An Ehrenfeucht-Fraïssé Game Approach to Collapse Results in Database Theory

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Abstract

We present a new Ehrenfeucht-Fraïssé game approach to collapse results in database theory. We show that, in principle, every natural generic collapse result may be proved via a translation of winning strategies for the duplicator in an Ehrenfeucht-Fraïssé game. Following this approach we can deal with certain infinite databases where previous, highly involved methods fail. We prove the natural generic collapse for \( \mathbb{Z} \)-embeddable databases over any linearly ordered context structure with arbitrary monadic predicates, and for \( \mathbb{N} \)-embeddable databases over the context structure \( \langle \mathbb{R}, <, +, \text{Groups}, \text{Mon}_Q \rangle \), where \text{Groups} is the collection of all subgroups of \( \langle \mathbb{R}, + \rangle \) that contain the set of integers and \text{Mon}_Q is the collection of all subsets of a particular infinite set \( Q \) of natural numbers. This, in particular, implies the collapse for arbitrary databases over \( \langle \mathbb{N}, <, +, \text{Mon}_Q \rangle \) and for \( \mathbb{N} \)-embeddable databases over \( \langle \mathbb{R}, <, +, \mathbb{Z}, Q \rangle \). I.e., first-order logic with \( < \) can express the same order-generic queries as first-order logic with \( <, +, \) etc.

Restricting the complexity of the formulas that may be used to formulate queries to Boolean combinations of purely existential first-order formulas, we even obtain the collapse for \( \mathbb{N} \)-embeddable databases over any linearly ordered context structure with arbitrary predicates. Finally, we develop the notion of \( \mathbb{N} \)-representable databases, which is a natural generalization of the notion of finitely representable databases. We show that natural generic collapse results for \( \mathbb{N} \)-embeddable databases can be lifted to the larger class of \( \mathbb{N} \)-representable databases.

To obtain, in particular, the collapse result for \( \langle \mathbb{N}, <, +, \text{Mon}_Q \rangle \), we explicitly construct a winning strategy for the duplicator in the presence of the built-in addition relation \(+\). This, as a side product, also leads to an Ehrenfeucht-Fraïssé game proof of the theorem of Ginsburg and Spanier, stating that the spectra of FO\((<, +)\)-sentences are semi-linear.

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1 Introduction

One of the issues in database theory that have attracted much interest in recent years is the study of relational databases that are embedded in a fixed, infinite context structure. This occurs, e.g., in current applications such as spatial or temporal databases, where data are represented by (natural or real) numbers, and where databases can be modelled as constraint databases. For a recent comprehensive survey see [21].

In many applications the numerical values only serve as identifiers that are exchangeable. If this is the case, queries commute with any permutation of the context universe; such queries are called generic. If the context universe is linearly ordered, a query may refer to the ordering. In this setting it is more appropriate to consider queries which commute with every order-preserving (i.e.,
strictly increasing) mapping. Such queries are called \textit{order-generic}. A basic way of expressing order-generic queries is by first-order formulas that make use of the linear ordering and of the database relations.

It is a reasonable question whether the use of the additional predicates of the context structure allows first-order logic to express more order-generic queries than the linear ordering alone. In some situations this question can be answered “yes”, e.g., if the context structure is $\langle \mathbb{N}, <, +, \times \rangle$. In other situations the question must be answered “no”, e.g., if the context structure is $\langle \mathbb{N}, <, + \rangle$ — such results are then called \textit{collapse results}, because first-order logic with the additional predicates collapses to first-order logic with linear ordering alone. A recent and comprehensive overview of this area of research is given in [23].

In classical database theory attention usually is restricted to finite databases. In this setting Benedikt et al. [6] have obtained a strong collapse result: \textit{First-order logic has the natural generic collapse for finite databases over o-minimal context structures}. This means that if the context structure has a certain property called \textit{o-minimality}, then for every order-generic first-order formula $\varphi$ which uses the additional predicates, there is a formula with linear ordering alone which is equivalent to $\varphi$ on all finite databases. In [2] this result was generalized to context structures that have \textit{finite VC-dimension}, a property that, e.g., the structures $\langle \mathbb{N}, <, + \rangle$, $\langle \mathbb{Q}, <, + \rangle$, $\langle \mathbb{R}, <, +, \times, \text{Exp} \rangle$ have. The proofs for these results are rather involved; in particular the proof of [2] uses non-standard models and hyperfinite structures.

The present paper proposes a new \textit{Ehrenfeucht-Fra"issé game} approach to collapse results. We show that, in principle, every natural generic collapse result can be proved via a translation of winning strategies for the duplicator in an Ehrenfeucht-Fra"issé game. Following this approach we can deal with certain \textit{infinite} databases where previous, highly involved methods fail. We prove the natural generic collapse for $\mathbb{Z}$-embeddable databases over linearly ordered context structures with arbitrary \textit{monadic} predicates, and for $\mathbb{N}$-embeddable databases over the context structure $\langle \mathbb{R}, <, +, \text{Mon}_Q, \text{Groups} \rangle$, where $\text{Groups}$ is the collection of all subgroups of $\langle \mathbb{R}, + \rangle$ that contain the set of integers and $\text{Mon}_Q$ is the collection of all subsets of a particular infinite set $Q$ of natural numbers. This, in particular, implies the collapse for arbitrary databases over $\langle \mathbb{N}, <, +, \text{Mon}_Q \rangle$ and for $\mathbb{N}$-embeddable databases over $\langle \mathbb{R}, <, +, \mathbb{Z}, \mathbb{Q} \rangle$. I.e., first-order logic with $<$ can express the same order-generic queries as first-order logic with $<, +$, etc.

Restricting the complexity of the formulas that may be used to formulate queries to Boolean combinations of purely existential first-order formulas, we also obtain the collapse for $\mathbb{N}$-embeddable databases over linearly ordered context structures with \textit{arbitrary} predicates.

Finally, we develop the notion of $\mathbb{N}$-\textit{representable} databases, which is a natural generalization of the notion of \textit{finitely representable} databases of [4] (also known as \textit{dense order constraint databases}). We show that natural generic collapse results for $\mathbb{N}$-embeddable databases can be lifted to the larger class of $\mathbb{N}$-representable databases.

Apart from the collapse results obtained with the method of the translation of winning strategies, the exposition of explicit strategies for the duplicator in the Ehrenfeucht-Fra"issé game is interesting in its own right. In particular, to obtain the collapse result for $\langle \mathbb{N}, <, +, \text{Mon}_Q \rangle$, we explicitly construct a winning strategy for the duplicator in the presence of the built-in addition relation $+$. This, as a side product, also leads to an Ehrenfeucht-Fra"issé game proof of the theorem of Ginsburg and Spanier, stating that the spectra of $\text{FO}(<, +)$-sentences are semi-linear.

The present paper contains results of the author’s dissertation [28]. It combines and extends the results of the conference contributions [22, 27]. The paper is structured as follows: In Section 2 we fix the basic notations used throughout the
Section 2 gives a brief introduction to collapse considerations in database theory, recalls known results of [6, 5, 2, 7, 8, 23], and summarizes the collapse results obtained in the present paper. In Section 3, we present the translation of strategies for the Ehrenfeucht-Fraïssé game as a new method for obtaining collapse results, and we show that, at least in principle, all natural generic collapse results can be proved by this method. In the Sections 5 and 6, we show that the translation of strategies is indeed possible for context structures with monadic built-in predicates and for context structures with the addition relation + and some particular monadic predicates, respectively. Restricting attention to Boolean combinations of purely existential first-order logic, we show in Section 7 that the translation of strategies is possible even for arbitrary context structures. Section 8 proves that natural generic collapse results for $\mathbb{N}$-embeddable databases can be lifted to the larger class of $\mathbb{N}$-representable databases. Finally, in Section 9, we summarize our results and point out further questions.

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2 Basic Notations

We use $\mathbb{Z}$ for the set of integers, $\mathbb{N} := \{0, 1, 2, \ldots\}$ for the set of natural numbers, $\mathbb{N}_{>0}$ for the set of positive natural numbers, $\mathbb{Q}$ for the set of rational numbers, $\mathbb{R}$ for the set of real numbers, and $\mathbb{R}_{\geq0}$ for the set of nonnegative real numbers.

For $a, b \in \mathbb{Z}$ we write $a \mid b$ to express that $a$ divides $b$, i.e., that $b = c \cdot a$ for some $c \in \mathbb{Z}$. For $n \in \mathbb{N}_{>0}$ the symbol $\equiv_n$ denotes the congruence relation modulo $n$, i.e., for $a, b \in \mathbb{Z}$ we have $a \equiv_n b$ iff $n \mid a - b$. This relation can be extended to real numbers $r, s \in \mathbb{R}$ via $r \equiv_n s$ iff $r - s = z \cdot n$ for some $z \in \mathbb{Z}$. For $r \in \mathbb{R}$ we write $\lfloor r \rfloor$ to denote the largest integer $\geq r$, and we write $\lceil r \rceil$ for the smallest integer $\geq r$. $|r|$ denotes the absolute value of $r$, i.e., $|r| = r$ if $r \geq 0$, and $-r$ otherwise. For $r, s \in \mathbb{R}$ we write $\text{int}[r, s]$ to denote the closed interval $\{x \in \mathbb{R} : r \leq x \leq s\}$. Analogously, we write $\text{int}(r, s)$ for the open interval $\{x \in \mathbb{R} : r < x < s\}$, $\text{int}[r, s)$ for the half open interval $\{x \in \mathbb{R} : r \leq x < s\}$, and $\text{int}(r, s]$ for the half open interval $\{x \in \mathbb{R} : r < x \leq s\}$. We write $a_1, \ldots, a_m \mapsto b_1, \ldots, b_m$ to denote the mapping $f$ with domain $\{a_1, \ldots, a_m\}$ and range $\{b_1, \ldots, b_m\}$ which satisfies $f(a_i) = b_i$ for all $i \in \{1, \ldots, m\}$. Depending on the particular context, we use $\vec{a}$ as abbreviation for a sequence $a_1, \ldots, a_m$ or a tuple $(a_1, \ldots, a_m)$. Accordingly, if $f$ is a mapping defined on all elements in $\vec{a}$, we write $f(\vec{a})$ to denote the sequence $(f(a_1), \ldots, f(a_m))$ or the tuple $(f(a_1), \ldots, f(a_m))$. If $R$ is an $m$-ary relation on the domain of $f$, we write $f(R)$ to denote the relation $\{f(\vec{a}) : \vec{a} \in R\}$. Instead of $\vec{a} \in R$ we often write $R(\vec{a})$.

A signature $\sigma$ consists of constant symbols and relation symbols. Each relation symbol $R \in \sigma$ has a fixed arity $\text{ar}(R) \in \mathbb{N}_{>0}$. Whenever we refer to some “$R \in \sigma$” we implicitly assume that $R$ is a relation symbol. Analogously, “$c \in \sigma$” means that $c$ is a constant symbol. Throughout this paper we adopt the convention that whenever a signature is denoted by the symbol $\tau$, then it is a finite set of relation symbols and constant symbols.

A $\sigma$-structure $\mathcal{A} = \langle A, \sigma^A \rangle$ consists of an arbitrary set $A$ which is called the universe of $\mathcal{A}$, and a set $\sigma^A$ that contains an interpretation $c^A \in A$ for each $c \in \sigma$, and an interpretation $R^A \subseteq A^{\text{ar}(R)}$ for each $R \in \sigma$. Sometimes we want to restrict our attention to $\sigma$-structures over a
particular universe \( U \). In these cases we speak of \( \langle U, \sigma \rangle \)-structures. An isomorphism \( \pi \) between two \( \sigma \)-structures \( A = \langle A, \sigma^A \rangle \) and \( B = \langle B, \sigma^B \rangle \) is a bijective mapping \( \pi : A \to B \) such that \( \pi(c^A) = c^B \) for each \( c \in \sigma \), and \( R^A(\bar{a}) \text{ iff } R^B(\pi(\bar{a})) \) for each \( R \in \sigma \). An automorphism of \( A \) is an isomorphism between \( A \) and \( A \). A partial isomorphism between \( A \) and \( B \) is a mapping \( \pi' : A' \to B' \) such that \( A' \subseteq A \) and \( B' \subseteq B \) contain all constants of \( A \) and \( B \) and \( \pi' \) is an isomorphism between the induced substructures obtained by restricting \( A \) and \( B \) to the universes \( A' \) and \( B' \).

We use the usual notation concerning first-order logic (cf., e.g., \([18, 11, 23]\)). In particular, we write \( \text{FO}(\sigma) \) to denote the set of all formulas of first-order logic over the signature \( \sigma \). Note that our notion of signatures does not include the use of function symbols. Therefore, when used in the context of formulas, arithmetic predicates such as +, \( \times \), \( \text{Exp} \), \( \text{Bit} \) are always interpreted by relations, i.e., + (respectively, \( \times \), \( \text{Exp} \)) denotes the ternary relation consisting of all triples \( (a, b, c) \) such that \( a + b = c \) (respectively, \( a \cdot b = c \), \( a^b = c \)); and \( \text{Bit} \) is the binary relation consisting of all tuples \( (a, i) \) such that the \( i \)-th bit in the binary expansion of \( a \) is 1, i.e., \( \lfloor a/2^i \rfloor \) is odd.

3 Collapse Results in Database Theory

Detailed information on the foundations of databases can be found in the textbook \([9]\). For a well-written concise survey on database theory we refer to the paper \([30]\). A detailed and very recent overview of collapse results on finite databases is given in \([23]\). More information can also be found in the book \([21]\). Not aiming at comprehensiveness, the present section of this paper gives a brief introduction to concepts, questions, and results in database theory that are related to collapse considerations. Furthermore, we summarize the collapse results that are obtained in this paper (see Section 3.4).

3.1 Databases and Queries

In relational database theory a database is modelled as a relational structure over a fixed possibly infinite universe \( U \). A database over \( U \) hence is a \( \langle U, \rho \rangle \)-structure \( A = \langle U, \rho^A \rangle \), for some signature \( \rho \) that consists of a finite number of relation symbols. In database theory such a relational signature \( \rho \) is often called the database schema. The active domain of \( A \), \( \text{adom}(A) \) for short, is the set of all elements in \( U \) that belong to (at least) one of \( A \)'s relations. I.e., \( U \) is the set of all potential database elements, whereas \( \text{adom}(A) \) is the set of all elements that indeed occur in the database relations.

A Boolean query is a “question” or, more formally, a mapping \( Q \) that assigns to each \( \langle U, \rho \rangle \)-structure \( A \) an answer “yes” or “no”. Examples of such queries are

1. Does the unary relation \( R_1 \) contain at least 3 elements?
2. Does the active domain have even cardinality?
3. Are all elements in \( R_1 \) smaller than all elements in \( R_2 \)?

Often one also considers \( k \)-ary queries which yield as answers \( k \)-ary relations over \( U \). Examples of such queries are

4. What are the elements in the active domain?
5. What is the transitive closure of the binary relation \( R_3 \)?

6. Which elements belong to \( R_1 \) and are smaller than all elements in \( R_2 \)?

For well-defined queries one demands the following consistency criterion

(CC): On identical databases, a query must produce identical answers.

Usually, two databases \( A = (\mathbb{U}, \rho^A) \) and \( B = (\mathbb{U}, \rho^B) \) are assumed to be “identical” iff they are isomorphic, i.e., iff there is a permutation \( \pi \) of \( \mathbb{U} \) such that \( \pi(\rho^A) = \rho^B \). Queries that satisfy (CC) are called generic queries. If \( Q \) is a Boolean query, this means that \( Q(A) = Q(\pi(A)) \); if \( Q \) is a \( k \)-ary query it means that \( \pi(Q(A)) = Q(\pi(A)) \), for all permutations \( \pi \) of \( \mathbb{U} \).

A basic way of expressing queries is by first-order formulas. I.e., a \( \langle \mathbb{U}, \rho \rangle \)-sentence \( \varphi(x_1, \ldots, x_k) \) with \( k \) free variables expresses a \( k \)-ary query. For example, if \( \rho \) consists of two unary relations \( R_1 \) and \( R_2 \) and a binary relation \( R_3 \), then the queries 1. and 4. can be expressed as follows:

\[
\varphi_1 \quad := \quad \exists x_1 \exists x_2 \exists x_3 \left( R_1(x_1) \land R_1(x_2) \land R_1(x_3) \land x_1 \neq x_2 \land x_2 \neq x_3 \land x_3 \neq x_1 \right)
\]
\[
\varphi_4(x) \quad := \quad R_1(x) \lor R_2(x) \lor \exists y R_3(x, y) \lor \exists y R_3(y, x).
\]

To avoid the distinction between Boolean queries and \( k \)-ary queries, we will not consider relational database schemas \( \rho \) but, instead, signatures \( \tau \) that consist of a finite number of relation symbols and constant symbols. This allows us to restrict our attention to Boolean queries in the following way: A \( \langle \mathbb{U}, \rho \rangle \)-formula \( \varphi(x_1, \ldots, x_k) \) with \( k \) free variables \( x_1, \ldots, x_k \) can be viewed as a \( FO(\rho) \)-sentence for \( \tau := \rho \cup \{ x_1, \ldots, x_k \} \). In general, any \( k \)-ary query \( Q \) on \( \langle \mathbb{U}, \rho \rangle \)-structures corresponds to the Boolean query \( Q' \) on \( \langle \mathbb{U}, \tau \rangle \)-structures that yields the answer “yes” for a \( \langle \mathbb{U}, \tau \rangle \)-structure \( A = (\mathbb{U}, \rho^A, a_1, \ldots, a_k) \) if and only if \( \{ a_1, \ldots, a_k \} \) belongs to the \( k \)-ary relation that \( Q \) defines on the structure \( \langle \mathbb{U}, \rho^A \rangle \).

From now on we will, without loss of generality, consider Boolean queries rather than \( k \)-ary queries. Signatures \( \tau \) will always consist of a finite number of relation symbols and constant symbols. The name \( \langle \mathbb{U}, \tau \rangle \)-database will be used as a synonym for \( \langle \mathbb{U}, \tau \rangle \)-structure. The active domain of a \( \langle \mathbb{U}, \tau \rangle \)-structure is defined as follows:

**3.1 Definition (Active Domain \( adom(A) \)).**

Let \( \tau \) be a signature and let \( A = (\mathbb{U}, \tau^A) \) be a \( \tau \)-structure. The active domain of \( A \), for short: \( adom(A) \), is the set of all elements in \( \mathbb{U} \) that occur in \( \tau^A \). I.e., \( adom(A) \) is the smallest set \( A \subseteq \mathbb{U} \) that contains the constants \( c^A \), for all \( c \in \tau \), and that satisfies \( R^A \subseteq A^{ar(R)} \), for all \( R \in \tau \).

It is obvious that all \( FO(\tau) \)-definable queries are generic. However, there are generic queries, e.g., the queries 2. and 5. above, that are not expressible in \( FO(\tau) \) (cf., e.g., [11] or [18]). To express more queries, one may allow the formulas to use extra information which is not explicitly part of the database, such as a linear ordering \( < \) or arithmetic predicates \( + \) and \( \times \). In this framework, for example query 2. for \( \mathbb{U} := \mathbb{N} \) can be expressed in \( FO(<, +, \times, \tau) \) via the formula

\[
\varphi_2 \quad := \quad \exists y \exists z \left( (\forall x \varphi_4(x) \leftrightarrow \varphi_{\text{Bit}}(y, x)) \land \varphi_{\text{BitSum}}(y, z) \land (\exists u u+u=z) \right).
\]
Here, $y$ encodes the active domain and $z$ is the cardinality of the active domain; $\varphi_{\text{Bit}}(y, x)$ and $\varphi_{\text{BitSum}}(y, z)$ are FO($+, \times$)-formulas expressing that the $x$-th bit of the binary representation of $y$ is 1, and that $z$ is the number of ones in the binary representation of $y$, respectively (cf., e.g., the textbook [18]).

The additional predicates such as $<, +, \ldots$ are viewed as built-in predicates associated with the universe $U$ of potential database elements. In other words, $(U, <, +, \ldots)$ is viewed as the context structure in which the $(U, \tau)$-databases live. In general, we use the following notation:

3.2 Definition (Context Structure $(U, <, \mathcal{B}ip)$).

A context structure consists of an infinite universe $U$, a linear ordering $<$ (i.e., $<$ is a binary relation on $U$ that is transitive, total, and antisymmetric), and a (possibly infinite) class $\mathcal{B}ip$ of relations on $U$.

Given a context structure $(U, <, \mathcal{B}ip)$ and a set $S \subseteq U$, we shortly write $(S, <, \mathcal{B}ip)$ to denote the induced substructure of $(U, <, \mathcal{B}ip)$ with universe $S$.

3.2 Finding an Adequate Notion of Genericity

When dealing with $(U, \tau)$-databases that live in a context structure $(U, <, \mathcal{B}ip)$ one has to revisit the concept of genericity. Paredaens, Van den Bussche, and Van Gucht [25] (see also [20]) pointed out that the adequate notion of genericity depends on the particular context (or, the geometry) in which the information stored in a database is interpreted. For the particular context considered in the present paper, this can be explained as follows: Recall that the consistency criterion (CC) demands that a generic Boolean query produces the same answer for a database $A = (U, \tau^A)$ as for its isomorphic image $\pi(A) = (U, \pi(\tau^A))$, for every permutation $\pi$ of $U$. Under this restrictive view, the above queries 3. and 6. are not generic. Nevertheless, these queries do make sense when having in mind temporal databases that store, e.g., the chronological order of events. With this interpretation, query 3. asks whether the task $R_1$ was finished before the task $R_2$ began.

When the linear ordering of the database elements is relevant, it seems adequate to call two $(U, \tau)$-databases $A$ and $B$ “identical” iff they are isomorphic via a $<$-preserving mapping. Precisely, several different notions are conceivable:

1. $A$ and $B$ are called order-isomorphic iff the linearly ordered structures $(U, <, \tau^A)$ and $(U, <, \tau^B)$ are isomorphic in the usual sense. Queries that produce identical answers for order-isomorphic databases are known as order-generic queries (cf., e.g., [5, 6, 7]).

For dense linear orderings such as $(\mathbb{R}, <)$ or $(\mathbb{Q}, <)$ this notion of genericity seems adequate. However, for discrete orderings like $(\mathbb{Z}, <)$ or $(\mathbb{N}, <)$ the above notion of order-genericity is too liberal and, equivalently, the notion of order-isomorphy is too restrictive: The identity function is the only order-isomorphism on $(\mathbb{N}, <)$, and consequently no two different databases are assumed to be “identical with respect to the linear ordering”. For a good formalization of what it means to be “identical with respect to the linear ordering” it seems reasonable to consider the active domain of the databases rather than the whole context universe $U$:

2. $A$ and $B$ are called locally order-isomorphic iff the linearly ordered structures $(\text{adom}(A), <, \tau^A)$ and $(\text{adom}(B), <, \tau^B)$ are isomorphic in the usual sense. Queries that produce identical answers for locally order-isomorphic databases are known as locally generic queries (cf., e.g., [5, 6, 7]).
When restricting attention to databases whose active domain is finite, the above notion of \textit{local order-isomorphism} perfectly catches the intuitive understanding of being “identical with respect to the linear ordering”. Moreover, it is not difficult to see (cf., [6] Proposition 1) that for the context structures \( \langle \mathbb{R}, < \rangle \), \( \langle \mathbb{Q}, < \rangle \), and in general for any doubly transitive linear ordering \( \langle \mathbb{U}, < \rangle \), the notions of \textit{order-isomorphism} and \textit{local order-isomorphism} coincide.

But what about databases with an infinite active domain? For example, let \( \mathbb{U} := \mathbb{R} \) and let \( \tau \) consist of a single unary relation symbol \( S \). Consider the \( \langle \mathbb{R}, \tau \rangle \)-structures \( A \) and \( B \) with \( S^A := \{a_1 < a_2 < \cdots \} \) where \( a_n := 1 - \frac{1}{n} \), and \( S^B := \{b_1 < b_2 < \cdots \} \) where \( b_n := n \), for all \( n \in \mathbb{N} \). Clearly, \( \langle \text{dom}(A), <, \tau^A \rangle \) is isomorphic to \( \langle \text{dom}(B), <, \tau^B \rangle \), and thus \( A \) and \( B \) are locally order-isomorphic. But would we intuitively say that \( A \) and \( B \) are “identical with respect to the linear ordering”? — Not really, since \( \text{dom}(A) \) has an upper bound in the context universe \( \mathbb{R} \) whereas \( \text{dom}(B) \) has not. Here it seems adequate to take into account not \( \text{dom}(A) \) but its \textit{closure}

\[
\text{dom}(A) := \text{dom}(A) \cup \{x \in \mathbb{R} : x \text{ is an accumulation point of } \text{dom}(A)\}.
\]

To catch the intuitive meaning of being “identical with respect to the linear ordering” we therefore propose the following formalization: Two \( \langle \mathbb{R}, \tau \rangle \)-structures \( A \) and \( B \) are called \textit{\( < \)-isomorphic} iff the linearly ordered structures \( \langle \text{dom}(A), <, \tau^A \rangle \) and \( \langle \text{dom}(B), <, \tau^B \rangle \) are isomorphic in the usual sense. For the context universe \( \mathbb{Q} \) rather than \( \mathbb{R} \) it seems appropriate to demand that \( \langle \text{dom}(A), <, \tau^A \rangle \) and \( \langle \text{dom}(B), <, \tau^B \rangle \) are isomorphic via a mapping that maps accumulation points in \( \mathbb{Q} \) on accumulation points in \( \mathbb{Q} \), and that maps accumulation points in \( \mathbb{R} \setminus \mathbb{Q} \) on accumulation points in \( \mathbb{R} \setminus \mathbb{Q} \).

For an arbitrary linearly ordered context universe \( \mathbb{U} \) we propose the following generalization: Let \( \langle \mathbb{U}, < \rangle \) be a Dedekind completion of \( \langle \mathbb{U}, < \rangle \). I.e., \( \mathbb{U} \subseteq \overline{\mathbb{U}} \), and every set \( A \subseteq \mathbb{U} \) that has an upper bound (respectively, a lower bound) in \( \mathbb{U} \) with respect to \( < \), has a unique greatest lower bound (respectively, greatest lower bound) in \( \overline{\mathbb{U}} \). For example, \( \langle \mathbb{R}, < \rangle \) is a Dedekind completion of \( \langle \mathbb{Q}, < \rangle \) and of \( \langle \mathbb{N}, < \rangle \), and \( \langle \mathbb{N}, < \rangle \) is a Dedekind completion of \( \langle \mathbb{N}, < \rangle \). The \textit{closure} of a set \( A \subseteq \mathbb{U} \) is the set \( \overline{A} \) that consists of all elements of \( A \) and all elements \( x \in \overline{\mathbb{U}} \) which are a least upper bound or a greatest lower bound of some subset \( A' \subseteq A \).

3.3 Definition \textit{(\( < \)-isomorphism, \( < \)-genericity).}

Let \( \langle \mathbb{U}, < \rangle \) be a linearly ordered context structure, and let \( \langle \overline{\mathbb{U}}, < \rangle \) be its Dedekind completion. Let \( \tau \) be a signature. Two \( \langle \mathbb{U}, \tau \rangle \)-structures \( A \) and \( B \) are called \textit{\( < \)-isomorphic} iff the structures

\[
\langle \text{dom}(A), <, \tau^A, \overline{\text{dom}(A)} \setminus \mathbb{U} \rangle \quad \text{and} \quad \langle \text{dom}(B), <, \tau^B, \overline{\text{dom}(B)} \setminus \mathbb{U} \rangle
\]

are isomorphic in the usual sense.

A Boolean query \( Q \) is called \textit{\( < \)-generic} on a \( \langle \mathbb{U}, \tau \rangle \)-structure \( A \) if and only if \( Q(A) = Q(B) \) for all \( \langle \mathbb{U}, \tau \rangle \)-structures \( B \) that are \textit{\( < \)-isomorphic} to \( A \). Accordingly, if \( \mathcal{K} \) is a class of \( \langle \mathbb{U}, \tau \rangle \)-structures, then we say that \( Q \) is \textit{\( < \)-generic} on \( \mathcal{K} \) iff it is \textit{\( < \)-generic} on every \( A \in \mathcal{K} \). \qed

In particular, the notions \( < \)-isomorphism and \( < \)-genericity coincide with the notions \textit{order-isomorphism} and \textit{order-genericity} if \( \mathbb{U} = \mathbb{R} \) or \( \mathbb{Q} \), and they coincide with the notions \textit{local order-isomorphism} and \textit{local genericity} if \( \mathbb{U} = \mathbb{N} \) or \( \mathbb{Z} \). This further indicates that these notions are adequate and uniform formalizations of what it means for databases to be “identical with respect to the ordering” and what it means for queries to produce consistent answers for “identical” databases. The following notion gives us an alternative characterization of \( < \)-isomorphism and \( < \)-genericity:
3.4 Definition (\(<\)-preserving mapping).
Let \(\langle U, < \rangle\) and \(\langle V, < \rangle\) be linearly ordered structures, and let \(\langle U, < \rangle\) and \(\langle V, < \rangle\) be their Dedekind completions. Let \(U \subseteq V\), let \(\alpha : U \to V\), and let \(V := \alpha(U)\).
The mapping \(\alpha\) is called \(<\>-preserving if it can be extended to an isomorphism between the structures \(\langle U, <, U \setminus U \rangle\) and \(\langle V, <, V \setminus V \rangle\).

\[\square\]

It is straightforward to see the following:

3.5 Remarks (\(<\>-isomorphy, \(<\>-genericity).)

(a) Two \(\langle U, \tau \rangle\)-structures \(A\) and \(B\) are \(<\>-isomorphic if and only if there is a \(<\>-preserving mapping \(\alpha : \text{adom}(A) \to U\) such that \(\alpha(\tau_A) = \tau_B\).

Consequently, a Boolean query \(Q\) is \(<\>-generic on \(A = \langle U, \tau_A \rangle\) if and only if \(Q(\langle U, \tau_A \rangle) = Q(\langle U, \alpha(\tau_A) \rangle) for all \(<\>-preserving mappings \(\alpha : \text{adom}(A) \to U\).

(b) If, in particular, \(U\) and \(V\) are \(\mathbb{N}\) or \(\mathbb{Z}\), then a mapping \(\alpha : U \to V\) is \(<\>-preserving if and only if it is strictly increasing, i.e., \(u < u\prime\) iff \(\alpha(u) < \alpha(u)\prime\) (for all \(u, u\prime \in U\)).

\[\square\]

3.3 Collapse Results for \(<\>-Generic Queries

Given a context structure \(\langle U, <, \mathcal{Bip} \rangle\) and a signature \(\tau\) we will consider the following query languages: Let \(Q\) be a Boolean query on \(\langle U, \tau \rangle\)-structures and let \(\mathcal{C}\) be a class of \(\langle U, \tau \rangle\)-structures.

We say that, on structures in \(\mathcal{C}\), \(Q\) is expressible in

- \(\text{FO}(\langle, \mathcal{Bip}, \tau \rangle)\) iff there is a \(\text{FO}(\langle, \mathcal{Bip}, \tau \rangle)-sentence \varphi\) such that
  \[\langle U, <, \mathcal{Bip}, \tau_A \rangle \models \varphi\iff Q(A) = \text{“yes”}\]
  is true for all \(A = \langle U, \tau_A \rangle\) in \(\mathcal{C}\). One speaks of natural semantics, since quantification ranges, in the natural way, over the whole universe \(U\). If \(\mathcal{Bip}\) is empty we simply write \(\text{FO}(\langle, \rangle)\).

- active domain \(\text{FO}(\langle, \mathcal{Bip}\rangle), for short: \text{FO}_{\text{adom}}(\langle, \mathcal{Bip}\rangle), iff there is a \(\text{FO}(\langle, \mathcal{Bip}, \tau \rangle)-sentence \varphi\) such that
  \[\text{adom}(A), <, \mathcal{Bip}, \tau_A \rangle \models \varphi\iff Q(A) = \text{“yes”}\]
  is true for all \(A = \langle U, \tau_A \rangle\) in \(\mathcal{C}\). One speaks of active domain semantics, since quantification is restricted to the active domain. If \(\mathcal{Bip}\) is empty we simply write \(\text{FO}_{\text{adom}}(\langle, \rangle)\).

It should be clear that all queries expressible in \(\text{FO}_{\text{adom}}(\langle, \rangle)\) are \(<\>-generic. Figure \[\text{I}\] illustrates the obvious inclusions concerning the expressive power of the above query languages.

It is an interesting question whether, for a particular context structure \(\langle U, <, \mathcal{Bip} \rangle\) and a particular class \(\mathcal{C}\) of \(\langle U, \tau \rangle\)-structures

- the predicates in \(\mathcal{Bip}\) allow to express more \(<\>-generic queries than the linear ordering alone
- the quantification over all elements in the context universe \(U\) allows to express more \(<\>-generic queries than the active domain quantification alone.
One speaks of a collapse result if the apparently stronger language is no more expressive than the apparently weaker one. For the particular class \( C_{\text{fin}} \) of all structures whose active domain is finite, strong collapse results are known. A comprehensive overview of such results can be found in [7, 23]. Here we in particular want to mention the following:

In [6] it was shown that for all linearly ordered context universes \( U \) and for the class \( \text{Arb} \) of arbitrary, i.e., all, predicates on \( U \), we have

\[
\text{\(<\text{-generic FO}_{\text{adom}}(<, \text{Arb}) = \text{FO}_{\text{adom}}(<)\) on } C_{\text{fin}} \text{ over } U.
\]

This means that every query \( Q \) that is \(<\text{-generic on } C_{\text{fin}} \) and that is \( \text{FO}_{\text{adom}}(<, \text{Arb})\)-expressible on \( C_{\text{fin}} \), is also \( \text{FO}_{\text{adom}}(<)\)-expressible on \( C_{\text{fin}} \). I.e., when quantification is restricted to the active domain, arbitrary built-in predicates do not help first-order logic to express \(<\text{-generic queries over finite databases. This result is known as the active generic collapse (over finite databases).}

Also the so-called natural generic collapse has been investigated. Various different conditions on the context structure \( \langle U, <, \text{Bip} \rangle \) are known which guarantee that

\[
\text{\(<\text{-generic FO}(<, \text{Bip}) = \text{<\text{-generic FO}(<)\) on } C_{\text{fin}} \text{ over } U.}
\]

In particular, Benedikt et al., Belegradek et al., and Baldwin and Benedikt have shown that the natural generic collapse over finite databases holds if the context structure is o-minimal [3], has the Isolation Property [5], or has finite Vapnik-Chervonenkis (VC) dimension [2] (see also [9]). Context structures which satisfy (at least) one of these conditions are, for example, \( \langle \mathbb{N}, <, + \rangle \), \( \langle \mathbb{Q}, <, + \rangle \), \( \langle \mathbb{R}, <, +, \times, \text{Exp} \rangle \), and \( \langle U, <, \text{Mon} \rangle \) for any linearly ordered \( U \) and the class \( \text{Mon} \) of all monadic predicates on \( U \) (cf., e.g., the survey [7]). Indeed, in [7] it is mentioned that the notion of finite VC-dimension coincides with the notion \( \text{NIP} \) of structures that lack the independence property \( \cite{2} \) and that these two notions include all context structures for which the natural generic collapse over \( C_{\text{fin}} \) is known by now (and, in particular, they include all o-minimal structures and all structures that have the Isolation Property). The following definition of finite VC-dimension is basically taken from [7]:

**3.6 Definition (Finite VC-Dimension).** Let \( \langle U, <, \text{Bip} \rangle \) be a context structure.

(a) Let \( \varphi(\bar{x}, \bar{y}) \) be a \( \text{FO}(<, \text{Bip}) \)-formula, and let \( n_{\bar{x}} \) and \( n_{\bar{y}} \) be the lengths of the tuples \( \bar{x} \) and \( \bar{y} \), respectively.

---

1. Considering \( C_{\text{fin}} \), one even obtains the collapse to \( \text{FO}_{\text{adom}}(<) \).
2. but not necessarily “only if”
3. In fact, the correspondence between NIP and finite VC-dimension easily follows from the definition of NIP in \cite{4} Definition 2.2] and the definition of finite VC-dimension as presented in Definition 3.6.
• For every \( \bar{a} \in \mathbb{U}^n \) the formula \( \varphi(\bar{x}, \bar{y}) \) defines the relation
\[
R_{\varphi(\bar{x}, \bar{y})} := \{ \bar{b} \in \mathbb{U}^n : \langle \mathbb{U}, <, \text{Bip} \rangle \models \varphi(\bar{b}, \bar{a}) \}.
\]
• The formula \( \varphi(\bar{x}, \bar{y}) \) defines the following family of relations on \( \mathbb{U} \):
\[
F_{\varphi(\bar{x}, \bar{y})} := \{ R_{\varphi(\bar{x}, \bar{y})} : \bar{a} \in \mathbb{U}^n \}.
\]
• A set \( B \subseteq \mathbb{U}^n \) is shattered by \( F_{\varphi(\bar{x}, \bar{y})} \) iff \( \{ B \cap R : R \in F_{\varphi(\bar{x}, \bar{y})} \} = \{ X : X \subseteq B \} \).

