ON THE OPTIMAL CONTROL PROBLEMS WITH CHARACTERISTIC TIME CONTROL CONSTRAINTS

CHANGJUN YU
Shanghai University
Shanghai, 200444, China

SHUXUAN SU* AND YANQIN BAI
Shanghai University
Shanghai, 200444, China

(Communicated by Kok Lay Teo)

Abstract. In this paper, we consider a class of optimal control problems with control constraints on a set of characteristic time instants. By applying the control parameterization technique, these constraints are imposed on the subdomains that contain the characteristic time points. The values of the control functions as well as the lengths for their corresponding subdomains become decision variables. Time-scaling transformation is an effective technique to optimize the length of each subdomain in a new time horizon. However, the characteristic time instants in the original time horizon become variable time instants in the new time horizon, and hence the control constraints imposed on these characteristic time points are difficult to be formulated in the new time horizon. We propose a surrogate condition and show that the characteristic time control constraints will be satisfied once the surrogate condition holds. Moreover, this surrogate condition is easy to formulate in the new time horizon. The resulting approximate problem can be readily solved by many existing computational methods for solving constrained optimal control problems. Finally, we conclude this paper by solving two examples.

1. Introduction. Optimal control concerns with finding a control strategy for a real-world dynamic system such that a given performance function is minimized subject to some physical constraints. We consider a class of optimal control problems with control constraints at a set of characteristic time points. Such problems arise naturally in the area of thermodynamic, such as the design of optimal quantum control [24], fast transport and expansion or compression of cold atoms [4, 8, 29] and optical solitons compression [25].

In [28], the optimal shortcuts to adiabaticity for a quantum piston is formulated as a free terminal time optimal control problem. Its aim is to manipulate the control function such that the piston expands from the initial state to the target

2020 Mathematics Subject Classification. Primary: 90C30, 90-08, 34H05.
Key words and phrases. Optimal control, characteristic time, control constraints, control parameterization, time-scaling transformation.

This work is supported by National Natural Science Foundation of China(NSFC), Grant No.11871039 and Science and Technology Commission of Shanghai Municipality(STCSM), Grant No. 20JC1413900.

* Corresponding author: Shuxuan Su.
state in minimum time, subject to adiabaticity constraint. Furthermore, the control functions are required to satisfy some specific conditions at the initial and the undetermined terminal time of the process. The optimal solution is obtained by applying Pontryagin’s Maximum Principle [2, 12]. However, for constrained nonlinear optimal control problems, it is generally very difficult to be solved analytically. Thus, numerical methods are indispensable for solving these practical constrained optimal control problems [13, 20, 32].

Control parameterization method is a popular approach for solving general constrained optimal control problems [7, 22, 33, 37, 38]. This method involves approximating the control function by a piecewise constant function. Then, the original optimal control problem becomes an approximate optimization problem with the heights of the piecewise constant function and the corresponding switching times being the decision variables to be optimized. The approximated problem can then be solved by gradient-based constrained optimization methods [3, 10, 26].

Under the framework of control parameterization, each section of the piecewise constant function will be used for a period of time. Thus, each characteristic time control constraint can be imposed on the interval that contains this particular characteristic time instant. However, as mentioned in [17, 18, 35], there are some deficiencies associated with the switching times being taken as decision variables.

In order to overcome the numerical difficulties caused by directly optimizing the control switching times, the time-scaling transformation technique is developed in [14]. This transformation maps the variable switching times in the original time horizon to fixed switching times in the new time horizon through the introduction of additional control variables called time-scaling controls. Then, the transformed optimal control problem becomes an optimal control problem with the heights of the piecewise constant controls and the lengths of the durations of the controls being taken as decision variables.

However, the time scaling transformation is not directly applicable to optimal control problem subject to the constraints imposed on the control at a set of characteristic times. This is because after applying the time-scaling transformation, those fixed characteristic time points become variable time points in the new time horizon [16, 34, 36], which leads to numerical difficulties in practical computation.

In this paper, we consider a class of optimal control problem with characteristic time control constraints. By applying the control parameterization method in conjunction with the time-scaling transformation, we obtain an approximate optimization problem. However, the characteristic time points in the original time scale become variable time points in the new time scale. In fact, these time points are depending on the duration of control vector. Hence, the control constraints imposed on these time points are very difficult to be formulated, this leads to numerical difficulties in practical computation. In order to overcome this difficulty, we propose a surrogate condition to replace the characteristic time control constraints, the surrogate condition is much easier to formulate. The satisfaction of the surrogate condition ensures that the characteristic time control constraints will be satisfied. The resulting approximate optimization problem can be readily solved by existing optimal control solver, i.e., MISER [9].

The paper is organized as follows. In Section 2, we formulate the optimal control problem governed by nonlinear dynamic system, subject to control constraints imposed on characteristic time points. A solution procedure is developed in Section 3 to solve this problem. We derive an implementable sufficient condition to
ensure that the characteristic time control constraints are satisfied. In Section 4, we present two examples to demonstrate the effectiveness of the solution methods for solving this type of optimal control problem. Finally, we provide the conclusion of the paper in Section 5.

2. Problem formulation. Consider the following dynamic system

$$\dot{x}(t) = f(x(t), u(t)), \quad t \in [t_0, t_f],$$

with the initial condition

$$x(t_0) = x^0,$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^\top \in \mathbb{R}^n$ is the state vector, $x^0$ is a given initial state, $u(t) = [u_1(t), u_2(t), \ldots, u_r(t)]^\top \in \mathbb{R}^r$ is the control vector and $f: \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ is a given function, $t_f$ stands for the terminal time, it can be pre-given or unknown.

