Fourier Multipliers on a Vector-Valued Function Space

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Abstract
We study multiplier theorems on a vector-valued function space, which is a generalization of the results of Calderón and Torchinsky, and Grafakos, He, Honzík, and Nguyen, and an improvement of the result of Triebel. For $0 < p < \infty$ and $0 < q \leq \infty$ we obtain that if $r > \frac{d}{s - (d/\min(1,p,q) - d)}$, then

$$\left\| \{ (m_k \hat{f}_k)^\vee \}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)} \lesssim_{p,q} \sup_{l \in \mathbb{Z}} \left\| m_l(2^l \cdot) \right\|_{L^r_\ell(\mathbb{R}^d)} \left\{ f_k \right\}_{k \in \mathbb{Z}} \| f_k \|_{L^p(\ell^q)}, \quad f_k \in \mathcal{E}(A2^k),$$

under the condition $\max(|d/p - d/2|, |d/q - d/2|) < s < d/\min(1, p, q)$. An extension to $p = \infty$ will be additionally considered in the scale of Triebel–Lizorkin space. Our result is sharp in the sense that the Sobolev space in the above estimate cannot be replaced by Sobolev spaces $L^r_\ell$ with $r < \frac{d}{s - (d/\min(1,p,q) - d)}$.

Keywords Hörmander’s multiplier theorem · Vector-valued function space · Littlewood–Paley theory · Triebel–Lizorkin space

Mathematics Subject Classification Primary 42B15 · 42B25 · 42B35

1 Introduction and Main Results

Let $S(\mathbb{R}^d)$ denote the Schwartz space and $S'(\mathbb{R}^d)$ the space of tempered distributions. For the Fourier transform of $f \in S(\mathbb{R}^d)$ we use the definition $\hat{f}(\xi) :=$
\[ \int_{\mathbb{R}^d} f(x) e^{-2\pi i (x, \xi)} \, dx \] and write \( f^\vee(\xi) := \hat{f}(-\xi) \) the inverse Fourier transform of \( f \). We also extend these transforms to the space of tempered distributions.

For \( m \in L^\infty(\mathbb{R}^d) \) the multiplier operator \( T_m \) is defined by \( T_m f(x) := (m \hat{f})^\vee(x) \) for \( f \in S(\mathbb{R}^d) \). The classical Mikhlin multiplier theorem [16] states that if a function \( m \) satisfies

\[ |\partial_\xi^\beta m(\xi)| \lesssim |\xi|^{-|\beta|} \]

for all multi-indices \( \beta \) with \( |\beta| \leq \left[ \frac{d}{2} \right] + 1 \), then the operator \( T_m \) is bounded in \( L^p(\mathbb{R}^d) \) for \( 1 < p < \infty \). In [14] Hörmander sharpened the result of Mikhlin, using the weaker condition

\[ \sup_{l \in \mathbb{Z}} \|m(2^l \cdot) \hat{\phi}\|_{L^2(\mathbb{R}^d)} < \infty \] (1.1)

for \( s > d/2 \), where \( L^2_s(\mathbb{R}^d) \) denotes the standard fractional Sobolev space on \( \mathbb{R}^d \) and \( \phi \) is a Schwartz function on \( \mathbb{R}^d \), which generates a Littlewood–Paley partition of unity via a dyadic dilation, defined in Sect. 2. Calderón and Torchinsky [3] proved that if (1.1) holds for \( s > d/p - d/2 \), then \( m \) is a Fourier multiplier of Hardy space \( H^p(\mathbb{R}^d) \) for \( 0 < p \leq 1 \). A different proof was given by Taibleson and Weiss [23]. It turns out that the condition \( s > d/\min(1, p) - d/2 \) is optimal for the boundedness to hold and it is natural to ask whether (1.1) can be weakened by replacing \( L^2_s(\mathbb{R}^d) \) by other function spaces. Baernstein and Sawyer [1] obtained endpoint \( H^p(\mathbb{R}^d) \) estimates by using Herz space conditions for \((m(2^j \cdot) \hat{\phi})^\vee\) and these estimates were improved and extended to Triebel–Lizorkin spaces by Seeger [21] and Park [18]. On the other hand, for \( 1 < p < \infty \), using an interpolation method, Calderón and Torchinsky [3] replaced \( L^2_s(\mathbb{R}^d) \) in (1.1) by \( L^r_s(\mathbb{R}^d) \) for the \( L^p \)-boundedness to hold and the assumption in their result was replaced by a weaker one by Grafakos, He, Honzík, and Nguyen [11]. Let \((I - \Delta)^{s/2}\) be the inhomogeneous fractional Laplacian operator, explicitly given by

\[ (I - \Delta)^{s/2} f := ((1 + 4\pi^2 \cdot |^2)^{s/2} \hat{f})^\vee \]

and let \( L^r_s(\mathbb{R}^d) \) be the space containing tempered distributions \( f \), defined on \( \mathbb{R}^d \), for which the norm

\[ \|f\|_{L^r_s(\mathbb{R}^d)} := \|(I - \Delta)^{s/2} f\|_{L^r(\mathbb{R}^d)} \]

is finite.

**Theorem 1.1** Let \( 1 < p < \infty \) and \( |d/p - d/2| < s < d \). Suppose that

\[ \sup_{l \in \mathbb{Z}} \|m(2^l \cdot) \hat{\phi}\|_{L^r_s(\mathbb{R}^d)} < \infty \quad \text{for } r > d/s. \]

Then \( T_m \) is bounded in \( L^p(\mathbb{R}^d) \).
We also refer to [12,13] for further improvement of the multiplier theorem by using Lorentz space conditions.

A vector-valued version of Hörmander’s multiplier theorem was studied by Triebel [24, 26, 2.4.9]. For \( r > 0 \) let \( E(r) \) denote the space of all distributions whose Fourier transform is supported in \( \{ \xi \in \mathbb{R}^d : |\xi| \leq 2r \} \). Let \( A > 0 \). For \( 0 < p < \infty \) and \( 0 < q \leq \infty \) or for \( p = q = \infty \) we define

\[
L^p_{A}(\ell^q) := \{ \{ f_k \}_{k \in \mathbb{Z}} \subset S' : f_k \in E(A2^k), \| \{ f_k \}_{k \in \mathbb{Z}} \|_{L^p(\ell^q)} < \infty \}.
\]

To give a rigorous definition of the space, we recall that for each \( f_k \in E(A2^k) \)

\[
f_k = f_k * \Pi_k \quad \text{in the sense of tempered distribution}
\]

where \( \Pi_k \) is a Schwartz function whose Fourier transform is equal to 1 on the ball of radius \( A2^{k+1} \), centered at 0 and is supported in a larger ball. Since convolution between a tempered distribution and a Schwartz function is a smooth function, \( f_k * \Pi_k \) is actually a smooth function and thus, the norm \( \| \{ f_k \}_{k \in \mathbb{Z}} \|_{L^p(\ell^q)} \) can be interpreted as

\[
\| \{ f_k \}_{k \in \mathbb{Z}} \|_{L^p(\ell^q)} = \| \{ f_k * \Pi_k \}_{k \in \mathbb{Z}} \|_{L^p(\ell^q)}.
\]

In the rest of this paper, we think of \( f_k \in E(A2^k) \) as a smooth function \( f_k * \Pi_k \).

Then \( L^p_{A}(\ell^q) \) is a quasi-Banach space (Banach space if \( p, q \geq 1 \)) with a (quasi-)norm \( \| \cdot \|_{L^p(\ell^q)} \) (see [26] for more details).

**Theorem 1.2** Let \( 0 < p < \infty \), \( 0 < q \leq \infty \), and \( A > 0 \). Suppose \( f_k \in E(A2^k) \) for each \( k \in \mathbb{N} \), and \( \{ m_k \}_{k \in \mathbb{N}} \) satisfies

\[
\sup_{l \in \mathbb{N}} \left\| m_l(2^l \cdot) \right\|_{L^2_{\ell^d}(\mathbb{R}^d)} < \infty
\]

for

\[
s > \begin{cases} 
  d / \min(1, p, q) - d/2 & \text{if } q < \infty \\
  d/p + d/2 & \text{if } q = \infty
\end{cases}.
\]

Then

\[
\| \{ m_k \hat{f}_k \} \|_{L^p_{\ell^q}} \lesssim_{p, q} \sup_{l \in \mathbb{N}} \left\| m_l(2^l \cdot) \right\|_{L^2_{\ell^d}} \{ f_k \}_{k \in \mathbb{N}} \|_{L^p(\ell^q)}.
\]

It was first proved that if (1.2) holds for \( s > d/2 \), then (1.3) works for \( 1 < p, q < \infty \), by using Hörmander’s multiplier theorem. For the case \( 0 < p < \infty \) and \( 0 < q \leq \infty \), it is easy to obtain that (1.3) is true under the assumption (1.2) with \( s > d/2 + d / \min(p, q) \). Then a complex interpolation method is applied to derive \( s > d / \min(1, p, q) - d/2 \) for general \( 0 < p, q < \infty \). However, the method cannot be applied to the endpoint case \( q = \infty \), which is stronger than seemingly “natural” condition \( s > d / \min(1, p) - d/2 \).
The aim of this paper is to provide an improvement of Theorem 1.2, which would be actually a vector-valued extension of Theorem 1.1 in the full range \(0 < p \leq \infty\). Let

\[
\tau^{(s,p)} := \frac{d}{s - (d/\min(1, p) - d)}, \quad \tau^{(s,p,q)} := \frac{d}{s - (d/\min(1, p, q) - d)}.
\]

For \(m := \{m_k\}_{k \in \mathbb{Z}}\), throughout this work we will use the notation:

\[
\mathcal{L}^r_s[m] := \sup_{l \in \mathbb{Z}} \|m_l(2^l \cdot)\|_{L^r_l(\mathbb{R}^d)}.
\]

**Theorem 1.3** Let \(0 < p < \infty\) and \(0 < q \leq \infty\), \(A > 0\), and

\[
\max \left( \left| \frac{d}{p} - \frac{d}{2} \right|, \left| \frac{d}{q} - \frac{d}{2} \right| \right) < s < d/\min(1, p, q).
\]

Suppose \(f \in \mathcal{E}(A2^k)\) for each \(k \in \mathbb{Z}\) and \(m := \{m_k\}_{k \in \mathbb{Z}}\) satisfies

\[
\mathcal{L}^r_s[m] < \infty \quad \text{for } r > \tau^{(s,p,q)}.
\]

Then

\[
\left\| \left\{ (m_k \hat{f}_k)^\vee \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)} \lesssim_{p,q} \mathcal{L}^r_s[m] \left\| \left\{ f_k \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)}.
\]  

(1.4)

Moreover, the inequality also holds for \(p = q = \infty\).

Theorem 1.3 can be extended to the case \(p = \infty\) and \(0 < q < \infty\) in the scale of Triebel–Lizorkin space. To describe this, let \(D\) denote the collection of all dyadic cubes in \(\mathbb{R}^d\) and for each \(P \in D\) let \(\ell(P)\) be the side length of \(P\).

**Theorem 1.4** Let \(0 < q < \infty\), \(A > 0\), \(\mu \in \mathbb{Z}\), and

\[
\left| \frac{d}{q} - \frac{d}{2} \right| < s < d/\min(1, q).
\]

Suppose \(f \in \mathcal{E}(A2^k)\) for each \(k \in \mathbb{Z}\) and \(m := \{m_k\}_{k \in \mathbb{Z}}\) satisfies

\[
\mathcal{L}^r_s[m] < \infty, \quad \text{for } r > \tau^{(s,q)}.
\]

Then

\[
\sup_{P \in D, \ell(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 \ell(P)}^\infty |(m_k \hat{f}_k)^\vee (x)|^q \, dx \right)^{1/q} \leq_{q} \mathcal{L}^r_s[m] \sup_{P \in D, \ell(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 \ell(P)}^\infty |f_k(x)|^q \, dx \right)^{1/q}
\]

uniformly in \(\mu\).
As a corollary of the two theorems, we can prove the $\hat{F}_p^{\alpha,q}$-boundedness of the operator $T_m$, which is a generalization of Theorem 1.1 and an improvement of the result in [25].

**Corollary 1.5** Let $0 < p, q \leq \infty$ and $\alpha \in \mathbb{R}$. Suppose

$$\max \left( \left| \frac{d}{p} - d/2 \right|, \left| \frac{d}{q} - d/2 \right| \right) < s < d / \min (1, p, q)$$

and $m \in L^\infty(\mathbb{R}^d)$ satisfies

$$\sup_{l \in \mathbb{Z}} \| m(2^l \cdot) \hat{\varphi} \|_{L^r_s(\mathbb{R}^d)} < \infty \quad \text{for} \quad r > \tau^{(s,p,q)}.$$ 

Then

$$\| T_m f \|_{\hat{F}_p^{\alpha,q}(\mathbb{R}^d)} \lesssim \sup_{l \in \mathbb{Z}} \| m(2^l \cdot) \hat{\varphi} \|_{L^r_s(\mathbb{R}^d)} \| f \|_{\hat{F}_p^{\alpha,q}(\mathbb{R}^d)}.$$ 

This follows from setting $m_k = m \hat{\varphi}_k$ and $f_k = 2^{\alpha k} \hat{\varphi}_k * f$ where $\hat{\varphi}_k := \Phi_{k-1} + \Phi_k + \Phi_{k+1}$. The detailed proof is omitted as standard arguments are applicable. We refer the reader to Sect. 2 for the definition of Triebel–Lizorkin spaces $\hat{F}_p^{\alpha,q}(\mathbb{R}^d)$. As the space $\hat{F}_p^{\alpha,q}$ is a generalization of many function spaces such as a Lebesgue space, a Hardy space and a bounded mean oscillation (BMO) space, Corollary 1.5 also implies the boundedness of $T_m$ on such function spaces.

It turns out that the condition $s > |d/p - d/2|$ is optimal for the $L^p$-boundedness to hold in Theorem 1.1 and the proof can be found in Slavíková [22]. Moreover, Grafakos and Park [12] recently proved that the condition $r > d/s$ should be also necessary in the theorem, using properties of Bessel potentials, which will be described in (9.6) later. We now consider the sharpness of the condition $r > \tau^{(s,p,q)}$ in Theorem 1.3. Our claim is that (1.4) fails for $r = \tau^{(s,p,q)}$.

**Theorem 1.6** Let $0 < p < \infty$, $0 < q \leq \infty$, and $d / \min (1, p, q) - d < s < d / \min (1, p, q)$. Then there exists $m := \{ m_k \}_{k \in \mathbb{Z}}$ such that $\mathcal{L}^{(s,p,q)}_s [m] < \infty$, but (1.4) does not hold.

