Certain Fractional Proportional Integral Inequalities via Convex Functions

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Abstract: The goal of this article is to establish some fractional proportional integral inequalities for convex functions by employing proportional fractional integral operators. In addition, we establish some classical integral inequalities as the special cases of our main findings.

Keywords: convex function; fractional integrals; proportional fractional integrals; inequalities; Qi inequality

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1. Introduction

Integral inequalities play a vital role in the field of fractional differential equations. In the past few decades, researchers have paid their valuable consideration to this area. The significant developments in this area have been investigated, for example, [1–3], and [4] (cf. references cited therein). In [5], Ngo et al. established the following inequalities

\[ \int_0^1 g^{\sigma+1}(t)dt \geq \int_0^1 t^\sigma g(t)dt \]  \hspace{1cm} (1)

and

\[ \int_0^1 g^{\sigma+1}(t)dt \geq \int_0^1 tg^{\sigma}(t)dt, \]  \hspace{1cm} (2)

where \( \sigma > 0 \) and the positive continuous function \( g \) on \([0, 1]\) such that

\[ \int_x^1 g(t)dt \geq \int_x^1 t dt, x \in [0, 1]. \]

Later on, Liu et al. [6] established the following inequalities

\[ \int_a^b g^{\sigma+\gamma}(t)dt \geq \int_a^b (t-a)^\sigma g^{\gamma}(t)dt, \]  \hspace{1cm} (3)
where \( \sigma > 0, \gamma > 0 \), and the positive continuous \( g \) on \([a, b]\) is such that

\[
\int_a^b g^\delta(t)dt \geq \int_a^b (t-a)^\delta dt, \delta = \min(1, \gamma), t \in [a, b].
\]

Liu et al. [7] derived two theorems for integral inequalities as follows:

**Theorem 1.** Suppose that the functions \( f_1 \) and \( g_1 \) are positive and continuous on \([a, b]\), \((a < b)\) with \( f_1 \leq g_1 \) on \([a, b]\) such that the function \( \frac{f_1}{g_1} \) \((g_1 \neq 0)\) is decreasing and the function \( f_1 \) is increasing. Assume that the function \( \Phi \) is a convex with \( \Phi(0) = 0 \). Then, the following inequality holds

\[
\frac{\int_a^b f_1(t)dt}{\int_a^b g_1(t)dt} \geq \frac{\int_a^b \Phi(f_1(t))dt}{\int_a^b \Phi(g_1(t))dt}.
\]

**Theorem 2.** Suppose that the functions \( f_1, f_2, \) and \( f_3 \) be positive and continuous on \([a, b]\), \((a < b)\) with \( f_1 \leq f_2 \) on \([a, b]\) such that the function \( \frac{f_1}{f_2} \) \((f_2 \neq 0)\) is decreasing and the functions \( f_1 \) and \( f_3 \) are increasing. Assume that the function \( \Phi \) is a convex with \( \Phi(0) = 0 \). Then, the following inequality holds

\[
\frac{\int_a^b f_1(t)dt}{\int_a^b f_2(t)dt} \geq \frac{\int_a^b \Phi(f_1(t)) f_3(t)dt}{\int_a^b \Phi(f_2(t)) f_3(t)dt}.
\]

The inequalities in Equations (1)–(3) and their various generalizations have gained attention of the researchers [8–12].

Furthermore, the research of fractional integral inequalities is also of prominent importance. In [13,14], the authors presented some weighted Grüss type and new inequalities involving Riemann–Liouville (R-L) fractional integrals. In [15], Nisar et al. introduced many inequalities for extended gamma and confluent hypergeometric \( k \)-functions. Certain Gronwall inequalities for R-L and Hadamard \( k \)-fractional derivatives with applications are observed in [16]. The inequalities concerning the generalized \((k, \rho)\)-fractional integral operators can be seen in [17].

The generalized fractional integral and Grüss type inequalities via generalized fractional integrals can be found in [18,19]. In [20], the authors examined the \((k, s)\)-R-L fractional integral and its applications. In [21], the authors presented generalized Hermite–Hadamard type inequalities through fractional integral operators. Dahmani [22] introduced some classes of fractional integral inequalities by employing a family of \( n \) positive functions. Further the applications of fractional integral inequalities can be found [23,24].

