TYPE II BLOW UP FOR THE FOUR DIMENSIONAL ENERGY CRITICAL SEMI LINEAR HEAT EQUATION

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Abstract. We consider the energy critical four dimensional semi linear heat equation \( \partial_t u - \Delta u - u^3 = 0 \). We show the existence of type II finite time blow up solutions and give a sharp description of the corresponding singularity formation. These solutions concentrate a universal bubble of energy in the critical topology \( u(t, r) - \frac{1}{\lambda} Q \left( \frac{r}{\lambda(t)} \right) \rightarrow u^* \) in \( \dot{H}^1 \)
where the blow up profile is given by the Talenti Aubin soliton
\[ Q(r) = \frac{1}{1 + \frac{r^2}{4}}. \]
and with speed
\[ \lambda(t) \sim \frac{T - t}{|\log(T - t)|^2} \text{ as } t \rightarrow T. \]

Our approach uses a robust energy method approach developed for the study of geometrical dispersive problems [19], [17], and lies in the continuation of the study of the energy critical harmonic heat flow [20] and the energy critical four dimensional wave equation [5].

1. Introduction

1.1. Setting of the problem. We consider in this paper the energy critical semi linear heat equation
\[
\partial_t u - \Delta u - u^3 = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^4
\]
which is the energy critical four dimensional version of the more general problem
\[
\partial_t u - \Delta u - u^p = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad p \geq 2^* - 1
\]
where
\[ 2^* = \frac{2N}{N - 2} \]
is the Sobolev exponent. There is an important literature devoted to the qualitative description of solutions to (1.2), and we refer to [19], [11] for a complete introduction to the history of the problem. For radial data, two type of blow up regimes are typically expected: type I blow up which corresponds to a self similar blow up, and type II blow up which displays excited blow up speeds. Such kind of type II blow up solutions were exhibited for the first time by Herrero and Velazquez [4] using matching asymptotic procedures for a large value of \( p \), and the corresponding regime displays a polynomial type blow up speed. A major breakthrough is achieved by Matano and Merle in [9], [10], where the non existence of type II blow up is shown for
\[ 2^* - 1 < p < p_c \]
where
\[ p_c = \begin{cases} +\infty & \text{for } N \leq 10 \\ 1 + \frac{4}{N-4-2\sqrt{N-1}} & \text{for } N \geq 11 \end{cases} \]
and the existence of type II blow up for \( p > p_c \) is proved. More precisely, such solutions are obtained as threshold dynamics between well known type I blow up solutions and global dissipative dynamics. A complete classification of these type II regimes is then completed in [11] where quantized blow up speeds are exhibited with polynomial rates.

These results leave completely open the question of existence of type II blow up in the energy critical setting. In fact, in the energy critical setting and even for the parabolic problem, the maximum principle does not seem to yield enough information to control a type II blow up. The criticality of the problem is reflected by the fact that the total dissipated energy
\[ E(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(t,x)|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^N} u^{p+1}(t,x) dx \]  
(1.3)
is left invariant by the scaling symmetry of the problem
\[ u_\lambda(t,x) = \lambda^{\frac{N-2}{2}} u(\lambda^2 t, \lambda x), \quad E(u_\lambda(t)) = E(u(\lambda^2 t)). \]
The study of critical problems has attracted a considerable attention for the past ten years in the dispersive community, in particular the study of the mass critical nonlinear Schrödinger equation [18], [12], [13], [14], [15], [16] and geometric problems like wave maps, Schrödinger maps and the harmonic heat flow [7], [21], [19], [17], [20]. In particular, a robust energy approach is developed in [19], [17] to construct type II blow up solutions in the energy critical setting. This strategy is implemented in the parabolic setting in [20] and led to the construction of stable blow up dynamics with sharp asymptotics on the singularity formation for the harmonic heat flow. Note that more type II regimes for dispersive problems were obtained in [7], [8] but rely on the construction of non smooth solutions and a procedure of backwards in time integration of the flow from the singularity which are both non suitable for parabolic problems.

1.2. Statement of the result. We carry out in this paper the program which was implemented in [5] to adapt the study of the geometric wave equation in [19] to the semi linear cubic four dimensional wave equation. The main difficulty is the fact that the energy (1.3) is non definite positive, and this induces a non positive eigenvalue in the spectrum of the linearized operator close to the Talenti-Aubin stationary solution
\[ Q(r) = \frac{1}{1 + \frac{r^2}{8}}, \quad \text{(1.4)} \]
which is the unique up to scaling radially symmetric solution to the stationary problem
\[ \Delta Q + Q^3 = 0, \quad Q(r) = \frac{1}{1 + \frac{r^2}{8}}. \quad \text{(1.5)} \]
This requires building our set of initial data on a suitable codimension one set, and we similarly claim in the continuation of [20] the existence of a type II blow up dynamics for the energy critical four dimensional problem:

Theorem 1.1 (Existence of type II blow up in dimension \( N = 4 \)). Let \( Q \) be the Talenti Aubin soliton (1.5). Then \( \forall \alpha^* > 0 \), there exists a radially symmetric initial
data $u_0 \in H^1(\mathbb{R}^4)$ with
\begin{equation}
E(Q) < E(u_0) < E(Q) + \alpha^*
\end{equation}

such that the corresponding solution to the energy critical focusing parabolic equation (1.1) blows up in finite time $T = T(u_0) < \infty$ in a type II regime according to the following dynamics: there exist $u^* \in H^1$ such that:
\begin{equation}
\nabla \left[ u(t,x) - \frac{1}{\lambda(t)} Q \left( \frac{x}{\lambda(t)} \right) \right] \to \nabla u^* \text{ in } L^2 \text{ as } t \to T
\end{equation}

at the speed
\begin{equation}
\lambda(t) = c(u_0) \left( 1 + o(1) \right) \frac{T - t}{|\log(T - t)|^2} \text{ as } t \to T
\end{equation}

for some $c(u_0) > 0$. Moreover, there holds the regularity of the asymptotic profile:
\begin{equation}
\Delta u^* \in L^2
\end{equation}

Comments on the result

1. In dimension four, the choice $p = 2^* - 1 = 3$ is therefore the only one for which a type II blow up occurs for radial data. We have decided to focus onto the four dimensional case for the case of simplicity but the construction we propose could be addressed in a much more general setting. Let us insist also that it does not rely on the maximum principle and may therefore be addressed in the non radial setting and for more complicated systems.

2. The blow up speed (1.8) is also the one obtained for the energy critical harmonic heat flow in [20]. Following the heuristics developed in [1], we conjecture the existence of a sequence of quantized blow up speeds with polynomial rates corrected by suitable logarithmic factors, and (1.8) is the fundamental which corresponds from the proof to a codimension one in some weak sense manifold of initial data.

The main open problem after this work is to obtain a complete classification of type II blow up for the energy critical problem, both in the radially symmetric case and the non symmetric case.

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Notations We introduce the differential operator
\[ \Lambda f = f + y \cdot \nabla f \text{ (energy critical scaling)}. \]

Given a positive number $b > 0$, we let
\begin{equation}
B_0 = \frac{1}{\sqrt{b}}, \quad B_1 = \frac{|\log b|}{\sqrt{b}}.
\end{equation}

Given a parameter $\lambda > 0$, we let
\[ u_\lambda(r) = \frac{1}{\lambda} u(y) \text{ with } y = \frac{r}{\lambda}. \]

We let $\chi$ as a smooth cut off function with
\[ \chi(y) = \begin{cases} 1 & \text{for } y \leq 1, \\ 0 & \text{for } y \geq 2. \end{cases} \]
We shall systematically omit the measure in all radial two dimensional integrals and note:

\[ \int f = \int_0^{+\infty} f(r)r^3 dr. \]

2. Construction of an explicit approximate solution

The aim of this section is to construct an approximate blow up solution of (1.1), which is close to the ground state \( Q \). This approximate solution will be the dominant part of the blow up profile inside the parabolic zone. We adapt the strategy developed in [17], [19], [20].

Let \( u \) be a stationary solution of (1.1). Let \( \lambda > 0 \). Then, \( \frac{1}{\lambda} u(\xi) \) is also a solution. Consider now that \( \lambda \) is no more a constant, but depends of time. Thus, we obtain the following equation:

\[ \lambda(t)^2 \partial_t u \left( \frac{r}{\lambda(t)} \right) - \lambda(t)\lambda'(t) \Lambda u \left( \frac{r}{\lambda(t)} \right) - \Delta u \left( \frac{r}{\lambda(t)} \right) - u^3 \left( \frac{r}{\lambda(t)} \right) = 0. \quad (2.1) \]

We then define a rescaled time

\[ s = \int_0^t \frac{d\tau}{\lambda^2(\tau)}. \quad (2.2) \]

Remark that if \( \lambda(t) \) verify the law (1.3) defined in the theorem 1.1, then \( s(t) \) is a bijection between \([0,T]\) and \( \mathbb{R}^+ \). We also let the rescaled variable \( y(t) = \frac{r}{\lambda(t)} \). The equation (2.1) becomes, using the new variables:

\[ \partial_s u - \frac{\lambda_s}{\lambda} \Lambda u - \Delta u - u^3 = 0. \quad (2.3) \]

As well as the parameter \( \lambda(s) \), we define a new parameter \( b(s) \) such that:

\[ b = -\frac{\lambda_s}{\lambda}(1 + o(1)) , \quad (2.4) \]

\[ b_s = -b^2(1 + o(1)) . \quad (2.5) \]

The modulation laws (2.4) and (2.5) will be justified thereafter. First, in the following subsection, we consider that

\[ b_s = -b^2 \quad \text{and} \quad b + \frac{\lambda_s}{\lambda} = 0, \quad (2.6) \]

\( b \) being positive.

2.1. Construction of explicit approximate blow up profiles.

Proposition 2.1 (Construction of the approximate profile). Let \( M > 0 \) enough large. Then, there exists a small enough universal constant \( b^*(M) \), such that the following holds. Let \( b \in [0, b^*(M)] \). Then there exists profiles \( T_1, T_2 \) and \( T_3 \), such that

\[ Q_b(y) = Q(y) + bT_1(y) + b^2T_2(y) + b^3T_3(y) = Q(y) + \alpha(y) \]

generates an error

\[ \Psi_b = -b^2(T_1 + 2bT_2) - \Delta Q_b - (Q_b)^3 + b\Delta Q_b \quad (2.7) \]

which satisfies

(i) Weighted bounds:

\[ \int_{y \leq 2B_1} |H\Psi_b|^2 \lesssim b^4 |\log b|^2 , \quad (2.8) \]
\[
\int_{y \leq 2B_1} \frac{1}{1 + y^8} |\Psi_b|^2 \lesssim b^6, \quad (2.9)
\]
\[
\int_{y \leq 2B_1} |H^2 \Psi_b|^2 \lesssim \frac{b^6}{|\log b|^2}. \quad (2.10)
\]

(ii) Flux computation: Let \( \Phi_M \) be given by (3.2), then:
\[
\frac{\langle H \Psi_b, \Phi_M \rangle}{\langle \Lambda Q, \Phi_M \rangle} = \frac{-2b^2}{|\log b|} + O \left( \frac{b^2}{|\log b|^2} \right). \quad (2.11)
\]

**Remark 2.2.** From the proof, the profiles \((T_i)_{1 \leq i \leq 3}\) display a lower order dependence in \(b\).

**Proof of Proposition 2.1**

**Step 1** Computation of the error

We expand \( Q^3_b \) and formulate the error \( \Psi_b \) as a polynomial expression in \(b\):

\[
Q^3_b = Q^3 + 3bQ^2T_1 + b^2 (3Q^2T_2 + 3QT_1^2) + b^3 (3Q^2T_3 + 6QT_1T_2 + T_1^3)
\]
\[+ R_1(T_1, T_2, T_3),\]

where \( R_1(T_1, T_2, T_3) \) is polynomial in \((T_i)_{1 \leq i \leq 3}\) and contains the terms of power \((b^j)_{j \geq 4}\). Hence,

\[
\Psi_b = b(HT_1 + \Lambda Q)
\]
\[+ b^2 (HT_2 - T_1 + \Lambda T_1 - 3QT_1^2)
\]
\[+ b^3 (HT_3 - 2T_2 + \Lambda T_2 - 6QT_1T_2 - T_1^3)
\]
\[+ b^4 \Lambda T_3 + R_1(T_1, T_2, T_3) \quad (2.12)
\]

with

\[
H = -\Delta - 3Q^2 = -\Delta - V. \quad (2.13)
\]

Moreover,

\[
V(y) = \frac{3}{\left(1 + \frac{y^2}{\pi}\right)^2}, \quad (2.14)
\]

which yields

\[
V(y) = \begin{cases} 
3 + O \left( \frac{y^2}{\pi} \right) & \text{as } y \to 0, \\
\frac{192}{y^4} + O \left( \frac{1}{y^6} \right) & \text{as } y \to +\infty,
\end{cases} \quad (2.15)
\]

and

\[
\Lambda V = \frac{-192(3y^2 - 8)}{(y^2 + 8)^3}. \quad (2.16)
\]

**Step 2** Construction of \(T_1\)

The spectral structure of the Schrödinger operator \(H\) is well known: it has a well localized non positive eigenvalue

\[
H \psi = -\zeta \psi, \quad \zeta > 0,
\]

and a resonance at the origin induced by the energy critical scaling symmetry:

\[
H \Lambda Q = 0, \quad \Lambda Q \notin L^2(\mathbb{R}^4).
\]
Hence the Green’s functions of $H$ are explicit and the other solution to $HT = 0$ for $y > 0$ is given by:

$$\Gamma(y) = -\Lambda Q \int_1^y \frac{dx}{x^2(\Lambda Q(x))^2} = \frac{y^2 - 8}{(y^2 + 8)^2} \left( \frac{y^2}{16} + 6\log y - \frac{583}{112} - \frac{4}{y^2} \right) - \frac{64}{(y^2 + 8)^2},$$

which yields

$$\Gamma(y) = \begin{cases} 
O \left( \frac{1}{y^2} \right) & \text{as } y \to 0, \\
\frac{1}{16} + O \left( \frac{\log y}{y^2} \right) & \text{as } y \to +\infty.
\end{cases} \quad (2.17)$$

We may thus invert $H$ explicitly and the smooth solutions at the origin of

$$Hu = f$$

are given by

$$u = \Gamma(y) \int_0^y f \Lambda Q - \Lambda Q(y) \int_0^y f \Gamma + c\Lambda Q(y), \quad c \in \mathbb{R}. \quad (2.18)$$

We let $T_1$ be the solution of

$$HT_1 + \Lambda Q = 0, \quad (2.19)$$
canceled in zero, which means that we choose the constant $c = 0$ in (2.18). There holds the behaviors at $r \to +\infty$

$$\Lambda^i T_1(y) = -4 \left( \log y - \frac{1}{2} + i \right) + O \left( \frac{(\log y)^2}{y^2} \right), \quad \text{for } 0 \leq i \leq 2. \quad (2.20)$$

There holds the behaviors at $r \to 0$

$$\Lambda^i T_1 = O \left( \frac{y^2}{y} \right), \quad \text{for } 0 \leq i \leq 2. \quad (2.21)$$

Hence, for $0 \leq i \leq 2$

$$\|\Lambda^i T_1\|_{L^\infty_{\nu \leq 2B}} \lesssim \log b,$n

$$HA^i T_1 \sim \frac{8}{y^2}, \quad \text{when } y \to \infty.$$

**Step 3** Construction of the radiation $\Sigma_b$

First, we can notice that the choice $b_s = -b^2$ allows to cancel the log $y$ growth of the expression $-T_1 + \Lambda T_1 - 3QT_1^2$. We now construct a radiation term according two specifications. First, it must compensate the 1-growth of the last expression. Moreover the error induced by this term inside the parabolic zone $y \lesssim B_0$ must be sufficiently small in order not to perturb the dynamics of the blow-up.

Let

$$c_b = \frac{64}{\int \chi_{B_0} \Lambda Q^2} = \frac{2}{|\log b|} \left( 1 + O \left( \frac{1}{|\log b|} \right) \right) \quad (2.22)$$

and

$$d_b = c_b \int_0^{B_0} \chi_{B_0} \Lambda Q \Gamma = O \left( \frac{1}{b|\log b|} \right). \quad (2.23)$$

Let $\Sigma_b$ be the solution to

$$H \Sigma_b = c_b \chi_{B_0} \Lambda Q + d_b H[(1 - \chi_{3B_0}) \Lambda Q] \quad (2.24)$$
given by
\[ \Sigma_b(y) = \Gamma(y) \int_0^y c_b \chi \frac{\Lambda Q}{4} (\Lambda Q)^2 - \Lambda Q(y) \int_0^y c_b \chi \frac{\Lambda Q}{4} \Gamma \Lambda Q + d_b (1 - \chi \Lambda_0) \Lambda Q. \]

The choice of the constant \( c_b \) and \( d_b \) yield:
\[ \Sigma_b \begin{cases} c_b T_1 & \text{for } y \leq \frac{B_0}{4} \\ 64 \Gamma & \text{for } y \geq 6B_0 \end{cases} \]
\[ \Lambda \Sigma_b \begin{cases} c_b \Lambda T_1 & \text{for } y \leq \frac{B_0}{4} \\ 64 \Lambda \Gamma & \text{for } y \geq 6B_0. \end{cases} \]

Then the estimate for \( \Sigma_b \) and \( \Lambda \Sigma_b \) for \( 6B_0 \leq y \leq 2B_1 \),
\[ \Sigma_b(y) = 4 + O \left( \frac{\log y}{y^2} \right) \]
\[ \Lambda \Sigma_b(y) = 4 + O \left( \frac{\log y}{y^2} \right), \]

which fits the first criterion, that we fixed to construct the radiation. For \( \frac{B_0}{4} \leq y \leq 6B_0 \), we have:
\[ \Sigma_b(y) = c_b \left( \frac{1}{16} + O \left( \frac{\log y}{y^2} \right) \right) \left[ \int_0^y \chi \frac{\Lambda Q}{4} (\Lambda Q)^2 \right] - c_b \Lambda Q(y) \int_1^y O(x)dx 
= 4 \int_0^y \chi \frac{\Lambda Q}{4} (\Lambda Q)^2 + O \left( \frac{1}{|\log b|} \right). \]

\[ \Lambda \Sigma_b(y) = \Lambda \Gamma(y) \int_0^y c_b \chi \frac{\Lambda Q}{4} (\Lambda Q)^2 - \Lambda^2 Q(y) \int_0^y c_b \chi \frac{\Lambda Q}{4} \Gamma \Lambda Q 
= c_b \left( \frac{1}{16} + O \left( \frac{\log y}{y^2} \right) \right) \left[ \int_0^y \chi \frac{\Lambda Q}{4} (\Lambda Q)^2 \right] - c_b \Lambda^2 Q(y) \int_1^y O(x)dx 
= 4 \int_0^y \chi \frac{\Lambda Q}{4} (\Lambda Q)^2 + O \left( \frac{1}{|\log b|} \right). \]

Similarly,
\[ \Lambda^2 \Sigma_b(y) = 4 \int_0^y \chi \frac{\Lambda Q}{4} (\Lambda Q)^2 + O \left( \frac{1}{|\log b|} \right). \]

The equation (2.24) and the cancellation \( H \Lambda Q = 0 \) yield the bounds:
\[ \int |H \Sigma_b|^2 \lessapprox \frac{1}{|\log b|}, \quad \int \frac{1}{1 + y^8} |\Sigma_b|^2 \lessapprox b^2, \quad \int |H^2 \Sigma_b|^2 \lessapprox \frac{b^2}{|\log b|^2}. \]

We will see that these bounds respect the second criterion for the conception of the radiation. Furthermore we will see the importance of the term \( c_b \), which modify the modulation equation of \( b \). It becomes:
\[ b_s = -b^2 \left( 1 + \frac{2}{|\log b|} \right). \]

After reintegration, this equation gives the expected blow-up speed (1.8).

