Multi-agent formation control for circumnavigation of dynamic shapes

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Abstract—The problem of multi-agent formation control for target tracking is considered in this paper. The target is an irregular dynamic shape approximated by a circle with moving centre and varying radius. It is assumed that there are \( n \) agents and one of them is capable of measuring both the distance to the boundary of the target and to its centre. All the agents must circumnavigate the boundary of the target while forming a regular polygon. Protocols are designed for two cases: limit bounded tracking with no actuator disturbances and finite-time tracking considering actuator disturbances. One simulated example is provided to verify the usage of one of the control protocols designed in this paper.

I. INTRODUCTION

The use of unmanned vehicles has achieved higher levels of precision and accuracy in many research expeditions as in [1]. It is more cost efficient, when compared to typical research expeditions. This approach is particularly relevant in challenging or hazardous environments, and if real-time data exchange is required for conducting the research process as in [2].

In this paper we consider a circumnavigation problem of a moving shape on a planar surface. This problem has many applications, especially in marine engineering, for instance the tracking of oil spills, algal blooms, plumes, frontal zones in the ocean as well as toxic clouds. These shapes are detected via satellite and may require constant tracking for real-time data collection.

Target tracking and multi-agent formation has a long history [3], [4], [5], [6], [7], [8]. A closely related work [9] proposed one adaptive protocol to circumnavigate around a moving point, the fish tracking using AUVs.

However, for applications like the one presented in this paper, the target to track is not defined as a moving point but as a time varying irregular shape. We assume this shape may be approximated by a moving circle on a 2D surface. Nevertheless, its radius will still vary with time. Therefore the available literature does not present a solution to this specific problem.

There are several methods to solve the tracking of these irregular dynamic shapes. One of them would be via satellite. The advantage of this method would be providing an image of the target location. But the disadvantages are its low frequency of measurements, namely the satellites that are not geosynchronous can only measure a specific area of the earth periodically. Also their resolution may be poor and there could be low visibility due to atmosphere phenomena. Hence our approach to the problem is relying on a multi-agent formation.

Multi-agent formation includes several agents with similar or differing sensing equipment such as imaging sensors. The agents may be, for instance, drones with different altitudes and fields of view or surface agents. This formation may include a satellite for a wider view to collect different type of data. Having so, we may find a symbiotic relationship between a team of surface agents or drones and a satellite.

The remaining sections of this paper are organised as follows. In Section II some necessary preliminaries are recalled. In Section III the main problem of interest is formulated. The main results are presented in Section IV where two protocols are designed, and their proofs are presented. The first one is proved to have asymptotic convergence while the second has finite time convergence taking into account actuator disturbances as well. Some simulations presenting the performance of the proposed algorithm are given in Section V. Concluding remarks and future directions come in Section VI.

Notations. The notations used in this paper are fairly standard. \( \mathbb{C}^n \) denotes \( n \)-dimensional complex space. \( I_n \) is \( n \) dimensional identity matrix. With \( \mathbb{R}_-, \mathbb{R}_+, \mathbb{R}_{\geq 0} \) and \( \mathbb{R}_{\leq 0} \) we denote the sets of negative, positive, non-negative, non-positive real numbers, respectively. \( \| \cdot \|_p \) denotes the \( \ell_p \)-norm and the \( \ell_2 \)-norm is denoted simply as \( \| \cdot \| \) without a subscript. We denote \( \mathbb{I}_n \) and \( 0_n \) as the column vectors containing only ones and zeros in \( \mathbb{R}^n \), respectively. The \( i \)th row and \( j \)th column of a matrix \( M \) are denoted as \( M_{i,} \) and \( M_{,,j} \), respectively. The scalar sign function is defined as

\[
\text{sign}(a) = \begin{cases} 
1 & \text{if } a > 0, \\
0 & \text{if } a = 0, \\
-1 & \text{if } a < 0.
\end{cases}
\]

\( L^2(\mathbb{C}^n) \) denotes the space of functions \( f : \mathbb{R}_+ \rightarrow \mathbb{C}^n \) which are square-Lebesgue-integrable over any finite interval.

We define matrix \( E \) as the two dimensional rotation matrix,

\[
E = \begin{bmatrix} 
0 & 1 \\
-1 & 0
\end{bmatrix}.
\]

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II. PRELIMINARIES

In this section, we briefly review some results from graph theory, adaptive control and differential inclusions that will be used in this paper.

