On a Class of Planar Graphs with Straight-Line Grid Drawings on Linear Area

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Abstract

A straight-line grid drawing of a planar graph $G$ is a drawing of $G$ on an integer grid such that each vertex is drawn as a grid point and each edge is drawn as a straight-line segment without edge crossings. It is well known that a planar graph of $n$ vertices admits a straight-line grid drawing on a grid of area $O(n^2)$. A lower bound of $\Omega(n^2)$ on the area-requirement for straight-line grid drawings of certain planar graphs are also known. In this paper, we introduce a fairly large class of planar graphs which admits a straight-line grid drawing on a grid of area $O(n)$. We give a linear-time algorithm to find such a drawing. Our new class of planar graphs, which we call “doughnut graphs,” is a subclass of 5-connected planar graphs. We show several interesting properties of “doughnut graphs” in this paper. One can easily observe that any spanning subgraph of a “doughnut graph” also admits a straight-line grid drawing with linear area. But the recognition of a spanning subgraph of a “doughnut graph” seems to be a non-trivial problem, since the recognition of a spanning subgraph of a given graph is an NP-complete problem in general. We establish a necessary and sufficient condition for a 4-connected planar graph $G$ to be a spanning subgraph of a “doughnut graph.” We also give a linear-time algorithm to augment a 4-connected planar graph $G$ to a “doughnut graph” if $G$ satisfies the necessary and sufficient condition.
1 Introduction

Recently automatic aesthetic drawings of graphs have created intense interest due to their broad applications in computer networks, VLSI layout, information visualization etc., and as a consequence a number of drawing styles have come out [4, 11, 14, 16]. A classical and widely studied drawing style is the “straight-line drawing” of a planar graph. A straight-line drawing of a planar graph $G$ is a drawing of $G$ such that each vertex is drawn as a point and each edge is drawn as a straight-line segment without edge crossings. A straight-line grid drawing of a planar graph $G$ is a straight-line drawing of $G$ on an integer grid such that each vertex is drawn as a grid point as shown in Figure 1(b).

![Figure 1: (a) A planar graph $G$, (b) a straight-line grid drawing of $G$ with area $O(n^2)$, (c) a doughnut embedding of $G$ and (d) a straight-line grid drawing of $G$ with area $O(n)$.

Wagner [19], Fary [6] and Stein [18] independently proved that every planar graph $G$ has a straight-line drawing. Their proofs immediately yield polynomial-
time algorithms to find a straight-line drawing of a given plane graph. However, the area of a rectangle enclosing a drawing on an integer grid obtained by these algorithms is not bounded by any polynomial of the number \(n\) of vertices in \(G\). In fact, to obtain a drawing of area bounded by a polynomial remained as an open problem for long time. In 1990, de Fraysseix et al. [3] and Schnyder [17] showed by two different methods that every planar graph of \(n \geq 3\) vertices has a straight-line drawing on an integer grid of size \((2n - 4) \times (n - 2)\) and \((n - 2) \times (n - 2)\), respectively. Figure 1(b) illustrates a straight-line grid drawing of the graph \(G\) in Figure 1(a) with area \(O(n^2)\). A natural question arises: what is the minimum size of a grid required for a straight-line drawing? de Fraysseix et al. showed that, for each \(n \geq 3\), there exists a plane graph of \(n\) vertices, for example nested triangles, which needs a grid size of at least \([2(n - 1)/3] \times [2(n - 1)/3]\) for any grid drawing [2, 3]. Recently Frati and Patrignani showed that \(n^2/9 + \Omega(n)\) area is necessary for any planar straight-line drawing of a nested triangles graph [7]. (Note that a plane graph is a planar graph with a given embedding.) It has been conjectured that every plane graph of \(n\) vertices has a grid drawing on a \([2n/3] \times [2n/3]\) grid, but it is still an open problem. For some restricted classes of graphs, more compact straight-line grid drawings are known. For example, a 4-connected plane graph \(G\) having at least four vertices on the outer face has a straight-line grid drawing with area \(((n/2) - 1) \times (\lfloor n/2 \rfloor)\) [15]. Garg and Rusu showed that an \(n\)-node binary tree has a planar straight-line grid drawing with area \(O(n)\) [9]. Although trees admit straight-line grid drawings with linear area, it is generally thought that triangulations may require a grid of quadratic size. Hence finding nontrivial classes of planar graphs of \(n\) vertices richer than trees that admit straight-line grid drawings with area \(o(n^2)\) is posted as an open problem in [1]. Garg and Rusu showed that an outerplanar graph with \(n\) vertices and maximum degree \(d\) has a planar straight-line drawing with area \(O(dn^{1.48})\) [10]. Recently Di Battista and Frati showed that a “balanced” outerplanar graph of \(n\) vertices has a straight-line grid drawing with area \(O(n)\) and a general outerplanar graph of \(n\) vertices has a straight-line grid drawing with area \(O(n^{1.48})\) [5].

In this paper, we introduce a new class of planar graphs which has a straight-line grid drawing on a grid of area \(O(n)\). We give a linear-time algorithm to find such a drawing. Our new class of planar graphs is a subclass of 5-connected planar graphs, and we call the class “doughnut graphs” since a graph in this class has a doughnut-like embedding as illustrated in Figure 1(c). In an embedding of a “doughnut graph” of \(n\) vertices, there are two vertex-disjoint faces each having exactly \(n/4\) vertices and each of all the other faces has exactly three vertices. Figure 1(a) illustrates a “doughnut graph” of 16 vertices where each of the two faces \(F_1\) and \(F_2\) contains four vertices and each of all other faces contains exactly three vertices. Figure 1(c) illustrates a doughnut-like embedding of \(G\) where \(F_1\) is embedded as the outer face and \(F_2\) is embedded as an inner face. A straight-line grid drawing of \(G\) with area \(O(n)\) is illustrated in Figure 1(d). The outerplanarity of a “doughnut graph” is 3. Thus “doughnut graphs” introduce a subclass of 3-outerplanar graphs that admits straight-line grid drawing with linear area. One can easily observe that any spanning subgraph of a “dough-
nut graph” also admits a straight-line grid drawing with linear area. But the recognition of a spanning subgraph of a given graph is an NP-complete problem in general. We establish a necessary and sufficient condition for a 4-connected planar graph to be a spanning subgraph of a “doughnut graph.” We also provide a linear-time algorithm to augment a 4-connected graph \( G \) to a “doughnut graph” if \( G \) satisfies the necessary and sufficient condition. This gives us a new class of graphs which is a subclass of 4-connected planar graphs that admits straight-line grid drawings with linear area.

The remainder of the paper is organized as follows. In Section 2, we give some definitions. Section 3 provides some properties of the class of “doughnut graphs.” Section 4 deals with straight-line grid drawings of “doughnut graphs.” Section 5 provides the characterization for a 4-connected planar graph to be a spanning subgraph of a “doughnut graph.” Finally Section 6 concludes the paper. Early versions of this paper have been presented at [12] and [13].

2 Preliminaries

In this section we give some definitions.

