Curious terminal turn of rolling rings

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We report an unexpected reverse spiral turn in the final stage of the motion of rolling rings. It is well known that spinning disks have definite centers of rotation until they stop. While a spinning ring starts its motion with a similar kinematics, moving along a cycloidal path prograde with the direction of its rigid body rotation, the mean trajectory of its center of mass develops an inflection point so that the ring makes a spiral turn and revolves in a retrograde direction around a new center. Using high speed imaging and numerical simulations of models featuring a rolling rigid body, we show that the hollow geometry of a ring tunes the rotational air drag resistance so that the frictional force at the contact point with the ground changes its direction at the inflection point and puts the ring on a retrograde spiral trajectory. Our findings have potential applications in designing topologically new surface-effect flying objects capable of performing complex reorientation and translational maneuvers.

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It is a common experience to spin a coin or a thin disk on a table and observe its rolling motion. As the coin keeps rolling, its inclination angle with respect to the table decreases while it generates a sound of higher and higher frequency before stopping. According to the equations of motion of a rolling rigid body with non-holonomic constraints, the spin rate must diverge to infinity when the disk rests on the table. In real world experiments, however, the spin of the disk vanishes within a finite duration of time. Both theoretical and experimental studies suggest that the finite life-time of this process is due to a combination of air drag and slippage that drain the disk’s kinetic energy, but an accurate model of dissipative mechanisms is still unknown.

Increasing the thickness of the disk changes the dynamics because of the existence of an unstable, inverted-pendulum-like, static equilibrium. Nevertheless, the center of mass of the disk with the global position vector \( \mathbf{r}_C \) always moves on a spiral trajectory for low inclination angles, while the orbital angular momentum vector \( \mathbf{L} = \mathbf{r}_G \times \mathbf{r}_C \) per unit mass is almost aligned with the angular velocity \( \omega \) of the disk and we have \( \mathbf{L} \cdot \omega > 0 \). Here \( \mathbf{r}_G \) is the position vector of the center of mass with respect to the contact point of the body with the surface. We call this spiraling motion a prograde turn. One expects a similar behavior for a ring, but experiments reveal a new type of motion, which we investigate in this report.

We describe the rotation of a ring of the outer radius \( R \), width \( h \), thickness \( w \), and mass \( m \) by a set of 3-1-2 Euler angles \( (\phi, \theta, \psi) \) as shown in Figure 1(a). The unit vectors \( (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \) are along the principal axes of the ring, \( \mathbf{e}_1 \) is always parallel to the surface of the table, and \( \mathbf{e}_2 \) is along the symmetry axis of the ring. The angular velocity of the ring thus becomes \( \omega = \dot{\theta} \mathbf{e}_1 + (\dot{\phi} \sin \theta + \dot{\psi}) \mathbf{e}_2 + \dot{\phi} \cos \theta \mathbf{e}_3 \). We denote the inertia tensor of the ring by \( \mathbf{I} \) and its angular momentum with respect to the center of mass by \( \mathbf{L}_G = \mathbf{I} \cdot \omega \). The equations of the coupled roto-translatory motion thus read

\[
\mathbf{I} \cdot \dot{\omega} + \mathbf{L} \times \mathbf{L}_G = -m \mathbf{r}_G \times \mathbf{F} \quad \mathbf{r}_G = (h/2) \mathbf{e}_2 + R \mathbf{e}_3, 
\]

\[
m \dot{\mathbf{r}}_C = \mathbf{F} - mg (\sin \theta \mathbf{e}_2 + \cos \theta \mathbf{e}_3),
\]

where \( g \) is the constant of gravity, \( \mathbf{r}_C \) is the global position vector of the center of mass, \( \mathbf{F} \) is the boundary force at the contact point of the ring and the table, and \( \Omega = \omega - \psi \mathbf{e}_2 \). Throughout our study we assume that the ring is in pure rolling condition and the constraint \( \mathbf{r}_C = \dot{\mathbf{r}}_C = \mathbf{r}_G \times \mathbf{r}_C \) holds. Equations (4) and (5) can therefore be combined to obtain the evolutionary equa-

![Figure 1](image.png)

