MaxWeight With Discounted UCB:
A Provably Stable Scheduling Policy for Nonstationary Multi-Server Systems With Unknown Statistics

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Abstract
Multi-server queueing systems are widely used models for job scheduling in machine learning, wireless networks, and crowdsourcing. This paper considers a multi-server system with multiple servers and multiple types of jobs. The system maintains a separate queue for each type of jobs. For each time slot, each available server picks a job from a queue and then serves the job until it is complete. The arrival rates of the queues and the mean service times are unknown and even nonstationary. We propose the MaxWeight with discounted upper confidence bound (UCB) algorithm, which simultaneously learns the statistics and schedules jobs to servers. We prove that the proposed algorithm can stabilize the queues when the arrival rates are strictly within the service capacity region. Specifically, we prove that the queue lengths are bounded in the mean under the assumption that the mean service times change relatively slowly over time and the arrival rates are bounded away from the capacity region by a constant whose value depends on the discount factor used in the discounted UCB. Simulation results confirm that the proposed algorithm can stabilize the queues and that it outperforms MaxWeight with empirical mean and MaxWeight with discounted empirical mean. The proposed algorithm is also better than MaxWeight with UCB in the nonstationary setting.

1 Introduction
A multi-server system is a system with multiple servers for serving jobs of different types as shown in Figure 1. An incoming job can be served by one of the servers and the service time depends on both the server capacity and the job type. Multi-server systems have been used to model many real-world applications in communication and computer systems such as load balancing in a cloud-computing cluster, packet scheduling in multi-channel wireless networks, packet switches, etc. One of the best known algorithms for scheduling in multi-server systems is the celebrated MaxWeight algorithm proposed in [22]. Let $Q_i(t)$ denote the number of type-$i$ jobs waiting to be served and $1/\mu_{i,j}$ denote the mean service time of serving a type-$i$ job at server $j$. When server $j$ is available, MaxWeight schedules a type $i_j$ job to server $j$ such that

$$i_j^* \in \arg \max Q_i(t) \mu_{i,j}.$$ 

The MaxWeight algorithm is provably throughput optimal [22], i.e., the queues are stable in the mean as long as the arrivals are within the service capacity region. Besides throughput optimality,
MaxWeight has also superior delay performance and has been shown to be heavy-traffic delay optimal in various settings [21, 1] [13, 9, 4, 16].

Figure 1: A multi-server system with three servers and three types of jobs

A key assumption behind the MaxWeight algorithm is that the scheduler knows the mean service rates $\mu_{i,j}$ for all $i$ and $j$. This assumption is becoming increasingly problematic in emerging applications such as cloud computing and crowdsourcing due to either high variability of jobs (such as complex machine learning tasks) or servers (such as human experts in crowdsourcing). In these emerging applications, the mean service rates need to be learned while making scheduling decisions. Therefore, learning and scheduling are coupled and jointly determine the performance of the system.

A straightforward idea to learn the mean service rates is using the sample average, i.e., replacing $\mu_{i,j}$ with $1/\bar{s}_{i,j}$ where $\bar{s}_{i,j}$ is the empirical mean of the service time of type-$i$ jobs at server $i$, based on the jobs completed at server $i$ so far. However, because of the coupling between learning and scheduling, this approach is unstable as shown in the following example.

Counter Example

Consider a multi-server system with two servers and two job types with the following statistics:

$$\Pr(S_{i,j} = 100) = 0.01 \quad \text{for} \quad i = j$$
and

$$\Pr(S_{i,j} = 1) = 0.99 \quad \text{for} \quad i \neq j,$$

where $S_{i,j}$ is the service time of type $i$ jobs at server $j$. We further assume the following job arrival process: $A_i(t) = 1$ for any $i$ and any $t = 1, 3, \cdots$ and $A_i(t) = 0$ for any $i$ and $t = 2, 4, \cdots$. We next consider the queue lengths over time under MaxWeight with empirical mean, and assume the algorithm uses $\hat{\mu}_{i,j} = 1$ as a default value if there is no data sample for $S_{i,j}$.

- **Time slot 1:** A type-$i$ job is scheduled at server $i$ and $S_{i,i} = 100$ for $i = 1, 2$ which occurs with probability 0.01.
- **Time slot 101:** Both queues have 49 jobs. We have estimated $\hat{\mu}_{i,i} = 0.01$ and $\hat{\mu}_{i,j} = 1 (i \neq j)$ as the default value. The algorithm now schedules a type-$i$ job to server $j$ for $j \neq i$.
- **Time slot 111:** Both queues have 53 jobs. The estimated service rates are $\hat{\mu}_{i,i} = 0.01$ and $\hat{\mu}_{i,j} = 0.1$ for $i \neq j$. Based on MaxWeight with mean-service-rate, the scheduler schedules type-$i$ jobs to server $j$ for $i \neq j$.
- **Time slot $> 111$:** Since $S_{i,j}$ is a constant for $i \neq j$, the estimated service rates do not change after the jobs are completed. Since the estimated service rates do not change as long as type-$i$ jobs are scheduled on server $j$ such that $i \neq j$, the schedule decisions also remain the same such that type-$i$ jobs are continuously scheduled to server $j$ for $i \neq j$. Since it takes 10 time slots to finish a job and there is a job arrival every two slots, both queues go to infinity.
Note that if we schedule type-i jobs to server i, the mean queue lengths are bounded because in this case, the mean service time is 1.99 time slots and the arrival rate is one job every two time slots. From the example above, we can see that the problem of using empirical mean is that the initial bad samples led to a poor estimation of $\mu_{i,j}$, which led to poor scheduling decisions. Since the scheduler only gets new samples from the served jobs, it was not able to correct the wrong estimate of $\hat{\mu}_{i,i} = 0.01$ when type-i jobs are no long routed to server i after time slot 101. Therefore, the system was “locked in” in a state with poor estimation and wrong scheduling decisions, which led to instability.

To overcome this problem, as in multi-armed bandit problems, we should encourage exploration: since the service rate of a server for a particular job can be estimated only by repeatedly scheduling jobs on all jobs, we should occasionally schedule jobs even on servers whose service rates are estimated to be small to overcome poor estimates due to randomness. For example, as in online learning, we can add an exploration bonus $b_{i,j}$, e.g., the upper confidence bound (UCB), to the empirical mean $\hat{\mu}_{i,j}$. Indeed, there have been a sequence of recent studies that study job scheduling in multi-server systems as an online learning problem (multi-armed bandits or linear bandits). We now review different categories of prior work along these lines and place our work in the context of the prior work:

**Queue-blind Algorithms:** Queue blind algorithms do not take queue lengths into consideration at all when making scheduling decisions. In one line of work, the performance metric is the total reward received from serving jobs \[14, 15\]; however, for such algorithms, the queue lengths can potentially blow up to infinity asymptotically, which means that finite-time bounds for queue lengths can be excessively large and thus, such algorithms cannot be used in practice. Another line of work in the context of queue-blind scheduling algorithms addresses stability by assuming that the arrival rates of each type of job is known. They then use well-known scheduling algorithms such as $c\mu$-rule \[12\] or weighted random routing \[2\] or utility-based joint learning and scheduling \[8\]. The drawback of such algorithms is that queue lengths can still be excessive large even if the queue lengths do not blow up to infinity asymptotically. The reason is the knowledge of queue lengths can encourage a phenomenon called resource pooling which leads to greater efficiency. While we will not spend too much space explaining the concept of resource pooling, we hope that the following example clarifies the situation. Suppose you visit a grocery store and are not allowed to look at the queue lengths at each checkout lane before joining the checkout line. Then, some checkout lines can be excessively long, while others may even be totally empty. On the other hand, in practice, we look at the length of each checkout line and join the shortest one, which results in much better delay performance.

**Queue-Aware Algorithms:** In early work on the problem \[17, 11, 13, 24\], a fraction of time is allocated to probing the servers and the rest of the time is used to exploit this information. In the context of our problem, we would end up exploring all (job type, server) pairs the same number of times which is wasteful. On the other hand, exploration and exploitation are decoupled in such a forced exploration, which makes it easier to derive analytical derivation of performance bounds . If one uses optimistic exploration such as UCB or related algorithms, the queue length information and the UCB-style estimation are coupled, which makes it difficult to analyze the system. Two approaches to decoupling UCB-style estimators have been studied prior to our paper: (a) In the work by \[20\], the algorithm proceeds in frames (a frame is a collection of contiguous time slots), where the queue length information is frozen at the beginning of each frame and UCB is used to estimate the service rates of the servers; additionally, UCB is reset at the end of each frame, and (b) In the work by \[5\], a schedule is fixed throughout each phase and thus, UCB is only executed for the jobs which are scheduled in that frame. The correlation between queues and UCB is more complicated here than in the algorithm of \[20\], which requires more sophisticated analysis to conclude stability. Our paper does not explicitly decouple exploration and exploitation. In fact, we continuously update the UCB bonuses and perform scheduling at each time instant but we use a version of UCB tailored to nonstationary environments. This allows our algorithm to quickly adapt to changes even in stationary settings, in addition to having the advantage of being able to handle nonstationary environments. On the other hand, the fact that the schedule and UCB are updated at every time step means that we
require a new analysis of stability. In particular, unlike prior work, our approach requires the use of concentration results for self-normalized means from [6]. In addition to differences in the algorithms and analysis, we also note other key differences between our paper and theirs [20, 5]: [20] considers scheduling in a general conflict graph, which includes our multi-server model as a special case. [5] considers a general multi-agent setting that includes the centralized case as a special case. Both [20] and [5] assume the system is stationary but [5] allows dynamic arrivals and departures of queues while our paper studies a nonstationary, centralized setting that includes the stationary setting with a fixed set of queues as a special case.

The main contributions of this paper are summarized below.

- **Theoretical Results:** We introduce MaxWeight with discounted UCB in this paper. Discounted UCB was first proposed for nonstationary bandit problems [10]. For our problem, with a revised discounted UCB, the values of UCB bonuses depend on limited past history, instead of the entire history, which allows us to decouple the queue lengths and UCB bonuses. We establish the queue stability of MaxWeight with discounted UCB for more challenging non-stationary environments where the arrival rates and service rates may change over time.

- **Methodology:** Our analysis is based on the traditional Lyapunov drift analysis. However, there are several difficulties due to joint scheduling and learning. For example, the estimated mean service time is the discounted sum of previous service times divided by the sum of the discount coefficients and the summation is taken over the time slots in which there is job completion, which themselves are random variables depending on the scheduling and learning algorithm. To deal with this difficulty, we first transform the summation into a summation over the time slots in which there is a job starting, and then use a Hoeffding-type inequality for self-normalized means with a random number of summands [6, Theorem 18] to obtain a concentration bound. Another difficulty is to bound the discounted number of times server \( j \) serves type-\( i \) jobs. Our method is to divide the interval into sub-intervals of carefully chosen lengths so that the discount coefficients can be lower bounded by a constant in each sub-interval. We believe these ideas can be useful for analyzing other joint learning and scheduling algorithms as well.

- **Numerical Studies:** We compare the proposed algorithm with several baselines, including using empirical mean, discounted empirical mean, and UCB as the estimated service rates. Our results show that both MaxWeight with empirical mean or discounted empirical mean are unstable. While MaxWeight with UCB is stable for stationary environments, queue lengths have wild oscillation in nonstationary environments and are much larger than MaxWeight with discounted UCB.

2 Model

We consider a multi-server system with \( J \) servers, indexed with \( j \in \{1, 2, \ldots, J\} \), and \( I \) types of jobs, indexed with \( i \in \{1, 2, \ldots, I\} \). The system maintains a separate queue for each type of jobs, as shown in Figure 1.

We consider a discrete-time system. The job arrivals at queue \( i \) are denoted by \( (A_i(t))_{t \geq 0} \) where \( t \) denotes the time slot. Assume that \( (A_i(t))_{t \geq 0} \) are independent with mean \( E[A_i(t)] = \lambda_i(t) \) and are bounded, i.e., \( A_i(t) \leq U_A \) for all \( i \) and \( t \). Let \( A(t) := (A_i(t))_{t \geq 1} \).

We say a server is available in time slot \( t \) if the server is not serving any job at the beginning of time slot \( t \); otherwise, we say the server is busy. At the beginning of each time slot, each available server picks a job from the queues. Note that each server can serve at most one job at a time and can start to serve another job only after finishing the current job, i.e., the job scheduling is nonpreemptive. When a job from queue \( i \) (job of type \( i \)) is picked by server \( j \) in time slot \( t \), it requires \( S_{i,j}(t) \) time slots to finish serving the job. For any \( i, j \), \( (S_{i,j}(t))_{t \geq 0} \) are independent random variables with unknown
mean \( E[S_{i,j}(t)] = \frac{1}{\mu_{i,j}(t)} \) and are bounded, i.e., \( S_{i,j}(t) \leq U_S \) for all \( i, j \) and \( t \). \( A_i(t) \) and \( S_{i,j}(t) \) for different \( i,j \) are also independent. Let \( S(t) := (S_{i,j}(t))_{i=1,\ldots,J,j=1,\ldots,J} \). Note that we assume \( \lambda \) and \( \mu \) are time-varying to model nonstationary environments, and the value of \( S_{i,j}(t) \) is generated at time slot \( t \) and will not change after that.

If server \( j \) is available and picks queue \( i \) in time slot \( t \) or if server \( j \) is busy serving queue \( i \) in time slot \( t \), we say server \( j \) is scheduled to queue \( i \) in time slot \( t \). Let \( I_j(t) \) denote the queue to which server \( j \) is scheduled in time slot \( t \). Define a waiting queue \( \tilde{Q}_i(t) \) for each job type \( i \). A job of type \( i \) joins the waiting queue \( \tilde{Q}_i(t) \) when it arrives, and leaves the waiting queue \( \tilde{Q}_i(t) \) when it is picked by a server under the algorithm. If an available server \( j \) picks queue \( i \) in time slot \( t \) and there is no job in the waiting queue \( i \), i.e., \( \tilde{Q}_i(t) + A_i(t) = 0 \), we say server \( j \) is idling in time slot \( t \) and the server \( j \) will be available in the next time slot. Let \( \eta_j(t) \) denote whether server \( j \) is idling in time slot \( t \), defined by:

\[
\eta_j(t) = \begin{cases} 
0, & \text{server } j \text{ is idling in time slot } t; \\
1, & \text{otherwise}.
\end{cases}
\]

Define \( 1_{i,j}(t) \) to be an indicator function such that \( 1_{i,j}(t) = 1 \) if \( I_j(t) = i \) and server \( j \) finishes serving the job of type \( i \) at the end of time slot \( t \), or if \( I_j(t) = i \) and server \( j \) is idling.

Let \( Q_i(t) \) denote the actual queue length of jobs at queue \( i \) at the beginning of time slot \( t \) so \( Q_i(t) \) is the total number of type-\( i \) jobs in the system, which is different from the waiting queue length \( \tilde{Q}_i(t) \). A job leaves the actual queue only when it is completed. Then we have the following queue dynamics:

\[
Q_i(t + 1) = Q_i(t) + A_i(t) - \sum_j 1_{i,j}(t)\eta_j(t).
\]

Our objective is to find an efficient learning and scheduling algorithm to stabilize \( Q_i(t) \) for all \( i \). In each time slot, the scheduling algorithm decides which queue to serve for each available server. The sequence of events that may occur within a time slot are shown in Fig. 2, where \( Q(t) := (Q_i(t))_{i=1,\ldots,J} \).

![Figure 2: The sequence of possible events during each time slot.](image)

### 3 Algorithm — MaxWeight with Discounted-UCB

Let \( \hat{s}_j(i)(t) \) be output of the scheduling algorithm if server \( j \) is available in time slot \( t \). We propose MaxWeight with Discounted-UCB algorithm, which combines the MaxWeight scheduling algorithm [23] with discounted UCB [10] for learning the service statistics. The algorithm is presented in Algorithm 1 where we use the convention that \( 0/0 = 0 \) and ties are broken arbitrary.