In the last few decades, the researchers have paid their valuable consideration to the field of fractional calculus. This field has received more attention from various researchers due to its wide applications in various fields. In the growth of fractional calculus, researchers concentrate to develop several fractional integral operators and their applications in distinct fields (see, e.g., [25–33]). Zaher et al. [34] presented a new fractional nonlocal model.

Such types of these new fractional integral operators promote the future study to develop certain new approaches to unify the fractional operators and secure fractional integral inequalities. Especially, several striking inequalities, properties, and applicability for the fractional conformable integrals and derivatives are recently studied by various researchers. We refer the interesting readers to the works by [35–44], and [45]. The applications of conformable derivative can be found in [46–49] (cf. references cited therein).
2. Preliminaries

Jarad et al. [50] proposed the following left and right generalized proportional integral operators, which are sequentially defined by

\[
\left( aJ\xi,\delta f \right)(\tau) = \frac{1}{\delta^\xi \Gamma(\xi)} \int_a^\tau \exp\left[\frac{\delta - 1}{\delta}(\tau - t)\right](\tau - t)^{\xi - 1}f(t)dt, a < \tau
\]

(4)

and

\[
\left( J\xi,\delta b f \right)(\tau) = \frac{1}{\delta^\xi \Gamma(\xi)} \int_b^\tau \exp\left[\frac{\delta - 1}{\delta}(t - \tau)\right](t - \tau)^{\xi - 1}f(t)dt, \tau < b,
\]

(5)

where the proportional index \(\delta \in (0, 1]\) and \(\xi \in \mathbb{C}\) with \(\text{Re}(\xi) > 0\), and \(\Gamma(\tau)\) is the well-know gamma function defined by \(\Gamma(\tau) = \int_0^\infty t^{\tau - 1}e^{-t}dt\) [51–53].

**Remark 1.** Setting \(\delta = 1\) in Equations (4) and (5), we obtain the following left and right R-L:

\[
\left( aJ\xi f \right)(\tau) = \frac{1}{\Gamma(\xi)} \int_a^\tau (\tau - t)^{\xi - 1}f(t)dt, a < \tau,
\]

and

\[
\left( J\xi b f \right)(\tau) = \frac{1}{\Gamma(\xi)} \int_\tau^b (t - \tau)^{\xi - 1}f(t)dt, \tau < b,
\]

where \(\xi \in \mathbb{C}\) with \(\text{Re}(\xi) > 0\).

Recently, the generalized proportional derivative, and integral operators are established and studied in [54,55]. Certain new classes of integral inequalities for a class of \(n (n \in \mathbb{N})\) positive continuous and decreasing functions on \([a, b]\) via generalized proportional fractional integrals can be found in the work of Rahman et al. [56]. The generalized Hadamard proportional fractional integrals and certain inequalities for convex functions by employing were recently proposed by Rahman et al. [57]. The bounds of proportional integrals in the sense of another function can be found in the work of Rahman et al. [58].

3. Main Results

In this section, we establish proportional fractional integral inequalities for convex functions by employing proportional fractional integral operators.

**Theorem 3.** Suppose that the functions \(f\) and \(g\) are positive and continuous on the interval \([a, b]\), \(a < b\) and \(f \leq g\) on \([a, b]\). If the function \(\frac{f}{g}\) \(\not= 0\) is decreasing and the function \(f\) is increasing on \([a, b]\), then, for any convex function \(\Phi\) with \(\Phi(0) = 0\), the following inequality satisfies the proportional fractional integral operator given by Equation (4)

\[
\frac{aJ^{\xi,\delta}[f(\tau)]}{aJ^{\xi,\delta}[g(\tau)]} \geq \frac{aJ^{\xi,\delta}[\Phi(f(\tau))]}{aJ^{\xi,\delta}[\Phi(g(\tau))]},
\]

(6)

where \(\delta \in (0, 1], \xi \in \mathbb{C}\) with \(\text{Re}(\xi) > 0\).
Proof. Since $\Phi$ is convex function with $\Phi(0) = 0$, the function $\frac{\Phi(t)}{t}$ is increasing. As $f$ is increasing, the function $\frac{\Phi(f(t))}{f(t)}$ is also increasing. Obviously, $\frac{\Phi(t)}{t}$ is decreasing function. Thus, for all $\rho, \theta \in [a, b]$, we have

$$
\left( \frac{\Phi(f(\rho))}{f(\rho)} - \frac{\Phi(f(\theta))}{f(\theta)} \right) \left( \frac{f(\theta)}{g(\theta)} - \frac{f(\rho)}{g(\rho)} \right) \geq 0.
$$

