NONLINEAR STABILITY OF PLANAR VORTEX PATCHES IN BOUNDED DOMAINS

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Abstract. We prove nonlinear stability of planar vortex patches concentrating at a strict local minimum point of the Robin function in a bounded domain. These vortex patches are stationary solutions of the 2-D incompressible Euler equations. This is achieved by showing that they are strict local maximizers of the kinetic energy among isovortical patches.

1. Introduction

In this paper we consider the incompressible inviscid flow without external force in the plane. The fluid motion is governed by the following Euler equations:

\[
\begin{aligned}
\nabla \cdot \mathbf{v} &= 0, \\
\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} &= -\nabla P,
\end{aligned}
\]

(1.1)

where \( \mathbf{v} = (v_1, v_2) \) is the velocity field and \( P \) is the pressure. Here, we assume the fluid density is one.

Let \( D \subset \mathbb{R}^2 \) be a bounded and simply-connected domain with smooth boundary. When the fluid moves in \( D \), the impermeability boundary condition is usually imposed:

\[
\mathbf{v} \cdot \mathbf{n} = 0,
\]

(1.2)

where \( \mathbf{n} \) is the outward unit normal of \( \partial D \). By introducing the vorticity function \( \omega = \partial_1 v_2 - \partial_2 v_1 \) and using the identity \( \frac{1}{2} \nabla |\mathbf{v}|^2 = (\mathbf{v} \cdot \nabla)\mathbf{v} + J\mathbf{v}\omega \), the second equation of (1.1) becomes

\[
\mathbf{v}_t + \nabla (\frac{1}{2} |\mathbf{v}|^2 + P) - J\mathbf{v}\omega = 0,
\]

(1.3)

where \( J(v_1, v_2) = (v_2, -v_1) \) denotes clockwise rotation through \( \frac{\pi}{2} \). Taking the curl in (1.3) gives

\[
\omega_t + \mathbf{v} \cdot \nabla \omega = 0.
\]

(1.4)

Since \( \mathbf{v} \) is divergence-free and \( D \) is simply-connected, \( \mathbf{v} \) can be written as \( \mathbf{v} = J\nabla \psi \) where the stream function \( \psi \) is defined as follows:

\[
\begin{aligned}
-\Delta \psi &= \omega \quad \text{in } D, \\
\psi &= \text{constant} \quad \text{on } \partial D.
\end{aligned}
\]

(1.5)
Without loss of generality, in this paper we assume that \( \psi \) vanishes on \( \partial D \) by adding a constant, so
\[
\psi(x) = \int_D G(x, y) \omega(y) dy,
\]
where \( G \) is the Green function for \(-\Delta\) in \( D \) with zero boundary condition
\[
G(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} - h(x, y), \quad x, y \in D.
\]

By using the notation \( \partial(\psi, \omega) \equiv \partial_1 \psi \partial_2 \omega - \partial_2 \psi \partial_1 \omega \), (1.4) can be written as
\[
\omega_t + \partial(\omega, \psi) = 0. \tag{1.6}
\]
(Integrating by parts gives the following weak form of (1.6):
\[
\int_D \omega(x, 0) \xi(x, 0) dx + \int_0^{+\infty} \int_D \omega(\xi_t + \partial(\xi, \psi)) dx dt = 0 \tag{1.7}
\]
for all \( \xi \in C_0^\infty(D \times [0, +\infty)) \). By Yudovich \[32\], for initial vorticity \( \omega(x, 0) \in L^\infty(D) \) there is a unique solution to (1.7) and \( \omega(t, x) \in L^\infty(D \times (0, +\infty)) \cap C([0, +\infty); L^p(D)), \forall p \in [1, +\infty) \). Moreover the flow is area-preserving, so the distribution function of \( \omega(x, t) \) does not change with time, i.e.
\[
|\{x \in D | \omega(x, t) > a\}| = |\{x \in D | \omega(x, 0) > a\}|, \quad \text{for all } a \in \mathbb{R}, \ t \geq 0, \tag{1.8}
\]
where \( |A| \) denotes the measure of a set \( A \subset \mathbb{R}^2 \). For convenience, we also write \( \omega(x, t) \) as \( \omega_t \).

For fixed \( t \), we call \( \omega_t \) a vortex patch if it has the form \( \omega_t = \lambda I_A \), where \( A \) is any measurable set in \( D \), \( \lambda \in \mathbb{R}^1 \) represents the vorticity strength and \( I_A \) denotes the characteristic function of \( A \), i.e., \( I_A(x) = 1 \) in \( A \) and \( I_A = 0 \) elsewhere. As a consequence of (1.8), we know that if \( \omega_0 \) is a vortex patch, then \( \omega_t \) is a vortex patch for all \( t \geq 0 \).

A vortex patch solution of (1.7) is called steady if it does not depend on time. Thus for any steady vortex patch \( \omega_t \), we have
\[
\int_D \omega \partial(\xi, \psi) dx dt = 0, \quad \text{for all } \xi \in C_0^\infty(D). \tag{1.9}
\]

For a vortex patch flow, if the fluid moves along the boundary of the vortex patch, then the shape of the vortex patch does not change with time, hence the flow must be steady. To be precise, we have the following lemma, the proof of which is left for Section 3:

**Lemma 1.1.** A vortex patch \( \omega \) is steady, if it satisfies:
\[
(i) \quad \omega = \lambda I_{\{\psi > \mu\}}, \quad \text{for some } \mu \in \mathbb{R}^1,
\]
\[
(ii) \quad \partial(\psi > \mu) \text{ is } C^1.
\]

Here \( \{\psi > \mu\} \equiv \{x \in D | \psi(x) > \mu\} \), and similar notations will be used in the sequel.

Note that the converse of Lemma 1.1 is also true: if the vortex patch flow is steady, then the fluid must move along the boundary of the vortex patch, which requires the stream function to be constant on this boundary.

In this paper, we mainly consider steady vortex patches satisfying (i) in Lemma 1.1.

Our main purpose is to prove the nonlinear stability of steady vortex patches. Here by nonlinear stability we mean Liapunov type:
Definition 1.2. A steady vortex patch $\omega$ is called stable, if for any $\varepsilon > 0$, there exists $\delta > 0$, such that for any $\omega_0 \in F_\omega, |\omega_0 - \omega|_{L^1} < \delta$, we have $|\omega_t - \omega|_{L^1} < \varepsilon$ for all $t \geq 0$, where $\omega_t(x) = \omega(x, t)$ is the solution of (1.7) with initial vorticity $\omega_0$.

