CIRCLE AND LINE BUNDLES OVER GENERALIZED WEYL ALGEBRAS

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Abstract. Strongly \( \mathbb{Z} \)-graded algebras or principal circle bundles and associated line bundles or invertible bimodules over a class of generalized Weyl algebras \( B(p; q, 0) \) (over a ring of polynomials in one variable) are constructed. The Chern-Connes pairing between the cyclic cohomology of \( B(p; q, 0) \) and the isomorphism classes of sections of associated line bundles over \( B(p; q, 0) \) is computed thus demonstrating that these bundles, which are labeled by integers, are non-trivial and mutually non-isomorphic. The constructed strongly \( \mathbb{Z} \)-graded algebras are shown to have Hochschild cohomology reminiscent of that of Calabi-Yau algebras. The paper is supplemented by an observation that a grading by an Abelian group in the middle of a short exact sequence is strong if and only if the induced gradings by the outer groups in the sequence are strong.

1. Introduction

This paper is devoted to studies of some aspects of non-commutative geometry of degree-one generalized Weyl algebras \([3]\) or rank-one hyperbolic algebras \([23]\) over the polynomial ring in one variable. In the complex case with an appropriately chosen \( * \)-structure such algebras can be interpreted as coordinate algebras of non-commutative surfaces, smooth if the defining polynomial has no repeated roots. To be specific and fix the notation, we consider algebras \( B(p; q, r) \) over a field \( K \) of characteristic zero, labeled by \( q, r \in K \), \( q \neq 0 \) and a non-zero polynomial \( p \) in variable \( z \). \( B(p; q, r) \) are generated by \( x, y, z \) subject to the relations

\[
xy = p(qz + r), \quad yx = p(z), \quad xz = (qz + r)x, \quad yz = q^{-1}(z - r)y. \tag{1.1}
\]

In terminology of Bavula, these are (all) degree-one generalized Weyl algebras over the ring \( \mathbb{K}[z] \); \( p \) is an element of \( \mathbb{K}[z] \) and \( \sigma(z) = qz + r \) is an automorphism of \( \mathbb{K}[z] \) that form the defining data of such an algebra \([3]\). Algebras \( B(p; q, r) \) include examples of continuing and new interest in non-commutative geometry such as quantum spheres \([25]\), non-commutative deformations of type-A Kleinian singularities \([18]\) and quantum weighted projective spaces \([7], [6]\).

The main results of this article can be summarized as follows. When \( r = 0 \) and \( 0 \) is a root of \( p \) with multiplicity \( k \) we construct a strongly \( \mathbb{Z} \)-graded algebra \([24]\) (or a principal \( \mathbb{KZ} \)-comodule algebra \([9]\) or a quantum principal circle bundle) \( A(p; q_{\pm})^{(k)} \) with \( B(p; q, 0) \), \( q = q_{+}, q_{-} \), as its degree-zero part. As for every strongly \( \mathbb{Z} \)-graded algebra, the degree \( n \) component \( A(p; q_{\pm})^{(k)}_{n} \) of \( A(p; q_{\pm})^{(k)} \) is a finitely generated projective

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invertible module over $\mathcal{B}(p; q, 0)$, so it plays the role of a line bundle over the non-commutative space represented by $\mathcal{B}(p; q, 0)$. The idempotents for these modules can be explicitly described. We show that if $p$ has at least one non-zero root and $q$ is not a root of unity, then $\mathcal{B}(p; q, r)$ admits a non-zero cyclic trace. Combining such a trace with the idempotents constructed earlier, the Chern-Connes pairing \cite{11} can be computed thus establishing that if $p$ has a root 0 (of multiplicity $k$) and another root, $r = 0$ and $q$ is not a root of unity, then the Chern number of $\mathcal{A}(p; q^\pm)^{(k)}_n$ is $-n$. Consequently, $\mathcal{B}(p; q, 0)$-modules $\mathcal{A}(p; q^\pm)^{(k)}_n$ and $\mathcal{A}(p; q^\pm)^{(k)}_m$ are not isomorphic for $m \neq n$. This makes $\mathcal{A}(p; q^\pm)^{(k)}_p$ an example of what we term a non-degenerate projectively graded algebra.

Finally, we look at homological properties of $\mathcal{A}(p; q^\pm)^{(k)}$. Exploring the results of \cite{21} and employing methods of \cite{22}, we calculate Hochschild cohomology of $\mathcal{A}(p; q^\pm)^{(1)}$ with values in its enveloping algebra, in the case in which $p$ has no repeated roots. This turns out to be trivial except degree 3, where it is isomorphic to $\mathcal{A}(p; q^\pm)^{(1)}$ as a bimodule twisted by an algebra endomorphism of $\mathcal{A}(p; q^\pm)^{(1)}$.

The paper is supplemented by an appendix in which it is shown that given an exact sequence of Abelian groups and an algebra graded by the middle group in the sequence, the grading is strong if and only if the induced gradings by the outer groups in the sequence are strong.

2. Preliminaries

2.1. Conventions and notation. All algebras considered in this paper are associative, unital algebras over a field $\mathbb{K}$ of characteristic 0. Unadorned tensor products are over $\mathbb{K}$. The identity of a $\mathbb{K}$-algebra $\mathcal{A}$ is denoted by 1, while $\mathcal{A}^\text{op}$ denotes the algebra with the multiplication opposite to that of $\mathcal{A}$, and $\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^\text{op}$ is the enveloping algebra of $\mathcal{A}$.

Given algebra endomorphisms $\mu, \nu$ of $\mathcal{A}$ and an $\mathcal{A}$-bimodule $M$, we write $^\mu M^\nu$ for the $\mathcal{A}$-bimodule that is isomorphic to $M$ as a vector space, but has $\mathcal{A}$-multiplications twisted by $\mu$ and $\nu$, i.e.

$$a \cdot m \cdot b = \mu(a) m \nu(b), \quad \text{for all } a, b \in \mathcal{A}, \ m \in M. \quad (2.1)$$

Note that if $\mu$ is an automorphism, then $^\mu M^\nu \cong M^\nu \mu$ as bimodules.

