Linear-Quadratic Mean Field Social Optimization with a Major Player

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Abstract

This paper considers a linear-quadratic (LQ) mean field control problem involving a major player and a large number of minor players, where the dynamics and costs depend on random parameters. The objective is to optimize a social cost as a weighted sum of the individual costs under decentralized information. We apply the person-by-person optimality principle in team decision theory to the finite population model to construct two limiting variational problems whose solutions, subject to the requirement of consistent mean field approximations, yield a system of forward-backward stochastic differential equations (FBSDEs). We show the existence and uniqueness of a solution to the FBSDEs and obtain decentralized strategies nearly achieving social optimality in the original large but finite population model.

Keywords: Mean field control, mean field approximation, person-by-person optimality, social optimum, decentralized control

Abbreviated title: Mean field social optimization

1 Introduction

Mean field dynamic decision problems have been extensively studied in the literature [5, 9, 11, 19, 20, 21, 29, 30, 40], and a central goal is to obtain decentralized strategies based on limited information for individual agents. In a noncooperative game theoretic context, decentralized solutions are developed in [19, 21] by applying consistent mean field approximations.

In a basic mean field decision model, all players (or agents) have comparably small influence and may be called peers. A modified modeling framework is to introduce one or a few major players interacting with a large number of minor players. Traditionally, models differentiating the strength of players have been studied in cooperative game theory, and they are customarily called mixed games with the players according called mixed players [16]; such literature only dealt with static models. The work [17] investigates an LQ mean field game involving a major player. The consideration of

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major and minor players in mean field control has attracted considerable interest addressing different nonlinear modeling aspects \[8, 12, 35\]. See \[4, 33\] for extension to hierarchical games, and \[27\] for the analysis of evolutionary games and their deterministic mean field limit under a principle agent.

On the other hand, cooperation in dynamic multi-agent decision problems is traditionally a well studied subject. For general cooperative differential games using various optimality notions, see \[39, 41, 44\]. Naturally, cooperative decision making in mean field models is of interest, especially from the point of view of addressing complexity \[20, 40\]. Such decision problems may be referred to as mean field teams for which the decision makers will also be called players or agents. The work \[20\] introduced an LQ social optimization problem where all the agents cooperatively minimize a social cost as the sum of their individual costs, and it shows that the consistency based approach in mean field games may be extended to this model by combining with a person-by-person optimality principle in team decision theory \[21, 42\]. The central result is the so-called social optimality theorem which states that the optimality loss of the obtained decentralized strategies becomes negligible when the population size goes to infinity \[20\]. The social optimum may be regarded as a specific Pareto optimum for the constituent agents. A mean field team is studied in \[43\] where a Markov jump parameter appears as a common source of randomness for all agents. An LQ mean field team is formulated in \[2\] by assuming mean field (i.e. the average state of the population) sharing for a given population size \(N\), which gives an optimal control problem with special partial state information. In a mixed player setting, \[8\] considers a nonlinear diffusion model and assumes that all minor players act as a team to minimize a common cost against the major player. Optimal control of McKean–Vlasov dynamics is analyzed in \[28\] and under some conditions it is shown that the optimal solution may be interpreted as the limit of the social optimum solution of \(N\)-players as \(N \to \infty\). Cooperative mean field control has applications in economic theory \[36\], collective motion control \[11, 38\], and power grids \[13\]. Furthermore, social optima are useful for studying efficiency of mean field games by providing a performance benchmark \[3, 18\].

For mean field teams with mixed players, the analysis in an LQ framework has been formulated in our earlier work \[22\], where partial analysis was presented by applying a state space augmentation technique to characterize the dynamics of the random mean field evolution. Later, \[23\] re-examined the problem by applying the person-by-person optimality principle adopted for the peer model in \[20\]. This paper further generalizes the model by including coupling in dynamics and random coefficients while \[22\] only considers cost coupling and deterministic coefficients. Specifically, the model parameters now depend on the Brownian motion of the major player. This suggests that the major player serves as a common source of randomness for all players, which has connections with mean field games with common noise \[5, 10, 11\]. In fact, the stochastic control literature \[7, 37\] has considered a similar randomness structure where the system coefficients depend on a smaller filtration, and such modeling has applications in finance \[26, 31\].

As in \[23\], our solution is to extend the person-by-person optimality argument of \[20\] to the current setting to deal with random mean field approximations due to the presence of the major player, and we solve two variational problems resulting from the major-minor player interactions. The linear backward stochastic differential equation (BSDE) \[6, 32\] technique adopted in this paper can treat the random mean field and coefficients in a unified manner. As it turns out, the consideration of the coupling in dynamics will necessitate delicate handling of a two-scale variational problem for the minor player. Note that for the person-by-person optimality principle only one player has control perturbation.
This feature is similar to mean field games where the equilibrium is tested by unilateral strategy changes. However, our performance characterization of social optimality must allow simultaneous control variations. The optimal control nature of our problem shares some similarity with mean field type optimal control \[14, 15\]. However, the later involves only a single decision maker which directly controls the state mean.

Throughout this paper, we use \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) to denote an underlying filtered probability space. Let \(S^n\) be the Euclidean space of \(n \times n\) real and symmetric matrices, \(S^n_+\) its subset of positive semi-definite matrices, and \(I_k\) the \(k \times k\) identity matrix. The Banach space \(L^2_{\mathcal{F}}(0, T; \mathbb{R}^k)\) consists of all \(\mathbb{R}^k\)-valued \(\mathcal{F}_t\)-adapted square integrable processes \(\{v(t), 0 \leq t \leq T\}\) with the norm \(\|v\|_{L^2_{\mathcal{F}}} = (E \int_0^T |v(t)|^2 dt)^{1/2}\). The Banach space \(L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^k)\) consists of all \(\mathbb{R}^k\)-valued \(\mathcal{F}_t\)-adapted essentially bounded processes \(\{v(t), 0 \leq t \leq T\}\) with the norm \(\|v\|_{L^\infty_{\mathcal{F}}} = \text{ess sup}_{t, \omega} |v(t)|\). We can similarly define such spaces with other choices of the filtration and the Euclidean space. Given a symmetric matrix \(M \geq 0\), the quadratic form \(z^T M z\) may be denoted as \(|z|^2_M\). For a matrix \(Z\), \(Z^\text{col}_i\) stands for the \(i\)th column of \(Z\). Some variables (such as \(X_0^N(t), u_i^N(t)\)) with a superscript of star are used for limiting models after taking mean field approximations. Let \(\{F_t^W, t \geq 0\}\) be the filtration by a Brownian motion \(\{W(t), t \geq 0\}\). We use \(C\) (or \(C_1\), etc.) to denote a generic constant which is independent of the population size \(N + 1\) and may change from place to place.

The organization of the paper is as follows. Section 2 formulates the social optimization problem with a major player. Sections 3 and 4 introduce two variational problems with random parameters for the major player and a representative minor player, respectively. The existence and uniqueness of the mean field social optimum solution is presented in section 5. An asymptotic social optimality theorem is established in section 6. Section 7 concludes the paper.

## 2 The Mean Field Social Optimization Model

Consider the LQ mean field decision model with a major player \(\mathcal{A}_0\) and minor players \(\{\mathcal{A}_i, 1 \leq i \leq N\}\). At time \(t \geq 0\), the states of \(\mathcal{A}_0\) and \(\mathcal{A}_i\) are, respectively, denoted by \(X_0^N(t)\) and \(X_i^N(t)\), \(1 \leq i \leq N\). The dynamics of the \(N + 1\) players are given by a system of linear stochastic differential equations (SDEs):

\[
\begin{align*}
dX_0^N(t) &= \left[A_0(t)X_0^N(t) + B_0(t)u_0^N(t) + F_0(t)X^{(N)}(t)\right] dt + D_0(t)dW_0(t), \\
dX_i^N(t) &= \left[A(t)X_i^N(t) + B(t)u_i^N(t) + F(t)X^{(N)}(t) + G(t)X_0^N(t)\right] dt + D(t)dW_i(t),
\end{align*}
\]

\[1 \leq i \leq N, \quad (2.1)\]

where \(X^{(N)}(t) = (1/N) \sum_{i=1}^{N} X_i^N(t)\) is the coupling term. The states \(X_0^N, X_i^N\) and controls \(u_0^N, u_i^N\) are, respectively, \(n\) and \(n_1\) dimensional vectors. The initial states \(X_0^N(0) = z_0, X_i^N(0) = x_i^0\), \(1 \leq i \leq N\), are deterministic. The coefficients in the dynamics are random. The noise processes \(W_0, W_i\) are \(n_2\) dimensional independent standard Brownian motions adapted to \(\mathcal{F}_t\). We choose \(\mathcal{F}_t\) as the \(\sigma\)-algebra \(\mathcal{F}_t^W := \sigma(W_j(\tau), 0 \leq j \leq N, \tau \leq t)\). Denote \(W_0 = [W_{01}, \ldots, W_{0n_2}]^T, F_0^W := \sigma(W_0(\tau), \tau \leq t)\), and \(F_t^W := \sigma(W_0(\tau), W_i(\tau), \tau \leq t)\).
For $0 \leq j \leq N$, denote $u_{-j}^N = (u_0, \ldots, u_{j-1}, u_{j+1}, \ldots, u_N)$. The cost for $A_0$ is given by

$$J_0(u_0^N, u_{-j}^N) = \mathbb{E} \int_0^T \left\{ |X_0^N(t) - \Psi_0(X^{(N)}(t))|^2_{Q_0} + (u_0^N(t))^T R_0(t) u_0^N(t) \right\} dt + E|X_0^N(T) - H_0, fX^{(N)}(T) - \eta_0f|_{Q_0}^2, \quad (2.3)$$

where $\Psi_0(X^{(N)}(t)) = H_0(t)X_0^N(t) + \eta_0(t)$. The cost for $A_i$, $1 \leq i \leq N$, is given by

$$J_i(u_i^N, u_{-i}^N) = \mathbb{E} \int_0^T \left\{ |X_i^N(t) - \Psi(X_i^N(t), X^{(N)}(t))|^2_{Q(t)} + (u_i^N(t))^T R(t) u_i^N(t) \right\} dt + E|X_i^N(T) - H_1fX_i^N(T) - H_2fX^{(N)}(T) - \eta_f|_{Q_i}^2, \quad (2.4)$$

where $\Psi(X_i^N(t), X^{(N)}(t)) = H_1(t)X_i^N(t) + H_2(t)X^{(N)}(t) + \eta(t)$. The terms $H_1(t)X_i^N(t)$ and $H_1fX_i^N(T)$ indicate the strong influence of the major player. Also, the parameters in the two costs are random.

Below we list the stochastic parameter processes

$$\{A_0(t), B_0(t), F_0(t), D_0(t), A(t), B(t), F(t), G(t), D(t), 0 \leq t \leq T\}, \quad (2.5)$$

$$\{H_0(t), H_1(t), H_2(t), Q_0(t), Q(t), R_0(t), R(t), \eta_0(t), \eta(t), 0 \leq t \leq T\}. \quad (2.6)$$

We introduce the standing assumptions for this paper.

(A1) We have

$$A_0, F_0, A, F, G, H_0, H_1, H_2 \in L^\infty_T \mathcal{F}_0(0, T; \mathbb{R}^{n \times n}),$$

$$B_0, B \in L^\infty_T \mathcal{F}_0(0, T; \mathbb{R}^{n \times m}), \quad D_0, D \in L^2_T \mathcal{F}_0(0, T; \mathbb{R}^{n \times n}),$$

$$\eta_0, \eta \in L^2_T \mathcal{F}_0(0, T; \mathbb{R}^n),$$

and

$$Q_0, Q \in L^\infty_T \mathcal{F}_0(0, T; S^n), \quad Q_0(t) \in S^n_+, \quad Q(t) \in S^n_+, \quad \forall t \in [0, T],$$

$$R_0, R \in L^\infty_T \mathcal{F}_0(0, T; S^{m^2}), \quad R_0(t) \geq c_1 I_n, \quad R(t) \geq c_1 I_n, \quad \forall t \in [0, T],$$

where $c_1 > 0$ is a fixed deterministic constant.

(A2) The terminal cost parameters

$$H_0f, Q_0f, H_1f, H_2f, Q_f, \quad (2.7)$$

are $\mathcal{F}_T$-measurable and essentially bounded, and $Q_0f, Q_f$ are $S^n_+$-valued. $\eta_0f$ and $\eta_f$ are $\mathcal{F}_T$-measurable and square integrable.

(A3) There exists a constant $c_2 > 0$ independent of $N$ such that $\sup_{j \geq 0} |x_{j0}^N| \leq c_2$ for the initial states, and $\lim_{N \to \infty} x_{0}^{(N)} = m_0$, where $x_{0}^{(N)} = \frac{1}{N} \sum_{i=1}^{N} x_{i0}^N$.

By (A1)–(A2), there exists a fixed constant $c_3$ such that

$$\text{ess sup}_{t, \omega} |\psi(t)| \leq c_3, \quad \text{ess sup}_{\omega} |\psi_f| \leq c_3,$$

where $\psi(t)$ (resp., $\psi_f$) stands for any entry in (2.5), (2.6) (resp., (2.7)).

