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The number of independent Traces and Supertraces on the Symplectic Reflection Algebra $H_{1,\eta}(\Gamma\backslash S_N)$

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Symplectic reflection algebra $H_{1,\eta}(G)$ has a $T(G)$-dimensional space of traces whereas, when considered as a superalgebra with a natural parity, it has an $S(G)$-dimensional space of supertraces. The values of $T(G)$ and $S(G)$ depend on the symplectic reflection group $G$ and do not depend on the parameter $\eta$.

In this paper, the values $T(G)$ and $S(G)$ are explicitly calculated for the groups $G = \Gamma \wr S_N$, where $\Gamma$ is a finite subgroup of $Sp(2,\mathbb{C})$.

1. Introduction

Let $V := \mathbb{C}^{2N}$, let $G \subset Sp(2N,\mathbb{C})$ be a finite group generated by symplectic reflections. In [11], it was shown that Symplectic Reflection Algebra $H_{1,\eta}(G)$ has $T(G)$ independent traces, where $T(G)$ is the number of conjugacy classes of elements without eigenvalue 1 belonging to the group $G \subset Sp(2N) \subset \text{End}(V)$, and that the algebra $H_{1,\eta}(G)$, considered as a superalgebra with a natural parity, has $S(G)$ independent supertraces, where $S(G)$ is the number of conjugacy classes of elements without eigenvalue $-1$ belonging to $G \subset Sp(2N) \subset \text{End}(V)$. Hereafter, speaking about spectrum, eigenvalues and eigenvectors, the rank of an element of the group algebra $\mathbb{C}[G]$ of the group $G$, etc., we have in mind the representation of the group algebra $\mathbb{C}[G]$ in the space $V$. Besides, we denote all the units in groups, algebras, etc., by 1, and $c\cdot1$ by $c$ for any number $c$.

Apart from a few cases, there are two families of groups generated by symplectic reflections, see [7] and also [9], [2], [5]:

Family 1): $G$ is a complex reflection group acting on $\mathfrak{r} \oplus \mathfrak{r}^*$, where $\mathfrak{r}$ is the space of reflection representation. In this case, $G$ is a direct product of several groups from the following set of Coxeter groups

$$A_n (n \geq 1), \quad B_n = C_n (n \geq 2), \quad D_n (n \geq 3), \quad E_6, \quad E_7, \quad E_8, \quad F_4, \quad G_2, \quad H_3, \quad H_4, \quad I_2(n) (n \geq 5, n \neq 6). \quad (1.1)$$

Family 2): $G = \Gamma \wr S_N$, which means here $G = \Gamma^N \rtimes S_N$ acting on $(\mathbb{C}^2)^N$, where $\Gamma$ is a finite subgroup of $Sp(2,\mathbb{C})$. 

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For groups $G$ from the set (1.1), the list of values $T(G)$ and $S(G)$ is given in [10].

In this work, we give the values of $T(G)$ and $S(G)$ for the 2nd family. Namely, we found the generating functions

$$t(\Gamma, x) := \sum_{N=0}^{\infty} T(\Gamma \wr S_N) x^N,$$

(1.2)

$$s(\Gamma, x) := \sum_{N=0}^{\infty} S(\Gamma \wr S_N) x^N$$

(1.3)

for each finite subgroup $\Gamma \subset Sp(2, \mathbb{C})$, see Theorem 5.1.

All needed definitions are given in the Section 2; the structure, conjugacy classes and characteristic polynomials of the groups $\Gamma \wr S_N$ are described in Section 3.

To include the case $N = 0$ in consideration in formulas (1.2)–(1.3), it is natural to set $\Gamma \wr S_0 := \{E\}$ and, since dim $V = 0$, to set $H_{1, \eta}(\Gamma \wr S_0) := \mathbb{C}[\{E\}]$, where $\{E\}$ is the group containing only one element $E$.

Applying the definitions given in Section 2 to the algebra $H_{1, \eta}(\Gamma \wr S_0)$ we deduce that

a) if the algebra $H_{1, \eta}(\Gamma \wr S_0) = \mathbb{C}[\{E\}]$ is considered as superalgebra, it has only a trivial parity $\pi \equiv 0$;

b) the algebra $H_{1, \eta}(\Gamma \wr S_0) = \mathbb{C}[\{E\}]$ has 1-dimensional space of traces and 1-dimensional space of supertraces; these spaces coincide;

c) it is natural to set $T(\Gamma \wr S_0) = S(\Gamma \wr S_0) = 1$;

d) the algebra $H_{1, \eta}(\Gamma \wr S_0) = \mathbb{C}[\{E\}]$ contains two Klein operators (i.e., elements satisfying conditions (2.1)–(2.3)), namely, 1 and $-1$.

