ORDERS OF STRONG AND WEAK AVERAGING PRINCIPLE FOR MULTISCALE SPDES DRIVEN BY $\alpha$-STABLE PROCESS

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Abstract. In this paper, the averaging principle is studied for a class of multiscale stochastic partial differential equations driven by $\alpha$-stable process, where $\alpha \in (1, 2)$. Using the technique of Poisson equation, the orders of strong and weak convergence are given $1 - \frac{1}{\alpha}$ and $1 - r$ for any $r \in (0, 1)$ respectively. The main results extend Wiener noise considered by Bréhier in [6] and Ge et al. in [17] to $\alpha$-stable process, and the finite dimensional case considered by Sun et al. in [39] to the infinite dimensional case.

1. Introduction

Many systems change involving slow and fast components in the natural world. For instance, dynamics of chemical reaction networks often take place on notably different time scales, from the order of nanoseconds ($10^{-9}$ s) to the order of several days; When you are looking at the interaction between temperature and climate, it is found that the daily temperature changes more rapidly, while climate changes are relatively slow. People always call this kind of system as the multiscale system or slow-fast system. Multiscale models have wide applications in various fields, such as nonlinear oscillations, chemical kinetics, biology, climate dynamics, see e.g. [2, 45] and the references therein.

Multiscale systems often show characteristics that do not conform to common sense, and the complexity of this kind system makes the traditional single theory no longer applicable, so the study of multiscale models system becomes inevitable and necessary. The mathematical methods people use are often referred to as the methods of averaging and of homogenization, see e.g. [12, 30] and the references therein.

The averaging principle for multiscale models describes the asymptotic behavior of the slow component as the scale parameter $\varepsilon \to 0$. Bogoliubov and Mitropolsky [3] first studied the averaging principle for the deterministic systems. Khasminskii [23] established an averaging principle for the stochastic differential equations driven by Wiener noise. Since these pioneering works, many people have studied averaging principles for various stochastic systems, see e.g. [18, 20, 21, 24, 25, 26, 27, 35, 42, 50] for stochastic differential equations (SDEs), and see e.g. [4, 6, 7, 8, 9, 11, 13, 14, 15, 16, 32, 40, 43, 47] for stochastic partial differential equations (SPDEs).

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In this paper, we consider the following slow-fast stochastic system on a Hilbert space $H$:

$$
\begin{align*}
&dX_t^\varepsilon = [AX_t^\varepsilon + B(X_t^\varepsilon, Y_t^\varepsilon)]dt + dL_t, \quad X_0^\varepsilon = x \in H, \\
&dY_t^\varepsilon = \frac{1}{\varepsilon}[AY_t^\varepsilon + F(X_t^\varepsilon, Y_t^\varepsilon)]dt + \frac{1}{\varepsilon^{1/\alpha}}dZ_t, \quad Y_0^\varepsilon = y \in H,
\end{align*}
$$

(1.1)

where $\varepsilon > 0$ is a small parameter describing the ratio of time scales between the slow component $X^\varepsilon$ and fast component $Y^\varepsilon$. $A$ is a selfadjoint operator, measurable functions $B, F : H \times H \to H$ satisfy some appropriate conditions, and $\{L_t\}_{t \geq 0}$ and $\{Z_t\}_{t \geq 0}$ are mutually independent cylindrical $\alpha$-stable processes with $\alpha \in (1, 2)$, which are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

The strong averaging principle for such stochastic systems (1.1) has attracted some attention recently. For instance, Bao et al. [1] proved the strong averaging principle for two-time scale SPDEs driven by $\alpha$-stable noise. The authors have proved the strong averaging principle for stochastic Ginzburg-Landau equation, stochastic Burgers equations and a class of SPDEs with Hölder coefficients in [41, 10] and [38] respectively. However, the key technique used in these mentioned papers was based on the Khasminskii’s time discretization, thus no satisfactory convergence order was obtained. Meanwhile, studying the convergence rate is an interesting and important topic in multiscale system. For instance, Bréhier [5, 6] used the convergence rate to construct the efficient numerical schemes, based on the Heterogeneous Multiscale Methods.

The order of convergence for slow-fast stochastic systems has been studied extensively. The technique of Khasminskii’s time discretization is frequently used to study the strong convergence rate (see e.g. [4, 19, 26, 35]), while the method of asymptotic expansion of solutions of Kolmogorov equations in the parameter $\varepsilon$ is used to study the weak convergence rate (see e.g. [4, 11, 14, 24, 49]). Recently, the technique of Poisson equation is widely used to study the strong and weak convergence rates, see e.g. [6, 17, 35, 36, 39, 46]. For more applications of Poisson equation, see e.g. [28, 29, 37] and references therein.

The aim of this paper is first establish the strong convergence rates of stochastic system (1.1). More precisely, for any $(x, y) \in H^0 \times H$ with $\eta \in (0, 1)$, $T > 0$ and $1 \leq p < \alpha$, one tries to prove that

$$
\sup_{t \in [0, T]} \mathbb{E}|X_t^\varepsilon - \bar{X}_t|^p \leq C\varepsilon^{(1 - \frac{1}{\alpha})p},
$$

where $C$ is a constant depending on $T, \|x\|_\eta, \|y\|_p$ and $\bar{X}$ is the solution of the corresponding averaged equation (see Eq. (2.7) below).

Secondly, we continuous to study the weak convergence rates of stochastic system (1.1). More precisely, for some fixed test function $\phi$, then for any $(x, y) \in H \times H$, $T > 0$ and $r \in (0, 1)$, one tries to prove that

$$
|\mathbb{E}\phi(X_t^\varepsilon) - \mathbb{E}\phi(\bar{X}_t)| \leq C\varepsilon^{1-r},
$$

where $C$ is a constant depending on $T, \|x\|, \|y\|, r$.

In contrast to the existing works [6, 17], due to the Wiener noise is considered there, thus the solution has finite second moment usually. However the solution here does not has finite second moment due to the $\alpha$-stable noise, hence some methods developed there do not work in this situation. In order to overcome this difficulty, we shall estimate the solution of...
the corresponding Poisson equation more carefully, meanwhile the accurate treatment of the \(\alpha\)-stable process is provided in the proof.

In contrast to the existing work \cite{39}, the strong and weak convergence rates for slow-fast SDEs driven by \(\alpha\)-stable noise are obtained there, we here extend the case of finite dimension to infinite dimension essentially. However, we have to overcome some non-trivial difficulties in the infinite dimensional case. For example the presentation of term \(AX^\varepsilon_t\), the method of Galerkin approximation and the smoothing properties of the semigroup \(e^{tA}\) will be used to deal with a serious of difficulties arising from the unbounded operator \(A\).

Another contribution of this paper is to fill a gap in \cite{1} partially. As stated in \cite[Remark 3.3]{1}, "for the technical reason, it seems hard to show Theorem 3.1 without the uniform boundedness of the nonlinearity", where "Theorem 3.1" means the strong averaging principle holds and "the nonlinearity" means the coefficient \(B\). In fact, the essential reason is that the method used in \cite{1} is the classical Khasminskii’s time discretization, which highly depends on the square calculation in the proof, hence the finite second moment of the solution \(X^\varepsilon_t\) is required usually. But the solution \(X^\varepsilon_t\) for system (1.1) only has finite \(p\)-th moment \((0 < p < \alpha)\), the uniform boundedness of \(B\) is used to weaken the finite second moment to finite first moment. However, the technique of Poisson equation is used to remove the condition of uniform boundedness of \(B\), but some bounded conditions of second and third derivatives for the coefficients are assumed. Moreover, the optimal strong averaging convergence order is obtained here.

The organization of this paper is as follows. In the next section, some notations and assumptions are introduced. Then we state our main results. Section 3 is devoted to study the regularity of the solution of the corresponding Poisson equation. The detailed proofs of strong and weak convergence rates are provided in Sections 4 and 5 respectively. The final section is the appendix, where we give some a-priori estimates of the solution, and study the Galerkin approximation of the system (1.1) and the finite dimensional approximation of the frozen equation.

We note that throughout this paper \(C, C_p, C_T\) and \(C_{p,T}\) denote positive constants which may change from line to line, where the subscript \(p, T\) are used to emphasize that the constants depend on \(p, T\).

2. Notations and main results

2.1. Notations and assumptions. We introduce some notation used throughout this paper. \(H\) is Hilbert space with inner product \(\langle \cdot, \cdot \rangle\) and norm \(| \cdot |\). \(\mathbb{N}_+\) stands for the collection of all the positive integers.

\(\mathcal{B}(H)\) denotes the collection of all measurable functions \(\varphi(x) : H \to \mathbb{R}\). For any \(k \in \mathbb{N}_+\),

\[ C^k(H) := \{ \varphi \in \mathcal{B}(H) : \varphi \text{ and all its Fréchet derivatives up to order } k \text{ are continuous} \}, \]
\[ C^k_b(H) := \{ \varphi \in C^k(H) : \text{ for } 1 \leq i \leq k, \text{ all } i\text{-th Fréchet derivatives of } \varphi \text{ are bounded} \}. \]

For any \(\varphi \in C^3(H)\), by the Riesz representation theorem, we often identify the first Fréchet derivative \(D\varphi(x) \in \mathcal{L}(H, \mathbb{R}) \cong H\), the second derivative \(D^2\varphi(x)\) as a linear operator in \(\mathcal{L}(H, H)\) and the third derivative \(D^3\varphi(x)\) as a linear operator in \(\mathcal{L}(H, \mathcal{L}(H, H))\), i.e.,

\[ D\varphi(x) \cdot h_1 = \langle D\varphi(x), h_1 \rangle, \quad h_1 \in H, \]
where \( D^k \varphi(x) \cdot h \) is the \( k \)-th directional derivative of \( \varphi \) in the direction \((h_1, \ldots, h_k)\), for \( k = 1, 2, 3 \).

A selfadjoint operator \( A \) satisfies \( \lambda_ne_n = -\lambda_ne_n \) with \( \lambda_n > 0 \) and \( \lambda_n \uparrow \infty \), as \( n \uparrow \infty \), where \( \{e_n\}_{n \geq 1} \subset \mathcal{D}(A) \) is a complete orthonormal basis of \( H \). For any \( s \in \mathbb{R} \), we define

\[
H^s := \mathcal{D}((-A)^{s/2}) := \left\{ u = \sum_{k \in \mathbb{N}_+} u_ke_k : u_k \in \mathbb{R}, \sum_{k \in \mathbb{N}_+} \lambda_k^s u_k^2 < \infty \right\}
\]

and

\[
(-A)^{s/2} u := \sum_{k \in \mathbb{N}_+} \lambda_k^{s/2} u_k e_k, \quad u \in \mathcal{D}((-A)^{s/2})
\]

with the associated norm \( \|u\|_s := \|(-A)^{s/2} u\| = \left(\sum_{k \in \mathbb{N}_+} \lambda_k^s u_k^2\right)^{1/2} \). It is easy to see \( \|\cdot\|_0 = |\cdot| \).

The following smoothing properties of the semigroup \( e^{tA} \) (see [4, Proposition 2.4]) will be used quite often later in this paper:

\[
\|e^{tA} x\|_{\sigma_2} \leq C_{\sigma_1, \sigma_2} t^{-\frac{\sigma_2 - \sigma_1}{2}} e^{-\frac{\lambda_1 t}{2}} \|x\|_{\sigma_1}, \quad x \in H^{\sigma_2}, \sigma_1 \leq \sigma_2, \ t > 0,
\]

\[
|e^{tA} x - x| \leq C_s t^2 \|x\|_{\sigma}, \quad x \in H^{\sigma}, \sigma > 0, \ t \geq 0.
\]

Let \( \{L_t\}_{t \geq 0} \) and \( \{Z_t\}_{t \geq 0} \) be mutually independent cylindrical \( \alpha \)-stable processes, where \( \alpha \in (1, 2) \), i.e.,

\[
L_t = \sum_{k \in \mathbb{N}_+} \beta_k L^k_t e_k, \quad Z_t = \sum_{k \in \mathbb{N}_+} \gamma_k Z^k_t e_k, \quad t \geq 0,
\]

where \( \{\beta_k\}_{k \in \mathbb{N}_+} \) and \( \{\gamma_k\}_{k \in \mathbb{N}_+} \) are two given sequence of positive numbers, \( \{L^k_t\}_{n \geq 1} \) and \( \{Z^k_n\}_{n \geq 1} \) are two sequences of independent one dimensional rotationally symmetric \( \alpha \)-stable processes with \( \alpha \in (1, 2) \) satisfying for any \( k \in \mathbb{N}_+ \) and \( t \geq 0 \),

\[
\mathbb{E}[e^{iL^k_{kh\Gamma}}] = \mathbb{E}[e^{iZ^k_{kh\Gamma}}] = e^{-t|h|^\alpha}, \quad h \in \mathbb{R}.
\]

For \( t > 0, k \in \mathbb{N}_+ \), and \( \Gamma \in \mathcal{B}(\mathbb{R} \setminus \{0\}) \), the Poisson random measure associated with \( L^k \) and \( Z^k \) are defined by

\[
N^{1,k}([0, t], \Gamma) = \sum_{0 \leq s \leq t} 1_{\Gamma}(L^k_s - L^k_{s-}), \quad N^{2,k}([0, t], \Gamma) = \sum_{0 \leq s \leq t} 1_{\Gamma}(Z^k_s - Z^k_{s-})
\]

and the corresponding compensated Poisson random measures are given by

\[
\tilde{N}^{i,k}([0, t], \Gamma) = N^{i,k}([0, t], \Gamma) - t \nu(\Gamma), \quad i = 1, 2,
\]

where \( \nu(dy) = \frac{c_\alpha}{|y|^{\alpha+1}} dy \) is the Lévy measure with \( c_\alpha > 0 \).

By Lévy-Itô’s decomposition and the symmetry of the Lévy measure \( \nu \), one has

\[
L^k_t = \int_{|x| \leq c} x \tilde{N}^{1,k}([0, t], dx) + \int_{|x| > c} x N^{1,k}([0, t], dx),
\]

\[
Z^k_t = \int_{|x| \leq c} x \tilde{N}^{2,k}([0, t], dx) + \int_{|x| > c} x N^{2,k}([0, t], dx),
\]

where \( c > 0 \). We also assume that \( \{L^n_t\}_{n \geq 1} \) and \( \{Z^n_t\}_{n \geq 1} \) are independent.
Now, we assume the following conditions on the coefficients $B, F : H \times H \to H$ throughout the paper:

**A1.** $B$ and $F$ are Lipschitz continuous, i.e., there exist positive constants $L_F$ and $C$ such that for any $x_1, x_2, y_1, y_2 \in H$,

\[
|B(x_1, y_1) - B(x_2, y_2)| \leq C(|x_1 - x_2| + |y_1 - y_2|),
\]
\[
|F(x_1, y_2) - F(x_2, y_2)| \leq C|x_1 - x_2| + L_F|y_1 - y_2|.
\]

**A2.** Assume that $\lambda_1 - L_F > 0$, $\sum_{k \in \mathbb{N}_+} \beta_k^{\alpha} \lambda_k^{\alpha-1} < \infty$ and $\sum_{k \in \mathbb{N}_+} \gamma_k^{\alpha} < \infty$.

**A3.** Assume that there exists $\kappa_1 \in (0, 2)$ such that the following directional derivatives are well-defined and satisfy:

\[
\begin{align*}
|D_x B(x, y) \cdot h| & \leq C|h| \quad \text{and} \quad |D_y B(x, y) \cdot h| \leq C|h|, \quad \forall x, y, h \in H, \\
|D_{xx} B(x, y) \cdot (h, k)| & \leq C|h||k|^{\kappa_1}, \quad \forall x, y, h \in H, k \in H^{\kappa_1}, \\
|D_{yy} B(x, y) \cdot (h, k)| & \leq C|h||k|^{\kappa_1}, \quad \forall x, y, h \in H, k \in H^{\kappa_1}, \\
|D_{xy} B(x, y) \cdot (h, k)| & \leq C|h||k|^{\kappa_1}, \quad \forall x, y, h \in H, k \in H^{\kappa_1}, \\
|D_{xy} B(x, y) \cdot (h, k)| & \leq C|h||k|^{\kappa_1}, \quad \forall x, y, h \in H, k \in H^{\kappa_1},
\end{align*}
\]

where $D_x B(x, y) \cdot h$ is the directional derivative of $B(x, y)$ in the direction $h$ with respective to $x$, other notations can be interpreted similarly. The above properties in (2.3) also hold for operator $F$.

**Remark 2.1.** Suppose that the assumption **A1** holds, $\sum_{k \in \mathbb{N}_+} \beta_k^{\alpha} / \lambda_k < \infty$ and $\sum_{k \in \mathbb{N}_+} \gamma_k^{\alpha} / \lambda_k < \infty$. As [33] did, we have that, for $\varepsilon > 0$ and $(x, y) \in H \times H$, the system (1.1) admits a unique mild solution $(X^\varepsilon_t, Y^\varepsilon_t) \in H \times H$, i.e., $\mathbb{P}$-a.s.,

\[
\begin{align*}
X_t^\varepsilon &= e^{tA}x + \int_0^t e^{(t-s)A}B(X_s^\varepsilon, Y_s^\varepsilon)ds + \int_0^t e^{(t-s)A}dL_s, \\
Y_t^\varepsilon &= e^{tA/\varepsilon}y - \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon}F(X_s^\varepsilon, Y_s^\varepsilon)ds + \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}dZ_s.
\end{align*}
\]

**Remark 2.2.** The condition $\lambda_1 - L_F > 0$ in assumption **A2** is called the strong dissipative condition, which is used to prove the existence and uniqueness of the invariant measures and the exponential ergodicity of the transition semigroup of the frozen equation. The condition $\sum_{k \in \mathbb{N}_+} \gamma_k^{\alpha} < \infty$ in assumption **A2** are necessary when applying Itô’s formula for the solution $(X^\varepsilon, Y^\varepsilon)$. While the condition $\sum_{k \in \mathbb{N}_+} \beta_k^{\alpha} \lambda_k^{\alpha-1} < \infty$ is used to control $\mathbb{E} \| \int_0^t e^{(t-s)A}dL_s \|_2$. For a more general result see [34, Lemma 4.1], i.e., if $\sum_{k \in \mathbb{N}_+} \frac{\beta_k^{\alpha}}{\lambda_k^{1-\alpha/2}} < \infty$ holds for some $\theta \geq 0$, then we have for any $0 < p < \alpha$,

\[
\sup_{t \geq 0} \mathbb{E} \left\| \int_0^t e^{(t-s)A}dL_s \right\|_\theta^p \leq C_{\alpha,p} \left( \sum_{k \in \mathbb{N}_+} \frac{\beta_k^{\alpha}}{\lambda_k^{1-\alpha/2}} \right)^{p/\alpha}.
\]
Remark 2.3. Here we give an example that the conditions in assumption A3 hold. Let $H := \{ g \in L^2(D) : g(\xi) = 0, \xi \in \partial \mathcal{D} \}$, where $\mathcal{D} = (0,1)^d$ for $d = 1, 2, 3$, $\partial \mathcal{D}$ is the boundary of domain $\mathcal{D}$ and $A := \Delta := \sum_{i=1}^{d} \partial_{\xi_i}^2$ be the Laplacian operator. The coefficient $B$ is defined to be the Nemytskii operator associated with a function $b : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, i.e., $B(x,y)(\xi) := b(x(\xi), y(\xi))$. Then the following directional derivatives are well-defined and belong to $H$,

\[
\begin{align*}
D_x B(x,y) \cdot h &= \partial_x b(x,y) h \quad \text{and} \quad D_y B(x,y) \cdot h = \partial_y b(x,y) h, \quad \forall x,y, h \in H, \\
D_{xx} B(x,y) \cdot (h,k) &= \partial_{xx} b(x,y) hk, \quad \forall x,y, h, k \in L^\infty(0,1), \\
D_{yy} B(x,y) \cdot (h,k) &= \partial_{yy} b(x,y) hk, \quad \forall x,y, h, k \in L^\infty(0,1), \\
D_{xy} B(x,y) \cdot (h,k) &= \partial_{xy} b(x,y) hk, \quad \forall x,y, h, k \in L^\infty(0,1), \\
D_{xxy} B(x,y) \cdot (h,k,l) &= \partial_{xxy} b(x,y) hkl, \quad \forall x,y, h, k, l \in L^\infty(0,1), \\
D_{yy} B(x,y) \cdot (h,k,l) &= \partial_{yy} b(x,y) hkl, \quad \forall x,y, h, k, l \in L^\infty(0,1), \\
D_{xxy} B(x,y) \cdot (h,k,l) &= \partial_{xxy} b(x,y) hkl, \quad \forall x,y, h, k, l \in L^\infty(0,1),
\end{align*}
\]

where all the partial derivatives of $b$ appear above are uniformly bounded by assumption.

