BINDING ENERGY FOR HYDROGEN-LIKE ATOMS IN THE NELSON MODEL WITHOUT CUTOFFS

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Abstract. In the Nelson model particles interact through a scalar massless field. For hydrogen-like atoms there is a nucleus of infinite mass and charge $Z e$, $Z > 0$, fixed at the origin and an electron of mass $m$ and charge $e$. This system forms a bound state with binding energy $E_{\text{bin}} = me^4Z^2/2$ to leading order in $e$. We investigate the radiative corrections to the binding energy and prove upper and lower bounds which imply that $E_{\text{bin}} = me^4Z^2/2 + c_0e^6 + O(e^7 \ln e)$ with explicit coefficient $c_0$ and independent of the ultraviolet cutoff. $c_0$ can be computed by perturbation theory, which however is only formal since for the Nelson Hamiltonian the smallest eigenvalue sits exactly at the bottom of the continuous spectrum.

1. Introduction

As a very famous result in the early days of constructive quantum field theory, E. Nelson proved that for charges coupled to a scalar massless Bose field the ultraviolet cutoff can be removed at the expense of an infinite energy renormalization [N]. In our contribution we study Nelson’s model for the case of a hydrogen-like atom. It consists of a nucleus of infinite mass, nailed down at the origin, carrying a charge $Ze$, $Z > 0$, and a quantum particle, called electron for simplicity, of mass $m$ carrying charge $e$. Note that in the Nelson model charges of equal sign attract each other. Without restriction we may set $e \geq 0$. We also use natural units, for which $m = 1/2$, $\hbar = 1$, $c = 1$. Thus $e$ remains as the only parameter in the model. Our goal is to obtain precise estimates on the binding energy and thus to prove that Nelson’s renormalized Hamiltonian is in agreement with the experimental fact of small radiative corrections.

On the physical level the binding energy is computed formally through perturbation theory. Since in the Nelson Hamiltonian the ground state is not separated by a gap from the continuous spectrum, there is no hope to justify such a procedure mathematically à la Kato [K]. In fact, as to be shown, the binding energy is not analytic in $e$ near $e = 0$. To have a substitute for the formal perturbation theory, we will develop a method which yields upper and lower bounds on the binding energy. In principle,

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our scheme can be pushed to arbitrary order. In the present contribution we include the first radiative correction of order $e^6$ with an error $O(e^7 \log e)$. In [HI, CH] a similar scheme has been established. Here we advance in two central issues. Firstly for the binding energy the iteration scheme has to incorporate an external potential. Secondly, we have to make sure that the bounds are uniform in the ultraviolet cutoff. As an extra bonus, the principles underlying the theory in [CH] are stated more clearly and we believe that in the present form the iteration scheme can be applied directly to other models of a similar structure.

The Hilbert space for the electron is $L^2(\mathbb{R}^3) = \mathcal{H}_p$ and the one for the scalar Bose field the symmetric Fock space $\mathcal{F}$ over $L^2(\mathbb{R}^3)$ as one-particle space. The coupled system has $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{F}$ as state space. Its scalar product is denoted by $(\cdot, \cdot)_{\mathcal{H}}$. We omit the index if obvious from the context. On $L^2(\mathbb{R}^3)$ the canonical pair is the multiplication operator $x$ and $-ip = \nabla_x$. The Bose field on $\mathcal{F}$ is given through the creation and annihilation operators, $a^*(f)$, $a(g)$, which are densely defined for test functions $f, g \in L^2(\mathbb{R}^3)$. The field satisfies the canonical commutation relations

\begin{equation}
[a(f), a^*(g)] = (f, g)_{L^2}, \quad [a(f), a(g)] = 0, \quad [a^*(f), a^*(g)] = 0. \tag{1.1}
\end{equation}

With this notation the Hamiltonian for the particle reads

\begin{equation}
H_{at} = p^2 - \frac{Ze^2}{4\pi|x|} \tag{1.2}
\end{equation}

and the one for the field is given by

\begin{equation}
H_f = \int dk \omega(k) a^*(k)a(k), \quad \omega(k) = |k|, \tag{1.3}
\end{equation}

i.e., $H_f$ is the second quantization of $\omega$ considered as multiplication operator on $L^2(\mathbb{R}^3,dk)$.

The coupling is mediated through $a(\varphi)$, $a^*(\varphi)$ with a special choice of $\varphi$. Notationally it is convenient to have a distinguished symbol. We set

\begin{equation}
A = \int dk \chi(k) \frac{1}{\sqrt{2\omega}} \frac{k}{\omega + k^2} a(k), \quad A = a(\varphi) \quad \text{with} \quad \varphi = \chi(k) \frac{1}{\sqrt{2\omega}} \frac{k}{\omega + k^2}, \tag{1.4}
\end{equation}

where $\chi(k) = (2\pi)^{-3/2}$ for $|k| < \Lambda$ and $\chi(k) = 0$ for $|k| > \Lambda$. For $\Lambda = \infty$ the test function appearing in $\chi$ fails to be in $L^2$ because of logarithmic divergence at $k = \infty$. Thus we have to keep $\Lambda < \infty$ in intermediate steps and make sure that all estimates are uniform in $\Lambda$. $A$ is a 3-vector. In expressions as $A^*A$, $pA$ we really should write $A^* \cdot A$, $p \cdot A$. Our convention is that in strings of 3-vectors the scalar product is understood in pairs, e.g., $AAA^*p$ means $(A \cdot A)^0(A^* \cdot p)$.

The renormalized Nelson Hamiltonian $H_{\text{ren}}$ in case of hydrogen-like atoms is explained in [HHS]. We use $H_{\text{ren}}$ as our starting point with the small
modification that $H_{\text{ren}}$ is unitarily transformed to

$$H = e^{ixP_t}H_{\text{ren}}e^{-ixP_t},$$

where $P_t$ denotes the momentum of the Bose field,

$$P_t = \int dk k a^*(k)a(k).$$

Physically, $p$ acquires then the meaning of the total momentum, i.e., the momentum of particle + field, rather than the momentum of the particle by itself. The starting Hamiltonian is thus, for $Z \geq 0$,

$$H_\Lambda = p^2 - \frac{Ze^2}{4\pi|x|} + H_I + P_t^2 - 2pP_t - 2e(A^*(p - P_t) + (p - P_t)A)$$

$$+ e^2(A^*A^* + AA + 2A^*A).$$

$A$ depends on the cutoff $\Lambda$, which is not indicated explicitly in our notation.

In the following we will need a smallness condition on $e$ which is summarized as $|e| < e_0$ with suitable $e_0$ fixed throughout. $e_0$ has its origin from several sources. It is needed for the self-adjointness of $H_\Lambda$, for the existence of a ground state, and in the lower bound estimate for the ground state energy of $H_\Lambda$. In each case $e_0$ can be computed, $e_0 = O(1)$ in our units, but to actually carry out the integrations would not add to the clarity of the paper.

If $|e| < e_0$, the interaction part of (1.7) is bounded with a bound less than 1 relative to $H_\Lambda$ at $e = 0$. At $\Lambda = \infty$, $H_\infty$ is relatively form bounded with a bound less than 1. Thus $H_\Lambda, H_\infty$ define self-adjoint operators by the KLMN theorem [RS]. We set

$$E^Z_\Lambda(e) = \inf \text{spec}(H_\Lambda).$$

As proved in [BFS], $E^Z_\Lambda(e)$ is an eigenvalue of $H_\Lambda$. It persists as $\Lambda \to \infty$ [HRS]. Note that $E^Z_\Lambda(e) = E^Z_\Lambda(-e)$, since $H_\Lambda$ at $\pm e$ are unitarily equivalent.

The binding energy is the minimal energy required in ionizing the atom. Let $T_\Lambda(p)$ denote the operator $H_\Lambda$ for $Z = 0$. $p$ appears now only as a parameter and it is known that $E(p) = \inf \text{spec}(T_\Lambda(p))$ achieves its minimum at $p = 0$ [E]. Thus we define

$$T_\Lambda(0) = T_\Lambda = H_I + P_t^2 + 2e(A^*P_t + P_tA) + e^2(A^*A^* + AA + 2A^*A)$$

and the self-energy

$$E^0_\Lambda(e) = \inf \text{spec}(T_\Lambda).$$

As $\Lambda \to \infty$, $H_\Lambda \to H_\infty$ and $T_\Lambda \to T_\infty$ in the norm-resolvent sense. In particular, this ensures that $\lim_{\Lambda \to \infty} E^Z_\Lambda(e) = E^Z_\infty(e)$ and $\lim_{\Lambda \to \infty} E^0_\Lambda(e) = E^0_\infty(e)$.

**DEFINITION 1.** The binding energy of $H$ is the difference

$$E_{\text{bin}}(e) = E^0_\infty(e) - E^Z_\infty(e).$$

It satisfies $E_{\text{bin}}(e) \geq 0$. 

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Perturbation theory means to Taylor expand $E^Z_\infty(e)$ in a power series in $e$. In such a computation one first leaves $-Ze^2/4\pi |x|$ unexpanded and expands only in the coupling to the field. To lowest order one finds thereby, as to be expected,

$$E^{(0)}_{\text{bin}} = -E_{\text{at}},$$

(1.12)

where $E_{\text{at}}$ is the ground state energy of $p^2 - Ze^2/4\pi |x|$, $E_{\text{at}} = -\frac{1}{4}(Ze^2/4\pi)^2$. The first radiative correction is obtained as

$$E^{(1)}_{\text{bin}} = -E_{\text{at}}(1 + (e^2/6\pi^2)).$$

(1.13)

Such a computation is only formal, since the ground state eigenvalue of $H_\Lambda$ and of $T_\Lambda$ sits at the bottom of the continuous spectrum. The required differentiability is not ensured through the general theory of linear operators. Still, as our main result, we confirm the formal perturbation theory by proving suitable upper and lower bounds.

**THEOREM 1.** Let $0 < e < e_0$ and let $E_{\text{bin}}(e)$ as given in Definition 1. Then there exists constants $c_+, c_-$, independent of $e$, such that

$$c_- e^7 \log\left(\frac{1}{e}\right) \leq E_{\text{bin}}(e) - (-E_{\text{at}})(1 + \frac{e^2}{6\pi^2}) \leq c_+ e^7.$$  

(1.14)

**Remark.** If one reintroduces the mass $m$ of the electron, our estimate states that

$$E_{\text{bin}}(e) = \frac{1}{2} m(Ze^2/4\pi)^2(1 + \frac{e^2}{6\pi^2}) + O(e^7 \log(1/e)).$$

(1.15)

Physically, energies are calibrated in units of the effective mass $m_{\text{eff}}$, which is defined as the inverse curvature of $E(p)$ at $p = 0$, see above Eq. (1.9). We extend in (1.15) by $m_{\text{eff}}$ and formally expand the ratio $m/m_{\text{eff}}$ in $e$ with the result

$$\frac{m}{m_{\text{eff}}} = 1 - \frac{e^2}{6\pi^2} + O(e^4).$$

(1.16)

Thus in (1.15) the relative $O(e^2)$-corrections cancel precisely. We conjecture that, after mass renormalization, for hydrogen like atoms the radiative correction decreases the binding energy and is of the form $E_{\text{bin}}(e) = -m_{\text{eff}}E_{\text{at}}(1 - O(e^6 \log(1/e)))$. This conjecture is strongly supported by the Lamb shift calculations, see, e.g. [HS, Eq. (5.18) and (5.23)].