In Algorithm 1, the discount factor \( \gamma \) is fixed beforehand. \( g(\gamma) \approx 8/\pi \) can be easily computed using numerical methods. Larger \( \gamma \) implies a larger \( g(\gamma) \). If server \( j \) is currently serving a type-\( i \) job, \( M_{i,j}(t) \) is the service time the job has received by time slot \( t \) (not including time slot \( t \)); otherwise, \( M_{i,j}(t) = 0 \). \( \tilde{N}_{i,j}(t) \) is the discounted number of type-\( i \) jobs served by server \( j \) by time slot \( t \). \( \hat{\phi}_{i,j}(t) \) is
Algorithm 1: MaxWeight with Discounted-UCB

1: Initialize:
2: \( \hat{N}_{i,j}(0) = 0, \hat{\phi}_{i,j}(0) = 0, \hat{\mu}_{i,j}(0) = 1, b_{i,j}(0) = 1, M_{i,j}(0) = 0 \) for all \( i, j; \gamma \in (0, 1) \).
3: Define \( g(\gamma) \) such that \( \gamma = 1 - \frac{8 \log g(\gamma)}{g(\gamma)} \).
4: for \( t = 0 \) to infinity do
5: \( \text{if } t \neq 0 \text{ then} \)
6: \( \text{for } i = 1, \ldots, I \text{ and } j = 1, \ldots, J \text{ do} \)
7: \( \text{if } I_j(t-1) = i \text{ then} \)
8: \( M_{i,j}(t) = M_{i,j}(t-1) + 1 \) // the number of time slots already served
9: \( \hat{N}_{i,j}(t) = \gamma \hat{N}_{i,j}(t-1) + \gamma^M_{i,j}(t-1) I_{i,j}(t-1) \eta_j(t-1) \)
10: \( \hat{\phi}_{i,j}(t) = \gamma \hat{\phi}_{i,j}(t-1) + \gamma^M_{i,j}(t-1) \hat{\phi}_{i,j}(t-1) \eta_j(t-1) M_{i,j}(t) \)
11: \( \hat{\mu}_{i,j}(t) = \frac{\hat{N}_{i,j}(t)}{\hat{\phi}_{i,j}(t)} / 1/\hat{\mu}_{i,j}(t) \) is an estimate of the mean service time
12: \( b_{i,j}(t) = \min \left\{ c_1 U_S \sqrt{\frac{\log g(\gamma)}{\hat{N}_{i,j}(t)}}, 1 \right\} \) // UCB bonus term, where \( c_1 > 0 \) is a constant
13: \( \text{if } I_{i,j}(t-1) = 1 \text{ then} \)
14: \( M_{i,j}(t) = 0 \) // if the server becomes available, reset the counter.
15: \( \text{for } j = 1, \ldots, J \text{ do} \)
16: \( \text{if server } j \text{ is available then} \)
17: \( \hat{\tau}_j^*(t) = \arg \max_i Q_i(t) (\hat{\mu}_{i,j}(t) + b_{i,j}(t)) \) // server \( j \) picks \( \hat{\tau}_j^*(t) \)

the discounted number of time slots used by server \( j \) for serving type-\( i \) jobs by time slot \( t \). We update \( \hat{N}_{i,j}(t) \) and \( \hat{\phi}_{i,j}(t) \) at the beginning of every time slot as shown in Line 9 and Line 10, respectively. Specifically, for each time slot, if the job has not yet finished or the server is idling, we simply multiply \( \hat{N}_{i,j}(t-1) \) and \( \hat{\phi}_{i,j}(t-1) \) by a discount factor \( \gamma \). If the server is not idling and the job has finished, we update \( \hat{N}_{i,j}(t-1) \) by multiplying a discount factor and adding a number \( \gamma^M_{i,j}(t-1) \), and we update \( \hat{\phi}_{i,j}(t-1) \) by multiplying a discount factor and adding a discounted service time. The discount \( \gamma^M_{i,j}(t-1) \) actually means that the service time is discounted starting from the time when the job starts. This update is slightly different from the discounted UCB in \([10]\) and is needed for a technical reason. Then we obtain \( \hat{\mu}_{i,j}(t) \), an estimate of the mean service rate, as shown in Line 11. For each available server, we pick the queue with the largest product of queue length and UCB for the service rate, as shown in Line 17.

In a stationary environment, the use of discounted average instead of simple average reduces the influence of previous service times on the current estimate, and weakens the dependence between queue lengths and UCB bonuses. In a nonstationary environment, it ensures that the estimation process can adapt to the nonstationary service rate since the discount factor reduces the influence of previous service times on the current estimate. UCB helps with the exploration of the service times for different servers and job types. The MaxWeight algorithm is known to be throughput optimal \([19]\). These ideas are combined in the proposed algorithm.

4 Main Result

In this section, we will present our main result. We consider Algorithm 1 with a sufficiently large \( \gamma \) such that \( g(\gamma) \geq \max \{ e^5, 8U_S \} \). We make the following assumption on the time-varying mean service time and rate:

**Assumption 1.** \( \mu_{i,j}(t) \) satisfies the following two conditions:
(1) For any $i, j$ and any $t_a, t_b$ such that $t_a \neq t_b$ and $|t_a - t_b| \leq 2g(\gamma)$,
\[ \left| \frac{1}{\mu_{i,j}(t_a)} - \frac{1}{\mu_{i,j}(t_b)} \right| \leq \frac{1}{g(\gamma)} \left( \frac{1}{\gamma} \right)^{|t_a - t_b| - 1} ; \]

(2) There exists an absolute constant $p > 0$ such that for any $i, j$ and any $t_a, t_b$ such that $|t_a - t_b| \leq U_S$,
\[ |\mu_{i,j}(t_a) - \mu_{i,j}(t_b)| \leq \frac{1}{[g(\gamma)]^p}. \]

**Remark 1.** Note that in the first condition in Assumption $\mathcal{A}$, $\frac{1}{\gamma} > 1$, so the allowable change of the mean service time increases exponentially with respect to the time difference. Therefore, the second condition in Assumption $\mathcal{A}$ will be dominating for large $|t_a - t_b|$. Recall that $g(\gamma) \approx \frac{8}{\gamma^2}$, so the bound in condition (2) is roughly equivalent to that the maximum change that can occur when serving a job is $(\frac{1}{\gamma})^p$ for some $p > 0$ (note that $U_S$ is an upper bound on the service times). This bound increases as $\gamma$ decreases because the algorithm can quickly adapt by aggressively discounting the past samples.

For the nonstationary system considered in this paper, we introduce the following definition $C(W)$ for the capacity region:
\[
C(W) = \left\{ (\lambda(t))_{t \geq 0} : \text{there exists } (\alpha(t))_{t \geq 0} \text{ such that } \sum_{i} \alpha_{i,j}(t) \leq 1 \text{ for all } j, t \geq 0, \right. \\
\left. \quad \text{and for all } i, t \geq 0, \text{ there exists } w(t) \text{ such that } 1 \leq w(t) \leq W \right. \\
\left. \quad \text{and } \sum_{\tau=t}^{t+w(t)-1} \lambda_i(\tau) \leq \sum_{\tau=t}^{t+w(t)-1} \sum_{j} \alpha_{i,j}(\tau)\mu_{i,j}(\tau) \right\} \tag{2}
\]
where $\lambda_i(t) := (\lambda_i(t))_{t=1}^{t}$, $\alpha_i(t) := (\alpha_{i,j}(t))_{j=1}^{J}$, and $W \geq 1$ is a constant. Note that $C(W_1) \subseteq C(W_2)$ if $W_1 \leq W_2$. This capacity region means that for any time $t$ and queue $i$, there exists a time window such that the sum of mean arrival rates over this time window is less than the sum of appropriately allocated service rates. If $(\alpha(t))_{t \geq 0}$ is given, then a randomized scheduling algorithm using $(\alpha(t))_{t \geq 0}$ guarantees that the service rate received by queue $i$ is at least as large as the arrival rate. Note that if $W = 1$ and $\lambda$ and $\mu$ are time-invariant, then this definition reduces to the capacity region definition for the stationary setting $[19]$. We assume that the arrival satisfies that $\lambda + \delta \mathbf{1} \in C(W)$, where we assume that $W \leq \frac{g(\gamma)}{2}$. We present the following theorem which shows that the MaxWeight with Discounted-UCB algorithm can stabilize the queues with such arrivals.

**Theorem 1.** Consider Algorithm $\mathcal{A}$ with $c_4 = 4$ and $g(\gamma) = \max\{e^{\gamma}, 8U_S\}$. Under Assumption $\mathcal{A}$ for arrivals that satisfy $\lambda + \delta \mathbf{1} \in C(W)$, where $W \leq \frac{g(\gamma)}{2}$ and $\delta \geq \frac{804JU_2^2\log g(\gamma)}{[g(\gamma)]^{\min\{\frac{1}{2}, p\}}}$, we have
\[
\limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=1}^{t} \mathbb{E} \left[ \sum_{j} Q_{i}(\tau) \right] \leq \frac{1642J^2U_2^2U_3^2\log g(\gamma)}{\delta}.
\]

From Theorem $\mathcal{A}$ we know that if $g(\gamma)$ is sufficiently large, i.e., $\gamma$ is sufficiently close to 1, then $\delta$ can be arbitrarily close to zero and Algorithm $\mathcal{A}$ can stabilize the queues with arrivals in the capacity region $C(\frac{g(\gamma)}{2})$.

## 5 Proofs

In this section, we will present the proof of Theorem $\mathcal{A}$. Before we present the proof, we define some additional notations. Define $H(t)$ to be
\[
H(t) := (\hat{Q}(t), M(t), \hat{N}(t), \phi(t)),
\]
where $\hat{Q}(t) := (\hat{Q}_i(t))_i$, $M(t) := (M_{i,j}(t))_{i,j}$, $\hat{N}(t) := (\hat{N}_{i,j}(t))_{i,j}$, and $\phi(t) := (\phi_{i,j}(t))_{i,j}$. If server $j$ is not available at the beginning of time slot $t$, i.e., $\sum_i M_{i,j}(t) > 0$, we let $\hat{y}_j^*(t) = 0$. \[7\]
5.1 Proof Roadmap

In this subsection, we present the proof roadmap. Firstly, we will divide the time horizon into intervals. We want each interval to be the concatenation of the windows $w(t)$ in the definition of capacity region so that we can apply the inequality \[2\] to bound the arrivals. We also want to make sure that the length of each interval is approximately $g(\gamma)$ so that the estimates of the mean service times in the current interval will “forget” the old samples in previous intervals due to the discount factor. Let $[t_k, t_{k+1})$ denote the $k^{th}$ interval and define the length to be $D_k = t_{k+1} - t_k$.

Secondly, we consider the Lyapunov function $\sum_i Q_i(t_k)$ and analyze the Lyapunov drift for each interval $[t_k, t_{k+1} + D_{k+1}]$ given the queue length $Q(t_k)$ and $H(t_k)$ at the beginning of the previous interval. The drift can be upper bounded by the difference between two terms, $\sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \sum_i Q_i(t_k + \tau)A_i(t_k + \tau)$ and $\sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \sum_i Q_i(t_k + \tau)\sum_j I_{i,j}(t_k + \tau)$, where the former term is related to the arrivals and the latter term is related to the services. The arrival term can be upper bounded using the inequality \[2\] in the definition of the capacity region. The upper bound contains a positive term $\sum_j \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \max_i Q_i(t_k)\mu_{i,j}(t_k + \tau)$, which can be later canceled out by the lower bound of the service term.

Next, we want to find a lower bound on the service term, which takes several steps. The first step is to prove a concentration result (Lemma \[4\]) regarding the deviation of the estimates of the mean service rates $\hat{\mu}_{i,j}(t)$ from the true mean service rates $\mu_{i,j}(t)$. This result cannot be proved by simply using the Hoeffding inequality and the union bound like in the traditional analysis of UCB algorithms. There are three main difficulties. First, the probability is conditioned on the queue length in the previous interval, which is related to the service times before the previous interval. Thanks to the discount factor $\gamma$ and the length $g(\gamma)$ of each interval, the contribution of the service times before the previous interval to the current estimate $\hat{\phi}_{i,j}(t)$ is negligible and can be bounded. Another difficulty is that $\hat{\phi}_{i,j}(t)$ is the discounted sum of previous service times and the summation is over the time slots in which there is job completion. Those time slots are random variables, which implies that the discount coefficients of those service times are also random. To deal with this difficulty, we first transform the summation into a summation over the time slots in which there is a job starting, and then use a Hoeffding-type inequality for self-normalized means with a random number of summands \[6\], Theorem 18 \[7\] to obtain a concentration bound. Another issue we need to consider is that the mean service times are time-varying and the estimate of the mean service time in the current time slot is based on the actual service times in previous time slots. We utilize the first condition in Assumption \[1\] to solve this time-varying issue.

The next step is to transform each summand in the service term into the product of two terms, $Q_{f_j(t_k + \tau)}(t_k + \tau)\mu_{f_j(t_k + \tau),j}(f_j(t_k + \tau))$ and $\frac{z_{i,j}(t_k + \tau)}{\mu_{f_j(t_k + \tau),j}(f_j(t_k + \tau))}$, where $f_j(t)$ denotes the starting time of the job that is being served at server $j$ in time slot $t$. Under the above mentioned high probability concentration event (see \[14\] and Lemma \[4\]), we next bound the product of queue length and mean service rate, i.e., $Q_{f_j(t_k + \tau)}(t_k + \tau)\mu_{f_j(t_k + \tau),j}(f_j(t_k + \tau))$. The main idea is to first utilize Line \[17\] in Algorithm \[1\] and then bound the sum of UCB bonus terms (Lemma \[3\]). In this way, we can get a lower bound which contains a term $\max_i Q_i(t_k)\mu_{i,j}(f_j(t_k + \tau))$, which can be later canceled out with the term $\sum_{\tau = D_k + D_{k+1}}^{D_k + D_{k+1} - 1} \max_i Q_i(t_k)\mu_{i,j}(t_k + \tau)$ in the upper bound of the arrival term. The main difficulty of bounding the sum of UCB bonus terms (proving Lemma \[3\]) is to get a lower bound for $\hat{N}_{i,j}(t)$. Our method is to divide the interval into log $g(\gamma)$ sub-intervals with length $\frac{g(\gamma)}{\log g(\gamma)}$ so that the discount coefficients can be lower bounded by a constant in each sub-interval.

The next step is to bound the weighted sum of job completion indicators, i.e., $\sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \max_i Q_i(t_k)\mu_{i,j}(f_j(t_k + \tau))$, where $\mu_{f_j(t_k + \tau),f_j(t_k + \tau)}$ are mean service times. Intuitively, if we replace the mean service times with actual service times, the sum should not change too much, which is formally stated in Lemma \[7\]. Lemma \[7\] can be proved using the same Hoeffding-type inequality for self-normalized means. The sum with actual service times is close to the sum of the weights in
each time slot, i.e., \( \sum_{\tau = D_k}^{D_k + D_k + 1 - 1} \max_i Q_i(t_k) \mu_{i,j}(t_k + \tau) \), if \( \mu_{i,j} \) does not change too much within the duration of each service (the second condition in Assumption 1).

Last, we combine the bounds for the arrival term and the service term and then do a telescoping sum over the intervals, which gives us a negative Lyapunov drift for large queue length. Hence, we finally obtain the result in Theorem 1. See Fig. 3 for a summary.

\[ \text{Lyapunov drift (3)} \leq \text{Arrival term (5)} \leq \text{Upper bound (13)} \leq \text{Service term (6)} \leq \text{Lower bound (45)} \]

Canceling out the term \( \sum_j \sum_\tau = D_k D_k + D_k + 1 - 1 \max_i Q_i(t_k) \mu_{i,j}(t_k + \tau) \), if \( \mu_{i,j} \) does not change too much within the duration of each service (the second condition in Assumption 1).

Last, we combine the bounds for the arrival term and the service term and then do a telescoping sum over the intervals, which gives us a negative Lyapunov drift for large queue length. Hence, we finally obtain the result in Theorem 1. See Fig. 3 for a summary.

**Figure 3:** The proof roadmap of Theorem 1.

In the following subsections, we will present the details of the proof of Theorem 1. The proof of all the lemmas can be found in the appendices.

### 5.2 Dividing the Time Horizon

Firstly, we want to divide the time horizon into intervals. Let \( T := g(\gamma) \) for ease of notation. We assume \( \frac{T}{2} \) is an integer without loss of generality. For any time slot \( \tau \), there exists a \( w(\tau) \) according to the capacity region definition (2). Let \( \tau_0 := t \), \( \tau_1 := \tau_{l-1}(t) + w(\tau_{l-1}(t)) \) for \( l \geq 1 \). Define \( D(t) \) such that

\[
D(t) = \min_n \left[ \frac{1}{n} \sum_{l=0}^{n} w(\tau_l(t)) \right] \quad \text{s.t.} \quad \sum_{l=0}^{n} w(\tau_l(t)) \geq \frac{T}{2}.
\]

Denote by \( n^*(t) \) the optimal solution to the above optimization problem. Note that \( n^*(t) \) and \( D(t) \) are fixed numbers rather than random variables for a given \( t \). We have the following upper and lower bounds for \( D(t) \):

**Lemma 1.** Suppose \( W \leq \frac{T}{2} \). Then \( \frac{T}{2} \leq D(t) \leq \frac{T}{2} + W \leq T \) for any \( t \).

