UNCOUNTABLY MANY ARCS IN $S^3$ WHOSE COMPLEMENTS HAVE NON-ISOMORPHIC, INDECOMPOSABLE FUNDAMENTAL GROUPS

ROBERT MYERS
Department of Mathematics
Oklahoma State University
Stillwater, OK 74078 USA
Email: myersr@math.okstate.edu

ABSTRACT
An uncountable collection of arcs in $S^3$ is constructed, each member of which is wild precisely at its endpoints, such that the fundamental groups of their complements are non-trivial, pairwise non-isomorphic, and indecomposable with respect to free products. The fundamental group of the complement of a certain Fox-Artin arc is also shown to be indecomposable.

Keywords: wild arc, wild embedding, indecomposable group, knot, 3-manifold.

1. Introduction
At the 1996 Workshop in Geometric Topology F. D. Ancel [1] posed the following questions:

Question 1.1. Let $A$ be the Fox-Artin arc in $S^3$ which is pictured in Figure 1. Is $\pi_1(S^3 - A)$ indecomposable with respect to free products?

Figure 1: The Fox-Artin arc $A$
**Question 1.2.** Are there infinitely (uncountably?) many wild arcs $A_i$ in $S^3$ such that $\pi_1(S^3 - A_i)$ and $\pi_1(S^3 - A_j)$ are non-isomorphic for $i \neq j$? 

Fox and Artin [3] proved that $\pi_1(S^3 - A)$ is non-trivial. (A is actually the mirror image of their Example 1.1.) At the workshop Ancel remarked that an incorrect proof that it is indecomposable had been published by Rosłaniec [15]. He also noted that an affirmative answer to Question 1.1 would give an affirmative answer to the countable case of Question 1.2 by concatenating finitely many copies of $A$; the resulting groups are free products of copies of $\pi_1(S^3 - A)$ and so would be non-isomorphic [3, Vol. II, p. 27]. These examples would have a finite but unbounded number of wild points.

In this paper we answer these two questions in the affirmative. In particular, regarding Question 1.2 we construct an uncountable family of arcs $A_i$ such that the fundamental groups $\pi_1(S^3 - A_i)$ are non-isomorphic for distinct indices and also are indecomposable and non-trivial. Moreover each arc is wild precisely at its endpoints.

We remark that if the fundamental group of the complement of an arc in $S^3$ is non-trivial, then it is not finitely generated [3, Corollary 2.6].

Ancel also posed the following question, to which one can of course add the question of indecomposability. As of this writing these questions remain open, but it seems likely that affirmative answers could be obtained by the methods of this paper.

**Question 1.3.** Let $B$ be the wild arc in the solid torus $V$ pictured in Figure 2. Suppose $k_i : V \to S^3$ is a knotted embedding such that $\pi_1(S^3 - k_i(V))$ is not isomorphic to $\pi_1(S^3 - k_j(V))$ for $i \neq j$. Is $\pi_1(S^3 - k_i(B))$ not isomorphic to $\pi_1(S^3 - k_j(V))$ for $i \neq j$?

The paper is organized as follows. In section 2 we give a criterion for the fundamental group of a non-compact 3-manifold to be indecomposable and non-trivial. In section 3 we prove that the exterior of the Fox-Artin arc satisfies this criterion. In section 4 we prove a lemma about embeddings of torus knot groups in torus knot groups. In section 5 we construct the uncountable family of arcs mentioned above and verify its properties.

The author thanks Bill Banks for drawing the Fox-Artin arc which is used in Figures 1, 2, and 3.
2. A Criterion for Indecomposability

Recall that a group $G$ is decomposable if it is a free product $K \ast L$, where $K$ and $L$ are non-trivial. $G$ is indecomposable if it is not decomposable.

**Lemma 2.1.** Let $\{H_k\}_{k \geq 0}$ be a sequence of non-trivial, non-infinite-cyclic, indecomposable subgroups of $G$ such that $H_k \subseteq H_{k+1}$ for all $k \geq 0$ and $G = \bigcup_{k=0}^{\infty} H_k$. Then $G$ is indecomposable.

**Proof.** Suppose $G = K \ast L$, where $K$ and $L$ are non-trivial. Then no non-trivial element of $K$ is conjugate to an element of $L$. This can be seen as follows. Let $N$ be the normal closure of $K$ in $G$. Let $p : G \to G/N$ be the natural projection. Then there is an isomorphism $q : G/N \to L$ such that the restriction of $q \circ p$ to $L$ is the identity of $L$ [11], pp. 101–102]. But $q \circ p$ sends any conjugate of an element of $K$ to the trivial element of $L$.

By the Kurosh subgroup theorem [5, 10] any subgroup of $G$ is a free product of a free group and conjugates of subgroups of $K$ and of $L$. Since $H_0$ is indecomposable and non-infinite-cyclic we may thus assume that it is conjugate to a subgroup of $K$. Similarly $H_1$ must be conjugate to a subgroup of $K$ or of $L$. The latter cannot happen since then some non-trivial element of $K$ would be conjugate to an element of $L$. Continuing in this fashion we get that each $H_k$ is conjugate to a subgroup of $K$. This implies that $G$ cannot be the union of the $H_k$ since the non-trivial elements of $L$ are excluded. □

We now consider fundamental groups of non-compact 3-manifolds. For basic definitions in 3-manifold topology we refer to [3] and [4]. A 3-manifold $M$ is $\partial$-irreducible if $\partial M$ is incompressible in $M$. Let $S$ and $S'$ be compact surfaces such that $S$ is properly embedded in $M$ and $S'$ either is properly embedded in $M$ or lies in $\partial M$. Then $S$ and $S'$ are parallel in $M$ if there is an embedding of $S \times [0, 1] \times [0, 1]$ in $M$ (called a parallelism from $S$ to $S'$) such that $S \times \{0\} = S, S \times \{1\} = S'$, and $(\partial S) \times [0, 1]$ lies in $\partial M$. If $S'$ lies in $\partial M$ then $S$ is $\partial$-parallel in $M$. The topological interior of $N$ in $M$ is denoted by $Int N$.

