Convex geometry and the Erdős–Ginzburg–Ziv problem

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Abstract

Denote by $s(\mathbb{F}_p^d)$ the Erdős–Ginzburg–Ziv constant of $\mathbb{F}_p^d$, that is, the minimum $s$ such that any sequence of $s$ vectors in $\mathbb{F}_p^d$ contains $p$ vectors whose sum is zero. Let $w(\mathbb{F}_p^d)$ be the maximum size of a sequence of vectors $v_1, \ldots, v_s \in \mathbb{F}_p^d$ such that for any integers $\alpha_1, \ldots, \alpha_s \geq 0$ with sum $p$ we have $\alpha_1 v_1 + \ldots + \alpha_s v_s \neq 0$ unless $\alpha_i = p$ for some $i$.

In 1995, Alon–Dubiner proved that $s(\mathbb{F}_p^d)$ grows linearly in $p$ when $d$ is fixed. In this work, we determine the constant of linearity: for fixed $d$ and growing $p$ we show that $s(\mathbb{F}_p^d) \sim w(\mathbb{F}_p^d)p$. Furthermore, for any $p$ and $d$ we show that $w(\mathbb{F}_p^d) \leq \frac{(2d-1)}{d} + 1$. In particular, $s(\mathbb{F}_p^d) \leq 4^d p$ for all sufficiently large $p$ and fixed $d$.

1 Introduction

1.1 History and new upper bound

In 1961, Erdős, Ginzburg and Ziv [10] showed that among any $2n-1$ integers one can always select exactly $n$ whose sum is divisible by $n$. Harborth [13] considered a higher-dimensional generalization of this problem: for given natural numbers $n, d$, what is the minimum number $s$ such that among any $s$ points in the integer lattice $\mathbb{Z}^d$ there are $n$ points whose centroid is also a lattice point? Equivalently, if we consider points of the lattice $\mathbb{Z}^d$ modulo $n$ then the quantity $s$ is the maximum size of a multi-set of points in $\mathbb{Z}_n^d$ such that the sum of any $n$ of them is not congruent to 0 modulo $n$. In light of the latter interpretation, the number $s$ is denoted by $s(\mathbb{Z}_n^d)$ and called the Erdős–Ginzburg–Ziv constant of the group $\mathbb{Z}_n^d$. Note that points are allowed to coincide in this definition. The problem of determining $s(\mathbb{Z}_n^d)$ for various $n$ and $d$ has received considerable attention but the precise value of $s(\mathbb{Z}_n^d)$ is still unknown for the majority of parameters $(n, d)$. One can also define the Erdős–Ginzburg–Ziv constant of an arbitrary finite abelian group $G$, see [12] for details and various generalizations.

Confirming a conjecture of Kemnitz [14], Reiher [18] showed that $s(\mathbb{Z}_n^2) = 4n - 3$ for any $n \geq 2$. In [1], Alon and Dubiner showed that for any $n$ and $d$ we have

$$s(\mathbb{Z}_n^d) \leq (Cd \log d)^d n$$

\[1\]

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for some absolute constant $C > 0$. In particular, if we fix $d$ and let $n \to \infty$ then $s(\mathbb{Z}_n^d)$ grows linearly with $n$. On the other hand, it is not hard to see that $s(\mathbb{Z}_n^d) \geq 2^d(n - 1) + 1$. Indeed, consider the vertices of the boolean cube \{0, 1\}^d where each vertex taken with multiplicity $n - 1$, then this set has no $n$ elements that sum up to 0 in $\mathbb{Z}_n^d$. The best known lower bound on $s(\mathbb{Z}_n^d)$ is due to Edel \[5\]:

$$s(\mathbb{Z}_n^d) \geq 96^{\lfloor d/6 \rfloor} (n - 1) + 1 \approx 2.139^d n,$$  \hspace{1cm} (2)

which holds for all odd $n$. The corresponding set of points generalizes the boolean cube construction, namely it is a cartesian product of \lfloor d/6 \rfloor copies of a certain explicitly constructed set $A \subset \mathbb{Z}^6$ of cardinality 96 where each point is taken with multiplicity $n - 1$. Note that the condition that $n$ is odd is necessary: for $n = 2^k$ we have \[13\] that $s(\mathbb{Z}_n^d) = 2^d(n - 1) + 1$ holds for all $d$.

The case when $n = p$ is a prime number is of particular interest. On the one hand, the vector space structure on $\mathbb{Z}_p^d \cong \mathbb{F}_p^d$ significantly simplifies analysis but the problem is still highly non-trivial. On the other hand, as it was observed for $d = 1$ in \[10\], one can deduce upper bounds on $s(\mathbb{Z}_n^d)$ from upper bounds on $s(\mathbb{F}_p^d)$ using simple induction on the prime decomposition of $n$. This paper focuses on the case when the dimension $d$ is fixed and $n = p$ is a very large prime number. We remark that the complementary case, i.e. when the prime $p$ is fixed and the dimension $d$ is large is also of great interest and, in a way, even more intriguing. The current best bounds are $s(\mathbb{F}_p^d) \leq 2.756^d$ for $p = 3$ which was proved by Ellenberg–Gijswijt in a breakthrough paper \[7\] and $s(\mathbb{F}_p^d) \leq C_p(2\sqrt{p})^d$ for $p > 5$ due to Sauermann \[19\]. Note that the best known lower bound in this regime is also \[2\] which creates a significant gap between the bases of the exponents. We refer to \[19\] and references therein for the history and state of art in this question. After the release of this paper, Sauermann and the author \[20\] showed that for fixed $p$ and large $d$ we have $s(\mathbb{F}_p^d) \leq D_{p,\varepsilon}(C_{p,\varepsilon} p)^d$ for any fixed $\varepsilon > 0$.

The main result of the present paper is an improvement of the Alon–Dubiner bound \[1\] for fixed $d$ and sufficiently large primes $p$.

**Theorem 1.1.** Let $d \geq 1$ and $p > p_0(d)$ be a sufficiently large prime number. Then we have

$$s(\mathbb{F}_p^d) \leq 4^d p.$$ \hspace{1cm} (3)

More generally, if all prime divisors of $n > 1$ are larger than $p_0(d)$ then we have $s(\mathbb{Z}_n^d) \leq 4^d n$.

Unfortunately, the condition that $p > p_0$ is necessary for our arguments and cannot be removed. Note that by a classical argument from \[10\], the bound for composite $n$ in Theorem 1.1 (essentially) follows from the corresponding bound for primes.

**Multiplicity $p - 1$ sets.** As we discussed above, taking each vertex of the boolean cube (or a certain more general set) with multiplicity $p - 1$ leads to a lower bound construction for $s(\mathbb{F}_p^d)$. It is natural to ask what is the best possible lower bound construction of this form? For $d \geq 1$ and a prime $p$ define the function $w(\mathbb{F}_p^d)$ as the maximum number $s$ such that there are vectors $v_1, \ldots, v_s \in \mathbb{F}_p^d$ with the property that for any non-negative integers $\alpha_1, \ldots, \alpha_s$ with sum $p$ we have $\alpha_1 v_1 + \ldots + \alpha_s v_s \equiv 0 \pmod{p}$ if and only if $\alpha_i = p$ for some $i$. For brevity, let us call any
set of vectors $X = \{v_1, \ldots, v_s\}$ satisfying this property $p$-hollow (a justification for this name will become more clear later).

Note that if $X$ is $p$-hollow, then taking each element of $X$ with multiplicity $p - 1$ results in a multiset not containing $p$ vectors with zero sum. This implies that for all $p$ and $d$ we have

$$s(F^d_p) \geq w(F^d_p)(p - 1) + 1. \quad (4)$$

It is easy to see that the Cartesian product of two $p$-hollow sets is again $p$-hollow, so any lower bound on $w(F^d_p)$ for some fixed $d_0$ extends to a lower bound for all $d \geq d_0$ by the product construction. In fact, all known lower bounds to $s(F^d_p)$ follow from this observation combined with (4) and in particular (2) follows from $w(F^6_p) \geq 96$ for all $p > 2$. In [12], Gao–Geroldinger conjectured that equality holds in (4). We confirm their conjecture asymptotically as $p \to \infty$.

**Theorem 1.2.** For any fixed $d \geq 1$ and $p \to \infty$, we have $s(F^d_p) = w(F^d_p)p + o(p)$.

Using the slice rank method, Naslund [16] showed that $w(F^d_p) \leq 4d - 1$. So it follows that Theorem 1.1 is in fact a consequence of Theorem 1.2. We have the following slight improvement of the slice rank bound:

**Proposition 1.3.** For any $d \geq 1$ and any prime $p$, we have $w(F^d_p) \leq \left(\frac{2d - 1}{d}\right) + 1$.

Note that $w(F^1_p) = 2 = \left(\frac{1}{1}\right) + 1$ and $w(F^2_p) = 4 = \left(\frac{3}{2}\right) + 1$ so the bound in Theorem 1.3 is achieved for $d = 1, 2$. On the other hand, for $d = 3$ it can be shown that $w(F^3_p) = 9$ for large $p$ while Theorem 1.3 only gives an upper bound of 11.

**Notation.** We use the asymptotic notation $A \gg B$ to denote that $A \geq cB$ for some constant $c > 0$, possibly depending on other parameters. The set of natural numbers $\mathbb{N}$ is the set $\{0, 1, \ldots\}$. A multiset $X \subset A$ of some set $A$ is an unordered sequence of elements of $A$, possibly with repetitions. Two elements of $X$ are said to be distinct if they are on different positions in the sequence (even though they may coincide as elements of $A$).

### 1.2 Connection to convex geometry

The main new ingredient in the proof Theorem 1.2 is a certain connection of the Erdős–Ginzburg–Ziv problem to convex geometry. Recall that original formulation of the question by Harborth is stated in terms of centroids of $n$ points in $\mathbb{Z}^d$, so it is natural to expect that tools from convex geometry may be useful to tackle the problem. On the other hand, there does not seem to be a direct way to employ this idea and, to the authors knowledge, convex geometry has not been used in the study of Erdős–Ginzburg–Ziv problem and related zero sum problems before.

Throughout this paper, a polytope $P \subset \mathbb{Q}^d$ is the convex hull of a finite set of points in $\mathbb{Q}^d$. A lattice $\Lambda \subset \mathbb{Q}^d$ is an affine image of the set $\mathbb{Z}^r \subset \mathbb{Q}^r$ for some $r \leq d$. We define a notion of integer points of polytopes in the following way.

**Definition 1.4 (Integer point).** Let $P \subset \mathbb{Q}^d$ be a polytope and let $q \in P$. Let $\Gamma \subset P$ be the minimal face of $P$ containing the point $q$ and let $\Lambda$ be the minimal lattice which contains all vertices of $\Gamma$. We say that $q$ is an integer point of $P$ if $q \in \Lambda$. 


Let us say a couple of words on why this notion is natural. If we have a polytope \( P \subset \mathbb{Q}^d \) then one might say that a point \( q \in P \) is an integer point if we simply have \( q \in \mathbb{Z}^d \). However this notion depends on the choice of the integer lattice \( \mathbb{Z}^d \) and so is not an ‘intrinsic’ property of \( P \) and \( q \). To fix this, we can modify the definition: let \( \Lambda \) be the lattice spanned by the vertices of \( P \) and say that a point \( q \in P \) is integral if we simply have \( q \in \Lambda \). This definition clearly does not depend on the lattice \( \mathbb{Z}^n \) and is closer to what we want. On the other hand, this notion has a problem: if \( \Gamma \subset P \) is a face of \( P \) and \( q \in \Gamma \) is a point then the properties of \( q \) being integral with respect to \( P \) and \( \Gamma \) are not the same. It is easy to construct examples where \( q \) is an integer point of \( P \) but not of \( \Gamma \). So our notion of integer points is not invariant under passing to a face of \( P \). To fix this, we introduce the additional step and choose the minimal face \( \Gamma \) containing \( q \), then define the lattice \( \Lambda \) spanned by the vertices of \( \Gamma \) and say that \( q \) is an integer point of \( P \) if \( q \in \Lambda \). This definition accomplishes both the invariance under the change of basis of \( \mathbb{Q}^d \) and passing to a face.

We say that a polytope \( P \subset \mathbb{Q}^d \) is a \textit{hollow} polytope if \( P \) does not have any integer points besides its vertices. For \( d \geq 1 \), let \( L(d) \) be the maximum number of vertices in a hollow polytope \( P \subset \mathbb{Q}^d \). It turns out that vertices of a hollow polytope precisely correspond to \( p \)-hollow sets modulo almost all primes \( p \).

**Proposition 1.5.** Let \( P \subset \mathbb{Q}^d \) be a hollow polytope and suppose that the vertices \( X \) of \( P \) lie in \( \mathbb{Z}^d \). Then for all but a finite list of primes \( p \), the reduction of \( X \) modulo \( p \) is a \( p \)-hollow set. In particular, for \( d \geq 1 \) and sufficiently large primes \( p \geq p_0(d) \) we have \( w(\mathbb{F}_p^d) \geq L(d) \).

Note that the list of forbidden primes can be written explicitly in terms of \( P \), see Section 2.2 for details and the proof.

As a matter of fact, all known lower bound constructions for \( p \)-hollow sets are actually coming from constructions of hollow polytopes, even though the notion of hollow polytopes has not been given explicitly in the zero sum set literature before. In particular, Elsholtz [8] showed that \( L(3) \geq 9 \), Edel [5] and Elsholtz [9] showed that \( L(4) \geq 20 \), and in [6] Edel showed that \( L(5) \geq 42 \), \( L(6) \geq 96 \), \( L(7) \geq 196 \). Note that the lower bound \( L(6) \geq 96 \) and a product construction give the best known asymptotic lower bounds on \( w(\mathbb{Z}_n^d) \) and \( s(\mathbb{Z}_n^d) \) cited previously. In light of this, it seems reasonable to expect that the converse of Proposition 1.5 should also be true.

**Conjecture 1.6.** For \( d \geq 1 \) and all sufficiently large primes \( p \) we have \( w(\mathbb{F}_p^d) = L(d) \).

It is an easy exercise to check this for \( d = 1, 2 \) and with some extra work one can show that \( w(\mathbb{F}_p^3) = L(3) = 9 \). But the next special case \( d = 4 \) seems to be out of reach and even computing \( L(4) \) seems unfeasible.

All known lower bound constructions suggest that the case when the multiset \( X \subset \mathbb{F}_p^d \) is contained in a box \([-K, K]^d \) of bounded size \( K \) plays a special role in the Erdős–Ginzburg–Ziv problem. Namely, the exact value of the optimal constant \( C \) in the bound \( s(\mathbb{F}_p^d) \leq Cp \) for growing \( p \) should come from this special case. As a first step towards the proof of Theorem 1.2 and as a way to explain our key ideas we show the following.
**Theorem 1.7.** Fix $d, K \geq 1$, $\varepsilon > 0$ and let $p > p_0(d, K, \varepsilon)$ be a prime. Suppose that $X \subset [-K, K]^d$ is a sequence of at least $(L(d) + \varepsilon)p$ elements. Then $X$ contains $p$ elements with zero sum.

Note that the constant $L(d)$ in the above result is tight as can be seen by the lower bound constructions for the Erdős–Ginzburg–Ziv problem discussed above. This result can be thought of as a variant of Theorem 1.2 in the important special case when $X$ is bounded. The proof of Theorem 1.7 heavily relies on ideas from convex geometry and we give a detailed overview of these ideas in Section 1.4. We note that even though Theorem 1.7 does not follow directly from Theorem 1.2 since we are unable to show that $\mu(\mathbb{F}_p^d) = L(d)$ for large $p$, the proof of the former can be easily extracted from the proof of the latter. Because of that, we do not present a complete proof of Theorem 1.7 in this paper and give an outline to demonstrate the main ideas only.

### 1.3 A structural result

The proof of Theorem 1.2 is based on the ideas from the proof of Theorem 1.7 (which we discuss later in Section 1.4) but requires new ingredients to go through. Indeed, a large set $X \subset \mathbb{F}_p^d$ without $p$ elements with zero sum is not necessarily contained (or almost contained) in a box $[-K, K]^d$ of bounded size in any coordinate system as we show by the following construction.

Let us take $1 \leq d' < d$ and consider a multiset $X' \subset \mathbb{F}_p^{d'}$ without $p$ elements with zero sum, for example, constructed from a hollow polytope $P \subset \mathbb{Q}^{d'}$ or a $p$-hollow set in $\mathbb{F}_p^{d'}$. Let $h: \mathbb{F}_p^d \to \mathbb{F}_p^{d-d'}$ be a uniformly random function and let

$$X = \{(x, h(x)), \ x \in X\} \subset \mathbb{F}_p^{d'} \times \mathbb{F}_p^{d-d'} \cong \mathbb{F}_p^d.$$ 

It is then easy to see that $X$ also does not contain $p$ elements with zero sum and, on the other hand, the size of intersection of $X$ with an affine image of the box $[-K, K]^d$ is tiny and does not give much information about $X$.

To encapsulate constructions like this, we need a more detailed structural description of $X$. Roughly speaking, for a given multiset $X \subset \mathbb{F}_p^d$ we want to find a basis of $\mathbb{F}_p^d$ and some $0 \leq d' \leq d$ such that $X$ is ‘bounded’ on the first $d'$ coordinates and ‘random’ on the last $d - d'$ coordinates. Then we should expect that if $X$ does not contain $p$ elements with zero sum then this fact can be already seen by considering the projection $X' \subset [-K, K]^{d'}$ of $X$ on the first $d'$ coordinates. Indeed, this can be seen by the following heuristic argument. For the sake of contradiction, suppose that the projection $X'$ has some elements $x'_1, \ldots, x'_p \in X'$ with sum zero. We want to show that we can lift the elements $x'_i$ to some elements $x_i \in X$ such that $y = x_1 + \ldots + x_p = 0$. Note that the sum $y$ is already zero on the first $d'$ coordinates and so we need to choose $x_i$-s in such a way that the last coordinates are zero as well. Note that $X'$ is contained in a box $[-K, K]^{d'}$ where $K$ and $d$ are fixed and $p$ is large. In particular, we expect that any element $x' \in X'$ should have at least $\frac{|X'|}{(2K)^{d'}} \gg p$ preimages in $X$. By our assumption, these preimages are distributed quite randomly in $\{x'\} \times \mathbb{F}_p^{d-d'}$ and so the set of all possible sums $y = x_1 + \ldots + x_p$ should be rather uniformly distributed on the last $d - d'$ coordinates and, in particular, we can...
find $x_i$-s such that all last coordinates are zero. This gives $p$ elements $x_1, \ldots, x_p \in X$ with zero sum, contradicting the initial assumption.

So we conclude that if $X$ has no $p$ elements with zero sum then so does its projection $X'$. But $X'$ is bounded and so we can apply Theorem 1.7 to $X'$ and conclude that $|X| = |X'| \leq (L(d) + \varepsilon)p$. This is roughly how we are going to eventually prove Theorem 1.2, however the actual argument is more involved (and we only get the constant $m(F_p^d)$ in the upper bound instead of $L(d)$). As it turns out, splitting coordinates into two parts where $X$ is ‘bounded’ and ‘random’ is not sufficient to show that the projection $X'$ has no $p$ elements with zero sum. The reason for that hides behind a precise notion of ‘random’ that we need to use. Roughly speaking, it says that $X$ is not concentrated on any strip of width $K' \gg K$ around a hyperplane $H \subset \mathbb{F}_p^d$ except for the hyperplanes $H$ coming from the first $d'$ coordinates (on which $X$ is in fact concentrated). This condition always implies that we can easily find subsets of $X$ of size $p$ whose sum is zero on the last $d - d'$ coordinates. However, if we start with a collection of elements $x'_1, \ldots, x'_p \in X'$ which sum to zero on the first $d'$ coordinates, then we are only allowed to choose elements from the subset $\tilde{X} \subset X$ consisting of elements $x$ of the form $(x'_i, \tilde{x}) \in X$ for some $i$ and $\tilde{x}$. We do not have control on exactly which set $\tilde{X}$ we get and it is quite possible that the ‘randomness’ condition may very well be violated for $\tilde{X}$ and the lifting procedure cannot be performed.

To fix this problem one might try to apply the same structural decomposition to the set $\tilde{X}$, namely, consider a new coordinate system where $\tilde{X}$ is bounded on the first $d''$ coordinates and ‘random’ on the rest, for some new $d'' > d'$, and then try to run the lifting procedure on the set $\tilde{X}$ again with respect to this new coordinate system. The problem with this is however that we do not know whether the sum $x'_1 + \ldots + x'_p$ is zero on the first $d''$ coordinates since our first step guaranteed this only for the first $d'$ coordinates. It is tempting to try to apply the first step of the argument again with $d'$ replaced by $d''$ and with the set $X$ replaced by $\tilde{X}$. But the problem with this approach is that the new set $\tilde{X}$ might be too small for the first step of the argument to work directly, indeed, we only know that $\tilde{X}$ contains at least $p$ elements which is way too small. So, in some sense, the first step of the proof where we find elements $x'_1, \ldots, x'_p \in X'$ which sum up to zero on the first $d'$ coordinates should take into account all these issues with sets $\tilde{X}$ being not random enough for the lifting procedure to go through. Namely, we want to make sure that whenever we find the collection of vectors $x'_1, \ldots, x'_p$ it is not only the case that their sum is zero on the first $d'$ coordinates but it is also zero on the first $d''$ coordinates after we apply the structural decomposition to the corresponding set $\tilde{X}$.

One of our main technical results, Theorem 4.9, which we call Flag Decomposition Lemma is designed exactly for this purpose. Section 4 is devoted to formulating and proving this result. Namely, we define certain poset-structures, called ‘convex flags’, which are composed of many subspaces in $\mathbb{F}_p^d$ and polytopes in $\mathbb{Q}^d$. We use these structures to decompose an arbitrary set of points $X \subset \mathbb{F}_p^d$ into pieces with several useful properties. Using these properties and the convex flag structure, we can adapt the ideas from the proof of Theorem 1.7 to obtain a collection of points whose sum is zero on the ‘bounded’ part of the decomposition. Then the properties of decomposition will give us sufficient randomness conditions on the remaining set of coordinates to run the lifting argument. In the end, this leads to the desired upper bound on the size of $X$. In particular, this strategy requires us to perform our convex geometry argument in an
abstract poset setting. The precise statement is Theorem 3.13 which we call Helly’s Theorem for Convex Flags and devote Section 3 to state and prove it.

In Section 5 we perform the lifting procedure step of the argument. In many ways, this part of the proof is closely related to the Alon–Dubiner’s \cite{1} proof of the linear upper bound $s(F_d) \leq C_d p$. Specifically, their argument corresponds precisely to the case when $d' = 0$ in the above considerations, that is, when the set $X$ is entirely random looking. In this case, the delicate convex geometric obstructions are no longer present and one can show much stronger bounds on $|X|$. Namely, as long as $X$ is ‘random-looking’ enough and $|X| \geq (1 + \varepsilon)p$ holds for some fixed $\varepsilon > 0$, one can already find $p$ points with zero sum inside of $X$. To show this, Alon–Dubiner \cite{1} used tools from additive combinatorics and spectral graph theory and, roughly speaking, show that, not only zero, but in fact any element of $F_d$ can be expressed as a sum of $p$ elements of $X$. They then deduce a linear upper bound on the size of $X$ by observing that if the conditions for this argument are not satisfied then for some hyperplane $H$ we have $|H \cap X| \gg |X|$. So one can replace $X$ with $X \cap H$ and use induction on $d$ to finish the proof. In our situation, we use these additive combinatorics tools to lift the elements $x_1', \ldots, x_p' \in X'$ to elements of $X$ so that their sum is zero on the last $d - d'$ coordinates. Our argument requires some modifications since we are more restricted with the choice of elements $x_i \in X$ which we can use for set expansion but the main idea and structure of this part of our argument is similar to that of Alon and Dubiner.

In Section 6 we prove an auxiliary convex geometry statement which allows us to find $p$ points with zero sum using some geometric properties of $X$. We explain this in further detail in Section 1.4 where we give an outline of the proof of Theorem 1.7 about the special case when $X$ is contained in a box of bounded size.

In Section 7 we put everything together and prove Theorem 1.2.

1.4 Outline of the proof of Theorem 1.7

Fix $d, K \geq 1, \varepsilon > 0$ and let $p$ be a large prime. We consider a multiset $X \subset [-K, K]^d$ of size $(L(d) + \varepsilon)p$. We want to find $p$ elements of $X$ which sum to zero modulo $p$. Let us rephrase this problem in a more convenient form. Let $w : [-K, K]^d \to \mathbb{N}$ denote the indicator function of $X$, then we want to find a point $q \in [-K, K]^d$ with integer coordinates and non-negative integer coefficients $a_x$, for $x \in [-K, K]^d$, such that:

$$
\sum_{x \in [-K, K]^d} a_x = p, \quad (5)
$$

$$
q = \frac{1}{p} \sum_{x \in [-K, K]^d} a_x x, \quad (6)
$$

$$
a_x \leq w(x), \text{ for any } x \in [-K, K]^d. \quad (7)
$$

For a function $w : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ with finite support, a point $q \in \mathbb{R}^d$ and $\theta \in [0, 1]$ we say that $q$ is a $\theta$-central for $w$ if for any half-space $H^+ \subset \mathbb{R}^d$ which contains $q$ we have

$$
\sum_{x \in H^+} w(x) \geq \theta \sum_{x \in \mathbb{R}^d} w(x),
$$
i.e. the half-space $H^+$ contains at least a $\theta$-fraction of the weight of $w$.

First, we consider a fractional relaxation of (5)-(7), namely, we allow coefficients $a_x$ to not necessarily be integers. We observe that one can find the desired coefficients $a_x$ provided that the point $q$ on the left hand side of (6) is $\theta$-central for $w$ with some parameter $\theta$. Indeed, note that if $q$ is $\theta$-central for $w$ for some $\theta > 0$ then it follows that $q$ belongs to the convex hull of the support of $w$. So there exists a convex combination with coefficient vector $(b_x)$, where $x \in \text{supp } w$, such that $q = \sum_x b_x x$. It turns out that we can control the magnitude of the coefficients $b_x$ in terms of $\theta$:

**Proposition 1.8.** Let $w : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ be a function such that $\sum_x w(x) = W$ and let $q$ be a $\theta$-central point for $w$ for some $\theta > 0$. Then there exist real coefficients $b_x \geq 0$ such that: $\sum b_x = 1$, $q = \sum_x b_x x$ and for every $x$ we have $b_x \leq \theta^{-1} \frac{w(x)}{W}$.