Here we recall some terminologies from graph theory [10]. Let $G = (\mathcal{V}, \mathcal{E})$ be a directed graph with node set $\mathcal{V} = \{v_1, \ldots, v_n\}$, edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge of $G$ is denoted by $e_{ij} := (v_i, v_j)$ and we write $\mathcal{I} = \{1, 2, \ldots, n\}$. The set of neighbours of node $v_i$ is denoted by $N_i := \{v_j \in \mathcal{V} : e_{ji} \in \mathcal{E}\}$. A directed path from node $v_i$ to node $v_j$ is a chain of edges from $\mathcal{E}$ such that the first edge starts from $v_i$, the last edge ends at $v_j$ and every edge in between starts where the previous edge ends. A directed path is called a directed ring if the initial node and ending node are coincident.

The incidence matrix of a digraph is denoted as $B \in \mathbb{R}^{n \times m}$, with $B_{ij} = -1$ if the $j$th edge is towards vertex $i$, and equal to 1 if the $j$th edge is originating from vertex $i$, and 0 otherwise.

Persistent excitation plays a key role in establishing parameter convergence in adaptive identification [11], [12].

**Definition 1.** [12] The function $f \in \mathcal{L}_2^2(\mathbb{C}^n)$ is said to be persistently exciting (p.e.) iff there exist positive constants $\varepsilon_1, T$ such that for all $\tau > 0$,

$$\int_{\tau}^{\tau + T} f(t)f(t)^* dt > \varepsilon_1 I_n.$$ 

$T$ will be termed an excitation period of $f$.

In the remainder of this section, we discuss Filippov solutions [13]. Consider the system

$$\dot{x} = f(x,t) \quad (3)$$

where $x(t) \in \mathcal{D} \subset \mathbb{R}^n$ denotes the state vector, $f : \mathcal{D} \times [0, \infty) \rightarrow \mathbb{R}^n$ is Lebesgue measurable and essentially locally bounded, uniformly in $t$ and $\mathcal{D}$ is an open and connected set.

**Definition 2** (Filippov solution [14], [13]). A function $x : [0, \infty) \rightarrow \mathbb{R}^n$ is called a solution of (3) on the interval $[0, \infty)$ if $x(t)$ is absolutely continuous and for almost all $t \in [0, \infty)$

$$\dot{x} \in \mathcal{F}[f](x(t), t) \quad (4)$$

where $\mathcal{F}[f](x(t), t)$ is an upper semi-continuous, nonempty, compact and convex valued map on $\mathcal{D}$, defined as

$$\mathcal{F}[f](x(t), t) := \bigcap_{\delta > 0} \bigcap_{\mu(S) = 0} \mathcal{C}(f(B(x, \delta)) \setminus S, t), \quad (5)$$

where $S$ is a subset of $\mathbb{R}^n$, $\mu$ denotes the Lebesgue measure, $B(x, \delta)$ is the ball centered at $x$ with radius $\delta$ and $\mathcal{C}(X)$ denotes the convex closure of a set $X$.

If $f$ is continuous at $x$, then $\mathcal{F}[f](x)$ contains only the point $f(x)$.

Given a locally Lipschitz function $V : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ in $(x,t)$, the generalised gradient $\partial V$ is defined by

$$\partial V(x,t) := \text{co}\left\{ \lim_{i \rightarrow \infty} \nabla V(x_i, t_i) \mid (x_i, t_i) \rightarrow (x,t), \right. \left. (x_i, t_i) \notin S \cup \Omega_f \right\},$$

where $\nabla$ is the gradient operator, $\Omega_f \subset \mathbb{R}^n \times [0, \infty)$ is the set of points where $V$ fails to be differentiable and $S \subset \mathbb{R}^n \times [0, \infty)$ is a set of measure zero that can be arbitrarily chosen to simplify the computation, since the resulting set $\partial V(x,t)$ is independent of the choice of $S$ [15].

