\(\kappa\)-deformations of \(D = 4\) Weyl and conformal symmetries

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Abstract

We provide first explicite examples of quantum deformations of \(D = 4\) conformal algebra with mass-like deformation parameters, in applications to quantum gravity effects related with Planck mass. It is shown that one of the classical \(r\)-matrices defined on the Borel subalgebra of \(sl(4)\) with \(o(4,2)\) reality conditions describes the light-cone \(\kappa\)-deformation of \(D = 4\) Poincaré algebra. We embed this deformation into the three-parameter family of generalized \(\kappa\)-deformations, with \(r\)-matrices depending additionally on the dilatation generator. Using the extended Jordanian twists framework we describe these deformations in the form of noncocommutative Hopf algebra. We describe also another four-parameter class of generalized \(\kappa\)-deformations, which is obtained by continuous deformation of distinguished \(\kappa\)-deformation of \(D = 4\) Weyl algebra, called here the standard \(\kappa\)-deformation of Weyl algebra.

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I. Introduction.

There are two basic classes of deformations of $D = 4$ relativistic symmetries:

i) $q$-deformations (see e.g. [1-3]), with Drinfeld-Jimbo deformations [4,5] of the Lorentz subalgebra. The basic relations describing noncommutative space-time coordinates are quadratic [6,7] and the deformation parameter is dimensionless. It can be shown [8,9] that in order to obtain the $q$-deformation of $D = 4$ Poincaré algebra with Hopf algebra structure one should introduce additional generators, i.e. consider 11-dimensional quantum Weyl algebra $U_q(W)$ ($W \equiv$Poincaré algebra $P^{3;1} \oplus$ dilatation generator $D$, where $P^{3;1} = (P_\mu, M_{\mu\nu})$). There is a natural Hopf algebra embedding of the $q$-deformed Weyl algebra $U_q(W)$ into the Drinfeld-Jimbo deformation $U_q(o(4, 2))$ of $D = 4$ conformal algebra

$$U_q(o(4, 2)) \supset U_q(P^{3;1} \oplus D) \quad (1)$$

The $q$-deformation of $D = 4$ conformal algebra was studied by several authors (see e.g. [10-13]).

ii) $\kappa$-deformations [14-21] with deformation parameter $\kappa^{-1}$ introducing fundamental dimensionfull length parameter $^1$. The basic relations describing $\kappa$-deformed space-time are Lie-algebraic, and in the general case take the form

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\kappa} (a_\mu \hat{x}_\nu - \hat{x}_\mu a_\nu) \quad (2)$$

where $a_\mu$ is a constant four-vector in Minkowski space, determining the quantum direction $\hat{y} = a_\mu \hat{x}_\mu$. In fact due to the choice of the four-vector $a_\mu$ one obtains the following three particular cases of $\kappa$-deformation:

ii1) $a_\mu a^\mu = 1$ defining time-like $\kappa$-deformations. The standard $\kappa$-deformation [17-19] obtained for $a_\mu = (1, 0, 0, 0), (i, j = 1, 2, 3)$ leads to relations

$$[\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa} \hat{x}_i \quad [\hat{x}_i, \hat{x}_j] = 0 \quad (3)$$

ii2) $a_\mu a^\mu = -1$ defining tachyonic $\kappa$-deformations. For $a_\mu = (0, 1, 0, 0), (r, s = 1, 2, )$ one gets

$$[\hat{x}_3, \hat{x}_0] = \frac{i}{\kappa} \hat{x}_0 \quad [\hat{x}_3, \hat{x}_r] = \frac{i}{\kappa} \hat{x}_r \quad (4)$$

Such a deformation leads to the deformation of nonrelativistic symmetries, with classical time parameter.

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1 If we put $\hbar = c = 1$, the parameter $\Lambda = \kappa^{-1}$ is the fundamental length parameter, and $\kappa$ describes the fundamental mass. For undeformed case $\kappa \rightarrow \infty$.

2 The $\kappa$-deformation for arbitrary choice of the four vector $a_\mu$ is equivalent to the description of standard $\kappa$-deformation in space-time with arbitrary symmetric metric tensor, firstly given in [20,21]
\( a_\mu a^\mu = 0 \) providing light-cone \( \kappa \)-deformations. For \( a_\mu = \frac{1}{\sqrt{2}}(1, 1, 0, 0) \) one obtains [20-22]

\[
\begin{aligned}
[\hat{x}_-, \hat{x}_+] &= \frac{i}{\kappa} \hat{x}_+ \\
[\hat{x}_-, \hat{x}_r] &= 0 \\
[\hat{x}_r, \hat{x}_s] &= 0 \\
[\hat{x}_r, \hat{x}_s] &= 0
\end{aligned}
\]

where \( \hat{x}_\pm = \frac{1}{\sqrt{2}}(\hat{x}_0 \pm \hat{x}_3) \) \((x_+ = \tilde{a}_\mu \tilde{x}_\mu \), where \( \tilde{a}_\mu \tilde{a}_\mu = 0 \) and \( a_\mu \tilde{a}_\mu = 1 \). The first quantum deformation with quantum-deformed direction on the light cone was proposed in [23] under the name of the null-plane quantum Poincaré algebra. It was shown [22] that after suitable choice of basis the quantum algebra presented in [23] can be identified with choice ii3) of \( a_\mu \)-dependent \( \kappa \)-deformation, given in [20].