Note that $H^{4/2+\eta} \subset L^\infty(\mathcal{D})$ for any $\eta > 0$(see [6, (6)]), thus it is easy to see that $B$ satisfy assumption A3 with any $\kappa_1 \in (1/2, 2)$ for $d = 1$, $\kappa_1 \in (1, 2)$ for $d = 2$ and $\kappa_1 \in (3/2, 2)$ for $d = 3$.

2.2. **Main results.** Let $\mu^x$ be the unique invariant measure of the transition semigroup of the frozen equation

\[
\begin{align*}
\begin{cases}
\frac{dY_t}{dt} = [AY_t + F(x,Y_t)] dt + dZ_t, \\
Y_0 = y \in H
\end{cases}
\end{align*}
\]

(2.6)

and define $\bar{B}(x) := \int_H B(x,y) \mu^x(dy)$. Let $\bar{X}$ be the solution of the corresponding averaged equation:

\[
\begin{align*}
\begin{cases}
\frac{d\bar{X}_t}{dt} = [A\bar{X}_t + \bar{B}(\bar{X}_t)] dt + dL_t, \\
\bar{X}_0 = x \in H.
\end{cases}
\end{align*}
\]

(2.7)

Here we state our main results.

**Theorem 2.4. (Strong convergence rate)** Suppose that assumptions A1 and A3 hold. Then for any initial values $(x,y) \in H^{\eta} \times H$ with $\eta \in (0,1)$, $T > 0$, $1 \leq p < \alpha$ and small enough $\epsilon, \delta > 0$, we have

\[
\sup_{t \in [0,T]} \mathbb{E}|X_t^x - \bar{X}_t|^p \leq C_{p,T,\delta} \left[ 1 + ||x||_\eta^{(1+\delta)p} + ||y||_\eta^{(1+\delta)p} \right]^{(1-\frac{1}{\alpha})p}.
\]

(2.8)

**Remark 2.5.** The result (2.8) above implies that the strong convergence order is $1 - \frac{1}{\alpha}$, which is the optimal order in the strong sense usually (see [39, Example 2.2]). Meanwhile, when $\alpha \uparrow 2$, this order $1 - \frac{1}{\alpha} \uparrow \frac{1}{2}$, which is in accord with the optimal order 1/2 in the case of Wiener noise (see [6, 17, 36]). Note that we do not assume the boundedness of $B$, thus it gives a positive answer to [1, Remark 3.3].

**Theorem 2.6. (Weak convergence rate)** Suppose that assumptions A1-A3 hold. Moreover, $\sup_{x,y} |B(x,y)| < \infty$ and there exists $\kappa_2 \in (0,2)$ such that the following directional derivatives are well-defined and satisfy:

\[
|D_{xxy} B(x,y) \cdot (h,k,l)| \leq C ||h||_{\kappa_2} ||k||_{\kappa_2} ||l||_{\kappa_2}, \quad \forall x,y, h \in H, k, l \in H^{\kappa_2},
\]

(2.9)
\[ |D_{xxx} F(x, y) \cdot (h, k, l)| \leq C |h||k||l|_{\kappa_2}, \quad \forall x, y, h \in H, k, l \in H^{\kappa_2}. \]  

(2.10)

Then for any test function \( \phi \in C^3_0(H) \), initial values \( (x, y) \in H \times H, T > 0, r \in (0, 1) \) and \( \varepsilon > 0 \), we have

\[
\sup_{t \in [0, T]} \left| \mathbb{E} \phi(X^\varepsilon_t) - \mathbb{E} \phi(\bar{X}_t) \right| \leq C_{r,T,\delta} \left[ 1 + |x|^{1+\delta} + |y|^{1+\delta} \right] \varepsilon^{1-r},
\]

(2.11)

where \( C_{r,T,\delta} \) is a constant depends on \( r, T, \delta \) and \( \lim_{\varepsilon \to 0} C_{r,T,\delta} = \infty \).

**Remark 2.7.** The result (2.11) implies that the weak convergence rate is \( 1 - r \) with any \( r \in (0, 1) \). Comparing with the theorem 2.4, the stronger regularity of the coefficients \( B, F \) are assumed, while initial value \( x \in H \) and the improved convergence order is obtained. It is worthy to point that the boundedness of the \( B \) is assumed for the reason of the solution does not has finite second moment. Meanwhile, it fails to obtain the expected weak convergence order 1 (see [39]).

**Remark 2.8.** Here we give an example that some additional conditions in Theorem (2.6) hold. Recall the notations in Remark 2.3. Obviously \( \sup_{x,y \in H} |B(x, y)| < \infty \) by assuming \( \sup_{x,y \in \mathbb{R}} |b(x, y)| < \infty \). Next we check the condition (2.10). Assume that \( \sup_{x,y \in \mathbb{R}} |\partial_{xxx} b(x, y)| < \infty \). Then the following directional derivatives are well-defined and belong to \( H \),

\[
D_{xxx} B(x, y) \cdot (h, k, l) = \partial_{xxx} b(x, y) hkl, \quad \forall x, y, h \in H, k, l \in L^\infty(0, 1).
\]

Note that \( H^{d/2+\eta} \subset L^\infty(\mathcal{D}) \) for any \( \eta > 0 \), thus it is easy to see that

\[
|D_{xxx} B(x, y) \cdot (h, k, l)| = |\partial_{xxx} b(x, y) hkl| \leq C |h||k||l|_{\kappa_2},
\]

where \( \kappa_2 \in (1/2, 2) \) for \( d = 1 \), \( \kappa_2 \in (1, 2) \) for \( d = 2 \) and \( \kappa_2 \in (3/2, 2) \) for \( d = 3 \). The assumption (2.10) can be handled similarly.

### 3. The Poisson Equation for Nonlocal Operator

Since the drift coefficient \( B \) may not be bounded and the solution \( X^\varepsilon_t \) does not has finite second moment, the classical Khasminskii’s time discretization dose not work in this situation (see [1, Remark 3.3]). We shall use the technique of Poisson equation to obtain the strong and weak convergence rates for system (1.1). Meanwhile, note that the operator \( A \) is not a bounded operator and \( H \ni x \) may not belong to \( \mathcal{D}(-A) \), we use Galerkin approximation to reduce the infinite dimensional problem to a finite dimension firstly, then we will take the limit finally, i.e., considering

\[
\begin{aligned}
& dX^m_t = [AX^m_t + B^m(X^m_t, Y^m_t)]dt + d\tilde{L}^m_t, \quad X^m_0 = x^m \in H_m, \\
& dY^m_t = \frac{1}{\varepsilon}[AY^m_t + F^m(X^m_t, Y^m_t)]dt + \frac{1}{\varepsilon^{1/2}}d\tilde{Z}^m_t, \quad Y^m_0 = y^m \in H_m,
\end{aligned}
\]

(3.1)

where \( m \in \mathbb{N}_+, H_m := \text{span}\{e_k; 1 \leq k \leq m\} \), \( \pi_m \) is the orthogonal projection of \( H \) onto \( H_m \), \( x^m := \pi_m x, y^m := \pi_m y \) and

\[
B^m(x, y) := \pi_m B(x, y), \quad F^m(x, y) := \pi_m F(x, y),
\]

\[
\tilde{L}^m_t := \sum_{k=1}^{m} \beta_k L^k_t e_k, \quad \tilde{Z}^m_t := \sum_{k=1}^{m} \gamma_k Z^k_t e_k.
\]
Similarly, we consider the following approximation to the averaged equation (2.7):

\[
\begin{align*}
    d\tilde{X}^m_t &= \left[ A\tilde{X}^m_t + \tilde{B}^m(\tilde{X}^m_t) \right] dt + d\tilde{L}^m_t, \\
    \tilde{X}^m_0 &= x^m,
\end{align*}
\]

where \( \tilde{B}^m(x) := \int_{H_m} B^m(x,y)\mu^{x,m}(dy) \), and \( \mu^{x,m} \) is the unique invariant measure of the transition semigroup of the following frozen equation:

\[
\begin{align*}
    \left\{ \begin{array}{l}
    dY^{x,y,m}_t = [AY^{x,y,m}_t + F^m(x, Y^{x,y,m}_t)]dt + d\tilde{Z}^m_t, \\
    Y^{x,y,m}_0 = y \in H_m.
    \end{array} \right.
\end{align*}
\]

Before we use the technique of Poisson equation, we need to do some preparations.

3.1. The frozen equation. For fixed \( x \in H \) and \( m \in \mathbb{N}_+ \), we recall the finite dimensional frozen equation (3.3). Note that \( F^m(x, \cdot) \) is Lipschitz continuous, then it is easy to show that for any initial value \( y \in H_m \), equation (3.3) has a unique mild solution \( \{Y^{x,y,m}_t\}_{t \geq 0} \) in \( H_m \), i.e., \( \mathbb{P} \)-a.s.,

\[
Y^{x,y,m}_t = e^{tA}y + \int_0^t e^{(t-s)A}F^m(x, Y^{x,y,m}_s)ds + \int_0^t e^{(t-s)A}d\tilde{Z}^m_s.
\]

Moreover, the solution \( \{Y^{x,y,m}_t\}_{t \geq 0} \) is a time homogeneous Markov process. Let \( \{P^{x,m}_t\}_{t \geq 0} \) be its transition semigroup, i.e., for any bounded measurable function \( \varphi : H_m \to \mathbb{R} \),

\[
P^{x,m}_t \varphi(y) := \mathbb{E} \varphi(Y^{x,y,m}) \quad y \in H_m, t \geq 0.
\]

Before studying the asymptotic behavior of \( \{P^{x,m}_t\}_{t \geq 0} \), we prove the following lemmas.

**Lemma 3.1.** For any \( p \in [1, \alpha) \), there exists \( C_p > 0 \) such that

\[
\sup_{m \geq 1} \mathbb{E}|Y^{x,y,m}_t|^p \leq e^{-\lambda_1 t}|y|^p + C_p(1 + |x|^p).
\]

**Proof.** By a straightforward computation, it is easy to see

\[
|Y^{x,y,m}_t| \leq e^{-\lambda_1 t}|y| + \int_0^t e^{-\lambda_1 (t-s)}|F^m(x, Y^{x,y,m}_s)|ds + \int_0^t e^{(t-s)A}d\tilde{Z}_s.
\]

Then by Minkowski’s inequality, we have for any \( p \in (1, \alpha) \) and \( t \geq 0 \),

\[
(\mathbb{E}|Y^{x,y,m}_t|^p)^{1/p} \leq e^{-\lambda_1 t}|y| + \int_0^t e^{-\lambda_1 (t-s)} \left[ L_F (\mathbb{E}|Y^{x,y,m}_s|^p)^{1/p} + C|x| + C \right] ds \\
+ \left( \mathbb{E} \left( \int_0^t e^{(t-s)A}d\tilde{Z}_s \right)^p \right)^{1/p}.
\]

By condition \( L_F < \lambda_1 \) in assumption \( \textbf{A2} \), we obtain

\[
\mathbb{E}|Y^{x,y,m}_t|^p \leq e^{-\lambda_1 t}|y|^p + C_p(1 + |x|^p) + C_p \sup_{t \geq 0} \mathbb{E} \left( \int_0^t e^{(t-s)A}d\tilde{Z}_s \right)^p.
\]
Note that condition $\sum_{k=1}^{\infty} \gamma_k^\alpha < \infty$ implies $\sum_{k=1}^{\infty} \frac{\gamma_k^\alpha}{\lambda_k} < \infty$. Then by (2.5), we get
\[
\sup_{t \geq 0} \mathbb{E} \left| \int_0^t e^{(l-s)A} dZ_s \right|^p \leq C_{\alpha,p} \left( \sum_{k=1}^{\infty} \frac{\gamma_k^\alpha}{\lambda_k} \right)^{p/\alpha}. \tag{3.6}
\]
Hence, (3.5) and (3.6) implies that (3.4) holds. The proof is complete. \(\square\)

**Lemma 3.2.** For any $t > 0$ and $x_1, x_2 \in H, y_1, y_2 \in H_m$, there exists $C > 0$ such that
\[
\sup_{m \geq 1} \left| Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m} \right| \leq e^{-\frac{(\lambda_1 - L_F)t}{2}} |y_1 - y_2| + C|x_1 - x_2|. \tag{3.7}
\]

**Proof.** For any $x_1, x_2 \in H, y_1, y_2 \in H_m$, note that
\[
\frac{d}{dt} (Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m}) = A(Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m}) + [F^m(x_1, Y_t^{x_1, y_1, m}) - F^m(x_2, Y_t^{x_2, y_2, m})].
\]
Multiplying $2(Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m})$ in both sides, we obtain
\[
\frac{d}{dt} |Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m}|^2 = -2\|Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m}\|^2 + 2(F^m(x_1, Y_t^{x_1, y_1, m}) - F^m(x_2, Y_t^{x_2, y_2, m}), Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m}).
\]
Then by $L_F < \lambda_1$ in assumption **A2** and Young’s inequality, we get
\[
\frac{d}{dt} |Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m}|^2 \leq -2\lambda_1 |Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m}|^2 + 2L_F |Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m}|^2
\]
\[
+ C|x_1 - x_2| (Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m})
\]
\[
\leq -(\lambda_1 - L_F) |Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m}|^2 + C|x_1 - x_2|^2.
\]

By comparison theorem, we have for any $t > 0$,
\[
|Y_t^{x_1, y_1, m} - Y_t^{x_2, y_2, m}|^2 \leq e^{-(\lambda_1 - L_F)t} |y_1 - y_2|^2 + C|x_1 - x_2|^2. \tag{3.8}
\]
The proof is complete. \(\square\)

Under the condition $\lambda_1 - L_F > 0$, it is well known that the transition semigroup $\{P_t^{x,m}\}_{t \geq 0}$ admits a unique invariant measure $\mu^{x,m}$ (see e.g. [44, Theorem 1.1]). Using (3.4), it is easy to check that for any $p \in [1, \alpha)$ and $x \in H$, $\sup_{m \geq 1} \int_{H_m} |z|^p \mu^{x,m}(dz) \leq C_p(1 + |x|^p)$. \tag{3.9}

Furthermore, we shall prove the following exponential ergodicity for the transition semigroup $\{P_t^{x,m}\}_{t \geq 0}$, which plays an important role in studying the regularity of the solution of the Poisson equation.

**Proposition 3.3.** For any Lipschitz continuous function $G : H \to H$, then we have for any $t > 0$,
\[
\sup_{x \in H} \left| P_t^{x,m}G(y) - \mu^{x,m}(G) \right| \leq C\|G\|_{\text{Lip}} e^{-\frac{(\lambda_1 - L_F)t}{2}} (1 + |x| + |y|), \tag{3.10}
\]
where $C > 0$ and $\|G\|_{\text{Lip}} := \sup_{x \neq y \in H} \frac{|G(x) - G(y)|}{|x - y|}$.\]
Proof. By the definition of invariant measure \( \mu^{x,m} \) and (3.7), we have for any \( t > 0 \),

\[
|P_{t}^{x,m}G(y) - \mu^{x,m}(G)| = \left| \mathbb{E}G(Y_{t}^{x,y,m}) - \int_{H_{m}} G(z) \mu^{x,m}(dz) \right|
\leq \left| \int_{H_{m}} [\mathbb{E}G(Y_{t}^{x,y,m}) - \mathbb{E}G(Y_{t}^{x,z,m})] \mu^{x,m}(dz) \right|
\leq \|G\|_{\text{Lip}} \int_{H_{m}} \mathbb{E}|Y_{t}^{x,y,m} - Y_{t}^{x,z,m}| \mu^{x,m}(dz)
\leq \|G\|_{\text{Lip}} e^{\frac{\lambda_1 - \lambda_2}{2}t} \int_{H_{m}} |y - z| \mu^{x,m}(dz)
\leq \|G\|_{\text{Lip}} e^{\frac{\lambda_1 - \lambda_2}{2}t} [1 + |x| + |y|],
\]
where the last inequality is a consequence of (3.9). The proof is complete.

\[\square\]

3.2. The regularity of solution of the Poisson equation. By the preparation in above subsection, this subsection is devoted to study the following Poisson equation:

\[-L_{2}^{m}(x)\Phi_{m}(x, y) = B^{m}(x, y) - \bar{B}^{m}(x), \quad x, y \in H_{m}, \tag{3.11}\]

where \( L_{2}^{m}(x) \) is the infinitesimal generator of the transition semigroup of the finite dimensional frozen equation (3.3), i.e.,

\[
L_{2}^{m}(x)\Phi_{m}(x, y) = D_{y}\Phi_{m}(x, y) \cdot (Ay + F^{m}(x, y)) + \sum_{k=1}^{m} \gamma_{k} \int_{\mathbb{R}} \Phi_{m}(x, y + e_{k}z) - \Phi_{m}(x, y) - (D_{y}\Phi_{m}(x, y), e_{k}z) 1_{\{|z| \leq 1\}} \nu(dz). \tag{3.12}\]

Note that \( L_{2}(x) \) is the infinitesimal generator of the transition semigroup of the frozen process \( \{Y_{t}^{x,y,m}\}_{t \geq 0} \), we define

\[
\Phi_{m}(x, y) := \int_{0}^{\infty} \left[ \mathbb{E}B^{m}(x, Y_{t}^{x,y,m}) - \bar{B}^{m}(x) \right] dt. \tag{3.13}\]

It is easy to check (3.13) solves equation (3.11). The following is the regularity of the solution \( \Phi_{m}(x, y) \) with respect to parameters, which will play an important role in the proof of our main results. The regularity of the solution of the Poisson equation with respect to parameters have been study in some references, see e.g. [29, 35, 39].

Proposition 3.4. For any \( \delta \in (0, 1] \), there exists \( C, C_{\delta} > 0 \) such that for any \( x, y, h, k \in H_{m} \),

\[
\sup_{m \geq 1} |\Phi_{m}(x, y)| \leq C(1 + |x| + |y|); \tag{3.14}\]
\[
\sup_{m \geq 1} |D_{y}\Phi_{m}(x, y) \cdot h| \leq C|h|; \tag{3.15}\]
\[
\sup_{m \geq 1} |D_{x}\Phi_{m}(x, y) \cdot h| \leq C_{\delta}(1 + |x|^{\delta} + |y|^{\delta})|h|; \tag{3.16}\]
\[
\sup_{m \geq 1} |D_{xx}\Phi_{m}(x, y) \cdot (h, k)| \leq C_{\delta}(1 + |x|^{\delta} + |y|^{\delta})|h||k|_{\kappa_{1}}, \tag{3.17}\]

where \( \kappa_{1} \) is the constant in assumption A3.
Remark 3.5. It is crucial that the parameter $\delta > 0$ in (3.16) and (3.17), which plays a key role in solving the difficulty caused by the solution does not have finite second moment.

Proof. The proof is divided into three steps.

Step 1. By Proposition 3.3, we get

$$|\Phi_m(x, y)| \leq \int_0^\infty |\mathbb{E}B^m(x, Y_t^{x,y,m}) - \tilde{B}^m(x)| dt$$

$$\leq C(1 + |y|) \int_0^\infty e^{-\frac{(\lambda_1 - LF)t}{2} t} dt \leq C(1 + |y|).$$

So the first estimate in (3.14) holds.