Given a set-valued map $\mathcal{T} : \mathbb{R}^n \times [0, \infty) \rightarrow 2^{\mathbb{R}^n}$, the set-valued Lie derivative $\mathcal{L}_T V$ of a locally Lipschitz function $V : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ with respect to $\mathcal{T}$ at $(x,t)$ is defined as

$$\mathcal{L}_T V(x,t) := \{a \in \mathbb{R} \mid \forall \zeta \in \partial V(x,t) \}.$$ 

III. PROBLEM STATEMENT

In this paper, we consider $n$ identical agents initialised at position $p_i(0), i \in \mathcal{I}$, which form a counterclockwise directed ring on the surface. We denote the angle between agent $i$ and agent $i + 1$, i.e., its counterclockwise neighbour as $\beta_i$ for $i \in \{1, \ldots, n - 1\}$. And $\beta_n$ is the angle between the nth and the first agent. Notice that in this case,

$$\beta_i(0) \geq 0, \quad \text{and} \quad \sum_{i=1}^{n} \beta_i(0) = 2\pi. \quad (6)$$

The kinematic of the agents are of the form

$$\dot{x}_i = u_i, \quad i \in \mathcal{I}, \quad (7)$$

where $x_i = (p_i^T, \beta_i) \in \mathbb{R}^2 \times [0, 2\pi)$ is a vector contains the position $p_i = (x_i, y_i)^T \in \mathbb{R}^2$ and the angle $\beta_i$, and $u_i \in \mathbb{R}^3$ is the control input that needs to be designed.

It is desired that these agents are capable of circumnavigating some irregular shape, i.e., moving along the boundaries, that varies with time. In this paper, however, we assume that these shapes may be roughly approximated by a moving circle. Thus, we consider a target of the form

$$x_t(c(t), r(t)) \in \mathbb{R}^3, \quad (8)$$

where $c(t) = (x_t(t), y_t(t))$ and $r(t)$ are the centre and the radius of the circle, respectively. Here we assume that agent $i$ is capable of measuring the distance to the centre and the boundary of the target circle, which are denoted as

$$D_i(t) = \|c(t) - p_i(t)\| \quad D_i^2(t) = |r(t) - D_i(t)|, \quad (9)$$

respectively. Note that even though only one of the agents is able to measure these values, we will use this notation throughout this paper to refer to the distances of each agent to the target.
When the target is stationary, i.e., $c$ and $r$ are constant, the circumnavigation of the moving circles is achieved if the agents
1) move in a counterclockwise direction on the boundary of the time-varying target,
2) and form a regular polygon formation while rotating, i.e., $\beta_i = \frac{2\pi}{n}$.

More precisely, we say that the circumnavigation is achieved in a finite bound and in finite-time, respectively, if the previous aim is satisfied for $t \to \infty$ and for $t \geq T$ with a finite $T$.

For the case with time-varying target, we assume that $||\dot{c}|| \leq \varepsilon_1$ and $|\dot{r}| \leq \varepsilon_2$ for some positive constant $\varepsilon_1$ and $\varepsilon_2$.

Now we are ready to pose the problem of interest that will be solved in the following sections.

**Problem 1.** Design the estimator of $c(t)$ and $r(t)$ when only the distance measures, i.e., $\{\| \hat{c} \|, |\dot{r}| \}$, are available to one of the agents, and design the control input $u_i$ for all the agents such that for
\begin{align}
||\dot{c}|| & \leq \varepsilon_1, \\
|\dot{r}| & \leq \varepsilon_2,
\end{align}

there exists $K_1$ and $K_2$ positive satisfying
\begin{align}
\lim \sup_{t \to \infty} ||\hat{c} - c|| & \leq K_1 \varepsilon_1, \\
\lim \sup_{t \to \infty} ||p_i - c|| - r(t) & \leq K_2 \varepsilon_2.
\end{align}

Throughout the rest of the paper, we denote the estimation of the target as
\[
\hat{x}_t(\hat{c}, \hat{r}) \in \mathbb{R}^3,
\]
where $\hat{c} = (\hat{x}_t, \hat{y}_t)$ and $\hat{r}$.

**IV. MAIN RESULTS**

Here we present two solutions for Problem 1. In the first case, circumnavigation using distance measurements, we design a protocol with limited boundedness and no actuator disturbances. In the second case, finite time circumnavigation we design a protocol which achieves finite-time tracking acknowledging actuator disturbances.

**A. Circumnavigation using distance measurements**

In this case, we consider $n$ agents $x_i(t)$ at positions $p_i(t)$ and we assume only one of them is capable of measuring its distance $D^c(t)$ to the target boundary as well as its distance $D(t)$ to the target centre. Then, it should estimate $x_i(c(t), r(t))$ from its distance measures $D^c(t), D(t)$ and share the information with the other agents. Each agent will calculate its direction and velocity taking into account its angle $\beta_i(t)$ to the next agent as well as its distance to the target centre and boundary, obtained with the estimates of the target.