The binding energy is the difference between the ground state energy of $H_\Lambda$ and the self-energy, i.e., the ground state energy of $T_\Lambda$. Thus to prove Theorem 1 we need upper and lower bounds on $E^Z_\infty$, respectively $E^Z_0$. In fact at order $e^6$ in this difference all terms except for a single one cancel. The cases $Z = 0$ and $Z \neq 0$ are handled by the same technique. The basic idea is to use the perturbative ground state as a backbone. The upper bound is easy and a straightforward application of the variational formula. The real effort lies in the lower bound, where we employ sharp operator estimates for the carefully corrected perturbative ground state.
The method used here was originally developed in [H1] in the context of the Pauli-Fierz Hamiltonian, including spin, with an ultraviolet cutoff. Catto and Hainzl [CH] refined the method and extend the result to higher order corrections. Further applications of the method are [H2, HVV, HS].

To give a brief outline: In Section 2 bounds on the self-energy are established and in Section 3 we prove the corresponding bounds for $E_{\Lambda}^\Omega$. The bounds are uniform in the cutoff $\Lambda$. Using these bounds we derive in Section 4 the estimate claimed in (1.14). The chain of arguments for the lower bounds is somewhat lengthy. Not to interrupt the main line we collect all the required operator norm estimates in the Appendices A - C. Some of them are stated only for completeness, while others have not been established before.

2. Self-energy

In this Section we establish bounds on $E_{\Lambda}^\Omega$. It is convenient to have the shorthands

$$D_t = P_t^2 + H_t, \quad L = D_t + 2e^2A^*A. \quad (2.1)$$

**Theorem 2.** Let $\Omega$ denote the vacuum vector in $\mathcal{F}$. Then

$$E_{\Lambda}^0(e) = -e^4(\Omega, AAD_t^{-1}A^*A^*\Omega) - 4e^6(\Omega, AAD_t^{-1}P_tAD_t^{-1}A^*P_tD_t^{-1}A^*A^*\Omega) - 4e^6(\Omega, AAD_t^{-1}A^*P_tD_t^{-1}P_tAD_t^{-1}A^*A^*\Omega) + 2e^6(\Omega, AAD_t^{-1}A^*AD_t^{-1}A^*A^*\Omega) + O(e^7), \quad (2.2)$$

where the error term $O(e^7)$ is bounded uniformly in $\Lambda$.

**Proof.** Upper bound: In the following the expression $r \leq O(e^m)$ means that there exists a positive constant $c$, independent of $\Lambda$ and $e$ for $|e| \leq e_0$, such that $r \leq ce^m$.

We choose as trial function the perturbative ground state

$$\Psi = \Omega \oplus (-2e^3D_t^{-1}P_tAD_t^{-1}A^*A^*\Omega) \oplus (-e^2D_t^{-1}A^*A^*\Omega) \oplus (-2e^3D_t^{-1}A^*P_tD_t^{-1}A^*A^*\Omega). \quad (2.3)$$

**Lower bound:** As an approximate ground state (a.g.s.) we denote any $\Psi \in \mathcal{F}$, such that $\|\Psi\| = 1$ and

$$(\Psi, T_\Lambda \Psi) \leq -e^4(\Omega, AAD_t^{-1}A^*A^*\Omega) + O(e^6). \quad (2.4)$$

Such states exist as can be seen from the upper bound. We first achieve an a priori bound on the kinetic energy of $\Psi$.

**Lemma 1.** Let $\Psi$ be an a.g.s., then

$$(\Psi, D_t\Psi) \leq O(e^4). \quad (2.5)$$
Proof. We have

\[
(\Psi, T_\lambda \Psi) \geq \frac{1}{2} (\Psi, D_t \Psi) + \frac{1}{2} (\Psi, D_t \Psi) + 2 \Re (\Psi, [2eA^* P_t + e^2 A^* A^*] \Psi)
\]

\[
\geq \frac{1}{2} (\Psi, D_t \Psi) - 2\|2eD_t^{-1/2} A^* P_t \Psi + e^2 D_t^{-1/2} A^* A^* \Psi\|^2 \tag{2.6}
\]

\[
\geq \frac{1}{2} (\Psi, D_t \Psi) - O(e^2) \|P_t \Psi\|^2 - O(e^4) [\|\Psi\|^2 + \|H_t^{1/2} \Psi\|^2],
\]

since we know from Lemma 13 of Appendix C that \(\|D_t^{-1/2} A^*\| \leq c\) and \(AAD_t^{-1} A^* A^* \leq c(1 + H_t)\). Here and throughout the paper \(c\) will denote a generic constant, independent of \(\lambda\) and \(e\). For \(e\) sufficiently small together with \(\ref{2.3}\) we arrive at the assertion of the lemma. \(\square\)

With our notation we can rewrite

\[
(\Psi, T_\lambda \Psi) = (\Psi, L \Psi) + 2 \Re (\Psi, [2eA^* P_t + e^2 A^* A^*] \Psi). \tag{2.7}
\]

Observe that \(L\) is invertible on \((1 - P_t)F\). Therefore we obtain the identity

\[
(\Psi, T_\lambda \Psi) = -\|2eL^{-1/2} A^* P_t \Psi + e^2 L^{-1/2} A^* A^* \Psi\|^2 + \|L^{1/2} h\|^2, \tag{2.8}
\]

with

\[
h = \Psi + [2eL^{-1} A^* P_t + e^2 L^{-1} A^* A^*] \Psi = \Psi + F^* \Psi. \tag{2.9}
\]

This notation turns out to be very convenient. In fact this idea allowed \(\ref{2.2}\) to recover higher order corrections. In the following we will implicitly show that \(h\) is small in the sense \((h, D_t h) \leq O(e^6)\) which implies that \(\Psi\) has to be close to the perturbative ground state \(\ref{2.1}\). Notice that \(\ref{2.9}\) immediately yields \((h, D_t h) \leq O(e^4)\).

In the first term on the r.h.s. of \(\ref{2.8}\) we insert \(\Psi = h - F^* \Psi\), hence

\[
(\Psi, T_\lambda \Psi) = -\|2eL^{-1/2} A^* P_t h - 2eL^{-1/2} A^* P_t F^* \Psi
\]

\[
+ e^2 L^{-1/2} A^* A^* \Psi\|^2 + \|L^{1/2} h\|^2
\]

\[
= -4e^2 \|L^{-1/2} A^* P_t h\|^2 - 4e^2 \|L^{-1/2} A^* P_t F^* \Psi\|^2
\]

\[
- e^4 \|L^{-1/2} A^* A^* \Psi\|^2 + 2 \Re \left[ 4e^2 (h, P_t AL^{-1} A^* P_t F^* \Psi)
\]

\[
- 2e^3 (\Psi, FP_t AL^{-1} A^* \Psi) + 2e^3 (h, P_t AL^{-1} A^* A^* \Psi) \right]
\]

\[
+ \|L^{1/2} h\|^2.
\]

First we estimate the diagonal terms. By Lemma 13

\[
e^2 \|L^{-1/2} A^* P_t h\|^2 \leq e^2 \|P_t h\|^2 \|L^{-1/2} A^*\|^2 \leq O(e^2) \|P_t h\|^2. \tag{2.11}
\]
Slightly more care is needed for the second term,

\[
e^2 \|L^{-1/2} A^* P_L F^* \Psi\|^2 = 4e^4 \|L^{-1/2} A^* P_L L^{-1} A^* P_L \Psi\|^2 \\
+ 2e^5 \Re(\Psi, P_L AL^{-1} P_L AL^{-1} A^* A^* \Psi) \\
+ e^6 \|L^{-1/2} A^* P_L L^{-1} A^* A^* \Psi\|^2. \tag{2.12}
\]

The first term on the r.h.s. of (2.12) we estimate by

\[
e^4 \|P_L \Psi\|^2 \|L^{-1/2} A^* P_L L^{-1} A^*\|^2 \leq O(e^8), \tag{2.13}
\]

because of

\[
\|L^{-1/2} A^* P_L L^{-1} A^*\| \leq \|L^{-1/2} A^*\| \|P_L L^{-1}\| \|L^{-1/2} A^*\|,
\]

(notice \(\|P_L L^{-1}\| \leq \|P_L D_L^{-1}\| \leq 1\), Lemma 13 and (2.5).

Applying in a similar way Lemma 13 to the second term on the r.h.s. of (2.12) we obtain

\[
e^5 \|\Psi, P_L AL^{-1} P_L AL^{-1} A^* P_L L^{-1} A^* A^* \Psi\| \\
\leq O(e^5) \|P_L \Psi\| \|(1 + H_L)^{1/2} \| \leq O(e^7). \tag{2.14}
\]

Obviously,

\[
e^6 \|L^{-1/2} A^* P_L L^{-1} A^* A^* \Psi\|^2 \leq e^6 \|D_L^{-1/2} A^* P_L L^{-1} A^* A^* \Psi\|^2. \tag{2.15}
\]

Recall the resolvent equation

\[
\frac{1}{L} = \frac{1}{D_L} - 2e^2 \frac{1}{D_L} A^* A \frac{1}{D_L} + 4e^4 \frac{1}{D_L} A^* A \frac{1}{L} A^* A \frac{1}{D_L}. \tag{2.16}
\]

Consequently, using Lemma 13

\[
\|D_L^{-1/2} A^* P_L L^{-1} A^* A^* \Psi\| \leq \|D_L^{-1/2} A^* P_L D_L^{-1} A^* A^* \Psi\| \\
+ 2e^2 \|D_L^{-1/2} A^* P_L D_L^{-1} A^* AD_L^{-1} A^* A^* \Psi\| \\
+ 4e^4 \|D_L^{-1/2} A^* P_L D_L^{-1} A^* AL^{-1} A^* AD_L^{-1} A^* A^* \Psi\| \\
\leq \|D_L^{-1/2} A^* P_L D_L^{-1} A^* A^* \Psi\| + O(e^2) \|(1 + H_L)^{1/2} \| + O(e^4) \|(1 + H_L)^{1/2} \|. \tag{2.17}
\]

By means of Lemma 13

\[
(\Psi, AAD_L^{-1} P_L AL^{-1} A^* P_L D_L^{-1} A^* A^* \Psi) \\
\leq \langle \Omega, AAD_L^{-1} P_L AD_L^{-1} A^* P_L D_L^{-1} A^* A^* \Omega \rangle \|\Psi\|^2 + c(\Psi, D_L \Psi). \tag{2.18}
\]

Thus we arrive at

\[
e^2 \|L^{-1/2} A^* P_L F^* \Psi\|^2 \leq e^6 \langle \Omega, AAD_L^{-1} P_L AD_L^{-1} A^* P_L D_L^{-1} A^* A^* \Omega \rangle + O(e^8). \tag{2.19}
\]
Using again (2.10) we obtain
\[ e^4 \| L^{-1/2} A^* A^* \Psi \|^2 = e^4 (\Psi, A A D^{-1}_1 A^* A^* \Psi) \]
\[ - 2e^6 (\Psi, A A D^{-1}_1 A^* A D^{-1}_1 A^* A^* \Psi) + 4e^8 \| L^{-1/2} A A D^{-1}_1 A^* A^* \Psi \|^2. \] (2.20)

According to Lemma 2 (cf. [H1, Lemma 1])
\[ (\Psi, A A D^{-1}_1 A^* A^* \Psi) \leq (\Omega, A A D^{-1}_1 A^* A^* \Omega) \| \Psi \|^2 + c(\Psi, D_1 \Psi). \] (2.21)

and to Lemma 5
\[ (\Psi, A A D^{-1}_1 A^* A D^{-1}_1 A^* A^* \Psi) \geq (\Omega, A A D^{-1}_1 A^* A D^{-1}_1 A^* A^* \Omega) \| \Psi \|^2 - c(\Psi, D_1 \Psi). \] (2.22)

Since
\[ \| L^{-1/2} A A D^{-1}_1 A^* A^* \Psi \| \leq \| L^{-1/2} A^* \| \| A D^{-1}_1 \| \| D^{-1}_1 \| A^* A^* \Psi \|
\leq c \| (1 + H_1)^{-1/2} \| \] (2.23)

and using our a priori bound (2.5), we arrive at
\[ e^4 \| L^{-1/2} A^* A^* \Psi \|^2 \leq e^4 (\Omega, A A D^{-1}_1 A^* A^* \Omega)
- 2e^6 (\Omega, A A D^{-1}_1 A^* A D^{-1}_1 A^* A^* \Omega) + O(e^8). \] (2.24)

