**KMS weights on graph $C^*$-algebras II**

Factor types and ground states

Klaus Thomsen

1. **Introduction**

The study of KMS states on graph $C^*$-algebras has a long story which began well before the notion of a graph $C^*$-algebra was coined, starting perhaps with the paper [PS] by Powers and Sakai, or the paper [OP] by Olesen and Pedersen. For finite graphs we have now a complete description of all KMS states for generalized gauge actions, thanks to the efforts of many mathematicians. See [aHLRS] and [CT] for the most recent and most comprehensive account. It seems therefore now time to consider gauge actions and generalised gauge actions on graph $C^*$-algebras of infinite graphs, and the first investigations in this direction was performed by Carlsen and Larsen in [CL], building on the closely related work by Exel and Laca in [EL], and by the author in [Th3], [Th4], [Th5]. The two approaches are very different and complement each other nicely.

The present work is a continuation of [Th5] where the gauge invariant KMS weights of generalised gauge actions were investigated, culminating in a complete description of the KMS weights for the gauge action on the $C^*$-algebra of a strongly connected graph with at most countably many exits. The structure turned out to be very rich; in fact, to a agree comparable with the structure constructed more than 30 years ago by Bratteli, Elliott and Kishimoto, [BEK], based on the classification of AF-algebras. However, the factor types of the extremal KMS weights identified in [Th5] were not determined, nor was the ground states of the systems obtained by restricting the gauge actions to corners given by vertexes in the graph. It is the purpose of the present paper to provide this information. At the same time we make an effort to obtain results valid for actions more complicated than the gauge action. It will be shown by example that by considering actions other than the gauge action, the variety of factor types becomes much greater.

It follows from [Th5] that there are three kinds of extremal KMS weights for the gauge action when the graph is strongly connected and only has countably many exits: Boundary KMS weights coming from infinite emitters in the graph, a conservative KMS weight coming from a positive eigenvector for the adjacency matrix of the graph with the smallest possible eigenvalue, and dissipative KMS weights coming from exits in the graph. The three types are not always all present, but when the graph has at most countably many exits they comprise all the extremal KMS weights. The factor type of the extremal boundary KMS weights are invaribly $I_\infty$, for all generalized gauge actions, and we focus therefore on the two other kinds of KMS weights since they exhibit a much more varied behaviour. In Section 3 we determine the factor type of the essentially unique conservative KMS weight for the
factor types and ground states

A gauge action which exists when the graph is recurrent with finite Gurevich entropy. The method of proof is taken from [Th3], but it is only successful because of the conservative nature of the corresponding measure which was established in [Th5]. The result is that the factor type of this weight is determined by the global period $d_G$ and the Gurevich entropy $h(G)$ of the graph; more precisely its factor type is $III_\lambda$ where $\lambda = e^{-d_G h(G)}$. In Section 4 we determine the factor types of the extremal KMS weights for the gauge action coming from exits in the graph, and they turn out to be of type $I_\infty$ when the associated measure on the space of infinite paths in the graph is atomic, and of type $II_\infty$ when it is not atomic.

Like the results, the methods used in Section 3 and Section 4 are very different. Furthermore, the methods extend to other actions, and in Section 4 we can in fact handle all generalized gauge actions. The key lemma here is valid for all these actions and shows how the factor arising from an exit in the graph is closely related to an Araki-Woods factor naturally defined from the exit. The method and results of Section 3 extend also well beyond the gauge action, and we pursue this higher generality because it appears that the variation of factor types becomes much richer for more general actions.

In the last section we describe the ground states of a generalized gauge action restricted to a corner defined from a vertex in the graph which no longer needs to be strongly connected. The method is an adaptation of a method from [Th2], and as often before (e.g. in [LR], [Th2], [CL] and [Ka]) the ground states turn out to be parametrised by the states of a $C^*$-algebra which in this case is a sub-quotient of the graph $C^*$-algebra determined by the data used to define the one-parameter group. Its structure can be quite complicated. We exhibit an example of a strongly connected row-finite graph where the ground states of the gauge action restricted to a corner are parametrised by the state space of the CAR algebra. The ground states for generalized gauge actions on the graph $C^*$-algebra itself were identified, under mild conditions, by Carlsen and Larsen in [CL]. There are considerably fewer of those, and in fact none unless the graph has sinks or infinite emitters; at least not when the function on the edges defining the action is strictly positive. In contrast, on a corner of the graph $C^*$-algebra given by a vertex in a strongly connected infinite graph, there are always at least one ground state for the gauge action.

2. Preparations

Let $G$ be a directed graph with vertexes $V$ and edges $E$. We assume that $G$ is countable in the sense that $V$ and $E$ are both countable sets. We let $r$ and $s$ denote the maps $r : E \to V$ and $s : E \to V$ which associate to an edge $e \in E$ its target vertex $r(e)$ and source vertex $s(e)$, respectively. A vertex $v$ is an infinite emitter when $s^{-1}(v)$ contains infinitely many edges and a sink when $s^{-1}(v)$ is empty. The union of sinks and infinite emitters constitute a set which will be denoted by $V_\infty$. Except for the last section on ground states we will only consider graphs that are strongly connected in the sense that for all vertexes $v, w \in V$ there is a finite path in $G$ which starts at $v$ and ends at $w$. In particular, there are no sinks. The graph $C^*$-algebra $C^*(G)$ is by definition the universal $C^*$-algebra generated by a collection $S_e, e \in E$, of partial isometries and a collection $P_v, v \in V$, of orthogonal projections subject to the conditions that

1) $S_e^* S_e = P_{r(e)}, \forall e \in E,$
2) \(\sum_{e \in F} s_e^* s_e \leq P_v\) for every finite subset \(F \subseteq s^{-1}(v)\) and all \(v \in V\), and
3) \(P_v = \sum_{e \in s^{-1}(v)} s_e^* s_e\), \(\forall v \in V \setminus V_\infty\).

The generalised gauge actions we consider can be defined using the universal property. Let \(F : E \to \mathbb{R}\) be any map. There is a one-parameter group \(\alpha^F = (\alpha^F_t)_{t \in \mathbb{R}}\) on \(C^*(G)\) defined by the requirements that

\[ \alpha^F_t(P_v) = P_v \text{ and } \alpha^F_t(S_e) = e^{iF(e)t}S_e \]

for all vertexes \(v\), all edges \(e\) and all real numbers \(t\).

For the present purpose it is crucial that \(C^*(G)\) can also be realised as the (reduced) \(C^*\)-algebra of an \(\acute{e}tale\) groupoid \(G\) constructed by A. Paterson in [Pa]. Let \(P_f(G)\) and \(P(G)\) denote the set of finite and infinite paths in \(G\), respectively. The range and source maps, \(r\) and \(s\), extend in the natural way to \(P_f(G)\); the source map also to \(P(G)\). A vertex \(v \in V\) will be considered as a finite path of length 0 and we set \(r(v) = s(v) = v\) when \(v\) is considered as an element of \(P_f(G)\). The unit space \(\Omega_G\) of \(G\) is the union \(\Omega_G = P(G) \cup Q(G)\), where

\[ Q(G) = \{ p \in P_f(G) : r(p) \in V_\infty \} \]

is the set of finite paths that terminate at a vertex in \(V_\infty\). In particular, \(V_\infty \subseteq Q(G)\) because vertices are considered to be finite paths of length 0. For any \(p \in P_f(G)\), let \(|p|\) denote the length of \(p\). When \(|p| \geq 1\), set

\[ Z(p) = \{ q \in \Omega_G : |q| \geq |p|, q_i = p_i, \; i = 1, 2, \ldots, |p| \} \]

and

\[ Z(v) = \{ q \in \Omega_G : s(q) = v \} \]

when \(v \in V\). When \(\nu \in P_f(G)\) and \(F\) is a finite subset of \(P_f(G)\), set

\[ Z_F(\nu) = Z(\nu) \setminus \left( \bigcup_{\mu \in F} Z(\mu) \right). \tag{2.1} \]

The sets \(Z_F(\nu)\) form a basis of compact and open subsets for a locally compact Hausdorff topology on \(\Omega_G\). When \(\mu \in P_f(G)\) and \(x \in \Omega_G\), we can define the concatenation \(\mu x \in \Omega_G\) in the obvious way when \(r(\mu) = s(x)\). The groupoid \(G\) consists of the elements in \(\Omega_G \times \mathbb{Z} \times \Omega_G\) of the form

\((\mu x, |\mu| - |\mu'|, \mu' x)\),

for some \(x \in \Omega_G\) and some \(\mu, \mu' \in P_f(G)\). The product in \(G\) is defined by