Given an automorphism $\sigma$ of an algebra $\mathcal{A}$, the skew polynomial ring over $\mathcal{A}$ in an indeterminate $z$ is denoted by $\mathcal{A}[z; \sigma]$. This consists of all polynomials in $z$ with coefficients from $\mathcal{A}$, subject to the relation $az = z\sigma(a)$, for all $a \in \mathcal{A}$.

2.2. $\mathbb{Z}$-graded algebras and noncommutative circle and line bundles. A $\mathbb{Z}$-graded algebra is an algebra $\mathcal{A}$ which decomposes into a direct sum of vector spaces $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n$ such that $\mathcal{A}_m \mathcal{A}_n \subseteq \mathcal{A}_{m+n}$, for all $m, n \in \mathbb{Z}$. It follows immediately that $\mathcal{A}_0$ is a subalgebra of $\mathcal{A}$ and each of the $\mathcal{A}_n$ is an $\mathcal{A}_0$-bimodule by restriction of multiplication of $\mathcal{A}$. A $\mathbb{Z}$-graded algebra $\mathcal{A}$ is said to be strongly graded, if $\mathcal{A}_m \mathcal{A}_n = \mathcal{A}_{m+n}$. As explained in \cite{24} Section AI.3.2 a $\mathbb{Z}$-graded algebra $\mathcal{A}$ is strongly graded if and only if there exist $\omega = \sum_i \omega_i' \otimes \omega_i'' \in \mathcal{A}_{-1} \otimes \mathcal{A}_1$ and $\bar{\omega} = \sum_i \bar{\omega}_i' \otimes \bar{\omega}_i'' \in \mathcal{A}_1 \otimes \mathcal{A}_{-1}$ such that

$$\sum_i \omega_i' \omega_i'' = \sum_i \bar{\omega}_i' \bar{\omega}_i'' = 1. \quad (2.2)$$
Starting with such \( \omega, \tilde{\omega} \) one constructs inductively elements \( \omega(n) \in A_{-n} \otimes A_n \) as

\[
\omega(0) = 1 \otimes 1, \quad \omega(n) = \begin{cases} \sum_i \omega'_i \omega(n-1) \omega''_i & \text{if } n > 0, \\ \sum_i \omega'_i \tilde{\omega}(n+1) \omega''_i & \text{if } n < 0. \end{cases}
\] (2.3)

Due to (2.2), the evaluation of the multiplication on these elements yields the identity element of \( A \). This implies that \( A_0 = A_n A_{-n} \), for all \( n \), and, consequently, \( A_{m+n} = A_m A_n \) as required.

The equality \( A_0 = A_n A_{-n} \) translates into an \( A_0 \)-bimodule isomorphism \( A_0 \cong A_n \otimes A_0 \), therefore each of the \( A_n \) is an invertible \( A_0 \)-bimodule (with the inverse \( A_{-n} \)), and hence in particular it is finitely generated and projective as a left and right \( A_0 \)-module. The entries of the idempotent matrix \( E(n) \) are expressible in terms of the \( \omega(n) = \sum_i \omega'(n)_i \otimes \omega''(n)_i \) given by (2.3) as

\[
E(n)_{ij} = \omega''(n)_i \omega'(n)_j. \] (2.4)

Since \( \omega''(n)_i \in A_n \) and \( \omega'(n)_j \in A_{-n} \), \( E(n)_{ij} \in A_0 \). The idempotent property of \( E(n) \) is guaranteed by (2.2). The left \( A_0 \)-module isomorphism \( A_n \to A_0^N E(n) \), where \( N \) is the size of \( E(n) \), is given by \( a \mapsto (\sum_i a \omega'(n)_i E(n)_{ij})_j \).

Note in passing that due to the iterative definition of the \( \omega(n) \), equations (2.3) are equivalent to

\[
\omega(0) = 1 \otimes 1, \quad \omega(n) = \begin{cases} \sum_i \omega_i(n-1) \omega_i(n-1)'' & \text{if } n > 0, \\ \sum_i \omega_i(n+1) \tilde{\omega}_i n+1)'' & \text{if } n < 0. \end{cases}
\] (2.5)

Strongly \( \mathbb{Z} \)-graded algebras are examples of principal comodule algebras, which in the context of non-commutative geometry play the role of coordinate algebras of principal fibre bundles \([10],[9]\). In the \( \mathbb{Z} \)-graded case, the coacting Hopf algebra or the fibre is the group algebra of \( \mathbb{Z} \) or, equivalently, coordinate algebra of the circle, hence strongly \( \mathbb{Z} \)-graded algebras represent circle principal bundles in noncommutative geometry. The elements \( \omega(n) \) combine into a function from the group algebra of \( \mathbb{Z} \) to \( A \otimes A \) which in the geometric context is interpreted as a (strong) connection form \([13],[12]\). In a general principal comodule algebra the existence of a (strong) connection form ensures that associated bundles are finitely generated projective modules with idempotents which in the circle bundle case take the form (2.4); see \([9]\). Also in this case, bundles associated to one-dimensional representations of the circle group coincide with the \( A_n \), and since the latter are invertible bimodules they can be given a genuine interpretation of sections of line bundles \([4]\).

