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Modeling gasdynamic vortex cooling

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We aim at studying gasdynamic vortex cooling in an analytically solvable, thermodynamically consistent model that can explain limitations on the cooling efficiency. To this end, we study a radial plus axial flow between two (co-axial) rotating permeable cylinders. Full account is taken of compressibility, viscosity and heat conductivity. For a weak inward radial flow the model qualitatively describes the vortex cooling effect—both in terms of temperature and the stagnation enthalpy decrease—seen in short uniflow vortex (Ranque) tubes. The cooling is not done due to an external work, and its efficiency is defined as the ratio of the lowest temperature reached adiabatically (for the given pressure gradient) to the actually reached lowest temperature. We show that for the vortex cooling the efficiency is strictly smaller than 1, but we found a cooling effect, where the efficiency can be larger than 1. This cooling effect is achieved for the outward radial flow, it is partly geometric, and is also based on heat conductivity.

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I. INTRODUCTION

Air swirling through a cylindrical tube achieves temperature separation: next to the swirling axis the temperature is lower than the input temperature \( T_0 \), while far from the axis it is higher than \( T_0 \). This is the vortex cooling-heating effect discovered by G. Ranque more than 80 years ago [1, 2]; see [3–6] for reviews. A temperature separation without cooling was observed also in highly-pressurized water [7]. An overall cooling (both output temperatures lower than \( T_0 \)) was seen for certain vortex tubes [8].

Coolers based on the Ranque effect are convenient in specific applications, e.g. because they do not have moving parts. However, their efficiency is smaller than 1. Much effort was devoted to increasing it, but the best efficiency is still \( \approx 0.6 \) [4, 6].

The flow inside of the Ranque tube is highly complex: it is essentially three-dimensional and turbulent. The effect comes in two versions: counter-flow and uniflow [3–6]. In counter-flow vortex tubes the output flows are collected from two different ends of the cylinder, the cold air comes out from the end closer to the injection point, while the hot air from the opposite end of the cylinder [3–6]. Such tubes have both radial [3, 4, 9] and axial temperature separation [5, 10–13]. In the uniflow situation the air is injected circumferentially at one end of the tube, and both output flows are collected from the opposite end [3, 14]. This radial temperature separation takes place close to the injection point of the air [15].

The full theory of the Ranque effect is elusive; there are several different approaches that attempt to describe the complex three-dimensional flow inside of the tube [16–26]. In particular, it is unclear what are the minimal ingredients needed to describe the effect.

Given the complexity of the original Ranque effect, and the necessity of understanding general limitations on the efficiency of gasdynamic cooling, it is desirable to come up with simpler cooling set-ups, where the complexities of the original Ranque effect are deliberately omitted. The quest for such a simplification was posed already in Ref. [27], where Savino and Ragsdal reported on an experimental realization of a short uniflow tube, where the flow is injected from the surface via permeable rotating wall, and the colder air is collected from the axis [27]. Axial separation of temperature is absent, and the whole outgoing flow is cooled [27].

Guided by this experiment, we aim at understanding the phenomenon of vortex—and more general gasodynamic—cooling on a possibly simple theoretical model. We focus on a compressible angular flow between two rotating cylinders plus a radial motion via permeable cylinder walls; see Fig. 1. We work with a compressible flow, because experimental angular velocities are nearly sonic [4, 27]. Moreover, once we are interested by thermodynamic aspects (i.e. cooling efficiency), it is desirable to work in the compressible situation, where the thermodynamic description of the flow is complete and consistent\textsuperscript{1}. We assume that the flow is viscous, because (according to the Bernoulli’s theorem) the adiabatic motion of the fluid does not predict cooling in terms of stagnation enthalpy. However, precisely such cooling is observed experimentally [27]\textsuperscript{2}. Hence viscosity is important [6], and then heat-conductivity is to be accounted for simultaneously with viscosity, because the Prandtl number of air is close to one both in laminar and turbulent regimes.

Our first result is that in the stationary regime of a

\textsuperscript{1} The incompressible limit is singular from the viewpoint of thermodynamics [28]. Despite of the widespread usage of this limit, its consistent thermodynamics was developed only recently [28].

\textsuperscript{2} Thus adiabatic theories of the Ranque effect [16–20] do not describe the full cooling effect [21].
weak radial flow and a quasi-solid angular (vortical) motion, the model predicts cooling both in terms of thermodynamic temperature and stagnation enthalpy. The efficiency of this cooling is smaller than 1. The finding is in a qualitative agreement with experiment [27]. The agreement is achieved by using effective (turbulent or eddy) values of viscosity and heat-conductivity in the laminar flow model. Such an approach is well-known [29]. (At any rate, applications of turbulence theories are qualitative [22–24].) It is crude, but it does allow a theoretical understanding of the cooling effect.

The model predicts a stronger cooling effect for (radially) outward flow of fluid. The unique feature of this effect is that its cooling efficiency is larger than 1, i.e. the pressure gradient is employed more efficiently than for the adiabatic effect. This effect agrees with the second law, and it is possible due to heat-conductivity.

Scenarios of cooling studied here do not amount to refrigeration, i.e. they are not achieved due to investing an external work. Naturally, they are also not due to low-temperature boundary baths; to ensure this we need to pay a special attention to boundary conditions. Hence their efficiency is defined as the ratio of the lowest temperature reached adiabatically (for the given pressure gradient) to the actually reached lowest temperature.

Cylindrical vortices with radial flow were already studied in Refs. [23, 25, 26, 30–37], but the problem of finding cooling scenarios with proper boundary conditions was (to our knowledge) not posed. Dombrau [25] and later on Pengelley [26] studied the problem precisely having the same purpose as we: to get a solvable model for vortex cooling. But they did not account for boundary conditions and heat conduction and thus did not obtain proper cooling. Pengelley proposed a necessary condition for cooling that relates to the work done by viscous forces [26]. Below we show that under certain additional limitations this condition is indeed able to produce cooling. Refs. [22–24] employed simplified turbulence theories of various types that account for radial heat conductivity and viscous vortex motion. A related, but more complete turbulent theory that also accounts for axial motion was given in Ref. [30]. Refs. [31, 33, 35, 36] focus on a laminar flow in the incompressible limit (but they account for axial motion). As our analysis shows, compressibility need not be large, but retaining it—and hence allowing for the proper coupling between thermodynamics and mechanics—is necessary for the proper theoretical description of cooling. Ref. [32] did not employ the incompressible limit, but studied the problem without the outer cylinder. Several studies on convective heat transfer between concentric cylinders are reviewed in [38–40].

The paper is organized as follows. Next section defines the problem and sets notations and dimensionless parameters. There we also discuss general limitations (in particular, on the cooling efficiency) imposed by the first and second laws. Section III focuses on the definition of cooling, which is not trivial (especially for permeable walls) and thus demands clarifications. Cooling scenarios of inward radial flow are studied in section IV. The extent to which this scenario agrees with experiments is discussed in section V. Section VI discusses the cooling of outward radial flow and shows that its efficiency is larger than one. We summarize in the last section. Several technical questions are relegated to Appendices.

II. THE MODEL

A. Navier-Stokes equation

The flow between two rotating concentric cylinders is described via cylindric coordinates \((r, \phi, z)\). The flow is characterized by the velocity components \((v_r, v_\phi, v_z)\) and the pressure \(p\). The problem is governed by the Navier-Stokes equations:

\[
\begin{align*}
\rho(v_r \frac{dv_r}{dr} - \frac{v_\phi^2}{r}) &= \frac{dp}{dr} + \left(\frac{\zeta}{4} + \frac{4\eta}{3}\right) \frac{d}{dr} \left[ \frac{1}{r} \frac{d(rv_r)}{dr} \right], \\
\rho(v_r \frac{dv_\phi}{dr} + \frac{v_r v_\phi}{r}) &= \eta \left( \frac{1}{r} \frac{dv_r}{dr} + \frac{1}{r} \left(\frac{dv_\phi}{dr} - \frac{v_\phi}{r^2}\right) \right),
\end{align*}
\]

where \(p\) is pressure, \(\eta\) and \(\zeta\) are viscosities, \(\rho\) is the mass density; see Table I. We assume that all the involved quantities depend only on \(r\), e.g., \(v_\phi = \tilde{v}(r)\). We also assume that \(v_z = 0\), since in the context of our problem it is useless to keep \(v_z \neq 0\), if it is a function of \(r\) only. See Fig. 1 for a schematic representation of the flow.

In the stationary regime the Navier-Stokes equations for \(v_r\) and \(v_\phi\) read [29]:

\[
\begin{align*}
\partial_r \rho v_r + \tilde{\nabla} (\rho v_r^2) &= \frac{1}{r} \frac{d}{dr} (\rho rv_r) = 0, \\
\rho rv_r + c &= \text{const},
\end{align*}
\]

where \(c\) (a positive or negative constant) characterizes the radial flow. Eqs. (2, 4) transform to

\[
\eta r^2 \frac{d^2 v_\phi}{dr^2} + (\eta - c) \frac{dv_\phi}{dr} - (\eta + c)v_\phi = 0.
\]

This equation is linear over \(v_\phi\). Its two independent solutions are obtained by putting \(v_\phi \propto r^a\) into (5). The latter produces a quadratic equation for \(a\). This equation has two solutions \(a = -1\) and \(a = 1 + \frac{\zeta}{4\eta}\).

We now impose boundary conditions on (resp.) inner and outer cylinder

\[
v_1 \equiv v_\phi(r_1), \quad v_2 \equiv v_\phi(r_2),
\]

where \(r_2 > r_1\). Then (5) and (6) are solved as a linear combination of \(a = -1\) and \(a = 1 + \frac{\zeta}{4\eta}\). Solutions of (5):

\[
\begin{align*}
v_\phi(r) &= v_2 \hat{v}_\phi(x), \quad x \equiv r/r_2, \\
\hat{v}_\phi(x) &= [(1 - \alpha)x^{-1} + \alpha x^{1+\kappa}], \\
\eta \alpha &\equiv \frac{c}{\eta}, \quad \alpha \equiv \frac{1 - (v_1 r_1)/(v_2 r_2)}{1 - (r_1/r_2)^{2+\kappa}},
\end{align*}
\]
where we introduced the dimensionless coordinate \( x \); see Table I. Eq. (8) is a weighted sum of two contributions: potential vortex 1/\( x \) and the quasi-solid vortex \( x^{1+k} \). The weight \( \alpha \) can hold both \( \alpha > 1 \) and \( \alpha < 0 \).

### B. Energy equation

The fluid energy equation reads [29]

\[
\partial_t \left( \frac{\rho \vec{v}^2}{2} + \rho \varepsilon \right) + \nabla \cdot (\rho \vec{v} (\vec{v}^2 + \varepsilon)) + p \vec{v} + \vec{\mu} - \lambda \nabla T = 0, \tag{10}
\]

where \( \frac{\rho \vec{v}^2}{2} + \rho \varepsilon \) is the energy density (kinetic energy plus internal energy), \( \rho \vec{v} (\vec{v}^2 + \varepsilon) \) is the advective energy flux, \( p \vec{v} \) is the pressure-driven energy flux, \( T \) is temperature (measured in Kelvins), \( \nabla (\lambda \nabla T) \) is the heat flow due to heat conductivity \( \lambda \) (we assume that \( \lambda \) does not depend on \( p, \rho \), and \( T \)), \( \mu_k = -\sum_j v_j \sigma_{jk} \) is the energy flow due to viscosity, and \( \sigma_{jk} \) is the stress tensor [29].