\((\mu x, |\mu| - |\mu'|, \mu' x)(\nu y, |\nu| - |\nu'|, \nu' y) = (\mu x, |\mu| + |\nu| - |\mu'| - |\nu'|, \nu' y)\),

when \(\nu' y = \nu y\), and the involution by \((\mu x, |\mu| - |\mu'|, \mu' x)^{-1} = (\mu' x, |\mu'| - |\mu|, \mu x)\). To describe the topology on \(G\), let \(Z_F(\mu)\) and \(Z_F(\mu')\) be two sets of the form (2.1) with \(r(\mu) = r(\mu')\). The topology we shall consider has as a basis the sets of the form

\[ \{(\mu x, |\mu| - |\mu'|, \mu' x) : \mu x \in Z_F(\mu), \; \mu' x \in Z_F(\mu') \}. \tag{2.2} \]

With this topology \(G\) becomes an \(\acute{e}tale\) locally compact Hausdorff groupoid and we can consider the reduced \(C^\ast\)-algebra \(C^\ast_r(G)\) as in [Re1]. As shown by Paterson in [Pa] there is an isomorphism \(C^*(G) \to C^*_r(G)\) which sends \(S_e\) to \(1_e\), where \(1_e\) is the characteristic function of the compact and open set

\[ \{(ex, 1, r(e)x) : x \in \Omega_G \} \subseteq G, \]
and $P_v$ to $1_v$, where $1_v$ is the characteristic function of the compact and open set $$\{(vx, 0, vx) : x \in \Omega_G\} \subseteq \mathcal{G}.$$ In the following we use the identification $C^*(G) = C^*_r(G)$ and identify $\Omega_G$ with the unit space of $\mathcal{G}$ via the embedding $\Omega_G \ni x \mapsto (x, 0, x)$.

By describing the generalised gauge action $\alpha^F$ in the groupoid picture, it is seen that it is a special case of actions considered by Renault in [Re1]. Specifically we extend $F$ to a function $F : P_I(G) \to \mathbb{R}$ such that $$F(p_1p_2 \cdots p_n) = \sum_{i=1}^n F(p_i)$$ when $p = p_1p_2 \cdots p_n$ is a path of length $n \geq 1$ in $G$, and $F(v) = 0$ when $v \in V$. We can then define a continuous function $c_F : \mathcal{G} \to \mathbb{R}$ such that $$c_F(u x, |u| - |u'|, u' x) = F(u) - F(u').$$ Since $c_F$ is a continuous homomorphism it gives rise to a continuous one-parameter automorphism group $\alpha^F$ on $C^*_r(G)$ defined such that $$\alpha^F_t(f)(\gamma) = e^{itc_F(\gamma)} f(\gamma)$$ when $f \in C_c(G)$, cf. [Re1]. When $F$ is constant 1 this action is known as the gauge action on $C^*(G)$ and we denote it by $\gamma$.

Recall, [KV1], [Th3], that a weight $\psi$ on the $C^*$-algebra $A$ is proper when it is non-zero, densely defined and lower semi-continuous. For such a weight, set $$\mathcal{N}_\psi = \{a \in A : \psi(a^* a) < \infty\}.$$ Let $\alpha : \mathbb{R} \to \text{Aut} A$ be a point-wise norm-continuous one-parameter group of automorphisms on $A$. Let $\beta \in \mathbb{R}$. Following Combes, [Co], we say that a proper weight $\psi$ on $A$ is a $\beta$-KMS weight for $\alpha$ when

i) $\psi \circ \alpha_t = \psi$ for all $t \in \mathbb{R}$, and

ii) for every pair $a, b \in \mathcal{N}_\psi \cap \mathcal{N}_\psi^*$ there is a continuous and bounded function $F$ defined on the closed strip $D_\beta$ in $\mathbb{C}$ consisting of the numbers $z \in \mathbb{C}$ whose imaginary part lies between 0 and $\beta$, and is holomorphic in the interior of the strip and satisfies that $$F(t) = \psi(a\alpha_t(b)), \quad F(t + i\beta) = \psi(\alpha_t(b)a)$$ for all $t \in \mathbb{R}$. [1]

A $\beta$-KMS weight $\psi$ on $A$ is extremal when the only $\beta$-KMS weights $\varphi$ on $A$ with the property that $\varphi(a^* a) \leq \psi(a^* a)$ for all $a \in A$ are the scalar multiples of $\psi$, viz. $\varphi = t\psi$ for some $t > 0$.

It was shown in [Th3] that all gauge invariant KMS weights $\psi$ for $\alpha^F$ are given by a regular Borel measure $m$ on $\Omega_G$ in the sense that $$\psi(a) = \int_{\Omega_G} P(a) \, dm, \quad (2.3)$$ where $P : C^*(G) \to C_0(\Omega_G)$ is the canonical conditional expectation. [Re1]. Under a certain condition on $F$, spelled out in (3.1) below, the KMS weights for $\alpha^F$ are automatically gauge invariant and hence they all come from measures on $\Omega_G$. A

---

1As in [Th1] we apply the definition from [C] for the action $\alpha_{-t}$ in order to use the same sign convention as in [BR], for example.
measure \( m \) corresponding to a \( \beta \)-KMS measure via (2.3) will be called a \( \beta \)-KMS measure. The measure associated to an extremal KMS weight is either supported on \( P(G) \), in which case it is harmonic, or on \( Q(G) \) and it is then a boundary KMS weight. See [CL] and [Th5] for more on this dichotomy.

Given a weight \( \psi \) on a \( C^* \)-algebra \( A \) there is a GNS-type construction consisting of a Hilbert space \( H_\psi \), a linear map \( \Lambda_\psi : \mathcal{N}_\psi \to H_\psi \) with dense range and a non-degenerate representation \( \pi_\psi \) of \( A \) on \( H_\psi \) such that

\[
\bullet \, \psi(b^*a) = \langle \Lambda_\psi(a), \Lambda_\psi(b) \rangle, \quad a, b \in \mathcal{N}_\psi, \quad \text{and}
\bullet \, \pi_\psi(a)\Lambda_\psi(b) = \Lambda_\psi(ab), \quad a \in A, \quad b \in \mathcal{N}_\psi,
\]

cf. [KV1], [KV2]. As observed in Lemma 4.4 in [Th3] the von Neumann algebra \( \pi_\psi(A)'' \) is a factor when \( \psi \) is extremal, and the von Neumann algebra type of \( \pi_\psi(A)'' \) is the factor type of \( \psi \).

As mentioned in the introduction, the extremal boundary KMS weights of a generalized gauge action on a graph \( C^* \)-algebra are of factor type \( I_\infty \), at least when the graph is strongly connected. This can be seen for example by showing that the factor in question contains an abelian projection and can not be finite, in much the same way as in the proof of Proposition 5.1 in [Th2]. We leave the details to the reader and focus here on the extremal harmonic KMS weights.

3. The conservative KMS weight

When the graph \( G \) is strongly connected and recurrent, and the Gurevich entropy \( h(G) \) of \( G \) is finite, there is a \( h(G) \)-KMS weight \( \psi \) for the gauge action on \( C^*(G) \) which is unique up to multiplication by scalars, cf. [EFW] for the finite case and Proposition 4.16 in [Th5] for the infinite case. In particular \( \psi \) is extremal and in this section we determine its factor type by using the method from Section 4.2 of [Th3]. The conservative nature on the measure on \( P(G) \) given by the weight is the crucial additional information which will allow us to be more conclusive than in [Th3].

Let \( \psi \) be an extremal \( \beta \)-KMS weight for \( \alpha^F \). It follows from Section 2.2 in [KV1] that \( \psi \) extends to a normal semi-finite faithful weight \( \tilde{\psi} \) on \( M = \pi_\psi(C^*(G))'' \) such that \( \psi = \tilde{\psi} \circ \pi_\psi \), and that the modular group on \( M \) corresponding to \( \tilde{\psi} \) is the one-parameter group \( \theta \) given by

\[
\theta_t = \tilde{\alpha}^F_{-\beta t},
\]

where \( \tilde{\alpha}^F \) is the \( \sigma \)-weakly continuous extension of \( \alpha^F \) defined such that \( \tilde{\alpha}^F_t \circ \pi_\psi = \pi_\psi \circ \alpha^F_t \). To simplify the notation in the following, let \( N \subseteq M \) be the fixed point algebra of \( \theta \), viz. \( N = M^\theta \), and suppress the representation \( \pi_\psi \) so that, in particular, \( 1_v \in C_c(G) \) will now also denote the projection \( \pi_\psi(1_v) \in M \). For \( f \in L^1(\mathbb{R}) \) define a linear map \( \sigma_f : M \to M \) such that

\[
\sigma_f(a) = \int_{\mathbb{R}} f(t)\theta_t(a) \, dt.
\]

For every vertex \( v \in V \) and every central projection \( e \) in \( 1_vN_1v \), set

\[
\text{Sp}(eMe) = \bigcap \{ Z(f) : f \in L^1(\mathbb{R}), \, \sigma_f(eMe) = \{0\} \},
\]

where

\[
Z(f) = \left\{ r \in \mathbb{R} : \int_{\mathbb{R}} e^{itr} f(t) \, dt = 0 \right\}.
\]
Then the invariant $\Gamma(M)$ introduced by Connes, [C], can be expressed as the intersection
\[ \Gamma(M) = \bigcap e \text{Sp}(eMe), \]
where we take the intersection over all non-zero central projections $e$ in $1_v N_1_v$. In particular, the intersection does not depend on which vertex $v$ we use.

