Universal quantum information compression and degrees of prior knowledge

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Abstract

We describe a universal information compression scheme that compresses any pure quantum i.i.d. source asymptotically to its von Neumann entropy, with no prior knowledge of the structure of the source. We introduce a diagonalisation procedure that enables any classical compression algorithm to be utilised in a quantum context. Our scheme is then based on the corresponding quantum translation of the classical Lempel–Ziv algorithm. Our methods lead to a conceptually simple way of estimating the entropy of a source in terms of the measurement of an associated length parameter while maintaining high fidelity for long blocks. As a by-product we also estimate the eigenbasis of the source. Since our scheme is based on the Lempel–Ziv method, it can be applied also to target sequences that are not i.i.d.

1 Introduction

In addition to its evident utility in practical communication issues, the concept of information compression provides a bridge between the abstract theory of information and concrete physics – it characterises the minimal physical resources (of various appropriate kinds) that are necessary and sufficient to faithfully encode or represent information.

In the case of quantum information, the study of optimal compression rates is especially interesting as it relates directly to non-orthogonality of states [1] and entanglement, and thus provides a new tool for investigating foundational properties of these uniquely quantum features. Almost all work to date on quantum information compression has studied compression properties of a so-called independent identically distributed source (i.i.d. source). (However see [10] for an interesting non-i.i.d. situation.) Let $E = \{ |\sigma_i\rangle ; p_i \}$ be an ensemble of (pure) quantum signal states $|\sigma_i\rangle$ with assigned probabilities $p_i$. An i.i.d. source comprises an unending sequence of states chosen independently from $E$. For each integer $n$ we have an ensemble of signal blocks of length $n$. Writing $I = i_1 \ldots i_n$ the states are $|\sigma_I\rangle = |\sigma_{i_1}\rangle \otimes \ldots \otimes |\sigma_{i_n}\rangle$ with probabilities $p_I = p_{i_1} \ldots p_{i_n}$. Let $H$ (with dimension $d$) denote the Hilbert space of single signals and let $Q_\alpha$ denote the space of all mixed states of $\alpha$ qubits (or the smallest integer greater than $\alpha$ if $\alpha$ is not an integer). Then $n$-blocks $|\sigma_I\rangle$ are in $H^{\otimes n}$ and in $Q_{n \log d}$. (In this paper log will denote logarithms to base 2). To define the notion of compression we first introduce the fidelity

$$F(|\psi\rangle\langle\psi|, \rho) = \langle \psi | \rho | \psi \rangle$$

between any pure and mixed state. More generally if $\rho$ and $\omega$ are mixed we define fidelity by [11, 12]

$$F(\rho, \omega) = (\text{tr} \sqrt{\sqrt{\omega} \rho \sqrt{\omega}})^2.$$
The von Neumann entropy $S$ of an ensemble $E$ is defined by

$$S = -\text{tr} \rho \log \rho$$

where $\rho = \sum_i p_i |\sigma_i\rangle\langle\sigma_i|$ is the overall density matrix of the signals.

An encoding–decoding scheme for blocks of length $n$, to $\alpha$ qubits per signal and average fidelity $1 - \epsilon$, is defined by the following ingredients:

(i) An encoding operation $E_n : \mathcal{H}^n \to Q_{n\alpha}$ which is a completely positive trace preserving (CPTP) map. $E_n(|\sigma_I\rangle)$ is a (mixed) state of $n\alpha$ qubits called the encoded or compressed version of $|\sigma_I\rangle$.

(ii) A decoding operation $D_n : Q_{n\alpha} \to Q_{n\log d}$ which is also a CPTP map. We write $\tilde{\sigma}_I = D_n E_n(|\sigma_I\rangle)$ and call it the decoded version of $|\sigma_I\rangle$. Note that $\tilde{\sigma}_I$ is generally a mixed state.

(iii) The average fidelity between $|\sigma_I\rangle$ and $\tilde{\sigma}_I$ is $1 - \epsilon$:

$$\sum_I p_I F(|\sigma_I\rangle, \tilde{\sigma}_I) = 1 - \epsilon$$

We say that the source $E$ may be compressed to $\alpha$ qubits per signal if the following condition is satisfied: for all $\epsilon > 0$ there is an $n_0$ such that for all blocks of length $n > n_0$ there is an encoding–decoding scheme for blocks of length $n$ to $\alpha$ qubits per signal and average fidelity at least $1 - \epsilon$.

The above definitions are motivated by source coding for i.i.d. sources in Shannon’s classical information theory (cf. [13] for an exposition). Indeed if the signal states are mutually orthogonal and the coding/decoding operations are classical probabilistic processes, then we regain the standard classical theory. The quantum generalisation of Shannon’s source coding theorem is Schumacher’s quantum source coding theorem [3, 14, 15, 16], stating that the optimal compression rate is the von Neumann entropy $S$ of the signal ensemble. More precisely, if $\alpha \neq S$ then $E$ may be compressed to $\alpha$ qubits per signal if $\alpha > S$. In these source coding theorems it is assumed that we have knowledge of the signal ensemble states $|\sigma_i\rangle$ and their prior probabilities $p_i$. (Actually knowledge of the density matrix $\rho = \sum_i p_i |\sigma_i\rangle\langle\sigma_i|$ alone suffices).

The question of universal compression concerns a situation in which we have only partial, or even no knowledge, about the i.i.d. source $E$. We may even go further and ask about compressing a target sequence from a source that is not even assumed to be i.i.d. (but perhaps has other properties e.g. a Markovian source). Thus universal compression may be studied in the presence of varying degrees of prior knowledge about the source. In this paper we will consider universal compression of i.i.d. sources. However in contrast to all other quantum compression schemes proposed to date, our methods can also be applied to non-i.i.d. sources in a natural way, in various situations (that will become clear in our exposition below).

