Measurement-device-independent quantification of irreducible high-dimensional entanglement

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(Dated: April 9, 2019)

The certification and quantification of entanglement are of great importance in characterizing entangled systems. Recently it is pointed out that the quantum correlation of higher dimensional bipartite states can be simulated with a sequence of lower dimensional states and sequential measurements. Such scheme may render some entanglement tests unreliable—the observed entanglement may not be a genuine high-dimensional one. Here we show that a recently proposed quantitative measurement-device-independent entanglement witness (QMDIEW) protocol is naturally robust against such scheme, thus can be used as an irreducible high-dimensional entanglement quantifier and irreducible dimension witness. We then experimentally demonstrate this protocol on a 3-dimensional bipartite state, observing an entanglement that exceeds the bound when the cheating scheme with arbitrary 2-dimensional states is used. The result certifies our system is entangled in dimension (at least) \( d = 3 \).

Introduction.— One of the core problems of quantum theory lies in the certification and quantification of quantum entanglement, which is the key resource in various quantum information processes [1–3]. Great efforts have been made to develop quantum entanglement, see Ref. [4, 5] for a review. Imperfect measurement schemes, however, may lead to incorrect estimation of the entanglement under investigation, thus invalidate these methods in practical scenarios [6].

Such imperfect entanglement detection tasks can be described as a three-party game, in which Charlie, a referee, aims to certify the shared entanglement between two parties, Alice and Bob. However, neither of Alice and Bob is reliable, they may be unable to operate on their share state as Charlie required, or may even deliberately adopt cheating schemes to declare more entanglement than they actually possess. Therefore Charlie must find methods to avoid these to happen. A solution comes from the field of device-independent (DI) quantum tests [7–9], where Alice’s and Bob’s experiment setups are treated as a black box. Charlie tells the black box what measurements to perform by giving them classical labels, and the black box returns the measurement outputs. In a DI entanglement detection game, Charlie does not need to know what measurements are performed, but only need to check the correlations on the inputs and outputs using some Bell-like inequalities.

However, Alice and Bob are still possible to cheat in a DI entanglement detection game. One of their schemes is based on local operation and classical communication (LOCC), which leads to the locality loophole in Bell tests [10]. Another one lies in the fact that, as pointed out in a recent work, quantum correlations of high-dimensional states may be simulated by sequential measurements on a sequence of lower-dimensional states with the assistance of classical feedforward [11]. This makes it possible to attain a quantity of entanglement that can only be obtained in a higher-dimensional system with lower-dimensional states in a Bell-type test. We call such scheme “simulation with sequential lower-dimensional states” (SSLDS), and say Charlie can detect irreducible entanglement if the test he uses is robust against SSLDS. A common reason that these schemes work is Alice and Bob can get exact information of the black box’s inputs. For example, the feedforward process in SSLDS scheme requires the knowledge of “what measurement is going to be simulated” to decide the choice of the measurement sequence.

A potential way to solve such problems, called measurement device independent (MDI) quantum test [12], was introduced based on semiquantum scenarios [13]. Instead of using classical inputs, Charlie sends Alice and Bob quantum states from a nonorthogonal set. With a compromise of trusting these quantum input states, Charlie prevents Alice and Bob from distinguishing what the inputs are. Such MDI tests have been proved to be robust against LOCC. One may ask intuitively, can we find a MDI test that is also robust to SSLDS scheme? Or alternatively, can we construct a MDI game where Charlie can make sure that the entanglement he observes is irreducible?

Here we show that a recently proposed MDI protocol, called quantitative measurement device independent entanglement witness (QMDIEW) in Ref. [14, 15] is intrinsically

![FIG. 1. In a DI entanglement detection test, both the quantum state under detected and the measurement setup are viewed as a black box with classical inputs (labeled as x, y) and classical outputs (labeled as a, b). In a MDI scenario, the inputs are replaced with trusted quantum states (\( \tau_x, \tau_y \)) while the outputs are still classical.](image-url)
robust against the SSLDS scheme, thus provides a method to quantify irreducible entanglement in a MDI way. We prove that even if the SSLDS scheme with a sequence of \(m\)-dimensional bipartite states are used, the witnessed entanglement in the QMDIEW will never exceed the upper bound of a \(m\)-dimensional system. This means that the shared system is truly entangled in dimension at least \(m + 1\) when its quantified entanglement exceeds the upper bound of \(m\)-dimensional systems. The “dimension” here is exactly the concept “irreducible dimension” for bipartite systems in Ref. [11]. So, our result shows that the QMDIEW can witness irreducible dimension in a MDI scenario. We then experimentally demonstrate the protocol on a two-qutrit system and observe a lower bound of its generalized robustness (GR) [16] that exceeds the value of arbitrary 2-dimensional systems. Our experiment shows the existence of irreducible high-dimensional entanglement for the first time, marking an important step beyond previous experimental demonstration of MDIEW on qubit systems [17–19]. Also, we demonstrate the noise-resistant feature of the protocol by certifying the entanglement of 3-dimensional Bell states in noisy environment.

Framework of QMDIEW.— We briefly introduce the construction of the QMDIEW, defined as a MDIEW whose expectation on a state provides a lower bound on its entanglement. In a semiquantum scenario, Alice and Bob share a bipartite state \(\rho_{AB}\), with their measurement devices are treated as black boxes. The black-box-like devices receive pairs of quantum states \(\tau_x\) (for Alice’s), \(\tau_y\) (for Bob’s) from Charlie who prepares the input states chosen randomly from a given set \(\{\tau_1, \ldots, \tau_n\}\). To certify the entanglement of the shared state, Alice (Bob) performs a joint measurement (described by the POVM \(\{A_a\}\) or \(\{B_b\}\)) on her (his) own system and input states. Correlations in this case write

\[
P(a, b|\tau_x, \tau_y) = Tr[A_a \otimes B_b(\tau_x \otimes \tau_y)] ,
\]

(1)

where, \(a, b\) is the outcome of \(A_a\) \((B_b)\). The system above can also be described with an effective POVM \(\{\Pi_{ab}\}\) acting on input states \(\tau_x \otimes \tau_y\). In this case, correlations in Eq. (1) can be generally expressed as

\[
P(a, b|\tau_x, \tau_y) = Tr[\Pi_{ab}(\tau_x \otimes \tau_y)].
\]