For the rest of the paper, for a stochastic process $\{Z(t), 0 \leq t \leq T\}$ appearing in various equations and equalities, we may write $Z$ for $Z(t)$ by suppressing the time variable $t$ for which the interpretation should be clear from the context. For instance, we often drop $t$ in $A_0(t), B_0(t), X_i^N(t), Q(t), etc.$

Throughout the paper, we denote $Y^{(N)} = \frac{1}{N} \sum_{i=1}^{N} Y_i$, and $Y_{-i}^{(N)} = \frac{1}{N} \sum_{j \neq i}^{N} Y_j$ for $N$ vectors $Y_1, \ldots, Y_N$. 

4
2.1 The mean field social optimization problem

For the mean field social optimization problem, we attempt to minimize the following social cost

$$J_{soc}^{(N)}(u) = J_0 + \frac{\lambda}{N} \sum_{k=1}^{N} J_k,$$

where $u^N = (u_0^N, u_1^N, \ldots, u_N^N)$ and $\lambda > 0$. It is necessary to introduce the scaling factor $\lambda/N$ in order to obtain a well defined limiting problem when $N$ tends to infinity. In view of the dynamics and costs of the $N + 1$ players, $J_0$ and $J_i$, $i \geq 1$, are generally of the same order of magnitude. If $\lambda/N$ were replaced by 1, the limiting control problem would be too insensitive to the performance of the major player and become inappropriate.

For the model of $N + 1$ players, let the optimal control be denoted by

$$\tilde{u}^N = (\tilde{u}_0^N, \tilde{u}_1^N, \ldots, \tilde{u}_N^N),$$

where each $\tilde{u}_j$ belongs to $L^2_T(0, T; \mathbb{R}^{n_1})$. Since the optimal control problem minimizing $J_{soc}^{(N)}$ is a strictly convex optimization problem with $J_{soc}^{(N)} \to \infty$ as $\|u^N\|_{L^2_T} \to \infty$, such $\tilde{u}^N$ exists and is unique. However, this solution is not what we desire to obtain since each player needs centralized information. Instead, it will serve as a starting point for designing decentralized strategies.

3 The Major Player’s Variational Problem

Consider the variation $\tilde{u}^N \in L^2_T(0, T; \mathbb{R}^{n_1})$ and $u_0^N = \tilde{u}_0^N + \bar{u}_0^N$. Let the state processes $(\tilde{X}_j^N)_{j=0}^{N}$ correspond to $(\tilde{u}_j^N)_{j=0}^{N}$, and $(X_j^N)_{j=0}^{N}$ correspond to $(u_0^N, \bar{u}_0^N, \bar{u}_1^N, \ldots, \bar{u}_N^N)$. Write $X_j^N = \tilde{X}_j^N + \bar{X}_j^N$ for $0 \leq j \leq N$, where $\tilde{X}_j^N$ is the state variation of player $A_j$. Then

$$dX_0^N(t) = (A_0X_0^N + F_0X^N(t) + B_0u_0^N)dt + D_0dW_0(t),$$

$$dX^N(t) = [(A + F)X^N(t) + B\tilde{u}^N(t)]dt + \frac{D}{\sqrt{N}} \sum_{i=1}^{N} dW_i(t),$$

and

$$d\tilde{X}_0^N(t) = (A_0\tilde{X}_0^N + F_0\tilde{X}^N(t) + B_0\bar{u}_0^N)dt,$$

$$d\tilde{X}^N(t) = [(A + F)\tilde{X}^N(t) + G\bar{X}_0^N]dt,$$

where $\tilde{X}_0^N(0) = \tilde{X}^N(0) = 0$. Note that we have followed the convention of dropping the time variable $t$ in various places. It can be checked that $\tilde{X}_i^N = \tilde{X}^N$ on $[0, T]$ for all $1 \leq i \leq N$. Denote

$$\delta L_0^\tilde{N}(t) = \left\{ [X_0^N - (H_0\tilde{X}^N + \eta_0)]^TQ_0(X_0^N - H_0\tilde{X}^N) + (\tilde{u}_0^N)^TR_0\bar{u}_0^N + \lambda[(I - H_2)\tilde{X}^N - H_1\tilde{X}_0^N - \eta]^TQ[(I - H_2)\tilde{X}^N - H_1\tilde{X}_0^N] \right\}(t),$$

and

$$\delta L_0^N = \left\{ [X_0^N - (H_0f\tilde{X}^N + \eta_0f)]^TQ_0f(X_0^N - H_0f\tilde{X}^N) + \lambda[(I - H_2f)\tilde{X}^N - H_1f\tilde{X}_0^N - \eta_f]^TQ_f[(I - H_2f)\tilde{X}^N - H_1f\tilde{X}_0^N] \right\}(T).$$
The first variation of the social cost is given by
\[ \delta J_0 + \frac{\lambda}{N} \sum_{i=1}^{N} \delta J_i = 2E \int_0^T \delta L_0^N(t) dt + 2E \delta L_0^{Nf}, \]
which is a linear functional of \( \tilde{u}_0^N \). We have the first order variational condition:

**Lemma 3.1** We have
\[ E \int_0^T \delta L_0^N(t) dt + E \delta L_0^{Nf} = 0, \quad \forall \tilde{u}_0^N \in L_2^2(0, T; \mathbb{R}^{n_1}). \]

*Proof.* We prove by using the so called person-by-person optimality principle [21]. Take a constant \( \epsilon \) and let \( \tilde{u}_0^N \) be fixed. Then consider the control \((\tilde{u}_0^N + \epsilon \tilde{u}_0^N, \tilde{u}_1^N, \ldots, \tilde{u}_N^N)\) for the players. It follows that
\[ J_{soc}^{(N)}(\tilde{u}_0^N + \epsilon \tilde{u}_0^N, \tilde{u}_1^N, \ldots, \tilde{u}_N^N) \geq J_{soc}^{(N)}(\tilde{u}_0^N, \tilde{u}_1^N, \ldots, \tilde{u}_N^N) \]
for all \( \epsilon \), and the lemma follows from elementary estimates of the left hand side after an expansion around \( \tilde{u}_0 \). \( \square \)

### 3.1 The limiting variational problem for the major player

Consider the limiting model
\[ dX_0^*(t) = (A_0X_0^* + B_0u_0^* + F_0m)dt + D_0dW_0(t), \]
\[ dm(t) = [(A + F)m + B\tilde{u} + GX_0^*]dt, \]
where \( X_0^*(0) = X_0^N(0), m(0) = m_0, \) and \( \tilde{u} \in L_2^2(W_0(0, T; \mathbb{R}^{n_1})). \) Here \( \tilde{u} \) and \( m \) are used to approximate \( \hat{u}^{(N)} \) and \( X^{(N)} \) for large \( N \), respectively. Note that each \( \tilde{u}_j \in L_2^2(0, T; \mathbb{R}^{n_1}) \) is a centralized control in that it depends on all Brownian motions. However, as \( N \to \infty \), we expect the randomness originated in \((W_1, \ldots, W_N)\) will be averaged out. This has motivated the consideration of \( \tilde{u} \in L_2^2(W_0(0, T; \mathbb{R}^{n_1})). \)

For a particular control \( \tilde{u}_0^* \in L_2^2(W_0(0, T; \mathbb{R}^{n_1}) \), let the associated state process be
\[ d\tilde{X}_0^*(t) = (A_0\tilde{X}_0^* + B_0\tilde{u}_0^* + F_0\tilde{m})dt + D_0dW_0(t), \]
\[ d\tilde{m}(t) = [(A + F)\tilde{m} + B\tilde{u} + G\tilde{X}_0^*]dt, \]
where \( \tilde{X}_0^*(0) = X_0^N(0) \) and \( \tilde{m}(0) = m_0 \). The state variations read
\[ d\tilde{X}_0^*(t) = (A_0\tilde{X}_0^* + F_0\tilde{m} + B_0\tilde{u}_0^*)dt, \]
\[ d\tilde{m}(t) = [(A + F)\tilde{m} + G\tilde{X}_0^*]dt, \]
where \( \tilde{X}_0^*(0) = 0, \tilde{m}(0) = 0, \) and \( \tilde{u}_0^* \in L_2^2(W_0(0, T; \mathbb{R}^{n_1}) \) is the control variation.

Denote
\[ K_0(t) = -Q_0H_0 - \lambda H_1^TQ(I - H_2), \]
\[ M_0(t) = Q_0 + \lambda H_1^TQH_1, \]
\[ M(t) = H_1^TQ_0H_0 + \lambda(I - H_2)^TQ(I - H_2), \]
\[ \nu_0(t) = \lambda H_1^TQ\eta - Q_0\eta_0, \]
\[ \nu(t) = H_1^TQ\eta_0 + \lambda H_2^TQ\eta - \lambda Q\eta, \]
\[ R_\lambda(t) = \lambda R, \]
\[ \eta(t) = \int_0^t \eta(s)ds, \]
where the time variable \( t \) in various places of the right hand sides is suppressed. For the terminal cost, similarly define

\[
K_{0f} = -Q_{0f}H_{0f} - \lambda H_{1f}^T Q_f (I - H_{2f}),
\]

\[
M_{0f} = Q_{0f} + \lambda H_{1f}^T Q_f H_{1f},
\]

\[
M_f = H_{0f}^T Q_{0f} H_{0f} + \lambda (I - H_{2f})^T Q_f (I - H_{2f}),
\]

\[
\nu_{0f} = \lambda H_{1f}^T Q_f \eta_f - Q_{0f} \eta_{0f},
\]

\[
\nu_f = H_{0f}^T Q_{0f} \eta_{0f} + \lambda H_{2f}^T Q_f \eta_f - \lambda Q_f \eta_f.
\]

Define

\[
\delta L^*_0(t) = \left\{ [\dot{X}_0^* - (H_0 \dot{m} + \eta_0)]^T Q_0 (\dot{X}_0^* - H_0 \dot{m}) + (\dot{u}_0^*)^T R_0 \dot{u}_0^* + \lambda \left( (I - H_2) \dot{m}_f - H_1 \dot{X}_0^* - \eta_f^T Q_0 (I - H_2) \dot{m}_f - H_1 \dot{X}_0^* \right) \right\}(t)
\]

\[
= \left\{ (\dot{X}_0^*)^T (M_0 \dot{X}_0^* + K_0 \dot{m}_f + \nu_0) + \dot{m}_f^T (K_0^T \dot{X}_0^* + M_0 \dot{m}_f + \nu) + (\dot{u}_0^*)^T R_0 \dot{u}_0^* \right\}(t),
\]

\[
\delta L^*_{0f} = \left\{ (\dot{X}_0^*)^T (M_0 \dot{X}_0^* + K_0 \dot{m}_f + \nu_0) + \dot{m}_f^T (K_0^T \dot{X}_0^* + M_0 \dot{m}_f + \nu_f) \right\}(T),
\]

which are intended to approximate \( \delta L^N_0(t) \) and \( \delta L^N_{0f} \), respectively.

**Variational Problem (I) VP–(I):** Find \( \tilde{u}_0^* \in L^2_{fw_0}(0, T; \mathbb{R}^{n_1}) \) such that

\[
E \int_0^T \delta L^*_0(t) dt + E \delta L^*_{0f} = 0, \quad \forall \tilde{u}_0^* \in L^2_{fw_0}(0, T; \mathbb{R}^{n_1}). \tag{3.1}
\]

We call \( \tilde{u}_0^* \) or \( (\tilde{u}_0^*, \tilde{X}_0^*, \tilde{m}_f) \) a solution of VP–(I).

Suppose \((\tilde{u}_0^*, \tilde{X}_0^*, \tilde{m}_f)\) is a solution to VP–(I). We introduce the backward stochastic differential equations (BSDEs)

\[
dp_0(t) = (M_0 \tilde{X}_0^* + K_0 \tilde{m}_f - A_0^T p_0 - G^T p + \nu_0) dt + \xi_0 dW_0(t), \tag{3.2}
\]

\[
dp(t) = [K_0^T \tilde{X}_0^* + M_0 \tilde{m}_f - F_0^T p_0 - (A + F)^T p + \nu] dt + \xi dW_0(t), \tag{3.3}
\]

where

\[
p_0(T) = -(M_0 f \tilde{X}_0^*(T) + K_0 f \tilde{m}_f(T) + \nu_0), \quad p(T) = -(K_0^T f \tilde{X}_0^*(T) + M_0 f \tilde{m}(T) + \nu_f). \tag{3.4}
\]

**Lemma 3.2** If \((\tilde{u}_0^*, \tilde{X}_0^*, \tilde{m}_f)\) is a solution to VP–(I), then \( (3.2) - (3.3) \) has a unique solution \((p_0, p, \xi_0, \xi)\) in \( L^2_{fw_0}(0, T; \mathbb{R}^{2n}) \times L^2_{fw_0}(0, T; \mathbb{R}^{2n \times n_2}) \), and \( \tilde{u}_0^*(t) = R_0^{-1}(t) B_0(t)p_0(t) \).

