The Mathieu conjecture for $SU(2)$ reduced to an abelian conjecture

Michael Müger and Lars Tuset

October 17, 2023

Abstract

We reduce the Mathieu conjecture for $SU(2)$ to a conjecture about moments of Laurent polynomials in two variables with single variable polynomial coefficients.

1 Introduction

O. Mathieu conjectured [4] that for complex valued regular (=finite type) functions $f, g$ on any connected compact Lie group with normalized Haar integral $\int$, the vanishing of $\int f^n$ for all positive integers $n$ implies $\int f^n g = 0$ for all large enough $n$. He then proved that this conjecture implies O.-H. Keller’s notorious Jacobian conjecture.

Motivated by work of Dings and Koelink [1] we reduce the Mathieu conjecture for $SU(2)$ to a conjecture of a more abelian nature about moments of Laurent polynomials in two variables with single variable polynomial coefficients. Generalized further in the natural fashion, this ‘xz-conjecture’ says that if $f(x, z) = \sum_m c_m(x)z^m$ is a Laurent polynomial in several $z$-variables with polynomial coefficients $c_m$ in several $x$-variables satisfying $\int f^n = 0$ for all positive integer $n$, where $z$ is integrated over the torus and $x$ over a cube, then $0$ is not in the convex hull of the set of multi-indices $m$ for which $c_m \neq 0$.

In the absence of $x$-variables, our conjecture reduces to a result proven by Duistermaat and van der Kallen [2] as part of their proof of the Mathieu conjecture in the abelian case. On the other hand, the $xz$-conjecture with one $x$ variable and no $z$’s is known to hold, see [5] and references therein. For the moment, the $xz$-conjecture remains open already for one $z$ and one $x$. Towards the end of this paper we explain that the natural inductive approach to proving it in this case fails due to the topological ‘worm problem’. We also include a trivial generalization of the approach of Dings and Koelink to any connected compact Lie group.

2 The $SU(2)$-case

In what follows, we equip $\mathbb{T} = \{ z \in \mathbb{C} \mid |z| = 1 \}$ with the normalized Haar measure $\frac{1}{2\pi i} \frac{dz}{z}$ and $[0, 1]$ with Lebesgue measure. Products $\mathbb{T}^k \times [0, 1]^l$ will carry the obvious product measure.

\[ \int x^n f(x) dx \]
Define maps

\[ \alpha : \quad SU(2) \to \mathbb{C}^4, \quad \begin{pmatrix} a & c \\ b & d \end{pmatrix} \mapsto (a, b, c, d), \]

\[ \beta : \quad \mathbb{T}^2 \times [0, 1] \to \mathbb{C}^4, \quad (z_1, z_2, x) \mapsto ((1 - x)z_2, xz_1, -z_1^{-1}, z_2^{-1}). \]

A function \( f : SU(2) \to \mathbb{C} \) is called regular if \( f = P \circ \alpha \) for some \( P \in \mathbb{C}[a, b, c, d] \). (That \( P \) is not uniquely determined will not be an issue.) Denoting the space of regular functions by \( R \), the maps \( \Lambda : \mathbb{C}[a, b, c, d] \to R, P \mapsto P \circ \alpha \) and \( \Pi : \mathbb{C}[a, b, c, d] \to C(\mathbb{T}^2 \times [0, 1], \mathbb{C}), P \mapsto P \circ \beta \) clearly are ring homomorphisms.

2.1 Lemma For each \( P \in \mathbb{C}[a, b, c, d] \) we have

\[ \int_{SU(2)} P \circ \alpha = \int_{\mathbb{T}^2 \times [0, 1]} P \circ \beta. \]  

(2.1)

Proof. It suffices to prove this for monomials \( P = a^{n_1}b^{n_2}c^{n_3}d^{n_4} \). For the l.h.s. of (2.1), like Dings and Koelink \([1]\) we use a classical integration formula for \( SU(2) \) \([6\) Ch. III, Sect. 6.1\):

\[ \int_{SU(2)} f = \frac{1}{16\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_{-2\pi}^{2\pi} f(F(\phi, \theta, \psi)) \sin \theta \, d\psi \, d\theta \, d\phi, \]

where

\[ F(\phi, \theta, \psi) = \left( \begin{array}{ccc} \cos \frac{\theta}{2} e^{i(\phi + \psi)/2} & i \sin \frac{\theta}{2} e^{i(\phi - \psi)/2} \\ i \sin \frac{\theta}{2} e^{-i(\phi - \psi)/2} & \cos \frac{\theta}{2} e^{-i(\phi + \psi)/2} \end{array} \right). \]