A classical i.i.d. source is fully characterised just by its probability distribution $\{p_i\}$ of signals. In a quantum i.i.d. source, for the purpose of studying the action of the encoding and decoding maps, each signal state may be taken to be in the mixed state $\rho = \sum_i p_i |\sigma_i\rangle\langle\sigma_i|$. Hence a quantum i.i.d. source is fully characterised by the classical probability distribution $\{\lambda_i\}$ of the eigenvalues of $\rho$ together with the specification of a
corresponding orthonormal eigenbasis \( \{|\lambda_i\rangle\} \). The distribution \( \{\lambda_i\} \) is the direct analogue of the distribution \( \{p_i\} \) of a classical source and the extra freedom in the quantum case, of the orientation of the eigenbasis makes the problem of universal quantum compression inherently more difficult than its classical counterpart.

Before presenting our main results we give a brief overview of existing work on universal quantum information compression. The basic technique of so-called Schumacher compression [3, 14] used in the Schumacher source coding theorem utilises the typical subspace of \( \rho \). This construction requires knowledge of both the eigenvalues and eigenvectors of \( \rho \). As such, it does not appear to offer any generalisation to a universal compression scheme, with a prior knowledge of anything less than full knowledge of the source. In [17] Jozsa et al. presented a universal compression scheme for quantum i.i.d. sources, requiring a prior knowledge of an upper bound \( S_0 > S \) on the von Neumann entropy of the source but requiring no prior knowledge of the orientation of the eigenbasis of the source. The scheme compressed the source to \( S_0 \) qubits per signal (in contrast to the optimal \( S \) in the case that the source is known). Hence if the von Neumann entropy (or set of eigenvalues) of the source is known then this scheme is universal, with no prior knowledge of the eigenbasis.

In [6] and [18] a quantum analogue of Huffman variable length coding was developed. The techniques of Schumacher and Westmoreland in [6] motivated the formulation of our two stage compression model below. Although the Schumacher–Westmoreland scheme is not presented as a universal scheme, if we adjoin the method of “smearing” measurements used by Hayashi and Matsumoto [7, 8] (and also described and used by us below) then the scheme can be made universal for a situation in which the orientation of the eigenbasis is known but the eigenvalue distribution is unknown.

The first fully universal compression scheme for quantum i.i.d. sources was presented by Hayashi and Matsumoto [7]. This scheme compresses any quantum i.i.d. source to its von Neumann entropy \( S \), requiring no prior knowledge of the eigenbasis or eigenvalues of the source density matrix. Their method is based on the scheme of [17] supplemented by an estimation of the eigenvalues of the source and hence of \( S \).

In this paper we present an alternative fully universal quantum compression scheme with various novel features [9]. In classical information theory [13] there exists a variety of schemes for universal classical information compression. Some of these, such as the Lempel–Ziv method, apply even to situations in which the target string does not come from an i.i.d. source. Below we will introduce a “diagonalisation procedure” which effectively enables any such classical scheme to be transferred into the context of quantum compression. Then utilising the measurement smearing technique of [7], in conjunction with a further iterative procedure, we will achieve universal quantum information compression. In contrast to the scheme of [7] our scheme will include an estimation of the eigenbasis orientation, and we estimate the source entropy \( S \) via a conceptually simpler estimation of a block length parameter, whose knowledge is equivalent to that of \( S \). Our scheme will be based on transferring the classical Lempel–Ziv scheme into a quantum context. Hence (in contrast to any other existing scheme) it will be applicable even to sources that are not assumed to be i.i.d., although the question of optimality of the achieved compression rate in these more general situations remains to be explored.
2 Two-Stage Compression Model

Let $\mathcal{E} = \{\sigma_i : p_i\}$ be the signal ensemble of a quantum i.i.d. source and let $\rho = \sum_i p_i |\sigma_i\rangle\langle\sigma_i|$. In the signal state space let $\{e_i\}$ be a fixed chosen basis called the computational basis. The process of compressing blocks of length $n$ of the source will in general consist of some unitary manipulations of the state $\rho^\otimes n$ (and we also include the possible adjoining of ancillary qubits in a standard state), and some operations which are not unitary (i.e. discarding qubits or measurement operations). We know from the study of quantum circuits [4] that the order of operations can be re-arranged to put all the non-unitary steps at the end. Thus the whole procedure can be naturally divided into two separate stages. The first, consisting of all the unitary manipulations of the state, will be called the “condensation” stage. This stage takes the form of an algorithm to be performed by a quantum computer. Naturally, it cannot decrease the total length of the sequence.

At the end of the condensation stage, the string should have been manipulated in such a way that the first $nR_c$ qubits of output contain a faithful representation of the data in the input (for some “condensation rate” $S(\rho) \leq R_c \leq 1$). The remainder of the qubits are in a state asymptotically independent of the input (for simplicity, we assume the state $|0\rangle$). We call these qubits “blank”.

The second stage consists of the measurement operations. This is where actual compression takes place, since now the dimension of the Hilbert space in which the state lives can be reduced. It is called the “truncation” stage, since it tends to involve removing the “blank” qubits at the end. In truncation the length of the string is reduced to $nR_t$ for some $R_c \leq R_t \leq 1$. Asymptotic independence between the “blanks” and the data qubits is equivalent to saying that this truncation can be performed with fidelity $F \to 1$ as $n \to \infty$.

Determining exactly where the truncation cut is to be made is in general a difficult task. Previous compression schemes have relied upon given prior knowledge of the source to do this. In the present case we do not assume that such information is given a priori.

3 Lempel-Ziv Algorithm

For the condensation stage of our compression scheme we will utilise the basic formalism of the classical universal Lempel–Ziv compression scheme, transferred to a quantum context. The precise details of the Lempel–Ziv method will not be required but we give an outline of the method. In fact any other classical scheme that is universal for classical i.i.d. sources could be used.

