Abstract. Let $G$ be a Lie group, $H$ a closed subgroup and $M$ the homogeneous space $G/H$. Each representation $\Psi$ of $H$ determines a $G$-equivariant principal bundle $\mathcal{P}$ on $M$ endowed with a $G$-invariant connection. We consider subgroups $\mathcal{G}$ of the diffeomorphism group $\text{Diff}(M)$, such that, each vector field $Z \in \text{Lie}(\mathcal{G})$ admits a lift to a preserving connection vector field on $\mathcal{P}$. We prove that \# $\pi_1(\mathcal{G}) \geq \# \Psi(Z(G))$. This relation is applicable to subgroups $\mathcal{G}$ of the Hamiltonian groups of the flag varieties of a semisimple group $G$.

Let $M_\Delta$ be the toric manifold determined by the Delzant polytope $\Delta$. We put $\varphi_b$ for the loop in the Hamiltonian group of $M_\Delta$ defined by the lattice vector $b$. We give a sufficient condition, in terms of the mass center of $\Delta$, for the loops $\varphi_b$ and $\varphi_\tilde{b}$ to be homotopically inequivalent.

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1. Introduction

In this note we will concern with the homotopy of some Lie subgroups $\mathcal{G}$ of $\text{Diff}(M)$, the diffeomorphism group of a manifold $M$ in two cases: (I) When $M$ is a homogeneous space. (II) When $M$ is a symplectic toric manifold.

As it is well-known, the lifting of loops in a space $X$ to paths in a fibration over $X$ is sometimes used for studying the homotopy of $X$. Here, we will also lift loops in $\mathcal{G}$ to isotopies in appropriate principal fiber bundles on $M$, in order to deduce consequences about $\pi_1(\mathcal{G})$.

Case (I). Given a connected Lie group $G$. Let $(H, \Psi)$ be a pair consisting of a closed subgroup $H$ of $G$ and a representation $\Psi$ of $H$. By means the representation and a complement $l$ of $h$ in $g$ (see (2.1)), we will construct a $G$-equivariant principal fibre bundle $\mathcal{P}$ over $M = G/H$ and a $G$-invariant connection $\Omega$ on it.

We will consider groups $\mathcal{G}$ of $\text{Diff}(M)$ such that each vector field $Z \in \text{Lie}(\mathcal{G})$ admits a lift $U$ to a vector field on $\mathcal{P}$ with the following properties: $U$ is invariant under the natural right action on $\mathcal{P}$ and it is an infinitesimal symmetry of $\Omega$ (the usual horizontal lift does not satisfy the last property). These properties allow us to lift a loop $\psi = \{\psi_t\}_{t \in [0,1]}$ in the group $\mathcal{G}$ to an isotopy $\{F_t\}_{t \in [0,1]}$ in $\mathcal{P}$, whose final element $F_1$ is a preserving connection gauge transformation of $\mathcal{P}$. The comparison of the gauge transformations, obtained by lifting different loops in $\mathcal{G}$, will permit us to distinguish some non-homotopic loops. For the sake of brevity, we will say that a subalgebra $\mathfrak{k}$ of $\mathfrak{X}(M)$ admits a lift to $\mathcal{P}$ if its elements admit lifts to vector fields on $\mathcal{P}$ with the mentioned properties.

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If $\Psi$ is a representation of $H$ on the vector space $W$, then the above mentioned bundle $\mathcal{P}$ is a $\text{GL}(W)$-principal bundle and the gauge transformation $F_1$ associated to a loop $\psi$ in $\mathcal{G}$ is defined by a map

\begin{equation}
(1.1) \quad h(\psi) : \mathcal{P} \to \text{GL}(W).
\end{equation}

We will prove that $h(\psi)$ is constant on each orbit $\{F_1(p) \mid t \in [0, 1]\}$ (Proposition 5). This fact will be important for characterizing the homotopy class of some loops in $\mathcal{G}$.

In the case that the left multiplication of points in $M = G/H$ by each $g \in G$ is an element of $\mathcal{G}$ and $Z(G)$ -the center of $G$- is a subgroup of $H$, then a path $\{g_t\}_{t \in [0, 1]}$ in $G$ with initial point at the identity element $e$ of $G$ and final point $g_1$ in $Z(G)$ determines a loop $\varphi$ in $\mathcal{G}$. These loops $\varphi$ enjoy nice properties:

(a) The corresponding gauge function $h(\varphi)$ is constant and equal to $\Psi(g_1) \in \text{GL}(W)$ (Proposition 7).

(b) Under certain hypotheses, we will construct for each $g_1 \in Z(G)$ a particular path $\{g_t\}$ in $G$, such that for the corresponding loop $\varphi$ in $\mathcal{G}$ the following property holds: If $\{\varphi^s\}_s$ is a deformation of the loop $\varphi$ in $\mathcal{G}$, then for all $s$ the map $h(\varphi^s)$ is the constant map that takes the value $\Psi(g_1)$. Thus, $\# \pi_1(\mathcal{G}) \geq \# \Psi(Z(G))$ (see Theorem 12).

The case when $\Psi$ is a 1-dimensional representation presents three related properties:

(i) The curvature $K$ of the connection $\Omega$ on $\mathcal{P}$ projects a closed 2-form $\omega$ on $M$; thus, $\omega$ defines a presymplectic structure on $M$. (Obviously, the form $\omega$ can also be defined directly from the 1-dimensional representation $\Psi$ of $H$ and the complement $l$ of $\mathfrak{h}$).

(ii) For any loop $\psi$ in a group $\mathcal{G}$ whose Lie algebra admits a lift to $\mathcal{P}$, the map (1.1) is constant, i.e., the gauge transformation $F_1$ determined by $\psi$ is simply the multiplication by a constant $\Theta(\psi)$ (see Proposition 14).

(iii) For each vector field $Z$ belonging to $\text{Lie}(\mathcal{G})$, there exists a function $J_Z$ on $M$ such that $dJ_Z = -\iota_Z \omega$. That is, $J_Z$ is a Hamiltonian function for $Z$ relative to $\omega$.

If the first homology group $H_1(M, \mathbb{Z})$ vanishes, then $\Theta(\psi)$ is an “action integral” (relative to the presymplectic structure $\omega$) around the loop $\psi$ (see 1.17). However, since the functions $J_Z$ will be not “normalized”, this “action integral” is not invariant under deformations of the loop $\psi$ in $\mathcal{G}$.

We will consider the algebra $\mathfrak{X}_\omega$ consisting of all the vector fields $Z$ on $M$, such that $\iota_Z \omega$ is an exact 1-form; i.e., the $\omega$-Hamiltonian vector fields on $M$. The respective connected Lie group is denoted by $\mathcal{H}_\omega(M)$. If it is possible to choose for each $Z \in \mathfrak{X}_\omega$ an $\omega$-Hamiltonian function $J_Z$, so that, $J_Z$ depends continuously on $Z$, then the algebra $\mathfrak{X}_\omega$ admits a lift to $\mathcal{P}$. In this case, we will prove that $\# \pi_1(\mathcal{H}_\omega(M)) \geq \# \Psi(Z(G))$ (Theorem 18).

When $G$ is a linear complex semisimple Lie group, the flag manifold $\mathcal{F}$ defined by a parabolic subgroup can be identified with a quotient $U_\mathbb{K}/H$, where $U_\mathbb{R}$ is a real compact form of $G$ and $H$ a closed subgroup of $U_\mathbb{R}$. If $\Psi$ is a 1-dimensional representation of $H$, then $\# \Psi(Z(U_\mathbb{R}))$ is also a lower bound for the cardinal of $\pi_1(\mathcal{H}_\omega(\mathcal{F}))$ (see Theorem 19).

The derivative of the character $\Psi$ is a covector of $\mathfrak{h}$ that can be extended in a trivial way to an element $\eta \in u^*_\mathbb{R}$, using the complement $l$ of $\mathfrak{h}$. In the particular case that the stabilizer of $\eta$ (for the coadjoint action of $U_\mathbb{R}$) is $H$, the manifold $\mathcal{F}$
is the coadjoint orbit of $\eta$, the form $\omega$ is the Kirillov symplectic structure $\omega$ [1] up to a constant factor and $H_\omega(F)$ is the Hamiltonian group Ham$(F, \omega)$ [2]. Under that hypothesis, $\# \Psi(Z(U))$ is a lower bound for the cardinal of $\pi_1(\text{Ham}(F, \omega))$ (Corollary 20). Here the manifold $F$ is compact and $\omega$ is symplectic, thus, the functions $J_Z$ can be normalized. This fact allows us to prove the invariance of $\Theta(\psi)$ under deformations of the loop $\psi$ in Ham$(F, \omega)$; thus, we have a group homomorphism

$$\Theta : \pi_1(\text{Ham}(F, \omega)) \longrightarrow U(1).$$

From this property, it is possible to obtain an alternative proof of Corollary 20 (see the last Remark in Subsection 1.2).

Case (II). In general, the Hamiltonian vector fields on quantizable manifolds [3] admit lifting to vector fields on the prequantum bundles, which are infinitesimal symmetries of the connection. We will consider the case of quantizable toric manifolds. Let $M$ be such a manifold and $T$ the torus whose action on $M$ defines the toric structure. Each element $b$ in the integer lattice of $t$ determines a loop $\varphi_b$ in the Hamiltonian group of $M$. The corresponding gauge transformation $F_1$, on the prequantum bundle, can be expressed in a simple way in terms of the mass center of the moment polytope of the manifold. This fact will allow us to prove Theorem 22 which gives a sufficient condition for $\varphi_b$ and $\varphi_{\tilde{b}}$ not be homotopic in the Hamiltonian group of $M$. Next, we apply Theorem 22 to the cases when $M$ is the projective space $\mathbb{C}P^n$, a Hirzebruch surface and the 1-point blow up of $\mathbb{C}P^n$.

If the moment polytope of a not necessarily quantizable toric manifold has $d$ facets, then the cohomology of the manifold is generated by the Chern classes $c_1, \ldots, c_d$ of $d$ line bundles. When the cohomology class of the symplectic structure is of the form $r \sum_j n_j c_j$, with $n_j \in \mathbb{Z}$ and $r$ a real number, we will prove Proposition 24 which is the adaptation of Theorem 22 to this type of manifolds.

The paper is organized as follows. In Section 2 from the representation $\Psi$ of $H$ on a vector space $W$ and a complement $l$ of $h$ in $g$, we construct the GL$(W)$-principal fibre bundle $P$ over $M$ and the $G$-invariant connection $\Omega$.

