The Independent Chip Model and Risk Aversion

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Abstract

We consider the Independent Chip Model (ICM) for expected value in poker tournaments. Our first result is that participating in a fair bet with one other player will always lower one’s expected value under this model. Our second result is that the expected value for players not participating in a fair bet between two players always increases. We show that neither result necessarily holds for a fair bet among three or more players.

1 Introduction

The analysis of expected value in poker tournaments is more complex than for cash games. By this, we mean that chips in a cash game are equivalent to cash. On the other hand, the expected value of chips in a tournament are related to expected value in cash in a nonlinear way.

In a typical (freezeout) tournament, each player begins with the same number of chips. Once a player is out of chips, he or she is out of the tournament. Play continues until one player has all of the chips. In most tournaments, however, the winning player gets only a portion of the prize money. The last player eliminated gets second place money, the next-to-last eliminated gets third place money, and so forth. Effectively, the winner is forced to give back some of the chips he or she has won. Consequently, most
players should play in a somewhat risk averse manner, the extent depending on his or her ability.

2 Modeling Poker Tournaments

In models of poker tournaments where all players have “equal abilities” and “equal opportunities,” the probability a player finishes first equals the fraction of the total chips in play that he or she holds. The model where a player wins or loses a single chip with probability 1/2 each is the standard Gambler’s Ruin or random walk problem going back to Huygens [10]. In fact, he further considers constant, but unequal, probabilities of winning or losing, which can be interpreted as introducing skill into the model. See also [4].

The probability of finishing first equals the fraction of the total chips in play held by the player much more generally, for instance if a player’s expected gain in chips is zero for each hand. If the player’s proportion of all chips is \( x \) and probability of winning the tournament is \( f(x) \), then

\[
(1 - f(x))(-x) + f(x)(1 - x) = 0,
\]

from which we see \( f(x) = x \). Henke [8] tested this model on data from World Poker Tour final tables. There was reasonable agreement between theory and data. Nevertheless, the model modestly overestimated the probabilities of small stacks winning and modestly underestimated the probabilities of large stacks winning. This could be due to flaws in the model or to differences in the skill of players that led to the disparities in stack sizes.

In contrast, the probability of finishing second, third, and so forth is very dependent on the model. Even in the particular model where two players are chosen at random and each wins or loses a single chip from the other with probability 1/2, a player’s probability of finishing second will depend not just on the fraction of chips held by each player but on the actual number of chips held by the players. In a more general setting, Swan and Bruss [11] give a recursive solution for the probability a particular player is the first to go broke in terms of Markov processes and unfolding. Unfortunately, it is not a computationally practical method for the repeated calculations needed to analyze poker tournaments. In the case of three players, the problem is a discrete Dirichlet problem on a triangle. Ferguson [5] solved the players’ probabilities of finishing first, second, and third for Brownian motion — the
limit as the number of chips increases to infinity. Employing a Schwarz-
Christoffel transformation, he expresses the answer in terms of the inverse
of the incomplete beta function, so it is not easy to actually compute these
limiting probabilities. Due to the specialized techniques, even this does not
generalize to more than three players [1].

In the early to middle stages of a tournament, the primary factor in
determining a player’s expected cash winnings is his or her chip count. Thus,
one could model expected winnings as a function of the fraction of the total
number of chips the player holds. There are two especially simple models of
this. One can use the biased random walk of single steps to model expected
winnings rather than the probability of finishing first. Alternatively, Chen
and Ankenman, [2] and [3] propose a model to estimate the probability of
finishing first by assuming the probability of doubling one’s chips before going
g a broke is constant. Again, one can instead consider expected winnings under
the same assumption. It would be appropriate to call these the small-pot
model and the big-pot model. They were developed with some preliminary
comparisons with data from online poker tournaments in [7]. Although skill
is naturally incorporated into these models, neither would be appropriate late
in a poker tournament when the number of chips held by a player’s opponents
becomes a critical factor in both determining the player’s expected winnings
and deciding on the optimal play in each hand of the tournament.

