ON SUM OF HECKE EIGENVALUE SQUARES OVER PRIMES
IN VERY SHORT INTERVALS

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Abstract. Let $\eta > 0$ be a fixed positive number, let $N$ be a sufficiently
large number. In this paper, we study the second moment of the sum of Hecke
eigenvalues over primes in short intervals (whose length is $\eta \log N$) on average
(with some weights) over the family of weight $k$ holomorphic Hecke cusp forms.
We also generalize the above result to Hecke-Maass cusp forms for $SL(2,\mathbb{Z})$
and $SL(3,\mathbb{Z})$. By applying the Hardy-Littlewood prime 2-tuples conjecture,
we calculate the exact values of the mean values.

1. Introduction

Let $\mathbb{H} = \{ z = x + iy | x \in \mathbb{R}, y \in (0, \infty) \}, G = SL(2,\mathbb{Z})$. Define $j_{\gamma}(z) = (cz + d)^{-1}$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. When a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfies

$$f(\gamma z) = j_{\gamma}(z)^k f(z)$$

for all $\gamma \in SL(2,\mathbb{Z})$, it is called a modular form of weight $k$. It is well known that any modular form $f(z)$ has a Fourier expansion at the cusp $\infty$

$$f(z) = \sum_{n=0}^{\infty} b_n e(nz)$$

(1.1)

where $e(z) = e^{2\pi iz}$, and the normalized Fourier coefficient $a(n)$ of $f(z)$ is defined by

$$a(n) := b_n n^{-\frac{k-1}{2}}.$$

(1.2)

The set of all modular forms of fixed weight $k$ is a vector space and we use $M_k$
to denote this. And we denote the set of all modular forms in $M_k$ which have a
zero constant term by $C_k$. Let $T_n$ be the n-th Hecke operator on $C_k$, which means

$$(T_n f)(z) := \frac{1}{\sqrt{n}} \sum_{ad=n \ b(\text{mod } d)} f\left(\frac{az+b}{d}\right)$$

for all $f \in C_k$.

It is known that there is an orthonormal basis of $C_k$ which consists of eigen-
functions for all Hecke operators $T_n$, and these are called Hecke cusp forms. In

2020 Mathematics Subject Classification. Primary 11F30; Secondary 11N05, 11F72.
Key words and phrases. Hecke eigenvalue; Primes; Kuznetsov Formula.
this paper, we use $S_k$ to refer this orthonormal basis of $C_k$. When $f$ is a Hecke
cusp form, the eigenvalues $\lambda_f(n)$ of the $n$-th Hecke operator satisfy

$$a(n) = a(1)\lambda_f(n).$$

For details, see [4, Chapter 14].

For convenience, we assume that $k$ is always even natural number and summing
over the index $p$ denotes summing over primes. $P$ denotes the set of all prime
numbers.

For Hecke-Maass cusp forms $\psi$ for $SL(2,\mathbb{Z})$, Y. Motohashi [5] proved that
there exist constants $c_0, \theta_0 > 0$ such that uniformly for $(\log N)^{-\frac{1}{2}} \leq \theta \leq \theta_0$,

$$\sum_{N - y \leq p \leq N} \lambda_\psi(p)^2 = \frac{y}{\log N} (1 + O(e^{-\frac{\theta}{10}})), y = N^{1-\theta}$$

for sufficiently big $N$. Note that the main term $\frac{y}{\log N}$ in (1.3) is also the main
terms of the number of primes in $[N - y, N]$. For the number of prime numbers
in short intervals $[N - y, N]$, there are many results for various $y$ (see [6]). For
the very short range such as $y = \eta \log N$ for any fixed $\eta > 0$, P. X. Gallagher
proved that by assuming the Hardy-Littlewood prime $j$-tuples conjecture and
using the method of moments, the number of primes in $[N - y, N]$ has a Poisson
distribution $P(\eta)$.

Let $\pi_{\vec{d}}(N)$ be the number of positive integers $n \leq N$ such that $n - d_1, n -
d_2, ..., n - d_j$ are all prime where $\vec{d} = (d_1, d_2, ..., d_j) \in \mathbb{N}^j$ and $1 \leq d_1 < d_2 < \ldots <
d_j \leq y$. And let $\pi(N)$ be the number of primes $p \leq N$. The Hardy-Littlewood
prime $j$-tuples conjecture claims that

$$\pi_{\vec{d}}(N) \sim \prod_p \frac{p^{j-1}}{(p-1)^j} (p - v_d(p)) \frac{N}{\log^j(N)}$$

where $v_d(p)$ is the number of distinct residue classes in $\{d_1, d_2, ..., d_j\}$ modulo $p$.
When $y = \eta \log N$ for some fixed positive number $\eta$, P. X. Gallagher showed that
by assuming (1.4) for all $j \in \mathbb{N}$,

