Abstract—The beam alignment (BA) problem consists in accurately aligning the transmitter and receiver beams to establish a reliable communication link in wireless communication systems. Existing BA methods search the entire beam space to identify the optimal transmit-receive beam pair. This incurs a significant latency when the number of antennas is large. In this work, we develop a bandit-based fast BA algorithm to reduce BA latency for millimeter-wave (mmWave) communications. Our algorithm is named Two-Phase Heteroscedastic Track-and-Stop (2PHITS&S). We first formulate the BA problem as a pure exploration problem in multi-armed bandits in which the objective is to minimize the required number of time steps given a certain fixed confidence level. By taking advantage of the correlation structure among beams that the information from nearby beams is similar and the heteroscedastic property that the variance of the reward of an arm (beam) is related to its mean, the proposed algorithm groups all beams into several beam sets such that the optimal beam set is first selected and the optimal beam is identified in this set after that. Theoretical analysis and simulation results on synthetic and semi-practical channel data demonstrate the clear superiority of the proposed algorithm vis-à-vis other baseline competitors.

Index Terms—Beam alignment, beam selection, mmWave communication, multi-armed bandits.

I. INTRODUCTION

In millimeter-wave (mmWave) communications, the beams at both the transmitter and receiver are narrow directional. The beam alignment (BA) problem consists in ensuring that the transmitter and receiver beams are accurately aligned to establish a reliable communication link [2], [3].

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An optimal transmitter (receiver) beam or a pair of transmitter and receiver beam is required to be selected to maximize the overall capacity, throughput, SNR or diversity. To achieve this goal, a naïve exhaustive search method scans all the beam space and hence causes significant BA latency. Specifically, there are two fundamental challenges to implement BA: (1) the amount of additional channel state information (CSI) corresponding to each beam (pair) is large; (2) the frequency of the channel measurement [4], [5] is high. These challenges become even more difficult to overcome when a large number of antennas are employed at both of the transmitter and receiver. Moreover, the change of antenna orientation, the effect of transmitter/receiver mobility, and the dynamic nature of the wireless channel will result in previously found optimal beam pairs to be suboptimal over time, which further exacerbates the latency problem [6], [7].

Therefore, the design of fast BA algorithms is of paramount importance, and has stimulated intense research interest. By utilizing the sparsity of the mmWave channel, Marzi et al. [8] incorporated compressed sensing techniques in the BA problem to reduce the beam measurement complexity. Wang et al. [9] developed a fast-discovery multi-resolution beam search method, which first probes the wide beam and continues to narrow beams until the optimal arm is identified. However, the beam resolution needs to be adjusted at each step. Besides, various forms of side information, e.g., location information [10], out-of-band measurements [11], and dedicated short-range communication [12], have also been used to reduce the required number of probing beams.

Due to its inherent ability to balance between exploration and exploitation, multi-armed bandit (MAB) theory has been recently utilized in wireless communication field, such as channel access in cognitive network [13], channel allocation [14] and adaptive modulation/coding [15], etc., and harnessed to address the BA problems; see [16], [17], [18], [19], [20], [21], [22]. In a single-user MIMO system, the work [16] employed the canonical upper confidence bound (UCB) bandit algorithm in beam selection, where the instantaneous full CSI is not required. The work [17] applied linear Thompson sampling (LTS) to address the beam selection problem. For mmWave vehicular systems, Sim et al. [18] developed an online learning algorithm to address the problem of beam selection with environment-awareness based on contextual bandit theory. The proposed method explores different beams over time while accounting for contextual information (i.e., vehicles’ direction of arrival). Similarly, Hashemi et al. [19] proposed to maximize the directivity gain...
(i.e., received energy) of the BA policy within a time period using the contextual information (i.e., the inherent correlation and unimodality properties), and formulate the BA problem as contextual bandits. Wu et al. [20] provided a method to quickly and accurately align beams in multi-path channels for a point-to-point mmwave system, which takes advantage of the correlation structure among beams such that the information from nearby beams is extracted to identify the optimal beam, instead of searching the entire beam space. In [21], Va et al. used a UCB-based framework to develop the online learning algorithm for beam pair selection and refinement, where the algorithm first learns coarse beam directions in some predefined beam codebook, and then fine-tunes the identified directions to match the peak of the power angular spectrum at that position. Hussain et al. [22] also designed a novel beam pair alignment scheme based on Bayesian multi-armed bandits, with the goal of maximizing the alignment probability and the data-communication throughput.

In this paper, different from existing methods, we first formulate the BA problem as a pure exploration problem in the theory of MABs, called bandit BA problem, with the objective of minimizing the required time steps to find the optimal beam pair with a prescribed probability of success (confidence level). Second, we derive a lower bound on the expected time complexity of any algorithm designed to solve the BA problem. Third, we present a bandit-based fast BA algorithm, which we name Two-Phase Heteroscedastic Track-and-Stop (2PHT&S). We do so by exploiting the correlation among beams and the heteroscedastic property that the variance of the reward of an arm (beam) is linearly related to its mean. Instead of searching over the entire beam space, the proposed algorithm groups all beams into several beam sets such that the optimal beam set is first selected and the optimal beam is identified within this set during the second phase. We derive an upper bound on the time complexity of the proposed 2PHT&S algorithm. Finally, extensive numerical results demonstrate that 2PHT&S significantly reduces the required number of time steps to identify the optimal beam compared to existing algorithms in both simulated and semi-practical channel data scenarios.

The remainder of this paper is organized as follows. The system model and problem setup are presented in Section II. Section III proposes our bandit-based fast BA algorithm and its time complexity upper bound. Simulation results are given in Section IV, and Section V concludes this paper.

II. PROBLEM FORMULATION

In this section, we first describe the system model and propose our problem setup, after that we provide a generic lower bound of the time complexity of the problem.

A. System Model

As shown in Fig. 1, we consider a massive mmWave MISO system, where a base station (BS) equipped with $N$ transmit antennas serves a single-antenna user. According to [23], each transmitting beam is selected from a predefined analog beamforming codebook $C$ of size $K$, which can be defined as

$$C \triangleq \{ \mathbf{f}_k = a(-1 + 2k/K) \mid k = 0, 1, 2, \ldots, K-1 \}$$

with $a(\cdot)$ denoting the array response vector. For a typical uniform linear array, it holds that $a(x) = \frac{1}{\sqrt{N}} [1, e^{j\frac{2\pi}{N}dx}, e^{j\frac{2\pi}{N}2dx}, \ldots, e^{j\frac{2\pi}{N}(N-1)dx}]^\mathsf{H} \in \mathbb{C}^{N \times 1}$, where $\lambda$ is the wavelength and $d$ is the antenna spacing. Note that if $K \leq N$, the beams in the codebook $C$ are linearly independent of one another. Based on channel measurement studies done in [24], [25], the mmWave channel follows a multipath channel model, and the number of propagation paths is small.

We consider a quasi-static channel, where the channel stays unchanged for a period covariance interval which consists of $T$ time slots. As shown in Fig. 2, each covariance interval can be divided into two phases, the BA phase (which consists of $T_B$ time slots) and the data transmission phase (which consists of $T_D = T - T_B$ time slots). With the selected beam $\mathbf{f}$, the effective achievable rate (EAR) of each covariance interval

$$R_{\text{eff}} := \left(1 - \frac{T_B}{T_D}\right) \log \left(1 + \frac{|\mathbf{h}^\mathsf{H}\mathbf{f}|^2}{\sigma^2}\right)$$

is widely accepted as a metric to measure the throughput performance, where $p$ and $\sigma^2$ represent the transmit power and noise variance respectively. We observe that the throughput performance increases with the decreasing $T_B/T_D$ and increasing $|\mathbf{h}^\mathsf{H}\mathbf{f}|^2$. As such, the time allocated for selecting the optimal beam $\mathbf{f}^* = \arg \max_{\mathbf{f} \in C} |\mathbf{h}^\mathsf{H}\mathbf{f}|^2$ should be minimized for higher system throughput, which is the motivation of this work.

In the BA phase, the BS selects one beam from the codebook $C$ and transmits pilot signals to the user with this beam. Without loss of generality, we set the pilot signal to be $1$. Then the received power can be expressed as

$$R(\mathbf{f}_k) = |\sqrt{\eta}\mathbf{h}^\mathsf{H}\mathbf{f}_k + n|^2 = p|\mathbf{h}^\mathsf{H}\mathbf{f}_k|^2 + |n|^2 + 2\sqrt{\eta} Re(\mathbf{h}^\mathsf{H}\mathbf{f}_k n^*)$$

where $n^*$ represents the complex conjugate of the complex Gaussian noise $n \sim \mathcal{CN}(0, \sigma^2)$. Note that the BS can observe the received power from the user through a feedback link, and this assumption is also adopted in other studies such as [20], [21]. We can see that $R(\mathbf{f}_k)$ is a random variable which is the sum of a Gamma random variable $|n|^2 \sim \mathcal{G}(1,1/\sigma^2)$ and a Gaussian random variable $p|\mathbf{h}^\mathsf{H}\mathbf{f}_k|^2 + 2\sqrt{\eta} Re(\mathbf{h}^\mathsf{H}\mathbf{f}_k n^*) \sim \mathcal{N}(p|\mathbf{h}^\mathsf{H}\mathbf{f}_k|^2, 2p|\mathbf{h}^\mathsf{H}\mathbf{f}_k|^2\sigma^2)$. Fig. 3 shows the average received power associated with different beams over the codebook space when $K = 120$, $N = 64$, $p = 26$ dBm and the power