See Section 6 for a proof. In our application, we have $b_x = \frac{a_x}{p}$ and we need $a_x$ to satisfy $a_x \leq w(x)$. So the relaxed version of (5)-(7) would follow from Proposition 1.8 if $q$ is a $\theta$-central point for $w$ with $\theta = \frac{p}{|X|}$.

To solve the original question we have two problems:

(i) make sure that the coefficients $a_x = pb_x$ are integers,

(ii) construct a point $q \in \mathbb{Z}^d$ which is $\theta$-central for $w$ for a suitable $\theta$,

To guarantee (i) we need to require an additional condition on $q$. Indeed, it is very possible that for some choices of $w : \mathbb{Z}^d \to \mathbb{N}$ and $q \in \mathbb{Z}^d$ no convex combination $b_x$ as in Proposition 1.8 such that $pb_x$ are integers, exists. For example, if the function $w$ is supported on a sublattice $\Lambda \subset \mathbb{Z}^d$ and we have $q \in \mathbb{Z}^d \setminus \Lambda$, then it is easy to check that if for some prime $p$ we have $pb_x \in \mathbb{Z}$ then $p$ must divide the index $|\mathbb{Z}^d/\Lambda|$. So to overcome this obstacle we need to put the following restriction:

$q$ belongs to the minimal lattice spanned by the support of $w$. \hspace{1cm} (8)

This condition however is not sufficient either for the following reason. Suppose that $d = 2$, let $w$ be supported on the points $(0, 0), (2, 0), (0, 1), (1, 1) \in \mathbb{Z}^2$ and take $q = (1, 0)$. Then it is clear that the support of $w$ spans the whole lattice $\mathbb{Z}^2$ and $q \in \mathbb{Z}^2$. However, for any $p > 2$ there does not exist a convex combination of the form $q = \sum_{x \in \text{supp } w} \frac{a_x}{p}$. Indeed, $q$ lies on the boundary of the support of $w$ and so only points $(0, 0)$ and $(2, 0)$ can be used in the convex combination. But then we run into the previous problem where $q$ does not belong to the lattice spanned by these points. So we need to refine (6) as follows. Let $P$ be the convex hull of the support of $w$ and let $\Gamma$ be the minimal face containing $q$, then we need

$q$ belongs to the minimal lattice spanned by the support of $w|_{\Gamma}$. \hspace{1cm} (9)

This condition turns out to be sufficient (up to some minor conditions on $w$). Thus, if for some constant $\theta > 0$ we can show that for a given function $w$ there exists a $\theta$-central point
$q \in \mathbb{Z}^d$ satisfying (9) then for large $p$ there exist coefficients $a_x$ satisfying (5)-(7) provided that $|X| > (1+\varepsilon)^{\frac{\theta}{d}}$ (the $\varepsilon$-error term comes from rounding coefficients when passing from a fractional to an integral solution). This would then imply the upper bound of $(1+\varepsilon)^{\frac{\theta}{d}}$ on the special case of the Erdős–Ginzburg–Ziv problem.

So far we reduced our problem to constructing of a $\theta$-central point $q$ with an additional integral property (9). Again, we start the discussion with relaxed versions of these conditions and then do the necessary adjustments to get what we need. The classical Centerpoint Theorem in convex geometry gives us a way to construct central points for arbitrary functions on $\mathbb{R}^d$.

**Theorem 1.9** (Centerpoint Theorem). Let $w : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ be a function with finite support, then there exists a $\frac{1}{d+1}$-central point $q \in \mathbb{R}^d$ for $w$.

This result is a consequence of another classical convex geometry result, Helly’s Theorem:

**Theorem 1.10** (Helly’s Theorem). Let $\mathcal{F}$ be a collection of compact convex sets in $\mathbb{R}^d$ such that every $d + 1$ of them share a common point. Then all sets in $\mathcal{F}$ share a common point.

Note that the number $d + 1$ in the statement cannot be lowered since otherwise we can take $\mathcal{F}$ to be the collection of faces of a $d$-dimensional simplex $\Delta = \text{conv}\{0, e_1, \ldots, e_d\} \subset \mathbb{R}^d$.

**Proof of Theorem 1.9** Let $\mathcal{F}$ be the collection of all closed half-spaces $H^+$ such that $w(H^+) > \frac{d}{d+1}w(\mathbb{R}^d)$ (to make the sets compact we can intersect $H^+$ with a ball of sufficiently large radius). Now by the pigeonhole principle, any $d+1$ halfspaces in $\mathcal{F}$ share a common point in the support of $w$. So by Helly’s theorem there exists a point $q$ belonging to all half-planes in $\mathcal{F}$. This point is $\frac{1}{d+1}$-central for $w$.

In order to construct central points $q$ with additional properties, we need to prove more refined versions of Helly’s Theorem, however, the deduction of the corresponding Centerpoint Theorem always proceeds in the same way as in the above argument.

Now to get a central point satisfying (9) (which in itself is not enough for us but closer to the condition (9) which we really want) we can use the following integer Helly’s Theorem due to Doignon [4]:

**Theorem 1.11** (Integer Helly’s Theorem). Let $\mathcal{F}$ be a collection of compact convex sets in $\mathbb{R}^d$ such that every $2^d$ of them share a common point $q \in \mathbb{Z}^d$. Then all sets in $\mathcal{F}$ share a common point $q \in \mathbb{Z}^d$.

Note that the constant $2^d$ in the above is tight: let $\mathcal{F}$ consists of convex sets $F = \text{conv}(\{0, 1\}^d \setminus \{x\})$ over all vertices $x$ of the boolean cube $\{0,1\}^d$. Then any $2^d - 1$ of sets in $\mathcal{F}$ share an integer vector but the intersection of all of them is disjoint from $\mathbb{Z}^d$. Using this result, we can find a $2^{-d}$-central point $q$ lying in the lattice spanned by the support of $w$. If this point happens to lie in the interior of the convex hull of the support of $w$ then the second part of the argument goes through and we get $|X| < (1 + \varepsilon)2^dp$. Recall however that for $d \geq 3$ the Erdős–Ginzburg–Ziv constant is significantly larger than $2^d$, so this strategy has no chance of working without modifying Theorem 1.11 in some way.
To illustrate that it is not always possible to find a point \(q\) in Theorem 1.11 which lies in the interior of \(\text{conv} (\text{supp} w)\), let us consider the following example coming from the lower bound \(s(P_3^d) \geq 9(p - 1) + 1\). Consider the following collection of 9 points in \(\mathbb{Z}^3\):

\[
S = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0), \\
(2, 0, 1), (2, 1, 1), (0, 2, 1), (1, 2, 1), \\
(2, 2, 2)\},
\]

let \(w\) be the characteristic function of \(S\). The minimal lattice containing \(S\) is \(\mathbb{Z}^3\) and the \(\frac{1}{8}\)-central point \(q\) guaranteed by Theorem 1.11 is \(q = (1, 1, 1)\). Indeed, any half-space containing \(q\) contains at least 2 points of \(S\). The point \(q\) does not belong to the interior of \(P = \text{conv} S\): it lies on a face \(\Gamma \subset P\) given by

\[
\Gamma = \text{conv} \{(0, 0, 0), (2, 2, 2), (2, 0, 1), (0, 2, 1)\}.
\]

Note that the minimal lattice \(\Lambda\) containing the 4 points above is smaller than the intersection of \(\mathbb{Z}^3\) with the affine hull of \(\Gamma\) and the central point \(q\) does not belong to it. So our argument so far breaks down on this example and indeed this example provides a lower bound on the Erdős–Ginzburg–Ziv constant. Moreover, note that the polytope \(P\) is in fact a hollow polytope. This is not a coincidence and it turns out that hollow polytopes play a similar role for our variant of Helly’s Theorem as the role of a simplex \(\Delta\) for Theorem 1.10 and the role of the boolean cube is not a coincidence and it turns out that hollow polytopes play a similar role for our variant of Ginzburg–Ziv constant. Moreover, note that the polytope \(P\) far breaks down on this example and indeed this example provides a lower bound on the Erdős–Γ support. Then there exists a face \(\Gamma \subset P\) contains at least 2 points of \(S\). The point \(q\) does not belong to the interior of \(P = \text{conv} S\): it lies on a face \(\Gamma \subset P\) given by

\[
\Gamma = \text{conv} \{(0, 0, 0), (2, 2, 2), (2, 0, 1), (0, 2, 1)\}.
\]

By adapting the proof of Theorem 1.11 from [4] we can prove a variant of Helly’s Theorem which achieves (9):

**Theorem 1.12.** Let \(P \subset \mathbb{Q}^d\) be a polytope and let \(w : P \to \mathbb{R}_{>0}\) be a function with finite support. Then there exists a face \(\Gamma \subset P\) and a point \(q\) in the interior of \(\Gamma\) such that \(q\) is \(\frac{1}{L(d)}\)-central for \(w\) and \(q\) belongs to the lattice spanned by the support of \(w|\Gamma\).

To obtain (9) we apply this theorem to \(P = \text{conv} (\text{supp} w)\). The constant \(L(d)\) in Theorem 1.12 is tight: similarly to the previous examples, let \(P\) be a hollow polytope with \(L(d)\) vertices and define \(\mathcal{F}\) to be the family of sets \(\text{conv} (S \setminus \{x\})\) where \(S\) is the set of vertices of \(P\) and \(x\) ranges over \(X\).

We prove Theorem 1.12 in Section 3.3. Now we finally have all tools to prove Theorem 1.7. We start with an arbitrary set \(X \subset [-K, K]^d\) of size at least \((1 + \varepsilon)L(d)p\) and let \(w\) be its characteristic function. After pruning \(X\) a bit to remove all elements with very small multiplicity, we let \(P = \text{conv} (X)\) and apply Theorem 1.12 to \(P\) and \(w\). We obtain some face \(\Gamma\) and a point \(q\) in the interior of \(\Gamma\) which is \(\theta\)-central for \(w\), with \(\theta = \frac{1}{L(d)}\), and lies in the lattice spanned by \(X \cap \Gamma\). Using Proposition 1.8 and the lattice condition we conclude that, for large enough \(p\), there are non-zero coefficients \(b_x = \frac{a_x}{p}\) with sum 1 such that \(q = \sum_{x \in X} b_x x\) and for each \(x \in X\) we have \(a_x \in \mathbb{N}\) and

\[
a_x = pb_x \leq p(1 + \varepsilon)\theta^{-1} \frac{w(x)}{|X|} = \frac{(1 + \varepsilon)L(d)p}{|X|} w(x) \leq w(x),
\]

so we obtain the desired \(p\) elements of \(X\) which sum to zero modulo \(p\).
2 Proofs of Proposition 1.3 and Proposition 1.5

2.1 Proof of Proposition 1.3

We argue indirectly. Assume that there are vectors \( v_1, \ldots, v_n \in \mathbb{F}_p^d \), with \( n \geq \binom{2d-1}{d} + 2 \) such that for any non-negative integers \( \alpha_1, \ldots, \alpha_n \) whose sum is \( p \), we have \( \sum \alpha_i v_i = 0 \) if and only if \( \alpha_i = p \) for some \( i \). Note that this condition implies that vectors \( v_i \) are pairwise distinct. Let \( S = \{ v_1, \ldots, v_n \} \).

Claim 2.1. There is a nonzero function \( h : \{1, \ldots, n\} \to \mathbb{F}_p \) such that \( h(n) = 0 \) and for any polynomial \( f \in \mathbb{F}_p[x_1, \ldots, x_d] \) of degree at most \( d - 1 \) we have

\[
\sum_{i=1}^{n} h(i) f(v_i) = 0.
\]

Proof. Recall that the dimension of the linear space of polynomials with \( \mathbb{F}_p \)-coefficients of degree at most \( d - 1 \) is equal to \( \binom{2d-1}{d} \). So the desired function \( h \) is a solution of a system consisting of \( \binom{2d-1}{d} \) + 1 linear equations in \( n \geq \binom{2d-1}{d} + 2 \) variables.

For \( i = 1, \ldots, p \) and \( j = 1, \ldots, d \), let \( y_{i,j} \) be a set of variables. Let \( y_i \) be the \( d \)-dimensional vector \( (y_{i,1}, \ldots, y_{i,d})^T \). Consider the following polynomial in \( p \times d \) variables:

\[
F(y_1, \ldots, y_p) = \prod_{j=1}^{d} \left( 1 - \left( \sum_{i=1}^{p} y_{i,j} \right)^{p-1} \right). \tag{10}
\]

Note that if we substitute in \( P \) some vectors \( y_i \in \mathbb{F}_p^d \) then \( F(y_1, \ldots, y_p) = 1 \) if \( y_1 + \ldots + y_p = 0 \) and it equals 0 otherwise. So if we consider a sequence \( v_1, \ldots, v_p \) of \( p \) elements of \( S \) then \( F(v_1, \ldots, v_p) = 1 \) if \( i_1 = \ldots = i_p \) and \( F(v_1, \ldots, v_p) = 0 \) otherwise.

Now we define a function \( \Phi : \{1, \ldots, n\} \to \mathbb{F}_p \) by:

\[
\Phi(t) = \sum_{i_1, \ldots, i_{p-1} \in [n]} h(i_1) \ldots h(i_{p-1}) F(v_{i_1}, \ldots, v_{i_{p-1}}, v_t). \tag{11}
\]

Let us compute \( \Phi(t) \) in two different ways and arrive at a contradiction. On the one hand, \( F(v_1, \ldots, v_{i_{p-1}}, v_t) \) is zero unless \( v_{i_1} = \ldots = v_{i_{p-1}} = v_t \) so

\[
\Phi(t) \equiv h(t)^{p-1} \pmod{p}. \tag{12}
\]

On the other hand, \( F(y_1, \ldots, y_p) \) is a polynomial in variables \( y_{i,j} \) of degree \( d(p-1) \) and so it can be expressed as a linear combination of monomials of the form \( m_1(y_1)m_2(y_2) \ldots m_p(y_p) \) where \( m_i \in \mathbb{Z}[x_1, \ldots, x_d] \) and \( \sum_{i=1}^{p} \deg m_i \leq (p - 1)d \). Restricting the sum \( \Phi \) on a fixed monomial we obtain:

\[
\sum_{i_1, \ldots, i_{p-1} \in [n]} h(i_1) \ldots h(i_{p-1}) m_1(v_{i_1}) m_2(v_{i_1}) \ldots m_{p-1}(v_{i_1}) m_p(v_t) = m_p(v_t) \prod_{j=1}^{p-1} \left( \sum_{i=1}^{n} h(i)m_j(v_i) \right). \tag{13}
\]
So by Claim 2.1, if \( \deg m_j \leq d - 1 \) for some \( j \leq p - 1 \) then the corresponding multiple in (13) must be zero. Otherwise, \( \deg m_j \geq d \) for all \( j \leq p - 1 \). But this implies that \( \deg m_p = 0 \), that is, \( m_p \) is a constant function. Thus, in any case the expression (13) does not depend on \( t \). However, by the construction of \( h \) and (12) we have \( \Phi(n) \equiv 0 \pmod{p} \) and \( \Phi(t) \) is not zero for some \( t \in \{1, \ldots, n\} \) because \( h \) is a non-zero function by Claim 2.1. This contradiction completes the proof.

### 2.2 Proof of Proposition 1.5

We begin with a different characterization of integer points of polytopes. For a polytope \( P \subseteq \mathbb{Q}^d \) we denote by \( \Lambda(P) \) the minimal by inclusion lattice containing the vertices of \( P \). We say that a prime \( p \) if good for \( P \) is for any face \( \Gamma \subseteq P \) the quotient group \( \Lambda(P)/\Lambda(\Gamma) \) has no elements of order \( p \).

**Claim 2.2.** Let \( P \subseteq \mathbb{Q}^d \) be a polytope and let \( q \in P \) be a point. Let \( q_1, \ldots, q_s \) be the vertices of \( P \). The following assertions are equivalent:

(i) The point \( q \) is an integer point of \( P \).

(ii) There exists a constant \( n_0(P) \) such that for all numbers \( n > n_0(P) \) there are nonnegative integer coefficients \( \alpha_1, \ldots, \alpha_s \) such that:

\[
\sum_{i=1}^{s} \alpha_i q_i = nq, \quad \sum_{i=1}^{s} \alpha_i = n. \tag{14}
\]

(iii) The point \( q \) belongs to the minimal lattice containing points \( q_1, \ldots, q_s \) and Condition 2 holds for some \( n = p \), where \( p \) is a good prime for \( P \).

**Proof.** If \( q \) is a vertex of \( P \) then there is nothing to prove so for the rest of the proof we may assume that \( q \) is not a vertex of \( P \).

(i)\(\Rightarrow\)(ii). By replacing \( P \) with the minimal face containing the point \( q \) we reduce to the case when \( q \) is an interior point of \( P \). Then implies that there exists a convex combination

\[
(q, 1) = \sum_{i=1}^{s} \beta_i(q_i, 1),
\]

where all coefficients \( \beta_i \) are positive rational numbers. Let \( m_0 \) be the least common multiple of the denominators of \( \beta_i \). Since there are only a bounded number of points in \( \Lambda(P) \cap P \), \( m_0 \) is bounded by a constant \( m_0(P) \). Then we can write \( \beta_i = b_i/m_0 \) for some positive integers \( b_i \).

Since \( q \) belongs to \( \Lambda(P) \), there is an integer affine combination

\[
\sum_{i=1}^{s} c_i(q_i, 1) = (q, 1), \tag{15}
\]
where \( c_i \in \mathbb{Z} \) are integer coefficients. Let \( K = \max |c_i| \), again we have \( K \leq K_0(P) \) for some constant \( K_0(P) \). We claim that one can now take \( n_0(P) = 2K_0(P)m_0(P)^2 \). Indeed, consider an arbitrary number \( n > 2Km_0^2 \). Write \( n = m_0k + r \) for some \( 0 \leq r < m_0 \) and define the coefficients by \( \alpha_i = kb_i + rc_i \). Then we have

\[
\sum_{i=1}^s \alpha_i(q_i, 1) = k \sum_{i=1}^s b_i(q_i, 1) + r \sum_{i=1}^s c_i(q_i, 1) = (km_0 + r)(q, 1) = n(q, 1), \tag{16}
\]

and for any \( i \) we have \( \alpha_i = kb_i + rc_i \geq k - rK \geq [n/m_0] - Km_0 > 0 \) by the choice of \( n \). Thus, the coefficients \( \alpha_i \) satisfy (ii).

(ii) \( \Rightarrow \) (iii). All primes \( p \geq 2 \) except for a finite collection are good for \( P \), so we can take \( p \) to be any good prime larger than \( n_0(P) \) and apply (ii).

(iii) \( \Rightarrow \) (ii). Let \( \Gamma \) be the minimal face of \( P \) containing \( q \). By a shift of coordinates we may assume that the origin 0 lies in \( \Lambda(\Gamma) \) so that it becomes a linear lattice, not just an affine one. By our assumption, \( q \in \Lambda(P) \) and there is a good for \( P \) prime \( p \) and nonnegative integer coefficients \( \alpha_1, \ldots, \alpha_s \) such that

\[
\sum_{i=1}^s \alpha_i(q_i, 1) = p(q, 1).
\]

Since \( q \in \Gamma \), we have \( \alpha_i = 0 \) for all \( i \) such that \( q_i \notin \Gamma \). Let \( \Lambda \) be the intersection of \( \Lambda(P) \) with the affine hull of \( \Gamma \). Then the quotient \( G = \Lambda/\Lambda(\Gamma) \) is a finite group and the definition of a good prime implies that \( p \) is coprime to \( |G| \). Let \( b > 1 \) be an integer such that \( pb = 1 \ (\text{mod } |G|) \), then we get

\[
(q, 1) = pb(q, 1) - (pb - 1)(q, 1) = \sum_{i=1}^s \alpha_i b(q_i, 1) - \frac{pb-1}{|G|}(|G|q, 1). \tag{17}
\]

By the definition of \( \Lambda(\Gamma) \), the points \( q_i \) with \( \alpha_i > 0 \) belong to \( \Lambda(\Gamma) \). By Lagrange’s theorem the point \( |G|q \) belongs to \( \Lambda(\Gamma) \) so, by (17), the point \( q \) lies in \( \Lambda(\Gamma) \) as well. This completes the last implication and the claim is proved. \( \square \)

Now we are ready to prove Proposition 1.5. Let \( P \subset \mathbb{Q}^d \) be a hollow polytope with \( L(d) \) vertices. After rescaling \( P \) we may assume that \( P \subset \mathbb{Z}^d \) and \( \mathbb{Z}^d \) is the minimal lattice containing the vertices of \( P \). Denote the vertices of \( P \) by \( q_1, \ldots, q_s \). Let \( p \) be a good prime for \( P \), then we can view the vertices of \( P \) as a subset in \( \mathbb{F}_p^d \). If \( P \) modulo \( p \) has a zero-sum \( \sum \alpha_i q_i \equiv 0 \ (\text{mod } p) \) for some non-negative integers \( \alpha_i \) whose sum is \( p \) then the point \( q = \frac{1}{p} \sum \alpha_i q_i \) belongs to \( P \cap \mathbb{Z}^d \). Then by Claim 2.2, \( q \) is an integer point of \( P \). Since \( P \) is hollow we must have \( q = q_i \) for some \( i \) which implies that \( \alpha_i = p \). We conclude that \( \mathbf{w}(\mathbb{F}_p^d) \geq L(d) \) for all primes \( p \) which are good for \( P \), in particular this is true for sufficiently large primes \( p \).
3 Convex flags and Helly’s theorem

3.1 Basic notions

In this Section we are going to define a certain generalizations of polytopes which we call convex flags. Convex flags will be a convenient way to describe the combinatorial structures appearing during the proof of Theorem 1.2. We start with explaining how a polytope P can be viewed as a convex flag and after that we will give a general definition.

Recall that a polytope P in \( \mathbb{Q}^d \) is a convex hull of a finite, non-empty set of points of \( \mathbb{Q}^d \). Note that the dimension of P may be less than d. For a polytope P in \( \mathbb{Q}^d \) let \( \mathcal{P}(P) \) be the set of all faces of P (including P itself but excluding the “empty” face) with the partial order induced by inclusion.

Note that for any set of faces \( S \subseteq \mathcal{P}(P) \) there is a unique minimal face \( \Gamma \in \mathcal{P}(P) \) which contains all faces from \( S \). Based on this observation, we call an arbitrary (finite) poset \( \mathcal{P} \) convex if every subset \( S \subseteq \mathcal{P} \) has a supremum \( \sup S \). That is, the set of all \( x \in \mathcal{P} \) such that \( y \preceq x \) for any \( y \in S \) has the minimum element \( \Gamma \).

Let \( P_1, P_2 \) be arbitrary polytopes in some \( \mathbb{Q} \)-spaces \( A_1 \) and \( A_2 \). An affine map \( \psi : A_1 \to A_2 \) is called a map of polytopes \( P_1 \) and \( P_2 \) if \( \psi(P_1) \subseteq P_2 \). Clearly, a composition of maps of polytopes is again a map of polytopes. Note that \( \psi \) is not assumed to be neither injective nor surjective.

Let \( P_1 \) be a face of \( P_2 \) then the corresponding inclusion map \( \psi_{P_2,P_1} \) is a map of polytopes \( P_1 \) and \( P_2 \). So we can equip the set \( \mathcal{P}(P) \) of faces of a polytope \( P \) with the following structure: for any pair \( x \preceq y \in \mathcal{P}(P) \) we consider the corresponding inclusion map \( \psi_{y,x} \). We thus encoded the structure of the original polytope \( P \) in terms of its faces and inclusion maps between them.

If we now allow maps \( \psi_{y,x} \) to not be necessarily injective and replace \( \mathcal{P}(P) \) by an arbitrary convex poset \( \mathcal{P} \) then we arrive at the notion of a convex flag.

**Definition 3.1** (Convex flag). Let \( (\mathcal{P}, \preceq) \) be a convex partially ordered set. Suppose that for any \( x \in \mathcal{P} \) there is a polytope \( P_x \subseteq A_x \) embedded in a \( \mathbb{Q} \)-space \( A_x \) and for any \( y \preceq x \) there is a map \( \psi_{x,y} : A_y \to A_x \) of polytopes \( P_x \) and \( P_y \) with the property that for any chain \( z \preceq y \preceq x \) we have \( \psi_{x,z} = \psi_{x,y} \psi_{y,z} \). In particular, \( \psi_{x,x} \) is the identity map of \( A_x \).

When we say that \( \mathcal{P} \) is a convex flag, we mean that \( \mathcal{P} \) is a convex poset and we fixed corresponding polytopes \( P_x \subseteq A_x \) and maps \( \psi_{x,y} \). As mentioned above, any polytope \( P \) gives rise of a convex flag, which we are going to denote by \( \mathcal{P}(P) \). Let us provide some other examples of convex flags.

**Example 3.2** (Binary tree, Figure 1). Let \( \mathcal{P} \) be the set of strings \( a_1a_2\ldots a_i \) consisting of 0-s and 1-s and of length \( i \leq d \) (including the empty string). For strings \( s_1, s_2 \) we have \( s_1 \preceq s_2 \) if \( s_1 \) is an initial segment of \( s_2 \). Note that \( |\mathcal{P}| = 2^{d+1} - 1 \).

For any \( s \in \mathcal{P} \) let \( A_s = \mathbb{Q} \) and \( P_s = [0,1] \). Let \( s \in \mathcal{P} \) and \( s' = s a \) be a successor of \( s \), where \( a \in \{0,1\} \). We define the map \( \psi_{s,s a} : [0,1] \to [0,1] \) to be the projection on the point \( a \in [0,1] \).

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1This terminology is not standard. In literature, posets which have this property are called usually *upper semilattices* but we prefer to use a simpler and more intuitive term instead.
Example 3.3 (Sunflower, Figure 2). Let $\mathcal{P} = \{a, b_1, \ldots, b_n, c_1, \ldots, c_n\}$. Here $a$ is the maximum element of $\mathcal{P}$ while elements $b_i$ and $c_i$ are ordered as follows: we have $c_i < b_i$ and $c_i < b_{i+1}$ (with indexes taken modulo $n$). Let $P_a \subset \mathbb{R}^2$ be an arbitrary $n$-gon and let $E_1, \ldots, E_n$ be the edges of $P_a$ labeled in a cyclic order. Let $v_{i-1}, v_i$ be the vertices of the edge $E_i$.

