THE SUPERMARKET MODEL WITH ARRIVAL RATE TENDING TO ONE

GRAHAM BRIGHTWELL AND MALWINA J. LUCZAK

Abstract. In the supermarket model, there are $n$ queues, each with a single server. Customers arrive in a Poisson process with arrival rate $\lambda n$, where $\lambda = \lambda(n) \in (0, 1)$. Upon arrival, a customer selects $d = d(n)$ servers uniformly at random, and joins the queue of a least-loaded server amongst those chosen. Service times are independent exponentially distributed random variables with mean 1. In this paper, we analyse the behaviour of the supermarket model in a regime where $\lambda(n)$ tends to 1, and $d(n)$ tends to infinity, as $n \to \infty$. For suitable triples $(n, d, \lambda)$, we identify a subset $\mathcal{N}$ of the state space where the process remains for a long time in equilibrium. We further show that the process is rapidly mixing when started in $\mathcal{N}$, and give bounds on the speed of mixing for more general initial conditions.

1. Introduction

The supermarket model is a Markov chain model for a dynamic load-balancing process. There are $n$ servers, and customers arrive according to a Poisson process with rate $\lambda = \lambda(n) < 1$. On arrival, a customer inspects $d = d(n)$ queues, chosen uniformly at random with replacement, and joins a shortest queue among those inspected (in case of a tie, the first shortest queue in the list is joined). Each server serves one customer at a time, and service times are iid random variables, with an exponential distribution of mean 1.

A number of authors [11, 12, 15, 2, 3, 7, 5, 6, 1] have studied the supermarket model, as well as various extensions, e.g., to the setting of a Jackson network [9] and to a version with one queue saved in memory [13, 8].

Previous work has concentrated on the case where $\lambda$ and $d$ are held fixed as $n$ tends to infinity. As with other related models, there is a dramatic change when $d$ is increased from 1 to 2: if $d = 1$, the maximum queue length in equilibrium is of order $\log n$, while if $d$ is a constant at least 2, then the maximum queue length in equilibrium is of order $\log \log n / \log d$.

Luczak and McDiarmid [5] prove that, for fixed $\lambda$ and $d$, the sequence of Markov chains indexed by $n$ is rapidly mixing: as $n \to \infty$, the time for...
the system to converge to equilibrium is of order \( \log n \), provided the initial state has not too many customers and no very long queue. Also, they show that, for \( d \geq 2 \), with probability tending to 1 as \( n \to \infty \), in the equilibrium distribution the maximum queue length takes one of at most 2 values, and that these values are \( \log \log n / \log d + O(1) \).

Consider the infinite system of differential equations

\[
\frac{dv_k(t)}{dt} = \lambda(v_{k-1}(t)^d - v_k(t)^d) - (v_k(t) - v_{k+1}(t)), \quad k \geq 1, \tag{1.1}
\]

where \( v_0(t) = 1 \) for all \( t \). For an initial condition \( v(0) \) such that \( 1 \geq v_1(0) \geq v_2(0) \geq \ldots \geq 0 \) and \( v_k(0) \to 0 \) as \( k \to \infty \), there is a unique solution \( v(t) \) for \( t \geq 0 \).

In equilibrium, with high probability, the proportion of queues of length at least \( k \) is close to \( \pi(k) \) for each \( k \geq 1 \), over time intervals of length polynomial in \( n \); see [2, 3, 5, 6].

In this paper, we extend the above results about equilibrium behaviour and rapid mixing to some regimes where \( \lambda(n) \to 1 \) and \( d(n) \to \infty \) as \( n \to \infty \). For \( \lambda \) and \( d \) functions of \( n \), there is no single limiting differential equation (1.1), but rather a sequence of approximating differential equations, each with their own solutions and fixed points. In this paper, we do not address the question of whether such approximations to the evolution of the process are valid in generality, focussing solely on equilibrium behaviour and the time to reach equilibrium. We show that, for a wide range of triples \((n, d, \lambda)\), the maximum queue length in equilibrium is equal to

\[
k = k^{\lambda d} = \left\lfloor \frac{\log(1 - \lambda)^{-1}}{\log d} \right\rfloor
\]

over long stretches of time, with high probability. Moreover, in equilibrium, with high probability, most queues have length exactly \( k \), and we are able to estimate precisely the numbers of queues of each smaller length. In other words, this is a regime where we have "nearly exact" load balancing between the servers. As we shall discuss later, this is still consistent with the principle that the proportion of queues of length at least \( k \) is close to \( \pi(k) \) for each \( k \).

We further prove that the mixing time from a "good" state is at most of order \( kd^{k-1} \log n \), and we show that this is roughly best possible. We also prove general bounds on the mixing time, in terms of the initial number
of customers and the initial maximum queue length, and show that these bounds are also roughly best possible.

We will shortly state our main results precisely, but first we describe the supermarket model more carefully. In fact, we describe a natural discrete-time version of the process, which we shall work with throughout; as is standard, one may convert results about the discrete time version to the continuous model, with the understanding that one unit of time in the continuous model corresponds to about \((1 + \lambda)n\) steps of the discrete model.

A queue-lengths vector is an \(n\)-tuple \((x(1), \ldots, x(n))\) whose entries are non-negative integers. If \(x(j) = i\), we say that queue \(j\) has length \(i\), or that there are \(i\) customers in queue \(j\); we think of these customers as in positions \(1, \ldots, i\) in the queue. We use similar terminology throughout; for instance, to say that a customer arrives and joins queue \(j\) means that \(x(j)\) increases by 1, and to say that a customer in queue \(j\) departs or is served means that \(x(j)\) decreases by 1. Given a queue-lengths vector \(x\), we write \(\|x\|_1 = \sum_{j=1}^n x(j)\) to denote the total number of customers in state \(x\), and \(\|x\|_\infty = \max x(j)\) to denote the maximum queue length in state \(x\).

For each \(i \geq 0\), and each \(x \in \mathbb{Z}_+^n\), we define \(u_i(x)\) to be the proportion of queues in \(x\) with length at least \(i\). So \(u_0(x) = 1\) for all \(x\), and, for each fixed \(x\), the \(u_i(x)\) form a non-increasing sequence of multiples of \(1/n\), such that \(u_i(x) = 0\) eventually. The sequence \((u_i(x))_{i \geq 0}\) captures the “profile” of a queue-lengths vector \(x\), and we shall describe various sets of queue-lengths vectors, and functions of the queue-lengths vector, in terms of the \(u_i(x)\).

For positive integers \(n\) and \(d\), and \(\lambda \in (0, 1)\), we now define the \((n, d, \lambda)\)-supermarket process. This process is a discrete-time Markov chain \((X_t)\), whose state space is the set \(\mathbb{Z}_+^n\) of queue-lengths vectors, and where transitions occur at non-negative integer times. Each transition is either a customer arrival, with probability \(\lambda/(1 + \lambda)\), or a potential departure, with probability \(1/(1 + \lambda)\). If there is a potential departure, then a queue \(K\) is selected uniformly at random from \(\{1, \ldots, n\}\): if there is a customer in queue \(K\), then they are served and depart the system. If there is an arrival, then \(d\) queues are selected uniformly at random, with replacement, from \(\{1, \ldots, n\}\), and the arriving customer joins a shortest queue among those selected. To be precise, a \(d\)-tuple \((K_1, \ldots, K_d)\) is selected, and the customer joins queue \(k = K_j\), where \(j\) is the least index such that \(x(K_j)\) is minimal among \(\{x(K_1), \ldots, x(K_d)\}\).

For \(x \in \mathbb{Z}_+^n\), \((X_t^x)\) denotes a copy of the \((n, d, \lambda)\)-supermarket process \((X_t)\) where \(X_0 = x\) a.s., although when it is clear from the context we shall prefer to use the simpler notation \((X_t)\). Throughout, we let \((Y_t)\) denote a copy of the process in equilibrium. Of course, the processes depend on the parameters \((n, d, \lambda)\), but we suppress this dependence in the notation. Throughout the paper, we use \((\mathcal{F}_t)\) to denote the natural filtration of the process \((X_t)\). We use the notation \(\mathbb{P}(\cdot)\) freely to denote probability in whatever space we are working in. As before, for \(\lambda \in (0, 1)\) and \(d \in \mathbb{N}\), we set \(k^{\lambda, d} = \lceil \log(1 - \lambda)^{-1} / \log d \rceil\).
We now state our main results. First, we describe sets of queue-lengths vectors $N^\varepsilon(n, d, \lambda, k)$: our aim is to prove that, for suitable values of $n$, $d$ and $\lambda$, $k = k^\lambda_d$, and appropriately small $\varepsilon$, an equilibrium copy of the $(n, d, \lambda)$-supermarket process spends almost all of its time in the set $N^\varepsilon(n, d, \lambda, k)$.

For $\varepsilon \in (0, 1/10]$, $\lambda \in (0, 1)$, and positive integers $n$, $d$ and $k$, let $N^\varepsilon = N^\varepsilon(n, d, \lambda, k)$ be the set of all queue-lengths vectors $x$ such that: $u_{k+1}(x) = 0$ and, for $1 \leq j \leq k$, 

$$(1 - 5\varepsilon)(1 - \lambda)(\lambda d)^{j-1} \leq 1 - u_j(x) \leq (1 + 5\varepsilon)(1 - \lambda)(\lambda d)^{j-1}.$$  

So, for $x \in N^\varepsilon$, we have the following.

(a) There are no queues of length $k + 1$ or greater.

(b) For $1 \leq j \leq k$, the number of queues of length less than $j$ is $n(1 - u_j(x))$, which lies between $(1 - 5\varepsilon)n(1 - \lambda)(\lambda d)^{j-1}$ and $(1 + 5\varepsilon)n(1 - \lambda)(\lambda d)^{j-1}$.

(c) In particular, the number of queues of length less than $k$ is at most $(1 + 5\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}$. We shall work under assumptions guaranteeing that $d^{k-1}(1 - \lambda) \to 0$ as $n \to \infty$, so that the number of queues of length less than $k$ is $o(n)$, and so the proportion of queues of length exactly $k$ tends to 1 as $n \to \infty$.

(d) For $1 \leq j \leq k - 1$, the number of queues of length exactly $j$ is $n(u_j(x) - u_{j+1}(x))$. Provided $\varepsilon \lambda d \geq 2$, this quantity lies between $(1 - 6\varepsilon)n(1 - \lambda)(\lambda d)^j$ and $(1 + 6\varepsilon)n(1 - \lambda)(\lambda d)^j$.

**Theorem 1.1.** For each $\varepsilon > 0$, there exists $C$ such that the following holds for all sufficiently large $n$, and all $\lambda$ with

$$\frac{C \log^2 n}{\sqrt{n}} \leq 1 - \lambda \leq \frac{1}{C \log^2 n}.$$  

For each positive integer $d$ with $(1 - \lambda)^{-1} > d \geq 4Ck_{\lambda,d} \log^2 n$ and

$$(2(1 - \lambda)^{-1} \log^2 n)^{1/k_{\lambda,d}} \leq d \leq \left(\frac{(1 - \lambda)^{-1}}{Ck_{\lambda,d}}\right)^{1/(k_{\lambda,d} - 1)},$$  

a copy $Y_t$ of the $(n, d, \lambda)$-supermarket process in equilibrium satisfies

$$\mathbb{P}\left(\exists t \in [0, e^{\frac{1}{2} \log^2 n}], Y_t \notin N^\varepsilon(n, d, \lambda, k_{\lambda,d})\right) \leq e^{-\frac{1}{4} \log^2 n}.$$  

Observe that, from the definition of $k_{\lambda,d}$, we have $d^{k_{\lambda,d} - 1} < (1 - \lambda)^{-1} \leq d^{k_{\lambda,d}}$. We interpret $[L.3]$ as saying that these inequalities are true with something to spare. In other words, $[L.3]$ will hold whenever $\log(1 - \lambda)^{-1}/\log d$ is not too close to an integer. If that is the case, then the conclusion of the theorem implies that, for the $(n, d, \lambda)$-supermarket process in equilibrium, with high probability the maximum queue length is $k_{\lambda,d}$ and most queues have exactly this length.

In fact, we shall prove the following result, which – as we shall show – implies Theorem [L.4].
Theorem 1.2. Suppose the natural numbers \( n, d \) and \( k \), and the real numbers \( \lambda \) and \( \varepsilon \) in \((0,1)\) satisfy: \( k \geq 2 \), \( d^k (1 - \lambda) \geq 2 \log^2 n \),
\[
\frac{1}{10} \geq \varepsilon \geq \max \left\{ \frac{150k}{\sqrt{d}}, 100k(1 - \lambda)d^k - 1, \frac{10 \sqrt{6k \log n (1 - \lambda)^{-1}}}{\sqrt{nd}} \right\}.
\]
Then a copy \((Y_t)\) of the \((n, d, \lambda)\)-supermarket process in equilibrium satisfies
\[
P \left( \exists \ t \in [0, \sqrt[3]{\log^2 n}], Y_t \notin \mathcal{A}^\varepsilon (n, d, \lambda, k) \right) \leq e^{-\frac{1}{4} \log^2 n}.
\]

On the surface, this theorem may appear to apply for every value of \( n \); however, the conditions above can only be satisfied if \( n \) is at least \( 10^{15} \). This, as well as other consequences of the assumptions above, is shown in Lemma 6.1.

Proof of Theorem 1.1. In order to prove Theorem 1.1, we may assume that \( \varepsilon \leq 1/10 \). We shall show that, given \( \varepsilon \in (0, 1/10] \), a suitable \( C \) can be found so that, whenever \( n \), \( d \) and \( \lambda \) satisfy the assumptions of Theorem 1.1, then \((n, d, \varepsilon, k, \lambda)\) satisfies all the conditions of Theorem 1.2.

The conditions \( d^{k \lambda/d} (1 - \lambda) \geq 2 \log^2 n \) and \( k \lambda/d \geq 2 \) (which is equivalent to \((1 - \lambda)^{-1} > d\)) follow automatically from the assumptions of Theorem 1.1. We also have, directly from the second inequality in (1.3), that \( \varepsilon \geq \sqrt{\log(1 - \lambda)/Ck} \), provided we choose \( C \geq 100 \varepsilon^{-1} \).

To see the remaining conditions, we first note that \( k \lambda/d \leq \log(1 - \lambda)^{-1} \leq \log n \), for \( n \) sufficiently large: the first inequality is from the definition of \( k \lambda/d \) and the second is from the lower bound on \( 1 - \lambda \), for \( n \) sufficiently large.

Now \( \varepsilon \sqrt{d} \geq 2 \varepsilon \sqrt{Ck \lambda/d \log n} \geq 2 \varepsilon \sqrt{Ck^{3/2}} \), using the lower bound on \( d \). We now deduce that \( \varepsilon \sqrt{d} \geq 150k \), provided \( \sqrt{C} \geq 75 \varepsilon^{-1} \).

Finally \( \frac{(1 - \lambda)^{-1}}{\sqrt{n}} \leq \frac{1}{C \log^2 n} \), and \( k \lambda/d \leq \log n \), so
\[
\frac{10 \sqrt{6k \lambda/d \log n (1 - \lambda)^{-1}}}{\sqrt{nd}} \leq \frac{10 \sqrt{6} \log n}{\sqrt{d}} = \frac{10 \sqrt{6}}{C \sqrt{d}} \leq \varepsilon,
\]
provided \( C \geq 10 \sqrt{6} \varepsilon^{-1} \).

In summary, provided we choose \( C \geq 75^2 \varepsilon^{-2} \), all the conditions of Theorem 1.2 are satisfied.

The conditions \( d^k (1 - \lambda) \geq 2 \log^2 n \) and \( 100k(1 - \lambda)d^k \leq \varepsilon \) imply that \( d \geq 200 \varepsilon^{-1} k \log^2 n \). Provided \((1 - \lambda)^{-1}\) is large compared with \( \log^2 n \), “most” values of \( d \) above this minimum fall into one of the ranges between \((2(1 - \lambda)^{-1} \log^2 n)^{1/k}\) and \((\frac{(1 - \lambda)^{-1}}{Ck})^{1/(k-1)}\), with only small transitional ranges around \((1 - \lambda)^{-1/k}\), for \( k \) an integer, left uncovered by the result.

For values of \( d \) in these transitional ranges, our results say nothing. However, in these ranges, we can make use of a coupling result in [11] (see also [2]). For \( d < d' \), there is a coupling of the \((n, d', \lambda)\)-supermarket process and the \((n, d, \lambda)\)-supermarket process such that, for all times \( t \geq 0 \), and
for each $j$, the number of customers in position at least $j$ in their queue at time $t$ in the $(n, d', \lambda)$-supermarket process is at most the corresponding number in the $(n, d, \lambda)$-supermarket process, provided this is true at time 0. This implies that, in equilibrium, the number of customers in position at least $j$ in the $(n, d', \lambda)$-supermarket process in equilibrium is stochastically at most the corresponding number in the $(n, d, \lambda)$-supermarket process. For instance, if in the equilibrium $(n, d, \lambda)$-supermarket process, the maximum queue length is at most $k$ with high probability, then the same is true for the equilibrium $(n, d', \lambda)$-supermarket process.

Also, if $\lambda < \lambda'$, then there is a coupling of the $(n, d, \lambda)$- and $(n, d, \lambda')$-supermarket processes, so that at each time, each queue in the $(n, d, \lambda)$-supermarket process is no longer than in the $(n, d, \lambda')$-supermarket process, provided this is true at time 0. So, for instance, if at a given time there are at least $m$ queues with length $k$ in the $(n, d, \lambda)$-supermarket process, then there are also at least $m$ queues with length at least $k$ in the $(n, d, \lambda')$-supermarket process.

To illustrate our results in some special cases, first suppose $\lambda = \lambda(n) = 1 - n^{-\alpha}$, and $d = n^{\beta}$, for some real numbers $\alpha, \beta \in (0, 1)$; then $k_{\lambda, d} = \lceil \alpha/\beta \rceil$. It is easy to check that the conditions of Theorem 1.2 are satisfied with $k = k_{\lambda, d}$ and $\varepsilon = n^{-\delta}$, provided: $\alpha/\beta$ is not an integer, $\alpha > \beta$ (so that $k \geq 2$), $2\alpha < 1 + \beta$ (so that $(1 - \lambda)^{-1}/\sqrt{nd} = n^{\alpha-(1+\beta)/2}$ tends to zero), and

$$\delta < \min \left( \frac{\beta}{2}, \left\lceil \frac{\alpha}{\beta} \right\rceil \beta - \alpha, \alpha - \frac{1 + \beta}{2} \right),$$

provided $n$ is sufficiently large. The conclusions of Theorem 1.2 then hold, so in equilibrium the process spends almost all of the time in $N^{\varepsilon} = N^{\varepsilon}(n, d, \lambda, k)$.

For $x \in N^{\varepsilon}$, the maximum queue-length is $k$, and the number of queues of length less than $k$ is given by $n(1 - u_k(x))$, which lies between

$$(1 - 5n^{-\delta})n^{1-\alpha + \lfloor \alpha/\beta \rfloor \beta} \quad \text{and} \quad (1 + 5n^{-\delta})n^{1-\alpha + \lfloor \alpha/\beta \rfloor \beta}$$

If $\alpha/\beta$ is equal to an integer $k \geq 2$, then we cannot expect as strong a conclusion to hold. However, by comparing with the process for slightly lower, and slightly higher, values of $\lambda$, we see that the maximum queue length in equilibrium is a.s. either $k$ or $k + 1$, and that most queues have length either $k$ or $k + 1$.

If we take $\lambda = 1 - n^{-\alpha}$ for some constant $\alpha \in (0, 1/2)$, and $d$ tending to infinity more slowly than a power of $n$, but with $d \geq C \log^3 n/\log \log n$ for a suitably large constant $C$, then again Theorem 1.2 applies, with $k = k_{\lambda, d} = \lceil \log(1 - \lambda)^{-1}/\log d \rceil = \lceil \alpha \log n/\log d \rceil$, provided that $\alpha \log n/\log d$ is not too close to an integer. In this case, $k_{\lambda, d}$ tends to infinity with $n$, and can be as large as $(\alpha/3 - \delta) \log n/\log \log n$, for $\delta$ any positive constant and $n$ sufficiently large. Specifically, suppose that $k = \gamma \log n/\log \log n$, with $\gamma < \alpha/3$. Then it is straightforward to check that the conditions of Theorem 1.2 are satisfied if $d = (\log n)^{(\alpha + \theta \log \log n/\log n)/\gamma}$, which is equivalent to $d^k = n^{\alpha(\log n)^{\theta}}$, for $2 < \theta < \alpha/\gamma - 1$. Here we can take $\varepsilon$ to be $(\log n)^{-\delta}$ for a
small enough $\delta$. In this range, even though $k$ is fairly large, the conclusion is that, with high probability in equilibrium, almost all queues have length exactly $k$, and there are no longer queues.

Next, suppose $\lambda = 1 - (\log n)^{-\alpha}$, for some fixed $\alpha > 2$. For such a value of $\lambda$, Theorem 1.2 requires that $d \geq 200e^{-1}k\log^2 n \geq \log^2 n$, and that $d^{k-1} \leq \varepsilon(1 - \lambda)^{-1}/100k \leq (1 - \lambda)^{-1} = (\log n)^\alpha$, which implies that $\alpha/(k - 1) < 2$, or $k < 1/2\alpha + 1$.

In other words, for such values of $\lambda$, we only have results giving conditions under which the maximum queue length $k = k_{\lambda,d}$ is a constant, with $2 \leq k < 1/2\alpha + 1$. If $d = (\log n)^\beta$, where $(\alpha + 2)/k < \beta < \alpha/(k - 1)$, then the conditions of Theorem 1.2 do apply for sufficiently large $n$, where we may take $\varepsilon = (\log n)^{-\delta}$ for a suitably small $\delta$.

If $\varepsilon \leq 1/10$ and $\lambda = 1 - \frac{1}{C\log^2 n}$, for $C > 160000\varepsilon^{-2}$, and $d = c\log^2 n$, for $\sqrt{2C} \leq c \leq \varepsilon C/200$, then the conditions of Theorem 1.2 apply for $k = 2$ and this value of $\varepsilon$, for sufficiently large $n$. This is the range of applicability of our results with the slowest rate of convergence of $\lambda$ to 1.

As mentioned earlier, and explained in more detail in Section 2, our results are in line with a more general hypothesis: for a very wide range of parameter values, the maximum queue length of the $(n, d, \lambda)$-supermarket model in equilibrium is within 1 of the largest $k$ such that

$$\lambda^{1 + d + \cdots + d^{k-1}} > \frac{1}{n}.$$  

This general hypothesis holds when $\lambda$ and $d$ are constants: see [5]. It is also valid for the range where $\lambda$ is fixed and $d \to \infty$: see [1].

Another range not covered by Theorem 1.1 is that where $d \geq (1 - \lambda)^{-1}$, where one should expect the maximum queue length $k$ to be equal to 1. For this range, our techniques can be used, but there are several places where we would need to pick out $k = 1$ as a special case and treat it separately. Rather than do this, we refer the interested reader to the PhD thesis [1] of Marianne Fairthorne, which contains a detailed treatment of this case. The authors, with Fairthorne, intend to write this result up for publication elsewhere.

We also prove various rapid mixing results. For $x \in \mathbb{Z}_+^n$, let $\mathcal{L}(X^x_t)$ denote the law at time $t$ of the $(n, d, \lambda)$-supermarket process $(X^x_t)$ started in state $x$. Also let $\Pi$ denote the stationary distribution of the $(n, d, \lambda)$-supermarket process.

**Theorem 1.3.** Suppose that $n, d, k, \lambda$ and $\varepsilon = \frac{1}{100}$ satisfy the conditions of Theorem 1.2. Let $x$ be a queue-lengths vector in $\mathcal{N}^\varepsilon(n, d, \lambda, k)$. Then, for $t \geq 0$,

$$d_{TV}(\mathcal{L}(X^x_t), \Pi) \leq n \left( 2e^{-\frac{1}{4}\log^2 n} + 4\exp\left(\frac{t}{1600kd^{k-1}n}\right) \right).$$

In other words, for a copy of the process started in a state in $\mathcal{N}^\varepsilon(n, d, \lambda, k)$, with $\varepsilon = \frac{1}{100}$, the mixing time is of order $kd^{k-1}n \log n \leq (1 - \lambda)^{-1}n \log n \leq n^2$.
Formally, rapid mixing is often defined to be mixing in $O(n \log n)$ steps, and this does not meet that criterion, but in fact this result is nearly best possible: we show that mixing, starting from states in $N^\varepsilon(n, d, \lambda, k)$, requires $\Omega(d^{k-1}n)$ steps.

From states not in $N^\varepsilon$, we cannot expect to have rapid mixing in general. For instance, suppose we start from a state $x$ with number of customers $\|x\|_1 = gn$, where $g$ is much larger than $k\lambda,d$. The expected decrease in the number of customers at each step of the chain is at most $1 - \lambda^{1+\lambda}$, so mixing takes $\Omega((1 - \lambda)^{-1} gn)$ steps. Similarly, if we start with one long queue, of length $\|x\|_\infty = \ell$ much greater than $k\lambda,d$, then mixing takes $\Omega(\ell n)$ steps, to allow time for the long queue to empty out. We prove the following result, giving a nearly best-possible mixing time for $(X_t^x)$ in terms of $\|x\|_1$ and $\|x\|_\infty$.

**Theorem 1.4.** Suppose that $(n, d, \lambda, k, \frac{1}{60})$ satisfies the hypotheses of Theorem 1.2, and let $x$ be any queue-lengths vector. Let

$$q = (6000kn + 4320\|x\|_1)(1 - \lambda)^{-1} + 8n\|x\|_\infty$$

and suppose that $q \leq \frac{1}{2} e^\frac{1}{3} \log^2 n$. Then, for $t \geq 2q$, we have

$$d_{TV}(\mathcal{L}(X_t^x), \Pi) \leq 2\|x\|_1 \left(2e^{-\frac{1}{4} \log^2 n} + 4 \exp \left(-\frac{t}{3200kd^{k-1}n}\right)\right).$$

The supermarket model is an instance of a model whose behaviour has been fully analysed even though there are an unbounded number of variables that need to be tracked – namely, the proportions $u_i(X_t)$. While what we achieve in this paper is similar to what is achieved by Luczak and McDiarmid in [5] for the case where $\lambda$ and $d$ are fixed as $n \to \infty$, only some of the techniques of that paper can be used here, as we now explain.

The proofs in [5] rely on a coupling of copies of the supermarket process where the distance between coupled copies does not increase in time. This coupling is, in particular, used to establish concentration of measure, over a long time period, for Lipschitz functions of the queue-lengths vector; this result is valid for any values of $(n, d, \lambda)$, and in particular in our setting. Fast coalescence of coupled copies, and hence rapid mixing, is shown by comparing the behaviour of the $(n, d, \lambda)$-process with the $(n, 1, \lambda)$-process, which is easy to analyse. This then also implies concentration of measure for Lipschitz functions in equilibrium, and that the profile of the equilibrium process is well concentrated around the fixed point $\pi$ of the equations (1.1).

The coupling from [5] also underlies the proofs in the present paper. However, in our regime, comparisons with the $(n, 1, \lambda)$-process are too crude. Thus we cannot show that the coupled copies coalesce quickly enough, until we know something about the profiles of the copies, in particular that their maximum queue lengths are small. Our approach is to investigate the equilibrium distribution first, as well as the time for a copy of the process from a fairly general starting state to reach a “good” set of states in which the
equilibrium copy spends most of its time. Having done this, we then prove rapid mixing in a very similar way to the proof in [5].

