Probing Hamiltonian field redefinition on the nontrivial conformal Algebra

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The compatibility between the conformal symmetry and the closure of conformal algebras is discussed on the nonlinear sigma model. The present approach, above the basis of field redefinition employed in the Hamiltonian scheme, attempts the method of quantisation with intuitive picture. As a general field theoretic treatment, the consistency is ensured by means of the interesting features which are observed in the historical studies for the gauge-invariant conformal symmetry. The identification of conformal anomaly is also shown coincident with the conventional one approached within the path-integral formulation.

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I. INTRODUCTION

A generic quantisation hopefully endows with model-independent form-preservation rule for operator algebras such that more physical significance of the quantised system can be directly inferred from its classical version. However, the simplest expectation that the classical and quantised operator algebras only deviate slightly, remains naïve since complexity arises, occasionally. For example, Schwinger terms [1, 2] have been demonstrated for the current algebra in general. The violation of the Jacobi identity has also been observed in the quark model [3, 4]. Moreover, the third cocycle is predicted for the quantum mechanics in the presence of magnetic monopole [5, 6, 7]. While the conventional quantisation is basically limited in an ‘a priori renormalisation scheme, considerable extension concerns field redefinition which is ubiquitous in the path integral approaches [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Yet, concreteness can be achieved better in the Hamiltonian scheme [25] where the field redefinitions are governed under the preservation of the canonical commutators. Following this line, Ref. [26] has concluded that the consistent quantisation must not entail arbitrary allowable configurations, where the intriguing role of the treatment of the nonlinear sigma model is to stress of the importance consistent quantisation over the superficial phenomenological correspondence. Such conclusion is sophisticate in the high energy physics [27, 28]: the consistent anomaly-free condition asserts treating the fermionic pairs equal-footingly, the so-called safe representation, in the standard model. Here, we continue implementing such Hamiltonian field redefinition to the operator algebras on the nonlinear sigma model.

In practise, one chooses to disregard the lacking of intuitive phenomenological correspondence owing to the non-linearity. Let the formulation stand on the scalar field prescription but allow it for the field redefinitions order by order. At the tree level, the canonical energy-momentum tensor is found symmetric and traceless, and the conformal invariance is sufficed [24, 30, 31, 32]. As an additional feature, the form of the canonical energy-momentum tensor coincides with the conventional stress energy tensor obtained at the flat space-time in absence of the dilaton coupling. Such additional feature has been disproved [28] at higher loop level since the improvement terms for achieving the symmetric and traceless energy-momentum tensor are shown nontrivial. However, consequently, the conserved improved energy-momentum tensor implies the form of conformal anomaly coinciding with the two-loop Weyl anomaly conditions as given explicitly in, e.g., Refs. [21, 22, 23, 24]. It has also been shown feasible when the torsion term and the dilaton [23, 33] are considered. For the former, the two-loop Weyl anomaly conditions are consistently coupled to the covariant torsion tensors, and the dependence of the residue renormalisation scheme in anomaly conditions, stressed in Ref. [23], is also observed. For the latter, the dilaton part of the conformal anomaly can be generated in the Hamiltonian scheme simply by means of counter-terms, even though it is poorly prescribed for the flat space-time. To this end, apart from the energy-momentum tensor invariance properties continue to impinge on the operator algebras as follows. Symmetry criterion requires the conserved Noether currents, while, in contrast, the generators for an internal symmetry transformation can exhibit the closed structure in absence of the symmetry [28]. Therefore,
whether the algebra is closed or not is not to encode the symmetry property but to comprise information peculiar to the field variable $s$ in use. Accordingly, it is not to argued model-independently, whether the closed Virasoro algebra preserves in absence of conformal anomaly for the nonlinear sigma model, and the explicit derivation remains essential. Our work is organised as follows.

In Section II the Noetherian process is reconsidered in conjunction with field redefinition. It is natural to interpret the form-ambiguity of operator algebras by means of the prescription of configuration space. The fact that the principle of invariance permits such flexibility basically supports the historical super-potential construction for the current operators. However, it can be shown that the total divergence terms does fully specify the corresponding ambiguity, and the uniqueness of charge operators must be questioned beyond this scope. The conclusion is also inspired by the examples based on Ref. [31, 32].

Section III revisits the previous formulation but with more technical points relevant. The canonical quantisation here follows the line of Ref. [25, 36, 37, 38, 39] in the normal order prescription but is equipped with the convention of vacuum provided by Ref. [25]. It is shown that the energy-momentum tensor must be improved at the two loop level via the construction suggested in the literature [29, 31, 32]. The conservation of the improved energy-momentum tensor then determines the conformal anomaly correctly.

