Exact ground states of one-dimensional long-range random-field Ising magnets

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(Dated: May 7, 2014)

We investigate the one-dimensional long-range random-field Ising magnet with Gaussian distribution of the random fields. In this model, a ferromagnetic bond between two spins is placed with a probability \( p \sim r^{-1-\sigma} \), where \( r \) is the distance between these spins and \( \sigma \) is a parameter to control the effective dimension of the model. Exact ground states at zero temperature are calculated for system sizes up to \( L = 2^{19} \) via graph theoretical algorithms for four different values of \( \sigma \in \{0.25, 0.4, 0.5, 1.0\} \) while varying the strength \( h \) of the random fields. For each of these values several independent physical observables are calculated, i.e., magnetization, Binder parameter, susceptibility and a specific-heat-like quantity. The ferromagnet-paramagnet transitions at critical values \( h_c(\sigma) \) as well as the corresponding critical exponents are obtained. The results agree well with theory, but for \( \sigma = 1/2 \) the critical random-field strength \( h_c > 0 \) in contrast to what was expected from analytical studies.

INTRODUCTION

The critical behavior of spin systems with quenched disorder is even today far from being well understood in contrast to pure models. Such a system with quenched disorder is the random-field Ising model (RFIM), where the spins interact ferromagnetically with each other and additionally a quenched random field with strength \( h \) acts locally on the spins. In short-range models, it is known that the proposed equivalence of the interaction strength \( d \sim r^{-d-\sigma} \) of the critical behavior of a \( d \)-dimensional RFIM and a \((d-2)\)-dimensional pure ferromagnet does not exist. A lower critical dimension of \( d_c = 3 \) for the RFIM resulting from the \( d \rightarrow (d-2) \)-rule was shown to be wrong. The correct value of \( d_c = 2 \) was found by Imry and Ma using their famous domain-wall argument and later proven mathematically by Bricmont and Kupiainen.

A generalization of the short-range model are random-field Ising magnets with long-range interactions \( J(r) \sim r^{-d-\sigma} \), the interaction strength \( J \) decays like a power-law in the distance \( r \). The exponent \( \sigma \) allows the tuning of the effective dimensionality of the model, allowing also for non-integer dimensions. Similar long-range spin glass models, i.e., with bond disorder, have been studied recently quite intensively for the case of the fully connected model as well as for the diluted case. For the random-field Ising model, it turned out that the proposed \( d \rightarrow (d-\sigma) \)-equivalence is analogous to the \( d \rightarrow (d-2) \)-rule for short-range models, is wrong at higher orders of the pertubative expansion. A related model is the ferromagnetic hierarchical spin model introduced by Dyson, where the interaction strength decays exponentially with the level of the hierarchy. This model is solvable with exact renormalization and the hierarchical couplings are equivalent to long-range power-law couplings in real space. Because of this equivalence, the critical behavior of the Dyson hierarchical model with random fields is expected to be the same as for one-dimensional long-range models with power-law interactions.

Further analyses of the RFIM with long-range interactions with renormalization-group theory or with mathematical tools have been performed. These references suggest that there is no phase transition at zero temperature at a finite value of the disorder strength for \( \sigma = 1/2 \) (and above). This result is obtained by a scaling argument similar to the Imry-Ma argument. Here, we use a slightly different model, where the couplings are random and only present with a certain probability but the interaction strength \( J \) has a fixed value. A central question is to find out whether there is a finite-disorder phase transition for the model studied here at zero temperature for the borderline case \( \sigma = 1/2 \). For comparison we also consider few other selected values of \( \sigma \). In parallel and independently of our work, the same question was tackled via considering the Binder parameter only. For the present work, we consider beyond this a full set of independent physical quantities, also involving the susceptibility and a specific-heat-like quantity, to study the disorder-driven phase transitions and to obtain complete sets of critical exponents.

The outline of this article is the following: First, the model is described, second the procedure to obtain a ground state for a given realization of the disorder is briefly outlined and third the physical observables and their expected scaling behaviors are explained. Next, results for the four investigated values of \( \sigma \) are presented. Last, a conclusion which includes a comparison of the results with scaling relations and an outlook is drawn.

