Decidability of the Monadic Shallow Linear
First-Order Fragment with Straight Dismatching
Constraints

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Abstract. The monadic shallow linear Horn fragment is well-known to
be decidable and has many application, e.g., in security protocol anal-
ysis, tree automata, or abstraction refinement. It was a long standing
open problem how to extend the fragment to the non-Horn case, pre-
serving decidability, that would, e.g., enable to express non-determinism
in protocols. We prove decidability of the non-Horn monadic shallow lin-
er fragment via ordered resolution further extended with dismatching
constraints and discuss some applications of the new decidable fragment.

1 Introduction

Motivated by the automatic analysis of security protocols, the monadic shallow
linear Horn (MSLH) fragment was shown to be decidable in \cite{16}. In addition to
the restriction to monadic Horn clauses, the main restriction of the fragment is
positive literals of the form $S(f(x_1, \ldots, x_n))$ or $S(x)$ where all $x_i$ are different,
i.e., shallow linear terms. The fragment can be finitely saturated by superpo-
sition (ordered resolution) where negative literals with non-variable arguments
are always selected. As a result, productive clauses with respect to the superpo-
sition model operator $I_N$ have the form $S_1(x_1), \ldots, S_n(x_n) \rightarrow S(f(x_1, \ldots, x_n))$.
Therefore, the models of saturated MSLH clause sets can both be represented by
tree automata \cite{4} and shallow linear sort theories \cite{6}. The models are typically
infinite. The decidability result of MSLH clauses was rediscovered in the context
of tree automata research \cite{5} where in addition DEXPTIME-completeness of the
MSLH fragment was shown. The fragment was further extended by disequality
constraints \cite{9,10} still motivated by security protocol analysis \cite{11}. Although from
a complexity point of view, the difference between Horn clause fragments and
the respective non-Horn clause fragments is typically reflected by membership
in the deterministic vs. the non-deterministic respective complexity fragment,
for monadic shallow linear clauses so far there was not even a decidability result
for the non-Horn case.

The results of this paper close this gap. We show the monadic shallow linear
non-Horn (MSL) clause fragment to be decidable by superposition (ordered res-
olution). From a security protocol application point of view, non-Horn clauses
enable a natural representation of non-determinism. Our second extension to the
fragment are unit clauses with disequations of the form $s \not\approx t$, where $s$ and $t$ are not unifiable. Due to the employed superposition calculus, such disequations do not influence saturation of an MSL clause set, but have an effect on potential models. They can rule out identification of syntactically different ground terms as it is, e.g., desired in the security protocol context for syntactically different messages or nonces. Our third extension to the fragment are straight dismatching constraints. These constraints are incomparable to the disequity constraints mentioned above \[9,10\]. They do not strictly increase the expressiveness of the MSL theory, but enable up to exponentially more compact saturations. For example, the constrained clause $(S(x), T(y) \Rightarrow S(f(x, y)); y \neq f(x', f(a, y'))) \Rightarrow S(x), T(a) \Rightarrow S(f(x, a)); S(x), T(b) \Rightarrow S(f(x, b)); \ldots; S(x), T(f(b, y')) \Rightarrow S(f(x, f(b, y'))); S(x), T(f(f(x', y''), y'))) \Rightarrow S(f(x, f(f(x'', y''), y'))) \Rightarrow S(f(x, f(x'', y'', y'))) \Rightarrow S(f(x, f(x'', y'', y')), y')) \Rightarrow S(f(x, f(x'', y'', y')))$ where the nested terms in the positive literals have to be factored out by the introduction of further predicates to be eventually mapped to satisfiability equivalent MSL clauses. For example, the first clause is replaced by the two MSL clauses $S(x), T(a), R(y) \Rightarrow S(f(x, y))$ and $R(a)$ where $R$ is a fresh monadic predicate. Altogether, the resulting MSL(SDC) fragment is shown to be decidable in Section \[3\].

The introduction of straight dismatching constraints (SDCs) enables an improved refinement step of our approximation refinement calculus \[13\]. Before, several clauses were needed to rule out a specific instance of a clause in an unsatisfiable core. For example, if due to a linearity approximation from clause $S(x), T(x) \Rightarrow S(f(x, x))$ to $S(x), T(x), S(y), T(y) \Rightarrow S(f(x, y))$ an instance \{ $x$ $\Rightarrow$ $f(a, x')$, $y$ $\Rightarrow$ $f(b, y')$ \} is used in the proof, before \[13\] several clauses were needed to replace $S(x), T(x) \Rightarrow S(f(x, x))$ in a refinement step in order to rule out this instance. With straight dismatching constraints the clause $S(x), T(x) \Rightarrow S(f(x, x))$ is replaced by the two clauses $S(f(a, x), T(f(a, x)) \Rightarrow S(f(f(a, x), f(a, x)))$ and $(S(x), T(x) \Rightarrow S(f(x, x)); x \neq f(a, y))$. For the improved approximation refinement approach (FO-AR) presented in this paper, any refinement step results in just two clauses, a modified existing one and an additional clause, in general, see Section \[3\]. The additional expressiveness of constraint clauses comes almost for free, because necessary computations, like, e.g., checking emptiness of SDCs, can all be done in polynomial time, see Section \[2\].

In addition to the extension of the known MSLH decidability result and the improved approximation refinement calculus FO-AR, we discuss in Section \[5\] the potential of the MSL(SDC) fragment in the context of FO-AR. Theorem \[2\] and its prototypical implementation in SPASS-AR \(\text{http://www.mpi-inf.mpg.de/fileadmin/inf/rgi/spass-ar}\). It turns out that for clause sets containing certain structures, FO-AR is superior to ordered resolution/superposition and instance generating methods. The paper ends with a discussion on challenges and future research directions, Section \[6\]. In addition to our workshop paper \[13\] this paper contains the decidability results for the MSL(SDC) fragment together with an improved refinement of \[13\] and experiments based on the above mentioned implementation.
2 First-Order Clauses with Straight Dismatching Constraints: MSL(SDC)

We consider a standard first-order language where letters \(v, w, x, y, z\) denote variables, \(f, g, h\) functions, \(a, b, c\) constants, \(s, t\) terms, \(p, q, r\) positions and Greek letters \(\sigma, \tau, \rho, \delta\) are used for substitutions. \(S, P, Q, R\) denote predicates, \(\approx\) denotes equality, \(A, B\) atoms, \(E, L\) literals, \(C, D\) clauses, \(N\) clause sets and \(V\) sets of variables. \(\mathcal{T}\) is the complement of \(L\). The signature \(\Sigma = (\mathcal{F}, \mathcal{P})\) consists of two disjoint, non-empty, in general infinite sets of function and predicate symbols \(\mathcal{F}\) and \(\mathcal{P}\), respectively. The set of all \emph{terms} over variables \(V\) is \(\mathcal{T}(\mathcal{F}, V)\). If there are no variables, then terms, literals and clauses are called \emph{ground}, respectively. A \emph{substitution} \(\sigma\) is denoted by pairs \(\{x \mapsto t\}\) and its update at \(x\) by \(\sigma[x \mapsto t]\). A substitution \(\sigma\) is a \emph{grounding} substitution for \(V\) if \(x\sigma\) is ground for every variable \(x \in V\).

The set of \emph{free} variables of an atom \(A\) (term \(t\)) denoted by \(\text{vars}(A)\) (\(\text{vars}(t)\)). A \emph{position} is a sequence of positive integers, where \(\varepsilon\) denotes the empty position. As usual \(t[p]_s\) denotes the term at position \(p\) in \(t\) and \(t[s]_p\) the replacement of the term in \(t\) at position \(p\) with \(s\) where \(p \in \text{pos}(t)\). These notions are extended to literals and multiple positions.

A predicate with exactly one argument is called \emph{monadic}. A term is \emph{complex} if it is not a variable and \emph{shallow} if it has at most depth one. It is called \emph{linear} if there are no duplicate variable occurrences. A literal, where every argument term is shallow, is also called \emph{shallow}. A term \(f(s_1, \ldots, s_n)\) is called \emph{straight}, if \(f(s_1, \ldots, s_n)\) is linear and all arguments are variables except for at most one straight argument term \(s_i\).

A \emph{clause} is a multiset of literals which we write as an implication \(\Gamma \rightarrow \Delta\) where the atoms in the multiset \(\Delta\) (the \emph{succeent}) denote the positive literals and the atoms in the multiset \(\Gamma\) (the \emph{antecedent}) the negative literals. We write \(\Box\) for the empty clause. If \(\Gamma\) is empty we omit \(\rightarrow\), e.g., we can write \(P(x)\) as an alternative of \(\rightarrow P(x)\) whereas if \(\Delta\) is empty \(\rightarrow\) is always shown. We abbreviate disjoint set union with sequencing, for example, we write \(\Gamma, \Gamma' \rightarrow \Delta, L\) instead of \(\Gamma \cup \Gamma' \rightarrow \Delta \cup \{L\}\). A clause \(E, E, \Gamma \rightarrow \Delta\) is equivalent to \(E, \Gamma \rightarrow \Delta\) and we call them equal \emph{modulo duplicate literal elimination}. If every term in \(\Delta\) is shallow, the clause is called \emph{positive shallow}. If all atoms in \(\Delta\) are linear and variable disjoint, the clause is called \emph{positive linear}. A clause \(\Gamma \rightarrow \Delta\) is called an \emph{MSL clause}, if it is (i) positive shallow and linear, (ii) all occurring predicates are monadic, (iii) no equations occur in \(\Delta\), and (iv) no equations occur in \(\Gamma\) or \(\Gamma = \{s \approx t\}\) and \(\Delta\) is empty where \(s\) and \(t\) are not unifiable. The first-order fragment consisting of MSL clauses we call \emph{MSL}. Note that adding clauses \(\Gamma, s \approx t \rightarrow \Delta\) where \(\Gamma, \Delta\) are non-empty and \(s, t\) are not unifiable does not increase the expressiveness of the MSL fragment. If \(s \approx t\) is selected then there is no possible inference with this clause and it does not contribute to the model (see Section 3).

An \emph{atom ordering} \(\prec\) is an irreflexive, well-founded, total ordering on ground atoms. Any atom ordering \(\prec\) is lifted to literals by representing \(\boxtimes\) and \(\neg\) as multisets \(\{A\}\) and \(\{A, \neg A\}\), respectively. The multiset extension of the literal ordering induces an ordering on ground clauses. The clause ordering is compatible
with the atom ordering; if the maximal atom in \( C \) is greater than the maximal atom in \( D \) then \( D \prec C \). We use \( \prec \) simultaneously to denote an atom ordering and its multiset, literal, and clause extensions. For a ground clause set \( N \) and clause \( C \), the set \( N^\prec C = \{ D \in N \mid D \prec C \} \) denotes the clauses of \( N \) smaller than \( C \).

A Herbrand interpretation \( I \) is a possibly infinite set of ground atoms. A ground atom \( A \) is called true in \( I \) if \( A \in I \) and false, otherwise. \( I \) is said to satisfy a ground clause \( C = \Gamma \rightarrow \Delta \), denoted by \( I \models C \), if \( \Delta \cap I \neq \emptyset \) or \( \Gamma \nsubseteq I \). A non-ground clause \( C \) is satisfied by \( I \) if \( I \models C \sigma \) for every grounding substitution \( \sigma \). An interpretation \( I \) is called a model of \( N \), \( I \models N \), if \( I \models C \) for every \( C \in N \).

A model \( I \) of \( N \) is considered minimal with respect to set inclusion, i.e., if there is no model \( I' \) with \( I' \subset I \) and \( I' \models N \). A set of clauses \( N \) is satisfiable, if there exists a model that satisfies \( N \). Otherwise, the set is unsatisfiable.

A disequation \( t \neq s \) is an atomic straight dismatching constraint if \( s \) and \( t \) are variable disjoint terms and \( s \) is straight. A straight dismatching constraint \( \pi \) is a conjunction of atomic constraints. Given a substitution \( \sigma \), \( \pi \sigma = \bigwedge_{i \in I} t_i \sigma \neq s_i \), \( \text{ivar}(\pi) = \bigcup_{i \in I} \text{vars}(t_i) \) are the left-hand variables of \( \pi \) and the depth of \( \pi \) is the maximal term depth of the \( s_i \). A solution of \( \pi \) is a grounding substitution \( \delta \) such that for all \( i \in I \), \( t_i \delta \) is not an instance of \( s_i \). A dismatching constraint is solvable if it has a solution and unsolvable, otherwise. Whether a straight dismatching constraint is solvable is decidable in linear-logarithmic time [14]. We further define \( \top \) and \( \bot \) as dismatching constraints where all grounding substitutions are (not) a solution, respectively.

We define constraint normalization \( \pi \downarrow \) as the normal form of the following rewriting rules over constraints.

\[
\begin{align*}
\pi \land f(s_1, \ldots, s_n) \neq y & \Rightarrow \bot \\
\pi \land f(s_1, \ldots, s_n) \neq f(y_1, \ldots, y_n) & \Rightarrow \bot \\
\pi \land f(s_1, \ldots, s_n) \neq f(t_1, \ldots, t_n) & \Rightarrow \pi \land s_i \neq t_i \quad \text{if} \ t_i \text{ is complex} \\
\pi \land f(s_1, \ldots, s_n) \neq g(t_1, \ldots, t_m) & \Rightarrow \pi \\
\pi \land x \neq t \land x \neq \pi \sigma & \Rightarrow \pi \land x \neq t
\end{align*}
\]

Note that the depth of \( \pi \downarrow \) is less or equal to the depth of \( \pi \) and that both have the same solutions.