**Lemma 2.2.** Let $W$ be a connected, non-compact 3-manifold which can be expressed as the union $W = \bigcup_{n=-\infty}^{\infty} X_n$ of compact, connected, irreducible, $\partial$-irreducible 3-manifolds $X_n$ such that $X_m \cap X_n = \emptyset$ for $|n - m| > 1$ and $X_n \cap X_{n+1} = \partial X_n \cap \partial X_{n+1}$ is a compact, connected surface which is incompressible in $X_n$ and in $X_{n+1}$ and is not a disk. Then $\pi_1(W)$ is non-trivial and indecomposable.

**Proof.** Standard arguments show that $Y_k = \bigcup_{n=-k}^{k} X_n$ is irreducible and $\partial$-irreducible. It follows that $\pi_1(Y_k)$ is non-trivial, non-infinite-cyclic, and indecomposable [5, Theorem 5.2, Lemma 6.6]. The incompressibility of each $X_n \cap X_{n+1}$ shows that $\pi_1(Y_k)$ injects into $\pi_1(M)$. We now apply Lemma 2.1. □

3. The Fox-Artin Arc

**Theorem 3.1.** $\pi_1(S^3 - A)$ is indecomposable, where $A$ is the Fox-Artin arc in Figure 1.

**Proof.** Let $N$ be a tapered regular neighborhood of $A$. Thus $N$ is a 3-ball containing $A$ such that $A \cap \partial N = \partial A$. $A$ is isotopic in $\bar{N}$ rel $\partial A$ to a diameter of $N$, and $N$ is tamely embedded in $S^3$ except at $\partial A$. Let $W = S^3 - (Int N \cup \partial A)$. (We call $W$ the exterior of $A$. We also use this term for the closure of the complement of a regular neighborhood of

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a tame submanifold of a manifold.) Then $\pi_1(W) \cong \pi_1(S^3 - A)$, and $\partial W = \partial N - \partial A$ is homeomorphic to an open annulus $S^1 \times \mathbb{R}$. It suffices to show that $W$ satisfies the hypotheses of Lemma 2.2. In the figures which follow we do not explicitly draw $N$, but its presence should be understood.

$S^3 - \partial A$ can be parametrized by $S^2 \times \mathbb{R}$ in such a way that $A$ meets each $S^2 \times [m, m+1]$, $m \in \mathbb{Z}$, in three arcs as indicated in Figure 3.

Figure 3: $S^3 - \partial A$ parametrized as $S^2 \times \mathbb{R}$

It is natural to consider the exterior of the union of these three arcs in $S^2 \times [m, m+1]$ and to regard $W$ as the union of these exteriors. Unfortunately these manifolds are cubes with two handles and so are not $\partial$-irreducible. Instead we take $S^2 \times [2n-1, 2n+1]$, $n \in \mathbb{Z}$, which also meets $A$ in three arcs, and let $X_n$ be the exterior of their union. The generic copy $X$ of $X_n$ is then the exterior of the union of the three arcs $\alpha$, $\beta$, and $\gamma$ in $S^2 \times [-1, 1]$ as indicated in Figure 4.

Figure 4: The arcs $\alpha$, $\beta$, and $\gamma$ in $S^2 \times [-1, 1]$

Since no component of $X \cap (S^2 \times \{-1, 1\})$ or of the closure of $\partial X - (S^2 \times \{-1, 1\})$ is a disk it suffices to prove the following.

**Lemma 3.2.** $X$ is irreducible and $\partial$-irreducible.

**Proof.** Irreducibility follows from the Schönflies theorem together with the fact that $X$ is a compact, connected submanifold of $S^3$ with connected boundary.

The strategy for proving $\partial$-irreducibility is to exhibit $X$ as a double covering space of a solid torus $V$ branched over a certain properly embedded arc $\delta$ in $V$. If $\partial X$ were compressible, then by the $\mathbb{Z}_2$ case [4] of the equivariant loop theorem [11] there would be a compressing disk $\tilde{D}$ for $\partial X$ such that either $\tau(\tilde{D}) \cap \tilde{D} = \emptyset$ or $\tau(\tilde{D}) = \tilde{D}$, where $\tau$ is the non-trivial covering translation. Let $D$ be the image of $\tilde{D}$ in $V$. In the first case $D$ would...
miss $\delta$. In the second case we could assume that $D$ would meet $\delta$ in a single transverse intersection point, since otherwise $\tilde{D}$ would contain the fixed point set $\tilde{\delta}$ of $\tau$, and we could reduce to the first case by replacing $\tilde{D}$ by a nearby parallel disk. In both cases $D$ would be a compressing disk for $\partial V$ in $V$ since if $\partial D = \partial E$ for some disk $E$ in $\partial V$, then the preimage of $E$ in $X$ would have a component $\tilde{E}$ with $\partial \tilde{E} = \partial \tilde{D}$. The proof is completed by showing that no such disk $D$ exists.

By sliding one endpoint of each of $\alpha$ and of $\beta$ onto $\gamma$ we see that $X$ is homeomorphic to the exterior of the graph $\omega$ in $S^2 \times [-1, 1]$ shown in Figure 5.

![Figure 5: The graph $\omega$ in $S^2 \times [-1, 1]$](image)

This in turn is homeomorphic to the exterior $\tilde{V}$ of the graph $\tilde{\theta}$ in $S^3$ shown in Figure 6.

![Figure 6: The graph $\tilde{\theta}$ in $S^3$](image)

This graph is invariant under the order two rotation $\tau$ about the simple closed curve $\tilde{\rho}$. This involution defines a branched double covering $q : S^3 \rightarrow S^3$. The images $\tilde{\theta}$ and $\rho$ of $\tilde{\theta}$ and $\tilde{\rho}$ are shown in Figure 7.

