On the mathematical origin of quantum space-time

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Abstract An Euclidean topological space $E$ is homeomorphic to the subset of $\delta$-functions of the space $\mathcal{D}'(E)$ of Schwartz distributions on $E$. Herewith, any smooth function of compact support on $E$ is extended onto $\mathcal{D}'(E)$. One can think of these extensions as sui generis quantum deformations. In quantum models, one therefore should replace integration of functions over $E$ with that over $\mathcal{D}'(E)$.

A space-time in field theory, except noncommutative field theory, is traditionally described as a finite-dimensional smooth manifold, locally homeomorphic to an Euclidean topological space $E = \mathbb{R}^n$. The following fact (Proposition 1) enables us to think that a space-time might be a wider space of Schwartz distributions on $E$.

Let $E = \mathbb{R}^n$ be an Euclidean topological space. Let $\mathcal{D}(E)$ be the nuclear space of smooth complex functions of compact support on $E$. Its topological dual $\mathcal{D}'(E)$ is the space of Schwartz distributions on $E$, provided with the weak* topology [1, 2]. Since $\mathcal{D}(E)$ is reflexive and the strong topology on $\mathcal{D}'(E)$ is equivalent to the weak* one, $\mathcal{D}(E)$ is the topological dual of $\mathcal{D}'(E)$. Therefore, any continuous form on $\mathcal{D}'(E)$ is completely determined by its restriction

$$\langle \phi, \delta_x \rangle = \int \phi(x')\delta(x - x')d^n x = \phi(x), \quad x \in E,$$

to the subset $T_\delta(E) \subset \mathcal{D}'(E)$ of $\delta$-functions.

**Proposition 1.** The assignment

$$s_\delta : E \ni x \to \delta_x \in \mathcal{D}'(E) \quad (1)$$

is a homeomorphism of $E$ onto the subset $T_\delta(E) \subset \mathcal{D}'(E)$ of $\delta$-functions endowed with the relative topology (see Appendix for the proof).

As a consequence, $T_\delta(E)$ is isomorphic to the topological vector space $E$ with respect to the operations $\delta_x \oplus \delta_{x'} = \delta_{x+x'}$, $\lambda \odot \delta_x = \delta_{\lambda x}$. Moreover, the injection $E \to T_\delta(E) \subset \mathcal{D}'(E)$ is smooth [3]. Therefore, we can identify $E$ with a topological subspace $E = T_\delta(E)$ of the...
space of Schwartz distributions. Herewith, any smooth function $\phi$ of compact support on $E = T_\delta(E)$ is extended to a continuous form

$$\tilde{\phi}(w) = \langle \phi, w \rangle, \quad w \in \mathcal{D}'(E),$$

(2)
on the space of Schwartz distributions $\mathcal{D}'(E)$. One can think of this extension as being a quantum deformation of $\phi$ as follows.

The space $\mathcal{D}(E)$ is a dense subset of the Schwartz space $S(E)$ of smooth complex functions of rapid decrease on $E$. Moreover, the injection $\mathcal{D}(E) \hookrightarrow S(E)$ is continuous. The topological dual of $S(E)$ is the space $S'(E)$ of tempered distributions, which is a subset of the space $\mathcal{D}'(E)$ of Schwartz distributions. In QFT, one considers the Borchers algebra

$$A_S = \mathbb{C} \oplus S(E) \oplus S(E \oplus E) \oplus \cdots \oplus S(\oplus E) \oplus \cdots,$$

(3)
treated as a quantum algebra of scalar fields [4, 5]. Being provided with the inductive limit topology, the algebra $A_S$ (3) is an involutive nuclear barreled LF-algebra [6]. It follows that a linear form $f$ on $A_S$ is continuous iff its restriction $f_k$ to each $S(\oplus E)$ is well [1]. Therefore any continuous positive form on $A_S$ is represented by a family of tempered distributions $W_k \in S'(\oplus^k E), k = 1, \ldots,$ such that

$$f_k(\phi(x_1, \ldots, x_k)) = \int W_k(x_1, \ldots, x_k)\phi(x_1, \ldots, x_k)d^n x_1 \cdots d^n x_k, \quad \phi \in S(\oplus^k E).$$

(4)

For instance, the states of scalar quantum fields on the Minkowski space $\mathbb{R}^4$ are described by the Wightman functions $W_k \subset S'(\mathbb{R}^{4k})$ [2].