Here and in the sequel $F_\omega$ denotes the rearrangement class of $\omega$, that is

$$F_\omega \triangleq \{v \text{ measurable} \mid |\{v > t\}| = |\{\omega > t\}|, \forall t \in \mathbb{R}^1\}.$$ 

The definition of nonlinear stability in this paper has been used in [31], which is natural for vortex patches. In this paper, we prove that a steady vortex patch near a strict local minimum point of the Robin function (defined in Section 2) must be stable. This can be regarded as a desingularized version of the stability for the vortex model, see Remark 2.9 in Section 2. The precise statements of our results will be given in Section 2.

In history, the stability of steady Euler flow has been studied by many authors, see for example [2, 3, 11, 20, 21, 23, 27, 31]. The first stability result is due to Kelvin in [20], where he established the linear stability for circular vortex patches in the plane. Later Love in [23] proved the linear stability of a rotating Kirchhoff elliptical vortex patch. Another excellent work is due to Arnold in [2, 3]. He proved the nonlinear stability for smooth steady Euler flow which is in fact a nonlinear version of the classical Rayleigh inflection point criterion for linear stability of shear flows in a channel. Moreover he asserted in [4] that a steady flow can be seen as a constrained critical point of the kinetic energy; if this critical point is a non-degenerate extreme, it should be stable. Unfortunately, it seems that his method does not apply to vortex patches, since there is strong discontinuity in the vorticity.

In 1985, based on the conservation of the kinetic energy, Wan and Pulvirenti in [31] proved the nonlinear stability for circular vortex patches in an open disk. They established a relative variational principle for the kinetic energy and turned the $L^1$ perturbation problem into a $C^1$ perturbation problem. This idea is used in this paper to prove Lemma 3.1 in Section 3. The key point in their paper is that for a circle, the Green function is explicitly known and rotationally invariant, then the $C^1$ perturbation case can be handled by spectral analysis of a negative definite operator. In 1987, Y. Tang in [27] proved the nonlinear stability of both circular vortex patches and rotating elliptic vortex patches in the plane by using the same idea as in [31]. But it seems difficult to extend their results to vortex patches in general bounded domains, since in such cases the Green functions have no explicit expression and the symmetry may be lost.

In 2005, Burton in [11] proved that a steady vortex flow as the strict local maximizer of the kinetic energy relative to an isovortical surface is stable in some $L^p$ norm. But only in few cases can the strictness be verified.

Recently Cao et al in [15] have established the local uniqueness for a single vortex patch concentrating at a non-degenerate critical point of the Robin function. This is achieved by considering the following free boundary problem

$$
\begin{align*}
-\Delta \psi &= \lambda I_{\{\psi > \mu\}} \quad \text{in } D, \\
\psi &= 0 \quad \text{on } \partial D,
\end{align*}
(1.10)
$$
where $\mu_{\lambda} \in \mathbb{R}^1$ is unknown, $\lambda|\{\psi > \mu_{\lambda}\}| = 1$. They proved that if the vortex set $\{\psi > \mu_{\lambda}\}$ shrinks to a non-degenerate critical point of the Robin function as $\lambda \to +\infty$, then the solution of (1.10) is unique provided $\lambda$ is large enough. The precise statement will be given in Theorem 2.12 in Section 2. This local uniqueness result is used in this paper to verify strictness for the local maximizer of the kinetic energy.

This paper is organized as follows. In Section 2 we give some known results that will be used later and then state our main theorems. In section 3 we prove the main theorems. In section 4 we give the complete construction of a single vortex patch near a given strict local minimum point of the Robin function.

2. Some Known Facts and Main Results

Let us start with some notations. In this paper $D$ always denotes a bounded and simply-connected domain in $\mathbb{R}^2$ with smooth boundary, $G$ is the Green function defined in Section 1, and the Robin function of $D$ is defined as

$$H(x) \triangleq h(x, x).$$

Recall that $h$ is the regular part of the Green function. Note that one can prove $H(x) \to +\infty$ as $x \to \partial D$ by the maximum principle (see Lemma 2.1 in [13] for example), so $H$ attains its minimum in $D$. The kinetic energy of a vortex flow is defined as

$$E(\omega) \triangleq \frac{1}{2} \int_D \int_D G(x, y)\omega(x)\omega(y)dx dy,$$

where $\omega$ is the vorticity of the flow. For vortex patch flow the kinetic energy is conserved, see Theorem 14 in [11].

2.1. Existence Of Steady Vortex Patches. We state two results on the existence of steady vortex patches.

The first one is due to Turkington. Define

$$M_\lambda \triangleq \{\omega \in L^\infty(D)| \ 0 \leq \omega \leq \lambda, \int_D \omega(x)dx = 1\},$$

where $\lambda$ is positive and large enough so that $M_\lambda$ is not empty. Let $\varepsilon$ be the positive number such that $\lambda\pi\varepsilon^2 = 1$.

**Theorem 2.1 ([28]).** There exists $\omega_{\lambda} \in M_\lambda$ such that $E(\omega_{\lambda}) = \sup_{\omega \in M_\lambda} E(\omega)$, $\omega_{\lambda}$ satisfies (1.10), and $\omega_{\lambda} = \lambda I_{\{\psi_{\lambda} > \mu_{\lambda}\}}$, where $\psi_{\lambda}$ is the stream function of $\omega_{\lambda}$ and $\mu_{\lambda}$ is a real number depending on $\omega_{\lambda}$. Furthermore, $\omega_{\lambda}$ shrinks to a global minimum point of $H$ as $\lambda \to +\infty$. More precisely, $\text{diam}(\text{supp}\omega_{\lambda}) \leq C\varepsilon$ and $\int_D x\omega(\lambda)dx \to x_0$ as $\lambda \to +\infty$, where $C$ is a positive number independent of $\lambda$ and $x_0$ is a global minimum point of $H$.

Here $\text{supp}\omega_{\lambda} = \{\omega_{\lambda} \neq 0\}$ and $\text{diam}(\text{supp}\omega_{\lambda})$ denotes the diameter of $\text{supp}\omega_{\lambda}$, i.e.,

$$\text{diam}(\text{supp}\omega_{\lambda}) = \sup_{x, y \in \text{supp}\omega_{\lambda}} |x - y|.$$
Remark 2.2. We don’t know whether $\omega_\lambda$ is unique when $\lambda$ is not large. It’s true when $D$ is an open disk, see [11]. We conjecture that it is also true for convex domains. Besides, there may be several global minimum points of $H$, but we don’t know which one $\omega_\lambda$ will shrink to.

Now we turn to the other existence theorem. By a similar procedure, we can construct a single steady vortex patch near a given strict local minimum point of $H$. Elcrat and Miller in [17] did this for exterior domains. Here we extend their result to general bounded domains.