I.e., for every \( X \subseteq B \) there is an \( \bar{a}_X \in \mathbb{U}^n \) such that for all \( \bar{b} \in B \) we have
\[
\bar{b} \in X \iff (\mathbb{U}, <, \text{Bip}) \models \varphi(\bar{b}, \bar{a}_X).
\]
• The family \( F_{\varphi(\bar{x}, \bar{y})} \) has finite VC-dimension iff there exists a number \( m_{\varphi(\bar{x}, \bar{y})} \in \mathbb{N} \) such that the following is true for all \( B \subseteq \mathbb{U}^n \):
If \( B \) is shattered by \( F_{\varphi(\bar{x}, \bar{y})} \), then \( |B| \leq m_{\varphi(\bar{x}, \bar{y})} \).

(b) \( \langle \mathbb{U}, <, \text{Bip} \rangle \) has finite VC-dimension if and only if \( F_{\varphi(\bar{x}, \bar{y})} \) has finite VC-dimension, for every \( \text{FO}(<, \text{Bip}) \)-formula \( \varphi(\bar{x}, \bar{y}) \).

According to \([23]\), the following result of \([2, 3]\) is the most general natural generic collapse theorem that is known by now for the class \( \mathcal{C}_{\text{fin}} \) of all finite databases.

3.7 Theorem (Baldwin, Benedikt).
If \( \langle \mathbb{U}, <, \text{Bip} \rangle \) is a context structure that has finite VC-dimension then
\[
\text{<}-\text{generic } \text{FO}(<, \text{Bip}) = \text{FO}_{\text{adom}}(<) \text{ on } \mathcal{C}_{\text{fin}} \text{ over } \mathbb{U}.
\]

On the other hand, it is straightforward to see (cf., e.g., \([7]\)) that the natural generic collapse does not hold for all context structures:

3.8 Facts (No Collapse for \( \langle \mathbb{N}, <, +, \times \rangle \) and \( \langle \mathbb{N}, <, +, \text{Squares} \rangle \)).

(a) For the context structure \( \langle \mathbb{N}, <, +, \times \rangle \) and any class \( \mathcal{C} \supseteq \mathcal{C}_{\text{fin}} \) we have
\[
\text{<}-\text{generic } \text{FO}(<, +, \times) \neq \text{<}-\text{generic } \text{FO}(<) \text{ on } \mathcal{C} \text{ over } \mathbb{N}.
\]
To see this consider the query \( Q_{\text{even}} : "\text{Does the active domain have even cardinality?}"."

Obviously, this query is <generic on all structures. Furthermore, in Section 3.1 we already saw that \( Q_{\text{even}} \) is expressible in \( \text{FO}(<, +, \times) \) but not in \( \text{FO}(<) \) over \( \mathbb{N} \).

(b) For the context structure \( \langle \mathbb{N}, <, +, \text{Squares} \rangle \), where \( \text{Squares} := \{ n^2 : n \in \mathbb{N} \} \), we have
\[
\text{<}-\text{generic } \text{FO}(<, +, \text{Squares}) \neq \text{<}-\text{generic } \text{FO}(<) \text{ on } \mathcal{C} \text{ over } \mathbb{N}.
\]
To see this use (a) and recall that \( \times \) is first-order definable in \( \langle \mathbb{N}, <, +, \text{Squares} \rangle \) (cf., e.g., the survey \([3]\)).

The collapse results mentioned so far all deal with the class \( \mathcal{C}_{\text{fin}} \) of databases whose active domain is finite. Belegradek et al. \([5]\) investigated finitely representable databases, i.e., databases whose relations, essentially, consist of a finite number of multidimensional rectangles in the context universe \( \mathbb{U} \). They showed, for every context structure \( \langle \mathbb{U}, <, \text{Bip} \rangle \), that a natural generic collapse on \( \mathcal{C}_{\text{fin}} \) over \( \mathbb{U} \) can be lifted to a natural generic collapse on the larger class \( \mathcal{C}_{\text{fin,rep}} \) of all finitely representable databases over \( \mathbb{U} \). We will further concentrate on this result in Section 3.8.

But what happens for the class \( \mathcal{C}_{\text{arb}} \) of arbitrary, i.e., all, structures? Can collapse results be lifted from \( \mathcal{C}_{\text{fin}} \) to \( \mathcal{C}_{\text{arb}} \)? — Not in general! Recall from the (already mentioned) result of \([4]\) for o-minimal structures that
\[
\text{<}-\text{generic } \text{FO}(<, +) = \text{<}-\text{generic } \text{FO}(<) \text{ on } \mathcal{C}_{\text{fin}} \text{ over } \mathbb{Q}.
\]
However, in [5, Theorems 3.3 and 3.4] it was shown that
\[
<\text{-generic FO}(<,+) \neq <\text{-generic FO}(<) \quad \text{on } \mathcal{C}_{\text{arb}} \text{ over } \mathbb{Q}.
\]
I.e., the natural generic collapse is valid for finite but not for arbitrary databases over the context structure \(\langle \mathbb{Q},<,+ \rangle\).

On the other hand, in [22] it was shown that the collapse does hold for arbitrary databases over the context structure \(\langle \mathbb{N},<,+ \rangle\). To the author’s knowledge this is the only collapse result known so far for the class \(\mathcal{C}_{\text{arb}}\) of arbitrary databases, and there are no publications other than [22, 27] that show the natural generic collapse for classes of databases larger than \(\mathcal{C}_{\text{fin}}\) and \(\mathcal{C}_{\text{fin.rep}}\). In the subsequent sections of this paper we will obtain these and other collapse results for such larger classes of databases. Precisely, our collapse results are of the following kind:

**3.9 Definition (Collapse Result).**

Let \(\langle U,<,\mathfrak{Bip} \rangle\) be a context structure, and let \(\mathcal{C}\) be a class of structures over the universe \(U\). We write
\[
<\text{-generic FO}(<,\mathfrak{Bip}) = \text{FO}_{\text{adom}}(<) \quad \text{on } \mathcal{C} \text{ over } U
\]
if and only if the following is true:

For every signature \(\tau\) and every \(\text{FO}(<,\mathfrak{Bip},\tau)\)-sentence \(\varphi\) there is a \(\text{FO}(<,\tau)\)-sentence \(\varphi'\) such that
\[
\langle U,<,\mathfrak{Bip},\tau^A \rangle \models \varphi \iff \langle \text{adom}(A),<,\tau^A \rangle \models \varphi'
\]
is true for all \(\langle U,\tau \rangle\)-structures \(A \in \mathcal{C}\) on which the query defined by \(\varphi\) is \(<\text{-generic}\).
For convenience we will henceforth say \(\varphi\) is \(<\text{-generic}\) on \(A\) to express that the query defined by \(\varphi\) is \(<\text{-generic}\) on \(A\).

The collapse result for any logic \(F\) other than \(\text{FO}\) is defined in the analogous way, replacing \(\text{FO}\) with \(F\) in the above definition.

Let us mention a technical detail: The “traditional” definition of collapse for the class \(\mathcal{C}_{\text{fin}}\) states the following: If a sentence \(\varphi \in \text{FO}(<,\mathfrak{Bip},\tau)\) is \(<\text{-generic}\) on \(\mathcal{C}_{\text{fin}}\), then it can be replaced by a \(\varphi' \in \text{FO}(<,\tau)\) that is equivalent to \(\varphi\) on \(\mathcal{C}_{\text{fin}}\). Just replacing \(\mathcal{C}_{\text{fin}}\) with \(\mathcal{C}_{\text{arb}}\) in this definition would reduce the set of formulas \(\varphi\) to which the collapse applies, because there certainly are formulas \(\varphi\) that are \(<\text{-generic}\) on \(\mathcal{C}_{\text{fin}}\) but not on \(\mathcal{C}_{\text{arb}}\). The above Definition 3.9 circumvents this problem by stating that any \(\varphi\) can be replaced by a \(\varphi'\) that is equivalent to \(\varphi\) on all databases on which \(\varphi\) is \(<\text{-generic}\).

We will in particular deal with the following classes of databases:

**3.10 Definition (finite, \(\mathbb{N}\)-embeddable, \(\mathbb{Z}\)-embeddable).**

Let \(\langle U,< \rangle\) be a linearly ordered structure and let \(\langle \overline{U},< \rangle\) be its Dedekind completion. Let \(\tau\) be a signature. A \(\langle U,\tau \rangle\)-structure \(A\) is called

- finite if \(\text{adom}(A)\) is finite.

-- Recall that signatures \(\tau\) always consist of a finite number of relation symbols and constant symbols.
• $\mathbb{N}$-embeddable iff there is a $<$-preserving mapping $\alpha : \text{dom}(A) \rightarrow \mathbb{N}$. I.e., $\text{dom}(A)$ is finite or $\text{dom}(A)$ is of the form $\{a_1 < a_2 < \cdots \}$ and has no accumulation points in $\mathbb{U}$. In particular, all $(\mathbb{N}, \tau)$-structures are $\mathbb{N}$-embeddable.

• $\mathbb{Z}$-embeddable iff there is a $<$-preserving mapping $\alpha : \text{dom}(A) \rightarrow \mathbb{Z}$. I.e., $\text{dom}(A)$ has no accumulation points and is $\mathbb{N}$-embeddable or of one of the forms $\{ \cdots < a_{-2} < a_{-1} < a_1 < a_2 < \cdots \}$ or $\{ a_{-1} > a_{-2} > \cdots \}$. In particular, all $(\mathbb{Z}, \tau)$-structures are $\mathbb{Z}$-embeddable.

We use $\mathcal{G}_{\text{fin}}, \mathcal{G}_{\text{N-emb}}, \mathcal{G}_{\text{Z-emb}},$ and $\mathcal{G}_{\text{arb}}$, respectively, to denote the classes of all finite, $\mathbb{N}$-embeddable, $\mathbb{Z}$-embeddable, and arbitrary (i.e., all) structures, respectively.

### 3.4 Collapse Results Obtained in this Paper

In Section 5 we will consider context structures which have as built-in predicates the class $\mathcal{M}_{\text{mon}}$ of all monadic, i.e., unary, relations over the context universe. Our result is

• $<\text{-generic } \mathcal{FO}(<, \mathcal{M}_{\text{mon}}) = \mathcal{FO}_{\text{dom}}(<)$ on $\mathcal{G}_{\text{Z-emb}}$ over $\mathbb{U}$,
  for any linearly ordered infinite context universe $\mathbb{U}$.

In particular, for $\mathbb{U} = \mathbb{N}$ and $\mathbb{U} = \mathbb{Z}$ this implies the collapse over $\mathcal{G}_{\text{arb}}$.

In Section 6 we will investigate context structures with built-in addition $+$, and we will prove the result of [23] and several extensions of that result. Precisely, we will expose an infinite set $Q \subseteq \mathbb{N}$ (which is not $\mathcal{FO}(<, +)$-definable) and show, for the class $\mathcal{M}_{\text{mon}}Q$ of all subsets of $Q$, that

• $<\text{-generic } \mathcal{FO}(<, +, Q, \mathcal{M}_{\text{mon}}Q) = \mathcal{FO}_{\text{dom}}(<)$ on $\mathcal{G}_{\text{arb}}$ over $\mathbb{N}$, and

• $<\text{-generic } \mathcal{FO}(<, +, Q, \mathcal{M}_{\text{mon}}Q, \mathcal{Groups}) = \mathcal{FO}_{\text{dom}}(<)$ on $\mathcal{G}_{\text{Z-emb}}$ over $\mathbb{R}$, where $\mathcal{Groups}$ is the class of all subsets of $\mathbb{R}$ that contain the number 1 and that are groups with respect to $+$.

In particular, this implies the natural generic collapse on $\mathcal{G}_{\text{Z-emb}}$ for the context structures $\langle \mathbb{N}, <, +, Q \rangle$, $\langle \mathbb{Q}, <, + \rangle$, $\langle \mathbb{Q}, <, +, \mathbb{Z} \rangle$, and $\langle \mathbb{R}, <, +, \mathbb{Z}, Q \rangle$. The collapse for the context structure $\langle \mathbb{N}, <, +, Q \rangle$ is remarkable since we know from Fact 3.8 (b) that the collapse does not hold when replacing the set $Q$ with the set $\text{Squares}$ of all square numbers.

In Section 7 we will look at the restriction of first-order logic to the class $\mathcal{BC}(\mathcal{EFO})$, i.e., to Boolean combinations of purely existential FO-formulas. As built-in predicates we will consider the class $\mathcal{A}\mathcal{rb}$ of arbitrary, i.e., all, relations. We will show that

• $<\text{-generic } \mathcal{BC}(\mathcal{EFO})(<, \mathcal{A}\mathcal{rb}) = \mathcal{BC}(\mathcal{EFO})_{\text{dom}}(<)$ on $\mathcal{G}_{\text{arb}}$ over $\mathbb{U}$,
  for any linearly ordered infinite context universe $\mathbb{U}$.

In particular, for $\mathbb{U} = \mathbb{N}$ this implies the collapse on $\mathcal{G}_{\text{arb}}$.

In Section 8 we will present the result from [27] which, in the spirit of [5]’s lifting from $\mathcal{G}_{\text{fin}}$ to $\mathcal{G}_{\text{fin,rep}}$, allows to lift collapse results from $\mathcal{G}_{\text{N-emb}}$ to a class $\mathcal{G}_{\text{fin,rep}}$ that is a proper extension of the class $\mathcal{G}_{\text{fin,rep}}$.

The proof method used in this paper for obtaining the collapse results over $\mathcal{G}_{\text{N-emb}}$ and $\mathcal{G}_{\text{Z-emb}}$ is considerably different from the methods used so far for proving collapse results in database theory:
The proofs in [6, 5, 2] are via model theory and use non-standard, hyperfinite structures. So far, no elementary proof of Theorem 3.7, stating that the collapse is valid over all context structures that have finite VC-dimension, is known.

An elementary and constructive proof of the results of [6] for o-minimal context structures was given by Benedikt and Libkin in [8] (see also [7, 23]). There, the natural generic collapse over o-minimal context structures is proved by a combination of the natural active collapse and the active generic collapse. In [23, Proposition 6.10] also an elementary proof for the particular (non o-minimal) context structure \( \langle \mathbb{N}, <, + \rangle \) is sketched.

In Section 4 we will present a specific notion of the translation of strategies for the Ehrenfeucht-Fraïssé game which allows us to prove collapse results. Apart from the collapse results obtained with this method, the exposition of explicit strategies for the Ehrenfeucht-Fraïssé game is interesting in its own right. Let us emphasize that the present paper investigates collapse results from the point of view of mathematical logic. That is, we want to gain a deeper understanding of the expressive power, or the expressive weakness, of first-order logic with certain built-in predicates, and we want to construct explicit winning strategies for the Ehrenfeucht-Fraïssé game in the presence of built-in predicates.

Those readers who are mainly interested in database theory or computer science as such, may have the objection that \( \mathbb{N} \)-embeddable structures in general cannot be represented in the finite and thus cannot be used as input for an algorithm. In this context we want to mention a line of research that considers recursive structures [16], i.e., structures where every relation is computable by an algorithm that decides whether or not an input tuple belongs to the respective relation. Of course, our collapse results for the classes \( \mathcal{C}_{arb} \) or \( \mathcal{C}_{\mathbb{N}emb} \) are still applicable when restricting attention to recursive structures in \( \mathcal{C}_{arb} \) or \( \mathcal{C}_{\mathbb{N}emb} \).

4 An Ehrenfeucht-Fraïssé Game Approach

In this section we present the translation of strategies for the Ehrenfeucht-Fraïssé game as a method for proving collapse results in database theory. We show that, in principle, all collapse results of the kind fixed in Definition 3.9 can be proved via Ehrenfeucht-Fraïssé games.

4.1 The Ehrenfeucht-Fraïssé Game for FO

Ehrenfeucht-Fraïssé games, for short: EF-games, were invented by Ehrenfeucht and Fraïssé in [4]. These combinatorial games are particularly useful for investigating what can, and what cannot, be expressed in various logics. A well-written survey on EF-games is, e.g., given by Fagin in [13]. More details can be found in the textbooks [18, 11]. In the present section we will concentrate on the classical, first-order \( r \)-round EF-game, which is defined as follows.

Let \( \tau \) be a signature and let \( r \) be a natural number. The \( r \)-round EF-game is played by two players, the spoiler and the duplicator, on two \( \tau \)-structures \( \mathcal{A} \) and \( \mathcal{B} \). The spoiler’s intention is to show a difference between the two structures, while the duplicator tries to make them look alike. There is a fixed number \( r \) of rounds. Each round \( i \in \{1, \ldots, r\} \) is played as follows: First, the spoiler chooses either an element \( a_i \) in the universe of \( \mathcal{A} \) or an element \( b_i \) in the universe of \( \mathcal{B} \). Afterwards, the duplicator chooses an element in the other structure. I.e., she chooses either an element \( b_i \) in the universe of \( \mathcal{B} \), if the spoiler’s move was in \( \mathcal{A} \), or an element \( a_i \) in the universe
of \( \mathcal{A} \), if the spoiler’s move was in \( \mathcal{B} \). After \( r \) rounds the game finishes with elements \( a_1, \ldots, a_r \) chosen in \( \mathcal{A} \) and \( b_1, \ldots, b_r \) chosen in \( \mathcal{B} \).

The duplicator has won the game if, restricted to the chosen elements and the interpretations of the constant symbols, the structures \( \mathcal{A} \) and \( \mathcal{B} \) are indistinguishable with respect to \( \{=\} \cup \tau \). Precisely, this means that the mapping \( \pi \) defined via

\[
\pi : \begin{cases} 
    c^A &\mapsto c^B \quad \text{for all constant symbols } c \in \tau \\
    a_i &\mapsto b_i \quad \text{for all } i \in \{1, \ldots, r\}
\end{cases}
\]

is a partial isomorphism between \( \mathcal{A} \) and \( \mathcal{B} \). Otherwise, the spoiler has won the game.

Since the game is finite, one of the two players must have a winning strategy, i.e., he or she can always win the game, no matter how the other player plays. We say that the duplicator wins the \( r \)-round EF-game on \( \mathcal{A} \) and \( \mathcal{B} \) and we write \( \mathcal{A} \approx^r \mathcal{B} \) iff the duplicator has a winning strategy in the \( r \)-round EF-game on \( \mathcal{A} \) and \( \mathcal{B} \). It is straightforward to see that, for every signature \( \tau \), the relation \( \approx^r \) is an equivalence relation on the set of all \( \tau \)-structures.

The fundamental use of the game comes from the fact that it characterizes first-order logic as follows (cf., e.g., [13, 18, 11]):

4.1 Theorem (Ehrenfeucht, Fraïssé). Let \( \tau \) be a signature.

(a) Let \( r \in \mathbb{N} \) and let \( \mathcal{A} \) and \( \mathcal{B} \) be \( \tau \)-structures. \( \mathcal{A} \approx^r \mathcal{B} \) if and only if \( \mathcal{A} \) and \( \mathcal{B} \) satisfy the same FO(\( \tau \))-sentences of quantifier depth at most \( r \).

(b) Let \( \mathcal{K} \) be a class of \( \tau \)-structures and let \( \mathcal{L} \subseteq \mathcal{K} \). The following are equivalent:

(i) \( \mathcal{L} \) is not FO(\( \tau \))-definable in \( \mathcal{K} \), i.e., there is no FO(\( \tau \))-sentence \( \varphi \) such that “\( \mathcal{A} \models \varphi \) iff \( \mathcal{A} \in \mathcal{L} \)” is true for all \( \mathcal{A} \in \mathcal{K} \).

(ii) For each \( r \in \mathbb{N} \) there are \( \mathcal{A}, \mathcal{B} \in \mathcal{K} \) such that \( \mathcal{A} \in \mathcal{L} \), \( \mathcal{B} \notin \mathcal{L} \), and \( \mathcal{A} \approx^r \mathcal{B} \). □

4.2 Remark. It is well-known (cf., e.g., [18, Exercise 6.11]) that for a fixed (finite) signature \( \tau \) there are only finitely many inequivalent FO(\( \tau \))-sentences of quantifier depth at most \( r \). Consequently, due to Theorem 4.1 (a), the relation \( \approx^r \) has only finitely many equivalence classes on the set of all \( \tau \)-structures — and each equivalence class can be defined by a FO(\( \tau \))-sentence of quantifier depth at most \( r \). More precisely: Let \( c = c(r, \tau) \in \mathbb{N} \) be the number of equivalence classes. There are FO(\( \tau \))-sentences \( \varphi_1, \ldots, \varphi_c \) of quantifier depth at most \( r \), such that

- each \( \tau \)-structure \( \mathcal{A} \) satisfies exactly one of the sentences \( \varphi_1, \ldots, \varphi_c \), and
- two \( \tau \)-structures \( \mathcal{A} \) and \( \mathcal{B} \) satisfy the same sentence from \( \varphi_1, \ldots, \varphi_c \) if and only if \( \mathcal{A} \approx^r \mathcal{B} \).

The formulas defining the equivalence classes are also known as Hintikka formulas. □

4.2 Using EF-Games for Collapse Results

4.3 Definition (Translation of Strategies). Let \( \langle U, <, \text{Bip} \rangle \) be a context structure and let \( \mathcal{C} \) be a class of structures over the universe \( U \). We say that
The duplicator can translate strategies for the FO_{adom(<)}-game into strategies for the FO(<, Bip)-game on ℵ over U if and only if the following is true:

For every finite set Bip′ ⊆ Bip, for every signature τ, and for every number k ∈ N there is a number r(k) ∈ N such that the following is true for all ⟨U, τ⟩-structures A, B ∈ ℵ: If the duplicator wins the r(k)-round FO_{adom(<)}-game on A and B, i.e., if ⟨adom(A), <, τ^A⟩ ≈_{r(k)} ⟨adom(B), <, τ^B⟩, then there are <,τ-preserving mappings α : adom(A) → U and β : adom(B) → U such that the duplicator wins the k-round FO(<, Bip′)-game on α(A) and β(B), i.e., ⟨U, <, Bip′, α(τ^A)⟩ ≈_k ⟨U, <, Bip′, β(τ^B)⟩.

Due to the specific notion of collapse result fixed in Definition 3.9 we obtain that all collapse results can be proved via the translation of strategies.

4.4 Theorem (Translation of Strategies ⇔ Collapse Result).

Let ⟨U, <, Bip⟩ be a context structure, and let ℵ be a class of structures over the universe U. The following are equivalent:

(a) The duplicator can translate strategies for the FO_{adom(<)}-game into strategies for the FO(<, Bip)-game on ℵ over U.

(b) <,τ-generic FO(<, Bip) = FO_{adom(<)} on ℵ over U.

Proof. (a)⇒(b): Let τ be a signature, let ϕ be a FO(<, Bip, τ)-sentence, and let ℵ be the set of all ⟨U, τ⟩-structures in ℵ on which ϕ is <,τ-generic.

We need to show that there is a FO(<, τ)-sentence ϕ′ such that

⟨U, <, Bip, τ^A⟩ |= ϕ  iff  ⟨adom(A), <, τ^A⟩ |= ϕ′

is true for all structures A = ⟨U, τ^A⟩ in ℵ. For the sake of contradiction, we assume that such a FO(<, τ)-sentence ϕ′ does not exist. This means that the class

ℓ′ := { ⟨adom(A), <, τ^A⟩ : A ∈ ℵ and ⟨U, <, Bip, τ^A⟩ |= ϕ }\n
is not FO(<, τ)-definable in ℵ′ := { ⟨adom(A), <, τ^A⟩ : A ∈ ℵ } . Hence, for every r ∈ N, Theorem 2.4[1] gives us structures A′_r, B′_r ∈ ℵ′ such that A^r′, B^r′ ∈ ℵ′, and A^r′ ≈_r B^r′. I.e., for every r ∈ N, there are structures A^r, B^r ∈ ℵ such that ⟨U, <, Bip, τ^A^r⟩ |= ϕ, ⟨U, <, Bip, τ^B^r⟩ |= ϕ, and ⟨adom(A^r), <, τ^A^r⟩ ≈_r ⟨adom(B^r), <, τ^B^r⟩.

Let us now make use of the presumption that the duplicator can translate strategies for the FO_{adom(<)}-game into strategies for the FO(<, Bip)-game on structures in ℵ. Let Bip′ be the finite set of relations from Bip that occur in ϕ, let k be the quantifier depth of ϕ, and let r := r(k) be chosen according to Definition 4.3. Thus, there are <,τ-preserving mappings α : adom(A^r) → U and β : adom(B^r) → U such that ⟨U, <, Bip′, α(τ^A^r)⟩ ≈_k ⟨U, <, Bip′, β(τ^B^r)⟩.

However, since ϕ is <,τ-generic on A and on B, we have that

⟨U, <, Bip′, α(τ^A^r)⟩ |= ϕ  and  ⟨U, <, Bip′, β(τ^B^r)⟩ ⊭ ϕ.

3Recall that signatures τ always consist of a finite number of relation symbols and constant symbols.
This is a contradiction to Theorem 4.1 which states that structures that are equivalent with respect to \( \approx_k \) do satisfy the same first-order sentences of quantifier depth \( k \).

Altogether, the proof of \("(a)\Rightarrow(b)"\) is complete.

\((b)\Rightarrow(a)\): Let \( \mathcal{Bip}' \) be a finite subset of \( \mathcal{Bip} \), let \( \tau \) be a signature, and let \( k \in \mathbb{N} \). From Remark 4.2 we know that the relation \( \approx_k \) has only a finite number \( c \in \mathbb{N} \) of equivalence classes on the set of all \((<, \mathcal{Bip}', \tau)\)-structures; and these equivalence classes can be described by \( \mathcal{FO}(<, \mathcal{Bip}', \tau) \)-sentences \( \varphi_1, \ldots, \varphi_c \) of quantifier depth at most \( k \). I.e., each structure \( \mathcal{A} \) satisfies exactly one of the sentences \( \varphi_1, \ldots, \varphi_c \), and two structures \( \mathcal{A} \) and \( \mathcal{B} \) satisfy the same sentence from \( \varphi_1, \ldots, \varphi_c \) iff \( \mathcal{A} \approx_k \mathcal{B} \).

We will consider all possible disjunctions of the formulas \( \varphi_i \). I.e., for each \( I \subseteq \{1, \ldots, c\} \) we define

\[ \varphi_I := \bigvee_{i \in I} \varphi_i \]

From the presumption we know that \(<\)-generic \( \mathcal{FO}(<, \mathcal{Bip}) = \mathcal{FO}_{adom(<)} \) on \( \mathcal{C} \) over \( U \). I.e., for each sentence \( \varphi_I \) there is a \( \mathcal{FO}(<, \tau) \)-sentence \( \varphi'_I \) such that

\[ (\ast) \quad \langle U, <, \mathcal{Bip}, \tau^A \rangle \models \varphi_I \iff \langle \text{adom}(A), <, \tau^A \rangle \models \varphi'_I \]

is true for all \((U, \tau)\)-structures \( \mathcal{A} \in \mathcal{C} \) on which \( \varphi_I \) is \(<\)-generic.

Choose \( r(k) \in \mathbb{N} \) to be the maximum quantifier depth of the sentences \( \varphi'_I \). Let \( \mathcal{A} = \langle U, \tau^A \rangle \) and \( \mathcal{B} = \langle U, \tau^B \rangle \) be structures in \( \mathcal{C} \) with \( \langle \text{adom}(\mathcal{A}), <, \tau^A \rangle \approx_{r(k)} \langle \text{adom}(\mathcal{B}), <, \tau^B \rangle \). Our aim is now to find \(<\)-preserving mappings \( \alpha : \text{adom}(\mathcal{A}) \rightarrow U \) and \( \beta : \text{adom}(\mathcal{B}) \rightarrow U \) such that \( \langle U, <, \mathcal{Bip}'(\alpha(\tau^A)) \rangle \approx_k \langle U, <, \mathcal{Bip}'(\beta(\tau^B)) \rangle \).

To this end, let \( I \) be the set of all those \( i \in \{1, \ldots, c\} \) for which there exists a \(<\)-preserving mapping \( \alpha_i : \text{adom}(\mathcal{A}) \rightarrow U \) such that \( \langle U, <, \mathcal{Bip}'(\alpha_i(\tau^A)) \rangle \models \varphi_i \).

Furthermore, let \( J \) be the according set for \( \mathcal{B} \) instead of \( \mathcal{A} \).

If \( I \cap J \neq \emptyset \), then there exists an \( i \in \{1, \ldots, c\} \) and \(<\)-preserving mappings \( \alpha : \text{adom}(\mathcal{A}) \rightarrow U \) and \( \beta : \text{adom}(\mathcal{B}) \rightarrow U \) such that \( \langle U, <, \mathcal{Bip}'(\alpha(\tau^A)) \rangle \models \varphi_i \) and \( \langle U, <, \mathcal{Bip}'(\beta(\tau^B)) \rangle \models \varphi_i \).

From the choice of \( \varphi_1, \ldots, \varphi_c \), we know that \( \langle U, <, \mathcal{Bip}'(\alpha_1(\tau^A)) \rangle \) and \( \langle U, <, \mathcal{Bip}'(\beta_1(\tau^B)) \rangle \) must belong to the same equivalence class of \( \approx_k \). I.e., \( \langle U, <, \mathcal{Bip}'(\alpha_1(\tau^A)) \rangle \approx_k \langle U, <, \mathcal{Bip}'(\beta_1(\tau^B)) \rangle \).

All that remains to show is that indeed \( I \cap J \neq \emptyset \).

For the sake of contradiction, let us assume that \( I \cap J = \emptyset \). Note that the set \( I \) is defined in such a way that the formula \( \varphi_I \) is \(<\)-generic on \( \mathcal{A} \). Furthermore, if \( I \cap J = \emptyset \), then \( \varphi_I \) is \(<\)-generic on \( \mathcal{B} \), too, and we have \( \langle U, <, \mathcal{Bip}'(\tau^A) \rangle \models \varphi_I \) and \( \langle U, <, \mathcal{Bip}'(\tau^B) \rangle \not\models \varphi_I \).

Thus, from \((\ast)\) we obtain a \( \mathcal{FO}(<, \tau) \)-formula \( \varphi'_I \) of quantifier depth at most \( r(k) \), such that \( \langle \text{adom}(\mathcal{A}), <, \tau^A \rangle \models \varphi'_I \) and \( \langle \text{adom}(\mathcal{B}), <, \tau^B \rangle \not\models \varphi'_I \). However, \( \mathcal{A} \) and \( \mathcal{B} \) were chosen in such a way that \( \langle \text{adom}(\mathcal{A}), <, \tau^A \rangle \approx_{r(k)} \langle \text{adom}(\mathcal{B}), <, \tau^B \rangle \), which is a contradiction to Theorem 4.1. Altogether, this completes the proof of Theorem 4.4.

In the following two sections we will show how the duplicator can translate strategies for the \( \mathcal{FO}_{adom(<)} \)-game into strategies for the \( \mathcal{FO}(<, +) \)-game and the \( \mathcal{FO}(<, \mathcal{M}on) \)-game, where \( \mathcal{M}on \) is the class of all monadic relations. Via Theorem 4.4 these translations of strategies will directly give us the according collapse results. Apart from the results themselves, the exposition of explicit strategies for the EF-game will be interesting in its own right.
4.3 A Lemma Useful for the Sections 5 and 6

Before concentrating on the translation proofs for $\text{FO}(\cdot, \text{Mon})$ and $\text{FO}(\cdot, +)$, we first show the following easy lemma that will help us avoid some annoying case distinctions within our proofs.

4.5 Lemma. Let $P := \{p_1 < p_2 < p_3 < \cdots\}$ be a countable, infinitely increasing sequence. Let $\tau$ be a signature, and let $A$ and $B$ be two $\mathbb{N}$-embeddable $\tau$-structures over linearly ordered universes. Furthermore, let $\alpha : \text{dom}(A) \to P$ and $\beta : \text{dom}(B) \to P$ map, for every $j$, the $j$-th smallest element in $\text{dom}(A)$ and $\text{dom}(B)$, respectively, onto the position $p_j$. Let $r \in \mathbb{N}$ and $r \geq 2$.
If $\langle \text{dom}(A), <, \tau^A \rangle \approx_r \langle \text{dom}(B), <, \tau^B \rangle$, then also $A := \langle P, <, \alpha^A \rangle \approx_r \langle P, <, \beta^B \rangle =: B$.

Proof. Since $r \geq 2$, one can easily see that $\text{dom}(A)$ and $\text{dom}(B)$ are either both finite or both infinite.

First consider the case where $\text{dom}(A)$ and $\text{dom}(B)$ are both infinite. Then, $\alpha$ is an isomorphism between $\langle \text{dom}(A), <, \tau^A \rangle$ and $A$, and $\beta$ is an isomorphism between $\langle \text{dom}(B), <, \tau^B \rangle$ and $B$. This obviously implies that $A \approx_r B$.

There remains the case where $\text{dom}(A)$ and $\text{dom}(B)$ are both finite. Let $m$ and $n$ denote the cardinalities of $\text{dom}(A)$ and $\text{dom}(B)$, respectively. From our presumption we know that the duplicator has a winning strategy in the $r$-round EF-game on $\langle \text{dom}(A), <, \tau^A \rangle$ and $\langle \text{dom}(B), <, \tau^B \rangle$. Henceforth, this game will be called the small game.

We now describe a winning strategy for the duplicator in the big game, i.e., in the $r$-round EF-game on $A$ and $B$. An illustration of this strategy is given in Figure 2.

In each round $i \in \{1, \ldots, r\}$ of the big game we proceed as follows: If the spoiler chooses an
element $a_i$ in the universe of $\mathfrak{A}$, we distinguish between two cases. (If he chooses an element $b_i$ in the universe of $\mathfrak{B}$, we proceed in the according way, interchanging the roles of $\mathfrak{A}$ and $\mathfrak{B}$.)

**Case 1:** $a_i \notin \alpha(\text{adom}(\mathfrak{A}))$, i.e., $a_i = p_{n+d_i}$ for some $d_i \in \mathbb{N}_{>0}$. In this case the duplicator chooses $b_i := p_{n+d_i}$.

**Case 2:** $a_i \in \alpha(\text{adom}(\mathfrak{A}))$, i.e., $a_i = \alpha(a_i)$ for some $a_i \in \text{adom}(\mathfrak{A})$. In this case we define $a_i$ to be a move for a “virtual spoiler” in the $i$-th round of the small game on $\langle \text{adom}(\mathfrak{A}), <, \tau^A \rangle$. A “virtual duplicator” who plays according to her winning strategy in the small game will find some answer $b_i$ in $\langle \text{adom}(\mathfrak{B}), <, \tau^B \rangle$. We can translate this answer into a move $b_i$ for the duplicator in the big game via $b_i := \beta(b_i)$.

After $r$ rounds, the “virtual duplicator” has won the small game; and it is straightforward to check that the duplicator has also won the big game.

Altogether, this completes the proof of Lemma [4.5].

5 How to Win the Game for $\text{FO}(<, \text{Mon})$

In this section we concentrate on the class $\text{Mon}$ of monadic, i.e., unary, built-in predicates. We consider the context structure $\langle \mathbb{U}, <, \text{Mon} \rangle$, for any linearly ordered infinite universe $\mathbb{U}$; and we explicitly describe how the duplicator can translate strategies for the $\text{FO}_{\text{adom}}(<)$-game into strategies for the $\text{FO}(<, \text{Mon})$-game on $\mathbb{Z}$-embeddable structures over $\mathbb{U}$. The overall proof idea is an adaption and extension of a proof developed by several researchers in the context of the Crane Beach conjecture [4] for the specific context of finite strings instead of arbitrary structures.

5.1 Theorem (FO$(<, \text{Mon})$-game for $\mathbb{Z}$-embeddable structures).

Let $\langle \mathbb{U}, < \rangle$ be a linearly ordered infinite structure, and let $\text{Mon}$ be the class of all monadic predicates on $\mathbb{U}$.

The duplicator can translate strategies for the $\text{FO}_{\text{adom}}(<)$-game into strategies for the $\text{FO}(<, \text{Mon})$-game on $\mathbb{Z}$-embeddable structures over $\mathbb{U}$. □

**Proof.** Let $\text{Mon}'$ be a finite subset of $\text{Mon}$, and let $\tau$ be a signature. For every number $k \in \mathbb{N}_{>0}$ of rounds for the $\text{FO}(<, \text{Mon}')$-game we choose $r(k) := k+1$ to be the according number of rounds for the $\text{FO}_{\text{adom}}(<)$-game.

Now let $\mathcal{A} = \langle \mathbb{U}, \tau^A \rangle$ and $\mathcal{B} = \langle \mathbb{U}, \tau^B \rangle$ be two $\mathbb{Z}$-embeddable structures on which the duplicator wins the $(k+1)$-round $\text{FO}_{\text{adom}}(<)$-game, i.e.,

$$\langle \text{adom}(\mathcal{A}), <, \tau^A \rangle \approx_{k+1} \langle \text{adom}(\mathcal{B}), <, \tau^B \rangle.$$

Our aim is to find $<$-preserving mappings $\alpha: \text{adom}(\mathcal{A}) \rightarrow \mathbb{U}$ and $\beta: \text{adom}(\mathcal{B}) \rightarrow \mathbb{U}$ such that the duplicator wins the $k$-round $\text{FO}(<, \text{Mon}')$-game on $\alpha(\mathcal{A})$ and $\beta(\mathcal{B})$, i.e., $\langle \mathbb{U}, <, \text{Mon}', \alpha(\tau^A) \rangle \approx_k \langle \mathbb{U}, <, \text{Mon}', \beta(\tau^B) \rangle$.