Next we treat the off-diagonal terms. Using Lemma 13 yields
\[ e^2 |(h, P_1 A L^{-1} A^* P_1 F^* \Psi)|
\leq O(e^3) \| P_1 h \| \| P_1 \Psi \| + O(e^4) \| P_1 h \| \| (1 + H_1)^{1/2} \| \leq O(e) \| P_1 h \|^2 + O(e^7). \] (2.25)

By definition of \( F \)
\[ e^3 (\Psi, F P_1 A L^{-1} A^* A^* \Psi)
= 2e^4 (\Psi, P_1 A L^{-1} P_1 A L^{-1} A^* A^* \Psi) + e^5 (\Psi, A A L^{-1} P_1 A L^{-1} A^* A^* \Psi). \] (2.26)

Concerning the first term on the r.h.s., we insert for the left vector of the inner product \( \Psi = h - F^* \Psi \). By Lemma 13 we have on the one hand
\[ e^4 |(h, P_1 A L^{-1} P_1 A L^{-1} A^* A^* \Psi)|
\leq O(e^4) \| P_1 h \| \| (1 + H_1)^{1/2} \| \leq O(e) \| P_1 h \|^2 + O(e^7) \] (2.27)
and on the other hand
\[ e^4 |(\Psi, F P_1 A L^{-1} P_1 A L^{-1} A^* A^* \Psi)| \leq O(e^5) \| P_1 \Psi \| \| (1 + H_1)^{1/2} \|
+ e^6 |(\Psi, A A L^{-1} P_1 A L^{-1} P_1 A L^{-1} A^* A^* \Psi)|. \] (2.28)

To the remaining term in (2.28) we apply the resolvent equation, the estimates in Lemma 13 as well as Lemma 6 (cf. [CH, Appendix C]), which states that
\[ |(\Psi, A A D^{-1}_1 P_1 A D^{-1}_1 A^* A^* \Psi)| \leq c \| \Psi \| \| H^{-1/2}_1 \|, \] (2.29)
and
\[ e^5 \langle \Psi, AAD_{\Gamma}^{-1} P_t AD_{\Gamma}^{-1} A^* A^* \Psi \rangle \leq O(e^5) \| \Psi \| \| H_1^{1/2} \Psi \|. \] (2.30)

Thus we have gained
\[ e^3 \langle \Psi, FP_t AL^{-1} A^* A^* \Psi \rangle \leq O(e) \| P_t h \|^2 + O(e^7). \] (2.31)

Assembling all together we conclude
\[
(\Psi, T_\Lambda \Psi) \geq -e^4(\Omega, AAD_{\Gamma}^{-1} A^* A^* \Omega) - 4e^6(\Omega, AAD_{\Gamma}^{-1} P_t AD_{\Gamma}^{-1} A^* P_t D_{\Gamma}^{-1} A^* A^* \Omega) \\
+ 2e^6(\Omega, AAD_{\Gamma}^{-1} A^* AD_{\Gamma}^{-1} A^* A^* \Omega) + \| L^{1/2} h \|^2 \\
+ 2\Re(h, 2e^3 P_t AL^{-1} A^* A^* \Psi) - O(e) \| P_t h \|^2 - O(e^7). \] (2.32)

We further use the identity
\[
\| L^{1/2} h \|^2 + 2\Re(h, 2e^3 P_t AL^{-1} A^* A^* \Psi) \\
= -4 \| e^3 L^{-1/2} P_t AL^{-1} A^* A^* \Psi \|^2 + \| L^{1/2} \hat{h} \|^2 \] (2.33)

with \( \hat{h} = h + 2e^3 L^{-1} P_t AL^{-1} A^* A^* \Psi = h + G^* \Psi \). (2.34)

By means of (2.33) together with Lemma 4 we further estimate
\[
(\Psi, T_\Lambda \Psi) \geq -e^4(\Omega, AAD_{\Gamma}^{-1} A^* A^* \Omega) \\
- 4e^6(\Omega, AAD_{\Gamma}^{-1} P_t AD_{\Gamma}^{-1} A^* P_t D_{\Gamma}^{-1} A^* A^* \Omega) \\
+ 2e^6(\Omega, AAD_{\Gamma}^{-1} A^* AD_{\Gamma}^{-1} A^* A^* \Omega) - O(e^6)(\Psi, D_t \Psi) + \| L^{1/2} \hat{h} \|^2 - O(e) \| P_t \hat{h} \|^2 - O(e) \| G^* \Psi \|^2 - O(e^7) \\
\geq -e^4(\Omega, AAD_{\Gamma}^{-1} A^* A^* \Omega) - 4e^6(\Omega, AAD_{\Gamma}^{-1} P_t AD_{\Gamma}^{-1} A^* P_t D_{\Gamma}^{-1} A^* A^* \Omega) \\
+ 2e^6(\Omega, AAD_{\Gamma}^{-1} A^* AD_{\Gamma}^{-1} A^* A^* \Omega) \\
- 4e^6(\Omega, AAD_{\Gamma}^{-1} A^* P_t D_{\Gamma}^{-1} P_t AD_{\Gamma}^{-1} A^* A^* \Omega) - O(e^7), \] (2.35)

for \( e \) small enough such that
\[
\| L^{1/2} \hat{h} \|^2 - O(e) \| P_t \hat{h} \|^2 \geq \| P_t \hat{h} \|^2 (1 - O(e)) \geq 0, \] (2.36)

which concludes the proof of Theorem 2 \( \square \)

3. Ground state energy

In the following we denote \( V = -\frac{e^2}{4\pi|x|} \) and \( \phi = \phi_0 \otimes \Omega \), where \( \phi_0 \) is the ground state of the Schrödinger operator \( p^2 + V \), with ground state energy \( E_{at} \), i.e.,
\[
(p^2 + V)\phi_0 = E_{at} \phi_0. \] (3.1)
Since $E_{at} = -\frac{1}{4(4\pi)^2}Z^2 e^4$, we observe, e.g., by virial theorem, $\|p\phi\|^2 = O(e^4)$.

For convenience we introduce the notation

\[
P = p - P_t, \\
B = P^2 + V - E_{at} + H_t, \\
K = B + 2e^2 A^* A.
\]

**THEOREM 3.**

\[
E_\Lambda^Z(e) = E_{at} - 4e^2(\phi, pAB^{-1}A^* p\phi) - e^4(\phi, AAB^{-1} A^* A^* \phi) \\
- 4e^6(\phi, AAB^{-1} P_t AB^{-1} A^* P_t B^{-1} A^* A^* \phi) \\
- 4e^6(\phi, AAB^{-1} A^* P_t B^{-1} P_t AB^{-1} A^* A^* \phi) \\
+ 2e^6(\phi, AAB^{-1} A^* AB^{-1} A^* A^* \phi) + O(e^7(1/e)) \tag{3.2}
\]

uniformly in $\Lambda$.

3.1. **Upper bound.** As in Section 2 we use the perturbative ground state

\[
\Psi = \phi \oplus (-2eB^{-1} A^* p\phi - 2e^2B^{-1} PAB^{-1} A^* A^* \phi) \\
\oplus (4e^2 B^{-1} A^* P^2 B^{-1} A^* p\phi - 2e^2 B^{-1} A^* A^* \phi) \oplus (-2e^3 B^{-1} A^* P^2 B^{-1} A^* A^* \phi)
\] \tag{3.3}

Apart from error terms which are at least of order $O(e^7)$ we obtain

\[
(\Psi, H\Psi) = E_{at} - 4e^2(\phi, pAB^{-1} A^* p\phi) - e^4(\phi, AAB^{-1} A^* A^* \phi) \\
+ 2e^6(\phi, AAB^{-1} A^* AB^{-1} A^* A^* \phi) - 4e^6(\phi, AAB^{-1} PAB^{-1} A^* P^2 B^{-1} A^* A^* \phi) \\
- 4e^6(\phi, AAB^{-1} A^* P^2 B^{-1} PAB^{-1} A^* A^* \phi) \\
+ 8e^4 R(\phi, pAB^{-1} PAB^{-1} A^* A^* \phi) + O(e^7). \tag{3.4}
\]

The last three terms can be simplified further by taking advantage of the fact that we deal with the Coulomb potential $V = -\frac{2}{4\pi|x|}$. Namely we insert $P = p - P_t$ and show that all terms resulting from the summand $p$ are of higher order. For this purpose we transform canonically as

\[
x \rightarrow x/e^2, \quad p \rightarrow e^2 p, \tag{3.5}
\]
i.e., through the unitary $U_e$ as

\[
U_e p^2 U_e^* = e^4 p^2, \quad U_e(p^2 - \frac{e^2}{4\pi|x|})U_e^* = e^4(p^2 - \frac{1}{4\pi|x|}). \tag{3.6}
\]

By means of that transformation we estimate, e.g.,

\[
(\phi, AAB^{-1} pAB^{-1} A^* pB^{-1} A^* A^* \phi) = \\
(U_e\phi, U_e AAB^{-1} pAB^{-1} A^* pB^{-1} A^* A^* U_e^* U_e\phi) \leq \\
\|\phi\|^2 \|B_e^{-1/2} A^* e^2 p^2 B_e^{-1} A^* A^* \| \leq O(e^4), \tag{3.7}
\]
where \( B_e = (e^2 p - P_1)^2 + e^2 V - E_{\text{at}} + H_t \). Using similar estimates, together with Schwarz inequality, we see that in the last three terms in (3.4), apart from higher order terms, only expressions involving \( P_1 \) play a role.

Finally notice that
\[
|\langle \phi, pAB^{-1}PAB^{-1}A^*A^*\phi \rangle| = e^2 |\langle U_e \phi, pAB_e^{-1}PAB_e^{-1}A^*A^*U_e \phi \rangle| 
\leq O(e^4), \quad (3.8)
\]
which follows from expanding \( 1/B_e \) and the fact that the lowest order term vanishes, since \( (U_e \phi, pU_e \phi) = 0 \).

3.2. Lower bound. We recall the convention on \( O(e^m) \) from Section 2. The proof of the lower bound proceeds in analogy to Theorem 2. The decisive difference is that we deal now with operators \( B, K \), and \( P \) which do not commute which means that we have to be a bit more carefully in our estimates. Apart from that the strategy is not altered.

As in Section 2 we consider an a.g.s. satisfying
\[
(\Psi, (H - E_{\text{at}})\Psi) \leq O(e^4). \quad (3.9)
\]
Because \( E_{\text{at}} = O(e^4) \) we conclude the bound
\[
(\Psi, [P^2 + H_t]\Psi) \leq O(e^4). \quad (3.10)
\]
Note that the existence of a true ground state is not needed for the argument. We can write
\[
(\Psi, H\Psi) = E_{\text{at}} \|\Psi\|^2 + (\Psi, K\Psi) + 2\Re(\Psi, 2eA^*P\Psi + e^2 A^*A^*\Psi). \quad (3.11)
\]
Following the same scheme as in (2.8) we obtain the identity
\[
(\Psi, H\Psi) = E_{\text{at}} \|\Psi\|^2 - \|2eK^{-1/2}A^*P\Psi + e^2 K^{-1/2}A^*A^*\Psi\|^2 + \|K^{1/2}h\|^2, \quad (3.12)
\]
with
\[
h = \Psi + 2eK^{-1}A^*P\Psi + e^2 K^{-1}A^*A^*\Psi = \Psi + F^*\Psi. \quad (3.13)
\]
Notice that \( h \) also fulfills \( (h, [P^2 + H_t]h) \leq O(e^4) \).