Remark that the assumption $d / \min (1, p, q) - d < s < d / \min (1, p, q)$ is clearly weaker than $\max \left( \left| \frac{d}{p} - d/2 \right|, \left| \frac{d}{q} - d/2 \right| \right) < s < d / \min (1, p, q)$ in Theorem 1.3.

We first study Theorem 1.4, using a proper separation of $f_k$ and $F_\infty$-variants of Peetre’s maximal inequality, introduced by the author [17]. For the proof of Theorem 1.3, the case $0 < p = q \leq \infty$ can be handled in an easy way via the $L^p$-boundedness of $T_{mk}$, which is stated in Lemma 6.1, and thus our interest will be given to the case $p \neq q$. For the case $0 < p \leq 1$ and $p < q \leq \infty$ we will establish a discrete characterization of $L_A^p(\ell^q)$ by using the $\varphi$-transform of Frazier and Jawerth [7–10] and apply atomic decomposition of discrete function space $\hat{F}_p^{0,q}$ in [10], which is analogous to the atomic decomposition of $H^p(\mathbb{R}^d)$. When $0 < q \leq 1$ and $q < p < \infty$, the
proof relies on a characterization of $L^p_A(\ell^q)$ by a dyadic version of the Fefferman–Stein sharp maximal function [6]. The remaining case $1 < p < \infty$ and $1 < q \leq \infty$ follows from a combination of complex interpolation techniques in Proposition 5.1 and duality arguments in Lemma 4.1. The central idea to prove Theorem 1.6 is a follows from a combination of complex interpolation techniques in Proposition 5.1 Sect. 5 we present a complex interpolation theorem for multipliers on

dualize the function space $L^p$ and this enables us to assume that

$$\| (m_k \hat{\Psi}_k) (2^k \cdot) \|_{L^q_\ell(\mathbb{R}^d)} = \| m_k (2^k \cdot) \hat{\Psi}_0 \|_{L^q_\ell(\mathbb{R}^d)} \lesssim \| m_k (2^k \cdot) \|_{L^q_\ell(\mathbb{R}^d)}$$

and this enables us to assume that

$$\text{Supp}(m_k) \subset \{ \xi \in \mathbb{R}^d : |\xi| \leq 2^k \}$$

(1.5)

in the proof. With this assumption, we can write $(m_k \hat{f}_k)^\vee (x) = m_k^\vee * f_k(x)$.

This paper is organized as follows. Section 2 is dedicated to preliminaries, introducing definitions and general properties which will be used in our proofs. Two characterizations of $L^p_A(\ell^q)$ will be given in Sect. 3.1, and by using one of them we dualize the function space $L^p_A(\ell^q)$ for $1 < p < \infty$ and $1 < q < \infty$ in Sect. 4. In Sect. 5 we present a complex interpolation theorem for multipliers on $L^p_A(\ell^q)$, based on the idea of Triebel [26, 2.4.9]. Section 6 contains a lemma which will play a fundamental role in the proof of both Theorems 1.3 and 1.4. The proof of Theorems 1.3, 1.4, and 1.6 will be provided in the last three sections.

Notations We use standard notations. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Denote by $\mathbb{Z}$ and $\mathbb{R}$ the set of all integers and the set of all real numbers, respectively. Let $D$ stand for the set of all dyadic cubes in $\mathbb{R}^d$ as above and for each $k \in \mathbb{Z}$, let $D_k$ be the subset of $D$ consisting of the cubes with side length $2^{-k}$. We use the symbol $X \lesssim Y$ to indicate that $X \leq CY$ for some constant $C > 0$, possibly different at each occurrence, and $X \approx Y$ if $x \lesssim y$ and $y \lesssim x$ simultaneously.

2 Preliminaries

2.1 Function Spaces

Let $\Phi_0$ be a Schwartz function so that $\text{Supp}(\hat{\Phi}_0) \subset \{ \xi \in \mathbb{R}^d : |\xi| \leq 2 \}$ and $\hat{\Phi}_0(\xi) = 1$ for $|\xi| \leq 1$ and define $\phi := \Phi_0 - 2^{-d} \Phi_0(2^{-1} \cdot)$ and $\phi_k := 2^{kd} \phi(2^k \cdot)$. Springer
Then \( \{ \phi_k \}_{k \in \mathbb{Z}} \) forms a (homogeneous) Littlewood–Paley partition of unity. That is, \( \text{Supp}(\hat{\phi}_k) \subset \{ \xi \in \mathbb{R}^d : 2^{k-1} \leq |\xi| \leq 2^{k+1} \} \) and \( \sum_{k \in \mathbb{Z}} \hat{\phi}_k(\xi) = 1 \) for \( \xi \neq 0 \).

For \( 0 < p, q \leq \infty \) and \( \alpha \in \mathbb{R} \), the (homogeneous) Triebel–Lizorkin space \( \dot{F}^{\alpha,q}_p(\mathbb{R}^d) \) is defined by the collection of all \( f \in S'/\mathcal{P} \) (tempered distribution modulo polynomials) such that

\[
\| f \|_{\dot{F}^{\alpha,q}_p(\mathbb{R}^d)} := \left\| \left\{ 2^{\alpha k} \phi_k \ast f \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)} < \infty, \quad 0 < p < \infty \text{ or } p = q = \infty,
\]

\[
\| f \|_{\dot{F}^{\alpha,\infty}_p(\mathbb{R}^d)} := \sup_{P \in \mathcal{D}} \left( \frac{1}{|P|} \int_P \sum_{k=-\log_2 \mathcal{L}(P)}^{\infty} 2^{\alpha k} \| \phi_k \ast f(x) \|_q \, dx \right)^{1/q}, \quad 0 < q < \infty
\]

where the supremum is taken over all dyadic cubes in \( \mathbb{R}^d \). Then these spaces provide a general framework that unifies classical function spaces:

- Hardy space \( \dot{F}^{0,2}_p(\mathbb{R}^d) = H^p(\mathbb{R}^d) \) if \( 1 < p < \infty \).
- Hardy-Sobolev space \( \dot{F}^{\alpha,2}_p(\mathbb{R}^d) = H^p_\alpha(\mathbb{R}^d) \) if \( 0 < p < \infty \).
- \( BMO \) space \( \dot{F}^{0,2}_\infty(\mathbb{R}^d) = BMO(\mathbb{R}^d) \).
- \( \text{Sobolev-BMO} \) \( \dot{F}^{\alpha,2}_\infty(\mathbb{R}^d) = BMO_\alpha(\mathbb{R}^d) \).

Note that \( H^p(\mathbb{R}^d) = L^p(\mathbb{R}^d) \) if \( 1 < p < \infty \).

### 2.2 Maximal Inequalities

A crucial tool in the theory of function spaces is the maximal inequalities of Fefferman and Stein [5] and Peetre [20].

Let \( \mathcal{M} \) be the Hardy–Littlewood maximal operator, defined by

\[
\mathcal{M} f(x) := \sup_{Q: x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy
\]

where the supremum is taken over all cubes containing \( x \), and for \( 0 < t < \infty \) let \( \mathcal{M}_t f := (\mathcal{M}(|f|^t))^{1/t} \). Then the Fefferman-Stein vector-valued maximal inequality [5] states that for \( 0 < r < p, q < \infty \),

\[
\left\| \left( \sum_k (\mathcal{M}_r f_k)^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| \left( \sum_k |f_k|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^d)}.
\]  

(2.1)

The inequality (2.1) also holds for \( 0 < p \leq \infty \) and \( q = \infty \).

For \( k \in \mathbb{Z} \) and \( \sigma > 0 \) we now define the Peetre maximal operator \( \mathcal{M}_{\sigma,2^k} \) by the formula

\[
\mathcal{M}_{\sigma,2^k} f(x) := \sup_{y \in \mathbb{R}^d} \frac{|f(x - y)|}{(1 + 2^k |y|)^\sigma}.
\]
It is known, see [20], that for \( f \in \mathcal{E}(A2^k) \),
\[
\mathfrak{M}_{d/r,2^k} f(x) \lesssim_A \mathcal{M}_r f(x) \quad \text{uniformly in } k.
\] (2.2)

Then (2.1) and (2.2) yield the following maximal inequality: Suppose \( f_k \in \mathcal{E}(A2^k) \) for some \( A > 0 \). Then for \( 0 < p < \infty \) or \( p = q = \infty \), we have
\[
\left\| \{ \mathfrak{M}_{\sigma,2^k} f_k \}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)} \lesssim_A \left\| \{ f_k \}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)}. \quad (2.3)
\]

if \( \sigma > d/\min(p,q) \).
Furthermore, a \( \hat{F}_\infty \)-version of (2.3) is recently given by the author [17]: Suppose \( f_k \in \mathcal{E}(A2^k) \) for some \( A > 0 \). Then for \( 0 < q < \infty \) and \( \mu \in \mathbb{Z} \), we have
\[
\sup_{P \in \mathcal{D}_\mu} \left( \frac{1}{|P|} \int_P \left| \sum_{k = -\log_2 \ell(P)}^{\infty} (\mathfrak{M}_{\sigma,2^k} f_k(x))^q \right| dx \right)^{1/q} \lesssim \sup_{P \in \mathcal{D}_\mu} \left( \frac{1}{|P|} \int_P \left| \sum_{k = -\log_2 \ell(P)}^{\infty} |f_k(x)|^q \right| dx \right)^{1/q} \quad (2.4)
\]
uniformly in \( \mu \) if \( \sigma > d/q \). We remark that (2.4) does not hold when \( \mathfrak{M}_{\sigma,2^k} f_k \) is replaced by \( \mathcal{M}_r f_k \) for all \( 0 < r < \infty \).

As an application of (2.4), we have
\[
\left\| \{ f_k \}_{k \geq \mu} \right\|_{L^\infty(\ell^\infty)} \lesssim \sup_{P \in \mathcal{D}: \ell(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \left| \sum_{k = -\log_2 \ell(P)}^{\infty} |f_k(x)|^q \right| dx \right)^{1/q}. \quad (2.5)
\]
See [17] for more details.

### 2.3 \( \varphi \)-Transform in \( \hat{F}_p^{0,q} \)

For a sequence of complex numbers \( b := \{ b_Q \}_{Q \in \mathcal{D}} \) we define
\[
\| b \|_{f_p^{0,q}} := \left\| g^q(b) \right\|_{L^p(\mathbb{R}^d)}, \quad 0 < p < \infty \quad \text{or} \quad p = q = \infty
\]
\[
\| b \|_{f_\infty^{0,q}} := \sup_{P \in \mathcal{D}} \left( \frac{1}{|P|} \int_P \sum_{Q \subset P, Q \in \mathcal{D}} (|b_Q||Q|^{-1/2} \chi_Q(x))^q dx \right)^{1/q}, \quad 0 < q < \infty
\]
where
\[
g^q(b)(x) := \left\| \{ |b_Q||Q|^{-1/2} \chi_Q(x) \} \right\|_{\ell^q}.
\]

Then the Triebel–Lizorkin space \( \hat{F}_p^{0,q}(\mathbb{R}^d) \) can be characterized by the discrete function space \( f_p^{0,q} \): For \( Q \in \mathcal{D} \) let \( x_Q \) be the lower left corner of \( Q \). Every \( f \in \hat{F}_p^{0,q}(\mathbb{R}^d) \)
can be written as

\[ f = \sum_{Q \in \mathcal{D}} b_Q \varphi_Q \quad \text{in } S'/\mathcal{P} \]

where \( \varphi_k \) and \( \widetilde{\varphi}_k \) are Schwartz functions with localized frequency, involving Littlewood–Paley decomposition, \( \varphi_Q^0(x) := |Q|^{1/2} \varphi_k(x - x_Q), \widetilde{\varphi}_Q^0(x) := |Q|^{1/2} \widetilde{\varphi}_k(x - x_Q) \) for each \( Q \in \mathcal{D}_k \), and \( b_Q := \langle f, \widetilde{\varphi}_Q^0 \rangle \). To be specific, since \( \sum_{k \in \mathbb{Z}} \widehat{\varphi}_k(\xi) \widehat{\widetilde{\varphi}_k}(\xi) = 1 \) for \( \xi \neq 0 \), we have \( f = \sum_{k \in \mathbb{Z}} \varphi_k \ast \widetilde{\varphi}_k \ast f \) in \( S'/\mathcal{P} \) and for each \( k \in \mathbb{Z} \)

\[ \varphi_k \ast \widetilde{\varphi}_k \ast f(x) = \sum_{Q \in \mathcal{D}_k} b_Q \varphi_Q^0(x). \quad (2.6) \]

Moreover, in the case, we have

\[ \|b\|_{\dot{F}^{0,q}_p} \lesssim \|f\|_{\dot{F}^{0,q}_p(\mathbb{R}^d)}. \quad (2.7) \]

The converse estimate is also true. For any sequence \( \mathbf{b} := \{b_Q\}_{Q \in \mathcal{D}} \) of complex numbers satisfying \( \|\mathbf{b}\|_{\dot{F}^{0,q}_p} < \infty \),

\[ f(x) := \sum_{Q \in \mathcal{D}} b_Q \varphi_Q^0(x) \]

belongs to \( \dot{F}^{0,q}_p \) and indeed,

\[ \|f\|_{\dot{F}^{0,q}_p(\mathbb{R}^d)} \lesssim \|\mathbf{b}\|_{\dot{F}^{0,q}_p}. \quad (2.8) \]

See [7,8] for more details.

### 2.4 Atomic Decomposition of \( \dot{f}^{0,q}_p \)

Let \( 0 < p \leq 1 \) and \( p \leq q \leq \infty \). A sequence of complex numbers \( \mathbf{r} := \{r_Q\}_{Q \in \mathcal{D}} \) is called an \( \infty \)-atom for \( \dot{f}^{0,q}_p \) if there exists \( Q_0 \in \mathcal{D} \) such that

\[ r_Q = 0 \quad \text{if} \quad Q \not\subset Q_0 \]

and

\[ \|g^q(\mathbf{r})\|_{L^\infty(\mathbb{R}^d)} \leq |Q_0|^{-1/p}. \quad (2.9) \]

Then the following atomic decomposition of \( \dot{f}^{0,q}_p \) holds:
Lemma 2.1 \cite{9,10} Suppose $0 < p \leq 1$, $p \leq q \leq \infty$, and $b := \{b_Q\}_{Q \in \mathcal{D}} \in \ell^0_p$. Then there exist $C_{p,q} > 0$, a sequence of scalars $\{\lambda_j\}$, and a sequence of $\infty$-atoms $r_j = \{r_j,Q\}_{Q \in \mathcal{D}}$ for $\ell^0_p$ so that

$$b = \{b_Q\}_{Q \in \mathcal{D}} = \sum_{j=1}^{\infty} \lambda_j \{r_j,Q\}_{Q \in \mathcal{D}} = \sum_{j=1}^{\infty} \lambda_j r_j,$$

and

$$\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} \leq C_{p,q} \|b\|_{\ell^0_p}.$$ 

Moreover, it follows that

$$\|b\|_{\ell^0_p} \approx \inf \left\{ \left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{1/p} : b = \sum_{j=1}^{\infty} \lambda_j r_j, \ r_j \text{ is a sequence of } \infty - \text{atoms for } \ell^0_p \right\}.$$ 

3 Characterizations of $L^p_A(\ell^q)$

As mentioned in Sect. 1, we assume $A = 2^{-2}$.

