Radiative Symmetry Breaking and Dynamical Origin of Cosmological Constant in $\phi^4$ Theory with Non-Linear Curvature Coupling

T Inagaki
Information Media Center, Hiroshima University,
1-7-1 Kagamiyama, Higashi-Hiroshima, Hiroshima 739-8521, JAPAN
E-mail: inagaki@hiroshima-u.ac.jp

Abstract. A scalar self-interacting theory non-linearly coupled with some power of the curvature have a possibility to explain the current smallness of the cosmological constant. In Ref.[1] a symmetry property of the model has been studied in an arbitrary dimensions and a solution of the cosmological constant problem has been found in four-dimensions. Here one concentrate on a massless scalar field in the four-dimensional Friedmann-Robertson-Walker (FRW) spacetime with flat spatial part. One apply the same analysis as Ref.[1] and show the phase structure of radiative symmetry breaking. One also review a dynamical resolution of the cosmological constant problem.

PACS numbers: 04.62.+v, 11.30.Qc, 98.80.Cq
1. Introduction

One of the most important and mysterious problem in the present cosmology is the origin of the dark energy. It has an anti-gravity contribution to accelerate the universe expansion. Even in particle physics much interest has been paid to the problem and new field theoretical models are considered to explain the acceleration of our Universe. A simple candidate of the dark energy is found in the vacuum energy or cosmological constant. However, the scale of the dark energy is about 120 orders of magnitude smaller than the Planck scale. We can not avoid fine tuning to obtain a suitable scale vacuum energy in most of particle physics models.

A interesting possibility to solve the dark energy problem is found in a modification of the gravity-matter coupling. S. Nojiri and S. D. Odintsov have proposed the class of dark energy models where dark energy field couples with some power of the curvature \( \phi^4 \) (see also related model in \[3\]). The model naturally resolves the problem of dark energy dominance in the current universe. In Ref.\[1\] the symmetry properties of such models have been studied on the example of scalar self-interacting theory with non-linear curvature coupling in arbitrary dimensions. It is found that the phase structure of the model strongly depends on the curvature power in 3.8 spacetime dimensions.

In the present paper we focus on the four-dimensional limit of the theory and continue the study of scalar self-interacting theory with non-linear curvature coupling. It should be noted that the theory is not standard one, in the sense that it is not multiplicatively renormalizable in curved spacetime \[4\]. We regards it as an effective theory steaming from a more fundamental theory at Planck scale. In our analysis we neglect the quantum gravity effects. We also assume that the spacetime curved slowly and neglect the higher order terms about curvature. It is appropriate for the study of quantum effects in the inflationary universe and also in late-time, dark energy universe.

First we introduce the scalar theory non-linearly coupled with some power of the curvature. To find the properties at the weak curvature limit we apply the Riemann normal coordinate expansion and evaluate the effective Lagrangian. The one-loop effective Lagrangian is obtained in close analogy with multiplicatively renormalizable theories. Next we consider the massless scalar field and investigate radiative symmetry breaking which is caused by the loop corrections of the scalar field. In our theory the expectation value of the scalar field corresponds to an order parameter of radiative symmetry breaking. We evaluate the effective Lagrangian numerically in four dimensions and find the phase structure of the theory in four dimensions. Finally we show the solution of the Einstein equation and discuss dynamical resolution of the cosmological constant problem.

2. \( \phi^4 \) theory with non-linear curvature coupling

As is well-known, \( \phi^4 \) theory is one of the simplest models where the spontaneous symmetry breaking takes place. It is more instructive to consider the \( \phi^4 \) theory as
a prototype model to study an influence of a non-linear curvature coupling. Here we extend the $\phi^4$ theory to include a coupling with some power of curvature:

$$S = \int d^D x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R + \left( \frac{R}{M^2} \right)^\alpha L_d \right].$$

(1)

where $M$ is an arbitrary mass scale, $g$ is the determinant of the metric tensor $g_{\mu\nu}$ and $L_d$ is the ordinary Lagrangian density of the $\phi^4$ theory,

$$L_d(\phi_0) = \frac{1}{2} \phi_{0\mu} \phi_{0\mu} - \frac{\xi_0 R}{2} \phi_0^2 + \frac{\mu_0^2}{2} \phi_0^2 - \frac{\lambda_0}{4!} \phi_0^4,$$

(2)

where $\phi_0$ is a real scalar field. Our sign convention for the metric is (+−−−⋯).

The action (1) is invariant under the discrete transformation, $\phi \to -\phi$. A non-vanishing expectation value for the field $\phi$ breaks this discrete $Z_2$ symmetry spontaneously. The expectation value for $\phi$ is found by observing the stationary point of the effective action $\Gamma[\phi]$. In a constant curvature spacetime the non-linear curvature coupling disappears by the transformation

$$\phi \to \left( \frac{R}{M^2} \right)^{-\alpha/2} \phi, \quad \lambda_0 \to \left( \frac{R}{M^2} \right)^\alpha \lambda_0.$$  

(3)

Hence a non-trivial effect of the non-linear curvature coupling will be found only in a non-static and/or inhomogeneous spacetime.