For any $h \in H_m$, we have

$$D_y \Phi_m(x, y) \cdot h = \int_0^\infty \mathbb{E}[D_y B^m(x, Y_t^{x,y,m}) \cdot D^h_t Y^{x,y,m}] dt,$$

where $D^h_t Y^{x,y,m}$ is the derivative of $Y_t^{x,y,m}$ with respect to $y$ in the direction $h$, which satisfies

$$\begin{cases}
D^h_t Y_t^{x,y,m} = AD^h_t Y_t^{x,y,m} dt + D_y F^m(x, Y_t^{x,y}) \cdot D^h_t Y_t^{x,y,m} dt, \\
D^h_0 Y_t^{x,y} = h.
\end{cases}$$

(3.18)

Then by $\lambda_1 - L_F > 0$ and condition (2.3), it is easy to see

$$\sup_{x,y \in H_m} |D^h_t Y_t^{x,y,m}| \leq C e^{-\frac{(\lambda_1 - LF)t}{2} |h|},$$

and for any $\eta \in (0, 2)$ and $t > 0$,

$$\sup_{x,y \in H_m} \|D^h_t Y_t^{x,y,m}\|_{\eta} \leq C e^{-\frac{(\lambda_1 - LF)t}{2} (1 + t^{-\eta/2}) |h|}.$$ (3.19)

Thus it follows

$$\sup_{x,y \in H_m} |D_y \Phi_m(x, y) \cdot h| \leq C |h|.$$ (3.20)

So (3.16) holds.

Now, we define

$$\tilde{B}^m_{t_0}(x, y, t) := \hat{B}^m(x, y, t) - \hat{B}^m(x, y, t + t_0),$$

where $\hat{B}^m(x, y, t) := \mathbb{E}B^m(x, Y_t^{x,y})$. Note that (3.10) implies

$$\lim_{t_0 \to \infty} \tilde{B}^m_{t_0}(x, y, t) = \mathbb{E}[B^m(x, Y_t^{x,y,m})] - \hat{B}^m(x).$$

So in order to prove (3.16) and (3.17), it suffices to show that there exists $C > 0$ such that for any $\delta \in (0, 1], t_0 > 0$, $t \geq 0$, $x, y \in H$,

$$|D_x \tilde{B}^m_{t_0}(x, y, t) \cdot h| \leq C e^{-\frac{(\lambda_1 - LF)t_0}{4} (1 + t^{-\frac{\eta_1 - \delta}{2}})} (1 + |x|^{\delta} + |y|^{\delta}) |h|,$$

(3.21)

$$|D_{xx} \tilde{B}^m_{t_0}(x, y, t) \cdot (h, k)| \leq C e^{-\frac{(\lambda_1 - LF)t_0}{4} (1 + t^{-\frac{\eta_1 - \delta}{2}})} (1 + |x|^{\delta} + |y|^{\delta}) |h| \|k\|_{\kappa_1},$$

(3.22)

which will be proved in step 2 and step 3 respectively.

Step 2. In this step, we intend to prove (3.21). It follows from the Markov property,

$$\tilde{B}^m_{t_0}(x, y, t) = \hat{B}^m(x, y, t) - \mathbb{E}B^m(x, Y_{t+t_0}^{x,y,m})$$

$$= \hat{B}(x, y, t) - \mathbb{E}\{\mathbb{E}[B^m(x, Y_{t+t_0}^{x,y,m})|\mathcal{F}_{t_0}]\}.$$
\[
= \hat{B}(x, y, t) - E\hat{B}^m(x, Y_{t_0}^{x,y,m}, t).
\]

Then we obtain
\[
D_{x} \hat{B}^m_{t_0}(x, y, t) \cdot h = D_{x} \hat{B}^m(x, y, t) \cdot h - ED_{x} \hat{B}^m(x, Y_{t_0}^{x,y,m}, t) \cdot h
- E \left[ D_{x} \hat{B}^m(x, Y_{t_0}^{x,y,m}, t) \cdot D_{x}^{h} Y_{t_0}^{x,y,m} \right],
\]
(3.23)

where \(D_{x}^{h} Y_{t}^{x,y,m}\) is the derivative of \(Y_{t}^{x,y,m}\) with respect to \(x\) in the direction \(h\), which satisfies
\[
\begin{align*}
d D_{x}^{h} Y_{t}^{x,y,m} &= \left[ AD_{x}^{h} Y_{t}^{x,y,m} + D_{x} F^{m}(x, Y_{t}^{x,y,m}) \cdot h + D_{y} F^{m}(x, Y_{t}^{x,y,m}) \cdot D_{x}^{h} Y_{t}^{x,y,m} \right] dt, \\
D_{x}^{h} Y_{t_0}^{x,y,m} &= 0.
\end{align*}
\]

By (2.1) and (2.3), we can easily obtain for any \(\eta \in [0, 2)\)
\[
\sup_{t \geq 0, x, y \in H_{m}} \| D_{x}^{h} Y_{t}^{x,y,m} \|_{\eta} \leq C |h|.
\]
(3.24)

Note that
\[
D_{y} \hat{B}^m(x, y, t) \cdot h = E \left[ D_{y} B^m(x, Y_{t}^{x,y,m}) \cdot D_{y}^{h} Y_{t}^{x,y,m} \right],
D_{x} \hat{B}^m(x, y, t) \cdot h = E \left[ D_{x} B^m(x, Y_{t}^{x,y,m}) \cdot h + D_{y} B^m(x, Y_{t}^{x,y,m}) \cdot D_{x}^{h} Y_{t}^{x,y,m} \right].
\]

Then by (2.3), (3.24) and (3.19), we get
\[
\sup_{t \geq 0, x, y \in H_{m}} \| D_{x} \hat{B}^m(x, y, t) \cdot h \| \leq C e^{-\frac{(\lambda_{1} - LF_{4}) t}{2}} |h|,
\]
(3.25)

\[
\sup_{t \geq 0, x, y \in H_{m}} \| D_{y} \hat{B}^m(x, y, t) \cdot h \| \leq C |h|.
\]
(3.26)

Next if we can prove that for any \(t > 0, h, k \in H_{m}\),
\[
\sup_{x, y \in H_{m}} | D_{xy} \hat{B}^m(x, y, t) \cdot (h, k) | \leq C e^{-\frac{(\lambda_{1} - LF_{4}) t}{2}} (1 + t^{-\frac{\delta}{2}}) |h||k|,
\]
(3.27)

which combines with (3.26) and (3.4) we obtain for any \(\delta \in (0, 1]\)
\[
| D_{x} \hat{B}^m(x, y, t) \cdot h | \leq E \left[ D_{x} B^m(x, Y_{t_0}^{x,y,m}, t) \cdot h \right]
\]
\[
\leq C |h|^{1-\delta} \left[ E \left| \int_{0}^{1} D_{xy} \hat{B}^m(x, \xi y + (1 - \xi) Y_{t_0}^{x,y,m}, t) \cdot (h, y - Y_{t_0}^{x,y,m}) d\xi \right| \right]^{\delta}
\]
\[
\leq C e^{-\frac{(\lambda_{1} - LF_{4}) t}{4}} (1 + t^{-\frac{\delta}{2}}) |h|.
\]
(3.28)

Thus by (3.25) and (3.28), we get
\[
| D_{x} \hat{B}^m_{t_0}(x, y, t) \cdot h | \leq C e^{-\frac{(\lambda_{1} - LF_{4}) t}{2}} (1 + |x|^{\delta} + |y|^{\delta}) |h| + C e^{-\frac{(\lambda_{1} - LF_{4}) t}{2}} |h|
\]
\[
\leq C e^{-\frac{(\lambda_{1} - LF_{4}) t}{2}} (1 + |x|^{\delta} + |y|^{\delta}) |h|,
\]
which proves (3.21).

Now, we are in a position to prove (3.27). Note that
\[
| D_{xy} \hat{B}^m(x, y, t) \cdot (h, k) | = | D_{xy} [EB^{m}(x, Y_{t}^{x,y,m})] \cdot (h, k) |
\]
\[
= | D_{y} [ED_{x} B^{m}(x, Y_{t}^{x,y,m}) + ED_{y} B^{m}(x, Y_{t}^{x,y,m}) \cdot D_{x} Y_{t}^{x,y,m}] \cdot (h, k) |
\]
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\[\leq \mathbb{E} \left[ D_{xy} B^m(x, Y_{t}^{x,y,m}) \cdot (D^{(h,k)} Y_{x,y,m}) \right] + \mathbb{E} \left[ D_y B^m(x, Y_{t}^{x,y,m}) \cdot D_x^{(h,k)} Y_{x,y,m} \right] + \mathbb{E} \left[ D_{yy} B^m(x, Y_{t}^{x,y,m}) \cdot (D_x^{(h,k)} Y_{x,y,m}) \right],\]

where \(D_x^{(h,k)} Y_{x,y,m}\) is the second order derivative of \(Y_{x,y,m}\) in the direction \((h, k) \in H_m \times H_m\) (once with respect to \(x\) in the direction \(h \in H_m\) and once with respect to \(y\) in the direction \(k \in H_m\)), which satisfies

\[
\begin{align*}
\begin{cases}
  dD_x^{(h,k)} Y_{x,y,m} &= \left[ AD^{(h,k)} Y_{x,y,m} + D_{xx} F^m(x, Y_{t}^{x,y,m}) \cdot (h, D^{(h,k)} Y_{x,y,m}) \\
  &+ D_{yy} F^m(x, Y_{t}^{x,y,m}) \cdot (D_x^{(h,k)} Y_{x,y,m}) + D_y F^m(x, Y_{t}^{x,y,m}) \cdot D_x^{(h,k)} Y_{x,y,m} \right] dt,
  \\
  D_x^{(h,k)} Y_{0,x,y,m} &= 0.
\end{cases}
\end{align*}
\]

By (2.3), (3.20), (3.24) and \(\lambda_1 - L_F > 0\), it is easy to prove for any \(\eta \in [0, 2)\) and \(t > 0\),

\[
\sup_{x,y \in H_m} \left\| D_x^{(h,k)} Y_{x,y,m} \right\| \eta \leq C e^{- \frac{(\lambda_1 - L_F) t}{4} + \frac{1}{2} (1 + t^{-\eta})} \|h\| |k|.
\]

Hence, (3.27) holds by assumption \(A_3\), (3.24), (3.20) and (3.29).

**Step 3.** In this step, we intend to prove (3.22). Recall that

\[
D_x \hat{B}^m(x, y, t) \cdot h = D_x \hat{B}^m(x, y, t) \cdot h - \mathbb{E} \left[ D_x \hat{B}^m(x, Y_{t0}^{x,y,m}, t) \cdot h \right] - \mathbb{E} \left[ D_y \hat{B}^m(x, Y_{t0}^{x,y,m}, t) \cdot D_x^{(h,k)} Y_{x,y,m} \right].
\]

Then it is easy to see

\[
\begin{align*}
D_{xx} \hat{B}^m(x, y, t) \cdot (h, k) &= \left[ D_{xx} \hat{B}^m(x, y, t) \cdot (h, k) - \mathbb{E} D_{xx} \hat{B}^m(x, Y_{t0}^{x,y,m}, t) \cdot (h, k) \right] \\
&- \mathbb{E} \left[ D_{xy} \hat{B}^m(x, Y_{t0}^{x,y,m}, t) \cdot (h, D^{(h,k)} Y_{x,y,m}) + D_{yx} \hat{B}^m(x, Y_{t0}^{x,y,m}, t) \cdot (D^{(h,k)} Y_{x,y,m}, k) \right] \\
&- \mathbb{E} \left[ D_{yy} \hat{B}^m(x, Y_{t0}^{x,y,m}, t) \cdot (D_x^{(h,k)} Y_{x,y,m}, D_y^{(h,k)} Y_{x,y,m}) \right] \\
&- \mathbb{E} \left[ D_y \hat{B}^m(x, Y_{t0}^{x,y,m}, t) \cdot D_x^{(h,k)} Y_{x,y,m} \right] := \sum_{i=1}^{4} J_i,
\end{align*}
\]

where \(D_x^{(h,k)} Y_{x,y,m}\) is the second derivative of \(Y_{x,y,m}\) with respect to \(x\) towards the direction \((h, k) \in H \times H\), which satisfies

\[
\begin{align*}
\begin{cases}
  dD_x^{(h,k)} Y_{x,y,m} &= \left[ AD_x^{(h,k)} Y_{x,y,m} + D_{xx} F^m(x, Y_{t}^{x,y,m}) \cdot (h, k) + D_{xy} F^m(x, Y_{t}^{x,y,m}) \cdot (h, D^{(h,k)} Y_{x,y,m}) + D_{yx} F^m(x, Y_{t}^{x,y,m}) \cdot (D^{(h,k)} Y_{x,y,m}, k) + D_{yy} F^m(x, Y_{t}^{x,y,m}) \cdot (D_x^{(h,k)} Y_{x,y,m}, D_y^{(h,k)} Y_{x,y,m}) \\
  &+ D_y F^m(x, Y_{t}^{x,y,m}) \cdot D_x^{(h,k)} Y_{x,y,m} \right] dt,
  \\
  D_x^{(h,k)} Y_{0,x,y,m} &= 0.
\end{cases}
\end{align*}
\]

For the term \(J_1\), note that

\[
D_x \hat{B}^m(x, y, t) \cdot h = \mathbb{E} \left[ D_x B^m(x, Y_{t}^{x,y,m}) \cdot h \right] + \mathbb{E} \left[ D_y B^m(x, Y_{t}^{x,y,m}) \cdot D_x^{(h,k)} Y_{x,y,m} \right],
\]

which implies

\[
D_{xx} \hat{B}^m(x, y, t) \cdot (h, k) = \mathbb{E} \left[ D_{xx} B^m(x, Y_{t}^{x,y,m}) \cdot (h, k) \right] + \mathbb{E} \left[ D_{xy} B^m(x, Y_{t}^{x,y,m}) \cdot (h, D^{(h,k)} Y_{x,y,m}) \right] + \mathbb{E} \left[ D_{yx} B^m(x, Y_{t}^{x,y,m}) \cdot (D^{(h,k)} Y_{x,y,m}, k) \right] + \mathbb{E} \left[ D_{yy} B^m(x, Y_{t}^{x,y,m}) \cdot (D_x^{(h,k)} Y_{x,y,m}, D_y^{(h,k)} Y_{x,y,m}) \right] + \mathbb{E} \left[ D_y B^m(x, Y_{t}^{x,y,m}) \cdot D_x^{(h,k)} Y_{x,y,m} \right].
\]
\[ + \mathbb{E}\left[D_{xy}B^m(x, Y_t^{x,y}) \cdot (D_x^hY_t^{x,y,m}, k)\right] + \mathbb{E}\left[D_{yy}B^m(x, Y_t^{x,y}) \cdot (D_x^hY_t^{x,y,m}, D_y^kY_t^{x,y,m})\right] + \mathbb{E}\left[D_yB^m(x, Y_t^{x,y}) \cdot D_{xx}^kY_t^{x,y,m}\right]. \]

Then it follows
\[
D_{xxy}\hat{B}^m(x, y, t) \cdot (h, k, l)
\]
\[
= \mathbb{E}\left[D_{xxy}B^m(x, Y_t^{x,y,m}) \cdot (h, k, D_y^lY_t^{x,y,m}) + D_{xyy}B^m(x, Y_t^{x,y,m}) \cdot (h, D_x^hY_t^{x,y,m}, D_y^lY_t^{x,y,m}) + D_{xy}B^m(x, Y_t^{x,y,m}) \cdot (h, D_x^hY_t^{x,y,m}, D_y^kY_t^{x,y,m}) + D_{yy}B^m(x, Y_t^{x,y,m}) \cdot (h, D_x^hY_t^{x,y,m}, D_y^kY_t^{x,y,m}) + D_{yy}B^m(x, Y_t^{x,y,m}) \cdot (h, D_x^hY_t^{x,y,m}, D_y^kY_t^{x,y,m}) + D_yB^m(x, Y_t^{x,y,m}) \cdot D_{xx}^kY_t^{x,y,m}\right],
\]

where \(D_{xxy}^hY_t^{x,y,m}\) is the third order derivative of \(Y_t^{x,y,m}\) (twice with respect to \(x\) in the direction \((h, k)\) \(\in H_m \times H_m\) and once with respect to \(y\) in the direction \(l \in H_m\).

By assumption A\textsubscript{3}, (3.24) and (3.20), it is easy to prove that
\[
\sup_{t \geq 0, x, y \in H_m} |D_{xx}^hY_t^{x,y,m}| \leq C|h||k|, \tag{3.30}
\]
\[
\sup_{x, y \in H_m} |D_{xxy}^hY_t^{x,y,m}| \leq Ce^{-(\frac{\lambda_1-L_p}{4}t)}(1 + t^{-\frac{\sigma}{2}})|h||k||l|.
\]

which combine with (3.20), (3.24), (3.29) and assumption A\textsubscript{3}, we get
\[
|D_{xxy}\hat{B}^m(x, y, t) \cdot (h, k)| \leq C|h||k|_{\kappa_1}, \tag{3.31}
\]
\[
|D_{xxy}\hat{B}^m(x, y, t) \cdot (h, k, l)| \leq Ce^{-(\frac{\lambda_1-L_p}{4}t)}(1 + t^{-\frac{\sigma}{2}})|h||k|_{\kappa_1}|l|. \tag{3.32}
\]

Then by (3.31), (3.32) and similar as we did in proving (3.28), we obtain for any \(\delta \in (0, 1],\)
\[
J_1 \leq Ce^{-(\frac{\lambda_1-L_p}{4}t)}(1 + t^{-\frac{\sigma}{2}})|h||k|_{\kappa_1}E|y - Y_{t_0}^{x,y,m}|^\delta
\leq Ce^{-(\frac{\lambda_1-L_p}{4}t)}(1 + t^{-\frac{\sigma}{2}})|h||k|_{\kappa_1}(1 + |x|^\delta + |y|^\delta). \tag{3.33}
\]

For the term \(J_2\), note that
\[
D_{xy}\hat{B}^m(x, y, t) \cdot (h, k) = \mathbb{E}\left[D_{xy}B^m(x, Y_t^{x,y,m}) \cdot (h, D_y^kY_t^{x,y,m})\right] + \mathbb{E}\left[D_yB^m(x, Y_t^{x,y,m}) \cdot D_{xy}^hY_t^{x,y,m}\right] + \mathbb{E}\left[D_{yy}B^m(x, Y_t^{x,y,m}) \cdot (D_x^hY_t^{x,y,m}, D_y^kY_t^{x,y,m})\right].
\]

Then by assumption A\textsubscript{3}, (3.20), (3.24) and (3.29), we have
\[
\sup_{x, y \in H} |D_{xy}\hat{B}^m(x, y, t) \cdot (h, k)| \leq Ce^{-(\frac{\lambda_1-L_p}{4}t)}(1 + t^{-\frac{\sigma}{2}})|h||k|.
\]

Thus it follows
\[
J_2 \leq Ce^{-(\frac{\lambda_1-L_p}{4}t)}(1 + t^{-\frac{\sigma}{2}})|h||k|. \tag{3.34}
\]
For the term $J_3$, by a similar argument as in estimating $J_2$, we have
\[
\sup_{x,y\in H_m} |D_{yy} \tilde{B}^m(x, y, t) \cdot (h, k)| \leq C e^{-\frac{(\lambda_1 - Lp)t}{4}} (1 + t^{-\frac{p}{2}})|h||k|.
\]
Hence, it is easy to see that
\[
J_3 \leq C e^{-\frac{(\lambda_1 - Lp)t}{4}} (1 + t^{-\frac{p}{2}})|h||k|.
\tag{3.35}
\]
For the term $J_4$, by (3.30) and (3.25), we easily get
\[
J_4 \leq C e^{-\frac{(\lambda_1 - Lp)t}{2}} |h||k|.
\tag{3.36}
\]
Finally, (3.22) can be easily obtained by combining (3.33)-(3.36). The proof is complete. \qed

4. Strong convergence order

This section is devoted to prove Theorem 2.4. We first study the well-posedness of equation (3.2) which approximates to the averaged equation. Then we give the detailed proof of Theorem 2.4.