With the assumptions and in the stationary regime the energy flux is \( c_v / r \), where \( c_v \) is a constant [cf. (4)]:

\[
c_v = cE - rv_v \sigma_{rr} - rv_\phi \sigma_{\phi \phi} - \lambda r \frac{dT}{dr}, \tag{11}
\]

\[
E = \frac{v_v^2 + v_\phi^2}{2} + \varepsilon + \frac{p}{\rho}, \tag{12}
\]

\[
\sigma_{rr} = 2\eta \frac{dv_v}{dr} + (\xi - \frac{2\eta}{3}) \frac{1}{r} \frac{d(rv_v)}{dr}, \tag{13}
\]

\[
\sigma_{\phi \phi} = \eta \left( \frac{dv_\phi}{dr} - \frac{v_v}{r} \right). \tag{14}
\]

Here \( E \) is the full energy (kinetic + internal + potential) per unit of mass; \( -\lambda \frac{dT}{dr} \) is the heat flux due to the radial temperature gradient; \( \sigma_{rr} \) and \( \sigma_{\phi \phi} \) are the components of the stress tensor [29]; \( v_v, \sigma_{rr} \) (\( v_\phi, \sigma_{\phi \phi} \)) is the rate of radial (angular) work done by viscous forces. Eqs. (11–14) express the first law for the radial flow.

Eqs. (1) and (11) become closed after specifying the thermodynamic state equation; we choose it by assuming that the fluid holds the ideal gas laws [see Appendix A]:

\[
p = R\rho T / \mu, \tag{15}
\]

\[
(\rho \varepsilon + p) / \rho = c_v T, \tag{16}
\]

\[
c_v = \hat{c}_v R / \mu, \tag{17}
\]

where \( c_v > 0 \) is the (constant) heat capacity at fixed pressure, and \( \hat{c}_v \) is a dimensionless number of order 1 (e.g. \( \hat{c}_v \approx 3.5 \) for air); see Table I. \( R = 8.314 \) J/K is the gas constant and \( \mu \) is the molar mass (29 g for air).

### C. Dimensionless parameters and variables

Employing (13–17) in (11), and (15, 16) in (1), we end up with the following dimensionless form of (respectively) (11) and (1):

\[
\left(\frac{\kappa}{2} + 1\right) \hat{T}^2 + x b \hat{v}_\phi \hat{v}_\phi + b \hat{T} - x \hat{T}' - \beta
\]

\[
+\left(\frac{\kappa}{2} + 2\right) \frac{w^2}{x^2} - (\chi + \frac{4}{3}) \frac{w w'}{x} = 0, \tag{18}
\]

\[
(\chi + \frac{4}{3}) \frac{w}{x} - (\kappa + \frac{4}{3}) \frac{w'}{x} + \kappa \hat{w} x^2
\]

\[
+ \frac{\kappa \hat{v}^2}{w} - \frac{b x}{c_v} \left( \hat{T} / w' \right) = 0, \tag{19}
\]

where \( x = \frac{r}{r_2} \) [cf. (7, 9)], prime means \( \frac{d}{dx} \), e.g.

\[
\hat{v}' = \frac{d\hat{v}_\phi}{dx}, \tag{20}
\]

and where we introduced [cf. Table I]:

\[
\hat{T} = \frac{\lambda}{v_2^2} T, \quad w = \frac{x v_r}{v_2}, \tag{21}
\]

\[
b = \frac{c c_p}{\chi}, \quad \kappa = \frac{\varepsilon}{\eta}, \tag{22}
\]

\[
\chi = \frac{\varepsilon}{\eta}, \quad \beta = \frac{c_v}{\eta \mu^2}. \tag{23}
\]

Here \( |\kappa| \) is the Reynolds number related to the radial flow, while \( b/\kappa \) is the Prandtl number. These and other dimensional and dimensionless parameters of the systems are discussed in Table I. The angular Mach number \( Ma \) of the outer cylinder reads via the above parameters as

\[
Ma = \sqrt{\frac{(\hat{c}_p - 1) \kappa}{b \hat{T}^2}} = \frac{|v_2|}{v_{\text{sound}}}, \tag{24}
\]

where \( v_{\text{sound}} = \sqrt{\frac{c_p T}{(\hat{c}_p - 1)}} = \sqrt{\frac{c_p T}{\hat{c}_p - 1}} \) is the speed of sound for the ideal gas; see (15) and (29). The constant \( \beta \) in (18) can be related to \( \hat{T}'(1) \) via \( \hat{v}_\phi'(1) \) [see (8)]:

\[
\hat{T}'(1) = -\beta + b \hat{T}(1) + (2 + \frac{\kappa}{2}) w^2(1) - (\chi + \frac{4}{3}) w(1) w'(1) - \alpha (2 + \kappa) + 2 + \frac{\kappa}{2}, \tag{25}
\]

### D. Scaling of temperature

The scaling over \( v_2 \) employed in (21) and in (7) does have a physical meaning, since below we show that cooling (i.e. temperature decrease) relates to the angular motion of the cylinders. The dimensionless temperature \( \hat{T} \) is reasonable, also because with typical numbers of experimental vortex cooling [4, 27], we get \( \hat{T} \sim 1 \). Indeed, using (21) we get

\[
T(r_2) = \hat{T}(1) v_{\text{sound}}^2 P r \frac{Ma}{c_p}, \tag{26}
\]

where \( v_{\text{sound}} \) is the sound velocity, and where \( Pr = \eta c_p / \lambda \) and \( Ma = v_\phi(r_2) / v_{\text{sound}} \) are the (resp.) Prandtl and...
Thus room temperature $T(r_2) \simeq \bar{T}(1) \times 100$. (27)

Thus room temperature $T(r_2) \simeq 300$ K means $\bar{T}(1) \simeq 3$. However, the above scaling of the dimensionless temperature $\bar{T}$ is not applicable for $v_2 \to 0$. Then we should change $v_2 \to v_1$ in (21, 23) and take instead of (7)

$$v_0(\bar{r}) = v_1 \hat{v}_0(\bar{x}), \quad \hat{v}_0(\bar{x}) = \frac{\bar{x}^{-1} - x_0^{1+\kappa}}{x_0^{-1} - x_0^{1+\kappa}}, \quad x_0 = \frac{r_1}{r_2} \tag{28}$$

3 This relation holds both for the laminar regime and in the fully developed turbulence regime [29, 45]. For the former (latter) we employ molecular (turbulent) values of heat-conductivity and viscosity; see section V.

### Table I: Variables and parameters.

| Variable/Parameter | Defined in Eq. | Description |
|--------------------|---------------|-------------|
| $r_2 > r_1$ | (6) | Radii of the coaxial cylinders |
| $x = r/r_2$ | (7) | Dimensionless radial distance |
| $x_0 = r_1/r_2$ | (28) | Dimensionless ratio of the radii |
| $\phi = v(r)$, $\phi = v_0(r)$ | (1.2) | Angular velocities of the coaxial cylinders |
| $w = x_0 v_0/v_2$ | (21) | Dimensionless radial velocity |
| $\theta$ | (1.2) | Mass density. Under normal conditions for air: $\theta = 1.2$ kg/m$^3$ |
| $\rho$ | (1.2) | Pressure |
| $\rho \theta$ | (10) | Internal energy density |
| $\rho \phi$ | (39, 41) | Internal entropy density |
| $\kappa$ | (4) | Isobaric heat capacity. For air: $\kappa = 1.004 \frac{J}{kg \cdot K}$ |
| $c_p$ | (4) | Isometric heat capacity. For: $c_p = 10^4 \frac{J}{kg \cdot K}$ |
| $\phi$ | (17) | Dimensionless isobaric heat-capacity; $\phi = c_p/(\phi_0 - 1) = c_p/c_v$ is the ratio of isobaric and isochoric heat-capacities |
| $\phi$ | (1.2) | Viscosities. Molecular value of air: $\phi_{mol} = 1.8 \times 10^{-5} \frac{kg}{m \cdot s}$ |
| $\chi = \phi/\lambda$ | (23) | Ratio of viscosities |
| $\beta$ | (3.11) | $|\beta|$ is the analogue of the Reynolds number related to the radial flow of energy |
| $\beta$ | (23, 11) | $|\beta|$ is the analogue of the Reynolds number related to the radial flow of energy |
| $\text{Pr} = \beta/\lambda = \eta c_p/\lambda$ | (26) | Prandtl number |
| $U$ | (34) | Stagnation enthalpy |

### E. First law

Using (21–23), the energy balance (11) can be written in terms of dimensionless, local rates of energy $\bar{E}$, radial work $\bar{W}_r$, angular work $\bar{W}_\phi$ and heat $\bar{Q}$:

$$\bar{E}(\bar{x}) = \frac{c}{\eta \nu_1^2} \left[ \frac{v_2^2(\bar{r}) + v_\phi^2(\bar{r})}{2} + c_p T(\bar{r}) \right]$$

$$= \eta \bar{T} + \frac{\kappa}{2} \left( \frac{\nu_1^2 + \frac{u^2}{x^2}}{x^2} \right), \tag{29}$$

$$\bar{W}_r(\bar{x}) = -\frac{r}{\eta \nu_2^2} v_\phi(\bar{r}) \sigma r(\bar{r}) = \hat{v}_\phi(\bar{x}) \left[ \hat{v}_\phi(\bar{x}) - x \hat{v}_\phi(\bar{x}) \right], \tag{30}$$

$$\bar{W}_\phi(\bar{x}) = -\frac{r}{\eta \nu_2^2} v_r(\bar{r}) \sigma r(\bar{r}) = \frac{2u^2}{x^2} - (\chi + \frac{4}{3} \frac{w u'}{x}), \tag{31}$$

$$\bar{Q}(\bar{x}) = -\frac{r \lambda}{\eta \nu_2^2} T'(\bar{r}) = -x \hat{T}'(\bar{x}), \tag{32}$$

where we employed (21–23) and (6–9).

The first law reads from (11):

$$\Delta \bar{E} + \Delta \bar{W}_r + \Delta \bar{W}_\phi + \Delta \bar{Q} = 0, \tag{33}$$

where $\Delta \bar{E} \equiv \bar{E}(1) - \bar{E}(0)$ etc. Note that $\Delta (\bar{W}_r + \bar{W}_\phi) > 0$ means that the system does work on the external...
sources which immerse the fluid into the system and rotate the cylinders, i.e. the work is extracted $^4$. We stress that this model of cooling is not completely autonomous, since it contains moving boundaries. The condition $\Delta \tilde{W}_{r} + \Delta \tilde{W}_{\phi} \geq 0$ means that cooling (if it shown to exist) is not due to external forces that move boundaries.

Let us also give the dimensionless form of the stagnation enthalpy:

$$\tilde{U} = \frac{1}{\rho} \left( \frac{\dot{v}^2}{2} + \frac{P}{\rho} + \varepsilon \right) = \frac{\dot{v}^2}{2} + \frac{\ddot{w}^2}{2\alpha^2} + b\frac{T}{\kappa},$$

(34)

which differs from (29) by the factor $c/\eta$ only.

