 31} based on Luttinger liquid theory. These theories require a knowledge of the coupling constants describing the interaction.

We have shown in detail, how a physical pair interaction between fermions in three spatial dimension determines three effective coupling functions $V_a$, $V_b$, and $V_c$ for the interaction of the fermions when they are confined to a quasi one-dimensional harmonic trap.

The coupling functions $V_a$ and $V_b$ describe the forward scattering, while $V_c$ quantifies the backward scattering. Their respective values depend significantly on the range $d_{3D}$ of the original pair interaction, the reference length being the inverse of the one-dimensional Fermi wave number.

For $2k_Fd_{3D} \ll 1$ and distinguishable fermions, i.e., for the inter-component interaction in a quantum gas mixture, e.g. for the van der Waals interaction, forward and backward scattering contribute about equally. The effective one-dimensional potential is also short ranged and the equivalent of the s-wave approximation is applicable.

For dipole–dipole interaction where $d_{3D}$ is not defined, the effective one-dimensional potential acquires a range $d_{1D} \approx 1/k_F$ inside the trap.

For distinguishable fermions, backward scattering becomes suppressed in comparison with forward scattering with increasing range of interaction.

These results change when the fermions are identical. The contact potential ("s-wave scattering") then does not contribute. This has a surprising effect on the effective coupling functions: To a good approximation, only backscattering survives, i.e., $V_a$ and $V_b$ can be neglected in comparison with $V_c$ even if this backscattering coefficient is significantly reduced in the case of short range interactions.

For the dipole-dipole interaction, however, it is of the same order as for the full dipole–dipole interaction. In reality, only the electric dipole–dipole interaction between identical fermionic molecules with a permanent electric dipole moment \cite{37} can become relevant.

Finally we mention that dipolar fermionic molecules $^{40}$K-$^{87}$Rb were discussed \cite{19} as possible candidates for fermionic superfluidity.

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Figure captions

Figure 1. Reduced interaction coefficient $V^{(s)}(1220, 1214; 1200, n)/\langle \alpha(V) \rangle$ for "s-wave" scattering in the quasi one-dimensional harmonic trap. Three contributions at $n = 1194$ ($V_b$), $n = 1206$ ($V_c$), and $n = 1234$ ($V_a$) dominate. Coefficients with an odd sum of arguments vanish.

Figure 2. Reduced interaction coefficients $V^{(p)}(1220, 1214; 1200, n)/\langle \alpha^3 z^2 V \rangle$ for "p-wave" scattering versus $n$ for the quasi one-dimensional harmonic trap. Backscattering coefficient $V_c^{(p)} < 0$ at $n = 1206$ dominates. Coefficients with an odd sum of arguments vanish. Background coefficients as seen in figure 1 almost disappeared.

Figure 3. Reduced interaction coefficients $V(107, 103; 105, n)/|\tilde{V}_{1\text{eff DD}}(0)|$ for the full dipole–dipole interaction between atoms in different hyperfine states versus number $n$ of oscillator state. Dipoles are oriented parallel to the trap axis and a filling factor of $\approx 1$ is assumed. Backscattering coefficient $V_c$ at $n = 109$ is smaller than the forward scattering coefficients $V_a$ and $V_b$, which also dominate the other interaction coefficients. Coefficients with an odd sum of arguments vanish.

Figure 4. Reduced interaction coefficients $V^{(\text{non-s})}(107, 103; 105, n)/|\tilde{V}_{1\text{eff DD}}(0)|$ for the dipole–dipole interaction between identical atoms versus number $n$ of oscillator state. Dipoles are oriented parallel to the trap axis and a filling factor of $\approx 1$ is assumed. Backscattering coefficient $V_c^{(\text{non-s})}$ at $n = 109$ dominates all other interaction coefficients including the forward scattering coefficients. Moreover, background coefficients are much smaller in comparison with the main peak than in figure 3. The inset shows the real space version of the effective one-dimensional potential $V_{1\text{eff DD}}$ versus $z$. Coefficients with an odd sum of arguments vanish.

Appendix: Model interaction in the box

Equations (20) and (22) allow easy numerical evaluation, but are not transparent. We, therefore, study a solvable model for the interaction coefficients. We consider the one-dimensional potential (cf. also [42] for a related analysis)

$$V(z) = \frac{g}{2d} \text{ for } |z| \leq d \text{ and } V(z) = 0 \text{ for } |z| > d,$$

(A.1)
normalized according to
\[ \int_{-\infty}^{\infty} dz \, V(z) = g. \]

The fermions are confined to a box of length \( L \gg d \) with infinitely high walls. The single particle eigenstates are then
\[ \psi_m(z) = \sqrt{\frac{2}{L}} \sin \frac{m\pi z}{L}, \quad m = 1, 2, 3, \ldots. \] (A.2)