Let \( G = (V, E) \) be a connected simple graph with vertex set \( V \) and edge set \( E \). Throughout the paper, we denote by \( n \) the number of vertices in \( G \), that is, \( n = |V| \), and denote by \( m \) the number of edges in \( G \), that is, \( m = |E| \). An edge joining vertices \( u \) and \( v \) is denoted by \((u, v)\). The degree of a vertex \( v \), denoted by \( d(v) \), is the number of edges incident to \( v \) in \( G \). \( G \) is called \( r \)-regular if every vertex of \( G \) has degree \( r \). We call a vertex \( v \) a neighbor of a vertex \( u \) in \( G \) if \( G \) has an edge \((u, v)\). The connectivity \( \kappa(G) \) of a graph \( G \) is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph \( K_1 \). \( G \) is called \( k \)-connected if \( \kappa(G) \geq k \). We call a vertex of \( G \) a cut-vertex of \( G \) if its removal results in a disconnected or single-vertex graph. For \( W \subseteq V \), we denote by \( G - W \) the graph obtained from \( G \) by deleting all vertices in \( W \) and all edges incident to them. A cut-set of \( G \) is a set \( S \subseteq V(G) \) such that \( G - S \) has more than one component or \( G - S \) is a single vertex graph. A path in \( G \) is an ordered list of distinct vertices \( v_1, v_2, \ldots, v_q \in V \) such that \((v_{i-1}, v_i) \in E \) for all \( 2 \leq i \leq q \). Vertices \( v_1 \) and \( v_q \) are end-vertices of the path \( v_1, v_2, \ldots, v_q \). Two paths are vertex-disjoint if they do not share any common vertex except their end vertices. The length of a path is the number of edges on the path. We call a path \( P \) an even path if the number of edges on \( P \) is even. We call a path \( P \) an odd path if the number of edges on \( P \) is odd.

A graph is \textit{planar} if it can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident. A \textit{plane graph} is a planar graph with a fixed embedding. A plane graph \( G \) divides the plane into connected regions called \textit{faces}. A bounded region is called an \textit{inner face} and the unbounded region is called the \textit{outer face}. For a face \( F \) in \( G \) we denote by \( V(F) \) the set of vertices of \( G \) on the boundary of face \( F \). Two faces \( F_1 \) and \( F_2 \) are vertex-disjoint if \( V(F_1) \cap V(F_2) = \emptyset \). Let \( F \) be a face in a plane graph \( G \) with \( n \geq 3 \). If the boundary of \( F \) has exactly three vertices...
then we call $F$ a triangulated face. One can divide a face $F$ of $p$ ($p \geq 3$) vertices into $p - 2$ triangulated faces by adding $p - 3$ extra edges. The operation above is called triangulating a face. If every face of a graph is triangulated, then the graph is called a triangulated plane graph. We can obtain a triangulated plane graph $G'$ from a non-triangulated plane graph $G$ by triangulating all faces of $G$.

A maximal planar graph is one to which no edge can be added without losing planarity. Thus the boundary of every face of $G$ is a triangle in any embedding of a maximal planar graph $G$ with $n \geq 3$, and hence an embedding of a maximal planar graph is often called a triangulated plane graph. It can be derived from Euler’s formula for planar graphs that if $G$ is a maximal planar graph with $n$ vertices and $m$ edges then $m = 3n - 6$, for more details see [16]. We call a face a quadrangle face if the face has exactly four vertices.

For any 3-connected planar graph the following fact holds.

Fact 1 Let $G$ be a 3-connected planar graph and let $\Gamma$ and $\Gamma'$ be any two planar embeddings of $G$. Then any facial cycle of $\Gamma$ is a facial cycle of $\Gamma'$ and vice versa.

Let $G$ be a 5-connected planar graph, let $\Gamma$ be any planar embedding of $G$ and let $p$ be an integer such that $p \geq 4$. We call $G$ a $p$-doughnut graph if the following conditions ($d_1$) and ($d_2$) hold:

($d_1$) $\Gamma$ has two vertex-disjoint faces each of which has exactly $p$ vertices, and all the other faces of $\Gamma$ has exactly three vertices; and

($d_2$) $G$ has the minimum number of vertices satisfying condition ($d_1$).

In general, we call a $p$-doughnut graph for $p \geq 4$ a doughnut graph. Since a doughnut graph is a 5-connected planar graph, Fact 1 implies that the decomposition of a doughnut graph into its facial cycles is unique. Throughout the paper we often mention faces of a doughnut graph $G$ without mentioning its planar embedding where the description of the faces is valid for any planar embedding of $G$.

3 Properties of Doughnut Graphs

In this section we will show some properties of a $p$-doughnut graph. We have the following lemma on the number of vertices of a graph satisfying condition ($d_1$).

Lemma 2 Let $G$ be a 5-connected planar graph, let $\Gamma$ be any planar embedding of $G$, and let $p$ be an integer such that $p \geq 4$. Assume that $\Gamma$ has two vertex-disjoint faces each of which has exactly $p$ vertices, and all the other faces of $\Gamma$ has exactly three vertices. Then $G$ has at least $4p$ vertices.

Proof: Let $F_1$ and $F_2$ be the two faces of $\Gamma$ each of which contains exactly $p$ vertices. Let $x$ be the number of vertices in $G$ which are neither on $F_1$ nor on $F_2$. Then $G$ has $x + 2p$ vertices.
We calculate the number of edges in \( G \) as follows. Faces \( F_1 \) and \( F_2 \) of \( \Gamma \) are not triangulated since \( p \geq 4 \). If we triangulate \( F_1 \) and \( F_2 \) of \( \Gamma \) then the resulting graph \( G' \) is a maximal planar graph. Using Euler’s formula, \( G' \) has exactly \( 3(x + 2p) - 6 = 3x + 6p - 6 \) edges. To triangulate each of \( F_1 \) and \( F_2 \), we need to add \( p - 3 \) edges and hence the number of edges in \( G \) is exactly

\[
(3x + 6p - 6) - 2(p - 3) = 3x + 4p.
\]

Since \( G \) is 5-connected, using the degree-sum formula, we get \( 2(3x + 4p) \geq 5(x + 2p) \). This relation implies

\[
x \geq 2p.
\]

Therefore \( G \) has at least \( 4p \) vertices. \( \blacksquare \).

Lemma 2 implies that a \( p \)-doughnut graph has \( 4p \) or more vertices. We now show that \( 4p \) vertices are sufficient to construct a \( p \)-doughnut graph as in the following lemma.

**Lemma 3** For an integer \( p \), \( p \geq 4 \), one can construct a \( p \)-doughnut graph \( G \) with \( 4p \) vertices.

To prove Lemma 3, we first construct a planar embedding \( \Gamma \) of \( G \) with \( 4p \) vertices by the construction **Construct-Doughnut** given below and then show that \( G \) is a \( p \)-doughnut graph.

**Construct-Doughnut.** Let \( C_1, C_2, C_3 \) be three vertex-disjoint cycles such that \( C_1 \) contains \( p \) vertices, \( C_2 \) contains \( 2p \) vertices and \( C_3 \) contains \( p \) vertices. Let \( x_1, x_2, \ldots, x_p \) be the vertices on \( C_1 \), \( y_1, y_2, \ldots, y_p \) be the vertices on \( C_3 \), and \( z_1, z_2, \ldots, z_{2p} \) be the vertices on \( C_2 \). Let \( R_1, R_2 \) and \( R_3 \) be three concentric circles on a plane with radius \( r_1, r_2 \) and \( r_3 \), respectively, such that \( r_1 > r_2 > r_3 \). We embed \( C_1, C_2 \) and \( C_3 \) on \( R_1, R_2 \) and \( R_3 \) respectively, as follows. We put the vertices \( x_1, x_2, \ldots, x_p \) of \( C_1 \) on \( R_1 \) in clockwise order such that \( x_1 \) is put on the leftmost position among the vertices \( x_1, x_2, \ldots, x_p \). Similarly, we put vertices \( z_1, z_2, \ldots, z_{2p} \) of \( C_2 \) on \( R_2 \) and \( y_1, y_2, \ldots, y_p \) of \( C_3 \) on \( R_3 \). We add edges between the vertices on \( C_1 \) and \( C_2 \), and between the vertices on \( C_2 \) and \( C_3 \) as follows. We have two cases to consider.