**F. Angular work**

The work (30) done by rotating cylinders can be calculated in a closed form from (7, 9, 14):

$$\Delta \tilde{W}_{\phi} = 2(1 - \alpha) - \kappa \alpha - \left[ \frac{1 - \alpha}{x_0} + \alpha x_0^{1 + \kappa} \right] \right] \times$$

$$\left[ \frac{2(1 - \alpha)}{x_0} - \alpha x_0^{1 + \kappa} \right], \quad x_0 \equiv \frac{r_1}{r_2}. \quad (35)$$

Now radially outward flow means $\kappa \geq 0$ or $c \geq 0$; see (23). For this case we checked numerically that $\Delta \tilde{W}_{\phi} \leq 0$, i.e. the cylinders always invest work. In particular, $\Delta \tilde{W}_{\phi} = 2(1 - \alpha)^2(1 - x_0^2) < 0$ for $\kappa = 0$. For inward flow $\kappa < 0$ there are situations, where $\Delta \tilde{W}_{\phi} > 0$, i.e. the work is extracted. Note from (7, 8) that $\kappa < 0$ means the angular velocity $\ddot{w}/r$ is a decreasing function of $r$.

**G. Cooling efficiency**

Any cooling process that is due to a pressure gradient can be usefully compared with the thermodynamic entropy-conserving (adiabatic) process, where the same pressure is employed for cooling. Let $(p_{\text{in}}, T_{\text{in}})$ and $(p_{\text{out}}, T_{\text{out}})$ be, respectively, the input and output pressure and temperature, and cooling $T_{\text{out}} < T_{\text{in}}$ is achieved due to $p_{\text{out}} < p_{\text{in}}$. For the considered ideal-gas model, the lowest temperature $T_{\text{out, ad}}$ reached adiabatically reads [see Appendix A]:

$$\frac{T_{\text{out, ad}}}{T_{\text{in}}} = \left( \frac{p_{\text{out}}}{p_{\text{in}}} \right)^{1/c_p},$$

(36)

where $c_p$ is defined in (16); see also Table I.

Hence one defines the cooling efficiency [41]:

$$\xi = \frac{T_{\text{out, ad}}}{T_{\text{out}}},$$

(37)

which has the standard meaning of efficiency (result over effort), since the achieved result of cooling is related with $1/T_{\text{out}}$. The pressure difference is a resource and it is quantified by $1/T_{\text{out, ad}}$, hence definition (37) $^5$.

When quantifying cooling, people sometimes employ the Hilsch efficiency [2, 4]:

$$\xi_H = \frac{T_{\text{in}} - T_{\text{out}}}{T_{\text{in}} - T_{\text{out, ad}}},$$

(38)

The meaning of $\xi_H$ differs from that of $\xi$, because $\xi_H$ directly accounts for the input temperature $T_{\text{in}}$. But they are related. As shown by (37), for $T_{\text{in}} - T_{\text{out, ad}} > 0$ (a natural condition for cooling), $\xi < 1$ ($\xi > 1$) implies $\xi_H < 1$ ($\xi_H > 1$). However, $\xi_H$ is less fundamental than $\xi$, since it does not appear directly in the efficiency bound imposed by the second law. We discuss this bound now.

**H. Second law bound for cooling efficiency**

The entropy balance of the fluid reads [29]

$$\partial_t (\rho s) = -\nabla [\rho \dot{w} \nabla T] + s_{\text{prod}}, \quad (39)$$

where $\rho s$ is the entropy density, $\rho \dot{w} \nabla T$ and $-\nabla [\rho \dot{w} \nabla T]$ are, respectively, advective and thermal entropy flux. The entropy production $s_{\text{prod}} > 0$ is positive due to viscosity and heat conduction $^6$. In the stationary situation $\partial_t (\rho s) = 0$, and (39) reads:

$$\frac{d}{dr} \left( c_s \frac{\lambda_r dT}{T} \right) = r s_{\text{prod}},$$

(40)

where we used (4). The ideal-gas entropy is [cf. Appendix A]:

$$s = c_p \left( \frac{1}{c_p} \ln[p] + \ln[T] - \ln \left( \frac{M}{R} \right) \right),$$

(41)

where we employed (15–17).

We consider two particular cases of the adiabatic process (36):

$$r_{\text{in}} = r_2, \quad r_{\text{out}} = r_1, \quad c < 0, \quad (42)$$

$$r_{\text{in}} = r_1, \quad r_{\text{out}} = r_2, \quad c > 0. \quad (43)$$

$^4$ Note from (10) that the integral $\int_{\partial V} d\dot{s} \vec{\mu}$, over a closed surface $\partial V$ is the work done by viscosity forces on the substance enclosed into $\partial V$. The sign of the work is determined as follows: with the normal vector $\vec{s}$ of $\partial V$ pointing outside, the integral contributes into $-\partial_t \int_{\partial V} dV (\dot{\gamma}^2 + \rho c)$; see (10). Hence a positive $\int_{\partial V} d\dot{s} \vec{\mu}$ means that the fluid in $V$ does work on external sources.

$^5$ Eq. (37) is different from the coefficient of performance (COP) of refrigerators, which is defined via the ratio of the heat transferred in refrigeration over the external work performed to achieve this transfer. We do not need the COP, since in our set-ups the cooling is not achieved due to external work.

$^6$ The entropy production reads [29]:

$$s_{\text{prod}} = \frac{\dot{w}}{c_T} \left( \frac{\partial s}{\partial \sigma_T} + \frac{\partial s}{\partial \sigma_j} - \frac{2 \sqrt{s} \nabla \dot{\gamma}^2}{\sigma_T} + \frac{s}{2} \left( \frac{\nabla \dot{\gamma}}{\sigma_T} + \frac{\dot{\gamma}^2}{\sigma_T} \right) \right) > 0.$$
Now (37, 40, 42, 43) imply:

$$s(r_2) - s(r_1) = \text{sign}[c] c_p \ln|\xi|.$$  \hfill (44)

Integrating (40) over $r$ for $r_1 < r < r_2$, using $s_{\text{prod}} > 0$, (41), (36) and (44), we get from (42, 43) an upper bound for the efficiency (37) that applies to both (42) and (43):

$$|b| \ln|\xi| \leq \frac{r_1}{T(r_1)} \frac{dT(r_1)}{dr} - \frac{r_2}{T(r_2)} \frac{dT(r_2)}{dr},$$  \hfill (45)

where $|b|$ is the Peclet number; see (22) and Table I. The right-hand-side of (45) is non-zero due to heat conduction. Hence if the heat-conduction is neglected (i.e. $\lambda = 0$) the cooling efficiency holds $\xi < 1$; see (45). We stress again that the inequality in (45) is due to positivity of the entropy production: $s_{\text{prod}} > 0$.

III. BOUNDARY CONDITIONS FOR COOLING AND FOR PERMEABLE WALLS

When studying cooling due to a confined gasodynamic flow one should exclude physically uninteresting cases, where the fluid is cooled due to cold thermal baths attached to boundaries or due to external work done by external forces.

Cooling demands that the temperature of the (radially) incoming fluid is larger than the temperature of the outgoing fluid. If there is a low-temperature boundary bath, the flow should be thermally isolated from it (adiabatic cooling). Whenever the radial flow is absent, thermal isolation is ensured by imposing vanishing heat flux at boundaries, e.g. at the outer boundary:

$$\frac{dT(r_2)}{dr} = 0.$$  \hfill (46)

If there is a flow through boundaries (i.e. permeable or porous walls), (46) does not hold, because there is a heat conductivity due to the fluid at the boundary.

In addition to known conditions for continuity of temperature and heat flux [29], there is now a specific condition to be satisfied on the adiabatic, permeable surface. To understand the origin of this condition, let us “decompose” the macroscopically homogeneous, permeable adiabatic outer surface into holes and solid parts. Recall that $(r, \phi, z)$ are the cylindrical coordinates. Now for $(\phi, z) \in \text{hole}$ and a small positive $\delta$, we get that $\frac{d}{dr}T(r_2 - \delta; \phi, z)$ stays finite for $\delta \to 0+$. For $(\phi, z) \in \text{solid}$, $\frac{d}{dr}T(r_2 - \delta; \phi, z)$ goes to zero for $\delta \to 0$. Hence $|\frac{d}{dr}T(r_2 - \delta)| > |\frac{d}{dr}T(r_2)|$ after averaging over $(\phi, z)$ that recovers the macroscopically homogeneous permeable wall. Hence instead of (46) we obtain the following boundary condition

$$\frac{d(T(r_2) - \delta)}{dr} > \frac{dT(r_2)}{dr}, \quad \text{sign} \left( \frac{dT(r_2)}{dr} \right) \frac{d^2T(r_2)}{dr^2} < 0,$$  \hfill (47)

where $\text{sign}[a] = 1$ if $a \geq 0$ and $\text{sign}[a] = -1$ if $a < 0$, and where the second inequality in (47) follows from the first one under $\frac{d}{dr}T(r_2) \neq 0$ and $\delta \to 0+$. Naturally, the first inequality in (47) also holds for $\frac{dT}{dr} = 0$, i.e. for an adiabatic wall.

Likewise, we have for the thermally isolated inner wall (for $\delta \to 0+$):

$$\left| \frac{dT(r_1 + \delta)}{dr} - \frac{dT(r_1)}{dr} \right| \geq \frac{d^2T(r_1)}{dr^2} > 0,$$  \hfill (48)

We stress that in the present model (with or without the radial flow) it is trivial to get arbitrary low temperatures in between of two cylinders. But generally these temperature profiles do not hold the boundary conditions (47) or (48), i.e. such scenarios of low temperatures do not constitute proper cooling, since they require that low temperatures pre-exist via boundary baths. In particular, Appendix B works out the Couette flow (laminar flow between 2 infinite rotating cylinders without radial motion) showing that the inhomogeneous temperature profile generated in this flow does not constitute cooling.

Conditions similar to (47, 48) are deduced for the radial velocity $v_r$ on a partially permeable wall. This is similar to the previous case in that $v_r = 0$ for an impermeable wall; see (46). A derivation analogous to that of (47, 48) produces [cf. (4)]

$$\text{sign}[c] \frac{dv_r(r_2)}{dr} < 0,$$  \hfill (49)
$$\text{sign}[c] \frac{dv_r(r_1)}{dr} > 0,$$  \hfill (50)

for the outer and inner wall, respectively.

In the present model there are no solutions that support conditions (47, 48) and (49, 50) for both inner and outer permeable walls. Thus we should put them on the wall from which the cold flow is coming out (to ensure that low temperatures do not exist before cooling), and left the other boundary as a control surface assuming that both the velocity and temperature on this surface are given.

IV. COOLING OF INWARD FLOW

A. Temperature profile for a weak radial flow

Recall from (21) and (9) that $\kappa$ and $w(x)$ are different dimensionless quantities although both are non-zero due to radial flow. We assume that $w(x)$ and its derivatives are small. Hence factors $(\frac{\kappa}{x} + 2)^{\frac{w}{x^2}}$ and $(\chi + \frac{\kappa}{x})^{\frac{w}{x^2}}$ are neglected in (18). This can be done provided that $x$ is not very small. But the influence of the radial flow on the vortex characteristics is not neglected, i.e. $\kappa \neq 0$; see (6–9). Now the remainder of (18), i.e. $(\frac{\kappa}{x} + 2)^{\frac{w}{x^2}}$
\[ x \hat{v}_\phi \hat{\nabla}_\phi + b \dot{T} - x \dot{T}' - \beta = 0 \] can be solved explicitly as
\[ \dot{T}(x) = g(x) + \frac{\beta}{b} + x^b C, \quad (51) \]
where \( C \) is a constant, and where
\[ g(x) \equiv \frac{2x^2 \alpha (\alpha - 1)}{b - \kappa} - \frac{(1 - \alpha)^2 (4 + \kappa)}{2(2 + b)x^2} - \frac{\kappa \alpha^2 x^{2+2\kappa}}{2(2 - b + 2\kappa)}. \quad (52) \]

Now \( \beta \) and \( C \) in (51) are conveniently expressed via \( \dot{T}'(1) \) and \( \dot{T}'(1) \), and the temperature profile reads from (51):
\[ \dot{T}(x) - \dot{T}(1) = \frac{x^b - 1}{b} \left[ \dot{T}'(1) - g'(1) \right] + g(x) - g(1), \quad (53) \]
The approximation that led to (53, 52) is confirmed by solving numerically full equations (18, 19); see Figs. 2 and 3.