It appears that one can introduce the quantum deformation of Poincaré group \( P^{3;1}(a_\mu) \) with the translation sector described by the relations (2) and the corresponding dual quantum Poincaré algebra \( U_\kappa(P^{3;1}; a_\mu) \) [20,21]. One can show\(^3\) that only in the case ii3) for the light-cone \( \kappa \)-deformation, the corresponding classical \( r \)-matrix \( r \in P^{3;1} \otimes P^{3;1} \) satisfies the classical Y-B equation\(^4\). For the choice of the four-vector \( a_\mu = \frac{1}{\sqrt{2}}(1, 1, 0, 0) \) it takes the form

\[
r = \frac{1}{2} (L_3 \wedge P_+ - (M_1 - L_2) \wedge P_2 + (M_2 + L_1) \wedge P_1)
\]

where \( M_i \) describe three space rotations, \( L_i \) represent three Lorentz boosts and \( P_\pm = P_0 \pm P_3 \). The Poincaré algebra \( P^{3;1} \) is the subalgebra of \( D = 4 \) Weyl algebra \( W \) and \( D = 4 \) conformal algebra \( C^{3;1} \simeq o(4, 2) \), therefore the classical \( r \)-matrix (6) is an example of the solution of the classical Y-B equation for \( D = 4 \) Weyl and conformal algebras. It follows therefore that the \( r \)-matrix (6) describes infinitesimally the quantum deformation \( U_\kappa(W) \) of \( D = 4 \) Weyl algebra (see [20]) as well as the quantum deformation \( U_\kappa(o(4, 2)) \) of \( D = 4 \) conformal algebra.

The aim of this paper is to describe the large family of generalized \( \kappa \)-deformations of \( D = 4 \) Weyl and conformal algebras in the form of Hopf algebras. One can show that the classical \( r \)-matrix (6) is defined on the Borel subalgebra \( B_+(o(4, 2)) \) of the Lie algebra \( o(4, 2) \), described as the real form of the complex Lie algebra \( sl(4) \). In section 2 we consider the Cartan-Weyl basis for \( o(4, 2) \) algebra and a general family of classical \( \hat{r} \)-matrices satisfying CYBE and belonging to the tensor square of \( B_+(o(4, 2)) \). We obtain two classes of \( r \)-matrices, with three and four arbitrary parameters, all

\(^3\)For an arbitrary \( a_\mu \) the classical \( \hat{r} \)-matrix is given in [20], where \( g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \) corresponds to the light-like choice of \( a_\mu \).

\(^4\)This result shown in [20] is also contained implicitly in [23].
introducing deformation parameters with the dimension of inverse mass \(^5\). We shall quantize all these solutions of CYBE by generalized extended Jordanian twists, which define classical universal enveloping \(D = 4\) Weyl and conformal algebras with twisted coproducts. The physical \(D = 4\) conformal basis and the expressions in terms of conformal basis of extended Jordanian twists [25-27] are given in section 3. A new basic \(\kappa\)-deformation of \(D = 4\) Weyl algebra will be considered in some detail. In section 4 we shall present comments on possible physical applications.

We would like to mention that our paper extends to the \(D = 4\) case the results known already for \(D = 1, D = 2\) and \(D = 3\). For \(D = 1\) it was shown in [28] that the so-called Jordanian deformation of \(sl(2; C)\) [29,30] by imposing the real form selecting \(sl(2, R) \simeq o(2, 1)\) generators describes the \(\kappa\)-deformation of \(D = 1\) conformal algebra. Because for \(D = 2\) conformal algebra \(O(2, 2) \simeq O_+ (2, 1) \bigoplus O_-(2, 1)\), the results in [28] do extend in straightforward way, with a pair of \(D = 1\) \(\kappa\)-deformations for two \(O_+ (2, 1)\) and \(O_-(2, 1)\) subalgebras described by two fundamental mass parameters \(\kappa_{\pm}\).

An example of \(\kappa\)-deformation of \(D = 3\) conformal algebra \(o(3, 2)\) has been given by Herranz [31], which can be treated as \(\kappa\)-deformation od \(D = 4\) anti-de-Sitter algebra, with two dimensionfull parameters: the first, \(\kappa\), possibly equal to Planck mass, related with quantum gravity effects at ultra-short distances and the second, represented by anti-de-Sitter radius \(R\), characterizing very large cosmological distances \(^6\). All these deformations of \(D < 4\) conformal algebras are generated by classical \(r\)-matrices having support in corresponding Weyl algebras. In this paper for the case \(D = 4\) we describe also the multiparameter \(\kappa\)-deformations of conformal algebras as induced by the \(\kappa\)-deformations of Weyl algebra.

II. Classical \(D = 4\) conformal \(r\)-matrices and \(\kappa\)-deformations.

a) \(sl(4; c) \cong so(6; c)\) Lie algebra in the Cartan-Weyl basis.

