REPRESENTATION FORMULAE AND MONOTONICITY OF THE GENERALIZED $k$-BESSEL FUNCTIONS

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Abstract. This paper introduces and studies a generalization of the $k$-Bessel function of order $\nu$ given by

$$W_{\nu,c}^k(x) := \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k(rk + \nu + k)} \frac{x^{2r+k}}{r!}.$$  

Representation formulae are derived for $W_{\nu,c}^k$. Further the function $W_{\nu,c}^k$ is shown to be a solution of a second order differential equation. Monotonicity and log-convexity properties for the generalized $k$-Bessel function $W_{\nu,c}^k$ are investigated, particularly in the case $c = -1$. Several inequalities, including the Turán-type inequality are established.

1. Introductions

Motivated with the repeated appearance of the expression

$$x(x+k)(x+2k) \ldots (x+(n-1)k)$$

in the combinatorics of creation and annihilation operators [7, 8] and the perturbative computation of Feynman integrals, see [3], a generalization of the well-known Pochhammer symbols is given in [6] as

$$(x)_{n,k} := x(x+k)(x+2k) \ldots (x+(n-1)k),$$

for all $k > 0$ and called it as Pochhammer $k$-symbol. The closely associated functions relate with the Pochhammer symbols are the gamma and beta functions. Thus it is natural to introduce about $k$-gamma and $k$-beta function. The $k$-gamma functions, denoted as $\Gamma_k$, is studied in [6], and defined by

$$\Gamma_k(x) := \int_0^\infty t^{x-1} e^{-\frac{t}{k}} dt, \quad (1.1)$$

for Re$(x) > 0$. Several properties of the $k$-gamma functions and it’s applications to generalize other related functions like as $k$-beta functions, $k$-digamma functions, can be seen in the articles [6, 16, 17] and references therein.

The $k$-digamma functions defined by $\Psi_k := \Gamma'_k/\Gamma_k$ is studied in [17]. This functions have the series representation as

$$\Psi_k(t) := \frac{\log(k) - \gamma_1}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk + t)} \quad (1.2)$$

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where $\gamma_1$ is the Euler-Mascheroni constant.

A calculation yields

$$
\Psi_k'(t) = \sum_{n=0}^{\infty} \frac{1}{(nk + t)^2}, \quad k > 0 \quad \text{and} \quad t > 0.
$$

(1.3)

Clearly, $\Psi_k$ is increasing on $(0, \infty)$.

The Bessel function of order $p$ is given by

$$
J_p(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k + p + 1) \Gamma(k + 1)} \left(\frac{x}{2}\right)^{2k+p},
$$

(1.4)

is a particular solution of the Bessel differential equation

$$
x^2 y''(x) + xy'(x) + (x^2 - p^2)y(x) = 0.
$$

(1.5)

Here $\Gamma$ denotes the gamma function. A solution of the modified Bessel equation

$$
x^2 y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = 0.
$$

(1.6)

yields the modified Bessel function

$$
I_\nu(x) := \sum_{k=0}^{\infty} \frac{1}{\Gamma(k + \nu + 1) \Gamma(k + 1)} \left(\frac{x}{2}\right)^{2k+\nu}.
$$

(1.7)

The Bessel function has gone through several generalizations and investigations, notably in [3, 9]. In [3], generalized Bessel function defined on the complex plane, and obtained sufficient conditions for it to be univalent, starlike, close-to-convex, and convex. This generalization is given by the power series

$$
W_{p,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k}{\Gamma(k + \frac{3}{2}) \Gamma(k + p + \frac{b+2}{2})} \left(\frac{z}{2}\right)^{2k+b}, \quad p, b, c \in \mathbb{C}.
$$

(1.8)

In this article we will consider the function defined by the series

$$
W_{p,c}(x) := \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(rk + \nu + k)r!} \left(\frac{x}{2}\right)^{2r+k}.
$$

(1.9)

where $k > 0$, $\nu > -1$, and $c \in \mathbb{R}$. Since for $k \rightarrow 1$, the $k$-Bessel functions $W_{1,c}$ reduce to the classical Bessel function $J_{\nu}$, while $W_{0,-1}$ is equivalent to the modified Bessel function $I_{\nu}$.