2.3. The algebras \( B(p;q,r) \) and other generalized Weyl algebras. Let \( A \) be an algebra, \( \sigma \) an automorphism of \( A \) and \( p \) an element of the centre of \( A \). A degree-one generalized Weyl algebra over \( A \) is an algebraic extension \( A(p,\sigma) \) of \( A \) obtained by supplementing \( A \) with additional generators \( x, y \) subject to the following relations

\[
xy = \sigma(p), \quad yx = p, \quad xa = \sigma(a)x, \quad ya = \sigma^{-1}(a)y; \] (2.6)

see \([3]\). The algebras \( A(p,\sigma) \) share many properties with \( A \), in particular, if \( A \) is a Noetherian algebra, so is \( A(p,\sigma) \), and if \( A \) has no zero-divisors and \( p \neq 0 \), \( A(p,\sigma) \) does not have zero-divisors either.
If $\mathcal{A} = \mathbb{K}[z]$, then every automorphism $\sigma$ of $\mathcal{A}$ necessarily takes the form $\sigma(z) = qz + r$, for $q, r \in \mathbb{K}$, $q \neq 0$. Hence any generalized Weyl algebra over $\mathbb{K}[z]$ coincides with an algebra $\mathcal{B}(p; q, r)$ described in Introduction. The basis for $\mathcal{B}(p; q, r)$ is given by monomials $x^k z^l$ and $y^k z^l$. The Hochschild cohomology of algebras $\mathcal{B}(p; q, r)$ was computed in [13, 26], while in [21] it has been established that all the $\mathcal{B}(p; q, r)$ are twisted Calabi-Yau algebras provided $p$ has no repeated roots.

In the complex case $\mathbb{K} = \mathbb{C}$, $\mathcal{B}(p; q, r)$ can be made into $\ast$-algebras by setting $z^* = z$, $y^* = x$ as long as $q, r$ are real and $p$ has real coefficients. In the case $q \neq 1$ and $p\left(\frac{r}{1-q}\right) \geq 0$, $\mathcal{B}(p; q, r)$ has one-dimensional $\ast$-representations $\pi_\lambda$, labelled by numbers $\lambda$ of modulus one, and given by

$$\pi_\lambda(z) = \frac{r}{1-q}, \quad \pi_\lambda(x) = \lambda \sqrt{p\left(\frac{r}{1-q}\right)}.$$

Furthermore, if $q \in (0, 1)$ and $r = 0$, every real root $\zeta$ of $p$ such that $p(q^k \zeta) > 0$, for all positive integers $k$, defines an infinite-dimensional bounded $\ast$-representation $\pi_\zeta$ on the Hilbert space with orthonormal basis $e_k$, $k \in \mathbb{N}$,

$$\pi_\zeta(z)e_k = q^k \zeta e_k, \quad \pi_\zeta(x)e_k = \sqrt{p(q^k \zeta)} e_{k-1}.$$

Quantum spheres [25] and quantum weighted projective spaces [7, 6] are examples of $\ast$-algebras $\mathcal{B}(p; q, r)$.

### 3. Non-degenerate Projectively Graded Algebras

In this section we construct $\mathbb{Z}$-graded algebras, whose degree-zero part coincides with $\mathcal{B}(p; q, r)$, and which fall within a specific category of graded algebras which is recorded in the following

**Definition 3.1.**

(1) A $\mathbb{Z}$-graded algebra $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}}\mathcal{A}_n$ is said to be **projectively graded** if, for all $n \in \mathbb{Z}$, $\mathcal{A}_n$ is a projective left $\mathcal{A}_0$-module.

(2) A projectively graded algebra $\mathcal{A} = \bigoplus_{n \in \mathbb{Z}}\mathcal{A}_n$ is said to be **non-degenerate**, if

(a) for all $n \in \mathbb{Z} \setminus \{0\}$, $\mathcal{A}_n$ is a finitely generated non-free left $\mathcal{A}_0$-module;

(b) for all $m, n \in \mathbb{Z}$, if $m \neq n$, then $\mathcal{A}_m \not\cong \mathcal{A}_n$ as left $\mathcal{A}_0$-modules.

A strongly $\mathbb{Z}$-graded algebra $\mathcal{A}$ is projectively graded but it is not necessarily non-degenerate (despite the fact that all the $\mathcal{A}_n$ are finitely generated). There is an easy test which allows one to detect the failure of satisfying condition (2)(a) in Definition 3.1.

**Lemma 3.2.** In a strongly $\mathbb{Z}$-graded algebra $\mathcal{A}$, $\mathcal{A}_n \cong \mathcal{A}_0$ as left (resp. right) $\mathcal{A}_0$-modules if and only if there exists a unit $u \in \mathcal{A}_n$.

**Proof.** If $u$ is a unit in $\mathcal{A}_n$, then $u^{-1} \in \mathcal{A}_{-n}$, since $1 \in \mathcal{A}_0 = \mathcal{A}_n \mathcal{A}_{-n}$. This allows one to define mutually inverse isomorphisms of left $\mathcal{A}_0$-modules by

$$\phi : \mathcal{A}_n \rightarrow \mathcal{A}_0, \quad a \mapsto au^{-1}, \quad \phi^{-1} : \mathcal{A}_0 \rightarrow \mathcal{A}_n, \quad b \mapsto bu.$$

Conversely, given an isomorphism $\phi : \mathcal{A}_n \rightarrow \mathcal{A}_0$, the element $\phi^{-1}(1) \in \mathcal{A}_n$ is the required unit. $\Box$
Corollary 3.3. In a strongly \( \mathbb{Z} \)-graded algebra \( \mathcal{A} \) in which \( \mathcal{A}_0 \) has the invariant basis number property, \( \mathcal{A}_n \) is a free \( \mathcal{A}_0 \)-module if and only if there is a unit in \( \mathcal{A}_n \).

Proof. If \( \mathcal{A}_n \cong \mathcal{A}_0^k \), then, since \( \mathcal{A}_{-n} \) is its inverse, also \( \mathcal{A}_{-n} \cong \mathcal{A}_0^l \), for some \( l \). Therefore,
\[
\mathcal{A}_0 \cong \mathcal{A}_0^k \otimes \mathcal{A}_0^l \cong \mathcal{A}_0^{k+l-1},
\]
hence \( k = l = 1 \) by the IBN property, and the assertion follows by Lemma 3.2 \( \square \)

One way of proving that a strongly graded \( \mathbb{Z} \)-algebra \( \mathcal{A} \) is a non-degenerate projectively graded algebra is to use the Chern-Connes pairing between the cyclic cohomology and \( K \)-theory of \( \mathcal{A}_0 \). This method allows one not only to determine whether the components of \( \mathcal{A} \) are non-free but also to establish that they are not mutually isomorphic. It was successfully applied recently in [9] for algebras arising as total spaces of circle bundles over Heegaard quantum weighted projective lines, and earlier in [10] to prove that the coordinate algebra of the quantum group \( SU_q (2) \) is a non-degenerate projectively graded algebra over the coordinate algebra of the quantum standard sphere \( S^2_q \), and a similar statement for mirror quantum spheres [17]. The method is based on evaluating a cyclic trace on \( \mathcal{A}_0 \) at traces of idempotents for the \( \mathcal{A}_n \). As recalled in Section 2.2 there is an explicit formula for the latter. The construction of a cyclic trace depends more heavily on the structure of \( \mathcal{A}_0 \) and we will present it for the generalized Weyl algebras \( \mathcal{B}(p; q, r) \) described in Introduction.