**Step 4 Construction of** \( T_2 \)

Define
\[ \Sigma_2 = \Sigma_b + T_1 - \Lambda T_1 - 3QT_1^2. \]
The profile $T_2$ will be defined later as the suitable output of $H$ for the argument $\Sigma_2$. Estimate $\Sigma_2$ before the choice of $T_2$. For $y \leq 1$,  
\[ \Sigma_2 \lesssim y^2. \]  
(2.32)

For $1 \leq y \leq 6B_0$  
\[ \Sigma_2 = 4 \left( \int_0^y \chi_{B_0}^2 (AQ)^2 \right) + O \left( \frac{(\log y)^2}{y^2} \right) + O \left( \frac{1}{|\log b|} \right) \lesssim \frac{1 + \log(y\sqrt{b})}{|\log b|}. \]  
(2.33)

According to the choice of the modulation law for $b$ and the conception of the radiation, we have, for $y \geq 6B_0$  
\[ \Sigma_2 \lesssim \frac{(\log y)^2}{y^2}. \]  
(2.34)

Hence, we obtain:  
\[ |\Sigma_2| \lesssim \frac{y^2}{1 + y^2} \left( \sum_{y \leq 1} + \frac{1 + \log(y\sqrt{b})}{|\log b|} \sum_{1 \leq y \leq 6B_0} \right) + \frac{(\log y)^2}{y^2} \sum_{y \geq 6B_0}. \]  
(2.35)

We have the same bound for $\Lambda \Sigma_2$ and for $\Lambda^2 \Sigma_2$. We now let $T_2$ be the solution to  
\[ HT_2 = \Sigma_2 \]  
(2.36)
given by  
\[ T_2(y) = \Gamma(y) \int_0^y \Sigma_2 AQ - AQ(y) \int_0^y \Sigma_2 \Gamma. \]  
(2.37)

We derive from (2.33) the bounds:  
\[ \forall y \leq 2B_1, \quad |\Lambda^i T_2(y)| \lesssim \frac{y^4}{1 + y^2} \left( \sum_{y \leq 1} + \frac{1}{b|\log b|} \sum_{1 \leq y \geq 1} \right), \quad 0 \leq i \leq 1 \]  
(2.38)

\[ |T_2(y)| \lesssim y^2. \]  
(2.39)

With an explicit calculus, we prove that for any function $f$:  
\[ H \Lambda f = 2Hf + \Lambda Hf - \Lambda Vf. \]  
(2.40)

Hence,  
\[ H(\Lambda T_2) = 2\Sigma_2 + \Lambda \Sigma_2 - \Lambda VT_2 \]  
(2.41)

and  
\[ |H(\Lambda T_2)| \lesssim \frac{y^2}{1 + y^2} \left( \sum_{y \leq 1} + \frac{1 + \log(y\sqrt{b})}{|\log b|} \sum_{1 \leq y \leq 6B_0} \right) + \frac{(\log y)^2}{y^2} \sum_{y \geq 6B_0}. \]  
(2.42)

**Step 5** Construction of $T_3$

In the same way as before, we define  
\[ \Sigma_3 = -2T_2 + \Lambda T_2 - 6QT_1 T_2 - T_1^3. \]  
(2.43)

Notice that we haven’t to conceive a second radiation term. We estimate from (2.20) and (2.38)  
\[ \forall y \leq 2B_1, \quad |\Sigma_3(y)| \lesssim \frac{y^4}{1 + y^2} \left( \sum_{y \leq 1} + \frac{1}{b|\log b|} \sum_{y \geq 1} \right) \]  
(2.44)
and
\[ \forall y \leq 2B_1, \ |\Lambda \Sigma_3(y)| \lesssim \frac{y^4}{1+y^4} \left( 1_{y \leq 1} + \frac{1}{b|\log b|} 1_{y \geq 1} \right). \tag{2.45} \]

We then let \( T_3 \) be the solution to
\[ HT_3 = \Sigma_3 \tag{2.46} \]
given by:
\[ T_3(y) = \Gamma(y) \int_0^y \Sigma_3 \Lambda Q - \Lambda Q(y) \int_0^y \Sigma_3 \Gamma. \tag{2.47} \]

Hence,
\[ \Lambda T_3(y) = \Lambda \Gamma(y) \int_0^y \Sigma_3 \Lambda Q - \Lambda^2 Q(y) \int_0^y \Sigma_3 \Gamma. \tag{2.48} \]

We estimate from (2.44)
\[ \forall y \leq 2B_1, \ |\Lambda^i T_3(y)| \lesssim \frac{y^6}{1+y^4} \left( 1_{y \leq 1} + \frac{1}{b|\log b|} 1_{y \geq 1} \right), \ 0 \leq i \leq 1 \tag{2.49} \]
\[ |T_3(y)| \lesssim y^2 (1 + y^2). \tag{2.50} \]

Finally with (2.40),
\[ H(\Lambda T_3) = 2\Sigma_3 + \Lambda \Sigma_3 - \Lambda V T_3 \tag{2.51} \]

and
\[ \forall y \leq 2B_1, \ |H \Lambda T_3(y)| \lesssim \frac{y^4}{1+y^4} \left( 1_{y \leq 1} + \frac{1}{b|\log b|} 1_{y \geq 1} \right). \tag{2.52} \]

We have thus the bounds for \( i = 0, 1 \), using (2.52), (2.49), (2.44) and (2.45):
\[ \int_{y \leq 2B_1} |H \Lambda^i T_3|^2 \lesssim \int_{y \leq 2B_1} \frac{1}{b^2|\log b|^2} \lesssim \frac{B_1^4}{b^4}, \tag{2.53} \]
\[ \int_{y \leq 2B_1} \frac{1}{1+y^8} |\Lambda^i T_3|^2 \lesssim \frac{1}{b^2|\log b|^2} \int_{y \leq 2B_1} \frac{1}{1+y^4} \lesssim \frac{1}{b^2}. \tag{2.54} \]

The crucial bounds to control the error at \( H^4 \) level is:
\[ \int_{y \leq 2B_1} |H^2 \Lambda^i T_3|^2 \lesssim \frac{1}{b^2|\log b|^2} \quad \text{for} \quad 0 \leq i \leq 1. \tag{2.55} \]

We now prove this bound. A rough estimate looses the huge gain \( \frac{1}{|\log b|} \), and we need to be more precise. From (2.51),
\[ H^2(\Lambda T_3) = H(2\Sigma_3 + \Lambda \Sigma_3) + O \left( \frac{1}{1+y^2} \right). \tag{2.56} \]

We use again (2.40), together with (2.43), (2.39), to obtain:
\[ H \Sigma_3 = -2HT_2 + H \Lambda T_2 + O \left( \frac{\log y^3}{y^2} \right) = \Lambda \Sigma_2 + O \left( \frac{\log y^3}{y^2} \right), \]
\[ H \Lambda \Sigma_3 = 2\Lambda \Sigma_2 + \Lambda^2 \Sigma_2 + O \left( \frac{\log y^3}{y^2} \right) \]
and injecting this into (2.56) with (2.35) yields:
\[ \int_{y \leq 2B_1} |H^2(\Lambda T_3)|^2 \lesssim \int_{y \leq 2B_1} |\frac{y^2}{1+y^2} \left( 1_{y \leq 1} + \frac{1 + \log(y \sqrt{b})}{|\log b|} 1_{1 \leq y \leq 6B_0} \right) + \frac{(\log y)^2}{y^2} 1_{y \geq 6B_0}|^2 \]
\[ \lesssim \frac{1}{b^2|\log b|^2} \]
and (2.55) is proved.

**Step 6** Estimate the error

We are in position to estimate the error \( \Psi_b \). According to our construction, we have from (2.12)

\[
\Psi_b = b^2 \Sigma_b + b^4 \Delta T_3 + R_1(T_1, T_2, T_3)
\]

We then study the last term, the others being already estimated. The bounds (2.29), (2.39) and (2.50) yield the bound for \( y \leq 2B_1 \), \( 0 \leq i \leq 4 \), and \( 0 \leq j \leq 5 \):

\[
\left| \frac{d R_1(y)}{dy} \right| \lesssim b^4 \left( y^{4-i} 1_{y \leq 1} + y^4 y^{2(j+1)-i} \left( 1 + |\log y|^2 \right) 1_{y \geq 1} \right).
\]

Hence:

\[
\int_{y \leq 2B_1} |H R_1|^2 \lesssim b^8 |\log b|^C \int_{y \leq 2B_1} 1 \lesssim b^6 |\log b|^C,
\]

\[
\int_{y \leq 2B_1} \frac{1 + |\log y|^2}{1 + y^4} |H R_1|^2 \lesssim b^8 |\log b|^C \int_{y \leq 2B_1} \frac{1 + |\log y|^2}{1 + y^4} \lesssim b^8 |\log b|^C,
\]

\[
\int_{y \leq 2B_1} |H^2 R_1|^2 + |R_1|^2 \lesssim b^8 |\log b|^C \int_{y \leq 2B_1} \frac{1}{1 + y^4} \lesssim b^8 |\log b|^C.
\]

Injecting these bounds together with (2.29), (2.39), (2.50) and (2.55) into (2.57) yields (2.58). We now prove the flux computation (2.11), which will be helpful for the improved modulation equations.

\[
\frac{(H\Psi_b, \Phi_M)}{(\Lambda Q, \Phi_M)} = \frac{1}{(\Lambda Q, \Phi_M)} \left[ (-b^2 c_b \chi_{b_T} \Lambda Q, \Phi_M) + O(C(M) b^3) \right] = -c_b b^2 + O(C(M) b^3) = -\frac{2b^2}{|\log b|} + O\left( \frac{b^2}{|\log b|^2} \right).
\]

Here we recall that \( M \) large enough being chosen, we assume \( |b| < b^*(M) \) so that the above claim make sense. This concludes the proof of Proposition 2.1.

2.2. Localization of the profile. Taking a careful look at the profiles \((T_i)_{1 \leq i \leq 3}\), we can notice that for \( y \gg B_1 \), \( Q \) is negligible compared to \( b T_1 + b^2 T_2 + b^3 T_3 \). Obviously, this doesn’t make sense, because we look for a solution close to \( Q \). So, we must localize the profiles, with cut-off smooth functions. For technical reasons, we use two localizations: one at \( B_1 \), another one at \( B_0 \).

**Proposition 2.3** (Localization of the profile near \( B_1 \)). Let a \( C^1 \) map \( s \mapsto b(s) \) defined on \([0, s_0]\) with a priori bound \( \forall s \in [0, s_0] \),

\[
0 < b(s) < b^*(M), \quad |b_s| \leq 10b^2.
\]

Let the localized profile

\[
\tilde{Q}_b(s, y) = Q + b \tilde{T}_1 + b^2 \tilde{T}_2 + b^3 \tilde{T}_3 = Q + \tilde{\alpha}
\]

where

\[
\tilde{T}_i = \chi_{B_1} T_i, \quad 1 \leq i \leq 3.
\]

Then

\[
\partial_s \tilde{Q}_b - \Delta \tilde{Q}_b - \frac{\lambda_s}{\lambda} \Lambda \tilde{Q}_b - \tilde{Q}_b^3 = \text{Mod}(t) + \tilde{\Psi}_b
\]

(2.59)
with
\[
\text{Mod}(t) = - \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{Q}_b + (b_s + b^2)(T_1 + 2bT_2) \tag{2.60}
\]
and where \( \tilde{\Psi}_b \) satisfies the bounds on \([0, s_0]\):
(i) Weighted bounds:
\[
\int |H\tilde{\Psi}_b|^2 \lesssim b^4 |\log b|^2, \tag{2.61}
\]
\[
\int \frac{1}{1 + y^8} |\tilde{\Psi}_b|^2 \lesssim b^6, \tag{2.62}
\]
\[
\int |H^2\tilde{\Psi}_b|^2 \lesssim \frac{b^6}{|\log b|^2}. \tag{2.63}
\]
(ii) Flux computation: Let \( \Phi_M \) be given by (3.2), then:
\[
(H\tilde{\Psi}_b, \Phi_M) = \frac{-2b^2}{|\log b|} \log b + O \left( \frac{b^2}{|\log b|^2} \right). \tag{2.64}
\]
We introduce a second localization at \( B_0 \) which will be relevant for \( H^2 \) control, see the proof of Proposition 5.3.

**Proposition 2.4** (Second localization). Let a \( C^1 \) map \( s \mapsto b(s) \) defined on \([0, s_0]\) with a priori bound (2.58). Let the localized profile
\[
\tilde{Q}_b(s, y) = Q + b\tilde{T}_1 + b^2\tilde{T}_2 + b^3\tilde{T}_3 = Q + \tilde{\alpha} \tag{2.65}
\]
where
\[
\tilde{T}_i = \chi_{B_0} T_i, \quad 1 \leq i \leq 3.
\]
Let the radiation:
\[
\zeta_b = \tilde{\alpha} - \alpha \tag{2.66}
\]
and the error
\[
\partial_s \tilde{Q}_b - \Delta \tilde{Q}_b - \frac{\lambda_s}{\lambda} \Lambda \tilde{Q}_b - \tilde{Q}_b^3 = \text{Mod}(t) + \tilde{\Psi}_b
\]
with
\[
\text{Mod}(t) = - \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{Q}_b + (b_s + b^2)(T_1 + 2bT_2). \tag{2.67}
\]
Then there holds the bounds:
\[
\left\| \partial^i_s \zeta_b \right\|_{L^\infty} \lesssim b^{2+i}|\log b|^C, \tag{2.68}
\]
\[
\int |H\zeta_b|^2 \lesssim b^2 |\log b|^2, \quad \Sigma_{i=0}^2 \int \frac{|\partial^i_y \zeta_b|^2}{1 + y^{2(3-i)}} \lesssim b^3 |\log b|^C, \tag{2.69}
\]
\[
\int |H^2\zeta_b|^2 + \Sigma_{i=0}^2 \int \frac{|\partial^i_y \zeta_b|^2}{1 + y^{8-2i}} \lesssim b^4 |\log b|^C, \tag{2.70}
\]
\[
\text{Supp}(H\tilde{\Psi}_b) \subset [0, 2B_0] \quad \text{and} \quad \int |H\tilde{\Psi}_b|^2 \lesssim b^4 |\log b|^2. \tag{2.71}
\]
The proof follows similar lines as in [20] and is displayed for the reader’s convenience in Appendix C.
3. Presentation of possible solution of Theorem 1.1

3.1. Uniqueness of the decomposition. We now look for a solution of (1.1) \( u \), which we will decompose in the form of:

\[
u = \left( \hat{Q}_{b(t)} + \varepsilon(t) \right)_{\lambda(t)}.
\]

(3.1)

Naturally, we must fix constrains to obtain the uniqueness of this decomposition. Moreover it’s crucial that the radiation term \( \varepsilon \) doesn’t perturb the modulation equation (2.30) found during the construction. We will see in the subsection devoted to the modulation equations that it’s the case if we have the following inequality:

\[
\int |H^2(\varepsilon(t))|^2 \lesssim \frac{b^4(t)}{\log(b(t))^2}.
\]

To control sharply the radiation term \( \varepsilon \), it appears then that this one ought to be orthogonal to the kernel of \( H^2 \). The smooth solutions of \( H^2 f = 0 \) are situated in \( \text{Span}(\Lambda Q, T_1) \). But neither \( \Lambda Q \) nor \( T_1 \) are in \( L^2(\mathbb{R}^4) \). Therefore, we use an approximation of the kernel, localizing both directions, with the smooth cut-off function \( \chi_M \), where \( M > 0 \) is an enough large constant. More precisely, we let the direction:

\[
\Phi_M = \chi_M \Lambda Q - c_M H(\chi_M \Lambda Q)
\]

(3.2)

with

\[
c_M = \frac{(\chi_M \Lambda Q, T_1)}{(H(\chi_M \Lambda Q), T_1)} = c_M \frac{M^2}{4} (1 + o_M \to +\infty(1)).
\]

The second term, which is a corrective term, makes the orthogonality between \( \Phi_M \) and \( T_1 \), and the orthogonality between \( \Lambda Q \) and \( H \Phi_M \), because \( H \) is a self-adjoint operator. Furthermore, in accordance with conception of this direction, we have:

\[
\int |\Phi_M|^2 \lesssim |\log M|,
\]

(3.3)

and the scalar products

\[
(\Lambda Q, \Phi_M) = (HT_1, \Phi_M) = (\chi_M \Lambda Q, \Lambda Q) = 64 \log M (1 + o_M \to +\infty(1)).
\]

(3.4)

In the appendix, we argue that we have coercive estimates for the operators \( H \) and \( H^2 \) under additional orthogonality conditions. As a consequence, we fix for the radiation term \( \varepsilon \) the orthogonality conditions:

\[
(\varepsilon(t), \Phi_M) = (\varepsilon(t), H \Phi_M) = 0.
\]

(3.5)

From a standard argument based on the implicit function theorem, these constrains give us the existence and the uniqueness of the decomposition (3.1). First, we have:

\[
(b, \lambda) \to (u, \Phi_M) = \left( \hat{Q}_{b(t)}(\lambda(t)), \Phi_M \right)
\]

is a \( C^1 \) map and thus:

\[
\begin{vmatrix}
\frac{\partial}{\partial \lambda} (\hat{Q}_b)_\lambda, \Phi_M & \frac{\partial}{\partial \lambda} (\hat{Q}_b)_\lambda, \Phi_M \\
\frac{\partial}{\partial b} (\hat{Q}_b)_\lambda, H \Phi_M & \frac{\partial}{\partial b} (\hat{Q}_b)_\lambda, H \Phi_M
\end{vmatrix}_{\lambda=1, b=0} = \begin{vmatrix}
-\Lambda Q, \Phi_M \\
0
\end{vmatrix} = -(\Lambda Q, \Phi_M)^2 \neq 0.
\]

(3.6)

We used the orthogonality conditions mentioned at the moment of the conception of \( \Phi_M \) and the following equality:

\[
\left. \left( \frac{\partial}{\partial \lambda} (\hat{Q}_b)_\lambda, \frac{\partial}{\partial b} (\hat{Q}_b)_\lambda \right) \right|_{\lambda=1, b=0} = -(\Lambda Q, T_1).
\]
As long as the solution remains in a fixed small neighbourhood of \( Q \) for the norm \( \dot{H}^1 \) what will be ensured for a suitable set of initial data, the implicit function theorem ensures the existence and uniqueness of the decomposition (3.1).

3.2. **Partial differential equation verified by the radiation and suitable energies.** From now on, we always use the last decomposition. Moreover depending on whether we have the need of original variables, or rescaled variables, we shall notice the radiation term namely:

\[
    u = \frac{1}{\lambda(t)} \left( \tilde{Q}_b + \varepsilon \right) \left( t, \frac{r}{\lambda(t)} \right) = \frac{1}{\lambda(t)} \tilde{Q}_b(t) \left( t, \frac{r}{\lambda(t)} \right) + w(t,r). \tag{3.7}
\]

We use the occasion to recall the correspondence between both systems of variables

\[
    s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)} \quad \text{and} \quad y = \frac{r}{\lambda(t)}. \]

We give also the rescaling formulas

\[
    u(t,r) = \frac{1}{\lambda} v(s,y), \quad \partial_t u = \frac{1}{\lambda^2} \left( \partial_s v - \frac{\lambda_s}{\lambda} \Lambda v \right). \]

We then can inject the decomposition (3.7) with the rescaled variables in the equation (1.1) using the one of \( \tilde{Q}_b \) (2.59) and obtain the following:

\[
    \partial_s \varepsilon - \frac{\lambda_s}{\lambda} \Lambda \varepsilon + H \varepsilon = F - \text{Mod} = F, \tag{3.8}
\]

where we remind that

\[
    H = -\Delta - V, \quad \text{Mod}(t) = - \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{Q}_b + (b_2 + b^2)(\tilde{T}_1 + 2b \tilde{T}_2)
\]

and where we noticed

\[
    F = -\tilde{\Psi}_b + L(\varepsilon) + N(\varepsilon), \tag{3.9}
\]

where \( L \) is a linear operator coming from the difference between \( H \) and \( H_{B_1} \):

\[
    L(\varepsilon) = H \varepsilon - H_{B_1} \varepsilon = 3(\tilde{Q}_b^2 - Q^2) \varepsilon \tag{3.10}
\]

with

\[
    H_{B_1} = -\Delta - 3\tilde{Q}_b^2
\]

and a last purely nonlinear term:

\[
    N(\varepsilon) = 3\tilde{Q}_b \varepsilon^2 + \varepsilon^3. \tag{3.11}
\]

It’s important to remark that we used here the localization of the profiles near \( B_1 \). At the end of this subsection, we will introduce in the same way some new operators with the second localization near \( B_0 \). Before rewriting (3.8) with the original variables, introduce the suitable norms for our study:

- Energy bound

\[
    \mathcal{E}_1 = \int |\nabla \varepsilon|^2 \tag{3.12}
\]

- Higher order Sobolev norms

\[
    \mathcal{E}_2 = \int |H \varepsilon|^2 = \int |\varepsilon_2|^2, \quad \mathcal{E}_4 = \int |H^2 \varepsilon|^2 = \int |\varepsilon_4|^2 \tag{3.13}
\]

with \( \varepsilon_i = H^i \varepsilon \) for \( i \in \{2, 4\} \).
To work with the original variables, we would recall that we have the following notation:

\[ f_\lambda(y) = \frac{1}{\lambda} f\left( \frac{r}{\lambda} \right). \]

Furthermore we must adapt this notation for the potential term namely because of its quadratic nature:

\[ \tilde{V}(y) = \frac{1}{\lambda^2} V\left( \frac{r}{\lambda} \right). \]

Then (3.8) becomes:

\[ \partial_t w + H_\lambda w = \frac{1}{\lambda^2} F_\lambda. \quad (3.14) \]

We define the same both functions:

\[ w_i = H_\lambda w \quad \text{for} \quad i \in \{2; 4\}, \]

which verify respectively:

\[ \partial_t w_2 + H_\lambda w_2 = -\partial_t \tilde{V} w + H_\lambda \left( \frac{1}{\lambda^2} F_\lambda \right), \quad (3.15) \]

\[ \partial_t w_4 + H_\lambda w_4 = -\partial_t \tilde{V} w_2 - H_\lambda \left( \partial_t \tilde{V} w \right) + H_\lambda^2 \left( \frac{1}{\lambda^2} F_\lambda \right). \quad (3.16) \]

We have also by substitution:

\[ \lambda^2 E_2 = \int |Hw|^2, \quad \lambda^6 E_4 = \int |H^2 w|^2. \quad (3.17) \]