Thus, we call the function $W_{p,c}$ as the generalized $k$-Bessel functions. The basic properties about the $k$-Bessel functions can be seen in the work by Ghelot et. al. [10–12].

Turán [18] proved that the Legendre polynomials $P_n(x)$ satisfy the determinantal inequality

$$
\begin{vmatrix}
P_n(x) & P_{n+1}(x) \\
P_{n+1}(x) & P_{n+2}(x)
\end{vmatrix} \leq 0, \quad -1 \leq x \leq 1
$$

(1.10)

where $n = 0, 1, 2, \ldots$ and equality occurs only if $x = \pm 1$. The above classical result has been extended in several directions, for example, ultraspherical polynomials, Laguerre and Hermite polynomials, Bessel functions of the first kind, modified Bessel functions, Polygamma etc. Karlin and Szeg [13] named determinants such in (1.10) as Turánians.
In Section 2, representation formulae and few recurrence relation for \( W_{\nu,c}^k \) will be derived. More importantly, the function \( W_{\nu,c}^k \) is shown to be a solution of a certain differential equation of second order, which reduces to (1.5) and (1.6) in the case \( k = 1 \) and for particular values of \( c \). At the end of the Section 2 two type integral representations of \( W_{\nu,c}^k \) are also given.

Section 3 is devoted to the investigation of monotonicity and log-convexity properties involving the function \( W_{\nu,c}^k \), as well as the ratio between two \( k \)-Bessel functions of different order. As a consequence, Turán-type inequalities are deduced.

2. REPRESENTATIONS FOR THE \( k \)-BESSEL FUNCTION

2.1. DIFFERENTIAL EQUATION. In this section we will find the differential equations corresponding to the functions \( W_{\nu,c}^k \).

Differentiating both side of (1.9) with respect to \( x \), it follows that

\[
x d \frac{d}{dx} W_{\nu,c}^k(x) = \sum_{r=0}^{\infty} \frac{(-c)^r (2r + \nu + k)}{\Gamma_k(rk + \nu + k)r!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{k}}.
\] (2.1)

Recall that the \( k \)-gamma function satisfy the relation \( \Gamma_k(z+k) = z\Gamma_k(z) \). Now differentiate (2.1) with respect to \( x \) and then using this property yields

\[
x^2 \frac{d^2}{dx^2} W_{\nu,c}^k(x) + \frac{d}{dx} W_{\nu,c}^k(x)
\]

\[
= \sum_{r=0}^{\infty} \frac{(-c)^r (2r + \nu + k)^2}{\Gamma_k(rk + \nu + k)r!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{k}}
\]

\[
= \sum_{r=0}^{\infty} \frac{(-c)^r 4r (r + \nu)(\nu + k)}{\Gamma_k(rk + \nu + k)r!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{k}} + \frac{\nu^2}{k^2} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma_k(rk + \nu + k)r!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{k}}
\]

\[
= \frac{4}{k^2} \sum_{r=1}^{\infty} \frac{(-c)^r}{\Gamma_k(rk + \nu)(r-1)!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{k}} + \frac{\nu^2}{k^2} W_{\nu,c}^k(x)
\]

Thus we have the following result.

**Proposition 2.1.** Let \( k > 0 \) and \( \nu > -k \). Then the function \( W_{\nu,c}^k \) is the solution of the homogeneous differential equation

\[
x^2 \frac{d^2y}{dx^2} + x^{-1} \frac{dy}{dx} + \frac{1}{k^2} \left( c k - \frac{\nu^2}{x^2} \right) y = 0.
\] (2.2)

2.2. RECURRENCE RELATIONS. From (2.1), we have

\[
x \frac{d}{dx} W_{\nu,c}^k(x) = \frac{1}{k} \sum_{r=0}^{\infty} \frac{(-c)^r (2rk + \nu)}{\Gamma_k(rk + \nu + k)r!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{k}}
\]

\[
= \frac{\nu}{k} \sum_{r=0}^{\infty} \frac{(-c)^r (2rk + \nu)}{\Gamma_k(rk + \nu + k)r!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{k}} + 2 \sum_{r=1}^{\infty} \frac{(-c)^r}{\Gamma_k(rk + \nu + k)(r-1)!} \left( \frac{x}{2} \right)^{2r + \frac{\nu}{k}}
\]
Thus, we have the difference equation

\[ x \frac{d}{dx} W_{\nu,c}^k(x) = \frac{\nu}{k} W_{\nu,c}^k(x) - x c W_{\nu+k,c}^k(x). \]  

