THE BIRATIONAL TYPE OF THE MODULI SPACE OF EVEN SPIN CURVES

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The moduli space \( S_g \) of smooth spin curves parameterizes pairs \([C, \eta]\), where \([C] \in M_g\) is a curve of genus \(g\) and \(\eta \in \text{Pic}^{g-1}(C)\) is a theta-characteristic. The finite forgetful map \(\pi : S_g \to M_g\) has degree \(2^{2g}\) and \(S_g\) is a disjoint union of two connected components \(S_g^+\) and \(S_g^-\) of relative degrees \(2^{g-1}(2^g + 1)\) and \(2^{g-1}(2^g - 1)\) corresponding to even and odd theta-characteristics respectively. A compactification \(\overline{S}_g\) of \(S_g\) over \(\overline{M}_g\) is obtained by considering the coarse moduli space of the stack of stable spin curves of genus \(g\) (cf. [C], [CCC] and [AJ]). The projection \(S_g \to M_g\) extends to a finite branched covering \(\pi : \overline{S}_g \to \overline{M}_g\). In this paper we determine the Kodaira dimension of \(S_g^+\):

**Theorem 0.1.** The moduli space \(\overline{S}_g^+\) of even spin curves is a variety of general type for \(g > 8\) and it is uniruled for \(g < 8\). The Kodaira dimension of \(\overline{S}_8^+\) is non-negative \(^1\).

It was classically known that \(\overline{S}_2^+\) is rational. The Scorza map establishes a birational isomorphism between \(\overline{S}_3^+\) and \(M_3\), cf. [DK], hence \(\overline{S}_3^+\) is rational. Very recently, Takagi and Zucconi [TZ] showed that \(\overline{S}_4^+\) is rational as well. Theorem 0.1 can be compared to [FL] Theorem 0.3: The moduli space \(R_g\) of Prym varieties of dimension \(g-1\) (that is, non-trivial square roots of \(O_C\) for each \([C] \in M_g\)) is of general type when \(g > 13\) and \(g \neq 15\). On the other hand \(\overline{R}_g\) is unirational for \(g < 8\). Surprisingly, the problem of determining the Kodaira dimension has a much shorter solution for \(\overline{S}_g^+\) than for \(\overline{R}_g\) and our results are complete.

We describe the strategy to prove that \(\overline{S}_g^+\) is of general type for a given \(g\). We denote by \(\lambda = \pi^*(\lambda) \in \text{Pic}(\overline{S}_g^+)\) the pull-back of the Hodge class and by \(\alpha_0, \beta_0 \in \text{Pic}(\overline{S}_g^+)\) and \(\alpha_i, \beta_i \in \text{Pic}(\overline{S}_g^+)\) for \(1 \leq i \leq \lfloor g/2 \rfloor\) boundary divisor classes such that

\[
\pi^*(\delta_0) = \alpha_0 + 2\beta_0 \quad \text{and} \quad \pi^*(\delta_i) = \alpha_i + \beta_i \quad \text{for} \quad 1 \leq i \leq \lfloor g/2 \rfloor
\]

(see Section 2 for precise definitions). Using Riemann-Hurwitz and [HM] we find that

\[
K_{\overline{S}_g^+} \equiv \pi^*(K_{\overline{M}_g}) + \beta_0 \equiv 13\lambda - 2\alpha_0 - 3\beta_0 - 2 \sum_{i=1}^{\lfloor g/2 \rfloor} (\alpha_i + \beta_i) - (\alpha_1 + \beta_1).
\]

We prove that \(K_{\overline{S}_g^+}\) is a big \(\mathbb{Q}\)-divisor class by comparing it against the class of the closure in \(\overline{S}_g^+\) of the divisor \(\Theta_{\text{null}}\) on \(S_g^+\) of non-vanishing even theta characteristics:

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\(^1\)Building on the results of this paper, we have proved quite recently in joint work with A. Verra, that \(\kappa(\overline{S}_8^+) = 0\). Details will appear later.
Theorem 0.2. The closure in $\overline{S}_g^+$ of the divisor $\Theta_{\text{null}} := \{ [C, \eta] \in S_g^+ : H^0(C, \eta) \neq 0 \}$ of non-vanishing even theta characteristics has class equal to

$$\Theta_{\text{null}} = \frac{1}{4} \lambda - \frac{1}{16} \alpha_0 - \frac{1}{2} \sum_{i=1}^{[g/2]} \beta_i \in \text{Pic}(\overline{S}_g^+).$$

Note that the coefficients of $\beta_0$ and $\alpha_i$ for $1 \leq i \leq [g/2]$ in the expansion of $\Theta_{\text{null}}$ are equal to 0. To prove Theorem 0.2, one can use test curves on $\overline{S}_g^+$ or alternatively, realize $\Theta_{\text{null}}$ as the push-forward of the degeneracy locus of a map of vector bundles of the same rank defined over a certain Hurwitz scheme covering $S_g^+$ and use [F1] and [F2] to compute the class of this locus. Then we use [FP] Theorem 1.1, to construct for each genus $3 \leq g \leq 22$ an effective divisor class $D \equiv a\lambda - \sum_{i=0}^{[g/2]} b_i \delta_i \in \text{Eff}(\overline{M}_g)$ with coefficients satisfying the inequalities

$$\frac{a}{b_0} \leq \begin{cases} 6 + \frac{12}{g+1}, & \text{if } g+1 \text{ is composite} \\ 7, & \text{if } g = 10 \\ \frac{6k^2 + k - 6}{(k-1)}, & \text{if } g = 2k - 2 \geq 4 \end{cases}$$

and $b_i/b_0 \geq 4/3$ for $1 \leq i \leq [g/2]$. When $g + 1$ is composite we choose for $D$ the closure of the Brill-Noether divisor of curves with a $g_d^r$, that is, $\overline{M}_{g,d}^r := \{ [C] \in M_g : G_d^r(C) \neq \emptyset \}$ in case when the Brill-Noether number $\rho(g, r, d) = -1$, and then cf. [EH2]

$$\overline{M}_{g,d}^r \equiv c_{g,d,r} \left( (g+3)\lambda - \frac{g+1}{6} \delta_0 - \sum_{i=1}^{[g/2]} i(g - i)\delta_i \right) \in \text{Pic}(\overline{M}_g).$$

