POSITIVE RADIAL SOLUTIONS OF A NONLINEAR BOUNDARY VALUE PROBLEM

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Abstract. In this work we study the following quasilinear elliptic equation:

\[
\begin{aligned}
-\text{div} \left( \frac{|x|^\alpha \nabla u}{(a(|x|) + g(u))^\gamma} \right) &= |x|^\beta u^p & \text{in } \Omega \\
u &= 0 & \text{on } \partial\Omega
\end{aligned}
\]

where \(a\) is a positive continuous function, \(g\) is a nonnegative and nondecreasing continuous function, \(\Omega = B_R\), is the ball of radius \(R > 0\) centered at the origin in \(\mathbb{R}^N\), \(N \geq 3\) and, the constants \(\alpha, \beta \in \mathbb{R}\), \(\gamma \in (0, 1)\) and \(p > 1\).

We derive a new Liouville type result for a kind of "broken equation". This result together with blow-up techniques, a priori estimates and a fixed-point result of Krasnosel’skiĭ, allow us to ensure the existence of a positive radial solution. In this paper we also obtain a non–existence result, proven through a variation of the Pohozaev identity.

1. Introduction. Consider the following quasilinear problem:

\[
\begin{aligned}
-\text{div} \left( A(x, u) \nabla u \right) &= b(x, u) & \text{in } B_R \\
u &= 0 & \text{on } \partial B_R,
\end{aligned}
\]

(1.1)

where \(B_R\) denotes the open ball of radius \(R > 0\) centered at the origin in \(\mathbb{R}^N\), with \(N \geq 3\). In this paper we study the existence and non–existence of positive radial solutions, for the case when \(A(x, u) = \frac{|x|^\alpha}{(a(|x|) + g(u))^\gamma} \), \(b(x, u) = |x|^\beta u^p\), \(p > 1\),

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\[ \gamma \in (0, 1), \alpha, \beta \geq 0 \text{ with } g \text{ and } a \text{ continuous functions.} \text{ In other words, here we study} \]

\[
\begin{cases}
- \text{div} \left( |x|^\alpha \nabla u \right) = |x|^\beta u^p & \text{in } B_R \\
u = 0 & \text{on } \partial B_R
\end{cases}
\]

\[ (1.2) \]

We will say that \( u \) is a solution of \((1.2)\) if \( u \in C^1(B_R) \cap C^0(\overline{B_R}) \) and solve the equation in Problem \((1.2)\) in the weak sense.

In 1996, Clement, de Figueiredo and Mitidieri \([7]\), under the assumption that \( \gamma = 0 \) and \( N + \alpha - 2 > 0 \) and \( \beta - \alpha + 2 > 0 \), using variational techniques, proved the existence of positive radial solutions for this class of quasilinear elliptic equations. In addition, they defined the critical exponent associated to \((1.2)\) by

\[ p^* = \frac{N + 2\beta - \alpha + 2}{N + \alpha - 2}. \]

\[ (1.4) \]

In 2003, Alvino, Boccardo, Ferone, Orsina and Trombetti, (see \([1]\)), studied the existence of positive solutions of Problem \((1.1)\) where \( b \) is independent on the \( u \) variable and the main part having degenerated coercivity. They proved that the existence and regularity of solutions, depending on the summability of the datum \( b \). Subsequently, Boccardo (see \([3]\)) considered a nonlinearity \( b \) belonging to \( L^m(\Omega) \), with \( 1 \leq m \leq \frac{N}{2} \) (independent on \( u \)), and using approximation techniques, proved the existence of an entropy solution. In \([4]\), Boccardo and Brezis studied the case that the datum \( b \) belongs to \( L^m(\Omega) \), with \( m > \frac{N}{2} \).

We have noted that in the literature, the problems of the form \((1.1)\), for example, when \( \gamma = 0 \) are studied using variational methods. However, when \( \gamma \neq 0 \) it is not possible to use those methods.

We emphasize that, in our case, the operators have degenerated coercivity and nonlinearity depends on \( x \) and \( u \). More precisely, by considering coefficients such as those in the Problems studied by Boccardo (see examples \([1, 2, 3, 4, 5, 6]\), among others).

Note that our approach, which is nonvariational, allows us to extend the class of operators studied initially by Clement, de Figueiredo and Mitidieri \([7]\).

On the other hand, regarding the non–existence of positive solutions for problems like \((1.1)\), in general, when the nonlinearity has a powerlike behavior to infinity, there is an extensive literature dealing with this kind of problems. Among them, we mention the work of Gidas and Spruck \([17]\), where they consider the non–existence of positive solutions of problem

\[ \Delta u + u^p = 0 \]

\[ (1.5) \]

where \( 1 < p < \frac{N+2}{2} \), either in \( \mathbb{R}^N \) or in the half space, see also \([16]\). In the case of an exterior domain Bidaut-Veron \([12]\) proved that if \( 1 < p < \frac{N}{N-2} \) then \((1.5)\) has no nontrivial solution. Now, in the case of degenerate elliptic equations in the form of

\[ - \Delta_m u = u^p \]

\[ (1.6) \]

with \( 1 < m < N \), Mitidieri and Pohozaev proved in \([20]\) that the problem \((1.6)\) has no positive supersolution in \( \mathbb{R}^N \) if \( p \in (m-1, m_\ast - 1) \), where \( m_\ast = (N-1)m/(N-m) \). For an exterior domain, Bidaut-Veron and Pohozaev \([13]\) proved that \((1.6)\) has no positive supersolution if \( p \in (1, m_\ast - 1) \).
Other sharp results are established in Serrin-Zou ([22]). They have proved that the problem (1.6) in $\mathbb{R}^N$ or in an exterior domain has a positive supersolution if, and only if, $p > m^* - 1$. They have also considered $m-$superlinear nonlinearities which are $m-$subcritical.

In the class of radial solutions, Liouville Theorems have been obtained by several authors, among them we mention the works of Kawano, Ni and Yotsutani [18] and Clent, de Figuieredo and Mitidieri [14], who have covered this study when $\gamma = 0$.

Now we consider the hypotheses:

(H$_0$) We suppose $\gamma \in (0, 1)$ and $\alpha, \beta \in \mathbb{R}$ such that $N + \alpha - 2 > 0$, $N + \beta > 0$ and $\beta - \alpha + 1 > 0$.

(H$_1$) We suppose that $1 \leq p + 1 \leq 2(1 - \gamma) \frac{(N + \beta)}{N + \alpha - 2}$.

(H$_2$) $g : [0, +\infty[ \to [0, +\infty]$ is a continuous nondecreasing function verifying

$$\lim_{v \to +\infty} \frac{g(v)}{v} = 1,$$

and $a : [0, +\infty[ \to [0, +\infty]$ is a continuous function satisfying

$$c_1 \leq a(|x|) \leq c_2$$

for all $x \in B_R$,

where $c_1$ and $c_2$ are positive constants.

Now let us state our main results:

Theorem 1.1. Assume hypotheses (H$_0$), (H$_1$) and (H$_2$). Problem (1.2) has at least one positive radial solution.

A delicate matter is to establish an Derrick-Pohozaev type inequality (see for example [9]) which allows us to get a non–existence results.

Theorem 1.2. Suppose hypothesis (H$_0$) and that $g : [0, +\infty[ \to [0, +\infty]$ is a $C^1$ function. Then Problem (1.2) has no positive radial solutions for $p \geq p^*$.

Finally, we use the algebraic properties of the solutions, to obtain a second result on non–existence of solution

Theorem 1.3. Assume (H$_2$) and $\beta - \alpha + 2 \leq 0$, then there are not radial positive solutions of the Problem (1.2).

