Number of vertices in graphs with locally small chromatic number and large chromatic number

Ilya I. Bogdanov∗

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Abstract

We discuss the minimal number of vertices in a graph with a large chromatic number such that each ball of a fixed radius in it has a small chromatic number. It is shown that for every graph $G$ on $\sim ((n + rc)/(c + rc))^{r+1}$ vertices such that each ball of radius $r$ is properly $c$-colorable, we have $\chi(G) \leq n$.

1 Introduction

Let $G = (V, E)$ be a graph (with no loops or multiple edges). By $d(u, v) = d_G(u, v)$ we denote the distance between the vertices $u, v \in V$. A subset $V_1 \subseteq V$ is independent if none of the edges has both endpoints in $V_1$. The chromatic number $\chi(G)$ of $G$ is the minimal number of colors in a proper coloring of $G$, that is — the minimal number of parts in a partition of $V$ into independent subsets.

Definition 1.1. Let $r$ be a nonnegative integer. The ball of radius $r$ with center $v \in V$ is the set $U_r(v, G) = \{u \in G : d(u, v) \leq r\}$. For $r \geq 1$, the $r$-local chromatic number $\ell \chi_r(G)$ of a graph $G$ is the maximal chromatic number of a ball of radius $r$ in $G$.

Notice that even for $r = 1$ our definition of the local chromatic number is quite different from that introduced by Erdős et al. in [3].

By a well-known result of Erdős [3], for every integer $g > 2$ and every $n > n_0(g)$, there exists a graph on $n^{2g+1}$ vertices of girth $g$ and chromatic number greater than $n$; thus for every $r$ there exist a graph $G$ with $\ell \chi_r(G) = 2$ and arbitrarily large $\chi(G)$. Later Erdős [4] conjectured that for every positive integer $s$ there exists a constant $c_s$ such that the chromatic number of each graph $G$ having $N$ vertices and containing no odd cycles of length less than $c_s N^{1/s}$ does not exceed $s + 1$. This conjecture was proved by Kierstead, Szemerédi, and Trotter [6]. In fact, they have proved a more general result; we will formulate this result in terms of the following notion.

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Definition 1.2. Let $n$, $r$, and $c$ be positive integers. Denote by $f_c(n, r)$ the maximal integer $f$ with the following property: If $G$ is a graph on $f$ vertices and $\ell\chi_r(G) \leq c$ then $\chi(G) \leq n$.

Then the aforementioned result can be formulated as

$$f_c(k(c-1)+1, r) \geq \left\lceil \frac{r}{2k} \right\rceil^k,$$  \hspace{1cm} (1.1)$$

while the result by Erdős yields

$$f_c(n, r) \leq f_2(n, r) < n^{4r+5} \quad \text{for all } n > n_0(r). \hspace{1cm} (1.2)$$

Several examples (cf., for instance, [10, 11]) show that the estimate (1.1) has a correct order in $r$ for $c = 2$. In [2] we show that this order is sharp even for all $c$. Namely, it is shown that

$$f_c(k(c-1), r) < \frac{(2rc+1)^k - 1}{2r}.$$  \hspace{1cm} (1.3)$$

On the other hand, the estimate (1.1) does not work for $n \gtrsim (c-1)r$. For $c = 2$, Berlov and the author [12] obtained the estimate

$$f_2(n, r) \geq \frac{(n + r + 1)(n + r + 2) \cdots (n + 2r + 1)}{2^r(r+1)^{r+1}}.$$  \hspace{1cm} (1.3)$$

It is worth mentioning that for several specific series of parameters there exist almost tight bounds of $f_c(n, r)$. Firstly, asymptotics of $f_2(n, 1)$ is tightly connected with the asymptotics of Ramsey numbers $R(n, 3)$. In the papers of Ajtai, Komlós, and Szemerédi [1] and Kim [8] it is shown that $c_1 \frac{n^2}{\log n} \leq R(n, 3) \leq c_2 \frac{n^2}{\log n}$ for some absolute constants $c_1$, $c_2$. One can check that these results imply the bounds

$$c_3 n^2 \log n \leq f(n, 2) \leq c_4 n^2 \log n$$

for some absolute constants $c_3$, $c_4$.

The asymptotics of $f_2(3, r)$ is also well investigated. From the generalized Mycielski construction by Stiebitz [11] it follows that $f_2(3, r) < 2r^2 + 5r + 4$. On the other hand, Jiang [7] showed that $f_2(3, r) \geq (r-1)^2$.