Let $P_{b_i} \subset \mathbb{R}^2$ be an arbitrary polygon which has a pair of parallel edges $F_{i0}, F_{i1} \subset P_{b_i}$. For every $i = 1, \ldots, n$, let $P_{c_i} = [0, 1]$. Now we define maps between polygons $P_a, P_{b_i}, P_{c_i}$. The map $\psi_{a,b_i} : P_{b_i} \to P_c$ is a projection of $P_{b_i}$ along its edges $F_{i0}$ and $F_{i1}$ onto the edge $E_i$. In particular, we have $\psi_{a,b_i}(F_{i0}) = v_{i-1}$ and $\psi_{a,b_i}(F_{i1}) = v_i$. Now let $\psi_{b_i,c_i} : P_{c_i} \to P_{b_i}$ be an arbitrary affine map such that $\psi_{b_i,c_i}(P_{c_i}) \subset F_{i1}$. Similarly, let $\psi_{b_i,c_{i-1}} : P_{c_{i-1}} \to P_{b_i}$ be an arbitrary affine map such that $\psi_{b_i,c_{i-1}}(P_{c_{i-1}}) \subset F_{i0}$.

The map $\psi_{a,c_i} : P_{c_i} \to P_a$ is now defined uniquely: we let $\psi_{a,c_i}(x) = v_i$ for every $x \in P_{c_i}$. This definition implies that we have $\psi_{x,z} = \psi_{x,y}\psi_{y,z}$ for all $x, y, z \in \mathcal{P}$ since the only triples $x, y, z$ for which this equality does not follow automatically are $(x, y, z) = (a, b_i, c_i)$ or $(a, b_i, c_{i-1})$. Therefore, we defined a convex flag structure on $\mathcal{P}$.

The name “sunflower” comes from the following interpretation of $\mathcal{P}$: $P_a$ is the “core” of the sunflower $\mathcal{P}$ and $P_{b_i}$-s are the “petals” which are glued together along edges $P_{c_i}$ and attached to $P_a$ at edges $E_i$.

We may also allow $F_{i0}$ or $F_{i1}$ to degenerate into single vertex and the resulting structure on $\mathcal{P}$ will also form a convex flag.

Now we translate the usual definitions of points and linear functions to this new setting.

Definition 3.4 (Linear functions). A linear functional $\xi$ on a convex flag $\mathcal{P}$ is a linear function $\xi_x : A_x \to \mathbb{R}$ for some $x \in \mathcal{P}$. The domain $\mathcal{D}_x$ of $\xi$ is the set $\mathcal{P}_x := \{y \in \mathcal{P} \mid y \leq x\}$. For any
Figure 2: Sunflower for $n = 4$
point \( q \in \mathcal{A}_y \), where \( y \in \mathcal{D}_\xi \) we define \( \xi_y(q) := \xi_x \psi_{x,y}(q) \).

For \( x \in \mathcal{P} \) we denote \( \mathcal{P}^x := \{ y \in \mathcal{P} \mid x \preceq y \} \). Note that since \( \mathcal{P} \) is a convex poset, for any \( x_1, \ldots, x_n \in \mathcal{P} \) the set \( \mathcal{P}^{x_1} \cap \ldots \cap \mathcal{P}^{x_n} \) also has the form \( \mathcal{P}^x \) for some \( x \in \mathcal{P} \). Namely, we take \( x = \sup\{x_1, \ldots, x_n\} \). In particular, this intersection is non-empty.

**Definition 3.5 (Points).** A point \( q \) of a convex flag \( \mathcal{P} \) is a point \( q_x \in \mathcal{P}_x \) for some \( x \in \mathcal{P} \) together with its images \( q_y = \psi_{y,x} q_x \) for all \( y \in \mathcal{P}^x \). We denote by \( \mathcal{D}^q := \mathcal{P}^x \) the domain of \( q \). The expression \( \inf \mathcal{D}^q := x \) denotes the minimum element \( x \) of \( \mathcal{D}^q \).

If for a linear function \( \xi \) and a point \( q \) the sets \( \mathcal{D}^\xi \) and \( \mathcal{D}^q \) intersect then we can define the value \( \xi(q) \) to be equal to \( \xi_x(q_x) \) for some \( x \in \mathcal{D}^\xi \cap \mathcal{D}^q \). It follows from our definitions that this number does not actually depend on \( x \).

For a set of points \( q_1, \ldots, q_n \) of a convex flag \( \mathcal{P} \) and non-negative coefficients \( \alpha_1, \ldots, \alpha_n \) with sum 1 we define the convex combination

\[
q = \sum_{i: \alpha_i > 0} \alpha_i q_i.
\] (18)

For a set of points \( S \) of a convex flag \( \mathcal{P} \) we define the convex hull \( \text{conv} S \) to be the set of all points \( q \) which can be expressed as a convex combination of points from a set \( S \).

Now we introduce the notion of lattices in convex flags.

**Definition 3.6 (Lattice).** A lattice \( \Lambda \) in a convex flag \( \mathcal{P} \) is a collection of lattices \( \Lambda_x \subset \mathcal{A}_x \) for all \( x \in \mathcal{P} \) such that for any \( x \preceq y \) we have \( \psi_{y,x} \Lambda_x \subset \Lambda_y \).

In what follows, we will usually work with a fixed convex flag \( \mathcal{P} \) and a lattice \( \Lambda \) on \( \mathcal{P} \). For shorthand, we will refer to a pair convex flag \( \mathcal{P} \) and a lattice \( \Lambda \) in \( \mathcal{P} \) as “convex flag \( (\mathcal{P}, \Lambda) \)”.

A point \( q \) belongs to the lattice \( \Lambda \) if \( q_x \in \Lambda_x \) for any \( x \in \mathcal{D}^q \). Equivalently, \( q \) belongs to \( \Lambda \) if \( q_x \in \Lambda_x \), where \( x = \inf \mathcal{D}^q \). We denote the fact that \( q \) belongs to \( \Lambda \) by the expression \( q \in \Lambda \) and we will call \( q \) an integer point of the convex flag \( (\mathcal{P}, \Lambda) \).

### 3.2 Helly’s theorem

Fix a convex flag \( (\mathcal{P}, \Lambda) \) with a lattice \( \Lambda \). Let \( \Omega \) be a set of points of the convex flag \( \mathcal{P} \) which is closed under convex combinations (i.e. \( \Omega = \text{conv} \Omega \)). Points \( q \in \Omega \) will be called proper points of the convex flag \( (\mathcal{P}, \Lambda) \). Until the end of this section we suppose that a set \( \Omega \) of proper points on \( (\mathcal{P}, \Lambda) \) is fixed but we often omit it from the notation.

**Definition 3.7 (Helly constant).** For a convex flag \( (\mathcal{P}, \Lambda) \) with a set of proper points \( \Omega \), define the Helly constant \( L(\mathcal{P}, \Lambda, \Omega) \) as the maximum size \( L \) of a collection of proper integer points \( q_1, \ldots, q_L \) with the following property. Consider a convex combination

\[
q = \sum_{i=1}^{L} \alpha_i q_i.
\]
and suppose that \( \mathbf{q} \in \Lambda \). Then \( \alpha_i = 1 \) for some \( i \).

We should point out that the last condition is not equivalent to saying that \( \mathbf{q} = \mathbf{q}_i \) for some \( i \). For brevity, we will usually omit \( \Omega \) from the notation and write \( L(\mathcal{P}, \Lambda) \) instead of \( L(\mathcal{P}, \Lambda, \Omega) \).

**Example 3.8.** Let \( \mathcal{P} = \{x\} \) be a one-element poset, let \( P_x \subset \mathbb{Q}^d \) be a polytope, \( \Lambda = \mathbb{Z}^d \) and \( \Omega \) is the set of all points of \( P_x \). Then we have \( L(\mathcal{P}, \Lambda) \leq 2^d \). Indeed, if we have some points \( q_1, \ldots, q_{2^d+1} \in P_x \cap \Lambda \) then by pigeonhole principle there are indices \( i \neq j \) such that \( q_i = q_j \) (mod 2) and so \( q = \frac{1}{2}q_i + \frac{1}{2}q_j \) belongs to \( \Lambda \) violating the condition in Definition 3.7. If \( P_x \) contains the boolean cube \( \{0,1\}^d \) then the Helly constant of \( (\mathcal{P}, \Lambda) \) equals to \( 2^d \).

**Example 3.9.** Let \( P \subset \mathbb{Q}^d \) be a polytope and consider the corresponding convex flag \( \mathcal{P} = \mathcal{P}(P) \). Let \( \Omega \) be the set of points \( \mathbf{q} \) of \( \mathcal{P} \) such that \( \text{inf} \mathcal{D}^\mathbf{q} \) is the minimum face of \( \mathcal{P} \) which contains \( \mathbf{q} \). So the set of proper points \( \Omega \) is in one-to-one correspondence with the set of points of \( P \). For a face \( \Gamma \) of \( P \) let \( \Lambda_\Gamma \) be the minimal lattice containing the vertices of \( \Gamma \).

Note that the integer points of the convex flag \( (\mathcal{P}, \Lambda) \) are in bijection with integer points of the polytope \( P \), according to Definition 1.4. Then if \( P \) is a hollow polytope then \( L(\mathcal{P}, \Lambda) \) is at most the number of vertices of \( P \) which, in turn, is at most \( L(d) \). In fact, we show later that inequality \( L(\mathcal{P}, \Lambda) \leq L(d) \) holds for any polytope \( P \subset \mathbb{Q}^d \).

For the usual notion of convexity in \( \mathbb{Q}^d \) we have the Hahn–Banach theorem: for any finite set \( S \) and \( \mathbf{q} \notin \text{conv} \mathcal{S} \) there exists a linear function \( \xi \) ‘separating’ \( \mathbf{q} \) from \( S \). However this is no longer the case in the setting of convex flags. With this in mind, we define a second notion of convex hull:

**Definition 3.10 (Weak convex hull).** For a set of points \( S \) of \( (\mathcal{P}, \Lambda) \) we define the *weak convex hull* \( \text{w-conv}(S) \) of \( S \) to be the set of points \( \mathbf{q} \) such that for any linear function \( \xi \) such that \( \xi(\mathbf{q}) \) is defined, there is a point \( \mathbf{s} \in S \) such that

\[
\xi(\mathbf{s}) \geq \xi(\mathbf{q}).
\]

Let \( \mathbf{q}, \mathbf{q}' \) be a pair of points of a convex flag \( (\mathcal{P}, \Lambda) \). We say that \( \mathbf{q} \) is a projection of the point \( \mathbf{q}' \) if \( \mathcal{D}^\mathbf{q} \subset \mathcal{D}^\mathbf{q}' \) and \( \mathbf{q}_x = \mathbf{q}'_x \) for any \( x \in \mathcal{D}^\mathbf{q} \).

**Proposition 3.11.** For an arbitrary set of points \( S \) and \( \mathbf{q} \), we have \( \mathbf{q} \in \text{w-conv}(S) \) if and only if there exists \( \mathbf{q}' \in \text{conv}(S) \) such that \( \mathbf{q} \) is a projection of \( \mathbf{q}' \).

**Proof.** Take \( \mathbf{q} \in \text{w-conv}(S) \) and let \( x = \text{inf} \mathcal{D}^\mathbf{q} \). Let \( X \subset P_x \) be the set of points \( \mathbf{q}'_x \in \mathbb{A}_x \) over all \( \mathbf{q}' \in \text{conv}(S) \) defined over \( x \). Then \( X \) is a convex subset of \( P_x \). Note that if \( \mathbf{q}_x \notin X \) then by the usual Hahn–Banach theorem there is a linear function \( \xi_x \) such that \( \xi_x(\mathbf{q}_x) > \xi_x(\mathbf{s}) \) for any \( s \in X \). Let \( \xi \) be the unique linear function on \( \mathcal{P} \) extending \( \xi_x \) and note that we obtain a contradiction with \( \mathbf{q} \in \text{w-conv}(S) \). We conclude that \( \mathbf{q}_x \in X \). So there is some \( \mathbf{q}' \in \text{conv}(S) \) such that \( \mathbf{q}'_x = \mathbf{q}_x \). In other words, \( \mathbf{q} \) is a projection of \( \mathbf{q}' \).

The inverse implication can be checked directly. Indeed, replacing a point \( \mathbf{q}' \in \text{conv}(S) \) by a projection \( \mathbf{q} \) only decreases the number of conditions one has to satisfy.
A set of points $S$ is in weakly convex position if no point of $S$ belongs to the weak convex hull of other points.

**Example 3.12.** Let $P = [0, 1]$ and consider the convex flag $\mathcal{P} = \mathcal{P}(P)$. Let $0, 1$ be the points of $\mathcal{P}$ such that $D^0 = \{[0, 1], \{0\}\}$, $D^1 = \{[0, 1], \{1\}\}$. Let $0^\prime, 1^\prime$ be points such that $0^\prime_P = 0_P = 0$ and $1^\prime_P = 1_P = 1$ but $D^{0^\prime} = D^{1^\prime} = \{[0, 1]\}$. So points $0^\prime$ and $1^\prime$ are projections of $0$ and $1$ respectively.

Then the set $S = \{0, 1, 0^\prime, 1^\prime\}$ is in convex position but not in weakly convex position. Indeed, the point $0^\prime$ belongs to the weak convex hull of $0$ but cannot be expressed as a convex combination of $0$, $1$ and $1^\prime$. We also have $0^\prime = \frac{1}{2}0 + \frac{1}{2}1$.

The following result explains why we call $L(\mathcal{P}, \Lambda)$ a Helly constant.

**Theorem 3.13** (Helly’s Theorem for Convex Flags). Let $\langle \mathcal{P}, \Lambda \rangle$ be a convex flag with a set of proper points $\Omega$ and denote $L = L(\mathcal{P}, \Lambda, \Omega)$ its Helly constant. Let $\mathcal{F}$ be a collection of sets $F \subset \Omega$ with the property that for any $F_1, \ldots, F_L \in \mathcal{F}$ there exists an integer proper point $q$ such that $q \in \bigcap_{i=1}^L w\text{-conv}(F_i)$. Then there exists an integer proper point $q \in \bigcap_{F \in \mathcal{F}} w\text{-conv}(F)$.

Let us emphasize the fact that we cannot take $q$ to be in the intersection of convex hulls $\text{conv}(F)$ but only weak convex hulls $w\text{-conv}(F)$. On the other hand, we can still guarantee that $q$ is a proper point. Recall that $\Omega$ is only assumed to be closed under convex combinations and it will not be the case in the applications that $\Omega = w\text{-conv}(\Omega)$.

**Proof.** As in the standard proof of the Helly’s Theorem, we proceed by induction on the size of the family $\mathcal{F}$. The base case $|\mathcal{F}| \leq L$ follows from the assumption of the theorem. Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a family of size $n > L$ satisfying the assumption of Theorem 3.13. By induction, for any $i = 1, \ldots, n$ there is an integer proper point $q_i$ such that

$$q_i \in \bigcap_{j=1, j \neq i}^n w\text{-conv}(F_j).$$

Denote $S = \{q_1, \ldots, q_n\}$, we are going to show that for any set of proper integer points $S$ of size $n$ there is a proper integer point $q$ such that

$$q \in \bigcap_{i=1}^n w\text{-conv}(S \setminus \{q_i\}).$$

Note that (19) implies that $q$ belongs to the intersection of weak convex hulls of all sets from $\mathcal{F}$. So the proof of Theorem 3.13 is reduced to showing (19).

Suppose that (19) does not hold for some set $S = \{q_1, \ldots, q_n\}$. The fact that none of the points $q = q_i$ satisfy (19) implies that $S$ is in weakly convex position. Since there are only finitely many integer proper points in $\mathcal{P}$ we may also assume $S$ to be a minimal counterexample to (19), that is, the set $w\text{-conv}(S)$ is minimal by inclusion among all possible counterexamples $S$.

\[\text{The following argument is based on [4, Proof of Proposition 4.2]}

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Let $Q$ be the set of all integer proper points $q$ such that $q = \sum_{i=1}^{n} \alpha_i q_i$ for some coefficients $0 \leq \alpha_i < 1$ and $\sum \alpha_i = 1$. Since $|S| = n > L = L(\mathcal{P}, \Lambda, \Omega)$, there is a convex combination $q = \sum_{i=1}^{n} \alpha_i q_i$ such that $q$ is integral and $\alpha_i < 1$ are non-zero. This means that the set $Q$ is non-empty.

Claim 3.14. If $q \in Q$ is a projection of $q_j$ for some $j$ then $q$ satisfies (19).

Proof. Since $q$ is a projection of $q_j$ it already implies that $q$ belongs to all weak convex hulls (19) expect for maybe $\text{w-conv}(S \setminus \{q_j\})$.

Write $q = \sum_{i=1}^{n} \alpha_i q_i$ for some convex combination $\alpha_i < 1$ and consider a point $q'$ defined as follows:

$$q' = \sum_{i \neq j} \frac{\alpha_i}{1-\alpha_j} q_i;$$

this is a well-defined convex combination since we have $\alpha_j < 1$ and we have $q = \alpha_j q_j + (1-\alpha_j)q'$. This and the fact that $q$ is a projection of $q$ imply that $q$ is a projection of $q'$. But $q' \in \text{conv}(S \setminus \{q_j\})$, so (19) follows. 

By Claim 3.14 we may assume that $Q$ does not contain points which are projections of some of $q_i$-s. Now let $r \in Q$ be a point which belongs to the maximum number of sets $\text{w-conv}(S \setminus \{q_i\})$ and let $I \subset [n]$ denote the set of all such $i$. If $I = [n]$ then $r$ satisfies (19) so for sake of contradiction we may assume that $I \neq [n]$.

Claim 3.15. If for some $j$ we have $r \notin \text{w-conv}(S \setminus \{q_j\})$ then the set $S' = S \setminus \{q_j\} \cup \{r\}$ is in weakly convex position and $\text{w-conv}(S')$ is a proper subset of $\text{w-conv}(S)$.

Proof. Suppose that $S'$ is not weakly convex. Then for some $i \neq j$ we have

$$q_i \in \text{w-conv}(S \cup \{r\} \setminus \{q_j, q_i\}) \subset \text{w-conv}(S \cup \{r\} \setminus \{q_i\}).$$

So by Proposition 3.11 there exists a point $q'_i \in \text{conv}(S \cup \{r\} \setminus \{q_i\})$ such that $q_i$ is a projection of $q'_i$. So there is a convex combination

$$q'_i = \sum_{t \neq i} \alpha_t q_t + \beta r$$

for some non-negative $\alpha_t, \beta$. Note that $\beta > 0$ because the set $S$ is weakly convex. Since $r \in \text{w-conv}(S)$ there is $r' \in \text{conv} S$ such that $r$ is a projection of $r'$. Now consider the point

$$q''_i = \sum_{t \neq i} \alpha_t q_t + \beta r'.$$

Then $q_i$ is a projection of $q''_i$ and the point $q''_i$ lies in $\text{conv}(S)$. Since the set $S$ is weakly convex there exists a linear function $\xi$ such that $\xi(q_i)$ is defined and $\xi(q_i) < \xi(q_i)$ holds for all $t \neq i$ for which this is defined.

Since $\xi$ is defined on $q_i$, it is also defined on $q''_i$ and thus on $r'$ and all $q_t$ such that $\alpha_t \neq 0$. 


If the coefficient of $q_i$ in the expression of $q'_i$ is non-zero then $D^{q''} \subset D^q$. Unless $r'$ is a projection of $q_i$ we have $\xi(r') < \xi(q_i)$. This however implies that

$$\xi(q_i) = \xi(q'_i) = \sum_{t \neq i, \alpha_t > 0} \alpha_t \xi(q_t) + \beta \xi(r') < \xi(q_i),$$

a contradiction. So $r'$ is a projection of $q_i$ and in particular $r$ is a projection of $q_i$. But this contradicts our assumption that no element of $Q$ is a projection of some $q_i$.

Now we show that $w\text{-conv}(S \setminus \{q_j\} \cup \{r\})$ is strictly contained in $w\text{-conv}(S)$. In fact, $q_j \not\in w\text{-conv}(S \setminus \{q_j\} \cup \{r\})$ holds. Indeed, this follows from the argument above applied to $j = i$. \qed

We conclude that $S'$ is a weakly convex set of size $n$ which is strictly smaller than $S$. So by the minimality of $S$ there exists an integer proper point $s$ which belongs to the intersection:

$$s \in w\text{-conv}(S \setminus \{q_j\}) \cap \bigcap_{i \neq j} w\text{-conv}(S \cup \{r\} \setminus \{q_j, q_i\}). \quad (21)$$

On the other hand, if $i \in I$ then $r \in w\text{-conv}(S \setminus \{q_i\})$ and so

$$s \in w\text{-conv}(S \cup \{r\} \setminus \{q_j, q_i\}) \subset w\text{-conv}(S \setminus \{q_i\}).$$

We conclude that the point $s$ belongs to $w\text{-conv}(S \setminus q_i)$ for all $i \in I \cup \{j\}$, contradicting the choice of $r$. That means that our assumption $I \neq [n]$ is false and there exists a point satisfying (19). This completes the proof of Helly’s theorem. \qed

Recall that the usual Helly’s theorem in $\mathbb{R}^d$ implies a centerpoint theorem. The convex flag analogue of this result will play a crucial role in the proof of Theorem 1.2.

**Corollary 3.16** (Centerpoint Theorem). Let $(\mathcal{P}, \Lambda)$ be a convex flag with a set of proper points $\Omega$. Let $\{q_1, \ldots, q_n\}$ be a set of pairwise distinct proper integer points of $\mathcal{P}$ and let $\omega_1, \ldots, \omega_n$ be non-negative weights with $\sum \omega_i = \omega$. Then there exists an integer proper point $q$ of $\mathcal{P}$ such that for any linear function $\xi$ with $D_\xi \cap D_q \neq \emptyset$ we have

$$\sum_{i: \beta(q_i) > \xi(q)} \omega_i \geq \frac{\omega}{L(\mathcal{P}, \Lambda, \Omega)}, \quad (22)$$

where the sum is taken over all $i$ such that $D_\xi \cap D_q \neq \emptyset$ and $\xi(q_i) > \xi(q)$.

**Proof.** For a linear function $\xi$ such that $D_\xi \cap D_q \neq \emptyset$ and a real number $\alpha$ let $S_{\xi, \alpha} \subset \{q_1, \ldots, q_n\}$ be the set of points $q_i$ such that $\xi(q_i) \leq \alpha$ or the value $\xi(q_i)$ is not defined (i.e. $D_\xi \cap D_q = \emptyset$). Let $\mathcal{F}$ be the family of all sets $S_{\xi, \alpha}$ such that

$$\sum_{q_i \in S_{\xi, \alpha}} \omega_i > \omega \frac{L(\mathcal{P}, \Lambda) - 1}{L(\mathcal{P}, \Lambda)}. \quad (23)$$

By the pigeonhole principle, any $L(\mathcal{P}, \Lambda)$ sets from $\mathcal{F}$ share a common element $q_i$ for some $i$ which is in particular an integer proper point of $\mathcal{P}$. So, by Theorem 3.13 there exists an integer
proper point \( q \) which lies in the intersection of weak convex hulls of all sets from \( \mathcal{F} \). Let us check that the conclusion of the Corollary 3.16 holds for this point. Let \( \xi \) be a linear function satisfying \( D_\xi \cap D^a \neq \emptyset \). For any \( \varepsilon > 0 \) let \( \alpha = \xi(q) - \varepsilon \). Then by Definition 3.10 \( q \) does not belong to \( w\text{-conv}(S_{\xi,\alpha}) \) since it is separated from this set by the linear function \( \xi \). So the set \( S_{\xi,\alpha} \) does not belong to the family \( \mathcal{F} \). But this means that (23) does not hold and so

\[
\sum_{i: \, q_i \notin S_{\xi,\alpha}} \omega_i \geq \frac{\omega}{L(\mathcal{P}, \Lambda)},
\]

(24)

For \( \varepsilon \) small enough, (24) coincides with (22) and so we are done. \( \square \)

### 3.3 Application to polytopes: proof of Theorem 1.12

From Theorem 3.16 we can derive a centerpoint theorem for integer points of polytopes mentioned in Section 1.4. For convenience, we restate the result here.

**Theorem 3.17.** Let \( P \subset \mathbb{Q}^d \) be a polytope and let \( w: P \to \mathbb{R}_{\geq 0} \) be a function with finite support. Then there exists a face \( \Gamma \subset P \) and a point \( q \) in the interior of \( \Gamma \) such that \( q \) is \( \frac{1}{L(d)} \)-central for \( w \) and \( q \) belongs to the lattice spanned by the support of \( w|_{\Gamma} \).

**Proof.** Denote by \( S \) the support of \( w \). Let \( \mathcal{P} = \mathcal{P}(P) \) be the convex flag corresponding to the polytope \( P \) and let \( \Lambda \) be a lattice on \( \mathcal{P} \) defined as follows: for a face \( \Gamma \subset P \) we let \( \Lambda_{\Gamma} \subset \Lambda_{\Gamma} \) be the minimal lattice containing the set \( S \cap \Gamma \). Let \( \Omega \) be the set of proper points of \( \mathcal{P} \) as defined in Example 3.9. Recall that the proper points of \( \mathcal{P} \) are in one-to-one correspondence with points of \( P \). In particular, if \( q \) is an integer proper point of \( \mathcal{P} \) and \( q \) is the corresponding point in \( P \) then \( q \) belongs to the minimal lattice \( \Lambda_{\Gamma} \) where \( \Gamma \) is the minimal face containing \( q \).

So by Corollary 3.16, the statement of the theorem follows from the upper bound \( L(\mathcal{P}, \Lambda, \Omega) \leq L(d) \) on the Helly constant of \( \mathcal{P} \). We check this inequality using Definition 3.7. For \( n > L(d) \) let \( q_1, \ldots, q_n \) be integer proper points of \( (\mathcal{P}, \Lambda) \). If \( q_i = q_j \) for some \( i \neq j \) then \( q = \frac{1}{2} q_i + \frac{1}{2} q_j \) is an integer point and we are done. So we may assume that all points \( q_i \) are pairwise distinct.

For each \( i \) let \( q_i \) be the point of \( P \) corresponding to \( q_i \). If the set \( \{q_1, \ldots, q_n\} \) is not in convex position there exists a convex combination of the form

\[
q_i = \sum_{j \neq i} \alpha_j q_j,
\]

for some \( 0 \leq \alpha_j < 1 \). Since \( q_i \) is an integer point this gives the desired convex combination.

Now we may assume that \( q_1, \ldots, q_n \) are in convex position. Since the polytope \( Q = \text{conv} \{q_1, \ldots, q_n\} \) has \( n > L(d) \) vertices, there is an integer point \( q \in Q \) which is not a vertex of \( Q \). Write \( q = \sum \alpha_i q_i \), with \( 0 \leq \alpha_i < 1 \) and \( \sum \alpha_i = 1 \) and let \( q \) be the corresponding proper point of the convex flag \( \mathcal{P} \). Clearly we have \( q = \sum \alpha_i q_i \). We want to show that \( q \) is an integer point of \( \mathcal{P} \).