The Cartan-Weyl basis for \(sl(4)\) is defined by the simple root generators \(e_{\pm i}, (i = 1, 2, 3)\) and the composite generators:

\[
e_4 = [e_1, e_2], \quad e_{-4} = [e_{-2}, e_{-1}], \quad (7)
\]
\[
e_5 = [e_2, e_3], \quad e_{-5} = [e_{-3}, e_{-2}], \quad (8)
\]
\[
e_6 = [e_1, e_5], \quad e_{-6} = [e_{-5}, e_{-1}]. \quad (9)
\]

Let \(h_i (i = 1, 2, 3)\) form the linear basis of the Cartan subalgebra and introduce the

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\(^5\)We shall call these deformations the generalized \(\kappa\)-deformations, in accordance with our dichotomic split of quantum deformations of space-time symmetries into \(q\)-deformations (dimensionless) and \(\kappa\)-deformations (dimensionfull).

\(^6\)One can show that the \(\kappa\)-deformation of \(D = 4\) anti-de-Sitter algebra considered in [31] provides in the contraction limit \(R \rightarrow \infty\) the light-cone \(\kappa\)-deformation of Poincaré algebra, with classical \(r\)-matrix (6).
elements
\[ h_4 = h_1 + h_2, \quad h_5 = h_2 + h_3, \quad h_6 = h_1 + h_2 + h_3. \] (10)
The Lie algebra \( sl(4) \) can be completely described as follows:
\[
[h_A, e_{\pm B}] = \pm \alpha_{AB} e_{\pm B}, \quad [e_A, e_{-B}] = \delta_{AB} h_B,
\] (11)
\((A = 1, 2, ..., 6; \text{no summation over repeated indices!})\). Here \( \alpha_{AB} \) is the extended symmetric Cartan matrix given by the formula (see e.g.[13])
\[
\alpha_{AB} = \begin{pmatrix}
2 & -1 & 0 & 1 & -1 & 1 \\
-1 & 2 & -1 & 1 & 1 & 0 \\
0 & -1 & 2 & -1 & 1 & 1 \\
1 & 1 & -1 & 2 & 0 & 1 \\
-1 & 1 & 1 & 0 & 2 & 1 \\
1 & 0 & 1 & 1 & 1 & 2
\end{pmatrix}
\] (12)
The remaining part of \( sl(4) \) Lie algebra is obtained from the Serre relations and for the positive root generators it takes the form:
\[
[e_1, e_3] = [e_1, e_4] = [e_1, e_6] = 0,
\]
\[
[e_2, e_4] = [e_2, e_5] = [e_2, e_6] = 0,
\]
\[
[e_4, e_5] = [e_4, e_6] = [e_5, e_6] = 0,
\]
\[
[e_3, e_5] = [e_3, e_6] = 0, \quad [e_3, e_4] = e_6.
\] (13)
The real forms of \( sl(4; c) \) are well-known (see e.g.[13,33]). We shall consider the one which selects the \( D = 4 \) real conformal algebra \((sl(4) \to su(2, 2) \simeq o(4, 2))\) and map the Borel subalgebras \( B_+(o(4, 2)) \rightarrow B_+ (o(4, 2)) \). We obtain
\[
h_1^+ = -h_3, \quad h_2^+ = -h_2,
\]
\[
e_1^+ = \epsilon e_3, \quad e_2^+ = -e_2,
\]
\[
e_4^+ = -\epsilon e_5, \quad e_6^+ = -e_6,
\] (14)
where \( \epsilon = \pm 1 \).

b) Classical \( r \)-matrices for \( o(4, 2) \).

We shall consider the following class of classical eight-dimensional \( r \)-matrices\(^7\) for \( sl(4, c) \) having their support in the Borel subalgebra \( B_+ \):
\[
r_+ = H_2 \wedge e_2 + H_6 \wedge e_6 + c e_2 \wedge e_6 + +e_1 \wedge e_5 - e_3 \wedge e_4
\] (15)
\(^7\)Following [33,34] one can introduce [32] the classical \( r \)-matrices for \( sl(4) \) satisfying CYBE with the support in 10-dimensional and 12-dimensional parabolic subalgebras, but these \( r \)-matrices contain the generators from both Borel algebras and are not compatible with \( o(4, 2) \) reality conditions (14).
where $H_2$ and $H_6$ are the linear combinations of Cartan generators with complex coefficients
\[ H_2 = b_1 h_1 + b_2 h_2 + b_3 h_3, \quad H_6 = a_1 h_1 + a_2 h_2 + a_3 h_3. \] (16)

We found the following two families of solutions for CYBE:

i) Let $H_2 \neq 0$. In this case one gets the four-parameter family of classical $r$-matrices, obtained by imposing three conditions on seven parameters $a_i, b_i, c$ ($i=1,2,3$):
\[ a_1 + a_3 = 1, \quad b_1 + b_3 = 0, \quad a_2 = \frac{1}{2}. \] (17)

ii) If $H_2 = 0$ one obtains only one condition
\[ a_1 + a_3 = 1, \] (18)

i.e. the three-parameter set of $r$-matrices.