**Case 1:** \( k \) is even in \( z_k \).

In this case, we add two edges \((z_k, x_{k/2}), (z_k, x_i)\) between \( C_2 \) and \( C_1 \), and one edge \((z_k, y_i)\) between \( C_2 \) and \( C_3 \) where \( i = 1 \) if \( k = 2p \), and \( i = k/2 + 1 \) otherwise.

**Case 2:** \( k \) is odd in \( z_k \).

In this case, we add two edges \((z_k, y_{(k/2)}), (z_k, y_i)\) between \( C_2 \) and \( C_3 \), and one edge \((z_k, x_{(k/2)})\) between \( C_1 \) and \( C_2 \) where \( i = 1 \) if \( k = 2p - 1 \), and \( i = \lceil k/2 \rceil + 1 \) otherwise.

We thus constructed a planar embedding \( \Gamma \) of \( G \). Figure 2 illustrates the construction above for the case of \( p = 4 \).

We have the following lemma on the construction **Construct-Doughnut**.
Lemma 4 Let \( \Gamma \) be the plane graph of \( 4p \) vertices obtained by the construction \textbf{Construct-Doughnut}. Then \( \Gamma \) has exactly two vertex-disjoint faces \( F_1 \) and \( F_2 \) each of which has exactly \( p \) vertices, and the rest of the faces are triangulated.

\textbf{Proof:} The construction of \( \Gamma \) implies that cycle \( C_1 \) is the boundary of the outer face \( F_1 \) of \( \Gamma \) and cycle \( C_3 \) is the boundary of an inner face \( F_2 \). Each of \( F_1 \) and \( F_2 \) has exactly \( p \) vertices. Clearly the two faces \( F_1 \) and \( F_2 \) of \( \Gamma \) are vertex-disjoint. Thus it is remained to show that the rest of the faces of \( \Gamma \) are triangulated. The rest of the faces can be divided into two groups; (i) faces having vertices on both the cycles \( C_1 \) and \( C_2 \), and (ii) faces having the vertices on both the cycles \( C_2 \) and \( C_3 \).

We only prove that each face in group (i) is triangulated, since the proof for group (ii) is similar.

From our construction each vertex \( z_i \) with even \( i \) has exactly two neighbors on \( C_1 \), and the two neighbors of \( z_i \) on \( C_1 \) are consecutive. Hence we get a triangulated face for each \( z_i \) with even \( i \) which contains \( z_i \) and the two neighbors of \( z_i \) on \( C_1 \).

We now show that the remaining faces in group (i) are triangulated. Clearly each of the remaining faces in group (i) must contain a vertex \( z_i \) with odd \( i \) since a vertex on a face in group (i) is either on \( C_1 \) or on \( C_2 \) and a vertex on \( C_2 \) has at most two neighbors on \( C_1 \). Let \( z_i, z_{i+1} \) and \( z_{i+2} \) be three consecutive vertices on \( C_2 \) with even \( i \). Then \( z_i \) and \( z_{i+2} \) has a common neighbor \( x \) on \( C_1 \). One can observe from our construction that \( x \) is also the only neighbor of \( z_{i+1} \) on \( C_1 \). Then exactly two faces in group (i) contain \( z_{i+1} \) and the two faces are triangulated. This implies that for each \( z_i \) on \( C_2 \) with odd \( i \) there are exactly two faces in group (i) which contain \( z_i \), and the two faces are triangulated.

\( \textbf{Q.E.D.} \)
We are ready to prove Lemma 3.

Proof of Lemma 3

We construct a planar embedding \( \Gamma \) of a graph \( G \) with \( 4p \) vertices by the construction \textit{Construct-Doughnut} and show that \( G \) is a \( p \)-doughnut graph. To prove this claim we need to prove that \( G \) satisfies the following properties (a)–(c):

(a) the graph \( G \) is a 5-connected planar graph;

(b) any planar embedding \( \Gamma' \) of \( G \) has exactly two vertex-disjoint faces each of which has exactly \( p \) vertices, and all the other faces are triangulated; and

(c) \( G \) has the minimum number of vertices satisfying (a) and (b).

(a) \( G \) is a planar graph since it has a planar embedding \( \Gamma \) as illustrated in Figure 2(c). To prove that \( G \) is 5-connected, we show that the size of any cut-set of \( G \) is 5 or more. We first show that \( G \) is 5-regular. From the construction, one can easily see that each of the vertex of \( C_2 \) has exactly three neighbors in \( V(C_1) \cup V(C_3) \). Hence the degree of each vertex of \( C_2 \) is exactly 5. We only prove that the degree of each vertex of \( C_1 \) is exactly 5 since the proof is similar for the vertices of \( C_3 \). Each even index vertex \( v \) of \( C_2 \) has two neighbors on \( C_1 \) and the two neighbors of \( v \) are consecutive on \( C_1 \) by construction. Each vertex \( u \) of \( C_1 \) has at most two even index neighbors on \( C_2 \), since \( C_2 \) has \( p \) even index vertices, \( C_1 \) has \( p \) vertices, and \( \Gamma \) is a planar embedding. Assume that a vertex \( u \) of \( C_1 \) has two even index neighbors \( y_i \) and \( y_{i+2} \) on \( C_2 \). Since \( \Gamma \) is a planar embedding \( y_{i+1} \) can have only one neighbor on \( C_1 \) which is \( u \). Thus a vertex \( u \) on \( C_1 \) has at most three neighbors on \( C_2 \). Since there are exactly \( 3p \) edges each of which has one end point on \( C_1 \) and the other on \( C_2 \), and a vertex on \( C_1 \) has at most three neighbors on \( C_2 \), each vertex of \( C_1 \) has exactly three neighbors on \( C_2 \). Hence the degree of a vertex on \( C_1 \) is 5. Therefore \( G \) is 5-regular. We next show that the size of any cut-set of \( G \) is 5 or more. Assume for a contradiction that \( G \) has a cut-set of less than five vertices. In such a case, \( G \) would have a vertex of degree less than five, a contradiction. (Note that \( G \) is 5-regular, the vertices of \( G \) lie on three vertex disjoint cycles \( C_1, C_2 \) and \( C_3 \), none of the vertices of \( C_1 \) has a neighbor on \( C_3 \), each of the faces of \( G \) is triangulated except faces \( F_1 \) and \( F_2 \).

(b) By Lemma 4, \( G \) has a planar embedding \( \Gamma \) such that \( \Gamma \) has exactly two vertex-disjoint faces \( F_1 \) and \( F_2 \) each of which has exactly \( p \) vertices, and the rest of the faces are triangulated. Since \( G \) is 5-connected, Fact 4 implies that any planar embedding \( \Gamma' \) of \( G \) has exactly two vertex-disjoint faces each of which has exactly \( p \) vertices, and all the other faces are triangulated.

(c) We have constructed the graph \( G \) with \( 4p \) vertices and proofs for (a) and (b) imply that \( G \) satisfies properties (a) and (b). \( G \) is a 5-connected planar graph and hence satisfies condition \((d_1)\) of the definition of a \( p \)-doughnut graph. By Lemma 2 \( 4p \) is the minimum number of vertices of such a graph. \( \Box \).
Condition (d2) of the definition of a p-doughnut graph and Lemmas 2 and 3 imply that a p-doughnut graph \( G \) has exactly 4\( p \) vertices. Then the value of \( x \) in Eq. (2) is 2\( p \) in \( G \). By Eq. (1), \( G \) has exactly 3\( x \) + 4\( p \) = 10\( p \) edges. Since \( G \) is 5-connected, every vertex has degree 5 or more. Then the degree-sum formula implies that every vertex of \( G \) has degree exactly 5. Thus the following theorem holds.

