We investigate the generation of electromagnetic and gravitational radiation in the vicinity of a perturbed Schwarzschild black hole. The gravitational perturbations and the electromagnetic field are studied by solving the Teukolsky master equation with sources, which we take to be locally charged, radially infalling, matter. Our results show that, in addition to the gravitational wave generated as the matter falls into the black hole, there is also a burst of electromagnetic radiation.

This electromagnetic field has a characteristic set of quasinormal frequencies, and the gravitational radiation has the quasinormal frequencies of a Schwarzschild black hole. This scenario allows us to compare the gravitational and electromagnetic signals that are generated by a common source.

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I. INTRODUCTION

In recent years there has been a significant amount of work performed, globally, in an effort to detect gravitational radiation. It is expected that within the next few years interferometric detectors will be able to detect the gravitational waves emitted from events like binary black hole, or neutron star, mergers and exploding, or collapsing, stars [1]. One of the most challenging problems for the detectors, putting aside the technical obstacles, is the extraction of the waveform from the large amount of noise generated in the detection process.

Several models for electromagnetic signals correlated with the gravitational wave emissions in astrophysical systems have been proposed [2]. In this paper we describe a process in which the emission of gravitational waves would be accompanied by an electromagnetic signal, the infall of charged particles onto a black hole. Since information carried gravitationally is complementary to that carried electromagnetically, we can learn a great deal about the generating event and its environment.

For example, because the companion gravitational wave passes through matter almost unaffected, its properties would describe the conditions at which both signals were initially emitted. One could then derive an almost redshift independent signature, see [3, 4] and references therein.

The program of simulating multi-messenger signals has been considered for some time in different scenarios, and by means of several techniques. Concerning black hole spacetimes, the dynamics of black holes immersed in a magnetic field was considered for example in [5–8]. In [9], a study of the coupling of axial gravitational and electromagnetic perturbations for black holes and neutron stars was performed. It was found that for neutron stars the ratio of the energy carried by the electromagnetic waves to the energy carried by gravitational waves is proportional to the square of the magnetic field of the neutron star. The same authors extended their study in [10] to consider polar gravitational perturbations since these may drive longer emission rates. Moreover, a program to study, numerically, the collision of charged black holes was initiated in [11, 12]. In [13] it was found that there is a scaling between the gravitational signal and the electromagnetic one, where the scale factor is driven by the charge-mass ratio of the black holes.

Current estimates suggest that the gravitational wave signal most likely to be observed is the chirp from the inspiral of two compact objects, such as neutron stars or black holes in a close binary. It seems reasonable to expect that, after the coalescence of a compact binary, an accretion disk forms around the remnant black hole, and that
this accretion disk persists for a long period of time, emitting gravitational as well as electromagnetic radiation. The latter because the disks surrounding black holes are sources for magnetic fields, as was described in the seminal work of Shakura and Sunyaev [14].

One can make further assumptions about the charge transport in the disk. For example one might also assume that the negative charges in the disk travel faster than the positive charges, as discussed in [15], which would result in an effective motion of charged matter. In our model, building on our previous results in [16][18], we will take the volume elements to be charged, and we make the simplifying assumption that they move along geodesics: i.e. we neglect the back-reaction. As we show below, such matter does indeed generate a gravitational signal as well as an electromagnetic one, once it reaches the black hole horizon. We then attempt to determine correlations between the two signals.

The paper is organized as follows: In section II we present the gravitational perturbation equations, as well as the Maxwell equations within the Newman Penrose null tetrad formalism. In section III we focus on the Schwarzschild background, described in horizon penetrating coordinates and write down the outgoing perturbation and electromagnetic equations explicitly. In section IV we introduce the numerical procedure to solve the equations and present the resulting waveforms of both signals. Finally in section V we give some concluding remarks. We adopt geometric units, \( c = G = 1 \), and the metric signature \((-,+,+,+))\).

II. FOUNDATIONS: NEWMAN PENROSE FORMALISM

In [19] Teukolsky used the spinor formulation introduced by Newman and Penrose [20][21] to find an equation that describes neutrino, electromagnetic, and scalar fields, as well as gravitational perturbations, on a Kerr background. The main idea behind this formulation lies in the choice of a basis consisting of null vectors.

A complete review of the subject can be found in [22] and [23]. Teukolsky’s master equation which describes a perturbed black hole, includes a term describing the source of the perturbation. In this work we take the source of the perturbation to be a set of infalling particles following geodesics in a non-spherical distribution.