Pick a vertex $v$ in $G$ and note that
\[ \{ \beta F(\mu) - \beta F(\mu') : \mu, \mu' \in P_f(G), r(\mu) = r(\mu') = s(\mu) = s(\mu') = v \} \]
is a subgroup of $\mathbb{R}$ which does not depend on the vertex $v$ since $G$ is strongly connected. Let $R_{G,F}$ be the closure in $\mathbb{R}$ of this subgroup.

**Lemma 3.1.** Assume that $G$ is strongly connected. Let $\psi$ be an extremal $\beta$-KMS weight for $\alpha^F$. Then $\pi_{\psi}(C^*(G))^\prime$ is a hyperfinite factor and
\[ \Gamma(\pi_{\psi}(C^*(G))^\prime) \subseteq R_{G,F}. \] (3.1)

**Proof.** $M = \pi_{\psi}(C^*(G))^\prime$ is hyperfinite because $C^*(G)$ is nuclear. We show that
\[ \mathbb{R} \backslash R_{G,F} \subseteq \mathbb{R} \backslash \Gamma(M). \]

Let $r \in \mathbb{R} \backslash R_{G,F}$ and choose a function $f \in L^1(\mathbb{R})$ whose Fourier transform $\hat{f}$ satisfies that $\hat{f}(t) = 0$ for all $t \in R_{G,F}$ and $\hat{f}(r) \neq 0$. Assume that $h \in C_c(G)$ is supported on a set of the form
\[ \{(\mu x, |\mu| - |\mu'|, \mu' x) : x \in \Omega_G \}, \] (3.2)
where $\mu, \mu' \in P_f(G)$ satisfy that $s(\mu) = s(\mu') = v$ and $r(\mu) = r(\mu')$. Then
\[ \sigma_f(h) = \int_{\mathbb{R}} f(t) \theta_v(h) \, dt = \hat{f}(\beta(F(\mu') - F(\mu)) h = 0 \]
because $\beta(F(\mu') - F(\mu)) \in R_{G,F}$. Note that elements of the form $1_v h 1_v$ span a strongly dense subspace of $1_v M_1_v$ and conclude that $\sigma_f(1_v M_1_v) = \{0\}$. Since $\hat{f}(r) \neq 0$ we conclude that $r \notin \Gamma(M)$. 

As in [Th5] we introduce the matrix $A(\beta)$ over $V$ such that
\[ A(\beta)_{vw} = \sum_{e \in s^{-1}(v) \cap r^{-1}(w)} e^{-\beta F(e)}. \] (3.3)

It was shown in [Th5] that there are no gauge invariant $\beta$-KMS weights for $\alpha^F$ unless all powers of $A(\beta)$ are finite and $\limsup_n (A(\beta)^n)_{vw}^\frac{1}{n} \leq 1$ for one and hence all vertexes $v \in V$. When these conditions hold and $A(\beta)$ is 1-recurrent, in the sense that
\[ \sum_{n=0}^{\infty} A(\beta)^n_{vw} = \infty, \]
there is a unique ray of gauge invariant $\beta$-KMS weights for the action $\alpha^F$. For $G$ finite this is a result of Enomoto, Fujii and Watatani, [EFW], when $F = 1$, and a result of Exel and Lace when $F$ is either strictly positive or strictly negative, [EL]. The general case follows from Theorem 4.14 and Proposition 4.16 in [Th5]. Furthermore, it was shown that these KMS weights are harmonic and that the associated measure on $P(G)$ is conservative. The following result determines their factor type under a certain condition on $F$. 


**Theorem 3.2.** Assume that $G$ is strongly connected and that $F$ satisfies the following condition:

$$\nu \in P_f(G), \ |\nu| \geq 1, \ s(\nu) = r(\nu) \Rightarrow F(\nu) \neq 0.$$  

(3.4)

Let $\psi$ be a $\beta$-KMS weight for $\alpha^F$, $\beta \neq 0$, whose associated measure is supported and conservative on $P(G)$. Then $\pi_\psi(C^*(G))''$ is a hyperfinite factor and

$$\Gamma(\pi_\psi(C^*(G))'') = R_{G,F}.$$  

**Proof.** Note, first of all, that under the present assumptions the real-valued homomorphism $c_F$ on $G$ defined from $F$ satisfies the two conditions on $c_0$ in Theorem 2.2 of [Th1]. This implies that all KMS weights of $\alpha^F$ are gauge invariant. Furthermore, since the measure associated with $\psi$ is supported and conservative on $P(G)$, it follows from Theorem 4.11, Theorem 4.14 and Proposition 4.16 in [Th5] that up to multiplication by scalars $\psi$ is the only $\beta$-KMS weight for $\alpha^F$. In particular, $\psi$ is extremal and $M = \pi_\psi(C^*(G))''$ is a factor.

By Lemma 3.1 and the discussion preceding it, it suffices here to consider a non-zero central projection $q \in 1_v N_1 v$ for some vertex $v \in V$, and show that $R_{G,F} \subseteq \text{Sp}(qMq)$. To this end consider two finite paths $l^\pm$ in $G$ such that $s(l^+) = r(l^-) = v$. It suffices to show that $\beta(F(l^+) - F(l^-)) \in \text{Sp}(qMq)$. Let $\omega$ be the state on $1_v M_1 v$ given by $\omega(a) = \psi(1_v)^{-1} \psi(a)$. Then $\omega$ is a faithful normal state which is a trace on $1_v N_1 v$ and we consider the corresponding 2-norm

$$\|a\|_v = \sqrt{\omega(a^* a)}.$$  

By Kaplansky’s density theorem there is an element $f \in 1_v C_c(G) 1_v$ such that $0 \leq f \leq 1$ and $\|q - f\|_v$ is as small as we want. Since $f$ can be approximated in norm by a linear combination of characteristic functions of sets of the form $[3.2]$ we may as well assume that $f$ is such a linear combination. Then the limit

$$g = \lim_{R \to \infty} \frac{1}{R} \int_0^R \alpha_t^F(f) \, dt$$

exists in norm, and $g \in 1_v C_c(G) 1_v$ is supported on the open sub groupoid $\mathcal{R}$ of $G$ consisting of the elements $(\mu x, |\mu| - |\mu'|)$ with $F(\mu) = F(\mu')$. Note that

$$\left\|q - \frac{1}{R} \int_0^R \alpha_t^F(f) \, dt\right\|_v = \left\|\frac{1}{R} \int_0^R \hat{\alpha}_t^F(q - f) \, dt\right\|_v \leq \|q - f\|_v$$

by Kadison’s Schwarz inequality. We may therefore assume that $f \in 1_v C_c(\mathcal{R}) 1_v$. The groupoid $\mathcal{R}$ has trivial isotropy groups since we assume that (3.1) holds, and $\mathcal{R}$ is therefore an étale equivalence relation. It follows therefore from (the proof of) Lemma 2.24 in [Th1] that there is a sequence

$$\{d_j^n : j = 1, 2, \cdots, N_n\}, \ n = 1, 2, \cdots$$

of non-negative elements in $C(Z(\nu))$ such that $\sum_{j=1}^{N_n} d_j^n = 1_v$ and

$$P(f) = \lim_{n \to \infty} \sum_{j=1}^{N_n} d_j^n f d_j^n,$$
where \( P \) is the conditional expectation \( P : \mathcal{C}^*_v(R)1_v \to C(Z(v)) \) and the limit is norm-convergent. Note that
\[
q = \sum_{j=1}^{N_n} d^m_j q d^m_j
\]
for all \( n \) because \( q \) is central in \( 1_v N_1 v \), and that
\[
\| q - P(f) \|_v = \lim_{n \to \infty} \left\| \sum_{j=1}^{N_n} d^m_j (q - f) d^m_j \right\|_v \leq \| q - f \|_v.
\]