2. Preliminaries

2.1. Traces

Let $\mathcal{A}$ be an associative superalgebra with parity $\pi$. All expressions of linear algebra are given for homogenous elements only and are supposed to be extended to inhomogeneous elements via linearity.

A linear complex-valued function $str$ on $\mathcal{A}$ is called a supertrace, if

$$str(fg) = (-1)^{\pi(f)\pi(g)} str(gf) \text{ for all } f, g \in \mathcal{A}.$$ 

A linear complex-valued function $tr$ on $\mathcal{A}$ is called a trace, if

$$tr(fg) = tr(gf) \text{ for all } f, g \in \mathcal{A}.$$ 

The element $K \in \mathcal{A}$ is called a Klein operator, if

$$\pi(K) = 0,$$

(2.1)

$$K^2 = 1,$$

(2.2)

$$K f = (-1)^{\pi(f)} f K \text{ for all } f \in \mathcal{A}.$$ 

(2.3)

Any Klein operator, if exists, establishes an isomorphism between the space of traces on $\mathcal{A}$ and the space of supertraces on $\mathcal{A}$.
Namely, if \( f \mapsto \text{tr}(f) \) is a trace, then \( f \mapsto \text{tr}(fK^{1+i\pi(f)}) \) is a supertrace, and if \( f \mapsto \text{str}(f) \) is a supertrace, then \( f \mapsto \text{str}(fK^{1+i\pi(f)}) \) is a trace.

### 2.2. Symplectic reflection group

Let \( V = \mathbb{C}^{2N} \) be endowed with a non-degenerate anti-symmetric \( \text{Sp}(2N) \)-invariant bilinear form \( \omega(\cdot, \cdot) \), let the vectors \( e_i \in V \), where \( i = 1, \ldots, 2N \), constitute a basis in \( V \).

The matrix \( (\omega_{ij}) := \omega(e_i, e_j) \) is anti-symmetric and non-degenerate.

Let \( x^i \) be the coordinates of \( x \in V \), i.e., \( x = e_i x^i \). Then \( \omega(x, y) = \omega_{ij} x^i y^j \) for any \( x, y \in V \). The indices \( i \) are lowered and raised by means of the forms \( (\omega_{ij}) \) and \( (\omega^{ij}) \), where \( \omega_{ij} \omega^{ij} = \delta^j_i \).

**Definition 2.1.** The element \( R \in \text{Sp}(2N) \subset \text{End} V \) is called a **symplectic reflection**, if \( \text{rank}(R-1) = 2 \).

**Definition 2.2.** Any finite subgroup \( G \) of \( \text{Sp}(2N) \) generated by a set of symplectic reflections is called a **symplectic reflection group**.

In what follows, \( G \) stands for a symplectic reflection group, and \( \mathcal{G} \) stands for the set of all symplectic reflections in \( G \).

Let \( R \in \mathcal{G} \). Set

\[
V_R := \text{Im}(R-1),
\]

\[
Z_R := \text{Ker}(R-1). \tag{2.4}
\]

Clearly, \( V_R \) and \( Z_R \) are symplectically perpendicular, i.e., \( \omega(V_R, Z_R) = 0 \), and \( V = V_R \oplus Z_R \).

So, let \( x = x_{v_R} + x_{z_R} \) for any \( x \in V \), where \( x_{v_R} \in V_R \) and \( x_{z_R} \in Z_R \). Set

\[
\omega_R(x, y) := \omega(x_{v_R}, y_{v_R}). \tag{2.5}
\]

### 2.3. Symplectic reflection algebra (following [3])

Let \( \mathbb{C}[G] \) be the **group algebra** of \( G \), i.e., the set of all linear combinations \( \sum_{g \in G} \alpha_g \bar{g} \), where \( \alpha_g \in \mathbb{C} \).