A pair of a clause and a constraint \((C; \pi)\) is called a constrained clause. Given a substitution \( \sigma \), \((C; \pi)\sigma = (C\sigma; \pi\sigma) \). \( C\delta \) is called a ground clause of \((C; \pi)\) if \( \delta \) is a solution of \( \pi \). \( G((C; \pi)) \) is the set of ground instances of \((C; \pi)\). If \( G((C; \pi)) \subseteq G((C'; \pi')) \), then \((C; \pi)\) is an instance of \((C'; \pi')\). If \( G((C; \pi)) = G((C'; \pi')) \), then \((C; \pi)\) and \((C'; \pi')\) are called variants. A Herbrand interpretation \( I \) satisfies \((C; \pi)\), if \( I \models G((C; \pi)) \). A constrained clause \((C; \pi)\) is called redundant in \( N \) if for every \( D \in G((C; \pi)) \), there exist \( D_1, \ldots, D_n \) in \( G(N)^\prec D \) such that \( D_1, \ldots, D_n \vdash D \). A constrained clause \((C'; \pi')\) is called a condensation of \((C; \pi)\) if \( C' \subset C \) and there exists a substitution \( \sigma \) such that, \( \pi\sigma = \pi' \), \( \pi' \subseteq \pi \), and for all \( L \in C \) there is an \( L' \in C' \) with \( L\sigma = L' \).

An MSL clause with straight dismatching constraints is called an MSL(SDC) clause with MSL(SDC) being the respective first-order fragment. Note that any
clause set \( N \) can be transformed into an equivalent constrained clause set by changing each \( C \in N \) to \((C; \top)\).

### 3 Decidability of the MSL(SDC) fragment

In the following we will show that the satisfiability of the MSL(SDC) fragment is decidable. For this purpose we will define ordered resolution with selection on constrained clauses [14] and show that with an appropriate ordering and selection function, saturation of an MSL(SDC) clause set terminates.

For the rest of this section we assume an atom ordering \( \prec \) such that a literal \( \lnot Q(s) \) is not greater than a literal \( P(t[s]_p) \), where \( p \neq \varepsilon \). For example, an LPO with a precedence where functions are larger than predicates or a KBO where all symbols have weight one have this property.

**Definition 1** (sel). Given an MSL(SDC) clause \((C; \pi) = (S_1(t_1), \ldots, S_n(t_n) \rightarrow P_1(s_1), \ldots, P_m(s_m); \pi)\). The Superposition Selection function sel is defined by \( S_i(t_i) \in \text{sel}(C) \) if (1) \( t_i \) is not a variable or (2) \( t_1, \ldots, t_n \) are variables and \( t_i \notin \text{vars}(s_1, \ldots, s_m) \) or (3) \( \{t_1, \ldots, t_n\} \subseteq \text{vars}(s_1, \ldots, s_m) \) and for some \( 1 \leq j \leq m \), \( s_j = t_i \).

The selection function sel (Definition 1) ensures that a clause \( \Gamma \rightarrow \Delta \) can only be resolved on a positive literal if \( \Gamma \) contains only variables, which also appear in \( \Delta \) at a non-top position. For example, in the clause \( P(f(x)), P(x), Q(z) \rightarrow Q(x), R(f(y)) \) sel selects \( P(f(x)) \) because \( f(x) \) is not a variable. In \( P(x), Q(z) \rightarrow Q(x), R(f(y)) \) sel selects \( Q(z) \) because rule (1) doesn’t apply and \( z \notin \{x, y\} \). In \( P(x), Q(y) \rightarrow Q(x), R(f(y)) \) sel selects \( P(x) \) because rules (1) and (2) don’t apply and the argument term of \( P(x) \) is also the argument of \( Q(x) \). Then in \( P(x), Q(y) \rightarrow Q(f(x)), R(f(y)) \) nothing is selected as no rule applies and furthermore note that \( P(x) \) and \( Q(y) \) are not maximal. Note that given an MSL(SDC) clause \((C; \pi) = (S_1(t_1), \ldots, S_n(t_n) \rightarrow P_1(s_1), \ldots, P_m(s_m); \pi)\), if some \( S_i(t_i) \) is maximal in \( C \), then at least one literal is selected.

**Definition 2.** A literal \( A \) is called [strictly] maximal in a constrained clause \((C \lor A; \pi)\) if and only if there exists a solution \( \delta \) of \( \pi \) such that for all literals \( B \) in \( C \), \( B\delta \preceq A\delta \) [\( B\delta \prec A\delta \)].

**Definition 3** (SDC-Resolution).

\[
\frac{(\Gamma_1 \rightarrow \Delta_1, A; \pi_1) \quad (\Gamma_2, B \rightarrow \Delta_2; \pi_2)}{((\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2)\sigma; (\pi_1 \land \pi_2)|\sigma)} \quad \text{if}
\]

1. \( \sigma = \text{mgu}(A, B) \);
2. \( A\sigma \) is strictly maximal in \((\Gamma_1 \rightarrow \Delta_1, A; \pi_1)\) and \( \text{sel}(\Gamma_1 \rightarrow \Delta_1, A) = \emptyset \);
3. \( B \in \text{sel}(\Gamma_2, B \rightarrow \Delta_2) \)
   or \( \text{sel}(\Gamma_2, B \rightarrow \Delta_2) = \emptyset \) and \( \lnot B\sigma \) maximal in \((\Gamma_2, B \rightarrow \Delta_2; \pi_2)\sigma \);
4. \( (\pi_1 \land \pi_2)|\sigma \) is solvable.
Definition 4 (SDC-Factoring).

\[
\frac{(Γ \rightarrow Δ, A, B ; π)}{((Γ \rightarrow Δ, A)σ; πσ\downarrow)} , \text{ if}
\]

1. \( σ = \text{mgu}(A, B) \);
2. \( \sigma(Γ \rightarrow Δ, A, B) = \emptyset \);
3. \( Aσ \) is maximal in \((Γ \rightarrow Δ, A, B; π)σ \)
4. \( πσ\downarrow \) is solvable.

Note that while the above rules do not operate on equations, we can actually allow unit clauses that consist of non-unifiable disequations, i.e., clauses \( s \approx t \rightarrow \) where \( s \) and \( t \) are not unifiable. There are no potential superposition inferences on such clauses as long as there are no positive equations. So resolution and factoring suffice for completeness. Nevertheless, clauses such as \( s \approx t \rightarrow \) affect the models of satisfiable problems. Constrained Resolution and Factoring are sound.

Lemma 1 (Soundness). SDC-Resolution and SDC-Factoring are sound.

Proof. Let \((Γ_1, Γ_2 \rightarrow Δ_1, Δ_2)σδ\) be a ground instance of \(((Γ_1, Γ_2 \rightarrow Δ_1, Δ_2)σ; (π_1 \land π_2)σ)\). Then, \( δ \) is a solution of \((π_1 \land π_2)σ\) and \( σδ \) is a solution of \( π_1 \) and \( π_2 \). Hence, \((Γ_1 \rightarrow Δ_1, A)σδ\) and \((Γ_2, B \rightarrow Δ_2)σδ\) are ground instances of \((Γ_1 \rightarrow Δ_1, A; π_1)\) and \((Γ_2, B \rightarrow Δ_2; π_2)\), respectively. Because \( Aσδ = Bσδ \), if \((Γ_1 \rightarrow Δ_1, A)σδ\) and \((Γ_2, B \rightarrow Δ_2)σδ\) are satisfied, then \((Γ_1, Γ_2 \rightarrow Δ_1, Δ_2)σδ\) is also satisfied. Therefore, SDC-Resolution is sound. Let \((Γ \rightarrow Δ, A)σδ\) be a ground instance of \(((Γ \rightarrow Δ, A)σ; πσ)\). Then, \( δ \) is a solution of \( πσ \) and \( σδ \) is a solution of \( π \). Hence, \((Γ \rightarrow Δ, A, B)σδ\) is a ground instance of \((Γ \rightarrow Δ, A, B; π)\). Because \( Aσδ = Bσδ \), if \((Γ \rightarrow Δ, A, B)σδ\) is satisfied, then \((Γ \rightarrow Δ, A)σδ\) is also satisfied. Therefore, SDC-Factoring is sound. \( \Box \)

Definition 5 (Saturation). A constrained clause set \( N \) is called saturated up to redundancy, if for every inference between clauses in \( N \) the result \( (R; π) \) is either redundant in \( N \) or \( G((R; π)) \) is a ground instance of \( N \).

Note that our redundancy notion includes condensation and the condition \( G((R; π)) \) is a ground instance of \( N \) allows ignoring variants of clauses.

Lemma 2. Let constrained clause \((C'; π')\) be a condensation of constrained clause \((C; π)\). Then, (i)\((C; π) \vdash (C'; π')\) and (ii)\((C; π) \vdash \{ (C'; π') \}\).

Proof. Let \( σ \) be a substitution such that \( C' \subset C \), \( πσ = π' \), \( π' \subset π \), and for all \( L \in C \) there is a \( L' \in C' \) with \( Lσ = L' \).

(i) Let \( C'σ \in G((C'; π')) \). Then \( σδ \) is a solution of \( π \) and hence \( Cσδ \in G((C; π)) \). Let \( Lσδ \in I \) for some \( L \in C \) and thus \( L'σδ \in I' \) for some \( L' \in C' \) with \( Lσ = L' \). Therefore, \( I' \vdash C'σδ \). Since \( I' \) and \( C'σδ \) were arbitrary, \( (C; π) \vdash (C'; π') \).

(ii) Let \( Cσ \in G((C; π)) \). Because \( π' \subset π \), \( δ \) is a solution of \( π' \) and hence, \( C'σδ \in G((C'; π')) \). Therefore, since \( C'σδ \subset Cσδ \), \( C'σδ \in G((C'; π')) \approx Cδ \) and \( Cσδ \vdash Cδ \). \( \Box \)
Definition 6 (Partial Minimal Model Construction). Given a constrained clause set \( N \), an ordering \(<\) and the selection function \( \text{sel} \), we construct an interpretation \( \mathcal{I}_N \) for \( N \), called a partial model, inductively as follows:

\[
\mathcal{I}_C := \bigcup_{D \in \mathcal{G}(N)} \delta_D, \text{ where } C \in \mathcal{G}(N)
\]

\[
\delta_D := \begin{cases} 
\{A\} & \text{if } D = \Gamma \rightarrow \Delta, A \\
\emptyset & \text{otherwise}
\end{cases}
\]

\[
\mathcal{I}_N := \bigcup_{C \in \mathcal{G}(N)} \delta_C
\]

Clauses \( D \) with \( \delta_D \neq \emptyset \) are called productive.

Lemma 3 (Ordered SDC Resolution Completeness). Let \( N \) be a constrained clause set saturated up to redundancy by ordered SDC-resolution with selection. Then \( N \) is unsatisfiable, if and only if \( \square \in \mathcal{G}(N) \). If \( \square \notin \mathcal{G}(N) \) then \( \mathcal{I}_N \models N \).

Proof. Assume \( N \) is unsatisfiable but \( \square \notin \mathcal{G}(N) \). For the partial model \( \mathcal{I}_N \), there exists a minimal false clause \( C \sigma \in \mathcal{G}((C; \pi)) \) for some \( (C; \pi) \in N \).

\( C \sigma \) is not productive, because otherwise \( \mathcal{I}_N \models C \sigma \). Hence, either \( \text{sel}(C) \neq \emptyset \) or no positive literal in \( C \sigma \) is strictly maximal. Assume \( C = I_2, B \rightarrow \Delta_2 \) with \( B \in \text{sel}(C) \) or \( \neg B \sigma \) maximal. Then, \( B \sigma \in \mathcal{I}_C \sigma \) and there exists a ground instance \( (I_1 \rightarrow \Delta_1, A) \tau = D \tau \prec C \sigma \) of some clause \( (D; \pi') \in N \), which produces \( A \tau = B \sigma \). Therefore, there exists a \( \rho = \text{mgu}(A, B) \) and ground substitution \( \delta \) such that \( C \sigma = C \rho \delta \), \( D \tau = D \rho \delta \). Since \( \rho \delta = \sigma \) is a solution of \( \pi \) and \( \pi' \), \( \delta \) is a solution of \( \pi \land \pi' \). Under these conditions, SDC-Resolution can be applied to \( (I_1 \rightarrow \Delta_1, A \pi') \) and \( (I_2, B \rightarrow \Delta_2; \pi) \). Their resolvent \( (R; \pi_R) = ((I_1, I_2 \rightarrow \Delta_1, \Delta_2) \rho; \pi \land \pi') \rho \) is either redundant in \( N \) or \( \mathcal{G}((R; \pi_R)) \subseteq \mathcal{G}(N) \). Its ground instance \( R \delta \) is false in \( \mathcal{I}_N \) and \( R \delta \prec C \sigma \). If \( (R; \pi_R) \) is redundant in \( N \), there exist \( C_1, \ldots, C_n \) in \( \mathcal{G}(N) \) with \( C_1, \ldots, C_n \models R \delta \). Because \( C_i \prec R \delta \prec C \sigma \), \( \mathcal{I}_N \models C_i \) and hence \( \mathcal{I}_N \models R \delta \), which contradicts \( \mathcal{I}_N \not\models R \delta \). Otherwise, if \( \mathcal{G}((R; \pi_R)) \subseteq \mathcal{G}(N) \), then \( R \delta \in \mathcal{G}(N) \), which contradicts \( C \sigma \) being minimal false.