Before closing this part, we are now getting interested us in the localization near \( B_0 \). Using the definition of the radiation \( \zeta_b \), we obtain the new unique decomposition:

\[ u = (\hat{Q}_b + \hat{\varepsilon})_\lambda \quad \text{ie} \quad \hat{\varepsilon} = \varepsilon + \zeta_b. \quad (3.18) \]

Thus, with this localization, we have:

\[ \partial_s \hat{\varepsilon} - \frac{\lambda_s}{\lambda} \hat{\Lambda} \hat{\varepsilon} + \hat{H} \hat{\varepsilon} = \hat{F} - \hat{Mod} = \hat{F}, \quad (3.19) \]

where we remind that

\[ \hat{H} = -\Delta - \hat{V}, \quad \hat{Mod}(t) = -\left( \frac{\lambda_s}{\lambda} + b \right) \Delta \hat{Q}_b + (b_n + b^2)(\hat{T}_1 + 2b\hat{T}_2) \]

and where we have noticed

\[ \hat{F} = -\hat{\Psi}_b + \hat{L}(\hat{\varepsilon}) + \hat{N}(\hat{\varepsilon}), \quad (3.20) \]

where \( \hat{L} \) is a linear operator coming from the difference between \( \hat{H} \) and \( \hat{H}_{B_1} \):

\[ \hat{L}(\hat{\varepsilon}) = \hat{H} \hat{\varepsilon} - \hat{H}_{B_1} \hat{\varepsilon} = 3(\hat{Q}_b^2 - \hat{Q}_b^2) \hat{\varepsilon} \quad (3.21) \]

with

\[ \hat{H}_{B_1} = -\Delta - 3\hat{Q}_b^2 \]

and a last purely nonlinear term:

\[ \hat{N}(\hat{\varepsilon}) = 3\hat{Q}_b \hat{\varepsilon}^2 + \hat{\varepsilon}^3. \quad (3.22) \]

We define likewise the operators \( \hat{\varepsilon}_2, \hat{w}_2 \) and \( \hat{w}_2 \), which come of course respectively from \( \varepsilon_2, w, \) and \( w_2 \). The energy at \( H^2 \) level becomes:

\[ \hat{E}_2 = \int |\hat{\varepsilon}_2|^2. \quad (3.23) \]
Moreover, with the bounds of the radiation (2.09), we can measure the difference between both energies at \( H^2 \) level.

\[
\dot{E}_2 \lesssim E_2 + \int |H\zeta_b|^2 \lesssim E_2 + b^2 |\log b|^2.
\]

(3.24)

Finally, \( \hat{w}_2 \) verifies the following partial differential equation:

\[
\partial_t \hat{w}_2 + \hat{H}_\lambda \hat{w}_2 = -\partial_t \tilde{V}_\hat{w} + \hat{H}_\lambda \left( \frac{1}{\lambda^2} \hat{F}_\lambda \right).
\]

(3.25)

3.3. Modulation equations. With the choice of orthogonality conditions (3.5), we can now measure the error made taking \( b = -\frac{\lambda_s}{\lambda} \) and \( b_s = -b^2 \left( 1 + \frac{2}{|\log b|} \right) \). This estimations are in the core of our proof. This demonstration is the same of in [20], with the exception of a very small difference with the linear and non linear operators \( L(\varepsilon) \) and \( N(\varepsilon) \), which truly brings any difficulty, because of the same interpolation bounds. In view of the importance to this lemma in our proof of the theorem 1.1, we considered useful to give integrally again this demonstration.

**Lemma 3.1** (Modulation equations). There holds the bound on the modulation parameters:

\[
\left| \frac{\lambda_s}{\lambda} + b \right| \lesssim \frac{b^2}{|\log b|} + \frac{1}{\sqrt{\log M}} \sqrt{\mathcal{E}_4},
\]

(3.26)

\[
\left| b_s + b^2 \left( 1 + \frac{2}{|\log b|} \right) \right| \lesssim \frac{1}{\sqrt{\log M}} \left( \sqrt{\mathcal{E}_4} + \frac{b^2}{|\log b|} \right).
\]

(3.27)

**Remark 3.2.** Note that this implies in the bootstrap the rough bounds:

\[
|b_s| + \left| \frac{\lambda_s}{\lambda} + b \right| \leq 2b^2.
\]

(3.28)

and in particular (2.58) holds.

**Proof of Lemma 3.1**

**Step 1** Law for \( b \).

Let

\[
V(t) = |b_s + b^2| + \left| \frac{\lambda_s}{\lambda} + b \right|.
\]

We take the inner product of (3.8) with \( H\Phi_M \) and estimate each terms.

\[
(\partial_s \varepsilon, H\Phi_M) + (H\varepsilon, H\Phi_M) - \left( \frac{\lambda_s}{\lambda} \Delta \varepsilon, H\Phi_M \right)
\]

\[
= - (\tilde{\Psi}_b, H\Phi_M) - (\text{Mod}(t), H\Phi_M) + (L(\varepsilon), H\Phi_M) + (N(\varepsilon), H\Phi_M).
\]

First according to our choice of orthogonality (3.5)

\[
(\partial_s \varepsilon, H\Phi_M) = (H\varepsilon, \Phi_M) = 0.
\]

(3.29)

Then

\[
(H\varepsilon, H\Phi_M) = (H^2 \varepsilon, \Phi_M) \lesssim \|H^2 \varepsilon\|_{L^2} \|\Phi_M\|_{L^2} \lesssim \sqrt{\mathcal{E}_4 \log M}.
\]

(3.30)
From the construction of the profile, (2.60) and the localization \( \text{Supp}(\Phi_M) \subset [0, 2M] \) from (3.2):

\[
(H(\text{Mod}(t)), \Phi_M) = -\left( b + \frac{\lambda_s}{\lambda} \right) \left( H\Lambda Q_b, \Phi_M \right) + (b_s + b^2) \left( H\left( \tilde{T}_1 + 2b\tilde{T}_2 \right), \Phi_M \right)
\]

Using the Hardy bounds of Appendix B:

\[
\left| \left( -\frac{\lambda_s}{\lambda} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), H\Phi_M \right) \right| \lesssim C(M)b(\varepsilon_4 + |V(t)|).
\]

Next, we compute from (3.4) and the orthogonality (3.3):

\[
b_s + b^2 = \frac{\langle \Psi_b, H\Phi_M \rangle}{\langle \Lambda Q, \Phi_M \rangle} + O \left( \frac{\sqrt{\log M\varepsilon_4}}{\log M} \right) + O(C(M)b|V(t)|)
\]

and (5.27) is proved.

**Step 2** Degeneracy of the law for \( \lambda \).

Now we take the inner product of (3.8) with \( \Phi_M \) and obtain:

\[
(\text{Mod}(t), \Phi_M) = -\langle \Psi_b, \Phi_M \rangle - (\partial_b \varepsilon + H\varepsilon, \Phi_M) - \left( -\frac{\lambda_s}{\lambda} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), \Phi_M \right).
\]

From our choice of orthogonality conditions (3.5):

\[
(\partial_b \varepsilon + H\varepsilon, \Phi_M) = 0.
\]

In the same way as the last step, using the Hardy bounds of Appendix B:

\[
\left| \left( -\frac{\lambda_s}{\lambda} \Lambda \varepsilon + L(\varepsilon) + N(\varepsilon), \Phi_M \right) \right| \lesssim C(M)b\sqrt{\varepsilon_4}.
\]

Next, we compute from (3.4) and the orthogonality (3.3):

\[
(\text{Mod}(t), \Phi_M) = -\left( b + \frac{\lambda_s}{\lambda} \right) \left( \Lambda Q_b, \Phi_M \right) + (b_s + b^2) \left( \tilde{T}_1 + 2b\tilde{T}_2, \Phi_M \right)
\]

and observe the cancellation from (2.25), (3.3):

\[
\left| \left( \Psi_b, \Phi_M \right) \right| \lesssim b^3|(\Sigma_b, \Phi_M)| + O(C(M)b^3) = c_b b^3|(T_1, \Phi_M)| + O(C(M)b^3) = O(C(M)b^3).
\]

We thus obtain the modulation equation for scaling:

\[
\left| \left( \frac{\lambda_s}{\lambda} + b \right) \right| \lesssim b^3C(M) + bC(M)O \left( \sqrt{\varepsilon_4} + |V(t)| \right).
\]

With (5.24), we obtain the bound

\[
|V(t)| \lesssim \frac{b^2}{|\log b|} + \frac{1}{\sqrt{\log M}} \sqrt{\varepsilon_4}.
\]

Injecting this bound in (3.31) implies the refined bound (3.27). This concludes the proof of Lemma 3.1.
3.4. Proof of the Theorem [1.1] In this section, we conclude the proof of Theorem [1.1] assuming the following a priori bounds on the solution on its maximum time interval of existence $[0, T)$, $0 < T \leq +\infty$:

- **Energy estimates**

  $$\forall t \in [0, T]\; \mathcal{E}_1(t) \leq \delta(b^*), \quad \mathcal{E}_2(t) \lesssim b(t)^2 |\log b(t)|^5, \quad \mathcal{E}_4(t) \lesssim \frac{b(t)^4}{|\log b(t)|^2}$$  \hspace{1cm} (3.32)

- **Link between both laws $b(t)$ and $\lambda(t)$**: there exist $\alpha_1, \alpha_2 > 0$ such that:

  $$C(u_0)b(t)|\log b(t)|^{\alpha_1} \leq \lambda(t) \leq C'(u_0)b(t)|\log b(t)|^{\alpha_2}$$  \hspace{1cm} (3.33)

The heart of our analysis in section 4 will be to produce such kind of solutions. We now assume (3.32) and (3.33) and prove Theorem 1.1. The proof adapts the argument in [20] which we sketch for the convenience of the reader.

**Step 1** Finite time blow up.

Let $T \leq +\infty$ be the life time of $u$. From (3.32), (3.33),

$$-\frac{d}{dt}\sqrt{\lambda} = -\frac{1}{2\lambda\sqrt{\lambda}} \frac{\lambda_s}{\lambda} \geq \frac{b}{\lambda\sqrt{\lambda}} \gtrsim C(u_0) > 0$$

and thus $\lambda$ touches zero in finite time which implies

$$T < +\infty.$$  \hspace{1cm} (3.34)

The bounds (3.32) and standard $H^4$ local well posedness theory ensure that blow up corresponds to

$$\lambda(t) \to 0 \quad \text{as} \quad t \to T$$  \hspace{1cm} (3.35)

**Step 2** Derivation of the sharp blow up speed.

To begin, the modulation laws become with the bound (3.32):

$$\left| b + \frac{\lambda_s}{\lambda} \right| \lesssim \frac{b^2}{|\log b|}$$  \hspace{1cm} (3.36)

$$\left| b_s + b^2 \left(1 + \frac{2}{|\log b|}\right) \right| \lesssim \frac{1}{\sqrt{\log M}} \frac{b^2}{|\log b|^2}. $$  \hspace{1cm} (3.37)

We have defined $M$ as a enough large constant. Let

$$B_\delta = \frac{1}{b^3}.$$  \hspace{1cm} (3.38)

Since the variation of $b(s)$ is very small, we can consider in the first time that it’s possible to take $M = B_\delta$ in (3.37). Assuming this, prove (1.3), and demonstrate afterwards that the made error is negligible for $\delta$ enough small. We have thus that (3.37) becomes:

$$\left| b_s + b^2 \left(1 + \frac{2}{|\log b|}\right) \right| \lesssim \frac{1}{\sqrt{\log B_\delta}} \frac{b^2}{|\log b|} \lesssim \frac{b^2}{|\log b|^2}. $$  \hspace{1cm} (3.39)
We now integrate this in time using \( \lim_{s \to +\infty} b(s) = 0 \):
\[
b(s) = \frac{1}{s} - \frac{2}{s \log s} + O\left(\frac{1}{s \log s^{3/2}}\right).
\]
(3.40)
Using this decomposition of \( b(s) \) in the modulation equation (3.36), we conclude:
\[
-\frac{\lambda_s}{\lambda} = \frac{1}{s} - \frac{2}{s \log s} + O\left(\frac{1}{s \log s^{3/2}}\right).
\]
We rewrite this as
\[
\left| \frac{d}{ds} \log \left( \frac{s \lambda(s)}{(\log s)^2} \right) \right| \lesssim \frac{1}{s \log s^{3/2}}
\]
and thus integrating in time yields the existence of \( \kappa(u_0) > 0 \) such that:
\[
\frac{s \lambda(s)}{(\log s)^2} = \frac{1}{\kappa(u_0)} \left[ 1 + O\left(\frac{1}{\log \lambda}\right) \right].
\]
Taking the log yields the bound
\[
|\log \lambda| = |\log s| \left[ 1 + O\left(\frac{|\log \lambda|}{\log s}\right) \right]
\]
and thus
\[
\frac{1}{s} = \kappa(u_0) \frac{\lambda}{|\log \lambda|^2} (1 + o(1)).
\]
Injecting this into (3.34) yields:
\[
-\lambda_\lambda t = -\frac{\lambda_s}{\lambda} = \frac{1}{s} (1 + o(1)) = \kappa(u_0) \frac{\lambda}{|\log \lambda|^2} (1 + o(1))
\]
and thus
\[
-|\log \lambda|^2 \lambda_\lambda t = \kappa(u_0)(1 + o(1)).
\]
Integrating from \( t \) to \( T \) with \( \lambda(T) = 0 \) yields
\[
\lambda(t) = \kappa(u_0) \frac{T - t}{|\log (T - t)|^2} \left[ 1 + o(1) \right],
\]
and (1.8) is proved.

Prove now that the made error is negligible. Indeed, we take the inner product of (3.36) with \( H \chi_{B_s} \Lambda Q \) and obtain:
\[
\frac{d}{ds} \left\{ (H \varepsilon, \chi_{B_s} \Lambda Q) \right\} - (H \varepsilon, \partial_s \chi_{B_s} \Lambda Q) + \frac{\lambda_s}{\lambda} (\chi_{B_s} \Lambda Q, H \Lambda e) + (H^2 \varepsilon, \chi_{B_s} \Lambda Q) = \left( H \left[ -\tilde{\Psi}_b + L(\varepsilon) - N(\varepsilon) - Mod \right], \chi_{B_s} \Lambda Q \right).
\]
(3.42)
We must estimate all terms in this identity. First, for \( \delta \) small enough, we have the rough bound:
\[
|(H \varepsilon, \partial_s \chi_{B_s} \Lambda Q)| + |\frac{\lambda_s}{\lambda} (\chi_{B_s} \Lambda Q, H \Lambda e)| + |(H[L(\varepsilon) - N(\varepsilon)], \chi_{B_s} \Lambda Q)| \lesssim \frac{b}{b^{3/2}} \sqrt{E_4} \lesssim \frac{b^2}{|\log b|^2}.
\]
For the linear term, we have immediately:
\[
|(H^2 \varepsilon, \chi_{B_s} \Lambda Q)| \lesssim \sqrt{E_4} \sqrt{|\log b|} \lesssim \frac{b^2}{\sqrt{|\log b|}}.
\]
The $\tilde{\Psi}_b$ term is computed from (2.57):

$$(-H\tilde{\Psi}_b, \chi_{B_b}\Psi_b) = -b^2 (H\Sigma_b, \chi_{B_b}\Psi_b) + O\left(\frac{b^3}{b^{C_6}}\right) = b^2 \tilde{c}_b (\Lambda Q, \chi_{B_b}\Lambda Q) + O\left(\frac{b^2}{|\log b|^2}\right).$$

From (2.60), we have the following estimate for the modulation term:

$$(-H Mod, \chi_{B_b}\Lambda Q) = \left(\frac{\lambda^3}{\chi^3} + b\right) (H\Lambda\tilde{Q}_b, \chi_{B_b}\Lambda Q) - (b_s + b^2)(H(\tilde{T}_1 + 2b\tilde{T}_2), \chi_{B_b}\Lambda Q)$$

$$= (b_s + b^2)(\Lambda Q, \chi_{B_b}\Lambda Q) + O\left(\frac{b}{b^{C_6}}|\log b|\right).$$

We now inject the estimates into (3.42) and obtain:

$$(b_s + b^2)(\Lambda Q, \chi_{B_b}\Lambda Q) = \frac{d}{ds} \{(H\varepsilon, \chi_{B_b}\Lambda Q)\} - c_4 b^2 (\Lambda Q, \chi_{B_b}\Lambda Q) + O\left(\frac{b^2}{|\log b|^2}\right)$$

which we rewrite using (2.22) and an integration by parts in time:

$$\frac{d}{ds} \left\{ b - \frac{(H\varepsilon, \chi_{B_b}\Lambda Q)}{(\Lambda Q, \chi_{B_b}\Lambda Q)} \right\} + b^2 \left(1 + \frac{2}{|\log b|}\right)$$

$$= O\left(\frac{b^2}{|\log b|^2}\right) + (H\varepsilon, \chi_{B_b}\Lambda Q) \frac{(\Lambda Q, \partial_s \chi_{B_b}\Lambda Q) + O\left(\frac{b^2}{|\log b|^2}\right)}{(\Lambda Q, \chi_{B_b}\Lambda Q)^2}.$$

We now estimate:

$$\left| (H\varepsilon, \chi_{B_b}\Lambda Q) \frac{(\Lambda Q, \partial_s \chi_{B_b}\Lambda Q)}{(\Lambda Q, \chi_{B_b}\Lambda Q)^2} \right| \lesssim \frac{\sqrt{\varepsilon_4}}{b^{C_6}} |b_s| \lesssim b^3,$$

$$\left| \frac{(H\varepsilon, \chi_{B_b}\Lambda Q)}{(\Lambda Q, \chi_{B_b}\Lambda Q)} \right| \lesssim \frac{\sqrt{\varepsilon_4}}{b^{C_6}} \lesssim b^2.$$

We inject these bounds into (3.43) and conclude that the difference between $b$ and $\tilde{b}$ is given by

$$\tilde{b} = b - \frac{(H\varepsilon, \chi_{B_b}\Lambda Q)}{(\Lambda Q, \chi_{B_b}\Lambda Q)} = b + O\left(\frac{b^2}{|\log b|^2}\right)$$

satisfies the pointwise differential control:

$$\left| \bar{b}_s + \tilde{b}_s \left(1 + \frac{2}{|\log b|}\right) \right| \lesssim \frac{\tilde{b}^2}{|\log b|^2},$$

which we rewrite:

$$\frac{\tilde{b}_s}{\tilde{b}^2} \left(1 + \frac{2}{|\log b|}\right) + 1 = O\left(\frac{1}{|\log b|^2}\right).$$

We now integrate this in time using $\lim_{s \to +\infty} \tilde{b}(s) = 0$ from (3.35), (3.44) and get:

$$\tilde{b}(s) = \frac{1}{s} - \frac{2}{s \log s} + O\left(\frac{1}{s|\log s|^2}\right)$$

and thus from (3.44):

$$b(s) = \frac{1}{s} - \frac{2}{s \log s} + O\left(\frac{1}{s|\log s|^2}\right).$$

This concludes the proof.
Step 3 Quantization of the focused energy.

We now turn to the proof of (1.7), (1.9) and adapt the strategy in [16]. We shall need the following bound, which is a direct consequence of our construction and (3.32):

$$\forall t \in [0, T), \|\Delta \tilde{u}(t, x)\|_{L^2} \leq C(v_0).$$

(3.46)

where

$$\tilde{u}(t, x) = u(t, x) - \frac{1}{\lambda(t)} Q \left( \frac{x}{\lambda(t)} \right)$$

(3.47)

The regularity of $v(t, x)$ outside the origin is a standard consequence of parabolic regularity. Hence there exists $u^* \in \dot{H}^1$ such that

$$\forall R > 0, \nabla u(t) \to \nabla u^* \text{ in } L^2(|x| \geq R) \text{ as } t \to T.$$

Moreover, $v$ is $\dot{H}^1$ bounded by decrease of energy, and thus recalling the decomposition (3.47) and the uniform bound (3.46):

$$\nabla \tilde{u}(t) \to \nabla u^* \text{ in } L^2 \text{ and } \Delta u^* \in L^2$$

which concludes the proof of (1.7), (1.9). This concludes the proof of Theorem 1.1.