$$\sum_{n \leq N} (\pi(n) - \pi(n - y))^j = \sum_{n \leq N} \sum_{n - y < p_1, ..., p_j \leq n} 1$$

$$= \sum_{r=1}^j \sigma(j, r) \sum_{\vec{d}} \pi_{\vec{d}}(N)$$

$$= N(m_j(\eta) + o(1))$$
where $\sigma(j, r)$ is the number of maps form $\{1, 2, ..., j\}$ onto $\{1, ..., r\}$, the inner sum over $\tilde{d}$ means, $\tilde{d} = (d_1, d_2, ..., d_r) \in \mathbb{N}^r$, $1 \leq d_1 < d_2 < ... < d_r \leq y$. And

$$m_j(\eta) = \sum_{r=1}^{j} \eta^r S(r, j)$$

where $S(r, j)$ is the Stirling number of the second kind (the number of ways to partition a set $\{1, 2, 3, ..., j\}$ into $j$ nonempty unlabelled subsets). For each $1 \leq r \leq j$, $\eta^j S(r, j)$ corresponds to $\sigma(j, r)$ times the sum of $\pi_{\tilde{d}}(N)$ over $\tilde{d}$ where $\tilde{d} = (d_1, d_2, ..., d_r) \in \mathbb{N}^r$, $1 \leq d_1 < d_2 < ... < d_r \leq y$. Because of (1.3), one might wonder whether $\sum_{p \leq N} \lambda_f(p)^2$ acts similar to $\pi(N)$ in the sense of (1.5). In this paper, we show that

$$\sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{n \leq N} \left( \sum_{n - \eta \log N \leq p \leq n} \lambda_f(p)^2 \right)^2$$

(1.6)

$$= \sum_{f \in S_k} \frac{1}{\|f\|^2} \left( \sum_{n \leq N} \left( (\pi(n) - \pi(n - y))^2 + \eta + o_{\epsilon}(1) \right) \right)$$

for sufficiently big $k$ (depends on $N$) and sufficiently big $N$, where $\|f\|$ is the Petersson norm of $f$ over $C_k$. (1.6) suggests that for intervals which have constant multiple of the average prime-gap length, $\lambda_f(p)^2$ act slightly different from primes. And by (1.5) (assuming the Hardy-Littlewood 2-tuples conjecture), the right-hand side of (1.6) is

$$\sum_{f \in S_k} \frac{N}{\|f\|^2} \left( m_2(\eta) + \eta + o_{\epsilon}(1) \right),$$

and by applying the Petersson Trace formula (see Lemma 2.2, (2.5), this is

$$N \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \left( m_2(\eta) + \eta + o_{\epsilon}(1) \right).$$

**Remark 1.1.** For $j = 2$, $m_2(\eta) = \eta^2 + \eta$. For the average of higher powers ($j > 2$) as (1.6), one might expect that

$$\sum_{n \leq N} \left( \sum_{n - \eta \log N \leq p \leq n} \lambda_f(p)^2 \right)^j$$

$$\sim \sum_{n \leq N} \left( (\pi(n) - \pi(n - y))^2 \right) + \left( \frac{(2j)!}{(j!)^2(j+1)} - S(j, j) \right) \eta^j$$

for some different range of $k$ (in the sense of (1.7)). For this, we need the $2j$-th power versions of Lemma 2.3

$$\sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{p \leq N} (\lambda_f(p))^{2j} = \sum_{f \in S_k} \frac{1}{\|f\|^2} \left( \left( \frac{2j}{(j!)^2(j+1)} + o(1) \right) \frac{N}{\log N} \right),$$

and this is easily deduced from Hecke relations (see (2.1)).

The idea of the proof of (1.6) is very simple. By applying Hecke relations, we separate 1 from some powers of $\lambda_f(n)$, then using the Petersson trace formula.
(Lemma 2.1) to treat non-constant terms. Note that the average difference \( \eta \) in (1.6) comes from the digonal terms. Due to this simplicity, we get similar results like (1.6) for Hecke-Maass cusp forms for \( SL(2, \mathbb{Z}) \) and \( SL(3, \mathbb{Z}) \) (for the background of Hecke-Maass cusp form, see [3, Chapter 3, Chapter 6]). For these, we only need the following results come from direct applications of the Kuznetsov formula.