Now, assume \( \text{sel}(C) = \emptyset \) and \( C = \Gamma \rightarrow \Delta, B \) with \( B \sigma \) maximal. Then, \( C = \Gamma \rightarrow \Delta', A, B \) with \( A \sigma = B \sigma \). Therefore, there exists a \( \rho = \text{mgu}(A, B) \) and ground substitution \( \delta \) such that \( C \sigma = C \rho \delta \) and \( \rho \delta \) is a solution of \( \pi \). Hence, \( \delta \) is a solution of \( \pi \rho \). Under these conditions, SDC-Factoring can be applied to \( (\Gamma \rightarrow \Delta', A, B; \pi) \). The result \( (R; \pi_R) = ((\Gamma \rightarrow \Delta', A) \rho; \pi \rho) \) is either redundant in \( N \) or \( \mathcal{G}((R; \pi_R)) \subseteq \mathcal{G}(N) \). Its ground instance \( R \delta \) is false in \( \mathcal{I}_N \) and \( R \delta \prec C \sigma \). If \( (R; \pi_R) \) is redundant in \( N \), there exist \( C_1, \ldots, C_n \) in \( \mathcal{G}(N) \) with \( C_1, \ldots, C_n \models R \delta \). Because \( C_i \prec R \delta \prec C \sigma \), \( \mathcal{I}_N \models C_i \) and hence \( \mathcal{I}_N \models R \delta \), which contradicts \( \mathcal{I}_N \not\models R \delta \). Otherwise, if \( \mathcal{G}((R; \pi_R)) \subseteq \mathcal{G}(N) \), then \( R \delta \in \mathcal{G}(N) \), which contradicts \( C \sigma \) being minimal false.

Therefore, if \( \square \notin \mathcal{G}(N) \), no minimal false clause exists and \( \mathcal{I}_N \models N \). \( \square \)
Lemma 4. Let $N$ be a set of MSL(SDC) clauses without variants or condensations and $\Sigma$ a finite signature. $N$ is finite if there exists an integer $d$ such that for every $(C; \pi) \in N$, $\text{depth}(\pi) \leq d$ and

1. $C = S_1(x_1), \ldots, S_n(x_n), S'_1(t), \ldots, S'_m(t) \rightarrow \Delta$ or
2. $C = S_1(x_1), \ldots, S_n(x_n), S'_1(t), \ldots, S'_m(t) \rightarrow S(t), \Delta$

with $t$ shallow and linear, and $\text{vars}(t) \cap \text{vars}(\Delta) = \emptyset$.

Proof. Let $(C; \pi) \in N$. $(C; \pi)$ can be separated into variable disjoint components $(\Gamma_1, \ldots, \Gamma_n \rightarrow \Delta_1, \ldots, \Delta_n; \pi_1 \land \ldots \land \pi_n)$, where $|\Delta_i| \leq 1$ and $\text{lvar}(\pi_i) \subseteq \text{vars}(\Gamma_i \rightarrow \Delta_i)$. For each positive literal $P(s) \in \Delta$ there is a fragment

(A) $(S_1(x_1), \ldots, S_k(x_k) \rightarrow P(s); \pi')$

with $\{x_1, \ldots, x_k\} \subseteq \text{vars}(s)$. If $m > 0$, there is another fragment

(B) $(S_1(x_1), \ldots, S_k(x_k), S'_1(t), \ldots, S'_m(t) \rightarrow; \pi')$

or

(C) $(S_1(x_1), \ldots, S_k(x_k), S'_1(t), \ldots, S'_m(t) \rightarrow S(t); \pi')$

with $\{x_1, \ldots, x_k\} \subseteq \text{vars}(t)$, respectively. Lastly, for each variable $x \in \text{vars}(C)$ with $x \notin \text{vars}(t) \cup \text{vars}(\Delta)$ there is a fragment

(D) $(S_1(x), \ldots, S_k(x) \rightarrow; \pi')$.

Since there are only finitely many terms $s$ with $\text{depth}(s) \leq d$ modulo renaming, there are only finitely many atomic constraints $x \neq s$ for a given variable $x$ different up to renaming $s$. Thus, a normal constraint can only contain finitely many combinations of subconstraints $\bigwedge_{i \in I} x \neq s_i$ without some $s_i$ being an instance of another $s_j$. Therefore, for a fixed set of variables $x_1, \ldots, x_k$, there are only finitely many constraints $\pi = \bigwedge_{i \in I} z_i \neq s_i$ with $\text{lvar}(\pi) \subseteq \{x_1, \ldots, x_k\}$ up to variants.

Since the number of predicates, function symbols, and their ranks is finite, the number of possible shallow and linear atoms $S(t)$ different up to variants is finite. For a given shallow and linear $t$, there exist only finitely many clauses of the form $(S_1(t), \ldots, S_n(t) \rightarrow S(t); \pi)$ or $(S_1(t), \ldots, S_n(t) \rightarrow; \pi)$ with $\text{lvar}(\pi) \subseteq \text{vars}(t)$ modulo condensation and variants. For a fixed set of variables $x_1, \ldots, x_k$, there exist only finitely many clauses of the form $(S_1(y_1), \ldots, S_k(y_k) \rightarrow; \pi)$ with $\{y_1, \ldots, y_l\} \cup \text{lvar}(\pi) \subseteq \{x_1, \ldots, x_k\}$ modulo condensation and variants. Therefore, there are only finitely many distinct clauses of each form (A)-(D) without variants or condensations.

If in the clause $(C; \pi) = (\Gamma_1, \ldots, \Gamma_n \rightarrow \Delta_1, \ldots, \Delta_n; \pi_1 \land \ldots \land \pi_n)$ for some $i \neq j$, $(\Gamma_i \rightarrow \Delta_i; \pi_i)$ is a variant of $(\Gamma_j \rightarrow \Delta_j; \pi_j)$, then $(C; \pi)$ has a condensation and is therefore not part of $N$. Hence, there can be only finitely many different $(C; \pi)$ without variants or condensations and thus $N$ is finite. \qed
Lemma 5 (Finite Saturation). Let $N$ be an MSL(SDC) clause set. Then $N$ can be finitely saturated up to redundancy by sdc-resolution with selection function sel.

Proof. The general idea is that given the way sel is defined the clauses involved in constrained resolution and factoring can only fall into certain patterns. Any result of such inferences then is either strictly smaller than one of its parents by some terminating measure or falls into a set of clauses that is bounded by Lemma 4. Thus, there can be only finitely many inferences before $N$ is saturated.

Let $d$ be an upper bound on the depth of constraints found in $N$ and $\Sigma$ be the finite signature consisting of the function and predicate symbols occurring in $N$. Let $(\Gamma_1 \rightarrow \Delta_1, S(t) ; \pi_1)$ and $(\Gamma_2, S(t') \rightarrow \Delta_2 ; \pi_2)$ be clauses in $N$ where sdc-resolution applies with $\sigma = \text{mgu}(S(t), S(t'))$ and resolvent $R = ((\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2)\sigma; (\pi_1 \land \pi_2)\downarrow)$.

Because no literal is selected by sel, $\Gamma_1 \rightarrow \Delta_1, S(t)$ can match only one of two patterns:

$(A) \quad S_1(x_1), \ldots, S_n(x_n) \rightarrow S(f(y_1, \ldots, y_k)), \Delta$

where $t = f(y_1, \ldots, y_k)$ and $\{x_1, \ldots, x_n\} \subseteq \{y_1, \ldots, y_k\} \cup \text{vars}(\Delta)$.

$(B) \quad S_1(x_1), \ldots, S_n(x_n) \rightarrow S(y), \Delta$

where $t = y$ and $x_1, \ldots, x_n$ are variables in $\text{vars}(\Delta)$, i.e., $y$ occurs only once.

The literal $S(t')$ is selected by sel in $\Gamma_2, S(t') \rightarrow \Delta_2$, and therefore $\Gamma_2, S(t') \rightarrow \Delta_2$ can match only one of the following three patterns:

$(1) \quad S(f(t_1, \ldots, t_k)), \Gamma' \rightarrow \Delta'$

$(2) \quad S(y'), \Gamma' \rightarrow \Delta'$ where $\Gamma'$ has no function terms and $y \notin \text{vars}(\Delta')$.

$(3) \quad S(y'), \Gamma' \rightarrow S'(y'), \Delta'$ where $\Gamma'$ has no function terms.

This means that the clausal part $(\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2)\sigma$ of $R$ has one of six forms:

$(A1) \quad S_1(x_1)\sigma, \ldots, S_n(x_n)\sigma, \Gamma' \rightarrow \Delta, \Delta'$ with $\sigma = \{y_1 \mapsto t_1, \ldots\}$.

$\Delta\sigma = \Delta$ because $S(f(y_1, \ldots, y_k))$ and $\Delta$ do not share variables.

$(B1) \quad S_1(x_1), \ldots, S_n(x_n), \Gamma' \rightarrow \Delta, \Delta'$.

The substitution $\{y \mapsto f(t_1, \ldots, t_k)\}$ is irrelevant since $S(y)$ is the only literal with variable $y$.

$(A2) \quad S_1(x_1), \ldots, S_n(x_n), \Gamma'\tau \rightarrow \Delta, \Delta'$ with $\tau = \{y' \mapsto f(y_1, \ldots, y_k)\}$.
\[ \Delta' \tau = \Delta' \text{ because } y' \notin \text{vars}(\Delta'). \]

(B2) \[ S_1(x_1), \ldots, S_n(x_n), \Gamma' \rightarrow \Delta, \Delta'. \]

(A3) \[ S_1(x_1), \ldots, S_n(x_n), \Gamma' \tau \rightarrow S'(f(y_1, \ldots, y_k)), \Delta, \Delta' \text{ with } \tau = \{ y \mapsto f(y_1, \ldots, y_k) \}. \]

\[ \Delta' \tau = \Delta' \text{ because } y' \notin \text{vars}(\Delta'). \]

(B3) \[ S_1(x_1), \ldots, S_n(x_n), \Gamma' \rightarrow S'(y'), \Delta, \Delta'. \]

In the constraint \((\pi_1 \land \pi_2)\sigma_\downarrow\), the maximal depth of the subconstraints is less or equal to the maximal depth of \(\pi_1\) or \(\pi_2\). Hence, \(d\) is also an upper bound on the constraint of the resolvent. In each case, the resolvent is again an MSL(SDC) clause.

In the first and second case, the multiset of term depths of the negative literals in \(R\) is strictly smaller than for the right parent. In both, the \(\Gamma\) is the same between the right parent and the resolvent. Only the \(f(t_1, \ldots, t_k)\) term is replaced by \(x_1\sigma, \ldots, x_n\sigma\) and \(x_1, \ldots, x_n\) respectively. In the first case, the depth of the \(x_i\sigma\) is either zero if \(x_i \notin \{ y_1, \ldots, y_k \}\) or at least one less than \(f(t_1, \ldots, t_k)\) since \(x_i\sigma = t_i\). In the second case, the \(x_i\) have depth zero which is strictly smaller than the depth of \(f(t_1, \ldots, t_k)\). Since the multiset ordering on natural numbers is terminating, the first and second case can only be applied finitely many times by constrained resolution.

In the third to sixth case \(R\) is of the form \((S_1(x_1), \ldots, S_i(x_i), S'_i(t), \ldots, S'_m(t) \rightarrow \Delta; \pi)\) or \((S_1(x_1), \ldots, S_i(x_i), S'_i(t), \ldots, S'_m(t) \rightarrow S(t)), \Delta; \pi)\) with \(t = f(y_1, \ldots, y_k)\). By Lemma [4] there are only finitely many such clauses after condensation and removal of variants. Therefore, these four cases can only apply only finitely many times during saturation.

Let \((\Gamma \rightarrow \Delta, S(t), S(t'); \pi)\) be a clause in \(N\) where sdc-factoring applies with \(\sigma = \text{mgu}(S(t), S(t'))\) and \(R = ((\Gamma \rightarrow \Delta, S(t))\sigma; \pi\sigma_\downarrow)\). Because in \((\Gamma \rightarrow \Delta, S(t), S(t')\) no literal is selected, \((\Gamma \rightarrow \Delta, S(t), S(t'))\) and \((\Gamma \rightarrow \Delta, S(t))\sigma\) can only match one of three patterns.

(A) \[ S_1(x_1), \ldots, S_n(x_n) \rightarrow S(f(y_1, \ldots, y_k)), S(f(z_1, \ldots, z_1)), \Delta \]

where \(t = f(y_1, \ldots, y_k)\), \(t' = f(z_1, \ldots, z_k)\), and \(\{ x_1, \ldots, x_n \} \subseteq \{ y_1, \ldots, y_k \} \cup \{ z_1, \ldots, z_1 \} \cup \text{vars}(\Delta)\). The result is

\[ S_1(x_1)\sigma, \ldots, S_n(x_n)\sigma \rightarrow S(f(y_1, \ldots, y_k)), \Delta \text{ with } \sigma = \{ z_1 \mapsto y_1, \ldots \}. \]

(B) \[ S_1(x_1), \ldots, S_n(x_n) \rightarrow S(f(y_1, \ldots, y_k)), S(z), \Delta \]

where \(t = f(y_1, \ldots, y_k)\), \(t' = z\) and \(\{ x_1, \ldots, x_n \} \subseteq \{ y_1, \ldots, y_k \} \cup \text{vars}(\Delta)\), i.e., \(z\) occurs only once. The result is

\[ S_1(x_1), \ldots, S_n(x_n) \rightarrow S(f(y_1, \ldots, y_k)), \Delta. \]

(C) \[ S_1(x_1), \ldots, S_n(x_n) \rightarrow S(y), S(z), \Delta \]
where \( t = y, t' = z \) and \( \{x_1, \ldots, x_n\} \subseteq \text{vars}(\Delta) \), i.e., \( y \) and \( z \) occur only once. The result is

\[
S_1(x_1), \ldots, S_n(x_n) \rightarrow S(y), \Delta.
\]

In the new constraint \( \pi \sigma \downarrow \) the maximal depth of the subconstraints is less or equal to the maximal depth of \( \pi \). Hence \( d \) is also an upper bound on the constraint of the resolvent. In each case, the resolvent is again an MSL(SDC) clause.