![Fig. 1. A mmWave massive MISO system.](image-url)
the non-line-of-sight (NLoS) path.) and the power of line-of-sight (LoS) path is around 3 dB larger than that of
\[ \mu f \]
the beam
\[ \text{similar.} \]
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value is the difference between the noise variances. Since the
\[ \text{larger as the increase of the noise variance, and the improved} \]
paths. It can be observed that all the received powers become
\[ \text{of LoS path is around 3 dB larger than those of the NLoS} \]
paths. It can be observed that all the received powers become larger as the increase of the noise variance, and the improved value is the difference between the noise variances. Since the noise variance is much smaller than the transmit power, the variable \( R(f_k) \) is approximately a Gaussian random variable with mean
\[ \mu_k = \mu |H^H f_k|^2 \text{ and variance } \sigma_k^2 = 2|\mu| |H^H f_k|^2 \sigma^2, \]
i.e.,
\[ r_k = \mu |H^H f_k|^2 + 2 \sqrt{\mu} R(H^H f_k n^*). \]
The optimal beam is defined as the beam that has the largest value of \( r_k \), i.e.,
\[ k^* = \arg \max_k \mu |H^H f_k|^2. \]
Let \( J \) be the correlation length of arms, which is related to the number of beams \( N \) and the size of the codebook \( K \) as follows:
\[ J = 2 \left[ \frac{K}{N} \right] - 1. \]
Then the received power using the beam \( f_k \) and the nearby beam \( f_i, |k - i| \leq J/2 \) are similar. A new \( \downarrow \)-lower resolution beam codebook \( C_{(J)} \) can be constructed by grouping the nearby beams in the codebook \( C \), i.e.,
\[ C_{(J)} \triangleq \left\{ b_g = \frac{\sum_{k=J(g-1)+1}^{Jg} f_k}{\sum_{k=J(g-1)+1}^{Jg} |f_k|^2} : g = 0, 1, \ldots, G - 1 \right\}, \]
where the beams \( b_g \)'s are normalized (to unit \( \ell_2 \) norm) to mimic reality in a phased array. If we choose a beam \( b_g \) in the codebook \( C_{(J)} \), the received power can be expressed as
\[ R_g = \mu |H^H b_g|^2 + 2 \sqrt{\mu} R(H^H b_g n^*). \]
Since the received powers associated with the nearby beams in the codebook \( C \) have means that are close to one another, \( R_g \) has large mean if \( r_k, k \in \{(g-1)J + 1, \ldots, gJ\} \) have large means.

B. Problem Setup

To introduce the idea of bandit to address the BA problem, we consider a bandit model where the beam \( f_k \) is regarded as the base arm \( k \), and the received power using this beam is regarded as the reward of the base arm \( k \). The means of the rewards when arms are pulled have the following property:

Property 1: Let \( \mu = (\mu_1, \ldots, \mu_K) \), and let \( \mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots \geq \mu_K \) be the sorted sequence of means. Then we assume that these means have the following properties:

1. The means of the reward associated with arms \( k \) and \( i \), where \(|i - k| \leq J/2\), are close.
2. The rewards are sparse. In particular, there are only \( LJ \) arms that have high rewards; the other \( K - LJ \) arms have rewards that are close to zero.
3. The variance of the reward of an arm is related to its mean as follows:
\[ \sigma_k^2 = 2\mu_k \sigma^2. \]

As shown in Fig. 3, Property 1.1 holds because the beam codebook is designed such that beams that are nearby have rewards that are close. Property 1.2 is due to the fact that there are only a few main paths in the considered channel model, which has been corroborated by studies in channel measurements [26]. Finally, Property 1.3 holds because the mean of reward associated with base arm \( k \) follows a heteroscedastic Gaussian distribution \( N(\mu_k, 2\mu_k \sigma^2) \) (see (1)). Furthermore, in this work, we focus on a challenging case with large noise, which satisfies that
\[ \sigma^2 > \max_{k \in [K]} \frac{(\mu_1 - \mu_k)^2}{4 \mu_k - 2 \mu_1 - 2 \mu_k \ln \left( \frac{\mu_k}{\mu_1} \right)}, \]
and this condition is assumed to hold in our theoretical analyses. In the simulations, we also numerically check that this condition holds.

Since a large number of base arms have mean rewards that are close to zero (Property 1.3), it is clearly a waste of BA overhead to search for the optimal arm among all the base arms. To overcome this problem, we propose to group the base arms by utilizing Property 1.1 which says that the means of the reward associated with nearby arms are similar. As such, we can formulate the BA problem as a novel MAB problem, called the bandit BA problem.
In a bandit problem with $K$ base arms, the base arm $k$ is associated with the beam $f_k$. We let $[K] = \{1, \ldots, K\}$ and $\binom{[K]}{\leq J}$ be the set of all non-empty consecutive tuples of length $\leq J$ whose element belongs to the set $[K]$, e.g., if $J = 2$ and $K = 6$,

$$\binom{[6]}{\leq 2} = \{\{1\}, \{1, 2\}, \{2\}, \{2, 3\}, \{3\}, \{3, 4\}, \{4\}, \{4, 5\}, \{5\}, \{5, 6\}, \{6\}\},$$

and we consider each tuple as a $(K, J)$-super arm. Moreover, the super arm associated with $b_j \in C_j$ is a subset of $\binom{[K]}{\leq J}$, e.g., $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\} \subset \binom{[6]}{\leq 2}$. At time step $t$, we choose an action (or a $(K, J)$-super arm) $A(t) \in \binom{[K]}{\leq J}$, and observe the reward $R(t)$ which is related to the base arms included in $A(t)$, i.e.,

$$R(t) = F\left(\sum_{k \in A(t)} f_k, p, h, n_t \right),$$

(3)

where $F(f, p, h, n) = p|h|^2 + 2\sqrt{pR(hn^2)}$, $n$ is a complex Gaussian random variable with mean zero and variance $\sigma^2$. We can conclude that for an arbitrary $(K, J)$-super arm $A \in \binom{[K]}{\leq J}$, the empirical mean $R_A$ also follows a heteroscedastic Gaussian distribution, i.e., $R_A \sim \mathcal{N}(\mu_A, 2\mu_A\sigma^2)$, where $\mu_A = p|h|^2 \sum_{k \in A} f_k|^2$.

To identify the optimal base arm, an agent uses an algorithm $\pi$ that decides which super arms to pull, the time $\tau^\pi$ to stop pulling, and which base arm $k^\pi$ to choose eventually. An algorithm consists of a triple $\pi := (\pi_1, \pi_2, \pi_3, J)$ in which:

- a sampling rule $\pi_1$ determining the $(K, J)$-super arm $A(t) \in \binom{[K]}{\leq J}$ to pull at time step $t$ based on the observation history and the arm history $\{A(1), R(1), A(2), R(2), \ldots, A(t-1), R(t-1)\}$;
- a stopping rule leading to a stopping time $\tau^\pi$ satisfying $\mathbb{P}(\tau^\pi < \infty) = 1$;
- a recommendation rule $\pi_3$ that outputs a base arm $k^\pi \in [K]$.

We define the time complexity of $\pi$ as $\tau^\pi$. In the fixed-confidence setting, a maximum risk $\delta \in (0, 1)$ is fixed. We say an algorithm $\pi$ is $(\delta, J)$-approximately correct (PAC) if $\mathbb{P}(\tau^\pi = k^\pi) \geq 1 - \delta$. The purpose is to design a $(\delta, J)$-PAC algorithm $\pi$ that minimizes the expected stopping time $\mathbb{E}[\tau^\pi]$.

### C. Lower Bound on the Time Complexity

Denote by $\nu$ a $K$-armed heteroscedastic Gaussian bandit instance, $\nu = \{\mathcal{N}(\mu_1, 2\mu_1\sigma^2), \ldots, \mathcal{N}(\mu_K, \mathcal{N}(\mu_1, 2\mu_1\sigma^2))\}$. Let $\mathcal{V}$ represent the set of $K$-armed heteroscedastic Gaussian bandit instances such that each bandit instance $\nu \in \mathcal{V}$ has a unique optimal arm: for each $\nu \in \mathcal{V}$, there exists an arm $A^*(\nu)$ such that $\mu^\ast_{A^*(\nu)} \geq \max_k \mu^\ast_k : k \neq A^*(\nu)$. Define $\mathcal{W}_K = \{w \in \mathbb{R}^K_+ : \sum_{k=1}^K w_k = 1\}$, $\mathcal{V} = \{u \in \mathcal{V} : A^*(u) \neq A^*(\nu)\}$, and $T_k(t)$ as the times the base arm $k$ is pulled until time step $t$. Moreover, the KL divergence between two heteroscedastic Gaussian distributions, i.e., $\mathcal{N}(\mu_1, 2\mu_1\sigma^2)$ and $\mathcal{N}(\mu_2, 2\mu_2\sigma^2)$, can be calculated as

$$D_{\text{HG}}(\mu_1, \mu_2) = \frac{1}{2} \ln \left(\frac{\mu_1}{\mu_2} + \frac{\mu_2}{\mu_1} + \frac{\mu_1 - \mu_2}{4\mu_1\sigma^2} - 1\right).$$

According to Theorem 1 of [27], we can obtain the general lower bound of this problem as follows.

**Theorem 1:** For any $(\delta, J)$-PAC algorithm where $\delta \in (0, 1)$, it holds that

$$\mathbb{E}_{\nu}[T] \geq c^\ast(\nu) \ln \left(\frac{1}{\delta}\right),$$

where

$$c^\ast(\nu)^{-1} = \sup_{w \in \mathcal{W}_K} \inf_{u \in \mathcal{V}(\nu)} \left(\sum_{k=1}^K w_k D_{\text{HG}}(\mu^u_k, \mu^u_k)^2\right).$$

### III. OUR ALGORITHM: 2PHT&S

In this section, we first state the performance baseline of this work, i.e., the original Track-and-Stop (T&S) algorithm [27], then we elaborate the design of the proposed 2PHT&S. We then upper bound its time complexity in Theorem 2.

#### A. A Baseline Algorithm

Note that the original Track-and-Stop (T&S) [27] is the state-of-the-art best arm identification algorithm with fixed confidence for exponential bandits, which can be modified to solve the bandit BA problem with low sample complexity, resulting in a fast BA solution.