Here we adopt the Riemann normal coordinate expansion [5, 6, 7] and evaluate the one-loop effective Lagrangian. It is divergent in four dimensions. To obtain the finite result we have to renormalize the theory. Here we impose the following renormalization conditions

$$\frac{\partial^2 \Gamma[\phi]}{\partial \phi^2} \bigg|_{\phi=0} = \left( \frac{R}{M^2} \right)^\alpha (\mu_r^2 - \xi_r R), \quad \frac{\partial^4 \Gamma[\phi]}{\partial \phi^4} \bigg|_{\phi=M} = -\left( \frac{R}{M^2} \right)^\alpha \lambda_r,$$

(4)

where $M$ is the renormalization scales. From these conditions one obtains the renormalized parameters $\mu_r$, $\xi_r$ and $\lambda_r$. By using these renormalized parameters the effective Lagrangian density reads at the four-dimensional limit,

$$L_{eff}^4 = \frac{1}{2\kappa^2} R + \left( \frac{R}{M^2} \right)^\alpha \left( \frac{1}{2} \phi_{\mu} \phi^{\mu} - \frac{\xi_r R}{2} \phi^2 + \frac{\mu_r^2}{2} \phi^2 - \frac{\lambda_r}{4!} \phi^4 \right)$$

$$+ \frac{\hbar}{128\pi^2} \left[ \frac{\lambda_r M^2}{\chi^2(M^2)} - \frac{\lambda_r^2 M^4}{6 \chi^2(M^2)} \right] + \lambda_r \phi^2 \chi^2(0)$$

$$+ \frac{3}{4} \lambda_r^2 \phi^4 - 2(\chi^2(\phi))^2 \ln \frac{\chi^2(\phi^2)}{\chi^2(0)} - \frac{1}{2} \lambda_r^2 \phi^4 \ln \frac{\chi^2(0)}{\chi^2(M^2)}$$

$$\left\{ \frac{\lambda_r^2}{4} \phi^4 \left( \frac{1}{\chi^2(M^2)} - \frac{2\lambda_r M^2}{\chi^2(M^2)^2} + \frac{2}{3} \frac{\lambda_r^2 M^4}{(\chi^2(M^2))^2} \right) \right\} \text{tr}\xi.$$

(5)

with

$$\text{tr}\xi = \alpha \left( \frac{\Box R}{R} - \frac{R_{\mu\nu} R^{\mu\nu}}{R^2} \right),$$

(6)
\[ \chi^2(\phi^2) = -\mu_0^2 + \lambda_0 \phi^2 + \left(\xi_0 - \frac{1}{6}\right) R - \frac{\alpha}{2} \left(\frac{\alpha}{2} - 1\right) \frac{R^\mu R_\mu}{R^2} - \frac{\alpha}{2} \Box R, \]

Therefore the ultra-violet divergences disappear after the one-loop renormalization. Of course, the question of dependence from the regularization in such effective models remains at higher order.

3. Radiative symmetry breaking in four-dimensional FRW spacetime

It is expected that the non-linear curvature coupling in our model leads to non-trivial consequences for the phase structure in a non-static spacetime. It is more interesting to consider the radiative symmetry breaking at \( \mu_r = 0 \). In this case spontaneous symmetry breaking can not take place on the classical level of the theory. It is expected that the radiative correction plays an essential role and the curvature effect is more important for symmetry properties.

Here we consider the model in the spatially flat FRW spacetime in four dimensions. It is defined by the metric

\[ ds^2 = dt^2 - a^2(t) \left[ dr^2 + r^2 d\Omega_2 \right]. \]

The time dependence of the scale factor is assumed to be \( a(t) = a_0 t^{h_0} \). All mass scales are normalized by an arbitrary mass scale \( M \) and \( \hbar = 1 \).

First we consider the stationary and spatially homogeneous \( \phi \). In this case the kinetic term of \( \phi \) disappears. One can define the effective potential by the opposite sign of the effective Lagrangian, \( V(\phi) \equiv -L_{eff}^{4D} \). The expectation value of the field \( \phi \) is determined by observing the minimum of the effective potential. The effective potential develops a small imaginary part for a negative \( \chi(\phi) \). Below we evaluate a real part of the effective potential to find the ground state.