Some of the terms in the lower bound are logarithmically infrared divergent. In this case we replace \( H_t \) by \( H_t + e^7 \), which causes the additional error \(-e^7\|\Psi\|^2\) in the r.h.s. of (3.10). Also the bound acquires a logarithmic correction.

We now insert
\[
\Psi = h - 2eK^{-1}A^*P\Psi - e^2 K^{-1}A^*A^*\Psi, \quad (3.14)
\]
in (3.11) in order to obtain
\[
(\Psi, H\Psi) = E_{\text{at}} - \| - 4e^2 K^{-1/2}A^*PK^{-1}A^*P\Psi \\
- 2e^3 K^{-1/2}A^*PK^{-1}A^*A^*\Psi + 2eK^{-1/2}A^*Ph \\
+ e^2 K^{-1/2}A^*A^*\Psi\|^2 + \|K^{1/2}h\|^2, \quad (3.15)
\]
recall that $\|\Psi\| = 1$. Multiplying out the norm we observe that (3.14) is equal to

\[
E_{at} + \|K^{1/2}h\|^2 = -4e^2(h, PAK^{-1}A^*Ph) - e^4(AAK^{-1}A^*A^*\Psi) - 16e^4(\Psi, PAK^{-1}PAK^{-1}A^*PK^{-1}A^*P\Psi) - 4e^6(AAK^{-1}PAK^{-1}A^*PK^{-1}A^*A^*\Psi) + 2\Re\left[4e^4(\Psi, PAK^{-1}PAK^{-1}A^*A^*\Psi)\right] + 4e^4(\Psi, AAK^{-1}PAK^{-1}A^*Ph) + 2e^5(AAK^{-1}PAK^{-1}A^*A^*\Psi) - 2e^3(h, PAK^{-1}A^*A^*\Psi) - 8e^5(\Psi, PAK^{-1}PAK^{-1}A^*PK^{-1}A^*A^*\Psi) - 8e^3(h, PAK^{-1}PAK^{-1}A^*Ph))
\]

(3.15)\(\underline{\quad}\) (3.19)\(\underline{\quad}\) (3.20)\(\underline{\quad}\) (3.21)\(\underline{\quad}\) (3.22)\(\underline{\quad}\) (3.23)\(\underline{\quad}\) (3.24)

Applying Lemma 15 (iii) together with (3.9) we immediately obtain

\[
|3.17| \leq O(e^{4\ln(1/e)}\|P\Psi\|^2 \leq O(e^{8\ln(1/e)})
\]

(3.25)

and by Lemma 15 (iii) and (iv), the bounds

\[
|3.23| \leq O(e^{5\ln(1/e)}\|P\Psi\|\|1 + Hf\|^{1/2}\Psi|| \leq O(e^{7\ln(1/e)}),
\]

respectively, by Lemma 14 (i) and Lemma 15 (iii)

\[
|3.24| \leq O(e^3(\|P\Psi\|\|PPh\|) \leq O(e^7).
\]

(3.26)

Additionally by Lemma 12 we can bound (3.21) by

\[
|3.21| = 2e^5|(\Psi, AAK^{-1}PAK^{-1}A^*A^*\Psi)| \leq O(e^{5\ln(1/e)}\|P\Psi\|\|H^{1/2}_f\| \leq O(e^{7\ln(1/e)}). (3.26)
\]

In the remaining terms, apart from (3.20) and (3.22), we insert again $\Psi = h - 2eK^{-1}A^*P\Psi - e^2K^{-1}A^*A^*\Psi$.

Applying our inequalities in Lemma 14 and 15 we infer

\[
(\Psi, H\Psi) \geq -4e^2(h, PAK^{-1}A^*Ph) - e^4(h, AAK^{-1}A^*A^*h) - 4e^6(h, AAK^{-1}PAK^{-1}A^*PK^{-1}A^*A^*h) + 2e^4\Re(h, 4PAK^{-1}PAK^{-1}A^*A^*h) - 2e^3\Re(h, 2PAK^{-1}A^*A^*\Psi) + 2e^4\Re(h, 4PAK^{-1}A^*PK^{-1}A^*A^*\Psi) + \|K^{1/2}h\|^2 + E_{at} - O(e^7 \ln(1/e)).
\]

(3.27)
Neglecting first the terms \( E_{at} - \mathcal{O}(e^7 \ln(1/e)) \) we rewrite (3.27) in the short-hand
\[
(h, [K + R]h) - 2\Re(h, [2e^3PAK^{-1}A^*A^* - 4e^4PAK^{-1}A^*PK^{-1}A^*A^*]\Psi),
\]
where
\[
R = -4e^2PAK^{-1}A^*P - e^4AAK^{-1}A^*A^* + 8e^4\Re[PAK^{-1}PAK^{-1}A^*A^*] - 4e^6AAK^{-1}PAK^{-1}A^*PK^{-1}A^*A^*. \tag{3.29}
\]
Since, due to the Lemmas in Appendix 13, \( R \) is relatively bounded to \( K \), we conclude that for \( e \) small enough
\[
K + R \geq -Ce^4 := -\mu. \tag{3.30}
\]
In fact, by Lemma 7 to 10 and Lemma 15 we see that for \( e \) small enough, i.e., for those \( e \) such that \( H_{\ell} \) in \( K \) dominates the error terms from \( R \),
\[
K + R \geq (1 - ce^2)P^2 + V - c'e^4 \geq -Ce^4,
\]
for appropriate constants which implies (3.30).
Therefore (3.28) can be rewritten as
\[
(3.28) = -\mu ||h||^2 + (h, [K + R + \mu]h) - 2\Re(h, [2e^3PAK^{-1}A^*A^* - 4PAK^{-1}A^*PK^{-1}A^*A^*]\Psi)
\]
\[
= -\mu ||h||^2 + ||[K + R + \mu]^{1/2}h||^2 - 4e^6||[K + R + \mu]^{-1/2}[PAK^{-1}A^*A^* - 2ePAK^{-1}A^*PK^{-1}A^*A^*]\Psi||^2. \tag{3.31}
\]
with
\[
\tilde{h} = h + 2e^3[K + R + \mu]^{-1}PAK^{-1}A^*A^*\Psi
- 4e^4[K + R + \mu]^{-1}PAK^{-1}A^*PK^{-1}A^*A^*\Psi. \tag{3.32}
\]
Using (3.32), Lemma 14 and 15 together with the fact that \( \mu \) is of order \( e^4 \) we estimate
\[
(3.32) = -\mu ||h||^2 + ||[K + R + \mu]^{1/2}h||^2
- 4e^6||[K + R + \mu]^{-1/2}[PAK^{-1}A^*A^* - 2ePAK^{-1}A^*PK^{-1}A^*A^*]\Psi||^2
\geq ||[K + R]^{1/2}\tilde{h}||^2 - 4e^6||[K + R + \mu]^{-1/2}PAK^{-1}A^*A^*\Psi||^2 - \mathcal{O}(e^7 \ln(1/e)). \tag{3.33}
\]
Apart from errors of order \( \mathcal{O}(e^7) \) we can set \( \Psi = \tilde{h} \) and \( h = h \). Consequently,
\[
(\Psi, H\Psi) \geq E_{at}
+ (\tilde{h}, [K + R]\tilde{h}) - 4e^6(\tilde{h}, AAK^{-1}A^*PK^{-1}PAK^{-1}A^*A^*\tilde{h})
- \mathcal{O}(e^7 \ln(1/e)), \tag{3.34}
\]
which leads to

\[(\Psi, H\Psi) \geq E_{at} + (\hbar, K\hbar) - \left(4e^2(\hbar, PAK^{-1}A^*P\hbar) + e^4(\hbar, AAK^{-1}A^*A^*\hbar) + 4e^6(\hbar, AAK^{-1}PAK^{-1}A^*A^*\hbar) + 4e^6(\hbar, AAK^{-1}A^*PK^{-1}PAK^{-1}A^*A^*\hbar) - 8e^4\Re(\hbar, PAK^{-1}PAK^{-1}A^*A^*\hbar)\right) - O(e^7 \ln(1/e)). \quad (3.35)\]

Next, we extract the $e^22A^*A$-term. To this aim recall $K = B + e^22A^*A$, use the resolvent equation \[2.16\], the operator inequalities in Lemma \[14\] and Lemma \[15\] We obtain

\[(\Psi, H\Psi) \geq E_{at} + (\hbar, B\hbar) - \left(4e^2(\hbar, PAB^{-1}A^*P\hbar) + e^4(\hbar, AAB^{-1}A^*A^*\hbar) - 2e^6(\hbar, AAB^{-1}A^*A^*AB^{-1}A^*A^*\hbar) + 4e^6(\hbar, AAB^{-1}PAB^{-1}A^*PB^{-1}A^*A^*\hbar) + 4e^6(\hbar, AAB^{-1}A^*PB^{-1}PAB^{-1}A^*A^*\hbar) - 8e^4\Re(\hbar, PAB^{-1}PAB^{-1}A^*A^*\hbar)\right) - O(e^7 \ln(1/e)). \quad (3.36)\]

To the terms inside the bracket we apply now Lemma \[7\] to \[10\]. The error terms corresponding to these Lemmas are bounded from below by

\[-O(e^4 \ln(1/e))\left(\|P\hbar\|^2 + (\hbar, H_{at}h)\right).

Recall $B = P^2 + V - E_{at} + H_{at}$. Therefore the error $-O(e^4 \ln(1/e))(\hbar, H_{at}h)$ is controlled by $\hbar, H_{at}h$ for $e$ small enough. Since $\|P\hbar\|^2 \leq O(e^4)$, we infer

\[(\Psi, H\Psi) \geq E_{at} + (\hbar, [P^2 + V - E_{at}]h) - \left(4e^2(\hbar, PAB^{-1}A^*P\hbar) + e^4(\hbar, AAB^{-1}A^*A^*\hbar) - 2e^6(\hbar, AAB^{-1}AB^{-1}A^*AB^{-1}A^*A^*\hbar) + 4e^6(\hbar, B^{-1}PAB^{-1}A^*PB^{-1}A^*A^*\hbar) + 4e^6(\hbar, B^{-1}A^*PB^{-1}PAB^{-1}A^*A^*\hbar) - 8e^4\Re(\varphi^i PB^{-1}\varphi \cdot P\hbar, B^{-1}h\varphi^i \cdot \varphi)\right) - O(e^7 \ln(1/e)) \quad (3.37)\]

where we used the notation

\[[\hbar\varphi^i \cdot \varphi]_{n+2} = \hbar_n(x, k_1, \ldots, k_n)\varphi(k_{n+1})\varphi(k_{n+2})\]

as introduced in \[3.6\]. By Lemma \[17\] (i) and Lemma \[8\] the first and the last term in the bracket are bounded by $(e^2 + e^4)\|P\hbar\|^2 + e^4\|\hbar\|^2$. Therefore they are relatively bounded with respect to $P^2 + V - E_{at}$. Due to Lemmas \[7\] to \[9\] the other terms in the bracket are bounded. Since $P^2 + V - E_{at}$ has 0 as isolated eigenvalue, we are now in the favorable position to apply Kato's perturbation theory \[K\].
To illustrate, for simplicity, we concentrate on one term inside the bracket, e.g., the term corresponding to $e^4(h^\varphi \cdot \varphi, B^{-1}h^\varphi \cdot \varphi)$. In other words we search for the ground state energy of

$$\langle \bar{h}, [P^2 + V - E_{at}]\bar{h} \rangle - e^4(h^\varphi \cdot \varphi, B^{-1}h^\varphi \cdot \varphi). \quad (3.38)$$

Recall $\phi$ is the unique ground state of $P^2 + V - E$ at with eigenvalue 0 therefore due to Kato

$$= -e^4(\phi, AAB^{-1}A^*A^*\phi)\|\bar{h}\|^2 + O(e^8)\|\bar{h}\|^2, \quad (3.39)$$

since

$$= (\phi, AAB^{-1}A^*A^*\phi). \quad (3.40)$$

Remark that from Lemma 14 and (3.9) together with definition (3.12) and (3.32) we obtain

$$\|\bar{h}\|^2 - 1 \leq O(e^3 \log(1/e)).$$

Consequently

$$= -e^4(\phi, AAB^{-1}A^*A^*\phi) + O(e^7 \log(1/e)). \quad (3.41)$$

Using this strategy for each term in the bracket of (3.37) and noticing $\|P\phi\|^2 = O(e^4)$, we obtain an equation equivalent to (3.4), this time with an error of order $O(e^7 \log(1/e))$. Finally we use the considerations from the upper bound and conclude the proof of Theorem 3.