In order to obtain the classical $r$-matrices for $o(4,2)$ one should impose the reality conditions $r_+ = (r_+)^+$ (notice that $(A \otimes B)^+ = A^+ \otimes B^+$) by the relations (14). One obtains the following conditions on seven complex parameters $a_i, b_i, (i=1,2,3), c$
\[ a_1 = a_3^*, \quad a_2 = a_2^*, \quad b_1 = b_3^*, \quad b_2 = b_2^*, \quad c = c^*. \] (19)

Imposing the conditions (17) one obtains the first four-parameter family of classical $o(4,2)$ $r$-matrices ($\alpha, \beta, \gamma, \delta$ real)
\[ r_+^{(1)}(\alpha, \beta, \gamma, \delta) = i(h_1 - h_3) \land (\alpha e_2 + \beta e_6) + \delta h_2 \land e_2 + \gamma e_2 \land e_6 + \hat{r} \] (20)
where $\hat{r} \equiv r_+^{(1)}(0,0,0,0)$ has the form
\[ \hat{r} = r + \frac{1}{2} h_2 \land e_6 = \frac{1}{2}(h_1 + h_2 + h_3) \land e_6 + e_1 \land e_5 - e_3 \land e_4 \] (21)

We shall show that the generators occurring in (20) belong to $D = 4$ Weyl algebra and will be called the standard $\kappa$-Weyl $r$-matrix. It should be pointed out that the $D = 1 \kappa$-deformation considered in [28] and $D = 3 \kappa$-deformation from [31] are of such a type.

The second class of $o(4,2)$ $r$-matrices is obtained by imposing the conditions (18,19). One gets ($\beta, \gamma, \rho$ real)
\[ r_+^{(2)}(\beta, \gamma, \rho) = i\beta(h_1 - h_3) \land e_6 + \rho h_2 \land e_6 + \gamma e_2 \land e_6 + r \] (22)
where $r \equiv r_+^{(2)}(0,0,0)$ is the light-cone $\kappa$-Poincarè $r$-matrix (6) in the Cartan-Weyl basis \(^8\)
\[ r = \frac{1}{2}(h_1 + h_3) \land e_6 + e_1 \land e_5 - e_3 \land e_4 \] (23)

\(^8\)This statement follows from formulae (37) in section 3.
The set $r^{(2)}_{+}(\beta, \gamma, \rho)$ contains the $r$-matrix (21) as a limit: $r^{(2)}_{+}(0, 0, \frac{1}{2}) = \hat{r}$. To avoid this reduction we shall assume that $\rho \neq \frac{1}{2}$.

To see what parameters in classical $r$-matrices are essential one should consider the group of outer and inner automorphisms of the Borel subalgebra $B_{+}$, belonging to $O(4, 2)$. Applying the automorphism generated by $e_2$ to the classical $r$-matrices $r^{(2)}_{+}$ (see (22)) one gets

$$\exp[\text{ad}^\otimes(\alpha_2 e_2)]r^{(2)}_{+} = r^{(2)}_{+} + \alpha_2(1 - 2\rho)e_2 \wedge e_6$$

(24)

i.e. the parameter $\gamma$ is shifted ($\gamma \to \gamma' = \gamma + \alpha_2(1 - 2\rho)$) and while $\rho \neq \frac{1}{2}$ it can be removed. One can check that do not exist the $O(4, 2)$ automorphisms which can be used to eliminate or scale independently the parameters $\beta$ and $\rho$. Similarly it can be shown that $O(4, 2)$ automorphisms can not eliminate any of four parameters $\alpha, \beta, \gamma, \delta$ in (20). We remind that for matrices $r^{(1)}_{+}$ we have $\rho = \frac{1}{2}$ and parameter $\gamma$ can not be eliminated through shift (24), however it can be scaled.

**c) Quantum deformations by twisting classical Lie algebras.**

Consider the universal enveloping algebra $U(g)$ of a Lie algebra $g$ as a Hopf algebra with the comultiplication $\Delta^{(0)}$ generated by the primitive coproducts of generators in $g$. The parametric solution $\mathcal{F}(\xi) = \sum f_1^{(1)} \otimes f_1^{(2)} \in U(g) \otimes U(g)$ of the twist equations [35]

$$\mathcal{F}_{12}(\Delta^{(0)} \otimes 1)\mathcal{F} = \mathcal{F}_{23}(1 \otimes \Delta^{(0)})\mathcal{F},$$

$$\epsilon \otimes \text{id})\mathcal{F} = (\text{id} \otimes \epsilon)\mathcal{F} = 1 \otimes 1.$$ (25)

(26)

defines the deformed (twisted) Hopf algebra $U_{\mathcal{F}}(g)$ with the unchanged multiplication, unit and counit (as in $U(g)$), the twisted comultiplication and antipode defined by the relations

$$\Delta_{\mathcal{F}}(u) = \mathcal{F}\Delta^{(0)}(u)\mathcal{F}^{-1}, \quad u \in U(g),$$

$$S_{\mathcal{F}}(u) = -uvu^{-1}, \quad v = \sum f_i^{(1)}(S^{(0)}f_i^{(2)}).$$ (27)

The twisted algebra $U_{\mathcal{F}}(g)$ is triangular, with the universal $\mathcal{R}$-matrix

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{F}^{-1}.$$ (28)

When $\mathcal{F}$ is a smooth function of $\xi$ and $\lim_{\xi \to 0}\mathcal{F} = 1 \otimes 1$ then in the neighborhood of the origin the $\mathcal{R}$-matrix can be presented as

$$\mathcal{R}_{\mathcal{F}} = 1 \otimes 1 + \xi r_{\mathcal{F}} + o(\xi),$$

(29)

where $r_{\mathcal{F}}$ is the skewsymmetric classical $r$-matrix corresponding to the twist $\mathcal{F}$. 