Figure 8 shows a regular neighborhood $R$ of $\theta$ in $S^3$ and the arc $\delta = \rho \cap (S^3 - \text{Int } R)$. Figure 9 shows $R$ straightened by an isotopy to a standard solid torus. Figure 10 moves the point at $\infty$ to a finite point. Figure 11 displays the solid torus $V = S^3 - \text{Int } R$ containing $\delta$. 

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Figure 7: The quotient graph $\theta$ in $S^3$

Figure 8: The regular neighborhood $R$ of $\theta$
Figure 9: $R$ isotoped to a standard solid torus

Figure 10: $\delta$ isotoped off the point at $\infty$
Lemma 3.3. There is no meridinal disk \( D \) in \( V \) such that \( D \cap \delta \) is either empty or a single transverse intersection point.

Proof. Let \( U \) be a regular neighborhood of \( \delta \) in \( V \). Let \( F = \partial V - \text{Int}(U \cap \partial V) \) and \( M = V - \text{Int} U \). It suffices to show that \( F \) is incompressible in \( M \) and that there is no properly embedded incompressible annulus \( G \) in \( M \) with one boundary component in the frontier (topological boundary) \( C = Fr U \) of \( U \) in \( V \) and the other a curve in \( F \) which bounds a meridinal disk \( D \) in \( V \) with \( D \cap M = G \). Let \( E \) be the meridinal disk shown in Figure 11. It meets \( U \) in a pair of disks and so meets \( M \) in a disk with two holes \( S \). Let \( V_0 \) be the 3-ball obtained by splitting \( V \) along \( E \) and \( M \) the 3-manifold obtained by splitting \( M \) along \( S \). Then \( E \) splits \( \delta \) into three arcs \( \delta_0, \delta_1, \text{ and } \delta_2 \), \( U \) into the regular neighborhoods \( U_0, U_1, \text{ and } U_2 \) of these arcs, \( C \) into the three annuli \( C_0, C_1, \text{ and } C_2 \), and \( F \) into the surface \( F_0 \). See Figure 12. Let \( S_0 \) and \( S_1 \) be the copies of \( S \) in \( M_0 \) which are identified to obtain \( S \), where \( S_0 \) meets \( C_0 \) and \( S_1 \) meets \( C_1 \) and \( C_2 \).

Let \( K \) be the disk in \( M_0 \) shown in Figure 12. Its boundary consists of one arc each in \( F_0, S_1, C_1, \text{ and } C_2 \). Splitting \( M_0 \) along \( K \) gives a 3-manifold \( M_1 \) which is homeomorphic to \( (S_0 \cup C_0) \times [0, 1] \) with \( S_0 \cup C_0 = (S_0 \cup C_0) \times \{0\} \). See Figure 13. \( M_0 \) is then obtained by attaching a 1-handle with cocore \( K \) to \( (S_0 \cup C_0) \times \{1\} \), so it is irreducible.

We first show that \( S \) is incompressible in \( M \). It suffices to show that \( S_0 \) and \( S_1 \) are each incompressible in \( M_0 \). The first of these follows from our description above of \( M_0 \) as a product \( I \)-bundle with a 1-handle attached. The second follows from homology considerations.

We next show that \( F_0 \) is incompressible in \( M_0 \). Suppose \( L \) is a compressing disk. Then \( \partial L \) separates one non-empty set of components of \( \partial F_0 \) from another. The seven possible partitions are all ruled out by a combination of homology arguments and the incompressibility of \( S_0 \).
We now show that $S$ is $\partial$-incompressible rel $F$ in $M$. This means that whenever $L$ is a disk in $M$ such that $L \cap S$ is a properly embedded arc $\lambda$ in $S$ and $L \cap \partial M$ is an arc $\mu$ in $F$ such that $\lambda \cap \mu = \partial \lambda = \partial \mu$ and $\partial L = \lambda \cup \mu$, then there is an arc $\nu$ in $\partial S$ and a disk $L'$ in $F$ such that $\mu \cap \nu = \partial \mu = \partial \nu$ and $\partial L' = \mu \cup \nu$. It suffices to prove that $S_0$ and $S_1$ are $\partial$-incompressible rel $F_0$ in $M_0$.

For $S_0$ this follows from homology considerations and the incompressibility of $F_0$ in $M_0$. For $S_1$ similar arguments reduce the problem to the case in which $\partial L = \lambda \cup \mu$ where $\lambda$ is an arc in $S_1$ such that $\partial \lambda$ lies in $S_1 \cap F_0$ and $\lambda$ separates $S_1 \cap C_1$ from $S_1 \cap C_2$ on $S_1$ and $\mu$ is an arc in $F_0$ separating $F_0 \cap C_1$ from $F_0 \cap C_2$.

Isotop $L$ so that $K$ and $L$ are in general position and the arcs $K \cap S_1$ and $L \cap S_1$ meet in a single transverse intersection point. Then there is an arc $\xi$ in $K \cap L$ joining this point to a point in $K \cap F_0$. Since $M_0$ is irreducible we may assume that in addition $K \cap L$ contains no simple closed curves. The intersection then consists of $\xi$ and possibly some arcs $\eta$ with

Figure 12: $M$ split along $S$ to obtain $M_0$

Figure 13: $M_0$ split along $K$ to obtain $M_1 \approx (S_0 \cup C_0) \times [0, 1]$
∂η in $K \cap F_0$. Assume η is outermost on $L$. Let $ζ$ be an arc in $∂L$ such that $ζ \cup η$ bounds a disk $L_0$ in $L$ whose interior misses $K$. Let $ε$ be the arc on $K \cap F_0$ with $∂ε = ∂η = ζ$. There is a disk $K_0$ in $K$ such that $∂K_0 = η \cup ε$. Then $K_0 \cap L_0 = η$ and $K_0 \cup L_0$ is a disk with boundary $ζ \cup ε$. Since $F_0$ is incompressible in $M_0$ this curve bounds a disk $F_1$ in $F_0$. Since $M_0$ is irreducible $K_0 \cup L_0 \cup F_1$ bounds a 3-ball $B_0$ in $M_0$. Note that $ξ \cap B_0 = ∅$. An isometry of $L$ which moves $L_0$ across $B_0$ to $K_0$ and then off $K_0$ removes $η$ and possibly other components of $K \cap L$ but does not affect $ξ$.