An axiomatic definition of the algebras $\mathfrak{X}$ which admit a lift to $P$ is given in Section 3. In Subsection 3.1, we study the continuity of the lifting of certain curves in $M$ to curves in $P$. The lift of isotopies in $\mathcal{G}$ to isotopies in the fibre bundle, when $\mathcal{G}$ is any subgroup of Diff$(M)$ whose subalgebra satisfies the axioms required to the subalgebras $\mathfrak{X}$, is studied in Subsection 3.2. In Subsection 3.3, we prove Theorem 12.

Section 4 is concerned only with the case when $\Psi$ is 1-dimensional. In Subsection 4.1, we give an explicit expression for $\Theta(\psi)$ and discuss its invariance under deformations of the loop $\psi$. In Subsection 4.2, we introduce the algebra $\mathfrak{X}_\omega$ and prove Theorems 18 and 19.

Section 5 treats with Case (II). We consider subgroups $\mathcal{G}$ of Diff$(M)$, where $M$ is a toric manifold. In Subsection 5.1, we assume that the toric manifold is quantizable and we prove Theorem 22. Subsection 5.2 concerns with toric manifolds not necessarily quantizable. In this subsection we prove Proposition 24.

Other approaches. The results presented in this article are a small contribution to the study of the homotopy of subgroups of the diffeomorphism group of the homogeneous spaces and the toric manifolds, specially the homotopy of Hamiltonian groups.
The homotopy type of the Hamiltonian groups of symplectic manifolds is only completely known in a few particular cases: When the dimension of the manifold is 2, and when the manifold is a ruled complex surface \[4\]. In the first case, the connected component of the identity map in the diffeomorphism group is homotopy equivalent to the component of the identity in the symplectomorphism group; using this fact, the topology of the Hamiltonian group of surfaces can be deduced from the one of the diffeomorphism group (see \([5\) page 52\]). In the second case, the positivity of intersections of \(J\)-holomorphic spheres in 4-manifolds played a crucial role in the proof of results about the homotopy type of the corresponding Hamiltonian groups \([6\), \(7\) \(8\].

For a general symplectic manifold \(N\), given a loop \(\psi\) in the Hamiltonian group of \(N\), the Maslov index of the linearized flow \(\psi_t\) gives rise to a map

\[
\pi_1(\text{Ham}(N)) \longrightarrow \mathbb{Z}/2\mathbb{Z},
\]

\(C\) being the minimal Chern number of \(N\) on spheres. Obviously, the map \(1.3\) can distinguish a maximum of \(2C\) elements in \(\pi_1(\text{Ham}(N))\). In \([9\), we obtained lower bounds for the cardinal of the fundamental group of some Hamiltonian groups, by means of the evaluation of the map \(1.3\) over certain elements of its domain. In particular, Theorem 6 of \([9\) gives a lower bound for the cardinal of \(\pi_1(\text{Ham}(\mathcal{O}))\), where \(\mathcal{O}\) is a coadjoint orbit of \(\text{SU}(n+1)\) diffeomorphic to the quotient of \(\text{SL}(n+1, \mathbb{C})\) by a parabolic subgroup. Of course, that lower bound is \(\leq 2C\). For the particular case when the parabolic subgroup is a Borel subgroup, i.e. \(\mathcal{O}\) is the flag variety of \(\text{sl}(n+1, \mathbb{C})\), the mentioned lower bound is less than or equal to 4, since the minimal Chern number of this manifold is 2 \([10\) page 117\). On the other hand, the lower bound for \(\#\pi_1(\text{Ham}(F, \Psi))\) given in Corollary 20 of the present article, when the group \(G\) is \(\text{SL}(n+1, \mathbb{C})\), depends on \(n\) and \(\Psi\). In the case that \(\Psi\) is injective on \(Z(\text{SU}(n+1))\), this lower bound is \(n+1\).

Given a symplectic manifold \(N\), there is a group homomorphism

\[
I : \pi_1(\text{Ham}(N)) \longrightarrow \mathbb{R},
\]

defined through characteristic classes of the Hamiltonian fibre bundle on \(S^2\) which determine each loop in \(\text{Ham}(N)\) (see \([11\]). As \(I\) is a group homomorphism, it vanishes on any element of finite order; so, \(I\) is not appropriate to detect these elements in \(\pi_1(\text{Ham}(N))\). But studying the homomorphism \(1.4\), we proved the existence of infinite cyclic subgroups in \(\pi_1(\text{Ham}(M))\), when \(M\) is a toric manifold (see \([11\) Theorem 1.2\]). Also for toric manifolds, McDuff and Tolman proved some results relative to the corresponding Hamiltonian groups, by developing the concept of mass linear pair (see for example \([12\) Proposition 1.22\)).

In \([13\), we considered a linear semisimple Lie group \(G\), a compact Cartan subgroup \(T\) and a discrete series representation \(\pi\) of \(G\), determined by an element \(\phi\) in weight lattice of \(t\). By means of these data, we constructed a principal bundle \(P\) on \(G/T\) with connection. As here, we considered subgroups of \(\text{Diff}(G/T)\) such that the vector fields in its Lie algebras admit a lift to vector fields on \(P\), which are infinitesimal symmetries of the connection. We sketched a proof of a result that is a particular case of Theorem \([12\) above mentioned. The present article is, in some extent, the generalization and formalization of results sketched in \([13\).
The principal bundle $P$

In this section, we introduce some notations that will be used in the sequel. We will consider pairs $(H, \Psi)$ such that, either

(A) $H$ is an abelian connected closed Lie subgroup of $G$ containing $Z(G)$ and $\Psi$ a representation of $H$ in a finite dimensional vector space, or

(B) $H$ is a connected closed Lie subgroup of $G$ containing $Z(G)$ and $\Psi$ a 1-dimensional representation of $H$.

We denote by $M$ the quotient $G/H$. Given $A \in \mathfrak{g}$, we put $X_A$ for the vector field on $M$ generated by $A$, through the left $G$-action.

From now on, we assume that there exists a vector subspace $l \subset \mathfrak{g}$, such that

\[ \mathfrak{g} = \mathfrak{h} \oplus l \]

and $t \cdot l = l$, for all $t \in H$; here the dot means the adjoint action. Henceforth, we assume that such a complement $l$ of $\mathfrak{h}$ has been fixed. The $\mathfrak{h}$-component of an element $B \in \mathfrak{g}$ will be denoted $B_0$. Obviously,

\[ (t \cdot B)_0 = t \cdot B_0, \quad \text{for all } t \in H \text{ and } B \in \mathfrak{g}. \]

We will denote by $[l, l]$ the space spanned by all elements of the form $[A, B]_0$, with $A, B \in l$.

$\Psi$ is a group homomorphism $\Psi : H \to \text{GL}(W)$, where $W$ is a complex vector space of finite dimension in the case (A) and with dimension 1 in the case (B). Its derivative will be denoted by $\Psi' : \mathfrak{h} \to \mathfrak{gl}(W)$.

For each $A \in \mathfrak{g}$, we set

\[ f_A(gH) := \Psi'((g^{-1} \cdot A)_0). \]

Proposition 1. Formula (2.3) defines a map $f_A : M \to \mathfrak{gl}(W)$.

Proof. In case (A), it is direct consequence of (2.2) together with the fact $t \cdot B_0 = B_0$, for all $t \in H$, since $H$ is abelian.

For the proof in case (B), it is sufficient to take into account (2.2) and that $\Psi'(t \cdot B_0) = \Psi'(B_0)$, since dimension of $\Psi$ is 1.

In the case (A), from the fact that $\Psi'$ is a Lie algebra homomorphism and $\mathfrak{h}$ is abelian, it follows that

\[ [f_A, f_B]_{\mathfrak{gl}} = 0, \]

where $[\cdot, \cdot]_{\mathfrak{gl}}$ is the commutator in $\mathfrak{gl}(W)$. Obviously, (2.4) is also valid for the case (B).

It is also easy to check that for any $A, C \in \mathfrak{g}$

\[ X_A(f_C) = -f_{[A, C]}. \]

We denote by $\mathcal{P}$ the following principal $\text{GL}(W)$-bundle over $M$

\[ \mathcal{P} = (G \times \text{GL}(W))/\sim \xrightarrow{\pi} M, \]

where $(g, \alpha) \sim (gt, \Psi(t^{-1})\alpha)$, for all $g \in G, \alpha \in \text{GL}(W)$ and $t \in H$. The projection map from $\mathcal{P}$ to $M$ will be denoted by $\pi$, and the $\text{GL}(W)$-right action on $\mathcal{P}$ by $R$.

$Y_A$ will denote the vector field determined by $A \in \mathfrak{g}$ through the left $G$-action on $\mathcal{P}$. The vertical vector field defined by $y \in \mathfrak{gl}(W)$ will be denoted $V_y$. 

The mapping \( f_A \) can be lifted to a map \( f_A \) on \( \mathcal{P} \) by putting
\[
(2.6) \quad f_A([g, \alpha]) = \alpha^{-1} \circ f_A(gH) \circ \alpha,
\]
where \([g, \alpha]\) denotes the element of \( \mathcal{P} \) determined by the pair \((g, \alpha) \in G \times \text{GL}(W)\).

Since \( \Psi(t)f_A(gH) = f_A(gH)\Psi(t) \), \( f_A \) is well-defined. From (2.6), it turns out
\[
(2.7) \quad Y_A(f_C) = -f_{[A, C]}, \quad V_y(p)(f_A) = -[y, f_A(p)]_{\text{gl}}.
\]

In case (B), (2.6) reduces to \( f_A([g, \alpha]) = f_A(gH) \) and (2.7)(ii) to \( V_y(f_A) = 0 \), obviously.

We denote by \( \Omega \) the \( G \)-invariant connection on \( \mathcal{P} \) determined by the condition
\[
(2.8) \quad \Omega(Y_A) = f_A.
\]

The horizontal lift \( X_A^\sharp \) of the vector field \( X_A \) is
\[
(2.9) \quad X_A^\sharp([g, \alpha]) = Y_A([g, \alpha]) + V_y([g, \alpha]),
\]
with \( y := -f_A([g, \alpha]) \). We denote by \( D \) the corresponding covariant derivative and by \( K \) the curvature of this connection. Using the structure equation, (2.4), (2.7) and (2.8), it is straightforward to check that
\[
(2.10) \quad K(Y_A, Y_C) = -f_{[A, C]}.
\]
From (2.7), (2.9) and (2.11), it follows
\[
(2.11) \quad Df_C = -K(Y_C, \cdot).
\]

3. Algebras admitting a lift to \( \mathcal{P} \)

From now on, we assume that the spaces of \( C^1 \) functions between two manifolds are equipped with the Whitney \( C^1 \)-topology, unless explicit mention is made to the contrary. In particular, the Lie subgroups of the group of diffeomorphisms of a manifold and the corresponding Lie algebras will be endowed with this topology.