**Lemma 4.1.** Equation (3.2) exists a unique mild solution $\tilde{X}_t^m$ satisfies
\[
\tilde{X}_t^m = e^{tA}x^m + \int_0^t e^{(t-s)A} \tilde{B}^m(\tilde{X}_s^m)ds + \int_0^t e^{(t-s)A} d\tilde{\Gamma}_s.
\tag{4.1}
\]
Moreover, for any $x \in H$, $T > 0$ and $1 \leq p < \alpha$, there exists a constant $C_{p,T} > 0$ such that
\[
\sup_{m \geq 1, t \in [0, T]} \mathbb{E}|\tilde{X}_t^m|^p \leq C_{p,T}(1 + |x|^p).
\tag{4.2}
\]
**Proof.** It is sufficient to check that the $\tilde{B}^m$ is Lipschitz continuous, then (3.2) admits a unique mild solution $\tilde{X}_t^m$. The estimate (4.2) can be proved by the same argument as in the proof of Lemma 6.2 in the appendix.

Indeed, for any $x_1, x_2 \in H_m$ and $t > 0$, by Proposition 3.10 and (3.7), we have
\[
|\tilde{B}^m(x_1) - \tilde{B}^m(x_2)| = \left| \int_{H_m} B^m(x_1, y) \mu^{x_1, m}(dy) - \int_{H_m} B^m(x_2, y) \mu^{x_2, m}(dy) \right|
\leq \left| \mathbb{E}B^m(x_1, Y_t^{x_1, 0, m}) - \int_{H_m} B^m(x_1, z) \mu^{x_1, m}(dz) \right|
+ \left| \mathbb{E}B^m(x_2, Y_t^{x_2, 0, m}) - \int_{H_m} B^m(x_2, z) \mu^{x_2, m}(dz) \right|
+ \left| \mathbb{E}B^m(x_1, Y_t^{x_1, 0, m}) - \mathbb{E}B^m(x_2, Y_t^{x_2, 0, m}) \right|
\leq C e^{-\frac{(\lambda_1 - Lp)t}{2}} + C \left( |x_1 - x_2| + \mathbb{E}|Y_t^{x_1, 0, m} - Y_t^{x_2, 0, m}| \right)
\leq C e^{-\frac{(\lambda_1 - Lp)t}{2}} + C|x_1 - x_2|.
\]
As a result, the proof is completed by letting $t \to \infty$. \qed

**Remark 4.2.** By a similar argument above, it is easy to prove that the averaged coefficient $\bar{B}$ is also Lipschitz continuous. As a consequence, equation (2.7) admits a unique mild solution $\tilde{X}_t$. 
Now we are in a position to prove Theorem 2.4.

**Proof of Theorem 2.4.** It is easy to see that for any $T > 0$, $p \in [1, \alpha)$ and $m \in \mathbb{N}_+$,

\[
\sup_{t \in [0, T]} \mathbb{E}|X_t^\varepsilon - \bar{X}_t|^p \leq C_p \sup_{t \in [0, T]} \mathbb{E}|X_t^{m, \varepsilon} - X_t^\varepsilon|^p + C_p \sup_{t \in [0, T]} \mathbb{E}|\bar{X}_t^m - \bar{X}_t|^p.
\]

By Lemmas 6.2 and 6.6, it follows

\[
\lim_{m \to \infty} \sup_{t \in [0, T]} \mathbb{E}|X_t^{m, \varepsilon} - X_t^\varepsilon|^p = 0, \quad \lim_{m \to \infty} \sup_{t \in [0, T]} \mathbb{E}|\bar{X}_t^m - \bar{X}_t|^p = 0.
\]

Thus it is sufficient to prove the for any $(x, y) \in H^0 \times H$ with $\eta \in (0, 1)$, $T > 0$ and small enough $\varepsilon, \delta > 0$, there exists a positive constant $C_{p,T,\delta}$ independent of $m$ such that

\[
\sup_{t \in [0, T]} \mathbb{E}|X_t^{m, \varepsilon} - \bar{X}_t^m|^p \leq C_{p,T,\delta}(1 + \|x\|_{t}^{(1+\delta)p} + \|y\|_{t}^{(1+\delta)p}e^{p(1-\alpha)}),
\]

which will proved by the following three steps.

**Step 1.** Using the formulation of the mild solutions $X_t^{m, \varepsilon}$ and $\bar{X}_t^m$, we have for any $t > 0$,

\[
X_t^{m, \varepsilon} - \bar{X}_t^m = \int_0^t e^{(t-s)A} \left[ B^m(X_s^{m, \varepsilon}, Y_s^{m, \varepsilon}) - \bar{B}^m(\bar{X}_s^m) \right] ds.
\]

Note that the averaged coefficient $\bar{B}^m$ has been proved that it is Lipschitz continuous in Lemma 4.1. Then we get for any $T > 0$,

\[
\sup_{t \in [0, T]} \mathbb{E}|X_t^{m, \varepsilon} - \bar{X}_t^m|^p \leq C_p \sup_{t \in [0, T]} \mathbb{E} \left| \int_0^t e^{(t-s)A} \left[ B^m(X_s^{m, \varepsilon}, Y_s^{m, \varepsilon}) - \bar{B}^m(X_s^{m, \varepsilon}) \right] ds \right|^p + C_p \mathbb{E} \int_0^T |X_t^{m, \varepsilon} - \bar{X}_t^m|^p dt.
\]

By Gronwall’s inequality, it follows

\[
\sup_{t \in [0, T]} \mathbb{E}|X_t^{m, \varepsilon} - \bar{X}_t^m|^p \leq C_{p,T} \sup_{t \in [0, T]} \mathbb{E} \left| \int_0^t e^{(t-s)A} \left[ B^m(X_s^{m, \varepsilon}, Y_s^{m, \varepsilon}) - \bar{B}^m(X_s^{m, \varepsilon}) \right] ds \right|^p.
\]

By Proposition 3.4, the following Poisson equation

\[
-\mathcal{L}_2^m(x)(\Phi_m)(x, y) = B^m(x, y) - \bar{B}^m(x)
\]

exists a solution $\Phi_m(x, y)$ satisfies (3.14)-(3.17).

By applying Itô’s formula,

\[
\Phi_m(X_t^{m, \varepsilon}, Y_t^{m, \varepsilon}) = e^{tA} \Phi_m(x^m, y^m) + \int_0^t (-A)e^{(t-s)A} \Phi_m(X_s^{m, \varepsilon}, Y_s^{m, \varepsilon}) ds
\]

\[
+ \int_0^t e^{(t-s)A} \mathcal{L}_1^m(Y_s^{m, \varepsilon}) \Phi_m(X_s^{m, \varepsilon}, Y_s^{m, \varepsilon}) ds + M_{t}^{m, \varepsilon, 1} + M_{t}^{m, \varepsilon, 2}
\]

\[
+ \frac{1}{\varepsilon} \int_0^t e^{(t-s)A} \mathcal{L}_2^{m, \varepsilon}(X_s^{m, \varepsilon}) \Phi_m(X_s^{m, \varepsilon}, Y_s^{m, \varepsilon}) ds,
\]

where $\mathcal{L}_1^m(y)$, $M_{t}^{m, \varepsilon, 1}$ and $M_{t}^{m, \varepsilon, 2}$ are defined as follows:

\[
\mathcal{L}_1^m(y) := D_x \Phi_m(x, y) \cdot (Ax + B^m(x, y))
\]
$$+ \sum_{k=1}^{m} \beta_k \int_{\mathbb{R}} \Phi_m(x + e_k z, y) - \Phi_m(x, y) - D_x \Phi_m(x, y) \cdot (e_k z) 1_{\{|z| \leq 1\}} \nu(dz)$$

$$M_{1}^{t, m, \varepsilon, 1} := \sum_{k=1}^{m} \int_{0}^{t} \int_{\mathbb{R}} e^{(t-s)A} \left[ \Phi_m(X^{m, \varepsilon}_{s-} + z \beta_k e_k, Y^{\varepsilon}_{s-}) - \Phi_m(X^{m, \varepsilon}_{s-}, Y^{\varepsilon}_{s-}) \right] \tilde{N}^{1,k}(ds, dz),$$

$$M_{1}^{t, m, \varepsilon, 2} := \sum_{k=1}^{m} \int_{0}^{t} \int_{\mathbb{R}} e^{(t-s)A} \left[ \Phi_m(X^{m, \varepsilon}_{s-}, Y^{m, \varepsilon}_{s-} + \varepsilon^{-1/\alpha} \gamma_k e_k) - \Phi_m(X^{m, \varepsilon}_{s-}, Y^{m, \varepsilon}_{s-}) \right] \tilde{N}^{2,k}(ds, dz).$$

Then it follows

$$\int_{0}^{t} e^{(t-s)A} \mathcal{L}_{2}^{m}(X^{m, \varepsilon}_{s}, Y^{m, \varepsilon}_{s}) ds$$

$$= \varepsilon \left[ e^{tA} \Phi_m(x^{m, y^{m}}) - \Phi_m(X^{m, \varepsilon}_{t}, Y^{m, \varepsilon}_{t}) + \int_{0}^{t} (-A) e^{(t-s)A} \Phi_m(X^{m, \varepsilon}_{t}, Y^{m, \varepsilon}_{t}) ds + M_{1}^{t, m, \varepsilon, 1} + M_{1}^{t, m, \varepsilon, 2} + \int_{0}^{t} e^{(t-s)A} \mathcal{L}_{1}^{m}(Y^{m, \varepsilon}_{s}) \Phi_m(X^{m, \varepsilon}_{s}, Y^{m, \varepsilon}_{s}) ds \right]. \quad (4.7)$$

By (4.4), (4.5) and (4.7), we get

$$\sup_{t \in [0, T]} \mathbb{E}[X^{m, \varepsilon}_{t} - X^{m}_{t}] \leq \mathbb{E}[X^{m, \varepsilon}_{T} - X^{m}_{T}] \leq \mathbb{E}[\mathcal{L}^{m}(Y^{m, \varepsilon}_{T})] \leq \mathbb{E}[\mathcal{L}^{m}(Y^{m, \varepsilon}_{T})]$$

$$\leq C_{p, T} \varepsilon \frac{5}{p} \sum_{k=1}^{5} \Lambda_{k}^{m}(T). \quad (4.8)$$

**Step 2.** In this step, we first estimate the terms \( \Lambda_{1}^{m}(T) - \Lambda_{4}^{m}(T) \). By (3.14), (6.4) and (6.5), we have

$$\Lambda_{1}^{m}(T) \leq C \left( 1 + |x|^{p} + |y|^{p} + \sup_{t \in [0, T]} \mathbb{E}|X^{m, \varepsilon}_{t}|^{p} + \sup_{t \in [0, T]} \mathbb{E}|Y^{m, \varepsilon}_{t}|^{p} \right)$$

$$\leq C_{T} (1 + |x|^{p} + |y|^{p}). \quad (4.9)$$

By (3.14), (6.4), (6.5) and Lemma 6.3, for any \( \eta \in (0, 1) \) we have

$$\Lambda_{2}^{m}(T) \leq C_{p} \sup_{t \in [0, T]} \mathbb{E} \left| \int_{0}^{t} (-A) e^{(t-s)A} \left[ \Phi_m(X^{m, \varepsilon}_{s}, Y^{m, \varepsilon}_{s}) - \Phi_m(X^{m, \varepsilon}_{t}, Y^{m, \varepsilon}_{t}) \right] ds \right|^{p}$$

$$+ C_{p} \sup_{t \in [0, T]} \mathbb{E} \left| \int_{0}^{t} (-A) e^{(t-s)A} \Phi_m(X^{m, \varepsilon}_{t}, Y^{m, \varepsilon}_{t}) ds \right|^{p}$$

$$\leq C_{p} \sup_{t \in [0, T]} \left| \int_{0}^{t} (t-s)^{-1} \mathbb{E}\left| \Phi_m(X^{m, \varepsilon}_{s}, Y^{m, \varepsilon}_{s}) - \Phi_m(X^{m, \varepsilon}_{t}, Y^{m, \varepsilon}_{t}) \right|^{p} ds \right|^{1/p}$$

$$+ C_{p} \sup_{t \in [0, T]} \mathbb{E} \left| \Phi_m(X^{m, \varepsilon}_{t}, Y^{m, \varepsilon}_{t}) \right|^{p}.$$
\[ C_p \sup_{t \in [0, T]} \left| \int_0^t (t-s)^{-1} \mathbb{E} \left( (1 + |X_{s,t}^{\varepsilon,m}|^\delta + |X_{s,t}^{\varepsilon,m}|^\delta + |Y_{s,t}^{\varepsilon,m}|^\delta) |X_{s,t}^{\varepsilon,m} - X_{s,t}^{\varepsilon,m}|^p \right) \right| ds \]

\[ + C_p \sup_{t \in [0, T]} \left| \int_0^t (t-s)^{-1} \mathbb{E} |Y_{s,t}^{\varepsilon,m} - Y_{s,t}^{\varepsilon,m}|^p \right| ds \]

\[ + C_p \sup_{t \in [0, T]} \mathbb{E} |\Phi_m(X_{t}^{\varepsilon,m}, Y_{t}^{\varepsilon,m})|^p \]

\[ \leq C_{p,T} (1 + |x|^p((1+\delta) + |y|^{(1+\delta)}) \sup_{t \in [0, T]} \left| \int_0^t (t-s)^{-1} (t-s)^{\frac{\delta}{\varepsilon}} \right| ds \]

\[ + C_{p,T} (1 + |x|^p + |y|^p) \sup_{t \in [0, T]} \left| \int_0^t (t-s)^{-1} \left( \frac{t-s}{\varepsilon} \left( \frac{1}{s^{\frac{\delta}{\varepsilon}}} \right) \right) \right| ds \]

\[ \leq C_{p,T} (1 + |x|^{p(1+\delta)} + |y|^{p(1+\delta)}) \varepsilon^{-\frac{p}{\delta}}. \quad (4.10) \]

For the term \( \Lambda_3^{\varepsilon,m}(T) \). By Burkholder-Davis-Gundy’s inequality, it follows for any \( p \in [1, \alpha) \),

\[ \Lambda_3^{\varepsilon,m}(T) \leq C_p \mathbb{E} \left[ \sum_{k=1}^m \int_0^T \mathbb{I}_{|\Phi_m(X_{s_k}^{\varepsilon,m} + \varepsilon_k, Y_{s_k}^{\varepsilon,m}) - \Phi_m(X_{s_k}^{\varepsilon,m}, Y_{s_k}^{\varepsilon,m})|^2 N_{1,k}(ds, dz) \right]^{p/2} \]

\[ \leq C_p \mathbb{E} \left[ \sum_{k=1}^m \int_0^T \mathbb{I}_{|\Phi_m(X_{s_k}^{\varepsilon,m} + \varepsilon_k, Y_{s_k}^{\varepsilon,m}) - \Phi_m(X_{s_k}^{\varepsilon,m}, Y_{s_k}^{\varepsilon,m})|^2 N_{1,k}(ds, dz) \right]^{p/2} \]

\[ + C_p \mathbb{E} \left[ \sum_{k=1}^m \int_0^T \mathbb{I}_{|\Phi_m(X_{s_k}^{\varepsilon,m} + \varepsilon_k, Y_{s_k}^{\varepsilon,m}) - \Phi_m(X_{s_k}^{\varepsilon,m}, Y_{s_k}^{\varepsilon,m})|^p N_{1,k}(ds, dz) \right]^{p/2} \]

By (3.16), for any \( \delta \in (0, 1] \) we have

\[ |\Phi_m(X_{s_k}^{\varepsilon,m} + \varepsilon_k, Y_{s_k}^{\varepsilon,m}) - \Phi_m(X_{s_k}^{\varepsilon,m}, Y_{s_k}^{\varepsilon,m})| = \left| \int_0^1 D_x \Phi_m(X_{s_k}^{\varepsilon,m} + \xi \varepsilon_k, Y_{s_k}^{\varepsilon,m}) \cdot (\varepsilon_k) d\xi \right| \]

\[ \leq C_\delta \int_0^1 (1 + |X_{s_k}^{\varepsilon,m} + \xi \varepsilon_k|^\delta + |Y_{s_k}^{\varepsilon,m}|^\delta)(dz)|\varepsilon_k| \]

\[ \leq C_\delta (1 + |X_{s_k}^{\varepsilon,m}|^\delta + |Y_{s_k}^{\varepsilon,m}|^\delta + |z|^\delta |z|. \]

Note that for small enough \( \delta < \frac{p}{p-1} \), it easy to see

\[ \int_{|z| \leq 1} |z|^2 \nu(dz) < \infty, \int_{|z| > 1} |z|^{p(1+\delta)} \nu(dz) < \infty, \]
which combing with the condition $\sum_{k=1}^{\infty} \beta_k^\alpha < \infty$ we obtain

$$
\Lambda_{3,m}^\varepsilon(T) \leq C_{p,\delta} \left[ \sum_{k=1}^{m} \beta_k^\alpha \int_0^T \int_{|z| \leq 1} E(1 + |X_s^m|^{2\delta} + |Y_s^m|^{2\delta} + |z|^{2\delta})|z|^2 \nu(dz)ds \right]^{p/2} \\
+ C_{p,\delta} \left[ \sum_{k=1}^{m} \beta_k^\alpha \int_0^T \int_{|z| > 1} E(1 + |X_s^m|^{p\delta} + |Y_s^m|^{p\delta} + |z|^{p\delta})|z|^p \nu(dz)ds \right]^{p/2} \\
\leq C_{p,T,\delta}(1 + |x|^{p\delta} + |y|^{p\delta}). \tag{4.11}
$$

For the term $\Lambda_4^\varepsilon_m(T)$. Similar as we did in estimating $\Lambda_3^\varepsilon_m(T)$. It is easy to see that

$$
\Lambda_4^\varepsilon_m(T) \leq C_{p} \lim_{m \to \infty} \left[ \sum_{k=1}^{m} \int_0^T \int_{|z| \leq 1} \Phi_m(X_{s-}, Y_{s-}^{m,\varepsilon} + \varepsilon^{-1/\alpha} \gamma_k z \varepsilon_k) - \Phi_m(X_{s-}^{m,\varepsilon}, Y_{s-}^{m,\varepsilon}) |2N^{2,k}(ds, dz) \right]^{p/2} \\
+ C_{p} \lim_{m \to \infty} \left[ \sum_{k=1}^{m} \int_0^T \int_{|z| > 1} \Phi_m(X_{s-}^m, Y_{s-}^{m,\varepsilon} + \varepsilon^{-1/\alpha} \gamma_k z \varepsilon_k) - \Phi_m(X_{s-}^{m,\varepsilon}, Y_{s-}^{m,\varepsilon}) |pN^{2,k}(ds, dz) \right]^{p/2} \\
\leq C_{p} \lim_{m \to \infty} \left[ \sum_{k=1}^{m} \gamma_k^\alpha E \int_0^T \int_{|z| \leq 1} \Phi_m(X_{s}^{m,\varepsilon}, Y_{s}^{m,\varepsilon} + \varepsilon^{-1/\alpha} \gamma_k z \varepsilon_k) - \Phi_m(X_{s}^{m,\varepsilon}, Y_{s}^{m,\varepsilon}) |2\nu(dz)ds \right]^{p/2} \\
+ C_{p} \lim_{m \to \infty} \left[ \sum_{k=1}^{m} \gamma_k^\alpha E \int_0^T \int_{|z| > 1} \Phi_m(X_{s}^{m,\varepsilon}, Y_{s}^{m,\varepsilon} + \varepsilon^{-1/\alpha} \gamma_k z \varepsilon_k) - \Phi_m(X_{s}^{m,\varepsilon}, Y_{s}^{m,\varepsilon}) |p\nu(dz)ds \right].
$$

By (3.21), we have

$$
|\Phi_m(X_{s}^{m,\varepsilon}, Y_{s}^{m,\varepsilon} + \varepsilon^{-1/\alpha} \gamma_k z \varepsilon_k) - \Phi_m(X_{s}^{m,\varepsilon}, Y_{s}^{m,\varepsilon})| = \left| \int_0^1 D_y \Phi_m(X_{s}^{m,\varepsilon}, Y_{s}^{m,\varepsilon} + \varepsilon^{-1/\alpha} \gamma_k z \varepsilon_k) \cdot (\varepsilon^{-1/\alpha} \gamma_k z \varepsilon_k) d\xi \right| \leq C_{\varepsilon^{-1/\alpha}} |z|.
$$