If we were rigorous, we would write \( \bar{g} \) to distinguish \( g \) considered as an element of \( G \subset \text{End}(V) \) from the same element \( \bar{g} \in \mathbb{C}[G] \) considered as an element of the group algebra. The addition in \( \mathbb{C}[G] \) is defined as follows:

\[
\sum_{g \in G} \alpha_g \bar{g} + \sum_{g \in G} \beta_g \bar{g} = \sum_{g \in G} (\alpha_g + \beta_g) \bar{g}
\]

and the multiplication is defined by setting \( \bar{g}_1 \bar{g}_2 = \bar{g}_1 g_2 \). In what follows, however, we abuse notation and omit the bar sign over elements of the group algebra.

Let \( \eta \) be a function on \( \mathcal{G} \), i.e., a set of constants \( \eta_R \in \mathbb{R} \) such that \( \eta_{R_1} = \eta_{R_2} \), if \( R_1 \) and \( R_2 \) belong to one conjugacy class of \( G \).

**Definition 2.3.** The algebra \( H_{t, \eta}(G) \), where \( t \in \mathbb{C} \), is an associative algebra with unit 1; it is the algebra \( \mathbb{C}[V] \) of (noncommutative) polynomials in the elements of \( V \) with coefficients in the group algebra \( \mathbb{C}[G] \) subject to the relations

\[
gx = g(x)g \quad \text{for any} \quad g \in G \text{ and } x \in V, \quad \text{where} \quad g(x) = e_i g^j x^j \quad \text{for} \quad x = e_ix^i; \tag{2.7}
\]

\[
[x, y] = t\omega(x, y) + \sum_{R \in \mathcal{G}} \eta_R \omega_R(x, y) R \quad \text{for any} \quad x, y \in V. \tag{2.8}
\]

The algebra \( H_{t, \eta}(G) \) is called a **symplectic reflection algebra**, see [3].
The commutation relations (2.8) suggest to define the parity $\pi$ by setting:

$$\pi(x) = 1, \quad \pi(g) = 0 \quad \text{for any } x \in V, \text{ and } g \in G,$$

enabling one to consider $H_{1, \eta}(G)$ as an associative superalgebra.

We consider the case $t \neq 0$ only, for any such $t$ it is equivalent to the case $t = 1$.

Let $A$ and $B$ be superalgebras such that $A$ is a $B$-module. We say that the superalgebra $A \ast B$ is a crossed product of $A$ and $B$, if $A \ast B = A \otimes B$ as a superspace and

$$(a_1 \otimes b_1) \ast (a_2 \otimes b_2) = a_1 b_1(a_2) \otimes b_1 b_2,$$

see [14]. The element $b_1(a_2)$ may include a sign factor imposed by the Sign Rule, see [1], p. 45.

The (super)algebra $H_{1, \eta}(G)$ is a deformation of the crossed product of the Weyl algebra $W_N$ and the group algebra of a finite subgroup $G \subset \text{Sp}(2N)$ generated by symplectic reflections.

2.4. The number of independent traces and supertraces on the symplectic reflection algebras

Theorem 2.1 ([11]). Let the symplectic reflection group $G \subset \text{End}(V)$ have $T_G$ conjugacy classes without eigenvalue 1 and $S_G$ conjugacy classes without eigenvalue $-1$.

Then the algebra $H_{1, \eta}(G)$ has $T_G$ independent traces whereas $H_{1, \eta}(G)$ considered as a superalgebra, see (2.9), has $S(G) = S_G$ independent supertraces.

Proposition 2.1. Let $G_1 \subset \text{End}(V_1)$, $G_2 \subset \text{End}(V_2)$ and $G = G_1 \times G_2 \subset \text{End}(V_1 \oplus V_2)$ be symplectic reflection groups. Then $T(G) = T(G_1)T(G_2)$ and $S(G) = S(G_1)S(G_2)$.

Proof follows from evident relations $T_G = T_{G_1} T_{G_2}$, $S_G = S_{G_1} S_{G_2}$ and Theorem 2.1.

Proposition 2.2. If there exists a $K \in G$ such that $K|_V = -1$, then $K$ is a Klein operator.

