Some control variates for exotic options

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Abstract. There are no known exact formulas for the valuation of a number of exotic options, and this is particularly true for options under discrete monitoring and for American style options. Therefore, one usually recourses to a Monte Carlo Simulation approach, amongst other numerical methods, to estimate the value of these options. The problem which then arises with this method is one of variance reduction. Control variates are often used, and we present some results for the optimization of these control variables, for the valuation of Asian and lookback options. An inequality on functions of correlations useful for comparing estimators in variance reduction procedures is also provided.

1. The problem of exotic options valuations
One important class of exotic options consists of path-dependent (or history-dependent) derivatives. They have a payoff that depends not just on the final price of the underlying asset, but also on the path followed by this price up to maturity. Asian options and lookback options are typical examples of path-dependent options. In some instances, numerical methods are the only means available to value these exotic options, and in addition to Monte Carlo methods, other commonly used numerical methods include the finite difference and the finite element methods, the trinomial tree method, and some times the Markov Chain method (see [1, 2, 3, 4, 5], and the references therein). Until the last decade, Monte Carlo method has been considered as costly and unreliable, but based on innovative techniques which are now a topic of current research, they’ve yielded more promising results. In this context, Barraquand and Martineau [3], amongst others, obtained some interesting results.

Consider an option on a given security having \( n \) days to maturity. Let \( S_d(j) \) denote the closing price of the security at the end of day \( j \) (for \( j = 1, \ldots, n \)). Assume also that the risk free interest rate is \( r \) per year, and denote by \( N \) the number of trading days in a year. With these notation, the payoff from an asian option is either given by

\[
Y = e^{-rn/N} \left( S_d(n) - \frac{1}{n} \sum_{j=1}^{n} s_d(j) \right)^+, \tag{1.1}
\]

where the strike price \( K \) is taken as the average of all prices up to the expiry date, or
\[ Y = e^{-rn/N} \left( \sum_{j=1}^{n} \frac{S_d(j)}{n} - K \right)^+, \]  

(1.2)

when the underlying’s terminal price is taken as the average of all prices up to maturity. The payoff from a lookback option is given by

\[ Y = e^{-rn/N} \left( S_d(n) - \text{Min}_{j=1,...,n} S_d(j) \right)^+. \]  

(1.3)

The main problem with using Monte Carlo Simulation to value path-dependent derivatives is that the computation time necessary to achieve a reasonable level of accuracy can be excessively high. Control variate techniques are some of the tools available for implementing the necessary variance reduction procedures, which can lead to dramatic savings in computation time. Our focus in this paper will be on the applications of these techniques to the valuation of Asian and lookback options.

2. Monte carlo simulations and variance reduction procedures

Suppose that the price \( S(t) \) of a given security follows a risk neutral geometric Brownian motion with constant parameters \( \mu \) and \( \sigma^2 \). This means in particular that for all \( y \geq 0 \), and \( t \geq 0 \), \( \ln(S(t + y)/S(y)) \) has a normal distribution with mean \( \mu t \) and standard deviation \( \sigma \sqrt{t} \), and that \( \mu = r - \sigma^2/2 \), where \( r \) represents the risk-free interest rate.

We denote as usual by \( S(0) \) the initial price of the security, and by \( S_d(j) \) the price of the security at the end of day \( j \) (for \( j = 1, \ldots, n \)), where \( n \) is the number of days to maturity. Set

\[ X(i) = \ln \left( \frac{S_d(i)}{S_d(i-1)} \right), \quad (i = 1, \ldots, n). \]  

(2.1)

The random variables \( X(1), \ldots, X(n) \) are independent and identically distributed with mean \( \mu/N \) and variance \( \sigma^2/N \). That is, \( X(i) \sim \Phi(\mu/N, \sigma^2/N) \). A straightforward calculation using (2.1) shows that

\[ S_d(i) = S_d(i - 1)e^{X(i)} \]  

(2.2)

\[ = S(0)e^{X(1)+\ldots+X(i)} \]  

(2.3)