Let \( \Gamma \) and \( \Gamma' \) be the minimal faces of \( P \) and \( Q \), respectively, containing the point \( q \). In particular, we have \( D^a = D_{\Gamma} \). Then \( \Gamma' \subset \Gamma \) and \( q \) belongs to the minimal lattice \( \Lambda' \) containing the set \( S' = \{q_i \mid q_i \in \Gamma'\} \). The set \( S' \) is contained in the lattice \( \Lambda' \) and so we have \( q_{\Gamma} \in \Lambda' \subset \Lambda_{\Gamma} \).
Therefore, the point \( q \) belongs to the lattice \( \Lambda \) and we conclude that \( L(P, \Lambda, \Omega) \leq L(d) \) as desired.

4 Flag Decomposition

4.1 The statement

In this section we formulate and prove the Flag Decomposition Lemma which is a certain structural result about arbitrary subsets of \( \mathbb{F}_p^d \) and which will play a crucial role in the proof of Theorem 1.2. To state the result we need some additional notation and terminology.

Recall that a convex flag \((P, \Lambda)\) is a collection of data consisting of spaces \( A_x \), convex polytopes \( P_x \subset A_x \), lattices \( \Lambda_x \subset A_x \) and connecting polytope maps \( \psi_{y,x} : A_x \to A_y \) for all \( x \preceq y \).

Let \( V = \mathbb{F}_p^d \) be a vector space over \( \mathbb{F}_p \) for some prime \( p > 2 \). Let \( V^* \) denote the space of linear functions on \( V \), including functions with constant term. For a function \( f : V \to \mathbb{R}_{\geq 0} \) and for a subset \( S \subset V \) we denote \( f(S) := \sum_{v \in S} f(v) \). Recall that for a linear function \( \xi \in V^* \) on \( V \) a \( K \)-slab \( H(\xi, K) \) is the set of points \( v \in V \) such that \( \xi(v) \in [-K, K] \).

**Definition 4.1 (Thinness and thickness).** A function \( f : V \to \mathbb{R}_{\geq 0} \) is called \((K, \varepsilon)\)-thin along a linear function \( \xi \in V^* \) if
\[
f(H(\xi, K)) \geq (1 - \varepsilon) f(V),
\]
and \( f \) is called \((K, \varepsilon)\)-thick along \( \xi \) otherwise.

The next definition relates convex flags with vector spaces over \( \mathbb{F}_p \). For a lattice \( \Lambda \) and a prime \( p \) we denote by \( \Lambda/p\Lambda \) the set of equivalence classes of points of \( \Lambda \) with respect to the relation \( x \sim_p y \) if and only if \( \frac{x + (p-1)y}{p} \in \Lambda \). Note that \( \Lambda/p\Lambda \) has precisely \( p^{\dim \Lambda} \) and can be identified with an affine space over \( \mathbb{F}_p \).

**Definition 4.2 (\( \mathbb{F}_p \)-Representation).** Let \((P, \Lambda)\) be a convex flag and \( V \) be a vector space over \( \mathbb{F}_p \). A representation \( \varphi \) of the flag \((P, \Lambda)\) in \( V \) is a collection of affine subspaces \( V_x \subset V \), for \( x \in P \), and affine surjective maps \( \varphi_x : V_x \to \Lambda_x/p\Lambda_x \) such that for any \( x \preceq y \) we have \( V_x \subset V_y \) and \( \varphi_y = \psi_{y,x} \varphi_x \).

For brevity, we denote the fact that \( \varphi \) is a representation of \((P, \Lambda)\) in \( V \) as: \( \varphi : V \to (P, \Lambda) \). The corresponding affine subspaces and maps will be always denoted by symbols \( V_x \) and \( \varphi_x \), possibly with some superscripts in case we work with multiple representations at once.

An affine basis of a lattice \( \Lambda \subset \mathbb{Q}^d \) is an origin point \( o \in \Lambda \) and a set of linearly independent vectors \( e_1, \ldots, e_l \) such that \( \Lambda = \langle o + \sum \lambda_i e_i \mid \lambda \in \mathbb{Z} \rangle \). Given an affine basis \( E \) of a lattice \( \Lambda \subset \mathbb{Q}^d \) we can define a lifting map \( \gamma = \gamma_E : \Lambda/p\Lambda \to \Lambda \) as follows: any equivalence class \( [v] \in \Lambda/p\Lambda \) contains a unique vector \( v' \) whose coordinates in the basis \( E \) belong to the set \( \{-\frac{p-1}{2}, \ldots, \frac{p-1}{2}\} \). If \( E \) is an affine basis of \( \Lambda \) and \( q \in \Lambda \) then we denote by \( \|q\|_{\infty,E} \) the largest absolute value of coordinates of \( q \) in the basis \( E \). We extend these definitions to the setting of convex flags.
Definition 4.3 (Basis). Let \( \Lambda \) be a lattice on a convex flag \( \mathcal{P} \). A basis \( E \) of the lattice \( \Lambda \) is a collection of affine bases \( E_x \) of \( \Lambda_x \) for \( x \in \mathcal{P} \). Let \( K : \mathcal{P} \to \mathbb{N} \) be a decreasing function, that is, for any \( x < y \) we have \( K(x) \geq K(y) \). We say that \( E \) is \( K \)-bounded if for any \( x \in \mathcal{P} \) and \( q \in P_x \cap \Lambda_x \) we have \( \|q\|_{\infty, E_x} \leq K(x) \).

Definition 4.4 (Flag decomposition). Let \( f : V \to \mathbb{N} \) be a function from an affine space over \( \mathbb{F}_p \) to non-negative integers. A flag decomposition \( \Phi \) of \( f \) is the following collection of data:

- A convex flag \( (\mathcal{P}, \Lambda) \) and a representation \( \varphi : (\mathcal{P}, \Lambda) \to V \),
- A collection of functions \( f_x : V_x \to \mathbb{N} \), \( x \in \mathcal{P} \), such that \( f^\Phi = \sum_{x \in \mathcal{P}} f_x \) is at most \( f \) pointwise.
- A basis \( E \) of \( \Lambda \) with the following property. For \( x \in \mathcal{P} \) let \( \hat{f}_x : \Lambda_x \to \mathbb{N} \) be a function such that for \( q \in \Lambda_x \) with \( \|q\|_{\infty, E_x} \leq \frac{p-1}{2} \) we have
  \[
  \hat{f}_x(q) = \sum_{y \leq x} f_y(\varphi_x^{-1}[q])
  \]
  where \([q] \in \Lambda_x/p\Lambda_x\) denotes the class of \( q \). If \( \|q\|_{\infty, E_x} > \frac{p-1}{2} \) then we put \( \hat{f}_x(q) = 0 \). In this notation, we require that \( P_x \) is the convex hull of all points \( q \in \Lambda_x \) such that \( \hat{f}_x(q) \neq 0 \).

For a function \( K : \mathcal{P} \to \mathbb{N} \) we call a flag decomposition \( K \)-bounded if the corresponding basis \( E \) is \( K \)-bounded.

Roughly speaking, a flag decomposition \( \Phi \) of \( f \) is a way to express an arbitrary function \( f : V \to \mathbb{N} \) as a sum of functions \( f_x \), \( x \in \mathcal{P} \), and an ‘error’ term \( (f - f^\Phi) \) (which we want to be ‘small’). The functions \( f_x \) are equipped with an additional structure: \( f_x \) is supported on a subspace \( V_x \subset V \) and after applying a surjective map \( V_x \to \Lambda_x/p\Lambda_x \) and lifting to the integer lattice \( \Lambda_x \) the support of \( f_x \) (and all \( f_y \), \( y \leq x \)) defines a convex polytope \( P_x \) (which we want to have bounded size, hence the notion of a \( K \)-bounded decomposition).

For \( x \in \mathcal{P} \) let us denote \( f_{\leq x} : V_x \to \mathbb{N} \) the sum \( f_{\leq x} = \sum_{y \leq x} f_y \). In particular, we have \( f^\Phi = f_{\leq \sup \mathcal{P}} \). For an integer point \( q \) of \( \mathcal{P} \) we define \( \hat{f}(q) \) to be equal to \( \hat{f}_x(q_x) \) where \( x = \inf \mathcal{D}^q \). For a subset \( S \subset \Lambda_x \) we denote by \( \hat{f}_x(S) \) the sum \( \sum_{q \in S} \hat{f}_x(q) \).

We say that a flag decomposition \( \Phi \) of \( f \) is minimal if for any \( x \in \mathcal{P} \) the affine space \( V_x \) is spanned by the support of \( f_{\leq x} \) and \( \Lambda_x \) is the minimal lattice containing the support of \( \hat{f}_x \).

In Section 3 we introduced a notion of proper points of a convex flag. Given a flag decomposition, there is a natural way to define a set of proper points.

Definition 4.5 (Proper points). Let \( \Phi \) be a flag decomposition of a function \( f \). Let \( \Omega_0 \) be the set of points \( q \) of \( \mathcal{P} \) such that \( \hat{f}(q) > 0 \) and let \( \Omega = \text{conv}(\Omega_0) \). The set \( \Omega \) is called the set of proper points of the flag decomposition \( \Phi \).

Let \( \mathcal{P} \) be a convex flag with a set of proper points \( \Omega \). Let \( x \in \mathcal{P} \) and \( \Gamma \) be a face of \( P_x \). Define an element \( x_\Gamma \in \mathcal{P} \) as follows:

\[
x_\Gamma := \sup_{q : q_x \in \Gamma} \inf \mathcal{D}^q,
\]  

(25)
where the supremum is taken over all proper points $q$ which are defined over $x$ and $q_x \in \Gamma$. In particular, we have $x_q \leq x$. We say that a face $\Gamma \subset P_x$ is good if $\psi_{x,x_t}(P_{x_t}) \subset \Gamma$. We say that an element $x \in \mathcal{P}$ is reduced if there exists a proper point $q$ such that $\inf \mathcal{D}^q = x$. It is easy to see that $x$ is reduced if and only if $x_{P_x} = x$. We say that a convex flag $(\mathcal{P}, \Omega)$ is reduced if every element $x \in \mathcal{P}$ is reduced.

**Definition 4.6 (Large face).** Let $\Phi$ be a flag decomposition, $\varepsilon > 0$ and $x \in \mathcal{P}$. A face $\Gamma \subset P_x$ is called $\varepsilon$-large if $f_x(\Gamma \cap \Lambda_x) \geq \varepsilon f^\Phi(V)$ and for any proper face $\Gamma' \subset \Gamma$ we have $f_x(\Gamma' \cap \Lambda_x) \leq (1 - \varepsilon)f_x(\Gamma \cap \Lambda_x)$.

An element $x \in \mathcal{P}$ is called $\varepsilon$-large if $f_x(P_x \cap \Lambda_x) \geq \varepsilon f^\Phi(V)$. Note that the fact that $x$ is $\varepsilon$-large does not imply that $P_x$ is $\varepsilon$-large.

For a flag decomposition $\Phi$ and a reduced element $x \in \mathcal{P}$ define the gap function $G(x)$ of $x$ to be the minimum of non-zero values of the function $\hat{f}_x(q)$ over all $q \in \Lambda_x$.

With these definitions we can now state the properties which we would like our flag decomposition to satisfy.

**Definition 4.7 (Complete element).** Let $\Phi$ be a flag decomposition of $f$, $\delta > 0$, $T > 0$ and let $x \in \mathcal{P}$. Then $x$ is called $(T, \delta)$-complete if for any linear function $\xi \in V^*_x$, which is not constant on fibers of $\varphi_x$, the function $f_{\leq x}$ is $(T(x), \delta)$-thick along $\xi$.

**Definition 4.8 (Complete decomposition).** Let $\varepsilon, \delta > 0$, let $\Phi$ be a flag decomposition and $T: \mathcal{P} \to \mathbb{N}$ be a function. A flag decomposition $\Phi$ is called $(T, \varepsilon, \delta)$-complete if $\Phi$ is reduced, minimal and

- any $\varepsilon$-large element $x \in \mathcal{P}$ is $(T(x), \delta)$-complete,
- for any $x \in \mathcal{P}$ any $\varepsilon$-large face $\Gamma \subset P_x$ is good.

Now we are ready to formulate the main result of this section. We say that a function $g: \mathbb{N} \to \mathbb{N}$ is increasing if $g(n) > n$ for all $x \in \mathbb{N}$.

**Theorem 4.9 (Flag Decomposition Lemma).** Let $\varepsilon > 0$ and let $g: \mathbb{N} \to \mathbb{N}$ be an increasing function. Then there are constants $p_0(d, \varepsilon, g)$ and $\delta \gg_{d, \varepsilon} 1$ such that the following holds for all primes $p > p_0(d, \varepsilon, g)$.

Let $V = \mathbb{F}_p^d$ and let $f: V \to \mathbb{N}$ be an arbitrary function. Then there exists a flag decomposition $\Phi$ of $f$ and functions $T, K: \mathcal{P} \to \mathbb{N}$ such that $\Phi$ is $K$-bounded and $(T, \varepsilon, \delta)$-complete. For any $x \in \mathcal{P}$ we have $T(x) \geq g(K(x))$, $K(x) \ll_{g, d, \varepsilon} 1$ and $G(x) \geq \delta^3 K(x)^{-d} f(V)$. We have $f^\Phi(V) \geq (1 - \varepsilon)f(V)$ and $|\mathcal{P}| \ll_{d, \varepsilon} 1$.

In Sections 4.2, 4.3, 4.5 we introduce several operations on flag decompositions and then we apply them in Section 4.4 as a black-box so the content of Sections 4.2, 4.5 is not going to be needed outside this section.

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4.2 Clean-up lemmas

This section contains simple operations on flag decompositions which would allow us to make
them reduced, make the gap function separated from 0 and optimize the lattice Λ. The com-
pleteness properties of decompositions are going to be preserved under these operations.

In what follows, we are going to work with multiple flag decompositions of the same function
at once. To avoid the notational clutter we will use following convention: all objects related
to a flag decomposition will be denoted by the same letters and superscripts will be added
to distinguish these objects between different decompositions. For instance the convex flag
corresponding to a flag decomposition Φ will be denoted by (P′, Λ′) and similarly for other
objects.

For convenience let us recall that a flag decomposition Φ of f consists of the following data:

- A convex poset P, a set Ω of proper points of P, functions T, K : P → N,
- For y ≤ x, we have a map ψx,y : A_y → A_x,
- For x ∈ P we have a space A_x, a polytope P_x ⊂ A_x, a lattice Λ_x ⊂ A_x, an affine basis E_x
  of Λ_x,
- For x ∈ P we have a subspace V_x ⊂ V, a surjective map ϕx : V_x → Λ_x/pΛ_x and a function
  f_x : V_x → N.

For a flag decomposition Φ of f. Let Pr ed be the set of reduced elements of P, that is, the elements
x ∈ P such that if we consider the set Q of all points q supported on x then
\[ \bigcap_{q \in Q} D^q = P^x. \]

Claim 4.10. The poset P red ⊂ P is convex.

Proof. We need to check that for any x, y ∈ P red the set \{x, y\} has a supremum in P red. Let
z = sup(x, y) ∈ P red and define z′ = z_P red that is:
\[ z' = \sup_{q : z \in D^q} \inf D^q, \]
where the supremum is taken over all proper points q which are defined on the element z. Any
point q which is supported on x or y is also supported on z and so we have x_{P red}, y_{P red} ≤ z′. Since
x = x_{P red} and y = y_{P red} this implies that z′ is an upper bound for \{x, y\}. But z′ ≤ z and so we
must have z′ = z and hence z ∈ P red. This shows that P red is a convex poset.

Now we can restrict all the data of the flag decomposition Φ on the convex subposet P red ⊂ P.
We claim that the resulting collection of data Φ red is a flag decomposition of f. The transition
from Φ to Φ red does not affect the numerical parameters of the decomposition but makes it
reduced.

Lemma 4.11 (Reduced decomposition). In the notation above, Φ red is a flag decomposition of
f such that:
• $\Phi^{\text{red}}$ is reduced and $K$-bounded, and $f^{\Phi^{\text{red}}} = f^{\Phi}$,

• If $x \in P^{\text{red}}$ is a $(T, \delta)$-complete element in $\Phi$ then $x$ is $(T, \delta)$-complete in $\Phi^{\text{red}}$,

• Let $x \in P^{\text{red}}$ and $\Gamma \subset P_x$ be a face. Then $x$ or $\Gamma$ is $\varepsilon$-large in $\Phi$ if and only if it is $\varepsilon$-large in $\Phi^{\text{red}}$, if $\Gamma$ is good in $\Phi$ then $\Gamma$ is good in $\Phi^{\text{red}}$,

• For any $x \in P^{\text{red}}$ we have $G^{\text{red}}(x) = G(x)$.

Proof. Note that if $x \notin P^{\text{red}}$ then there are no proper points $q$ of $P$ with $\inf D^q = x$. By the definition of the set of proper points of $\Phi$ we get that $f_x(q) = 0$ for any $q \in \Lambda_x$ and so the function $f_x$ is 0 at every point of $V_x$. This implies that $f^{\Phi^{\text{red}}} = f^{\Phi}$.

Note that all properties of $\Phi$ which depend only on structures associated with a particular element $x \in P$ hold automatically for $\Phi^{\text{red}}$. In particular, $(T, \delta)$-completeness is a property of the function $f_{x,x}$ which coincides on both $\Phi$ and $\Phi^{\text{red}}$. Similarly we can check the properties of being $\varepsilon$-large, $K$-bounded and that $G^{\text{red}}(x) = G(x)$.

It remains to check that if $\Gamma \subset P_x$ is good in $\Phi$ then it is good in $\Phi^{\text{red}}$. Note that $x_\Gamma$ is a reduced element: if a point $q$ is supported on $x_\Gamma$ then since $\psi_{x_\Gamma,P_{x_\Gamma}} \subset \Gamma$, we get $q_{x_\Gamma} \in \Gamma$. Thus, a point $q$ is supported on $x_\Gamma$ if and only if $q_{x_\Gamma} \in \Gamma$. So the supremums in the definitions of elements $x_{P_{x_\Gamma}}$ and $x_\Gamma$ are taken over the same set of points which implies $x_{P_{x_\Gamma}} = x_\Gamma$ and $x_{\Gamma}$ is reduced. So we have $x^{\text{red}}_{x_\Gamma} = x_\Gamma \leq x$ and $\psi_{x_\Gamma,x^{\text{red}}_{x_\Gamma}} P_{x_\Gamma} \subset \Gamma$. We conclude that $\Gamma$ is good. \hfill \square

The next lemma shows that one can always modify the functions $f_x$ a bit to make the gap function $G^x$ separated from 0. The parameters of the flag decomposition do not change significantly after this operation.

Lemma 4.12 (Creating large gap). Let $\Phi$ be a reduced flag decomposition of a function $f$. For any $\alpha > 0$ there exists a convex subposet $P' \subset P$ and a reduced $K$-bounded flag decomposition $\Phi'$ of $f$ and with poset $P'$ with the following properties:

• For any $x \in P'$ the objects $V_x, \Lambda_x, E_x, \varphi_x$ stay the same and we have $f'_x \leq f_x$ pointwise and $P'_x \subset P_x$,

• For any $x \in P'$ we have $G'(x) \geq \alpha(2K(x) + 1)^{-\dim V} |P|^{-1} f^\Phi(V)$, where $G'$ denotes the gap function of $\Phi'$,

• $f^{\Phi'}(V) \geq (1 - \alpha) f^\Phi(V)$,

• If an element $x \in P'$ is $\varepsilon$-large and $(T, \delta)$-complete in $P$ for some $T, \varepsilon, \delta$ then $x$ is $(T, \delta - \frac{\alpha}{\varepsilon})$-complete in $\Phi'$.

• If $x \in P'$ and a face $\Gamma \subset P_x$ is good in $\Phi$ and $\Gamma' = \Gamma \cap P'_x$ is non-empty then $\Gamma'$ is good in $\Phi'$.
Proof. Denote \( f'_x = f_x \) for all \( x \in \mathcal{P} \) and perform the following procedure to the collection of functions \( \{f'_x\} \). Suppose that there exists \( x \in \mathcal{P} \) and a point \( q \in \Lambda_x \) such that

\[
0 < \hat{f}'_x(q) \leq \alpha(2K(d) + 1)^{-d}|\mathcal{P}|^{-1}f^\Phi(V),
\]

where \( \hat{f}'_x(q) \) denotes the sum of \( f'_y(\varphi_x^{-1}[q]) \) over all \( y \leq x \). In this case, for each \( y \leq x \) define a new function \( f''_y \):

\[
f''_y(v) = \begin{cases} f'_y(v), & \varphi_x(v) \neq [q]; \\ 0, & \varphi_x(v) = [q], \end{cases}
\]

where \([q]\) denotes the class of the point \( q \) in \( \Lambda_x/p\Lambda_x \). Replace \( f'_y \) by the function \( f''_y \) and repeat the step until there are no \( x \in \mathcal{P} \) and \( q \in \Lambda_x \) satisfying the inequality above. Note that each pair \((x, q)\) can appear at most once so this process eventually terminates at some collection of functions \( f'_x, x \in \mathcal{P} \). First, observe that the functions \( f'_x \) defined as above satisfy the desired gap condition. It remains to define the corresponding flag decomposition with functions \( f'_x \) and verify the rest of the properties of the lemma.

With functions \( \hat{f}'_x \) already defined for all \( x \in \mathcal{P} \), we let \( P'_x \) to be the convex hull of the support of \( \hat{f}'_x \). Then clearly \( P'_x \subseteq P_x \) and we have \( f'_x \leq f_x \) pointwise. It is clear from definition that for any \( y \leq x \) we have \( \psi_{x,y}P'_y \subseteq P'_x \) so this defines a convex flag \( \mathcal{P}' \) on the same poset \( \mathcal{P} \). By keeping \( \Lambda_x, E_x \) unchanged we get a lattice \( \Lambda' \) on \( \mathcal{P}' \) and that \( (\mathcal{P}', \Lambda') \) is \( K \)-bounded. Using the same functions \( \varphi_x \) and spaces \( V_x \) we thus can define a flag decomposition \( \Phi' \) of \( f \).

Note that each step of the procedure decreases \( f^{\Phi'}(V) \) by at most the right hand side of (26) and since there are at most \( (2K(x) + 1)^{\dim V} \) points in the cube \([-K(x), K(x)]^d\), each element \( x \in \mathcal{P} \) decreases \( f^{\Phi'}(V) \) by at most \( \alpha|\mathcal{P}|^{-1}f^\Phi(V) \) in total over all steps of the procedure. Since there are \(|\mathcal{P}| \) elements in total, we conclude that \( f^{\Phi'}(V) \geq (1 - \alpha)f^\Phi(V) \) after the last step.

Note that the set of proper points \( \Omega' \) of \( \Phi' \) is contained in the set of proper points \( \Omega \) of \( \Phi \). So if \( \Gamma \) is a face of \( P_x \), such that \( \Gamma' = P'_x \cap \Gamma \) is non-empty then the supremum in the definition of \( x'_{\Gamma'} \) is taken over a subset of points from \( \Omega \) supported on \( \Gamma \). So we have \( x'_{\Gamma'} \preceq x_\Gamma \) which implies that

\[ \psi_{x,x'_{\Gamma'}}P'_{x'_{\Gamma'}} \subset \psi_{x,x_{\Gamma}}P_{x_{\Gamma}} \subset \Gamma, \]

and so \( \Gamma' \) is good in \( \Phi' \).

Let \( x \in \mathcal{P} \) be an \( \varepsilon \)-large and \((T, \delta)\)-complete element for some \( T, \varepsilon, \delta \) and let \( \xi \in V_x^* \) be a linear function which is not constant on fibers of \( \varphi_x \). Then we have

\[ f_{x}(V_x \setminus H(\xi, T)) \geq \delta f_{<x}(V_x), \]

then since \( x \) is \( \varepsilon \)-large, i.e. \( f_{x}(V_x) \geq \varepsilon f^\Phi(V) \) we obtain

\[ f'_{x}(V_x \setminus H(\xi, T)) \geq f_{x}(V_x \setminus H(\xi, T)) - \alpha f^\Phi(V) \geq \left( \delta - \frac{\alpha}{\varepsilon} \right) f_{x}(V_x) \]

so we get that \( x \) is \((T, \delta - \alpha/\varepsilon)\)-complete. We checked all the required properties except that \( \Phi' \) is reduced. By Lemma 4.11 we conclude that \( \Phi^{\text{red}} \) satisfies all properties of the lemma and we are done. □
Lastly, we show that provided that \( p \) is large enough, we can always modify a flag decomposition in order to make it minimal. Namely, given a flag decomposition \( \Phi \) of \( f \), for \( x \in \mathcal{P} \) let \( V_x^\text{min} \) be the minimal affine subspace containing the support of the function \( f_{\leq x} \) and let \( \Lambda_x^\text{min} \subset \Lambda_x \) be the minimal lattice containing the support of \( f_x \).

**Lemma 4.13** (Minimal decomposition). Let \( \Phi \) be a \( K \)-bounded flag decomposition of a function \( f \). If \( p \gg_{K,d} 1 \), then there exists a minimal flag decomposition \( \Phi^\text{min} \) of \( f \) on the convex flag \( \mathcal{P} \), with functions \( f_x \), lattice \( \Lambda^\text{min} = (\Lambda^\text{min}) \) and subspaces \( V_x^\text{min} \). Moreover, \( \Phi' \) is \( K' \)-bounded where \( K' : \mathcal{P} \to \mathbb{N} \) satisfies \( K'(x) \leq A_d(K(x)) \) for all \( x \in \mathcal{P} \) and some function \( A_d \) depending only on \( d = \dim V \).

In particular, \( f_{\Phi^\text{min}}(V) = f_{\Phi}(V) \), the good faces of \( \Phi^\text{min} \) and \( \Phi \) are the same and any \((T, \delta)\)-complete element \( x \in \mathcal{P} \) in \( \Phi \) is \((T, \delta)\)-complete in \( \Phi' \).

**Proof.** In order to define a flag decomposition \( \Phi^\text{min} \) we need to construct maps \( \varphi^\text{min}_{x,y} : V_x^\text{min} \to \Lambda_x^\text{min}/p\Lambda_x^\text{min} \) and define an affine basis \( E_x^\text{min} \) of \( \Lambda_x^\text{min} \).