Furthermore, in each case the clause is of the form \( (S_1(x_1), \ldots, S_l(x_l) \rightarrow \Delta; \pi) \). By Lemma 4, there are only finitely many such clauses after condensation and removal of variants. Therefore, these three cases can apply only finitely many times during saturation.

\[\Box\]

**Theorem 1 (MSL(SDC) Decidability).** Satisfiability of the MSL(SDC) first-order fragment is decidable.

**Proof.** Follows from Lemma 5 and 3.

### 4 Approximation and Refinement

In the following, we show how decidability of the MSL(SDC) fragment can be used to improve the approximation refinement calculus presented in [13]. As mentioned before, one motivation to use dismatching constraints is that for unconstrained clause the refinement adds quadratically many new clauses to the approximation. In contrast, with constrained clauses the same can be accomplished with adding just a single new clause. This extension is rather simple as constraints are treated the same as the antecedent literals in the clause. Furthermore we now present refinement as a separate transformation rule. The second change is the removal of the Horn approximation rule. We have shown in Section 3 that the restriction to Horn clauses is not required anymore for decidability. Instead, the linear and shallow approximations are extended to non-Horn clauses.

Starting from a constrained clause set \( N \) the transformation is parametrized by a single monadic projection predicate \( T \), fresh to \( N \) and for each non-monadic predicate \( P \) a separate projection function \( f_P \) fresh to \( N \).

**Definition 7.** Given a predicate \( P \), projection predicate \( T \), and projection function \( f_P \), define the injective function \( \mu^P_T(P(\vec{t})) := T(f_P(\vec{t})) \) and \( \mu^P_T(Q(\vec{s})) := Q(\vec{s}) \) for \( P \neq Q \). The function is extended to constrained clauses, clause sets and interpretations. Given a signature \( \Sigma \) with non-monadic predicates \( P_1, \ldots, P_n \), define \( \mu^\Sigma_T(N) = \mu^T_P(\ldots(\mu^T_{P_n}(N))\ldots) \) and \( \mu^\Sigma_T(I) = \mu^T_P(\ldots(\mu^T_{P_n}(I))\ldots) \).

The approximation consists of individual transformation rules \( N \Rightarrow N' \) that are non-deterministically applied. The clauses in \( N \) are called the original clauses while the clauses in \( N' \) are the approximated clauses.
Monadic \( N \Rightarrow_{\text{MO}} \mu^{\mathcal{P}}_N(N) \) 

provided \( P \) is a non-monadic predicate in the signature of \( N \).

Shallow 
\[ N \cup \{(\Gamma \rightarrow E[s]_p, \Delta; \pi)\} \Rightarrow_{\text{SH}} \]
\[ N \cup \{(S(x), \Gamma_1 \rightarrow E[p/x], \Delta_1; \pi); (\Gamma_r \rightarrow S(s), \Delta_r; \pi)\} \]

provided \( s \) is complex, \( |p| = 2 \), \( x \) and \( S \) fresh, \( \Gamma_1(x \mapsto s) \cup \Gamma_r = \Gamma \), \( \Delta_1 \cup \Delta_r = \Delta \), \( \{Q(y) \in \Gamma \mid y \in \text{vars}(E[p/x], \Delta_1)\} \subseteq \Gamma_r \), \( \{Q(y) \in \Gamma \mid y \in \text{vars}(s, \Delta_r)\} \subseteq \Gamma_r \).

Linear 1 
\[ N \cup \{(\Gamma \rightarrow \Delta, E'[x]_p, E[x]_q; \pi)\} \Rightarrow_{\text{LI}} \]
\[ N \cup \{(\Gamma \sigma, \Gamma \rightarrow \Delta, E'[x]_p, E[q/x']\pi \land \pi\sigma)\} \]

provided \( x' \) is fresh and \( \sigma = \{x \mapsto x'\} \).

Linear 2 
\[ N \cup \{(\Gamma \rightarrow \Delta, E[x]_p, E[x]_q; \pi)\} \Rightarrow_{\text{LI}} \]
\[ N \cup \{(\Gamma \sigma, \Gamma \rightarrow \Delta, E[q/x']\pi \land \pi\sigma)\} \]

provided \( x' \) is fresh, \( p \neq q \) and \( \sigma = \{x \mapsto x'\} \).

Refinement 
\[ N \cup \{(C, \pi)\} \Rightarrow_{\text{Ref}} N \cup \{(C; \pi \land x \neq t), (C; \pi)\{x \mapsto t\}\} \]

provided \( x \in \text{vars}(C) \), \( t \) straight and \( \text{vars}(t) \cap \text{vars}(C, \pi) = \emptyset \).

Note that variables are not renamed unless explicitly stated in the rule. This means that original clauses and their approximated counterparts share variable names. We use this to trace the origin of variables in the approximation. Assume that in the end all clauses are renamed to be variable disjoint before the calculus is applied to the clause set.

To reach the MSL(SDC) fragment the refinement transformation is strictly optional. However, the satisfiability equivalent transformation \( N \Rightarrow_{\text{Ref}} N' \) is used to achieve a more fine-grained over-approximation of \( N \), see below.

In the shallow transformation, \( \Gamma \) and \( \Delta \) are separated into \( \Gamma_1 \), \( \Gamma_r \), \( \Delta_1 \), and \( \Delta_r \), respectively. The separation can be almost arbitrarily chosen as long as no atom from \( \Gamma \), \( \Delta \) is skipped. However, the goal is to minimize the set of shared variables, \( \text{vars}(\Gamma_r, s, \Delta_r) \cap \text{vars}(\Gamma_1, E[p/x], \Delta_1) \). because if there are no shared variables, the shallow transformation is satisfiability equivalent. The conditions on \( \Gamma_1 \) and \( \Gamma_r \) ensure that \( S(x) \) atoms are not separated from the respective positive occurrence of \( x \) in subsequent shallow transformation applications.

Consider the clause \( Q(f(x), y) \rightarrow P(g(f(x), y)) \). The simple shallow transformation \( S(x'), Q(f(x), y) \rightarrow P(g(x', y)); S(f(x)) \) is not satisfiability equivalent – nor with any alternative partitioning of \( \Gamma \). However, by replacing the occurrence of the extraction term \( f(x) \) in \( Q(f(x), y) \) with the fresh variable \( x' \), the approximation \( S(x'), Q(x, y) \rightarrow P(g(x', y)); S(f(x)) \) is satisfiability equivalent. Therefore, we allow the extraction of \( s \) from the terms in \( \Gamma_1 \) and require \( \Gamma_1 \{x \mapsto s\} \cup \Gamma_r = \Gamma \).

We consider Linear 1 and Linear 2 as two cases of the same linear transformation rule. Their only difference is whether the two occurrences of \( x \) are in the same or different literals. The duplication of literals and constraints in \( \Gamma \) and \( \pi \) is not needed if \( x \) does not occur in \( \Gamma \) or \( \pi \).
Further, consider a linear transformation \( N \cup \{(C; \pi)\} \Rightarrow_{\text{SH}} N \cup \{(C_a; \pi_a)\} \), where a fresh variable \( x' \) replaces an occurrence of a non-linear variable \( x \) in \((C; \pi)\). Then, \((C_a; \pi_a)x' \mapsto x\) is equal to \((C; \pi)\) modulo duplicate literal elimination. A similar property can be observed of a resolvent of \((C_l; \pi)\) and \((C_r; \pi)\) resulting from a shallow transformation \( N \cup \{(C; \pi)\} \Rightarrow_{\text{SH}} N \cup \{(C_l; \pi), (C_r; \pi)\} \).

Note that by construction, \((C_l; \pi)\) and \((C_r; \pi)\) are not necessarily variable disjoint. To simulate standard resolution, we need to rename at least the shared variables in one of them.

**Definition 8 (\(\Rightarrow_{\text{AP}}\)).** We define \(\Rightarrow_{\text{AP}}\) as the priority rewrite system \([7]\) consisting of \(\Rightarrow_{\text{Ref}}, \Rightarrow_{\text{MO}}, \Rightarrow_{\text{SH}}\) and \(\Rightarrow_{\text{LI}}\) with priority \(\Rightarrow_{\text{Ref}} > \Rightarrow_{\text{MO}} > \Rightarrow_{\text{SH}} > \Rightarrow_{\text{LI}}\), where \(\Rightarrow_{\text{Ref}}\) is only finitely many times.

**Lemma 6 (\(\Rightarrow_{\text{AP}}\) is a Terminating Over-Approximation).** The approximation rules are terminating over-approximations: (i) \(\Rightarrow_{\text{AP}}\) terminates, (ii) the linear transformation is an over-approximation, (iii) the shallow transformation is an over-approximation, (iv) the monadic transformation is an over-approximation, (v) the refinement transformation is an over-approximation.

**Proof.** (i) The transformations can be considered sequentially, because of the imposed rule priority. There are, by definition, only finitely many refinements at the beginning of an approximation \(\Rightarrow_{\text{AP}}\). The monadic transformation strictly reduces the number of non-monadic atoms. The shallow transformation strictly reduces the number of distinct variable occurrences in positive literals. Hence \(\Rightarrow_{\text{AP}}\) terminates.

(ii) Let \(N \cup \{(C; \pi)\} \Rightarrow_{\text{LI}} N \cup \{(C_a; \pi_a)\}\) where an occurrence of a variable \(x\) in \((C; \pi)\) is replaced by a fresh \(x'\). As \((C_a; \pi_a)x' \mapsto x\) is equal to \((C; \pi)\) modulo duplicate literal elimination, \(I \models (C; \pi)\) if \(I \models (C_a; \pi_a)\). Therefore, the linear transformation is an over-approximation.

(iii) Let \(N \cup \{(C; \pi)\} \Rightarrow_{\text{SH}} N \cup \{(C_l; \pi_l), (C_r; \pi_r)\} \) and \((C_a; \pi_a)\) be the shallow \(\rho\)-resolvent. As \((C_a; \pi_a)\rho^{-1}\) equals \((C; \pi)\) modulo duplicate literal elimination, \(I \models (C; \pi)\) if \(I \models (C_l; \pi_l), (C_r; \pi_r)\). Therefore, the shallow transformation is an over-approximation.

(iv) Let \(N \Rightarrow_{\text{MO}} \mu_P(N) = N'\). Then, \(N = \mu_P^{-1}(N')\). Let \(I\) be a model of \(N'\) and \((C; \pi) \in N\). Since \(\mu_P((C; \pi)) \in N'\), \(I \models \mu_P((C; \pi))\) and thus, \(\mu_P^{-1}(I) \models (C; \pi)\). Hence, \(\mu_P^{-1}(I)\) is a model of \(N\). Therefore, the monadic transformation is an over-approximation. Actually, it is a satisfiability preserving transformation.

(v) Let \(N \cup \{(C; \pi)\} \Rightarrow_{\text{Ref}} N \cup \{(C; \pi \land x \neq t), (C; \pi)x \mapsto t\}\). Let \(C \delta \in \mathcal{G}((C; \pi))\). If \(x\delta\) is not an instance of \(t\), then \(\delta\) is a solution of \(\pi \land x \neq t\) and \(C \delta \in \mathcal{G}((C; \pi \land x \neq t))\). Otherwise, \(\delta = \{x \mapsto t\} \delta'\) for some substitution \(\delta'\). Then, \(\delta\) is a solution of \(\pi x \mapsto t\) and thus, \(C \delta = C \{x \mapsto t\} \delta' \in \mathcal{G}((C; \pi \land x \neq t) \cup \mathcal{G}((C; \pi \land x \neq t) \cup \mathcal{G}((C; \pi)x \mapsto t))\). Hence, \(\mathcal{G}((C; \pi)) \subseteq \mathcal{G}((C; \pi \land x \neq t)) \cup \mathcal{G}((C; \pi)x \mapsto t))\). Therefore, if \(I\) is a model of \(N \cup \{(C; \pi \land x \neq t), (C; \pi)x \mapsto t\}\), then \(I\) is also a model of \(N \cup \{(C; \pi)\}\). \(\square\)

Note that both \(\Rightarrow_{\text{Ref}}\) and \(\Rightarrow_{\text{MO}}\) are also under-approximations and are therefore satisfiability equivalent transformations.
Corollary 1. If $N \vdash^*_{AP} N'$ and $N'$ is satisfied by a model $\mathcal{I}$, then $\mu_{\mathcal{I}}^{-1}(\mathcal{I})$ is a model of $N$.

Proof. Follows from Lemma 8 (ii)-(v).}

On the basis of $\Rightarrow_{AP}$ we can define an ancestor relation $\Rightarrow$ that relates clauses, literal occurrences, variables with respect to approximation. This relation is needed in order to figure out the exact clause, literal, variable for refinement.

**Definition 9 (The Shallow Resolvent).** Let $N \cup \{(C; \pi)\} \Rightarrow_{SH} N \cup \{(C_l; \pi_l), (C_r; \pi_r)\}$ with $C = \Gamma \rightarrow E[s]_t, \Delta, C_l = S(x), \Gamma_l \rightarrow E[p/x], \Delta_l$ and $C_r = \Gamma_r \rightarrow E[s], \Delta_r$. Let $x_1, \ldots, x_n$ be the variables shared between $C_l$ and $C_r$ and $\rho = \{x_1 \mapsto x'_1, \ldots, x_n \mapsto x'_n\}$ be a variable renaming with $x'_1, \ldots, x'_n$ fresh in $C_l$ and $C_r$. We define $(\Gamma_l\{x \mapsto s\}, \Gamma_r \mapsto E[p/s], \Delta_l, \Delta_r; \pi \wedge \pi)$ as the shallow $\rho$-resolvent.