For \( f = P = a^{n_1}b^{n_2}c^{n_3}d^{n_4} \) this becomes

\[ \int_{SU(2)} P \circ \alpha = \frac{2^{n_2+n_3}}{16\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_{-2\pi}^{2\pi} F(\phi, \theta, \psi) \sin \theta \, d\theta \, d\phi \int_{-2\pi}^{2\pi} e^{i\frac{\psi}{2}(n_1-n_2+n_3-n_4)} \, d\psi. \]

Now, \( \int_{-2\pi}^{2\pi} e^{i\frac{\psi}{2}(n_1-n_2+n_3-n_4)} \, d\psi = 4\pi \delta_{n_1-n_2+n_3-n_4,0} \). When this is non-zero, we have \( n_1 - n_4 = n_3 - n_2 \), so that \( n_1 - n_2 + n_3 - n_4 \) is even, so that the integration \( \int_{-2\pi}^{2\pi} \cdots \, d\phi \) gives a factor \( 2\pi \delta_{n_1-n_2+n_3-n_4,0} \). The combination of \( n_1 + n_2 - n_3 - n_4 = 0 \) and \( n_1 - n_2 + n_3 - n_4 = 0 \) is equivalent to \( n_1 = n_4 \land n_2 = n_3 \). Thus

\[ \int_{SU(2)} P \circ \alpha = \frac{(-1)^{n_2} \delta_{n_1,n_4} \delta_{n_2,n_3}}{2} \int_0^{2\pi} \cos^{2n_1} \frac{\theta}{2} \sin^{2n_2} \frac{\theta}{2} \sin \theta \, d\theta. \]

With \( x = \sin^2 \frac{\theta}{2} \) we have \( \frac{dx}{d\theta} = 2 \sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2} \cdot \frac{1}{2} = \frac{\sin \theta}{2\theta} \), thus \( \sin \theta \, d\theta = 2 \, dx \), and with \( \cos^2 \frac{\theta}{2} = 1 - \sin^2 \frac{\theta}{2} = 1 - x \) we have

\[ \int_{SU(2)} P \circ \alpha = (-1)^{n_2} \delta_{n_1,n_4} \delta_{n_2,n_3} \int_0^1 (1 - x)^{n_1} x^{n_2} \, dx, \]

(where the x-integral is Euler’s function \( \beta(n_1 + 1, n_2 + 1) \)). On the other hand,

\[ \int_{\mathbb{T}^2 \times [0, 1]} P \circ \beta = \int_{\mathbb{T}^2 \times [0, 1]} ((1 - x)z_2)^{n_1} (xz_1)^{n_2} (-z_1^{-1})^{n_3} (z_2^{-1})^{n_4} = (-1)^{n_3} \int_0^1 (1 - x)^{n_1} x^{n_2} \, dx \int_{\mathbb{T}^2} z_1^{n_2-n_3} z_2^{-n_1-n_4} = (-1)^{n_3} \delta_{n_1,n_4} \delta_{n_2,n_3} \int_0^1 (1 - x)^{n_1} x^{n_2} \, dx, \]

\[ = (-1)^{n_3} \delta_{n_1,n_4} \delta_{n_2,n_3} \int_0^1 (1 - x)^{n_1} x^{n_2} \, dx, \]

\[ = (-1)^{n_3} \delta_{n_1,n_4} \delta_{n_2,n_3} \int_0^1 (1 - x)^{n_1} x^{n_2} \, dx, \]

\[ = (-1)^{n_3} \delta_{n_1,n_4} \delta_{n_2,n_3} \int_0^1 (1 - x)^{n_1} x^{n_2} \, dx, \]
and comparing the two integrals completes the proof. ■

2.2 Definition Let \( k, l \in \mathbb{N}_0 = \{0, 1, \ldots \} \) and \( f \in \mathbb{C}[x_1, \ldots, x_l, z_1, z_1^{-1}, \ldots, z_k, z_k^{-1}] \). Considering \( f \) as a Laurent polynomial \( \sum_{\mathbf{m} \in \mathbb{Z}^k} c_{\mathbf{m}} z^\mathbf{m} \) in \( z_1, \ldots, z_k \) with coefficients \( c_{\mathbf{m}} \in \mathbb{C}[x_1, \ldots, x_l] \), we define the spectrum of \( f \) as

\[
\text{Sp}(f) = \{ \mathbf{m} \in \mathbb{Z}^k \mid c_{\mathbf{m}} \neq 0 \}.
\]

2.3 Conjecture (xz-conjecture) Let \( k, l \in \mathbb{N}_0 \) and \( f \in \mathbb{C}[x_1, \ldots, x_l, z_1, z_1^{-1}, \ldots, z_k, z_k^{-1}] \). If \( \int_{[0,1]^l \times \mathbb{T}^k} f^n = 0 \) for all \( n \in \mathbb{N} \), then \( \mathbf{0} \) is not in the convex hull of \( \text{Sp}(f) \subset \mathbb{R}^k \).