**Proof.** From the linear BSDEs, we can solve a unique solution \((p_0, p, \xi_0, \xi)\). It follows from Itô’s formula that

\[
d[p^T(t) \tilde{X}_0^*(t) + p(t) \tilde{m}(t)]
\]

\[
= p_0^T(A_0 \tilde{X}_0^* + F_0 \tilde{m} + B_0 \tilde{u}_0^*) dt + p^T[(A + F) \tilde{m} + G \tilde{X}_0^*] dt
\]

\[
+ (\tilde{X}_0^*)^T (M_0 \tilde{X}_0^* + K_0 \tilde{m}_f - A_0^T p_0 - G^T p + \nu_0) dt + (\tilde{X}_0^*)^T \xi_0 dW_0(t)
\]

\[
+ \dot{m}_f^T [K_0^T \tilde{X}_0^* + M_0 \tilde{m}_f - F_0^T p_0 - (A + F)^T p + \nu] dt + \dot{m}_f^T \xi dW_0(t).
\]

7
Since $\tilde{X}_i^*(0) = \tilde{m}(0) = 0$, this implies
\[
E\left[p_0^T(T)\tilde{X}_0^*(T) + p^T(T)\tilde{m}(T)\right] = E \int_0^T \left[p_0^T B_0 \tilde{u}_0^* + (\tilde{X}_0^* + K_0 \tilde{m} + \nu_0)^T (M_0 \tilde{X}_0^* + M \tilde{m} + \nu)\right] dt.
\]
(3.5)

It follows from (3.1) and (3.5) that for any $\tilde{u}_0^* \in L^2_{\mathcal{F}w_0}(0, T; \mathbb{R}^{n_1})$,
\[
E \int_0^T (\tilde{u}_0^*)^T (B_0^T p_0 - R_0 \tilde{u}_0^*) dt = 0.
\]
(3.6)

The lemma follows. □

Given $\tilde{u} \in L^2_{\mathcal{F}w_0}(0, T; \mathbb{R}^{n_1})$, denote the forward-backward stochastic differential equation (FBSDE)
\[
\begin{cases}
\begin{aligned}
dX_0^*(t) &= (A_0 \tilde{X}_0^* + B_0 R_0^{-1} B_0^T p_0 + F_0 \tilde{m}) dt + D_0 dW_0(t), \\
\tilde{m}(t) &= [(A + F) \tilde{m} + B \tilde{u}] dt,
\end{aligned}
\end{cases}
\]
dp_0(t) = \left(M_0 \tilde{X}_0^* + K_0 \tilde{m} - A_0^T p_0 - G^T p + \nu_0\right) dt + \xi_0 dW_0(t),
\]
(3.7)

where $\tilde{X}_0^*(0) = X_0^N(0)$, $\tilde{m}(0) = m_0$, $p_0(T) = -(M_0 \tilde{X}_0^*(T) + K_0 \tilde{m}(T) + \nu_0)$, $p(T) = -(K_0^T \tilde{X}_0^*(T) + M \tilde{m}(T) + \nu_f)$.

To analyze (3.7), we introduce the notation:
\[
X_0 = \begin{bmatrix} \tilde{X}_0^* \\ \tilde{m} \end{bmatrix}, \quad Y_0 = \begin{bmatrix} p_0 \\ p \end{bmatrix}, \quad Z_0 = \begin{bmatrix} \xi_0 \\ \xi \end{bmatrix}, \quad A_0 = \begin{bmatrix} A_0 & F_0 \\ G & A + F \end{bmatrix}, \quad B_0 = \begin{bmatrix} B_0 \\ 0 \end{bmatrix},
\]
(3.8)

\[
\mathbb{E} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad \mathbb{D}_0 = \begin{bmatrix} D_0 \\ 0 \end{bmatrix}, \quad \mathbb{Q}_0 = \begin{bmatrix} M_0 & K_0 \\ K_0^T & M \end{bmatrix}, \quad \mathbb{v}_0 = \begin{bmatrix} \nu_0 \\ \nu \end{bmatrix},
\]
(3.9)

\[
\mathbb{Q}_{0f} = \begin{bmatrix} M_{0f} & K_{0f}^T \\ K_{0f} & M_f \end{bmatrix}, \quad \mathbb{v}_{0f} = \begin{bmatrix} \nu_{0f} \\ \nu_f \end{bmatrix}.
\]
(3.10)

**Lemma 3.3** $\mathbb{Q}_0(t)$ and $\mathbb{Q}_{0f}$ are positive semi-definite for all $0 \leq t \leq T$.

**Proof.** Since $\mathbb{Q}_0(t), Q(t)$ are symmetric and $\mathbb{Q}_0(t), Q(t) \succeq 0$, we can write $\mathbb{Q}_0(t) = U_0^T(t) U_0(t)$ and $Q(t) = U^T(t) U(t)$ for some $\mathbb{R}^{n \times n}$-valued random matrices $U_0(t), U(t)$. Denote
\[
U_0(t) = \begin{bmatrix} U_0 & -U_0(t) H_0 \\ 0 & 0 \end{bmatrix}(t), \quad U(t) = \sqrt{\lambda} \begin{bmatrix} U H_1 & -U (I - H_2) \\ 0 & 0 \end{bmatrix}(t).
\]

It is clear that $\mathbb{Q}_0(t)$ is symmetric and
\[
\mathbb{Q}_0(t) = \mathbb{U}_0^T(t) \mathbb{U}_0(t) + \mathbb{U}^T(t) \mathbb{U}(t) \succeq 0.
\]
The case of $\mathbb{Q}_{0f}$ can be similarly checked. □
Theorem 3.4 i) For any \( u \in L^2_{\mathcal{F}^T_0}(0,T;\mathbb{R}^{n_1}) \), \((3.7)\) has a unique solution \((\hat{X}^*_0, \hat{m}, p_0, p, \xi_0, \xi)\) in \( L^2_{\mathcal{F}^T_0}(0,T;\mathbb{R}^{n_1}) \times L^2_{\mathcal{F}^T_0}(0,T;\mathbb{R}^{2n_1\times n_2}) \).

ii) VP–(I) has a unique solution given by
\[
\hat{u}_0^*(t) = R_0^{-1}(t)B_0^T(t)p_0(t).
\]

Proof. i) We rewrite \((3.7)\) in the form
\[
\begin{aligned}
dX_0(t) &= (A_0 X_0 + B_0 \hat{u}_0 + F_0 X^N_0 + G X^N_0)dt + D_0 dW_0(t), \\
dY_0(t) &= (Q_0 X_0 + \theta^T(t) X_0 + v_0)dt + Z_0(t)dW_0(t),
\end{aligned}
\]

where \( Y_0(T) = -Q_0 X_0(T) - v_0T \). Under (A1)–(A2) and in view of Lemma 3.3, we apply Lemma A.2 to obtain the existence and uniqueness of a solution.

ii) We solve \((3.7)\) and choose \( \hat{u}_0^* \) by \((3.11)\). Such a control \( \hat{u}_0^* \) ensures \((3.6)\) while \((3.5)\) still holds; this further implies \((3.1)\). Hence, \( \hat{u}_0^* \) is a solution to VP–(I).

On the other hand, if \((\hat{u}_0^*, \hat{X}_0, \hat{m})\) is a solution to VP–(I) such that \((3.1)\) holds, by Lemma 3.2, \((3.7)\) holds and is uniquely solved, and uniqueness of \( \hat{u}_0^* \) follows from its representation in Lemma 3.2. \(\square\)

4 The Minor Player’s Variational Problem

Recall that the control \( \hat{u} \) yields state processes \( \hat{X}_j^N, j = 0, \ldots, N \). Now consider the control \((u_i^N, \hat{u}_{-i}^N)\) for a fixed \( i \geq 1 \). Note that the state of player \( A_j, 1 \leq j \neq i \leq N \), is affected even if only \( \hat{u}_i^N \) changes to \( u_i^N \). Under \((u_i^N, \hat{u}_{-i}^N)\), the state process of player \( A_j, 1 \leq j \neq i \leq N \), is
\[
dX_j^N(t) = (AX_j^N + B\hat{u}_j^N + FX^{(N)} + GX^N_0)dt + DdW_j(t).
\]

Thus,
\[
dX_0^N(t) = \left( A_0 X_0^N + B_0 \hat{u}_0^N + F_0 X^{(N)}_0 + \frac{1}{N} F_0 X^N_i \right)dt + D_0 dW_0(t), \\
dX^{(N)}_{-i}(t) = \left[ (A + F) X^{(N)}_j + B\hat{u}_j^N + \frac{1}{N} FX^N_i + GX^N_0 \right]dt \\
\quad + \frac{1}{N} \sum_{j \neq i} dW_j(t) - \frac{1}{N} \left( FX^{(N)}_{-i} + \frac{1}{N} FX^N_i + GX^N_0 \right)dt, \\
dX_i^N(t) = \left( AX_i^N + Bu_i^N + FX^{(N)} + GX^N_0 \right)dt + DdW_i(t) + \frac{1}{N} FX^N_i dt,
\]

where we use the notation \( Y^{(N)}_{-i} = \frac{1}{N} \sum_{j \neq i} Y_j \).

Let \( \hat{X}_j^N, 0 \leq j \leq N \), denote the state variations caused by \( \hat{u}_i^N \). The state variation of \( A_j, 1 \leq j \neq i \leq N \), is
\[
d\hat{X}_j^N(t) = \left( AX_j^N + F\hat{X}^{(N)}_{-i} + \frac{1}{N} FX^N_i + G\hat{X}_0^N \right)dt,
\]

where \( \hat{X}_j^N(0) = 0 \). This implies \( \hat{X}_j^N = \hat{X}_{j'}^N \) for all \( 1 \leq j, j' \neq i \). Now,
\[
\begin{aligned}
d\hat{X}_0^N(t) &= \left( A_0 \hat{X}_0^N + F_0 \hat{X}^{(N)}_{-i} + \hat{F}_0 X^N_i \right)dt, \\
d\hat{X}^{(N)}_{-i}(t) &= \left[ (A + F) \hat{X}^{(N)}_{-i} + F \frac{\hat{X}^N_i}{N} + G\hat{X}_0^N \right]dt - \frac{1}{N} \left( F\hat{X}^{(N)}_{-i} + F \frac{X^N_i}{N} + G\hat{X}_0^N \right)dt, \\
d\hat{X}^N_i(t) &= \left( AX^N_i + Bu^N_i \right)dt + \left( F\hat{X}^{(N)}_i + F \frac{X^N_i}{N} + G\hat{X}_0^N \right)dt,
\end{aligned}
\]

(4.1)
where $\tilde{X}_0^N(0) = \tilde{X}_{-i}^{(N)}(0) = \tilde{X}_i^N(0) = 0$, and $\tilde{u}_i^N \in L^2_T(0,T;\mathbb{R}^n)$.

**Lemma 4.1** There exists a constant $C$ independent of $N$ such that

$$\sup_{0 \leq t \leq T} E\left(\left|\tilde{X}_0^N(t)\right|^2 + \left|\tilde{X}_{-i}^{(N)}(t)\right|^2 + \frac{1}{N^2}\left|\tilde{X}_i^N(t)\right|^2\right) \leq \frac{C}{N^2} E \int_0^T \left|\tilde{u}_i^N(t)\right|^2 dt.$$ 

**Proof.** By solving the linear ODE of $(\tilde{X}_0^N, \tilde{X}_{-i}^{(N)}, \tilde{X}_i^N/N)$, we first have a uniform bound estimate on the fundamental solution matrix on $[0,T]$ and next obtain the estimate

$$\sup_{0 \leq t \leq T} (|\tilde{X}_0^N(t)| + |\tilde{X}_{-i}^{(N)}(t)| + |\tilde{X}_i^N(t)/N|) \leq \frac{C}{N} \int_0^T |\tilde{u}_i^N(s)| ds. \quad (4.2)$$

The lemma follows by applying Schwarz inequality. \qed

**Remark 4.2** It is seen that $(\tilde{X}_0^N, \tilde{X}_{-i}^{(N)})$ and $\tilde{X}_i^N$ have two different scales.

When the control changes from $(\tilde{u}_i^N, \tilde{u}_{-i}^N)$ to $(\tilde{u}_i^N + \tilde{u}_i^N, \tilde{u}_{-i}^N)$, the first variations of the costs have the following form

$$\frac{1}{2} \delta J_0 = E \int_0^T \chi_0(t) dt + E \chi_{0f}, \quad \frac{\lambda}{2N} \delta J_1 = E \int_0^T \chi_1(t) dt + E \chi_{1f},$$

$$\frac{\lambda}{2N} \sum_{1 \leq j \neq i} \delta J_j = E \int_0^T \chi_{-i}(t) dt + E \chi_{-if},$$

where

$$\chi_0 = \left[\tilde{X}_0^N - (H_0 \tilde{X}^{(N)} + \eta_0)\right]^T Q_0 (\tilde{X}_0^N - H_0 \tilde{X}_{-i}^{(N)} - \frac{1}{N} H_0 \tilde{X}_i^N),$$

$$\chi_1 = \left[\tilde{X}_i^N - (H_1 \tilde{X}_0^N + H_2 \tilde{X}^{(N)} + \eta)\right]^T \frac{1}{N} \lambda Q \left(\tilde{X}_i^N - H_1 \tilde{X}_0^N - H_2 \tilde{X}_{-i}^{(N)} - \frac{1}{N} H_2 \tilde{X}_i^N\right)$$

$$+ \left(\tilde{u}_i^N\right)^T \frac{1}{N} \lambda R \tilde{u}_i^N,$$

$$\chi_{-i} = \left[(I - H_2) \tilde{X}^{(N)} - H_1 \tilde{X}_0^N - \eta\right]^T \lambda Q \left[(I - H_2) \tilde{X}_{-i}^{(N)} - H_1 \tilde{X}_0^N - \frac{1}{N} H_2 \tilde{X}_i^N\right] + \mathcal{E}_1^N,$$

and

$$\chi_{0f} = \left[\tilde{X}_0^N - (H_{0f} \tilde{X}^{(N)} + \eta_{0f})\right]^T Q_{0f} (\tilde{X}_0^N - H_{0f} \tilde{X}_{-i}^{(N)} - \frac{1}{N} H_{0f} \tilde{X}_i^N)(T),$$

$$\chi_{1f} = \left[\tilde{X}_i^N - (H_{1f} \tilde{X}_0^N + H_{2f} \tilde{X}^{(N)} + \eta_f)\right]^T \frac{1}{N} \lambda Q_f \left(\tilde{X}_i^N - H_{1f} \tilde{X}_0^N - H_{2f} \tilde{X}_{-i}^{(N)} - \frac{1}{N} H_{2f} \tilde{X}_i^N\right)(T),$$

$$\chi_{-if} = \left[(I - H_{2f}) \tilde{X}^{(N)} - H_{1f} \tilde{X}_0^N - \eta_f\right]^T \lambda Q_f \left[(I - H_{2f}) \tilde{X}_{-i}^{(N)} - H_{1f} \tilde{X}_0^N - \frac{1}{N} H_{2f} \tilde{X}_i^N\right](T) + \mathcal{E}_f^N.$$ 