Proposition 3.4. Let \( \mathcal{B}(p; q, r) \) be a generalized Weyl algebra given by generators and relations (1.1). If \( q \) is not a root of unity and \( p \) has at least one non-zero root, then \( \mathcal{B}(p; q, r) \) admits a non-trivial trace (cyclic cocycle).

Proof. Let \( \zeta \) be a non-zero root of \( p \). We will construct a \( \mathbb{K} \)-linear map \( \hat{\tau}_\zeta : \mathbb{K}[z] \to \mathbb{K} \), such that \( \hat{\tau}_\zeta(1) = 0 \) and, for all polynomials \( f \in \mathbb{K}[z] \) of degree at least one,
\[
\hat{\tau}_\zeta(f(z)) - \hat{\tau}_\zeta(f(qz + r)) = f(\zeta).
\]
Suppose that
\[
\hat{\tau}_\zeta(z^n) = \frac{1}{1 - q^n} \sum_{i=1}^{n} t^n_i \zeta^i.
\]
We need to find coefficients \( t^n_i \) such that
\[
\hat{\tau}_\zeta(z^n) - \hat{\tau}_\zeta((qz + r)^n) = \zeta^n.
\]
Comparing respective powers of \( \zeta \), one easily derives an inductive formula for the \( t^n_i \),
\[
t^n_n = 1, \quad t^n_{n-k} = \sum_{i=1}^{k} \binom{n}{i} r^i q^{n-i} t^n_{n-k}.
\]
Define a \( \mathbb{K} \)-linear map \( \tau_\zeta : \mathcal{B}(p; q, r) \to \mathbb{K} \),
\[
\tau_\zeta(x^m z^n) = \begin{cases} \hat{\tau}_\zeta(z^n) & \text{if } m = 0, n \neq 0, \\ 0 & \text{otherwise}, \end{cases} \quad \tau_\zeta(y^m z^n) = \begin{cases} \hat{\tau}_\zeta(z^n) & \text{if } m = 0, n \neq 0, \\ 0 & \text{otherwise}. \end{cases}
\]
Then, by the definition of \( \tau_\zeta \),
\[
\tau_\zeta(xz) = 0 = \tau_\zeta(zx), \quad \tau_\zeta(yz) = 0 = \tau_\zeta(zy).
\]
Next, consider the algebra automorphism $\sigma : \mathbb{K}[z] \to \mathbb{K}[z]$, $z \mapsto qz + r$, and define, for all $n$,

$$s_n(z) = \prod_{m=0}^{n-1} \sigma^m(p(z)).$$

The relations \((\ref{1.1})\) (cf. \((\ref{2.6})\)) imply that, for all $n \in \mathbb{N}$, $y^n x^n = s_n(z)$ and $x^n y^n = \sigma^n(s_n(z))$. Since, for all $l = 0, \ldots, n - 1$, the polynomial $\sigma^l(s_n(z))$ has a root $\zeta$, the definition of $\tau_\zeta$ and equation \((\ref{3.1})\) yield

$$\tau_\zeta(x^n y^n) = \hat{\tau}_\zeta(\sigma^n(s_n(z))) = \hat{\tau}_\zeta(\sigma^{n-1}(s_n(z))) = \ldots = \hat{\tau}_\zeta(s_n(z)) = \tau_\zeta(y^n x^n).$$

Hence $\tau_\zeta$ defined in \((\ref{3.3})\) has the property $\tau_\zeta(ab) = \tau_\zeta(ba)$, for all $a, b \in \mathcal{B}(p; q, r)$, i.e. it is a cyclic cocycle (trace) on $\mathcal{B}(p; q, r)$, as required.

\[\Box\]

\textbf{Corollary 3.5.} For $\mathcal{B}(p; q, 0)$ in which $q$ is not a root of unity and $p$ has at least one non-zero root, say $\zeta$, the trace $\tau_\zeta$ constructed in Proposition 3.4 comes out as

$$\tau_\zeta(x^m z^n) = \begin{cases} \frac{c_n}{1-q^n} & \text{if } m = 0, n \neq 0, \\ 0 & \text{otherwise,} \end{cases} \quad \tau_\zeta(y^m z^n) = \begin{cases} \frac{c_n}{1-q^n} & \text{if } m = 0, n \neq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

\textbf{Proof.} If $r = 0$, the inductive formula \((\ref{3.2})\) implies that for $t^m_{n-k} = 0$ if $k \neq 0$, while $t^n_0 = 1$. \[\Box\]

We are now in position to construct quantum circle bundles over $\mathcal{B}(p; q, 0)$ and establish their non-degeneracy as projectively $\mathbb{Z}$-graded algebras. Fix a non-zero polynomial $p(z)$ with a root 0 of multiplicity $k$, and let $\tilde{p}(z)$ be obtained from $p(z)$ by factoring out $z^k$, i.e.