3.1 Characterization of $L^p_A(\ell^q)$ by Using a Method of $\varphi$-Transform

We will study properties of $\{f_k\}_{k \in \mathbb{Z}} \in L^p_A(\ell^q)$, which are analogous to (2.6), (2.7), and (2.8).

Suppose that $\Psi_0 \in S(\mathbb{R}^d)$ satisfies

$$\text{Supp}(\hat{\Psi}_0) \subset \{\xi : |\xi| \leq 1\} \quad \text{and} \quad \hat{\Psi}_0(\xi) = 1 \quad \text{for } |\xi| \leq 1/2.$$ 

For each $k \in \mathbb{Z}$ and $Q \in \mathcal{D}_k$ let $\Psi_k := 2^{kd} \Psi_0(2^k \cdot)$ and

$$\Psi^Q(x) := |Q|^{1/2} \Psi_k(x - x_Q)$$

where $x_Q$ denotes the lower left corner of the cube $Q$ as before.

Lemma 3.1 Let $0 < p < \infty$ or $p = q = \infty$.

(1) Assume $f_k \in \mathcal{E}(2^{k-2})$ for each $k \in \mathbb{Z}$. Then there exists a sequence of complex numbers $b := \{b_Q\}_{Q \in \mathcal{D}}$ such that

$$f_k(x) = \sum_{Q \in \mathcal{D}_k} b_Q \Psi^Q(x) \quad \text{and} \quad \|b\|_{\ell^0_p} \lesssim \|\{f_k\}_{k \in \mathbb{Z}}\|_{L^p(\ell^q)}.$$ 

\(\square\) Springer
(2) For any sequence \( b = \{b_Q\}_{Q \in D} \) of complex numbers satisfying \( \|b\|_{j_p^0} < \infty \),

\[
f_K(x) := \sum_{Q \in D_k} b_Q \Psi_Q(x)
\]

satisfies

\[
\left\| \{f_K\}_{k \in \mathbb{Z}} \right\|_{L^p(q')} \leq \|b\|_{j_p^0}.
\]

(3.1)

For the case \( p = \infty \) and \( 0 < q < \infty \) we introduce

\[
\|b\|_{j_\infty^{0,q}(\mu)} := \sup_{P \in D: \ell(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{Q \in D, Q \subset P} (|b_Q||Q|^{-1/2} \chi_Q(x))^q \right)^{1/q}
\]

for \( \mu \in \mathbb{Z} \).

Lemma 3.2 Let \( 0 < q < \infty \) and \( \mu \in \mathbb{Z} \).

(1) Assume \( f_k \in \mathcal{E}(2^{k-1}) \) for each \( k \geq \mu \). Then there exists a sequence of complex numbers \( b := \{b_Q\}_{Q \in D, \ell(Q) \leq 2^{-\mu}} \) such that

\[
f_K(x) = \sum_{Q \in D_k} b_Q \Psi_Q(x)
\]

and

\[
\|b\|_{j_\infty^{0,q}(\mu)} \leq \sup_{P \in D: \ell(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 \ell(P)}^{\infty} |f_K(x)|^q \right)^{1/q}.
\]

(2) For any sequence \( b := \{b_Q\}_{Q \in D, \ell(Q) \leq 2^{-\mu}} \) of complex numbers satisfying

\[
\|b\|_{j_\infty^{0,q}(\mu)} < \infty,
\]

\[
f_K(x) = \sum_{Q \in D_k} b_Q \Psi_Q(x)
\]

satisfies

\[
\sup_{P \in D: \ell(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 \ell(P)}^{\infty} |f_K(x)|^q \right)^{1/q} \leq \|b\|_{j_\infty^{0,q}(\mu)}.
\]

Proof of Lemma 3.1 (1) Since \( \text{Supp} (\hat{f}_k(2^{k-1} \cdot)) \subset \{||\xi|| \leq 1/2\} \), \( \hat{f}_k \) admits the decomposition

\[
\hat{f}_k(\xi) = 2^{-kd} \sum_{l \in \mathbb{Z}^d} f_k(2^{-k}l) e^{-2\pi i (2^{-k}l \cdot, \xi)}.
\]
using a scaling argument and the Fourier series representation of \( \hat{f}_k(2^k \cdot) \). Then we have

\[
    f_k(x) = (\hat{f}_k \Psi_k) \vee (x) = 2^{-kd} \sum_{l \in \mathbb{Z}^d} f_k(2^{-k}l) \Psi_k(x - 2^{-k}l)
\]

\[
    = \sum_{l \in \mathbb{Z}^d} 2^{-kd/2} f_k(2^{-k}l) 2^{-kd/2} \Psi_k(x - 2^{-k}l).
\]

(3.2)

For any \( Q \in \mathcal{D}_k \) we write

\[
    Q = Q_{k,l} := \{ x \in \mathbb{R}^d : 2^{-k}l_i \leq x_i \leq 2^{-k}(l_i + 1), \ i = 1, \ldots, d \}
\]

where \( l := (l_1, \ldots, l_d) \in \mathbb{Z}^d \). That is, \( Q_{k,l} \) is the dyadic cube, contained in \( \mathcal{D}_k \), whose lower left corner is \( x_{Q_{k,l}} = 2^{-k}l \). Now we use the notations

\[
    b_{Q_{k,l}} := 2^{-kd/2} f_k(2^{-k}l) = |Q_{k,l}|^{1/2} f_k(x_{Q_{k,l}}),
\]

\[
    \Psi_{Q_{k,l}}(x) := 2^{-kd/2} \Psi_k(x - 2^{-k}l) = |Q_{k,l}|^{1/2} \Psi_k(x - x_{Q_{k,l}}).
\]

Then (3.2) can be expressed as

\[
    f_k(x) = \sum_{Q \in \mathcal{D}_k} b_Q \Psi_Q(x).
\]

(3.3)

In addition, for almost every \( x \in \mathbb{R}^d \) there exists the unique dyadic cube \( Q_0 \in \mathcal{D}_k \) whose interior contains \( x \), and this yields

\[
    \sum_{Q \in \mathcal{D}_k} \| b_Q \| Q \|^{-1/2} \chi_Q(x) = \| b_{Q_0} \| Q_0 \|^{-1/2} = |f_k(x_{Q_0})| \lesssim \mathcal{M}_{\sigma, 2^k} f_k(x) \quad \text{for almost every } x.
\]

(3.4)

Here, the inequality holds due to the fact that

\[
    \sup_{y \in Q} |f_k(y)| \lesssim \inf_{y \in Q} \mathcal{M}_{\sigma, 2^k} f_k(y) \quad \text{uniformly in } Q \in \mathcal{D}_k,
\]

(3.5)

which is valid even for \( f_k \) without Fourier support condition. Then we can easily see that for \( \sigma > d / \min (p, q) \), using (3.4) and (2.3),

\[
    \| \mathbf{b} \|_{L_P^{p,q}} = \left\| \left\{ |b_Q| Q \right\} \right\|_{L_P^{(p,q)}} = \left\| \sum_{Q \in \mathcal{D}_k} |b_Q| Q \|^{1/2} \chi_Q \right\|_{L_P^{(p,q)}} \lesssim \left\| \left\{ \mathcal{M}_{\sigma, 2^k} f_k \right\} \right\|_{L_P^{(p,q)}} \lesssim \left\| \left\{ f_k \right\} \right\|_{L_P^{(p,q)}},
\]

as desired.
(2) For a given $b := \{b_Q\}_{Q \in D}$ and $k \in \mathbb{Z}$ let

$$f_k(x) := \sum_{Q \in D_k} b_Q \Psi^Q(x).$$

Setting

$$E^k_0(x) := \{Q \in D_k : |x - x_Q| < 2^{-k}\}$$
$$E^k_j(x) := \{Q \in D_k : 2^{-k+j-1} \leq |x - x_Q| < 2^{-k+j}\}, \quad j \in \mathbb{N}$$

for each $k \in \mathbb{Z}$ and $x \in \mathbb{R}^d$, we can write

$$|f_k(x)| \leq \sum_{j=0}^{\infty} \sum_{Q \in E^k_j(x)} |b_Q| |\Psi^Q(x)|.$$

Choose $0 < \epsilon < \min(1, p, q)$ and $M > d/\epsilon$. Observe that $|\Psi^Q(x)| \lesssim M 2^{-jM} |Q|^{-1/2}$ on $E^k_j$ and then the embedding $\ell^\epsilon \hookrightarrow \ell^1$ shows that

$$|f_k(x)| \lesssim \sum_{j=0}^{\infty} 2^{-jM} \left( \sum_{Q \in E^k_j(x)} (|b_Q| |Q|^{-1/2})^\epsilon \right)^{1/\epsilon} \approx \sum_{j=0}^{\infty} 2^{-j(M-d/\epsilon)} \left( \frac{1}{2^{-kd/2}} \int_{\mathbb{R}^d} \sum_{Q \in E^k_j(x)} (|b_Q| |Q|^{-1/2} \chi^Q(y))^\epsilon dy \right)^{1/\epsilon} \lesssim M \epsilon \left( \sum_{Q \in D_k} |b_Q| |Q|^{-1/2} \chi^Q(x) \right).$$

Finally, as a result of the maximal inequality (2.1), we obtain

$$\|\{f_k\}_{k \in \mathbb{Z}}\|_{L^p(\ell^q)} \lesssim \left\{ \sum_{Q \in D_k} |b_Q| |Q|^{-1/2} \chi^Q \right\}_{k \in \mathbb{Z}} \|_{L^p(\ell^q)} = \|b\|_{f^0,q},$$

as required.

Q.E.D.

Proof of Lemma 3.2 (1) The proof is very similar to that of Lemma 3.1. Indeed, using (3.3), (3.4), and (2.4) with $\sigma > d/q$, it can be verified that

$$\|b\|_{f_0,q}^{\mu} = \sup_{P \in D : \ell(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 \ell(P)}^{\infty} \left( \sum_{Q \in D_k} |b_Q| |Q|^{-1/2} \chi^Q(x) \right)^q dx \right)^{1/q} \lesssim \sup_{P \in D : \ell(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k=-\log_2 \ell(P)}^{\infty} \left( M_{\sigma,2^k} f_k(x) \right)^q dx \right)^{1/q} \left( M_{\sigma,2^k} f_k(x) \right)^{1/q}.$$
\[ \|b\|_{f_0^q, q} = \sup_{P \in D : \ell(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k = -\log_2 \ell(P)}^{\infty} |f_k(x)|^q \, dx \right)^{1/q}. \] (3.6)

Let
\[ f_k(x) := \sum_{Q \in D_k} b_Q \psi_Q(x) \]
and choose \( M > d / \min (1, q) \). Using Hölder’s inequality if \( q > 1 \) or the embedding \( \ell^q \hookrightarrow \ell^1 \) if \( q \leq 1 \), we obtain
\[
|f_k(x)| \lesssim M \sum_{Q \in D_k} |b_Q| |Q|^{-1/2} \frac{1}{(1 + 2^k |x - x_Q|)^{2M}} \\
\lesssim \left( \sum_{Q \in D_k} \left( |b_Q| |Q|^{-1/2} \right)^q \frac{1}{(1 + 2^k |x - x_Q|)^{Mq}} \right)^{1/q},
\]
which further implies that
\[
\sup_{P \in D : \ell(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_{P} \sum_{k = -\log_2 \ell(P)}^{\infty} |f_k(x)|^q \, dx \right)^{1/q} \\
\lesssim \sup_{P \in D : \ell(P) \leq 2^{-\mu}} \left( \sum_{k = -\log_2 \ell(P)}^{\infty} \sum_{Q \in D_k} (|b_Q| |Q|^{-1/2})^q \frac{1}{|P|} \int_{P} \frac{1}{(1 + 2^k |x - x_Q|)^{Mq}} \, dx \right)^{1/q}. 
\]

For each \( P \in D \) and \( m \in \mathbb{Z}^d \) let \( P + \ell(P)m := \{ x + \ell(P)m : x \in P \} \) and denote by \( D_k(P, m) \) the subfamily of \( D_k \) that contains any dyadic cubes belonging to \( P + \ell(P)m \). Then in the last expression we decompose
\[
\sum_{Q \in D_k} = \sum_{m \in \mathbb{Z}^d, |m| \leq 2d} \sum_{Q \in D_k(P, m)} + \sum_{m \in \mathbb{Z}^d, |m| > 2d} \sum_{Q \in D_k(P, m)} =: I^P_{k, M} + J^P_{k, M}
\]
which is possible because \( P \) and \( Q \)’s are dyadic cubes with \( \ell(Q) = 2^{-k} \leq \ell(P) \).