6
By a nonlinear change of basis in $\mathfrak{g}$ one can modify the twisted coproducts and locate part of the deformation in the algebraic sector.

d) **Twisting elements and real forms of Hopf algebras**

The generators of physical symmetries are described by self-adjoint (real) generators and it is desirable to have a real form of quantized Lie algebras. Real forms of complex simple Lie algebra $\mathfrak{g}$ are described by the involutive automorphism $\Phi : U(\hat{\mathfrak{g}}) \to U(\hat{\mathfrak{g}})$ with the following properties

$$
\begin{align*}
\Phi^2 &= \text{id}, & \Phi(1) &= 1, \\
\Phi(X \cdot Y) &= \Phi(Y) \cdot \Phi(X), & \Phi(\mu X + \nu Y) &= \mu^* \Phi(X) + \nu^* \Phi(Y).
\end{align*}
$$

(30)

The Hopf algebra $(U(\hat{\mathfrak{g}}), \cdot, \Delta, S, \eta, \varepsilon, \Phi)$ that in general has nonprimitive costructure is a standard real Hopf algebra if $\Phi$ satisfies (30) and additionally

$$
\Delta(\Phi(X)) = \Phi(\Delta(X))
$$

(31)

where $\Phi(X \otimes Y) = \Phi(X) \otimes \Phi(Y)$. This implies

$$(\Phi \circ S)^2 = 1.
$$

(32)

In particular for $\Delta_{\mathcal{F}}$ given by (27) one gets

$$
\Phi(\Delta_{\mathcal{F}}(X)) = \Phi(\mathcal{F}^{-1})\Delta^{(0)}(\Phi(X))\Phi(\mathcal{F}),
$$

(33)

where we used the relation (31) for $\Delta = \Delta^0$ (the comultiplication induced by primitive coproducts of generators). From (33) one gets (instead of (31))

$$
\Delta_{\mathcal{F}}(\Phi(X)) = \mathcal{F}\Delta^{(0)}(\Phi(X))\mathcal{F}^{-1} = J\Phi(\Delta_{\mathcal{F}}(X))J^{-1},
$$

(34)

where

$$
\mathcal{F} \cdot \Phi(\mathcal{F}) = J.
$$

(35)

Therefore we see that the standard definition of real form for quantum group is valid only if the twisting element $\mathcal{F}$ is ”$\Phi$-unitary”, i.e. $[\Delta_{\mathcal{F}}(X), J] = 0$ for any $X \in U(\hat{\mathfrak{g}})$. In such a case one can say that the twist $\mathcal{F}$ is compatible with the involution $\Phi$ (with the relation (30) satisfied). Due to the formulae (35) the element $J$ is ”$\Phi$-unitary”, i.e. $[J \cdot \Phi(J^{-1}), \Delta_{\mathcal{F}}(X)] = 0$.

It is known that for $\mathfrak{sl}(n; c)$ the extended Jordanian twists are naturally compatible with the involution providing the real form $\mathfrak{sl}(n; R)$ (see [25], sec. 5). In the general case the involutions defining other real forms are not compatible with extended Jordanian twists.
e) Twists for $\kappa$-deformed conformal algebra

The general approach to the solution of twist equations for the Lie algebra $sl(n)$ was presented in [25, 27]. Here we shall apply these results for the case $n = 4$.

The two-parameter family of classical $r$-matrices (22) that contains the light-cone $\kappa$-Poincarè $r$-matrix (we recall that in (22) one can put $\gamma = 0$) leads to the following three-parameter twisting element:

$$F_\xi(\beta, \varrho) = \exp[\xi e_1 \otimes e_5(1 + \xi e_6)^{(2i\beta - \varrho)}] \exp[\xi e_4 \otimes e_3(1 + \xi e_6)^{(2i\beta + \varrho - 1)}] \exp[i\beta(h_1 - h_3) + \varrho h_2 + \frac{1}{2}(h_1 + h_3)] \otimes \sigma_6(\xi)]$$

where (we recall that $\rho \neq \frac{1}{2}$)

$$\sigma_k = \ln(1 + \xi e_k).$$

and the reality condition $\xi^* = \pm \xi$ copies the reality condition for $e_k$ (see (14)) i.e.