**FIG. 1.** The geometry of a rolling ring and its trajectory in the absence of dissipative effects. (a) The geometry of a ring spun on a horizontal table. The dashed line is perpendicular to the surface of the table. The origin of the coordinate frame defined by \( (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \) coincides with the center of mass of the ring, point \( C \). The point \( A \) is the center of the upper circular edge. (b) The cycloid-type quasi-periodic trajectory of \( r_4(t) \) projected on the surface of the table. We have set the initial conditions to \( \theta_0 = 0.55 \) rad, \( \phi_0 = 4.5 \) and \( \psi_0 = 0 \). All other conditions have been set to zero.
tions of angular velocities:

\[ \mathbf{I} \cdot \ddot{\mathbf{\gamma}} - m \mathbf{r}_G \times (\mathbf{r}_G \times \ddot{\mathbf{\gamma}}) = -\mathbf{\Omega} \times \mathbf{L}_G \\
- m \mathbf{r}_G \times (\mathbf{\Omega} \times \mathbf{v}_G) + mg[R \sin \theta - (h/2) \cos \theta] \mathbf{e}_1 \] (3)

It is almost impossible to track the motion of the center of mass experimentally. We therefore use the center of the top circular edge of the ring (point A in Fig. 1(a)), with the position vector \( \mathbf{r}_A = \mathbf{r}_C + (h/2) \mathbf{e}_2 \), for measuring the position and velocity of the ring. The velocity of point A is related to the speed of the center of mass through \( \mathbf{v}_A = \mathbf{r}_A = \mathbf{r}_C + \mathbf{\Omega} \times (h/2) \mathbf{e}_2 \). We normalize all lengths and position vectors to the mean radius \( R - w/2 \). Accelerations have been normalized so that the initial value of \( R\dot{\varphi}^2 \) at \( t = 0 \) equals the experimental value \( \approx 20.2g \).

Integration of equation (3) for initial conditions \( \dot{\theta}_0 = 0 \) and \( \theta_0 > \arctan(h/(2R)) \) show that the center of mass of the ring moves on a generally quasi-periodic cycloidal orbit. A typical quasi-periodic orbit is shown in Fig. 1(b) for \( R = 1.025 \), \( h = 0.88 \) and \( w = 0.05 \), which correspond to the ring in our experiments discussed below. The size of the inner turning loop of cycloids is a function of \( \psi_0/\phi_0 \) and \( \theta_0 \). Such orbits, however, are not observed in real world experiments. Spinning a wedding ring on a glass or wooden table shows that the motion is composed of two prominent phases. In the first phase, the ring spins and travels similar to the prograde turn of a coin/disk, but in contrast with a disk that continues prograde spiraling until its resting position, it abruptly makes a retrograde spiral turn before stopping (see supplementary video 1 on http://youtu.be/TbROdgrwaYI).

The retrograde turn does not belong to the phase space structure of equation (3), nor is it observed in spinning disks.

To understand the ring dynamics, we prepared a high-speed imaging set-up and spun a ring of \( R = 20.66 \) mm, \( w = 1 \) mm and \( h = 18 \) mm on a polished and waxed wooden table. We rotated and released the ring by hand, but assured that the initial conditions satisfy \( \theta_0 \approx 0 \) and \( \theta_0 > \arctan(h/(2R)) \). To trace the translational and rotational motions, we put four marks in a cross configuration at the top circular edge of the ring, and stored their coordinates (in pixels) while filming the motion (Fig. 2(a)). The centroid of these marks has the position vector \( \mathbf{r}_A \). Figs 2a,b and supplementary video 2 show the projection of the trajectory of \( \mathbf{r}_A(t) \) on the surface of the table for one of our experiments. The Euler angles \( \theta \) and \( \psi \) can be computed from the formulae

\[ 1 + \sin^2(\theta) = (L_{13}^2 + L_{24}^2) / D^2 \]

and \( 1 + \sin^2(\psi) [\sin^2(\theta) - 1] = L_{13}^2 / D \), where \( L_{13} \) and \( L_{24} \) are the apparent distances between the points 1 and 3, and 2 and 4, respectively, and \( D = 2R - w \) is the mean diameter of the ring. Our experimental error level in computing \( \mathbf{r}_A(t) \) has been \( \approx 5\% \) because the ring is not always along the line of sight of the camera, and the image is distorted as the ring rolls. The mean error threshold is roughly the measured value of \( 1 - (L_{13} + L_{24})/(2D) \) after the stopping of the ring.

The trajectories of point A displayed in Figs 2(a,b) and supplementary video 2 unveil unique features of the ring’s motion. An initial prograde turning phase occurs along cycloidal curves similar to what we observe in Fig. 1(b). As time elapses, the inner turning loops of cycloids shrink and evolve to cuspy turning points that connect half-circle-shape arcs. The radii of half-circle steps decrease and the motion becomes directional along the arrow until a retrograde spiral turn begins at an inflection point.