For $g = 10$ we take the closure of the divisor $K_{10} := \{ [C] \in M_{10} : C \text{ lies on a K3 surface} \}$ (cf. [FP] Theorem 1.6). In the remaining cases, when necessarily $g = 2k - 2$, we choose for $D$ the Gieseker-Petri divisor $\overline{P}_{1, g,k}$ consisting of those curves $[C] \in M_g$ such that there exists a pencil $A \in W_1^1(C)$ such that the multiplication map

$$\mu_0(A) : H^0(C, A) \otimes H^0(C, K_C \otimes A^\vee) \rightarrow H^0(C, K_C)$$

is not an isomorphism, see [EH2], [F2]. Having chosen $D$, we form the $\mathbb{Q}$-linear combination of divisor classes

$$8 \cdot \Theta_{\text{null}} + \frac{3}{2b_0} \cdot \pi^*(D) = \left( 2 + \frac{3a}{2b_0} \right) \lambda - 2\alpha_0 - 3\beta_0 - \sum_{i=1}^{[g/2]} \frac{3b_i}{2b_0} \alpha_i - \sum_{i=1}^{[g/2]} \left( 4 + \frac{3b_i}{2} \right) \beta_i \in \text{Pic}(\overline{S}_g^+),$$

from which we can write

$$K_{\overline{S}_g^+} = \nu_g \cdot \lambda + 8\Theta_{\text{null}} + \frac{3}{2b_0} \cdot \pi^*(D) + \sum_{i=1}^{[g/2]} \left( c_i \cdot \alpha_i + c_i' \cdot \beta_i \right),$$

where $c_i, c_i' \geq 0$. Moreover $\nu_g > 0$ precisely when $g \geq 9$, while $\nu_8 = 0$. Since the class $\lambda \in \text{Pic}(\overline{S}_g^+)$ is big and nef, we obtain that $K_{\overline{S}_g^+}$ is a big $\mathbb{Q}$-divisor class on the normal variety $\overline{S}_g^+$ as soon as $g > 8$. It is proved in [Lud] that for $g \geq 4$ pluricanonical forms defined on $\overline{S}_{g, \text{reg}}$ extend to any resolution of singularities $\overline{S}_{g}^+ \rightarrow \overline{S}_g^+$, which shows that $\overline{S}_g^+$ is of general type whenever $\nu_g > 0$ and completes the proof of Theorem 0.1 for $g \geq 8$. 

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When $g \leq 7$ we show that $K_{ \mathcal{S}^+_{g}} \not\in \text{Eff}(\mathcal{S}^+_{g})$ by constructing a covering curve $R \subset \mathcal{S}^+_{g}$ such that $R \cdot K_{ \mathcal{S}^+_{g}} < 0$, cf. Theorem 1.2 We then use [BDPP] to conclude that $\mathcal{S}^+_{g}$ is uniruled.

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1. The stack of spin curves

We review a few facts about Cornalba’s compactification $\pi : \overline{S}_g \to \overline{M}_g$, see [C]. If $X$ is a nodal curve, a smooth rational component $E \subset X$ is said to be exceptional if $\#(E \cap X - E) = 2$. The curve $X$ is said to be quasi-stable if $\#(E \cap X - E) \geq 2$ for any smooth rational component $E \subset X$, and moreover any two exceptional components of $X$ are disjoint. A quasi-stable curve is obtained from a stable curve by blowing-up each node at most once. We denote by $[\text{st}(X)] \in \overline{M}_g$ the stable model of $X$.

**Definition 1.1.** A spin curve of genus $g$ consists of a triple $(X, \eta, \beta)$, where $X$ is a genus $g$ quasi-stable curve, $\eta \in \text{Pic}^{g-1}(X)$ is a line bundle of degree $g - 1$ such that $\eta_E = O_E(1)$ for every exceptional component $E \subset X$, and $\beta : \eta^{\otimes 2} \to \omega_X$ is a sheaf homomorphism which is generically non-zero along each non-exceptional component of $X$.

A family of spin curves over a base scheme $S$ consists of a triple $(X \to S, \eta, \beta)$, where $f : X' \to S$ is a flat family of quasi-stable curves, $\eta \in \text{Pic}(X')$ is a line bundle and $\beta : \eta^{\otimes 2} \to \omega_{X'}$ is a sheaf homomorphism, such that for every point $s \in S$ the restriction $(X_s, \eta_{X_s}, \beta_{X_s} : \eta_{X_s}^{\otimes 2} \to \omega_{X_s})$ is a spin curve.