MODEL

We study one-dimensional random-field Ising magnets with power-law diluted interactions. Instead of all-to-all coupling, where the interaction strength decays with a power law in the distance, we use diluted interactions with fixed coupling strength, which recently
have been used for spin glasses [13, 14]. The Hamiltonian of the model used here is
\[ \mathcal{H} = -J \sum_{i<j} \varepsilon_{ij} S_i S_j - \sum_i (B_i + H) S_i, \]
where \( J > 0 \) is the ferromagnetic coupling strength and the \( S_i = \pm 1 \) are Ising spins distributed on a ring with circumference \( L \) (cf. Fig. 1). \( B_i \) are the local random fields drawn from a Gaussian distribution with zero mean:
\[ p(B_i) = \frac{1}{\sqrt{2\pi h^2}} \exp \left( -\frac{B_i^2}{2h^2} \right), \]
where the width \( h \) of the distribution controls the disorder strength. The external homogeneous field of the model used here is \( H = 0 \) except for the determination of the susceptibility, where small fields are needed, for technical reasons. The critical behavior of a Gaussian RFIM along the phase boundary is controlled by the zero-temperature fixed point [26]. Therefore, it is convenient to study the RFIM at \( T = 0 \) and to alter the random field strength \( h \) to cross the phase boundary (see arrow in Fig. 1). For the calculation of the exact ground state at \( T = 0 \) for a given realisation the undirected graph is mapped to a directed network [27]. The maximum flow on this network is then calculated using a Push-and-Relabel algorithm [28], whereof an efficient implementation exists in the LEDA-library [29]. These algorithms have a polynomial running time [30] and are faster than Monte-Carlo simulations (see e.g. [31]), because no equilibration time is needed and the ground state is exact. After one has obtained the maximum flow, the directed network is mapped back to a ground-state spin configuration.

More details about the mapping to a directed network can be found in [25].

OBSERVABLES

After obtaining the spin configuration of a ground state, we calculate physical quantities of interest. First, we fix \( H = 0 \) and use \( H > 0 \) only for the calculation of the susceptibility. The average magnetization per spin is given by
\[ m = \langle |M| \rangle_h = \left\lfloor \frac{1}{N} \sum_i S_i \right\rfloor_h, \]
where \( N \equiv L \) is the number of spins and \( \lfloor \cdot \rfloor_h \) denotes average over disorder. This averaging for fixed \( h \) is performed over different realisations of graphs and random

FIG. 1: Left: One-dimensional spin-ring with \( L = 12 \) Ising spins. Right: Phase diagram of the Gaussian RFIM (corresponding to \( \mathcal{P}^w \)), where \( \mathcal{P}^w \) denotes the ferromagnetic and \( \mathcal{P}^p \) the paramagnetic phase, both separated by the phase boundary.

The universality class of the model can be changed by varying \( \sigma \). For \( 0 < \sigma < 1/3 \) the critical exponents assume their mean-field (MF) values and for \( 1/3 < \sigma < 1/2 \) the model is assumed to be in the non-MF region [20]. If \( \sigma \geq 1/2 \), one expects no phase transition [20, 21, 22], i.e. the critical random-field strength \( h_c = 0 \) for \( T = 0 \).

The MF values [17, 20, 21] of the critical exponents are \( \alpha = 0, \beta = 1/2, \gamma = 1 \) and \( \nu = 1/\sigma \). In the non-MF domain, i.e. \( 1/3 < \sigma < 1/2 \) the correlation length exponent \( \nu \) is not known exactly, so only the relations [20]
\[ \frac{2-\alpha}{\nu} = 1 - \sigma, \quad \frac{\beta}{\nu} = \frac{1}{2} - \sigma, \quad \frac{\gamma}{\nu} = \sigma \]
are known analytically exact. But if, e.g., \( \alpha \) is known \( (\alpha = 0 \) seems plausible from the results presented below), the first relation in Eqs. (1) allows the determination of \( \nu \) and thus of the other exponents.

Here, we focus on \( \sigma = 0.25 \), which belongs to the MF region, \( \sigma = 0.4 \) corresponding to the non-MF domain, \( \sigma = 1/2 \) right at the predicted border between non-MF region and the domain without a phase transition and \( \sigma = 1 \) from the \( h_c = 0 \) region.