The classical Lempel-Ziv compression scheme [2, 13] asymptotically compresses the output of an i.i.d. source with unknown probability distribution to $H$ bits per signal, where $H$ is the Shannon entropy of the distribution. It depends upon the fact that at any time in the decoding process, there is a significant quantity of data that is known to both sender and receiver. By making reference to this data as a shared resource, the sender can more efficiently transmit further signals from the same source. The encoder scans the sequence, building up a dictionary of subsequences in such a way that each new entry in the dictionary is a 1-bit extension of some previous word. When the whole sequence has been parsed in this way, this internal structure of the dictionary is transmitted as a list of references; instead
of sending a whole subsequence, its position in the dictionary is transmitted, plus the single extra bit. In the limit, these references are logarithmically shorter than the subsequences they represent. The decoder reverses this procedure, building up the encoder’s dictionary from the list of references. The original sequence is reconstructed simply by concatenating the words of the dictionary. The Lempel-Ziv code is therefore lossless (i.e. has fidelity 1) and the compression rate $H$ is achieved as an average value over all possible inputs.

Bennett [5] showed that it is possible to implement any classical algorithm reversibly, with only a polynomial increase in time and space resources. We can therefore construct a reversible version of the Lempel-Ziv algorithm (or any other classical universal compression algorithm), which may be run on a quantum computer.

The resulting algorithm treats the orthonormal states of the computational basis as if they were classical signals. For sources which are not diagonal in the computational basis the input sequence can be regarded as a superposition of “pseudo-classical” sequences, each of which is operated upon independently. However as we will show below, the action of the quantum implementation of the Lempel-Ziv algorithm on such non-diagonal i.i.d. quantum sources is simply to condense them to a rate asymptotically approaching $H$ qubits per signal, where $H \geq S$ is the Shannon entropy of a suitable probability distribution. This feature of $H$ being generally greater than $S$, embodies the difficulty arising from a mismatch between the eigenbasis of the source and the computational basis of the computer. As part of our main result we will show how this difficulty can be overcome.

In order to help us examine the effect of running the algorithm, we note that the quantum implementation (via Bennett’s result) of any classical deterministic algorithm merely enacts a permutation of the set of all strings of computational basis states at each step. That is, each state $|e_I⟩ = |e_{i_1}⟩ \otimes \ldots \otimes |e_{i_n}⟩$ is mapped to some $|e_{P(I)}⟩$, where $P$ is a permutation on the set of sequences $I = i_1 \ldots i_n$; no superposition or probabilistic mixing is created. This action, denoted:

$$|e_I⟩ \xrightarrow{C} |e_{P(I)}⟩$$

is much more restricted than that of an arbitrary unitary transformation:

$$|e_I⟩ \xrightarrow{U} \sum_{J} b_{IJ} |e_{J}⟩,$$

and the restriction will be important for us later (cf Theorem 1 below).

4 Condensation rate with mismatched bases

If we knew the eigenbasis of the source density matrix, we could simply set the computer to use this basis as its computational basis, and analysis of the output from the algorithm would be relatively simple. But since we are aiming to achieve fully universal compression, we must assume that we do not know the source’s eigenvectors.

We denote the computational basis by $B_C = \{|e_i⟩\}$, and denote the eigenbasis of the source by $B_S = \{|\lambda_i⟩\}$. Our first task is to study the effect of the algorithm on an i.i.d. source whose eigenbasis $B_S$ does not coincide with the computational basis $B_C$. We do this by introducing a hypothetical diagonalisation procedure which has the effect of making the source appear diagonal in the computational basis.
4.1 The Diagonalisation procedure

Given any orthonormal basis $\mathcal{B} = \{|i\rangle\}$ with $n$ states, we define

$$D : |i\rangle |j\rangle \rightarrow |i\rangle |j \oplus i\rangle \quad \forall i, j \in \mathcal{B}$$

(4)

where $\oplus$ denotes addition mod $n$. This operation is commonly used in quantum information processing, particularly in the special case of $j = 0$:

$$D : |i\rangle |0\rangle \rightarrow |i\rangle |i\rangle \quad \forall i \in \mathcal{B}$$

where it serves as a duplication operation. Note that this only copies basis states, and not superpositions of them - there is no conflict with the No-Cloning theorem. The action of $D$ on a superposition is an entangled state:

$$|\psi\rangle |0\rangle = \sum_i a_i |i\rangle |0\rangle \rightarrow \sum_i a_i |i\rangle |i\rangle .$$

(5)

This is a fatal feature for those who would like to clone quantum information, but it will be the key to solving our problem of mismatched bases.

If we apply $D$ to each signal state in the input sequence, we will produce a duplication (relative to the basis $\mathcal{B}$) of this input sequence, entangled with the original. If we allow the computer only to operate on the original, or only on the duplicate, and not to make joint operations on both, then the state addressed is described by tracing out one of the systems. From the RHS of (5) we see that the reduced state of either system is $\sum_i |a_i|^2 |i\rangle\langle i|$ which is always diagonal in the basis $\mathcal{B}$.

If we take $\mathcal{B}$ to be the computational basis of the computer and only address each part of the duplication separately, the computer will act on an input that is diagonal in its computational basis. Note that everything so far is done coherently. Although the computer is now addressing a mixed state, no measurements have been carried out, and no information has been lost.

Now, imagine allowing the computer to address each part of the duplication in turn; it carries out the algorithm on the first sequence, leaving the duplicate unchanged, then repeats the process on the duplicate, leaving the first part undisturbed. Alternatively, we can imagine building a computer twice as large, partitioned into two sides, each of which works simultaneously on a single copy of the sequence. Finally apply $D^{-1}$ at each signal position in the resultant state.

