Generic canonical form of pairs of matrices with zeros

Tat’yana N. Gaiduk
Department of Physics and Mathematics
Chernigov Pedagogical University, Chernigov, Ukraine

Vladimir V. Sergeichuk∗
Institute of Mathematics
Tereshchenkivska 3, Kiev, Ukraine

Abstract
We consider a family of pairs of $m \times p$ and $m \times q$ matrices, in which some entries are required to be zero and the others are arbitrary, with respect to transformations $(A, B) \mapsto (SAR_1, SBR_2)$ with nonsingular $S$, $R_1$, and $R_2$. We prove that almost all of these pairs reduce to the same pair $(A_0, B_0)$ from this family, except for pairs whose arbitrary entries are zeros of a certain polynomial. The polynomial and the pair $(A_0, B_0)$ are constructed by a combinatorial method based on properties of a certain graph.

AMS classification: 15A21

Keywords: Structured matrices; Parametric matrices; Canonical forms

1 Introduction and main results
Let $\mathcal{A} : U_1 \to V$ and $\mathcal{B} : U_2 \to V$ be linear mappings of vector spaces over an arbitrary field $\mathbb{F}$. Changing the bases of the vector spaces, we may reduce

---

∗Corresponding author. Partially supported by NSF grant DMS-0070503. E-mail address: sergeich@imath.kiev.ua
the matrices $A$ and $B$ of these mappings by transformations

$$(A, B) \mapsto (SAR_1, SBR_2) \quad \text{with nonsingular } S, \ R_1, \ \text{and} \ R_2. \quad (1)$$

A canonical form of $(A, B)$ for these transformations is

$$\begin{pmatrix}
I_r & 0 & 0 \\
0 & I_s & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & I_r & 0 \\
0 & 0 & 0 \\
I_t & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad (2)$$

where $I_r$ denotes the $r$-by-$r$ identity matrix and $r$, $s$, and $t$ are determined by the equalities $r + s = \text{rank } A$, $r + t = \text{rank } B$, and $r + s + t = \text{rank } [A \mid B]$ (see Lemma 3).

We consider a family of pairs $(A, B)$, in which $n$ entries $a_1, \ldots, a_n$ are arbitrary and the others are required to be zero. We prove that there exists a nonzero polynomial $f(x_1, \ldots, x_n)$ such that all pairs $(A, B)$ with $f(a_1, \ldots, a_n) \neq 0$ reduce to the same pair $(A_{\text{gen}}, B_{\text{gen}})$ from this family. The pair $(A_{\text{gen}}, B_{\text{gen}})$ has the form (2) up to permutations of columns and simultaneous permutations of rows in $A$ and $B$. Following [6], we call $(A_{\text{gen}}, B_{\text{gen}})$ a generic canonical form of the family (this notion has no sense if $\mathbb{F}$ is a finite field). We give a combinatorial method of finding $f(x_1, \ldots, x_n)$ and $(A_{\text{gen}}, B_{\text{gen}})$.

### 1.1 Generic canonical form of matrices with zeros

Since the rows of $A$ and $B$ in (1) are transformed by the same matrix $S$, we represent the pair $(A, B)$ by the block matrix $M = [A \mid B]$, which will be called a bipartite matrix. A family of bipartite matrices, in which some entries are zero and the others are arbitrary, may be given by a matrix

$$M(x) = [A(x) \mid B(x)], \quad x = (x_1, \ldots, x_n), \quad (3)$$

whose $n$ entries are unknowns $x_1, \ldots, x_n$ and the others are zero. For instance,

$$M(x) = \begin{bmatrix}
0 & 0 & x_4 & x_7 & 0 \\
x_1 & 0 & x_5 & 0 & 0 \\
0 & x_2 & 0 & 0 & x_9 \\
0 & x_3 & x_6 & x_8 & 0
\end{bmatrix}, \quad (4)$$

2
gives the family \( \{M(a) \mid a \in \mathbb{F}^n\} \).

Considering (3) as a matrix over the field
\[
\mathbb{K} = \left\{ \frac{f(x_1, \ldots, x_n)}{g(x_1, \ldots, x_n)} \mid f, g \in \mathbb{F}[x_1, \ldots, x_n] \text{ and } g \neq 0 \right\}
\]
of rational functions (its elements are quotients of polynomials), we put
\[
r_A = \text{rank}_\mathbb{K} A(x), \quad r_B = \text{rank}_\mathbb{K} B(x), \quad r_M = \text{rank}_\mathbb{K} M(x).
\]

The following theorem is proved in Section 2.