4. Proof of Theorem 1

To complete the proof of Theorem 1 we only have to work out the leading terms in (2.2) and (3.2) and to show that the difference agrees with (1.14) up to errors of order $e^7$. For this purpose we use the resolvent expansion

$$\frac{1}{B} = \frac{1}{Q} - \frac{1}{Q} b \frac{1}{Q} + \frac{1}{Q} b \frac{1}{B} \frac{1}{Q} \quad (4.1)$$

with $Q = p^2 + V - E_{at} + H_f + P_t^2$ and $b = -2pP_t$. (4.1) is inserted in (3.2). The terms linear in $p$ vanish and the quadratic terms are of order $O(e^8)$, since $(\phi_0, p^2\phi_0) = -2E_{at}$. Thus only the term $Q^{-1}$ remains. Comparing it with (2.2) we note that all terms in Eq. (2.2) are canceled. The only contribution remaining is then

$$E^0_\Lambda - E^2_\Lambda = -E_{at} + 4e^2(\phi, pAQ^{-1}A^*p\phi) + O(e^7 \log e). \quad (4.2)$$

The scalar product in (4.2) reads, to lowest order,

$$-E_{at} \frac{4}{3} e^2(2\pi)^{-3} \int_{|k| \leq \Lambda} dk \frac{1}{2|k|} k^2(|k| + k^2)^{-3}. \quad (4.3)$$

Taking the limit $\Lambda \to \infty$, using that all error bounds are uniform in $\Lambda$, proves (1.14).
Appendix A. Sharp estimates needed for Theorem 2

We collect sharp inequalities as used in the proof of Theorem 2 and proceed analogously to [H1, CH], with the slight difference that we have to take care of the uniform boundedness of the error terms in the cutoff Λ.

For this aim notice that for \( s \in (0,1) \)

\[
\int dk \left| \frac{\varphi(k)}{|k|^s} \right|^2 \leq C \frac{s}{(1-s)}, \tag{A.1}
\]

where the constant \( C \) is independent of the cutoff. For later purposes we also define

\[
c_I = \int \left| \frac{\varphi(k)}{|k|^{1/2}} \right|^2 dk, \quad c_{II} = \int \frac{\varphi(k)}{|k|} dk. \tag{A.2}
\]

Recall that

\[
D_f = P_f^2 + H_f.
\]

**Lemma 2.**

\[
(\Psi, AAD_f^{-1} A^* A^* \Psi) \leq (\Omega, AAD_f^{-1} A^* A^* \Omega) \|\Psi\|^2 + c(\Psi, D_f \Psi) \tag{A.3}
\]

with \( c \) uniformly bounded in \( \Lambda \).

**Proof.** The proof follows [H1, Lemma 1]. Fix the photon number \( n \) and recall

\[
[A^* A^* \psi_n]_{n+2} = \frac{1}{\sqrt{(n+2)(n+1)}} \sum_{j=1}^{n+2} \sum_{i=1}^{n+2} \varphi(k_j) \cdot \varphi(k_i) \times
\]

\[
\times \psi_n(k_1, \ldots, k_j, \ldots, k_i, \ldots, k_{n+2}), \tag{A.4}
\]

where \( k_j \) indicates that the \( j \)-th variable is omitted. Using permutation symmetry we distinguish between three different terms,

\[
(\psi_n, AAD_f^{-1} A^* A^* \psi_n) = I_n + II_n + III_n, \tag{A.5}
\]

which result naturally once we insert Equation (A.4) into (A.5) and have in mind that the l.h.s. of (A.5) can be written as

\[
(A^* A^* \psi_n, [P_f^2 + H_f]^{-1} A^* A^* \psi_n). \tag{A.6}
\]

The most important diagonal term reads

\[
I_n = 2 \int \frac{[\varphi(k_1) \cdot \varphi(k_2)]^2 |\psi_n(k_3, \ldots, k_{n+2})|^2}{|\sum_{i=1}^{n+2} k_i|^2 + \sum_{i=1}^{n+2} |k_i|} dk_1 \ldots dk_{n+2}. \tag{A.7}
\]

If we set \( Q = |\sum_{i=3}^{n+2} k_i|^2 + |k_1 + k_2|^2 + \sum_{i=1}^{n+2} |k_i| \) and \( b = -2 \left[ \sum_{i=3}^{n+2} k_i \right] \left[ k_1 + k_2 \right] \) and use the expansion (A.1) then we see that the second term vanishes.
when integrating over $k_1, k_2$. Therefore, with $Q \geq |k_1 + k_2|^2 + |k_1| + |k_2|$ and $Q + b \geq |k_1| + |k_2|$ we arrive at

$$I_n \leq 2\|\psi_n\|^2 \int \frac{[\varphi(k_1) \cdot \varphi(k_2)]^2}{|k_1 + k_2|^2 + |k_1| + |k_2|} dk_1 dk_2$$

$$+ 4 \int \frac{|\varphi(k_1)|^2 |\varphi(k_2)|^2 |k_1 + k_2|^2}{(|k_1 + k_2|^2 + |k_1| + |k_2|)^2 (|k_1| + |k_2|)}$$

$$\times \left( \sum_{i=3}^{n+2} k_i^2 |\psi_n(k_3, \ldots, k_{n+2})|^2 dk_1 \ldots dk_{n+2} \right)$$

$$\leq (\Omega, AAD^{-1} A^* A^* \Omega)\|\psi_n\|^2 + c_f^2 \|P_f \psi_n\|^2. \quad (A.8)$$

Furthermore, by use of Schwarz inequality,

$$II_n \leq n \int \frac{[\varphi(k_1)]^2 |\varphi(k_2)||\varphi(k_1)||\varphi(k_{n+2})|}{\sum_{i=1}^{n+2} |k_i|}$$

$$\times |\psi_n(k_3, \ldots, k_{n+2})||\psi_n(k_2, \ldots, k_{n+1})| dk_1 \ldots dk_{n+2}$$

$$\leq \int \frac{[\varphi(k_1)]^2}{|k_1|} dk_1 \left( \frac{|\varphi(k_2)|}{|k_2|^{1/2}} |k_{n+2}|^{1/2} |\psi_n(k_3, \ldots, k_{n+2})| \right) \times$$

$$\times \frac{|\varphi(k_{n+2})|}{|k_{n+2}|^{1/2}} |k_2|^{1/2} |\psi_n(k_2, \ldots, k_{n+1})| \right)$$

$$\leq c_{II}^2 (\psi_n, H_f \psi_n). \quad (A.9)$$

For the third term we use Schwarz again to obtain

$$III_n \leq n^2 \int \frac{[\varphi(k_1)]^2 |\varphi(k_2)||\varphi(k_{n+1})||\varphi(k_{n+2})|}{\sum_{i=1}^{n+2} |k_i|}$$

$$\times |\psi_n(k_3, \ldots, k_{n+2})||\psi_n(k_1, \ldots, k_n)| dk_1 \ldots dk_{n+2}$$

$$\leq n^2 \left( \frac{[\varphi(k_1)]^2 |\varphi(k_2)|}{|k_1|^{1/2}} |k_{n+1}|^{1/2} |k_{n+2}|^{1/2} |\psi_n(k_3, \ldots, k_{n+2})|, \right)$$

$$\times |k_1|^{1/2} |k_2|^{1/2} |\psi_n(k_1, \ldots, k_n)|$$

$$\times \frac{|\varphi(k_{n+1})| |\varphi(k_{n+2})|}{|k_{n+1}|^{1/2} |k_{n+2}|^{1/2}}$$

$$\leq c_{III}^2 n \int |k_{n+2}| \sum_{i=3}^{n+1} \frac{|\psi_n(k_3, \ldots, k_{n+2})|^2}{|k_i|} dk_3 \ldots dk_{n+2}$$

$$\leq c_{III}^2 n \int |k_{n+2}| |\psi_n|^2 dk_3 \ldots dk_{n+2} = c_{III}^2 (\psi_n, H_f \psi_n). \quad (A.10)$$

By summing over the photon number $n$ we arrive at the statement of the Lemma.

All following lemmas are proven by a scheme similar to Lemma 2. To shorten the calculations we introduce the operator $|A|$, which is defined by
replacing $\varphi$ in $A$ by $|\varphi|$, i.e.,

$$|A| = \int |\varphi(k)|a(k)dk.$$  \hfill (A.11)

$|A|^*$ denotes its operator adjoint. In essence by (A.9) one has

$$|A|^*|A| \leq c_A H_f$$  \hfill (A.12)

with $c_A = \int \frac{|\varphi(k)|^2}{|k|}dk$. Similar methods were used in [H1, CH]. In addition, in order to simplify the notation, we introduce

$$P_j^l = \sum_{i=j}^l k_i, \quad H_j^l = \sum_{i=j}^l |k_i|.$$  \hfill (A.13)

**LEMMA 3.**

$$(\Psi, AAD^{-1}_f P_l A D^{-1}_f A^* P_l D^{-1}_f A^* A^* \Psi)$$

$$\leq (\Omega, AAD^{-1}_f P_l A D^{-1}_f A^* P_l D^{-1}_f A^* A^* \Omega) \|\Psi\|^2 + c(\Psi, D_l \Psi)$$  \hfill (A.14)

with $c$ uniformly bounded in $\Lambda$.

**Proof.** Following the scheme of Lemma 2 we can now distinguish between four different terms, since there are three photons created.

The diagonal and most interesting part reads

$$I_n = \int \frac{[\varphi(k_{n+1}) \cdot \varphi(k_{n+2})]^2 [\varphi(k_{n+3}) \cdot P_{n+2}^l]^2 |\psi_n(k_1, \ldots, k_n)|^2}{[(P_{n+2}^l)^2 + H_{n+2}^l][P_{n+2}^l]^2 + H_{n+2}^l]} dk_1 \ldots dk_{n+3}$$

$$\leq \int \frac{[\varphi(k_{n+1}) \cdot \varphi(k_{n+2})]^2 [\varphi(k_{n+3}) \cdot P_{n+2}^l]^2 |\psi_n(k_1, \ldots, k_n)|^2}{[(P_{n+2}^l)^2 + H_{n+1}^l + 2P_{n+1}^l(P_{n+2}^l)^2 + H_{n+1}^l + 2P_{n+1}^l(P_{n+2}^l)^2]} dk_1 \ldots dk_{n+3}.$$  \hfill (A.15)

In order to expand the denominator we write

$$\frac{1}{(Q_1 + b_1)^2(Q_2 + b_2)}$$

$$= \left[ \frac{1}{Q_1^2} - \frac{2b_1}{Q_1^2(Q_1 + b_1)} + \frac{b_1^2}{Q_1^2(Q_1 + b_1)^2} \right] \left[ \frac{1}{Q_2} - \frac{b_2}{Q_2(Q_2 + b_2)} \right]$$

$$= \frac{1}{Q_1Q_2} + M$$  \hfill (A.16)

with $Q_1 = (P_{n+2}^l)^2 + H_{n+2}^l$, $b_1 = 2P_{n+1}^lP_{n+2}^l$ and the equivalent expression for $Q_2, b_2$. 