Define

$$U := \{v = [v_1, \ldots, v_r]^\top : a^L_k \leq v_k \leq a^U_k, \quad k = 1, \ldots, r\},$$

where $a^L_k$ and $a^U_k$, $k = 1, \ldots, r$, are given constants such that $a^L_k < a^U_k$. We suppose that the following equation is satisfied:

$$u(\tau_j) = \xi_j, \quad j = 1, \ldots, m,$$

where $\tau_j$ is referred to as characteristic time such that $t_0 < \tau_1 < \cdots < \tau_m < t_f$, and $\xi_j, \quad j = 1, \ldots, m$ are given vectors in $\mathbb{R}^r$. Let $u: [t_0, t_f] \to \mathbb{R}^r$ be a piecewise continuous function which is right-continuous at the points of discontinuity. For any $t \in [t_0, t_f]$, if $u(t) \in U$, then it is called an admissible control. Let $U$ denote the class of all such admissible controls. For a given $u \in U$, let $x(\cdot|u)$ be the corresponding solution of system dynamics (1).

Now, let us consider following optimal control problem with characteristic time control constraints.

**Problem (P).** Given dynamic system (1), find an admissible control $u \in U$ to minimize the following cost functional

$$G_0(u) = \Phi_0(x(t_f)) + \int_{t_0}^{t_f} L_0(x(t), u(t)) dt,$$

subject to the canonical equality and inequality constraints

$$G_i(u) = \Phi_i(x(t_f)) + \int_{t_0}^{t_f} L_i(x(t), u(t)) dt \begin{cases} 0, & i = 1, \ldots, N_e, \\ \geq 0, & i = N_e + 1, \ldots, N, \end{cases}$$

and characteristic time control constraints (2), where $\Phi_i: \mathbb{R}^n \to \mathbb{R}$, $i = 0, \ldots, N$, and $L_i: \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$, $i = 0, \ldots, N$, are given real-valued functions.

Throughout this paper, we assume that the following assumptions are satisfied.

**Assumption 2.1.** There exists a positive constant $L_1 > 0$ such that

$$\|f(\eta, \xi)\| \leq L_1 (1 + \|\eta\| + \|\xi\|), \quad (\eta, \xi) \in \mathbb{R}^n \times \mathbb{R}^r,$$

where $\|\cdot\|$ denotes the Euclidean norm.

**Assumption 2.2.** $f: \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ is twice continuously differentiable.
Assumption 2.3. $\Phi_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 0, \ldots, N$, and $\mathcal{L}_i : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}$, $i = 0, \ldots, N$, are continuously differentiable with respect to each of their arguments.

Note that under Assumptions 2.1-2.3, the solution of the dynamic system (1) is assured (see [1]).

3. Problem procedure.

3.1. Control parameterization. We subdivide the time horizon $[t_0, t_f]$ into $p \geq 1$ subintervals. Let $t_l$, $l = 0, 1, \ldots, p-1$, be the partition points such that the following conditions are satisfied:

$$t_0 < t_1 < t_2 < \cdots < t_{p-1} < t_p = t_f. \quad (5)$$

Let $\Delta$ denote the set of all vectors $\delta = [t_1, \ldots, t_{p-1}]^T$ such that (5) is satisfied. Furthermore, let the control $u$ be approximated by a piecewise constant function as follows:

$$u(t) \approx u^p(t) = \sum_{l=1}^{p} \sigma^l \chi_{[t_{l-1}, t_l)}(t), \ t \in [t_0, t_f], \quad (6)$$

where $\sigma^l = [\sigma^l_1, \ldots, \sigma^l_j]^T$ is the value of the control on the $l$-th subinterval, $\sigma = [(\sigma^1)^T, \ldots, (\sigma^p)^T]^T \in \mathbb{R}^{p \times r}$ is called the control vector, and $\chi$ is the indicator function such that:

$$\chi_{[t_{l-1}, t_l)}(t) = \begin{cases} 1, & \text{if } t \in [t_{l-1}, t_l), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\Sigma$ denote the set of all such control vectors $\sigma \in \mathbb{R}^{p \times r}$. It is obvious that for each $j = 1, \ldots, m$, one can always find a unique $i \in \{1, \ldots, p\}$ such that $\tau_j \in [t_{i-1}, t_i)$, and hence the characteristic time control constraints (2) become

$$\sigma^i = \zeta_j. \quad (7)$$

The dynamic system (1) becomes

$$\dot{x}(t) = \sum_{l=1}^{p} f(x(t)|u^p(\cdot|\delta, \sigma)), \sigma^l) \chi_{[t_{l-1}, t_l)}(t), \ t \in [t_0, t_f], \quad (8a)$$

$$x(t_0) = x^0. \quad (8b)$$

Let $x(\cdot|\delta, \sigma)$ denote the solution of (8) corresponding to $汗 (\delta, \sigma) \in \Delta \times \Sigma$.

Problem $(P)$ becomes an optimal parameter selection problem given below.