Thus we may assume that $K \cap L = ξ$. We now split $M_0$ along $K$ to obtain $M_1$, as before. This splits $L$ into disks $L_0$ and $L_1$ either of which we can take as a compressing disk for $(S_0 \cup C_0) \times \{0\}$ in $M_1 = (S_0 \cup C_0) \times [0, 1]$. This contradiction completes the proof that $S_0$ and $S_1$ are $∂$-incompressible rel $F_0$ in $M_0$ and hence that $S$ is $∂$-incompressible rel $F$ in $M$.

Now suppose that $D$ is a compressing disk for $F$ in $M$. Put $D$ in general position with respect to $S$ so that $D \cap S$ has a minimal number of components. By the incompressibility of $S$ and the irreducibility of $M$ none of them are simple closed curves. Since $S$ is $∂$-incompressible rel $F$ in $M$ none of them can be arcs, so $D \cap S = ∅$. Since $F_0$ is incompressible in $M_0$ we have that $D$ cannot exist.

Finally suppose that $G$ is an incompressible annulus in $M$ with one boundary component in $C$ and the other a curve in $F$ which bounds a meridinal disk $D$ of $V$ such that $D \cap M = G$. We may assume that the first boundary component misses $S$, that $G$ is in general position with respect to $S$ and that among all such annuli in its isotopy class $G \cap S$ has a minimal number of components. Then none of these components is a simple closed curve which bounds a disk in $S$ or in $G$ or is an arc joining the two components of $∂G$.

Suppose some component $κ$ of $G \cap S$ is a simple closed curve. Then we may assume that $κ$ and $G \cap C$ form the boundary of a subannulus $G_0$ of $G$ which lies in $M_0$. If $κ$ lies in $S_0$, then for homological reasons $G \cap C$ must lie in $C_0$. We can isotop $G_0$ so that it misses $K$. Hence $G_0$ lies in $M_1 = (S_0 \cup C_0) \times [0, 1]$. By Corollary 3.2$G_0$ is parallel to an annulus in $(S_0 \cup C_0) \times \{0\}$ and so $κ$ can be removed by an isotopy, contradicting minimality. If $κ$ lies in $S_1$, then for homological reasons $G \cap C$ must be in $C_1$ or $C_2$, say $C_1$. Let $M_2 = M_0 \cup U_0$. Then $M_2$ is homeomorphic to $S_1 \times [0, 1]$ with $S_1 = S_1 \times \{1\}$. Now $G_0$ is incompressible in $M_2$ and can be isotoped keeping $κ$ fixed to an annulus $G'_0$ such that $∂G'_0$ lies in $S_1$. It then follows from Corollary 3.2$G'_0$ is parallel to an annulus in $S_1$ and hence $G_0$ is $∂$-parallel in $M_2$. Since this parallelism does not meet $U_0$ we have that $G_0$ is $∂$-parallel in $M_0$. It follows that $κ$ can be removed by an isotopy, again contradicting minimality.

Hence any component of $G \cap S$ must be an arc whose boundary lies in $F \cap S$. Since $S$ is $∂$-incompressible rel $F$ in $M$ and $S$ is incompressible in $M$ any outermost such arc can be removed by an isotopy. Thus $G \cap S = ∅$, and we may regard $G$ as lying in $M_0$. For homological reasons $G \cap C$ must lie in $C_1$ or $C_2$, say $C_1$. Since $D$ is a meridinal disk of $V$ we must have for homological reasons that $∂D$ splits $F_0$ into two components such that one contains $F_0 \cap S_0$ and $F_0 \cap C_1$ and the other contains $F_0 \cap S_1$ and $F_0 \cap C_2$. Let $M'_1 = M_0 \cup U_1$. Then $M'_1$ is homeomorphic to $(S_0 \cup C_0) \times [0, 1]$ with $S_0 \cup C_0 = (S_0 \cup C_0) \times \{0\}$. So $D$ is a compressing disk for $∂M'_1 = (S_0 \cup C_0)$ in $K$. This contradiction completes the proof of Lemma 3.3.$□$

This completes the proof of Lemma 3.2.$□$

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This completes the proof of Theorem 3.1. □

4. Embeddings of Torus Knot Groups

In this section we prove a technical result concerning embeddings of torus knot groups in torus knot groups which will be used in the next section to distinguish among the fundamental groups of the complements of a certain uncountable collection of arcs. Recall that the fundamental group of the complement of a \((p, q)\) torus knot is the group \(G_{p,q} = \langle x, y \mid x^p = y^q \rangle\).

Lemma 4.1. Let \(p, q, r, s\) be primes such that \(p < q < r < s\). Then \(G_{p,q}\) embeds in \(G_{r,s}\) if and only if \(p = r = q = s\).

Proof. Let \(Z(G)\) denote the center of the group \(G\). Recall that \(Z(G_{p,q})\) is an infinite cyclic group generated by \(x^p\) and that \(G/Z(G_{p,q}) \cong \mathbb{Z}_p * \mathbb{Z}_q\). Recall also that a free product of two non-trivial groups has trivial center and that any element of finite order in a free product is conjugate to an element of one of the factors. (See [8] pp. 140–141, 100–101.)

We may assume that \(G_{p,q}\) is a subgroup of \(G_{r,s}\). Let \(K = G_{p,q} \cap Z(G_{r,s})\). Then \(K\) is a subgroup of \(Z(G_{p,q})\) and is the kernel of the restriction of the natural projection \(G_{r,s} \to \mathbb{Z}_r * \mathbb{Z}_s\) to \(G_{p,q}\). If \(u \in G_{r,s}\), then let \(\bar{u}\) denote its image in \(\mathbb{Z}_r * \mathbb{Z}_s\).