The most important term is the one involving $\frac{1}{Q_1 Q_2}$, i.e.,

$$\int \frac{[\varphi(k_{n+1}) \varphi(k_{n+2})]^2 [\varphi(k_{n+3}) \cdot (P_{n+1}^{n+2} + P_1^n)]^2 |\psi_n(k_1, \ldots, k_n)|^2}{[(P_{n+1}^{n+2})^2 + H_{n+1}^{n+2}][P_{n+1}^{n+3}]^2 + H_{n+1}^{n+3}} \, dk_1 \ldots dk_{n+3}$$

$$\leq (\Omega, A A D_1^{-1} P_1 A D_1^{-1} A^* P_1 A D_1^{-1} A^* \Omega) |\psi_n|^2$$

$$+ \int \frac{|\varphi(k_1)|^2 |\varphi(k_2)|^2 |\varphi(k_3)|^2}{(H_1^2)^2 H_1^3} \, dk_1 dk_2 dk_3 \, |\psi_n, P_1^2 \psi_n|$$

$$+ 2 \int \frac{|\varphi(k_1)|^2 |\varphi(k_2)|^2 |\varphi(k_3)|^2}{(H_1^2)^{1/2} H_1^3} \, dk_1 dk_2 dk_3 \, (\psi_n, H_1 \psi_n), \quad (A.17)$$

where we used

$$\frac{P_{n+1}^{n+2}}{(P_{n+1}^{n+2})^2 + H_{n+1}^{n+2}} \leq \frac{1}{2(H_{n+1}^{n+2})^{1/2}}$$

and then changed variables to simplify the notation. Observe

$$\int \frac{|\varphi(k_1)|^2 |\varphi(k_2)|^2 |\varphi(k_3)|^2}{(H_1^2)^2 H_1^3} \, dk_1 dk_2 dk_3 \leq c_{II1}^2$$

and

$$\int \frac{|\varphi(k_1)|^2 |\varphi(k_2)|^2 |\varphi(k_3)|^2}{(H_1^2)^{1/2} H_1^3} \, dk_1 dk_2 dk_3 \leq \int \frac{|\varphi(k_1)|^2 |\varphi(k_2)|^2 |\varphi(k_3)|^2}{|k_1|^{1/4} |k_2|^{1/4} |k_3|} \, dk_1 dk_2 dk_3$$

which are obviously uniformly bounded.

Estimating the terms involving $M$ works similar to the two last terms. It is a straightforward but lengthy calculation, hence skipped.

In $I_{1n}$, where only one index differs, we meet the term

$$n \int \frac{[\varphi(k_{n+1}) \cdot \varphi(k_{n+2})]^2 [\varphi(k_{n+3}) \cdot P_{1}^{n+2}] |\varphi(k_1) \cdot P_2^{n+3}| \times}{[(P_1^{n+2})^2 + H_1^{n+2}][P_2^{n+3}]^2 + H_2^{n+3}][P_1^{n+3}]^2 + H_1^{n+3}} \times |\psi_n(k_1, \ldots, k_n)||\psi_n(k_2, \ldots, k_n, k_{n+3})| \, dk_1 \ldots dk_{n+3}$$

$$\leq c_{II}^2 |\psi_n|, |A|||A|\psi_n| \leq c_{II}^2 c_A(\psi_n, H_1 \psi_n) \quad (A.18)$$

and the term

$$n \int \frac{|\varphi(k_{n+1})||\varphi(k_{n+2})||\varphi(k_1)||\varphi(k_{n+3}) \cdot P_1^{n+2}|^2 \times}{[(P_1^{n+2})^2 + H_1^{n+2}][P_1^{n+3}]^2 + H_1^{n+3}} \times |\psi_n(k_1, \ldots, k_n)||\psi_n(k_2, \ldots, k_n, k_{n+2})| \, dk_1 \ldots dk_{n+3}$$

$$\leq c_{II}^2 (\psi_n, |A|^*A|\psi_n) \leq c_{II}^2 c_A(\psi_n, H_1 \psi_n), \quad (A.19)$$

where we used $|P_1| \leq H_1$. 
Finally we look at the term \( III_n \) where all indices differ, i.e.,
\[
\begin{align*}
I_n &= 2 \int \frac{[\varphi(k_1) \cdot \varphi(k_2)][\varphi(k_1) \cdot \varphi(k_2)][P_{1+1}^n \cdot \varphi(k_1)][P_{2+1}^n \cdot \varphi(k_1)] \times}{[(P_1^{n+2})^2 + H_1^{n+2}][(P_2^{n+2})^2 + H_1^{n+2}][(P_2^{n+2})^2 + H_2^{n+2}]} \times |\psi_n(k_3, \ldots, k_{n+2})|^2 \, dk_1 dk_2 \ldots dk_{n+2},
\end{align*}
\]
where \( P_1 = \bar{k_1} + \sum_{i=2}^l k_i \) and \( H_1 = |\bar{k_1}| + \sum_{i=2}^l |k_i| \).

We decompose as in (A.16) and the main part is estimated like
\[
2 \int \frac{[\varphi(k_1) \cdot \varphi(k_2)][\varphi(k_1) \cdot \varphi(k_2)][(P_1^2 + P_3^{n+2}) \cdot \varphi(k_1)][(P_2^2 + P_3^{n+2}) \cdot \varphi(k_1)] \times}{[(P_1^{n+2})^2 + H_1^{n+2}][(P_2^{n+2})^2 + H_1^{n+2}][(P_2^{n+2})^2 + H_2^{n+2}]} \times |\psi_n(k_3, \ldots, k_{n+2})|^2 \, dk_1 dk_2 \ldots dk_{n+2} \\
\leq (\Omega, AAD_{\Gamma}^{-1}P_1A^*D_{\Gamma}^{-1}AP_1D_{\Gamma}^{-1}A^*A^*\Omega)\|\psi_n\|^2 + c_{I1}(\psi_n, H_1\psi_n).
\] (A.23)

The remaining terms of the diagonal part are bounded analogously to the error terms in the previous inequality, whereas the off-diagonal terms are estimated like in the previous lemmas. \( \square \)

**LEMMA 4.**
\[
(\Psi, AAD_{\Gamma}^{-1}P_1A^*D_{\Gamma}^{-1}AP_1D_{\Gamma}^{-1}A^*A^*\Psi) \\
\leq (\Omega, AAD_{\Gamma}^{-1}P_1A^*D_{\Gamma}^{-1}AP_1D_{\Gamma}^{-1}A^*A^*\Omega)\|\psi_n\|^2 + c(\Psi, D_{\Gamma}\Psi) \quad (A.21)
\]
with \( c \) uniformly bounded in \( \Lambda \).

**Proof.** The diagonal term looks like
\[
I_n = 2 \int \frac{[\varphi(k_1) \cdot \varphi(k_2)][\varphi(k_1) \cdot \varphi(k_2)][P_{1+1}^n \cdot \varphi(k_1)][P_{2+1}^n \cdot \varphi(k_1)] \times}{[(P_1^{n+2})^2 + H_1^{n+2}][(P_2^{n+2})^2 + H_1^{n+2}][(P_2^{n+2})^2 + H_2^{n+2}]} \times |\psi_n(k_3, \ldots, k_{n+2})|^2 \, dk_1 dk_2 \ldots dk_{n+2},
\] (A.22)

We decompose as in (A.16) and the main part is estimated like
\[
2 \int \frac{[\varphi(k_1) \cdot \varphi(k_2)][\varphi(k_1) \cdot \varphi(k_2)][(P_1^2 + P_3^{n+2}) \cdot \varphi(k_1)][(P_2^2 + P_3^{n+2}) \cdot \varphi(k_1)] \times}{[(P_1^{n+2})^2 + H_1^{n+2}][(P_2^{n+2})^2 + H_1^{n+2}][(P_2^{n+2})^2 + H_2^{n+2}]} \times |\psi_n(k_3, \ldots, k_{n+2})|^2 \, dk_1 dk_2 \ldots dk_{n+2} \\
\leq (\Omega, AAD_{\Gamma}^{-1}P_1A^*D_{\Gamma}^{-1}AP_1D_{\Gamma}^{-1}A^*A^*\Omega)\|\psi_n\|^2 + c_{I1}(\psi_n, H_1\psi_n).
\] (A.23)

**LEMMA 5.**
\[
(\Psi, AAD_{\Gamma}^{-1}A^*A_{\Gamma}^{-1}A^*A^*\Psi) \\
\geq (\Omega, AAD_{\Gamma}^{-1}A^*A_{\Gamma}^{-1}A^*A^*\Omega)\|\psi_n\|^2 - c(\Psi, D_{\Gamma}\Psi) \quad (A.24)
\]
with \( c \) uniformly bounded in \( \Lambda \).
Proof. Since we now look for a lower bound, we have to be a little bit more careful when treating the diagonal part

\[
I_n = 2 \int \frac{[\varphi(k_1) \cdot \varphi(k_2)] [\varphi(\bar{k}_1) \cdot \varphi(\bar{k}_2)] [\varphi(k_1) \cdot \varphi(k_1)]}{([P_1^{n+2}]^2 + H_1^{n+2}) ([P_1^{n+2}]^2 + H_1^{n+2})} \times
\]

\[
\frac{1}{[Q + b][Q + b]} [\varphi_n(k_3, \ldots, k_{n+2})]^2 d\bar{k}_1 d\bar{k}_1 \ldots d\bar{k}_{n+2}
\]

\[
= 2 \int \frac{[\varphi(k_1) \cdot \varphi(k_2)] [\varphi(\bar{k}_1) \cdot \varphi(\bar{k}_2)] [\varphi(k_1) \cdot \varphi(k_1)]}{[Q + b][Q + b]} \times
\]

\[
[Q + b][Q + b] [\varphi_n(k_3, \ldots, k_{n+2})]^2 d\bar{k}_1 d\bar{k}_1 \ldots d\bar{k}_{n+2}, \quad (A.25)
\]

with

\[
Q = (P_1^2)^2 + H_1^2, \quad b = (P_1^{n+2})^2 + H_3^{n+2} + 2P_1^2 P_3^{n+2}
\]

and the equivalent expression for \( \tilde{Q}, \tilde{b} \) replacing \( k_1 \) by \( \bar{k}_1 \). Using

\[
\frac{1}{Q + b} \frac{1}{Q + b} = \left[ \frac{1}{Q} - \frac{b}{Q(Q + b)} \right] \left[ \frac{1}{Q} - \frac{\tilde{b}}{Q(Q + \tilde{b})} \right]
\]

\[
= \frac{1}{QQ} - \frac{b}{QQ(Q + b)} - \frac{b}{QQ(Q + b)} + \frac{bb}{QQ(Q + b)(Q + b)}, \quad (A.26)
\]

due to the symmetry of the two terms in the middle, we get immediately

\[
I_n \geq 2 \int \frac{[\varphi(k_1) \cdot \varphi(k_2)] [\varphi(\bar{k}_1) \cdot \varphi(\bar{k}_2)] [\varphi(k_1) \cdot \varphi(k_1)]}{QQ} \times
\]

\[
[Q + b][Q + b] [\varphi_n(k_3, \ldots, k_{n+2})]^2 d\bar{k}_1 d\bar{k}_1 \ldots d\bar{k}_{n+2}
\]

\[
- 6 \int \frac{[\varphi(k_1)]^2 [\varphi(k_2)]^2 [\varphi(\bar{k}_1)]^2 [b]}{QQ(Q + b)} [\varphi_n(k_3, \ldots, k_{n+2})]^2 d\bar{k}_1 d\bar{k}_1 \ldots d\bar{k}_{n+2}
\]

\[
\geq (\Omega, AAD^{-1}A^* AD^{-1}A^* A^* \Omega) \| \psi_n \|^2
\]

\[
- c^2_I(\psi_n, D_I \psi_n) - c^2_I(c_I(\psi_n, H_I \psi_n), (A.27)
\]