**Theorem 1.** Let \( M(x) = [A(x) \mid B(x)] \) be a matrix whose \( n \) entries are unknowns \( x_1, \ldots, x_n \) and the others are zero. Then there exists a nonzero polynomial
\[
f(x) = \sum c_i x_1^{m_{i1}} \cdots x_n^{m_{in}}
\]
such that all matrices of the family
\[
\mathcal{M}_f = \{M(a) \mid a \in \mathbb{F}^n \text{ and } f(a) \neq 0\}
\]
reduce by transformations \( [A \mid B] \mapsto [SAR_1 \mid SBR_2] \) with nonsingular \( S, R_1, \) and \( R_2 \) to the same matrix
\[
M_{\text{gen}} = [A_{\text{gen}} \mid B_{\text{gen}}] \in \mathcal{M}_f.
\]

Up to a permutation of columns within \( A_{\text{gen}} \) and \( B_{\text{gen}} \) and a permutation of rows, the matrix \( M_{\text{gen}} \) has the form
\[
\begin{bmatrix}
I_r & 0 & 0 & 0 & I_r & 0 \\
0 & I_s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]
which is uniquely determined by \( M(x) \) due to the equalities
\[
r + s = r_A, \quad r + t = r_B, \quad r + s + t = r_M \quad \text{(see (6))}.
\]

We call \( M_{\text{gen}} \) a **generic canonical form** of the family \( \{M(a) \mid a \in \mathbb{F}^n\} \) because \( M(a) \) reduces to \( M_{\text{gen}} \) for all \( a \in \mathbb{F}^n \) except for those in the proper algebraic variety \( \{a \in \mathbb{F}^n \mid f(a) = 0\} \).
1.2 A combinatorial method

The polynomial \( f(x) \) and the matrix \( M_{\text{gen}} \) can be constructed by a combinatorial method: we represent the matrix \( M(x) = [A(x) \mid B(x)] \) by a graph and study its subgraphs. Similar methods were applied in [2, 4, 5, 6] to square matrices up to similarity and to pencils of matrices.

The graph is defined as follows. Its vertices are

\[ 1, \ldots, m, 1^-, \ldots, p^-, 1^+, \ldots, q^+, \]  

where \( m \times p \) and \( m \times q \) are the sizes of \( A(x) \) and \( B(x) \). Its edges

\[ \alpha_1, \ldots, \alpha_n \]  

are determined by the unknowns \( x_1, \ldots, x_n \): if \( x_i \) is the \((i, j)\) entry of \( A(x) \) then \( \alpha_i : i \rightarrow j^- \) (that is, \( \alpha_i \) links the vertices \( i \) and \( j^- \)), and if \( x_i \) is the \((i, j)\) entry of \( B(x) \) then \( \alpha_i : i \rightarrow j^+ \). The edges between \( \{1, \ldots, m\} \) and \( \{1^-, \ldots, p^-\} \) are called left edges, and the edges between \( \{1, \ldots, m\} \) and \( \{1^+, \ldots, q^+\} \) are called right edges.

For example, the matrix (4) is represented by the graph

![Graph representation of matrix](image)

with the left edges \( \alpha_1, \alpha_2, \alpha_3 \) and the right edges \( \alpha_4, \alpha_5, \ldots, \alpha_9 \).

Each subset \( S \) in the set of edges (11) can be given by the characteristic vector

\[ \varepsilon_S = (e_1, \ldots, e_n), \quad e_i = \begin{cases} 1 & \text{if } \alpha_i \in S, \\ 0 & \text{otherwise.} \end{cases} \]

By a matchbox we mean a set of edges (matches) that have no common vertices. The size of a matchbox \( S \) is the number of its matches; since each row and each column of \( M(\varepsilon_S) \) have at most one 1 and the other entries are zero,

\[ \text{size } S = \text{rank } M(\varepsilon_S). \]
A matchbox is left (right) if all its matches are left (right). Such a matchbox is said to be largest if it has the maximal size among all left (right) matchboxes. For example, the subgraph

\[ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\bullet & \bullet & \bullet & \bullet \\
1^- & 2^- & 1^+ & 2^+ & 3^+ \\
\end{array} \]

of (12) is formed by the largest left and right matchboxes
\[ A = \{2\rightarrow 1^-, \ 3\rightarrow 2^-\} \quad \text{and} \quad B = \{1\rightarrow 2^+, \ 2\rightarrow 1^+, \ 3\rightarrow 3^+\}. \quad (14) \]

For a left matchbox \( A \) and a right matchbox \( B \), we denote by \( A \uplus B \) the matchbox obtained from \( A \cup B \) by removing all matches of \( B \) that have common vertices with matches of \( A \). For example,
\[ A \uplus B = \{2\rightarrow 1^-, \ 3\rightarrow 2^-, \ 1\rightarrow 2^+\} \quad (15) \]

for the matchboxes (14).