Remark 1.4. Observe that when $\gamma = 0$ we obtain the same critical value introduced in [7], see (1.4)

Remark 1.5. In the case that $g(u) = u$ and $a(|x|) = 1$, performing a simple change of variable Problem (1.2) turns into

$$\begin{cases} -\text{div}(|x|^\alpha \nabla v) = (1 - \gamma)|x|^\beta (v^{\frac{1}{1-\gamma}} - 1)^p & \text{in } \Omega \\ v = 0 & \text{on } \partial \Omega. \end{cases}$$

To prove our results, associated to our equation which is not variational, we will consider a parametrized truncated problem and we will obtain a priori bounds. Then, using a fixed–point argument, we prove the existence of a positive solution for the truncated problem. Finally, for some range of the parameter, Liouville type theorems show that the solution of the parametrized truncated problem is in fact a solution of the original problem. This study it’s a delicate one because it requires studying the limiting problem, including a Liouville type theorems for a broken equation.
Note that all of the results above are valid for strictly positive nonlinearities in \((0, \infty)\), which behave like powers to infinity or at zero, see e.g. \([11, 15]\) and references therein. In our case, we will need Liouville type results for the following equations:

\[
\begin{aligned}
- \text{div} (|x|^\alpha a_0(u) \nabla u) &= |x|^{\beta} u^p, & \text{in } \mathbb{R}^n \\
\end{aligned}
\]  

(1.7)

and

\[
\begin{aligned}
- \text{div} (|x|^\alpha a_0(u) \nabla u) &= |x|^{\beta} u^p, & \text{in } \mathbb{R}^n \\
\end{aligned}
\]

(1.8)

on \(\partial B_{s_0}\),

where

\[
a_0(u) = \begin{cases} 
1 & ; |x| \leq s_0, \\
1/d |x|^\gamma & ; |x| \geq s_0.
\end{cases}
\]

For more details, see Theorem 2.1 and 2.3. Note that the equations above are limit problems to the truncated equations \(((A)^{k}_\xi)\) given in Section 3 and Section 5.

This manuscript is organized as follows. In Section 2 we prove some Liouville type results. In Section 3, we truncate the problem \((3.1)\) to obtain a priori estimates for a family of parameterized truncated problems (see problem \(((A)^{k}_\xi)\)). In Section 4, we prove the existence of positive solution for the truncated Problem \((A)_{\xi}^k\). In the Section 5, we show that, for a large enough \(k\), the solutions of the truncation problem are radial solutions of Problem \((1.1)\), which proves our main result (Theorem 1.1). Finally in Section 5, we show some non–existence results of positive radial solutions of Problem \((1.1)\).

2. Liouville type theorems. In this section, we prove some the Liouville type results, which will be used to obtain a priori estimates for the auxiliary problem \(((A)^{k}_\xi)\) (See section 3).

Let \(s_0 \geq 0\) and \(\lambda > 0\), we consider the following Problem

\[
\begin{aligned}
- \left(y^{N+\alpha-1}u'(y)\right)' &= \lambda y^{N+\beta-1}u^p(y), & y &\in (s_0, \infty) \\
u(s_0) &= 1.
\end{aligned}
\]

(2.1)

**Lemma 2.1.** Let \(u \in C^2(0, +\infty)\) be a nonnegative solution of \((2.1)\), where \(\alpha, \beta\) and \(p\) satisfy \((H_0)\) and \((H_1)\) respectively. Then, the function \(y u'(y) + \rho u(y)\) is nonnegative and nonincreasing for \(\rho = N + \alpha - 2\). In particular, the function \(y^\rho u(y)\) is nondecreasing on \((s_0, +\infty)\).

**Proof:** The equation in \((2.1)\), we know that \(- (y^{N+\alpha-1}u'(y))' \geq 0\) for all \(y \in (s_0, +\infty)\) then

\[
y^{N+\alpha-2} [(N + \alpha - 1)u'(y) + yu''(y)] \leq 0
\]

for \(y \in (s_0, +\infty)\).

Observe that

\[
y u'(y) + \rho u(y)\]

\(= (N + \alpha - 1)u'(y) + y u''(y) \leq 0\) for all \(y \in (s_0, +\infty)\),

then, \(y u'(y) + \rho u(y)\) is a nonincreasing function in \((s_0, +\infty)\).

On the other hand, if we suppose that exists \(y_0 > s_0\) such that \(y_0 u'(y_0) + \rho u(y_0) < m_0\), with \(m_0 < 0\). We have that \(y u'(y) + \rho u(y) < m_0\) for all \(y \geq y_0\).
Then \( u'(y) < m_0 y^{-1} \), and integrating from \( y_0 \) to \( y \), we have that
\[
 u(y) - u(y_0) < m_0 \ln \left( \frac{y}{y_0} \right).
\]
Thus
\[
 \lim_{y \to \infty} u(y) = -\infty,
\]
which is impossible for a nonnegative function. Thus, we have proved that function \( y u'(y) + \rho u(y) \) is nonnegative and nonincreasing on \((s_0, +\infty)\).

Since the function \( y u'(y) + \rho u(y) \) is nonnegative, we have that
\[
(y^\rho u(y))' = y^{\rho - 1}(y u'(y) + \rho u(y)) \geq 0 \quad \text{for all} \quad y > s_0.
\]

Now, we will show a result of a priori estimates for solutions of Problem 2.1, which will be fundamental to obtain Liouville type results.

**Lemma 2.2.** Assume \((H_0)\) and \((H_1)\), let \( u \) be a nonnegative solution of \((2.1)\). Then, there is a positive constant \( C(N, \alpha, \beta) \) such that
\[
y^{N+\beta} u^{p+1}(y) \leq C(N, \alpha, \beta) y^{\frac{p(N+\alpha-2)-(N+2\beta-\alpha-2)}{p-1}} ,
\]
and
\[
\lambda \left[ \frac{(N + \beta)}{p + 1} - \frac{N + \alpha - 2}{2} \right] \int_{s_0}^{y} s^{N+\beta-1} u^{p+1}(s) \, ds
\]
\[
= \frac{y^{N+\alpha-1}}{2} u'(y)(yu'(y) + (N + \alpha - 2)u(y))
\]
\[
- \frac{s_0^{N+\alpha-1} u'(s_0)}{2} [s_0 u'(s_0) + (N + \alpha - 2)] + \frac{\lambda}{p+1} \left[ y^{N+\beta} u^{p+1}(t) - s_0^{N+\beta} \right]
\]