Then by condition $\sum_{k=1}^{\infty} \gamma_k^\alpha < \infty$ we obtain

$$
\Lambda_{4,m}^\varepsilon(T) \leq C_{p} \varepsilon^{-p/\alpha} \lim_{m \to \infty} \left[ \sum_{k=1}^{m} \gamma_k^\alpha \int_0^T \int_{|z| \leq 1} |z|^2 \nu(dz)ds \right]^{p/2} \\
+ C_{p} \varepsilon^{-p/\alpha} \lim_{m \to \infty} \left[ \sum_{k=1}^{m} \gamma_k^\alpha E \int_0^T \int_{|z| > 1} |z|^p \nu(dz)ds \right]^{p/2} \\
\leq C_{p,T} \varepsilon^{-p/\alpha}. \tag{4.12}
$$

**Step 3.** In this step, we estimate the term $\Lambda_5^\varepsilon_m(T)$. It is easy to see

$$
\Lambda_5^\varepsilon_m(T) \leq C_p \sup_{t \in [0,T]} E \left[ \int_0^t e^{(t-s)A} \left[ D_x \Phi_m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) \cdot AX_s^{m,\varepsilon} \right] ds \right]^p
$$
$$+ C_p \sup_{t \in [0,T]} \mathbb{E} \left[ \int_0^t e^{(t-s)A} \left[ D_x \Phi_m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) \cdot B^m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) \right] ds \right]^p$$

$$+ C_p \sup_{t \in [0,T]} \mathbb{E} \left[ \sum_{k=1}^m \beta_k^\alpha \int_0^t \int_\mathbb{R} e^{(t-s)A} \left[ \Phi_m(x + e_k z, y) - \Phi_m(x, y) \right] \cdot (-D_x \Phi_m(x, y) \cdot (e_k z)) \cdot \nu(dz) ds \right]^p := \sum_{i=1}^3 \Lambda_{5i}(T).$$

By (6.13), (3.16), (2.5), (6.4), (6.5) and Minkowski’s inequality, we get for any $\gamma \in (0, 1)$ and $\eta \in (0, 1)$,

$$\Lambda_{51}^{\varepsilon, m}(T) \leq C_{p, \delta} \sup_{t \in [0,T]} \mathbb{E} \left[ \int_0^t \|X_s^{m,\varepsilon}\|_2 (1 + |X_s^{m,\varepsilon}|^\delta + |Y_s^{m,\varepsilon}|^\delta) ds \right]^p$$

$$\leq C_{p, \delta} \sup_{t \in [0,T]} \left[ \mathbb{E} \left( \|X_s^{m,\varepsilon}\|_2^p (1 + |X_s^{m,\varepsilon}|^{\delta p} + |Y_s^{m,\varepsilon}|^{\delta p}) \right) \right]^{1/p} ds$$

$$\leq C_{p, \delta} \sup_{t \in [0,T]} \left[ \mathbb{E} \left( \|X_s^{m,\varepsilon}\|_2^{p/2} \right)^{1/p'} \left[ \mathbb{E}(1 + |X_s^{m,\varepsilon}|^{\delta p'} + |Y_s^{m,\varepsilon}|^{\delta p'}) \right] \right]^{1/p} ds$$

$$\leq C_{p, T, \delta}(1 + ||x||_\gamma^{(1+\delta)p} + |y|^{(1+\delta)p}) \sup_{t \in [0,T]} \left[ \int_0^t s^{-1+\gamma} ds + \varepsilon^{-\gamma} \right] \right]^{1/p} ds$$

where $p < p' < \alpha$ and $\delta$ is small enough such that $\frac{\delta p'}{p'} - p \leq 1$.

By a similar argument above, we have for $\delta$ is small enough such that $\delta < \frac{\alpha}{p} - 1$,

$$\Lambda_{52}^{\varepsilon, m}(T) \leq C_{p, \delta} \mathbb{E} \int_0^T \left| B^m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) \right|^p (1 + |X_s^{m,\varepsilon}|^{\delta p} + |Y_s^{m,\varepsilon}|^{\delta p}) ds$$

$$\leq C_{p, \delta} \mathbb{E} \int_0^T (1 + |X_s^{m,\varepsilon}|^p + |Y_s^{m,\varepsilon}|^p) (1 + |X_s^{m,\varepsilon}|^{\delta p} + |Y_s^{m,\varepsilon}|^{\delta p}) ds$$

$$\leq C_{p, T, \delta}(1 + ||x||_\gamma^{(1+\delta)p} + |y|^{(1+\delta)p}).$$

By (3.16) and (3.17), we have for $\delta$ is small enough such that $\delta < \alpha - 1$,

$$\Lambda_{53}^{\varepsilon, m}(T) \leq C_p \mathbb{E} \left[ \sum_{k=1}^m \beta_k^\alpha \int_0^T \int_{|z| \leq \tilde{c}_k} |\Phi_m(X_s^{m,\varepsilon} + e_k z, Y_s^{m,\varepsilon}) - \Phi_m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon})| \cdot (e_k z) \cdot |\nu(dz)| ds \right]^p$$

$$+ C_p \mathbb{E} \left[ \sum_{k=1}^m \beta_k^\alpha \int_0^T \int_{|z| > \tilde{c}_k} |\Phi_m(X_s^{m,\varepsilon} + e_k z, Y_s^{m,\varepsilon}) - \Phi_m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon})| \cdot (e_k z) \cdot |\nu(dz)| ds \right]^p$$

$$\leq C_p \mathbb{E} \left[ \sum_{k=1}^m \beta_k^\alpha \int_0^T \int_{|z| \leq \tilde{c}_k} |\Phi_m(X_s^{m,\varepsilon} + e_k z, Y_s^{m,\varepsilon}) - \Phi_m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon})| \cdot |\nu(dz)| ds \right]^p$$

$$+ C_p \mathbb{E} \left[ \sum_{k=1}^m \beta_k^\alpha \int_0^T \int_{|z| > \tilde{c}_k} |\Phi_m(X_s^{m,\varepsilon} + e_k z, Y_s^{m,\varepsilon}) - \Phi_m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon})| \cdot |\nu(dz)| ds \right]^p$$

$$\leq C_p \left( \sum_{k=1}^m \beta_k^\alpha \frac{\alpha}{\varepsilon^2} \tilde{c}_k^{2-\alpha} \right)^p \mathbb{E} \left[ \int_0^T (1 + |X_s^{m,\varepsilon}|^p + |Y_s^{m,\varepsilon}|^p) ds \right]^p$$
where \( \tilde{c}_k := \lambda_k^{-\frac{1}{(1-\delta)}} \). Then by \( \sum_{k \in \mathbb{N}_+} \frac{\alpha}{\lambda_k^m} < \infty \), it follows
\[
\Lambda_{53}^{\varepsilon, m}(T) \leq C_{p,T,\delta}(1 + |x| + |y|) \left( \sum_{k=1}^{\infty} \beta_k^\alpha \lambda_k^{\frac{\alpha(1-\delta)-1}{2(1-\delta)}} \right)^p
\]
\[
\leq C_{p,T,\delta}(1 + |x|^p + |y|^p) \left( \sum_{k=1}^{\infty} \beta_k^\alpha \lambda_k^{\frac{\alpha(1-\delta)-1}{2(1-\delta)}} \right)^p
\]
\[
\leq C_{p,T,\delta}(1 + |x|^p + |y|^p) \left( \sum_{k=1}^{\infty} \beta_k^\alpha \lambda_k^{\alpha-1} \right)^p
\]
\[
\leq C_{p,T,\delta}(1 + |x|^p + |y|^p).
\]
As a result, for small enough \( \delta > 0 \) we have
\[
\Lambda_{5}^{\varepsilon, m}(T) \leq C_{p,T,\delta}(1 + |x|^{(1+\delta)p} + |y|^{(1+\delta)p}).
\]
(Hence, (4.3) holds by combining (4.9)-(4.13). The proof is complete.

5. Weak convergence order

This section is devoted to prove Theorem 2.6. We still consider the problem in finite dimension firstly, then passing the limit to the infinite dimensional case.

For a test function \( \phi \in C^3_b(H) \), we have for any \( t \geq 0 \),
\[
\left| \mathbb{E} \phi \left( X_t^\varepsilon \right) - \mathbb{E} \phi \left( \tilde{X}_t \right) \right| \leq \left| \mathbb{E} \phi \left( X_t^\varepsilon \right) - \mathbb{E} \phi \left( X_{t}^{m,\varepsilon} \right) \right| + \left| \mathbb{E} \phi \left( X_{t}^{m,\varepsilon} \right) - \mathbb{E} \phi \left( \tilde{X}_t \right) \right|
\]
\[
+ \left| \mathbb{E} \phi \left( X_{t}^{m,\varepsilon} \right) - \mathbb{E} \phi \left( X_t^m \right) \right|.
\]
By Lemmas 6.2 and 6.6 in the appendix, it is easy to see that
\[
\lim_{m \to \infty} \sup_{t \in [0,T]} \left| \mathbb{E} \phi \left( X_t^\varepsilon \right) - \mathbb{E} \phi \left( X_t^{m,\varepsilon} \right) \right| = 0,
\]
\[
\lim_{m \to \infty} \sup_{t \in [0,T]} \left| \mathbb{E} \phi \left( X_{t}^{m,\varepsilon} \right) - \mathbb{E} \phi \left( \tilde{X}_t \right) \right| = 0.
\]
Then the proof will be completed if we can show that there exists a positive constant \( C \) independent of \( m \) such that
\[
\sup_{t \in [0,T]} \left| \mathbb{E} \phi \left( X_t^{m,\varepsilon} \right) - \mathbb{E} \phi \left( X_t^m \right) \right| \leq C\varepsilon.
\]
Now we are in a position to prove Theorem 2.6.

Proof of Theorem 2.6. We will divide the proof into three steps.

**Step 1.** We first introduce the following Kolmogorov equation in finite dimension:
\[
\begin{cases}
\partial_t u_m(t, x) = \mathcal{L}_1^{m} u_m(t, x), & t \in [0, T], \\
u(0, x) = \phi(x), & x \in H_m,
\end{cases}
\]
where \( \phi \in C^3_b(H) \) and \( \mathcal{L}_1^{m} \) is the infinitesimal generator of the transition semigroup of the averaged equation (3.2), which is given by
\[
\mathcal{L}_1^{m} \phi(x) := D_x \phi(x) \cdot [Ax + \tilde{B}^{m}(x)].
\]
\[ + \sum_{k=1}^{m} \beta_k \int_{\mathbb{R}} \left[ \phi(x + e_kz) - \phi(x) - \langle D_x \phi(x), e_k z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right] \nu(dz). \]

Note that
\[ \tilde{B}^m(x) := \int_{H^m} B^m(x, y) \mu^x, m(dy) = \lim_{t \to \infty} \mathbb{E} B^m(x, Y_t^x, y, m). \]

By assumptions A3, conditions (2.9) and (2.10), then through a straightforward computation, it is easy to check that
\[ |D \tilde{B}^m(x) \cdot h| \leq C|h| \quad \forall x, h \in H^m, \]
\[ |D^2 \tilde{B}^m(x) \cdot (h, k)| \leq C|h||k|_{\kappa_1}, \quad \forall x, h, k \in H^m, \]
\[ |D^3 \tilde{B}^m(x) \cdot (h, k, l)| \leq C|h||k||l|_{\kappa_1}, \quad \forall x, h, k, l \in H^m, \]
where \( \kappa_1 \) is the constant in assumption A3. As a consequence, (5.2) has a unique solution \( u_m \) given by
\[ u_m(t, x) = \mathbb{E} \phi(\bar{X}_t^m(x)), \quad t \in [0, T]. \]

Furthermore, for any \( h, k, l \in H^m \), we have
\[ \sup_{x \in H^m, m \geq 1} |D_x u_m(t, x) \cdot h| \leq C_T|h|, \quad t \in [0, T], \] (5.3)
\[ \sup_{x \in H^m, m \geq 1} |D_{xx} u_m(t, x) \cdot (h, k)| \leq C_T|h||k|, \quad t \in [0, T]. \] (5.4)
\[ \sup_{x \in H^m, m \geq 1} |D_{xxx} u_m(t, x) \cdot (h, k, l)| \leq C_T|h||k||l|, \quad t \in [0, T], \] (5.5)
\[ \sup_{m \geq 1} |\partial_t(D_x u_m(t, x)) \cdot h| \leq C_T t^{-1}(1 + |x|)|h|, \quad x \in H^m, t \in (0, T]. \] (5.6)

Indeed, note that for any \( h, k, l \in H^m \),
\[ D_x u_m(t, x) \cdot h = \mathbb{E}[D \phi(\bar{X}_t^m) \cdot \eta_t^{h, m}(x)], \]
\[ D_{xx} u_m(t, x) \cdot (h, k) = \mathbb{E} \left[ D^2 \phi(\bar{X}_t^m) \cdot (\eta_t^{h, m}(x), \eta_t^{k, m}(x)) \right] \]
\[ + \mathbb{E} \left[ D^3 \phi(\bar{X}_t^m) \cdot (\eta_t^{h, l, m}(x), \eta_t^{l, m}(x)) \right], \]
\[ D_{xxx} u_m(t, x) \cdot (h, k, l) = \mathbb{E} \left[ D^3 \phi(\bar{X}_t^m) \cdot (\eta_t^{h, m}(x), \eta_t^{k, m}(x), \eta_t^{l, m}(x)) \right] \]
\[ + \mathbb{E} \left[ D^2 \phi(\bar{X}_t^m) \cdot (\zeta_t^{h, k, m}(x), \eta_t^{l, m}(x)) \right] + \mathbb{E} \left[ D^2 \phi(\bar{X}_t^m) \cdot (\zeta_t^{h, k, m}(x), \eta_t^{h, m}(x)) \right], \]

where \( \eta_t^{h, m}(x) := D_x \bar{X}_t^m(x) \cdot h \) satisfies
\[ \left\{ \begin{array}{l}
    d\eta_t^{h, m}(x) = A \eta_t^{h, m}(x) dt + \tilde{B}^m(\bar{X}_t^m) \cdot \eta_t^{h, m}(x) dt, \\
    \eta_0^{h, m}(x) = h,
\end{array} \right. \]
\( \zeta_t^{h, k, m}(x) := D_{xx} \bar{X}_t^m(x) \cdot (h, k) \) satisfies
\[ \left\{ \begin{array}{l}
    d\zeta_t^{h, k, m}(x) = [A_{(t)}^{h, k, m}(x) + \tilde{B}^m(\bar{X}_t^m) \cdot \zeta_t^{h, k, m}(x) + \tilde{B}^m(\bar{X}_t^m) \cdot (\eta_t^{h, m}(x), \eta_t^{k, m}(x)))] dt, \\
    \zeta_0^{h, k, m}(x) = 0
\end{array} \right. \]
and \( \chi^{h,k,l,m}_t(x) := D_{xxx} \bar{X}^m_t(x) \cdot (h, k, l) \) satisfies

\[
\begin{aligned}
&d\chi^{h,k,l,m}_t(x) = \\
&\begin{cases}
-A \chi^{h,k,l,m}_t(x) + D \bar{B}^m(\bar{X}^m_t) \cdot \chi^{h,k,l,m}_t(x) + D^2 \bar{B}^m(\bar{X}^m_t) \cdot (\chi^{h,k,l,m}_t(x), \eta^{l,m}_t(x)) \\
+ D^2 \bar{B}^m(\bar{X}^m_t) \cdot (\eta^{h,m}_t(x), \eta^{k,m}_t(x)) + D^2 \bar{B}^m(\bar{X}^m_t) \cdot (\eta^{l,m}_t(x), \zeta^{l,m}_t(x)) \\
+ D^3 \bar{B}^m(\bar{X}^m_t) \cdot (\eta^{h,m}_t(x), \eta^{k,m}_t(x), \eta^{l,m}_t(x)) 
\end{cases} dt,
\end{aligned}
\]

\( \chi^{h,k,l,m}_0(x) = 0. \)

By a straightforward computation, we obtain for any \( t \in (0, T) \) and \( \theta \in [0, 2) \),

\[
\| \eta^{h,m}_t(x) \|_\theta \leq C_T t^{-\theta/2} |h|, \quad \| \eta^{h,m}_t(x) \|_2 \leq C_T t^{-1} (1 + |x|) |h|, \quad (5.7)
\]

\[
|\zeta^{h,k,m}_t(x)| \leq C_T |h||k|, \quad (5.8)
\]

\[
|\chi^{h,k,l,m}_t(x)| \leq C_T |h||k||l|, \quad (5.9)
\]

Hence, it is easy to see (5.7)-(5.9) imply (5.3)-(5.5) hold.

By Itô’s formula and taking expectation, we have

\[
\mathbb{E}[D\phi(\bar{X}^m_t) \cdot \eta^{h,m}_t(x)] = D\phi(x) \cdot h + \int_0^t \mathbb{E} \left[ D^2 \phi(\bar{X}^m_s) \cdot (\eta^{h,m}_s(x), A \bar{X}^m_s + \bar{B}^m(\bar{X}^m_s)) \right] ds + \\
+ \int_0^t \mathbb{E}(\delta \phi(\bar{X}^m_s), A \eta^{h,m}_s(x) + D \bar{B}^m(\bar{X}^m_s) \cdot \eta^{h,m}_s(x)) ds + \\
+ \sum_{k=1}^m \beta^\alpha_k \mathbb{E} \left[ \int_0^t \int_\mathbb{R} (\phi(\bar{X}^m_s + e_k z) - \phi(\bar{X}^m_s) - (D^2 \phi(\bar{X}^m_s) \cdot e_k z) 1_{|z| \leq 1}) \eta^{h,m}_s(x) \nu(dz) ds \right],
\]

which implies

\[
\partial_t (D_x u_m(t, x)) \cdot h = \mathbb{E} \left[ D^2 \phi(\bar{X}^m_t) \cdot (\eta^{h,m}_t(x), A \bar{X}^m_t + \bar{B}^m(\bar{X}^m_t)) \right] + \\
+ \mathbb{E}(\delta \phi(\bar{X}^m_t), A \eta^{h,m}_t(x) + D \bar{B}^m(\bar{X}^m_t) \cdot \eta^{h,m}_t(x)) + \\
+ \sum_{k=1}^m \beta^\alpha_k \mathbb{E} \left[ \int_\mathbb{R} (\phi(\bar{X}^m_t + e_k z) - \phi(\bar{X}^m_t) - (D^2 \phi(\bar{X}^m_t) \cdot e_k z) 1_{|z| \leq 1}) \eta^{h,m}_t(x) \nu(dz) ds \right].
\]

By (4.2), (5.7) and (6.11), we get for any \( t \in (0, T) \),

\[
|\partial_t (D_x u_m(t, x)) \cdot h| \leq C \mathbb{E} \left[ \| \eta^{h,m}_t(x) \|_2 + |\eta^{h,m}_t(x)|(|\bar{X}^m_t|_2 + |\bar{X}^m_t| + 1) \right] + \\
+ C \sum_{k=1}^m \beta^\alpha_k \mathbb{E} |\eta^{h,m}_t(x)| \left[ \int_{|z| \leq 1} |z|^2 \nu(dz) + \int_{|z| > 1} |z| \nu(dz) \right] \leq C_T t^{-1} |h|(1 + |x|),
\]

which proves (5.6).