Let $(C_a; \pi_a)$ be the shallow $\rho$-resolvent of $N \cup \{(C; \pi)\} \Rightarrow_{SH} N \cup \{(C_l; \pi_l), (C_r; \pi_r)\}$. Note that for any two ground instances $C_l \delta_l$ and $C_r \delta_r$, their resolvent is a ground instance of $(C_a; \pi_a)$. Furthermore, using the reverse substitution $\rho^{-1} = \{x'_1 \mapsto x_1, \ldots, x'_n \mapsto x_n\}, (C_a; \pi_a)\rho^{-1} = (\Gamma_l\{x \mapsto s\}, \Gamma_r \rightarrow E[s], \Delta_l, \Delta_r; \pi \wedge \pi)$ is equal to $(C; \pi)$ modulo duplicate literal elimination. This is because, $\Delta_l \cup \Delta_r = \Delta$ and $\Gamma_l\{x \mapsto s\} \cup \Gamma_r = \Gamma$ by definition of $\Rightarrow_{SH}$ and $\pi \wedge \pi$ is equivalent to $\pi$.

Next, we establish parent relations that link original and approximated clauses, as well as their variables and literals. Together the parent, variable and literal relations will allow us to not only trace any approximated clause back to their origin, but also predict what consequences changes to the original set will have on its approximations.

For the following definitions, we assume that clause and literal sets are lists and that $\mu_{\mathcal{I}}^T$ and substitutions act as mappings. This means we can uniquely identify clauses and literals by their position in those lists. Further, for every shallow transformation $N \Rightarrow_{SH} N'$, we will also include the shallow resolvent in the parent relation as if it were a member of $N'$.

**Definition 10 (Parent Clause).** For an approximation step $N \Rightarrow_{AP} N'$ and two clauses $(C; \pi) \in N$ and $(C'; \pi') \in N'$, we define $[(C; \pi), N] \Rightarrow_{\mathcal{A}} [(C'; \pi'), N']$ expressing that $(C; \pi)$ in $N$ is the parent clause of $(C'; \pi')$ in $N'$:

If $N \Rightarrow_{\mathcal{A}}^* \mu_{\mathcal{I}}^T(N)$, then

$[(C; \pi), N] \Rightarrow_{\mathcal{A}} [(C; \pi), \mu_{\mathcal{I}}^T(N)]$ for all $(C; \pi) \in N$.

If $N = N'' \cup \{(C; \pi)\} \Rightarrow_{SH} N'' \cup \{(C_l; \pi_l), (C_r; \pi_r)\} = N'$, then

$[(D, \pi'), N] \Rightarrow_{\mathcal{A}} [(D, \pi'), N']$ for all $(D, \pi') \in N''$ and

$[(C, \pi), N] \Rightarrow_{\mathcal{A}} [(C_l; \pi_l), N']$ and

$[(C, \pi), N] \Rightarrow_{\mathcal{A}} [(C_r; \pi_r), N']$ and

$[(C, \pi), N] \Rightarrow_{\mathcal{A}} [(C_a; \pi_a), N']$ for any shallow resolvent $(C_a; \pi_a)$.

If $N = N'' \cup \{(C; \pi)\} \Rightarrow_{LI} N'' \cup \{(C_a; \pi_a)\} = N'$, then

$[(D, \pi'), N] \Rightarrow_{\mathcal{A}} [(D, \pi'), N']$ for all $(D, \pi') \in N''$ and

$[(C, \pi), N] \Rightarrow_{\mathcal{A}} [(C_a, \pi_a), N']$.

If $N = N'' \cup \{(C; \pi)\} \Rightarrow_{Ref} N'' \cup \{(C; \pi \land x \neq t), (C; \pi)\{x \mapsto t\}\} = N'$, then
([D, π′], N) ⇒A [([D, π′], N′) for all (D, π′) ∈ N′,
[(C, π), N] ⇒A [(C; π ∧ x ̸= t), N′] and
[(C, π), N] ⇒A [(C; πx → t), N′].

Definition 11 (Parent Variable). Let N ⇒AP N′ be an approximation step and
[(C; π), N] ⇒A [(C′; π′), N′]. For two variables x and y, we define [x, (C; π), N] ⇒A
[y, (C′; π′), N′] expressing that x ∈ vars(C) is the parent variable of y ∈ vars(C′):
If x ∈ vars((C; π)) ∩ vars((C′; π′)), then
[x, (C; π), N] ⇒A [x, (C′; π′), N′].
If N ⇒SH N′ and (C′, π′) is the shallow ρ-resolvent,
[xi, (C; π), N] ⇒A [xi, π, (C′; π′), N′] for each xi in the domain of ρ.
If N ⇒LI N′, C = Γ → Δ[xp,q] and C′ = Γ → Δ[q/x′], then
[x, (C; π), N] ⇒A [x′, (C′; π′), N′].

Note that if N ⇒SH N′ and x is the fresh extraction variable in (C; π), then
x has no parent variable. For literals, we actually further specify the relation on
the positions within literals of a clause (C; π) using pairs (L, r) of literals and
positions. We write (L, r) ∈ C to denote that (L, r) is a literal position in (C; π)
if L ∈ C and r ∈ pos(L). Note that a literal position (L, r) in (C; π) corresponds
to the term L[r].

Definition 12 (Parent literal position). Let N ⇒AP N′ be an approximation step and
[(C; π), N] ⇒A [(C′; π′), N′]. For two literal positions (L, r) and
(L′, r′), we define [r, L, (C; π), N] ⇒A [r′, L′, (C′; π′), N′] expressing that (L, r)
in (C; π) is the parent literal position of (L′, r′) in (C′; π′):
If (C; π) = (C′; π′), then
[r, L, (C; π), N] ⇒A [r, L, (C′; π′), N′] for all (L, r) ∈ C.
If N ⇒Ref N′ and (C′, π′) = (C; π ∧ x ̸= t), then
[r, L, (C; π), N] ⇒A [r, L, (C′; π′), N′] for all (L, r) ∈ C.
If N ⇒Ref N′ and (C′, π′) = (C; π)x → t, then
[r, L, (C, π), N] ⇒A [r, Lx → t, (C′; π′), N′] for all (L, r) ∈ C.
If N ⇒MO µp(N) = N′, then
[e, P(t)], (C; π), N] ⇒A [e, T(p(t)), (C′; π′), N′] for all P(t) ∈ C and
[r, P(t)], (C; π), N] ⇒A [r, T(p(t)), (C′; π′), N′] for all (P(t), r) ∈ C.
If N ⇒SH N′, C = Γ → E[s], Δ and C′ = S(x), Γ1 → E[p/x], Δ1, then
[r, E[s], (C; π), N] ⇒A [r, E[p/x], (C′; π′), N′] for all r ∈ pos(E[p/x]),
[r, E[s], (C, π), N] ⇒A [r, S(x), (C′; π′), N′] for all r ∈ pos(S(x)),
[r, L(x → s), (C, π), N] ⇒A [r, L, (C′; π′), N′] for all (L, r) ∈ Γ1,
[r, L, (C, π), N] ⇒A [r, L, (C′; π′), N′] for all (L, r) ∈ Δ1.
If N ⇒SH N′, C = Γ → E[s], Δ and C′ = Γr → S(s), Δr, then
[r, E[s], (C; π), N] ⇒A [r, S(s), (C′; π′), N′],
[r, pr, E[s], (C; π), N] ⇒A [1, r, S(s), (C′; π′), N′] for all r ∈ pos(s) and
[r, L, (C, π), N] ⇒A [r, L, (C′; π′), N′] for all (L, r) ∈ Γr ∪ Δr.
If N ⇒SH N′, C = Γ → E[s], Δ and (C′, π′) is the shallow ρ-resolvent, then
[r, E[s], (C; π), N] ⇒A [r, E[p/sp], (C′; π′), N′] for all r ∈ pos(E[p/sp]),
[r, L(x → s), (C; π), N] ⇒A [r, Lx → sp, (C′; π′), N′] for all (L, r) ∈ Γ, 
[r, L, (C; π), N] ⇒A [r, Lp, (C′; π′), N′] for all (L, r) ∈ Γ ∪ Δr, and
If \( N (\delta) \) have no corresponding equivalent in \( N \) (a solvable constraint)

Compared to our previous calculus \(^1\), the lifting process stays relatively the

Definition 13 (Conflicting Core).

The transitive closures of each parent relation are called ancestor relations.
The over-approximation of a clause set \( N \) can introduce resolution refutations
that have no corresponding equivalent in \( N \) which we consider a lifting failure.
Compared to our previous calculus \(^9\), the lifting process stays relatively the
same with the exception that there is no case for the removed Horn transfor-
mation. We only update the definition of conflict cores to consider constrained

Clauses.

Definition 13 (Conflicting Core). A finite set of unconstrained clauses and
a solvable constraint \((N^\perp; \pi)\) are a conflicting core if \(N^\perp \delta\) is unsatisfiable for
all solutions \( \delta \) of \( \pi \) over \( \text{vars}(N^\perp) \cup \text{lvar}(\pi) \). A conflicting core \((N^\perp; \pi)\) is a
conflicting core of the constrained clause set \( N \) if for every \( C \in N^\perp \) there is a
clause \((C', \pi') \in N\) such that \((C; \pi)\) is an instance of \((C'; \pi')\) modulo duplicate

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\[ [r, L, (C; \pi), N] \Rightarrow [r, L, (C'; \pi'), N'] \text{ for all } (L, r) \in \Delta. \]

If \( N \Rightarrow_{\text{LI}} N' \), \( C = \Gamma \Rightarrow \Delta, E' \left[ x \right]_p, E \left[ x \right]_q \) and \( C' = \Gamma \Rightarrow \Delta, E' \left[ x \right]_p, E \left[ q/x' \right] \),

\[ [r, E' \left[ x \right]_p, (C; \pi), N] \Rightarrow [r, E' \left[ x \right]_p, (C'; \pi'), N'] \text{ for all } r \in \text{pos}(E' \left[ x \right]_p). \]

\[ [r, E \left[ x \right]_q, (C; \pi), N] \Rightarrow [r, E' \left[ q/x' \right], (C'; \pi'), N'] \text{ for all } r \in \text{pos}(E \left[ q/x' \right]). \]

\[ [r, L, (C; \pi), N] \Rightarrow [r, L \{ x \mapsto x' \}, (C'; \pi'), N'] \text{ for all } (L, r) \in \Gamma, \]

\[ [r, L, (C; \pi), N] \Rightarrow [r, L, (C'; \pi'), N'] \text{ for all } (L, r) \in \Delta. \]

\[ \Gamma \Rightarrow E \left[ s \right]_p, \Delta \] shallow left

\[ \Gamma \Rightarrow E \left[ s \right]_p, \Delta \] shallow right

\[ \Gamma \Rightarrow E \left[ s \right]_p, \Delta \] shallow resolvent

linear 1

linear 2

Fig. 1. Visual representation of the parent literal position relation (Definition 12)
literal elimination. \((C'; \pi')\) is then called the instance clause of \((C; \pi)\) in \((N'\uplus; \pi)\). We call \((N'\uplus; \pi)\) complete if for every clause \(C \in N'\downarrow\) and literal \(L \in C\), there exists a clause \(D \in N'\downarrow\) with \(\overline{L} \in D\).

A conflicting core \((N'\downarrow; \pi)\) can always be completed by removing clauses in \(N'\downarrow\) that violate the condition. Since they contain literals that can never be resolved away, such clauses do not contribute to an unsatisfiability proof of \(N\).

We discuss the potential lifting failures and the corresponding refinements only for the linear and shallow case because lifting the satisfiability equivalent monadic and refinement transformations always succeeds. To reiterate from our previous work: In the linear case, there exists a clause in the conflicting core that is not an instance of the original clauses. In the shallow case, there exists a pair of clauses whose resolvent is not an instance of the original clauses. We combine these two cases by introducing the notion of a lift-conflict.

**Definition 14 (Conflict).** Let \(N \cup \{(C, \pi)\} \Rightarrow_{LI} N \cup \{(C_a, \pi_a)\}\) and \(N'\downarrow\) be a complete ground conflicting core of \(N \cup \{(C_a, \pi_a)\}\). We call a conflict clause \(C_c \in N'\downarrow\) with the instance clause \((C_a, \pi_a)\) a lift-conflict if \(C_c\) is not an instance of \((C, \pi)\) modulo duplicate literal elimination. Then, \(C_c\) is an instance of \((C_a, \pi_a)\), which we call the conflict clause of \(C_c\).

Let \(N \cup \{(C, \pi)\} \Rightarrow_{SH} N \cup \{(C_l, \pi_l), (C_r, \pi_r)\}\), \((C_a; \pi_a)\) be the shallow resolvent and \(N'\downarrow\) be a complete ground conflicting core of \(N \cup \{(C_l, \pi_l), (C_r, \pi_r)\}\). We call the resolvent \(C_c\) of \(C_l\delta_l \in N'\downarrow\) and \(C_r\delta_r \in N'\downarrow\) a lift-conflict if \(C_c\) is not an instance of \((C, \pi)\) modulo duplicate literal elimination. Then, \(C_c\) is an instance of \((C_a; \pi_a)\), which we call the conflict clause of \(C_c\).

The goal of refinement is to change the original parent clause in such a way that is both satisfiability equivalent and prevents the lift-conflict after approximation. Solving the refined approximation will then either necessarily produce a complete saturation or a new refutation proof because its conflicting core has to be different. For this purpose, we use the refinement transformation to segment the original parent clause \((C; \pi)\) into two parts \((C; \pi \land x \neq t)\) and \((C; \pi)\{\substitute x \rightarrow t\}\).