We will show below that this combined process will leave a final state in the first register that is identical to the result of simply applying the algorithm to the given input sequence with no diagonalisation operations being applied. Since in the alternative process (involving the operation $D$) the algorithm acts only on states diagonal in the computational basis, we can use this equivalence to give a simple derivation of the condensation properties that result when the computational basis and eigenbasis of the source are not matched.

Consider any classical deterministic algorithm which has been formulated in a reversible way and implemented on a quantum computer. Thus any step $C$ of the algorithm is a permutation of the computational basis states, in the sense of Eq. (3). We wish to prove
that the following diagram has a “pseudo-commutation” i.e. that the two ways of going around the loop give the same result:

\[
\begin{align*}
|e_I\rangle |0\rangle &\xrightarrow{\hat{D}} |e_I\rangle |e_I\rangle \xrightarrow{U \otimes U} U \otimes U \sum_{JK} b_{IJ} b_{IK} |e_J\rangle |e_K\rangle \\
&\xrightarrow{\hat{D}^{-1}} \sum_{JK} b_{IJ} b_{IK} |e_J\rangle |e_K \oplus e_J\rangle
\end{align*}
\]

where \( \hat{D} = D^{\otimes n} \), the operation \( D \) applied at each position in the sequence.

**Theorem 1:** For any unitary operation \( C \), the equality \( \hat{D}^{-1} (C \otimes C) \hat{D} = (C \otimes I) \) is satisfied iff \( C \) enacts a permutation on the computational basis states.

**Proof:** If we write \( C \) as the unitary operation \( U : |e_I\rangle \rightarrow \sum_J b_{IJ} |e_J\rangle \), we get:

\[
|e_I\rangle |0\rangle \xrightarrow{\hat{D}} |e_I\rangle |e_I\rangle \xrightarrow{U \otimes U} \sum_{JK} b_{IJ} b_{IK} |e_J\rangle |e_K\rangle \xrightarrow{\hat{D}^{-1}} \sum_{JK} b_{IJ} b_{IK} |e_J\rangle |e_K \oplus e_J\rangle
\]

where \( \oplus \) denotes subtraction mod \( n \). For this to be of the form \( \sum_J b_{IJ} |e_J\rangle |0\rangle \), it is necessary that \( |e_K\rangle \) and \( |e_J\rangle \) are always identical i.e. \( U \) must map each \( |e_I\rangle \) to a multiple of a unique \( |e_J\rangle \) (and not a superposition):

\[
|e_I\rangle \xrightarrow{U} e^{i\alpha_I} |e_J\rangle.
\]

Finally substituting this form of \( U \) into Eq. (5) and equating with \( U \otimes I |e_I\rangle |0\rangle \) gives \( \alpha_I = 0 \). Hence \( U \) must be a permutation as claimed. Conversely if \( C \) is a permutation then an easy calculation shows that \( \hat{D}^{-1} (C \otimes C) \hat{D} = (C \otimes I) \) holds.

The final state arrived at is the same whether we insert the diagonalisation operation or not. The duplication process can therefore be considered merely a mathematical convenience to save us from having to directly analyse the action of the condensation algorithm on non-diagonal inputs.

### 4.2 Rate of Condensation

In the special case of \( B_C = B_S \), the computer views the source as emitting classical signals \( \{|e_i\rangle\} \) with probability distribution \( \{p_i\} \), where the \( p_i \) are simply the eigenvalues of the source density matrix. Thus the asymptotic condensation rate is the von Neumann entropy of the source, \( S(\rho) \). Therefore, in this special case, the Lempel-Ziv algorithm achieves asymptotically optimal condensation.

If \( B_C \neq B_S \) then by Theorem 1 the condensation effect of the algorithm acting directly on the source \( \mathcal{E} \) can be found by considering the reduced state of the duplication of \( \mathcal{E} \). The rate achieved will be the same as the condensation rate of a source diagonal in the computational basis \( B_C \), but whose signal probabilities are given by the eigenvalues of this reduced density matrix.
If $\lambda_i$ and $|\lambda_i\rangle$ are the eigenvalues and eigenvectors of $E$ then we can expand the eigenvectors in the computational basis $\{|e_i\rangle\}$ as

$$|\lambda_i\rangle = \sum_j a_{ij} |e_j\rangle$$

and the source density matrix becomes

$$\rho = \sum_i \lambda_i |\lambda_i\rangle \langle \lambda_i| = \sum_{ijk} \lambda_i a_{ij} a_{ik}^* |e_j\rangle \langle e_k|$$

When we append the ancilla and apply $D$, we get

$$\rho \xrightarrow{D} \sum_{ijk} \lambda_i a_{ij} a_{ik}^* |e_j\rangle \langle e_k|_A \otimes |e_j\rangle \langle e_k|_B$$

and taking the partial trace over system $B$ gives the reduced density matrix:

$$\rho' = \sum_j \left( \sum_i \lambda_i |a_{ij}|^2 \right) |e_j\rangle \langle e_j|$$

This density matrix, which is diagonal in the computational basis, describes the distribution of states addressed by the computer if the duplication process is carried out. We therefore refer to $\rho'$ as the effective source density matrix.