To show anything about the equilibrium distribution, we would like to examine the trajectory of the vector \( u(X_t) \), whose components are the \( u_i(X_t) \) for \( i \geq 1 \). This seems difficult to do directly, but we perform a change of variables and analyse instead a collection of just \( k = k_{\lambda,d} \) functions \( Q_1(X_t), \ldots, Q_k(X_t) \). These are linear functions of \( u_1(X_t), \ldots, u_k(X_t) \), with the property that the drift of each \( Q_j(X_t) \) can be written, approximately, in terms of \( Q_j(X_t) \) and \( Q_{j+1}(X_t) \) only. Exceptionally, the drift of \( Q_k(X_t) \) is written in terms of \( Q_k(X_t) \) and \( u_{k+1}(X_t) \). The particular forms of the \( Q_j \) are chosen by considering the Perron-Frobenius eigenvalues of certain matrices \( M_k \) derived from the drifts of the \( u_j(x) \). Making this change of variables allows us to consider one function \( Q_j(X_t) \) at a time, and show that each in turn drifts towards its equilibrium mean (which is derived from the fixed point \( \pi \) of (1.1)), and we are thus able to prove enough about the trajectory of the \( Q_j(X_t) \) to show that, starting from any reasonable state, with high probability the chain soon enters a good set of states where, in particular, \( u_{k+1}(X_t) = 0 \), and so the maximum queue length is at most \( k \). We also show that, with high probability, the chain remains in this good set of states for a long time, which implies that the equilibrium copy spends the vast majority of its time in this set. The argument from [5] about coalescence of coupled copies can be used to show rapid mixing from this good set of states. The drift of the function \( Q_k \) to its equilibrium is slower than that of any other \( Q_j \), and its drift rate, \( 1/(\lambda d)^{k-1}n \), is approximately the spectral gap of the Markov chain \( (X_t) \), and hence determines the speed of mixing.

The structure of the paper is as follows. In Section 2, we expand on the discussion above, and motivate the definitions of the functions \( Q_j : \mathbb{Z}_+^n \to \mathbb{R} \), which are fundamental to the proof. In Section 3, we give a number of results about the long-term behaviour of random walks with drifts, including several variants on results from [5]. In Section 4, we describe the key coupling from [5], and use it to prove some results about the maximum queue length and number of customers. In Section 5, we discuss in detail the drifts of the functions \( Q_j \). The proof of Theorem 1.2 starts in Section 6, where we show how to derive a closely related result from a sequence of lemmas. These lemmas are proved in Sections 7–9. In Section 10, we complete the proof of Theorem 1.2. We prove our results on mixing times in Section 11.

2. Heuristics

In this section, we set out the intuition behind our results and proofs. As before, let \( (Y_t) \) be an equilibrium copy of the \((n, d, \lambda)\)-supermarket process. Guided by the results in [1, 5], we start by supposing that, for each \( i \geq 1 \), \( u_i(Y_t) \) is well-concentrated around its expectation \( u_i \), and seeing what that
implies about the \( u_i \). We have
\[
\Delta u_i(Y_t) = \mathbb{E}[u_i(Y_{t+1}) - u_i(Y_t) \mid Y_t] = \frac{1}{n(1 + \lambda)}[\lambda u_{i-1}(Y_t)^d - \lambda u_i(Y_t)^d - u_i(Y_t) + u_{i+1}(Y_t)].
\]

To see this, observe that, for \( i \geq 1 \), conditioned on \( Y_t \), the probability that the event at time \( t + 1 \) is an arrival to a queue of length exactly \( i - 1 \), increasing \( u_i \) by \( 1/n \), is \( \frac{\lambda}{1 + \lambda} (u_{i-1}(Y_t)^d - u_i(Y_t)^d) \), while the probability that the event is a departure from a queue of length exactly \( i \), decreasing \( u_i \) by \( 1/n \), is \( \frac{1}{1 + \lambda} (u_i(Y_t) - u_{i+1}(Y_t)) \). Note that \( u_0 \) is identically equal to 1.

Taking expectations on both sides, and setting them to 0, we see that, since \( (Y_t) \) is in equilibrium,
\[
0 = \mathbb{E}[u_i(Y_{t+1}) - u_i(Y_t)] = \frac{1}{n(1 + \lambda)} \mathbb{E}[(\lambda u_{i-1}(Y_t)^d - \lambda u_i(Y_t)^d - u_i(Y_t) + u_{i+1}(Y_t)] \]
\[
\simeq \frac{1}{n(1 + \lambda)} [\lambda u_{i-1}^d - \lambda u_i^d - u_i + u_{i+1}],
\]
where the approximations \( \mathbb{E} u_i(Y_t)^d \simeq u_i^d \) and \( \mathbb{E} u_{i-1}(Y_t)^d \simeq u_{i-1}^d \) are justified because of our assumption that \( u_i(Y_t) \) and \( u_{i-1}(Y_t) \) are well-concentrated around their respective means \( u_i \) and \( u_{i-1} \).

The system of equations
\[
\begin{align*}
0 &= \lambda \hat{u}_{i-1}^d - \lambda \hat{u}_i^d - \hat{u}_i + \hat{u}_{i+1} \quad (i = 1, 2, \ldots) \\
1 &= \hat{u}_0
\end{align*}
\]
has a unique solution with \( \hat{u}_i \to 0 \) as \( i \to \infty \), namely:
\[
\hat{u}_i = \lambda^{1+\cdots+d^{i-1}} \quad (i = 0, 1, \ldots).
\]

See [5] and the references therein for details.

By analogy with [5], and motivated by (2.3), if the \( u_i(Y_t) \) are well concentrated, we expect that \( u_i \approx \hat{u}_i \), for each \( i \), and moreover that the values of \( u_i(Y_t) \) remain close to the corresponding \( \hat{u}_i \) for long periods of time. In the regime of Theorem 1.2
\[
\log \hat{u}_i = \log(1 - (1 - \lambda))(1 + \cdots + d^{i-1}) = -(1 - \lambda)d^{i-1}(1 + O(1 - \lambda))(1 + O(1/d)) = -(1 - \lambda)d^{i-1}(1 + O(1/\log^2 n)),
\]
for each \( i \geq 1 \). In particular, \( \hat{u}_{k+1} \) is much smaller than \( 1/n \) – recall that \( d^k(1 - \lambda) \geq 2\log^2 n \). One part of our goal is to show that indeed, in equilibrium, for a long period of time there is no queue of length greater than \( k \).

On the other hand, for \( i \leq k \), our assumptions on \( \lambda \) and \( d \) imply that \( \hat{u}_i \) is close to 1 – recall that \( d^{k-1}(1 - \lambda) \leq \varepsilon/100k \). This suggests that, in equilibrium, most queues have length exactly \( k \). Moreover, \( \hat{u}_k^d = 1 - o(1) \) for \( i < k \), so that \( 1 - \hat{u}_i^d \approx d(1 - \hat{u}_i) \), whereas \( \hat{u}_k^d = o(1) \). We then obtain
Making the approximations that the drift of this function satisfies

\( j \) case of this function to show that \((1 + \lambda) \Delta Q(x)\) directly appears to be challenging. Instead, we could then analyse the trajectory of this function to show that \((u_1(x), \ldots, u_k(x))\) stays close to \(\tilde{u}\) for a long period. We have been unable to find such a function, and indeed analysing the evolution of the \(u_j(X_t)\) directly appears to be challenging. Instead, we work with a sequence of functions \(Q_j(x), j = 1, \ldots, k\), each of the form \(Q_j(x) = n \sum_{i=1}^{j} \gamma_{j,i}(1 - u_i(x))\), where the \(\gamma_{j,i}\) are positive real coefficients. This sequence of functions has the property that the drift of each \(Q_j(x)\) can be written (approximately) in terms of \(Q_j(x)\) itself and \(Q_{j+1}(x)\).

Let us see how these coefficients should be chosen, starting with the special case \(j = k\), where we write \(\beta_i\) for \(\gamma_{k,i}\). Consider a function of the form \(Q_k(x) = n \sum_{i=1}^{k} \beta_i (1 - u_i(x))\). As in the argument leading to (2.1), we have that the drift of this function satisfies

\[
(1 + \lambda) \Delta Q_k(x) = -(1 + \lambda) n \sum_{i=1}^{k} \beta_i \Delta u_i(x)
\]

\[
= -\sum_{i=1}^{k} \beta_i [\lambda u_{i-1}(x)^d - \lambda u_i(x)^d - u_i(x) + u_{i+1}(x)]
\]

\[
= \sum_{i=1}^{k} \beta_i [\lambda (1 - u_{i-1}(x)^d) - \lambda (1 - u_i(x)^d) - (1 - u_i(x)) + (1 - u_{i+1}(x))].
\]

Making the approximations \(u_{k+1}(x) \approx 0\), \(u_k(x)^d \approx 0\), and \(1 - u_i(x)^d \approx d(1 - u_i(x))\) for \(i = 1, \ldots, k - 1\), and rearranging, we arrive at

\[
(1 + \lambda) \Delta Q_k(x) \approx \beta_k (1 - \lambda) + (\beta_{k-1} - \beta_k)(1 - u_k(x))
\]

\[
+ \sum_{i=1}^{k-1} [\lambda d(\beta_{i+1} - \beta_i) - \beta_i + \beta_{i-1}](1 - u_i(x)).
\]
We set $\beta_0 = 0$ for convenience of writing the above expression. This calculation is done carefully, with precise inequalities, in Lemma 5.1 below. We would like to choose the $\beta_i$ so that the vector

$$(\lambda d(\beta_2 - \beta_1) - \beta_1 + \beta_0, \ldots, \lambda d(\beta_k - \beta_{k-1}) - \beta_{k-1} + \beta_{k-2}, \beta_{k-1} - \beta_k)$$

is a (negative) multiple of $(\beta_1, \ldots, \beta_{k-1}, \beta_k)$. This would entail

$$(1 + \lambda)\Delta Q_k(x) \simeq \beta_k(1 - \lambda) - \mu Q_k(x),$$

for some (positive) $\mu$, which in turn would mean that $Q_k$ drifts towards a value of $\beta_k(1 - \lambda)/\mu$, which should be very close to $n(1 - \lambda)(\lambda d)^{k-1}$ if $Q_k$ is above this value then it drifts down, whereas if $Q_k$ is below then it drifts up.

The point is that these drifts can be bounded below in magnitude, regardless of the precise values of the $u_i$ that go towards making up $Q_k$. What we need is for $(\beta_1, \ldots, \beta_{k-1}, \beta_k)$ to be a left eigenvector of the $k \times k$ matrix

$$M_k = \begin{pmatrix}
-\lambda d - 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\lambda d & -\lambda d - 1 & 1 & \cdots & 0 & 0 & 0 \\
0 & \lambda d & -\lambda d - 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\lambda d - 1 & 1 & 0 \\
0 & 0 & 0 & \cdots & \lambda d & -\lambda d - 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & \lambda d & -1
\end{pmatrix},$$

or, equivalently, of the matrix

$$M_k' = M_k + (\lambda d + 1)I_k = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\lambda d & 0 & 1 & \cdots & 0 & 0 & 0 \\
0 & \lambda d & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & \lambda d & 0 & 1 \\
0 & 0 & 0 & \cdots & 0 & \lambda d & \lambda d
\end{pmatrix}.$$
For $1 \leq j < k$, a similar analysis reveals that, if $Q_j(x) = n \sum_{i=1}^j \gamma_{j,i}(1 - u_i)$, then

$$(1 + \lambda) \Delta Q_j(x) \simeq \sum_{i=1}^j (1-u_i(x)) [\gamma_{j,i-1} + \lambda d \gamma_{j,i+1} - (\lambda d + 1) \gamma_{j,i}] + (1-u_{j+1}(x)).$$

(See the proof of Lemma 5.2.) We think of $1-u_{j+1}(x)$ as an “external” term (which in practice will be very close to $Q_{j+1}(x)/n$), which will determine the value towards which $Q_j$ drifts. We would like the rest of the expression to be a negative multiple of $Q_j(x)$. For this we need $(\gamma_{j,1}, \ldots, \gamma_{j,j})$ to be a left eigenvector of the $j \times j$ matrix

$$M_j = \begin{pmatrix}
-\lambda d - 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\lambda d & -\lambda d - 1 & 1 & \cdots & 0 & 0 & 0 \\
0 & \lambda d & -\lambda d - 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -\lambda d - 1 & 1 & 0 \\
0 & 0 & 0 & \cdots & \lambda d & -\lambda d - 1 & 1 \\
0 & 0 & 0 & \cdots & 0 & \lambda d & -\lambda d - 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\end{pmatrix},$$

or, equivalently, of the matrix

$$M'_j = M_j + (\lambda d + 1)I_j = \begin{pmatrix}
0 & \lambda d & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & \lambda d & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \lambda d & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 & \lambda d \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\end{pmatrix}.$$

These matrices are tridiagonal Toeplitz matrices, and there is an exact formula for the eigenvalues and eigenvectors. (See, for instance, Example 7.2.5 in [10].) The Perron-Frobenius eigenvalue of $M'_j$ is $2\sqrt{\lambda d} \cos \left( \frac{\pi j + 1}{j+1} \right)$, with the left eigenvector $(\gamma_{j,1}, \ldots, \gamma_{j,j})$ given by

$$\gamma_{j,i} = (\lambda d)^{(j-i)/2} \frac{\sin \left( \frac{j\pi}{j+1} \right)}{\sin \left( \frac{j\pi}{j+1} \right)}.$$

This means that the largest eigenvalue of $M_j$ is $-\lambda d + O(\sqrt{\lambda d})$, so that we obtain

$$(1 + \lambda) \Delta Q_j(x) \simeq -\lambda d \frac{Q_j(x)}{n} + \frac{Q_{j+1}(x)}{n} \; (1 \leq j < k),$$
meaning that $Q_j(x)$ will drift to a value close to $Q_{j+1}(x)/\lambda d$. The choices of coefficients ensure that, if the $u_j(x)$ are all near to $\tilde{u}_j$, then

$$Q_j(x) \simeq n(1 - \lambda) \sum_{i=1}^{j} \frac{\sin\left(\frac{j\pi}{j+1}\right)}{\sin\left(\frac{i\pi}{j+1}\right)} \left(\lambda d\right)^{i-1+(j-i)/2},$$

and the top term $i = j$ dominates the rest of the sum, provided $\lambda d$ is large, so $Q_j(x) \simeq (1 - u_j(x))$: this is also true for $j = k$. Thus the relationship $Q_j \simeq Q_{j+1}/\lambda d$ is as we would expect.

This means that, if $Q_{j+1}(X_t)$ remains in an interval around $\tilde{Q}_{j+1} := n(1 - \lambda)(\lambda d)^j$ for a long time, then $Q_j(X_t)$ will enter some interval around $\tilde{Q}_j$ within a short time, and stay there for a long time. We can then conduct the analysis for each $Q_j$ in turn, starting with $j = k$, to show that indeed all the $Q_j(X_t)$ quickly become close to $\tilde{Q}_j$, and stay close for a long time. This will then imply that the $u_j(X_t)$ all become and remain close to $\tilde{u}_j$.

A subsidiary application of this same technique forms another important step in the proofs (see the proof of Lemma 6.5(1)). If we do not assume that $u_{k+1}(x)$ is zero, but instead build this term into our calculations, we obtain the approximation

$$(1 + \lambda)\Delta Q_k(x) \simeq (1 - \lambda - u_{k+1}(x)) - \frac{Q_k(x)}{(\lambda d)^{k-1}n}.$$

If $u_{k+1}(X_t)$ remains above $\varepsilon(1 - \lambda)$, for some $\varepsilon > 0$, for a long time, this drift equation tells us that $Q_k$ drifts down into an interval whose upper end is below the value $\tilde{Q}_k$, and then each of the $Q_j$ in turn drift down into intervals whose upper ends are below the corresponding $\tilde{Q}_j$, and remain there. For $j = 1$, this means that the number of empty queues is at most $(1 - \delta)(1 - \lambda)n$, for some positive $\delta$, for a long period of time; this results in a persistent drift down in the total number of customers (since the departure rate is bounded below by $n - (1 - \delta)(1 - \lambda)n = \lambda n + \delta(1 - \lambda)n$ while the arrival rate is $\lambda n$), and this is not possible.

3. Random Walks with Drifts

In this section, we prove some general results about the long-term behaviour of real-valued functions of a Markov chain with bounds on the drift.

We start with two lemmas concerning random walks with a drift, both adapted from lemmas introduced in Luczak and McDiarmid [5]. In each case, we assume that we have a sequence $(R_t)$ of real-valued random variables on some probability space. On some “good” event, the jumps $Z_t = R_t - R_{t-1}$ have magnitude at most 1, and expectation at most $-\mu < 0$. The first lemma shows that, on the good event, with high probability, such a random walk, started at some value $r_0$, hits a lower value $r_1$ after not too many more than $(r_0 - r_1)/\mu$ steps.
Lemma 3.1. Let $\varphi_0 \subseteq \varphi_1 \subseteq \ldots \subseteq \varphi_m$ be a filtration, and let $Z_1, \ldots, Z_m$ be random variables taking values in $[-1, 1]$ such that each $Z_i$ is $\varphi_i$-measurable. Let $E_0, E_1, \ldots, E_{m-1}$ be events where $E_i \in \varphi_i$ for each $i$, and let $E = \bigcap_{i=0}^{m-1} E_i$. Fix $v \in (0, 1)$, and let $r_0, r_1 \in \mathbb{R}$ be such that $r_0 > r_1$ and $vm \geq 2(r_0 - r_1)$. Set $R_0 = r_0$ and, for each integer $t > 0$, let $R_t = R_0 + \sum_{i=1}^{t} Z_i$.

Suppose that, for each $i = 1, \ldots, m$,

$$
\mathbb{E}(Z_i \mid \varphi_{i-1}) \leq -v \text{ on } E_{i-1} \cap \{R_{i-1} > r_1\}.
$$

Then

$$
\mathbb{P}(E \cap \{R_t > r_1 \ \forall t \in \{1, \ldots, m\}\}) \leq \exp\left(-\frac{v^2 m}{8}\right).
$$

Proof. We first prove the lemma assuming the inequalities $\mathbb{E}(Z_i \mid \varphi_{i-1}) \leq -v$ hold almost surely, that is ignoring the events $E_{i-1} \cap \{R_{i-1} > r_1\}$. We shall then see how to incorporate these events.

We can couple the $Z_i$ with random variables $V_i$ taking values in $[-1, 1]$ such that $\mathbb{E}[V_i \mid \varphi_{i-1}] = -v$ for each $i$ and $\mathbb{P}(Z_i \leq V_i) = 1$ for each $i$: to do this, we define $V_i$ as follows. Take a $U[0, 1]$ random variable $W_i$, independent of the $Z_i$ and the other $W_j$, and set $p_i = (v + 1)/(1 - \mathbb{E}[Z_i \mid \varphi_{i-1}]) \in [0, 1]$. Now set $V_i = Z_i \mathbb{1}_{W_i \leq p_i} + \mathbb{1}_{W_i > p_i}$. Note that indeed $|V_i| \leq 1$ for each $i$ and $\mathbb{P}(V_i \geq Z_i) = 1$. Furthermore,

$$
\mathbb{E}[V_i \mid \varphi_{i-1}] = p_i \mathbb{E}[Z_i \mid \varphi_{i-1}] + (1 - p_i) = -p_i (1 - \mathbb{E}[Z_i \mid \varphi_{i-1}]) + 1 = -v.
$$

For each $t \geq 0$, let $S_t = \sum_{i=1}^{t} V_i$, so $S_t \geq \sum_{i=1}^{t} Z_i = R_t - R_0$, set $\mu_t = -vt = \mathbb{E}S_t$, and note that $(S_t - \mu_t)$ is a martingale. By the Hoeffding-Azuma inequality, $\mathbb{P}(S_t \geq \mu_t + y) \leq \exp(-y^2/2t)$. Thus, if $a = r_0 - r_1 \leq \frac{1}{2}vm$,

$$
\mathbb{P}(R_t > r_1 \ \forall t \in \{1, \ldots, m\} \mid R_0 = r_0) \leq \mathbb{P}(R_m - R_0 > -a) \\
\leq \mathbb{P}(S_m > -a) \\
\leq \exp\left(-\frac{(vm - a)^2}{2m}\right) \\
\leq \exp\left(-\frac{v^2 m}{8}\right).
$$

Now let us return to the full lemma as stated, with the events $E_i$. For each $i = 0, 1, \ldots, m - 1$, let $F_i = E_i \cap \{R_i > r_1\}$, and for each $i = 1, \ldots, m$, let $\tilde{Z}_i = Z_i \mathbb{1}_{F_{i-1}} - 1_{\overline{F}_i}$. Let $\tilde{R}_0 = R_0$ and, for $t = 1, \ldots, m$, let $\tilde{R}_t = \tilde{R}_0 + \sum_{i=1}^{t} \tilde{Z}_i$. Then $\mathbb{E}(\tilde{Z}_i \mid \varphi_{i-1}) \leq -v$. Hence, by what we have just proved applied to the $\tilde{Z}_i$,

$$
\mathbb{P}(E \cap \{R_t > r_1 \ \forall t \in \{1, \ldots, m\}\} \mid R_0 = r_0) \\
= \mathbb{P}(E \cap \{\tilde{R}_t > r_1 \ \forall t \in \{1, \ldots, m\}\} \mid \tilde{R}_0 = r_0) \\
\leq \mathbb{P}(\tilde{R}_t > r_1 \ \forall t \in \{1, \ldots, m\}) \\
\leq \exp\left(-\frac{v^2 m}{8}\right),
$$

as required. \qed
The next lemma states that, if a discrete-time 1-dimensional random walk \((H_t)\), starting at \(h_0\) and making jumps of size at most 1, has negative drift whenever it lies in the interval \([h_0 - b, h_0 + a]\), then it is unlikely to “cross against the drift” and make its first exit from the interval at the upper end.

**Lemma 3.2.** Let \(h_0, a\) and \(b\) be positive real numbers. Let \(v \in (0, 1]\). Let \(\varphi_0 \subseteq \varphi_1 \subseteq \ldots\) be a filtration, and let \(Z_1, Z_2, \ldots\) be random variables taking values in \([-1, 1]\) such that each \(Z_i\) is \(\varphi_i\)-measurable. Let \(E_0, E_1, \ldots\) be events where \(E_i \in \varphi_i\) for each \(i\). Let \(H_0\) be \(\varphi_0\)-measurable, and, for each integer \(t > 0\), let \(H_t = H_0 + \sum_{i=1}^{t} Z_i\). Assume for each \(i = 1, \ldots\),

\[
\mathbb{E}(Z_i \mid \varphi_{i-1}) \leq -v \text{ on } E_{i-1} \cap \{h_0 - b \leq H_{i-1} < h_0 + a\}.
\]

Let

\[T = \inf\{t \geq 1 : H_t \in (-\infty, h_0 - b) \cup [h_0 + a, \infty)\}, \text{ and } E = \bigcap_{i=0}^{T-1} E_i.\]

Then, on the event that \(H_0 = h_0\),

\[
\mathbb{P}(E \cap \{H_T \geq h_0 + a\} \mid \varphi_0) \leq e^{-2va}.
\]

**Proof.** Let us first ignore the events \(E_i\).

Note that, for any function \(f\) convex on \([-1, 1]\), we have

\[
f(z) \leq \frac{f(1) - f(-1)}{2} z + \frac{f(1) + f(-1)}{2}, \quad z \in [-1, 1],
\]

so, for each \(i\),

\[
\mathbb{E}[f(Z_i) \mid \varphi_{i-1}] \leq \mathbb{E}
\left[
Z_i\frac{f(1) - f(-1)}{2} + \frac{f(1) + f(-1)}{2} \mid \varphi_{i-1}
\right]
= \frac{f(1) - f(-1)}{2} \mathbb{E}[Z_i \mid \varphi_{i-1}] + \frac{f(1) + f(-1)}{2}.
\]

Let \(M_t = \left(\frac{1 + v}{1 - v}\right)^{H_t}\), for each \(t \in \mathbb{Z}_+\), and note that \(f(z) = \left(\frac{1 + v}{1 - v}\right)^z\) is convex on \([-1, 1]\). Then, for each \(t \in \mathbb{N}\),

\[
\mathbb{E}[M_t \mid \varphi_{t-1}]
= M_{t-1} \mathbb{E}
\left[
\left(\frac{1 + v}{1 - v}\right)^{Z_t} \mid \varphi_{t-1}\right]
\leq M_{t-1} \left(\frac{1 + v}{2(1-v)} - \frac{1 - v}{2(1+v)}\right) \mathbb{E}[Z_t \mid \varphi_{t-1}] + \frac{1 + v}{2(1-v)} + \frac{1 - v}{2(1+v)}
\leq M_{t-1} \left(-v\left(\frac{1 + v}{2(1-v)} - \frac{1 - v}{2(1+v)}\right) + \frac{1 + v}{2(1-v)} + \frac{1 - v}{2(1+v)}\right)
= M_{t-1} \left(\frac{1 - v^2}{2(1-v)} + \frac{1 - v^2}{2(1+v)}\right)
= M_{t-1},
\]

so \((M_t)\) is a supermartingale. We deduce that, on the event that \(H_0 = h_0\),

\[
\mathbb{E}[M_t \mid \varphi_0] \leq \mathbb{E}[M_0 \mid \varphi_0] = \left(\frac{1 + v}{1 - v}\right)^{h_0}.
\]
Thus, by the optional stopping theorem, on the event that $H_0 = h_0$,
\[
\left(\frac{1 + v}{1 - v}\right)^{h_0} \geq \mathbb{E}[M_T \mid \varphi_0] \\
\geq \mathbb{P}(H_T \geq h_0 + a \mid \varphi_0) \left(\frac{1 + v}{1 - v}\right)^{h_0+a} + \mathbb{E}\left[\left(\frac{1 + v}{1 - v}\right)^{H_T} \mathbbm{1}_{(H_T < h_0 - b)} \mid \varphi_0\right].
\]
Using the elementary inequality $(1 + v)/(1 - v) \geq e^{2v}$ for $0 \leq v < 1$, we deduce that, on the event that $H_0 = h_0$,
\[
\mathbb{P}(H_T \geq h_0 + a \mid \varphi_0) \leq \left(\frac{1 + v}{1 - v}\right)^{-a} \leq e^{-2va},
\]
which yields the result in the case without the events $E_i$.

Now let us incorporate the events $E_i$, and consider the full lemma as stated. For each $i = 0, 1, \ldots$, let $F_i = E_i \cap \{h_0 - b \leq H_i < h_0 + a\} \in \varphi_i$ and $\tilde{Z}_i = Z_i \mathbbm{1}_{T_{i-1}} - \mathbbm{1}_{T_i}$. Let $\tilde{H}_i$ and $\tilde{T}_i$ be defined in the obvious way. Then $\tilde{Z}_i$ is $\varphi_i$-measurable, $\tilde{Z}_i \in [-1, 1]$ and $\mathbb{E}[\tilde{Z}_i \mid \varphi_{i-1}] \leq -v$.

On the event $\bigcap_{i=0}^{T-1} F_i = \bigcap_{i=0}^{T-1} E_i = E$, we have $H_T = \tilde{H}_T$, and so, applying the first part of the proof to the $\tilde{Z}_i$,
\[
\mathbb{P}(E \cap \{H_T \geq h_0 + a\} \mid \varphi_0) \leq \mathbb{P}(\tilde{H}_T \geq h_0 + a \mid \varphi_0) \leq e^{-2va},
\]
on the event that $H_0 = h_0$, as required.

We now use the two lemmas above to prove a result about real-valued functions of a Markov chain, that we shall use repeatedly in our proofs.

For a real-valued function $F$ defined on the set $\mathbb{Z}_+^n$ of queue-lengths vectors, a copy $(X_t)$ of the $(n, d, \lambda)$-supermarket process, and $x \in \mathbb{Z}_+^n$, we define
\[
\Delta F(x) := \mathbb{E}[F(X_{t+1}) - F(X_t) \mid X_t = x],
\]
and call this the drift of $F$ (at $x$). Similarly, we shall also use the notation $\Delta F(X_t)$ to denote the random variable $\mathbb{E}[F(X_{t+1}) - F(X_t) \mid X_t]$.