A. Gravitational perturbation

The Einstein equations, in the Newman - Penrose formalism, consist of five equations for the Weyl scalars \( \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4 \). We are interested in the scalar \( \Psi_4 \) that reflects the outgoing radiation. This scalar is given by

\[
\Psi_4 = -C_{\mu\nu\lambda\tau} k^\mu m^\nu k^\lambda m^\tau,
\]

where \( C_{\mu\nu\lambda\tau} \) is the Weyl tensor and \( k^\mu, m^\nu \) are two null vectors; the first one along the null cone, pointing inwards, and the second one lying in the plane perpendicular to the light cone. In [19] Teukolsky derived a decoupled equation for the perturbation of \( \Psi_4 \) in terms of the spin coefficients valid for any Type D metric. The equation for the perturbation \( \Psi_4 \) has the form:

\[
[(\Delta - 4 \mu s - \mu s^* - 3 \gamma s + \gamma s^*) (D + \rho s - 4 \epsilon s) - (\delta^s - 3 \alpha s - \beta s^* - 4 \pi s + \tau s^*) (\delta - 4 \beta s + \tau s)]
+ 3\Psi_2 \Psi_4^{(1)} = -4\pi T_4,
\]

where \( \Delta, D \) and \( \delta \) are derivative operators projected along the null directions of the tetrad components, \( \mu, \gamma, \rho, \epsilon, \beta, \pi, \tau \) are the spin coefficients in the Schwarzschild metric, \( * \) denotes the complex conjugate, and \( \Psi_2 \equiv -C_{\mu\nu\lambda\tau} l^\mu m^\nu m^\lambda m^\tau \) is the only non zero component of the background metric. The null vector \( l^\mu \) points outwards along the light cone. The source term \( T_4 \), is composed of operators acting on the projections of the stress-energy tensor \( T_{\mu\nu} \) that triggers the gravitational response along the tetrad:

\[
T_4 = \hat{T}^k k T_k k + \hat{T}^m m^* T_k m^* + \hat{T}^m m^* T_m^m m^* m^*.
\]

The operators \( \hat{T}^{ab} \) have the explicit form

\[
\hat{T}^{kk} = -[\delta^s + 3 \alpha s + \beta s^* + 4 \pi s - \tau s^*] (\delta^s - 2 \alpha s - 2 \beta s^* + \tau s^*),
\]

\[
\hat{T}^k m^* = (\Delta - 4 \mu s - \mu s^* - 3 \gamma s + \gamma s^*) (\delta^s - 2 \alpha s + \tau s^*)
+ (\delta^s - 3 \alpha s - \beta s^* - 4 \pi s + \tau s^*) (\Delta - 2 \mu s^* - \gamma s),
\]

\[
\hat{T}^m m^* = -[(\Delta - 4 \mu s - \mu s^* - 3 \gamma s + \gamma s^*) (\Delta - \mu s^* - 2 \gamma s + 2 \gamma s^*).
\]

and \( T_{kk} = T_{\mu\nu} k^\mu k^\nu, T_{km} = T_{\mu\nu} k^\mu m^\nu, T_{m^m m^*} T_{\mu\nu} m^* m^\nu \) are the projections of the tensor \( T_{\mu\nu} \), on the corresponding null vector.
The electromagnetic field, in the Newman Penrose formulation, is expressed in terms of three complex quantities which are projections of the Faraday tensor $F_{\mu \nu}$:

$$\phi_0 \equiv F_{\mu \nu} l^\mu m^\nu, \quad \phi_1 \equiv \frac{1}{2} F_{\mu \nu} (l^\mu k^\nu + m^\mu m^\nu), \quad \phi_2 \equiv F_{\mu \nu} m^\mu k^\nu. \quad (2.5)$$

The scalars $\phi_0$ and $\phi_2$ represent the ingoing and outgoing electromagnetic components. It turns out that $\phi_2$ is the electromagnetic counterpart to the outgoing gravitational radiation. Notice that, in the case of the electromagnetic field, we are describing the actual field, which is already taken to be a perturbation on the background.

We then obtain a set of equations for the scalars $\phi$'s by projecting the Maxwell equations $F^{\mu \nu} \rho_\mu = 4 \pi J^\mu$ onto the null tetrad. Furthermore, following [19] the equation for $\phi_2$ can be decoupled to get an equation for the outgoing component of the electromagnetic field:

$$\left[ (-\Delta + 2 \mu_s + \mu_s^* + \gamma_s - \gamma_s^*) \left( -\Delta - \rho_s + 2 \epsilon_s \right) - (-\delta^s + \alpha_s + \beta_s^* + 2 \pi_s - \tau_s^*) \left( -\delta + 2 \beta_s - \tau_s \right) \right] \phi_2 = 4 \pi J_2, \quad (2.6)$$

where the source term $J_2$ is expressed in terms of operators acting on the projections of the current vector along the null tetrad

$$J_2 = (-\Delta + 2 \mu_s + \mu_s^* + \gamma_s - \gamma_s^*) J_{m^*} - (-\delta^s + \alpha_s + \beta_s^* + 2 \pi_s - \tau_s^*) J_k. \quad (2.7)$$

Here $J_k, J_m$ are the projections of the current vector $J_\mu$, on the respective null vector. In this way, we arrive at two equations describing the gravitational, Eq. (2.2) and electromagnetic Eq. (2.6) outgoing radiation, for a given source described by $T_{\mu \nu}$ and $J^\mu$.