2.4 Remark The conjecture is trivially true for \( k = l = 0 \). For \( l = 0 \), in which case the \( c_{\mathbf{m}} \) are just numbers, it was proven in [2]. For \( k = 0 \) and all \( l \) it was proven in [3] and again by the authors [5], using ideas from the proof in [2] for \( l = 0, k = 1 \). To the best of the authors’ knowledge it is open for all other \((k, l)\). See Remark 2.6 for comments on a failed attempt at proving it for \( k = l = 1 \).

2.5 Theorem The case \( k = 2, l = 1 \) of the xz-conjecture implies the Mathieu conjecture for \( SU(2) \).

Proof. Let \( f \) be a regular function on \( SU(2) \) such that \( \int f^n = 0 \) for all \( n \in \mathbb{N} \). Pick \( P \in \mathbb{C}[a, b, c, d] \) such that \( f = P \circ \alpha \). With the lemma, we have

\[
\int_{\mathbb{T}^2 \times [0,1]} (P \circ \beta)^n = \int_{\mathbb{T}^2 \times [0,1]} P^n \circ \beta = \int \sum_{\mathbf{m} \in \mathbb{Z}^k} (P^n \circ \alpha)^n = \int f^n = 0 \quad \forall n \in \mathbb{N}.
\]

Since \( P \circ \beta \in \mathbb{C}[x, z_1, z_1^{-1}, z_2, z_2^{-1}] \), the xz-conjecture (for \( k = 2, l = 1 \)) implies that \( \mathbf{0} \) is not in the convex hull of \( \text{Sp}(P \circ \beta) \subset \mathbb{Z}^2 \). By a classical result, there is a straight line in \( \mathbb{R}^2 \) separating \( \mathbf{0} \) from \( \text{Sp}(P \circ \beta) \). This is equivalent to the existence of \( \mathbf{v} \in \mathbb{R}^2 \setminus \{\mathbf{0}\} \) such that \( \mathbf{v} \cdot \mathbf{m} \geq 1 \) for all \( \mathbf{m} \in \text{Sp}(P \circ \beta) \). This implies \( \mathbf{v} \cdot \mathbf{m} \geq n \) for all \( \mathbf{m} \in \text{Sp}((P \circ \beta)^n) \). As a consequence, \( \text{Sp}((P \circ \beta)^n) \) moves off to infinity as \( n \to \infty \), in the sense of becoming disjoint from every finite subset of \( \mathbb{Z}^2 \) for \( n \) large enough.

If now \( g \) is another regular function on \( SU(2) \) and \( Q \) a polynomial such that \( g = Q \circ \alpha \), for each \( n \in \mathbb{N} \) we have

\[
\int f^n g = \int (P \circ \alpha)^n (Q \circ \alpha) = \int (P^n Q) \circ \alpha = \int_{\mathbb{T}^2 \times [0,1]} (P^n Q) \circ \beta = \int_{\mathbb{T}^2 \times [0,1]} (P \circ \beta)^n (Q \circ \beta).
\]

By the above, \( \text{Sp}((P \circ \beta)^n) \) will be disjoint from the finite set \( -\text{Sp}(Q \circ \beta) \) for all large enough \( n \), so that the \( z \)-integrations give zero for all \( x \in [0,1] \). Thus \( \int f^n g = 0 \), proving the Mathieu conjecture for \( SU(2) \). ■

2.6 Remark We briefly report on a failed attempt to prove the xz-conjecture for one \( x \) and one \( z \) by adapting the approach to Laurent polynomials in one variable \( z \) pursued by Duistermaat and van der Kallen. Namely for \( f = f(z; x) \) with \( 0 \) in the convex hull of \( \text{Sp}(f) \), consider the generating function

\[
F(t) = \sum_{n=1}^{\infty} t^{n-1} \int_0^1 dx \int_{\mathbb{T}} f^n = (2\pi i)^{-1} \int_0^1 dx \int_C \frac{f(z; x)}{1-tf(z; x)} \frac{dz}{z},
\]