To describe locally the map $\pi : \overline{S}_g \to \overline{M}_g$ we follow [C] Section 5. We fix $[X, \eta, \beta] \in \overline{S}_g$ and set $C := \text{st}(X)$. We denote by $E_1, \ldots, E_r$ the exceptional components of $X$ and by $p_1, \ldots, p_r \in C_{\text{sing}}$ the nodes which are images of exceptional components. The automorphism group of $(X, \eta, \beta)$ fits in the exact sequence of groups

$$1 \longrightarrow \text{Aut}_0(X, \eta, \beta) \longrightarrow \text{Aut}(X, \eta, \beta) \longrightarrow \text{Aut}(C).$$

We denote by $\mathcal{C}^{3g-3}_r$ the versal deformation space of $(X, \eta, \beta)$ where for $1 \leq i \leq r$ the locus $(\tau_i = 0) \subset \mathcal{C}^{3g-3}_r$ corresponds to spin curves in which the component $E_i \subset X$ persists. Similarly, we denote by $\mathcal{C}^{3g-3}_t = \text{Ext}^1(\Omega_C, O_C)$ the versal deformation space of $C$ and denote by $(t_i = 0) \subset \mathcal{C}^{3g-3}_t$ the locus where the node $p_i \in C$ is not smoothed. Then around the point $[X, \eta, \beta]$, the morphism $\pi : \overline{S}_g \to \overline{M}_g$ is locally given by the map

$$\frac{\mathcal{C}^{3g-3}_r}{\text{Aut}(X, \eta, \beta)} \longrightarrow \frac{\mathcal{C}^{3g-3}_t}{\text{Aut}(C)}, \quad t_i = \tau_i^2 \quad (1 \leq i \leq r) \text{ and } t_i = \tau_i \quad (r + 1 \leq i \leq 3g - 3).$$

From now on we specialize to the case of even spin curves and describe the boundary of $\mathcal{S}^+_{g}$. In the process we determine the ramification of the finite covering $\pi : \mathcal{S}^+_{g} \to \overline{M}_g$.

1.1. The boundary divisors of $\mathcal{S}^+_{g}$

If $[X, \eta, \beta] \in \pi^{-1}([C \cup_D D])$ where $[C, y] \in \mathcal{M}_{i, 1}$ and $[D, y] \in \mathcal{M}_{g-i, 1}$, then necessarily $X := C \cup_{y_1} E \cup_{y_2} D$, where $E$ is an exceptional component such that $C \cap E = \{y_1\}$ and $D \cap E = \{y_2\}$. Moreover

$$\eta = (\eta_C, \eta_D, \eta_E = O_E(1)) \in \text{Pic}^{g-1}(X),$$
We start with a fixed \( \eta \), the closure of the locus corresponding to pairs \( ([C, y, \eta_C], [D, y, \eta_D]) \in \mathcal{S}_{g-i,1}^{+} \times \mathcal{S}_{g-i,1}^{+} \) and by \( B_i \subset \mathcal{S}_{g}^{+} \), the closure of the locus corresponding to pairs \( ([C, y, \eta_C], [D, y, \eta_D]) \in \mathcal{S}_{i,1}^{-} \times \mathcal{S}_{g-i,1}^{-} \).

For a general point \( [X, \eta, \beta] \in A_i \cup B_i \), we have that \( \text{Aut}_0(X, \eta, \beta) = \text{Aut}(X, \eta, \beta) = \mathbb{Z}_2 \). Using (I), the map \( \mathbb{C}_3^{g-3} \to \mathbb{C}_3^{g-3} \) is given by \( t_1 = \tau_i^2 \) and \( t_i = \tau_i \) for \( i \geq 2 \).

Furthermore, \( \text{Aut}_0(X, \eta, \beta) \) acts on \( \mathbb{C}_3^{g-3} \) via \( (\tau_1, \tau_2, \ldots, \tau_{g-3}) \mapsto (-\tau_1, \tau_2, \ldots, \tau_{g-3}) \). It follows that \( \Delta_i \subset \mathcal{M}_g \) is not a branch divisor for \( \pi : \mathcal{S}_{g}^{+} \to \mathcal{M}_g \) and if \( \alpha_i = [A_i] \in \text{Pic}(\mathcal{S}_{g}^{+}) \) and \( \beta_i = [B_i] \in \text{Pic}(\mathcal{S}_{g}^{+}) \), then for \( 1 \leq i \leq [g/2] \) we have the relation

\[
(2) \quad \pi^*(\delta_i) = \alpha_i + \beta_i.
\]

Moreover, \( \pi_*(\alpha_i) = 2^{g-2}(2^i+1)(2^{g-i}+1)\delta_i \) and \( \pi_*(\beta_i) = 2^{g-2}(2^i-1)(2^{g-i}+1)\delta_i \).

For a point \( [X, \eta, \beta] \) such that \( s(t)(X) = C_{yy} := C/y \sim q \), with \( [C, y, q] \in \mathcal{M}_{g-1,2} \), there are two possibilities depending on whether \( X \) possesses an exceptional component or not. If \( X = C_{yy} \) and \( \eta_C := \nu(\eta) \) where \( \nu : C \to X \) denotes the normalization map, then \( \eta_{g-2}^{\circ} = K_C(y + q) \). For each choice of \( \eta_C \in \text{Pic}^{g-1}(C) \) as above, there is precisely one choice of gluing the fibres \( \eta_C(y) \) and \( \eta_C(q) \) such that \( h^0(X, \eta) = 0 \) mod 2. We denote by \( A_0 \) the closure in \( \mathcal{S}_{g}^{+} \) of the locus of points \( [C_{yy}, \eta_C] \in \mathcal{S}(y + q) \) as above and clearly \( \deg(A_0/\Delta_0) = 2^{g-2} \).