The constant \( A \) where the width \( h \) of the distribution controls the disorder strength. The external homogeneous field is \( H = 0 \) except for the determination of the susceptibility, where small fields are needed, for technical reasons. The critical behavior of a Gaussian RFIM along the phase boundary is controlled by the zero-temperature fixed point [26]. Therefore, it is convenient to study the RFIM at \( T = 0 \) and to alter the random field strength \( h \) to cross the phase boundary (see arrow in Fig. 1). For the calculation of the exact ground state at \( T = 0 \) for a given realisation the undirected graph is mapped to a directed network [27]. The maximum flow on this network is then calculated using a Push-and-Relabel algorithm [28], whereof an efficient implementation exists in the LEDA-library [29]. These algorithms have a polynomial running time [30] and are faster than Monte-Carlo simulations (see e.g. [31]), because no equilibration time is needed and the ground state is exact. After one has obtained the maximum flow, the directed network is mapped back to a ground-state spin configuration.

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fields \( \{B_i\} \), where for each configuration of long-range bonds one random-field realisation is used.

From finite-size scaling (FSS) (see e.g. [32]) one knows that near the critical point \( h_c \) the magnetization should scale like

\[ m(h) = L^{-\beta/\nu} \tilde{m}([h - h_c] L^{1/\nu}), \]

with some scaling function \( \tilde{m} \).

The Binder cumulant [33] is calculated via

\[ g(L, h) = \frac{1}{2} \left( 3 - \frac{[M^4]_h}{[M^2]_h^2} \right), \tag{3} \]

where in comparison to the original quantity the thermal average is omitted, because \( T = 0 \) and the ground state is nondegenerate for a Gaussian RFIM.

Close to the critical point, being a dimension-less quantity, the Binder parameter is assumed to have the following scaling behavior:

\[ g(L, h) = \tilde{g}([h - h_c] L^{1/\nu}). \]

To determine a specific-heat-like quantity [34] at \( T = 0 \) we measure the bond energy

\[ E_J = \frac{1}{N} \sum_{i<j} \varepsilon_{ij} S_i S_j. \]

Now, we are able to differentiate \( E_J \) numerically with respect to \( h \) by calculating a finite central difference

\[ C \left( \frac{h_1 + h_2}{2} \right) = \frac{[E_J(h_1)]_h - [E_J(h_2)]_h}{h_1 - h_2}, \tag{4} \]

which results in the specific-heat-like quantity \( C \). The values \( h_1 \) and \( h_2 \) are two consecutive values of the random-field strength \( h \), which have to be chosen appropriately.

The scaling behavior of the singular part of the specific-heat-like quantity is

\[ C(h) = L^{\alpha/\nu} \tilde{C}([h - h_c] L^{1/\nu}). \tag{5} \]

The disconnected susceptibility is given by

\[ \chi_{\text{dis}} = L^d \langle M^2 \rangle_h, \tag{6} \]

in which \( d = 1 \) in our case.

Finite-size scaling predicts for the disconnected susceptibility

\[ \chi_{\text{dis}}(h) = L^{7/\nu} \tilde{\chi}_{\text{dis}}([h - h_c] L^{1/\nu}). \tag{7} \]

For the determination of the susceptibility five different field strengths \( H_n = n \cdot H_L \) with \( n \in \{0, 4\} \) of the homogeneous external field are applied to the system for each realisation and each value of \( h \). A parabolic fit (for details see [33]) to the datapoints yields the zero-field susceptibility

\[ \chi = \frac{\text{dm}}{dH} \bigg|_{H=0}, \]

which is given by the slope of the parabola at \( H = 0 \).

The scaling behavior for the susceptibility is expected to be

\[ \chi(h) = L^{\gamma/\nu} \tilde{\chi}([h - h_c] L^{1/\nu}). \]

RESULTS

Next, we present the simulation results for the different values of \( \sigma \in \{0.25, 0.4, 0.5, 1\} \). System sizes from \( L = 2^6 = 64 \) up to \( L = 2^{19} = 524288 \) spins and \( 10^3 \) to \( 10^6 \) samples were used. All shown data points are averages over the given number of samples and the statistical errors result from the bootstrap resampling method [36]. The average number of long-range bonds per node is fixed to \( z = 6 \). For the determination of the susceptibility, the applied field stride \( H_L \) of the homogeneous field is shown in Tab. I.