**Theorem 1** Let \( G \) be a p-doughnut graph. Then \( G \) is 5-regular and has exactly 4\( p \) vertices.

For a cycle \( C \) in a plane graph \( G \), we denote by \( G(C) \) the plane subgraph of \( G \) inside \( C \) excluding \( C \). Let \( C_1, C_2 \) and \( C_3 \) be three vertex-disjoint cycles in a planar graph \( G \) such that \( V(C_1) \cup V(C_2) \cup V(C_3) = V(G) \). Then we call a planar embedding \( \Gamma \) of \( G \) a doughnut embedding of \( G \) if \( C_1 \) is the outer face and \( C_3 \) is an inner face of \( \Gamma \), \( G(C_1) \) contains \( C_2 \) and \( G(C_2) \) contains \( C_3 \). We call \( C_1 \) the outer cycle, \( C_2 \) the middle cycle and \( C_3 \) the inner cycle of \( \Gamma \). Next we show that a p-doughnut graph has a doughnut embedding. To prove the claim we need the following lemmas.

**Lemma 5** Let \( G \) be a p-doughnut graph. Let \( F_1 \) and \( F_2 \) be the two faces of \( G \) each of which contains exactly \( p \) vertices. Then \( G - \{ V(F_1) \cup V(F_2) \} \) is connected and contains a cycle.

**Proof:** Since \( G \) is 5-connected, \( G' = G - \{ V(F_1) \cup V(F_2) \} \) is connected; otherwise, \( G \) would have a cut-set of 4 vertices - two of them are on \( F_1 \) and the other two are on \( F_2 \), a contradiction. Clearly \( G' \) has exactly 2\( p \) vertices. Since \( G \) is 5-regular and has exactly 4\( p \) vertices by Theorem 1, one can observe following the degree-sum formula that \( G' \) contains at least 2\( p \) edges; if there is no edge between a vertex of \( F_1 \) and a vertex of \( F_2 \) in \( G \) then \( G' \) contains exactly 2\( p \) edges, otherwise \( G' \) contains more than 2\( p \) edges. Since \( G' \) is connected, has 2\( p \) vertices and has at least 2\( p \) edges, \( G' \) must have a cycle.

**Q.E.D.**

**Lemma 6** Let \( G \) be a p-doughnut graph. Let \( F_1 \) and \( F_2 \) be the two faces of \( G \) each of which contains exactly \( p \) vertices. Let \( \Gamma \) be a planar embedding of \( G \) such that \( F_1 \) is embedded as the outer face. Let \( C \) be a cycle in \( G - \{ V(F_1) \cup V(F_2) \} \). Then \( G(C) \) in \( \Gamma \) contains \( F_2 \).

**Proof:** Assume that \( G(C) \) does not contain \( F_2 \) in \( \Gamma \). Since \( F_1 \) is embedded as the outer face of \( \Gamma \), \( F_2 \) will be an inner face of \( \Gamma \) as illustrated in Figure 3. Then there would be edge crossings in \( \Gamma \) among the edges from the vertices on \( C \) to the vertices on \( F_1 \) and \( F_2 \) as illustrated in Figure 3 a contradiction to the assumption that \( \Gamma \) is a planar embedding of \( G \). (Note that \( G \) is 5-connected, 5-regular and has 10\( p \) edges.) Therefore \( G(C) \) contains \( F_2 \).

**Q.E.D.**

**Lemma 7** Let \( G \) be a p-doughnut graph. Let \( F_1 \) and \( F_2 \) be the two faces of \( G \) each of which contains exactly \( p \) vertices. Then the following (a) - (c) hold.

(a) There is no edge between a vertex of \( F_1 \) and a vertex of \( F_2 \).
Figure 3: An embedding $\Gamma'$ of $G$ where $F_1$ is embedded as the outer face and $G(C')$ does not contain $F_2$.

(b) $G - \{V(F_1) \cup V(F_2)\}$ contains exactly $2p$ edges and has exactly one cycle.

(c) All the vertices of $G - \{V(F_1) \cup V(F_2)\}$ are contained in a single cycle.

Proof: (a) Let $\Gamma$ be a planar embedding of $G$ such that $F_1$ is embedded as the outer face. Let $C$ be a cycle in $G - \{V(F_1) \cup V(F_2)\}$. Since $G(C)$ contains $F_2$ by Lemma 6, there is no edge between a vertex of $F_1$ and a vertex of $F_2$; otherwise, an edge between a vertex of $F_1$ and a vertex of $F_2$ would cross an edge on $C$, a contradiction to the assumption that $\Gamma$ is a planar embedding of $G$.

(b) Since there is no edge between a vertex of $F_1$ and a vertex of $F_2$ by Lemma 7(a), $G - \{V(F_1) \cup V(F_2)\}$ contains exactly $2p$ edges as mentioned in the proof of Lemma 5. Since $G - \{V(F_1) \cup V(F_2)\}$ is connected, contains exactly $2p$ vertices and has exactly $2p$ edges, $G - \{V(F_1) \cup V(F_2)\}$ contains exactly one cycle.

(c) Let $\Gamma$ be a planar embedding of $G$ such that $F_1$ is embedded as the outer face. Let $C$ be a cycle in $G - \{V(F_1) \cup V(F_2)\}$. By Lemma 7(b), $C$ is the only cycle in $G - \{V(F_1) \cup V(F_2)\}$. Assume that the cycle $C$ does not contain all the vertices of $G - \{V(F_1) \cup V(F_2)\}$. Then there is at least a vertex in $G - \{V(F_1) \cup V(F_2)\}$ whose degree is one in $G - \{V(F_1) \cup V(F_2)\}$. Let $v$ be a vertex of degree one in $G - \{V(F_1) \cup V(F_2)\}$. We may assume that the vertex $v$ is outside of the cycle $C$ in $\Gamma$ since the proof is similar if $v$ is inside of the cycle $C$. Then the four neighbors of $v$ must be on $F_1$, since $G - \{V(F_1) \cup V(F_2)\}$ contains exactly one cycle by Lemma 7(b), the vertex $v$ is outside of the cycle $C$, and $G(C)$ contains $F_2$ by Lemma 6. Then either $G$ would not be 5-regular or $\Gamma$ would not be a planar embedding of $G$, a contradiction. Hence all the vertices of $G - \{V(F_1) \cup V(F_2)\}$ are contained in cycle $C$.

We now prove the following theorem.

Theorem 2 A $p$-doughnut graph always has a doughnut embedding.
Proof: Let $F_1$ and $F_2$ be two faces of $G$ each of which contains exactly $p$ vertices. Let $\Gamma$ be a planar embedding of $G$ such that $F_1$ is embedded as the outer face. By Lemma 7(c), all the vertices of $G - \{V(F_1) \cup V(F_2)\}$ are contained in a single cycle $C$. By Lemma 6, $G(C)$ contains $F_2$. Then in $\Gamma$, $G(F_1)$ contains $C$ and $G(C)$ contains $F_2$, and hence $\Gamma$ is a doughnut embedding. Q.E.D.

A 1-outerplanar graph is an embedded planar graph where all vertices are on the outer face. It is also called 1-outerplane graph. An embedded graph is a $k$-outerplane ($k > 1$) if the embedded graph obtained by removing all vertices of the outer face is a $(k - 1)$-outerplane graph. A graph is $k$-outerplanar if it admits a $k$-outerplanar embedding. A planar graph $G$ has outerplanarity $k$ ($k > 0$) if it is $k$-outerplanar and it is not $j$-outerplanar for $0 < j < k$.

In the rest of this section, we show that the outerplanarity of a $p$-doughnut graph $G$ is 3. Since none of the faces of $G$ contains all vertices of $G$, $G$ does not admit 1-outerplanar embedding. We thus need to show that $G$ does not admit a 2-outerplanar embedding. We have the following fact.

Fact 8 A graph $G$ having outerplanarity 2 has a cut-set of four or less vertices.