If \( X = C \cup_{(y, q)} E \) where \( E \) is an exceptional component, then \( \eta_C := \eta \circ \mathcal{O}_C \) is a theta-characteristic on \( C \). Since \( H^0(X, \omega) \cong H^0(C, \omega_C) \), it follows that \( [C, \eta_C] \in \mathcal{S}_{g-1}^{+} \). For \( [C, y, q] \in \mathcal{M}_{g-1,2} \) sufficiently generic we have that \( \text{Aut}(X, \eta, \beta) = \text{Aut}(C) = [\mathcal{I}_C] \), and then from (I) it follows that \( \pi \) is simply branched over such points. We denote by \( B_0 \subset \mathcal{S}_{g}^{+} \) the closure of the locus of points \( [C \cup_{(y, q)} E, \eta_C] \in \mathcal{S}_{g}^{+}, \eta_E = \mathcal{O}_E(1) \). If \( \alpha_0 = [A_0] \in \text{Pic}(\mathcal{S}_{g}^{+}) \) and \( \beta_0 = [B_0] \in \text{Pic}(\mathcal{S}_{g}^{+}) \), we then have the relation

\[
(3) \quad \pi^*(\delta_0) = \alpha_0 + 2\delta_0.
\]

Note that \( \pi_*(\alpha_0) = 2^{g-2}\delta_0 \) and \( \pi_*(\beta_0) = 2^{g-2}(2g-1)\delta_0 \).

1.2. The uniruledness of \( \mathcal{S}_{g}^{+} \) for small \( g \).

We employ a simple negativity argument to determine \( k(\mathcal{S}_{g}^{+}) \) for small genus.

Using an analogous idea we showed that similarly, for the moduli space of Prym curves, one has that \( k(\mathcal{R}_{g}) = -\infty \) for \( g < 8 \), cf. [FL] Theorem 0.7.

**Theorem 1.2.** For \( g < 8 \), the space \( \mathcal{S}_{g}^{+} \) is uniruled.

**Proof.** We start with a fixed \( K3 \) surface \( S \) carrying a Lefschetz pencil of curves of genus \( g \). This induces a fibration \( f : \text{Bl}_{2}(S) \to \mathbb{P}^1 \) and then we set \( B := \left( m_f \right)_{*}^{r}(\mathbb{P}^1) \subset \mathcal{M}_g \), where \( m_f : \mathbb{P}^1 \to \mathcal{M}_g \) is the moduli map \( m_f(t) := [f^{-1}(t)] \). We have the following well-known formulas on \( \mathcal{M}_g \) (cf. [PP] Lemma 2.4):

\[
B \cdot \lambda = g + 1, \quad B \cdot \delta_0 = 6g + 18, \quad \text{and} \quad B \cdot \delta_i = 0 \quad \text{for} \quad i \geq 1.
\]

We lift \( B \) to a pencil \( R \subset \mathcal{S}_{g}^{+} \) of spin curves by taking

\[
R := B \times \mathcal{M}_g \mathcal{S}_{g}^{+} = \{ [C_t, \eta_{C_t}] \in \mathcal{S}_{g}^{+} : [C_t] \in B, \eta_{C_t} \in \text{Pic}^{g-1}(C_t), t \in \mathbb{P}^1 \} \subset \mathcal{S}_{g}^{+}.
\]
Using \(3\) one computes the intersection numbers with the generators of \(\text{Pic}(\overline{S}_g^+)\):
\[
R \cdot \lambda = (g+1)2^{g-1}(2^g + 1), \quad R \cdot \alpha_0 = (6g + 18)2^{g-2} \quad \text{and} \quad R \cdot \beta_0 = (6g + 18)2^{g-2}(2^{g-1} + 1).
\]
Furthermore, \(R\) is disjoint from all the remaining boundary classes of \(\overline{S}_g^+\), that is, \(R \cdot \alpha_i = R \cdot \beta_i = 0\) for \(1 \leq i \leq \lfloor g/2 \rfloor\). One verifies that \(R \cdot K_{\overline{S}_g^+} < 0\) precisely when \(g \leq 7\). Since \(R\) is a covering curve for \(\overline{S}_g^+\) in the range \(g \leq 7\), we find that \(K_{\overline{S}_g^+}\) is not pseudo-effective, that is, \(K_{\overline{S}_g^+} \in \text{Eff}(\overline{S}_g^+)\). Pseudo-effectiveness of the canonical bundle is a birational property for normal varieties, therefore the canonical bundle of any smooth model of \(\overline{S}_g^+\) lies outside the pseudo-effective cone as well. One can apply [BDPP] Corollary 0.3, to conclude that \(\overline{S}_g^+\) is uniruled for \(g \leq 7\). 

### 2. The geometry of the divisor \(\overline{\Theta}_{\text{null}}\)

We compute the class of the divisor \(\overline{\Theta}_{\text{null}}\) using test curves. The same calculation can be carried out using techniques developed in [FL], [F2] to calculate push-forwards of tautological classes from stacks of limit linear series.