**Lemma 1.2.** Let \( \{ \psi_j \} \) be an orthonormal basis of Hecke-Maass cusp forms for \( SL(2, \mathbb{Z}) \), and let \( \frac{1}{2} + t_j^2 \) be the Laplace eigenvalue of \( \psi_j \). Let \( p, q \) be distinct primes. Let \( T > 1 \), let \( \epsilon > 0 \) be fixed small positive number. Then

\[
\sum_j \lambda_{\psi_j}(p^2) \frac{\zeta(2)}{L(1, \text{sym}^2 \psi_j)} e^{-t_j T} = O_\epsilon(T^{1+\epsilon}p^\epsilon + p^{1+\epsilon}),
\]

\[
\sum_j \lambda_{\psi_j}(p^2)^2 \frac{\zeta(2)}{L(1, \text{sym}^2 \psi_j)} e^{-t_j T} = \frac{T^2}{6} + O_\epsilon(T^{1+\epsilon}p^\epsilon + p^{1+\epsilon}),
\]

\[
\sum_j \frac{\zeta(2)}{L(1, \text{sym}^2 \psi_j)} e^{-t_j T} = \frac{T^2}{6} + O_\epsilon(T^{1+\epsilon}),
\]

\[
\sum_j \lambda_{\psi_j}(p^2) \lambda_{\psi_j}(q^2) \frac{\zeta(2)}{L(1, \text{sym}^2 \psi_j)} e^{-t_j T} = O_\epsilon(T^{1+\epsilon}(pq)^\epsilon + (pq)^{1+\epsilon})
\]

where the summation over \( j \) denotes the summation over all \( \psi_j \).

**Proof.** See [1] Lemma 1. \( \square \)

By applying Lemma 1.2 with the methods in the proof of Proposition 3.1, we get the following theorem.

**Theorem 1.3.** Assume the Hardy-Littlewood prime 2-tuples conjecture. Let \( \epsilon > 0, \eta > 0 \) be fixed positive numbers. Let \( N \) be a sufficiently big number, and \( y = \eta \log N \). Let \( T \) be a positive number such that \( N^{1+\epsilon} = o_\epsilon(T) \). Then

\[
\frac{1}{N} \sum_j \frac{\zeta(2)}{L(1, \text{sym}^2 \psi_j)} e^{-t_j T} \sum_{n \leq N} \left( \sum_{n-\eta \log N \leq p \leq n} \lambda_{\psi_j}(p^2) \right)^2
\]

\[
= \sum_j \frac{\zeta(2)}{L(1, \text{sym}^2 \psi_j)} e^{-t_j T} \left( m_2(\eta) + \eta + o_\epsilon(1) \right)
\]

\[
= \frac{T^2}{6} \left( m_2(\eta) + \eta + o_\epsilon(1) \right).
\]

Let \( \{ \phi_j \} \) be an orthonormal basis of Hecke-Maass cusp forms for \( SL(3, \mathbb{Z}) \), and let \( \{ A_j(n, 1) \} \) be the Hecke eigenvalues. Let \( v_j \) be the Laplacian eigenvalue of \( \phi_j \). By Hecke relations,

\[
|A_j(p, 1)|^2 = A_j(p, p) + 1,
\]

\[
|A_j(p, 1)|^4 = A_j(p, p)^2 + 2A_j(p, p) + 1
\]

(1.9)
(see [3, Theorem 6.4.11]). We need the following result comes from the GL(3)-Kuznetsov formula.

**Lemma 1.4.** Let \( p, q \) be distinct primes, let \( T > 1 \). Then
\[
\sum_{j} \frac{A_j(p,p)}{\text{Res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)} e^{-\frac{v_j}{T^2}} = \frac{\sqrt{3}}{2^7 \pi^2} T^5 + O_\epsilon(p^{4+\epsilon} T^{\frac{37}{8} + \epsilon}),
\]
\[
\sum_{j} \frac{A_j(p,p)}{\text{Res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)} e^{-\frac{v_j}{T^2}} = O_\epsilon(p^{2+\epsilon} T^{\frac{37}{8} + \epsilon}),
\]
\[
\sum_{j} \frac{1}{\text{Res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)} e^{-\frac{v_j}{T^2}} = \frac{\sqrt{3}}{2^7 \pi^2} T^5 + O_\epsilon(T^{\frac{37}{8} + \epsilon})
\]
\[
\sum_{j} \frac{A_j(p,p)A_j(q,q)}{\text{Res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)} e^{-\frac{v_j}{T^2}} = O_\epsilon((pq)^{2+\epsilon} T^{\frac{37}{8} + \epsilon}).
\]

where the summation over \( j \) denotes the summation over all \( \phi_j \), \( \text{Res} \) denotes the residue at \( s = 1 \).

**Proof.** See [1, Theorem 5]. \( \square \)

By applying Lemma 1.4 with the methods in the proof of Proposition 3.1, we get the following theorem.