Motivated by [9], we propose the following adaptive estimation of the radius $r(t)$ of the target. Observe that
\[
\frac{d}{dt}(\dot{r}^2) = 2D(D - \dot{r}).
\]
Assume the estimation of $\dot{r}$ is denoted as $\dot{\hat{r}}$, we have
\[
\frac{1}{2} \left( \frac{d}{dt}D^2 - \frac{d}{dt}D^2 \right) + \dot{\hat{r}} = \dot{\hat{r}}(\dot{r} - \hat{r}).
\]
Then for some positive constant $\gamma$ the dynamic
\[
\dot{\hat{r}} = -\gamma \dot{\hat{r}} \left[ \frac{1}{2} \left( \frac{d}{dt}D^2 - \frac{d}{dt}D^2 \right) + \dot{\hat{r}} \right]
\]
can estimate the real parameter $\hat{r}$ under some persistent excitation condition. Indeed, in this case
\[
\frac{d}{dt}(\hat{r} - \hat{r}) = -\gamma (\dot{\hat{r}})^2(\hat{r} - \hat{r}),
\]
which converge to zero exponentially if $\dot{\hat{r}}$ is p.e..

**Remark 1.** Recall Definition 1 on persistent excitation. This means that for the persistently exciting condition to apply, the agent must move in a trajectory that is not confined to a straight line in the 2D space. As referred in [9], "The agent cannot simply head straight towards the target but must execute a richer class of motion. There is a need to reconcile the p.e. requirement with the circumnavigation objective."

However, the implementation of (18) needs the derivative of $D^c$ and $D$ which is not realistic. Thus, for some positive constant $\alpha$ we adopt the state variable filtering and then design the estimator as follows
\begin{align}
\dot{z}_1 &= -\alpha z_1(t) + \frac{1}{2} D^2 \\
\eta(t) &= \dot{z}_1 \\
\dot{z}_2 &= -\alpha z_2(t) + \frac{1}{2} D^2 \\
m(t) &= \dot{z}_2 \\
\dot{z}_3 &= -\alpha z_3(t) + D \\
V(t) &= \dot{z}_3
\end{align}
with initial conditions $z_1(0) = z_2(0) = z_3(0) = 0$. Now together the above dynamics, the estimator for $\hat{r}$ is given as
\[
\dot{\hat{r}} = -\gamma V[\eta - m + V\hat{r}].
\]

Now we need to know $c(t)$ but we only know $D_i(t)$ and $D_i^c(t)$. Thus, we must use again adaptive estimation for the centre $c(t)$ of the target.

Observe that
\[
\frac{d}{dt}D(t)^2 = 2p_i^T(t)(p_i(t) - \bar{c}).
\]
Assume the estimation of $\bar{c}$ is denoted as $\hat{c}$, we have
\[
\frac{1}{2} \left( \frac{d}{dt}D^2(t) - \frac{d}{dt}p_i(t) \right) + \hat{p}_i^T(t)(\bar{c} - \hat{p}_i(t)) = \hat{p}_i(t)(\bar{c} - \hat{c}).
\]
Then the dynamic
\[
\dot{\hat{c}}(t) = -\gamma \dot{p}_i(t) + \frac{1}{2} \left( D^2(t) - \frac{d}{dt} \| p_i(t) \|^2 \right) + \dot{p}_i^T(t) \dot{c}(t)
\]
\[\text{(29)}\]
can estimate the real parameter \( \hat{c} \) under some persistent excitation condition. Indeed, in this case
\[
\frac{d}{dt} \left( \hat{c}(t) - c \right) = -\gamma \dot{p}_i^2(t)(\hat{c}(t) - c),
\]
\[\text{(30)}\]
which converge to zero exponentially if \( \dot{p}_i(t) \) is p.e..

Recall that for the persistently exciting condition to apply, the agent must move in a trajectory that is not confined to a straight line in the 2D space. As referred in [9], "The agent cannot simply head straight towards the target but must execute a richer class of motion. There is a need to reconcile the p.e. requirement with the circumnavigation objective."