Let $\nu$ represent a $K$-armed Gaussian bandit instance, $\nu = \{\mathcal{N}(\mu'_1, \sigma'_1^2), \mathcal{N}(\mu'_2, \sigma'_2^2), \ldots, \mathcal{N}(\mu'_K, \sigma'_K^2)\}$. Given the fixed $\delta$, the time complexity of T&S, $\mathbb{E}[T^{\text{T&S}}]$ satisfies

$$\lim_{\delta \to 0} \frac{\mathbb{E}[T^{\text{T&S}}]}{\log(1/\delta)} = T^\ast(\nu),$$

where

$$T^\ast(\nu)^{-1} := \sup_{w \in \mathcal{W}_K} \inf_{u \in \mathcal{V}} \left(\sum_{k=1}^K w_k D(\mathcal{N}(\mu'_k, \sigma'_k^2), \mathcal{N}(\mu^u_k, \sigma^u_k^2))^2\right).$$

and the KL-divergence $D(\mathcal{N}(\mu'_k, \sigma'_k^2), \mathcal{N}(\mu^u_k, \sigma^u_k^2)) = \ln \left(\frac{\sigma^u_k}{\sigma'_k} + \frac{(\sigma'_k)^2 + (\mu'_k - \mu^u_k)^2}{2\sigma^u_k^2}\right) - \frac{1}{2}$.

Furthermore, when we apply T&S to the $K$-armed heteroscedastic Gaussian bandit instance $\nu \in \mathcal{V}$ under consideration, we have

$$\lim_{\delta \to 0} \frac{\mathbb{E}[T^{\text{T&S}}]}{\log(1/\delta)} \leq T^u_\ast(\nu),$$

where

$$T^u_\ast(\nu) = \frac{8\mu'_u(\nu)^2(\sigma^u)^2}{\Delta^2_{\nu, \min}} + \sum_{k \neq A^*(\nu)} \frac{8\mu'_u(\nu)^2(\sigma^2)^2}{\Delta^2_{\nu, k}},$$

and the gaps are $\Delta_{\nu, k} := \mu^u_k - \mu^u_k, k \neq A^*(\nu)$ and $\Delta^2_{\nu, \min} := \min_{k \neq A^*(\nu)} \mu^u_k - \mu^2_k$. 

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We describe them in the following. “grouped” reward can be observed in one time step. neighboring beams can be grouped and the corresponding by existing algorithms, i.e., Property 1 and the fact that exploit the prior knowledge which have not been considered of the BA problem. The main idea of the proposed algorithm is to achieve a smaller time complexity by leveraging the structure.

For our

In Phase II, we construct a base arm set (including two

According to (3), the reward associated with super arm $S_g$ can be written as $R_g(t) = F(\sum_{k \in S_g} \bar{f}_k, p, h, n_t)$. Second, we search for the optimal super arm, $g^* = \arg\max_{g \in [G]} \mathbb{E}[R_g(t)]$, with a high probability of at least $1 - \delta_1$, where $\delta_1$ denotes the maximum allowable risk in Phase I.

In Phase II, we construct a base arm set (including two super arms) and select the optimal base arm from this base arm set. Considering the case shown in Fig. 5, where the optimal base arm lies in the right edge of the super arm $S_g$, and the second optimal base arm lies in the left edge of the super arm $S_{g+2}$, the mean of the reward of the super arm $S_{g+1}$ may be higher than that of the super arm $S_g$. This observation results in a higher difficulty of identifying the optimal super arm. Therefore, to improve the selection accuracy, we construct a base arm set, i.e., the combination of the optimal super arm and either one of its (left or right) neighbors that has the larger mean. Next, we select the optimal base arm in this base arm set with a certain fixed confidence probability of at least $1 - \delta_2$.

Given the fixed maximum risk $\delta$, risks $\delta_1$ and $\delta_2$ should satisfy the constraint $\delta = \delta_1 + \delta_2$. The proposed 2PHT&S is detailed in Algorithm 1, where $\hat{\mu}^s = [\hat{\mu}_1^b, \ldots, \hat{\mu}_J^b]$ and $\hat{\mu}^b = [\hat{\mu}_{S_1}^b(1), \ldots, \hat{\mu}_{S_J}^b(2,J)]$ represent the empirical means of super arms and base arms included in $S_i$, respectively. In both of Phase I and Phase II of 2PHT&S, an improved T&S algorithm exploiting the heteroscedastic property, which is referred to as Heteroscedastic Track-and-Stop (HT&S), is leveraged to search for the optimal super arm and the optimal base arm, respectively.

We now introduce the HT&S algorithm. Define a set of $I \in \mathbb{N}$ super arms $\mathcal{I} = \{S_1, \ldots, S_I\}$ where the reward $R_i(t)$ of super arm $S_i$ is related to the beams $k \in S_i$ as follows: $R_i(t) = F(\sum_{k \in S_i} \bar{f}_k, p, h, n_t)$. Given $\mathcal{I}$ and the fixed $\delta$, HT&S outputs all the empirical means, $\hat{\mu}^s = [\hat{\mu}_1^s, \ldots, \hat{\mu}_J^s]$, and the required number of time steps $\tau$. Let $A(t)$ represent the index of the super arm pulled in time step $t$, i.e., in time step $t$ the super arm $S_{A(t)}$ is selected by the sampling rule.

1Note that since the two phases are independent, the probability of selecting the optimal base arm can be expressed as $(1 - \delta_1)(1 - \delta_2) = 1 - \delta_1 - \delta_2 + \delta_1\delta_2$. Then, the error probability is $\delta_1 + \delta_2 - \delta_1\delta_2$, which is less than $\delta_1 + \delta_2$ since the two phases are independent. For simplicity, for the given fixed confidence $\delta$, we set $\delta_1 + \delta_2 = \delta$. 

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**Fig. 4.** Illustration of the proposed algorithms.

**Fig. 5.** Illustration of a case that the optimal base arm does not lie in the optimal super arm.
We observe the following reward associated with the super arm $S_{A(t)}$ is
\[
R(t) = F \left( \sum_{k \in S_{A(t)}} f_k, p, h, c(t) \right).
\]
Accordingly, we update the number of times each arm is selected up to time step $t$ and their empirical mean as
\[
T_i(t) = \sum_{a=1}^{t} \mathbb{1} \{ A(a) = i \} \quad \text{and} \quad \hat{\mu}_i(t) = \frac{1}{T_i(t)} \sum_{a=1}^{t} R(a) \mathbb{1} \{ A(a) = i \}. \tag{7}
\]
The search is terminated when the stopping rule (delineated in (8) below) is satisfied. Define the heteroscedastic Gaussian bandit instance at time step $t$ as $\tilde{\nu}_t = \{ N(\hat{\mu}_1(t), 2\hat{\mu}_1(t)\sigma^2), \ldots, N(\hat{\mu}_K(t), 2\hat{\mu}_K(t)\sigma^2) \}$. The stopping rule and sampling rule, which exploit the heteroscedastic property, form the core of HT&S:

1) **Stopping Rule**: We choose the threshold as $\beta(t, \delta, \alpha) = \ln(\alpha t / \delta)$, and the stopping rule as
\[
\tau_0 = \min \{ t \in \mathbb{N} : Z(t) \geq \beta(t, \delta, \alpha) \}, \tag{8}
\]
where $Z(t)$ is defined in (9), shown at the bottom of the page, and
\[
q_i(t) = \frac{T_A(\nu_i(t)) + T_i(t)}{T_A(\nu_i(t))\hat{\mu}_i(t) + T_i(t)\mu_A^{\star}(\nu_i(t))\hat{\mu}_i(t)}, \quad A'(\nu_i) = \arg \max_{i \in [t]} \{ \hat{\mu}_i(t) : i \neq A'(\nu_i) \}. \tag{10}
\]
Please refer to (16)–(17) in Appendix C for more details of the derivation of (a) in (9).

**Lemma 1**: Let $\nu$ be a heteroscedastic Gaussian bandit instance. Let $\delta \in (0, 1)$ and $\alpha > 1$. Using the above stopping rule in (8) with the threshold $\beta(t, \delta, \alpha) = \ln(\alpha t / \delta)$ ensures that
\[
\mathbb{P}_n(\tau_0 < \infty, A_{T_0} \neq A^{\star}(\nu)) \leq \delta.
\]

**Proof**: Please refer to Appendix A. □

2) **Sampling Rule**: The sampling rule can be summarized by $Q(t)$ defined in (11), shown at the bottom of the next page, where $\tilde{w}(t) \in \mathcal{W}_I$ is the set of parameters to be updated at time step $t$. Define
\[
f_{Y,\tilde{x},t}(x_i) := \frac{1}{2} \ln \left( \frac{\hat{\mu}_i(t) + x_i\mu_A^{\nu}(\nu)(t)}{(1 + x_i)\hat{\mu}_i(t)} \right) + \frac{x_i}{2} \ln \left( \frac{\hat{\mu}_i(t) + x_i\mu_A^{\nu}(\nu)(t)}{(1 + x_i)\mu_A^{\nu}(\nu)} \right) + \frac{x_i(\mu_A^{\nu}(\nu)(t) + \hat{\mu}_i(t))(\mu_A^{\nu}(\nu)(t) - \hat{\mu}_i(t))^2}{4\sigma^2(\hat{\mu}_i(t) + x_i\mu_A^{\nu}(\nu)(t))},
\]
and
\[
f_{X,\tilde{x},t}(y) := \frac{1}{\sum_{a=1}^{t} f_{X,a,t}(y)} f_{X,a,t}(y).
\]
\[
f_{Y,\tilde{x},t}(x_i) = \frac{1}{\sum_{a=1}^{t} f_{X,a,t}(y)} \sum_{a=1}^{t} f_{X,a,t}(y), \tag{12}
\]
\[
Z(t) = \min_{\nu_i \neq \hat{\nu}_i} \inf_{u \in \mathcal{A}(\nu_i)} \left\{ T_A(\nu_i(t))D_HG(\hat{\mu}_A^{\nu}(\nu_i(t)), \mu_A^{\nu}(\nu_i)) + T_i(t)D_HG(\hat{\mu}_i(t), \mu_i^{\nu}) \right\}, \tag{9}
\]
where \( y^*(t) \) is the unique solution of the equation

\[
\sum_{i \neq A^*(\mu)} y_i f_{X_i,t}(y) - f_{X,t}(y) = 1.
\]

Note that the choice of \( \tilde{w}^*(t) \) is determined according to the optimization problem in (4); see the technical details in Appendix C.