$$(\xi e_k)^* = \xi e_k,$$ therefore in (37) $\xi^* = -\xi$. In this expression $\xi$ plays the role of an "over-all" deformation parameter and provides the expansion

$$F_\xi(\beta, \varrho) = 1 + \xi r^{ns}(\beta, \varrho) + O(\xi^2).$$

Where the element $r^{ns} \in g \otimes g$ can be obtained from (22) by substituting the wedge product $(A \wedge B = A \otimes B - B \otimes A)$ by the tensor product: $(A \wedge B \rightarrow A \otimes B)$ and setting $\gamma = 0$. If we assume that in (36) $\beta = \varrho = 0$ we obtain the formula for the twisting element $F_\xi$ for the light-cone $\kappa$-Poincarè deformation

$$F_\xi \equiv F_\xi(0, 0) = \exp(\xi e_1 \otimes e_5)\exp[\xi e_4 \otimes e_3(1 + \xi e_6)^{-1}]\exp[\frac{i}{2}(h_1 + h_3)] \otimes \sigma_6(\xi)].$$

The four-parameter family of infinitesimal deformations described by classical $r$-matrices (20) leads to the following six-parameter family of twists

$$F_{\xi, \xi'}(\alpha, \beta, \gamma, \delta) = \exp[\gamma\sigma_2(\xi') \otimes \sigma_6(\xi)]\exp[(\frac{i\alpha}{2\delta}(h_1 - h_2) + \frac{1}{2}h_2) \otimes \sigma_2(\xi')] \exp[\xi e_1 \otimes e_5(1 + \xi e_6)^{(2i\beta - \frac{1}{2})}] \exp[\xi e_4 \otimes e_3(1 + \xi e_6)^{(2i\beta - \frac{1}{2})}] \exp[i\beta(h_1 - h_3) + \frac{1}{2}(h_1 + h_2 + h_3)] \otimes \sigma_6(\xi).$$

where $\xi^* = -\xi$, $(\xi')^* = -\xi'$. The first factor in (40) contributes to the classical $r$-matrix due to the relation

$$\exp[\frac{\gamma}{\xi} \sigma_6(\xi) \otimes \sigma_2(\xi')] = \exp[\gamma\xi(e_6 - \xi e_6^2/2 + O(\xi^2)) \otimes (e_2 - \xi' e_2^2/2 + O(\xi'^2))].$$

Both parametric families of twists (36) and (40) start with the similar extended Jordanian twists. The difference between the families $F_\xi$ and $F_{\xi, \xi'}$ is that the second
of them is the full chain of twists containing two links with the additional Reshetikin factor while the first is the single extended twist [25,26]. The intersection of these families is the two-parametric set $\mathcal{F}_\xi(\beta, 1/2)$.

Due to the "matreshka" effect [27] the deformation parameters $\xi$ and $\xi'$ are independent. To pass to the quasi-classical limit one must choose in the space of parameters the one-dimensional smooth curve. We shall consider here the case where $\xi'$ is proportional to $\xi$. In particular for $\xi' = 2\delta\xi$ and $\xi$ as the overall deformation parameter we obtain the classical $r$-matrix (20) as the leading term in the expansion of the universal $R$-matrix $\mathcal{R}(\xi) = (\hat{\mathcal{F}}_\xi)_{21}(\hat{\mathcal{F}}_\xi)_{12}^{-1}$. The twist $\hat{\mathcal{F}}_\xi \equiv \mathcal{F}_\xi(0, 0, 0, 0)$ for $\kappa$-Weyl deformation described by the classical $r$-matrix $\hat{r}$ (see (20)) is obtained from (40) at $\beta = 0$ at particular limits $\gamma \to 0, \xi' \to 0$ as well as $\alpha \to 0, \delta \to 0$ and is given by the formula

$$
\hat{\mathcal{F}}_\xi = \exp[\xi_1 \otimes e_5 (1 + \xi_6)] \exp[\xi_4 \otimes e_3 (1 + \xi_6)] \exp[\frac{1}{2} (h_1 + h_2 + h_3)] \otimes \sigma_6(\xi) \quad (42)
$$

III. $D = 4$ conformal $\kappa$-deformations described by twists.

i) Physical basis for $D = 4$ conformal algebra and classical conformal $r$-matrices. The $D = 4$ conformal Lie algebra is described by 15 anti-hermitean generators $J_{AB}$ ($J^\dagger_{AB} = -J_{BA}$; $A, B = 0, 1, 2, 3, 4, 5$) satisfying the relation

$$
[J_{AB}, J_{CD}] = \eta_{AD} J_{BC} + \eta_{BC} J_{AD} - \eta_{BD} J_{AC} - \eta_{AC} J_{BD} \quad (43)
$$

with $\eta_{AB} = \text{diag}(-1, 1, 1, 1, -1)$. The physical basis is given by the relations

$$
P_\mu = \frac{1}{\sqrt{2}} (M_{5\mu} + M_{4\mu}), \quad K_\mu = \frac{1}{\sqrt{2}} (M_{5\mu} - M_{4\mu}),
$$

$$
M_i = \frac{1}{2} \epsilon_{ijk} J_{jk}, \quad L_i = M_{0i}, \quad D = M_{45} \quad (44)
$$

where $\mu = 0, 1, 2, 3$ and $i, j = 1, 2, 3$. All the generators (44) are also anti-hermitean. For convenience we introduce also the the generators $P_\pm = P_0 \pm P_3$ as well as $M_\pm = M_1 \pm iM_2$ ($(M_\pm)^+ = -M_\mp$) and $L_\pm = L_1 \pm iL_2$ ($(L_\pm)^+ = -L_\mp$). The Cartan-Weyl basis is described in terms of $D = 4$ conformal generators ($M_i, N_i, D, P_i, P_0, K_i, K_0$) as follows