Note that the condition $(\ast)$ gives us that, in particular, $\text{adom}(\mathcal{A})$ has a lower bound (respectively, an upper bound) if and only if $\text{adom}(\mathcal{B})$ has. Since $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{Z}$-embeddable, we know that they have no accumulation points and that exactly one of the following four cases is valid:

**Case I:** $\text{adom}(\mathcal{A}) = \{u_1 < u_2 < \cdots\}$ and $\text{adom}(\mathcal{B})$ are infinitely increasing.

**Case II:** $\text{adom}(\mathcal{A}) = \{u_1 > u_2 > \cdots\}$ and $\text{adom}(\mathcal{B})$ are infinitely decreasing.


Case III: \( \text{dom}(A) = \{ \cdots < u_{-2} < u_{-1} < u_1 < u_2 < \cdots \} \) and \( \text{dom}(B) \) are infinite in both directions.

Case IV: \( \text{dom}(A) \) and \( \text{dom}(B) \) are finite.

Let us first concentrate on Case I, i.e., on the case where \( \text{dom}(A) \) and \( \text{dom}(B) \) are infinitely increasing. Let \( u_1 < u_2 < \cdots \) such that \( \text{dom}(A) = \{ u_1, u_2, \ldots \} \).

**Step 1:** We first choose a suitable subsequence \( p_1 < p_2 < \cdots \) of \( u_1 < u_2 < \cdots \) onto which the active domain elements of \( A \) and \( B \) will be moved via \( \prec \)-preserving mappings \( \alpha \) and \( \beta \). To find this sequence, we use the following theorem from Ramsey Theory. A well-presented introduction to Ramsey Theory as well as a proof of the Ramsey Theorem can be found in Diestel’s textbook \([\text{[2]} \text{, Section 9]}\).

**5.2 Theorem (Ramsey).** Let \( G = \langle V, E \rangle \) be the graph with vertex set \( V = \{ u_1, u_2, \ldots \} \) and edge set \( E = \{ (u_i, u_j) \in V^2 : i < j \} \). Let \( C \) be a finite set, and let each edge \( (u_i, u_j) \) of \( G \) be colored with an element \( \text{col}(u_i, u_j) \in C \).

There exists an infinite monochromatic path, i.e., there is an infinite sequence \( p_1 < p_2 < \cdots \) in \( V \), such that \( \text{col}(p_1, p_2) = \text{col}(p_2, p_3) = \cdots = \text{col}(p_i, p_j) \), for all \( i < j \). □

We choose the following coloring: The edge \( (u_i, u_j) \) is colored with the \( k \)-type of the substructure of \( (U, \prec, \text{Mon}') \) with universe \( \{ u_i \} \subseteq U : u_i \leq u < u_j \}. \). I.e., we choose \( \text{col}(u_i, u_j) := k \)-type \( u_i, u_j \), where \( k \)-type \( u_i, u_j \) is the equivalence class of the structure \( \{ u_i, u_j \}, u_i \prec, \text{Mon}' \) with respect to the relation \( \approx_k \). According to Remark 4.2, the number of \( k \)-types is finite, and hence the Ramsey Theorem 5.2 gives us an infinite monochromatic path, i.e., an infinite sequence \( p_1 < p_2 < \cdots \) in \( V \) such that \( \text{col}(p_1, p_2) = \text{col}(p_2, p_3) = \cdots = \text{col}(p_i, p_j) \), for all \( i < j \). Note that, by definition, \( k \)-type \( p_j, p_{j+1} \) \( k \)-type \( p_j, p_{j+1} \) means that

\( (**): \quad \langle p_j, p_{j+1}, p_j, \prec, \text{Mon}' \rangle \equiv_k \langle p_j, p_{j+1}, p_j, \prec, \text{Mon}' \rangle \).

The positions \( p_1, p_2, \ldots \) will be called “special positions”, and the set of all special positions will be denoted \( P \). We define \( \alpha \) and \( \beta \) to be the \( \prec \)-preserving mappings that move the active domain elements of \( A \) and \( B \) onto the “special positions”. Precisely, \( \alpha : \text{dom}(A) \to P \) and \( \beta : \text{dom}(B) \to P \) map, for every \( j \), the \( j \)-th smallest element of \( \text{dom}(A) \) and \( \text{dom}(B) \), respectively, onto the position \( p_j \).

From the presumption (*) and from Lemma 4.3 we obtain that a “virtual duplicator” has a winning strategy for the \( k \)-round EF-game on \( \mathcal{A}' := (P, \prec, \alpha(p_n)) \) and \( \mathcal{B}' := (P, \prec, \beta(p_n)) \). I.e., we know that \( \mathcal{A}' := (P, \prec, \alpha(p_n)) \equiv_k (P, \prec, \beta(p_n)) \equiv : \mathcal{B}' \). Henceforth, this game will be called the \( \prec \)-game (on \( \mathcal{A}' \) and \( \mathcal{B}' \)).

**Step 2:** We now describe a winning strategy for the duplicator in the \( k \)-round FO(\( <, \text{Mon}' \))

The positions \( p_1, p_2, \ldots \) will be called “special positions”, and the set of all special positions will be denoted \( P \). We define \( \alpha \) and \( \beta \) to be the \( \prec \)-preserving mappings that move the active domain elements of \( A \) and \( B \) onto the “special positions”. Precisely, \( \alpha : \text{dom}(A) \to P \) and \( \beta : \text{dom}(B) \to P \) map, for every \( j \), the \( j \)-th smallest element of \( \text{dom}(A) \) and \( \text{dom}(B) \), respectively, onto the position \( p_j \).

From the presumption (*) and from Lemma 4.3 we obtain that a “virtual duplicator” has a winning strategy for the \( k \)-round EF-game on \( \mathcal{A}' := (P, \prec, \alpha(p_n)) \) and \( \mathcal{B}' := (P, \prec, \beta(p_n)) \). I.e., we know that \( \mathcal{A}' := (P, \prec, \alpha(p_n)) \equiv_k (P, \prec, \beta(p_n)) \equiv : \mathcal{B}' \). Henceforth, this game will be called the \( \prec \)-game (on \( \mathcal{A}' \) and \( \mathcal{B}' \)).

In each round \( i \in \{ 1, \ldots, k \} \) of the \( \text{Mon}' \)-game we proceed as follows: If the spoiler chooses an element \( a_i \) in the universe of \( \mathcal{A} \), we distinguish between two cases. (If he chooses an element \( b_i \) in the universe of \( \mathcal{B} \), we proceed in the according way, interchanging the roles of \( \mathcal{A} \) and \( \mathcal{B} \).)
Case 1: \( a_i \) is smaller than the smallest “special position”, i.e., \( a_i < p_1 \). In this case, the duplicator chooses the identical element in the universe of \( \mathfrak{B} \), i.e., she chooses \( b_i := a_i < p_1 \).

Case 2: \( a_i \geq p_1 \). In this case there exists a \( j \in \mathbb{N}_0 \) such that \( a_i \in [p_j, p_{j+1}) \) (note that we essentially use here that \( P \) has no accumulation points in \( \overline{U} \)). The position \( p_j \) represents the interval \( [p_j, p_{j+1}) \) to which the spoiler’s choice \( a_i \) belongs. We define \( a'_i := p_j \) to be a move for a “virtual spoiler” in the \( i \)-th round of the \( < \)-game on \( \mathfrak{A}' \). A “virtual duplicator” who plays according to her winning strategy in the \( < \)-game will find some answer \( b'_i \) in \( \mathfrak{B}' \). Let \( j' \in \mathbb{N}_0 \) such that \( b'_i = p_{j'} \).

The duplicator in the \( \text{Mon}' \)-game will choose some \( b_i \) in \( \mathfrak{B} \) that lies in the interval \( [p_{j'}, p_{j'+1}) \). But which element in this interval shall she choose? — Here we make use of the fact that another “virtual duplicator” wins the game \((**\text{)}\) on the intervals \( [p_j, p_{j+1}) \) and \( [p_{j'}, p_{j'+1}) \): Let \( a_1, \ldots, a_s \) be those elements among \( a_1, \ldots, a_{i-1} \) that lie in the interval \( [p_j, p_{j+1}) \). By induction we know that \( \{s_1, \ldots, s_t\} = \{s \in \{1, \ldots, i-1\} : a_s \in [p_j, p_{j+1})\} \) is a special position in \( \mathfrak{B} \). By induction with \((**\text{)}\) we know that

\[
\langle [p_j, p_{j+1}), p_j, <, \text{Mon}', a_{i-1} \rangle \approx_{k-i+1} \langle [p_{j'}, p_{j'+1}), p_{j'}, <, \text{Mon}', b_{i-1} \rangle.
\]

For \( i=1 \) this is true because of \((**\text{)}\); for \( i>1 \) this follows from the duplicator’s choices in the previous rounds. Since \( a_i \in [p_j, p_{j+1}) \), a “virtual duplicator” in the game \((**\text{)}\) can choose a suitable \( b_i \in [p_{j'}, p_{j'+1}) \), such that

\[
\langle [p_j, p_{j+1}), p_j, <, \text{Mon}', a_{i-1}, a_i \rangle \approx_{k-i} \langle [p_{j'}, p_{j'+1}), p_{j'}, <, \text{Mon}', b_{i-1}, b_i \rangle.
\]

We choose exactly this \( b_i \) to be the answer of the duplicator in the \( i \)-th round of the \( \text{Mon}' \)-game on \( \mathfrak{B} \).
After \( k \) rounds we know that the “virtual duplicator” has won the \(<\)-game (on \( \mathcal{A}' \) and \( \mathcal{B}' \)) as well as all the interval games (\( \ast \ast \)). It is straightforward (although tedious) to check that the duplicator has also won the \( \text{Mon}' \)-game (on \( \mathcal{A} \) and \( \mathcal{B} \)). This completes the proof of Theorem 5.1 for Case I, i.e., for the case that \( \text{adom}(\mathcal{A}) \) and \( \text{adom}(\mathcal{B}) \) are infinitely increasing.

Case II, i.e., the case where \( \text{adom}(\mathcal{A}) \) and \( \text{adom}(\mathcal{B}) \) are infinitely decreasing, is symmetric to Case I.

Let us now concentrate on Case III, i.e., the case where \( \text{adom}(\mathcal{A}) \) and \( \text{adom}(\mathcal{B}) \) are infinite in both directions. Let \( \text{adom}(\mathcal{A}) = \{ \cdots < u_{-2} < u_{-1} < u_1 < u_2 < \cdots \} \). The problem here is that the Ramsey Theorem 5.2 gives us one infinite monochromatic increasing path \( p_1 < p_2 < \cdots \), and another infinite monochromatic decreasing path \( p_{-1} > p_{-2} > \cdots \). However, these two paths do not necessarily have the same color. Imagine, e.g., that all edges on the increasing path are colored “blue” and all edges on the decreasing path are colored “red”. We therefore have to carefully decide which part of the original structure is mapped onto the “blue” path and which part is mapped onto the “red” path. To this end, let a “virtual spoiler” choose an element \( a_{\text{blue}} \) in \( \text{adom}(\mathcal{A}) \) in the first round of the game (\( \ast \)). A “virtual duplicator” who wins the game (\( \ast \)) can answer with an element \( b_{\text{blue}} \) in \( \text{adom}(\mathcal{B}) \) such that

\[
\ast' \quad \langle \text{adom}(\mathcal{A}), <, \tau^A, a_{\text{blue}} \rangle \approx_k \langle \text{adom}(\mathcal{B}), <, \tau^B, b_{\text{blue}} \rangle.
\]

The idea is now to map the active domain elements of \( \mathcal{A} \) which are \( \geq a_{\text{blue}} \) (and the active domain elements of \( \mathcal{B} \) which are \( \geq b_{\text{blue}} \)) onto an increasing “blue” path and which are colored “blue” and all edges on the decreasing path are colored “red”. We therefore have to carefully decide which part of the original structure is mapped onto the “blue” path and which part is mapped onto the “red” path. To this end, let a “virtual spoiler” choose an element \( a_{\text{blue}} \) in \( \text{adom}(\mathcal{A}) \) in the first round of the game (\( \ast \)). A “virtual duplicator” who wins the game (\( \ast \)) can answer with an element \( b_{\text{blue}} \) in \( \text{adom}(\mathcal{B}) \) such that

\[
\ast' \quad \langle \text{adom}(\mathcal{A}), <, \tau^A, a_{\text{blue}} \rangle \approx_k \langle \text{adom}(\mathcal{B}), <, \tau^B, b_{\text{blue}} \rangle.
\]

The idea is now to map the active domain elements of \( \mathcal{A} \) which are \( \geq a_{\text{blue}} \) (and the active domain elements of \( \mathcal{B} \) which are \( \geq b_{\text{blue}} \)) onto an increasing “blue” path and which part is mapped onto the “red” path. To this end, let a “virtual spoiler” choose an element \( a_{\text{blue}} \) in \( \text{adom}(\mathcal{A}) \) in the first round of the game (\( \ast \)). A “virtual duplicator” who wins the game (\( \ast \)) can answer with an element \( b_{\text{blue}} \) in \( \text{adom}(\mathcal{B}) \) such that

\[
\ast' \quad \langle \text{adom}(\mathcal{A}), <, \tau^A, a_{\text{blue}} \rangle \approx_k \langle \text{adom}(\mathcal{B}), <, \tau^B, b_{\text{blue}} \rangle.
\]

The idea is now to map the active domain elements of \( \mathcal{A} \) which are \( \geq a_{\text{blue}} \) (and the active domain elements of \( \mathcal{B} \) which are \( \geq b_{\text{blue}} \)) onto an increasing “blue” path and which part is mapped onto the “red” path. To this end, let a “virtual spoiler” choose an element \( a_{\text{blue}} \) in \( \text{adom}(\mathcal{A}) \) in the first round of the game (\( \ast \)). A “virtual duplicator” who wins the game (\( \ast \)) can answer with an element \( b_{\text{blue}} \) in \( \text{adom}(\mathcal{B}) \) such that

\[
\ast' \quad \langle \text{adom}(\mathcal{A}), <, \tau^A, a_{\text{blue}} \rangle \approx_k \langle \text{adom}(\mathcal{B}), <, \tau^B, b_{\text{blue}} \rangle.
\]

The idea is now to map the active domain elements of \( \mathcal{A} \) which are \( \geq a_{\text{blue}} \) (and the active domain elements of \( \mathcal{B} \) which are \( \geq b_{\text{blue}} \)) onto an increasing “blue” path and which part is mapped onto the “red” path. To this end, let a “virtual spoiler” choose an element \( a_{\text{blue}} \) in \( \text{adom}(\mathcal{A}) \) in the first round of the game (\( \ast \)). A “virtual duplicator” who wins the game (\( \ast \)) can answer with an element \( b_{\text{blue}} \) in \( \text{adom}(\mathcal{B}) \) such that

\[
\ast' \quad \langle \text{adom}(\mathcal{A}), <, \tau^A, a_{\text{blue}} \rangle \approx_k \langle \text{adom}(\mathcal{B}), <, \tau^B, b_{\text{blue}} \rangle.
\]
Case IV, i.e., the case where $\text{dom}(A)$ and $\text{dom}(B)$ are finite, can be treated in a similar way as Case I. However, unlike in the previous cases, we cannot take the “special positions” $p_1 < p_2 < \cdots$ from the active domain of $\mathcal{A}$, since $\text{dom}(A)$ is finite. However, since $\mathcal{U}$ is infinite, there must exist an infinite increasing sequence $u_1 < u_2 < \cdots$ or an infinite decreasing sequence $u_{-1} > u_- > \cdots$ (see Fact 5.3 below). If we have an infinite increasing sequence $u_1 < u_2 < \cdots$, we can proceed in the same way as in Case I to obtain an infinite subsequence $p_1 < p_2 < \cdots$ such that $k$-type $[p_1, p_2) = k$-type $[p_j, p_{j+1})$, for all $j \in \mathbb{N}_{>0}$. Define $\alpha$ and $\beta$ to be the $\langle \cdot \rangle$-preserving mappings which move the active domain elements of $\mathcal{A}$ and $\mathcal{B}$ onto the “special positions” $p_1 < p_2 < \cdots$. The rest of the proof is identical to the proof for Case I. The case where we have an infinite decreasing sequence in $\mathcal{U}$ is symmetric to the case where we have an infinite increasing sequence in $\mathcal{U}$.

Altogether, this completes the proof of Theorem 5.1.

In the above proof we used the following well-known fact from Analysis:

5.3 Fact. Let $(\mathcal{U}, \langle \rangle)$ be a linearly ordered infinite structure. There exists an infinitely increasing sequence $u_1 < u_2 < \cdots$ or an infinitely decreasing sequence $u_1 > u_2 > \cdots$ of elements in $\mathcal{U}$.

To conclude the investigation of the class $\mathfrak{M}_0$ of monadic predicates, let us mention that several generalizations of the notion of $\mathbb{Z}$-embeddable structures are conceivable, to which the proof of Theorem 5.1 can be generalized — e.g.: structures whose active domain is of the form $u_1 < u_2 < u_3 < \cdots$ where $u_1 < u_2 < u_3 < \cdots$ is infinitely increasing, $v_1 > v_2 > v_3 > \cdots$ is infinitely decreasing, and $u_i < v_j$ for all $i, j \in \mathbb{N}_{>0}$.

It remains open whether Theorem 5.1 is still valid when replacing “$\mathbb{Z}$-embeddable structures” with “arbitrary structures”.

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6 How to Win the Game for $\text{FO}(<, +, Q)$

In this section we concentrate on context structures with built-in addition relation $+$. We show that the duplicator can translate strategies for the $\text{FO}_{\text{adom}}(<)$-game into strategies for the $\text{FO}(<, +)$-game on arbitrary structures over $\mathbb{N}$. We even obtain the following extension of this result: We enrich the context structures $\langle \mathbb{N}, <, + \rangle$ and $\langle \mathbb{Z}, <, + \rangle$ with a set $Q \subseteq \mathbb{N}$ which is not definable in $\text{FO}(<, +)$. We expose certain conditions $W(\omega)$ and show that the duplicator can translate strategies for the $\text{FO}_{\text{adom}}(<)$-game into strategies for the $\text{FO}(<, +, Q)$-game on arbitrary structures over $\mathbb{N}$ and on $\mathbb{N}$-embeddable structures over $\mathbb{Z}$, whenever $Q$ satisfies the conditions $W(\omega)$. This possibility of translating strategies for the augmented context structure $\langle \mathbb{N}, <, +, Q \rangle$ is notable especially in the light of Fact 3.8 (b) which (together with Theorem 4.4) tells us that the translation is not possible when replacing $Q$ with the set $\text{Squares}$ of all square numbers.

In Section 6.3 we transfer the translation result to $\mathbb{N}$-embeddable structures over the context structure $\langle \mathbb{R}, <, +, Q, \text{Groups} \rangle$, where $\text{Groups}$ is the class of all subsets of $\mathbb{R}$ that contain the number 1 and that are groups with respect to $+$. In particular, this implies the translation result for the context structures $\langle Q, <, + \rangle$, $\langle Q, <, +, \mathbb{Z} \rangle$, and $\langle \mathbb{R}, <, +, \mathbb{Z}, Q \rangle$. In Section 6.4 we present some variations and consequences of the translation proofs, including the result that even all subsets of $Q$ may be added as built-in predicates.

Since the duplicator’s strategy in the $\text{FO}(<, +, Q)$-game is rather involved, we first concentrate on a basic case which, as a side product, will give us an EF-game proof of the theorem of Ginsburg and Spanier, stating that the spectra of $\text{FO}(<, +)$-sentences are semi-linear.

6.1 A Basic Case of the $\text{FO}(<, +)$-Game over $\mathbb{Z}$

Assume that we are given a number $n \in \mathbb{N}$ and two structures $\mathfrak{A} := \langle \mathbb{Z}, <, +, a_1, \ldots, a_n \rangle$ and $\mathfrak{B} := \langle \mathbb{Z}, <, +, b_1, \ldots, b_n \rangle$. The aim of this section is to find, for each $k \in \mathbb{N}_{>0}$, a list $W(k)$ of conditions such that the duplicator wins the $k$-round EF-game on $\mathfrak{A}$ and $\mathfrak{B}$ whenever $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ satisfy the conditions $W(k)$. This question has been considered before:

- In the textbook [17, Section 3.3] such conditions were formulated, aiming at a proof for the decidability of Presburger arithmetic.
- Ruhl [21] obtained according conditions for the (more difficult) $k$-round EF-game for first-order logic with unary counting quantifiers and addition.
- Lynch [24] developed a winning strategy for the duplicator in the $k$-round $\text{FO}(<, +, P_k)$-game, for a suitable set $P_k$ of natural numbers.
- Lautemann and the author of the present paper [22] extended Lynch’s method in order to show that the duplicator can translate strategies for the $\text{FO}_{\text{adom}}(<)$-game into strategies for the $\text{FO}(<, +)$-game; we will prove (an extension of) this in the following Section 6.2.

All the above references are written in a top-down manner, i.e., they first formulate the (very involved) conditions, and afterwards they prove that the conditions indeed lead to a winning strategy for the duplicator. However, it remains unclear how one can find such conditions and why they need to be chosen in the way they are. In the present section we try to answer this question by developing the conditions in a bottom-up manner.

We start with $k = 1$. In the unique round of the EF-game elements $a_{n+1}$ and $b_{n+1}$ are chosen.
in \( \mathfrak{A} \) and \( \mathfrak{B} \) — and afterwards the duplicator shall have won the game. I.e., for all \( \mu, \nu, \eta \in \{1, \ldots, n+1\} \) we shall have

\[
a_\mu < a_\nu \text{ iff } b_\mu < b_\nu \quad \text{and} \quad a_\mu + a_\nu = a_\eta \text{ iff } b_\mu + b_\nu = b_\eta.
\]

What conditions do these atoms impose on \( a_{n+1} \) and \( b_{n+1} \)?

Let us have a look at all atoms that involve \( a_{n+1} \), and let us solve these atoms for \( a_{n+1} \):

| atoms involving \( a_{n+1} \) | solved for \( a_{n+1} \) |
|-----------------------------|-----------------------------|
| \( a_{n+1} = a_\mu \)       | \( a_{n+1} = a_\mu \)       |
| \( a_{n+1} < a_\mu \)       | \( a_{n+1} < a_\mu \)       |
| \( a_{n+1} > a_\mu \)       | \( a_{n+1} > a_\mu \)       |
| \( a_\mu + a_\nu = a_{n+1} \) | \( a_{n+1} = a_\mu + a_\nu \) |
| \( a_{n+1} + a_\nu = a_\mu \) | \( a_{n+1} = a_\mu - a_\nu \) |
| \( a_\mu + a_\nu = a_{n+1} \) | \( a_{n+1} = 2a_\mu \)       |
| \( a_{n+1} + a_{n+1} = a_\mu \) | \( a_{n+1} = \frac{1}{2}a_\mu \) |
| \( a_{n+1} + a_{n+1} = a_{n+1} \) | \( a_{n+1} = 0 \)            |
| \( a_{n+1} + a_\mu = a_\mu \) | \( a_{n+1} = 0 \)            |
| \( a_{n+1} + a_\mu = a_{n+1} \) | no condition on \( a_{n+1} \) |

On the righthand side of the equations “\( a_{n+1} = \cdots \)” we have terms, or linear combinations, of the form \( d_1a_\mu + d_2a_\nu \), where \( \mu, \nu \in \{1, \ldots, n\} \), \( \mu \neq \nu \), \( d_1, d_2 \in \mathbb{Q}[2] \) := \( \{ \frac{u}{u'} : u, u' \in \mathbb{Z}, u' \neq 0, |u|, |u'| \leq 2 \} \). Each such linear combination \( s \) evaluates to a real number \( \overline{s} \). Let \( S \) be the set of all these linear combinations, and let \( T \) be the according set of linear combinations obtained from replacing \( a_1, \ldots, a_n \) with \( b_1, \ldots, b_n \). I.e., if \( s \in S \) is of the form \( d_1a_\mu + d_2a_\nu \), then the linear combination \( t := d_1b_\mu + d_2b_\nu \) is the according element of \( T \) that corresponds to \( s \). The evaluations of these linear combinations are distributed over the real numbers. An illustration is given in Figure 4.

![Figure 4](https://example.com/figure4.png)

**Figure 4:** The evaluations of all linear combinations \( s \) in \( S \) and all linear combinations \( t \) in \( T \). Integers are represented by strokes.

Certainly, if the spoiler chooses \( a_{n+1} = \overline{s} \), for some \( s \in S \), then the duplicator should answer \( b_{n+1} := \overline{t} \), for the corresponding \( t \in T \). Similarly, if the spoiler’s choice \( a_{n+1} \) lies strictly between \( \overline{s_1} \) and \( \overline{s_2} \), for \( s_1, s_2 \in S \), then the duplicator should answer a \( b_{n+1} \) that lies strictly between \( \overline{t_1} \) and \( \overline{t_2} \), for the corresponding \( t_1, t_2 \in T \). Obviously, the duplicator wins if the following conditions are satisfied:
Theorem 6.13 we will essentially need that the duplicator’s strategy works for all conditions. Precisely, the following procedure leads to a winning strategy for the duplicator: If the spoiler chooses $a_n + 1 \in \mathbb{Z}$ in $\mathcal{A}$ such that $a_n + 1 \neq \mathfrak{a} + f$ for some $s \in S$ and $f \in \text{int} \left[ -\frac{1}{2}, \frac{1}{2} \right] \subseteq \mathbb{R}$, then the duplicator answers $b_n + 1 \equiv t + f$, where $t$ is the according linear combination that corresponds to $s$. The gap parameter $f$ is added here to ensure that there is an integer between $\mathfrak{a}$ and $\mathfrak{a}_2$ if and only if there is an integer between $t_1$ and $t_2$. Indeed, it would suffice to restrict attention to rational $f \in \mathbb{Q}[2]$. However, later in this section, in the proof of Theorem 6.13 we will essentially need that the duplicator’s strategy works for all real numbers $f \in \text{int} \left[ -\frac{1}{2}, \frac{1}{2} \right] \subseteq \mathbb{R}$.

Certainly, the duplicator will win if the following two conditions are satisfied:

1. $\frac{1}{2} < \mathfrak{a}_1 + f < \mathfrak{a}_2 + h$ \iff \( t_1 + f < t_2 + h \)
   for all $s_1, s_2 \in S$ and the corresponding $t_1, t_2 \in T$, and all $f, h \in \text{int} \left[ -\frac{1}{2}, \frac{1}{2} \right]$. 

2. $\mathfrak{a} + f \in \mathbb{Z}$ \iff \( t + f \in \mathbb{Z} \)
   for all $s \in S$ and the corresponding $t \in T$, and all $f \in \text{int} \left[ -\frac{1}{2}, \frac{1}{2} \right]$. 

Since the denominator of a coefficient $d$ in a linear combination $s$ is either 1 or 2 or $-1$ or $-2$, condition (2.) is equivalent to the condition $\mathfrak{a}_v \equiv b_v \pmod{2}$ for all $v \in \{1, \ldots, n\}$.

If the spoiler chooses $a_n + 1 \in \mathbb{Z}$ in $\mathcal{A}$ such that $a_n + 1 \neq \mathfrak{a} + f$ for all $s \in S$ and all $f \in \text{int} \left[ -\frac{1}{2}, \frac{1}{2} \right]$, then determine the interval w.r.t. $S$ to which $a_n + 1$ belongs. I.e., choose $s_-, s_+ \in S$ such that $\mathfrak{a}_- < a_n + 1 < \mathfrak{a}_+$ and, for all $s \in S$, $\mathfrak{a} \leq \mathfrak{a}_-$ or $\mathfrak{a} \geq \mathfrak{a}_+$. Now, the duplicator takes her answer $b_n + 1$ from the corresponding interval in $\mathcal{B}$. I.e., she chooses the linear combinations $t_-, t_+ \in T$ that correspond to $s_-, s_+$, and she answers with an arbitrary $b_n + 1 \in \mathbb{Z}$ such that $\mathfrak{a}_- < b_n + 1 < \mathfrak{a}_+$. Such an integer does really exist, because we know that $\mathfrak{a}_- + \frac{1}{2} < a_n + 1 < \mathfrak{a}_+ - \frac{1}{2}$ and, due to condition (1.), $\mathfrak{a}_- + \frac{1}{2} < \mathfrak{a}_+ - \frac{1}{2}$, i.e., $\mathfrak{a}_- - \mathfrak{a}_+ > 1$.

What we have seen is the following:

\textbf{6.1 Lemma} \( \left( W(1) \Rightarrow \approx_1 \right) \).

Let $n \in \mathbb{N}$, let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Z}$, and let $\mathcal{A} := (\mathbb{Z}, < , a_1, \ldots, a_n)$ and $\mathcal{B} := (\mathbb{Z}, < , + , b_1, \ldots, b_n)$. The duplicator has a winning strategy in the 1-round EF-game on $\mathcal{A}$ and $\mathcal{B}$ if the following conditions $W(1)$ are satisfied:

\begin{itemize}
  \item[(*)] $a_v \equiv_2 b_v$ for all $v \in \{1, \ldots, n\}$, and
  \item[(**)] for all $f, h \in \text{int} \left[ -\frac{1}{2}, \frac{1}{2} \right] \subseteq \mathbb{R}$, for all $v_1, v_2, \mu_1, \mu_2 \in \{1, \ldots, n\}$ with $v_1 \neq v_2$ and $\mu_1 \neq \mu_2$, and all $d_1, d_2, e_1, e_2 \in \mathbb{Q}[2] := \{ \frac{u}{u'} : u, u' \in \mathbb{Z}, u' \neq 0, |u|, |u'| \leq 2 \}$, we have $d_1 a_{v_1} + d_2 a_{v_2} + f < e_1 a_{\mu_1} + e_2 a_{\mu_2} + h$
  \begin{itemize}
    \item if and only if $d_1 b_{v_1} + d_2 b_{v_2} + f < e_1 b_{\mu_1} + e_2 b_{\mu_2} + h$. \quad \square
  \end{itemize}
\end{itemize}

\textsuperscript{6}Recall from Section \( \mathfrak{a} \equiv_2 \) denotes the congruence relation modulo 2.
Let us now concentrate on the 2-round EF-game on $A := (\mathbb{Z}, <, +, a_1, \ldots, a_n)$ and $B := (\mathbb{Z}, <, +, b_1, \ldots, b_n)$. Our aim is to find a list $W(2)$ of conditions that enable the duplicator to play the first round in such a way that afterwards the conditions $W(1)$ are satisfied. From Lemma 6.1 we then obtain that the duplicator can play the remaining round in such a way that she wins the game.

In general, by induction on $k$, we will find a list $W(k+1)$ of conditions that enable the duplicator to play the first round in such a way that afterwards the conditions $W(k)$ are satisfied. To this end we consider the following generalization of the conditions $W(1)$.

6.2 Definition ((l, c, g)-Combinations; Conditions $C(m, l, c, g)$).

Let $m, l, c \in \mathbb{N}_{>0}$ and $g \in \mathbb{R}_{>0}$. Here,

- $m$ is the modulus with respect to which $a_\nu$ and $b_\nu$ shall be congruent,
- $l$ is the maximum length of the linear combinations under consideration,
- $c$ is the maximum size of the numerator and the denominator of the coefficients occurring in linear combinations, and
- $g$ is the maximum size of the gap parameters that are respected by the linear combinations.

Let $n \in \mathbb{N}_{>0}$ and let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Z}$.

An $(l, c, g)$-combination over $a_1, \ldots, a_n$ is a formal sum, or a linear combination, of the form $\sum_{\nu=1}^{l'} d_\nu a_\nu + f$, where $l' \leq l$, $\nu_1, \ldots, \nu_{l'}$ are pairwise distinct elements in $\{1, \ldots, n\}$, $d_1, \ldots, d_{l'} \in \mathbb{Q}[c] := \left\{ \frac{u}{v} : u, u' \in \mathbb{Z}, u' \neq 0, |u|, |u'| \leq c \right\}$, and $f \in \text{int}[-g, g] \subseteq \mathbb{R}$.

Every $(l, c, g)$-combination $s$ evaluates to a real number $\bar{s}$.

Given an $(l, c, g)$-combination $s$ over $a_1, \ldots, a_n$, the according $(l, c, g)$-combination $t$ over $b_1, \ldots, b_n$ that corresponds to $s$ is obtained by replacing every $a_\nu$ in $s$ with $b_\nu$. I.e., if $s = \sum_{\nu=1}^{l'} d_\nu a_\nu + f$, then $t = \sum_{\nu=1}^{l'} d_\nu b_\nu + f$.

We say that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ satisfy the conditions $C(m, l, c, g)$ if and only if

(*) $a_\nu \equiv_m b_\nu$ for all $\nu \in \{1, \ldots, n\}$, and

(**) for all $(l, c, g)$-combinations $s_1$ and $s_2$ over $a_1, \ldots, a_n$ and the corresponding $(l, c, g)$-combinations $t_1$ and $t_2$ over $b_1, \ldots, b_n$ we have

$\bar{s}_1 < \bar{s}_2$ if and only if $\bar{t}_1 < \bar{t}_2$.

In particular, the conditions $W(1)$ are exactly the conditions $C(2, 2, 2, \frac{1}{2})$.

Our aim is now to find, for given parameters $m, l, c, g$, new parameters $\tilde{m}, \tilde{l}, \tilde{c}, \tilde{g}$ such that the following is true: If the conditions $C(\tilde{m}, \tilde{l}, \tilde{c}, \tilde{g})$ are satisfied at the beginning, then the duplicator can play one round of the EF-game in such a way that afterwards the conditions $C(m, l, c, g)$ are satisfied.

To this end, let $n \in \mathbb{N}$, let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Z}$, and let $A := (\mathbb{Z}, <, +, a_1, \ldots, a_n)$ and $B := (\mathbb{Z}, <, +, b_1, \ldots, b_n)$. In one round of the EF-game elements $a_{n+1}$ and $b_{n+1}$ are chosen in $A$ and $B$, and afterwards the conditions $C(m, l, c, g)$ shall be satisfied by $a_1, \ldots, a_{n+1}$ and $b_1, \ldots, b_{n+1}$. I.e.,
\(\ast\) \(a_\nu \equiv_m b_\nu\) for all \(\nu \in \{1, \ldots, n+1\}\), and

\(\ast\ast\) for all \((l, c, g)\)-combinations over \(a_1, \ldots, a_{n+1}\) of the form \(\sum_{i=1}^{l'} d_i a_{\nu_i} + f\) and \(\sum_{i=1}^{l''} e_i a_{\mu_i} + h\) we have

\[
\sum_{i=1}^{l'} d_i a_{\nu_i} + f < \sum_{i=1}^{l''} e_i a_{\mu_i} + h
\]

if and only if

\[
\sum_{i=1}^{l'} d_i b_{\nu_i} + f < \sum_{i=1}^{l''} e_i b_{\mu_i} + h.
\]

What conditions do the inequalities of \(\ast\ast\) impose on \(a_{n+1}\) and \(b_{n+1}\)? To answer this question, we have a look at all inequalities that involve \(a_{n+1}\) and we solve them for \(a_{n+1}\). Let, for example, \(\nu_1 = \mu_1 = n+1\), let \(d_1 > e_1\), let \(\nu_2 = \mu_2 \neq n+1\), and let the indices \(\nu_3, \ldots, \nu_{l'}, \mu_3, \ldots, \mu_{l''}\) be pairwise distinct (and different from \(n+1\) and \(\nu_2\)). In this case, we have

\[
\sum_{i=1}^{l'} d_i a_{\nu_i} + f < \sum_{i=1}^{l''} e_i a_{\mu_i} + h
\]

if and only if

\[
(d_1 - e_1) a_{n+1} < (e_2 - d_2) a_{\nu_2} + \sum_{i=3}^{l''} e_i a_{\mu_i} - \sum_{i=3}^{l'} d_i a_{\nu_i} + (h - f)
\]

if and only if

\(\ast\ast\ast\) : \(a_{n+1} < \frac{e_2 - d_2}{d_1 - e_1} a_{\nu_2} + \sum_{i=3}^{l''} \frac{e_i}{d_1 - e_1} a_{\mu_i} + \sum_{i=3}^{l'} \frac{d_i}{d_1 - e_1} a_{\nu_i} + \frac{h - f}{d_1 - e_1}\).