So far the weak radial flow approximation amounted to neglecting the radial velocity \( v_r \) in the energy equation (18), but retaining it in the angular Navier-Stokes equation (2); see also (6–9). If we neglect the radial velocity \( v_r \), also in the radial Navier-Stokes equation (1) [or equivalently in (19)] we obtain
\[ \rho(r) v_\phi^2(r)/r = dp/dr. \quad (54) \]
This known equation is solved as
\[ \frac{p(x)}{p(1)} = \exp \left[ -\frac{\kappa \hat{\rho}}{b} \int_x^1 dy \frac{\hat{v}_\phi^2(y)}{y \dot{T}(y)} \right], \quad (55) \]
where \( \dot{T}(x) \) and \( \hat{v}_\phi(x) \) are given by (53, 52) and (7, 8), respectively. Eq. (55) shows that the (dimensionless) pressure is a monotonically increasing function of \( x \).

There are cases, where (53, 52) are valid, but (54) [and (55)] is not. In section VI we show an important example of this type, where even if \( w(x) \to 0 \) is imposed in the vicinity of \( x = 1 \), it does not hold for \( x < 1 \), because \( w(x) \) grows fast.

7 Let us mention the simplest (but incorrect) argument for the Ranque effect. Setting \( \rho(r) = \) constant in (54) and using the ideal gas law \( T(r) \propto \rho(r) \), shows that \( T(r) \) is an increasing function of \( r \) (i.e. a radial temperature separation is achieved), formally resembling the Ranque effect. The problem with this argument is that imposes a constant \( \rho \). This may look formally consistent with other equations, but it is incorrect, e.g. because it applies also for \( v_r = 0 \) (no radial motion whatsoever), while our detailed analysis of this \( v_r = 0 \) situation shows that no cooling scenarios are possible, because the proper boundary conditions are not satisfied; see Appendix B.

B. Boundary conditions for inward radial flow

For inward flow \( c \leq 0 \) (hence \( b \leq 0 \) and \( \kappa \leq 0 \)) we study cooling scenarios, where the higher temperature fluid enters into the system through the outer boundary at \( x = r/r_2 = 1 \). Now for adiabatic boundary conditions, the lower temperature fluid leaves the system through the inner thermally isolated boundary at \( x = x_0 < 1 \) [cf. (48)]:
\[ \dot{T}(1) > \dot{T}(x_0), \quad \dot{T}'(x_0) > 0. \quad (56) \]

No specific conditions are put at \( x = 1 \), i.e. it is taken as a control surface.

The adiabatic boundary condition (56) relates to the isothermal situation \([x_0 \leq x \leq 1]\)
\[ \dot{T}(1) = \dot{T}(x_0) > \dot{T}(x), \quad (57) \]
where \( \dot{T}(x) \) assumes a minimum at some \( x = x_{\text{min}} \). Whenever (57) holds, one can take \( x_0 \geq x_{\text{min}} \) and this produce an example of (56).

Eq. (57) does not refer to a practically useful situation, since no cold fluid really comes out. Nevertheless, it is interesting, since the expected of behavior of the temperature is that it is larger inside of the fluid, i.e. for \( \dot{T}(1) = \dot{T}(x_0) \) we expect \( \dot{T}(x) > \dot{T}(1) = \dot{T}(x_0) \) [29]. (The expectation is also confirmed by the example of the Couette flow in Appendix B.) This is because viscosity—which dissipates energy in the bulk of the fluid—generates heat that must be transported out of the boundaries [29]. The expected behavior holds for \( c > 0 \). But there are isothermal and adiabatic cooling scenarios for \( c < 0 \); see Figs. 2 and 3. We now turn to discussing them.

C. Cooling via quasi-solid vortex

Let us start with a quasi-solid vortex in (7, 8):
\[ \alpha = 1 \quad \text{or} \quad v_1/v_2 = (r_1/r_2)^{1+\kappa}. \quad (58) \]
The temperature for this situation reads from (53, 52):
\[ \dot{T}(x) - \dot{T}(1) = \frac{[\dot{T}'(1) + \frac{\kappa}{b}][x^b - 1]}{b} + \frac{\kappa(x^b - x^{2+2\kappa})}{2(2 + 2\kappa) - b}. \quad (59) \]
Let us see to which extent (59) can hold condition (57).

Now \( \dot{T}'(x_{\text{min}}) = 0 \) leads to
\[ x_{\text{min}}^{2-b+2\kappa} = 1 + \frac{(2 - b + 2\kappa)\dot{T}'(1)}{\kappa(1 + \kappa)}, \quad (60) \]
\[ \dot{T}'(x_{\text{min}}) = -\kappa(1 + \kappa)x_{\text{min}}^{2\kappa}. \quad (61) \]

Eq. (60) means that \( \dot{T}'(x_{\text{min}}) = 0 \) has only one solution. Since this solution ought to be a minimum [cf. (57)], we
have to require \( \hat{T}'(1) > 0 \) that together with \( 0 \leq x \equiv \frac{\kappa}{2} < 1 \) leads from (60, 61) to \( \kappa(1 + \kappa) < 0 \) and \( 2(1 + \kappa) > b \), or equivalently to

\[
-1 < \kappa < 0, \quad 0 < \hat{T}'(1) < -\frac{\kappa(1 + \kappa)}{2(1 + \kappa) - b}. \tag{62}
\]

Thus under conditions (62)—and naturally \( x_0 \) sufficiently smaller than \( x_{\text{min}} \)—we get an isothermal cooling scenario (57). Taking \( x_0 \gtrsim x_{\text{min}} \) we get instead an example of the adiabatic scenario (56); cf. the discussion after (57).

Eq. (58, 62) imply a quasi-solid vortex that is frequently observed experimentally. Examples of the above cooling scenario are presented in Figs. 2 and 3 for isothermal and adiabatic scenarios, respectively. Naturally, the cooling takes place both in terms of (thermodynamic) temperature \( \hat{T} \) and stagnation enthalpy \( \hat{U} \). We shall see below that conditions (58, 62) are sufficiently representative, i.e. more general cooling scenarios implied from (53) are close to those predicted by (58, 62).

D. Magnitude and efficiency of cooling

Both adiabatic and isothermal cooling scenarios lead to relatively weak effects in the sense of

\[
\frac{\hat{T}(1) - \hat{T}(x_{\text{min}})}{\text{min}[1, T(1)]} \approx 0.01 - 0.1. \tag{63}
\]

The cooling magnitude in terms of stagnation enthalpy is larger; see Figs. 2–4.

The efficiency (37) of cooling under condition (56) is smaller than 1:

\[
\xi < 1. \tag{64}
\]

Whenever \( \frac{d}{dr} T(r_1) \) is sufficiently small, (64) follows directly from the second law bound (45), where \( \frac{d}{dr} T(r_2) > 0 \); cf. (42). Otherwise, (64) is confirmed numerically; see Figs. 2–4. Hence the Hilsch efficiency (38) also holds \( \xi_H < 1 \), as shown by Figs. 2–4.

For both isothermal and adiabatic cooling scenarios we obtain from (44, 64) for the entropy difference:

\[
s(r_2) - s(r_1) = c_p \ln[\xi] < 0. \tag{65}
\]

Hence the final entropy is always larger than the initial one: \( s(r_1) > s(r_2) \).

E. Work and energetics

As shown by (35, 58), the work done by rotating cylinders is positive under conditions (62):

\[
\Delta \hat{W}_\phi = -\kappa(1 - x_0^{2+2\kappa}) > 0, \tag{66}
\]

which means that the work is extracted. The work \( \hat{W}_r \) done by radial external forces is small, but negative (i.e. it is invested), and the overall work is positive; see Figs. 2 and 3. Thus the set-up does not demand an external investment of work \(^8\); cooling takes place due to the initial pressure larger than the final one, \( p(1) > p(x_0) \). In other words, cooling takes place due to the initial potential energy of the fluid.

Under isothermal boundary conditions both thermal baths (at \( x = 1 \) and \( x = x_0 \), respectively) provide heat to the system. Using (57, 58) (and the fact that \( w(x) \) is assumed to be small), we get from (29, 33)

\[
\Delta \hat{E} = \Delta \hat{E}_\text{kin} = \frac{\kappa}{2}(1 - \hat{v}_0^2(x_0)), \tag{67}
\]

which is negative due to (58). Hence we also get cooling in terms of the stagnation enthalpy; see (34, 29).

Eqs. (66, 67) are consistent with the first law (33), which states (whenever the latter holds one can obtain an adiabatic cooling scenario that does not reduce to the isothermal case

\[
\alpha \text{ and } -\hat{v}_0(x_0), \text{ are assumed to be small}), \text{ we get from (29, 33)}
\]

\[
\Delta \hat{E} = \Delta \hat{E}_\text{kin} = \frac{\kappa}{2}(1 - \hat{v}_0^2(x_0)), \tag{67}
\]

which is negative due to (58). Hence we also get cooling in terms of the stagnation enthalpy; see (34, 29).

The features hold for other cases of isothermal and adiabatic cooling. Fig. 4 shows an isothermal scenario with \( \alpha = -0.5 \), where \( \hat{v}_0(x) \) is again a concave, increasing function of \( x \). Fig. 3 demonstrates an adiabatic cooling scenario that does not reduce to the isothermal case (whenever the latter holds one can obtain an adiabatic scenario by taking \( x_0 > x_{\text{min}} \)).

Let us write from (11, 29–32)

\[
0 = \left[-\hat{E}(x) + x \hat{T}'(x) - \hat{W}_\phi(x) - \hat{W}_r(x) \right]' \tag{68}
\]

In (68) we assume that isothermal boundary conditions hold for \( c < 0 \); hence \( b \) and \( \kappa \) are negative. Now \( \left[-\hat{E}(x)' \right] > 0 \) for \( x_{\text{min}} \lesssim x \), because this means cooling in terms of the stagnation enthalpy. One also has \( \left[x \hat{T}'(x)' \right] > 0 \) for \( x_{\text{min}} \approx x \). It appears also that \( \left[-\hat{W}_r(x)' \right] > 0 \) has the same sign as \( \left[-\hat{E}(x)' \right] > 0 \). Moreover, it quickly prevails over other factors; this is why for \( c < 0 \) cooling exists only in the weak radial flow situation, where \( \hat{W}_r \rightarrow 0 \).