Proof of this lemma can be found in Appendix A. Let \( t_0 = 0 \) and \( t_k = t_{k-1} + D(t_{k-1}) \) for \( k \geq 1 \). Let \( D_k := D(t_k) \) for simplicity. Then the time horizon can be divided into intervals with length \( D_0, D_1, \ldots, D_k, \ldots \), where the \( k \)th interval is \( [t_k, t_{k+1}] \). We remark that this partition of the time horizon into time intervals is for the analysis only. The proposed algorithm does not need to know this partition and does not use the time interval information for scheduling and learning.
5.3 Decomposing the Lyapunov Drift

Consider the Lyapunov function \( L(t) := \sum_i Q_i^2(t) \). We first consider the Lyapunov drift for the interval \([t_{k+1}, t_{k+1} + D_{k+1}]\) given the queue length \( Q(t_k) \) and \( H(t_k) \). We analyze the drift conditioned on \( Q(t_k) \) and \( H(t_k) \) instead of \( Q(t_{k+1}) \) and \( H(t_{k+1}) \) to weaken the dependence of the UCB bonuses and the estimated service rates on the conditional values. We have

\[
E \left[ L(t_{k+1} + D_{k+1}) - L(t_{k+1}) \mid Q(t_k) = q, H(t_k) = h \right] = E \left[ L(t_k + D_k + D_{k+1}) - L(t_k + D_k) \mid Q(t_k) = q, H(t_k) = h \right] \nonumber \\
= \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} E \left[ L(t_k + \tau + 1) - L(t_k + \tau) \mid Q(t_k) = q, H(t_k) = h \right].
\]

We first look at each term in the summation above. Note that by the queue dynamic (1) we can obtain the following upper bound for \( Q_i(t + 1) \):

**Lemma 2.** For any \( i, t \), \( Q_i(t + 1) \leq \max \left\{ J, Q_i(t) + A_i(t) - \sum_j 1_{i,j}(t) \right\} \).

Proof of this lemma can be found in Appendix B. Denote by \( \hat{E}_{tk}[\cdot] \) the conditional expectation \( E[\cdot \mid Q(t_k) = q, H(t_k) = h] \). By Lemma 2, we have

\[
\hat{E}_{tk} \left[ L(t_k + \tau + 1) - L(t_k + \tau) \right] = \hat{E}_{tk} \left[ \sum_i (Q_i^2(t_k + \tau + 1) - Q_i^2(t_k + \tau)) \right] \nonumber \\
\leq \hat{E}_{tk} \left[ \sum_i \max \left\{ J^2, (Q_i(t_k + \tau) + A_i(t_k + \tau) - \sum_j 1_{i,j}(t_k + \tau))^2 \right\} - Q_i^2(t_k + \tau) \right] \nonumber \\
\leq \hat{E}_{tk} \left[ \sum_i J^2 + (Q_i(t_k + \tau) + A_i(t_k + \tau) - \sum_j 1_{i,j}(t_k + \tau))^2 - Q_i^2(t_k + \tau) \right] \nonumber \\
= \hat{E}_{tk} \left[ \sum_i 2Q_i(t_k + \tau)(A_i(t_k + \tau) - \sum_j 1_{i,j}(t_k + \tau)) \right] + \hat{E}_{tk} \left[ \sum_i (A_i(t_k + \tau) - \sum_j 1_{i,j}(t_k + \tau))^2 \right] + IJ^2
\]

(4)

where the second inequality is due to the fact that \( \max\{x^2, y^2\} \leq x^2 + y^2 \), and the second term in the last line can be bounded as follows:

\[
\sum_i (A_i(t_k + \tau) - \sum_j 1_{i,j}(t_k + \tau))^2 \leq \sum_i \left( \max\left\{ A_i(t_k + \tau), \sum_j 1_{i,j}(t_k + \tau) \right\} \right)^2 
\leq \sum_i (A_i(t_k + \tau))^2 + \sum_j (\sum_j 1_{i,j}(t_k + \tau))^2 \leq IU_A^2 + \left[ \sum_i \sum_j 1_{i,j}(t_k + \tau) \right]^2 \leq IU_A^2 + J^2
\]

where the last two steps are due to the fact that \( A_i(t_k + \tau) \leq U_A \) and \( \sum_i \sum_j 1_{i,j}(t_k + \tau) \leq J \). Hence, from (4), we have

\[
\hat{E}_{tk} \left[ L(t_k + \tau + 1) - L(t_k + \tau) \right] \nonumber \\
\leq \hat{E}_{tk} \left[ \sum_i 2Q_i(t_k + \tau)A_i(t_k + \tau) \right] - \hat{E}_{tk} \left[ \sum_i 2Q_i(t_k + \tau) \sum_j 1_{i,j}(t_k + \tau) \right] + IU_A^2 + J^2 + IJ^2.
\]
Substituting the above inequality into (5), we have
\[ E \left[ L(t_k + D_k + D_{k+1}) - L(t_k + D_k) \right] | Q(t_k) = q, H(t_k) = h ] \]
\[ \leq \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_{t_k} \left[ \sum_i 2Q_i(t_k + \tau)A_i(t_k + \tau) \right] \]
\[ - \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_{t_k} \left[ \sum_i 2Q_i(t_k + \tau) \sum_j 1_{i,j}(t_k + \tau) \right] + (I\mu_i^2 + J^2 + IJ^2)T \]
where the inequality uses the upper bound on \( D_{k+1} \) in Lemma 1. We will next find the bounds for the arrival term (5) and the service term (6).

### 5.4 Bounding the Arrival Term

We first analyze the arrival term (5). We have
\[ (5) = \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_{t_k} \left[ E \left[ \sum_i 2Q_i(t_k + \tau)A_i(t_k + \tau) \right] | Q(t_k + \tau), Q(t_k) = q, H(t_k) = h \right] \]
\[ = \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_{t_k} \left[ \sum_i 2Q_i(t_k + \tau)E \left[ A_i(t_k + \tau) \right] | Q(t_k + \tau), Q(t_k) = q, H(t_k) = h \right] \]
\[ = \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_{t_k} \left[ \sum_i 2Q_i(t_k + \tau)\lambda_i(t_k + \tau) \right] , \]
where the first equality is by law of iterated expectation and the last equality is due to the fact that \( A_i(t_k + \tau) \) is independent of \( Q(t_k + \tau), Q(t_k) \), and \( H(t_k) \). By adding and subtracting \( \delta \), we have
\[ (5) = \sum_{i} 2Q_i(t_k + \tau)(\lambda_i(t_k + \tau) + \delta) - 2\delta \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_{t_k} \left[ \sum_i Q_i(t_k + \tau) \right] . \]

By the queue dynamics (11) and the bounds on the arrival rate and service rate, we have the following bounds on the difference between queue lengths in two different time slots:

**Lemma 3.** For any \( t, \tau \geq 0 \), we have

1. \( Q_i(t) - J\tau \leq Q_i(t + \tau) \leq Q_i(t) + \tau U_A; \)
2. \( \sum_i Q_i(t + \tau) \geq \sum_i Q_i(t) - J\tau. \)

Proof of this lemma can be found in Appendix By Lemma 3, we have
\[ Q_i(t_k + \tau) \leq Q_i(t_k) + \tau U_A \]
where the last inequality holds since \( \tau \leq D_k + D_{k+1} - 1 \leq 2T \) by Lemma 1. Then, substituting \( (8) \) into (7), we have
\[ (5) \leq \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \sum_i 2Q_i(t_k + \tau)(\lambda_i(t_k + \tau) + \delta) + \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} 4\delta \sum_i \hat{E}_{t_k} \left[ \sum_i Q_i(t_k + \tau) \right] . \]
Since $\lambda + \delta 1 \in C(W)$, by the definitions of $t_{k+1}$ and $D_{k+1}$, we have

$$D_{k}+D_{k+1}-1 \sum_{\tau=t_{k}}^{D_{k}+D_{k+1}-1} (\lambda_{i}(t_{k} + \tau) + \delta) = D_{k}+D_{k+1}-1 \sum_{\tau=t_{k}+D_{k}}^{D_{k}+D_{k+1}-1} (\lambda_{i}(t_{k} + \tau) + \delta) = D_{k}+D_{k+1}-1 \sum_{\tau=t_{k}+D_{k}}^{D_{k}+D_{k+1}-1} (\lambda_{i}(t_{k} + \tau) + \delta).$$

Then by the definitions of $\tau_{l}(t_{k+1})$ and $C(W)$, we can bound the above term as follows:

$$D_{k}+D_{k+1}-1 \sum_{\tau=t_{k}}^{D_{k}+D_{k+1}-1} (\lambda_{i}(t_{k} + \tau) + \delta) = n^{*}(t_{k+1}) \sum_{\tau=t_{k}+D_{k}}^{D_{k}+D_{k+1}-1} (\lambda_{i}(t_{k} + \tau) + \delta) = n^{*}(t_{k+1}) \sum_{\tau=t_{k}+D_{k}}^{D_{k}+D_{k+1}-1} (\lambda_{i}(t_{k} + \tau) + \delta).$$

In the same way, we can transform the double summations back to a single summation to obtain:

$$D_{k}+D_{k+1}-1 \sum_{\tau=t_{k}}^{D_{k}+D_{k+1}-1} (\lambda_{i}(t_{k} + \tau) + \delta) \leq D_{k}+D_{k+1}-1 \sum_{\tau=t_{k}}^{D_{k}+D_{k+1}-1} \alpha_{i,j}(t_{k} + \tau) \mu_{i,j}(t_{k} + \tau).$$

Substituting the above bound back into (10), we have

$$\leq \sum_{\tau=D_{k}}^{D_{k}+D_{k+1}-1} \sum_{i} 2q_{l} \sum_{j} \alpha_{i,j}(t_{k} + \tau) \mu_{i,j}(t_{k} + \tau) + 4T \sum_{\tau=D_{k}}^{D_{k}+D_{k+1}-1} \sum_{i} \sum_{j} \alpha_{i,j}(t_{k} + \tau) \mu_{i,j}(t_{k} + \tau) - 2 \delta \sum_{\tau=D_{k}}^{D_{k}+D_{k+1}-1} E_{t_{k}} \left[ \sum_{i} Q_{i}(t_{k} + \tau) \right].$$

Note that

$$\sum_{\tau=D_{k}}^{D_{k}+D_{k+1}-1} \sum_{i} \sum_{j} \alpha_{i,j}(t_{k} + \tau) \mu_{i,j}(t_{k} + \tau) \leq \sum_{\tau=D_{k}}^{D_{k}+D_{k+1}-1} \sum_{i} \max_{j} \mu_{i,j}(t_{k} + \tau) \leq JD_{k+1} \leq JT,$$

where the first inequality is by $\sum_{i} \alpha_{i,j}(t_{k} + \tau) \leq 1$, the second inequality is by $\mu_{i,j}(t_{k} + \tau) \leq 1$, and the last inequality is by Lemma [1]. Note that

$$\sum_{i} q_{l} \alpha_{i,j}(t_{k} + \tau) \mu_{i,j}(t_{k} + \tau) \leq \max_{i} q_{l} \mu_{i,j}(t_{k} + \tau) \sum_{i} \alpha_{i,j}(t_{k} + \tau) \leq \max_{i} q_{l} \mu_{i,j}(t_{k} + \tau),$$

where the last inequality is by $\sum_{i} \alpha_{i,j}(t_{k} + \tau) \leq 1$. Substituting (11) and (12) into (10), we have

$$\leq 2 \sum_{\tau=D_{k}}^{D_{k}+D_{k+1}-1} \max_{i} q_{l} \mu_{i,j}(t_{k} + \tau) + 4T^{2}J \sum_{\tau=D_{k}}^{D_{k}+D_{k+1}-1} E_{t_{k}} \left[ \sum_{i} Q_{i}(t_{k} + \tau) \right].$$

5.5 Bounding the Service Term

Now we analyze the service term (10). Let us first fix a server $j$. We want to lower bound the term

$$\sum_{\tau=D_{k}}^{D_{k}+D_{k+1}-1} E_{t_{k}} \left[ \sum_{i} Q_{i}(t_{k} + \tau) \Omega_{i,j}(t_{k} + \tau) \right].$$
5.5.1 Adding the Concentration to the Condition

We first define a concentration event as follows:

\[
\mathcal{E}_{i,j}^{k} := \left\{ \text{for all } i, \tau \in \left[ D_k - \frac{T}{8}, D_k + D_{k+1} - 1 \right], |\hat{\mu}_{i,j}(t_k + \tau) - \mu_{i,j}(t_k + \tau)| \leq b_{i,j}(t_k + \tau) \right\}. \tag{14}
\]

Then we have

**Lemma 4.** For any \( k \geq 0 \), we have \( \Pr(\mathcal{E}_{i,j}^{k} \mid Q(t_k) = q, H(t_k) = h) \leq \frac{1}{T^2} \).

Proof of this lemma can be found in Appendix D. Denote by \( \hat{P}_k(\cdot) \) the conditional probability \( \Pr(\cdot \mid Q(t_k) = q, H(t_k) = h) \). Denote by \( \hat{E}_t_{k} \left[ \mid \mathcal{E}_{i,j}^{k} \right] \) the conditional expectation \( E[\cdot \mid Q(t_k) = q, H(t_k) = h, \mathcal{E}_{i,j}^{k}] \). Then by Lemma 4 we have

\[
\sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_t_{k} \left[ \sum_{i} Q_i(t_k + \tau)1_{i,j}(t_k + \tau) \mid \mathcal{E}_{i,j}^{k} \right] \\
\geq \hat{P}_k(\mathcal{E}_{i,j}^{k}) \hat{E}_t_{k} \left[ \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \sum_{i} Q_i(t_k + \tau)1_{i,j}(t_k + \tau) \mid \mathcal{E}_{i,j}^{k} \right] \\
\geq \left( 1 - \frac{I}{T^2} \right) \hat{E}_t_{k} \left[ \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \sum_{i} Q_i(t_k + \tau)1_{i,j}(t_k + \tau) \mid \mathcal{E}_{i,j}^{k} \right]. \tag{15}
\]

Using the bound (8) and the bound on \( D_{k+1} \) in Lemma 1, we have

\[
\hat{E}_t_{k} \left[ \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \sum_{i} Q_i(t_k + \tau)1_{i,j}(t_k + \tau) \mid \mathcal{E}_{i,j}^{k} \right] \leq T \sum_{i} q_i + 2U_A T^2 I. \tag{16}
\]

By Lemma 3 and Lemma 1 we can obtain the following bound on \( \sum_i q_i \):

**Lemma 5.** \( \sum_i q_i \leq \frac{1}{D_{k+1}} \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_t_{k} \left[ \sum_{i} Q_i(t_k + \tau) + 2JT \right] \).

Proof of this lemma can be found in Appendix E. By (16), Lemma 5, and Lemma 1 we have

\[
\hat{E}_t_{k} \left[ \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \sum_{i} Q_i(t_k + \tau)1_{i,j}(t_k + \tau) \mid \mathcal{E}_{i,j}^{k} \right] \\
\leq 2 \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_t_{k} \left[ \sum_{i} Q_i(t_k + \tau) \right] + 2JT^2 + 2U_A T^2 I, \tag{17}
\]

Substituting (17) into (15), we have

\[
\sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_t_{k} \left[ \sum_{i} Q_i(t_k + \tau)1_{i,j}(t_k + \tau) \right] \\
\geq \hat{E}_t_{k} \left[ \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \sum_{i} Q_i(t_k + \tau)1_{i,j}(t_k + \tau) \mid \mathcal{E}_{i,j}^{k} \right] \\
- \frac{2I}{T^2} \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_t_{k} \left[ \sum_{i} Q_i(t_k + \tau) \right] - 2IJ - 2U_A I^2. \tag{18}
\]
Next, we want to lower bound the term $\hat{E}_{t_k} \left[ \sum_{\tau = D_k}^{D_k + D_k + 1 - 1} \sum_i Q_i(t_k + \tau) \mathbb{1}_{i,j}(t_k + \tau) | \mathcal{E}_{t_k,j} \right]$ in (18) using $q$ and $\sum_{\tau = D_k}^{D_k + D_k + 1 - 1} \hat{E}_{t_k} \left[ \sum_i Q_i(t_k + \tau) \right]$. Note that $\mathbb{1}_{i,j}(t_k + \tau) = 1$ can only happen for the queue to which server $j$ is scheduled in time slot $t_k + \tau$, i.e., the queue $I_j(t_k + \tau)$. Hence, we have

$$
\hat{E}_{t_k} \left[ \sum_{\tau = D_k}^{D_k + D_k + 1 - 1} \sum_i Q_i(t_k + \tau) \mathbb{1}_{i,j}(t_k + \tau) | \mathcal{E}_{t_k,j} \right] = \hat{E}_{t_k} \left[ \sum_{\tau = D_k}^{D_k + D_k + 1 - 1} Q_{I_j(t_k + \tau)}(t_k + \tau) \mathbb{1}_{I_j(t_k + \tau), j}(t_k + \tau) | \mathcal{E}_{t_k,j} \right].
$$