It follows that

$$
\frac{\Phi(f(\rho))}{f(\rho)} f(\theta) - \frac{\Phi(f(\theta))}{f(\theta)} f(\rho) = \frac{\Phi(f(\theta))}{f(\theta)} f(\theta) g(\theta) - \frac{\Phi(f(\rho))}{f(\rho)} f(\rho) g(\theta) \geq 0.
$$

Multiplying Equation (7) by $g(\rho)g(\theta)$, we have

$$
\frac{\Phi(f(\rho))}{f(\rho)} f(\theta) g(\rho) + \frac{\Phi(f(\theta))}{f(\theta)} f(\rho) g(\rho) - \frac{\Phi(f(\theta))}{f(\theta)} f(\theta) g(\rho) - \frac{\Phi(f(\rho))}{f(\rho)} f(\rho) g(\theta) \geq 0. \quad (8)
$$

Multiplying Equation (8) by $\frac{1}{\delta \Gamma(\xi)} \exp\left[\frac{\delta - 1}{\delta} (\tau - \rho)\right] (\tau - \rho)^{-\xi - 1}$, and integrating with respect to $\rho$ over $[a, \tau], a < \tau \leq b$, we have

$$
\frac{1}{\delta \Gamma(\xi)} \int_a^\tau \exp\left[\frac{\delta - 1}{\delta} (\tau - \rho)\right] (\tau - \rho)^{-\xi - 1} \frac{\Phi(f(\rho))}{f(\rho)} f(\theta) g(\rho) d\rho + \frac{1}{\delta \Gamma(\xi)} \int_a^\tau \exp\left[\frac{\delta - 1}{\delta} (\tau - \rho)\right] (\tau - \rho)^{-\xi - 1} \frac{\Phi(f(\theta))}{f(\theta)} f(\rho) g(\theta) d\rho
$$

$$
- \frac{1}{\delta \Gamma(\xi)} \int_a^\tau \exp\left[\frac{\delta - 1}{\delta} (\tau - \rho)\right] (\tau - \rho)^{-\xi - 1} \frac{\Phi(f(\theta))}{f(\theta)} f(\theta) g(\rho) d\rho
$$

$$
- \frac{1}{\delta \Gamma(\xi)} \int_a^\tau \exp\left[\frac{\delta - 1}{\delta} (\tau - \rho)\right] (\tau - \rho)^{-\xi - 1} \frac{\Phi(f(\rho))}{f(\rho)} f(\rho) g(\theta) d\rho \geq 0.
$$

Then, it follows that

$$
f(\theta) a J^{\xi,\delta}_a \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) \right) + \left( \frac{\Phi(f(\theta))}{f(\theta)} g(\theta) \right) a J^{\xi,\delta}_a \left( f(\tau) \right)
$$

$$
- \left( \frac{\Phi(f(\theta))}{f(\theta)} f(\theta) \right) a J^{\xi,\delta}_a \left( g(\tau) \right) - g(\theta) a J^{\xi,\delta}_a \left( \frac{\Phi(f(\tau))}{f(\tau)} \right) \geq 0. \quad (9)
$$

Again, multiplying both sides of Equation (9) by $\frac{1}{\delta \Gamma(\xi)} \exp\left[\frac{\delta - 1}{\delta} (\tau - \rho)\right] (\tau - \rho)^{-\xi - 1}$, and integrating the resultant inequality with respect to $\theta$ over $[a, \tau], a < \tau \leq b$, we get

$$
a J^{\xi,\delta}_a \left( f(\tau) \right) a J^{\xi,\delta}_a \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) \right) + a J^{\xi,\delta}_a \left( \frac{\Phi(f(\theta))}{f(\theta)} g(\theta) \right) a J^{\xi,\delta}_a \left( f(\tau) \right)
$$

$$
\geq a J^{\xi,\delta}_a \left( g(\tau) \right) a J^{\xi,\delta}_a \left( \Phi(f(\tau)) \right) + a J^{\xi,\delta}_a \left( f(\tau) \right) a J^{\xi,\delta}_a \left( g(\tau) \right).
$$

It follows that

$$
a J^{\xi,\delta}_a \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) \right) \geq a J^{\xi,\delta}_a \left( \frac{\Phi(f(\tau))}{f(\tau)} \right). \quad (10)
$$