**Lemma 3.3.** Let $h, v, c, \rho \geq 2$, $m$ and $s$ be positive real numbers with $vm \geq 2(c - h)$. Let $(X_t)_{t \geq 0}$ be a discrete-time Markov process with state-space $\mathcal{X}$, adapted to the filtration $(\varphi_t)_{t \geq 0}$. Let $S$ be a subset of $\mathcal{X}$, and let $F$ be a real-valued function on $\mathcal{X}$ such that, for all $x \in S$ with $F(x) \geq h$,
\[
\Delta F(x) \leq -v,
\]
and for all $t \geq 0$, $|F(X_{t+1}) - F(X_t)| \leq 1$ a.s. Let $T^*$ be any stopping time, and suppose that $F(X_{T^*}) \leq c$ a.s.

Let
\[
T_0 = \inf\{t \geq T^* : X_t \notin S\}, \\
T_1 = \inf\{t \geq T^* : F(X_t) \leq h\}, \\
T_2 = \inf\{t > T_1 : F(X_t) \geq h + \rho\}.
\]

Then
(i) $\mathbb{P}(T_1 \wedge T_0 > T^* + m) \leq \exp(-\sigma^2 m/8)$;
(ii) $\mathbb{P}(T_2 \leq s \wedge T_0) \leq s \exp(-\rho v)$. 

When we use the lemma, \( m \) will be much smaller than \( s \), and moreover with high probability \( T^* \) will be much smaller than \( s \), and also \( P(T_0 \leq s) \) will be small. In these circumstances, the lemma allows us to conclude that, with high probability, \( F(X_t) \) decreases from its value at \( T^* \) (at most \( c \)) to below \( h \) in at most a further \( m \) steps, and does not increase back above \( h + \rho \) before time \( s \). We shall sometimes use the conclusion of (ii) in the weaker form \( P(T_2 \leq s < T_0) \leq s \exp(-\rho v) \).

**Proof.** We start by proving the lemma in the special case where the stopping time \( T^* \) is equal to 0.

For (i), we apply Lemma 3.1. The filtration \( \varphi_0 \subseteq \varphi_1 \subseteq \cdots \subseteq \varphi_m \) will be the initial segment of the filtration \( (\varphi_t)_{t \geq 0} \). For \( t \geq 1 \), we set \( Z_t = F(X_t) - F(X_{t-1}) \), so that \( R_t := R_0 + \sum_{i=1}^{t} Z_i = F(X_t) \). For \( t \geq 0 \), we set \( E_t \) to be the event that \( T_0 > t \) (i.e., \( X_i \in \mathcal{S} \) for all \( i \) with \( 0 \leq i \leq t \)), so \( E = \cap_{i=0}^{m-1} E_i \) is the event that \( T_0 \geq m \). We set \( r_0 = F(X_0) \leq c \), and \( r_1 = h \). We may assume that \( r_0 > r_1 \); otherwise \( T_1 = 0 \) and there is nothing to prove.

On the event \( E_{i-1} \cap \{ R_{i-1} > r_1 \} \), we have \( X_{i-1} \in \mathcal{S} \) and \( F(X_{i-1}) > r_1 = h \), so \( E(Z_i \mid \varphi_{i-1}) = \Delta F(X_{i-1}) \leq -v \). Thus, noting that \( vm \geq 2(r_0 - r_1) \) by our assumption on \( m \), we see that the conditions of Lemma 3.1 are satisfied. The event that \( R_t > r_1 \) for all \( t = 1, \ldots, m \) is the event that \( T_1 > m \), so

\[
P(T_1 \wedge T_0 > m) \leq P(\{T_1 > m\} \cap \{T_0 \geq m\}) \leq e^{-v^2m/8},
\]

as required for (i).

We move on to (ii). For each time \( r \in \{0, \ldots, s-1\} \), set

\[
T(r) = \min\{t \geq 0 : F(X_{r+t}) \notin [h, h + \rho]\}.
\]

We say that \( r \) is a departure point if: \( T_1 \leq r \), \( F(X_r) \in [h, h+1) \), \( F(X_{r+T(r)}) \geq h + \rho \), and \( r + T(r) \leq s \wedge T_0 \). To say that \( T_2 \leq s \wedge T_0 \) means that \( F(X_t) \) crosses from its value, at most \( h \), at time \( T_1 \), up to a value at least \( h + \rho \), taking steps of size at most 1, by time \( s \wedge T_0 \). This is equivalent to saying that there is at least one departure point \( r \in [0, s) \). Therefore

\[
P(T_2 \leq s \wedge T_0)
\]

\[
\leq \sum_{r=0}^{s-1} P\left(\{T_1 \leq r\} \cap \{F(X_r) \in [h, h+1]\} \cap \{F(X_{r+T(r)}) \geq h + \rho\} \cap \{r + T(r) \leq s \wedge T_0\}\right)
\]

\[
= \sum_{r=0}^{s-1} E\left[1_{\{T_1 \leq r\}} 1\{F(X_r) \in [h, h+1]\} E\left[1\{F(X_{r+T(r)}) \geq h + \rho\} 1\{r+T(r) \leq s \wedge T_0\} \mid \varphi_r\right]\right].
\]

Fix any \( r \in [0, s) \). We claim that, for any \( h_0 \in [h, h+1) \), on the \( \varphi_r \)-measurable event that \( F(X_r) = h_0 \), the conditional expectation

\[
E\left[1\{F(X_{r+T(r)}) \geq h + \rho\} 1\{r+T(r) \leq s \wedge T_0\} \mid \varphi_r\right]
\]
is at most $e^{-\rho v}$. This will imply that each term of the sum above is at most $e^{-\rho v}$, and so that $\mathbb{P}(T_2 \leq s \land T_0) \leq s \exp(-\rho v)$, as required.

To prove the claim, we use Lemma 3.2. We consider the re-indexed process $(X'_t) = (X_{t+T})$; by the Markov property, this is a Markov chain with the same transition probabilities as $(X_t)$, and initial state $X'_0 = X_r$ with $F(X'_0) = h_0$. We set $\varphi'_i = \varphi_{r+i}$ for each $i$, so that $(X'_t)$ is adapted to the filtration $(\varphi'_t)$. Let $Z_i = F(X'_t) - F(X'_{t-1})$, so that $|Z_i| \leq 1$, and $Z_i$ is $\varphi'_i$-measurable, for each $i$. Set $H_0 = h_0 = F(X'_0)$, so that $H_t = H_0 + \sum_{i=1}^{t} Z_i = F(X'_t)$. We set $a = h + \rho - h_0 \geq \rho - 1 \geq \rho/2$, and $b = h_0 - h$. Thus the event \{ $h_0 - b \leq H_{i-1} < h_0 + a \}$ translates to \{ $h \leq F(X'_{i-1}) < h + \rho \}$, and the event $\{ H_{T(r)} \geq h_0 + a \}$ translates to $F(X_{T(r)} + T) \geq h + \rho$.

For $i = 0, \ldots, s$, set $E_i = \{ r + i \leq s \land T_0 \}$, noting that this event is in $\varphi'_i$, and that $E := \bigcap_{i=0}^{T(r)-1} E_i = \{ r + T(r) \leq s \land T_0 \}$. On the event $E_i \land \{ h_0 - b \leq H_{i-1} < h_0 + a \}$, we have $F(X'_{i-1}) \geq h$, and $X'_{i-1} = X_{r+i-1} \in S$, and therefore $E(Z_i \mid \varphi'_{i-1}) = \Delta F(X'_{i-1}) \leq -v$. From Lemma 3.2, we now conclude that, on the event $F(X_r) = h_0$,

$$
\mathbb{P} \left( \{ F(X_{r+T(r)} + T) \geq h + \rho \} \cap \{ r + T(r) \leq s \land T_0 \} \mid \varphi_r \right)
\leq \mathbb{P} \left( E \cap \{ H_{T(r)} \geq h_0 + a \} \right)
\leq e^{-2va}
\leq e^{-\rho v},
$$

as required. This completes the proof in the special case where $T^* = 0$.

We now proceed to the general case. Suppose then that the hypotheses of the lemma are satisfied, with stopping time $T^*$. We apply the result we have just proved to the process $(X'_t) = (X_{T^*+t})$. By the strong Markov property, $(X'_t)$ is also a Markov process, adapted to the filtration $(\varphi'_{t})_{t \geq 0} = (\varphi_{T^*+t})_{t \geq 0}$. The condition that $F(X_{T^*}) \leq c$ is equivalent to $F(X'_0) \leq c$. Set:

- $T'_0 = \inf \{ t \geq 0 : X'_t \notin S \} = \inf \{ t \geq 0 : X_{T^*+t} \notin S \} = T_0 - T^*$
- $T'_1 = \inf \{ t \geq 0 : F(X'_t) \leq h \} = \inf \{ t \geq 0 : F(X_{T^*+t}) \leq h \} = T_1 - T^*$
- $T'_2 = \inf \{ t > T'_1 : F(X'_t) \geq h + \rho \} = \inf \{ t > T'_1 : F(X_{T^*+t}) \geq h + \rho \} = T_2 - T^*$,

and note that these are all stopping times with respect to the filtration $(\varphi'_t)$. The special case of the result (with $T^* = 0$) now tells us that:

(i) $\mathbb{P}(T_1 \land T_0 > T^* + m) = \mathbb{P}((T^* + T'_1) \land (T^* + T'_0) > T^* + m)$

$= \mathbb{P}(T'_1 \land T'_0 > m)$

$\leq \exp(-\nu^2 m/8)$;

(ii) $\mathbb{P}(T_2 \leq s \land T_0) = \mathbb{P}(T^* + T'_2 \leq s \land (T^* + T'_0))$

$\leq \mathbb{P}(T^* + T'_2 \leq (T^* + s) \land (T^* + T'_0))$

$= \mathbb{P}(T'_2 \leq s \land T'_0)$

$\leq \exp(-\rho v)$. 

In both cases, these are the desired results.

We shall also make use of a “reversed” version of Lemma 3.3 where \( \Delta F(x) \geq v \) for all \( x \) in some “good” set \( S \) with \( F(x) \leq h \). The result and proof are practically identical to Lemma 3.3, changing the directions of inequalities where necessary, and using “reversed” versions of Lemmas 3.1 and 3.2.

The next lemma is a more precise version of Lemma 2.2 in [5]. We omit the proof, which is exactly as in [5], except that we track more carefully the values of the various constants appearing in that proof, and separate out the effects of the two occurrences of \( \delta \) in that theorem.

**Lemma 3.4.** Let \( (\varphi_i)_{i \geq 0} \) be a filtration. Let \( Z_1, Z_2, \ldots \) be \( \{0, \pm 1\} \)-valued random variables, where each \( Z_i \) is \( \varphi_i \)-measurable. Let \( S_0 \geq 0 \) a.s., and for each positive integer \( j \) let \( S_j = S_0 + \sum_{i=1}^j Z_i \). Let \( A_0, A_1, \ldots \) be events, where each \( A_i \) is \( \varphi_i \)-measurable.

Suppose that there is a positive integer \( k_0 \) and a constant \( \delta \) with \( 0 < \delta < 1/2 \) such that

\[
P(Z_i = -1 \mid \varphi_{i-1} \geq \delta) \geq \delta \quad \text{on} \quad A_{i-1} \cap \{S_{i-1} \in \{1, \ldots, k_0 - 1\}\}
\]

and

\[
P(Z_i = -1 \mid \varphi_{i-1} \geq \delta) \geq 3/4 \quad \text{on} \quad A_{i-1} \cap \{S_{i-1} \geq k_0\}.
\]

Then, for each positive integer \( m \)

\[
P\left( \bigcap_{i=1}^m \{S_i \neq 0\} \cap \bigcap_{i=0}^{m-1} A_i \right) \leq P(S_0 > \lfloor m/16 \rfloor) + 3 \exp\left(-\frac{\delta^{k_0-1}k_0^m}{200k_0^{-m}}\right).
\]

Several times we shall use the fact that, if \( Z \) is a binomial or Poisson random variable with mean \( \mu \), then for each \( 0 \leq \epsilon \leq 1 \) we have

\[
P(Z - \mu \leq -\epsilon \mu) \leq e^{-(1/2)\epsilon^2 \mu}.
\]

(3.1)

4. Coupling

We now introduce a natural coupling of copies of the \((n, d, \lambda)\)-supermarket process \((X^x_t)\) with different initial states \( x \). The coupling is a natural adaptation to discrete time of that in [5].

We describe the coupling in terms of three sequences of random variables. There is an iid sequence \( V = (V_1, V_2, \ldots) \) of \( 0 \)-1 random variables where each \( V_i \) takes value 1 with probability \( \lambda/(1 + \lambda) \); \( V_i = 1 \) if and only if time \( i \) is an arrival. Corresponding to every time \( i \) there is also an ordered list \( D_i \) of \( d \) queue indices, each chosen uniformly at random with replacement. Let \( D = (D_1, D_2, \ldots) \). Furthermore, corresponding to every time \( i \) there is a uniformly chosen queue index \( \hat{D}_i \). Let \( \hat{D} = (\hat{D}_1, \hat{D}_2, \ldots) \). At time \( i \), \( D_i \) will be used if \( Z_i = 1 \), and there will be an arrival to the first shortest queue in \( D_i \); otherwise, there will be a departure from the queue with index \( \hat{D}_i \), if that queue is currently non-empty.
Suppose that we are given a realisation \((v, d, \tilde{d})\) of \((V, D, \tilde{D})\). For each possible initial queue-lengths vector \(x \in \mathbb{Z}_+^n\), this realisation yields a deterministic process \((x_t)\) with \(x_0 = x\): let us write \(x_t = s_t(x; v, d, \tilde{d})\). Then, for each \(x \in \mathbb{Z}_+^n\), the process \(s_t(x; V, D, \tilde{D})\) has the distribution of the \((n, d, \lambda)\)-supermarket process \(X^t\) with initial state \(x\). In this way, we construct copies \((X^t)\) of the \((n, d, \lambda)\)-supermarket process for each possible starting state \(x\) on a single probability space. When we treat more than one such copy at the same time, we always work in this probability space, and we let \(\mathbb{P}(\cdot)\) denote the corresponding coupling measure.

We shall use the following lemma, which is a discrete-time analogue of Lemma 2.3 in [5] and is proved in exactly the same way.

**Lemma 4.1.** Fix any triple \(z, d, \tilde{d}\) as above, and for each queue-lengths vector \(x\) write \(s_t(x)\) for \(s_t(x; z, d, \tilde{d})\). Then, for each \(x, y \in \mathbb{Z}_+^n\), both \(|s_t(x) - s_t(y)|\) and \(|s_t(x) - s_t(y)|\) are nonincreasing; and further, if \(0 \leq t < t'\) and \(s_t(x) \leq s_t(y)\), then \(s_{t'}(x) \leq s_{t'}(y)\).

For a queue-lengths vector \(x\), let \(|x|_\infty = \max x(i)\) denote the maximum length of a queue in \(x\), and \(|x|_1 = \sum_{i=1}^n x(i)\) denote the total number of customers. Given positive real numbers \(\ell\) and \(g\), we set

\[
\mathcal{A}_0(\ell, g) = \{x: |x|_\infty \leq \ell \text{ and } |x|_1 \leq gn\}; \\
\mathcal{A}_1(\ell, g) = \{x: |x|_\infty \leq 3\ell \text{ and } |x|_1 \leq 3gn\}.
\]

We also set

\[
\ell^* = \log^2 n(1 - \lambda)^{-1}, \quad g^* = 2(1 - \lambda)^{-1}, \quad \mathcal{A}_0^* = \mathcal{A}_0(\ell^*, g^*), \quad \mathcal{A}_1^* = \mathcal{A}_1(\ell^*, g^*).
\]

The next result tells us that the \((n, d, \lambda)\)-supermarket process \((Y_t)\) in equilibrium is very unlikely to be outside the set \(\mathcal{A}_0^*\) for any \(d\). This is accomplished by proving the result for \(d = 1\), when the process is easy to analyse explicitly, and then using coupling in \(d\) to deduce the result for all \(d\). Of course, the result is actually extremely weak for all \(d > 1\), and later we shall show a much stronger result whenever the various parameters of the model satisfy the conditions of Theorem [1.2]; the importance of the lemma below is that it gets us started and enables us to say something about where the equilibrium of the process lives.

**Lemma 4.2.** Let \((Y_t)\) be a copy of the \((n, d, \lambda)\)-supermarket process in equilibrium. Then \(\mathbb{P}(Y_t \notin \mathcal{A}_0^*) \leq 2ne^{-\log^2 n} \).

**Proof.** Let \(\tilde{Y}\) denote a stationary copy of the \((n, 1, \lambda)\)-supermarket process, in which each arriving customer joins a uniform random queue. Then the queue lengths \(\tilde{Y}_t(j)\) are independent geometric random variables with mean \(\lambda/(1 - \lambda)\), where \(\mathbb{P}(\tilde{Y}_t(j) = k) = (1 - \lambda)\lambda^k\) for \(k = 0, 1, 2, \ldots\). Therefore, \(\mathbb{P}(|\tilde{Y}_t|_\infty \geq k) \leq n\lambda^k\), and also it can easily be checked that \(\mathbb{P}\left(|\tilde{Y}_t|_1 \geq 2n(1 - \lambda)^{-1}\right) \leq e^{-n/4}\).
As mentioned in Section 11 there is a coupling between supermarket processes with different values of \( d \), which can be used to show that the equilibrium copy \((Y_t)\) of the \((n, d, \lambda)\)-supermarket process, for any \( d \), also satisfies \( P(\|Y_t\|_1 \geq 2n(1 - \lambda)^{-1}) \leq e^{-n/4} \) and \( P(\|Y_t\|_\infty \geq \log^2 n(1 - \lambda)^{-1}) \leq n\lambda \log^2 n(1 - \lambda)^{-1} \leq ne^{-\log^2 n} \), as required. \( \square \)

Next we prove a very crude concentration of measure result: if the process \((Y_t)\) in equilibrium is concentrated inside some set \( A_0(\ell, g) \), and we start a copy \((X^\ell_t)\) of the process at a state \( x \in A_0(\ell, g) \), then the process \((X^\ell_t)\) is unlikely to leave the larger set \( A_1(\ell, g) \) over a long period of time.

**Lemma 4.3.** Let \( \ell \) and \( g \) be natural numbers and \( x \) a queue-lengths vector in \( A_0(\ell, g) \). Then for any natural number \( s \),
\[
P(\exists t \in [0, s], X^\ell_t \not\in A_1(\ell, g)) \leq P(\exists t \in [0, s], Y_t \not\in A_0(\ell, g)).
\]

**Proof.** By Lemma 4.1 we can couple \((X^\ell_t)\) and \((Y_t)\) in such a way that \( \|X^\ell_t - Y_t\|_1 \) and \( \|X^\ell_t - Y_t\|_\infty \) are both non-increasing, and hence that, for each \( t \geq 0 \),
\[
\|X^\ell_t\|_1 \leq \|X^\ell_t - Y_t\|_1 + \|Y_t\|_1 \\
\leq \|x - Y_0\|_1 + \|Y_t\|_1 \\
\leq \|x\|_1 + \|Y_0\|_1 + \|Y_t\|_1 \\
\leq gn + \|Y_0\|_1 + \|Y_t\|_1,
\]
and similarly
\[
\|X^\ell_t\|_\infty \leq \ell + \|Y_0\|_\infty + \|Y_t\|_\infty.
\]
We deduce that, for each \( t \geq 0 \),
\[
\{X^\ell_t \not\in A_1(\ell, g)\} = \{\|X^\ell_t\|_1 > 3gn\} \cup \{\|X^\ell_t\|_\infty > \ell\} \\
\subseteq \{\|Y_0\|_1 > gn\} \cup \{\|Y_t\|_1 > gn\} \\
\quad \cup \{\|Y_0\|_\infty > \ell\} \cup \{\|Y_t\|_\infty > \ell\} \\
= \{Y_0 \not\in A_0(\ell, g)\} \cup \{Y_t \not\in A_0(\ell, g)\}.
\]
The result now follows immediately. \( \square \)

We shall use Lemma 4.3 later for general values of \( \ell \) and \( g \), but for now we note the following immediate consequence of the previous two lemmas.

**Lemma 4.4.** Let \( x \) be any queue-lengths vector in \( A_0^* \), and let \( T^\downarrow_A = \inf\{t : X^\ell_t \not\in A_t^*\} \). Then, for \( n \geq 2000 \),
\[
P(T^\downarrow_A \leq e^{\frac{1}{2}\log^2 n}) \leq e^{-\frac{1}{2}\log^2 n}.
\]

**Proof.** The probability in question is \( P(\exists t \in [0, e^{\frac{1}{2}\log^2 n}], X^\ell_t \not\in A_t^*) \) which, by Lemma 4.3 and Lemma 4.2 is at most
\[
P(\exists t \in [0, e^{\frac{1}{2}\log^2 n}], Y_t \not\in A_0^*) \leq (e^{\frac{1}{2}\log^2 n} + 1) P(Y_t \not\in A_0^*) \leq 3ne^{-\frac{1}{2}\log^2 n},
\]
which, for \( n \geq 2000 \), is at most \( e^{-\frac{1}{2}\log^2 n} \), as required. \( \square \)
5. Functions and Drifts

We now start the detailed proofs. The results in this section will be used in the course of the proof of Theorem 1.2 and we could assume that all the conditions of Theorem 1.2 hold; however, for this section all that is necessary is that $k \geq 2$ and $\lambda d \geq 4$.

As explained in Section 2, we will consider a sequence of functions $Q_k$, $Q_{k-1}$, ..., $Q_1$ defined on the set $\mathbb{Z}_+^n$ of queue-lengths vectors. We now give precise definitions of these functions, along with another function $P_{k-1}$, and derive some of their properties.

As in Section 2, let $Q_k$ be the function defined on the set $\mathbb{Z}_+^n$ of all queue-lengths vectors by

$$Q_k(x) = n \sum_{i=1}^{k} \beta_i (1 - u_i(x)),$$

where, for $i = 1, \ldots, k$,

$$\beta_i = 1 - \frac{1}{(\lambda d)^i} - \frac{i - 1}{(\lambda d)^k}.$$

It is also convenient to set $\beta_0 = 0$. Evidently $\beta_i < 1$ for each $i$, an inequality we shall use freely in future. We also note that $\beta_i$ is increasing in $i$, and that $\beta_k = 1 - k(\lambda d)^{-k}$.

Let

$$P_{k-1}(x) = n \sum_{i=1}^{k-1} (1 - u_i(x)).$$

Also, for $j = 1, \ldots, k - 1$, we let

$$Q_j(x) = n \sum_{i=1}^{j} \gamma_{j,i} (1 - u_i(x)),$$

where the coefficients $\gamma_{j,i}$ are given by

$$\gamma_{j,i} = (\lambda d)^{(j-i)/2} \frac{\sin(i\pi/j+1)}{\sin(j\pi/j+1)}.$$

Consistent with the expression above, we also define $\gamma_{j,0} = \gamma_{j,j+1} = 0$. It can easily be checked that, for each $i = 1, \ldots, j - 1$, and for each $j = 1, \ldots, k - 1$,

$$\lambda d \gamma_{j,i+1} + \gamma_{j,i-1} = 2 \sqrt{\lambda d} \cos \left( \frac{\pi}{j+1} \right) \gamma_{j,i}.$$

This is equivalent to saying that the $\gamma_{j,i}$ form eigenvectors of the tridiagonal Toeplitz matrices $M_j$ given in Section 2.

We will need some bounds on the sizes of the $Q_j(x)$. Observe that $\gamma_{j,j} = 1$ for each $j$, while generally we have

$$1 \leq \frac{\sin(i\pi/(j+1))}{\sin(j\pi/(j+1))} = \frac{\sin(i\pi/(j+1))}{\sin(\pi/(j+1))} \leq i,$$
since the sine function is concave on \([0, \pi]\). Thus
\[
(\lambda d)^{(j-i)/2} \leq \gamma_{j,i} \leq i(\lambda d)^{(j-i)/2},
\]
and therefore
\[
Q_j(x) \leq n \sum_{i=1}^{j} i(\lambda d)^{(j-i)/2} \leq 2n(\lambda d)^{(j-1)/2},
\]
provided \(\lambda d \geq 4\). We also note at this point that changing one component \(x(\ell)\) of \(x\) by \(\pm 1\) changes \(Q_j(x)\) by at most \(\gamma_{j,1} = (\lambda d)^{(j-1)/2}\).

It can readily be checked that, for \(j \geq 1\), the function
\[
f(i) = \sin\left(\frac{i\pi}{j+2}\right) / \sin\left(\frac{i\pi}{j+1}\right)
\]
is increasing over the range \([1, j]\), and so we have, for \(1 \leq i \leq j \leq k-2\):
\[
\frac{\gamma_{j+1,i}}{\gamma_{j,i}} = \sqrt{\lambda d} \frac{\sin(i\pi/(j+2))\sin(\pi/(j+1))}{\sin(i\pi/(j+1))\sin(\pi/(j+2))} \\
\leq \sqrt{\lambda d} \frac{\sin(j\pi/(j+2))\sin(\pi/(j+1))}{\sin(j\pi/(j+1))\sin(\pi/(j+2))} \\
= \frac{\sqrt{\lambda d}}{\pi/(j+2)} \\
\leq 2\sqrt{\lambda d}.
\]
A consequence is that, for \(j = 1, \ldots, k-2\), and any \(x \in \mathbb{Z}_+^n\),
\[
\frac{Q_{j+1}(x)}{n} = (1 - u_{j+1}(x)) + \sum_{i=1}^{j} \gamma_{j+1,i}(1 - u_i(x)) \\
\leq (1 - u_{j+1}(x)) + \sum_{i=1}^{j} 2\sqrt{\lambda d}\gamma_{j,i}(1 - u_i(x)) \\
\leq (1 - u_{j+1}(x)) + 2\sqrt{\lambda d}Q_j(x) / n.
\]
For \(j = k-1\), we have the stronger inequality that, for any \(x \in \mathbb{Z}_+^n\),
\[
\frac{Q_k(x)}{n} \leq \sum_{i=1}^{k} (1 - u_i(x)) \leq (1 - u_k(x)) + \frac{Q_{k-1}(x)}{n}.
\]
We now prove the following result about the drift of the function \(Q_k(x)\): roughly speaking, we wish to show that it is approximately equal to
\[
\frac{1}{1 + \lambda} \left(1 - \lambda - u_{k+1}(x) - \frac{1}{(\lambda d)^{k-1}} \frac{Q_k(x)}{n}\right).
\]
Lemma 5.1. For any state \( x \in \mathbb{Z}_+ \),

\[
(1 + \lambda) \Delta Q_k(x) \leq \beta_k \left( (1 - \lambda) - u_{k+1}(x) + \lambda \exp(-dQ_k(x)/kn) \right) - \frac{1}{(\lambda d)^{k-1}} \frac{Q_k(x)}{n} \left( 1 - \frac{2}{\lambda d} \right),
\]

\[
(1 + \lambda) \Delta Q_k(x) \geq \beta_k \left( (1 - \lambda) - u_{k+1}(x) \right) - \frac{1}{(\lambda d)^{k-1}} \frac{Q_k(x)}{n} \left( \frac{\beta_k}{n} \right)^2 \frac{1}{(\lambda d)^{k-3}}.
\]

Proof. As in (2.1), we have that, for \( i = 1, \ldots, k \),

\[
\Delta u_i(x) = \frac{1}{n(1 + \lambda)} \left( \lambda u_{i-1}(x)^d - \lambda u_i(x)^d - u_i(x) + u_{i+1}(x) \right).
\]

and that \( u_0 \) is identically equal to 1. We deduce that

\[
\Delta Q_k(x) = -n \sum_{i=1}^k \beta_i \Delta u_i(x) = \frac{1}{1 + \lambda} \sum_{i=1}^k \beta_i \left( -\lambda u_{i-1}(x)^d + \lambda u_i(x)^d + u_i(x) - u_{i+1}(x) \right).
\]

We rearrange the formula above as follows:

\[
(1 + \lambda) \Delta Q_k(x) = \beta_k \left( (1 - \lambda) + \lambda u_k(x)^d - u_{k+1}(x) + \lambda(1 - u_{k-1}(x)^d) - (1 - u_k(x)) \right)
\]

\[
+ \sum_{i=1}^{k-1} \beta_i \left( \lambda(1 - u_{i-1}(x)^d) - \lambda(1 - u_i(x)^d) - (1 - u_i(x)) + (1 - u_{i+1}(x)) \right)
\]

\[
= \beta_k \left( (1 - \lambda) + \lambda u_k(x)^d - u_{k+1}(x) \right)
\]

\[
+ \lambda \sum_{i=1}^{k-1} (\beta_{i+1} - \beta_i)(1 - u_i(x)^d) - \sum_{i=1}^k (\beta_i - \beta_{i-1})(1 - u_i(x)).
\]

Here we have used the facts that \( \beta_0 = 0 \) and \( 1 - u_0(x) = 0 \).

