THE DUALITY OPERATION IN THE CHARACTER RING
OF A FINITE CHEVALLEY GROUP

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It is possible (as in [4]) to define a duality operation \( \xi \mapsto \xi^* \) in the ring of virtual characters of an arbitrary finite group with a split \((B, N)\)-pair of characteristic \( p \). Such a group arises as the fixed points under a Frobenius map of a connected reductive algebraic group, defined over a finite field [1]. This paper contains statements of several general properties of the duality map \( \xi \mapsto \xi^* \) and two related operations (see \$2\$ and \$4\$). The duality map \( \xi \mapsto \xi^* \) generalizes the construction in [2] of the Steinberg character, and interacts well with the organization of the characters from the point of view of cuspidal characters (\$6\$). It is hoped that there is also a useful interaction with the Deligne-Lusztig virtual characters \( R_T^G \theta \). Partial results have been obtained in this direction (\$5\$). Detailed proofs will appear elsewhere.

1. Let \( G \) be a finite group with split \((B, N)\)-pair of characteristic \( p \). Let \((W, R)\) be the Coxeter system, and let \( P_j = L_j V_j \) be the standard parabolic subgroup corresponding to \( J \subseteq R \), with \( V_j = O_p(P_j) \) (see [3] for definitions and notations). Let \( \text{char}(G) \) denote the ring of virtual characters of \( G \), and \( \text{Irr}(G) \) the set of irreducible characters of \( G \), all taken in the complex field. For \( J \subseteq R \) and \( \xi \in \text{char}(G) \), define

\[
\xi_{(P_j/V_j)} = \sum_{\Im} \xi_{(P_j/V_j)} \sim
\]

where \( \sim \) denotes extension to \( P_j \) via the projection \( P_j \rightarrow L_j \cong P_j/V_j \), and the sum is over all \( \lambda \in \text{Irr}(L_j) \). Let \( \xi_{(P_j)} = \xi_{(P_j/V_j)} \sim \). The duality map is then defined by:

1.2 DEFINITION. \( \xi^* = \sum_{J \subseteq R} (-1)^{|J|} \xi_{(P_j)} \), for all \( \xi \in \text{char}(G) \).

2. The truncation map \( \xi \mapsto \xi_{(P_j/V_j)} \) and the map \( \lambda \mapsto \lambda^G \) behave in much the same way as ordinary restriction and induction. The following basic properties follow directly from the structure theorems [3].

2.1 FROBENIUS RECIPROCITY. Let \( \xi \in \text{char}(G) \) and \( \lambda \in \text{char}(L_j) \). Then

\[ \xi \mapsto \xi_{(P_j)} \text{ and } \lambda \mapsto \lambda^G \text{ behave in much the same way as ordinary restriction and induction.} \]
If $K \subseteq J \subseteq R$, let $Q_K$ be the standard parabolic subgroup $P_K \cap L_J$ of $L_J$ and let $V_{J,K} = O_p(Q_K) = L_J \cap V_K$. Then if $\xi \in \text{char}(G)$ and $\xi \in \text{char}(L_J)$, we have

$$((\xi(P_J/V_J)))(Q_K/V_{J,K}) = \xi(P_K/V_K),$$

and

$$((\lambda L_J)^G = \lambda^G.$$
4. The first main result relates duality and the operations \( \xi \rightarrow \xi_{(p_f/V_f)} \) and \( \lambda \rightarrow \widetilde{\lambda}^G \). Part (1) is Theorem 1.3 of [4].

**Theorem.** (1) \( (\xi^*)_{{(p_f/V_f)}} = (\xi_{(p_f/V_f)})^* \) for \( J \subseteq R, \xi \in \text{char}(G) \)

(2) \( (\widetilde{\lambda}^G)^* = (\lambda^*)_{-G} \) for \( J \subseteq R, \lambda \in \text{char}(L_J) \).

We provide a sketch of the proof of (2). Let \( J_1 = J \). Using 2.4, 2.2, and then Lemma 3.1 (noting that \( L_{K_1} = \cup L_{K_2} \) by Proposition 2.6 of [3]) we have

\[
(\widetilde{\lambda}^G)^* = \sum_{J_2 \subseteq R} (-1)^{|J_2|} \sum_{w \in \mathcal{W}_{J_1}, J_2} \lambda(Q_{K_1}/V_{J_1}, K_1)_{-G}
\]

The proof is then completed by applying Lemma 3.2 and 2.2.

**4.2 Theorem.** The map \( \xi \rightarrow \xi^* \), from \( \text{char}(G) \rightarrow \text{char}(G) \) is an isometry of order two. In particular, \( \xi^{**} = \xi \) and \( \pm \xi^* \in \text{Irr}(G) \), whenever \( \xi \in \text{Irr}(G) \).