We have a natural map \( \iota_x : \Lambda_x^\text{min}/p\Lambda_x^\text{min} \to \Lambda_x/p\Lambda_x \) which sends an equivalence class \([q]\) in \( \Lambda_x^\text{min}/p\Lambda_x^\text{min} \) to the unique class \( \iota_x([q]) \) in \( \Lambda_x/p\Lambda_x \) which contains it set-theoretically. In the basis \( E_x \), the sublattice \( \Lambda_x^\text{min} \) is defined by a finite collection of points with coordinates bounded by \( K(x) \). So if \( p \) is large enough compared to \( K(x) \) and \( \dim \Lambda_x \leq d \) then the quotient \( \Lambda_x/\Lambda_x^\text{min} \) has no \( p \)-torsion and so the map \( \iota_x \) is an injective affine map over \( \mathbb{F}_p \). Let \( \varphi^\text{min}_x \) be the composition of \( \varphi_x \) with \( \iota_x^{-1} \). It then follows that \( \varphi^\text{min}_{x,y} \) holds for all \( y \leq x \) and \( \varphi^\text{min}_x \) is a surjective map since its image contains the support of the function \( \hat{f}_x \). Since \( \Lambda_x^\text{min} \subset \Lambda_x \) is defined by a collection of points with coordinates at most \( K(x) \), one can find an affine basis \( E_x^\text{min} \) of \( \Lambda_x^\text{min} \) such that the support of \( \hat{f}_x \) is \( A_d(K(x)) \)-bounded with respect to \( E_x^\text{min} \) for some function \( A_d \) depending on \( d \) only. Lastly, we need to check that the functions \( \hat{f}_x \) and \( \hat{f}_x^\text{min} \) coincide with this choice of the basis \( E_x^\text{min} \). Indeed, this follows from the norms induced by \( E_x^\text{min} \) and \( E_x|_{\Lambda_x^\text{min}} \) are equivalent up to a constant depending only on \( K(x) \) and \( d \) and so any point \( q \) satisfying \( \|q\|_{\infty, E_x} \leq K(x) \) automatically satisfies \( \|q\|_{\infty, E_x^\text{min}} \leq \frac{p-1}{2} \). This finishes the proof. \( \square \)

### 4.3 Refinements

A flag decomposition whose existence is guaranteed by Theorem 4.9 has the property that all large faces are good and all large elements are complete. The constructions in this section will allow us to refine a current flag decomposition \( \Phi \) and make it so that a given face becomes good or a given element becomes complete. At the same time, all properties of the decomposition change in a controllable manner. Iterating this process will eventually bring us to the flag decomposition in Theorem 4.9.

For a poset \( \mathcal{P} \) and an element \( x \in \mathcal{P} \) it will be convenient to define an auxiliary poset \( \mathcal{P}[x] \) as follows. As a set, \( \mathcal{P}[x] = \mathcal{P} \times \{1\} \cup \mathcal{P}_x \times \{0\} \), where we denote \( \mathcal{P}_x = \{ y : y \preceq x \} \) and for \((y, \alpha), (y', \alpha') \in \mathcal{P}_x \) we have \((y, \alpha) \preceq (y', \alpha') \) if \( y \preceq y' \) in \( \mathcal{P} \) and \( \alpha \leq \alpha' \). Note that if \( \mathcal{P} \) is a convex poset then \( \mathcal{P}[x] \) is convex as well.

Recall that a face \( \Gamma \subset P_x \) is good if \( \psi_{x,t} P_{x,t} \subset \Gamma \). Then if a face \( \Gamma \) is not good and we want to
fix that then we can try to add a new element $x'_\Gamma$ to the poset $P$ so that it plays a role of $x_\Gamma$ in the new flag decomposition. More precisely we have the following construction.

**Lemma 4.14 (Good faces).** Let $\Phi$ be a $K$-bounded flag decomposition of $f$, let $x \in P$ and $\Gamma \subset P_x$ be a face. Suppose that $p \gg_{K,d} 1$. Then there exists a flag decomposition $\Phi[x]$ on the poset $P[x]$ with the following properties:

- For any $(y, 1) \in P[x]$ the objects $V_{(y,1)}, \Lambda_{(y,1)}, P_{(y,1)}, E_{(y,1)}, \varphi_{(y,1)}$ coincide with the corresponding objects for $y \in P$.
- We have $f^{\Phi[x]}(V) = f^{\Phi}(V)$ and $f_y = f_{(y,1)} + f_{(y,0)}$ for all $y \leq x$.
- $\Phi[x]$ is $K$-bounded, where $K : P[x] \to \mathbb{N}$ satisfies $K((y,1)) = K(y)$ for any $y \in P$ and $K((y,0)) \leq A_d(K(y))$ for all $y \leq x$ and some increasing function $A_d : \mathbb{N} \to \mathbb{N}$ depending only on $d = \dim V$.
- If $y \in P$ is $(T, \delta)$-complete in $\Phi$ then $(y, 1) \in P[x]$ is $(T, \delta)$-complete in $\Phi[x]$. If $\Gamma' \subset P_y$ is good in $\Phi$ for some $y \in P$ then $\Gamma' \subset P_{(y,1)}$ is good in $\Phi[x]$.
- The face $\Gamma \subset P_{(x,1)}$ is good in $\Phi[x]$.

**Proof.** For each $y \leq x$ we let $\Lambda_{(y,0)} \subset \Lambda_y$ be the lattice obtained as intersection of $\Lambda_y$ with the affine hull of the set of points $q \in \Lambda_y$ such that $\psi_{x,y}(q) \in \Gamma$ and $\hat{f}_y(q) > 0$. Since the quotient $\Lambda_y/\Lambda_{(y,0)}$ has no torsion, the natural map $\theta_y : \Lambda_{(y,0)}/p\Lambda_{(y,0)} \to \Lambda_y/p\Lambda_y$ of affine spaces over $\mathbb{F}_p$ is injective. Let $V_{(y,0)} \subset V_y$ be the preimage of $\Lambda_{(y,0)}$ in $V_x$, that is, $V_{(y,0)} = \varphi_y^{-1}\Im \theta_y$. So we get a map $\varphi_{(y,0)} : V_{(y,0)} \to \Lambda_{(y,0)}/p\Lambda_{(y,0)}$ by restriction of the map $\varphi_y$ on the subspace $V_{(y,0)}$. For $y \leq y' \leq x$ it is clear that the map $\psi_{y',y} : \Lambda_{(y,0)}$ maps $\Lambda_{(y,0)}$ into $\Lambda_{(y',0)}$.

Note that the support of the function $\hat{f}_y$ is $K(y)$-bounded in the basis $E_y$ and the sublattice $\Lambda_{(y,0)}$ is defined by a subset of the support of $\hat{f}_y$. So, by a compactness argument, there is some function $A_d$ and an affine basis $E_{(y,0)}$ of $\Lambda_{(y,0)}$ such that the restriction of $\hat{f}_y$ on $\Lambda_{(y,0)}$ is $A_d(K(y))$-bounded in $E_{(y,0)}$. For $y \leq x$ we define $\hat{f}_{(y,0)}$ as the restriction of $f_y$ on the subspace $V_{(y,0)} \subset V_y$ and $f_{(y,1)} = f_y - f_{(y,0)}$. Since $p$ is assumed to be sufficiently large with respect to $K(y)$ and $d$, it follows that the restriction of $\hat{f}_y$ on $\Lambda_{(y,0)}$ coincides with the function $\hat{f}_{(y,0)}$ defined by the collection of functions $\{f_{(y',0)}\}$. Indeed, we use the fact that the $l_\infty$-norms defined by the bases $E_y$ and $E_{(y,0)}$ are equivalent up to a constant depending only on $d$ and $K(y)$. Finally, let $P_{(y,0)}$ be the convex hull of the support of $\hat{f}_{(y,0)}$.

Let $\Lambda_{(y,0)} = \Lambda_{(y,1)} = \Lambda_y$ and let $\psi_{(y,1),(y,0)}$ be the identity map. We now described all data required to define a flag decomposition $\Phi[x]$ of $f$ on the poset $P[x]$ and it remains to verify the claimed properties of this construction. The first three bullet points in Lemma 4.14 follow directly from the construction. Since this construction leaves all structures of points $y \in P$ unchanged and $\hat{f}_{(y,1)} = \hat{f}_y$, the completeness and goodness properties for elements $(y, 1) \in P[x]$ follow automatically, which verifies the fourth point.

Lastly, we check that $\Gamma$ is a good face in $\Phi[x]$. Note that by construction we have $P_{(x,0)} = \Gamma$ and so $\psi_{(x,0),(x,1)}P_{(x,0)} \subset \Gamma$. Thus, it is enough to check that for any proper point $q$ such that
\( \mathbf{q}_{(x,1)} \in \Gamma \) we have \((x, 0) \in D_{\mathbf{q}} \). By definition of the set of proper points of a flag decomposition, any proper point \( \mathbf{q} \) is a convex combination \( \mathbf{q} = \sum_{i=1}^{n} \alpha_i \mathbf{q}_i \) of integer points \( \mathbf{q}_i \) such that \( \hat{f}(\mathbf{q}_i) > 0 \). Let \((y_i, a_i) \in \mathcal{P}[x] \) be the element \( \inf D_{\mathbf{q}i} \) and let \( \mathbf{q}_i = \mathbf{q}_i \in \Lambda_{(y_i, a_i)} \). Then we have
\[
\hat{f}_{(y_i, a_i)}(\mathbf{q}_i) = \hat{f}(\mathbf{q}_i) > 0.
\]

On the other hand, if we have \( \mathbf{q}_{(x,1)} \in \Gamma \) then for every \( i = 1, \ldots, n \) we get \((x, 1) \in D_{\mathbf{q}i} \) and \( \mathbf{q}_{(x,1)} \in \Gamma \). That is, \( \psi_{x,y} \mathbf{q}_i \in \Gamma \) and so \( \mathbf{q}_i \) belongs to the lattice \( \Lambda_{(y,0)} = \psi_{x,y}^{-1} \Theta \). This means that the preimage of the class \( [\mathbf{q}_i] \in \Lambda_{y_i}/p\Lambda_{y_i} \) under the function \( \varphi_{y_i} \) belongs to the space \( V_{(y,0)} = V_{y_i} \cap U \). By definition, the function \( f_{(y,1)} \) has zero support on \( V_{(y,1)} \) and so we get
\[
\hat{f}_{(y,1)}(\mathbf{q}_i) = f_{(y,1)}(\varphi_{y_i}^{-1}[\mathbf{q}_i]) = 0,
\]

which combined with \([27]\) implies that \( a_i = 0 \) and, thus, \((x, 0) \in D_{\mathbf{q}i} \), as desired. We conclude that \( \Gamma \) is a good face in \( \Phi[x] \).

The second operation allows us to make a particular element \( x \in \mathcal{P} \) to be \((T, \delta)\)-complete. Recall that \( x \) is \((T, \delta)\)-complete if the function \( f_{\leq x} \) is \((T, \delta)\)-thick along any linear function \( \xi \) which is not constant on fibers of \( \varphi_x \). The basic idea behind this construction is that if \( x \) is not thick along some \( \xi \) then we can make \( f_{\leq x} \) to be supported on a strip of width \( T \) by removing a few elements from its support and then use this strip to modify the flag decomposition.

**Proposition 4.15** (Complete elements). Let \( \Phi \) be a minimal \( K \)-bounded flag decomposition of \( f \), let \( x \in \mathcal{P} \) and fix an increasing function \( g : \mathbb{N} \to \mathbb{N} \) and \( \delta > 0 \). Suppose that \( p \gg_{d,K,g} 1 \). Then there exists a flag decomposition \( \Phi[x] \) on the poset \( \mathcal{P}[x] \) with the following properties:

- For any \( y \in \mathcal{P} \) the objects \( V_{(y,1)}, \Lambda_{(y,1)}, E_{(y,1)}, \varphi_{(y,1)} \) coincide with the corresponding objects for \( y \in \mathcal{P} \) and we have \( P_{(y,1)} \subset P_y \).
- We have \( f^{\Phi[x]}(V) \geq (1 - 3^{d+1}x) f^\Phi(V) \) and for any \( y \leq x \) we have \( f_{(y,1)} = 0 \).
- \( \Phi[x] \) is \( K \)-bounded, where \( K : \mathcal{P}[x] \to \mathbb{N} \) satisfies \( K(y, 1) = K(y) \) and for \( y \leq x \) we have \( K((y,0)) \leq \max\{g^d(K(x)), K(y)\} \).
- If \( y \in \mathcal{P} \) is \((T, \alpha')\)-complete in \( \Phi \) then \( (y, 1) \in \mathcal{P}[x] \) is \((T, \alpha')\)-complete in \( \Phi[x] \) where
\[
\alpha' \geq \alpha - 3^{d+1}x \frac{f_{\leq x}(V)}{f_{\leq y}(V)}.
\]
- If \( \Gamma \subset P_y \) is good in \( \Phi \) for some \( y \in \mathcal{P} \) and \( \Gamma' = \Gamma \cap P_{(y,1)} \) is non-empty then \( \Gamma' \subset P_{(y,1)} \) is good in \( \Phi[x] \).
- The element \((x, 0)\) is \((g(K(x,0)), \delta)\)-complete in \( \Phi[x] \).

**Proof.** Let \( W \subset V_x^* \) be the space of linear functions \( \xi \) on \( V_x \) which are constant on fibers of \( \varphi_x \). In other words, any \( \xi \in W \) has the form \( \xi(v) = \eta \varphi_x(v) \) for a linear function \( \eta : \Lambda_x/p\Lambda_x \to \mathbb{R}_p \). Note that \( W \) contains the 1-dimensional subspace of constant functions. Recall that \( f_{\leq x} = \sum_{y \leq x} f_y \)

and consider a maximal sequence of linear functions \( \xi_1, \ldots, \xi_k \in V_x^* \) such that:
The function $f_{\leq x}$ is $(g^i(K(x)), 3^i\delta)$-thin along $\xi_i$,

- The dimension of the space $W' = \langle W, \xi_1, \ldots, \xi_k \rangle$ equals $\dim W + k$.

By definition, for any $\xi \not\in W'$, the function $f_{\leq x}$ is $(g^{k+1}(K(x)), 3^{k+1}\delta)$-thick along $\xi$. Let $\Pi \subset V_x$ be the set of vectors $v$ such that $\xi_i(v) \in [-g^i(K(x)), g^i(K(x))]$, for all $i = 1, \ldots, k$. For $y \preceq x$ define $f_{(y,0)}$ to be the restriction of $f_y$ on $\Pi$ and let $f_{(y,1)} = 0$. For $y \not\preceq x$ we let $f_{(y,1)} = f_y$.

For $y \preceq x$ put $A_{(y,0)} = A_y \times \mathbb{Q}^k$ and define a new lattice $\Lambda_{(y,0)} \subset A_y \times \mathbb{Z}^k$ to be the minimal lattice containing vectors of the form:

$$((\tilde{\varphi}_y(v), \tilde{\xi}_1(v), \ldots, \tilde{\xi}_k(v)) \in A_y \times \mathbb{Z}^k,$$

where $v \in V_y$ is such that $f_{\preceq (y,0)}(v) > 0$, $\tilde{\varphi}_y(v) \in A_y$ denotes the unique lifting of $\varphi_y(v)$ such that $\|\tilde{\varphi}_y(v)\|_{\infty, E_y} \leq K(y) < p/2$ and similarly $\tilde{\xi}_i(v) \in [-g^i(K(x)), g^i(K(x))]$ is the lifting of the element $\xi_i(v) \in \mathbb{F}_p$.

Now we can define a natural map $\varphi_{(y,0)} : V_y \to \Lambda_{(y,0)}/p\Lambda_{(y,0)}$ sending $v$ to the vector

$$((\varphi_y(v), \xi_1(v), \ldots, \xi_k(v)) \in A_y/pA_y \times \mathbb{F}_{p}^k$$

which then can be identified with an element of $\Lambda_{(y,0)}/p\Lambda_{(y,0)}$. More precisely, we use the fact that $\Phi$ is minimal, so that the support of $f_{\preceq (y,0)} = f_{\preceq y}$ affinely spans $V_y$ and so $\varphi_{(y,0)}$ can be first defined on the support of $f_{\preceq (y,0)}$ in the obvious way and then extended by linearity on the whole space $V_y$.

Note that the lattice $\Lambda_{(y,0)} \subset A_y \times \mathbb{Z}^k$ is defined by vectors with coordinates bounded by $K(y)$ and $g^k(K(x))$ and so one can find an affine basis $E_{(y,0)}$ in which the support of $\hat{f}_{(y,0)}$ is $A_d(K(y), g^d(K(x)))$-bounded for some function $A_d$ depending on $d$ only. For $y' \preceq y \preceq x$ define a map $\psi_{(y,0),(y',0)} : A_{(y',0)} \to A_{(y,0)}$ as $\psi_y$ on the first coordinate and by identity on the second coordinate, this maps the lattice $\Lambda_{(y',0)}$ into $\Lambda_{(y,0)}$.

For all $y \in P$ we let $A_{(y,1)} = A_y$, $A_{(y,1)} = A_y$ and $E_{(y,1)} = E_y$. Let $\psi_{(y,1),(y,0)} : A_{(y,0)} \to A_y$ to be the projection on the first coordinate. Define the functions $\hat{f}_{(y,0)}, f_{(y,1)}$ and bases $E_{(y,\alpha)}$ defined above and let $P_{(y,0)}$ and $P_{(y,1)}$ be the convex hulls of the supports of these functions.

We claim that we constructed a flag decomposition $\Phi[x]$ of $f$ on the poset $P[x]$. Indeed, the axioms of a convex flag and of a flag decomposition are satisfied by this construction. In remains to verify the properties of $\Phi[x]$ claimed in the statement of the lemma. The first bullet point follow directly from the construction. Since $f_{\preceq x}$ is $(g^i(K(x)), 3^i\delta)$-thin along $\xi_i$, we get

$$f^{\Phi}(V) - f^{\Phi[x]}(V) \preceq f_{\preceq x}(V_x \setminus \Pi) \leq \sum_{i=1}^k f_{\preceq x}(V_x \setminus H(\xi_i, g^i(K(x)))) \leq$$

$$\leq \sum_{i=1}^k 3^i\delta f_{\preceq x}(V_x) \leq \frac{1}{2} 3^{k+1}\delta f_{\preceq x}(V_x),$$

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which implies the second point. For any \( y \preceq x \) the polytope \( P_{(y,0)} \) is contained in the product \( P_y \times [-g^k(K(x)), g^k(K(x))]^k \) and so \( P_{(y,0)} \) is max\{\( K(y), g^k(K(x)) \}\)-bounded with respect to the basis \( E_{(y,0)} \), so the third point is proved.

Let \( y \in \mathcal{P} \) be a \((T, \alpha)\)-complete element of \( \Phi \). Let \( \xi \) be any function not constant on fibers of \( \varphi_{(y,1)} = \varphi_y \) then by definition we have \( f_{\preceq y}(V_y \setminus H(\xi, T)) \geq \alpha f_{\preceq y}(V_y) \) which gives
\[
 f_{\preceq (y,1)}(V_y \setminus H(\xi, T)) \geq \alpha f_{\preceq y}(V_y) - 3^{d+1}\delta f_{\preceq x}(V_x) \geq \alpha' f_{\preceq y}(V_y),
\]
and so \( (y,1) \) is \((T, \alpha')\)-complete in \( \Phi|_x \).

Suppose that \( \Gamma \subset P_y \) is a good face in \( \Phi \) for some \( y \in \mathcal{P} \) and \( \Gamma' = \Gamma \cap P_{(y,1)} \) is non-empty. Any proper point \( q \) of \( \Phi|x \) such that \( q_{(y,1)} \in \Gamma \) corresponds to a proper point \( q' \) of \( \Phi \), indeed, we define \( \mathcal{D}' = \{ z \in \mathcal{P} \mid (z,1) \in \mathcal{D}' \} \) and let \( q'_x = q_{(x,1)} \) for any \( x \in \mathcal{D}' \). So the supremum in the definition of the element \((y,1)_{\Gamma'} \) is taken over a subset of the supremum in the definition of \( y_{\Gamma} \) and so we have
\[
 \psi_{(y,1),(y,1)}(y_{\Gamma(1)} P_{(y,1)} \subset \psi_{(y,1),(y,1)} P_{(y,1)} \subset \Gamma \cap P_{(y,1)},
\]
so \( \Gamma' \) is a good face in \( \Phi|x \), as desired.

Let us check that \((x,0)\) is \((g^{k+1}(K(x)), \delta)\)-complete in \( \Phi|x \). Let \( \xi \in V^*_x = V^*_x \) be a linear function not constant on fibers of \( \varphi_{(x,0)} \). This is equivalent to the condition that \( \xi \notin W' = \langle W, \zeta_1, \ldots, \zeta_k \rangle \), so \( f_{\preceq x} \) is \((g^{k+1}(K(x)), 3^{k+1}\delta)\)-thick along \( \xi \) and so using the bound on \( f_{\Phi}(V) - f_{\Phi|x}(V) \) we get
\[
 f_{\preceq (x,0)}(V_x \setminus H(\xi, g^{k+1}(K(x)))) \geq f_{\preceq x}(V_x \setminus H(\xi, g^{k+1}(K(x)))) - \frac{1}{2} 3^{k+1}\delta f_{\preceq x}(V_x) \geq \frac{1}{2} 3^{k+1}\delta f_{\preceq x}(V_x) \geq \delta f_{\preceq (x,0)}(V_x),
\]
and the proof is complete. \( \square \)

### 4.4 Preliminaries

In this section we collect some additional results needed in the proof of Theorem 4.9. Let \( P \subset \mathbb{Q}^d \) be a polytope, \( \mu \) be a finite measure on \( P \) and \( \varepsilon > 0 \). A face \( \Gamma \subset P \) is called \( \varepsilon\)-large with respect to \( P \) and \( \mu \) if \( \mu(\Gamma) \geq \varepsilon \mu(P) \) and for any proper face \( \Gamma' \subset \Gamma \) we have \( \mu(\Gamma') \leq (1 - \varepsilon)\mu(\Gamma) \).

**Proposition 4.16.** Let \( P_1 \supset P_2 \supset \ldots \supset P_N \) be a sequence of polytopes in \( \mathbb{Q}^d \), let \( \mu \) be a finite measure on \( \mathbb{Q}^d \) and let \( \varepsilon > 0 \). Suppose that \( \mu(P_N) \geq \varepsilon \mu(P_1) \). Suppose that for \( i = 1, \ldots, N \), \( \Gamma_i \subset P_i \) is an \( \varepsilon\)-large face with respect to \( P_i \) and \( \mu \) and that for any \( 1 \leq i < j \leq N \) we have \( \Gamma_i \cap P_j = \Gamma_j \). Then we have \( N \leq (\varepsilon^{-3} + d + 2)^{d+2} \).

**Proof.** For an integer \( t \geq 1 \), let \( N_t \) be the maximum number \( n \) such that there exists a set of \( t \) affinely independent points of \( \mathbb{Q}^d \) which is contained in at least \( N_t \) faces \( \Gamma_i \). Note that
\[
 \sum_{i=1}^{N} \mu(\Gamma_i) \geq \sum_{i=1}^{N} \varepsilon \mu(P_i) \geq \varepsilon^2 N \mu(P_1),
\]
and the proof is complete.
so by the pigeonhole principle, there exists a point \( q \in P_1 \) which belongs to at least \( \lfloor \varepsilon^2 N \rfloor \) faces \( \Gamma_1 \). In particular, we get \( N_1 \geq \lfloor \varepsilon^2 N \rfloor \). On the other hand, since there are no \( d + 2 \) affinely independent points in \( \mathbb{Q}^d \), we trivially have \( N_{d+2} = 0 \). Now let \( 1 \leq t \leq d + 1 \) be arbitrary and consider a \( t \)-element affinely independent set \( S = \{ q_1, \ldots, q_t \} \) which is contained in \( N_t \) faces \( \Gamma_{i_1}, \ldots, \Gamma_{i_N} \) for some indices \( i_1 < \ldots < i_N \). For each \( j = 1, \ldots, N_t \), let \( \Gamma_{ij} \) be the minimal face of \( P_{ij} \) containing \( S \). For any \( j \leq j' \) we have \( \Gamma_{ij} \subseteq \Gamma_{ij'} \). Note that if for some \( j' \) we have \( \Gamma_{ij} = \Gamma_{ij'} \), then for any \( j < j' \) we get \( \Gamma_{ij} \subseteq \Gamma_{ij'} \subseteq \Gamma_{ij} \cap P_{ij} \).

By the assumption, we have \( \Gamma_{ij} \neq \Gamma_{ij'} \cap P_{ij} \), therefore, \( \Gamma_{ij} \) is a proper face in \( \Gamma_{ij} \cap P_{ij} \) and, in particular, we get that \( \dim \Gamma_{ij} < \dim \Gamma_{ij} \). Thus, there are at most \( d + 1 \) indices \( j \in [N_t] \) such that \( \Gamma_{ij} = \Gamma_{ij}' \). Denote by \( J \subseteq [N_t] \) the set of all indices \( j \) such that \( \Gamma_{ij} \neq \Gamma_{ij}' \).

Let \( j \in J \) and note that since \( \Gamma_{ij} \) is \( \varepsilon \)-large, we have

\[
\mu(\Gamma_{ij} \setminus \Gamma_{ij}') \geq \varepsilon \mu(\Gamma_{ij}) \geq \varepsilon^3 \mu(P_{ij}) \geq \varepsilon^3 \mu(P_1).
\]

Thus, \( \sum_{j \in J} \mu(\Gamma_{ij} \setminus \Gamma_{ij}') \geq \varepsilon^3 (N_t - d - 1) \mu(P_1) \) and by pigeonhole principle there exists a point \( q \) belonging to at least \( M = \lfloor \varepsilon^3 (N_t - d - 1) \rfloor \) sets \( \Gamma_{ij} \setminus \Gamma_{ij}' \), \( j \in J \). Suppose that \( M > 0 \) and let \( j \) be such an index. Then \( q \) does not belong to the affine hull \( V \) of the set \( S \). Indeed, otherwise \( q \in V \cap P_{ij} \) and so \( q \) lies in the minimal face of \( P_{ij} \) containing the set \( S \), that is \( q \in \Gamma_{ij}' \), which is a contradiction. We conclude that the set \( S' = S \cup \{ q \} \) is affinely independent and is contained in \( M \) faces \( \Gamma_i \) for \( 1 \leq i \leq N \). So we get \( N_{t+1} \geq M = \lfloor \varepsilon^3 (N_t - d - 1) \rfloor \) which implies that

\[
\varepsilon^{-3}(N_{t+1} + 1) + d + 1 \geq N_t
\]

holds for all \( t = 1, \ldots, d + 1 \). Using \( N_{d+2} = 0 \) and chaining these inequalities together, we get an upper bound \( N_1 \leq (\varepsilon^{-3} + d + 2)^{d+1} \). Combined with the bound \( N \leq \varepsilon^{-2}(N_1 + 1) \) and some simplifications, this leads to the desired estimate on \( N \).