Now, for \( 1 \leq i \leq k \), we have \( 1 - u_k(x) \geq 1 - u_i(x) \) and \( \beta_k \geq \beta_i \), and so \( \beta_k (1 - u_k(x)) \geq Q_k(x)/kn \), and hence

\[
0 \leq u_k(x)^d \leq \left( 1 - \frac{Q_k(x)}{kn} \right)^d \leq \exp(-dQ_k(x)/kn).
\]

In order to estimate the terms constituting the two sums, we note the inequalities

\[
d(1 - u) - \left( \frac{d}{2} \right) (1 - u)^2 \leq 1 - u^d \leq d(1 - u).
\]
To obtain our upper bound on $\Delta Q_k(x)$, we apply the inequality $1 - u_i(x)^d \leq d(1 - u_i(x))$ for each $i = 1, \ldots, k - 1$. Since also

$$\beta_{i+1} - \beta_i = \frac{1}{(\lambda d)^i} - \frac{1}{(\lambda d)^{i+1}} - \frac{1}{(\lambda d)^k} > 0,$$

for $i = 0, \ldots, k - 1$, we have

$$\lambda \sum_{i=1}^{k-1} (\beta_{i+1} - \beta_i)(1 - u_i(x)^d) - \sum_{i=1}^{k} (\beta_i - \beta_{i-1})(1 - u_i(x))$$

$$\leq \lambda d \sum_{i=1}^{k-1} (\beta_{i+1} - \beta_i)(1 - u_i(x)) - \sum_{i=1}^{k} (\beta_i - \beta_{i-1})(1 - u_i(x))$$

$$= -\left[ \frac{1}{(\lambda d)^k} - \frac{2}{(\lambda d)^k} \right] (1 - u_k(x))$$

$$+ \sum_{i=1}^{k-1} \left[ \frac{\lambda d}{(\lambda d)^i} - \frac{\lambda d}{(\lambda d)^{i+1}} - \frac{\lambda d}{(\lambda d)^k} - \frac{1}{(\lambda d)^i} + \frac{1}{(\lambda d)^k} \right] (1 - u_i(x))$$

$$= -\frac{1}{(\lambda d)^{k-1}} \left[ \left(1 - \frac{2}{\lambda d}\right) \left(1 - u_k(x)\right) + \sum_{i=1}^{k-1} \left(1 - \frac{1}{\lambda d}\right) (1 - u_i(x)) \right]$$

$$\leq -\frac{1}{(\lambda d)^{k-1}} \frac{Q_k(x)}{n} \left(1 - \frac{2}{\lambda d}\right).$$

This establishes the required upper bound on $(1 + \lambda)\Delta Q_k(x)$. The calculation works because the $\beta_i$ are the entries of a good approximation to the dominant eigenvector of the matrix $M_k$ defined in Section 2.

For the lower bound, the previous calculation, and the bound $1 - u_i(x)^d \geq d(1 - u) - \left(\frac{d}{2}\right)(1 - u)^2$, lead us to

$$\lambda \sum_{i=1}^{k-1} (\beta_{i+1} - \beta_i)(1 - u_i(x)^d) - \sum_{i=1}^{k} (\beta_i - \beta_{i-1})(1 - u_i(x))$$

$$\geq -\lambda \left(\frac{d}{2}\right) \sum_{i=1}^{k-1} (\beta_{i+1} - \beta_i)(1 - u_i(x))^2$$

$$- \frac{1}{(\lambda d)^{k-1}} \left[ \left(1 - \frac{2}{\lambda d}\right) \left(1 - u_k(x)\right) + \sum_{i=1}^{k-1} \left(1 - \frac{1}{\lambda d}\right) (1 - u_i(x)) \right]$$

$$\geq -\lambda \left(\frac{d}{2}\right) \sum_{i=1}^{k-1} (\beta_{i+1} - \beta_i)(1 - u_i(x))^2 - \frac{1}{(\lambda d)^{k-1}} \frac{Q_k(x)}{n}.$$

Here we used the fact that $1 - 1/(\lambda d) \leq \beta_i$ for each $i$. It remains to show that

$$\lambda \left(\frac{d}{2}\right) \sum_{i=1}^{k-1} (\beta_{i+1} - \beta_i)(1 - u_i(x))^2 \leq \left(\frac{Q_k-1(x)}{n}\right)^2 \frac{1}{(\lambda d)^{k-3}}.$$
We observe that
\[
\left( \frac{Q_{k-1}(x)}{n} \right)^2 = \left( \sum_{i=1}^{k-1} (\lambda d)^{(k-1)-i} \frac{\sin \left( \frac{ix}{k} \right)}{\sin \left( \frac{(k-1)}{k} \pi \right)} (1 - u_i(x)) \right)^2
\]
\[
\geq \sum_{i=1}^{k-1} (\lambda d)^{(k-1)-i} (1 - u_i(x))^2
\]
\[
\geq (\lambda d)^{(k-1)} \sum_{i=1}^{k-1} (\beta_{i+1} - \beta_i)(1 - u_i(x))^2,
\]
which implies the required inequality. \(\square\)

We prove a similar result for the functions \(Q_j(x), 1 \leq j \leq k - 1\). Ideally, the drift bounds would be expressed in terms of \(Q_j(x)\) itself and \(Q_{j+1}(x)\); however, there is a complication. In the upper bound, there appears a term which can be bounded above by \(\lambda d \sum_{i=1}^{j} \gamma_{j,i} (1 - u_i(x))^2\), and we would like to show that this is small compared with \(\lambda d \sum_{i=1}^{k} \gamma_{j,i} (1 - u_i(x))^2\). This is true if \(1 - u_j(x) \ll 1/d\), but in general we cannot assume this. We bound this term above, very crudely, by
\[
\lambda \left( \frac{d}{2} \right) \left( \sum_{i=1}^{k-1} (1 - u_i(x)) \right) \left( \sum_{i=1}^{j} \gamma_{j,i} (1 - u_i(x)) \right) = \lambda \left( \frac{d}{2} \right) \frac{P_{k-1}(x)Q_j(x)}{n^2};
\]
we use the function \(P_{k-1}\) here because its drifts are relatively easy to handle.

**Lemma 5.2.** Fix \(j\) with \(1 \leq j \leq k - 1\). For any state \(x \in \mathbb{Z}_n^+\), we have
\[
(1 + \lambda) \Delta Q_j(x) \leq -\lambda d \frac{Q_j(x)}{n} \left( 1 - \frac{2}{\sqrt{\lambda d}} \frac{dP_{k-1}(x)}{n} \right) + \frac{Q_{j+1}(x)}{n},
\]
\[
(1 + \lambda) \Delta Q_j(x) \geq -\lambda d \frac{Q_j(x)}{n} \left( 1 + \frac{2}{\sqrt{\lambda d}} \right) + \frac{Q_{j+1}(x)}{n}.
\]

**Proof.** We begin by calculating
\[
(1 + \lambda) \Delta Q_j(x) = \sum_{i=1}^{j} \gamma_{j,i} \left( -\lambda u_{i-1}(x)^d + \lambda u_i(x)^d + u_i(x) - u_{i+1}(x) \right)
\]
\[
= \sum_{i=1}^{j} \gamma_{j,i} \left( \lambda(1 - u_{i-1}(x)^d) - \lambda(1 - u_i(x)^d) \right)
\]
\[
+ \sum_{i=1}^{j} \gamma_{j,i} \left( -(1 - u_i(x)) + (1 - u_{i+1}(x)) \right).
\]
Rearranging now gives

\[(1 + \lambda)\Delta Q_j(x) = \sum_{i=1}^{j} (\gamma_{j,i} - \gamma_{j,i})(1 - u_i(x)) - \lambda \sum_{i=1}^{j} (\gamma_{j,i} - \gamma_{j,i+1})(1 - u_i(x)) + \gamma_{j,j}(1 - u_{j+1}(x)).\]

Recall that \(\gamma_{j,0} = \gamma_{j,j+1} = 0\), and note that \(\gamma_{j,1} > \gamma_{j,2} > \cdots > \gamma_{j,j} = 1\).

As before, we proceed by approximating \(1 - u_i(x)^d\) by \(d(1 - u_i(x))\), for \(i \leq j\). Using first that \(1 - u_i(x)^d \leq d(1 - u_i(x))\) for each \(i\), we have

\[(1 + \lambda)\Delta Q_j(x) \geq \sum_{i=1}^{j} (\gamma_{j,i} - \gamma_{j,i})(1 - u_i(x)) - \lambda d \sum_{i=1}^{j} (\gamma_{j,i} - \gamma_{j,i+1})(1 - u_i(x))
+ (1 - u_{j+1}(x)) \]

\[= \sum_{i=1}^{j} (1 - u_i(x)) [\gamma_{j,i-1} + \lambda d \gamma_{j,i+1} - (\lambda d + 1) \gamma_{j,i}] + (1 - u_{j+1}(x)) \]

\[= -\sum_{i=1}^{j} (1 - u_i(x)) \gamma_{j,i} \left[ \lambda d + 1 - 2\sqrt{\lambda d} \cos \left( \frac{\pi}{j+1} \right) \right] + (1 - u_{j+1}(x)) \]

\[= -\left[ \lambda d + 1 - 2\sqrt{\lambda d} \cos \left( \frac{\pi}{j+1} \right) \right] \frac{Q_j(x)}{n} + (1 - u_{j+1}(x)) \]

\[\geq -\lambda d \frac{Q_j(x)}{n} + \frac{Q_{j+1}(x)}{n} - 2\sqrt{\lambda d} \frac{Q_{j+1}(x)}{n},\]

as claimed. In the last line above, we used Exercise 2, as well as the inequality \(2\sqrt{\lambda d} \cos(\pi/(j+1)) \geq \sqrt{2\lambda d} \geq 1\), valid since \(\lambda d \geq 4\).

For the upper bound, we use the facts that \(1 - u_{j+1}(x) \leq \frac{Q_{j+1}(x)}{n}\) and \(1 - u_i(x)^d \geq d(1 - u_i(x)) - \binom{d}{2}(1 - u_i(x))^2\), to obtain

\[(1 + \lambda)\Delta Q_j(x) \leq -\left[ \lambda d + 1 - 2\sqrt{\lambda d} \cos \left( \frac{\pi}{j+1} \right) \right] \frac{Q_j(x)}{n} + (1 - u_{j+1}(x)) \]

\[+ \lambda \binom{d}{2} \sum_{i=1}^{j} (\gamma_{j,i} - \gamma_{j,i+1})(1 - u_i(x))^2 \]

\[\leq -\lambda d \frac{Q_j(x)}{n} \left( 1 - \frac{2}{\sqrt{\lambda d}} \right) + \frac{Q_{j+1}(x)}{n} + \frac{P_{k-1}(x)}{n} \lambda \binom{d}{2} \sum_{i=1}^{j} \gamma_{j,i}(1 - u_i(x)) \]

\[\leq -\lambda d \frac{Q_j(x)}{n} \left( 1 - \frac{2}{\sqrt{\lambda d}} \right) + \frac{Q_{j+1}(x)}{n} + \frac{P_{k-1}(x)Q_j(x)}{n^2} \lambda \binom{d}{2},\]

as claimed. \(\square\)
Next we prove a similar result for the function $P_{k-1}$. For this function, we need only a fairly crude upper bound on the drift.

**Lemma 5.3.** For any state $x \in \mathbb{Z}_+^n$, we have

$$(1 + \lambda)\Delta P_{k-1}(x) \leq -\lambda \left(1 - \exp\left(-\frac{dP_{k-1}(x)}{(k-1)n}\right)\right) + \frac{Q_k(x)}{n}.$$ 

**Proof.** The calculation this time is simpler: we have

$\Delta P_{k-1}(x) = (1 + \lambda)E[P_{k-1}(X_{t+1}) - P_{k-1}(X_t) | X_t = x]$

$$= -\sum_{i=1}^{k-1} (\lambda u_{i-1}(x)^d - \lambda u_i(x)^d - u_i(x) + u_{i+1}(x))$$

$$= \lambda u_{k-1}(x)^d - \lambda - u_k(x) + u_1(x)$$

$$= -\lambda(1 - u_{k-1}(x)^d) + (1 - u_k(x)) - (1 - u_1(x))$$

$$\leq -\lambda(1 - u_{k-1}(x)^d) + (1 - u_k(x)).$$

We have $1 - u_k(x) \leq \frac{Q_k(x)}{n}$ and $1 - u_{k-1}(x) \geq \frac{1}{k-1} \frac{P_{k-1}(x)}{n}$, so

$$u_{k-1}(x)^d \leq \left(1 - \frac{1}{k-1} \frac{P_{k-1}(x)}{n}\right)^d$$

$$\leq \exp\left(-\frac{dP_{k-1}(x)}{(k-1)n}\right),$$

which gives the required bound. \qed

6. **Hitting Times and Exit Times**

At this point, we begin the proof of Theorem 1.2. Accordingly, from now on we fix values of $n, d, k \in \mathbb{N}$, $\lambda, \epsilon \in (0, 1)$ such that:

$$d^k(1 - \lambda) \geq 2 \log^2 n,$$  \hspace{1cm} (6.1)

$$k \geq 2,$$  \hspace{1cm} (6.2)

$$\epsilon \leq \frac{1}{10},$$  \hspace{1cm} (6.3)

$$\epsilon \sqrt{d} \geq 150k,$$  \hspace{1cm} (6.4)

$$\epsilon \geq 100k(1 - \lambda)d^{k-1},$$  \hspace{1cm} (6.5)

$$\epsilon^2 dn(1 - \lambda)^2 \geq 600k^2 \log^2 n.$$  \hspace{1cm} (6.6)

We explore some consequences of these assumptions.
Lemma 6.1. For positive integers \( n, d, k \), and real numbers \( \lambda \in (0,1) \), \( \varepsilon \in (0,1) \) satisfying (6.1) to (6.6) above, we also have:

\[
\begin{align*}
\epsilon &\geq 10^{15}, \quad (6.7) \\
\varepsilon d &\geq 200k \log^2 n, \quad (6.8) \\
\log^2 n (1 - \lambda) &\leq \frac{\varepsilon^2}{80000}, \quad (6.9) \\
k &= \left\lceil \frac{\log (1 - \lambda) - 1}{\log d} \right\rceil, \quad (6.10) \\
\varepsilon^3 n (1 - \lambda) &\geq 60000k^3 \log^2 n d^{k - 2}, \quad (6.11) \\
\varepsilon n (1 - \lambda) &\geq 60000, \quad (6.12) \\
k &\leq \log n, \quad (6.13) \\
d^k (1 - \lambda) &\geq 2k \log n, \quad (6.14) \\
\lambda^k &\geq 9/10, \quad (6.15) \\
\beta_k &= 1 - \frac{k}{(\lambda d)^k} \geq 1 - \varepsilon/2, \quad (6.16) \\
\sum_{i=1}^{\infty} \frac{1}{(\lambda d)^i} &\leq \sum_{i=1}^{\infty} \frac{1}{(\lambda d)^{i/2}} \leq \frac{\varepsilon}{2k}. \quad (6.17)
\end{align*}
\]

Proof. Squaring (6.5) and multiplying by (6.6) gives

\[\varepsilon^4 n \geq 6 \times 10^6 k^4 \log^2 n d^{2k - 3}.\]

Using (6.3) and (6.2) now yields \( n/\log^2 n \geq 9.6 \times 10^{11} \), which implies (6.7).

(6.8) follows from multiplying (6.1) and (6.5). Multiplying by (6.5) again gives \( \varepsilon^2 \geq 20000k^2 (1 - \lambda) \log^2 n d^{k - 2} \), which implies (6.9) via (6.2).

(6.10) is equivalent to \( d^k (1 - \lambda) \geq 1 > d^{k - 1} (1 - \lambda) \), and these inequalities are clear from (6.1) and (6.5).

Multiplying (6.5) and (6.6) gives (6.11), and the weaker version (6.12).

It follows from (6.8) and (6.10) that \( k \leq \log (1 - \lambda)^{-1} \), and from (6.12) that \( (1 - \lambda)^{-1} \leq n \). (6.13) follows, and also (6.14) now follows from (6.1).

(6.15) follows from

\[\lambda^k = (1 - (1 - \lambda))^k \geq 1 - k (1 - \lambda) \geq 1 - \frac{\varepsilon}{100d},\]

where at the end we used (6.5) and (6.2).

(6.16) is equivalent to \( 2k \leq \varepsilon (\lambda d)^k \), which follows comfortably from (6.8).

From (6.4) and (6.15), we have that

\[\frac{1}{\sqrt{\lambda d}} \leq \sqrt{\frac{10}{9}} \frac{\varepsilon}{150k} \leq \frac{\varepsilon}{100k},\]

which comfortably implies (6.17). \( \square \)
We shall mention explicitly each time we use one of the inequalities (6.1)–(6.17). Exceptionally, we will not mention (6.7); on several occasions we note that an inequality holds for large enough $n$, and $n \geq 10^{15}$ will always suffice.

We define a sequence of pairs of subsets of $\mathbb{Z}^n_{\geq 0}$. Each pair consists of a set $S_0$ in which some inequality holds, and a set $S_1$ in which a looser version of the inequality holds: we also demand that $S_0$ and $S_1$ be subsets of the previous set $R_1$ in the sequence. Associated with each pair $(S_0, S_1)$ in the sequence is a hitting time

$$T_S = \inf\{t \geq T_R : X_t \in S_0\},$$

where $(R_0, R_1)$ is the previous pair in the sequence, and an exit time

$$T^*_S = \inf\{t \geq T_S : X_t \notin S_1\}.$$

Our aim in each case is to prove that, with high probability, unless the previous exit time $T^*_R$ occurs early, $T_S$ is unlikely to be larger than some quantity $m_S$ whose order is to be thought of as polynomial in $n$. More precisely, if we start in a state in $A(\ell, g)$, then the sum of all the $m_S$ is of order at most the maximum of $kn(1-\lambda)^{-1}$, $gn(1-\lambda)^{-1}$ and $\ell n$; note that $k \leq \log n$ and $(1-\lambda)^{-1} \leq n$ (see (6.13) and (6.12)), so if $\ell$ and $g$ are bounded by a polynomial in $n$, then so are all the $m_S$.

Throughout the proof, we set

$$s_0 = e^{\frac{1}{4}\log^2 n}.$$

We shall also prove that, again with high probability, each exit time $T^*_S$ is at least $s_0$, which is larger than the sum of all the terms $m_S$. For convenience, we shall not be too precise about our error probabilities, and simply declare them all to be at most

$$1/s_0 = e^{-\frac{1}{4}\log^2 n},$$

or some small multiple of $1/s_0$.

We fix, for the moment, a pair of positive real numbers $\ell \geq k$ and $g \geq k$. We set $q(\ell, g) = (23k + 72g)\varepsilon^{-1}n(1-\lambda)^{-1} + 8\ell n$, and we make the (mild) assumption that $\ell$ and $g$ are chosen so that $q(\ell, g) \leq s_0/2$.

The first pair of sets in our sequence will be as defined earlier:

$$A_0 = A_0(\ell, g) = \{x : \|x\|_{\infty} \leq \ell \text{ and } \|x\|_1 \leq gn\},$$

$$A_1 = A_1(\ell, g) = \{x : \|x\|_{\infty} \leq 3\ell \text{ and } \|x\|_1 \leq 3gn\},$$

and we adopt the hypothesis that $X_0 = x_0$ almost surely, where $x_0$ is a fixed state in $A_0 = A_0(\ell, g)$, so that $T_A := \min\{t \geq 0 : X_t \in A_0\} = 0$.

For $\ell = \ell^* = \log^2 n(1-\lambda)^{-1}$ and $g = g^* = 2(1-\lambda)^{-1}$, Lemma 4.4 tells us that indeed the exit time $T^*_A = \inf\{t > 0 : X_t \notin A_1^*\}$ is unlikely to be less than $s_0$. For smaller values of $\ell$ and $g$, we do not know this a priori.

The sets we define are dependent on the chosen values of $n$, $d$, $k$, $\lambda$ and $\varepsilon$, as well as on $\ell$ and $g$. For the most part, we drop reference to this dependence from the notation. However, later in the paper we shall need
to vary $\varepsilon$ while keeping all other parameters fixed; in this case, we shall use the notation (e.g.) $B_0^j$ to emphasise the dependence.

We define:

$B_0 = \{x : Q_k(x) \leq (1 + \varepsilon)n(1 - \lambda)(\lambda d)^{k-1}\} \cap A_1,$

$B_1 = \{x : Q_k(x) \leq (1 + 2\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}\} \cap A_1,$

$C_0 = \{x : P_{k-1}(x) \leq 2kn(1 - \lambda)(\lambda d)^{k-2}\} \cap B_1,$

$C_1 = \{x : P_{k-1}(x) \leq 3kn(1 - \lambda)(\lambda d)^{k-2}\} \cap B_1,$

$D_0 = \{x : Q_{k-1}(x) \leq (1 + 4\varepsilon)n(1 - \lambda)(\lambda d)^{k-2}\} \cap C_1,$

$D_1 = \{x : Q_{k-1}(x) \leq (1 + 5\varepsilon)n(1 - \lambda)(\lambda d)^{k-2}\} \cap C_1,$

$E_0 = \{x : u_{k+1}(x) \leq \varepsilon(1 - \lambda) \text{ and } Q_k(x) \geq (1 - 3\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}\} \cap D_1,$

$E_1 = \{x : u_{k+1}(x) \leq \varepsilon(1 - \lambda) \text{ and } Q_k(x) \geq (1 - 4\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}\} \cap D_1.$

Next we have a sequence of pairs of sets, indexed by $j = k - 1, \ldots, 1$:

$G_0^j = \{x : \left[1 - (4 + \frac{k - j - 1/2}{k})\varepsilon\right]n(1 - \lambda)(\lambda d)^{j-1} \leq Q_j(x) \leq \left[1 + (4 + \frac{k - j - 1/2}{k})\varepsilon\right]n(1 - \lambda)(\lambda d)^{j-1}\} \cap G_1^{j+1},$

$G_1^j = \{x : \left[1 - (4 + \frac{k - j}{k})\varepsilon\right]n(1 - \lambda)(\lambda d)^{j-1} \leq Q_j(x) \leq \left[1 + (4 + \frac{k - j}{k})\varepsilon\right]n(1 - \lambda)(\lambda d)^{j-1}\} \cap G_1^{j+1},$

where we declare $G_k^j$ to be equal to $E_1$. Finally, departing slightly from our pattern, we define

$\mathcal{H} = \mathcal{H}_0 = \mathcal{H}_1 = \{x : u_{k+1}(x) = 0\} \cap G_1^1.$

The hitting times and exit times are all defined in accordance with the pattern given. For instance $T_B = \inf\{t : X_t \in B_0\}$, $T_B^1 = \inf\{t > T_B : X_t \notin B_1\}$, and $T_C = \inf\{t \geq T_B : X_t \in C_0\}$. We also set $T_G^k = T_C$ and $T_G^{k+1} = T_C$, in accordance with the notion that the set pair $(G_0^{k-1}, G_1^{k-1})$ follows $(E_0, E_1)$ in the sequence.

Initially, the sets above all depend on the values of $\ell$ and $g$ defining the initial pair of sets $(A_0, A_1)$, since all the sets are intersected with $A_1$. However, since states in $\mathcal{H}$ have no queue of length $k + 1$ or greater, we have $\mathcal{H} \subseteq A_0(k, k) \subseteq A_1(\ell, g)$ for all $\ell, g \geq k$, and so the set $\mathcal{H}$ does not depend on $\ell$ and $g$, provided these parameters are each at least $k$.

We now state a sequence of lemmas. Throughout, we assume that $X_0 = x_0$ a.s., where $x_0$ is an arbitrary state in $A_0 = A_0(\ell, g)$.

**Lemma 6.2.** Let $m_B = 8k\varepsilon^{-1}n(1 - \lambda)^{-1}$.

1. $\mathbb{P}(T_B \wedge T_A^1 \geq m_B) \leq 1/s_0$.
2. $\mathbb{P}(T_B^1 \leq s_0 < T_A^1) \leq 1/s_0$. 
Lemma 6.3. Let $m_C = 12kn(1 - \lambda)^{-1}(\lambda d)^{1-k}$.

(1) $\mathbb{P}(T_C \cap T_B^t \geq T_B + m_C) \leq 1/s_0$.
(2) $\mathbb{P}(T_C^t \leq s_0 < T_B^t) \leq 1/s_0$.

Lemma 6.4. Let $m_D = 8\epsilon^{-1}n(1 - \lambda)^{-1}(\lambda d)^{-k/2}$.

(1) $\mathbb{P}(T_D \cap T_C^t \geq T_C + m_D) \leq 1/s_0$.
(2) $\mathbb{P}(T_D^t \leq s_0 < T_C^t) \leq 1/s_0$.

Lemma 6.5. Let $m_G = m_G(g) = (13k + 72g)\epsilon^{-1}n(1 - \lambda)^{-1}$.

(1) $\mathbb{P}(T_G \cap T_{G_j}^t \geq T_G + m_G) \leq 1/s_0$.
(2) $\mathbb{P}(T_G^t \leq s_0 < T_{G_j}^t) \leq 1/s_0$.

Lemma 6.6. Let $m_S = m_S(\ell) = n(8\ell + 32\log^2 n)$.

(1) $\mathbb{P}(T_H \cap T_{G_1}^t \geq T_{G_1} + m_H) \leq 1/s_0$.
(2) $\mathbb{P}(T_H^t \leq s_0 < T_{G_1}^t) \leq 1/s_0$.

We note here that $q(\ell, g) = (23k + 72g)\epsilon^{-1}n(1 - \lambda)^{-1} + 8\ell n$ is larger than all the constants $m_B, m_C, \ldots$ appearing in the lemmas, so these constants are all at most $s_0/2$. Combining the lemmas gives the following result.

Proposition 6.8. For any $x_0 \in A_0 = A_0(\ell, g)$, and a copy $(X_t)$ of the process with $X_0 = x_0$ a.s., we have

$$\mathbb{P}(X_t \in \mathcal{H} \text{ for all } t \in [q(\ell, g), s_0]) \geq 1 - \frac{2k + 8}{s_0} - \mathbb{P}(T_A^t \leq s_0).$$

Proof. The idea is that, with high probability, either the chain $(X_t)$ exits $\mathcal{A}_1(\ell, g)$ before time $s_0$, or the chain enters each of the sets $B_0, \ldots, \mathcal{H}_0$ in turn, within time $q(\ell, g)$, and does not exit any of the sets $\mathcal{A}_1, \ldots, \mathcal{H}_1$ before time $s_0$, which is what we need.