**Proof:** Assume that \( u \) is a nonnegative solution of Problem \((2.1)\), integrating equation \((2.1)\) from \( y \) to \( t \), with \( y \geq s_0 \), we obtain
\[
-t^{N+\alpha-1} u'(t) - y^{N+\alpha-1} u'(y) = \lambda \int_{y}^{t} s^{N+\beta-1} u^{p}(s) \, ds.
\]
Thus, by Lemma 2.1, \( u'(y) \leq 0 \) and
\[
t^{N+\alpha-1} |u'(t)| \geq \lambda \int_{y}^{t} s^{N+\beta-1} u^{p}(s) \, ds
\]
\[
= \lambda \int_{y}^{t} s^{N+\beta-1-\rho p} (s^\rho u(s))^p \, ds
\]
\[
\geq \lambda (y^\rho u(y))^p \int_{y}^{t} s^{N+\beta-1-\rho p} \, ds
\]
\[
= \lambda y^{\rho p} u^p(y) \frac{t^{N+\beta-\rho p} - y^{N+\beta-\rho p}}{N + \beta - \rho p}
\]
Since \( \rho u(t) \geq -tu'(t) = t|u'(t)| \), we obtain
\[
\rho t^{N+\alpha-2} u(t) \geq \lambda y^{\rho p} u^p(y) \frac{t^{N+\beta-\rho p} - y^{N+\beta-\rho p}}{N + \beta - \rho p}
\]
Taking \( t = 2y \) and using the fact that \( u \) is a positive and decreasing function, we have
\[
\rho(2y)^{N+\alpha-2}u(y) \geq \lambda y^\rho p u^p(y) \left( \frac{(2y)^{N+\beta-\rho p} - y^{N+\beta-\rho p}}{N + \beta - \rho p} \right) = \lambda y^{N+\beta} u^p(y) \frac{2^{N+\beta-\rho p} - 1}{N + \beta - \rho p},
\]
and so, from the last inequality, we have
\[
\rho u^{p-1}(y) \leq \lambda^{-1} y^{-(\beta-\alpha+2)} 2^{N+\alpha-2} \left[ \frac{2^{N+\beta-\rho p} - 1}{N + \beta - \rho p} \right]. \tag{2.3}
\]
Thus,
\[
y^{N+\beta} u^{p+1}(y) \leq C(N, \alpha, \beta) y^{\frac{\rho(N+\alpha-2) + (N+\beta-\alpha+2)}{p+1}}, \text{ for all } y > s_0.
\]
Moreover, multiplying the equation in problem (2.1) by \( y u'(y) \) and integrating from \( s_0 \) to \( y \), we obtain
\[
\lambda \int_{s_0}^{y} s^{N+\beta} u^p(s) u'(s) \, ds \\
= - \int_{s_0}^{y} \left( s^{N+\alpha-1} u'(s) \right)' s u'(s) \, ds \\
= -y^{N+\alpha} u'(y)^2 + s_{0}^{N+\alpha} u'(s_0)^2 + \int_{s_0}^{y} s^{N+\alpha-1} u'(s)^2 \, ds + \int_{s_0}^{y} s^{N+\alpha} u''(s) u'(s) \, ds.
\]
Observe that
\[
\int_{s_0}^{y} s^{N+\alpha} u''(s) u'(s) \, ds = \frac{y^{N+\alpha}}{2} u'(y)^2 - \frac{s_{0}^{N+\alpha}}{2} u'(s_0)^2 - \frac{N + \alpha - 2}{2} \int_{s_0}^{y} s^{N+\alpha-1} u'(s)^2 \, ds.
\]
Thus, combining the last two equations, we obtain
\[
\lambda \int_{s_0}^{y} s^{N+\beta} u^p(s) u'(s) \, ds \\
= -\frac{y^{N+\alpha}}{2} u'(y)^2 + \frac{s_{0}^{N+\alpha}}{2} u'(s_0)^2 - \frac{N + \alpha - 2}{2} \int_{s_0}^{y} s^{N+\alpha-1} u'(s)^2 \, ds. \tag{2.4}
\]
On the other hand, we note that
\[
\lambda \int_{s_0}^{y} s^{N+\beta} u^p(s) u'(s) \, ds \\
= \frac{\lambda}{p+1} \left[ y^{N+\beta} u^{p+1}(y) - s_{0}^{N+\beta} - (N + \beta) \int_{0}^{y} s^{N+\beta-1} u^{p+1}(s) \, ds \right]. \tag{2.5}
\]
From equations (2.4) and (2.5), we obtain
\[
-\frac{y^{N+\alpha}}{2} u'(y)^2 + \frac{s_{0}^{N+\alpha}}{2} u'(s_0)^2 - \frac{N + \alpha - 2}{2} \int_{s_0}^{y} s^{N+\alpha-1} u'(s)^2 \, ds \\
= \frac{\lambda}{p+1} \left[ y^{N+\beta} u^{p+1}(y) - s_{0}^{N+\beta} - (N + \beta) \int_{0}^{y} s^{N+\beta-1} u^{p+1}(s) \, ds \right].
\]
Notice that, multiplying equation in (2.1) by \( u \) and integrating from \( s_0 \) to \( y \), we obtain
\[
\int_{s_0}^{y} s^{N+\alpha-1} u'(s)^2 \, ds = y^{N+\alpha-1} u'(y) u(y) - s_{0}^{N+\alpha-1} u'(s_0) + \lambda \int_{s_0}^{y} s^{N+\beta-1} u^{p+1}(s) \, ds.
\]
Then, by the last two equalities, we have
\[- \frac{y^{N+\alpha}}{2} u'(y)^2 + \frac{s_0^{N+\alpha}}{2} u'(s_0)^2\]
\[- \frac{N + \alpha - 2}{2} \left[ y^{N+\beta} u^p y + s_0^{N+\beta} u'(s_0) + \lambda \int_{s_0}^y s^{N+\beta-1} u^{p+1}(s) \, ds \right] \]
\[= \frac{\lambda}{p+1} \left[ y^{N+\beta} u^{p+1}(y) - s_0^{N+\beta} - (N + \beta) \int_{s_0}^y s^{N+\beta-1} u^{p+1}(s) \, ds \right].\]
Hence, we obtain
\[\lambda \left[ \frac{(N + \beta)}{p+1} - \frac{N + \alpha - 2}{2} \right] \int_{s_0}^y s^{N+\beta-1} u^{p+1}(s) \, ds \]
\[= \frac{y^{N+\alpha-1}}{2} u'(y) \left[ y u'(y) + (N + \alpha - 2) u(y) \right] \]
\[- \frac{s_0^{N+\alpha-1} u'(s_0)}{2} \left[ s_0 u'(s_0) + (N + \alpha - 2) \right] + \frac{\lambda}{p+1} \left[ y^{N+\beta} u^{p+1}(y) - s_0^{N+\beta} \right].\]

**Theorem 2.3.** Assume the hypotheses \((H_0), (H_1)\) and \(\lambda > 0\). Then, the Problem
\[-(y^{N+\alpha-1} u')' = \lambda y^{N+\beta-1} u^p, \quad y \in (0, \infty), \]
\[u(0) = 1, \quad u'(0) = 0 \quad (2.6)\]
does not have a positive solution.

**Proof.** Notice that by hypothesis \((H_1)\), we have that \(N+2\beta - \alpha + 2 - p(N+\alpha - 2) > 0\), and by a priori estimate of Lemma 2.2, we obtain
\[\lim_{t \to \infty} t^{N+\beta} u^{p+1}(t) = 0. \quad (2.7)\]
By Lemma 2.2 with \(s_0 = 0\), we have that
\[\lambda \left[ \frac{(N + \beta)}{p+1} - \frac{N + \alpha - 2}{2} \right] \int_{0}^y s^{N+\beta-1} u^{p+1}(s) \, ds \]
\[= \frac{y^{N+\alpha-1}}{2} u'(y) \left[ y u'(y) + (N + \alpha - 2) u(y) \right] + \frac{\lambda}{p+1} \frac{u^{p+1}(y) y^{N+\beta}}{p+1} \]
Where, by Lemma 2.1, we know that
\[u'(y) \frac{y^{N+\alpha-1}}{2} \left[ y u'(y) + (N + \alpha - 2) u(y) \right] \leq 0, \quad (2.8)\]
then
\[\left[ \frac{(N + \beta)}{p+1} - \frac{N + \alpha - 2}{2} \right] \int_{0}^y s^{N+\beta-1} u^{p+1}(s) \, ds \leq \frac{u^{p+1}(y) y^{N+\beta}}{p+1}.\]
The last inequality, we obtain
\[\int_{0}^{\infty} s^{N+\beta-1} u^{p+1}(s) \, ds \leq 0,\]
which is absurd, since \(u\) is a positive solution. \(\square\)

Now, we announce and prove a second result of non–existence, which will be fundamental in this paper.
Theorem 2.4. Assuming the hypotheses \((H_0), (H_1)\) and \((H_2)\), the following problem

\[
(P) \begin{cases}
-(y^{N+\alpha-1}u')' = y^{N+\beta-1}u^p, & y \in (0, s_0) \\
u(0) = 1, & u'(0) = 0 \\
- \left( \frac{y^{N+\alpha-1}u'(y)}{d^u(y)} \right)' = y^{N+\beta-1}u^p(y), & y \in (s_0, \infty) \\
u(s_0) = d^{-1}
\end{cases}
\]

where \(d > 1\), does not have nonnegative solutions.