To better distinguish the differences between the trajectories of rings and disks, we repeated our experiment for an aluminum disk of diameter \( D = 63.5 \) mm and width \( h = 6.14 \) mm, and recorded its trajectory. Figs 2(c,d) and the supplementary video 3 show the inspiraling motion of the disk’s center of mass. This is a generic behavior of all rolling disks, regardless of their thickness.

Using the coordinates of the four markers on the ring, we have computed the magnitude of the velocity \( \mathbf{v}_p = \mathbf{v}_A - (\mathbf{v}_A \cdot \mathbf{e}_1) \mathbf{e}_1 \), which is parallel to the surface of the table, and plotted it in Fig. 3 versus the frame.
number $n$. Here $e_\perp = \sin(\theta)e_2 + \cos(\theta)e_3$ is the unit vector normal to the surface. At the highest ($\theta = \theta_{\text{min}}$) and lowest ($\theta = \theta_{\text{max}}$) vertical positions of the center of mass, $v_\parallel$ becomes identical to $v_A$. It is seen that the envelope of the velocity profile declines according to a double power-law with a shallow decline up to and after the retrograde turn, followed by a steep fall and termination of the motion.

We have observed the retrograde turn in almost all rings. Although several mechanisms like rolling friction, slippage [4], air drag [7], and even elastic vibrations [14] can be held responsible for the phenomenon, our numerical simulations show that including only the air drag fully captures the physics of the retrograde turn. In the presence of external drag torques, equation (4) takes the form

$$\mathbf{J} \cdot \mathbf{\dot{\omega}} = \mathbf{f}(\omega, \theta) + \mathbf{T}_{\text{drag}},$$

where $\mathbf{J}$ is a constant matrix, $\mathbf{f}$ is a vector function of the angular velocity $\omega$ and the Euler angle $\theta$, and $\mathbf{T}_{\text{drag}}$ is the resultant drag-induced torque. Let us define the Reynolds number as $Re = 2R[C_2]R/v_\alpha$, where $v_\alpha$ is the kinematic viscosity of the air, and assume $\mathbf{T}_{\text{drag}} = \mathbf{D} \cdot \mathbf{\omega}$ so that the drag matrix $\mathbf{D}$ is generally a function of $Re$, Euler’s angles and their rates. According to the velocity data of Fig. [3] the Reynolds number satisfies $Re \lesssim 800$. Consequently, the airflow is most likely in laminar regime with no or weak separation of the flow at the boundaries. It is therefore reasonable to assume that the influence of the drag torque on the components of $\omega$ is separable and the matrix $\mathbf{J}^{-1} \cdot \mathbf{D}$ is diagonal so that

$$\mathbf{\tau} = \mathbf{J}^{-1} \cdot \mathbf{D} \cdot \mathbf{\omega} = -\sum_{i=1}^{3} C_i |\omega_i| \omega_i e_i, \quad \omega_i = \omega \cdot e_i.$$

Here the constants $C_i$ ($i = 1, 2, 3$) are rotational drag coefficients, and implicitly depend on $Re$ and the reference area of the ring exposed to airflow. If the ring was far from any wall/surface, the rotational symmetry about the $e_2$-axis would imply $C_1 = C_3$, but for rolling rings this identity does not necessarily hold.

Equations (4) and (5) yield $\mathbf{\dot{\omega}} = \mathbf{J}^{-1} \cdot \mathbf{f}(\omega, \theta) + \mathbf{\tau}$. We numerically integrate this equation using the initial conditions that we measure at the first inner turning point of Fig. [2(b)]. In that specific position, the angular velocity $\dot{\theta}$ vanishes, and we find $\theta_{\text{min}} \approx 0.55$ rad, $\phi \approx -0.38$ rad, $\psi \approx 0$, and $\phi = v_A/|R \sin(\theta_{\text{min}}) - (h/2) \cos(\theta_{\text{min}})| \approx 4.5$. The computed initial angular velocities are dimensionless. Without loss of generality, we assume $\psi(0) = 0$. The initial velocity of the center of mass is calculated using the rolling condition. To the best of our knowledge, the drag coefficients of a ring have not been measured or tabulated so far. Therefore, we constrain the parameter space $(C_1, C_2, C_3)$ by generating all orbits that resemble the experimental trajectory displayed in Fig. [2(b)]. We find the best match between theoretical and experimental trajectories by setting $C_1 \approx 0.03, C_2 \approx 0.063$, and $C_3 \approx 0.085$. The projection of the simulated trajectory of $r_A(t)$ on the surface has been demonstrated in Fig. [4(a)] together with the experimental trajectory. According to our numerical computations, the topology of the trajectory was not sensitive to the variations of $C_1$ over the range $0.01 \lesssim C_1 \lesssim 0.1$ and we observed only minor differences in the location and size of the terminal spiral feature.