Problems $(P(p))$. Given dynamic system (8), choose a pair $(\delta, \sigma) \in \Delta \times \Sigma$ such that the cost functional

$$G_0^p(\delta, \sigma) = \Phi_0(x(t_f)) + \sum_{l=1}^{p} \int_{t_{l-1}}^{t_l} \mathcal{L}_0(x(t)|u^p(\cdot|\delta, \sigma)), \sigma^l)dt \quad (9)$$

is minimized subject to the canonical equality and inequality constraints

$$G_i^p(\delta, \sigma) = \Phi_i(x(t_f)) + \sum_{l=1}^{p} \int_{t_{l-1}}^{t_l} \mathcal{L}_i(x(t)|u^p(\cdot|\delta, \sigma)), \sigma^l)dt \begin{cases} = 0, & i = 1, \ldots, N_e, \\ \geq 0, & i = N_e + 1, \ldots, N, \end{cases} \quad (10)$$

and the characteristic time control constraints (7).
3.2. Time-scaling transformation. After applying control parameterization to Problem \( (P) \), the control is approximated by a piecewise constant function with its control heights and switching times being taken as the decision variables. To overcome the difficulties [17] caused by directly optimizing the control switching times, in this paper, we apply the well-known time-scaling transformation [14, 21]. It works by mapping the variable switching time points in the original time horizon into fixed time points in a new time horizon \([0, p]\).

First, for any \( \mathbf{t} = [t_1, \ldots, t_{p-1}]^T \in \Delta \), define
\[
\Theta := \{ \mathbf{t} = [\theta_1, \ldots, \theta_{p-1}]^T \in \mathbb{R}^{p-1} : \theta_{l} = t_l - t_{l-1} > 0, \; l = 1, \ldots, p - 1 \}.
\]

Clearly, \( \theta_l \) is the duration between the switching times for the control vector \( \mathbf{\sigma}^l \), \( l = 1, \ldots, p - 1 \), a vector in \( \Theta \) is called admissible duration vector. For any \( \mathbf{t} \in \Theta \), it is easy to see that \( \theta_1 + \cdots + \theta_p = t_f - t_0 \).

The time-scaling transformation is a one-to-one mapping between \( [0, p] \) and the new time horizon, which is referred to as the time-scaling function, 
\[
\delta(s) = \frac{dt(s)}{ds} = \theta_l, \; s \in [l - 1, l), \tag{11}
\]
with the initial and terminal condition
\[
t(0) = t_0, \tag{12}
\]
\[
t(p) = t_f. \tag{13}
\]

By integrating (11)-(13), we obtain the relationship between the original time horizon and the new time horizon, which is referred to as the time-scaling function,
\[
t = \mu(s|\mathbf{t}) = t_0 + \sum_{l=1}^{[s]} \theta_l + \theta_{l+[s]+1} (s - [s]), \; s \in [0, p], \tag{14}
\]
where \([\cdot]\) denotes the floor function.

Define a new state vector \( \mathbf{v}(s) = \mathbf{x}(\mu(s)) \). Applying the time-scaling transformation to system (8) gives
\[
\dot{\mathbf{y}}(s) = \frac{d}{ds} \mathbf{x}(\mu(s)) = \theta_l \mathbf{f}(\mathbf{v}(s), \mathbf{\sigma}^l), \; s \in [l - 1, l), \; l = 1, \ldots, p.
\]

Thus, we obtain the following new dynamic system in the new time horizon:
\[
\dot{\mathbf{y}}(s) = \theta_l \mathbf{f}(\mathbf{y}(s), \mathbf{\sigma}^l), s \in [l - 1, l), \; l = 1, \ldots, p, \tag{15a}
\]
\[
\mathbf{y}(0) = \mathbf{x}^0, \tag{15b}
\]
where \( \mathbf{t} \in \Theta \), \( \mathbf{\sigma} \in \Sigma \), \( \mathbf{y}(s) := [y_{1}(s), \ldots, y_{n}(s)]^T \in \mathbb{R}^n \). Under Assumptions 2.2 and 2.3, system (15) has a unique solution for each admissible pair \((\mathbf{t}, \mathbf{\sigma})\). Let \( \mathbf{v}(\cdot|\mathbf{t}, \mathbf{\sigma}) \) denote the solution of (15).

Now, for each admissible duration vector \( \mathbf{t} \in \Theta \), define \( \mathbf{v}(\mathbf{t}) = [\mu(1), \ldots, \mu(p - 1)]^T \). Since for each \( l = 1, \ldots, p \), \( \mu(l - 1) < \mu(l) \), it is clearly that \( \mathbf{v}(\mathbf{t}) \) is an admissible switching time vector for Problem \((P(p))\). For each \((\mathbf{t}, \mathbf{\sigma}) \in \Theta \times \Sigma\), let \( \mathbf{x}(\cdot|\mathbf{v}(\mathbf{t}), \mathbf{\sigma}) \) be the corresponding state trajectory of system (8). From [19], it is clear that
\[
\mathbf{x}(t|\mathbf{v}(\mathbf{t}), \mathbf{\sigma}) = \mathbf{y}(s|\mathbf{t}, \mathbf{\sigma}), \; s \in [0, p].
\]
Given a \((\theta, \sigma) \in \Theta \times \Sigma\), one can solve the system of differential equations (15a) using the initial condition (15b) to obtain \(y(s|\theta, \sigma)\). Then, the objective and constraint functionals for Problem \((P(p))\) become

\[
J_i(\theta, \sigma) = \Phi_i(y(p|\theta, \sigma)) + \sum_{l=1}^{p} \int_{l-1}^{l} \theta_i \mathcal{L}_i(y(s|\theta, \sigma), \sigma') ds, \tag{16}
\]

where \(\Phi_i : \mathbb{R}^n \rightarrow \mathbb{R}\) and \(\mathcal{L}_i : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}\) are as defined in Section 3.1. Define

\[
\mathcal{F} = \{(\theta, \sigma) \in \Theta \times \Sigma : J_i(\theta, \sigma) = 0, i = 1, \ldots, N_e \text{ and } J_i(\theta, \sigma) \geq 0, i = N_e + 1, \ldots, N\}. \tag{17}
\]

**Remark 3.1.** By the definition of the time scaling function, it is clear that for any time instant \(t' \in [t_{l-1}, t_l]\), \(l = 1, \ldots, p\), in the original time horizon, the corresponding time instant in the new time horizon \(s'\) lies in the interval of \([l-1, l]\) and vice-versa.