Any state of $A_S$ is also a state of its subalgebra

$$A_D = \mathbb{C} \oplus \mathcal{D}(E) \oplus \mathcal{D}(E \oplus E) \oplus \cdots \oplus \mathcal{D}(\oplus E) \oplus \cdots.$$

This quantization can be treated as follows. Given a function $\phi \in \mathcal{D}(\oplus^k E)$ on $\oplus^k E$, we have its quantum deformation

$$\hat{\phi} = \phi + f_k(\phi) \subset C^\infty(\oplus^k E).$$

(5)

Let $\oplus^k E$ be identified to the subspace $T_\delta(\oplus^k E) \subset \mathcal{D}'(\oplus^k E)$ of $\delta$-functions on $\oplus^k E$. Then the quantum deformation $\hat{\phi}$ (5) of $\phi$ comes from the extension of $\phi$ onto $\mathcal{D}'(\oplus^k E)$ by the formula

$$\hat{\phi}(z) = \phi(z + W_k), \quad z + W_k \in S'(\oplus^k E) \subset \mathcal{D}'(\oplus^k E).$$

Generalizing this construction, let us consider a continuous injection

$$s : \oplus^k E \ni z \to s_z \in \mathcal{D}'(\oplus^k E)$$

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and a continuous function

\[ s_\phi : k \oplus E \ni z \mapsto s_z(\phi) \in \mathbb{C}. \]

for any \( \phi \in \mathcal{D}(k \oplus E) \). For instance, the map \( s_\delta \) (1) where \( s_\delta,\phi = \phi \) is of this type. Given a function \( \phi \in \mathcal{D}(k \oplus E) \), we agree to call

\[ \hat{\phi} = \phi + s_\phi, \quad \hat{\phi}(z) = \phi(z) + s_z(\phi) = \phi(z + s_z) \]

(6)

the quantum deformation of \( \phi \) and to treat it as a function on the quantum space \( \hat{E} = (s_\delta + s)(E) \subset \mathcal{D}'(E) \).

For instance, let \( \phi(x, y) \in \mathcal{D}(E \oplus E) \) be a symmetric function on \( E \oplus E \). Then its quantum deformation (6) obeys the commutation relation

\[ \hat{\phi}(x, y) - \hat{\phi}(y, x) = \langle \phi, s_{x,y} - s_{y,x} \rangle. \]

Let \( E \) be coordinated by \( (x^\lambda) \), and let us consider a function \( x^1 x^2 \) on \( E \), though it is not of compact support. Let us choose a map \( s \) such that all distributions \( s_x, x \in E \), are of compact support. Its quantum deformation is \( \hat{x}^{1} x^{2} = x^{1} x^{2} + s_x(x^{1} x^{2}) \). It is readily observed that \( \hat{x}^{1} x^{2} - \hat{x}^{2} x^{1} = 0 \), i.e., coordinates on a quantum space commute with each other, in contrast to a space in noncommutative field theory.

Bearing in mind quantum deformations \( \hat{\phi} \) (2) of functions \( \phi \) on \( E \), one should replace integration of functions over \( E \) with that over \( \mathcal{D}'(E) \). Here, we summarize the relevant material on integration over the space of Schwartz distributions \( \mathcal{D}'(E) \).

I. Due to the homeomorphism (1), the space \( T_\delta(E) \) is provided with the measure \( d^nx \), invariant with respect to translations \( \delta_x \to \delta_{x+a} \).

II. The space \( M(E, \mathbb{C}) \) of measures on \( E \) is the topological dual of the space \( K(E, \mathbb{C}) \) of continuous functions of compact support on \( E \) endowed with the inductive limit topology (see Appendix). The space \( M(E, \mathbb{C}) \) is provided with the weak* topology. It is homeomorphic to a subspace of \( \mathcal{D}'(E) \) provided with the relative topology. It follows that, for any measure \( \nu \) on \( E \), there exists an element \( w_\nu \in \mathcal{D}'(E) \) and the Dirac measure \( \varepsilon_\nu \) of support at \( w_\nu \) such that, for each \( \phi \in \mathcal{D}(E) \), we have

\[ \int_E \phi \nu(x) = \langle \phi, w_\nu \rangle = \int_{\mathcal{D}'(E)} \langle \phi, w \rangle \varepsilon_\nu(w). \]

Let \( T_x \subset \mathcal{D}'(E) \) denote a subspace of point measures \( \lambda \delta_x, \lambda \in \mathbb{C} \), on \( E = T_\delta(E) \). It is a Banach space with respect to the norm \( ||\lambda \delta_x|| = |\lambda| \). Let us consider the direct product

\[ T(E) = \prod_{x \in E} T_x. \]
By analogy with the notion of a Hilbert integral [7], we define the Banach space integral 
\( (T(E), L(E), d^m x) \) where \( L(E) \) is a set of fields 
\( \tilde{\phi} : E \ni x \to \phi_x \delta_x \in T(E) \)

such that:

- the range of \( L(E) \) is a vector subspace of the direct product \( T(E) \) (7);
- there is a countable set \( \{ \varphi^i \} \) of elements of \( L(E) \) such that, for any \( x \in E \), the set \( \{ \varphi_x^i \} \) is total in \( T_x \);
- the function \( x \to ||\varphi_x|| = |\varphi_x| \) is \( d^m x \)-integrable for any \( \varphi \in L(E) \).