The aim of this paper is to extend this estimate for larger values of $c$. For the convenience, we use the notation $n^F = n(n + 1) \cdots (n + k - 1)$. We prove the following results.

Theorem 1.1. For all positive integer $n$, $r$, and $c > 1$ we have

$$f_c(n, r) \geq \frac{(n/c + r/2)^{r+1}}{(r+1)^{r+1}}.$$  \hspace{1cm} (1.4)$$

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2 Main result

For a graph \( G = (V, E) \) and a subset \( V_1 \subseteq V \), we denote by \( G[V_1] \) the induced subgraph of \( G \) on the set \( V_1 \). For \( v \in V \), we denote by \( S_r(v, G) = \{ u \in V \mid d_G(u, v) = r \} \) the sphere with radius \( r \) and center \( v \). In particular, \( S_0(v, G) = U_0(v, G) = \{ v \} \). Denote also by \( \partial^\text{out}_r V_1 = \{ u \in V \setminus V_1 \mid \exists v \in V_1: (u, v) \in E \} \) the outer boundary of a subset \( V_1 \subseteq V \). In particular, \( S_r(v, G) = \partial^\text{out}_r U_{r-1}(v, G) \).

Our estimate is based on the following lemma.

Lemma 2.1. For every graph \( G = (V, E) \) and every positive integer \( r \), there exists a decomposition \( V = U \sqcup N \) such that each connected component of \( U \) lies in some ball in \( G \) of radius \( r \), and
\[
|N| \leq \frac{r+1}{r\sqrt{|V|}}|V|.
\]

Proof. Set \( v = |V| \). We will construct inductively a sequence of partitions of \( V \) into nonintersecting parts,
\[
V = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_s \sqcup N_s \sqcup V_s,
\]
such that the following conditions are satisfied:

(i) for all \( i = 1, \ldots, s \) we have \( \partial^\text{out}_r U_i \subseteq N_i \); moreover, \( \partial^\text{out}_r V_s \subseteq N_s \);

(ii) for every \( i = 1, 2, \ldots, s \) the graph \( G[U_i] \) is contained in some ball in \( G \) of radius \( r \);

(iii) \((r+1)(|U_1| + \cdots + |U_s|) \geq |N_s|\).

For the base case \( s = 0 \), we may set \( V_0 = V \), \( N_0 = \emptyset \) (there are no sets \( U_i \) in this case).

For the induction step, suppose that the partition \( V = U_1 \sqcup U_2 \sqcup \cdots \sqcup U_{s-1} \sqcup N_{s-1} \sqcup V_{s-1} \) has been constructed, and assume that the set \( V_{s-1} \) is nonempty. Consider the graph \( G_{s-1} = G[V_{s-1}] \) and choose an arbitrary vertex \( u \in V_{s-1} \). Now consider the sets
\[
U_0(u, G_{s-1}) = \{ u \}, \quad U_1(u, G_{s-1}), \quad \ldots, \quad U_{r+1}(u, G_{s-1}).
\]

One of the ratios
\[
\frac{|U_1(u, G_{s-1})|}{|U_0(u, G_{s-1})|}, \quad \frac{|U_2(u, G_{s-1})|}{|U_1(u, G_{s-1})|}, \quad \ldots, \quad \frac{|U_{r+1}(u, G_{s-1})|}{|U_r(u, G_{s-1})|}
\]
does not exceed \( \frac{r+1}{\sqrt{v}} \), since the product of these ratios is
\[
|U_{r+1}(u, G_{s-1})| \leq v.
\]

So, let us choose \( 1 \leq m \leq r + 1 \) such that
\[
\frac{|U_m(u, G_{s-1})|}{|U_{m-1}(u, G_{s-1})|} \leq \frac{r+1}{\sqrt{v}}.
\]

Now we set
\[
U_s = U_{m-1}(u, G_{s-1}), \quad N_s = N_{s-1} \cup S_m(u, G_{s-1}), \quad V_s = V_{s-1} \setminus U_m(u, G_{s-1}).
\]
Since the condition (i) was satisfied on the previous step, we have
\[(\partial_G V_s) \cup (\partial_G U_s) \subseteq (\partial_G V_{s-1}) \cup S_m(u, G_{s-1}) \subseteq N_s\]
so this condition also holds now. The condition (ii) is satisfied trivially. Finally, the choice of \(m\) and the condition (iii) for the previous step imply that
\[
\frac{r'}{\sqrt{v}} \cdot |U_s| = \frac{r'}{\sqrt{v}} \cdot |U_{m-1}(u, G_{s-1})| \geq |U_m(u, G_{s-1})|,
\]
and hence
\[
\left(\frac{r'}{\sqrt{v}} - 1\right)(|U_1| + \cdots + |U_s|) \geq |N_{s-1}|
\]
and hence
\[
\left(\frac{r'}{\sqrt{v}} - 1\right)(|U_1| + \cdots + |U_s|) \geq |N_{s-1}| + |U_m(u, G_{s-1})| - |U_{m-1}(u, G_{s-1})| = |N_s|.
\]
Thus, the condition (iii) also holds on this step.