Proof. Suppose \(u\) is a nonnegative solution of \((P)\). Considering the change

\[
v(y) = \frac{u^{1-\gamma}(y)}{d^{1-\gamma}}, \quad y \in (s_0, \infty),
\]

is easy to see that \(v\), is a nonnegative solution of the following problem

\[
\begin{cases}
-(y^{N+\alpha-1}v'(y))' = \lambda y^{N+\beta-1}v^p(y), & y \in (s_0, \infty) \\
v(s_0) = 1 \quad \text{and} \quad v'(s_0) = (1 - \gamma)d'v'(s_0).
\end{cases}
\]

where \(\lambda = (1 - \gamma)d^{1-p}\). Since \(0 \leq \frac{p}{1-\gamma} < \frac{N+2\beta-\alpha+2}{N+\alpha-2}\), from Lemma 2.2, we know that

\[
y^{N+\beta}v^\frac{p+1-\gamma}{p+1}(y) \leq C(N, \alpha, \beta)y^\frac{p(N+\alpha-2)-(1-\gamma)(N+2\beta-\alpha+2)}{p+1-\gamma} \quad \text{for all } y > s_0,
\]

and

\[
\lambda \left[ \frac{(1-\gamma)(N+\beta)}{p+1-\gamma} - \frac{N+\alpha-2}{2} \right] \int_{s_0}^{y} s^{N+\beta-1}v^{\frac{p+1-\gamma}{p+1}}(s) \, ds
\]

\[
= \frac{y^{N+\alpha-1}}{2}v'(y)[gv'(y) + (N + \alpha - 2)v(y)] - \frac{s_0^{N+\alpha-1}v'(s_0)}{2}[s_0v'(s_0) + (N + \alpha - 2)] + \frac{\lambda(1-\gamma)}{p+1-\gamma} \left[ y^{N+\beta}v^\frac{p+1-\gamma}{p+1}(y) - s_0^{N+\beta} \right]
\]

(2.12)

Since \(v\) is a decreasing function and by Lemma 2.1, we obtain

\[
v'(y)\frac{y^{N+\alpha-1}}{2} [gv'(y) + (N + \alpha - 2)v(y)] \leq 0 \quad \text{for all } y \in (s_0, \infty).
\]

(2.13)

Notice that, by inequality in (2.11) is easy to see that

\[
y^{N+\beta}v^\frac{p+1-\gamma}{p+1}(y) \to 0.
\]

(2.14)

Also note that by hypothesis \((H_1)\), we have

\[
\lambda \left[ \frac{(1-\gamma)(N+\beta)}{p+1-\gamma} - \frac{N+\alpha-2}{2} \right] \int_{s_0}^{y} s^{N+\beta-1}v^{\frac{p+1-\gamma}{p+1}}(s) \, ds \geq 0.
\]

(2.15)

Now, from equation (2.12) to facilitate the notation, we define

\[
L := \frac{s_0^{N+\alpha-1}v'(s_0)}{2}[s_0v'(s_0) + (N + \alpha - 2)] + \frac{\lambda(1-\gamma)}{p+1-\gamma}s_0^{N+\beta}.
\]

Where it’s easy to verify that by returning to the variable \(u\), we obtain

\[
L = \frac{(1-\gamma)d'v'(s_0)s_0^{N+\alpha-1}}{2}((1-\gamma)du'(s_0) + (N + \alpha - 2)) + \frac{d(1-\gamma)^2}{p+1-\gamma}s_0^{N+\beta}
\]
Since \( u \) equation (2.19) in (2.18), we have from the last inequality we have

\[
\begin{align*}
L - d & = \left( \frac{N + \alpha - 2}{2} \right) u'(s_0) u(s_0) s_0^{N+\alpha-1} + \frac{1 - \gamma}{p + 1 - \gamma} s_0^{N+\beta}.
\end{align*}
\]

(2.16)

Our objective is to study the sign of \( L \), for this, we observe that multiplying the equation (P) by \( u \) and integrating from 0 to \( s_0 \), we obtain

\[
\begin{align*}
\left( \frac{N + \beta}{p + 1} - \frac{N + \alpha - 2}{2} \right) \int_0^{s_0} y^{N+\beta-1} u^{p+1}(y)dy = u^{p+1}(s_0) s_0^{N+\beta} + u'(s_0) s_0^{N+\alpha-1} + \frac{1 - \gamma}{p + 1 - \gamma} s_0^{N+\beta}.
\end{align*}
\]

(2.17)

Then, using equation (2.17) in (2.16), we obtain

\[
\begin{align*}
L = d(1 - \gamma) \left( \frac{N + \beta}{p + 1} - \frac{N + \alpha - 2}{2} \right) \int_0^{s_0} y^{N+\beta-1} u^{p+1}(y)dy - u^{p+1}(s_0) s_0^{N+\beta} + u'(s_0) s_0^{N+\alpha-1} + \frac{1 - \gamma}{p + 1 - \gamma} s_0^{N+\beta}.
\end{align*}
\]

(2.18)

Since \( u \) is a positive nonincreasing function, solution of (P), we have

\[
\begin{align*}
 u(s_0) u'(s_0) s_0^{N+\alpha-1} &= - \int_0^{s_0} s^{N+\beta-1} u^p(s) u(s_0) ds \geq - \int_0^{s_0} s^{N+\beta-1} u^{p+1}(s) ds.
\end{align*}
\]

(2.19)

Now, using equation (2.19) in (2.18), we have

\[
\begin{align*}
\frac{L}{d(1 - \gamma)} & \geq d(1 - \gamma) \left( \frac{N + \beta}{p + 1} - \frac{N + \alpha - 2}{2} \right) \int_0^{s_0} y^{N+\beta-1} u^{p+1}(y)dy - u^{p+1}(s_0) s_0^{N+\beta} + u'(s_0) s_0^{N+\alpha-1} + \frac{1 - \gamma}{p + 1 - \gamma} s_0^{N+\beta} \\
& \quad - \gamma(\alpha - 2) \int_0^{s_0} y^{N+\beta-1} u^{p+1}(y)dy + \frac{1 - \gamma}{p + 1 - \gamma} s_0^{N+\beta} \\
& = \left( d(1 - \gamma) \left( \frac{N + \beta}{p + 1} - \frac{N + \alpha - 2}{2} \right) \right) \int_0^{s_0} y^{N+\beta-1} u^{p+1}(y)dy \\
& \quad + \gamma d(1 - \gamma) \left( \frac{1}{p + 1 - \gamma} - \frac{u^p(s_0)}{p + 1} \right) \\
& = d \left( \frac{N + \beta}{p + 1} - \frac{N + \alpha - 2}{2} \right) \int_0^{s_0} y^{N+\beta-1} u^{p+1}(y)dy \\
& \quad + \gamma(d - 1) \left( \frac{N + \alpha - 2}{2} \right) \int_0^{s_0} y^{N+\beta-1} u^{p+1}(y)dy \\
& \quad + (1 - \gamma) s_0^{N+\beta} \frac{(p + 1)(1 - u^p(s_0)) + \gamma u^p(s_0)}{(p + 1 - \gamma)(p + 1)}.
\end{align*}
\]

(2.20)

Then, since \( u \) is a nonnegative function, \( d > 1 \) and hypotheses (\( H_0 \)) and (\( H_1 \)), from the last inequality we have \( L > 0 \).
Finally, in (2.12) by inequalities (2.11), (2.13) and (2.14), we obtain
\[(1 - \gamma)d^\gamma \left( \frac{(N + \beta)(1 - \gamma) - N + \alpha - 2}{2} \right) \lim_{t \to \infty} \int_0^\infty s^{N + \beta - 1} u_t^{p + 1} \ ds < 0,\]
which contradicts inequality (2.15). Therefore, we conclude that Problem (P) has no nonnegative solution.