**Step 2.** Let \( \tilde{u}^t_m(s, x) := u_m(t - s, x), s \in [0, t] \), by Itô’s formula, we have

\[
\tilde{u}^t_m(t, X^m(s, x)) = \tilde{u}^t_m(0, x) + \int_0^t \partial_s \tilde{u}^t_m(s, X^m(s, x)) ds + \int_0^t \mathcal{Z}^m_s(Y^m(s, x)) \tilde{u}^t_m(s, X^m(s, x)) ds + \tilde{M}^t_m,
\]

where

\[
\mathcal{Z}^m_s(Y^m(s, x)) = \mathcal{Z}^m_s(Y^m(s, x)) \cdot h, (5.10)
\]
where $\tilde{M}_t^m$ is defined as follows,

$$\tilde{M}_t^m := \sum_{k=1}^{m} \beta_k^t \int_0^t \int_{\mathbb{R}} \tilde{u}^t_m(s, X_{s-}^{m,\varepsilon} + z\varepsilon_k) - \tilde{u}^t_m(s, X_{s-}^{m,\varepsilon})\tilde{N}^1(k)(dz, ds).$$

Note that

$$\tilde{u}^t_m(t, X_t^{m,\varepsilon}) = \phi(X_t^{m,\varepsilon}), \tilde{u}^t_m(0, x) = \mathbb{E}\phi(X_t^{m,\varepsilon}), \partial_s \tilde{u}^t_m(s, X_s^{m,\varepsilon}) = -\mathcal{L}_{1}^{m} \tilde{u}^t_m(s, X_s^{m,\varepsilon}).$$

It follows for any $t \in [0, T]$,

$$\left|\mathbb{E}\phi(X_t^{m,\varepsilon}) - \mathbb{E}\phi(\tilde{X}_t^m)\right| = \left|\mathbb{E} \int_0^t -\mathcal{L}_{1}^{m} \tilde{u}^t_m(s, X_s^{m,\varepsilon})ds + \mathbb{E} \int_0^t \mathcal{L}_{1}^{m}(Y_s^{m,\varepsilon}) \tilde{u}^t_m(s, X_s^{m,\varepsilon})ds\right|$$

$$= \left|\mathbb{E} \int_0^t D_x \tilde{u}^t_m(s, X_s^{m,\varepsilon}) \cdot [B^m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) - \tilde{B}^m(X_s^{m,\varepsilon})]ds\right|$$

Define $\rho(\varepsilon) := \varepsilon^{1-r}$ for $r \in (0, 1)$. If $t \leq 2\rho(\varepsilon)$, then by (5.3) and the boundedness of $B$,

$$\left|\mathbb{E}\phi(X_t^{m,\varepsilon}) - \mathbb{E}\phi(\tilde{X}_t^m)\right| \leq C_T \rho(\varepsilon). \quad (5.10)$$

If $t > 2\rho(\varepsilon)$,

$$\left|\mathbb{E}\phi(X_t^{m,\varepsilon}) - \mathbb{E}\phi(\tilde{X}_t^m)\right| \leq \left|\mathbb{E} \int_0^{t-\rho(\varepsilon)} D_x \tilde{u}^t_m(s, X_s^{m,\varepsilon}) \cdot [B^m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) - \tilde{B}^m(X_s^{m,\varepsilon})]ds\right|$$

$$+ \left|\mathbb{E} \int_{t-\rho(\varepsilon)}^t D_x \tilde{u}^t_m(s, X_s^{m,\varepsilon}) \cdot [B^m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) - \tilde{B}^m(X_s^{m,\varepsilon})]ds\right|$$

$$+ \left|\mathbb{E} \int_0^{\rho(\varepsilon)} D_x \tilde{u}^t_m(s, X_s^{m,\varepsilon}) \cdot [B^m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) - \tilde{B}^m(X_s^{m,\varepsilon})]ds\right|$$

$$\leq \left|\mathbb{E} \int_0^{t-\rho(\varepsilon)} D_x \tilde{u}^t_m(s, X_s^{m,\varepsilon}) \cdot [B^m(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) - \tilde{B}^m(X_s^{m,\varepsilon})]ds\right|$$

$$+ C_T \rho(\varepsilon). \quad (5.11)$$

For any $s \in [0, t], x, y \in H_m$, define

$$G^t_m(s, x, y) := D_x \tilde{u}^t_m(s, x) \cdot B^m(x, y)$$

$$\bar{G}^t_m(s, x) := \int_{H_m} G^t_m(s, x, y)\mu^{x,m}(dy) = D_x \tilde{u}^t_m(s, x) \cdot \bar{B}^m(x).$$

By an argument similar to that used in the proof of Proposition 3.4, we construct

$$\bar{\Phi}^t_m(s, x, y) := \int_0^\infty \mathbb{E}G^t_m(s, x, Y^x_{r,y,m}) - \bar{G}^t_m(s, x)dr, \quad s \in [0, t], x, y \in H_m,$$

which is a solution of the following Poisson equation:

$$-\mathcal{L}_{2}^{m}(x)\bar{\Phi}^t_m(s, x, y) = G^t_m(s, x, y) - \bar{G}^t_m(s, x). \quad (5.12)$$

Moreover, for any $T > 0$, $t \in [0, T]$, $\delta \in (0, 1]$, there exists $C_T, C_T, \delta > 0$ such that the following estimates hold:

$$\sup_{m \geq 1} |\partial_s \bar{\Phi}^t_m(s, x, y)| \leq C_{T, \delta}(t - s)^{-1}(1 + |x|)(1 + |x|^{\delta} + |y|^{\delta}), \quad s \in (0, t]; \quad (5.13)$$

$$\sup_{s \in [0, t], x \in H_m} |\bar{\Phi}^t_m(s, x, y)| \leq C_T (1 + |x| + |y|); \quad (5.14)$$

$$\sup_{s \in [0, t], x \in H_m} |D_x \bar{\Phi}^t_m(s, x, y) \cdot h| \leq C_{T, \delta} (1 + |x|^{\delta} + |y|^{\delta})|h|; \quad (5.15)$$
\begin{align}
\sup_{s \in [0, t], x \in H^{m, m \geq 1}} |D_{x_{x}} \tilde{\Phi}_{m}^{t}(s, x, y) \cdot (h, k)| & \leq C_{T} (1 + |x| + |y|) |h||k|_{\kappa_{1}}. 
\end{align}

We here only give the proof of (5.13), and the proofs of (5.14)-(5.16) are omitted since it follows almost the same argument in Proposition 3.4.

In fact, by (3.10), (5.6) and the boundedness of $B$, we have for small enough $\delta > 0$,

\begin{align}
|\partial_{s} \tilde{\Phi}_{m}^{t}(s, x, y)| & \leq \int_{0}^{\infty} \partial_{s} D_{x_{x}} \tilde{\Phi}_{m}^{t}(s, x) \cdot \left[ \mathbb{E} B^{m}(x, Y_{r}^{x, y, m}) - \tilde{B}^{m}(x) \right] ds \\
& \leq C_{T} (t - s)^{-1} (1 + |x|) \int_{0}^{\infty} |\mathbb{E} B^{m}(x, Y_{r}^{x, y, m}) - \tilde{B}^{m}(x)| ds \\
& \leq C_{T} (t - s)^{-1} (1 + |x|) \int_{0}^{\infty} |\mathbb{E} B^{m}(x, Y_{r}^{x, y, m}) - \tilde{B}^{m}(x)| ds \\
& \leq C_{T} (t - s)^{-1} (1 + |x|) (1 + |x|^{\delta} + |y|^{\delta} \int_{0}^{\infty} e^{-\frac{(t-s)^{\delta}}{2}} ds \\
& \leq C_{T} (t - s)^{-1} (1 + |x|) (1 + |x|^{\delta} + |y|^{\delta}).
\end{align}

**Step 3.** Applying Itô’s formula and taking expectation, we get for any $t \in [2 \rho(\varepsilon), T]$,

\begin{align}
\mathbb{E} \tilde{\Phi}_{m}^{t}(t - \rho(\varepsilon), X_{t-\rho(\varepsilon)}^{m, \varepsilon}, Y_{t-\rho(\varepsilon)}^{m, \varepsilon}) &= \mathbb{E} \tilde{\Phi}_{m}^{t}(\rho(\varepsilon), X_{\rho(\varepsilon)}^{m, \varepsilon}, Y_{\rho(\varepsilon)}^{m, \varepsilon}) + \mathbb{E} \int_{\rho(\varepsilon)}^{t-\rho(\varepsilon)} \partial_{s} \tilde{\Phi}_{m}^{t}(s, X_{s}^{m, \varepsilon}, Y_{s}^{m, \varepsilon}) ds \\
&+ \mathbb{E} \int_{\rho(\varepsilon)}^{t-\rho(\varepsilon)} \mathcal{L}_{1}^{m}(Y_{s}^{m, \varepsilon}) \tilde{\Phi}_{m}^{t}(s, X_{s}^{m, \varepsilon}, Y_{s}^{m, \varepsilon}) ds + \frac{1}{\varepsilon} \mathbb{E} \int_{\rho(\varepsilon)}^{t-\rho(\varepsilon)} \mathcal{L}_{2}^{m}(X_{s}^{m, \varepsilon}) \tilde{\Phi}_{m}^{t}(s, X_{s}^{m, \varepsilon}, Y_{s}^{m, \varepsilon}) ds,
\end{align}

which implies

\begin{align}
- \mathbb{E} \int_{\rho(\varepsilon)}^{t-\rho(\varepsilon)} \mathcal{L}_{2}^{m}(X_{s}^{m, \varepsilon}) \tilde{\Phi}_{m}^{t}(s, X_{s}^{m, \varepsilon}, Y_{s}^{m, \varepsilon}) ds \\
= \varepsilon \left[ \mathbb{E} \tilde{\Phi}_{m}^{t}(t - \rho(\varepsilon), X_{t-\rho(\varepsilon)}^{m, \varepsilon}, Y_{t-\rho(\varepsilon)}^{m, \varepsilon}) - \mathbb{E} \tilde{\Phi}_{m}^{t}(\rho(\varepsilon), X_{\rho(\varepsilon)}^{m, \varepsilon}, Y_{\rho(\varepsilon)}^{m, \varepsilon}) + \mathbb{E} \int_{\rho(\varepsilon)}^{t-\rho(\varepsilon)} \partial_{s} \tilde{\Phi}_{m}^{t}(s, X_{s}^{m, \varepsilon}, Y_{s}^{m, \varepsilon}) ds \\
+ \mathbb{E} \int_{\rho(\varepsilon)}^{t-\rho(\varepsilon)} \mathcal{L}_{1}^{m}(Y_{s}^{m, \varepsilon}) \tilde{\Phi}_{m}^{t}(s, X_{s}^{m, \varepsilon}, Y_{s}^{m, \varepsilon}) ds \right].
\end{align}

Combining (5.11), (5.12) and (5.17), we get for any $t \in [2 \rho(\varepsilon), T]$,

\begin{align}
\left| \mathbb{E} \phi(X_{t}^{m, \varepsilon}) - \mathbb{E} \phi(\tilde{X}_{t}^{m}) \right| & \leq C_{T} \rho(\varepsilon) + \mathbb{E} \int_{\rho(\varepsilon)}^{t-\rho(\varepsilon)} \mathcal{L}_{2}^{m}(X_{s}^{m, \varepsilon}) \tilde{\Phi}_{m}^{t}(s, X_{s}^{m, \varepsilon}, Y_{s}^{m, \varepsilon}) ds \\
& \leq C_{T} \rho(\varepsilon) + \varepsilon \left[ \left| \mathbb{E} \tilde{\Phi}_{m}^{t}(\rho(\varepsilon), X_{\rho(\varepsilon)}^{m, \varepsilon}, Y_{\rho(\varepsilon)}^{m, \varepsilon}) \right| + \left| \mathbb{E} \tilde{\Phi}_{m}^{t}(t - \rho(\varepsilon), X_{t-\rho(\varepsilon)}^{m, \varepsilon}, Y_{t-\rho(\varepsilon)}^{m, \varepsilon}) \right| \\
+ \mathbb{E} \int_{\rho(\varepsilon)}^{t-\rho(\varepsilon)} \partial_{s} \tilde{\Phi}_{m}^{t}(s, X_{s}^{m, \varepsilon}, Y_{s}^{m, \varepsilon}) ds \right] + \mathbb{E} \int_{\rho(\varepsilon)}^{t-\rho(\varepsilon)} \mathcal{L}_{2}^{m}(X_{s}^{m, \varepsilon}) \tilde{\Phi}_{m}^{t}(s, X_{s}^{m, \varepsilon}, Y_{s}^{m, \varepsilon}) ds \right] ds \\
:= C_{T} \rho(\varepsilon) + \varepsilon \sum_{k=1}^{4} \tilde{A}^{m, \varepsilon}(t).
\end{align}

For the terms $\tilde{A}^{m, \varepsilon}(t)$ and $\tilde{A}^{m, \varepsilon}(t)$. By estimates (5.14) and (6.5), it is easy to get

\begin{align}
\tilde{A}^{m, \varepsilon}(t) + \tilde{A}^{m, \varepsilon}(t) & \leq C_{T} \sup_{t \in [0, T]} \mathbb{E}(1 + |X_{t}^{m, \varepsilon}| + |Y_{t}^{m, \varepsilon}|) \leq C_{T} (1 + |x| + |y|).
\end{align}
For the terms $\tilde{\Lambda}_3^{m,\varepsilon}(t)$. By (5.13), (6.4) and (6.5), we have
\[
\tilde{\Lambda}_3^{m,\varepsilon}(t) \leq C_{T,\delta} \int_0^{t-\rho(\varepsilon)} (t-s)^{-1} \mathbb{E} \left[ (1 + |X_s^{m,\varepsilon}|)(1 + |X_s^{m,\varepsilon}|^\delta + |Y_s^{m,\varepsilon}|) \right] ds \\
\leq C_{T,\delta} \ln \left( \frac{T}{\rho(\varepsilon)} \right) (1 + |x|^{1+\delta} + |y|^{1+\delta}). \tag{5.19}
\]

For the terms $\tilde{\Lambda}_4^{m,\varepsilon}(t)$. It is easy to see
\[
\tilde{\Lambda}_4^{m,\varepsilon}(t) \leq \int_{\rho(\varepsilon)}^t \mathbb{E} \left| D_x \tilde{\Phi}_m^t (X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) \cdot AX_s^{m,\varepsilon} \right| ds \\
+ \int_{\rho(\varepsilon)}^t \mathbb{E} \left| D_x \tilde{\Phi}_m^t (X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) \cdot B_m (X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) \right| ds \\
+ \int_0^t \mathbb{E} \sum_{k=1}^m \beta_k^x \int_{\mathbb{R}} \left[ \tilde{\Phi}_m^t (X_s^{m,\varepsilon} + e_k z, Y_s^{m,\varepsilon}) - \tilde{\Phi}_m^t (X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) \right] \\
-D_x \tilde{\Phi}_m^t (X_s^{m,\varepsilon}, Y_s^{m,\varepsilon}) \cdot (e_k z) \mathbb{1}_{|z| \leq 1} \nu(dz) \right| ds := \sum_{i=1}^3 \tilde{\Lambda}_{4i}^{m,\varepsilon}(t). \tag{5.20}
\]

By (5.15) and (6.13), it is easy to see for any $r \in (0, 1)$
\[
\tilde{\Lambda}_{41}^{m,\varepsilon}(t) \leq C_{T,\delta} \mathbb{E} \int_{\rho(\varepsilon)}^T \|X_s^{m,\varepsilon}\|_2 (1 + |X_s^{m,\varepsilon}|^\delta + |Y_s^{m,\varepsilon}|) ds \\
\leq C_{T,\delta} \int_{\rho(\varepsilon)}^T \mathbb{E} \left[ \|X_s^{m,\varepsilon}\|^{p_0} + |Y_s^{m,\varepsilon}|^{\frac{p_0}{p}} \right] \left( 1 + |X_s^{m,\varepsilon}|^{\frac{p_0}{p}} + |Y_s^{m,\varepsilon}|^{\frac{p_0}{p}} \right) \frac{e^{-r}}{p} ds \\
\leq C_T \int_{\rho(\varepsilon)}^T (s^{-1} + \varepsilon^{-r})(1 + |x|^{1+\delta} + |y|^{1+\delta}) ds \\
\leq C_T \left[ \ln \left( \frac{T}{\rho(\varepsilon)} \right) + \varepsilon^{-r} \right] (1 + |x|^{1+\delta} + |y|^{1+\delta}). \tag{5.21}
\]

By (5.15), (5.16), (6.4) and (6.5), we have
\[
\tilde{\Lambda}_{42}^{m,\varepsilon}(t) \leq C_T \mathbb{E} \int_0^T (1 + |X_s^{m,\varepsilon}| + |Y_s^{m,\varepsilon}|)(1 + |Y_s^{m,\varepsilon}|^\delta) ds \\
\leq C_T (1 + |x|^{1+\delta} + |y|^{1+\delta}). \tag{5.22}
\]

By following the same argument in the estimating $\Lambda_{33}^{m,\varepsilon}(T)$, we have for any $\delta < \alpha - 1$,
\[
\tilde{\Lambda}_{43}^{m,\varepsilon}(t) \leq C_{\delta, T} (1 + |x| + |y|). \tag{5.23}
\]

Finally, combining estimates (5.10), (5.11), (5.18)-(5.23), we final obtain
\[
\sup_{t \in [0,T], m \geq 1} \left| \mathbb{E} \phi(X_t^{m,\varepsilon}) - \mathbb{E} \phi(X_t^{\tilde{m}}) \right| \\
\leq C_T \varepsilon^{1-r} (1 + |x|^{1+\delta} + |y|^{1+\delta}) + C_T \varepsilon \ln \left( \frac{T}{\varepsilon^{1-r}} \right) (1 + |x|^{1+\delta} + |y|^{1+\delta}) \\
\leq C_{T,\varepsilon}^{1-r} (1 + |x|^{1+\delta} + |y|^{1+\delta}).
\]

The proof is complete.
6. Appendix

In this section, we give some a priori estimates of the solution \((X_t^\varepsilon, Y_t^\varepsilon)\) (see Lemma 6.1), which is used to study the Galerkin approximation of the system (1.1) (see Lemma 6.2). Then we study the increment of the time of solution \((X_t^\varepsilon, Y_t^\varepsilon)\) (see Lemma 6.3). Finally, the finite dimensional approximation of the frozen equation (2.6) is given (see Lemma 6.6).

**Lemma 6.1.** For any \(x, y \in H, 1 \leq p < \alpha\) and \(T > 0\), there exist constants \(C_{p,T}, C_T > 0\) such that the solution \((X_t^\varepsilon, Y_t^\varepsilon)\) of system (1.1) satisfies

\[
\sup_{t \in [0,T]} \mathbb{E}|X_t^\varepsilon|^p \leq C_{p,T}(1 + |x|^p + |y|^p), \quad \forall \varepsilon > 0, \tag{6.1}
\]

\[
\sup_{t \in [0,T]} \mathbb{E}|Y_t^\varepsilon|^p \leq C_p(1 + |x|^p + |y|^p), \quad \forall \varepsilon > 0. \tag{6.2}
\]

**Proof.** Define \(\tilde{Z}_t := \frac{1}{\varepsilon^{1/\alpha}} Z_{t\varepsilon}\), which is also a cylindrical \(\alpha\)-stable process. Then by [33, (4.12)], for any \(p \in (1, \alpha)\),

\[
\mathbb{E} \left| e^{(t-s)A/\varepsilon} Z_s \right|^p = \mathbb{E} \left( \int_0^{t/\varepsilon} e^{(t/\varepsilon-s)A} dZ_s \right)^p \leq C \left( \sum_{k=1}^{\infty} \frac{1 - e^{-\alpha \lambda_k t/\varepsilon}}{\alpha \lambda_k} \right)^{p/\alpha} \leq C \left( \sum_{k=1}^{\infty} \frac{\gamma_k^\alpha}{\alpha \lambda_k} \right)^{p/\alpha} < \infty.
\]

Then by Minkowski’s inequality, we get for any \(p \in (1, \alpha)\) and \(0 < t \leq T\),

\[
(\mathbb{E}|Y_t^\varepsilon|^p)^{1/p} \leq |e^{tA/\varepsilon} y| + \left[ \mathbb{E} \left( \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon} F(X_s^\varepsilon, Y_s^\varepsilon) ds \right)^p \right]^{1/p}.
\]

Thus by \(L_F < \lambda_1\), it follows

\[
(\mathbb{E}|X_t^\varepsilon|^p)^{1/p} \leq (\mathbb{E}|Y_t^\varepsilon|^p)^{1/p} + C, \quad \forall \varepsilon > 0
\]

Therefore, for any \(1 < p < \alpha\), by (6.3) and (2.5) in Remark 2.2, we obtain that

\[
\sup_{t \in [0,T]} \mathbb{E}|X_t^\varepsilon|^p \leq C_p |x|^p + C_p \int_0^T \mathbb{E}|X_s^\varepsilon|^p ds + C_p \int_0^T \mathbb{E}|Y_s^\varepsilon|^p ds + C_p \sup_{t \in [0,T]} \mathbb{E} \left| \int_0^t e^{(t-s)A} dL_s \right|^p \leq C_p (1 + |x|^p + |y|^p) + C_p \int_0^T \mathbb{E}|X_s^\varepsilon|^p ds.
\]
Then by Gronwall’s inequality, we have
\[
\sup_{t \in [0, T]} \mathbb{E}|X_t^\varepsilon|^p \leq C_{p, T}(1 + |x|^p + |y|^p),
\]
which also implies that (6.2) holds easily. The proof is complete. \qed

Recall the Galerkin approximation (3.1) of system (1.1). We have the following approximation.