In practice, the p.e. condition can be verified by checking whether on each interval of a fixed length, there are at least three time points at which the agent positions are sufficiently removed from any single straight line. [16]

However, the implementation of (18) needs the derivative of \( p_i(t) \) and \( D(t) \) which is not realistic. Therefore we use the previously defined equation (23) for \( D(t) \) and redefine it as \( \eta_2(t) = \dot{z}_2 \) and add the following
\[
\dot{z}_4 = -\alpha z_4(t) + \frac{1}{2} p_i^T(t) p_i(t)
\]
\[\text{(31)}\]
\[
m_2(t) = \dot{z}_4
\]
\[\text{(32)}\]
\[
\dot{z}_5 = -\alpha z_5(t) + p_i(t)
\]
\[\text{(33)}\]
\[
V_2(t) = \dot{z}_5
\]
\[\text{(34)}\]
with initial conditions \( z_4(0) = z_5(0) = 0 \). Now together the above dynamics, the estimator for \( \hat{c} \) is given as
\[
\dot{\hat{c}} = -\gamma V_2 [\eta_2 - m_2 + V_2^T \dot{\hat{c}}].
\]
\[\text{(35)}\]

Now, we want to obtain \( \dot{p}_i(t) \) using the previously measured and estimated variables. The total velocity of each agent comprises of two sub-tasks: approaching the target and circumnavigating it. Therefore we define the direction of each agents towards the centre of the target as the bearing \( \psi_i(t) \),
\[
\psi_i(t) = \frac{c(t) - p_i(t)}{D_i(t)}.
\]
\[\text{(36)}\]

The first sub-task is equivalent to the bearing \( \psi_i(t) \) and the second one is equivalent to its perpendicular, \( E\psi_i(t) \). The weight for the approaching sub-task should be equivalent to how distant the agent is to the boundary of the target and the weight for the circumnavigating sub-task should be equivalent to how distant from the next agent it is. Also, we introduce a constant positive scalar \( \alpha \) that will guarantee a minimum circumnavigation velocity.

Therefore, together with the dynamic of \( \beta_i \) which is set to be standard consensus protocol, the control law for each
agent \( i \) is
\[
\dot{p}_i = \dot{\hat{c}} + ((D_i - \dot{\hat{r}}) - \dot{\hat{r}}) \psi_i + (\alpha + \beta_i) E\psi_i,
\]
\[
\dot{\hat{r}} = \beta_{\hat{r}} - \beta_{\hat{r}1} - \beta_{\hat{r}1},
\]
\[\text{(37)}\]

**Theorem 1.** Suppose \( \dot{p}_i(t) \) and \( D(t) \) are p.e., \( \| \hat{c} \| \leq \varepsilon_1 \), and \( |\hat{r}| \leq \varepsilon_2 \). Consider the closed-loop (37), then there exists \( K_1 \) and \( K_2 \) such that for \( \| \hat{c} \| \leq \varepsilon_1 \) and \( |\hat{r}| \leq \varepsilon_2 \), circumnavigation of the moving circle can be achieved up to a bounded error, i.e.
\[
\lim_{t \to \infty} \| \hat{c} - c \| \leq K_1 \varepsilon_1,
\]
\[\text{(38)}\]
\[
\lim_{t \to \infty} \| p_i - c - r(t) \| \leq K_2 \varepsilon_2.
\]
\[\text{(39)}\]

**Proof.** The proof is divided into three parts.

1) In the first part, we shall show that the angle between the agents will converge to the average consensus for \( n \) agents, \( \beta_i = \frac{2\pi}{n} \).

From the dynamic of \( \beta_i \), we have that
\[
\sum_{i=1}^{n} \dot{\beta}_i(t) = 0.
\]
\[\text{(40)}\]

Therefore, for some constant \( k \) we get
\[
\sum_{i=1}^{n} \beta_i(t) = k.
\]
\[\text{(41)}\]

Then having \( \sum_{i=1}^{n} \beta_i(t) \) initialized as \( 2\pi \), it will remain \( 2\pi \). Now we can use Theorem 1 from [17] to prove the angles \( \beta_i \) converge to the average consensus, i.e., \( \beta_i \rightarrow \frac{2\pi}{n} \).

2) Next, we prove that for a stationary target, all agents reach the boundary of the moving circles asymptotically, \( \lim_{t \to \infty} D_i(t) = r(t) \).

