Riccati-type equations, 
generalised WZNW equations, 
and multidimensional Toda systems

L. A. Ferreira*, J. F. Gomes*, A. V. Razumov†
M. V. Saveliev*, A. H. Zimerman*

*Instituto de Física Teórica - IFT/UNESP
Rua Pamplona 145, 01405-900, São Paulo - SP, Brazil
laf@ift.unesp.br, jfg@ift.unesp.br, saveliev@ift.unesp.br, ahz@ift.unesp.br

†Institute for High Energy Physics
142284 Protvino, Moscow Region, Russia
razumov@mx.ihep.su

Abstract

We associate to an arbitrary \( \mathbb{Z} \)-gradation of the Lie algebra of a Lie group a system of Riccati-type first order differential equations. The particular cases under consideration are the ordinary Riccati and the matrix Riccati equations. The multidimensional extension of these equations is given. The generalisation of the associated Redheffer–Reid differential systems appears in a natural way. The connection between the Toda systems and the Riccati-type equations in lower and higher dimensions is established. Within this context the integrability problem for those equations is studied. As an illustration, some examples of the integrable multidimensional Riccati-type equations related to the maximally nonabelian Toda systems are given.

\(^{1}\)On leave of absence from the Institute for High Energy Physics, 142284 Protvino, Moscow Region, Russia, saveliev@mx.ihep.su
1 Introduction

To the present time there is a great number of papers in mathematics and physics devoted to various aspects of the matrix differential Riccati equation proposed in twenties by Radon in the context of the Lagrange variational problem. In particular, this equation has been discussed in connection with the oscillation of the solutions to systems of linear differential equations, Lie group and differential geometry aspects of the theory of analytic functions of several complex variables in classical domains, the probability theory, computation schemes. For a systematic account of the development in the theory of the matrix differential Riccati equation up to seventies see, for example, the survey [1]. More recently there appeared papers where this equation was considered as a Bäcklund-type transformation for some integrable systems of differential geometry, in particular, for the Lamé and the Bourlet equations, and a relevant superposition principle for the equation has been studied on the basis of the theory of Lie algebras, see, for example, [2] and references therein. The matrix Riccati equation also arises as equation of motion on Grassmann manifolds and on homogeneous spaces attached to the Hartree–Fock–Bogoliubov problem, see, for example, [3] and references therein; and in some other subjects of applied mathematics and physics such as optimal control theory, plasma, etc., see, for example, [4]. Continued–fraction solutions to the matrix differential Riccati equation were constructed in [5], based on a sequence of substitutions with the coefficients satisfying a matrix generalisation of the Volterra-type equations which in turn provide a Bäcklund transformation for the corresponding matrix version of the Toda lattice. In papers [6] the matrix differential Riccati equation occurs in the steepest descent solution to the total least squares problem as a flow on Grassmannians via the Brockett double bracket commutator equation; in the special case of projective space this is the Toda lattice flow in Moser’s variables.

In the present paper we investigate the equations associated with an arbitrary $\mathbb{Z}$-gradation of the Lie algebra $\mathfrak{g}$ of a Lie group $G$. For the case $G = \text{GL}(2, \mathbb{C})$ and the principal gradation of $\mathfrak{gl}(2, \mathbb{C})$ this is the ordinary Riccati equation, for the case $G = \text{GL}(n, \mathbb{C})$ and some special $\mathbb{Z}$-gradation of $\mathfrak{gl}(n, \mathbb{C})$ we get the matrix Riccati equation. The underlying group-algebraic structure allows us to give a unifying approach to the investigation of the integrability problem for the equations under consideration which we call the Riccati-type equations.

We also give a multidimensional generalisation of the Riccati-type equations and discuss their integrability.

It appeared very useful for the study of ordinary matrix Riccati equations to associate with them the so-called Redheffer–Reid differential system [7]. In our approach the corresponding generalisation of such systems appears in a natural way. The associated Redheffer–Reid system can be considered as the constraints providing some reduction of the Wess–Zumino–Novikov–Witten (WZNW) equations. From the other hand, it is well known that the Toda-type systems can be also obtained by the appropriate reduction of the WZNW equations, see, for example, [8]. This implies the deep connection of the Toda-type systems and the Riccati-type equations. In particular, under the relevant constraints the Riccati-type equations play the role of a Bäcklund map for the Toda systems, and, in a sense, are a generalisation of the Volterra equations.

Some years ago there appeared a remarkable generalisation [9] of the Wess–Zumino–Novikov–Witten (WZNW) equations. The associated Redheffer–Reid system in the multidi-

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[1] [2] [3] [4] [5] [6] [7] [8] [9]
mensional case can be considered again as the constraints imposed on the solutions of those equations. We show that in the same way as in two dimensional case, the appropriate reduction of the multidimensional WZNW equations leads to the multidimensional Toda systems \cite{10}, in particular to the equations \cite{11} describing topological and antitopological fusion.²

The multidimensional Toda systems are integrable for the relevant integration data with the general solution being determined by the corresponding arbitrary mappings in accordance with the integration scheme developed in \cite{10}. Therefore the integrability problem for the multidimensional Riccati-type equations can be studied, in particular, on the basis of that fact. As an illustration of the general construction we discuss in detail some examples related to the maximally nonabelian Toda systems \cite{12}.

Analogously to the Toda systems one can construct higher grading generalisations in the sense of \cite{13, 14} for the multidimensional Riccati-type equations.

2 One dimensional Riccati–type equations

Let \( G \) be a connected Lie group and \( \mathfrak{g} \) be its Lie algebra. Without any loss of generality we assume that \( G \) is a matrix Lie group, otherwise we replace \( G \) by its image under some faithful representation of \( G \). For any fixed mapping \( \lambda : \mathbb{R} \rightarrow \mathfrak{g} \) consider the equation

\[
\psi^{-1} \frac{d\psi}{dx} = \lambda
\]

(2.1)

for the mapping \( \psi : \mathbb{R} \rightarrow G \). Certainly one can use the complex plane \( \mathbb{C} \) instead of the real line \( \mathbb{R} \).

Suppose that the Lie algebra \( \mathfrak{g} \) is endowed with a \( \mathbb{Z} \)-gradation, \( \mathfrak{g} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_m \).

Define the following nilpotent subalgebras of \( \mathfrak{g} \):

\[
\mathfrak{g}_{<0} = \bigoplus_{m < 0} \mathfrak{g}_m, \quad \mathfrak{g}_0 = \bigoplus_{m = 0} \mathfrak{g}_m, \quad \mathfrak{g}_{>0} = \bigoplus_{m > 0} \mathfrak{g}_m,
\]

and represent the mapping \( \lambda \) in the form

\[
\lambda = \lambda_{<0} + \lambda_0 + \lambda_{>0},
\]

where the mappings \( \lambda_{<0}, \lambda_0 \) and \( \lambda_{>0} \) take values in \( \mathfrak{g}_{<0}, \mathfrak{g}_0 \) and \( \mathfrak{g}_{>0} \) respectively.