We first see that
\begin{align*}
&\left( \sum_{k=-\log_2 \ell(P)}^{\infty} J_{k,M}^P \right)^{1/q} \\
&\lesssim \sum_{m \in \mathbb{Z}^d, |m| \leq 2d} \left( \frac{1}{|P|} \sum_{k=-\log_2 \ell(P)}^{\infty} \sum_{Q \in \mathcal{D}_k(P,m)} (|b_Q||Q|^{-1/2})^{q} |Q| \right)^{1/q} \\
&\lesssim \sup_{R \in \mathcal{D} : \ell(R) = \ell(P)} \left( \frac{1}{|R|} \sum_{k=-\log_2 \ell(R)}^{\infty} \sum_{Q \in \mathcal{D}_k, Q \subset R} (|b_Q||Q|^{-1/2})^{q} |Q| \right)^{1/q}.
\end{align*}

On the other hand, if $|m| > 2d$ and $Q \in \mathcal{D}_k(P,m)$ then

$$|x - x_Q| \gtrsim \ell(P)|m|,$$

and therefore

$$J_{k,M}^P \lesssim \sum_{m \in \mathbb{Z}^d, |m| > 2d} \frac{1}{|m|^M} \frac{1}{2^k M^q} \frac{1}{\ell(P) M^q} \sum_{Q \in \mathcal{D}_k(P,m)} (|b_Q||Q|^{-1/2})^q.$$

Now we apply the triangle inequality if $q \geq 1$ or $\ell^q \hookrightarrow \ell^1$ if $q < 1$ to obtain that

$$\left( \sum_{k=-\log_2 \ell(P)}^{\infty} J_{k,M}^P \right)^{\min(1,q)/q} \lesssim \sum_{m \in \mathbb{Z}^d, |m| > 2d} \frac{1}{|m|^M \min(1,q)} \left( \sum_{k=-\log_2 \ell(P)}^{\infty} \frac{1}{2^k M^q} \frac{1}{\ell(P) M^q} \sum_{Q \in \mathcal{D}_k, Q \subset P + m \ell(P)} (|b_Q||Q|^{-1/2})^q \right)^{\min(1,q)/q}.$$

Since $M \min(1,q) > d$ and $2^k \ell(P) \geq 1$, the above expression is bounded by

$$\sum_{m \in \mathbb{Z}^d, |m| > 2d} \frac{1}{|m|^M \min(1,q)} \left( \frac{1}{|P|} \sum_{k=-\log_2 \ell(P)}^{\infty} \sum_{Q \in \mathcal{D}_k, Q \subset P + m \ell(P)} (|b_Q||Q|^{-1/2})^q |Q| \right)^{\min(1,q)/q}$$

$$\lesssim \sup_{R \in \mathcal{D} : \ell(R) = \ell(P)} \left( \frac{1}{|R|} \sum_{k=-\log_2 \ell(R)}^{\infty} \sum_{Q \in \mathcal{D}_k, Q \subset R} (|b_Q||Q|^{-1/2})^q |Q| \right)^{\min(1,q)/q}.$$
\[
\sup_{P \in D: \ell(P) \leq 2^{-\mu}} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 \ell(P)}^{\infty} |f_k(x)|^q dx \right)^{1/q} \\
\lesssim \sup_{R \in D: \ell(R) \leq 2^{-\mu}} \left( \frac{1}{|R|} \sum_{k = -\log_2 \ell(R)}^{\infty} \sum_{Q \in D, Q \subset R} \left( |b_Q| |Q|^{-1/2} |Q| \right)^{1/q} \right)^{1/q} \leq \|b\|_{f^0_q(\mu)}.
\]

\[\Box\]

### 3.2 Characterization of \( L^p_{A}(\ell^q) \) by Using a Sharp Maximal Function

Given a locally integrable function \( f \) on \( \mathbb{R}^d \) the Fefferman-Stein sharp maximal function \( f^{\#} \) is defined by

\[
f^{\#}(x) := \sup_{P: x \in P} \frac{1}{|P|} \int_P |f(y) - f_P| dy
\]

where \( f_P := \frac{1}{|P|} \int_P f(z) dz \) and the supremum is taken over all cubes \( P \) containing \( x \) (not necessarily dyadic cubes). Then a fundamental inequality of Fefferman and Stein [6] says that for \( 1 < p < \infty \) and \( 1 \leq p_0 \leq p \), if \( f \in L^{p_0}(\mathbb{R}^d) \), then we have

\[
\|Mf\|_{L^p(\mathbb{R}^d)} \lesssim \|f^{\#}\|_{L^p(\mathbb{R}^d)}. \tag{3.7}
\]

Using this result, it can be proved that for \( 0 < q < p < \infty \),

\[
\|f\|_{\ell^q_{p_0}(\mathbb{R}^d)} \approx \left\| \sup_{P: x \in P} \left( \frac{1}{|P|} \int_P \sum_{k = -\log_2 \ell(P)}^{\infty} |\phi_k \ast f(y)|^q dy \right)^{1/q} \right\|_{L^p(x)} \tag{3.8}
\]

where the supremum in the \( L^p \)-norm is taken over all cubes containing \( x \). See [19], [21, Proposition 6.1 and 6.2] for more details.

By following the proof of the estimate (3.7) in [6] we can actually replace the maximal functions by dyadic maximal ones. For a locally integrable function \( f \) we define the dyadic maximal function

\[
M^{(d)} f(x) := \sup_{P \in D: x \in P} \frac{1}{|P|} \int_P |f(y)| dy,
\]

and the dyadic sharp maximal function

\[
M^{\#} f(x) := \sup_{P \in D: x \in P} \frac{1}{|P|} \int_P |f(y) - f_P| dy
\]

where the supremums are taken over all dyadic cubes \( P \) containing \( x \). Then for \( 1 < p < \infty \), \( 1 \leq p_0 \leq p \), and \( f \in L^{p_0} \) we have

\[
\|M^{(d)} f\|_{L^p(\mathbb{R}^d)} \lesssim_p \|M^{\#} f\|_{L^p(\mathbb{R}^d)}. \tag{3.9}
\]
We now provide a characterization of $L^p_A(\ell^q)$ for $0 < q < p < \infty$, which is the analogue of (3.8).

**Lemma 3.3** Let $0 < q < p < \infty$. Suppose $f_k \in E(2^{k-2})$ for each $k \in \mathbb{Z}$. Then

$$\| \{ f_k \}_{k \in \mathbb{Z}} \|_{L^p(\ell^q)} \approx \left\| \sup_{P \in \mathcal{D}, x \in P} \left( \frac{1}{|P|} \int_{P} \sum_{k = -\log_2 \ell(P)}^{\infty} |f_k(y)|^q \, dy \right)^{1/q} \right\|_{L^p(x)} \tag{3.10}$$

where the supremum is taken over all dyadic cubes containing $x$.

The proof of the above lemma is almost the same as that of [19, Lemma 2.3], and for completeness we give a brief proof here.

**Proof** The direction “$\gtrsim$” is immediate because the right-hand side of (3.10) is bounded by $\| M_q (\| \{ f_k \}_{k \in \mathbb{Z}} \|_{\ell^q}) \|_{L^p(\mathbb{R}^d)}$ and the $L^p$-boundedness of $M_q$ yields the desired estimate.

For the opposite direction, using (3.9), the left-hand side of (3.10) is smaller than a constant times

$$\left\| M^{\sharp} \left( \sum_{k \in \mathbb{Z}} |f_k|^q \right) \right\|_{L^{p/q}(\mathbb{R}^d)} \tag{3.11}$$

and the sharp maximal function can be controlled by the sum of

$$\sup_{P \in \mathcal{D}, x \in P} \frac{1}{|P|} \int_{P} \frac{1}{|P|} \int_{P} \sum_{k = -\infty}^{\log_2 \ell(P) - 1} |f_k(y) - f_k(z)|^q \, dz \, dy.$$
for some $\psi_k \in S(\mathbb{R}^d)$ with $\text{Supp}(\hat{\psi}_k) \subset \{ \xi \in \mathbb{R}^d : |\xi| \lesssim 2^k \}$. Moreover, (3.5) implies that for any $\sigma > 0$

$$\sup_{w \in Q_P} |\psi_k| \ast |f_k|(w) \lesssim_{\sigma} \inf_{w \in Q_P} M_{\sigma, 2^k} (|\psi_k| \ast |f_k|)(w) \lesssim \inf_{w \in Q_P} M_{\sigma, 2^k} (M_{\sigma, 2^k} f_k)(w) \lesssim \inf_{w \in Q_P} M_{\sigma, 2^k} f_k(w)$$

and this yields

$$M^q \left( \{ f_k \}_{k \in \mathbb{Z}} \right) \lesssim \sup_{P \in \mathcal{D}_x \times P} \sum_{k = -\infty}^{-\log_2 \ell(P) - 1} (2^k \ell(P))^q \left( \inf_{w \in Q_P} M_{\sigma, 2^k} f_k(w) \right)^q$$

$$\lesssim \sup_{P \in \mathcal{D}_x \times P} \sup_{k \in \mathbb{Z}} \inf_{w \in Q_P} (M_{\sigma, 2^k} f_k(w))^q.$$ 

We observe that for each $Q_P \in \mathcal{D}_k$, the infimum over $w \in O_P$ in the preceding expression is less than

$$\inf_{w \in Q_P} \sum_{l = -\log_2 \ell(Q_P)}^{\infty} (M_{\sigma, 2^l} f_l(w))^q \leq \frac{1}{|Q_P|} \int_{Q_P} \sum_{l = -\log_2 \ell(Q_P)}^{\infty} (M_{\sigma, 2^l} f_l(w))^q dw$$

$$\leq \sup_{Q \in \mathcal{D}_x} \frac{1}{|Q|} \int_Q \sum_{l = -\log_2 \ell(Q)}^{\infty} (M_{\sigma, 2^l} f_l(w))^q dw$$

since $x \in P \subset Q_P$. Choosing $\sigma > p, q$, the last expression can be further controlled by

$$\sup_{Q \in \mathcal{D}_x} \frac{1}{|Q|} \int_Q \sum_{l = -\log_2 \ell(Q)}^{\infty} |f_l(w)|^q dw.$$ 

The proof of this estimate is contained in [19, Lemma 2.2] and we omit it here. This completes the proof of (3.11).

\[ \square \]

4 Dualization of $L_p^{q_p}(\ell^q)$ via a Discrete Function Space $f_p^{0, q}$

Suppose $1 < p < \infty$ and $1 \leq q < \infty$. Let $1 < p' < \infty$ and $1 < q' \leq \infty$ be the Hölder conjugates of $p$ and $q$, respectively. Then it is known, see [9], that the dual of $f_p^{0, q}$ is $f_p^{0, q'}$. Indeed, for $\{ b_Q \}_{Q \in \mathcal{D}} \in f_p^{0, q'}$

$$\| \{ b_Q \}_{Q \in \mathcal{D}} \|_{f_p^{0, q}} = \sup_{Q \in \mathcal{D}} \| b_Q \|_{f_p^{0, q}} = \sup_{Q \in \mathcal{D}} \sum_{Q \in \mathcal{D}} b_Q r_Q.$$ 

(4.1)
In this section, we dualize $L^p_A(\ell^q)$ through the relationship between the vector-valued space $L^p_A(\ell^q)$ and the discrete space $f^0_{p,q}$ in Lemma 3.1.

For any $(f_k)_{k \in \mathbb{Z}} \in L^p_A(\ell^q)$ and $Q \in \mathcal{D}$ we define the operator $\mathcal{U}_Q$ by

$$\mathcal{U}_Q(\{f_k\}_{k \in \mathbb{Z}}) := |Q|^{1/2} f_{-\log_2 \epsilon(Q)}(x_Q)$$

where we recall that $x_Q$ is the lower left corner of $Q \in \mathcal{D}$. Furthermore, for any $(r_Q)_{Q \in \mathcal{D}} \in f^0_{p,q}$ and $k \in \mathbb{Z}$ we define the operator $\mathcal{V}^\Psi_k (r_Q)_Q \in \mathcal{D}$ by

$$\mathcal{V}^\Psi_k (\{r_Q\}_{Q \in \mathcal{D}})(x) := \sum_{Q \in \mathcal{D}_k} r_Q x_Q. $$

Then for each $k \in \mathbb{Z}$

$$\mathcal{V}^\Psi_k (\{\mathcal{U}_Q(\{f_l\}_{l \in \mathbb{Z}})\}_Q \in \mathcal{D})(x) = f_k(x)$$

and it follows from Lemma 3.1 (2) that

$$\|\{\mathcal{V}^\Psi_k (r_Q)_Q \in \mathcal{D}\}_{k \in \mathbb{Z}}\|_{L^p(\ell^q)} \lesssim \sup_{\|r_Q\|_{f^0_p,q} \leq 1} \left| \int_{\mathbb{R}^d} \sum_{Q \in \mathcal{D}_k} f_k(x) \mathcal{V}^\Psi_k (r_Q)_Q (x) dx \right|. \tag{4.2}$$

**Lemma 4.1** Let $1 < p < \infty$ and $1 \leq q < \infty$. Suppose $f_k \in \mathcal{E}(2^{k-2})$ for $k \in \mathbb{Z}$. Then

$$\|\{f_k\}_{k \in \mathbb{Z}}\|_{L^p(\ell^q)} \lesssim \sup_{\|r_Q\|_{f^0_p,q} \leq 1} \left| \int_{\mathbb{R}^d} \sum_{Q \in \mathcal{D}_k} f_k(x) \mathcal{V}^\Psi_k (r_Q)_Q (x) dx \right|. $$

**Proof** By using Lemma 3.1 and (4.1)

$$\|\{f_k\}_{k \in \mathbb{Z}}\|_{L^p(\ell^q)} = \|\{\mathcal{U}_Q(\{f_l\}_{l \in \mathbb{Z}})\}_Q \in \mathcal{D}\|_{f^0_p,q}$$

Moreover, for each $k \in \mathbb{Z}$

$$\sum_{Q \in \mathcal{D}_k} \mathcal{U}_Q(\{f_l\}_{l \in \mathbb{Z}}) r_Q = \sum_{Q \in \mathcal{D}_k} 2^{-kd/2} f_k(x_Q) r_Q = \sum_{Q \in \mathcal{D}_k} 2^{-kd/2} \tilde{\Psi}_k(x) * f_k(x_Q) r_Q$$

$$= \int_{\mathbb{R}^d} f_k(x) \left( \sum_{Q \in \mathcal{D}_k} r_Q 2^{-kd/2} \tilde{\Psi}_k(x - x_Q) \right) dx$$

$$= \int_{\mathbb{R}^d} f_k(x) \mathcal{V}^\Psi_k (r_Q)_Q (x) dx$$

where $\tilde{\Psi}_k := \Psi_k(-\cdot)$, and this proves the lemma. \qed
5 Complex Interpolation Theorem for Multipliers on $L^p_A(\ell^q)$

In this section, we obtain an interpolation theorem for multipliers on $L^p_A(\ell^q)$ by using the complex method of Triebel [26, 2.4.9], which is a generalization of the well-known results of Calderón [2] and Calderón and Torchinsky [3].

Let $\Omega := \{z \in \mathbb{C} : 0 < Re(z) < 1\}$ be a strip in the complex plane $\mathbb{C}$ and $\overline{\Omega}$ denote its closure. We say that the mapping $z \mapsto f^z \in S'(\mathbb{R}^n)$ is a $S'$-analytic function in $\Omega$ if the following properties are satisfied:

1. For any $\varphi \in S(\mathbb{R}^n)$ with compact support, $g(x, z) := (\varphi \hat{f}^z)(x)$ is a uniformly continuous and bounded function in $\mathbb{R}^n \times \overline{\Omega}$.

2. For any $\varphi \in S(\mathbb{R}^n)$ with compact support and any fixed $x \in \mathbb{R}^n$, $h_x(z) := (\varphi \hat{f}^z)(x)$ is an analytic function in $\Omega$.