Using \( P(f) \) in the place of \( f \) we may therefore assume that \( f \in C(Z(v)) \) and that there are finite paths \( \nu_i \in P_f(G) \), finite collections of paths \( F_i \subseteq P_f(G) \) and numbers \( 0 < t_i \leq 1, \ i = 1, 2, \ldots, N \), such that \( Z_{F_i}(\nu_i) \cap Z_{F_j}(\nu_j) = \emptyset \) when \( i \neq j \) and
\[
f = \sum_{i=1}^{N} t_i 1_{Z_{F_i}(\nu_i)}.
\]
Fix \( i \) for a moment. There is a finite or countably infinite set \( H_i' \subseteq P_f(G) \) such that
\[
Z_{F_i}(\nu_i) = \sqcup_{\nu \in H_i'} Z(\nu).
\]
It follows from (2.3) that there is a Borel probability measure \( m \) on \( Z(v) \) such that
\[
\omega(g) = \int_{Z(v)} P(g) \, dm
\]
when \( g \in 1_v C^*(G)1_v \). By Theorem 4.11 in [Th5] \( m \) is supported on
\[
\{(x_i)_{i=1}^\infty \in P(G) : s(x_1) = s(x_i) = v \text{ for infinitely many } i\}
\]
and we can therefore choose a finite set \( H_i \subseteq P_f(G) \) such that \( r(\nu) = s(\nu) = v \) for all \( \nu \in H_i \);
\[
\sqcup_{\nu \in H_i} Z(\nu) \subseteq Z_{F_i}(\nu_i)
\]
and
\[
\left\| 1_{Z_{F_i}(\nu_i)} - \sum_{\nu \in H_i} 1_{Z(\nu)} \right\|_v = \sqrt{m(Z_{F_i}(\nu_i) \setminus \sqcup_{\nu \in H_i} Z(\nu))}
\]
is as small as we want. By exchanging \( \sum_{\nu \in H_i} 1_{Z(\nu)} \) for \( 1_{Z_{F_i}(\nu_i)} \) for each \( i \), we can therefore assume that there are finite paths \( \nu_i \in P_f(G) \) and numbers \( 0 \leq t_i \leq 1, \ i = 1, 2, \ldots, M' \), such that \( s(\nu_i) = r(\nu_i) = v \) for all \( i \),
\[
f = \sum_{i=1}^{M'} t_i 1_{Z(\nu_i)},
\]
and \( Z(\nu_i) \cap Z(\nu_j) = \emptyset \) when \( i \neq j \). Finally, since \( q \) is a projection a standard argument, as in the proof of Lemma 12.2.3 in [KR], allows us select a subset of the \( \nu_i \)'s to arrange, after a renumbering, that
\[
p = \sum_{i=1}^{M} 1_{Z(\nu_i)}
\]
is a projection in \( C(Z(v)) \) such that
\[
\| q - p \|_v \leq \epsilon,
\]
where
\[
q = \sum_{j=1}^{N_n} d^m_j q d^m_j
\]
where \( \epsilon > 0 \) is as small as we need. We choose \( \epsilon > 0 \) so small that
\[
e^{-F(t^+)\beta} \epsilon + e^{-F(F(t^-) - F(t^+))\beta} \epsilon + 2 \epsilon < e^{-F(t^+)\beta} \omega(q).
\]
For each \( \nu \) from \([3.3]\) we let \( w_i^\pm \in 1_v C_c(G) 1_v \) be the characteristic function of the compact and open set
\[\{(\nu x, -|l^\pm|, \nu^l x) : x \in \Omega_G\}\]
in \( G \). Each \( w_i^\pm \) is a partial isometry such that
\begin{itemize}
  \item[a)] \( w_i^\pm w_i^\pm* = 1_{Z(\nu)} \),
  \item[b)] \( w_i^\pm* w_i^\pm = 1_{Z(\nu^l)} \leq 1_{Z(\nu)} \), and
  \item[c)] \( \alpha_t^F (w_i^\pm) = \epsilon^{-iF(t^\pm)} w_i^\pm \) for all \( t \in \mathbb{R} \).
\end{itemize}
It follows from Lemma 2.9 in \([Th5]\) that
\begin{itemize}
  \item[d)] \( m(Z(\nu l^\pm)) = e^{-\beta F(t^\pm)} m(Z(\nu)) \) for all \( i \).
\end{itemize}
Set \( w_\pm = \sum_{i=1}^M w_i^\pm \) and note that \( w_\pm \) are both partial isometries. It follows from b) that \( w_+ p = w_+ \) and therefore from \([3.6]\) that
\[\omega(q w_+ w_- q w_-* w_+) \geq \omega(w_+^* w_- q w_-* w_) - 2 \epsilon.
\] Since \( \psi \) is a \( \beta \)-KMS weight, it follows from c) that
\[\omega(w_+^* w_- q w_-* w_+) = e^{-(F(t^+) - F(t^-))\beta} \omega(q w_-^* w_+ w_+^* w_-),
\] which thanks to a) is the same as
\[e^{-(F(t^+) - F(t^-))\beta} \omega(q w_-^* w_-).
\]
Using \([3.6]\) again we find that
\[\omega(q w_-^* w_-) \geq \omega(p w_-^* w_-) - \epsilon.
\] It follows from b) that \( \omega(p w_-^* w_-) = \omega(w_-^* w_-) \) while b), d) and a) imply that
\[\omega(w_-^* w_-) = m \left( \cup_i Z(\nu_i^l) \right) = e^{-F(t^-)} \omega(p).
\] Since \( \omega(p) \geq \omega(q) - \epsilon \) by \([3.6]\) we can put everything together and conclude that
\[\omega(q w_+^* w_- q w_-* w_+ q) \geq e^{-F(t^+)\beta} \omega(q) - e^{-F(t^+)\beta} \epsilon - e^{-F(t^+) - F(t^-)\beta} \epsilon - 2 \epsilon.
\] It follows therefore from the choice of \( \epsilon \) that \( \omega(q w_+^* w_- q w_-* w_+ q) > 0 \). By c),
\[\tilde{\alpha}_t^F (q w_+^* w_- q) = e^{i(F(t^+) - F(t^-)) t} q w_+^* w_- q
\]
and hence \( \theta_t(q w_+^* w_- q) = e^{-it\beta(F(t^+) - F(t^-))} q w_+^* w_- q \) for all \( t \in \mathbb{R} \). Since \( q w_+^* w_- q \neq 0 \) it follows now from Lemme 2.3.6 in \([C]\) that \( \beta(F(t^+) - F(t^-)) \in \text{Sp}(q M q) \), as required.

\[\square\]

A part of the assumption in Theorem \(3.2\) is that \( \beta \neq 0 \), and both the proof and the conclusion fails in general if \( \beta = 0 \). However, a 0-KMS weight is a densely defined trace, and it can not exist when \( C^*(G) \) is simple and purely infinite, which it is when \( G \) is strongly connected unless \( G \) only consists of a single finite loop. This is therefore the only case ruled out by the assumption \( \beta \neq 0 \).

By using Theorem \(3.2\) the factor type of the KMS weight \( \psi \) can be determined from the rule summarised by Pedersen in 8.15.11 of \([Pe]\). When \( G \) is finite Theorem \(3.2\) recovers the results of Okayasu from \([O]\). As pointed in \([Th5]\) there can be KMS states on \( C^*(G) \) when \( G \) is finite, also for actions that are not covered by Okayasu’s
work. In such cases the factor type can be determined from Theorem 3.2 because the condition (3.4) is satisfied, cf. Proposition 4.15 in [Th5].

To formulate the result for the gauge action, let the global period $d_G$ of $G$ be the greatest common divisor of the set

$$\{ |\nu| \geq 1 : \nu \in P_f(G), \ s(\nu) = r(\nu) = v \}$$

for some vertex $v$; the number is independent of which vertex we choose. Since $G$ is strongly connected the global period is the same as the $d_G$ introduced for arbitrary graphs in Section 4.2 of [Th3].

**Corollary 3.3.** Assume that $G$ is strongly connected, recurrent and with finite Gurevich entropy $h(G) > 0$. Let $\psi$ be the essentially unique $h(G)$-KMS weight for the gauge action on $C^*(G)$. The factor type of $\psi$ is $\text{III}_\lambda$ where $\lambda = e^{-d_G h(G)}$.

**Proof.** $\psi$ exists under the given assumptions and the corresponding measure is conservative by [Th5]. Hence Theorem 3.2 applies and the result follows because

$$R_{G,F} = d_G h(G) \mathbb{Z}$$

since $F = 1$. \qed

The condition $h(G) > 0$ rules out only the case where $G$ is a single finite loop.