Let us have a close look at the coefficients on the righthand side of the last inequality \(\ast\ast\ast\):

We know that \(d_i, e_i \in \mathbb{Q}[c]\), i.e., that \(d_i = \frac{u_i}{v_i}\) and \(e_i = \frac{u_i}{v_i}\) for suitable integers \(u_i, u'_i, v_i, v'_i\) with \(|u_i|, |u'_i|, |v_i|, |v'_i| \leq c\). Hence, \(d_1 - e_1 = \frac{u_1}{v_1} - \frac{u_2}{v_2} = \frac{u_1 v_2 - u_2 v_1}{u_1 v_1}\). In particular, \(\frac{e_2 - d_2}{d_1 - e_1} = \frac{v_2 u_2' - u_2 v_2'}{u_1 v_1' - v_1 u_1'} \in \mathbb{Q}[2c^4]\). Obviously, also the other coefficients \(\frac{e_i}{d_1 - e_1}\) and \(\frac{d_i}{d_1 - e_1}\) belong to \(\mathbb{Q}[2c^4]\). Similarly, the gap parameter \(\frac{h - f}{d_1 - e_1}\) belongs to \(\text{int} [-2gc^2, 2gc^2] \subseteq \mathbb{R}\), because

\[
\left| \frac{h - f}{d_1 - e_1} \right| = |h - f| \cdot \left| \frac{u_1' v_2' - u_2' v_1'}{u_1 v_1' - v_1 u_1'} \right| \leq (|h| + |f|) \cdot |u_1'| \cdot |v_2'| \leq 2gc^2.
\]

Altogether, the righthand side of the inequality \(\ast\ast\ast\) is a \((2l-1, 2c^4, 2gc^2)\)-combination over \(a_1, \ldots, a_n\).

Indeed, one can easily see that every inequality of \(\ast\ast\) that involves \(a_{n+1}\)

- is either, for \(\kappa \in \{<, >\}\), equivalent to an inequality of the form “\(a_{n+1} \kappa \cdots\)”, the righthand side of which is a \((2l-1, 2c^4, 2gc^2)\)-combination over \(a_1, \ldots, a_n\), or
- does not impose any condition on \(a_{n+1}\) at all.

Let \(S\) be the set of all \((2l-1, 2c^4, 2gc^2)\)-combinations over \(a_1, \ldots, a_n\), and let \(T\) be the according set of \((2l-1, 2c^4, 2gc^2)\)-combinations for \(b_1, \ldots, b_n\) instead of \(a_1, \ldots, a_n\). If \(s \in S\) is of the form \(\sum_{i=1}^{l'} d_i a_{\nu_i} + f\), then \(t := \sum_{i=1}^{l'} d_i b_{\nu_i} + f\) is the according element in \(T\) that corresponds to \(s\). The evaluations \(\bar{s}\) (for all \(s \in S\)) and \(\bar{t}\) (for all \(t \in T\)) of these linear combinations are distributed
over the real numbers.

Certainly, if the spoiler chooses \( a_{n+1} = \bar{s} \), for some \( s \in S \), then the duplicator should answer \( b_{n+1} := \bar{t} \), for the corresponding \( t \in T \). Similarly, if the spoiler’s choice \( a_{n+1} \) lies strictly between \( \overline{\mathbb{R}} \) and \( \mathbb{R} \), for \( s_1, s_2 \in \mathbb{S} \), then the duplicator should answer a \( b_{n+1} \) that lies strictly between \( \mathbb{T}_1 \) and \( \mathbb{T}_2 \) and that belongs to the same residue class modulo \( m \) as \( a_{n+1} \) (here, \( t_1 \) and \( t_2 \) are the according linear combinations that correspond to \( s_1 \) and \( s_2 \)). Afterwards, \( a_1, \ldots, a_{n+1} \) and \( b_1, \ldots, b_{n+1} \) satisfy the conditions \( C(m, l, c, g) \), if the following is true:

- The numbers \( \bar{s} \), for all \( s \in S \), are ordered in the same way as the corresponding numbers \( \bar{t} \), for all \( t \in T \),
- \( \bar{s} \equiv_m \bar{t} \), for every \( s \in S \) and the corresponding \( t \in T \), and
- for every \( r \in \{0, \ldots, m-1\} \), there is an integer \( a \) between \( \overline{\mathbb{R}} \) and \( \mathbb{R} \) with \( a \equiv_m r \) if and only if there is an integer \( b \) between \( \mathbb{T}_1 \) and \( \mathbb{T}_2 \) with \( b \equiv_m r \) (for all \( s_1, s_2 \in S \) and the corresponding \( t_1, t_2 \in T \)).

Precisely, the following procedure leads to a successful strategy for the duplicator:

If the spoiler chooses \( a_{n+1} \in \mathbb{Z} \) in \( \mathbb{A} \) such that \( a_{n+1} = \bar{s} + f' \), for some \( s \in S \) and \( f' \in \text{int} [-\frac{m}{2}, \frac{m}{2}] \subseteq \mathbb{R} \), then the duplicator answers \( b_{n+1} := \bar{t} + f' \), where \( t \in T \) is the according linear combination that corresponds to \( s \). The gap parameter \( f' \) is added here to ensure, for every \( r \in \{0, \ldots, m-1\} \), that there is an integer \( a \) between \( \overline{\mathbb{R}} \) and \( \mathbb{R} \) with \( a \equiv_m r \) if and only if there is an integer \( b \) between \( \mathbb{T}_1 \) and \( \mathbb{T}_2 \) with \( b \equiv_m r \).

Certainly, the conditions \( C(m, l, c, g) \) are satisfied if the following is true:

1. \( \overline{\mathbb{R}} + f' < \mathbb{R} + h' \) \iff \( \mathbb{T}_1 + f' < \mathbb{T}_2 + h' \)
   for all \( s_1, s_2 \in S \) and the corresponding \( t_1, t_2 \in T \), and all \( f', h' \in \text{int} [-\frac{m}{2}, \frac{m}{2}] \).
2. \( \bar{s} + f' \equiv_m \bar{t} + f' \)
   for all \( s \in S \) and the corresponding \( t \in T \), and all \( f' \in \text{int} [-\frac{m}{2}, \frac{m}{2}] \).

As explained below, condition (2.) can be replaced by the condition\footnote{Recall that \( \text{lcm}\{n_1, \ldots, n_k\} \) denotes the least common multiple of \( n_1, \ldots, n_k \).}

\( 2') \ a_\nu \equiv_m \text{lcm}\{1, \ldots, 2c^4\} b_\nu \quad \text{for all} \ \nu \in \{1, \ldots, n\} \).

This can be seen as follows: Let \( s \) be of the form \( \sum_{i=1}^{l'} d_i a_{\nu_i} + f \). We know that all the coefficients \( d_i \) belong to \( \mathbb{Q}[2c^4] \). I.e., \( d_i = \frac{a_{\nu_i}}{u_i} \) with \( u_i, u'_i \in \mathbb{Z}, u'_i \neq 0 \), and \( |u_i|, |u'_i| \leq 2c^4 \). By definition of \( \equiv_m \) we have \( \overline{s} + f' \equiv_m \bar{t} + f' \) if and only if there is an integer \( z \in \mathbb{Z} \) such that \( \overline{s} - \bar{t} = m \cdot z \). Of course, \( \overline{s} - \bar{t} = \sum_{i=1}^{l'} d_i (a_{\nu_i} - b_{\nu_i}) = \sum_{i=1}^{l'} u_i \cdot \frac{a_{\nu_i} - b_{\nu_i}}{u'_i} \).

Now, if \( a_{\nu_i} \equiv_m \text{lcm}\{1, \ldots, 2c^4\} b_{\nu_i} \), then \( a_{\nu_i} - b_{\nu_i} = z_i \cdot m \cdot \text{lcm}\{1, \ldots, 2c^4\} \) for a suitable \( z_i \in \mathbb{Z} \).

Thus, \( \overline{s} - \bar{t} = \sum_{i=1}^{l'} u_i \cdot z_i \cdot m \cdot \text{lcm}\{1, \ldots, 2c^4\} = \sum_{i=1}^{l'} u_i \cdot z_i \cdot \frac{1}{u'_i} \cdot \text{lcm}\{1, \ldots, 2c^4\} \). Since \( |u'_i| \in \{1, \ldots, 2c^4\} \), we thus have found the desired integer \( z := \sum_{i=1}^{l'} u_i \cdot z_i \cdot \frac{1}{u'_i} \cdot \text{lcm}\{1, \ldots, 2c^4\} \) with \( \overline{s} - \bar{t} = m \cdot z \). Altogether, this gives us that \( \overline{s} + f' \equiv_m \bar{t} + f' \). I.e., condition (2.) follows from condition (2.’).

(Indeed, one can easily see that both conditions are equivalent).
If the spoiler chooses \( a_{n+1} \in \mathbb{A} \) such that \( a_{n+1} \neq s + f' \) for all \( s \in S \) and all \( f' \in \text{int} \left[ -\frac{n}{2}, \frac{n}{2} \right] \), then determine the interval w.r.t. \( S \) to which \( a_{n+1} \) belongs. I.e., choose \( s_-, s_+ \in S \) such that \( \frac{s_-}{s_+} < a_{n+1} < \frac{s_+}{s_-} \) and, for all \( s \in S \), \( s \leq s_- \) or \( \frac{s_-}{s_+} \geq s \).

Now, the duplicator takes her answer \( b_{n+1} \) from the corresponding interval in \( \mathbb{B} \). I.e., let \( t_-, t_+ \in T \) be the according linear combinations that correspond to \( s_-, s_+ \). The element \( b_{n+1} \in \mathbb{Z} \) is chosen such that \( \frac{t_-}{t_+} < b_{n+1} < \frac{t_+}{t_-} \) and \( b_{n+1} \equiv_m a_{n+1} \). Such an integer does really exist, because we know that \( \frac{t_-}{t_+} < \frac{m}{s} < \frac{t_+}{t_-} \) and, due to condition (1.), \( \frac{t_-}{t_+} > t_+ - \frac{m}{t_-} \), i.e., \( t_+ - t_- > m \).

What we have seen is that the conditions (1.) and (2.)’ enable the duplicator to win the one-round \( k \)-round EF-game in such a way that afterwards the conditions \( C(m, l, c, g) \) are satisfied. Note that the conditions (1.) and (2.)’ are exactly the conditions \( C(\tilde{m}, \tilde{l}, \tilde{c}, \tilde{g}) \), with parameters \( \tilde{m}, \tilde{l}, \tilde{c}, \tilde{g} \) as defined in the following lemma that sums up what we have obtained so far:

**6.3 Lemma** \( (C(\tilde{m}, \tilde{l}, \tilde{c}, \tilde{g}) \Rightarrow C(m, l, c, g)) \). Let \( m, l, c \in \mathbb{N}_{>0} \) and let \( g \in \mathbb{R}_{>0} \). Define

\[
\begin{align*}
\tilde{m} &:= m \cdot \text{lcm}\{1, \ldots, 2c^4\}, \\
\tilde{l} &:= 2l - 1, \\
\tilde{c} &:= 2c^4, \\
\tilde{g} &:= 2g c^2 + \frac{m}{2}.
\end{align*}
\]

Let \( n \in \mathbb{N} \) and let \( a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Z} \).

Let \( \mathbb{A} := (\mathbb{Z}, <, +, a_1, \ldots, a_n) \) and let \( \mathbb{B} := (\mathbb{Z}, <, +, b_1, \ldots, b_n) \).

If \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) satisfy the conditions \( C(\tilde{m}, \tilde{l}, \tilde{c}, \tilde{g}) \), then the duplicator can play one round in the EF-game in which integers \( a_{n+1} \) and \( b_{n+1} \) are chosen in \( \mathbb{A} \) and \( \mathbb{B} \) in such a way that afterwards the conditions \( C(m, l, c, g) \) are satisfied by \( a_1, \ldots, a_{n+1} \) and \( b_1, \ldots, b_{n+1} \).

Using the Lemmas 6.1 and 6.3, we can easily formulate, for every \( k \in \mathbb{N}_{>0} \), conditions \( W(k) \) which enable the duplicator to win the \( k \)-round EF-game:

**6.4 Theorem** \( (W(k) \Rightarrow \approx_k) \). By induction on \( k \) we define the functions

\[
\begin{align*}
m(1) &:= 2, & m(k+1) &:= m(k) \cdot \text{lcm}\{1, \ldots, 2c(k)^4\}, \\
l(1) &:= 2, & l(k+1) &:= 2l(k) - 1, \\
c(1) &:= 2, & c(k+1) &:= 2c(k)^4, \\
g(1) &:= \frac{1}{2}, & g(k+1) &:= 2g(k) c(k)^2 + \frac{m(k)}{2}.
\end{align*}
\]

We define \( W(k) \) to be exactly the conditions \( C(a(k), l(k), c(k), g(k)) \).

Let \( n \in \mathbb{N} \), let \( a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Z} \), and let \( \mathbb{A} := (\mathbb{Z}, <, +, a_1, \ldots, a_n) \) and \( \mathbb{B} := (\mathbb{Z}, <, +, b_1, \ldots, b_n) \).

If \( a_1, \ldots, a_n \) and \( b_1, \ldots, b_n \) satisfy the conditions \( W(k) \), then the duplicator has a winning strategy in the \( k \)-round EF-game on \( \mathbb{A} \) and \( \mathbb{B} \).

The duplicator’s strategy is summarized in Figure 5, where \( m(0) := 1 \).

**Proof.** By induction on \( k \). The induction start is established in Lemma 6.1. The induction step from \( k \) to \( k+1 \) follows from Lemma 6.3.
How to play the $i$-th round (for $i \in \{1, \ldots, k\}$)

In the $i$-th round, elements $a_{n+i}$ and $b_{n+i}$ will be chosen in $\mathfrak{A}$ and $\mathfrak{B}$.

Let $S$ be the set of all \((1(k-i+1), c(k-i+1), g(k-i+1) - \frac{n(k-i)}{2})\)-combinations over $a_1, \ldots, a_{n+i-1}$, and let $T$ be the according set of linear combinations over $b_1, \ldots, b_{n+i-1}$.

We consider the case where the spoiler chooses an element $a_{n+i}$ in $\mathfrak{A}$.
(The case where he chooses an element $b_{n+i}$ in $\mathfrak{B}$ is symmetric.)

To find her answer $b_{n+i}$, the duplicator distinguishes between two cases:

- If $a_{n+i} = \bar{s} + f'$ for some $s \in S$ and $f' \in \text{int} \left(-\frac{n(k-1)}{2}, \frac{n(k-1)}{2}\right)$, then the duplicator answers $b_{n+i} := \bar{t} + f'$, where $t$ is the according element in $T$ that corresponds to $s$.

- If $a_{n+i} \neq \bar{s} + f'$ for all $s \in S$ and all $f' \in \text{int} \left(-\frac{n(k-1)}{2}, \frac{n(k-1)}{2}\right)$, then the duplicator determines $s_- \in S$ such that $\bar{s} \leq a_{n+i} < \bar{s_-}$ and, for all $s \in S$, $\bar{s} \leq s$ or $\bar{s} \geq s$.

She chooses $t_-, t_+$ to be the according elements in $T$ that correspond to $s_-, s_+$, and she answers an arbitrary $b_{n+i}$ with $b_- < b_{n+i} < b_+$ and $a_{n+i} \equiv_{n(k-i)} b_{n+i}$.

At the end of the $i$-th round, the duplicator knows that $a_1, \ldots, a_{n+i}$ and $b_1, \ldots, b_{n+i}$ satisfy the conditions $W(k-i)$.

Figure 5: The duplicator’s winning strategy in the $k$-round EF-game on $\mathfrak{A} = \langle \mathbb{Z}, <, +, a_1, \ldots, a_n \rangle$ and $\mathfrak{B} = \langle \mathbb{Z}, <, +, b_1, \ldots, b_n \rangle$, where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ satisfy the conditions $W(k) := C(\langle m(k), 1(k), c(k), g(k) \rangle)$.

Let us mention that Theorem 6.4 gives us an EF-game proof of the theorem of Ginsburg and Spanier, stating that the spectra of $\text{FO(<,+)}$-sentences are semi-linear.

6.5 Corollary ($\text{FO(<,+)}$-sentences have semi-linear spectra).

Let $k \in \mathbb{N}_{>0}$ and let $\varphi$ be a $\text{FO(<,+)}$-sentence of quantifier depth at most $k$.

The spectrum $\text{Spec}(\varphi) := \{N \in \mathbb{N}_{>0} : \langle \{0, \ldots, N\}, <, + \rangle \models \varphi \}$ is semi-linear with parameters $N_0 := 2 g(k) c(k)^2$ and $p := m(k)$. I.e., “$N \in \text{Spec}(\varphi)$ iff $N+p \in \text{Spec}(\varphi)$” is true for all $N > N_0$.

\[ \square \]

Proof. Let $N > N_0 := 2 g(k) c(k)^2$ and let $p := m(k)$.

We use Theorem 6.4 for $n = 2$, $a_1 = 0$, $a_2 = N$, $b_1 = 0$, and $b_2 = N+p = N+m(k)$.

It is straightforward to verify that $a_1$, $a_2$ and $b_1$, $b_2$ satisfy the conditions $W(k)$. Therefore, Theorem 6.4 gives us a winning strategy for the duplicator in the $k$-round EF-game on $\langle \mathbb{Z}, <, +, a_1, a_2 \rangle$ and $\langle \mathbb{Z}, <, +, b_1, b_2 \rangle$. I.e., we have $\langle \mathbb{Z}, <, +, 0, N \rangle \approx_k \langle \mathbb{Z}, <, +, 0, N+p \rangle$. This implies that $\langle \{0, \ldots, N\}, <, +, 0, N \rangle \approx_k \langle \{0, \ldots, N+p\}, <, +, 0, N+p \rangle$. Since $\varphi$ is of quantifier depth at most $k$, Theorem 6.4 gives us that $\langle \{0, \ldots, N\}, <, + \rangle \models \varphi$ iff $\langle \{0, \ldots, N+p\}, <, + \rangle \models \varphi$, i.e., $N \in \text{Spec}(\varphi)$ iff $N+p \in \text{Spec}(\varphi)$.

\[ \blacksquare \]
6.2 The \( FO(<, +, Q) \)-Game over \( \mathbb{N} \) and \( \mathbb{Z} \)

The aim of this section is to show that the duplicator can translate strategies for the \( \text{FO}_{\text{dom}}(<) \)-
game into strategies for the \( \text{FO}(<, +, Q) \)-game on arbitrary structures over \( \mathbb{N} \). Here, \( Q \) is an
infinite subset of \( \mathbb{N} \) that satisfies certain conditions \( W(\omega) \).
Our proof is based on Lynch’s proof of his following theorem from [24].

6.2.1 Lynch’s Theorem and his Proof Idea

6.6 Theorem ([24, Theorem 3.7]). For every \( k \in \mathbb{N}_{>0} \) there exists a number \( d(k) \in \mathbb{N} \) and
an infinite set \( P_k \subseteq \mathbb{N} \) such that, for all sets \( A, B \subseteq P_k \), the following holds: If \( |A| = |B| \)
or \( d(k) < |A|, |B| < \infty \), then the duplicator wins the \( k \)-round EF-game on \( \langle \mathbb{N}, <, +, A \rangle \) and
\( \langle \mathbb{N}, <, +, B \rangle \). □

Unfortunately, neither the statement nor the proof of Lynch’s theorem gives us directly what we
need for translating strategies for the \( \text{FO}_{\text{dom}}(<) \)-game into strategies for the \( \text{FO}(<, +, Q) \)-game.

Going through Lynch’s proof in detail, we will modify and extend his notation and his reasoning
in a way appropriate for obtaining our translation results.

To illustrate the overall proof idea, let us first try to explain intuitively Lynch’s proof method.
For simplicity, we concentrate on subsets \( A, B \subseteq P_k \) of the same size and discuss what the
duplicator has to do in order to win the \( k \)-round EF-game on \( \mathcal{A} := \langle \mathbb{N}, <, +, A \rangle \) and \( \mathcal{B} := \langle \mathbb{N}, <, +, B \rangle \).

Assume that after \( i-1 \) rounds, the elements \( a_1, \ldots, a_{i-1} \) have been chosen in \( \mathcal{A} \), and the
elements \( b_1, \ldots, b_{i-1} \) have been chosen in \( \mathcal{B} \). In the \( i \)-th round let the spoiler choose some element
\( a_i \) in \( \mathcal{A} \).

In the previous Section 6.1 we have seen that, in order to win, the duplicator should play in such a
way that after the \( i \)-th round the conditions \( W(k-i) \) are satisfied by \( a_1, \ldots, a_i \) and \( b_1, \ldots, b_i \), i.e.,
she should follow the strategy described in Figure 5. In particular, this means that if \( a_i = \pi + f' \)
for a suitable linear combination \( s \) over \( a_1, \ldots, a_{i-1} \), then she should answer \( b_i := \pi + f' \), for
the corresponding linear combination \( t \) over \( b_1, \ldots, b_{i-1} \). However, in the present situation we
also have the sets \( A \) and \( B \) which must be respected, i.e., we need that \( a_i \in A \) if and only if
\( b_i \in B \). This means that, for any linear combination \( s \), we need to have \( \pi + f' \in A \) if and only if
\( \pi + f' \in B \).

To solve this problem we demand that \( A, B \subseteq P_k \), where \( P_k \) satisfies the conditions \( W(k) \) in
the following uniform way: For all sequences \( p_1 < \cdots < p_{\nu(k)} \) and \( q_1 < \cdots < q_{\nu(k)} \) in \( P_k \),
the conditions \( W(k) \) are satisfied by \( p_1, \ldots, p_{\nu(k)} \) and \( q_1, \ldots, q_{\nu(k)} \). Instead of considering linear
combinations \( s \) only over \( a_1, \ldots, a_{i-1} \), we now consider linear combinations over \( A, a_1, \ldots, a_{i-1} \).

If \( s \) is such a linear combination, in which elements \( p_1, \ldots, \nu \) from \( A \) occur, then the according
linear combination \( t \) over \( b_1, \ldots, b_{i-1} \) is obtained by replacing \( a_1, \ldots, a_{i-1} \) with \( b_1, \ldots, b_{i-1} \),
and replacing \( p_1, \ldots, \nu \) with \( q_1, \ldots, q_{\nu} \), where \( q_{\nu} \) is the \( j \)-th smallest element in \( B \) whenever \( p_{\nu} \)

is the \( j \)-th smallest element in \( A \). (Recall that here we assume that \( |A| = |B| \); in the more difficult
case where \( |A|, |B| > d(k) \) we will make use of an EF-game on \( \langle A, < \rangle \) and \( \langle B, < \rangle \) to find suitable
\( q_1, \ldots, q_{\nu} \) in \( B \) that fit for the elements \( p_1, \ldots, \nu \) in \( A \).)

Now, the duplicator’s strategy in the \( i \)-th round can be described as follows:

- If \( a_i = \pi + f' \) for some \( f(k-i+1), c(k-i+1), g(k-i+1) - \frac{m(k-i) + 1}{2} \)-combination \( s \) over
  \( A, a_1, \ldots, a_{i-1} \) and some \( f' \in \text{int} [-\frac{m(k-i)}{2}, +\frac{m(k-i)}{2}] \), then the duplicator chooses \( b_i := \pi + f' \),
where \( t \) is the corresponding combination over \( B, b_1, \ldots, b_{i-1} \).
In particular, we get that \( a_i \in A \) iff \( b_i \in B \).

- If no such \( s \) and \( f' \) exist, then let \( s_- \) and \( s_+ \) be the \( (1(k-i+1), c(k-i+1), g(k-i+1) - \frac{p(k-1)}{q(k-1)}) \)-combinations that approximate \( a_i \) from below and from above as closely as possible; and let \( t_- \) and \( t_+ \) be the corresponding combinations over \( B, b_1, \ldots, b_{i-1} \). The duplicator chooses an arbitrary \( b_i \) that lies strictly between \( t_- \) and \( t_+ \) with \( b_i \equiv a(k-i) \) \( a_i \). In particular, we know that \( a_i \not\in A \) and \( b_i \not\in B \).

As we will see below, this leads to a successful strategy for the duplicator.

6.2.2 The Translation of Strategies
To formally state our precise translation result, we need the following generalized version of Definition 6.2.

6.7 Definition ((l, c, g)-Combination, Correspondence, \( C(m, l, c, g), W(k) \)).
Let \( m, l, c \in \mathbb{N}_{>0} \) and \( g \in \mathbb{R}_{\geq 0} \).
Let \( P \subseteq \mathbb{N} \) be infinite, let \( n \in \mathbb{N} \), and let \( a_1, \ldots, a_n, b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{Z} \).

(a) (l, c, g)-Combination \( s \):
An \((l, c, g)\)-combination \( s \) over \( P, a_1, \ldots, a_n \) is a formal sum (or: a linear combination) of the form \( \sum_{i=1}^{l} d_ix_i + f \), where \( l \leq l, x_1, \ldots, x_{l}\) are pairwise distinct elements in \( P \cup \{a_1, \ldots, a_n\} \), \( d_1, \ldots, d_{l} \in \mathbb{Q}[c] \), and \( f \in \text{int}[-g, +g] \subseteq \mathbb{R} \).
Every such linear combination \( s \) evaluates to a real number \( \bar{s} \).
The elements \( x_1, \ldots, x_{l} \) are called the terms of \( s \).

(b) Correspondence \( \pi \):
A correspondence between \( P, a_1, \ldots, a_n \) and \( P, b_1, \ldots, b_n \) is a partial mapping \( \pi \) from \( P \cup \{a_1, \ldots, a_n\} \) to \( P \cup \{b_1, \ldots, b_n\} \) which satisfies the following conditions:
- \( \pi \) is \( < \)-preserving on \( P \),
- \( \pi(a_\nu) = b_\nu \), for all \( \nu \in \{1, \ldots, n\} \), and
- \( x \in P \) iff \( \pi(x) \in P \), for all elements \( x \) on which \( \pi \) is defined.

If \( \pi \) is such a correspondence, and if \( s = \sum_{i=1}^{l} d_ix_i + f \) is an \((l, c, g)\)-combination over \( P, a_1, \ldots, a_n \) whose terms are in the domain of \( \pi \), then we write \( \pi(s) \) to denote the \((l, c, g)\)-combination over \( P, b_1, \ldots, b_n \) obtained by replacing every term in \( s \) with its image under \( \pi \), i.e., \( \pi(s) := \sum_{i=1}^{l} d_i \pi(x_i) + f \).

(c) Conditions \( C(m, l, c, g) \):
We say that \( P, \epsilon_1, \ldots, \epsilon_n \) satisfy the conditions \( C(m, l, c, g) \) if and only if
\( (*) \) \( \quad p \equiv_m q \) for all \( p, q \in P \), and
\( (**) \) for all \((l, c, g)\)-combinations \( s_1 \) and \( s_2 \) over \( P, \epsilon_1, \ldots, \epsilon_n \) and for every correspondence \( \pi \) between \( P, \epsilon_1, \ldots, \epsilon_n \) and \( P, \epsilon_1, \ldots, \epsilon_n \) which is defined on all the terms of \( s_1 \) and \( s_2 \), we have \( \pi(s_1) < \pi(s_2) \) iff \( \pi(s_1) < \pi(s_2) \).
(d) **Conditions \( W(k) \) (for \( k \in \mathbb{N}_{>0} \)):**

We say that \( P, \epsilon_1, \ldots, \epsilon_n \) satisfy the conditions \( W(k) \) if and only if they satisfy the conditions \( C(m(k), 1(k), c(k), g(k)) \). Here, the functions \( m, 1, c, g \) are chosen as defined in Theorem 6.4.

As the following lemma shows, there do exist infinite sets \( P \subseteq \mathbb{N} \) that satisfy the conditions \( C(m, l, c, g) \).

6.8 Lemma. Let \( m, l, c, g \in \mathbb{N}_{>0} \) and \( g \in \mathbb{R}_{>0} \).

(a) If \( p_0 < p_1 < p_2 < \cdots \) is a sequence of natural numbers that satisfies

\[
\begin{align*}
p_0 & \geq 0, \\
p_i & \geq (2l - 1) \cdot 2c^3 \cdot p_{i-1} + 2gc^2, & \text{for all } i \in \mathbb{N}_{>0}, \text{ and} \\
p_i & = m \cdot p_{i+1}, & \text{for all } i \in \mathbb{N}_{>0},
\end{align*}
\]

then the set \( P := \{ p_1, p_2, \ldots \} \) satisfies the conditions \( C(m, l, c, g) \). Moreover, the conditions \( C(m, l, c, g) \) are satisfied even by \( P, \epsilon_1, \ldots, \epsilon_n \), for arbitrary \( n \in \mathbb{N} \) and \( \epsilon_1, \ldots, \epsilon_n \in \{0, \ldots, p_0\} \).

(b) There is an infinite set \( Q = \{ q_0 < q_1 < q_2 < \cdots \} \) such that, for every \( k \in \mathbb{N}_{>0} \), the conditions \( W(k) \) are satisfied by \( Q, q_0, \ldots, q_k-1 \). One example of such a set is given via

\[
\begin{align*}
q_0 & := 0, \\
q_i & := m(i) \cdot \left( (2l(i) - 1) \cdot 2c(i)^3 \cdot q_{i-1} + 2g(i) c(i)^2 \right), & \text{for all } i \in \mathbb{N}_{>0}.
\end{align*}
\]

Obviously, this set \( Q \) is not semi-linear and hence, due to the theorem of Ginsburg and Spanier (cf., Corollary 6.3), not definable in \( FO(<, +) \).

**Proof.** (a): It is obvious that the congruence condition \((*)\) is satisfied. We thus concentrate on condition \((**)*\). Let \( \pi \) be a correspondence between \( P, \epsilon_1, \ldots, \epsilon_n \) and \( P, \epsilon_1, \ldots, \epsilon_n \), and let \( s_1 \) and \( s_2 \) be \((l, c, g)\)-combinations over \( P, \epsilon_1, \ldots, \epsilon_n \) whose terms are in the domain of \( \pi \). We need to show that \( \pi(s_1) < \pi(s_2) \) iff \( \pi(s_1) < \pi(s_2) \).

Let \( s_1 = \sum_{i=1}^l d_i x_i + f \) and \( s_2 = \sum_{j=1}^l d'_j x'_j + f' \). By definition we know that \( x_1, \ldots, x_l \) (resp., \( x'_1, \ldots, x'_l \)) are pairwise distinct elements in \( P \cup \{ \epsilon_1, \ldots, \epsilon_n \} \). Hence, \( \{ x_1, \ldots, x_l, x'_1, \ldots, x'_l \} \) consists of \( l' \) pairwise distinct elements \( z_1, \ldots, z_{l'} \), for some \( l' \leq 2l \). Obviously,

\[
\begin{align*}
\pi(s_2) - \pi(s_1) & = \sum_{j=1}^l d'_j x'_j - \sum_{i=1}^l d_i x_i + (f' - f) \\
& = \sum_{r=1}^{l'} e_r z_r + h,
\end{align*}
\]

where \( h := f' - f \), and if \( z_r = x'_j = x_i \) then \( e_r := d'_j - d_i \), if \( z_r = x'_j \neq x_i \) for all \( i \), then \( e_r := d'_j \), and if \( z_r = x_i \neq x'_j \) for all \( j \), then \( e_r := -d_i \).

Since \( d_i, d'_j \in \mathbb{Q}[c] \) and \( |f|, |f'| \leq g \), one can easily see that

\[
(*) \quad l' \leq 2l, \quad |h| \leq 2g, \quad |e_r| \leq 2c, \text{ and } e_r = 0 \text{ or } |e_r| > \frac{1}{c}.
\]

In case that \( e_r = 0 \) for all \( r \), we have \( \pi(s_2) - \pi(s_1) = h = \pi(s_2) - \pi(s_1) \), and thus \( \pi(s_1) < \pi(s_2) \) iff \( \pi(s_1) < \pi(s_2) \). We can hence concentrate on the case where at least one of the coefficients \( e_r \) is different from 0. Without loss of generality we may assume that there is an \( l'' \) with \( 1 \leq l'' \leq l' \),
such that $e_r \neq 0$ for all $r \leq l''$, and $e_r = 0$ for all $r > l''$. Furthermore, we may assume that $z_1 > \cdots > z_{l''}$.

If $z_1$ is not an element in $P$, then all $z_r$ must be elements in $\{c_1, \ldots, c_n\}$, and hence $\pi(z_r) = z_r$ for all $r$. In particular, this means that $\pi(s_2) - \pi(s_1) = \pi(s_2) - \pi(s_1)$, and thus $\pi(s_1) < \pi(s_2)$.

It remains to consider the case where $z_1$ is an element in $P$. In this case we have $z_1 = p_i$ for some $i \in \mathbb{N}_{>0}$, and $z_2, \ldots, z_{l''} \leq p_{i-1}$. Furthermore, $\pi(z_1) = p_j$ for some $j \in \mathbb{N}$, and $\pi(z_2), \ldots, \pi(z_{l''}) \leq p_{j-1}$. Of course, we have

$$\pi(s_2) - \pi(s_1) = \sum_{r=1}^{l''} e_r z_r + h \leq e_1 p_i + \left| \sum_{r=2}^{l''} e_r z_r + h \right|.$$

Moreover, due to ($\bullet$) we have

$$\left| \sum_{r=2}^{l''} e_r z_r + h \right| \leq \sum_{r=2}^{l''} |e_r| p_{i-1} + |h| \leq (2l - 1) \cdot 2 c \cdot p_{i-1} + 2 g \leq \frac{2g}{c}.$$

This gives us that $\pi(s_2) - \pi(s_1) \leq e_1 p_i + \frac{2g}{c}$ and $\pi(s_2) - \pi(s_1) \geq e_1 p_i - \frac{2g}{c}$. Since $\pi$ is a correspondence, the same reasoning leads to the analogous result for $\pi(s_2) - \pi(s_1)$. I.e., we have

$$(e_1 - \frac{1}{c}) \cdot p_i \leq \pi(s_2) - \pi(s_1) \leq (e_1 + \frac{1}{c}) \cdot p_i$$

and

$$(e_1 - \frac{1}{c}) \cdot p_j \leq \pi(s_2) - \pi(s_1) \leq (e_1 + \frac{1}{c}) \cdot p_j.$$

Due to ($\bullet$) we know that $|e_1| > \frac{1}{c}$. Hence we have either $(e_1 - \frac{1}{c}) > 0$, implying that $\pi(s_2) - \pi(s_1) > 0$ and $\pi(s_2) - \pi(s_1) > 0$, or $(e_1 + \frac{1}{c}) < 0$, implying that $\pi(s_2) - \pi(s_1) < 0$ and $\pi(s_2) - \pi(s_1) < 0$.

This gives us that $\pi(s_1) < \pi(s_2)$ if $\pi(s_1) < \pi(s_2)$, and the proof of part (a) is complete.

(b): Let $k \in \mathbb{N}_{>0}$. Define $m := \overline{m}(k)$, $l := \overline{1}(k)$, $c := \overline{c}(k)$, and $g := \overline{g}(k)$. Furthermore, define $n := k$ and $(c_1, \ldots, c_n) := (q_0, \ldots, q_{k-1})$. We consider the sequence $p_0 < p_1 < p_2 < \cdots$ given, for all $i \in \mathbb{N}$, via $p_i := q_{k-1+i}$. Let $P := \{p_1, p_2, \ldots\} = Q \setminus \{q_0, \ldots, q_{k-1}\}$. From Definition 6.7 one can directly see that $Q$, $q_0, \ldots, q_{k-1}$ satisfies the conditions $W(k)$ if and only if $P, c_1, \ldots, c_n$ satisfies the conditions $C(m, l, c, g)$. We can thus make use of part (a). Of course, $c_1, \ldots, c_n \leq q_{k-1} = p_0$. Furthermore, it is straightforward to check that the sequence $p_0 < p_1 < p_2 < \cdots$ satisfies the conditions formulated in part (a): The congruence condition is satisfied since, for $i \in \mathbb{N}_{>0}$, $p_i = q_{k-1+i}$ is a multiple of $m(k+1+i)$ which itself is a multiple of $m(k) = m$. Hence, $p_i \equiv m$ for all $i \in \mathbb{N}_{>0}$. The growth condition formulated in part (a) is satisfied since the functions $m$, $l$, $c$, and $g$ are increasing. From part (a) we therefore obtain that $P, c_1, \ldots, c_n$ satisfies the conditions $C(m, l, c, g)$. Altogether, this completes the proof of Lemma 6.8.

6.9 Definition (Conditions $W(\omega)$).

Let $Q = \{q_0 < q_1 < q_2 < \cdots\} \subseteq \mathbb{N}$ be an infinite set of natural numbers.

We say that $Q$ satisfies the conditions $W(\omega)$ if and only if the following is true: For every $k \in \mathbb{N}_{>0}$ there exists an $n_k \in \mathbb{N}_{>0}$ such that the conditions $W(k)$ (cf. Definition 6.7) are satisfied by $Q$, $q_0, \ldots, q_{n_k-1}$. An example of such a set $Q$ is given in Lemma 6.8(b).

$\square$
We are now ready to state the main result of this section:

**6.10 Theorem (FO\(<\), +, Q)-game over \(\mathbb{N}\) and \(\mathbb{Z}\).**

Let \(Q = \{q_0 < q_1 < q_2 < \cdots\} \subseteq \mathbb{N}\) satisfy the conditions \(W(\omega)\).
The duplicator can translate strategies for the \(\text{FO}_{\text{dom}}(<)-\text{game}\) into strategies for the \(\text{FO}(<, +, Q)\)-game on \(\mathcal{C}_{\text{arb}}\) over \(\mathbb{N}\) and on \(\mathcal{C}_{\text{emb}}\) over \(\mathbb{Z}\).

The above theorem is a direct consequence of the following technical result:

**6.11 Proposition.** Let \(\tau\) be a signature, let \(k \in \mathbb{N}_{>0}\) be a number of rounds for the "\(+\)-game". The according number \(r(k)\) of rounds for the "\(<\)-game" is inductively defined via \(r(0) := 1\) and, for all \(j \in \mathbb{N}\), \(r(j+1) := r(j) + 2 \cdot 1(j+1)\).