Denote by \( G_{<0}, G_0 \) and \( G_{>0} \) the connected Lie subgroups of \( G \) corresponding to the subalgebras \( \mathfrak{g}_{<0}, \mathfrak{g}_0 \) and \( \mathfrak{g}_{>0} \) respectively. Under the appropriate assumptions for an element \( a \in G \) belonging to some dense subset of \( G \) it is valid the generalised Gauss decomposition

\[
a = a_{<0} a_0 a_{>0},
\]

(2.2)

²It is rather clear that the the multidimensional systems suggested in \cite{10} become two dimensional equations only under a relevant reduction. Moreover, arbitrary mappings determining the general solution to these equations are not necessarily factorised to the products of mappings each depending on one coordinate only. One can easily get convinced of it just by the examples considered there in detail.
where \( a_{<0} \in G_{<0}, a_0 \in G_0 \) and \( a_{>0} \in G_{>0} \). For the mapping \( \psi \) we can write
\[
\psi = \psi_{<0} \psi_0 \psi_{>0},
\] (2.3)
where the mapping \( \psi_{<0} \) takes values in \( G_{<0} \), the mapping \( \psi_0 \) takes values in \( G_0 \) and the mapping \( \psi_{>0} \) takes values in \( G_{>0} \). Using the Gauss decomposition (2.3) of the mapping \( \psi \) rewrite equation (2.1) as
\[
\psi_{>0}^{-1} \left( \psi_{<0}^{-1} \frac{d\psi_{<0}}{dx} \right) \psi_{>0} + \psi_{>0}^{-1} \frac{d\psi_{>0}}{dx} = \lambda,
\] (2.4)
where \( \psi_{<0} = \psi_{<0} \psi_0 \). From (2.4) it follows that
\[
\psi_{>0}^{-1} \frac{d\psi_{<0}}{dx} + \frac{d\psi_{>0}}{dx} \psi_{>0}^{-1} = \psi_{>0} \lambda \psi_{>0}^{-1},
\]
and hence
\[
\psi_{>0}^{-1} \frac{d\psi_{<0}}{dx} = (\psi_{>0} \lambda \psi_{>0}^{-1})_{\leq 0},
\] (2.5)
where the subscript \( \leq 0 \) denotes the corresponding component with respect to the decomposition
\[
g = g_{\leq 0} \oplus g_{>0} = (g_{<0} \oplus g_0) \oplus g_{>0}.
\]
Substituting (2.5) into (2.4) one gets
\[
\psi_{>0}^{-1} \frac{d\psi_{>0}}{dx} = \lambda - \psi_{>0}^{-1} (\psi_{>0} \lambda \psi_{>0}^{-1})_{\leq 0} \psi_{>0}
\]
that can be rewritten as
\[
\frac{d\psi_{>0}}{dx} \psi_{>0}^{-1} = (\psi_{>0} \lambda \psi_{>0}^{-1})_{>0}.
\] (2.6)
By the reasons which are clear from what follows we call this equation for the mapping \( \psi_{>0} \) a Riccati-type equation.

The formal integration of equation (2.6) can be performed in the following way. Consider (2.1) as a linear differential equation for the mapping \( \psi \):
\[
\frac{d\psi}{dx} = \psi \lambda.
\] (2.7)
Find the solution of this equation with the initial condition \( \psi(0) = a \), where \( a \) is a constant element of the Lie group \( G \). Using now the Gauss decomposition (2.3) of the mapping \( \psi \) we find the solution of equation (2.6) with the initial condition \( \psi_{>0} = a_{>0} \), where \( a_{>0} \) is the positive grade component of \( a \) arising from the Gauss decomposition (2.3). It is clear that in order to obtain the general solution of equation (2.6) it suffices to consider elements \( a \) belonging to the Lie subgroup \( G_{>0} \). Then the solution of (2.6) is expressed in terms of solution of (2.7). Note that the solution of equation (2.7) with the initial condition \( \psi(0) = a \) can be obtained from the solution with the initial condition \( \psi(0) = e \), where \( e \) is the unit element of \( G \), by left multiplication by \( a \).
Thus we have shown that one can associate a Ricatti-type equation to any  $\mathbb{Z}$-gradation of a Lie group. The integration of this equations is reduced to integration of some matrix system of first order linear differential equations.

Let now $\chi$ be some mapping from $\mathbb{R}$ to $G$. It is clear that if the mapping $\psi$ satisfies equation (2.6), then the mapping $\psi' = \psi \chi^{-1}$ satisfies the equation

$$\frac{d\psi'}{dx} = \psi' \lambda',$$

where

$$\lambda' = \chi \lambda \chi^{-1} - \frac{d\chi}{dx} \chi^{-1}.$$

(2.8)

If $\chi$ is a mapping from $\mathbb{R}$ to $G_0$, then the corresponding component

$$\psi'_{>0} = \chi \psi_{>0} \chi^{-1}$$

of the mapping $\psi'$ satisfies the Ricatti-type equation (2.6) with $\lambda$ replaced by $\lambda'$. In this,

$$\lambda'_0 = \chi \lambda_0 \chi^{-1} - \frac{d\chi}{dx} \chi^{-1},$$

and it is clear that we can choose the mapping $\chi$ so that $\lambda'_0$ vanishes.

Another interesting possibility arises when $\chi$ is a mapping from $\mathbb{R}$ to $G_{>0}$. Let us choose a mapping $\chi$ such that $\lambda'_{>0} = 0$. From (2.8) it follows that this case is realised if and only if

$$\frac{d\chi}{dx} \chi^{-1} = (\chi \lambda \chi^{-1})_{>0},$$

i.e., $\chi$ should satisfy the Riccati-type equation (2.8). Thus, having a particular solution of the Riccati-type equation, its general solution can be constructed from the general solution of the equation with $\lambda_{>0} = 0$. As will be shown below, for this case the Riccati-type equation can be solved in a quite simple way.

### 3 Simplest example

Consider first the case of the Lie group $\text{GL}(n, \mathbb{C})$, $n \geq 2$ and represent $n$ as the sum of two positive integers $n_1$ and $n_2$. For the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ there is a $\mathbb{Z}$-gradation where arbitrary elements $x_{<0}$, $x_0$ and $x_{>0}$ of the subalgebras $\mathfrak{g}_{<0}$, $\mathfrak{g}_{>0}$ and $\mathfrak{g}_0$ have the form

$$x_{<0} = \begin{pmatrix} 0 & (x_{<0})_{21} & 0 \\ (x_{<0})_{21} & 0 & 0 \end{pmatrix}, \quad x_0 = \begin{pmatrix} (x_0)_{11} & 0 & (x_0)_{22} \\ 0 & (x_0)_{22} & 0 \end{pmatrix}, \quad x_{>0} = \begin{pmatrix} 0 & (x_{>0})_{12} \\ 0 & 0 \end{pmatrix}.$$ 