Suppose \(K = Z(G_{p,q})\). Then we have an embedding \(\mathbb{Z}_p * \mathbb{Z}_q \to \mathbb{Z}_r * \mathbb{Z}_s\). Since \(\bar{x}\) has order \(p\) it must be conjugate to an element of \(\mathbb{Z}_r\) or of \(\mathbb{Z}_s\), hence \(p\) divides \(r\) or \(p\) divides \(s\), hence since \(r\) and \(s\) are prime we have \(p = r\) or \(p = s\). Similarly \(q = r\) or \(q = s\). Since \(p < q\) and \(r < s\) we must have \(p = r = q = s\).

Now suppose that \(K\) is a proper subgroup of \(Z(G_{p,q})\). Then it is generated by \(x^p k\) for some \(k \geq 0, k \neq 1\). Let \(G_{p,q,k} = G_{p,q}/K\). It embeds in \(\mathbb{Z}_r * \mathbb{Z}_s\) and has presentation \(\langle x, y \mid x^p = y^q, x^p k = 1 \rangle\). By the Kurosh subgroup theorem [3], \(G_{p,q,k}\) must be a free product of cyclic groups and so must either be cyclic or have trivial center. It thus suffices to show that neither of these is the case.

For \(k = 0\) this group is just \(G_{p,q}\), and we are done. So assume \(k \geq 2\). Define functions \(f, g : \mathbb{Z}_{pqk} \to \mathbb{Z}_{pqk}\) by \(f(n) = n + q \mod pqk\) and \(g(n) = n + p \mod pqk\). Then \(f\) and \(g\) are one to one and so may be regarded as elements of the symmetric group \(S_{pqk}\).

Define \(\psi : G_{p,q,k} \to S_{pqk}\) by \(\psi(\bar{x}) = f\) and \(\psi(\bar{y}) = g\). Then \(\psi\) is well defined because \(f(q) = n + pq = g(q) = n + qp\) and \(f(p) = n + qk = n + pk\). Since \(\psi(\bar{x}^p) = f^p \neq \text{id}\) we have that \(Z(G_{p,q,k})\) is non-trivial. Since \(G_{p,q,k}\) maps onto \(\mathbb{Z}_p * \mathbb{Z}_q\) it is non-cyclic, and so we are done. □

5. Uncountably Many Arcs

Theorem 5.1. There are uncountably many arcs \(A_i\) in \(S^3\) such that:

(1) \(\pi_1(S^3 - A_i)\) is indecomposable and non-trivial.

(2) \(\pi_1(S^3 - A_i)\) and \(\pi_1(S^3 - A_j)\) are isomorphic if and only if \(i = j\).

(3) \(A_i\) is wildly embedded precisely at its endpoints.
Proof. We first outline the proof and then fill in the details with a sequence of lemmas.

The construction of the $A_i$ will have a pattern similar to that of the Fox-Artin arc. $S^3 - \partial A_i$ will be parametrized as $S^2 \times \mathbb{R}$, and for each integer $n$ we will have that $A_i$ meets $S^2 \times [n, n+1]$ in three properly embedded arcs $\alpha_n$, $\beta_n$, and $\gamma_n$, where $\alpha_n$ runs from $S^2 \times \{n\}$ to itself, $\beta_n$ runs from $S^2 \times \{n\}$ to itself, and $\gamma_n$ runs from $S^2 \times \{n\}$ to $S^2 \times \{n+1\}$. These arcs will be chosen so that the exterior $X_n$ of $\alpha_n \cup \beta_n \cup \gamma_n$ in $S^2 \times [n, n+1]$ is irreducible and $\partial$-irreducible. Hence by Lemma 2.2 we will have that $\pi_1(S^3 - A_i)$ is indecomposable and non-trivial. Thus $A_i$ will be wild. It will clearly be tame at points not in $\partial A_i$. It will be wild at both endpoints since otherwise its complement would be simply connected. (Any meridian of the arc would bound a disk consisting of an annulus which follows the arc to a tame endpoint and is then capped off by a disk behind it. In fact it can be shown as in [4, Example 1.2] that $S^3 - A_i$ would be homeomorphic to $\mathbb{R}^3$.)

A map is $\pi_1$-injective if it induces an injection on fundamental groups; the same term is applied to a submanifold if its inclusion map has this property. The arcs will be chosen so that the interior of $X_n$ will contain a $\pi_1$-injective submanifold $Q_n$, which is homeomorphic to the exterior of a $(p_n, q_n)$ torus knot in $S^3$, where $p_n$ and $q_n$ are primes with $p_n < q_n$. It will follow from the $\partial$-irreducibility of all the $X_n$ that $\pi_1(S^3 - A_i)$ will have a subgroup isomorphic to $\pi_1(Q_n)$. Moreover it will be shown that any subgroup of $\pi_1(S^3 - A_i)$ which is isomorphic to a $(p, q)$ torus knot group for primes $p$ and $q$ with $p < q$ must be isomorphic to one of the $\pi_1(Q_n)$. We then let $J$ be the set of all pairs of primes $(p, q)$ with $p < q$ and let $2^J$ be the set of all subsets of $J$. For each non-empty $i \in 2^J$ we construct an arc $A_i$ as above such that the $(p, q)$ torus knot subgroups of $\pi_1(S^3 - A_i)$ with $(p, q) \in J$ are precisely those for which $(p, q) \in i$. It follows that $\pi_1(S^3 - A_i)$ and $\pi_1(S^3 - A_j)$ are isomorphic if and only if $i = j$. Since $2^J$ is uncountable we will be done.