### 4. The KMS weights from exits

We retain in this section the assumption that $G$ is strongly connected. As in [Th5] an exit path in $G$ is a sequence $t = (t_i)_{i=1}^\infty$ of vertexes such that there is an edge $e_i$ with $s(e_i) = t_i$ and $r(e_i) = t_{i+1}$ for all $i$, and such that $\lim_{i \to \infty} t_i = \infty$ in the sense that $t_i$ eventually leave any finite subset of vertexes. For a given exit path $t$ and a real number $\beta$ we set

$$t^\beta(i) = A(\beta)_{t_1 t_2} A(\beta)_{t_2 t_3} \cdots A(\beta)_{t_{i-1} t_i}$$

where $A(\beta)$ is the matrix (3.3). Then $t$ is $\beta$-summable when the limit

$$\lim_{i \to \infty} t^\beta(i)^{-1} \sum_{n=0}^\infty A(\beta)^n_{v t_i}$$

is finite for one, and hence for all vertexes $v \in V$. As shown in [Th5] a $\beta$-summable exit gives rise to a harmonic $\beta$-KMS measure $m_t$ determined by the condition that

$$m_t(Z(v)) = \lim_{i \to \infty} t^\beta(i)^{-1} \sum_{n=0}^\infty A(\beta)^n_{v t_i}$$

for every vertex $v$. It follows from Corollary 5.4 in [Th5] that the corresponding KMS weight $\psi_t$ on $C^*(G)$ is extremal among the gauge invariant KMS weights. But in fact, $\psi_t$ is extremal among all KMS weights when $G$ is strongly connected. This follows from a combination of Proposition 2.6 in [Th5] with Theorem 1.3 in [N] by observing that the isotropy group in $G$ of an element in $P(G)$ which is not pre-periodic is trivial. We can therefore talk about the factor type of $\psi_t$ without further assumptions on $F$.

To determine the factor type of $\psi_t$, set

$$k_n = \# s^{-1}(t_n) \cap r^{-1}(t_{n+1})$$
which is finite, cf. [Th5]. Choose a numbering $e_1^n, e_2^n, \ldots, e_k^n$ of the edges in $s^{-1}(t_n) \cap r^{-1}(t_{n+1})$, and let $\omega_n$ be the state on $M_{k_n}(\mathbb{C})$ given by

$$\omega_n(a) = \frac{\text{Tr}(e^{-\beta H} a)}{\text{Tr}(e^{-\beta H})},$$

(4.1)

where $H = \text{diag}(F(e_1^n), F(e_2^n), \ldots, F(e_k^n))$. Then $\omega = \otimes_{n=1}^\infty \omega_n$ is a state on the UHF-algebra

$$A = \otimes_{n=1}^\infty M_{k_n}(\mathbb{C}),$$

(4.2)

and we set

$$\mathcal{R}(t) = \pi(\omega(A))''.$$

Note that $\mathcal{R}(t)$ is an Araki-Woods factor, cf. [AW].

**Lemma 4.1.** Let $G$ be a strongly connected graph and $t$ a $\beta$-summable exit path in $G$, and let $\psi_t$ be the corresponding $\beta$-KMS weight for the generalised gauge action $\alpha^F$ on $C^*(G)$. There is a projection $p \in \pi_{\psi_t}(C^*(G))''$ in the fixed point algebra of $\hat{\alpha}^F$ such that

$$p\pi_{\psi_t}(C^*(G))''p \simeq \mathcal{R}(t).$$

**Proof.** Set $M = \pi_{\psi_t}(C^*(G))''$. As in the proof of Theorem 3.2 we suppress the representation $\pi_{\psi_t}$ in the notation. For each $n \in \mathbb{N}$ let

$$A_n = \{\xi \in P_f(G) : \xi = e_1 e_2 \cdots e_{n-1}, s(e_i) = t_i, i = 1, 2, \ldots, n-1, r(e_n) = t_n\}.$$

When $\xi, \xi' \in A_n$, let $1_{\xi,\xi'} \in C_c(\mathcal{G})$ be the characteristic function of

$$\{(\xi, 0, \xi') : x \in \Omega_G\}.$$

Set $q_n = \sum_{\xi, \xi' \in A_n} 1_{\xi, \xi'} \in C_c(\omega_G)$. Then $q_n \geq q_{n+1}$ and we define $p \in M$ to the limit

$$p = \lim_{n \to \infty} q_n$$

in the strong operator topology. If we let $\hat{\psi}_t$ denote the normal weight on $M$ which extends $\psi_t$, we find that

$$\hat{\psi}_t(p) = \lim_{n \to \infty} \psi_t(q_n) = m_t(\pi^{-1}(t)) = 1,$$

cf. Lemma 5.3 in [Th5]. In particular, $p \neq 0$, and $\hat{\psi}_t$ is a state on $pM p$. Since $C_0(\omega_G) \subseteq C^*(G)$ is in the fixed point algebra of $\alpha^F$ it follows that $p$ is fixed by $\hat{\alpha}^F$. Set

$$p_{\xi,\xi'} = p1_{\xi,\xi'} p.$$

A calculation in the $*$-algebra $C_c(\mathcal{G})$ shows that

$$q_m 1_{\xi,\xi} q_m q_n 1_{\xi,\xi} q_m = \begin{cases} 0, & \text{when } \xi_2 \neq \xi_3, \\ q_m 1_{\xi_1,\xi_3} q_m, & \text{when } \xi_2 = \xi_3, \end{cases}$$

provided $m > n$. It follows that $\{p_{\xi,\xi'} : \xi, \xi' \in A_n\}$ is the set of matrix units in a unital $C^*$-subalgebra $\mathcal{M}_n$ of dimension $\#A_n$ in $pM p$. Observe that

$$p_{\xi,\xi'} = \sum_e p_{\xi,\xi'}^e$$

when we sum over all edges $e \in s^{-1}(t_n) \cap r^{-1}(t_{n+1})$. It follows that $\mathcal{M}_n \subseteq \mathcal{M}_{n+1}$. Note also that $pM p$ is the closure in the strong operator topology of

$$p1_{t_n} C_c(\mathcal{G}) 1_{t_n} p.$$
By definition of the topology of \( \mathcal{G} \) any \( f \in 1_tC_c(\mathcal{G})1_t \) can be approximated in norm by finite linear combinations of functions that are the characteristic function of a set \( B \) of the form (2.2) with \( s(\mu) = s(\mu') = t_1 \). Note that \( q_m1_Bq_m = 0 \) when \( m > \max\{|\mu|,|\mu'|\} \) and \( |\mu| \neq |\mu'| \). So assume that \( |\mu| = |\mu'| = k \). Then \( q_m1_Bq_m = 0 \) for \( m > k \) unless \( \mu, \mu' \in A_k \). So assume that \( \mu, \mu' \in A_k \). When \( m \) is larger than \( m_0 = k + \max\{|\nu| : \nu \in F \cup F'\} \) we see that there are subsets \( F_1, F_2 \subseteq A_{m_0} \) such that
\[
q_m1_Bq_m = \sum_{\xi \in F_1, \xi' \in F_2} q_m1_{\xi,\xi'}q_m.
\]
It follows first that \( p1_Bp \in \mathcal{M}_{m_0} \) and then that \( \bigcup_n \mathcal{M}_n \) is dense in \( pMp \) for the strong operator topology.

Using the same numbering of the edges in \( s^{-1}(t_n) \cap r^{-1}(t_{n+1}) \) as above we set
\[
u_t^n = \text{diag} \left( e^{iF(e_1^n)\mu}, e^{iF(e_2^n)\mu}, \ldots, e^{iF(e_{n_k}^n)\mu} \right) \in M_{kn}(\mathbb{C}).
\]
The norm closure \( \bigcup_n \mathcal{M}_n \) is a copy of the UHF-algebra \( A \) from (1.2) and in this picture the restriction of the automorphism group \( \tilde{\alpha}_F \) to \( \bigcup_n \mathcal{M}_n \) is the tensor product action
\[
\otimes_{n=1}^\infty \text{Ad} \nu_t^n, \ t \in \mathbb{R}.
\]
The tensor product state \( \omega = \otimes_{n=1}^\infty \omega_n \) is a \( \beta \)-KMS state for \( \otimes_{n=1}^\infty \text{Ad} \nu_t^n \) on \( A \), and the same is the restriction of \( \psi_t \) to \( \bigcup_n \mathcal{M}_n \). The two states must therefore agree under the identification \( \bigcup_n \mathcal{M}_n = A \), thanks to uniqueness of the \( \beta \)-KMS state, [K].