$$
e_1 = \frac{1}{2} (M_+ + iL_+), \quad e_{-1} = \frac{1}{2} (-M_- - iL_-),
$$

$$
e_2 = \frac{1}{2} P_-, \quad e_{-2} = -\frac{1}{2} K_+,
$$

$$
e_3 = -\frac{1}{2} (M_- - iL_-), \quad e_{-3} = \frac{1}{2} (M_+ - iL_+),
$$

$$
e_4 = \frac{i}{2} (P_1 + iP_2), \quad e_{-4} = -\frac{i}{2} (K_1 - iK_2),
$$

$$
e_5 = -\frac{i}{2} (P_1 - iP_2), \quad e_{-5} = \frac{i}{2} (K_1 + iK_2),
\[ e_6 = \frac{1}{2} P_+ \quad e_{-6} = -\frac{1}{2} K_-, \]
\[ h_1 = \bar{L}_3 - i M_3, \quad h_2 = -L_3 - D, \quad h_3 = L_3 + i M_3. \]  

(45)

The classical \( r \)-matrix (23) describing light-like \( \kappa \)-deformation is given respectively by the formula (6), with the "overall" deformation parameter \( \xi = \frac{2i}{\kappa} \) carrying the inverse mass dimension, i.e. we obtain the following expansion for universal \( R \)-matrices

\[ R_\kappa = 1 + \frac{1}{\kappa}(L_3 \wedge P_+ - (M_1 - L_2) \wedge P_2 + (M_2 + L_1) \wedge P_1) + O\left(\frac{1}{\kappa^2}\right) \]  

(46)

Similarly, the \( \kappa \)-Weyl \( r \)-matrix (21) provides the expansion

\[ \hat{R}_\kappa = 1 + \frac{1}{\kappa}[\frac{1}{2}(L_3 - D) \wedge P_+ - (M_1 - L_2) \wedge P_2 + (M_2 + L_1) \wedge P_1] + O\left(\frac{1}{\kappa^2}\right) \]  

(47)

where the term linear in \( \frac{1}{\kappa} \) describes the \( \kappa \)-Weyl \( r \)-matrix in physical basis (45).

Denoting

\[ \tilde{E}_1 = L_1 + M_2, \quad \tilde{E}_2 = L_2 - M_1, \quad \tilde{E}_3 = L_3 \]  

(48)

one gets for the light-cone \( \kappa \)-Poincarè \( r \)-matrix the following six-dimensional carrier algebra

\[ [\tilde{E}_1, \tilde{E}_2] = 0, \quad [\tilde{E}_3, \tilde{E}_r] = \tilde{E}_r, \quad [\tilde{E}_r, P_+] = 0, \quad [\tilde{E}_r, P_s] = \delta_{rs} P_+, \quad [\tilde{E}_3, P_+] = P_+, \quad [\tilde{E}_3, P_s] = 0 \]  

(49)

(50)

where \( r, s = 1, 2 \). If we introduce the notation

\[ \hat{E}_1 = L_1 + M_2, \quad \hat{E}_2 = L_2 - M_1, \quad \hat{E}_3 = \frac{1}{2}(L_3 - D) \]  

(51)

the carrier algebra for the \( \kappa \)-Weyl \( r \)-matrix is the following

\[ [\hat{E}_1, \hat{E}_2] = 0, \quad [\hat{E}_3, \hat{E}_r] = \hat{E}_r, \quad [\hat{E}_r, P_+] = 0, \quad [\hat{E}_r, P_s] = \delta_{rs} P_+, \quad [\hat{E}_3, P_+] = P_+, \quad [\hat{E}_3, P_s] = \frac{1}{2} P_s \]  

(52)

(53)

where \( r, s = 1, 2 \).

It follows from relations (45) that in the formulae (21) and (23) the parameters \( \alpha, \beta, \delta, \varrho \) have the dimension of inverse mass and the parameter \( \gamma \), occurring only in (20), describes the "soft deformation" with the dimension of the inverse mass square. We see therefore that all deformations of conformal algebra considered in this paper are the ones with dimensionfull deformation parameters i.e. describe generalized \( \kappa \)-deformations (see footnote 5).

In order to characterize physically the deformations provided by the twists (39) and (42) we shall consider the twisted coproducts for the four-momentum generators. Using the twist (36) one obtains the following formulae:

\[ \Delta \mathcal{F}_\xi(e_6) = e_6 \otimes e^{\sigma_6(\xi)} + 1 \otimes e_6, \]
\[
\begin{align*}
\Delta \mathcal{F}_\xi(e_2) &= e_2 \otimes e^{(2q-1)\sigma_6(\xi)} + 1 \otimes e_2, \\
\Delta \mathcal{F}_\xi(e_4) &= e_4 \otimes e^{(2i\beta+q-1)\sigma_6(\xi)} + 1 \otimes e_4, \\
\Delta \mathcal{F}_\xi(e_5) &= e_5 \otimes e^{(-2i\beta+q)\sigma_6(\xi)} + e^{\sigma_6(\xi)} \otimes e_5,
\end{align*}
\]