Define a mapping $f_j$ that maps a time slot to another time slot such that if $y = f_j(x)$ then $x$ is the time slot when server $j$ picked the job that was being served at server in time slot $x$. If server $j$ was idling in time slot $x$, then let $f_j(x) = x$. That is, $f_j(x) := \max \{t : t \leq x, j^*(t) = I_j(x)\}$. Then

$$
\hat{E}_{t_k} \left[ \sum_{\tau = D_k}^{D_k + D_k + 1 - 1} \sum_i Q_i(t_k + \tau) \mathbb{1}_{i,j}(t_k + \tau) | \mathcal{E}_{t_k,j} \right] = \hat{E}_{t_k} \left[ \sum_{\tau = D_k}^{D_k + D_k + 1 - 1} Q_{I_j(t_k + \tau)}(t_k + \tau) \mathbb{1}_{I_j(t_k + \tau), j}(t_k + \tau) \frac{\mathbb{1}_{I_j(t_k + \tau), j}(t_k + \tau)}{\mu_{I_j(t_k + \tau), j}(f_j(t_k + \tau))} | \mathcal{E}_{t_k,j} \right].
$$

(19)

5.5.2 Bounding the Product of Queue Length and Service Rate

We next want to lower bound the term $Q_{I_j(t_k + \tau)}(t_k + \tau) \mu_{I_j(t_k + \tau), j}(f_j(t_k + \tau))$ in (19). The following analysis is conditioned on the concentration event $\mathcal{E}_{t_k,j}$. Since $U_S \leq \frac{T}{8}$, we have $f_j(t_k + \tau) \geq t_k + \tau - U_S \geq t_k + D_k - U_S \geq t_k + D_k - \frac{T}{8}$. Also note that $f_j(t_k + \tau) \leq t_k + \tau \leq t_k + D_k + D_k + 1 - 1$. Hence, we have

$$
f_j(t_k + \tau) \in \left[ t_k + D_k - \frac{T}{8}, t_k + D_k + D_k + 1 - 1 \right] \subseteq [t_k, t_k + 2T - 1],
$$

(20)

where the inclusion is by Lemma 1. Then, by (20) and the definition of the concentration event $\mathcal{E}_{t_k,j}$ in (14), we have

$$
Q_{I_j(t_k + \tau)}(t_k + \tau) \mu_{I_j(t_k + \tau), j}(f_j(t_k + \tau)) \geq Q_{I_j(t_k + \tau)}(t_k + \tau) \left( \hat{\mu}_{I_j(t_k + \tau), j}(f_j(t_k + \tau)) - b_{I_j(t_k + \tau), j}(f_j(t_k + \tau)) \right)
$$

$$
= Q_{I_j(t_k + \tau)}(t_k + \tau) \left( \hat{\mu}_{I_j(t_k + \tau), j}(f_j(t_k + \tau)) + b_{I_j(t_k + \tau), j}(f_j(t_k + \tau)) \right)
$$

$$
- 2Q_{I_j(t_k + \tau)}(t_k + \tau) b_{I_j(t_k + \tau), j}(f_j(t_k + \tau)).
$$

Note that by Lemma 3 and the fact that $t_k + \tau - f_j(t_k + \tau) \leq U_S \leq \frac{T}{8}$, we have $Q_{I_j(t_k + \tau)}(t_k + \tau) \geq Q_{I_j(t_k + \tau)}(f_j(t_k + \tau)) - \frac{JT}{8}$. Hence, we have

$$
Q_{I_j(t_k + \tau)}(t_k + \tau) \mu_{I_j(t_k + \tau), j}(f_j(t_k + \tau)) \geq Q_{I_j(t_k + \tau)}(f_j(t_k + \tau)) \left( \hat{\mu}_{I_j(t_k + \tau), j}(f_j(t_k + \tau)) + b_{I_j(t_k + \tau), j}(f_j(t_k + \tau)) \right) - \frac{JT}{4}
$$

$$
- 2Q_{I_j(t_k + \tau)}(t_k + \tau) b_{I_j(t_k + \tau), j}(f_j(t_k + \tau))
$$

(21)

since $\hat{\mu}_{I_j(t_k + \tau), j}(f_j(t_k + \tau)) + b_{I_j(t_k + \tau), j}(f_j(t_k + \tau)) \leq 2$. By Line 17 in Algorithm 1, we have

$$
Q_{I_j(t_k + \tau)}(f_j(t_k + \tau)) \left( \hat{\mu}_{I_j(t_k + \tau), j}(f_j(t_k + \tau)) + b_{I_j(t_k + \tau), j}(f_j(t_k + \tau)) \right)
$$

$$
= \max_i Q_i(f_j(t_k + \tau)) \left( \hat{\mu}_{i,j}(f_j(t_k + \tau)) + b_{i,j}(f_j(t_k + \tau)) \right).
$$

(22)
Combining (21) and (22), we have
\[ Q_{I_j(t_k+\tau)}(t_k+\tau)\mu_{I_j(t_k+\tau),j}(f_j(t_k+\tau)) \]
\[ \geq \max_i Q_i(f_j(t_k+\tau))(\mu_{i,j}(f_j(t_k+\tau)) + b_{i,j}(f_j(t_k+\tau))) \]
\[ - \frac{JT}{4} - 2Q_{I_j(t_k+\tau)}(t_k+\tau)b_{I_j(t_k+\tau),j}(f_j(t_k+\tau)) \]
\[ \geq \max_i Q_i(f_j(t_k+\tau))\mu_{i,j}(f_j(t_k+\tau)) - \frac{JT}{4} - 2Q_{I_j(t_k+\tau)}(t_k+\tau)b_{I_j(t_k+\tau),j}(f_j(t_k+\tau)), \]
where the last inequality is due to (20) and the fact that
\[ \text{Combining (21) and (22), we have} \]
\[ \hat{E}_{t_k} \left[ \sum_{\tau=0}^{D_{k+1}+1} Q_i(t_k+\tau)1_{i,j}(t_k+\tau) | E_{t_k,j} \right] \]
\[ \geq \hat{E}_{t_k} \left[ \sum_{\tau=0}^{D_{k+1}+1} \left( \max_i q_i \mu_{i,j}(f_j(t_k+\tau)) - \frac{9JT}{4} \right) \right. \]
\[ - 2Q_{I_j(t_k+\tau)}(t_k+\tau)b_{I_j(t_k+\tau),j}(f_j(t_k+\tau)) \]
\[ \frac{1}{\mu_{I_j(t_k+\tau),j}(f_j(t_k+\tau)) | E_{t_k,j}} \]
\[ \hat{Q}_{ij}(f_j(t_k + \tau)) \eta_j(f_j(t_k + \tau)) \] Also note that \( 0 \leq Q_{ij}(f_j(t_k + \tau)) - \hat{Q}_{ij}(f_j(t_k + \tau)) \leq J \) by definition. Hence, we have
\[
Q_{ij}(f_j(t_k + \tau)) \leq \hat{Q}_{ij}(f_j(t_k + \tau)) \eta_j(f_j(t_k + \tau)) + J
\leq Q_{ij}(f_j(t_k + \tau)) \eta_j(f_j(t_k + \tau)) + J.
\] (28)

Substituting the bound (28) into (27) and using Lemma 6, we have
\[
\hat{E}_{tk} \left[ \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} Q_{ij}(f_j(t_k + \tau)) b_{ij}(f_j(t_k + \tau)) | I_{ij}(t_k + \tau) | \mathcal{E}_{tk} \right] 
\leq \left( \sum_i q_i \right) \hat{E}_{tk} \left[ \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} b_{ij}(f_j(t_k + \tau)) \eta_j(f_j(t_k + \tau)) | I_{ij}(t_k + \tau) | \mathcal{E}_{tk} \right] 
+ JT + \frac{U_A T^2}{8}.
\] (30)

The sum of UCB bonus terms in each interval can be bounded by \( O(\sqrt{T} \log T) \), shown as follows:

**Lemma 6.** For any \( j \) and \( k \geq 0 \),
\[
\sum_{\tau = D_k}^{D_k + D_{k+1} - 1} b_{ij}(f_j(t_k + \tau)) \eta_j(f_j(t_k + \tau)) | I_{ij}(t_k + \tau) | \leq 99 I_S \sqrt{T} \log T.
\]

Proof of this lemma can be found in Appendix F. Hence, by Lemma 6, (30), and (26), we have
\[
(26) \geq - (198 I_S^2 \sqrt{T} \log T) \sum_i q_i - 4T^2 U_S U_A - 2U_S JT - \frac{U_S U_A T^2}{4}.
\] (31)

### 5.5.4 Bounding the Weighted Sum of Job Completion Indicators

We next look at the term in (25):
\[
\hat{E}_{tk} \left[ \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \max_i q_i \mu_{ij}(f_j(t_k + \tau)) | I_{ij}(t_k + \tau) | \mathcal{E}_{tk} \right]
\]
Let \( v_j(t) := \max_i q_i \mu_{i,j}(t) \) for any time slot \( t \). By law of total expectation, we have

\[
\hat{E}_{t_k} \left[ \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} v_j(f_j(t_k + \tau)) \frac{\mathbb{1}_{I_{j}(t_k + \tau),j}(t_k + \tau)}{\mu_{I_{j}(t_k + \tau),j}(f_j(t_k + \tau))} | E_{t_k} \right] \\
\geq \hat{E}_{t_k} \left[ \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} v_j(f_j(t_k + \tau)) \frac{\mathbb{1}_{I_{j}(t_k + \tau),j}(t_k + \tau)}{\mu_{I_{j}(t_k + \tau),j}(f_j(t_k + \tau))} \right] - \hat{P}_{t_k}(E_{t_k}) \hat{E}_{t_k} \left[ \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} v_j(f_j(t_k + \tau)) \frac{\mathbb{1}_{I_{j}(t_k + \tau),j}(t_k + \tau)}{\mu_{I_{j}(t_k + \tau),j}(f_j(t_k + \tau))} E_{t_k}^c \right] \\
\geq \sum_{i=1}^{I} \epsilon_{t_k} \left[ D_k + D_{k+1} - 1 \right] v_j(f_j(t_k + \tau)) \frac{\mathbb{1}_{I_{j}(t_k + \tau),j}(t_k + \tau)}{\mu_{I_{j}(t_k + \tau),j}(f_j(t_k + \tau))} - \frac{D_{k+1} U_S \sum_i q_i}{T^2}, \tag{32}
\]

where the last inequality is by Lemma 4 and the facts that \( v_j(t) \leq \max_i q_i \leq \sum_i q_i \) and \( \frac{1}{\mu_{i,j}(t)} \leq U_S \) for any \( i, j, t \). We can write the first term of (32) in a different form by summing over the time slots in which the jobs start, i.e.,

\[
\sum_{i=1}^{I} \hat{E}_{t_k} \left[ \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} v_j(f_j(t_k + \tau)) \frac{\mathbb{1}_{I_{j}(t_k + \tau),j}(t_k + \tau)}{\mu_{I_{j}(t_k + \tau),j}(f_j(t_k + \tau))} \right] \\
\geq \sum_{i=1}^{I} \hat{E}_{t_k} \left[ D_k + D_{k+1} - 1 \right] v_j(f_j(t_k + \tau)) \frac{\mathbb{1}_{I_{j}(t_k + \tau),j}(t_k + \tau)}{\mu_{I_{j}(t_k + \tau),j}(f_j(t_k + \tau))} - U_S \sum_i q_i, \tag{33}
\]

where the inequality holds since the last job starting before \( t_k + D_k + D_{k+1} \) may not finish before \( t_k + D_k + D_{k+1} \) and the first job finishing at or after \( t_k + D_k \) may not start at or after \( t_k + D_k \), and we also use the fact that \( v_j(t) \leq \sum_i q_i \) and \( 1/\mu_{i,j}(t) \leq U_S \) for all \( t \). Define \( X_{i,j}(t) \) such that

\[
X_{i,j}(t) := \begin{cases} 
S_{i,j}(t), & \text{if } \eta_j(t) = 1 \text{ (not idling)}; \\
1, & \text{if } \eta_j(t) = 0 \text{ (idling)}. 
\end{cases}
\]

Then since \( \mu_{i,j}(t_k + \tau) \leq 1 \), we have

\[
\sum_{i=1}^{I} \hat{E}_{t_k} \left[ D_k + D_{k+1} - 1 \right] v_j(f_j(t_k + \tau)) \frac{\mathbb{1}_{I_{j}(t_k + \tau),j}(t_k + \tau)}{\mu_{I_{j}(t_k + \tau),j}(f_j(t_k + \tau))} \\
\geq \sum_{i=1}^{I} \hat{E}_{t_k} \left[ D_k + D_{k+1} - 1 \right] v_j(t_k + \tau) \mathbb{1}_{I_{j}(t_k + \tau),j}(t_k + \tau) E[X_{i,j}(t_k + \tau)] \tag{34}
\]

From (33) and (34), we have

\[
\sum_{i=1}^{I} \hat{E}_{t_k} \left[ D_k + D_{k+1} - 1 \right] v_j(f_j(t_k + \tau)) \frac{\mathbb{1}_{I_{j}(t_k + \tau),j}(t_k + \tau)}{\mu_{I_{j}(t_k + \tau),j}(f_j(t_k + \tau))} \\
\geq \sum_{i=1}^{I} \hat{E}_{t_k} \left[ D_k + D_{k+1} - 1 \right] v_j(t_k + \tau) \mathbb{1}_{I_{j}(t_k + \tau),j}(t_k + \tau) - \frac{U_S \sum_i q_i}{T^2} \\
\sum_{i=1}^{I} \hat{E}_{t_k} \left[ D_k + D_{k+1} - 1 \right] v_j(t_k + \tau) \mathbb{1}_{I_{j}(t_k + \tau),j}(t_k + \tau) E[X_{i,j}(t_k + \tau)] - \frac{D_{k+1} U_S \sum_i q_i}{T^2} \\
\geq \sum_{i=1}^{I} \hat{E}_{t_k} \left[ D_k + D_{k+1} - 1 \right] v_j(t_k + \tau) \mathbb{1}_{I_{j}(t_k + \tau),j}(t_k + \tau) X_{i,j}(t_k + \tau) - \frac{D_{k+1} U_S \sum_i q_i}{T^2} \tag{35}
\]

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Note that the term $\sum_{i=1}^{I} \hat{E}_{tk} \left[ \sum_{\tau=D_k}^{D_k+D_k+1-1} v_j(t_k + \tau) \mathbb{1}_{i,j}^{\tau}(t_k+\tau) = i \right] X_{i,j}(t_k + \tau)$ in (33) can be rewritten using $f_j$ in the following way:

$$
\sum_{i=1}^{I} \hat{E}_{tk} \left[ \sum_{\tau=D_k}^{D_k+D_k+1-1} v_j(t_k + \tau) \mathbb{1}_{i,j}^{\tau}(t_k+\tau) = i \right] X_{i,j}(t_k + \tau) = \hat{E}_{tk} \left[ \sum_{\tau=\tau_{\text{start}}}^{\tau_{\text{end}}} v_j(f_j(t_k + \tau)) \right],
$$

where $t_k + \tau_{\text{start}}$ is the starting (or idling) time of the first schedule that starts at or after $t_k + D_k$ and $t_k + \tau_{\text{end}}$ is the finishing (or idling) time of the last schedule that starts at or before $t_k + D_k + D_k+1 - 1$. By the facts that $t_k + \tau_{\text{start}} < t_k + D_k + U_S$, $t_k + \tau_{\text{end}} \geq t_k + D_k + D_k+1 - 1$, and $v_j(t) \leq \sum_i q_i$, we have

$$
\hat{E}_{tk} \left[ \sum_{\tau=\tau_{\text{start}}}^{\tau_{\text{end}}} v_j(f_j(t_k + \tau)) \right] \geq \hat{E}_{tk} \left[ \sum_{\tau=D_k}^{D_k+D_k+1-1} v_j(f_j(t_k + \tau)) \right] - U_S \sum_i q_i
$$

$$
\geq \sum_{\tau=D_k}^{D_k+D_k+1-1} \max_i q_i \left( \mu_{i,j}(t_k + \tau) - \frac{1}{T_p} \right) - U_S \sum_i q_i
$$

$$
\geq \sum_{\tau=D_k}^{D_k+D_k+1-1} \max_i q_i \mu_{i,j}(t_k + \tau) - \left( \frac{D_k+1}{T_p} + U_S \right) \sum_i q_i,
$$

(37)

where the first inequality uses the fact that $t_k + \tau - f_j(t_k + \tau) \leq U_S$ and the second condition in Assumption 1. Then, combining (32), (35), (36), and (37), we have

$$
\hat{E}_{tk} \left[ \sum_{\tau=D_k}^{D_k+D_k+1-1} v_j(t_k + \tau) \mathbb{1}_{i,j}^{\tau}(t_k+\tau) \right] \mathbb{1}_{f_j(t_k + \tau) = j} \left[ \sum_{\tau=D_k}^{D_k+D_k+1-1} v_j(t_k + \tau) \mathbb{1}_{i,j}^{\tau}(t_k+\tau) = i \right] (E[X_{i,j}(t_k + \tau)] - X_{i,j}(t_k + \tau))
$$

$$
\geq \sum_{i=1}^{I} \hat{E}_{tk} \left[ \sum_{\tau=D_k}^{D_k+D_k+1-1} \max_i q_i \mu_{i,j}(t_k + \tau) - \left( \frac{D_k+1}{T_p} + 2U_S + \frac{D_k+1}{T_p} \right) \sum_i q_i \right].
$$