In Section IV the status of the conformal algebras is clarified and the corresponding algebras are explicitly derived. The generators are obtained as the standard Bessel-Hagen form for the conformal currents [29, 31, 40]. The corresponding Fourier mode expansion of the improved energy-momentum tensor reflects the conventional holomorphic or antiholomorphic expressions [29, 31, 41, 42] in the conventional two-dimensional theories. The resultant algebras of generators, in fact, do not reproduce the well-known Virasoro version. The central extensions do not appear as c-numbers in contrast to the ones appropriate for Dirac’s simple conjecture [43]. The anomalous contribution is fully characterised in terms of the improvement terms which signify strong dependence on the field variables after field redefinitions. However, the conservation of the conformal currents remain ensured by the anomaly-free conditions. Moreover, the basic points in Section II appear sound for it.

More discussions and conclusion are presented in Sec. V.

II. SOME GENERAL FEATURES OF THE CURRENT AND CHARGE

In general, a symmetry property subject to a certain external transformation of type $T$ is prescribed by the invariance of action, say,

$$\delta_T S = \int \sum \delta_T \varphi \frac{\delta S}{\delta \varphi},$$

where the notation $\int \sum$ denotes the functional integral over the parameter space of variation. However, it remains flexible to determine the formulation in terms of distinct sets of field variables. In each case, the dynamical equations may end up with various fashions. Although, such generic aspect contrasts with the conventional formulation for the field theory, which relies on a specific choice of field variables, considerable benefits in analysis can be resulted in.

The form ambiguity for the current operators can be intuitively interpreted as an issue of the standard Noetherian processes equipped with distinct sets of field variables. It may turn out too restricted to decode types of ambiguity for the current operators solely by means of the super-potential [29, 31, 32]. Suppose, on the contrary, it remains true, then certain divergence term would furnish the relationship, say,

$$J^\mu_T|_B = J^\mu_T|_A + \partial_\lambda X^{\lambda \mu},$$

by allowing only indistinguishable charges, where $X^{\lambda \mu} = -X^{\mu \lambda}$. Note that $J^\mu_T|_B$ and $J^\mu_T|_A$ are the currents appropriate for the choices $\varphi_B$ and $\varphi_A$, respectively. Equation (2) cannot hold true in general among distinct sets of field variables for the following argument. Consider, for example, additional symmetry are implemented by transforming $\varphi_B \rightarrow \varphi_B^\nu$. Then, the action can be as well parametrised in terms of the new variable $\varphi_B^\nu$; the Noether current $J^\mu_T|_B^\nu$ can be explicitly deduced. However, $J^\mu_T|_B^\nu$ now represents a composite transform of $T$ and the element of $U$ which cannot be equipped with the same charge as the one for $J^\mu_T|_B$. It turns out that the choice of field variables in use in fact pertains to the physical significance for the system, an issue beyond the action itself. Thus, the requirement on the form preservation of the quantised version of algebras of generators is eventually unexpected either.

One may specialise the flexibility discussed above to the interest for the space-time symmetry of particular types. Above the framework of conventional formulation, the symmetric and traceless canonical energy-momentum tensor denotes the conformal symmetry for the system. Such condition must be relaxed subject to the possibility of choosing field variables independently. It is sufficient to have the improved energy-momentum tensor of symmetric and traceless form by means of

$$\tilde{T}_{\alpha \beta} = T_{\alpha \beta} + W_{\alpha \beta} = T_{\alpha \beta} + \partial^\gamma X_{\gamma \alpha \beta},$$

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$$\tilde{T}_{\alpha \beta} = T_{\alpha \beta} + W_{\alpha \beta} = T_{\alpha \beta} + \partial^\gamma X_{\gamma \alpha \beta},$$

and

$$\tilde{T}_{\alpha \beta} = T_{\alpha \beta} + W_{\alpha \beta} = T_{\alpha \beta} + \partial^\gamma X_{\gamma \alpha \beta}.$$
for a conformally invariant system [29, 32], where the so-called improvement term $W_{\alpha\beta}$ is formally provided by the total divergence and $X_{\gamma\alpha\beta} = -X_{\alpha\gamma\beta}$. Note that Eq. (3) is a straightforward extension of Eq. (2). The situation for the conformal symmetry to be signified by the the symmetric and traceless canonical energy-momentum tensor can be modified by the quantisation involving higher order; the same condition must be imposed on the improved one appeared in Eq. (3) instead. This is because the consistent quantisation needs not prohibit the modification on field content by the field redefinitions. Moreover, it is shown [29] that the canonical energy-momentum tensor cannot be a symmetric one unless the field variable is spinless. Therefore, once the the symmetric property of the canonical energy-momentum tensor is not lost during quantisation, it simply highlights that the theory does no more solely consist of scalar fields. In order to be sufficiently generic, the conformal current of the Bessel-Hagen form must be considered, say,