**Definition 1.** We say that a subalgebra \( X \) of \( X(M) \), the algebra consisting of the vector fields on \( M \), admits a lift to \( \mathcal{P} \), if for each \( Z \in X \) there is a map \( a(Z) : \mathcal{P} \to \text{gl}(W) \), satisfying the following conditions:

(i) \( a(Z)(p\beta) = \beta^{-1} \circ a(Z)(p) \circ \beta \), for all \( \beta \in \text{GL}(W) \) and all \( p \in \mathcal{P} \).
(ii) \( Da(Z) = -K(Z^\sharp, \cdot) \), where \( Z^\sharp \) is the horizontal lift of \( Z \).
(iii) The map
\[
Z \in X \mapsto a(Z) \in C^\infty(\mathcal{P}, \text{gl}(W))
\]
is continuous with respect to the \( C^0 \) Whitney topologies in \( X \) and in \( C^\infty(\mathcal{P}, \text{gl}(W)) \).

(iv) If \( X_A \in X \), then \( a(X_A) = f_A \).

Note that, by (2.9) and (2.11), condition (ii) is consistent with (iv). Moreover, the Lie algebra
\[
\{X_A | A \in \mathfrak{g}\}
\]
adopts a lift to \( \mathcal{P} \).

Given \( Z \in X \), we set \( U(Z) \) for the vector field
\[
(3.2) \quad U(Z) := Z^\sharp + V_y(Z).
\]

The following proposition is a consequence of condition (i) in the above definition.

**Proposition 2.** The vector field \( U(Z) \) is invariant under the right translations in \( \mathcal{P} \); that is \( (R_\beta)_*(U(Z)) = U(Z) \), for all \( \beta \in \text{GL}(W) \).
Since $\Omega$ is a pseudo tensorial form of type $\text{Ad}$, from Proposition[2] it follows
\begin{equation}
(3.3) \quad \Omega(U(Z)) = \text{Ad}(\beta^{-1})(\Omega(U(Z)));
\end{equation}
for any $\beta \in \text{GL}(W)$.

**Proposition 3.** The Lie derivative of $\Omega$ along $U(Z)$ vanishes.

**Proof.** We put $U$ for $U(Z)$ and $a$ for $a(Z)$. By Cartan’s formula, it is sufficient to check that
\[ d(\iota_U \Omega) + \iota_U (d\Omega) = 0. \]
By (3.2), $d(\iota_U \Omega) = da$. Thus, we need to prove that 1-form on $P$
\begin{equation}
(3.4) \quad da + \iota_U (d\Omega)
\end{equation}
vanishes.

Given an arbitrary point $p \in P$, we will prove that the 1-form $\Omega$ applied to a horizontal vector $E$ at $p$ vanishes. By (ii) in Definition 1, $da(E) = -K(Z^t, E)$. Next, we extend $E$ to a horizontal field; by the structure equation
\[(\iota_U (d\Omega))(E) = K(U, E) - [\Omega(U), \Omega(E)] = K(Z^t, E).\]
So, $\Omega$ vanishes on any horizontal vector.

Now consider the vertical vector $V_\varphi(p)$, with $\varphi \in \text{gl}(W)$. Again by the structure equation together with (3.2)
\[(\iota_U (d\Omega))(V_\varphi(p)) = -[\Omega p(U), \Omega(V_\varphi(p))] = -[a(p), \varphi].\]
On the other hand, by (i) in Definition 1, $da(V_\varphi(p)) = -[\varphi, a(p)]$. Thus, (3.3) also vanishes on vertical vectors. \qed

3.1. **Lifting of curves in $M$ to curves in $P$.** All the curves on $M$ and on $P$ considered in this subsection are defined on the interval $[0, 1]$. Furthermore, $X$ denotes a given algebra that admits a lift to $P$.

Let $\xi(t)$ be a curve in $M$ with $\xi(0) = \bar{e} := eH$ and $p$ a given point in $\pi^{-1}(\bar{e})$. Let us assume that there exists a vector field $Z \in X$, such that $Z(\xi(t)) = \xi(t)$, for all $t$. Then we can consider the integral curve $\dot{\xi}(t)$ of $U(Z)$ such that $\xi(0) = p$. By (3.2), the curve $\dot{\xi}(t)$ in $P$ can be considered as a lifting (no horizontal lift) of $\xi(t)$. In the following paragraphs, we will show the continuity of this lifting as a consequence of property (iii) in Definition 1.

Let $\{\xi^c\}_{c \in [0, 1]}$ be a deformation of $\xi = \xi^0$. That is, a family of curves $\xi^c : [0, 1] \to M$ with initial point at $\bar{e}$ (i.e., $\xi^c(0) = \bar{e}$), satisfying the following conditions:

(a) The vectors $\dot{\xi}^c(t) = \frac{d}{dt}\xi^c(t)$ are elements of a vector field $Z^c \in X$; that is, $Z^c(\xi^c(t)) = \dot{\xi}^c(t)$, for all $t$.

(b) $\{\xi^c\}_{c \in [0, 1]}$ is a continuous family in the set $C^1([0, 1], M)$ endowed with the $C^1$ Whitney topology.

For the sake of simplicity, we assume that there exists a chart $(V, \rho)$ on $M$, such that $\xi^c([0, 1]) \subset V$, for all $c \in [0, 1]$. The condition (b) involves:

$\alpha$) For any $\epsilon_1 > 0$, there exists $\delta_1 > 0$ such that for all $t \in [0, 1]$
\[ ||\xi^c(t) - \xi(t)|| < \epsilon_1, \text{ if } c < \delta_1. \]
Here $\xi^c(t)$ denotes the coordinates of the point $\xi^c(t)$ in the chart $(V, \rho)$.

$\beta$) For any $\epsilon_2 > 0$, there exists $\delta_2 > 0$ such that for all $t \in [0, 1]$
\[ ||Z^c(\xi^c(t)) - Z(\xi(t))|| < \epsilon_2, \text{ if } c < \delta_2. \]
Here $Z^c(\xi^c(t))$ means the coordinates of the corresponding vector in the mentioned chart.

From property (a), together with the fact that the curves are defined on a compact interval and that $\xi^c(0) = \bar{e}$, one deduces the equivalence between properties (a) and (β).

By $\xi^c$ we denote the lift of $\xi^c$ at the point $p$. That is, the integral curve of $U^c := U(Z^c)$ with initial point at $p$.

The continuity of the horizontal lift of vectors fields on $M$ to vector fields in $P$, together with the condition (iii) in Definition 1 and (β), imply the following property: For any $\epsilon > 0$, there exists $\delta > 0$ such that for all $t \in [0, 1]$

$$||U^c(\tilde{\xi}(t)) - U(\tilde{\xi}(t))|| < \epsilon, \text{ if } c < \delta.$$  

In this formula $U^c$ and $U$ denote the coordinates of the respective vector fields in a chart of $P$ that contains the curves $\{\tilde{\xi}(t)\}$.

On the other hand, an equivalence similar to the one between (a) and (β) holds for the curves $\tilde{\xi}^c$ in $P$ and the vectors fields $U^c$.

In summary, we have proved the following proposition:

**Proposition 4.** With the above notations, given a neighborhood $O$ of $\tilde{\xi}$ in the $C^1$ Whitney topology of $C^1([0, 1], P)$, there is $\delta > 0$ such that $\tilde{\xi}^c \in O$, for all $c < \delta$.

3.2. Lift of isotopies in $M$ to isotopies in $P$. Let $\{Z_t\}_{t \in [0, 1]}$ be a time-dependent vector field on $M$, with $Z_t \in \mathfrak{X}$. Let $\psi_t$ be the isotopy of $M$ defined by

$$(3.5) \quad \frac{d\psi_t}{dt} = Z_t \circ \psi_t, \quad \psi_0 = \text{Id}_M.$$  

We have the time-dependent vector field $U_t := U(Z_t)$ on $P$ and the corresponding flow $F_t$:

$$(3.6) \quad \frac{dF_t(p)}{dt} = U_t(F_t(p)), \quad F_0 = \text{Id}_P.$$  

By the definition of $U(Z_t)$, it follows that $\pi(F_t(p)) = \psi_t(\pi(p))$, for all $t$, where $p$ is any point of $P$. Thus, if we put $\xi(t) = \psi_t(\bar{e})$, its lift at the point $p \in \pi^{-1}(\bar{e})$ is the curve $\tilde{\xi}(t) = F_t(p)$.

From Proposition 4 we deduce that the diffeomorphism $F_t$ preserves the connection form $\Omega$; that is, $F_t^* \Omega = \Omega$.

Let us assume that the above isotopy $\psi = \{\psi_t\}_{t \in [0, 1]}$ satisfies $\psi_0 = \psi_1 = \text{Id}_M$; that is, $\psi$ is a loop of diffeomorphisms of $M$. Then $F_1$ is a diffeomorphism of $P$ over the identity, i.e., a gauge transformation of $P$. Thus, there exists a map $h : P \rightarrow \text{GL}(\mathbb{W})$ satisfying $h(p\beta) = \beta^{-1}h(p)\beta$ and such that

$$(3.7) \quad F_1(p) = ph(p), \text{ for all } p \in P.$$  

Moreover, $F_1^* (\Omega) = h^{-1} dh + \text{Ad}(h^{-1}) \circ \Omega$.

Since $\Omega = F_1^* (\Omega)$,  

$$(3.8) \quad \Omega = h^{-1} dh + \text{Ad}(h^{-1}) \circ \Omega.$$  

Equality $\text{(3.8)}$ applied to $U_t$ together with $\text{(3.6)}$ give

$$(3.9) \quad dh(U_t) = 0.$$  

Thus, we have proved the following proposition:
Proposition 5. Let \( \psi = \{ \psi_t \}_{t \in [0, 1]} \) be a loop of diffeomorphisms of \( M \) generated by a time-dependent vector field \( Z_t \in \mathfrak{X} \). The map \( h : P \to \text{GL}(W) \) defined by (3.7) is constant on the orbit \( \{ F_t(p) \mid t \in [0, 1] \} \) of any point \( p \in P \).