(2)

It is proved that the POVM \(\{\Pi_{ab}\}\) can be used to recover the ensemble which can be extracted from the setup and whose entanglement provides a lower bound on the entanglement of the shared state \(\rho_{AB}\) [15]. That is, the entanglement of \(\rho_{AB}\) under an arbitrary convex entanglement \(E\) [20] is lower bounded by applying the measure \(E\) on the set of operators \(\{\Pi_{ab}\}\); formulated as

\[
Q = \frac{1}{d_X d_Y} \sum_{ab} \mathbb{E}(\Pi_{ab}) \leq E(\rho_{AB}) ,
\]

(3)

where \(d_X\) (\(d_Y\)) is the dimension of the Hilbert space \(\mathcal{H}_X\) (\(\mathcal{H}_Y\)) of Alice’s (Bob’s) input state and \(Q\) is the lower bound obtained in this method. The entire process from raw correlations in Eq. (2) to a quantified entanglement \(Q\) (a lower bound) of \(\rho_{AB}\) can be described as a semidefinite programming (SDP [21]) and a QMDIEW can be recovered from its dual program.

Robustness of QMDIEW against SSLDS.— By adopting SSLDS scheme, Alice and Bob may convince Charlie that they have a higher entanglement which they do not actually possess (see SM [22] for more details). In Ref. [11] an additional test to exclude such scheme was given, but only for the two-qubit case. Now we show that QMDIEW is naturally robust against SSLDS scheme. Suppose Alice and Bob want to confirm Charlie that they have a bipartite state entangled in dimension \(m\) in a QMDIEW game, however, they only have a source that can produce a sequence of bipartite states entangled in dimension \(m\) \((m < n)\) one after one, which allows them to make measurements sequentially. Under such restriction, the operations of Alice and Bob can summed up as two steps: sequential measurements step and decision step. After receiving the quantum input \(\tau_x\), Alice begins the sequential measurement step (Bob being analogy). She makes a joint measurement on \(\tau_x\) and her part of the first \(m\)-dimensional bipartite state, obtaining an outcome \(a_1\), then she makes joint measurement on the post-measurement input system and her second state, obtaining an outcome \(a_2\), after a sequence of \(k\) measurements, Alice obtains a sequence of outcomes \(S_a = (a_1, a_2, \ldots, a_k)\) (Bob obtains a sequence \(S_b\) analogously). Then in the decision step, Alice and Bob decide what outcome \(a, b\) they would report to Charlie according to some distribution \(p(a, b|S_a, S_b)\) \((\sum_{a, b} p(a, b|S_a, S_b) = 1, \forall S_a, S_b)\). Let \(P_{S_a, S_b|\tau_x, \tau_y}\) be the correlation of their sequential measurement process, then the the correlation \(P_{SSLDS}^m(a, b|\tau_x, \tau_y)\) Charlie actually get is

\[
P_{SSLDS}^m(a, b|\tau_x, \tau_y) = \sum_{S_a, S_b} p(a, b|S_a, S_b) \cdot P_{S_a, S_b|\tau_x, \tau_y} .
\]

(4)

However, we demonstrate that in such case, the entanglement Charlie finally detected in QMDIEW can not exceed \(E^m\), the upper bound of \(m\)-dimensional states. The demonstration includes two theorems, whose proofs are left in supplementary [22].

Theorem 1. In a SSLDS scheme where a sequence of \(m\)-dimensional bipartite states and sequential measurements are used to simulate the correlation of some \(n\)-dimensional bipartite state, correlation \(P_{S_a, S_b|\tau_x, \tau_y}\) satisfies

\[
P_{S_a, S_b|\tau_x, \tau_y} = Tr[(A^m_{S_a} \otimes B^m_{S_b})(\tau_x \otimes \rho_{AB}^{m} \otimes \tau_y)] ,
\]

(5)

where \(\rho_{AB}^m\) is some quantum state in \(\mathcal{H}_A^m \otimes \mathcal{H}_B^m\), \(A^m_{S_a}\) and \(B^m_{S_b}\) are effective measurement operators acting on spaces \(\mathcal{H}_A^m\) and \(\mathcal{H}_B^m\), satisfying \(E(A^m_{S_a}) \leq Tr(A^m_{S_a} \otimes E^m)\), \(E(B^m_{S_b}) \leq Tr(B^m_{S_b} \otimes E^m)\).

Theorem 1 indicates that in a SSLDS scheme, the effective measurement operators Alice and Bob can simulate is strongly restricted. Let \(Q_{SSLDS}^m\) be the witnessed entanglement by Charlie, which is reconstructed by \(P_{S_a, S_b|\tau_x, \tau_y}\) using Eqs. (2), (7) and (4), then we have theorem 2.
 FIG. 2. Experimental setup. The setup contains 3 modules: the Source module, the Trust module, and the BSM module. In the Source module, 3-dimensional maximally entangled state $|\phi\rangle = \frac{1}{\sqrt{3}} (|HH\rangle_u + |HH\rangle_l + |VV\rangle_l)$ (the subscripts $u$ and $l$ denote upper and lower path respectively) and maximal mixed state $I/9$ are prepared respectively and the two states are then combined with different proportions at the beam splitters. The photon pair rate was around 2500 counts/s with a detection efficiency of 0.222. In the Trust module, arbitrary three dimensional pure state can be prepared by expanding the distributed photons’ path degree of freedom. High quality of the input state is ensured by highly controllable operations HWPS and BDs. In the BSM module, projector onto arbitrary one of the nine 3-dimensional Bell state can be constructed by setting angles of HWPs and blocking proper paths. BD-beam displacer, PBS-polarizing beam splitter, HWPS-half wave plate, QWP-quarter wave plate.