$$J_\alpha = \delta_c x^\beta \tilde{T}_{\alpha\beta},$$  \hspace{1cm} (4)

where $\delta_c x^\beta$ denotes the conformal displacements for the coordinates. Meanwhile, the ambiguity for the operator algebras must be reduced after the quantisation, an effect realised by the field redefinition.

To this end, seemingly independent approaches of conformal currents performed in the two (abelian) and more dimensional Yang-Mill theories [34, 35] also shed some light to our study. In those cases, the Noetherian process directly applied to the conformal transformation fails to provide gauge-invariant conformal currents. To furnish such structure, the compensation by means of the gauge transformation for the vector potential has been concluded as essential. The issues bear interesting resemblance to our analysis as follows. The deviation for the charges caused by changing field variables is now evident. The corresponding compensation of transform also reflects the structure of our general argument after Eq. (2). In addition, the comparison for the case at hand with the examples coupled with vector potential is adequate since the loss of symmetric canonical energy-momentum tensor signifies the existence of field variables with nontrivial spin structure. Moreover, the conformal algebras for the composite transformation do not close [11, 29, 34], rendering the corresponding finite version of transform path-dependent [34, 45].

III. OBTAINING THE CONFORMAL SYMMETRY

For the action of the nonlinear sigma model, one is referred to

$$S = -\frac{T}{2} \int d\sigma d\tau \eta^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X),$$ \hspace{1cm} (5)

where $G_{\mu\nu}$ denotes the metric of the target space, $\eta^{\alpha\beta}$ is the metric of the flat world sheet with the signature (-1,1), and $T$ is often called the string tension.

The classical algebras start with the conventional choice of field variable denoted by $X$ in the Hamiltonian scheme. Formally, the typical components of the canonical energy-momentum tensor can be derived as

$$T_{00}|_X = \frac{1}{2T} \Pi_0 \Pi_\nu G^{\mu\nu} + \frac{T}{2} \dot{X}^\mu \dot{X}^\nu G_{\mu\nu},$$ \hspace{1cm} (6)

$$T_{01}|_X = \Pi_\mu \dot{X}^\mu,$$  \hspace{1cm} (7)

where $\Pi^\mu$ denotes the conjugate momentum for the field variable $X$. The spatial derivative is denoted as $\dot{F}(\sigma) = \partial_\sigma F(\sigma)$ throughout this work. Note that the Poisson brackets of the components given in Eqs. (6) and (7) develop a simple closed structure for the classical algebras. Moreover, the energy-momentum tensor is traceless and symmetric in both its indices, namely,

$$T^\alpha_\alpha|_X \equiv T_{00}|_X - T_{11}|_X \equiv 0,$$ \hspace{1cm} (8)

$$T_{01}|_X \equiv T_{10}|_X,$$ \hspace{1cm} (9)

which automatically fulfill the conformal symmetry conditions at the classical level.

For quantisation, the normal ordering is introduced in order to subtract the tadpoles. Hence, the operator products exhibit the standard decomposition

$$A(\sigma)B(\bar{\sigma}) = :A(\sigma)B(\bar{\sigma}):+A(\sigma)B(\bar{\sigma}),$$ \hspace{1cm} (10)

where the normal ordered product is delimited by a pair of double dots and the Wick-contraction is underlined. The Fourier expansion is employed, say,

$$O^m = \frac{1}{2} \int_0^\pi d\sigma e^{-i m \sigma} O(\sigma) = -i \int d\xi \xi^{-m-1} O(\sigma) = \int_m O(\sigma)$$ \hspace{1cm} (11)
where $T^m$, $B^n$ are considered as functionals of the fundamental conjugate pair $\Pi$ and $X$. Note that the upper indices $m$ and $n$ in Eqs. (11) and (12) simply stand for the corresponding Fourier mode and should not be confused with the power of the objects. The contracted term on the R.H.S. of Eq. (12) can be fully determined, once the fundamental contraction for $\Pi$ and $X$ is fixed by a proper order prescription. Upon renormalising the vacuum with the convention of Ref. [25], the fundamental contraction reads

$$ \Pi\mu(\sigma)X^\nu(\tilde{\sigma}) = -\frac{i}{\pi} \delta^\nu_\mu \frac{\tilde{\xi}}{\xi - \tilde{\xi}} $$

subject to the condition $|\xi| < |\tilde{\xi}|$. This renders the quantised Hamiltonian at the tree level formally unaltered in contrast to the classical one [25].