For \(g \geq 9\), Harer [H] has showed that \(H^2(S_g^+, \mathbb{Q}) \cong \mathbb{Q}\). The range for which this result holds has been recently improved to \(g \geq 5\) in [F1]. In particular, it follows that \(\text{Pic}(S_g^+)\) is generated by the classes \(\lambda, \alpha_i, \beta_i\) for \(i = 0, \ldots, \lfloor g/2 \rfloor\). Thus we can expand the divisor class \(\overline{\Theta}_{\text{null}}\) in terms of the generators of the Picard group

\[
\overline{\Theta}_{\text{null}} \equiv \lambda \cdot \lambda - \alpha_0 \cdot \alpha_0 - \beta_0 \cdot \beta_0 - \sum_{i=1}^{\lfloor g/2 \rfloor} (\bar{\alpha}_i \cdot \alpha_i + \bar{\beta}_i \cdot \beta_i) \in \text{Pic}(S_g^+)\mathbb{Q},
\]

and determine the coefficients \(\bar{\lambda}, \bar{\alpha}_0, \bar{\beta}_0, \bar{\alpha}_i\) and \(\bar{\beta}_i \in \mathbb{Q}\) for \(1 \leq i \leq \lfloor g/2 \rfloor\).

#### Remark 2.1

To show that the class \(\overline{\Theta}_{\text{null}} \in \text{Pic}(S_g^+)\mathbb{Q}\) is a multiple of \(\lambda\) and thus, the expansion \(\mathbb{H}\) makes sense for all \(g \geq 3\), one does not need to know that \(\text{Pic}(S_g^+)\mathbb{Q}\) is infinite cyclic. For instance, for even \(g = 2k - 2 \geq 4\), we note that, via the base point free pencil trick, \([C, \eta] \in \overline{\Theta}_{\text{null}}\) if and only if the multiplication map
\[
\mu_C(A, \eta) : H^0(C, A) \otimes H^0(C, A \otimes \eta) \to H^0(C, A \otimes^2 \eta)
\]
is not an isomorphism for a base point free pencil \(A \in W^1_K(C)\). We set \(\tilde{M}_g\) to be the open subvariety consisting of curves \([C] \in M_g\) such that \(W^1_K(C) = \emptyset\) and denote by \(\sigma : \mathcal{G}_k^1 \to \tilde{M}_g\) the Hurwitz scheme of pencils \(g^1_k\) and by
\[
\tau : \mathcal{G}_k^1 \times \tilde{M}_g S^+_g \to S^+_g, \quad u : \mathcal{G}_k^1 \times \tilde{M}_g S^+_g \to \mathcal{G}_k^1
\]
the (generically finite) projections. Then \(\overline{\Theta}_{\text{null}} = \tau_* (Z)\), where
\[
Z = \{([A, C, \eta]) \in \mathcal{G}_k^1 \times \tilde{M}_g S^+_g : \mu_C(A, \eta) \text{ is not injective}\}.
\]

Via this determinantal presentation, the class of the divisor \(Z\) is expressible as a combination of \(\tau^*(\lambda), u^*(a), u^*(b)\), where \(a, b \in \text{Pic}(\mathcal{G}_k^1)\) are the tautological classes defined in e.g. [FL] p.15. Since \(\tau_* (u^*(a)) = \tau^*(\sigma^*(a))\) (and similarly for the class \(b\)), the conclusion follows. For odd genus \(g = 2k - 1\), one uses a similar argument replacing \(\mathcal{G}_k^1\) with any generically finite covering of \(M_g\) given by a Hurwitz scheme (for instance, we take the space of pencils \(g^1_{k+1}\) with a triple ramification point).
We start the proof of Theorem 2.2 by determining the coefficients of $\alpha_i$ and $\beta_i$ ($i \geq 1$) in the expansion of $[\overline{\Sigma}_{\text{null}}]$.

**Theorem 2.2.** We fix integers $g \geq 3$ and $1 \leq i \leq \lfloor g/2 \rfloor$. The coefficient of $\alpha_i$ in the expansion of $[\overline{\Sigma}_{\text{null}}]$ equals 0, while the coefficient of $\beta_i$ equals $-1/2$. That is, $\bar{\alpha}_i = 0$ and $\bar{\beta}_i = 1/2$.

**Proof.** For each integer $2 \leq i \leq g - 1$, we fix general curves $[C] \in \mathcal{M}_i$ and $[D, q] \in \mathcal{M}_{g-i, 1}$ and consider the test curve $C^i := \{C \cup_{y=q} D\}_{y \in C} \subset \Delta_i \subset \mathcal{M}_g$. We lift $C^i$ to test curves $F_i \subset A_i$ and $G_i \subset B_i$ inside $\overline{\mathcal{S}}_g^+$ constructed as follows. We fix even (resp. odd) theta-characteristics $\eta_C^+ \in \text{Pic}^{i-1}(C)$ and $\eta_D^+ \in \text{Pic}^{g-i-1}(D)$ (resp. $\eta_C^- \in \text{Pic}^{i-1}(C)$ and $\eta_D^- \in \text{Pic}^{g-i-1}(D)$).

If $E \cong \mathbb{P}^1$ is an exceptional component, we define the family $F_i$ (resp. $G_i$) as consisting of spin curves

$$F_i := \{t := [C \cup_y E \cup_q D, \eta_C = \eta_C^+, \eta_E = \mathcal{O}_E(1), \eta_D = \eta_D^+] \in \overline{\mathcal{S}}_g^+: y \in C\}$$

and

$$G_i := \{t := [C \cup_y E \cup_q D, \eta_C = \eta_C^-, \eta_E = \mathcal{O}_E(1), \eta_D = \eta_D^-] \in \overline{\mathcal{S}}_g^+: y \in C\}.$$

Since $\pi_*(F_i) = \pi_*(G_i) = C^i$, clearly $F_i \cdot \alpha_i = C^i \cdot \delta_i = 2 - 2i$, $F_i \cdot \beta_i = 0$ and $F_i$ has intersection number 0 with all other generators of Pic($\overline{\mathcal{S}}_g^+$). Similarly

$$G_i \cdot \beta_i = 2 - 2i, G_i \cdot \alpha_i = 0, G_i \cdot \lambda = 0,$$

and $G_i$ does not intersect the remaining boundary classes in $\overline{\mathcal{S}}_g^+$.