To state the theorem, we need some notations slightly different. Here and in the sequel $x_1 \in D$ is a strict local minimum point of $H$. Without loss of generality, we assume that $x_1$ is the unique minimum point of $H$ on $\overline{B_r(x_1)}$ for some $r$ small. Define

$$N_\lambda \triangleq \{\omega \in L^\infty(D) | 0 \leq \omega \leq \lambda, \int_D \omega(x)dx = 1, \text{supp}(\omega) \subset B_r(x_1)\}.$$ 

Theorem 2.3. There exists $\omega_\lambda \in N_\lambda$ such that $E(\omega_\lambda) = \sup_{\omega \in N_\lambda} E(\omega)$, and $\omega_\lambda$ has the form $\omega_\lambda = \lambda I_{\{\psi_\lambda > \mu_\lambda\} \cap B_r(x_1)}$, where $\psi_\lambda$ is the stream function of $\omega_\lambda$ and $\mu_\lambda$ is a real number depending on $\omega_\lambda$. Furthermore, $\omega_\lambda$ shrinks to $x_1$ as $\lambda \to +\infty$. More precisely, $\text{diam}(\text{supp}\omega_\lambda) \leq C\varepsilon$ for some $C$ independent of $\lambda$ and $\int_D \omega_\lambda(x)dx \to x_1$ as $\lambda \to +\infty$. If $\lambda$ is large enough, $\omega_\lambda$ is steady, i.e., $\omega_\lambda$ satisfies (1.9).

Remark 2.4. In Theorem 2.3 $\omega_\lambda$ satisfies $\omega_\lambda = \lambda I_{\{\psi_\lambda > \mu_\lambda\}}$ if $\lambda$ is large, since the support of $\omega_\lambda$ shrinks to $x_1$ as $\lambda \to +\infty$. But when $\lambda$ is not large, $\{\psi_\lambda > \mu_\lambda\}$ may not be included in $B_r(x_1)$, and we don’t know whether $\omega_\lambda$ is steady in this case.

The complete proof of Theorem 2.3 will be given in Section 4.

2.2. The Main Theorem. Now we can state our stability theorem.

Theorem 2.5. Assume that $\omega_\lambda$ is a family of steady vortex patches satisfying the following properties:

(i) $\omega_\lambda = \lambda I_{\{\psi_\lambda > \mu_\lambda\}}$, where $\psi_\lambda$ is the stream function of $\omega_\lambda$ and $\mu_\lambda$ is a number depending on $\omega_\lambda$,

(ii) $\int_D \omega_\lambda = 1$,

(iii) the support of $\omega_\lambda$ shrinks to $x^*$ as $\lambda \to +\infty$, where $x^*$ is a strict local minimum point and non-degenerate isolated critical point of $H$.

Then $\omega_\lambda$ is stable provided $\lambda$ is large enough.

Here the support of $\omega_\lambda$ shrinking to $x^*$ means that for any neighbourhood of $x^*$, the support of $\omega_\lambda$ is contained in this neighbourhood if $\lambda$ is large enough.

As a consequence of Theorem 2.5 we can easily prove that the steady vortex patches obtained in Theorem 2.1 and Theorem 2.3 are stable if $\lambda$ is large enough.

Corollary 2.6. In Theorem 2.1 assume that the minimum point of $H$ is unique, which is also a non-degenerate isolated critical point, then $\omega_\lambda$ is stable provided $\lambda$ is large enough.

Corollary 2.7. In Theorem 2.3 assume that $x_1$ is also a non-degenerate isolated critical point, then $\omega_\lambda$ is stable provided $\lambda$ is large enough.
Remark 2.8. By [13], for a convex domain the Robin function is strictly convex, thus the minimum point must be unique and also a non-degenerate isolated critical point.

Remark 2.9. Our results are closely related to the vortex model (see [24], Chapter 4), which is a simplified model describing the fluid motion when the vorticity is sufficiently concentrated. In the vortex model, the vorticity is regarded as a delta measure called a vortex whose location is determined by the following Kirchhoff-Routh equation:

\[
\begin{aligned}
\frac{dx(t)}{dt} &= -\frac{1}{2} J \nabla H(x(t)), \\
x(0) &= x_0.
\end{aligned}
\]  

(2.2)

It’s easy to see that

\[
\frac{dH(x(t))}{dt} = 0, 
\]  

(2.3)

that is, the vortex moves along the level curve of \( H \). So if \( x^* \) is a strict local minimum point of \( H \), it must be stable by choosing \( H \) as the Liapunov function (see [24], Theorem 1.4). In such a way, our result can be interpreted as a desingularized version of the stability for the vortex model.

The proof of Theorem 2.5 will be given in Section 3.

2.3. Two Key Results Needed. For the proof of Theorem 2.5 given in Section 3, we need two key results. One is the stability result due to Burton in [11] and the other one is the local uniqueness given by Cao et al in [15].

In [11], Burton proved that steady vortex flow as strict local maximizer of the kinetic energy on an isovortical surface is stable in some \( L^p \) norm. In the case of vortex patches, his result can be stated as follows:

**Theorem 2.10.** Let \( \omega \) be a vortex patch in \( D \). If \( \omega \) is a strict local maximizer of the kinetic energy \( E \) relative to \( F_\omega \), then it is steady and stable.

Here \( \omega \) being a strict local maximizer relative to \( F_\omega \) means that there exists some \( \delta_0 > 0 \), such that for any \( \bar{\omega} \in F_\omega \), \( 0 < |\bar{\omega} - \omega|_{L^1} < \delta_0 \), we have \( E(\bar{\omega}) < E(\omega) \).

**Remark 2.11.** Theorem 2.10 can be very general. According to [11], any steady vortex flow as strict local maximizer or minimizer relative to the rearrangement class is stable in some \( L^p \) norm. But in the case of vortex patches Theorem 2.10 is enough for our use.

The most important condition in Theorem 2.10 is “strictness” which requires the vortex patch to be a non-degenerate extreme of the kinetic energy. In [11], only one example is given as strict global maximizer, i.e., \( D \) is an open disc and \( \omega \) is a non-negative radially symmetric decreasing function. For general maximizer, especially local maximizer, the strictness condition is hard to verify. Roughly speaking, “strictness” is equivalent to uniqueness in some sense, which is usually a more difficult problem than existence.

Recently Cao et al in [15] proved a local uniqueness result when the vortex patch is sufficiently concentrated:
Theorem 2.12. Assume that $\omega_\lambda$ is a family of vortex patches satisfying the following properties:

(i) $\omega_\lambda = \lambda I_{\{\psi_\lambda > \mu_\lambda\}}$, where $\psi_\lambda$ is the stream function of $\omega_\lambda$ and $\mu_\lambda$ is a number depending on $\omega_\lambda$,

(ii) $\int_D \omega_\lambda = 1$,

(iii) the support of $\omega_\lambda$ shrinks to $x^* \in D$.