Let \( \Phi \) be a flag decomposition of a function \( f : V = \mathbb{F}_p^d \to \mathbb{N} \). For an element \( x \in \mathcal{P} \) define a function \( l_\Phi(x) \in [d]^2 \), which we call em level of \( x \), to be the pair \( (\text{codim } V_x, \text{dim } \Lambda_x) \). We view \([d]^2\) as a linearly ordered set with respect to the lexicographical order \( \preceq_{\text{lex}} \). Note that for any \( y \preceq x \) we always have \( l_\Phi(y) \succeq_{\text{lex}} l_\Phi(x) \), that is, either \( \text{codim } V_y \geq \text{codim } V_x \) or \( \text{dim } V_y \geq \text{dim } V_x \) and \( \text{dim } \Lambda_y \geq \text{dim } \Lambda_x \). Moreover, we have \( l_\Phi(x) = l_\Phi(y) \) only if \( V_x = V_y \), lattices \( \Lambda_x \) and \( \Lambda_y \) have equal dimensions and \( \psi_{x,y} \) is an injection.

**Observation 4.17.** Let \( \Phi \) be a flag decomposition of \( f \), let \( x \in \mathcal{P} \) and let \( \Gamma \subset P_x \) be a proper face of \( P_x \). Let \( \Phi[x] \) be a flag decomposition given by Lemma 4.14 applied to the face \( \Gamma \). Then we have \( l_{\Phi[x]}(y,0) \succeq_{\text{lex}} l_\Phi(x) \) for all \( y \preceq x \). Moreover, for any \( y \in \mathcal{P} \) we have \( l_{\Phi[x]}(y,1) = l_\Phi(y) \).

**Proof.** Note that the proof of Lemma 4.14 in fact gives that \( (x,1) \Gamma \preceq (x,0), \psi(x,1),(x,0)P_{x,0} \subset \Gamma \) and the image of \( \Lambda(x,0) \) is contained in the affine hull of \( \Gamma \). The space \( V_{x,0} \) is defined as the preimage of the lattice \( \Lambda(x,0) \) obtained as the intersection of \( \Lambda_x \) with the affine hull of \( \Gamma \). It \( \Gamma \) is a proper face then it follows that \( \dim V_{x,0} < \dim V_x \). This implies that \( l_{\Phi[x]}(y,0) \succeq_{\text{lex}} l_\Phi(x,0) \preceq_{\text{lex}} l_\Phi(x) \) for all \( y \preceq x \).

The last assertion follows from the fact that \( V_{y,1} = V_y \) and \( \Lambda(y,1) = \Lambda_y \) for \( y \in \mathcal{P} \).
Observation 4.18. Let $\Phi$ be a minimal flag decomposition of $f$, let $x \in P$ and suppose that $x$ is not $(g(K(x)), \delta)$-complete in $\Phi$. Let $\Phi[x]$ be a flag decomposition given by Lemma 4.15 applied to the element $x$ and the same $g, \delta$. Then we have $l_{\Phi[x]}(y, 0) \succeq_{\text{lex}} l_{\Phi}(x)$ for all $y \preceq x$. Moreover, for any $y \in P$ we have $l_{\Phi[x]}(y, 0) = l_{\Phi}(y)$.

Proof. Since $x$ is not $(g(K(x)), \delta)$-complete, in the proof of Lemma 4.15 we have $k \geq 1$. In the proof of Lemma 4.15 we defined $\Lambda(x, 0)$ as a minimal lattice in $\Lambda_x \times \mathbb{Z}^k$ containing all points of the form $(\hat{\varphi}_x(v), \xi_1(v), \ldots, \xi_k(v))$ over all $v$ such that $f_{\preceq x}(v) > 0$. By assumption, points $v$ with $f_{\preceq x}(v) > 0$ span $V_x$ and, by construction, the map $(\varphi_x, \xi_1, \ldots, \xi_k) : V_x \to \Lambda_x / p\Lambda_x \times \mathbb{F}_p^k$ is surjective. It follows that the map

$$\Lambda(x, 0) / p\Lambda(x, 0) \to \Lambda_x / p\Lambda_x \times \mathbb{F}_p^k$$

is an isomorphism and in particular, $\dim \Lambda(x, 0) > \dim \Lambda_x$. Since $\dim V(x, 0) = \dim V_x$ we conclude that $l_{\Phi[x]}((x, 0)) \succeq_{\text{lex}} l_{\Phi}(x)$.

The last assertion follows from the fact that $V(y, 1) = V_y$ and $\Lambda(y, 1) = \Lambda_y$ for $y \in P$.

Observation 4.19. Let $\Phi$ be a flag decomposition of $f$ and let $\Phi'$ be a result of application of either of the lemmas 4.11, 4.12, 4.13. Then for any $x \in P'$ we have $l_{\Phi'}(x) \succeq_{\text{lex}} l_{\Phi}(x)$.

Proof. In Lemma 4.11 we only remove elements from $P$ so the value of $l_{\Phi}(x)$ is not affected for the remaining elements. Similarly, Lemma 4.12 does not affect $l_{\Phi}(x)$. In Lemma 4.13 we replace $\Lambda_x$ and $V_x$ by the minimal lattice and subspace respectively which contain supports of $\hat{f}_x$ and $f_{\preceq x}$. We note that if $\dim \Lambda_x^{\min} < \dim \Lambda_x$ then we also have $\dim V_x^{\min} < \dim V_x$. So either both dimensions stay the same, or $\dim V_x$ decreases. This implies that $l_{\Phi^{\min}}(x) \succeq_{\text{lex}} l_{\Phi}(x)$ holds for all $x \in P'$.

Lastly, we will use the following result from Ramsey theory.

Claim 4.20. For integers $N, k \geq 1$ let $\chi : [N] \to [k]$ be a coloring of the set of first $N$ natural numbers in $k$ colors. Let $h : \mathbb{N} \to \mathbb{N}$ be any function. If $N \gg_{h, k} 1$ then for some $l \in [k]$ there exists an interval $J = [j_0, j_1] \subset [N]$ such that $\chi(j) \in [l, k]$ for any $j \in J$ and $\chi(j) = l$ for at least $h(j_0)$ elements $j \in J$.

Proof. By König’s tree lemma, it is enough to prove the analogous statement with $[N]$ replaced by $\mathbb{N}$. If $\chi : \mathbb{N} \to [k]$ is an arbitrary coloring then we let $l$ to be the least element of $[k]$ such that color $l$ appears in $\chi$ infinitely many times. Since all colors $l' < l$ appear only finitely many times there is some $j_0 \in \mathbb{N}$ such that $\chi(j) \geq l$ for any $j \geq j_0$. Now let $j_1 \geq j_0$ be an element such that the interval $[j_0, j_1]$ contains at least $h(j_0)$ elements $j$ such that $\chi(j) = l$. Then $J = [j_0, j_1]$ is the desired interval.

### 4.5 Proof of Flag Decomposition Lemma

Now we turn to the proof of Theorem 4.9. Let $f : V = \mathbb{F}_p^d \to \mathbb{N}$ be an arbitrary non-zero function, $\varepsilon > 0$ and $g$ is an increasing function such that $p \gg_{d, \varepsilon, g} 1$. We are going to apply the lemmas from the previous sections repeatedly to build the desired flag decomposition.
Let $\delta_0 \gg_{d, \varepsilon} 1$ be sufficiently small and for $i \geq 0$ denote $\delta_i = 3^{-2^i} \delta_0$ and $\varepsilon_i = \varepsilon + 2^{-i} \varepsilon$.

Initialization. Let $\Phi^0$ be a flag decomposition of $f$ defined in the following way. Let $\mathcal{P}^0 = \{x_0\}$ be a 1 element poset, let $V_{x_0}^0 = V$, $f_{x_0}^0 = f$, let be $A_{x_0}^0$ a 0-dimensional space and $A_{x_0}^0 = \Lambda_{x_0}^0$. Since $\Lambda_{x_0}^0 / pA_{x_0}^0$ consists of a single element, there is a unique map $\varphi_{x_0}^0 : V_{x_0}^0 \rightarrow \Lambda_{x_0}^0 / pA_{x_0}^0$. Finally let $E_{x_0}^0$ be the only affine basis of $A_{x_0}^0$ which consists of a single origin point and let $P_{x_0}^0 = A_{x_0}^0$ be the 1-point polytope.

Step $i$. Suppose that we are given a flag decomposition $\Phi^{i-1}$ of $f$. We construct a new flag decomposition $\Phi^i$ or finish the process according to the following cases. In each case we assume that all previous cases do not apply.

(i) If $\Phi^{i-1}$ is not reduced then let $\Phi^i = \Phi^{i-1,\text{red}}$ and proceed to step $i$. Otherwise, if $\Phi^{i-1}$ is not minimal, let $\Phi^i = \Phi^{i-1,\text{min}}$ and proceed to step $i + 1$. Otherwise, if there exists some $x \in \mathcal{P}^{i-1}$ such that $G(x) < \delta_i^2 \varepsilon K_{i-1}(x)^{-d} f(V)$ then apply Lemma 4.12 to $\Phi^{i-1}$ and $\alpha = \delta_i^2 \varepsilon$, let $\Phi^i$ be the resulting flag decomposition and proceed to step $i + 1$.

(ii) Suppose that there is an element $x \in \mathcal{P}^{i-1}$ and a face $\Gamma \subset P_x$ such that $\Gamma$ is $\varepsilon$-large and not good in $\Phi^{i-1}$. Choose such $x$ so that $l_{\Phi^{i-1}}(x)$ is minimal possible and apply Lemma 4.14 to $\Phi^{i-1}$ with parameters $x$ and $\Gamma$. Let $\Phi^i$ be the resulting flag decomposition and proceed to step $i + 1$.

(iii) Suppose that the previous case does not apply and there exists an $\varepsilon$-large element $x \in \mathcal{P}^{i-1}$ which is not $(g(K_{i-1}(x)), \delta_i)$-complete. Choose such $x$ so that $l_{\Phi^{i-1}}(x)$ is minimal possible and apply Lemma 4.15 to $\Phi^{i-1}$ with parameters $x$ and $g$, $\delta_i$. Let $\Phi^i$ be the resulting flag decomposition and proceed to step $i + 1$.

(iv) If none of the above applies, and stop the procedure and let $\Phi = \Phi^{i-1}$.

We are going to show that the procedure stops in $N \ll_{d, \varepsilon} 1$ steps. Let us first see how this would imply Theorem 4.9. Indeed, let $\Phi$ denote the resulting flag decomposition of $f$ after the procedure stops in $N \ll_{d, \varepsilon} 1$ steps. Denote $\delta = \delta_N \gg_{d, \varepsilon} 1$. Since (i) is not applicable, $\Phi$ is reduced, minimal and $G(x) \geq \delta^3 K(x)^d f(V)$. Since (ii) is not applicable, any $\varepsilon$-large face of $\Phi$ is good. Since (iii) is not applicable, every $\varepsilon$-large element $x \in \mathcal{P}$ is $(g(K(x)), \delta_N)$-complete. We conclude that $\Phi$ is $(T, \varepsilon, \delta)$-complete for some function $T : \mathcal{P} \rightarrow \mathbb{N}$ such that $T(x) \geq g(K(x))$ for all $x \in \mathcal{P}$.

At each step the number of elements of $\mathcal{P}$ increases by a factor of at most 2, so we have $|\mathcal{P}| = |\mathcal{P}_N| \leq 2^N \ll_{d, \varepsilon} 1$. At each step in which (i) is applied, by Lemma 4.12 we have $f^{\Phi^i}(V) \geq (1 - \delta_i^2 \varepsilon) f^{\Phi^{i-1}}(V)$, for each step in which (ii) is applied by Lemma 4.15 we have $f^{\Phi^i}(V) \geq (1 - 3^{d+1} \delta_i) f^{\Phi^{i-1}}(V)$ and the value of $f^{\Phi^{i-1}}(V)$ does not change when we apply (ii) by Lemma 4.14. It is easy to see that with our choice of parameters we get

$$f^{\Phi}(V) \geq f^{\Phi_N}(V) \geq (1 - \varepsilon) f^{\Phi_0}(V) = (1 - \varepsilon) f(V).$$
We checked all properties claimed in Theorem 4.9 and it remains to show that the procedure above terminates in \( N \ll d, \varepsilon \) steps.

Assume that the process did not stop in \( N \) steps and let us arrive at a contradiction provided that \( N \) is sufficiently large. Given any element \( l \in [d]^2 \) we denote \( \bar{l} = (d + 1)l_1 + l_2 \in [(d + 1)^2] \) so that it defines an embedding of the lexicographical order on \([d]^2\) in \( \mathbb{N} \). Let \( h : \mathbb{N} \to \mathbb{N} \) be an increasing function depending on \( d, \varepsilon \) which will be determined and define a coloring \( \chi \) of the interval \([N]\) as follows:

- If (i) is applied at step \( i \) and 'reduced elements' lemma was applied then we let \( \chi(i) = 2(d + 1)^2 + 2 \),
- If (i) is applied at step \( i \) and 'minimal lattice' lemma was applied then we let \( \chi(i) = 2(d + 1)^2 + 1 \),
- If (i) is applied at step \( i \) and 'large gaps' lemma was applied then we let \( \chi(i) = 2(d + 1)^2 \),
- If (ii) is applied to an element \( x \) at step \( i \) then we let \( \chi(i) = 2\bar{l} \Phi_{i-1}(x) + 1 \),
- If (iii) is applied to an element \( x \) at step \( i \) then we let \( \chi(i) = 2\bar{l} \Phi_{i-1}(x) \).

If \( N \) is large enough compared to \( d \) and \( h \), then Claim 4.20 implies that there is some \( c \in \mathbb{N} \) and \( 1 \leq j_0 < j_1 \leq N \) such that \( \chi(j) \geq c \) for any \( j \in [j_0, j_1] \) and \( \chi(j) = c \) for at least \( h(j_0) \) elements \( j \in [j_0, j_1] \). For each value of \( c \) we are going to show that \( h(j_0) \) cannot be arbitrarily large and thus arrive at a contradiction.

**Case** \( c = 2(d + 1)^2 + 1 \) and \( c = 2(d + 1)^2 + 2 \). In this case for any \( j \in [j_0, j_1] \) we apply (i) and only invoke 'reduced elements' and 'minimal lattice' subcases. Observe that the properties of \( \Phi \) being reduced or minimal are preserved by applications of Lemma 4.11 and Lemma 4.13. So the interval \([j_0, j_1]\) contains at most 2 elements in this case and we conclude that \( h(j_0) \leq 2 \).

**Case** \( c = 2(d + 1)^2 \). In this case, (i) is applied at step \( j \) for any \( j \in [j_0, j_1] \). Note that in lemmas 4.11 and 4.13 the function \( K(x) \) can only increase. So if \( G(x) \geq \delta_j^3 K_{j-1}(x) - d f(V) \) is true for all \( x \in P \) at some step \( j \in [j_0, j_1] \) then it remains true for all \( j' \in [j, j_1] \). Thus, the 'large gap' sub-case of (i) is applied at most once on the interval \([j_0, j_1]\). On the other hand, Claim 4.20 guarantees that there are at least \( h(j_0) \) such values, a contradiction.

**Case** \( c = 2\bar{l} + 1 \). In this case, for any \( j \in [j_0, j_1] \) only the following operations could be performed:

- any subcase of (i),
- case (ii) applied to an element \( x \) with \( l_{\Phi_{j-1}}(x) \geq \text{lex } \bar{l} \), moreover, this case was applied to at least \( h(j_0) \) elements \( x \) with \( l_{\Phi_{j-1}}(x) = \bar{l} \),

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Note that none of these operations increases the number of elements \( x \in P^j \) with \( l_{\Phi_j}(x) \leq \bar{l} \).
Indeed, for fixed \( x \) the sequence of elements \( l_{\Phi_j}(x) \) is increasing in \( j \) (where we identify \( x \in P^j \)
with elements of the form \((x, 1) \in P^{j+1}\), \((x, 1) \in P^{j+2}\) and so on) and by Observations 4.17 and 4.18 the new added elements to \( P^j \) always have level strictly larger than \( \bar{l} \).
Since we have \(|P^{j_0}| \leq 2^{j_0}\), there are at most \( 2^{j_0} \) elements \( x \) with level at most \( \bar{l} \) appearing in any of the posets \( P^j, j \in [j_0, j_1] \) and, in fact, all of these elements already appear in \( P^{j_0} \).
By pigeonhole principle, there exists an element \( x \in P^{j_0} \) with level \( \bar{l} \) such that case (ii) was applied to \( x \) and some \( \varepsilon_j \)-large face of \( P^j_x \) for at least \( M = [h(j_0)2^{-j_0}] \) different indices \( j \in [j_0, j_1] \). Let \( i_1 < \ldots < i_M \) be the list of such indices and for \( t = 1, \ldots, M \) let \( \Gamma_t \subset P^{i_{t}}_x \) be the \( \varepsilon \)-large face to which case (ii) was applied. In particular, \( \Gamma_t \) is not good in \( \Phi^{i_{t}-1} \) and it is good in \( \Phi^{i_{t}} \).
Since \( \Gamma_t \) is \( \varepsilon \)-large in \( \Phi^{i_{t}-1} \), the weight of the function \( \hat{f}^{i_{t}-1}_x \) on \( \Gamma_t \) is at least \( \varepsilon f^{\Phi^{i_{t}-1}}(V) \), it follows that for all \( j \geq i_t \), the restriction of \( \hat{f}^{j}_x \) on \( \Gamma_t \) is non-zero since the total weight removed from \( f \) at all these steps is smaller than \( \varepsilon f^{\Phi^{i_{t}-1}}(V) \). In particular, for every \( j \in [i_t, j_1] \) we have \( \Gamma_t \cap P^{j}_x \neq \emptyset \). Thus, our ‘clean-up’ and ‘refinement’ lemmas imply that \( \Gamma_t \cap P^{i_{t}}_x \) is a good face in \( \Phi^j \) for all \( j \in [i_t, j_1] \). But for any \( t' > t \), \( \Gamma_{t'} \) is not good in \( \Phi^{i_{t'}-1} \) so we get \( \Gamma_{t'} \neq \Gamma_t \cap P^{i_{t'}-1}_x \).
Let \( \mu \) be the measure on \( \Lambda_x \) corresponding to the function \( \hat{f}^{j}_x \). Then using the inequalities between \( \hat{f}^{j}_x(\Lambda_x) \) for different \( j \in [j_0, j_1] \) one can show that for every \( t = 1, \ldots, M \), the face \( \Gamma_t \subset P^{i_{t}-1}_x \) is \( \varepsilon/2 \)-large with respect to \( P^{i_{t}-1}_x \) and the measure \( \mu \). Thus, Proposition 4.16 can be applied and we get that \( M \ll_{d, \varepsilon} 2^{j_0} \) and so we arrive at a contradiction provided that \( h \) grows fast enough (depending on \( \varepsilon, d \) only).

**Case** \( c = 2\bar{l} \). In this case, for any \( j \in [j_0, j_1] \) only the following operations could be performed:

- any subcase of (i),
- case (ii) applied to an element \( x \) with \( l_{\Phi_{j-1}}(x) \succeq_{\text{lex}} \bar{l} \),
- case (iii) applied to an element \( x \) with \( l_{\Phi_{j-1}}(x) \succeq_{\text{lex}} \bar{l} \), moreover, this case was applied to at least \( h(j_0) \) elements \( x \) with \( l_{\Phi_{j-1}}(x) = \bar{l} \).

If case (iii) is applied to some element \( x \) at step \( j \in [j_0, j_1] \) then by Lemma 4.15 \( x \) (which we identify with the element \((x, 0)\)) is not a reduced element in \( \Phi^j \) and so \( x \) is removed from \( P^j \) on the next step of the procedure. Thus, case (iii) can be applied to any element \( x \) only once. Since the number of elements on level at most \( \bar{l} \) does not increase on the interval \([j_0, j_1]\), we conclude that \( h(j_0) \leq 2^{j_0} \) which is a contradiction provided that \( h \) grows fast enough.
5 Relative set expansion

5.1 Additive combinatorics tools

For a non-constant linear function $\xi : \mathbb{F}_p^d \to \mathbb{F}_p$ and a positive integer $K > 0$ we define a $K$-slab $H(\xi, K)$ to be the set $\{ v \in \mathbb{F}_p^d : \xi(v) \in [-K, K] \}$.

**Definition 5.1.** Let $K \geq 1$ be an integer and $\varepsilon > 0$. We say that a multiset $X \subset \mathbb{F}_p^d$ is $(K, \varepsilon)$-thick along $\xi$ if we have $|X \cap H(\xi, K)| \leq (1 - \varepsilon)|X|$. We say that $X$ is $(K, \varepsilon)$-thick if $X$ is $(K, \varepsilon)$-thick along $\xi$ for any linear function $\xi$. Otherwise we say that $X$ is $(K, \varepsilon)$-thin along $\xi$ and $(K, \varepsilon)$-thin, respectively.

A $K$-slab $H = H(\xi, K)$ is called centrally symmetric if the linear function $\xi$ has no constant term.

The next two lemmas are similar to the main tools Alon and Dubiner [1] Propositions 2.4 and 2.1, respectively] used in their proof of the bound (1). The statements that we need are slightly different from their analogues in [1], so for the readers convenience we include full proofs.

**Lemma 5.2.** Fix $d, K \geq 1$ and $\varepsilon > 0$ and a prime $p$ such that $p > 100K$. Let $A$ be a sequence of elements of $\mathbb{F}_p^d$ and suppose that any centrally symmetric $K$-slab contains at most $(1 - \varepsilon)|A|$ members of $A$. Then, for every subset $Y \subset \mathbb{F}_p^d$ of at most $p^d/2$ elements there is an element $a \in A$ such that $|(Y+a) \cup Y| \geq (1 + K\varepsilon/c_0)|Y|$. Here one can take $c_0 = 100$.

**Proof.** Let $m = \lceil 200K \rceil$. Let $f : \mathbb{F}_p^d \to \mathbb{Z}$ be the characteristic function of the sequence $A$ and for some integer $m \geq 1$ define a function $f_m$ as follows:

$$f_m(x) = \sum_{t=1}^{m} f(x/t).$$

Consider a weighted graph $G$ on $\mathbb{F}_p^d$, where vectors $x, y \in \mathbb{F}_p^d$ are connected by an edge of weight $f_m(x - y) + f_m(y - x)$. Then $G$ is a regular graph of degree $\Delta = 2m|A|$. Let $\lambda_2(G)$ denote the second largest eigenvalue of the graph $G$. By Alon–Milman inequality [2], for any set $Y \subset \mathbb{F}_p^d$ of size at most $p^d/2$ we have

$$E(Y, \overline{Y}) \geq (\Delta - \lambda_2(G)) \frac{|Y||\overline{Y}|}{p^d} \geq (\Delta - \lambda_2(G))|Y|/2,$$

where $E(Y, Z)$ denotes the weight of edges between $Y$ and $Z$. Suppose that we have shown that $\lambda_2(G) \leq (1 - \varepsilon/2)\Delta$, then the above inequality gives

$$E(Y, \overline{Y}) \geq \varepsilon m|A||Y|/2.$$

The graph $G$ is a union (with multiplicities) of matchings $\{(x, x + ta), x \in \mathbb{F}_p^d\}$ over all possible choices $a \in A$ and $t = 1, \ldots, m$. So by pigeonhole principle, there exists $a \in A$ and $t \in [1, m]$ such that

$$|(Y + ta) \setminus Y| \geq \varepsilon|Y|/4.$$

Since we are working with affine spaces we allow $\xi$ to have a constant term.
On the other hand, we have a simple inequality
\[ |(Y + ta) \setminus Y| \leq t|(Y + a) \setminus Y| \leq m(Y + a) \setminus Y|, \]
and so together these bounds imply that \(|Y \cup (Y + a)| \geq (1 + \frac{eK}{100p})|Y|\), as desired.

So, to finish the proof it remains to check that we indeed have the upper bound \(\lambda_2(G) \leq (1 - \varepsilon/2)\Delta\) on the second eigenvalue of \(G\). Recall that the eigenvalues of the weighted Cayley graph \(G\) are given by the (normalized) Fourier coefficients of the function \(f_m(x) + f_m(-x)\).

For an arbitrary function \(h : \mathbb{F}_p^d \to \mathbb{C}\) and a linear function \(\xi : \mathbb{F}_p^d \to \mathbb{F}_p\) define the Fourier coefficient
\[
\hat{h}(\xi) = \sum_{x \in \mathbb{F}_p^d} h(x)e(\xi(x)),
\]
where for shortcut we denoted \(e(y) = e^{\frac{2\pi i y}{p}}\). The normalization in (28) is not quite standard but it is more convenient for our purposes: the second largest eigenvalue of \(G\) is now precisely
\[
\lambda_2(G) = \max_{\xi \neq 0} \left( \hat{f}_m(\xi) + \overline{\hat{f}_m(\xi)} \right) \leq 2 \max_{\xi \neq 0} |\hat{f}_m(\xi)|.
\]

Applying (28) to \(f_m\) we get
\[
\hat{f}_m(\xi) = \sum_x f_m(x)e(\xi(x)) = \sum_x \sum_{t=1}^m f(x/t)e(\xi(x)) = \sum_x \left( f(x) \cdot \sum_{t=1}^m e(t\xi(x)) \right).
\]

Summing over the geometric progression, for \(\xi(x) \neq 0\), we get
\[
\hat{f}_m(\xi) = \sum_x f(x)e(\xi(x)) \frac{1 - e(m\xi(x))}{1 - e(\xi(x))}.
\]

For \(x \in \mathbb{F}_p^d\) such that \(\xi(x) \in [-K, K]\) we bound the term on the right hand side by \(mf(x)\). For \(x \in \mathbb{F}_p^d\) such that \(\xi(x) \notin [-K, K]\) we have
\[
\left| e(\xi(x)) \frac{1 - e(m\xi(x))}{1 - e(\xi(x))} \right| \leq \frac{2}{|1 - e^{2\pi i K}|} \leq \frac{20p}{K},
\]
and so by the triangle inequality we can estimate \(\hat{f}_m(\xi)\) for \(\xi \neq 0\):
\[
|\hat{f}_m(\xi)| \leq |A \cap H(\xi, K)|m + |A \setminus H(\xi, K)| \frac{20p}{K},
\]
here the first term accounts for the contribution of vectors \(x\) such that \(\xi(x) \in [-K, K]\) and the second term accounts for \(x\) such that \(\xi(x) \notin [-K, K]\). Recall that we picked \(m = \frac{40p}{K}\) and by the assumption on the set \(A\) we have \(|A \setminus H(\xi, K)| \geq \varepsilon|A|\). So we have
\[
|\hat{f}_m(\xi)| \leq |A \cap H(\xi, K)|m + |A \setminus H(\xi, K)|m/2 \leq |A|m(1 - \varepsilon/2).
\]

This gives us the desired bound on the second eigenvalue of \(G\) and completes the proof. \(\square\)
Lemma 5.3. Let $A \subset \mathbb{F}_p^d$ be a non-empty subset such that $|A| = x^d \leq (p/2)^d$. Let $E$ be a basis of $\mathbb{F}_p^d$. Then, there is an element $v \in E$ such that $|A \cup (A + v)| \geq (x + \frac{1}{3d})^d$.