Next we determine $F_i \cap \overline{\Sigma}_{\text{null}}$. Assume that a point $t \in F_i$ lies in $\overline{\Sigma}_{\text{null}}$. Then there exists a family of even spin curves $(t : \mathcal{X} \to S, \eta, \beta)$, where $S = \text{Spec}(R)$, with $R$ being a discrete valuation ring and $\mathcal{X}$ is a smooth surface, such that, if $0, \xi \in S$ denote the special and the generic point of $S$ respectively and $X_\xi$ is the generic fibre of $t$, then

$$h^0(X_\xi, \eta_\xi) \geq 2, h^0(X_\xi, \eta_\xi) \equiv 0 \mod 2, \eta_\xi^* \cong \omega_{X_\xi}$$

and $(f^{-1}(0), \eta_{f^{-1}(0)}) = t \in \overline{\mathcal{S}}_g^+$.

Following the procedure described in [EH1] p. 347-351, this data produces a limit linear series $g_{g-1}$ on $C \cup D$, say

$$l := (l_C = (L_C, V_C), l_D = (L_D, V_D)) \in G_{g-1}^1(C) \times G_{g-1}^1(D),$$

such that the underlying line bundles $L_C$ and $L_D$ respectively, are obtained from the line bundle $(\eta_C^+, \eta_E, \eta_D^+)$ by dropping the $E$-aspect and then tensoring the line bundles $\eta_C^+$ and $\eta_D^+$ by line bundles supported at the points $y \in C$ and $q \in D$ respectively. For degree reasons, it follows that $L_C = \eta_C^+ \otimes \mathcal{O}_C((g - i)y)$ and $L_D = \eta_D^+ \otimes \mathcal{O}_D(iq)$. Since both $C$ and $D$ are general in their respective moduli spaces, we have that $H^0(C, \eta_C^+) = 0$ and $H^0(D, \eta_D^+) = 0$. In particular $a_1^C(y) \leq g - i - 1$ and $a_0^D(q) < a_1^D(q) \leq i - 1$, hence $a_1^C(y) + a_0^D(q) \leq g - 2$, which contradicts the definition of a limit $g_{g-1}^1$. Thus $F_i \cap \overline{\Sigma}_{\text{null}} = \emptyset$. This implies that $\bar{\alpha}_i = 0$, for all $1 \leq i \leq \lfloor g/2 \rfloor$ (for $i = 1$, one uses instead the curve $F_{g-1} \subset A_1$ to reach the same conclusion).

Assume that $t \in G_i \cap \overline{\Sigma}_{\text{null}}$. By the same argument as above, retaining also the notation, there is an induced limit linear series on $C \cup D$,

$$(l_C, l_D) \in G_{g-1}^1(C) \times G_{g-1}^1(D),$$


where $L_C = \eta_C^- \otimes O_C((g - i)y)$ and $L_D = \eta_D^- \otimes O_D(iy)$. Since $[C] \in \mathcal{M}_i$ and $[D, q] \in \mathcal{M}_{g-1, i}$ are both general, we may assume that $h^0(D, \eta_D^+) = h^0(C, \eta_C^+) = 1$, $q \notin \text{supp}(\eta_D^-)$ and that $\text{supp}(\eta_C^-)$ consists of $i - 1$ distinct points. In particular $a^D_0(q) \leq i$, hence $a^C_0(y) \geq g - 1 - a^D_0(q) \geq g - i - 1$. Since $h^0(C, \eta_C^+) = 1$, it follows that one has in fact equality, that is, $a^C_0(y) = g - i - 1$ and then necessarily $a^D_0(q) = i$.

Similarly, $a^C_1(y) \leq g - i + 1$ (otherwise div($\eta_C^-$) $\geq 2y$, that is, $\text{supp}(\eta_C^-)$ would be non-reduced, a contradiction), thus $a^D_0(q) \geq i - 2$, and the last two inequalities must be equalities as well (one uses that $h^0(D, L_D \otimes O_D(-(i - 1)q)) = h^0(D, \eta_D^- \otimes O_D(q)) = 1$, that is, $a^D_0(q) < i - 1$). Since $a^C_1(y) = g - i + 1$, we find that $y \in \text{supp}(\eta_C^-)$.

To sum up, we have showed that $(l_C, l_D)$ is a refined limit $g_{g-1}^1$ and in fact

\[(5) \quad l_D = |\eta_D^+ \otimes O_D(2q)| + (i - 2)q \in C_{g-1}^1(D), \quad l_C = |\eta_C^- \otimes O_C(y)| + (g - i - 1)y \in C_{g-1}^1(C),\]

hence $a^D_0(q) = (i - 2, i)$ and $a^C_0(y) = (g - i - 1, g - i + 1)$.