The eigenvalues $\mu_j$ of $\rho'$ are thus given by $\mu_j = \sum_i \lambda_i |a_{ij}|^2$, where the $a_{ij}$ are the coefficients in the expansion of source eigenstate $|\lambda_i\rangle$. We may write these coefficients as $\langle e_j| \lambda_i\rangle$, and therefore the $j^{th}$ eigenvalue of $\rho'$ is:

$$\sum_i \lambda_i \langle e_j| \lambda_i\rangle \langle \lambda_i| e_j\rangle = \langle e_j| \rho |e_j\rangle$$

i.e. the diagonal matrix elements of $\rho$ when $\rho$ is written in the computational basis. Inserting this expression into the formula for Shannon entropy gives:

$$R_c(\rho, B_C) = - \sum_{|e_j\rangle \in B_C} \langle e_j| \rho |e_j\rangle \log \langle e_j| \rho |e_j\rangle$$

This is the general formula for the asymptotic rate of condensation for the source $E$ achieved when we work in an arbitrary computational basis $B_C$. In the special case $B_C = B_S$, the formula reduces to $S(\rho)$, as expected. Furthermore since the matrix $[a_{ij}]$ represents the transition between two orthonormal bases, it is a unitary matrix and then $[r_{jk}] \equiv [|a_{kj}|^2]$ is doubly stochastic. Consequently the eigenvalues $\mu_j = \sum_k r_{jk} \lambda_k$ of $\rho'$ are a doubly stochastic transform of the eigenvalues of $\rho$, and by a monotonicity theorem for entropy we have $S(\rho') \geq S(\rho)$. Thus the algorithm with a mismatch of bases acts as a standard condensation process but incurs a loss of optimality of the achieved condensation rate as quantified by the above formulae.
5 Truncation

In the truncation stage we apply the measurements which actually reduce the physical resources occupied by the quantum information. The difficulty of truncation is in identifying how many qubits are “blanks” and may safely be discarded. In doing this, we are effectively estimating the condensation rate achieved, $R_c$, which depends upon $\rho$ and is therefore not known \textit{a priori}.

Before treating the truncation problem itself, we will first consider a simpler idealised situation. We are presented with a sequence of $n$ qubits, and told that it is composed of two parts: the first part is a sequence of $k$ maximally mixed qubits (each in the state $\frac{1}{2}$), for some integer $0 \leq k \leq n$, while the remaining $n - k$ qubits are in the zero state $|0\rangle$. Our task is to determine the value of $k$ as accurately as possible, in such a manner that, when $n \to \infty$ (keeping the fraction $\frac{k}{n}$ constant) the global fidelity of the state remains arbitrarily high, $F \to 1$.

We can argue that the state described above resembles the output from the condensation algorithm. The output state does consist, approximately, of two distinct substrings: a “data” part to be preserved and a “blank” part, whose qubits are all in the state $|0\rangle$ (except for a small “tail” at the end of the data.) The size of the “data” section is an unknown fraction $\frac{k}{n}$ of the total length $n$; for large $n$, this fraction is approximately equal to $R_c(\rho, B_C)$, which is independent of $n$.

Bearing in mind these approximations, we can assert that a solution to the simplified problem will get us most of the way to a solution of the truncation problem itself. Afterwards, we will weaken the assumptions to something more realistic, whilst preserving the solution.

Following [6] we define a projector

$$\Pi_l = I_1 \otimes \cdots \otimes I_{l-1} \otimes |0^{l+1 \ldots n} \rangle \langle 0^{l+1 \ldots n}|$$

which acts on a sequence of $n$ qubits, projecting onto the subspace in which the last $(n - l)$ qubits are in the state $|0\rangle$. To locate the position $k$ of the boundary we will develop a strategy that involves applying a sequence of (suitably smeared) $\Pi_l$'s with decreasing $l$ values.

If we apply this projector to the sequence at some position $l$, it will (in general) tell us whether the “boundary” position $k$ lies to the left or to the right of $l$. If the projector is to the right of the boundary then it will certainly project into its positive subspace, and cause no disturbance. However, if it lies to the left of the boundary then it will certainly cause some disturbance, whichever outcome is obtained. The closer to the boundary it lies, the greater the disturbance caused. The probabilities and magnitudes of disturbance depend upon the number of maximally mixed qubits to the right of the projector, which we denote $s$. The projector is then $\Pi_l = \Pi_{k-s}$.

If a projector $\Pi_{k-s}$ projects to its positive subspace (giving outcome “1”), then all qubits to the right of the projector (including those that were maximally mixed) are set to the zero state. Since $s$ qubits were maximally mixed, the probability to project to $|0\rangle^{\otimes s}$ is simply $2^{-s}$. The result of this disturbance is so great that, as an approximation, we can assign a fidelity of zero to the resultant state. This is a valid thing to do, since we are only looking
for a lower bound on the fidelity achieved. We will also need to derive an upper bound on the probability of such an “error” in our procedure.

Conversely, if \( \Pi_{k-s} \) projects to its perpendicular subspace (outcome “0”), the maximally mixed qubits to the right of the projector are projected away from zero. This is also a disturbance, although in general a smaller one. The probability of this is \((1 - 2^{-s})\). Using Eq. (2) the fidelity after the disturbance is readily seen to be \((1 - 2^{-s})\). The outcome of the projector tells us that we have found a position to the left of the boundary, and no more projections should be made - the process terminates.

Therefore, a projector located \( s \) places to the left of the boundary maintains a fidelity of at least \((1 - 2^{-s})^2\), and contributes \(2^{-s}\) to the error probability (i.e. the probability of not registering the presence of the boundary).

No strategy in which we simply make measurements with \( \Pi_l \) projectors can safely give us the information required. Wherever we choose to apply the projectors, there can always be cases of \( l \) values which lie very close to the left of the boundary value \( k \) (i.e. \( s \) is very small), and therefore cause a large disturbance. Admittedly, the probability of such an event is low if \( k \) is chosen at random. That is, for most values of \( k \) (i.e. most sources) such a strategy would work well. However, probabilistic success is not good enough for universal compression, which must give high fidelity for all sources, not just the majority of them.