It follows that
\[
pMp \simeq \pi_\omega(A)'' = \mathcal{R}(t).
\]

We will say that the exit path \( t = (t_n)_{n=1}^\infty \) is slim when \( s^{-1}(t_i) \cap r^{-1}(t_{i+1}) \) only contains one edge for all \( i \) large enough.

**Theorem 4.2.** Let \( G \) be a strongly connected graph and \( t \) an exit path in \( G \) which is \( \beta \)-summable for the gauge action, and let \( \psi_t \) be the corresponding \( \beta \)-KMS weight for the gauge action on \( C^*(G) \). The factor type of \( \psi_t \) is \( I_\infty \) when \( t \) is slim and \( II_\infty \) factor when it is not.

**Proof.** The gauge action arises by choosing \( F \) to be constant 1. It follows that \( \mathcal{R}(t) \) is a type I factor when \( t \) is slim and the hyper finite \( II_1 \) factor otherwise. It follows therefore from Lemma 4.1 that \( \pi_{\psi_t}(C^*(G))'' \) is type I when \( t \) is slim and type II otherwise. To complete the proof we need to show that \( M \) is not finite. To this end choose for each \( n \geq 2 \) a path \( \mu_n \) in \( G \) such that \( s(\mu_n) = t_n \) and \( r(\mu_n) = t_1 \). The characteristic function \( V_n \) of the set
\[
\{(\mu_n,x,|\mu_n|,x) : x \in Z(t_1)\}
\]
is an element of \( C_c(\mathcal{G}) \) such that \( V_n^*V_n = 1_{t_1} \) and \( V_nV_n^* \leq 1_{t_n} \). In particular, the \( V_n \)'s are non-zero partial isometries in \( M \) and since \( t \) is an exit, an infinite sub collection of them will have orthogonal ranges. It follows that \( M \) is not finite. \( \square \)

The dichotomy in Corollary 4.2 can also be formulated in terms of the exit measure \( m_t \). Indeed, \( t \) is slim if and only if \( m_t \) is an atomic measure on \( P(G) \). The constructions in [15,5] show that both possibilities occur in abundance.
It follows from Lemma 2.3.3 in [C] that in the setting of Lemma 4.1 we have the identity
\[ \Gamma(\pi_\psi(C^*(G)))'' = \Gamma(R(t)). \]
Therefore Lemma 4.1 also applies to determine the factor type of the extremal \( \beta \)-KMS weights coming from an exit for more general actions. The following gives an example of this.

**Example 4.3.** Let \( G \) be the following graph.

\[
\begin{array}{c}
\cdots \\
& t_1 \nonumber \\
& \downarrow \\
& t_2 \nonumber \\
& \downarrow \\
& t_3 \nonumber \\
& \downarrow \\
& t_4 \nonumber \\
& \downarrow \\
& t_5 \nonumber \\
& \downarrow \\
& \cdots \\
\end{array}
\]

\[ (4.3) \]

For each \( n \in \mathbb{N} \) there are two edges \( e^n_1 \) and \( e^n_2 \) such that \( s(e^n_i) = t_n, r(e^n_i) = t_{n+1}, \) \( i = 1, 2 \). For each \( n \) we set \( F(e^n_i) = a_i \), where \( a_1, a_2 \) are positive real numbers with \( a_1 > a_2 \), and we set \( F(e) = 0 \) for all other edges \( e \). Let \( \beta \in \mathbb{R} \) and consider the matrix \( A(\beta) \) from (3.3). Recall, [Th5], that a harmonic vector for \( A(\beta) \) is a vector \( \psi = (\psi_v)_{v \in V} \) such that \( \psi_v > 0 \) and \( \psi_v = \sum_{w \in V} A(\beta)_{vw} \psi_w \) for all \( v \in V \). Set
\[
x_\beta = e^{-\beta a_1} + e^{-\beta a_2}.
\]
Consider a vector \( \psi = (\psi_v)_{v \in V} \) with \( \psi_{t_1} = 1 \). Then \( \psi \) is a harmonic vector for \( A(\beta) \) if and only if \( \psi_v > 0 \) for all \( v \in V \), and
\[
\begin{align*}
& a) \quad \psi_{d_i} = 1 \text{ for all } i, \\
& b) \quad \psi_{t_2} = x_\beta^{-1}, \text{ and} \\
& c) \quad \psi_{t_{k+1}} = x_\beta^{-1}(\psi_{t_k} - 1), \quad k \geq 2.
\end{align*}
\]
It follows that a harmonic vector exists if and only if \( \beta \geq \beta_0 \), where \( \beta_0 \) is the real number with the property that
\[
x_{\beta_0} = e^{-\beta_0 a_1} + e^{-\beta_0 a_2} = \frac{1}{2}.
\]
By using the Criterion 2 in Corollary 4 on page 372 in [V] we see that \( A(\beta) \) is 1-transient for \( \beta > \beta_0 \) and 1-recurrent for \( \beta = \beta_0 \). It follows then from Theorem 5.6 in [Th5] that there is a unique \( \beta \)-KMS weight \( \psi_\beta \) for \( \alpha^F \), up to multiplication by scalars, for all \( \beta \geq \beta_0 \); for \( \beta > \beta_0 \) it is given by the exit measure coming from the unique exit in \( G \) and for \( \beta = \beta_0 \) it is given by a conservative measure on \( P(G) \). By Lemma 4.1 the factor type of \( \psi_\beta \) is \( III_\lambda \) where \( \lambda = e^{-\beta(a_1 - a_2)} \) when \( \beta > \beta_0 \). \( \pi_{\psi_\beta}(C^*_\alpha(G))'' \) is a Powers factor in this case. Concerning the factor type of \( \psi_{\beta_0} \) we note that the condition on \( F \) in Theorem 3.2 is satisfied and conclude that
\[
\Gamma(\pi_{\psi_{\beta_0}}(C^*_\alpha(G)))'' = \beta_0 a_1 \mathbb{Z} + \beta_0 a_2 \mathbb{Z}.
\]
Thus \( \psi_{\beta_0} \) is of type \( III_1 \) if \( \frac{a_1}{a_2} \) is irrational. In other cases its type can be determined from 8.15.11 of [Pe].
5. Ground states

In this section the graph $G$ can be an arbitrary countable oriented graph and $F: E \to \mathbb{R}$ an arbitrary function. We consider the generalized gauge action $\alpha^F$ restricted to a corner $1_v C^*(G)1_v$, where $v$ is a fixed but arbitrary vertex in the graph. Recall that a state $\omega$ on $1_v C^*(G)1_v$ is a ground state for $\alpha^F$ restricted to $1_v C^*(G)1_v$ when

$$-i\omega(\alpha^F(a)) \geq 0$$

for all elements $a$ in the domain of $\delta$, the infinitesimal generator of (the restriction of) $\alpha^F$, cf. [BR]. To describe the ground states observe first that the fix point algebra of $\alpha^F$ is the $C^*$-algebra of the open sub-groupoid $\mathcal{F} = \{((\mu x, |\mu| - |\mu'|, \mu' x) : x \in \Omega_G, F(\mu) = F(\mu')\}$ of $G$. The conditional expectation $Q : C^*(G) \to C^*_r(\mathcal{F})$ extending the restriction map $C_c(G) \to C_c(\mathcal{F})$ can be described as a limit:

$$Q(a) = \lim_{R \to \infty} \frac{1}{R} \int_0^R \alpha^F_t(a) \, dt,$$

(5.1)

cf. the proof of Theorem 2.2 in [Th3].

When $x \in \Omega_G$, $z \in P_f(G)$, write $z \subseteq x$ when $x[|z|] = z$ or $|z| = 0$ and $z = s(x)$. An element $x \in \Omega_G$ is an $F$-geodesic when the following holds:

$$z, z' \in P_f(G), z \subseteq x, s(z') = s(z), r(z') = r(z) \Rightarrow F(z') \geq F(z).$$

We denote the set of $F$-geodesics in $\Omega_G$ by $\text{Geo}(F,G)$. We leave the proof of the following observation to the reader.