Let \(n \in \mathbb{N}\), let \(\bar{\epsilon} := \epsilon_1, \ldots, \epsilon_n \in \mathbb{N}\), and let \(P := \{p_1 < p_2 < \cdots\} \subseteq \mathbb{N}\) be an infinite set such that \(P, \bar{\epsilon}\) satisfies the conditions \(W(k)\) and such that, for all \(\nu \in \{1, \ldots, n\}\), \(\epsilon_\nu\) is smaller than the smallest element in \(P\). Let \(A\) and \(B\) be two \(\mathbb{N}\)-embeddable \(\tau\)-structures, and let \(\alpha : \text{dom}(A) \rightarrow P\) map, for every \(j\), the \(j\)-th smallest element in \(\text{dom}(A)\) onto the position \(p_j\). Accordingly, let \(\beta : \text{dom}(B) \rightarrow P\) map, for every \(j\), the \(j\)-th smallest element in \(\text{dom}(B)\) onto the position \(p_j\).

If \((\text{dom}(A), <, \tau^A) \approx_{r(k)} (\text{dom}(B), <, \tau^B)\), then \((\mathbb{Z}, <, +, P, \bar{\epsilon}, \alpha(\tau^A)) \approx_k (\mathbb{Z}, <, +, P, \bar{\epsilon}, \beta(\tau^B))\).

Before proving Proposition 6.11, let us first show that it enables us to prove Theorem 6.10.

**Proof of Theorem 6.10.**

Let \(\tau\) be a signature and let \(k \in \mathbb{N}_{>0}\) be the number of rounds for the \(\text{FO}(<, +, Q)\)-game. Choose the number \(r(k)\) of rounds for the \(\text{FO}_{\text{dom}}(<)\)-game as given in Proposition 6.11. Choose \(P := Q \setminus \{q_0, \ldots, q_{n_k-1}\}\), choose \(n := n_k+1\), and \(\bar{\epsilon} := 0, q_0, \ldots, q_{n_k-1}\). From the presumption we know that \(Q\) satisfies the conditions \(W(\omega)\), and thus \(Q, q_0, \ldots, q_{n_k-1}\) satisfies the conditions \(W(k)\). From Definition 6.11 one can directly see that this implies that also \(P, \bar{\epsilon}\) satisfies the conditions \(W(k)\). Let \(A\) and \(B\) be two \(\mathbb{N}\)-embeddable \(\tau\)-structures with \((\text{dom}(A), <, \tau^A) \approx_{r(k)} (\text{dom}(B), <, \tau^B)\), then Proposition 6.11 gives us \(<\)-preserving mappings \(\alpha, \beta\) such that \((\mathbb{Z}, <, +, P, \bar{\epsilon}, \alpha(\tau^A)) \approx_k (\mathbb{Z}, <, +, P, \bar{\epsilon}, \beta(\tau^B))\). Since \(\bar{\epsilon} = 0, q_0, \ldots, q_{n_k-1}\) and \(Q = P \cup \{q_0, \ldots, q_{n_k-1}\}\), this in particular implies that \((\mathbb{N}, <, +, Q, \alpha(\tau^A)) \approx_k (\mathbb{N}, <, +, Q, \beta(\tau^B))\), and hence \((\mathbb{N}, <, +, Q, \alpha(\tau^A)) \approx_k (\mathbb{N}, <, +, Q, \beta(\tau^B))\). Altogether, this completes the proof of Theorem 6.10, both, for \(\mathbb{N}\)-embeddable structures over \(\mathbb{Z}\), and for arbitrary structures over \(\mathbb{N}\).

We will now concentrate on the proof of Proposition 6.11.

**Proof of Proposition 6.11.**

Let \(\tau\) be a signature, let \(k \in \mathbb{N}_{>0}\), and let \(r(k)\) be the according number defined in Proposition 6.11.

Let \(n \in \mathbb{N}\), let \(\bar{\epsilon} := \epsilon_1, \ldots, \epsilon_n \in \mathbb{N}\), and let \(P := \{p_1 < p_2 < \cdots\} \subseteq \mathbb{N}\) be an infinite set such that \(P, \bar{\epsilon}\) satisfies the conditions \(W(k)\) and such that, for all \(\nu \in \{1, \ldots, n\}\), \(\epsilon_\nu\) is smaller than the smallest element in \(P\). Let \(A\) and \(B\) be two \(\mathbb{N}\)-embeddable \(\tau\)-structures, and let \(\alpha : \text{dom}(A) \rightarrow P\) map, for every \(j\), the \(j\)-th smallest element in \(\text{dom}(A)\) onto the position \(p_j\). Accordingly, let

\*From Lemma 5.8.a we know how to construct such \(P, \bar{\epsilon}\).
\( \beta : \text{dom}(B) \to P \) map, for every \( j \), the \( j \)-th smallest element in \( \text{dom}(B) \) onto the position \( p_j \).

We assume that \( \langle \text{dom}(A), \prec, \tau^A \rangle \approx_k \langle \text{dom}(B), \prec, \tau^B \rangle \), i.e., the duplicator wins the \( \tau(k) \)-round FO\(_{\text{dom}}(\prec) \)-game on \( A \) and \( B \). From Lemma 5.3 we obtain that \( A := \langle P, \prec, \alpha(\tau^A) \rangle \approx_k \langle \mathcal{Z}, \prec, +, P, \tilde{c}, \beta(\tau^B) \rangle := \mathcal{B} \).

Our aim is to show that \( \mathcal{A} := \langle \mathcal{Z}, \prec, +, P, \tilde{c}, \alpha(\tau^A) \rangle \approx_k \langle \mathcal{Z}, \prec, +, P, \tilde{c}, \beta(\tau^B) \rangle := \mathcal{B} \).

Henceforth, the game on \( \mathcal{A} \) and \( \mathcal{B} \) will be called the \( \prec \)-game, and the game on \( \mathcal{A} \) and \( \mathcal{B} \) will be called the \( + \)-game.

For each round \( i \in \{1, \ldots, k\} \) of the \( + \)-game we use \( a_i \) and \( b_i \), respectively, to denote the element chosen in that round in \( \mathcal{A} \) and \( \mathcal{B} \). We will translate each move of the spoiler in the \( + \)-game, say \( a_i \) (if he chooses in \( \mathcal{A} \)), into a number of moves \( a'_i, \ldots, a'_n \) for a “virtual spoiler” in the \( \prec \)-game in \( \mathcal{A} \). Then we can find the answers \( b'_i, \ldots, b'_m \) of a “virtual duplicator” who plays according to her winning strategy in the \( \prec \)-game. Afterwards, we translate these answers into a move \( b_i \) for the duplicator in the \( + \)-game. (The case where the spoiler chooses \( b_i \) in \( \mathcal{B} \) is symmetric.)

As abbreviation we use \( \bar{a}'_i \) to denote the sequence \( a'_1, \ldots, a'_n \), and we use \( \bar{b}'_i \) to denote the sequence \( b'_1, \ldots, b'_m \). A partial mapping from \( \mathcal{Z} \) to \( \mathcal{Z} \) is called \( \tau \)-respecting iff it is a partial isomorphism between the structures \( \langle \mathcal{Z}, \alpha(\tau^A) \rangle \) and \( \langle \mathcal{Z}, \beta(\tau^B) \rangle \).

We show that the duplicator can play the \( + \)-game in such a way that the following conditions hold at the end of each round \( i \), for \( i \in \{0, \ldots, k\} \):

1. \( \langle \mathcal{A}', \bar{a}'_i, \ldots, \bar{a}'_i \rangle \approx_{\tau(k-i)} \langle \mathcal{B}', \bar{b}'_i, \ldots, \bar{b}'_i \rangle .
2. \( a_i = a(k-i) \) \( b_i \) (if \( i \neq 0 \)).
3. The following mapping
   \[
   \pi_i : \begin{cases} 
   \alpha(c^A) &\mapsto \beta(c^B) \quad &\text{for all constant symbols } c \in \tau \\
   \tilde{c}_i &\mapsto \tilde{c}_i \\
   a'_\nu &\mapsto \bar{b}'_\nu \quad &\text{for all } \nu \in \{1, \ldots, i\} \\
   \bar{a}'_\nu &\mapsto \bar{b}'_\nu \quad &\text{for all } \nu \in \{1, \ldots, i\}
   \end{cases}
   \]
   is a \( \tau \)-respecting correspondence between \( P, \tilde{c}, a_1, \ldots, a_i \) and \( P, \tilde{c}, b_1, \ldots, b_i \).
4. If \( i \neq 0 \), then for every \( (k-i+1), c(k-i+1), g(k-i+1) - \frac{n(k-1)}{2} \)-combination \( t \) over \( P, \tilde{c}, a_1, \ldots, a_{i-1} \), and for every extension \( \pi \) of \( \pi_i \) which is \( \prec \)-preserving on \( P \) and which is defined on all the terms of \( t \), we have
   \[ a_i < \mathcal{T} \quad \text{if and only if} \quad b_i < \pi(t). \]
5. If \( i \neq k \), then for all \( (k-i), c(k-i), g(k-i) \)-combinations \( s_1 \) and \( s_2 \) over \( P, \tilde{c}, a_1, \ldots, a_i \) and for every extension \( \pi \) of \( \pi_i \) which is \( \prec \)-preserving on \( P \) and which is defined on all the terms of \( s_1 \) and \( s_2 \), we have
   \[ \pi(s_1) < \pi(s_2) \quad \text{if and only if} \quad \pi(s_1) < \pi(s_2). \]

The following can be seen easily:

Claim 1. If the conditions (3) and (4) are satisfied for \( i = k \) and condition (5) is satisfied for
i = k - 1, then the mapping \( \pi_k \) is a partial isomorphism between \( \mathfrak{A} \) and \( \mathfrak{B} \) and hence the duplicator has won the k-round +-game on \( \mathfrak{A} \) and \( \mathfrak{B} \).

\[ \square \]

**Proof.** Recall that \( \mathfrak{A} = (\mathbb{Z}, \prec, +, P, \mathfrak{c}, \alpha(\pi^A)) \) and \( \mathfrak{B} = (\mathbb{Z}, \prec, +, P, \mathfrak{c}, \beta(\pi^B)) \). From condition (3) (for \( i := k \)) we know that the mapping \( \pi := \pi_k \) is a \( \pi \)-respecting correspondence between \( P, \mathfrak{c}, a_1, \ldots, a_k \) and \( P, \mathfrak{c}, b_1, \ldots, b_k \). In particular, this means that \( \pi \) is a partial isomorphism between \( (\mathbb{Z}, \alpha(\pi^A)) \) and \( (\mathbb{Z}, \beta(\pi^B)) \), and that \( \pi \) is defined. All that remains to be done is to show that for all \( x, y, z \) in the domain of \( \pi \) we have \( x \prec y \iff \pi(x) < \pi(y) \) and \( x + y = z \iff \pi(x) + \pi(y) = \pi(z) \).

In order to prove that \( x \prec y \iff \pi(x) < \pi(y) \) we distinguish between three cases: If \( x = y = a_k \) then, certainly, \( x = y \) and \( \pi(x) = \pi(y) \). If \( x \) and \( y \) are both different from \( a_k \), then \( s_1 := x \) and \( s_2 := y \) can be viewed as \((1, 1, 0)\)-combinations over \( P, \mathfrak{c}, a_1, \ldots, a_{k-1} \). Hence, condition (5) (for \( i := k - 1 \)) gives us that \( x + y \prec \pi(x) + \pi(y) \). If either \( x \) or \( y \) is equal to \( a_k \), then condition (4) (for \( i := k \)) gives us that \( x \prec y \iff \pi(x) < \pi(y) \).

In order to prove that \( x + y = z \iff \pi(x) + \pi(y) = \pi(z) \) we distinguish between three cases: If \( z = a_k \) and either \( x \) or \( y \) is equal to \( a_k \), then, certainly, \( x + y = z \iff \pi(x) + \pi(y) = \pi(z) \). If \( x, y, z \) are different from \( a_k \), then it is straightforward to define \((2, 2, 0)\)-combinations \( s_1 \) and \( s_2 \) over \( P, \mathfrak{c}, a_1, \ldots, a_{k-1} \) such that \( x + y = z \iff s_1 = s_2 \) and \( \pi(x) + \pi(y) = \pi(z) \iff \pi(s_1) = \pi(s_2) \). Hence, condition (5) (for \( i := k - 1 \)) gives us that \( x + y = z \iff \pi(x) + \pi(y) = \pi(z) \). In all remaining cases it is straightforward to define a \((2, 2, 0)\)-combination \( t \) over \( P, \mathfrak{c}, a_1, \ldots, a_{k-1} \) such that \( x + y = z \iff a_k = t \) and \( \pi(x) + \pi(y) = \pi(z) \iff b_k = \pi(t) \). Condition (4) (for \( i := k \)) then gives us that \( x + y = z \iff \pi(x) + \pi(y) = \pi(z) \).

Altogether, the proof of Claim 1 is complete.

From our presumptions we know that the conditions (1)–(5) are satisfied for \( i = 0 \). For the induction step from \( i - 1 \) to \( i \in \{1, \ldots, k\} \) we assume that (1)–(5) hold for \( i - 1 \). We show that in the \( i \)-th round the duplicator can play in such a way that (1)–(5) hold for \( i \). Let us assume that the spoiler chooses \( a_i \) in \( \mathfrak{A} \). (The case where he chooses \( b_i \) in \( \mathfrak{B} \) is symmetric.)

The duplicator’s strategy in the \( i \)-th round is similar to the strategy described in Figure 3. First, she determines two linear combinations \( s_- \) and \( s_+ \) over \( P, \mathfrak{c}, a_1, \ldots, a_{i-1} \) which approximate \( a_i \) from below and from above as closely as possible. For the precise choice of \( s_- \) and \( s_+ \) she distinguishes between three cases:

(I) \( a_i \in P \cup \{\mathfrak{c}, a_1, \ldots, a_{i-1}\} \) then \( s_- := s_+ := a_i \).

(II) Otherwise, if \( a_i = \mathfrak{c} + f' \) for some \( (1k-i+1), c(k-i+1), g(k-i+1) - \frac{n(k-i)}{2} \)-combination \( s \) over \( P, \mathfrak{c}, a_1, \ldots, a_{i-1} \) and some \( f' \in \text{int}[\frac{n(k-i)}{2}, \frac{m(k-i)}{2}] \), then \( s_- := s_+ := s + f' \).

(III) Otherwise, let \( s_- \) and \( s_+ \) be the \((1k-i+1), c(k-i+1), g(k-i+1) - \frac{n(k-i)}{2} \)-combinations over \( P, \mathfrak{c}, a_1, \ldots, a_{i-1} \) that approximate \( a_i \) from below and from above as closely as possible. I.e., \( \mathfrak{c} - a_i < \mathfrak{c} - s_+ \), and for all \((1k-i+1), c(k-i+1), g(k-i+1) - \frac{n(k-i)}{2} \)-combinations \( s \) we have \( \mathfrak{c} - a_i < \mathfrak{c} - s \) or \( \mathfrak{c} - s < \mathfrak{c} - s_+ \). In particular, since case (II) does not apply, we know that \( \mathfrak{c} - \frac{n(k-i)}{2} < a_i < \mathfrak{c} - \frac{n(k-i)}{2} \), and hence \( \mathfrak{c} - \frac{n(k-i)}{2} > m(k-i) \).

In all three cases, \( s_- \) and \( s_+ \) are \((1k-i+1), c(k-i+1), g(k-i+1) \)-combinations over
Let \( \tilde{a}_i \) be those pairwise distinct terms of \( s_- \) and \( s_+ \) that belong to \( P \). In particular, we know that \( n_i \leq 2 \cdot 1(k-i+1) \). The elements \( \tilde{a}_i \) are the moves for a “virtual spoiler” in the \( \langle \pi \rangle \)-game. From condition (1) (for \( i=1 \)) we know that \( (\tilde{a}_i', a_1', \ldots, a_{n_i}') \approx_{r(k-i+1)} (\pi, \tilde{b}_1', \ldots, \tilde{b}_{n_i-1}'). \) Thus, a “virtual duplicator” can find answers \( \tilde{b}_i := b_{i,1}', \ldots, b_{i,n_i} \) such that \( (\tilde{a}_i', a_1', \ldots, a_{n_i}') \approx_{r(k-i+1)-n_i} (\pi, \tilde{b}_1', \ldots, \tilde{b}_{n_i-1}'). \) Since \( n_i \leq 2 \cdot 1(k-i+1) \), and since the function \( r \) was defined in such a way that \( r(k-i+1) = r(k-i) + 2 \cdot 1(k-i+1) \), we know that \( r(k-i+1) - n_i \geq r(k-i) \), and hence condition (1) is satisfied for \( i \).

Let \( \tilde{\pi}_i \) be the extension of the mapping \( \pi_{i-1} \) in \( \pi_i' \) via \( \tilde{a}_i' \mapsto \tilde{b}_i' \). It should be clear that, due to condition (3) (for \( i=1 \)), \( \tilde{\pi}_i \) is a \( r \)-respecting correspondence between \( P, \tilde{\pi}, a_1, \ldots, a_{n_i} \) and \( P, \tilde{\pi}, b_1, \ldots, b_{n_i-1} \).

For her choice of \( b_i \) in \( \mathcal{B} \), the duplicator makes use of the following:

**Claim 2.**

(a) \( \frac{m(k-i)}{m(k-i)} = \tilde{\pi}_i(s_-) \).

(b) If \( \frac{m(k-i)}{m(k-i)} > m(k-i) \) then \( \frac{\tilde{\pi}_i(s_+)}{\tilde{\pi}_i(s_-)} > m(k-i) \). □

**Proof.** (a): We know that \( s_- \) is a \( (1(k-i+1), c(k-i+1), g(k-i+1)) \)-combination over \( P, \tilde{\pi}, a_1, \ldots, a_{n_i} \). In particular, \( s_- = \sum_{\nu=1}^u d_\nu x_\nu + f \), where \( d_\nu \in \mathbb{Q}[c(k-i+1)] \), i.e., \( d_\nu = \frac{s_{-\nu}}{u_\nu} \) for \( u_\nu \neq 0 \) and \( |u_\nu| \in \{0, \ldots, c(k-i+1)\} \). In order to show that \( \tilde{\pi}_i(s_-) = \frac{m(k-i)}{m(k-i)} \), we need to find some \( z \in \mathbb{Z} \) such that \( \frac{s_{-\nu}}{u_\nu} = \tilde{\pi}_i(s_-) \).

Of course, \( \tilde{x}_- \equiv \frac{s_{-\nu}}{u_\nu} \tilde{x}_+ \equiv \frac{m(k-i)}{m(k-i)} \tilde{x}_+ \). From the presumption that \( P, \tilde{\pi} \) satisfies the conditions \( W(k) \) and from condition (2) (for \( i=1 \)) we know for all the \( x_\nu \) that \( x_\nu = \tilde{\pi}_i(x_\nu) = \tilde{\pi}_i(z_\nu) \). Since, there exists \( z_\nu \in \mathbb{Z} \) such that \( x_\nu = \tilde{\pi}_i(x_\nu) = z_\nu \cdot m(k-i+1) \). By the definition of \( m \) we know that \( m(k-i+1) = m(k-i) \cdot \text{lcm}(1, \ldots, c(k-i+1)) \). Hence, \( \frac{s_{-\nu}}{u_\nu} = \frac{m(k-i)}{m(k-i)} \cdot \frac{\text{lcm}(1, \ldots, c(k-i+1))}{u_\nu} \). This gives us the desired integer \( z := \sum_{\nu=1}^u z_\nu \cdot m(k-i) \cdot \frac{\text{lcm}(1, \ldots, c(k-i+1))}{u_\nu} \), such that \( \tilde{x}_- \equiv \frac{s_{-\nu}}{u_\nu} \tilde{x}_+ \equiv z \cdot m(k-i) \).

(b): Since \( \frac{s_{-\nu}}{u_\nu} > m(k-i) \), we know that \( s_- \) and \( s_+ \) must have been chosen according to case (III) and must hence be \( (1(k-i+1), c(k-i+1), g(k-i+1) - \frac{\text{lcm}(1, \ldots, c(k-i+1))}{2}) \)-combinations. Let \( h := \tilde{\pi}_i(s_-) - \tilde{\pi}_i(s_-) \). We need to show that \( h > m(k-i) \).

Suppose that, on the contrary, \( h \leq m(k-i) \). Then, \( s_1 := s_+ + h \) and \( s_2 := s_- - h \) are \( (1(k-i+1), c(k-i+1), g(k-i+1)) \)-combinations with \( \tilde{\pi}_i(s_1) = \tilde{\pi}_i(s_2) \). From condition (5) (for \( i=1 \)) we obtain that \( \tilde{x}_+ = \tilde{x}_- \) and hence \( \tilde{x}_+ = \tilde{x}_- \approx h \leq m(k-i) \). This is a contradiction to our presumption that \( \tilde{x}_+ \approx \tilde{x}_- \approx m(k-i) \). Altogether, the proof of Claim 2 is complete. □

The duplicator chooses \( b_i \) in \( \mathcal{B} \) as follows:

- If \( s_i = s_- \) then \( b_i := \tilde{\pi}_i(s_-) \).

According to Claim 2 (a) we have \( a_i \equiv m(k-i) b_i \). In particular, since \( a_i \in \mathbb{Z} \), this implies that \( b_i \in \mathbb{Z} \).
• If \( a_i \neq \overline{s_-} \) then \( s_- \) and \( s_+ \) must have been chosen according to case (III). In particular, we know that \( \overline{s_-} - \overline{s_-} > m(k-i) \).

According to Claim 2(b) we have \( \overline{\pi_i(s_+)} - \overline{\pi_i(s_-)} > m(k-i) \). Thus there exists a \( b_i \in \mathbb{Z} \) with \( a_i \equiv_{m(k-i)} b_i \).

In both cases, condition (2) is satisfied for \( i \).

In order to show that condition (3) is satisfied for \( i \), we distinguish between case (I) on the one hand and the cases (II) and (III) on the other hand, and we make use of the fact that we already know that \( \pi_i \) is a \( \tau \)-respecting correspondence between \( P, \pi_i, a_1, \ldots, a_{i-1} \) and \( P, \pi_i, b_1, \ldots, b_{i-1} \).

In case (I) we know that \( a_i \in P \cup \{ \overline{c}, a_1, \ldots, a_{i-1} \} \) and that \( s_- = a_i \). In particular, \( a_i \) lies in the domain of \( \pi_i \). As described above, the duplicator chooses \( b_i := \overline{\pi_i(s_-)} = \overline{\pi_i(a_i)} \).

Hence, \( \pi_i \) is exactly the mapping \( \pi_i \) considered in condition (3); and certainly, \( \pi_i \) is a \( \tau \)-respecting correspondence between \( P, \pi_i, a_1, \ldots, a_i \) and \( P, \pi_i, b_1, \ldots, b_i \).

In the cases (II) and (III) we know that \( a_i \notin P \cup \{ \overline{c}, a_1, \ldots, a_{i-1} \} \). In particular, \( a_i \) is not in the domain of \( \pi_i \). Thus we can extend \( \pi_i \) to \( \pi_j \) via \( a_i \mapsto b_i \). If we can show that \( b_i \notin P \), then \( \pi_i \) inherits from \( \pi_j \) that it is \( \tau \)-respecting, that it is \( \prec \)-preserving on \( P \), and that it satisfies, for all elements \( x \) on which it is defined, that \( x \in P \iff \pi_i(x) \in P \). I.e., we obtain that \( \pi_i \) is a \( \tau \)-respecting correspondence between \( P, \pi_i, a_1, \ldots, a_i \) and \( P, \pi_i, b_1, \ldots, b_i \).

It remains to show that \( b_i \notin P \). For the sake of contradiction, assume that \( b_i \in P \). From condition (1) (for \( i \)) we know that \( \langle \overline{\mathcal{A}}, \overline{a'_1}, \ldots, \overline{a'_i} \rangle \equiv_{r(k-i)} \langle \overline{\mathcal{B}}, \overline{b'_1}, \ldots, \overline{b'_i} \rangle \). Furthermore, \( r(k-i) > r(0) = 1 \), and hence the “virtual duplicator” can win (at least) one more round of the game. In this round let the “virtual spoiler” choose \( b_i \) in \( \overline{\mathcal{B}} \) (this is possible since we assume that \( b_i \in P \)).

The “virtual duplicator” can find some \( p \in \overline{\mathcal{A}} \) (i.e., \( p \in P \)) such that \( \langle \overline{\mathcal{A}}, \overline{a'_1}, \ldots, \overline{a'_i}, p \rangle \equiv_0 \langle \overline{\mathcal{B}}, \overline{b'_1}, \ldots, \overline{b'_i}, b_i \rangle \). Hence, the extension \( \pi \) of \( \pi_i \) via \( p \mapsto b_i \) must be \( \prec \)-preserving on \( P \). In particular, condition (5) (for \( i-1 \)) can be applied to the mapping \( \pi \). Furthermore, we have \( \pi(s_-) = \overline{\pi_i(s_-)} \) and \( \pi(s_+) = \overline{\pi_i(s_+)} \); and \( p \) can be viewed as a \( \langle 1(k-i+1), c(k-i+1), g(k-i+1) - \frac{m(k-i)}{2} \rangle \)-combination over \( P, \pi, a_1, \ldots, a_{i-1} \).

In condition (II) we know that \( a_i = \overline{s_-} \) and \( b_i = \overline{\pi_i(s_-)} \). I.e., we have \( \overline{\pi(p)} = b_i = \overline{\pi(s_-)} \). From condition (3) (for \( i-1 \)) we obtain that \( p = \overline{s_-} = a_i \), which is a contradiction to \( a_i \notin P \).

In condition (III) we know that \( \overline{s_-} < a_i < \overline{s_+} \) and \( \pi(s_-) < b_i = \pi(p) < \pi(s_+) \). From condition (3) (for \( i-1 \)) we obtain that \( \overline{s_-} < p < \overline{s_+} \). This is a contradiction to the choice of \( s_- \) and \( s_+ \) according to case (III). In the cases (II) and (III) we thus must have \( b_i \notin P \).

Altogether, we have seen that condition (3) is satisfied for \( i \).

In order to show that condition (4) is satisfied for \( i \), let \( t \) be a \( \langle 1(k-i+1), c(k-i+1), g(k-i+1) - \frac{m(k-i)}{2} \rangle \)-combination over \( P, \pi, a_1, \ldots, a_{i-1} \), and let \( \pi \) be an extension of \( \pi_i \) which is \( \prec \)-preserving on \( P \) and which is defined on all the terms of \( t \). We need to show that \( a_i < \overline{\pi(t)} \) if and only if \( b_i < \overline{\pi(t)} \).

For the “if” direction we assume that \( a_i < \overline{\pi(t)} \), and we show that \( b_i < \overline{\pi(t)} \).

From the choice of \( s_- \) we know that \( \overline{s_-} > \overline{\pi(t)} \). Condition (5) (for \( i-1 \)) gives us that \( \overline{\pi(s_-)} > \overline{\pi(t)} \).

Furthermore, from the choice of \( b_i \) we know that \( b_i \geq \overline{\pi_i(s_-)} = \overline{\pi(s_-)} \). Hence, \( b_i \geq \overline{\pi(t)} \).

For the “only if” direction we assume that \( a_i < \overline{\pi(t)} \), and we show that \( b_i < \overline{\pi(t)} \).

In the case that \( a_i = \overline{s_-} \) we have that \( b_i = \overline{\pi_i(s_-)} = \overline{\pi(s_-)} \) and that \( \overline{s_-} < \overline{\pi(t)} \). Condition (5) (for \( i-1 \)) gives us that \( \pi(s_-) < \pi(t) \), and hence \( b_i < \overline{\pi(t)} \).

In case that \( a_i \neq \overline{s_-} \) we know that \( s_- \) and \( s_+ \) must have been chosen according to case (III),
This, in particular, implies that $a_i < \frac{\pi(s_+)}{\pi(s_+)} \leq \frac{7}{2}$. Condition (5) (for $i=1$) gives us that $\frac{\pi(s_+)}{\pi(s_+)} \leq \frac{\pi(t)}{\pi(t)}$. Furthermore, from the choice of $b_i$ we know that $b_i < \frac{\pi_i(s_+)}{\pi_i(s_+)} = \frac{\pi(s_+)}{\pi(s_+)}$. Hence, $b_i < \frac{\pi(t)}{\pi(t)}$.

Altogether, we obtain that condition (4) is satisfied for $i$.

To show that condition (5) is satisfied for $i$ (if $i \neq k$), let $s_1$ and $s_2$ be $(1(k-i), c(k-i), g(k-i))$-combinations over $P, \tilde{c}, a_1, \ldots, a_i$, and let $\pi$ be an extension of $\pi_i$ which is $<\!$-preserving on $P$ and which is defined on all the terms of $s_1$ and $s_2$. We have to show that $\frac{\pi(s_1)}{\pi(s_2)} < \frac{\pi(t)}{\pi(t)}$ if and only if $\frac{\pi(s_1)}{\pi(s_2)} < \frac{\pi(t)}{\pi(t)}$.

Let $s_1 = \sum_{i=1}^l d_i x_i + f$ and $s_2 = \sum_{j=1}^l d'_j x'_j + f'$. By definition we know that $x_1, \ldots, x_l$ (resp., $x'_1, \ldots, x'_l$) are pairwise distinct elements in $P \cup \{\tilde{c}, a_1, \ldots, a_i\}$. Hence, $\{x_1, \ldots, x_l, x'_1, \ldots, x'_l\}$ consists of $l'$ pairwise distinct elements $z_1, \ldots, z_{l'}$, for some $l'$ with $l' \leq 2l = 21(k-i)$. Obviously,

$$\frac{\pi(s_1)}{\pi(s_2)} = \frac{\sum_{i=1}^l d_i x_i - \sum_{j=1}^l d'_j x'_j + (f-f')}{\sum_{r=1}^{l'} e_r z_r + h},$$

where $h := f-f'$, and if $z_r = x_i = x'_j$ then $e_r := d_i - d'_j$, if $z_r = x_i \neq x'_j$ for all $j$, then $e_r := d_i$, and if $z_r = x'_j \neq x_i$ for all $i$, then $e_r := -d'_j$.

Since $d_i, d'_j \in \mathbb{Q}[c(k-i)]$ and $|f|, |f'| \leq g(k-i)$, one can easily see that

$$(*) : \quad \frac{e_r}{u_r} \quad \text{for} \quad u_r, u'_r \in \mathbb{Z} \quad \text{with} \quad |u_r| \leq 2c(k-i)^2 \quad \text{and} \quad |u'_r| \leq c(k-i)^2.$$

In case that $e_r = 0$ for all $r$, we have $\frac{\pi(s_1)}{\pi(s_2)} = \frac{\pi(s_1) - \pi(s_2)}{\pi(t)}$, and thus $\frac{\pi(s_1)}{\pi(s_2)} < \frac{\pi(t)}{\pi(t)}$ iff $\pi(s_1) < \pi(s_2)$.

We can hence concentrate on the case where at least one of the coefficients $e_r$ is different from 0. Without loss of generality we may assume that there is an $l''$ with $1 \leq l'' \leq l'$, such that $e_r \neq 0$ for all $r \leq l''$, and $e_r = 0$ for all $r > l''$. Furthermore, we may assume that if $a_i \in \{z_1, \ldots, z_{l''}\}$, then $a_i = z_1$.

Define $t_1 := z_1$ and $t_2 := \sum_{r=1}^{l''} \frac{e_r}{u_r} \cdot z_r + \frac{h}{e_1}$. It is straightforward to see that $t_2$ is a $(1(k-i+1), \{c(k-i+1), g(k-i+1) - \frac{c(k-i)}{2}\})$-combination over $P, \tilde{c}, a_1, \ldots, a_{l''-1}$. From $(*)$ we obtain $l'' - 1 \leq 21(k-i) - 1 = 21(k-i+1)$, and $\frac{e_r}{u_r} \in \mathbb{Q}[c(k-i)]$, where $2c(k-i)^4 = c(k-i+1)$, and $\frac{h}{e_1} \leq 2g(k-i)$, where $2g(k-i) c(k-i)^2 = g(k-i+1) - \frac{c(k-i)}{2}$.

In case that $t_1 = a_i$, we can apply condition (4) (for $i$); and otherwise we can apply condition (5) (for $i-1$) to obtain that $\frac{\pi(t_1)}{\pi(t_2)} < \frac{\pi(t_1)}{\pi(t_2)}$. Of course, this in particular gives us

$$(a) : \quad \frac{e_1 \cdot \pi(t_1)}{\pi(t_2)} < \frac{e_1 \cdot \pi(t_1)}{\pi(t_2)} \quad \text{iff} \quad e_1 \cdot \pi(t_1) < e_1 \cdot \pi(t_2).$$

Furthermore, we know that $\pi(s_1) < \pi(s_2)$ if and only if $\frac{\pi(s_1)}{\pi(s_2)} < \frac{\pi(t)}{\pi(t)}$. In other words, we have

$$(b) : \quad \frac{\pi(s_1)}{\pi(s_2)} < \frac{\pi(t)}{\pi(t)} \quad \text{iff} \quad e_1 \cdot \pi(t_1) < e_1 \cdot \pi(t_2).$$

Analogously, $\pi(s_1) < \pi(s_2)$ if and only if $\pi(s_1) - \pi(s_2) < 0$ if and only if $\sum_{r=1}^{l''} e_r z_r + h < 0$ if and only if $e_1 z_1$.

$$(c) : \quad \frac{\pi(s_1)}{\pi(s_2)} < \frac{\pi(t)}{\pi(t)} \quad \text{iff} \quad e_1 \cdot \pi(t_1) < e_1 \cdot \pi(t_2).$$
Altogether, (a), (b), and (c) give us that $s_1 < s_2$ iff $\pi(s_1) < \pi(s_2)$.
We hence obtain that condition (5) if satisfied for $i$.

Summing up, we have shown that the conditions (1)–(5) hold for $i=0$. Furthermore, we have shown for each $i \in \{1, \ldots, k\}$, that if they hold for $i-1$, then the duplicator can play in such a way that they hold for $i$. In particular, we conclude that the duplicator can play in such a way that the conditions (1)–(5) hold for all $i \in \{0, \ldots, k\}$. According to Claim 1 she thus has a winning strategy in the $k$-round $+\wedge$-game on $\mathfrak{A}$ and $\mathfrak{B}$.

This completes our proof of Proposition 6.11. $\blacksquare$

In fact, the proof of Proposition 6.11 shows the following result which is stronger but also more technical than Theorem 6.10. We will use this result in the following Section 6.3 in order to transfer the translation result to context structures whose universe is the set $\mathbb{R}$ of real numbers.

6.12 Proposition. Let $Q = \{q_0 < q_1 < q_2 < \cdots \} \subseteq \mathbb{N}$ satisfy the conditions $W(\omega)$ (cf., Definition 6.9). Let $m, l, c, g \in \mathbb{N}_{>0}$ and $g \in \mathbb{R}_{>0}$.

For every number $k \in \mathbb{N}_{>0}$ of rounds for the $\text{FO}(<, +, Q)$-game there is a number $r_{(m,l,c,g)}(k) \in \mathbb{N}$ of rounds for the $\text{FO}_{\text{dom}}(<)$-game such that the following is true for every signature $\tau$ and for all $\mathbb{N}$-embeddable $\tau$-structures $\mathfrak{A}$ and $\mathfrak{B}$: If
\[
\langle \text{dom}(\mathfrak{A}), <, \tau^A \rangle \approx_{r_{(m,l,c,g)}(k)} \langle \text{dom}(\mathfrak{B}), <, \tau^B \rangle,
\]
then there are $<$-preserving mappings $\alpha : \text{dom}(\mathfrak{A}) \rightarrow Q$ and $\beta : \text{dom}(\mathfrak{B}) \rightarrow Q$ such that the duplicator wins the $k$-round $\text{EF}$-game on
\[
\mathfrak{A} := \langle \mathbb{Z}, <, +, 0, Q, \alpha(\tau^A) \rangle \quad \text{and} \quad \mathfrak{B} := \langle \mathbb{Z}, <, +, 0, Q, \beta(\tau^B) \rangle
\]
in such a way that after the $k$-th round the following holds true:

- Let, for every $i \in \{1, \ldots, k\}$, $a_i$ and $b_i$ be the elements chosen in the $i$-th round in $\mathfrak{A}$ and $\mathfrak{B}$.
- Furthermore, let $\pi$ be the mapping defined via
  \[
  \pi : \{ \begin{array}{c}
  \alpha(c^A) \mapsto \beta(c^B) \quad \text{for all constant symbols } c \in \tau \\
  a_i \mapsto b_i \quad \text{for all } i \in \{1, \ldots, k\}
  \end{array} \}.
  \]

Then we have
- $x \equiv_m \pi(x)$, for every $x$ in the domain of $\pi$, and
- $s_1 < s_2$ iff $\pi(s_1) < \pi(s_2)$, for all $(l, c, g)$-combinations $s_1$ and $s_2$ over the domain of $\pi$. $\square$

Proof. Since the functions $1$, $c$, $g$ are increasing, we can find some $k_0 \in \mathbb{N}_{>0}$ such that $1(k_0) \geq l$, $c(k_0) \geq c$, $g(k_0) \geq g$, and $c(k_0) \geq m$. In particular, this also gives us that $m \mid m(k_0)$, because $m(k_0) = m(k_0-1) \cdot \text{lcm}\{1, \ldots, c(k_0)\}$. I.e., we have
\[
(*) : \quad 1(k_0) \geq l, \quad c(k_0) \geq c, \quad g(k_0) \geq g, \quad \text{and} \quad m \mid m(k_0).
\]

Let $r$ be the function defined in Proposition 6.11. Define the function $r_{(m,l,c,g)}(k)$ via $r_{(m,l,c,g)}(k) := r(k+k_0)$, for every $k \in \mathbb{N}$. Let $\vec{c} := 0, q_0, \ldots, q_{n_k+k_{0-1}}$ and $P := Q \setminus \{\vec{c}\}$. From the presumption
we know that $P, \bar{c}$ satisfies the conditions $W(k + k_0)$ and that all elements in $\bar{c}$ are smaller than the smallest element in $P$.