$$p(z) = z^k \tilde{p}(z). \quad (3.4)$$

Fix a non-zero $q \in \mathbb{K}$ and let $q_+ q_- = q$. With these data we associate an algebra $\mathcal{A}(p; q_\pm)$ generated by $z_\pm$ and $x_\pm$ subject to relations

$$z_+ z_- = z_- z_+, \quad x_+ x_- = \tilde{p}(z_+ z_-), \quad x_- x_+ = \tilde{p}(q z_+ z_-),$$

$$x_+ z_\pm = q_\pm^1 z_\pm x_+, \quad x_- z_\pm = q_\pm z_\pm x_- \quad (3.5)$$

The algebra $\mathcal{A}(p; q_\pm)$ can be understood as a generalized Weyl algebra over the polynomial ring $\mathbb{K}[z_+, z_-]$. We view it as a $\mathbb{Z}$-graded algebra with degrees of $z_\pm$ being equal to $\pm 1$, and that of $x_\pm$ being equal to $\pm k$. Define the subalgebra of $\mathcal{A}(p; q_\pm)$, by

$$\mathcal{A}(p; q_\pm)^{(k)} := \bigoplus_{n \in \mathbb{Z}} \mathcal{A}(p; q_\pm)_{nk}, \quad (3.6)$$

i.e. as spanned by all elements of $\mathcal{A}(p; q_\pm)$ of degree that is a multiple of $k$. $\mathcal{A}(p; q_\pm)^{(k)}$ is a $\mathbb{Z}$-graded algebra: $a \in \mathcal{A}(p; q_\pm)^{(k)}$ has degree $n$ if it has a degree $kn$ in $\mathcal{A}(p; q_\pm)$.

\textbf{Theorem 3.6.} Let $p$ be a non-zero polynomial with a root 0 of multiplicity $k$, and let $\mathcal{A}(p; q_\pm)^{(k)}$ be the $\mathbb{Z}$-graded algebra defined by \((3.6)\).

1. $\mathcal{A}(p; q_\pm)^{(k)}$ is a strongly $\mathbb{Z}$-graded algebra with the degree-zero part equal to $\mathcal{B}(p; q, 0)$, where $q = q_+ q_-$. 

2. If $q$ is not a root of unity, then $\mathcal{A}(p; q_\pm)^{(k)}$ is a non-degenerate projectively graded algebra if and only if $p$ has at least one non-zero root.
Proof. (1) One easily verifies that the degree-zero part of $\mathcal{A}(p; q_\pm)^{(k)}$ is generated by $x := x_- z_+^k$, $y := z_+^k x_+$ and $z := z_+ z_-$, and that these satisfy relations (1.1). We will continue to write $x$ for $z_+ z_-$. Define polynomial $\tilde{p}(z)$ by

$$\tilde{p}(z) = \tilde{p}(0) - z \tilde{p}(z).$$

(3.7)

Note that $\tilde{p}(0) \neq 0$, so we can consider

$$\omega := \frac{1}{\tilde{p}(0)^k} \left(q^k \tilde{p}(q) z_-^{k} \otimes z_+^{k} + x_- \otimes \left(\sum_{i=0}^{k-1} z^i \tilde{p}(z)^i \frac{\tilde{p}(0)^{k-i-1}}{\tilde{p}(z)} \right) x_+ \right).$$

(3.8)

A simple computation which uses relations \((3.5)\),

$$\frac{1}{\tilde{p}(0)^k} \left(x_+ \tilde{p}(0)^k - z \tilde{p}(z) \right)$$

(3.9)

reveals that $\omega$ satisfies condition (2.2). In a similar way one checks that

$$\tilde{\omega} := \frac{1}{\tilde{p}(0)^k} \left(\tilde{p}(z)^k z_+^k \otimes z_-^k + x_+ \otimes \tilde{p}(0)^k - q^k z_+^k \tilde{p}(q) z_-^k \right) = 1,$$

satisfies the other condition in (2.2). Therefore, $\mathcal{A}(p; q_\pm)^{(k)}$ is a strongly $\mathbb{Z}$-graded algebra as claimed.

(2) If $p$ has no roots other than 0, then the second and third of relations (3.5) imply that $x_\pm$ are units, hence, for all positive $n$, $x_\pm^n$ are units in $\mathcal{A}(p; q_\pm)^{(k)}_{\pm n}$, so $\mathcal{A}(p; q_\pm)^{(k)}_{\pm n} \cong B(p; q, 0)$ by Lemma 3.2. Consequently $\mathcal{A}(p; q_\pm)^{(k)}$ is a degenerate projectively graded algebra.

Assume that $p$ has non-zero roots. To prove that all the $\mathcal{A}(p; q_\pm)^{(k)}_n$ are non-free (except for $n = 0$) and mutually non-isomorphic we will pick a non-zero root $\zeta$ of $p$ and compute the value of the cyclic cocycle $\tau_\zeta$ (3.3) on the trace of the idempotent $E(n)$ given by (2.4). We deal only with the $n$ positive case, the other case is similar.

Let

$$e_n := \text{Tr } E(n) = \sum_{i} \omega(n)_i \omega(n)_i'.$$

Exploring (3.8) and (2.5) as well as the defining relations (3.5) of $\mathcal{A}(p; q_\pm)_n$, we observe that $e_n$ is a polynomial in $z$ (independent of $x_\pm$), which, for non-negative $n$ is given by the following recursive formula:

$$e_{n+1}(z) = \frac{1}{\tilde{p}(0)^k} \left(\tilde{p}(0)^k - z_+^k \tilde{p}(z) \right) x_+ e_n(z) x_-^k + q^k z_+^k e_n(z) \tilde{p}(q) z_-^k.$$ 

(3.10)
With the help of (3.5) and (3.7) this can be evaluated further to give
\[ e_{n+1}(z) = \frac{1}{\hat{p}(0)^k} \left( (\hat{p}(0)^k - \hat{p}(z)^k z^k) e_n(q^{-1} z) - (\hat{p}(0)^k - q^k \hat{p}(q z)^k z^k) e_n(z) \right) + e_n(z). \] (3.11)

In particular,
\[ e_1(z) = \frac{1}{\hat{p}(0)^k} (q^k \hat{p}(q z)^k - \hat{p}(z)^k) z^k + 1, \] (3.12)
hence \( e_1(0) = 1 \). A simple inductive argument which uses (3.11) proves that in fact \( e_n(0) = 1 \), for all positive \( n \). We will show that \( \tau_\zeta(e_n(z)) = -n \).

