Partial autocorrelation parameterisation of models with unit roots on the unit circle

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1 Introduction

Let \( \{ Y_t \} \) be a time series whose evolution can be described by the equation

\[
U(B)\phi(B)Y_t = \theta(B)\varepsilon_t.
\]

(1)

Tiao and Tsay (1983) refer to this model as a nonstationary ARMA model. Huang and Anh (1990) call this model autoregressive unit root moving average (ARUMA), see also Woodward et al. (2017). Here \( \{ \varepsilon_t \} \) is white noise, \( B \) is the backward shift operator and all roots of the polynomials \( \phi(z) \) and \( \theta(z) \) are outside the unit circle. The nonstationary part is specified by the polynomial \( U(z) = 1 - U_1 z - U_2 z^2 - \cdots - U_d z^d \) whose all roots have moduli 1 (i.e., lie on the unit circle). Traditionally the polynomial \( U(z) \) does not have coefficients to be estimated. This is the case, for example, for the familiar ARIMA and seasonal ARIMA (SARIMA) models obtained when \( U(z) = (1 - B)^d \) and \( U(z) = (1 - B)^d (1 - B^s)^{d_s} \), respectively.

Tiao and Tsay (1983) and Tsay and Tiao (1984) study (iterative) ordinary least squares procedures for estimation of such models and, in particular, show how the unit roots can be estimated consistently.

For time series data it is typical to consider whether seasonal trends appear. This behaviour is easily captured by the existing models by allowing further polynomials to appear in the model with the appropriate power transformation of \( B \) to account for the seasonality. Standard example is the SARIMA class of models, mentioned above. The operator \( (1 - B^s)^{d_s} \) however is sometimes too crude and may be impractical when the number of seasons, \( s \), is large or in the case of multiple seasons. A more flexible class of models is obtained by replacing it with a operator containing only some
harmonics of $1/s$. With a seasonal extension, we refer to this class of models as \textit{SARUMA}. Here is a symbolic representation:

$$U_s(B^s)U(B)\phi_s(B^s)\phi(B)Y_t = \theta_s(B^s)\theta(B)\varepsilon_t, \quad (2)$$

where $U_s(z)$ is a seasonal polynomial of degree $d_s$ where all roots are unit, $\phi_s(z)$ is a seasonal autoregressive polynomial of degree $p_s$, $\theta_s(z)$ is a seasonal moving average polynomial of degree $q_s$ such that all roots of $\phi_s(z)$ and $\theta_s(z)$ lie outside the unit circle. The remaining terms are as in Equation 1. We also require that there are no common roots between the $\phi_s(z^s)\phi(z)$ and $\theta_s(z^s)\theta(z)$ components of the model. In practice, it is sometimes useful to factor $U(z)$ and $U_s(z)$ into further factors in order to obtain more meaningful and/or manageable models.

In principle the SARUMA model can be written in the form of model (1) by expanding $U_s(B^s)U(B)$ and estimate it using the OLS method of Tsay and Tiao (1984) but this loses any parsimony that might be achievable otherwise.

Parameterisations of stationary models through partial autocorrelations are widely used in the stationary case but for unit root models partial autocorrelations are not defined. Nevertheless, we show that partial autocorrelations equal to $\pm 1$ naturally describe multiplicative ARUMA models and neatly fit with the standard practice of fitting ARIMA models. We continue to call them partial autocorrelations though they do not have the usual statistical interpretation and are purely a parameterisation of the polynomial on the left-hand side of Equation (2).

The transformation from partial autocorrelations to polynomial coefficients is unique, so residuals and sums of squares are easily available and estimation is possible.

In this paper we obtain the algebraic properties of the partial autocorrelations in the context of unit roots. The main result is that if a partial autocorrelation sequence contains some values equal to 1 or $-1$, then it can be split at these values into sequences each of which represents the partial autocorrelations of a factor of the overall polynomial on the left-hand side of the model. A separate paper will discuss the details of the estimation procedure and its properties. An implementation is provided by Boshnakov and Halliday (2022, function \texttt{sarima}).