![FIG. 2: Average magnetization as a function of random-field strength \( h \) for different system sizes \( L \) and \( \sigma = 1/4 \). Data points are averaged over at least \( 10^3 \) samples and error bars result from 30 bootstrap samples. Lines are guides to the eyes only.](image)

**Mean-field region \( \sigma = 0.25 \)**

Figure 2 shows the average magnetization per spin calculated by formula (2) as a function of disorder strength
$h$. For small $h$ the system is in the ferromagnetic ordered phase, where $m(h) \approx 1$ and for larger values of the random-field strength the system is in the paramagnetic phase, where $h \to 0$. With increasing $L$ the curves get steeper suggesting a phase transition at a critical value of $h_c \approx 5$.

To determine this critical random-field strength more accurate, we calculate the Binder parameter, given in equation (4). Finite-size scaling theory predicts an intersection of the curves for the Binder cumulant for different system sizes at the critical point $h_c$. This can be seen in Fig. 3 from which we estimate $h_c \approx 5.1$.

Next, we investigate the specific-heat-like quantity $C$, where we choose $h$ values with distance $h_1 - h_2 = 0.1$ in equation (4). Figure 4 shows the peaks of $C$ close to the critical point for different system sizes. One can observe that with increasing system size $L$ the peak height grows as well as the peak position shifts to larger values of $h$.

This impression is confirmed by Fig. 5. Apparently, both the peak heights and the peak positions behave like a power-law with added constant as a function of the number of spins $L$: In fact we tested three different possible behaviors of the peak heights of the specific-heat-like quantity:

$$C^\log_{\max}(L) = a + b \ln L,$$

$$C^\text{alg}_{\max}(L) = c \cdot (1 + d \cdot L^k),$$

$$C^\text{corr alg}_{\max}(L) = c_2 L^{\alpha/\nu} \cdot (1 + d_2 \cdot L^{k_2}),$$

a logarithmic divergence, an algebraic behavior and an algebraic function with a correction term.

All fits are least-squares fits with a reduced chisquare of $\chi^2_{\text{red}} = \sum_i ((y_i - f(x_i))/\Delta_i)^2/n_{\text{df}}$, where the degrees of freedom of the fit are $n_{\text{df}} = n - n_{\text{param}}$, which is the difference between the number of datapoints $n$ and the number of parameters $n_{\text{param}}$ in the fit-function $f$. The datapoints $(x_i, y_i \pm \Delta_i)$ have an error of $\Delta_i$.

The logarithmic fit yields a reduced chisquare of $\chi^2_{\text{red}} \approx 200$ for system sizes $L > 256$ and $\chi^2_{\text{red}} \approx 118$ for $L > 512$, which is quite bad. A better result is obtained with the algebraic fit where $\chi^2_{\text{red}} = 6.9$ ($L > 256$) or $\chi^2_{\text{red}} = 4.2$ for $L > 512$, which is o.k. Because of these fits, a logarithmic divergence of the specific-heat-like quantity can be excluded. The fit by equation (10) does not converge for values $\alpha/\nu > 0$, so that we conclude $\alpha/\nu = 0$.

The peak positions are fitted by an algebraic function

$$h_{\max}(L) = h_c + a_2 \cdot L^{-1/\nu}.$$  \hspace{1cm} (11)

Due to the change of curvature of the data, see inset of Fig. 5 for the fit only system sizes $L > 2048$ were used. The fit by formula (11) gives $\chi^2_{\text{red}} = 9.6$, $h_c = 5.13 \pm 0.10$ and $1/\nu = 0.215 \pm 0.071$.

Figure 6 shows the maxima of the zero-field susceptibility $\chi$, where the smallest external fields $H_L$, were used to determine this quantity are given in Tab. 1.

It seems that the larger the system size $L$, the larger the peak height of $\chi$ and the (slightly) more the peak position is at larger values of $h$. This behavior is shown in Fig. 7 where the maxima are expected to increase like

$$\chi_{\max}(L) = a_3 \cdot L^{\gamma/\nu}.$$  \hspace{1cm} (12)
A fit to the data with fixed mean-field value $\gamma/\nu = 0.25$ yields a reduced chisquare of $\chi_{red}^2 \approx 5000$. If one takes finite-size corrections into account and uses

$$\chi_{\text{max}}(L) = a_4 \cdot L^{\gamma/\nu} \cdot (1 + d_3 \cdot L^{k_3})$$

(13)

again with fixed mean-field value $\gamma/\nu = 0.25$ and $k_3 = -2 \cdot 10^{-3} \pm 7 \cdot 10^{-3}$, one gets $\chi_{\text{red}}^2 = 0.5$. This reduced chisquare value is much smaller than for a fit without corrections. Thus, the mean-field value of $\gamma/\nu$ seems to be appropriate.