Remembering the definitions of \( \tau_\zeta \) and \( \hat{\tau}_\zeta \), specifically that \( \hat{\tau}_\zeta(1) = 0 \), we can use (3.12) to compute
\[ \tau_\zeta(e_1(z)) = \frac{1}{\hat{p}(0)^k} \left( \hat{\tau}_\zeta \left( (q z)^k \hat{p}(q z)^k - z^k \hat{p}(z)^k \right) \right). \]

Note that since \( \zeta \) is a non-zero root of \( p(z) \) it is also a root of \( \hat{p}(z) \), hence
\[ \zeta \hat{p}(\zeta) = \hat{p}(0). \] (3.13)

Furthermore, \( z\hat{p}(z) \) is a polynomial of degree at least one, so the formula (3.11) can be applied thus yielding
\[ \tau_\zeta(e_1(z)) = -\frac{1}{\hat{p}(0)^k} (\zeta \hat{p}(\zeta))^k = -1, \] by (3.13). Assume inductively that \( \tau_\zeta(e_m(z)) = -m \). Then, using (3.11) and (3.1) one finds
\[ \tau_\zeta(e_{m+1}(z)) = \frac{1}{\hat{p}(0)^k} \hat{\tau}_\zeta \left( (\hat{p}(0)^k - \hat{p}(z)^k z^k) e_m(q^{-1} z) - \hat{p}(0)^k \right) \]
\[ -\frac{1}{\hat{p}(0)^k} \hat{\tau}_\zeta \left( (\hat{p}(0)^k - q^k \hat{p}(q z)^k z^k) e_m(z) - \hat{p}(0)^k \right) + \hat{\tau}_\zeta(e_m(z)) \]
\[ = \frac{1}{\hat{p}(0)^k} \left( (\hat{p}(0)^k - \hat{p}(\zeta)^k \zeta^k) e_m(q^{-1} \zeta) - \hat{p}(0)^k \right) - m = -m - 1. \]

Therefore,
\[ \tau_\zeta(e_n(z)) = -n. \] (3.14)

As explained in [11, Section III.3] the formula (3.14) does not depend on the choice of the idempotent representing the isomorphism class of \( A(p; q_{\pm})^{(k)} \) in the algebraic \( K \)-group \( K_0(A(p; q_{\pm})^{(k)}) \). The index formula (3.14) determines the Chern-Connes pairing between even cyclic cohomology and the \( K_0 \)-group. Since the numbers \( \tau_\zeta(e_n(z)) \) are not zero and distinguish between different \( n \), all the modules \( A(p; q_{\pm})^{(k)} \) are not free for all positive \( n \) and they are mutually non-isomorphic. Similar arguments confirm that the index formula (3.14) remains true also for negative values of \( n \). Therefore, \( A(p; q_{\pm})^{(k)} \) is a non-degenerate projectively graded algebra. \( \Box \)
Example 3.7. The quantum weighted projective space or the quantum spindle algebra \( \mathcal{O}(\mathbb{WP}_q(k, l)) \) \cite{7} is a generalized Weyl algebra \( \mathcal{B}(p; q^d, 0) \), where

\[
p(z) = z^k \prod_{i=0}^{l-1} (1 - q^{-2i}z).
\]

The associated algebra \( \mathcal{A}(p; q^l) \), with \( q^l = q^l \) coincides with the coordinate algebra of the quantum lens space \( \mathcal{O}(L_q(kl; k, l)) \) \cite{19}. Hence the first part of Theorem 3.6 recovers \cite[Theorem 3.3]{7} in the case \( k = 1 \) and \cite[Proposition 6.5]{11} for all other values of \( k \). The index pairing calculations presented in the proof of the second part of Theorem 3.6 confirm and extend those in \cite[Section 7.2]{11}.

Remark 3.8. Note that the algebra \( \mathcal{A}(p; q^l) \) is not strongly graded, unless, of course \( k = 1 \). The degree-one submodule of \( \mathcal{A}(p; q^l) \) is generated by \( x_\pm z_{\pm}^{k+1} \) and \( z_+ \), while the degree minus one submodule is generated by \( x_\pm z_{\pm}^{k-1} \) and \( z_- \). Every combination of the elements from the first group multiplied by the elements from the second will produce a term with at least one \( z_\pm \). Such terms cannot combine to produce the identity element of \( \mathcal{A}(p; q^l) \).

Recall that the Hochschild cohomology \( H^l(\mathcal{A}, M) \) of an algebra \( \mathcal{A} \) with values in an \( \mathcal{A} \)-bimodule \( M \) understood as a left \( \mathcal{A}^e \)-module is defined as the right derived functor \( \text{Ext}^l_{\mathcal{A}^e}(\mathcal{A}, M) \) of the Hom-functor. Since \( \mathcal{A}^e \) is an \( \mathcal{A}^e \)-bimodule, the Hochschild cohomology groups \( \text{Ext}^l_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e) \) inherit an \( \mathcal{A} \)-bimodule structure from the right \( \mathcal{A}^e \)-module structure of \( \mathcal{A}^e \) (or the inner bimodule structure of \( \mathcal{A}^e \)).

Proposition 3.9. Let \( p(z) \) be a non-zero polynomial with a simple root \( 0 \), let \( \bar{p}(z) \) be given by \((3.3)\) with \( k = 1 \), and let \( \mathcal{A}(p; q^l) \) be the algebra given by generators and relations \((3.5)\). If the polynomial \( p \) has no repeated roots, then there exists an algebra endomorphism \( \kappa : \mathcal{A}(p; q^l) \to \mathcal{A}(p; q^l) \) such that

\[
H^l(\mathcal{A}(p; q^l), \mathcal{A}(p; q^l)^e) \cong \begin{cases} 
0, & \text{if } l \neq 3, \\
\mathcal{A}(p; q^l)^e, & \text{if } l = 3,
\end{cases}
\]

as \( \mathcal{A}(p; q^l) \)-bimodules. The superscript \( \kappa \) indicates twisting of the regular right \( \mathcal{A}(p; q^l) \)-module structure as in \((2.1)\).