In the above,

$$\mathcal{E}_1^N = - \frac{\lambda}{N^2} \tilde{X}_i^N)^T Q \left[(I - H_2) \tilde{X}_{-i}^{(N)} - H_1 \tilde{X}_0^N - \frac{1}{N} H_2 \tilde{X}_i^N + \frac{1}{N} H_2 \tilde{X}_{-i}^{(N)}\right]$$

$$+ \frac{\lambda}{N^2 - 1} (\tilde{X}_i^N)^T Q \tilde{X}_i^{(N)}$$

$$- \frac{\lambda}{N} (H_1 \tilde{X}_0^N + H_2 \tilde{X}^{(N)} + \eta)^T Q \left(H_1 \tilde{X}_0^N + H_2 \tilde{X}_{-i}^{(N)} + \frac{1}{N} H_2 \tilde{X}_i^N\right). \quad (4.3)$$
See appendix B for the derivation of \((4.3)\). The derivation of \(E_j^N\) is similar and omitted here. We may regard \(E_1^N\) as a higher order term relative to the first term in \(\chi_{-i}\). Specifically, by Lemma 4.1 we have

\[
E|E_1^N(t)| = O\left(\frac{1}{N^2} \left( E[|\tilde{X}_i^N(t)|^2 + |\tilde{X}^{(N)}(t)|^2 + |\tilde{X}_0^N(t)|^2] \right)^{1/2} \left( E \int_0^T |\tilde{u}_i^N(t)|^2 dt \right)^{1/2} \right).
\]

We may give a similar upper bound for \(E|E_j^N|\) by using \(\left( E[|\tilde{X}_i^N(T)|^2 + |\tilde{X}^{(N)}(T)|^2 + |\tilde{X}_0^N(T)|^2] \right)^{1/2}\) in place of the middle factor of \(O(\cdot)\) above.

**Proposition 4.3** We have

\[
E \int_0^T (\chi_0 + \chi_i + \chi_{-i}) dt + E(\chi_{0f} + \chi_{if} + \chi_{-if}) = 0, \quad \forall \tilde{u}_i^N \in L^2_B(0, T; \mathbb{R}^{n_i}).
\]

**Proof.** The proof is similar to that of Lemma 3.1 and we omit the detail. \(\square\)

It can be shown that

\[
\chi_0 + \chi_i + \chi_{-i} = \left[ \tilde{X}_0^{N} - (H_0 \tilde{X}^{(N)} + \eta_0) \right] \!^T \! Q_0 \left( \tilde{X}_0^{N} - H_0 \tilde{X}_i^{(N)} - \frac{1}{N} H_0 \tilde{X}_i^{N} \right)
+ \left[ \tilde{X}_i^{N} - (H_1 \tilde{X}_0^{N} + H_2 \tilde{X}^{(N)} + \eta) \right] \!^T \! \frac{1}{N} \lambda Q \tilde{X}_i^{N} + (\tilde{u}_i^{N}) \!^T \! \frac{1}{N} \lambda R \tilde{u}_i^{N}
+ \left[ (I - H_2) \tilde{X}^{(N)} - H_1 \tilde{X}_0^{N} - \eta \right] \!^T \! \lambda Q \left[ (I - H_2) \tilde{X}_i^{(N)} - H_1 \tilde{X}_0^{N} - \frac{1}{N} H_2 \tilde{X}_i^{N} \right]
+ \mathcal{E}_2^N,
\]

where \(\mathcal{E}_2^N\) can again be treated as a higher order term, and we may similarly rewrite \(\chi_{0f} + \chi_{if} + \chi_{-if}\).

### 4.1 Limiting variational problem for the minor player

Consider

\[
\begin{align*}
d\tilde{X}_0^*(t) &= (A_0 \tilde{X}_0^* + B_0 \tilde{u}_0^* + F_0 \tilde{m}) dt + D_0 dW_0(t), \\
d\tilde{m}(t) &= ((A + F) \tilde{m} + B \tilde{u} + G \tilde{X}_0^*) dt, \\
d\tilde{X}_i^*(t) &= (A \tilde{X}_i^* + B \tilde{u}_i^* + F \tilde{m} + G \tilde{X}_0^*) dt + D dW_i(t),
\end{align*}
\]

where \(\tilde{u}_0^*\) has been determined from the solution of \(\text{VP}-(1)\), and \(\tilde{u}_i^* \in L^2_{\tilde{w}_0 \cdot w_i}(0, T; \mathbb{R}^{n_i}).\)

Denote the state variational equations

\[
\begin{cases}
d\tilde{X}_0^*(t) = \left( A_0 \tilde{X}_0^* + F_0 \tilde{m} + \frac{1}{N} F_0 \tilde{X}_i^* \right) dt, \\
d\tilde{m}(t) = \left[ (A + F) \tilde{m} + \frac{1}{N} F \tilde{X}_i^* + G \tilde{X}_0^* \right] dt, \\
d\tilde{X}_i^*(t) = (A \tilde{X}_i^* + B \tilde{u}_i^*) dt,
\end{cases}
\]

where \(\tilde{X}_0^*(0) = \tilde{m}(0) = \tilde{X}_i^*(0) = 0\), and \(\tilde{u}_i^* \in L^2_{\tilde{w}_0 \cdot w_i}(0, T; \mathbb{R}^{n_i})\).

**Remark 4.4** We see that \((\tilde{X}_0^*, \tilde{m})\) and \(\tilde{X}_i^*\) have different scales when \(N\) increases, which is similar to the case of \((\tilde{X}_0^N, \tilde{X}_i^{(N)})\) and \(\tilde{X}_i^N\).
We give some motivation for introducing the two variational equations in (4.4) containing the $1/N$ factor. For large $N$, if the perturbation $\tilde{u}_i^N$ is small, $J_{\text{soc}}(N)$ has a change by the order of $(1/N)(E \int_0^T |\tilde{u}_i^N|^2 ds)^{\frac{1}{2}}$. Thus we need to look at the optimizing behavior at a finer scale. For this reason the two $1/N$ scaled terms in (4.4) are significant, and as it turns out below, they ensure that $(\tilde{X}_0^*, \tilde{m})$ provides a good approximation to $(\tilde{X}_0^N, \tilde{X}_{-i}^N)$.

For approximating $\chi_0 + \chi_i + \chi_{-i}$, denote
\[
\delta L_i^*(t) = \left\{ [\tilde{X}_0^* - (H_0\tilde{m} + \eta_0)]^T Q_0 \left( \tilde{X}_0^* - H_0\tilde{m} - \frac{1}{N} H_0 \tilde{X}_i^* \right) + [\tilde{X}_i^* - (H_1 \tilde{X}_0^* + H_2 \tilde{m} + \eta)]^T \frac{1}{N} \lambda Q \tilde{X}_i^* + (\tilde{u}_i^*)^T \frac{1}{N} \lambda R \tilde{u}_i^* + \left[ (I - H_2) \tilde{m} - H_1 \tilde{X}_0^* - \eta \right]^T \lambda Q \left[ (I - H_2) \tilde{m} - H_1 \tilde{X}_0^* - \frac{1}{N} H_2 \tilde{X}_i^* \right] \right\}(t)
\]
\[
= (\tilde{X}_0^*)^T \left( M_0 \tilde{X}_0^* + K_0 \tilde{m} + \nu_0 \right) + \tilde{m}^T \left( K_{0f}^T \tilde{X}_0^* + M_f \tilde{m} + \nu_f \right) + \frac{\lambda}{N} \left( K_{0f}^T \tilde{X}_0^* + (M_f - \lambda Q \tilde{m}) + \nu_f + (\tilde{u}_i^*)^T \frac{1}{N} \lambda R \tilde{u}_i^* \right).
\]

In parallel to $\delta L_i^*(t)$, we introduce a term in the variational term
\[
\delta L_{i_f}^* = \left\{ (\tilde{X}_0^*)^T (M_{0f} \tilde{X}_0^* + K_{0f} \tilde{m} + \nu_{0f}) + \tilde{m}^T \left( K_{0f}^T \tilde{X}_0^* + M_f \tilde{m} + \nu_f \right) + \lambda N \left( K_{0f}^T \tilde{X}_0^* + (M_f - \lambda Q \tilde{m}) + \nu_f + (\tilde{u}_i^*)^T \frac{1}{N} \lambda R \tilde{u}_i^* \right) \right\}(T).
\]

**Variational Problem II VP–(II):** Find $\tilde{u}_i^* \in L^2_{x,w_0}(0, T; \mathbb{R}^{n_1})$ such that
\[
E \int_0^T \delta L_i^* dt + E \delta L_{i_f}^* = 0, \quad \forall \tilde{u}_i^* \in L^2_{x,w_0}(0, T; \mathbb{R}^{n_1}).
\]

The variational condition in VP–(II) may be regarded as an approximation of the person-by-person optimality property as stated in Proposition 4.3. The proposition below gives insights into the limiting variational problem VP–(II) and provides a justification for the form of (4.4).

**Proposition 4.5** Let $\tilde{u}_i^N = \tilde{u}_i^* = v$ in (4.1) and (4.4) for some fixed $v \in L^2_{x}(0, T; \mathbb{R}^{n_1})$. Then for some constant $C$ we have
\[
\sup_{0 \leq t \leq T} E \left[ |\tilde{X}_0^N(t) - \tilde{X}_i^*(t)|^2 + |\tilde{X}_{-i}^N(t) - \tilde{m}(t)|^2 + \left| \frac{1}{N} \tilde{X}_i^* - \frac{1}{N} \tilde{X}_i^N(t) \right|^2 \right] \leq \frac{C}{N^4}.
\]

**Proof.** Denote $\delta_0(t) = \tilde{X}_0^N - \tilde{X}_0^*$, $\delta_{-i}(t) = \tilde{X}_{-i}^N - \tilde{m}$, and $\delta_i(t) = \tilde{X}_i^N - \tilde{X}_i^*$. Then we write
\[
d\delta_0(t) = \left( A_0 \delta_0 + F_0 \delta_{-i} + F_0 \frac{\lambda}{N} \right) dt,
\]
\[
d\delta_{-i}(t) = \left[ (A + F) \delta_{-i} + F_0 \frac{\lambda}{N} + G \delta_0 \right] dt - \frac{1}{N} \left( F \tilde{X}_{-i}^N + F \tilde{X}_{-i}^N + G \tilde{X}_0^N \right) dt,
\]
\[
d\delta_i(t) = A_0 \delta_i dt + \frac{1}{N} \left( F \tilde{X}_{-i}^N + F \tilde{X}_{-i}^N + G \tilde{X}_0^N \right) dt,
\]
where $\delta_0(0) = \delta_{-i}(0) = \delta_i(0) = 0$. By assumption (A1),
\[
\sup_{0 \leq t \leq T} |\delta_0 + \delta_{-i} + (\delta_i/N)| \leq \frac{C}{N} \sup_{0 \leq t \leq T} (|\tilde{X}_{-i}^N| + |\tilde{X}_i^N| + |\tilde{X}_0^N|).
\]
Recalling (4.2), the proposition follows. □
For VP–(II) and the associated variational equations in (4.4), we try to identify adjoint processes \((q_0, q, q_i)\) such that the equality in VP–(II) is expressed only in terms of \(\hat{u}_i^*\) and \((\hat{X}_0^*, \hat{m}, \hat{X}_i^*, \hat{u}_i^*)\). Denote
\[
\begin{align*}
 dq_0(t) &= \psi_{11} dt + \psi_{12} dW_0(t) + \psi_{13} dW_i(t), \\
 dq(t) &= \psi_{21} dt + \psi_{22} dW_0(t) + \psi_{23} dW_i(t), \\
 dq_i(t) &= \psi_{31} dt + \psi_{32} dW_0(t) + \psi_{33} dW_i(t),
\end{align*}
\]
where \(\psi_{jk}\) and \(q_0(T), q(T), q_i(T)\) are to be determined. After elementary although tedious computations, it turns out that \((q_0, q)\) and \((p_0, p)\) in (3.7) are determined by exactly the same equations and terminal conditions. Thus, we may use the adjoint processes \((p_0, p, q_i)\) with the equation of \(q_i\) appropriately determined.