Values of \( X(1), \ldots, X(n) \) can be generated by a computer and Eq. (2.2)) or (2.3) can be used to sample a random path for the price \( S \) of the security. The valuation simulation for an asian option whose payoff \( Y \) is as in Eq. (1.2) can thus be implemented by generating for each simulation run \( j \) the corresponding payoff \( Y_j = e^{-rn/N} \left( \sum_{j=1}^{n} \frac{S_d(j)}{n} - K \right)^+ \) of the derivative. The next step then consists in computing the average mean \( \bar{Y} = \frac{1}{R} \sum_{1}^{R} Y_j \), where \( R \) is the total number of simulation runs. The value of \( \bar{Y} \) represents an estimate of the exact cost \( Y \) of the derivative. We have

\[ E[\bar{Y}] = E[Y] \]  

(2.4)

and

\[ \text{Var}(\bar{Y}) = \text{Var}(Y)/R. \]  

(2.5)

The last equality, Eq. (2.5), shows that the accuracy of \( \bar{Y} \) improves with the number of simulations.
The simulation procedure is similar for lookback and for most of the path-dependent derivatives.

The drawbacks with Monte Carlo simulation is that the amount of time necessary to achieve a reasonable accuracy can be unacceptably high. As remedial measures, a number of variance reduction techniques are available, and they can lead to dramatic savings in computation time. These include the antithetic variable technique, the importance sampling, the stratified sampling, and the control variate technique. All these procedures and many others are described by Hull [6], Clewlow and Strickland [7], and Boyle [8].

2.1. Control variates

Suppose that the payoff $Y$ from an option on a given security is of any of the forms given by the Eqs. (1.1)-(1.3). With the notation of the previous section, set

$$V = \sum_{i=1}^{n} \alpha_i X(i) \quad (2.6)$$

where $\alpha_i$ are some weights to be determined for the control variate $V$, and take

$$W = Y + c(V - E(V)) \quad (2.7)$$

as the new estimator for $Y$. We have

$$E[V] = \sum_{i=1}^{n} \alpha_i E[X(i)] = \sum_{i=1}^{n} \frac{\alpha_i}{N} \left( r - \frac{\sigma^2}{2} \right),$$

so that

$$W = Y + \sum_{i=1}^{n} \beta_i \left[ X(i) - \left( r - \frac{\sigma^2}{2} \right) / N \right], \quad \text{where } \beta_i = c \alpha_i. \quad (2.8)$$

On the other hand, we have

$$Var(W) = Var(Y) + \sum_{i=1}^{n} \beta_i^2 Var(X(i)) + 2 \sum_{i=1}^{n} \beta_i Cov(Y, X(i))$$

and it readily follows that the values of $\beta_i$ that minimize $W$ are given by

$$\beta_i^* = - \frac{Cov(Y, X(i))}{Var(X(i))} \quad (2.9)$$

In terms of these optimal values of the $\beta_i$, the smallest possible value for $Var(W)$ is given after simplification by

$$\frac{Var(W)}{Var(Y)} = 1 - \sum_i Corr^2(Y, X(i)) \quad (2.10)$$

We have thus proven the following result

**Theorem 1.** Let $V = \sum_i \alpha_i X(i)$, for some arbitrary weights $\alpha_i$ and for $i = 1, \ldots, n$. Let $W = Y + c(V - E[V])$ be an estimator of $Y$ for some constant $c$.

(a) The optimal variance reduction is achieved with this control variable $V$ for the values of $\alpha_i$ and $c$ such that

$$cc_i = - \frac{Cov(Y, X(i))}{Var(X(i))}, \quad (i = 1, \ldots, n)$$
\[ (\text{b}) \quad \text{The optimal variance reduction thus obtained is given by (2.10).} \]

Remark.

(i) It follows from part (a) of the theorem that a control variate of the form \( V = \sum_i \alpha_i X(i) \) gives rise to an optimal reduction if and only if \( \alpha_i = -\lambda \frac{\text{cov}(Y, X(i))}{\text{var}(X(i))} \) for all \( i = 1, \ldots, n \) and for a nonzero constant \( \lambda \).