Now, since $f \leq g$ on $[a, b]$ and $\frac{\Phi(t)}{t}$ is an increasing function, for $\rho \in [a, \tau], a < \tau \leq b$, we have

$$
\frac{\Phi(f(\rho))}{f(\rho)} \leq \frac{\Phi(g(\rho))}{g(\rho)}. \quad (11)
$$
Multiplying both sides of Equation (11) by \( \frac{1}{\delta^\xi \Gamma(\xi)} \exp\left[\frac{\delta-1}{\delta}(\tau - \rho)\right](\tau - \rho)^{-1}g(\rho) \) and integrating the resultant inequality with respect to \( \rho \) over \([a, \tau], a < \tau \leq b\), we get

\[
\frac{1}{\delta^\xi \Gamma(\xi)} \int_a^\tau \exp\left[\frac{\delta-1}{\delta}(\tau - \rho)\right](\tau - \rho)^{-1} \Phi(f(\rho)) f(\rho) g(\rho) d\rho \\
\leq \frac{1}{\delta^\xi \Gamma(\xi)} \int_a^\tau \exp\left[\frac{\delta-1}{\delta}(\tau - \rho)\right](\tau - \rho)^{-1} \Phi(g(\rho)) g(\rho) d\rho,
\]

which, in view of Equation (4), can be written as

\[
aJ_{\xi, \delta}^\xi \left( \frac{\Phi(f(\tau))}{f(\tau)} \right) g(\tau) \leq aJ_{\xi, \delta}^\xi \left( \frac{\Phi(g(\tau))}{g(\tau)} \right). \tag{12}
\]

Hence, from Equations (10) and (12), we get Equation (6). \( \square \)

**Remark 2.** Applying Theorem 3 for \( \delta = 1 \), we get Theorem 3.1 proved by [59].

**Remark 3.** Applying Theorem 3 for \( \xi = \delta = 1 \) and \( x = b \), we get Theorem 1.

**Theorem 4.** Suppose that the functions \( f \) and \( g \) are positive and continuous on \([a, b], (a < b) \) and \( f \leq g \) on \([a, b] \). If the function \( \frac{f(\tau)}{g(\tau)} \) is decreasing and the function \( f \) is increasing on \([a, b] \), then, for any convex function \( \Phi \) with \( \Phi(0) = 0 \), the following inequality satisfies the proportional fractional integral operator given by Equation (4)

\[
aJ_{\xi, \delta}^\xi \left[ f(\tau) \right] aJ_{\lambda, \delta}^\lambda \left[ \Phi(g(\tau)) \right] + aJ_{\xi, \delta}^\xi \left[ f(\tau) \right] aJ_{\lambda, \delta}^\lambda \left[ \Phi(g(\tau)) \right] + aJ_{\lambda, \delta}^\xi \left[ \Phi(g(\tau)) \right] \geq 1,
\]

where \( \delta \in (0, 1], \xi, \lambda \in \mathbb{C} \) with \( \text{Re}(\xi) > 0 \) and \( \text{Re}(\lambda) > 0 \).

**Proof.** Since \( \Phi \) is convex function with \( \Phi(0) = 0 \), the function \( \frac{f(\tau)}{g(\tau)} \) is increasing. As \( f \) is increasing, the function \( \frac{\Phi(f(\tau))}{f(\tau)} \) is also increasing. Clearly, the function \( \frac{f(\tau)}{g(\tau)} \) is decreasing for all \( \rho, \theta \in [a, \tau], a < \tau \leq b \). Multiplying Equation (9) by \( \frac{1}{\delta^\xi \Gamma(\xi)} \exp\left[\frac{\delta-1}{\delta}(\tau - \theta)\right](\tau - \theta)^{-1} \) and integrating the resultant inequality with respect to \( \theta \) over \([a, \tau], a < \tau \leq b\), we get

\[
aJ_{\lambda, \delta}^\xi \left( f(\tau) \right) aJ_{\xi, \delta}^\xi \left( \Phi(f(\tau)) \right) g(\tau) \geq aJ_{\xi, \delta}^\xi \left( g(\tau) \right) \frac{\Phi(f(\tau))}{f(\tau)} f(\tau) + aJ_{\lambda, \delta}^\xi \left( \Phi(f(\tau)) \right) g(\tau) f(\tau). \tag{13}
\]