We next recall some terminology. Let $M$ be a compact, connected, orientable 3-manifold. We say that $M$ is anatoidal if every properly embedded, incompressible torus $S^1 \times S^1$ in $M$ is $\partial$-parallel in $M$ and is anannular if every properly embedded, incompressible annulus $S^1 \times [0, 1]$ in $M$ is $\partial$-parallel in $M$. If $M$ is irreducible, $\partial$-irreducible, anannular and atoroidal, contains a 2-sided, properly embedded incompressible surface, and is not a 3-ball, then $M$ is excellent; the same term is applied to a compact, properly embedded 1-manifold in a compact 3-manifold $P$ if its exterior in $P$ has these properties.

**Lemma 5.2.** Let $Y'$ and $Y''$ be excellent 3-manifolds. Suppose $Y = Y' \cup Y''$, where $S = Y' \cap Y'' = \partial Y' \cap \partial Y''$ is a compact surface such that $S$ is incompressible in $Y'$ and in $Y''$, $\partial Y' - \text{Int} S$ is incompressible in $Y'$, $\partial Y'' - \text{Int} S$ is incompressible in $Y''$, and each component of $S$ has negative Euler characteristic. Then $Y$ is excellent.

**Proof.** This is [4, Lemma 2.1]. □

We now construct the arcs. Let $R$ be an unknotted solid torus in the interior of $S^2 \times [0, 1]$. Let $P = S^2 \times [0, 1] - \text{Int} R$. (We say that $R$ is unknotted if there is a properly embedded disk $E$ in $P$ such that $\partial E \subseteq \partial R$ and a meridinal disk $D$ of $R$ such that $\partial D$ and $\partial E$ meet transversely in a single point.)

**Lemma 5.3.** There exist disjoint properly embedded arcs $\alpha$, $\beta$, and $\gamma$ in $P$ such that $\partial \alpha \subseteq S^2 \times \{0\}$, $\partial \beta \subseteq S^2 \times \{1\}$, $\gamma$ has one endpoint in $S^2 \times \{0\}$ and the other in $S^2 \times \{1\}$, and $\alpha \cup \beta \cup \gamma$ is excellent.
Proof. Let \( \alpha', \beta', \) and \( \gamma' \) be any arcs in \( P \) whose boundaries satisfy the given conditions. By [14, Theorem 1.1] any compact, properly embedded 1-manifold in a compact, connected, orientable 3-manifold which meets each 2-sphere boundary component in at least two points is homotopic relative its boundary to a properly embedded 1-manifold which is excellent. Let \( \alpha, \beta, \) and \( \gamma \) be the respective components of this new 1-manifold.

For those who prefer a more concrete construction of such arcs we give an alternative proof at the end of this section.

Now let \( Q \) be the exterior of a \( (p, q) \) torus knot in \( S^3 \), where \( (p, q) \in J \). Glue \( P \) and \( Q \) together by identifying \( \partial P \) with \( \partial Q \) in such a way that \( \partial E \) is identified with a meridian of \( \partial Q \). Then the union of \( Q \) and a regular neighborhood of \( E \) in \( P \) is a 3-ball, and so \( P \cup Q \) is homeomorphic to \( S^2 \times [0, 1] \). Let \( Y \) be the exterior of \( \alpha \cup \beta \cup \gamma \) in \( P \) and \( X = Y \cup Q \). It follows from the irreducibility and \( \partial \)-irreducibility of \( Y \) and of \( Q \) that \( X \) is irreducible and \( \partial \)-irreducible and that \( Q \) is \( \pi_1 \)-injective in \( X \).

We now repeat this construction using \( (n, n) \) torus knots with \( (p, q) \in i \) to obtain \( \alpha_n, \beta_n, \gamma_n, P_n, Q_n, Y_n, \) and \( X_n \) contained in \( S^2 \times [n, n + 1] \). We construct an arc \( A_i \) by identifying the endpoints of the arcs so that the arcs occur in the sequence \( \ldots, \gamma_n, \alpha_{n+1}, \beta_n, \gamma_{n+1}, \ldots \) on \( A_i \). The exterior \( W_i \) of \( A_i \) then satisfies the hypotheses of Lemma 2.2, and so \( \pi_1(S^3 - A_i) \) is indecomposable and non-trivial. Moreover the incompressibility of each \( X_n \cap X_{n+1} \) implies that each \( Q_n \) is \( \pi_1 \)-injective in \( W_i \).

We next review some characteristic submanifold theory [6, 7, 8], following [7] but restricting attention to the special case which we will need. We first refine our notion of parallel surfaces. A pair \( (M, F) \) is an irreducible 3-manifold pair if \( M \) is a compact, orientable, irreducible 3-manifold and \( F \) is a compact, incompressible surface in \( \partial M \). Let \( S \) and \( S' \) be disjoint compact surfaces in \( M \) such that \( S \) is properly embedded in \( M, S' \) is either properly embedded in \( M \) or contained in \( \partial M \), and \( \partial S \cup \partial S' \) is contained in \( F \). We say that \( S \) and \( S' \) are parallel in \( (M, F) \) if there is a parallelism \( S \times [0, 1] \) from \( S \) to \( S' \) such that \( (\partial S) \times [0, 1] \) is contained in \( F \); if \( S' \subseteq F \) we say that \( S \) is \( F \)-parallel. Our old definitions of “parallel” and “\( \partial \)-parallel” in \( M \) correspond to the case of \( F = \partial M \).

The characteristic pair of the irreducible 3-manifold pair \( (M, \partial M) \) is a certain irreducible 3-manifold pair \( (\Sigma, \Phi) \) such that \( \Sigma \subseteq M \) and \( \Sigma \cap \partial M = \Phi \). For its definition and proof of existence see [8, Chapter V]. We will limit our discussion to two basic issues: using \( (\Sigma, \Phi) \) and recognizing \( (\Sigma, \Phi) \). The property we will use is that any \( \pi_1 \)-injective map from a Seifert fibered space with non-cyclic fundamental group into \( M \) which is not homotopic to a map whose image lies in \( \partial M \) must be homotopic to a map whose image lies in \( \Sigma \) [8, p. 138].