To prove that the intersection between $G_i$ and $\overline{\text{null}}$ is transversal, we follow closely \cite{EH1} Lemma 3.4 (see especially the Remark on p. 45): The restriction $\overline{\text{null}} |_{G_i}$ is isomorphic, as a scheme, to the variety $\tau : \Sigma_{g-1}(G_i) \to G_i$ of limit linear series $g_{g-1}$ on the curves of compact type $\{ C \cup y - q : D : y \in C \}$, whose $C$ and $D$-aspects are obtained by twisting suitably at $y \in C$ and $q \in D$ the fixed theta-characteristics $\eta_C^-$ and $\eta_D^-$ respectively. Following the description of the scheme structure of this moduli space given in \cite{EH2} Theorem 3.3 over an arbitrary base, we find that because $G_i$ consists entirely of singular spin curves of compact type, the scheme $\Sigma_{g-1}(G_i)$ splits as a product of the corresponding moduli spaces of $C$ and $D$-aspects respectively of the limits $g_{g-1}$. By direct calculation we have showed that $\Sigma_{g-1}(G_i) \cong \text{supp}(\eta_C^-) \times \{ l_D \}$. Since $\text{supp}(\eta_C^-)$ is a reduced 0-dimensional scheme, we obtain that $\overline{\text{null}} |_{G_i}$ is everywhere reduced. It follows that $G_i \cdot \overline{\text{null}} = \# \text{supp}(\eta_C^-) = i - 1$ and thus $\beta_i = (G_i \cdot \overline{\text{null}}) / (2i - 2)$. This argument does not work for $i = 1$, when one uses instead the intersection of $\overline{\text{null}}$ with $G_{g-1}$, and this finishes the proof.

Next we construct two pencils in $\Sigma_g^+$ which are lifts of the standard degree 12 pencil of elliptic tails in $\overline{\mathcal{M}}_g$. We fix a general pointed curve $[C, q] \in \mathcal{M}_{g-1, 1}$ and a pencil $f : \text{Bl}_0(\mathbf{P}^2) \to \mathbf{P}^1$ of plane cubics together with a section $\sigma : \mathbf{P}^1 \to \text{Bl}_0(\mathbf{P}^2)$ induced by one of the base points. We then consider the pencil $R := \{ (C \cup y - q, \sigma^{-1}(\lambda)) \}_{\lambda \in \mathbf{P}^1} \subset \overline{\mathcal{M}}_g$.

We fix an odd theta-characteristic $\eta_C^- \in \text{Pic}^{g-2}(C)$ such that $q \notin \text{supp}(\eta_C^-)$ and $E \cong \mathbf{P}^1$ will again denote an exceptional component. We define the family

$$F_0 := \{ (C \cup E, \sigma^{-1}(\lambda), \eta_C = \eta_C^-, \eta_E = \mathcal{O}(1), \eta_{f-1}(\lambda) = \mathcal{O}(\lambda) : \lambda \in \mathbf{P}^1) \} \subset \Sigma_g^+.$$

Since $F_0 \cap A_1 = \emptyset$, we find that $F_0 \cdot \beta_1 = \pi(F_0) \cdot \delta_1 = -1$. Similarly, $F_0 \cdot \lambda = \pi(F_0) \cdot \lambda = 1$ and obviously $F_0 \cdot \alpha_i = F_0 \cdot \beta_i = 0$ for $2 \leq i \leq [g/2]$. For each of the 12 points $\lambda_{\infty} \in \mathbf{P}^1$ corresponding to singular fibres of $R$, the associated $\eta_{\infty} \in \text{Pic}^{g-1}(C \cup E \cup f^{-1}(\lambda_{\infty}))$ are actual line bundles on $C \cup E \cup f^{-1}(\lambda_{\infty})$ (that is, we do not have to blow-up the extra node). Thus we obtain that $F_0 \cdot \beta_0 = 0$, therefore $F_0 \cdot \alpha_0 = \pi(F_0) \cdot \delta_0 = 12$. 

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We also fix an even theta-characteristic \( \eta_C^+ \in \Pic^{g-2}(C) \) and consider the degree 3 branched covering \( \gamma : S_1^+ \to \overline{M}_{1,1} \) forgetting the spin structure. We define the pencil
\[
G_0 := \{ [C \cup q \cup 0(f^{-1}(\lambda), \eta_C = \eta_C^+, \eta_E = \mathcal{O}_E(1), \eta_{f^{-1}(\lambda)} \in \gamma^{-1}[f^{-1}(\lambda)] : \lambda \in \mathbb{P}^1 \} \subset S_g^+.
\]
Since \( \pi_+(G_0) = 3R \), we have that \( G_0 \cdot \lambda = 3 \). Obviously \( G_0 \cdot \beta_0 = G_0 \cdot \beta_1 = 0 \), hence \( G_0 \cdot \alpha_1 = \pi_+(G_0) \cdot \delta_1 = -3 \). The map \( \gamma : S_1^+ \to \overline{M}_{1,1} \) is simply ramified over the point corresponding to \( j \)-invariant \( \infty \). Hence, \( G_0 \cdot \alpha_0 = 12 \) and \( G_0 \cdot \beta_0 = 12 \), which is consistent with formula (3).

The last pencil we construct lies in the boundary divisor \( B_0 \subset S_g^+ \); Setting \( E \cong \mathbb{P}^1 \) for an exceptional component, we define
\[
H_0 := \{(C \cup (y,q) E, \eta_C = \eta_C^+, \eta_E = \mathcal{O}_E(1)) : y \in C \} \subset S_g^+.
\]
The fibre of \( H_0 \) over the point \( y = q \in C \) is the even spin curve
\[
[C \cup q E' \cup q'' E''] E, \eta_C = \eta_C^+, \eta_E = \mathcal{O}_E'(1), \eta_E = \mathcal{O}_E(1), \eta_{E''} = \mathcal{O}_E''(-1)],
\]
having as stable model \( [C \cup q E_\infty] \), where \( E_\infty := E''/y'' \sim q'' \) is the rational nodal curve corresponding to \( j = \infty \). Here \( E', E'' \) are rational curves, \( E' \cap E'' \subset \{ q \} \), \( E \cap E'' = \{ q'' \} \) and the stabilization map for \( C \cup E \cup E' \cup E'' \) contracts the components \( E' \) and \( E \), while identifying \( q'' \) and \( y'' \).