For every matchbox
\[ \mathcal{S} = \{i_1\rightarrow j_1^-; \ldots; \ i_{\alpha}\rightarrow j_\alpha^-; \ i_{\alpha+1}\rightarrow k_{1}^+; \ldots; \ i_{\alpha+\beta}\rightarrow k_{\beta}^+\}, \]
we denote by \( \mu_{\mathcal{S}}(x) \) the minor of order \( \alpha + \beta \) in \( M(x) = [A(x) \mid B(x)] \) whose matrix belongs to the rows numbered \( i_1, \ldots, i_{\alpha+\beta} \), to the columns of \( A(x) \) numbered \( j_1, \ldots, j_\alpha \), and to the columns of \( B(x) \) numbered \( k_1, \ldots, k_\beta \). For example, the matchbox (15) determines the minor
\[ \mu_{A \uplus B}(x) = \begin{vmatrix} 0 & 0 & x_7 \\ x_1 & 0 & 0 \\ 0 & x_2 & 0 \end{vmatrix} = x_1x_2x_7 \quad \text{in (14)}. \]

The next theorem will be proved in Section 2.

**Theorem 2.** The generic canonical form \( M_{\text{gen}} \) and the polynomial \( f(x) \) from Theorem 1 may be constructed as follows. We represent \( M(x) \) by the graph.
Among pairs consisting of a largest left matchbox and a largest right matchbox, we choose a pair \((A, B)\) with the minimal number \(v(A, B)\) of common vertices, and take

\[
M_{\text{gen}} = M(\varepsilon_{A \cup B}), \quad f(x) = f_{AB}(x),
\]

where \(f_{AB}(x)\) is the lowest common multiple of \(\mu_A(x)\), \(\mu_B(x)\), and \(\mu_{A \cap B}(x)\):

\[
f_{AB}(x) = \text{LCM}\{\mu_A(x), \mu_B(x), \mu_{A \cap B}(x)\}.
\]

Up to permutations of columns within \(A_{\text{gen}}\) and \(B_{\text{gen}}\) and a permutation of rows, the matrix \(M(\varepsilon_{A \cup B})\) has the form (9) with

\[
r = v(A, B), \quad s = \text{size } A - r, \quad \text{and} \quad t = \text{size } B - r.
\]

1.3 An example

Let us apply Theorems 1 and 2 to the family given by the matrix (4) with the graph (12). The matchboxes (14) do not satisfy the condition \(s\) of Theorem 2 because they have two common vertices ‘2’ and ‘3’. This number is not minimal since the largest matchboxes

\[
A = \{2-1^-, 3-2^-\}, \quad B = \{1-1^+, 3-3^+, 4-2^+\}
\]

forming the graph

have a single common vertex ‘3’. The matchboxes (19) satisfy the conditions of Theorem 2 since there is no pair of largest matchboxes without common vertices.

The conditions of Theorem 2 also hold for the largest matchboxes

\[
A' = \{2-1^-, 4-2^-\}, \quad B' = \{1-2^+, 2-1^+, 3-3^+\}
\]
forming the graph

since they have a single common vertex too.

For these pairs of matchboxes, we have

\[
\mathcal{A} \cup \mathcal{B} = \{2-1^-, 3-2^-, 1-1^+, 4-2^+\},
\]

\[
f_{\mathcal{A}\mathcal{B}}(x) = \text{LCM}\{x_1x_2, x_9(x_6x_7 - x_4x_8), x_1x_2(x_4x_8 - x_6x_7)\}
\]

and

\[
\mathcal{A}' \cup \mathcal{B}' = \{2-1^-, 4-2^-, 1-2^+, 3-3^+\},
\]

\[
f_{\mathcal{A}'\mathcal{B}'}(x) = \text{LCM}\{x_1x_3, -x_5x_7x_9, -x_1x_3x_7x_9\}.
\]

By Theorems 1 and 2,

\[
\begin{bmatrix}
0 & 0 & a_4 & a_7 & 0 \\
a_1 & 0 & a_5 & 0 & 0 \\
0 & a_2 & 0 & 0 & a_9 \\
0 & a_3 & a_6 & a_8 & 0
\end{bmatrix}
\]

with \(a_1, \ldots, a_9 \in \mathbb{F}\)