We will recognize \( \Sigma \) by recognizing its components and using the Splitting Theorem [8, p. 157] to recognize the frontier \( Fr \Sigma \) of \( \Sigma \) in \( M \). The components \( (\sigma, \varphi) \) of \( (\Sigma, \Phi) \) are Seifert pairs, i.e. \( \sigma \) is either an \( I \)-bundle over a compact surface with \( \varphi \) the associated \( O \)-bundle or \( \sigma \) is a Seifert fibered space with \( \varphi \) a union of fibers in \( \partial \sigma \). One of the properties we will need is that the inclusion map from \( (\sigma, \varphi) \) into \( (M, \partial M) \) is not homotopic as a map of pairs to a map whose image lies in \( \Sigma - \sigma \). Also the components of \( Fr \Sigma \) are incompressible annuli and tori none of which is \( \partial \)-parallel in \( M \) though some components may be parallel in \( (M, \partial M) \) to each other. (See the examples in [8, Chapter IX].) A union \( Fr^* \Sigma \) of components of \( Fr \Sigma \) such that no two components of \( Fr^* \Sigma \) are parallel in \( (M, \partial M) \) to each other and \( Fr^* \Sigma \) is maximal with respect to inclusion among all such unions is called a reduction of \( Fr \Sigma \). We call the components of \( Fr \Sigma - Fr^* \Sigma \) redundant components of
If \( \mathcal{T} \) is minimal with respect to inclusion among all compact, properly embedded surfaces in \( M \) satisfying (a) and (b), then by the Splitting Theorem \( \mathcal{T} \) is isotopic to \( Fr^* \Sigma \).

Now let \( M_k = \bigcup_{n=-k}^k X_n \) and \( C_k = \bigcup_{n=-k}^k Q_n \).

**Lemma 5.4.** \( (M_k, \partial M_k) \) is an irreducible 3-manifold pair, and its characteristic pair \( (\Sigma, \Phi) = (C_k, \emptyset) \).

**Proof.** The irreducibility and \( \partial \)-irreducibility of \( M_k \) and the incompressibility of \( \partial C_k \) in \( M_k \) follow from the irreducibility and \( \partial \)-irreducibility of the \( X_n \), the incompressibility of the \( X_n \cap X_{n+1} \) in \( X_n \) and in \( X_{n+1} \), and the incompressibility of \( \partial Q_n \) in \( X_n \).

Let \( \mathcal{T} = \partial C_k \). Since \( \partial M_k \) is a surface of genus two no component of \( \mathcal{T} \) is \( \partial \)-parallel in \( M_k \). The components of \( (M'_{k}, \partial M_k) \) are the \( (Q_n, \emptyset) \) and \( (Z, \partial M_k) \), where \( Z = \bigcup_{n=-k}^k Y_n \).

Each \( Q_n \) is a Seifert fibered space. By Lemma 5.2 we have that \( Z \) is excellent and therefore \( (Z, \partial M_k) \) is a simple pair. Thus \( \mathcal{T} \) satisfies properties (a) and (b). Deleting any components of \( \mathcal{T} \) gives a surface which splits \( M_k \) into components one of which, say \( N \), is the union of \( Z \) and some of the \( Q_n \). Now \( N \) is not Seifert fibered since it contains \( \partial M_k \). It is not an I-bundle over a compact surface \( S \) since \( S \) would be covered by \( \partial M_k \), and so \( \pi_1(S) \cong \pi_1(N) \) could not contain the \( \mathbb{Z} \oplus \mathbb{Z} \) subgroup \( \pi_1(\partial Q_n) \). Finally \( (N, \partial M_k) \) is not a simple pair because \( \partial Q_n \) is not \( \partial \)-parallel in \( N \). Thus \( \mathcal{T} \) is minimal with respect to inclusion among surfaces satisfying (a) and (b). So by the Splitting Theorem \( \mathcal{T} = Fr^* \Sigma \).

By arguments similar to those applied above to \( N \) we have that \( (Z, \partial M_k) \) is not a Seifert pair. So if there are no redundant components we must have \( (\Sigma, \Phi) = (C_k, \emptyset) \), and we are done.

Suppose there is a redundant component. Then it must be a torus which is parallel in \( (M_k, \partial M_k) \) to \( \partial Q_n \) for some \( n \); denote it by \( T_n \). Thus we may assume that there is an embedding of \( T_n \times [0,1] \) in \( M_k \) such that \( T_n \times [0,1] \) meets \( Q_n \) in \( T_n \times \{0\} = \partial Q_n \), \( T_n \times \{1\} = T_n \), and \( T_n \times \{0,1\} \) contains all other redundant tori which are parallel to \( \partial Q_n \). If there are such extra redundant tori, then they are isotopic in \( T_n \times [0,1] \) to tori of the form \( T_n \times \{t\} \) [17] Corollary 3.2. It follows that there is some component \( \sigma \) of \( \Sigma \) of the form \( T_n \times [r,s] \). Its inclusion map into \( M_k \) is homotopic to a map whose image lies in \( \Sigma - \sigma \), contradicting one of the properties of \( \Sigma \).

Thus there are no extra redundant tori. Now let \( Z' \) be the closure of the complement in \( Z \) of the union of all the products \( T_n \times [0,1] \). Then \( Z' \) is homeomorphic to \( Z \), and so \( (Z', \partial M_k) \) is a simple pair which is not Seifert pair. Thus \( T_n \times [0,1] \) is a component of \( \Sigma \), and \( (Q_n, \emptyset) \) is a simple pair. Now in fact \( (Q_n, \emptyset) \) actually is a simple pair. However, it is also a Seifert fibered space with non-cyclic fundamental group. Its inclusion map cannot be homotopic to a map whose image lies in \( \partial M_k \) because \( \pi_1(M_k) \) has no \( \mathbb{Z} \oplus \mathbb{Z} \) subgroups.
Thus it must be homotopic to a map whose image lies in some component $\sigma$ of $\Sigma$. In particular the image lies in the complement of $Q_n$.