**Theorem 2.** In a QMDIEW game, when the quantum input set $\{\tau_x, \tau_y\}$ is tomographically complete, there is

$$Q^m_{SSLDS} \leq E^m.$$  

(6)

Theorem 2 gives an important result that the entanglement $Q^m_{SSLDS}$ Alice and Bob can simulate with an $m$-dimensional SSLDS scheme can not exceed the bound of $E^m$ of $m$-dimensional bipartite states. Therefore, if the entanglement Charlie obtains exceeds the bound of $m$-dimensional bipartite states, he can be confirmed that Alice and Bob own bipartite states of dimension at least $m + 1$. Therefore we proved that Charlie can exclude the effect of SSLDS without any additional tests, making it attractive to demonstrate irreducible high-dimensional entanglement in the lab.

**Experimental protocol.**— We consider the experimental observation of irreducible high-dimensional entanglement with a simplest system composed of a two-qutrit state. To simulate 3-dimensional entanglement states, Alice and Bob should adopt SSLDS scheme of two-qubit states. In the QMDIEW game, however, Charlie can identify the system is truly entangled in dimension 3 as long as he observes its entanglement exceeding the upper bound of two-qubit systems. Here, the entanglement measure is chosen to be GR, whose value is upper bounded by $n - 1$ for bipartite n-dimensional systems [16].

In our experiment, the state is chosen to be one of the maximally 3-dimensional entangled states, i.e. $\rho_{AB} = |\phi\rangle\langle\phi|$, with $|\phi\rangle = \frac{1}{\sqrt{3}} (|00\rangle + |11\rangle + |22\rangle)$. After the distribution of $\rho_{AB}$, Charlie prepares input states $\tau_x$ and $\tau_y$ randomly selected from a set $S = \{|0\rangle, |1\rangle, |0\rangle + |1\rangle, |0\rangle + i|1\rangle, |0\rangle + |2\rangle, |1\rangle + |2\rangle\}$ [23] and sends them to Alice and Bob. Then Alice and Bob choose to perform POVM $\{A_a\}$ and $\{B_b\}$ consisting of the projectors onto all nine Bell states on the distributed qutrit and input state. The obtained correlations $P(ab|\tau_x, \tau_y)$ are reported to Charlie who then performs a regularization process to find out the closest regularized distribution $P_r(ab|\tau_x, \tau_y)$ and thus corrects the inconsistencies of raw $P(ab|\tau_x, \tau_y)$ caused by noise and finite statistics [17, 22]. Based on $P_r(ab|\tau_x, \tau_y)$, Charlie can finally calculate the GR of $\rho_{AB}$ and recover a QMDIEW. With the quantified GR of $\rho_{AB}$, Charlie can also determine a lower bound on its irreducible dimension of entanglement according to Theorem 2.

As a byproduct, we also apply the QMDIEW to several noisy states $\rho_{\text{noise}} = \rho_{AB} + \frac{1-p}{2} I$ $(p \in (0, 1))$ with the same input set $S$, but only collect 3 outcomes of BSM each side [24]. Although incomplete data collection decreases the quantity of entanglement Charlie can observed, this will not invalidate its efficiency to determine if the state under tested is entangled [17–19]. We thus demonstrate the availability of the protocol under noisy environment, informationally incomplete inputs and measurement outcomes.

We realize this procedure using linear optics. The whole experimental setup is illustrated in Fig. 2 and can be divided into three modules: the Source module for state preparation (orange), the Trust module for trusted inputs preparation (blue) and the BSM for three dimensional Bell state measurement.
In this paper, we displayed the power of a recently reported QMDIEW protocol for high-dimensional systems by showing that it can witness irreducible entanglement, which can not be simulated by lower-dimensional systems. Our results shows that QMDIEW not only can be used to quantify how much entanglement there is, but also can help to determine the genuine dimension that the entanglement exists in. As the simplest application, we demonstrated the high-dimensional QMDIEW protocol with a 3-level bipartite entangled state and quantified a lower bound of its GR that exceeds the bound of 2-dimensional systems, presenting the existence of irreducible 3-dimensional entanglement.

Our work here may provide inspiration for some future works. From a theoretical point of view, further exploration into the robustness of MDI tests is possible. For instance, we may consider a more general sequential scheme that simulates a higher entanglement by a sequence lower entangled states, and discuss the robustness of MDI tests against it. From a practical point of view, our experiment leads to important applications based on high dimensional entangled systems, such as randomness generation [26] and high dimen-

Results analysis.— We calculate the quantity of GR of d-dimensional Bell states in a noisy environment, i.e. $\rho_{\text{noise}}$, where only white noise is considered for simplicity. Fig. 3 shows the results of entanglement quantification, where GR as an entanglement measure varies with different Bell state fraction $p$. The results show that with same fraction $p$, 3-dimensional states (red line) are more robust than 2-dimensional ones (blue line). When $p = 1$, the values of GR reach maximums for both 3- and 2-dimensional cases and are 2 and 1 respectively. The black dot is the experimental result of the state $\rho_{AB}$ and the obtained value of GR is $1.632 \pm 0.017$ which exceeds the bound of arbitrary 2-dimensional system by more than 37 standard deviations. The deviation of the GR value from its theoretical predictions are due to imperfect state preparation and measurement error. Our experimental result of GR requires a fidelity of our state to ideal $\rho_{AB}$ of approximate $f = 0.985$. The experimental fidelity $f_{\text{real}} = 0.986 \pm 0.002$ matches well with its theoretical expectation [25].

Meanwhile, Fig. 3 indicates that GR of the two-qutrit state $\rho_{\text{noise}}$ with $p > 5/8$ exceeds the upper bound of two-qubit systems, implying our scheme can be used to detect the irreducible entanglement for bipartite states. In our case, the observed GR confirms that our system is genuinely entangled at least in dimension $d=3$, which means it cannot be simulated with qubit systems through SSLDS scheme. In other words, we demonstrate an irreducible 3-dimensional state with out characterizations on the measurement apparatus.