Proof: Deletion of all vertices on the outer face from a 2-outerplane graph leaves a 1-outerplane graph. Since all vertices of a 1-outerplane graph are on the outer face, a 1-outerplane graph has a cut-set of at most two vertices. Then one can observe that a graph $G$ having outerplanarity 2 has a cut-set of four or less vertices.

Since $G$ is 5-connected graph, $G$ has no cut-set of four or less vertices. Hence by Fact 8, the graph $G$ has outerplanarity greater than 2. Thus the following lemma holds.

Lemma 9 Let $G$ be a $p$-doughnut graph for $p \geq 4$. Then $G$ is neither a 1-outerplanar graph nor a 2-outerplanar graph.

We now prove the following theorem.

Theorem 3 The outerplanarity of a $p$-doughnut graph $G$ is 3.

Proof: A doughnut embedding of $G$ immediately implies that $G$ has a 3-outerplanar embedding. By Lemma 9, $G$ is neither a 1-outerplanar graph nor a 2-outerplanar graph. Therefore the outerplanarity of a $p$-doughnut graph is 3. Q.E.D.

4 Drawings of Doughnut Graphs

In this section we give a linear-time algorithm for finding a straight-line grid drawing of a doughnut graph on a grid of linear area.

Let $G$ be a $p$-doughnut graph. Then $G$ has a doughnut embedding by Theorem 2. Let $\Gamma$ be a doughnut embedding of $G$ as illustrated in Figure 4(a). Let $C_1$, $C_2$ and $C_3$ be the outer cycle, the middle cycle and the inner cycle of $\Gamma$, respectively. We have the following facts.
Fact 10 Let $G$ be a $p$-doughnut graph and let $\Gamma$ be a doughnut embedding of $G$. Let $C_1$, $C_2$ and $C_3$ be the outer cycle, the middle cycle and the inner cycle of $\Gamma$, respectively. For any two consecutive vertices $z_i, z_{i+1}$ on $C_2$, one of $z_i, z_{i+1}$ has exactly one neighbor on $C_1$ and the other has exactly two neighbors on $C_1$.

Fact 11 Let $G$ be a $p$-doughnut graph and let $\Gamma$ be a doughnut embedding of $G$. Let $C_1$, $C_2$ and $C_3$ be the outer cycle, the middle cycle and the inner cycle of $\Gamma$, respectively. Let $z_i$ be a vertex of $C_2$, then either the following (a) or (b) holds.

(a) $z_i$ has exactly one neighbor on $C_1$ and exactly two neighbors on $C_3$.

(b) $z_i$ has exactly one neighbor on $C_3$ and exactly two neighbors on $C_1$.

Before describing our algorithm we need some definitions. Let $z_i$ be a vertex of $C_2$ such that $z_i$ has two neighbors on $C_1$. Let $x$ and $x'$ be the two neighbors of $z_i$ on $C_1$ such that $x'$ is the counter clockwise next vertex to $x$ on $C_1$. We call $x$ the left neighbor of $z_i$ on $C_1$ and $x'$ the right neighbor of $z_i$ on $C_1$. Similarly we define the left neighbor and the right neighbor of $z_i$ on $C_3$ if a vertex $z_i$ on $C_2$ has two neighbors on $C_3$. We are now ready to describe our algorithm.

We embed $C_1$, $C_2$ and $C_3$ on three nested rectangles $R_1$, $R_2$ and $R_3$, respectively on a grid as illustrated in Figure 4(b). We draw rectangle $R_1$ on grid with four corners on grid point $(0, 0)$, $(p+1, 0)$, $(p+1, 5)$ and $(0, 5)$. Similarly the four corners of $R_2$ are $(1, 1)$, $(p, 1)$, $(p, 4)$, $(1, 4)$ and the four corners of $R_3$ are $(2, 2)$, $(p-1, 2)$, $(p-1, 3)$, $(2, 3)$.

We first embed $C_2$ on $R_2$ as follows. Let $z_1, z_2, \ldots, z_{2p}$ be the vertices on $C_2$ in counter clockwise order such that $z_1$ has exactly one neighbor on $C_1$. We put $z_1$ on $(1, 1)$, $z_p$ on $(p, 1)$, $z_{p+1}$ on $(p, 4)$ and $z_{2p}$ on $(1, 4)$. We put the other vertices of $C_2$ on grid points of $R_2$ preserving the relative positions of vertices of $C_2$.

We next put vertices of $C_1$ on $R_1$ as follows. Let $x_1$ be the neighbor of $z_1$ on $C_1$ and let $x_1, x_2, \ldots, x_p$ be the vertices of $C_1$ in counter clockwise order. We put $x_1$ on $(0, 0)$ and $x_p$ on $(0, 5)$. Since $z_1$ has exactly one neighbor on $C_1$, by Fact 10 $z_{2p}$ has exactly two neighbors on $C_1$. Since $z_1$ and $z_{2p}$ are on a triangulated face of $G$ having vertices on both $C_1$ and $C_2$, $x_1$ is a neighbor of $z_{2p}$. One can easily observe that $x_p$ is the other neighbor of $z_{2p}$ on $C_1$. Clearly the edges $(x_1, z_1)$, $(x_1, z_{2p})$, $(x_p, z_{2p})$ can be drawn as straight-line segments without edge crossings as illustrated in Figure 4(b). We next put neighbors of $z_p$ and $z_{p+1}$. Let $x_i$ be the neighbor of $z_p$ on $C_1$ if $z_p$ has exactly one neighbor on $C_1$, otherwise let $x_i$ be the left neighbor of $z_p$ on $C_1$. We put $x_i$ on $(p+1, 0)$ and $x_{i+1}$ on $(p+1, 5)$. In case of $z_p$ has exactly one neighbor on $C_1$, by Fact 10 $z_{p+1}$ has two neighbors on $C_1$, and $x_i$ and $x_{i+1}$ are the two neighbors of $z_{p+1}$ on $C_1$. Clearly the edges $(z_p, x_i), (z_{p+1}, x_i)$ and $(z_{p+1}, x_{i+1})$ can be drawn as straight-line segments without edge crossings as illustrated in Figure 4(b). In case of $z_p$ has exactly two neighbors $x_i$ and $x_{i+1}$ on $C_1$, the edges between neighbors of $z_p$ and $z_{p+1}$ on $C_1$ can be drawn without edge crossings as illustrated in Figure 4(b).

We put the other vertices of $C_1$ on grid points of $R_1$ arbitrarily preserving their relative positions on $C_1$.
Figure 4: (a) A doughnut embedding of a $p$-doughnut graph of $G$, (b) edges between four corner vertices of $R_1$ and $R_2$ are drawn as straight-line segments, (c) edges between vertices on $R_1$ and $R_2$ are drawn, (d) edges between four corner vertices of $R_2$ and $R_3$ are drawn as straight-line segments, and (e) a straight-line grid drawing of $G$. 
One can observe that all the edges of $G$ connecting vertices in \{z_2, z_3, \ldots, z_{p-1}\} to vertices in \{x_2, x_3, \ldots, x_{i-1}\}, and connecting vertices in \{z_{p+2}, z_{p+2}, \ldots, z_{2p-1}\} to vertices in \{x_{i+2}, x_{i+3}, \ldots, x_{p-1}\} can be drawn as straight-line segments without edge crossings. See Figure 4(c).