In Fig. 1 we illustrate the behavior of the effective potential for a conformal and a minimal gravitational coupling \( \xi = 1/6 \) and \( \xi = 0 \) respectively. It should be noted that the theory is conformally invariant only when \( \alpha = 0 \). Behaviors of the minimal of the effective potential are drawn in Figs. 2 and 3 with varying \( \lambda \) and \( h_0 \). First order phase transition is observed in Figs. 2 and 3. However, the radiative correction has only a small effect in an ordinary theory. The expectation value generated by the radiative symmetry breaking is extremely small at \( \alpha = 0 \), as is shown in Fig. 2. For a positive \( \alpha \) the non-linear curvature coupling enhances radiative corrections as curvature decreases. Thus the expectation value \( \langle \phi \rangle \) becomes extremely larger in comparison with that for the case \( \alpha = 0 \). In Fig. 3 we observe two steps of the transition at \( h_0 = 1.2 \). In the case of negative \( \alpha \) the radiative correction is suppressed as the curvature decreases and then radiative symmetry breaking does not occur.

As is shown in Figs. 2 and 3, the expectation value \( \langle \phi \rangle \) has non-negligible time dependence, especially for \( \alpha = 1 \). Hence we next consider the spatially homogeneous but time dependent \( \phi \) at \( \alpha = 1 \). The time evolution of \( \langle \phi \rangle \) is described by the equation
of motion:
\[ \frac{1}{\sqrt{-g}} \delta S = - \frac{\partial V(\phi)}{\partial \phi} - \left( \frac{R}{M^2} \right)^\alpha \left[ \dddot{\phi} + \frac{3}{a} \dot{\phi} - \frac{2\alpha}{t} \dot{\phi} \right] = 0. \]  
(9)

To find an exact solution one needs to solve this equation for a general time-dependent form of $\phi$. However, it is instructive to consider the solution of Eq. (9) for a special form of $\phi$. In the present paper it is assumed that

\[ \phi(t) = \langle \phi(t) \rangle = vt^x, \quad \phi^{\mu\nu} \phi_{\mu\nu} = \frac{x^2}{t^2} \langle \phi(t) \rangle^2, \]  
(10)

where $v$ is a constant parameter. Here we fix the parameter $x$ at 1/2 or 1 and numerically solve the equation of motion (9).

The solution is shown in Fig. 4. As is seen in Fig. 4, we observe the first order phase transition again. In this case the mass scale $v$ depends on the time $t$ again. However, we observe that the mass scale $v$ is almost static around $tM \sim 60$ for $x = 1$ and larger $t$ for $x = 1/2$. It is expected that there is a solution with gradually decreasing $x$ after the first order transition. It is also interesting that the $\xi$ dependence of $v$ is smaller than results for a stationary $\phi$. 

Figure 1. Behaviour of the effective potential for $h_0 = 2$, $\mu_r = 0$, $\lambda = 1$ and $D = 4$. 

(a) $\alpha = 0$, $\xi = \xi_{\text{conformal}}$  
(b) $\alpha = 1$, $\xi = \xi_{\text{conformal}}$  
(c) $\alpha = 0$, $\xi = 0$  
(d) $\alpha = 1$, $\xi = 0$
\( \phi^4 \) Theory with Non-Linear Curvature Coupling

(a) \( \alpha = 0, \xi = \xi_{\text{conformal}} \)

(b) \( \alpha = 1, \xi = \xi_{\text{conformal}} \)

(c) \( \alpha = 0, \xi = 0 \)

(d) \( \alpha = 1, \xi = 0 \)

Figure 2. Behaviour of the mass gap for \( h_0 = 2, \mu_r = 0 \) and \( D = 4 \).

(a) \( \xi = \xi_{\text{conformal}} \)

(b) \( \xi = 0 \)

Figure 3. Behaviour of the mass gap for \( \alpha = 1, \mu_r = 0, \lambda = 1 \) and \( D = 4 \).
4. Resolution of the cosmological constant problem

There are some proposal for solving the cosmological constant problem dynamically \[8, 9\]. One of the possible solutions was pointed out by Mukouyama and Randall \[10\]. Here we consider the scalar theory non-linearly coupled with the curvature (1) in the four dimensional FRW metric with flat spatial part. We apply the analysis by Mukouyama and Randall to our model. A solvable case is found at \(\alpha = -1\) \[1\].