To avoid this problem we adopt a method of “smearing” measurements that was used by Hayashi and Matsumoto \([\text{[4]}\)\]. We define a POVM using an equally-weighted average of a set of projectors \( \Pi_l \), each offset by a different amount from a common basepoint:

\[
\bar{\Pi}_l = \frac{1}{Y} \sum_{i=0}^{Y-1} \Pi_{l+i}
\]  

(12)

where \( Y \), the number of projectors in each POVM “cluster”, is some parameter we are free to choose. The POVM elements are then \( \{ \bar{\Pi}_l, I - \bar{\Pi}_l \} \). Physically we may interpret this POVM as applying a random choice, \( \Pi_m \), of the projectors \( \Pi_l, \ldots, \Pi_{l+Y} \) (chosen with equal prior probabilities \( 1/Y \)) and then forgetting the value of \( m \). Thus for any given \( k \) value and any choice of \( l \), the probability that \( \Pi_m \) is close to \( k \) (and hence causes a large disturbance) is only \( O(1/Y) \) which can be kept small by choosing \( Y \) large enough.

We will use POVMs based on \( \bar{\Pi}_{xL} \) for some integer \( L \) and \( x \in \{0, 1, \ldots, \frac{n}{L} \} \) (i.e. moving in steps of \( L \)). For simplicity we assume \( L > Y \). We apply the POVMs in decreasing order (i.e starting with the one furthest to the right).

If all the projectors in the cluster are to the right of the boundary value \( k \), then whichever one is chosen it is sure to project into its positive subspace, since all the qubits to the right of that position are certainly zeroes. Thus no disturbance is caused, and an outcome of “1” is guaranteed. Given this outcome, we move to the next lower POVM (i.e decrease \( n \) by one), and measure again. However, these measurements only provide upper bounds on the value of \( k \). To obtain a lower bound on \( k \) as well, we must make a measurement in which some projectors in the cluster lie to the left of the boundary, in which case some disturbance is inevitably caused.

The expected disturbance depends on \( k \) only through the value of \( k \) (mod \( L \)). It is clear that, since the action of projectors to the right of the boundary is entirely deterministic and non-disruptive, we would observe just the same success rate if the value of \( k \) were decreased
or increased by \( L \) - there would simply be one more or one less POVM applied to the right. We can therefore subsequently ignore the value of \( k \) itself, and consider only \( k \) (mod \( L \)), which we denote \( K \). In principle we must consider each case individually, from \( K = 0 \) through to \( K = L - 1 \), but we find that they divide up into two classes.

### 5.1 Disturbance Bounds

**Fidelity when \( K \geq Y \)**

When \( K \geq Y \), we know that all the projectors in the POVM lie to the left of the boundary. The fidelity in this case is therefore an average over the behaviour of each of these projectors. The separation \( s \) ranges from \( K \), when the leftmost projector is chosen, to \( K - (Y - 1) \), when the rightmost projector is chosen. The average is therefore:

\[
F(K \geq Y) = \frac{1}{Y} \sum_{i=0}^{Y-1} \left(1 - 2^{-i(K-i)}\right)^2
\]

It can easily be seen that the above argument depends only on the fact that all projectors in the POVM are to the left of the boundary, and therefore holds for any value of \( K \geq Y \). Expanding the square and neglecting the small squared term we get a simple lower bound

\[
F(K \geq Y) \geq 1 - \frac{2}{Y}
\]

in this case.

**Fidelity when \( K < Y \)**

In those cases where \( K < Y \), the above argument does not go through. The boundary now lies within the “cluster”. Some of the projectors lie to the left of the boundary, but there are others on the boundary and to the right. The projectors to the left can be treated in the same way as above: each has probability \( \frac{1}{Y} \) of being chosen; \( s \) ranges from \( K \) (for the leftmost projector) to 1 (for the projector immediately to the left of the boundary). This gives the first term in the formula below.

When a projector to the right of the boundary is chosen, it will give the outcome “1” with certainty, and we will move to the next lowest POVM, whose projectors lie around \( L \) places further to the left. All these projectors are therefore to the left of the boundary (since \( L > Y \)). The fidelity contributed in this case is the average over that cluster, which is given in the same way as in the \( K \geq Y \) case, but now with \( s \) running from \( K + L \) to \( K + L - (Y - 1) \). Finally, there are \( Y - K \) projectors to the right of the boundary, each with probability \( \frac{1}{Y} \) of being chosen, which gives the weight on the second term.

Putting this all together gives us:

\[
F(K < Y) = \frac{1}{Y} \sum_{s=1}^{K} \left(1 - 2^{-s}\right)^2
+ \left(\frac{Y - K}{Y}\right) \frac{1}{Y} \sum_{i=0}^{Y-1} \left(1 - 2^{-(K+L-i)}\right)^2
\]

\[
11
\]
As in the previous case we expand the squares and neglect the small square terms:

\[
F(K < Y) \geq \frac{1}{Y} \left( \sum_{i=1}^{K} 1 \right) - \frac{1}{Y} \left( \sum_{i=1}^{K} 2^{1-s} \right) + \\
\left( \frac{Y - K}{Y^2} \right) \left( \sum_{i=0}^{Y-1} 1 \right) - \left( \frac{Y - K}{Y^2} \right) \left( \sum_{i=0}^{Y-1} 2^{1-(K+i)} \right)
\]

\[
= K \frac{Y}{Y} - 2 \frac{1}{Y} (1 - 2^{-K}) + \left( \frac{Y - K}{Y} \right) \left( 2^{1-(K+L)} \right) (2Y - 1)
\]

Re-arranging gives:

\[
F(K < Y) \geq 1 - \left( 1 - 2^{-K} \right) - \left( \frac{Y - K}{Y} \right) \left( \frac{2^{Y} - 1}{2^{K+L}} \right)
\]

and so

\[
F(K < Y) \geq 1 - \frac{2}{Y}
\]

Thus the same lower bound on fidelity applies for all possible values of \(K\), and can be considered a worst case value for fidelity.

Therefore, if we choose \(Y\) large enough (recalling that \(n\) can be arbitrarily large), we can always guarantee that the fidelity of the truncation process is greater than \(1 - \epsilon\) for any given \(\epsilon > 0\).