The fits to the peak positions of the susceptibility are shown in the inset of Fig. 4. A fit by formula (11) with fixed mean-field value $1/\nu = 0.25$ yields $\chi_{red}^2 = 47.3$ with free fit parameter $h_c = 4.933 \pm 0.039$.

A fit with correction term

$$h_{\text{max}}(L) = h_{c2} + a_5 \cdot L^{-1/\nu} \cdot (1 + d_4 \cdot L^{k_4})$$

(14)

and again fixed $1/\nu = 0.25$ gives $\chi_{\text{red}}^2 = 6.5$. This value is smaller than for a fit without corrections, so we keep the chosen value $1/\nu = 0.25$. Further parameters of the fit by equation (14) are $h_{c2} = 5.09 \pm 0.09$ and $k_4 = -0.52 \pm 0.50$.

Next, we perform data collapses of the observables to obtain estimates for the critical exponents with another independent approach. For the determination of the best collapse we used a python script [37]. Figure 5 shows the collapse for the Binder cumulant with parameters $h_c = 5.110 \pm 0.027$ and $1/\nu = 0.262 \pm 0.035$. Thus, the value of $\nu$ is very close to the mean-field value $16, 17, 20, 21$ $1/\nu = 0.25$. The quality of the collapse is very high below the critical point. Above the critical point, only the two smallest system sizes exhibit a notable deviation from a joint scaling curve, which can be attributed to finite-size corrections to scaling.

The data collapse of the magnetization is presented in Fig. 6. The parameters of the collapse, which has a high quality around the phase transition $h - h_c \approx 0$, have the following values $h_c = 5.127 \pm 0.014, 1/\nu = 0.252 \pm 0.016$.

| $L$ | $h_c$ | $N_{\text{samp}}$ |
|-----|-------|------------------|
| 64  | 0.300 | 100              |
| 128 | 0.065 | 10               |
| 256 | 0.050 | 5                |
| 512 | 0.039 | 5                |
| 1024| 0.030 | 5                |
| 2048| 0.023 | 5                |
| 4096| 0.018 | 5                |
| 8192| 0.014 | 5                |
| 16384| 0.011 | 5               |

TABLE I: System sizes $L$, smallest external fields $h_c$, and number of samples $N_{\text{samp}}$ which are used to determine the susceptibility for the given values of $\sigma$. 

FIG. 5: Double logarithmic plot of the peak heights of the specific-heat-like quantity $C$ as a function of system size $L$ for $\sigma = 1/4$. Dotted line denotes logarithmic fit (5) for $L > 256$ with parameters $a = 0.33(7)$, $b = 0.15(1)$ and dash-dotted line is an algebraic fit (9) also for $L > 256$ with parameters $c = 2.49(7)$, $d = -1.52(9)$ and $k = -0.17(1)$. Inset: Peak positions of $C$ as a function of $L$. Dash-dotted line denotes a fit by Eqs. (11), where $h_c = 5.13$, $a_2 = -2.7$ and $1/\nu = 0.215$.

FIG. 6: Susceptibility $\chi$ averaged over at least $10^5$ samples with error bars resulting from 30 bootstrap samples as a function of random-field strength $h$ for different system sizes $L$ and $\sigma = 1/4$. Dashed lines are example fits for three system sizes with a Gaussian and additional sigmoidal term to obtain the maxima of $\chi$. Note that for $L = 64$ and $L = 128$ $\chi$-values up to $h = 8$ were used to determine the maxima, but are omitted here for clarity of the plot.
and $\beta/\nu = 0.132 \pm 0.015$. This means $\beta = 0.524 \pm 0.093$, which is compatible within error bars with the mean-field value.

To check our results for the specific-heat-like quantity, we used equation (9) to collapse the data. The result is shown in Fig. 10, where the important parameters $h_c = 5.092 \pm 0.030$, $1/\nu = 0.251 \pm 0.022$ and $k = 0.078 \pm 0.013$ were used for the data collapse.