Let $0 < p_0, p_1, q_0, q_1 < \infty$. Then we define $F \left( L^p_A(\ell^{q_0}), L^p_A(\ell^{q_1}) \right)$ to be the collection of all systems $f^z := \{f^z_k\}_{k \in \mathbb{Z}}$ such that each $f^z_k$ is a $S'$-analytic function in $\Omega$,

$$f^z = \{f^z_k\}_{k \in \mathbb{Z}} \in L^p_A(\ell^{q_0}), \quad f^{1+iz} = \{f^{1+iz}_k\}_{k \in \mathbb{Z}} \in L^p_A(\ell^{q_1}) \quad \text{for any} \quad t \in \mathbb{R},$$

and

$$\sup_{t \in \mathbb{R}} \|f^{1+iz}\|_{L^p_A(\ell^{q_1})} < \infty \quad \text{for each} \quad l = 1, 2.$$

Moreover, for $f^z \in F \left( L^p_A(\ell^{q_0}), L^p_A(\ell^{q_1}) \right)$,

$$\|f^z\|_{F \left( L^p_A(\ell^{q_0}), L^p_A(\ell^{q_1}) \right)} := \max \left( \sup_{t \in \mathbb{R}} \|f^{it}\|_{L^p_A(\ell^{q_0})}, \sup_{t \in \mathbb{R}} \|f^{1+it}\|_{L^p_A(\ell^{q_1})} \right).$$

For $0 < \theta < 1$ the intermediate space $(L^p_A(\ell^{q_0}), L^p_A(\ell^{q_1}))_{\theta}$ is defined by

$$(L^p_A(\ell^{q_0}), L^p_A(\ell^{q_1}))_{\theta} := \{ f_k \}_{k \in \mathbb{Z}} : \exists f^z = \{ f^z_k \}_{z \in \mathbb{Z}} \in F \left( L^p_A(\ell^{q_0}), L^p_A(\ell^{q_1}) \right) \text{ such that } f_k = f^z_k \}
$$

and the (quasi-)norm in the space is

$$\|f_k\|_{(L^p_A(\ell^{q_0}), L^p_A(\ell^{q_1}))_{\theta}} := \inf_{f^z \in F \left( L^p_A(\ell^{q_0}), L^p_A(\ell^{q_1}) \right) : f_k = f^z_k} \|f^z\|_{F \left( L^p_A(\ell^{q_0}), L^p_A(\ell^{q_1}) \right)}$$

where the infimum is taken over all admissible systems $f^z = \{ f^z_k \}_{k \in \mathbb{Z}} \in F \left( L^p_A(\ell^{q_0}), L^p_A(\ell^{q_1}) \right)$ such that $f_k = f^z_k$. It is known, see [26, 2.4.9], that for any $0 < p_0, p_1, q_0, q_1 < \infty$ and $0 < \theta < 1$

$$(L^p_A(\ell^{q_0}), L^p_A(\ell^{q_1}))_{\theta} = L^p_A(\ell^q) \quad (5.1)$$

when $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. 

(Springer)
Proposition 5.1 Let \( 0 < p_0, p_1, q_0, q_1 < \infty, s_0, s_1 \geq 0, \) and \( 1 < r_0, r_1 < \infty. \) Suppose that for any \( \{g_k\}_{k \in \mathbb{Z}} \in L_{A}^{p_0}(\ell^{q_0}) \) and \( \{h_k\}_{k \in \mathbb{Z}} \in L_{A}^{p_1}(\ell^{q_1}), \)

\[
\left\| \left\{ m_k^{\vee} * g_k \right\}_{k \in \mathbb{Z}} \right\|_{L^{p_0}(\ell^{q_0})} \lesssim L_{S_{0}}^{r_0}[m] \left\| g_k \right\|_{L^{p_0}(\ell^{q_0})}, \tag{5.2}
\]

\[
\left\| \left\{ m_k^{\vee} * h_k \right\}_{k \in \mathbb{Z}} \right\|_{L^{p_1}(\ell^{q_1})} \lesssim L_{S_{1}}^{r_1}[m] \left\| h_k \right\|_{L^{p_1}(\ell^{q_1})}. \tag{5.3}
\]

Then for any \( 0 < \theta < 1 \) and \( p, q, r, s \) satisfying

\[
1/p = (1-\theta)/p_0 + \theta/p_1, \quad 1/q = (1-\theta)/q_0 + \theta/q_1, \tag{5.4}
\]

\[
1/r = (1-\theta)/r_0 + \theta/r_1, \quad s = (1-\theta)s_0 + \theta s_1, \tag{5.5}
\]

and \( \{f_k\}_{k \in \mathbb{Z}} \in L_{A}^{p}(\ell^{q}), \) we have

\[
\left\| \left\{ m_k^{\vee} * f_k \right\}_{k \in \mathbb{Z}} \right\|_{L^{p}(\ell^{q})} \lesssim L_{S}[m] \left\| f_k \right\|_{L^{p}(\ell^{q})}. \]

**Proof** Suppose \( p, q, r, s \) satisfy (5.4) and (5.5), and \( \{f_k\}_{k \in \mathbb{Z}} \in L_{A}^{p}(\ell^{q}), \) Then, due to (5.1), for any \( \epsilon > 0 \) there exists \( f^\epsilon = \{f_k^\epsilon\} \in (L_{A}^{p_0}(\ell^{q_0}), L_{A}^{p_1}(\ell^{q_1}))_{\theta} \) such that \( f_k = f_k^\epsilon \) and

\[
\left\| f^\epsilon \right\|_{F(L_{A}^{p_0}(\ell^{q_0}), L_{A}^{p_1}(\ell^{q_1}))} < \left\| \{f_k\}_{k \in \mathbb{Z}} \right\|_{(L_{A}^{p_0}(\ell^{q_0}), L_{A}^{p_1}(\ell^{q_1}))_{\theta}} + \epsilon.
\]

Now let

\[
\sigma_{k,s} := (I-\Delta)^{s/2}(m_k(2^k \cdot))
\]

and

\[
\sigma_{k,s}^\epsilon := \left( L_{S}[m] \right)^{1-r\left(\frac{1}{r_0} + \frac{s}{s_1}\right)} \frac{(1+\theta)^{d/2+1}}{(1+z)^{d/2+1}} \left( I-\Delta \right)^{-\frac{\theta(1-s)+s_1}{2}} \left\| \sigma_{k,s} \right\|_{r\left(\frac{1}{r_0} + \frac{s}{s_1}\right)} \epsilon^{i \text{Arg}(\sigma_{k,s})} (./2^k)
\]

where \( \text{Arg}(\sigma_{k,s}) \) means the argument of \( \sigma_{k,s} \). Then we note that \( \sigma_{k,s}^\epsilon = m_k \) and \( F_k^\epsilon := \langle \sigma_{k,s}^\epsilon \rangle \) is an \( S(\mathbb{R}^d) \)-analytic function in \( \Omega \). Moreover,

\[
\left\| \left\{ m_k^{\vee} * f_k \right\}_{k \in \mathbb{Z}} \right\|_{L^{p}(\ell^{q})} \approx \left\| \left\{ \sigma_{k,s}^{\theta} \right\}^{\vee} \right\|_{L^{p_0}(\ell^{q_0}), L_{A}^{p_1}(\ell^{q_1})_{\theta}}
\]

\[
= \left\| \{F_k^{\theta} \}_{k \in \mathbb{Z}} \right\|_{L_{A}^{p_0}(\ell^{q_0}), L_{A}^{p_1}(\ell^{q_1})_{\theta}} \leq \left\| \{F_k^{\epsilon} \}_{k \in \mathbb{Z}} \right\|_{F(L_{A}^{p_0}(\ell^{q_0}), L_{A}^{p_1}(\ell^{q_1}))}
\]

\[
= \max \left( \sup_{t \in \mathbb{R}} \left\| F_k^{\epsilon(t)} \right\|_{L^{p_0}(\ell^{q_0})}, \sup_{t \in \mathbb{R}} \left\| F_k^{1+i\epsilon(t)} \right\|_{L^{p_1}(\ell^{q_1})} \right).
\]

From (5.2),
then we are done by using (5.1) and taking ϵ > 0.

Suppose that (1.5) holds. Then for 1 < r_0 < r_1 < ∞ and s ≥ 0 we have

\[ \| f_k^{it} \|_{L^0(\mathbb{R}^d)} = \sup_{j \in \mathbb{R}} \| \sigma_{j,s}^{it} \|_{L^0(\mathbb{R}^d)} \]

and similarly, thanks to (5.3),

\[ \| f_k^{it} \|_{L^1(\mathbb{R}^d)} \leq \sup_{j \in \mathbb{R}} \| \sigma_{j,s}^{it} \|_{L^1(\mathbb{R}^d)} \]

Therefore, once we prove

\[ \| \sigma_{j,s}^{it} \|_{L^0(\mathbb{R}^d)}, \| \sigma_{j,s}^{it} \|_{L^1(\mathbb{R}^d)} \leq C_s[m], \quad \text{uniformly in } j \in \mathbb{Z}, \text{(5.6)} \]

then we are done by using (5.1) and taking ϵ → 0.

Let us prove (5.6). By using Hörmander’s multiplier theorem, \( \| \sigma_{j,s}^{it} \|_{L^0(\mathbb{R}^d)} \) is controlled by a constant times

\[
(L_{s}[m])^{1 - \frac{r}{r_0} \frac{1}{(1 + |t|)^{d/2+1}}} \| (I - \Delta)^{\frac{it(t_0 - t)}{2}} (|\sigma_{j,s} |^{\frac{r}{r_0} - i tr(\frac{1}{r_0} - \frac{1}{r_1}) e^{i \text{Arg}(\sigma_{j,s})}) \|_{L^0(\mathbb{R}^d)}
\]

On the other hand, \( \| \sigma_{j,s}^{it} \|_{L^1(\mathbb{R}^d)} \) is less than a constant multiple of

\[
(L_s'[m])^{1 - \frac{r}{r_1} \frac{1}{(1 + |t|)^{d/2+1}}} \| (I - \Delta)^{\frac{it(t_0 - t)}{2}} (|\sigma_{j,s} |^{\frac{r}{r_1} - i tr(\frac{1}{r_1} - \frac{1}{r_1}) e^{i \text{Arg}(\sigma_{j,s})}) \|_{L^1(\mathbb{R}^d)}
\]

which finishes the proof of (5.6).

\[ \square \]

6 The Key Lemma

Suppose that (1.5) holds. Then for 1 < r_0 < r_1 < ∞ and s ≥ 0 we have

\[ \| m_k(2^k \cdot) \|_{L^0(\mathbb{R}^d)} \lesssim \| m_k(2^k \cdot) \|_{L^1(\mathbb{R}^d)}. \]

The proof of this will be given in “Appendix”. Now the principal ingredient in the proof of Theorems 1.3 and 1.4 is the following lemma:
Lemma 6.1 Suppose $0 < p \leq \infty$ and $k \in \mathbb{Z}$. Suppose $f_k \in \mathcal{E}(2^{k-2})$ and \{m_k\}_{k \in \mathbb{Z}} satisfies (1.5). Then for

$$|d/p - d/2| < s < d/\min(1, p) \quad \text{and} \quad r > \tau^{(s, p)},$$

we have

$$\left\| m_k^\vee \ast f_k \right\|_{L^p(\mathbb{R}^d)} \lesssim \left\| m_k(2^k \cdot) \right\|_{L^p(\mathbb{R}^d)} \| f_k \|_{L^p(\mathbb{R}^d)} \quad \text{uniformly in } k.$$
by a factor of 9. Note that $P^*$ is a union of some dyadic cubes near $P$. Then we decompose

$$
\left( \frac{1}{|P|} \int_P \sum_{k=v}^{\infty} \left| m_k^* \ast f_k(x) \right|^q \, dx \right)^{1/q} \lesssim \left( \frac{1}{|P|} \int_P \sum_{k=v}^{\infty} \left| m_k^* \ast (\chi_{P^*} f_k)(x) \right|^q \, dx \right)^{1/q}
$$

\[ + \left( \frac{1}{|P|} \int_P \sum_{k=v}^{\infty} \left| m_k^* \ast (\chi_{(P^*)^c} f_k)(x) \right|^q \, dx \right)^{1/q} \]

$$
=: \mathcal{U}_P + \mathcal{V}_P.
$$

We observe that, due to (1.5),

$$
m_k^* \ast (\chi_{P^*} f_k) = m_k^* \ast \Psi_{k+1} \ast (\chi_{P^*} f_k) \quad (7.1)
$$

and then $\mathcal{U}_P$ is estimated by

$$
\left( \frac{1}{|P|} \sum_{k=v}^{\infty} \left\| m_k^* \ast \Psi_{k+1} \ast (\chi_{P^*} f_k) \right\|_{L^q(\mathbb{R}^d)}^q \right)^{1/q} \lesssim \mathcal{L}_q^q[m] \left( \frac{1}{|P|} \sum_{k=v}^{\infty} \left\| \Psi_{k+1} \ast (\chi_{P^*} f_k) \right\|_{L^q(\mathbb{R}^d)}^q \right)^{1/q},
$$

due to Lemma 6.1. We now claim that for any $\sigma > 0$

$$
\left\| \Psi_{k+1} \ast (\chi_{P^*} f_k) \right\|_{L^q(\mathbb{R}^d)} \lesssim \sigma \left( \int_{P^*} \left( \left\langle m_{\sigma, 2^k} f_k \right\rangle^q \right)^{1/q} \right).
$$

(7.2)

This follows immediately from Young’s inequality if $q \geq 1$. For $0 < q < 1$, we write

$$
\left\| \Psi_{k+1} \ast (\chi_{P^*} f_k) \right\|_{L^q(\mathbb{R}^d)}^q = \sum_{Q \in \mathcal{D}_k, Q \subset P^*} \left\| \Psi_{k+1} \ast (\chi_{Q} f_k) \right\|_{L^q(\mathbb{R}^d)}^q
$$

\[ \leq \sum_{Q \in \mathcal{D}_k, Q \subset P^*} \left\| f_k \right\|_{L^{q\infty}(Q)} \int_{\mathbb{R}^d} \left( \int_{Q} \left| \Psi_{k+1}(x - y) \right| \, dy \right) \, dx. \]

The integral in the preceding expression can be estimated, using Hölder’s inequality with $1/q > 1$, by

$$
\left( \int_{\mathbb{R}^d} \left( \frac{1}{\left(1 + 2^k |x - c_Q| \right)^{M/(1-q)}} \, dx \right)^{1-q} \right)^{1/q}
$$

$$
\left( \int_{Q} \left( \int_{\mathbb{R}^d} \left( \frac{1}{\left(1 + 2^k |x - c_Q| \right)^{M/q}} \, dy \right) \right)^q \, dx \right)^q.
$$
which is clearly smaller than a constant multiple of $2^{-kd}$ for sufficiently large $M > 0$. This, together with (3.5), yields that

$$
\| \Psi_{k+1} \ast (\chi_{P^*} f_k) \|_{L^q(\mathbb{R}^d)}^q \lesssim \sum_{Q \in D_k, Q \subset P^*} 2^{-kd} \inf_{y \in Q} (\mathcal{M}_{\sigma,2^k} f_k(y))^q \\
\leq \sum_{Q \in D_k, Q \subset P^*} \int_Q (\mathcal{M}_{\sigma,2^k} f_k(y))^q \, dy = \int_{P^*} (\mathcal{M}_{\sigma,2^k} f_k(y))^q \, dy
$$

and we finally arrive at the desired estimate (7.2). Therefore we have

$$
\mathcal{U}_P \lesssim \mathcal{L}_s^r[m] \left( \frac{1}{|P|} \int_P \sup_{R \in D \subset \ell(R) \leq 2^{-\eta}} \left( \frac{1}{|R|} \int_R \sum_{k=v}^{\infty} (\mathcal{M}_{\sigma,2^k} f_k(y))^q \, dy \right)^{1/q} \right). 
$$