Then $x^* \in D$ and $x^*$ must be a critical point of $H$. Furthermore, if $x^*$ is a non-degenerate and isolated critical point, then $\omega_\lambda$ is locally unique when $\lambda$ is large.

Remark 2.13. The local uniqueness in Theorem 2.12 has another equivalent statement: let $x^*$ be a non-degenerate isolated critical point of $H$, then there exists $\lambda_0 > 0, r_0 > 0$, such that the following problem has a unique solution:

\[
\begin{align*}
\omega &= \lambda I_{\{\psi > \mu\}} \text{ for some } \mu \in \mathbb{R}, \\
\int_D \omega &= 1, \\
\psi(x) &= \int_D G(x, y) \omega(y) dy, \\
\lambda &= \lambda_0, \\
\{\psi > \mu\} &\subset B_{r_0}(x^*).
\end{align*}
\]

3. Proof of the Main Results

In this section, we first establish some preliminary results for the proof of Theorem 2.5. We begin by proving Lemma 1.1 given in Section 1.

Proof of Lemma 1.1. Denote $A = \{\psi > \mu\}$. Then $A$ is a bounded domain with $C^1$ boundary. We apply the Green formula on $A$

\[
\int_D \omega \partial(\psi, \xi) = \lambda \int_A \partial_1 \psi \partial_2 \xi - \partial_2 \psi \partial_1 \xi
\]

\[
= \lambda \int_A \partial_2 (\partial_1 \psi \xi) - \partial_1 (\partial_2 \psi \xi)
\]

\[
= -\lambda \int_{\partial A} \partial_1 \psi \xi dx + \partial_2 \psi \xi dy
\]

\[
= 0,
\]

for any $\xi \in C_0^\infty(D)$. Here $\partial_1 \psi dx + \partial_2 \psi dy = 0$ since $\psi$ is constant on $\partial A$.

Now we turn to the proof of our stability results. When the steady vortex patch is not only a local but also a global strict maximizer, for example the steady vortex patch constructed in Theorem 2.11, stability can be easily obtained by combining Theorem 2.10 and Theorem 2.12. For reader’s convenience, we give a short proof of Corollary 2.6 independently, although it can be proved as a particular case of Theorem 2.5.
Proof of Corollary 2.6. According to Theorem 2.10, it suffices to prove that \( \omega_\lambda \) is a strict global maximizer (thus a strict local maximizer) of \( E \) on \( F_{\omega_\lambda} \). Since \( F_{\omega_\lambda} \subset M_\lambda \) and \( \omega_\lambda \in F_{\omega_\lambda} \), we know that \( \omega_\lambda \) is a maximizer of \( E \) on \( F_{\omega_\lambda} \).

Now we prove that \( \omega_\lambda \) is in fact a strict maximizer if \( \lambda \) is large. Suppose that there is another maximizer of \( E \) on \( F_{\omega_\lambda} \), say \( \bar{\omega}_\lambda \), \( \bar{\omega}_\lambda \in F_{\omega_\lambda} \), \( E(\omega_\lambda) = E(\bar{\omega}_\lambda) \), we show that \( \omega_\lambda \equiv \bar{\omega}_\lambda \).

First \( \bar{\omega}_\lambda \) must be a maximizer of \( E \) on \( M_\lambda \), so by Theorem 2.11 \( \bar{\omega}_\lambda \) satisfies \( \bar{\omega}_\lambda = \lambda \mathcal{I}_{\{\psi_0 > \mu\}} \) for some \( \mu \in \mathbb{R}^1 \) and \( \omega_\lambda \) will shrink to the unique minimum of \( H \) as \( \lambda \to +\infty \). Then Theorem 2.12 implies \( \bar{\omega}_\lambda \equiv \omega_\lambda \) provided \( \lambda \) is large. This is the desired result. \( \square \)

To prove Theorem 2.5, we give a more general stability criterion for steady vortex patches, i.e. Lemma 3.1 below.

Let \( \omega \) be a vortex patch enclosed by a \( C^1 \) closed curve denoted by \( \gamma_\omega \) (thus \( \gamma_\omega \) is a planar set). A \( \delta \) neighbourhood of \( \gamma_\omega \) is defined by:

\[
\gamma_{\omega,\delta} \triangleq \{ x \in \mathbb{R}^2 | \text{dist}(x, \gamma_\omega) < \delta \}. \tag{3.1}
\]

Lemma 3.1. Let \( \omega_0 \) be a vortex patch in \( D \) satisfying the following conditions:

(C1) \( \omega_0 \) has the form \( \omega_0 = \lambda \mathcal{I}_{\{\psi_0 > \mu\}} \) for some \( \mu > 0 \), where \( \psi_0 = \int_D G(x,y)\omega_0(y)dy \);

(C2) \( \partial\{\psi_0 > \mu\} \) is a \( C^1 \) closed curve and \( \partial \psi_0 / \partial n < 0 \) on this curve, where \( n \) is the outward unit normal of \( \partial \{\psi_0 > \mu\} \);

(C3) there exists \( \delta > 0 \), such that if \( \omega_1 \in F_{\omega_0} \) is another vortex patch (not necessarily steady) enclosed by a \( C^1 \) simple curve and \( \gamma_{\omega_1} \subset \gamma_{\omega_0,\delta} \), then \( E(\omega_1) \leq E(\omega_0) \), the equality holds if and only if \( \omega_0 \equiv \omega_1 \).

Then \( \omega_0 \) is steady and stable.

Proof. By Lemma 1.4, \( \omega_0 \) is steady. To prove its stability, it suffices to prove that \( \omega_0 \) is a strict local maximizer of \( E \) on \( F_{\omega_0} \) by Theorem 2.10. We show this by contradiction in the following.

Suppose that \( \omega_0 \) is not a strict local maximizer of \( E \) on \( F_{\omega_0} \), then we can take a sequence \( \{\omega_n\} \), \( \omega_n \in F_{\omega_0}, 0 < |\omega_n - \omega_0|_{L^1} < \frac{1}{n} \), and

\[
E(\omega_n) \geq E(\omega_0). \tag{3.2}
\]

For such a sequence, we have the following claim:

Claim: If \( n \) is large, there exists a unique \( \nu_n > 0 \), such that

(i) \( \partial\{\psi_n > \nu_n\} \) is a \( C^1 \) closed curve, where \( \psi_n \) is the stream function of \( \omega_n \),

(ii) \( |\{\nu_n > \nu_n\}| = |\{\psi_0 > \mu\}| \),

(iii) for each \( \delta > 0 \), \( \partial\{\psi_n > \nu_n\} \subset \gamma_{\omega_0,\delta} \) provided \( n \) is large enough.