(38)

Next let us look at the first term of (38). Note that the differences $E[X_{i,j}(t_k + \tau)] - X_{i,j}(t_k + \tau)$ at the idling time slots are zero by definition. Hence,

$$
\sum_{i=1}^{I} \hat{E}_{tk} \left[ \sum_{\tau=D_k}^{D_k+D_k+1-1} \max_i q_i \mu_{i,j}(t_k + \tau) - \left( \frac{D_k+1}{T_p} + 2U_S + \frac{D_k+1}{T_p} \right) \sum_i q_i \right].
$$

(39)

Consider the following concentration event

$$
\mathcal{E}_{X_{t_k,i,j}} := \left\{ \sum_{\tau=D_k}^{D_k+D_k+1-1} v_j(t_k + \tau) \mathbb{1}_{i,j}^{\tau}(t_k+\tau) \geq \hat{E}_{tk} \left[ \sum_{\tau=D_k}^{D_k+D_k+1-1} v_j(t_k + \tau) \mathbb{1}_{i,j}^{\tau}(t_k+\tau) \right] \right\}.
$$

(40)

We have the following lemma:

**Lemma 7.** For any $k \geq 0$, $i \in \{1, \ldots, I\}$, $j \in \{1, \ldots, J\}$, we have $\hat{P}_{tk} \left( \mathcal{E}_{X_{t_k,i,j}} \right) \leq \frac{1}{T^2}$.
where the last inequality is by Lemma 7. Combining (25), (38), (39), and (41), we have
\[
\sum_{i=1}^{l} \hat{E}_{tk} \left[ \sum_{\tau=D_k}^{D_k+D_{k+1}-1} v_j(t_k + \tau) \mathbb{1}_{i,i}(t_k + \tau) \left( E[X_{i,j}(t_k + \tau)] - X_{i,j}(t_k + \tau) \right) \right] \\
\geq \sum_{i=1}^{l} \left[ \hat{P}_{tk} (\mathcal{E}_{X,t_k,i,j}) \left( -U_S\sqrt{2T\log T} \sum_i q_i \right) + \hat{P}_{tk} (\mathcal{E}_{X,t_k,i,j}) \left( -D_{k+1}U_S \sum_i q_i \right) \right] \\
\geq - \left( IU_S\sqrt{2T\log T} + \frac{D_{k+1}U_S}{T^2} \right) \sum_i q_i,
\]
(41)
where the last inequality is by Lemma 7. Combining (25), (38), (39), and (41), we have
\[
(25) = \hat{E}_{tk} \left[ \sum_{\tau=D_k}^{D_k+D_{k+1}-1} v_j(t_k(t_k + \tau) - \sum_{i=1}^{l} \hat{E}_{tk} \left[ \sum_{\tau=D_k}^{D_k+D_{k+1}-1} v_j(t_k(t_k + \tau) - \sum_{i=1}^{l} q_i \mu_{i,j}(t_k + \tau) - \left( \frac{D_{k+1}}{T^p} + 2U_S + IU_S\sqrt{2T\log T} + \frac{2D_{k+1}U_S}{T^2} \right) \sum_i q_i - \frac{9JT^2U_S}{4} \right] \\
\geq \sum_{\tau=D_k}^{D_k+D_{k+1}-1} \max_i q_i \mu_{i,j}(t_k + \tau) - \left( \frac{D_{k+1}}{T^p} + 2U_S + IU_S\sqrt{2T\log T} + \frac{2D_{k+1}U_S}{T^2} \right) \sum_i q_i - \frac{9JT^2U_S}{4} \right] \\
\geq \sum_{\tau=D_k}^{D_k+D_{k+1}-1} \max_i q_i \mu_{i,j}(t_k + \tau) - \frac{401U^2_S\log T}{T^{\min\{\frac{1}{2},p\}}} \sum_{\tau=D_k}^{D_k+D_{k+1}-1} \tilde{E}_{tk} \left[ \sum_i Q_i(t_k + \tau) \right] - \frac{407IU^2_SU_AT^2}{4} \right].
\]
(42)
Combining (25), (26), (42), and (31) and using Lemma 5 on \( \sum_i q_i \) and Lemma 1 on \( D_{k+1} \), we have
\[
\sum_{\tau=D_k}^{D_k+D_{k+1}-1} \max_i q_i \mu_{i,j}(t_k + \tau) - \left( \frac{D_{k+1}}{T^p} + 2U_S + IU_S\sqrt{2T\log T} + \frac{2D_{k+1}U_S}{T^2} \right) \sum_i q_i - \frac{9JT^2U_S}{4} \right] \\
\geq \sum_{\tau=D_k}^{D_k+D_{k+1}-1} \max_i q_i \mu_{i,j}(t_k + \tau) - \frac{401U^2_S\log T}{T^{\min\{\frac{1}{2},p\}}} \sum_{\tau=D_k}^{D_k+D_{k+1}-1} \tilde{E}_{tk} \left[ \sum_i Q_i(t_k + \tau) \right] - \frac{407IU^2_SU_AT^2}{4} \right].
\]
(43)
Substituting (43) into (18), we have
\[
\sum_{\tau=D_k}^{D_k+D_{k+1}-1} \tilde{E}_{tk} \left[ \sum_i Q_i(t_k + \tau) \mathbb{1}_{i,i}(t_k + \tau) \right] \\
\geq \sum_{\tau=D_k}^{D_k+D_{k+1}-1} \max_i q_i \mu_{i,j}(t_k + \tau) - \frac{402U^2_S\log T}{T^{\min\{\frac{1}{2},p\}}} \sum_{\tau=D_k}^{D_k+D_{k+1}-1} \tilde{E}_{tk} \left[ \sum_i Q_i(t_k + \tau) \right] - \frac{408IJU^2_SU_AT^2}{4}.
\]
(44)
Substituting (44) into (6), we have
\[
(6) \leq - \frac{804IJU^2_S\log T}{T^{\min\{\frac{1}{2},p\}}} \sum_{\tau=D_k}^{D_k+D_{k+1}-1} \tilde{E}_{tk} \left[ \sum_i Q_i(t_k + \tau) \right] + 817J^2U^2_SU_AT^2.
\]
(45)
5.6 Telescoping Sum

Combining (5), (6), (13), and (45), we have

\[ E \left[ L(t_k + D_k + D_{k+1}) - L(t_k + D_k) \mid Q(t_k) = q, H(t_k) = h \right] \]
\[ \leq \left( \frac{804JU^2 \log T}{T^{\min(\frac{1}{2}, p)}} - 2\delta \right) \sum_{\tau=D_k}^{D_k+D_{k+1}-1} \hat{E}_k \left[ \sum_i Q_i(t_k + \tau) \right] + 821T^2 J^2 U_S^2 U_A^2 T^2 \]
\[ \leq -\delta \sum_{\tau=D_k}^{D_k+D_{k+1}-1} E \left[ \sum_i Q_i(t_k + \tau) \mid Q(t_k) = q, H(t_k) = h \right] + 821T^2 J^2 U_S^2 U_A^2 T^2 \]

since \( \frac{804JU^2 \log T}{T^{\min(\frac{1}{2}, p)}} \leq \delta \).

Taking expectation on both sides and summing over \( k = 0, 1, \ldots, K - 1 \), we have

\[ E \left[ L \left( \sum_{k=0}^{K} D_k \right) - L(D_0) \right] \leq -\delta \sum_{\tau=D_0}^{K-1} E \left[ \sum_i Q_i(\tau) \right] + 821K^2 J^2 U_S^2 U_A^2 T^2. \]

Hence, we have

\[ \sum_{\tau=D_0}^{K-1} E \left[ \sum_i Q_i(\tau) \right] \leq \frac{1}{\delta} E \left[ -L \left( \sum_{k=0}^{K} D_k \right) + L(D_0) \right] + \frac{821K^2 J^2 U_S^2 U_A^2 T^2}{\delta} \]

Dividing both sides by \( \sum_{k=1}^{K} D_k \), we have

\[ \frac{1}{\sum_{k=1}^{K} D_k} \sum_{\tau=D_0}^{K-1} E \left[ \sum_i Q_i(\tau) \right] \leq \frac{1}{\delta \sum_{k=1}^{K} D_k} E \left[ -L \left( \sum_{k=0}^{K} D_k \right) + L(D_0) \right] + \frac{821K^2 J^2 U_S^2 U_A^2 T^2}{\delta \sum_{k=1}^{K} D_k} \]
\[ \leq \frac{1}{\delta \sum_{k=1}^{K} D_k} E \left[ L(D_0) \right] + \frac{1642J^2 U_S^2 U_A^2 T}{\delta}, \]

where the last inequality uses Lemma [1]. Letting \( K \to \infty \), we can conclude that

\[ \limsup_{t \to \infty} \frac{1}{t} \sum_{\tau=1}^{t} E \left[ \sum_i Q_i(\tau) \right] \leq \frac{1642J^2 U_S^2 U_A^2 T}{\delta}. \]

6 Simulation Results

In this section, we evaluate the proposed algorithm numerically through simulation. We compare the proposed MaxWeight with discounted UCB algorithm with several baselines, including three MaxWeight algorithms using empirical mean (MaxWeight with EM), discounted empirical mean (MaxWeight with discounted EM), and UCB (MaxWeight with UCB) as the estimated service rates.

In the simulation, we assume that the arrival \( A_i(t) \) follows a Bernoulli distribution and the service time \( S_{ij}(t) \) takes value in \{1, 2\} \( (U_S = 2) \). We consider a system with 10 job types and 10 servers. We compare the algorithms in the following four settings, stationary, nonstationary aperiodic, nonstationary periodic, and nonstationary periodic with a larger period. More details about the settings and parameters can be found in Appendix [1]. The results are shown in Fig. [4].

Fig. [4a] shows the results for the stationary setting, where the mean arrival rates and the mean service rates are time-invariant. As seen in the figure, the queue lengths of MaxWeight with EM and MaxWeight with discounted EM increase very fast and unstable and exceed 100 after time slot 30 and 16, respectively, while both MaxWeight with discounted UCB and MaxWeight with UCB are stable. The reason is that the empirical mean method lacks exploration so the system may “locks in” in a
state with poor estimation and wrong scheduling decision. We also present the same figure with a larger range of Y-axis in Appendix [H] which shows the missing parts of the curves.

Fig. 4b shows the results for the nonstationary aperiodic setting, where the mean service rates are time-varying and aperiodic. We have the same observation for MaxWeight with EM and MaxWeight with discounted EM as in the stationary setting. Their queue lengths quickly exceed 140. However, MaxWeight with UCB is unstable in this setting, because the mean service rates are changing over time but the algorithm is still making scheduling decisions based on the outdated information. Thanks to the discount factor, MaxWeight with discounted UCB is stable and performs the best.

Fig. 4c and Fig. 4d show the results for the nonstationary periodic settings, where the mean arrival rates and the mean service rates are time-varying and periodic. In both figures, we have the same observation for MaxWeight with EM and MaxWeight with discounted EM as in the other settings. Their queue lengths quickly exceed the Y-axis limits. Compared with MaxWeight with discounted UCB, the queue length of MaxWeight with UCB is larger. Note that the period of the setting of Fig. 4d is 10 times as large as that of Fig. 4c. As seen in the figures, the queue length of MaxWeight with UCB has wild oscillation and the amplitude of the oscillation becomes much larger when the period becomes larger, while the amplitude of the oscillation for MaxWeight with discounted UCB remains approximately the same. The reason is that the proposed algorithm can quickly adapt to the changing statistics thanks to the discount factor.

7 Conclusions

This paper considered scheduling in multi-server queueing systems with unknown arrival and service statistics, and proposed a new scheduling algorithm, MaxWeight with discounted UCB. Based on the Lyapunov drift analysis and concentration inequalities of self-normalized means, we proved that MaxWeight with discounted UCB guarantees queue stability (in the mean) when the arrival rates are strictly within the service capacity region. This result holds both for stationary systems and nonstationary systems.
(a) Stationary. The queue lengths of \textit{MaxWeight with EM} and \textit{MaxWeight with discounted EM} exceed 100 after time slot 30 and 16, respectively.

(b) Nonstationary aperiodic. The queue lengths of \textit{MaxWeight with EM} and \textit{MaxWeight with discounted EM} exceed 140 after time slot 54 and 34, respectively.

(c) Nonstationary periodic: period=400 for mean arrival rates, period=800 for mean service rates. Both the queue length of \textit{MaxWeight with EM} and that of \textit{MaxWeight with discounted EM} exceed 120 after time slot 34.

(d) Nonstationary periodic: period=4000 for mean arrival rates, period=8000 for mean service rates. The queue lengths of \textit{MaxWeight with EM} and \textit{MaxWeight with discounted EM} exceed 800 after time slot 227 and 205, respectively.

Figure 4: Simulation results.
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A Proof of Lemma 1

Lemma 1. Suppose $W \leq \frac{T}{2}$. Then $\frac{T}{2} \leq D(t) \leq \frac{T}{2} + W \leq T$ for any $t$.

Proof. Recall the definition of $D(t)$:

$$D(t) = \min_{n} \sum_{l=0}^{n} w(\tau_l(t))$$

s.t. $\sum_{l=0}^{n} w(\tau_l(t)) \geq \frac{T}{2}$.

Recall that $n^*(t)$ is the optimal solution to the above optimization problem. Note that $D(t) = \sum_{l=0}^{n^*(t)} w(\tau_l(t)) \geq \frac{T}{2}$ and $\sum_{l=0}^{n^*(t)-1} w(\tau_l(t)) < \frac{T}{2}$. Hence, we have

$$D(t) = \sum_{l=0}^{n^*(t)} w(\tau_l(t)) = \sum_{l=0}^{n^*(t)-1} w(\tau_l(t)) + w(\tau_{n^*(t)}(t)) \leq \frac{T}{2} + W,$$

where the last inequality is due to the bound $w(\tau) \leq W$ for any $\tau$. Therefore, for any $t$, we have

$$\frac{T}{2} \leq D(t) \leq \frac{T}{2} + W \leq T,$$

where the last inequality is by $W \leq \frac{T}{2}$.

\[ \square \]

B Proof of Lemma 2

Lemma 2. For any $i, t$, $Q_i(t+1) \leq \max \left\{ J, Q_i(t) + A_i(t) - \sum_j \mathbb{1}_{i,j}(t) \right\}$.

Proof. Fix $i$ and $t$. Consider two cases. The first case is that there exists $j$ such that $\eta_j(t) = 0$ (server $j$ is idling) and $I_j(t) = i$. The second case is that for all servers $j$, $\eta_j(t) = 1$ or $I_j(t) \neq i$.

Notice that for the first case we must have

$$\hat{Q}_i(t) + A_i(t) = 0$$

since server $j$ is scheduled to $i$ and is idling. Hence, we have

$$Q_i(t) + A_i(t) \leq \hat{Q}_i(t) + J + A_i(t) = J.$$

Hence, by the queue dynamics (1) and the above inequality, we have

$$Q_i(t+1) \leq Q_i(t) + A_i(t) \leq J$$

for the first case. For the second case, we have

$$Q_i(t+1) = Q_i(t) + A_i(t) - \sum_j \mathbb{1}_{i,j}(t) \eta_j(t)$$

$$= Q_i(t) + A_i(t) - \sum_j \mathbb{1}_{i,j}(t),$$

where the second inequality holds since for any server $j$, either $\eta_j(t) = 1$ or $\mathbb{1}_{i,j}(t) = 0$.

Combining the two cases, we obtain that for any $i, t$,

$$Q_i(t+1) \leq \max \left\{ J, Q_i(t) + A_i(t) - \sum_j \mathbb{1}_{i,j}(t) \right\}.$$

\[ \square \]

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C Proof of Lemma [3]

Lemma 3. For any \( t, i, \tau \geq 0 \), we have

1. \( Q_i(t) - J\tau \leq Q_i(t + \tau) \leq Q_i(t) + \tau U_A \);
2. \( \sum_i Q_i(t + \tau) \geq \sum_i Q_i(t) - J\tau \).

Proof. (1) holds since \( Q_i(t) \) can increase at most \( U_A \) and can decrease at most \( J \) for each time slot by the queue dynamics (1).