Here $(x_{<0})_{21}$ is an $n_2 \times n_1$ matrix, $(x_{>0})_{12}$ is an $n_1 \times n_2$ matrix, $(x_0)_{11}$ and $(x_0)_{22}$ are $n_1 \times n_1$ and $n_2 \times n_2$ matrices respectively. The corresponding subgroups $G_{<0}$, $G_{>0}$ and $G_0$ are formed by the matrices

$$a_{<0} = \begin{pmatrix} I_{n_1} & 0 \\ (a_{<0})_{21} & I_{n_2} \end{pmatrix}, \quad a_0 = \begin{pmatrix} (a_0)_{11} & 0 \\ 0 & (a_0)_{22} \end{pmatrix}, \quad a_{>0} = \begin{pmatrix} I_{n_1} & (a_{>0})_{12} \\ 0 & I_{n_2} \end{pmatrix}.$$
Here \((a_{<0})_{21}\) is an arbitrary \(n_2 \times n_1\) matrix, \((a_{>0})_{12}\) is an arbitrary \(n_1 \times n_2\) matrix, \((a_0)_{11}\) and \((a_0)_{22}\) are arbitrary nondegenerate \(n_1 \times n_1\) and \(n_2 \times n_2\) matrices respectively. The Gauss decomposition (2.2) of an element

\[
a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}
\]
is given by the relations

\[
\begin{align*}
(a_0)_{11} &= a_{11}, & (a_{>0})_{12} &= a_{11}^{-1}a_{12}, \\
(a_{<0})_{21} &= a_{21}a_{11}^{-1}, & (a_0)_{22} &= a_{22} - a_{21}a_{11}^{-1}a_{12}.
\end{align*}
\]

(3.1)

(3.2)

Parametrizing the mapping \(\lambda\) as

\[
\lambda = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

and \(\psi_{>0}\) as

\[
\psi_{>0} = \begin{pmatrix} I_{n_1} & U \\ 0 & I_{n_2} \end{pmatrix},
\]

(3.3)

one easily sees that equation (2.6) takes in the case under consideration the form

\[
\frac{dU}{dx} = B - AU + UD - UCU.
\]

(3.4)

In the case \(n = 2, n_1 = n_2 = 1\), we have the usual Riccati equation. For \(n = 2m, n_1 = n_2 = m\), we come to the so-called matrix Riccati equation. This justifies our choice for the name of equation (2.6) in general case.

### 3.1 Case \(B = 0\)

If \(C = 0\) then equation (3.4) is linear. In the case \(B = 0\), under the conditions \(n_1 = n_2\) and \(\det U(x) \neq 0\) for any \(x\), the substitution \(V = U^{-1}\) leads to the linear equation

\[
\frac{dV}{dx} = VA - DV + C.
\]

Nevertheless, it is instructive to consider the procedure of obtaining the general solution to equation (3.4) for \(B = 0\). Recall that having a particular solution to the Riccati-type equation, we can reduce the consideration to the case where \(\lambda_{>0} = 0\). For the equation in question this is equivalent to the requirement \(B = 0\).

First, find the mapping \(\chi : \mathbb{R} \rightarrow G_0\) such that transformation (2.8) would give \(\lambda_0' = 0\). Parametrising \(\chi\) as

\[
\chi = \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix},
\]

one comes to the following equations for \(R\) and \(Q\):

\[
\frac{dQ}{dx} = QA, \quad \frac{dR}{dx} = RD.
\]
Therefore we can choose

\[ Q(x) = P \exp \left( \int_0^x A(x') \, dx' \right), \quad R(x) = P \exp \left( \int_0^x D(x') \, dx' \right), \quad (3.5) \]

where the symbol \( P \exp(\cdot) \) denotes the path ordered exponential (multiplicative integral).

Now solve the equation

\[ \frac{d\psi'}{dx} = \psi' \lambda', \]

where

\[ \lambda' = \begin{pmatrix} 0 & 0 \\ C' & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ R \cdot C Q^{-1} & 0 \end{pmatrix}. \]

The solution of this equation with the initial condition \( \psi'(0) = I_n \) is

\[ \psi(x) = \begin{pmatrix} \begin{pmatrix} I_{n_1} & 0 \\ S(x) & I_{n_2} \end{pmatrix} \end{pmatrix} \]

with

\[ S(x) = \int_0^x R(x') C(x') Q^{-1}(x') \, dx'. \quad (3.6) \]

Hence, the solution of equation (2.7) with the initial condition \( \psi(0) = I_n \) is given by

\[ \psi = \begin{pmatrix} \begin{pmatrix} Q & 0 \\ SQ & R \end{pmatrix} \end{pmatrix}. \]

To obtain the general solution of the equation under consideration we should have the solution of equation (2.7) with the initial condition

\[ \psi(0) = \begin{pmatrix} \begin{pmatrix} I_{n_1} & m \\ 0 & I_{n_2} \end{pmatrix} \end{pmatrix}, \quad (3.7) \]

where \( m \) is an arbitrary \( n_1 \times n_2 \) matrix. Such a solution is represented as

\[ \psi = \begin{pmatrix} \begin{pmatrix} (I_{n_1} + mS)Q & mR \\ SQ & R \end{pmatrix} \end{pmatrix}. \]

Now, using (3.1) we conclude that the general solution to equation (3.4) in the case \( B = 0 \) is

\[ U = Q^{-1}(I_{n_1} + mS)^{-1}mR, \]

where \( Q, R \) and \( S \) are given by relations (3.5) and (3.6).

Thus we see that in the case when \( \lambda \) is a block upper or lower triangular matrix the Riccati-type equation (3.4) can be explicitly integrated. Actually if \( \lambda \) is a constant mapping we can reduce it by a similarity transformation to the block upper or lower triangular form and solve the corresponding Riccati-type equation. The solution of the initial equation is obtained then by some algebraic calculations.
3.2 The case $A = 0$ and $D = 0$

Representing the mapping $\psi$ in the form

$$\psi = \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$$

one easily sees that equation (2.7) is equivalent to the system

$$\begin{align*}
\frac{d\psi_{11}}{dx} &= \psi_{12}C, \quad \frac{d\psi_{12}}{dx} = \psi_{11}B, \\
\frac{d\psi_{21}}{dx} &= \psi_{22}C, \quad \frac{d\psi_{22}}{dx} = \psi_{21}B.
\end{align*}$$

(3.8)

(3.9)

3.2.1 The case $C = B$

Consider the case $C = B$; that is certainly possible only if $n_1 = n_2$. In this case we can rewrite equations (3.8) and (3.9) as

$$\begin{align*}
\frac{d(\psi_{11} + \psi_{12})}{dx} &= (\psi_{11} + \psi_{12})B, \\
\frac{d(\psi_{11} - \psi_{12})}{dx} &= -(\psi_{11} - \psi_{12})B, \\
\frac{d(\psi_{22} + \psi_{21})}{dx} &= (\psi_{22} + \psi_{21})B, \\
\frac{d(\psi_{22} - \psi_{21})}{dx} &= -(\psi_{22} - \psi_{21})B.
\end{align*}$$

Hence, the solution of equation (2.7) with the initial condition $\psi(0) = I_n$ is given by

$$\psi = \frac{1}{2} \begin{pmatrix} F + H & F - H \\ F - H & F + H \end{pmatrix},$$

where

$$F(x) = P \exp \left( \int_0^x B(x') \, dx' \right), \quad H(x) = P \exp \left( - \int_0^x B(x') \, dx' \right).$$