So we consider the function \( W_i(t) := D_i(t) - r(t) \) with time derivative given as
\[
\dot{W}_i = \frac{(c - p_i) \dot{c} - \hat{c} \dot{c}_i}{D_i} - \dot{\hat{r}} = - (D_i - r - \dot{r}) - \dot{\hat{r}} = - (D_i - \dot{r}).
\]

Note that the dynamics of \( W_i \) are in the format \( \dot{W}_i = -W_i \) which is converging exponentially.

3) Finally, we want to prove, for a moving target, that (38) and (39) holds. The proof for boundedness of the center \( c \) (38) can be found on [9], Proposition 7.1. The proof for boundedness of the radius, however, needs to be derived in this paper. We can define the error of the measurement of \( r \) in the following manner:
\[
\tilde{r} = \hat{r} - \hat{r}
\]
\[\text{(42)}\]
rate the uncertainty of the estimations of $x$ where inclusion $B$. Finite-time circumnavigation (37).

which is more robust, by allowing actuator disturbances, than and doesn’t achieve finite time tracking. Therefore, in the is then proved to converge asymptotically. However, it is

for $\dot{x} = \hat{r} - \hat{r} = -\gamma V [\eta - m + V \hat{r}] - \hat{r} = -\gamma V [\eta - m + V \hat{r}] - \hat{r} = -\gamma V^2 \hat{r} - \gamma V [\eta - m + V r] - \hat{r} = -\gamma V^2 \hat{r} + G(t)$

Where $G(t) = -\gamma V [\eta - m + V r] - \hat{r}$. We know that $|G(t)| < k_2 \epsilon$ for some $k, \epsilon > 0$ because $V$ and $\hat{r}$ are bounded and that $|\eta - m + V r| < k \epsilon$ we can prove that for a lyapunov function $W = \frac{1}{2} e^T \hat{r}$ we get

$$W = \dot{W} = \dot{\hat{r}}^T \hat{r} = -\gamma V^2 \hat{r} + \dot{G}(t)$$

$$= -\gamma V^2 \hat{r}^T + \dot{r}^T G(t)$$

$$\leq -\gamma V^2 \hat{r}^T + k_1 \epsilon \| \hat{r} \|$$

Then we get that for $\dot{W} \leq 0$ to hold, $-\gamma V^2 \hat{r}^T + k_1 \epsilon \| \hat{r} \| \leq 0$ must hold. So, we have that $W \geq k_1 \epsilon \| \hat{r} \|$. This solution is then proved to converge asymptotically. However, it is not robust as it doesn’t account for actuator disturbances and doesn’t achieve finite time tracking. Therefore, in the following subsection, we propose one finite-time controller which is more robust, by allowing actuator disturbances, than (37).

B. Finite-time circumnavigation

In this subsection, we suppose that the signal $c(t), r(t)$ are available to the agents. This could be reasonable assumption when finite-time estimators/observers exist [18]. We propose one protocol which achieves finite-time tracking when there are actuator disturbances. The controller is based on proportional-derivative (PD) mechanism. More precisely, the kinematic of $i$th agent is given as

$$\dot{p}_i = \dot{c} + d_{i_1} + (\alpha + \beta_i)E\psi_i + \left[ k_i \text{sign}(D - r) - \hat{r} + d_{i_2}\right] \psi_i$$

for $i \in I$, and

$$\dot{\beta}_i = \text{sign}(\beta_{i+1} - \beta_i), \quad i \in \{1, \ldots, n - 1\}$$

$$\dot{\beta}_n = \text{sign}(\beta_1 - \beta_n)$$

where $k_i$ is a positive gain, the continuous functions $d_{i_1}(t) \in \mathbb{R}^2$ and $d_{i_2}(t) \in \mathbb{R}$ are bounded disturbance which incorporate the uncertainty of the estimations of $x(t)$ and $r(t)$, sign is the sign function.

Since the discontinuity is introduced by the sign function, we understand the solution of (45) and (46) in the sense of Filippov which is the solutions of the following differential inclusion

$$\dot{p}_i \in \dot{c} + d_{i_1} + (\alpha + \beta_i)E\psi_i + \left[ k_i F[\text{sign}](D - r) - \hat{r} + d_{i_2}\right] \psi_i$$

for $i \in I$, and

$$\dot{\beta}_i \in F[\text{sign}](\beta_{i+1} - \beta_i), \quad i \in \{1, \ldots, n - 1\}$$

$$\dot{\beta}_n \in F[\text{sign}](\beta_1 - \beta_n)$$

where we have used Theorem 1 in [19].