(2) holds since the total queue length can decrease by at most \( J \) for each time slot. This is because there are \( J \) servers in total and each server can serve at most one job at a time.

\[ \square \]

D Proof of Lemma [4]

Lemma 4. For any \( k \geq 0 \), we have

\[
\hat{P}_k(\mathcal{E}^*_t) := \Pr(\mathcal{E}^*_t | Q(t_k) = q, H(t_k) = h) \leq \frac{I}{T^2},
\]

where

\[
\mathcal{E}^*_t := \left\{ \text{for all } i, \tau \in \left[ D_k - \frac{T}{8}, D_k + D_{k+1} - 1 \right], |\bar{\mu}_{i,j}(t_k + \tau) - \mu_{i,j}(t_k + \tau)| \leq b_{i,j}(t_k + \tau) \right\}.
\]

Proof. We define another event as follows:

\[
\mathcal{E}^\prime_{t_k} := \left\{ \text{for all } i, \tau \in \left[ D_k - \frac{T}{8}, D_k + D_{k+1} - 1 \right], \left| \frac{1}{\bar{\mu}_{i,j}(t_k + \tau)} - \frac{1}{\mu_{i,j}(t_k + \tau)} \right| \leq \sqrt{\frac{4U^2 \log T}{N_{i,j}(t_k + \tau)}} \right\}.
\]

We first show that \( \mathcal{E}^\prime_{t_k} \subseteq \mathcal{E}_{t_k} \). Suppose we have

\[
\left| \frac{1}{\bar{\mu}_{i,j}(t_k + \tau)} - \frac{1}{\mu_{i,j}(t_k + \tau)} \right| \leq \sqrt{\frac{4U^2 \log T}{N_{i,j}(t_k + \tau)}}.
\]

Then

\[
\frac{1}{\bar{\mu}_{i,j}(t_k + \tau)} - \frac{1}{\mu_{i,j}(t_k + \tau)} \leq \frac{1}{N_{i,j}(t_k + \tau)} \leq \frac{1}{\bar{\mu}_{i,j}(t_k + \tau)} + \frac{4U^2 \log T}{N_{i,j}(t_k + \tau)},
\]

which implies that

\[
\frac{1}{\mu_{i,j}(t_k + \tau)} + \sqrt{\frac{4U^2 \log T}{N_{i,j}(t_k + \tau)}} \leq \mu_{i,j}(t_k + \tau) \leq \frac{1}{\mu_{i,j}(t_k + \tau)} + \sqrt{\frac{4U^2 \log T}{N_{i,j}(t_k + \tau)}},
\]

due to the fact that \( \mu_{i,j}(t_k + \tau) \leq 1 \). Note that we also have

\[
\frac{1}{\mu_{i,j}(t_k + \tau)} + \sqrt{\frac{4U^2 \log T}{N_{i,j}(t_k + \tau)}} \leq \frac{1}{\bar{\mu}_{i,j}(t_k + \tau)} = \bar{\mu}_{i,j}(t_k + \tau) \leq \frac{1}{\mu_{i,j}(t_k + \tau)} + \sqrt{\frac{4U^2 \log T}{N_{i,j}(t_k + \tau)}},
\]
due to the fact that \( \hat{\mu}_{i,j}(t_k + \tau) \leq 1 \). Hence,

\[
|\hat{\mu}_{i,j}(t_k + \tau) - \mu_{i,j}(t_k + \tau)| \leq \max \left\{ \frac{1}{\mu_{i,j}(t_k + \tau)} - \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}}, 1 \right\} - \min \left\{ \frac{1}{\mu_{i,j}(t_k + \tau)} + \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}}, 1 \right\}
\]

\[
= \max \left\{ \frac{1}{\mu_{i,j}(t_k + \tau)} - \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}}, 1 \right\} - \min \left\{ \frac{1}{\mu_{i,j}(t_k + \tau)} + \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}}, 1 \right\}
\]

\[
= \max \left\{ \frac{1}{\mu_{i,j}(t_k + \tau)} - \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}}, 1 \right\} - \min \left\{ \frac{1}{\mu_{i,j}(t_k + \tau)} + \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}}, 1 \right\}
\]

\[
\leq \max \left\{ \frac{1}{\mu_{i,j}(t_k + \tau)} - \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}}, 1 \right\} \leq 2 \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}},
\]

where the last inequality is due to the fact that \( \frac{1}{\mu_{i,j}(t_k + \tau)} \geq 1 \). Hence, we have

\[
|\hat{\mu}_{i,j}(t_k + \tau) - \mu_{i,j}(t_k + \tau)| \leq \min \left\{ 2 \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}}, 1 \right\} = b_{i,j}(t_k + \tau)
\]

since \( \hat{\mu}_{i,j}(t_k + \tau), \mu_{i,j}(t_k + \tau) \in [0, 1] \). \( \mathcal{E}_{t_k,i,j}^c \subseteq \mathcal{E}_{t_k,i,j} \) is proved.

Next, it remains to show that

\[
P_t \left( \mathcal{E}_{t_k,i,j}^c \right) \leq \frac{1}{T^2}.
\]

Consider the event

\[
\mathcal{E}'_{t_k,i,j} : = \left\{ \frac{1}{|\hat{\mu}_{i,j}(t_k + \tau) - \mu_{i,j}(t_k + \tau)|} \leq \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}} \right\}.
\]

We have

\[
\hat{P}_t \left( \mathcal{E}_{t_k,i,j}^c \right) = \hat{P}_t \left( \left\{ \frac{1}{\hat{\mu}_{i,j}(t_k + \tau)} - \frac{1}{\mu_{i,j}(t_k + \tau)} \right\} > \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}} \right)
\]

\[
= \hat{P}_t \left( \left\{ \frac{1}{\hat{\mu}_{i,j}(t_k + \tau)} - \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}}, \hat{N}_{i,j}(t_k + \tau) > 4 \log T \right\} \right)
\]

\[
+ \hat{P}_t \left( \left\{ \frac{1}{\hat{\mu}_{i,j}(t_k + \tau)} - \frac{1}{\mu_{i,j}(t_k + \tau)} \right\} > \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}}, \hat{N}_{i,j}(t_k + \tau) \leq 4 \log T \right) \right).
\]

Since

\[
\hat{P}_t \left( \left\{ \frac{1}{\hat{\mu}_{i,j}(t_k + \tau)} - \frac{1}{\mu_{i,j}(t_k + \tau)} \right\} > \sqrt{\frac{4U_2^2 \log T}{N_{i,j}(t_k + \tau)}}, \hat{N}_{i,j}(t_k + \tau) \leq 4 \log T \right) \right) \right)
\]

\[
\leq \hat{P}_t \left( \left\{ \frac{1}{\hat{\mu}_{i,j}(t_k + \tau)} - \frac{1}{\mu_{i,j}(t_k + \tau)} \right\} > \sqrt{U_2} \right) = 0,
\]

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we then have
\[
\hat{P}_k \left( E_{i,j}^{k} \right) = \hat{P}_k \left( \frac{1}{\mu_{i,j}(t_k + \tau)} - \frac{1}{\mu_{i,j}(t_k + \tau)} \right) > \sqrt{\frac{4U^2_3 \log T}{\hat{N}_{i,j}(t_k + \tau)}}, \hat{N}_{i,j}(t_k + \tau) > 4 \log T \right).
\]
\[
= \hat{P}_k \left( \frac{\gamma \phi_{i,j}(t_k + \tau) - \hat{N}_{i,j}(t_k + \tau)}{\mu_{i,j}(t_k + \tau)} \right) > \sqrt{4\hat{N}_{i,j}(t_k + \tau)U_3^2 \log T}, \hat{N}_{i,j}(t_k + \tau) > 4 \log T \right). \tag{46}
\]

Define a random mapping \( f_j \) that maps a time slot to another time slot such that if \( y = f_j(x) \) then \( y \) is the time slot when server \( j \) picked the job that was being served at server \( j \) in time slot \( x \). If server \( j \) was idling in time slot \( x \), then let \( f_j(x) = x \). That is,
\[
f_j(x) := \max \left\{ t : t \leq x, \hat{i}_j^\times(t) = I_j(x) \right\}.
\]

Let \( M_1 \) be such that \( t_k + M_1 \) is the first time slot when server \( j \) picked queue \( i \) at or after \( t_k \), i.e.,
\[
M_1 := \min \left\{ m : m \geq 0, \hat{i}_j^\times(t_k + m) = i \right\}.
\]

Let \( M_2 := M_1 + (S_{i,j}(t_k + M_1) - 1)\eta_j(t_k + M_1) \). Then \( t_k + M_2 \) is the time slot when the job picked by server \( j \) at time \( t_k + M_1 \) was completed or if server \( j \) was idling because the selected waiting queue is empty at \( t_k + M_1 \), \( M_2 = M_1 \). Hence, \( t_k + M_1 = f_j(t_k + M_2) \). Note that \( M_1, M_2 \) are random variables. Then according to the algorithm, we have
\[
\gamma \phi_{i,j}(t_k + \tau)
\]
\[
= \gamma^{\tau - M_2} \phi_{i,j}(t_k + M_2)
\]
\[
+ \sum_{m=M_2+1}^{\tau} \gamma^{\tau - m + M_{i,j}(t_k + m - 1)} \eta_j(t_k + m - 1) \left[ M_{i,j}(t_k + m - 1) + 1 \right]
\]
\[
= \gamma^{\tau - M_2} \phi_{i,j}(t_k + M_2)
\]
\[
+ \sum_{m=M_2+1}^{\tau} \gamma^{\tau - m + S_{i,j}(f_j(t_k + m - 1)) - 1} \eta_j(f_j(t_k + m - 1)) S_{i,j}(f_j(t_k + m - 1)),
\]
where the summation only includes the time slots when there is job completion of queue \( i \) at server \( j \). This can be transformed into summing over the time slots when server \( j \) is available and picks queue \( i \), i.e.,
\[
\gamma \phi_{i,j}(t_k + \tau)
\]
\[
= \gamma^{\tau - M_2} \phi_{i,j}(t_k + M_2) + \sum_{m=M_2+1}^{\tau} \gamma^{\tau - m} \eta_j(t_k + m - 1) S_{i,j}(t_k + m - 1)
\]
where \( M_3 \) is a random variable such that \( t_k + M_3 = f_j(t_k + \tau) \). Since there is no job of type \( i \) starting at server \( j \) in the time interval \([t_k, t_k + M_1 - 1] \), we have
\[
\gamma \phi_{i,j}(t_k + \tau)
\]
\[
= \gamma^{\tau - M_2} \phi_{i,j}(t_k + M_2) + \sum_{m=1}^{M_3} \gamma^{\tau - m} \eta_j(t_k + m - 1) S_{i,j}(t_k + m - 1)
\]
We claim that there is no job completion of type \( i \) at server \( j \) in the time interval \([t_k + \frac{T}{8}, t_k + M_2 - 1]\) if \( M_2 - 1 \geq \frac{T}{8} \). We can prove it by contradiction. Suppose there is a job of type \( i \) that was completed
at server $j$ in $[t_k + \frac{T}{8}, t_k + M_2 - 1]$. Then the job must start at or after $t_k$ since $U_S \leq \frac{T}{8}$. Also, the job must start before $t_k + M_1$ since there is another job at server $j$ starting at $t_k + M_1$ and finishes at $t_k + M_2$ by the definition of $M_1$ and $M_2$. Therefore, the job that was completed at server $j$ in $[t_k + \frac{T}{8}, t_k + M_2 - 1]$ should start in the time interval $[t_k, t_k + M_1 - 1]$. However, by the definition of $M_1$, there should not be any job of type $i$ starting at server $j$ in $[t_k, t_k + M_1 - 1]$, which is a contradiction. Based on the claim, if $M_2 - 1 \geq \frac{T}{8}$, since there is no job completion, from the algorithm we know

$$
\hat{\phi}_{i,j}(t_k + M_2) = \gamma^{M_2 - \frac{T}{8}} \hat{\phi}_{i,j} \left( t_k + \frac{T}{8} \right) = \gamma^{M_2 - \min \left\{ \frac{T}{8}, M_2 \right\}} \hat{\phi}_{i,j} \left( t_k + \min \left\{ \frac{T}{8}, M_2 \right\} \right)
$$

where the last equality holds since $\min \{ \frac{T}{8}, M_2 \} = \frac{T}{8}$. If $M_2 - 1 < \frac{T}{8}$, then we have

$$
\hat{\phi}_{i,j}(t_k + M_2) = \gamma^{M_2 - \min \left\{ \frac{T}{8}, M_2 \right\}} \hat{\phi}_{i,j} \left( t_k + \min \left\{ \frac{T}{8}, M_2 \right\} \right)
$$

since $\min \{ \frac{T}{8}, M_2 \} = M_2$. Combining the above two cases, we have

$$
\gamma^{T - M_2} \hat{\phi}_{i,j}(t_k + M_2) = \gamma^{T - \min \{ \frac{T}{8}, M_2 \}} \hat{\phi}_{i,j} \left( t_k + \min \left\{ \frac{T}{8}, M_2 \right\} \right)
$$

We want to upper bound this term. Since

$$
\hat{\phi}_{i,j} \left( t_k + \min \left\{ \frac{T}{8}, M_2 \right\} \right) \leq U_S \sum_{t=0}^{\infty} \gamma^t = \frac{T U_S}{8 \log T},
$$

we have

$$
\gamma^{T - \min \{ \frac{T}{8}, M_2 \}} \hat{\phi}_{i,j} \left( t_k + \min \left\{ \frac{T}{8}, M_2 \right\} \right) \leq \left( 1 - \frac{8 \log T}{T} \right)^T \frac{T U_S}{8 \log T} \leq \left( 1 - \frac{2 \log T}{T} \right)^T \frac{T U_S}{8 \log T} \leq \exp(-2 \log T) \frac{T U_S}{8 \log T} = \frac{U_T}{T \log T},
$$

where the second inequality holds since $\tau \geq D_k - \frac{T}{8}$, the third inequality uses the bound on $D_k$ in Lemma 1 and the last inequality is due to the fact that $(1 - \frac{x}{n})^n \leq \exp(-x)$ for any $x \leq n$ and $n \in \mathbb{N}$. Therefore,

$$
\hat{\phi}_{i,j}(t_k + \tau) \leq \frac{U_S}{8 T \log T} + \sum_{m=1}^{M_1} \gamma^{T - m} \tilde{\xi}_{i,j}^{\tau}(t_k + m - 1) \eta_j(t_k + m - 1) S_{i,j}(t_k + m - 1).
$$

(47)

Similarly, we have

$$
\tilde{N}_{i,j}(t_k + \tau) \leq \frac{1}{8 T \log T} + \sum_{m=1}^{M_3} \gamma^{T - m} \tilde{\xi}_{i,j}^{\tau}(t_k + m - 1) \eta_j(t_k + m - 1).
$$

(48)

Note that $E[S_{i,j}(t_k + \tau)] = \frac{1}{\mu_{i,j}(t_k + \tau)}$. Let $\epsilon_m := \tilde{\xi}_{i,j}^{\tau}(t_k + m - 1) \eta_j(t_k + m - 1)$. Substituting [47]
and $\text{(48)}$ into $\text{(46)}$, we have

$$
\hat{P}_k \left( \mathcal{E}_{t_k,i,j,\tau}^c \right) \\
= \hat{P}_k \left( \phi_{i,j}(t_k + \tau) - E[S_{i,j}(t_k + \tau)] - \hat{N}_{i,j}(t_k + \tau) > \sqrt{4\hat{N}_{i,j}(t_k + \tau)U_3^2 \log T}, \hat{N}_{i,j}(t_k + \tau) > 4 \log T \right) \\
\leq \hat{P}_k \left( \sum_{m=1}^{M_3} \gamma^{-m} \epsilon_m (S_{i,j}(t_k + m - 1) - E[S_{i,j}(t_k + m - 1)]) \\
- \sum_{m=1}^{M_3} \gamma^{-m} \epsilon_m E[S_{i,j}(t_k + \tau)] \\
+ \sum_{m=1}^{M_3} \gamma^{-m} \epsilon_m |E[S_{i,j}(t_k + m - 1)] - E[S_{i,j}(t_k + \tau)]| > \sqrt{4\hat{N}_{i,j}(t_k + \tau)U_3^2 \log T} - \frac{U_S}{8T \log T}, \hat{N}_{i,j}(t_k + \tau) > 4 \log T \right),
$$

where in the last inequality we add and subtract the term $E[S_{i,j}(t_k + m - 1)]$ and use the triangle inequality. We note that that in a stationary system, $E[S_{i,j}(t)]$ is a constant and the last step is not needed. Recall Assumption $\text{II}$ on the time-varying service time. We have

$$
|E[S_{i,j}(t_k + m - 1)] - E[S_{i,j}(t_k + \tau)]| \leq \frac{1}{T} \left( \frac{1}{\gamma} \right)^{-m}.
$$

Hence, we have

$$
\hat{P}_k \left( \mathcal{E}_{t_k,i,j,\tau}^c \right) \leq \hat{P}_k \left( \sum_{m=1}^{M_3} \gamma^{-m} \epsilon_m (S_{i,j}(t_k + m - 1) - E[S_{i,j}(t_k + m - 1)]) \\
> \sqrt{4\hat{N}_{i,j}(t_k + \tau)U_3^2 \log T} - \frac{U_S}{8T \log T}, \hat{N}_{i,j}(t_k + \tau) > 4 \log T \right). \quad (49)
$$