The solution of equation (2.7) with the initial condition of form (3.7) is

$$\psi = \frac{1}{2} \begin{pmatrix} F + H + m(F - H) & F - H + m(F + H) \\ F - H & F + H \end{pmatrix};$$

therefore, the general solution to the Riccati-type equation under consideration can be written as

$$U = (F + H + m(F - H))^{-1}(F - H + m(F + H)).$$

3.2.2 The case of constant $B$ and $C$

As we noted above, the general solution to the Riccati-type equations (1.4) for the case of constant mapping $\lambda$ can be obtained by a reduction of $\lambda$ to the block upper or lower triangular form. Nevertheless, it is interesting to consider the particular case of constant $\lambda$ when the general solution has the most simple form.
Suppose that \( n_1 = n_2 \) and that \( B \) and \( C \) are constant nondegenerate matrices. In this case the solution of equation (2.7) with the initial condition \( \psi(0) = I_n \) is
\[
\psi(x) = \begin{pmatrix}
\cosh(\sqrt{BCx}) & \sinh(\sqrt{BCx})
\end{pmatrix} \begin{pmatrix}
\sinh(\sqrt{CBx})\sqrt{CBB^{-1}} & \cosh(\sqrt{CBx})
\end{pmatrix},
\]
and for the general solution one has
\[
U(x) = \left( \cosh(\sqrt{BCx}) + m\sinh(\sqrt{CBx})\sqrt{CBB^{-1}} \right)^{-1}
\times \left( \sinh(\sqrt{BCx})\sqrt{BCC^{-1}} + m\cosh(\sqrt{CBx}) \right).
\]
It should be noted here that the expression for \( U(x) \) does not actually contain square roots of matrices that can be easily seen from the corresponding expansions into the power series.

4 A further example

The next example is based on another \( \mathbb{Z} \)-gradation of the Lie algebra \( \mathfrak{gl}(n, \mathbb{C}) \). Here one represents \( n \) as the sum of three positive integers \( n_1, n_2 \) and \( n_3 \) and consider an element \( x \) of \( \mathfrak{gl}(n, \mathbb{C}) \) as a \( 3 \times 3 \) block matrix \((x_{rs})\) with \( x_{rs} \) being an \( n_r \times n_s \) matrix. The subspace \( \mathfrak{g}_m \) is formed by the block matrices \( x = (x_{rs}) \) where only the blocks \( x_{rs} \) with \( s - r = m \) are different from zero. Arbitrary elements \( x_{<0}, x_0 \) and \( x_{>0} \) of the subalgebras \( \mathfrak{g}_{<0}, \mathfrak{g}_0 \) and \( \mathfrak{g}_{>0} \) have the form
\[
x_{<0} = \begin{pmatrix}
0 & 0 & 0 \\
(x_{<0})_{21} & 0 & 0 \\
(x_{<0})_{31} & (x_{<0})_{32} & 0
\end{pmatrix}, \quad x_{>0} = \begin{pmatrix}
0 & (x_{>0})_{12} & (x_{>0})_{13} \\
0 & 0 & (x_{>0})_{23} \\
0 & 0 & 0
\end{pmatrix},
\]
\[
x_0 = \begin{pmatrix}
(x_0)_{11} & 0 & 0 \\
0 & (x_0)_{22} & 0 \\
0 & 0 & (x_0)_{33}
\end{pmatrix}.
\]
The subgroups \( G_{<0}, G_0 \) and \( G_{>0} \) are formed by the nondegenerate matrices
\[
a_{<0} = \begin{pmatrix}
I_{n_1} & 0 & 0 \\
(a_{<0})_{21} & I_{n_2} & 0 \\
(a_{<0})_{31} & (a_{<0})_{32} & I_{n_3}
\end{pmatrix}, \quad a_{>0} = \begin{pmatrix}
I_{n_1} & (a_{>0})_{12} & (a_{>0})_{13} \\
0 & I_{n_2} & (a_{>0})_{23} \\
0 & 0 & I_{n_3}
\end{pmatrix},
\]
\[
a_0 = \begin{pmatrix}
(a_0)_{11} & 0 & 0 \\
0 & (a_0)_{22} & 0 \\
0 & 0 & (a_0)_{33}
\end{pmatrix}.
\]
The Gauss decomposition of an element \( a \in \text{GL}(n, \mathbb{C}) \) is determined by the relations
\[
(a_{<0})_{21} = a_{21}a_{11}^{-1}, \quad (a_{<0})_{31} = a_{31}a_{11}^{-1},
\]
\[
(a_{<0})_{32} = (a_{32} - a_{31}a_{11}^{-1}a_{12})(a_{22} - a_{21}a_{11}^{-1}a_{12})^{-1},
\]
\[
(a_0)_{11} = a_{11}, \quad (a_0)_{22} = a_{22} - a_{21}a_{11}^{-1}a_{12}.
\]
\[
(a_0)_{33} = a_{33} - a_{31}a_{11}^{-1}a_{13} - (a_{32} - a_{31}a_{11}^{-1}a_{12})(a_{22} - a_{21}a_{11}^{-1}a_{12})^{-1}(a_{23} - a_{21}a_{11}^{-1}a_{13}),
(a_{>0})_{12} = a_{11}^{-1}a_{12}, \quad (a_{>0})_{13} = a_{11}^{-1}a_{13},
(a_{>0})_{23} = (a_{22} - a_{21}a_{11}^{-1}a_{12})^{-1}(a_{23} - a_{21}a_{11}^{-1}a_{13}).
\]

We parametrise the mapping \( \lambda \) as
\[
\lambda = \begin{pmatrix}
A_{11} & B_{12} & B_{13} \\
C_{21} & A_{22} & B_{23} \\
C_{31} & C_{32} & A_{33}
\end{pmatrix}
\]
and the mapping \( \psi_{>0} \) as
\[
\psi_{>0} = \begin{pmatrix}
I_{n_1} & U_{12} & U_{13} \\
0 & I_{n_2} & U_{23} \\
0 & 0 & I_{n_3}
\end{pmatrix}.
\]

After some algebra one sees that the Riccati-type equations for the case under consideration is
\[
\frac{dU_{12}}{dx} = B_{12} - A_{11}U_{12} + U_{12}A_{22} + U_{13}C_{32} - U_{12}C_{21}U_{12} - U_{13}C_{31}U_{12},
\]
\[
\frac{dU_{23}}{dx} = B_{23} - A_{22}U_{23} + U_{23}A_{33} - C_{21}U_{13}
+ C_{21}U_{12}U_{23} - U_{23}C_{31}U_{13} - U_{23}C_{32}U_{23} + U_{23}C_{31}U_{12}U_{23},
\]
\[
\frac{dU_{13}}{dx} = B_{13} - A_{11}U_{13} + U_{13}A_{33} + U_{12}B_{23} - U_{12}C_{21}U_{13} - U_{13}C_{31}U_{13}.
\]

Consider the case where \( B_{rs} = 0 \). Here by transformation (2.8) we can reduce our equations to the case where additionally \( A_{rs} = 0 \). In the latter case the solution of equation (2.7) with the initial condition \( \psi(0) = I_n \) has the form
\[
\psi = \begin{pmatrix}
I_{n_1} & 0 & 0 \\
S_{21} & I_{n_2} & 0 \\
S_{31} & S_{32} & I_{n_3}
\end{pmatrix},
\]
where
\[
S_{21}(x) = \int_0^x C_{21}(x') \, dx',
\]
\[
S_{31}(x) = \int_0^x \left( C_{31}(x') + \int_0^{x'} C_{32}(x'') \, dx'' \right) C_{21}(x') \, dx',
\]
\[
S_{32}(x) = \int_0^x C_{32}(x') \, dx'.
\]