## III. SCHWARZSCHILD BACKGROUND

We will write the gravitational perturbation equation for $\Psi^{(1)}_4$ and the Maxwell equation for $\phi_2$ in a background given by a static black hole. We take this to be the Schwarzschild metric, written in Kerr-Schild coordinates in order to avoid geometric singularities at the horizon.

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)dr^2 + \frac{4M}{r}dtdr + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \quad (3.1)$$

and define the tetrad components as

$$l^\mu = \left(\frac{1}{2} + \frac{1}{2}, \frac{1}{2}, \frac{1}{2} - \frac{M}{r}, 0, 0\right), \quad k^\mu = (1, -1, 0, 0), \quad m^\mu = \frac{1}{r \sqrt{2}}(0, 0, 1, i \csc \theta). \quad (3.2)$$

The normalization is such that $l^\mu k_\mu = -m^\mu m^\mu = -1$. The directional derivatives are $D = l^\mu \partial_\mu, \Delta = k^\mu \partial_\mu$ and $\delta = m^\mu \partial_\mu$. By direct substitution, we get the non-zero spin coefficients for this metric:

$$\mu_s = \frac{1}{r}, \quad \rho_s = \frac{r - 2M}{2 r^2}, \quad \epsilon_s = -\frac{M}{2 r^2}, \quad \alpha_s = \frac{\cot \theta}{2 \sqrt{2} r}, \quad \beta_s = -\alpha_s. \quad (3.3)$$

The non-vanishing Weyl component $\Psi_2 = \frac{M}{r^3}$.

### A. Gravitational radiation equation

The substitution of the directional derivatives and the spin coefficients into yields the differential equation:

$$\Box_{tr} \Psi^{(1)}_4 = 16\pi r T_4, \quad (3.4)$$

where the radial-temporal operator is

$$\Box_{tr} = -\left(1 + \frac{2M}{r}\right) \frac{\partial^2}{\partial t^2} + \left(1 - \frac{2M}{r}\right) \frac{\partial^2}{\partial r^2} + \frac{4M}{r} \frac{\partial^2}{\partial t \partial r} + 2 \left(\frac{2}{r} + \frac{M}{r^2}\right) \frac{\partial}{\partial t} + 2 \left(\frac{2}{r} - \frac{M}{r^2}\right) \frac{\partial}{\partial r} + \frac{2M}{r^3}. \quad (3.5)$$
and the angular part
\[
\Box^{-2} = \frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + \cot \theta \frac{\partial}{\partial \theta} - \frac{4i}{\sin^2 \theta} \frac{\partial}{\partial \varphi} - 2 \left( \frac{1 + \cos^2 \theta}{\sin^2 \theta} \right). \tag{3.6}
\]

Following [16], we assume that \( \Psi_4^{(1)} \) may be written in terms of spherical harmonics of spin weight \( s = -2 \) as
\[
\Psi_4^{(1)} = \sum_{\ell m} \frac{R^G_{\ell m}(t, r)}{r} Y_{-2, \ell m}(\theta, \phi), \tag{3.7}
\]
where \( \ell = 0, 1, 2 \ldots \) and \( |m| < \ell \). It can be seen, for instance in [22, 24], that \( Y_{-2, \ell m} \) are eigenfunctions of the angular operator (3.6)
\[
\Box^{-2} Y_{-2, \ell m} = - (\ell - 1) (\ell + 2) Y_{-2, \ell m}. \tag{3.8}
\]

After replacing the angular operator in (3.4) by its eigenvalue, multiplying by the complex conjugate \( Y_{-2, \ell m} \) and integrating over the sphere, Eq. (3.4) we arrive at an ordinary differential equation for each radial function
\[
- \left( 1 + \frac{2M}{r} \right) \frac{\partial^2 R_{\ell m}^G}{\partial t^2} + \left( 1 - \frac{2M}{r} \right) \frac{\partial^2 R_{\ell m}^G}{\partial r^2} + 4M \frac{\partial^2 R_{\ell m}^G}{\partial r \partial t} + 2 \left( \frac{2}{r} + \frac{M}{r^2} \right) \frac{\partial R_{\ell m}^G}{\partial t} + 2 \left( \frac{2}{r} - \frac{M}{r^2} \right) \frac{\partial R_{\ell m}^G}{\partial r} = \frac{16 \pi r T_{\ell m}}{r} \tag{3.9}
\]

The source term is \( T_{\ell m} = \int T_4 Y_{-2, \ell m} \sin \theta d\theta d\varphi \).