## 2 Levinson-Durbin algorithm and its inverse

The use of partial autocorrelations as a parameterisation of autoregressive (AR) stationary models and stable filters is well established. For stationary AR models there is a one-to-one map between the autoregressive parameters and the partial autocorrelations. The partial autocorrelations have a clear statistical meaning in this case. The one-to-one map allows to think of the partial autocorrelations also as an alternative way to parameterise the coefficients of the associated autoregressive polynomial.
For a stationary process \( \{X_t\} \), let \( \phi_1^{(n)}, \ldots, \phi_n^{(n)} \) be the partial prediction coefficients for the best linear predictor, \( \phi_1^{(n)} Y_t + \cdots + \phi_n^{(n)} Y_{t-n+1} \), of \( Y_{t+1} \) based on the latest available \( n \) observations. Let \( \beta_1, \beta_2, \ldots \) be the partial autocorrelations. It is convenient to define \( \beta_0 = 1 \). Consider also the polynomial

\[
1 - \phi_1^{(n)} z - \cdots - \phi_n^{(n)} z^n.
\]

The statistical meaning of the partial autocorrelations and partial prediction coefficients is not really needed for the exposition below but gives context.

The Levinson-Durbin recursions (Brockwell and Davis, 1991) can be used to compute the partial prediction coefficients from the partial autocorrelations, as follows:

\[
\begin{align*}
\phi_n^{(n)} &= \beta_n \\
\phi_k^{(n)} &= \phi_k^{(n-1)} - \beta_n \phi_{n-k}^{(n-1)} \\
&\quad \text{for } k = 1, \ldots, n-1
\end{align*}
\]

It is evident that the transformation from partial autocorrelations to partial coefficients is uniquely defined without the need to put restrictions on \( \beta_1, \ldots, \beta_n \). Note that, strictly speaking, the Levinson-Durbin algorithm contains an additional step at each \( n \) for computing the partial autocorrelation from autocorrelations, which we don’t need since we start with partial autocorrelations.

The recursions can be arranged in reverse order to compute the partial autocorrelations from the partial coefficients \( \phi_1^{(m)}, \ldots, \phi_m^{(m)} \):

\[
\begin{align*}
\beta_n &= \phi_n^{(n)} \\
\phi_k^{(n-1)} &= (\phi_k^{(n)} + \beta_n \phi_{n-k}^{(n)})/(1 - \beta_n^2) \\
&\quad \text{for } k = 1, \ldots, n-1
\end{align*}
\]

At the end we have \( \beta_1, \ldots, \beta_n \). Detailed discussion of several variants of the Levinson-Durbin algorithm is given by Porat (1994).

Of course, the inverse recursion will work only if \( |\beta_k| \neq 1 \) for \( k = 1, \ldots, n \). In that case the relationship between the two sets of coefficients is one-to-one. The case \( |\beta_k| > 1 \) is of interest to us here. Our aim is to show that allowing some of the partial autocorrelations to be equal to one provides a very natural parameterisation for models with arbitrary unit roots, including seasonal ARIMA models. Since partial autocorrelations uniquely determine the filter coefficients, this means that residuals can be computed and so a non-linear least squares estimation of the unit root filter can be performed.
Some further insight can be obtained by noticing that the equations are paired for $k$ and $n - k$:

\[
\begin{align*}
\phi_k^{(n)} &= \phi_{k}^{(n-1)} - \beta_n \phi_{n-k}^{(n-1)} \\
\phi_{n-k}^{(n)} &= \phi_{n-k}^{(n-1)} - \beta_n \phi_k^{(n-1)}
\end{align*}
\]

$k = 1, \ldots, \lfloor n/2 \rfloor$.

If $n$ is even and $k = n/2 = n - k$ the two equations can be reduced to

\[
\phi_n^{(n)} = \phi_{n/2}^{(n-1)} - \beta_n \phi_{n/2}^{(n-1)} = \phi_{n/2}^{(n-1)} (1 - \beta_n).
\]

In particular, if $\beta_n = 1$ then $\phi_n^{(n)} = 0$ and if $\beta_n = -1$ then $\phi_n^{(n)} = 2\phi_{n/2}^{(n-1)}$. It is also obvious that when $k \neq n/2$ that $\phi_k^{(n)} = -\phi_{n-k}^{(n)}$ when $\beta_n = 1$ and $\phi_k^{(n)} = \phi_{n-k}^{(n)}$ when $\beta_n = -1$. For example, when $n = 2$, the above gives $\phi_1^{(2)} = 0$ if $\beta_2 = 1$ and the polynomial must be $1 - z^2$. When $\beta_2 = -1$ then $\phi_1^{(2)} = 2\phi_1^{(1)} = 2\beta_1$ and the polynomial is $1 - 2\beta_1 z + z^2$, which generates a pair of complex roots.