Lemma 3.10. Let \( \mathcal{A} \) be an algebra, let \( \omega \) be a normal regular element of \( \mathcal{A} \) (i.e. \( \omega \) is not a zero-divisor and is such that \( \mathcal{A}\omega = \omega \mathcal{A} \)), and let \( \mathcal{B} := \mathcal{A}/\omega \mathcal{A} \). If there exists an algebra endomorphism \( \nu : \mathcal{A} \to \mathcal{A} \) such that

\[
\text{Ext}^l_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e) = \begin{cases} 
0, & \text{if } l \neq d + 1, \\
\mathcal{A}^e, & \text{if } l = d + 1,
\end{cases}
\]

as \( \mathcal{A} \)-bimodules, then there exists an algebra endomorphism \( \kappa : \mathcal{B} \to \mathcal{B} \), such that

\[
\text{Ext}^l_{\mathcal{B}^e}(\mathcal{B}, \mathcal{B}^e) \cong \begin{cases} 
0, & \text{if } l \neq d, \\
\mathcal{B}^e, & \text{if } l = d,
\end{cases}
\]

as \( \mathcal{B} \)-bimodules.
Proof. We will adapt the proof of [21, Proposition 4.4] (see also the proof of [20, 4.4. Corollary]) by incorporating a twist into it and extending it to this more general situation and calculate \( \text{Ext}_{\mathcal{B}^e}(\mathcal{B}, \mathcal{B}^e) \).

Let \( \pi : \mathcal{A} \rightarrow \mathcal{B} \) be the canonical epimorphism. Whenever we view a \( \mathcal{B} \)-bimodule \( M \) as an \( \mathcal{A} \)-bimodule via \( \pi \), we write \( M_{\pi} \).

Since \( \omega \) is not a zero divisor (a regular element of \( \mathcal{A} \)), it defines an automorphism \( \mu \) of \( \mathcal{A} \), by \( a\omega = \omega \mu(a) \). The definition of \( \mathcal{B} \) is encoded in the following short exact sequence of \( \mathcal{A} \)-bimodule maps

\[
0 \longrightarrow \mathcal{A}^{\mu -1} \xrightarrow{r_{\omega}} \mathcal{A} \longrightarrow \mathcal{B}_{\pi} \longrightarrow 0,
\]

where \( r_{\omega} \) denotes the right multiplication by \( \omega \). Since \( \text{Ext}^{i}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e) = 0 \) unless \( i = d+1 \) in which case \( \text{Ext}^{d}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e) = \mathcal{A}^{\nu} \), the application of \( \text{Ext}^{i}_{\mathcal{A}^e}(-, \mathcal{A}^e) \) to (3.16) yields a long exact sequence, whose nontrivial part is

\[
0 \longrightarrow \text{Ext}^{d+1}_{\mathcal{A}^e}(\mathcal{B}_{\pi}, \mathcal{A}^e) \longrightarrow \mathcal{A}^{\nu} \xrightarrow{l_{\omega}} \mathcal{A}^{\nu \mu} \longrightarrow \text{Ext}^{d+2}_{\mathcal{A}^e}(\mathcal{B}_{\pi}, \mathcal{A}^e) \longrightarrow 0,
\]

where \( l_{\omega} \) is the left multiplication by \( \omega \). Since \( l_{\omega} \) is a monomorphism and coker \( l_{\omega} = \mathcal{B} \) we obtain

\[
\text{Ext}^{i}_{\mathcal{A}^e}(\mathcal{B}_{\pi}, \mathcal{A}^e) = \begin{cases} 
0, & \text{if } l \neq d + 2, \\
\mathcal{B}^{\nu \mu}_{\pi}, & \text{if } l = d + 2.
\end{cases}
\]

Note that \( \mathcal{B}^{\nu} = \mathcal{B} \otimes \mathcal{A}^{\nu} \) and \( \mathcal{B} \otimes \mathcal{A}^{\nu} = \mathcal{A}^{\nu} \otimes \chi \mathcal{B} \). Both 1 \( \otimes \omega \) and \( \omega \otimes 1 \) are normal regular elements, inducing automorphisms \( id_{\mathcal{B}} \otimes \mu^{-1} = \mu \otimes id_{\mathcal{A}^{\nu}} \), respectively. This allows one to use the twisted version of the Rees lemma (see [2, Lemma 1.2] or [20, Section 3.4]) to compute the following chain of isomorphisms of \( \mathcal{A} \)-bimodules

\[
\text{Ext}^{i}_{\mathcal{B}^e}(\mathcal{B}, \mathcal{B}^e)_{\pi} \cong \mu \text{Ext}^{i+1}_{\mathcal{B}^e \otimes \mathcal{A}^{\nu} \otimes \mathcal{B}^e}(\mathcal{B}, \mathcal{B} \otimes \mathcal{A}^{\nu}) \cong \mu \text{Ext}^{i+2}_{\mathcal{A}^e}((\mathcal{B}_{\pi}, \mathcal{A}^e)_{\mu^{-1}},
\]

where \( \mathcal{B} \) and its modules are always understood as \( \mathcal{A} \)-modules via the map \( \pi \) and the indicated automorphisms of \( \mathcal{A} \). Therefore,

\[
\text{Ext}^{i}_{\mathcal{B}^e}(\mathcal{B}, \mathcal{B}^e)_{\pi} \cong \begin{cases} 
0, & \text{if } l \neq d, \\
\mu \mathcal{B}_{\pi}^{\nu \mu}, & \text{if } l = d,
\end{cases}
\]

(3.17)

as \( \mathcal{A} \)-bimodules. Write \( \theta : \text{Ext}^{d}_{\mathcal{B}^e}(\mathcal{B}, \mathcal{B}^e)_{\pi} \rightarrow \mathcal{B}^{\nu \mu}_{\pi} \), for the isomorphism (3.17). Since \( \pi \) is an epimorphism, \( \theta \) is an isomorphism of left \( \mathcal{B} \)-modules \( \text{Ext}^{d}_{\mathcal{B}^e}(\mathcal{B}, \mathcal{B}^e) \rightarrow \mathcal{B} \). The right \( \mathcal{B} \)-module structures induce an endomorphism

\[
\kappa : \mathcal{B} \rightarrow \mathcal{B}, \quad b \mapsto \theta(\theta^{-1}(1)b),
\]

that makes \( \theta \) an isomorphism of \( \mathcal{B} \)-bimodules \( \text{Ext}^{d}_{\mathcal{B}^e}(\mathcal{B}, \mathcal{B}^e) \rightarrow \mathcal{B}^e \), as required. \( \square \)

Note in passing that assumption (3.15) is satisfied by rigid Gorenstein \([5]\), hence in particular by twisted Calabi-Yau algebras \([14]\) of dimension \( d + 1 \).