In order to obtain a numerical solution of this second order equation we transform it into a first order system by defining a set auxiliary variables:
\[
\psi^{\ell G} = \partial_t R_{\ell}^G, \quad \pi^{\ell G} = \left( 1 + \frac{2M}{r} \right) \partial_t R_{\ell}^G - \frac{2M}{r} \psi^{\ell G}, \tag{3.10}
\]
which yields
\[
\partial_t R_{\ell}^G = \frac{1}{r + 2M} \left( r \pi_{\ell}^G + 2M \psi_{\ell}^G \right), \tag{3.11}
\]
\[
\partial_t \psi_{\ell}^G = \partial_r \left( \frac{1}{r + 2M} \left( r \pi_{\ell}^G + 2M \psi_{\ell}^G \right) \right) = \frac{1}{r + 2M} \left( r \partial_r \pi_{\ell}^G + 2M \partial_r \psi_{\ell}^G \right) + \frac{2M}{(r + 2M)^2} \left( \pi_{\ell}^G - \psi_{\ell}^G \right), \tag{3.12}
\]
\[
\partial_t \pi_{\ell m}^G = \frac{1}{r + 2M} \left( 2M \partial_r \pi_{\ell m}^G + r \partial_r \psi_{\ell m}^G \right) + \frac{2}{(r + 2M)^2} \left( \left( 2r^2 + 5Mr + 4M^2 \right) \pi_{\ell m}^G + (r + 4M) \left( 2r + 3M \right) \psi_{\ell m}^G \right) + \frac{2M}{r^3} \left( \ell - 1 \right) \left( \ell + 2 \right) R_{\ell m}^G - 16 \pi r T_{\ell m}. \tag{3.13}
\]

This system is solved numerically to obtain \( \pi_\ell, \psi_\ell \) and \( R_\ell \), as the response to \( T_{\ell m} \). In section IV we show the numerical results.

**B. Electromagnetic field equation**

Substituting the directional derivatives and the spin coefficients (3.3) into equation (2.6) for \( \phi_2 \) we obtain
\[
\left[ \Box^\phi_{tr} + \frac{1}{r^2} \Box^-_{\theta \varphi} \right] r \phi_2 = -8 \pi r J_2, \tag{3.14}
\]
is given by $u$ moving particles induce an electromagnetic field, but this field does not backreact on the spacetime. The four velocity

$$J^\mu_{\ell m} = \delta^\mu_0 \delta_\ell_m,$$

We assume that the particles are falling radially into the black hole, and we recall that in our approximation the energy tensor and the conservation of the four current

$$\text{continuity equation}$$

charge-mass ratio throughout the cloud. Then the electric current induced by the motion of the particles is

$$\text{density is given by}$$

$$\rho = \frac{e}{m}$$

After replacing this expansion in (3.14) we get an equation for each mode

$$\square_{tr} = \left(1 + \frac{2M}{r}\right) \frac{\partial^2}{\partial t^2} + \left(1 - \frac{2M}{r}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{4M}{r} \frac{\partial^2}{\partial t \partial r} + \frac{2}{r} \frac{\partial}{\partial t} + \frac{2}{r} \frac{\partial}{\partial r}\right),$$

As was done for the gravitational perturbation, we expand $\phi_2$ in spherical harmonics. However, in this case we use a spin weight $s = -1$. The choice of the spin weight is not arbitrary; in fact it is given by the spin of the field considered [21, 24].

$$\phi_2 = \sum_{\ell} \frac{R_{E,\ell m}^E(t, r)}{r} Y_{\ell, -1}(\theta, \varphi).$$

After replacing this expansion in (3.14) we get an equation for each mode

$$\begin{align*}
- \left(1 + \frac{2M}{r}\right) \frac{\partial^2 R_{E,\ell m}^E(t, r)}{\partial t^2} &+ \left(1 - \frac{2M}{r}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{4M}{r} \frac{\partial^2}{\partial t \partial r} + \frac{2}{r} \frac{\partial}{\partial t} + \frac{2}{r} \frac{\partial}{\partial r}\right) \\
+ \frac{2}{r} \frac{\partial R_{E,\ell m}^E(t, r)}{\partial r} &- \left(\frac{\ell(\ell + 1)}{r^2}\right) R_{E,\ell m}^E(t, r) = -8\pi r J_{\ell m}^E,
\end{align*}$$

where the source term is expressed as $J_{\ell m}^E = \int J_2 Y_{\ell m} \sin \theta \, d\theta \, d\varphi$.