Lemma 5.1. $\text{Geo}(F,G) \cap Z(v)$ is closed in $Z(v)$, and $\mathcal{F}$-invariant in the sense that

$$(x, k, y) \in \mathcal{F}, x \in \text{Geo}(F,G) \cap Z(v), y \in Z(v) \Rightarrow y \in \text{Geo}(F,G).$$

Set $\text{Geo}_0(F,G) = \text{Geo}(F,G) \cap Z(v)$. It follows from Lemma[5.1] that the reduction $\mathcal{F}|_{\text{Geo}_0(F,G)}$ of $\mathcal{F}$ to $\text{Geo}_0(F,G)$ is an étale groupoid in itself and that there is a surjective $*$-homomorphism

$$R : 1_v C^*_r(\mathcal{F})_1 \to C^*_r(\mathcal{F}|_{\text{Geo}_0(F,G)})$$

extending the map $1_v C_c(\mathcal{F})_1 \to C_c(\mathcal{F}|_{\text{Geo}_0(F,G)})$ given by restriction.

Lemma 5.2. $-iR \circ Q(f^*\delta(f)) \geq 0$ for all $a$ in the domain of $\delta$.

Proof. Since $1_v C_c(G)_1$ is a core for $\delta$, cf. Corollary 3.1.7 in [BR], it suffices to show that

$$-iR \circ Q(f^*\delta(f)) \geq 0$$

when $f \in 1_v C_c(G)_1$. There is a finite collection of paths $\mu_i, \nu_i \in P_f(G), i = 1, 2, \ldots, m$, such that $s(\mu_i) = s(\nu_i) = v$ for all $i$ and $f = \sum_{k=1}^m f_k$, where

$$\text{supp } f_k \subseteq \{(\mu_k, |\mu_k| - |\nu_k|, \nu_k x) : x \in \Omega_G\}.$$ 

Set

$$A = \{F(\mu_k) - F(\nu_k) : k = 1, 2, \ldots, m\}.$$ 

Then $f = \sum_{a \in A} h_a$, where $h_a \in 1_v C_c(G)_1$ has support in

$$\bigcup_{F(\mu_k) - F(\nu_k) = a} \{(\mu_k x, |\mu_k| - |\nu_k|, \nu_k x) : x \in \Omega_G\}.$$
Factor Types and Ground States

In particular, \( \alpha_t^F(h_a) = e^{iat}h_a \) for all \( t \in \mathbb{R} \). It follows that

\[
-iQ(f^*\delta(f)) = \sum_{a,b \in A} bQ(h_a^*h_b) = \sum_{a \in A} ah_a^*h_a.
\]

Note that

\[
(x, k, y) \in \mathbb{G}, \ s(x) = v, \ y \in \text{Geo}_v(F, G) \Rightarrow h_a(x, k, y) = 0
\]

when \( a < 0 \). It follows that \( R(h_a^*h_a) = 0 \) when \( a < 0 \) and hence

\[
R(\sum_{a \in A} ah_a^*h_a) = R(\sum_{a \geq 0} ah_a^*h_a) \geq 0.
\]

\[ \square \]

**Theorem 5.3.** The map \( \omega \mapsto \omega \circ R \circ Q \) is an affine homeomorphism from the state space of \( C^*_r(F|_{\text{Geo}_v(F, G)}) \) onto the set of ground states for the restriction of \( \alpha^F \) to the corner \( 1_v C^*_r(G)1_v \).

**Proof.** Let \( \tau \) be a ground state for the gauge action on \( 1_v C^*_r(G)1_v \). Since \( \tau \) is \( \alpha^F \)-invariant by Proposition 5.3.19 in [BR] it follows that \( \tau = \tau \circ Q \). Thanks to Lemma 5.2 it suffices therefore to show that the restriction of \( \tau \) to \( 1_v C^*_r(F)1_v \) factorises through \( R \), i.e. that \( \tau \) annihilates the kernel of \( R \), which is the reduction of \( F \) to \( Z(v) \setminus \text{Geo}_v(F, G) \). (This fact follows from Remark 4.10 in [Re2] provided we know that \( F|_{\text{Geo}_v(F, G)} \) is amenable. That this is true follows from Theorem 4.2 in [Pa] by use of results from [AD-DR].) There is an approximate unit for \( C^*_r(F|_{Z(v) \setminus \text{Geo}_v(F, G)}) \) in \( C_c(Z(v) \setminus \text{Geo}_v(F, G)) \) and it suffices therefore to show that \( \tau(f) = 0 \) when \( f \in C_c(Z(v) \setminus \text{Geo}_v(F, G)) \). Let \( x \in \text{supp} \ f \). Since \( x \notin \text{Geo}_v(F, G) \) there is an element \( h \in C_c(\mathbb{G}) \), supported in a bi-section of \( \mathbb{G} \), such that \( \alpha^F_t(h) = e^{iat}h \) for all \( t \in \mathbb{R} \), where \( a < 0 \), and such that \( h^*h \in C_c(Z(v) \setminus \text{Geo}_v(F, G)) \) is constant 1 in a neighborhood of \( x \). Now

\[
0 \leq -i\tau(h^*\delta(h)) = ah^*h,
\]

because \( \tau \) is a ground state, and it follows that \( \tau(h^*h) = 0 \) since \( a < 0 \). As \( x \in \text{supp} \ f \) was arbitrary, a standard partition of unity argument shows that \( f \) is a finite sum, \( f = \sum_i g_i f \), where \( g_i \in C_c(Z(v) \setminus \text{Geo}_v(F, G)) \) are non-negative functions in the kernel of \( \tau \). Hence \( \tau(f) = 0 \), as required. \[ \square \]

To formulate what we get from Theorem 5.3 concerning the ground states of the gauge action, we say that an element \( x \in \Omega_v \) is a geodesic when \( x \in V_\infty \) or the following holds:

\[
y \in \Omega_v, \ s(y_1) = s(x_1), \ r(y_j) = r(x_i) \Rightarrow j \geq i.
\]

We denote the set of geodesics in \( \Omega_v \) by \( \text{Geo}(G) \), and set \( \text{Geo}_v(G) = \text{Geo}(G) \cap Z(v) \). Set

\[
\mathcal{R} = \{(\mu x, 0, \mu' x) \in \mathbb{G} : x \in \Omega_v\},
\]

which is the open sub-groupoid of \( \mathbb{G} \) supporting the fixed point algebra of the gauge action.

**Corollary 5.4.** The set of ground states of the gauge action on \( 1_v C^*_r(G)1_v \) is affinely homeomorphic to the state space of \( C^*_r(\mathcal{R}|_{\text{Geo}_v(G)}) \).
Remark 5.5. Following Connes and Marcolli, [CM], several authors have singled out the so-called $KMS_\infty$-states among the ground states for various one-parameter groups. A ground state is a $KMS_\infty$-state when it is the limit in the weak* topology of a sequence of $\beta_n$-KMS states, where $\lim_{n \to \infty} \beta_n = \infty$. By combining the description of the ground states in Theorem 5.3 with the observation that KMS states for $\alpha^F$ on the corner $1_v C^*(G) 1_v$ are trace states on $C^*_r (F)$, it follows that a ground state can only be a $KMS_\infty$-state if it is a trace. Presently it is not clear if this is the only condition in general.

If $G$ is a strongly connected graph with infinitely many vertexes, the set $Geo_v(G)$ is not empty for any vertex $v$. Indeed, if there is an infinite emitter $v_0$ in $V$, a path in $G$ from $v$ to $v_0$ of minimal length is an element in $Geo_v(G) \cap Q(G)$, and if there are no infinite emitters in $G$, the proof of Lemma 7.5 in [Th5] will give us an element of $Geo_v(G) \cap P(G)$.

Example 5.6. The following graph $G$ was considered in Example 5.1 in [Th3].

Let $\alpha$ be the real root of the polynomial $x^3 - x - 3$. As shown in [Th3] there are three extremal rays of $\beta$-KMS weights for the gauge action on $C^*(G)$ when $\beta > \log \alpha$ and a unique ray of $\log \alpha$-KMS weights. In fact, the last ray consists of finite $\log \alpha$-KMS weights that may be normalised to a $\log \alpha$-KMS state. For $\beta > \log \alpha$ the weights are not finite. The factor type of the KMS weights were not determined in [Th3], but it follows from Theorem 4.2 that the factor type of the $\log \alpha$-KMS state is $III_{\alpha-1}$ while the other extremal $\beta$-KMS weights are all of type $I_\infty$ by Theorem 4.2.
Let $v$ be the central vertex in $G$; the one labelled 1. Then $\text{Geo}_v(G)$ consists of three easily identified paths in $G$ and

$$C^*_r(\mathcal{R}|_{\text{Geo}_v(G)}) \simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}.$$ 

By Corollary 5.4, the set of ground states for the gauge action on $1_vC^*(G)1_v$ is a triangle.