**Theorem 1.5.** Assume the Hardy-Littlewood prime 2-tuples conjecture. Let \( \epsilon > 0, \eta > 0 \) be fixed positive numbers. Let \( N \) be a sufficiently big number, and \( y = \eta \log N \). Let \( T \) be a positive number such that \( N^{\frac{37}{8} + 2\epsilon} = o_\epsilon(T) \). Then
\[
\frac{1}{N} \sum_{j} \frac{1}{\text{Res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)} e^{-\frac{v_j}{T^2}} \sum_{n \leq N} \left( \sum_{n - \eta \log N \leq \rho \leq n} |A_j(p,1)|^2 \right)^2
\]
\[
= \sum_{j} \frac{1}{\text{Res}_{s=1} L(s, \phi_j \times \bar{\phi}_j)} e^{-\frac{v_j}{T^2}} (m_2(\eta) + \eta + o_\epsilon(1))
\]
\[
= \frac{\sqrt{3}}{2^7 \pi^2} T^5 (m_2(\eta) + \eta + o_\epsilon(1)).
\]

**2. LEMMAS**

By Hecke relations, we have
\[
\lambda_f(p)^2 = \lambda_f(p^2) + 1.
\]

Later, we need to deal with some off-diagonal terms \( \lambda_f(p^2) \lambda_f(q^2) \) where \( p \in P, q \in P, q \neq p \) or \( p \in P, q = 1 \). For this, we will use the following lemma.

**Lemma 2.1.** (Trace formula) For any two natural numbers \( m \) and \( n \),
\[
\sum_{f \in S_k} \frac{\lambda_f(n)\lambda_f(m)}{||f||^2} = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \delta(m-n)
\]
\[
+ O\left( \frac{(4\pi)^{k-1}}{\Gamma(k-1)} \left( (\log(3mn))^2 d((m,n))(mn)^{\frac{1}{2}} \right) \right)
\]
where $\delta$ is the delta function, $d$ is the divisor function, the implied constant is absolute, and $\|f\|$ is the Petersson norm of $f$ over $C_k$.

Proof. See [4, Corollary 14.24, Theorem 16.7].

By Lemma 2.1, we get the following Lemma.

**Lemma 2.2.** Let $p$ be a prime. Let $f \in S_k$. Then

(2.3) \[ \sum_{f \in S_k} \frac{\lambda_f(p^2)}{\|f\|^2} = O\left( \frac{(4\pi)^{k-1}}{\Gamma(k-1)} (\log (3p^2))^2 \frac{p^{\frac{k}{2}}}{k^2} \right), \]

(2.4) \[ \sum_{f \in S_k} \frac{\lambda_f(p^2)^2}{\|f\|^2} = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} + O\left( \frac{(4\pi)^{k-1}}{\Gamma(k-1)} (\log (3p^4))^2 \frac{p}{k^2} \right), \]

(2.5) \[ \sum_{f \in S_k} \frac{1}{\|f\|^2} = \frac{(4\pi)^{k-1}}{\Gamma(k-1)} + O\left( \frac{(4\pi)^{k-1}}{k^2 \Gamma(k-1)} \right). \]

Proof. For (2.3), put $n = p^2, m = 1$ in (2.2). For (2.4), put $m = n = p^2$ in (2.2). For (2.5), put $m = n = 1$ in (2.2).

**Lemma 2.3.** Let $N > 0$ be sufficiently big. Then

\[
\sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{p \leq N} (\lambda_f(p))^4 = \sum_{f \in S_k} \frac{1}{\|f\|^2} \left( 2 + O\left( \frac{N(\log N)^2}{k^2} \right) \right) \left( \sum_{p \leq N} 1 \right)
\]

as $N \to \infty$.

Proof. By Hecke relations, $\lambda_f(p)^4 = 1 + 2\lambda_f(p^2) + \lambda_f(p^2)^2$. From the main terms in (2.4) and (2.5), we get

\[
\sum_{f \in S_k} \frac{2}{\|f\|^2} \left( \sum_{p \leq N} 1 \right),
\]

and the remainder terms come from the error terms in (2.3), (2.4), (2.5).

**3. Main Theorem**

For convenience, let

(3.1) \[ \sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{n \leq N} A_f(n, y)^2 = \sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{n \leq N} \left( (\pi(n - y) - \pi(n))^2 + \eta + o(1) \right). \]