Continuing the construction in this manner, we will eventually come to the partition with \(V_s = \emptyset\) since the value of \(|V_s|\) strictly decreases. As the result, we obtain the partition \(V = U_1 \cup U_2 \cup \cdots \cup U_s \cup N_s\) such that \(|N_s| \leq \left(\frac{r'}{\sqrt{v}} - 1\right)(|U_1| + \cdots + |U_s|)\). So, setting \(U = U_1 \cup \cdots \cup U_s\) and \(N = N_s\) we get
\[
\frac{r'}{\sqrt{v}} \cdot |N| \leq \left(\frac{r'}{\sqrt{v}} - 1\right)|U| + \left(\frac{r'}{\sqrt{v}} - 1\right)|N| = |V|\left(\frac{r'}{\sqrt{v}} - 1\right),
\]
or \(|N| \leq |V| \frac{r'}{\sqrt{v}} - 1\), as required. \(\square\)

**Corollary 2.1.** Setting \(v = f_c(n, r) + 1\), we have
\[
v \geq \frac{r'}{\sqrt{v}} \left(f_c(n - 2, k) + 1\right).
\] \(\text{(2.2)}\)

**Proof.** Let \(G = (V, E)\) be a graph on \(v\) vertices such that \(\ell_{\chi_r}(G) \leq c\) although \(\chi(G) > n\). Applying Lemma 2.1 we get a decomposition \(V = U \cup N\) such that \(G[U]\) has a proper coloring in \(c\) colors. So, \(G[N]\) cannot be properly colored in \(n - c\) colors, hence \(|N| \geq f_c(n - c, r)\), hence the relation (2.1) yields (2.2). \(\square\)

The next proposition shows how to make an explicit estimate for \(f_c(n, r)\) from Corollary 2.1.

**Proposition 2.1.** Suppose that for some integer \(n_0 \geq 1\) and real \(a\) the inequality
\[
f(m, k) \geq \frac{(a + m/c)^{r+1}}{(r + 1)^{r+1}} - 1
\] \(\text{(2.3)}\)
holds for \(m = n_0\). Then the same estimate holds for all integer \(m \geq n_0\) with \(m - n_0 \equiv 0\) \((\text{mod } c)\).
Proof. We use the Induction on $m$ with step $c$; the base case holds by the conditions of the proposition.

Assume now that (2.3) holds for $m = n - c$ but not for $m = n$. Denote $v = f_c(n, r) + 1$; then we have

$$r+1\sqrt{v} < r+1\sqrt{\frac{(a+n/c+r)^{r+1}}{(r+1)^{r+1}}} = \frac{a+n/c+r}{r+1}$$

and hence

$$r+1\sqrt{v} - 1 > \frac{a+n/c+r}{a+n/c-1}.$$ 

By (2.2) this yields

$$v \geq \frac{a+n/c+r}{a+n/c-1} \cdot \frac{(a+(n-c)/c)^{r+1}}{(r+1)^{r+1}} = \frac{(a+n/c)^{r+1}}{(r+1)^{r+1}},$$

which contradicts our assumption. Thus the induction step is proved.

Proof of Theorem 1.1. In view of Proposition 2.1 it suffices to check (1.4) for all $n \leq c$. Trivially, we have $f_c(n, r) \geq 1$. On the other hand, by the AM–GM inequality we have

$$\left(\frac{n}{c} + \frac{r}{2}\right)^{\frac{r+1}{r+1}} \leq \left(\frac{n}{c} + r\right)^{\frac{r+1}{r+1}} \leq (r+1)^{r+1},$$

thus

$$f_c(n, r) \geq 1 \geq \frac{(n/c + r/2)^{r+1}}{(r+1)^{r+1}},$$

as required.

Remark. For large values of parameters, one may use the larger value of $a$ in Proposition 2.1 for instance, by the use of (1.1).

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