Let \( \{ q_t \mid t \in [0, 1] \} \) be a \( C^1 \) curve in \( G \) with initial point at the identity element \( e \); for brevity such a curve will be called a path in \( G \). It determines the isotopy \( \varphi = \{ \varphi_t \}_{t \in [0, 1]} \) of \( M \), with

\[
\varphi_t(gH) = g_t H.
\]

We put \( \{ A_t \} \) for the corresponding velocity curve; that is, \( A_t \in \mathfrak{g} \) is defined by

\[
A_t = g_t g_t^{-1}.
\]

Let us assume that \( X_{A_t} \in \mathfrak{X} \). This family gives rise to the time-dependent vector field on \( P \), \( U_t := U(X_{A_t}) \), which in turn defines a flow \( F_t \) on \( P \) (the flows in \( P \) determined by paths in \( G \) will be denoted in boldface). Obviously, the lift of the curve \( \varphi_t(\bar{e}) \) at the point \( p \in \pi^{-1}(\bar{e}) \) is \( F_t(p) \). From (iv) in Definition 1 together with (2.9), it follows \( U(X_{A_t}) = Y_{A_t} \). So,

\[
\frac{dF_t}{dt} = Y_{A_t} \circ F_t, \quad F_0 = \text{Id}_P.
\]

Lemma 6. The bundle diffeomorphism \( F_t \) defined in (3.12) is the left multiplication by \( g_t \) in \( P \).

Proof. Given \( p \in P \), by (3.11)

\[
\left. \frac{d}{du} \right|_{u=t} g_u p = \left. \frac{d}{du} \right|_{u=t} g_u g_t^{-1} g_t p = Y_{A_t}(g_t p).
\]

If \( g_1 \) is an element of the center of \( G \), then \( \varphi_0 = \varphi_1 \) and \( F_1 \) is a gauge transformation of \( P \).

Henceforth, we assume that \( \Psi(g_1) \) is a scalar operator for any \( g_1 \in Z(G) \). This condition holds trivially in case (B). The case (A) with this additional condition will be designed as (A').

Proposition 7. Let \( g_t \) be a path in \( G \), such that \( X_{A_t} \in \mathfrak{X} \), where \( A_t = g_t g_t^{-1} \). If \( g_t \in Z(G), \) then \( F_1(p) = p\Psi(g_1), \) for all \( p \in P \).

Proof. By Lemma 6

\[
F_1([g, \alpha]) = [g_1 g, \alpha] = [gg_1, \alpha] = [g, \Psi(g_1)\alpha] = [g, \alpha]\Psi(g_1),
\]

since \( \Psi(g_1) \) is a scalar operator.

Remark. Note that paths in \( G \) with the same final point in \( Z(G) \) determine the same gauge transformation \( F_1 \).

3.3. Subgroups of \( \text{Diff}(M) \). Let \( G \) be a connected Lie subgroup of \( \text{Diff}(M) \) such that:

\[
\begin{align*}
\text{(i) } & \text{Lie}(G) \text{ admits a lift to } P, \\
\text{(ii) } & \text{If } \{ g_t \} \text{ is a path in } G, \text{ then the isotopy } \{ \varphi_t \} \text{ is contained in } G.
\end{align*}
\]

We will prove that

\[
\# \pi_1(G) \geq \# \{ \Psi(g) \mid g \in Z(G) \},
\]

but we need to introduce some notations.
Let \( \{g_t\} \) be a path in \( G \) such that \( g_1 \in Z(G) \). Let \( \{\zeta^s\}_s \) be a deformation of the loop \( \varphi \) in \( G \). That is, for each \( s \), \( \zeta^s \) is a loop in \( G \) at \( \text{Id}_M \), with \( \zeta^0 = \varphi \). We also assume that \( s \mapsto \zeta^s \) is a continuous map (considering \( G \) equipped with the \( C^1 \)-topology, as we said).

For each \( s \), we have the corresponding time-dependent vector field \( Z_h^s \in \text{Lie}(G) \), given by
\[
\frac{d\zeta^s}{dt} = Z_h^s \circ \zeta^s.
\]
The respective time-dependent vector field on \( P \), \( U_h^s := (Z_h^s)^t + V_{\alpha(Z_h^s)} \) determines the corresponding flow \( F_t^s \). As above, \( F_t^s \) is a gauge transformation, which can be written
\[
\text{(3.14)} \quad F_t^s(q) = qh^s(q), \quad \text{for all } q \in P,
\]
with \( h^s \) constant on the orbits \( \{F_t^s(q) \mid t \in [0, 1]\} \).

Fixed \( s \), we can also consider the loop \( \xi^s \) in \( M \), obtained by evaluating \( \zeta^s_t \) at the point \( \bar{e} \)
\[
\text{(3.15)} \quad \{\xi^s(t) := \zeta^s_t(\bar{e}) \mid t \in [0, 1]\}.
\]
Obviously, the lifting of this loop to \( P \) at a point \( p \in \pi^{-1}(\bar{e}) \) is just the curve \( F_t^s(p) \); i.e.,
\[
\text{(3.16)} \quad \dot{\xi}^s(t) = F_t^s(p).
\]

In the statement of the following theorem we refer to the \( H \)-principal bundle
\[
\text{(3.17)} \quad H \rightarrow G \overset{\text{pr}}{\rightarrow} M = G/H.
\]
We assume that this bundle is endowed with the invariant connection [14, page 103] determined by the splitting [21]. It is well-known that the Lie algebra of the holonomy group \( \text{Hol}_e \) at \( e \in G \) (of this invariant connection) is \([l, l]_0 \) (see [14, Theorem 11.1, page 103]).

**Theorem 8.** Let \( \{g_t\}_{t \in [0, 1]} \) be a path in \( G \) which is a horizontal curve with respect the invariant connection in \( G \overset{\text{pr}}{\rightarrow} M \) and such that its final point belongs to \( Z(G) \). If \( \{\zeta^s\}_s \) is a deformation of the loop \( \varphi \) in \( G \), then the gauge transformation \( F_t^s \) of \( P \) defined by \( \zeta^s \) satisfies
\[
\text{\( (3.17) \quad F_t^s(p) = p\Psi(g_1), \)}
\]
for all \( p \in \pi^{-1}(\bar{e}) \).

**Proof.** Fix an arbitrary point \( p \in P \), such that \( \pi(p) = \bar{e} \). The idea of the proof is to construct a path in \( G \), with final point at \( g_1 \), that defines a loop in \( M \) as close to the evaluation curve \( \xi^s \) (see (3.15)) as we wish. Then the lifting of the loop at the point \( p \) and the one of \( \xi^s \), give rise to the constants \( \Psi(g_1) \) and \( h^s(p) \), whose difference will be as small as we wish.

We have the following closed curves in \( M \):
\[
\text{(3.18)} \quad \{\varphi_t(\bar{e}) \mid t \in [0, 1]\}, \quad \xi^s = \{\xi^s(t) \mid t \in [0, 1]\}.
\]

(1) **Horizontal lifting from \( M \) to \( G \).** The invariant connection on the principal bundle \( G \rightarrow M \) determines the horizontal lifting of the curves (3.18). We denote by \( v^a \) the curve in \( G \) horizontal lift of \( \xi^a \) at the point \( e \in G \). By hypothesis, \( g_t \) is the horizontal lift of \( \{\varphi_t(\bar{e}) = g_t\bar{e}\} \) at the point \( e \). In particular, \( v^0_1 = g_t \).
Fixed $s$, a value of the parameter of deformation, since the horizontal lifting is a continuous operation,

$$S := \{ v^s_t | a \in [0, s], t \in [0, 1] \}$$

is a piece of surface in $G$. The element $g_1$, final point of $v^0$, belongs to this connected surface in $G$.

Given $0 < \epsilon < 1$, let $\{ \mu_t | t \in [1 - \epsilon, 1] \}$ be a curve in $S$ with $\mu_{1-\epsilon} = v^s_{1-\epsilon}$ and final point at $g_1$. The curve $\mu$ can be chosen so that its juxtaposition with the restriction of $v^s$ to $[0, 1 - \epsilon]$ is a $C^1$ curve in $G$ (see Figure 1). The path in $G$ defined by this juxtaposition will be denoted $\{ g'_t | t \in [0, 1] \}$; its final point is $g_1$.

By taking $\epsilon$ sufficiently small we can get a curve $\{ g'_t \bar{e} = \text{pr}(g'_t) | t \in [0, 1] \}$ in $M$ contained in an arbitrary neighborhood of the curve $\xi^s$ in the $C^1$ topology of $C^1([0, 1], M)$. Thus, we have proved the following lemma:

**Lemma 9.** Given a neighborhood of the curve $\xi^s$ in the $C^1$ topology of $C^1([0, 1], M)$, there is a path $\{ g'_t \}$ in $G$, with final point at $g_1$, such that the loop $\{ g'_t \bar{e} \}$ is contained in that neighborhood.

**Figure 1.** The path $\{ g'_t \}$ is the juxtaposition of $v^s_{[0, 1-\epsilon]}$ and $\mu_t$.

**II) Lift from $M$ to $P$.** As we said, $p$ will be an arbitrary point of $\pi^{-1}(\bar{e})$. The loop of diffeomorphisms $\zeta^s$ defines the corresponding flow $F^s_t$ and $F^s_1$ is determined by $h^s$ (see (3.14)).

We can also consider $\tilde{\xi}^s$, the lifting to $P$ at the point $p$ of the loop in $\xi^s$. By (3.14) and (3.16)

$$\tilde{\xi}^s(1) = F^s_1(p) = ph^s(p).$$

On the other hand, the path $\{ g'_t \}$ has final point at $g_1 \in Z(G)$. It determines the respective flow $F'_t$ and, according to Proposition 7, $F'_t$ is the gauge transformation defined by the constant $\Psi(g_1)$

$$F'_1(q) = q\Psi(g_1).$$

We have also the lifting to $P$ at the point $p$ of the loop $\{ g'_t \bar{e} \}$; this lift is, of course, the curve

$$\{ F'_t(p) | t \in [0, 1] \}$$
(III) Continuity of the lifting. Let $O$ be an arbitrary neighborhood of the curve $\xi^s$ in the $C^1$ topology of $C^1([0,1],\mathcal{P})$. By the continuity of the lifting stated in Proposition 4 together with Lemma 9, there are curves of the form $\{g'_t\}$, where $\{g'_t\}$ is a path in $G$ with final point at $g_1$, whose lifting (3.21) to belong to $O$. Hence, $\xi^s(1) = ph^s(p)$ is a point of $\mathcal{P}$ as close to one point of the type $F'_t(p)$ as we wish. But, by (3.20), all the points $F'_t(p)$ are equal to $p\Psi(g_1)$ (independently of the path $\{g'_t\}$). Therefore, the constant $h^s(p)$ is $\Psi(g_1)$; that is, $F'_1(p) = p\Psi(g_1)$, for all $p \in \pi^{-1}(\bar{e})$.