Using the explicit expressions for the Gauss decomposition given in this section, we find that the solution to the Riccati-type equation under consideration with the initial condition
\[
\psi_{>0}(0) = \begin{pmatrix}
I_{n_1} & m_{12} & m_{13} \\
0 & I_{n_2} & m_{23} \\
0 & 0 & I_{n_3}
\end{pmatrix}
\]
is determined by the relations

\[ U_{12} = (I_{n_1} + m_{12}S_{21} + m_{13}S_{31})^{-1}(m_{12} + m_{13}S_{32}), \]
\[ U_{13} = (I_{n_1} + m_{12}S_{21} + m_{13}S_{31})^{-1}m_{13}, \]
\[ U_{23} = (I_{n_2} + m_{23}S_{32} - (S_{21} + m_{23}S_{31})(I_{n_1} + m_{12}S_{21} + m_{13}S_{31})^{-1}m_{13}^{-1}m_{23} - (S_{21} + m_{23}S_{31})(I_{n_1} + m_{12}S_{21} + m_{13}S_{31})^{-1}) \times (m_{23} - (S_{21} + m_{23}S_{31})(I_{n_1} + m_{12}S_{21} + m_{13}S_{31})^{-1}m_{13}^{-1}). \]

The above consideration can be directly generalised to the case of the Z-gradation of \( \mathfrak{gl}(n, \mathbb{C}) \) which leads to the natural representation of \( n \times n \) matrices as \( p \times p \) block matrices. The corresponding equations look more and more complicated. Nevertheless, at least for the case of constant mappings \( \lambda \) and for the case of block upper or lower triangular mappings \( \lambda \), they can be explicitly integrated. Actually these gradations exhaust in a sense all possible Z-gradations of the Lie algebra \( \mathfrak{gl}(n, \mathbb{C}) \) \[12, 17\].

### 5 Multidimensional Riccati-type equations

Let now \( \lambda_i, i = 1, \ldots, d \) be some \( \mathfrak{g} \)-valued functions on \( \mathbb{R}^d \) whose standard coordinates are denoted by \( x^i \). Consider the following system of equations for a mapping \( \psi \) from \( \mathbb{R}^d \) to the Lie group \( G \):

\[ \partial_i \psi = \lambda_i \psi, \quad (5.1) \]

where \( \partial_i = \partial / \partial x^i \). The integrability conditions for system (5.1) look as

\[ \partial_i \lambda_j - \partial_j \lambda_i + [\lambda_i, \lambda_j] = 0. \quad (5.2) \]

Similarly to the one dimensional case we obtain the following equations for the component \( \psi > 0 \) entering the Gauss decomposition of type (2.3):

\[ \partial_i \psi > 0 \psi^{-1} > 0 = (\psi > 0 \lambda_i \psi^{-1} > 0) > 0. \quad (5.3) \]

We call these equations \textit{multidimensional Riccati-type equations}. The integration of equations (5.3) is again reduced to the integration of linear system (5.1).

The transformation (2.8), where \( \chi \) is a mapping from \( \mathbb{R}^d \) to \( G_0 \), cannot be used now to get the Riccati-type equations with \( \lambda_0 = 0 \). Indeed, to this end we should solve the equations

\[ \chi^{-1} \partial_i \chi = (\lambda_i)_0. \quad (5.4) \]

The integrability conditions for these equations do not in general follow from (5.2). However, for the case \( (\lambda_i)_> > 0 = 0 \) relations (5.4) are consequence of relations (5.2) and we can, with the help of transformation (2.8), reduce these equations to the case where \( (\lambda_i)_0 = 0 \). Note that in the multidimensional case it is again possible to use transformation (2.8), where \( \chi \) is some solution of the Riccati-type equations, to reduce the equations to the case where \( (\lambda_i)_> > 0 = 0 \).

When \( \lambda_i \) are constant mappings, conditions (5.2) imply that the matrices \( \lambda_i \) commute. Here, by a similarity transformation, we can reduce \( \lambda_i \) to a triangular form. In such case,
and not only for constant $\lambda_i$, the multidimensional Riccati-type equations can be integrated by a procedure similar to one used in the one dimensional case.

As a concrete example consider the Lie group $GL(n, \mathbb{C})$ with the gradation of its Lie algebra described in section 3. Parametrising the mappings $\lambda_i$ as

$$
\lambda_i = \begin{pmatrix}
A_i & B_i \\
C_i & D_i
\end{pmatrix}
$$

and using for the mapping $\psi_{>0}$ parametrisation (5.3) we come to the following multidimensional Riccati-type equations:

$$
\partial_i U = B_i - A_i U + U D_i - U C_i U.
$$

(5.5)

When $A_i = 0$ and $B_i = 0$ conditions (5.2) become as

$$
\partial_i C_j - \partial_j C_i = 0;
$$

hence, there exists a mapping $S$ such that $C_i = \partial_i S$. Then, the general solution of equations (5.3) has the form

$$
U = (I_{n_1} + mS)^{-1} m,
$$

where $m$ is an arbitrary $n_1 \times n_2$ matrix.

6 Generalised WZNW equations and multidimensional Toda equations

Consider the space $\mathbb{R}^{2d}$ as a differential manifold and denote the standard coordinates on $\mathbb{R}^{2d}$ by $z^{-i}$, $z^{+i}$, $i = 1, \ldots, d$. Let $\psi$ be a mapping from $\mathbb{R}^{2d}$ to the Lie group $G$, which satisfies the equations

$$
\partial_{+j}(\psi^{-1} \partial_{-i}\psi) = 0,
$$

that can be equivalently rewritten as

$$
\partial_{-i}(\partial_{+j}\psi \psi^{-1}) = 0.
$$

Here and in what follows we use the notations $\partial_{-i} = \partial/\partial z^{-i}$ and $\partial_{+j} = \partial/\partial z^{+j}$. In accordance with [9] we call equations (6.1) the generalised WZNW equations. It is well-known that the two dimensional Toda equations can be considered as reductions of the WZNW equations; for a review we refer the reader to some remarkable papers [8], and for the affine case to [16]. Let us show that in multidimensional situation the appropriate reductions of the generalised WZWN equations give the multidimensional Toda equations recently proposed and investigated in [10].

It is clear that the $\mathfrak{g}$-valued mappings

$$
\iota_{-i} = \psi^{-1} \partial_{-i}\psi, \quad \iota_{+j} = -\partial_{+j}\psi \psi^{-1}
$$

(6.2)

satisfy the relations

$$
\partial_{+j}\iota_{-i} = 0, \quad \partial_{-i}\iota_{+j} = 0.
$$

(6.3)
Moreover, the mappings \( \iota_i - \iota_j \) and \( \iota_i + \iota_j \) satisfy, by construction, the following zero curvature conditions:

\[
\partial_{-j} \iota_i - \partial_{-i} \iota_j + [\iota_i, \iota_j] = 0, \quad \partial_{++} \iota_i - \partial_{++} \iota_i + [\iota_i, \iota_i] = 0.
\]

The reduction in question is realised by imposing on the mapping \( \psi \) the constraints

\[
(\psi^{-1} \partial_{-i} \psi)_{<0} = c_{-i}, \quad (\partial_{++} \psi \psi^{-1})_{>0} = -c_{++},
\]

where \( c_{-i} \) and \( c_{++} \) are some fixed mappings taking values in the subspaces \( \mathfrak{g}_{-1} \) and \( \mathfrak{g}_{+1} \) respectively. In other words, one imposes the restrictions

\[
(t_{-i})_{<0} = c_{-i}, \quad (t_{++})_{>0} = c_{++}.
\]