Choosing $\sigma > d/q$ and applying the maximal inequality (2.4), we conclude that

$$
\mathcal{U}_P \lesssim \mathcal{L}_s^r[m] \sup_{R \in D \subset \ell(R) \leq 2^{-\eta}} \left( \frac{1}{|R|} \int_R \sum_{k=v}^{\infty} |f_k(x)|^q \, dx \right)^{1/q}. 
$$

To estimate $\mathcal{V}_P$ we note that $r > \tau^{(s,q)}$ implies that $s - d/r > d/\min(1, q) - d$ and there exists $\epsilon > 0$ so that $s - \epsilon - d/r > d/\min(1, q) - d \geq 0$. Then we see that for $x \in P$

$$
|m_k^\nu \ast (\chi_{P^*} f_k)(x)| \leq \int_{|z| \geq \ell(P)} |m_k^\nu(z)| |f_k(x - z)| \, dz \\
\leq \mathcal{M}_{\sigma,2^k} f_k(x) \int_{|z| \geq \ell(P)} (1 + 2^k|z|)^{\nu} |m_k^\nu(z)| \, dz
$$

and the integral is less than a constant times

$$
\left( \int_{|z| \geq 2^k \ell(P)} \frac{1}{|z|^{(s-\epsilon)e-r}} \, dz \right)^{1/r} \| (1 + 4\pi^2 | \cdot |^2)^{\nu/2} (m_k(2^k \cdot))^\nu \|_{L^r(\mathbb{R}^d)} \lesssim 2^{-(k-v)(s-\epsilon-d/r)} \mathcal{L}_s^r[m]
$$

by applying Hölder’s inequality and the Hausdorff–Young inequality. This proves that

$$
\mathcal{V}_P \lesssim \mathcal{L}_s^r[m] \left( \frac{1}{|P|} \int_P \sum_{k=v}^{\infty} 2^{-q(k-v)(s-\epsilon-d/r)} \| \mathcal{M}_{\sigma,2^k} f_k(x) \|_{L^q(\mathbb{R}^d)} \right)^{1/q} \\
\lesssim \mathcal{L}_s^r[m] \left( \| \mathcal{M}_{\sigma,2^k} f_k \|_{L^\infty(\mathbb{R}^d)} \right)^{1/q} \lesssim \mathcal{L}_s^r[m] \left( \| f_k \|_{L^\infty(\mathbb{R}^d)} \right)^{1/q}
$$
\[ \lesssim \mathcal{L}_s^r [m] \sup_{R \in \mathcal{D} : r(R) \leq 2^{-\nu}} \left( \frac{1}{|R|} \int_{P} \sum_{k=\log_2 r(R)}^{\infty} |f_k(x)|^q \, dx \right)^{1/q} \]  

(7.4)

where the maximal inequality (2.3) and the embedding (2.5) are applied.

By taking the supremum of \( U_P \) and \( V_P \) over all dyadic cubes \( P \) whose side length is less than or equal to \( 2^{-\mu} \), the proof of Theorem 1.4 is complete.

8 Proof of Theorem 1.3

A straightforward application of Lemma 6.1 proves the special case \( 0 < p = q \leq \infty \) and therefore we work only with the case \( p \neq q \) and \( 0 < p < \infty \).

8.1 The Case \( 0 < p \leq 1 \) and \( p < q \leq \infty \)

Assume \( d/p - d/2 < s < d/p \). Then \( 1 < \tau(s,p) < 2 \) and we may assume \( \tau (s,p) < r < 2 \) because of (6.1). According to Lemmas 3.1 and 2.1, if \( \text{Supp}(\hat{f}_k) \subset \{ \xi : |\xi| \leq 2^{k-1} \} \) for each \( k \in \mathbb{Z} \), then there exist \( \{ b_Q \}_{Q \in \mathcal{D}} \in \mathcal{F}_0^{0,q} \), a sequence of scalars \( \{ \lambda_j \} \), and a sequence of \( \infty \)-atoms \( \{ r_j, Q \} \) for \( \mathcal{F}_0^{0,q} \) such that

\[
 f_k(x) = \sum_{Q \in \mathcal{D}_k} b_Q \Psi_Q (x) = \sum_{j=1}^{\infty} \lambda_j \sum_{Q \in \mathcal{D}_k} r_{j,Q} \Psi_Q (x), \quad k \in \mathbb{Z},
\]

and

\[
 \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \lesssim \| b \|_{\mathcal{F}_p^{0,q}} \lesssim \| \{ f_k \}_{k \in \mathbb{Z}} \|_{L^p(\ell^q)}.
\]

Then by applying \( \ell^p \leftrightarrow \ell^1 \) and Minkowski’s inequality with \( q/p > 1 \), we have

\[
 \| \{ m_k^\vee \ast f_k \}_{k \in \mathbb{Z}} \|_{L^p(\ell^q)} \lesssim \left( \sum_{j=1}^{\infty} |\lambda_j|^p \right)^{1/p} \sup_{n \geq 1} \left\| \left\{ m_k^\vee \left( \sum_{Q \in \mathcal{D}_k} r_{n,Q} \Psi_Q \right) \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)}
\]

\[
 \lesssim \left\| \{ f_k \}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)} \sup_{n \geq 1} \left\| \left\{ m_k^\vee \left( \sum_{Q \in \mathcal{D}_k} r_{n,Q} \Psi_Q \right) \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)}.
\]

Therefore, it suffices to show that the supremum in the above expression is dominated by a constant times \( \mathcal{L}_s^r [m] \), which is equivalent to

\[
 \left\| \left\{ m_k^\vee \ast A_{Q_0,k} \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)} \lesssim \mathcal{L}_s^r [m] \text{ uniformly in } Q_0
\]
where \( \{ r_Q \} \) is an \( \infty \)-atom for \( f_p^{0,q} \) associated with \( Q_0 \in \mathcal{D} \) and

\[
A_{Q_0,k}(x) := \sum_{Q \in \mathcal{D}_k, Q \subset Q_0} r_Q \Psi^Q(x).
\]

Suppose \( Q_0 \in \mathcal{D}_v \) for some \( v \in \mathbb{Z} \). Then the condition \( Q \subset Q_0 \) ensures that \( A_{Q_0,k} \) vanishes unless \( v \leq k \), and thus our actual goal now is to prove

\[
\left\| \left\{ m_k^\vee \ast A_{Q_0,k} \right\}_{k \geq v} \right\|_{L^p(\ell^q)} \lesssim L'_s [m] \quad \text{uniformly in } v \text{ and } Q_0. \tag{8.1}
\]

We observe that for \( x \in \mathbb{R}^d \)

\[
\left\| \left\{ |r_Q| \|Q|^{-1/2} \chi_Q(x) \right\}_{Q \subset Q_0} \right\|_{\ell^q} \leq |Q_0|^{-1/p} \tag{8.2}
\]

and for \( 0 < t < \infty \)

\[
\left\| A_{Q_0,k} \right\|_{L^t(\mathbb{R}^d)} \lesssim \sum_{Q \in \mathcal{D}_k, Q \subset Q_0} |r_Q| \|Q|^{-1/2} \chi_Q \|_{L^t(\mathbb{R}^d)} \leq |Q_0|^{-1/p+1/t} \tag{8.3}
\]

by using the argument in (3.1) and the estimate (2.9). Moreover,

\[
\text{Supp}(\widehat{A_{Q_0,k}}) = \text{Supp}(\widehat{\Psi_k}) \subset \{ \xi : ||\xi|| \leq 2^k \}.
\]

Let \( Q_0^* \) and \( Q_0^{**} \) denote the concentric dilates of \( Q_0 \) with side length \( 9 \ell(Q_0) \) and \( 81 \ell(Q_0) \), respectively. Then we write

\[
\left\| \left\{ m_k^\vee \ast A_{Q_0,k} \right\}_{k \geq v} \right\|_{L^p(\ell^q)} \lesssim \left( \int_{Q_0^*} \left\| \left\{ m_k^\vee \ast A_{Q_0,k}(x) \right\}_{k \geq v} \right\|_{\ell^q}^pdx \right)^{1/p} + \left( \int_{(Q_0^{**})^c} \left\| \left\{ m_k^\vee \ast A_{Q_0,k}(x) \right\}_{k \geq v} \right\|_{\ell^q}^pdx \right)^{1/p}. \tag{8.4}
\]

Using Hölder’s inequality and Lemma 6.1 with \( \tau^{(s,q)} \leq \tau^{(s,p)} < r \) and

\[
|d/q - d/2| < s - (d/p - d/\min(1,q)) < d/\min(1,q),
\]

the first one is controlled by

\[
|Q_0^{**}|^{1/p-1/q} \left\| \left\{ m_k^\vee \ast A_{Q_0,k} \right\}_{k \geq v} \right\|_{L^q(\ell^q)} \lesssim \sup_{l \in \mathbb{Z}} \left\| m_l(2^l \cdot) \right\|_{L^r \cap (k-(d/p-d/\min(1,q))(\mathbb{R}^d) \mid Q_0 \mid^{1/p-1/q} \left\| A_{Q_0,k} \right\|_{L^q(\ell^q)}
\]

and we see that, from (3.1) and (8.2),

\[
\left\| \left\{ A_{Q_0,k} \right\}_{k \geq v} \right\|_{L^q(\ell^q)} \lesssim \left\{ r_Q \right\}_{Q \in \mathcal{D}, Q \subset Q_0} \left| f_q^{0,q} \right| \lesssim |Q_0|^{-1/p+1/q}.
\]
Now using the embedding $L^r_s(\mathbb{R}^d) \hookrightarrow L^r_{s-(d/p-d/\min(1,q))}(\mathbb{R}^d)$, we obtain
\[
\sup_{l \in \mathbb{Z}} \left\| m^l(\mathcal{Q}^l \cdot) \right\|_{L^r_{s-(d/p-d/\min(1,q))}(\mathbb{R}^d)} \lesssim \mathcal{L}^r_s[m],
\]
which finishes the proof of
\[
\left( \int_{Q_0^*} \left\| \{m_k^\nu * A_{Q_0,k}(x)\}_{k \geq v} \right\|_{L^p((Q_0^*)^c)}^p dx \right)^{1/p} \lesssim \mathcal{L}^r_s[m].
\]
To handle the term (8.4) we make use of the embedding $\ell^p \hookrightarrow \ell^q$ to obtain
\[
(8.4) \leq \left( \sum_{k=v}^{\infty} \left\| m_k^\nu * A_{Q_0,k} \right\|_{L^p((Q_0^*)^c)}^p \right)^{1/p}.
\]
Then, writing
\[
\left\| m_k^\nu * A_{Q_0,k} \right\|_{L^p((Q_0^*)^c)}^p \leq \left\| m_k^\nu * (A_{Q_0,k} * Q_0) \right\|_{L^p((Q_0^*)^c)}^p + \left\| m_k^\nu * (A_{Q_0,k} * (Q_0^*)^c) \right\|_{L^p((Q_0^*)^c)}^p,
\]
the proof of (8.1) will be complete once we establish the estimates that for some $\delta > 0$
\[
\left\| m_k^\nu * (A_{Q_0,k} * Q_0) \right\|_{L^p((Q_0^*)^c)} \lesssim 2^{-\delta(k-v)} \mathcal{L}^r_s[m], \tag{8.5}
\]
\[
\left\| m_k^\nu * (A_{Q_0,k} * (Q_0^*)^c) \right\|_{L^p((Q_0^*)^c)} \lesssim 2^{-\delta(k-v)} \mathcal{L}^r_s[m]. \tag{8.6}
\]
It follows from the embedding $\ell^p \hookrightarrow \ell^1$ that
\[
\left\| m_k^\nu * (A_{Q_0,k} * Q_0) \right\|_{L^p((Q_0^*)^c)} \leq \left( \sum_{Q \in \mathcal{D}_k, Q \subset Q_0^*} \int_{(Q_0^*)^c} \left| m_k^\nu * (A_{Q_0,k} * Q_0) \right|^p dx \right)^{1/p}
\]
\[
\leq \left( \sum_{Q \in \mathcal{D}_k, Q \subset Q_0^*} \left\| A_{Q_0,k} \right\|_{L^\infty(Q)} \int_{(Q_0^*)^c} \left( \int_Q \left| m_k^\nu (x - y) \right|^p dy \right)^p dx \right)^{1/p}.
\]
We notice that the assumption $r > \tau^{(s,p)}$ is equivalent to $s > d/r + d/p - d$ and therefore there exists $M > d(1-p)$ such that $s > d/r + M/p > d/r + d/p - d$. Recall that $x_0$ denotes the left lower corner of $Q \in \mathcal{D}$ and observe that for $Q \subset Q_0^*$
\[
\int_{(Q_0^*)^c} \left( \int_Q \left| m_k^\nu (x - y) \right|^p dy \right)^p dx
\]
\[
\lesssim 2^{-kM} \ell(Q_0)^{-M+d(1-p)} \left( \int_Q \int_{(Q_0^*)^c} \left( 1 + 2^k |x - x_0| \right)^{M/p} \left| m_k^\nu (x - y) \right| dx dy \right)^p
\]
\[\square\]
\[
\lesssim 2^{-k(M+pd)} \ell(Q_0)^{-M+4(d-1)p} \left( \int_{\mathbb{R}^d} \left( 1 + 2^k |y| \right)^{M/p} |m_k^\vee(y)| dy \right)^p
\]

where we utilized Hölder’s inequality if \(0 < p < 1\) and the fact that \(|x-x_Q| \lesssim |x-y|\) for \(x \in (Q_0^*)^c\) and \(y \in Q \subset Q_0^c\). Moreover, Hölder’s inequality with \(r > 1\) and the Hausdorff–Young inequality yield that

\[
\left( \int_{\mathbb{R}^d} \left( 1 + 2^k |y| \right)^{M/p} |m_k^\vee(y)| dy \right)^p = \left( \int_{\mathbb{R}^d} \left( 1 + |y|^2 \right)^{M/p} \left( m_k(2^k \cdot)^\vee(y) \right) dy \right)^p
\]

\[
\lesssim \| (1 + 4\pi^2 |y|^2)^{s/2} (m_k(2^k \cdot)^\vee(y)) \|_{L^p(\mathbb{R}^d)}^p
\]

\[
\lesssim \mathcal{L}_s^r[m].
\]

Furthermore, (3.5) proves that for \(\sigma > d/p\)

\[
\| A_{Q_0,k} \|_{L^\infty(Q)} \lesssim \inf_{y \in Q} \mathcal{M}_{\sigma,2k} A_{Q_0,k}(y) \lesssim 2^{kd/p} \| \mathcal{M}_{\sigma,2k} A_{Q_0,k} \|_{L^p(Q)}.
\]

Consequently,

\[
\| m_k^\vee \ast (A_{Q_0,k} \chi_{Q_0^*}) \|_{L^p((Q_0^*)^c)} \lesssim 2^{-(k-v)(M/p-(d/p-d))} \mathcal{L}_s^r[m] \| \mathcal{M}_{\sigma,2k} A_{Q_0,k} \|_{L^p(Q_0)}
\]

\[
\lesssim 2^{-(k-v)(M/p-(d/p-d))} \mathcal{L}_s^r[m]
\]

where we applied (2.3) with \(\sigma > d/p\) and (8.3) to obtain \(\| \mathcal{M}_{\sigma,2k} A_{Q_0,k} \|_{L^p(Q_0)} \lesssim 1\).