Corollary 10. Let $\{g_t\}$ and $\{\tilde{g}_t\}$ be paths in $G$ satisfying the hypotheses of Theorem 8. If $\Psi(g_1) \neq \Psi(\tilde{g}_1)$, then the corresponding loops $\varphi$ and $\tilde{\varphi}$ are not homotopic in $G$.

If $Z(G) \subset \text{Hol}_e$, each element of $Z(G)$ can be joined to $e$ by a horizontal curve in $G$ (horizontal with respect to the invariant connection).

The following proposition gives a sufficient condition for $Z(G) \subset \text{Hol}_e$, when $\mathfrak{g}$ is a perfect Lie algebra; i.e. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Proposition 11. If $\mathfrak{g}$ is a perfect Lie algebra and $H$ is abelian, then $Z(G) \subset \text{Hol}_e$.

Proof. Since $\mathfrak{h}$ is abelian and $\mathfrak{l}$ is invariant under $\text{Ad}(H)$,

\begin{equation}
\mathfrak{h} \oplus \mathfrak{l} = \mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{l}_1 + [\mathfrak{l}, \mathfrak{l}],
\end{equation}

with $\mathfrak{l}_1 \subset \mathfrak{l}$. By taking the $\mathfrak{h}$-component in (3.22), we obtain $\mathfrak{h} = [\mathfrak{l}, \mathfrak{l}_0]$; as $H$ is connected, $H \subset \text{Hol}_e$. By hypothesis, $Z(G)$ is a subgroup of $H$ in case (A) and in case (B).

We denote by $\exp([\mathfrak{l}, \mathfrak{l}_0])$ the connected subgroup of $G$ whose Lie algebra is $[\mathfrak{l}, \mathfrak{l}_0]$. Henceforth, we assume that the following property holds

\begin{equation}
Z(G) \subset \exp([\mathfrak{l}, \mathfrak{l}_0]).
\end{equation}

From Corollary 10 it follows the following theorem:

Theorem 12. Let $G$ be a Lie group, and $(H, \Psi)$ a pair satisfying either condition $(A')$ or condition (B), and $\mathfrak{l}$ an $\text{Ad}(H)$-invariant complement of $\mathfrak{h}$ for which (3.23) is valid. If $\mathcal{G}$ is a connected Lie subgroup of $\text{Diff}(M)$ for which (3.11) holds, then

\[ \# \pi_1(\mathcal{G}) \geq \# \Psi(Z(G)). \]

Given $g \in G$, we denote by $\mathfrak{g}$ the diffeomorphism of $M$ defined by left action of $g$; that is, $\mathfrak{g}(bH) = gbH$. We put $G$ for the subgroup of $\text{Diff}(M)$ consisting of all diffeomorphisms $\mathfrak{g}$. Obviously, the group $\mathcal{G} \subset \text{Diff}(M)$ satisfies (3.11) (ii) if $G \subset \mathcal{G}$.

Each closed path in $G$ gives rise to a loop in $G$, and homotopic closed paths in $G$ induce homotopic loops in $G$. However, there are loops in $G$ which are not defined by closed paths in $G$. Evidently, $\text{Lie}(G)$ is the Lie algebra (3.1). This Lie algebra satisfies the conditions required to the algebras $\mathfrak{X}$ corresponding to any representation $\Psi$ of $H$. Thus, we have the following corollary of Theorem 12.

Corollary 13. If $H$, $\Psi$ and $\mathfrak{l}$ satisfy the hypotheses of Theorem 12, then

\[ \# \pi_1(G) \geq \# \Psi(Z(G)). \]
4. Hamiltonian groups

In this Section we assume that the pair \((H, \Psi)\) belongs to case (B). When \(\Psi\) is a 1-dimensional representation, the curvature \(\mathbf{K}\) projects a presymplectic structure \(\omega\) on \(M\), as we said in the Introduction.

In Subsection 4.1 we will consider the \(\omega\)-action integral around the loop of diffeomorphisms generated by a time-dependent vector field in an algebra which admits a lift to \(\mathcal{P}\). We will relate this action integral with the gauge transformation associated with that loop.

In Subsection 4.2, we consider the algebra \(\mathfrak{X}_\omega\) of all \(\omega\)-Hamiltonian vector fields on \(M\). We will give conditions for \(\mathfrak{X}_\omega\) admitting a lift to \(\mathcal{P}\) and we will apply Theorem 12 to this particular case.

4.1. Action integral. When \(\dim W = 1\), Proposition 5 can be stated as:

**Proposition 14.** Let \(\psi = \{\psi_t\}_{t \in [0, 1]}\) be a loop of diffeomorphisms of \(M\) generated by a time-dependent vector field \(Z_t\) that belongs to a Lie algebra \(\mathfrak{X}\) admitting a lift to \(\mathcal{P}\). If \(\dim W = 1\), then there exists a complex number \(\Theta(\psi)\) such that,

\[
F_1(p) = p\Theta(\psi),
\]

for all \(p \in \mathcal{P}\).

*Proof.* As dimension of \(W\) is 1, (3.8) reduces to \(dh = 0\). So, \(h\) is constant on \(\mathcal{P}\). \(\square\)

Let \(Z\) be a vector field in an algebra \(\mathfrak{X}\) that admits a lift to \(\mathcal{P}\); now, by property (i) in Definition 1, \(a(Z)\) projects a function \(J_Z\) on \(M\) defined by

\[
J_Z(p) = a(Z)(p).
\]

Analogously, the curvature \(\mathbf{K}\) projects a closed 2-form \(\omega\) on \(M\); thus, \(\omega\) is a presymplectic structure on \(M\). The condition (ii) in Definition 1 gives rise to

\[
dJ_Z = -\omega(\mathbf{K}, \cdot).
\]

Note that \(Z\) determines \(J_Z\) by (4.2), up to an additive constant. However, if \(Z = X_A\), then condition (iv) in Definition 1 implies that \(J_X = f_A\).

Next, we will express the number \(\Theta(\psi)\) defined in Proposition 14 as a “generalized action integral” on the presymplectic manifold \((M, \omega)\).

Let \(\psi\) be the loop considered in Proposition 14. Given \(x \in M\), we put \(\{x_t\}\) for the closed evaluation curve \(\{\psi_t(x)\}\). For \(p \in \pi^{-1}(x)\), we set \(p_t := F_t(p)\), thus, \(\pi(p_t) = x_t\). Let \(\sigma\) be a section of the principal bundle \(\mathcal{P}\), such that \(x\) belongs to the domain of \(\sigma\) and \(\sigma(x) = p\). We define the complex number \(\theta_t\) by the relation

\[
F_t(p) = \sigma(x_t)\theta_t.
\]

Obviously \(\theta_0 = 1\) and \(\theta_1 = \Theta(\psi)\). Deriving (4.4) with respect to \(t\),

\[
\frac{dF_t(p)}{dt} = (R_{\theta_t})_* (\sigma_*(Z_t(x_t))) + V_{\theta_t/\theta_0}(p_t).
\]

Taking into account that

\[
\sigma_*(Z_t(x_t)) = Z_t^\sharp(\sigma(x_t)) + V_y(\sigma(x_t)),
\]

with \(y = \Omega(\sigma_*(Z_t(x_t))) = \alpha(Z_t(x_t))\), where \(\alpha\) is the connection form in the trivialization defined by \(\sigma\). Then,

\[
\frac{dF_t(p)}{dt} = Z_t^\sharp(p_t) + V_y(p_t) + V_{\theta_t/\theta_0}(p_t).
\]
On the other hand, by (3.6), 
\[
\frac{dF_t(p)}{dt} = Z_t^i(p_t) + V_{a_t}(p_t),
\]
where \(a_t := a(Z_t(p_t))\). From (4.5), (4.6) and (4.2), it follows
\[
\dot{\theta} = -\alpha(Z_t(x_t)) + J Z_{x_t}.
\]
Hence,
\[
\Theta(\psi) = \exp\left( -\int_0^1 \alpha(Z_t(x_t)) + \int_0^1 J Z_{x_t} dt \right).
\]

If \(H_1(M, \mathbb{Z}) = 0\), there exists a 2-chain \(C\) in \(M\) whose boundary is the closed curve \(x_t\). In this case, by Stokes’ theorem and taking into account that the curvature \(K\) projects on \(M\) the closed 2-form \(\omega\), we have the following proposition:

**Proposition 15.** Let \(\psi\) be the loop of Proposition 14 and \(x\) a point of \(M\). If \(H_1(M, \mathbb{Z}) = 0\), then
\[
\Theta(\psi) = \exp\left( -\int_C \omega + \int_0^1 J Z_{x_t} dt \right),
\]
where \(C\) is a 2-chain whose boundary is the closed curve \(\{x_t := \psi_t(x)\}\).

From the equality (4.7), one deduces that the right hand side of this equation is independent of the point \(x\). It has the form of the exponential of an action integral (see the last Remark of Subsection 4.2).

In general, \(\Theta\) is not invariant under deformations of the loop \(\psi\), essentially because the \(J^i\) are not normalized. More precisely, let \(\kappa^s\) be a family of loops in \(\text{Diff}(M)\) at \(\text{Id}_M\), each of which satisfies the hypotheses of Proposition 14 and such that \(\kappa^0 = \psi\). Then we have the corresponding time-dependent vector fields \(Z^s_t \in \mathfrak{X}\), the maps \(J_{Z_t^s}\) with \(J_{Z_0^s} = J_{Z_t}\) and the corresponding invariants \(\Theta^s := \Theta(\kappa^s)\) (defined by means of the \(J_{Z_t^s}\))
\[
\Theta^s = \exp\left( -\int_C \omega + \int_0^1 J Z^s_{x_t}(\kappa^s_t(x)) dt \right),
\]
where \(C^s\) is a 2-chain such that \(\partial C^s\) is the curve \(\{\kappa^s_t(x)\}_t\).