For the data collapse of the susceptibility, which is shown in Fig. 11 formula (13) was used to scale the y-axis instead of a simple power-law. This correction made the collapse rather insensitive to the choice of $\nu$, hence we used the known mean-field value, which was anyway confirmed by the results presented above. The important parameters of the collapse are $h_c = 5.00 \pm 0.18$, $1/\nu = 0.25$ (fixed) and $\gamma/\nu = 0.237 \pm 0.052$. Finally, the data collapse of the disconnected susceptibility (not shown) for system sizes $L = 2048$ up to $L = 131072$ yields $h_c = 5.307 \pm 0.040$, $1/\nu = 0.252 \pm 0.014$ and $\gamma/\nu = 0.502 \pm 0.064$.

A summary of the results for all critical exponents is shown in Table I. We have obtained these values by averaging the results obtained by different methods, respectively. The error bars are chosen such that they include the values obtained by the different methods. This should account for systematical errors, in particular corrections to scaling.
Collapse was performed for system sizes $L = 16384$. Function to scale y-axis is formula (13) with $h_a(\sigma) = 1$ for performed simulations and analyses in the same way as discussed above. The results are summarized in Tab. II. In particular, $\gamma/\nu = 4.5(1)$ and $\beta/\nu = 0.53(13)$. One can see an intersection of all curves close to $h_c \approx 4.45$ indicating a phase transition at this point. The inset presents the data collapse of the Binder cumulant which guides to the eyes only. Inset: Data collapse of the Binder cumulant for $\sigma = 0.4$ and system sizes $L = 2048$ up to $L = 32768$. Smaller sizes are shown for comparison.

**Non-mean-field region $\sigma = 0.4$**

For the non-mean field region, we expect still a clear phase transition but with different exponents. We have performed simulations and analyses in the same way as for $\sigma = 0.25$. For brevity, we omit most plots, since they look similar as for the mean-field case.

As an example, Fig. 12 shows the Binder parameter as a function of the disorder strength $h$ for $\sigma = 0.4$. One

![FIG. 10: Data collapse of the specific-heat-like quantity for $\sigma = 1/4$. System sizes from $L = 2048$ up to $L = 131072$ were used for the collapse. Function to scale y-axis is Eq. (10) with $c = 1$.](image1)

![FIG. 11: Data collapse of the susceptibility for $\sigma = 1/4$. Collapse was performed for system sizes $L = 512$ up to $L = 16384$. Function to scale y-axis is formula (13) with $a_4 = 1$.](image2)

![FIG. 12: Binder parameter as a function of random-field strength $h$ for different system sizes $L$ and $\sigma = 0.4$. Lines are guides to the eyes only. Inset: Data collapse of the Binder cumulant for $\sigma = 0.4$ and system sizes $L = 2048$ up to $L = 32768$. Smaller sizes are shown for comparison.](image3)
0.29(3) agrees with $1/\nu = 0.316(9)$ from reference [24] (for $\rho = 1.4$ in the cited paper).

**Borderline case $\sigma = 0.5$**

The value of $\sigma = 1/2$ was conjectured to correspond to the lower critical dimension with $h_c = 0$ [22,23]. Nevertheless, right at the critical value $\sigma = \sigma_c$, the behavior could also correspond to $h_c > 0$. We investigated this issue in the same way as for the cases $\sigma < 0.5$.

The curves of the Binder cumulant (Fig. 13) for different system sizes do not show a clear intersection. This could be a hint towards $h_c = 0$.

Thus, our results clearly support $h_c > 0$ for $\sigma = 0.5$. Recent results which support our findings were provided by [11], where the Dyson hierarchical random-field model (cf. ref. [20]) for $\sigma = 1/2$ was investigated numerically for system sizes up to $L = 2^{21}$. These results strongly indicate that the magnetization converges for system sizes $L \rightarrow \infty$ to one common curve at $h_c > 0$. In reference [24], Binder cumulants of a one-dimensional RFIM on a Lévy lattice are studied. Finite-size scaling analysis of the Binder parameter at the value $\sigma = 1/2$ (corresponding to $\rho = 3/2$ in the cited paper) yielded $(h/J)_c \approx 2.31(5) > 0$

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![FIG. 13: Binder cumulant as a function of the random field strength for $\sigma = 1/2$. No clear intersection of the curves for different system sizes $L$ can be determined. Lines are guides to the eyes only.](image1.png)