3 The Independent Chip Model (ICM)

The best-known model for tournaments between players of (roughly) equal
abilities is the Independent Chip Model (ICM). Although it did not arise from
any model for the movement of chips from player to player, this model is often
used by serious poker players for analysis of the late stages of tournaments.
Under the ICM, a player’s probability of finishing first is the fraction of the
total chips in play he or she holds. Recursively, the conditional probability of
finishing in kth position given the k − 1 players finishing 1st through (k − 1)st
is the fraction of the chips held by the player once the chips of the (k − 1)
players finishing higher have been taken out of play. Thus, if the fractions of
chips held by players 1 through k are x_1, . . . , x_k, the probability they finish
1st through kth is

\[ p_k(x_1, \ldots, x_k) = \frac{x_1 x_2 \cdots x_k}{(1 - x_1)(1 - x_1 - x_2) \cdots (1 - x_1 - x_2 - \ldots - x_{k-1})}. \]
Ganzfried and Sandholm [6] have used this model to analyze the effects of position in a simulated three-player Texas hold’em tournament under “jam/fold” strategies. This author has data from online single table tournaments that, at first glance, suggest the ICM is a reasonable approximation of the probabilities in the aggregate. We mention in passing that the ICM is essentially the model used since 1990 to determine the first three picks in the National Basketball Association draft.

4 Risk Aversion Under the ICM

We introduce notation we will use in all that follows. Let \( q_k(x; y; z_1, \ldots, z_k) \) denote the probability, under the ICM, that a given player with fraction \( y \) of the chips finishes first, one with fraction \( x \) finishes somewhere among the top \( k + 2 \) players, with the remaining \( k \) places taken by given players with fractions \( z_1, \ldots, z_k \), who finish in this relative order. Thus, by definition,

\[
q_k(x; y; z_1, \ldots, z_k) = p_{k+2}(y, z_1, \ldots, z_k) + \sum_{i=1}^{k} p_{k+2}(y, z_i, \ldots, z_k, x) + \sum_{i=1}^{k-1} p_{k+2}(y, z_i, \ldots, z_k, 1 - x - y - z_i - \cdots - z_k).
\]

We now assume the identity for \( k \) and prove it for \( k + 1 \). Applying the

**Lemma 1.** For every integer \( k \geq 0 \),

\[
q_k(x; y; z_1, \ldots, z_k) = \frac{y}{x+y} p_{k+1}(x+y, z_1, \ldots, z_k) - p_{k+2}(y, z_1, \ldots, z_k, 1 - x - y - z_1 - \cdots - z_k).
\]

**Proof of Lemma.** We note that the second term is the probability that players with chip fractions \( y, z_1, \ldots, z_k \) finish in the first \( k + 1 \) places and that anyone other than the player with fraction \( x \) finishes in \((k + 2)\)nd place.

We prove the lemma by induction. For \( k = 0 \), we have

\[
q_0(x; y) = \frac{y}{1-y} = \frac{y}{x+y}(x+y) - \frac{y(1-x-y)}{1-y}.
\]

We now assume the identity for \( k \) and prove it for \( k + 1 \). Applying the
inductive hypothesis yields

\[ q_{k+1}(x; y; z_1, \ldots, z_{k+1}) \]
\[ = q_k(x; y; z_1, \ldots, z_k) \frac{z_{k+1}}{1 - x - y - z_1 - \cdots - z_k} + p_{k+3}(y, z_1, \ldots, z_{k+1}, x) \]
\[ = \left[ \frac{y}{x + y} p_{k+1}(x + y, z_1, \ldots, z_k) \right. \]
\[ - p_{k+2}(y, z_1, \ldots, z_k, 1 - x - y - z_1 - \cdots - z_k) \left. \right] \]
\[ \times \frac{z_{k+1}}{1 - x - y - z_1 - \cdots - z_k} + p_{k+3}(y, z_1, \ldots, z_{k+1}, x) \]
\[ = \frac{y}{x + y} p_{k+2}(x + y, z_1, \ldots, z_{k+1}) - p_{k+2}(y, z_1, \ldots, z_k, z_{k+1}) \]
\[ + p_{k+3}(y, z_1, \ldots, z_{k+1}, x) \]
\[ = \frac{y}{x + y} p_{k+2}(x + y, z_1, \ldots, z_{k+1}) \]
\[ - p_{k+3}(y, z_1, \ldots, z_{k+1}, 1 - x - y - z_1 - \cdots - z_{k+1}), \]
as desired.

Lemma 2. Let \( f > 0 \), \( f' \geq 0 \), \( f'' \geq 0 \) and \( g > 0 \), \( g' > 0 \), \( g'' > 0 \). Then \( fg > 0 \), \( (fg)' > 0 \), \( (fg)'' > 0 \).

Proof of Lemma. The lemma is an immediate consequence of the product rule.