**Error probability when \(K \geq Y\)**

We saw above that the probability of a projector \(\Pi_l\) at position \(l = k - s\) projecting into its positive subspace, and thus setting \(s\) maximally mixed qubits to zero, is \(p_e(s) = 2^{-s}\). We can therefore write the expected error probability (averaging over choice of projector) for any value of \(K\).

When \(K \geq Y\), and the boundary lies outside the cluster, we simply average \(p_e(s)\) over the projectors in the cluster:

\[
P_e(K \geq Y) = \frac{1}{Y} \sum_{i=0}^{Y-1} 2^{-(K+i)} = \frac{1}{Y} \left( \frac{2^Y - 1}{2^K} \right)
\]

Therefore we have an upper bound on the error probability in this case:

\[
P_e(K \geq Y) \leq \frac{1}{Y}
\]

**Error probability when \(K < Y\)**

When the boundary lies within the cluster, an argument similar to that used for fidelity applies. The projectors in the cluster which lie to the left of the boundary each have probability \(\frac{1}{Y}\) of being chosen, so we average over their contributions. Additionally, we
have a $\frac{Y-K}{Y}$ probability of choosing one of the projectors which lie to the right of the boundary; if this happens, we move to the next POVM to the left, and average over the error probabilities for the projectors in that cluster. This gives us:

$$P_e(K < Y) = \frac{1}{Y} \sum_{s=1}^{K} p_e(s) + \left(\frac{Y - K}{Y}\right) \frac{1}{Y} \sum_{i=0}^{Y-1} p_e(K + L - i)$$

(18)

Now expanding the sums and re-arranging:

$$= \frac{1}{Y} \sum_{s=1}^{K} 2^{-s} + \left(\frac{Y - K}{Y^2}\right) 2^{-K-L} \sum_{i=0}^{Y-1} 2^i$$

$$= \frac{1}{Y} (1 - 2^{-K}) + \left(\frac{Y - K}{Y^2}\right) \frac{2^Y - 1}{2^{K+L}}$$

$$= \frac{1}{Y} \left[ 1 - 2^{-K} + 2^{-K} \left(\frac{Y - K}{Y}\right) \left(\frac{2^Y - 1}{2^{L+K}}\right) \right]$$

so

$$P_e(K < Y) \leq \frac{1}{Y}$$

(19)

and we have a worst case error probability for all possible values of $K$. If we can make $Y$ large enough, the probability that we will erroneously project a qubit to $|0\rangle$ can be made smaller than any given $\epsilon > 0$.

**Uncertainty in $k$**

We now determine how much information we can expect to obtain from this procedure. We continue to apply POVMs until a measurement gives the “0” outcome, saying that we have found a subsequence that is not all zeroes. When this result is obtained, (and assuming that we have not made an error), we know the range of values that $k$ could have.

Given a “0” outcome from some POVM, the boundary cannot lie to the left of this POVM (or else it would certainly have given a “1” outcome instead). Also, it can lie no further to the right than the rightmost edge of the previous POVM - for if the boundary were there, the previous POVM would already have given a “0” outcome (or else made an error).

The position of $k$ is therefore constrained to the $L+Y$ positions between the outermost edges of the last two POVMs. That is as much information as we can obtain.

The constraints of high fidelity and low error probability require us to make $Y$ large. However, the only constraint on $L$ is the initial assumption that $L > Y$ (and it is easy to see that removing this constraint does not reduce the uncertainty). We therefore pick $L = Y + 1$, and so have a final uncertainty in $k$ of the order of $2Y$. Thus given any fixed $\epsilon > 0$ we can choose $Y = O(\frac{1}{\epsilon})$ (independent of $n$) and hence for all suitably large $n$ we can learn the value of $k/n$ with an uncertainty of $2Y/n$ (which tends to 0 as $n \to \infty$) while maintaining a fidelity of $1 - O(\epsilon)$ and error probability $O(\epsilon)$.
5.2 Weakening the Assumptions

In the simplification we assumed that each qubit in the first section was maximally mixed, and every qubit in the second section was in state $|0\rangle$; in reality, this will not be the case. These assumptions were not used directly in the argument, but only to put bounds on the behaviour of the fidelities and projector probabilities. We can therefore weaken the assumptions to something more realistic, whilst preserving the inequalities derived.

We assume that there exists a number $k$ such that:

1. A projector $\Pi_{k+s}$ to the right of that position has a probability to project into its perpendicular subspace that decreases exponentially with $s$; the drop in fidelity when this happens decreases exponentially with $s$.

2. A projector $\Pi_{k-s}$ to the left of that position has a probability to project into its positive subspace that decreases exponentially with $s$; the drop in fidelity when this happens decreases exponentially with $s$.

These assumptions effectively say that the components with lengths significantly greater or less than $R_c$ have exponentially small probability, which we would expect to be a property of classical i.i.d. sources (and even some non-i.i.d. sources) condensed by a classical universal scheme such as the Lempel–Ziv scheme, and hence of our condensation scheme too. With these weaker assumptions in place, the derivations of the bounds on fidelity and error probability can be repeated.

Thus the procedure defined in this section allows us to truncate the condensed quantum sequence, removing virtually all the “blank” qubits (except for a small constant number independent of the sequence length $n$), whilst leaving the fidelity bounded toward unity. The procedure does not depend upon any prior knowledge of the source, and can be applied to any sequence for which one can make the above assumptions — in particular, those sequences which are produced as output from our condensation algorithm.

6 Learning the Eigenbasis

Combining the Condensation and Truncation procedures described above, we see that, with knowledge of the eigenbasis of the source density matrix, but no information about its eigenvalues, optimal compression to $S(\rho)$ qubits per signal can be attained asymptotically. For large enough $n$ the overhead in the achieved rate, $\frac{Y}{n}$ can be made smaller than any prescribed $\delta > 0$.