The characteristic time instants in the new time horizon are closely related to the duration of control vector \(\theta\). It is given in the following theorem.

**Theorem 3.1.** For a given characteristic time \(\tau_j \in [t_0, t_f]\), \(j = 1, \ldots, m\) and \(\theta \in \Theta\), let \(s_j\) be the corresponding time instant in the new time horizon. Then, there exists a unique integer \(\kappa(s_j|\theta)\) such that

\[
\tau_j \in [t_0 + \sum_{i=1}^{\kappa(s_j|\theta)} \theta_i, t_0 + \sum_{i=1}^{\kappa(s_j|\theta) + 1} \theta_i), \tag{18}
\]

and

\[
s_j \in [\kappa(s_j|\theta), \kappa(s_j|\theta) + 1]. \tag{19}
\]

**Proof.** For simplicity, we use \(\mu(\cdot)\) and \(\kappa(\cdot)\) to denote \(\mu(|\theta|)\) and \(\kappa(|\theta|)\), respectively. Recall the definition of the time-scaling function, we have \(t_i = \mu(i)\). For any \(\tau'_j \in [t_0, t_f]\), \(t_0, t_1, t_2, \ldots, t_{p-1}, t_f\), and for any \(i, j \in \{1, 2, \ldots, p-1\}\), \(i \neq j\), we have \([t_{i-1}, t_i] \cap [t_{j-1}, t_j] = \emptyset\). Then, there exists a unique \(\kappa(s_j) \in \{1, 2, \ldots, p-1\}\) such that \(\tau'_j \in [t_{\kappa(s_j)}, t_{\kappa(s_j) + 1}]\). This indicates that for a given \(\theta \in \Theta\), there exists a unique \(\kappa(s_j) \in \{1, 2, \ldots, p-1\}\) such that \(\tau'_j \in [\mu(\kappa(s_j)), \mu(\kappa(s_j) + 1)]\). From Remark 3.1, it follows that \(s'_j \in [\kappa(s_j), \kappa(s_j) + 1]\). This completes the proof. \(\square\)

By Theorem 3.1, the characteristic time control constraints (7) become

\[
\sigma^{\kappa(s_j|\theta) + 1} = \zeta_j, j = 1, \ldots, m, \tag{20}
\]

in the new time horizon. Then, Problem \((P(p))\) defined in the new time horizon could be stated formally as follows.

**Problem \((\hat{P}(p))\).** Given system (15), find a feasible pair \((\sigma, \theta) \in \mathcal{F}\) such that the cost functional \(J_0\) is minimized over \(\mathcal{F}\) subject to the characteristic time control constraints (20).
3.3. Surrogate condition for the characteristic time control constraints. The relationship for \( t_j \) and \( s_j \) is depicted in Fig.1. Clearly, the characteristic time points in the new time horizon depend on the duration vector \( \mathbf{\theta} \). This leads to numerical difficulty when dealing with the control constraint (20) in practical computation. More specifically, since the index \( \kappa(s_j|\mathbf{\theta}) + 1 \) is a function of \( \mathbf{\theta} \), there is no prior information which component of \( \mathbf{\sigma} \) should be equal to \( \zeta_j \), and hence the control constraints (20) cannot be handled in practice. Therefore, we develop a surrogate condition for the characteristic time control constraint, the satisfaction of the surrogate condition ensures that the characteristic time control constraints are always satisfied.

**Remark 3.2.** Any given time point \( t_j \in [t_0, t_f] \) can be written as a convex combination of the initial time \( t_0 \) and the terminal time \( t_f \), i.e.

\[
\tau_j = t_0 + \alpha_j(t_f - t_0), \quad \alpha_j \in [0, 1],
\]

where \( \alpha_j \) is fixed.

The surrogate condition is given as a theorem below.

**Theorem 3.2.** For each fixed \( \alpha_j, j = 1, \ldots, m \), we suppose that the number of partitions of the control \( p \) is chosen such that \( |\alpha_j| \geq \max\{1 - \frac{1}{\alpha_j}, \frac{1}{\alpha_j}\} \), then the constraints (20) can be written as \( \mathbf{\sigma}^{\kappa(s_j|\mathbf{\theta}) + 1} = \zeta_j, s_j \in [[\alpha_j], [\alpha_j] + 1) \). If the following inequalities are satisfied by the duration vector \( \mathbf{\theta} \in \Theta \):

\[
\frac{\sum_{l=1}^{\alpha_j} \theta_l}{p} \leq \frac{\sum_{l=1}^{\alpha_j} \theta_l}{p}, \quad j = 1, \ldots, m.
\]

Then, the characteristic time control constraints (20) are satisfied.