From (6.3) and (6.4) it follows that we should consider only the mappings \( c_{-i} \) and \( c_{++} \) which satisfy the conditions

\[
\partial_{++} c_{-i} = 0, \quad \partial_{-i} c_{++} = 0,
\]

\[
[c_{-i}, c_{-j}] = 0, \quad [c_{++}, c_{++}] = 0.
\]

Using the Gauss decomposition (2.3) we have

\[
\psi^{-1} \partial_{-i} \psi = \psi^{-1}_{>0} \psi^{-1}_{<0} (\psi^{-1}_{<0} \partial_{-i} \psi_{<0}) \psi_{>0} + \psi^{-1}_{>0} (\psi^{-1}_{>0} \partial_{-i} \psi_{>0}) \psi_{<0} + \psi^{-1}_{>0} \partial_{-i} \psi_{>0}.
\]

Taking into account the first equality of (6.3), one sees that

\[
\psi^{-1}_{>0} (\psi^{-1}_{<0} \partial_{-i} \psi_{<0}) \psi_{>0} = c_{-i}.
\]

Similarly one obtains the equality

\[
\partial_{++} \psi \psi^{-1} = \partial_{++} \psi_{<0} \psi^{-1}_{<0} + \psi_{<0} (\partial_{++} \psi_{<0} \psi^{-1}_{<0}) + \psi_{<0} (\partial_{++} \psi_{<0} \psi_{>0} \psi^{-1}_{>0}) \psi^{-1}_{<0} + \psi_{>0} \partial_{++} \psi_{>0} \psi^{-1}_{>0} = -c_{++}.
\]

Let us use now the observation that the generalised WZNW equations can be considered as the zero curvature condition for the connection on the trivial principal fibre bundle \( \mathbb{R}^{2d} \times G \) determined by the \( \mathfrak{g} \)-valued 1-form \( \rho \) on \( \mathbb{R}^{2d} \) with the components

\[
\rho_{-i} = \psi^{-1} \partial_{-i} \psi, \quad \rho_{++} = 0.
\]

After the gauge transformation of the form \( \psi^{-1} \) generated by the mapping \( \psi^{-1}_{>0} \) we come to the connection form \( \omega \) with the components

\[
\omega_{-i} = \psi^{-1}_{>0} (\psi^{-1}_{<0} \partial_{-i} \psi_{<0}) \psi_{>0} + \psi^{-1}_{>0} \partial_{-i} \psi_{>0}, \quad \omega_{++} = \psi_{>0} \partial_{++} \psi^{-1}_{>0}.
\]
Since the zero curvature condition is invariant with respect to gauge transformations, we conclude that the generalised WZNW equations are equivalent to zero curvature condition for the form $\omega$. Using (6.8), (6.9) and denoting $\psi_0$ by $\gamma$ we see that

$$
\omega_{-i} = c_{-i} + \gamma^{-1} \partial_{-i} \gamma, \quad \omega_{+i} = \gamma^{-1} c_{+i} \gamma.
$$

(6.10)

It is exactly the components of the form whose zero curvature condition leads to multidimensional Toda equations [10] having the following explicit form

$$
\partial_{-i} (\gamma c_{-j} \gamma^{-1}) = \partial_{-j} (\gamma c_{-i} \gamma^{-1}), \quad (6.11)
$$

$$
\partial_{+j} (\gamma^{-1} \partial_{-i} \gamma) = [c_{-i}, \gamma^{-1} c_{+j} \gamma], \quad (6.12)
$$

$$
\partial_{+i} (\gamma^{-1} c_{+j} \gamma) = \partial_{+j} (\gamma^{-1} c_{+i} \gamma). \quad (6.13)
$$

Thus, if a mapping $\psi$ satisfies the generalised WZNW equations (6.1) and constraints (6.5), then its component $\psi_0$, entering the Gauss decomposition (2.3), satisfies multidimensional Toda equations (6.11)–(6.13). On the other hand, assume that $\gamma$ is a solution of the multidimensional Toda equations (6.11)–(6.13); then putting $\psi_0 = \gamma$ and choosing some $\psi_0$ and $\psi_0$ which satisfy (6.8) and (6.9), respectively, one can construct the solution

$$
\psi = \psi_0 \psi_0 \psi_0
$$

of the generalised WZNW equation submitted to constraints (6.3). The explicit construction of the mappings $\psi_0$ and $\psi_0$ from a given solution of Toda equation for the two-dimensional case was considered in [17]. Below we give the generalisation of such construction to the multidimensional case.

First recall the procedure of obtaining the general solution to multidimensional Toda equations [10]. Let $\gamma_-$ and $\gamma_+$ be some mappings from $\mathbb{R}^{2d}$ to $G_0$ satisfying the conditions

$$
\partial_{+i} \gamma_- = 0, \quad \partial_{-i} \gamma_+ = 0.
$$

Consider the equations

$$
\mu_-^{-1} \partial_{-i} \mu_- = \gamma_- c_{-i} \gamma_-^{-1}, \quad \mu_+^{-1} \partial_{+i} \mu_+ = \gamma_+ c_{+i} \gamma_+^{-1}, \quad (6.14)
$$

where $\mu_-$ and $\mu_+$ obey the conditions

$$
\partial_{+i} \mu_- = 0, \quad \partial_{-i} \mu_+ = 0.
$$

The integrability conditions for equations (6.14) are

$$
\partial_{-i} (\gamma_- c_{-j} \gamma_-^{-1}) - \partial_{-j} (\gamma_- c_{-i} \gamma_-^{-1}) = 0, \quad \partial_{+i} (\gamma_+ c_{+j} \gamma_+^{-1}) - \partial_{+j} (\gamma_+ c_{+i} \gamma_+^{-1}) = 0.
$$

Hence, the mappings $\gamma_-$ and $\gamma_+$ cannot be arbitrary. Suppose that the above integrability conditions are satisfied and solve equations (6.14). Consider the Gauss decomposition

$$
\mu_-^{-1} \mu_- = \nu_- \eta \mu_+^{-1}, \quad (6.15)
$$

3In [10] there was considered the case of constant $c_{-i}$ and $c_{+i}$. The generalisation to the case of arbitrary $c_{-i}$ and $c_{+i}$ satisfying (6.6) and (6.7) is straightforward.
where the mapping $\nu_-$ takes values in $G_{<0}$, the mapping $\eta$ takes values in $G_0$ and the mapping $\nu_+$ takes values in $G_{>0}$. It can be shown [10] that the mapping

$$\gamma = \gamma_+^{-1}\eta\gamma_-$$

(6.16)
satisfies the multidimensional Toda equations (6.11)–(6.13).