Proof of the Claim: By the definition of \( \psi_n \) and \( \psi_0 \), they satisfy the following equations:

\[
\begin{aligned}
-\Delta \psi_n &= \omega_n, \\
-\Delta \psi_0 &= \omega_0.
\end{aligned} \tag{3.3}
\]
For vortex patches, \(|\omega_n - \omega_0|_{L^1} \to 0\) implies \(|\omega_n - \omega_0|_{L^p} \to 0\) for any \(1 \leq p < +\infty\). Then by \(L^p\) estimates, \(|\psi_n - \psi_0|_{W^{2,p}} \to 0\) for any \(1 < p < +\infty\). Choosing \(p\) large, by the Sobolev embedding \(W^{2,p}(D) \hookrightarrow C^{1,\alpha}(\overline{D})\) for some \(\alpha \in (0, 1)\), we have

\[
|\psi_n - \psi_0|_{C^1} \to 0. \tag{3.4}
\]

By (C2) we can take \(\delta_0 > 0\) small, such that the set \(\{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}\) is an annulus domain and \(\frac{\partial \psi_n}{\partial n} < 0\) on each closed curve \(\{\psi_0 = a\}, \mu - \delta_0 \leq a \leq \mu + \delta_0\). This is true by the continuity of \(\psi_0\) and \(\nabla \psi_0\). Since \(|\psi_n - \psi_0|_{C^1} \to 0\), we have \(\frac{\partial \psi_n}{\partial n} < 0\) on each curve \(\{\psi_0 = a\}, \mu - \delta_0 \leq a < \mu + \delta_0\) if \(n\) is large enough. Thus \(\nabla \psi_n \neq 0\) in the annulus domain \(\{\mu - \delta_0 \leq \psi_0 \leq \mu + \delta_0\}\).

Now choose \(\varepsilon < \delta_0\) small and define \(M \triangleq \max_{\{\psi_0 = \mu - \varepsilon\}} \psi_n, m \triangleq \min_{\{\psi_0 = \mu + \varepsilon\}} \psi_n\). By the implicit function theorem, \(\{\psi_n = M\}\) and \(\{\psi_n = m\}\) are both \(C^1\) curves locally. By the properties that \(\psi_n\) strictly increases along the direction \(\nabla \psi_n\) in the annulus domain \(\{\mu - \delta_0 \leq \psi_0 \leq \mu + \delta_0\}\), the curve \(\{\psi_n = M\}\) cannot go outside \(\{\psi_0 \geq \mu - \varepsilon\}\).

In fact, suppose that there exists \(x_0 \in \{\psi_0 < \mu - \varepsilon\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}\) and \(\psi_n(x_0) = M\), we can find \(x_1 \in \{\psi_0 = \mu - \varepsilon\}\) by solving the following ODE:

\[
\begin{cases}
\frac{dx(t)}{dt} = -\nabla \psi_0(x), \\
x(0) = x_0.
\end{cases}
\tag{3.5}
\]

Since \(\frac{\partial \psi_n}{\partial n} < 0\), we have \(\psi_n(x_1) > \psi_n(x_0)\). But by the definition of \(M\), \(\psi_n(x_1) \leq M = \psi_n(x_0)\), which is a contradiction.

On the other hand, by taking \(n\) large enough the curve \(\{\psi_n = M\}\) can not enter \(\{\psi_0 > \mu\}\). In fact, suppose that there exists \(x_0 \in \{\psi_0 = \mu\}\) and \(x_0 \in \{\psi_n = M\}\), then we have \(\psi_n(x_0) = M\). But by (3.4) \(\psi_n(x_0) \to \mu\) and \(M \to \mu - \varepsilon\) as \(n \to +\infty\). So by taking \(n\) large enough we have \(\psi_n(x_0) > M\), which is a contradiction.

So \(\{\psi_n = M\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}\) must be a \(C^1\) closed curve and

\[
\{\psi_0 > \mu\} \subset \{\psi_n > M\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\} \subset \{\psi_0 > \mu - \varepsilon\}. \tag{3.6}
\]

Similarly \(\{\psi_n = m\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}\) must be a \(C^1\) closed curve and

\[
\{\psi_0 > \mu + \varepsilon\} \subset \{\psi_n > m\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\} \subset \{\psi_0 > \mu\}.
\tag{3.7}
\]

so

\[
|\{\psi_n > m\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}| \leq |\{\psi_0 > \mu\}| \leq |\{\psi_n > M\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}|. \tag{3.8}
\]

By the continuity of \(\psi_n\) we can choose \(\nu_n \in [m, M]\) such that \(\{|\psi_n > \nu_n\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\}| = |\{\psi_0 > \mu\}|\), moreover \(\{\psi_n = \nu_n\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\} \subset \{\mu - \varepsilon < \psi_0 < \mu + \varepsilon\}\). Note that such \(\nu_n\) must be unique because \(\frac{\partial \psi_n(x)}{\partial n} < 0\) on the curve \(\{\psi_n = \nu_n\}\).

By (3.4) \(\sup_{\{\psi_0 < \mu - \delta_0\}} \psi_n < \nu_n\) if \(n\) is large enough and by strong maximum principle \(\inf_{\{\psi_0 > \mu + \delta_0\}} \psi_n > \nu_n\), so \(\{\psi_n > \nu_n\} \cap \{\mu - \delta_0 < \psi_0 < \mu + \delta_0\} = \{\psi_n > \nu_n\}\) if \(n\) is large enough.
That is, for any \( \varepsilon > 0 \), if \( n \) is large, we can choose a unique \( \nu_n \), such that \( \partial \{ \psi_n > \nu_n \} = \{ \psi_n = \nu_n \} \) is a \( C^1 \) closed curve, \( |\{ \psi_n > \nu_n \}| = |\{ \psi_0 > \mu \}| \) and \( \{ \psi_n = \nu_n \} \subset \{ \mu - \varepsilon < \psi_0 < \mu + \varepsilon \} \). Hence the statements in the claim are proved.