**Proof.** Let \( \mathbf{\theta} \in \Theta \) be a duration vector satisfying (22). Suppose that \( \mathbf{\sigma}^{\kappa(s_j|\mathbf{\theta}) + 1} = \zeta_j, s_j \in [[\alpha_j], [\alpha_j] + 1), \) it suffices to show that \( \alpha_j(t_f - t_0) \leq \sum_{l=1}^{\alpha_j} \theta_l, \sum_{l=1}^{\alpha_j + 1} \theta_l \) for then the constraints (20), is satisfied in the original time horizon. From the
right-hand-side inequality of (22), we get
\[
\alpha_j(t_f - t_0) = \alpha_j \sum_{j=1}^{p} \tilde{\theta}_j = \alpha_j \sum_{l=1}^{[\alpha_j]} \tilde{\theta}_l + \alpha_j \sum_{l=[\alpha_j]+1}^{p} \tilde{\theta}_l \\
\leq \alpha_j \sum_{l=1}^{[\alpha_j]} \tilde{\theta}_l + \alpha_j \sum_{l=[\alpha_j]+1}^{p} \tilde{\theta}_l + \alpha_j \sum_{l=[\alpha_j]+2}^{p} \tilde{\theta}_l \\
= \sum_{l=1}^{[\alpha_j]} \tilde{\theta}_l + \alpha_j \sum_{l=[\alpha_j]+1}^{p} \tilde{\theta}_l \\
< \sum_{l=1}^{[\alpha_j]} \tilde{\theta}_l + \sum_{l=[\alpha_j]+1}^{p+1} \tilde{\theta}_l.
\]

On the other hand, from the left-hand-side inequality of (22), we get
\[
\alpha_j(t_f - t_0) = \alpha_j \sum_{l=1}^{p} \tilde{\theta}_l = \alpha_j \sum_{l=1}^{[\alpha_j]} \tilde{\theta}_l + \alpha_j \sum_{l=[\alpha_j]+1}^{p} \tilde{\theta}_l \\
\geq \alpha_j \sum_{l=1}^{[\alpha_j]} \tilde{\theta}_l + (1 - \alpha_j) \sum_{l=[\alpha_j]+1}^{p} \tilde{\theta}_l \\
= \sum_{l=1}^{[\alpha_j]} \tilde{\theta}_l.
\]

Thus, by the definition of the time-scaling function \( \mu \), we have \( t_0 + \alpha_j(t_f - t_0) \in \left[ t_0 + \sum_{l=1}^{[\alpha_j]} \tilde{\theta}_l, t_0 + \sum_{l=1}^{[\alpha_j]+1} \tilde{\theta}_l \right) \). Hence, the constraint (20) is satisfied. The proof is completed. \( \square \)

Theorem 3.2 implies that the unique integer \( \kappa(s_j|\theta) \) in the constraints (20) can be determined by \( \alpha_j \) and the number of partitions of the control \( p \). Then, the new problem can be stated as follows.

**Problem (Q).** Given system (15), choose a feasible pair \((\theta, \sigma) \in F\), such that the cost functional \( J_0(\theta, \sigma) \) is minimized over \( F \) subject to
\[
\sum_{l=1}^{p} \tilde{\theta}_l = t_f - t_0
\]
and satisfying the inequality condition (22).

Problem (Q) is a standard optimal control problem and hence it can be solved by optimal control methods and hence by optimal control solvers. In fact, the solving procedure of Problem (P) can also be used to the case that the free-terminal time optimal control problems with the control constraints at the characteristic time points. We denote the terminal time as \( t_T \). In this case, for the characteristic time instants, it is required that \( \tau_j < t_T \) for all \( j = 1, \ldots, m \). However, since \( t_T \) is undetermined, if there exists \( j \) such that \( t_T < \tau_j \), then the corresponding control constraint at \( \tau_j \) is not well defined. To ensure the characteristic time control constraints are well-defined, the characteristic time points \( \tau_j \) have to somehow be related to the undetermined terminal time \( t_T \). Hence, we have to redefine the
characteristic time points as the relative intermediate time points by the convex combination of the initial time instant and the terminal time instant. Its control constraints become the relative intermediate time control constraints. Furthermore, the sufficient condition for this type of control constraints is still available to handle the relative intermediate time control constraints. Then, this problem is solvable. In the next section, we will solve two practical examples by using the optimal control solver Miser [9, 35] to demonstrate that the solving method for the two cases of problems are effective.

4. Numerical Examples.

4.1. Problem 1: optimal control problem with fixed terminal time. Consider the following dynamic system:

\[ \dot{x}(t) = u(t), \]
\[ x(0) = 1, \]

with terminal state constraint

\[ x(20) = 0.75 \]

and control vector satisfies

\[ u(0) = -1, \ u(10) = 0, \ u(20) = 1.5 \]

The objective functional

\[ G_0 = \int_0^{20} \{x^2(t) + u^2(t)\} dt, \]

is to be minimized.

Problem 1 is solved by the traditional control parameterization method with the partition points equally distributed over the time horizon, as well as the newly developed time-scaling transformation method. The numerical results obtained are listed in Table 1. From Table 1, we observe that the optimal cost decreases with the increase in the partition number. Furthermore, it is important to observe that with the same partition number, the optimal cost of the problem obtained by using the newly developed time-scaling transformation method is significantly lower than that obtained by using the traditional control parameterization method.