(54)

If we consider the light-cone \(\kappa\)-Poincarè deformation (see (46)) we observe from (54) that after the substitution (45) all the four-momentum components in Poincarè algebra basis are endowed with nonprimitive coproducts. In order to obtain primitive coproduct for the light-cone energy \(P'_+\) we should introduce the following nonlinear transformation of \(P'_+\)-variable\(^9\):

\[
\frac{1}{2} \xi P'_+ = \sigma_6(\xi) = \ln(1 + \frac{1}{2} \xi P_+)
\]

(55)

Substituting \(\xi = \frac{2i}{\kappa}\) one gets for the light-cone \(\kappa\)-deformation the following four-momentum coproducts:

\[
\begin{align*}
\Delta \mathcal{F}_\xi(P'_+) &= P'_+ \otimes 1 + 1 \otimes P'_+, \\
\Delta \mathcal{F}_\xi(P_-) &= P_- \otimes e^{-iP_+/\kappa} + 1 \otimes P_- \\
\Delta \mathcal{F}_\xi(P_1 + iP_2) &= (P_1 + iP_2) \otimes e^{-iP_+/\kappa} + 1 \otimes (P_1 + iP_2), \\
\Delta \mathcal{F}_\xi(P_1 - iP_2) &= (P_1 - iP_2) \otimes 1 + e^{iP_+/\kappa} \otimes (P_1 - iP_2),
\end{align*}
\]

(56)

The new basis with light-cone energy generator given by (55) leads to the deformation of the commutators between Abelian four-momentum generators \((P_1, P_2, P_-, P'_+)\) and classical Lorentz generators \(M_{\mu\nu} = (M_1, N_1)\):

\[
\begin{align*}
[M_1, P'_+] &= P_2 e^{-iP_+/\kappa}, & [M_2, P'_+] &= -P_1 e^{-iP_+/\kappa}, \\
[M_3, P'_+] &= 0, \\
[L_1, P'_+] &= P_1 e^{-iP_+/\kappa}, & [L_2, P'_+] &= P_2 e^{-iP_+/\kappa}, \\
[L_3, P'_+] &= -i\kappa(1 - e^{-iP_+/\kappa}).
\end{align*}
\]

(57)

It should be noticed that the coproduct (56) for \(P'_+\) remains primitive for any choice of the parameters \(\beta\) and \(q\) in the twist formulae (35).

It is easy to see that in terms of the generators \(P_1, P_2, P_-, P'_+\) the mass Casimir takes the form

\[
C_2 = P_1^2 + P_2^2 - \frac{2}{\xi}(1 - e^{4\xi P'_+}) P_-
\]

(58)

The coproduct formulae for the twist (40) which are analogous to the ones given by the relations (54) look as follows:

\[
\Delta \mathcal{F}_{\xi, \xi'}(e_6) = e_6 \otimes e^{\sigma_6(\xi)} + 1 \otimes e_6,
\]

\(^9\)It is interesting to observe that the relation (55) coincides with the deformation map describing \(\kappa\)-deformed Weyl algebra in [36] (see formula (14)). We thank Piotr Kosiński for providing this observation.

11
If we introduce the expression (55) and redefine the energy $P_-$

$$\frac{1}{2} \xi P'_- = \sigma_2(\xi) = \ln(1 + \frac{1}{2} \xi P_-) \quad (60)$$

the two coproducts, for the generators $P'_6$ and $P'_3$, remain primitive. If $\alpha = \beta = 0$ the formulae (59) describe the coproducts of four-momenta generated by the $\kappa$-Weyl twist (42).

In order to obtain the quantum $D = 4$ conformal algebra with more suitable deformation of the algebra and coalgebra sectors one should at least supplement (55) and (60) with nonlinear transformation of three-momenta (see also [36]). The choice of "optimal" nonlinear basis in $U(o(4,2))$ describing the $\kappa$-deformed $D = 4$ conformal algebras $U_\kappa(o(4,2))$ is under consideration.

### IV. Conclusions.

It is well-known that in conformal-invariant theories all mass-like parameters should be put equal to zero; on the other hand the Einstein gravity action is proportional to the dimensionful Newton coupling constant defining the Planck mass $M_{PL}$ (or Planck length $\lambda_{PL}$). It is therefore natural to assume that due to the gravitational effects in $D = 4$ conformal-invariant field theory (e.g. non-Abelian gauge theory) should appear conformal-breaking terms, proportional to inverse powers of the Planck mass. The $\kappa$-deformations of the conformal symmetry algebra leads to a corresponding algebraic scheme of broken conformal symmetries with the appearance of the fundamental mass parameters, which physically should be related with the Planck mass.

The advantage of our scheme is the description of broken Weyl and conformal symmetries by exact quantum symmetries that permits to use the algebraic techniques of quantum groups and quantum Lie algebras. In particular a genuine (Hopf-algebraic) quantum deformation of symmetry algebra permits to construct the consistent representation theory for irreducible as well as for tensor product representations, what should be useful in passing from quantum-mechanical to quantum field-theoretic description of models with broken conformal symmetry.

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