We define a fair bet for a player to be a random variable \( W \) (for wager) that is not identically 0 and whose expected value in chips is 0. We may let \( W \) stand for either a player’s gain or loss. In our context of poker tournaments, \( W \) will be expressed as a fraction of the chips in play and can take on only finitely many values. Here we include subsequent bets in the hand as part of the expected value. Thus we are interpreting the wager, which may be either initiated or accepted by the player, to be the possible gain or loss in chips over the course of the rest of the hand.

Our first result is the following.

Theorem 1. Suppose a tournament has prize money for \( n \)th place which is at least that for \((n + 1)\)st place and that at least one player still in the tournament will not earn as much as second place prize money. Under the
Independent Chip Model, any fair bet in which only one other player can gain or lose chips in the hand being played will lower the player’s expected prize money.

Proof. We will first break down the expected prize winnings under the ICM into a sum of simpler terms, each of which is either linear or concave down, allowing us to conclude Theorem [1] by convexity.

Consider a tournament paying prize money \( m_1 \geq m_2 \geq \ldots \geq m_n \) for finishing first, second, \ldots, nth. Our first reduction is view this as \( n \) simultaneous sub-tournaments, the first a winner-take-all paying \( m_1 - m_2 \) for first place, the second paying \( m_2 - m_3 \) to the first and second place finishers, through one paying \( m_n \) to each of the top \( n \) finishers. It will suffice to prove that, by participating in a fair bet, a player’s expected winnings will not increase in any of these sub-tournaments and will lose expected value in at least one of them.

Denote the player in question by \( A \) and the opponent involved in the bet as \( B \). Let \( A \) have fraction \( x \) of the total number of chips in play, let \( B \) have fraction \( y \), and let \( w \) denote the fraction of all chips \( A \) loses on the bet (negative when \( A \) wins). We will use \( u_i \) and \( z_i \) as needed to denote the fraction of chips held by other players.

In any sub-tournament where all players get the same prize money (including those with prize 0), \( A \)'s expected winnings are that amount regardless of wagers.

For the winner-take-all sub-tournament, \( A \)'s expected value participating in the wager is

\[
(m_1 - m_2) E[x - w] = (m_1 - m_2) x,
\]

i.e. \( A \)'s expected value hasn’t changed.

All remaining sub-tournaments, of which there is at least one, satisfy the conditions of the theorem. Thus, it suffices to prove the theorem for those tournaments where each of at least two winners gets a prize of 1 and at least one nonwinner gets 0. After losing a wager \( w \), the probability \( A \) finishes in \( m \)th place behind players other than \( B \) having chip fractions \( u_1, \ldots, u_{m-1} \) is

\[
p_m(u_1, \ldots, u_{m-1}, x - w).
\]

This is linear in \( w \), so

\[
E[p_m(u_1, \ldots, u_{m-1}, x - w)] = p_m(u_1, \ldots, u_{m-1}, x).
\]
On the other hand, in those scenarios in which player $B$ finishes ahead of player $A$, with both among the winners, the dependence on $w$ is nonlinear. We partition all such cases by fixing the first $m$ finishers with $B$ in $m$th place and fixing the relative positions of all other finishers except player $A$. Denote the fraction of chips held by the first $m - 1$ players by $u_1, \ldots, u_{m-1}$ and those of the remaining $k$ players other than $A$ or $B$ by $z_1, \ldots, z_k$, where $k = n - m - 1$. Setting $\Delta = 1 - u_1 - \cdots - u_{m-1}$, player $A$’s expected winnings may be written as
\[
 p_{m-1}(u_1, \ldots, u_{m-1}) \cdot q_k((x - w)/\Delta; (y + w)/\Delta; z_1/\Delta, \ldots, z_k/\Delta).
\]
We may drop the leading term, $p_{m-1}(u_1, \ldots, u_{m-1})$, and rescale units, dividing $w, x, y,$ and $z_i$ by $\Delta$. This leaves us needing to show the concavity of the simpler expression $q_k(x - w; y + w; z_1, \ldots, z_k)$.

Applying Lemma 1, we see that
\[
 q_k(x - w; y + w; z_1, \ldots, z_k) = \frac{y + w}{x + y} p_{k+1}(x + y, z_1, \ldots, z_k)
 - p_{k+2}(y + w, z_1, \ldots, z_k, 1 - x - y - z_1 - \cdots - z_k).
\]
The first term is linear in $w$. The latter expands to
\[
 - \frac{(y + w) z_1 z_2 \cdots z_k (1 - x - y - z_1 - \cdots - z_k)}{(1 - y - w)(1 - y - w - z_1) \cdots (1 - y - w - z_1 - \cdots - z_k)}.
\]

Thus it suffices to show that
\[
 g_k(w) = \frac{y + w}{(1 - y - w)(1 - y - w - z_1) \cdots (1 - y - w - z_1 - \cdots - z_k)}
\]
has positive second derivative. Observe that, for the range of relevant wagers, $-(1 - \min\{x, y\}) \leq w \leq (1 - \min\{x, y\})$, $1/(1 - y - w - z_1 - \cdots - z_j)$ satisfies the conditions of Lemma 2 as does
\[
 g_0(w) = \frac{y + w}{1 - y - w} = \frac{1}{1 - y - w} - 1.
\]
By induction, $g_k > 0$, $g_k' > 0$, $g_k'' > 0$ for all $k$.