The interaction coefficients according to equation (9) are given (up to a small correction of order \( d \)) by
\[ V_{\text{box}}(m, p; q, n; d) = \frac{2g}{dL^2} \int_0^L dz_1 \sin \frac{m\pi z_1}{L} \sin \frac{q\pi z_1}{L} \int_{z_1-d}^{z_1+d} dz_2 \sin \frac{p\pi z_2}{L} \sin \frac{n\pi z_2}{L} \] (A.3)
\[ = \frac{1}{2L} \left( \frac{\sin(p-n)\pi d/L}{(p-n)\pi d/L} (\delta_{m+p,q+n} + \delta_{m+n,p+q}) + \frac{\sin(p+n)\pi d/L}{(p+n)\pi d/L} \delta_{m+q,p+n} \right). \]

All coefficients, except those with \( m \pm p \pm q \pm n = 0 \),
(A.4)
vanish due to local translational invariance inside the box. But near the Fermi surface \((m, p, q, n \approx N \gg 1)\) only the three processes shown in equation (A.3) remain: two with small and one with large momentum transfer \( \Delta p \) as was discussed in Section 2.

In the notation of equation (11), this can be written as
\[ V_{\text{box}}(m, p; q, n; d) = V_a \delta_{m+q+p+n} + V_b \delta_{m+n+p+q} + V_c \delta_{m+q+p+n}. \] (A.5)

Again, for processes near the Fermi edge and \( d \ll L \) we approximate
\[ V_a = V_b \approx \frac{g}{2L} \left( 1 - \frac{(p-n)^2 \pi^2 d^2}{6L^2} \right), \] (A.6)
\[ V_c \approx \frac{g}{2L} \left( \frac{\sin(2N\pi d/L)}{2N\pi d/L} \right). \] (A.7)

The coefficients depend on the differences between the indices only \((V_{a,b})\) or are constant with respect to the indices \((V_c)\). We note in passing that the interaction coefficients \( V_{\text{box}}(m, p; q, n; d) \) show perfectly the behaviour which allows application of the Bosonization method, see \[30\].
In the limit $d \to 0$ the potential (A.1) becomes a contact potential $g \delta(z)$ with pure $s$-wave scattering and all three types of coefficients are equal and constant

$$V_{a,b,c} \to V_{a,b,c}^{(s)} = \frac{g}{2L}. \quad (A.8)$$

However, when the range $d$ of the potential (A.1) increases and becomes larger than the average distance $L/N$ of the atoms in the box the backscattering coefficient $V_c$ decreases while $V_{a,b}$ remain constant as long as $d \ll L$.

As discussed in Section 3 above, any fully symmetric part common to all interaction coefficients does not contribute to the interaction $\hat{V}_3$ according to equation (1) and may therefore be removed. Doing so we subtract the fully symmetric potential $V_{\text{box}}(m,p; q, n; d = 0)$ (A.8) from $V_{\text{box}}(m, p; q, n; d)$ (A.5, A.6, A.7) with the result that only a backscattering term which we denote by $V_{c}^{(p)}$ remains

$$V_{\text{box}}(m, p; q, n; d) \to V_{c}^{(p)} \delta_{m+q,p+n} \quad (A.9)$$

$$V_{c}^{(p)} = \frac{g}{2L} \left( \frac{\sin(2\pi Nd/L)}{2\pi Nd/L} - 1 \right) \delta_{m+q,p+n} \quad (A.10)$$

$$\approx -\frac{g}{3L} k_F^2 d^2 \delta_{m+q,p+n},$$

where the latter approximation is valid for $k_F d \ll 1$ ($k_F = \pi N/L$). The sign of $V_{c}^{(p)}$ is opposite to the sign of $V_{c}^{(s)}$ in accordance with the results for the harmonic trap, compare figures 1 and 2.

In equations (21) and (23) we found in the case of the harmonic trap for the scaling of the coefficients with the particle number $N$: $V_{a,b,c}^{(s)} \sim N^{-0.3}$ and $V_{c}^{(p)} \sim N^{0.5}$. To compare with equations (A.8) and (A.9) where the coefficients are given in terms of $L$ and $k_F$ we recall that in the harmonic trap $k_F$ and $L$ are proportional to $\sqrt{2N}$. We substitute these relations into the box coefficients and get

$$V_{a,b,c}^{(s)} \sim N^{-1/2}, \quad V_{c}^{(p)} \sim N^{1/2}. \quad (A.11)$$

We conclude that in spite of the crude character of our model the agreement of the coefficients $V_{a,b,c}$ in the box and in the harmonic trap is quite good. The background terms for the momentum non-conserving processes in the harmonic trap are, of course, absent in the box.

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One must expect that a renormalization of the bare values occurs. For practical applications, this may not be a serious problem because only the central coupling constants which describe the interaction in the mass and composition channels are relevant. As in the Luttinger model these are treated as adjustable parameters.

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