We find that \( H_0 \cdot \lambda = 0, H_0 \cdot \alpha_1 = H_0 \cdot \beta_0 = 0 \) for \( 2 \leq i \leq \lfloor g/2 \rfloor \). Moreover \( H_0 \cdot \alpha_0 = 0 \), hence \( H_0 \cdot \beta_0 = \frac{1}{2} \pi_+(H_0) \cdot \delta_0 = 1 - g \). Finally, \( H_0 \cdot \alpha_1 = 1 \) and \( H_0 \cdot \beta_0 = 0 \).

**Theorem 2.3.** If \( F_0, G_0, H_0 \subset \overline{S}_g^+ \) are the families of spin curves defined above, then
\[
F_0 \cdot \overline{\Theta}_{\text{null}} = G_0 \cdot \overline{\Theta}_{\text{null}} = H_0 \cdot \overline{\Theta}_{\text{null}} = 0.
\]

**Proof.** From the limit linear series argument in the proof of Theorem 2.2 we get that the assumption \( F_0 \cap \overline{\Theta}_{\text{null}} \neq \emptyset \) implies that \( q \in \text{supp}(\eta_C) \), a contradiction. Similarly, we have that \( G_0 \cap \overline{\Theta}_{\text{null}} \neq \emptyset \) because \( \{ C \} \in \mathcal{M}_{g-1} \) can be assumed to have no even theta-characteristics \( \eta_C^+ \in \Pic^{g-2}(C) \) with \( h^0(C, \eta_C^+) \geq 2 \), that is \( \{ C, \eta_C^+ \} \notin \overline{\Theta}_{\text{null}} \subset \overline{S}_g^+ \). Finally, we assume that there exists a point \( \{ X := C \cup (y,q) E, \eta_C = \eta_C^+, \eta_E = \mathcal{O}_E(1) \} \in H_0 \cap \overline{\Theta}_{\text{null}} \). Then certainly \( h^0(X, \eta_X) \geq 2 \) and from the Mayer-Vietoris sequence on \( X \) we find that
\[
h^0(X, \eta_X) = \text{Ker}\{ H^0(C, \eta_C) \oplus H^0(E, \mathcal{O}_E(1)) \to C^2_{y,q} \},
\]
hence \( h^0(C, \eta_C) = h^0(X, \eta_X) \geq 2 \). This contradicts the assumption that \( \{ C \} \in \mathcal{M}_{g-1} \) is general. A similar argument works for the special point in \( H_0 \cap \pi^{-1}(\Delta_1) \), hence \( H_0 \cdot \overline{\Theta}_{\text{null}} = 0 \). \( \Box \)

**Proof of Theorem 0.2.** Looking at the expansion of \( \overline{\Theta}_{\text{null}} \), Theorem 2.3 gives the relations
\[
F_0 \cdot \overline{\Theta}_{\text{null}} = \hat{\lambda} - 12\hat{\alpha}_0 + \hat{\beta}_1 = 0, \quad G_0 \cdot \overline{\Theta}_{\text{null}} = 3\hat{\lambda} - 12\hat{\alpha}_0 - 12\hat{\beta}_0 + 3\hat{\alpha}_1 = 0
\]
and \( H_0 \cdot \overline{\Theta}_{\text{null}} = (g - 1)\hat{\beta}_0 - \hat{\alpha}_1 = 0 \).

Since we have already computed \( \hat{\alpha}_i = 0 \) and \( \hat{\beta}_i = 1/2 \) for \( 1 \leq i \leq \lfloor g/2 \rfloor \), (cf. Theorem 2.2), we obtain that \( \hat{\lambda} = 1/4, \hat{\alpha}_0 = 1/16 \) and \( \hat{\beta}_0 = 0 \). This completes the proof. \( \Box \)

A consequence of Theorem 0.2 is a new proof of the main result from [1]:

\[ \text{[1]} \]
Theorem 2.4. If $\mathcal{M}_g^1$ is the locus of curves $[C] \in \mathcal{M}_g$ with a vanishing theta-null then its closure has class equal to

$$\overline{\mathcal{M}_g^1} \equiv 2g-3 \left( (2g+1)\lambda - 2g^{-3}\delta_0 - \sum_{i=1}^{[g/2]} (2g-i-1)(2i-1)\delta_i \right) \in \text{Pic}(\overline{\mathcal{M}_g}).$$

Proof. We use the scheme-theoretic equality $\pi_*(\Theta_{\text{null}}) = \overline{\mathcal{M}_g^1}$ as well as the formulas

$$\pi_*(\lambda) = 2g^{-1}(2g+1)\lambda, \quad \pi_*(\alpha_0) = 2^{2g-2}\delta_0, \quad \pi_*(\beta_0) = 2^{g-2}(2g+1)\delta_0,$$

$$\pi_*(\alpha_i) = 2^{g-2}(2i+1)(2g-i+1)\delta_i$$

and

$$\pi_*(\beta_i) = 2^{g-2}(2i-1)(2g-i-1)\delta_i$$

valid for $1 \leq i \leq [g/2].$ \hfill $\Box$

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