**Lemma 6.2.** For any \( \varepsilon > 0 \), \((x, y) \in H \times H \) and \( m \in \mathbb{N}_+ \), system (3.1) has a unique mild solution \((X_t^{m, \varepsilon}, Y_t^{m, \varepsilon}) \in H \times H \), i.e., \( \mathbb{P} \)-a.s.,
\[
\begin{align*}
X_t^{m, \varepsilon} &= e^{tA}x^m + \int_0^t e^{(t-s)A}B^m(X_s^{m, \varepsilon}, Y_s^{m, \varepsilon})ds + \int_0^t e^{(t-s)A}d\bar{L}_s^m, \\
Y_t^{m, \varepsilon} &= e^{tA/\varepsilon}y^m + \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon}F^m(X_s^{m, \varepsilon}, Y_s^{m, \varepsilon})ds + \frac{1}{\varepsilon^{1/2}} \int_0^t e^{(t-s)A/\varepsilon}d\bar{Z}_s^m. 
\end{align*}
\]
Moreover, for any \( 1 \leq p < \alpha \) and \( T > 0 \), there exist constants \( C_{p, T}, C_T > 0 \) such that for any \( \varepsilon > 0 \),
\[
\begin{align*}
\sup_{m \geq 1, t \in [0, T]} \mathbb{E}|X_t^{m, \varepsilon}|^p &\leq C_{p, T}(1 + |x|^p + |y|^p); \\
\mathbb{E} |Y_t^{m, \varepsilon}|^p &\leq C_{p, T}(1 + |x|^p + |y|^p); \\
\lim_{m \to \infty} \sup_{t \in [0, T]} \mathbb{E}|X_t^{m, \varepsilon} - X_t^\varepsilon|^p = 0, \quad \forall \varepsilon > 0.
\end{align*}
\]

**Proof.** Under the assumptions **A1** and **A2**, it is easy to show that the existence and uniqueness of the mild solution of system (3.1). The estimates (6.4) and (6.5) can be proved by following the same argument as in the proof of Lemma 6.1. Next, we prove the approximation (6.6). It is easy to see that for any \( t > 0 \),
\[
X_t^{m, \varepsilon} - X_t^\varepsilon = e^{tA}(x^m - x) + \int_0^t e^{(t-s)A}(\pi_m - I)B(X_s^\varepsilon, Y_s^\varepsilon)ds
\]
\[
+ \int_0^t e^{(t-s)A}[B^m(X_s^{m, \varepsilon}, Y_s^{m, \varepsilon}) - B^m(X_s^\varepsilon, Y_s^\varepsilon)]ds + \left[ \int_0^t e^{(t-s)A}d\bar{L}_s^m - \int_0^t e^{(t-s)A}dL_s \right].
\]
Then for any \( T > 0, p \in [1, \alpha) \), we get
\[
\sup_{t \in [0, T]} \mathbb{E}|X_t^{m, \varepsilon} - X_t^\varepsilon|^p \leq C_p |x^m - x|^p + C_p \int_0^T \mathbb{E}|(\pi_m - I)B(X_s^\varepsilon, Y_s^\varepsilon)|^p ds
\]
\[
+ C_p \mathbb{E} \left[ \int_0^T |B^m(X_s^{m, \varepsilon}, Y_s^{m, \varepsilon}) - B^m(X_s^\varepsilon, Y_s^\varepsilon)| ds \right]^p
\]
\[
+ C_p \mathbb{E} \left( \int_0^T e^{(t-s)A}d\bar{L}_s^m - \int_0^t e^{(t-s)A}dL_s \right)^p
\]
\[
\leq C_p |x^m - x|^p + C_p \int_0^T \mathbb{E}|(\pi_m - I)B(X_s^\varepsilon, Y_s^\varepsilon)|^p ds
\]
\[
+ C_p \mathbb{E} \left( \int_0^T |X_s^{m, \varepsilon} - X_s^\varepsilon| + |Y_s^{m, \varepsilon} - Y_s^\varepsilon| ds \right)^p
\]
\[
+ C_p \mathbb{E} \left( \int_0^T e^{(t-s)A}d\bar{L}_s^m - \int_0^t e^{(t-s)A}dL_s \right)^p.
\]
On the other hand,

\[
Y^{m,\varepsilon}_t - Y^\varepsilon_t = e^{tA/\varepsilon}(y^m - y) + \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon}(\pi_m - I)F(X^\varepsilon_{s}, Y^\varepsilon_s)ds \\
+ \frac{1}{\varepsilon} \int_0^t e^{(t-s)A/\varepsilon}[F^m(X^{m,\varepsilon}_{s}, Y^{m,\varepsilon}_s) - F^m(X^\varepsilon_{s}, Y^\varepsilon_s)]ds \\
+ \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}d\bar{Z}^m_s - \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}dZ_s.
\]

It is clear that for any \( T > 0 \),

\[
\int_0^T |Y^{m,\varepsilon}_t - Y^\varepsilon_t|dt \leq C|y^m - y| + \frac{1}{\varepsilon} \int_0^T \int_0^t e^{\frac{\lambda_1(t-s)}{\varepsilon}}|\pi_m - I|F(X^\varepsilon_{s}, Y^\varepsilon_s)|dsdt \\
+ \frac{1}{\varepsilon} \int_0^T \int_0^t e^{\frac{\lambda_1(t-s)}{\varepsilon}}|F^m(X^{m,\varepsilon}_{s}, Y^{m,\varepsilon}_s) - F^m(X^\varepsilon_{s}, Y^\varepsilon_s)|dsdt \\
+ \int_0^T \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}d\bar{Z}^m_s - \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}dZ_s |dt \\
\leq CT|y^m - y| + \frac{1}{\varepsilon} \int_0^T |\pi_m - I|F(X^\varepsilon_{s}, Y^\varepsilon_s)|ds \\
+ \frac{1}{\lambda_1} \int_0^T \int_0^t (C|X^{m,\varepsilon}_{s} - X^\varepsilon_s| + LF|Y^{m,\varepsilon}_s - Y^\varepsilon_s|) e^{\frac{-\lambda(t-s)}{\varepsilon}}dsdt \\
+ \int_0^T \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}d\bar{Z}^m_s - \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}dZ_s |dt.
\]

By the condition \( \lambda_1 > L_F \) in assumption \textbf{A2}, it follows

\[
\int_0^T |Y^{m,\varepsilon}_t - Y^\varepsilon_t|dt \leq C|y^m - y| + \frac{C}{\lambda_1} \int_0^T |\pi_m - I|F(X^\varepsilon_{s}, Y^\varepsilon_s)|ds + \frac{C}{\lambda_1} \int_0^T |X^{m,\varepsilon}_{s} - X^\varepsilon_s|ds \\
+ C \int_0^T \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}d\bar{Z}^m_s - \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}dZ_s |dt. \tag{6.8}
\]

Then by (6.7) and (6.8), we obtain

\[
\sup_{t \in [0, T]} \mathbb{E}|X^{m,\varepsilon}_t - X^\varepsilon_t|^p \leq C_p|x^m - x|^p + C_p|y^m - y|^p + C_{p,T}\mathbb{E} \int_0^T |X^{m,\varepsilon}_{s} - X^\varepsilon_s|^pds \\
+ C_{p,T}\mathbb{E} \int_0^T |\pi_m - I|B(X^\varepsilon_{s}, Y^\varepsilon_s)|^pds + C_{p,T}\mathbb{E} \int_0^T |\pi_m - I|F(X^\varepsilon_{s}, Y^\varepsilon_s)|^pds \\
+ C_{p,T} \int_0^T \mathbb{E} \left| \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}d\bar{Z}^m_s - \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}dZ_s \right|^p |dt \\
+ C_p \sup_{t \in [0, T]} \mathbb{E} \left| \int_0^t e^{(t-s)A}d\bar{Z}^m_s - \int_0^t e^{(t-s)A}dL_s \right|^p.
\]
The Gronwall’s inequality implies
\[
\sup_{t \in [0,T]} \mathbb{E}|X_t^{m,\varepsilon} - X_t^\varepsilon|^p \\
\leq C_{p,T} \left( |x^m - x|^p + |y^m - y|^p \right) + C_{p,T} \int_0^T \mathbb{E}|(\pi_m - I)B(X_s^\varepsilon, Y_s^\varepsilon)|^p ds \\
+ C_{p,T} \int_0^T \mathbb{E}|(\pi_m - I)F(X_s^\varepsilon, Y_s^\varepsilon)|^p ds + C_{p,T} \int_0^T \mathbb{E} \left( \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A^{\varepsilon}} d\bar{Z}_s^{m} - \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A^{\varepsilon}} dZ_s \right)^p dt \\
+ C_{p,T} \sup_{t \in [0,T]} \mathbb{E} \left| \int_0^t e^{(t-s)A^{\varepsilon}} d\bar{L}_s^{m} - \int_0^t e^{(t-s)A^{\varepsilon}} dL_s \right|^p.
\]

It is clear that as \( m \to \infty \),
\[
|x^m - x|^p \to 0, \quad |y^m - y|^p \to 0.
\]

By the a priori estimate of \((X_s^\varepsilon, Y_s^\varepsilon)\) and the dominated convergence theorem,
\[
\lim_{m \to \infty} \int_0^T \mathbb{E}|(\pi_m - I)B(X_s^\varepsilon, Y_s^\varepsilon)|^p ds = \int_0^T \mathbb{E} \lim_{m \to \infty} |(\pi_m - I)B(X_s^\varepsilon, Y_s^\varepsilon)|^p ds = 0,
\]
\[
\lim_{m \to \infty} \int_0^T \mathbb{E}|(\pi_m - I)F(X_s^\varepsilon, Y_s^\varepsilon)|^p ds = \int_0^T \mathbb{E} \lim_{m \to \infty} |(\pi_m - I)F(X_s^\varepsilon, Y_s^\varepsilon)|^p ds = 0.
\]

By assumption \( \textbf{A2} \) and Remark 2.2, as \( m \to \infty \),
\[
\lim_{m \to \infty} \sup_{t \in [0,T]} \mathbb{E}|X_t^{m,\varepsilon} - X_t^\varepsilon|^p = 0.
\]

The proof is complete. \( \square \)

**Lemma 6.3.** For any \((x, y) \in H \times H\), \( m \in \mathbb{N}_+\), \( p \in [1, \alpha) \) and \( T > 0 \), there exist constants \( C_{p,T}, C_T > 0 \) such that any \( \varepsilon \in (0,1) \), \( \eta \in (0,1) \) and \( 0 < s \leq t < T \),
\[
\sup_{m \geq 1} \mathbb{E}|X_t^{m,\varepsilon} - X_s^{m,\varepsilon}|^p \leq C_T(t-s)^\frac{\varepsilon}{2} s^{-\frac{\eta}{2}} (1 + |x| + |y|),
\]
\[
\sup_{m \geq 1} \mathbb{E}|Y_t^{m,\varepsilon} - Y_s^{m,\varepsilon}|^p \leq C_T \left( \frac{t-s}{\varepsilon} \right)^\frac{\eta}{2} s^{-\frac{\eta}{2}} (1 + |x| + |y|).
\]

**Proof.** By (2.1), (2.5) and Minkowski’s inequality, we get for any \( p \in (1, \alpha) \), \( \eta \in (0,1) \) and \( 0 < t \leq T \),
\[
\mathbb{E}\|X_t^{m,\varepsilon}\|_\eta \leq C_p \|e^{tA} x\|_\eta + C_p \mathbb{E} \left( \int_0^t \|e^{(t-s)A} B_t^{m}(X_s^{m,\varepsilon}, Y_s^{m,\varepsilon})\|_s ds \right)^p + C_p \mathbb{E} \left\| \int_0^t e^{(t-s)A} d\bar{L}_s^{m} \right\|_\eta^p
\]
\[
\leq C_p t^{-\frac{\eta}{2}} |x|^p + C_p \left[ \int_0^t (t-s)^{-\eta/2} \mathbb{E}(1 + |X_s^{m,\varepsilon}|^p + |Y_s^{m,\varepsilon}|^p)^{1/p} ds \right]^p + C_p
\]
\[
\leq C_{p,T} t^{-\frac{\eta}{2}} (1 + |x|^p + |y|^p).
\]
Note that
\[ X^{m,\varepsilon}_t = e^{(t-s)A}X^{m,\varepsilon}_s + \int_s^t e^{(t-r)A}B^m(X^{m,\varepsilon}_r, Y^{m,\varepsilon}_r)dr + \int_s^t e^{(t-r)A}d\tilde{L}^m_r. \]

It follows from (2.2) that for any 0 < s ≤ t ≤ T,
\[
\mathbb{E}|X^{m,\varepsilon}_t - X^{m,\varepsilon}_s|^p \leq C_p(t-s)^{\frac{p}{p+1}}\mathbb{E}\|X^{m,\varepsilon}_s\|_p^p + C_p\left(\int_s^t \mathbb{E}\|B^m(X^{m,\varepsilon}_r, Y^{m,\varepsilon}_r)\|^{1/p}_r dr\right)^p
\]
\[+ C_p\left[\sum_{k=1}^{\infty} \frac{\beta_k\left(1-e^{-\lambda_k(t-s)}\right)^{\gamma_k}}{\lambda_k}\right]^{p/\alpha}
\]
\[\leq C_p(T)(t-s)^{\frac{p}{p+1}}(1+|x|^p + |y|^p) + C(t-s)^p(1+|x|^p + |y|^p)
\]
\[+ C_p(t-s)^{\frac{p}{p+1}}\left[\sum_{k=1}^{\infty} \frac{\beta_k\left(1-\lambda_k^{1-\gamma_k}\right)^{\gamma_k}}{\lambda_k}\right]^{p/\alpha}
\]
\[\leq C_p(T)(t-s)^{\frac{p}{p+1}}(1+|x|^p + |y|^p).
\]

Define \( \tilde{Z}_t := \frac{1}{\varepsilon^{1/\alpha}}Z_{t\varepsilon} \), which is also a cylindrical \( \alpha \)-stable process. Then by (2.5), for any \( \eta \in (0,1) \) we have
\[
\mathbb{E}\left\| \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}dZ_s \right\|_\eta^p = \mathbb{E}\left\| \int_0^{t/\varepsilon} e^{(t-s)A}d\tilde{Z}_s \right\|_\eta^p
\]
\[\leq C \left(\sum_k \gamma_k^{\frac{\alpha}{\alpha - 1}} \frac{1 - \lambda_k^{1/\varepsilon}}{\lambda_k^{1-\alpha \eta/2}} \right)^{p/\alpha}
\]
\[= C \left(\sum_k \frac{\alpha \frac{\gamma_k^{\alpha}}{\lambda_k^{1-\alpha \eta/2}}}{\lambda_k^{1-\alpha \eta/2}} \right)^{p/\alpha}
\]
\[\leq C\alpha \left(\sum_k \frac{\gamma_k^{\alpha}}{\lambda_k^{1-\alpha \eta/2}} \right)^{p/\alpha} \leq C_{\alpha,p}.
\]

Similarly, by (2.1) and Minkowski’s inequality, we get for any \( \eta \in (0,1) \) and 0 < t ≤ T,
\[
\mathbb{E}\|Y^{m,\varepsilon}_t\|_\eta^p \leq C_p\|e^{tA/\varepsilon}y\|_\eta^p + C_p\mathbb{E}\left(\int_0^t \left\| e^{(t-s)A}F^m(X^{m,\varepsilon}_s, Y^{m,\varepsilon}_s) \right\|_\eta ds\right)^p
\]
\[+ C_p\mathbb{E}\left\| \frac{1}{\varepsilon^{1/\alpha}} \int_0^t e^{(t-s)A/\varepsilon}d\tilde{Z}^m_s \right\|_\eta^p
\]
\[\leq C_p\left(\frac{t}{\varepsilon}\right)^{\frac{p}{p+1}} |y|^p + C \left[\frac{1}{\varepsilon} \int_0^t \left(\frac{t-s}{\varepsilon}\right)^{-\frac{\eta}{2}} e^{-\frac{(t-s)\lambda_1}{s \varepsilon}} [\mathbb{E}(1 + |X^{m,\varepsilon}_s|^p + |Y^{m,\varepsilon}_s|^p)]^{1/p} ds \right]^{p/\alpha}
\]
\[+ C_{\alpha,p}
\]
\[\leq C T t^{-\frac{p}{p+1}}(1+|x|^p + |y|^p).
\]

Note that
\[ Y^{m,\varepsilon}_t = e^{(t-s)A}X^{m,\varepsilon}_s + \frac{1}{\varepsilon} \int_s^t e^{(t-r)A}F^m(X^{m,\varepsilon}_r, Y^{m,\varepsilon}_r)dr + \frac{1}{\varepsilon^{1/\alpha}} \int_s^t e^{(t-r)A}d\tilde{Z}^m_r.
\]

Thus by (2.2), it follows that for any 0 < s < t ≤ T,
\[ \mathbb{E}|Y^{m,\varepsilon}_t - Y^{m,\varepsilon}_s|^p \]
\[
\leq C_p \mathbb{E} \left[ e^{\frac{(t-s)A}{\varepsilon}} Y_{s}^{m,\varepsilon} - Y_{s}^{m,\varepsilon} \right|^p + C_p \mathbb{E} \left[ \frac{1}{\varepsilon} \int_{s}^{t} e^{\frac{(t-s)A}{\varepsilon}} F_{2}^{m}(X_{r}^{m,\varepsilon}, Y_{r}^{m,\varepsilon}) \, dr \right]^p \\
+ C_p \mathbb{E} \left[ \frac{1}{\varepsilon^{1/\alpha}} \int_{s}^{t} e^{\frac{(t-s)A}{\varepsilon}} d\tilde{Z}_{r}^{m} \right]^p \\
\leq C_p \left( \frac{t-s}{\varepsilon} \right)^{\frac{pn}{2}} \mathbb{E} \left[ Y_{s}^{m,\varepsilon} \right|^p_{\eta} + C_p \left[ \frac{1}{\varepsilon} \int_{s}^{t} e^{\frac{(t-s)A}{\varepsilon}} \left[ \mathbb{E}(1 + \left| X_{r}^{m,\varepsilon} \right|^p + \left| Y_{r}^{m,\varepsilon} \right|^p) \right]^{1/p} \, dr \right]^p \\
+ C_p \left[ \sum_{k=1}^{\infty} \frac{\gamma_k^\alpha (1 - e^{-\alpha \lambda_k (t-s)/\varepsilon})}{\lambda_k} \right]^{p/\alpha} \\
\leq C_{p,T} \left( \frac{t-s}{\varepsilon} \right)^{\frac{pn}{2}} \left( \frac{1}{\varepsilon} \right)^{p/\alpha} \left( \sum_{k=1}^{\infty} \frac{\gamma_k^\alpha}{\lambda_k^{1-\alpha n/2}} \right)^{p/\alpha} \\
\leq C_{p,T} \left( \frac{t-s}{\varepsilon} \right)^{\frac{pn}{2}} \left( 1 + \left| x \right|^p + \left| y \right|^p \right),
\]
where we use the fact that \(1 - e^{-x} \leq C x^{\alpha n/2}\), for any \(x > 0\). The proof is complete. \(\square\)