It is worth noting that the stopping rule and sampling rule of HT&S are different from those of original T&S, and both of them are derived from the lower bound given in Theorem 1 based on the heteroscedastic property.

### C. Time Complexity Analysis of 2PHT&S

**Theorem 2:** Let

\[
s = \{N(\mu_0^a, 2\mu_1^a \sigma_1^2), \ldots, N(\mu_G^a, 2\mu_G^a \sigma_2^2)\} \text{ and } b = \{N(\mu_{S_1(1)}^b, 2\mu_{S_1(1)}^b \sigma_1^2), \ldots, N(\mu_{S_1(2,J)}^b, 2\mu_{S_1(2,J)}^b \sigma_1^2)\}
\]

be heteroscedastic Gaussian bandit instances studied in Phases I and II of 2PHT&S, where \( \mu_g^a = p|\ln|\sum_{k \in S_g} f_k|^2 |\).

Using the proposed stopping rule and the sampling rule, we have

\[
\limsup_{\delta \to 0} \frac{\mathbb{E}[\tau]}{\ln(1/\delta)} \leq c_u^{\ast}(s) + c_u^{\ast}(b),
\]

where

\[
c_u^{\ast}(s) = \frac{\mu_A^s(s)}{2 \sum_{i=1}^{G} \mu_i^s} \ln \left( \frac{2 \mu_A^s(s)}{\mu_A^s + \mu_A^s(s)} \right)
+ \frac{\mu_A^s(s)}{2 \sum_{i=1}^{G} \mu_i^s} \ln \left( \frac{2 \mu_A^s(s)}{\mu_A^s + \mu_A^s(s)} \right)
+ \frac{(\mu_A^s(s) - \mu_A^s(s))^2}{8 \sigma^2 \sum_{i=1}^{G} \mu_i^s} \left( \frac{\mu_A^s + \mu_A^s(s)}{2 \sum_{i=1}^{G} \mu_i^s} \right)^{-1},
\]

and

\[
c_u^{\ast}(b) = \frac{\mu_B^b(b)}{2 \sum_{i \in S_1} \mu_i^b} \ln \left( \frac{2 \mu_B^b(b)}{\mu_B^b + \mu_B^b(b)} \right)
+ \frac{\mu_B^b(b)}{2 \sum_{i \in S_1} \mu_i^b} \ln \left( \frac{2 \mu_B^b(b)}{\mu_B^b + \mu_B^b(b)} \right)
+ \frac{(\mu_B^b(b) - \mu_B^b(b))^2}{8 \sigma^2 \sum_{i \in S_1} \mu_i^b} \left( \frac{\mu_B^b + \mu_B^b(b)}{2 \sum_{i \in S_1} \mu_i^b} \right)^{-1}.
\]

**Proof:** We first obtain the upper bound of the time complexity achieved by HT&S in Lemma 2.

**Lemma 2:** For a heteroscedastic Gaussian bandit instance \( \nu \), Algorithm 2 ensures that for any \( \alpha \geq 1, t_0 > 0, \epsilon > 0 \) and \( \delta > 0 \), there exists three constants \( \Gamma_1 = \Gamma_1(\epsilon, t_0, 1) \), \( \Gamma_2 = \Gamma_2(\nu, \epsilon) \) and \( \Gamma_3 = \Gamma_3(\nu, \epsilon) \) such that

\[
\mathbb{E}[\tau] \leq \Gamma_1 + \alpha c_u^{\ast}(\nu) \left[ \ln \left( \frac{\alpha c_u^{\ast}(\nu)}{\delta} \right) + \ln \left( \frac{\alpha c_u^{\ast}(\nu)}{\delta} \right) \right]
+ \sum_{T=1}^{\infty} \Gamma_2 T \exp \left( -\Gamma_3 T^{1/8} \right).
\]

**Proof:** See the proof in Appendix D. \( \Box \)

Using HT&S in Phase I and Phase II, the overall required number of time steps of Algorithm 1, i.e., \( \tau \), can be expressed as

\[
\mathbb{E}[\tau] = \mathbb{E}[\tau_1] + \mathbb{E}[\tau_2] \\
\leq \alpha_1 c_u^{\ast}(s) \ln \left( \frac{1}{\delta_1} \right) + \alpha_2 c_u^{\ast}(b) \ln \left( \frac{1}{\delta_2} \right) + M^s + M^b,
\]

where

\[
M^s = \Gamma_1^s + \alpha_1 c_u^{\ast}(s) \left[ \ln(\alpha_1 e) + \ln \left( \frac{\alpha_1 c_u^{\ast}(s)}{\delta_1} \right) \right]
+ \sum_{T=1}^{\infty} \Gamma_2^s T \exp \left( -\Gamma_3^s T^{1/8} \right),
\]

\[
M^b = \Gamma_1^b + \alpha_2 c_u^{\ast}(b) \left[ \ln(\alpha_2 e) + \ln \left( \frac{\alpha_2 c_u^{\ast}(b)}{\delta_2} \right) \right]
+ \sum_{T=1}^{\infty} \Gamma_2^b T \exp \left( -\Gamma_3^b T^{1/8} \right).
\]

For simplicity, we fix \( \alpha_1 = \alpha_2 = 1 \) and \( \delta_1 = \delta_2 = \frac{1}{2} \delta \), then (13) can be rewritten as

\[
\mathbb{E}[\tau] \leq c_u^{\ast}(s) \ln \left( \frac{1}{2\delta} \right) + c_u^{\ast}(b) \ln \left( \frac{1}{2\delta} \right) + M^s + M^b.
\]

When we normalize by \( \ln(1/\delta) \) and let \( \delta \) tend to zero, we obtain (24) as desired.

When \( \delta \) tends to 0, we obtain

\[
\limsup_{\delta \to 0} \frac{\mathbb{E}[\tau]}{\ln(1/\delta)} \leq c_u^{\ast}(s) + c_u^{\ast}(b),
\]

which concludes the proof.

To summarize, we present in Table I the time complexities of the proposed 2PHT&S algorithm and its baseline, the original T&S algorithm [27], with fixed \( \alpha = \alpha_1 = \alpha_2 = 1 \). In Table I, the terms \( c_u^{\ast}(s) \) and \( c_u^{\ast}(b) \) result from the time complexities of Phase I and Phase II, respectively. These constants respectively represent the difficulties of finding the optimal super arm in Phase I and the optimal base arm in Phase II. There are two main differences between 2PHT&S and T&S. First, the proposed 2PHT&S algorithm contains two phases. In the first phase, the BS searches for the optimal super arm (beam set), and in the second phase, the BS searches for the optimal base arm (beam) among the beams in the optimal beam set and its neighbor with higher mean reward. However, the vanilla T&S algorithm directly searches for the optimal base arm (beam) among the whole beam space, which may be prohibitive large. Second, the heteroscedastic property is explicitly considered in the design of the sampling rule and the stopping rule of the 2PHT&S algorithm, but it is not exploited by the original T&S algorithm. Furthermore, we would like to note that the computational complexity of T&S algorithm scales linearly with the number of beams \( K \), while that of the

\[
Q(t) = \begin{cases} 
\arg\min_i T_i(t-1), & \text{if } \min_i T_i(t-1) \leq \sqrt{t}, \\
\arg\max_i T_i(t-1) - T_i(t-1), & \text{otherwise}.
\end{cases}
\]
The proposed 2PHT&S algorithm scales linearly with the number of beam sets $G$ and the number of beams in each beam set $J$. Because $G$ and $J$ are usually much smaller than $K$, in general, the proposed 2PHT&S algorithm can significantly reduce the computational complexity as compared to the original T&S algorithm.

IV. Simulation Results

We consider a massive mmWave MISO system, where a BS equipped with $N = 64$ transmit antennas serves a single-antenna user. The size of codebook is set as $K = 120$ and the correlation length is set to $J = 2 \lceil \frac{K}{N} \rceil - 1$. Note that the mmWave channel is sparse, and hence we set the maximum number of channel paths as 3, which consists of one dominant LoS path and two NLoS paths. In addition, according to practical in-field measurements, NLoS paths suffer around 1 dB more path loss than the LoS path. The noise variance is fixed to $\sigma^2 = -80$ dBm. The code to reproduce all the experiments can be found at this Github link (https://github.com/YiWei0129/Fast-beam-alignment).

In the simulation, we compare the proposed 2PHT&S with three other bandit algorithms, i.e., the original T&S [27], two-phase T&S (2PT&S), where the search process is also divided into two phases as the proposed algorithm and the original T&S is used in each phase, and the proposed HT&S. Note that 2PT&S also exploits the correlation between the nearby beams (Property 1.1), but it does not take the heteroscedasticity into account. The algorithms we consider are all designed to search the optimal beam/base arm with high probability using as few time steps as possible. Moreover, we also investigate the 2PHT&S algorithm with an overlapping scheme, i.e., in Phase II, we construct the base arm set with $2J$ base arms where first $J$ base arms overlap with the last $J$ base arms of the super arm to the left of the selected super arm in Phase I, and the last $J$ base arms overlap with the first $J$ base arms of the super arm to the right of the selected super arm. We call this scheme 2PHT&S (overlapping). Finally, we also consider the classical BA algorithms in wireless communications, i.e., exhaustive BA (EBA) algorithm and the hierarchical exhaustive BA (HEBA) algorithm as performance baselines.

Additionally, note that in a typical mmWave communication setting, one may experience channel coherence times of 35 $\mu$s when the system is deployed at the carrier frequency 28 GHz. In this setting, $T = 35$ $\mu$s, each time slot is 2.5 $n$s, and there are approximately 14000 time slots in each coherence time period. In practice, another link with different frequency can be used for feedback, thus no delay will result.

2In the HEBA algorithm, the search phase is divided into two phases. In each phase, the EBA algorithm is employed.
As compared to the results in Table II, the time complexities of the algorithms under consideration in Scenario 2 are higher. Note that if the amplitudes of the largest path and the second largest path are close, the time complexities of all algorithms will increase.