3. The group $\Gamma \setminus S_N$

3.1. Finite subgroups of $\text{Sp}(2, \mathbb{C})$

The complete list of the finite subgroups $\Gamma \subset \text{Sp}(2, \mathbb{C})$ is as follows, see, e.g., [15]:

| $\Gamma$                  | Order | Presence of $-1$                  | The number of conjugacy classes $C(\Gamma)$ |
|---------------------------|-------|-----------------------------------|--------------------------------------------|
| Cyclic group $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ | $n$   | yes, if $n$ is even; no, if $n$ is odd | $n$                                        |
| Binary dihedral group $\mathbb{D}_n$                  | $4n$  | yes                              | $n + 3$                                    |
| Binary tetrahedral group $\mathbb{I}$                  | 24    | yes                              | 7                                          |
| Binary octahedral group $\mathbb{O}$                  | 48    | yes                              | 8                                          |
| Binary icosahedral group $\mathbb{I}$                  | 120   | yes                              | 9                                          |

It is easy to see that each of these groups, except $\mathbb{Z}_{2k+1}$, has $C(\Gamma) - 1$ conjugacy classes without $+1$ in the spectrum and has $C(\Gamma) - 1$ conjugacy classes without $-1$ in the spectrum. The group $\mathbb{Z}_{2k+1}$ has $C(\mathbb{Z}_{2k+1}) - 1$ conjugacy classes without $+1$ in the spectrum and it has $C(\mathbb{Z}_{2k+1})$ conjugacy classes without $-1$ in the spectrum.
3.2. Symplectic reflections in $\Gamma \wr S_N$ (following [4])

Let $V = \mathbb{C}^{2N}$ and let the symplectic form $\omega$ have the shape

$$\omega := \begin{pmatrix} \varpi & \varpi & \cdots & \varpi \\ \varpi & \cdots & \cdots & \varpi \\ \vdots & \vdots & \ddots & \vdots \\ \varpi & \cdots & \cdots & \varpi \end{pmatrix},$$

where $\varpi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

The elements of the group $\Gamma \wr S_N$ have the form of $N \times N$ block matrix with $2 \times 2$ blocks. Consider the following elements of $\Gamma \wr S_N$

$$(D_{g,i})_{kl} := \begin{cases} g, & \text{if } k = l = i, \\ 1, & \text{if } k = l \neq i, \\ 0, & \text{otherwise}, \end{cases}$$

$$(K_{ij})_{kl} := \begin{cases} \delta_{kl}, & \text{if } k, l \neq i, k, l \neq j, \\ \delta_{ki} \delta_{lj} + \delta_{kj} \delta_{li}, & \text{otherwise}, \end{cases}$$

$$S_{g,ij} := D_{g,i}D_{g^{-1},j}K_{ij},$$

where $i, j = 1, \ldots, N, i \neq j, 1 \neq g \in \Gamma$. It is clear that $K_{ij} = K_{ji}$ and $S_{g,ij} = S_{g^{-1},ji}$.

The complete set of symplectic reflections in $\Gamma \wr S_N$ consists of $D_{g,i}, K_{ij}$ and $S_{g,ij}$, where $1 \leq i < j \leq N$ and $1 \neq g \in \Gamma$. This set generates the group $\Gamma \wr S_N$.

The symplectic reflections $K_{ij}$ and $S_{g,ij}$ lie in one conjugacy class for all $i \neq j$ and $g \neq 1$; the elements $D_{g,i}$ ($g \neq 1$) and $D_{h,j}$ ($h \neq 1$) lie in one conjugacy class, if $g$ and $h$ are conjugate in $\Gamma$. So, the algebra $H_{1,\eta}(\Gamma \wr S_N)$ depends on $C(\Gamma)$ parameters $\eta$, if $N \geq 2$, and on $C(\Gamma) - 1$ parameters, if $N = 1$. Here $C(\Gamma)$ is the number of conjugacy classes in $\Gamma$ including the class $\{1\}$.

3.3. Conjugacy classes (following [13])

Further, the elements of the group $\Gamma \wr S_N$ can be represented in the form $D\sigma$ where $D \in \Gamma^N$ is a diagonal $N \times N$ block matrix, each block being a $2 \times 2$-matrix, and $\sigma$ is $N \times N$ block matrix of permutation each block being a $2 \times 2$-matrix.