Now it follows from [7, Squeezing Theorem, p. 139] or [3, Theorem IX.12] that $Q_n$ is actually isotopic to a submanifold of $\sigma$. This fact can be used to contradict our knowledge of the structure of $Z'$. We choose, however, to give the following somewhat more direct argument.

Let $p : \tilde{M}_k \to M_k$ be the covering map corresponding to $\pi_1(Q_n)$. There is a component $\tilde{Q}_n$ of $p^{-1}(Q_n)$ such that the restriction $\tilde{Q}_n \to Q_n$ of $p$ is a homeomorphism and $\pi_1(\tilde{Q}_n) \to \pi_1(M_k)$ is an isomorphism. It follows that $\pi_1(\partial \tilde{Q}_n) \to \pi_1(M_k - Int \tilde{Q}_n)$ is an isomorphism. Now the homotopy of $Q_n$ into its complement lifts to a homotopy of $\tilde{Q}_n$ into $\tilde{M}_k - Int \tilde{Q}_n$. This implies that $\pi_1(Q_n)$ is abelian, which is not the case. \hfill $\Box$

We now suppose that $\pi_1(S^3 - A_i)$ and $\pi_1(S^3 - A_j)$ are isomorphic. Then $\pi_1(W_i)$ and $\pi_1(W_j)$ are isomorphic, where $W_i$ and $W_j$ are the exteriors of $A_i$ and $A_j$, respectively. Since these spaces are irreducible and orientable, the sphere theorem implies that they are aspherical. Hence there is a map $h : W_j \to W_i$ such that $h_* : \pi_1(W_j) \to \pi_1(W_i)$ is an isomorphism. We then restrict $h$ to a $(p,q)$ torus knot space arising in the construction of $A_j$. This map is $\pi_1$-injective. Its image lies in some $M_k$. Since $\pi_1(\partial M_k)$ has no $\mathbb{Z} \oplus \mathbb{Z}$ subgroups Lemma 5.4 implies that it is homotopic to a map whose image lies in some $(r,s)$ torus knot space arising in the construction of $A_i$. By Lemma 4.1 we have that $(p,q) = (r,s)$. Thus $j \subseteq i$. The symmetric argument shows that $i \subseteq j$, concluding the proof of Theorem 5.1. \hfill $\Box$

**Alternative Proof of Lemma 5.3.** Figure 14 shows a three component tangle in a 3-ball. Figure 15 shows a two component tangle in a 3-ball. By [13, Proposition 4.1] and [12, Proposition 4.1] these two tangles are excellent. Let $Y'$ and $Y''$ be their respective exteriors.

We glue $Y'$ and $Y''$ together as indicated in Figure 16 to obtain the exterior $Y$ of the union of the arcs $\alpha$, $\beta$, and $\gamma$ in the space $P$ obtained by removing the interior of an unknotted solid torus $R$ contained in the interior of $S^2 \times [0,1]$. $S = Y' \cap Y'' = \partial Y' \cap \partial Y''$ has two components; each is a disk with two holes. Since a compact surface contained in an incompressible boundary component of a compact 3-manifold is incompressible if none
of the components of its complement in the boundary component has closure a disk, we have that \( S \) is incompressible in \( Y' \) and in \( Y'' \). We now apply Lemma 5.2 to conclude that \( Y \) is excellent. □

![Figure 16: The three arcs \( \alpha, \beta, \) and \( \gamma \) in \((S^2 \times [0, 1]) - \text{Int} R\)](image-url)

References

[1] F. D. Ancel, et. al., *Problem Session*, Proc. 13th Annual Workshop in Geometric Topology, Colorado College, Colorado Springs, CO (June 13–15, 1996) 50–52.
[2] R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. (2) 49 (1948) 979–990.

[3] D. E. Galewski, J. G. Hollingsworth, and D. R. McMillan, Jr., *On the fundamental group and homotopy type of open 3-manifolds*, General Topology and Appl. 2 (1972) 299–313.

[4] C. McA. Gordon and R. A. Litherland, *Incompressible surfaces in branched coverings*, in *The Smith Conjecture*, Pure Appl. Math. 112, Academic Press, Orlando, FL (1984) 139–152.

[5] J. Hempel, *3-Manifolds*, Ann. of Math. Studies 86, Princeton (1976).

[6] W. Jaco, *Lectures on three-manifold topology*, CBMS Regional Conference Series in Math. 43, Amer. Math. Soc. (1980).

[7] W. Jaco and P. Shalen, *Seifert fibered spaces in 3-manifolds*, Memoirs Amer. Math. Soc. 21 No. 220 (1979).

[8] K. Johansson, *Homotopy equivalences of 3-manifolds with boundaries*, Lecture Notes in Math. 761, Springer-Verlag (1979).

[9] A. G. Kurosh, *The theory of groups*. Translated from the Russian and edited by K. A. Hirsch. 2nd English ed. 2 volumes Chelsea Publishing Co., New York (1960).

[10] W. S. Massey, *Algebraic Topology: An Introduction*, Graduate Texts in Mathematics 56, Springer-Verlag, New York-Heidelberg (1977).

[11] W. H. Meeks, III and S. T. Yau, *The equivariant Dehn’s lemma and loop theorem*, Comment. Math. Helv. 56 (1981) 225–239.

[12] R. Myers, *Simple knots in compact, orientable 3-manifolds*, Trans. Amer. Math. Soc. 273 (1982) 75-91.

[13] R. Myers, *Homology cobordisms, link concordances, and hyperbolic 3-manifolds*, Trans. Amer. Math. Soc. 278 (1983) 271-288.

[14] R. Myers, *Excellent 1-manifolds in compact 3-manifolds*, Topology Appl. 49 (1993) 115–127.

[15] H. Roslaniec, *On decomposition spaces of $E^n$ with a small homeomorphism group*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976) 901–904.

[16] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. 87 (1968) 56–88.