On the other hand, the variation of \(\kappa^s_t(x)\) with respect to \(s\) defines the time-dependent vector field \(B_t\)
\[
B_t(\kappa^s_t(x)) := \frac{\partial}{\partial s} \kappa^s_t(x).
\]
Hence,
\[
\frac{1}{\Theta(\psi)} \left. \frac{d\Theta^s}{ds} \right|_{s=0} = \int_0^1 \omega(Z_t(x_t), B_t(x_t)) dt + \int_0^1 B_t(x_t)(J_{Z_t}) dt + \int_0^1 J_t(x_t).
\]
where
\[
J_t := \left. \frac{d}{ds} \right|_{s=0} J_{Z_t}.
\]
Since $\nu_Z, \omega = -dJ_Z,$

\[ (4.9) \quad \frac{1}{\Theta(\psi)} \frac{d\Theta^{s^*}}{ds} \bigg|_{s=0} = \int_0^1 J_t(x_t) dt. \]

Thus, we have the following proposition:

**Proposition 16.** Let $\{\kappa^s\}_s$ be a deformation of the loop $\psi$, where each $\kappa^s$ is generated by a time-dependent vector field belonging to $\mathfrak{X}$. Then

\[ \int_0^1 J_t(x_t) dt \]

is independent of the point point $x$, and it equals

\[ \frac{1}{\Theta(\psi)} \frac{d\Theta(\kappa^s)}{ds} \bigg|_{s=0}. \]

Let $\varphi$ be the loop in the statement of Theorem 8. After Proposition 14, Theorem 8 asserts (when $\dim W = 1$) the following:

**Proposition 17.** Let $\varphi$ be the loop in the statement of Theorem 8 and $\{\zeta^s\}_s$ a deformation of $\varphi$ in $\mathcal{G}$, then

\[ \frac{d\Theta(\zeta^s)}{ds} \bigg|_{s=0} = 0. \]

As we will see in the last Remark at the end of this Section, when $G$ is semisimple, $M$ is a flag manifold of $\mathfrak{g}$ and $\omega$ the Kirillov symplectic form, the constant (4.9) vanishes for any deformation in the Hamiltonian group of a given Hamiltonian loop $\psi$. In more precise terms, in this case, it is possible to modify the maps $a(Z)$, whose existence is postulated in the definition of $\mathfrak{X}$, so that: (a) The new maps satisfy the conditions imposed in Definition 1. (b) For the new maps $J_Z$, integral (4.9) is zero.

### 4.2. Presymplectic structure.

By (2.10) the 2-form $\omega$, projection of $\mathbf{K}$ on $M$, satisfies

\[ (4.10) \quad \omega(X_A, X_B) = -f_{[A, B]}. \]

This relation allows to define the form $\omega$ by means the representation $\Psi$ and the complement $I$ directly, without resorting to the principal bundle $\mathcal{P}$. From (4.10), using the Jacobi identity, one can prove directly that $\omega$ is a closed form (see [15, page 6]).

We put

\[ \mathfrak{X}_\omega := \{ Z \in \mathfrak{X}(M) | \omega(Z, .) \text{ is exact} \}. \]

Let $\mathcal{H}_\omega(M)$ denote the group consisting of the time-1 maps of the isotopies in $M$ generated by a time-dependent vector fields $Z_t \in \mathfrak{X}_\omega$. If $\omega$ is a symplectic form, $\mathcal{H}_\omega(M)$ is the corresponding Hamiltonian group $\text{Ham}(M, \omega)$.

By (2.5),

\[ (4.11) \quad \omega(X_A, .) = -df_A. \]

Thus, the vector field $X_A$ belongs to $\mathfrak{X}_\omega$ and $G \subset \mathcal{H}_\omega(M)$. 
Theorem 18. Let us assume that $G$, $H$, $\Psi$ and $I$ satisfy the hypotheses of Theorem \[12\] and $\dim \Psi = 1$. If for each $Z \in \mathfrak{X}_\omega$ it is possible to choose a function $J_Z$ on $M$, such that: (a) $\omega(Z, \cdot) = -dJ_Z$; (b) the map $Z \mapsto J_Z$ is continuous; c) $J_{X_A} = f_A$, for all $A \in \mathfrak{g}$, then $\# \pi_1(\mathcal{H}_\omega(M)) \geq \# \Psi(Z(G))$.

Proof. For $Z \in \mathfrak{X}_\omega$, we set

\[ a(Z) : [g, \alpha] \in \mathcal{P} \mapsto J_Z(gH) \in \mathbb{C}. \]

On the other hand, the relation (a) is equivalent to $Da(Z) = -K(Z, \cdot)$. Hence, $\mathfrak{X}_\omega$ admits a lift to $\mathcal{P}$. The theorem follows from Theorem \[12\] \qed

When $G$ is a complex semisimple Lie group and $P$ a parabolic subgroup, we put $\mathcal{F}$ for the flag manifold $G/P$. Then $\mathcal{F} \cong U_\mathbb{R}/H$, where $U_\mathbb{R}$ is a real compact form of $G$ and $H = P \cap U_\mathbb{R}$. Since $U_\mathbb{R}$ and $H$ are compact groups, they are unimodular. Hence, $\mathcal{F} = U_\mathbb{R}/H$ has a nonzero left invariant Borel measure $d\mu$ (see [16, Theorem 8.36]). The subgroup $H$ contains a maximal torus of $U_\mathbb{R}$, so, $Z(G) \subset H$, and by means of the system of roots associated to this torus, it is possible to define a complement $\mathfrak{l}$ of $\mathfrak{h}$ in $\mathfrak{u}_\mathbb{R}$, invariant under $\text{Ad}(H)$.

Given $\Psi$ a representation of $H$ of dimension 1 (as $H$ is compact, we can assume that $\Psi$ is unitary), we have the respective presymplectic form $\omega$ and $\mathfrak{X}_\omega$. In this case, a choice of the functions $J_Z$ which satisfy the conditions (a)-(c) in the statement of Theorem \[18\] can be carried out as follows. Given $Z \in \mathfrak{X}_\omega$, we fix $J_Z$ imposing the following normalization condition

\[ \int_{\mathcal{F}} J_Z d\mu = 0. \]

On the other hand, denoting with $l_g$ the left translation on $U_\mathbb{R}$ defined by $g \in U_\mathbb{R}$, for $A \in \mathfrak{u}_\mathbb{R}$

\[ I(A) := \int_{\mathcal{F}} f_A d\mu = \int_{\mathcal{F}} f_A \circ l_g d\mu = \int_{\mathcal{F}} f_{g^{-1} \cdot A} d\mu = I(g^{-1} \cdot A). \]

Hence $I[A, B] = 0$, for all $B \in \mathfrak{u}_\mathbb{R}$. Thus, $I$ vanishes on the commutator ideal $[\mathfrak{u}_\mathbb{R}, \mathfrak{u}_\mathbb{R}]$, that coincides with $\mathfrak{u}_\mathbb{R}$. That is, $f_A$ satisfies the normalization condition (4.13).

From the preceding result together with Theorem \[18\] it follows the following theorem:

Theorem 19. Let $\mathcal{F}$ be the flag manifold $U_\mathbb{R}/H$, where $U_\mathbb{R}$ is a compact real form of the complex semisimple Lie group $G$. Let $\Psi$ be a 1-dimensional representation of $H$ and $\omega$ the presymplectic form determined by $\Psi$ and $I$. If $\mathcal{G}$ is any subgroup of $\mathcal{H}_\omega(\mathcal{F})$ satisfying (3.13) (ii), then

\[ \# \pi_1(\mathcal{G}) \geq \# \Psi(Z(U_\mathbb{R})). \]

The linear map $\Psi'$ can be extended to an element $\eta$ of $\mathfrak{u}_\mathbb{R}^*$ by putting $\Psi'|_1 = 0$. If the stabilizer of $\eta$ with respect to the coadjoint action is precisely $H$, then $\mathcal{F}$ is the coadjoint orbit associated to $\eta$ and $\omega$ defines a symplectic structure on $\mathcal{F}$, namely $-2\pi i \omega$, where $\omega$ is the Kirillov structure. Under this hypothesis, $\mathcal{H}_\omega(\mathcal{F})$ is the Hamiltonian group $\text{Ham}(\mathcal{F}, \omega)$ of the coadjoint orbit of $\eta$. In this case, as invariant measure $d\mu$ we can take the one determined by $\omega^n$, with $2n = \dim \mathcal{F}$. 

\[ \mathbf{□} \]
Corollary 20. Under the hypotheses of Theorem 19 if \( H \) is the stabilizer of \( \eta \), then

\[
\# \pi_1(\text{Ham}(F, \varpi)) \geq \# \Psi(Z(U_R)).
\]

Remark. For \( U_R = \text{SU}(2) \), and \( H = U(1) \subset \text{SU}(2) \), the corresponding flag manifold is \( \mathbb{C}P^1 \). For \([z_0 : z_1] \in \mathbb{C}P^1 \) with \( z_0 \neq 0 \), we put \( x + iy = z_1/z_0 \). It is straightforward to verify that the vector fields \( X_C \) and \( X_D \) defined by the matrices of \( \text{su}(2) \)

\[
C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & di \\ di & 0 \end{pmatrix},
\]

take at the point \((x = 0, y = 0)\) the values \( X_C = -c \partial_x \), \( X_D = d \partial_y \).

Let \( \Psi \) be the character of \( H \) defined by \( \Psi(\text{diagonal}(e^{ai}, e^{-ai})) = e^{ai} \). Then

\[
\varpi_{[1:0]}(X_C, X_D) = \frac{i}{2\pi} \Psi'( [C, D]) = -cd/\pi.
\]

By Corollary 20

\[
(4.14) \# \pi_1(\text{Ham}(\mathbb{C}P^1, \varpi)) \geq 2.
\]

We denote by \( \sigma \) the Fubini-Study form on \( \mathbb{C}P^1 \), then \( \sigma_{[1:0]}(\partial_x, \partial_y) = 1/\pi \). By the invariance of \( \sigma \) and \( \varpi \) under the action of \( \text{SU}(2) \), we conclude that \( \varpi = \sigma \). So, \( (4.14) \) is consistent with the fact that \( \pi_1(\text{Ham}(\mathbb{C}P^1, \sigma)) \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \) [5 page 52].

Remark. The result stated in Corollary 20 can be deduced using the invariancy of the action integral under loop deformations. More precisely, let \( \psi \) be the loop in \( \text{Ham}(F, \varpi) \) at the identity, generated by a time-dependent vector field \( Z_t \) on \( F \).