The results of the QMDIEW of 3-dimensional isotropic states $\rho_{\text{noise}}$ are shown in Fig. 4, where the witness value varies as a function of the parameter $p$. The blue line and black dots represent the theory expectation and experimental results respectively and they match very well. When $p < 1/4$, the witness value is zero, which means the states are separable. Our experiment can certify almost all entangled $\rho_{\text{noise}}$. Theoretically the entanglement of state $\rho_{\text{noise}}$ should vary linearly with $p$, the nonlinearity of our results stems from incomplete information we obtained using tomographically incomplete input set $S$ and incomplete BSM. If the input set is tomographically complete, the result would be linear as the red line in Fig. 4. If complete BSM being implemented, the result would agree with the one in Fig. 3. The error bars of the parameter $p$ are due to the inaccuracy of the ratio of $\rho_{AB}$ to $I/9$ in the state $\rho_{\text{noise}}$. All of the error bars are estimated by Monte Carlo simulation.

Conclusion.— In this paper, we displayed the power of a recently reported QMDIEW protocol for high-dimensional systems by showing that it can witness irreducible entanglement, which can not be simulated by lower-dimensional systems. Our results shows that QMDIEW not only can be used to quantify how much entanglement there is, but also can help to determine the genuine dimension that the entanglement exists in. As the simplest application, we demonstrated the high-dimensional QMDIEW protocol with a 3-level bipartite entangled state and quantified a lower bound of its GR that exceeded the bound of 2-dimensional systems, presenting the existence of irreducible 3-dimensional entanglement.

FIG. 3. GR varies as a function of the weights $p$. The red and blue lines are calculated theoretical results for 3- and 2-dimensional isotropic states. For three dimensional case, GR rises linearly from 0 to 2 when $p$ varies from 1/4 to 1, while it rises from 0 to 1 with $p$ varies from 1/3 to 1 for 2-dimensional case. The black dot is our experimental result which is $1.632 \pm 0.017$. Error bar is simulated by Monte Carlo algorithm. The different colour of background indicates the irreducible dimension $d$ of the system is at least. Red means $d = 2$, green means $d = 3$, blue means $d = 4$.

FIG. 4. Witness values for two-qutrit isotropic states vary with different parameter $p$. The blue line and black dots represent the theoretical and experimental results. The red line shows the result when a tomographically complete input set is used. The error bars of horizontal axis are due to inaccuracy of the ratio of $\rho_{AB}$ to $I$ in the state $\rho_{\text{noise}}$ and the error bars of the vertical axis are calculated using Monte Carlo simulation and a Poissonian noise on the detection rate.
sional quantum key distribution [27], without assumptions on the measurement apparatus.

We thank D. Rosset and X.-H Li for valuable discussions. This work was supported by the National Key Research and Development Program of China (No. 2017YFA0304100, No. 2016YFA0301300 and No. 2016YFA0301700), NSFC (Nos. 11774335, 61327901, 11874345, 11504253), the Key Research Program of Frontier Sciences, CAS (No. QYZDY-SSW-SLH003), the Fundamental Research Funds for the Central Universities, and Anhui Initiative in Quantum Information Technologies (Nos. AHY020100, AHY060300, AHY080000).

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**SUPPLEMENTARY MATERIAL**

**Supplementary Note 1: SDP for entanglement quantification**

The semidefinite programming (SDP) [21] we use for entanglement quantification is as follow:

\[
\begin{align*}
\text{minimize} & \quad Q = \frac{1}{d_X d_Y} \sum_{a,b} \mathcal{E}(\Pi_{ab}), \\
\text{over} & \quad \Pi_{ab} \in \text{Herm}_+ \mathcal{H}_X \otimes \mathcal{H}_Y, \forall a,b, \\
\text{subject to} & \quad \text{tr}(\Pi_{ab} \tau_x \otimes \tau_y) = P(a,b|\tau_x,\tau_y), \forall abxy,
\end{align*}
\]

where \(d_X (d_Y)\) is the dimension of the Hilbert space \(\mathcal{H}_X (\mathcal{H}_Y)\) of Alice’s (Bob’s) input state. The optimized solution \(Q\) obtained in SDP (7) is the lower bound of entanglement under measure \(\mathcal{E}\).

**Supplementary Note 2: data regularization**

Due to the existence of experimental imperfections, the correlations \(P(a,b|\tau_x,\tau_y)\) we obtained through experiment may not be able to run SDP (7) successfully. For example, there may be no valid operators \(\Pi_{ab}\) that can realize \(P(a,b|\tau_x,\tau_y)\) in SDP.
some projectors in $M$ (7). So the correlations $P(a, b|\tau_x, \tau_y)$ must be re-regularized. We find a set of re-regularized correlations $P_r(a, b|\tau_x, \tau_y)$ using another SDP:

$$\begin{align*}
\text{minimize} & \quad D(\hat{P}, \hat{P}_r), \\
\text{over} & \quad \Pi_{ab} \in \text{Herm}_n, H_X \otimes H_Y, \forall ab, \\
\text{subject to} & \quad \text{tr}(\Pi_{ab} \tau_x \otimes \tau_y) = P_r(a, b|\tau_x, \tau_y), \forall abxy,
\end{align*}$$

where $\hat{P}$ and $\hat{P}_r$ are vectors of dimensions $n_A \times n_B \times d_X \times d_Y$, whose components are $P(a, b|\tau_x, \tau_y)$ and $P_r(a, b|\tau_x, \tau_y)$ of different $a, b, \tau_x, \tau_y$. $D(\hat{P}, \hat{P}_r)$ is the Euclidean distance of $\hat{P}$ and $\hat{P}_r$.

**Supplementary Note 3: SSLDS scheme**

In this note we give a more detailed description to the SSLDS scheme, an show that it can be used to simulate higher-dimensional entanglement with bipartite states of lower dimension.