In what follows we show how polynomials can be separated after the occurrence of a partial autocorrelation value of unit magnitude and show that sequence of partial autocorrelations ending with a unit value produces a polynomial that contains only roots on the unit circle. This methodology can be used to define each polynomial in Equation (2).

### 3 Parameterisation using partial autocorrelations

Let $\beta_k$, $k = 1, 2, \ldots$, be a sequence of partial autocorrelations. Define polynomials $P_n(z)$ by

\[
P_n(z) = \sum_{k=1}^{n} \phi_k^{(n)} z^k, \quad \text{for } n = 1, 2, \ldots, \quad P_0(z) = 0,
\]

where $\phi_k^{(n)}$ are the partial coefficients obtained from $\beta_1, \ldots, \beta_n$, using Equations (3)–(4). Our main interest is in the positions of the zeroes of the polynomials

\[
\Phi_n(z) = 1 - P_n(z) = 1 - \sum_{k=1}^{n} \phi_k^{(n)} z^k, \quad \text{for } n = 0, 1, 2, \ldots.
\]

It is well known that if the coefficients of the polynomial $\Phi_n(z)$ are obtained from partial autocorrelations $\beta_1, \ldots, \beta_n$, such that $|\beta_i| < 1$ for $i = 1, \ldots, n$, then all zeroes of the polynomial $\Phi_n(z)$ are outside the unit circle (i.e., have moduli greater than 1). In particular, their product has modulus larger than 1.

What happens if $|\beta_i| < 1$ for $i = 1, \ldots, n - 1$, but $\beta_n = \pm 1$? We formulate the result as a lemma. It is hardly new but not easily available.
Lemma 1. If $|\beta_i| < 1$ for $i = 1, \ldots, n-1$, $\beta_n = \pm 1$, then all zeroes, $z_1, \ldots, z_n$, of the polynomial $\Phi_n(z) = 1 - P_n(z)$ are on the unit circle (i.e., $|z_i| = 1$ for $i = 1, \ldots, n$).

One way to show this is to notice that in that case the Vietta formulas imply that the product of the zeroes of $\Phi_n(z)$ is $\pm 1$. Then let $\beta_n^{(i)} \rightarrow \beta_n$, $|\beta_n^{(i)}| < 1$ for $i = 1, 2, \ldots$ and consider the sequence of polynomials $\Phi_n^{(i)}(z)$, $i = 1, 2, \ldots$. Since the zeroes of polynomials are continuous functions of their coefficients, and hence the partial autocorrelations, the zeroes of $\Phi_n^{(i)}(z)$ converge to the zeroes of $\Phi_n(z)$. But all zeroes of $\Phi_n^{(i)}(z)$ are strictly outside the unit circle, so their limits (the zeroes of $\Phi_n(z)$) are outside or on the unit circle. This means that their product can be equal to $1$ only if all of them have modulus $1$.

The following relation between the polynomials $P_n(z)$ can be obtained from the Levinson-Durbin recursions. Let $n \geq 2$. For general $z$, multiply Equation (4) by $z^k$ for $k = 1, \ldots, n-1$, and sum to obtain

$$\sum_{k=1}^{n-1} \phi_k^{(n)} z^k = \sum_{k=1}^{n-1} \phi_k^{(n-1)} z^k - \beta_n \sum_{k=1}^{n-1} \phi_{n-k}^{(n-1)} z^k.$$  

Using the definition of the polynomial $P_n(z)$ and $\beta_n = \phi_n^{(n)}$, this can be written as

$$P_n(z) - \beta_n z^n = P_{n-1}(z) - \beta_n z^n P_{n-1}(z^{-1}),$$

which after rearranging becomes

$$(1 - P_n(z)) = (1 - P_{n-1}(z)) - \beta_n z^n (1 - P_{n-1}(z^{-1})). \tag{8}$$

The above equation was derived for $n \geq 2$ but it holds also, trivially, for $n = 1$. Note that the coefficients of the polynomial $z^n (1 - P_{n-1}(z^{-1}))$ are those of $(1 - P_n(z))$ in reverse order.