Now, since \( f \leq g \) on \([a, b] \) and \( \frac{\Phi(\tau)}{f(\tau)} \) is an increasing function, for \( \rho \in [a, \tau], a < \tau \leq b \), we have

\[
\frac{\Phi(f(\rho))}{f(\rho)} \leq \frac{\Phi(g(\rho))}{g(\rho)}, \tag{14}
\]

Multiplying both sides of Equation (14) by \( \frac{1}{\delta^\xi \Gamma(\xi)} \exp\left[\frac{\delta-1}{\delta}(\tau - \rho)\right] f(\rho) \) and integrating the resultant inequality with respect to \( \rho \) over \([a, \tau], a < \tau \leq b\), we get

\[
\frac{1}{\delta^\xi \Gamma(\xi)} \int_a^\tau \exp\left[\frac{\delta-1}{\delta}(\tau - \rho)\right](\tau - \rho)^{-1} \Phi(f(\rho)) f(\rho) g(\rho) d\rho \\
\leq \frac{1}{\delta^\xi \Gamma(\xi)} \int_a^\tau \exp\left[\frac{\delta-1}{\delta}(\tau - \rho)\right](\tau - \rho)^{-1} \Phi(g(\rho)) g(\rho) d\rho,
\]
which, in view of Equation (4), can be written as

\[ aJ^{\frac{\xi}{\delta}, \delta} \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) \right) \leq aJ^{\frac{\xi}{\delta}, \delta} \left( \Phi(g(\tau)) \right). \]  

(15)

Similarly, one can obtain

\[ aJ^{\frac{\lambda}{\delta}, \delta} \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) \right) \leq aJ^{\frac{\lambda}{\delta}, \delta} \left( \Phi(g(\tau)) \right). \]  

(16)

Hence, from Equations (12), (13), (15), and (16), we get the desired result. □

**Remark 4.** Setting \( \xi = \lambda \), Theorem 4 leads to Theorem 3.

**Remark 5.** Applying Theorem 4 for \( \delta = 1 \), we get Theorem 3.3 proved by Dahmani [59].

**Theorem 5.** Suppose that the functions \( f, h, \) and \( g \) are positive and continuous on \([a, b], (a < b)\) and \( f \leq h \) on \([a, b] \). If the function \( \frac{f}{\tau} \) is decreasing and the functions \( f \) and \( h \) are increasing on \([a, b] \), then, for any convex function \( \Phi \) with \( \Phi(0) = 0 \), the following inequality satisfies the proportional fractional integral operator given by Equation (4)

\[
\frac{aJ^{\xi, \delta} [f(\tau)]}{aJ^{\lambda, \delta} [g(\tau)]} \geq \frac{aJ^{\xi, \delta} [\Phi(f(\tau)) h(\tau)]}{aJ^{\lambda, \delta} [\Phi(g(\tau)) h(\tau)]},
\]

where \( \delta \in (0, 1], \xi \in \mathbb{C} \) with \( \text{Re}(\xi) > 0 \).

**Proof.** Since \( \Phi \) is convex function such that \( \Phi(0) = 0 \), the function \( \frac{\Phi(f(\tau))}{f(\tau)} \) is increasing. As the function \( f \) is increasing, \( \frac{\Phi(f(\tau))}{f(\tau)} \) is also increasing. Clearly, the function \( \frac{f(\tau)}{g(\tau)} \) is decreasing for all \( \rho, \theta \in [a, \tau], a < \tau \leq b \).

\[
\left( \frac{\Phi(f(\rho))}{f(\rho)} h(\rho) - \frac{\Phi(f(\theta))}{f(\theta)} h(\theta) \right) (f(\theta)g(\rho) - f(\rho)g(\theta)) \geq 0.
\]

It follows that

\[
\frac{\Phi(f(\rho))h(\rho)}{f(\rho)} f(\theta)g(\rho) + \frac{\Phi(f(\theta))h(\theta)}{f(\theta)} f(\rho)g(\theta) - \frac{\Phi(f(\theta))h(\theta)}{f(\theta)} f(\rho)g(\rho) - \frac{\Phi(f(\rho))h(\rho)}{f(\rho)} f(\theta)g(\theta) \geq 0. \]  

(17)