In Ref. [11], it is pointed out that the correlation of some higher dimensional bipartite states in a device-dependent dimensional witness (DIDW) can be simulated with sequential measurements on a sequence of lower-dimensional states. The ququart case is given as an example, that with standard binary encoding on a ququart:

$$\begin{align*}
|0\rangle & \rightarrow |00\rangle, |1\rangle \rightarrow |01\rangle, \\
|2\rangle & \rightarrow |10\rangle, |3\rangle \rightarrow |11\rangle,
\end{align*}$$

a two-ququart maximally entangled state (MES) can be mapped into the product of two two-qubit MESs:

$$|\Psi\rangle \rightarrow |\phi_{A_1B_1}\rangle \otimes |\phi_{A_2B_2}\rangle,$$

where $|\Psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle + |22\rangle + |33\rangle)$, $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Also, some projective measurement operators (projectors) of local higher dimensional system can be encoded into sequential projective measurements on lower dimensional systems, and the choice of the measurements in the sequence depends on what the higher dimensional projector is and the outcomes of previous measurements in the sequence (classical information and classical feedforward). Under such encoding, the correlation $P(a, b|x, y)$ of some higher dimensional bipartite states can be simulated, therefore the higher dimensional violation of some Bell-type inequality can be reached by lower dimensional states. An example is given under one of the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequalities, CGLMP$_4$. It is shown that one can obtain a violation greater than $I_4 = 0.315$, which lower bounds the dimension of the observed system to two-ququarts, by sequential measurements on two two-ququart MESs. From the result we can deduce that, in a DI entanglement witness game, if Alice and Bob has two two-ququart MESs, they can convince Charlie that they have the entanglement of a two-ququart MES. Therefore, the DI quantum tests are not reliable either as a dimensional witness or an entanglement witness. Although a solution is given in Ref. [11] to exclude the existence of SSLDS scheme, but it requires additional tests, and is only given for special case (two-ququart states).

**Supplementary Note 4: proof of theorem 1**

**Proof.** Since the bipartite states entangled in dimension $m$ are upper bounded by $m$-dimensional maximally entangled states, which are all pure states. So we can just consider the case where each bipartite states of Alice and Bob are pure states in Hilbert space of dimension $m \times m$. Let $H_X$ and $H_Y$ denote the $n$ dimensional Hilbert space of the input states, $H_{A_i}$ and $H_{B_i}$ denote the $m$ dimensional Hilbert space of the $i$th pair of states Alice and Bob prepare. In the sequential measurement step, Alice and Bob make sequential joint measurements on the states in $H_{A_i} \otimes H_X$ and $H_{B_i} \otimes H_Y$. Let $M_{X_i}, M_{Y_i}, M_{A_i}, M_{B_i}$ denote the spaces of Hermitian operators acting on $H_X, H_Y, H_{A_i}, H_{B_i}$, respectively, and $M_{X \otimes A_i}$ denote the Hermitian operators acting on $H_X \otimes H_{A_i}$ ($M_{Y \otimes B_i}$ being analogy). In the $i$th round of the sequential measurement, Alice and Bob perform general positive operator valued measures (POVM) $M_{xa_i} \in M_{X \otimes A_i}$, and $M_{yb_i} \in M_{Y \otimes B_i}$. In Ref. [30], it is pointed out that that any POVM can be written in a “fine grained” form where measurement operators are shrunk projective measurement operators (projectors) onto a set of non-renormalized, non-orthogonal pure states. Therefore, without lost of generality, the measurement operator of Alice and Bob can be written as $M_{xa} = u_{xa}, M_{xa},$ and $M_{yb} = u_{yb}, M_{yb},$ where $0 \leq u_{xa}, u_{yb} \leq 1$, $M_{xa}$ and $M_{yb}$ are some projectors in $M_{X \otimes A_i}$ and $M_{Y \otimes B_i}$.

Suppose Alice and Bob use a sequence of length $k$ to simulate correlation $P(a, b|\tau_x, \tau_y)$ under each pair of inputs $(\tau_x, \tau_y)$. We denote the quantum state they have by $\rho_{AB} = \rho_{A_1B_1} \otimes \rho_{A_2B_2} \otimes \cdots \otimes \rho_{A_kB_k}$, the sequences of their outcomes by $S_a = (a_1, a_2, \ldots, a_k)$, $S_b = (b_1, b_2, \ldots, b_k)$. Then, correlation $P_{S_a, S_b|\tau_x, \tau_y}$ can be written as the product of $k$ correlations $P_{a_i, b_i|\tau_x, \tau_y}$, that is

$$P_{S_a, S_b|\tau_x, \tau_y} = \prod_{i=1}^{k} P_{a_i, b_i|\tau_x, \tau_y}.$$
The correlation for the first-round measurements is

\[ P_{a_1b_1|x_{a_1},u_{y_{b_1}}} = u_{x_{a_1}} \cdot u_{y_{b_1}} \cdot \text{Tr}(\mathbb{M}_{xa_1} \otimes I_{A_j} \otimes M_{yb_1} \otimes I_{B_j})(\tau_x \otimes \rho_{AB} \otimes \tau_y), \]

where \( I_{A_j} (I_{B_j}) \) are identities in spaces \( \mathcal{M}_{A_j} (\mathcal{M}_{B_j}) \). We let \( M_{xya,b_1} = \mathbb{M}_{xa_1} \otimes I_{A_j} \otimes M_{yb_1} \otimes I_{B_j} \), and the post-measurement system after the first round is

\[ \rho_{a_1b_1|x_{a_1},u_{y_{b_1}}} = u_{x_{a_1}} \cdot u_{y_{b_1}} \cdot \frac{M_{xya,b_1}(\tau_x \otimes \rho_{AB} \otimes \tau_y)M^\dagger_{xya,b_1}}{P_{a_1b_1|x_{a_1},u_{y_{b_1}}}}. \]

And after the second round the correlation becomes

\[ P_{a_2b_2|x_{a_2},u_{y_{b_2}}} = u_{x_{a_2}} \cdot u_{y_{b_2}} \cdot \text{Tr}(M_{xya,b_2}\rho_{a_1b_1|x_{a_1},u_{y_{b_1}}}). \]