4. Description of the initial data and bootstrap

The proof of the Theorem 1.1 consists now in the demonstration of the existence of initial data, close to $Q$, whose the timing will be in agreement with the assumed bounds (3.32) and (3.33). We have already seen the condition of smallness of $b$ to assure the uniqueness of the decomposition, through the implicit function theorem:

$$0 < b(0) < b^*(M) \ll 1.$$  

(4.1)

To be large regarding the constrains (3.32), we fix the initial generous bounds namely:

$$|E_1(0)| \leq b(0)^2,$$

(4.2)

and

$$|E_2(0)| + |E_4(0)| \leq b(0)^{10}.$$  

(4.3)

Moreover, there is a crucial difference compared to [20]. The linear operator $H$ possesses a negative direction $\psi$, source of instability, which can be harmful to the blow up dynamics, if we don’t control this. Therefore, we manage this as in [5]. We note

$$\kappa(t) = (\varepsilon(t), \psi),$$

(4.4)

and

$$a^+ = \kappa(0) = (\varepsilon(0), \psi).$$

(4.5)

We impose that:

$$|a^+| \leq \frac{2b(0)^{\frac{5}{7}}}{|\log b(0)|}.$$  

(4.6)

The propagation of regularity by the parabolic heat flow ensures that these estimates hold on some time interval $[0, t_1)$ together with the regularity $(\lambda, b) \in C^1([0, t_1), R_+^* \times R)$. Given a large enough universal constant $K > 0$ -independent of $M^*$, we assume on $[0, t_1)$:

- Control of $b(t)$:

$$0 < b(t) < 10b(0).$$

(4.7)
• Control of the radiation:
\[ \int |\nabla \varepsilon(t)|^2 \leq 10 \sqrt{b(0)}, \quad (4.8) \]
\[ |\mathcal{E}_2(t)| \leq K b^2(t) \log(b(t))^{5}, \quad (4.9) \]
\[ |\mathcal{E}_4(t)| \leq K \frac{b^4(t)}{\log(b(t))^2}. \quad (4.10) \]
• A priori bound on the unstable mode
\[ |\kappa(t)| \leq 2 \sqrt{b(0)} \log(b(0))^{2}. \quad (4.11) \]

We may describe the bootstrap regime as follows:

**Definition 4.1** (Exit time). Given \( a^+ \in \left[-2 \sqrt{b(0)}, 2 \sqrt{b(0)} \right] \), we let \( T(a^+) \) be the life time of the solution to (1.1) with initial data (4.1), (4.2), (4.3) and (4.6), and \( T_1(a^+) > 0 \) be the supremum of \( T \in (0, T(a^+)) \), such that for all \( t \in [0; T] \), the estimates (4.7), (4.8), (4.9), (4.10) and (4.11) hold.

The existence of blow up solutions in the regime described by Theorem 1.1 now follows from the following:

**Proposition 4.2.** There exists \( a^+ \in \left[-2 \sqrt{b(0)}, 2 \sqrt{b(0)} \right] \) such that
\[ T_1(a^+) = T(a^+) \quad (4.12) \]
and then corresponding solution of (1.1) blows up in finite time in the regime described by Theorem 1.1.

We shall use the same strategy as in [19], [17], [5], and [20]. We will process in three times:

• First, we shall derive of suitable Lyapounov functionals at Sobolev respectively \( \dot{H}^4 \) and \( \dot{H}^2 \) levels. That is the most difficult part of the proof, particularly because of the estimates of non linear terms, for whose we must make a sharp study. Moreover, we shall see that it was crucial that \( |a^+| \lesssim \frac{b(0)^{\frac{5}{2}}}{\log(b(0))} \).

• Secondly, we will reintegrate this functionals, to obtain improved bounds for \( \mathcal{E}_2 \) and \( \mathcal{E}_4 \). The bounds (4.7) is a direct consequence of the energy decrease. Thus, only the last bounds (4.11) for the unstable direction can be the cause of an exit time less than the life time of the solution.

• Finally, we will study the dynamics of the unstable mode, and see that we can choose a \( a^+ \) to obtain Proposition 4.2. To ensure the existence of this solution, it is important that \( |a^+| \gtrsim g(b(0)) \) with \( \frac{\log(b(x))}{b(x)^{\frac{3}{2}}} = o(g(x)) \) as \( x \to 0 \), hence the choice for the bounds of \( a^+ \) in (4.6).

5. Lyapounov monotonicities

5.1. At \( \dot{H}^4 \) level.

**Proposition 5.1** (Lyapounov monotonicity \( \dot{H}^4 \)). There holds:
\[ \frac{d}{dt} \left\{ \frac{1}{\lambda^6} \left[ \mathcal{E}_4 + O \left( \frac{b^4}{\log(b(t))} \right) \right] \right\} \leq C \frac{b}{\lambda^8} \left[ \frac{\mathcal{E}_4}{\sqrt{\log M}} + \frac{\mathcal{E}_4^2}{\log(b(t))} + \frac{\mathcal{E}_4}{\log(b(t))} \right] \quad (5.1) \]
for some universal constant $C > 0$ independent of $M$ and of the bootstrap constant $K$ in (4.7), (4.8), (4.9), (4.10), provided $b'(M)$ in (4.1) has been chosen small enough.

**Proof of Proposition 5.1**

We recall the partial differential equations satisfied by $w_2$ and $w_4$:

$$
\partial_t w_2 + H_\lambda w_2 = -\partial_t \tilde{V} w + H_\lambda \left( \frac{1}{\lambda^2} F_\lambda \right) .
$$

(5.2)

$$
\partial_t w_4 + H_\lambda w_4 = -\partial_t \tilde{V} w_2 - H_\lambda \left( \partial_t \tilde{V} w \right) + H^2_\lambda \left( \frac{1}{\lambda^2} F_\lambda \right) .
$$

(5.3)

Moreover, we recall the action of time derivatives on rescaling:

$$
\partial_t w_\lambda(r) = \frac{1}{\lambda^2} \left( \partial_t v - \frac{\lambda}{\lambda} \Lambda v \right) .
$$

**Step 1 Energy identity**

**Lemma 5.2** (Energy identity $\dot{H}^4$).

$$
\frac{1}{2} \frac{d}{dt} \int \left\{ w_4^2 - 2 \partial_t \tilde{V} w w_4 \right\} = -\int w_4 H_\lambda w_4 + \int \left( \partial_t \tilde{V} \right)^2 \ w_4 w - \int \partial_t \tilde{V} w_4 w + \int H_\lambda \left( \partial_t \tilde{V} w \right) \partial_t \tilde{V} w + \int w_4 H^2_\lambda \frac{1}{\lambda^2} F_\lambda - \int H_\lambda \left( \partial_t \tilde{V} w \right) H_\lambda \left( \frac{1}{\lambda^2} F_\lambda \right) - \int \partial_t \tilde{V} w_4 \frac{1}{\lambda^2} F_\lambda .
$$

**Proof of the Lemma 5.2.** We propose here a simplification with respect to the algebra in [20]. Dissipation also allows us to sign some terms and avoid the study of suitable quadratic forms as in [5]. We compute the energy identity:

$$
\frac{1}{2} \frac{d}{dt} \int w_4^2 = \int w_4 \partial_t w_4
$$

$$
= \int w_4 \left( -H_\lambda w_4 - \partial_t \tilde{V} w_2 - H_\lambda \left( \partial_t \tilde{V} w \right) + H^2_\lambda \frac{1}{\lambda^2} F_\lambda \right) .
$$

We now treat separately the second and the third term.

$$
- \int w_4 \partial_t \tilde{V} w_2
$$

$$
= \int \partial_t \tilde{V} w_2 \left( \partial_t w_2 + \partial_t \tilde{V} w - H_\lambda \left( \frac{1}{\lambda^2} F_\lambda \right) \right)
$$

$$
= \int \left( \partial_t \tilde{V} \right)^2 w_2 w - \int \partial_t \tilde{V} w_2 H_\lambda \left( \frac{1}{\lambda^2} F_\lambda \right) + \int \partial_t \tilde{V} \partial_t \left( \frac{w^2_2}{2} \right)
$$

$$
= \int \left( \partial_t \tilde{V} \right)^2 w_2 w - \int \partial_t \tilde{V} w_2 H_\lambda \left( \frac{1}{\lambda^2} F_\lambda \right) - \frac{1}{2} \int \partial_t \tilde{V} w_2^2 + \frac{1}{2} \frac{d}{dt} \left( \int \partial_t \tilde{V} w_2^2 \right) .
$$
Now
\[- \int w_4 H_\lambda \left( \partial_t \tilde{V} w \right) = \int H_\lambda \left( \partial_t \tilde{V} w \right) \left( \partial_t w_2 + \partial_t \tilde{V} w - H_\lambda \left( \frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \]
\[= \int H_\lambda \left( \partial_t \tilde{V} w \right) \left( \partial_t \tilde{V} w - H_\lambda \left( \frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \]
\[+ \frac{d}{dt} \int H_\lambda \left( \partial_t \tilde{V} w \right) w_2 - \int \partial_t \left[ H_\lambda \left( \partial_t \tilde{V} w \right) \right] w_2.\]

The last term becomes
\[- \int \partial_t H_\lambda \left( \partial_t \tilde{V} w \right) w_2 = \int \left( \partial_t \tilde{V} \right)^2 w_2 w - \int \partial_t \tilde{V} w w_4 - \int \partial_t \tilde{V} \partial_t w w_4,\]
and
\[- \int \partial_t \tilde{V} \partial_t w w_4 = \int \partial_t \tilde{V} \left( w_2 - \frac{1}{\lambda^2} \mathcal{F}_\lambda \right) w_4 \]
\[= - \int \partial_t \tilde{V} \frac{1}{\lambda^2} \mathcal{F}_\lambda w_4 + \int \partial_t \tilde{V} w_2 \left( - \partial_t w_2 - \partial_t \tilde{V} w + H_\lambda \left( \frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \]
\[= - \int \partial_t \tilde{V} \frac{1}{\lambda^2} \mathcal{F}_\lambda w_4 + \int \partial_t \tilde{V} w_2 \left( - \partial_t \tilde{V} w + H_\lambda \left( \frac{1}{\lambda^2} \mathcal{F}_\lambda \right) \right) \]
\[- \frac{1}{2} \frac{d}{dt} \left( \int \partial_t \tilde{V} w_2^2 \right) + \frac{1}{2} \int \partial_t \tilde{V} w_2^2.\]

In the following steps, we estimate each terms of Lemma 5.2 in order to prove Proposition 5.1.

**Step 2 Lower order quadratic terms**

We have from (2.16) and (5.4), and the modulation equations, the bounds:
\[|\partial_t \tilde{V}| \lesssim \frac{b}{\lambda^8} \frac{1}{1 + y^4} \quad |\partial_t \tilde{V}| \lesssim \frac{b}{\lambda^6} \frac{1}{1 + y^2}. \quad (5.4)\]

Using (A.3), we obtain
\[- \int H_\lambda w_4 w_4 = - \frac{1}{\lambda^8} \int H \varepsilon_4 \varepsilon_4 \lesssim \frac{1}{\lambda^8} \left( \int \varepsilon_4 \varepsilon_4 \right)^2 \lesssim \frac{\varsigma^2}{\lambda^8} \left( \int \varepsilon \varepsilon \right)^2 \]
\[\lesssim \frac{\lambda^2}{\lambda^8} \lesssim \frac{b}{\lambda^8} \frac{b^4}{|\log b|^2}.\]

Remark that:
\[\int H_\lambda \left( \partial_t \tilde{V} w \right) \partial_t \tilde{V} = \left( \partial_t \tilde{V} \right)^2 w_2 w - \int \Delta \left( \partial_t \tilde{V} \right) \partial_t \tilde{V} w^2 - 2 \int \partial_t \tilde{V} \partial_t \tilde{V} \partial_t w w.\]

We treat now the two following terms:
\[\left| 2 \int \left( \partial_t \tilde{V} \right)^2 w_2 w - \int \partial_t \tilde{V} w w_4 - \int \Delta \left( \partial_t \tilde{V} \right) \partial_t \tilde{V} w^2 - 2 \int \partial_t \tilde{V} \partial_t \tilde{V} \partial_t w \right| \]
\[\lesssim \frac{b^2}{\lambda^8} \int \left( \frac{\varepsilon \varepsilon_2}{1 + y^8} + \frac{\varepsilon \varepsilon_4}{1 + y^4} + \frac{\varepsilon^2}{1 + y^{10}} + \frac{\varepsilon \partial_y \varepsilon}{1 + y^9} \right) \lesssim \frac{b^2}{\lambda^8} b^4 |\log b|^C.\]
The last inequality comes from Cauchy-Schwarz and the bounds (B.1) and (B.3).

Finally, we estimate the boundary term in time

\[
\left| \int \partial_t \tilde{V} w w_4 \right| \lesssim \frac{b}{\lambda^6} \left( \int \frac{\varepsilon^2}{1 + y^8} \right)^{\frac{1}{2}} \left( \int \varepsilon^2_4 \right)^{\frac{1}{2}} \lesssim \frac{b^4}{\lambda^6} |\log b|^C \lesssim \sqrt{b} \frac{b^4}{|\log b|^2}.
\]

**Step 3** Further use of dissipation

First, we claim the following bounds:

\[
\int \frac{1}{1 + y^8} x^2 \lesssim \left[ \frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M} \right], \tag{5.5}
\]

\[
\int |H^2 F|^2 \lesssim b^2 \left[ \frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M} \right]. \tag{5.6}
\]

Thus,

\[
\left| \int w_4 H_\lambda^2 \frac{1}{\lambda^2} F_\lambda - \int H_\lambda \left( \partial_t \tilde{V} w \right) H_\lambda \left( \frac{1}{\lambda^2} F_\lambda \right) - \int \partial_t \tilde{V} w_4 \frac{1}{\lambda^2} F_\lambda \right| 
\]

\[
\lesssim \frac{1}{\lambda^8} \left\{ \int \left( \varepsilon^2 + b^2 \frac{\varepsilon^2}{1 + y^8} \right)^{\frac{1}{2}} \left( \int |H^2 F|^2 \right)^{\frac{1}{2}} + b \left( \int \varepsilon^2_4 \right)^{\frac{1}{2}} \left( \int \frac{1}{1 + y^8} x^2 \right)^{\frac{1}{2}} \right\}
\]

\[
\lesssim \frac{b}{\lambda^8} \left[ \frac{\mathcal{E}_4}{\sqrt{\log M}} + \frac{b^4}{|\log b|^2} + \frac{b^2}{|\log b|^{\frac{1}{2}}} \sqrt{\mathcal{E}_4} \right].
\]

This concludes the proof of the Proposition 5.1. We now turn of the proof of (5.5) and (5.6). We recall that

\[
\mathcal{F} = -\tilde{\Psi}_b - \text{Mod}(t) + L(\varepsilon) + N(\varepsilon).
\]

**Step 4** $\tilde{\Psi}_b$ terms.

The contribution of the $\tilde{\Psi}_b$ terms in (5.5) and (5.6) has already been proved in the Lemma 2.3. For that matter, the construction of the approximated solution by the profiles $(T_i)_{1 \leq i \leq 3}$ has been made to obtain these good estimates.

**Step 5** $\text{Mod}(t)$ terms.

Recall the definition (2.60) of $\text{Mod}(t)$.

\[
\text{Mod}(t) = - \left( \frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{Q}_b + (b_s + b^2)(\tilde{T}_1 + 2b \tilde{T}_2).
\]

With the modulation equation (3.26) and (3.27), we have:

\[
\left| \frac{\lambda_s}{\lambda} + b \right|^2 + |b_s + b^2|^2 \lesssim \frac{b^4}{|\log b|^2} + \frac{\mathcal{E}_4}{\log M}.
\]

But

\[
\int \frac{1}{1 + y^8}|\Lambda \tilde{Q}_b|^2 + \int \frac{1}{1 + y^8}|\tilde{T}_1 + 2b \tilde{T}_2|^2 \lesssim 1.
\]

Now,

\[
\int |H^2 \Lambda \tilde{Q}_b|^2 + \int |H^2 (\tilde{T}_1 + 2b \tilde{T}_2)|^2 \lesssim b^2. \tag{5.7}
\]
Indeed:

\[ \int |H^2 \tilde{T}_1|^2 \lesssim \int_{B_1 \leq y \leq 2B_1} \left| \frac{\log y}{y^4} \right|^2 \lesssim \frac{|\log b|^2}{B_1^4} \lesssim b^2 \quad (5.8) \]

There is a whole proof of the estimate for \( \tilde{T}_2 \) in [20]. Here is a summary of this demonstration. With the definition of \( \tilde{T}_2 \), and the bound (2.38) of \( T_2 \), we have:

\[ \int |H^2 \tilde{T}_2|^2 \lesssim \left[ \int_{B_1 \leq y \leq 2B_1} \left| \frac{y^2}{y'\!^2} \right|^2 + \int_{y \leq 2B_1} |H \Sigma_2|^2 \right] \lesssim 1 + \int_{y \leq 2B_1} |H \Sigma_2|^2. \quad (5.9) \]

From the construction of the radiation \( \Sigma_b \) and the definition of \( \Sigma_2 \), we compute:

\[ H \Sigma_2 = H \Sigma_b + H(T_1 - \Lambda T_1) + O \left( \frac{y|\log y|^2}{1 + y^5} \right) \]

\[ = \frac{1}{|\log b|} O \left( \frac{1}{1 + y^2} 1_{y \leq 3B_0} \right) + H(T_1 - \Lambda T_1) + O \left( \frac{y|\log y|^2}{1 + y^5} \right). \]

But

\[ HT_1 - H \Lambda T_1 = HT_1 - \left( 2HT_1 + \Lambda HT_1 - \frac{\Lambda V}{y^2} T_1 \right) \]

\[ = O \left( \frac{|\log y|}{1 + y^2} \right). \]

We thus conclude:

\[ \int |H \Sigma_2|^2 \lesssim \frac{1}{|\log b|} \int_{y \leq 2B_0} \frac{1}{1 + y^4} + \int_{y \leq 2B_1} \frac{|\log y|^4}{1 + y^8} \lesssim 1, \]

and the contribution of \( Mod(t) \) terms to (5.5) and (5.6) are small enough.

**Step 6** Small linear term \( L(\varepsilon) \).

We recall the expression of \( L(\varepsilon) \):

\[ L(\varepsilon) = 3 \left( \tilde{Q}_b^2 - Q^2 \right) \varepsilon. \]

We have, with the rough bounds (2.20), (2.39) and (2.50):

\[ \left| 3 \left( \tilde{Q}_b^2 - Q^2 \right) \right| \lesssim b \]

and, with (B.3)

\[ \int \frac{1}{1 + y^8} |L(\varepsilon)|^2 \lesssim b^2 \int \frac{1}{1 + y^8} \varepsilon^2 \lesssim \frac{b^4}{|\log b|^2}. \]

Let us study the second estimate. In order to do that, let

\[ g = 3 \left( \tilde{Q}_b^2 - Q^2 \right). \]

We have the following bound:

\[ |H^3 \left( \partial_y^i g \right) | \lesssim b^2 \frac{y^k \left( 1 + |\log y| \right)}{1 + y^2 + |2j + k|}, \quad 0 \leq i \leq 4, \quad 0 \leq j \leq 1 \quad (5.11) \]

where

\[ k = \max \{0; 2 - i - 2j\}. \]
So, with the bounds (5.11) and those of the Lemma B.1 we obtain:

\[
\int |H^2(L(\varepsilon))|^2 = \int |H^2(g\varepsilon)|^2 = \int |H (gH\varepsilon - \varepsilon \Delta g - \partial_y g \partial_y \varepsilon) |^2
\]

\[
\lesssim \int |gH^2(\varepsilon)|^2 + \int |\Delta gH\varepsilon|^2 + \int |\partial_y g \partial_y (H\varepsilon)|^2
\]

\[
+ \int |H(\partial_y g)\partial_y \varepsilon|^2 + \int |\partial_y g \Delta (\partial_y \varepsilon)|^2 + \int |\partial_{yy} g \partial_{yy} \varepsilon|^2
\]

\[
\lesssim \frac{b^6}{|\log b|^2}.
\]

In order to improve the redaction, we didn’t have developed the sharp estimation for each term. The method to use is the same one that we are going to use now for the nonlinear terms.