Let \((\hat{X}_0^*, \hat{m}, p_0, p)\) be solved first. After the above procedure of constructing the adjoint processes, we introduce the equation system
\[
\begin{align*}
 d\hat{X}_0^*(t) &= (A_0 \hat{X}_0^* + B_0 R_0^{-1} B_0^T p_0 + F_0 \hat{m}) dt + D_0 dW_0(t), \\
 dr(t) &= [(A + F) \hat{m} + B\hat{u} + G \hat{X}_0^*] dt, \\
 d\hat{X}_i^*(t) &= (A \hat{X}_i^* + BR_\lambda^{-1} B^T q_i + F \hat{m} + G \hat{X}_0^*) dt + D_i dW_i(t), \\
 dp_0(t) &= (M_0 \hat{X}_0^* + K_0 \hat{m} - A_0^T p_0 - G^T p + \nu_0) dt + \xi_0(t) dW_0(t), \\
 dp(t) &= [K_0^T \hat{X}_0^* + M \hat{m} - F_0^T p_0 - (A + F)^T p + \nu(t)] dt + \xi(t) dW_0(t), \\
 dq_i(t) &= [K_0^T \hat{X}_0^* + (M - \lambda Q) \hat{m} + \lambda Q \hat{X}_i^* - F_0^T p_0 - F^T p - A^T q_i + \nu(t)] dt + \zeta_{ai}(t) dW_0(t) + \zeta_{bi}(t) dW_i(t),
\end{align*}
\]
where
\[
\begin{align*}
 \hat{X}_0^*(0) &= X_0^N(0), \quad \hat{m}(0) = m_0, \quad \hat{X}_i^*(0) = X_i^N(0), \\
p_0(T) &= -(M_0 \hat{X}_0^*(T) + K_0 \hat{m}(T) + \nu_0), \quad p(T) = -(K_0^T \hat{X}_0^*(T) + M \hat{m}(T) + \nu_f), \\
q_i(T) &= -(K_0^T \hat{X}_0^*(T) + (M_f - \lambda Q_f) \hat{m}(T) + \lambda Q_f \hat{X}_i^*(T) + \nu_f).
\end{align*}
\]

**Theorem 4.6** Given \(\bar{u} \in L^2_{\mathcal{F}, w_0}(0, T; \mathbb{R}^m)\), (4.6) has a unique solution
\[
(\hat{X}_0^*, \hat{m}, \hat{X}_i^*, p_0, p, q_i, \xi_0, \xi, \zeta_{ai}, \zeta_{bi})
\]
such that
\[
(\hat{X}_0^*, \hat{m}, p_0, p, \xi_0, \xi) \in L^2_{\mathcal{F}, w_0}(0, T; \mathbb{R}^n) \times L^2_{\mathcal{F}, w_0}(0, T; \mathbb{R}^{2n \times n_2}), \\
(\hat{X}_i^*, q_i, \zeta_{ai}, \zeta_{bi}) \in L^2_{\mathcal{F}, w_0}(0, T; \mathbb{R}^{n}) \times L^2_{\mathcal{F}, w_0}(0, T; \mathbb{R}^{2n \times n_2}),
\]
and VP–(II) has a unique solution given by
\[
\hat{u}_i^*(t) = R_\lambda^{-1} B^T q_i(t). \quad (4.7)
\]

**Proof.** Note that \((\hat{X}_0^*, \hat{m}, p_0, p, \xi_0, \xi)\) is uniquely determined by Theorem 3.3. To proceed, denote
\[
\begin{align*}
 \chi_1(t) &= G \hat{X}_0^* + F \hat{m}, \\
 \chi_2(t) &= K_0^T \hat{X}_0^* + (M - \lambda Q) \hat{m} - F_0^T p_0 - F^T p + \nu.
\end{align*}
\]
We rewrite
\[
\begin{aligned}
    d\hat{X}_t^i(t) &= (A\hat{X}_t^i + BR_\lambda^{-1}B^T q_i + \chi_1)dt + Dw_i(t), \\
    dq_i(t) &= (\lambda Q\hat{X}_t^i - A^T q_i + \chi_2)dt + \zeta_{ai}(t)dW_0(t) + \zeta_{bi}(t)dW_i(t),
\end{aligned}
\]  
(4.8)
for which we obtain a unique solution by using Lemma A.2 with the vector Brownian motion \((W_0, W_i)\).

We proceed to show that (4.7) is a solution to VP–(II), where the associated state processes are \(\hat{X}_0^i, \hat{m}, \hat{X}_i^*\). Applying Itô’s formula to \(d[(\hat{X}_0^i)^T p_0 + \hat{m}^T p + (\hat{X}_i^*/N)^T q_i]\) gives the relation
\[
E[(\hat{X}_0^i)^T p_0 + \hat{m}^T p + (\hat{X}_i^*/N)^T q_i](T) = E\int_0^T \psi(\hat{X}_0^i, \hat{m}, \hat{X}_i^*, p_0, p, q_i)dt, \tag{4.9}
\]
where the integrand may be easily determined. Combining (4.9) with (4.7), we can show that \(\hat{u}_i^*\) satisfies the variational condition in VP–(II). The proof of uniqueness is similar to part ii) of Theorem 3.4. This is done by showing that a solution to VP–(II) is necessarily represented as (4.7) via solving (4.6). □

To further analyze (4.8), we introduce the backward stochastic Riccati differential equation (BSRDE)
\[
-dP_\lambda(t) = (P_\lambda A + A^T P_\lambda - P_\lambda BR_\lambda^{-1} B^T P_\lambda + \lambda Q)dt - \sum_{k=1}^{n_2} \Psi_k(t)dW_{0k}(t), \tag{4.10}
\]
\[P_\lambda(T) = \lambda Qf.\]
By Lemma A.1 we solve a unique \(P_\lambda \geq 0\) in \(L^\infty_{F\mathcal{W}_0}(0, T; S^n)\) with \(\Psi_k \in L^2_{F\mathcal{W}_0}(0, T; S^n)\). Denote the BSDE
\[
d\phi(t) = (P_\lambda BR_\lambda^{-1}B^T - A^T)\phi + P_\lambda \chi_1 + \chi_2 dt + \Lambda_0 dW_0(t),
\]
where \(\phi(T) = -(K_0 f_0 \hat{X}_0^*(T) + (M_f - \lambda Qf)\hat{m}(T) + \nu_f)\). We obtain a unique solution \((\phi, \Lambda_0)\) in \(L^2_{F\mathcal{W}_0}(0, T; R^n) \times L^2_{F\mathcal{W}_0}(0, T; R^{n \times n_2})\).

**Lemma 4.7** We have \(\zeta_{ai} = \Lambda_0 - [\psi_1 \hat{X}_1^*, \ldots, \psi_{n_2} \hat{X}_{n_2}^*]\) and \(\zeta_{bi} = \zeta_b := -P_\lambda D\).

**Proof.** By the method in proving Lemma A.2 we can show \(q_i\) is in fact given by \(-P_\lambda \hat{X}_i^* + \phi\). We further obtain the relation
\[P_\lambda D + \zeta_{bi} = 0, \quad \zeta_{ai} = \Lambda_0 - [\psi_1 \hat{X}_1^*, \ldots, \psi_{n_2} \hat{X}_{n_2}^*].\]
The lemma follows. □

5 Mean Field Social Optimum Solution

5.1 Consistency condition

So far we have assumed that \(\bar{u}(t) \in L^2_{F\mathcal{W}_0}(0, T; R^{n_1})\), as the approximation of \(\bar{u}^{(N)}(t)\), is known for solving the variational problems VP–(I) and VP–(II). Below we introduce a procedure to determine \(\bar{u}\).

Let VP–(II) be solved for \(i = 1, \ldots, N\), so that (4.6) determines
\[
\hat{u}_i^*(t) = R_\lambda^{-1} B^T q_i(t), \quad 1 \leq i \leq N.
\]
Denote
\[ q^{(N)}(t) = \frac{1}{N} \sum_{i=1}^{N} q_i(t), \quad \hat{u}^{(N)}(t) = \frac{1}{N} \sum_{i=1}^{N} \hat{u}_i^*(t), \quad \hat{X}^{(N)}(t) = \frac{1}{N} \sum_{i=1}^{N} \hat{X}_i^*(t). \]

It is plausible to approximate \( \hat{u} \) by \( \hat{u}^{(N)}(t) = R^{-1}_{\lambda} B^T q^{(N)}(t) \). We obtain
\[
dq^{(N)}(t) = \left\{ K_0^T \hat{X}_0^* + (M - \lambda Q) \hat{m} + \lambda Q \hat{X}^{(N)} - F_0^T p_0 - F^T p - A^T q^{(N)} + \nu \right\} dt + \zeta_o^{(N)} dW_0(t) + \frac{1}{N} \sum_{i=1}^{N} \zeta_b dW_i(t),
\]
where \( \zeta_o^{(N)} = \frac{1}{N} \sum_{i=1}^{N} \zeta_{a_i} \) and \( \zeta_0 \) is given in Lemma 4.7.

Recall that \( \hat{m} \) was introduced to approximate \( \hat{X}^{(N)}(T) \). Also, in view of Lemma 4.7, let \( \zeta_a^{(N)}(t) \) be approximated by \( \zeta_a(t) \). When \( N \to \infty \), the above equation of \( q^{(N)} \) is approximated by
\[
d\bar{q}(t) = \left\{ K_0^T \hat{X}_0^* + M \hat{m} - F_0^T p_0 - F^T p - A^T \bar{q} + \nu \right\} dt + \zeta_a dW_0(t),
\]
where \( \bar{q}(T) = -(K_0^T \hat{X}_0^*(T) + M \hat{m}(T) + \nu_f) \). We solve a unique solution \( (\bar{q}, \zeta_a) \in L^2_{\mathcal{F}w_0}(0,T; \mathbb{R}^n) \times L^2_{\mathcal{F}w_0}(0,T; \mathbb{R}^{n \times n_2}) \). The proof of the next lemma is straightforward and omitted here.

**Lemma 5.1** We have \( \bar{q}(t) = p(t) \) on \( [0,T] \).

Now we introduce the following consistency condition
\[
\bar{u} = R^{-1}_{\lambda} B^T p. \tag{5.1}
\]

We note that a fixed point property is embodied in (5.1). A similar situation also arises in mean field games [19, 21]. Given a general \( \bar{u}' \in L^2_{\mathcal{F}w_0}(0,T; \mathbb{R}^{n_1}) \), we solve VP–(I) to obtain a well-defined adjoint process \( p \in L^2_{\mathcal{F}w_0}(0,T; \mathbb{R}^{n_1}) \), and we use an operator to denote \( \Gamma(\bar{u}') = R^{-1}_{\lambda} B^T p \). So (5.1) is equivalent to the fixed point relation \( \bar{u} = \Gamma(\bar{u}). \)

Typically in a mean field game with mixed players, one determines the consistency condition by combining the solutions of the two optimization problems of the major player and a representative minor player [17, 34, 35]. For the present problem, indeed we may determine \( \bar{u} \) via \( \bar{q} \) after solving VP–(II). However, now \( \bar{p} \) and \( p \) coincide, and for this reason (5.1) is determined by the solution of VP–(I) alone.

### 5.2 The system of FBSDEs

Substituting \( \bar{u} \) above into (1.6), we introduce the new system
\[
\begin{align*}
\left\{ 
\begin{array}{l}
d\hat{X}_0^*(t) = (A_0 \hat{X}_0^* + F_0 \hat{m} + B_0 R^{-1}_\lambda B_0^T p_0) dt + D_0 dW_0(t), \\
d\hat{m}(t) = [G \hat{X}_0^* + (A + F) \hat{m} + BR^{-1}_\lambda B^T p] dt, \\
dp_0(t) = (M_0 \hat{X}_0^* + K_0 \hat{m} - A_0^T p_0 - G^T p + \nu_0) dt + \xi_0(t) dW_0(t), \\
dp(t) = [K_0^T \hat{X}_0^* + M \hat{m} - F_0^T p_0 - (A + F)^T p + \nu] dt + \xi(t) dW_0(t),
\end{array}
\right.
\end{align*}
\]

(5.2)
where \( \hat{X}_0^*(0) = X_0^N(0) \), \( \hat{m}(0) = m_0 \), \( p_0(T) = -(M_{0f}\hat{X}_0^*(T) + K_{0f}\hat{m}(T) + \nu_{0f}) \), \( p(T) = -(K_{0f}^T\hat{X}_0^*(T) + M_f\hat{m}(T) + \nu_f) \); and its solution is used to define

\[
\hat{\chi}_1(t) = G\hat{X}_0^*(t) + F\hat{m}(t),
\]
\[
\hat{\chi}_2(t) = K_f^T\hat{X}_0^*(t) + (M - \lambda Q)\hat{m}(t) - F_f^Tp_0(t) - F^Tp(t) + \nu.
\]

Note that (5.2) differs from (4.6) due to the elimination of \( \hat{u} \) by the consistency condition. To distinguish the associated processes, we use the new notation \( \hat{X}_0^* \) and \( \hat{m} \) in (5.2) in place of \( \hat{X}_0^* \) and \( \hat{m} \). However, the variables \( p_0, p \) are reused for the adjoint processes, and their identification should be clear from the context.