Let \( L(E) = L^2(E, d^m x) \) be the space of complex square \( d^m x \)-integrable functions on \( E \). Clearly, \( D(E) \subset L(E) \), and there is an injection \( L(E) \to M(E, \mathbb{C}) \subset D'(E) \) such that 
\[
\varphi(\phi) = \int \phi(x') \varphi_x \delta(x' - x) d^m x d^m x'.
\]

Therefore, let
\[
\int \phi_x \delta_x d^m x
\]
de note the image of \( \varphi \) in \( D'(E) \). Then any \( d^m x \)-equivalent measure \( \nu = c^2 d^m x \) (where \( c \in L^2(E, d^m x) \) is strictly positive almost everywhere on \( E \)) defines the corresponding element
\[
w_\nu = \int c^2(x) \delta_x d^m x
\]
of \( D'(E) \). For instance, if \( \nu = d^m x \), we have \( \varphi_x = 1 \) and
\[
w_\nu = \int \delta_x d^m x.
\]

**III.** Let \( Q \) be an arbitrary nuclear space (e.g., \( D(E), S(E) \)) and \( Q' \) its topological dual (e.g., \( D'(E), S'(E) \)). A complex function \( Z(q) \) on \( Q \) is called positive-definite if \( Z(0) = 1 \) and
\[
\sum_{i,j} Z(q_i - q_j) \bar{\lambda}_i \lambda_j \geq 0
\]
for any finite set \( q_1, \ldots, q_m \) of elements of \( Q \) and arbitrary complex numbers \( \lambda_1, \ldots, \lambda_m \).

In accordance with the well-known Bochner theorem for nuclear spaces [8, 9, 10], any continuous positive-definite function \( Z(q) \) on a nuclear space \( Q \) is the Fourier transform 
\[
Z(q) = \int \exp[i \langle q, w \rangle] \mu(w)
\]
(8)
of a positive measure $\mu$ of total mass 1 on the dual $Q'$ of $Q$, and *vice versa.*

Note that there is no translationally-invariant measure on $Q'$. Let a nuclear space $Q$ be provided with a separately continuous non-degenerate Hermitian form $\langle \cdot | \cdot \rangle$. In the case of $Q = D(E)$, we have

$$\langle \phi | \phi' \rangle = \int \overline{\phi} d^nx.$$ 

Let $w_q, q \in Q$, be an element of $Q'$ given by the condition $\langle q', w_q \rangle = \langle q'|q \rangle$ for all $q' \in Q$. These elements form the image of the monomorphism $Q \to Q'$ determined by the Hermitian form $\langle \cdot | \cdot \rangle$ on $Q$. If a measure $\mu$ in (8) remains equivalent under translations

$$Q' \ni w \mapsto w + w_q \in Q', \quad \forall w_q \in Q \subset Q',$$

in $Q'$, it is called translationally quasi-invariant. However, it does not remains equivalent under an arbitrary translation in $Q'$, unless $Q$ is finite-dimensional.

Gaussian measures exemplify translationally quasi-invariant measures on the dual $Q'$ of a nuclear space $Q$. The Fourier transform of a Gaussian measure reads

$$Z(q) = \exp \left[ -\frac{1}{2} B(q) \right],$$

where $B(q)$ is a seminorm on $Q'$ called the covariance form. Let $\mu_K$ be a Gaussian measure on $Q'$ whose Fourier transform

$$Z_K(q) = \exp \left[ -\frac{1}{2} B_K(q) \right]$$

is characterized by the covariance form $B_K(q) = \langle K^{-1}q | K^{-1}q \rangle$, where $K$ is a bounded invertible operator in the Hilbert completion $\tilde{Q}$ of $Q$ with respect to the Hermitian form $\langle \cdot | \cdot \rangle$. The Gaussian measure $\mu_K$ is translationally quasi-invariant. It is equivalent $\mu$ if

$$\text{Tr}(1 - \frac{1}{2} KK^+) < \infty.$$ 

For instance, the Gaussian measures $\mu$ and $\mu'$ possessing the Fourier transforms

$$Z(q) = \exp[-\lambda^2 \langle q|q \rangle], \quad Z(q) = \exp[-\lambda'^2 \langle q|q \rangle], \quad \lambda, \lambda' \in \mathbb{R},$$

are not equivalent if $\lambda \neq \lambda'$.