The optimal controls obtained by using these two different methods are shown in Fig.2. The two respective optimal state trajectories for the case of \( p = 9 \) are shown in Fig.3. In this case, the unique subinterval in the new time horizon that contains the characteristic time point is \([4,5]\).

| Knots | \( G_0^* \) |
|-------|-------------|
| Traditional control parameterization | New time-scaling transformation |
| \( p = 5 \) | 110.8 | 1.9531 |
| \( p = 7 \) | 42.34 | 1.6671 |
| \( p = 9 \) | 21.79 | 1.6053 |

Table 1. Optimal costs for Problem 1 obtained using the two different methods.
4.2. Problem 2: optimal control problem with variable free-terminal time. We consider a practical problem in the transportation and expansion of atoms for Bose-Einstein condensates (BEC) by shortcut to adiabaticity (STA) schemes [24, 31]. The states of atoms are described by the Gross-Pitaevskii equation (GPE) [23].
The cigar-shaped harmonic trapping potential (one-dimensional GPE) [27] in the x-direction is:

\[ i\psi_t + \frac{1}{2} \psi_{xx} + g(t)\psi|\psi|^2 - \frac{1}{2} \omega(t)^2 (x - x_0)\psi = 0, \]

(23)

where \( \psi(x, t) \) represents wavefunction of the condensates, \( g(t) \) represents the atom-atom interaction, \( \omega(t) \) is the time-dependent trapping frequency, \( i \) denotes the imaginary number. Note that \( \psi(x, t) \) is the solution of (23).

The scaling solution of (23) can be approximated by a Gaussian ansatz of the form [11]:

\[ \psi(x, t) = A(t) \exp\left\{-\frac{|x - q(t)|^2}{2a(t)^2} + ib(t)[x - q(t)]^2 + ic(t)[x - q(t)]\right\}, \]

(24)

where \( A(t), \ a(t), \ b(t), \ c(t), \ q(t) \) represent the amplitude, width, chirp, velocity, position of the wavefunction, respectively. The two parameters \( a(t) \) and \( q(t) \) satisfy the following equations [8, 15, 29]:

\[ \frac{d^2}{dt^2} a(t) + a\omega^2(t) = \frac{1}{a^3} + \frac{gN}{\sqrt{2\pi}a^2}, \]

(25)

\[ \frac{d^2}{dt^2} q(t) + \omega^2(t)[q - q_0(t)] = 0. \]

(26)

where \( q_0(t) \) denotes the center position of the trap.

For the purpose of achieving atoms expansion, we need to control the frequency \( \omega(t) \) to achieve adiabatic-like evolution [28] from initial frequency \( \omega(0) = \omega_0 \) to final frequency \( \omega(t_T) = \omega_T \), such that the following boundary conditions [4, 5] are satisfied:

\[ a(0) = \omega_0, \quad a(t_T) = \sqrt{\omega_0/\omega_T}, \]

(27)

\[ \frac{d}{dt} a(t)|_{t=0} = \frac{d^2}{dt^2} a(t)|_{t=0} = 0, \]

(28)

\[ \frac{d}{dt} a(t)|_{t=t_T} = \frac{d^2}{dt^2} a(t)|_{t=t_T} = 0. \]

(29)

For the transport of atoms, we control the position of the trap, and the boundary conditions below are imposed [6, 30]:

\[ q(0) = q_0(0) = 0, \quad q(t_T) = q_0(t_T) = d(d > 0), \]

(30)

\[ \frac{d}{dt} q(t)|_{t=0} = \frac{d^2}{dt^2} q(t)|_{t=0} = 0, \]

(31)

\[ \frac{d}{dt} q(t)|_{t=t_T} = \frac{d^2}{dt^2} q(t)|_{t=t_T} = 0, \]

(32)

Now, we consider the trap expansion for the atoms expansion to change from \( \omega_0 = 1 \) to \( \omega_T = 0.01 \) and its position to change from \( q(0) = 0 \) to \( q(t_T) = 5 \). Applying the optimal control theory, it can be formulated as the following optimal control problem. To control this process precisely, the relative intermediate control constraint is imposed. We set \( z_1(t) = a(t), \ z_2(t) = \frac{d}{dt} a(t), \ z_3(t) = q(t), \ z_4(t) = \frac{d}{dt} q(t) \), the control functions \( \nu_1(t) = \omega^2(t), \ \nu_2(t) = q(t) - q_0(t) \), and let \( gN = 0.01, \ q_0(t_T) = d = 5 \). Then, with these variables, (25)-(26) become

\[ \dot{z}_1(t) = z_2(t), \]

\[ \dot{z}_2(t) = -\nu_1(t)z_1(t) + \frac{1}{z_1^4(t)} + \frac{1}{100\sqrt{2\pi}z_2^2(t)}. \]
\[ \dot{z}_3(t) = z_4(t), \]
\[ \dot{z}_4(t) = -\nu_1(t)\nu_2(t), \]

with the boundary conditions at \( t = 0 \) and \( t = t_T \) given by
\[ z_1(0) = 1, \quad z_1(t_T) = 10, \]
\[ z_2(0) = 0, \quad z_2(t_T) = 0, \]
\[ z_3(0) = 0, \quad z_3(t_T) = 5, \]
\[ z_4(0) = 0, \quad z_4(t_T) = 0. \]

The constraints on the control functions \( \nu_1(t) \) and \( \nu_2(t) \) are:
\[ |\nu_1(t)| \leq 1, \quad |\nu_2(t)| \leq 1; \]
and
\[ \nu_1(0) = 1, \quad \nu_1(t_T) = 0.0001, \]
\[ \nu_2(0) = 0, \quad \nu_2(t_T) = 0, \]

and the relative intermediate control constraint
\[ \nu_1\left(\frac{t_T}{2}\right) = 0.1. \]

The goal is to minimize the cost functional
\[ J_0 = \int_0^{t_T} 1dt = t_T, \]
for the given dynamic system and subject to the specific control constraints.