![FIG. 14: Peak positions of the specific-heat-like quantity as a function of system size for $\sigma = 1/2$. Dotted line is a fit by Eq. (11) with parameters $h_c = 3.899$, $a_2 = 2.20$ and $1/\nu = 0.307$. Dashed-dotted line is a fit by Eq. (13) with $h_{c2} = 3.898$, $a_5 = 2.13$, $1/\nu = 0.302$, $d_4 = 11$ and $k_3 = -1$. Horizontal line denotes $h_c = 3.899$. Inset: Peak positions of the susceptibility as a function of system size $L$. Dotted line is a fit by Eq. (11), where $h_c = 3.869$. $a_2 = 5.84$ and $1/\nu = 0.316$. Horizontal line denotes $h_c = 3.869$.](image2.png)

A possible reason for the discrepancy between analytical studies [22,23] and the numerical results is that for the models studied analytically the interaction strength $J(r)$ decays like a power-law in the distance $r$: $J(r) \sim r^{-\kappa}$. In this case, for a power-law exponent $1.5 \leq \kappa \leq 2$ the randomness dominates and there is only a trivial phase transition i.e. $h_c = 0$ [23]. But in the proof in reference [22] one precondition is that the interaction strength needs to be translation-invariant and non-random, which is not the case for the model studied here. Hence, this proof might not apply to our model as we used random interactions in contrast to regular long-range bonds with varying coupling strengths, which were used in references [20,22,23].

Finally note that also the data points of the magneti-
Region without non-trivial phase transition  $\sigma = 1.0$

Finally we turn to the case $\sigma = 1$ where we expect no phase transition. Fig. 15 shows the Binder parameter for various system sizes. One can see that for any value $h > 0$ to $m = 1$ for $h = 0$, meaning $\beta = 0$. Nevertheless, for the specific heat-like quantity and the susceptibilities, we could study (not shown here) the behavior when approaching $h = 0$ in the same way as for the previously discussed values of $\sigma$. This results in $\nu = 0.40(9)$, $\alpha \approx 0$, $\gamma = 2.19(55)$ and $\bar{\gamma} = 2.5(5)$, as shown in Tab. 11.

![Figure 15: Binder cumulant for $\sigma = 1.0$ averaged over at least $10^5$ realisations. Lines are guides to the eyes only.](image)

CONCLUSION/OUTLOOK

We have studied exact ground states of one-dimensional ($d = 1$) long-range random-field Ising magnets. The strength of the long range interaction is given by $J(r) \sim r^{-d-\sigma}$. Since polynomial-time running algorithms exist, based on a mapping to the maximum-flow problem, we could study large systems numerically with a high number of random samples. We studied the model for different values of $\sigma$, which are representatives for the different expected behavior of the model.

Table 11 summarizes the obtained values of the critical point and the critical exponents in comparison with the expected values from theory. In the mean-field case for $\sigma = 0.25$ the critical exponents agree well within error bars with the theoretical values. The critical point is consistent with values found for the Dyson hierarchical version 20 of the RFIM. In the non-mean-field region for $\sigma = 0.4$, the exponents also agree well with theory. The critical point $h_c$ does not agree with the one found in reference 20, but these points are anyway non-universal.

In the borderline case $\sigma = 0.5$, in particular the critical point $h_c > 0$ does not agree with some analytical work 22, 23, where a value of $h_c = 0$ was predicted for $\sigma = 0.5$. Nevertheless, $h_c > 0$ agrees with results of a recent work 24, which was performed independently and in parallel to our work. In the cited work, an Imry-Ma argument is given and also calculations of exact ground states were carried out, but it was restricted to the analysis of the Binder cumulant. Nevertheless, all measured exponents agree with theory, if one assumes the theory (cf. Eqs. 11) for $1/3 \leq \sigma < 1/2$ to be valid also at $\sigma = 1/2$. Note that the value of $\beta$ is off by a few error bars, but for values close to zero, one would have to go to large system sizes to see the limiting behavior.

For $\sigma = 1$, the measured critical point $h_c = 0$ agrees with theory as well as the value for $\beta$. Nevertheless, the expected jumps 40 in the magnetization as $\beta = 0$ were not observed. As usual for first-order transitions, a real jump can be expected to be visible only in the thermodynamic limit, i.e., for huge system sizes.

The found value of the correlation length exponent $\nu$ does agree with theory within two error bars, where $\nu = 1/2$ is predicted for $\sigma = 1$. Both values for $\gamma$ and $\bar{\gamma}$ are compatible with the expected values if the error bars are taken into account.