For example, consider \(N\) and its linear transformation \(N'\).

\[
P(a, b) \Rightarrow \Rightarrow_{LI} \Rightarrow_{AP} \Rightarrow_{AP} \Rightarrow_{AP} \Rightarrow_{AP} \Rightarrow_{AP}
\]

The ground conflicting core of \(N'\) is

\[
P(a, b) \Rightarrow \Rightarrow_{AP} \Rightarrow_{AP} \Rightarrow_{AP} \Rightarrow_{AP} \Rightarrow_{AP}
\]

Because \(P(a, b)\) is not an instance of \(P(x, x)\), lifting fails. \(P(a, b)\) is the lift-conflict. Specifically \(\{x \rightarrow a\}\) and \(\{x \rightarrow b\}\) are conflicting substitutions for the parent variable \(x\). We pick \(\{x \rightarrow a\}\) to segment \(P(x, x)\) into \((P(x, x); x \neq a)\) and \(P(x, x)\{x \rightarrow a\}\). Now, any descendant of \((P(x, x); x \neq a)\) cannot have \(a\) at the position of the first \(x\), and any descendant of \((P(x, x)\{x \rightarrow a\})\) must have an \(a\) at the position of the second \(x\). Thus, \(P(a, b)\) is excluded in both cases and no longer appears as a lift-conflict.

To show that the lift-conflict will not reappear in the general case, we use that the conflict clause and its ancestors have strong ties between their term structures and constraints.
Definition 15 (Constrained Term Skeleton). The constrained term skeleton of a term $t$ under constraint $\pi$, $\text{skt}(t, \pi)$, is defined as the normal form of the following transformation:

$$(t[x]_{p,q}; \pi) \Rightarrow_{\text{skt}} (t[q/x']; \pi \land \pi\{x \mapsto x'\})$$

where $p \neq q$ and $x'$ is fresh.

The constrained term skeleton of a term $t$ is essentially a linear version of $t$ where the restrictions on each variable position imposed by $\pi$ are preserved. For $(t, \pi)$ and a solution $\delta$ of $\pi$, $t\delta$ is called a ground instance of $(t, \pi)$.

Lemma 7. Let $N_0 \Rightarrow_{\text{AP}} N_k$, $(C_k; \pi_k)$ in $N$ with the ancestor clause $(C_0; \pi_0) \in N_0$ and $(N_k^\uparrow; \top)$ be a complete ground conflicting core of $N_k$. Let $\delta$ be a solution of $\pi_k$ such that $C_k\delta$ is in $N_k^\uparrow$. If $(L', q')$ is a literal position in $(C_k; \pi_k)$ with the ancestor $(L, q)$ in $(C_0, \pi_0)$, then (i) $L'\delta|_{q'}$ is an instance of $\text{skt}(L|_{q_0}, \pi_0)$, (ii) $q = q'$ if $L$ and $L'$ have the same predicate, and (iii) if $L'|_{q'} = x$ and there exists an ancestor variable $y$ of $x$ in $(C_0, \pi_0)$, then $L|_q = y$.

Proof. By induction on the length of the approximation $N_0 \Rightarrow_{\text{AP}} N_k$.

The base case $N_k = N_0$, is trivial. Let $N_0 = N \cup \{(C; \pi)\} \Rightarrow_{\text{Sh}} N \cup \{(C_1; \pi_1), (C_2; \pi_2)\} = N_k$, $(C_k; \pi_k)$ be the shallow $\rho$-resolvent and $C_k\delta$ be the resolvent of two instances of $(C_1; \pi_1)$ and $(C_2; \pi_2)$ in $N_k^\uparrow$. Then, $(C_k; \pi_k)\rho^{-1}$ is equal to $(C; \pi)$ modulo duplicate literal elimination. Thus, by definition $(L, q) = (L', q')\rho^{-1}$. Therefore, (i) $L'\delta|_{q'}$ is an instance of $\text{skt}(L|_{q_0}, \pi_0)$, (ii) $q = q'$ if $L$ and $L'$ have the same predicate, and (iii) if $L'|_{q'} = x$ and there is an ancestor variable $y$ of $x$ in $(C_0, \pi_0)$, then $L|_q = y$.

Now, let $N_0 \Rightarrow_{\text{AP}} N_1 \Rightarrow_{\text{AP}} N_k$. Since $(L', p)$ has an ancestor literal position in $(C_0, \pi_0)$, the ancestor clause of $(C_k; \pi_k)$ in $N_1$, $(C_1, \pi_1)$, contains the the ancestor literal position $(L_1, q_1)$, which has $(L, q)$ as its parent literal position. By the induction hypothesis on $N_1 \Rightarrow_{\text{AP}} N_k$, (i) $L'\delta|_{q'}$ is an instance of $\text{skt}(L|_{q_1}, \pi_1)$, (ii) $q_1 = q'$ if $L_1$ and $L'$ have the same predicate, and (iii) if $L'|_{q'} = x$ and there is an ancestor variable $y_1$ of $x$ in $(C_1, \pi_1)$, then $L_1|_{q_0} = y_1$.

Let $N_0 = N \cup \{(C; \pi)\} \Rightarrow_{\text{Ref}} N \cup \{(C; \pi \land x \neq t), (C; \pi\{x \mapsto t\})\} = N_1$. If $(C_1, \pi_1)$ is neither $(C; \pi \land x \neq t)$ nor $(C; \pi\{x \mapsto t\})$, then trivially $(C_0, \pi_0) = (C_1, \pi_1)$. Otherwise, $(C_1, \pi_1) = (C; \pi \land x \neq t)$ or $(C_1, \pi_1) = (C; \pi\{x \mapsto t\})$. Then $(L_1, q_1) = (L, q)$ or $(L_1, q_1) = (L, q)\{x \mapsto t\}$. In either case, (i) $L'\delta|_{q'}$ is an instance of $\text{skt}(L|_{q_0}, \pi_0)$, (ii) $q = q'$ if $L$ and $L'$ have the same predicate, and (iii) if $L'|_{q'} = x$ and there exists an ancestor variable $y$ of $x$ in $(C_0, \pi_0)$, then $L|_{q} = y$.

Let $N_0 \Rightarrow_{\text{MO}} \mu_p(N) = N_1$. If $P$ is not the predicate of $L$, then trivially $(L, q) = (L_1, q_1)$. If $P$ is the predicate of $L$, then $(L, q) = (P(t_1, \ldots, t_n), q)$ and $(L_1, q_1) = (T(f_p(t_1, \ldots, t_n)), 1.q)$. Thus, (i) $L'\delta|_{q'}$ is an instance of $\text{skt}(L|_{q_0}, \pi_0) = \text{skt}(T(f_p(t_1, \ldots, t_n)|_{1, q_0})$. (ii) The predicate of $L'$ is $P$ by definition. (iii) Let $L'|_{q'} = x$ and $y$ be the ancestor variable of $x$ in $(C_0, \pi_0)$. Then, $y$ is also the ancestor variable of $x$ in $(C_1, \pi_1)$ and $L_1|_{q_0} = y$. Therefore, $L|_{q} = P(t_1, \ldots, t_n)|_{q} = T(f_p(t_1, \ldots, t_n)|_{1, q} = L_1|_{q_1} = y$.

Let $N_0 = N \cup \{(C; \pi)\} \Rightarrow_{\text{L1}} N \cup \{(C_a; \pi_a)\} = N_1$ where an occurrence of a variable $x$ is replaced by a fresh $x'$. If $(C_1, \pi_1) \neq (C_0, \pi_0)$, then trivially $(C_0, \pi_0) = (C_1, \pi_1)$. Otherwise, $(C_1, \pi_1) = (C_0, \pi_0) = (C, \pi)$. By definition, $(L, q) = (L_1\{x' \mapsto x\}, q_1)$ and $\pi_0 = \pi_1\{x' \mapsto x\}$. Thus, $\text{skt}(L|_{q_0}, \pi_0) = ...$
respectively. \( L' \delta' \) is an instance of \( \text{skt}(L_1, \pi_0) \). Since \( L \) and \( L_1 \) have the same predicate and \( q = q_1, q = q' \) if \( L \) and \( L' \) have the same predicate. Let \( L' \delta' = z \) and \( y \) be the ancestor variable of \( z \) in \((C_1, \pi_1)\). If \( y \neq x' \), then \( y \) is the ancestor variable of \( z \) in \((C_0, \pi_0)\) and \( L_0 q = L_1 \{x' \mapsto x\} |_{q_0} = y_1 \). Otherwise, \( x \) is the ancestor variable of \( z \) in \((C_0, \pi_0)\) and \( L_0 q = L_1 \{x' \mapsto x\} |_{q_1} = x \).

Let \( N_0 = N \cup \{(C; \pi)\} \Rightarrow_S N \cup \{(C_1; \pi_1), (C_2; \pi_2)\} = N_1 \) where a term \( s \) is extracted from a positive literal \( Q(s'[s]_{p}) \) via introduction of fresh predicate \( S \) and variable \( x \). If \((C_1, \pi_1)\) is neither \((C_1; \pi_1)\) nor \((C_2; \pi_2)\), then trivially \((C_0, \pi_0) = (C_1, \pi_1)\).

If \((C_1, \pi_1) = (C_1; \pi_1)\) and \( L_1 = S(x) \), then \((C_0, \pi_0) = (C; \pi)\) and \((Q(s'[s]_{p}), q_1)\) in \((C; \pi)\) and ancestor literal position of \((L', q')\) in \((C_k, \pi_k)\). If \( q_1 \) is not a position at or above \( p \), the subterm at \( p \) is irrelevant and thus \( \text{skt}(Q(s'[s]_{p}), q_1, \pi) = \text{skt}(Q(s'[s]_{p}), q_1, \pi_1) \). Otherwise, let \( r \) be a position such that \( qr = 1.p \). Since \([p] = 2\), no following shallow transformation step extracts a subterm of \( s'[s]_{p} \) containing \( x \). Thus by definition of \( \Rightarrow_A \), \( L' = Q(t'[v]_{[x,p]}) \) and \( C_k \) also contains the negative literal \( S(x) \).

Let \( S(x) \delta = S(t) \). Analogously to the previous case, \( t \) is an instance of \( \text{skt}(s, \pi_1) \).

Combined with \( L' \delta' \) being an instance of \( \text{skt}(L_1 q_1, \pi_1) = \text{skt}(Q(s'[s]_{p}), q_1, \pi_1) \) and \( L' \delta'_{[1,p]} = t \), \( L' \delta'_{[1]} \) is an instance of \( \text{skt}(Q(s'[s]_{p}), q_1, \pi) \). Since \( L_1 \) and \( L_1 \) have the same predicate and \( q = q_1, q = q' \) if \( L \) and \( L' \) have the same predicate. Let \( L' \delta' = z \) and \( y \) be the ancestor variable of \( z \) in \((C_k, \pi_k)\). Since \( x \) has no parent, \( y \neq x \) and \( y \) in \((C_0, \pi_0)\) is the ancestor variable of \( z \). Therefore, \( Q(s'[s]_{p}) |_{q_1} = y \) because \( Q(s'[s]_{p}) |_{q_1} = y \).

If \((C_1, \pi_1) = (C_2; \pi_2)\) and \( L_1 = S(s) \), let \( q_1 = 1.q' \). Then, \((C_0, \pi_0) = (C; \pi)\) and \((L, q) = (Q(s'[s]_{p}), 1.q'_{p})\) in \((C_0, \pi_0)\) is the parent literal position of \((L_1, q_1)\) in \((C_1, \pi_1)\). Thus, \( L' \delta'_{q'} \) is an instance of \( \text{skt}((Q(s'[s]_{p}), 1.q'_{p}), \pi) = \text{skt}(s_{q'_1}, \pi) = \text{skt}(L_1 q_1, \pi_1) \). Because \( S \) is fresh, \( Q \) is not the predicate of \( L' \). Let \( L' \delta'_{q'} = z \) and \( y \) be the ancestor variable of \( z \) in \((C_k, \pi_k)\). Then, \( y \) in \((C_0, \pi_0)\) is the ancestor variable of \( z \) and \( Q(s'[s]_{p}) |_{q} = s_{q'_1} = y \) because \( s_{q'_1} = L_1 q_1 = y \).

Otherwise, \((L_1, q_1)\) in \((C_0, \pi_0)\) is the parent literal position of \((L_1, q_1)\) in \((C_1, \pi_1)\), by definition. Then, \( \text{skt}(L_1, \pi) = \text{skt}(L_1, \pi_1) \) or \( \text{skt}(L_1, \pi) = \text{skt}(L_1, \pi_r) \), respectively.

Next, we define the notion of descendants and descendant relations to connect lift-conflicts in ground conflicting cores with their corresponding ancestor clauses. The goal, hereby, is that if a ground clause \( D \) is not a descendant of a
clause in \( N \), then it can never appear in a conflicting core of an approximation of \( N \).

**Definition 16 (Descendants).** Let \( N \Rightarrow_{AP} N' \), \([(C; \pi), N] \Rightarrow_{A} [(C'; \pi'), N'] \) and \( D \) be a ground instance of \((C'; \pi')\). Then, we call \( D \) a descendant of \((C; \pi)\) and define the \([(C; \pi), N] \Rightarrow_{A} [(C'; \pi'), N']\)-descendant relation \( \Rightarrow_{D} \) that maps literals in \( D \) to literal positions in \((C; \pi)\) using the following rule:

\[
L' \delta \Rightarrow_{D} (L, r) \text{ if } L' \delta \in D \text{ and } [r, L, (C; \pi), N] \Rightarrow_{A} [\varepsilon, L', (C'; \pi'), N']
\]

For the descendant relations it is of importance to note that while there are potentially infinite ways that a lift-conflict \( C_c \) can be a descendant of an original clause \((C; \pi)\), there are only finitely many distinct descendant relations over \( C_c \) and \((C; \pi)\). This means, if a refinement transformation can prevent one distinct descendant relation without generating new distinct descendant relations (Lemma 8), a finite number of refinement steps can remove the lift-conflict \( C_c \) from the descendants of \((C; \pi)\) (Lemma 9). Thereby, preventing any conflicting cores containing \( C_c \) from being found again.