Third, we consider a more realistic channel model from the 3GPP TR 38.901 standard, i.e., Scenario 3. The channel coefficients are generated by the QuaDRiGa simulator [28], which extends the popular Wireless World Initiative for New Radio (WINNER) channel model with new features to enhance its realism. In Fig. 7, the base arm 86 is the optimal arm to be chosen, and the super arm 29, which is related to the base arms 85, 86, and 87, has the largest means of the reward among super arms of size 3. Table IV presents the estimated average time complexities of the considered algorithms in Scenario 3 when $\text{SNR} \in \{78, 80, 82, 84\}$ dB. It can be observed that similar to the previous channel models, the proposed 2PHT&S algorithm is clearly superior in terms of time complexity compared to the other competing algorithms.

Finally, to validate the effectiveness of the proposed 2PHT&S, we investigate a practical scenario in a city, i.e., Scenario 4, which is shown in Fig. 8, where the BS and user are located at (573m, 622m, 41m) and (603m, 630m, 43m), respectively. For this scenario, we generate the semi-practical channel data in the ray-tracing setups using the software Wireless InSite. Note that Wireless InSite is a professional electromagnetic simulation tool that models the physical characteristics of irregular terrain and urban building features, performs the electromagnetic calculations, and then evaluates the signal propagation characteristics. As can be seen in Fig. 9, the base arm 119 is the optimal arm, and the super arm 40, which is related to the base arm 118, 119, and 120, has the largest means of the rewards among super arms of size 3.

![Fig. 7. Mean of the reward of each base arm and super arm in Scenario 3 ($p = 0$ dBm).](image-url)
Table V presents the estimated average time complexities of the considered algorithms in Scenario 4 when SNR \( \in \{70, 74, 78, 82\} \) dB. It can be observed that similar to the previous simulated scenarios, the proposed 2PHT&S outperforms the other algorithms in this more practical scenario, which implies that the proposed algorithm is also effective in practice.

### APPENDIX A

#### PROOF OF LEMMA 1

**Proof:** Let \( \mu'_i \) be the mean of the reward of the arm \( i \) in the heteroscedastic Gaussian bandit instance \( \nu \). First, for every arm \( i_1 \) and \( i_2 \) where \( \mu'_{i_1} < \mu'_{i_2} \) and \( \mu'_{i_1}(t) > \mu'_{i_2}(t) \), we define

\[
Z_{i_1, i_2}(t) = \inf_{\mu'_1 < \mu'_2} \left\{ T_{i_1}(t) D_{HG}(\mu'_{i_1}(t), \mu'_{i_1}) + T_{i_2}(t) D_{HG}(\mu'_{i_2}(t), \mu'_{i_2}) \right\}
\]

Then, according to the definition of \( Z(t) \) given in (9), we can rewrite \( Z(t) \) as

\[
Z(t) = \min_{i \neq A^*(\nu)} \inf_{\mu'_i} \left\{ T_{A^*(\nu)}(t) D_{HG}(\mu'_{A^*(\nu)}(t), \mu'_{A^*(\nu)}) + T_{i}(t) D_{HG}(\mu'_{i}(t), \mu'_{i}) \right\}
\]

\[
= \min_{i \in [I]} Z_{A^*(\nu), i}(t).
\]

**Lemma 3:** For all \( \psi \geq I + 1 \) and \( t \geq 0 \), it holds that

\[
\mathbb{P} \left( \sum_{i=1}^{I} T_{i}(t) D_{HG}(\mu'_{i}(t), \mu'_{i}) \geq \psi \right) \leq e^{I+1-\psi} \left( \frac{\psi \ln \psi}{I} \right)^I.
\]

**Proof:** See the proof in Appendix B. \( \square \)

Now consider

\[
\mathbb{P}(t_3 < \infty, A_{t_3} \neq A^*(\nu))
\]

\[
\leq \mathbb{P} \left( \exists i \in A^*(\nu), \exists t > 0 : \mu'_{i}(t) > \mu'_{A^*(\nu)}(t) \right)
\]

\[
+ \mathbb{P} \left( \exists t > 0 : \exists i \in [I] \setminus A^*(\nu) : Z_{A^*(\nu), i}(t) > \beta(t, \delta, \alpha) \right)
\]

\[
\leq \mathbb{P} \left( \exists t > 0 : \sum_{i=1}^{I} T_{i}(t) D_{HG}(\mu'_{i}(t), \mu'_{i}) \geq \beta(t, \delta, \alpha) \right)
\]

\[
\leq \sum_{t=1}^{\infty} e^{I+1-\psi} \left( \frac{\psi \ln \psi}{I} \right)^I e^{-\beta(t, \delta, \alpha)}
\]

### V. CONCLUSION

In this work, we developed a bandit-based fast BA algorithm 2PHT&S to reduce BA latency for mmWave communications. By taking advantage of the correlation structure among beams that the information from nearby beams are similar and the heteroscedastic property that the variance of the reward of an arm (beam) is related to its mean, the proposed algorithm groups all beams into several beam sets such that the optimal beam set is first selected and the optimal beam is identified in this set. The proposed 2PHT&S is shown to theoretically and empirically perform much better than its baseline competitors.
where (a) holds using the union bound and Lemma 3. Hence, with \( \beta(t, \delta, \alpha) = \ln(\alpha/\delta) \), by choosing an \( \alpha \) satisfying
\[
\sum_{i=1}^{\infty} e^{i+1} t^{i+1} \ln^2(\alpha/\delta) \ln(t)^i \leq \alpha,
\]
we obtain \( \mathbb{P}(\tau_\delta < \infty, A_{\tau_\delta} \neq A^*(\nu)) \leq \delta. \)

**Appendix B**
**Proof of Lemma 3**

In this section, we prove Lemma 3 assuming that Lemma 4 holds, and then we complete the argument by proving Lemma 4.

**Lemma 4: Fix an I-armed heteroscedastic Gaussian bandit \( \nu \) and let \( 1 \leq i \leq t \). Let \( \eta > 0 \). Define the event:
\[
E = \bigcap_{i=1}^{t} \{ \tilde{I}_i \leq T_i(t) \leq (1+\eta)\tilde{I}_i \}.
\]
(15)

For \( \psi \geq (1+\eta)I \), it holds that
\[
\mathbb{P}\left( \bigcap_{i=1}^{t} S_i(t) D_HG(\tilde{\mu}_t(t), \mu_t^*) \geq \psi \right) \leq \left( \frac{\psi e}{T} \right)^I e^{-\psi/(1+\eta)}.
\]

**Proof:** See the proof in Appendix B-B. \( \square \)

**A. Proof of Lemma 3**

**Proof:** Let \( \psi \geq I + 1 \) and \( \eta > 0 \). Define \( M = \lfloor \ln(t)/\ln(1+\eta) \rfloor \), and set \( M = \{1, \ldots, M\} \) and events \( A \) and \( B_m \) as
\[
A = \left\{ \sum_{i=1}^{t} T_i(t) D_HG(\tilde{\mu}_t(t), \mu_t^*) \geq \psi \right\},
\]
\[
B_m = \bigcap_{i=1}^{t} \{ (1+\eta)^{m-1} \leq T_i(t) \leq (1+\eta)^m \}, \quad \forall m \in M.
\]

Since \( A = \bigcup_{m \in M}(A \cap B_m) \), we have \( \mathbb{P}(A) \leq \sum_{m \in M} \mathbb{P}(A \cap B_m) \). Using Lemma 4, we obtain for all \( m \in M \):
\[
\mathbb{P}(A \cap B_m) \leq \left( \frac{\psi e}{T} \right)^I e^{-\psi/(1+\eta)}.
\]

Since \( |M| = M! \), we have
\[
\mathbb{P}(A) \leq M! \mathbb{P}(A \cap B_m) \leq \left( \frac{M \psi e}{T} \right)^I e^{-\psi/(1+\eta)}.
\]

With the choice \( \eta = 1/(\psi - 1) \) and the inequality \( \ln(1+\eta) \geq 1-1/(1+\eta) = 1/\psi \), it holds that
\[
\mathbb{P}(A) \leq e^{-\psi \left( \frac{\psi \ln(t)}{T} \right)^I} e^{I+1},
\]
which concludes the proof.

**B. Proof of Lemma 4**

**Proof:** Define the event
\[
\mathcal{F} = \left\{ \mathbb{E}_{i=1}^{T} T_i(t) D_HG(\tilde{\mu}_t(t), \mu_t^*) \geq \psi \right\}.
\]

Let \( \zeta_i \in (\mathbb{R}^+)^I \), \( T \in \mathbb{N} \) and \( x_i(t) \) such that (i) if there exists \( 0 \leq x \leq \mu_t^* \) such that \( T D_HG(x, \mu_t^*) = \zeta_i \), then \( x_i(T) = x \), (ii) else \( x_i(T) = 0 \). Since \( dD_HG(x, \mu_t^*)/dx \leq 0 \) for \( 0 \leq x \leq \mu_t^* \), \( D_HG(x, \mu_t^*) \) is a monotonically decreasing function of \( x \) and furthermore \( x_i(T) \) is a monotonically increasing function of \( T \). As a result, if \( T_i(t) D_HG(\tilde{\mu}_t(t), \mu_t^*) \geq \zeta_i \), we have
\[
\mu_t^* \leq x_i(T_i(t)) \leq x_i((1+\eta)\tilde{I}_i),
\]
where (a) holds due to the monotonicity of \( x_i(T) \), and (b) is due to the definition of event \( E \) in (15).