The field redefinition can be implemented by writing the generic canonical energy-momentum tensor as

$$ T_{\alpha\beta} = T_{\alpha\beta}|_X + T_{\alpha\beta}^{III} + T_{\alpha\beta}^{IV} + \cdots, \tag{14} $$

where $T_{\alpha\beta}|_X$’s ($\alpha = 0, 1$) corresponds to the canonical energy-momentum tensor obtained from the tree level. The counterterms are known to be

$$ (T_{00}^{II})^m = \frac{-i(m+1)}{2\pi T} \int_m \Pi\Gamma, \tag{15} $$

$$ (T_{00}^{III})^m = \frac{1}{4\pi^2 T} \int_m \Pi G(\mathcal{R} + \alpha) \tilde{X} - \frac{m(m+1)(k+1)}{12\pi^2 T} \int_m tr(G\mathcal{R} + G\alpha), \tag{16} $$

$$ (T_{00}^{IV})^m = \frac{i(m+1)}{8\pi^2 T} \int_m Q\tilde{X}, \tag{17} $$

$$ (T_{00}^{IV})^m = \frac{i(m+1)}{2\pi} \int_m \Gamma\tilde{X}, \tag{18} $$

while $T_{01}^{III}$ and $T_{01}^{IV}$ remain trivial. The explicit manipulation picks up the zeroth order of antisymmetric coupling from the algebras of the canonical energy-momentum tensor [26] and shows

\[
\begin{align*}
[ (T_{00})^m, (T_{00})^n ] &= (n-m)(T_{00})^{m+n} + \frac{D}{6}(m^3 - m)\delta_{m+n,0} \\
&\quad + \frac{(n-m)}{2\pi} \int_{n+m} \tilde{X}\mathcal{R}\tilde{X} + \frac{(n-m)}{8\pi^2 T^2} \int_{n+m} \Pi G\mathcal{U}\tilde{X} \\
&\quad + \frac{i(n-m)(n+m+1)(k+1)}{24\pi^2 T^2} \int_{n+m} \Pi G\partial tr(G\mathcal{R}) \\
&\quad + \frac{(m^3 - m)}{16\pi^2 T^2} \mathcal{T}^{n+m} + \frac{(n-1)(n+1)(m+1)}{16\pi^2 T^2} \mathcal{S}^{n+m} \\
&\quad + O(5) + \cdots, \tag{19}
\end{align*}
\]

\[
\begin{align*}
[ (T_{00})^m, (T_{01})^n ] &= (n-m)(T_{00})^{n+m} \\
&\quad + \frac{m(m+1)(n+1)(2-k) - m(m^2 - 1)k}{12\pi^2 T} \int_{n+m} tr(G\mathcal{R}) \\
&\quad + O(5) + \cdots, \tag{20}
\end{align*}
\]

\[
\begin{align*}
[ (T_{01})^m, (T_{01})^n ] &= (n-m)(T_{01})^{n+m} + \frac{D}{6}(m^3 - m)\delta_{m+n,0} \tag{21}
\end{align*}
\]

with

\[
\mathcal{U}_{\mu\nu} = \left\{ \partial_\mu G^{a\tau} \partial_\nu (G(\alpha + \mathcal{R}))_\tau^{\rho} \xi_{\mu} - \partial_\mu G^{a\tau} \partial_\nu (G(\alpha + \mathcal{R}))_\tau^{\rho} + \partial_\mu G^{a\tau} \partial_\nu (G(\alpha + \mathcal{R}))_\tau^{\rho} \right\} G_{\nu\nu}
\]

\[
- \left\{ \partial_\mu \partial_\lambda G^{a\tau} \partial_\nu G^{b\rho} + \partial_\mu G^{a\tau} \partial_\nu \Gamma^0 \right\} G_{\nu\nu}
\]
order for the field redefinition. This problem is remedied by the consideration of Eq. (3) which renders the conformal matrix \( U \) with respect to the field variables.

where the functions \( \tilde{q} \) are consistent, once the symmetry is recovered, \( \tilde{\theta} \) can be made covariant as well, namely,

\[
I = \frac{2}{3} \partial \rho (G(\alpha + \mathcal{R}))^\mu \partial \mu G^{\rho \nu} - \frac{1}{6} \{ \partial \lambda \partial \rho G^{\mu \nu} \partial \lambda \partial \mu G^{\rho \nu} + \text{tr} (G(\alpha + \mathcal{R})G(\alpha + \mathcal{R})) \},
\]