In order to prove Theorem 4.2, one first proves that \( (\xi_1, \xi_2)_G = (\xi_1^*, \xi_2^*)_G \). It then suffices to prove \( \xi^{**} = \xi \). The key is to apply Theorem 4.1 part (1) to the expression for \( \xi^{**} \). We have

\[
\xi^{**} = \sum_{J \subseteq R} (-1)^{|J|} \xi_{(p_f/V_f)}^*_{-G}
\]

\[= \sum_{J \subseteq R} (-1)^{|J|} \sum_{K \subseteq J} (-1)^{|K|} \xi_{(p_K)^G}
\]

using 2.2. To finish the proof, note that \( \sum (-1)^{|J|} \) summed over all \( J \) such that \( K \subseteq J \subseteq R \) is zero unless \( K = R \).

5. It is clear that \( \xi^* = (-1)^{|R|} \xi \) for any cuspidal \( \xi \in \text{Irr}(G) \). Thus by applying Theorem 4.1 part (2) we have:

**5.1 Corollary.** Let \( \lambda \in \text{Irr}(L_\lambda) \) be cuspidal. Then \( (\widetilde{\lambda}^G)^* = (-1)^{|R|}\widetilde{\lambda}^G \).

Thus duality permutes (up to sign) the components of \( \widetilde{\lambda}^G \). We can thus determine the “sign” of \( \xi^* \) as follows: \( (-1)^{|J|} \xi^* \) is in \( \text{Irr}(G) \) if \( \xi \in \text{Irr}(G) \) is a component of \( \widetilde{\lambda}^G, \lambda \in \text{Irr}(L_J) \) cuspidal. In particular, \( \xi \rightarrow \xi^* \) permutes the principal series characters, i.e. the components of \( \widetilde{\lambda}^G, \lambda \in \text{Irr}(L_G) \). A more explicit result is known for the components \( \xi_{\varphi, q} \) of \( \Gamma_{B(q)} \) in a system of groups \( \{G(q)\} \) of type \( (W, R) \). Specifically, \( \xi_{\varphi, q}^* = \xi_{e\varphi, q} \) where \( e \) is the sign character of \( \mathcal{W} \) ([4]).
Finally, consider the case $G = G^F$ where $G$ is a reductive algebraic group and $F : G \to G$ is a Frobenius map over $F_q$. Let $R_T^G \theta$ denote the Deligne-Lusztig generalized character of $G$ (a $F$-stable maximal torus of $G$, $\theta$ a linear character of $T^F$). It is natural to ask whether

\[(5.2) \quad (R_T^G \theta)^* = \pm R_T^G \theta\]

holds. The following suggests the answer is yes.

\[(5.3) \quad (R_T^G \theta)^*(s) = \pm R_T^G \theta(s)\]

for semisimple elements $s$ of $G$. The $\pm$ sign in 5.3 does not depend on the particular element $s$ of $G$. The proof of 5.3 uses several results of [5]. (Note added in proof: The conjecture 5.2 has been proved by G. Lusztig.)

5.4 Example. Let $G = G^F$ as above, with (relative) Coxeter system $(W, R)$. Let $V$ be the set of unipotent elements of $G$ and let $\epsilon_V$ be the characteristic function of $V$. A recent result of Springer (Theorem 1 of [6])\(^1\) shows

\[\epsilon_V = q^d \sum_{J \subseteq R} (-1)^{|J|} |P_J|^{-1} 1_{G,J}^G\]

where $d = \dim(G/B)$, $B$ a Borel subgroup of $G$. Applying Theorems 4.1 and 4.2 we have:

5.5 Theorem. (1) $\epsilon_V^* = (q^d/|G|) \rho_G$ where $\rho_G$ is the regular character of $G$.

(2) For $\xi \in \text{Irr}(G)$,

\[\frac{1}{\xi(1)} \sum_{\nu \in \nu} \xi(\nu) = q^d (\xi^*(1)/\xi(1)).\]

(3) For $\xi \in \text{Irr}(G), |\xi^*(1)|_p = \xi(1)_p$, where $p$ is the characteristic of $F_q$ and $n_p$ is the $p'$ part of $n$.

(4) For $\xi \in \text{Irr}(G), 1/\xi(1) \sum_{\nu \in \nu} \xi(\nu)$ is, up to sign, a power of $p$.

Part (4) of Theorem 5.5 confirms a special case of a conjecture of Macdonald (see [6]), namely the case when $q = p$ is prime.

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\(^1\)The author is indebted to T. A. Springer for communicating both his results in [6] and the suggestion of G. Lusztig of combining them with duality.
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