(see (4)) reduces to the matrix

\[
M(\varepsilon_{A\cup B}) = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

if \(f_{\mathcal{A}\mathcal{B}}(a) = a_1a_2a_9(a_4a_8 - a_6a_7) \neq 0\)

and to the matrix

\[
M(\varepsilon_{A'\cup B'}) = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

if \(f_{\mathcal{A}'\mathcal{B}'}(a) = a_1a_3a_5a_7a_9 \neq 0\).
Up to permutations of columns within vertical strips and permutations of rows, these matrices have the form

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\] (see (9)).

2 Proof of Theorems 1 and 2

2.1 Bipartite matrices

The canonical form of a pair for transformations (1) is well known, see [1, Sect. 1.2]. We recall it since we will use it in the proof of Theorems 1 and 2.

Clearly, \((A, B)\) reduces to \((A', B')\) by transformations (1) if and only if \([A | B]\) reduces to \([A' | B']\) by a sequence of

(i) elementary row-transformations in \([A | B]\),

(ii) elementary column-transformations in \(A\), and

(iii) elementary column-transformations in \(B\).

Lemma 3. Every bipartite matrix \(M = [A | B]\) over a field \(F\) reduces by transformations (i)–(iii) to the form

\[
\begin{bmatrix}
I_r & 0 & 0 & I_r & 0 \\
0 & I_s & 0 & 0 & 0 \\
0 & 0 & I_t & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\] (20)

determined by the equalities

\[
\begin{align*}
r + s &= \text{rank} A,
\quad r + t &= \text{rank} B,
\quad r + s + t &= \text{rank} M.
\end{align*}
\] (21)

Proof. By transformations (i) and (ii), we reduce \(M\) to the form

\[
\begin{bmatrix}
I_h & 0 & B_1 \\
0 & 0 & B_2
\end{bmatrix},
\]
and then by elementary row-transformations within the second horizontal strip and by transformations (iii) to the form

\[
\begin{bmatrix}
I_h & 0 & B_3 & B_4 \\
0 & 0 & I_t & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Adding linear combinations of rows of \(I_t\) to rows of \(B_3\) by transformations (i), we “kill” all non-zero entries of \(B_3\):

\[
\begin{bmatrix}
I_h & 0 & 0 & B_4 \\
0 & 0 & I_t & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

At last, we reduce \(B_4\) to \(I_r \oplus 0\) by elementary transformations. The row-transformations with \(B_4\) have “spoiled” the block \(I_h\), but we restore it by the inverse column-transformations (ii) and obtain the matrix (20) with \(r + s = h\).

Since the transformations (i)–(iii) with \(M = [A \mid B]\) preserve the ranks of \(M\), \(A\), and \(B\), we have the equalities (21). This implies the uniqueness of (20) since \(s = \text{rank} M - \text{rank} B\), \(t = \text{rank} M - \text{rank} A\), and \(r = \text{rank} A + \text{rank} B - \text{rank} M\).

\[\square\]

### 2.2 Reduction of bipartite matrices by permutations of rows and columns

In this section we consider a bipartite matrix \(M = [A \mid B]\) with respect to permutations of rows and columns.

**Lemma 4.** Every bipartite matrix \([A \mid B]\) with linearly independent columns reduces by a permutation of rows to the form

\[
\begin{bmatrix}
A' & \cdot \\
\cdot & B'
\end{bmatrix},
\]

where \(A'\) and \(B'\) are nonsingular square blocks and the points denote unspecified blocks.

**Proof.** By permutations of rows we reduce \([A \mid B]\) to the form

\[
\begin{bmatrix}
A_1 & B_1 \\
\cdot & \cdot
\end{bmatrix}
\]

9
with a nonsingular square matrix \([A_1 \mid B_1]\). Laplace’s theorem (see [3, The-orem 2.4.1]) states that the determinant of \([A_1 \mid B_1]\) is equal to the sum of
products of the minors whose matrices belong to the rows of \(A_1\) by their co-
factors (belonging to \(B_1\)). One of these summands is nonzero since \([A_1 \mid B_1]\)
is nonsingular. We collect the rows of the minor from this summand at the
top and obtain the matrix (22).