Theorem 2. For any positive gain $k_i$, satisfying $k_i > \| d_{i_1} \| + |d_{i_2}(t)|, \quad i \in I$. Finite-time tracking is achieved.

Proof. The proof is divided into two parts.

1) In the first part, we shall show that the angle between the agents will converge to the average consensus in finite time. Notice that the underlying graph is a directed ring, then by Theorem 9 in [20], the Filippov solutions of (49) and (50) are converging to the set

$$0_n \in \times_{i=1}^n \mathcal{F}[\beta_t](-B_{ij} \beta)$$

asymptotically, where $\times$ denotes the Cartesian product and $B$ is the incidence matrix of the directed ring. Furthermore, since

$$I^T \dot{\beta} \in \{0\},$$

we have the angles $\beta_i$ are converging to the average consensus, i.e., $\beta \to \frac{\bar{n}}{n} 1$. Moreover, the finite-time convergence can be seen that for Lyapunov functions $V_1 = \max_i \beta_i$ and $V_2 = \min_i \beta_i$, we have the Lie-derivative $\mathcal{L}V_1(t)$ and $\mathcal{L}V_2(t)$ with respect to (49) and (50) are equal to $-1$ and $1$ for almost all time $t$, respectively. Then the finite-time convergence follows by Proposition 4 in [21].

2) Next we shall prove that all the agent will reach the boundary of the moving circles in finite time if $k_i > \| d_{i_1} \| + |d_{i_2}(t)|$. Consider the function $W_i(t) := D_i(t) - r(t)$ with time derivative given as

$$\mathcal{L} W_i = \frac{(c - p_i)^T(c - p_i) - \dot{r}}{D_i}$$

$$= \frac{(c - p_i)^T(-d_{i_1})}{D_i}$$

$$- k_i F[\text{sign}](D_i - r) - d_{i_2}.$$
V. SIMULATIONS

In this section, we present simulations of both protocols designed in section IV. For the first result we used the derived method for estimation of the target (26) and (35) and the controlling protocol for the agents (37). We also needed to generate a moving target. We did so according to the system of equations (52)

\[
\begin{align*}
\dot{x}(t) &= \alpha_1(t) + 0.5 \\
\dot{y}(t) &= \alpha_2(t) + 0.5 \\
\dot{r}(t) &= \alpha_3(t)
\end{align*}
\]

Being \(\alpha_i(t)\) a random scalar drawn from the standard normal distribution for \(i = 1, 2, 3\). For this generated target we got the following results. We can see the agents circumnavigating the moving target in Fig 1. This gives us a more practical idea of how the agents behave in their target-tracking mission. Through their paths we can infer how the target behaved - varying radius and moving centre. Note that

\[
\alpha_i(t) = \frac{\text{random scalar from standard normal distribution}}{\text{standard normal distribution}}
\]

in Fig 2 we can get a graphical idea of the estimation errors through the comparison of the real and estimated target. Also, in the left bottom picture we can see the distance of each target to the boundary of the target - the perfect tracking would result in a 0 for all agents, for every iteration.

In the second result we used the designed controlling protocol for the agents (48), (49) and (50). Also for this result we needed to generate a moving target. Since we are only representing the target radius in Fig 3 we will also provide only the target radius we generated in equation (53)

\[
\dot{r}(t) = 0.1\sin(t) + 0.2
\]

In Fig 3 we can see the distance of one agent to the varying radius of the target. Note how finite time tracking is guaranteed throughout the movement. First, the agent moves almost in a straight line towards the target’s boundary and then tracks it switching its control signal constantly.

VI. CONCLUSIONS

In this paper, we considered the circumnavigation problem of moving circles with varying radius. Our solution relied on \(n\) agents for circumnavigation but only one for measurements. The measurements taken were the distance of this agent to the boundary of the target as well as its centre. Two protocols were proposed and their stability was proved for given conditions. The first guarantees asymptotic convergence for \(\hat{p}_i(t)\) and \(\hat{D}(t)\) p.e.. The second guarantees finite time tracking having some constraints on the actuator disturbances.

Future work includes the circumnavigation of this same irregular shape but without assuming it can be approximated by a moving circle. Also, having this moving circle, we would like to explore how we can do the circumnavigation with access to only the distance to the target boundary. This might be achieved exploiting the number of agents available and assuming they are all able to measure this distance.

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