On Quasisymmetric Functions with Two Bordering Variables

Alexander Zhang
Mentor: Andrey Khesin

October 17-18, 2020
MIT PRIMES Conference
The extended natural numbers: the set $\mathbb{N} \cup \{0, \infty\}$, denoted by $\mathcal{N}$. Its total order is given by $0 \prec 1 \prec 2 \prec \cdots \prec \infty$. 
Introductory Definitions

- **The extended natural numbers**: the set $\mathbb{N} \cup \{0, \infty\}$, denoted by $\mathcal{N}$. Its total order is given by $0 \prec 1 \prec 2 \prec \cdots \prec \infty$.

- $\mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]]$: Ring of formal power series in the variables $x_0, x_1, x_2, \ldots, x_\infty$ over $\mathbb{Z}$. Essentially the ring of polynomials except elements may contain infinitely many monomials.

- Denote by $m(p)$ the coefficient of the monomial $m$ in the power series $p$.

- Example: $\left[ x_3 y \right] (x + 7y + 13x_3y) = 13$. 
**Introductory Definitions**

- The **extended natural numbers**: the set $\mathbb{N} \cup \{0, \infty\}$, denoted by $\mathcal{N}$. Its total order is given by $0 \prec 1 \prec 2 \prec \cdots \prec \infty$.

- $\mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]]$: Ring of formal power series in the variables $x_0, x_1, x_2, \ldots, x_\infty$ over $\mathbb{Z}$. Essentially the ring of polynomials except elements may contain infinitely many monomials.
  
  - Example: $x_0 + x_1 + x_2 + \cdots + x_\infty = \sum_{i \in \mathcal{N}} x_i$. 

$\mathbf{x}_0$ and $\mathbf{x}_\infty$ are bordering variables; $\mathbf{x}_i$ for all $i \in \mathbb{N}$ are natural variables.
The extended natural numbers: the set \( \mathbb{N} \cup \{0, \infty\} \), denoted by \( \mathcal{N} \). Its total order is given by \( 0 \prec 1 \prec 2 \prec \cdots \prec \infty \).

\( \mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]] \): Ring of formal power series in the variables \( x_0, x_1, x_2, \ldots, x_\infty \) over \( \mathbb{Z} \). Essentially the ring of polynomials except elements may contain infinitely many monomials.

Example: \( x_0 + x_1 + x_2 + \cdots + x_\infty = \sum_{i \in \mathcal{N}} x_i \).

\( x_0 \) and \( x_\infty \) are bordering variables; \( x_i \) for all \( i \in \mathbb{N} \) are natural variables.
The extended natural numbers: the set \( \mathbb{N} \cup \{0, \infty\} \), denoted by \( \mathcal{N} \). Its total order is given by 
\( 0 \prec 1 \prec 2 \prec \cdots \prec \infty \).

\[ \mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]]: \text{Ring of formal power series in the variables } x_0, x_1, x_2, \ldots, x_\infty \text{ over } \mathbb{Z}. \text{ Essentially the ring of polynomials except elements may contain infinitely many monomials.} \]

Example: \( x_0 + x_1 + x_2 + \cdots + x_\infty = \sum_{i \in \mathcal{N}} x_i \).

\( x_0 \) and \( x_\infty \) are bordering variables; \( x_i \) for all \( i \in \mathbb{N} \) are natural variables.

Denote by \( [m](p) \) the coefficient of the monomial \( m \) in the power series \( p \).
The **extended natural numbers**: the set $\mathbb{N} \cup \{0, \infty\}$, denoted by $\mathcal{N}$. Its total order is given by $0 \prec 1 \prec 2 \prec \cdots \prec \infty$.

$\mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]]$: Ring of formal power series in the variables $x_0, x_1, x_2, \ldots, x_\infty$ over $\mathbb{Z}$. Essentially the ring of polynomials except elements may contain infinitely many monomials.

Example: $x_0 + x_1 + x_2 + \cdots + x_\infty = \sum_{i \in \mathcal{N}} x_i$.

$x_0$ and $x_\infty$ are **bordering variables**; $x_i$ for all $i \in \mathbb{N}$ are **natural variables**.

Denote by $[m](p)$ the coefficient of the monomial $m$ in the power series $p$.

Example: $[x^3y](x + 7y + 13x^3y) = 13$. 
The **ring of quasisymmetric functions** (QSym): the ring of power series $f$ in the natural variables $x_1, x_2, \ldots$ such that

$$[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}] (f) = [x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}] (f)$$

for all $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$. 