3. A priori estimates. In this section we use the Liouville type results obtained in Section 2, to get a priori estimates for the radial solutions of Problem (1.1).

Notice that the radial problem associated with Problem (1.1) is as follows
\[
\begin{align*}
- \left( \frac{r^{N + \alpha - 1} v'}{(a(r) + g(v))^{\gamma}} \right)' &= r^{N + \beta - 1} v^p & \text{for } & r < R \\
v'(0) &= 0, & v(R) &= 0,
\end{align*}
\]
where the coefficient $\frac{1}{(a(r) + g(v))^{\gamma}}$ is not bounded from below. In order to overcome these difficulties, for each $k \in \mathbb{N}$, we introduce the functions $T_k(s) := \max\{-k, \min\{k, s\}\}$ and $g_k(s) := (g \circ T_k)(s)$ and consider the family of truncated problems parametrized by $\xi \geq 0$,
\[
\begin{align*}
- \left( \frac{r^{N + \alpha - 1} v'}{(a(r) + g_k(v))^{\gamma}} \right)' &= r^{N + \beta - 1} \left( v^p + \frac{\xi}{h(\|v\|_{\infty})} \right) & \text{for } & r < R \\
v'(0) &= 0, & v(R) &= 0,
\end{align*}
\]
where
\[
h(t) = \begin{cases} 
t^{q-p} & \text{if } t \geq 1 \\ 1 & \text{if } 0 \leq t \leq 1, \end{cases}
\]
and $q > p$.

The following is a result of a priori bound for the auxiliary problem:

**Theorem 3.1.** Assume hypotheses $(H_0)$, $(H_1)$ and $(H_2)$. Then there is a positive constant $C_k$ which depends only on $k$, such that
\[\|v\|_{\infty} \leq C_k,\]
for every solution $v$ of Problem $(A)^\xi_k$.

**Proof.** Let $k \in \mathbb{N}$ be fixed and assume by contradiction that there is a sequence of positive solutions $\{v_n\}_n$ of Problem $(A)^\xi_k$, so that $\|v_n\|_{\infty} \to +\infty$ when $n \to +\infty$.

Note that by using the following changes of variables
\[y = \frac{z_n}{t_n} r,\]
and defining
\[w_n(y) = \frac{v_n(r)}{t_n},\]
and defining
\[w_n(y) = \frac{v_n(r)}{t_n},\]
where \( t_n := \|w_n\|_\infty \) and \( z_n = (a(0) + g(k))^{\frac{\gamma}{\beta - \alpha}} t_n^{\frac{\beta - \alpha + 1}{\beta - \alpha + 2}} \), for a large \( n \), the function \( w_n \) is a solution of the following problem:

\[
- \left( \frac{y^{N+\alpha} w_n'(y)}{(a(t_n z_n^{-1} y) + g_k(t_n w_n(y)))^\gamma} \right)' = \frac{\beta - \alpha + 1}{z_n^{\beta - \alpha + 2} y^{N+\beta-1} \left( t_n^\beta w_n^\beta(y) + \xi t_n^{\beta-p-1} \right)} \text{ for } y < \frac{z_n R}{t_n},
\]

\[
w_n(0) = 0, \quad w_n(0) = 1,
\]

\[
w_n\left( \frac{R z_n}{t_n} \right) = 0.
\]

(3.4)

Observe that the function \( w_n \) verifies the equation

\[
- (N + \alpha - 1) y^{N+\alpha-2} \left( \frac{w_n'(y)}{(a(t_n z_n^{-1} y) + g_k(t_n w_n(y)))^\gamma} \right)'
\]

\[
- y^{N+\alpha-1} \left( \frac{w_n'(y)}{(a(t_n z_n^{-1} y) + g_k(t_n w_n(y)))^\gamma} \right)'
\]

\[
= \frac{\beta - \alpha + 1}{z_n^{\beta - \alpha + 2} y^{N+\beta-1} \left( t_n^\beta w_n^\beta(y) + \xi t_n^{\beta-p-1} \right)},
\]

thus, since \( w_n \) is a decreasing function, we have

\[
- \left( \frac{w_n'(y)}{(a(t_n z_n^{-1} y) + g_k(t_n w_n(y)))^\gamma} \right)'
\]

\[
\leq \frac{\beta - \alpha + 1}{z_n^{\beta - \alpha + 2} y^{\beta-\alpha} \left( t_n^\beta w_n^\beta(y) + \xi t_n^{\beta-p} \right)}.
\]

Replacing \( z_n \) and integrating the last inequality from 0 to \( y \), we obtain

\[
-w_n'(y) \leq \frac{(c_2 + g_k(t_n w_n(y)))^\gamma}{(a(0) + g(k))^\gamma} \int_0^y \tau^{\beta-\alpha} \left( w_n^\beta(\tau) + \xi \frac{t_n^\beta}{t_n^{\beta-p}} \right) d\tau.
\]

Since \( w_n'(y) < 0 \) for all \( y \in (0, \frac{R z_n}{t_n}) \), we have

\[
|w_n'(y)| \leq 2 \int_0^y \tau^{\beta-\alpha} d\tau.
\]

(3.5)

By (3.5), we get that \( w_n' \) is uniformly bounded in compact intervals, this is, for each \( M \in \mathbb{R} \), there is a constant \( C(M) > 0 \) so that

\[
w_n'(y) \leq C(M), \quad \text{for all } n \in \mathbb{N}, \quad \text{and for all } y \in [0, M].
\]

(3.6)

Thus the sequence \( \{w_n\}_n \) is equicontinuous in \([0, M]\). Since this sequence is uniformly bounded, by Ascoli Arzelà’s theorem, \( \{w_n\}_n \) contains a convergent subsequence (which we still denote by \( \{w_n\}_n \)), such that \( w_n \to w \) in \( C[0, M] \).

Since every function \( w_n \), it’s solution of the equation

\[
- \left( \frac{y^{N+\alpha-1} w_n'(y)}{(a(t_n z_n^{-1} y) + g_k(t_n w_n(y)))^\gamma} \right)' = \frac{y^{\beta+\gamma} - 1}{(a(0) + g(k))^\gamma} \left( w_n^\beta(y) + \xi t_n^{\beta-p-1} \right) \text{ for } y < \frac{R z_n}{t_n},
\]

\[
w_n(0) = 0, \quad w_n(0) = 1,
\]

\[
w_n\left( \frac{R z_n}{t_n} \right) = 0,
\]

(3.7)
From the properties of the function $w_n$, it is easy to see that $t_n w_n \to \infty$. Then, for $k$ sufficiently large, the solution satisfies the following integral equation
\[
-w_n(y) + 1 = \int_0^y \frac{1}{s^{N+\alpha-1}} \left( \frac{a(t_n^{-1} s) + g(k)}{a(0) + g(k)} \right)^\gamma \int_0^s \tau^{N+\beta-1} w_n^p(\tau) \, d\tau \, ds, \tag{3.8}
\]
for each $y \in [0, M]$.