In general, the polynomials $1 - P_n(z)$, $n = 1, 2, \ldots$, do not have common zeroes. A remarkable exception, particularly important for unit root models, is given by the following lemma. It shows that if $z_0$ is such that it and $z_0^{-1}$ are both zeroes of the polynomial $1 - P_m(z)$, then they are also zeroes of the polynomials $1 - P_n(z)$ for all $n \geq m$.

Lemma 2. Let $z_0$ be such that $1 - P_m(z_0) = 0$ and $1 - P_m(z_0^{-1}) = 0$ for some $m \in \mathbb{Z}^+$. Then $1 - P_n(z_0) = 0$ and $1 - P_n(z_0^{-1}) = 0$ for any $n \geq m$.

Proof. Setting $n = m + 1$ in Equation (8) gives

$$(1 - P_{m+1}(z)) = (1 - P_m(z)) - \beta_{m+1} z^{m+1} (1 - P_m(z^{-1})). \tag{9}$$
If $z = z_0$ or $z_0^{-1}$, then both terms on the right-hand side of the last equation are zero, by the assumptions of the lemma. Hence, the left-hand side is also zero, i.e. $1 - P_{m+1}(z_0) = 0$ and $1 - P_{m+1}(z_0^{-1}) = 0$. So, the claim of the lemma holds for $n = m + 1$. But Equation (8) holds also for $n > m + 1$, so the proof can be completed by induction.

The following corollary concerning roots on the unit circle is of primary interest for our purposes. Indeed, complex roots of polynomials with real coefficients come in complex conjugate pairs. Moreover, if $|z_0| = 1$ then $z_0^{-1} = \bar{z}_0$. So, in this case $1 - P_m(z_0) = 0$ implies $1 - P_m(z_0^{-1}) = 0$ and we have:

**Lemma 3.** If $|z_0| = 1$ and $1 - P_m(z_0) = 0$ then $1 - P_n(z_0) = 0$ and $1 - P_n(z_0^{-1}) = 0$ for any $n \geq m$.

A useful consequence of Lemma 3 is the following result.

**Lemma 4.** If all roots, $z_1, \ldots, z_m$, of the polynomial $1 - P_m(z)$ are on the unit circle (i.e., $|z_i| = 1$ for $i = 1, \ldots, m$), then $1 - P_m(z)$ is a factor of $1 - P_n(z)$ for any $n \geq m$.

**Proof.** Since the roots have moduli equal to 1 and $1 - P_{m-1}(z)$ has real coefficients, it follows from Lemma 3 that $z_1, \ldots, z_m$ are roots of $1 - P_n(z)$ for all $n \geq m$, hence the result.

Lemma 4 shows that if $P_n(z)$ is the polynomial generated from the partial autocorrelation sequence $\beta_1, \ldots, \beta_m, \beta_{m+1}, \ldots, \beta_n$, where $\beta_m = \pm 1$ and $|\beta_{m+i}| < 1$ for $i = 1, \ldots, n - m$, then $1 - P_n(z) = (1 - P_m(z))(1 - T(z))$, where $T(z)$ is some polynomial. It turns out that $\beta_{m+1}, \ldots, \beta_n$ are, up to possible sign changes, the partial autocorrelations generating the polynomial $T(z)$. Our main result in this section states the complete result.

**Theorem 1** (Main result). Let $\beta_1, \ldots, \beta_m, \beta_{m+1}, \ldots$, be partial autocorrelations, such that $|\beta_i| \leq 1$, for $i = 1, \ldots, m - 1$, $\beta_m = \pm 1$, and $|\beta_{m+i}| \leq 1$ for $i \geq 1$. Let $P_n(z)$ be the polynomials defined by Equation (7). Let also $\gamma_i = (-1)^{d_+} \beta_{m+i}$, $i \geq 1$, where $d_+$ is the number of zeroes of $1 - P_m(z)$ equal to $+1$.

Then, for each $n \geq m + 1$, $(1 - P_n(z)) = (1 - P_m(z))(1 - Q_{n-m}(z))$, where the polynomial $Q_{n-m}(z)$ is generated from the partial autocorrelations $\gamma_1, \ldots, \gamma_{n-m}$.