The product has the form:

$$(D_1 \sigma_1)(D_2 \sigma_2) = D_3 \sigma_3$$

where $\sigma_3 = \sigma_1 \sigma_2$ and $D_3 = D_1 \sigma_1 D_2 \sigma_1^{-1}$. 
Fix an element \( g_0 = D_0 \sigma_0 \). Since the permutation \( \sigma_0 \) is a product of cycles, there exists a permutation \( \sigma' \) such that
\[
\sigma' \sigma_0 (\sigma')^{-1} = \begin{pmatrix}
    c_1 \\
    c_2 \\
    \vdots \\
    c_y
\end{pmatrix}
\]
where \( c_k \) are the cycles of length \( L_k \), \( \sum_k L_k = N \), \( c_k = \begin{pmatrix}
    0 & 1 & 0 & \cdots & 0 \\
    0 & 0 & 1 & \cdots & 0 \\
    0 & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & 1 \\
    1 & 0 & 0 & \cdots & 0
\end{pmatrix} \) (3.5)

The element \( \sigma'D_0 \sigma_0 (\sigma')^{-1} \) has the form
\[
\sigma'D_0 \sigma_0 (\sigma')^{-1} = \begin{pmatrix}
    D_1 c_1 \\
    D_2 c_2 \\
    \vdots \\
    D_s c_s
\end{pmatrix}
\]

where \( D_k \) is an \( L_k \times L_k \) diagonal block matrix, each block being a \( 2 \times 2 \)-matrix:
\[
D_k = \begin{pmatrix}
    g^k_1 \\
    g^k_2 \\
    \vdots \\
    g^k_{L_k}
\end{pmatrix}, \quad g^k_i \in \Gamma.
\]

Next, consider diagonal block matrices \( H_k = \text{diag}(h^k_1, h^k_2, \ldots, h^k_{L_k}) \) and the elements
\[
H_k D_k c_k H_k^{-1} = \begin{pmatrix}
    h^k_1 g^k_1 \left( h^k_2 \right)^{-1} & h^k_2 g^k_2 \left( h^k_3 \right)^{-1} & \cdots & h^k_{L_k} g^k_{L_k} \left( h^k_1 \right)^{-1} \\
    h^k_2 g^k_2 \left( h^k_3 \right)^{-1} & h^k_3 g^k_3 \left( h^k_4 \right)^{-1} & \cdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    h^k_{L_k} g^k_{L_k} \left( h^k_1 \right)^{-1} & \cdots & \cdots & h^k_{L_k} g^k_{L_k} \left( h^k_{L_k} \right)^{-1}
\end{pmatrix} c_k.
\]

For any element \( h^k_1 \in \Gamma \), one can choose
\[
h^k_2 = h^k_1 g^k_1, \quad h^k_3 = h^k_2 g^k_2, \ldots, \quad h^k_{L_k} = h^k_{L_k-1} g^k_{L_k-1}
\]
such that
\[
H_k D_k c_k H_k^{-1} = \begin{pmatrix}
    1 \\
    1 \\
    \vdots \\
    h^k_1 g^k_1 g^k_2 \cdots g^k_{L_k} \left( h^k_1 \right)^{-1}
\end{pmatrix} c_k.
\]
So, each conjugacy class of $\Gamma \wr S_N$ is described by the set of cycles in the decomposition (3.5), (3.6) of $\sigma_0$, where each cycle is marked by some conjugacy class of $\Gamma$.

The cycle of length $r$ marked by the conjugacy class $\alpha$ of $\Gamma$ with representative $g_\alpha \in \Gamma$ has the shape:

$$A_{\alpha,r} := \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ g_\alpha & g_\alpha & \ldots & g_\alpha \end{pmatrix}$$

where $\alpha = 1, \ldots, C(\Gamma)$ and $r = 1, 2, \ldots$; the matrix $A_{\alpha,r}$ and the cycle $c^r$ are the $r \times r$ block matrices, each block being a $2 \times 2$ matrix.

So, each element $g \in \Gamma \wr S_N$ is conjugate to the element of the shape

$$\begin{pmatrix} A_{\alpha_1,r_1} & & \\ & A_{\alpha_2,r_2} & \\ & & \ddots \\ & & & A_{\alpha_r,r_r} \end{pmatrix}$$

It is convenient to describe the conjugacy class of $\Gamma \wr S_N$ with representative (3.8) by the set of nonnegative integers $p^r_{\alpha}$, where $r = 1, 2, 3, \ldots$, and $\alpha = 1, \ldots, C(\Gamma)$, such that

$$\sum_{\alpha,r} r p^r_{\alpha} = N.$$

The value $p^r_{\alpha}$ for some conjugacy class is the number of cycles $A_{\alpha,r}$ of length $r$ in the decomposition (3.8) marked by the conjugacy class $\alpha$ of the group $\Gamma$.