Let $\tau$ be a signature and let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{N}$-embeddable $\tau$-structures such that $(\text{adom}(\mathcal{A}), <, \tau^A) \cong_{r(k + k_0)} (\text{adom}(\mathcal{B}), <, \tau^B)$. Let $\alpha : \text{adom}(\mathcal{A}) \rightarrow P$ and $\beta : \text{adom}(\mathcal{B}) \rightarrow P$ map, for every $j$, the $j$-th smallest element of $\mathcal{A}$ respectively $\mathcal{B}$ onto the $j$-th smallest element in $P$. In the proof of Proposition 6.11 we have seen that the duplicator can win the $(k + k_0)$-round EF-game on $\mathfrak{A} := \langle \mathbb{Z}, <, +, P, c, \alpha(\tau^A) \rangle$ and $\mathfrak{B} := \langle \mathbb{Z}, <, +, P, c, \beta(\tau^B) \rangle$ in such a way that after the $k$-th round condition (5) is satisfied for $i = k$ and condition (2) is satisfied for all $i \in \{1, \ldots, k\}$. In particular, for the mapping $\pi$ defined in the formulation of Proposition 6.12 this means that

- $x \equiv_{n(k_0)} \pi(x)$, for every $x$ in the domain of $\pi$, and

- $\overline{s_1} < \overline{s_2}$ iff $\pi(s_1) < \pi(s_2)$, for all $(1(k_0), c(k_0), g(k_0))$-combinations $s_1$ and $s_2$ over the domain of $\pi$.

Due to $\ast$, this completes the proof of Proposition 6.12. \hfill $\blacksquare$

### 6.3 The $\text{FO}(<, +, Q, \text{Groups})$-Game over $\mathbb{R}$

In the previous Section 6.3 we investigated the context universes $\mathbb{N}$ and $\mathbb{Z}$, and showed that the duplicator can translate strategies for the $\text{FO}_{\text{adom}}(<)$-game into strategies for the $\text{FO}(<, +, Q)$-game on arbitrary structures over $\mathbb{N}$ and $\mathbb{N}$-embeddable structures over $\mathbb{Z}$ (cf., Theorem 5.10).

In the present section we transfer these results to the context universes $\mathbb{Q}$ and $\mathbb{R}$. As a consequence of Proposition 6.12 we obtain the following:

#### 6.13 Theorem (FO($<, +, Q, \text{Groups}$)-game over $\mathbb{R}$).

Let $Q \subseteq \mathbb{N}$ satisfy the conditions $W(\omega)$. Let $\text{Groups}$ consist of all sets $G \subseteq \mathbb{R}$ where $1 \in G$ and $(G, +)$ is a subgroup of $(\mathbb{R}, +)$. The duplicator can translate strategies for the $\text{FO}_{\text{adom}}(<)$-game into strategies for the $\text{FO}(<, +, Q, \text{Groups})$-game over $\mathbb{R}$.

In particular, this implies that the duplicator can translate strategies for the $\text{FO}_{\text{adom}}(<)$-game into strategies for the $\text{FO}(<, +, Q, \mathbb{Z}, \mathbb{Q})$-game on $\mathbb{N}$-embeddable structures over $\mathbb{Q}$ and over $\mathbb{R}$. \hfill $\square$

**Proof.** Let $k \in \mathbb{N}_{>0}$ be a number of rounds for the $\text{FO}(<, +, Q, \text{Groups})$-game. We define the according number $r(k)$ of rounds for the $\text{FO}_{\text{adom}}(<)$-game via $r(k) := r_{(1, 2, 2, 2)}(k)$, where $r_{(1, 2, 2, 2)}$ is the function from Proposition 6.12 for $m = 1$ and $l = c = g = 2$.

Let $\tau$ be a signature and let $\mathcal{A}$ and $\mathcal{B}$ be two $\mathbb{N}$-embeddable $(\mathbb{R}, \tau)$-structures such that $(\text{adom}(\mathcal{A}), <, \tau^A) \cong_{r(k)} (\text{adom}(\mathcal{B}), <, \tau^B)$. From Proposition 6.12 we obtain $<$-preserving mappings $\alpha : \text{adom}(\mathcal{A}) \rightarrow Q$ and $\beta : \text{adom}(\mathcal{B}) \rightarrow Q$ such that the duplicator can win the $k$-round EF-game on $\mathfrak{A}_\mathbb{Z} := \langle \mathbb{Z}, <, +, Q, \alpha(\tau^A) \rangle$ and $\mathfrak{B}_\mathbb{Z} := \langle \mathbb{Z}, <, +, Q, \beta(\tau^B) \rangle$ in such a way that after the $k$-th round the conditions formulated in Proposition 6.12 are satisfied. Henceforth, this game on $\mathfrak{A}_\mathbb{Z}$ and $\mathfrak{B}_\mathbb{Z}$ will be called the $\mathbb{Z}$-game.

Our aim is to show that the duplicator wins the $k$-round EF-game on $\mathfrak{A}_\mathbb{R} := \langle \mathbb{R}, <, +, Q, \text{Groups}, \alpha(\tau^A) \rangle$ and $\mathfrak{B}_\mathbb{R} := \langle \mathbb{R}, <, +, Q, \text{Groups}, \beta(\tau^B) \rangle$. Henceforth, the game on $\mathfrak{A}_\mathbb{R}$ and $\mathfrak{B}_\mathbb{R}$ will be called the $\mathbb{R}$-game.

In order to win the $\mathbb{R}$-game, the duplicator plays according to the strategy illustrated in Figure 8.
For the \( i \)-th round (for every \( i \in \{1, \ldots, k\} \)) this precisely means the following:
Assume that the spoiler chooses an element \( a_i \in A_R \) (the case where he chooses \( b_i \in B_R \) is symmetric). We translate the spoiler’s move \( a_i \) into a move \( a'_i \in A_Z \) for a “virtual spoiler” in the \( Z \)-game via \( a'_i := \lfloor a_i \rfloor \). In particular, we know that \( a_i = a'_i + f_i \) for some \( f_i \in \text{int} \langle 0, 1 \rangle \subseteq \mathbb{R} \).

Now, let \( b'_i \in B_Z \) be the answer of a “virtual duplicator” who plays according to her winning strategy in the \( Z \)-game. We can translate this answer into a move \( b_i \) for the duplicator in the \( R \)-game via \( b_i := b'_i + f_i \).

It is straightforward to see that after \( k \) rounds the duplicator has won the \( R \)-game: We need to show that the mapping \( \pi \) defined via

\[
\pi : \begin{cases} 
\alpha(c^A) & \mapsto \beta(c^B) \\
 a_i & \mapsto b_i 
\end{cases} \quad \text{for all constant symbols } c \in \tau 
\]

is a partial isomorphism between \( A_R \) and \( B_R \). We already know that the “virtual duplicator” has won the \( Z \)-game and that even the conditions formulated in Proposition 6.12 are satisfied. I.e., for the mapping \( \pi' \) defined via

\[
\pi' : \begin{cases} 
\alpha(c^A) & \mapsto \beta(c^B) \\
 a'_i & \mapsto b'_i 
\end{cases} \quad \text{for all constant symbols } c \in \tau 
\]

we know that

\((*)\) \( \pi' \) is a partial isomorphism between \( A_Z \) and \( B_Z \), and

\((**\) \( \overline{s_1} < \overline{s_2} \iff \pi'(s_1) < \pi'(s_2) \) \) is true for all \( (2, 2, 2) \)-combinations \( s_1 \) and \( s_2 \) over the domain of \( \pi' \).

Furthermore, we know that \( a_i = a'_i + f_i \) and \( b_i = b'_i + f_i \) for \( a'_i, b'_i \in \mathbb{Z} \) and \( f_i \in \text{int} \langle 0, 1 \rangle \subseteq \mathbb{R} \).

This, in particular, gives us that \( a_i \in \mathbb{Z} \) iff \( b_i \in \mathbb{Z} \) and, in general, for every \( G \in \text{Groups} \), that \( a_i \in G \) iff \( b_i \in G \). Together with (*) we furthermore obtain that \( a_i \in Q \) iff \( b_i \in Q \), and that \( \pi \) is a partial isomorphism between \( \langle \mathbb{R}, Q, \text{Groups}, \alpha(G^A) \rangle \) and \( \langle \mathbb{R}, Q, \text{Groups}, \beta(G^B) \rangle \).
All that remains to be done is to show that “$x < y$ iff $\pi(x) < \pi(y)$” and “$x + y = z$ iff $\pi(x) + \pi(y) = \pi(z)$” are true for all $x, y, z$ in the domain of $\pi$. In order to show this, consider the integers $x' := [x]$, $y' := [y]$, and $z' := [z]$, and choose $f, g, h \in \text{int} [0, 1] \subseteq \mathbb{R}$ such that $x = x' + f$, $y = y' + g$, and $z = z' + h$. Obviously, $x', y', z'$ must belong to the domain of $\pi'$, and we must have $\pi(x) = \pi'(x') + f$, $\pi(y) = \pi'(y') + g$, and $\pi(z) = \pi'(z') + h$. Due to (***) we know that $x' + f < y' + g$ iff $\pi'(x') + f < \pi'(y') + g$. This, in particular, gives us that $x < y$ iff $\pi(x) < \pi(y)$. Furthermore, (***) gives us that $x' + y' + (f + g) = z' + h$ iff $\pi'(x') + \pi'(y') + (f + g) = \pi'(z') + h$. In other words, we obtain that $x + y = z$ iff $\pi(x) + \pi(y) = \pi(z)$. Altogether, the proof of Theorem 6.13 is complete. \[ \square \]

6.4 Variations

6.4.1 More Built-In Predicates: $\text{Mon}_Q$ 

With Theorem 6.10 we obtained the translation result for every context structure $< \mathbb{Z}, <, +, Q >$ where $Q$ satisfies the conditions $W(\omega)$. Making use of the method for monadic predicates described in Section 5, we may add all subsets of $Q$ as built-in predicates:

6.14 Theorem. 

Let $Q \subseteq \mathbb{N}$ satisfy the conditions $W(\omega)$. Let $\text{Mon}_Q$ be the class of all subsets of $Q$.

(a) The duplicator can translate strategies for the $\text{FO}_{\text{adom}}(<)$-game into strategies for the $\text{FO}(<, +, Q, \text{Mon}_Q)$-game on arbitrary structures over $\mathbb{N}$ and on $\mathbb{N}$-embeddable structures over $\mathbb{Z}$.

(b) The duplicator can translate strategies for the $\text{FO}_{\text{adom}}(<)$-game into strategies for the $\text{FO}(<, +, Q, \text{Mon}_Q, \text{Groups})$-game on $\mathbb{N}$-embeddable structures over $\mathbb{N}$.

Proof (sketch). (b) can be obtained from (a) (respectively, from the according variant of Proposition 6.12) in the same way as Theorem 6.13 was obtained from Theorem 6.10. Part (a) is a direct consequence of the following variant of Proposition 6.11:

6.15 Proposition. Let $k, n \in \mathbb{N}$, let $\tilde{\mathcal{C}} := c_1, \ldots, c_n \in \mathbb{N}$, and let $P := \{ p_1 < p_2 < p_3 < \cdots \} \subseteq \mathbb{N}$ be an infinite set such that $P, \tilde{\mathcal{C}}$ satisfies the conditions $W(k)$ and such that, for all $\nu \in \{1, \ldots, n\}$, $c_\nu$ is smaller than the smallest element in $P$. Let $\text{Mon}_P$ be the class of all subsets of $P$. There is a number $r(k) \in \mathbb{N}$ such that the following is true for all finite subsets $\text{Mon}_P^r$, of $\text{Mon}_P$, and for all signatures $\tau$:

If $\mathcal{A}$ and $\mathcal{B}$ are $\mathbb{N}$-embeddable $\tau$-structures with $\langle \text{adom}(\mathcal{A}), <, \tau^\mathcal{A} > \approx_{r(k)} \langle \text{adom}(\mathcal{B}), <, \tau^\mathcal{B} \rangle$, then there are $\tau$-preserving mappings $\alpha : \text{adom}(\mathcal{A}) \rightarrow P$ and $\beta : \text{adom}(\mathcal{B}) \rightarrow P$ such that $\langle \mathbb{Z}, <, +, P, \tilde{\mathcal{C}}, \text{Mon}_P^r, \alpha(\tau^\mathcal{A}) \rangle \approx_{r(k)} \langle \mathbb{Z}, <, +, P, \tilde{\mathcal{C}}, \text{Mon}_P, \beta(\tau^\mathcal{B}) \rangle$. \[ \square \]

For the proof of Proposition 6.15 choose $r(k) := r_1(r_2(k))$, where $r_2$ is the function $r$ obtained from Proposition 6.11, and $r_1$ is the function $r$ obtained from Theorem 5.4. Assume we are given two $\mathbb{N}$-embeddable $\tau$-structures $\mathcal{A}$ and $\mathcal{B}$ with $\langle \text{adom}(\mathcal{A}), <, \tau^\mathcal{A} \rangle \approx_{r_1(r_2(k))} \langle \text{adom}(\mathcal{B}), <, \tau^\mathcal{B} \rangle$. Theorem 5.4 (for $U := P$) gives us $<\text{-preserving mappings} \alpha : \text{adom}(\mathcal{A}) \rightarrow P$ and $\beta : \text{adom}(\mathcal{B}) \rightarrow P$ such that $\langle P, <, \text{Mon}_P^r, \alpha(\tau^\mathcal{A}) \rangle \approx_{r_2(k)} \langle P, <, \text{Mon}_P, \beta(\tau^\mathcal{B}) \rangle$. In the proof of Proposition 6.11 we considered the structures $\mathcal{A}' := \langle P, <, \alpha(\tau^\mathcal{A}) \rangle \approx_{r_2(k)} \langle P, <, \beta(\tau^\mathcal{B}) \rangle =: \mathcal{B}'$. Instead of $\mathcal{A}'$ and $\mathcal{B}'$ we now use the structures $\mathcal{A}'' := \langle P, <, \text{Mon}_P^r, \alpha(\tau^\mathcal{A}) \rangle \approx_{r_2(k)} \langle P, <, \text{Mon}_P, \beta(\tau^\mathcal{B}) \rangle =: \mathcal{B}''$. In the proof of Proposition 6.11 we replace $\mathcal{A}'$ and $\mathcal{B}'$
with $\mathfrak{A}'$ and $\mathfrak{B}'$. This gives us the desired result that $\langle \mathbb{Z}, <, +, P, \tilde{\varepsilon}, \mathfrak{Mon}\rangle, \alpha(\tau^A) \rangle \approx_k \langle \mathbb{Z}, <, +, P, \tilde{\varepsilon}, \mathfrak{Mon}\rangle, \beta(\tau^B) \rangle$. Altogether, this proves Proposition 6.13 and completes the proof sketch of Theorem 6.14.

\section*{6.4.2 A Question of Belegradek et al.}

Considering the natural generic collapse over finite databases, Belegradek et al. asked in the conclusion of \cite{3}: “How much higher than $+$ in $\langle \mathbb{Z}, < \rangle$ can we go?” Our Theorem 6.14 gives an answer: We still obtain the natural generic collapse when adding a set $Q$ that satisfies the conditions $W(\omega)$ and when adding all subsets of $Q$ as built-in predicates.

Furthermore, Belegradek et al. conjectured the following: “If, for some class $\mathfrak{Bip}$ of built-in predicates, $<\text{-}generic\ FO(<, +, \mathfrak{Bip}) \neq \mathfrak{FO}_{\text{dom}}(<)$ on $\mathcal{C}_{\text{fin}}$ over $\mathbb{Z}$, then the first-order theory of $\langle \mathbb{Z}, <, +, \mathfrak{Bip} \rangle$ is undecidable.” Our result shows that the converse of this conjecture is not true: Let $Q$ be the set obtained in Lemma 6.8 (b), and let $\tilde{Q}$ be an undecidable subset of $Q$. E.g., $\tilde{Q}$ can be chosen to contain, for every $n \in \mathbb{N}_{>0}$, the $n$-th largest element in $Q$ if and only if the $n$-th Turing machine halts with empty input. Clearly, the first-order theory of $\langle \mathbb{Z}, <, +, \tilde{Q} \rangle$ is undecidable. On the other hand, $\tilde{Q}$ satisfies the conditions $W(\omega)$, and hence Theorem 6.10 gives us that $<\text{-}generic\ FO(<, +, \tilde{Q}) = \mathfrak{FO}_{\text{dom}}(<)$ on $\mathcal{C}_{\text{fin}}$ over $\mathbb{Z}$.

\section*{6.4.3 More Structures: $\mathbb{Z}$-embeddable Structures?}

Theorem 6.10 states the translation result for $\mathbb{N}$-embeddable structures over the context structure $\langle \mathbb{Z}, <, +, Q \rangle$. It remains open whether the translation is possible also for $\mathbb{Z}$-embeddable structures. The main reason why our proof does not work for all $\mathbb{Z}$-embeddable structures is that there does not exist a set $P$ which satisfies the conditions $W(k)$ and which is infinite in both directions (this easily follows from the definition of the conditions $W(k)$).

However, with some modification, our proof of Proposition 6.11 shows the following:

\section*{6.16 Theorem.} Let $Q \subseteq \mathbb{N}$ satisfy the conditions $W(\omega)$. Let Inv be the binary relation which connects each number with its additive inverse, i.e., Inv$(x, y)$ iff $x > 0$ and $y = -x$.

(a) The duplicator can translate strategies for the $\FO(<, \text{Inv})$-game into strategies for the $\FO(<, +, Q)$-game on arbitrary structures over $\mathbb{Z}$.

(b) The duplicator can translate strategies for the $\FO(<, \text{Inv})$-game into strategies for the $\FO(<, +, Q, \text{Groups})$-game on $\mathbb{Z}$-embeddable structures over $\mathbb{R}$.

\section*{Proof (sketch).} It should be clear that (b) can be obtained from (a) (respectively, from the according variant of Proposition 6.12) in the same way as Theorem 6.13 was obtained from Theorem 6.10.

Part (a) can be proved as follows: Let $k$ be a number of rounds for the $\FO(<, +, Q)$-game, and let $r(k)$ and $P, \tilde{\varepsilon}$ be chosen as in the proof of Theorem 5.10. Assume we are given two $\langle \mathbb{Z}, \tau \rangle$-structures $\mathfrak{A}$ and $\mathfrak{B}$ with $\langle \mathbb{Z}, <, \text{Inv}, \tau^{A} \rangle \approx_{r(k)} \langle \mathbb{Z}, <, \text{Inv}, \tau^{B} \rangle$. In the proof of Proposition 6.11 we considered structures $\mathfrak{A}' := \langle P, <, \alpha(\tau^{A}) \rangle \approx_{r(k)} \langle P, <, \beta(\tau^{B}) \rangle =: \mathfrak{B}'$. Instead, we now consider the following $<\text{-}preserving$ mappings $\alpha$ and $\beta$: The mapping $\alpha$ is defined via $\alpha(0) := 0$, $\alpha(n) := p_n$, and $\alpha(-n) := -p_n$, for all $n \in \mathbb{N}_{>0}$. Here, we assume that $\alpha$ is even on all $\mathbb{N}$-embeddable databases over $\mathbb{Z}$.
the universe of $r$ game for spoiler’s move was in “classical” EF-game that is suitable for characterizing the logic structures section is that the duplicator can translate strategies for the existential we write $(3.)$ The relation Bip 7.1 The EF-Game for $BC$

In the previous sections we concentrated on the EF-game for $FO$. In the present section we restrict our attention to the sublogic $BC(EFO)$, consisting of the Boolean combinations of purely existential first-order formulas. We introduce the single-round $r$-move game as a variant of the “classical” EF-game that is suitable for characterizing the logic $BC(EFO)$. The main result of this section is that the duplicator can translate strategies for the $BC(EFO)_{adom}(\prec)$-game into strategies for the $BC(EFO)(\prec, \mathcal{B}ip)$-game on $\mathbb{N}$-embeddable structures over every context structure $\langle U, \prec, \mathcal{B}ip \rangle$.

7.1 The EF-Game for $BC(EFO)$

In the same way as the “classical” EF-game characterizes the logic $FO$, the following variant of the EF-game characterizes the logic $BC(EFO)$.

Let $\tau$ be a signature and let $r$ be a natural number. The single-round $r$-move game on two $\tau$-structures $A$ and $B$ is played as follows: First, the spoiler chooses either $r$ elements $a_1, \ldots, a_r$ in the universe of $A$, or $r$ elements $b_1, \ldots, b_r$ in the universe of $B$. Afterwards, the duplicator chooses $r$ elements in the other structure. I.e., she chooses either $r$ elements $b_1, \ldots, b_r$ in the universe of $B$, if the spoiler’s move was in $A$, or she chooses $r$ elements $a_1, \ldots, a_r$ in the universe of $A$, if the spoiler’s move was in $B$.

The winning condition is identical to the winning condition in the “classical” $r$-round EF-game for $FO$. We say that the duplicator wins the single-round $r$-move game on $A$ and $B$, and we write $A \sim_r B$, if and only if the duplicator has a winning strategy in the single-round $r$-move game on $A$ and $B$. It is straightforward to see that, for every signature $\tau$, the relation $\sim_r$ is an equivalence relation on the set of all $\tau$-structures. By the standard argumentation (see, e.g., the textbooks [13, 11]) one obtains the according variants of Theorem [2] and Remark [12]. I.e.:

1. $A \sim_r B$ if and only if $A$ and $B$ cannot be distinguished by $BC(EFO)(\tau)$-sentences of quantifier depth $\leq r$.

2. A class $\mathcal{L}$ of $\tau$-structures is not $BC(EFO)(\tau)$-definable in $\mathcal{K}$ if and only if for every $r \in \mathbb{N}$ there are structures $A_r, B_r \in \mathcal{K}$ with $A_r \in \mathcal{L}$ and $B_r \notin \mathcal{L}$ and $A_r \sim_r B_r$.

3. The relation $\sim_r$ has only finitely many equivalence classes on the set of all $\tau$-structures; and each such equivalence class is definable by a $BC(EFO)(\tau)$-sentence of quantifier depth $\leq r$.

It is straightforward to modify Definition [12] in such a way that it serves for proving a collapse of the form $\prec$-generic $BC(EFO)(\prec, \mathcal{B}ip) = BC(EFO)_{adom}(\prec)$ on $\mathcal{G}$ over $U$. 

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7.1 Definition (Translation of Strategies for BC(EFO)).
Let \( \langle U, <, \mathcal{B}_p \rangle \) be a context structure, and let \( \mathcal{C} \) be a class of structures over the universe \( U \). We say that

the duplicator can translate strategies for the \( \text{BC}(\text{EFO})_{\text{adom}}(\langle \cdot \rangle) \)-game into strategies for the \( \text{BC}(\text{EFO})(\langle \cdot, \mathcal{B}_p \rangle) \)-game on \( \mathcal{C} \) over \( U \)

if and only if the following is true:

For every finite set \( \mathcal{B}_p' \subseteq \mathcal{B}_p \), for every signature \( \tau \), and for every number \( k \in \mathbb{N} \) there is a number \( r(k) \in \mathbb{N} \) such that the following is true for all \( \langle U, \tau \rangle \)-structures \( A, B \in \mathcal{C} \): If the duplicator wins the single-round \( r(k) \)-move BC(EFO)_{\text{adom}}(\langle \cdot \rangle)-game on \( A \) and \( B \), i.e., if \( \langle \text{adom}(A), <, \tau^A \rangle \sim_{r(k)} \langle \text{adom}(B), <, \tau^B \rangle \), then there are \( < \)-preserving mappings \( \alpha : \text{adom}(A) \to U \) and \( \beta : \text{adom}(B) \to U \) such that the duplicator wins the single-round \( k \)-move BC(EFO)(\langle \cdot, \mathcal{B}_p' \rangle)-game on \( \alpha(A) \) and \( \beta(B) \), i.e., \( \langle U, <, \mathcal{B}_p', \alpha(\tau^A) \rangle \sim_k \langle U, <, \mathcal{B}_p', \beta(\tau^B) \rangle \).

\( \square \)

Replacing FO with BC(EFO) and replacing \( \approx_r \) with \( \sim_r \) in the proof of Theorem 4.4, we directly obtain the following:

7.2 Theorem (Translation of Strategies \( \iff \) Collapse Result).
Let \( \langle U, <, \mathcal{B}_p \rangle \) be a context structure, and let \( \mathcal{C} \) be a class of structures over the universe \( U \). The following are equivalent:

(a) The duplicator can translate strategies for the \( \text{BC}(\text{EFO})_{\text{adom}}(\langle \cdot \rangle) \)-game into strategies for the \( \text{BC}(\text{EFO})(\langle \cdot, \mathcal{B}_p \rangle) \)-game on \( \mathcal{C} \) over \( U \).

(b) \( < \)-generic \( \text{BC}(\text{EFO})(\langle \cdot, \mathcal{B}_p \rangle) = \text{BC}(\text{EFO})_{\text{adom}}(\langle \cdot \rangle) \) on \( \mathcal{C} \) over \( U \).

\( \square \)

In Section 7.3 below we will show that the duplicator can indeed translate strategies for the \( \text{BC}(\text{EFO})_{\text{adom}}(\langle \cdot \rangle) \)-game into strategies for the \( \text{BC}(\text{EFO})(\langle \cdot, \mathcal{B}_p \rangle) \)-game on \( \mathcal{C}_{\text{emb}} \) over \( U \), for every linearly ordered infinite universe \( U \). However, we first show a lemma that will help us avoid some technical difficulties in the translation proof.

7.2 A Technical Lemma Similar to Lemma 4.5
The following lemma is an analogue of Lemma 4.5. Note, however, that the mappings \( \alpha \) and \( \beta \) now depend on the number \( r \) of moves in the game.

7.3 Lemma. Let \( P := \{p_1 < p_2 < p_3 < \cdots \} \) be a countable, infinitely increasing sequence, and let \( \text{succ}^P \) be the binary successor relation on \( P \), i.e., \( \text{succ}^P := \{(p_j, p_{j+1}) : j \in \mathbb{N}_{>0} \} \). Let \( \tau \) be a signature, and let \( A \) and \( B \) be two \( \mathbb{N} \)-embeddable \( \tau \)-structures over linearly ordered universes.

For every \( r \in \mathbb{N} \) there exist \( < \)-preserving mappings \( \alpha : \text{adom}(A) \to P \) and \( \beta : \text{adom}(B) \to P \) such that the following is true: If \( \langle \text{adom}(A), <, \tau^A \rangle \sim_r \langle \text{adom}(B), <, \tau^B \rangle \), then also \( \mathfrak{A} := \langle P, <, p_1, \text{succ}^P, \alpha(\tau^A) \rangle \sim_r \langle P, <, p_1, \text{succ}^P, \beta(\tau^B) \rangle =: \mathfrak{B} \).

\( \square \)

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Proof. The main idea is to define the mappings $\alpha$ and $\beta$ in such a way that there is a large gap between any two active domain elements. Precisely, given $P = \{p_1 < p_2 < p_3 < \cdots\}$ it suffices to move the active domain elements of $A$ and $B$ onto the positions $p_{2r} < p_{4r} < p_{6r} < \cdots$. I.e.: \(\alpha : \text{dom}(A) \to P\) and $\beta : \text{dom}(B) \to P$ map, for every $j$, the $j$-th smallest element in $\text{dom}(A)$ and $\text{dom}(B)$, respectively, onto the position $p_{2rj}$. From the presumption we know that a “virtual duplicator” wins the single-round $r$-move game on $(\text{dom}(A), <, \tau^A)$ and $(\text{dom}(B), <, \tau^B)$, i.e.,

\[(*) : \langle \text{dom}(A), <, \tau^A \rangle \sim_{r} \langle \text{dom}(B), <, \tau^B \rangle.\]

Obviously, this remains valid if $r$ is replaced with a number $s \leq r$. The game $(*)$ will henceforth be called the small game.

The aim is to find a winning strategy for the duplicator in the single-round $r$-move game on $\mathfrak{A} := \langle P, <, p_1, \text{succ}^P, \alpha(\tau^A) \rangle$ and $\mathfrak{B} := \langle P, <, p_1, \text{succ}^P, \beta(\tau^B) \rangle$. This game will henceforth be called the big game.

Assume that the spoiler chooses the elements $a_1, \ldots, a_r$ in the universe of $\mathfrak{A}$ (if he chooses the elements $b_1, \ldots, b_r$ in the universe of $\mathfrak{B}$, we can proceed in the according way, interchanging the roles of $\mathfrak{A}$ and $\mathfrak{B}$). Some — possibly all, or none — of the elements $a_1, \ldots, a_r$ belong to $\alpha(\text{dom}(A))$. Let $s$ be the number of these elements and let, without loss of generality, $a_1, \ldots, a_s \in \alpha(\text{dom}(A))$ and $a_{s+1}, \ldots, a_r \notin \alpha(\text{dom}(A))$. Furthermore, we may assume that $a_1 < \cdots < a_s$.

Of course there exist positions $a_1 < \cdots < a_s$ in $\text{dom}(A)$ such that $a_1 = \alpha(a_1), \ldots, a_s = \alpha(a_s)$. These elements $a_1, \ldots, a_s$ are the moves for a “virtual spoiler” in the small game. A “virtual duplicator” who plays according to her winning strategy in the small game will find answers $b_1 < \cdots < b_s$ in $\text{dom}(B)$. We can translate these answers into moves $b_1 < \cdots < b_s$ in $\mathfrak{B}$ via $b_1 := \beta(b_1), \ldots, b_s := \beta(b_s)$. The mapping $a_1, \ldots, a_s \mapsto b_1, \ldots, b_s$ obviously is a partial isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

The elements $b_1, \ldots, b_s$ will belong to the duplicator’s answers in the big game. However, the duplicator also has to find elements $b_{s+1}, \ldots, b_r \notin \beta(\text{dom}(B))$ such that, for all $\nu, \nu' \in \{1, \ldots, r\}$, we have

\[(**): b_\nu = p_1 \text{ iff } a_\nu = p_1, \quad b_\nu < b_{\nu'} \text{ iff } a_\nu < a_{\nu'}, \quad \text{succ}^P(b_\nu, b_{\nu'}) \text{ iff } \text{succ}^P(a_\nu, a_{\nu'}).\]

For every $i < s$, $b_i$ is of the form $p_{2rj}$ and $b_{i+1}$ is of the form $p_{2rj'}$ for suitable $j < j' \in \mathbb{N}_{>0}$. In particular, there are at least $2r-1$ different elements in $P$ between $b_i$ and $b_{i+1}$. Therefore, it is straightforward to find elements $b_{s+1}, \ldots, b_r$ such that the condition $(**)$ is satisfied by $b_1, \ldots, b_s, b_{s+1}, \ldots, b_r$. With these answers, the duplicator wins the big game, and hence the proof of Lemma 7.3 is complete.

### 7.3 How to Win the BC(EFO) (<, Arb)-Game

#### 7.4 Theorem (BC(EFO) (<, Arb)-Game over $\mathcal{U}$).

Let $(\mathcal{U}, <)$ be an infinite linearly ordered structure, and let Arb be the collection of arbitrary, i.e., all, predicates on $\mathcal{U}$. The duplicator can translate strategies for the BC(EFO)_{dom(<)}-game into strategies for the BC(EFO)_{dom(<)}-game on $\mathcal{U}_{\text{emb}}$ over $\mathcal{U}$. □
The overall proof idea is an adaption and extension of a proof developed in the context of the Crane Beach conjecture \[\text{[4]}\] for the specific context of finite strings instead of arbitrary structures.

We make use of the following variant of Ramsey’s Theorem:

**7.5 Theorem.** Let \(\langle U, \prec \rangle\) be an infinite linearly ordered structure. Let \(r \in \mathbb{N}_{>0}\) and let \(C_1, \ldots, C_r\) be finite sets. Each set \(C_h\) serves as a set of possible colors for \(h\)-element subsets of \(U\). I.e., for every \(h \in \{1, \ldots, r\}\), let every \(h\)-element subset \(Y_h = \{y_1 < \cdots < y_h\} \subseteq U\) be colored with an element \(\text{col}_h(Y_h) \in C_h\).

If \(\langle U, \prec \rangle\) contains an infinitely increasing sequence, then there exists an infinitely increasing set \(P = \{p_1 < p_2 < \cdots\} \subseteq U\) that satisfies the following condition (\(\star\)):

For every \(h \in \{1, \ldots, r\}\) there exists a color \(c_h \in C_h\) such that every \(h\)-element subset \(Y_h \subseteq P\) has the color \(\text{col}_h(Y_h) = c_h\).

Otherwise, if \(\langle U, \prec \rangle\) does not contain an infinitely increasing sequence, then there exists an infinitely decreasing set \(P = \{p_1 > p_2 > \cdots\} \subseteq U\) that satisfies the condition (\(\star\)). \(\square\)

**Proof.** The idea is to apply the following “classical” Ramsey Theorem successively for \(h = 1, 2, \ldots, r\).

**7.6 Theorem (Ramsey, cf., \[10, \text{Theorem 9.1.2}]\).** Let \(X\) be an infinite set and let \(h \in \mathbb{N}_{>0}\). Let \(C_h\) be a finite set such that every \(h\)-element set \(Y_h \subseteq X\) is colored with an element \(\text{col}_h(Y_h) \in C_h\).

There exists an infinite set \(X' \subseteq X\) and a color \(c_h \in C_h\) such that every \(h\)-element subset \(Y_h \subseteq X'\) has the color \(\text{col}_h(Y_h) = c_h\). \(\square\)

For the proof of Theorem 7.6, let us first assume that \(U\) contains a countable, infinitely increasing subset \(X_0\). For \(X := X_0\) and \(h := 1\), the above Ramsey Theorem 7.6 gives us an infinite set \(X_1 := X' \subseteq X_0\) and a color \(c_1 \in C_1\) such that all 1-element subsets of \(X_1\) have the color \(c_1\). Another application of the Ramsey Theorem for \(X := X_1\) and \(h := 2\) yields an infinite set \(X_2 \subseteq X_1\) and a color \(c_2 \in C_2\) such that all 2-element subsets of \(X_2\) have the color \(c_2\). Iterating this process for \(h = 1, 2, \ldots, r\) leads to sets \(X_1 \supseteq \cdots \supseteq X_r\) and to colors \(c_1 \in C_1, \ldots, c_r\) such that \(X_r\) is an infinitely increasing set and, for every \(h \in \{1, \ldots, r\}\), every \(h\)-element subset of \(X_r\) has the color \(c_h\). Consequently, the set \(P := X_r\) is the desired set of the form \(\{p_1 > p_2 > \cdots\}\) that satisfies the condition (\(\star\)).

It remains to consider the case where \(U\) does not contain a countable, infinitely increasing subset \(X_0\). In this case, since \(U\) is infinite and linearly ordered, there must exist an infinitely decreasing subset \(X_0\) (see Fact 5.3). Starting with this particular set \(X_0\), the same argumentation as above now leads to the desired set \(P := X_r\) of the form \(\{p_1 > p_2 > \cdots\}\). Altogether, this completes the proof of Theorem 7.6. \(\blacksquare\)

**Proof of Theorem 7.4 (BC(EFO)\(\prec, \mathfrak{Atb}\)\)-Game over \(U\).**

We concentrate on the case where \(\langle U, \prec \rangle\) contains an infinitely increasing sequence (at the end of the proof we will indicate how the arguments can be modified for the case that \(\langle U, \prec \rangle\) contains no such sequence).