Proof. The proof is based on a discrete version of the Loomis–Whitney inequality [15]:

Proposition 5.4. Let $A \subset \mathbb{R}^d$ be a finite set. Let $A_i$ be the projection of $A$ on the $i$-th coordinate hyperplane $\{(x_1, \ldots, x_d) \mid x_i = 0\}$. Then one has an inequality $|A|^{d-1} \leq \prod_{i=1}^d |A_i|$.

Let $A \subset \mathbb{F}_p^d$ and $|A| = x^d \leq (p/2)^d$. We may assume that $E$ is the standard basis of $\mathbb{F}_p^d$. Now consider the standard embedding of $\mathbb{F}_p^d$ in $\mathbb{Z}^d$ and apply Proposition 5.4 to the image of $A$. It follows that there is $i \in \{1, \ldots, d\}$ such that $|A_i| \geq x^{d-1}$. This means that at least $x^{d-1}$ lines of the form $l_v = \{v + te_i\} \subset \mathbb{F}_p^d$ intersect $A$. For any line $l_v$ intersecting $A$ we have either $|(A \cup (A + e_i)) \cap l_v| > |A \cap l_v|$ or $l_v \subset A$. But the number of lines $l_v$ contained in $A$ is at most $|A|/p$ so there are at least $x^{d-1} - x^{d-1}/p$ lines $l_v$ which intersect $A$ and not contained in it. Thus,

$$|(A + e_i) \setminus A| \geq x^{d-1} - x^{d-1}/p \geq x^{d-1}/2.$$ 

Finally, it is easy to verify that for any $x, d \geq 1$ the following inequality holds: $x^d + x^{d-1}/2 \geq (x + \frac{1}{3d})^d$. \hfill \square

5.2 Set Expansion argument

This section is devoted to a proof of the following technical result.

Theorem 5.5. Fix integers $d, t \geq 0, K \geq 1$ and $\delta > 0$. There exist integers $T_0, p_0 \geq 0$ depending on all these parameters such that the following holds for any integer $T \geq T_0$ and a prime $p \geq p_0$.

Let $S \subset [-K, K]^d$ be a set of integer vectors. Suppose that for every $q \in S$ there exists a multiset $X_q \subset \mathbb{F}_p^t$ and an integer $\alpha_q$ such that the following conditions hold. We have

$$\sum_{q \in S} \alpha_q q = 0 \quad \text{and} \quad \sum_{q \in S} \alpha_q = p,$$

and for any $q \in S$, we have $\delta p \leq \alpha_q \leq |X_q| - \delta p$. Let $X = \bigcup_{q \in S} \{q\} \times X_q \subset \mathbb{F}_p^{d+t}$, and let $\xi : \mathbb{F}_p^{d+t} \to \mathbb{F}_p$ be a linear function not constant on $\{0\} \times \mathbb{F}_p^t$. Suppose that $X$ is $(T, \delta)$-thick along any such function $\xi$.

Then $X$ contains $p$ distinct elements with sum zero modulo $p$.

Before we give a proof, let us discuss some special cases of Theorem 5.5. If we take $t = 0$ then the statement is trivial: the set $X$ is obtained from $S$ by taking each point $q$ with multiplicity $|X_q|$ and since we have $\alpha_q \leq |X_q|$ the coefficients $\alpha_q$ provide the desired $p$ elements with zero sum. If $d = 0$ and the set $S$ consists of a single point $q$ (in a 0-dimensional space) then $X = X_q$ is assumed to be $(T, \delta)$-thick along any non-zero linear function $\xi$ for some $T \geq T_0(\delta)$. This is more or less the setting of the argument of Alon–Dubiner [11] in which they prove a linear upper
bound on the Erdős–Ginzburg–Ziv function using additive combinatorics tools from Section 5.1. Our proof generalizes their argument to arbitrary values of paramaters $d, t$.

Finally, let us remark that our proof gives a slightly more general version of Theorem 5.5: if we replace the condition $\sum_{q \in S} \alpha_q = p$ with $\sum_{q \in S} \alpha_q = m$ for an arbitrary $m$ then, under the same conditions, $X$ contains $m$ distinct elements with sum zero modulo $p$.

Proof of Theorem 5.5. Fix the data as stated in the theorem. Let $\Lambda \subset \mathbb{Z}^S$ be the dependence lattice of the set of points $S \subset \mathbb{Z}^d$, namely,

$$\Lambda = \left\{ (\beta_q)_{q \in S} \mid \sum \beta_q q = 0, \sum \beta_q = 0, \beta_q \in \mathbb{Z} \right\}. \quad (29)$$

Note that $\Lambda$ is defined by a system of equations with coefficients bounded by $K$. Basic facts from linear algebra imply that $\Lambda$ we can choose a basis $e_1, \ldots, e_k$ of the lattice $\Lambda$ such that $\|e_i\|_{\infty} \leq K_1$ for all $i = 1, \ldots, k$ and some $K \ll K_1 d$. Here and in what follows we consider the $l_p$-norms $\| \cdot \|_p$ on spaces $\mathbb{R}^d$ and $\mathbb{R}^S$ taken with respect to the natural bases.

Let $K_2 \gg K_1$ be a sufficiently large function of $K, d$ and consider the set

$$\Phi = \{(\lambda, \lambda') \in \mathbb{N}^S \times \mathbb{N}^S \mid \lambda - \lambda' \in \Lambda, \|\lambda\|_\infty, \|\lambda'\|_\infty \leq K_2\}.$$

For $(\lambda, \lambda') \in \Phi$ we define $J^{\lambda, \lambda'}$ to be the family of all pairs $(J, J')$ where $J, J' \subset X$, $J \cap J' = \emptyset$ and for every $q \in S$ we have

$$|J \cap \{q\} \times X_q| = \lambda_q, \quad |J' \cap \{q\} \times X_q| = \lambda'_q. \quad (30)$$

For $(J, J') \in J^{\lambda, \lambda'}$ we denote

$$\sigma(J, J') = \sum_{x \in J} x - \sum_{x' \in J'} x' \in \mathbb{F}_p^{d+t} \quad (31)$$

Since we have $\lambda - \lambda' \in \Lambda$, for any $(J, J') \in J^{\lambda, \lambda'}$ we have $|J| = |J'|$ and by (29) and (30) the projection of $\sigma(J, J')$ on the first $d$ coordinates is 0. So we have $\sigma(J, J') \in \{0\} \times \mathbb{F}_p^d$. For $(\lambda, \lambda') \in \Phi$ define a function $\nu_{\lambda, \lambda'} : \mathbb{F}_p^d \to \mathbb{R}_{\geq 0}$ as follows:

$$\nu_{\lambda, \lambda'}(v) := \frac{|\{(J, J') \in J^{\lambda, \lambda'} : \sigma(J, J') = (0, v)\}|}{|J^{\lambda, \lambda'}|}, \quad (32)$$

so, in particular, we have $\nu_{\lambda, \lambda'}(\mathbb{F}_p^d) = 1$. Put $\nu = \sum_{(\lambda, \lambda') \in \Phi} \nu_{\lambda, \lambda'}$.

Lemma 5.6. The function $\nu : \mathbb{F}_p^d \to \mathbb{R}_{\geq 0}$ is $(T/K_3, \delta/K_3)$-thick along any centrally symmetric non-zero linear function $\xi$ on $\mathbb{F}_p^d$. Here the constant $K_3$ is a bounded function of $d, K, K_1, K_2, \delta$.

Proof. Suppose that there is a linear function $\xi$ such that $\nu$ is $(T', \delta')$-thin along $\xi$ and $\xi(0) = 0$ for some $T' > 1$ and $\delta' > 0$. Denote $H = H(\xi, B) \subset U$. Our strategy is to deduce that $X$ is $(T'', \delta'')$-thin along some linear function $\eta$ on $\mathbb{F}_p^{d+t}$ which coincides with $\xi$ on $\{0\} \times \mathbb{F}_p^d$. With the right choice of parameters, this will contradict our initial assumption on $X$ and thus prove the lemma.
Let \( \Phi' \subset \Phi \) be the set of pairs \((\lambda, \lambda') \in \Phi\) such that \( \nu_{\lambda, \lambda'} \) is \((T', 2\delta')\)-thin along \( \xi \). It follows that

\[
\nu(\mathbb{F}_p^q)\delta \geq \nu(\mathbb{F}_p^q \setminus H) = \sum_{(\lambda, \lambda') \in \Phi} \nu_{\lambda, \lambda'}(\mathbb{F}_p^q \setminus H) \geq \sum_{(\lambda, \lambda') \in \Phi \setminus \Phi'} \nu_{\lambda, \lambda'}(\mathbb{F}_p^q)2\delta',
\]

so, since \( \nu_{\lambda, \lambda'}(\mathbb{F}_p^q) = 1 \) we get

\[
|\Phi'| \geq \frac{1}{2}|\Phi|.
\] (33)

As a first step, we show that the values of \( \xi \) on sets \( X_q \subset \mathbb{F}_p^q \) should be concentrated on short intervals.

**Proposition 5.7.** For any \( q \in S \) there exists a number \( r_q \in \mathbb{F}_p \) such that \( \xi(x) - r_q \in [-2T', 2T'] \) for all but at most \( 6\delta'|X_q| \) elements \( x \in X_q \).

**Proof.** We claim that there exists \((\lambda, \lambda') \in \Phi'\) such that \((\lambda_q, \lambda'_q) \neq (0, 0)\). Indeed, the set of \((\lambda, \lambda_q) \in \Phi\) such that \((\lambda_q, \lambda'_q) = (0, 0)\) is contained in \( \Phi \cap V \) for some hyperplane \( V \subset \mathbb{R}^S \times \mathbb{R}^S \). Since \((\lambda, \lambda) \in \Phi\) for any \( \|\lambda\|_{\infty} \leq K_3 \), the set \( \Phi \) is not contained in \( V \). It follows that if we choose \( K_2 \) sufficiently large compared to \( d, K, K_1 \) then \( |\Phi \cap V| \leq 0.1|\Phi| \). By (33), we conclude that \( \Phi' \not\subset V \) and there exists \((\lambda, \lambda') \in \Phi'\) such that \((\lambda_q, \lambda'_q) \neq (0, 0)\). Without loss of generality let us assume that \( \lambda_q \neq 0 \).

Let \( G \) be a graph on the vertex set \( X_q \) where elements \( x, x' \in X_q \) are connected by an edge if \( \xi(x) - \xi(x') \not\in [-2T', 2T'] \). Note that if the independence number of \( G \) is at least \((1 - 6\delta')|X_q|\) then the statement of the proposition follows (take \( r_q = \xi(x) \) for any member \( x \) of the independent set). So we may assume that \( G \) has no independent set of size \((1 - 6\delta')|X_q|\). So we can find \( \ell = [3\delta'|X_q|/\lambda_q] \) pairwise disjoint edges \((x_1, y_1), \ldots, (x_\ell, y_\ell)\) in \( G \).

For \( x \in X_q \) denote \( J_x \subset J^{\lambda, \lambda'} \) the set of pairs \((J, J')\) such that \( x \in J \). By symmetry, we have

\[
|J_x| = \frac{\lambda_q}{|X_q|}|J^{\lambda, \lambda'}| \quad \text{and for any distinct } x, y \in X_q \text{ we have } |J_x \cap J_y| \leq \left( \frac{\lambda_q}{|X_q|} \right)^2 |J^{\lambda, \lambda'}|.
\]

For some \( x, y \), let \((J, J') \in J_x \setminus J_y \), then if we let \( J'' = J \setminus \{x\} \cup \{y\} \) then \((J'', J') \in J_y \setminus J_x \) and by (31)

\[
\sigma(J, J') - \sigma(J'', J') = \xi(x) - \xi(y).
\]

So if \((x, y)\) is an edge in \( G \) then one of the sums \( \sigma(J, J') \) or \( \sigma(J'', J') \) does not belong to the strip \( H(\xi, T') \). Let \( \mathcal{I} \) be the set of pairs \((J, J') \in J^{\lambda, \lambda'} \) with \( \sigma(J, J') \not\in [-T', T'] \). So for any edge \((x, y) \in E(G)\) we get at least \(|J_x \setminus J_y|\) such pairs \((J, J') \in J_x \cup J_y \). So, using the Bonferroni inequality we get

\[
|\mathcal{I}| \geq \ell \sum_{i=1}^\ell |J_{x_i} \setminus J_{y_i}| - \sum_{i<j} |(J_{x_i} \cup J_{y_i}) \cap (J_{x_j} \cup J_{y_j})| \geq \ell \frac{\lambda_q}{|X_q|} - 2\ell^2 \left( \frac{\lambda_q}{|X_q|} \right)^2 |J^{\lambda, \lambda'}|.
\]

Since \( \ell = [3\delta'|X_q|/\lambda_q] \gg 1 \), it follows that the right hand side is at least \( 2\delta'|J^{\lambda, \lambda'}| \) (provided that \( \delta' < 0.01 \)). This contradicts the assumption that \((\lambda, \lambda') \in \Phi'\), i.e. that \( \nu_{\lambda, \lambda'} \) is \((T', 2\delta')\)-thin along \( \xi \). This concludes the proof of the proposition. \( \square \)
For \( q \in S \) denote \( Z_q \subset X_q \) the subset of elements \( x \) such that \( \xi(x) - r_q \in [-2T', 2T'] \). By Proposition 5.7 we have \( |Z_q| \geq (1 - 6\delta')|X_q| \). Let \( Z = \bigcup_{q \in S}\{q\} \times Z_q \) and define a family \( \mathcal{J}^{\lambda, \lambda'} \) consisting of all pairs \( (J, J') \in \mathcal{J}^{\lambda, \lambda'} \) such that \( J, J' \subset Z \). Then one can easily check

\[
|\mathcal{J}^{\lambda, \lambda'}|/|\mathcal{J}^{\lambda, \lambda'}| \geq 1 - 20K_2|S|\delta' \geq 0.5,
\]

where the last inequality holds provided that \( \delta'|S|K_2 \leq 0.01 \). So if \( (\lambda, \lambda') \in \Phi' \) then there exists a pair \( (J, J') \in \mathcal{J}^{\lambda, \lambda'} \) such that \( \xi(\sigma(J, J')) \in [-T', T'] \). Expanding the definition of \( \sigma \) and using \( \xi(x) - r_q \in [-2T', 2T'] \) for \( x \in Z_q \) gives

\[
\sum_{q \in S}(\lambda_q - \lambda'_q)r_q \in [-K'T', K'T'] \quad (\text{mod } p)
\]

where \( K' = 10|S|K_2 \), which holds for any \( (\lambda, \lambda') \in \Phi' \). By (33), there exist \( I, M \ll_{K_2, S} 1 \) such that any vector in \( \Phi \) can be expressed as a sum of at most \( M \) vectors in \( \Phi' \) divided by \( I \) (pick a maximal linearly independent collection in \( \Phi' \) and use triangle inequality). So we get that for every \( (\lambda, \lambda') \in \Phi \) we have

\[
I \sum_{q \in S}(\lambda_q - \lambda'_q)r_q \in [-K''T', K''T'] \quad (\text{mod } p),
\]

(34)

where \( K'' = MK' \). Let \( S' \subset S \) be a minimal subset whose affine hull coincides with the affine hull of \( S \). Choose \( a \in \mathbb{Z}^d, b \in \mathbb{Z} \) such that \( r_q = \langle a, q \rangle + b \) (mod \( p \)) for all \( q \in S' \). Since the set \( S' \) is affinely independent and \( p \) is large enough compared to \( K \), we can always solve this system modulo \( p \).

For any \( q \in S \setminus S' \) there exists a unique up to a constant vector \( u(q) \in \Lambda \) with support in \( S' \cup \{q\} \) and \( u(q)_q \neq 0 \). Since \( S \subset [-K, K]^d \), we can choose this vector in such a way that \( \|u(q)\|_{\infty} \ll_{K, d} 1 \). Since \( K_2 \) is taken sufficiently large, we have \( \|u(q)\|_{\infty} \leq K_2 \). So we can apply (34) to the pair \( (u(q), 0) \in \Phi \) and obtain

\[
Iu(q)_q r_q + I \sum_{q' \in S'}u(q)_{q'}r_{q'} \in [-K'''T', K'''T'] \quad (\text{mod } p),
\]

where \( K''' = K_2|S|K'' \). On the other hand, since \( u(q) \in \Lambda \), we have

\[
0 = u(q)_q(\langle a, q \rangle + b) + \sum_{q' \in S'}u(q)_{q'}(\langle a, q' \rangle + b) = u(q)_q(\langle a, q \rangle + b) + \sum_{q' \in S'}u(q)_{q'} r_{q'},
\]

where in the last transition we used the definition of \( a \) and \( b \).

Let \( \tilde{I} = IK_2! \). By subtracting the two equations above with an appropriate coefficient, we conclude that for any \( q \in S \) we have

\[
\tilde{I}(r_q - \langle a, q \rangle - b) \in [-\tilde{K}T', \tilde{K}T'] \quad (\text{mod } p).
\]

Recall that \( \xi(x) - r_q \in [-2T', 2T'] \) for every \( x \in Z_q \), so by the triangle inequality we get for every \( x \in X_q \)

\[
\tilde{I}\xi(x) - \tilde{I}(\langle a, q \rangle - b) \in [-\tilde{K}'T', \tilde{K}'T'] \quad (\text{mod } p),
\]

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where \( \tilde{K}' = \tilde{K} + 2\tilde{I} \). Let \( \mathbb{F}_p^d \times \mathbb{F}_p^t \to \mathbb{F}_p \) be a function defines as

\[
\eta(y, x) = \tilde{I}(\xi(x) - \langle a, y \rangle - b),
\]

where \( y \in \mathbb{F}_p^d, x \in \mathbb{F}_p^t \). Since \( \xi \neq 0 \) and \( p > \tilde{I} \), the restriction of \( \eta \) on \{0\} \times \mathbb{F}_p^t \) is non-constant. So by our assumption, the set \( X \) is \((T, \delta)\)-thick along \( \eta \). On the other hand, we showed that for any \( \nu = (q, x) \in \mathbb{Z}_p^t \subset \mathbb{Z} \) we have \( \eta(\nu) \in [-\tilde{K}'T', \tilde{K}'T'] \). Recall that by Proposition 5.7 we have \( |Z| \geq (1 - 6\delta')|X| \). If we choose \( T' = T/\tilde{K}' \) and \( \delta' = \min\{\delta/6, 0.01|S|^{-1}K_2^{-1}\} \) then these conditions contradict each other (and the second term in the minimum makes the argument go through). This completes the proof of the lemma.

Denote \( T' = T/K_3, \delta' = \delta/K_3 \) so that \( \nu \) is \((T', \delta')\)-thick as in Lemma 5.6. Now we can apply Lemmas 5.2 and 5.3 to the function \( \nu \). Let \( J = \bigcup_{(\lambda, \lambda') \in \Phi} J_{\lambda, \lambda'} \). For a set \( J \subset \mathbb{F}_p^{d+t} \) we denote \( \sigma(J) = \sum_{x \in J} x \).

**Proposition 5.8.** For any \( c \ll K,d,\delta \) there is a sequence of pairs \((J_i, J'_i) \in J \) for \( i = 1, \ldots, cp \) such that:

1. For any \( i \neq j \) sets \( J_i \cup J'_i \) and \( J_j \cup J'_j \) are disjoint.
2. The sum of cardinalities of all these sets is at most \( 2K_2cp \).
3. Let \( M_i = \{\sigma(J_i), \sigma(J'_i)\} \subset \mathbb{F}_p^{d+t} \) then we have

\[
|M_1 + \ldots + M_{cp}| \geq \left( \frac{cp}{3t} \right)^t.
\]

**Proof.** First we note the second conclusion trivial: since \(|J| + |J'| \leq 2K_2\) for any \((J, J') \in J\) the sum of cardinalities of sets \( J_i, J'_i \) is at most \( 2K_2cp \).

Using the thickness of \( \nu \) and simple union bounds one can find at least \( j \gg K,d,\delta \) \( p \) linear bases \( B_1, \ldots, B_j \subset \mathbb{F}_p^t \) with the property that the \( i \)-th basis \( B_i \) has the form

\[
\{\sigma(J_{i,k}, J'_{i,k})\}_{k=1}^t,
\]

where \( \{ (J_{i,k}, J'_{i,k})\}_{i,k=1,1}^{j,t} \) is a collection of pairs from \( J \) such that all these pairs are pairwise disjoint. By iterative application of Lemma 5.3 we can choose some pairs \((J_{i,k_1}, J'_{i,k_1}) \) for \( i = 1, \ldots, j \) which satisfy

\[
|\{0, \sigma(J_{1,k_1}, J'_{1,k_1})\} + \ldots + \{0, \sigma(J_{j,k_j}, J'_{j,k_j})\}| \geq \left( \frac{j}{3d} \right)^t.
\]

The Minkowski sum in the statement of the proposition is a shift of (36) and so the proposition holds for any \( c \ll j/p \).

In the next proposition we continue the process of adding new pairs to the sequence \((J_i, J'_i) \) but we will invoke Lemma 5.2 instead of Lemma 5.3. Let \( Y = M_1 + \ldots + M_{cp} \subset \mathbb{F}_p^{d+t} \).
Proposition 5.9. For some $c \gg K, d, \delta$ there is a sequence of pairs $(J_i, J'_i) \in J$ for $i = cp + 1, \ldots, cp + l$ for some $l \leq cp$ such that:

1. For any $1 \leq i \neq j \leq cp + l$ sets $J_i \cup J'_i$ and $J'_j \cup J'_j$ are disjoint.
2. The sum of cardinalities of all these sets is at most $0.1\delta p$.
3. For $i = cp + 1, \ldots, cp + l$ let $M_i = \{\sigma(J_i), \sigma(J'_i)\}$. Then we have
\[
|Y + M_{cp+1} + \ldots + M_{cp+l}| \geq p^l/2.
\] (37)

Proof. As in the previous proposition, the bound on the sum of cardinalities follows if we take $c < 0.01\delta/K_2$.

We construct pairs $(J_i, J'_i)$ one by one. At step $1 \leq j \leq cp$ we consider the set
\[
Y_j = Y + M_1 + \ldots + M_{j-1}
\]
and consider a function $\nu_j$ defined analogously to $\nu$ but where we remove elements already appearing in sets $J_i, J'_i$ for $i < j$. Since the union of all these sets has size at most $2K_2cp \leq 2K_2c|X|$, the function $\nu_j$ is $(T', \delta'/2)$-thick provided that $c$ is small enough in terms of $\delta', K_2$ and $|S|$.

If $|Y_j| \leq p^l/2$ we can apply Lemma 5.2 to the set $Y_j$ and the function $\nu_j$ and obtain a pair $(J_j, J'_j) \in J_j$ (where $J_j$ is defined in analogy to $J$) such that
\[
|Y_j \cup (Y_j + \sigma(J_j, J'_j))| \geq \left(1 + \frac{T'\delta'}{Cp}\right)|Y_j|,
\] (38)
for some absolute constant $C$. After this we can repeat this argument with $j + 1$ instead of $j$.

The procedure above can stop only in two cases: if for some $j \leq cp$ we get $|Y_j| \geq p^l/2$ which completes the proof, or if we reach $j = cp$. In the latter case we get by (38):
\[
|Y_{cp+1}| \geq \left(1 + \frac{T'\delta'}{Cp}\right)^{cp}|Y_1| \gg e^{\frac{cT'\delta'}{3t}} \left(\frac{c}{3t}\right)^t p'.
\]

However we have $T'\delta' \geq T\delta/K_2$ and if we take $T$ large enough compared to $c, \delta, K, K_3, d, t$ then the right hand side exceeds 1 which is absurd. This completes the proof.  

By removing all constructed pairs from $X$ and applying the propositions above once again, we can construct another sequence of at most $\tilde{j} \leq 2cp$ pairs $(\tilde{J}_i, \tilde{J}'_i)$ which are pairwise disjoint and disjoint from the previously constructed sets, have the sum of sizes at most $0.1\delta p$ and satisfy $|\tilde{M}_1 + \ldots + \tilde{M}_j| \geq p^l/2$ (where $\tilde{M}_i = \{\sigma(\tilde{J}_i, \tilde{J}'_i)\}$). Considering the union of these two sequences, applying the easy part of the Cauchy–Davenport theorem and after relabeling indices we arrive at

Corollary 5.10. There is a set of $j \leq 4cp$ pairs $(J_i, J'_i) \in J$, $i = 1, \ldots, j$, such that:

1. The sets $J_i \cup J'_i$, $i = 1, \ldots, j$ are pairwise disjoint.
2. The sum of cardinalities of all these sets is at most $0.2\delta p$.