More formally, consider the following list of events concerning the various stopping times we have defined:

$$E_1 = \{T_A^t > s_0\}, \quad E_2 = \{T_B \leq m_B\}, \quad E_3 = \{T_B^t > s_0\},$$
$$E_4 = \{T_C \leq m_B + m_C\}, \quad E_5 = \{T_C^t > s_0\}, \quad E_6 = \{T_D \leq m_B + m_C + m_D\},$$
$$E_7 = \{T_D^t > s_0\}, \quad E_8 = \{T_E \leq m_B + \cdots + m_G\}, \quad E_9 = \{T_E^t > s_0\},$$
$$E_{10} = \{T_{G_{j-1}} \leq m_B + \cdots + m_G + m_G\}, \quad E_{11} = \{T_{G_{j-1}}^t > s_0\}, \quad \ldots,$$
$$E_{2k+6} = \{T_{G_1} \leq m_B + \cdots + (k - 1)m_G\}, \quad E_{2k+7} = \{T_{G_1}^t > s_0\},$$
$$E_{2k+8} = \{T_H \leq m_B + \cdots + (k - 1)m_G + m_H\}, \quad E_{2k+9} = \{T_H^t > s_0\}.$$
If $E_{2k+8}$ holds, then
\[
T_H \leq m_B + m_C + m_D + m_E + (k-1)m_G + m_H \\
\leq 8k\varepsilon^{-1}n(1-\lambda)^{-1} + 12kn(1-\lambda)^{-1}(\lambda d)^{1-k} \\
+ 8\varepsilon^{-1}n(1-\lambda)^{-1}(\lambda d)^{-k/2} + (13k + 72g)\varepsilon^{-1}n(1-\lambda)^{-1} \\
+ 32(k-1)k\varepsilon^{-1}n(1-\lambda)^{-1}(\lambda d)^{-1} + n(8\varepsilon + 32\log^2 n) \\
\leq k\varepsilon^{-1}n(1-\lambda)^{-1}(8 + \frac{12\varepsilon}{\lambda d} + \frac{8}{\lambda d} + 13 + \frac{32(k-1)}{\lambda d} \\
+ 32\varepsilon\log^2 n(1-\lambda)) + 72g\varepsilon^{-1}n(1-\lambda)^{-1} + 8\varepsilon n \\
\leq \varepsilon^{-1}n(1-\lambda)^{-1}(23k + 72g) + 8\varepsilon n \\
= q(\ell, g),
\]
where we used (6.4) and (6.15) to tell us that $\frac{32(k-1)+8+12\varepsilon}{\lambda d} \leq 1$, and (6.9) to show that $32\varepsilon\log^2 n(1-\lambda) \leq 1$. Therefore, if $E = \bigcap_{j=1}^{2k+9} E_j$ holds, then in particular $E_{2k+8}$ and $E_{2k+9}$ hold, which implies that $X_t \in H$ for $q(\ell, g) \leq t \leq s_0$. Thus $E$ is contained in the event \( \{X_t \in H \text{ for all } t \in [q(\ell, g), s_0]\} \), and it suffices to show that $\mathbb{P}(E) \leq \frac{2k+8}{s_0} + \mathbb{P}(E_1)$. We write
\[
\mathbb{P}(E) = \mathbb{P}(E_1) + \sum_{j=2}^{2k+9} \mathbb{P} \left( E_j \cap \bigcap_{i=1}^{j-1} E_i \right),
\]
and now we see that it suffices to prove that each of the terms $\mathbb{P} \left( E_j \cap \bigcap_{i=1}^{j-1} E_i \right)$ is at most $1/s_0$.

We show how to derive the first few of these inequalities from Lemmas (6.2, 6.7) first have
\[
\mathbb{P}(E_2 \cap E_1) = \mathbb{P}(T_A^\uparrow > s_0, T_B > m_B) \leq \mathbb{P}(T_B \land T_A^\uparrow \geq m_B) \leq 1/s_0
\]
by Lemma (6.2)(1). Then we have
\[
\mathbb{P}(E_3 \cap E_1 \cap E_2) \leq \mathbb{P}(E_3 \cap E_1) = \mathbb{P}(T_B^\uparrow \leq s_0 < T_A^\uparrow) \leq 1/s_0
\]
by Lemma (6.2)(2). Next we have, using the fact that $m_B + m_C \leq s_0$,
\[
\mathbb{P}(E_4 \cap E_1 \cap E_2 \cap E_3) \leq \mathbb{P}(E_4 \cap E_2 \cap E_3) \\
= \mathbb{P}(T_B^\uparrow > s_0, T_B \leq m_B, T_C > m_B + m_C) \\
\leq \mathbb{P}(T_C \land T_B^\uparrow > m_B + m_C, T_B \leq m_B) \\
\leq \mathbb{P}(T_C \land T_B^\uparrow > T_B + m_C) \\
\leq 1/s_0,
\]
by Lemma (6.3)(1). For $j = 5, \ldots, 2k+9$, the upper bound on $\mathbb{P} \left( E_j \cap \bigcap_{i=1}^{j-1} E_i \right)$ follows either as for $j = 3$ or as for $j = 4$: it is important here that $m_B + m_C + m_D + m_E + (k-1)m_G + m_H \leq q(\ell, g) \leq s_0$. □

We now have the following consequence for an equilibrium copy $(Y_t)$ of the $(n,d,\lambda)$-supermarket process.
Corollary 6.9. \( \mathbb{P}(Y_t \in \mathcal{H} \text{ for all } t \in [0, s_0]) \geq 1 - (4k + 20)/s_0 \geq 1 - e^{-\frac{1}{4} \log^2 n}, \text{ for } n \geq 1000. \)

**Proof.** Recall the definitions of \( \ell^*, g^*, \mathcal{A}_0^* \) and \( \mathcal{A}_1^* \) from Section 4. Set also \( q^* = q(\ell^*, g^*) \), and note that \( q^* \leq s_0/2 \), with plenty to spare. From Lemma 4.2, we have that \( \mathbb{P}(Y_0 \notin \mathcal{A}_0^*) \leq ne^{-\log^2 n} \leq e^{-\frac{1}{4} \log^2 n} = 1/s_0, \) since \( n \geq 5 \). Also, from Lemma 4.3, for a copy \( (X_t^0) \) of the process starting in a state \( x \in \mathcal{A}_0^* \), we have that \( \mathbb{P}(T_A^\dagger < s_0) \leq 1/s_0 \). We now have

\[
\mathbb{P}(Y_t \notin \mathcal{H} \text{ for some } t \in [0, s_0/2]) \\
= \mathbb{P}(Y_t \notin \mathcal{H} \text{ for some } t \in [q^*, q^* + s_0/2]) \\
\leq \mathbb{P}(Y_t \notin \mathcal{H} \text{ for some } t \in [q^*, q^* + s_0/2] \mid Y_0 \in \mathcal{A}_0^*) + \mathbb{P}(Y_0 \notin \mathcal{A}_0^*) \\
\leq \sup_{x \in \mathcal{A}_0^*} \mathbb{P}(X_t^x \notin \mathcal{H} \text{ for some } t \in [q^*, s_0]) + 1/s_0 \\
\leq \frac{2k + 8}{s_0} + \frac{1}{s_0} = \frac{2k + 10}{s_0},
\]

by Proposition 6.8. Hence \( \mathbb{P}(Y_t \notin \mathcal{H} \text{ for some } t \in [0, s_0]) \leq (4k + 20)/s_0 \).

For the final inequality, note that \( (4k + 20)/s_0 \leq (4 \log n + 20)e^{-\frac{1}{4} \log^2 n} < e^{-\frac{1}{4} \log^2 n} \) for \( n \geq 1000. \)

We can now use the result above to prove the following more explicit version of Proposition 6.8.

**Theorem 6.10.** Suppose that \( \ell \) and \( g \) are at least \( k \), and that \( q(\ell, g) \leq s_0/2 \). Let \( x_0 \) be any queue-lengths vector in \( \mathcal{A}_0(\ell, g) \), and suppose that \( X_0 = x_0 \) a.s. Then we have

\[
\mathbb{P}(X_t \in \mathcal{H} \text{ for all } t \in [q(\ell, g), s_0]) \geq 1 - \frac{6k + 28}{s_0}.
\]

**Proof.** We apply, successively, Proposition 6.8, Lemma 4.3 and Corollary 6.9 to obtain that

\[
\mathbb{P}(X_t \in \mathcal{H} \text{ for all } t \in [q(\ell, g), s_0]) \\
\geq 1 - \frac{2k + 8}{s_0} - \mathbb{P}(T_A^\dagger \leq s_0) \\
= 1 - \frac{2k + 8}{s_0} - \mathbb{P}(\exists t \in [0, s_0], X_t \notin \mathcal{A}_1(\ell, g)) \\
\geq 1 - \frac{2k + 8}{s_0} - \mathbb{P}(\exists t \in [0, s_0], Y_t \notin \mathcal{A}_0(\ell, g)) \\
\geq 1 - \frac{2k + 8}{s_0} - \mathbb{P}(\exists t \in [0, s_0], Y_t \notin \mathcal{H}) \\
\geq 1 - \frac{2k + 8}{s_0} - \frac{4k + 20}{s_0},
\]

as required. \( \square \)
In the next sections, we shall prove Lemmas 6.2 to 6.7. Then we show that $H \subseteq \mathcal{N}^\varepsilon(n,d,\lambda,k)$. Theorem 1.2 will then follow from Corollary 6.9, since $s_0/2 = \frac{1}{2} \varepsilon^{\frac{1}{3} \log^2 n} > e^{\frac{1}{4} \log^2 n}$ for $n \geq 18$.

We draw one further conclusion from the results in this section. Suppose that $(X_t)$ starts in a state $x_0$ in the set

$$I = A_0^* \cap B_0 \cap C_0 \cap D_0 \cap E_0 \cap \bigcap_{j=1}^{k-1} G_j^0 \cap H_0.$$ 

Then all the hitting times $T_B, T_C, T_D, T_E$ and $T_H$ are equal to 0. In the notation of the proof of Proposition 6.8 this implies that the events $E_j$ for $j$ even occur with probability 1. Also, by Lemma 4.4, $P(E_1) \leq \frac{1}{s_0}$. So following the proof of Proposition 6.8 yields the result below.

**Theorem 6.11.** Suppose $x_0 \in I$, and $X_0 = x_0$ a.s. Then

$$P(X_t \in H \text{ for all } t \in [0,s_0]) \geq 1 - (k + 5)/s_0.$$ 

We shall explore the consequences of this result further in Section 10.

7. Proofs of Lemmas 6.2, 6.3 and 6.4

In this section, we prove the first three of the sequence of lemmas stated in the previous section, and also derive tighter inequalities on the drifts of the functions $Q_j(x)$ for $x \in D_1$. The proofs of the three lemmas are all straightforward applications of Lemma 3.3 and all similar to one another.

**Proof of Lemma 6.2**

*Proof.* We apply Lemma 3.3. We set $(\varphi_t) = (\mathcal{F}_t)$, the natural filtration of the process, and also: $F = Q_k, S = A_1$, $h = (1 + \varepsilon)(1 - \lambda)n(\lambda d)^{k-1}, \rho = \varepsilon(1 - \lambda)n(\lambda d)^{k-1}$, $m = m_B = 8k\varepsilon^{-1}n(1 - \lambda)^{-1}, s = s_0 = e^{\frac{1}{3} \log^2 n}$ and $T^* = 0$. We have $\rho \geq 60\lambda(\lambda d)^{k-1} \geq 2$, by (6.12) and (6.15). It is also clear that $Q_k(x) \leq c := kn$ for any $x \in \mathbb{Z}_n^+$. We note also that $Q_k$ takes jumps of size at most 1.

Suppose now that $Q_k(x) \geq h$. Then

$$\exp\left(-\frac{dQ_k(X_t)}{kn}\right) \leq \exp\left(-\frac{(1 - \lambda)(\lambda d)^k}{k}\right).$$

Now we have, using first (6.15) and (6.14) and then (6.12), that

$$\frac{(1 - \lambda)(\lambda d)^k}{k} \geq \frac{9}{5} \log n > -\log\left(\frac{\varepsilon}{60\lambda(1 - \lambda)}\right).$$

Thus we have

$$\exp\left(-\frac{dQ_k(X_t)}{kn}\right) \leq \frac{\varepsilon}{60\lambda(1 - \lambda)}.$$
Hence, by Lemma 5.1 for $x$ with $Q_k(x) \geq h$, we have
\[
(1 + \lambda)\Delta Q_k(x) \leq \beta_k \left( (1 - \lambda) - u_{k+1}(x) + \lambda \exp(-dQ_k(x)/kn) \right)
- \frac{1}{(\lambda d)^{k-1}} \frac{Q_k(x)}{n} \left( 1 - \frac{2}{\lambda d} \right),
\]
\[
\leq \beta_k \left( (1 - \lambda) + \frac{\varepsilon(1 - \lambda)}{60000} \right) - (1 + \varepsilon)(1 - \lambda) \left( 1 - \frac{2}{\lambda d} \right)
\leq (1 - \lambda) \left[ 1 + \frac{\varepsilon}{60000} - 1 - \varepsilon + (1 + \varepsilon)\frac{2}{\lambda d} \right]
\leq -(1 - \lambda)\frac{\varepsilon}{2},
\]
where at the end we used the fact that $\frac{2}{\lambda d} \leq \frac{\varepsilon}{8}$, which follows comfortably from (6.14). So $\Delta Q_k(x) \leq -(1 - \lambda)\varepsilon/4 := -v$. Note that $m_Bv = 2c$.

We have now verified that the conditions of Lemma 3.3 are satisfied, for the given values of the parameters. As in the lemma, we have $T_0 = T^*_A$, $T_1 = \inf\{t : Q_k(X_t) \leq h\}$ and $T_2 = \inf\{t > T_1 : Q_k(X_t) \geq h + \rho\}$.

It need not be the case that $T_1 = T_B$, since $X_{T_1}$ need not be in $A_1$.

However, we do have $T_1 \wedge T^*_A = T_B \wedge T^*_A$ and thus
\[
\mathbb{P}(T_B \wedge T^*_A > m_B) = \mathbb{P}(T_1 \wedge T^*_A > m_B)
\leq \exp(-v^2m_B/8)
= \exp(-\varepsilon kn(1 - \lambda)/16)
\leq \exp(-3750\varepsilon^{-2}k^4 \log^2 n d^{-k-2})
\leq 1/s_0.
\]

In the penultimate line, we used (6.11); in the final line, all we needed was that $3750\varepsilon^{-2}k^4 \log^2 d^{-k-2} \geq \frac{1}{3} \log^2 n$, which is true with plenty to spare.

Also the events $T_2 \leq s_0 < T^*_A$ and $T^*_B \leq s_0 < T^*_A$ coincide, so we have
\[
\mathbb{P}(T^*_B \leq s_0 < T^*_A) \leq \mathbb{P}(T_2 \leq s_0 < T^*_A)
\leq s \exp(-\rho v)
= s_0 \exp(-\varepsilon^2(1 - \lambda)^2 n(\lambda d)^{k-1}/6)
\leq s_0 \exp(-90k^2 \log^2 n d^{-k-2})
\leq 1/s_0,
\]
as required, where in the penultimate line we used (6.12) and (6.15). \hfill \Box

**Proof of Lemma 6.3**

Proof. Again we apply Lemma 3.3 to the Markov process $(X_t)$ with its natural filtration. Set $F = P_{k-1}$, $S = B_1$,
\[
h = 2kn(1 - \lambda)(\lambda d)^{k-2}, \quad \rho = kn(1 - \lambda)(\lambda d)^{k-2},
m = m_C = 12kn(1 - \lambda)^{-1}(\lambda d)^{1-k}, \quad \text{and} \quad s = s_0.
\]
Set $T^* = T_B$. It is again clear from (6.12) that $\rho \geq 2$, and also that $P_{k-1}$ takes jumps of size at
most 1, and that \( P_{k-1}(x) \leq c := kn \) for all \( x \in \mathbb{Z}_n^2 \). Here \( T_0 = T_B^\dagger \), \( T_1 = \inf\{ t \geq T_B : P_{k-1}(X_t) \leq h \} \), and \( T_2 = \inf\{ t > T_1 : P_{k-1}(X_t) \geq h + \rho \} \).

For \( x \in B_1 \) with \( P_{k-1}(x) \geq h \), we have \( Q_k(x) \leq (1 + 2\varepsilon)n(1 - \lambda)(\lambda d)^{k-1} \) and so, by Lemma 5.3
\[
(1 + \lambda)\Delta P_{k-1}(x) \\
\leq -\lambda \left( 1 - \exp \left( -\frac{dP_{k-1}(x)}{(k-1)n} \right) \right) + \frac{Q_k(x)}{n} \\
\leq -\lambda \left( 1 - \exp \left( -2d(1 - \lambda)(\lambda d)^{k-2} \right) \right) + (1 + 2\varepsilon)(1 - \lambda)(\lambda d)^{k-1}.
\]

Now we have \( y = 2d(1 - \lambda)(\lambda d)^{k-2} \leq \frac{50}{90} \leq 1/6 \), by (6.5); it is easy to check that \( e^{-y} \leq 1 - \frac{5}{6}y \) for \( 0 \leq y \leq 1/6 \). Also \( 1 + 2\varepsilon < 4/3 \) from (6.3), so \( \lambda d^k \geq 1/3 \).

\[
(1 + \lambda)\Delta P_{k-1}(x) \leq -\frac{5}{3}(1 - \lambda)(\lambda d)^{k-1} + \frac{4}{3}(1 - \lambda)(\lambda d)^{k-1} \\
= -\frac{1}{3}(1 - \lambda)(\lambda d)^{k-1},
\]
We conclude that, for such \( x \), \( \Delta P_{k-1}(x) \leq -\frac{1}{6}(1 - \lambda)(\lambda d)^{k-1} := -v \). Note that \( m_C v = 2c \).

As in the previous lemma, it need not be the case that \( T_1 = T_C \), since \( X_{T_1} \) need not be in \( B_1 \), so we may have \( T_C > T_1 \). However, we do have \( T_1 \land T_B^\dagger = T_C \land T_B^\dagger \). From Lemma 5.3 we obtain, using also (6.11),
\[
\mathbb{P}(T_C \land T_B^\dagger > T_B + m_C) = \mathbb{P}(T_1 \land T_0 > T_B + m_C) \\
\leq \exp(-v^2 m_C/8) \\
= \exp(-kn(1 - \lambda)(\lambda d)^{k-1}/24) \\
\leq \exp\left(-2500\varepsilon^{-3}nk^4\log^2 n\right) \\
\leq 1/s_0.
\]
Similarly, the events \( T_2 \leq s_0 < T_B^\dagger \) and \( T_C^\dagger \leq s_0 < T_B^\dagger \) coincide, and so
\[
\mathbb{P}(T_C^\dagger \leq s_0 < T_B^\dagger) = \mathbb{P}(T_2 \leq s_0 < T_0) \\
\leq s_0 \exp(-\rho v) \\
= s_0 \exp(-kn(1 - \lambda)^2(\lambda d)^{2k-3}/6) \\
\leq s_0 \exp\left(-90\varepsilon^{-2}(\lambda d)^{2k-4}k^3\log^2 n\right) \\
\leq 1/s_0.
\]
as required. Here, we used (6.6) and (6.15). \(\square\)

**Sketch of proof of Lemma 6.3**

**Proof.** The basic plan for this proof is the same as for the previous two lemmas, but here we have to take account of the fact that \( Q_{k-1} \) can take jumps of size up to \( (\lambda d)^{k-2}/2 \), and accordingly we apply Lemma 3.3 to the “scaled” function \( F(x) = Q'_{k-1}(x) = Q_{k-1}(x)/(\lambda d)^{(k-2)/2} \).
Apart from this, the proof is identical in structure to that of Lemma 5.3, and we give only the key calculation. For \( x \in C_1 \) with \( Q_{k-1}^t(x) \geq h = (1 + 4\varepsilon)n(1 - \lambda)(\lambda d)^{(k-2)/2} \), we have \( Q_k(x) \leq (1 + 2\varepsilon)n(1 - \lambda)(\lambda d)^{k-1} \), \( P_{k-1}(x) \leq 3kn(1 - \lambda)(\lambda d)^{k-2} \) and \( Q_{k-1}(x) \geq (1 + 4\varepsilon)n(1 - \lambda)(\lambda d)^{k-2} \). Thus, by Lemma 5.2 with \( j = k - 1 \), we have

\[
(1 + \lambda)\Delta Q_{k-1}(x) \\
\leq -\lambda d\frac{Q_{k-1}(x)}{n} \left( 1 - \frac{2}{\sqrt{\lambda d}} - \frac{dP_{k-1}(x)}{n} \right) + \frac{Q_k(x)}{n},
\]

\[
\leq -\lambda d(1 + 4\varepsilon)(1 - \lambda)(\lambda d)^{k-2} \left( 1 - \frac{2}{\sqrt{\lambda d}} - 3kd(1 - \lambda)(\lambda d)^{k-2} \right)
\]

\[
+ (1 + 2\varepsilon)(1 - \lambda)(\lambda d)^{k-1}
\]

\[
= -(1 - \lambda)(\lambda d)^{k-1} \left[(1 + 4\varepsilon) \left( 1 - \frac{2}{\sqrt{\lambda d}} - 3kd(1 - \lambda)(\lambda d)^{k-2} \right) - (1 + 2\varepsilon) \right]
\]

\[
\leq -(1 - \lambda)(\lambda d)^{k-1} \left[(1 + 4\varepsilon)(1 - \frac{\varepsilon}{50} - \frac{3\varepsilon}{100}) - (1 + 2\varepsilon) \right]
\]

\[
\leq -\varepsilon(1 - \lambda)(\lambda d)^{k-1}.
\]

In the penultimate line, we used (6.4) and (6.15), giving that \( \varepsilon\sqrt{\lambda d} \geq 100 \), and also (6.5), giving that \( \varepsilon \geq 100k(1 - \lambda)\lambda^{k-2}d^{k-1} \). Thus, for such \( x \), the drift in the scaled chain satisfies \( \Delta Q'_{k-1}(x) \leq -\frac{1}{2}\varepsilon(1 - \lambda)(\lambda d)^{k/2} := -v \).

Now \( m_cv = 4n \), and \( Q'_{k-1}(x) \leq 2n \) for all \( x \) by (5.2).

It is now straightforward to derive the result. \( \square \)

---

A queue-lengths vector \( x \in D_1 \) satisfies the three inequalities:

\[
Q_k(x) \leq (1 + 2\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}, \quad (7.1)
\]

\[
P_{k-1}(x) \leq 3kn(1 - \lambda)(\lambda d)^{k-2}, \quad (7.2)
\]

\[
Q_{k-1}(x) \leq (1 + 5\varepsilon)n(1 - \lambda)(\lambda d)^{k-2};
\]

in fact the second of these is redundant, as \( P_{k-1}(x) \leq Q_{k-1}(x) \leq 2n(1 - \lambda)(\lambda d)^{k-2} \) for all \( x \in \mathbb{Z}^n_+ \). Substituting these bounds into the bounds of Lemmas 5.1 and 5.2, we obtain the following.

**Lemma 7.1.** For \( x \in D_1 \), we have

\[
(1 + \lambda)\Delta Q_k(x) \leq \beta_k(1 - \lambda - u_{k+1}(x)) - \frac{Q_k(x)}{n(\lambda d)^{k-1}}
\]

\[
+ \exp(-dQ_k(x)/kn) + \frac{\varepsilon}{6}(1 - \lambda),
\]

\[
(1 + \lambda)\Delta Q_k(x) \geq \beta_k(1 - \lambda - u_{k+1}(x)) - \frac{Q_k(x)}{n(\lambda d)^{k-1}} - \frac{\varepsilon}{6}(1 - \lambda),
\]
and, for \(1 \leq j \leq k - 1\),

\[
(1 + \lambda) \Delta Q_j(x) \leq -\lambda d Q_j(x) \frac{n}{n} \left(1 - \frac{\epsilon}{25k}\right) + \frac{Q_{j+1}(x)}{n},
\]

\[
(1 + \lambda) \Delta Q_j(x) \geq -\lambda d Q_j(x) \frac{n}{n} \left(1 + \frac{\epsilon}{50k}\right) + \frac{Q_{j+1}(x)}{n}.
\]

Proof. For \(x \in D_1\), we combine the upper bound for \(Q_k(x)\) in Lemma 5.1 with (7.1), and obtain

\[
(1 + \lambda) \Delta Q_k(x) \leq \beta_k (1 - \lambda - u_{k+1}(x)) - \frac{Q_k(x)}{n(\lambda d)^k} \frac{1}{1 - \lambda} + \frac{Q_{k+1}(x)}{n(\lambda d)^k} + \frac{Q_k(x)}{n(\lambda d)^k} \frac{1}{1 + \lambda}.
\]

Here we used also the facts that \(\beta_k < 1\) and \(\lambda < 1\). Using (6.3) and (6.15), we have \(\frac{\lambda}{\lambda d}(1 + 2\epsilon)(1 - \lambda)\). To deduce the upper and lower bounds on \((1 + \lambda) \Delta Q_j(x)\) from Lemma 5.2, we first note that \(2/\sqrt{\lambda d} \leq \epsilon/50k\), by (6.14) and (6.15). This already gives the upper bound; for the lower bound, we also observe that

\[
dP_{k-1}(x) \leq \frac{dQ_{k-1}(x)}{n} \leq 2(1 - \lambda)d^{k-1} \leq \frac{\epsilon}{50k},
\]

using (7.2) and (6.5). \(\square\)

8. PROOF OF LEMMA 6.5

This section is devoted to the rather more complex proof of Lemma 6.5. First, we prove a statement stronger than part (1) of the lemma. We set

\(K = \{x : u_{k+1}(x) \leq \epsilon(1 - \lambda)\) and \(Q_k(x) \geq n(1 - \epsilon/3)(1 - \lambda)(\lambda d)^{k-1}\} \cap D_1\); \(W_K = \inf\{t \geq T_D : X_t \in K\}\).

Note that \(K \subseteq E_0\), so to prove Lemma 6.5(1) it suffices to prove that

\[
\mathbb{P}(W_K \wedge T_D^\dagger \geq T_D + m\epsilon) \leq 1/s_0.
\]
We prove this result on the assumption that \( T_D = 0 \) (i.e., that \( x_0 \in A_0 \cap B_0 \cap C_0 \cap D_0 \)). The general case follows immediately by applying the result for \( T_D = 0 \) to the shifted process \((X'_t) = (X_{T_D+t})\), using the strong Markov property. So our task is to show that \( P(W_{K\cap T_D} \geq m\varepsilon) \leq 1/s_0 \), where \( W_K = \text{inf}\{t \geq 0 : X_t \in K\} \), whenever \( X_0 = x_0 \) a.s., for any \( x_0 \in A_0 \cap B_0 \cap C_0 \cap D_0 \).

We define the following further sets, hitting times and exit times. We set
\[
L^{k+1}_1 = D_1 \setminus K
\]
\[
= \{ x : u_{k+1}(x) > \varepsilon(1-\lambda) \text{ or } Q_k(x) < n(1 - \frac{\varepsilon}{3})(1-\lambda)(\lambda d)^{k-1} \} \cap D_1,
\]
\[
W_{L^{k+1}} = 0 \text{ and } W^{\dagger\downarrow}_{L^{k+1}} = \text{inf}\{t \geq 0 : X_t \notin L^{k+1}_1\} = W_{K\cap T_D}^\dagger. \text{ Also, for } j = k, \ldots, 1, \text{ let}
\]
\[
L^{\downarrow\dagger}_j = \left\{ x : Q_j(x) \leq n(1-\lambda)(\lambda d)^{j-1}(1 - \frac{\varepsilon}{6} - \frac{j\varepsilon}{6k}) \right\} \cap L^{\downarrow\dagger}_j + 1;
\]
\[
L^{\dagger\downarrow}_1 = \left\{ x : Q_j(x) \leq n(1-\lambda)(\lambda d)^{j-1}(1 - \frac{\varepsilon}{6} - \frac{j\varepsilon}{6k} + \frac{\varepsilon}{24k}) \right\} \cap L^{\dagger\downarrow}_j + 1;
\]
\[
W_{L^\downarrow} = \inf\{ t \geq W_{L^{j+1}} : X_t \in L^\downarrow_j \};
\]
\[
W_{L^\dagger} = \inf\{ t \geq W_{L^\dagger j} : X_t \notin L^\dagger_j \}.
\]

Our goal is to show that \( P(W_{L^{k+1}}^\dagger < m\varepsilon) \geq 1 - 1/s_0 \). If \( x_0 \in K \), then \( W_{L^{k+1}}^\dagger = 0 \) and we are done, so we may assume that \( x_0 \notin K \), and hence that \( x_0 \in L^{k+1}_1 \). Thus Lemma 6.5(1) follows from the proposition below.