**Proposition 3.1.** Let $\epsilon > 0, \eta > 0$ be fixed positive numbers. Let $N$ be a sufficiently big number, and $y = \eta \log N$. Let $k$ be an even number such that $N^{2+\epsilon} = o(k)$. Then

\[
\sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{n \leq N} A_f(n, y)^2 = \sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{n \leq N} \left( (\pi(n - y) - \pi(n))^2 + \eta + o(1) \right).
\]
Proof. By the definition (3.1),
\[
\sum_{n \leq N} A_f(N, y)^2 = \sum_{n \leq N} \sum_{1 \leq d_1, d_2 \leq y} \lambda_f(n - d_1)^2 1_{n - d_1 \in P} \lambda_f(n - d_2)^2 1_{n - d_2 \in P}.
\]
Let’s split the inner sum into diagonal terms and off-diagonal terms
\[
\sum_{n - y \leq p \leq n} \lambda_f(p)^4 + 2 \sum_{1 \leq d_1 < d_2 \leq y} \lambda_f(n - d_1)^2 1_{n - d_1 \in P} \lambda_f(n - d_2)^2 1_{n - d_2 \in P}.
\]
First, let’s consider the diagonal terms
\[
\sum_{n \leq N} \sum_{n - y \leq p \leq n} \lambda_f(p)^4.
\]
By Hecke relations,
\[
\lambda_f(p)^4 = \lambda_f(p^2)^2 + 2\lambda_f(p^2) + 1.
\]
By the assumption \(N^{2+\epsilon} = o_k(N),\) Lemma 2.3,
\[
\sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{n \leq N} \sum_{n - y \leq p \leq n} (\lambda_f(p))^4
\]
\[
= \sum_{f \in S_k} \frac{1}{\|f\|^2} (2 + o(1)) \left( \sum_{n \leq N} \sum_{n - y \leq p \leq n} 1 \right)
\]
\[
= \sum_{f \in S_k} \frac{1}{\|f\|^2} (1 + o(1)) \left( \sum_{n \leq N} \sum_{n - y \leq p \leq n} 1 \right) + \sum_{f \in S_k} \frac{1}{\|f\|^2} \left( \sum_{n \leq N} \sum_{n - y \leq p \leq n} 1 \right)
\]
\[
= \sum_{f \in S_k} \frac{1}{\|f\|^2} (1 + o(1)) \left( \sum_{n \leq N} \sum_{n - y \leq p \leq n} 1 \right) + \sum_{f \in S_k} \frac{1}{\|f\|^2} \sigma(2, 1) \sum_{\tilde{d}} \pi_{\tilde{d}}(N)
\]
where the summation over \(\tilde{d}\) means the sum over \(\tilde{d} = (d) \in \mathbb{N}, 1 \leq d \leq y.\) Let’s consider the off-diagonal terms
\[
\sum_{n \leq N} \sum_{1 \leq d_1 < d_2 \leq y} \lambda_f(n - d_1)^2 \lambda_f(n - d_2)^2 1_{n - d_1, n - d_2 \in P}.
\]
By (2.1),
\[
\lambda_f(n - d_1)^2 \lambda_f(n - d_2)^2 1_{n - d_1, n - d_2 \in P} = \lambda_f((n - d_1)^2) 1_{n - d_1 \in P} + \lambda_f((n - d_2)^2) 1_{n - d_2 \in P}
\]
\[
+ \lambda_f((n - d_1)^2) \lambda_f((n - d_2)^2) 1_{n - d_1, n - d_2 \in P}
\]
\[
+ 1_{n - d_1, n - d_2 \in P}.
\]
From \(1_{n - d_1, n - d_2 \in P}\) in the above equation, we get
\[
2 \sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{n \leq N} \sum_{1 \leq d_1 < d_2 \leq y} 1_{n - d_1, n - d_2 \in P} = \sum_{f \in S_k} \frac{1}{\|f\|^2} \sigma(2, 2) \sum_{\tilde{d}} \pi_{\tilde{d}}(N)
\]
where the summation over \( \vec{d} \) means the summation over \( \vec{d} = (d_1, d_2) \in \mathbb{N}^2, 1 \leq d_1 < d_2 \leq y \). By (2.3),

\[
(3.6) \quad \sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{n \leq N, 1 \leq d_1 < d_2 \leq y} \lambda_f((n - d_1)^2)1_{n-d_1 \in P} + \lambda_f((n - d_2)^2)1_{n-d_2 \in P} = \sum_{n \leq N, 1 \leq d_1 < d_2 \leq y} \sum_{f \in S_k} \frac{1}{\|f\|^2} \lambda_f((n - d_1)^2)1_{n-d_1 \in P} + \lambda_f((n - d_2)^2)1_{n-d_2 \in P} = O_k \left( \frac{(4\pi)^{k-1}}{\Gamma(k-1)k^2} N^{\frac{1}{2}+\epsilon}y^2 \right).
\]

By the assumption \( N^{\frac{1}{2}+\epsilon} = o_k(k) \) and (2.5),

\[
\frac{(4\pi)^{k-1}}{\Gamma(k-1)k^2} N^{\frac{1}{2}+\epsilon} = o \left( \sum_{f \in S_k} \frac{1}{\|f\|^2} \right).
\]