(2.3)

Again rewrite (2.1) as

\[
x \frac{d}{dx} W_{\nu,c}^k(x) = \frac{1}{k} \sum_{r=0}^{\infty} \frac{(-c)^r (2r+2\nu-k)}{\Gamma(rk+\nu+k)!} \left( \frac{x}{2} \right)^{2r+\frac{k}{2}} \]

\[
= -\frac{\nu}{k} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(rk+\nu+k)!} \left( \frac{x}{2} \right)^{2r+\frac{k}{2}} + 2 \sum_{r=0}^{\infty} \frac{(-c)^r (r+\nu+k)}{\Gamma(rk+\nu+k)!} \left( \frac{x}{2} \right)^{2r+\frac{k}{2}} 
\]

\[
= -\frac{\nu}{k} W_{\nu,c}^k(x) + \frac{x}{k} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(rk+\nu+k)!} \left( \frac{x}{2} \right)^{2r+\frac{k}{2}} 
\]

This gives us the second difference equation as

\[
x \frac{d}{dx} W_{\nu,c}^k(x) = \frac{x}{k} W_{\nu-k,c}^k(x) - \frac{\nu}{k} W_{\nu,c}^k(x). \]  

(2.4)

Thus (2.3) and (2.4) leads to the following recurrence relations.

**Proposition 2.2.** Let \( k > 0 \) and \( \nu > -k \). Then

\[
2\nu W_{\nu,c}^k(x) = x W_{\nu-k,c}^k(x) + x c W_{\nu+k,c}^k(x), \]  

(2.5)

\[
W_{\nu-k,c}^k(x) = \frac{2}{x} \sum_{r=0}^{\infty} (-1)^r (\nu+2rk) W_{\nu+2rk,c}^k(x). \]  

(2.6)

\[
\frac{d}{dx} \left( x^{\nu} W_{\nu,c}^k(x) \right) = \frac{x^{\nu}}{k} W_{\nu-k,c}^k(x) \]  

(2.7)

\[
\frac{d}{dx} \left( x^{-\nu} W_{\nu,c}^k(x) \right) = -cx^{-\nu} W_{\nu+k,c}^k(x) \]  

(2.8)

\[
\frac{d^m}{dx^m} \left( W_{\nu,c}^k(x) \right) = \frac{1}{2m^k} \sum_{n=0}^{m} (-1)^n \binom{m}{n} c^n k^m W_{\nu-mk+2nk,c}^k(x) \quad \text{for all} \quad m \in \mathbb{N}. \]  

(2.9)

**Proof.** The relation (2.5) follows by subtracting (2.4) from (2.3).

Next to establish (2.6), let's rewrite (2.5) as

\[
W_{\nu-k,c}^k(x) + c k W_{\nu+k,c}^k(x) = 2 \frac{\nu}{x} W_{\nu,c}^k(x). \]  

(2.10)

Now multiply both side of (2.10) by \(-ck\) and replace \( \nu \) by \( \nu + 2k \). Then we have

\[
-ck W_{\nu+k,c}^k(x) - c^2 k^2 W_{\nu+3k,c}^k(x) = -2ck \frac{\nu + 2k}{x} W_{\nu+2k,c}^k(x). \]  

(2.11)
Similarly, multiplying both side of (2.10) by $c^2 k^2$ and replacing $\nu$ by $\nu + 4k$, we get
\[
c^2 k^2 W_{\nu+3k,c}(x) + c^3 k^3 W_{\nu+5k,c}(x) = 2c^2 k^2 \nu + 4k \frac{d}{dx} W_{\nu+4k,c}(x).
\] (2.12)

If we continue like the above and addition them leads to (2.6).