**Lemma 5.** Every bipartite matrix \([A \mid B]\) reduces by permutations of rows
and permutations of columns in \(A\) and \(B\) to the form

\[
\begin{bmatrix}
X_r & \cdots & Y_r \\
\cdot & Z_s & \cdots \\
\cdot & \cdot & T_t \\
\cdot & \cdot & \cdot
\end{bmatrix},
\]

where \(X_r, Y_r, Z_s,\) and \(T_t\) are nonsingular \(r \times r, r \times r, s \times s,\) and \(t \times t\) blocks
in which all diagonal entries are nonzero and

\[r + s = \text{rank} A, \quad r + t = \text{rank} B, \quad r + s + t = \text{rank} [A \mid B].\] (24)

**Proof.** Denote

\[\rho_A = \text{rank} A, \quad \rho_B = \text{rank} B, \quad \rho_M = \text{rank} [A \mid B].\]

We first reduce \([A \mid B]\) by a permutation of columns to the form \([\cdots A_1 \mid B]\),
where \(A_1\) has \(\rho_A\) columns and they are linearly independent. Then we reduce
it to the form \([\cdots A_1 \mid B_1 \cdots]\), where \([A_1 \mid B_1]\) has \(\rho_M\) columns and they are
linearly independent.

Lemma 4 to \([A_1 \mid B_1]\) ensures that the matrix \([\cdots A_1 \mid B_1 \cdots]\) reduces by
a permutation of rows to the form

\[
\begin{bmatrix}
\cdot & A_2 \\
\cdot & \cdots \\
\cdot & B_2 \\
\cdot & \cdots \\
\cdot & \cdots
\end{bmatrix}
\]

with nonsingular square matrices \(A_2\) and \(B_2\).

Rearranging rows of the first strip and breaking it into two substrips, we reduce (25) to the form

\[
\begin{bmatrix}
\cdot & A_3 \\
\cdot & A_4 \\
\cdot & B_3 \\
\cdot & \cdots \\
\cdot & B_2 \\
\cdot & \cdots
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cdot & \cdots \\
\cdot & \cdots \\
\cdot & \cdots
\end{bmatrix}
\]

(26)
where the matrices

$$
\begin{bmatrix}
A_3 \\
A_4
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
B_3 \\
B_2
\end{bmatrix}
$$

have linearly independent rows. Lemma 4 to their transposes insures that (26) reduces by permutations of columns to the form

$$
\rho_A \left\{ \begin{bmatrix}
\cdot & Z & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & X & Y \\
\cdot & \cdot & \cdot & \cdot & T
\end{bmatrix} \right\} \rho_B \left\{ \begin{bmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{bmatrix} \right\} \rho_M
$$

(27)

with nonsingular $X$, $Y$, $Z$, and $T$. If an $n$-by-$n$ matrix has a nonzero determinant, then one of its $n!$ summands is nonzero, and we may dispose the entries of this summand along the main diagonal by a permutation of columns. In this manner we make nonzero the diagonal entries of $X$, $Y$, $Z$, and $T$. At last, we reduce (27) to the form (23) by permutations of rows and columns. \(\square\)

### 2.3 Proof of Theorems 1 and 2

In this section $M(x) = [A(x) | B(x)]$ is the matrix (3), $A$ and $B$ are the matchboxes from Theorem 2, and $r_A$, $r_B$, $r_M$ are the numbers (6).

**Lemma 6.**

$$
\text{size } A = r_A, \quad \text{size } B = r_B, \quad \text{size } A \cup B = r_M.
$$

(28)

**Proof.** By Lemma 5, the matrix $M(x)$ over the field $\mathbb{K}$ of rational functions (3) reduces by permutations of rows and by permutations of columns within $A(x)$ and $B(x)$ to a matrix $N(x)$ of the form (23), in which by (24)

$$
r + s = r_A, \quad r + t = r_B, \quad r + s + t = r_M.
$$

(29)

The diagonal entries of $X_r$, $Y_r$, $Z_s$, and $T_t$ are all nonzero, and hence they are independent unknowns; replacing them by 1 and the other unknowns by 0, we obtain the matrix

$$
N(a) = \begin{bmatrix}
I_r & 0 & 0 & 0 & I_r & 0 \\
0 & I_s & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_t & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad a \in \{0, 1\}^n.
$$

(30)
The inverse permutations of rows and columns reduce $N(x)$ to $M(x)$, and hence $N(a)$ to $M(a)$. As follows from (30), $a = \varepsilon_{A'\cup B'}$, where $A'$ is a left matchbox, $B'$ is a right matchbox, and by (29)

$$\text{size } A' = r_A, \quad \text{size } B' = r_B, \quad \text{size } A' \cup B' = r_M.$$  