Thus for holding (68) and achieving cooling we need

\[
\hat{W}_\phi(x) > 0, \tag{69}
\]

which—using (30, 7, 8)—is equivalent to

\[
0 > 4(1 - \alpha)^2 + x^{2+\kappa}(1 - \alpha)\kappa(\kappa - 2) \tag{70}
\]

\[
+ 2x^{4+2\kappa}\kappa^2(\kappa + 1).
\]

---

8 We mention another scenario of cooling, which is realized for the inward flow and the potential vortex \( \hat{v}_\phi(x) = \frac{1}{x}; \text{ see (8)} \) with \( \alpha = 0 \). This scenario is less interesting, since it is driven by external investment of work \( \hat{W}_\phi' \), as seen from (69, 70), while its efficiency and magnitude hold the same constraints (64, 63). An interesting point of this scenario is that it is accompanied by the kinetic energy that increases in the direction of the flow: \( \alpha \hat{v}_\phi^2 + a \hat{T}(x) \leq 0 \) in (68). Appendix D studies details of this scenario.
Hence the validity of (69) (at least for certain values of \( x \)), i.e. the positivity of work, is a necessary condition for both isothermal and adiabatic cooling in the regime \( c < 0 \). This condition was obtained in [26], but its necessary character was not properly stressed, in particular, because the heat conductivity and boundary conditions necessary for cooling were neglected. In particular, (69) can lead to mistakes if it is taken as a sufficient condition for cooling.

\[ T_0(x) - T_0(1) = \frac{x^b - 1}{b} \left[ \tilde{T}_0^2(1) - \epsilon g'(1) \right] + \epsilon \left( g(x) - g(1) \right), \quad \epsilon \equiv \frac{\nu_2^2}{\nu_{2,0}^2}, \quad \text{(71)} \]

where \( g(x) \) is still defined by (52), i.e. it does not have any parametric dependence on \( \nu_2 \) or on \( \epsilon \). According to (71), a larger \( \epsilon \) means a bigger deviation of \( \nu_2 \) from its reference values.

Fig. 6 shows temperature profiles (71) for different values of \( \epsilon \) and for parameters given by (58), (62). For given values of parameters, there is a value of \( \nu_2 \), where the cooling effect is maximal, i.e. the lowest temperature is reached. For the parameters of Fig. 6 this critical value is found from \( \epsilon = 0.8 \). When \( \nu_2 \) decreases from this critical value, the cooling effect ceases to exist, since the cooling boundary conditions—as given by (56) or (57)—cannot hold anymore, i.e. the curve in Fig. 6 that corresponds to \( \epsilon = 0.78 \) is void of physical meaning. When \( \nu_2 \) increases from the critical value, the magnitude of vortex cooling—as measured by the lowest temperature reached—monotonously decreases. In particular, for a larger \( \epsilon \), we need to take a larger \( x_0 = r_1/r_2 \) (i.e. \( x_0 \rightarrow 1 \)) to achieve cooling; see Fig. 6.

V. RELATIONS WITH EXPERIMENTS

A. Effective viscosity

The actual flow in vortex tubes is highly turbulent [4]. Hence if one uses hydrodynamic equations (in particular, Navier-Stockes equations) with constant values of viscosities \( \eta \) and \( \zeta \) and heat-conductivity \( \lambda \), it is at very least necessary to employ there effective (i.e. turbulent or nonmolecular) estimates for these parameters [29, 45].

Taking into account that the considered flow is confined and inhomogeneous (i.e. there are radial and angular flows) we choose to estimate the turbulent viscosity \( \eta \) via the Nusselt’s formula [44], which was originally proposed for estimating the turbulent viscosity in pipes. Ref. [45] found that this formula applies for describing compressible turbulence in a sufficiently wide range of Reynolds numbers. The formula reads [44, 45]:

\[ \eta = 0.15 \eta_{\text{mol}} \left[ \rho \nu_0 l / \eta_{\text{mol}} \right]^{3/4}, \quad \text{(72)} \]

where \( \nu_0 \) is the characteristic value of the angular velocity, and \( l \) is the characteristic length, and \( \eta_{\text{mol}} \) is the molecular viscosity: \( \eta_{\text{mol}} = 1.8 \times 10^{-5} \text{ kg/(m/s)} \) for the air. Also, taking in (72) typical experimental parameters for vortex tubes [27]: \( \rho = 1.2 \text{ kg/m}^3, \nu_0 \approx \nu_{\text{sound}} = 331 \text{ m/s} \) and \( l = 0.1 \text{ m} \), we get

\[ \eta \sim 0.085 \text{ kg/(m/s)}, \quad \text{(73)} \]

which is several orders of magnitude larger than \( \eta_{\text{mol}} \). The estimate (73) roughly coincides with an estimate
given in [22] via the mixing-length formula [45]: \( \eta = \rho \ell^2 \frac{\partial v}{\partial r} \), where \( \ell = \frac{\beta \varepsilon}{(1 + \kappa)^2} \) is the mixing length and where \( \beta \) is a suitable numerical constant. It is this specific form of the mixing-length formula that can apply to compressible turbulence [45].

B. Comparison with experiment

Let us recall that the present model omits several physical factors that are met in realistic vortex tubes; see section I for a discussion of basic set-ups for vortex tubes. In particular, the axial motion is neglected, and the fluid is removed radially (in contrast to axial removal in vortex tubes). Hence the model set-up has one (not two) output temperatures, and the whole output is fluid is cooled (in contrast to the standard Ranque tube which has two output flows and achieves temperature separation [4]).

Recall relation (27) between dimensional and dimensionless velocities of air obtained for sonic velocities. Then the magnitude of cooling is predicted from (63) to be around 10 K. Note that best vortex tubes provide (starting from 300 K) a larger cooling of order 70-80 K [2–4]. Though such a stronger effect is lacking in the present model, we recall that in those cases only a part (e.g. \( \sim 20\% \) according to [2]) part of the overall flow is strongly cooled, the remaining part is heated up. In the present model the whole outgoing fluid is cooled.

The Hilsch efficiency \( \xi_H \) of cooling obtained in the present model is of order of 0.1 – 0.4; see Figs. 2–4. It also agrees with experiments, though not for the most efficient vortex tubes, where \( \xi_H \) can be as high 0.6 – 0.7 [2, 4].

The magnitude of the radial flow over the angular flow, expressed by

\[
w(1) = 10^{-3} - 10^{-4},
\]

also agrees with experimental measurements [4, 5], though it is to be stressed that these measurements were carried out for sufficiently long vortex tubes, where the axial velocities (neglected altogether in the present model) are definitely larger than the radial velocities. The quasi-solid vortex (58) for \( \bar{v}_\phi(x) \) is also seen experimentally, though the experimental results also indicate that for \( x \sim 1 \), \( \bar{v}_\phi(x) \) starts to decay, i.e. the quasi-solid vortex \( \bar{v}_\phi(x) \sim x^{1+\kappa} \) (for our model we took \( \kappa \sim 0.5 \)) changes towards the potential vortex \( \bar{v}_\phi(x) \sim x^{-1} \) [4, 5]. This change of \( \bar{v}_\phi(x) \) is given much importance in certain theories of vortex cooling [3, 4], but is not present here.

The input density is estimated from (4, 21):

\[
\rho(r_2) = \frac{|\kappa| \eta}{w(1) r_2 v_\phi(r_2)}. \tag{75}
\]

Estimating \( r_2 \simeq 0.1 \) m (reasonable value for the outer radius of a vortex tube), \( v_\phi(r_2) \simeq v_{\text{sound}} \) and \( \eta \) for air as \( \eta \simeq 0.085 \) kg/(m s) [see 73], we end up with \( \rho(r_2) \simeq 10^{-3} \) kg/(m s) in (75). Estimating from the present model \( w(1) \simeq 10^{-4} \) (and recalling \( p = R\rho T/\mu \)) we get that the input pressure is few times larger than the atmospheric pressure [1].

Altogether, given limitations of the present model, and complications of the flow in real vortex tubes, one can say that the model is in a fair qualitative agreement with experiments, though it is far from predicting (and explaining) the features of best vortex tubes, those providing the largest efficiency or the largest magnitude of cooling.

Savino and Ragsdal presented a simplified set-up of vortex cooling effect [27] that in several respects is similar to the present model. They studied two short (compared to the diameters) concentric cylinders; the length to diameter ratio was 0.1 and 0.5 for two different samples. (For traditional Ranque-Hilsch tubes the length to diameter ratio is 20–50). The rotating air enters radially from the whole outer permeable cylinder and leaves through the inner (smaller) cylinder. Rotational flow was created via the outer cylinder with the Mach number \( \approx 0.2 \). The velocity of this flow was much larger than that of the radial flow. The authors established here cooling effect in terms of the radial variation of the stagnation enthalpy \( ^9 \) (no data on thermodynamic temperature or velocities was given). The magnitude of this cooling effect is lower than predictions of the present model; cf. the stagnation enthalpy data in Figs. 2–3. They confirmed that the radial distribution of the stagnation enthalpy is established already near the end-wall of the tube and is not affected by the weak axial flow. In particular, the axial change of the stagnation enthalpy was much smaller than the radial one. (Hence it was legitimate to neglect the axial flow in the model.) They found that the experimental data can be described by (54), where pressure is balanced by the centrifugal force (this equation does not contain the viscosity explicitly). It was observed that the pressure decreases monotonically with the radius, as confirmed by (55) of the present model.

VI. COOLING OF OUTWARD FLOW

A. Conditions for cooling

Now we assume that \( c > 0 \) (i.e. \( \kappa > 0 \) and \( b > 0 \)) and the outer boundary of the system is thermally isolated in the sense of (47). Hence the outgoing fluid being colder than in-coming implies (for \( c > 0 \)) \( \bar{T}(x) < 0 \), and then (47) demands

\[
\bar{T}''(1) > 0, \tag{76}
\]

\( ^9 \) Recall that (due to the Bernoulli’s theorem) cooling in terms of the stagnation enthalpy cannot be explained via adiabatic fluid dynamics.
The full expression for $\dot{T}'(1)$ is worked out from (18, 19) [recall (25)]:

$$\dot{T}'(1) = \frac{\dot{W}(1)w(1)}{c_p w(1)} + \left[\frac{b(c_p - 1)}{c_p} - 1\right] \dot{T}'(1)$$

$$-(\chi + \frac{1}{3})\frac{w'(1)^2}{2} - (2w(1) - w'(1))^2 - (\alpha(2 + \kappa) - 2)^2.$$  

(77)

The first line (77) contains potentially positive terms, while all terms in (78) are non-positive. Hence (76) demands that (78) is sufficiently small, e.g. via $w'(1) \to 0$. Likewise, $\dot{T}'(1)$ cannot be very small.

If $w(x)$ and $w'(x)$ are sufficiently small, $\dot{T}(x)$ can be approximately determined from (53, 52). However, (54, 55) do not apply anymore, because the dimensionless pressure $\dot{p}(x) = \dot{T}'(x)/w(x)$ is now a decreasing function of $x$ for $x \in [x_0, 1]$. Thus, $w(x)$ is now essential in (19) and it is important for determining the energetics. The physical reason for this is that the work $\Delta \dot{W}$ [see (31)] done by viscous radial forces is relevant, as seen below.

Fig. 7 demonstrates the main outward-flow cooling scenario for $\alpha = 1$; cf. (8). The magnitude of cooling is now sizable

$$[\dot{T}(x_0) - \dot{T}'(1)]/\dot{T}(x_0) \geq 10.$$  

(79)

The temperature profile $\dot{T}(x)$ shown in Fig. 7 with a very good precision coincides with that found from (59, 60), where now $\dot{T}'(1) < 0$ and $x_{\text{min}}$ in (60) should be changed to $x_{\text{max}}$, because this is now the maximum of $\dot{T}(x)$. Then one should take $x_0 > x_{\text{max}}$ so that the temperature decreases for $x_0 < x < 1$.