Therefore, $q_k(x - w; y + w; z_1, \ldots, z_k)$ is concave down and by convexity,
\[
 E[q_k(x - w; y + w; z_1, \ldots, z_k)] < q_k(x; y; z_1, \ldots, z_k),
\]
completing the proof of Theorem 1. \qed
Theorem 1 is false for fair wagers among three or more players. With many players, counterexamples are unusual. On the other hand, they are easy to construct: start with a tournament paying two places and with three players, all participating in a fair wager. Barring the unlikely possibility that expected winnings for all three are unaffected by the wager, the expected winnings for at least one must increase. One could easily add one or more uninvolved players with very small chip stacks to the counterexample. We give another, explicit, counterexample following Theorem 2.

We move on to examine the impact of a fair wager on the expected winnings of players not involved in the bet.

**Theorem 2.** Suppose a tournament has prize money for \( n \)th place which is at least that for \((n+1)\)st place and that at least one player still in the tournament will not earn as much as second place prize money. Under the Independent Chip Model, the expected prize money of any player not involved in a fair bet between two players will increase.

*Proof.* The proof parallels that of Theorem 1. Let \( A \) and \( B \) denote the two players involved in the fair bet and let \( C \) denote one player who is not involved.

We break down \( C \)'s expected winnings into a sum of expected winnings from sub-tournaments paying 1 to each of it winners. The expected winnings of \( C \) in a winner-take-all sub-tournament is unaffected by a fair bet. Similarly, they are unaffected in scenarios when neither \( A \) nor \( B \) finishes ahead of \( C \). It suffices to prove that \( C \)'s expected winnings increase when \( B \) finishes ahead of both \( A \) and \( C \).

For sub-tournaments paying the top two finishers, \( C \)'s expected winnings when \( B \) finishes first are

\[
\frac{(y + w)u}{1 - y - w} = u g_0(w),
\]

which we have seen is concave up. In this case we can actually conclude that \( C \)'s expected winnings must increase for any fair wager among two or more players.

As in the proof of Theorem 1, we further break down to scenarios in which a fixed \( m - 1 \) other players finish ahead of \( B \), who finishes in \( m \)th place. Again, there is no loss of generality in assuming \( B \) finishes first, with the top \( k + 2 \) places paid, with \( k \geq 1 \).
Our final reduction is to scenarios where the order of the first \( k \) finishers other than \( A, B, \) or \( C \) are fixed. Let \( x, y, \) and \( u \) denote the respective fractions of all chips in play held by \( A, B, \) and \( C. \) Let the fractions of the other relevant \( k \) finishers be \( z_1, \ldots, z_k. \) Let \( w \) be the amount \( B \) wins on the fair wager. When \( A \) also finishes in the money, we’ll fix \( C \)’s position and “float” \( A \) as in the proof of Theorem 1 summing over the different positions for \( C. \) To these we’ll add those cases that \( A \) does not finish in the money by floating \( C. \) Noting that only \( k + 2 \) places are paid in this scenario, \( C \)’s expected winnings are

\[
\begin{align*}
[q_{k+1}(x - w; y + w; u, z_1, \ldots, z_k) & - p_{k+3}(y + w, u, z_1, \ldots, z_k, x - w)] \\
+ [q_{k+1}(x - w; y + w; z_1, u, z_2, \ldots, z_k) & - p_{k+3}(y + w, z_1, u, \ldots, z_k, x - w)] \\
+ \cdots + [q_{k+1}(x - w; y + w; z_1, \ldots, u, z_k) & - p_{k+3}(y + w, z_1, \ldots, u, z_k, x - w)] \\
& + q_k(u; y + w; z_1, \ldots, z_k).
\end{align*}
\]