**Proposition 8.1.** Let \( x_0 \) be any queue-lengths vector in \( L^{k+1}_1 \). For a copy \((X_t)\) of the \((n,d,\lambda)\)-supermarket process with \( X_0 = x_0 \) a.s., we have
\[
P(W_{L^{k+1}}^\dagger \geq m\varepsilon) \leq 1/s_0.
\]

For the proof of Proposition 8.1, we fix a state \( x_0 \in L^{k+1}_1 \), and work with a copy \((X_t)\) of the \((n,d,\lambda)\)-supermarket process where \( X_0 = x_0 \) a.s.

Our general plan for proving Proposition 8.1 is as follows. We suppose that the process \((X_t)\) stays inside \( L^{k+1}_1 = D_1 \setminus K \) over the interval \([0,m\varepsilon]\), with the aim of showing that this event has low probability. Observe that, if \( x \in L^{k+1}_1 \setminus L^0_0 \), then \( u_{k+1}(x) > \varepsilon(1-\lambda) \) and \( Q_k(x) > n(1 - \varepsilon/2)(1-\lambda)(\lambda d)^{k-1} \). This “excess” in \( u_{k+1} \) would result in a downward drift in \( Q_k(X_t) \), so if the process does not exit \( L^{k+1}_1 \) quickly, then it enters \( L^0_1 \) quickly, and stays in \( L^k_1 \) throughout the interval \([0,m\varepsilon]\): i.e., \( W_{L^k} \) is small and \( W_{L^k}^\dagger \) is large, with high probability. This means that \( Q_k(X_t) \) maintains a “deficit” compared to \( \tilde{Q}_k := n(1-\lambda)(\lambda d)^{k-1} \) until time \( m\varepsilon \). A deficit in \( Q_k(X_t) \) would lead to a deficit in each \( Q_j(X_t) \) in turn, compared to \( \tilde{Q}_j := n(1-\lambda)(\lambda d)^{j-1} \), for \( j = k-1, k-2, \ldots, 1 \): each \( W_{L^j} \) is small, and \( W_{L^j}^\dagger \) is large, with high probability. Finally, a deficit in \( \tilde{Q}_1(X_t) \) compared to \( \tilde{Q}_1 = n(1-\lambda) \) is unsustainable, as this would lead to a drift down in the total number of
customers over a long enough time interval to empty the entire system of customers. This would entail exiting the set \( B_1 \supseteq \mathcal{L}_1^{k+1} \), a contradiction.

**Lemma 8.2.**

1. \( \mathbb{P}(W_{t_k}^l \land W_{t_k+1}^m \geq 12k \varepsilon^{-1} n(1 - \lambda)^{-1}) \leq 1/12s_0. \)
2. \( \mathbb{P}(W_{t_k}^m < m \varepsilon \leq W_{t_k+1}^m) \leq 1/6s_0. \)

**Proof.** We apply Lemma 7.3 to the process \( (X_t) \), with its natural filtration, and the function \( F = Q_k \). We set \( h = (1 - \frac{\varepsilon}{3}) n(1 - \lambda)(\lambda d)^{k-1} \) and \( \rho = \frac{\varepsilon}{24} n(1 - \lambda)(\lambda d)^{k-1} \); it follows from (6.11) that \( \rho \geq 2 \). We also set \( S = \mathcal{L}_1^{k+1} \) and \( T^* = 0 \). We note that \( Q_k(x) \leq c := kn \) for every \( x \), and we take \( m = 12k \varepsilon^{-1} n(1 - \lambda)^{-1} \), and \( s = m \varepsilon - 1 \). Then \( T_0 = W_{t_k+1}^1 \), \( T_1 = \inf\{ t : Q_k(X_t) \leq h \} \) and \( T_2 = \inf\{ t > T_1 : Q_k(X_t) \geq h + \rho \} \), as in the lemma.

For \( x \in \mathcal{L}_1^{k+1} \) with \( Q_k(x) > h \), we have \( u_{k+1}(x) > \varepsilon(1 - \lambda) \) and \( x \in D_1 \). So Lemma 7.1 applies, and we have

\[
(1 + \lambda)\Delta Q_k(x)
\leq \beta_k(1 - \lambda - u_{k+1}(X_t)) - \frac{Q_k(x)}{n(\lambda d)^{k-1}} + \exp(-dQ_k(x)/kn) + \frac{\varepsilon}{6}(1 - \lambda)
\leq (1 - \lambda)(1 - \varepsilon) - (1 - \lambda)(1 - \frac{\varepsilon}{3})
+ \exp\left(-\frac{d(1 - \frac{\varepsilon}{3})(1 - \lambda)(\lambda d)^{k-1}}{k}\right) + \frac{\varepsilon}{6}(1 - \lambda)
\leq \frac{1}{2}\varepsilon(1 - \lambda) + \exp\left(-\frac{(1 - \frac{\varepsilon}{3})d^k(1 - \lambda)\lambda^{k-1}}{k}\right).
\]

From (6.3), (6.14) and (6.15), we have that

\[
\frac{(1 - \frac{\varepsilon}{3})d^k(1 - \lambda)\lambda^{k-1}}{k} \geq \frac{177}{100} \log n.
\]

Also \( e^{-\frac{177}{100} \log n} \leq 1/2n \leq (1 - \lambda)\varepsilon/6 \) for \( n \geq 2 \), using (6.12). So

\[
(1 + \lambda)\Delta Q_k(x) \leq -\frac{1}{3}\varepsilon(1 - \lambda)
\text{ and } \Delta Q_k(x) \leq -\frac{1}{6}\varepsilon(1 - \lambda) := -v,
\]

for such \( x \). Note that \( mv = 2c \). Hence we may apply Lemma 3.3.

As in earlier lemmas, we have \( T_1 \land W_{t_k+1}^1 = W_{t_k}^l \land W_{t_k+1}^m \), so we obtain

\[
\mathbb{P}(W_{t_k}^l \land W_{t_k+1}^m > m) = \mathbb{P}(T_1 \land T_0 > m)
\leq \exp(-v^2m/8)
= \exp(-\varepsilon kn(1 - \lambda)/24)
\leq \exp(-2500\varepsilon^{-2}k^4 \log^2 n d^{k-2})
< 1/12s_0,
\]

where we used (6.11).
Also the events \( W_{L_k}^t < m_{\varepsilon} \leq W_{L_{k+1}}^t \) and \( T_2 < m_{\varepsilon} \leq W_{L_{k+1}}^t \) coincide, and the second is equivalent to \( T_2 \leq s < W_{L_{k+1}}^t \) (since \( s = m_{\varepsilon} - 1 \)). So
\[
\mathbb{P}(W_{L_k}^t < m_{\varepsilon} \leq W_{L_{k+1}}^t) \leq \mathbb{P}(T_2 \leq s < T_0)
\leq s \exp(-\rho v)
= s \exp(-\varepsilon^2 n(1 - \lambda)^2(\lambda d)^{k-1}/144k)
\leq m_{\varepsilon} \exp(-\frac{15}{4}k \log^2 n d^{k-2})
\leq 1/6s_0,
\]
as required. Here we used (6.6) and (6.15).

The next lemma states that, if the process stays in some set \( L_1^{j+1} \) for a long time, then it quickly enters the “next” set \( L_0^j \).

**Lemma 8.3.** For each \( j = k - 1, \ldots, 1 \),
\begin{enumerate}
\item \( \mathbb{P}(W_{L_{j+1}}^t \land W_{L_j} > W_{L_{j+1}}^t + \varepsilon^{-1} n(1 - \lambda)^{-1}) \leq 1/3ks_0 \).
\item \( \mathbb{P}(W_{L_j}^t < m_{\varepsilon} \leq W_{L_{j+1}}^t) \leq 1/3ks_0 \).
\end{enumerate}

**Proof.** (Sketch) This proof is very similar to that of earlier lemmas, and we mention only a few points. As in Lemma 6.4, we apply Lemma 3.3 to the chain \( (Q^j(x)) \), with the second \( \epsilon \) being \( \varepsilon \). In the case \( j = 1 \), this is the place where practically the full strength of (6.6) is used.

We now prove a hitting time lemma for \( \|X_t\|_1 \), the total number of customers in the system at time \( t \). Let \( W_{\mathcal{M}} = \min\{t \geq W_{L^1} : \|X_t\|_1 = 0\} \).

**Lemma 8.4.**
\[
\mathbb{P}(W_{L^1}^t \land W_{\mathcal{M}} > W_{L^1} + 72\varepsilon^{-1} n(1 - \lambda)^{-1} \leq 1/12s_0.
\]

**Proof.** We apply Lemma 3.3(i) to the chain \( (X_t) \), with the filtration \( (\mathcal{F}_t) \), and the function \( F(x) = \|x\|_1 \), which takes jumps of size at most 1. Since
\( A_1(\ell, g) \supseteq L_1 \), we have \( \|X_0\|_1 \leq c := 3gn \). We also set \( S = L_1^* \), \( T^* = W_{L^1} \), \( h = 0 \) and \( m = 72g^{-1}n(1 - \lambda)^{-1} \).

Note that \( \|X_{t+1}\|_1 - \|X_t\|_1 \) is equal to \( +1 \) if the event at time \( t \) is an arrival, with probability \( \lambda/(1 + \lambda) \), and equal to \( -1 \) if the event is a potential departure from a non-empty queue, with probability \( u_1(X_t)/(1 + \lambda) \), so the drift \( \Delta\|x\|_1 \) is equal to \( \frac{1}{1 + \lambda}(\lambda - u_1(x)) \). For \( x \in L_1 \), we have

\[
1 - u_1(x) = \frac{Q_1(x)}{n} \leq (1 - \lambda) \left( 1 - \frac{\varepsilon}{6} - \frac{\varepsilon}{6k} + \frac{\varepsilon}{24k} \right) \leq (1 - \lambda) \left( 1 - \frac{\varepsilon}{6} \right).
\]

Hence, for \( x \in L_1 \),

\[
(1 + \lambda)\Delta\|x\|_1 = (1 - u_1(x)) - (1 - \lambda) \leq -\frac{\varepsilon}{6}(1 - \lambda),
\]

and so \( \Delta\|x\|_1 \leq -\frac{\varepsilon}{6}(1 - \lambda) := -\varepsilon \). Note that \( \varepsilon m = 2c \).

Hence we may apply Lemma 3.3(i). With \( T_0 \) and \( T_1 \) as in that lemma, we have \( T_0 = W_{L_1^*}^t \) and \( T_1 = W_M \), so we conclude that

\[
\mathbb{P}(W_{L_1^*}^t \land W_M \geq W_{L_1^*}^t + m) \leq \exp(-\varepsilon gn(1 - \lambda)/16) \leq \exp\left(-3750\varepsilon^{-2}gk^3\log^2 nd^{k-2}\right) \leq 1/12s_0,
\]

as required. Here we used (6.11). \( \square \)

We now combine Lemmas 8.2, 8.3 and 8.4 to prove Proposition 8.1.

Observe that, for a copy \((X_t)\) of the \((n, d, \lambda)\)-supermarket process starting in a state \( x_0 \in L_{k+1}^* \), exactly one of the following occurs:

\begin{enumerate}[(a)]
\item \( W_{L_{k+1}}^t < m_\varepsilon \),
\item not (a), and one of \( W_{L_k}^t \), \( W_{L_{k-1}}^t \), \ldots, \( W_{L_1}^t \) is less than \( m_\varepsilon \),
\item neither of the above, and \( W_{L_k}^t > 12k\varepsilon^{-1}n(1 - \lambda)^{-1} \),
\item none of the above, and \( W_{L_j}^t > W_{L_{j+1}}^t + \varepsilon^{-1}n(1 - \lambda)^{-1} \) for some \( j = k - 1, \ldots, 1 \),
\item none of the above, and \( W_M > W_{L_1}^t + 72g\varepsilon^{-1}n(1 - \lambda)^{-1} \),
\item none of the above, and \( W_M < m_\varepsilon \leq W_{L_{k+1}}^t \).
\end{enumerate}

Indeed, if none of (a)–(e) occurs, then \( W_{L_{k+1}}^t \geq m_\varepsilon \) since (a) fails, and also

\[
W_M = W_{L_k}^t + \sum_{j=1}^{k-1} (W_{L_j}^t - W_{L_{j+1}}^t) + (W_M - W_{L_1}^t) \leq 12k\varepsilon^{-1}n(1 - \lambda)^{-1} + (k - 1)\varepsilon^{-1}n(1 - \lambda)^{-1} + 72g\varepsilon^{-1}n(1 - \lambda)^{-1} \leq (13k + 72g)\varepsilon^{-1}n(1 - \lambda)^{-1} = m_\varepsilon.
\]
We now show that the probability of each of (b)–(f) is small. For (b), Lemmas 8.2(2) and 8.3(2) give that
\[
\Pr(W_{L^k}^t \wedge W_{L^k-1}^t \wedge \cdots \wedge W_{L^1}^t < m \leq W_{L^k+1}^t) \\
\leq \Pr(W_{L^k}^t < m \leq W_{L^k+1}^t) + \sum_{j=1}^{k-1} \Pr(W_{L^j}^t < m \leq W_{L^{j+1}}^t) \\
\leq \frac{1}{6s_0} + (k-1)\frac{1}{3ks_0} \\
\leq \frac{1}{2s_0},
\]
i.e., the probability of (b) is at most 1/2s_0. The probability of (c) is at most 1/12s_0 by Lemma 8.2(1). The probability of (d) is at most (k-1)s_0 \leq 1/3s_0 by Lemma 8.3(1). The probability of (e) is at most 1/12s_0 by Lemma 8.4. Finally, (f) is not possible, since at time W_M there are no customers in the system, so Q_k(X_{W_M}) > n, and thus W_M \geq T_{B^T}^t, but also T_{B^T}^t \geq W_{L^k+1}^t since L^{k+1}_1 \subseteq D_1 \subseteq B_1 by definition.

Thus the probability of (a), for a copy of the process starting in a state in L^{k+1}_1, is at least
\[
1 - \frac{1}{2s_0} - \frac{1}{12s_0} - \frac{1}{3s_0} - \frac{1}{12s_0} = 1 - \frac{1}{s_0},
\]
which is what we need to prove Proposition 8.1 and also Lemma 6.5(1).

Now we move to the proof of Lemma 6.5(2), stating that the exit time T_\varepsilon^t is large with high probability. There are two things to prove here. The first is that, if X_t \in \mathcal{E}_1, then it is very unlikely that, at time t + 1, a customer arrives and creates a queue of length k + 1. The second is that, once Q_k(X_t) has reached (1 - 3\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}, while u_{k+1}(X_t) is at most \varepsilon(1 - \lambda), Q_k is unlikely to “cross down against the drift” to (1 - 4\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}.

For t \geq 0, let L_t denote the event that, at time t, a customer arrives and joins a queue of length at least k (equivalently, the probability that the event is an arrival and that all the selected queues have length at least k). So L_t is the event that u_j(X_t) > u_j(X_{t-1}) for some j \geq k + 1.

**Lemma 8.5.** On the event that X_t \in \mathcal{E}_1, we have \(\Pr(L_{t+1} \mid F_t) < e^{-\log^2 n}\).

**Proof.** From the definition of L_t, we have \(\Pr(L_{t+1} \mid F_t) = \frac{1}{1+\lambda} u_k(X_t)^d \leq u_k(X_t)^d\). For \(x \in \mathcal{E}_1\), we have \(Q_k(x) \geq (1 - 4\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}\) and \(Q_{k-1}(x) \leq (1 + 5\varepsilon)n(1 - \lambda)(\lambda d)^{k-2} \leq \frac{1}{3} \varepsilon n(1 - \lambda)(\lambda d)^{k-1}\), using (6.4) with a lot to spare. Therefore, by (6.4), (6.3) and (6.15), we have
\[
1 - u_k(x) \geq \frac{Q_k(x)}{n} - \frac{Q_{k-1}(x)}{n} \geq (1 - \frac{13}{3})\varepsilon(1 - \lambda)(\lambda d)^{k-1} \geq \frac{1}{2}(1 - \lambda)d^{k-1}.
\]
Hence, from (6.1), we have, on the event that $X_t \in \mathcal{E}_1$,
\[
  u_k(X_t)^d \leq \left( 1 - \frac{1}{2}(1 - \lambda)d^{k-1} \right)^d \leq \exp \left( -\frac{1}{2}(1 - \lambda)d^{k} \right) \leq \exp(-\log^2 n),
\]
as required. \hfill \Box

Let $U^\dagger = \inf \{ t > T_E : u_{k+1}(X_t) > \varepsilon(1 - \lambda) \}$ and $V^\dagger = \inf \{ t > T_E : Q_k(X_t) < (1 - 4\varepsilon)n(1 - \lambda)(\lambda d)^{k-1} \}$, and note that $T_{E}^\dagger = T_D^\dagger \wedge U^\dagger \wedge V^\dagger$. We thus have
\[
  \mathbb{P}(T_{E}^\dagger \leq s_0 < T_D^\dagger) \leq \mathbb{P}(U^\dagger \leq s_0 \wedge T_D^\dagger \wedge V^\dagger) + \mathbb{P}(V^\dagger \leq s_0 \wedge T_D^\dagger \wedge U^\dagger).
\]

We claim that each of these last two probabilities is at most $1/2s_0$. For the first, we may apply Lemma 8.5. Observe that, if $U^\dagger = t + 1$, then the event $L_{t+1}$ occurs. We now have:
\[
  \mathbb{P}(U^\dagger \leq s_0 \wedge T_D^\dagger \wedge V^\dagger) = \sum_{t=0}^{s_0-1} \mathbb{P}(U^\dagger = t + 1 \leq T_D^\dagger \wedge V^\dagger)
\]
\[
  = \sum_{t=0}^{s_0-1} \mathbb{P}(U^\dagger = t + 1 \text{ and } X_t \in \mathcal{E}_1)
\]
\[
  = \sum_{t=0}^{s_0-1} \mathbb{E}[1_{\{X_t \in \mathcal{E}_1\}} \mathbb{E}(1_{\{U^\dagger = t+1\}} \mid \mathcal{F}_t)]
\]
\[
  \leq \sum_{t=0}^{s_0-1} \mathbb{E}[1_{\{X_t \in \mathcal{E}_1\}} \mathbb{E}(1_{L_{t+1}} \mid \mathcal{F}_t)].
\]

By Lemma 8.5, each term is at most $e^{-\log^2 n}$, and so we have
\[
  \mathbb{P}(U^\dagger \leq s_0 \wedge T_D^\dagger \wedge V^\dagger) \leq s_0 e^{-\log^2 n} < 1/2s_0,
\]
as claimed.

To obtain the other required inequality, we apply the reversed version of Lemma 6.3.2. We consider the process $(X_t)$, with its natural filtration, the function $F = Q_k$, and the set $\mathcal{S} = \{ x : u_{k+1}(x) \leq \varepsilon(1 - \lambda) \} \cap \mathcal{D}_1$. We set $h = (1 - 3\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}$ and $\rho = \varepsilon n(1 - \lambda)(\lambda d)^{k-1} \geq 2$. We also set $s = s_0$ and $T^* = T_E$. We have $T_0 = \inf \{ t \geq T_E : X_t \notin \mathcal{D}_1 \text{ or } u_{k+1}(X_t) > \varepsilon(1 - \lambda) \}$, so that $T_0 \geq T_D^\dagger \wedge U^\dagger$ (strict inequality occurs if $T_D^\dagger < T_E$). Also $T_1 = \inf \{ t \geq T_E : Q_k(X_t) \geq h \} = T_E$, and $T_2 = \inf \{ t > T_E : Q_k(X_t) \leq h - \rho \} = V^\dagger$.

Take $x \in \mathcal{S}$ with $Q_k(x) \leq h$. As $x \in \mathcal{D}_1$, we apply Lemma 7.3.1 to obtain
\[
  (1 + \lambda)\Delta Q_k(x) \geq \beta_k(1 - \lambda - u_{k+1}(x)) - \frac{Q_k(x)}{n(\lambda d)^{k-1}} - \frac{\varepsilon}{6}(1 - \lambda)
\]
\[
  \geq \beta_k(1 - \lambda)(1 - \varepsilon) - (1 - \lambda)(1 - 3\varepsilon) - \frac{\varepsilon}{6}(1 - \lambda)
\]
\[
  \geq (1 - \lambda) \left[ \left( 1 - \frac{\varepsilon}{2} \right)(1 - \varepsilon) - 1 + 3\varepsilon - \frac{\varepsilon}{6} \right]
\]
\[
  \geq \varepsilon(1 - \lambda),
\]
where we also used (6.16). This yields \( \Delta Q_k(x) \geq \frac{1}{2} \varepsilon (1 - \lambda) := v \), for such \( x \).

The reversed version of Lemma 3.3(2) gives that
\[
\mathbb{P}(V^+ \leq s_0 \land T_D^I \land U^+) \leq \mathbb{P}(T_2 \leq s_0 \land T_0) \\
\leq s_0 \exp(-pv) \\
= s_0 \exp(-\varepsilon^2 n(1 - \lambda)^2(\lambda d)^{k-1}/2) \\
\leq \frac{1}{2}s_0,
\]
as required. Here we used (6.6) and (6.15).

This completes the proof of Lemma 6.5.

9. Proofs of Lemmas 6.6 and 6.7

In this section, we prove the final two of our sequence of lemmas.

Proof of Lemma 6.6

**Proof.** Fix \( j \) with \( 1 \leq j \leq k - 1 \), and consider the state of the process at the hitting time \( T_{G^j} \). The hitting time \( T_{G^j} \) is the first time \( t \geq T_{G^j} \) such that \( Q_j(X_t) \) lies in the interval between 
\[
1 - (4 + \frac{k-j-1/2}{k}) \varepsilon \] and 
\[
1 + (4 + \frac{k-j-1/2}{k}) \varepsilon \] \( n(1 - \lambda)(\lambda d)^{j-1} \) and 
\[
1 - (4 + \frac{k-j-1/2}{k}) \varepsilon \] \( n(1 - \lambda)(\lambda d)^{j-1} \), and \( B_t \) be the event that \( Q_j(X_{T_{G^j}}) < 1 - (4 + \frac{k-j-1/2}{k}) \varepsilon \).

For part (1) of the lemma, we have to show that, on the event \( B_h \), with high probability \( Q_j(X_t) \) enters the interval from above within time \( m_G \), and also that, on the event \( B_t \), with high probability \( Q_j(X_t) \) enters the interval from below within time \( m_G \). These two results are essentially the same, and we give details only for the first. Of course, we have nothing to prove on the event that \( Q_j(X_t) \) is already in the interval.

We apply Lemma 3.3(i) to \( (X_t) \), with its natural filtration, and the scaled function \( F(x) = Q'_j(x) = Q_j(x)/(\lambda d)^{j-1/2} \). We take \( \mathcal{S} = G^{j+1}_1 \) and \( T' = T_{G_1} \). We set
\[
h = \left[ 1 + (4 + \frac{k-j-1/2}{k}) \varepsilon \right] n(1 - \lambda)(\lambda d)^{j-1/2},
\]
and \( m = m_G = 32k\varepsilon^{-1}n(1 - \lambda)^{-1}(\lambda d)^{-1} \). From (5.2), we have that \( Q'_j(x) \leq c := 2n \) for all \( x \). Also \( T_0 = T_{G^j}^I + 1 \) and \( T_1 = \inf \{ t \geq T_{G^j}^I : Q'_j(X_t) \leq h \} \).

For \( x \in G^{j+1}_1 \), we have
\[
Q_{j+1}(x) \leq \left[ 1 + (4 + \frac{k-j-1}{k}) \varepsilon \right] n(1 - \lambda)(\lambda d)^j.
\]
(This follows from the specification of \(G_t^{j+1}\) for \(j < k - 1\), and since \(G_t^j = E_1 \subseteq B_1\) for \(j = k - 1\).) If also \(Q'_j(x) \geq h\), we have

\[
Q_j(x) \geq \left[ 1 + (4 + \frac{k - j - 1/2}{k}) \varepsilon \right] n(1 - \lambda)(\lambda d)^{j-1}.
\]

Lemma 7.1 applies since \(x \in D_1\), so

\[
(1 + \lambda) \Delta Q_j(x) \leq -\lambda d \frac{Q_j(x)}{\varepsilon} \left(1 - \frac{\varepsilon}{25k}\right) + \frac{Q_{j+1}(x)}{\varepsilon} n
\]

\[
\leq - \left[1 + (4 + \frac{k - j - 1/2}{k}) \varepsilon \right] (1 - \lambda)(\lambda d)^j \left(1 - \frac{\varepsilon}{25k}\right)
\]

\[
+ \left[1 + (4 + \frac{k - j - 1}{k}) \varepsilon \right] (1 - \lambda)(\lambda d)^j
\]

\[
\leq - \frac{1}{4k} \varepsilon (1 - \lambda)(\lambda d)^j,
\]

and so \(\Delta Q'_j(x) \leq \frac{1}{8k} \varepsilon (1 - \lambda)(\lambda d)^{j+1/2} := -v\). Note that \(vmG \geq 2c\).

Lemma 3.3(i) now gives, using (6.11),

\[
\mathbb{P}(T_1 \wedge T_0 > T_{G_j} + m_G) \leq \exp(-v^2 m_G/8)
\]

\[
= \exp(-\varepsilon n(1 - \lambda)(\lambda d)^j / 16k)
\]

\[
\leq \exp(-3750 \varepsilon^{-2} k^2 \log^2 n)
\]

\[
\leq 1/2s_0.
\]

On the \(T_{G_j}\)-measurable event \(B_h\), the stopping times \(T_1 \wedge T_0\) and \(T_{G_j} \wedge T_0\) coincide, so we have

\[
\mathbb{P}(B_h \cap \{T_{G_j} \wedge T_{G_{j+1}}^\dagger > T_{G_{j+1}} + m_G\}) \leq 1/2s_0.
\]

Essentially exactly the same calculation gives

\[
\mathbb{P}(B_t \cap \{T_{G_j} \wedge T_{G_{j+1}}^\dagger > T_{G_{j+1}} + m_G\}) \leq 1/2s_0,
\]

and part (1) of the lemma now follows, for this value of \(j\).

To prove part (2) of the lemma, we need to show that, once \(X_t\) has reached \(G_t^j\), and while it remains in \(G_t^{j+1}\), the process is unlikely to leave the set \(G_t^j\) quickly. There are two separate things to prove: that \(Q_j(X_t)\) is unlikely to cross against the drift from \(1 + (4 + \frac{k - j - 1/2}{k}) \varepsilon \) \(n(1 - \lambda)(\lambda d)^{j-1}\) to \(1 + (4 + \frac{k - j}{k}) \varepsilon \) \(n(1 - \lambda)(\lambda d)^{j-1}\) before time \(s_0\), and also that \(Q_j(X_t)\) is unlikely to cross against the drift from \(1 - (4 + \frac{k - j - 1/2}{k}) \varepsilon \) \(n(1 - \lambda)(\lambda d)^{j-1}\) to \(1 - (4 + \frac{k - j}{k}) \varepsilon \) \(n(1 - \lambda)(\lambda d)^{j-1}\) before time \(s_0\). Again, the two calculations required here are essentially identical, and we shall concentrate on the first.