From the definition (1.9) it is clear that
\[
x^\nu W_{\nu,c}(x) = \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(rk + \nu + k)} \frac{2^{2r+\nu}}{r!} (x)^{2r+\nu}. \tag{2.13}
\]
The derivative of (2.13) with respect to $x$ yields
\[
\frac{d}{dx} (x^\nu W_{\nu,c}(x)) = \sum_{r=0}^{\infty} \frac{(-c)^r (2r + \nu) x^{2r+\nu-1}}{\Gamma(rk + \nu + k) 2^{2r+\nu} r!} (x)^{2r+\nu-1} = \frac{x^{\nu+1}}{k} W_{\nu-k,c}(x).
\]

Similarly,
\[
\frac{d}{dx} (x^{-\frac{\nu}{k}} W_{\nu,c}(x)) = \sum_{r=0}^{\infty} \frac{(-c)^r 2^r x^{2r+\nu-1}}{\Gamma(rk + \nu + k)(r+1)! 2^{2r+\nu} r!} (x)^{2r+\nu-1} = c x^{-\frac{\nu}{k}} W_{\nu+k,c}(x).
\]

The identity (2.19) can be proved by using mathematical induction on $m$. Recall that
\[
\binom{r}{k} = \binom{r}{0} = 1 \quad \text{and} \quad \binom{r}{n} + \binom{r}{n-1} = \binom{r+1}{n}.
\]
For $m = 1$, the proof of the identity (2.9) is equivalent to show that
\[
2k \frac{d}{dx} W_{\nu,c}(x) = W_{\nu-k,c}(x) - c k W_{\nu+k,c}(x). \tag{2.14}
\]
This above relation can be obtained by simple adding (2.3) and (2.4). Thus, identity (2.9) hold for $m = 1$.

Assume that the identity (2.9) also holds for any $m = r \geq 2$, i.e.
\[
\frac{d^r}{dx^r} (W_{\nu,c}(x)) = \frac{1}{2^m k^m} \sum_{n=0}^{r} (-1)^n \binom{r}{n} c^n k^n W_{\nu-rk+2nk,c}(x).
\]

This implies for $m = r + 1$,
\[
\frac{d^{r+1}}{dx^{r+1}} (W_{\nu,c}(x)) = \frac{1}{2^{r+1} k^{r+1}} \sum_{n=0}^{r} (-1)^n \binom{r}{n} c^n k^n \frac{d}{dx} W_{\nu-rk+2nk,c}(x).
\]
\[
= \frac{1}{2^{r+1} k^{r+1}} \sum_{n=0}^{r} (-1)^n \binom{r}{n} c^n k^n \left( W_{\nu-(r+1)k+2nk,c}(x) - c k W_{\nu-(r+1)k+2nk,c}(x) \right).
\]
Owing to [6], we have the identity \( \Gamma_k \) is defined by
\[
\Gamma_k = \int_0^1 t^{k-1}(1-t)^{\frac{\nu}{r}-1} dt.
\]

Now (1.9) and (2.18) together yield
\[
\mathcal{W}_{\nu,c}^k(x) = \frac{2}{\Gamma_k(\nu)} \left( \frac{x}{2} \right)^\frac{x}{r} \int_0^1 t(1-t^2)^{\frac{x}{2r}-1} \sum_{r=0}^{\infty} \frac{(-c)^r}{\Gamma(r+1)r!} \left( \frac{xt}{2\sqrt{k}} \right)^{2r} dt.
\]

where \( \mathcal{W}_{p,b,c} \) is defined in (1.8).
Now for the second integral representation, substitute $x = r + \frac{k}{2}$ and $y = \nu + \frac{k}{2}$ in (2.16). Then, (2.17) can be rewrite as

\[
\frac{1}{\Gamma_k(rk + \nu + k)} = \frac{2}{\Gamma_k\left((r + \frac{1}{2})k\right)} \times \frac{2}{\Gamma_k\left((\nu + \frac{k}{2})\right)} \int_0^1 t^{2r} (1 - t^2)^{\frac{\nu + k}{2}} dt.
\] (2.20)

Again the identity $\Gamma_k(kx) = k^{x-1} \Gamma(x)$ yields

\[
\Gamma_k\left(\left(r + \frac{1}{2}\right)k\right) = k^{r-\frac{1}{2}} \Gamma\left(r + \frac{1}{2}\right).
\] (2.21)

Further, the Legendre duplication formula (see [1, 2])

\[
\Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = \frac{2^{1-2z} \sqrt{\pi} \Gamma(2z)}{(2r)!}
\] (2.22)

shows that

\[
\Gamma\left(r + \frac{1}{2}\right) r! = r \Gamma\left(r + \frac{1}{2}\right) \Gamma(r) = \frac{\sqrt{\pi} (2r)!}{2^{2r}}.
\]