However, without knowing the source eigenbasis we can only compress asymptotically to a rate $R_c$ defined by Eq. (10). Although we do not know the eigenbasis in advance (by assumption), we now demonstrate that the above procedure can be used iteratively, to learn the eigenbasis of the source with arbitrarily small disturbance and hence achieve a fully universal compression scheme.

Since we can compress the original sequence while maintaining arbitrarily high fidelity and arbitrarily low probability of error, we can repeat the process with a different computational basis. We can continue iterating the compression-decompression process as often
as we choose; on each iteration the error probability and fidelity become worse, but still bounded toward zero and unity respectively with bounds determined by the value of $Y$ and the number of iterations. For any given number of iterations we can always choose $Y$ large enough to ensure that we meet any previously specified bounds.

More precisely, for any given $\delta > 0$ we may compress the source to $S(\rho) + \delta$ qubits per signal as follows (and we simultaneously obtain an estimate of the eigenbasis). Let $d$ be the dimension of the single signal space. The unitary group $U(d)$ viewed as a subset of $C^d$ inherits the standard euclidean distance $|U_1 - U_2|$ and it acts transitively on the set of all orthonormal bases in the signal space. For any density matrix $\rho$ and $U \in U(d)$ let $H(U, \rho)$ be the condensation rate of Eq. (10) when the mismatch between the computational basis and eigenbasis is given by $U$. Thus $H(U, \rho)$ is the Shannon entropy of the probability distribution $\mu_j$ given by the doubly stochastic transform of the eigenvalues $\lambda_i$ of $\rho$:

$$\mu_j = \sum_i |U_{ij}|^2 \lambda_i$$

and $H(I, \rho) = S(\rho)$ for any $\rho$.

Now for each given $\rho$ and $\delta$ let $\Upsilon(\rho, \delta)$ be the largest real number such that

$$|U - I| < \Upsilon(\rho, \delta) \quad \text{implies} \quad |S(\rho) - H(U, \rho)| < \delta.$$ 

Clearly from the definition, $\Upsilon(\rho, \delta) > 0$ for all $\rho$ and $\delta > 0$, and if we allow $\rho$ to vary over all density matrices then the quantity

$$\Upsilon(\delta) = \min_{\rho} \Upsilon(\rho, \delta)$$

will be strictly positive.

Next note that the group $U(d)$ is compact so there exists a finite mesh $M(\delta)$ of points $\{V_i\}$ in $U(d)$ with the property that for every $U \in U(d)$ there is a $V_i$ with $|U - V_i| < \delta$ (and hence $|U - V_i| < \Upsilon(\rho, \delta)$ for any $\rho$). Let $\{B_i\}$ be the corresponding set of bases obtained by applying the transformations of $M(\delta)$ to the computational basis. Now given any source (with unknown density matrix $\rho$), the above definitions will guarantee that the condensation rate $H(V_i, \rho)$ (i.e. obtained by using $B_i$ as the computational basis) will have the property that

$$|S(\rho) - H(V_i, \rho)| < \delta$$

for at least one $i = i_0$.

i.e. $H(V_{i_0}, \rho) < S(\rho) + \delta$.

Thus by iteratively compressing and decompressing sequentially relative to the finite list of bases $B_i$ we can choose the one giving the smallest condensation rate and hence compress the source to $S(\rho) + \delta$ qubits per signal. We also learn the identity of the minimal basis which then provides an estimate of the eigenbasis. Depending on the (fixed, finite) size of the set $M(\delta)$ we can choose the fidelity and error bounds sufficiently small (i.e. $Y$ and $n$ large enough) in each iteration to meet any prescribed bounds for the total process.

7 Concluding remarks

The argument presented here demonstrates that a pure quantum i.i.d. source can be optimally compressed (i.e. compressed asymptotically to its von Neumann entropy) with no
prior knowledge of the structure of the source. The possibility of such universal quantum compression has also been recently demonstrated by Hayashi and Matsumoto [7] but there are significant differences in the two approaches. We introduced a diagonalisation procedure that enables any classical algorithm to be utilised in a quantum context. For any classical compression algorithm this gives a simple method of determining the effect of the resulting quantum algorithm on sources that are not diagonal in the computational basis. This leads to a simplified method of estimating the entropy of an i.i.d. source in terms of an associated length parameter, while maintaining high fidelity for sufficiently long blocks of signals. As a by-product, we also estimate the eigenbasis of the source.

Our diagonalisation procedure may be applied to any classical algorithm and it may be interesting to explore its applicability in other cases (in addition to the Lempel–Ziv algorithm) and even beyond issues of information compression. The (not universal) quantum compression schemes of Schumacher [3] and Schumacher and Westmoreland [6] may be viewed as similar translations of classical algorithms, but in those cases knowledge of the source makes it unnecessary to invoke the diagonalisation analysis.

Our universal quantum compression scheme was based on a quantum implementation of the classical Lempel–Ziv algorithm. This classical algorithm is known to be applicable to a target sequence, such as literary text, that is not produced by an i.i.d. source. (Indeed the algorithm is widely used in practice for compressing computer files). Hence unlike any previously proposed quantum compression scheme, our scheme is also applicable to such more general target sequences of quantum states and it would be interesting to explore its performance for various kinds of non-i.i.d. sources.

Intuitively, the classical Lempel–Ziv algorithm operates by building up and continually improving a model of the source, based on increasing numbers of already received signals, which may then be used to reduce the resources needed in further transmissions. Our quantum translation merely mimicks this procedure in the computational basis and so it would be especially interesting to investigate whether a “truly quantum” extension of the Lempel–Ziv idea exists, in the context of quantum information, that is not tied to a particular basis.

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