which defines a holomorphic function for \( |t| \) small. In analogy to [2], as \( \lim_{z \to 0} f(z) = \infty \), the residue theorem and L’Hospital’s rule tell us that for such \( t \), we have

\[
F(t) = -\frac{1}{t} - \sum_j \frac{1}{t^2} \int_0^1 \frac{dz}{f'(\zeta_j; x) \zeta_j},
\]
The hope then is that various cases extend analytically to a function that is not identically zero by looking at its behavior as $t \to 0$. In the final round one needs then to carefully discuss the contributions to the integrals of the values of $f$ that the moments of $G$ cannot all vanish. But here the problem arises that we don’t have $C'$ even for simple $f$’s. Indeed, the critical values of $f(z; x) = c_{-1}(x)z^{-1} + c_0(x) + c_1(x)z$ are given by $\tau_{\pm}(x) = c_0(x) \pm 2\sqrt{c_{-1}(x)c_1(x)}$, and the specific choice of polynomials $c_j(x)$ given by $c_1(x) = c_{-1}(x) = 2x - 1 + i(1 - (2x - 1))^2$ and $c_0(x) = 2x - 1 - i(1 - (2x - 1)^2)$ produces curves or ‘worms’ $\tau_{\pm}([0,1])$ that enclose the origin completely, thus preventing any curve to reach the origin from infinity.

3 The Dings-Koelink approach

Let $G$ be a connected compact Lie group with maximal torus $T$, say of dimension $r$. Let $\hat{G}$ be the set of equivalence classes of irreducible unitary representations of $G$, and let $V_\sigma$ be the $G$-module of $\sigma \in \hat{G}$. Decompose $V_\sigma = \oplus_{m \in \mathbb{Z}} V_{\sigma,m}$ as a module over $T$. Let $f$ be a regular function on $G$. Then by definition we may write

$$f = \sum_{\sigma \in \hat{G}} \text{Tr}_{V_\sigma}(A_\sigma \pi_\sigma(\cdot)) = \sum_{\sigma \in \hat{G}} \sum_{m,m' \in \mathbb{T}} \text{Tr}_{V_\sigma}(A_{\sigma,m,m'} \pi_{\sigma,m,m'}(\cdot))$$

for only finitely many non-zero complex quadratic matrices $A_\sigma$ each of size $\text{dim}(V_\sigma)$. Consider the ‘spectrum’ of $f$ to be

$$X_f = \{(m, m') \in \mathbb{T} \times \widehat{T} \mid A_{\sigma,m,m'} \neq 0\} \subset \mathbb{R}^r \times \mathbb{R}^r.$$

Dings and Koelink then showed by using the multinomial formula and left- and right actions of $T$ on $G$, that if $(0,0)$ is not in the convex hull of $X_f$, then the moments $\int_G f^n(s) \, ds$ vanish for all non-negative integers $n$, where $ds$ is the Haar measure on $G$. By using the same trick once more, they also showed that if the converse (their Conjecture 4.1) of the previous statement holds for $G$, then the Mathieu conjecture holds for $G$. Thus it remains to show their conjecture:

*If all the moments of $f$ vanish, then $(0,0)$ is not in the convex hull of $X_f$."

One might of course ask what the relation between $X_f$ and $\text{Sp}(f)$ is when $G = SU(2)$, but we won’t discuss that here.

Acknowledgment. The authors thank the referees for comments simplifying the exposition. M. M. thanks L. T. and OsloMet for hospitality during two months spent there in 2019. His expenses were shared by OsloMet and Radboud University.

References

[1] T. Dings, E. Koelink: On the Mathieu conjecture for $SU(2)$. Indag. Math. 26, 219-224 (2015).
[2] J. J. Duistermaat, W. van der Kallen: Constant terms in powers of a Laurent polynomial. Indag. Math. 9, 221-231 (1998).

[3] J.P. Francoise, F. Pakovich, Y. Yomdin, W. Zhao: Moment vanishing problem and positivity: Some examples. Bull. Sci. math. 135, 10-32 (2011).

[4] O. Mathieu: Some conjectures about invariant theory and their applications. pp. 263-279 in: J. Alev, G. Cauchon (eds.): Algèbre non commutative, groupes quantiques et invariants. (Proceedings of the 7th Franco-Belgian Conference, Reims, June 26-30, 1995.). Soc. Math. France, 1997.

[5] M. Müger, L. Tuset: On the moments of a polynomial in one variable. Indag. Math. 31, 147-151 (2020).

[6] N. J. Vilenkin: Special functions and the theory of group representations. Amer. Math. Soc. Transl., 1968.