**Step 7** Nonlinear term $N(\varepsilon)$.

We recall the expression of $N(\varepsilon)$:

\[
N(\varepsilon) = 3\tilde{Q}_b \varepsilon^2 + \varepsilon^3.
\]

We have, with the rough bounds (2.20), (2.39) and (2.50):

\[
\left| 3 \left( \tilde{Q}_b(y) - Q(y) \right) \right| \lesssim |b|(1 + |\log y|)
\]

and with (B.6)

\[
\int \frac{1}{1 + y^8} \left| 3 \left( \tilde{Q}_b - Q + Q \right) \varepsilon^2 \right|^2 \lesssim b^2 \int \frac{1 + |\log y|^2}{1 + y^8} \varepsilon^4 + \int \varepsilon^4 \frac{1 + |\log y|^2}{1 + y^8}
\]

\[
\lesssim b^2 \|\varepsilon\|_{L^\infty}^4 \int \frac{1 + |\log y|^2}{1 + y^8} + \frac{\varepsilon^4}{1 + y^8} \int \frac{1}{1 + y^8} \lesssim \frac{b^4}{|\log b|^2}
\]

and to conclude:

\[
\int \frac{1}{1 + y^8} |\varepsilon^3|^2 \lesssim \|\varepsilon\|_{L^\infty}^6 \int \frac{1}{1 + y^8} \lesssim \frac{b^4}{|\log b|^2}.
\]

For the second bound, let us compute $H(\varepsilon^3)$

\[
H(\varepsilon^3) = \varepsilon^2 H(\varepsilon) - 2\partial_y (\varepsilon^2) \partial_y \varepsilon - \Delta (\varepsilon^2) \varepsilon
\]

\[
= \varepsilon^2 H(\varepsilon) - 2(\partial_y \varepsilon)^2 \varepsilon - \varepsilon \partial_{yy} \varepsilon + (\partial_y \varepsilon)^2 + 3 \frac{\varepsilon \partial_{yy} \varepsilon}{y} \varepsilon
\]

\[
= \varepsilon^2 H(\varepsilon) - 3(\partial_y \varepsilon)^2 \varepsilon - \varepsilon^2 \Delta \varepsilon
\]

\[
= 2\varepsilon^2 H(\varepsilon) + V \varepsilon^3 - 3\varepsilon (\partial_y \varepsilon)^2.
\]
Now, we treat each terms separately. First:

\[
\int |H(\varepsilon^2 H(\varepsilon))|^2 \lesssim \int \varepsilon^4 |H^2(\varepsilon)|^2 + \int \Delta(\varepsilon^2)^2 H(\varepsilon)^2 + \int |\partial_y(\varepsilon^2)|^2 |\partial_y H(\varepsilon)|^2
\]

\[
\lesssim \|\varepsilon\|_{L^\infty_y}^2 E_4 + \left( \|\varepsilon\|_{L^\infty_y}^2 \|\partial_y \varepsilon\|_{L^\infty_y}^2 + \|\partial_y \varepsilon\|_{L^\infty_y}^4 + \|\partial_y \varepsilon\|_{L^\infty_y}^2 \|\varepsilon\|_{L^\infty_y}^2 \right) E_2
\]

\[
+ \left( \|\varepsilon\|_{L^\infty_y}^2 \|y \partial_y \varepsilon\|_{L^\infty_y}^2 + \|\partial_y \varepsilon\|_{L^\infty_y}^2 + \|\partial_y \varepsilon\|_{L^\infty_y}^2 \|\varepsilon\|_{L^\infty_y}^2 \right) E_4
\]

\[
\lesssim \frac{b^6}{|\log b|^2}.
\]

Secondly, using that

\[
|\partial^i_y V| \lesssim \frac{1}{1 + y^{4+i}}, \quad 0 \leq i \leq 2,
\]

we have:

\[
\int |H(V^3)|^2 \lesssim \int |H(V)|^2 \varepsilon^6 + \int \Delta(\varepsilon^3)^2 V^2 + \int |\partial_y(\varepsilon^3)|^2 |\partial_y V|^2
\]

\[
\lesssim \|\varepsilon\|_{L^\infty}^2 \left( \|\frac{\varepsilon}{1 + y}\|_{L^\infty_y}^2 + \|\frac{\varepsilon}{1 + y^2}\|_{L^\infty_y}^2 \right) \int \frac{1}{1 + y^6}
\]

\[
+ \|\varepsilon\|_{L^\infty}^2 \int \frac{|\partial_y \varepsilon|^2}{1 + y^8} \lesssim \frac{b^6}{|\log b|^2}.
\]

Lastly:

\[
\int |H(\varepsilon(\partial_y \varepsilon))^2| \lesssim \int |H(\varepsilon)|^2 (|\partial_y \varepsilon|^2)^2 + \int \varepsilon^2 |\Delta((\partial_y \varepsilon)^2)|^2 + \int \partial_y \varepsilon^2 |\partial_y((\partial_y \varepsilon)^2)|^2
\]

\[
\lesssim \|\partial_y \varepsilon\|_{L^\infty_y}^2 E_2 + \int \varepsilon^2 |\Delta((\partial_y \varepsilon)^2)|^2 \lesssim \frac{b^6}{|\log b|^2} + \int \varepsilon^2 |\Delta((\partial_y \varepsilon)^2)|^2.
\]

But

\[
\int \varepsilon^2 |\Delta((\partial_y \varepsilon)^2)|^2
\]

\[= \int \varepsilon^2 \left( |\partial_{yy} \varepsilon|^2 + \partial_y \varepsilon \partial_y^3 \varepsilon + 3 \frac{\partial_y \varepsilon \partial_{yy} \varepsilon}{y} \right)^2
\]

\[= \int \varepsilon^2 \left( \partial_{yy} \varepsilon (H(\varepsilon) + V\varepsilon) + \partial_y \varepsilon \partial_y^3 \varepsilon \right)^2
\]

\[
\lesssim \|\varepsilon\|_{L^\infty}^2 \left( \|y \partial_y \varepsilon\|_{L^\infty_y}^2 E_4 + \|y \partial_y \varepsilon\|_{L^\infty_y}^2 E_2 + \|\varepsilon\|_{L^\infty}^2 \int |\partial_{yy} \varepsilon|^2 \right)
\]

\[
+ \|\varepsilon(1 + |\log y|^2)|_{L^\infty_y} \|y \partial_y \varepsilon\|_{L^\infty_y}^2 E_4 + \|\varepsilon\|_{L^\infty_y}^2 \|y \partial_y \varepsilon\|_{L^\infty_y}^2 \int |\partial_y^3 \varepsilon|^2
\]

\[
\lesssim \frac{b^6}{|\log b|^2}.
\]

Thus,

\[
\int |H^2(\varepsilon)^3|^2 \lesssim \frac{b^6}{|\log y|^2}.
\]

Treat now the other contribution of \(N(\varepsilon)\) in the bound (5.6). Let

\[
f = 3 \left( \frac{\tilde{Q}_b - Q}{\varepsilon} \right).
\]
We have the following bounds:

\[ |\partial_i y f| \lesssim \frac{b y^{2-i}(1 + |\log y|)}{1 + y^{2+i}}, \quad 0 \leq i \leq 2 \]  

\[ |H(\partial_j y f)| \lesssim \frac{b y^k}{1 + y^{2+j+k}} + \frac{1}{1 + y^{4+j}}, \quad 0 \leq j \leq 2 \]  

where

\[ k = \begin{cases} 
1 & \text{for } j = 1, \\
0 & \text{otherwise.} 
\end{cases} \]

Let us compute \( H(f^2) \)

\[ H(f^2) = H(\epsilon^2)f - \epsilon^2 \Delta f - 2 \partial_y f \partial_y (\epsilon^2) \]

\[ = H(\epsilon^2)f + \epsilon^2 H f - 2 \partial_y f \partial_y (\epsilon^2) + V f \epsilon^2 \]  

(5.16)

In the same way as the last proof, we treat each term separately. First:

\[ H(H(\epsilon^2)f) = H^2(\epsilon^2)f - \Delta f H(\epsilon^2) - \partial_y f \partial_y H(\epsilon^2). \]  

(5.17)

Let us estimate the three components using that:

\[ H(\epsilon^2) = 3 \epsilon H(\epsilon) + 2 V \epsilon^2 - 2 (\partial_y \epsilon)^2 \]  

(5.18)

and,

\[ H^2(\epsilon^2) = 3 (\epsilon H^2(\epsilon) + |H(\epsilon)|^2 + V \epsilon H(\epsilon) - 2 \partial_y \epsilon \partial_y H(\epsilon)) + 2 (HV \epsilon^2 - V \Delta (\epsilon^2) - 2 \partial_y V \partial_y (\epsilon^2)) \]

\[ - 2 \left( 3 \partial_y \epsilon H(\partial_y \epsilon) + 2 V (\partial_y \epsilon)^2 - 2 (\partial_y \epsilon)^2 \right) \]  

(5.19)

and moreover,

\[ H(\partial_y \epsilon) = \partial_y H(\epsilon) - \frac{\partial_y \epsilon}{y^2} - \partial_y V \epsilon \]  

(5.20)

(5.17), (5.18), (5.19) and (5.20) together with the bounds (5.14), (5.15), (5.12) and those of the Lemma B.1 imply:

\[ \int |H(H(\epsilon^2)f)|^2 \lesssim \frac{b^6}{|\log b|^2} \]  

(5.21)

Let us study the second term of (5.16) .

\[ H(\epsilon^2 H f) = \epsilon^2 H^2(f) - H f \Delta (\epsilon^2) - 2 \partial_y (H f) \partial_y (\epsilon^2). \]  

(5.22)

The bounds (5.14), (5.15), the Lemma B.1 yields

\[ \int |H(\epsilon^2 H f)|^2 \lesssim \frac{b^6}{|\log b|^2}. \]  

(5.23)

We estimate the two last terms in (5.16) in the same way. This concludes the proof of (5.6) and thus of Proposition 5.1.

5.2. At \( H^2 \) level. For the \( H^2 \) level, we use the profile \( \hat{Q}_b \) localized near \( B_0 \). The description of this localization and the estimates of the new error generated by these are given by the Lemma 2.4 and in the subsection 3.2. We recall the equation verified by \( \hat{w}_2 \):

\[ \partial_t \hat{w}_2 + \hat{H}_\lambda \hat{w}_2 = -\partial \hat{V} \hat{w} + \hat{H}_\lambda \left( \frac{1}{\lambda^2} \hat{F}_\lambda \right). \]  

(5.24)
The term of error $\hat{\Psi}$. For the last, we don’t use exactly the same strategy as for the control $W$. We next estimate from (2.20), (2.38):

\[
\frac{1}{2} \frac{d}{dt} \hat{\Psi} = \int \frac{1}{\lambda^2} \left[ -H' \hat{\psi} - \partial_t \hat{V} \hat{\psi} + H \left( \frac{1}{\lambda^2} \hat{\Psi} \right) \right] = - \int \hat{\psi} H' \hat{\psi} - \int \partial_t \hat{V} \hat{\psi} \hat{\psi} + \int \hat{\psi} H \left( \frac{1}{\lambda^2} \hat{\Psi} \right).
\]

(5.26)

We study each terms separately:

To begin, in agreement with (A.3), the a priori bound (4.11) on the unstable direction, and the bounds (2.70) we have

\[
|\partial_t \hat{V}| \leq \frac{b}{\lambda^4 + 1 + y^4}.
\]

Hence, using Cauchy-Schwarz, with the a priori bound (4.11) measuring the difference between the two energies at $\hat{H^2}$ level, and the bounds (2.70) and (B.1)

\[
\int |\partial_t \hat{V} \hat{\psi}| \leq \frac{b}{\lambda^4} \hat{\psi} \left( \int \left( |\hat{\psi}|^2 + |\hat{\psi}|^2 \right)^{\frac{1}{2}} \right) \leq \frac{b}{\lambda^4} \hat{\psi}^2 |\hat{\psi}|^2.
\]

(5.27)

The second term is a lower order quadratic term. Recall that:

\[
|\partial_t \hat{V}| \leq \frac{b}{\lambda^4 + 1 + y^4}.
\]

For the last, we don’t use exactly the same strategy as for the control $\hat{H^4}$. Indeed, for the term of error, the term of modulation, the global $L^2$ bounds for $\hat{\psi}$, that we dispose is too rough. We must then improve it, as both terms are localized for $y \leq 2B_0$. So:

\[
\int_{y \leq 2B_0} |\hat{\psi}|^2 \leq B_0 |\hat{\psi}|^2 \int \frac{|\hat{\psi}|^2}{(1 + y^4)|\log y|^2} + \int |H \hat{\psi}|^2 \leq C(M)b^2 + b^2 |\log b|^2 \leq b^3 |\log b|^2.
\]

(5.29)

The term of error $\hat{\Psi}_b$ is now estimated using (2.71) and the improved bound (5.29):

\[
|\hat{\psi}, H \hat{\Psi}_b | \lesssim \|H \hat{\Psi}_b \| \|\hat{\psi} \| \lesssim (b^4 |\log b|^2 b^2 |\log b|^2)^{\frac{1}{2}} \leq b^3 |\log b|^2.
\]

(5.30)

We next estimate from (2.20), (2.38):

\[
\int |H \hat{\Psi}_1|^2 \lesssim \int_{y \leq 2B_0} |\Lambda Q|^2 + \int_{B_0 \leq y \leq 2B_0} \frac{|\log y|^2}{y^2} \lesssim |\log b|^2,
\]

\[
\int |H \hat{\Psi}_2|^2 \lesssim \int_{y \leq 2B_0} \frac{y}{y^2 |\log b|^2} \lesssim \frac{1}{b^2 |\log b|^2}.
\]
and thus from (2.67), (3.26), (3.27):

\[
\int |\hat{H}^\text{Mod}(t)|^2 \lesssim \left| \frac{\lambda_s}{\lambda} + b \right|^2 \int |H\hat{Q}_b|^2 + |b_s + b|^2 \int |H(\hat{T}_1 + b\hat{T}_2)|^2 \\
\lesssim b^4 |\log b|^2 \lesssim b^4 |\log b|^2.
\]

Moreover, \( \text{Supp}(\hat{H}^\text{Mod}) \subset [0, 2B_0] \) and thus with (5.29):

\[
|\langle \hat{\varepsilon}_2, H^\text{Mod} \rangle| \lesssim (b^4 |\log b|^2 b^2 |\log b|^2)^{\frac{1}{2}} \lesssim b^3 |\log b|^2. \tag{5.31}
\]

We now claim the following bound for the small linear term, and the nonlinear term:

\[
\int |H\hat{L}(\hat{\varepsilon})|^2 + |H\hat{N}(\hat{\varepsilon})|^2 \lesssim b^5. \tag{5.32}
\]

Assume (5.32). Thus,

\[
|\langle \hat{\varepsilon}_2, H\hat{L}(\hat{\varepsilon}) \rangle| \lesssim b|\log b|^2 b^3 \lesssim b^3 |\log b|^2. \tag{5.33}
\]

(5.33), together with (5.27), (5.28), (5.30) and (5.31) concludes the proof of the Proposition 5.3.

**Proof of (5.32):**

We recall that:

\[
\hat{L}(\hat{\varepsilon}) = 3(\hat{Q}^2_b - \hat{Q}^2).
\]

In the same way as previously, we let:

\[
\hat{g} = 3(\hat{Q}^2_b - \hat{Q}^2),
\]

for which we have the bounds:

\[
\left| \frac{\hat{g}}{1 + |\log y|} + |y\partial_y \hat{g}| + |y^2 H\hat{g}| \right| \lesssim b \quad \text{for} \quad y \leq 1
\]

\[
|y\partial_y \hat{g}| + |y^2 H\hat{g}| \lesssim \frac{b}{1 + y^2} \quad \text{for} \quad y \geq 1.
\]

This bounds, and (2.69), (2.70), and whose of the Lemma B.1 together with the following decomposition:

\[
H\hat{L} = \hat{\varepsilon}H\hat{g} - \hat{g}\Delta \hat{\varepsilon} - 2\partial_y \hat{g}\partial_y \hat{\varepsilon}
\]

yield the hoped bound for the linear term. We estimate afterwards the nonlinear term. We know that:

\[
\hat{N}(\hat{\varepsilon}) = 3Q\hat{\varepsilon}^2 + 3\hat{f}\hat{\varepsilon}^2 + \hat{\varepsilon}^3.
\]

where we note

\[
\hat{f} = 3(\hat{Q}_b - Q)
\]

we are within the bounds of the profiles \((T_i)_{1 \leq i \leq 3}:

\[
\left| \frac{\hat{f}}{1 + |\log y|} + |y\partial_y \hat{f} + y^2 H\hat{f}| \right| \lesssim b \quad \text{for} \quad y \leq 1
\]

\[
|y\partial_y \hat{f} + y^2 H\hat{f}| \lesssim b \quad \text{for} \quad y \geq 1.
\]
With the estimates $L^\infty$ of the Lemma 1.1, of the second localisation (2.69) and (2.70), we have:

\[
\int |H \hat{N}(\hat{\varepsilon})|^2 \lesssim \frac{\Delta(\hat{\varepsilon})^2}{1 + y^2} + (\partial_y \hat{\varepsilon})^2 + \int \frac{\hat{\varepsilon}^4}{1 + y^8} + \|\hat{\varepsilon}\|_L^2 \int |H \hat{f}|^2 + |\Delta \hat{f}|^2 + |\partial_y \hat{f} |^2 \\
+ \|\hat{\varepsilon}\|_L^4 \|\hat{\varepsilon}\|_L^2 + |\Delta \hat{\varepsilon}|^2 + \|\hat{\varepsilon}\|_L^2 \|\partial_y \hat{\varepsilon}\|_L^2 \int |\partial_y \hat{\varepsilon}|^2 \\
\lesssim b^5.
\]

Proposition 5.3 is proved.

6. Proof of the proposition 4.2

6.1. Improved bound. The twice Lyapounov monotonicity properties give us the arguments to get better the priori bounds, under the a priori control (4.11) on the unstable direction.

Lemma 6.1 (Improved bounds under the a priori control (4.11)). Assume that $K$ in (4.7), (4.8), (4.9), and (4.10) has chosen large enough. Then, $\forall t \in [0, t_1]$:  

\[
0 \leq b(t) \leq 2b(0),
\]

\[
\int |\nabla \hat{\varepsilon}|^2 \leq \sqrt{b(0)},
\]

\[
|E_2(t)| \leq \frac{K}{2} b^2(t) \log b(t)^5,
\]

\[
|E_4(t)| \leq \frac{K}{2} \frac{b^4(t)}{|\log b(t)|^2}.
\]

Proof of the Lemma 6.1

Step 1 Positivity and smallness of $b(t)$

The proof of (6.1) is a direct consequence of the modulation equations. Indeed, this last equation (3.27) yields that

\[
b_s \leq 0
\]

We must now prove that $b(t)$ can’t be negative. We argue by contradiction. As $b(0) \geq 0$ and $b(t)$ is a continue function, we suppose that it exists such $t_0$ that $b(t_0) = 0$. With the modulation equations, we have that:

\[
|b_s| \leq 2b^2
\]

Hence, there exists $\delta$ such that $b(t) = 0$ on $[t_0 - \delta, t_0]$, and thus from (4.10), $\lambda(t) = \lambda(t_0)$ and $u(t) = Q_{\lambda(t_0)}$ on $[t_0 - \delta, t_0]$. Iterating on $\delta > 0$, we conclude that $u_0$ is initially a rescaling of $Q$, meaning a contradiction.

Step 2 Energy bound

(6.2) is a consequence of the decrease of energy. Indeed, let

\[
\hat{\varepsilon} = \hat{\alpha} + \varepsilon.
\]
Then
\[ E(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{4} \int |u|^4 \]
\[ = \frac{1}{2} \left\{ \int |\nabla Q|^2 + \int |\nabla \tilde{\epsilon}|^2 \right\} + \int \partial_y Q \partial_y \tilde{\epsilon} - \frac{1}{4} \int \left[ Q^4 + 4Q^3 \tilde{\epsilon} + 6Q^2 \tilde{\epsilon}^2 + 4Q \tilde{\epsilon}^3 + \tilde{\epsilon}^4 \right] \]
\[ = E(Q) + (H\tilde{\epsilon}, \tilde{\epsilon}) - \frac{1}{4} \int \left[ 4Q \tilde{\epsilon}^3 + \tilde{\epsilon}^4 \right]. \quad (6.8) \]

Now,
\[ (H\tilde{\epsilon}, \tilde{\epsilon}) = (H\varepsilon, \varepsilon) + (H\tilde{\alpha}, \tilde{\alpha}) + 2(\tilde{\alpha}, H\varepsilon) \]
\[ = (H\varepsilon, \varepsilon) + O(b|\log b|^C). \]

The last equality comes from Cauchy-Schwarz, the bound (4.9) for \( \varepsilon \) and the inequalities:
\[ \|\tilde{\alpha}\|^2_{L^2} \lesssim |\log b|^C \quad \|H\tilde{\alpha}\|^2_{L^2} \lesssim b^4|\log b|^C. \]

Moreover, using (A.10), the orthogonality conditions (3.5), and the a priori bound on the unstable direction (4.11)
\[ (H\varepsilon, \varepsilon) \geq c \int |\nabla \varepsilon|^2 - \frac{1}{c} [(\varepsilon, \Phi^M)^2 + (\varepsilon, \psi)^2] \gtrsim \int |\nabla \varepsilon|^2 + O\left(\frac{b^5}{|\log b|^2}\right). \]

Thus,
\[ (H\tilde{\epsilon}, \tilde{\epsilon}) \gtrsim \int |\nabla \varepsilon|^2 + O(b|\log b|^C). \quad (6.9) \]

Let us see the nonlinear terms. We recall that:
\[ \|\tilde{\epsilon}\|_{L^\infty} \lesssim \|\varepsilon\|_{L^\infty} + \|\tilde{\alpha}\|_{L^\infty} \lesssim b|\log b| \quad \left\|\frac{\tilde{\alpha}}{y}\right\|^2_{L^2} + \|\nabla \tilde{\alpha}\|^2_{L^2} \lesssim b^2|\log b|^C. \]

Like this, with the bound (4.3) and the fact that:
\[ \forall u \in H^1_{rad}(\mathbb{R}^4), \quad \int \frac{|u|^2}{y^2} \lesssim \int |\nabla u|^2, \quad \left(\int |u|^4\right)^{\frac{1}{4}} \lesssim \left(\int |\nabla u|^2\right)^{\frac{1}{2}}, \quad (6.10) \]

thus
\[ \int [4Q \tilde{\epsilon}^3 + \tilde{\epsilon}^4] \lesssim \|\tilde{\epsilon}\|_{L^\infty} \left(\int \frac{|\varepsilon|^2}{1+y^2} + \left\|\frac{\tilde{\alpha}}{y}\right\|_{L^2}^2 \right) + \left(\int |\nabla \varepsilon|^2\right)^2 + \left(\int |\nabla \tilde{\epsilon}|^2\right)^2 \]
\[ \lesssim \sqrt{b(0)} \int |\nabla \varepsilon|^2 + O(b|\log b|^C). \quad (6.11) \]

The first inequality of (6.10) comes from the Lemma (A.1). The second is a classical result of Sobolev, Gagliardo and Nirenberg in dimension \( N = 4 \). This result in general case and its proof is available in [2] with the Theorem IX.9. By construction,
\[ 0 \leq E(u) - E(Q) \lesssim b^2(0)|\log b(0)|^C. \quad (6.12) \]

Injecting (6.9), (6.11) and (6.12) into (6.8) concludes the proof of (6.2).