Here, we fix the Hamiltonian \( J_{Z_t} \) imposing the condition

\[
(4.15) \int_F J_{Z_t} \varpi^n = 0,
\]

for all \( t \). As in Subsection 4.1 we put \( x_t \) for the closed evaluation curve \( \psi_t(x) \), where \( x \) is a given point in \( F \). Since \( F \) is simply connected [17] page 33, there is a 2-chain \( C \) in \( F \) whose boundary is the curve \( \{x_t\} \). Thus, the right hand side of \( (4.7) \) is the exponential of the action integral around the loop \( \psi \) [18 19].

Let \( \kappa^* \) be a deformation of \( \psi \), as in Subsection 4.1. Since \( \kappa^*_t \) is a \( \varpi \)-symplectomorphism and

\[
(4.16) \int_F (J_{
\dot{\psi}_t} \circ \psi_t) \varpi^n = 0,
\]

where \( \dot{\psi}_t \) is the function defined in (4.8).

By Proposition 16 the function

\[
m := \int_0^1 (\dot{\psi}_t \circ \psi_t) dt : x \in F \longmapsto \int_0^1 \dot{\psi}_t(\psi_t(x)) dt,
\]

is constant. Hence, by (4.16)

\[
m \int_F \varpi^n = \int_F \left( \int_0^1 (J \circ \psi_t) dt \right) \varpi^n = \int_0^1 dt \left( \int_F (J \circ \psi_t) \varpi^n \right) = 0.
\]
Thus,

\[ m = \frac{1}{\Theta(\psi)} \frac{d\Theta}{ds} \bigg|_{s=0} = 0, \]

and we obtain the known fact that the action integral is invariant under deformations of the loop; that is, \( \Theta \) factors through the homotopy group, giving rise to the map \( (1.2) \). From \( (4.1) \), it follows that \( \Theta(\psi \star \tilde{\psi}) = \Theta(\psi)\Theta(\tilde{\psi}) \), where \( \star \) denotes the path product. Thus, \( (1.2) \) is a group homomorphism.

Now, using this invariance of \( \Theta \), we can deduce Corollary 20. Given \( g_1 \in Z(U_{\mathbb{R}}) \), let \( A \in \mathfrak{h} \), such that \( e^A = g_1 \). We set \( g_t = e^{tA} \); this path defines the corresponding loop \( \varphi \) in the Hamiltonian group and the respective Hamiltonian function is \( J_t = f_tA = t\Psi'(A) \). The curve \( \varphi_t(\bar{e}) \) reduces to \( \{\bar{e}\} \) and the corresponding value of \( \Theta \) given by \( (4.7) \) is \( \Theta(\varphi) = \Psi(g_1) \). Thus, the loops \( \varphi \) associated to two elements in \( Z(U_{\mathbb{R}}) \) on which \( \Psi \) takes different values, determine distinct elements in \( \pi_1(\text{Ham}(\mathcal{F}, \omega)) \).

5. Toric manifolds

In this section, we will consider symplectic toric manifolds and the group \( G \) will be the respective Hamiltonian group.

Let \( T \) be the torus \( (U(1))^n \), and \( \Delta \) a Delzant polytope in \( t^* \), whose mass center is denoted by \( \text{Cm}(\Delta) \). We denote by \( (M_\Delta, \omega_\Delta) \) the toric 2n-manifold, defined by the polytope \( \Delta \). We put \( \Phi \) for the corresponding moment map \( \Phi : M \to t^* \).

Since \( M_\Delta \) is simply connected, \( \text{Lie}(\text{Ham}(M_\Delta, \omega_\Delta)) \) consists of the vector fields \( Z \) such that \( d\iota_Z\omega_\Delta = 0 \). For such a \( Z \), we denote by \( f_Z \) the corresponding mean normalized Hamiltonian; i.e.

\[
(5.1) \quad df_Z = -\iota_Z\omega_\Delta, \quad \int_{M_\Delta} f_Z(\omega_\Delta)^n = 0.
\]

Let \( b \) be a vector of \( t \). We set \( X_b \) for the vector field on \( M_\Delta \) generated by \( b \) through the \( T \)-action; \( X_b \) is a Hamiltonian vector field and

\[
\text{d}\langle \Phi, b \rangle = -\omega_\Delta(X_b, \cdot).
\]

The respective normalized Hamiltonian is the function

\[
(5.2) \quad f_b := \langle \Phi, b \rangle - \langle \text{Cm}(\Delta), b \rangle,
\]

where

\[
(5.3) \quad \langle \text{Cm}(\Delta), b \rangle = \frac{\int_M \langle \Phi, b \rangle (\omega_\Delta)^n}{\int_M (\omega_\Delta)^n}.
\]

5.1. Quantizable manifolds. Identifying \( t^* \) with \( \mathbb{R}^n \), we can assume that \( * := (0, \ldots, 0) \) is a vertex of \( \Delta \). Under this assumption, the symplectic manifold \( (M_\Delta, \omega_\Delta) \) is quantizable if and only if the vertices of \( \Delta \) have integer coordinates.

In this subsection, we assume that \( (M_\Delta, \omega_\Delta) \) is quantizable. Then there exists a \( U(1) \) principal bundle \( L \xrightarrow{\pi} M_\Delta \) with a connection such that the curvature \( K \) projects a 2-form \( \omega \) on \( M_\Delta \), with

\[
(5.4) \quad \frac{i}{2\pi} \omega = \omega_\Delta.
\]
We will show that \( X \), the Lie algebra of the Ham\((M_\Delta, \omega_\Delta)\), “admits a lift” to \( L \), in the sense that it satisfies properties similar to the ones stated in Definition 1. For \( Z \in X \), we denote by \( a(Z) \) the map

\[
a(Z) : p \in L \mapsto -2\pi i f_Z(\text{pr}(p)) \in i \mathbb{R}
\]

For this map the following conditions hold:
(i) \( a(Z)(p\beta) = a(Z) \), for any \( \beta \in U(1) \).
(ii) From (5.4) together with (5.1), it follows \( Da(Z) = -K_1(Z^\sharp, .) \), where \( Z^\sharp \) is the horizontal lift of \( Z \).
(iii) The continuity of \( Z \mapsto a(Z) \) is guaranteed by conditions (5.1), which determine \( f \) uniquely.
(iv) Obviously, \( a(X_b)(p) = -2\pi i f_b(\text{pr}(p)) \).

As in Section 3, we define \( U(Z) = Z^\sharp + V_{a(Z)} \), where \( Z^\sharp \) is the horizontal lift of \( Z \) to \( L \). By the way, the vector field \( U(Z) \) is the operator assigned to the function \( f \) in Geometric Quantization (see [22, formula (3.29)]). Propositions 2 and 3 in Section 3 have an immediate translation to the present situation.

If \( \psi \) is the loop in Ham\((M_\Delta, \omega_\Delta)\) at the identity, generated by the family \( Z_t \in X \), then we have the corresponding isotopy \( F_t \) in \( L \) defined by the time-dependent vector field \( U(Z_t) \). As in Proposition 14, \( F_1 \) is the multiplication by a constant, say \( F_1(p) = p\Lambda(\psi) \).

Taking into account that \( H_1(M_\Delta, \mathbb{Z}) = 0 \), in the same way as in Proposition 15, we have:

**Proposition 21.** Let \( \psi \) be the loop in Ham\((M_\Delta, \omega_\Delta)\) generated by the family \( Z_t \in X \) and \( x \) an arbitrary point of \( M_\Delta \), then

\[
\Lambda(\psi) = \exp \left( 2\pi i \int_C \omega_\Delta - 2\pi i \int_0^1 f_{Z_t}(x_t) \, dt \right),
\]

where \( C \) is any 2-chain whose boundary is the curve \( \{ x_t = \psi_t(x) \} \).

As in the last Remark of Subsection 4.2, \( \Lambda \) defines a group homomorphism

\[
(5.5) \quad \Lambda : \pi_1(\text{Ham}(M_\Delta, \omega_\Delta)) \rightarrow U(1).
\]

Let \( b \) be a vector in the integer lattice of \( t \); it defines a loop \( \varphi_b \) in Ham\((M_\Delta, \omega_\Delta)\). The vertices of \( \Delta \) are the fixed points of the \( T \)-action, so \( \varphi_b(t)(*) = * \) for all \( t \). From (5.2), \( f_b(*) = -(\text{Cm}(\Delta), b) \), and Proposition 21 applied to the point \( * \) gives

\[
(5.6) \quad \Lambda(\varphi_b) = \exp \left( 2\pi i (\text{Cm}(\Delta), b) \right).
\]

By (5.5), we can state the following theorem:

**Theorem 22.** Let \( b, \hat{b} \) be vectors in the integer lattice of \( t \). If

\[
(5.7) \quad (\text{Cm}(\Delta), b - \hat{b}) \notin \mathbb{Z},
\]

then

\[
[\varphi_b] \neq [\varphi_{\hat{b}}] \in \pi_1(\text{Ham}(M_\Delta, \omega_\Delta)).
\]

As \((M_\Delta, \omega_\Delta)\) is quantizable, the coordinates of \( \text{Cm}(\Delta) \) are rational numbers, say \( \text{Cm}(\Delta) = (\frac{\alpha_1}{m}, \ldots, \frac{\alpha_n}{m}) \). From Theorem 22 we deduce the corollary:
Corollary 23. If $\text{g.c.d.}(r_1, \ldots, r_n) = 1$, then there exists $\varphi_{b_i}$, with $j = 1, \ldots, |r|$, such that for $i \neq j$
\[ [\varphi_{b_i}] \neq [\varphi_{b_j}] \in \pi_1(\text{Ham}(M_\Delta, \omega_\Delta)). \]

Remark. The values taken by the map $\Lambda$ admit an interpretation in terms of the integration of $T$-equivariant forms on $M_\Delta$. As $T$ is abelian, the map $f = \Phi - \text{Cm}(\Delta) : M \to \mathfrak{t}^*$ is $T$-equivariant. By [6,1](i), $\omega_\Delta - f$ is an $T$-equivariantly closed 2-form on $M_\Delta$ [23 page 111]. Given $q \in M_\Delta$ a fixed point for the $T$-action; let $m_{q_1, \ldots, m_{q_n}}$ denote the weights of the isotropy representation of $T$ on the tangent space $T_qM_\Delta$. The localization formula of equivariant cohomology [24] applied to the form $\alpha = \exp(2\pi i(\omega - f))$ gives, for a lattice vector $b$,
\[
\int_M \alpha(b) = \sum_q \frac{\exp(-2\pi i f_b(q))}{\prod_k m_{q,k}(b)},
\]
where $q$ runs over the set of fixed points of the $T$-action (assumed that all $m_{q,k}(b)$ are non zero).