Now we continue our proof of Lemma 3.1. Define \( \bar{\omega}_n = \lambda I_{\{ \psi_n > \nu_n \}} \), where \( \nu_n \) is the one in the above Claim. It is obvious that \( \bar{\omega}_n \in F_{\omega_n} \). Now we compare the energy of \( \bar{\omega}_n \) and \( \omega_n \):

\[
E(\bar{\omega}_n) - E(\omega_n) = \frac{1}{2} \int_D \bar{\omega}_n \bar{\psi}_n - \frac{1}{2} \int_D \omega_n \psi_n
\]

\[
= \int_D \psi_n (\bar{\omega}_n - \omega_n) + \frac{1}{2} \int_D (\bar{\psi}_n - \psi_n)(\bar{\omega}_n - \omega_n),
\]

where we use \( \int_D \psi_n \bar{\omega}_n = \int_D \bar{\psi}_n \omega_n \) by the symmetry of Green function. Integrating by parts we have

\[
\frac{1}{2} \int_D (\bar{\psi}_n - \psi_n)(\bar{\omega}_n - \omega_n) = \frac{1}{2} \int_D |\nabla (\bar{\psi}_n - \psi_n)|^2,
\]

so

\[
E(\bar{\omega}_n) - E(\omega_n) \geq \int_D \psi_n (\bar{\omega}_n - \omega_n) + \frac{1}{2} \int_D |\nabla (\bar{\psi}_n - \psi_n)|^2.
\]

Since \( |\{ \psi_n > \nu_n \}| = |\{ \psi_0 > \mu \}| \) and \( \bar{\omega}_n = \lambda I_{\{ \psi_n > \nu_n \}} \), the integral \( \int_D \psi_n \omega_n \) attains its maximum if and only if \( \omega_n = \bar{\omega}_n \). So we have \( \int_D \psi_n (\bar{\omega}_n - \omega_n) \geq 0 \), which means

\[
E(\bar{\omega}_n) \geq E(\omega_n),
\]

the equality holds if and only if \( \omega_n \equiv \bar{\omega}_n \). By (3.2) and (3.3) we have

\[
E(\bar{\omega}_n) \geq E(\omega_0).
\]

On the other hand, by (iii) in the above Claim we can take \( n \) large enough such that \( \gamma_{\omega_n} \subset \gamma_{\omega_0, \delta} \), then by (C3) we have

\[
\bar{\omega}_n \equiv \omega_0,
\]

which implies that

\[
E(\bar{\omega}_n) = E(\omega_0) = E(\omega_n),
\]

hence \( \omega_n \equiv \bar{\omega}_n \equiv \omega_0 \). This leads to a contradiction since \( |\omega_n - \omega_0|_{L^1} > 0 \) for each \( n \).

\( \square \)

Lemma 3.1 is a general stability criterion in which we do not require the vortex patch to be concentrated. But the conditions (C1) – (C3) are not easy to verify in general. In the next lemma we verify (C1) and (C2) for the steady vortex patch in Theorem 2.5 by using the asymptotic estimates in [15].

**Lemma 3.2.** Let \( \omega_\lambda \) be the steady vortex patch in Theorem 2.5, \( \psi_\lambda \) is the corresponding stream function. Then \( \{ \psi_\lambda > \mu_\lambda \} \) is a simply-connected domain with \( C^1 \) boundary, and \( \frac{\partial \psi_\lambda}{\partial n} < 0 \) on this boundary, provided \( \lambda \) is large enough.

**Proof.** See [15]. \( \square \)
Remark 3.3. There are two ways to construct steady vortex patches: the vorticity method and the stream function method. The vorticity method is the one stated in Theorem 2.1 and Theorem 2.3, i.e. by maximizing the kinetic energy on a suitable function class of vorticity, one can refer to [9, 10, 12, 17, 28, 29]. The stream function method is to construct solutions by solving a free boundary problem of a semilinear elliptic equation satisfied by the stream function, see [1, 7, 8, 14, 16, 25, 26] for example. In [16], a reduction method has been used to construct multiple vortex patches near a non-degenerate critical point of the Kirchhoff-Routh function. Theorem 2.12 shows that these two entirely different methods actually result in the same solution provided the vortex patch is sufficiently concentrated. So the steady vortex patch obtained in Theorem 2.3 has the same asymptotic behaviors as the one in [16].

Having made the preparation, we are ready to prove Theorem 2.5.

Proof of Theorem 2.5. It suffices to show that \( \omega_\lambda \) satisfies (C1) – (C3) in Lemma 3.1 (C1) and (C2) hold by Lemma 3.2. We verify (C3) next.

By the uniqueness, the steady vortex patch in Theorem 2.5 is the same as the one in Theorem 2.3 when \( \lambda \) is large, so we just consider the case \( \omega_\lambda \) is the one in Theorem 2.3, that is, it’s an energy maximizer on \( \mathcal{N}_\lambda \). Suppose that there exists another vortex patch \( \bar{\omega}_\lambda \) enclosed by a \( C^1 \) closed curve \( \gamma_{\bar{\omega}_\lambda} \), \( \gamma_{\bar{\omega}_\lambda} \subset \gamma_{\omega_\lambda, \delta} \) and \( E(\omega_\lambda) = E(\bar{\omega}_\lambda) \), it suffices to show that \( \omega_\lambda \equiv \bar{\omega}_\lambda \). Choosing \( \delta \) small such that \( \bar{\omega}_\lambda \in \mathcal{N}_\lambda \), we see that \( \bar{\omega}_\lambda \) is a maximizer of \( E \) on \( \mathcal{N}_\lambda \). Then by Theorem 2.3, \( \bar{\omega}_\lambda \) must satisfy \( \bar{\omega}_\lambda = \lambda I_{\{\psi_\lambda > \mu_\lambda \}} \) for some \( \bar{\mu}_\lambda \in \mathbb{R}^1 \) and shrink to \( x_1 \). As a consequence of Theorem 2.12, we have \( \omega_\lambda \equiv \bar{\omega}_\lambda \) if \( \lambda \) is large and \( \delta \) is small. Hence Theorem 2.5 is proved. \( \square \)

4. Proof of Theorem 2.3

In this section, we give the proof of Theorem 2.3 for completeness. The idea is basically from [28].

Lemma 4.1. There exists \( \omega_\lambda \in \mathcal{N}_\lambda \) such that \( E(\omega_\lambda) = \sup_{\omega \in \mathcal{N}_\lambda} E(\omega) \). Moreover, \( \omega_\lambda \) satisfies \( \omega_\lambda = \lambda I_{\{\psi_\lambda > \mu_\lambda \}} \cap B_r(x_1) \) for some \( \mu_\lambda \in \mathbb{R} \).