Since the matchboxes $A$ and $B$ are largest, $\text{size } A \geq r_A$ and $\text{size } B \geq r_B$. The minors $\mu_A(x)$ of $A(x)$ and $\mu_B(x)$ of $B(x)$ (defined in Section 1.2) are nonzero and their orders are equal to the sizes of $A$ and $B$, hence $\text{size } A \leq r_A$ and $\text{size } B \leq r_B$. We have

$$\text{size } A = \text{size } A' = r_A, \quad \text{size } B = \text{size } B' = r_B,$$

and so the matchboxes $A'$ and $B'$ are largest too. Because of the minimality of the number $v(A, B)$ of common vertices and since

$$\text{size } A \cup B = \text{size } A + \text{size } B - v(A, B),$$

we have

$$v(A, B) \leq v(A', B'), \quad \text{size } A \cup B \geq \text{size } A' \cup B' = r_M.$$  

In actuality the last inequality is an equality since the minor $\mu_{A\cup B}(x)$ of order $r_M$ is nonzero. \qed

**Lemma 7.** If $a \in \mathbb{F}^n$ and $f_{AB}(a) \neq 0$, then

$$\text{rank } A(a) = r_A, \quad \text{rank } B(a) = r_B, \quad \text{rank } M(a) = r_M.$$  

**Proof.** The matrix $A(a)$ has a nonzero minor $h(a)$, whose order is equal to the rank of $A(a)$. The corresponding minor $h(x)$ of $A(x)$ (belonging to the same rows and columns) is a nonzero polynomial, and so $\text{rank } A(a) \leq \text{rank } A(x) = r_A$. Analogously $\text{rank } B(a) \leq r_B$ and $\text{rank } M(a) \leq r_M$.

By (17), the minors $\mu_A(a)$ of $A(a)$, $\mu_B(a)$ of $B(a)$, and $\mu_{AB}(a)$ of $M(a)$ are nonzero. Their orders are equal to the sizes of $A$, $B$, and $A \cup B$, hence

$$\text{rank } A(a) \geq \text{size } A, \quad \text{rank } B(a) \geq \text{size } B, \quad \text{rank } M(a) \geq \text{size } A \cup B.$$  

This proves (32) due to (28). \qed
Let \( a \in \mathbb{F}^n \) and \( f_{AB}(a) \neq 0 \). By Lemma 3, \( M(a) \) reduces to the matrix (9), which is determined by (10) due to (21) and (32). The matrix \( M(\varepsilon_{A\cup B}) \) reduces by permutations of rows and columns to the same matrix (9) because (13) and (28) imply

\[
\begin{align*}
\text{rank} \, A(\varepsilon_{A\cup B}) &= \text{size} \, A = r_A, \\
\text{rank} \, B(\varepsilon_{A\cup B}) &= \text{size} \, B = r_B, \\
\text{rank} \, M(\varepsilon_{A\cup B}) &= \text{rank} \, M(\varepsilon_{A\cap B}) = \text{size} \, A \cup B = r_M.
\end{align*}
\]

Hence \( M(a) \) reduces to \( M(\varepsilon_{A\cup B}) \). This proves Theorem 1; we can take \( M_{\text{gen}} \) and \( f(x) \) as indicated in (16). This also proves Theorem 2; the equalities (18) follow from (33), (34), and (31).

Acknowledgements. Sergey V. Savchenko read the paper and made very important improvements and corrections. In fact, he is a coauthor.

References

[1] P. Gabriel and A.V. Roiter, *Representations of Finite-Dimensional Algebras*, Springer-Verlag, 1997.

[2] D. Hershkowitz, The relation between the Jordan structure of a matrix and its graph, *Linear Algebra Appl.* 184 (1993) 55–69.

[3] V.V. Prasolov, *Problems and Theorems in Linear Algebra*, Translations of mathematical monographs, v. 134, Amer. Math. Soc., 1996.

[4] K. Röbenack and K.J. Reinschke, Graph-theoretically determined Jordan-block structure of regular matrix pencils, *Linear Algebra Appl.* 263 (1997) 333–348.

[5] K. Röbenack and K.J. Reinschke, Digraph-based determination of Jordan block size structure of singular matrix pencils, *Linear Algebra Appl.* 275–276 (1998) 495–507.

[6] J.W. van der Woude, The generic canonical form of a regular structured matrix pencil, *Linear Algebra Appl.* 353 (2002) 267–288.