A clause \((C; \pi)\) can have two descendants that are the same except for the names of the \( S \)-predicates introduced by shallow transformations. Because the used approximation \( N \Rightarrow_{AP} N' \) is arbitrary and therefore also the choice of fresh \( S \)-predicates, if \( D \) is a descendant of \((C; \pi)\), then any clause \( D' \) equal to \( D \) up to a renaming of \( S \)-predicates is also a descendant of \((C; \pi)\). On the other hand, the actual important information about an \( S \)-predicate is which term it extracts. Two descendants of \((C; \pi)\) might be the exactly the same but their \( S \)-predicate extract different terms in \((C; \pi)\). For example, \( P(a) \rightarrow S(f(a)) \) is a descendant of \( P(x), P(y) \rightarrow Q(f(x), g(f(x))) \) but might extract either occurrence of \( f(x) \). These cases are distinguished by their respective descendant relations. In the example, we have either \( S(f(a)) \Rightarrow_{D} (Q(f(x), g(f(x))), 1) \) or \( S(f(a)) \Rightarrow_{D} (Q(f(x), g(f(x))), 2) \).

**Lemma 8.** Let \( N_0 = N \cup \{(C; \pi)\} \Rightarrow_{Ref} N \cup \{(C; \pi \land x \neq t), (C; \pi)\} \Rightarrow_{Ref} N_1 \) be a refinement transformation and \( D \) a ground clause. If there is a \([(C; \pi \land x \neq t), N_1] \Rightarrow_{A} [(C'; \pi'), N_2]\)- or \([(C; \pi)\} \Rightarrow_{Ref} N_1 \Rightarrow_{A} [(C'; \pi'), N_2]\)-descendant relation \( \Rightarrow_{D} \), then there is an equal \([(C; \pi), N_0] \Rightarrow_{A} [(C'; \pi'), N_2]\)-descendant relation \( \Rightarrow_{D} \).

**Proof.** Let \( L_D \) be a literal of \( D \) and \( L' \Rightarrow_{D} (L, r) \). If \( D \) is a descendant of \((C; \pi \land x \neq t)\), then \([r, L, (C; \pi \land x \neq t), N_1] \Rightarrow_{A} [\varepsilon, L', (C'; \pi'), N_2]\). Because \([r, L, (C; \pi), N_0] \Rightarrow_{A} [r, L, (C; \pi \land x \neq t), N_1], L' \Rightarrow_{D} (L, r) \). If \( D \) is a descendant of \((C; \pi)\} \Rightarrow_{D} \), the proof is analogous. \( \square \)

**Lemma 9 (Refinement).** Let \( N \Rightarrow_{AP} N' \) and \((N^\perp; \top)\) be a complete ground conflicting core of \( N' \). If \( C_c \in N^\perp \) is a lift-conflict, then there exists a finite refinement \( N \Rightarrow_{Ref} N_R \) such that for any approximation \( N_R \Rightarrow_{AP} N_R' \) and ground conflicting core \((N^\perp_R; \top)\) of \( N_R' \), \( C_c \) is not a lift-conflict in \((N^\perp_R; \top)\) modulo duplicate literal elimination.
Proof. Let \((C_a, \pi_a)\) be the conflict clause of \(C_c\) and \((C; \pi) \in N\) be the parent clause of \((C_a, \pi_a)\). \(C_c\) is a descendant of \((C; \pi)\) with the corresponding \([((C; \pi), N] \Rightarrow \Lambda [(C_a; \pi_a), N']\)-descendant relation \(\Rightarrow_0^\Lambda\). We apply induction on the number of distinct \(((C; \pi), N] \Rightarrow \Lambda [(C'; \pi'), N'']\)-descendant relations \(\Rightarrow_{C_c}\) for arbitrary approximations \(N \Rightarrow_{AP}\ N''\).

Since only the shallow and linear transformations can produce lift-conflicts, the clause \((C; \pi)\) is replaced by either a linearized clause \((C'; \pi')\) or two shallow clauses \((C_1; \pi)\) and \((C_2; \pi)\). Then, the conflict clause \((C_a; \pi_a)\) of \(C_c\) is either the linearized \((C'; \pi')\) or the resolvent of \((C_1; \pi)\) and \((C_2; \pi)\). In either case, \(C_c = C_a\delta\) for some solution \(\delta\) of \(\pi_a\). Furthermore, there exists a substitution \(\tau = \{x_1' \mapsto x_1, \ldots, x_n' \mapsto x_n\}\) such that \((C; \pi)\) and \((C_a; \pi_a)\tau\) are equal modulo duplicate literal elimination. That is, \(\tau = \{x' \mapsto x\}\) for a linear transformation and \(\tau = \rho^{-1}\) for shallow transformation (Definition 9).

Assume \(C_c = C_0\tau\sigma\) for some grounding substitution \(\sigma\), where \(\sigma\tau\sigma\) is a solution of \(\pi_a\). Thus, \(\sigma\) is a solution of \(\pi_a\tau\), which is equivalent to \(\pi\). Then, \(C_c\) is equal to \(C\sigma\) modulo duplicate literal elimination an instance of \((C; \pi)\), which contradicts with \(C_c\) being a lift-conflict. Hence, \(C_c = C_a\delta\) is not an instance of \(C_a\tau\) and thus \(x_i\delta \neq x'_i\delta\) for some \(x_i\) in the domain of \(\tau\).

Because \(x_i\delta\) and \(x'_i\delta\) are ground, there is a position \(p\) where \(x_i\delta|_p\) and \(x'_i\delta|_p\) have different function symbols. We construct the straight term \(t\) using the path from the root to \(p\) on \(x_i\delta\) with variables that are fresh in \((C, \pi)\). Then, we can use \(x_i\) and \(t\) to segment \((C; \pi)\) into \(\{C; \pi \land x_i \neq t\}\) and \(\{C; \pi\} \{x_i \mapsto t\}\) for the refinement \(N \Rightarrow_{Ref} N_R\). Note, that \(x_i\delta\) is a ground instance of \(t\), while \(x'_i\delta\) is not.

Let \((L_1', r_1')\) and \((L_2', r_2')\) in \((C_a, \pi_a)\) be literal positions of the variables \(x_i\) and \(x'_i\) in \(C_a\), and \((L_1, r_1)\) and \((L_2, r_2)\) in \((C, \pi)\) be the parent literal positions of \((L_1', r_1')\) and \((L_2', r_2')\), respectively. Because \((C_a, \pi_a)\tau\) is equal to \((C; \pi)\) modulo duplicate literal elimination, \(L_1|_{r_1} = L_2|_{r_2} = x_i\). Let \(N \Rightarrow_{Ref} N_1\) be the refinement where \((C; \pi)\) is segmented into \(\{C; \pi \land x_i \neq t\}\) and \(\{C; \pi\} \{x_i \mapsto t\}\).

By Lemma 8 all \([\{C; \pi \land x_i \neq t\}, N_1] \Rightarrow_{\Lambda} ([C'_a; \pi'_a], N_2]\)-descendant relations correspond to an equal \([\{C; \pi\}, N] \Rightarrow_{\Lambda} ([C'_a; \pi'_a], N_2]\)-descendant relation. Assume there is a \([\{C; \pi \land x_i \neq t\}, N_1] \Rightarrow_{\Lambda} ([C'_a; \pi'_a], N_2]\)-descendant relation \(\Rightarrow_{C_c}^1\) that is not distinct from \(\Rightarrow_{C_c}^0\). Because \(L_1|_{\delta} = 0_{C_c} (L_1, r)\) for some literal position \(L_1, r\) in \((C; \pi, \pi)\), which is the parent literal position of \((L_1', r)\) in \((C; \pi \land x_i \neq t)\), \(L_1|_{\delta} = 0_{C_c} (L_1, r)\). However, this contradicts Lemma 7 because \(x_i\delta\) is not an instance of \(\text{sft}(L_1|_{r_1}, \pi \land x_i \neq t) = \text{sft}(x_i, \pi \land x_i \neq t)\). The case that there is a \([\{C; \pi\} \{x_i \mapsto t\}, N_1] \Rightarrow_{\Lambda} ([C'_a; \pi'_a], N_2]\)-descendant relation that is not distinct from \(\Rightarrow_{C_c}^0\) is analogous using the argument that \(x'_i\delta\) is not an instance of \(\text{sft}(L_2\{x_i \mapsto t\}, \pi) = \text{sft}(t, \pi)\). Hence, there are strictly less distinct descendant relations over \(C_c\) and \((C; \pi \land x_i \neq t)\) or \((C; \pi) \{x \mapsto t\}\) than there are distinct descendant relations over \(C_c\) and \((C, \pi)\).

If there are no descendant relations, then \(C_c\) can no longer appear as a lift conflict. Otherwise, by the inductive hypothesis, there exists a finite refinement \(N \Rightarrow_{Ref} N_1 \Rightarrow_{Ref} N_R\) such that for any approximation \(N_R \Rightarrow_{AP} N''_R\) and ground
conflicting core $N'_R$ of $N'_R$. $C_c$ is not a lift-conflict in $N'_R$ modulo duplicate literal elimination.

Theorem 2 (Soundness and Completeness of FO-AR). Let $N$ be an unsatisfiable clause set and $N'$ its MSL(SDC) approximation: (i) if $N$ is unsatisfiable then there exists a conflicting core of $N'$ that can be lifted to a refutation in $N$; (ii) if $N'$ is satisfiable, then $N$ is satisfiable too.

Proof. (Idea) By Lemma 8 and Lemma 9, where the latter can be used to show that a core of $N'$ that cannot be lifted also excludes the respective instance for unsatisfiability of $N$.

Let $(C_a, \pi_a)$ be the conflict clause of $C_c$ and $(C; \pi) \in N$ be the parent clause of $(C_a, \pi_a)$. $C_c$ is a descendant of $(C; \pi)$ with the corresponding $\lambda_a ([C; \pi], N) \Rightarrow \lambda_a ([C_a; \pi_a], N')$. We apply induction on the number of distinct $\lambda_a ([C; \pi], N) \Rightarrow \lambda_a ([C'; \pi'], N'')$-descendant relations $\Rightarrow C_c$ for arbitrary approximations $N \Rightarrow \lambda N''$. Since only the shallow and linear transformations can produce lift-conflicts, the clause $(C; \pi)$ is replaced by either a linearized clause $(C'; \pi')$ or two shallow clauses $(C_1; \pi)$ and $(C_2; \pi)$. Then, the conflict clause $(C_a; \pi_a)$ of $C_c$ is either the linearized $(C'; \pi')$ or the resolvent of $(C_1; \pi)$ and $(C_2; \pi)$. In either case, $C_c = C_a \delta$ for some solution $\delta$ of $\pi_a$. Furthermore, there exists a substitution $\tau = \{x_1' \mapsto x_1, \ldots, x_n' \mapsto x_n\}$ such that $(C; \pi)$ and $(C_a; \pi_a)\tau$ are equal modulo duplicate literal elimination. That is, $\tau = \{x' \mapsto x\}$ for a linear transformation and $\tau = \rho^{-1}$ for shallow transformation (Definition 6).

Assume $C_c = C_a \tau \sigma$ for some grounding substitution $\sigma$, where $\tau \sigma$ is a solution of $\pi_a$. Thus, $\sigma$ is a solution of $\pi_a \tau$, which is equivalent to $\pi$. Then, $C_c$ is equal to $C \sigma$ modulo duplicate literal elimination an instance of $(C; \pi)$, which contradicts with $C_c$ being a lift-conflict. Hence, $C_c = C_a \delta$ is not an instance of $C_a \tau$ and thus, $x_1 \delta \neq x' \delta$ for some $x_1$ in the domain of $\tau$.

Because $x_1 \delta$ and $x'_1 \delta$ are ground, there is a position $p$ where $x_1 \delta[p]$ and $x'_1 \delta[p]$ have different function symbols. We construct the straight path $t$ using the path from the root to $p$ on $x_1 \delta$ with variables that are fresh in $(C; \pi)$. Then, we can use $x_1$ and $t$ to segment $(C; \pi)$ into $(C; \pi \land x_1 \neq t)$ and $(C; \pi \land x_1 \mapsto t)$ for the refinement $N \Rightarrow \sigma N R$. Note, that $x_1 \delta$ is a ground instance of $t$, while $x_1 \delta$ is not.

Let $(L_1, r_1)$ and $(L_2, r_2)$ in $(C_1, \pi_a)$ be literal positions of the variables $x_i$ and $x'_i$ in $C_a$, and $(L_1, r_1)$ and $(L_2, r_2)$ in $(C; \pi)$ be the parent literal positions of $(L'_1, r'_1)$ and $(L'_2, r'_2)$, respectively. Because $(C_a, \pi_a)\tau$ is equal to $(C; \pi)$ modulo duplicate literal elimination, $L_1|_{r_1} = L_2|_{r_2} = x_i$. Let $N \Rightarrow \sigma N R$ be the refinement where $(C; \pi)$ is segmented into $(C; \pi \land x_1 \neq t)$ and $(C; \pi \land x_1 \mapsto t)$.