This along with (2.20) and (2.21) reduce the series of $W_{k,c}^\nu(x)$ as

\[
W_{k,c}^\nu(x) = \frac{2\sqrt{k}}{\Gamma_k(\nu + \frac{k}{2})} \left(\frac{x}{2}\right)^{\frac{x}{2}} \int_0^1 (1 - t^2)^{\frac{\nu + k}{2} - \frac{3}{4}} \sum_{r=0}^\infty \frac{(-c)^r}{\Gamma(r+1)r!} \left(\frac{xt}{2\sqrt{k}}\right)^{2r} dt.
\] (2.23)

Finally for $c = \pm \alpha^2$, $\alpha \in \mathbb{R}$, the representation (2.23) respectively leads to

\[
W_{k,\alpha^2}^\nu(x) = \frac{2\sqrt{k}}{\sqrt{\pi} \Gamma_k(\nu + \frac{k}{2})} \left(\frac{x}{2}\right)^{\frac{x}{2}} \int_0^1 (1 - t^2)^{\frac{\nu + k}{2} - \frac{3}{4}} \sum_{r=0}^\infty \frac{(-\alpha)^r}{(2r)!} \left(\frac{xt}{\sqrt{k}}\right)^{2r} dt.
\] (2.24)

and

\[
W_{k,-\alpha^2}^\nu(x) = \frac{2\sqrt{k}}{\sqrt{\pi} \Gamma_k(\nu + \frac{k}{2})} \left(\frac{x}{2}\right)^{\frac{x}{2}} \int_0^1 (1 - t^2)^{\frac{\nu + k}{2} - \frac{1}{4}} \cosh\left(\frac{\alpha xt}{\sqrt{k}}\right) dt.
\] (2.25)

If $\nu = \frac{k}{2}$, then from (2.24) a computation give the relation between sine functions and generalized $k$-Bessel functions by

\[
\sin\left(\frac{\alpha x}{\sqrt{k}}\right) = \frac{\alpha}{k} \sqrt{\frac{\pi x}{2}} W_{k,\alpha^2}^\nu(x).
\]

Similarly, the relation

\[
\sinh\left(\frac{\alpha x}{\sqrt{k}}\right) = \frac{\alpha}{k} \sqrt{\frac{\pi x}{2}} W_{k,-\alpha^2}^\nu(x)
\]

can be derive from (2.25).
3. Monotonicity and log-convexity properties

This section is devoted for the discussion of the monotonicity and the log-convexity properties of the modified \( k \)-Bessel function \( \mathcal{I}_{\nu}^k = \Gamma_{\nu}^k \). As consequences of those results, we derive several functional inequalities for \( \Gamma_{\nu}^k \).

The following result of Biernacki and Krzyż [4] will be required.

**Lemma 3.1.** Consider the power series \( f(x) = \sum_{k=0}^{\infty} a_k x^k \) and \( g(x) = \sum_{k=0}^{\infty} b_k x^k \), where \( a_k \in \mathbb{R} \) and \( b_k > 0 \) for all \( k \). Further suppose that both series converge on \( |x| < r \). If the sequence \( \{a_k/b_k\}_{k \geq 0} \) is increasing (or decreasing), then the function \( x \mapsto f(x)/g(x) \) is also increasing (or decreasing) on \( (0, r) \).

The above lemma still holds when both \( f \) and \( g \) are even, or both are odd functions.

We now state and prove our main results in this section. Consider the functions

\[
\mathcal{I}_{\nu}^k(x) := \left( \frac{2}{\pi} \right)^{\frac{k}{2}} \Gamma_k(\nu + k) \Gamma_{\nu}^k(x) = \sum_{r=0}^{\infty} f_r(\nu)x^{2r},
\]

where,

\[
f_r(\nu) = \frac{\Gamma_k(\nu + k)}{\Gamma_k(rk + \nu + k)4^r r!}.
\]

Then we have the following properties.

**Theorem 3.1.** Let \( k > 0 \). Following results are true for the modified \( k \)-Bessel functions.

(a) If \( \nu \geq \mu > -k \), then the function \( x \mapsto \mathcal{I}_{\nu}^k(x)/\mathcal{I}_{\mu}^k(x) \) is increasing on \( \mathbb{R} \).