Since the function $a$ bounded and Lebesgue’s dominated theorem, $w$ satisfies the following integral equation
\[
1 - w(y) = \int_0^y \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} w^p(\tau) \, d\tau \, ds. \tag{3.9}
\]
Then, $w$ is a positive solution in $[0, M]$ to the following initial value problem
\[
-(y^{N+\alpha-1}w')' = \lambda_0 y^{N+\beta-1}w^p \quad \text{in} \quad (0, M)
\]
$w(0) = 1, \ w'(0) = 0.$

where $\lambda_0$ is a positive constant. By using a diagonal iterative scheme, as in the last part of the proof of Proposition 4.1 in [8], $w$ can be extended to all $\mathbb{R}_+$, as a positive solution of (2.6). Furthermore, using the argument of [8], it can be shown that $w$ is indeed a positive solution of class $C^2(0, +\infty)$ of (2.6).

Finally, for $k > 0$ fixed we conclude that there exists a constant $C_k > 0$ (independent of $\xi$) such that for every solutions of Problem $(A)^\xi_k$, verifies
\[
||v||_{\infty} \leq C_k.
\]

4. Existence of positive solution of the truncated Problem $(A)^\xi_k$. The existence of positive solution of the truncated Problem $(A)^\xi_k$, is based on the following theorem due to Krasnosel’skii (See [10], [19]).

**Theorem 4.1** (Krasnosel’skii). Let $C$ be a cone in a Banach space, and let $F : C \to C$ be a compact operator such that $F(0) = 0$. Suppose there exists $\delta > 0$ verifying

(a) $u \neq t F(u)$, for all $||u|| = \delta$ and $t \in [0, 1]$.

Suppose further that there is a compact homotopy $H : [0, 1] \times C \to C$ and $\eta > \delta$ such that:

(b) $F(u) = H(0, u)$, for all $u \in C$.

(c) $H(t, u) \neq u$, for all $||u|| = \eta$ and $t \in [0, 1]$.

(d) $H(1, u) \neq u$, for all $||u|| \leq \eta$.

Then $F$ has a fixed point $u_0$ verifying $\delta < ||u_0|| < \eta$.

In order to use this result, we consider the Banach space $X = C([0, 1], \mathbb{R})$ endowed with the $L^\infty$-norm, and the cone of nonnegative continuous functions given by
\[
C_1 = \{v \in C[0, R]; \ v \geq 0, \ v(R) = 0\}.
\]

Define also the operator $F : X \to X$ by
\[
(Fv)(r) = \int_r^R \frac{(a(s) + g_k(v)(s))^\gamma}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} v^p(\tau) \, d\tau \, ds. \tag{4.1}
\]

Note that a simple calculation shows that the fixed point of operator $F$ are the positive solutions of Problem $(A)^\xi_k$.

**Lemma 4.2.** The operator $F : X \to X$ defined by (4.1) is compact, and the cone $C_1$ is invariant under $F$. 
Theorem 4.3. Assume hypothesis $(H_0), (H_1)$ and $(H_2)$. Then there is a positive solution of the truncated Problem $(A)_0^0$.

Proof. To prove the existence of positive solution for the truncated problem, it’s satisfactory to show that $F$ has a fixed point. For this we will check the conditions of Theorem 4.1. Define the homotopy $H : [0,1] \times C_1 \rightarrow C_1$ by

$$H(t,v)(r) = \int_{r}^{R} \frac{(a(s) + g_k(v)(s))\gamma}{s^{N+\alpha-1}} \int_{0}^{s} \tau^{N+\beta-1} \left( v^p(\tau) + \frac{t\xi}{h(||v||)} \right) d\tau ds.$$ 

Note that $H(t,v)$ is a compact homotopy and that $H(0,v)(r) = F(v)(r)$, which verifies (b).

On the other hand, we have

$$tF(v)(r) \leq \left( c_2 + g(k) \right) ||v||_\infty^P \int_{r}^{R} \frac{1}{s^{N+\alpha-1}} \int_{0}^{s} \tau^{N+\beta-1} d\tau ds = \frac{\left( c_2 + g(k) \right)\gamma}{(N+\beta)(\beta-\alpha+2)} R^{\beta-\alpha+1} ||v||_\infty^P.$$ 

Taking $\delta = ||v||_\infty$ sufficiently small, such that

$$\frac{\left( c_2 + g(k) \right)\gamma}{(N+\beta)(\beta-\alpha+2)} R^{\beta-\alpha+1} \delta^{p-1} < 1,$$

we have $|tF(v)| < ||v||_\infty$, that is, $tF(v) \neq v$ for all $||v||_\infty = \delta$ and $t \in [0,1]$.

If we consider $\eta$ large such that $\eta > \max\{C_k, 1\}$, with $C_k$ given by Theorem 3.1, we have that $H(t,v) \neq v$ for all $||v||_\infty = \eta$ and $t \in [0,1]$.

Let $v \in C$ such that $H(1,v) = v$ and $||v|| \leq \eta$, then

$$||v||_\infty = \int_{0}^{R} \frac{(a(s) + g_k(v)(s))\gamma}{s^{N+\alpha-1}} \int_{0}^{s} \tau^{N+\beta-1} \left( v^p(\tau) + \frac{\xi}{||v||_\infty^{q-p}} \right) d\tau ds \geq \frac{c_1 \gamma R^{\beta-\alpha+2}}{(N+\beta)\eta^{q-p}}.$$ 

Then, choosing $\xi$ sufficiently large in the homotopy $H(t,u)$, we see that Condition (d) is satisfied by Theorem 3.1.

Hence, using Krasnosel’skii Theorem, we have that the operator $(4.1)$ has a fixed–point $v$ such that $\delta < ||v||_\infty < \eta$, which is solution of equation $(A)_k^0$.  

5. Proof of Theorem 1.1. In this section we will prove the existence of positive radial solution of Problem (1.1). Notice that if there is $k_0 \in \mathbb{N}$, such that $||v_{k_0}||_\infty \leq k_0$, where $v_{k_0}$ is a solution of Truncated Problem $(A)_{k_0}^0$, then $v_{k_0}$ is a radial solution of Problem (1.1).

Proof of Theorem 1.1. We will prove that there is $k_0 \in \mathbb{N}$, such that $||v_{k_0}||_\infty \leq k_0$, where $v_{k_0}$ is a solution of Truncated Problem $(A)_{k_0}^0$. For this, suppose by contradiction that $||v_k||_\infty > k$ for all $k \in \mathbb{N}$, where $v_k$ is a solution of $(A)_k^0$.  

Using a similar change of variables given in (3.2), where \( t_k := \|v_k\|_\infty \) and \( z_k = \left(1 + g(k)\right)\frac{\gamma}{\gamma + \beta - \alpha + 1} \), we see that the function \( w_k \) is a solution of the following problem

\[
-\left( \frac{y^{N+\alpha-1}w_k'(y)}{(a(t_k)z_k^{-1}y + g_k(t_kw_k(y)))^\gamma} \right)' = \frac{y^{N+\beta-1}}{(1 + g(k))^\gamma} w_k^p(y) \quad \text{for} \quad y < \frac{z_k R}{t_k},
\]

\[
w_k'(0) = 0, \quad w_k(0) = 1,
\]

\[
w_k\left(\frac{R z_k}{t_k}\right) = 0.
\]

**Remark 5.1.** Note that, by hypothesis \((H_1)\), we have \( \frac{R z_k}{t_k} \to \infty \) when \( k \to \infty \).

From the last equation, it is not difficult to see that, \( w_k'(y) < 0 \) for all \( y \in (0, \frac{z_k R}{t_k}) \) and

\[
|w_k'(y)| \leq 2 \int_0^y s^{\beta - \alpha} ds
\]

for all \( k \in \mathbb{N} \).