Note that in [13] the notation $m_r(\alpha)$ is used instead of $p^r_{\alpha}$ we use in this paper.

The restriction (3.9) can be omitted and can serve as definition of $N$ for each set of the numbers $p^r_{\alpha}$.

The number of conjugacy classes in $\Gamma \wr S_N$ is equal to

$$C(\Gamma \wr S_N) = \sum_{\sum p^r_{\alpha} = N} 1.$$

The generating function $c(\Gamma, x)$ of the number of conjugacy classes is defined as

$$c(\Gamma, x) := \sum_{N=0}^{\infty} C(\Gamma \wr S_N) x^N$$

and is equal to

$$c(\Gamma, x) = \sum_{p^r_{\alpha}} x^{\sum p^r_{\alpha}} = \sum_{p^r_{\alpha}=0}^{\infty} \prod_{r=1}^{\infty} \prod_{\alpha=1}^{C(\Gamma)} (x^r)^{p^r_{\alpha}} = \prod_{r=1}^{\infty} \prod_{\alpha=1}^{C(\Gamma)} \frac{1}{1-x^r} = (\Psi(x))^{C(\Gamma)},$$

where $\Psi(x)$ is the Euler function

$$\Psi(x) := \prod_{r=1}^{\infty} \frac{1}{1-x^r}.$$
3.4. Characteristic polynomials of conjugacy classes

Before seeking the generating functions \( t(\Gamma, x) \) and \( s(\Gamma, x) \), let us find the characteristic polynomial of the conjugacy class \( g \) of \( \Gamma \wr S_N \) identified by the set \( p^\alpha \).

Let \( P_M(\lambda) := \det(M - \lambda I) \) be the characteristic polynomial of the matrix \( M \). Then it is easy to see that

\[
P_{A_{\alpha,r}}(\lambda) = \det(A_{\alpha,r} - \lambda) = \det(g_{\alpha} - \lambda^r) = P_{g_{\alpha}}(\lambda^r),
\]

where the marked cycle \( A_{\alpha,r} \) is defined by (3.7).

Let \( g \in \Gamma \wr S_N \) be defined by Eq. (3.8).

Now, it is easy to show that

\[
P_{g}(\lambda) = \det(g - \lambda) = \prod_{\alpha,r} \det(A_{\alpha,r} - \lambda)
= \prod_{\alpha,r: p_{\alpha,r}^r \geq 1} (\det(g_{\alpha} - \lambda^r))^r,
\]

(3.10)

if \( g \) is a representative of the conjugacy class in \( \Gamma \wr S_N \) corresponding to the set \( p^\alpha \).

**Definition 3.1.** We call a conjugacy class \( t \)-admissible, if its representative \( g \in \Gamma \wr S_N \) is such that \( P_{g}(1) \neq 0 \).

**Definition 3.2.** We call a conjugacy class \( s \)-admissible, if its representative \( g \in \Gamma \wr S_N \) is such that \( P_{g}(-1) \neq 0 \).

**Definition 3.3.** We call a marked cycle \( A_{\alpha,r} \), see Eq. (3.7), \( t \)-admissible, if \( P_{A_{\alpha,r}}(1) \neq 0 \).

**Definition 3.4.** We call a marked cycle \( A_{\alpha,r} \), see Eq. (3.7), \( s \)-admissible, if \( P_{A_{\alpha,r}}(-1) \neq 0 \).

Equation (3.10) implies the following statements:

**Proposition 3.1.** The conjugacy class of \( \Gamma \wr S_N \) identified by the set \( p^\alpha \) is \( t \)-admissible, if and only if the marked cycle \( A_{\alpha,r} \) is \( t \)-admissible for any pair \( \alpha, r \) such that \( p_{\alpha,r}^r \neq 0 \).