Therefore, (3.6) is bounded by

\[
o_k \left( N \sum_{f \in S_k} \frac{1}{\|f\|^2} \right)
\]

for sufficiently big \( N \). By Lemma 2.1,

\[
(3.7) \quad \sum_{f \in S_k} \frac{1}{\|f\|^2} \lambda_f((n - d_1)^2)\lambda_f((n - d_2)^2)1_{n-d_1, n-d_2 \in P} = O \left( (\log N)^2 \frac{N(4\pi)^{k-1}}{\Gamma(k-1)k^2} \sum_{n \leq N} 1_{n-d_1, n-d_2 \in P} \right).
\]

Therefore by the assumption \( N^{\frac{1}{2}+\epsilon} = o_k(k) \) and (2.5),

\[
(3.8) \quad \sum_{1 \leq d_1 < d_2 \leq y} \sum_{n \leq N} \sum_{f \in S_k} \frac{1}{\|f\|^2} \lambda_f((n - d_1)^2)\lambda_f((n - d_2)^2)1_{n-d_1, n-d_2 \in P} = O \left( (\log N)^2 \frac{N(4\pi)^{k-1}}{\Gamma(k-1)k^2} \sum_{1 \leq d_1 < d_2 \leq y} \sum_{n \leq N} 1_{n-d_1, n-d_2 \in P} \right)
\]

\[
= o_k \left( \sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{1 \leq d_1 < d_2 \leq y} \sum_{n \leq N} 1_{n-d_1, n-d_2 \in P} \right).
\]

Hence,

\[
(3.9) \quad \sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{n \leq N} A_f(n, y)^2 = \left( \sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{n \leq N} (\pi(n - y) - \pi(n))^2 \right) \left( 1 + o(1) \right)
\]

\[
+ \sum_{f \in S_k} \frac{1}{\|f\|^2} \frac{yN}{\log N} \left( 1 + o(1) \right).
\]

By applying the Hardy-Littlewood prime 2-tuples conjecture (the last equation in (1.5)), we get the following theorem.
Theorem 3.2. Assume the Hardy-Littlewood prime 2-tuples conjecture. Let $\epsilon > 0, \eta > 0$ be fixed positive numbers. Let $N$ be a sufficiently big number, and $y = \eta \log N$. Let $k$ be an even number such that $N^{2+\epsilon} = o_{\epsilon}(k)$. Then

$$\frac{1}{N} \sum_{f \in S_k} \frac{1}{\|f\|^2} \sum_{n \leq N} \left( \sum_{n-y \leq p \leq n} \lambda_f(p)^2 \right)^2 = \sum_{f \in S_k} \frac{1}{\|f\|^2} (m_2(\eta) + \eta + o_{\epsilon}(1))$$

$$= \frac{(4\pi)^{k-1}}{\Gamma(k-1)} (m_2(\eta) + \eta + o_{\epsilon}(1)).$$

Remark 3.3. It is known that $|S_k| \sim \frac{k}{12}$. Let $\Lambda_f(p) = \frac{\Gamma(k-1)}{12\pi\|f\|^2(4\pi)^{k-1}}$. Then with the conditions in Theorem 3.2,

$$(3.10) \quad \frac{1}{N} \sum_{f \in S_k} \sum_{n \leq N} \left( \sum_{n-y \leq p \leq n} \Lambda_f(p)^2 \right)^2 = \sum_{f \in S_k} \left( m_2(\eta) + \eta + o_{\epsilon}(1) \right).$$

4. Proof of Theorem 1.3, Theorem 1.5

4.1. Proof of Theorem 1.3.

Proof. The proof is basically same as the proof of Theorem 3.2. For the diagonal terms, we just need to replace (3.4) with

$$\sum_j \sum_{n \leq N} \sum_{n-y \leq p \leq n} \lambda_{\psi_j}(p)^4 \frac{\zeta(2)}{L(1, \text{sym}^2 \psi_j)} e^{-\frac{t_j}{T}}$$

$$= \sum_j \frac{\zeta(2)}{L(1, \text{sym}^2 \psi_j)} e^{-\frac{t_j}{T}} (2 + o(1)) \left( \sum_{n \leq N} \sum_{n-y \leq p \leq n} 1 \right)$$