(b) The function \( \nu \mapsto \mathcal{I}_{\nu}^k(x)/\mathcal{I}_{\nu + k}^k(x) \) is increasing on \( (-k, \infty) \) that is, for \( \nu \geq \mu > -k \), the inequality

\[
\mathcal{I}_{\nu}^k(x) \mathcal{I}_{\nu + k}^k(x) \geq \mathcal{I}_{\nu + k}^k(x) \mathcal{I}_{\nu}^k(x)
\]

holds for each fixed \( x > 0 \) and \( k > 0 \).

(c) The function \( \nu \mapsto \mathcal{I}_{\nu}^k(x) \) is decreasing and log-convex on \( (-k, \infty) \) for each fixed \( x > 0 \).

**Proof.** (a). From \([3.1]\) it follows that

\[
\frac{\mathcal{I}_{\nu}^k(x)}{\mathcal{I}_{\mu}^k(x)} = \sum_{r=0}^{\infty} \frac{f_r(\nu)x^{2r}}{f_r(\mu)x^{2r}}.
\]

Denote \( w_r := f_r(\nu)/f_r(\mu) \). Then

\[
w_r = \frac{\Gamma_k(\nu + k)\Gamma_k(\mu + k + k)}{\Gamma_k(\mu + k)\Gamma_k(\nu + k + k)}.
\]

Now using the properties \( \Gamma_k(y + k) = y \Gamma_k(y) \) it can be shown that

\[
\frac{w_{r+1}}{w_r} = \frac{\Gamma_k(rk + \nu + k)\Gamma_k(\nu + k + 2k)}{\Gamma_k(rk + \mu + k)\Gamma_k(\nu + k + 2k)} \leq 1,
\]

in view of \( \nu \geq \mu > -k \). Now the result follows from the Lemma 3.1.
(b). Let $\nu \geq \mu > -k$. It follows from part (a) that
\[
\frac{d}{dx} \left( \frac{I^k_\nu(x)}{I^k_\mu(x)} \right) \geq 0
\]
on $(0, \infty)$. Thus
\[
(I^k_\nu(x))^' (I^k_\mu(x)) - (I^k_\nu(x))^' (I^k_\mu(x)) \geq 0.
\]
It now follows from (2.8) that
\[
\frac{x}{2} \left( I^k_{\nu+k}(x) - I^k_{\mu+k}(x) I^k_\mu(x) \right) \geq 0,
\]
whence $I^k_{\nu+k}/I^k_\nu$ is increasing for $\nu > -k$ and for some fixed $x > 0$.

(c). It is clear that for all $\nu > -k$,
\[
f_r(\nu) = \frac{\Gamma_k(\nu + k)}{\Gamma_k(rk + \nu + k)4^r r!} > 0.
\]
A logarithmic differentiation of $f_r(\nu)$ with respect to $\nu$ yields
\[
\frac{f'_r(\nu)}{f_r(\nu)} = \Psi_k(\nu + k) - \Psi_k(rk + \nu + k) \leq 0,
\]
in view of the fact that the $\Psi_k$ is increasing functions on $(-k, \infty)$. This implies that $f_r(\nu)$ is decreasing.

Thus for $\mu \geq \nu > -k$, it follows that
\[
\sum_{r=0}^\infty f_r(\nu)x^{2r} \geq \sum_{r=0}^\infty f_r(\mu)x^{2r},
\]
which is equivalent to say that the function $\nu \mapsto I^k_\nu$ is decreasing on $(-k, \infty)$ for some fixed $x > 0$.

The twice logarithmic differentiation of $f_r(\nu)$ yields
\[
\frac{\partial^2}{\partial \nu^2} \left( \log(f_r(\nu)) \right) = \Psi'_k(\nu + k) - \Psi'_k(rk + \nu + k)
\]
\[
= \sum_{n=0}^\infty \left( \frac{1}{(nk + \nu + k)^2} - \frac{1}{(nk + rk + \nu + k)^2} \right)
\]
\[
= \sum_{n=0}^\infty \frac{rk(2nk + rk + 2\nu + 2k)}{(nk + \nu + k)^2(nk + rk + \nu + k)^2} \geq 0.
\]
for all $k > 0$ and $\nu > -k$. Thus $\nu \mapsto f_r(\nu)$ is log-convex on $(-k, \infty)$. In view of the fact that sums of log-convex functions are also log-convex, it follows that $I^k_\nu$ is log-convex on $(-k, \infty)$ for each fixed $x > 0$. □
Remark 3.1. One of the most significance consequences of the Theorem \[3.1\] is the Turán-type inequality for the function $\mathcal{I}_\nu^k$. From the definition of log-convexity, it follows from Theorem \[3.1\] (c) that