Since the manifold $\mathbb{R}^{2d}$ is simply connected and the connection form $\omega$ satisfies the zero curvature condition, then there exists a mapping $\varphi : \mathbb{R}^{2d} \to G$ such that

$$\omega_{-i} = \varphi^{-1}\partial_{-i}\varphi, \quad \omega_{+i} = \varphi^{-1}\partial_{+i}\varphi.$$ 

As it was shown in [10], the general form of the mapping $\varphi$ corresponding to the solution of the multidimensional Toda equations constructed with the help of the above described procedure, is

$$\varphi = a\mu_+\nu_-\eta\gamma_- = a\mu_-\nu_+\gamma_-, \quad (6.17)$$

where $a$ is an arbitrary constant element of the Lie group $G$. Using (6.17) we have

$$\omega_{-i} = \varphi^{-1}\partial_{-i}\varphi = (\eta\gamma_-)^{-1}(\nu_-^{-1}\partial_{-i}\nu_-)\eta\gamma_- + (\eta\gamma_-)^{-1}\partial_{-i}(\eta\gamma_-).$$

Comparing this relation with the first equality in (6.10) and taking into account (6.16) we conclude that

$$(\gamma_+^{-1}\nu_-\gamma_+)^{-1}\partial_{-i}(\gamma_+^{-1}\nu_-\gamma_+) = \gamma_{c_i}^{-1}\gamma.$$ 

Thus we see that the general solution of equations (6.8) with $\psi_0 = \gamma$ can be written as

$$\psi_{<0} = \xi_-^{-1}\gamma_+^{-1}\nu_-\gamma_+, \quad (6.18)$$

where $\xi_-$ is an arbitrary mapping which takes values in $G_{<0}$ and satisfies the conditions

$$\partial_{-i}\xi_- = 0.$$ 

In a similar way we obtain the relation

$$\partial_{+i}(\gamma_+^{-1}\nu_+^{-1}\gamma_-)(\gamma_+^{-1}\nu_+\gamma_-) = -\gamma_{c_{+i}}^{-1}\gamma$$

which implies that the general solution of equations (6.9) with $\psi_0 = \gamma$ is given by

$$\psi_{>0} = \gamma_{-i}^{-1}\nu_+^{-1}\gamma_-\xi_+, \quad (6.19)$$

where $\xi_+$ is an arbitrary mapping which takes values in $G_{>0}$ and satisfies the conditions

$$\partial_{+i}\xi_+ = 0.$$ 

Using relations (6.18) and (6.19) we come to the following representation for the solution of the generalised WZNW equations corresponding to the solution of the multidimensional Toda equations $\psi_0 = \gamma$:

$$\psi = \psi_{<0}\psi_0\psi_{>0} = \xi_-^{-1}\gamma_+^{-1}\nu_-\eta\nu_+^{-1}\gamma_-\xi_+.$$ 

Due to relation (6.19) this representation is equivalent to

$$\psi = \xi_-^{-1}\gamma_+^{-1}\mu_+^{-1}\mu_-\gamma_-\xi_+.$$ 

In the next section we use this representation to construct some integrable classes of the multidimensional Riccati-type equations.
7 Multidimensional Toda systems and Riccati-type equations

Let $\lambda_{-i}$ and $\lambda_{+i}$, $i = 1, \ldots, d$, be some fixed mappings from the manifold $\mathbb{R}^{2d}$ to the Lie algebra $\mathfrak{g}$ which satisfy conditions

$$
\partial_{+j}\lambda_{-i} = 0, \quad \partial_{-j}\lambda_{+i} = 0.
$$

(7.1)

Consider the system of equations

$$
\partial_{-i}\psi = \psi\lambda_{-i}, \quad \partial_{+i}\psi = -\lambda_{+i}\psi,
$$

(7.2)

where $\psi$ is a mapping from $\mathbb{R}^{2d}$ to the Lie group $G$. The integrability conditions for this system is given by

$$
\partial_{-i}\lambda_{-j} - \partial_{-j}\lambda_{-i} + [\lambda_{-i}, \lambda_{-j}] = 0, \quad \partial_{+i}\lambda_{+j} - \partial_{+j}\lambda_{+i} + [\lambda_{+i}, \lambda_{+j}] = 0.
$$

(7.3)

It is clear that the mapping $\psi$ satisfies the generalised WZNW equations. Hence we can treat system (7.2) with the mappings $\lambda_{-i}$ and $\lambda_{+i}$ satisfying (7.1) and (7.3), as a reduction of the generalised WZNW equations similar to the reduction considered in the previous section. The difference is that in the previous section we fixed only the components $(t_{-i})_{<0}$ and $(t_{+i})_{>0}$ of the mappings $t_{-i}$ and $t_{+i}$ and did it in a quite special way, but here we fix the mappings $t_{-i}$ and $t_{+i}$ completely. It is easy to show that if the mapping $\psi$ satisfies equations (7.2) then the mappings $\psi_{<0}$ and $\psi_{>0}$ satisfy the multidimensional Riccati-type equations

$$
\partial_{+i}\psi_{<0}^{-1}\psi_{<0} = (\psi_{<0}\lambda_{+i}\psi_{<0})_{<0},
$$

(7.4)

$$
\partial_{-i}\psi_{>0}^{-1}\psi_{>0} = (\psi_{>0}\lambda_{-i}\psi_{>0}^{-1})_{>0}.
$$

(7.5)

Equations (7.2) are multidimensional generalisation of the so-called associated Redheffer–Reid system [7]. The investigation of that system is very useful for studying one dimensional Riccati and matrix Riccati equations, see for example [1]. We believe that our generalisation also play a significant role for the multidimensional Riccati-type equations. As a first application of such systems let us give a construction of some integrable class of the multidimensional Riccati-type equations.

Suppose now that the mappings $\lambda_{-i}$ and $\lambda_{+i}$ are that

$$
(\lambda_{-i})_{<0} = c_{-i}, \quad (\lambda_{+i})_{<0} = c_{+i},
$$

(7.6)

with the mappings $c_{-i}$ and $c_{+i}$ taking values in $\mathfrak{g}_{-1}$ and $\mathfrak{g}_{+1}$, respectively, and submitted to conditions (6.6) and (6.7). In this case the mapping $\gamma = \psi_0$ satisfies the multidimensional Toda equations (6.11)–(6.13). On the other hand, if we have a solution $\gamma$ of equations (6.11)–(6.13), then using results of the previous section we can find the general solution to equations (6.8) and (6.9), and construct the mapping $\psi$ which satisfies the generalised WZNW equations and constraints (6.3). This mapping, via equalities (7.2), generates some mappings $\lambda_{-i}$ and $\lambda_{+i}$ certainly satisfying constraints (7.3). Actually if we have the general solution to multidimensional Toda equations (6.11)–(6.13), then we get in this way the general form of the mappings $\lambda_{-i}$ and $\lambda_{+i}$ which satisfy the integrability conditions (7.3).
and constraints \( \{7.4\} \). Moreover, we have here the general solution to the multidimensional Riccati equations \( \{7.4\} \) and \( \{7.5\} \).

The explicit form of the mappings \( \lambda_{-i} \) and \( \lambda_{+i} \) obtained with the help of the above described procedure is

\[
\lambda_{-i} = \xi_{-i}^{-1} c_{-i} \xi_{+i} + \xi_{-i}^{-1} (\gamma_{-i} \partial_{-i} \gamma_{i}) \xi_{+i} + \xi_{-i}^{-1} \partial_{-i} \xi_{+i}, \\
\lambda_{+i} = \xi_{+i}^{-1} \partial_{+i} \xi_{+i} + \xi_{+i}^{-1} (\gamma_{+i} \partial_{+i} \gamma_{-i}) \xi_{+i} + \xi_{+i}^{-1} c_{+i} \xi_{-i},
\]

and the corresponding solutions of equations \( \{7.4\} \) and \( \{7.5\} \) are given by \( \{6.18\} \) and \( \{6.19\} \).