Multiplying Equation (17) by \( \frac{1}{\beta I(\xi)} \exp\left[\frac{\delta - 1}{\delta} (\tau - \rho)\right] (\tau - \rho)^{\delta - 1} \) and integrating the resultant inequality with respect to \( \rho \) over \([a, \tau], a < \tau \leq b \), we have

\[
\frac{1}{\beta I(\xi)} \int_a^\tau \exp\left[\frac{\delta - 1}{\delta} (\tau - \rho)\right] (\tau - \rho)^{\delta - 1} \frac{\Phi(f(\rho))}{f(\rho)} f(\theta)g(\rho)h(\rho) d\rho + \frac{1}{\beta I(\xi)} \int_a^\tau \exp\left[\frac{\delta - 1}{\delta} (\tau - \rho)\right] (\tau - \rho)^{\delta - 1} \frac{\Phi(f(\theta))}{f(\theta)} f(\rho)g(\theta)h(\theta) d\rho
\]

\[
- \frac{1}{\beta I(\xi)} \int_a^\tau \exp\left[\frac{\delta - 1}{\delta} (\tau - \rho)\right] (\tau - \rho)^{\delta - 1} \frac{\Phi(f(\theta))}{f(\theta)} f(\rho)g(\rho)h(\theta) d\rho - \frac{1}{\beta I(\xi)} \int_a^\tau \exp\left[\frac{\delta - 1}{\delta} (\tau - \rho)\right] (\tau - \rho)^{\delta - 1} \frac{\Phi(f(\rho))}{f(\rho)} f(\theta)h(\rho)g(\theta) d\rho \geq 0.
\]
It follows that
\[
\begin{align*}
&f(\theta) a^{\mathcal{H}}(\mathbf{f}(\tau))h(\tau) + \left( \frac{\Phi(f(\theta))}{f(\theta)} - g(\theta) h(\theta) \right) a^{\mathcal{H}}(f(\tau)) \\
&\quad - \left( \frac{\Phi(f(\theta))}{f(\theta)} - g(\theta) a^{\mathcal{H}}(f(\tau)) \right) a^{\mathcal{H}}(g(\tau)) - g(\theta) a^{\mathcal{H}}(f(\tau)) f(\tau) h(\tau) \geq 0.
\end{align*}
\]

(18)

Again, multiplying both sides of Equation (18) by \[\frac{1}{\delta \Gamma(\xi)} \exp[\frac{\delta - 1}{\delta} (\tau - \theta)] \right] (\tau - \theta)^{\zeta - 1} \] and integrating the resultant inequality with respect to \(\theta\) over \([a, \tau]\), we get
\[
a^{\mathcal{H}}(f(\tau)) a^{\mathcal{H}}(g(\tau)) + a^{\mathcal{H}}(f(\tau)) g(\tau) h(\tau) a^{\mathcal{H}}(f(\tau)) g(\tau) h(\tau) \geq a^{\mathcal{H}}(g(\tau)) a^{\mathcal{H}}(f(\tau)) h(\tau) + a^{\mathcal{H}}(f(\tau)) h(\tau) a^{\mathcal{H}}(g(\tau)).
\]

It follows that
\[
a^{\mathcal{H}}(f(\tau)) a^{\mathcal{H}}(g(\tau)) \geq a^{\mathcal{H}}(f(\tau)) h(\tau) a^{\mathcal{H}}(g(\tau)).
\]

(19)

In addition, since \(f \leq g\) on \([a, b]\) and \(\Phi(\tau)\) is an increasing function, for \(\eta, \theta \in [a, b]\), we have
\[
\frac{\Phi(f(\eta))}{f(\eta)} \leq \frac{\Phi(g(\eta))}{g(\eta)}.
\]

(20)

Multiplying both sides of Equation (20) by \[\frac{1}{\delta \Gamma(\xi)} \exp[\frac{\delta - 1}{\delta} (\tau - \eta)] \right] (\tau - \eta)^{\zeta - 1} \] and integrating the resultant inequality with respect to \(\eta\) over \([a, \tau]\), we get
\[
\frac{1}{\delta \Gamma(\xi)} \int_a^\tau \exp[\frac{\delta - 1}{\delta} (\tau - \eta)] \right] (\tau - \eta)^{\zeta - 1} \Phi(f(\eta)) d\eta
\leq \frac{1}{\delta \Gamma(\xi)} \int_a^\tau \exp[\frac{\delta - 1}{\delta} (\tau - \eta)] \right] (\tau - \eta)^{\zeta - 1} \Phi(g(\eta)) d\eta,
\]

which, in view of Equation (4), can be written as
\[
a^{\mathcal{H}}(f(\tau)) g(\tau) h(\tau) \leq a^{\mathcal{H}}(g(\tau)) h(\tau).
\]

(21)

Hence, from Equations (21) and (19), we obtain the required result. \(\square\)

Remark 6. Applying Theorem 5 for \(\delta = 1\), we get Theorem 3.5 proved by Dahmani [59].