We define the corresponding first order auxiliary variables as $\psi^E = \partial_t R^E_{\ell m}$ and $\pi^E_{\ell m} = (1 + \frac{2M}{r}) \partial_t R^E_{\ell m} - \frac{2M}{r} \psi^E$. The equations for $\partial_t R^E_{\ell m}$ and $\partial_t \psi^E$ have the same functional form as the gravitational case (3.11) and (3.12). The equation for $\pi^E_{\ell m}$ on the other hand is

$$\begin{align*}
\partial_t \pi^E_{\ell m} &= \frac{1}{r + 2M} \left[r \partial_r \psi^E_{\ell m} + 2M \partial_r \pi^E_{\ell m}\right] + \frac{4M^2}{r(r + 2M)^2} (\pi^E_{\ell m} - \psi^E_{\ell m}) \\
+ \frac{2}{r} \left[\frac{1}{r + 2M} (r \pi^E_{\ell m} + 2M \psi^E_{\ell m})\right] + \frac{2}{r} \psi^E_{\ell m} - \frac{\ell(\ell + 1)}{r^2} R^E_{\ell m} + 8\pi r J_{\ell m}^E.
\end{align*}$$

In the next section, we give a description of the accreting matter model.

C. Sources of the gravitational perturbation and the electromagnetic field

We shall take a collection of charged particles that behave like dust to be the source of the perturbation. For simplicity each particle has the same mass $m$ and charge $e$. We also assume that the particles can not be created or destroyed.

Let the number density in their rest frame be $n$ (the number of particles per unit of volume); then the rest mass density is given by $\rho = mn$. Let us further assume that the charged particles are of the same species, with a constant charge-mass ratio throughout the cloud. Then the electric current induced by the motion of the particles is

$$J^\mu_{\ell m} = q \rho u^\mu,$$

where $q$ is the charge to mass ratio $e/m$ and $u^\mu$ is the four velocity of the particles. The stress energy tensor of the system is given by

$$T_{\mu\nu} = \rho u_\mu u_\nu.$$

We assume that the particles are falling radially into the black hole, and we recall that in our approximation the moving particles induce an electromagnetic field, but this field does not backreact on the spacetime. The four velocity is given by $u^\mu = (u^0(t, r), u^1(t, r), 0, 0)$. The dynamics of the particles is governed by the conservation of the stress energy tensor and the conservation of the four current $J^\mu := \rho u^\mu$. The conservation of the mass four current, the continuity equation $J_{\mu\mu} = \rho = 0$ implies

$$\partial_t (\sqrt{-g} \rho u^0) + \partial_r (\sqrt{-g} \rho u^1) = 0.$$
The conservation of the stress energy tensor implies that the particles follow geodesics, \( u^\nu u_\nu = 0 \). Using the normalization of the four velocity \( u^\mu u_\mu = -1 \) the geodesics equation can be integrated to obtain the components of the four velocity,

\[
\begin{align*}
  u^1 &= \pm \sqrt{E^2 - 1 + 2 \frac{M}{r}} \\
  u^0 &= \frac{E r + 2 M u^1}{r - 2 M}
\end{align*}
\]  

(3.23)

where \( E \) is a constant of motion related to the energy of each particle. We choose the minus sign in the square root of \( u^1 \), indicating that the particles are in-falling into the black hole.

It has been shown in [16] that with the decomposition of the density in spherical harmonics:

\[
\rho = \sum \rho_{l,m}(t, r) Y^{l,m}_0(\theta, \phi)
\]  

(3.24)

it is possible to transform (3.22) into a set of equations for each mode

\[
\frac{\partial_t \rho_{l,m} + v^r \partial_r \rho_{l,m} + 2 \frac{E^2 - 1 + \frac{3M}{r}}{E^2 - 1 + \frac{2M}{r}} v^r \rho_{l,m}}{\sqrt{\frac{1}{\ell} (\ell + 1)}} = 0
\]  

(3.25)

where we used the 3 velocity defined as \( v^r = u^1/u^0 \). Notice that the configuration (3.24) is not necessarily spherically symmetric.

Let us focus on the electromagnetic source for \( \phi^2 \). Given the current vector (3.20), and the expansion (3.24) the only non-vanishing projection is along the vector \( k^\mu \) (Notice that \( k^\mu u_\mu = -(u^0 + u^1) \))

\[
J_2 = \frac{1}{r\sqrt{2}} (\partial_\theta - i \csc \theta \partial_\phi) J_k
\]  

(3.26)

\[
= \frac{q}{r\sqrt{2}} (u^0 + u^1) \sum \rho_{l,m} Y^{l,m}_0(\theta, \phi)
\]

\[
= -\frac{q}{r\sqrt{2}} (u^0 + u^1) \sqrt{\frac{1}{\ell} (\ell + 1)} \sum \rho_{l,m} Y^{l,m}_{-1}(\theta, \phi)
\]

where we have used the fact that the operator \( \eth-bar \), \( \eth \), lowers the spin weight of the spherical harmonics in the form:

\[
\eth_s Y^{l,m}_s = -\sqrt{(l + s) (l - s + 1)} Y^{l,m}_{s+1} .
\]  

(3.27)

Then, we integrate over the sphere to get

\[
J_{l,m} = -\frac{q}{r\sqrt{2}} (u^0 + u^1) \sqrt{\frac{1}{\ell} (\ell + 1)} \rho_{l,m} .
\]  

(3.28)

From here it is straightforward to see that the monopole does not excite the electromagnetic field. Notice also that it is enough to solve the radial-temporal components to get the waveform.