Consider the Lie group \( GL(n, \mathbb{C}) \) and the \( \mathbb{Z} \)-gradation of the Lie algebra \( gl(n, \mathbb{C}) \) discussed in section 3. Parametrise the mappings \( \gamma_{\pm} \) as

\[
\gamma_{\pm} = \begin{pmatrix} \beta_{\pm1} & 0 \\ 0 & \beta_{\pm2} \end{pmatrix}.
\]

The general form of the mappings \( c_{\mp i} \) is

\[
c_{-i} = \begin{pmatrix} 0 & 0 \\ X_{-i} & 0 \end{pmatrix}, \quad c_{+i} = \begin{pmatrix} 0 & X_{+i} \\ 0 & 0 \end{pmatrix},
\]

where the mappings \( X_{-i} \) and \( X_{+i} \) are arbitrary. The integrability conditions of equations \( \{6.14\} \) have now the form

\[
\partial_{-i} (\beta_{-2} X_{-j} \beta_{-1}^{-1}) - \partial_{-j} (\beta_{-2} X_{-i} \beta_{-1}^{-1}) = 0, \\
\partial_{+i} (\beta_{+1} X_{+j} \beta_{+1}^{-1}) - \partial_{-j} (\beta_{+1} X_{-i} \beta_{+1}^{-1}) = 0.
\]

For the mappings

\[
\lambda_{\mp i} = \begin{pmatrix} A_{\mp i} & B_{\mp i} \\ C_{\mp i} & D_{\mp i} \end{pmatrix}
\]

we obtain

\[
A_{-i} = \beta_{-1}^{-1} \partial_{-i} \beta_{-1} - (\xi_{+})_{12} X_{-i}, \\
B_{-i} = \beta_{-1}^{-1} \partial_{-i} \beta_{-1} (\xi_{+})_{12} - (\xi_{+})_{12} \beta_{-1}^{-1} \partial_{-i} \beta_{-2} - (\xi_{+})_{12} X_{-i} (\xi_{+})_{12} + \partial_{-i} (\xi_{+})_{12}, \\
C_{-i} = X_{-i}, \quad D_{-i} = \beta_{-2}^{-1} \partial_{-i} \beta_{-2} + X_{-i} (\xi_{+})_{12}, \\
A_{+i} = \beta_{+1}^{-1} \partial_{+i} \beta_{+1} + X_{+i} (\xi_{-})_{21}, \quad B_{+i} = X_{+i}, \\
C_{+i} = \beta_{+1}^{-1} \partial_{+i} \beta_{-1} (\xi_{-})_{21} - (\xi_{-})_{21} \beta_{+1}^{-1} \partial_{+i} \beta_{+1} - (\xi_{-})_{21} X_{+i} (\xi_{-})_{21} + \partial_{+i} (\xi_{-})_{21}, \\
D_{+i} = \beta_{+2}^{-1} \partial_{+i} \beta_{+2} - (\xi_{-})_{21} X_{+i},
\]

where \( (\xi_{+})_{12} \) and \( (\xi_{-})_{21} \) are the nontrivial blocks of the mappings \( \xi_{+} \) and \( \xi_{-} \).

In order to solve equations \( \{7.4\} \) and \( \{7.5\} \) one considers first equations \( \{6.14\} \). Next, one uses the Gauss decomposition \( \{6.15\} \) for finding the mappings \( \nu_{+1}^{-1} \) and \( \nu_{-} \). In the case under consideration

\[
\nu_{+1}^{-1} = \begin{pmatrix} I_{n_1} & -(I_{n_1} - (\mu_{+})_{12}(\mu_{-})_{21}^{-1}(\mu_{+})_{12}) \\ 0 & I_{n_2} \end{pmatrix}, \\
\nu_{-} = \begin{pmatrix} I_{n_1} & 0 \\ (\mu_{-})_{21}(I_{n_1} - (\mu_{+})_{12}(\mu_{+})_{21}^{-1} & I_{n_2} \end{pmatrix}.
\]
Finally, using (6.19) and (6.18) one arrives at the following expressions for nontrivial blocks
$(\psi_{>0})_{12} = U_-$ and $(\psi_{<0})_{21} = U_+$ of the mappings $\psi_{>0}$ and $\psi_{<0}$:

$$U_- = (\xi_+)^{12} - \beta_{-1}^{-1}(I_n_1 - (\mu_+)^{12}_1(\mu_-)^{21}_2)^{-1}(\mu_+)^{12}_1\beta_{-2},$$

$$U_+ = (\xi_-)^{21} + \beta_{+1}^{-1}(\mu_-)^{21}_2(I_n_1 - (\mu_+)^{12}_1(\mu_-)^{21}_2)^{-1}(\mu_-)^{21}_1\beta_{+2}.$$

It is clear that the dependence of $U_-$ and $U_+$ on $z^\pm_i$ and $z^-i$, respectively, is parametric, and the general solution of the equations can be written as

$$U_- = (\xi_+)^{12} - \beta_{-1}^{-1}(I_n_1 - m_-(\mu_-)^{21}_2)^{-1}m_-\beta_{-2},$$

$$U_+ = (\xi_-)^{21} + \beta_{+2}^{-1}(I_n_1 - (\mu_+)^{12}_1)^{-1}\beta_{+1},$$

where $m_-$ and $m_+$ are arbitrary constant matrices of dimensions $n_1 \times n_2$ and $n_2 \times n_1$ respectively.

We have said nothing yet about solving integrability conditions (7.7) and (7.8). In the general case the solution to these equations is not known. However, they can be solved in some particular cases. For example, let $n = d + 1$, $n_1 = d$ and $n_2 = 1$, and let the mappings $X_{\mp i}$ be defined by the relations

$$(X_{-i})_{1j} = \delta_{ij}, \quad (X_{+i})_{j1} = \delta_{ij}.$$

In this case the general solution [10] of integrability conditions (7.7) and (7.8) is

$$(\beta_{-1}^{-1})_{ij} = F_-\partial_iH_{-j}, \quad (\beta_{-2}^{-1})_{ij} = F_-,$$

$$(\beta_{+1})_{ij} = F_+\partial_{+j}H_{+i}, \quad (\beta_{+2})_{ij} = F_+,$$

where $F_+$ and $H_{\mp i}$ are arbitrary functions depending on the coordinates $z^\mp i$. For the blocks $(\mu_-)^{21}_1$ and $(\mu_+)^{12}_1$ one has

$$(\mu_-)^{21}_1 = H_-, \quad (\mu_+)^{12}_1 = H_+,$$

where $H_-$ and $H_+$ are $1 \times d$ and $d \times 1$ matrices formed by the functions $H_{-i}$ and $H_{+i}$ respectively. Now using the evident notations we can write expressions (7.9) and (7.10) as

$$U_{-i} = \xi_{+i} + \partial_{-i}\log(1 - H_-m_-), \quad U_{+i} = \xi_{-i} - \partial_{+i}\log(1 - m_+H_+).$$

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