Then (8.5) follows with \(\delta = M/p - (d/p - d) > 0\).

To verify (8.6) we see that, similar to (7.1), under the assumption (1.5),

\[
m_k^\vee \ast (A_{Q_0,k} \chi_{Q_0^*}) = m_k^\vee \ast \psi_{k+1} \ast (A_{Q_0,k} \chi_{Q_0^*})
\]

and, it follows from Lemma 6.1 that

\[
\| m_k^\vee \ast (A_{Q_0,k} \chi_{Q_0^*}) \|_{L^p(\mathbb{R}^d)} \lesssim \mathcal{L}_s^r[m] \| \psi_{k+1} \ast (A_{Q_0,k} \chi_{Q_0^*}) \|_{L^p(\mathbb{R}^d)}.
\]

In addition, for sufficiently large \(L > 0\),

\[
\| \psi_{k+1} \ast (A_{Q_0,k} \chi_{Q_0^*}) \|_{L^p(\mathbb{R}^d)} \lesssim L \left( \int_{\mathbb{R}^d} \left( \sum_{Q \subset Q_0} |r_Q| |Q|^{-1/2} \int_{(Q_0^*)^c} |\psi_{k+1}(x-y)| \frac{1}{(1 + 2^k |y-x_Q|)^{2L}} dy \right)^p dx \right)^{1/p}
\]

\[
\lesssim 2^{-kL} \left( \sum_{Q \subset Q_0} |r_Q| |Q|^{-1/2} \left( \int_{\mathbb{R}^d} \int_{(Q_0^*)^c} |\psi_{k+1}(x-y)| \frac{1}{|y-x_Q|^{L}} dy \right)^p dx \right)^{1/p}
\]

because \(|y-x_Q| \gtrsim \ell(Q_0)| and

\[
\frac{1}{(1 + 2^k |y-x_Q|)^{2L}} \lesssim (2^k \ell(Q_0))^{-L} \left( \frac{1 + 2^k |x_Q-x_Q_0|}{(1 + 2^k |y-x_Q|)^L} \right)^L \lesssim \frac{1}{(2^k |y-x_Q|)^L}
\]
for \( y \in (Q_0^*)^c \) and \( Q \subset Q_0 \). Due to (8.2), we have

\[
\sum_{Q \in D_k, Q \subseteq Q_0} |r_Q||Q|^{-1/2} \leq 2^{vd(1/p-1)}2^{kd}
\]

and, using Hölder’s inequality (if \( p < 1 \)), we obtain that

\[
\left( \int_{\mathbb{R}^d} \left( \int_{(Q_0^*)^c} \frac{|\Psi_{k+1}(x-y)|}{|y-x|_Q^L} dy \right)^p dx \right)^{1/p} \lesssim N 2^{-kd(1/p-1)} \int_{(Q_0^*)^c} \frac{1}{|y-x|_Q^L} \int_{\mathbb{R}^d} \left( 1 + 2^k|x-x_0| \right)^{N/p} |\Psi_{k+1}(x-y)| dx dy
\]

\[
\lesssim 2^{-kd(1/p-1)}2^{kN/p} \int_{(Q_0^*)^c} \frac{1}{|y-x|_Q^{L-N/p}} dy \lesssim L^N 2^{-kd(1/p-1)}2^{kN/p}2^{v(L-N/p-d)}
\]

for \( N > d(1-p) \) and \( L - N/p > d \).

Finally, we have

\[
\| \Psi_{k+1} \ast (A_{Q_0,k}x(Q_0^*)^c) \|_{L^p(\mathbb{R}^d)} \lesssim 2^{-(k-v)(L-N/p+d/p-2d)}
\]

and this leads to (8.6) with \( \delta = L - N/p + d/p - 2d > 0 \).

### 8.2 The Case \( 0 < q \leq 1 \) and \( q < p < \infty \)

Assume \( s > d/\min(1, q) - d/2 \) and \( r > \tau^{(s,q)} \). As in the proof of Theorem 1.4, we select \( \epsilon > 0 \) so that \( s - \epsilon - d/r > d/\min(1, q) - d \).

We first consider the case \( p > d/\epsilon \). In view of Lemma 3.3 we can write

\[
\left\| \left\{ m_k^\gamma \ast f_k \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\mathbb{R})} \lesssim \sup_{P, x \in P \subseteq \mathbb{D}} \left( \frac{1}{|P|} \int_P \sum_{k=-\log_2 \ell(P)}^{\infty} |m_k^\gamma \ast f_k(y)|^q dy \right)^{1/q}
\]

Now let \( x \in P \in \mathcal{D}_\nu \) for some \( \nu \in \mathbb{Z} \) and define \( P^* = 9P \) as before. Using (7.3),

\[
\left( \frac{1}{|P|} \int_P \sum_{k=1}^{\infty} |m_k^\gamma \ast \chi_{P^*}f_k(x)|^q dy \right)^{1/q} \lesssim L_s^\nu[m] \left( \frac{1}{|P^*|} \int_{P^*} \sum_{k=1}^{\infty} |m_{\sigma^2}f_k(y)|^q dy \right)^{1/q}
\]

\[
\lesssim L_s^\nu[m] \mathcal{M}_q \left( \left\| \chi_{P^*}f_k(\cdot) \right\|_{\ell^q(\nu)} \right)(x)
\]

for \( \sigma > d/q \). Then the \( L^p \) boundedness of \( \mathcal{M}_q \) and Peetre’s maximal inequality (2.3) yield that

\( \mathcal{S} \) Springer
The proof is based on a suitable use of the complex interpolation method in Proposition 8.3.

**Case 1**

This proves that for $\sup |m_k^\vee (x)|^q \leq C$, we have

$$\left\| m_k^\vee \right\|_{\ell^p} \leq C \left\| f_k \right\|_{\ell^q}.$$ 

Furthermore, it follows from (7.4) that

$$\left( \frac{1}{|P|} \sum_{k=-\log_2 |P|}^\infty \right)^{1/q} \left\| m_k^\vee \right\|_{\ell^p} \leq C \left\| f_k \right\|_{\ell^q}.$$

Then via the $L^p$ boundedness of $M_q$, (2.3) with $\epsilon > d/p$, and the embedding $\ell^q \hookrightarrow \ell^\infty$ we have

$$\left\| m_k^\vee \right\|_{\ell^p} \leq C \left\| f_k \right\|_{\ell^q}.$$

This proves that for $d/\epsilon < p < \infty$

$$\left\| m_k^\vee \right\|_{\ell^p} \leq C \left\| f_k \right\|_{\ell^q}.$$

The general case $q < p < \infty$ follows from the interpolation method in Proposition 5.1 between (8.7) and $L^q(\ell^q)$ estimate with the same values of $s$ and $r$.

### 8.3 The Case $1 < p < \infty$ and $1 < q < \infty$

The proof is based on a suitable use of the complex interpolation method in Proposition 5.1 and the duality property in Lemma 4.1.

**Step 1** We claim that for $2 < p < \infty$, $d/2 - d/p = d/p' - d/2 < s < d$, and $r > d/s$,

$$\left\| m_k^\vee \right\|_{\ell^p} \leq C \left\| f_k \right\|_{\ell^q}.$$

Choose $\epsilon > 0$ and $\tilde{p}$ such that $s > d/r + \epsilon$ and $\max (d/\epsilon, p) < \tilde{p} < \infty$. Then, by using Lemma 3.3 and the arguments used in obtaining (8.7), we can prove that

$$\left\| m_k^\vee \right\|_{\ell^p} \leq C \left\| f_k \right\|_{\ell^q}.$$

Now (8.8) follows from the interpolation with the $L^{p'}(\ell^{p'})$ boundedness with the same values of $r$ and $s$ because $p' < p < \tilde{p}$.

**Step 2** We prove that for $1 < p < 2$, $d/p - d/2 = d/2 - d/p' < s < d$, and $r > d/s$,

$$\left\| m_k^\vee \right\|_{\ell^p} \leq C \left\| f_k \right\|_{\ell^q}.$$
Suppose that \( \{ f_k \}_{k \in \mathbb{Z}} \in L^p_{\mathcal{D}}(\ell^p) \). By using Lemma 4.1, the left-hand side of (8.9) can be dualized and estimated by

\[
\sup_{\|r_Q\|_{\mathcal{D}} \leq 1} \left| \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} m_k^* \ast f_k(x) \mathcal{M}^q_k(\|r_Q\|_{\mathcal{D}})(x) \, dx \right|
\]

which can also be written as

\[
\sup_{\|r_Q\|_{\mathcal{D}} \leq 1} \left| \int_{\mathbb{R}^d} \sum_{k \in \mathbb{Z}} f_k(x) m_k^* \ast (\mathcal{M}^q_k(\|r_Q\|_{\mathcal{D}}))(x) \, dx \right|
\]

This is clearly majorized, using Hölder’s inequality, by

\[
\left\| \{ f_k \}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^p)} \sup_{\|r_Q\|_{\mathcal{D}} \leq 1} \left\| \{ m_k^* \ast (\mathcal{M}^q_k(\|r_Q\|_{\mathcal{D}})) \}_{k \in \mathbb{Z}} \right\|_{L^{p'}(\ell^p)}
\]

Moreover, the result in Step 1 and (4.2) yield that the \( L^{p'}(\ell^p) \)-norm in the above expression is smaller than a constant times

\[
\mathcal{L}^p_f[m] \left\| \{ \mathcal{M}^q_k(\|r_Q\|_{\mathcal{D}}) \}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^p)} \lesssim \mathcal{L}^p_f[m] \left\| \{ f_k \}_{k \in \mathbb{Z}} \right\|_{f_{p',p}}.
\]

which proves (8.9).

**Step 3** Let \( 1 < p < \infty \) and \( q \) is between \( p \) and \( p' \) so that \( |d/p - d/2| > |d/q - d/2| \). Suppose \( |d/p - d/2| < s < d \) and \( r > d/s \). We interpolate two cases \((p, p')\) and \((p, p)\) by using Proposition 5.1 with the same values of \( s \) and \( r \). Then we establish the estimate

\[
\left\| \{ m_k^* \ast f_k \}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)} \lesssim \mathcal{L}^p_f[m] \left\| \{ f_k \}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)}.
\]

**Step 4** Let \( 1 < q < \infty \) and \( p \) is between \( q \) and \( q' \) so that \( |d/q - d/2| > |d/p - d/2| \). Suppose \( |d/q - d/2| < s < d \) and \( r > d/s \). We interpolate two cases \((q', q)\) and \((q, q)\) by using Proposition 5.1 with the same values of \( s \) and \( r \). Then we have the estimate

\[
\left\| \{ m_k^* \ast f_k \}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)} \lesssim \mathcal{L}^p_f[m] \left\| \{ f_k \}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)}.
\]

**Step 5** Let \( 1 < p < \infty \) and \( q = 1 \). Suppose \( d/2 < s < d \) and \( r > d/s \). An argument similar to that used in Step 2, with Lemma 4.1 and the result for \( 1 < p < \infty \) and \( q = 1 \), leads to the desired estimate. We skip the details to avoid unnecessary repetition.
9 Proof of Theorem 1.6

We now describe the proof of Theorem 1.6, using the ideas in [4,12]. Suppose $0 < p < \infty$ or $p = q = \infty$.

9.1 Necessary Conditions for Vector-Valued Operator Inequalities

We investigate necessary conditions for the inequality that for $K \in \mathcal{E}(1)$,

$$\left\| \left\{ 2^k d K(2^k \cdot) * f_k \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)} \leq A \left\| \{ f_k \}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)}, \quad f_k \in \mathcal{E}(2^{k-1})$$

(9.1)

for some $A > 0$.

An immediate consequence is that

$$\|K\|_{L^p(\mathbb{R}^d)} \lesssim p A,$$

(9.2)

which follows from setting $f_0 = 4^d \Psi_0(4 \cdot)$ and $f_k = 0$ for $k \neq 0$ so that

$$\|K\|_{L^p(\mathbb{R}^d)} = \left\| \left\{ 2^{kd} K(2^k \cdot) * f_k \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)} \leq A \|4^d \Psi_0(4 \cdot)\|_{L^p(\mathbb{R}^d)} \lesssim A.$$

Moreover, it is known, see [4], that if (9.1) holds for $0 < q \leq p < \infty$, then

$$\|K\|_{L^q(\mathbb{R}^d)} \lesssim p, q A.$$  

(9.3)

Now we consider the case $1 < p, q < \infty$. Using the dualization argument in Lemma 4.1, which was used to obtain (8.9), the $L^p(\ell^q)$ boundedness also implies that

$$\left\| \left\{ 2^k d K(2^k \cdot) * f_k \right\}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q')} \lesssim \sup_{|r_Q| \leq 4^d, \|Q\|_{Q \in \mathcal{D}} \leq 1} \left\| \left\{ 2^k d K(2^k \cdot) * (4^d \Psi_0(4 \cdot))_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q)} \right\|_{L^p(\ell^q')} \lesssim A \left\| \{ f_k \}_{k \in \mathbb{Z}} \right\|_{L^p(\ell^q')} .$$

Therefore it is clear from (9.2) that

$$\|K\|_{L^p(\mathbb{R}^d)} \lesssim p A$$

and if $1 < p \leq q < \infty$ (that is, $1 < q' \leq p' < \infty$), then we have

$$\|K\|_{L^q(\mathbb{R}^d)} \lesssim p, q A$$

from the estimate (9.3).