We finally put the vertices of $C_3$ on $R_3$ as follows. Since $z_1$ has exactly one neighbor on $C_1$, by Fact 11(a), $z_1$ has exactly two neighbors on $C_3$. Then, by Fact 11(b), $z_{2p}$ has exactly one neighbor on $C_3$. Let $y_1, y_2, \ldots, y_p$ be the vertices on $C_3$ in counter clockwise order such that $y_1$ is the right neighbor of $z_1$. Then $y_p$ is the left neighbor of $z_1$. We put $y_1$ on (2, 2) and $y_p$ on (2, 3). Clearly the edges $(y_1, z_1)$, $(y_p, z_{2p})$, $(y_p, z_1)$ can be drawn as straight-line segments without edge crossings, as illustrated in Figure 4(d). We next put neighbors of $z_p$ and $z_{p+1}$ on $C_3$ as we have put the neighbors of $z_p$ and $z_{p+1}$ on $C_1$ at the other two corners of $R_3$ in a counter clockwise order as illustrated in Figure 4(d). We put the other vertices of $C_3$ on grid points of $R_3$ arbitrarily preserving their relative positions on $C_3$. It is not difficult to show that edges from the vertices on $C_2$ to the vertices on $C_3$ can be drawn as straight-line segments without edge crossings. Figure 4(e) illustrates the complete straight-line grid drawing of a $p$-doughnut graph.

The area requirement of the straight-line grid drawing of a $p$-doughnut graph $G$ is equal to the area of rectangle $R_1$ and the area of $R_1$ is $(p+1) \times 5 = (n/4 + 1) \times 5 = O(n)$, where $n$ is the number of vertices in $G$. Thus we have a straight-line grid drawing of a $p$-doughnut graph on a grid of linear area. Clearly the algorithm takes linear time. Thus the following theorem holds.

**Theorem 4** A doughnut graph $G$ of $n$ vertices has a straight-line grid drawing on a grid of area $O(n)$. Furthermore, the drawing of $G$ can be found in linear time.
5 Spanning Subgraphs of Doughnut Graphs

In Section 4, we have seen that a doughnut graph admits a straight-line grid drawing with linear area. One can easily observe that a spanning subgraph of a doughnut graph also admits a straight-line grid drawing with linear area. Figure 6(b) illustrates a straight-line grid drawing with linear area of a graph $G'$ in Figure 6(a) where $G'$ is a spanning subgraph of a doughnut graph $G$ in Figure 1(a). Using a transformation from the “subgraph isomorphism” problem [8], one can easily prove that the recognition of a spanning subgraph of a given graph is an NP-complete problem in general. Hence the recognition of a spanning subgraph of a doughnut graph seems to be a non-trivial problem. We thus restrict ourselves only to 4-connected planar graphs. In this section, we give a necessary and sufficient condition for a 4-connected planar graph to be a spanning subgraph of a doughnut graph as in the following theorem.

![Figure 6: (a) A spanning subgraph $G'$ of $G$ in Figure 1(a), and (b) a straight-line grid drawing of $G'$ with area $O(n)$.](image)

**Theorem 5** Let $G$ be a 4-connected planar graph with $4p$ vertices where $p > 4$ and let $\Delta(G) \leq 5$. Let $\Gamma$ be a planar embedding of $G$. Assume that $\Gamma$ has exactly two vertex disjoint faces $F_1$ and $F_2$ each of which has exactly $p$ vertices. Then $G$ is a spanning subgraph of a $p$-doughnut graph if and only if the following conditions (a) – (e) hold.

- (a) $G$ has no edge $(x, y)$ such that $x \in V(F_1)$ and $y \in V(F_2)$.
- (b) Every face $f$ of $\Gamma$ has at least one vertex $v \in \{V(F_1) \cup V(F_2)\}$.
- (c) For any vertex $x \notin \{V(F_1) \cup V(F_2)\}$, the total number of neighbors of $x$ on faces $F_1$ and $F_2$ are at most three.
- (d) Every face $f$ of $\Gamma$ except the faces $F_1$ and $F_2$ has either three or four vertices.
- (e) For any $x$-$y$ path $P$ such that $V(P) \cap \{V(F_1) \cup V(F_2)\} = \emptyset$ and $x$ has exactly two neighbors on face $F_1(F_2)$. Then the following conditions hold.
  - (i) If $P$ is even, then the vertex $y$ has at most two neighbors on face $F_1(F_2)$ and at most one neighbor on face $F_2(F_1)$.
(ii) If \( P \) is odd, then the vertex \( y \) has at most one neighbor on face \( F_1(F_2) \) and at most two neighbors on face \( F_2(F_1) \).

Fact [1] implies that the decomposition of a 4-connected planar graph \( G \) into its facial cycles is unique. Throughout the section we thus often mention faces of \( G \) without mentioning its planar embedding where the description of the faces is valid for any planar embedding of \( G \), since \( \kappa(G) \geq 4 \) for every graph \( G \) considered in this section.

Before proving the necessity of Theorem 5, we have the following fact.

**Fact 12** Let \( G \) be a 4-connected planar graph with \( 4p \) vertices where \( p > 4 \) and let \( \Delta(G) \leq 5 \). Assume that \( G \) has exactly two vertex disjoint faces \( F_1 \) and \( F_2 \) each of which has exactly \( p \) vertices. If \( G \) is a spanning subgraph of a doughnut graph then \( G \) can be augmented to a 5-connected 5-regular graph \( G' \) through triangulation of all the non-triangulated faces of \( G \) except the faces \( F_1 \) and \( F_2 \).

One can easily observe that the following fact holds from the construction given in Section 3.

**Fact 13** Let \( G \) be a doughnut graph, and let \( P \) be any \( x-y \) path such that \( V(P) \cap \{V(F_1) \cup V(F_2)\} = \emptyset \) and \( x \) has exactly two neighbors on face \( F_1(F_2) \). Then the following conditions (i) and (ii) hold.

(i) If \( P \) is even, then the vertex \( y \) has exactly two neighbors on face \( F_1(F_2) \) and exactly one neighbor on face \( F_2(F_1) \).

(ii) If \( P \) is odd, then the vertex \( y \) has exactly one neighbor on face \( F_1(F_2) \) and exactly two neighbors on face \( F_2(F_1) \).

We are ready to prove the necessity of Theorem 5.

**Proof for the Necessity of Theorem 5**

Assume that \( G \) is a spanning subgraph of a \( p \)-doughnut graph. Then by Theorem 1 \( G \) has \( 4p \) vertices. Clearly \( \Delta(G) \leq 5 \) and \( G \) satisfies the conditions (a), (b) and (c), otherwise \( G \) would not be a spanning subgraph of a doughnut graph. The necessity of condition (e) is obvious by Fact 13. Hence it is sufficient to prove the necessity of condition (d) only.

(d) \( G \) does not have any face of two or less vertices since \( G \) is a 4-connected planar graph. Then every face of \( G \) has three or more vertices. We now show that \( G \) has no face of more than four vertices. Assume for a contradiction that \( G \) has a face \( f \) of \( q \) vertices such that \( q > 4 \). Then \( f \) can be triangulated by adding \( q - 3 \) extra edges. These extra edges increase the degrees of \( q - 2 \) vertices, and the sum of the degrees will be increased by \( 2(q - 3) \). Using the pigeonhole principle, one can easily observe that there is a vertex among the \( q(> 4) \) vertices whose degree will be raised by at least 2 after a triangulation of \( f \). Then \( G' \) would have a vertex of degree six or more where \( G' \) is a graph obtained after triangulation of \( f \). Hence we cannot augment \( G \) to a 5-regular graph through triangulation of all the non-triangulated faces of \( G \) other than the faces \( F_1 \) and \( F_2 \). Therefore \( G \) cannot be a spanning subgraph of a doughnut
In the remaining of this section we give a constructive proof for the sufficiency of Theorem 5. Assume that \( G \) satisfies the conditions in Theorem 5. We have the following lemma.

**Lemma 14** Let \( G \) be a 4-connected planar graph satisfying the conditions in Theorem 5. Assume that all the faces of \( G \) except \( F_1 \) and \( F_2 \) are triangulated. Then \( G \) is a doughnut graph.