Example of a quasisymmetric function:

$$\sum_{i < j < k} x_i x_2 x_k + \sum_{i < j} x_i x_j.$$ 

Example of a non-quasisymmetric function:

$$f = x_1 x_2 x_3 + x_1 x_2 x_3^2 + x_1 x_2 x_3^3 + x_2 x_3 x_3^4.$$ 

because $[x_1 x_2 x_3^3] (f) \neq [x_1 x_2 x_3^5] (f)$. 

We will investigate power series quasisymmetric in $x_1, x_2, \ldots$ but with two bordering variables $x_0, x_\infty$. 


The **ring of quasisymmetric functions** (QSym): the ring of power series $f$ in the natural variables $x_1, x_2, \ldots$ such that

$$[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}](f) = [x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}](f)$$

for all $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

Example of a quasisymmetric function: $\sum_{i<j<k} 6x_i x_j^2 x_k^3 + \sum_{i<j} x_i x_j$. 
Quasisymmetric Functions

The ring of quasisymmetric functions (QSym): the ring of power series $f$ in the natural variables $x_1, x_2, \ldots$ such that

$$[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}](f) = [x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}](f)$$

for all $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

Example of a quasisymmetric function: $\sum_{i<j<k} 6x_i x_j^2 x_k^3 + \sum_{i<j} x_i x_j$.

Example of a non-quasisymmetric function:

$$f = x_1 x_2^2 x_3^3 + x_1 x_2^2 x_4^3 + x_1 x_3^2 x_4^3 + x_2 x_3^2 x_4^3$$

because $[x_1 x_2^2 x_3^3](f) \neq [x_1 x_2^2 x_5^3](f)$. 
The **ring of quasisymmetric functions** (QSym): the ring of power series $f$ in the natural variables $x_1, x_2, \ldots$ such that

$$[x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}](f) = [x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}](f)$$

for all $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

Example of a quasisymmetric function: $\sum_{i<j<k} 6x_i x_j^2 x_k^3 + \sum_{i<j} x_i x_j$.

Example of a non-quasisymmetric function:

$$f = x_1 x_2^2 x_3^3 + x_1 x_2^2 x_4^3 + x_1 x_3^2 x_4^3 + x_2 x_3^2 x_4^3$$

because $[x_1 x_2^2 x_3^3](f) \neq [x_1 x_2^2 x_5^3](f)$.

We will investigate power series quasisymmetric in $x_1, x_2, \ldots$ but with two bordering variables $x_0, x_\infty$. 

---

**Quasisymmetric Functions**
Let $n \in \mathbb{N} \cup \{0\}$, $[n] = \{1, 2, \ldots, n\}$, and $\Lambda \subseteq [n]$. Define $K_{n,\Lambda}$ as:

$K_{n,\Lambda} = \sum_{(g_1, g_2, \ldots, g_n) \in \mathbb{N}^n \mid 0 \preceq g_1 \preceq g_2 \preceq \cdots \preceq g_n \preceq \infty} |\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|$

The sum runs over all combinations with replacement of $n$ nondecreasing elements $g_1, g_2, \ldots, g_n \in \mathbb{N}$ such that for all $i \in \Lambda$, $g_i - 1 = g_i = g_i + 1$ is false, where $g_0 = 0$ and $g_{n+1} = \infty$.

Example: If $n = 5$ and $\Lambda = \{1, 4\}$, then the combinations $(0, 0, 1, 3, \infty)$ and $(2, 2, 7, 7, 7)$ are not included because they have $g_0 = g_1 = g_2$ and $g_3 = g_4 = g_5$, but $(0, 1, 5, 5, 8)$ is included.
Our Family of Power Series

Let \( n \in \mathbb{N} \cup \{0\} \), \([n] = \{1, 2, \ldots, n\}\), and \( \Lambda \subseteq [n] \). Define \( K_{n,\Lambda} \) as:

\[
K_{n,\Lambda} = \sum_{(g_1, g_2, \ldots, g_n) \in \mathbb{N}^n; \ 0 \leq g_1 \leq g_2 \leq \cdots \leq g_n \leq \infty; \ \text{no } i \in \Lambda \text{ satisfies } g_{i-1} = g_i = g_{i+1}} 2^{\left|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}\right|} x_{g_1} x_{g_2} \cdots x_{g_n}.
\]