We note that if $K \in \mathcal{E}(1)$, then Bernstein’s inequality shows that

$$\|K\|_{L^{r_1}(\mathbb{R}^d)} \lesssim \|K\|_{L^{r_0}(\mathbb{R}^d)} \quad \text{for} \quad r_0 < r_1.$$  

(9.4)
Therefore, we conclude that the following holds

**Lemma 9.1** Let $0 < p < \infty$ and $0 < q \leq \infty$. Suppose that $K \in \mathcal{E}(1)$. If (9.1) holds, then

$$\|K\|_{L_{\min(p,q,p',q')}^\infty(\mathbb{R}^d)} \lesssim_{p,q,d} \mathcal{A}$$

where we adhere to the standard convention that $p' = \infty$ for $p \leq 1$ and $q' = \infty$ for $q \leq 1$.

On the other hand, when $p \geq 1$, (9.1) implies that the convolution operator with $K$ is bounded in $L^p(\mathbb{R}^d)$. Indeed, for any $f \in L^p(\mathbb{R}^d)$ let

$$f_0 := 4^d \Psi_0(4\cdot) * f,$$

and $f_k := 0, \ k \neq 0$.

Then using the identity $K = 4^d \Psi_0(4\cdot) * K$, we have

$$\|K * f\|_{L^p(\mathbb{R}^d)} = \|\{2^kd K(2^k \cdot) * f_k\}_{k \in \mathbb{Z}}\|_{L^p(\mathbb{R}^d)} \leq \mathcal{A}\|f_0\|_{L^p(\mathbb{R}^d)} \lesssim \mathcal{A}\|f\|_{L^p(\mathbb{R}^d)}$$

where the last inequality follows from Young’s inequality with $p \geq 1$. Hence it follows that

$$\|\hat{K}\|_{L^\infty(\mathbb{R}^d)} \lesssim \mathcal{A}.$$

By additionally assuming that $K \in \mathcal{E}(1)$ is a nonnegative function, we obtain that

$$\|K\|_{L^1(\mathbb{R}^d)} = \hat{K}(0) \leq \|\hat{K}\|_{L^\infty(\mathbb{R}^d)} \lesssim \mathcal{A},$$

and this, together with (9.4), yields the following lemma.

**Lemma 9.2** Let $0 < p < \infty$ and $0 < q \leq \infty$. Suppose that $K \in \mathcal{E}(1)$ is a nonnegative function on $\mathbb{R}^d$. If (9.1) holds, then

$$\|K\|_{L_{\min(1,p,q)}^\infty(\mathbb{R}^d)} \lesssim_{p,q,d} \mathcal{A}.$$

**9.2 Construction of Examples**

Note that $s < d/\min(1, p, q)$ implies $\min(1, p, q) < \tau(s,p,q)$. Choosing

$$t := \frac{d}{\min(1, p, q)} \quad \text{and} \quad 2^{\tau(s,p,q)} < \gamma < \frac{2}{\min(1, p, q)}, \quad (9.5)$$

we define

$$\mathcal{H}^{(t,\gamma)}(x) := \frac{1}{(1 + 4\pi^2|x|^2)^{t/2}} \frac{1}{(1 + \ln(1 + 4\pi^2|x|^2))^{\gamma/2}}.$$
Then it is proved in [12] that
\[
\left| (I - \Delta)^{s/2} \hat{\mathcal{H}}^{(t,\gamma)}(\xi) \right| = \left| \hat{\mathcal{H}}^{(t-s,\gamma)}(\xi) \right| \\
\lesssim_{t,\gamma,d} \begin{cases} 
\frac{e^{-|\xi|/2}}{|\xi|} & \text{for } |\xi| > 1 \\
(1 + 2 \ln |\xi| - \gamma/2) & \text{for } |\xi| \leq 1
\end{cases}
\]
\tag{9.6}
\]
where \(d - t + s = s - d/\min(1, p, q) + d > 0\).

Let \(\eta \in S(\mathbb{R}^d)\) have the properties that \(\eta \geq 0\), \(\eta(x) \geq c > 0\) on \(\{x \in \mathbb{R}^d : |x| \leq 1/100\}\) for some \(c > 0\), and \(\text{Supp}(\hat{\eta}) \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 1/10\}\). We define
\[
K^{(t,\gamma)}(x) := \mathcal{H}^{(t,\gamma)} * \eta(x), \quad K_k^{(t,\gamma)}(x) := 2^{kd} K^{(t,\gamma)}(2^k x),
\]
and
\[
m_k^{(t,\gamma)} := \hat{K}_k^{(t,\gamma)}.
\]

We first observe that
\[
m_k^{(t,\gamma)}(2^k \xi) = \hat{K}_k^{(t,\gamma)}(\xi) = \hat{\mathcal{H}}^{(t,\gamma)}(\xi) \hat{\eta}(\xi)
\]
and this yields
\[
\mathcal{L}_{s}^{(s,p,q)}[m] = \left\| \hat{\mathcal{H}}^{(t,\gamma)} \hat{\eta} \right\|_{L^s_{s}^{(s,p,q)}} \lesssim \left\| (I - \Delta)^{s/2} \hat{\mathcal{H}}^{(t,\gamma)} \right\|_{L^s_{s}^{(s,p,q)}(\mathbb{R}^d)}
\]
where the Kato–Ponce inequality is applied. Then using (9.6), we obtain that
\[
\mathcal{L}_{s}^{(s,p,q)}[m] \lesssim 1 + \left( \int_{|\xi| \leq 1} \frac{1}{|\xi|^{(s,p,q)(d-t+s)}} \frac{1}{(1 + 2 \ln |\xi| - \gamma/2)^{s(p,q)\gamma/2}} d\xi \right)^{1/(s,p,q)}.
\]

Using a change of variables, the second term is estimated by a constant times
\[
\int_{1}^{\infty} \frac{1}{u} \frac{1}{(1 + 2 \ln u)^{(s,p,q)\gamma/2}} du < \infty
\]
because \(\tau^{(s,p,q)(d-t+s)} = d\) and \(\tau^{(s,p,q)\gamma/2} > 1\) with the choice of \(t\) and \(\gamma\) in (9.5). Finally, we have
\[
\mathcal{L}_{s}^{(s,p,q)}[m] \lesssim 1.
\]

Now we suppose (1.4) holds with \(m_k = m_k^{(t,\gamma)}\) and \(A = 2^{-2}\), which is equivalent to (9.1) with \(K = K^{(t,\gamma)}\) and \(A = \mathcal{L}_{s}^{(s,p,q)}[m]\). Then it follows from Lemma 9.2 that
\[
\|K^{(t,\gamma)}\|_{L^{\min(1,p,q)}(\mathbb{R}^d)} \lesssim \mathcal{L}_{s}^{(s,p,q)}[m] \lesssim 1.
\tag{9.7}
\]
since \( K(t, \gamma) \) is a nonnegative function. However,

\[
\| K(t, \gamma) \|_{L^{\min(1, p, q)}(\mathbb{R}^d)} = \| \mathcal{H}(t, \gamma) \ast \eta \|_{L^{\min(1, p, q)}(\mathbb{R}^d)} \gtrsim \| \mathcal{H}(t, \gamma) \|_{L^{\min(1, p, q)}(\mathbb{R}^d)}
\]

where the inequality follows from the fact that \( \mathcal{H}(t, \gamma), \eta \geq 0 \) and \( \mathcal{H}(t, \gamma)(x - y) \geq \mathcal{H}(t, \gamma)(x) \mathcal{H}(t, \gamma)(y) \). This yields that

\[
\| K(t, \gamma) \|_{L^{\min(1, p, q)}(\mathbb{R}^d)} \gtrsim \left( \int_{\mathbb{R}^d} \frac{1}{(1 + 4\pi^2|x|^2)^{d/2}} \frac{1}{(1 + \ln(1 + 4\pi^2|x|^2))^{\nu \min(1, p, q)/2}} \, dx \right)^{1/\min(1, p, q)} = \infty,
\]

since \( \nu \min(1, p, q)/2 < 1 \), which contradicts (9.7).

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**Appendix A. Proof of (6.1)**

(6.1) is a consequence of the following lemma:

**Lemma A.1** Let \( 1 < r_0 < r_1 < \infty \) and \( s \geq 0 \). Suppose that \( f \in L^{r_0}_s(\mathbb{R}^d) \) is supported in \( \{ x \in \mathbb{R}^d : |x| \leq B \} \) for some \( B > 0 \). Then \( f \in L^{r_0}_s(\mathbb{R}^d) \) and indeed,

\[
\| f \|_{L^{r_0}_s(\mathbb{R}^d)} \lesssim_s B^{d/r_0 - d/r_1} \| f \|_{L^{r_1}_s(\mathbb{R}^d)}.
\]

**Proof** Let \( \Gamma \in S(\mathbb{R}^d) \) satisfy \( \text{Supp}(\Gamma) \subset \{ x \in \mathbb{R}^d : |x| \leq 2B \} \) and \( \Gamma(x) = 1 \) for \( |x| \leq B \). Define the multiplication operator \( T \) by

\[
T g(x) := g(x) \Gamma(x) \quad \text{for} \quad g \in S(\mathbb{R}^d).
\]

Using Hölder’s inequality and the Kato–Ponce inequality [15], we obtain that for each \( n \in \mathbb{N}_0 \),

\[
\| T g \|_{L^{r_0}_n(\mathbb{R}^d)} \lesssim B^{d/r_0 - d/r_1} \| T g \|_{L^{r_1}_n(\mathbb{R}^d)} \lesssim_n B^{d/r_0 - d/r_1} \| g \|_{L^{r_1}_n(\mathbb{R}^d)}.
\]

Then we interpolate these estimates to extend to

\[
\| T g \|_{L^{r_0}_s(\mathbb{R}^d)} \lesssim B^{d/r_0 - d/r_1} \| g \|_{L^{r_1}_s(\mathbb{R}^d)} \quad \text{(A.1)}
\]

for all \( s \geq 0 \).

Now suppose \( g \in S(\mathbb{R}^d) \) has compact support in \( \{ x \in \mathbb{R}^d : |x| \leq B \} \) so that \( g = T g \). Then (A.1) implies that

\[
\| g \|_{L^{r_0}_s(\mathbb{R}^d)} \lesssim B^{d/r_0 - d/r_1} \| g \|_{L^{r_1}_s(\mathbb{R}^d)}.
\]
from which the desired result follows, using the density of $S(\mathbb{R}^d)$ in the two Banach spaces $L^r_s(\mathbb{R}^d)$ and $L^r_t(\mathbb{R}^d)$.

References

1. Baernstein, A., Sawyer, E.T.: Embedding and Multiplier Theorems for $H^p(\mathbb{R}^n)$ Memoirs of the American Mathematical Society, vol. 318. American Mathematical Society, Providence (1985)
2. Calderón, A.P.: Intermediate spaces and interpolation, the complex method. Stud. Math. 24, 113–190 (1964)
3. Calderón, A.P., Torchinsky, A.: Parabolic maximal functions associated with a distribution. II. Adv. Math. 24, 101–171 (1977)
4. Christ, M., Seeger, A.: Necessary conditions for vector-valued operator inequalities in harmonic analysis. Proc. Lond. Math. Soc. 93(3), 447–473 (2006)
5. Fefferman, C., Stein, E.M.: Some maximal inequalities. Am. J. Math. 93, 107–115 (1971)
6. Fefferman, C., Stein, E.M.: $H^p$ spaces of several variables. Acta Math. 129, 137–193 (1972)
7. Frazier, M., Jawerth, B.: Decomposition of Besov spaces. Indiana Univ. Math. J. 34, 777–799 (1985)
8. Frazier, M., Jawerth, B.: The $\psi$-Transform and Applications to Distribution Spaces, in Function Spaces and Applications. Lecture Notes in Mathematics, vol. 1302. Springer, New York (1988)
9. Frazier, M., Jawerth, B.: A discrete transform and decomposition of distribution spaces. J. Funct. Anal. 93, 34–170 (1990)
10. Frazier, M., Jawerth, B.: Applications of the $\phi$ and Wavelet Transforms to the Theory of Function Spaces. Wavelets and Their Applications, pp. 377–417. Jones and Bartlett, Boston (1992)
11. Grafakos, L., He, D., Honzik, P., Nguyen, H.V.: The Hörmander multiplier theorem I: the linear case revisited. Ill. J. Math. 61, 25–35 (2017)
12. Grafakos, L., Park, B.: Sharp Hardy space estimates for multipliers. To appear in Int. Math. Res. Not.
13. Grafakos, L., Slavíková, L.: A sharp version of the Hörmander multiplier theorem. Int. Math. Res. Not. 15, 4764–4783 (2019)
14. Hörmander, L.: Estimates for translation invariant operators in $L_p$ spaces. Acta Math. 104, 93–140 (1960)
15. Kato, T., Ponce, G.: Commutator estimates and the Euler and Navier–Stokes equations. Commun. Pure Appl. Math. 41, 891–907 (1988)
16. Mihlin, S.G.: On the multipliers of Fourier integrals. Dokl. Akad. Nauk SSSR (N. S.) 109, 701–703 (1956). Russian
17. Park, B.: Some maximal inequalities on Triebel–Lizorkin spaces for $p = \infty$. Math. Nachr. 292, 1137–1150 (2019)
18. Park, B.: Fourier multiplier theorems for Triebel–Lizorkin spaces. Math. Z. 293, 221–258 (2019)
19. Park, B.: Boundedness of pseudo-differential operators of type $(0, 0)$ on Triebel–Lizorkin and Besov spaces. Bull. Lond. Math. Soc. 51, 1039–1060 (2019)
20. Peetre, J.: On spaces of Triebel–Lizorkin type. Ark. Mat. 13, 123–130 (1975)
21. Seeger, A.: Remarks on singular convolution operators. Stud. Math. 97(2), 91–114 (1990)
22. Slavíková, L.: On the failure of the Hörmander multiplier theorem in a limiting case. Rev. Mat. Iberoam. 36, 1013–1020 (2020)
23. Taibleson, M., Weiss, G.: The molecular characterization of certain Hardy spaces. Astérisque 77, 67–151 (1980)
24. Triebel, H.: On $L_p(l_q)$-spaces of entire analytic functions of exponential type: complex interpolation and Fourier multipliers. The case $0 < p < \infty$, $0 < q < \infty$. J. Approx. Theory 28, 317–328 (1980)
25. Triebel, H.: Complex interpolation and Fourier multipliers for the spaces $B^s_{p,q}$ and $F^s_{p,q}$ of Besov–Hardy–Sobolev type?: The case $0 < p \leq \infty$, $0 < p \leq \infty$. Math. Z. 176, 495–510 (1981)
26. Triebel, H.: Theory of Function Spaces. Birkhauser, Basel (1983)

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