Comparing Eq. (13) and Eq. (14), we have

\[ P_{a_1b_1|x_{a_1},u_{y_{b_1}}} = u_{x_{a_1}} \cdot u_{x_{a_2}} \cdot u_{y_{b_1}} \cdot u_{y_{b_2}} \cdot \text{Tr}(\tilde{M}_{xya_2b_2}\tau_x \otimes \rho_{AB} \otimes \tau_y), \]

where \( \tilde{M}_{xya_2b_2} = M^\dagger_{xya_1b_1}M_{xya_2b_2}M_{xya_1b_1} \) is an effective measurement operator. Analogously, after \( k \) rounds of measurements, the correlation they can produce is

\[ P_{S_n S_{n+1} | \tau_x \tau_y} = \prod_{i=1}^{k} P_{a_1b_1|x_{a_1},u_{y_{b_1}}} = f_{x_{S_n}} \cdot f_{y_{S_{n+1}}} \cdot \text{Tr}(\tau_x \otimes \rho_{AB} \otimes \tau_y \tilde{M}_{xya_{S_n} S_{S_{n+1}}}), \]

where \( \tilde{M}_{xya_{S_n} S_{S_{n+1}}} = \prod_{i=1}^{k-1} M^\dagger_{xya_i b_i}M_{xya_k b_k}(\prod_{j=k-1}^{1} M_{xya_b}, b_j), f_{x_{S_n}} = u_{a_1} \cdot u_{a_2} \cdot \ldots \cdot u_{a_k}, f_{y_{S_{n+1}}} = u_{b_1} \cdot u_{b_2} \cdot \ldots \cdot u_{b_k}, \) and notice that

\[ \sum_{S_n} f_{x_{S_n}} = \sum_{S_{n+1}} f_{y_{S_{n+1}}} = 1. \]

Eq. (16) indicates that the correlation \( P_{S_n S_{n+1} | \tau_x \tau_y} \) can be obtained by some effective measurement operator \( \tilde{M}_{xya_{S_n} S_{S_{n+1}}} \) acting on the state \( \sigma_x \otimes \rho_{AB} \otimes \sigma_y \), and multiplied by some normalizing coefficients \( f_{x_{S_n}} \) and \( f_{y_{S_{n+1}}} \). Now we come to discuss the form of the effective measurement operators. Since every operator \( \tilde{M}_{xya_i b_i} \) is composed of tensor product of projectors \( \mathbb{M}_{xa_i} \), \( \mathbb{M}_{yb_i} \) in \( \mathcal{M}_X \otimes \mathcal{M}_{A_i} \), \( \mathcal{M}_Y \otimes \mathcal{M}_{B_i} \), and identities in other spaces, there is

\[ \tilde{M}_{xya_{S_n} S_{S_{n+1}}} = M^\dagger_{xya_1 b_1}M_{xya_2 b_2}M_{xya_1 b_1} = M_{xya_1 b_1} \pi_{a_1} \pi_{b_1}, \]

where \( \pi_{a_1} \) and \( \pi_{b_1} \) are positive semidefinite operators in \( \mathcal{M}_{A_1} \) and \( \mathcal{M}_{B_1} \), whose form depends on \( \mathbb{M}_{xa_1 b_1} \) and \( \mathbb{M}_{ya_1 b_1} \). Therefore,

\[ \tilde{M}_{xya_{S_n} S_{S_{n+1}}} = M_{xya_1 b_1} \prod_{j=2}^{k} \pi_{a_j} \pi_{b_j} = M_{xa_1} \otimes \prod_{j=2}^{k} \pi_{a_j} \otimes M_{yb_1} \otimes \prod_{j=2}^{k} \pi_{b_j}. \]

Since every \( \pi_{a_j} \) and \( \pi_{b_j} \) can be written as mixture of projectors, measurement \( M_{xa_1} \otimes \prod_{j=2}^{k} \pi_{a_j} \) and \( M_{yb_1} \otimes \prod_{j=2}^{k} \pi_{b_j} \) can be written as combinations of sequential projectors:

\[ M_{xa_1} \otimes \prod_{j=2}^{k} \pi_{a_j} = \sum_{l} c_{l} \tilde{A}_{S_a,l}, \]

\[ M_{yb_1} \otimes \prod_{j=2}^{k} \pi_{b_j} = \sum_{l'} c_{l'} \tilde{B}_{S_b,l'}, \]

where \( \sum_{l} c_{l} = \text{Tr}(M_{xa_1} \otimes \prod_{j=2}^{k} \pi_{a_j}) = 1, \sum_{l'} c_{l'} = 1, \) and

\[ \tilde{A}_{S_a,l} = M_{xa_1} \prod_{j=2}^{k} \sigma_{a_{j,l}}, \]

\[ \tilde{B}_{S_b,l'} = M_{yb_1} \prod_{j=2}^{k} \sigma_{b_{j,l'}}, \]
where \( \sigma_{a_i,.} \), \( \sigma_{b_j,.} \) are projectors on \( \mathcal{M}_{A_j} \) and \( \mathcal{M}_{B_j} \). Since both \( \tilde{A}_{S_{a,t}} \) and \( \tilde{B}_{S_{b,t'}} \) are projectors, they can be written as \( |\phi_{xS_a}| \langle \phi_{xS_a}| \) and \( |\phi_{yS_b}| \langle \phi_{yS_b}| \), where \( |\phi_{xS_a}| \) and \( |\phi_{yS_b}| \) are pure states in spaces \( \mathcal{H}_X \otimes \mathcal{H}_A \) and \( \mathcal{H}_Y \otimes \mathcal{H}_B \).

Now we decode them back into \( n \)-dimensional representation, we let \( \mathcal{H}_A \) and \( \mathcal{H}_B \) be the \( n \) dimensional Hilbert space for Alice and Bob respectively, and let the two pure states \( |\phi_{xS_a}| \in \mathcal{H}_X \otimes \mathcal{H}_A \) and \( |\phi_{yS_b}| \in \mathcal{H}_Y \otimes \mathcal{H}_B \) be the pure states after decoding.