**Proof.** Changing $n$ to $l$ in Equation (8) and summing from $m + 1$ to $n$ we obtain

$$\sum_{l=m+1}^{n} (1 - P_l(z)) = \sum_{l=m+1}^{n} (1 - P_{l-1}(z)) - \sum_{l=m+1}^{n} \beta_l z^l (1 - P_{l-1}(z^{-1})).$$

$$= \sum_{l=m}^{n-1} (1 - P_l(z)) - \sum_{l=m+1}^{n} \beta_l z^l (1 - P_{l-1}(z^{-1})).$$

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After cancelling the common terms in the two sides of the equation and rearranging, we get

\[(1 - P_n(z)) = (1 - P_m(z)) - \sum_{l=m+1}^{n} \beta_l z^l (1 - P_{l-1}(z^{-1})).\]

In particular, for \(m = 0\) we have

\[(1 - P_n(z)) = (1 - P_0(z)) - \sum_{l=1}^{n} \beta_l z^l (1 - P_{l-1}(z^{-1})).\]

By Lemma 1 all roots of the polynomial \(1 - P_m(z)\) are on the unit circle. Let \(d_+\) and \(d_-\) be the number of roots equal to +1 and -1, respectively. The remaining 2\(r\) roots are complex conjugate pairs, \(\alpha_i, \alpha_i^{-1}, i = 1, \ldots, r\), where \(\alpha_i^{-1}\) is the complex conjugate of \(\alpha_i\) since \(|\alpha_i| = 1\). Obviously, \(m = d_+ + d_- + 2r\). We have

\[1 - P_m(z) = (1 - z)^{d_+} (1 + z)^{d_-} \prod_{i=1}^{r} \left(1 - \frac{z}{\alpha_i}\right) (1 - \alpha_i z)\]

= \((1 - z)^{d_+} (1 + z)^{d_-} \prod_{i=1}^{r} \left(1 - \frac{1}{\alpha_i} + \alpha_i z + z^2\right).\]

From this we get

\[1 - P_m(z^{-1}) = (1 - z^{-1})^{d_+} (1 + z^{-1})^{d_-} \prod_{i=1}^{r} \left(1 - \frac{z^{-1}}{\alpha_i}\right) (1 - \alpha_i z^{-1}),\]

= \(z^{-d_+} (z - 1)^{d_+} z^{-d_-} (z + 1)^{d_-} \prod_{i=1}^{r} z^{-2} \left(z - \frac{1}{\alpha_i}\right) (z - \alpha_i)\)

= \(z^{-d_+} (z - 1)^{d_+} z^{-d_-} (z + 1)^{d_-} z^{-2r} \prod_{i=1}^{r} \left(z^2 - \frac{1}{\alpha_i} + \alpha_i z + 1\right)\)

= \(z^{-m}(-1)^{d_+} (1 - z)^{d_+} (1 + z)^{d_-} \prod_{i=1}^{r} \left(1 - \frac{z}{\alpha_i}\right) (1 - \alpha_i z)\)

= \(z^{-m}(-1)^{d_+} (1 - P_m(z))\) \hspace{1cm} (11)

Together with Equation (10) (with \(n = m + 1\)) this gives

\[1 - P_{m+1}(z) = 1 - P_m(z) - \beta_{m+1} z^{m+1} (z^{-m}(-1)^{d_+} (1 - P_m(z))),\]

= \((1 - P_m(z))(1 - (-1)^{d_+} \beta_{m+1} z)\).

= \((1 - P_m(z))(1 - \gamma_1 z)\).


Therefore, when \( n = m + 1 \), \( 1 - P_m(z) \) is a factor of \( 1 - P_{m+1}(z) \) and, moreover, we have the explicit factorisation with \( Q_1(z) = \gamma_1 z \).