(By Lemma 1.2 and the assumption $N^{1+\epsilon} = o_{\epsilon}(T)$). For the off-diagonal terms, we need to replace (3.6) with

$$\sum_j \frac{\zeta(2)}{L(1, \text{sym}^2 \psi_j)} \sum_{n \leq N} \sum_{1 \leq d_1 < d_2 \leq y} \lambda_{\psi_j}((n-d_1)^2)1_{n-d_1 \in P} + \lambda_{\psi_j}((n-d_2)^2)1_{n-d_2 \in P}$$

$$= O_{\epsilon} \left( T^{1+\epsilon}N^\epsilon + N^{1+\epsilon}y^2N \right).$$

And then, we need to replace (3.7) with

$$\sum_j \frac{\zeta(2)}{L(1, \text{sym}^2 \psi_j)} \lambda_{\psi_j}((n-d_1)^2)\lambda_{\psi_j}((n-d_2)^2)1_{n-d_1, n-d_2 \in P} = O_{\epsilon} (T^{1+\epsilon}N^\epsilon + N^{2+\epsilon}).$$
Therefore by the assumption \(N^{1+\epsilon} = o_\epsilon(T)\) and (4.3),
\[
(4.4) \quad \sum_{1 \leq d_1 < d_2 \leq y \leq N} \sum_j \frac{\zeta(2)e^{-\frac{t_j}{2}}}{L(1, \text{sym}^2 \psi_j)} \lambda_{\psi_j}((n-d_1)^2)\lambda_{\psi_j}((n-d_2)^2)1_{n-d_1, n-d_2 \in P} = O_{\epsilon}\left((T^{1+\epsilon} N^\epsilon + N^{2+\epsilon}) \sum_{1 \leq d_1 < d_2 \leq y \leq N} 1_{n-d_1, n-d_2 \in P}\right) = o_{\epsilon}\left(\sum_j \frac{\zeta(2)e^{-\frac{t_j}{2}}}{L(1, \text{sym}^2 \psi_j)} \sum_{1 \leq d_1 < d_2 \leq y \leq N} 1_{n-d_1, n-d_2 \in P}\right).
\]

Finally,
\[
(4.5) \quad \sum_j \frac{\zeta(2)e^{-\frac{t_j}{2}}}{L(1, \text{sym}^2 \psi_j)} \sum_{n \leq N} \left(\sum_{n-y \leq p \leq n} \lambda_{\psi_j}(p)^2\right)^2 = \sum_j \frac{\zeta(2)e^{-\frac{t_j}{2}}}{L(1, \text{sym}^2 \psi_j)} \sum_{n \leq N} ((\pi(n-y) - \pi(n))^2 + \eta + o_{\epsilon}(1)),
\]
and by the Hardy-Littlewood prime 2-tuples conjecture,
\[
\sum_{n \leq N} ((\pi(n-y) - \pi(n))^2 + \eta + o_{\epsilon}(1)) = N(m_2(\eta) + \eta + o_{\epsilon}(1)).
\]

\[\square\]

4.2. Proof of Theorem 1.5.

Proof. By squaring out,
\[
(4.6) \quad \sum_{n \leq N} \left(\sum_{n-y \leq p \leq n} |A_j(p, 1)|^2\right)^2 = \sum_{n \leq N} \sum_{1 \leq d_1 < d_2 \leq y} |A_j(n-d_1, 1)|^2 1_{n-d_1 \in P} |A_j(n-d_2, 1)|^2 1_{n-d_2 \in P}.
\]

We split the inner sum into digonal terms and off-diagonal terms
\[
(4.7) \quad \sum_{n-y \leq p \leq n} |A_j(p, 1)|^4 + 2 \sum_{1 \leq d_1 < d_2 \leq y} |A_j(n-d_1, 1)|^2 1_{n-d_1 \in P} |A_j(n-d_2, 1)|^2 1_{n-d_2 \in P}.
\]

First, let’s consider the digonal terms
\[
\sum_{n \leq N} \sum_{n-y \leq p \leq n} |A_j(p, 1)|^4.
\]

By (1.9), Lemma 1.4 and the assumption \(N^{\frac{32}{3}+2\epsilon} = o_\epsilon(T)\),
\[
(4.8) \quad \sum_j \frac{1}{\text{Res}} \frac{L(s, \phi_j \times \phi_j)}{L(s, \phi_j \times \phi_j)} e^{-\frac{t_j}{2}} \sum_{n \leq N} \sum_{n-y \leq p \leq n} |A_j(p, 1)|^4 = \sum_j \frac{1}{\text{Res}} \frac{L(s, \phi_j \times \phi_j)}{L(s, \phi_j \times \phi_j)} e^{-\frac{t_j}{2}} (2 + o(1)) \left(\sum_{n \leq N} \sum_{n-y \leq p \leq n} 1\right).
\]
Let’s consider the off-diagonal terms

\[ \sum_{n \leq N} \sum_{1 \leq d_1 < d_2 \leq y} |A_j(n - d_1, 1)|^2 1_{n-d_1 \in P} |A_j(n - d_2, 1)|^2 1_{n-d_2 \in P}. \]