We further introduce

\[
\begin{align*}
d\hat{X}_i^*(t) = \left[ A\hat{X}_i^* + BR_i^{-1}B^Tq_i + \hat{\chi}_1 \right] dt + DdW_i(t),
dq_i(t) = \left[ \lambda Q\hat{X}_i^* - A^Tq_i + \hat{\chi}_2 \right] dt + \zeta_{ai}(t)dW_0(t) + \zeta_{o}(t)dW_i(t),
\end{align*}
\]

where \( \hat{X}_i^*(0) = X_i^N(0) \) and \( q_i(T) = -(K_{fi}^T\hat{X}_0^*(T) + (M_f - \lambda Q_f)\hat{m}(T) + \lambda Q_f\hat{X}_i^*(T) + \nu_f) \).

**Theorem 5.2** The FBSDE (5.2) has a unique solution \( (\hat{X}_i^*, \hat{m}, p_0, p, \xi_0, \xi) \) in

\[
L^2_{\mathcal{F}w_0}(0, T; \mathbb{R}^{4n}) \times L^2_{\mathcal{F}w_0}(0, T; \mathbb{R}^{2n})
\]

and subsequently we uniquely solve (5.3) to obtain \( (\hat{X}_i^*, q_i, \zeta_{ai}, \zeta_{o}) \) in

\[
L^2_{\mathcal{F}w_0,w_i}(0, T; \mathbb{R}^{2n}) \times L^2_{\mathcal{F}w_0,w_i}(0, T; \mathbb{R}^{2n}).
\]

**Proof.** We follow the notation in (3.8)–(3.10) and further denote

\[
X_0 = \begin{bmatrix} \hat{X}_0^* \\ \hat{m} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_0 & 0 \\ 0 & B \end{bmatrix}, \quad R_1 = \begin{bmatrix} R_0 & 0 \\ 0 & R \end{bmatrix}.
\]

We rewrite the system (5.2) in the form

\[
\begin{align*}
dX_0(t) &= \left[ A_0X_0(t) + B_1R_1^{-1}B_1^TY_0(t) \right] dt + D_0dW_0(t), \\
dY_0(t) &= \left[ Q_0X_0(t) - A_0^TY_0(t) + v_0 \right] dt + Z(t)dW_0(t),
\end{align*}
\]

where \( Y_0(T) = -Q_{0f}X_0(T) - v_{0f} \). By Lemma A.2, we uniquely solve \((X_0, Y_0, Z)\), and subsequently (5.3). This completes the proof. \( \square \)

Since \( Q_0 \geq 0 \) and \( Q_{0f} \geq 0 \), let \((P \geq 0, \Psi_1, \ldots, \Psi_{n_2})\) be the unique solution to the BSRDE

\[
-dP(t) = \left[ P(t)A_0 + A_0^TP(t) - P(t)B_1R_1^{-1}B_1^TP(t) + Q_0 \right] dt - \sum_{k=1}^{n_2} \Psi_k(t)dW_{0k}(t), \quad P_0(T) = Q_{0f}.
\]

We further uniquely solve

\[
d\varphi(t) = \left[ -A_0^T\varphi + P(t)B_1R_1^{-1}B_1^T\varphi + v_0 + \sum_{k=1}^{n_2} \Phi_kdW_{0k} \right] dt + \Lambda dW_0(t)
\]

with the terminal condition \( \varphi(T) = -v_{0f} \). Then we can write \( Y_0(t) = -P(t)X_0(t) + \varphi(t) \).

For (5.3), denote

\[
\hat{X}^{(N)}(t) = \frac{1}{N} \sum_{i=1}^{N} \hat{X}_i^*(t), \quad q^{(N)}(t) = \frac{1}{N} \sum_{i=1}^{N} q_i(t), \quad 0 \leq t \leq T.
\]
Lemma 5.3 For (5.2) and (5.3), there exists a constant $C$ independent of $N$ such that

$$
\epsilon_{1,N} := \sup_{0 \leq t \leq T} E\left( |\hat{X}^*(N)(t) - \hat{m}(t)|^2 + |q^*(N)(t) - p(t)|^2 \right) \leq C \left( \frac{1}{N} + |x_0^{(N)} - m_0|^2 \right).
$$

Proof. Denote $y_N(t) = \hat{X}^*(N)(t) - \hat{m}(t)$ and $r_N(t) = q^*(N)(t) - p(t)$. We have

$$
dy_N(t) = \left[ Ay_N(t) + BR^{-1}_\lambda B^T r_N(t) \right] dt + \frac{D}{N} \sum_{i=1}^N dW_i(t),
$$

$$
dr_N(t) = \left[ \lambda Q y_N(t) - A^T r_N(t) \right] dt + (\zeta^{(N)}_a - \zeta) dW_0(t) + \frac{\zeta_b}{N} \sum_{i=1}^N dW_i(t),
$$

where $r_N(T) = q^*(N)(T) - p(T) = \lambda Q f(\hat{m}(T) - \hat{X}^*(N)(T)) = -\lambda Q f y_N(T)$. Let $(P_\lambda, \Psi_1, \ldots, \Psi_{n_2})$ be solved from (4.10). Writing $r_N(t) = -P_\lambda y_N(t) + \psi_N(t)$, we obtain

$$
d\psi_N(t) = \left( P_\lambda(t) BR^{-1}_\lambda B^T - A^T \right) \psi_N(t) dt + \sum_{k=1}^{n_2} (\Psi_k y_N + (\zeta^{(N)}_a - \zeta) \xi^{col}_k) dW_{0k} + \frac{P_\lambda D + \zeta_b}{N} \sum_{i=1}^N dW_i(t),
$$

where $\psi_N(T) = q^*(N)(T) - p(T) + \lambda Q y_N(T) = 0$. Note that $P_\lambda D + \zeta_b = 0$ by Lemma 4.7. Take $Z$ with $Z^{col}_N = \Psi_k y_N + (\zeta^{(N)}_a - \zeta) \xi^{col}_k$. Then $\psi_N(t) = 0$ and we further determine $\Psi_k y_N + (\zeta^{(N)}_a - \zeta) \xi^{col}_k = 0$. Next, by use of (A1) and $r_N(t) = -P_\lambda y_N(t)$, we directly estimate $\sup_{0 \leq t \leq T} E|y_N(t)|^2$, which further gives a bound on $\sup_{0 \leq t \leq T} E|r_N(t)|^2$ since $P_\lambda$ is an essentially bounded process. \qed

6 Asymptotic Social Optimality

Denote by $U_{centr}$ the set of centralized controls consisting of all $u = (u_0, u_1, \ldots, u_N)$, where each $u_j \in L^2_T(0, T; \mathbb{R}^{n_1})$. For a general $u \in U_{centr}$, let the corresponding state processes be $(X_0^N, X_1^N, \ldots, X_N^N)$. We have the following equations

$$
dX_0^N(t) = (A_0 X_0^N + B_0 u_0^N + F_0 X^N) dt + D_0 dW_0(t),
$$

$$
dX_i^N(t) = (A^N X_i^N + B_i^N X^N) dt + D dW_i(t), \quad 1 \leq i \leq N.
$$

We combine (5.2) and (5.3) to write the following FBSDE

$$
\begin{align*}
\frac{dX_0^*}{dt} &= (A_0 \hat{X}_0^* + F_0 \hat{m} + B_0 R_0^{-1} B^T_0 p_0) dt + D_0 dW_0(t), \\
\frac{dm}{dt} &= [G \hat{X}_0^* + (A + F) \hat{m} + BR_0^{-1} B^T p] dt, \\
\frac{d\hat{X}_0^*}{dt} &= (A \hat{X}_0^* + BR_0^{-1} B^T q_0 + F \hat{m} + G \hat{X}_0^*) dt + D dW_0(t), \\
\frac{dp_0}{dt} &= (M_0 \hat{X}_0^* + K_0 \hat{m} - A^T_0 p_0 - G^T p + \nu_0) dt + \xi_0 dW_0(t), \\
\frac{dp}{dt} &= [K_0^T \hat{X}_0^* + M \hat{m} - F^T_0 p_0 - (A + F)^T p + \nu] dt + \xi dW_0(t), \\
\frac{dq_0}{dt} &= [K_0^T \hat{X}_0^* + (M - \lambda Q) \hat{m} + \lambda Q \hat{X}_0^* - F^T_0 p_0 - F^T p - A^T q_0 + \nu] dt + \zeta_{at}(t) dW_0(t) + \zeta_{bz}(t) dW_i(t),
\end{align*}
$$

(6.1)
where $\bar{X}_0^*(0) = X_i^N(0)$, $\hat{m}(0) = m_0$, $\bar{X}_i^*(0) = X_i^N(0)$, $p_0(T) = -(M_{0f}\bar{X}_0^*(T) + K_{0f}\hat{m}(T) + \nu_{0f})$, $p(T) = -(K_{0f}^T\bar{X}_0^*(T) + M_f\hat{m}(T) + \nu_f)$, $q_i(T) = -(K_{iy}^T\bar{X}_0^*(T) + (M_f - \lambda Q_f)\hat{m}(T) + \lambda Q_f \bar{X}_i^*(T) + \nu_f)$.

We use Theorem 5.2 to determine the unique solution $(\bar{X}_0^*; \hat{m}, \bar{X}_i^*; p_0, p, q_i, \xi, \xi_{ai}, \zeta_0)$ for (6.1). Denote the set of individual controls

$$\hat{u}_i^N = R_{0_1}^{-1} B_0^T p_0, \quad \hat{u}_i^N = R_{X_i^N}^{-1} B_i^T q_i, \quad 1 \leq i \leq N.$$

For $\hat{u} = (\hat{u}_0^N, \hat{u}_1^N, \ldots, \hat{u}_N^N)$, let the corresponding state processes be $(\bar{X}_0^N, \bar{X}_1^N, \ldots, \bar{X}_N^N)$.

$$d\bar{X}_0^N(t) = (A_0 \bar{X}_0^N + B_0 \hat{u}_0^N + F_0 \bar{X}(N)) dt + D_0 dW_0(t),$$
$$d\bar{X}_i^N(t) = (A \bar{X}_i^N + B \hat{u}_i^N + F \bar{X}(N) + G \bar{X}_0^N) dt + D dW_i(t), \quad 1 \leq i \leq N,$$

where $\bar{X}_j^N(0) = X_j^N(0)$ for $0 \leq j \leq N$. It follows that

$$d\bar{X}(N)(t) = [(A + F) \bar{X}(N) + B \hat{u}(N) + G \bar{X}_0^N] dt + \frac{D}{N} \sum_{i=1}^{N} dW_i(t). \quad (6.2)$$

Denote $\bar{X}_j^N(t) = X_j^N(t) - \bar{X}_j^N(t)$, $\hat{u}_j^N(t) = u_j^N(t) - \hat{u}_j^N(t)$ for $0 \leq j \leq N$, and

$$\bar{X}(N) = \frac{1}{N} \sum_{i=1}^{N} \bar{X}_i^N, \quad \hat{u}(N) = \frac{1}{N} \sum_{i=1}^{N} \hat{u}_i^N.$$

We obtain

$$d\bar{X}_0^N(t) = (A_0 \bar{X}_0^N + B_0 \hat{u}_0^N + F_0 \bar{X}(N)) dt, \quad (6.3)$$
$$d\bar{X}_i^N(t) = (A \bar{X}_i^N + B \hat{u}_i^N + F \bar{X}(N) + G \bar{X}_0^N) dt, \quad 1 \leq i \leq N,$$
$$d\bar{X}(N)(t) = [(A + F) \bar{X}(N) + B \hat{u}(N) + G \bar{X}_0^N] dt, \quad (6.4)$$

where $\bar{X}_0^N(0) = \bar{X}_0^N(0) = \bar{X}(N)(0) = 0$.

Denote

$$\epsilon_{2,N} = \sup_{0 \leq t \leq T} E\left(|\bar{X}_0^N(t) - \bar{X}_0^*(t)|^2 + |\bar{X}(N)(t) - \hat{m}(t)|^2 + |\hat{u}(N)(t) - \hat{u}(t)|^2\right),$$

where $\hat{u} = R_{X_i^N}^{-1} B_i^T p$ and $p$ is given by (6.1).

**Lemma 6.1** We have

$$\epsilon_{2,N} = O\left(\frac{1}{N} + |x_0^N - m_0|^2\right).$$

**Proof.** Note that $\hat{u}(N)(t) - \hat{u}(t) = R_{X_i^N}^{-1} B_i^T (q(N)(t) - p(t))$. Under (A1), Lemma 5.3 implies

$$\sup_{0 \leq t \leq T} E|\hat{u}(N)(t) - \hat{u}(t)|^2 = O\left(\frac{1}{N} + |x_0^N - m_0|^2\right).$$

Denote

$$y_0(t) = \bar{X}_0^N(t) - \bar{X}_0^*(t), \quad y(N)(t) = \bar{X}(N)(t) - \bar{X}^*(N)(t).$$
Then \( y_0(t) \) and \( y_N(t) \) satisfy the following linear ODE:

\[
\frac{d}{dt} \begin{bmatrix} y_0(t) \\ y_N(t) \end{bmatrix} = \begin{bmatrix} A_0 & F_0 \\ G & A + F \end{bmatrix} \begin{bmatrix} y_0(t) \\ y_N(t) \end{bmatrix} + \begin{bmatrix} F_0(\hat{X}^{(N)}(t) - \hat{m}) \\ F(\hat{X}^{(N)}(t) - \hat{m}) \end{bmatrix}, \quad \begin{bmatrix} y_0(0) \\ y_N(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Since all the parameter processes are bounded and \( \sup_{0 \leq t \leq T} E|\hat{X}^{(N)}(t) - \hat{m}(t)|^2 \leq C(\frac{1}{\sqrt{N}} + |x_0^{(N)} - m_0|^2) \) by Lemma 5.3, the lemma follows. \( \square \)

Now we are ready to state the asymptotic social optimality theorem.