Since the manifold is quantizable and $b$ is in the integer lattice, for two fixed points $q$ and $q'$ of the $T$-action, $f_b(q) - f_b(q') \in \mathbb{Z}$. So, by [3.10],
\[
\int_M \alpha(b) = \Lambda(\varphi_b) \sum_q \left( \prod_k m_{q,k}(b) \right)^{-1}.
\]

By means of the localization theorem for $S^1$-actions, it is possible to deduce sufficient conditions for two $S^1$-actions $\varphi_b$ and $\varphi_b'$ not to be homotopically equivalent through a homotopy consisting of $S^1$-actions (see [23, Theorem 5]). The conclusion of Theorem 22 is obviously stronger, although it is only applicable to quantizable manifolds.

Examples. For some toric manifolds $M_\Delta$, we proved in [11] the existence of infinite cyclic subgroups in $\pi_1(\text{Ham}(M_\Delta))$, generated by lattice vectors $b$ such that $(\Delta, b)$ is not a mass linear pair. Below, in the Examples 2 and 3, we consider some of those manifolds and show that there are lattice vectors $b$ that define homotopically non trivial loops in $\text{Ham}(M_\Delta)$, although $(\Delta, b)$ are mass linear pairs [12].

Example 1: The projective space $\mathbb{C}P^n$. The manifold $M_\Delta$ associated to the standard simplex
\[
\Delta = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_i x_i \leq 1, 0 \leq x_i \right\},
\]
is $\mathbb{C}P^n$. The mass center $\text{Cm}(\Delta) = \left( \frac{1}{n+1}, \ldots, \frac{1}{n+1} \right)$. Denoting $b_j = (j, 0, \ldots, 0)$, then $\langle \text{Cm}(\Delta), b_j \rangle = \frac{j}{n+1}$. By Theorem 22 the loops $\varphi_{b_i}$, for $i = 0, 1, \ldots, n$ determine $n + 1$ different elements in $\pi_1(\text{Ham}(M_\Delta, \omega_\Delta))$. This result coincides with the one stated in [3, Theorem 5], which was deduced by calculations of Maslov indices. Furthermore, that lower bound for the cardinal of $\pi_1(\text{Ham}(M_\Delta, \omega_\Delta))$ is consistent with the following two facts: $\pi_1(\text{Ham}(\mathbb{C}P^1)) = \mathbb{Z}/2\mathbb{Z}$, and $\text{Ham}(\mathbb{C}P^2)$ has the homotopy type of $\text{PU}(3)$ [8].

Example 2: Hirzebruch surfaces. Given $r \in \mathbb{Z}_{>0}$ and $\tau, \lambda \in \mathbb{R}_{>0}$ with $\sigma := \tau - r\lambda > 0$. The polytope $\Delta$ in $\mathbb{R}^2$ defined by the vertices
\[(0, 0), (0, \lambda), (\tau, 0), (\sigma, \lambda)\]
when $rb$ takes the value

$$\langle \text{Cm}(\Delta), b \rangle = \left( \frac{3r^2 - 3r\tau\lambda + r^2\lambda^2}{3(2r - r\lambda)}, \frac{3\lambda r - 2r^2\lambda^2}{3(2r - r\lambda)} \right).$$

In [11, Corollary 4.2] and using the homomorphism $\text{I}_n$, we proved that $\varphi_b$ generates an infinite cyclic subgroup in $\pi_1(\text{Ham}(M_\Delta, \omega_\Delta))$, if $b = (b_1, b_2)$ satisfies $rb_1 \neq 2b_2$. The point was that the homomorphism $\text{I}_n$ vanishes on $[\varphi_b]$ precisely when $rb_1 = 2b_2$; in other words, when $(\Delta, b)$ is a mass linear pair.

However, when $(M_\Delta, \omega_\Delta)$ is quantizable, if $rb_1 = 2b_2$, then the map (5.5) on $\varphi_b$ takes the value

$$\Lambda([\varphi_b]) = \exp(2\pi i (\text{Cm}(\Delta), b)) = \exp(\pi ib_1\tau).$$

Thus, if $b_1\tau$ is odd and $rb_1$ even, then the lattice vector $b = (b_1, \frac{rb_1}{2})$ determines a loop homotopically non trivial in the Hamiltonian group.

In the context of the toric quantizable manifolds, the homomorphisms $\text{I}$ and $\Lambda$ have "complementary properties", in the following sense: The map (1.4) can not detect nontrivial elements of finite order in $\pi_1(\text{Ham}(M_\Delta, \omega_\Delta))$, because it is an $\mathbb{R}$-valued group homomorphism. By contrast, $\Lambda$ can not distinguish elements $[\varphi_b]$ of finite order from those of order infinite. In fact, as $\text{Cm}(\Delta)$ has rational coordinates, for any lattice vector $b$ there is an integer $m$, such that $\Lambda(|\varphi_b|^m) = 1$ (see (5.6)).

**Example 3:** One point blow-up of $\mathbb{C}P^n$. Given $\tau, \lambda \in \mathbb{Z}_{>0}$ with $\sigma := \tau - \lambda > 0$, we denote by $\Delta$ the following truncated simplex

$$\Delta = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_i x_i \leq \tau, \ 0 \leq x_i, \ x_n \leq \lambda \right\}.$$ 

The corresponding toric manifold $M_\Delta$ is the one point blow-up of $\mathbb{C}P^n$.

The mass center $\text{Cm}(\Delta)$ is the point

$$\text{Cm}(\Delta) = \frac{1}{\tau^n - \sigma^n} \left( \frac{\tau^{n+1} - \sigma^{n+1}}{n+1} w - \lambda \sigma^n e_n \right),$$

where $w = (1, \ldots, 1)$ and $e_n = (0, \ldots, 0, 1)$. In [11], we proved that the loop $\varphi_b$, defined by a lattice vector $b = (b_1, \ldots, b_n)$, with $nb_n \neq \sum_{i=1}^{n-1} b_i$, determines an element $[\varphi_b]$ in $\pi_1(\text{Ham}(M_\Delta, \omega_\Delta))$ of infinite order. This property is consequence of the non-vanishing of the group homomorphism $\text{I}$ on $[\varphi_b]$.

When $(\Delta, b)$ is a linear pair; i.e., $nb_n = \sum_{i=1}^{n-1} b_i$, (5.9)

$$\langle \text{Cm}(\Delta), b \rangle = \left( \frac{\tau^{n+1} - \sigma^{n+1} - \lambda \sigma^n}{\tau^n - \sigma^n} \right) b_n.$$ 

Hence, if (5.9) is no integer, then $\varphi_b$ is not contractible.

5.2. **Some non quantizable manifolds.** Let us assume that $\Delta$ has $d$ facets. Denoting with $n_j \in \mathfrak{t}$ the normalized conormals to the facets, then the correspondence $e_j \mapsto n_j$, where $\{e_j\}$ is the standard basis of $\mathbb{R}^d$ induces a group homomorphism

$$\mathbb{R}^d / \mathbb{Z}^d \longrightarrow T = \mathbb{R}^n / \mathbb{Z}^n,$$
whose kernel will be denoted by $N$ (see [21, Chapter 1]). The standard action of $\mathbb{R}^d/\mathbb{Z}^d$ on $\mathbb{C}^d$ gives rise to an action of $N$ on that space, which preserves the splitting
\[
\bigoplus_{j=1}^d \mathbb{C}e_j = \mathbb{C}^d.
\]

Let $\mathcal{L}_j$ denote the line bundle over $M_{\Delta}$ defined by the representation of $N$ on $\mathbb{C}e_j$. We denote by $c_j$ the Chern class of $\mathcal{L}_j$. The classes $c_1, \ldots, c_d$ generate the ring $H^\ast(M_{\Delta}, \mathbb{C})$ (see [23, Proposition 9.8.7]).

Let us assume that the cohomology class $[\omega_{\Delta}]$ of the symplectic structure satisfies
\[
[\omega_{\Delta}] = r \sum_{j=1}^d n_j c_j,
\]
with $r \in \mathbb{R}$ and $n_j \in \mathbb{Z}$. Obviously, the sum in (5.10) is the Chern class of the line bundle
\[
\mathcal{L} := \bigotimes_{j=1}^d \mathcal{L}_j^{\otimes n_j}.
\]

The manifold $(M_{\Delta}, \omega_{\Delta}/r)$ is a quantizable and there exists a connection on $\mathcal{L}$ such that its curvature $K$ projects on $M_{\Delta}$ the form $-2\pi i \omega_{\Delta}/r$.

Let $\mathfrak{X}$ denote the algebra of Hamiltonian vector fields on $(M_{\Delta}, \omega_{\Delta})$. As at the beginning of the section, $f_Z$ will be normalized Hamiltonian function associated to $Z \in \mathfrak{X}$; i.e. $f_Z$ satisfies (5.1). The map
\[
a(Z) : p \in \mathcal{L} \mapsto -\frac{2\pi i}{r} f_Z(\text{pr}(p)) \in \mathbb{C},
\]
is obviously invariant under the right translations in $\mathcal{L}$ and $Da(Z) = -K(Z^t, \cdot)$.

Given $\psi$ a loop in $\text{Ham}(M_{\Delta}, \omega_{\Delta})$ at the identity generated by a family $Z_t$ of vector fields, as in Subsection 5.1, the corresponding gauge transformation $F_1$ is the multiplication by a constant $\Lambda(\psi)$, with
\[
\Lambda(\psi) = \exp \left( \frac{2\pi i}{r} \int_C \omega_{\Delta} - \frac{2\pi i}{r} \int_0^1 f_Z(x_t) dt \right),
\]
where $C$ is any 2-chain whose boundary is $\{x_t = \psi_t(x)\}$. Hence, we have the following proposition:

**Proposition 24.** Assumed that $[\omega_{\Delta}]$ satisfies (5.10). If $b$ and $\tilde{b}$ are lattice vectors such that
\[
\langle Cm(\Delta), b - \tilde{b} \rangle \notin r\mathbb{Z},
\]
then
\[
[\phi_b] \neq [\phi_{\tilde{b}}] \in \pi_1(\text{Ham}(M_{\Delta}, \omega_{\Delta})).
\]

**Remark.** Let $\Delta'$ denote the polytope $\frac{1}{r} \Delta$. The manifold $(M_{\Delta'}, \omega_{\Delta'})$ is quantizable and the hypothesis (5.7) of Theorem 22 applied to $(M_{\Delta'}, \omega_{\Delta'})$ is equivalent to (5.11).
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