**Proposition 3.2.** The conjugacy class of \( \Gamma \wr S_N \) identified by the set \( p^\alpha \) is \( s \)-admissible, if and only if the marked cycle \( A_{\alpha,r} \) is \( s \)-admissible for any pair \( \alpha, r \) such that \( p_{\alpha,r}^r \neq 0 \).

Recall that \( g_{\alpha} \in \Gamma \subset Sp(2, \mathbb{C}) \), where \( \Gamma \) is a finite group. So \( \det g_{\alpha} = 1 \) and the Jordan normal form of \( g_{\alpha} \) is diagonal. This implies that if \( g_{\alpha} \) has \( +1 \) in its spectrum, then \( g_{\alpha} = 1 \) and if \( g_{\alpha} \) has \(-1 \) in its spectrum, then \( g_{\alpha} = -1 \). These facts together with Eq. (3.10) imply, in their turn, the following two propositions:

**Proposition 3.3.** The conjugacy class of \( \Gamma \wr S_N \) identified by the set \( p^\alpha \) is \( t \)-admissible if and only if \( g_{\alpha} \neq 1 \) for all \( \alpha, r \) with \( p_{\alpha,r}^r \neq 0 \).

**Proposition 3.4.** The conjugacy class of \( \Gamma \wr S_N \) identified by the set \( p^\alpha \) is \( s \)-admissible if and only if for any pair \( \alpha, r \) such that \( p_{\alpha,r}^r \neq 0 \), at least one of the next three conditions holds:

a) \( g_{\alpha} \neq -1 \) and \( g_{\alpha} \neq 1 \),
b) \( r \) is even and \( g_\alpha = -1 \).

c) \( r \) is odd and \( g_\alpha = 1 \).

Note that the three sets of pairs \((r, \alpha)\) defined by the cases a), b), c) in Proposition 3.4 have empty pair-wise intersections.

**Definition 3.5.** Let \( t_r(\Gamma) \) for \( r = 1, 2, \ldots \) be equal to the number of different \( \alpha \) such that \( A_{\alpha, r} \) is \( t \)-admissible.

Evidently, \( t_r(\Gamma) = C(\Gamma) - 1 \). \hspace{1cm} (3.11)

**Definition 3.6.** Let \( s_r(\Gamma) \) for \( r = 1, 2, \ldots \) be equal to the number of different \( \alpha \) such that \( A_{\alpha, r} \) is \( s \)-admissible.

Evidently, if \( \Gamma \ni -1 \), then \( s_r(\Gamma) = C(\Gamma) - 1 \). \hspace{1cm} (3.12)

and if \( \Gamma \not\ni -1 \), then

\[
s_r(\Gamma) = \begin{cases} 
C(\Gamma) - 1, & \text{if } r \text{ is even}, \\
C(\Gamma), & \text{if } r \text{ is odd}.
\end{cases} \hspace{1cm} (3.13)
\]

4. Combinatorial problem

Consider the following combinatorial problem (analogous problems are considered in [8]).

Suppose we have an unlimited supply of 1-gram colored weights for each of \( n_1 \) different colors, an unlimited supply of 2-gram colored weights for each of \( n_2 \) different colors, an unlimited supply of 3-gram colored weights for each of \( n_3 \) different colors, and so on. Let \( a_{n_1, \ldots, n_k}^N \) be the number of opportunities to choose weights from our set of total mass \( N \) grams.

The problem is to find generating function

\[
F_{n_1, \ldots, n_k, \ldots}(x) := \sum_{N=0}^{\infty} a_{n_1, \ldots, n_k, \ldots}^N x^N.
\]

This problem is exactly the problem we discussed earlier. Namely, now we say “\( r \)-gram weight” instead of cycle of length \( r \), and “the number of different colors \( n_r \)” instead of the number \( t_r \) (3.11) or \( s_r \) (3.12)–(3.13) of different \( \alpha \).