For the general case, let \( n > m + 1 \) and assume that the claim is true for all \( l < n \). Concentrate on the case \( l = n \) and let \( 1 - Q_{l-m}(z) \) represent the polynomial remaining after division of \( 1 - P_l(z) \) by \( 1 - P_m(z) \) for \( l > m \), so that

\[
1 - P_n(z) = (1 - P_m(z)) (1 - Q_{n-m}(z))
\]  

(12)

Starting from Equation (8), and with the help of Equation (11),

\[
1 - P_n(z) = 1 - P_m(z) - \sum_{l=m+1}^{n} \beta_l z^l \left( 1 - P_{l-1}(z^{-1}) \right)
\]

\[
= 1 - P_m(z) - \sum_{l=m+1}^{n} \beta_l z^l \left( 1 - P_m(z^{-1}) \right) (1 - Q_{l-1-m}(z^{-1}))
\]

\[
= 1 - P_m(z) - \sum_{l=m+1}^{n} \beta_l z^{l-m} (-1)^{d_l} (1 - P_m(z)) \left( 1 - Q_{l-1-m}(z^{-1}) \right)
\]

\[
= (1 - P_m(z)) \left( 1 - \sum_{l=m+1}^{n} (-1)^{d_l} \beta_l z^{l-m} (1 - Q_{l-1-m}(z^{-1})) \right),
\]

\[
= (1 - P_m(z)) \left( 1 - \sum_{l=m+1}^{n} \gamma_l z^{l-m} (1 - Q_{l-1-m}(z^{-1})) \right),
\]

\[
= (1 - P_m(z)) \left( 1 - \sum_{k=1}^{n-m} \gamma_k z^k (1 - Q_{k-1}(z^{-1})) \right),
\]  

(13)

Equation (13) shows that \( 1 - P_m(z) \) is a factor of \( 1 - P_n(z) \) for some \( n > m \) and moreover, by comparing it with Equation (12) we can see that

\[
1 - Q_{n-m}(z) = 1 - \sum_{l=1}^{n-m} \gamma_l z^l \left( 1 - Q_{l-1}(z^{-1}) \right),
\]

where \( \gamma_l = (-1)^{d_l} \beta_l \). Notice the similarities between this equation and Equation (10).

\( 1 - Q_{n-m}(z) \) is of the same form as the original polynomial \( 1 - P_n(z) \) except that the original partial autocorrelation coefficients \( \beta_k \) have been replaced by \( \gamma_k \).

By induction, the claim of the theorem is proved. \( \square \)

If there are more partial autocorrelations with modulus 1, Theorem 1 can be applied recursively to get a factorisation of the unit root polynomials.
Theorem 2. Let $m_1 < m_2 < \cdots < m_r$, be positive integers such that $|\beta_{m_i}| = 1$, $i = 1, \ldots, r$. Then for each $n \geq m_r + 1$

$$(1 - P_n(z)) = (1 - P_{m_1}(z))(1 - P_{m_2 - m_1}(z)) \cdots (1 - P_{m_r - m_{r-1}}(z))(1 - Q_{n-m_r}(z)),$$

where the polynomials $(1 - P_{m_i}(z))$ are obtained from the partial autocorrelations $\beta_i$, $i = m_{i-1} + 1, \ldots, m_i$ with adjusted signs as given by Theorem 1 (applied recursively) and the polynomial $Q_{n-m}(z)$ is generated from the partial autocorrelations $\gamma_1, \ldots, \gamma_{n-m}$.

There are a number of ways to use Theorem 1 in modelling. The most transparent and useful is given by the following result.

Theorem 3 (ARUMA result). Let $n > m$ and $\beta_1, \ldots, \beta_m, \beta_{m+1}, \ldots, \beta_n$, be partial autocorrelations, such that $|\beta_i| \leq 1$, for $i = 1, \ldots, m - 1$, $\beta_m = \pm 1$, and $|\beta_{m+i}| < 1$ for $i = 1, \ldots, n - m$. Let $P_n(z)$ be the polynomials defined by Equation (7).

Then $$(1 - P_n(z)) = (1 - P_m(z))(1 - Q_{n-m}(z)),$$

where all zeroes of $(1 - P_m(z))$ are on the unit circle and all zeroes of $(1 - Q_{n-m}(z))$ are outside the unit circle. Further, $1 - P_m(z)$ is generated by $\beta_1, \ldots, \beta_m$ and $1 - Q_{n-m}(z)$ by $\gamma_1, \ldots, \gamma_{n-m}$, where $\gamma_i = (-1)^{d_+} \beta_{m+i}$, $i = 1, \ldots, n - m$ and $d_+$ is the number of zeroes of $1 - P_m(z)$ equal to $+1$.