(ii) This theorem implies that for all \( \alpha_1, \ldots, \alpha_n \) and for every constant \( c \)

\[ \frac{\text{Var}(W)}{\text{Var}(Y)} \geq 1 - \sum_i \text{Corr}^2(Y, X(i)) \]

where

\[ W = Y + c \left( \sum_i \alpha_i X(i) - \alpha_i (r - \sigma^2/2) \right) \]

Ross [9] considers the problem of finding the values of \( \alpha_i \) (for \( i = 1, \ldots, n \)) for the best control variate of the form \( V = \sum_i \alpha_i X(i) \), and alternatively, the problem of finding the values of \( c_1, \ldots, c_n \) for the best estimator of \( Y \) of the form \( Y + \sum_{i=1}^n c_i (X(i) - r + \frac{\sigma^2/2}{N}) \). Theorem 1 gives an answer to this question, by determining the values of \( \alpha_i \) and \( c_i \) and by indicating precisely the optimal variance reduction that can be achieved.

**Theorem 2.** Let \( Y \) be a random variable and let \( X_1, \ldots, X_n \) be \( n \) independent random variables. Then for all numbers \( \alpha_1, \ldots, \alpha_n \)

\[ \text{Corr}^2 \left( Y, \sum_i \alpha_i X_i \right) \leq \sum_i \text{Corr}^2(Y, X_i). \]

**Proof.** Let \( V = \sum_i \alpha_i X_i \) and let \( W = Y + c(V - \text{E}[V]) \) be an estimator of \( Y \) for some constant \( c \). For any fixed values of the \( \alpha_i \) the optimal variance reduction can be shown similarly to (2.10) to be given by

\[ \frac{\text{Var}(W)}{\text{Var}(Y)} = 1 - \text{Corr}^2(Y, V) \]

This is not smaller than the optimal variance reduction obtained for all possible values of the \( \alpha_i \) and \( c \) given by Eq. (2.10). Consequently, \( 1 - \sum_i \text{Corr}^2(Y, X_i) \leq 1 - \text{Corr}^2(Y, \sum_i \alpha_i X_i) \); that is, \( \text{Corr}^2(Y, \sum_i \alpha_i X_i) \leq \sum_i \text{Corr}^2(Y, X_i) \).

The result stipulated by this theorem is certainly very important, but it is likely to be unknown. Indeed in statistics books that present the most comprehensive material on the topic of correlation coefficient between random variables, there is rarely any discussion of a relationship between functions of correlations (see [10] and [11]).

2.2. Applications

(i) If \( V = \sum_i \alpha_i X_i \) where the \( \alpha_i \) are constants independent of \( X_i \), then no estimator of \( Y \) of the form \( W = Y + \sum_i \alpha_i X_i - \text{E}[V] \) using \( V \) as control variable can lead to an optimal variance reduction, since by Theorem 1, the \( \alpha_i \) must depend on \( X_i \).

(ii) Let \( V = \ln(s_d(n)/S(0)) \). By the definition of the \( X(i) \), it readily follows that \( V = \sum_i X(i) \).

Thus by the preceding remark, taking \( V \) as control variable cannot give rise to the best estimator.
3. Conclusion
A generalization of this type of inequalities between functions of correlation coefficients is
desirable, to compare for instance an estimator obtained with a control variable of the form
\[ V = \sum_i \beta_i S_d(i), \]
with that obtained using a control variable of the form \( \sum_i \alpha_i X(i) \). In this
instance, one would need a general relationship between functions of correlation coefficients of
the type \( \text{Corr}^2(Y, \sum_i \beta_i S_d(i)) \) and \( \text{Corr}^2(Y, \sum_i \beta_i \ln(S_d(i)/S_d(i-1))) \). As stated in this paper,
no result on such type of relationship seems to be available in the literature. Finally, the results
obtained in this paper clearly apply to any random variable for which variance reduction is
required, and not only to the specific case of exotic options considered in this paper.

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