**Lemma 6.4.** For any \((x, y) \in H^\gamma \times H\) with \(\gamma \in [0, 1), \eta \in (0, 1), T > 0\) and \(p \in [1, \alpha)\), there exists a constant \(C_T\) such that for any \(\varepsilon \in (0, 1)\) and \(t \in (0, T)\), we have
\[
\sup_{m \geq 1} \left[ \mathbb{E} \left\| X_{t}^{m,\varepsilon} \right\|^p_{\gamma} \right]^{\frac{1}{p}} \leq C t^{-1 + \frac{\alpha}{2}} \left\| x \right\|_{\gamma} + C_T \varepsilon^{-\frac{\alpha}{2}} (1 + \left| x \right| + \left| y \right|). \tag{6.13}
\]

**Proof.** Note that for any \(t > 0\), we have
\[
X_{t}^{m,\varepsilon} = e^{tA} x + \int_{0}^{t} e^{\frac{(t-s)A}{\varepsilon}} B^{m}(X_{s}^{m,\varepsilon}, Y_{s}^{m,\varepsilon}) \, ds \\
+ \int_{0}^{t} e^{\frac{(t-s)A}{\varepsilon}} \left[ B^{m}(X_{s}^{m,\varepsilon}, Y_{s}^{m,\varepsilon}) - B^{m}(X_{t}^{m,\varepsilon}, Y_{t}^{m,\varepsilon}) \right] \, ds + \int_{0}^{t} e^{\frac{(t-s)A}{\varepsilon}} d\tilde{L}_{s}^{m} \\
:= \sum_{i=1}^{4} I_{i}.
\]
For the term \(I_{1}\), using (2.1), for any \(\gamma \in (0, 1)\) we have
\[
\left\| e^{tA} x \right\|_{2} \leq C t^{-1 + \frac{\alpha}{2}} \left\| x \right\|_{\gamma}. \tag{6.14}
\]
For the term \(I_{2}\), we have
\[
\left[ \mathbb{E} \left\| I_{2} \right\|_{2}^{p} \right]^{1/p} \leq \left( \mathbb{E} \left[ \left( e^{tA} - I \right) B^{m}(X_{t}^{m,\varepsilon}, Y_{t}^{m,\varepsilon}) \right]^{p} \right)^{1/p} \\
\leq C \left[ 1 + \left( \mathbb{E} \left\| X_{t}^{m,\varepsilon} \right\|^p \right)^{1/p} + \left( \mathbb{E} \left\| Y_{t}^{m,\varepsilon} \right\|^p \right)^{1/p} \right] \\
\leq C_{T} (1 + \left| x \right| + \left| y \right|).
\]
For the term \(I_{3}\), using Minkowski’s inequality and Lemma 6.3, we obtain
\[
\left[ \mathbb{E} \left\| I_{3} \right\|_{2}^{p} \right]^{1/p} \leq C \int_{0}^{t} \frac{1}{t-s} \left[ \mathbb{E} \left\| F_{1}^{m}(X_{s}^{m,\varepsilon}, Y_{s}^{m,\varepsilon}) - F_{1}^{m}(X_{t}^{m,\varepsilon}, Y_{t}^{m,\varepsilon}) \right\|^{p} \right]^{1/p} \, ds \\
\leq C \int_{0}^{t} \frac{1}{t-s} \left[ \mathbb{E} \left\| X_{s}^{m,\varepsilon} - X_{t}^{m,\varepsilon} \right\|^{p} + \mathbb{E} \left\| Y_{s}^{m,\varepsilon} - Y_{t}^{m,\varepsilon} \right\|^{p} \right]^{1/p} \, ds.
\]
Lemma 6.6. Proof. It is easy to see that for any $\bar{x}$ and $T > 0$,
\begin{align*}
&\leq C(1 + |x| + |y|) \int_0^T \frac{1}{t-s}(t-s)^{\frac{3}{2}-\frac{\beta}{2}} s^{-\frac{\beta}{2}} ds \\
&+ C(1 + |x| + |y|) \int_0^T \frac{1}{t-s} \left(\frac{t-s}{\varepsilon}\right)^{\frac{3}{2}-\frac{\beta}{2}} s^{-\frac{\beta}{2}} ds \\
&\leq C\varepsilon^{-n/2}(1 + |x| + |y|).
\end{align*}
(6.15)

For the term $I_4$, by (2.5) and assumption A2, we easily have
\begin{align*}
\|E|I_4|^{p/2}\|^{1/p} &\leq C_p \left( \sum_{k \in \mathbb{N}_+} \frac{\beta_k^\alpha}{\lambda_k^{-\alpha}} \right)^{1/\alpha} \leq C_p.
\end{align*}
(6.16)

Combining (6.14)–(6.16) yields the desired result. \hfill \Box

Remark 6.5. By the same argument above, we can easily prove that for any $(x, y) \in H \times H$, $T > 0$ and $p \in [1, \alpha)$, there exists a constant $C_T$ such that for any $t \in (0, T]$, we have
\begin{align*}
\sup_{m \geq 1} \left[ E\|\bar{X}_t^m - \bar{X}_t\|^{p/2}\right]^{1/p} &\leq C_T t^{-1}(1 + |x|).
\end{align*}
(6.17)

Recall the approximate equation (3.2) to the averaged equation (2.7). Note that $\bar{X}^m$ is not the Galerkin approximation of $\bar{X}$, hence we have to check its approximation carefully.

Lemma 6.6. For any $x \in H$, $T > 0$ and $p \in [1, \alpha)$, we have
\begin{align*}
\lim_{m \to \infty} \sup_{t \in [0, T]} E|\bar{X}_t^m - \bar{X}_t|^p = 0.
\end{align*}
(6.18)

Proof. It is easy to see that for any $t > 0$,
\begin{align*}
\bar{X}_t^m - \bar{X}_t &= e^{tA}(x^m - x) + \int_0^t e^{(t-s)A}(\pi_m - I)\bar{B}(\bar{X}_s) ds \\
&+ \int_0^t e^{(t-s)A} \left[ \bar{B}(\bar{X}_s) - \pi_m \bar{B}(\bar{X}_s) \right] ds + \left[ \int_0^t e^{(t-s)A} d\bar{L}_s^m - \int_0^t e^{(t-s)A} d\bar{L}_s \right].
\end{align*}

Then for any $T > 0$ and $p \in [1, \alpha)$, we have
\begin{align*}
\sup_{t \in [0, T]} E|\bar{X}_t^m - \bar{X}_t|^p &\leq C_p|x^m - x|^p + C_{p,T} \int_0^T E|\pi_m - I\bar{B}(\bar{X}_s)|^p ds + C_{p,T} \int_0^T E|\bar{B}(\bar{X}_s) - \pi_m \bar{B}(\bar{X}_s)|^p ds \\
&+ C_{p,T} \int_0^T |e^{(t-s)A}d\bar{L}_s^m - e^{(t-s)A}d\bar{L}_s|^p ds \\
&\leq C_p|x^m - x|^p + C_{p,T} \int_0^T E|\pi_m - I\bar{B}(\bar{X}_s)|^p ds + C_{p,T} \int_0^T E|\bar{X}_t^m - \bar{X}_t|^p dt \\
&+ C_{p,T} \int_0^T |e^{(t-s)A}d\bar{L}_s^m - e^{(t-s)A}d\bar{L}_s|^p. \hspace{1cm} (6.16)
\end{align*}

By Gronwall’s inequality, we get
\begin{align*}
\sup_{t \in [0, T]} E|\bar{X}_t^m - \bar{X}_t|^p &\leq C_{p,T}|x^m - x|^p + C_{p,T} \int_0^T E|\pi_m - I\bar{B}(\bar{X}_s)|^p ds.
\end{align*}
By the a priori estimate of \( \bar{X} \) and the dominated convergence theorem,

\[
\lim_{m \to \infty} \int_0^T \mathbb{E}|(\pi_m - I)\bar{B}(\bar{X}_s)|^p ds = 0, \quad (6.19)
\]

\[
\lim_{m \to \infty} \sup_{t \in [0,T]} \mathbb{E} \left| \int_0^t e^{(t-s)A}d\bar{L}_s^m - \int_0^t e^{(t-s)A}dL_s \right|^p = 0. \quad (6.20)
\]

Thus if we can prove for any \( x \in H \),

\[
\lim_{m \to \infty} |\bar{B}^m(x) - \pi_m \bar{B}(x)| = 0, \quad (6.21)
\]

Then by dominated convergence theorem, we get

\[
\lim_{m \to \infty} \mathbb{E} \int_0^T |\bar{B}^m(\bar{X}_s) - \pi_m \bar{B}(\bar{X}_s)|^p ds = 0. \quad (6.22)
\]

Hence, (6.18) holds by combining (6.19), (6.20) and (6.22).

In fact, for any \( t > 0 \),

\[
\mathbb{E}|Y_{t,x}^{x,0,m} - Y_{t,x}^{x,0}| \leq \int_0^t \mathbb{E} \left| (\pi_m - I)F(x, Y_{s,x}^{x,0}) \right| ds + \int_0^t \mathbb{E} \left| F^m(x, Y_{s,x}^{x,0,m}) - F^m(x, Y_{s,x}^{x,0}) \right| ds
\]

\[
+ \mathbb{E} \left| \int_0^t e^{(t-s)A}d\bar{Z}_s^m - \int_0^t e^{(t-s)A}dZ_s \right|.
\]

By Gronwall's inequality,

\[
\mathbb{E}|Y_{t,x}^{x,0,m} - Y_{t,x}^{x,0}| \leq e^{Ct} \left[ \int_0^t \mathbb{E} \left| (\pi_m - I)F(x, Y_{s,x}^{x,0}) \right| ds + \mathbb{E} \int_0^t e^{(t-s)A}d\bar{Z}_s^m - \int_0^t e^{(t-s)A}dZ_s \right].
\]

As a consequence, it is easy to see

\[
\lim_{m \to \infty} \mathbb{E}|Y_{t,x}^{x,0,m} - Y_{t,x}^{x,0}| = 0.
\]

By (3.10), we have for any \( t > 0 \),

\[
|\bar{B}^m(x) - \pi_m \bar{B}(x)| \leq |\bar{B}^m(x) - \mathbb{E}\pi_m B(x, Y_{t,x}^{x,0,m})| + |\mathbb{E}\pi_m B(x, Y_{t,x}^{x,0}) - \pi_m \bar{B}(x)|
\]

\[
+ |\pi_m B(x, Y_{t,x}^{x,0,m}) - \mathbb{E}\pi_m B(x, Y_{t,x}^{x,0})|
\]

\[
\leq Ce^{-(\lambda_1 - \lambda E)x^2} + \mathbb{E}|Y_{t,x}^{x,0,m} - Y_{t,x}^{x,0}|.
\]

Then by taking \( m \to \infty \) firstly, then \( t \to \infty \), we finally get (6.21). The proof is complete. \( \square \)

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References

[1] J. Bao, G. Yin, C. Yuan, Two-time-scale stochastic partial differential equations driven by \(\alpha\)-stable noises: averaging principles. *Bernoulli* 23(1) (2017) 645-669.

[2] R. Bertram, J. E. Rubin, Multi-timescale systems and fast-slow analysis. *Math. Biosci.* 287 (2017) 105-121.

[3] N.N. Bogoliubov, Y.A. Mitropolsky, *Asymptotic methods in the theory of Non-linear Oscillations*. Gordon and Breach Science Publishers, New York, 1961.

[4] C.E. Bréhier, Strong and weak orders in averaging for SPDEs. *Stochastic Process. Appl.* 122 (2012) 2553-2593.

[5] C. E. Bréhier, Analysis of an HMM time-discretization scheme for a system of stochastic PDEs. *SIAM J. Numer. Anal.* 51 (2013) 1185-1210.

[6] C.E. Bréhier, Orders of convergence in the averaging principle for SPDEs: the case of a stochastically forced slow component. *Stochastic Process. Appl.* 130 (2020) 3325-3368.

[7] S. Cerrai, A Khasminskii type averaging principle for stochastic reaction-diffusion equations. *Ann. Appl. Probab.* 19 (2009) 899-948.

[8] S. Cerrai, Averaging principle for systems of reaction-diffusion equations with polynomial nonlinearities perturbed by multiplicative noise, *SIAM J. Math. Anal.* 43 (2011) 2482-2518.

[9] S. Cerrai, A. Lunardi, Averaging principle for nonautonomous slow-fast systems of stochastic reaction-diffusion equations: the almost periodic case. *SIAM J. Math. Anal.* 49 (2017) 2843-2884.

[10] Y. Chen, Y. Shi, X. Sun, Averaging principle for slow-fast stochastic Burgers equation driven by \(\alpha\)-stable process. *Appl. Math. Lett.* 103 (2020) 106199.

[11] Z. Dong, X. Sun, H. Xiao and J. Zhai, Averaging principle for one dimensional stochastic Burgers equation. *J. Differential Equations* 265 (2018) 4749-4797.

[12] M. Freidlin, A.D. Wentzell, *Random perturbations of dynamical systems*. Third edition, Springer, Heidelberg, 2012. xxviii+458 pp.

[13] H. Fu, L. Wan, J. Liu, Strong convergence in averaging principle for stochastic hyperbolic-parabolic equations with two time-scales. *Stochastic Process. Appl.* 125 (2015) 3255-3279.

[14] H. Fu, L. Wan, J. Liu, X. Liu, Weak order in averaging principle for stochastic wave equation with a fast oscillation. *Stochastic Process. Appl.* 128 (2018) 2557-2580.

[15] P. Gao, Averaging principle for stochastic Korteweg-de Vries equation. *J. Differential Equations* 267 (2019) 6872-6909.

[16] P. Gao, Averaging principle for complex Ginzburg-Landau equation perturbed by mixing random forces. *SIAM J. Math. Anal.* 53 (1) (2021) 32-61.

[17] Y. Ge, X. Sun, Y. Xie, Optimal convergence rates in the averaging principle for slow-fast SPDEs driven by multiplicative noise. https://arxiv.org/abs/2101.09076

[18] D. Givon, I. G. Kevrekidis and R. Kupferman, Strong convergence of projective integration schemes for singularly perturbed stochastic differential systems. *Comm. Math. Sci.* 4 (2006) 707-729.

[19] D. Givon, Strong convergence rate for two-time-scale jump-diffusion stochastic differential systems. *SIAM J. Multiscale Model. Simul.* 6 (2007) 577-594.

[20] J. Golec, Stochastic averaging principle for systems with pathwise uniqueness. *Stochastic Anal. Appl.* 13 (1995) 307-322.

[21] M. Hairer, X. Li, Averaging dynamics driven by fractional Brownian motion. *Ann. Probab.* 48 (4) (2020) 1826-1860.

[22] E. Hausenblas, J. Seidler, Stochastic convolutions driven by martingales: maximal inequalities and exponential integrability. *Stoch. Anal. Appl.* 26 (2008) 98-119.

[23] R.Z. Khasminskii, On an averging principle for Itô stochastic differential equations. *Kibernetica* 4 (1968), 260-279.

[24] R. Z. Khasminskii and G. Yin, On averaging principles: an asymptotic expansion approach. *SIAM J. Math. Anal.* 35 (2004) 1534-1560.

[25] Y. Kifer, Diffusion approximation for slow motion in fully coupled averaging. *Probab. Theory Related Fields* 129 (2004) 157-181.

[26] D. Liu, Strong convergence of principle of averaging for multiscale stochastic dynamical systems. *Commun. Math. Sci.* 8 (2010) 999-1020.
[27] W. Liu, M. Röckner, X. Sun, Y. Xie, Averaging principle for slow-fast stochastic differential equations with time dependent locally Lipschitz coefficients. *J. Differential Equations* 268 (2020) 2910-2948.

[28] E. Pardoux and A. Yu. Veretennikov, On the Poisson equation and diffusion approximation. I. *Ann. Prob.* 29 (2001) 1061-1085.

[29] E. Pardoux and A. Yu. Veretennikov, On the Poisson equation and diffusion approximation. 2. *Ann. Prob.* 31 (2003) 1166-1192.

[30] G. A. Pavliotis and A. M. Stuart, *Multiscale methods: averaging and homogenization*. volume 53 of Texts in Applied Mathematics. Springer, New York, 2008.

[31] B. Pei, Y. Xu, J. Wu, Two-time-scales hyperbolic-parabolic equations driven by Poisson random measures: existence, uniqueness and averaging principles. *J. Math. Anal. Appl.* 477(1) (2017) 243-268.

[32] B. Pei, Y. Xu, G. Yin, Stochastic averaging for a class of two-time-scale systems of stochastic partial differential equations. *Nonlinear Anal.* 160 (2017) 159-176.

[33] E. Priola, J. Zabczyk, Structural properties of semilinear SPDEs driven by cylindrical stable processes. *Probab. Theory Related Fields* 149 (2011) 97-137.

[34] E. Priola, A. Shirikyan, L. Xu, and J. Zabczyk, Exponential ergodicity and regularity for equations with Lévy Noise. *Stochastic Process. Appl.* 122 (2012) 106-133.

[35] M. Röckner, X. Sun, Y. Xie, Strong convergence order for slow-fast McKean-Vlasov stochastic differential equations. *Ann. Inst. Henri Poincaré Probab. Stat.* 57(1) (2021) 547-576.

[36] M. Röckner, L. Xie, L. Yang, Asymptotic behavior of multiscale stochastic partial differential equations. *https://arxiv.org/abs/2010.14897*.

[37] M. Röckner, L. Xie, Diffusion approximation for fully coupled stochastic differential equations. *Ann. Probab.* 49 (3) (2021) 1205-1236.

[38] X. Sun, H. Xia, Y. Xie and X. Zhou, Strong averaging principle for a class of slow-fast singular SPDEs driven by α-stable process. To appear in *Front. Math. China* [https://arxiv.org/abs/2011.11988]

[39] X. Sun, L. Xie and Y. Xie, Strong and weak convergence rates for slow-fast stochastic differential equations driven by α-stable process, To appear in *Bernoulli*. [https://arxiv.org/abs/2004.02595].

[40] X. Sun and J. Zhai, Averaging principle for stochastic real Ginzburg-Landau equation driven by α-stable process. *Commun. Pure Appl. Anal.* 19 (2020) 1291-1319.

[41] A.Y. Veretennikov, On the averaging principle for systems of stochastic differential equations. *Math. USSR Sborn.* 69 (1991) 271-284.

[42] W. Wang, A.J. Roberts, Average and deviation for slow-fast stochastic partial differential equations. *J. Differential Equations* 253 (2012) 1265-1286.

[43] J. Wang, Exponential ergodicity and strong ergodicity for SDEs driven by symmetric α-stable process. *Appl. Math. Lett.* 26 (2013), 654-658.

[44] F. Wu, T. Tian, J.B. Rawlings, G. Yin, Approximate method for stochastic chemical kinetics with two-time scales by chemical Langevin equations. *J. Chem. Phys.* 144 (2016) 174112.

[45] L. Xie, L. Yang, Diffusion approximation for multi-scale stochastic reaction-diffusion equations. [https://arxiv.org/abs/2101.03917].

[46] J. Xu, Y. Miao, J. Liu, Strong averaging principle for slow-fast SPDEs with Poisson random measures. *Discrete Contin. Dyn. Syst. Ser. B* 20 (2015) 2233-2256.

[47] L. Xu, Ergodicity of the stochastic real Ginzburg-Landau equation driven by α-stable noises. *Stochastic Process. Appl.* 123 (2013) 3710-3736.

[48] B. Zhang, H. Fu, L. Wan, J. Liu, Weak order in averaging principle for stochastic differential equations with jumps. *Adv. Difference Equ.* 2018, Paper No. 197, 20 pp.

[49] Y. Zhang, Q. Huang, X. Wang, Z. Wang, J. Duan, Weak averaging principle for multiscale stochastic dynamical systems driven by α-stable processes. [https://arxiv.org/abs/2007.08408]
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