Now we demonstrate that \( |\phi_{xS_a}| \) and \( |\phi_{yS_b}| \) have Schmidt number no more than \( m \), we only discuss \( |\phi_{xS_a}| \), and the results are analogous for \( |\phi_{yS_b}| \). Since \( \mathcal{H}_A \) has dimension \( m \), an arbitrary pure state \( |\psi\rangle \in \mathcal{H}_X \otimes \mathcal{H}_A \) can be written as \( \sum_{i=1}^{m} \alpha_i |\psi_i\rangle \otimes |i\rangle \), where \( |\psi_i\rangle \) are some pure states in space \( \mathcal{H}_X \), \( \{ |i\rangle \} \) is a set of basis states of \( \mathcal{H}_A \), \( \sum_{i=1}^{m} |\alpha_i|^2 = 1 \). Therefore \( |\phi_{xS_a}| \) can be written as \( |\phi_{xS_a}\rangle = (\sum_{i=1}^{m} \alpha_i |\psi_i\rangle \otimes |i\rangle) \otimes |\phi_{a_i}\rangle \), where \( |\phi_{a_i}\rangle = |\phi_{a_i}\rangle \). Let \( |\mu_i\rangle \) denote the pure state \( |\mu_i\rangle = |\phi_{a_i}\rangle \), then we have

\[
|\phi_{xS_a}\rangle = \sum_{i=1}^{m} \alpha_i |\psi_i\rangle \otimes |\mu_i\rangle.
\]

We let states \( |\bar{\mu}_i\rangle \in \mathcal{H}_A \) denote the states \( |\mu_i\rangle \) after decoding, then

\[
|\phi_{xS_a}\rangle = \sum_{i=1}^{m} \alpha_i |\psi_i\rangle \otimes |\bar{\mu}_i\rangle.
\]

Since such binary encoding keeps the inner product unchanged, we have

\[
\langle \bar{\mu}_j | \bar{\mu}_i \rangle = \langle \mu_j | \mu_i \rangle = \delta_{ij},
\]

where \( \delta_{ij} \) is the Kronecker delta. Combining Eqs. (23) and (24) we know that \( \{ |\bar{\mu}_i\rangle \} \) can be seemed as a set of orthogonal basis that generates a subspace of dimension \( m \) of \( \mathcal{H}_A \), thus the Schmidt decomposition of \( |\phi_{xS_a}\rangle \) has at most \( m \) nonvanishing terms, the Schmidt number is at most \( m \).

Consequently, we now have \( \mathcal{E}(\langle \phi_{xS_a}|\phi_{xS_a}\rangle) \leq \mathcal{E}^m \), where \( \mathcal{E}^m \) is the upper entanglement bound of \( m \)-dimensional bipartite states. Since the total effective measurement \( A_{S_a}^m \) of Alice can be written as

\[
A_{S_a}^m = f_{xS_a} \cdot M_{ax_i} \otimes \pi_{a_j} = f_{xS_a} \cdot \sum_{l} c_l |\phi_{xS_a}\rangle \langle \phi_{xS_a}|,
\]

we have

\[
\mathcal{E}(A_{S_a}^m) = \mathcal{E}(f_{xS_a} \cdot \sum_{l} c_l |\phi_{xS_a}\rangle \langle \phi_{xS_a}|) \leq f_{xS_a} \cdot \sum_{l} c_l \cdot \mathcal{E}^m = Tr(A_{S_a}^m) \cdot \mathcal{E}^m.
\]

Analogously we have \( B_{S_b}^m = f_{yS_b} \cdot \sum_{l'} c_{l'} |\phi_{yS_b}\rangle \langle \phi_{yS_b}| \), and \( \mathcal{E}(B_{S_b}^m) \leq Tr(B_{S_b}) \cdot \mathcal{E}^m \). 

**Supplementary Note 5: proof of theorem 2**

First we propose a lemma before we prove theorem 2.

**Lemma S1** Let \( \{ M_l \} \) be a set of \( k \times k \) positive semidefinite matrices satisfying

\[
\max_i \{ \text{rank}(M_l) \} = r \quad (r \leq k),
\]

then any matrix \( A \) obtaining by

\[
A = \sum_{l} a_l M_l,
\]

where \( a_l \geq 0 \), is of rank at least \( r \).

**proof.** The proof of Lemma S1 is straightforward so we just state it short. Let \( M_0 \) be the matrix of rank \( r \) in \( \{ M_l \} \), if we represent \( M_0 \) in the eigenbasis of \( A \), there must be at least \( r \) positive diagonal elements. Since \( a_l \geq 0 \), and that the diagonal elements of positive semidefinite matrices are non-negative in any bases, we have \( A = \sum_{l \neq 0} a_l M_l + a_0 M_0 \) has at least \( r \) positive diagonal elements in its own eigenbasis. Therefore \( \text{rank}(A) \geq r \). 

Now we come to the proof of theorem 2.

**proof.** In theorem 1 we proved that effective measurements \( A_{S_a}^m \) and \( B_{S_b}^m \) can be written as the convex combination of some projectors whose entanglement do not exceed \( \mathcal{E}^m \), multiplied by some normalizing coefficients \( f_{S_a} \) and \( f_{S_b} \), then there is

\[
A_{S_a}^m \otimes B_{S_b}^m = f_{S_a} \cdot f_{S_b} \sum_{l,l'} c_{l} \cdot c_{l'} A_{S_a,l} \otimes B_{S_b,l'}
\]
where $\sum_i c_i = \sum_j c_j = 1$, $\sum_{S_a} f_{S_a} = \sum_{S_b} f_{S_b} = 1$. $A_{S_a}$ and $B_{S_b}$ are some projectors acting on $H_X \otimes H_A$ and $H_Y \otimes H_B$.

Due to the convexity of the entanglement measure $\mathcal{E}$, we know that the entanglement $Q_{SSLDS}^m$ is upper bounded by the cases where $A_{S_a}^m$ and $B_{S_b}^m$ are pure. So now we let $A_{S_a}^m$ and $B_{S_b}^m$ be the projectors with unit trace, acting on $H_X \otimes H_A$ and $H_Y \otimes H_B$, and let $O_{S_a S_b}$ denote the positive semidefinite operator obtained by

$$O_{S_a S_b} = Tr_{AB}[(A_{S_a}^m \otimes B_{S_b}^m)(I_X \otimes \rho_{AB} \otimes I_y)].$$

(30)

Under tomographically complete input sets $\{r_a\}$ and $\{r_y\}$, by comparing Eqs. (1) and (2) in the main text, we have

$$\Pi_{ab}^m = \sum_{S_a, S_b} p(a, b | S_a, S_b) \cdot O_{S_a S_b}.$$  

(31)