**Proof:** To prove the claim, we have to prove that 

(i) \( G \) is 5-connected, 
(ii) \( G \) has two vertex disjoint faces each of which has exactly \( p \), \( p > 4 \), vertices and all the other faces of \( G \) has exactly three vertices, and 
(iii) \( G \) has the minimum number of vertices satisfying the properties (i) and (ii).

(i) We first prove that \( G \) is a 5-regular graph. Every face of \( G \) except \( F_1 \) and \( F_2 \) is a triangle. Furthermore each of \( F_1 \) and \( F_2 \) has exactly \( p \), \( p > 4 \), vertices. Then \( G \) has \( 3(4p) - 6 - 2(p - 3) = 10p \) edges. Since none of the vertices of \( G \) has degree more than five and \( G \) has exactly \( 10p \) edges, each vertex of \( G \) has degree exactly five. We next prove that the vertices of \( G \) lie on three vertex-disjoint cycles \( C_1, C_2 \) and \( C_3 \) such that cycles \( C_1, C_2, C_3 \) contain exactly \( p \), \( 2p \) and \( p \) vertices, respectively. We take an embedding \( \Gamma \) of \( G \) such that \( F_1 \) is embedded as the outer face and \( F_2 \) is embedded as an inner face. We take the contour of face \( F_1 \) as cycle \( C_1 \) and contour of face \( F_2 \) as cycle \( C_3 \). Then each of \( C_1 \) and \( C_2 \) contains exactly \( p \), \( p > 4 \), vertices. Since \( G \) satisfies conditions (a), (b) and (c) in Theorem 5 and all the faces of \( G \) except \( F_1 \) and \( F_2 \) are triangulated, the rest \( 2p \) vertices of \( G \) form a cycle in \( \Gamma \). We take this cycle as \( C_2 \). \( G(C_2) \) contains \( C_3 \) since \( G \) satisfies condition (b) in Theorem 5. Clearly \( C_1, C_2 \) and \( C_3 \) are vertex-disjoint and cycles \( C_1, C_2, C_3 \) contain exactly \( p \), \( 2p \) and \( p \) vertices, respectively. We finally prove that \( G \) is 5-connected. Assume for a contradiction that \( G \) has a cut-set of less than five vertices. In such a case \( G \) would have a vertex of degree less than five, a contradiction.

(ii) The proof of this part is obvious since \( G \) has two vertex disjoint faces each of which has exactly \( p \) vertices and all the other faces of \( G \) has exactly three vertices.

(iii) The number of vertices of \( G \) is \( 4p \). Using Lemma 2 we can easily prove that the minimum number of vertices required to construct a graph \( G \) that satisfies the properties (i) and (ii) is \( 4p \).

Q.E.D.

We thus assume that \( G \) has a non-triangulated face \( f \) except faces \( F_1 \) and \( F_2 \). By condition (d) in Theorem 5, \( f \) is a quadrangle face. It is sufficient to show that we can augment the graph \( G \) to a doughnut graph by triangulating each of the quadrangle faces of \( G \). However, we cannot augment \( G \) to a doughnut graph by triangulating each quadrangle face arbitrarily. For example, the graph \( G \) in Figure 7(a) satisfies all the conditions in Theorem 5 and it has exactly one quadrangle face \( f_1(a,b,c,d) \). If we triangulate \( f_1 \) by adding an
edge \((a, c)\) as illustrated in Figure 7(b), the resulting graph \(G'\) would not be a doughnut graph since a doughnut graph does not have an edge \((a, c)\) such that \(a \in V(F_1)\) and \(c \in V(F_2)\). But if we triangulate \(f_1\) by adding an edge \((b, d)\) as illustrated in Figure 7(c), the resulting graph \(G'\) is a doughnut graph. Hence every triangulation of a quadrangle face is not always valid to augment \(G\) to a doughnut graph. We call a triangulation of a quadrangle face \(f\) of \(G\) a valid triangulation if the resulting graph \(G'\) obtained after the triangulation of \(f\) does not contradict any condition in Theorem 5. We call a vertex \(v\) on the contour of a quadrangle face \(f\) a good vertex if \(v\) is one of the end vertex of an edge which is added for a valid triangulation of \(f\).

![Figure 7](image1)

**Figure 7**: (a) \(f_1(a, b, c, d)\) is a quadrangle face, (b) the triangulation of \(f_1\) by adding the edge \((a, c)\) and (c) the triangulation of \(f_1\) by adding the edge \((b, d)\).

We call a quadrangle face \(f\) of \(G\) an \(\alpha\)-face if \(f\) contains at least one vertex from each of the faces \(F_1\) and \(F_2\). Otherwise, we call a quadrangle face \(f\) of \(G\) a \(\beta\)-face. In Figure 8, \(f_1(a, b, c, d)\) is an \(\alpha\)-face whereas \(f_2(p, q, r, s)\) is a \(\beta\)-face.

![Figure 8](image2)

**Figure 8**: \(f_1(a, b, c, d)\) is an \(\alpha\)-face and \(f_2(p, q, r, s)\) is a \(\beta\)-face.

In a valid triangulation of an \(\alpha\)-face \(f\) of \(G\) no edge is added between any two vertices \(x, y \in V(f)\) such that \(x \in V(F_1)\) and \(y \in V(F_2)\). Hence the following fact holds on an \(\alpha\)-face \(f\).

**Fact 15** Let \(G\) be a 4-connected planar graph satisfying the conditions in Theorem 5. Let \(f\) be an \(\alpha\)-face in \(G\). Then \(f\) admits a unique valid triangulation and the triangulation is obtained by adding an edge between two vertices those are not on \(F_1\) and \(F_2\).
Faces $f_1(a, b, c, d)$ and $f_2(p, q, r, s)$ in Figure 9(a) are two $\alpha$-faces and Figure 9(b) illustrates the valid triangulations of $f_1$ and $f_2$. Vertices $b$ and $d$ of $f_1$ and vertices $q$ and $s$ of $f_2$ are good vertices.

Figure 9: (a) $f_1(a, b, c, d)$ and $f_2(p, q, r, s)$ are two $\alpha$-faces, and (b) valid triangulations of $f_1$ and $f_2$.

We call a $\beta$-face a $\beta_1$-face if the face contains exactly one vertex either from $F_1$ or from $F_2$. Otherwise we call a $\beta$-face a $\beta_2$-face. In Figure 10, $f_1(a, b, c, d)$ is a $\beta_1$-face whereas $f_2(p, q, r, s)$ is a $\beta_2$-face. We call a vertex $v$ on the contour of a $\beta_1$-face $f$ a middle vertex of $f$ if the vertex is in the middle position among the three consecutive vertices other than the vertex on $F_1$ or $F_2$. In Figure 10, vertex $c$ of $f_1$ and vertex $r$ of $f_2$ are the middle vertices of $f_1$ and $f_2$, respectively.

Figure 10: $f_1(a, b, c, d)$ is a $\beta_1$-face and $f_2(p, q, r, s)$ is a $\beta_2$-face.

In a valid triangulation of a $\beta_1$-face $f$ of $G$ no edge is added between any two vertices $x, y \in V(f)$ such that $x, y \notin V(F_1) \cup V(F_2)$. Hence the following fact holds on a $\beta_1$-face $f$.

**Fact 16** Let $G$ be a 4-connected planar graph satisfying the conditions in Theorem 5. Let $f$ be a $\beta_1$-face of $G$. Then $f$ admits a unique valid triangulation and the triangulation is obtained by adding an edge between the vertex on $F_1$ or $F_2$ and the middle vertex.

Faces $f_1(a, b, c, d)$ and $f_2(p, q, r, s)$ in Figure 11(a) are two $\beta_1$-faces and Figure 11(b) illustrates the valid triangulations of $f_1$ and $f_2$. Vertices $a$ and $c$ of $f_1$ and vertices $p$ and $r$ of $f_2$ are good vertices.
Figure 11: (a) $f_1(a,b,c,d)$ and $f_2(p,q,r,s)$ are two $\beta_1$-faces, and (b) valid triangulations of $f_1$ and $f_2$.