**Theorem 6.2** We have

\[
\left| J_{\text{soc}}^{(N)}(\hat{u}) - \inf_{u \in \mathcal{U}_{\text{centr}}} J_{\text{soc}}^{(N)}(u) \right| = O\left( \frac{1}{\sqrt{N}} + |x_0^{(N)} - m_0| \right).
\]

The importance of the theorem comes from the fact that the set of decentralized individual controls \((\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_N)\) can optimize \( J_{\text{soc}}^{(N)}(u) \) with little optimality loss in comparison with centralized controls. The rest of this section is devoted to the proof of this theorem.

### 6.1 Some Technical Lemmas

Denote

\[
\begin{align*}
\Delta_0^N(t) &= (\hat{X}_0^N - \Psi_0(\hat{X}^{(N)}))^T Q_0(\hat{X}_0^N - H_0 \hat{X}^{(N)}), \\
\Delta_i^N(t) &= (\hat{X}_i^N - \Psi(\hat{X}_0^N, \hat{X}^{(N)}))^T Q(\hat{X}_i^N - H_1 \hat{X}_0^N - H_2 \hat{X}^{(N)}), \\
\Delta_{0f}^N &= \{ (\hat{X}_0^N - H_{0f} \hat{X}^{(N)} - \eta_{0f})^T Q_{0f}(\hat{X}_0^N - H_{0f} \hat{X}^{(N)}) \}(T), \\
\Delta_{if}^N &= \{ (\hat{X}_i^N - H_{1f} \hat{X}_0^N - H_{2f} \hat{X}^{(N)} - \eta_{if})^T Q_{if}(\hat{X}_i^N - H_{1f} \hat{X}_0^N - H_{2f} \hat{X}^{(N)}) \}(T).
\end{align*}
\]

**Lemma 6.3** For any \( u \in \mathcal{U}_{\text{centr}} \), we have

\[
J_{\text{soc}}^{(N)}(u) \geq J_{\text{soc}}^{(N)}(\hat{u}) + 2E \int_0^T \left[ \Delta_0^N(t) + \frac{\lambda}{N} \sum_{i=1}^N \Delta_i^N(t) + (\hat{u}_0^N)^T R_0 \hat{u}_0^N + \frac{\lambda}{N} \sum_{i=1}^N (\hat{u}_i^N)^T R_i \hat{u}_i^N \right] dt \\
+ 2E \left( \Delta_{0f}^N + \frac{\lambda}{N} \sum_{i=1}^N \Delta_{if}^N \right).
\]

**Proof.** We check the integrands of \( J_0 \) and \( J_i \) to obtain

\[
\begin{align*}
|X_0^N - \Psi(X^{(N)})|^2_{Q_0} + (\hat{u}_0^N)^T R_0 \hat{u}_0^N \\
= |\hat{X}_0^N - \Psi_0(\hat{X}^{(N)}) + \hat{X}_0^N - H_0 \hat{X}^{(N)}|^2_{Q_0} + |\hat{u}_0^N + \hat{u}_0^N|^2_{R_0} \\
\geq |\hat{X}_0^N - \Psi_0(\hat{X}^{(N)})|^2_{Q_0} + (\hat{u}_0^N)^T R_0 \hat{u}_0^N + 2\Delta_0^N + 2(\hat{u}_0^N)^T R_0 \hat{u}_0^N
\end{align*}
\]

and similarly,

\[
\begin{align*}
|X_i^N - \Psi(X_i^N, X^{(N)})|^2_{Q_i} + (\hat{u}_i^N)^T R_i \hat{u}_i^N \\
\geq |\hat{X}_i^N - \Psi(\hat{X}_0^N, \hat{X}^{(N)})|^2_{Q_i} + (\hat{u}_i^N)^T R_i \hat{u}_i^N + 2\Delta_i^N + 2(\hat{u}_i^N)^T R_i \hat{u}_i^N.
\end{align*}
\]

We further check the terminal costs in \( J_0 \) and \( J_i \) to obtain the estimate. \( \square \)
We give a prior estimate on $\tilde{X}_0$ and $\tilde{X}^{(N)}$. By elementary estimate we can show that there exists a constant $\hat{C}_0$ independent of $N$ such that
\[
J_{soc}^{(N)}(\hat{u}_0^N, \hat{u}_1^N, \ldots, \hat{u}_N^N) \leq \hat{C}_0.
\]
For the estimate below it suffices to consider a set of individual controls $u = (u_0^N, \ldots, u_N^N) \in U_{centr}$ such that
\[
J_{soc}^{(N)}(u_0^N, u_1^N, \ldots, u_N^N) \leq \hat{C}_0. \tag{6.5}
\]
Denote all $u$ satisfying (6.5) by the set $U_0$.

**Lemma 6.4** For all $u \in U_0$, there exists $C_1$ such that
\[
\sup_{0 \leq t \leq T} E\left( |\tilde{X}_0^N(t)|^2 + |\tilde{X}^{(N)}(t)|^2 \right) dt \leq C_1.
\]

**Proof.** By use of (A1)–(A3) and direct SDE estimates for (6.1) we can show that,
\[
\sup_{0 \leq t \leq T} E \int_0^T |\tilde{u}_j^N(t)|^2 dt \leq C.
\]
Thus for $u \in U_0$, we have
\[
E \int_0^T |\tilde{u}_j^N(t)|^2 dt \leq 2E \int_0^T \left( |u_j^N(t)|^2 + |\tilde{u}_j^N(t)|^2 \right) dt \leq 2E \int_0^T |u_j^N(t)|^2 dt + C, \quad 0 \leq j \leq N.
\]
Since $R_0(t), R(t) \geq c_1 I$ by (A1), (6.5) implies
\[
E \int_0^T \left( |\tilde{u}_0^N(t)|^2 + |\tilde{u}^{(N)}(t)|^2 \right) dt \leq 2E \int_0^T \left( |u_0^N(t)|^2 + \frac{1}{N} \sum_{i=1}^N |u_i^N(t)|^2 \right) dt + C \leq C_2,
\]
where $C_2$ depends on $\hat{C}_0$.

By (6.3) and (6.4), we obtain for any $0 \leq t \leq T$,
\[
|\tilde{X}_0^N(t)| + |\tilde{X}^{(N)}(t)| \leq C \int_0^T \left( |\tilde{u}_0^N(t)| + |\tilde{u}^{(N)}(t)| \right) dt,
\]
and by applying Schwarz inequality,
\[
E \left( |\tilde{X}_0^N(t)|^2 + |\tilde{X}^{(N)}(t)|^2 \right) \leq CE \int_0^T \left( |\tilde{u}_0^N(t)|^2 + |\tilde{u}^{(N)}(t)|^2 \right) dt \leq C_1.
\]
This completes the proof. \(\square\)

Denote
\[
\Theta(t) = \left\{ (\tilde{X}_0^N)^T(M_0 \tilde{X}_0^* + K_0 \tilde{m} + \nu_0) + \tilde{u}_0^N R \tilde{u}_0^N \right. \\
+ (\tilde{X}^{(N)})^T [K_0^T \tilde{X}_0^* + (M - \lambda Q) \tilde{m} + \nu] + \frac{\lambda}{N} \sum_{i=1}^N ((\tilde{X}_i^N)^T Q \tilde{X}_i^* + (\tilde{u}_i^N)^T R \tilde{u}_i^N) \right\}(t),
\]
and
\[
\Theta_f = \left\{ (\tilde{X}_0^N)^T(M_{0f} \tilde{X}_0^* + K_{0f} \tilde{m} + \nu_{0f}) \right. \\
+ (\tilde{X}^{(N)})^T [K_{0f}^T \tilde{X}_0^* + (M_f - \lambda Q_f) \tilde{m} + \nu_f] + \frac{\lambda}{N} \sum_{i=1}^N ((\tilde{X}_i^N)^T Q_f \tilde{X}_i^*) \right\}(T).
\]
Lemma 6.5 Suppose \( u \in U_{\text{centr}} \). Then

\[
E \int_0^T \Theta(t) dt + E \Theta_f + E \int_0^T [(\hat{X}^{(N)})^T F^T + (\hat{X}_0^N)^T G^T](q^{(N)} - p) dt = 0. \tag{6.6}
\]

If, in addition, \( u \in U_0 \), then

\[
\left| E \int_0^T \Theta(t) dt + E \Theta_f \right| \leq C \left( E \int_0^T |q^{(N)} - p|^2 dt \right)^{1/2}.
\]

Proof. We have

\[
d(p_0^T \hat{X}_0^N) = [(\hat{X}_0^N)^T (M_0 \hat{X}_0^* + K_0 \hat{m} - G^T p + \nu_0) + p_0^T (B_0 \tilde{u}_0^N + F_0 \hat{X}^{(N)})] dt
\]

\[
+ (\hat{X}_0^N)^T \xi_0 dW_0.
\]

Then

\[
E \int_0^T g_1(t) dt + E \{(\hat{X}_0^N(T))^T (M_0 f \hat{X}_0^*(T) + K_0f \hat{m}(T) + \nu_0 f) \} = 0, \tag{6.7}
\]

where

\[
g_1(t) = (\hat{X}_0^N)^T (M_0 \hat{X}_0^* + K_0 \hat{m} - G^T p + \nu_0) + p_0^T (B_0 \tilde{u}_0^N + F_0 \hat{X}^{(N)}).
\]

By checking \( d\left( \frac{1}{N} \sum_{i=1}^N q_i^T \hat{X}_i^N \right) \), we obtain

\[
E \int_0^T g_2 dt + \frac{1}{N} \sum_{i=1}^N E \{(\hat{X}_i^N(T))^T [K_0^T \hat{X}_i^* + (M - \lambda Q) \hat{m} - F_0^T p_0 - F^T p + \nu + F^T q^{(N)}] \}
\]

\[
+ (\hat{X}_0^N)^T G^T q^{(N)} + \frac{\lambda}{N} \sum_{i=1}^N (\hat{X}_i^N)^T Q \hat{X}_i^* + \frac{1}{N} \sum_{i=1}^N (\tilde{u}_i^N)^T B^T q_i.
\]

Then

\[
g_1(t) + g_2(t) = [(\hat{X}_0^N)^T (M_0 \hat{X}_0^* + K_0 \hat{m} + \nu_0) + \tilde{u}_0^N R \tilde{u}_0^N]
\]

\[
+ (\hat{X}^{(N)})^T [K_0^T \hat{X}_0^* + (M - \lambda Q) \hat{m} + \nu]
\]

\[
+ \frac{\lambda}{N} \sum_{i=1}^N (\hat{X}_i^N)^T Q \hat{X}_i^* + \frac{\lambda}{N} \sum_{i=1}^N (\tilde{u}_i^N)^T R \tilde{u}_i^N
\]

\[
+ (\hat{X}^{(N)})^T F^T (q^{(N)} - p) + (\hat{X}_0^N)^T G^T (q^{(N)} - p)
\]

\[
= \Theta + (\hat{X}^{(N)})^T F^T (q^{(N)} - p) + (\hat{X}_0^N)^T G^T (q^{(N)} - p).
\]

By (6.7)–(6.8), we derive (6.6). The remaining part follows by applying Schwarz theorem and Lemma 6.4. \( \square \)
6.2 Proof of Theorem 6.2

We have

\[
\Delta_0^N + \frac{\lambda}{N} \sum_{i=1}^{N} \Delta_i^N = (\hat{\lambda}^N)^T \left[ (Q_0 + \lambda H_1^T Q H_1) \hat{\lambda}^N - (Q_0 H_0 + \lambda H_1^T Q (I - H_2)) \hat{\lambda}^N + \lambda H_1^T Q \eta - Q_0 \eta \right] \\
+ (\hat{\lambda}^N)^T [H_0^T Q_0 H_0 - \lambda Q H_2 - \lambda H_2^T Q (I - H_2)] \hat{\lambda}^N \\
+ (\hat{\lambda}^N)^T (H_1^T Q H_1 - H_0^T Q_0 - \lambda Q H_1) \hat{\lambda}^N \\
+ (\hat{\lambda}^N)^T (\lambda H_2^T Q \eta + H_0^T Q_0 \eta_0) \\
+ \frac{\lambda}{N} \sum_{i=1}^{N} (\hat{\lambda}_i^N)^T Q \hat{\lambda}_i^N \\
= (\hat{\lambda}^N)^T (M_0 \hat{\lambda}_0^N + K_0 \hat{\lambda}^N + \nu_0) + (\hat{\lambda}^N)^T [(M - \lambda Q) \hat{\lambda}^N + K_0^T \hat{\lambda}_0^N + \nu] \\
+ \frac{\lambda}{N} \sum_{i=1}^{N} (\hat{\lambda}_i^N)^T Q \hat{\lambda}_i^N.
\]