Proof. The proof is exactly the same as Theorem 2.1 and Corollary 2.3 in [28] and we omit it therefore. \( \square \)

Now we estimate the size and location of the support of \( \omega_\lambda \) as \( \lambda \to +\infty \). This is somewhat different from [28], so we give a detailed proof. Define \( \zeta_\lambda = \psi_\lambda - \mu_\lambda, \Omega \triangleq \{\psi_\lambda > \mu_\lambda \} \cap B_r(x_1) \) which is called the vortex core, and \( T(\omega_\lambda) \triangleq \frac{1}{2} \int_D \zeta_\lambda \omega_\lambda \) which represents the kinetic energy of \( \omega_\lambda \) on \( \{\psi_\lambda > \mu_\lambda \} \).
Obviously we have the identity \( E(\omega_\lambda) = T(\omega_\lambda) + \frac{1}{2} \mu_\lambda \). Moreover integrating by parts gives
\[
T(\omega_\lambda) = \frac{1}{2} \int_D \zeta_\lambda \omega_\lambda = \frac{1}{2} \int_{\{\zeta_\lambda > 0\}} |\nabla \zeta_\lambda|^2 = \frac{1}{2} \int_D |\nabla \zeta_\lambda^+|^2.
\]

**Lemma 4.2.** \( T(\omega_\lambda) \leq C \), where \( C \) is a positive number independent of \( \lambda \).

**Proof.** Firstly by Hölder inequality, we have
\[
T(\omega_\lambda) = \frac{1}{2} \int_D \zeta_\lambda \omega_\lambda = \frac{1}{2} \lambda \int_\Omega \zeta_\lambda \leq \frac{1}{2} \lambda |\Omega|^{\frac{1}{2}} \left\{ \int_\Omega |\zeta_\lambda|^2 \right\}^{\frac{1}{2}} = \frac{1}{2} \lambda |\Omega|^{\frac{1}{2}} \left\{ \int_{B_r(x_1)} |\zeta_\lambda^+|^2 \right\}^{\frac{1}{2}}.
\]

Then by the Sobolev embedding \( W^{1,1}(B_r(x_1)) \hookrightarrow L^2(B_r(x_1)) \), we have
\[
\left\{ \int_{B_r(x_1)} |\zeta_\lambda^+|^2 \right\}^{\frac{1}{2}} \leq C \left( \int_{B_r(x_1)} \zeta_\lambda^+ + \int_{B_r(x_1)} |\nabla \zeta_\lambda^+| \right)
= C \left( \int_\Omega \zeta_\lambda^+ + \int_\Omega |\nabla \zeta_\lambda^+| \right).
\]

Here and in the sequel \( C \) denotes various positive numbers independent of \( \lambda \). So
\[
T(\omega_\lambda) \leq \frac{1}{2} C \lambda |\Omega|^{\frac{1}{2}} \int_\Omega \zeta_\lambda + \frac{1}{2} C \lambda |\Omega|^{\frac{1}{2}} \int_\Omega |\nabla \zeta_\lambda|
\leq CT(\omega_\lambda) |\Omega|^{\frac{1}{2}} + \frac{1}{2} C \lambda |\Omega| \left\{ \int_\Omega |\nabla \zeta_\lambda|^2 \right\}^{\frac{1}{2}}
\leq CT(\omega_\lambda) \lambda^{-\frac{1}{2}} + \frac{1}{2} C \left\{ T(\omega_\lambda) \right\}^{\frac{1}{2}},
\]
where we use \( \lambda |\Omega| = \int_D \omega_\lambda = 1 \). By choosing \( \lambda \) large enough such that \( C \lambda^{-\frac{1}{2}} < \frac{1}{2} \), we have \( T(\omega_\lambda) \leq C \). \( \square \)

**Lemma 4.3.** \( E(\omega_\lambda) \geq -\frac{1}{4\pi} \ln \varepsilon - C \) for some \( C > 0 \) independent of \( \lambda \), where \( \varepsilon = \frac{1}{\sqrt{\lambda} \pi} \).

**Proof.** Define \( \omega_\lambda = \lambda I_{B_r(x_1)} \). It’s easy to see that \( \omega_\lambda \in N_\lambda \), so we have \( E(\omega_\lambda) \geq E(\tilde{\omega}_\lambda) \). Now we calculate \( E(\tilde{\omega}_\lambda) \):
\[ E(\omega_\lambda) = \frac{1}{2} \int_D \int_D G(x, y) \omega_\lambda(x) \omega_\lambda(y) dx dy \]
\[ = - \frac{1}{4\pi} \int_D \int_D \ln |x - y| \omega_\lambda(x) \omega_\lambda(y) dx dy - \frac{1}{2} \int_D \int_D h(x, y) \omega_\lambda(x) \omega_\lambda(y) dx dy \]
\[ = - \frac{\lambda^2}{4\pi} \int_{B_\varepsilon(x_1)} \int_{B_\varepsilon(x_1)} \ln |x - y| dx dy - \frac{1}{2} \int_{B_\varepsilon(x_1)} \int_{B_\varepsilon(x_1)} h(x, y) dx dy. \]

Since \(|x - y| \leq 2\varepsilon\) for \(x, y \in B_\varepsilon(x_1)\), we have
\[ - \frac{\lambda^2}{4\pi} \int_{B_\varepsilon(x_1)} \int_{B_\varepsilon(x_1)} \ln |x - y| dx dy \geq - \frac{\lambda^2}{4\pi} \int_{B_\varepsilon(x_1)} \int_{B_\varepsilon(x_1)} \ln |2\varepsilon| dx dy \]
\[ = - \frac{1}{4\pi} \ln \varepsilon - \frac{1}{4\pi} \ln 2. \]

On the other hand, the integral \(\int_{B_\varepsilon(x_1)} \int_{B_\varepsilon(x_1)} h(x, y) dx dy\) converges to \(h(x_1, x_1)\), thus is bounded, as \(\lambda \to +\infty\). So
\[ E(\omega_\lambda) \geq - \frac{1}{4\pi} \ln \varepsilon - C \]
for some \(C\) independent of \(\lambda\).

From Lemma 4.2, Lemma 4.3 and the identity \(E(\omega_\lambda) = T(\omega_\lambda) + \frac{1}{2} \mu_\lambda\), we know \(\mu_\lambda \geq - \frac{1}{2\pi} \ln \varepsilon - C\). We will use this to estimate the size and location of the support of \(\omega_\lambda\).

**Lemma 4.4.** There exists some \(R_0 > 0\) such that \(\text{diam}(\text{supp}(\omega_\lambda)) < R_0\varepsilon\) when \(\lambda\) is large enough.

**Proof.** For any \(x \in \text{supp}(\omega_\lambda)\), we have \(\psi_\lambda(x) \geq \mu_\lambda\), that is
\[ \int_D G(x, y) w_\lambda(y) dy \geq - \frac{1}{2\pi} \ln \varepsilon - C. \]  \hfill (4.1)

Since \(h(x, y)\) is bounded from below, we have
\[ \int_D G(x, y) w_\lambda(y) dy \leq - \frac{1}{2\pi} \int_D \ln |x - y| \omega_\lambda(y) dy + C. \]  \hfill (4.2)

By (4.1), (4.2),
\[ \int_D \ln \frac{\varepsilon}{|x - y|} \omega_\lambda(y) dy \geq C \]
for some \(C \in \mathbb{R}^1\).