In a valid triangulation of a $\beta_2$-face $f$ of $G$ no edge is added between any two vertices $x,y \in V(f)$ where $x \in V(F_1)(V(F_2))$, $y \notin \{V(F_1) \cup V(F_1)\}$ and $G$ has either (i) an even $q$-$y$ path $P$ such that $q$ has exactly two neighbors on $F_2(F_1)$ and $V(P) \cap \{V(F_1) \cup V(F_2)\} = \emptyset$, or (ii) an odd $q$-$y$ path $P$ such that $q$ has exactly two neighbors on $F_1(F_2)$ and $V(P) \cap \{V(F_1) \cup V(F_2)\} = \emptyset$. Hence the following fact holds on a $\beta_2$-face.

\textbf{Fact 17} Let $G$ be a 4-connected planar graph satisfying the conditions in Theorem 5. Let $f$ be a $\beta_2$-face of $G$. Then $f$ admits a unique valid triangulation and the triangulation is obtained by adding an edge between a vertex on face $F_1$ or $F_2$ and a vertex $z \notin V(F_1) \cup V(F_2)$.

Face $f_1(a,b,c,d)$ in the graph in Figure 12(a) is a $\beta_2$-face and the graph has an even $u$-$d$ path $P$ such that $u$ has exactly two neighbors $g$ and $h$ on $F_2$, and $V(P) \cap \{V(F_1) \cup V(F_2)\} = \emptyset$. Figure 12(c) illustrates the valid triangulation of $f_1$. Vertices $a$ and $c$ are the good vertices of $f_1$. Face $f_2(l,m,n,o)$ in the graph in Figure 12(b) is a $\beta_2$-face and the graph has an odd $v$-$o$ path $P$ such that $v$ has exactly two neighbors $s$ and $t$ on $F_1$, and $V(P) \cap \{V(F_1) \cup V(F_2)\} = \emptyset$. Figure 12(d) illustrates the valid triangulation of $f_2$. Vertices $l$ and $n$ are the good vertices of $f_2$.

Before giving a proof for the sufficiency of Theorem 5 we need to prove the following Lemmas 18 and 19.

\textbf{Lemma 18} Let $G$ be a 4-connected planar graph satisfying the conditions in Theorem 5. Then any quadrangle face $f$ of $G$ admits a unique valid triangulation such that after triangulation $d(v) \leq 5$ holds for any vertex $v$ in the resulting graph.

\textbf{Proof:} By Facts 15, 16 and 17 $f$ admits a unique valid triangulation. Since a valid triangulation increases the degree of a good vertex by one, it is sufficient to show that each good vertex of $f$ has degree less than five in $G$. Assume for a contradiction that a good vertex $v$ has degree more than four in $G$. Then one can observe that $G$ would violate a condition in Theorem 5. Q.E.D.
Lemma 19 Let $G$ be a 4-connected planar graph satisfying the conditions in Theorem 5. Also assume that $G$ has quadrangle faces. Then no two quadrangle faces $f_1$ and $f_2$ have a common vertex which is a good vertex for both the faces $f_1$ and $f_2$.

Proof: Assume that $u$ is a common vertex between two quadrangle faces $f_1$ and $f_2$. If $u$ is neither a good vertex of $f_1$ nor a good vertex of $f_2$, then we have done. We thus assume that $u$ is a good vertex of $f_1$ or $f_2$. Without loss of generality, we assume that $u$ is a good vertex of $f_1$. Then $u$ is not a good vertex of $f_2$, otherwise $u$ would not be a common vertex of $f_1$ and $f_2$, a contradiction. Q.E.D.

Proof for the Sufficiency of Theorem 5

Assume that the graph $G$ satisfies all the conditions in Theorem 5. If all the faces of $G$ except $F_1$ and $F_2$ are triangulated, then $G$ is a doughnut graph by Lemma 14. Otherwise, we triangulate each quadrangle face of $G$, using its valid triangulation. Let $G'$ be the resulting graph. Lemmas 18 and 19 imply that $d(v) \leq 5$ for each vertex $v$ in $G'$. Then the graph $G'$ satisfies the conditions in Theorem 5 since $G$ satisfies the conditions in Theorem 5. $G'$ is obtained from $G$ using valid triangulations of quadrangle faces and $d(v) \leq 5$ for each vertex $v$ in $G'$. Hence $G'$ is a doughnut graph by Lemma 14. Therefore $G$ is a spanning subgraph of a doughnut graph. Q.E.D.

We now have the following lemma.

Lemma 20 Let $G$ be a 4-connected planar graph satisfying the conditions in Theorem 5. Then $G$ can be augmented to a doughnut graph in linear time.

Proof: We first embed $G$ such that $F_1$ is embedded as the outer face and $F_2$ is embedded as an inner face. We then triangulate each of the quadrangle faces of $G$ using its valid triangulation if $G$ has quadrangle faces. Let $G'$ be the resulting graph. As shown in the sufficiency proof of Theorem 5, $G'$ is a
doughnut graph. One can easily find all quadrangle faces of $G$ and perform their valid triangulations in linear time, hence $G'$ can be obtained in linear time.

\textbf{Q.E.D.}

In Theorem 5 we have given a necessary and sufficient condition for a 4-connected planar graph to be a spanning subgraph of a doughnut graph. As described in the proof of Lemma 20, we have provided a linear-time algorithm to augment a 4-connected planar graph $G$ to a doughnut graph if $G$ satisfies the conditions in Theorem 5. We have thus identified a subclass of 4-connected planar graphs that admits straight-line grid drawings with linear area as stated in the following theorem.

\textbf{Theorem 6} Let $G$ be a 4-connected planar graph satisfying the conditions in Theorem 5. Then $G$ admits a straight-line grid drawing on a grid of area $O(n)$. Furthermore, the drawing of $G$ can be found in linear time.

\textbf{Proof:} Using the method described in the proof of Lemma 20, we augment $G$ to a doughnut graph $G'$ by adding dummy edges (if required) in linear time. By Theorem 4, $G'$ admits a straight-line grid drawing on a grid of area $O(n)$. We finally obtain a drawing of $G$ from the drawing of $G'$ by deleting the dummy edges (if any) from the drawing of $G'$. By Lemma 20, $G$ can be augmented to a doughnut graph in linear time and by Theorem 4 a straight-line grid drawing of a doughnut graph can be found in linear time. Moreover, the dummy edges can also be deleted from the drawing of a doughnut graph in linear time. Hence the drawing of $G$ can be found in linear time. \textbf{Q.E.D.}

\section{Conclusion}

In this paper we introduced a new class of planar graphs, called doughnut graphs, which is a subclass of 5-connected planar graphs. A graph in this class has a straight-line grid drawing on a grid of linear area, and the drawing can be found in linear time. We showed that the outerplanarity of a doughnut graph is 3. Thus we identified a subclass of 3-outerplanar graphs that admits straight-line grid drawing with linear area. One can easily observe that any spanning subgraph of a doughnut graph also admits straight-line grid drawing with linear area. However, the recognition of a spanning subgraph of a doughnut graph seems to be a non-trivial problem. We established a necessary and sufficient condition for a 4-connected planar graph $G$ to be a spanning subgraph of a doughnut graph. We also gave a linear-time algorithm to augment a 4-connected planar graph $G$ to a doughnut graph if $G$ satisfies the necessary and sufficient condition. By introducing the necessary and sufficient condition, in fact, we have identified a subclass of 4-connected planar graphs that admits straight-line grid drawings with linear area. Finding other nontrivial classes of planar graphs that admit straight-line grid drawings on grids of linear area is also left as an open problem.
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