**Step 3** Control of \( E_4 \)
We argue similarly as in [20]. \( \forall \tau \in [0, t_1) \),
\[
\mathcal{E}_4(t) \leq 2 \left( \frac{\lambda(t)}{\lambda(0)} \right)^6 \mathcal{E}_4(0) + C \frac{b'(0)}{\| \log b(0) \|^2} \frac{b^4(t)}{\| \log b(t) \|^2} + \frac{b^4(t)}{\| \log b(t) \|^2} \tag{6.13}
\]
\[+ \left[ 1 + \frac{K}{\log M} + \sqrt{K} \right] \lambda^6(t) \int_0^t \frac{b}{\lambda^8 \| \log b \|^2} \, d\tau \]
for some universal constant \( C > 0 \) independent of \( M \).
Let us now consider two constants
\[
\alpha_1 = 2 - \frac{C_1}{\sqrt{\log M}}, \quad \alpha_2 = 2 + \frac{C_2}{\sqrt{\log M}} \tag{6.14}
\]
for some large enough universal constant \( C_1, C_2 \). We compute using the modulation equations (3.26), (3.27) and the bootstrap bound (4.10):
\[
\frac{d}{ds} \left\{ \frac{\| \log b \|^{\alpha_1} b}{\lambda} \right\} = \frac{\| \log b \|^{\alpha_1}}{\lambda} \left[ \left( 1 - \frac{\alpha_1}{\| \log b \|} \right) b' - \frac{\lambda s}{\lambda} \right]
\]
\[= \frac{\| \log b \|^{\alpha_1}}{\lambda} \left[ \left( 1 - \frac{\alpha_1}{\| \log b \|} \right) b' + b^2 + O \left( \frac{\lambda^3}{\| \log b \|} \right) \right]
\]
\[= \left( 1 - \frac{\alpha_1}{\| \log b \|} \right) \frac{\| \log b \|^{\alpha_1}}{\lambda} \left[ b' + b^2 \left( 1 + \frac{\alpha_1}{\| \log b \|} + O \left( \frac{1}{\| \log b \|} \right) \right) \right]
\]
\[\left\{ \begin{array}{l}
\leq 0 \text{ for } i = 1 \\
g \geq 0 \text{ for } i = 2.
\end{array} \right. \]
Integrating this from 0 to \( t \) yields:
\[
\mathcal{E}_4(0) \leq (b(t)\| \log b(t) \|^{\alpha_2}) \frac{\mathcal{E}_0}{(b(0)\| \log b(0) \|^{\alpha_2})^6} \leq \frac{b^4(t)}{\| \log b(t) \|^2}, \tag{6.15}
\]
This yields in particular using the initial bound (4.3) and the bound (4.7):
\[
C \left( \frac{\lambda(t)}{\lambda(0)} \right)^6 \mathcal{E}_4(0) \leq (b(t)\| \log b(t) \|^{\alpha_2}) \frac{\mathcal{E}_0}{(b(0)\| \log b(0) \|^{\alpha_2})^6} \leq \frac{b^4(t)}{\| \log b(t) \|^2}, \tag{6.16}
\]
\[
C \left( \frac{\lambda(t)}{\lambda(0)} \right)^6 \sqrt{\frac{b'(0)}{\| \log b(0) \|^2}} \leq \left( \frac{b(t)\| \log b(t) \|^{\alpha_2}}{b(0)\| \log b(0) \|^{\alpha_2}} \right)^6 \sqrt{\frac{b'(0)}{\| \log b(0) \|^2}}
\]
\[
\leq C (b(t))^{1 + \frac{4}{\alpha_2}} \leq \frac{b^4(t)}{\| \log b(t) \|^2}. \tag{6.17}
\]
We now compute explicitly using \( b = -\lambda \lambda_t + O \left( \frac{b^2}{\| \log b \|} \right) \) from (3.26):
\[
\int_0^t \frac{b}{\lambda^8 \| \log b \|^2} \, d\sigma = \frac{1}{6} \left[ \frac{b^4}{\lambda^6 \| \log b \|^2} \right]_0^t - \frac{1}{6} \int_0^t \frac{b}{\lambda^6 \| \log b \|^2} \left( 4 + \frac{2}{\| \log b \|} \right) \, d\tau
\]
\[+ O \left( \int_0^t \frac{b}{\lambda^6 \| \log b \|^2} \, d\tau \right)
\]
which implies using now \( |b' + b^2| \leq \frac{b^2}{\| \log b \|} \) from (3.27) and (4.10):
\[
\lambda^6(t) \int_0^t \frac{b}{\lambda^6 \| \log b \|^2} \, d\sigma \leq 1 + O \left( \frac{1}{\| \log b(t) \|} \right) \left( \frac{b^4(t)}{\| \log b(t) \|^2} \right).
\]
Injecting this together with (6.13), (6.17) into (6.13) yields
\[
\mathcal{E}_4(t) \leq C \left( \frac{b^4(t)}{\| \log b(t) \|^2} \right) \left[ 1 + \frac{K}{\log M} + \sqrt{K} \right].
\]
for some universal constant $C > 0$ independent of $K$ and $M$, and thus (6.3) follows for $K$ large enough independent of $M$.

**Step 4 Control of $\mathcal{E}_2$**

Similarly to the control of $\mathcal{E}_4$, we give the same proof as well as in [20]. We integrate the monotonicity formula (6.20) after recalling the estimate of the difference between $\hat{\mathcal{E}}_2$ and $\mathcal{E}_2$:

$$
\mathcal{E}_2(t) = \lambda^2(t)\|w_2(t)\|_{L^2}^2 \lesssim \|H\zeta(t)\|_{L^2}^2 + \lambda^2(t)\|\hat{w}_2(t)\|_{L^2}^2 \quad (6.18)
$$

$$
\lesssim b^4(t)|\log b(t)|^2 + \left(\frac{\lambda(t)}{\lambda(0)}\right)^2 [\mathcal{E}_2(0) + b^2(0)|\log b(0)|^2] + \lambda^2(t) \int_0^t \frac{b^3|\log b|^2}{\lambda^4(\tau)} d\tau.
$$

From (4.3), (6.15):

$$
\left(\frac{\lambda(t)}{\lambda(0)}\right)^2 [\mathcal{E}_2(0) + b^2(0)|\log b(0)|^2] \lesssim \frac{(b(0))^{10} + b^2(0)|\log b(0)|^2}{(b(0)|\log b(0)|^{\alpha_2})^2} b^2(t)|\log b(t)|^{2\alpha_2}
$$

$$
\lesssim b^2(t)|\log b(t)|^{4 + \frac{4}{\alpha_2}}.
$$

We now use the bound $b_s \lesssim -b^2$ and (6.15) to estimate:

$$
\lambda^2(t) \int_0^t \frac{b^3|\log b|^2}{\lambda^4(\tau)} d\tau \lesssim \lambda^2(t) \int_0^t \frac{-b_t|\log b|^2}{\lambda^4(\tau)} d\tau
$$

$$
\lesssim \left(\frac{\lambda(t)}{\lambda(0)}\right)^2 b^2(0)|\log b(0)|^{2\alpha_2} \int_0^t \frac{-b_t}{b|\log b|^{2\alpha_1-2}} d\tau
$$

$$
\lesssim \left(\frac{\lambda(t)}{\lambda(0)}\right)^2 b^2(0)|\log b(0)|^{2\alpha_1} \frac{1}{|\log b(0)|^{2\alpha_1-3}}
$$

$$
\lesssim b^2(t)|\log b(t)|^{2\alpha_2} \frac{|\log b(0)|^3}{|\log b(0)|^{2\alpha_2}} \lesssim b^2(t)|\log b(t)|^{4 + \frac{4}{\alpha_2}}.
$$

Injecting these bounds into (6.18) yields:

$$
\mathcal{E}_2(t) \lesssim b^2(t)|\log b(t)|^{4 + \frac{4}{\alpha_2}}
$$

and concludes the proof of (6.3).

6.2. **Dynamic of the unstable mode.** To conclude the Proposition 4.2 we must study the dynamic of the unstable mode, which is the object of this subsection. We recall that $\kappa(t) = (\varepsilon(t), \psi)$.

**Lemma 6.2** (Control of the unstable mode). There holds: for all $t \in [0, T_1(a^+)]$,

$$
\left|\frac{d\kappa}{ds} - \zeta\kappa\right| \leq \sqrt{b} \frac{b \frac{\beta}{\alpha}}{|\log b|}.
$$

(6.19)

**Proof of the Lemma 6.2**

We compute the equation satisfied by $\kappa$ by taking the inner product of (3.8) with the well localized direction $\psi$ to get:

$$
\frac{d\kappa}{ds} - \zeta\kappa = E(\varepsilon)
$$

(6.20)

with

$$
E(\varepsilon) = (-\tilde{\Psi}_b, \psi) + (L(\varepsilon), \psi) + (N(\varepsilon), \psi) - (Mod, \psi) + \frac{\lambda_s}{\lambda}(\Lambda\varepsilon, \psi).
$$

(6.21)
We now estimate all terms of RHS. We recall the exponential localization of $\psi$ as well as the orthogonality $(\psi, \Lambda Q) = 0$. To begin, using (2.63)

$$\left| (\tilde{\psi}_b, \psi) \right| = \left| \frac{1}{c^2} (H^2 \tilde{\psi}_b, \psi) \right| = \left| \frac{1}{c^2} (H^2 \tilde{\psi}_b, \psi) \right| \lesssim \left( \int |H^2 (\tilde{\psi}_b)|^2 \right)^{\frac{1}{2}} \lesssim \sqrt{b} \frac{\tilde{\psi}}{|\log b|}$$

From the definition (3.21) of $L(\varepsilon)$, we have the following bound:

$$|L(\varepsilon)| \lesssim by^{10}|\varepsilon|.$$  

Thus,

$$|(L(\varepsilon), \psi)| \lesssim b \left| \left( \frac{|\varepsilon|}{y^2 (1 + y^2) (1 + |\log y|)} (1 + y^{14}) (1 + |\log y|) \psi \right) \right| \lesssim \sqrt{b} \frac{\tilde{\psi}}{|\log b|}. \quad (6.23)$$

In the same way, with (3.11),

$$|N(\varepsilon)| \lesssim \left( by^{10}|\varepsilon| + \left\| \frac{\varepsilon}{1 + y} \right\|^2_{L^\infty} y^2 \right) |\varepsilon|.$$  

(3.9) and the fact that $\forall i, \|y_i \psi\|_{L^\infty} \lesssim 1$ yield:

$$|(N(\varepsilon), \psi)| \lesssim \sqrt{b} \frac{\tilde{\psi}}{|\log b|}. \quad (6.24)$$

With the notation of the Lemma 2.3, we have

$$|(Mod, \psi)| = \left| \left( -\left( \frac{\lambda_s}{\lambda} + b \right) \Lambda \tilde{\psi}_b + (b_s + b^2)(\tilde{T}_1 + 2b \tilde{T}_2), \psi \right) \right| \lesssim \left( \frac{\lambda_s}{\lambda} + b \right) |(\Lambda Q + \Lambda \tilde{a}, \psi)| + |b_s + b^2| \left( |\tilde{T}_1 + 2b \tilde{T}_2, \psi| \right).$$

But

$$|\Lambda \tilde{a}| + |2b \tilde{T}_2| \lesssim by^{10}$$

and

$$\left| (\tilde{T}_1, \psi) \right| = \left| \frac{-1}{\zeta} (\tilde{T}_1, H \psi) \right| = \left| \frac{-1}{\zeta} (H \tilde{T}_1, \psi) \right| \lesssim \int_{y \geq B_2} (H \tilde{T}_1 - \Lambda Q) \psi \lesssim b.$$  

Hence, with the modulation equations

$$|(Mod, \psi)| \lesssim \sqrt{b} \frac{\tilde{\psi}}{|\log b|}. \quad (6.25)$$

For the last term, we use (B.1).

$$\left| \frac{\lambda_s}{\lambda} \Lambda \varepsilon, \psi \right| = \left| \frac{\lambda_s}{\lambda} \left( \frac{\Lambda \varepsilon}{y^2 (1 + y^2) (1 + |\log y|)} (1 + y^{14}) \right) \right| \lesssim \sqrt{b} \frac{\tilde{\psi}}{|\log b|}. \quad (6.26)$$

This concludes the proof of the Lemma 6.2.
6.3. **Conclusion.** We have at our disposal all the elements to finish the proof using a rough argument which does not give a sharp information on the link between the initial data and the choice of $a^+$, and uniqueness for example is not covered at this stage. The keystone of the proof is the fact that the map:

$$
\begin{bmatrix}
-2 \frac{b_0^2}{|\log b_0|}; 2 \frac{b_0^2}{|\log b_0|}
\end{bmatrix} \to \mathbb{R}^+
$$

$$
a^+ \to T_1(a^+)
$$

is continuous as a consequence of the strictly outgoing behavior on exit \((6.20)\) defined by \((1.11)\). This classical argument is displayed in details in \([3]\), Lemma 6, in a more complicated setting and therefore left to the reader. Hence, we also have the continuity of the map

$$
\begin{bmatrix}
-2 \frac{b_0^2}{|\log b_0|}; 2 \frac{b_0^2}{|\log b_0|}
\end{bmatrix} \to \mathbb{R}^+
$$

$$
a^+ \to \kappa(T_1(a^+)).
$$

In agreement with the dynamics of the unstable mode found in the last subsection, we know that, for $a^+ = 2 \frac{b_0^2}{|\log b_0|}$:

$$
\frac{d}{ds}\kappa(0) = 2 \zeta \frac{b_0^2}{|\log b_0|} + O \left( \frac{b_0^3}{|\log b_0|} \right) > 0
$$

and then

$$
\kappa \left( T_1 \left( 2 \frac{b_0^2}{|\log b_0|} \right) \right) = 2 \frac{b_0^2}{|\log b_0|}.
$$

(6.27)

Likewise,

$$
\kappa \left( T_1 \left( -2 \frac{b_0^2}{|\log b_0|} \right) \right) = -2 \frac{b_0^2}{|\log b_0|}.
$$

(6.28)

By contuity, it exists $a^+ \in \left[ -2 \frac{b_0^2}{|\log b_0|}; 2 \frac{b_0^2}{|\log b_0|} \right]$ such that

$$
\kappa \left( T_1 \left( a^+ \right) \right) = 0.
$$

(6.29)

In addition, according to the definition of exit time and the Lemma 6.1 we have two choices. Either $|\kappa(T_1(a^+))| = 2 \frac{b(T_1(a^+))}{|\log b(T_1(a^+))|}$ or $T_1(a^+)$ is the life time of the solution. If the first possibility is the good one, the condition \((6.29)\) gives

$$
2 \frac{b(T_1(a^+))}{|\log b(T_1(a^+))|} = 0.
$$

As we have proved that $b(t) > 0$ for $t < T$, where $T$ is the life time of the solution, we have thus the second possibility. Notice that in this case, the two choices tally. This is exactly the Proposition 4.2.
Appendix A. $L^2$ coercivity estimates

In this appendix, we let prove at the first Hardy inequalities for functions $u \in H^2_{rad}({\mathbb R}^4)$. We will use afterwards this results to establish properties of weighted sub-coercivity for $H$ and $H^2$, which allow us to obtain coercive estimates for these operators under additional orthogonality conditions. This coercive estimates are crucial in our study. The proof lies in the continuation of the analysis in [5].

A.1. Hardy inequalities.

Lemma A.1. There exists a constant $C$ for which there holds, for any $v \in H^1_{rad}({\mathbb R}^4)$

$$\left[\int_{{\mathbb R}^4} \frac{v(y)^2}{y^2}\right]^{\frac{1}{2}} + \sup_{y \in {\mathbb R}^4} (|yv(y)|) \leq C \left[\int_{{\mathbb R}^4} |\nabla v(y)|^2\right]^{\frac{1}{2}}, \quad (A.1)$$

$$||v||_{L^\infty_{y \geq 1}} \leq \int_{y \geq 1} \left(\frac{|\nabla v|^2}{y^2} + \frac{|v|^2}{y^4}\right) + \int_{\frac{1}{2} \leq y \leq 1} \frac{|v|^2}{y^2} \quad (A.2)$$

Proof of the Lemma A.1. By integration-by-parts:

$$\int_{{\mathbb R}^4} \frac{v(y)^2}{y^2} = \left[\int_{{\mathbb R}^4} \frac{v(y)^2}{y^2}\right]_{0}^{\infty} - \int_{{\mathbb R}^4} \frac{v(y)\partial_y v(y)}{y} \leq \left[\int_{{\mathbb R}^4} \frac{v(y)^2}{y^2}\right]^{\frac{1}{2}} \left[\int |\nabla v(y)|^2\right]^{\frac{1}{2}}.$$  

Next:

$$|v(y)| \lesssim \int_y^{+\infty} |\partial_y v| \lesssim \frac{1}{y} \left(\int |\partial_y v|^2\right)^{\frac{1}{2}},$$

$$|v^2(y)| \lesssim \int_{1 \leq y \leq 2} |v|^2 + \int_{y \geq 1} \frac{|v||\partial_y v|}{y^3} \lesssim \left(\int_{y \geq 1} \frac{|\partial_y v|^2}{y^3}\right)^{\frac{1}{2}} \left(\int \frac{|v|^2}{y^4}\right)^{\frac{1}{2}}.$$  

This concludes the proof of Lemma A.1.

Lemma A.2 (Hardy inequalities). $\forall R > 2$, $\forall v \in H^2_{rad}({\mathbb R}^4)$, $\forall \gamma > 0$ there holds the following controls:

$$\int |\partial_{yy} v|^2 + \int |\partial_y v|^2 \lesssim \int (\Delta v)^2, \quad (A.3)$$

$$\int_{y \leq R} \frac{|v|^2}{y^4(1 + |\log y|)^2} \lesssim \int_{y \leq R} \frac{|\partial_y v|^2}{y^2} + \int_{1 \leq y \leq 2} |v|^2, \quad (A.4)$$

$$\int_{1 \leq y \leq R} \frac{|v|^2}{y^{1+\gamma}(1 + |\log y|)^2} \lesssim \int_{1 \leq y \leq R} \frac{|\nabla v|^2}{y^{2+\gamma}(1 + |\log y|)^2} + C_\gamma \int_{1 \leq y \leq 2} |v|^2 \quad (A.5)$$

Proof of the Lemma A.2. Let $v$ smooth and radially symmetric. (A.3) follows from the explicit formula after integration of parts

$$\int (\Delta v)^2 = \int \left(\partial_{yy} v + \frac{3}{y} \partial_y v\right)^2 = \int |\partial_{yy} v|^2 + 3 \int \frac{|\partial_y v|^2}{y^2}.$$  

To prove (A.4) and (A.5), from the one dimensional Sobolev embedding $H^1(1 \leq y \leq 2)$ in $L^\infty(1 \leq y \leq 2)$, we obtain

$$|v(1)|^2 \lesssim \int_{1 \leq y \leq 2} (|v|^2 + |\partial_y v|^2). \quad (A.6)$$
Let \( f(y) = -\frac{\epsilon_y}{y^{(1+\log y)^2}} \) so that \( \nabla . f = \frac{1}{y^{(1+\log y)^2}} \), and integrate by parts to get:

\[
\int_{1 \leq y \leq R} \frac{|v|^2}{y^4(1+|\log y|)^2} = \int_{1 \leq y \leq R} |v|^2 \nabla . f
\]

\[
= - \left[ \frac{|v|^2}{1+\log y} \right]_{1}^{R} + 2 \int_{1 \leq y \leq R} \frac{v \partial_y v}{y^3(1+\log y)}
\]

\[
\lesssim |v(1)|^2 + \left( \int_{y \leq R} \frac{|v|^2}{y^3(1+|\log y|)^2} \right) \left( \int_{y \leq R} \frac{|\partial_y v|^2}{y^2} \right)^{\frac{1}{2}}.
\]

(A.7)

Similarly, using \( \tilde{f}(y) = -\frac{\epsilon_y}{y^{(1-\log y)}} \), we get:

\[
\int_{\epsilon \leq y \leq 1} \frac{|v|^2}{y^4(1-\log y)^2} = \int_{\epsilon \leq y \leq 1} |v|^2 \nabla . f
\]

\[
= \left[ \frac{|v|^2}{1-\log y} \right]_{\epsilon}^{1} + 2 \int_{1 \leq y \leq R} \frac{v \partial_y v}{y^3(1-\log y)}
\]

\[
\lesssim |v(1)|^2 + \left( \int_{y \leq R} \frac{|v|^2}{y^3(1+|\log y|)^2} \right) \left( \int_{y \leq R} \frac{|\partial_y v|^2}{y^2} \right)^{\frac{1}{2}}.
\]

(A.8)

(A.6), (A.7) and (A.8) now yield (A.4). To prove (A.5), let \( \gamma > 0 \), and

\[
\tilde{f}(y) = -\frac{\epsilon_y}{y^{\gamma+3}(1+\log y)^2}
\]

so that for \( y \geq 1 \)

\[
\nabla . f(y) = \frac{1}{y^{\gamma+4}(1+\log y)^2} \left[ \gamma + \frac{2}{1+\log y} \right] \geq \frac{\gamma}{y^{\gamma+4}(1+\log y)^2}.
\]

We then integrate by parts to get:

\[
\gamma \int_{1 \leq y \leq R} \frac{|v|^2}{y^{4+\gamma}(1+|\log y|)^2} \leq \int_{1 \leq y \leq R} |v|^2 \nabla . f
\]

\[
\leq - \left[ \frac{|v|^2}{y^{4+\gamma}(1+|\log y|)^2} \right]_{1}^{R} + 2 \int_{1 \leq y \leq R} \frac{|v \partial_y v|}{y^{4+\gamma}(1+|\log y|)^2}
\]

\[
\leq C \int_{1 \leq y \leq 2} |v|^2 + 2 \left( \int_{y \leq R} \frac{|v|^2}{y^{2+\gamma}(1+|\log y|)^2} \right) \left( \int_{y \leq R} \frac{|\nabla v|^2}{y^{2+\gamma}(1+|\log y|)^2} \right)^{\frac{1}{2}}.
\]

and (A.5) follows.