Since in theorem 1 we have demonstrated the quantum states that projectors $A_{S_a}^m$ and $B_{S_b}^m$ project on have Schmidt numbers no more than $m$, by simple calculation we have

$$Tr_X(O_{S_a S_b}) = Tr_Y(O_{S_a S_b}) \leq m.$$  

(32)

As $O_{S_a S_b}$ are positive semidefinite operators, we can find a set of normalized quantum states $\{\sigma_{S_a S_b}\}$ satisfying

$$O_{S_a S_b} = t_{S_a S_b} \cdot \sigma_{S_a S_b}, \quad \forall S_a, S_b,$$

(33)

where $t_{S_a S_b} = Tr(O_{S_a S_b})$. We know that $\sigma_{S_a S_b}$ can always be written as the combination of some bipartite pure states, and we can also affirm that the Schmidt number of these pure states are no more than $m$, since if any of these pure states has Schmidt number $n$ ($n > m$), with Lemma S1 and simple calculation we will have $Tr_X(O_{S_a S_b}) \geq n$, which contradicts Eq. (32). Then, as bipartite pure states of Schmidt number $m$ has entanglement $E^m$, we have

$$E(\sigma_{S_a S_b}) \leq E^m, \quad \forall S_a, S_b.$$  

(34)

Finally, for the detected entanglement $Q$ is

$$Q = \frac{1}{d_X d_Y} \sum_{ab} E(\Pi_{ab}^m)$$

$$= \sum_{a, b} t_{S_a S_b} \cdot p(a, b | S_a, S_b) \cdot E(\sigma_{S_a S_b}),$$

(35)

where $\sum_{S_a, S_b} t_{S_a S_b} = 1$, $\sum_{a, b} p(a, b | S_a, S_b) = 1$, therefore we have $Q \leq E^m$. Since $Q$ is an upper bound of the detected entanglement $Q_{SSLDS}^m$, we have $Q_{SSLDS}^m \leq Q \leq E^m$. □

**Supplementary Note 6: experimental details**

The experimental setup is shown in Fig. 2 of the main text. A cw violet laser at 404nm was separated with a beam displacer (BD40), and then was incident to a Sagnac interferometer to pump a type-II cut ppKTP crystal to generate photon pairs at 808nm. Through adjustments of half- and quarter- plates(HWPs and QWPs), three dimensional pure state $|\phi\rangle = \frac{1}{\sqrt{3}}(|HH\rangle_u + |HH\rangle_l + |VV\rangle_l)$, i.e. $\rho_{AB}$, was prepared, where the subscripts $u$ and $l$ denote upper path and lower path (refer to our previous works [28, 29] for more details). To prepare the mixed state $\rho_{\text{noise}}$, another photon pair was used [17]. Setting the angles of HWPs in the Source module, the state $\frac{1}{\sqrt{3}}(|HH\rangle_u + |HH\rangle_l + |VV\rangle_l) \otimes (|HH\rangle_u + |HH\rangle_l + |VV\rangle_l)$ was obtained. Inserting four quartz, we destroyed the coherence of this state and obtained the mixed state $I/9$. Mixing the two states $\rho_{AB}$ and $I/9$ at beam splitters, we finally prepared the path-polarization hybrid entangled state $\rho_{\text{noise}}$ with tunable $p$ by adjusting the ratio of the two parts.

To prepare the input states $\{r_x, r_y\}$, Charlie expanded the dimension of photon’s path degree of freedom by separating the photons via HWP and BD20 (half the length of BD40) in the Trust modules as shown in Fig. 2. Encoded as $h_u$, $h_l$, and $v_l$, arbitrary trusted input states of the form $\beta_1|h_u\rangle + \beta_2|h_l\rangle + \beta_3|v_l\rangle$ could be prepared by adjusting the angles of the HWPs. The input states $r_x (r_y)$ are assumed to be trustable and thus should be prepared with high accuracy. This could be met for our scheme restricts in highly controllable operations of polarization elements (HWPs and BDs). Our method to prepare the input states consumes no extra degree of freedom of the photons which is indeed precious resource.

There are 9 Bell states for a 3-dimensional system, which can be divided into 3 categories (each contains 3 of them) as: category 1

$$|\Phi_1\rangle = \frac{1}{\sqrt{3}}(|00\rangle + e^{i\phi_0}|11\rangle + e^{i\phi_1}|22\rangle),$$

(36)
category 2

$$|\Phi_2\rangle = \frac{1}{\sqrt{3}} (|01\rangle + e^{i\varphi_0}|12\rangle + e^{i\varphi_1}|20\rangle),$$

(37)

category 3

$$|\Phi_3\rangle = \frac{1}{\sqrt{3}} (|02\rangle + e^{i\varphi_0}|10\rangle + e^{i\varphi_1}|21\rangle),$$

(38)

where $$(\varphi_0, \varphi_1) \in \{(0, 0), (2\pi/3, 4\pi/3), (4\pi/3, 8\pi/3)\}$$ for each category.

It remains a nodus to implement high dimensional BSM and seems unattainable without consuming auxiliary entangled resource, specially when two particles involved. Benefitting from QMDIEW’s robustness against locality loophole, we overcome the problem by providing a method which could be used to implement high dimensional BSM performed on two subspaces of a single photon (the distributed qutrit subspace and the trusted input state subspace in our case). The BSM modules in Fig. 2 can be used to perform BSM onto arbitrary one of nine Bell states of three dimension. To implement these BSMs, photons with proper path-polarization modes were recombined step by step. For instance, blocking the second, third and fifth arms and setting the angles of HWP1-9 at 0°, 45°, 0°, 45°, 0°, 0°, 45°, 0°, 45°, 0°, 22.5°, and 27.37°, the measurement onto $1/\sqrt{3}(|Hh\rangle_u + |Hh\rangle_l + |Vv\rangle_l)$ (one of the states in category 1) was obtained. For other two states in category 1, it could be achieved by inserting a 2λ/3, a 4λ/3 wave plate or a 8λ/3, a 8λ/3 wave plate at 0° in front of H8 and H9 respectively. Other BSMs can be achieved analogously. Recording the twofold coincidences between the single photon detectors of Alice and Bob, the correlations in Eq. (2) of the main text were then obtained.