If the function $\mathbb{R} \ni t \to Z(tq)$ is analytic on $\mathbb{R}$ at $t = 0$ for all $q \in Q$, then one can show that the function $\langle q|u \rangle$ on $Q'$ (e.g., the extension $\tilde{\phi}$ (2) of $\phi$ onto $D'(E)$) is square $\mu$-integrable for all $q \in Q$. Moreover, the correlation functions can be computed by the formula

$$\langle q_1 \cdots q_n \rangle = i^{-n} \frac{\partial}{\partial \alpha^1} \cdots \frac{\partial}{\partial \alpha^n} Z(\alpha^i q_i)|_{\alpha^i=0} = \int \langle q_1, w \rangle \cdots \langle q_n, w \rangle \mu(w).$$
In particular, an integral of the function $\tilde{\phi}$ (2) over $\mathcal{D}'(E)$ reads
\[
\int \tilde{\phi} \mu(w) = \int \langle \phi' w \rangle \mu(w) = i \frac{\partial}{\partial \alpha} Z(\alpha \phi).
\]

Appendix

Let $\mathcal{K}(E, \mathbb{C})$ be the space of continuous complex functions of compact support on $E = \mathbb{R}^n$. For each compact subset $K \subset E$, we have a seminorm
\[
p_K(\phi) = \sup_{x \in K} |\phi(x)|
\]
on $\mathcal{K}(E, \mathbb{C})$. These seminorms provide $\mathcal{K}(E, \mathbb{C})$ with the topology of compact convergence. At the same time, $\mathcal{K}(E, \mathbb{C})$ is a Banach space with respect to the norm
\[
\|f\| = \sup_{x \in E} |\phi(x)|.
\]
Its normed topology, called the topology of uniform convergence, is finer than the topology of compact convergence. The space $\mathcal{K}(E, \mathbb{C})$ can also be equipped with another topology, which is especially relevant to integration theory. For each compact subset $K \subset E$, let $\mathcal{K}_K(E, \mathbb{C})$ be the vector subspace of $\mathcal{K}(E, \mathbb{C})$ consisting of functions of support in $K$. Let $\mathcal{U}$ be the set of all absolutely convex absorbent subsets $U$ of $\mathcal{K}(E, \mathbb{C})$ such that, for every compact $K$, the set $U \cap \mathcal{K}_K(E, \mathbb{C})$ is a neighborhood of the origin in $\mathcal{K}_K(E, \mathbb{C})$ under the topology of uniform convergence on $K$. Then $\mathcal{U}$ is a base of neighborhoods for the inductive limit topology on $\mathcal{K}(E, \mathbb{C})$ [11]. This is the finest topology such that the injection $\mathcal{K}_K(E, \mathbb{C}) \rightarrow \mathcal{K}(E, \mathbb{C})$ is continuous for each $K$. The inductive limit topology is finer than the topology of uniform convergence and, consequently, the topology of compact converges. The space $M(E, \mathbb{C})$ of complex measures on $E$ is the topological dual of $\mathcal{K}(E, \mathbb{C})$, endowed with the inductive limit topology. The space $M(E, \mathbb{C})$ is provided with the weak* topology, and $\mathcal{K}(E, \mathbb{C})$ is its topological dual. The following holds [10].

Lemma 2. Let $\varepsilon_x$ denote the Dirac measure of support at a point $x \in E$. The assignment
\[
s_\varepsilon : E \ni x \rightarrow \varepsilon_x \in M(E, \mathbb{C})
\]
is a homeomorphism of $E$ onto the subset $T_\varepsilon \subset M(E, \mathbb{C})$ of Dirac measures endowed with the relative topology.

Of course, $\mathcal{D}(E) \subset \mathcal{K}(E, \mathbb{C})$, but the standard topology of $\mathcal{D}(E)$ is finer than its relative topology as a subset of $\mathcal{K}(E, \mathbb{C})$. Let $\mathcal{D}_R(E)$ denote $\mathcal{D}(E) \subset \mathcal{K}(E, \mathbb{C})$ provided with the relative topology, and let $\mathcal{D}'_R(E)$ be its topological dual endowed with the weak* topology.
Then $M(E, \mathbb{C})$ is homeomorphic to a subspace of $\mathcal{D}'_R(E)$ provided with the relative topology. At the same time, $\mathcal{D}'_R(E)$ is a subspace of $\mathcal{D}'(E)$ endowed with the relative topology. Thus, we have the morphisms

$$E \xrightarrow{s_\varepsilon} M(E, \mathbb{C}) \xrightarrow{-} \mathcal{D}'_R(E) \xrightarrow{-} \mathcal{D}'(E),$$

whose composition leads to the homeomorphism $x \to \varepsilon_x = \delta_x d^n x \to \delta_x \ (1)$. 

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