(22)

\[
S = \frac{1}{3} (\Gamma^\mu \partial \rho + G^{\mu \nu} \partial \nu) ((k + 1) \text{tr} (G \mathcal{R}) + \text{tr} (G \alpha)) - (G(\alpha + \mathcal{R}))^\rho \partial \rho \Gamma^\nu
\]

(23)

+ \partial \lambda \partial \rho \Gamma^\mu \partial \mu G^{\lambda \rho} + \partial \rho \Gamma^\mu \partial \mu G^{\lambda \rho} + G^{\lambda \rho} \partial \lambda Q_\rho + \Gamma^\rho Q_\mu,
\]

(24)

where \( \mathcal{R}_{\mu \nu} \) is the Ricci tensor and various contracted connections have been employed as

\[
\Gamma^\lambda = \Gamma_{\mu \nu} G^{\mu \nu}, \quad \tilde{\Gamma}_\mu = \Gamma_{\rho \mu}
\]

and the covariant derivative is defined as \( D_\mu V_\nu = \partial_\mu V_\nu + \Gamma^\lambda \partial_\mu V_\lambda \). Note that the expressions of Eqs. 15 - 20 have been compacted by letting the neighbouring indices contracted. Hence, objects appearing in the parenthesis of \( \text{tr} (\ldots) \) are contracted cyclically. In addition, the spatial derivatives are abbreviated in form of \( \bar{F}(\sigma) = \bar{\partial}_\sigma F(\sigma) \) through out this work. In Eqs. (19 - 21), corrections to higher orders are denoted with \( O(N) \), corresponding to terms of \( N \) derivatives with respect to the field variables.

The expressions of Eqs. 22 - 20 have also been partially shown in Ref. 25. The basic observation is that the matrix \( \mathcal{U}_{\mu \nu} \), containing antisymmetric part, can not be made covariant simply by extending counterterms with higher order for the field redefinition. This problem is remedied by the consideration of Eq. 3 which renders the conformal symmetry signified by the improved energy-momentum tensor instead. Resembling the criterion appropriate for the general choice of field variable, the conditions are, via the obtainable conserved improvement terms,

\[
\tilde{T}_\alpha = 0,
\]

(27)

\[
\tilde{T}_{\alpha \beta} = 0,
\]

(28)

\[
\bar{\partial}^\alpha \tilde{T}_{\alpha \beta} = 0.
\]

(29)

Accordingly, Eqs. 19 - 21 are identified as

\[
\begin{align*}
\begin{bmatrix} (T_{00})^m, (T_{00})^n \end{bmatrix} &= (n - m)(T_{01})^{n + m} - (n - m)(T_{01} - T_{10})^{n + m} + c_1(n, m) \\
&+ (n - m)(W_{01} - W_{10})^{n + m},
\end{align*}
\]

(30)

\[
\begin{align*}
\begin{bmatrix} (T_{01})^m, (T_{00})^n \end{bmatrix} &= (n - m)(T_{00})^{n + m} + m (T_{01})^{n + m} + c_2(n, m) \\
&- m (W_{11} - W_{00})^{n + m},
\end{align*}
\]

(31)

\[
\begin{align*}
\begin{bmatrix} (T_{01})^m, (T_{01})^n \end{bmatrix} &= (n - m)(T_{01})^{n + m} + c_3(n, m),
\end{align*}
\]

(32)

where the functions \( c_i(n, m) \) are restraint by the translational invariance. Although this obviously deviates from the closure version assumed by the classical formulation subject to the conventional field variable \( X \), the symmetry conditions can be achieved as well, namely,

\[
\begin{align*}
\begin{bmatrix} (T_{00})^m, (T_{00})^n \end{bmatrix} &= n (T_{01})^n - n (T_{01} - T_{10})^n, \\
\begin{bmatrix} (T_{01})^m, (T_{00})^n \end{bmatrix} &= -m (T_{00})^m - m (T_{00})^m,
\end{align*}
\]