Proof of Proposition 3.9. Consider \( \mathcal{B}(\overline{p}; q, 0) \), where \( q = q_{-}q_{+} \), and iterated skew polynomial algebra over \( \mathcal{B}(\overline{p}; q, 0) \),

\[
\mathcal{A} := \mathcal{B}(\overline{p}; q, 0)[z_{-}; \sigma_{-}][z_{+}; \sigma_{+}].
\]
where $\sigma_-$ is an automorphism of $\mathcal{B}(\tilde{p}; q, 0)$ and $\sigma_+$ of $\mathcal{B}(\tilde{p}; q, 0)[z_-; \sigma_-]$, given by

$$
\sigma_{\pm}(z) = z, \quad \sigma_{\pm}(x) = q_{\pm}x, \quad \sigma_{\pm}(y) = q_{\pm}^{-1}y, \quad \sigma_{+}(z_-) = z_-.
$$

Let $\omega = z_-z_+-z$. Note that $\omega$ is a normal regular element of $A$ and that $A(p; q_{\pm}) \cong A/A\omega$ (the isomorphism sends $x_-$ to the class of $x$ and $x_+$ to the class of $y$).

Since $\tilde{p}$ has no repeated roots, $\mathcal{B}(\tilde{p}; q, 0)$ is a twisted Calabi-Yau algebra of dimension 2 by [21, Theorem 4.5]. By [22, Theorem 0.2] $A$ is a twisted Calabi-Yau algebra of dimension 4, which, in particular means that $A$ satisfies assumption (3.15) with $d = 3$, and the assertion follows by Lemma 3.10.

\[\square\]

Appendix A. Strongly graded algebras and exact sequences of Abelian groups

The notion of a $\mathbb{Z}$-graded algebra is a special case of the notion of a group-graded algebra. Let $G$ be a group. A $G$-graded algebra $A$ decomposes into a direct sum of subspaces $A_g$ labelled by $g \in G$ such that $A_gA_h \subseteq A_{gh}$, for all $g, h \in G$. In case $A_gA_h = A_{gh}$, for all $g, h \in G$, $A$ is said to be strongly $G$-graded. We will write $|a|_G$ for the $G$-degree of $a \in A$.

Let $A = \bigoplus_{g \in G} A_g$ be a $G$-graded algebra. Any group epimorphism $\pi : G \to H$ induces an $H$-grading on $A$ by setting, for all $h \in H$,

$$
A_h = \bigoplus_{g \in \pi^{-1}(h)} A_g.
$$

Given a group monomorphism $\varphi : K \to G$, one can define a $K$-graded algebra

$$
A^{(K)} := \bigoplus_{k \in K} A_{\varphi(k)}.
$$

The aim of this appendix is to prove the following lemma, which supplements [21, Section A.I.3.1.b].

Lemma A.1. Consider a short exact sequence of Abelian groups:

$$
0 \longrightarrow K \xrightarrow{\varphi} G \xrightarrow{\pi} H \longrightarrow 0.
$$

A $G$-graded algebra $A$ is strongly graded if and only if the induced $H$-grading on $A$ and $K$-grading on $A^{(K)}$ are strong.

Proof. By [21, Section A.I.3.1.b], if $A$ is strongly $G$-graded then also the induced graded algebras are strongly graded.

In the converse direction, take any $g \in G$. Since the $H$-grading on $A$ induced by $\pi$ is strong, there exist $a_j, b_j \in A$ such that

$$
|a_j|_H = \pi(g) \quad \text{and} \quad \sum_j a_j b_j = 1.
$$

The exactness of the sequence implies that there exist $k_j \in K$, such that

$$
|a_j|_G = g + \varphi(k_j).
$$
Since the $\varphi$ induced $K$-grading is strong, there exist $a_{ij}, b_{ij} \in A$ such that

$$|a_{ij}|_K = -k_j \quad \text{and} \quad \sum_i a_{ij} b_{ij} = 1.$$ 

Note that the first condition above means that $|a_{ij}|_G = -\varphi(k_j)$, hence

$$|a_j a_{ij}|_G = |a_j|_G + |a_{ij}|_G = g \quad \text{and} \quad \sum_{i,j} a_j a_{ij} b_{ij} = \sum_j a_j b_j = 1,$$

which implies that $A$ is a strongly $G$-graded algebra, as required. □

In the context of algebras discussed in the main body of this paper, one can take $A = A(p; q^\pm), G = K = \mathbb{Z}, K = \mathbb{Z}/k\mathbb{Z}$, the epimorphism $\pi : \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}, n \mapsto n \mod k$, and its kernel monomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}, n \mapsto kn$. Then the $\mathbb{Z}$-graded subalgebra induced by $\varphi$ is $A_H = A(p; q^\pm)^{(k)}$. Since the $\mathbb{Z}$-grading of $A(p; q^\pm)$ is not strong (see Remark 3.8) and the $\mathbb{Z}$-grading of $A(p; q^\pm)^{(k)}$ is strong by Theorem 3.6 Lemma [A.1] implies that the $\mathbb{Z}/k\mathbb{Z}$-grading of $A(p; q^\pm)$ is not strong.

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