By (1.9),

\[ |A_j(n - d_1, 1)|^2 |A_j(n - d_2, 1)|^2 1_{n-d_1, n-d_2 \in P} = A_j(n - d_1, n - d_1) 1_{n-d_1 \in P} \]
\[ + A_j(n - d_2, n - d_2) 1_{n-d_2 \in P} \]
\[ + A_j(n - d_1, n - d_1) A_j(n - d_2, n - d_2) 1_{n-d_1, n-d_2 \in P} \]
\[ + 1_{n-d_1, n-d_2 \in P}. \]

From \(1_{n-d_1, n-d_2 \in P}\) in the above equation, we get

\[ 2 \sum_j \frac{\sum_{s=1}^{\sigma_0} e^{-\frac{v_j}{T_2}} \sum_{n \leq N} 1_{n-d_1, n-d_2 \in P}}{\prod_{s=1}^{\sigma_2} \pi_{\vec{d}}(N)}. \]

(4.9)

where the summation over \(\vec{d}\) means the summation over \(\vec{d} = (d_1, d_2) \in \mathbb{N}^2, 1 \leq d_1 < d_2 \leq y\). By Lemma 1.4,

\[ \sum_j \frac{e^{-\frac{v_j}{T_2}} \sum_{n \leq N} 1_{n-d_1, n-d_2 \in P}}{\prod_{s=1}^{\sigma_2} \pi_{\vec{d}}(N)} \]

(4.10)

\[ = \sigma_2(2,2) \sum_{\vec{d}} \pi_{\vec{d}}(N). \]

By the assumption \(N^{\frac{32}{3} + 2\epsilon} = o(T)\) and (1.10), (4.10) is bounded by

\[ \alpha(N \sum_j \frac{e^{-\frac{v_j}{T_2}}}{\prod_{s=1}^{\sigma_2} \pi_{\vec{d}}(N)}) \]

for sufficiently big \(N\). By Lemma 1.4,

\[ \sum_j \frac{e^{-\frac{v_j}{T_2}} A_j(n - d_1, n - d_1) A_j(n - d_2, n - d_2) 1_{n-d_1, n-d_2 \in P}}{\prod_{s=1}^{\sigma_2} \pi_{\vec{d}}(N)} \]

(4.11)

\[ = O_\epsilon(N^{4+\epsilon} T_2^{\frac{37}{3}} + 1_{n-d_1, n-d_2 \in P}). \]
Therefore by the assumption $N^{\frac{32}{32}+2\varepsilon} = o_\varepsilon(T)$,

(4.12)

$$\sum_{1 \leq d_1 < d_2 \leq y \leq n \leq N} \sum_j e^{-\frac{v_j}{T^2}} A_j(n - d_1, n - d_1) A_j(n - d_2, n - d_2) \frac{L(s, \phi_j \times \phi_j)}{\text{Res} \, L(s, \phi_j \times \phi_j)} \sum_{1 \leq d_1 < d_2 \leq y \leq n \leq N} 1_{n-d_1, n-d_2 \in P}$$

$$= O\left(N^{4+ \varepsilon} T^{\frac{37}{32} + \varepsilon} \sum_{1 \leq d_1 < d_2 \leq y \leq n \leq N} 1_{n-d_1, n-d_2 \in P}\right)$$

$$= o_\varepsilon\left(\sum_j \frac{\text{Res} \, L(s, \phi_j \times \phi_j)}{\sum_{1 \leq d_1 < d_2 \leq y \leq n \leq N} 1_{n-d_1, n-d_2 \in P}}\right).$$

Finally,

$$\sum_j \frac{\text{Res} \, L(s, \phi_j \times \phi_j)}{\sum_{n \leq N} \left(\sum_{n - y \leq p \leq n} |A_j(p, 1)|^2\right)^2}$$

(4.13)

$$= \left(\sum_j \frac{\text{Res} \, L(s, \phi_j \times \phi_j)}{\sum_{n \leq N} \left(\pi(n - y) - \pi(n)\right)^2}\right) (1 + o_\varepsilon(1))$$

$$+ \sum_j \frac{\text{Res} \, L(s, \phi_j \times \phi_j)}{\sum_{n \leq N} \log N} \frac{yN}{1 + o_\varepsilon(1)},$$

and by the Hardy-Littlewood prime 2-tuples conjecture,

$$\sum_{n \leq N} \left(\pi(n - y) - \pi(n)\right)^2 + \eta + o_\varepsilon(1) = N \left(m_2(\eta) + \eta + o_\varepsilon(1)\right).$$

\[\square\]

**Acknowledgements.** The author would like to thank his advisor Professor Xiaqing Li, for her constant support. The author would also like to thank Professor Milicevic for pointing out that the Hardy-Littlewood conjecture was not included in the statement of Theorem 3.1 in the previous version, and the referee for useful suggestions.

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