By Lemma 8 all $[\lambda_a ([C; \pi], N_1) \Rightarrow \lambda_a ([C_a; \pi_a], N_2)$-descendant relations correspond to an equal $\lambda_a ([C; \pi], N) \Rightarrow \lambda_a ([C_a; \pi_a], N_2)$-descendant relation. Assume there is a $\lambda_a ([C; \pi \land x_1 \neq t], N_1) \Rightarrow \lambda_a ([C_a; \pi_a], N_2)$-descendant relation that is not distinct from $\Rightarrow C_c$. Because $L_1|_r \Rightarrow_{C_c} (L_1, r)$ for some literal position $(L_1, r)$ in $(C; \pi)$, which is the parent literal position of $(L_1, r)$ in $(C; \pi \land x_1 \neq t)$, $L_1|_r \Rightarrow_{C_c} (L_1, r)$. However, this contradicts Lemma 7 because $x_1 \delta$ is not an instance of $\text{skt}((L_1, r), \pi \land x_1 \neq t) = \text{skt}(x_1, \pi \land x_1 \neq t)$. Theorem 2 is proved.
\[ x_i \neq t \). The case that there is a \([ (C; \pi) \{ x_i \mapsto t \}, N_1 ] \Rightarrow_{\Lambda}^* \ [(C'_\pi; \pi'_\pi), N_2 ] \]-descendant relation that is not distinct from \( \Rightarrow_{C^*}^0 \) is analogous using the argument that \( x'_i \delta \) is not an instance of \( \text{skt}(L_{\{ x_i \mapsto t \}})_{\pi_2}, \pi \). Hence, there are strictly less distinct descendant relations over \( C_c \) and \((C; \pi \land x \neq t) \) or \((C; \pi) \{ x \mapsto t \} \) than there are distinct descendant relations over \( C_c \) and \((C, \pi) \).

If there are no descendant relations, then \( C_c \) can no longer appear as a lift conflict. Otherwise, by the inductive hypothesis, there exists a finite refinement \( N \Rightarrow_{\text{Ref}} N_1 \Rightarrow_{\text{Ref}} N_R \) such that for any approximation \( N_R \Rightarrow_{\text{AP}} N_R' \) and ground conflicting core \( N_R^\perp \) of \( N_R' \), \( C_c \) is not a lift-conflict in \( N_R^\perp \) modulo duplicate literal elimination.

Actually, Lemma 9 can be used to define a fair strategy on refutations in \( N' \) in order to receive also a dynamically complete FO-AR calculus, following the ideas presented in [13].

In Lemma 8 we segment the conflict clause’s immediate parent clause. If the lifting later successfully passes this point, the refinement is lost and will be possibly repeated. Instead, we can refine any ancestor of the conflict clause as long as it contains the ancestor of the variable used in the refinement. By Lemma 7(iii), such an ancestor will contain the ancestor variable at the same positions. If we refine the ancestor in the original clause set, the refinement is permanent because lifting the refinement steps always succeeds. Only variables introduced by shallow transformation cannot be traced to the original clause set. However, these shallow variables are already linear and the partitioning in the shallow transformation can be chosen such that they are not shared variables. Assume a shallow variable \( y \), that is used to extract term \( t \), is a shared variable in the shallow transformation of \( \Gamma \Rightarrow E[p], \Delta \) into \( S(x), \Gamma_i \Rightarrow E[p/x], \Delta_i \) and \( \Gamma_r \Rightarrow S(s), \Delta_r \). Since \( \Delta_i \cup \Delta_r = \Delta \) is a partitioning, \( y \) can only appear in either \( E[p/x], \Delta_i \) or \( S(s), \Delta_r \). If \( y \in \text{vars}(E[p/x], \Delta_i) \) we instantiate \( \Gamma_r \) with \( \{ y \mapsto t \} \) and \( \Gamma_i \), otherwise. Now, \( y \) is no longer a shared variable.

The refinement Lemmas only guarantee a refinement for a given ground conflicting core. In practice, however, conflicting cores contain free variables. We can always generate a ground conflicting core by instantiating the free variables with ground terms. However, if we only exclude a single ground case via refinement, next time the new conflicting core will likely have overlap with the previous one. Instead, we can often remove all ground instances of a given conflict clause at once.

The simplest case is when unifying the conflict clause with the original clause fails because their instantiations differ at some equivalent positions. For example, consider \( N = \{ P(x, x); P(f(x, a), f(y, b)) \} \). It is satisfiable but the linear transformation is unsatisfiable with conflict clause \( P(f(x, a), f(y, b)) \) which is not unifiable with \( P(x, x) \), because the two terms \( f(x, a) \) and \( f(y, b) \) have different functions at the second argument. A refinement of \( P(x, x) \) is

\[
(P(x, x); x \neq f(v, a))
\]

\[
(P(f(x, a), f(x, a)); \top)
\]
While instances with refinement will continue to enumerate all which is not unifiable with signature \(\Sigma\) \cite{pire41}, for ordered-resolution/superposition, iProver 2.5 \cite{7} an implementation and state-of-the-art implementations, in particular SPASS 3.9 \cite{17}, Vamp-and instantiation-based methods fail. We argue both according to the respective calculations.

In the following we discuss several first-order clause structures for which FO-AR 5 Experiments without ever reaching a satisfiable approximation.

Lastly, we should mention that there are cases where the refinement process, trying to cover all ground instances of the conflict clause does not terminate. For example, consider the clause set \(N = \{P(x, x); (P(x, y) \rightarrow; x \neq a \land x \neq b \land y \neq c \land y \neq d)\}\) with signature \(\Sigma = \{a, b, c, d\}\). It is satisfiable but the linear transformation of \(N\) is unsatisfiable with conflict clause \(P(x, y); x \neq a \land x \neq b \land y \neq c \land y \neq d\). While \(P(x, x)\) and \(P(x, y)\) are unifiable, the resulting constraint \(x \neq a \land x \neq b \land x \neq c \land x \neq d\) has no solutions. A refinement of \(P(x, x)\) is

\[
(P(x, x); x \neq a \land x \neq b) \\
(P(a, a); \top) \\
(P(b, b); \top)
\]

\((P(x, y); x \neq a \land x \neq b \land y \neq c \land y \neq d)\) shares no ground instances with the approximations of the refined clauses.

While \(P(y, g(y))\) is not an instance of the refined approximation, it shares ground instances with \(P(g(x), g(x'))\). The new conflict clause is \(P(g(y), g(g(y)))\) and the refinement will continue to enumerate all \(P(g^i(x), g^i(x))\) instances of \(P(x, x)\) without ever reaching a satisfiable approximation.

5 Experiments

In the following we discuss several first-order clause structures for which FO-AR implemented in SPASS-AR immediately decides satisfiability but superposition and instantiation-based methods fail. We argue both according to the respective calculi and state-of-the-art implementations, in particular SPASS 3.9 \cite{17}, Vampire 4.1 \cite{15}, for ordered-resolution/superposition, iProver 2.5 \cite{7} an implement-
tation of Inst-Gen \(^8\), and Darwin v1.4.5 \(^2\) an implementation of the model evolution calculus \(^3\). All experiments were run on a 64-Bit Linux computer (Xeon(R) E5-2680, 2.70GHz, 256GB main memory). For Vampire and Darwin we chose the respective CASC setting, for iProver we set the schedule to “sat” and SPASS, SPASS-AR were used in default mode. Please note that Vampire and iProver are portfolio solvers including implementations of several different calculi whereas SPASS, SPASS-AR, and Darwin only implement superposition, FO-AR, and model evolution, respectively.

For the first example

\[ P(x, y) \rightarrow P(x, z), P(z, y); \quad P(a, a) \]

and second example,

\[ Q(x, x); \quad Q(v, w), P(x, y) \rightarrow P(x, v), P(w, y); \quad P(a, a) \]

the superposition calculus produces independently of the selection strategy and ordering an infinite number of clauses of form

\[ \rightarrow P(a, z_1), P(z_1, z_2), \ldots, P(z_n, a). \]

Using linear approximation, however, FO-AR replaces \[ P(x, y) \rightarrow P(x, z), P(z, y) \]
and \[ \rightarrow Q(x, x) \] with \[ P(x, y) \rightarrow P(x, z), P(z', y) \] and \[ \rightarrow Q(x, x') \], respectively. Consequently, ordered resolution derives \[ \rightarrow P(a, z_1), P(z_2, a) \] which subsumes any further inferences \[ \rightarrow P(a, z_1), P(z_2, z_3), P(z_4, a) \]. Hence, saturation of the approximation terminates immediately. Both examples belong to the Bernays-Schönfinkel fragment, so model evolution (Darwin) and Inst-Gen (iProver) can decide them as well. Note that the concrete behavior of superposition is not limited to the above examples but potentially occurs whenever there are variable chains in clauses.

On the third problem

\[ P(x, y) \rightarrow P(g(x), z); \quad P(a, a) \]

superposition derives all clauses of the form \[ \rightarrow P(g(\ldots g(a)\ldots), z) \]. With a shallow approximation of \[ P(x, y) \rightarrow P(g(x), z) \] into \[ S(v) \rightarrow P(v, z) \] and \[ P(x, y) \rightarrow S(g(x)) \], FO-AR (SPASS-AR) terminates after deriving \[ \rightarrow S(g(a)) \] and \[ S(x) \rightarrow S(g(x)) \]. Again, model evolution (Darwin) and Inst-Gen (iProver) can also solve this example.

The next example

\[ P(a); \quad P(f(a)) \rightarrow; \quad P(f(f(x))) \rightarrow P(x); \quad P(x) \rightarrow P(f(f(x))) \]

is already saturated under superposition. The same almost holds true for FO-AR, where \[ P(x) \rightarrow P(f(f(x))) \] is replaced by \[ S(x) \rightarrow P(f(x)) \] and \[ P(x) \rightarrow S(f(x)) \]. Then ordered resolution terminates after inferring \[ S(a) \rightarrow \] and \[ S(f(x)) \rightarrow P(x) \].

The Inst-Gen and model evolution calculi, however, fail. In either, a satisfying model is represented by a finite set of literals, i.e., a model of the propositional approximation for Inst-Gen and the trail of literals in case of model evolution. Therefore, there necessarily exists a literal \[ P(f^n(x)) \] or \[ \neg P(f^n(x)) \] with a maximal \( n \) in these models. This contradicts the actual model where either \[ P(f^n(a)) \] or \[ P(f^n(f(a))) \] is true. However, iProver can solve this problem using its built-in ordered resolution solver whereas Darwin does not terminate on this problem.

Lastly consider an example of the form

\[ f(x) \approx x \rightarrow; \quad f(f(x)) \approx x \rightarrow; \quad \ldots; \quad f^n(x) \approx x \rightarrow \]
which is trivially satisfiable, e.g., saturated by superposition, but any model has at least \(n+1\) domain elements. Therefore, adding these clauses to any satisfiable clause set containing \(f\) forces calculi that explicitly consider finite models to consider at least \(n+1\) elements. The performance of final model finders typically degrades in the number of different domain elements to be considered.

Combining each of these examples into one problem is then solvable by neither superposition, Inst-Gen, or model evolution and not practically solvable with increasing \(n\) via testing finite models. For example, we tested

\[
\begin{align*}
P(x, y) \rightarrow P(x, z), P(z, y); & \quad P(a, a); \quad P(f(a), y) \rightarrow; \\
P(f(f(x)), y) \rightarrow P(x, y); & \quad P(x, y) \rightarrow P(f(f(x)), y); \\
f(x) \approx x \rightarrow; & \ldots, f^n(x) \approx x \rightarrow;
\end{align*}
\]

for \(n = 11\) against SPASS, Vampire, iProver, and Darwin for more than one hour each without success. Only SPASS-AR solved it in less than one second.

### 6 Conclusion

The previous section showed FO-AR is superior to superposition, instantiation-based methods on certain classes of clause sets. Of course, there are also classes of clause sets where superposition and instantiation-based methods are superior to FO-AR, e.g., for unsatisfiable clause sets where the structure of the clause set forces FO-AR to enumerate failing ground instances due to the approximation in a bottom-up way.

Our prototypical implementation SPASS-AR cannot compete with systems such as iProver or Vampire on the respective CASC categories of the TPTP\(^{12}\). This is already due to the fact that they are all meanwhile portfolio solvers. For example, iProver contains an implementation of ordered resolution and Vampire an implementation of Inst-Gen. Our results, Section 5, however, show that these systems may benefit from FO-AR by adding it to their portfolio.

The \(\text{DEXPTIME}\)-completeness result for MSLH strongly suggest that both the MSLH and also our MSL(SDC) fragment have the finite model property. However, we are not aware of any proof. If MSL(DSC) has the finite model property, the finite model finding approaches are complete on MSL(SDC), whereas the models generated by FO-AR and superposition are typically infinite. It remains an open problem, even for fragments enjoying the finite model property, e.g., the first-order monadic fragment, to design a calculus that combines explicit finite model finding with a structural representation of infinite models. For classes that have no finite models this problem seems to become even more difficult. To the best of our knowledge, SPASS is currently the only prover that can show satisfiability of the clauses \(R(x, x) \rightarrow; R(x, y), R(y, z) \rightarrow R(x, z); R(x, g(x)) \rightarrow\) due to an implementation of chaining. It is unknown to us how the result of specific inferences for transitivity can be incorporated in any of the discussed calculi in a generic way such that it becomes available for classes of clauses including variable chains.

Finally, there are not many results on calculi that operate with respect to models containing positive equations. Even for fragments that are decidable with
equality, such as the Bernays-Schoenfinkel-Ramsey fragment or the monadic fragment with equality, there seem currently no convincing suggestions compared to the great amount of techniques for these fragments without equality. Adding positive equations to MSL(SDC) while keeping decidability is, to the best of our current knowledge, only possible for at most linear, shallow equations \( f(x_1, \ldots, x_n) \approx h(y_1, \ldots, y_n) \) [6]. However, approximation into such equations from an equational theory with nested term occurrences results typically in an almost trivial equational theory. So it cannot be expected to be very useful in practice.

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