Now taking \(R > 1\) to be determined, we have
\[ \int_{B_R(x)} \ln \frac{\varepsilon}{|x - y|} \omega_\lambda(y) dy + \int_{D \setminus B_R(x)} \ln \frac{\varepsilon}{|x - y|} \omega_\lambda(y) dy \geq C. \]  \hfill (4.3)
The first integral in (4.3) can be estimated by the rearrangement inequality,
\[
\int_{B_{R}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_\lambda(y)dy \leq \lambda \int_{B_{r}(x)} \ln \frac{\varepsilon}{|x-y|}dy = \frac{1}{2}
\]
So we have
\[
\int_{D \setminus B_{R}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_\lambda(y)dy \geq C.
\]
But
\[
\int_{D \setminus B_{R}(x)} \ln \frac{\varepsilon}{|x-y|} \omega_\lambda(y)dy \leq \int_{D \setminus B_{R}(x)} \ln \frac{1}{R} \omega_\lambda(y)dy,
\]
so we get
\[
\int_{D \setminus B_{R}(x)} \omega_\lambda(y)dy \leq \frac{C}{\ln R},
\]
which means that
\[
\int_{B_{R}(x)} \omega_\lambda(y)dy \geq 1 - \frac{C}{\ln R}.
\]
Choosing \(R\) large such that \(1 - \frac{C}{\ln R} > \frac{1}{2}\), since \(\int_{D} \omega_\lambda = 1\) and \(x \in \text{supp}(\omega_\lambda)\) is arbitrary, we have
\[
diam(\text{supp} \omega_\lambda) < 2R\varepsilon.
\]
The lemma is proved by taking \(R_0 = 2R\).

**Lemma 4.5.** \(\int_{D} x \omega_\lambda(x)dx \rightarrow x_1\) as \(\lambda \rightarrow +\infty\).

**Proof.** Denote \(x_\lambda \triangleq \int_{D} x \omega_\lambda(x)dx\), then \(x_\lambda \in B_r(x_1)\). For any sequence \(\{x_\lambda\}, \lambda \rightarrow +\infty\), there exists a subsequence \(\{x_{\lambda_j}\}\) such that \(x_{\lambda_j} \rightarrow x^* \in B_r(x_1)\). For simplicity, we still denote the subsequence by \(\{x_{\lambda_j}\}\). It suffices to show that \(x^* = x_1\).

Define \(\bar{\omega}_\lambda \triangleq \lambda I_{B_{r}(x_1)}\), we have \(E(\bar{\omega}_\lambda) \leq E(\omega_\lambda)\), which means
\[
\int_{D} \int_{D} \frac{1}{2\pi} \ln |x-y| \bar{\omega}_\lambda(x)\bar{\omega}_\lambda(y)dx\,dy = \int_{D} \int_{D} h(x, y)\bar{\omega}_\lambda(x)\bar{\omega}_\lambda(y)dx\,dy
\]
\[
\leq \int_{D} \int_{D} \frac{1}{2\pi} \ln |x-y| \omega_\lambda(x)\omega_\lambda(y)dx\,dy = \int_{D} \int_{D} h(x, y)\omega_\lambda(x)\omega_\lambda(y)dx\,dy.
\]
By Riesz rearrangement inequality (see [22], 3.7),
\[
\int_{D} \int_{D} \frac{1}{2\pi} \ln |x-y| \bar{\omega}_\lambda(x)\bar{\omega}_\lambda(y)dx\,dy \geq \int_{D} \int_{D} \frac{1}{2\pi} \ln |x-y| \omega_\lambda(x)\omega_\lambda(y)dx\,dy.
\]
So we have
\[
\int_{D} \int_{D} h(x, y)\bar{\omega}_\lambda(x)\bar{\omega}_\lambda(y)dx\,dy \geq \int_{D} \int_{D} h(x, y)\omega_\lambda(x)\omega_\lambda(y)dx\,dy.
\]
Letting \(k \rightarrow +\infty\), we have \(h(x^*, x^*) \leq h(x_1, x_1)\), i.e. \(H(x^*) \leq H(x_1)\). Since \(x_1\) is the unique minimum point of \(H(x)\) in \(B_r(x_1)\), we have \(x^* = x_1\). This is the desired result. \(\square\)
Lemma 4.6. $\omega_\lambda$ satisfies (1.9) provided $\lambda$ is large enough.

Proof. For any $\xi \in C_0^\infty(D)$, we define a family of $C^1$ transformations $\Phi_t(x), t \in (-\infty, +\infty)$, from $D$ to $D$ by the following dynamical system,

$$
\begin{align*}
\begin{cases}
\frac{d\Phi_t(x)}{dt} = J\nabla \xi(\Phi_t(x)), & t \in \mathbb{R}, \\
\Phi_0(x) = x,
\end{cases}
\end{align*}
$$

(4.4)

where $J$ denotes clockwise rotation through $\frac{\pi}{2}$ as before. Note that (4.4) is solvable for all $t$ since $J\nabla \xi$ is a smooth vector field with compact support in $D$. It’s easy to see that $J\nabla \xi$ is divergence-free, so by Liouville theorem (see [24], Appendix 1.1) $\Phi_t(x)$ is area-preserving.

Now define

$$\omega_t(x) \triangleq \omega_\lambda(\Phi_t(x)),
$$

(4.5)

we have $\omega_t \in F_{\omega_\lambda}$. Since $\text{supp}(\omega_\lambda)$ is away from $\partial B_{r}(x_1)$, we have $\omega_t \in N_\lambda$ if $|t|$ is small, so $\frac{dE(\omega_t)}{dt} = 0$. Expanding $E(\omega_t)$ at $t = 0$ gives

$$E(\omega_t) = \frac{1}{2} \int_D \int_D G(x,y)\omega_\lambda(\Phi_t(x))\omega_\lambda(\Phi_t(y))dxdy
$$

$$= \frac{1}{2} \int_D \int_D G(\Phi_{-t}(x), \Phi_{-t}(y))\omega_\lambda(x)\omega_\lambda(y)dxdy
$$

$$= E(\omega_\lambda) + t \int_D \omega_\lambda \partial(\psi_\lambda, \xi) + o(t),
$$

as $t \to 0$. So we have

$$\int_D \omega_\lambda \partial(\psi_\lambda, \xi) = 0,$$

which completes the proof. □

Proof of Theorem 2.3. It follows from Lemma 4.1, Lemma 4.4, Lemma 4.5 and Lemma 4.6. □

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