Next, we check the Rushbrooke equality 42 for the different values of $\sigma$:

$$\alpha + 2\beta + \gamma = 2.$$  \hspace{1cm} (15)

For $\sigma = 0.25$ one gets the value $\alpha + 2\beta + \gamma = 2.05(60)$, which fulfills equation (15) within the statistical error.

For $\sigma = 0.4$, formula (15) yields $\alpha + 2\beta + \gamma = 2.11(55)$, which is in good agreement with the expected value when the statistical error is taken into account. For the borderline case $\sigma = 0.5$ between non-mean-field region and the region without a non-trivial phase transition, one obtains $\alpha + 2\beta + \gamma = 2.18(99)$, which fulfills equation (15) within error bars. In the region, where $h_c = 0$ and thus $\sigma = 1$, one gets $\alpha + 2\beta + \gamma = 2.19(55)$, which satisfies the scaling relation (15) within the statistical error.

We now compare the theoretical and estimated values of the so-called droplet exponent $\theta$. In the mean-field
case one gets $\theta_{MF} = \gamma_{MF}/\nu_{MF} = 1/\nu_{MF} = \sigma$. For $\sigma = 0.25$, our measurement of $1/\nu = 0.25(3)$ fits well to the theoretical prediction, but does not allow for new insight since we have checked $1/\nu = \sigma$ already. In the non-mean-field region one obtains $\theta = (\bar{\gamma} - \gamma)/\nu$. For $\sigma = 0.4$ we obtain $(\bar{\gamma} - \gamma)/\nu = 0.38(26)$, which agrees well with a prediction $\theta = \sigma$ by Grinstein [16]. For $\sigma = 0.5$ we obtain $(\bar{\gamma} - \gamma)/\nu = 0.45(64)$, which also agrees with this conjecture. In the case $\sigma = 1$, we obtain $\theta = 0.13(46)$ which is compatible with $\theta = 0$ within error bars.

The conjecture $\theta = \sigma$ only holds for the Dyson hierarchical model [19]. It was shown later, that this prediction was perturbatively wrong at higher orders [17] for models with interaction strengths which decay like a power-law in the distance. However, the conjecture seems to be true for our model, as for $\sigma \in \{0.25, 0.4, 0.5\}$ the values for $\theta$ agree with the expected values within error bars.

In a two exponent scenario, the Schwartz-Soffer equation \( \bar{\gamma} = 2 \gamma \) would hold. For $\sigma = 0.25$ formula (16) is valid, when the statistical error is taken into account. In the cases $\sigma = 0.4$ and $\sigma = 0.5$, equation (16) is also fulfilled within statistical errors. For $\sigma = 1$ the Schwartz-Soffer equation does not hold.

To summarize, the critical exponents for the investigated values $\sigma \in \{0.25, 0.4, 0.5, 1\}$ agree well with theory, except for $\beta$ in the case $\sigma = 0.5$. This deviation might be due to too large system sizes which are needed to see the infinite-size behavior. The Rushbrooke equality is fulfilled for all studied values of $\sigma$. The droplet exponent $\theta$ agrees well with theory for all $\sigma \in \{0.25, 0.4, 0.5, 1\}$. The two-exponent scenario is supported by the confirmation of the Schwartz-Soffer equation for $\sigma \in \{0.25, 0.4, 0.5\}$.

For the critical case $\sigma = 1/2$, it was found that $h_c > 0$, for as other recent numerical studies on the Dyson hierarchical model [20] and for the same diluted model [21] as studied here. This in the first place seems to contradict theoretical predictions. But the model studied here is not in contradiction with theory, as it is different from the investigated models with respect to the coupling strengths.

For future studies, it could be of interest to study the same diluted long-range model on higher dimensional lattices. At least $d = 2$ and $d = 3$ should be accessible using the highly efficient maximum-flow algorithms used here.

ACKNOWLEDGEMENTS

We would like to thank M. Moore for suggesting the project to us. Furthermore, we thank him, C. Monthus, A. van Enter, A. P. Young, and T. Garel for helpful discussions.

The simulations were performed at the HERO cluster of the University of Oldenburg funded by the DFG (INST 184/108-1 FUGG) and the ministry of Science and Culture (MWK) of the Lower Saxony State.
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