Concerning \( II_n \) we obtain two types of terms, namely

\[
n \int \frac{[\varphi(k_{n+2})]^2 [\varphi(k_1)] [\varphi_n(k_2, \ldots, k_{n+1})]}{H_1^{n+2}} dk_{n+2} \times
\]

\[
\int \frac{[\varphi(k_1)] [\varphi(k_2)] [\varphi(\bar{k}_{n+2})] [\varphi_n(k_3, \ldots, k_{n+1}, \bar{k}_{n+2})]}{H_1^{n+2}} dk_{n+2} dk_{n+1} \leq c^2_I(\psi_n, |A|^* |A| |\psi_n|) \leq c^2_I c_A(\psi_n, H_I \psi_n) \quad (A.28)
\]

and

\[
n \int \frac{[\varphi(k_1)]^2 [\varphi(k_2)]^2 [\varphi(\bar{k}_{n+2})] [\varphi(k_{n+2})] [\varphi_n(k_3, \ldots, k_{n+1}, \bar{k}_{n+2})]}{H_1^{n+2} H_1^{n+2}} \times
\]

\[
[\varphi_n(k_3, \ldots, k_{n+2})] [\psi_n(k_3, \ldots, k_{n+1}, \bar{k}_{n+2})] dk_1 \ldots dk_{n+2} \leq c^2_I(\psi_n, |A|^* |A| |\psi_n|) \leq c^2_I c_A(\psi_n, H_I \psi_n). \quad (A.29)
\]
Concerning \(III_n\) we estimate
\[
\begin{align*}
n^2 \int \frac{\phi(k_1)|\phi(k_2)|\phi(k_3,\ldots,k_{n+2})}{H_1^{n+2}} & \times \frac{\phi(k_1)|\phi(k_{n+1})|\phi(k_{n+2})|\psi_n(k_2,\ldots,k_{n+1},k_{n+2})}{H_1^{n+2}} dk_1 \ldots dk_{n+2}d\bar{k}_{n+2} \\
& \leq c_{II}(|\psi_n|, |A|^*H_f^{-1/2}|A|^*A_f^{-1/2}|A||\psi_n|) \leq c_{II}c_A^2(\psi_n, H_f \psi_n) \tag{A.30}
\end{align*}
\]
as well as concerning \(III\_1\)
\[
\begin{align*}
n^3 \int \frac{\phi(k_1)|\phi(k_2)|\phi(k_3,\ldots,k_{n+2})}{H_1^{n+2}} & \times \frac{\phi(k_1)|\phi(k_{n+1})|\phi(k_{n+2})|\psi_n(k_2,\ldots,k_{n+1},k_{n+2})}{H_1^{n+2}} dk_1 \ldots dk_{n+2}d\bar{k}_{n+2} \\
& \leq (|\psi_n|, |A|^*H_f^{-1/2}|A|^*A_f^{-1/2}|A|^*A_f^{-1/2}|A||\psi_n|) \\
& \leq c_A^3(\psi_n, H_f \psi_n). \tag{A.31}
\end{align*}
\]

The next Lemma is similar to the ones explained in [CH] Appendix C.

**LEMMA 6.**
\[
\begin{align*}
(i) & \quad |(\Psi, AAD_f^{-1}P_f A^*A_f^{-1}P_f A^*A^*\Psi)| \leq c||\Psi||H_1^{1/2}\Psi||. \tag{A.32} \\
(ii) & \quad |(\Psi, AAD_f^{-1}P_f A^*A_f^{-1}A^*A^*\Psi)| \leq c||\Psi||H_1^{1/2}\Psi|| \tag{A.33}
\end{align*}
\]
with \(c\) uniformly bounded in \(\Lambda\).

*Proof.* We sketch the proof of (ii). (i) works analogously. The diagonal part reads
\[
\begin{align*}
n \int \frac{\phi(k_1)\phi(k_{n+1})P_1^{n+1}\phi(k_{n+2})\psi_{n-1}(k_1,\ldots,k_n)}{(P_1^{n+1})^2 + H_1^{n+1}} & \times \frac{\phi(k_{n+1})\phi(k_{n+2})\psi_n(k_1,\ldots,k_n)}{(P_1^{n+2})^2 + H_1^{n+2}} \\
& \leq c_{II}(\psi_{n-1}, |A||\psi_n|) \leq c_{II}c_A^{1/2}\psi_{n-1}||H_1^{1/2}\psi_n||. \tag{A.34}
\end{align*}
\]

By methods similar to the previous lemmas the off-diagonal terms are estimated by \(|A||\psi_{n-1}|, |A||H_f^{-1/2}|A||\psi_n|\), respectively by \(|A||H_f^{-1/2}|A||\psi_{n-1}|, |A||H_f^{-1/2}|A||H_f^{-1/2}|A||\psi_n|\).

**APPENDIX B. SHARP ESTIMATES NEEDED FOR THEOREM 3**

We introduce
\[
c(e) = \int \frac{|\phi(k)|^2}{|k||k| + e^2} dk \leq c_{II} \ln[1/e]. \tag{B.1}
\]
Furthermore recall that for all $0 \leq \varepsilon < 1$
\[ \varepsilon P^2 \leq (P^2 + V - E_{at}) + \varepsilon |E_{at}|/(1 - \varepsilon), \]  
from which we obtain
\[ P^2 \leq 2(P^2 + V - E_{at}) + c \]  
with $c = 2|E_{at}|$. Inserting $\varepsilon = H_f/(H_f - E_{at})$ in (B.2) (cf. [HS, Equation (4.16)]) shows
\[ P \frac{1}{P^2 + V - E_{at} + H_f} P \leq 1 + \frac{|E_{at}|}{H_f} \]  
as well as
\[ \frac{1}{P^2 + V - E_{at} + H_f} P \frac{1}{P^2 + V - E_{at} + H_f} \leq c \left( \frac{1}{H_f} + \frac{1}{H_f^2} \right). \]
In the following we deal with states of the form $h\varphi \cdot \varphi$, meaning we understand that as
\[ [h\varphi \cdot \varphi]_{n+2}(x, k_1, \ldots, k_n) = h_n(x, k_1, \ldots, k_n)\varphi(k_{n+1}) \cdot \varphi(k_{n+2}), \]  
where $h \in H$. This wave function is introduced for notational simplification. It is not symmetric in all variables. This does not matter, since all operations also hold for the case of general wave functions, once we extend the definition of $A$ as
\[ [A\psi]_{n-1}(x, k_1, \ldots, k_{n-1}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int \varphi(k_i)\psi_n(x, k_1, \ldots, k_i, \ldots, k_{n+2}) \]  
Recall that
\[ B = P^2 + V - E_{at} + H_f. \]

**LEMMA 7.**
\[ (h, AAB^{-1}A^*h) \leq (h \varphi \cdot \varphi, B^{-1}h\varphi \cdot \varphi) + c(h, H_f h), \]  
with
\[ (h \varphi \cdot \varphi, B^{-1}h\varphi \cdot \varphi) \leq c\|h\|^2, \]  
where the constants are uniformly bounded in the cutoff.

**Proof.** We fix again a photon number $n$. Recall, as noted in (A.4),
\[ [A^*A^* h_n]_{n+2} = \frac{1}{\sqrt{(n+2)(n+1)}} \sum_{j=1}^{n+2} \sum_{i=1}^{n+2} \varphi(k_j) \cdot \varphi(k_i) \times \]  
\[ \times h_n(k_1, \ldots, k_j, \ldots, k_i, \ldots, k_{n+2}). \]  
By symmetry we again distinguish three different terms, where the first, diagonal term, is simply given as
\[ (h_n \varphi \cdot \varphi, B^{-1}h_n \varphi \cdot \varphi). \]
We proceed in analogy to II/1 where we used Schwarz inequality, the fact that symmetry of \( h \) we rewrite it as

\[
\frac{\varphi(k_1)}{|k_1|^{1/2}} \cdot \varphi(k_{n+2}) |h_n(x, k_2, \ldots, k_{n+1})| \leq c^2_{II}(h_n, H_1 h_n). \quad (B.12)
\]

The second off-diagonal term is given by

\[
n^2 \left( \varphi(k_1) \cdot \varphi(k_2) h_n(x, k_3, \ldots, k_{n+2}), B^{-1} h_n(x, k_1, \ldots, k_n) \cdot \varphi(k_{n+1}) \cdot \varphi(k_{n+2}) \right). \quad (B.13)
\]

We rewrite it as

\[
n^2 \left( \frac{\varphi(k_1)}{|k_1|^{1/2}} \cdot \varphi(k_2) \right) |k_{n+1}|^{1/2} |k_{n+2}|^{1/2} h_n(x, k_3, \ldots, k_{n+2}), \frac{1}{B} \times
\]

\[
\times \frac{\varphi(k_{n+1})}{|k_{n+1}|^{1/2}} \cdot \varphi(k_{n+2}) \right) \leq n^2 \int \frac{\varphi(k_1)^2 \varphi(k_2)^2}{|k_1||k_2|} dk_1 dk_2 \int |h_n(x, k_3, \ldots, k_{n+2})|^2 \sum_{i=3}^{n+1} \frac{|h_i|}{|k_i|} dk_3 \ldots dk_{n+2}
\]

\[
\leq Cn \int |h_{n+2}| \sum_{i=3}^{n+1} \frac{|h(x, k_3, \ldots, k_{n+2})|^2}{|k_i|^2} dk_3 \ldots dk_{n+2} = C(h_n, H_1 h_n), \quad (B.14)
\]

where we used Schwarz inequality, the fact that \( 1/B \leq 1/H_1 \), and the symmetry of \( h_n(x, k_3, \ldots, k_{n+2})/(\sum_{i=3}^{n+1} |k_i|) \) in the variables \( k_3 \) to \( k_{n+1} \). Obviously \( \boxed{B.9} \) holds since \( (h \varphi \cdot \varphi, B^{-1} h \varphi \cdot \varphi) \leq c^2_H \).

**LEMMA 8.**

\[
|\langle h, PAB^{-1} PAB^{-1} A^* A^* h \rangle - \langle \varphi \cdot PB^{-1} \varphi \cdot Ph, B^{-1} h \varphi \cdot \varphi \rangle| \leq c(e)^{1/2} \|Ph\| \|H_1^{1/2} h\| \quad (B.15)
\]