B. Energetics, entropy and efficiency

We expectedly have

$$\dot{Q}(x_0) = -x_0 \dot{\dot{T}}'(x_0) > 0, \quad \dot{Q}(1) = -\dot{T}'(1) > 0,$$

(80)
i.e. the heat enters from the inner boundary $x = x_0$ and leaves at the outer boundary $x = 1$ due to the heat present at this boundary (but not via to the outer wall itself, which is thermally isolated due to $\dot{T}'(1) > 0$; see (48)). We see numerically that $Q_1 \lesssim Q_2$; see Fig. 7.

Now $\Delta \dot{W}_\phi < 0$ (rotating cylinders invest work), as we discussed after (30). But the radial external forces do extract work, $\Delta \dot{W}_r > 0$ as much that the total work is extracted [see Figs. 7, 8]:

$$\Delta \dot{W} = \Delta \dot{W}_r + \Delta \dot{W}_\phi > 0.$$  

(81)

Moreover, the overall kinetic energy (see (29)) also increases, $\Delta E_{\text{kin}} > 0$ (albeit slightly, as seen in Fig. 7) due to contribution $\frac{1}{2}(1 - x_0^{2+2\kappa})$ of the vortex.

The most interesting aspect of this cooling scenario is that the cooling efficiency (37) is larger than 1 [see Fig. 7]:

$$\xi \geq 1,$$  

(82)
i.e. the adiabatic process provides less cooling, since now the heat transfer from the system is essential; see after (45). Together with (82) also the Hilsch efficiency (38) is larger than 1: $\xi_H > 1$; see Fig. 7.

Eq. (82) is thermodynamically consistent. Recalling (39, 43), we see that the upper bound (45) amounts to $x_0 \dot{r}'(x_0) - \dot{T}'(1)$: This expression is positive; cf. (80).

Hence it is possible to have (82) provided that the entropy production is sufficiently small, which appears to be the case, as confirmed numerically. We stress again that $\xi > 1$ is possible due to heat conductivity; cf. the discussion after (45).

Due to (39) and (82), the entropy entering to the system is larger than the one that leaves it [cf. (65)]:

$$s(r_2) - s(r_1) = c_p \ln[1/\xi].$$  

(83)

Thus we get that (without any investment of overall external work) the temperature, stagnation enthalpy and entropy decrease, while the kinetic energy increases. The outgoing fluid is more ordered, since not only its thermal energy decreases, but also the kinetic energy increases.

C. Physical mechanisms of the effect and cooling without vortical motion

We saw around (68, 69) that for the inward flow cooling it is necessary to have an angular motion with the viscous forces doing work of the proper sign. Here the physical meaning of cooling can be clarified along the same lines. We get from (18) and (29–32)

$$0 = \left[\dot{E}(x) + \dot{Q}(x) + \dot{W}_{r}(x) + \dot{W}_\phi(x)\right].$$  

(84)

Now cooling implies that outgoing fluid has lower energy: $\dot{E}'(x) < 0$. Possible necessary conditions for cooling is provided by the heat conductivity: $\dot{Q}'(x) > 0$, and/or by the work done via radial viscosity: $\dot{W}_{r}'(x) > 0$. Figs. 7 and 8 show that both these conditions hold. The contribution of $\dot{Q}'(x) > 0$ is larger than that of $\dot{W}_{r}'(x) > 0$.

But the vortex contribution has the same sign as energy: $\dot{W}_\phi'(x) < 0$; cf. with (69). Hence a similar cooling scenario is also possible without vortex, i.e. for $\psi = 0$ in (18, 19). In fact, eliminating the angular motion almost does not change the temperature profile in Fig. 7. The main difference with the above situation is that the kinetic energy decreases, $\Delta E_{\text{kin}} < 0$, since the vortex motion is now absent.

Note that to eliminate the vortex from equations of motion, one should take $\psi = 0$ in (18, 19), and to suppress the last factor $-\alpha(2 + \kappa) + 2 + \frac{3}{3}$ in (25), as well as...
\[-(a(2 + \kappa) - 2)^2\] in (78). Definitions (21, 22, 23) of dimensionless parameters still apply, where now \(v_2 = \nu_{2.0}\) is an arbitrary characteristic velocity; see section IVG. It drops out from (18, 19).

A simple analytical description of the temperature profile can be obtained from (18) assuming that the change of \(w(x)\) can be neglected. (But note that the change of \(w(x)\) within (19) cannot be neglected.) Taking in (18) \(\dot{u}_o = 0\), \(w(x) = w(1)\) and \(w'(x) \to 0\), we get

\[
\hat{T}(x) - \hat{T}(1) = \gamma[1 - \frac{1}{x^2}] + [2\gamma - \hat{T}'(1)] \frac{1 - x^b}{b}, \quad \text{(85)}
\]

\[
\gamma = \frac{w(1)^2(4 + \kappa)}{2(2 + b)}. \quad \text{(86)}
\]

Now \(\hat{T}'(x) = 0\) is solved as \(x_{b, max}^{b+2} = \frac{2\gamma}{2\gamma - \hat{T}'(1)}\). This solution exists only for \(\hat{T}'(1) < 0\) (recall that \(b > 0\)) and it is a maximum of \(\hat{T}(x)\). Hence one should take \(x_0 > x_{b, max}\) in order to get a monotonic decrease of temperature \(\hat{T}(x)\) from \(x = x_0\) till \(x = 1\).

One feature of this cooling scenario is that once \(w(1)\) (the boundary condition for the outward flow) decreases, the solution of hydrodynamic equations (18, 19) ceases to exist below a certain critical value of \(w(1)\), because now \(w(x_0)\) becomes negative. Recall from (4) that a negative \(w(x_0)\) for \(c > 0\) is not acceptable, since it would mean a negative mass density \(\rho\). Thus, if \(w(1)\) is decreased, then \(c\) also has to decrease, to keep the solution physical.

Generally, the magnitude (79) of the cooling increases upon decreasing the radial flow \(c\), i.e. for \(c \to 0\). This can be seen from (85) or from Figs. 8. But this limit \(c \to 0\) is not useful, since it diminishes the cooling power \(b(\hat{T}(x_0) - \hat{T}(1))\).

D. Geometric aspects of the cooling effect

Note that the present cooling scenario (without vortex) is specific for the cylindrical geometry. Its traces are seen in 1d (plane geometry), but the cooling as such is negligibly weak there; see Appendix C.

To understand the origin of this effect, let us take the situation, where all the velocities vanish (hence \(c = 0\)). Due to the cylindrical geometry, (11) there is still a non-trivial stationary temperatures profile, \(c_e = -\lambda \hat{T}'_0\) that reads in terms of the dimensionless temperature \(\hat{T}\) and dimensionless length \(x = r/r_2\) \((x_0 \leq x \leq 1)\):

\[
\hat{T}_0(x) - \hat{T}_0(1) = \hat{T}_0'(1) \ln x. \quad \text{(87)}
\]

Now it should be clear that (87) does not describe as such any cooling effect. Indeed, for \(\hat{T}_0(1) \leq 0\) we should put a thermally isolated wall at \(x = 1\) (to avoid assuming the existence of even colder temperatures), which leads us to \(\hat{T}_0'(1) = 0\), and hence to a constant temperature profile.

However, there is a relation between (87) and the temperature profiles obtained above for \(c > 0\) and illustrated in Figs. 7, and 8:

\[
\hat{T}_0(x) > \hat{T}(x), \quad \text{(88)}
\]

if \(\hat{T}_0'(1) = \hat{T}'(1) < 0\) and \(\hat{T}_0(1) = \hat{T}(1), \quad \text{(89)}
\]

where we note from (87) that \(\hat{T}_0''(1) = -\hat{T}_0'(1)\), i.e. for \(\hat{T}_0'(1) = \hat{T}'(1) < 0\), \(\hat{T}_0'(1)\) agrees with the boundary condition (47) required for cooling.

Eqs. (88, 89)—which we verified numerically—show that the reported cooling effect is the modification of the formal temperature profile (87) to the physical, situation with \(c > 0\). In the 1d case (plane geometry) the general zero-velocity temperature profile (the analogue of (87)) is just \(T_0 = \text{const}\), which explains why the cooling effect is negligible there; see Appendix C.

VII. SUMMARY

We worked out a tractable model for describing gasodynamic cooling. The model extends the standard Couette flow between two coaxial cylinders by adding there a radial flow (hence demanding that cylinders are permeable). Only the radial dependence of relevant quantities is retained and the axial flow is neglected.

The model accounts for viscosity, heat-conductivity and compressibility; see section II. They are generally important for gasodynamic cooling, and there are at least two reasons for keeping each of them in the description. Viscosity is to be retained, (first) since we should achieve cooling also in terms of the stagnation enthalpy (as observed experimentally [27]), and (second) since due to turbulence the actual viscosity is much larger than its molecular value. Heat-conductivity is to be considered simultaneously with viscosity, since the Prandtl number of air is close to one. It is also important for ensuring boundary conditions of cooling. Compressibility is needed, because we need the proper relation between fluid mechanics and thermodynamics, and also because the involved angular velocities are sonic.

The emphasis of our study is not so much in describing details of the vortex cooling effect as observed in experimental examples of vortex tubes, but rather in showing how a hydrodynamic model can account for cooling via specific boundary conditions (see section III), and how already the simplest model can provide new (and thermodynamically consistent) predictions for cooling with efficiency larger than 1.

We show that the model predicts a vortex cooling effect for an inward radial flow; see section IV. Though the general cause of cooling is in the pressure gradient that drives the flow, the local cause is related to the work done by viscous forces. The cooling effect comes in two versions—adiabatic and isothermal—that are closely related, but differ from each other by the boundary conditions. In several ways the obtained cooling effect is
similar to what was experimentally seen in Ref. [27] for a short uniflow vortex tube. In accordance with experimental results, the model predicts that the efficiency of vortex cooling is generally smaller than 1 [see section 11G], though the concrete values for the efficiency and for the magnitude of cooling are lower than what was observed for best vortex tubes.

The model predicts as well a cooling effect that was (to our knowledge) so far not observed experimentally; see section VI. This effect is realized for an outward flow and it does not need an angular (vortical) motion. Its cooling efficiency is larger than one, i.e. for the given gradient of pressure, this cooling is more efficient than the adiabatic (i.e. entropy conserving) thermodynamic process. This cooling effect is consistent with the second law, and it is possible due to heat-conductivity. It has partly a geometric origin, since it is negligible for the plane geometry.

There is an experimental report on the Hilsch efficiency being larger than 1 for a counter-flow tube (where only a part of the air is cooled) [42]. However, this result was shown to be not reproducible in [6]. According to the author of Ref. [6], the report concerned externally cooled vortex tubes; cf. our discussion in section III. Hence we finalize by stressing that the issue of cooling with an efficiency larger than one is open.