The actual and simulated trajectories are similar in many aspects, including 9 and 15 cycles that they make, respectively, before the directional walk and retrograde turn phases. Their major differences are the long-lived feature. We suspect that the observed drift has been due to (i) uncertainties in calculating the initial angular velocities through the de-projection of the images and (ii) slippage at some cuspy turning points that has slightly changed the direction of $v_A$. For the existing discrepancy in the final spiral path we have the following explanation: as the motion of the ring slows down, the Reynolds number decreases and the drag coefficients increase. Consequently, the life-time of the spiral turn is shorter in reality. We would expect a better match with the experiment if the accurate profiles of the drag coefficients were known in terms of $Re$. The terminal stage of the ring’s spin is as complex as a disk, and the data of Fig. [3] show a finite time singularity.

A fundamental question is why disks do not make a retrograde turn like rings? This returns to differences in their aerodynamic properties near the ground: air can always flow through the central hole of the ring, but it is trapped and compressed between the disk and the ground, forming a thin supporting layer like a hydrodynamic bearing. Therefore, for rolling disks we expect $C_1/C_2 \gg 1$ and $C_3/C_2 \gg 1$. By taking the same initial conditions for the ring in our experiments, we used $C_2 = 0.063$ and $C_1 = C_3 = 0.2$, and found that the corresponding simulated trajectory of $r_A(t)$ (Fig. [4(b)]) is a single prograde spiral analogous to the experimentally measured trajectory of Fig. [2(d)]. This shows the role of
enhanced drag torque about the diameter in maintaining the prograde turn.

We have found that the evolution of the lateral component \( F_l = F \cdot \cos(\theta) \mathbf{e}_2 - \sin(\theta) \mathbf{e}_3 \) of the frictional force at the contact point is the dynamical origin of the retrograde turn. The ring maintains its motion on a trajectory as in Figs 1(b) and 2(d) if the lateral force satisfies \( F_l > 0 \) and supports the centrifugal acceleration needed for the prograde turn, especially when the center of mass passes through its lowest vertical position (with \( \theta = \theta_{\text{max}} \) and \( \dot{\theta} = 0 \)) at each cycle. At this point, the kinetic energy of the center of mass is maximum and its potential energy takes a minimum. We remark that the component \( F_l = \mathbf{F} \cdot \mathbf{e}_1 \) of \( \mathbf{F} \) is also caused by friction, but it helps the rolling and cannot balance the centrifugal acceleration at turning points. Our computations (Fig. 4(c)) show that because of drag torques, a local minimum that develops on the profile of \( F_l \) at \( \theta = \theta_{\text{max}} \) gradually becomes spiky and flips sign from positive to negative. As the ring experiences the strong negative kicks of \( F_l \), the centrifugal acceleration switches sign as well, and the ring starts to revolve around a new point by retrograde turning. This process does not happen for disks, for \( \omega_1 \) and \( \omega_3 \) decay quickly due to a large \( C_1 \) and \( C_3 \), and the orbital angular momentum \( \mathbf{L} = \mathbf{r}_G \times \mathbf{r}_C \) is dominated by the \( \mathbf{e}_2 \)-component. Consequently, \( F_l \) that supports the centrifugal acceleration remains positive as \( \theta \to \pi/2 \) (Fig. 4(c)). The coefficient of static friction for the surface on which we had spun our ring was \( \mu \approx 0.4 \). We computed the normal component of the contact force \( F_N = \mathbf{F} \cdot \mathbf{e}_\perp \) over the entire motion of the ring and found that the inequality \( \mu F_N > \sqrt{F_1^2 + F_3^2}/2 \) holds at all turning points with \( \theta = \theta_{\text{max}} \) and \( \dot{\theta} = 0 \). Therefore, slippage is not expected to play any major role in the qualitative features of the motion.

In summary, the aerodynamic interactions of spinning bodies can lead to complex, and sometimes unpredictable, results depending on the shape of the object and the initial conditions of its motion. Three well-known examples of spinning objects that significantly change their course of motion are the returning boomerang, soccer balls, and frisbees that fly along curved paths. Neither a boomerang nor a frisbee can move on curved trajectories without aerodynamic effects. Our finding for spinning rings is a new case where the frictional force and aerodynamic forces near the surface collaborate to change the course of motion.

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