**Lemma 5: Fix an I-armed heteroscedastic Gaussian bandit \( \nu \) and let \( 1 \leq i \leq t \). For any collection of \( 0 \leq b_i \leq \mu_t^*, i \in [I] \), we have
\[
\mathbb{P}\left( \cap_{i \in [I]} \{ \mu_t^* \leq b_i, \tilde{I}_i \leq T_i(t) \} \right) \leq \prod_{i=1}^{I} e^{-\zeta_i/(1+\eta)}.
\]

**Proof:** See Appendix B-C. With this lemma, we can deduce that
\[
\mathbb{P}\left( \cap_{i \in [I]} \{ \mu_t^* \leq b_i, \tilde{I}_i \leq T_i(t) \} \right) \leq \prod_{i=1}^{I} e^{-\zeta_i/(1+\eta)}.
\]

Lastly, by applying [29, Lemma 8], we have
\[
\mathbb{P}(\mathcal{F}) \leq \left( \frac{\psi e}{T(1+\eta)} \right)^I e^{-\psi/(1+\eta)} \leq \left( \frac{\psi e}{T} \right)^I e^{-\psi/(1+\eta)},
\]
which concludes the proof. \( \square \)

**C. Proof of Lemma 5**

**Proof:** According to the Gibbs variational principle [30], one has the following dual characterization of the KL divergence:
\[
D(F_1, F_2) = \sup_{\lambda \in \mathbb{R}} [\mathbb{E}_{F_1} (\ln(\lambda X_1) - \ln(\mathbb{E}_{F_2} [\ln(\lambda X_2)])]
\]
where the variables \( X_1 \sim F_1 \) and \( X_2 \sim F_2 \). If we let \( f(x) = \lambda x, \lambda \leq 0 \), for \( 0 \leq x \leq \mu_t^* \), we have
\[
D_HG(x, \mu_t^*) = \sup_{\lambda < 0} \{ \lambda x - \phi_i(\lambda) \},
\]
where \( \phi_i(\lambda) = \ln \mathbb{E}[\lambda e^{\lambda X_i}] = \ln(\mu_t^* e^{\lambda} + 1 - \mu_t^*) \) and \( X_i \sim N(\mu_t^*, 2\mu_t^* \sigma^2) \). Define the event \( Q = Q_1 \cap Q_2 \), where \( Q_1 = \cap_{i \in [I]} \{ \tilde{I}_i \leq T_i(t) \} \) and \( Q_2 = \cap_{i \in [I]} \{ \tilde{\mu}_t^*(t) \leq b_i \} \).

For all \( i \), let \( \lambda_i \leq 0 \) and define \( C(t) = \exp \left( \sum_{i \in [I]} \lambda_i \tilde{S}_i(t) - T_i(t) \phi_i(\lambda) \right) \), where \( \tilde{S}_i(t) = T_i(t) \tilde{\mu}_t^*(t) \). For all \( i \), we set...
\[ \lambda_i = \arg \max_{\lambda \leq \lambda_i} \{ \lambda b_i - \phi_i(\lambda) \} \quad \text{and} \quad \lambda_i b_i - \phi_i(\lambda_i) = D_{HGC}(b_i, \mu_i^u). \]

Then, we have
\[
P(Q) = P(\bigcap_{i \in [t]} \{ S_i(t) \leq T_i(t) b_i, Q_1 \}) \leq P\left( \sum_{i \in [t]} \lambda_i S_i(t) \geq \sum_{i \in [t]} \lambda_i T_i(t) b_i, Q_1 \right) \leq P(1_Q, \sum_{i \in [t]} \lambda_i S_i(t) \geq \sum_{i \in [t]} \lambda_i T_i(t) b_i) \leq P\left( 1_Q, \sum_{i \in [t]} \lambda_i S_i(t) \geq \sum_{i \in [t]} \lambda_i T_i(t) b_i, Q_1 \right) \leq P\left( 1_Q, C(t) \geq \sum_{i \in [t]} \alpha_i D_{HGC}(b_i, \mu_i^u) \right) \leq \mathbb{E}[1_Q, C(t)] \exp \left( - \sum_{i \in [t]} \alpha_i D_{HGC}(b_i, \mu_i^u) \right),
\]

where (a) is due to Markov’s inequality, (b) is due to the fact that \( \mathbb{E}[1_Q, C(t)] \leq \mathbb{E}[C(t)] \) and \( \mathbb{E}[C(t)] = 1 \) is shown in the proof of [29, Lemma 7]. This concludes the proof. \( \square \)

**Appendix C**

In this section, we first study the optimization problem (4), so as to better understand the choice of \( w^* \), then we provide an efficient method for computing \( w^* \).

By minimizing the KL divergence between the reward distributions of two bandits associated with arm \( A^*(\nu) \) and arm \( i \neq A^*(\nu) \), (4) can be transformed into the following optimization problem
\[
e^{-c^*(\nu) - 1} = \sup_{w \in W} \inf_{w \in W} \left( \sum_{i=1}^f w_i D_{HGC}(\mu_i^u, \mu_i^u) \right) = \sup_{w \in W \setminus \{ A^*(\nu) \}} \inf_{w \in W} \left( \sum_{\nu \in \{ A^*(\nu), i \}} w_\nu D_{HGC}(\mu_\nu^u, \mu_\nu^u) \right), \quad (16)
\]

It follows that
\[
w^* = \arg \max_{w \in W \setminus \{ A^*(\nu) \}} \inf_{w \in W} \left( \sum_{\nu \in \{ A^*(\nu), i \}} w_\nu D_{HGC}(\mu_\nu^u, \mu_\nu^u) \right).
\]

By using Lagrange multipliers, we can solve the optimization problem
\[
\min_{\mu_i^u \geq A^*(\nu), \mu_i^u \geq A^*(\nu)} w_{A^*(\nu)} \frac{1}{2} \ln \left( \frac{\mu_i^u}{\mu_i^u} \right) + \frac{1}{2} \frac{\mu_i^u}{\mu_i^u} + \frac{1}{4} \frac{\mu_i^u}{\mu_i^u} \right) - \frac{1}{2} \}
\]

and obtain the optimal values of \( \mu_i^u A^*(\nu), \mu_i^u \), i.e.,
\[
\mu_i^u A^*(\nu) = \mu_i^u = \frac{w_{A^*(\nu)} + w_i}{w_{A^*(\nu)}^u + w_i A^*(\nu)^u} \mu_i^u A^*(\nu)^u. \quad (17)
\]

By substituting (17) into (16), we have
\[
c^*(\nu)^{-1} = \sup_{w \in W \setminus \{ A^*(\nu) \}} \inf_{w \in W} \left( \frac{w_{A^*(\nu)}^u \ln \left( \frac{w_{A^*(\nu)}^u + w_i A^*(\nu)^u}{w_{A^*(\nu)}^u + w_i A^*(\nu)^u} \right)}{2} + \frac{w_i A^*(\nu)^u}{2} \ln \left( \frac{w_{A^*(\nu)}^u + w_i A^*(\nu)^u}{w_{A^*(\nu)}^u + w_i A^*(\nu)^u} \right) + \frac{w_{A^*(\nu)}^u w_i A^*(\nu)^u}{4 \sigma^2} \frac{(\mu_i^u A^*(\nu) - \mu_i^u)^2}{2} - \frac{1}{2} \left( w_{A^*(\nu)}^u + w_i A^*(\nu)^u \right) \right).
\]

With \( x_i^* = \frac{\mu_i^u A^*(\nu)}{\mu_i^u A^*(\nu)^u} \), we have
\[
w_{A^*(\nu)}^u = \frac{1}{1 + \sum_{\nu \neq A^*(\nu)} x_i^* a}, \quad x_i^* = \frac{x_i^*}{1 + \sum_{\nu \neq A^*(\nu) x_i^* a} a}.
\]

As a result, according to (18), \( x_1^*, \ldots, x_i^* \) belongs to the set
\[
\arg \max_{x_1, \ldots, x_i} \min_{x \in W} \frac{1}{1 + \sum_{\nu \neq A^*(\nu)} x_i a} f_{Y,i}(x_i). \quad (19)
\]

where
\[
f_{Y,i}(x_i) = \frac{1}{2} \ln \left( \frac{\mu_i^u + x_i A^*(\nu)^u}{1 + x_i A^*(\nu)^u} \right) + \frac{1}{2} \ln \left( \frac{\mu_i^u + x_i A^*(\nu)^u}{1 + x_i A^*(\nu)^u} \right) + \frac{x_i A^*(\nu)^u - \mu_i^u)^2}{4 \sigma^2} \frac{1 + x_i A^*(\nu)^u}{2}.
\]

By differentiating \( f_{Y,i}(x_i) \) with respect to \( x_i \), we have
\[
df_{Y,i} \frac{d}{dx_i} = \frac{\mu_i^u A^*(\nu) - \mu_i^u}{2(\mu_i^u + x_i A^*(\nu)^u)} + \frac{1}{2} \ln \left( \frac{\mu_i^u + x_i A^*(\nu)^u}{1 + x_i A^*(\nu)^u} \right) + \frac{\mu_i^u (\mu_i^u A^*(\nu) - \mu_i^u)^2}{4 \sigma^2 \mu_i^u A^*(\nu)^u} - 1 - 2.
\]

Since \( \frac{df_{Y,i}}{dx_i} \leq 0 \), \( \frac{dF_{Y,i}}{dx_i} \) is monotonically decreasing for all \( x_i \in [0, \infty) \). Furthermore, it holds that \( \lim_{x_i \to \infty} \frac{dF_{Y,i}}{dx_i} = -1/2 \). Because of (2), we have \( \frac{df_{Y,i}}{dx_i} \leq 0 \) holds for all \( x_i \in [0, \infty) \), \( f_{Y,i}(x_i) \) is a monotonically decreasing function. Then, we let
\[
y^* = f_{Y,1}(x_1^*) = f_{Y,2}(x_2^*) = \ldots = f_{Y,i}(x_i^*), \quad y^* = \begin{cases} 0, & y_1, \\ 0, \end{cases}
\]

based on which we then introduce the inverse function of \( f_{Y,i}(x_i) \), i.e., \( f_{X,i} = f_{Y,i}^{-1} \). According to (19), \( y^* \) belongs to the set
\[
\arg \min_{y \in [0, y_1]} \mathcal{F}_Y(y) \quad \text{with} \quad \mathcal{F}_Y(y) = \frac{y}{1 + \sum_{\nu \neq A^*(\nu)} f_{X,i}(y)}.
\]