Example: If \( n = 5 \) and \( \Lambda = \{1, 4\} \), then the combinations \((0, 0, 1, 3, \infty)\) and \((2, 2, 7, 7, 7)\) are not included because they have \( g_0 = g_1 = g_2 \) and \( g_3 = g_4 = g_5 \), but \((0, 1, 5, 5, 8)\) is included.
Our Family of Power Series

Let \( n \in \mathbb{N} \cup \{0\} \), \( [n] = \{1, 2, \ldots, n\} \), and \( \Lambda \subseteq [n] \). Define \( K_{n,\Lambda} \) as:

\[
K_{n,\Lambda} = \sum_{(g_1, g_2, \ldots, g_n) \in \mathbb{N}^n; \ 0 \leq g_1 \leq g_2 \leq \cdots \leq g_n \leq \infty; \ \text{no } i \in \Lambda \text{ satisfies } g_{i-1} = g_i = g_{i+1}} 2^{|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.
\]

The sum runs over all combinations with replacement of \( n \) nondecreasing elements \( g_1, g_2, \ldots, g_n \in \mathbb{N} \) such that for all \( i \in \Lambda \), \( g_{i-1} = g_i = g_{i+1} \) is false, where \( g_0 = 0 \) and \( g_{n+1} = \infty \).
Let \( n \in \mathbb{N} \cup \{0\} \), \( [n] = \{1, 2, \ldots, n\} \), and \( \Lambda \subseteq [n] \). Define \( K_{n,\Lambda} \) as:

\[
K_{n,\Lambda} = \sum_{(g_1, g_2, \ldots, g_n) \in \mathbb{N}^n; \atop 0 \leq g_1 \leq g_2 \leq \cdots \leq g_n \leq \infty; \atop \text{no } i \in \Lambda \text{ satisfies } g_{i-1} = g_i = g_{i+1}} 2^{|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.
\]

The sum runs over all combinations with replacement of \( n \) nondecreasing elements \( g_1, g_2, \ldots, g_n \in \mathbb{N} \) such that for all \( i \in \Lambda \), \( g_{i-1} = g_i = g_{i+1} \) is false, where \( g_0 = 0 \) and \( g_{n+1} = \infty \).
Let $n \in \mathbb{N} \cup \{0\}$, $[n] = \{1, 2, \ldots, n\}$, and $\Lambda \subseteq [n]$. Define $K_{n,\Lambda}$ as:

$$K_{n,\Lambda} = \sum_{(g_1, g_2, \ldots, g_n) \in \mathcal{N}^n; \ 0 \leq g_1 \leq g_2 \leq \ldots \leq g_n \leq \infty; \ \text{no } i \in \Lambda \text{ satisfies } g_i - 1 = g_i = g_{i+1}} 2^{|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

The sum runs over all combinations with replacement of $n$ nondecreasing elements $g_1, g_2, \ldots, g_n \in \mathcal{N}$ such that for all $i \in \Lambda$, $g_{i-1} = g_i = g_{i+1}$ is false, where $g_0 = 0$ and $g_{n+1} = \infty$.

Example: If $n = 5$ and $\Lambda = \{1, 4\}$, then the combinations $(0, 0, 1, 3, \infty)$ and $(2, 2, 7, 7, 7)$ are not included because they have $g_0 = g_1 = g_2$ and $g_3 = g_4 = g_5$, but $(0, 1, 5, 5, 8)$ is included.
Let $n \in \mathbb{N} \cup \{0\}$, $[n] = \{1, 2, \ldots, n\}$, and $\Lambda \subseteq [n]$. Define $K_{n,\Lambda}$ as:

$$K_{n,\Lambda} = \sum_{(g_1, g_2, \ldots, g_n) \in \mathbb{N}^n; \quad 0 \leq g_1 \leq g_2 \leq \ldots \leq g_n \leq \infty; \quad \text{no } i \in \Lambda \text{ satisfies } g_{i-1} = g_i = g_{i+1} \quad (\text{where } g_0 = 0 \text{ and } g_{n+1} = \infty)} \ 2^{|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$ 

Each summand of $K_{n,\Lambda}$ is the monomial $x_{g_1} x_{g_2} \cdots x_{g_n}$ multiplied by $2$ to the power of the number of distinct natural variables in $\{x_{g_1}, x_{g_2}, \ldots, x_{g_n}\}$. 

Example: If $n = 5$ and $(g_1, g_2, \ldots, g_n) = (0, 1, 5, 5, 7)$, then the corresponding summand $2^{|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}$ equals $8x^0x^1x^5x^7$. 