Since

\[
d(\hat{\lambda}_i^N(t) - \hat{\lambda}_i^*(t)) = \left[ A(\hat{\lambda}_i^N - \hat{\lambda}_i^*) + F(\hat{\lambda}_i^N) - \hat{m} + G(\hat{\lambda}_i^N - \hat{\lambda}_0^N) \right] dt,
\]

and \( \hat{\lambda}_i^N(0) - \hat{\lambda}_i^*(0) = 0 \) for \( 1 \leq i \leq N \), we obtain

\[
\hat{\lambda}_i^N(t) - \hat{\lambda}_i^*(t) = \hat{\lambda}^N(t) - \hat{\lambda}_i^*(t).
\]

So

\[
\Delta_0^N + \frac{\lambda}{N} \sum_{i=1}^{N} \Delta_i^N + (\hat{\lambda}_i^N)^T R_0 \hat{\lambda}_0^N + \frac{\lambda}{N} \sum_{i=1}^{N} (\hat{\lambda}_i^N)^T R \hat{\lambda}_i^N - \Theta \\
= (\hat{\lambda}^N)^T [M_0 (\hat{\lambda}_0^N - \hat{\lambda}_0^*) + K_0 (\hat{\lambda}^N) - \hat{m}] \\
+ (\hat{\lambda}^N)^T [K_0^T (\hat{\lambda}_0^N - \hat{\lambda}_0^*) + (M - \lambda Q) (\hat{\lambda}^N) - \hat{m}] \\
+ \frac{\lambda}{N} \sum_{i=1}^{N} (\hat{\lambda}_i^N)^T Q (\hat{\lambda}_i^N - \hat{\lambda}_i^*) \\
= (\hat{\lambda}^N)^T [M_0 (\hat{\lambda}_0^N - \hat{\lambda}_0^*) + K_0 (\hat{\lambda}^N) - \hat{m}] \\
+ (\hat{\lambda}^N)^T [K_0^T (\hat{\lambda}_0^N - \hat{\lambda}_0^*) + M (\hat{\lambda}^N) - \hat{m}) - \lambda Q (\hat{\lambda}_i^*(N) - \hat{m})].
\]

(6.9)

In a similar manner, we can show

\[
\Delta_0^N + \frac{\lambda}{N} \sum_{i=1}^{N} \Delta_i^N - \Theta_f = \left\{ (\hat{\lambda}^N)^T [M_{0f} (\hat{\lambda}_0^N - \hat{\lambda}_0^*) + K_{0f} (\hat{\lambda}^N) - \hat{m}] \\
+ (\hat{\lambda}^N)^T [K_{0f}^T (\hat{\lambda}_0^N - \hat{\lambda}_0^*) + M_f (\hat{\lambda}^N) - \hat{m}) - \lambda Q_f (\hat{\lambda}_i^{*(N)} - \hat{m})] \right\} (T).
\]

(6.10)

It follows from (6.9) and (6.10) that

\[
K_1 := E \int_0^T \left[ \Delta_0^N + \frac{\lambda}{N} \sum_{i=1}^{N} \Delta_i^N + (\hat{\lambda}_i^N)^T R_0 \hat{\lambda}_0^N + \frac{\lambda}{N} \sum_{i=1}^{N} (\hat{\lambda}_i^N)^T R \hat{\lambda}_i^N \right] dt \\
+ E(\Delta_0^N + \frac{\lambda}{N} \sum_{i=1}^{N} \Delta_i^N) \\
\leq \left| E \int_0^T \Theta(t) dt + E \Theta_f \right| + CE \int_0^T \phi_N(t) dt + CE \phi_N(T),
\]

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where $\phi_N(t) = \{(|\hat{X}_0^N(t)| + |\hat{X}^{(N)}|) (|\hat{X}_0^N - \hat{X}^N|) + |\hat{X}^{(N)} - \hat{m}| + |\hat{X}^{(N)} - \hat{m}|\}(t)$. Lemmas 6.4 and 6.5 imply that

$$K_1 \leq C(\epsilon_{1, N} + \epsilon_{2, N})^{1/2}.$$ 

By Lemma 6.3 and the above upper bound for $K_1$, for all $u \in U_0$, we have

$$J_{soc}^{(N)}(\hat{u}) \leq J_{soc}^{(N)}(u) + O((\epsilon_{1, N} + \epsilon_{2, N})^{1/2}),$$

which is automatically true when $u$ is not in $U_0$. Recalling Lemmas 5.3 and 6.1 we complete the proof. □

7 Conclusion

This paper studies an LQ mean field social optimization problem with mixed players. The solution is obtained by exploiting a person-by-person optimality principle and constructing two low dimensional limiting variational problems. This method derives an FBSDE system for the major player and a representative minor player. We prove the existence and uniqueness of the solution to the FBSDE and establish asymptotic social optimality for the resulting decentralized controls of the $N + 1$ players.

Appendix A

Lemma A.1 [37, 25] Assume

i) $\{\hat{W}(t) = [\hat{W}_1(t), \ldots, \hat{W}_l(t)]^T, t \geq 0\}$ is an $\mathbb{R}^l$-valued standard Brownian motion;

ii) $\{\hat{A}(t), \hat{B}(t), \hat{Q}(t), \hat{R}(t), 0 \leq t \leq T\}$ are $\mathcal{F}_t^\hat{W}$-adapted essentially bounded processes and are $\mathbb{R}^{k \times k}$, $\mathbb{R}^{k \times l}$, $S^k$, $S^{k_1}$-valued, respectively; $\hat{R}(t) \geq \alpha I$ for a deterministic constant $\alpha > 0$; and $\hat{Q}_f$ is $S^k$-valued, $\mathcal{F}_T^\hat{W}$-measurable, and essentially bounded.

Then the backward stochastic Riccati differential equation (BSRDE)

$$\begin{align*}
-dP(t) &= (\hat{A}^T P + P \hat{A} - P \hat{B}R^{-1}\hat{B}^T P + \hat{Q})(t) dt - \sum_{i=1}^l \Psi_i(t) d\hat{W}_i(t), \\
\hat{P}(T) &= \hat{Q}_f
\end{align*}$$

has a unique $\mathcal{F}_t^\hat{W}$-adapted solution $(P, \Psi_1, \ldots, \Psi_l)$ satisfying that $P$ is $S^k_+$-valued and essentially bounded, and that each $\Psi_i \in L^2_{\mathcal{F}_t^\hat{W}}(0, T; S^k)$.

More general forms of this Riccati equation were studied in [37, sec. 5], [25, sec. 2], where $\Psi_i$ also appears linearly in the drift term. The proof method was presented in [37, sec. 5] by applying quasi-linearization of the Riccati equation.

We further introduce the assumption

$$g, v \in L^2_{\mathcal{F}_t^\hat{W}}(0, T; \mathbb{R}^k), \quad \hat{D} \in L^2_{\mathcal{F}_t^\hat{W}}(0, T; \mathbb{R}^{k \times l}), \quad v_f \text{ is } \mathcal{F}_T^\hat{W}-\text{measurable, } E|v_f|^2 < \infty. \quad (A.1)$$

Consider the FBSDE

$$\begin{align*}
dX(t) &= (\hat{A}X + \hat{B}\hat{R}^{-1}\hat{B}Y + g)dt + \hat{D}d\hat{W}(t), \\
dY(t) &= (\hat{Q}X - \hat{A}^TY + v)dt + Zd\hat{W}(t),
\end{align*}$$

(A.2)
where $Y(T) = -\hat{Q}_f X(T) - v_f$ and $X(0) = x_0 \in \mathbb{R}^k$.

Denote the linear BSDE
\[
d\psi(t) = \left(-\hat{A}^T \psi + P \hat{B} \hat{R}^{-1} \hat{B}^T \psi + P g + v \right) dt + \Lambda d\hat{W}(t),
\]
where $\psi(T) = -v_f$. There exists a unique solution $(\psi, \Lambda) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^k) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k \times l})$.

**Lemma A.2** Suppose the assumptions in Lemma A.1 and (A.1) hold, then (A.2) has a unique solution $(X, Y, Z)$ in $L^2_{\mathcal{F}}(0, T; \mathbb{R}^k) \times L^2_{\mathcal{F}}(0, T; \mathbb{R}^{k \times l})$, and
\[
Y = -PX + \psi, \quad Z_i^{\text{col}} = \Lambda_i^{\text{col}} - P \hat{D}_i^{\text{col}} - \Psi_i X.
\]

**Proof.** To show existence, consider the SDE
\[
dx(t) = (\hat{A} X + \hat{B}^T \hat{R}^{-1} \hat{B} (-PX + \psi) + g) dt + \hat{D} d\hat{W}(t), \quad X(0) = x_0,
\]
which has a unique solution; we choose $Y = -PX + \psi$. By Itô’s formula, we derive
\[
dy = (\hat{Q} X - \hat{A}^T Y + v) dt + \sum_{i=1}^l [\Lambda_i^{\text{col}} - P \hat{D}_i^{\text{col}} - \Psi_i X] d\hat{W}_i.
\]
We choose $Z_i^{\text{col}} = [\Lambda_i^{\text{col}} - P \hat{D}_i^{\text{col}} - \Psi_i X]$ for all $i \leq l$. Then $(X, Y, Z)$ constructed above is a solution to (A.2).

To show uniqueness, suppose there is another solution $(X', Y', Z')$. Denote $\tilde{X} = X - X'$, $\tilde{Y} = Y - Y'$ and $\tilde{Z} = Z - Z'$. So $\tilde{Y}(T) = -\hat{Q}_f \tilde{X}(T)$. Denote $\tilde{Y} = -P \tilde{X} + \tilde{\varphi}$, where $\tilde{\varphi}$ is to be determined. By Itô’s formula,
\[
d\tilde{\varphi} = (-\hat{A}^T \tilde{\varphi} + P \hat{B} \hat{R}^{-1} \hat{B}^T \tilde{\varphi}) dt + \tilde{Z} d\hat{W} + \sum_{i=1}^l \Psi_i \tilde{X} d\hat{W}_i,
\]
where $\tilde{\varphi}(T) = 0$. Note that $\tilde{X}$ has been given and $(\tilde{\varphi}, \tilde{Z})$ is a solution to the above linear BSDE. We necessarily have $\tilde{\varphi} = 0$ and $\tilde{Z}_i^{\text{col}} = -\Psi_i \tilde{X}$. We can further show $\tilde{X} = \tilde{Y} = 0$ and $\tilde{Z} = 0$. This proves uniqueness. \qed

**Appendix B**

**Derivation of (4.3):** We have the first order cost variation: For $1 \leq j \neq i$,
\[
\frac{1}{N} \delta J_j = \left[ X_j^N - (H_1 X_0^N + H_2 X_i^{(N)} + \eta) \right]^T \frac{1}{N} Q \left( X_j^N - H_1 X_0^N - H_2 X_i^{(N)} - \frac{1}{N} H_2 X_i^N \right).
\]

By the fact that all $X_j$, $1 \leq j \neq i$, are equal, we calculate
\[
\Delta_1 := \sum_{1 \leq j \neq i} (X_j^N)^T \frac{1}{N} \lambda Q \left( X_j^N - H_1 X_0^N - H_2 X_i^{(N)} - \frac{1}{N} H_2 X_i^N \right)
\]
\[
= (\bar{X}_i^{(N)})^T \frac{1}{N} \lambda Q \left( \bar{X}_i^{(N)} - H_1 X_0^N - H_2 X_i^{(N)} - \frac{1}{N} H_2 X_i^N \right)
\]
\[
= (X_i^{(N)} - \frac{1}{N} X_i^N)^T \frac{1}{N} \lambda Q \left( \frac{N}{N-1} \bar{X}_i^{(N)} - H_1 X_0^N - H_2 X_i^{(N)} - \frac{1}{N} H_2 X_i^N \right)
\]
\[
= (X_i^{(N)} - \frac{1}{N} X_i^N)^T \frac{1}{N} \lambda Q \left( (I - H_2) \bar{X}_i^{(N)} - H_1 X_0^N - \frac{1}{N} H_2 X_i^N + \frac{1}{N-1} \bar{X}_i^{(N)} \right),
\]
\[ \Delta_2 := \sum_{1 \leq j \neq i} (H_1 \tilde{X}_i^N + H_2 \tilde{X}^{(N)} + \eta)^T \lambda Q \left( \tilde{X}_j^N - H_1 \tilde{X}_0^N - H_2 \tilde{X}^{(N)} - \frac{1}{N} H_2 \tilde{X}_i^N \right) \]
\[ = (H_1 \tilde{X}_0^N + H_2 \tilde{X}^{(N)} + \eta)^T \lambda Q \left( \tilde{X}_i^{(N)} - H_1 \tilde{X}_0^N - H_2 \tilde{X}^{(N)} - \frac{1}{N} H_2 \tilde{X}_i^N \right) \]
\[ = (H_1 \tilde{X}_0^N + H_2 \tilde{X}^{(N)} + \eta)^T \lambda Q \left( I - H_2 \right) \tilde{X}_i^{(N)} - H_1 \tilde{X}_0^N - \frac{1}{N} H_2 \tilde{X}_i^N + \frac{1}{N} (H_1 \tilde{X}_0^N + H_2 \tilde{X}^{(N)} + \frac{1}{N} H_2 \tilde{X}_i^N) \].

We may write
\[ \frac{\lambda}{2N} \sum_{j \neq i}^N \delta J_j = \Delta_1 - \Delta_2. \]

Subsequently, we determine the form of \( E_1^N \) as in (4.3).

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