By taking the derivative of \( \mathcal{F}_Y(y) \), we have
\[
\frac{d\mathcal{F}_Y}{dy} = \frac{1}{1 + \sum_{\nu \neq A^*(\nu)} f_{X,i}(y)} - \sum_{\nu \neq A^*(\nu)} \frac{y f_{X,i}(y)}{(1 + \sum_{\nu \neq A^*(\nu)} f_{X,i}(y))^2}.
\]
By setting \( \frac{dF_y}{dy} = 0 \), we obtain \( y \) such that \( F_Y(y) \) achieves its minimum, i.e.,
\[
\sum_{i \in \tilde{A}^*(\nu)} y f'_{X,i}(y) - f_{X,i}(y) = 1.
\]
Then we let \( F_{\nu}(y) = \sum_{i \notin \tilde{A}^*(\nu)} y f'_{X,i}(y) - f_{X,i}(y) \). Since \( f'_{X,i}(y) = \frac{1}{f_{X,i}(y)} \) and \( \frac{d^2 F_{\nu}}{dy^2} \leq 0 \), we have
\[
\frac{dF_{\nu}(y)}{dy} = \sum_{i \notin \tilde{A}^*(\nu)} y \frac{d^2 f_{X,i}(y)}{dy^2} = \sum_{i \notin \tilde{A}^*(\nu)} -\frac{y}{(f'_{X,i}(y))^2} \frac{d^2 f_{X,i}}{dy^2} \geq 0.
\]
Therefore, since \( F_{\nu}(y) \) is strictly increasing and \( F_{\nu}(y_a) = \infty \), \( F_{\nu}(y) = 1 \) must have a unique solution \( y^* \), which can be solved using the bisection method. Then, we can obtain \( w^*_i \) as
\[
w^*_i = \frac{f_{X,i}(y^*)}{\sum_{a=1}^n f_{X,a}(y^*)}.
\]
Then, by replacing the mean \( \mu^*_i \) with the empirical mean \( \hat{\mu}^*_i(t) \), we can obtain \( \hat{w}^*_i(t) \) as in (12).

**Appendix D**

**A. Proof of Lemma 2**

**Proof:** Assume a heteroscedastic Gaussian bandit instance \( \nu \) with \( \mu^*_1 \geq \mu^*_2 \geq \ldots \geq \mu^*_I \). There exists \( \xi = \xi(\epsilon) \leq (\mu^*_1 - \mu^*_2)/4 \) such that
\[
\mathcal{I}_\epsilon := [\mu^*_1 - \xi, \mu^*_I + \xi] \times \ldots \times [\mu^*_1 - \xi, \mu^*_I + \xi].
\]
Then, for a bandit model \( \hat{\nu}_t \in \mathcal{I}_\epsilon \) and \( t_0 > 0 \)
\[
\sup_{t \geq t_0} \max_i \left| \hat{w}^*_i(t) - w^*_i \right| \leq \epsilon.
\]
Furthermore, for all \( \hat{\nu}_t \in \mathcal{I}_\epsilon \), the empirical optimal arm is \( \tilde{A}^*(\hat{\nu}_t) = 1 \).

Let define \( h(T) = T^{1/4} \) and the event
\[
\mathcal{E}_T(\epsilon) = \bigcap_{t = h(T)}^T \{ \hat{\nu}_t \in \mathcal{I}_\epsilon \}
\]
in which it holds for \( t \geq h(T) \), \( A^*(\hat{\nu}_t) = 1 \). Then, let us rewrite \( Z(t) \) in (9) as
\[
Z(t) = \min_{i \neq 1} \left( T_i(t) D_{HG}(\mu^*_1(t), q_i(t)) + T_i(t) D_{HG}(\hat{\mu}^*_i(t), q_i(t)) \right)
\]
\[
= t f_Z(\hat{\nu}_t, \frac{T_i(t)}{t}) \xi_{i=1}^I,
\]
where \( q_i(t) \) is defined in (10), \( f_Z(\nu', \mathbf{w}') = \min_{i \neq 1} (w'_1 D_{HG}(\mu'_1, q'_i) + w'_i D_{HG}(\mu'_i, q'_i)) \) and
\[
q'_i = \frac{w'_1 + w'_i}{w'_1 \mu'_1 + w'_i \mu'_i} \mu'_i.
\]

**Lemma 6:** The sampling rule ensures that \( T_i(t) \geq \sqrt{T} \) and that for all \( \epsilon > 0 \) and \( t_0 > 0 \), there exists a constant \( t_\epsilon = \max \left\{ \left[ \frac{\alpha}{\delta \mu^*_1}, \frac{1}{\delta \mu^*_1}, \frac{1}{\mu^*_1} \right] \right\} \) such that
\[
\sup_{t \geq t_0} \max_i \left| \hat{w}^*_i(t) - w^*_i \right| \leq \epsilon
\]
\[
\implies \sup_{t \geq t_\epsilon} \max_i \left| \frac{T_i(t)}{t} - w^*_i \right| \leq 3(I-1)\epsilon.
\]

**Proof:** See the proof in Appendix D-B.

According to Lemma 6 and the definition of \( \mathcal{E}_T \), when \( T \geq t^*_\epsilon \), \( \mathcal{E}_T \) it holds that
\[
Z(t) \geq t C^*_\epsilon(\nu), \quad \forall t \geq \sqrt{T}.
\]
When \( T \geq t_\epsilon \), and on the event \( \mathcal{E}_T \), it holds that
\[
\min \left\{ \tau_\delta, T \right\}
\leq \sqrt{T} + \sum_{t = [\sqrt{T}]}^{T} \mathbb{I}_{(\tau_\delta > t)} \leq \sqrt{T} + \sum_{t = [\sqrt{T}]}^{T} \mathbb{I}_{(Z(t) \leq \beta(t, \delta, \alpha))}
\leq \sqrt{T} + \sum_{t = [\sqrt{T}]}^{T} \mathbb{I}_{(C^*_\epsilon(\nu) \leq \beta(t, \delta, \alpha))}
\leq \max \left\{ \sqrt{T}, \frac{\beta(t, \delta, \alpha)}{C^*_\epsilon(\nu)} \right\}
\]
Define
\[
T_0(\delta) = \inf \left\{ T \in \mathbb{N} : \max \left\{ \sqrt{T}, \frac{\beta(t, \delta, \alpha)}{C^*_\epsilon(\nu)} \right\} \leq T \right\}
= \inf \left\{ T \in \mathbb{N} : \frac{\beta(t, \delta, \alpha)}{C^*_\epsilon(\nu)} \leq T \right\}
= \inf \left\{ T \in \mathbb{N} : C^*_\epsilon(\nu)T \geq \ln(\alpha T/\delta) \right\}.
\]
Using [27, Lemma 18], we have
\[
T_0(\delta) \leq \frac{\alpha}{C^*_\epsilon(\nu)} \ln \left( \frac{\alpha^e}{\delta C^*_\epsilon(\nu)} \right) + \ln \left( \frac{\alpha}{\delta C^*_\epsilon(\nu)} \right).
\]

**Lemma 7:** There exist two constants \( \Gamma_2 \) and \( \Gamma_3 \) (which depend on \( \nu \) and \( \epsilon \)) such that
\[
\mathbb{P}(\mathcal{E}_T) \leq \Gamma_2 T \exp(-\Gamma_3 T^{1/8}).
\]

**Proof:** See the proof in Appendix D-C.
According to the definition of $C^*_r(\nu)$ in (20) and $c^*(\nu)^{-1}$ in (4), it holds that $C^*_r(\nu) \leq c^*(\nu)^{-1}$. As a result, we have

$$\mathbb{E}[\tau_6] = \sum_{T=1}^{\infty} \mathbb{P}(\tau_6 \geq T)$$

$$= \sum_{T=1}^{\infty} \mathbb{P}(\tau_6 \geq T) + \sum_{T=\max(T_0(\delta), t_*)+1}^{\infty} \mathbb{P}(\tau_6 \geq T)$$

$$\leq t_\epsilon + T_0(\delta) + \sum_{T=1}^{\infty} \frac{\tau_2 T \exp \left( -\Gamma_3 T^{1/8} \right)}{\delta}$$

$$\leq t_\epsilon + \alpha c^*(\nu) \left[ \ln \left( \frac{\alpha c^*(\nu)}{\delta} \right) + \ln \left( \frac{\alpha c^*(\nu)}{\delta} \right) \right]$$

$$+ \sum_{T=1}^{\infty} \frac{\tau_2 T \exp \left( -\Gamma_3 T^{1/8} \right)}{\delta}.$$

Lemma 8: Let $\nu$ represent a heteroscedastic Gaussian bandit instance. When $\nu$ has a unique optimal arm, then we have

$$c^*(\nu) \leq c^*_u(\nu),$$

where

$$c^*_u(\nu) = \left( \frac{\mu^*_{A^*_u(\nu)}}{2 \sum_{i=1}^{T} \mu^*_i} \ln \left( \frac{2 \mu^*_u(\nu)}{\mu^*_{A^*_u(\nu)} + \mu^*_A(\nu)} \right) \right)$$

$$\leq \frac{\tau_2 T \exp \left( -\Gamma_3 T^{1/8} \right)}{\delta}.$$

Proof: Please refer to Appendix D-D. □

Let $\nu$ represent a set of $I$-armed heteroscedastic Gaussian bandit instances. Note that

$$c^*(\nu)^{-1} = \sup_{w \in \mathcal{W}_I, \tau_1(i) \neq \tau_2(i), \nu^*_i \geq \nu^*_u} \left( \sum_{i=1}^{I} \frac{w_i D_{HG}(\mu^*_i, \mu^*_u)}{\nu^*_i} \right),$$

which is related to the choice of $\nu$ and $w$. According to Lemma 8, it holds that

$$\mathbb{E}[\tau_6] \leq t_\epsilon + \alpha c^*(\nu) \left[ \ln \left( \frac{\alpha c^*(\nu)}{\delta} \right) + \ln \left( \frac{\alpha c^*(\nu)}{\delta} \right) \right]$$

$$+ \sum_{T=1}^{\infty} \frac{\tau_2 T \exp \left( -\Gamma_3 T^{1/8} \right)}{\delta}.$$