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APPENDIX A: IDEAL GAS THERMODYNAMICS

We briefly recall ideal-gas formulas as applied in hydrodynamics. Thermodynamic relations of hydrodynamics are written for extensive quantities are divided by the overall number N of involved particles and by the mass m of a single particle. Thus the extensive ideal gas entropy

$$ S = k_B C_v \ln[p] + k_B C_p \ln[V/(Nm)], \quad (A1) $$

where $C_v$ and $C_p$ are heat-capacities and V is the volume, becomes

$$ s = \frac{S}{Nm} = \frac{k_B}{m} \frac{C_v}{N} \ln[p] + \frac{k_B}{m} \frac{C_p}{N} \ln[V/(Nm)] $$

$$ = \frac{k_B}{Nm} \hat{c}_v \ln[p] - \frac{k_B}{Nm} \hat{c}_p \ln[p], \quad (A2) $$

where $\rho =Nm/V$ is the mass density, $N_A$ is the Avogadro number, and where $\hat{c}_v$ and $\hat{c}_p$ are dimensionless numbers of order one:

$$ \hat{c}_p - \hat{c}_v = 1. \quad (A3) $$

After denoting

$$ k_B N_A = R, \quad N_A m = \mu, \quad (A4) $$

where $R = 8.314$ J/K is the gas constant, and $\mu$ is the molar mass, (A2) reads:

$$ s = c_v \ln[p] - c_p \ln[p], \quad (A5) $$

$$ c_v = (R/\mu) \hat{c}_v, \quad c_p = (R/\mu) \hat{c}_p. \quad (A6) $$

The full entropy $S$ (and similarly other extensive quantities) is obtained as $S = \int_V d^3 \rho \cdot s$. Noting that the temperature $T$ is measured in Kelvins, the ideal gas equation of state $pV = k_B N T$ becomes

$$ p = (R/\mu) \rho T. \quad (A7) $$

For purposes of dimensionless analysis, we write (A5) as

$$ s = c_p \left( \frac{1}{c_p} \ln[p] + \ln[T] - \ln \left[ \frac{\mu}{R} \right] \right). \quad (A8) $$

Eq. (A8) implies that if the pressure and temperature adiabatically (i.e. for a constant entropy) change as $p \rightarrow p'$ and $T \rightarrow T'$, then

$$ T'/T = (p'/p)^{1/c_p}. \quad (A9) $$

APPENDIX B: VORTEX FLOW WITHOUT RADIAL MOTION (COUETTE FLOW)

Consider the distribution of temperature inside of the vortex (7) when the radial motion is absent. This is one of standard problems of hydrodynamics (the Couette flow) and it is studied in many places; see e.g. [29, 43]. We reconsider this problem here, because we want to understand why specifically this situation does not contain any interesting stationary cooling scenario (contrary to remarks given in [43]). For

$$ v_r = c = w = \kappa = b = 0, \quad (B1) $$

we get ($\frac{5}{2} + 1) \hat{c}_v^2 - x \hat{c}_v \hat{c}_p + b T - x \hat{T}' - \beta = 0$ from (18). This equation integrates and determines temperature inside of the vortex ($x \equiv r/r_2$)

$$ \hat{T}(x) = \hat{T}(1) - (1 - \alpha)^2 \left( \frac{1}{x^2} - 1 \right) $$

$$ + (\hat{T}'(1) - 2(1 - \alpha)^2) \ln x, \quad x_0 \equiv \frac{r_1}{r_2} \leq x \leq 1, (B2) $$

where we employed (25, B1) for expressing $\beta$ via $\hat{T}'(1)$, and where $\alpha$ is given by (9) under $\kappa = 0$:

$$ \alpha = \frac{1 - (v_1 r_1)/(v_2 r_2)}{1 - (r_1/r_2)^2}. \quad (B3) $$
Note that taking the inner radius $r_1$ to zero, $r_1 \to 0$, does not lead to anything interesting: in this limit we get $\alpha \to 1$, (B2) implies $\hat{T}(x) = \hat{T}(1) + \hat{T}'(1) \ln x$, but we have to assume also $\hat{T}'(1) = 0$ for preventing the singularity at $x \to 0$. Then $\hat{T}(x)$ does not depend on $x$. Hence $r_1$ should be kept finite.

Interesting (stationary) cooling scenarios are those, where the low temperatures created inside of the fluid are not due to even lower temperatures imposed on its boundary. In particular, if one of the boundaries is left without thermal isolation—so there is an active thermal bath working at this boundary—then the inside temperature should be lower than the temperature of this bath.

Let us start with the case, where no thermal isolation is imposed for both boundaries. Then $\hat{T}(x)$ should have a local minimum at some $x \in (x_0, 1)$. Eq. (B2) shows that there is only one solution $x = x_{\text{max}}$ of $\hat{T}'(x) = 0$:

$$x_{\text{max}} = \sqrt{\frac{2(1 - \alpha)^2}{2(1 - \alpha)^2 - \hat{T}'(1)}}. \quad \text{(B4)}$$

If $0 < x_{\text{max}} < 1$ (for which it is necessary and sufficient that $\hat{T}'(1) < 0$), then $x_{\text{max}}$ is the local maximum (not minimum) of $\hat{T}(x)$. Hence we get no cooling for this case.

Next, let us thermally isolate the outer boundary: $\hat{T}'(1) = 0$. We get from (B2):

$$\hat{T}(x) - \hat{T}(1) = -(1 - \alpha)^2 \left(\frac{1}{x^2} - 1 - \ln \frac{1}{x^2}\right). \quad \text{(B5)}$$

Then $\hat{T}(x)$ is a monotonically increasing function of $x$, i.e. the inside temperature $\hat{T}(x)$ is larger than the (inner) bath temperature $\hat{T}(x_0)$. For a thermally isolated inner boundary, $\hat{T}'(x_0) = 0$, $\hat{T}(x)$ is a monotonically decreasing function of $x$ [see (B4)], i.e. again we get no interesting scenarios of cooling. There are no solutions when both boundaries are thermally isolated; see (B2).

The absence of interesting cooling scenarios is confirmed by looking at the total work produced by external forces that rotate the cylinders. It reads from (35) [with $\kappa = 0$]:

$$\hat{W}_{\phi} = 2(1 - \alpha)^2(1 - \frac{1}{x_0^2}) < 0. \quad \text{(B6)}$$

The negativity of (B6) means that the work is invested externally and dissipated for overcoming the viscous forces. This work leaves the system as heat.

Thus three regimes are impossible for the considered Couette flow:

- It cannot cool the fluid isothermally, i.e. when both boundaries are kept at the same temperature.
- It cannot cool the fluid adiabatically: no regime exists when one boundary is thermally isolated, while another one is subject to a thermal bath, and it is demanded to get the fluid colder than the active bath temperature.

Hence low-temperatures present in the system according to (B2) do not constitute any non-trivial cooling: they are due to the low-temperature bath present at one of boundaries. Put differently, for the Couette flow the active bath is the one with the lower temperature.

The above two conclusions seem to hold rather generally for stationary hydrodynamic systems without mass flow, though we so far did not get a general argument for their validity. At any rate they hold for the (generalized) Couette (sometimes also called Taylor-Dean) flow, where the fluid is subject to azimuthal driving with a volume force $\vec{j}$, where only the $\phi$-component $f_\phi(r)$ of $\vec{j}$ is non-zero, but it is an arbitrary function of $\phi$.

- The Couette flow cannot also function as a heat-engine, since irrespectively of the values of $\hat{T}(r_2)$ and $\hat{T}(r_1)$—the work is always dissipated; see (B6).

APPENDIX C: 1D EXAMPLE OF WEAK ADIABATIC COOLING

Consider a 1d flow (from left to right) between two permeable plates separated by distance $L$. Continuity of mass leads to

$$\rho v = c_1 = \text{const}. \quad \text{(C1)}$$

1d Navier-Stokes and energy equations read in dimensionless form

$$\kappa \hat{v}^2(x) + \frac{b}{\epsilon_p} \hat{T}(x) - (\chi + \frac{4}{3}) \hat{v}(x) \hat{v}'(x) = \gamma \hat{v}(x), \quad \text{(C2)}$$

$$\frac{\kappa}{2} \hat{v}^2(x) + b \hat{T}(x) - (\chi + \frac{4}{3}) \hat{v}(x) \hat{v}'(x) - \hat{T}' = \beta, \quad \text{(C3)}$$

where $0 \leq x \leq 1$, $L$ is the distance between two plates, $\beta$ and $\gamma$ are constants, and we introduced the following dimensionless parameters:

$$\hat{T} = \frac{\lambda}{\epsilon_p^2} T, \quad \hat{v}(x) = \frac{v(x)}{v(L)} \quad \text{(C4)}$$

$$b = \frac{c_1 c_p L}{\lambda}, \quad \kappa = \frac{c_1 L}{\eta}, \quad \text{(C5)}$$

$$\chi = \lambda \eta, \quad \hat{v} = \epsilon_p(\mu/R). \quad \text{(C6)}$$

The constants $\beta$ and $\gamma$ can be expressed via (respectively) $\hat{v}'(1)$ and $\hat{T}'(1)$:

$$\gamma = \kappa + \frac{b}{\epsilon_p} \hat{T}(1) - (\chi + \frac{4}{3}) \hat{v}'(1), \quad \text{(C7)}$$

$$\beta = \frac{1}{2} \kappa + b \hat{T}(1) - (\chi + \frac{4}{3}) \hat{v}'(1) - \hat{T}'(1). \quad \text{(C8)}$$

We obtain from (C4–C7):

$$\hat{T}'(1) = b \hat{T}'(1) - \gamma (\chi + \frac{4}{3}) \hat{v}'(1)^2 + \frac{b}{\epsilon_p} \hat{T}(1) \hat{v}'(1). \quad \text{(C9)}$$
Let us now look at conditions for adiabatic cooling. Now $c_1 > 0$ (hence $b > 0$ and $\kappa > 0$) and $\hat{v} > 0$ from (C1). Hence we look for
\[ \hat{T}(0) > \hat{T}(1), \quad \hat{T}''(1) > 0. \quad (C10) \]
The temperature profiles appear to be monotonic so that the first condition in (C10) can be written as $\hat{T}'(1) < 0$. Then the first and second term in the right-hand-side of (C9) are negative. Hence $T''(1) > 0$ can be satisfied only due to sufficiently large $b^c \hat{c}_p \hat{T}(1) \hat{v}'(1) > 0$. This implies limitations on $T'(1)$ (which cannot be sufficiently small) and on $|\hat{T}'(1)|$ (which cannot be sufficiently large).
Eventually, the adiabatic cooling appears to be a relatively small effect, though it is still possible in this model. For example, under $\chi = 10$, $b = \kappa = 10$, $\hat{c}_p = 3.5$, $\hat{T}(1) = 1$, $\hat{T}'(1) = -0.1$ and $\hat{v}'(1) = 0.1$ (we have $\hat{v}(1) = 1$ by definition) we get for the cooling magnitude $[\hat{T}(0) - \hat{T}(1)]/\hat{T}(0) = 0.037$.

**APPENDIX D: POTENTIAL VORTEX**

Another familiar type of vortex in (7, 8) is:
\[ \alpha = 0 \quad \text{or} \quad \frac{v_1}{v_2} = \frac{r_2}{r_1}. \quad (D1) \]
Eqs. (53, 52) imply
\[ \hat{T}(x) - \hat{T}(1) = \left[ \hat{T}'(1) - \frac{\kappa + 4}{b + 2} \frac{x^b - 1}{b} \right] + \frac{\kappa + 4}{2(b + 2)}(1 - x^{-2}). \quad (D2) \]
Now $\hat{T}'(x) = 0$ is solved as
\[ x^{-2b} = 1 - \frac{(b + 2)\hat{T}'(1)}{(w^x(1) + 1)(\kappa + 4)}. \quad (D3) \]
Hence the minimum of $\hat{T}(x)$ for $\hat{T}'(1) > 0$ can exist only for
\[ \kappa < -4, \quad (D4) \]
i.e. only for radially inward flowing fluid ($e < 0$).

Now the isothermal cooling is driven by the work done for rotating cylinders. Eqs. (35, D1) imply
\[ \hat{W}_\phi = 2(1 - x_0^{-2}) < 0, \quad (D5) \]
while the kinetic energy change is now larger than zero; see (67, D1). Due to this, $\hat{U}(x_0) > \hat{U}(1)$.