3. For $i = 1, \ldots, j$ let $M_i = \{\sigma(J_i), \sigma(J'_i)\}$, then for some $u_0 \in \mathbb{F}_p^d$ we have

$$M_1 + \ldots + M_j = \{u_0\} \times \mathbb{F}_p^d.$$  

(39)

Let $(\lambda_i, \lambda'_i) \in \Phi$ be the pair corresponding to sets $(J_i, J'_i)$. Note that by (39) we then have

$$u_0 = \sum_{q \in S} \sum_{i=1}^j \lambda_{i,q} q$$

(40)

since exactly $\lambda_{i,q}$ elements in $J_i$ have first $d$ coordinates equal to $q$. Let $B = \bigcup_{i=1}^j J_i$ and $B' = \bigcup_{i=1}^j J'_i$. Note that for every $q \in S$ we have

$$|B \cap (\{q\} \times X_q)| = \sum_{i=1}^j \lambda_{i,q}.$$  

(41)

Recall that $\delta p \leq \alpha_q \leq |X_q| - \delta p$ so for any $q \in S$ we have

$$0 \leq \alpha_q - |B \cap (\{q\} \times X_q)| \leq |X_q| - \delta p,$$

and there exists a subset $D_q \subset X_q$ of size exactly $\alpha_q - |B \cap X_q|$ and such that $\{q\} \times D_q$ is disjoint from $B \cup B'$. Let us denote $v_1 = \sum_{q \in S} \sum_{x \in D_q} x \in \mathbb{F}_p^d$, by Corollary 5.10 we can choose subsets $I_i \in \{J_i, J'_i\}$, $i = 1, \ldots, j$, such that

$$\sum_{i=1}^j \sigma(I_i) = (u_0, -v_1).$$

(42)

We claim that the disjoint union

$$Y = I_1 \cup \ldots \cup I_j \cup \bigcup_{q \in S} \{q\} \times D_q \subset X$$

consists of $p$ elements whose sum is zero. Indeed, we have $|I_i| = |J'_i| = |J_i|$ for every $i$ and so

$$|Y| = |I_1| + \ldots + |I_j| + \sum_{q \in S} |D_q| = |B| + \sum_{q \in S} (\alpha_q - |B \cap (\{q\} \times X_q)|) = \sum_{q \in S} \alpha_q = p.$$

Let $(w_0, w_1) = \sum_{y \in Y} y$, we need to show that $w_0 = 0$ and $w_1 = 0$. We have by (39), (40) and (42):

$$w_0 = u_0 + \sum_{q \in S} |D_q| q = u_0 + \sum_{q \in S} (\alpha_q - |B \cap (\{q\} \times X_q)|) q =$$

$$= u_0 - \sum_{q \in S} |B \cap (\{q\} \times X_q)| q = u_0 - \sum_{q \in S} \sum_{i=1}^j \lambda_{i,q} q = 0$$

By (42) and definition of $v_1$ we have $w_1 = -v_1 + v_1 = 0$. So the sum of elements of $Y$ is zero. Thus, the set $X$ contains $p$ distinct elements with zero sum and the theorem is proved.
6 Balanced convex combinations

In this section we give the last ingredient needed in the proof of Theorem 1.2. Let \( w : \mathbb{R}^d \to \mathbb{R}_{\geq 0} \) be a function with finite support. For a subset \( S \subset \mathbb{R}^d \) we denote by \( w(S) \) the sum \( \sum_{s \in S} w(s) \). We say that a point \( c \in \mathbb{R}^d \) is \( \theta \)-central for \( w \) if for any halfspace \( H^+ \) which contains \( c \) we have \( w(H^+) \geq \theta w(\mathbb{R}^d) \).

**Lemma 6.1.** Let \( \theta > 0 \), let \( w : \mathbb{R}^d \to \mathbb{R}_{\geq 0} \) be a function with finite support \( S \). Let \( \Lambda \) be the minimal lattice containing \( S \) and \( c \in \Lambda \cap \text{int}(\text{conv } S) \) be a \( \theta \)-central point for \( w \).

Then for any \( \varepsilon > 0 \) and all \( n > n_0(\varepsilon, w, c) \) there are non-negative integer coefficients \( \alpha_q \) for \( q \in S \) and \( \mu = \mu(\varepsilon, w, c) > 0 \) such that:

\[
\sum_{q \in S} \alpha_q = n, \quad \sum_{q \in S} \alpha_q q = nc, \quad (43)
\]

and for every \( q \in S \) we have

\[
\mu n \leq \alpha_q \leq (1 + \varepsilon) \frac{nw(q)}{\theta w(S)}. \quad (44)
\]

**Proof.** Without loss of generality, we may assume that \( c = 0 \), the set \( S \) spans \( \mathbb{R}^d \), \( \Lambda = \mathbb{Z}^d \) and that \( w(S) = \sum_{q \in S} w(q) = 1 \).

**Claim 6.2.** There are rational coefficients \( \beta_q \) such that:

\[
\sum_{q \in S} \beta_q q = 0, \quad \sum_{q \in S} \beta_q = 1,
\]

and \( \beta_q \in (0, \theta^{-1} \omega(q)) \) for any \( q \in S \).

**Proof.** Note that it is enough to find real coefficients \( \beta_q \) with properties described in the claim. The existence of rational coefficients would then follow automatically.

We denote by \( \mathbb{R}^S \) the space of all functions \( \xi : S \to \mathbb{R} \). This space is equipped this the natural scalar product \( \xi \cdot \eta = \sum_{q \in S} \xi(q) \eta(q) \). In what follows we identify \( \mathbb{R}^S \) with the dual space \( (\mathbb{R}^S)^* \) via this scalar product.

Let \( H \subset \mathbb{R}^S \) be the set of vectors \( (c_q)_{q \in S} \) such that \( \sum_{q \in S} c_q q = 0 \). Let \( \Omega \subset \mathbb{R}^S \) be the set of all functions \( v : S \to \mathbb{R} \) such that

\[
0 \leq v(q) \leq \theta^{-1} w(q) \sum_{q' \in S} v(q'),
\]

for any \( q \in S \). Note that if the intersection \( H \cap \text{int}(\Omega) \) is non-empty, then we are done: take any \( v \in H \cap \text{int}(\Omega) \) and define \( \beta_q = \frac{v(q)}{\sum_{q' \in S} v(q')} \).

Let us assume that \( H \cap \text{int}(\Omega) = \emptyset \) and arrive at a contradiction. Since \( H \) is a vector subspace and \( \text{int}(\Omega) \) is an open convex set, there exists a function \( \xi \in \mathbb{R}^S \) such that

\[
\xi(H) = 0 \quad \text{and} \quad \xi(\Omega) \geq 0.
\]
The first condition can be reformulated as $\xi \in H^\perp$. Note that the space $H^\perp$ is isomorphic to $\mathbb{R}^d$: given a function $\zeta \in H^\perp$ we define a linear function $\tilde{\zeta}$ on $\mathbb{R}^d$ by setting $\tilde{\zeta}(q) = \zeta(q)$ for $q \in S$ and extending $\tilde{\zeta}$ by linearity. The conditions that $S$ spans $\mathbb{R}^d$ and the linear equations defining $H^\perp$ imply that this definition is correct. Let $\xi$ be the linear function on $\mathbb{R}^d$ corresponding to $\xi$.

Let $\varepsilon_q$ be the element of the standard basis of $\mathbb{R}^S$ corresponding to $q \in S$ and denote $\sigma = \sum_{q \in S} \varepsilon_q$. The set $\Omega$ is defined as the set of vectors $v \in \mathbb{R}^S$ such that $\varepsilon_q \cdot v \geq 0$ and $(w(q)\sigma - \theta \varepsilon_q) \cdot v \geq 0$, \hspace{1cm} (45)

for all $q \in S$. By duality, the condition $\xi(\Omega) \geq 0$ is a non-negative linear combination of inequalities (45). Indeed, if not, then $\xi$ can be separated by a hyperplane from functions (45) in the space of all linear functions on $\mathbb{R}^S$. But this hyperplane will correspond to a point in $\Omega$ on which the value of $\xi$ is negative. So there are nonnegative real coefficients $a_q, b_q \geq 0$ such that

$$\xi = \sum_{q \in S} a_q \varepsilon_q + b_q (w(q)\sigma - \theta \varepsilon_q) = \sum_{q \in S} (a_q - \theta b_q) \varepsilon_q + \left( \sum_{q \in S} b_q w(q) \right) \sigma. \hspace{1cm} (46)$$

Let $I \subset S$ be the set of $q \in S$ such that $\xi \cdot \varepsilon_q \leq 0$. Since $c = 0$ is a $\theta$-central point for $w$ and $\xi \cdot \varepsilon_q = \tilde{\xi}(q)$ for all $q \in S$, we have

$$\sum_{q \in I} w(q) \geq \theta. \hspace{1cm} (47)$$

On the other hand, for any $q \in I$ by (46) we have

$$\xi(q) = (a_q - \theta b_q) + \left( \sum_{q' \in S} b_{q'} w(q') \right) \leq 0, \hspace{1cm} (48)$$

hence, by discarding $a_q$ from (48) we get

$$\theta b_q \geq \sum_{q' \in S} b_{q'} w(q').$$

Summing this over $q \in I$ with weights $w(q) > 0$ we obtain:

$$\theta \sum_{q \in I} b_q w(q) \geq \left( \sum_{q \in I} w(q) \right) \left( \sum_{q \in S} b_q w(q) \right) \geq \theta \left( \sum_{q \in S} b_q w(q) \right). \hspace{1cm} (49)$$

The sum on the left hand side is a subsum of the right hand side. Since $\theta > 0$ and $w(q) > 0$ for all $q$, it follows that an equality is attained in (48) for all $q \in I$. This however means that $\xi(q) \geq 0$ for every $q \in S$ and the point $c = 0$ lies on the boundary of conv $(S)$ which contradicts our assumption. We conclude that there cannot be such a function $\xi$ and hence $H \cap \text{int}(\Omega) \neq \emptyset$, as desired.

Take some rational coefficients $\beta_q$ provided by Claim 6.2, note that we can define them as functions of $w$ and $c$. Let $m$ be the least common multiple of denominators of $\beta_q$. Since $c = 0$
belongs to the minimal lattice of $S$ there is an integer vector $\delta \in \mathbb{Z}^S$ such that $\sum_{q \in S} \delta_q q = 0$ and $\sum_{q \in S} \delta_q = 1$. Let $C = \max_{q \in S} |\delta_q|$. Let us define the function $n_0 = n_0(\varepsilon, w, c)$ by
\[
n_0 = 2Cm^2 + \varepsilon^{-1}Cm\theta \max_{q \in S} w(q)^{-1},
\]
(note that $w(q) > 0$ for any $q \in S$ by assumption) and consider an arbitrary $n > n_0$. Write $n = am + r$ where $0 \leq r < m$ and define the coefficients by $\alpha_q = am\beta_q + r\delta_q$, note that $\alpha_q$ is an integer. Let us check that all required conditions are satisfied:
\[
\sum_{q \in S} \alpha_q q = \sum_{q \in S} am\beta_q q + r\delta_q q = amc + rc = nc,
\]
\[
\sum_{q \in S} \alpha_q = am + r = n,
\]
\[
\alpha_q = am\beta_q + r\delta_q \leq am\theta^{-1}w(q) + rC \leq n\theta^{-1}w(q)(1 + mCn^{-1}\theta w(q)^{-1}) < n\theta^{-1}w(q)(1 + \varepsilon),
\]
by a similar computation we obtain $\alpha_q > \mu n$ for some $\mu > 0$ not depending on $n$. Lemma 6.1 is proved.

\section{Proof of Theorem 1.2}

In this section we put everything together and prove our main result, Theorem 1.2. Since $s(F_p^d) \geq w(F_p^d)(p - 1) + 1$ for any $d$ and $p$, it is enough to prove that for any fixed $d \geq 1$, any $\varepsilon > 0$ and all sufficiently large primes $p > p_0(d, \varepsilon)$ the inequality
\[
s(F_p^d) \leq (w(F_p^d) + \varepsilon)p
\]
holds.

Let $X \subset F_p^d$ be a multiset of size at least $(w(F_p^d) + \varepsilon)p$ and let $f : F_p^d \to \mathbb{N}$ be the characteristic function of $X$. Let $g : \mathbb{N} \to \mathbb{N}$ be an increasing function which grows fast enough depending on $d, \varepsilon$ and denote $\varepsilon = 100^{-d}\varepsilon$. Apply Theorem 4.9 to the function $f$ with parameters $g$ and $\varepsilon$. So for some $\delta \gg_{d, \varepsilon} 1$ there exists a flag decomposition $\Phi$ of $f$ on a convex flag $(\mathcal{P}, \Lambda)$ and functions $T, K : \mathcal{P} \to \mathbb{N}$ such that $\Phi$ is $K$-bounded and $(T, \varepsilon, \delta)$-complete. For any $x \in \mathcal{P}$ we have $T(x) \geq g(K(x))$, $K(x) \ll_{g, d, \varepsilon} 1$ and $G(x) \gg_{d, K(x)} p$. We have
\[
f^{\Phi}(V) \geq (1 - \varepsilon)f(V) = (1 - \varepsilon)|X| \geq (w(F_p^d) + \varepsilon/2)p.
\]

\begin{proposition}
The Helly constant $L(\mathcal{P}, \Lambda, \Omega)$ of the convex flag $(\mathcal{P}, \Lambda)$ is at most $w(F_p^d)$.\end{proposition}

\begin{proof}
Recall that the set of proper points $\Omega$ of the flag decomposition $\Phi$ is defined as $\Omega = \text{conv}(\Omega_0)$ where $\Omega_0$ is the set of points $q$ of $\mathcal{P}$ such that $\hat{f}(q) > 0$.

Take arbitrary integer proper points $q_1, \ldots, q_n$ of the convex flag $\mathcal{P}$ for some $n > w(F_p^d)$. We need to show that there coefficients $\alpha_i \in [0, 1)$ summing to 1 such that the convex combination $q = \sum \alpha_i q_i$ an integer point of $(\mathcal{P}, \Lambda)$.

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Recall that for each $x \in P$ we are given an affine subspace $V_x \subset \mathbb{F}_p^d$ and an affine surjective map $\varphi_x : V_x \to \Lambda_x/p\Lambda_x$. For $i = 1, \ldots, n$, let $x_i = \inf D^{q_i}$ and let $q_i = q_{i,x_i}$ be the point on the lattice $\Lambda_{x_i}$ corresponding to $q_i$. Let $w_i \in V_i \subset \mathbb{F}_p^d$ be an arbitrary vector such that $\varphi_{x_i}(w_i)$ is congruent to $q_i$ modulo $p\Lambda_{x_i}$, such a vector exists since the map $\varphi_{x_i}$ is surjective.

Since $n > \mathfrak{w}(\mathbb{F}_p^d)$, by the definition of this function applied to the set $\{w_1, \ldots, w_n\}$ there are non-negative integer coefficients $\alpha_1, \ldots, \alpha_n$ such that

\[
\sum_{i=1}^{n} \alpha_i = p, \quad (50)
\]

\[
\sum_{i=1}^{n} \alpha_i w_i \equiv 0 \pmod{p}, \quad (51)
\]

and $\alpha_i < p$ for every $i$.

Let $q$ be the convex combination of points $q_1, \ldots, q_n$ with coefficients $\alpha_i/p$, i.e. $q = \sum_{i=1}^{n} \frac{\alpha_i}{p} q_i$.

By definition, $q$ is a point of the convex flag $P$ such that

\[
D^q = \bigcap_{i: \alpha_i \neq 0} D^{q_i}
\]

and for any $x \in D^q$ we have the following identity:

\[
q_x = \sum_{i=1}^{n} \frac{\alpha_i}{p} q_{i,x}. \quad (52)
\]

We claim that $q_x \in \Lambda_x$ for any $x \in D^q$. Indeed, for each $i$ such that $\alpha_i \neq 0$ we have $q_{i,x} = \psi_{x,x_i}(q_i) \in \Lambda_x$ and $\varphi_x(w_i) = \psi_{x,x_i} \varphi_{x_i}(w_i)$. So the point $q_{i,x}$ is congruent to $\varphi_x(w_i)$ modulo $p\Lambda_x$.

Pick a consistent origins in spaces $\Lambda_x/p\Lambda_x$ and $V_x$, then the fact that the convex combination $\sum_{\alpha_i \neq 0} \frac{\alpha_i}{p} q_{i,x}$ is an integer point of the lattice $\Lambda_x$ is equivalent to saying that the sum $s = \sum_{\alpha_i \neq 0} \alpha_i q_{i,x}$ is a zero element in the vector space $\Lambda_x/p\Lambda_x$ (this sum depends on the choice of an origin but the fact that it is zero does not). Finally, using the map $\varphi_x$, we get

\[
\sum_{\alpha_i \neq 0} \alpha_i q_{i,x} \equiv \sum_{\alpha_i \neq 0} \alpha_i \varphi_x(w_i) = \varphi_x \left( \sum_{i=1}^{n} \alpha_i w_i \right) \equiv 0. \quad (53)
\]

We conclude that $q$ is an integer point of the flag $(P, \Lambda)$. Since all $\alpha_i$ are less than $p$ this implies that $L(P, \Lambda, \Omega) \leq \mathfrak{w}(\mathbb{F}_p^d)$. \qed

Remark. If we assume that the original multiset $X \subset \mathbb{F}_p^d$ is in fact a genuine set without multiplicities then the bound in Proposition 7.1 can be refined to $L(P, \Lambda) \leq \mathfrak{w}(\mathbb{F}_p^{d-1})$ by observing that all maps $\varphi_x$ have at least one dimensional kernels and so one can pick vectors $w_i$ inside a generic hyperplane and apply definition of $\mathfrak{w}$ inside of it. By combining this with the rest of the proof one can show that $|X| \leq (1 + \epsilon) \mathfrak{w}(\mathbb{F}_p^{d-1}) p$ whenever $X$ is a set without $p$ elements with zero sum.
For a proper integer point \( q \) we assign a weight \( w_q \) defined as follows. Let \( x = \inf \mathcal{D}^q \) and put \( q = q_x \in \Lambda_x \) and denote by \([q]\) the class of \( q \) in \( \Lambda_x/p\Lambda_x \). Then we define

\[
w_q = f_x(\varphi^{-1}_x[q])
\]

(note that this notion is different from \( \hat{f}(q) \)). Let \( \mathcal{Q} \) be the set of all proper integer points \( q \) such that \( w_q > 0 \). Note that we have

\[
\sum_{q \in \mathcal{Q}} w_q = \sum_{x \in \mathcal{P}} f_x(V_x) = f^\Phi(\mathbb{F}^d_p) \geq (\mathfrak{w}(\mathbb{F}_p^d) + \epsilon/2)p.
\]

By Centerpoint Theorem (Corollary 3.16) applied to the convex flag \((\mathcal{P}, \Lambda, \Omega)\) and the point set \( \mathcal{Q} \) with weight function \( w \), there exists a proper integer point \( q_0 \) of \( \mathcal{P} \) such that for any linear function \( \xi \) with \( D\xi \cap Dq_0 \neq \emptyset \) we have

\[
\sum_{q \in \mathcal{Q}: \xi(q) > \xi(q_0)} \hat{f}_x(q) \geq \frac{f^\Phi(\mathbb{F}_p^d)}{L(\mathcal{P}, \Lambda, \Omega)}.
\]

Let \( S \subset \Lambda_x \) be the set of points such that \( \hat{f}_x(q) > 0 \), recall that, by definition we have \( P_x = \text{conv} S \). Then by (56) the point \( q_0 \) is a \( \theta \)-central point for the function \( \hat{f}_x \), where

\[
\theta = \frac{f^\Phi(\mathbb{F}_p^d)}{L(\mathcal{P}, \Lambda, \Omega)\hat{f}_x(S)}.
\]

Let \( K = K(x) \), recall that since \( \mathcal{P} \) is \( K \)-bounded, the set \( S \) is contained in a box \([-K, K]^{\dim \Lambda_x} \) in the coordinate system \( E_x \). Recall that one of the conclusions of Theorem 4.9 is that \( G(x) \geq \gamma p \) for some \( \gamma \gg \delta, K(x) \) 1, that is, for any \( q \in S \) we have \( \hat{f}_x(q) \geq \gamma p \). Let \( H = \varepsilon \gamma p \) and define a function \( \omega : S \rightarrow \mathbb{N} \) as

\[
\omega(q) = \left\lfloor \frac{\hat{f}_x(q)}{H} \right\rfloor
\]

for \( q \in S \). It is clear that for any set of points \( S' \) we have

\[
|\omega(S') - \frac{1}{H}\hat{f}_x(S')| \leq |S'| \leq \frac{\hat{f}_x(S)}{\gamma p} \leq \varepsilon \frac{1}{H}\hat{f}_x(S).
\]

Using this, it is easy to see that the point \( q_0 \) is \( (\theta - \varepsilon) \)-central for the function \( \omega \).
Note that for any \( q \in S \) we have \( \omega(q) \geq \hat{f}_x(q)/H - 1 > 0 \) and so the supports of \( \hat{f}_x \) and \( \omega \) coincide. One of the conditions of a \((T, \varepsilon, \delta)\)-completeness is that \( \Phi \) is a minimal flag decomposition, that is, that \( \Lambda_x \) is the minimal lattice containing the support \( S \) of \( \hat{f}_x \). We conclude that \( q_0 \) belongs to the minimal lattice containing the support of \( \omega \).

Finally, we claim that \( q_0 \) belongs to the interior of \( P_x \). For the sake of contradiction suppose that \( q_0 \in \text{int} \Gamma \) for some proper face \( \Gamma \subset P_x \). By taking \( \xi \) to be a linear function vanishing on \( \Gamma \) and negative on \( P_x \setminus \Gamma \), by (56) we get that \( \hat{f}_x(\Gamma) \geq f^\Phi(\mathbb{F}^d_p)/L(p, \Lambda_x, \alpha) \geq 4^{-d} f^\Phi(\mathbb{F}^d_p) \). Similarly, for any proper face \( \Gamma' \subset \Gamma \) we can find a linear function \( \xi \) vanishing on \( q_0 \) and negative on \( \Gamma' \) and on \( S \setminus \Gamma \), thus, giving \( \hat{f}_x(\Gamma \setminus \Gamma') \geq 4^{-d} f^\Phi(\mathbb{F}^d_p) \). These observations imply that \( \Gamma \) is a \( 4^{-d} \)-large face of \( P_x \). Since we have \( \varepsilon < 4^{-d} \), one of the conclusions of Theorem 4.9 tells us that \( \Gamma \subset P_x \) is a good face in the flag decomposition \( \Phi \). By definition, it means that \( \psi_{x,x_T}(P_{x_T}) \subset \Gamma \). In particular, we have \( x_T < x \). Since \( q_0 \) is a proper point, we have \( x_T \in \mathcal{D}^{q_0} \). But we defined \( x \) as the minimum element of \( \mathcal{D}^{q_0} \), a contradiction. We conclude that \( q_0 \) lies in the interior of \( P_x \).

We are in a position to apply Lemma 6.1. Indeed, the function \( \omega \) has a \((\theta - \varepsilon)\)-central point \( q \) which belongs to both the minimal lattice spanned by the support of \( \omega \) and to the interior of the convex hull of the support of \( \omega \). Thus, Lemma 6.1 may be applied to \( \omega, q, \theta - \varepsilon \) and any integer \( n > n_0(\varepsilon, \omega, q) \). Note that we may take \( p \) large enough to ensure that \( p > n_0(\varepsilon, \omega, q) \) holds for all possible choices of \( \omega \) and \( q \). Indeed, the set \( S \) is contained in a box \([-K, K]^{\dim \Lambda_x} \) where \( K \ll_{q,d,\varepsilon} 1 \) and \( \dim \Lambda_x \leq d \) and the function \( \omega \) takes values in the set of integers of size at most

\[
\frac{f^\Phi(\mathbb{F}^d_p)}{H} \leq |X| <_{\varepsilon/\gamma} p \ll_{K,d,\varepsilon} 1.
\]

So we can take \( p_0(d, \varepsilon) \) to be larger than \( n_0(\varepsilon, \omega, q) \) for all possible choices of parameters. So, Lemma 6.1 indeed applies and there is some \( \mu \gg_{\varepsilon, \omega, q} 1 \) and integer coefficients \( \alpha_q \), for \( q \in S \), such that:

\[
\sum_{q \in S} \alpha_q = p, \quad \sum_{q \in S} \alpha_q q = pq_0, \quad \mu p \leq \alpha_q \leq (1 + \varepsilon) \frac{p \omega(q)}{(\theta - \varepsilon)\omega(S)},
\]

Let \( d' = \dim \Lambda_x \) and denote \( \dim V_x = d' + t \). We can change a basis of \( V_x \) and identify it with \( \mathbb{F}^d_p \times \mathbb{F}^t_p \) in such a way that the map \( \varphi_x : V_x \to \Lambda_x/p\Lambda_x \) is projection onto the first \( d' \) coordinates and the reduction of the basis \( E_x \) of \( \Lambda_x \) modulo \( p \) gives the first \( d' \) elements of the standard basis of \( V_x \). Further, by making a shift and replacing \( K \) by \( 2K \) we may also assume that \( q_0 = 0 \), so that \( \sum_{q \in S} \alpha_q = 0 \). With this notation in mind, for each point \( q \in S \) define \( X_q \subset \mathbb{F}^t_p \) as the multiset corresponding to the function \( f_{\preceq x} \) restricted to the fiber \( \{q\} \times \mathbb{F}^t_p \). Let \( X' = \bigcup_{q \in S} \{q\} \times X_q \), or equivalently, \( X' \) is the multiset of the function \( f_{\preceq x} \).

By unravelling the definitions, we have \( |X_q| = \hat{f}_x(q) \) for every \( q \in S \) and \( |X'| = \hat{f}_x(S) \). So by (58) we have

\[
\alpha_q \leq \frac{(1 + \varepsilon)p \omega(q)}{\theta - \varepsilon \omega(S)} \leq (1 + 10^d \varepsilon)p\theta^{-1} \frac{\hat{f}_x(q)}{\hat{f}_x(S)} = (1 + 10^d \varepsilon)p\theta^{-1} \frac{|X_q|}{|X'|}.
\]
By \((49), (57)\) and Proposition \(7.1\) we get
\[
\alpha_q \leq (1 + 10^d \varepsilon) p \frac{\mu(\mathbb{F}_p^d) |X_q|}{f(\mathbb{F}_p^d)} \leq \frac{1 + 10^d \varepsilon}{1 + \varepsilon/2} |X_q| \leq (1 - \varepsilon/10) |X_q|.
\]

Finally, since the flag decomposition \(\Phi\) is \((T, \varepsilon, \delta)\)-complete, the element \(x\) is \((T, \delta)\)-complete: for any linear function \(\xi : V_x \to \mathbb{F}_p\) which is not constant on \(\{0\} \times \mathbb{F}_p^t\), the function \(f_{\leq x}\) is \((T, \delta)\)-thick along \(\xi\). Since \(f_{\leq x}\) is the characteristic function of \(X'\), the same condition holds for the set \(X'\) as well. Let \(\delta' = \min\{\mu, \varepsilon/10, \delta\}\) and observe that the collection of sets \(X_q\) and coefficients \(\alpha_q, q \in S\), satisfy the conditions of Theorem \(5.5\). For the theorem to apply we need to make sure that \(T > T_0(d', t, K, \delta')\) and \(p > p_0(d', t, K, \delta')\). Recall that Theorem \(4.9\) implies that we have \(T > g(K)\) where the function \(g\) can grow arbitrarily fast depending on parameters \(d\) and \(\varepsilon\).

Note that in our situation we have \(\delta' \gg_{K, d, \varepsilon} 1\) and so the function \(T_0(d', t, K, \delta')\) is bounded in terms of \(K, d, \varepsilon\) and therefore there exists a function \(g = g_{d, \varepsilon}\) such that \(g(K) > T_0(d', t, K, \delta')\).

At the start of the proof, we pick the function \(g\) so that this condition holds.

So we conclude that all the necessary conditions of Theorem \(5.5\) are satisfied. So the set \(X' \subset X\) contains \(p\) distinct elements with zero sum. Theorem \(1.2\) is proved.

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