Since $\tau = I + \tau_3$ according to Line 15 of Algorithm 2, it holds that

$$\mathbb{E}[\tau] = I + \mathbb{E}[\tau_6]$$

$$\leq I + t_\epsilon + \alpha c^*_u(\nu) \left[ \ln \left( \frac{\alpha c^*_u(\nu)}{\delta} \right) + \ln \left( \frac{\alpha c^*_u(\nu)}{\delta} \right) \right]$$

$$+ \sum_{T=1}^{\infty} \frac{\tau_2 T \exp \left( -\Gamma_3 T^{1/8} \right)}{\delta}.$$

The proof is concluded by letting $A = I + T_\epsilon$. □

B. Proof of Lemma 6

Proof: Fix $t_0^*$ such that when $t_0 \geq t_0^*$, it holds that

$$\forall \tau \geq t_0^*, \sqrt{\tau} \leq 2 \tau c + 1/\tau \leq \epsilon.$$

To satisfy the requirement $\sqrt{\tau} \leq 2 \tau c$, we fix $t \geq \max\{t_0, \frac{1}{2 \tau c}\}$. Thus, we let $t_0^* = \max\{t_0, \frac{1}{2 \tau c}\}$. According to [27, Lemma 17] and its proof in [27, Appendix B.2], by choosing $\tilde{\lambda}(i) = \tilde{w}_i^*(\nu)$ and $\lambda^* = \tilde{w}_i^*$, we have

$$\sup_{i} \left[ \frac{T_i(\nu)}{t} - w_i^*(\nu) \right] \leq \left( I - 1 \right) \max \left\{ 2 \epsilon + \frac{1}{t}, \frac{t_0^*}{t} \right\}$$

$$\leq \left( I - 1 \right) \max \left\{ 3 \epsilon, t_0^* \right\}.$$

As a result, when $t \geq \frac{t_0^*}{\epsilon^2}$, it holds that $\sup_{i} \left[ \frac{T_i(\nu)}{t} - w_i^*(\nu) \right] \leq 3(1 - 1/\epsilon)$. Let $t_\epsilon = \max \left\{ \frac{1}{\epsilon^2}, \frac{1}{3(1 - 1/\epsilon)} \right\}$, which concludes the proof. □

C. Proof of Lemma 7

Proof: First, we have

$$\mathbb{P}(\mathcal{E}_T^c) \leq \sum_{T=\max(T)}^{T} \mathbb{P}(\hat{\nu}^T \notin \mathcal{I}_c)$$

$$= \sum_{T=\max(T)}^{T} \sum_{T=\max(T)}^{T} \mathbb{P}(\hat{\mu}^T \leq \mu^*_\nu - \xi) + \mathbb{P}(\hat{\mu}^T \geq \mu^*_\nu + \xi).$$

According to Lemma 6, for each arm, we have

$$T_i(\nu) > \sqrt{t} - I, \quad \forall \tau \geq h(T).$$

Then, we have

$$\mathbb{P}(\hat{\mu}^T \leq \mu^*_\nu - \xi) \leq \sum_{m=\sqrt{t} - I}^{t} \mathbb{P}(\hat{\mu}^T \leq \mu^*_\nu - \xi)$$

$$\leq \sum_{m=\sqrt{t} - I}^{t} \exp(-m D_{HG}(\mu^*_\nu - \xi, \mu^*_\nu))$$

$$\leq \frac{e^{-\sqrt{t} - I} D_{HG}(\mu^*_\nu - \xi, \mu^*_\nu)}{1 - e^{-D_{HG}(\mu^*_\nu - \xi, \mu^*_\nu)}}.$$

where (a) is obtained due to (21), (b) and (c) holds due to the union bound and the Chernoff inequality. Due to the same reason, it holds that

$$\mathbb{P}(\hat{\mu}^T \geq \mu^*_\nu + \xi) \leq \frac{e^{-\sqrt{t} - I} D_{HG}(\mu^*_\nu + \xi, \mu^*_\nu)}{1 - e^{-D_{HG}(\mu^*_\nu + \xi, \mu^*_\nu)}}.$$

Then, define

$$\Gamma_2 \left( \sum_{i=1}^{I} \left( \frac{e^{i D_{HG}(\mu^*_\nu - \xi, \mu^*_\nu)}}{1 - e^{-D_{HG}(\mu^*_\nu - \xi, \mu^*_\nu)}} + \frac{e^{i D_{HG}(\mu^*_\nu + \xi, \mu^*_\nu)}}{1 - e^{-D_{HG}(\mu^*_\nu + \xi, \mu^*_\nu)}} \right) \right)$$

$$\Gamma_3 \left( \min_{i \in [I]} \left\{ D_{HG}(\mu^*_\nu - \xi, \mu^*_\nu), D_{HG}(\mu^*_\nu + \xi, \mu^*_\nu) \right\} \right),$$
we obtain
\[
\Pr(\mathcal{E}_T^c) \leq \sum_{t=h(T)}^{T} \sum_{i=1}^{I} \left[ \Pr(\mu_i^*(t) \leq \mu_i^* - \xi) + \Pr(\mu_i^*(t) \leq \mu_i^* + \xi) \right] \\
\leq \sum_{t=h(T)}^{T} \Gamma_2 \exp(-\sqrt{T} \Gamma_3) \leq \Gamma_2 T \exp(-\sqrt{h(T)} \Gamma_3) = \Gamma_2 T \exp(-\Gamma_3 T^{1/8}),
\]
which concludes the proof.

\section*{D. Proof of Lemma 8}

Proof: By setting \( \tilde{w} \in \mathcal{W}_I \) as \( \tilde{w}_i = \frac{\mu_i^*}{\sum_{a=1}^{I} \mu_a^*} \), according to (18), \( c^*(\nu)^{-1} \) satisfies
\[
c^*(\nu)^{-1} \geq \inf_{\nu \neq \nu^*} \left\{ \frac{\mu_{A^*(\nu)}^*}{2} \ln \left( \frac{2\mu_{A^*(\nu)}^*}{\mu_{A^*(\nu)}^* + \mu_{i}^*} \right) + \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^* - \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^* \right\} \\
+ \frac{\mu_i^*}{2} \ln \left( \frac{2\mu_{A^*(\nu)}^*}{\mu_{A^*(\nu)}^* + \mu_{i}^*} \right) + \frac{(\mu_{A^*(\nu)}^* - \mu_i^*)^2}{8\sigma^2 \sum_{a=1}^{I} \mu_a^* - \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^*} \right\}.
\]
(22)

Define
\[
F_{\mu}^{\nu}(\mu_{A^*(\nu)}^*) \triangleq \frac{\mu_{A^*(\nu)}^*}{2} \ln \left( \frac{2\mu_{A^*(\nu)}^*}{\mu_{A^*(\nu)}^* + \mu_{i}^*} \right) + \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^* - \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^* \right\} \\
+ \frac{\mu_i^*}{2} \ln \left( \frac{2\mu_{A^*(\nu)}^*}{\mu_{A^*(\nu)}^* + \mu_{i}^*} \right) + \frac{(\mu_{A^*(\nu)}^* - \mu_i^*)^2}{8\sigma^2 \sum_{a=1}^{I} \mu_a^* - \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^*} \right\}.
\]

By taking the derivative of \( \mu_{A^*(\nu)}^* \), we have
\[
\frac{dF_{\mu}^{\nu}(\mu_{A^*(\nu)}^*)}{d\mu_{A^*(\nu)}^*} = -\frac{\mu_{A^*(\nu)}^*}{2} \ln \left( \frac{2\mu_{A^*(\nu)}^*}{\mu_{A^*(\nu)}^* + \mu_{i}^*} \right) + \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^* - \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^* \right\} \\
+ \frac{\mu_i^*}{2} \ln \left( \frac{2\mu_{A^*(\nu)}^*}{\mu_{A^*(\nu)}^* + \mu_{i}^*} \right) + \frac{(\mu_{A^*(\nu)}^* - \mu_i^*)^2}{8\sigma^2 \sum_{a=1}^{I} \mu_a^* - \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^*} \right\}.
\]

Since \( \frac{dF_{\mu}^{\nu}(\mu_{A^*(\nu)}^*)}{d\mu_{A^*(\nu)}^*} < 0 \) holds, \( F_{\mu}^{\nu}(\mu_{A^*(\nu)}^*) \) is monotonically decreasing. Therefore, we conclude that when a suboptimal arm \( A^*(\nu) = \arg \min_{\nu \neq \nu^*} \mu_{A^*(\nu)}^* \) is selected, (22) can be rewritten as
\[
c^*(\nu)^{-1} \geq \frac{\mu_{A^*(\nu)}^*}{2} \ln \left( \frac{2\mu_{A^*(\nu)}^*}{\mu_{A^*(\nu)}^* + \mu_{i}^*} \right) + \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^* - \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^* \right\} \\
+ \frac{\mu_i^*}{2} \ln \left( \frac{2\mu_{A^*(\nu)}^*}{\mu_{A^*(\nu)}^* + \mu_{i}^*} \right) + \frac{(\mu_{A^*(\nu)}^* - \mu_i^*)^2}{8\sigma^2 \sum_{a=1}^{I} \mu_a^* - \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^*} \right\}.
\]
as such we have
\[
c^*(\nu) \leq \frac{\mu_{A^*(\nu)}^*}{2} \ln \left( \frac{2\mu_{A^*(\nu)}^*}{\mu_{A^*(\nu)}^* + \mu_{i}^*} \right) + \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^* - \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^* \right\} \\
+ \frac{\mu_i^*}{2} \ln \left( \frac{2\mu_{A^*(\nu)}^*}{\mu_{A^*(\nu)}^* + \mu_{i}^*} \right) + \frac{(\mu_{A^*(\nu)}^* - \mu_i^*)^2}{8\sigma^2 \sum_{a=1}^{I} \mu_a^* - \frac{\mu_i^*}{2} \sum_{a=1}^{I} \mu_a^*} \right\}.
\]
which concludes the proof.
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