(33)

which indicate that the divergence of the improved current is up to the quantities \( n(T_{01} - T_{10})^n \) and \( m(T_{00})^m \). To be concrete, once the symmetry is recovered, \( \tilde{T}_{\alpha \beta} \) must conserve and its symmetric and traceless property can be consistently achieved.
The peculiar structure of the improvement term $W_{\alpha \beta}$ in two dimensions also reflects the conventional construction as shown in Eq. 3. Basically, the corresponding construction is

$$(W_{00})^h = \frac{ih}{8\pi^2 T} \int_h \left( \dot{X}Y + \frac{1}{T} \Pi GY \right) - \frac{1}{8\pi^2 T} \int_h \bar{F} + \frac{h^2 - 1}{24\pi^3 T^2} \int_h \text{tr} \left( G(E_1 + E_3) \right),$$  

$$(W_{01})^h = \frac{ih}{8\pi^2 T} \int_h \left( \dot{X}Y + \frac{1}{T} \Pi GY \right) - \frac{1}{8\pi^2 T} \int_h F + \frac{h^2 - 1}{24\pi^3 T^2} \int_h \text{tr} \left( G(E_1 + E_3) \right) + \frac{h(h+1)}{16\pi^3 T^2} \int_h S,$$  

with

$$\bar{F} = \frac{-1}{8\pi^2 T} \left( \dot{X}\bar{E}_1 \dot{X} + \frac{1}{T} \Pi G\bar{E}_2 \dot{X} + \frac{1}{T^2} \Pi G\bar{E}_3 \Pi \right),$$  

$$F = \frac{-1}{8\pi^2 T} \left( \dot{X}\bar{E}_1 \dot{X} + \frac{1}{T} \Pi G\bar{E}_2 \dot{X} + \frac{1}{T^2} \Pi G\bar{E}_3 \Pi \right),$$

where $Y$, $\bar{E}_i$'s and $E_i$'s are considered as functional of the field variable $X$ of proper ranks, $Y$ is of $O(3)$ and $E_i$'s, $\bar{E}_i$ are of $O(4)$. The other two components of the improvement term are similarly determined by conservation laws as

$$(W_{10})^h = \frac{1}{h} \left[ (T_{00})^0, (W_{00})^h \right] = \frac{ih}{8\pi^2 T} \int_h \left( \frac{1}{T} \Pi GY + \dot{X}Y \right) + \frac{h^2 - 1}{24\pi^3 T^2} \int_h \text{tr} \left( G(E_1 + E_3) \right) + \frac{1}{8\pi^2 T^2} \int_h \Pi G \left( \frac{1}{2} (DY - (DY)^T) + (-2\bar{E}_1 - 2\bar{E}_2) \right) \dot{X} + \frac{1}{8\pi^2 T} \int_h \left( \frac{1}{T^2} \Pi G \left( \frac{1}{2} DY - \bar{E}_3 \right) \Pi \Pi + \dot{X} \left( -\frac{1}{2} DY - \bar{E}_2 \right) \dot{X} \right),$$  

$$(W_{11})^h = \frac{1}{h} \left[ (T_{00})^0, (W_{01})^h \right] = \frac{ih}{8\pi^2 T} \int_h \left( \frac{1}{T} \Pi GY + \dot{X}Y \right) + \frac{h^2 - 1}{24\pi^3 T^2} \int_h \text{tr} \left( G(E_1 + E_3) \right) + \frac{1}{8\pi^2 T^2} \int_h \Pi G \left( \frac{1}{2} (DY - (DY)^T) + (-2\alpha_1 E_3 - 2\alpha_2 \bar{E}_3) \right) \dot{X} + \frac{1}{8\pi^2 T} \int_h \left( \frac{1}{T^2} \Pi G \left( \frac{1}{2} DY - \alpha_3 \bar{E}_3 \right) \Pi \Pi + \dot{X} \left( -\frac{1}{2} DY - \alpha_4 \bar{E}_2 \right) \dot{X} \right).$$

Note that, in above, the undetermined constants $\alpha_i$'s and $\bar{\alpha}_i$'s ($i = 1..4$) respond to the choice of the $O(5)$ term for the single $[(T_{00})^0, O^n]$ type of contractions in the sense that

$$\int_m \int_n : \Pi G \Pi : \dot{X} \bar{E}_1 \dot{X} : = \frac{-n\pi}{2} \int_m : \Pi G \bar{E}_1 \dot{X} : + \frac{-i\pi}{4} \int_m : \Pi G \frac{d}{d\sigma} (\bar{E}_1 \dot{X}) :$$  

$$= \frac{n\pi}{2} \int_m : \Pi G \bar{E}_1 \dot{X} : + \frac{i\pi}{4} \int_m : \Pi G \frac{d}{d\sigma} (\Pi \Pi) \bar{E}_1 \dot{X} :$$  

$$= \frac{(\gamma - n) + (1 - \gamma) m}{2} \int_m : \Pi G \bar{E}_1 \dot{X} :$$  

$$+ \frac{-i\gamma \pi}{4} \int_m : \Pi G \frac{d}{d\sigma} (\bar{E}_1 \dot{X}) : + \frac{i(1 - \gamma) \pi}{4} \int_m : \Pi G \frac{d}{d\sigma} (\Pi \Pi) \bar{E}_1 \dot{X} :,$$  

where the first two equations deviate from each other only by a surface term and the generic form of the third equation can be linear combination of above two equations, weighted by the constant $\gamma$. 