In a similar way, using these expressions for the density and the four velocity in the stress energy tensor, Eq. (3.21), we compute the needed projections on the tetrad and act with the operators \( \hat{T}^{ab} \), from Eqs. (2.4). The details of this derivation were presented in [16]. The final result is that the source term for the gravitational perturbation takes the form

\[
T_4 = -\frac{(u^0 + u^1)^2}{2r^2} \sum \rho_{l,m}(t, r) \sqrt{\frac{1}{\ell} (\ell - 1) (\ell + 1) (\ell + 2)} Y^{l,m}_{-2} .
\]  

(3.29)

and consequently, after an integration over the solid angle, and taking into account the orthonormality of the spherical harmonics, we obtain the following expression for the gravitational source:

\[
T_{l,m} = -\frac{(u^0 + u^1)^2}{2r^2} \rho_{l,m}(t, r) \sqrt{\frac{1}{\ell} (\ell - 1) (\ell + 1) (\ell + 2)} .
\]  

(3.30)

Any initial matter distribution can be expanded in terms of spherical harmonics: According to (3.30) and (3.28) the monopole mode will not generate any gravitational or electromagnetic reaction; the dipole mode will generate
an electromagnetic but not a gravitational response, and the quadrupole and higher modes generate both type of responses.

Furthermore, each mode of matter excites only the corresponding electromagnetic and gravitational mode with same harmonic number $l$. The decomposition (3.30) and (3.28) allows us to study, with a 1 dimensional numerical code, any radially infalling charged dust matter distribution and its gravitational and electromagnetic reaction. Notice also that there is a degeneracy with respect to the $m$ modes. This degeneracy comes from the fact that the background is spherically symmetric and that the fluid movement is restricted to be radial.

D. Time evolution

We solve the equations for the gravitational perturbation with sources plus the equations for the electromagnetic field by using the Method Of Lines (MoL). The detailed description of the code is given in [16]. Here we just summarize the main aspects. The numerical code evolves the first order variables with a third order Runge Kutta integrator with a fourth order spatial stencil in a domain $r \in [r_{\text{min}}, r_{\text{max}}]$. Since we are using Kerr-Schild type coordinates $r_{\text{min}}$ lies inside the event horizon. Typically we choose $r_{\text{min}} = 1.5M$ and $r_{\text{max}} = 2000M$. As usual, we introduce a small sixth order dissipation to get rid off high frequency modes. At the boundaries we impose the condition that all the incoming waves (as given by the characteristic fields) vanish.

In all the simulations presented we use as initial data a gaussian packet in the density. This represents a non spherical shell of particles falling into the hole. We solve the equation for the rest mass density and look for the gravitational and electromagnetic response. The gravitational and electromagnetic waveforms are then extracted at a fixed $r = r_o$ radius.

IV. RESULTS AND DISCUSSIONS

In the following we shall report the behaviour of the gravitational signal and its electromagnetic counterpart when charged dust particles fall into the black hole.

We have considered for simplicity, a shell of matter described by a single spherical harmonic mode

$$\rho(t, r, \theta, \phi) = \rho_{\ell,m}(t, r)Y_{\ell,m}^{\ell,m}(\theta, \phi). \quad (4.1)$$

We will take the $\ell = 1, 2$ modes since these modes yields the main contribution to the quadrupole gravitational radiation and the dipolar electromagnetic radiation which are expected to be dominant. The modes with $\ell > 2$ will be relevant for higher gravitational and electromagnetic multipoles.

We have used as initial data for the radial distribution a Gaussian centred at $r_{\text{cg}}$:

$$\rho_{\ell,m}(0, r) = \rho_0 e^{-(r-r_{\text{cg}})^2/2\sigma^2}, \quad (4.2)$$

with $\rho_0 = 5 \times 10^{-3}$, $r_{\text{cg}} = 100M$ and $\sigma = 0.5M$, but we will vary this width in some experiments as described below. The gravitational and electromagnetic functions, $R^G, R^E$, are set to zero, as well as their time derivatives. Our results indicate that the effect of this choice on the gravitational and electromagnetic waveforms has negligible. Finally, the outer boundary was set far enough from the horizon to ensure that any possible incoming radiation has no effect on our results.