Let \(\mathfrak{Atb}\) be a finite subset of \(\mathfrak{Atb}\). Let \(\tau\) be a signature, and let \(\kappa \in \mathbb{N}\) be the number of constant symbols in \(\tau\). For every number \(k \in \mathbb{N}\) of moves in the BC(EFO)\(\prec, \mathfrak{Atb}\)-game we choose \(r := r(k) := 2k + \kappa\) to be the according number of moves in the BC(EFO)\(\text{dom}(\prec)\)-game.
Let $A = \langle U, \tau^A \rangle$ and $B = \langle U, \tau^B \rangle$ be two $\mathbb{N}$-embeddable structures on which the duplicator wins the single-round $r$-move $BC(EFO)_{\text{adom}(<)}$-game, i.e.,

$$(*) : \quad \langle \text{adom}(A), <, \tau^A \rangle \sim_r \langle \text{adom}(B), <, \tau^B \rangle.$$  

We have to find $<\!-$preserving mappings $\alpha : \text{adom}(A) \to U$ and $\beta : \text{adom}(B) \to U$ such that the duplicator wins the single-round $k$-move $BC(EFO)(<, \mathcal{Bip}')$-game on $\alpha(A)$ and $\beta(B)$, i.e.,

$$(\langle U, <, \mathcal{Bip}', \alpha(\tau^A) \rangle \sim_k \langle U, <, \mathcal{Bip}', \beta(\tau^B) \rangle).$$

**Step 1:** We first choose a suitable infinite set $P = \{ p_1 < p_2 < \cdots \}$ onto which the active domain elements of $A$ and $B$ will be moved via $<\!-$preserving mappings $\alpha$ and $\beta$. To find this set $P$ we use the above Ramsey Theorem 7.5. The precise choice of the sets of colors $C_1, \ldots, C_r$ is quite elaborate. For better accessibility of the proof it might be helpful to skip this at first reading, having seen the duplicator’s strategy for the single-round $k$-move $BC(EFO)(<, \mathcal{Bip}')$-game on $\alpha(A)$ and $\beta(B)$.

Let $h \in \{1, \ldots, r\}$ and let $Y_h = \{ a'_1 < \cdots < a'_h \} \subseteq U$ be an $h$-element subset of $U$. For every $(a_1, \ldots, a_k) \in U^k$ we define $\text{type}_{=, <, \mathcal{Bip}'}(a_1, \ldots, a_k, a'_1, \ldots, a'_h)$ to be the complete atomic type of $(a_1, \ldots, a_k, a'_1, \ldots, a'_h)$ with respect to the relations $\{=, <\} \cup \mathcal{Bip}'$. Precisely, this means the following: We use first-order variables $x_1, \ldots, x_k$ and $y_1, \ldots, y_h$, and we consider all atomic $(\{=, <\} \cup \mathcal{Bip}')$-formulas over these variables. $\text{type}_{=, <, \mathcal{Bip}'}(a_1, \ldots, a_k, a'_1, \ldots, a'_h)$ is defined to be the set of exactly those atomic formulas $\varphi$ that are satisfied when interpreting the variables $x_1, \ldots, x_k$ and $y_1, \ldots, y_h$ with the elements $a_1, \ldots, a_k$ and $a'_1, \ldots, a'_h$, respectively. It should be clear that $\text{type}_{=, <, \mathcal{Bip}'}(a_1, \ldots, a_k, a'_1, \ldots, a'_h) = \text{type}_{=, <, \mathcal{Bip}'}(b_1, \ldots, b_k, b'_1, \ldots, b'_h)$ if and only if the mapping $(a_1, \ldots, a_k, a'_1, \ldots, a'_h) \mapsto (b_1, \ldots, b_k, b'_1, \ldots, b'_h)$ is a partial automorphism of the structure $\langle U, <, \mathcal{Bip}' \rangle$.

To apply the Ramsey Theorem 7.5 we color every $h$-element set $Y_h = \{ a'_1 < \cdots < a'_h \} \subseteq U$ with the collection of all atomic types that are realizable with $a'_1, \ldots, a'_h$. Precisely, this means that

$$\text{col}_h(Y_h) := \{ \text{type}_{=, <, \mathcal{Bip}'}(a_1, \ldots, a_k, a'_1, \ldots, a'_h) : (a_1, \ldots, a_k) \in U^k \}.$$  

Since $\mathcal{Bip}'$ is finite, the number of complete atomic types over the variables $x_1, \ldots, x_k, y_1, \ldots, y_h$ is finite. Consequently, also the set of colors used for $h$-element subsets of $U$, i.e., the set $C_h := \{ \text{col}_h(Y_h) : Y_h \text{ is an } h\text{-element subset of } U \}$, must be finite.

We use these colorings $C_h$, for all $h \in \{1, \ldots, r\}$, and apply the Ramsey Theorem 7.5. Hence we obtain an infinitely increasing set $P = \{ p_1 < p_2 < \cdots \} \subseteq U$ that satisfies the following condition: For every $h \in \{1, \ldots, r\}$ there exists a color $c_h \in C_h$ such that every $h$-element subset $Y_h \subseteq P$ has the color $\text{col}_h(Y_h) = c_h$.

In the following we will use the elements of $P$ as “special positions” onto which the active domain of the given structures $A$ and $B$ will be moved.

**Step 2:** From the presumption $(*)$ and from Lemma 7.3 we obtain $<\!-$preserving mappings $\alpha : \text{adom}(A) \to P$ and $\beta : \text{adom}(B) \to P$ such that a “virtual duplicator” has a winning strategy for the single-round $r$-move game on $\langle P, <, p_1, \text{succ}^P, \alpha(\tau^A) \rangle$ and $\langle P, <, p_1, \text{succ}^P, \beta(\tau^B) \rangle$, i.e.,

$$(**) : \quad \exists' : \quad \langle P, <, p_1, \text{succ}^P, \alpha(\tau^A) \rangle \sim_r \langle P, <, p_1, \text{succ}^P, \beta(\tau^B) \rangle =: \mathcal{B}'.$$
Obviously, (***) remains valid if \( r \) is replaced by a number \( h \leq r \).

We now describe a winning strategy for the duplicator, showing that \( \mathfrak{A} := \langle \mathbb{U}, <, \mathfrak{Bip}', \alpha(\tau^A) \rangle \sim_k \langle \mathbb{U}, <, \mathfrak{Bip}', \beta(\tau^B) \rangle := \mathfrak{B} \). Assume that the spoiler chooses the elements \( \vec{a} = a_1, \ldots, a_k \) in the universe of \( \mathfrak{A} \) (if he chooses the elements \( \vec{b} = b_1, \ldots, b_k \) in the universe of \( \mathfrak{B} \), we can proceed in the according way, interchanging the roles of \( \mathfrak{A} \) and \( \mathfrak{B} \)). To find appropriate answers \( \vec{b} = b_1, \ldots, b_k \) for the duplicator, we proceed as follows: We determine, for every \( i \in \{1, \ldots, k\} \), the unique elements in \( P \) that are the closest elements to \( a_i \). Precisely, if \( p_j \leq a_i < p_{j+1} \) then \( p_j \) and \( p_{j+1} \) are these closest elements, and we fix the 2-element set \( I_i := \{p_j, p_{j+1}\} \). Accordingly, if \( a_i < p_1 \) then \( p_1 \) is the closest element, and we fix the singleton set \( I_i := \{p_1\} \).

Of course, the set \( I := I_1 \cup \cdots \cup I_k \) has cardinality \( \leq 2k \). Consequently, the union of \( I \) with the set of all constants of \( \mathfrak{A}' \) is a set of the form \( \{a'_{i_1} < \cdots < a'_{i_h}\} \subseteq P \), for a suitable \( h \leq 2k + \kappa = r \). The elements \( a'_{i_1}, \ldots, a'_{i_h} \) are the moves for a “virtual spoiler” in the game on \( \mathfrak{A}' \) and \( \mathfrak{B}' \). A “virtual duplicator” who plays according to her winning strategy in the game (***) will find answers \( b'_i < \cdots < b'_{i_h} \) in \( \mathfrak{B}' \).

Since \( \{a'_{i_1} < \cdots < a'_{i_h}\} \) and \( \{b'_{i_1} < \cdots < b'_{i_h}\} \) are \( h \)-element subsets of \( P \), and since \( P \) was chosen according to Step 1, they must have the same color \( c_h \in C_h \). Due to the particular definition of the colors, as fixed in Step 1, there hence must be elements \( b = b_1, \ldots, b_k \) in \( \mathbb{U} \) such that

\[
(***): \quad \text{type}_{=, <, \mathfrak{Bip}'}(\vec{b}, b'_1, \ldots, b'_h) = \text{type}_{=, <, \mathfrak{Bip}'}(\vec{a}, a'_1, \ldots, a'_h).
\]

We choose exactly these elements \( b_1, \ldots, b_k \) to be the duplicator’s answers in \( \mathfrak{B} \).

**Step 3:** It remains to verify that the duplicator has indeed won the game on \( \mathfrak{A} \) and \( \mathfrak{B} \). I.e., we have to show that the mapping \( \pi \) defined via

\[
\pi: \begin{cases} 
\alpha(c^A) & \mapsto \beta(c^B) \quad \text{for all constant symbols } c \in \tau \\
a_i & \mapsto b_i \quad \text{for all } i \in \{1, \ldots, k\}
\end{cases}
\]

is a partial isomorphism between the structures \( \mathfrak{A} = \langle \mathbb{U}, <, \mathfrak{Bip}', \alpha(\tau^A) \rangle \) and \( \mathfrak{B} = \langle \mathbb{U}, <, \mathfrak{Bip}', \beta(\tau^B) \rangle \).

**Claim 1:** \( \pi \) is a partial automorphism of \( \langle \mathbb{U}, <, \mathfrak{Bip}' \rangle \).

By definition, all the constants of \( \mathfrak{A} \) belong to the sequence \( a'_1, \ldots, a'_h \). Since the “virtual duplicator” wins the game (**), all the constants of \( \mathfrak{B} \) must occur in the sequence \( b'_1, \ldots, b'_h \). Consequently, the above property (***), tells us that \( \pi \) is a partial automorphism of \( \langle \mathbb{U}, <, \mathfrak{Bip}' \rangle \).

**Claim 2:** \( a_i \in P \) iff \( b_i \in P \) (for all \( i \in \{1, \ldots, k\} \)).

To show this, we will essentially use that the strategy of the “virtual duplicator” in the game (***) preserves the successor relation \( \text{succ}^\mathfrak{A} \) on \( P \).

For the “only if” direction let \( a_i \in P \), and show that \( b_i \in P \): Since \( a_i \in P = \{p_1 < p_2 < \cdots\} \), there is an index \( j \) such that \( a_i = p_j \). By the definition of the set \( \{a'_{i_1} < \cdots < a'_{i_h}\} \) we have \( a_i = p_j = a'_j \) for some \( j \in \{1, \ldots, h\} \). From (***) we obtain that \( b_i = b'_j \in P \).

For the “if” direction let \( a_i \not\in P \), and show that \( b_i \not\in P \): If \( a_i < p_1 \) then, by the definition of the set \( \{a'_{i_1} < \cdots < a'_{i_h}\} \), we have \( a_i < p_1 = a'_1 \). Since the “virtual duplicator” wins the game (**), we know that \( b'_1 = p_1 \). Furthermore, from (***) we obtain that \( b_i < b'_1 = p_1 \), and consequently, \( b_i \not\in P \).

If there is a \( j \) such that \( p_j < a_i < p_{j+1} \), then, by the definition of the set \( \{a'_{i_1} < \cdots < a'_{i_h}\} \), we
In particular, \( a_{\nu} \) and \( a_{\nu+1} \) are successors in \( P \), i.e., \( \text{succ}^P(a_{\nu}, a_{\nu+1}) \). Since the “virtual duplicator” wins the game \((**)\), we know that also \( \text{succ}^P(b_{\nu}, b_{\nu+1}) \). Furthermore, from (***) we obtain that \( b_{\nu} < b_1 < b_{\nu+1} \). In particular, this implies that \( b_1 \notin P \). Altogether, the proof of Claim 2 is complete.

All that remains to do is to consider the relations in \( \tau \). Let \( R \) be a relation symbol in \( \tau \) of arity, say, \( m \) and let \( \bar{x} := (x_1, \ldots, x_m) \) be in the domain of \( \pi \). We need to show that \( \bar{x} \in \alpha(R^A) \) iff \( \pi(\bar{x}) \in \beta(R^B) \). If at least one of the elements in \( \bar{x} \), say \( x_j \), does not belong to \( P \), then we know that \( \bar{x} \notin \alpha(R^A) \subseteq P^m \). From Claim 2 we furthermore know that also \( \pi(x_j) \) does not belong to \( P \). Consequently, also \( \pi(\bar{x}) \notin \beta(R^B) \subseteq P^m \). If all the elements in \( \bar{x} \) belong to \( P \), then the following is true: By the definition of the set \( \{a_1 < \cdots < a_h\} \) of moves for the “virtual spoiler” we know that all the elements in \( \bar{x} \) belong to \( \{a_1' < \cdots < a_h'\} \). I.e., there are indices \( i_1, \ldots, i_m \) such that \( (x_1, \ldots, x_m) = (a_{i_1}', \ldots, a_{i_m}') \). Since the “virtual duplicator” wins the game \((**)\), we know that \( (x_1, \ldots, x_m) = (a_{i_1}', \ldots, a_{i_m}') \in \alpha(R^A) \) iff \( (b_{i_1}', \ldots, b_{i_m}') \in \beta(R^B) \). Furthermore, from (***) we obtain that \( \pi(x_1), \ldots, \pi(x_m) = (b_{i_1}', \ldots, b_{i_m}') \). Consequently, we have shown that \( \bar{x} \in \alpha(R^A) \) iff \( \pi(\bar{x}) \in \beta(R^B) \).

Together with Claim 1 we obtain that \( \pi \) is a partial isomorphism between the structures \( \mathcal{A} \) and \( \mathcal{B} \), and thus the duplicator has won the single-round \( k \)-move game on \( \mathcal{A} \) and \( \mathcal{B} \). Altogether, this completes the proof of Theorem 7.4 for the case where the structure \( \langle \mathcal{U},< \rangle \) contains an infinitely increasing sequence.

For the remaining case where \( \langle \mathcal{U},< \rangle \) does not contain an infinitely increasing sequence, we know from Fact 5.3 that \( \mathcal{U} \) must contain an infinitely decreasing sequence. With the same coloring as in Step 1 above, the Ramsey Theorem 7.5 gives us an infinitely decreasing set \( P = \{p_1 > p_2 > \cdots \} \). Concerning the given \( \langle \mathcal{U},\tau \rangle \)-structures \( \mathcal{A} \) and \( \mathcal{B} \), we know that \( \mathcal{A} \) and \( \mathcal{B} \) are \( \mathbb{N} \)-embeddable. In particular, \( \text{dom}(\mathcal{A}) \) and \( \text{dom}(\mathcal{B}) \) must be finite, since otherwise they would constitute an infinitely increasing sequence in \( \mathcal{U} \). Consequently, it is possible to embed \( \mathcal{A} \) and \( \mathcal{B} \) in \( P \) in such a way that Lemma 7.3 is valid if replacing \( \text{succ}^P \) with the predecessor relation \( \text{pred}^P \). The rest can be taken almost verbatim from Step 2 and Step 3 above.

Altogether, the proof of Theorem 7.4 is complete.

8 How to Lift Collapse Results

In this section we develop the notion of \( \mathbb{N} \)-representable structures, which is a natural generalization of the notion of finitely representable (i.e., order constraint) databases. Following the spirit of 5’s lifting from finite to finitely representable databases, we show that any collapse result for first-order logic on \( \mathbb{N} \)-embeddable structures can be lifted to the analogous collapse result on \( \mathbb{N} \)-representable structures.

8.1 The Lifting Method

It is by now quite a common method in database theory to lift results from one class of databases to another. This lifting method can be described as follows:

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Known: A result for a class of “easy” databases.
Wanted: The analogous result for a class of “complicated” databases.
Method:

(1.) Show that all the relevant information about a “complicated” database can be represented by an “easy” database.

(2.) Show that the translation from the “complicated” to the “easy” database (and vice versa) can be performed in an appropriate way (e.g., via an efficient algorithm or via FO-formulas).

(3.) Use this to translate the known result for the “easy” databases into the desired result for the “complicated” databases.

In the literature the “easy” database which represents a “complicated” database is often called the invariant of the “complicated” database. Table 1 gives a listing of recent papers in which the lifting method has been used:

| “compl.” dbs | “easy” dbs | result for “easy” dbs | result for “compl.” dbs |
|--------------|------------|-----------------------|------------------------|
| planar spatial dbs | finite dbs | evaluation of fixpoint+counting queries | evaluation of top. FO(<)-queries over \( \mathbb{R} \) |
| region dbs | finite dbs | collapse from \(<\text{-gen.} FO(<, +, \times)\) to \( FO_{\text{dom}}(<) \) over \( \mathbb{R} \) | collapse from top. \( FO(<, +, \times) \) to top. \( FO(<) \) over \( \mathbb{R} \) |
| finitely representable dbs | finite dbs | logical characterization of complexity classes | complexity of query evaluation |
| finitely representable dbs | finite dbs | natural generic collapse over \( \langle \cup, <, \text{Bip} \rangle \) | natural generic collapse over \( \langle \cup, <, \text{Bip} \rangle \) |

Table 1: Some papers that use the lifting method.

Segoufin and Vianu \cite{29} represent a spatial database (of a certain kind) by a finite database called the topological invariant of the spatial database. They concentrate on the evaluation of topological \( FO(<) \)-queries against spatial databases over \( \mathbb{R} \). One of their results is that a topological query against the spatial database can be efficiently translated into a fixpoint+counting query against the topological invariant. This shows that efficient query evaluation for the topological invariants leads to efficient query evaluation for spatial databases.

Kuijpers and Van den Bussche \cite{19} show that all topological \( FO(<, +, \times) \)-queries over so-called (fully 2D) region databases over \( \mathbb{R} \) can already be expressed in \( FO(<) \). A crucial step in their proof is to represent region databases by finite databases, to which the natural generic collapse of \cite{6} applies, i.e., the collapse from \( <\text{-gen.} FO(<, +, \times) \) to \( FO_{\text{dom}}(<) \) on finite databases.
databases over \( \mathbb{R} \).

Belegradek, Stolboushkin, and Taitslin [5] and Grädel and Kreutzer [15] consider finitely representable databases (also known as order constraint databases), which are defined as follows:

8.1 Definition (finitely representable).
Let \( \langle U, < \rangle \) be a dense linear ordering without endpoints.\footnote{I.e., \( < \) is a linear ordering that has no maximal and no minimal element in \( U \), and for any two elements \( u, v \in U \) with \( u < v \) there is an element \( w \in U \) with \( u < w < v \).}

(a) A relation \( R \subseteq U^m \) is called finitely representable iff it can be explicitly defined by a \( \text{FO} \)-formula that makes use of the linear ordering and of finitely many constants in \( U \). Precisely this means that there are a number \( k \in \mathbb{N} \), elements \( s_1, \ldots, s_k \in U \), and a \( \text{FO}(<, s_1, \ldots, s_k) \)-formula \( \varphi(x_1, \ldots, x_m) \) such that \( R = \{ \bar{a} \in U^m : \langle U, <, s_1, \ldots, s_k \rangle \models \varphi(\bar{a}) \} \). Due to quantifier elimination \( \varphi \) can, without loss of generality, be chosen quantifier free.

(b) For a signature \( \tau \), a \( \langle U, \tau \rangle \)-structure \( A \) is called finitely representable iff each of \( A \)'s relations is.

We use \( \mathcal{C}_{\text{fin.rep}} \) to denote the class of all finitely representable structures.

In [5] and [15] it was shown that all the relevant information about a finitely representable database can be represented by a finite database, and that the translation from finitely representable to finite (and vice versa, in [5]) can be done by a first-order interpretation. Grädel and Kreutzer use this translation to carry over logical characterizations of complexity classes to results on the data complexity of query evaluation. They lift, e.g., the well-known logical characterization “\( \text{PTIME} = \text{FO} + \text{LFP} \) on ordered finite structures” to the result stating that the polynomial time computable queries against finitely representable databases are exactly the \( \text{FO} + \text{LFP} \)-definable queries. Belegradek, Stolboushkin, and Taitslin use their \( \text{FO} \)-translations from finitely representable databases to finite databases, and vice versa, to lift collapse results for finite databases to collapse results for finitely representable databases. Precisely, they obtain the following lifting theorem [5, Theorem 4.10]:

8.2 Theorem (BST’s lifting from finite to finitely representable).
Let \( \langle U, <, \mathcal{B}_{\text{Bip}} \rangle \) be a context structure. If \( <\text{-generic FO}(<, \mathcal{B}_{\text{Bip}}) = <\text{-generic FO}(<) \) on \( \mathcal{C}_{\text{fin}} \) over \( U \), then \( <\text{-generic FO}(<, \mathcal{B}_{\text{Bip}}) = <\text{-generic FO}(<) \) on \( \mathcal{C}_{\text{fin.rep}} \) over \( U \).

Note that the collapse to \( \text{FO}_{\text{adom}}(<) \) is not possible over \( \mathcal{C}_{\text{fin.rep}} \), since the \( <\)-generic query “Does the active domain have an upper bound in \( U \)?” is definable in \( \text{FO}(<) \), but not in \( \text{FO}_{\text{adom}}(<) \).

In the previous sections of this paper we obtained collapse results not only for the class \( \mathcal{C}_{\text{fin}} \), but even for the larger class \( \mathcal{C}_{\mathbb{N}\text{-emb}} \) of structures whose active domain is \( \mathbb{N} \)-embeddable. In the present section we will lift these collapse results to a larger class of structures that we call \( \mathbb{N} \)-representable. The resulting lifting theorem was presented in the conference contribution [27]. There, the according structures were called \( \omega \)-representable. The author now thinks that the name \( \mathbb{N} \)-representable is more appropriate.
8.2 A Generalization of Finitely Representable: \( \mathbb{N} \)-Representable

8.2.1 An Informal Approach

To find an adequate generalization, let us first point out what finitely representable structures look like. Let \( \tau \) consist, for the moment, of a single binary relation symbol, and let \( A = \langle U, R \rangle \) be a finitely representable \( \langle U, \tau \rangle \)-structure. This means that the relation \( R \subseteq U^2 \) is definable by a \( FO(<, s_1, \ldots, s_k) \)-formula \( \varphi_R(x_1, x_2) \). Due to quantifier elimination \( \varphi_R \) is, without loss of generality, a Boolean combination of atomic formulas over the relations \(<, =\), the variables \( x_1, x_2 \), and the constants \( s_1, \ldots, s_k \). In other words: The constants \( s_1, \ldots, s_k \), together with the diagonal \( "x_1 = x_2" \), impose a finite grid on the plane \( U^2 \); and the formula \( \varphi_R \) expresses, for each region \( M \) in the grid, whether \( M \subseteq R \) or \( M \cap R = \emptyset \). Such a relation \( R \) is illustrated in Figure 7.

In general, a binary relation \( R \) is definable in \( FO(<, s_1, \ldots, s_k) \) if and only if \( R \) is constant, in the sense of the following Definition 8.3, on all the regions of the grid that is defined by \( s_1, \ldots, s_k \) and the diagonal \( "x_1 = x_2" \).

![Figure 7: A finitely representable binary relation \( R \). The grey regions are those that belong to \( R \). Essentially, \( R \) consists of a finite number of “rectangular” regions.](image)

8.3 Definition \( (R \text{ constant on } M) \). Let \( m \in \mathbb{N}_{>0} \). We say that a relation \( R \subseteq U^m \) is constant on a set \( M \subseteq U^m \) if either all elements of \( M \) belong to \( R \) or no element of \( M \) belongs to \( R \). \( \Box \)

In the proof of their Lifting Theorem 8.2, Belegradek et al. represent a \( FO(<, s_1, \ldots, s_k) \)-definable \( \langle U, \tau \rangle \)-structure \( A \) by a structure \( \text{rep}(A) \) with active domain \( \{ s_1, \ldots, s_k \} \), and they show that the translations from \( A \) to \( \text{rep}(A) \), and vice versa, can be done via first-order interpretations. In their lifting theorem they have available the collapse over \( \mathcal{C}_{\text{fin}} \), i.e., the collapse over the representations \( \text{rep}(A) \), for \( A \in \mathcal{C}_{\text{fin}} \).

In the present situation we have available the collapse over \( \mathcal{C}_{\text{N-emb}} \). Thus, as representatives \( \text{rep}(A) \), we may use structures whose active domain is \( \mathbb{N} \)-embeddable, i.e., of the form \( \{ s_1 < s_2 < s_3 < \cdots \} \) and unbounded in \( U \). Of course, the constants \( s_1, s_2, s_3, \ldots \) and the diagonal \( "x_1 = x_2" \) impose an infinite grid on the plane \( U^2 \). Consequently, it seems reasonable to say that a relation \( R \subseteq U^2 \) is \( \mathbb{N} \)-representable via \( \{ s_1 < s_2 < \cdots \} \) if and only if \( R \) is constant on all the regions of the infinite grid that is defined by \( s_1, s_2, \ldots \) and the diagonal \( "x_1 = x_2" \). These relations...
are exactly the relations definable by infinitary Boolean combinations of atomic formulas over the relations <, =, the variables $x_1, x_2$, and the constants $s_1, s_2, \ldots$. We will see that we can even allow infinitary formulas with quantifiers, i.e., $L_{\infty\omega}(\langle, s_1, s_2, \ldots\rangle)$-formulas to define such relations.

8.2.2 Formalization: $L_{\infty\omega}$ and $\mathbb{N}$-Representable Structures

Infinitary logic $L_{\infty\omega}$ is defined in the same way as first-order logic, except that arbitrary (i.e., possibly infinite) disjunctions and conjunctions are allowed.

What we need in the present section is the following: Let $S$ be a possibly infinite set of constant symbols. The logic $L_{\infty\omega}(\langle, S)$ is given by the following clauses: It contains all atomic formulas $x=y$ and $x<y$, where $x$ and $y$ are variable symbols or elements in $S$. If it contains $\varphi$, then it contains also $\neg \varphi$. If it contains $\varphi$ and if $x$ is a variable symbol, then it contains also $\exists x \varphi$ and $\forall x \varphi$. If $\Phi$ is a (possibly infinite) set of $L_{\infty\omega}(\langle, S)$-formulas, then $\forall \Phi$ and $\bigwedge \Phi$ are formulas in $L_{\infty\omega}(\langle, S)$. The semantics is a direct extension of the semantics of first-order logic, where $\forall \Phi$ is true if there is some $\varphi \in \Phi$ which is true; and $\bigwedge \Phi$ is true if every $\varphi \in \Phi$ is true. We will always identify the set $S$ of constant symbols with a set $S \subseteq U$, where $U$ is the universe of the underlying context structure $(U, <, B \mathbb{H})$.

8.4 Definition ($\mathbb{N}$-representable). Let $(U, <)$ be a dense linear ordering without endpoints.

(a) A relation $R \subseteq U^m$ is called $\mathbb{N}$-representable iff it can be explicitly defined by a $L_{\infty\omega}$-formula that makes use of the linear ordering and of an $\mathbb{N}$-embeddable set of constants in $U$. Precisely this means that there are an $\mathbb{N}$-embeddable set $S = \{s_1 < s_2 < \cdots\} \subseteq U$ and a $L_{\infty\omega}(\langle, S)$-formula $\varphi(x_1, \ldots, x_m)$ such that $R = \{\bar{a} \in U^m : (U, <, s_1, s_2, \ldots) \models \varphi(\bar{a})\}$.

Below we will see that $\varphi$ can, without loss of generality, be chosen quantifier free and in the normal form obtained in Proposition $\ref{Proposition_8.3}$.

(b) For a signature $\tau$, a $(U, \tau)$-structure $A$ is called $\mathbb{N}$-representable iff each of $A$'s relations is.

We use $\mathcal{C}_{\mathbb{N}\text{-rep}}$ to denote the class of all $\mathbb{N}$-representable structures. $\square$

8.2.3 A Normal Form for $L_{\infty\omega}(\langle, S)$-Formulas

From now on let $(U, <)$ always be a dense linear ordering without endpoints.

It is well-known that $FO(\langle, S)$ allows quantifier elimination over $U$, for every set of constants $S \subseteq U$. In this section we show that also $L_{\infty\omega}(\langle, S)$ allows quantifier elimination over $U$, provided that $S$ is $\mathbb{N}$-embeddable. However, our aim is not only to show that $L_{\infty\omega}(\langle, S)$ allows quantifier elimination, but to give an explicit characterization of the quantifier free formulas.

Before giving the formalization of the quantifier elimination let us fix some notations: For the rest of this section let $S \subseteq U$ always be $\mathbb{N}$-embeddable. We write $S(i)$ to denote the $i$-th smallest element in $S$. For infinite $S$ we define $S(0) := -\infty$ and $N(S) := \mathbb{N}$. For finite $S$ we define $S(0) := -\infty$, $N(S) := \{0, \ldots, |S|\}$, and $S(|S|+1) := +\infty$. For $m \in \mathbb{N}_{\geq 0}$ and $\vec{r} = (i_1, \ldots, i_m) \in N(S)^m$ we define $S(\vec{i}) := (S(i_1), \ldots, S(i_m))$, and $Cube_{S;\vec{r}} := int[S(i_1), S(i_1+1)] \times \cdots \times int[S(i_m), S(i_m+1)]$ (where $int[-\infty, r) := \{r' \in U : r' < r\}$).

We say that $S(i)$ are the coordinates of the cube $Cube_{S;\vec{r}}$. Obviously, $U^m$ is the disjoint union of
the sets $\text{Cube}_{S,T}$ for all $\vec{t} \in N(S)^m$.

Let $\vec{a} = (a_1, \ldots, a_m) \in U^m$. The type $\text{Type}_{R,S,T}$ of $\vec{a}$ with respect to $\text{Cube}_{S,T}$ is the conjunction of all atoms in \{y_i=x_i, y_i<x_i, x_i=x_j, x_i<x_j : i,j \in \{1,\ldots,m\}, i \neq j\} which are satisfied if one interprets the variables $x_1, \ldots, x_m, y_1, \ldots, y_m$ by the elements $a_1, \ldots, a_m, S(i_1), \ldots, S(i_m)$.

I.e., $\text{type}_{\vec{a},S,T}$ describes the relative position of $\vec{a}$ with respect to $\text{Cube}_{S,T}$. We define $\text{types}_m$ to be the set of all complete conjunctions of atoms in \{y_i=x_i, y_i<x_i, x_i=x_j, x_i<x_j : i,j \in \{1,\ldots,m\}, i \neq j\}, i.e., the set of all conjunctions $t$ where, for all $i,j \in \{1,\ldots,m\}$ with $i \neq j$, either $y_i=x_i$ or $y_i<x_i$ occurs in $t$, and either $x_i=x_j$ or $x_i<x_j$ or $x_j<x_i$ occurs in $t$. Of course, $\text{types}_m$ is finite, and $\text{type}_{\vec{a},S,T} \in \text{types}_m$. Analogously, we define $\text{Types}_m$ to be the set of all types of $\text{types}_m$, i.e., $\text{Types}_m = \{T : T \subseteq \text{types}_m\}$. For a relation $R \subseteq U^m$ we define $\text{Type}_{R,S,T} := \{\text{type}_{\vec{a},S,T} : \vec{a} \in R \cap \text{Cube}_{S,T}\}$ to be the set of all types occurring in the restriction of $R$ to $\text{Cube}_{S,T}$. We say that $\text{Type}_{R,S,T}$ is the type of $\text{Cube}_{S,T}$ in $R$. Of course, $\text{Type}_{R,S,T} \in \text{Types}_m$.

In the formalization of the quantifier elimination we further use the following notation: If $\varphi$ is a $L_{\infty\omega}(<, S)$-formula with free variables $\bar{x} := x_1, \ldots, x_k$ and $\bar{y} := y_1, \ldots, y_m$, then we write $\varphi(\bar{y}/S(\bar{t}))$ to denote the formula one obtains by replacing the variables $y_1, \ldots, y_m$ by the elements $S(i_1), \ldots, S(i_m)$.

8.5 Proposition (Quantifier Elimination for $L_{\infty\omega}(<, S)$). Let $(U, <)$ be a dense linear ordering without endpoints, let $S \subseteq U$ be $\mathbb{N}$-embeddable, and let $m \in \mathbb{N}_{>0}$. Every formula $\varphi(x_1, \ldots, x_m)$ in $L_{\infty\omega}(<, S)$ is equivalent over $U$ to the formula

$$\hat{\varphi}(\bar{x}) := \bigvee_{r \in N(S)^m} \bigvee_{t \in \text{Type}_{R,S,T}} \left( t(\bar{y}/S(\bar{t})) \land \bigwedge_{j=1}^m S(i_j) \leq x_j < S(i_j+1) \right)$$

where $R \subseteq U^m$ is the relation defined by $\varphi(\bar{x})$, i.e.,

$$R = \{ \vec{a} \in U^m : \langle U, <, S(1), S(2), \ldots \rangle \models \varphi(\vec{a}) \}$$

$$= \{ \vec{a} \in U^m : \langle U, <, S(1), S(2), \ldots \rangle \models \hat{\varphi}(\vec{a}) \}.$$  

$\square$

Proof. The proof is similar to the quantifier elimination for $FO(<, S)$ over $U$. For simplicity, we write $N$ instead of $N(S)$.

(1): We first show that the proposition is valid in the special case where $\varphi$ is quantifier free. Let $\hat{R}$ be the relation defined by $\hat{\varphi}$. We need to show that $\hat{R} = R$. Let $\vec{a} \in U^m$, let $\vec{t} \in N^m$ such that $\vec{a} \in \text{Cube}_{S,T}$, and let $t := \text{Type}_{\vec{a},S,T}$. By definition we know that $t(\bar{y}/S(\bar{t}))$ is satisfied if one interprets $\bar{x}$ by $\vec{a}$.

For showing that $R \subseteq \hat{R}$, assume that $\vec{a} \in R$. From the definition of $\text{Type}_{R,S,T}$ we know that $t \in \text{Type}_{R,S,T}$. Hence, $\hat{\varphi}$ is satisfied if one interprets $\bar{x}$ by $\vec{a}$, i.e., $\vec{a} \in \hat{R}$.

For showing that $R \supseteq \hat{R}$, assume that $\vec{a} \in \hat{R}$, i.e., $\hat{\varphi}$ is satisfied when interpreting $\bar{x}$ by $\vec{a}$. Of course, $\vec{t}$ is the only element in $N^m$ with $\vec{a} \in \text{Cube}_{S,T}$, and $t$ is the only element in $\text{types}_m$ that is satisfied when interpreting $\bar{x}$ by $\vec{a}$ and $\bar{y}$ by $S(\bar{t})$. We conclude that $t$ must be an element of $\text{Type}_{R,S,T}$. Thus there must be some $\vec{b} \in R \cap \text{Cube}_{S,T}$ such that $\text{type}_{\vec{b},S,T} = t$. One can easily see that every atomic formula in

$$\{s=x_i, s<x_i, x_i=x_j, x_i<x_j : s \in S, i,j \in \{1,\ldots,m\}, i \neq j\}$$
is satisfied if one interprets \( \bar{x} \) by \( \bar{a} \) if and only if it is satisfied if one interprets \( \bar{x} \) by \( \bar{b} \). Since \( \varphi \) is a (possibly infinitary) Boolean combination of such atomic formulas, we conclude that \( \varphi \) is satisfied if one interprets \( \bar{x} \) by \( \bar{a} \) if and only if it is satisfied if one interprets \( \bar{x} \) by \( \bar{b} \). Since \( b \in R \) we hence obtain that also \( \bar{a} \in R \).

Altogether, we have shown that \( R = \bar{R} \), which completes our proof of (1).

(2): We now show that the proposition is valid in the special case where \( \varphi \) is of the form

\[
(*) : \; \exists x_{m+1} \left( \bigwedge_{i=1}^{p} x_{m+1} = u_i \right) \land \left( \bigwedge_{j=1}^{q} v_j < x_{m+1} \right) \land \left( \bigwedge_{k=1}^{r} x_{m+1} < w_k \right),
\]

where \( p, q, r \in \mathbb{N} \) and \( \{u_1, \ldots, u_q, v_1, \ldots, v_q, w_1, \ldots, w_r\} \subseteq \{x_1, \ldots, x_m\} \cup S \).

In case that \( p \neq 0 \), we can replace \( x_{m+1} \) by \( u_1 \) and obtain that \( \varphi \) is equivalent (over \( \mathbb{U} \)) to

\[
(\bigwedge_{i=1}^{p} u_1 = u_i) \land (\bigwedge_{j=1}^{q} v_j < u_1) \land (\bigwedge_{k=1}^{r} u_1 < w_k).
\]

In case that \( p = 0 \) and \( q \) and \( r \) are both different from \( 0 \), \( \varphi \) says that there exists an element which is larger than each \( v_j \) and smaller that each \( w_k \). Since \( < \) is dense, \( \varphi \) is equivalent (over \( \mathbb{U} \)) to

\[
\bigwedge_{j=1}^{q} v_j < w_k.
\]

In case that \( p = 0 \) and \( r = 0 \), \( \varphi \) says that there exists an element which is larger than each \( v_j \) — which is true since \( < \) has no endpoints. Analogously, in case that \( p = 0 \) and \( q = 0 \), \( \varphi \) says that there exists an element which is smaller than each \( w_k \) — which, again, is true in since \( < \) has no endpoints. Hence, in both cases \( \varphi \) is equivalent to a formula which is always true (e.g., the formula \( x_1 = x_1 \)).

Altogether, we have seen that a formula \( \varphi \) of the form (\( * \)) is equivalent to a quantifier free formula.

Thus we can use (1) to conclude that \( \varphi \) is equivalent to \( \bar{\varphi} \).

(3): We are now ready to show, by induction on the construction of \( \bar{\varphi} \), that the proposition is valid for all \( \varphi \) in \( L_{\infty\omega}(<, S) \).