Therefore, for any \( M > 0 \) there is a positive constant \( C(M) \) such that

\[
|w_k'(y)| < C(M) \quad \text{for all} \quad k \in \mathbb{N} \quad \text{and} \quad y \in [0, M].
\]

From the last inequality, we have the sequence \( \{w_k\}_k \) is equicontinuous. Since it is also uniformly bounded, by Ascoli Arzelà’s theorem, we have that \( \{w_k\}_k \) contains a convergent subsequence, which we denote again by \( \{w_k\}_k \), verifying

\[
w_k \to w \quad \text{in} \quad C[0, M].
\]

Now, we will study the **limiting problem** associated with the Problem \((B)_k\).

Since \( t_k > k \) for all \( k \in \mathbb{N} \), we have \( 0 < \frac{k}{t_k} < 1 \). Then, there are \( l \in [0, 1] \) and a subsequence, which we denote again by \( \{\frac{k}{t_k}\}_k \) such that \( \frac{k}{t_k} \to l \).

Note that, since \( w_k \) is a solution of Problem \((B)_k\), for each \( k \in \mathbb{N} \) there is \( s_k \in \left(0, \frac{z_k R}{t_k}\right) \) such that

\[
w_k(s_k) = \frac{k}{t_k}.
\]

Now, we analyze the **limit problem** associated with Problem \((B)_k\), depending on the value of \( l \).

1. If \( l = 0 \), that is \( \frac{k}{t_k} \to 0 \), it is easy to see that \( s_k \to +\infty \). Thus, for any \( M > 0 \) there is \( k_M \in \mathbb{N} \) such that \( s_k > M \) for all \( k \geq k_M \). Then, since \( w_k \) is a non-increasing function and by definition of \( g_k \), we have \( g_k(t_kw_k(y)) = g(k) \) for all \( y \in [0, M] \) and \( k \geq k_M \).

From equation in \((B)_k\), it is easy to see that, for \( k \geq k_M \) we have

\[
-w_k(y) + 1 = \int_0^y \frac{1}{s^{\gamma + \alpha - 1}} \left( a(t_k)z_k^{-1}s + g(k) \right)^\gamma \int_0^s s^{N+\beta-1} w_k^p(\tau) d\tau ds,
\]

for each \( y \in [0, M] \).

Since the function \( a \) bounded, by convergence in \((5.1)\) and using Lebesgue’s dominated theorem, \( w \) satisfies the following integral equation

\[
1 - w(y) = \int_0^y \frac{1}{s^{N+\alpha - 1}} \int_0^s s^{N+\beta-1} w^p(\tau) d\tau ds.
\]
Thus, \( w \) is a positive solution in \([0, M]\) of the initial value problem

\[
-(y^{N+\alpha-1}w')' = y^{N+\beta-1}w^p
\]

\[
w(0) = 1, \quad w'(0) = 0.
\]  

(5.5)

Using a diagonal iterative scheme, \( w \) can be extended to all \( \mathbb{R}_+ \), as a positive solution of (5.5), and using [8], it can be shown that \( w \) is indeed a positive solution of class \( C^2(0, +\infty) \) of (2.6). This is a contradiction with Theorem 2.3.

(2) If \( l = 1 \), that is \( \frac{k}{t_k} \to 1 \). Here, is natural to expect that \( s_k \to 0 \). Indeed, integrating from 0 to \( s_k \in ]0, M] \) as in (5.3), we have

\[
-k \cdot \frac{1}{t_k} = 1 - \frac{1}{g(k)} - \frac{\int_0^{s_k} (a(t_kz_k^{\gamma-1} + g_k(t_kw_k(s)))^{-\gamma} ds}{s^{N+\alpha-1}} \int_0^{s_k} \tau^{N+\alpha-1}w_k^p(\tau) d\tau ds
\]

\[
\geq \left( \frac{k}{t_k} \right)^{\beta-\alpha+2} \frac{s^\beta-\alpha+1}{(N+\beta)(\beta-\alpha+2)}.
\]

Hence, we obtain \( s_k \to 0 \) as \( k \to \infty \).

Note that, the equation in \((B)_k\), we have that \( w_k \) satisfies the following integral equation

\[
-w_k(y) + \frac{k}{t_k} = \int_0^{y} \left( a(t_kz_k^{\gamma-1} + g_k(t_kw_k(y)))^{-\gamma} \frac{1}{1 + g(k)} \int_0^{s_k} \tau^{N+\alpha-1}w_k^p(\tau) d\tau ds\right) dy,
\]

(5.6)

for all \( y \in [0, M] \).

By convergence in (5.1), hypothesis \((H_2)\), then using the Lebesgue’s dominated convergence theorem in the last equality, \( w \) satisfies

\[
w(y) + 1 - \frac{1}{s^{N+\alpha-1}} \int_0^{y} \tau^{N+\alpha-1}w^p(\tau) d\tau ds
\]

for all \( y \in [0, M] \).

Thus, \( w \) is a nonnegative nontrivial solution in \([0, M]\) of the initial value problem

\[
-(y^{N+\alpha-1}w')' = y^{N+\beta-1}w^p
\]

\[
w(0) = 1, \quad w'(0) = 0.
\]  

(5.7)

As in the previous case, \( w \) can be extended to all \( \mathbb{R}_+ \), as a positive solution of (5.7). Furthermore, using [8], it can be shown that \( w \) is indeed a positive solution of class \( C^2(0, +\infty) \) of (5.7).

Let us consider the change of variables given in (2.9), we have that \( u \in C^2(0, +\infty) \) is a positive solution of

\[
-(y^{N+\alpha-1}u')' = (1 - \gamma)y^{N+\beta-1}u^{\frac{\gamma}{\gamma-1}} w_k^p
\]

\[
u(0) = 1, \quad u'(0) = 0.
\]  

(5.8)

This is a absurd by Theorem 2.3.

(3) If \( 0 < l < 1 \). Since \( w_k \) a solution of Problem \((B)_k\) and \( w_k(\frac{z_kM}{t_k}) = 0 \), it is not difficult to verify that \( s_k \to \infty \). Then, there is a constant \( c_0 > 0 \), such that \( s_k \leq c_0 \) for all \( k \in \mathbb{N} \). Then by compactness, there is \( s_0 \in \mathbb{R}_+ \) and a subsequence, which we denote again by \( \{s_k\}_k \), such that \( s_k \to s_0 \).
Observe that for \( y \in (0, s_k) \) we have that \( g_k(t_k w_k(y)) = g(k) \). Then, we have that \( w_k \) satisfies the integral equation (5.3) in \((0, s_0)\). Proceeding as in case (1), we conclude that \( w \) is a positive solution of the following problem

\[
-(y^{N+\alpha-1}w')' = y^{N+\beta-1}w^p \quad \text{in } (0, s_0),
\]

\[
w(0) = 1, \quad w'(0) = 0.
\]

On the other hand, for each \( y \in (s_k, \infty) \), it is easy to see that \( w(y) > 0 \), then we have \( t_k w_k(y) \to \infty \) for each \( y \in (s_k, \infty) \). From the equation (5.6), by hypothesis \((H_2), (5.1)\) and Lebesgue’s dominated convergence theorem, we conclude that \( w \) satisfies de following integral equation

\[
-w(y) + l = \int_{s_k}^{y} \left( \frac{l-1}{s^{N+\alpha-1}} \right)^{\gamma} \int_{0}^{s} \tau^{N+\beta-1}w^p(\tau) d\tau ds \quad \text{for all } y \in (s_0, \infty).
\]

Thus, \( w \) is a nonnegative nontrivial solution in \((s_0, \infty)\) to the initial value problem

\[
-(\frac{y^{N+\alpha-1}w'(y)}{l-\gamma w(y)})' = y^{N+\beta-1}w^p(y), \quad y \in (s_0, \infty)
\]

Then, we have that \( w \) is nontrivial solution of \((P)\). This is absurd by Theorem 2.4.