In addition, modifications on the zeroth mode should be included such that the status for Hamiltonian and the translation generator on the space direction can remain. This amounts to

\[
(W_{00})^h = = (\bar{W}_{00})^h (1 - \delta_{h,0}),
\]

\[
(W_{01})^h = = (\bar{W}_{01})^h (1 - \delta_{h,0}),
\]

\[
(W_{10})^h = = (\bar{W}_{10})^h - (\bar{W}_{01})^h \delta_{h,0},
\]

\[
(W_{11})^h = = (\bar{W}_{11})^h - (\bar{W}_{00})^h \delta_{h,0}.
\]

Noteworthy is that

\[
\left( (T_{00})^0 \right)(W_{0\beta})^h - h(W_{1\beta})^h \right) = \left[ (T_{00})^0 \right] - h(W_{1\beta})^h), \forall h,
\]

leaving the derivations for Eqs. (39) and (40) unaltered. Moreover, since

\[
W_{01} - W_{10} = \bar{W}_{01} - \bar{W}_{10},
\]

\[
W_{00} - W_{11} = \bar{W}_{00} - \bar{W}_{11},
\]

considerable simplifications can be achieved by the observation that Eqs. (59) and (61) only depend on the difference between the improvement terms.

Subsequently, the improvement term is proposed such that the antisymmetric part of \(\Pi_G H X\) coupling in Eq. (19) is cancelled out via further ajouting the counterterms, \(T_{\mu\nu}^0\) to the expressions appropriate for Eqs. (15) - (18). Hence, consistency equations relating the known quantities in Eqs. (19) - (24) to the improvement terms are determined as

\[
\{(\bar{\mathcal{E}})_1\}_{\mu\nu} - \frac{1}{2} D(\mu \gamma \nu) - \alpha_4 (\bar{\mathcal{E}}_2)_{(\mu\nu)} = 0,
\]

\[
(\bar{\mathcal{E}})_2_{(\mu\nu)} - 2 \alpha_1 (\bar{\mathcal{E}}_1)_{(\mu\nu)} - 2 \alpha_2 (\bar{\mathcal{E}}_3)_{(\mu\nu)} = 0,
\]

\[
(\bar{\mathcal{E}})_3_{(\mu\nu)} - \alpha_3 (\bar{\mathcal{E}}_2)_{(\mu\nu)} + \frac{1}{2} D(\mu \gamma \nu) = 0,
\]

\[
(\mathcal{E})_{1\mu\nu} - D(\mu \gamma \nu) - (\bar{\alpha}_4 - \bar{\alpha}_3) \bar{\mathcal{E}}_2 - (\mathcal{E})_{(\mu\nu)} = \mathcal{K}_{\mu\nu},
\]

\[
(\bar{\mathcal{E}})_2_{\mu\nu} - \frac{1}{2} D(\mu \gamma \nu) = 0,
\]

\[
U_{\mu\nu} + (\bar{\mathcal{E}})_2_{\mu\nu} - \frac{1}{2} D(\mu \gamma \nu) = 0,
\]

\[
\text{tr} \left( G (\mathcal{E}_2 - 2 \alpha_1 \bar{\mathcal{E}}_1 - 2 \alpha_2 \bar{\mathcal{E}}_3) \right) = 3 (S - T) - \text{tr} (G U),
\]

\[
\frac{1}{2} D Y + \text{tr} \left( G (\mathcal{E}_3 - \bar{\alpha}_3 \bar{\mathcal{E}}_2) \right) = \text{tr} (G K),
\]

where \(\mathcal{K}_{\mu\nu}\) denotes a certain covariant functional of \(O(4)\) and the notation of covariant derivative \(D(\mu \gamma \nu)\) are employed for compacting. In addition, the choice of \(k = 2\) in Eq. (19) - (24) turns out to be the scheme of field redefinition appropriate for the Principle of Least Anomaly [20]. Hence,