As is shown in the previous section, the radiative correction is suppressed when the curvature is small for a negative \(\alpha\). The behavior of the scalar field \(\phi\) is found by solving the Einstein equation and the field equation for \(\phi\) simultaneously. A solution of these equations is given by

\[
H = \frac{h_0}{t}, \quad \phi = \frac{\phi_0}{t}, \quad (h_0 > 0), \quad \text{or} \quad H = \frac{h_0}{t_s - t}, \quad \phi = \frac{\phi_0}{t_s - t}, \quad (h_0 < 0). \quad (11)
\]

Substituting (11) into the Einstein equation and the field equation, we find the solution:

\[
\phi_0^2 = \frac{3}{\kappa^2 \left\{ \frac{8 - 9h_0}{24 (-h_0 + h_0^2)^2} - \frac{(4 - 7h_0) \xi}{(-h_0 + 2h_0^2) h_0} \right\}}, \quad (12)
\]
\[ \phi^4 \text{ Theory with Non-Linear Curvature Coupling} \]

\[ \lambda = -6\kappa^2 h_0 \left\{ 1 - 2 (1 - 2h_0) \xi \right\} \times \left\{ \frac{8 - 9h_0}{24(-h_0 + h_0^2)^2} - \frac{(4 - 7h_0) \xi}{(-h_0 + 2h_0^2) h_0} \right\}. \]  

(13)

For example we consider a minimal coupling case, \( \xi = 0 \), here. Since \( \phi_0^2 \) should be positive, one finds \( h_0 \leq 9/8 \).

If we choose \( h_0 = -1/60 \), it gives the state equation parameter \( w \) as

\[ w = -1 + \frac{2}{3h_0} = -1.025, \]  

(14)

It is consistent with the observed one. Therefore, with the proper choice of parameters we obtain the solution for the Einstein equation and the field equation.

As is clearly seen the Hubble rate \( H \) is suppressed as time runs. If we substitute the present age of the Universe \( 10^{10} \) year into \( t \) or \( t_s - t \), the observed value of \( H \) could be reproduced. It explains the smallness of the effective cosmological constant \( \Lambda \sim H^2 \). Then by properly choosing the parameters, we may obtain an exact solution for cosmological constant.

5. Concluding remarks

We have investigated the radiative symmetry breaking and discuss the dynamical resolution of the cosmological constant problem in the scalar self-interacting theory non-linearly coupled with some power of the curvature. We numerically evaluate the one-loop effective Lagrangian in the four dimensional FRW spacetime with flat spatial part at the weak curvature limit. The phase structure strongly depends on the sign of \( \alpha \). The \( \xi \) dependence of it is very weak. For a non-negative \( \alpha (\geq 0) \) we observed the first-order phase transition. Compared with the usual \( \phi^4 \) theory in weakly curved space, i.e. \( \alpha = 0 \), the expectation value \( \langle \phi \rangle \) is extremely enhanced for a positive \( \alpha \). In the case of a negative \( \alpha \) the radiative correction is suppressed as curvature decreases. No radiative symmetry breaking is observed for \( \alpha < 0 \). We apply the mechanism proposed in Ref. [10] to our model with negative \( \alpha \). Dynamical mechanism to solve the cosmological constant problem is naturally realized also for the class of models investigated in Ref. [1].

As the simplest model we consider radiative symmetry breaking of a discrete \( Z_2 \) symmetry. Breaking of the discrete symmetry must construct a domain wall structure in our universe. It is only a prototype model of the dark energy. We do not expect to explain all the problem in this simplest model. It is straightforward to perform the same analysis in a complex scalar theory which has a continuous \( U(1) \) symmetry. Then we can avoid the domain wall problem and find the same behaviors.

There are many directions where our approach may be generalized. In particularly, it is known that phase structure in the NJL-like model in curved spacetime is quite rich (for a review, see [11]). It is quite interesting to consider models with fermions and gauge bosons. From another point of view, the new matter-gravity coupling [2] may also be considered as a kind of modification of gravitation itself. It should be interesting to
calculate the one-loop effective action when the extreme gravity is formulated in Palatini form. That attracts us to further research works continuously.

Acknowledgments

The main part of this paper is based on the work [1]. The author benefited a lot from discussions with S. Nojiri and S. D. Odintsov.

References

[1] Inagaki T, Nojiri S and Odintsov S D 2005 JCAP 0506(2005)010.
[2] Nojiri S and Odintsov S D 2004 Phys. Lett. B 599 137.
[3] Abdalla M C B, Nojiri S and Odintsov S D 2005 Class. Quantum Grav. 22 L35.
[4] Buchbinder I L, Odintsov S D and Shapiro I L 1992 Effective Action in Quantum Gravity (IOP Publishing)
[5] Petrov A Z 1969 Einstein Space, (Pergamon, Oxford)
[6] Bunch T S and Parker L 1979 Phys. Rev. D 20 2499.
[7] Parker L and Toms D J 1984 Phys. Rev. D 29 1584.
[8] Dolgov A D and Kawasaki M 2003 Preprint astro-ph/0307442.
[9] Jackiw R, Nunez C and Pi S -Y 2005 Phys. Lett. A 347 47.
[10] Mukohyama S and Randall L 2004 Phys. Rev. Lett. 92 211302.
[11] Inagaki T, Muta T and Odintsov S D 1997 Prog. Theor. Phys. Suppl. 127 93.