A. Waveforms

Our numerical results confirm the expected proportionality of the wavefunction with the charge. We found that the electromagnetic radial wave function is homogeneous of degree 1 in the charge i.e. $R^E(t, r_o; \lambda q) = \lambda R^E(t, r_o; q)$, indeed we found that $R^E$ is linear in the charge $q$.

In Fig. 1 we display the radial waveforms as obtained at $r_o = 1000M$. In the left panel we display the dipolar component $\ell = 1$, and in the right panel, the quadrupolar $\ell = 2$. In the inset we plot the absolute value on a logarithmic scale to show the different stages of the signal: the initial burst due to the initial data, the quasinormal ringing and the tail. In these figures the waveforms have been rescaled to show the dependence of $R^E$ on $q$.

While the ring down frequency is determined by the quasinormal ringing, it is interesting to note that the amplitude can be derived from a scaling law. This behaviour might be related with another scaling observed in [13]. In that work, the authors considered the collision of two charged black holes with opposite charge $Q$ and mass $M$ such that
the resultant black hole was of the Schwarzschild type. It was found that the emitted electromagnetic waveform in the final black hole scales as \( Q(1 + Q^2/M^2)^{(1/2)} \). For a distant observer, however, the scenario is that there is an electromagnetic signal coming from an almost stationary black hole, very similar to our final set-up when the charged particles have fallen into the hole.

![Graph](image1)

**FIG. 1:** Electromagnetic signal for \( \ell = 1 \) and \( \ell = 2 \) modes. The waveforms have been rescaled to show the property \( R^E(t, r_o, \lambda q) = \lambda R^E(t, r_o; q) \).

A comparison between the electromagnetic and gravitational waveforms is shown in Fig. 2. The figure shows the quadrupolar modes for the gravitational signal and two representative cases for the electromagnetic signal with charge \( q = 0.2, 0.8 \). We find no sign of mixing between gravitational and electromagnetic frequencies. Each signal displays its characteristic ring down frequency. The inset shows the absolute value on a logarithmic scale. We found that

![Graph](image2)

**FIG. 2:** Radial profiles of the gravitational (\( R^G_\ell \)) and electromagnetic (\( R^E_\ell \)) signals for the quadrupolar \( \ell = 2 \) mode. The extraction radius is \( r_o = 1000M \). The inset panel shows the absolute value in a logarithmic scale to properly distinguish the different phases; the ringdown and the tail decay.

the electromagnetic waves have a specific frequency corresponding to the \( \ell = 1 \) quasinormal mode of electromagnetic
waves in a Schwarzschild background \cite{24,27}. The frequency of the gravitational waves on the other hand is the one associated to the quadrupolar $\ell = 2$ quasinormal mode \cite{28,31}. In order to find the frequencies we fit the data with a sinusoidal waveform. The numerical values of the corresponding frequencies are listed in table \ref{table1}. The values we find coincide with the values previously reported in \cite{25}. While it is known that quasinormal mode frequencies are in general complex, we were only interested on the oscillatory behaviour of the signal and not in their time decay. For black holes of a few solar masses $10M_\odot < M < 10^3M_\odot$ the electromagnetic frequencies are in the interval $8\text{Hz} \sim 800\text{Hz}$ whereas the gravitational waves produced for such a range of masses are in the $12\text{Hz}$ to $1.2\text{kHz}$ window. As has been pointed out in several works, quasinormal ringing can be used to determine the intrinsic properties of the black hole \cite{26}. Electromagnetic waves with such low frequencies however could easily be absorbed by the interstellar medium during its propagation and it will be almost impossible to directly detect them. One possible way to observe the electromagnetic waves is through the detection of its indirect effects on the medium, such as synchrotron radiation. In this scenario, however, more information is needed about the medium.

We studied the effect of the variation of the Gaussian half width in the gravitational and electromagnetic signals. The response of an isolated black hole to the width of Gaussian initial perturbations is well known. When the Gaussian is very broad, no quasinormal ringing could be seen in the scattered radiation. When the Gaussian is made thinner ringing occurs \cite{31,32}. Similar results were reported when the perturbation was caused by a Gaussian distribution of dust \cite{16,24,33}. Our results for charged dust particles point in the same direction for both electromagnetic and gravitational signals.

The waveforms for several values of the initial Gaussian width are plotted in figure \ref{figure3}. The left panel corresponds to the electromagnetic wave and the right to the gravitational one. The width of the Gaussian increases from top to bottom. From the figure it is clear that no ringing at all is seen for very broad Gaussians.

| $\ell$ | $M\omega_{GW}$ | $M\omega_{EM}$ |
|-------|----------------|----------------|
| 1     | -              | 0.248          |
| 2     | 0.373          | 0.457          |
| 3     | 0.598          | 0.655          |

TABLE I: Frequencies of the gravitational and electromagnetic waveforms obtained with a sinusoidal fit. The values match, up to the decimal figures shown, the quasi normal frequencies reported in \cite{25}.