Proof. The factorisation $(1 - P_n(z)) = (1 - P_m(z))(1 - Q_{n-m}(z))$ follows from Theorem 1. By Lemma 1 all zeroes of $(1 - P_m(z))$ are on the unit circle. Further, $(1 - Q_{n-m}(z))$ since by Theorem 1 they are generated by partial autocorrelations $|\gamma_i| < 1$, $i = 1, \ldots, n - m$, which have the same moduli as $\beta_{m+1}, \ldots, \beta_n$. \qed

Theorems 1 and 3 fit nicely with the standard practice of applying unit root and/or seasonal unit root filters (represented here by the polynomial $1 - P_m(z)$) to make a time series stationary and then fitting a stationary model to the filtered time series. The unit root filters are typically chosen in advance. Our results allow for estimating the unit root filter. In the simplest case, $\beta_m$ (where $m$ is as in Theorem 3) is fixed to $\pm 1$ and the remaining partial autocorrelations are estimated using nonlinear optimisation in the unit cube.

Recall that for the SARUMA model $\Phi(z) = 1 - P_n(z)$. From the results above, we know that $\Phi(z)$ decomposes into $(1 - P_m(z))(1 - Q_{n-m}(z))$ if all roots of $(1 - P_m(z))$ are on the unit circle. We express $(1 - P_m(z))$ as $U(z)$, the unit root polynomial. If no unit partial autocorrelation values remain in $(1 - Q_{n-m}(z))$ then this corresponds to the stationary $\phi(z)$. Otherwise, the unit root polynomials can be iteratively separated and stored as a product in $U(z)$. When $U(z)$ contains all nonstationary aspects of the model, the Levinson-Durbin recursion can be used to generate the coefficients of $U(z)$ by fixing the final coefficient to $\pm 1$. For example, say that $U(z)$ is of degree
The remaining partial autocorrelations can be used to estimate the coefficients in $\phi(z)$, starting from $\beta_{d+1}$ and after multiplication with $(-1)^{d+}$. Firstly assume, without loss of generality, that all seasonal polynomials can be dropped ($U_s(z) = \phi_s(z) = \theta_s(z) \equiv 1$). Then the resulting ARUMA model can be written

$$\Phi(B)Y_t = \theta(B)\epsilon_t.$$ (14)

Furthermore, define the polynomial $P_n(z)$ as

$$P_n(z) = \sum_{k=1}^{n} \phi_k^{(n)} z^k, \quad \text{for} \quad n = 1, 2, \ldots, \quad P_0(z) = 0,$$

so that $\Phi(z) = 1 - P_n(z)$ with $n = p + d$.

We will discuss the details and the properties of an estimation procedure for ARUMA models based on the results here in a separate paper. An implementation can be found in package ‘sarima’ (Boshnakov and Halliday, 2022).

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### A Stable polynomials

In signal processing, the partial autocorrelations (multiplied by $-1$) are known as *reflection coefficients* (RCs) and play an important role in determining the zero locations of a polynomial with complex coefficients. Let $p(z)$ denote such a polynomial of degree $n$, then

$$p(z) = \sum_{i=0}^{n} p_i z^i.$$ 

The polynomial is called *stable* if all roots of the polynomial lie outside the unit circle. The RCs contain the necessary information regarding the locations of roots with respect to the unit circle and the following theorem holds Bistritz (1996):

**Theorem 4.** A polynomial $p(z)$ with a well-defined set of RCs $\{\beta_k\}_{k=1}^{n}$, $|\beta_k| \neq 1$, has $\nu$ roots inside the unit circle and $n - \nu$ roots outside the unit circle, where $\nu$ can be calculated by counting the number of negative terms in the sequence

$$\nu = n - \{q_n, q_{n-1}, \ldots, q_1\}$$

whose members are defined by

$$q_k = \prod_{i=n}^{k} (1 - \beta_i^2), \quad k = n, \ldots, 1.$$ 

An immediate consequence of Theorem 4 is that necessary and sufficient conditions for stability (or $\nu = 0$) are

$$|\beta_k| < 1, \quad k = 1, \ldots, n.$$ 

The result is formulated for RCs but holds also for partial autocorrelations since it involves only their moduli and squares.