Then there is \( k_0 \in \mathbb{N} \), such that \( \|v_{k_0}\| \leq k_0 \), then \( v_{k_0} \) is a positive solution of Problem 3.1.  

6. **non–existence via Pohozaev type identity.** In this section, we prove Theorem 1.2 (see \([21]\)), i.e. we establish the non–existence of positive solutions of Problem (1.1). For this, using a variant of Pohozaev identity, we obtain the exponent \( p^* \) defined in (1.4), which for the case \( \gamma = 0 \) correspond to the critical exponent associated to the Laplacian operator.

**Proof Theorem 1.2.** Suppose \( v \in C^1(\Omega) \cap C^0(\bar{\Omega}) \) and \( \frac{r^{N+\alpha-1}v'}{(a(r) + g(v(r)))^\gamma} \in C^1(\Omega) \) be a radial solution of Problem (1.1). Then, multiplying (3.1) by \( rv'(r) \) and integrating on \([0, R]\), we find

\[
-\int_{0}^{R} \left( \frac{r^{N+\alpha-1}v'(r)}{(a(r) + g(v(r)))^\gamma} \right)' rv'(r) dr = \int_{0}^{R} r^{N+\beta}v^p(r)v'(r) dr,
\]

We rewrite the expression (6.1) as

\[
A = B.
\]

The term on the left is

\[
A := -r^{N+\alpha}v'^2(R) + \int_{0}^{R} \frac{r^{N+\alpha-1}v'^2(r)}{(a(r) + g(v(r)))^\gamma} dr + \int_{0}^{R} \frac{r^{N+\alpha}v''(r)v'(r)}{(a(r) + g(v(r)))^\gamma} dr
\]

\[
= -r^{N+\alpha}v'^2(R) + \int_{0}^{R} \frac{r^{N+\alpha-1}v'^2(r)}{(a(r) + g(v(r)))^\gamma} dr + A_1,
\]

with

\[
A_1 = \frac{1}{2} \int_{0}^{R} \frac{r^{N+\alpha}(v'^2(r))'}{(a(r) + g(v(r)))^\gamma} dr
\]

\[
= \frac{1}{2} r^{N+\alpha}v'^2(R) \left( \frac{N + \alpha}{(a(R) + g(0))^\gamma} - \frac{N + \alpha}{2} \int_{0}^{R} \frac{r^{N+\alpha-1}v'^2(r)}{(a(r) + g(v(r)))^\gamma} dr \right)
\]
Thus, if \( v \) is nonnegative and nontrivial solution, we have

\[
A = -\frac{1}{2} \frac{R^{N+\alpha} v' \sqrt{R}}{(a(R) + g(0))^\gamma} + \left( 1 - \frac{N + \alpha}{2} \right) \int_0^R \frac{r^{N+\alpha-1} v'^2(r)}{(a(r) + g(v(r)))^\gamma} \, dr + \frac{\gamma}{2} \int_0^R \frac{r^{N+\alpha} v'^2(r) g'(v(r))}{(a(r) + g(v(r)))^{\gamma+1}} \, dr.
\]

Then, we have

\[
B = \int_0^R r^{N+\beta} v^p(r) v'(r) \, dr = \frac{1}{p+1} \int_0^R r^{N+\beta} (v^{p+1}(r))' \, dr = \frac{1}{p+1} r^{N+\beta} v^{p+1}(r) \bigg|_0^R - \frac{N + \beta}{p+1} \int_0^R r^{N+\beta-1} v^{p+1}(r) \, dr.
\]

On the other hand, the right hand side of (6.1) is

\[
B = -\frac{N + \beta}{p+1} \int_0^R r^{N+\beta-1} v^{p+1}(r) \, dr.
\]

Similarly, if we multiply the equation (3.1) by \( v \) and integrating on \([0, R]\), getting

\[
\int_0^R r^{N+\beta-1} v^{p+1}(r) \, dr = \int_0^R \frac{r^{N+\alpha-1} v'^2(r)}{(a(r) + g(v(r)))^\gamma} \, dr.
\]

Combine (6.3) and (6.4), we obtain

\[
B = -\frac{N + \beta}{p+1} \int_0^R \frac{r^{N+\alpha-1} v'^2(r)}{(a(r) + g(v(r)))^\gamma} \, dr.
\]

Then, by last calculation and (6.2), we have

\[
\left( 1 - \frac{N + \alpha}{2} + \frac{N + \beta}{p+1} \right) \int_0^1 \frac{r^{N+\alpha-1} (v'(r))^2}{(a(r) + g(v(r)))^\gamma} \, dr = \frac{1}{2} \frac{R^{N+\alpha} v'^2(R)}{(a(R) + g(0))^\gamma} - \frac{\gamma}{2} \int_0^R \frac{r^{N+\alpha} v'^2(r) g'(v(r))}{(a(r) + g(v(r)))^{\gamma+1}} \, dr.
\]

Hence if \( v \) is nonnegative and nontrivial solution, we have

\[
\frac{1}{2} \frac{R^{N+\alpha} v'^2(R)}{(a(R) + g(0))^\gamma} - \frac{\gamma}{2} \int_0^R \frac{r^{N+\alpha} v'^2(r) g'(v(r))}{(a(r) + g(v(r)))^{\gamma+1}} \, dr > 0.
\]

Thus

\[
\left( 1 - \frac{N + \alpha}{2} + \frac{N + \beta}{p+1} \right) \int_0^1 \frac{r^{N+\alpha-1} (v'(r))^2}{(a(r) + g(v(r)))^\gamma} \, dr > 0,
\]

Which implies

\[
1 - \frac{N + \alpha}{2} + \frac{N + \beta}{p+1} > 0.
\]

Hence,

\[
p < \frac{N + 2\beta - \alpha + 2}{N + \alpha - 2}.
\]
**Proof of Theorem 1.3.** Suppose that \( v \in C[0, R] \cap C^1(0,1) \) is a radial solution of Problem (1.1). Thus, the function \( v \) verifies the integral equation

\[
v(r) = \int_r^R \frac{(a(s) + g(v(s)))^\gamma}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} v^p(\tau) d\tau ds.
\]

Since \( v \) is nonnegative, nontrivial, and decreasing function, we have that, given \( \varepsilon > 0 \) is small, there exists \( r_0 \in (0, \frac{R}{2}) \) such that

\[
v(r_0) > \varepsilon.
\]

Then, we have

\[
v(r) = \int_r^R \frac{(a(s) + g(v(s)))^\gamma}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} v^p(\tau) d\tau ds
\]
\[
\geq \varepsilon^p c_1^\gamma \int_r^{r_0} \frac{1}{s^{N+\alpha-1}} \int_0^s \tau^{N+\beta-1} d\tau ds
\]
\[
= \frac{\varepsilon^p c_1^\gamma}{N+\beta} \int_r^{r_0} s^{\beta-\alpha+1} ds.
\]

Since \( \beta - \alpha + 2 \leq 0 \), we have

\[
\lim_{r \to 0} v(r) = +\infty.
\]

This proves \( v \notin C[0, R] \cap C^1(0,1) \).

\( \square \)

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