**Proposition 4.1.**

\[
F_{n_1+m_1, n_2+m_2, \ldots, n_k+m_k, \ldots}(x) = F_{n_1, n_2, \ldots, n_k, \ldots}(x) \cdot F_{m_1, m_2, \ldots, m_k, \ldots}(x).
\]

**Proof.** To prove this proposition, it suffices to note that

\[
a_{n_1+m_1, n_2+m_2, \ldots, n_k+m_k, \ldots}^N = \sum_{M=0}^{N} a_{n_1, n_2, \ldots, n_k, \ldots}^M \cdot a_{m_1, m_2, \ldots, m_k, \ldots}^{N-M}.
\]

□
Introduce the functions
\[ f_i := F_{n_1^i, n_2^i, \ldots, n_k^i}, \quad \text{where } n_k^i = \delta_k^i. \]
Then
\[ f_i(x) = 1 + x^i + x^{2i} + x^{3i} + \ldots = \frac{1}{1-x^i}. \]

The next theorem follows from Proposition 4.1

**Theorem 4.1.**
\[ F_{n_1, n_2, \ldots, n_k, \ldots} = \prod_{i=1}^{\infty} (f_i)^{n_i}. \]

The function \( F_{1,1,\ldots} = \Psi(x) \) is the well-known Euler function, the generating function of the number of partitions of \( N \) into the sum of positive integers.

### 5. Generating functions \( t(\Gamma) \) and \( s(\Gamma) \)

**Theorem 5.1.** Set
\[ \Psi(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}, \quad (\text{Euler function}), \quad (5.1) \]
\[ \Phi(x) := \prod_{k=0}^{\infty} \frac{1}{1-x^{2k+1}}. \quad (5.2) \]

Let \( T(\Gamma \wr S_N) \) be the dimension of the space of traces on \( H_{\Gamma \wr S_N} \) and let \( S(\Gamma \wr S_N) \) be the dimension of the space of supertraces on \( H_{\Gamma \wr S_N} \) considered as a superalgebra.

Let
\[ t(\Gamma, x) := \sum_{N=0}^{\infty} T(\Gamma \wr S_N)x^N \quad \text{and} \quad s(\Gamma, x) := \sum_{N=0}^{\infty} S(\Gamma \wr S_N)x^N. \]

Then
\[ t(\Gamma, x) = (\Psi(x))^{C(\Gamma)-1}, \]
\[ t(\Gamma, x) = (\Psi(x))^{C(\Gamma)-1}, \quad \text{if } \Gamma \neq \mathbb{Z}_{2k+1}, \]
\[ t(\Gamma, x) = (\Psi(x))^{C(\Gamma)-1} \Phi(x), \quad \text{if } \Gamma = \mathbb{Z}_{2k+1}. \]

**Proof.** To prove Theorem 5.1, we apply Theorem 4.1 to the numbers (3.11)–(3.13) of admissible conjugacy classes. It is clear that
\[ t(\Gamma) = F_{t_1(\Gamma), t_2(\Gamma), \ldots} = \Psi^{C(\Gamma)-1}, \]
\[ s(\Gamma) = F_{s_1(\Gamma), s_2(\Gamma), s_3(\Gamma), \ldots} = \begin{cases} \Psi^{C(\Gamma)-1}, & \text{if } \Gamma \ni -1, \\ \Psi^{C(\Gamma)-1}\Phi, & \text{if } \Gamma \not\ni -1. \end{cases} \]

Observe that \( \Phi(x) = \sum_{i=0}^{\infty} O_N x_N \), where \( O_N \) is the number of partitions of \( N \) into the sum of odd positive integers, and \( O_N \) coincides with the number of independent supertraces on \( H_{1\eta}(S_N) \), see [12].
5.1. Inequality theorem

Theorem 5.2. Let \( G = \Gamma \wr S_N \). For each positive integer \( N \), the following statements hold:

\[
S(G) > 0, \\
S(G) \geq T(G), \\
S(G) = T(G) \text{ if and only if } H_{1,\eta}(G) \text{ contains a Klein operator.}
\]

Literally the same statements were proved for the groups \( G \) from Family 1) in [10], and hence these statements hold for the direct product of any finite number of groups from Family 1) and Family 2) defined on page 1.

Proof. Let \( \Gamma \neq Z_{2k+1} \). Since each finite group \( \Gamma \in Sp(2N, \mathbb{C}) \), except \( \Gamma = Z_{2k+1} \), contains \(-1\), the group \( \Gamma \wr S_N \) contains Klein operator \( K = \prod_{i=1}^{N} D_{-1,i} \).

There is no Klein operator in \( H_{1,\eta}(Z_{2k+1} \wr S_N) \) since for this algebra, \( S(Z_{2k+1} \wr S_N) > T(Z_{2k+1} \wr S_N) \), as it follows from Theorem 5.1.

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