A.2. Sub-positivity estimates with \( H \). The following lemma highlights the negative part of the operator \( H \). We recall that this operator possesses a unique nonpositive direction \( \psi \).

**Lemma A.3.** Let \( u \in H^2_{rad}(\mathbb{R}^4) \), then there exists a constant \( C > 0 \) such that:

\[
(Hu, u) \geq -C \left( u, \psi \right)^2.
\]

(A.9)

**Proof of the Lemma A.3.** Let \( u \in H^2_{rad}(\mathbb{R}^4) \). There exists an unique decomposition of \( u \):

\[
u = \kappa \psi + v
\]

with the orthogonality condition

\[
(\psi, v) = 0.
\]
By definition, we have
\[ \kappa = \frac{(u, \psi)}{(\psi, \psi)}. \]
Moreover, the uniqueness of the negative direction of H gives
\[ (Hv, v) \geq 0. \]
Thus,
\[ (Hu, u) = \kappa^2 (H\psi, \psi) + \kappa [(Hv, \psi) + (v, H\psi)] + (Hv, v) \]
\[ \geq \frac{(H\psi, \psi)}{(\psi, \psi)} (u, \psi)^2 \]
\[ \geq -\zeta (u, \psi) (u, \psi)^2. \]

A.3. Sub-coercivity estimates. In this subsection, we prove sub-coercivity estimates for \( H \) and \( H^2 \) which are the key to the proof of coercive estimates for these operators under additional orthogonality conditions.

Lemma A.4 (Sub-coercivity estimates with H). Let \( u \in H^2_{rad}(\mathbb{R}^4) \), then there exists constants \( \delta > 0, C > 0 \) such that:
\[ (H\varepsilon, \varepsilon) \geq c \int |\nabla \varepsilon|^2 \left( -\frac{1}{c} [(\varepsilon, \Phi_M)^2 + (\varepsilon, \psi)^2] \right), \quad (A.10) \]
\[ \int |\partial_y u|^2 + \int \frac{|\partial_y u|^2}{y^2} + \int \frac{u^2}{y^4(1 + |\log y|)^2} - C \left[ \int \frac{|\partial_y u|^2}{1 + y^4} \right] \lesssim \int (Hu)^2. \quad (A.11) \]

Proof of the Lemma A.4 (A.11) is a direct consequence of the inequalities (A.3), (A.4) and the following decomposition:
\[ \int (Hu)^2 = \int (\Delta u + Vu)^2 = \int (\Delta u)^2 - 2 \int V(\partial_y u)^2 + \int (\Delta V + V^2)u^2 \]
\[ \gtrsim \left[ \int (\Delta u)^2 + \int \frac{u^2}{1 + y^6} \right] - C \left[ \int \frac{|\partial_y u|^2}{1 + y^4} \right] \]
where we used that
\[ V(y) = \frac{192}{1 + y^4} \left[ 1 + O \left( \frac{1}{1 + y^2} \right) \right] \text{ as } y \to +\infty. \]

Lemma A.5 (Weighted sub-coercivity for H). Let \( u \in H^4_{rad}(\mathbb{R}^4) \), then exists a constant \( C \) such that:
\[ \int \frac{|u|^2}{y^4(1 + y^4)(1 + |\log y|)^2} + \int \frac{|\partial_y u|^2}{y^6(1 + |\log y|)^2} + \int \frac{|\partial_{yy} u|^2}{y^4(1 + |\log y|)^2} \]
\[ + \int \frac{|\partial^2_y u|^2}{y^2(1 + |\log y|)^2} + \int \frac{|\partial^4_y u|^2}{(1 + |\log y|)^2} \]
\[ - C \left[ \int \frac{|u|^2}{y^2(1 + y^6)(1 + |\log y|)^2} + \int \frac{|\partial_y u|^2}{y^4(1 + y^4)(1 + |\log y|)^2} \right] \]
\[ \lesssim \int \frac{|Hu|^2}{y^4(1 + |\log y|)^2} + \int \frac{|\partial_y Hu|^2}{y^2(1 + |\log y|)^2} + \int \frac{|\partial_{yy} Hu|^2}{(1 + |\log y|)^2}, \quad (A.12) \]
Remark A.6. Using (A.1), (A.4), and (A.5), \( u \in H^4_{\text{rad}}(\mathbb{R}^4) \) yields
\[
\int \frac{|u|^2}{y^4(1 + y^4)(1 + |\log y|)^2} + \int \frac{|\partial_y u|^2}{y^6(1 + |\log y|)^2} + \int \frac{|\partial_{yy} u|^2}{y^4(1 + |\log y|)^2}
\]
\[
+ \int \frac{|\partial_y^2 u|^2}{y^2(1 + |\log y|)^2} + \int \frac{(1 + |\log y|)^2}{(1 + |\log y|)^2}
\]
\[
+ \int \frac{|Hu|^2}{y^4(1 + |\log y|)^2} + \int \frac{|\partial_y H u|^2}{y^2(1 + |\log y|)^2} + \int \frac{|\partial_{yy} H u|^2}{(1 + |\log y|)^2} < \infty.
\]

Proof of Lemma A.3 Let \( \chi(y) \) be a smooth cut-off function with support in \( y \geq 1 \) and equal to 1 for \( y \geq 2 \).
\[
\int \chi \frac{|Hu|^2}{y^{4(1 + |\log y|)^2}} = \int \chi \frac{-\partial_y (y^3 \partial_y u) - y^3 V u}{y^{10}(1 + |\log y|)^2}
\]
\[
= \int \chi \frac{|\partial_y (y^3 \partial_y u)|}{y^{10}(1 + |\log y|)^2} + 2 \int \chi \frac{\partial_y (y^3 \partial_y u) V u}{y^7(1 + |\log y|)^2} + \int \chi \frac{V^2 u^2}{y^4(1 + |\log y|)^2}
\]
\[
= \int \chi \frac{|\partial_y (y^3 \partial_y u)|}{y^{10}(1 + |\log y|)^2} - 2 \int \chi \frac{V (\partial_y u)^2}{y^4(1 + |\log y|)^2} + \int \chi \frac{V^2 u^2}{y^4(1 + |\log y|)^2}
\]
\[
+ \int |u|^2 \Delta \left( \chi \frac{V}{y^4(1 + |\log y|)^2} \right).
\]

We now observe that for \( k \geq 0 \)
\[
|\partial_y V(y)| \lesssim \frac{1}{1 + y^{4+k}}
\]
and thus,
\[
|\partial_y V(y)| \lesssim \int \chi \frac{|u|^2}{y^{2(1 + y^3)(1 + |\log y|)^2}} + \int \chi \frac{|\partial_y u|^2}{y^4(1 + |\log y|)^2}
\]
\[
\lesssim \int \chi \frac{|u|^2}{y^{4(1 + |\log y|)^2}} + \int \chi \frac{|\partial_y u|^2}{y^4(1 + |\log y|)^2}.
\]
Hence,
\[
\int \chi \frac{|Hu|^2}{y^{4(1 + |\log y|)^2}} \gtrsim \int \chi \frac{y^{10}(1 + |\log y|)^2}{y^4(1 + |\log y|)^2}
\]
\[
- C \left( \int \frac{|u|^2}{y^2(1 + y^3)(1 + |\log y|)^2} + \int \frac{|\partial_y u|^2}{y^4(1 + y^4)(1 + |\log y|)^2} \right).
\]

We may apply twice the Hardy inequality [A.5] with \( \gamma = 8 \) and \( \gamma = 4 \) and get for a sufficiently large universal constant \( R \):
\[
\int \chi \frac{\partial_y (y^3 \partial_y u)^2}{y^{10}(1 + |\log y|)^2} \gtrsim \int_{y \geq R} \frac{|\partial_y u|^2}{y^6(1 + |\log y|)^2} - C \int \frac{|\partial_y u|^2}{y^4(1 + y^4)(1 + |\log y|)^2}
\]
\[
\gtrsim \int_{y \geq R} \frac{|u|^2}{y^2(1 + y^3)(1 + |\log y|)^2} - C \left( \int \frac{|u|^2}{y^2(1 + y^3)(1 + |\log y|)^2} + \int \frac{|\partial_y u|^2}{y^4(1 + y^4)(1 + |\log y|)^2} \right).
Hence
\[
\int_{y \geq 2} \frac{|u|^2}{y^4(1 + y^6)(1 + |\log y|)^2} + \int_{y \geq 2} \frac{|\partial_y u|^2}{y^4(1 + |\log y|)^2} + \int_{y \geq 2} \frac{|\partial_{yy} u|^2}{y^4(1 + |\log y|)^2} - C \left[ \int \frac{|u|^2}{y^4(1 + |\log y|)^2} + \int \frac{|\partial_y u|^2}{y^4(1 + |\log y|)^2} \right]
\]
\[
\lesssim \int \frac{\chi}{y^4(1 + |\log y|)^2} |Hu|^2.
\]
(A.13)

Now, the control of the third derivative for \( y \leq 1 \) follows from:
\[
\int \chi \frac{|\partial_y Hu|^2}{y^2(1 + |\log y|)^2} \geq \int \chi \frac{|\partial_y (\Delta u + Vu)|^2}{y^2(1 + |\log y|)^2} \gtrsim \int \chi \frac{|\partial_y^3 u|^2}{y^2(1 + |\log y|)^2} - C \left[ \int \frac{|u|^2}{y^2(1 + |\log y|)^2} + \int \frac{|\partial_y u|^2}{y^4(1 + |\log y|)^2} \right]
\]
and the fourth derivate from:
\[
\int \chi \frac{|\partial_{yy} Hu|^2}{y^2(1 + |\log y|)^2} \geq \int \chi \frac{|\partial_{yy} (\Delta u + Vu)|^2}{(1 + |\log y|)^2} \lesssim \int \chi \frac{|\partial_{yy}^3 u|^2}{y^4(1 + |\log y|)^2} - C \left[ \int \frac{|\partial_y u|^2}{y^4(1 + |\log y|)^2} + \int \frac{|\partial_{yy} u|^2}{y^4(1 + |\log y|)^2} \right].
\]
(A.14) (A.15)

(A.13) (A.14) and (A.16) yield (A.12), away from the origin. Let us study this control near the origin. Let \( \zeta = (1 - \chi)^\frac{1}{2} \). With the Lemma [A.4], we have that:
\[
\int \zeta^2 \frac{|Hu|^2}{y^4(1 + |\log y|^2)^2} \gtrsim \int \zeta^2 |Hu|^2 \gtrsim |Hu|^2 - C \int_{1 \leq y \leq 2} (|\partial_y u|^2 + |u|^2)
\]
\[
\gtrsim \int \frac{|\zeta u|^2}{y^4(1 + |\log y|^2)^2} - C \int \frac{|\zeta u|^2}{y^2} \gtrsim \int_{y \leq 1} \frac{|u|^2}{y^4(1 + |\log y|^2)^2} - C \int_{y \geq 2} \frac{|u|^2}{y^2}.
\]
Now, by definition, we have:
\[
Hu = -\frac{1}{y^3} \frac{\partial}{\partial_y} (y^3 \partial_y u) - Vu.
\]
Hence
\[
\partial_y u = -\frac{1}{y^3} \int_{\tau \leq y} (Vu + Hu).
\]
(A.17)

We then estimate from Cauchy-Schwarz and Fubini:
\[
\int_{y \leq 1} \frac{|\partial_y u|^2}{y^6(1 + |\log y|^2)^2} = \int_{y \leq 1} \frac{|\partial_y u|^2}{y^2(1 + |\log y|^2)^2} \left( \int_0^y |Vu + Hu|^2 \right) \leq \int_{y \leq 1} \frac{|\partial_y u|^2}{y^2(1 + |\log y|^2)} \left( \int_0^y |Vu(\tau)|^2 + |Hu(\tau)|^2 \right)
\]
\[
\lesssim \int_{y \leq 1} \frac{|Vu(\tau)|^2 + |Hu(\tau)|^2}{\tau^3} \left( \int_{\tau \leq y} \frac{1}{y^2(1 + |\log y|^2)^2} \right)
\]
\[
\lesssim \int_{y \leq 1} \frac{|Hu|^2}{y^4(1 + |\log y|)^2}.
\]
(A.18)
Finally, for the control of the other derivates near the origin:
\[
\partial_{yy} u = - Hu - 3 \frac{\partial_y u}{y} - Vu.
\]

Thus,
\[
\int_{y \leq 1} \frac{|\partial_{yy} u|^2}{y^4(1 + |\log y|^2)} \leq \int_{y \leq 1} \frac{|Hu|^2}{y^4(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|\partial_y u|^2}{y^6(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|u|^2}{y^4(1 + |\log y|^2)},
\]
(A.19)
\[
\int_{y \leq 1} \frac{|\partial_y^3 u|^2}{y^2(1 + |\log y|^2)} \leq \int_{y \leq 1} \frac{|\partial_y Hu|^2}{y^2(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|\partial_y u|^2}{y^4(1 + |\log y|^2)}
+ \int_{y \leq 1} \frac{|\partial_y u|^2}{y^6(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|u|^2}{y^4(1 + |\log y|^2)},
\]
(A.20)
\[
\int_{y \leq 1} \frac{|\partial_y^4 u|^2}{(1 + |\log y|^2)} \leq \int_{y \leq 1} \frac{|\partial_y Hu|^2}{(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|\partial_y u|^2}{y^4(1 + |\log y|^2)} + \int_{y \leq 1} \frac{|u|^2}{y^4(1 + |\log y|^2)}.
\]
(A.21)

This concludes the proof.

We now combine the results of Lemma A.4 and the Lemma A.5.

**Lemma A.7** (Sub-coercivity for $H^2$). Let $u \in H^4_\text{rad}(\mathbb{R}^4)$. Then,
\[
\int |H^2 u|^2 \geq \int \frac{|Hu|^2}{y^4(1 + |\log y|^2)} + \int \frac{|\partial_y Hu|^2}{y^2(1 + |\log y|^2)} + \int \frac{|\partial_y u|^2}{y^6(1 + |\log y|^2)} + \int \frac{|u|^2}{y^4(1 + |\log y|^2)},
\]
(A.22)
\[
\int \frac{|\partial_y^3 u|^2}{y^2(1 + |\log y|^2)} \geq \int \frac{|\partial_y^3 u|^2}{y^2(1 + |\log y|^2)} + \int \frac{|\partial_y Hu|^2}{y^2(1 + |\log y|^2)} + \int \frac{|\partial_y u|^2}{y^4(1 + |\log y|^2)}
+ \int \frac{|\partial_y^3 u|^2}{y^2(1 + |\log y|^2)} \int \frac{|\partial_y^3 u|^2}{y^2(1 + |\log y|^2)} - C \int \frac{|\partial_y u|^2}{y^2(1 + y^4)(1 + |\log y|^2)} - C \int \frac{|\partial_y u|^2}{y^2(1 + y^4)(1 + |\log y|^2)}
+ \int \frac{|\partial_y^4 u|^2}{y^2(1 + |\log y|^2)} \int \frac{|\partial_y^4 u|^2}{y^2(1 + |\log y|^2)} \leq C(M) \int |H^2 u|^2.
\]
(A.23)

**A.4. Coercivity of $H^2$.** We are now in position to derive the fundamental coercivity property of $H^2$ at the heart of our analysis.

**Lemma A.8** (Coercivity of $H^2$). Let $M \geq 1$ be a large enough universal constant. Let $\Phi_M$ be given by (3.2). Then there exists a universal constant $C(M) > 0$ such that for all $u \in H^4_\text{rad}(\mathbb{R}^4)$ satisfying the orthogonality conditions:
\[(u, \Phi_M) = 0, \quad (Hu, \Phi_M) = 0\]
there holds:
\[
\int \frac{|Hu|^2}{y^4(1 + |\log y|^2)} + \int \frac{|\partial_y Hu|^2}{y^2(1 + |\log y|^2)} + \int \frac{|\partial_y u|^2}{(1 + |\log y|^2)}
+ \int \frac{|\partial_y u|^2}{y^4(1 + |\log y|^2)} + \int \frac{|\partial_y u|^2}{y^6(1 + |\log y|^2)} + \int \frac{|\partial_y^3 u|^2}{y^8(1 + |\log y|^2)}
\]
\[
\int \frac{|\partial_y^4 u|^2}{y^2(1 + |\log y|^2)} + \int \frac{|\partial_y^4 u|^2}{y^2(1 + |\log y|^2)} \leq C(M) \int |H^2 u|^2.
\]
(A.23)
Proof of the Lemma \[A.3\] We argue by contradiction. Let \( M > 0 \) fixed and consider a normalized sequence \( u_n \)

\[
\int \frac{|H u_n|^2}{y^4(1 + |\log y|)^2} + \int \frac{|\partial_y H u_n|^2}{y^2(1 + |\log y|)^2} + \int \frac{|\log y H u_n|^2}{(1 + |\log y|)^2} \\
+ \int \frac{|u_n|^2}{y^4(1 + y^4)(1 + |\log y|)^2} + \int \frac{|\partial_y u_n|^2}{y^6(1 + |\log y|)^2} + \int \frac{|\partial_{yy} u_n|^2}{y^4(1 + |\log y|)^2} \\
+ \int \frac{|\partial^3_{yy} u_n|^2}{y^2(1 + |\log y|)^2} + \int \frac{|\partial_{yy} u_n|^2}{(1 + |\log y|)^2} = 1 \quad (A.24)
\]

satisfying the orthogonality conditions

\[
(u_n, \Phi_M) = 0, \quad (H u_n, \Phi_M) = 0
\]

and

\[
\int |H^2(u_n)|^2 \leq \frac{1}{n}. \quad (A.25)
\]

The normalization condition implies that the sequence \( u_n \) is uniformly bounded in \( H^1_{loc} \). As a consequence, we can assume that \( u_n \) weakly converges in \( H^1_{loc} \) to \( u_\infty \). Moreover, \( u_\infty \) satisfies the equation

\[
H^2 u_\infty = 0 \quad \text{for} \quad r \in (0).
\]

Integrating this ODE leads to

\[
Hu_\infty = \alpha \Lambda Q + \beta \Gamma \quad \text{for} \quad r > 0.
\]

Using the condition \( u_\infty \in H^1_{loc} \), we can determine that \( \beta = 0 \). Hence, the function \( u_\infty \) can be written in the form

\[
u_\infty = -\alpha T_1 + \gamma \Lambda Q + \delta \Gamma.
\]

The condition \( u_\infty \in H^1_{loc} \) yields that \( \delta = 0 \). Passing through the limit in the orthogonality conditions, using that \( u_n \) converges to \( u_\infty \) weakly in \( H^1_{loc} \), we conclude that \( u_\infty \) satisfies

\[
(u_\infty, \Phi_M) = 0, \quad (H u_\infty, \Phi_M) = 0.
\]

We may therefore determine the constant \( \alpha \) and \( \gamma \) using (3.2), (3.5) which yield \( \alpha = \gamma = 0 \) and thus \( u_\infty = 0 \).