Let $n \in \mathbb{N} \cup \{0\}$, $[n] = \{1, 2, \ldots, n\}$, and $\Lambda \subseteq [n]$. Define $K_{n, \Lambda}$ as:

$$K_{n, \Lambda} = \sum_{(g_1, g_2, \ldots, g_n) \in \mathbb{N}^n; \atop 0 \leq g_1 \leq g_2 \leq \cdots \leq g_n \leq \infty; \atop \text{no } i \in \Lambda \text{ satisfies } g_{i-1} = g_i = g_{i+1} \atop \text{(where } g_0 = 0 \text{ and } g_{n+1} = \infty)} 2^{|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

Each summand of $K_{n, \Lambda}$ is the monomial $x_{g_1} x_{g_2} \cdots x_{g_n}$ multiplied by 2 to the power of the number of distinct natural variables in $\{x_{g_1}, x_{g_2}, \ldots, x_{g_n}\}$.

Example: If $n = 5$ and $(g_1, g_2, \ldots, g_n) = (0, 1, 5, 5, 7)$, then the corresponding summand $2^{|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}$ equals $8x_0x_1x_5^2x_7$. 
Examples of \( K_{n,\Lambda} \)’s

\[
K_{n,\Lambda} = \sum_{(g_1, g_2, \ldots, g_n) \in \mathbb{N}^n; \quad 0 \leq g_1 \leq g_2 \leq \cdots \leq g_n \leq \infty; \quad \text{no } i \in \Lambda \text{ satisfies } g_{i-1} = g_i = g_{i+1}} \quad 2^{|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.
\]

When \( n = 1 \), \( K_{1,\{\}} = K_1 \), \( \{1\} = x_0 + \sum_{i \in \mathbb{N}} x_i + \sum_{i \in \mathbb{N}} x_i \). There are no restrictions on \( g_1 \) because \( 0 = g_1 = \infty \) is impossible.

When \( n = 2 \), \( K_{2,\{1\}} = \sum_{i \in \mathbb{N}} x_0 x_i \sum_{i \in \mathbb{N}} x_i \sum_{i \in \mathbb{N}} x_{i+1} + \sum_{i < j \in \mathbb{N}} x_i x_j + \sum_{i \in \mathbb{N}} x_{2i+1} \). The coefficient of each monomial depends on how many distinct natural variables it contains.
Examples of $K_{n,\Lambda}$’s

$$K_{n,\Lambda} = \sum_{(g_1, g_2, \ldots, g_n) \in \mathbb{N}^n; \quad \forall i \in \Lambda: g_{i-1} \leq g_i = g_{i+1} \left( g_0 = 0 \text{ and } g_{n+1} = \infty \right)} 2^{|\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}|} x_{g_1} x_{g_2} \cdots x_{g_n}.$$  

- When $n = 1$, $K_{1,\{\}} = K_{1,\{1\}} = x_0 + \sum_{i \in \mathbb{N}} 2x_i + x_\infty$. There are no restrictions on $g_1$ because $0 = g_1 = \infty$ is impossible.
Examples of $K_{n, \Lambda}$’s

$$K_{n, \Lambda} = \sum_{(g_1, g_2, \ldots, g_n) \in \mathbb{N}^n; \quad 0 \leq g_1 \leq g_2 \leq \cdots \leq g_n \leq \infty; \quad \text{no } i \in \Lambda \text{ satisfies } g_{i-1} = g_i = g_{i+1}} \sum_{\{g_1, g_2, \ldots, g_n\} \cap \mathbb{N}} x_{g_1} x_{g_2} \cdots x_{g_n}.$$

- When $n = 1$, $K_{1, \{\}} = K_{1, \{1\}} = x_0 + \sum_{i \in \mathbb{N}} 2x_i + x_\infty$. There are no restrictions on $g_1$ because $0 = g_1 = \infty$ is impossible.
- When $n = 2$,

$$K_{2, \{1\}} = \sum_{i \in \mathbb{N}} 2x_0 x_i + \sum_{i \in \mathbb{N}} 2x_i x_\infty + \sum_{i \in \mathbb{N}} 2x_i^2 + \sum_{i < j \in \mathbb{N}} 4x_i x_j + x_\infty^2.$$

The coefficient of each monomial depends on how many distinct natural variables it contains.
Main Theorem

Theorem

*The span of our family of power series, \( \text{span} \left( K_n, \Lambda \right)_{n \in \mathbb{N} \cup \{0\}; \ \Lambda \subseteq [n]} \), is a \( \mathbb{Z} \)-subalgebra of \( \mathbb{Z}[[x_0, x_1, x_2, \ldots, x_{\infty}]] \).*
Theorem

The span of our family of power series, \( \text{span} \left( K_n, \Lambda \right)_{n \in \mathbb{N} \cup \{0\}; \ \Lambda \subseteq [n]} \), is a \( \mathbb{Z} \)-subalgebra of \( \mathbb{Z}[[x_0, x_1, x_2, \ldots, x_\infty]] \).