Recall that $t_k + M_3 = f_j(t_k + \tau)$. Hence, $t_k + M_3 \in [t_k + \tau - U_S + 1, t_k + \tau]$. Since $U_S \leq T \frac{3}{8}$ and $\tau \in [D_k - \frac{T}{4}, D_k + D_{k+1} - 1]$, we have $D_k - \frac{T}{4} + 1 \leq M_3 \leq D_k + D_{k+1} - 1$. By the bound on $D_k$ and $D_{k+1}$ in Lemma $\text{I}$ we further have

$$
\frac{T}{4} + 1 \leq M_3 \leq 2T - 1. \quad (50)
$$

Hence, we have

$$
\frac{U_S}{8T \log T} + \frac{M_3}{T} \leq \frac{U_S}{8T \log T} + 2 \leq (2 - \sqrt{3})\sqrt{4U_3^2 (\log T)^2} \leq (2 - \sqrt{3})\sqrt{U_3^2 \hat{N}_{i,j}(t_k + \tau) \log T}, \quad (51)
$$

where the second inequality holds for $T > e^5$, and the last inequality holds when $\hat{N}_{i,j}(t_k + \tau) > 4 \log T$. Based on $\text{(51)}$, we can continue to bound $\text{(49)}$ and obtain

$$
\hat{P}_k \left( \mathcal{E}_{t_k,i,j,\tau}^c \right) \\
\leq \hat{P}_k \left( \sum_{m=1}^{M_3} \gamma^{-m} \epsilon_m (S_{i,j}(t_k + m - 1) - E[S_{i,j}(t_k + m - 1)]) > \sqrt{3\hat{N}_{i,j}(t_k + \tau)U_3^2 \log T} \right). \quad (52)
$$
Note that \( \tilde{N}_{i,j}(t_k + \tau) \geq \sum_{m=1}^{M_3} \gamma^{\tau-m} \eta^m_{i,j}(t_{k+m-1}) = \sum_{m=1}^{M_3} \gamma^{\tau-m} \epsilon_m \). Hence, we can further bound (52) as

\[
\hat{P}_k (E_{t_k, \tau}^{\text{rc}}) \leq \hat{P}_k \left( \frac{\sum_{m=1}^{M_3} \gamma^{\tau-m} \epsilon_m (S_{i,j}(t_{k+m-1}) - E[S_{i,j}(t_{k+m-1})])}{\sqrt{\sum_{m=1}^{M_3} \gamma^{\tau-m} \epsilon_m}} > \sqrt{3U_S^2 \log T} \right).
\]

Since \( M_3 \leq \tau \) and \( \gamma < 1 \), we have \( \sqrt{\gamma^{M_3-\tau}} \geq 1 \). Hence, we have

\[
\hat{P}_k (E_{t_k, \tau}^{\text{rc}}) \leq \hat{P}_k \left( \frac{\sum_{m=1}^{M_3} \gamma^{M_3-m} \epsilon_m (S_{i,j}(t_{k+m-1}) - E[S_{i,j}(t_{k+m-1})])}{\sqrt{\sum_{m=1}^{M_3} \gamma^{M_3-m} \epsilon_m}} > \sqrt{3U_S^2 \log T} \right)
\]

\[
= \hat{P}_k \left( \frac{\sum_{m=1}^{M_3} \gamma^{M_3-m} \epsilon_m (S_{i,j}(t_{k+m-1}) - E[S_{i,j}(t_{k+m-1})])}{\sqrt{\sum_{m=1}^{M_3} \gamma^{M_3-m} \epsilon_m}} > \sqrt{3U_S^2 \log T} \right)
\]

where we added the “2” in the last inequality because we want to use the Hoeffding-type inequality for self-normalized means [6, Theorem 18] later in the proof.

Consider the event \( \tilde{E}_{t_k, \tau} : = \bigcap_{\tau = D_k + D_k + 1}^{D_k + D_k + 1} E_{t_k, \tau}^{\text{rc}} \). Then from the result (53), we have

\[
\hat{P}_k (E_{t_k, \tau}^{\text{rc}}) \leq \hat{P}_k \left( \text{there exists } \tau \in \left[ D_k - \frac{T}{8}, D_k + D_k + 1 \right], \right.
\]

\[
\frac{\sum_{m=1}^{M_3} \gamma^{M_3-m} \epsilon_m (S_{i,j}(t_{k+m-1}) - E[S_{i,j}(t_{k+m-1})])}{\sqrt{\sum_{m=1}^{M_3} \gamma^{M_3-m} \epsilon_m}} > \sqrt{3U_S^2 \log T}
\]

\[
\leq \hat{P}_k \left( \text{there exists } M \in \left[ \frac{T}{4} + 1, 2T - 1 \right], \right.
\]

\[
\frac{\sum_{m=1}^{M} \gamma^{M-m} \epsilon_m (S_{i,j}(t_{k+m-1}) - E[S_{i,j}(t_{k+m-1})])}{\sqrt{\sum_{m=1}^{M} \gamma^{M-m} \epsilon_m}} > \sqrt{3U_S^2 \log T}
\]

\[
\leq \sum_{M = \frac{T}{4} + 1}^{2T-1} \hat{P}_k \left( \frac{\sum_{m=1}^{M} \gamma^{M-m} \epsilon_m (S_{i,j}(t_{k+m-1}) - E[S_{i,j}(t_{k+m-1})])}{\sqrt{\sum_{m=1}^{M} \gamma^{M-m} \epsilon_m}} > \sqrt{3U_S^2 \log T} \right). \quad (54)
\]

where the second inequality uses the bound (50) on \( M_3 \), and the last inequality uses the union bound.

Let us view the conditional probability \( \hat{P}_k \) as a new probability measure. Then \( \hat{E}_t \) is the expectation under this measure. Note that \( (S_{i,j}(t_k + m - 1))_{m=1}^{\infty} \) is a sequence of independent bounded random variables under this new measure since they are independent of \( Q(t_k) \) and \( H(t_k) \), which also implies that

\[
E[S_{i,j}(t_k + m - 1)] = \hat{E}_t [S_{i,j}(t_k + m - 1)]. \quad (55)
\]

Let \( \mathcal{F}_m \) defined as

\[
\mathcal{F}_m := \sigma ((S(t_k + n - 1))_{n=1}^{m}, (A(t_k + n - 1))_{n=1}^{m+1}, (Q(t_k + n - 1))_{n=1}^{m+1}, (H(t_k + n - 1))_{n=1}^{m+1})
\]

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where \( \sigma(\cdot) \) denotes the \( \sigma \)-algebra generated by the random variables. Note that

\[
\sigma(S_{i,j}(t_k), ..., S_{i,j}(t_k + m - 1)) \subset \mathcal{F}_m
\]

and for any \( n > m, S_{i,j}(t_k + n - 1) \) is independent of \( \mathcal{F}_m \). Recall that \( \epsilon_m := 1_{i_j^*(t_k + m - 1)} - \eta_j(t_k + m - 1) \).

Since the scheduling decision at time \( t_k + m - 1 \) is determined by \( Q(t_k + m - 1) \) and \( H(t_k + m - 1) \), \( 1_{i_j^*(t_k + m - 1)} = i \) is \( \mathcal{F}_{m-1} \)-measurable. Since \( \eta_j(t_k + m - 1) \) is determined by \( A(t_k + m - 1), Q(t_k + m - 1) \), and \( H(t_k + m - 1), \eta_j(t_k + m - 1) \) is also \( \mathcal{F}_{m-1} \)-measurable. Therefore, \( \epsilon_m \) is \( \mathcal{F}_{m-1} \)-measurable, i.e., \( \epsilon_m \) is a previsible (or predictable) sequence of Bernoulli random variables. We restate the Hoeffding-type inequality for self-normalized means in [6, Theorem 18], [7] in our setting as follows:

**Theorem 2** (Hoeffding-type inequality for self-normalized means, Theorem 18 in [8]).

Let \( (X_m)_{m \geq 1} \) be a sequence of nonnegative independent bounded random variables with \( X_m \in [0, B] \), \( \sigma(X_1, \ldots, X_m) \subset \mathcal{F}_m \), and for \( n > m, X_n \) is independent of \( \mathcal{F}_m \). For all integers \( M \) and all \( \beta > 0 \),

\[
\hat{P}_{t_k} \left( \frac{\sum_{m=1}^{M} \gamma^{M-m} X_m \epsilon_m - \sum_{m=1}^{M} \gamma^{M-m} \hat{E}_{t_k}[X_m] \epsilon_m}{\sqrt{\sum_{m=1}^{M} \gamma^{2(M-m)} \epsilon_m}} > \beta \right) \leq \exp \left( \frac{-2\beta^2}{B^2} \left( 1 - \frac{\epsilon^2}{16} \right) \right)
\]

for all \( \zeta > 0 \).

Applying Theorem 2 with \( X_m = S_{i,j}(t_k + m - 1), \beta = \sqrt{3U_3^2 \log T}, \) and \( B = U_s \), we have

\[
\hat{P}_{t_k} \left( \frac{\sum_{m=1}^{M} \gamma^{M-m} \epsilon_m (S_{i,j}(t_k + m - 1) - \hat{E}_{t_k}[S_{i,j}(t_k + m - 1)])}{\sqrt{\sum_{m=1}^{M} \gamma^{2(M-m)} \epsilon_m}} > \sqrt{3U_3^2 \log T} \right) \leq \exp \left( \frac{-6}{1 - \frac{\epsilon^2}{16}} \log T \right)
\]

for all \( \zeta > 0 \) and all positive integers \( M \), where the last inequality holds since \( \sum_{m=1}^{M} \gamma^{M-m} = \frac{1 - \gamma^M}{1 - \gamma} \leq \frac{T}{8 \log T} \). Although in [3] the bound is only proved for overestimation, the proof can be extended to show that the bound also holds for underestimation. Specifically, note that

\[
\hat{E}_{t_k} \left[ \hat{E}_{t_k}[S_{i,j}(t_k + m - 1)] - S_{i,j}(t_k + m - 1) \right] = 0
\]

and

\[
\hat{E}_{t_k}[S_{i,j}(t_k + m - 1)] - U_S \leq \hat{E}_{t_k}[S_{i,j}(t_k + m - 1)] - S_{i,j}(t_k + m - 1) \leq \hat{E}_{t_k}[S_{i,j}(t_k + m - 1)].
\]

Hence, considering the random variable \( \hat{E}_{t_k}[S_{i,j}(t_k + m - 1)] - S_{i,j}(t_k + m - 1) \), from [3] Lemma 8.1, for any \( \lambda > 0 \), we have

\[
\log \hat{E}_{t_k}[\exp(-\lambda S_{i,j}(t_k + m - 1))] \leq \frac{\lambda^2 U_3^2}{8} - \lambda \hat{E}_{t_k}[S_{i,j}(t_k + m - 1)].
\]
Hence, we can apply the same proof in [3] Theorem 18 by replacing $\lambda$ in the proof with $-\lambda$. Then for underestimation, we also have the same bound, i.e.,

$$
\hat{P}_t \left( \sum_{m=1}^{M} \gamma^{M-m} \epsilon_m \left( \hat{E}_{t_k} [S_{i,j}(t_k + m - 1)] - S_{i,j}(t_k + m - 1) \right) \right) > \sqrt{3U_S^2 \log T}
$$

\begin{align}
&\leq \left( \log \left( \frac{T}{8 \log T} \right) + 1 \right) \exp \left( -6 \left( 1 - \frac{\zeta^2}{16} \right) \log T \right)
\end{align}

for all $\zeta > 0$ and all positive integers $M$. Taking the union bound over underestimation and overestimation, we have

$$
\hat{P}_t \left( \left| \sum_{m=1}^{M} \gamma^{M-m} \epsilon_m \left( S_{i,j}(t_k + m - 1) - \hat{E}_{t_k} [S_{i,j}(t_k + m - 1)] \right) \right| \right) > \sqrt{3U_S^2 \log T} 
$$

\begin{align}
&\leq 2 \left( \log \left( \frac{T}{8 \log T} \right) + 1 \right) \exp \left( -6 \left( 1 - \frac{\zeta^2}{16} \right) \log T \right)
\end{align}

for all $\zeta > 0$ and all positive integers $M$. Setting $\zeta = 0.3$, we have

$$
\hat{P}_t \left( \left| \sum_{m=1}^{M} \gamma^{M-m} \epsilon_m \left( S_{i,j}(t_k + m - 1) - \hat{E}_{t_k} [S_{i,j}(t_k + m - 1)] \right) \right| \right) > \sqrt{3U_S^2 \log T} \leq \frac{2 \log T}{T^3 \log 1.3}
$$

(56)

for all $T \geq e^5$ and all positive integers $M$.

Combining (54), (55), and (56), we have

$$
\hat{P}_t (E^c_{t_k,i,j}) \leq \sum_{M=\frac{T}{4}+1}^{2T-1} \frac{2 \log T}{T^3 \log 1.3} \leq \frac{1}{T^2}
$$

for all $T \geq e^5$. Taking the union bound over $i$, we have

$$
\hat{P}_t (E^c_{t_k,j}) \leq \sum_{i=1}^{I} \hat{P}_t (E^c_{t_k,i,j}) \leq \frac{I}{T^2}.
$$

\[\square\]

### E Proof of Lemma 5

**Lemma 5.** $\sum_i q_i \leq \frac{1}{D_{k+1}} \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_t \left[ \sum_i Q_i(t_k + \tau) + 2JT \right]$.

**Proof.** By Lemma 3 we have

$$
\sum_i Q_i(t_k + \tau) \geq \sum_i Q_i(t_k) - J\tau \geq \sum_i Q_i(t_k) - 2JT,
$$

where the last inequality holds since $\tau \leq D_k + D_{k+1} - 1 \leq 2T$ by Lemma 1. Based on (57), we have

$$
\sum_i q_i \leq \frac{1}{D_{k+1}} \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} \hat{E}_t \left[ \sum_i Q_i(t_k + \tau) + 2JT \right].
$$

\[\square\]
F Proof of Lemma 6

Lemma 6. For any \( j \) and any \( k \geq 0 \),

\[
D_k + D_{k+1} - 1 \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} b_{i,j}(t_k + \tau) f_j(t_k + \tau) I_{i,j}(t_k + \tau) \leq 99U \sqrt{T \log T}.
\]

Proof. Notice that the summation can be rewritten as

\[
D_k + D_{k+1} - 1 \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} b_{i,j}(t_k + \tau) f_j(t_k + \tau) I_{i,j}(t_k + \tau) = \sum_{i} \left( \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} b_{i,j}(t_k + \tau) f_j(t_k + \tau) I_{i,j}(t_k + \tau) \right)_{i = i'}. \tag{58}
\]

The summation term for queue \( i \) can be rewritten as

\[
D_k + D_{k+1} - 1 \sum_{\tau = D_k}^{D_k + D_{k+1} - 1} b_{i,j}(t_k + \tau) f_j(t_k + \tau) I_{i,j}(t_k + \tau) = \sum_{n=1}^{N} b_{i,j}(\tau_n), \tag{59}
\]

where \( N \) is the total number of times when server \( j \) picks queue \( i \) such that server \( j \) is not idling and the completion time is in the time interval \([t_k + D_k, t_k + D_k + D_{k+1} - 1]\), and \( \tau_n \) is the time slot for the \( n^{th} \) time.