Note that the traditional control parameterization method fails to solve this problem. Table 2 shows the numerical results for different lower bounds with the partition numbers \( p = 5, 6, 7 \). From Table 2, we see that the optimal cost decreases with the increase in the partition number for the same lower bounds on \( \theta_l \). For the same \( p \), we observe that the optimal cost increases with the increase of the lower bounds on each \( \theta_l \). Hence, we choose smaller lower bounds on each \( \theta_l \) and finer partition for \( p \) to obtain the optimal solution. However, in practical experiments, the lower bound on \( \theta_l \) can’t be set to be infinitely small, because the control \( \nu(\frac{t_T}{4}) \) must last for a while at \( \frac{t_T}{4} \). The obtained results can be divided into two cases: (i) \( p \geq 7 \), the duration of \( \nu(\frac{t_T}{4}) \) is the lower bounds of the given \( \theta_l \); (ii) \( 3 \leq p < 7 \), the control \( \nu(\frac{t_T}{4}) \) will last for certain time which is longer than the lower bounds on \( \theta_l \). The obtained optimal controls are shown in Figs. 4, 5 and 6, and the corresponding state trajectories are displayed in Fig.7 for the case \( \theta_l \geq 0.05 \).

| Lower bounds on \( \theta_l \) \( (l = 1, \ldots, p) \) | \( J_0^* \) | \( p = 5 \) | \( p = 6 \) | \( p = 7 \) |
|---|---|---|---|---|
| \( \geq 0.05 \) | 12.189 | 8.113 | 4.572 |
| \( \geq 0.1 \) | 12.347 | 8.258 | 4.673 |
| \( \geq 0.5 \) | 13.147 | 9.442 | 5.506 |
| \( \geq 1 \) | 14.453 | 10.924 | 7.000 |

| Table 2. Optimal costs for Problem 2 using different lower bounds on each \( \theta_l \) with different partition \( p \). |
5. Conclusion. In this paper, we consider a class of optimal control problems with characteristic time control constraints. We proposed a unified computational solution procedure to solve this type of problem. Since the characteristic time control constraints are difficult to be satisfied in practical computation, a surrogate condition for the characteristic time control constraints have been developed to ensure that the control constraints at these particular time points are satisfied. In this way, we obtain a standard optimal control problem which can be solved by existing optimal control solver, such as MISER. Two numerical examples are given to illustrate the effectiveness of the proposed approach.
Figure 7. Optimal state trajectories for Problem 2 with different partition numbers.