Equivalently, the product of any \( K_n, \Lambda K_m, \Omega \) can be written as the sums and differences of several \( K_{n+m}, \Xi \)'s, where each \( \Xi \subseteq [n + m] \).
The span of our family of power series, \( \text{span} \left( K_n, \Lambda \right)_{n \in \mathbb{N} \cup \{0\}; \: \Lambda \subseteq [n]} \), is a \( \mathbb{Z} \)-subalgebra of \( \mathbb{Z}[x_0, x_1, x_2, \ldots, x_\infty] \).

Equivalently, the product of any \( K_n, \Lambda K_m, \Omega \) can be written as the sums and differences of several \( K_{n+m}, \Xi \)'s, where each \( \Xi \subseteq [n + m] \).

Example: When \( n = m = 2 \) and \( \Lambda = \Omega = \{1, 2\} \), the product \( K_{2, \{1,2\}} K_{2, \{1,2\}} \) can be written as

\[
K_{4, \{2\}} + 2K_{4, \{1,3\}} + 2K_{4, \{1,4\}} + K_{4, \{2,4\}} + K_{4, \{1,2,4\}} - K_{4, \{1,2\}}.
\]
Grinberg proved the theorem in 2018 for exterior peak sets $\Lambda$, $\Omega$; i.e. sets with no pair of consecutive integers. That result was key to proving the shuffle-compatibility of the exterior peak set statistic $E_{pk}$. 
Grinberg proved the theorem in 2018 for exterior peak sets $\Lambda$, $\Omega$; i.e. sets with no pair of consecutive integers. That result was key to proving the shuffle-compatibility of the exterior peak set statistic $\text{Epk}$.

In 2020, Grinberg later proved the theorem for $x_0 = x_\infty = 0$, in which case the power series are quasisymmetric.
Proof Outline

Consider similar family of functions, $L_{n, \Lambda}$’s, which can be handled more easily; prove the following equivalent theorem: the product of any $L_{n, \Lambda} L_{m, \Omega}$ can be written as the sums and differences of several $L_{n+m, \Xi}$’s, where each $\Xi \subseteq [n + m]$. 

"Zero out" the coefficients of monomials in $L_{n, \Lambda} L_{m, \Omega}$ by adding/subtracting $L_{n+m, \Xi}$’s in a specific order. Show that doing so results in the coefficients of every monomial becoming zero, thus proving the theorem.
Proof Outline

Consider similar family of functions, $L_n,\Lambda$’s, which can be handled more easily; prove the following equivalent theorem: the product of any $L_n,\Lambda L_m,\Omega$ can be written as the sums and differences of several $L_{n+m},\Xi$’s, where each $\Xi \subseteq [n + m]$.

“Zero out” the coefficients of monomials in $L_n,\Lambda L_m,\Omega$ by adding/subtracting $L_{n+m},\Xi$’s in a specific order.
Proof Outline

Consider similar family of functions, $L_{n,\Lambda}$’s, which can be handled more easily; prove the following equivalent theorem: the product of any $L_{n,\Lambda}L_{m,\Omega}$ can be written as the sums and differences of several $L_{n+m,\Xi}$’s, where each $\Xi \subseteq [n+m]$.

“Zero out” the coefficients of monomials in $L_{n,\Lambda}L_{m,\Omega}$ by adding/subtracting $L_{n+m,\Xi}$’s in a specific order.

Show that doing so results in the coefficients of every monomial becoming zero, thus proving the theorem.
D. Grinberg, Shuffle-compatible permutation statistics II: the exterior peak set. *Electron. J. Combin.* 25 (2018), Paper 17.

D. Grinberg, The eta-basis of QSym, 2020. Available at www.cip.ifi.lmu.de/~grinberg/algebra/etabasis.pdf.
Acknowledgements

- My mentor, Andrey Khesin
- Professor Darij Grinberg
- The MIT PRIMES program
- My aunt, for letting me use her WiFi