![Figure 3](image-url)
B. Energy radiated

In terms of the multipolar decomposition (3.7), the radiated gravitational energy flux is given by (19):

\[ P_{GW} = \frac{d}{dt} E_{GW} = \lim_{r \to \infty} \frac{1}{16\pi} \sum_{\ell,m} \int_{-\infty}^{\infty} dt' |R^G_{\ell}(t')|^2. \]  

(4.3)

The energy flux carried off by outgoing electromagnetic waves in terms of the multipolar decomposition (3.17) is given by

\[ P_{EM} = \frac{d}{dt} E_{EM} = \lim_{r \to \infty} \frac{1}{4\pi} \sum_{\ell,m} |R^E_{\ell}(t)|^2. \]  

(4.4)

We compute the total radiated energies by direct integration of Eqs. (4.3), (4.4). To calculate the total energy, we start the integration of the fluxes after some time in order to get rid of the radiation resulting from the initial data. The choice of the start time depends on the extraction radius, typically we take it to be \( t_s \sim 1400M \) for an extraction radius of \( r_o = 1000M \). The value of the energy does not change when we put the observers far away, indicating the validity of the far region approximation. In order to show the effect of the charge of the particles on both the gravitational and electromagnetic signals, we consider the quotient between the computed energies. In Fig. 4 we plot the ratio \( E_{EM}/E_{GW} \) versus the density of charge \( q \) for \( \ell = 2 \). We choose as initial data a gaussian pulse with \( \sigma = 0.5M \) and \( r_{cg} = 100M \). As expected, when the value of the charge increases the amount of electromagnetic energy also increases. For small values of the charge the gravitational energy dominates but as the charge grows, it is the electromagnetic energy which dominates. We found that there is a relation between the two energies and their quotient with a quadratic function of the form \( E_{EM}/E_{GW} = aq^2 \). We found that the numerical value of \( a \) depends on the initial width of the gaussian. In Ref. [16] it was shown that the energy of the gravitational wave varies monotonically with the width of the gaussian. This behaviour is related with the possibility of the initial data to induce quasinormal modes. Our result indicates that the electromagnetic energy varies in such a way that the quotient \( E_{EM}/E_{GW} \) depends quadratically on \( q \). We also have found that the value of \( a \), for a fixed width that produces quasinormal ringing, depends on the angular momentum number \( \ell \) with \( a = 12.417, 11.128, 10.928 \) for \( \ell = 2, 3, 4 \) respectively.

![FIG. 4: The ratio of the electromagnetic and gravitational radiated energy computed from the integration of the fluxes (dots). The continuous line is a quadratic fit \( E_{EM}/E_{GW} = 12.417q^2 \).](image)

V. CONCLUSIONS

In this paper we studied the combined, electromagnetic and gravitational, response due to a shell of charged particles falling radially, along geodesics, into a Schwarzschild black hole. Varying the angular distribution of the particles allowed us to produce changes in the multipolar components of the infalling matter, and to excite gravitational and electromagnetic emission.

In this way, we found that the falling matter triggers two signals, one electromagnetic and one gravitational. The gravitational signal displays the characteristic behaviour of a perturbed black hole; an oscillating phase after an initial
burst and a final power law decay. The electromagnetic waves show the same qualitative behaviour. However, we found no direct coupling between the electromagnetic waves. The gravitational signal corresponds to the quasinormal ringing and the electromagnetic corresponds to a electromagnetic field scattered from a Schwarzschild black hole. The absence of any type of coupling between the frequencies may be due to the fact that we neglected the interaction between the charge of the particles and the electromagnetic field in the background. We expect that the coupling will be manifest in a model which takes into account the backreaction of the spacetime.

We found that if the initial distribution of dust is very broad no quasinormal ringing appears. We studied the effect of the charge of the particles on the signals emitted and found that there is a linear dependence of the electromagnetic waveforms on the charge-mass ratio, $q$. We also found that the gravitational and electromagnetic energies are related by the square of $q$. Within our simplified model, if one were able to determine the electromagnetic signal one would then be able to infer properties of the gravitational wave, such as the frequency and the energy content once the charge was determined by other means. In order to achieve this goal more involved models of matter must be taken into account.

There are several directions in which the present study could be continued. It may be worthwhile to extend the analysis to include more realistic models for accretion disks, to consider not only vacuum spacetimes and to study the system presented in this manuscript in a rotating black hole scenario. Work in this direction is under way.

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