**with**

\[
|\langle \varphi \cdot PB^{-1} \varphi \cdot Ph, B^{-1} h \varphi \cdot \varphi \rangle| \leq c \|h\| \|Ph\|. \quad (B.16)
\]
Proof. We can estimate the first off-diagonal term by using Schwarz inequality and by a similar calculation as in (B.12),

\[ n(P, \varphi(k_{n+2})B^{-1}P, \varphi(k_{n+1})h_n(x, k_1, \ldots, k_n), B^{-1} \times \varphi(k_1) \varphi(k_{n+2})h_n(x, k_2, \ldots, k_{n+1})) \]

\[ \leq \left[ n \left( \varphi(k_{n+2})P B^{-1} \frac{\varphi(k_{n+1})}{|k_{n+1}|^{1/2}} Ph_n(x, k_1, \ldots, k_n)|k_1|^{1/2}, |k_{n+2}|^{-1} \times \varphi(k_{n+1})P B^{-1} \frac{\varphi(k_{n+1})}{|k_{n+1}|^{1/2}} Ph_n(x, k_1, \ldots, k_n)|k_1|^{1/2} \right) \right]^{1/2} \left[ c_{II}^2(h_n, Hf h_n) \right]^{1/2} \]

\[ \leq \left[ c_{II} n \left( Ph_n(x, k_1, \ldots, k_n)|k_1|^{1/2}, \frac{|\varphi(k_{n+1})|}{|k_{n+1}|^{1/2}}, [H_f^{-1} + H_f^{-2}] \times \varphi(k_{n+1})P B^{-1} \frac{|\varphi(k_{n+1})|}{|k_{n+1}|^{1/2}} Ph_n(x, k_1, \ldots, k_n)|k_1|^{1/2} \right) \right]^{1/2} \left[ c_{II}^2(h_n, Hf h_n) \right]^{1/2} \]

\[ \leq \left[ c_{II} c(e)n \left( Ph_n, \frac{|k_1|}{\sum_{i=1}^n |k_i|} Ph_n \right) \right]^{1/2} \left[ c_{II}^2(h_n, Hf h_n) \right]^{1/2} \]

\[ \leq c(e)^{1/2}||Ph_n||H_f^{1/2}h_n||, \quad (B.17) \]

where also (B.15) is used. For the second off-diagonal term we proceed similarly. Thereby, after Schwarz inequality, the more difficult term, suppressing the square root, can be bounded by

\[ n^2 \left( Ph_n(x, k_1, \ldots, k_n)|k_1|^{1/2} |k_2|^{1/2} \frac{|\varphi(k_{n+1})|}{|k_{n+1}|^{1/2}} \frac{|\varphi(k_{n+2})|}{|k_{n+2}|^{1/2}} [H_f^{-1} + H_f^{-2}] \times H_f^{-1} Ph_n(x, k_1, \ldots, k_n)|k_1|^{1/2} |k_2|^{1/2} \frac{|\varphi(k_{n+1})|}{|k_{n+1}|^{1/2}} \frac{|\varphi(k_{n+2})|}{|k_{n+2}|^{1/2}} \right) \]

\[ \leq c n^2(Ph_n, \frac{|k_1||k_2|}{\left( \sum_{i=1}^n |k_i| \right)^2} Ph_n) \leq c||Ph_n||^2, \quad (B.18) \]

where we used \( B \geq H_f \) and (B.15). The inequality (B.16) is obvious. \( \square \)

**Lemma 9.**

\[ |(h, AAB^{-1} A^* AB^{-1} A^* A^t h) - (h\varphi, B^{-1} A^* AB^{-1} h\varphi, \varphi)| \leq c(h, Hf h) \quad (B.19) \]

with

\[ (h\varphi, B^{-1} A^* AB^{-1} h\varphi, \varphi) \leq c||h||^2. \quad (B.20) \]

Proof. Denote \( S = B^{-1/2} A^* AB^{-1/2} \). Notice, due to Lemma 13, \( ||AB^{-1/2}|| \leq c_A^{-1/2} \) and consequently \( S \leq c \). The result follows by applying the proof of Lemma 7 to \( (h, AAB^{-1/2} SB^{-1/2} A^* A^t h) \). \( \square \)
**LEMMA 10.**

\[(h, AAB^{-1}PAB^{-1}A^*PB^{-1}A^*h) \leq (h \varphi \varphi, B^{-1}PAB^{-1}A^*PB^{-1}h \varphi \varphi) + (c + c(e))(h, H_I h), \quad (B.21)\]

with

\[(h \varphi \varphi, B^{-1}PAB^{-1}A^*PB^{-1}h \varphi \varphi) \leq c\|h\|^2. \quad (B.22)\]

**Proof.** Denote \(S = B^{-1}PAB^{-1}A^*PB^{-1}.\) Using Lemma 14 together with (B.5) we see that \(S \leq c[H_I^{-1} + H_I^{-2}].\) The off-diagonal terms can be estimated by using \(S.\) The terms corresponding to \(H_I^{-1}\) are treated as in Lemma 7 whereas the terms corresponding to \(H_I^{-2}\) can be bounded by similar methods (cf., e.g., the calculations of (B.17) and (B.18)) by \(c(e)(h_n, H_I h_n)\) with fixed but arbitrary photon number. \(\square\)

**LEMMA 11.**

\[(h, AAB^{-1}A^*PB^{-1}PAB^{-1}A^*A^*h) \leq (h \varphi \varphi, B^{-1}A^*PB^{-1}PAB^{-1}h \varphi \varphi) + (c + c(e))(h, H_I h) \quad (B.23)\]

with

\[(h \varphi \varphi, B^{-1}A^*PB^{-1}PAB^{-1}h \varphi \varphi) \leq c\|h\|^2. \quad (B.24)\]

**Proof.** The proof proceeds analogously to Lemma 10 since according to Lemma 13

\[B^{-1}A^*PB^{-1}PAB^{-1} \leq c[H_I^{-1} + H_I^{-2}] \quad (B.25)\]

also holds. \(\square\)

**LEMMA 12.**

\[|⟨Ψ, AAB^{-1}PAB^{-1}A^*A^*Ψ⟩| \leq c(e)\|Ψ\|H_I^{1/2}\|Ψ\|. \quad (B.26)\]

**Proof.** The proof proceeds similarly to Lemma 9 and [CH, Appendix C]. We demonstrate it on the “diagonal” term

\[
\sqrt{n}(\varphi(k_{n+2})\cdot PB^{-1}\varphi(k_n)\cdot \varphi(k_{n+1})\psi_{n-1}(x, k_1, \ldots, k_{n-1}), B^{-1} \times \\
\times \psi_n(x, k_1, \ldots, k_n)\varphi(k_{n+1})\varphi(k_{n+2}))
\]

\[
= \sqrt{n}(\varphi(k_{n+2})\cdot PB^{-1}\varphi(k_n)\cdot \varphi(k_{n+1})\psi_{n-1}(x, k_1, \ldots, k_{n-1}), B^{-1} \times \\
\times \psi_n(x, k_1, \ldots, k_n)|k_n|^{1/2}\varphi(k_{n+1})\varphi(k_{n+2})) \quad (B.27)
\]

Using (B.5), \(B \geq |k_{n+2}|\) together with Schwarz inequality, we bound

\[
|B(26)| \leq c_{II}(\psi_{n-1}(x, k_1, \ldots, k_{n-1})|k_n|^{1/2}\varphi(k_{n+1})|H_I^{-1} + H_I^{-2} | \times \\
\times \psi_{n-1}(x, k_1, \ldots, k_{n-1})|k_n|^{1/2}\varphi(k_{n+1}))^{1/2} c_I(\psi_n, H_I \psi_n)^{1/2}
\]

\[
\leq c_{II}c(e)^{1/2}(\psi_{n-1}||H_I^{1/2} \psi_n||. \quad (B.27)
\]
The remaining terms are covered in a similar fashion.

**Appendix C. Operator inequalities**

In this section we state and prove some operator inequalities used in the proof of Theorem 2 and 3.

We start with a simple but useful Lemma for our estimates employed in the proof of Theorem 2.

**Lemma 13.** In the sense of forms we have

\[(i) \quad AD_f^{-1}A^* \leq c, \quad (C.1)\]

\[(ii) \quad AAD_f^{-1}A^*A^* \leq c(1 + H_f). \quad (C.2)\]

Since \(L \geq D_f\) the above inequalities also hold for \(L\).

**Proof.** (i)

\[
\|D_f^{-1/2}A^*\| = \|AD_f^{-1/2}\| \leq \|AH_f^{-1/2}\| \leq c_A^{1/2}. \quad (C.3)
\]

(ii) Follows directly from the proof of Lemma 2, since \(I_n\) in (A.7) can be bounded by \(c_I^2\|\psi_n\|^2\). \(\square\)

The auxiliary Lemma for the proof of Theorem 3 is a bit more involved.

**Lemma 14.** In the sense of forms we have

\[(i) \quad AB^{-1}A^* \leq c, \quad (i)\]

\[(ii) \quad AB^{-2}A^* \leq c(e), \quad (ii)\]

\[(iii) \quad AAB^{-1}A^*A^* \leq c(1 + H_f), \quad (iii)\]

\[(iv) \quad AAB^{-2}A^*A^* \leq c, \quad (iv)\]

\[(v) \quad AAB^{-1}P^2B^{-1}A^*A^* \leq c(1 + H_f), \quad (v)\]

\[(vi) \quad AB^{-1}P^2B^{-1}A^* \leq c(e) + c \quad (vi)\]

\[(vii) \quad AB^{-1}A^*H_f^{-1}AB^{-1}A^* \leq cH_f^{-1} \quad (vii)\]

Since \(K \geq B\), the above inequalities also hold for \(K\).

**Proof.** (i) is a simple consequence of Lemma 13 (i), since \(B \geq H_f\).

(ii) Observe \(AB^{-2}A^* \leq AH_f^{-2}A^*\). The corresponding diagonal part is bounded by

\[
\left(\|\varphi(k_1)\|\psi_n(x, k_2, \ldots, k_{n+1})|H_f^{-2}\|\varphi(k_1)\|\psi_n(x, k_2, \ldots, k_{n+1})\|\right)
\leq \|\psi_n\|^2 \int \frac{||\varphi(k)||^2}{|k| + e^2} \frac{dk}{k} \leq c(e)\|\psi_n\|^2. \quad (C.4)
\]
The off-diagonal part is estimated by

\[
\begin{align*}
\text{(iii)} & \quad \text{is obvious.} \\
\text{(iv)} & \quad \text{The first two terms are treated similarly to (ii), only this time we have} \\
& \quad \text{the finite bounds } c_{II}^2 \text{ thanks to the fact there are two photons created.} \\
& \quad \text{The third term, where the indices in the created photons as well as in the} \\
& \quad \text{wave function } \psi_n \text{ are distinct, is estimated by} \\
& \quad n^2 \left( \frac{|\varphi(k_1)||\varphi(k_2)|}{|k_1|^{1/2}|k_2|^{1/2}} |\psi_n(x,k_1,k_2,k_{n+2})| \right) \\
& \quad \leq c_{II}^2 \int \sum_{i=3}^{n+2} |k_i| |\psi_n(x,k_3,k_4,\ldots,k_{n+2})|^2 \frac{dk_3 \ldots dk_{n+2}}{|\psi_n|^2} = c_{II}^2 |\psi_n|^2. \quad (C.6)
\end{align*}
\]

Observe that by means of \((B.5)\) together with Lemma 13 \((i) - (iv)\) we arrive at \((iv)\) and \((v)\).
\((vii)\) is an easy application of our method and can be guessed immediately, since \(AB^{-1}A^* \leq c\).

□

By means of Lemma 13 we can easily prove the operator inequalities used in the proof of Theorem 3.
Lemma 15.

(i) \[ \| B^{-1/2} A^* A B^{-1} A^* \Psi \| \leq \| \Psi \|(c + c(e))^{1/2} \]
(ii) \[ \| B^{-1/2} A^* A B^{-1} A^* \Psi \| \leq \|(1 + H_f)^{1/2} \Psi \|, \]
(iii) \[ \| B^{-1/2} A^* P B^{-1} A^* \Psi \| \leq \| \Psi \|(c + c(e))^{1/2}, \]
(iv) \[ \| B^{-1/2} A^* P B^{-1} A^* A^* \Psi \| \leq \|(1 + H_f)^{1/2} \Psi \|, \]
(v) \[ \| B^{-1/2} P A B^{-1} A^* A^* \Psi \| \leq \|(1 + H_f)^{1/2} \Psi \|, \]
(vi) \[ \| B^{-1/2} A^* P B^{-1} A^* B^{-1} A^* \Psi \| \leq \| \Psi \|(c + c(e))^{1/2} \]
(vii) \[ \| B^{-1/2} A^* P B^{-1} A^* B^{-1} A^* A^* \Psi \| \leq \|(1 + H_f)^{1/2} \Psi \|. \]

Since \( K \geq B \), the above inequalities also hold for \( K \).

Proof. (i) and (ii) are a simple consequence of Lemma 14 (i) – (iv). For (iii) and (iv) apply Lemma 14 (i) and (v), respectively (vi). For (v) apply \( PB^{-1} P \leq c(1 + H_f^{-1}) \). Furthermore use Lemma 14 (vii). This together with Lemma 14 (iii) and (iv) implies the inequality.

(vi) and (vii) are a direct consequence of Lemma 14. \( \square \)

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