**Example 5.7.** Consider the gauge action on $C^*(G)$ when $G$ is the graph (4.3). A vector $\psi_v$ indexed by the vertexes in $G$ with $\psi_{t_1} = 1$ is harmonic for $A(\beta)$ when $\psi_v > 0$ for all $v \in V$ and

1. $\psi_{d_n} = e^{-n\beta} \quad n = 1, 2, 3, \ldots,$
2. $\psi_{t_2} = \frac{1}{\alpha},$
3. $\psi_{t_{k+1}} = \frac{1}{2}(e^k\psi_{t_k} - e^{-k\beta}) \quad k \geq 2.$

It follows from this (and [Th5]) that there is a $\beta$-KMS weight $\psi_\beta$ for the gauge action on $C^*(G)$ if and only if $\beta \geq \frac{1}{2}\log \alpha$, where $\alpha \sim 1.61$ is the positive root of the polynomial $x^2 - 2x - 2$, and that it is unique up to multiplication by scalars. In addition, it follows from [Th5] that only the $\frac{\log \alpha}{2}$-KMS weight can arise from a conservative measure on $P(G)$, and by using the criterion from [V] one checks that it does. It follows then from Theorem 4.2 that $\psi_\beta$ is of factor type $III_\infty$ for all $\beta > \frac{\log \alpha}{2}$ while the factor type of $\psi_\beta$ is $III_{\alpha-1}$ when $\beta = \frac{\log \alpha}{2}$ by Theorem 3.2.

Concerning the ground states it follows from Corollary 5.4 that the ground states for the gauge action on the corner $1_{t_1}C^*(G)1_{t_1}$ are parametrised by the state space of the UHF algebra of type $2^\infty$, also known as the CAR algebra. For comparison, let $\alpha^F$ be the generalized gauge action on $C^*(G)$ considered in Example 4.3. The restriction of $\alpha^F$ to the same corner $1_{t_1}C^*(G)1_{t_1}$ has only one ground state by Theorem 5.3.

**Example 5.8.** By using amalgamations of graphs in much the same way as in [Th5], it is possible to build examples with other kinds of variation of the factor types. For example we can construct cases where there are extremal KMS weights of different factor types for the same value of $\beta$. For this consider the disjoint union of the graph (4.3) from Example 4.3 and the graph (5.6) from Example 5.6 and identify the vertex $t_1$ from (4.3) with the vertex 1 from (5.6) to get a strongly connected graph $H$. Consider the generalised gauge action on $C^*(H)$ obtained by taking $F(e)$ to be the value it has in Example 4.3 when $e$ comes from (4.3) and to be 1 when $e$ comes from (5.6). For the corresponding action $\alpha^F$ on $C^*(H)$ there are $\beta$-KMS weights if and only if $\beta \geq \beta_0$ where $\beta_0$ is the positive real number for which

$$\frac{3e^{-\beta_0}}{e^{2\beta_0} - 1} + \frac{e^{-\beta_0 a_1} + e^{-\beta_0 a_2}}{1 - e^{-\beta_0 a_1} - e^{-\beta_0 a_2}} = 1.$$ 

When $\beta > \beta_0$ there are four extremal rays of $\beta$-KMS weights, three of which are of type $I_\infty$ and one of type $III_\lambda$ where $\lambda = e^{-\beta(a_1-a_2)}$. There is only one ray of $\beta_0$-KMS weights, and by using the results of Vere-Jones as in Example 4.3 we can conclude from [Th5] that the corresponding measure on $P(H)$ is conservative. We can then use Theorem 4.2 to see that the $\Gamma$-invariant of the corresponding factor is

$$\beta_0 a_1\mathbb{Z} + \beta_0 a_2\mathbb{Z} + \beta_0\mathbb{Z}.$$ 

Its type depends therefore on the two parameters $a_1, a_2$, and can be determined from 8.15.11 of [Pe].
References

[A-DR] C. Anantharaman-Delaroche and J. Renault, Amenable groupoids, Monographies de L'Enseignement Mathematique, vol. 36, L'Enseignement Mathematique, Geneva, 2000.

[AW] H. Araki and E.J. Woods, A classification of factors, Publ. Res. Inst. Math. Sci. Ser. A 4 (1968), 51-130.

[BR] O. Bratteli and D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics I + II, Texts and Monographs in Physics, Springer Verlag, New York, Heidelberg, Berlin, 1979 and 1981.

[BEK] O. Bratteli, G. Elliott and A. Kishimoto, The temperature state space of a dynamical system I, J. Yokohama Univ. 28 (1980), 125-167.

[CL] T.M. Carlsen and N. Larsen, Partial actions and KMS states on relative graph $C^*$-algebras, arXiv:1311.0912.

[Co] F. Combes, Poids associé à une algèbre hilbertienne à gauche, Compos. Math. 23 (1971), 49-77.

[CT] J. Christensen and K. Thomsen, Finite digraphs and KMS states, arXiv, May 2015.

[C] A. Connes, Une classification des facteurs de type III, Ann. Sci. Ecole Norm. Sup. 6 (1973), 133-252.

[CM] A. Connes and M. Marcolli, Quantum statistical mechanics of Q-lattices, In Frontiers in Number Theory, Physics, and Geometry I, Springer-Verlag, 2006, pp 269-349.

[EFW] M. Enomoto, M. Fujii and Y. Watatani, KMS states for gauge action on $O_A$, Math. Japon. 29 (1984), 607-619.

[EL] R. Exel and M. Laca, Partial dynamical systems and the KMS condition, Comm. Math. Phys. 232 (2003), 223-277.

[aHLRS] A. an Huef, M. Laca, I. Raeburn and A. Sims, KMS states on the $C^*$-algebras of reducible graphs, Ergodic Th. & Dynam. Syst., to appear.

[Ka] E. T.A. Kakariadis, On Nica-Pimsner algebras of $C^*$-dynamical systems over $\mathbb{Z}^+_N$, arXiv:1411.4992.

[KR] R.V. Kadison and J.R. Ringrose, Fundamentals of the Theory of Operator Algebras II, Academic Press, London 1986.

[K] A. Kishimoto, On uniqueness of KMS states for one-dimensional quantum lattice systems, Comm. Math. Phys. 47 (1976), 167-170.

[KV1] J. Kustermans and S. Vaes, Weight theory for $C^*$-algebraic quantum groups, arXiv:990163.

[KV2] J. Kustermans and S. Vaes, Locally compact quantum groups, Ann. Scient. Éc. Norm. Sup. 33, (2000), 837-934.

[LR] M. Laca and I. Raeburn, Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers, Adv. Math. 225 (2010), 643-688.

[N] S. Neshveyev, KMS states on the $C^*$-algebras of non-principal groupoids, J. Operator Theory 70 (2013), 513-530.

[O] R. Okayasu. Type III factors arising from Cuntz-Krieger algebras, Proc. Amer. Math. Soc. 131 (2003), 2145-2153.

[OP] D. Olesen and G.K. Pedersen, Some $C^*$-dynamical systems with a single KMS state, Math. Scand. 42 (1978), 111-118.

[Pa] A.L.T. Paterson, Graph inverse semigroups, groupoids and their $C^*$-algebras, J. Operator Theory 48 (2002), 645-662.

[Pe] G. K. Pedersen, $C^*$-algebras and their automorphism groups, Academic Press, London, 1979.

[PS] R.T. Powers and S. Sakai, Existence of ground states and KMS states for approximately inner dynamics, Comm. Math. Phys. 39 (1975), 273-288.

[Re1] J. Renault, A Groupoid Approach to $C^*$-algebras, LNM 793, Springer Verlag, Berlin, Heidelberg, New York, 1980.

[Re2] J. Renault, The ideal structure of groupoid crossed product $C^*$-algebras, J. Oper.Theory 25 (1991), 3-36.

[Th1] K. Thomsen, Semi-étale groupoids and applications, Annales de l’Institute Fourier 60 (2010), 759-800.
[Th2] K. Thomsen, *Exact circle maps and KMS states*, Israel J. Math. **205** (2015), 397-420. DOI: 10.1007/s11856-014-1124-x

[Th3] K. Thomsen, *KMS weights on groupoid and graph C*-algebras*, J. Func. Analysis **266** (2014), 2959-2988.

[Th4] K. Thomsen, *Dissipative conformal measures on locally compact spaces*, Ergodic Th. & Dynam. Syst., to appear.

[Th5] K. Thomsen, *KMS weights on graph C*-algebras*, [arXiv:1409.3702](https://arxiv.org/abs/1409.3702)

[V] D. Vere-Jones, *Ergodic properties of non-negative matrices I*, Pacific J. Math. **22** (1967), 361-386.

E-mail address: matkt@math.au.dk

Department of Mathematics, Aarhus University, Ny Munkegade, 8000 Aarhus C, Denmark