Influence of a diffusion mass flux on instability development of surface shape of high-viscosity liquid

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Abstract. This work is devoted to theoretical investigation of instability development of surface shape of high-viscosity liquid under the influence of diffusion mass flux. If there is a local eminence on the surface it can lead to growth of perturbation of surface shape. Stability and instability of perturbation depends on factors which will prevail. Diffusion causes distortion of surface shape and dominance of capillary forces achieves the opposite effect.

1. Instability of flat shape surface
This work is divided into two parts. First, we consider a surface having the shape of a flat in unperturbed state. The mass stream can be calculated in quasi–steady–state approximation using quasi–steady–state equation of diffusion. Action of capillary forces and gravitation is taken from [1]. Fourier harmonica with wave number k of perturbation of a free surface of liquid change over time under the law:

\[ h_k = h_k(0) e^{a(k)t}, \quad a(k) = Vk\text{c}t\text{h}(kH) - \frac{1}{2\mu} \left( \sigma k + \frac{\rho g}{k} \right), \]

where \( V \) - upward velocity of the average level of liquid; \( H \) - thickness of diffusion boundary layer; \( \mu, \rho \) and \( \sigma \) - viscosity, density and coefficient of surface tension of liquid, \( g \) - acceleration of gravity.

With the wave numbers, where \( a(k) \) is less than zero, the perturbation decays, and this corresponds to conditions of stability. Wave numbers, for which \( a(k) \) is greater than zero, correspond to the regime of instability.

For the analysis of parameters of system we entered dimensionless quantities:

\[ \chi = \frac{k}{k^*}, \quad k^* = \frac{1}{\sqrt{\frac{\rho g}{\sigma}}}, \quad \gamma = k^*H, \quad \omega = \frac{\sigma}{2\mu V}. \]

In figure 1 areas in space of parameters \( (\gamma, \omega) \) corresponding to various modes of perturbation evolution are represented. If wave number is very small we will observe relaxation of the surface at any mode, but if wave number grows there are different options of surface perturbation development: at \( \omega>1 \) growth of wave number doesn’t matter, this is an area of modes of stability, at \( \omega<1 \) it will lead to distortion of surface shape. Figure A2 and figure B2 visually demonstrate behavior of function \( a(\chi) \) at certain values of \( \gamma \) and \( \omega \), taken from B and A areas respectively.

There are two specific modes. In the area C instability becomes apparent in the final range of wave numbers (figure C2). It corresponds to formation of wave structure on the surface with constant wavelength.
D area combines stability and instability manifestations in the final range of wave numbers (figure D2). It can be interpreted, as formation of wave structure with wavelength changing over time.

Figure 1. Areas in space of parameters ($\gamma$, $\omega$) corresponding to various modes of perturbation evolution: A - mode of instability, B – mode of stability, C – mode of local instability, D - mixed mode.

Figure 2. Behavior of function $a(\chi)$ at certain values of $\gamma$ and $\omega$, taken from different areas: a) mode of stability, $\gamma = 1.5$, $\omega = 1.5$; b) mode of instability, $\gamma = 1.5$, $\omega = 0.5$; c) mode of local instability, $\gamma = 0.3$, $\omega = 1.2$; d) mixed mode, $\gamma = 0.5$, $\omega = 0.985$. 

2. Instability of spherical shape surface

Second part of work devoted to spherical shape of surface. In particular, we observe the nanoparticle, which has the shape of a sphere in the zero approximation. For the description of a nanoparticle the liquid-drop model is used. The unperturbed surface is considered as an ideal sphere. For the description of a mass stream to a free surface we solved the quasi steady-state diffusion equation where concentration is a solution of the Laplace’s equation in spherical coordinates.

Action of capillary forces is taken from [2]. Perturbation of surface is expanded in series of Legendre polynomials:

\[ h(\theta, t) = \sum_{n=1}^{\infty} a_n(t) P_n(\cos \theta). \]

The coefficients of expansion change over time and have dependence on indexes \( n \), which is polynomial degree:

\[ a_n(t) = a_n(0) \exp \left( \frac{1}{R_0} \left( V(n + 1) - \frac{\sigma}{2\mu} \gamma_n \right) \right), \]

\[ a_n(0) = \frac{2n+1}{2} \int_0^\pi h_0(\theta) \sin \theta \ P_n(\cos \theta) \ d\theta, \]

where \( V \) is unperturbed velocity of sphere radius growth; \( \mu \) and \( \sigma \) - viscosity and coefficient of surface tension of nanoparticle; \( R_0 \) is initial radius; \( \gamma_n = n(n + 2)(2n + 1)/(2n^2 + 4n + 3) \); \( h_0(\theta) \) initial perturbation.

Big role in research of the equation of evolution of perturbation of a surface form is played by parameters. For the analysis of parameters entering under an exhibitor in coefficients at Legendre polynomials, it is better observe them as ratio \( \sigma/2\mu V \).

In the Figure 3 the line of ratio of parameters \( \sigma/2\mu V \) divided into areas is presented. Left area marked as B is an area of a ratio of parameters at which all harmonicas, since the first, are responsible for growth of perturbation development. Right area marked as A is an area of a ratio of parameters at which all harmonicas are responsible for a relaxation of the perturbation surface. Between these main areas there is located the infinite set of decreasing areas where the part of harmonicas starts being responsible for instability.

![Figure 3](image_url)

**Figure 3.** Line of ratio of parameters \( \sigma/2\mu V \) divided into areas: A is an area of a ratio of parameters at which all harmonicas are responsible for a relaxation of the perturbation surface; B is an area of a ratio of parameters at which all harmonicas are responsible for growth of perturbation development.
The value of a straight line below which the n-harmonica starts making a contribution to instability of perturbation development, can be calculated from:

\[ \sigma = \frac{(n+1)(2n^2 + 4n + 3)}{2\mu V n(n+2)(2n+1)}, \]

where \( n = 1, 2 \ldots \infty \).

Thus knowing one of parameters of system it is possible to pick up optimum their ratios for achievement of this or that effect.

It is important to mention that at \( n \to \infty \), if \( \frac{\sigma}{2\mu} > \frac{V}{\sigma} \), coefficients of expansion correspond to unstable harmonics and speed of instability development increases with index growth.

3. Summary
In this work the equations of evolution of perturbation of flat and spherical shape surface are received. Various modes of development of the perturbation surface concerning parameters are investigated. There are revealed modes at which the surface loses stability, and modes at which there is a relaxation of perturbation of a free surface.

The most actual is finding conditions for instability mode. Critical velocity must be bigger than surface tension coefficient divided into the doubled effective coefficient of viscosity.

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