If \( \varphi \) is quantifier free, the claim follows from (1). If \( \varphi \) is of the form \( \neg \psi \) or \( \bigvee \Phi \), the induction step is obvious. If \( \varphi \) is of the form \( \exists x_{m+1} \psi(x_1, \ldots, x_{m+1}) \) then we show

\[
(**) : \; \varphi \text{ is equivalent to a formula } \bigvee \Phi, \text{ where } \Phi \text{ is a set of formulas of the form } \xi \land \eta, \text{ such that } \xi(x_1, \ldots, x_m) \text{ is of the form } (*) \text{ and } \eta(x_1, \ldots, x_m) \text{ is quantifier free.}
\]

Making use of (\( ** \)) and (2), we obtain that \( \varphi \) is equivalent to the quantifier free formula \( \bigvee \{\xi \land \eta : \xi \land \eta \in \Phi\} \). According to (1) we thus conclude that \( \varphi \) is equivalent to \( \bar{\varphi} \).

It remains to show (\( ** \)). By the induction hypothesis, \( \psi \) is equivalent to \( \bar{\psi} \), which is, by definition, the disjunction of the formulas

\[
\chi_{\vec{i}, \bar{a}} := \text{type}_{\vec{a}; S; i}(\vec{y}/S(\bar{i})) \land \left( \bigwedge_{j=1}^{m+1} S(i_j) \leq x_j < S(i_j+1) \right),
\]

for all \( i \in N^{m+1} \) and all \( \bar{a} \in R \cap \text{Cube}_{S; \vec{x}} \). Since \( \varphi \) is equivalent to \( \exists x_{m+1} \psi \), it also is equivalent to the disjunction of the formulas \( \exists x_{m+1} \chi_{\vec{i}, \bar{a}} \).

We transform each \( \chi_{\vec{i}, \bar{a}} \) into a finite disjunction of finite conjunctions \( \lambda_{\vec{i}, \bar{a}, j} \) of unnegated atoms of the form \( u = v \) and \( u < v \), where \( u \) and \( v \) are distinct elements in \( \{x_1, \ldots, x_{m+1}\} \cup S \), as follows: For not necessarily distinct \( u \) and \( v \), we replace each negated atom of the form \( (\neg u=v) \)
by \((u < v \lor v < u)\), we replace each negated atom of the form \((-u < v)\) by \((v < u \lor u = u)\), and we replace each atom of the form \((u \leq v)\) by \((u < v \lor u = v)\). Afterwards we repeatedly use the distributive law \(\alpha \land (\beta \lor \gamma)\) is equivalent to \((\alpha \land \beta) \lor (\alpha \land \gamma)\), to transform \(\chi_{\vec{a},j}\) into a disjunction of conjunctions of unnegated atoms of the form \(u = v\) and \(u < v\). Finally, we remove each conjunction where there occurs an atom of the form \(u < u\); and in the remaining conjunctions we remove each atom of the form \(u = u\). This gives us that each \(\chi_{\vec{a},j}\) is equivalent to a finite disjunction of finite conjunctions \(\lambda_{\vec{a},j}\) of unnegated atoms of the form \(u = v\) and \(u < v\), where \(u\) and \(v\) are distinct elements in \(\{x_1, \ldots, x_{m+1}\} \cup S\).

Since \(\varphi\) is equivalent to the disjunction of the formulas \(\exists x_{m+1} \chi_{\vec{a},j}\), it is also equivalent to the disjunction of the formulas \(\exists x_{m+1} \lambda_{\vec{a},j}\) which do involve the variable \(x_{m+1}\), and let \(\eta_{\vec{a},j}\) be the conjunction of all other atoms in \(\lambda_{\vec{a},j}\). Clearly, \(\lambda_{\vec{a},j}\) is equivalent to \(\xi_{\vec{a},j} \land \eta_{\vec{a},j}\). Hence \(\varphi\) is equivalent to the disjunction of the formulas \(\exists x_{m+1} (\xi_{\vec{a},j} \land \eta_{\vec{a},j})\) which, in turn, is equivalent to the disjunction of the formulas \(\exists x_{m+1} \xi_{\vec{a},j}\) and \(\eta_{\vec{a},j}\). This means that \(\varphi\) is equivalent to the disjunction of the formulas \(\xi_{\vec{a},j} \land \eta_{\vec{a},j}\), where \(\xi_{\vec{a},j} := \exists x_{m+1} \xi_{\vec{a},j}\) is of the form \((*)\) and where \(\eta_{\vec{a},j}\) is quantifier free. This completes the proof of \((**)*\) and thus also the proof of Proposition 8.3. ■

8.3 The Lifting Theorem and its Proof

8.6 Theorem (Lifting from \(\mathbb{N}\)-embeddable to \(\mathbb{N}\)-representable).

Let \((U, <, \mathcal{B}ip)\) be a context structure where \(<\) is a dense linear ordering without endpoints. If \(\prec\)-generic \(\mathcal{F}O(\prec, \mathcal{B}ip) = \prec\)-generic \(\mathcal{F}O(\prec)\) on \(\mathcal{C}_{\mathbb{N}\text{-emb}}\) over \(U\), then \(\prec\)-generic \(\mathcal{F}O(\prec, \mathcal{B}ip) = \prec\)-generic \(\mathcal{F}O(\prec)\) on \(\mathcal{C}_{\mathbb{N}\text{-rep}}\) over \(U\).

The proof will be given throughout the following subsections: In Section 8.3.1 we show how all the relevant information about an \(\mathbb{N}\)-representable structure \(\mathcal{A}\) can be represented by an \(\mathbb{N}\)-embeddable structure \(\text{rep}(\mathcal{A})\). In Section 8.3.2 we show that the translation from \(\mathcal{A}\) to \(\text{rep}(\mathcal{A})\), and vice versa, can be done via first-order interpretations \(\Phi\) and \(\Phi'\). As shown in Section 8.3.3, this will enable us to prove Theorem 8.6. The overall proof idea is visualized in Figure 8.

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**Figure 8:** The overall proof idea for the Lifting Theorem 8.6.
Let us mention that the proof presented here does not work when replacing FO with the sublogic $BC(EFO)$. The main objection is that the FO-interpretations contain several alternations of quantifiers. It therefore remains open whether the Lifting Theorem can be proved for logics weaker than FO and, in particular, for $BC(EFO)$.

8.3.1 $\mathbb{N}$-Representations of Relations and Structures

8.7 Definition ($S$ sufficient for defining $R$).
Let $R \subseteq U^m$. A set $S \subseteq U$ is called sufficient for defining $R$ iff $S$ is $\mathbb{N}$-embeddable and $R$ is definable in $L_{\infty \omega}(\cdot, S)$ over $U$.

8.8 Remark ($S$ sufficient for defining $R$). From Proposition 8.3 we obtain that a $\mathbb{N}$-embeddable set $S \subseteq U$ is sufficient for defining $R$ if and only if $R$ is constant, in the sense of Definition 8.3, on the sets $\text{Cube}_{S;\vec{t}} := \{ \vec{b} \in \text{Cube}_{S;\vec{t}} : \text{type}_{\vec{b};S;\vec{t}} = t \}$, for all $\vec{t} \in N(S)^m$ and all $t \in \text{types}_m$.

Let $R \subseteq U^m$ be $\mathbb{N}$-representable and let $S \subseteq U$ be sufficient for defining $R$. From Remark 8.8 we know, for all $\vec{t} \in N(S)^m$ and all $t \in \text{types}_m$ that either $R \cap \text{Cube}_{S;\vec{t}} = \emptyset$ or $R \supseteq \text{Cube}_{S;\vec{t}}$. This means that if we know, for each $\vec{t} \in N(S)^m$ and each $t \in \text{types}_m$, whether or not $R$ contains an element of $\text{Cube}_{S;\vec{t}}$, then we can reconstruct the entire relation $R$.

For $i_j \neq 0$ we represent the interval $\text{int}[S(i_j), S(i_j+1)] \subseteq U$ by the element $S(i_j)$. Consequently, for $\vec{t} \in (N(S) \setminus \{0\})^m$, we can represent $\text{Cube}_{S;\vec{t}} \subseteq U^m$ by the tuple $S(\vec{t}) \in S^m$. The information whether or not $R$ contains an element of $\text{Cube}_{S;\vec{t}}$ can be represented by the relation

$$R_{S;\vec{t}} := \{ S(\vec{t}) : \vec{t} \in (N(S) \setminus \{0\})^m \text{ and } R \cap \text{Cube}_{S;\vec{t}} \neq \emptyset \}.$$ 

In general, we would like to represent every $\text{Cube}_{S;\vec{t}}$, for every $\vec{t} \in N(S)^m$, by a tuple in $S^m$. Unfortunately, the case where $i_j = 0$ must be treated separately, because $S(0) = -\infty \notin S$. There are various possibilities for solving this technical problem. Here we propose the following solution: Use $S(1)$ to represent the interval $\text{int}[S(0), S(1)]$. With every tuple $\vec{t} \in N(S)^m$ we associate a characteristic tuple $\text{char}(\vec{t}) := (c_1, \ldots, c_m) \in \{0, 1\}^m$ and a tuple $\vec{t}' \in (N(S) \setminus \{0\})^m$ via $c_j := 0$ and $i_j' := 1$ if $i_j = 0$, and $c_j := 1$ and $i_j' := i_j$ if $i_j \neq 0$. Now $\text{Cube}_{S;\vec{t}}$ can be represented by the tuple $S(\vec{t}') \in S^m$. The information whether or not $R$ contains an element of $\text{Cube}_{S;\vec{t}}$ can be represented by the relations

$$R_{S;\vec{t};\vec{a}} := \{ S(\vec{t}') : \vec{t}' \in N(S)^m, \text{ char}(\vec{t}) = \vec{a}, \text{ and } R \cap \text{Cube}_{S;\vec{t}} \neq \emptyset \},$$

for all $\vec{a} \in \{0, 1\}^m$. This leads to the following definition:

8.9 Definition ($\mathbb{N}$-Representation of a Relation).
Let $R \subseteq U^m$ be $\mathbb{N}$-representable, and let $S \subseteq U$ be sufficient for defining $R$.

(a) We represent the $m$-ary relation $R$ over $U$ by a finite number of $m$-ary relations over $S$ as follows: The $\mathbb{N}$-representation of $R$ with respect to $S$ is the collection

$$\text{rep}_S(R) := \{ R_{S;\vec{t};\vec{a}} \}_{\vec{t} \in \text{types}_m, \vec{a} \in \{0, 1\}^m},$$

where $R_{S;\vec{t};\vec{a}} := \{ S(\vec{t}') : \vec{t}' \in N(S)^m, \text{ char}(\vec{t}) = \vec{a}, \text{ and } R \cap \text{Cube}_{S;\vec{t}} \neq \emptyset \}$. Here, for $\vec{t} \in N(S)^m$ we define $\vec{t}'$ and $\text{char}(\vec{t})$ via $i_j' := 1$ and $(\text{char}(\vec{t}))_{i_j} := 0$ if $i_j = 0$, and $i_j := i_j$ and $(\text{char}(\vec{t}))_{i_j} := 1$ if $i_j \neq 0$.\nopagebreak
(b) For $\bar{x} \in \text{Cube}_{S;\bar{u}}$, we say that

- $\bar{u} := \text{char}(i)$ is the characteristic tuple of $\bar{x}$ w.r.t. $S$,
- $\bar{y} := S(i)$ is the representative of $\bar{x}$ w.r.t. $S$, and
- $t$ is the type of $\bar{x}$ w.r.t. $S$.

From Remark 8.8 we obtain that $\bar{x} \in R$ iff $\bar{y} \in R_{S;\bar{u}}$. □

We now transfer the notion of $N$-representation from relations to $\tau$-structures. Recall from Definition 8.4 that a $\langle U, \tau \rangle$-structure $A$ is called $N$-representable iff each of $A$’s relations is.

8.10 Definition (S sufficient for defining $A$).
Let $A$ be a $\langle U, \tau \rangle$-structure. A set $S \subseteq U$ is called sufficient for defining $A$ iff

- $S$ is $N$-embeddable,
- $c^A \in S$, for every constant symbol $c \in \tau$, and
- $S$ is sufficient for defining $R^A$, for every relation symbol $R \in \tau$. □

Let $A$ be a $\langle U, \tau \rangle$-structure and let $S$ be a set sufficient for defining $A$. According to Definition 8.3, each of $A$’s relations $R^A$ of arity, say, $m$ can be represented by a finite collection $\text{rep}_S(R^A) = \langle R^A_{S;\bar{u}} \rangle_{t \in \text{types}_m, \bar{u} \in \{0,1\}^m}$ of relations over $S$. I.e., $A$ can be represented by a structure $\text{rep}_S(A)$ with active domain $S$ as follows:

8.11 Definition ($N$-Representation of $A$). Let $\tau$ be a signature.

(a) The type extension $\tau'$ of $\tau$ is the signature which consists of

- the same constant symbols as $\tau$,
- a unary relation symbol $S$, and
- a relation symbol $R_{t;\bar{u}}$ of arity $m := \text{ar}(R)$, for every relation symbol $R \in \tau$, every $t \in \text{types}_m$, and every $\bar{u} \in \{0,1\}^m$.

(b) Let $A$ be an $N$-representable $\langle U, \tau \rangle$-structure and let $S$ be a set sufficient for defining $A$. We represent $A$ by the $\langle U, \tau' \rangle$-structure $\text{rep}_S(A)$ which satisfies

- $c^{\text{rep}_S(A)} = c^A$, for each $c \in \tau'$,
- $S^{\text{rep}_S(A)} = S$, for the unary relation symbol $S \in \tau'$, and
- $R^{\text{rep}_S(A)}_{t;\bar{u}} = R^A_{S;\bar{u}}$, for each $R \in \tau$ of arity $m := \text{ar}(R)$, each $t \in \text{types}_m$, and each $\bar{u} \in \{0,1\}^m$. □
8.3.2 FO-Interpretations

The concept of first-order interpretations (or, reductions) is well-known in mathematical logic (cf., e.g., [11]). In the present section we consider the following easy version:

8.12 Definition (FO-Interpretation of \( \sigma \) in \( \rho \)).

Let \( \sigma \) and \( \rho \) be signatures. A FO-interpretation of \( \sigma \) in \( \rho \) is a collection

\[
\Psi = \left\langle \left( \varphi_c(x) \right)_{c \in \sigma}, \left( \varphi_R(x_1, \ldots, x_{ar(R)}) \right)_{R \in \sigma} \right\rangle
\]

of FO(\( \rho \))-formulas. For every \( \langle \mathbb{U}, \rho \rangle \)-structure \( A \), the \( \langle \mathbb{U}, \sigma \rangle \)-structure \( \Psi(A) \) is given via

- \( \{c^{\Psi(A)}\} = \{a \in \mathbb{U} : A \models \varphi_c(a)\}, \) for each constant symbol \( c \in \sigma \),
- \( R^{\Psi(A)} = \{\bar{a} \in \mathbb{U}^{ar(R)} : A \models \varphi_R(\bar{a})\}, \) for each relation symbol \( R \in \sigma \).

The effect of a FO-interpretation is visualized in Figure 9.

Making use of a FO-interpretation of \( \sigma \) in \( \rho \), one can translate FO(\( \sigma \))-formulas into FO(\( \rho \))-formulas (cf., [11, Exercise 11.2.4]):

8.13 Lemma. Let \( \sigma \) and \( \rho \) be signatures and let \( \Psi \) be a FO-interpretation of \( \sigma \) in \( \rho \).

For every FO(\( \sigma \))-sentence \( \chi \) there is a FO(\( \rho \))-sentence \( \chi' \) such that "\( A \models \chi \) iff \( \Psi(A) \models \chi' \)" is true for every \( \langle \mathbb{U}, \rho \rangle \)-structure \( A \).

Proof. \( \chi' \) is obtained from \( \chi \) by replacing every atomic formula \( R(\bar{x}) \) (respectively, \( x=c \)) by the formula \( \varphi_R(\bar{x}) \) (respectively, by the formula \( \varphi_c(x) \)).

![Figure 9: The effect of a FO-interpretation \( \Phi \) of \( \sigma \) in \( \rho \). For every \( \rho \)-structure \( A \), \( \Psi \) defines a \( \sigma \)-structure \( \Psi(A) \). For every FO(\( \sigma \))-sentence \( \chi \), \( \Psi \) defines a FO(\( \rho \))-sentence \( \chi' \) such that \( A \models \chi \) iff \( \Psi(A) \models \chi' \).](image)

The following lemma shows that \( A \) is first-order definable in rep\(_S\)(\( A \)). In other words: All relevant information about \( A \) can be reconstructed from the structure rep\(_S\)(\( A \)) (if \( A \) is \( \mathbb{N} \)-representable and \( S \) is sufficient for defining \( A \)).

8.14 Lemma (\( A \hookrightarrow \Phi \) rep\(_S\)(\( A \))). There is a FO-interpretation \( \Phi \) of \( \tau \) in \( \tau' \cup \{<\} \) such that \( \Phi((\text{rep}\_S(A), <)) = A \), for every \( \mathbb{N} \)-representable \( \langle \mathbb{U}, \tau \rangle \)-structure \( A \) and every set \( S \) which is sufficient for defining \( A \).

Proof. For every constant symbol \( c \in \tau \) we define \( \varphi_c(x) := x=c \).

For every relation symbol \( R \in \tau \) of arity, say, \( m \) we construct a formula \( \varphi_R(\bar{x}) \) which expresses...
that $\bar{x} \in R$: From Definition 8.9 (b) we know that $\bar{x} \in R$ iff $\bar{y} \in R_{S;t,\bar{u}}$, where $\bar{y}$, $t$, and $\bar{u}$ are the representative, the type, and the characteristic tuple, respectively, of $\bar{x}$ w.r.t. $S$. It is straightforward to construct, for fixed $t \in \text{types}_m$ and $\bar{u} \in \{0,1\}^m$, a FO($\tau',<$)-formula $\psi_{t,\bar{u}}(\bar{x})$ which expresses that

- $\bar{x}$ has type $t$ w.r.t. $S$,
- $\bar{u}$ is the characteristic tuple of $\bar{x}$ w.r.t. $S$, and
- for the representative $\bar{y}$ of $\bar{x}$ w.r.t. $S$ it holds that $R_{t,\bar{u}}(\bar{y})$.

The disjunction of the formulas $\psi_{t,\bar{u}}(\bar{x})$, for all $t \in \text{types}_m$ and all $\bar{u} \in \{0,1\}^m$, gives us the desired formula $\varphi_R(\bar{x})$ which expresses that $\bar{x} \in R$.

We now want to show the converse of Lemma 8.14, i.e., we want to show that the $\mathbb{N}$-representation of $A$ is first-order definable in $A$. Up to now the $\mathbb{N}$-representation rep$_S(A)$ was parameterized by a set $S$ which is sufficient for defining $A$. For the current step we need the existence of a canonical, first-order definable set $S$. For this canonization we can use the following result of Grädel and Kreutzer [15, Definition 6 and Lemmas 7 and 8]:

8.15 Lemma (Canonical set $S_R$ sufficient for defining $R$; formula $\zeta_R(x)$).
Let $\langle \mathbb{U},< \rangle$ be a dense linear ordering without endpoints. Let $R \subseteq \mathbb{U}^m$ be $\mathbb{N}$-representable and let $S_R$ be the set of all elements $s \in \mathbb{U}$ which satisfy the following condition (*):

- $s$ is a $n$-tuple,
- $s$ is included in every set $S \subseteq \mathbb{U}$ which is sufficient for defining $R$,
- $S_R$ is sufficient for defining $R$.

The set $S_R$ is called the canonical set sufficient for defining $R$.

It is straightforward to formulate a FO($R,<$)-formula $\zeta_R(x)$ which expresses condition (*). Consequently we have, for every $\mathbb{N}$-representable $m$-ary relation $R$, that $S_R = \{ s \in \mathbb{U} : \langle \mathbb{U},R,< \rangle \models \zeta_R(s) \}$. □

8.16 Definition (Canonical Representation of $A$). Let $\tau$ be a signature and let $A$ be a $\mathbb{N}$-representable $\langle \mathbb{U},\tau \rangle$-structure. The set $S_A := \{ c^A : c \in \tau \} \cup \bigcup_{R \in \tau} S_{R^A}$ is called the canonical set sufficient for defining $A$. The representation rep($A$) := rep$_{S_A}(A)$ is called the canonical representation of $A$. □
8.17 Remark. It is straightforward to see that “\( \alpha(\text{rep}(A)) = \text{rep}(\alpha(A)) \)” is true for every \( \mathbb{N}\)-representable \((U, \tau)\)-structure \(A\) and for every \( \prec \)-preserving mapping \( \alpha : \text{dom}(A) \rightarrow U \). \( \square \)

We are now ready to prove the converse of Lemma 8.14.

8.18 Lemma \((A \xrightarrow{\Phi'} \text{rep}(A))\). There is a FO-interpretation \( \Phi' \) of \( \tau' \) in \( \tau \cup \{\prec\} \) such that \( \Phi'(\langle A, \prec \rangle) = \text{rep}(A) \), for every \( \mathbb{N}\)-representable \((U, \tau)\)-structure \(A\). \( \square \)

Proof. For every constant symbol \( c \in \tau' \) we define \( \varphi_c(x) := x = c \).

For every relation symbol \( R \in \tau \) let \( \zeta_R(x) \) be the formula from Lemma 8.15 describing the canonical set sufficient for defining \( R^A \). Obviously, the formula \( \varphi_S(x) := \bigvee_{c \in \tau} x = c \lor \bigvee_{R \in \tau} \zeta_R(x) \) describes the canonical set sufficient for defining \( A \).

For every relation symbol \( R_{t;\bar{u}} \in \tau' \) of arity, say, \( m \) we construct a formula \( \varphi_{R_{t;\bar{u}}}(\bar{y}) \) which expresses that \( \bar{y} \in R_{t;\bar{u}} \). We make use of Definition 8.9 (b). I.e., \( \varphi_{R_{t;\bar{u}}} \) states that \( y_1, \ldots, y_m \) satisfy \( \varphi_S \) and that there is some \( \bar{x} \) such that

- \( \bar{y} \) is the representative of \( \bar{x} \) w.r.t. \( S_A \),
- \( R(\bar{x}) \),
- \( \bar{x} \) has type \( t \) w.r.t. \( S_A \), and
- \( \bar{u} \) is the characteristic tuple of \( \bar{x} \) w.r.t. \( S_A \).

It is straightforward to formalize this in first-order logic. \( \square \)

8.3.3 The Proof of the Lifting Theorem

We are now ready to prove the lifting theorem, which allows to lift collapse results for \( \mathbb{N}\)-embeddable structures to collapse results for \( \mathbb{N}\)-representable structures. An illustration of the overall proof idea is given in Figure 8.

Proof of Theorem 8.6 (Lifting from \( \mathbb{N}\)-embeddable to \( \mathbb{N}\)-representable).

Let \((U, \prec, \mathcal{B}ip)\) be a context structure where \( \prec \) is a dense linear ordering without endpoints, and let \( \prec\text{-}\)generic \( FO(\prec, \mathcal{B}ip) = \prec\text{-}\)generic \( FO(\prec) \) on \( \mathcal{C}_{\mathbb{N}\text{-}emb} \) over \( U \). Our aim is to show that \( \prec\text{-}\)generic \( FO(\prec, \mathcal{B}ip) = \prec\text{-}\)generic \( FO(\prec) \) on \( \mathcal{C}_{\mathbb{N}\text{-}rep} \) over \( U \).

Let \( \tau \) be a signature, let \( \varphi \) be a \( FO(\tau, \prec, \mathcal{B}ip) \)-sentence, and let \( \mathcal{Y} \) be the class of all \( \mathbb{N}\)-representable \((U, \tau)\)-structures on which \( \varphi \) is \( \prec \)-generic. We need to find a \( FO(\tau, \prec) \)-sentence \( \psi \) such that, for all \( A \in \mathcal{X} \),

\[
\langle A, \prec \rangle \models \varphi \quad \text{iff} \quad \langle A, \prec \rangle \models \psi.
\]

Let \( \tau' \) be the type extension of \( \tau \). We first use Lemma 8.14, which gives us a FO-interpretation \( \Phi \) of \( \tau \) in \( \tau' \cup \{\prec\} \) such that \( \Phi(\langle \text{rep}(A), \prec \rangle) = A \), for all \( A \in \mathcal{X} \). From Lemma 8.13 we obtain a \( FO(\tau', \prec, \mathcal{B}ip) \)-sentence \( \varphi' \) such that, for all \( A \in \mathcal{X} \),

\[
\langle \text{rep}(A), \prec \rangle \models \varphi' \quad \text{iff} \quad \langle \Phi(\langle \text{rep}(A), \prec \rangle), \prec, \mathcal{B}ip \rangle \models \varphi
\]

\[
\text{iff} \quad \langle A, \prec, \mathcal{B}ip \rangle \models \varphi.
\]
From our presumption we know that the natural generic collapse over \((\mathbb{U}, <, \mathcal{Bip})\) is true for the class of \(\mathbb{N}\)-embeddable structures. Of course \(\text{rep}(\mathcal{A})\) is \(\mathbb{N}\)-embeddable. Furthermore, with Remark 8.17 we obtain that \(\varphi'\) is \(\prec\)-generic on \(\text{rep}(\mathcal{A})\), for all \(\mathcal{A} \in \mathcal{K}\). Hence there must be a \(\text{FO}(\tau', <)\)-sentence \(\psi'\) such that, for all \(\mathcal{A} \in \mathcal{K}\),

\[
\langle \text{rep}(\mathcal{A}), <, \mathcal{Bip} \rangle \models \varphi' \iff \langle \text{rep}(\mathcal{A}), < \rangle \models \psi'.
\]

We now use Lemma 8.18, which gives us a \(\text{FO}\)-interpretation \(\Phi'\) of \(\tau'\) in \(\tau \cup \{<\}\) such that \(\Phi'((\mathcal{A}, <)) = \text{rep}(\mathcal{A})\), for all \(\mathcal{A} \in \mathcal{K}\). According to Lemma 8.13, we can transform \(\psi'\) into a \(\text{FO}(\tau, <)\)-sentence \(\psi\) such that, for all \(\mathcal{A} \in \mathcal{K}\),

\[
\langle \mathcal{A}, < \rangle \models \psi \iff \langle \Phi'((\mathcal{A}, <)), < \rangle \models \psi' \iff \langle \text{rep}(\mathcal{A}), < \rangle \models \psi'.
\]

Obviously, \(\psi\) is the desired sentence, and hence the proof of Theorem 8.6 is complete.

### 8.4 \(\mathbb{Z}\)-Representable instead of \(\mathbb{N}\)-Representable

It is straightforward to modify the proof of Theorem 8.6 in such a way that collapse results for the class of \(\mathbb{Z}\)-embeddable structures can be lifted to the class \(\mathcal{C}_{\mathbb{Z}, \text{rep}}\) of structures which are \(\mathbb{Z}\)-representable in the following sense: A structure is called \(\mathbb{Z}\)-representable if all its relations are \(\mathbb{Z}\)-representable, i.e., definable in \(L_{\infty\omega}(<, S)\) for a \(\mathbb{Z}\)-embeddable set \(S\).

### 8.19 Corollary (Lifting from \(\mathbb{Z}\)-embeddable to \(\mathbb{Z}\)-representable).

Let \((\mathbb{U}, <, \mathcal{Bip})\) be a context structure where \(<\) is a dense linear ordering without endpoints. If \(-\text{-generic FO}(<, \mathcal{Bip}) = -\text{-generic FO}(<)\) on \(\mathcal{C}_{\mathbb{Z}, \text{emb}}\) over \(\mathbb{U}\) then \(-\text{-generic FO}(<, \mathcal{Bip}) = -\text{-generic FO}(<)\) on \(\mathcal{C}_{\mathbb{Z}, \text{rep}}\) over \(\mathbb{U}\). □

### 9 Conclusion and Open Questions

Aiming at natural generic collapse results for potentially infinite databases we developed the notion of \(-\text{-genericity}\) which coincides both, with the classical notion of \(\text{order-genericity}\) on the densely ordered context universes \(\mathbb{Q}\) and \(\mathbb{R}\) and with the notion of \(\text{local genericity}\) on the discretely ordered context universes \(\mathbb{N}\) and \(\mathbb{Z}\) (Definition 3.3). We presented the translation of winning strategies for the duplicator in the Ehrenfeucht-Fraïssé game as a new method for proving natural generic collapse results and showed that, at least in principle, all collapse results can be proved by this method (Theorem 4.4). In the Theorems 5.1, 6.14, and 7.4 we explicitly showed how the duplicator can translate winning strategies for the Ehrenfeucht-Fraïssé game in the presence of particular built-in predicates. Via Theorem 1.4 this directly gives us the following natural generic collapse results:

### 9.1 Corollary.

Let \(\mathcal{M}_{\text{Mon}}\) be the class of all built-in monadic predicates on the respective context universe. Let \(Q \subseteq \mathbb{N}\) satisfy the conditions \(W(\omega)\) (cf., Definition 5.9), and let \(\mathcal{M}_{\text{Mon}_Q}\) be the class of all subsets of \(Q\). Let \(\mathcal{G}_{\text{Groups}}\) be the class of all subsets of \(\mathbb{R}\) that contain the number 1 and that are groups with respect to +.

(a) \(-\text{generic FO}(<, \mathcal{M}_{\text{Mon}}) = \text{FO}_{\text{adom}}(<)\) on \(\mathcal{C}_{\mathbb{Z}, \text{emb}}\) over \(\mathbb{U}\). For any linearly ordered infinite context universe \(\mathbb{U}\). In particular, for \(\mathbb{U} = \mathbb{Z}\) this implies the natural generic collapse on arbitrary databases over \((\mathbb{Z}, <, \mathcal{M}_{\text{Mon}})\).
(b) \( <\text{-generic } \text{FO}(<, +, Q, \text{Mon}_Q) = \text{FO}_{\text{adom}}(<) \) on \( \mathcal{C}_{\text{arb}} \) over \( \mathbb{N} \) and on \( \mathcal{C}_{\text{N-emb}} \) over \( \mathbb{Z} \).

(c) \( <\text{-generic } \text{FO}(<, +, Q, \text{Mon}_Q, \text{Groups}) = \text{FO}_{\text{adom}}(<) \) on \( \mathcal{C}_{\text{N-emb}} \) over \( \mathbb{R} \). In particular, this implies the natural generic collapse on \( \mathbb{N} \)-embeddable databases over the context structures \( \langle \mathbb{R}, <, +, \mathbb{Z}, \mathbb{Q} \rangle \) and \( \langle \mathbb{Q}, <, +, \mathbb{Z} \rangle \).

(d) \( <\text{-generic } \text{BC}(\text{EFO})(<, \text{Bip}) = \text{BC}(\text{EFO})_{\text{adom}}(<) \) on \( \mathcal{C}_{\text{N-emb}} \) over \( \mathbb{U} \), for any linearly ordered infinite context structure \( \langle \mathbb{U}, <, \text{Bip} \rangle \). In particular, for \( \mathbb{U} = \mathbb{N} \) this implies the natural generic collapse for the logic \( \text{BC}(\text{EFO}) \) on arbitrary databases. □

Theorem 8.6 (and Corollary 8.19) allows us to lift collapse results from the class of \( \mathbb{N} \)-embeddable (respectively, \( \mathbb{Z} \)-embeddable) databases to the larger class of \( \mathbb{N} \)-representable (respectively, \( \mathbb{Z} \)-representable) databases, provided that the context universe is equipped with a dense linear orderings without endpoints.

9.2 Corollary.

(a) \( <\text{-generic } \text{FO}(<, \text{Mon}) = <\text{-generic } \text{FO}(<) \) on \( \mathcal{C}_{\text{Z-rep}} \) over \( \mathbb{U} \), if \( \langle \mathbb{U}, < \rangle \) is a dense linear ordering without endpoints.

(b) \( <\text{-generic } \text{FO}(<, +, Q, \text{Mon}_Q, \text{Groups}) = <\text{-generic } \text{FO}(<) \) on \( \mathcal{C}_{\text{Z-rep}} \) over \( \mathbb{R} \) and over \( \mathbb{Q} \).

I.e., the natural generic collapse is true for the class of all \( \mathbb{N} \)-representable databases over the context structures \( \langle \mathbb{Q}, <, +, \mathbb{Z} \rangle \), \( \langle \mathbb{R}, <, +, \mathbb{Z}, \mathbb{Q} \rangle \), and \( \langle \mathbb{R}, <, +, Q, \text{Mon}_Q, \text{Groups} \rangle \). □

In the present paper we investigated collapse results from a logical point of view. From the point of view of computer science, especially constructive collapse proofs are interesting, i.e., proofs which lead to a “collapse algorithm” that transforms a \( <\text{-generic } \) input formula \( \varphi \in \text{FO}(<, \text{Bip}) \) into an equivalent output formula \( \varphi' \in \text{FO}(<) \). Benedikt and Libkin [8] presented such an algorithm for the collapse from \( <\text{-generic } \text{FO}(<, \text{Bip}) \) to \( \text{FO}_{\text{adom}}(<) \) on the class \( \mathcal{C}_{\text{fin}} \) over \( \mathcal{O}_{\text{min}} \) context structures. Other deep natural generic collapse proofs for the class \( \mathcal{C}_{\text{fin}} \), such as the collapse results for context structures that have the Isolation Property [5] or finite VC-dimension [4], are non-constructive. Also, our Ehrenfeucht-Fra"{i}ssé game approach does not necessarily lead to a collapse algorithm. However, the lifting theorem 8.6 does preserve constructiveness. Precisely, this means the following: Assume that we are given an algorithm that produces, for every input sentence \( \varphi' \in \text{FO}(<, \text{Bip}) \), an output sentence \( \psi' \in \text{FO}(<) \) such that

\[ \langle \mathbb{U}, <, \text{Bip}, \tau^A \rangle \models \varphi' \iff \langle \mathbb{U}, <, \tau^A \rangle \models \psi' \]

is true for all \( \mathbb{N} \)-embeddable structures \( \langle \mathbb{U}, \tau^A \rangle \) on which \( \varphi' \) is \( <\text{-generic } \). Making use of this algorithm and of the \( \text{FO} \)-interpretations \( \Phi \) and \( \Phi' \) from the Lemmas 8.14 and 8.18, one directly obtains an algorithm that produces, for every input sentence \( \varphi \in \text{FO}(<, \text{Bip}) \), an output sentence \( \psi \in \text{FO}(<) \) such that

\[ \langle \mathbb{U}, <, \text{Bip}, \tau^A \rangle \models \varphi \iff \langle \mathbb{U}, <, \tau^A \rangle \models \psi \]

is true for all \( \mathbb{N} \)-representable structures \( \langle \mathbb{U}, \tau^A \rangle \) on which \( \varphi \) is \( <\text{-generic } \).
Open questions:

It remains open whether the natural generic collapse for $\mathbb{N}$-embeddable databases is valid over context structures other than $\langle \mathbb{U},<,\text{Mon} \rangle$, $\langle \mathbb{Z},<,+,\text{Q},\text{Mon}_Q \rangle$, $\langle \mathbb{R},<,+,\text{Q},\text{Mon}_Q,\text{Groups} \rangle$. For example: Is it valid over $\langle \mathbb{R},<,+,\times \rangle$, over all o-minimal context structures, or even over all context structures that have finite VC-dimension? In other words: Can Theorem 3.7 be generalized from $\mathcal{C}_{\text{fin}}$ to $\mathcal{C}_{\text{N-emb}}$ (or even to $\mathcal{C}_{\text{Z-emb}}$)? Recall, however, from Section 3 that it cannot be generalized to $\mathcal{C}_{\text{arb}}$ since the natural generic collapse is not valid for arbitrary databases over the context structure $\langle \mathbb{Q},<,\text{+} \rangle$.

We also may ask whether the collapse results proved in this paper remain valid for even larger classes of databases, e.g.: Is the collapse still valid for arbitrary databases over every context structure $\langle \mathbb{U},<,\text{Mon} \rangle$ where $\text{Mon}$ is the class of monadic predicates over $\mathbb{U}$? Is the collapse still valid for arbitrary databases over $\langle \mathbb{Z},<,\text{+} \rangle$ or for $\mathbb{Z}$-embeddable databases over $\langle \mathbb{R},<,+,\text{Q},\text{Mon}_Q,\text{Groups} \rangle$?

Another approach is to restrict the complexity of the formulas that may be used to formulate queries. We know that the collapse over the context structure $\langle \mathbb{N},<,+,\times \rangle$ is not valid for full first-order logic, but that it is valid for Boolean combinations of purely existential first-order formulas. It remains open how many quantifier alternations are necessary to defeat the collapse. A task to start with would be, e.g., to try to lift Theorem 7.4 from $\mathcal{C}(\text{EFO})$ to $\Sigma^0_2 \cap \Pi^0_2$.

Let us also mention a potential application concerning topological queries: Kuijpers and Van den Bussche [19] used the natural generic collapse on $\mathcal{C}_{\text{fin}}$ over $\langle \mathbb{R},<,+,\times \rangle$ to obtain a collapse result for topological first-order definable queries. One step of their proof is to encode spatial databases (of a certain kind) by finite databases, to which the natural generic collapse over $\langle \mathbb{R},<,+,\times \rangle$ can be applied. Here the question arises whether there is an interesting class of spatial databases that can be encoded by $\mathbb{N}$-embeddable structures in such a way that our collapse results for $\mathcal{C}_{\text{N-emb}}$ help to obtain collapse results for topological queries.

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