We apply Lemma 3.3(ii), again for the process \((X_t)\) with its natural filtration, and the scaled function \(F(x) = Q_j(x) / (\lambda d)^{(j-1)/2}\). We take the same values of parameters as above, and additionally set \(\rho = \frac{\varepsilon}{25} n(1 - \lambda)(\lambda d)^{(j-1)/2}\).
and \( s = s_0 \). Here \( T_2 = \inf\{ t > T_1 : Q'_j(X_t) \geq h + \rho \} \), and we have, using (6.6) and (6.15),

\[
\mathbb{P}(T_2 \leq s_0 < T_{Gj}^{\dagger}) \leq s_0 \exp(-\rho v) \\
= s_0 \exp(-\varepsilon^2 n(1 - \lambda)^2(\lambda d)^j / 16k) \\
\leq s_0 \exp\left(-\frac{135}{4} k \log^2 n \right) \\
\leq 1/2s_0.
\]

Setting \( U_2 = \inf\{ t > T_1 : Q_j(X_t) \leq \left[1 - (4 + \frac{k-j}{k})\varepsilon \right] n(1 - \lambda)(\lambda d)^j - 1 \} \), we have, similarly,

\[
\mathbb{P}(U_2 \leq s_0 < T_{Gj}^{\dagger}) \leq 1/2s_0.
\]

The events \( T_2 \land U_2 \leq s_0 < T_{Gj}^{\dagger} \) and \( T_{Gj}^{\dagger} \leq s_0 < T_{Gj}^{\dagger+1} \) coincide, so

\[
\mathbb{P}(T_{Gj}^{\dagger} \leq s_0 < T_{Gj}^{\dagger+1}) = \mathbb{P}(T_2 \leq s < T_0) + \mathbb{P}(U_2 \leq s < T_0) \\
\leq \frac{1}{2s_0} + \frac{1}{2s_0} = \frac{1}{s_0},
\]

as required for part (2) for this value of \( j \). \( \square \)

**Proof of Lemma 6.7**

**Proof.** We first prove part (1). For \( i = 1, \ldots, n \), let \( N_i \) be the number of potential departures from queue \( i \) over the time period between \( T_{Gj}^{\dagger} \) and \( T_{Gj}^{\dagger} + m_H \), so \( N_i \) is a binomial random variable with parameters \((m_H, 1/n(1 + \lambda))\). Recall that \( L_t \) is the event that, at time \( t \), a customer arrives and joins a queue of length \( k \) or longer, and observe that

\[
\mathbb{P}(T_H \land T_{Gj}^{\dagger} \geq T_{Gj}^{\dagger} + m_H) \\
\leq \mathbb{P}\left( \bigcup_{t=T_{Gj}^{\dagger}+1}^{T_{Gj}^{\dagger}+m_H} (L_t \cap \{X_{t-1} \in G_1^i\}) \right) + \mathbb{P}\left( \exists i, N_i < 3\ell \right).
\]

Indeed, at time \( T_{Gj}^{\dagger} \), the process is in \( A_1(\ell, g) \), and so there is no queue with more than \( 3\ell \) customers in it at that time. If there are at least \( 3\ell \) potential departures from each queue over the time interval, and \( \bigcup_{t=T_{Gj}^{\dagger}+1}^{T_{Gj}^{\dagger}+m_H} L_t \) does not occur, then by time \( T_{Gj}^{\dagger} + m_H \), every queue is reduced to length at most \( k \), and no new queue of length \( k+1 \) is created before \( T_{Gj}^{\dagger} + m_H \).
Now let \((X_t') = (X_{T_{G_1}+t})\), \((F'_t) = (F_{T_{G_1}+t})\) and \(L'_t = L_{T_{G_1}+t}\). We have:

\[
\mathbb{P} \left( \bigcup_{t=T_{G_1}+1}^{T_{G_1}+m_H} (L_t \cap \{X_{t-1} \in G_1^1\}) \right) = \mathbb{P} \left( \bigcup_{t=1}^{m_H} (L'_t \cap \{X'_{t-1} \in G_1^1\}) \right)
\leq \sum_{t=1}^{m_H} \mathbb{P}(L'_t \cap \{X'_{t-1} \in G_1^1\})
= \sum_{t=1}^{m_H} \mathbb{E} \left[ \mathbb{1}_{\{X'_{t-1} \in G_1^1\}} \mathbb{E}[1_{L'_t \mid F'_{t-1}}] \right]
\leq m_H e^{-\log^2 n}
\leq \frac{1}{2s_0},
\]

where we used the strong Markov property, and Lemma 8.5.

Recall that \(m_H = n(8\ell + 32\log^2 n)\), so that the mean \(\mu\) of each \(N_i\) is \(m_H/n(1 + \lambda) \geq 4\ell + 16\log^2 n\). By (3.1), with \(\epsilon = 1/4\), we have

\[
\mathbb{P}(N_i \leq 3\ell) \leq \mathbb{P}(N_i \leq \frac{3}{4}\mu) \leq e^{-\mu/32} \leq e^{-\frac{1}{2}\log^2 n}
\]

for each \(i\). Thus the probability that there are fewer than \(3\ell\) departures from any queue over the interval from \(T_{G_1}\) to \(T_{G_1} + m_H\) is at most \(ne^{-\frac{1}{2}\log^2 n} < 1/2s_0\), and part (1) follows.

For part (2), as above we have

\[
\mathbb{P} \left( \bigcup_{t=T_{G_1}+1}^{T_{G_1}+s_0} (L_t \cap \{X_{t-1} \in G_1^1\}) \right) \leq s_0 e^{-\log^2 n} \leq 1/s_0.
\]

Thus \(\mathbb{P}(T_H^1 \leq s_0 < T_{G_1}^1)\) is at most the probability that \(X_t\) exits the set \(H_1\) before time \(T_{G_1}^1 \wedge s_0\), necessarily by the creation of a new queue of length \(k + 1\), is at most \(1/s_0\), as required. \(\square\)

10. The sets \(H, I\) and \(N\)

One goal of this section is to show that \(H \subseteq N\), thus completing the proof of Theorem 1.2; see Corollary 6.9 and the remarks after. We also show that the set \(N\) is “path-connected”, a fact we shall need in the next section.

We continue to assume that \(n, d, k, \lambda\) and \(\epsilon\) satisfy the hypotheses of Theorem 1.2. For this section, it will be important to be explicit about the fact that the various sets we have defined depend on the value of the parameter \(\epsilon\); accordingly, we shall refer to our sets as (e.g.) \(H_\epsilon, I_\epsilon\) and \(N_\epsilon\). (Note that these sets do not depend on the values of \(g\) and \(\ell\) used in defining earlier sets in the sequence.)
By definition, the set $\mathcal{H}^\varepsilon$ consists of those queue-lengths vectors $x$ satisfying all of the following:

\[
\|x\|_\infty \leq 3 \log^2 n(1 - \lambda)^{-1}, \quad \|x\|_1 \leq 6n(1 - \lambda)^{-1}, \\
Q_k(x) \leq (1 + 2\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}, \quad P_k(x) \leq 3kn(1 - \lambda)(\lambda d)^{k-2}, \\
Q_{k-1}(x) \leq (1 + 5\varepsilon)n(1 - \lambda)(\lambda d)^{k-2}, \quad u_{k+1}(x) \leq \varepsilon(1 - \lambda), \\
Q_k(x) \geq (1 - 4\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}, \quad u_{k+1}(x) = 0, \\
Q_j(x) \geq \left[1 - (4 + \frac{k - j}{k})\varepsilon\right]n(1 - \lambda)(\lambda d)^{j-1} \quad (1 \leq j \leq k - 1), \\
Q_j(x) \leq \left[1 + (4 + \frac{k - j}{k})\varepsilon\right]n(1 - \lambda)(\lambda d)^{j-1} \quad (1 \leq j \leq k - 1).
\]

Evidently many of these conditions are redundant. The condition that $u_{k+1}(x) = 0$ implies not only that $u_{k+1}(x) \leq \varepsilon(1 - \lambda)$, and that $\|x\|_\infty \leq 3 \log^2 n(1 - \lambda)^{-1}$, but also that $\|x\|_1 \leq kn < 6n(1 - \lambda)^{-1}$. The upper bound on $P_{k-1}(x)$ is implied by $P_{k-1}(x) \leq Q_{k-1}(x) \leq (1 + 5\varepsilon)n(1 - \lambda)(\lambda d)^{k-2}$. Also, the earlier upper bound on $Q_{k-1}(x)$ is weaker than the final one. Thus $\mathcal{H}^\varepsilon$ consists of those queue-lengths vectors $x$ satisfying all of:

\[
u_{k+1}(x) = 0, \\
Q_k(x) \leq (1 + 2\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}, \\
Q_j(x) \geq \left[1 - (4 + \frac{k - j}{k})\varepsilon\right]n(1 - \lambda)(\lambda d)^{j-1} \quad (1 \leq j \leq k - 1), \\
Q_j(x) \leq \left[1 + (4 + \frac{k - j}{k})\varepsilon\right]n(1 - \lambda)(\lambda d)^{j-1} \quad (1 \leq j \leq k - 1).
\]

Similarly, $\mathcal{I}^\varepsilon$ consists of those queue-lengths vectors $x$ satisfying:

\[
\|x\|_\infty \leq \log^2 n(1 - \lambda)^{-1}, \quad \|x\|_1 \leq 2n(1 - \lambda)^{-1}, \\
Q_k(x) \leq (1 + \varepsilon)n(1 - \lambda)(\lambda d)^{k-1}, \quad P_k(x) \leq 2kn(1 - \lambda)(\lambda d)^{k-2}, \\
Q_{k-1}(x) \leq (1 + 4\varepsilon)n(1 - \lambda)(\lambda d)^{k-2}, \quad u_{k+1}(x) \leq \varepsilon(1 - \lambda), \\
Q_k(x) \geq (1 - 3\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}, \quad u_{k+1}(x) = 0, \\
Q_j(x) \geq \left[1 - (4 + \frac{k - j - 1/2}{k})\varepsilon\right]n(1 - \lambda)(\lambda d)^{j-1} \quad (1 \leq j \leq k - 1), \\
Q_j(x) \leq \left[1 + (4 + \frac{k - j - 1/2}{k})\varepsilon\right]n(1 - \lambda)(\lambda d)^{j-1} \quad (1 \leq j \leq k - 1).
\]

Removing redundancies, $\mathcal{I}^\varepsilon$ consists of those vectors $x$ such that:

\[
u_{k+1}(x) = 0, \\
Q_k(x) \leq (1 + \varepsilon)n(1 - \lambda)(\lambda d)^{k-1}, \\
Q_k(x) \geq (1 - 3\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}, \\
Q_j(x) \geq \left[1 - (4 + \frac{k - j - 1/2}{k})\varepsilon\right]n(1 - \lambda)(\lambda d)^{j-1} \quad (1 \leq j \leq k - 1), \\
Q_j(x) \leq \left[1 + (4 + \frac{k - j - 1/2}{k})\varepsilon\right]n(1 - \lambda)(\lambda d)^{j-1} \quad (1 \leq j \leq k - 1).
\]
Rather crudely, we have $I^\varepsilon \subseteq H^\varepsilon \subseteq I^{2\varepsilon}$, for any $\varepsilon > 0$.

Now we bring the sets $N^\varepsilon$ into the picture. Recall that

$$N^\varepsilon = \{ x : u_{k+1}(x) = 0, \quad (1 - 5\varepsilon)(1 - \lambda)(\lambda d)^{j-1} \leq 1 - u_j(x) \leq (1 + 5\varepsilon)(1 - \lambda)(\lambda d)^{j-1} \quad (j = 1, \ldots, k) \}.$$  

**Lemma 10.1.** For any $\varepsilon > 0$, $H^\varepsilon \subseteq N^\varepsilon \subseteq H^{3\varepsilon}$.

**Proof.** Take $x \in H^\varepsilon$: we check that $x \in N^\varepsilon$. We do have $u_{k+1}(x) = 0$. Note that $\frac{1}{\sqrt{\lambda d}} \leq \frac{\varepsilon}{135k}$, from (6.4) and (6.15). Using also (5.3), and that $Q_j(x) \leq 2n(1 - \lambda)(\lambda d)^{j-2}$ by (6.3), we have for $j = 1, \ldots, k - 1$,

$$1 - u_j(x) \geq \frac{Q_j(x)}{n} - 2\sqrt{\lambda d}Q_{j-1}(x) \geq \left[ 1 - (4 + \frac{k - j}{k})\varepsilon \right] (1 - \lambda)(\lambda d)^{j-1} - 4\sqrt{\lambda d}(1 - \lambda)(\lambda d)^{j-2} \geq (1 - \lambda)(\lambda d)^{j-1}\left[ 1 - (5 - \frac{1}{k})\varepsilon - 4\frac{\varepsilon}{135k} \right] \geq (1 - 5\varepsilon)(1 - \lambda)(\lambda d)^{j-1}.$$  

We also have

$$1 - u_j(x) \leq \frac{Q_j(x)}{n} \leq \left[ 1 + (4 + \frac{k - j}{k})\varepsilon \right] (1 - \lambda)(\lambda d)^{j-1} \quad (j = 1, \ldots, k - 1) \leq (1 + 5\varepsilon)(1 - \lambda)(\lambda d)^{j-1},$$

$$1 - u_k(x) \leq \frac{Q_k(x)}{n} \frac{1}{\beta_k} \leq (1 + 5\varepsilon)(1 - \lambda)(\lambda d)^{k-1},$$

using (6.10). So $x \in N^\varepsilon$, as required.

Now take $x \in N^\varepsilon$: we check that $x \in H^{3\varepsilon}$. We do have $u_{k+1}(x) = 0$. Also

$$Q_k(x) \leq n \sum_{j=1}^{k}(1 - u_j(x)) \leq (1 + 5\varepsilon)n(1 - \lambda)\sum_{j=1}^{k}(\lambda d)^{j-1} \leq (1 + 5\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}(1 + \sum_{i=1}^{k-1}\frac{1}{(\lambda d)^i}) < (1 + 6\varepsilon)n(1 - \lambda)(\lambda d)^{k-1},$$
THE SUPERMARKET MODEL WITH ARRIVAL RATE TENDING TO ONE

where we used (6.17). For $1 \leq j \leq k - 1$, we have, using (5.1) and (6.17),

$$Q_j(x) = n \sum_{i=1}^{j} \gamma_{j,i}(1 - u_i(x))$$

$$\leq n(1 - u_j(x)) + n \sum_{i=1}^{j-1} i(\lambda d)^{(j-i)/2}(1 - u_i(x))$$

$$\leq (1 + 5\varepsilon)n(1 - \lambda)(\lambda d)^{j-1} + nk \sum_{i=1}^{j-1} (\lambda d)^{(j-i)/2}(1 + 5\varepsilon)(1 - \lambda)(\lambda d)^{i-1}$$

$$\leq n(1 - \lambda)(\lambda d)^{j-1} \left[ 1 + 5\varepsilon + 2k \sum_{i=1}^{j-1} \frac{1}{(\lambda d)^{(j-i)/2}} \right]$$

$$\leq (1 + 6\varepsilon)n(1 - \lambda)(\lambda d)^{j-1}.$$  

For $1 \leq j \leq k - 1$, we have $Q_j(x) \geq n(1 - u_j(x)) \geq (1 - 5\varepsilon)n(1 - \lambda)(\lambda d)^{j-1}$, while also, using (6.16),

$$Q_k(x) \geq n\beta_k(1 - u_k(x))$$

$$\geq \beta_k(1 - 5\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}$$

$$\geq (1 - 6\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}.$$  

So indeed $x \in \mathcal{H}^{3\varepsilon}$.

Hence Theorem 6.10 and Corollary 6.9 hold with $\mathcal{H}$ replaced by $\mathcal{N}$. Moreover, we have the following analogue of Theorem 6.11. Here and subsequently, we require that $\varepsilon \leq 1/60$, to ensure that the conditions of Theorem 1.2 are met with $\varepsilon$ replaced by $6\varepsilon$.

**Theorem 10.2.** Take $\varepsilon \leq 1/60$, and $x_0 \in \mathcal{N}^{\varepsilon}$. Suppose $(X_t)$ is a copy of the $(n, d, \lambda)$-supermarket process in which $X_0 = x_0$ almost surely. Then

$$\mathbb{P}(X_t \in \mathcal{H}^{6\varepsilon} \text{ for all } t \in [0, s_0]) \geq 1 - (k + 5)/s_0,$$

and hence

$$\mathbb{P}(X_t \in \mathcal{N}^{6\varepsilon} \text{ for all } t \in [0, s_0]) \geq 1 - (k + 5)/s_0.$$

**Proof.** If $x_0 \in \mathcal{N}^{\varepsilon} \subseteq \mathcal{H}^{3\varepsilon} \subseteq \mathcal{T}^{6\varepsilon}$, then, by Theorem 6.11, with probability at least $1 - (k + 5)/s_0$, for all times $t \in [0, s_0]$, $X_t \in \mathcal{H}^{6\varepsilon} \subseteq \mathcal{N}^{6\varepsilon}$, as required.  

We say two queue-lengths vectors are *adjacent* if they differ by one customer in one queue. A *path* of length $m$ between two vectors $x$ and $y$ is a sequence $x = x_0x_1 \cdots x_m = y$ of queue-lengths vectors, with each pair $(x_i, x_{i+1})$ adjacent. The path is said to lie in a set $\mathcal{S}$ if each $x_i$ is in $\mathcal{S}$.

**Lemma 10.3.** Between any two queue-lengths vectors in $\mathcal{N}^{\varepsilon}$, there is a path of length at most $4n(1 - \lambda)(\lambda d)^{k-1}$ lying in $\mathcal{N}^{\varepsilon}$.
Proof. For \( j = 1, \ldots, k \), set \( u^*_j = \frac{1}{n}[n(1-(1-\lambda)(\lambda d)^{j-1})]. \) Now set

\[
P = \{ z : u^j_{k+1}(z) = 0, u^*_j = u^*_j, \ \text{for} \ j = 1, \ldots, k \}.
\]

In other words, \( P \) consists of those queue-lengths vectors \( z \) with no queues of length greater than \( k \), and such that the number \( nu^*_j(z) \) of queues of length at least \( j \) is equal to \( nu^*_j = [n(1-(1-\lambda)(\lambda d)^{j-1})], \) for each \( j = 1, \ldots, k \).

We think of \( P \) as forming the “centre” of the set \( \mathcal{N}^\varepsilon \). Our plan is to show that every queue-lengths vector in \( \mathcal{N}^\varepsilon \) is joined to some vector in \( P \) by a short path lying entirely in \( \mathcal{N}^\varepsilon \), and then to show that every two vectors in \( P \) are connected by a short path, again lying entirely in \( \mathcal{N}^\varepsilon \).

Let \( x \) be a queue-lengths vector in \( \mathcal{N}^\varepsilon \). We first show that there is a path within \( \mathcal{N}^\varepsilon \) of length at most \( 6k\varepsilon n(1-\lambda)(\lambda d)^{k-1} \) from \( x \) to a vector \( x' \) in \( P \), of the form \( x = x^{(k+1)} \ldots x^{(k)} \ldots x^{(k-1)} \ldots \ldots x^{(0)} = x' \), where:

(a) \( u^j_{k+1}(y) = 0 \) for all vectors \( y \) on the path,

(b) for each \( j = 1, \ldots, k \):

- all the vectors \( y \) on the section of the path from \( x \) up to \( x^{(j+1)} \) satisfy \( u^*_j(y) = u^*_j(x) \),

- all the vectors \( y \) on the section of the path from \( x^{(j)} \) to \( x' \) satisfy \( u^*_j(y) = u^*_j \),

- on the section of path between \( x^{(j+1)} \) and \( x^{(j)} \), the value of \( u^*_j(y) \) changes monotonically in steps of size \( 1/n \) from \( u^*_j(x) \) to \( u^*_j \).

Such a path will certainly lie in \( \mathcal{N}^\varepsilon \), since, for each \( j \) and each \( y \) on the path, \( u^*_j(y) \) lies between \( u^*_j(x) \) and \( u^*_j \).

To establish the existence of such a path, we explain how to construct each section individually. For each \( j \), assuming only that \( x^{(j+1)} \) is in \( \mathcal{N}^\varepsilon \), we show that there is a sequence of adjacent vectors \( y \), starting with \( x^{(j+1)} \), so that \( nu^*_j(y) \) changes monotonically from \( nu^*_j(x) \) to \( nu^*_j \) in steps of size \( 1 \) along the sequence, while each other \( u^*_i \) is constant along the sequence. If \( u^*_j(x^{(j+1)}) > u^*_j \), the number of queues of length exactly \( j \) in \( x^{(j+1)} \) is

\[
n(u^*_j(x^{j+1}) - u^*_j)
\geq n[u^*_j(x^{j+1}) - u^*_j] + [n(1 - (1 - \lambda)(\lambda d)^{j-1})] - n[1 - (1 - 5\varepsilon)(1 - \lambda)(\lambda d)^j]
\geq n[u^*_j(x^{j+1}) - u^*_j],
\]

so we may form a suitable sequence by taking \( n[u^*_j(x^{j+1}) - u^*_j] \) queues of length exactly \( j \) and removing one customer from each of these queues in turn. Similarly, if \( u^*_j(x^{j+1}) < u^*_j \), the number of queues of length exactly \( j - 1 \) in \( x^{(j+1)} \) is

\[
n(u^*_j - u^*_j(x^{(j+1)}))
\geq n[u^*_j - u^*_j(x^{(j+1)})] - [n(1 - (1 - \lambda)(\lambda d)^{j-1})] + n[1 - (1 + 5\varepsilon)(1 - \lambda)(\lambda d)^{j-2}]
\geq n[u^*_j - u^*_j(x^{(j+1)})],
\]
so we may form a suitable sequence by taking \(n[u_j^* - u_j(x^{j+1})]\) queues of length exactly \(j - 1\) and adding one customer to each of these queues in turn. Neither of these operations affects \(u_1\), the proportion of queues of length at least \(i\), for any value of \(i\) other than \(j\). The section of path from \(x^{(j+1)}\) and \(x^{(j)}\) has length \(n[u_j(x^{j+1}) - u_j^*] \leq 5\varepsilon n(1 - \lambda)(\lambda d)^{j-1}\). 

Thus we can construct a path from \(x\) to \(x'\) staying within \(N^\varepsilon\), and this path has length at most

\[
\sum_{j=1}^{k} 5\varepsilon n(1 - \lambda)(\lambda d)^{j-1} \leq 6\varepsilon n(1 - \lambda)(\lambda d)^{k-1},
\]

where we used (6.17).

We now show that there is a path of length at most \(3n(1 - \lambda)(\lambda d)^{k-1}\) between any two queue-lengths vectors \(x\) and \(x'\) in the “centre” \(P\) of \(N^\varepsilon\), staying within \(N^\varepsilon\). This path will be of the form \(x = x^{(0)} \cdots x^{(1)} \cdots x^{(k)} = x'\). Our path will have the following properties, for each \(j = 1, \ldots, k - 1\):

(a) for every vector \(y\) on the path from \(x^{(j)}\) to \(x^{(k)}\), the set of queues of length \(j - 1\) is the same in \(y\) as in \(x'\); thus the set of all queues of length at most \(j - 1\) is fixed from \(x^{(j)}\) to the end of the path \(x^{(k)}\), and hence so is their number \(n(1 - u_j(x')) = n(1 - u_j^*)\);

(b) along the path from \(x\) to \(x^{(j-1)}\), \(u_j(y)\) decreases monotonically from \(u_j^*\) to a value \(u_j(x^{(j-1)}) \geq u_j^*(1 - \varepsilon)\);

(c) along the section of path from \(x^{(j-1)}\) to \(x^{(j)}\), \(u_j(y)\) stays between \(u_j(x^{(j-1)})\) and \(u_j^* + 1/n\), until it becomes equal to \(u_j^*\) at \(x^{(j)}\).

With reference to (b) above, we shall actually show that: (b') for each \(j > i\), as \(y\) goes from \(x^{(i-1)}\) and \(x^{(i)}\) along the path, the value of \(nu_j(y)\) decreases monotonically by at most \(n(1 - u_j^*)\). Thus, for \(j = 2, \ldots, k\), the total decrease in \(u_j(y)\) as \(y\) goes from \(x\) to \(x^{(j-1)}\) is at most

\[
\sum_{i=1}^{j-1} n(1 - u_i^*) \leq n(1 - \lambda)^2 \sum_{i=1}^{j-1} (\lambda d)^{j-1} < \frac{1}{2} \varepsilon n(1 - \lambda)(\lambda d)^{j-1} < \varepsilon n(1 - u_j^*),
\]

where we used (6.17). So (b') implies (b).

To construct the section of path between \(x = x^{(0)}\) and \(x^{(1)}\), we consider queues of length 0: in both \(x\) and \(x'\), the number of these empty queues is equal to \(n(1 - u_1^*) = [n(1 - \lambda)]\). We let \(K_{\ell_1}, \ldots, K_{\ell_r}\) be the queues that are empty in \(x\) but not in \(x'\), and \(K_{\ell_1}, \ldots, K_{\ell_r}\) be the queues that are empty in \(x'\) but not in \(x\) so \(r \leq n(1 - u_1^*)\). The section of path from \(x^{(0)}\) to \(x^{(1)}\) is constructed by: adding a customer to queue \(K_{\ell_1}\), emptying out queue \(K_{\ell_1}\), adding a customer to queue \(K_{\ell_2}\), emptying out queue \(K_{\ell_2}\), and so on. In this way, for every vector on this section of path, the number of empty queues is within 1 of \(n(1 - u_1^*)\). Meanwhile, for \(j \geq 2\), as we go along the path from \(x^{(0)}\) to \(x^{(1)}\), the value of \(u_j(y)\) may be decreased (since some of the queues \(K_{\ell_i}\) may have length greater than \(j\) in \(x^{(0)}\): we decrease the lengths of these
queues to 0, without creating any new queues of length at least $j$), but by
at most $r \leq n(1 - u^*_j)$, as required.

We now proceed in the same way for the set of queues of each length
$j = 1, 2, \ldots, k - 1$ in turn. We describe the construction of the section of
path from $x(j)$ to $x(j+1)$. In $x(j)$, the set of queues of each length less than $j$
is the same as in $x'$, and also we have $u_{j+1}(x(j)) \leq u^*_{j+1}$, by the properties
of the construction up to this point. Let $K_1, \ldots, K_r$ be the queues
that have length exactly $j$ in $x(j)$ but are longer in $x'$, and $K_{ \ell_1}, \ldots, K_{ \ell_s}$ be the
queues that have length $j$ in $x'$ but are longer in $x(j)$. Note that $r - s = n[(1 - u_{j+1}(x(j))) - (1 - u^*_{j+1})] \geq 0$, and also that $s$ is at most the number
of queues of length $j$ in $x'$, which is at most $n(1 - u^*_{j+1})$. The section of
path from $x(j)$ to $x(j+1)$ is constructed by: adding a customer to queue
$K_i$, reducing the length of queue $K_{j+1}$ to $j$, adding a customer to queue
$K_i$, reducing the length of queue $K_{j+2}$ to $j$, and so on. At the end, we
add a customer to each of the remaining queues $K_{i+1}, \ldots, K_r$ in turn. In
this way, for each intermediate vector $y$ on this section of path, the number
$nu_{j+1}(y)$ of queues of length at least $j + 1$ is at least its value $nu_{j+1}(x(j))$
at the beginning of this section of path, and at most $nu^*_{j+1} + 1$ (this can
be achieved if $u_{j+1}(x(j)) = u^*_{j+1}$, in which case $nu_{j+1}(y)$ alternates between
$nu^*_{j+1}$ and $nu^*_{j+1} + 1$ along the section of path). For $h > j + 1$, $u_h(y)$ is
decreased as go along this section of the path, by at most $s \leq n(1 - u^*_{j+1})$,
as required.