From [20], we have \( f_j(t_k + \tau) \in [t_k, t_k + 2T - 1] \), and thus \( \tau_n \in [t_k, t_k + 2T - 1] \). Divide the interval \([t_k, t_k + 2T - 1]\) into parts where each part contains \( \left\lfloor \frac{T}{16 \log T} \right\rfloor \) samples. Then there are at most \([32 \log T]\) parts. Consider the \( m^{th} \) part such that

\[
\tau_n \in \left[ t_k + (m - 1) \left\lfloor \frac{T}{16 \log T} \right\rfloor, t_k + m \left\lfloor \frac{T}{16 \log T} \right\rfloor - 1 \right].
\]

Consider the summation in the \( m^{th} \) part:

\[
\sum_{n=1}^{N_m} b_{i,j}(\tau_{m,n}),
\]

where \( N_m \) is the total number of times in the \( m^{th} \) part such that \( \sum_m N_m = N \), and \( \tau_{m,n} \) is the time slot for the \( n^{th} \) time in the \( m^{th} \) part. If \( N_m = 0 \), then \( \sum_{n=1}^{N_m} b_{i,j}(\tau_{m,n}) = 0 \). We now consider the case where \( N_m > 0 \) and derive an upper bound for the summation. First, we know that

\[
\tau_{i,j}(\tau_{m,1}) \geq 0, \text{ and } b_{i,j}(\tau_{m,1}) \leq 1.
\]

Consider the contribution of the completion of the job starting at \( \tau_{m,1} \) to \( \tau_{i,j}(\tau_{m,2}) \). Since \( \tau_{m,2} - \tau_{m,1} \leq \left\lfloor \frac{T}{16 \log T} \right\rfloor \text{ and } \frac{T}{16 \log T} \geq 1 \), we have

\[
\tau_{i,j}(\tau_{m,2}) \geq \left( 1 - \frac{8 \log T}{T} \right)^{\tau_{m,2} - \tau_{m,1}} \geq \frac{1}{2},
\]

where the last inequality follows from the fact that \((1 - x)^y \geq 1 - xy\) for any \( x \in [0, 1] \) and \( y \geq 1 \). For a general \( n \), we have

\[
\tau_{i,j}(\tau_{m,n+1}) \geq \sum_{l=1}^{n} \left( 1 - \frac{8 \log T}{T} \right)^{\tau_{m,n+1} - \tau_{m,l}}.
\]

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Since $\tau_{m,n+1} - \tau_{m,l} \leq \left\lceil \frac{T}{16 \log T} \right\rceil$ for any $l$, we have
\[
\left(1 - \frac{8 \log T}{T}\right)^{\tau_{m,n+1} - \tau_{m,l} - 1} \geq \frac{1}{2},
\]
and thus
\[
\hat{N}_{i,j}(\tau_{m,n+1}) \geq \sum_{l=1}^{n} \left(1 - \frac{8 \log T}{T}\right)^{\tau_{m,n+1} - \tau_{m,l} - 1} \geq \frac{n}{2}.
\]
Hence, we have
\[
b_{i,j}(\tau_{m,n+1}) \leq \sqrt{\frac{16U_2^2 \log T}{\hat{N}_{i,j}(\tau_{m,n+1})}} \leq US\sqrt{\frac{32 \log T}{n}}.
\]
From the analysis above, we obtain
\[
\sum_{n=1}^{N_m} b_{i,j}(\tau_{m,n}) \leq 1 + \sum_{n=2}^{N_m} US\sqrt{\frac{32 \log T}{n-1}} \leq 1 + US\sqrt{128(N_m - 1) \log T} \leq 1 + \sqrt{8TU_S}
\]
where the last inequality follows from the fact that $N_m \leq \left\lceil \frac{T}{16 \log T} \right\rceil$.

Therefore, since there are at most $\lceil 32 \log T \rceil$ parts, we conclude that
\[
\sum_{n=1}^{N} b_{i,j}(\tau_n) \leq \lceil 32 \log T \rceil (1 + \sqrt{8TU_S}) \leq 99US\sqrt{T} \log T.
\]
Combining the above bound with (58) and (59), we have
\[
\frac{D_k + \hat{D}_{k+1} - 1}{D_k} \sum_{\tau = D_k}^{\hat{D}_{k+1} - 1} b_{ij}(t_k + \tau) \eta_j(t_k + \tau) (E[X_{i,j}(t_k + \tau)] - X_{i,j}(t_k + \tau)) \leq 99US\sqrt{T} \log T.
\]

\[\square\]

G Proof of Lemma [7]

Lemma 7. For any $k \geq 0$, $i \in \{1, \ldots, I\}$, $j \in \{1, \ldots, J\}$, we have
\[
\hat{P}_h(\mathcal{E}_{X,t_k,i,j}^c) := \Pr(\mathcal{E}_{X,t_k,i,j}^c | Q(t_k) = q, H(t_k) = h) \leq \frac{1}{T^2},
\]
where
\[
\mathcal{E}_{X,t_k,i,j} := \left\{ \sum_{\tau = D_k}^{D_k + \hat{D}_{k+1} - 1} v_j(t_k + \tau) \mathbb{1}_{j = i}(t_k + \tau) \eta_j(t_k + \tau) (E[X_{i,j}(t_k + \tau)] - X_{i,j}(t_k + \tau)) \geq -US\sqrt{2T \log T} \sum_i q_i \right\}.
\]

Proof. If server $j$ is not idling, then $X_{i,j}(t) = S_{i,j}(t)$ by definition. Hence, we have
\[
\mathcal{E}_{X,t_k,i,j} = \left\{ \sum_{\tau = D_k}^{D_k + \hat{D}_{k+1} - 1} v_j(t_k + \tau) \mathbb{1}_{j = i}(t_k + \tau) \eta_j(t_k + \tau) (E[S_{i,j}(t_k + \tau)] - S_{i,j}(t_k + \tau)) \geq -US\sqrt{2T \log T} \sum_i q_i \right\}
\]
\[
= \left\{ \sum_{m=1}^{D_k + \hat{D}_{k+1} - 1} v_j(t_k + D_k + m - 1) \mathbb{1}_{j = i}(t_k + D_k + m - 1) \eta_j(t_k + D_k + m - 1) \right\}
\]
\[
(\sum_{m=1}^{D_k + \hat{D}_{k+1} - 1} v_j(t_k + D_k + m - 1) \mathbb{1}_{j = i}(t_k + D_k + m - 1) \eta_j(t_k + D_k + m - 1) \geq -US\sqrt{2T \log T} \sum_i q_i \right\}.
\]
\[
\frac{D_k + \hat{D}_{k+1} - 1}{D_k} \sum_{\tau = D_k}^{\hat{D}_{k+1} - 1} b_{ij}(t_k + \tau) \eta_j(t_k + \tau) (E[X_{i,j}(t_k + \tau)] - X_{i,j}(t_k + \tau)) \leq 99US\sqrt{T} \log T.
\]
where $\epsilon_m := \mathbb{1}_{i^*(t_k + D_k + m - 1) = i} \eta_j(t_k + D_k + m - 1)$. Let $S_{i,j}^{(m)} := S_{i,j}(t_k + D_k + m - 1)$. Let $v_j^{(m)} := v_j(t_k + D_k + m - 1)$. Then

$$
\hat{P}_{t_k} \left( \mathcal{E}_{X, t_k, i, j} \right) = \hat{P}_{t_k} \left( \sum_{m=1}^{D_k+1} \epsilon_m v_j^{(m)} \left( E[S_{i,j}^{(m)}] - S_{i,j}^{(m)} \right) < - U \mathbb{S} \sqrt{2 T \log T} \sum_i q_i \right)
$$

$$
= \hat{P}_{t_k} \left( \sum_{m=1}^{D_k+1} \epsilon_m v_j^{(m)} \left( E[S_{i,j}^{(m)}] - S_{i,j}^{(m)} \right) \right) \sqrt{T} < - U \mathbb{S} \sqrt{2 T \log T} \sum_i q_i
$$

$$
\leq \hat{P}_{t_k} \left( \sum_{m=1}^{D_k+1} \epsilon_m v_j^{(m)} \left( E[S_{i,j}^{(m)}] - S_{i,j}^{(m)} \right) \right) \sqrt{T} \sum_{m=1}^{D_k+1} \epsilon_m < - U \mathbb{S} \sqrt{2 T \log T} \sum_i q_i
$$

$$
= \hat{P}_{t_k} \left( \sum_{m=1}^{D_k+1} \epsilon_m v_j^{(m)} \left( S_{i,j}^{(m)} - E[S_{i,j}^{(m)}] \right) \right) \sqrt{T} \sum_{m=1}^{D_k+1} \epsilon_m > U \mathbb{S} \sqrt{2 T \log T} \sum_i q_i \right), \quad (60)
$$

where the last inequality holds since $\sum_{m=1}^{D_k+1} \epsilon_m \leq D_k + 1 \leq T$ by Lemma 1.

Let us view the conditional probability $\hat{P}_{t_k}$ as a new probability measure. Then $\hat{E}_{t_k}$ is the expectation under this measure. Note that $\left( S_{i,j}^{(m)} \right)_{m=1}^\infty$ is a sequence of independent bounded random variables under this new measure since they are independent of $Q(t_k)$ and $H(t_k)$, which also implies that

$$
E \left[ S_{i,j}^{(m)} \right] = \hat{E}_{t_k} \left[ S_{i,j}^{(m)} \right].
$$

Note that given $Q(t_k)$ and $H(t_k)$, $v_j^{(m)}$ is a constant, which implies

$$
v_j^{(m)} \hat{E}_{t_k} \left[ S_{i,j}^{(m)} \right] = \hat{E}_{t_k} \left[ v_j^{(m)} S_{i,j}^{(m)} \right].
$$

Therefore, from (60), we have

$$
\hat{P}_{t_k} \left( \mathcal{E}_{X, t_k, i, j} \right) \leq \hat{P}_{t_k} \left( \sum_{m=1}^{D_k+1} \epsilon_m \left( v_j^{(m)} S_{i,j}^{(m)} - \hat{E}_{t_k} \left[ v_j^{(m)} S_{i,j}^{(m)} \right] \right) \right) \sqrt{T} \sum_{m=1}^{D_k+1} \epsilon_m > U \mathbb{S} \sqrt{2 T \log T} \sum_i q_i \right), \quad (61)
$$

Let $\mathcal{F}_m$ defined as

$$
\mathcal{F}_m := \sigma \left( (S(t_k + D_k + n - 1))_{n=1}^m, (A(t_k + D_k + n - 1))_{n=1}^{m+1}, (Q(t_k + D_k + n - 1))_{n=1}^{m+1}, (H(t_k + D_k + n - 1))_{n=1}^{m+1} \right)
$$

where $\sigma(\cdot)$ denotes the $\sigma$-algebra generated by the random variables. Note that

$$
\sigma \left( v_j^{(1)} S_{i,j}^{(1)}, v_j^{(2)} S_{i,j}^{(2)}, \ldots, v_j^{(m)} S_{i,j}^{(m)} \right) \subset \mathcal{F}_m
$$

and for any $n > m$, $v_j^{(n)} S_{i,j}^{(n)}$ is independent of $\mathcal{F}_m$. Also note that $(\epsilon_m)_{m=1}^\infty$ is a previsible (predictable) sequence of Bernoulli random variables, i.e., $\epsilon_m$ is $\mathcal{F}_{m-1}$-measurable. Note that $0 \leq v_j^{(n)} S_{i,j}^{(n)} \leq U \mathbb{S} \sum_i q_i$. Based on the above conditions, we can then apply Theorem 2 (Hoeffding-type inequality for self-normalized means, Theorem 18 in [6]) with $X_m = v_j^{(m)} S_{i,j}^{(m)}$, $\beta = U \mathbb{S} \sqrt{2 T \log T} \sum_i q_i$, and $B = U \mathbb{S} \sum_i q_i$, and
to obtain
\[
P_{tk} \left( \frac{\sum_{m=1}^{D_{k+1}} \epsilon_m \left( v_j^{(m)} s_{i,j}^{(m)} - \hat{E}_{tk} \left[ v_j^{(m)} s_{i,j}^{(m)} \right] \right)}{\sqrt{\sum_{m=1}^{D_{k+1}} \epsilon_m}} > U_S \sqrt{2 \log T \sum_i q_i} \right) 
\leq \left( \frac{\log D_{k+1}}{\log(1 + \zeta)} + 1 \right) \exp \left( -4 \left( 1 - \frac{\zeta^2}{16} \right) \log T \right).
\]

Setting \( \zeta = 0.3 \) and by the bound on \( D_{k+1} \) in Lemma 1, we have
\[
P_{tk} \left( \frac{\sum_{m=1}^{D_{k+1}} \epsilon_m \left( v_j^{(m)} s_{i,j}^{(m)} - \hat{E}_{tk} \left[ v_j^{(m)} s_{i,j}^{(m)} \right] \right)}{\sqrt{\sum_{m=1}^{D_{k+1}} \epsilon_m}} > U_S \sqrt{2 \log T \sum_i q_i} \right) \leq \frac{1}{T^2}
\]
for \( T \geq e^5 \). Substituting (62) into (61), the lemma is proved.

\[\square\]

### H Additional Details of the Simulation

In this section, we present more details of the simulation.

For MaxWeight with discounted UCB, we set \( g(\gamma) = 8192 \), \( \gamma = 0.9912 \), and \( c_1 = 0.25 \). For MaxWeight with discounted EM, we use the same \( \gamma \). For MaxWeight with UCB, we use the same \( c_1 \) and replace \( g(\gamma) \) in the UCB bonus term with \( t \) since there is no discount factor.

**For the stationary setting**, we set the mean arrival rates \( \lambda_i = 0.7 \) for all \( i \in \{1, 2, \ldots, 10\} \). We set the mean service rates as follows:

\[
\mu_{2k+1,2l+1} = 0.8, \quad \mu_{2k+1,2l+2} = 0.5, \quad \mu_{2k+2,2l+1} = 0.5, \quad \mu_{2k+2,2l+2} = 0.9,
\]
for all \( k, l \in \{0, 1, 2, 3, 4\} \).

**For the nonstationary aperiodic setting**, we set the mean arrival rates \( \lambda_i(t) = 0.6 \) for all \( i \in \{1, 2, \ldots, 10\} \) and all \( t \). We set the mean service rates as follows:

\[
\begin{align*}
\mu_{2k+1,2l+1}(t) &= (0.80000, 0.79999, \ldots, 0.50001); \\
\mu_{2k+1,2l+2}(t) &= (0.50000, 0.50001, \ldots, 0.79999); \\
\mu_{2k+2,2l+1}(t) &= (0.50000, \ldots, 0.50000, 0.50001, \ldots, 0.69999); \\
\mu_{2k+2,2l+2}(t) &= (0.90000, 0.89999, \ldots, 0.60001),
\end{align*}
\]
for all \( k, l \in \{0, 1, 2, 3, 4\} \) and the time horizon is 30000.

**For the nonstationary periodic setting**, we set the mean arrival rates

\[
\lambda_i(t) = (0.700, 0.699, \ldots, 0.501, 0.500, 0.501, \ldots, 0.699, 0.700, \ldots),
\]

one period
where the period is 400, for all \( i \in \{1, 2, \ldots, 10\} \). We set the mean service rates as follows:

\[
\mu_{2k+1,2l+1}(t) = (0.800, 0.799, \ldots, 0.501, 0.500, \ldots, 0.500, 0.501, \ldots, 0.799, 0.800, \ldots),
\]

repeat 201 times

\[
\text{one period}
\]

where period = 800;

\[
\mu_{2k+1,2l+2}(t) = (0.500, 0.501, \ldots, 0.899, 0.900, 0.899, \ldots, 0.501, 0.500, \ldots),
\]

one period

where period = 800;

\[
\mu_{2k+2,2l+1}(t) = (0.500, \ldots, 0.500, 0.501, \ldots, 0.799, 0.800, 0.799, \ldots, 0.501, 0.500, \ldots, 0.500, 0.500 \ldots),
\]

repeat 101 times

\[
\text{one period}
\]

repeat 100 times

where period = 800;

\[
\mu_{2k+2,2l+2}(t) = (0.900, 0.899, \ldots, 0.501, 0.500, 0.501, \ldots, 0.899, 0.900, \ldots),
\]

one period

where period = 800,

for all \( k, l \in \{0, 1, 2, 3, 4\} \).

**For the second nonstationary periodic setting with a larger period**, we set the mean arrival rates

\[
\lambda_i(t) = (0.7000, 0.6999, \ldots, 0.5001, 0.5000, 0.5001, \ldots, 0.6999, 0.7000, \ldots),
\]

one period

where the period is 4000, for all \( i \in \{1, 2, \ldots, 10\} \). We set the mean service rates as follows:

\[
\mu_{2k+1,2l+1}(t) = (0.8000, 0.7999, \ldots, 0.5001, 0.5000, \ldots, 0.5000, 0.5001, \ldots, 0.7999, 0.8000, \ldots),
\]

repeat 2001 times

\[
\text{one period}
\]

where period = 8000;

\[
\mu_{2k+1,2l+2}(t) = (0.5000, 0.5001, \ldots, 0.8999, 0.9000, 0.8999, \ldots, 0.5001, 0.5000, \ldots),
\]

one period

where period = 8000;

\[
\mu_{2k+2,2l+1}(t) = (0.5000, \ldots, 0.5000, 0.5001, \ldots, 0.7999, 0.8000, 0.7999, \ldots, 0.5001, 0.5000, \ldots, 0.5000, \ldots),
\]

repeat 1001 times

\[
\text{one period}
\]

repeat 1000 times

\[
0.5000 \ldots),
\]

where period = 8000;

\[
\mu_{2k+2,2l+2}(t) = (0.9000, 0.8999, \ldots, 0.5001, 0.5000, 0.5001, \ldots, 0.8999, 0.9000, \ldots)
\]

one period

where period = 8000,
for all $k,l \in \{0,1,2,3,4\}$.

We calculate the total queue length ($\sum_t Q_i(t)$) averaged over 1000 simulations. The shaded area in all the figures in this paper is 95% confidence interval. We present all the simulation results in Fig. 5, Fig. 6, Fig. 7, and Fig. 8 each with a zoom in view and a zoom out view. The zoom out view has a Y-axis with a larger range so that it can include the parts of curves that are missing in the zoom in view. Note that the zoom in view figures are also presented in Section 6 in the main text.

Figure 5: Stationary arrival rate and service rate.

Figure 6: Nonstationary service rate with aperiodic means. Here we plot one point every 10 time slots.
Figure 7: Nonstationary arrival rate and service rate with periodic means.

Figure 8: Nonstationary arrival rate and service rate with periodic means with a larger period. Here we plot one point every 10 time slots.