$$
\mathcal{I}_{\alpha_1+(1-\alpha)\nu_2}(x) \leq (\mathcal{I}_{\nu_1}^k)^\alpha(x) (\mathcal{I}_{\nu_2}^k)^{1-\alpha}(x),
$$

where $\alpha \in [0,1]$, $\nu_1, \nu_2 > -k$, and $x > 0$. Choose $\alpha = 1/2$, and for any $a \in \mathbb{R}$, let $\nu_1 = \nu - a$ and $\nu_2 = \nu + a$. Then the above inequality yields reverse Turán type inequality

$$
(\mathcal{I}_\nu^k(x))^2 - \mathcal{I}_{\nu-a}^k(x)\mathcal{I}_{\nu+a}^k(x) \leq 0
$$

(3.5)

for any $\nu \geq |a| - k$.

Our final result is based on the Chebyshev integral inequality \[15, p. 40\], which states the following: suppose $f$ and $g$ are two integrable functions and monotonic in the same sense (either both decreasing or both increasing). Let $q : (a,b) \to \mathbb{R}$ be a positive integrable function. Then

$$
\left(\int_a^b q(t)f(t)dt\right)\left(\int_a^b q(t)g(t)dt\right) \leq \left(\int_a^b q(t)dt\right)\left(\int_a^b q(t)f(t)g(t)dt\right).
$$

The inequality in (3.6) is reversed if $f$ and $g$ are monotonic but in the opposite sense.

Following function is required

$$
\mathcal{J}_\nu^k(x) := \left(\frac{2}{x}\right)^\frac{k}{2} \Gamma_k(\nu+k)J_\nu^k(x) = \sum_{r=0}^\infty g_r(\nu)x^{2r},
$$

(3.7)

where,

$$
g_r(\nu) = \frac{(-1)^r \Gamma_k(\nu+k)}{\Gamma_k(rk+\nu+k)4^r r!}.
$$

(3.8)

Theorem 3.2. Let $k > 0$. Then for $\nu \in (-3k/4,-k/2] \cup [k/2,\infty)$

$$
\mathcal{I}_\nu^k(x)\mathcal{I}_\nu^{k-\frac{k}{2}}(x) \leq \frac{\sqrt{k}}{x} \sinh\left(\frac{x}{k}\right) \mathcal{I}_\nu^{k-\frac{k}{2}}(x).
$$

(3.9)

and

$$
\mathcal{J}_\nu^k(x)\mathcal{J}_\nu^{k-\frac{k}{2}}(x) \leq \frac{\sqrt{k}}{x} \sinh\left(\frac{x}{k}\right) \mathcal{J}_\nu^{k-\frac{k}{2}}(x).
$$

(3.10)

The inequalities in (3.9) and (3.10) are reversed if $\nu \in (-k/2,k/2)$.

Proof. Define the functions $p$, $f$, and $g$ on $[0,1]$ as

$$
q(t) = \cos\left(\frac{xt}{\sqrt{k}}\right), \quad f(t) = (1-t^2)^{\nu/2+\frac{1}{2}}, \quad g(t) = (1-t^2)^{\nu/2+\frac{1}{2}}.
$$

Then

$$
\int_0^1 q(t)dt = \int_0^1 \cos\left(\frac{xt}{\sqrt{k}}\right)dt = \frac{\sqrt{k}}{x} \sin\left(\frac{x}{\sqrt{k}}\right);
$$

$$
\int_0^1 q(t)f(t)dt = \int_0^1 \cos\left(\frac{xt}{\sqrt{k}}\right)(1-t^2)^{\nu/2+\frac{1}{2}}dt = \mathcal{I}_\nu^k(x), \quad \text{if} \quad \nu \geq -k;
$$
\[
\int_0^1 q(t)g(t)dt = \int_0^1 \cos \left( \frac{x t}{\sqrt{k}} \right) (1 - t^2)^{\frac{\nu}{2} + \frac{1}{2}} dt = I_{\nu + k}^k(x), \quad \text{if} \quad \nu \geq -2k;
\]
\[
\int_0^1 q(t)f(t)g(t)dt = \int_0^1 \cos \left( \frac{x t}{\sqrt{k}} \right) (1 - t^2)^{\frac{2\nu}{k}} dt = I_{2\nu + \frac{1}{2}}^{2\nu + \frac{1}{2}}(x), \quad \text{if} \quad \nu \geq -\frac{3k}{4};
\]