Along this path from $x$ to $x'$ as a whole, for each queue $i$, its length
changes monotonically from $x(i)$ to $x'(i)$. So the total length of the path is
at most

$$
\sum_{i=1}^{n} |x(i) - x'(i)| \leq \sum_{i=1}^{n} (k - x(i)) + (k - x'(i))
= n \left( \sum_{i=1}^{k} (1 - u_k(x)) + \sum_{i=1}^{k} (1 - u_k(x')) \right)
= 2n \sum_{i=1}^{k} (1 - u^*_i)
\leq \frac{5}{2} n \sum_{i=1}^{k} (1 - \lambda)(\lambda d)^{i-1},
$$

which is at most $3n(1 - \lambda)(\lambda d)^{k-1}$ by (6.17).

Thus there is a path between any pair of states in $\mathcal{N}$ of length at most

$$
3n(1 - \lambda)(\lambda d)^{k-1} + 12 \varepsilon n(1 - \lambda)(\lambda d)^{k-1} \leq 4n(1 - \lambda)(\lambda d)^{k-1},
$$
as claimed. \qed
11. Rapid Mixing

Our aims in this section are to prove a variety of results about rapid mixing of the \((n, d, \lambda)-\)supermarket process. We continue to assume that the parameters \(n, d, k, \lambda, \text{and } \varepsilon\) of the model satisfy the conditions of Theorem 6.2 (or equivalently (6.1)–(6.6)). For this section, we make the stronger assumption that \(\varepsilon \leq 1/60\), so that \((n, d, k, \lambda, 6\varepsilon)\) also satisfies the conditions.

We first consider two copies of the process starting in adjacent states in \(A_0(\ell, g)\), coupled according to the coupling referred to in Lemma 4.1. The proof partly follows along the lines of the proof of Lemma 2.6 in [5].

We set
\[
q(\ell, g) = (23k + 72g)e^{-1}n(1 - \lambda)^{-1} + 8\ell n,
\]
as in Proposition 6.8. We assume throughout this section that \(q(\ell, g) \leq s_0/2\).

**Lemma 11.1.** Let \(x, y\) be a pair of adjacent states in \(A_0(\ell, g)\), with \(x(j_0) = y(j_0) - 1\) for some queue \(j_0\), and \(x(j) = y(j)\) for \(j \neq j_0\). Consider coupled copies \((X^x_t)\) and \((X^y_t)\) of the \((n, d, \lambda)\)-supermarket process, where \(X^x_0 = x\) and \(X^y_0 = y\). For all times \(t \geq 2q(\ell, g)\), we have
\[
\mathbb{E}\|X^x_t - X^y_t\|_1 = \mathbb{P}(X^x_t \neq X^y_t) \leq e^{-\frac{1}{4}\log^2 n} + 4\exp\left(-\frac{t}{3200kd^k - 1}n\right).
\]

**Proof.** By Lemma 4.1 \(X^x_t\) and \(X^y_t\) are always neighbours or equal, always \(X^x_t \leq X^y_t\), and if for some time \(s\) we have \(X^x_s = X^y_s\), then \(X^x_t = X^y_t\) for all \(t \geq s\). Thus in particular \(\mathbb{E}\|X^x_t - X^y_t\|_1 = \mathbb{P}(X^x_t \neq X^y_t)\).

Initially, the queue \(j_0\) is unbalanced, i.e., \(X^x_{j_0} \neq X^y_{j_0}\), and all other queues are balanced. Observe that the index of the unbalanced queue in the coupled pair of processes may change over time. Let \(W_t\) denote the longer of the unbalanced queue lengths at time \(t\), if there is such a queue, and let \(W_t = 0\) otherwise. The time for the two coupled processes to coalesce is the time \(T\) until \(W_t\) hits 0.

Let us first run \((X^x_t)\) and \((X^y_t)\) together using the coupling. Let \(T^x_{\mathcal{H}}\) and \(T^y_{\mathcal{H}}\) denote the times \(T_{\mathcal{H}}\), as defined in Section 6 for the two copies of the process, and set \(T^*_{\mathcal{H}} = T^x_{\mathcal{H}} \lor T^y_{\mathcal{H}}\). By Theorem 6.10 \(T^*_{\mathcal{H}} \leq q(\ell, g)\) with probability at least
\[
1 - \frac{2(6k + 28)}{s_0} = 1 - (12k + 56)e^{-\frac{1}{4}\log^2 n}
\]
\[
\geq 1 - (12\log n + 56)e^{-\frac{1}{4}\log^2 n}
\]
\[
\geq 1 - \frac{1}{3}e^{-\frac{1}{4}\log^2 n},
\]
where we used (6.13); the final inequality holds for \(n \geq 10000\).

We now track the performance of the coupling after time \(T^*_{\mathcal{H}}\). If the processes have coalesced by time \(T^*_{\mathcal{H}}\) (i.e., if \(T \leq T^*_{\mathcal{H}}\)), then we are done. Otherwise, \(X^{x}_{T^*_{\mathcal{H}}}\) and \(X^{y}_{T^*_{\mathcal{H}}}\) are still adjacent, and there is some random index...
In other words, on the event \( N_0 \) such that the queue \( J_0 \) is unbalanced, i.e., \( X_{T_{j_0}}(J_0) \neq X_{T_{j_0}}(J_0) \), and all other queues are balanced. Moreover, since \( u_{k+1}(x) = 0 \) for all \( x \in \mathcal{H} \), we have \( W_{T_{j_0}} \leq k \).

We shall use Lemma 3.4 to give a suitable upper bound on \( \mathbb{P}(W_t > 0) \). The idea is that, since, with high probability, both copies of the process remain in \( \mathcal{H} \) for a long time, the unbalanced queue length \( W_t \) will often be driven below \( k \), and then there is a chance of going all the way down to 0.

For each \( t \geq 0 \), let \( B_t \) be the event that \( X_t^y, X_t^x \in \mathcal{H} \) for all \( s \) with \( T_{j_0}^y \leq s \leq t - 1 \). It follows from Theorem 6.10 that \( \mathbb{P}(B_t) \leq (12k + 56)/s_0 \leq 1/2e^{-1/4\log^2 n} \) (as above), provided \( t \leq s_0 \).

Let \( N_r \) be the number of jumps of the longer unbalanced queue length in the first \( r \) steps after \( T_{j_0}^x \). Also set \( N = N_T \), the total number of these jumps, with \( N_T = 0 \) if \( T \leq T_{j_0}^x \). For \( j = 1, 2, \ldots \), let \( T_j \) be the time of the \( j \)th jump after \( T_{j_0}^x \) if \( N \geq j \), and otherwise set \( T_j = T_{j_0}^x \vee T \). Thus, if \( T_{j_0}^x < T \), we have \( T_j < T_1 < \cdots < T = T_N = T_{N+1} = \cdots \). If \( T_{j_0}^x \geq T \), then all of the \( T_j \) are equal to \( T_{j_0}^x \).

Let \( S_0 = y(J_0) = W_{T_{j_0}^x} \mathbb{1}_{\{T_{j_0}^x < T \}} \), the longer unbalanced queue length at time \( t = T \) if coalescence has not occurred. For each positive integer \( j \), let \( S_j = W_{T_j} \), which is either 0 or the longer of the unbalanced queue lengths at time \( T_j \), immediately after the \( j \)th arrival or departure at the unbalanced queue. Also, if \( N \geq j \), let \( Z_j \) be the \pm 1-valued random variable \( S_j - S_{j-1} \). For each non-negative integer \( j \), let \( \varphi_j \) be the \( \sigma \)-field \( \mathcal{F}_{T_{j+1}} \), of all events before time \( T_{j+1} \). Let also \( A_j \) be the \( \varphi_j \)-measurable event \( B_{T_{j+1}} \), that is the event that \( X_s^y, X_s^x \in \mathcal{H} \) for each \( s \) with \( T_{j_0}^x \leq s \leq T_{j+1} - 1 \).

We shall use Lemma 3.4. We shall take the sequences \( (\varphi_{j})_{j \geq 0} \), \( (Z_j)_{j \geq 0} \), \( (S_j)_{j \geq 0} \) and \( (A_j)_{j \geq 0} \) as defined above, and we set \( k_0 = k \) and \( \delta = 1/(\lambda d + 1) \).

Note first, at any time \( t < T \), the probability, conditioned on \( \mathcal{F}_t \), of an arrival to the longer of the unbalanced queues is at most \( d\lambda/n(1+\lambda) \), while the conditional probability of a departure from that queue is \( 1/n(1+\lambda) \). Therefore, on the event that \( N \geq j \), the probability, conditioned on \( \varphi_{j-1} \), that the event at time \( T_j \) is a departure from the longer unbalanced queue is at least

\[
\frac{1/n(1+\lambda)}{1/n(1+\lambda) + d\lambda/n(1+\lambda)} = \frac{1}{1+d\lambda} = \delta.
\]

In other words, on the event \( N \geq j \) we have \( \mathbb{P}(Z_j = -1 \mid \varphi_{j-1}) \geq \delta \).

We now show that, on the event \( \{N \geq j\} \cap A_{j-1} \cap \{S_{j-1} \geq k\} \), we have

\[
\mathbb{P}(Z_j = -1 \mid \varphi_{j-1}) \geq \frac{3}{4}.
\]

To see this, consider a time \( t \geq T_{j_0}^x \). On the event \( B_t \), we have \( X_t \in \mathcal{H} \subseteq \mathcal{E}_1 \), and so, by Lemma 3.5 the conditional probability \( \mathbb{P}(L_{t+1} \mid \mathcal{F}_t) \) that the event at time \( t + 1 \) is an arrival to a queue of length \( k \) or greater is at most \( e^{-\log^2 n} \). In particular, on the event \( B_t \cap \{W_{t-1} \geq k\} \), the conditional probability that the event at time \( t + 1 \) is an arrival joining the longer unbalanced queue is at most \( e^{-\log^2 n} \), while the conditional probability that
the event at time \( t + 1 \) is a departure from the longer unbalanced queue is \( 1/n(1 + \lambda) \). Therefore, on the event \( \{ N \geq j \} \cap A_{j-1} \cap \{ S_{j-1} \geq k \} \), we have

\[
\mathbb{P}(Z_j = -1 \mid \varphi_{j-1}) \geq \frac{1/n(\lambda + 1)}{1/n(\lambda + 1) + e^{-\log^2 n}} \geq \frac{3}{4},
\]

for \( n \geq 10 \).

We have now shown that \( S_m - S_0 \) can be written as a sum \( \sum_{i=1}^m Z_i \) for \( \{0, \pm 1\} \)-valued random variables \( Z_i \) that satisfy the conditions of Lemma 3.4 with \( k_0 = k \) and \( \delta = 1/(\lambda d + 1) \). (The argument above establishes this for \( m \leq N \): for \( m > N \), we have set \( Z_m = S_m = 0 \), which also meets the requirements of the lemma.) Note that

\[
\delta^{-(k-1)} = (\lambda d + 1)^{(k-1)} \leq d^{k-1}(1 + 1/d)^k \leq d^{k-1}e^{k/d} \leq d^{k-1}e^{\varepsilon/150\sqrt{d}} \leq 2d^{k-1},
\]

where we used (6.4). Hence, for \( m \geq 16k \),

\[
\mathbb{P} \left( \bigcap_{i=1}^m \{ S_i \neq 0 \} \cap \bigcap_{i=0}^{m-1} A_i \right) \leq \mathbb{P}(S_0 > [m/16]) + 3 \exp \left( -\frac{\delta^{k-1}m}{200k} \right) \\
\leq 0 + 3 \exp \left( -\frac{m}{400kd^{k-1}} \right).
\]

Here \( \mathbb{P}(\cdot) \) refers to the coupling measure in the probability space of Section 4 with coupled copies of the process for each possible starting state.

Let \( q = q(\ell, g) \), take \( r \) with \( 64kn \leq r \leq s_0 - q \) and let \( m = [r/4n] \geq 16k \). Since, at each time after \( T_\ast^H \) and before \( T \), a jump in the longer unbalanced queue occurs with probability at least \( 1/2n \) while the queue is nonempty, we have, by inequality (3.1), \( \mathbb{P}(\{ T > T_\ast^H + r \} \cap \{ N_r < m \} \} \leq e^{-r/16n} \). Also,

\[
\mathbb{P} \left( \{ N_r \geq m \} \cap \bigcup_{i=0}^{m-1} A_i \cap \{ T_\ast^H \leq q \} \right) \leq \mathbb{P}(B_{q+r}) \\
\leq \mathbb{P}(B_{s_0}) \\
\leq \frac{1}{3} e^{-\frac{1}{4} \log^2 n}.
\]

Now we have that

\[
\mathbb{P}(T > q + r) \leq \mathbb{P}(T_\ast^H > q) + \mathbb{P}(\{ T > T_\ast^H + r \} \cap \{ T_\ast^H \leq q \}) \\
\leq \mathbb{P}(T_\ast^H > q) + \mathbb{P}(\{ T > T_\ast^H + r \} \cap \{ N_r < m \}) \\
+ \mathbb{P} \left( \{ N_r \geq m \} \cap \bigcup_{i=0}^{m-1} A_i \cap \{ T_\ast^H \leq q \} \right) \\
+ \mathbb{P} \left( \{ N_r \geq m \} \cap \bigcap_{i=0}^{m-1} A_i \cap \bigcap_{i=1}^{m} \{ S_i \neq 0 \} \right).
\]
To see this, note that \( \{ N_r \geq m \} \cap \bigcup_{i=1}^{m} \{ S_i = 0 \} \subseteq \{ T \leq T^s + r \} \). Now we have
\[
\mathbb{P}(T > q + r) \leq \frac{1}{3} e^{-\frac{1}{4} \log^2 n} + e^{-r/16} + \frac{1}{3} e^{-\frac{1}{4} \log^2 n} + 3 \exp \left( -\frac{r}{1600kd^{k-1}n} \right)
\]
provided \( 64kn \leq r \leq s_0 - q \).

If \( 2q \leq t \leq s_0 \), then setting \( r = t - q \geq t/2 \) gives
\[
\mathbb{P}(T > t) \leq \frac{2}{3} e^{-\frac{1}{4} \log^2 n} + 4 \exp \left( -\frac{t}{3200kd^{k-1}n} \right),
\]
which gives the required result. For \( t > s_0 \), we have
\[
\mathbb{P}(T > t) \leq \mathbb{P}(T > s_0) \leq \frac{2}{3} e^{-\frac{1}{4} \log^2 n} + 4 \exp \left( -\frac{s_0 - q}{1600kd^{k-1}n} \right) \leq e^{-\frac{1}{4} \log^2 n},
\]
so the result holds in this case too.

**Theorem 11.2.** Let \((X^x_t)\) and \((X^y_t)\) be two copies of the \((n, d, \lambda)\)-supermarket process, starting in states \(x\) and \(y\) in \(A_0(\ell, g)\). Then, for \( t \geq 2q(\ell, g) \), we have
\[
\mathbb{E}\|X^x_t - X^y_t\|_1 \leq 2gn \left( e^{-\frac{1}{4} \log^2 n} + 4 \exp \left( -\frac{t}{3200kd^{k-1}n} \right) \right).
\]

**Proof.** Given two distinct states \(x\) and \(y\) in \(A_0(\ell, g)\), we can choose a path \(x = z_0, z_1, \ldots, z_m = y\) of adjacent states in \(A_0(\ell, g)\) from \(x\) down to the empty queue-lengths vector and back up to \(y\), where \( m = \|x\|_1 + \|y\|_1 \leq 2gn \).

By Lemma [11.1], for \( t \geq 2q(\ell, g) \),
\[
\mathbb{E}\|X^x_t - X^y_t\|_1 \leq \sum_{i=0}^{m-1} \mathbb{E}\|X^x_{t_i} - X^y_{t_i+1}\|_1 \leq 2gn \left( e^{-\frac{1}{4} \log^2 n} + 4 \exp \left( -\frac{t}{3200kd^{k-1}n} \right) \right),
\]
as required.

We saw in Corollary [6.9] that \( Y_t \in A_0(\ell, g) \) with probability at least \( 1 - e^{-\frac{1}{4} \log^2 n} \), whenever \( \ell, g \geq k \), where \((Y_t)\) is a copy of the \((n, d, \lambda)\)-supermarket process in equilibrium. Thus we have the following corollary.

**Corollary 11.3.** Take any \( \ell, g \geq k \), and let \((X^x_t)\) be a copy of the \((n, d, \lambda)\)-supermarket process with starting in a state \(x \in A_0(\ell, g)\). Also let \((Y_t)\) be a copy in equilibrium. Then, for \( t \geq 2q(\ell, g) \), we have
\[
d_{TV}(\mathcal{L}(X^x_t), \mathcal{L}(Y_t)) \leq 2gn \left( 2e^{-\frac{1}{4} \log^2 n} + 4 \exp \left( -\frac{t}{3200kd^{k-1}n} \right) \right).
\]
Proof. The total variation distance is at most the probability that $Y_0 \notin \mathcal{A}_0(\ell, g)$, plus the maximum, over all states $y \in \mathcal{A}_0(\ell, g)$, of $\mathbb{P}(X_t^x \neq X_t^y)$. By Corollary 6.9 and Theorem 11.2, this is at most

$$e^{-\frac{1}{4}\log^2 n} + 2gn \left( e^{-\frac{1}{4}\log^2 n} + 4\exp\left(-\frac{t}{3200kd^{k-1}n}\right) \right)$$

$$\leq 2gn \left( 2e^{-\frac{1}{4}\log^2 n} + 4\exp\left(-\frac{t}{3200kd^{k-1}n}\right) \right),$$

as claimed. \hfill \Box

This implies Theorem 11.4 on setting $\varepsilon = 1/60$ (the conclusion is independent of $\varepsilon$, and $q(\ell, g)$ is decreasing in $\varepsilon$, so it is best to take the highest legitimate value), $\ell = \max(k, \|x\|_\infty)$ and $g = \max(k, \|x\|_1/n)$, so that

$$q(\ell, g) \leq 60(23k + 72(k + \|x\|_1/n))n(1 - \lambda)^{-1} + 8(k + \|x\|_\infty)n$$

$$\leq 6000kn(1 - \lambda)^{-1} + 4320\|x\|_1(1 - \lambda)^{-1} + 8n\|x\|_\infty,$$

as in the statement of the theorem.

We interpret Theorem 11.4 as saying that we have mixing in time of order

$$\max\{kn(1 - \lambda)^{-1}, gn(1 - \lambda)^{-1}, \ell n, kd^{k-1}n \log n\},$$

where $q(\ell, g)$ is bounded as above, or alternatively of order

$$\max\{kn(1 - \lambda)^{-1}, gn(1 - \lambda)^{-1}, \ell n, kd^{k-1}n \log n\}.$$
copies \((X_t^x)\) and \((X_t^y)\) of the \((n,d,\lambda)\)-supermarket process. For all times \(t \geq 0\), we have

\[
E \|X_t^x - X_t^y\|_1 = P(X_t^x \neq X_t^y) \\
\leq e^{-\frac{1}{4} \log^2 n} + 4 \exp \left( -\frac{t}{1600kd^{k-1}n} \right).
\]

**Proof.** The proof is nearly identical to that of Lemma 11.1. Here, instead of starting by running the two copies of the process together until some time \(T^*\), we make use of Theorem 10.2, which tells us that, with probability at most \(1 - 2(k + 5)/s_0\), both \(X_t^x\) and \(X_t^y\) remain within \(H^{0e}\) throughout the interval \(0 \leq t \leq s_0\). We may thus repeat the proof of Lemma 11.1 with \(T^*\) and \(q = q(\ell, g)\) replaced by 0, and we obtain the result stated. \(\square\)

Exactly as before, we can use this result to deduce the following.

**Theorem 11.5.** Let \((X_t^x)\) and \((X_t^y)\) be two copies of the \((n,d,\lambda)\)-supermarket process with starting states \(x\) and \(y\) in \(N^e\). Then, for \(t \geq 0\), we have

\[
E \|X_t^x - X_t^y\|_1 \leq n \left( e^{-\frac{1}{4} \log^2 n} + 4 \exp \left( -\frac{t}{1600kd^{k-1}n} \right) \right).
\]

Note that the conclusion is independent of \(\varepsilon\), and the hypothesis is weakest when \(\varepsilon\) is as large as possible, namely \(\varepsilon = 1/60\).

**Proof.** Take any two queue-lengths vectors \(x\) and \(y\) in \(N^e\). By Lemma 10.3, there is a path between \(x = z_0z_1 \cdots z_m = y\) in \(N^e\) of length \(m \leq 4n(1 - \lambda)(\lambda d)^{k-1} \leq n\) between \(x\) and \(y\). The result now follows as in the proof of Theorem 11.2. \(\square\)

As before, since \(Y_0\) lies in \(H^e \subseteq N^e\) with probability at least \(1 - e^{-\frac{1}{4} \log^2 n}\), by Corollary 6.9 we may now deduce that the total variation distance \(d_{TV}(\mathcal{L}(X_t^x), \mathcal{L}(Y_t))\) is at most

\[
e^{-\frac{1}{4} \log^2 n} + n \left( e^{-\frac{1}{4} \log^2 n} + 4 \exp \left( -\frac{t}{1600kd^{k-1}n} \right) \right)
\]

whenever \(x \in N^e\). This result, with \(\varepsilon = 1/60\), is exactly the statement of Theorem 11.3.

Theorem 11.3 shows that, from states \(x \in N^e\), we have mixing to equilibrium in time of order \(kd^{k-1}n \log n\). We now indicate why this bound is approximately best possible, for any value of \(\varepsilon\) such that \((n,d,\lambda,k,\varepsilon)\) satisfies the conditions of Theorem 11.2 where also \(\varepsilon \leq 1/30\).

Note that there is a state \(z\) in \(\mathcal{I}^{3e} \subseteq H^{3e} \subseteq N^{3e}\) with \(Q_k(z) \leq (1 - 9\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}\). However, we know from Corollary 6.9 that \(P(Y_t \in H^{3e}) \geq 1 - e^{-\frac{1}{4} \log^2 n}\), so in order for \(d_{TV}(\mathcal{L}(X_t^x), \Pi)\) to be small, we need that \(Q_k(X_t^x) \geq (1 - 5\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}\) with high probability. Set \(t = n(\lambda d)^{k-1}\).
The supermarket model with arrival rate tending to one

For $x \in H^{3\varepsilon}$, we obtain from Lemma 7.1, with a calculation almost exactly as in Lemma 8.2, that

$$(1 + \lambda)\Delta Q_k(x) \leq (1 - \lambda)(1 + \varepsilon/2) - \frac{Q_k(x)}{n(\lambda d)^{k-1}} + \exp(-dQ_k(x)/kn)$$

$$\leq (1 - \lambda)(1 + \varepsilon/2 - (1 - \varepsilon)) + e^{-\frac{177}{50}\log n}$$

so $\Delta Q_k(x) \leq 2\varepsilon(1 - \lambda)$ also. We know from Theorem 6.11 that, with probability at least $1 - \frac{(k + 5)}{s_0}$, $X^x_s \in H^{3\varepsilon}$ for all $s = 0, \ldots, t - 1$, and we also have that $Q_k(x) \leq kn$ for every state $x$. It follows that

$$\mathbb{E}[Q_k(X^x_t)] = Q_k(z) + \sum_{s=0}^{t-1} \mathbb{E}[\mathbb{E}(\Delta Q_k(X^x_s) | \mathcal{F}_s)]$$

$$\leq (1 - 9\varepsilon)n(1 - \lambda)(\lambda d)^{k-1} + 2\varepsilon t(1 - \lambda) + kn\frac{k + 5}{s_0}$$

$$\leq (1 - 6\varepsilon)n(1 - \lambda)(\lambda d)^{k-1}.$$ 

A result from [5] (adapted for discrete time) states that, for some absolute constant $c$, for any 1-Lipschitz function $f$, any starting state $z$, any $t > 0$ and any $u \geq 0$,

$$\mathbb{P}(|f(X^x_t) - \mathbb{E}[f(X^x_t)]| \geq u) \leq ne^{-cu^2/(t+u)}.$$ 

Applying this with $f = Q_k$, $t = n(\lambda d)^{k-1}$ and $u = \varepsilon t(1 - \lambda)$, we find that

$$\mathbb{P}(Q_k(X^x_t) > (1 - 5\varepsilon)n(1 - \lambda)(\lambda d)^{k-1})$$

$$\leq \mathbb{P}(Q_k(X^x_t) - \mathbb{E}[Q_k(X^x_t)] > \varepsilon n(1 - \lambda)(\lambda d)^{k-1})$$

$$\leq ne^{-\varepsilon^2n(1-\lambda)^2(\lambda d)^{k-1}/2}$$

$$\leq ne^{-270ck^2n^2d^{k-2}},$$

using (6.6) and (6.15). Therefore the mixing time is at least $t = n(\lambda d)^{k-1}$.

References

[1] M. Fairthorne (2011) PhD Thesis, London School of Economics.
[2] C. Graham (2000) Chaoticity on path space for a queuing network with selection of the shortest queue among several. *J. Appl. Probab.* 37 198–201.
[3] C. Graham (2004) Functional central limit theorems for a large network in which customers join the shortest of several queues. *Probab. Theory Related Fields* 131 97–120.
[4] M.J. Luczak (2008) Concentration of measure and mixing of Markov chains. *Discrete Mathematics and Theor. Comp. Sci.* (Proceedings of the 5th Colloq. Mathem. Comp. Sci.) 95–120.
[5] M.J. Luczak and C. McDiarmid (2006) On the maximum queue length in the supermarket model. *Annals of Probability* 34 493–527.
[6] M.J. Luczak and C. McDiarmid (2007) Asymptotic distributions and chaos for the supermarket model. *Elec. Jour. Probab.* 12 75–99.
[7] M.J. Luczak and J.R. Norris (2005) Strong approximation for the supermarket model. *Ann. Appl. Probab.* 15 2038–2061.
[8] M.J. Luczak and J.R. Norris. Averaging over fast variables in the fluid limit for Markov chains: application to the supermarket model with memory. *Ann. Appl. Probab.*, to appear.

[9] J.B. Martin and Y.M. Suhov (1999) Fast Jackson networks. *Ann. Appl. Probab.* 9 854–870.

[10] C.D. Meyer (2000) *Matrix Analysis and Applied Linear Algebra*, SIAM.

[11] M. Mitzenmacher (1996) Load balancing and density dependent jump Markov processes. *Proc. 37th Ann. Symp. Found. Comp. Sci.* 213–222.

[12] M. Mitzenmacher (1996) The power of two choices in randomized load-balancing. PhD thesis, Berkeley [http://www.eec.harvard.edu/~michaelm/](http://www.eec.harvard.edu/~michaelm/).

[13] M. Mitzenmacher, B. Prabhakar and D. Shah (2002) Load-balancing with memory. *Proc. 43rd Ann. IEEE Symp. Found. Comp. Sci.* 799–808.

[14] S.R.E. Turner (1998) The effect of increasing routing choice on resource pooling. *Probab. Engrg. Inform. Sci.* 12 109–124.

[15] N.D. Vvedenskaya, R.L. Dobrushin and F.I. Karpelevich (1996) Queueing system with selection of the shortest of two queues: an asymptotic approach. *Prob. Inform. Transm.* 32 15–27.

Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom

E-mail address: g.r.brightwell@lse.ac.uk

URL: [http://www.maths.lse.ac.uk/Personal/graham/](http://www.maths.lse.ac.uk/Personal/graham/)

School of Mathematics and Statistics, University of Sheffield

E-mail address: m.luczak@sheffield.ac.uk