Remark 7. Applying Theorem 5 for \(\delta = \zeta = 1\) and \(x = b\), we get Theorem 2.

Theorem 8. Suppose that the functions \(f, h,\) and \(g\) are positive and continuous on \([a, b]\), \((a < b)\) and \(f \leq g\) on \([a, b]\). If the function \(\frac{f}{g}\) is decreasing and the functions \(f\) and \(h\) are increasing on \([a, b]\), then, for any convex function \(\Phi\) with \(\Phi(0) = 0\), the following inequality satisfies the proportional fractional integral operator given by Equation (4)
\[
\frac{a^{\mathcal{H}}[f(\tau)] + a^{\mathcal{H}}[g(\tau)] + a^{\mathcal{H}}[h(\tau)]}{a^{\mathcal{H}}[f(\tau)] + a^{\mathcal{H}}[g(\tau)] + a^{\mathcal{H}}[h(\tau)]} \geq 1,
\]

(22)

where \(\delta \in (0, 1], \zeta, \lambda \in \mathbb{C}\) with \(Re(\zeta) > 0\) and \(Re(\lambda) > 0\).
Proof. Multiplying both sides of Equation (18) by $\frac{1}{\delta(1-\delta)} \exp\left[\frac{\delta-1}{\delta} (\tau - \theta)\right] (\tau - \theta)^{1-\delta}$ and integrating the resultant inequality with respect to $\theta$ over $[a, \tau], a < \tau \leq b$, we get

$$\int_a^\tau \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) h(\tau) \right) d\tau \geq \int_a^\tau \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) h(\tau) \right) d\tau \geq \int_a^\tau \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) h(\tau) \right) d\tau.$$ (23)

Since $f \leq g$ on $[a, b]$ and $\frac{\Phi(\tau)}{\tau}$ is an increasing function, for $\eta, \theta \in [1, x], a < \tau \leq b$, we have

$$\frac{\Phi(f(\eta))}{f(\eta)} \leq \frac{\Phi(g(\eta))}{g(\eta)}.$$ (24)

Multiplying both sides of Equation (24) by $\frac{1}{\delta(1-\delta)} \exp\left[\frac{\delta-1}{\delta} (\tau - \eta)\right] (\tau - \eta)^{1-\delta} g(\eta) h(\eta), \eta \in [a, x], a < \tau \leq b$ and integrating the resultant inequality with respect to $\eta$ over $[a, \tau], a < \tau \leq b$, we get

$$\int_a^\tau \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) h(\tau) \right) d\tau \leq \int_a^\tau \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) h(\tau) \right) d\tau \leq \int_a^\tau \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) h(\tau) \right) d\tau.$$ (25)

Similarly, one can obtain

$$\int_a^\tau \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) h(\tau) \right) d\tau \leq \int_a^\tau \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) h(\tau) \right) d\tau \leq \int_a^\tau \left( \frac{\Phi(f(\tau))}{f(\tau)} g(\tau) h(\tau) \right) d\tau.$$ (26)

Hence, from Equations (23), (25), and (26), we obtain the required inequality in Equation (22).

Remark 8. If we consider $\zeta = \lambda$, then Theorem 6 leads to Theorem 5.

Remark 9. Applying Theorem 6 for $\delta = 1$, we get Theorem 3.7 of Dahmani [59].

4. Concluding Remarks

Some interesting integral inequalities for convex functions were presented by Liu et al. ([7] Theorems 9 and 10). Later, Dahmani [59] improved these integral inequalities by utilizing the R-L fractional integral operator. Here, we present some new fractional proportional integral inequalities for convex functions by utilizing the proportional fractional integrals. In fact, we established the inequalities presented in Theorem 1 and Theorem 2 using the fractional proportional integrals, which are nonlocal and their orders depend on two indices: $\delta$, which is the proportional index, and $\xi$, which is the iterated index.

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