The sub-coercitivity bound (A.22) together with (A.25) ensures:

\[
\frac{1}{n} \geq \int |H^2(u_n)|^2 \geq \int \frac{|H u_n|^2}{y^4(1 + |\log y|)^2} + \int \frac{|\partial_y H u_n|^2}{y^2(1 + |\log y|)^2} + \int \frac{|\log y H u_n|^2}{(1 + |\log y|)^2} \\
+ \int \frac{|u_n|^2}{y^4(1 + y^4)(1 + |\log y|)^2} + \int \frac{|\partial_y u_n|^2}{y^6(1 + |\log y|)^2} + \int \frac{|\partial_{yy} u_n|^2}{y^4(1 + |\log y|)^2} \\
+ \int \frac{|\partial^3_{yy} u_n|^2}{y^2(1 + |\log y|)^2} + \int \frac{|\partial_{yy} u_n|^2}{(1 + |\log y|)^2} - C \left[ \int \frac{|\partial_y H u_n|^2}{y^4(1 + y^4) + \int \frac{|H u_n|^2}{1 + y^4}} + \int \frac{|u_n|^2}{1 + y^4} + \int \frac{|\partial_y u_n|^2}{y^4(1 + y^4)(1 + |\log y|)^2} \right].
\]

Coupling this with the normalization condition we obtain that

\[
\int \frac{|\partial_y H u_n|^2}{1 + y^4} + \int \frac{H u_n^2}{1 + y^4} + \int \frac{|u_n|^2}{y^2(1 + y^8)(1 + |\log y|)^2} + \int \frac{|\partial_y u_n|^2}{y^4(1 + y^4)(1 + |\log y|)^2} \geq c
\]
Lemma A.9. Therefore let to the reader.

in [5] with a slight differently different orthogonality condition but the proof is the same and H, which follows from standard compactness argument. A complete proof is given established in the previous section, by the corresponding statement for the operator $u$ that the following holds true. Let $u_\infty$ be any compact subinterval of $y \in (0, \infty)$, we can pass to the limit to conclude

$$\int \frac{|\partial_y H u_\infty|^2}{1 + y^4} + \int \frac{H u_\infty}{1 + y^4} + \int \frac{|u_\infty|^2}{y^2(1 + y^8)(1 + |\log y|)^2} + \int \frac{|\partial_y u_\infty|^2}{y^4(1 + y^4)(1 + |\log y|)^2} \geq c.$$  

This contradicts the established identity $u_\infty = 0$ and concludes the proof of Lemma A.7.

A.5. Coercivity of $H$. We complement the coercivity property of the operator $H^2$, established in the previous section, by the corresponding statement for the operator $H$, which follows from standard compactness argument. A complete proof is given in [5] with a slightly different orthogonality condition but the proof is the same and therefore let to the reader.

Lemma A.9 (Coercivity of $H$). Let $M \geq 1$ fixed. Then there exists $c(M) > 0$ such that the following holds true. Let $u \in H^2_{\text{rad}}$ with

$$(u, \Phi_M) = 0$$

then

$$\int |\partial_y u|^2 + \int \frac{|\partial_y u|^2}{y^2} + \int \frac{u^2}{y^2(1 + |\log y|)^2} \leq c(M) \int |Hu|^2. \quad (A.26)$$

Appendix B. Interpolation estimates

In this section, we prove interpolation estimates for $\epsilon$ in the bootstrap regimes which are used all along the proof of Proposition 4.2. We recall the norm $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_4$, introduced in [5,13,16], together with their bootstrap bounds:

$$\mathcal{E}_1 = \int |\nabla \epsilon|^2 \leq K \delta(b^4),$$

$$\mathcal{E}_2 = \int |H \epsilon|^2 \leq K b^2(t)|\log b(t)|^5,$$

$$\mathcal{E}_4 = \int |H^2 \epsilon|^2 \leq K \frac{b^4(t)}{|\log b(t)|^2}.$$  

Lemma B.1 (Interpolation estimates). There holds -with constants a priori depending on $M$-

$$\int \frac{|H(\epsilon)|^2}{y^4(1 + |\log y|^2)} + \int \frac{|\partial_y H(\epsilon)|^2}{y^2(1 + |\log y|^2)} + \int \frac{|\partial_y y H(\epsilon)|^2}{1 + |\log y|^2} \lesssim \mathcal{E}_4, \quad 1 \leq i \leq 4,$$

$$\int \frac{|\epsilon|^2}{y^4(1 + |\log y|^2)} + \int \frac{|\partial_y \epsilon|^2}{y^8(1 + |\log y|^2)} \lesssim \mathcal{E}_2, \quad 1 \leq i \leq 2,$$

$$\int_{y \geq 1} \frac{1 + |\log y|^C}{y^{8-2i}(1 + |\log y|^2)} |\partial_y \epsilon|^2 \lesssim b^i |\log b|^C_i, \quad 0 \leq i \leq 2,$$

$$\int_{y \geq 1} \frac{1 + |\log y|^C}{y^{6-2i}(1 + |\log y|^2)} |\partial_y \epsilon|^2 \lesssim b^i |\log b|^C_i, \quad 0 \leq i \leq 2,$$

$$\int_{y \geq 1} \frac{1 + |\log y|^C}{y^{4-2i}(1 + |\log y|^2)} |\partial_y \epsilon|^2 \lesssim b^i |\log b|^C_i, \quad 0 \leq i \leq 1,$$  

for some positive constant $c > 0$. Since $u_\infty$ weakly converges to $u_\infty$ in $H^4_{\text{loc}}$ on any compact subinterval of $y \in (0, \infty)$, we can pass to the limit to conclude

$$\int \frac{|\partial_y H u_\infty|^2}{1 + y^4} + \int \frac{H u_\infty}{1 + y^4} + \int \frac{|u_\infty|^2}{y^2(1 + y^8)(1 + |\log y|)^2} + \int \frac{|\partial_y u_\infty|^2}{y^4(1 + y^4)(1 + |\log y|)^2} \geq c.$$
\begin{align}
    \|\varepsilon(1 + |\log y|^C)\|_{L^\infty_{y \geq 1}}^2 & \lesssim b^2|\log b|^{C_1(C)}, \quad \text{(B.6)} \\
    \|\varepsilon\|_{L^\infty_{y \leq 1}}^2 + \|\partial_y \varepsilon\|_{L^\infty_{y \leq 1}}^2 + \|y\partial_y \varepsilon\|_{L^\infty_{y \leq 1}}^2 & \lesssim \frac{b^4}{|\log b|^2}, \quad \text{(B.7)} \\
    \|y\partial_y \varepsilon\|_{L^\infty_{y \leq 1}}^2 & \lesssim b^2|\log b|^5, \quad \text{(B.8)} \\
    \left\| \frac{\varepsilon}{1 + y} \right\|_{L^\infty}^2 + \left\| \frac{\partial_y \varepsilon}{1 + y} \right\|_{L^\infty}^2 & \lesssim b^3|\log b|^C, \quad \text{(B.9)} \\
    \left\| \frac{\varepsilon}{1 + y^2} \right\|_{L^\infty_{y \geq 20}}^2 + \left\| \frac{\partial_y \varepsilon}{1 + y} \right\|_{L^\infty_{y \geq 20}}^2 + \|\partial_{yy} \varepsilon\|_{L^\infty_{y \geq 1}}^2 & \lesssim b^4|\log b|^C. \quad \text{(B.10)}
\end{align}

**Proof of the Lemma (B.1)**: Lemma A.8 and Lemma A.9 and definition of the norms \( \mathcal{E}_2 \) and \( \mathcal{E}_4 \).

To prove (B.3), we split the integral at \( y = B_0^4 \).

\[
    \int_{y \geq 1} \frac{1 + |\log y|^C}{y^{8-2t}(1 + |\log y|^2)} |\partial_y \varepsilon|^2 \\
    = \int_{1 \leq y \leq B_0^4} \frac{1 + |\log y|^C}{y^{8-2t}(1 + |\log y|^2)} |\partial_y \varepsilon|^2 + \int_{y \geq B_0^4} \frac{1 + |\log y|^C}{y^{8-2t}(1 + |\log y|^2)} |\partial_y \varepsilon|^2 \\
    \lesssim |\log b|^{C+2} \mathcal{E}_4 \\
    + \left\| \frac{1 + |\log y|^C}{y^2} \right\|_{L^\infty_{y \geq B_0^4}} \left( \int_{y \geq 1} \frac{|\partial_y \varepsilon|^2}{y^{8-2t}(1 + |\log y|^2)} \right)^{1/2} \left( \int_{y \geq 1} \frac{|\partial_y \varepsilon|^2}{y^{4-2t}(1 + |\log y|^2)} \right)^{1/2}.
\]

The bounds (B.3) and (B.2) concludes the proof. The bound (B.4) is a direct consequence of the last bound and the bootstrap bound for \( \mathcal{E}_2 \). Indeed:

\[
    \int_{y \geq 1} \frac{1 + |\log y|^C}{y^{6-2t}(1 + |\log y|^2)} |\partial_y \varepsilon|^2 \\
    \lesssim \left( \int_{y \geq 1} \frac{1 + |\log y|^2C}{y^{8-2t}(1 + |\log y|^2)} |\partial_y \varepsilon|^2 \right)^{1/2} \left( \int_{y \geq 1} \frac{|\partial_y \varepsilon|^2}{y^{4-2t}(1 + |\log y|^2)} \right)^{1/2}
\]

and (B.4) follows.

The proof of (B.5) is the same of (B.3) using the energy \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \). (B.6) comes from (A.2) and (B.3). Indeed:

\[
    \left\| \varepsilon(1 + |\log y|^C) \right\|_{L^\infty_{y \geq 1}}^2 \\
    \lesssim \int_{y \geq 1} \frac{|\partial_y \varepsilon|^2(1 + |\log y|^C)}{y^2} + \int_{y \geq 1} \frac{\varepsilon^2(1 + |\log y|^C)}{y^4} + \int_{\frac{1}{2} \leq y \leq 1} \frac{\varepsilon^2(1 + |\log y|^C)}{y^2} \\
    \lesssim b^2|\log b|^C.
\]

Let us prove (B.7). Let \( a \in [1; 2] \) such that:

\[
    |\varepsilon(a)| \lesssim \int_1^2 \frac{|\varepsilon(y)|}{y^3} \lesssim \sqrt{\mathcal{E}_4}. \quad \text{(B.11)}
\]

Then using Cauchy-Schwarz

\[
    \forall y \in [0; 1], |\varepsilon(y)| \lesssim |\varepsilon(a)| + \left| \int_a^y \frac{|\partial_y \varepsilon(y)|}{y^3} \right| \lesssim \sqrt{\mathcal{E}_4}.
\]
In the same way, let \( a \in [1; 2] \) such that:
\[
|\partial_y \varepsilon(a)| \lesssim \int_1^2 \frac{|\partial_y \varepsilon(y)|}{y^2} \lesssim \sqrt{E_4} \tag{B.12}
\]
and
\[
\forall y \in [0; 1], |\partial_y \varepsilon(y)| \lesssim |\partial_y \varepsilon(a)| + \left| \int_a^y \frac{|\partial_y \varepsilon(y)|}{y^3} \right| \lesssim \sqrt{E_4}.
\]
Finally, let \( a \in [1; 2] \) such that:
\[
|\partial_{yy} \varepsilon(a)| \lesssim \int_1^2 \frac{|\partial_{yy} \varepsilon(y)|}{y^2} \lesssim \sqrt{E_4} \tag{B.13}
\]
and
\[
\forall y \in [0; 1], |y \partial_{yy} \varepsilon(y)| \lesssim a |\partial_{yy} \varepsilon(a)| + \left| \int_a^y \left( \frac{|\partial_{yy} \varepsilon(y)|}{y^3} + \frac{|\partial_y^3 \varepsilon(y)|}{y^2} \right) \right| \lesssim \sqrt{E_4}.
\]
The bound \( (B.8) \) is a direct consequence of the Lemma \( A.1 \) and \( B.2 \).

We now prove \( (B.9) \) using \( (A.2) \), \( (B.2) \) and \( (B.3) \):
\[
\left\| \frac{\varepsilon}{y} \right\|_{L_y^\infty}^2 + \left\| \frac{\partial_y \varepsilon}{y} \right\|_{L_y^\infty}^2 \lesssim \int_{y \geq 1} \left( \frac{\varepsilon^2}{y^6} + \frac{|\partial_y \varepsilon|^2}{y^4} + \frac{|\partial_{yy} \varepsilon|^2}{y^2} \right) + \int_{\frac{1}{2} \leq y \leq 1} \left( \frac{|\varepsilon|^2}{y^4} + \frac{|\partial_y \varepsilon|^2}{y^2} \right)
\]
\[
\lesssim \sum_{i=0}^2 \left( \int_{y \geq 1} \frac{|\partial_y^i \varepsilon|^2}{y^{4-2i}(1 + |\log y|)^2} \right)^{\frac{1}{2}} \left( \int_{y \geq 1} \frac{|\partial_y^i \varepsilon|^2(1 + |\log y|^2)}{y^{8-2i}} \right)^{\frac{1}{2}} + E_4 \lesssim b^3 |\log b|^C.
\]
Similarly, we prove \( (B.10) \):
\[
\left\| \frac{\varepsilon}{y^2} \right\|_{L_y^\infty}^2 + \left\| \frac{\partial_y \varepsilon}{y} \right\|_{L_y^\infty}^2 + \left\| \partial_{yy} \varepsilon \right\|_{L_y^\infty}^2 \lesssim \int_{y \geq 1} \left( \frac{\varepsilon^2}{y^5} + \frac{|\partial_y \varepsilon|^2}{y^4} + \frac{|\partial_{yy} \varepsilon|^2}{y^2} \right) + \int_{\frac{1}{2} \leq y \leq 1} \left( \frac{|\varepsilon|^2}{y^6} + \frac{|\partial_y \varepsilon|^2}{y^4} + \frac{|\partial_{yy} \varepsilon|^2}{y^2} \right)
\]
\[
\lesssim b^4 |\log b|^C.
\]

Appendix C. Localization of the profile

In this appendix, we are going to give the important steps of the proof of the Proposition \( 2.4 \) and \( 2.4 \).

To begin, remark that the definition \( (2.59) \) of \( \tilde{\Psi}_b \) in the localization near \( B_1 \) gives two types of error. One is the result of the only localization. One is the effect of the time derivate. Indeed, we can rewrite \( \tilde{\Psi}_b \) as following:
\[
\tilde{\Psi}_b = \Psi_b^{(1)} + \tilde{R} \tag{C.1}
\]
where
\[
\Psi_b^{(1)} = -b^2 (\tilde{T}_1 + b \tilde{T}_2) - \Delta \tilde{Q}_b + b \Delta \tilde{Q}_b - (\tilde{Q}_b)^3 \tag{C.2}
\]
and
\[
\tilde{R} = b_s \left( 3b^2 \tilde{T}_3 + b \frac{\partial \tilde{T}_1}{\partial b} + b^2 \frac{\partial \tilde{T}_2}{\partial b} + b^3 \frac{\partial \tilde{T}_3}{\partial b} \right) \tag{C.3}
\]
We compute the action of localization which produces an error localized in \([B_1, 2B_1]\) up to the term \((1 - \chi_{B_1})\Lambda Q\):

\[
\Psi_b^{(1)} = \chi_{B_1} \Psi_b + b(1 - \chi_{B_1})\Lambda Q + b\Lambda \chi_{B_1} \alpha - \alpha \Delta \chi_{B_1} - 2\partial_y \chi_{B_1} \partial_y \alpha + \Lambda (Q + \chi_{B_1}\alpha)^3 - Q^3 - \chi_{B_1}((Q + \alpha)^3 - Q^3). \tag{C.4}
\]

We estimate from the rough bounds of \((T_i)_{1 \leq i \leq 3}\) and the choice of \(B_1\):

\[
\forall y \leq 2B_1, \quad |\alpha(y)| \lesssim \frac{b|\log y|}{y} + \frac{b|\log b|}{|\log y|} \lesssim b|\log y|
\]

and thus:

\[
|b(1 - \chi_{B_1})\Lambda Q + b\Lambda \chi_{B_1} \alpha - \alpha \Delta \chi_{B_1} - 2\partial_y \chi_{B_1} \partial_y \alpha| \lesssim \frac{b^2 \log y}{y^2} 1_{y \geq B_1} + 2b^2 \log y 1_{B_1 \leq y \leq 2B_1},
\]

\[
|((Q + \chi_{B_1}\alpha)^3 - Q^3 - \chi_{B_1}((Q + \alpha)^3 - Q^3)| \lesssim \frac{|\alpha(y)|}{y^2} 1_{B_1 \leq y \leq 2B_1},
\]

\[
\lesssim \frac{b \log y}{y^2} 1_{B_1 \leq y \leq 2B_1}
\]

Hence, \(\Psi_b^{(1)}\) verifies the bounds \((2.61), (2.62), (2.63)\). For the control of time derivates, we have to use that:

\[
\frac{\partial c_b}{\partial b} = O \left( \frac{1}{b|\log b|^2} \right), \quad \frac{\partial d_b}{\partial b} = O \left( \frac{1}{b^2|\log b|^2} \right), \tag{C.5}
\]

which are consequences of their definition, and that:

\[
\frac{\partial \Sigma_b}{\partial b} = O \left( \frac{1}{b|\log b|} 1_{y \leq \frac{b_0}{2}} + \frac{1}{y^2 b^2|\log b|^2} 1_{\frac{b_0}{2} \leq y \leq \frac{b_0}{2}} \right). \tag{C.6}
\]

Using \((C.5), (C.6)\) together the explicit formula of \((T_i)_{1 \leq i \leq 3}\) yield \((2.61), (2.62), (2.63)\) without difficulty. Only an estimate is more delicate, and requests more cancellation. Indeed, we must use that:

\[
H^2 \left( \frac{\partial T_2}{\partial b} \right) = H \left( \frac{\partial \Sigma_2}{\partial b} \right) = H \left( \frac{\partial \Sigma_b}{\partial b} \right),
\]

and similarly

\[
H^2 \left( \frac{\partial T_3}{\partial b} \right) = H \left( \frac{\partial \Sigma_3}{\partial b} \right) = \Lambda \left( \frac{\partial \Sigma_b}{\partial b} \right).
\]

The proof of \((2.64)\) is left to the reader. For the proof of Proposition 2.4 we just remark that, by definition:

\[
\zeta_b = (\chi_{B_1} - \chi_{B_0})(bT_1 + b^2T_2 + b^3T_3)
\]

This proof is afterwards the same as the previous one.

References

[1] Van den Berg, G.J.B.; Hubshof, J.; King, J., Formal asymptotics of bubbling in the harmonic map heat flow, SIAM J. Appl. Math. vol 63, 05, pp 1682-1717.
[2] Brezis, H., Analyse fonctionnelle, Masson (1983).
[3] Cote, R.; Martel, Y.; Merle, F., Construction of multisolitons solutions for the \(L^2\)-supercritical gKdV and NLS equations, arXiv:0910.2504 (2009)
[4] Herrero, M.A.; Velázquez, J.J.L.: Explosion de solutions des équations paraboliques semi-linéaires supercritiques. C. R. Acad. Sci. Paris 319, 141–145 (1994).
[5] Hillaret, M.; Raphaël, P., Smooth type II blow up solutions to the four dimensional energy Phys. (2010), 300, no 1, 205-242.
[6] Krieger, J.; Martel, Y.; Raphaël, P., Two-soliton solutions to the three-dimensional gravitational Hartree equation, Comm. Pure Appl. Math. 62 (2009), no. 11, 1501–1550.
[7] Krieger, J.; Schlag, W.; Tataru, D. Renormalization and blow up for charge one equivariant critical wave maps, Invent. Math. 171 (2008), no. 3, 543–615.
[8] Krieger, J.; Schlag, W., On the focusing critical semi-linear wave equation, Amer. J. Math. 129 (2007), no. 3, 843–913.
[9] Matano, H.; Merle, F., Classification of type I and type II behaviors for a supercritical non-linear heat equation. J. Funct. Anal. 256 (2009), no. 4, 992–1064.
[10] Matano, H.; Merle, F., On nonexistence of type II blowup for a supercritical nonlinear heat equation. Comm. Pure Appl. Math. 57 (2004), no. 11, 1501–1550.
[11] Mizoguchi, N., Rate of type II blowup for a semilinear heat equation, Math. Ann. 339 (2007), no. 4, 839–877.
[12] Merle, F.; Raphaël, P., Blow up dynamic and upper bound on the blow up rate for critical nonlinear Schrödinger equation, Ann. Math. 161 (2005), no. 1, 157-222.
[13] Merle, F.; Raphaël, P., Sharp upper bound on the blow up rate for critical nonlinear Schrödinger equation, Geom. Funct. Anal. 13 (2003), 591-642.
[14] Merle, F.; Raphaël, P., On universality of blow up profile for $L^2$ critical nonlinear Schrödinger equation. Invent. Math. 156, 565-672 (2004).
[15] Merle, F.; Raphaël, P., Sharp lower bound on the blow up rate for critical nonlinear Schrödinger equation, J. Amer. Math. Soc. 19 (2006), no. 1, 37-90.
[16] Merle, F.; Raphaël, P., Profiles and quantization of the blow up mass for critical nonlinear Schrödinger equation, Comm. Math. Phys. 253 (2005), no. 3, 675-704.
[17] Merle, F.; Raphaël, P.; Rodnianski, I., Blow up dynamics for smooth solutions to the energy critical Schrödinger map, preprint 2011.
[18] Perelman, G., On the blow up phenomenon for the critical nonlinear Schrödinger equation in 1D, Ann. Henri. Poincaré, 2 (2001), 605-673.
[19] Raphaël, P.; Rodnianski, I., Stable blow up dynamics for the critical corotational wave maps and equivariant Yang Mills problems, to appear in Prep. Math. IHES.
[20] Raphaël, P.; Schweyer, R., Stable blow up dynamics for the 1-corotational heat flow, preprint 2011.
[21] Rodnianski, I., Sterbenz, J., On the formation of singularities in the critical $O(3)$ $\sigma$-model, Ann. of Math. (2) 172 (2010), no. 1, 187-242.

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