We apply Lemma 1 to each of the differences. The first is

\[
q_{k+1}(x - w; y + w; u, z_1, \ldots, z_k) - p_{k+3}(y + w, u, z_1, \ldots, z_k, x - w)
= \frac{y + w}{x + y}p_{k+2}(x + y, u, z_1, \ldots, z_k)
- p_{k+3}(y + w, u, z_1, \ldots, z_k, 1 - x - y - u - z_1 - \cdots - z_k)
- p_{k+3}(y + w, u, z_1, \ldots, z_k, x - w)
= \frac{y + w}{x + y}p_{k+2}(x + y, u, z_1, \ldots, z_k) - p_{k+2}(y + w, u, z_1, \ldots, z_k),
\]

with similar expressions for the other terms. From the definition of \( q, \) we can express \( C \)’s expected winnings as

\[
\begin{align*}
& \frac{y + w}{x + y} [q_k(u; x + y; z_1, \ldots, z_k) - p_{k+2}(x + y, z_1, \ldots, z_k, u)] \\
& + p_{k+2}(y + w, z_1, \ldots, z_k, u).
\end{align*}
\]

The first term is linear in \( w. \) The second term is, essentially,

\[
z_1 \cdots z_k u q_{k+1}(w)
\]

in the notation of equation (2) and is thus concave up, completing the proof. □
As we saw in its proof, Theorem 2 holds for tournaments paying only two places for fair bets among three or more players, but is false in general. Even then, counterexamples are quite rare. In our counterexample below, the first three finishers win 1 unit. (Perhaps it’s a satellite tournament to earn entry into another tournament.) The bet has two equally likely outcomes: A wins 16 units, B and C each lose 8 units or A loses 16 units, B and C each win 8 units. The expected winnings, to four decimal places, are given in the table below.

| Player | Initial Chip Count | Initial Expected Winnings | Final Expected Winnings |
|--------|--------------------|--------------------------|------------------------|
| A      | 140                | 0.9952                   | 0.9914                 |
| B      | 10                 | 0.5256                   | 0.5316                 |
| C      | 10                 | 0.5256                   | 0.5316                 |
| D      | 50                 | 0.9536                   | 0.9455                 |

However, there is an interesting special case, with which we conclude this paper.

**Corollary 1.** Under the assumptions of Theorem 2, if two or more players each bet a fixed amount and the total of all bets is won by one of these players with probability proportional to the size of his or her wager, then the expected winnings of all players not involved in the bet increases. In particular, if the chips of two or more players are combined into a single player’s stack, the expected winnings of all other players increase.

**Proof.** The case of two players is a special case of Theorem 2. The general case can be realized as a sequence of such fair wagers between two players. □

**References**

[1] F. Thomas Bruss, Guy Louchard, and John W. Turner, On the N-Tower-Problem and Related Problems, Adv. Appl. Prob. 35:1 (2003), 278–294 (MR1975514).

[2] William Chen and Jerrod Ankenman, The Theory of Doubling Up, The Intelligent Gambler 23 (Spring/Summer 2005), 3–4.
[3] Bill Chen and Jerrod Ankenman, The Mathematics of Poker, ConJelCo LLC, 2006.

[4] William Feller, An Introduction to Probability Theory and Its Applications, vol. 1, 3rd Edition, Wiley, 1968.

[5] Tom Ferguson, Gambler’s Ruin in Three Dimensions, 1995, available at http://www.math.ucla.edu/~tom/papers/unpublished/gamblersruin.pdf.

[6] Sam Ganzfried and Tuomas Sandholm, An approximate jam/fold equilibrium for 3-player no-limit Texas hold’em tournaments, Proc. of 7th Int. Conf. on Autonomous Agents and Multiagent Systems (AAMAS 2008), Padgham, Parkes, Müller and Parsons (eds.), May, 12–16, 2008, Estoril, Portugal.

[7] George T. Gilbert, Racing Early in Tournaments, Two Plus Two Internet Magazine, 2:3 (March 2006). (Available from the author: g.gilbert@tcu.edu.)

[8] Tony Henke, Is poker different from flipping coins?, Masters Thesis, Washington University in St. Louis, 2007, available at http://economics.wustl.edu/conference/Honors/Henke_Thesis.pdf.

[9] Mason Malmuth, Settling Up in Tournaments: Part III, in Gambling Theory and Other Topics, Two Plus Two Publishing, 1994.

[10] Eddie Shoesmith, Huygens’ Solution to the Gambler’s Ruin Problem, Historia Mathematica 13 (1986), 157–167 (MR0851874).

[11] Yvik C. Swan and F. Thomas Bruss, A Matrix-Analytic Approach to the N-Player Ruin Problem, J. Appl. Prob. 43 (2006), 755–766 (MR2274798).