In other respects the two scenarios of cooling (quasisolid and potential) are similar to each other: both have roughly the same magnitude, both need small radial velocities and both have efficiency $\xi$ smaller than 1.
FIG. 1: The cross-section of the flow. Two coaxial permeable cylinders with radii $r_1$ and $r_2$ rotate with prescribed speeds. Solid arrows $\vec{v}_\phi = v_\phi \vec{e}_\phi$ and $\vec{v}_r = v_r \vec{e}_r$ refer to the velocity components of the flow for $r_1 < r < r_2$. Here the radial flow is directed from the outer (larger) cylinder to the inner (smaller) cylinder. Now $|\vec{v}_\phi| < |\vec{v}_r|$, since the radial flow has to be smaller for the cooling.

FIG. 2: Isothermal cooling with inward flow ($c < 0$) and quasi-solid vortex. Left (right) figure: dimensionless temperature $\hat{T}(x)$ (dimensionless stagnation enthalpy $\hat{U}(x)$) versus $x = r/r_2$ in the interval $x \in [x_0, 1]$, where $x_0 = 0.007347$. The curves are obtained from numerical solution of (18, 19) for $\alpha = 1$, $\kappa = 0.5$, $b = 1$, $\hat{c}_p = 3.5$, $\beta = -1$ ($\hat{T}'(1) = 0.124$), $\chi = 10$ and $\hat{T}(1) = 1.126$, $w(1) = 10^{-4}$, $w'(1) = 0$. These parameter values are consistent with experiments [27]; see (27), (62) and the discussion after it, (72), (74) and the discussion around it.

The minimal dimensionless temperature is $\hat{T}_{\text{min}} = \hat{T}(x_{\text{min}}) = 1.0223$, $x_{\text{min}} = 0.008891$. The energy values: $\Delta \hat{E} = -0.24699$, $\Delta \hat{W} = 0.48228$, $\Delta \hat{W}_r = -0.01405$, $\hat{Q}(x_0) = -x_0 \hat{T}'(x_0) = 0.11129$, $\hat{Q}(1) = -\hat{T}(1) = -0.124$.

The pressure $p(x)$ is monotonically increasing function of $x \in [x_0, 1]$. $\hat{T}_{\text{ad}}(x_{\text{min}}) = 0.73556$. The Hilsch efficiency $\xi_H = 0.26559$. Upon decreasing the initial temperature, $x_{\text{min}}$ increases, while both the magnitude and quality of cooling decrease, e.g. for $\hat{T}(1) = 1.16$ we get $x_{\text{min}} = 0.50597$, $\hat{T}_{\text{min}} = \hat{T}(x_{\text{min}}) = 1.13083$ and $\xi_H = 0.13016$. $w(x)$ (and hence $w(x)/x$) is monotonically decreasing function of $x \in [x_0, 1]$. Thus condition (50) holds for the inner wall. (Condition (49) for the outer wall cannot hold simultaneously with (50); this is a reason for not imposing a permeable wall at $r = r_2$. The latter is just a control surface within which we describe the flow.)
FIG. 3: Adiabatic cooling with inward flow and quasi-solid vortex.
Left (right) figure: dimensionless temperature $\hat{T}(x)$ (dimensionless stagnation enthalpy $\hat{U}(x)$) versus $x = r/r_2$ in the interval $x \in [x_0, 1]$, where $x_0 = 0.1$.
The curves are obtained from numerical solution of (18, 19) for $\alpha = 1$, $\kappa = -0.5$, $b = -0.5$, $\hat{c}_p = 3.5$, $\beta = -0.5$, $\chi = 10$ and $\hat{T}(1) = 1.175$, $w(1) = 10^{-4}$, $w'(1) = 0$.
The minimal temperature $\hat{T}(x_0) = 1.04233$. As required for an adiabatic, permeable inner boundary: $\hat{T}''(x_0) = 1.48034$.
The energy values: $\Delta E = 0.291278$, $\Delta W = 0.44866$, $\hat{Q}(x_0) = -x_0 T'(x_0) = -0.0051$, $\hat{Q}(1) = -T'(1) = -0.1625$.
Efficiency: $\hat{T}_{ad}(x_0) = 0.52012$; see (36, 42). The Hilsch efficiency $\xi_H = 0.2026$. Pressure $p(x)$ and $w(x)$ hold $p'(x) > 0$ and $w'(x) < 0$ for $x \in [x_0, 1]$.

FIG. 4: Isothermal cooling with inward flow and $\alpha = -0.5$ vortex.
Left (right) figure: dimensionless temperature $\hat{T}(x)$ (dimensionless stagnation enthalpy $\hat{U}(x)$) versus $x = r/r_2$ in the interval $x \in [x_0, 1]$, where $x_0 = 0.639472$.
The curves are obtained from numerical solution of (18, 19) for $\alpha = -0.5$, $\kappa = -4$, $b = -5$, $\hat{c}_p = 3.5$, $\beta = -7$, $\chi = 10$ and $\hat{T}(1) = 1.1115$, $w(1) = 0.01$, $w'(1) = 0$.
The minimal dimensionless temperature is $\hat{T}_{min} = 1.05166$ and it is reached for $x_{min} = 0.80077$.
The energy values: $\Delta E = -1.62148$, $\Delta W = 2.26812$, $\Delta W_r = -0.01403$, $\hat{Q}(x_0) = -x_0 T'(x_0) = 0.204143$, $\hat{Q}(1) = -T'(1) = -0.4425$.
The pressure $\hat{p}(x)$ ($w(x)$) is monotonically increasing (decreasing) functions of $x \in [x_0, 1]$. $\hat{T}_{ad}(x_0) = 0.952581$; see (36, 42).
The Hilsch efficiency $\xi_H = 0.376542$. 
FIG. 5: Dependence of the dimensionless temperature $\hat{T}(x)$ profiles [see (53)] on the radial Reynolds number, where the Prandtl number $Pr = b/\kappa = 2$ is fixed, the quasi-solid vortex conditions $\alpha = 1$ is obeyed [see (58)] and $\hat{T}'(1) = 0.1$. Full (blue) curves from top to bottom: $\kappa = -0.5, \kappa = -0.72$ and $\kappa = -0.725$. The first curve holds condition (56) for all $x_0$, the second one holds them for $x_0 < 0.21$, while the latter curve (with $\kappa = -0.725$) is void of physical meaning, since neither the adiabatic boundary condition (56), nor the isothermal condition (57) are satisfied for it.

Dashed (black) curves from top to bottom: $\kappa = -0.3, \kappa = -0.25$ and $\kappa = -0.15$. For the curve with $\kappa = -0.25$ the adiabatic boundary condition (56) holds for $x_0 > 0.447$. The whole curve with $\kappa = -0.15$ is void of the physical meaning, since neither (56), nor (57) hold.

FIG. 6: Dependence of temperature profiles $\hat{T}_0(x) - \hat{T}_0(1)$ on the rotation speed according to (71). A larger $\epsilon$ means a bigger deviation of $v_2$ from its reference values. From bottom to top: $\epsilon = 0.78$ (no cooling is present, since neither (56), nor (57) hold), $\epsilon = 0.81$, $\epsilon = 1$, $\epsilon = 1.5$ and $\epsilon = 3$. For other parameters we take [cf. Figs. 2 and 3]: $\hat{T}_0'(1) = 0.1$, $\alpha = 1$, $b = -1$, $\kappa = -0.5$. 
Eqs. (18, 19) are solved numerically for $\alpha = 1$, $\kappa = 0.1$, $b = 0.1$, $\tilde{c}_p = 3.5$, $\beta = 11$ ($\hat{T}'(1) = -10.7867$), $\chi = 10$, $\hat{T}(1) = 1.295$, $w(1) = 0.1$, and $w'(1) = -0.1$ in the range $x = \hat{r}/r_2 \in [x_0, 1]$, where $x_0 = 0.3$.

Left figure: dimensionless temperature $\hat{T}(x)$ versus $x = \hat{r}/r_2$ (the dimensionless stagnation enthalpy $\hat{U}(x)$ behaves similarly). A nearly identical temperature profile is obtained upon solving (18) with $w(x) = w(1)$ and $w'(x) = 0$.

Middle figure: the dimensionless energy $\hat{E}(x) - \hat{E}(1)$ (solid line), radial work $\hat{W}_r(x) - \hat{W}_r(1)$ (dashed line) and heat $\hat{Q}(x) - \hat{Q}(1)$ (bold line) versus $x$. Right figure: $w(x)$ versus $x$. Condition (49) hold. They are written as $w(x_0) > x_0 w'(x_0)$ and $w(1) > w'(1)$. The radial velocity $\propto w(x)/x$ is a monotonically decreasing function of $x$.

The minimal (maximal) dimensionless temperature is $\hat{T}(1) = 1.295$ ($\hat{T}(x_0) = 13.6389$) and $\hat{T}'(1) = 9.98$.

Energetics: $\Delta \hat{E} = -1.1925$, $\Delta \hat{E}_{\text{kin}} = 0.04188$, $\Delta \hat{W}_r = 0.21743$, $\Delta \hat{W} = 0.1245$, $\hat{Q}(x_0) = -x_0 \hat{Q}'(x_0) = 9.7187$, $\hat{Q}(1) = -\hat{T}'(1) = 10.885$.

Efficiency: $\hat{T}_{\text{ad}}(1) = 6.8718 > \hat{T}(1) = 1.295$ and the Hilsch efficiency $\xi_H = 1.8241$. The pressure $p(x)$ is a decreasing function of $x$.

The curves are obtained from numerical solution of (18, 19) for $\kappa = 1$, $b = 1$, $\tilde{c}_p = 3.5$, $\beta = 40$ ($\hat{T}'(1) = -36.5$), $\chi = 10$ and $\hat{T}(1) = 1$, $w(1) = 1$, $w'(1) = 0$.

Left figure: dimensionless temperature $\hat{T}(x)$ versus $x = \hat{r}/r_2$ in the interval $x \in [x_0, 1]$, where $x_0 = 0.4$ (dimensionless stagnation enthalpy $\hat{U}(x)$ behaves similarly).

Middle figure: the dimensionless energy $\hat{E}(x) - \hat{E}(1)$ (solid line), radial work $\hat{W}_r(x) - \hat{W}_r(1)$ (dashed line) and heat $\hat{Q}(x) - \hat{Q}(1)$ (bold line) versus $x$. Right figure: $w(x)$ versus $x$. Condition (49) hold; cf. the data for Fig. 7. It holds that $|w(x)/x| < 0$.

The minimal dimensionless temperature is $\hat{T}(1) = 1$ and $\hat{T}'(1) = 6.4293$.

Energetics: $\Delta \hat{E}_{\text{kin}} = -1.8809$, $\Delta \hat{W} = \hat{W}_r = 0.6676$, $\hat{Q}(x_0) = -x_0 \hat{Q}'(x_0) = 12.9503$, $\hat{Q}(1) = -\hat{T}'(1) = 36.5$.

Efficiency: $\hat{T}_{\text{ad}}(1) = 9.12373 > \hat{T}(1) = 1$ and the Hilsch efficiency $\xi_H = 1.5719$. The pressure $p(x)$ is a decreasing function of $x$. 

FIG. 7: Adiabatic cooling with weak outward radial flow and quasi-solid vortex.

FIG. 8: Adiabatic cooling with outward radial flow and without vortex motion ($\nu_0 = 0$).