Since the functions \( f \) and \( g \) both are decreasing for \( \nu \geq k/2 \) and both are increasing for \( \nu \in (-3k/4, -k/2] \), the inequality (3.10) yields (3.9). On the other hand if \( \nu \in (-k/2, k/2) \), the function \( f \) is increasing but \( g \) is decreasing, and hence the inequality in (3.9) is reversed.

Similarly, the inequality in (3.10) can be derived by using (3.6) by choosing
\[
q(t) = \cosh \left( \frac{x t}{\sqrt{k}} \right), \quad f(t) = (1 - t^2)^{\frac{\nu}{k} - \frac{1}{2}}, \quad g(t) = (1 - t^2)^{\frac{\nu}{k} + \frac{1}{2}}.
\]

\[\square\]

References

[1] M. Abramowitz and I. A. Stegun, *A Handbook of Mathematical Functions*, New York, (1965).
[2] G. E. Andrews, R. Askey and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, 71, Cambridge Univ. Press, Cambridge, 1999.
[3] Á. Baricz, Generalized Bessel functions of the first kind, Lecture Notes in Mathematics, vol. 1994. Springer-Verlag, Berlin, 2010.
[4] M. Biernacki and J. Krzyż, On the monotonicity of certain functionals in the theory of analytic functions, Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 9 (1955), 135–147 (1957).
[5] P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D. Morrison, E. Witten, Quantum fields and strings: A course for mathematicians, American Mathematical Society, 1999.
[6] R. Díaz and E. Pariguan, On hypergeometric functions and Pochhammer \( k \)-symbol, Divulg. Mat. 15 (2007), no. 2, 179–192.
[7] R. Díaz and E. Pariguan, Quantum symmetric functions, Comm. Algebra 33 (2005), no. 6, 1947–1978.
[8] R. Díaz and E. Pariguan, Symmetric quantum Weyl algebras, Ann. Math. Blaise Pascal 11 (2004), no. 2, 187–203.
[9] L. Galué, A generalized Bessel function, Integral Transforms Spec. Funct. 14 (2003), no. 5, 395–401.
[10] K. S. Gehlot, Differential Equation of \( k \)-Bessels Function and its Properties, Nonl. Analysis and Differential Equations, Vol. 2, 2014, no. 2, 61 – 67
[11] K. S. Gehlot, Recurrence Relations of \( k \)-Bessels function Thai J. Math., to appear: [http://thaijmath.in.cmu.ac.th/index.php/thaijmath/article/view/1042/717](http://thaijmath.in.cmu.ac.th/index.php/thaijmath/article/view/1042/717)
[12] K. S. Gehlot and S. D. Purohit, Integral representations of the $k$-Bessel’s function, Honam Math. J. 38 (2016), no. 1, 17–23.

[13] S. Karlin and G. Szegö, On certain determinants whose elements are orthogonal polynomials, J. Analyse Math. 8 (1960/1961), 1–157.

[14] A. Laforgia and P. Natalini, Turán-type inequalities for some special functions, JIPAM. J. Inequal. Pure Appl. Math. 7 (2006), no. 1, Article 22, 3 pp. (electronic).

[15] D.S. Mitrinović, Analytic inequalities, Springer, New York, 1970.

[16] S. Mubeen and M. Naz, G. Rahman, A note on k-hypergeometric differential equations, Journal of Inequalities and Special Functions, Volume 4 Issue 3 (2013), Pages 38-43.

[17] K. Nantomah and E. Prempeh, Some Inequalities for the k-Digamma Function, Mathematica Aeterna, Vol. 4, 2014, no. 5, 521 - 525.

[18] P. Turán, On the zeros of the polynomials of Legendre, Časopis Pěst. Mat. Fys. 75 (1950), 113–122.

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