REFERENCES

[1] N. U. Ahmed, Elements of Finite Dimensional Systems and Control Theory, Longman Scientific and Technical, 1988.
[2] M. Athans, Advances in Control Systems: Theory and Applications, vol. 11, Elsevier, 1966.
[3] S. Boyd and L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.
[4] X. Chen, A. Ruschhaupt, S. Schmidt, A. del Campo, D. Guery-Odelin and J. G. Muga, Fast optimal frictionless atom cooling in harmonic traps: Shortcut to adiabaticity, Physical Review Letters, 104 (2010), 063002.
[5] X. Chen, E. Torrontegui and J. G. Muga, Lewis-riesenfeld invariants and transitionless quantum driving, Physical Review A, 83 (2011), 062116.
[6] X. Chen, E. Torrontegui, D. Stefanatos, J.-S. Li and J. G. Muga, Optimal trajectories for efficient atomic transport without final excitation, Physical Review A, 84 (2011), 043415.
[7] Z. Gong, C. Liu and Y. Wang, Optimal control of switched systems with multiple time-delays and a cost on changing control, Journal of Industrial and Management Optimization, 14 (2018), 183–198.
[8] T.-Y. Huang, B. A. Malomed and X. Chen, Shortcuts to adiabaticity for an interacting bose–einstein condensate via exact solutions of the generalized ermakov equation, Chaos: An Interdisciplinary Journal of Nonlinear Science, 30 (2020), 053131, 10 pp.
[9] L. Jennings, K. L. Teo, M. Fisher, C. J. Goh, L. S. Jennings and M. E. Fisher, MISER3 version 2, Optimal Control Software: Theory and User Manual, vol. 1, The University of Western Australia, Australia, 1997.
[10] C. Jiang, K. Xie, C. Yu, M. Yu, H. Wang, Y. He and K. L. Teo, A sequential computational approach to optimal control problems for differential-algebraic systems based on efficient implicit runge-kutta integration, Applied Mathematical Modelling, 58 (2018), 313–330.
[11] Y. Kagan, E. L. Surkov and G. V. Shlyapnikov, Evolution of a bose-condensed gas under variations of the confining potential, Physical Review A, 54 (1996), R1753–R1756.
[12] D. E. Kirk, Optimal Control Theory: An Introduction, Courier Corporation, 2004.
[13] L. Kong, C. Yu, K. L. Teo and C. Yang, Robust real-time optimization for blending operation of alumina production, Journal of Industrial and Management Optimization, 13 (2017), 1149–1167.
[14] H. Lee, K. Teo, V. Rehbock and L. Jennings, Control parametrization enhancing technique for time optimal control problems, Dynamic Systems & Applications, 6 (1997), 243–262.
[15] J. Li, K. Sun and X. Chen, Shortcut to adiabatic control of soliton matter waves by tunable interaction, Scientific Reports, 6 (2016), 38258.
[16] L. Li, C. Yu, N. Zhang, Y. Bai and Z. Gao, A time-scaling technique for time-delay switched systems, Discrete and Continuous Dynamical Systems-S, 13 (2020), 1825–1843.
[17] Q. Lin, R. Loxton and K. L. Teo, The control parameterization method for nonlinear optimal control: A survey, Journal of Industrial and Management Optimization, 10 (2014), 275–309.
[18] Q. Lin, R. Loxton, K. L. Teo and Y. H. Wu, A new computational method for optimizing nonlinear impulsive systems, Dynamics of Continuous, Discrete and Impulsive Systems B, 18 (2011), 59–76.
[19] Q. Lin, R. Loxton, K. L. Teo and Y. H. Wu, Optimal control computation for nonlinear systems with state-dependent stopping criteria, Automatica, 48 (2012), 2116–2129.
[20] C. Liu, Z. Gong, E. Feng and H. Yin, Modelling and optimal control for nonlinear multistage dynamical system of microbial fed-batch culture, Journal of Industrial and Management Optimization, 5 (2009), 835–850.
[21] R. C. Loxton, K. L. Teo and V. Rehbock, Optimal control problems with multiple characteristic time points in the objective and constraints, Automatica, 44 (2008), 2923–2929.
[22] P. Mu, L. Wang and C. Liu, A control parameterization method to solve the fractional-order optimal control problem, Journal of Optimization Theory and Applications, 187 (2020), 234–247.
[23] J. G. Muga, X. Chen, A. Ruschhaupt and D. Guéryodelin, Frictionless dynamics of bose–einstein condensates under fast trap variations, Journal of Physics B: Atomic Molecular and Optical Physics, 42 (2009), 241001.
[24] J. G. Muga, X. Chen, E. Torrontegui, S. Ibáñez, I. Lizuain and A. Ruschhaupt, Shortcuts to quantum adiabatic processes, Journal of Physics Conference, 306 (2011), 012022.
[25] M. Nakazawa, K. Kurokawa, H. Kubota and E. Yamada, Observation of the trapping of an optical soliton by adiabatic gain narrowing and its escape, Physical Review Letters, 65 (1990), 1881–1884.
[26] J. Nocedal, S. Wright, T. Mikosch, S. Resnick and S. Robinson, Numerical Optimization, Springer series in operations research and financial engineering, 1999.
[27] L. Salasnich, A. Parola and L. Reatto, Effective wave equations for the dynamics of cigar-shaped and disk-shaped bose condensates, Physical Review A, 65 (2002), 043614.
[28] D. Stefanatos, Optimal shortcuts to adiabaticity for a quantum piston, Automatica, 49 (2013), 3079–3083.
[29] A. Tobalina, M. Palmero, S. Martínez-Garaot and J. G. Muga, Fast atom transport and launching in a nonrigid trap, Scientific Reports, 7 (2017), 5753.
[30] E. Torrontegui, S. Ibáñez, X. Chen, A. Ruschhaupt, D. Guéry-Odelin and J. G. Muga, Fast atomic transport without vibrational heating, Physical Review A, 83 (2011), 013415.
[31] E. Torrontegui, S. Ibáñez, S. Martínez-Garaot, M. Modugno, A. del Campo, D. Guéry-Odelin, A. Ruschhaupt, X. Chen and J. G. Muga, Shortcuts to adiabaticity, Advances in Atomic, Molecular, and Optical Physics, 62 (2013), 117–169.
[32] L. Wang, J. Yuan, C. Wu and X. Wang, Practical algorithm for stochastic optimal control problem about microbial fermentation in batch culture, Optimization Letters, 13 (2019), 527–541.
[33] Y. Wang, C. Yu and K. Teo, A new computational strategy for optimal control problem with a cost on changing control, Numerical Algebra, Control and Optimization, 6 (2016), 339–364.
[34] D. Wu, Y. Bai and C. Yu, A new computational approach for optimal control problems with multiple time-delay, Automatica, 101 (2019), 388–395.
[35] F. Yang, K. L. Teo, R. Loxton, V. Rehbock, B. Li, C. Yu and L. Jennings, Visual miser: An efficient user-friendly visual program for solving optimal control problems, Journal of Industrial and Management Optimization, 12 (2016), 781–810.
[36] C. Yu, Q. Lin, R. Loxton, K. L. Teo and G. Wang, A hybrid time-scaling transformation for time-delay optimal control problems, Journal of Optimization Theory and Applications, 169 (2016), 876–901.
[37] C. Yu, Y. Wang and L. Li, Smoothing spline via optimal control under uncertainty, *Applied Mathematical Modelling*, **58** (2018), 203–216.

[38] Y. Zhang, C. Yu, Y. Xu and Y. Bai, Minimizing almost smooth control variation in nonlinear optimal control problems, *Journal of Industrial and Management Optimization*, **16** (2020), 1663–1683.

Received June 2020; 1st revision October 2020, final revision December 2020.

E-mail address: yuchangjun@126.com
E-mail address: sushuxuan@outlook.com
E-mail address: yqbai@t.shu.edu.cn