\[
(T_{00})^h = \frac{-1}{16 \pi^2 T} \int \mathcal{X} \left( U + \mathcal{E}_2 - 2 \bar{\alpha}_1 \bar{\mathcal{E}}_1 - 2 \bar{\alpha}_2 \bar{\mathcal{E}}_3 \right) \mathcal{X} + \frac{-1}{8 \pi^2 T^2} \int \Pi \left( \mathcal{E}_3 - \bar{\alpha}_3 \bar{\mathcal{E}}_2 + \frac{1}{2} \left( D Y + (D Y)^T \right) \right) \mathcal{X},
\]

while no further counterterms for \(T_{01}\) are needed. Consequently,

\[
\left( (T_{00})^m , (T_{00})^n \right) = (n - m)(T_{00})^{n+m} + \frac{D}{6} (m^3 - m) \delta_{m+n,0}
\]

\[
+ \frac{(n - m)}{2\pi} \int \bar{\mathcal{X}} \left( \mathcal{R} + \frac{1}{4\pi T} \mathcal{K} \right) \mathcal{X}
\]

\[
+ (n - m) (W_{01} - W_{10})^{n+m}
\]

\[
+ O(5) + \cdots,
\]

\[
\left( (T_{00})^m , (T_{01})^n \right) = (n - m)(T_{00})^{n+m} - \frac{m(m^2 - 1)}{6\pi^2 T} \int \text{tr} (G \mathcal{R})
\]

\[
- m (W_{11} - W_{00})^{n+m}
\]

\[
+ O(5) + \cdots,
\]

\[
\left( (T_{01})^m , (T_{01})^n \right) = (n - m)(T_{01})^{n+m} + \frac{D}{6} (m^3 - m) \delta_{m+n,0}.
\]
In above, the consistent identifications for Eqs. (33) and (34) provide the anomaly conditions, up to $O(5)$, as

\[ n \left( \tilde{T}_{01} - \tilde{T}_{10} \right)^n = \frac{n}{2\pi} \int \tilde{X} \left( \mathcal{R} + \frac{1}{4\pi T} \mathcal{K} \right) \tilde{X}, \quad (61) \]

\[ m \left( \tilde{T}_a^a \right)^m = \frac{-m(m^2-1)}{6\pi^2 T} \int m \left( \mathcal{R} + \frac{1}{4\pi T} \mathcal{K} \right), \quad (62) \]

which suggest that the conformal symmetry can be recovered when the condition

\[ \mathcal{R}_{\mu\nu} + \frac{1}{4\pi T} \mathcal{K}_{\mu\nu} + O(6) + \cdots = 0 \quad (63) \]

is fulfilled. However, the expression can be cast more limited by virtue of the recursive use of its consistency conditions and the secondary constraints \[46\] thereof. $\mathcal{K}_{\mu\nu}$ only appears as a linear combinations of covariant quantities of $O(4)$, say,

\[ \mathcal{K}_{\mu\nu} = \sum_i \alpha_i \mathcal{K}^{(i)}_{\mu\nu}, \quad (64) \]

in which some of freedom can be further subtracted out by means of the consistent conditions of Eq. \[63\]. The corresponding conditions relate $O(4)$ to $O(6)$ appears as direct consequence of Eq. \[63\] and are given, for instance, as

\[ \mathcal{R}_{\mu\nu} \mathcal{R}^\lambda + \frac{1}{4\pi T} \mathcal{R}_{\mu\nu} \mathcal{K}^\lambda + \cdots = 0, \quad (65) \]

\[ \mathcal{R}_{\mu\nu} \mathcal{R}_{\rho\sigma} + \frac{1}{4\pi T} \mathcal{R}_{\mu\nu} \mathcal{K}_{\rho\sigma} + \cdots = 0, \quad (66) \]

\[ D^2 \mathcal{R}_{\mu\nu} + \frac{1}{4\pi T} D^2 \mathcal{K}_{\mu\nu} + \cdots = 0, \quad (67) \]

\[ D^\lambda D_\mu \mathcal{R}_{\lambda\nu} + \frac{1}{4\pi T} D^\lambda D_\mu \mathcal{K}_{\lambda\nu} + \cdots = 0, \quad (68) \]

where the first two are obtained via direct contractions and the latter two are derived from the secondary constraints \[46\]. While no other relevant quantities essentially enter, except for the dependence on the conventional Riemann $\mathcal{R}_{\mu\nu\rho\sigma}$, one can achieve,

\[ \mathcal{R}_{\mu\nu} + \frac{\alpha}{4\pi T} \mathcal{R}_{\mu\rho} \mathcal{R}^\rho_{\nu\lambda} + O(6) + \cdots = 0, \quad (69) \]

where $\alpha$ remains some undetermined constant. Noteworthy is about the object $D^\lambda D^\rho \mathcal{R}_{\mu\lambda\rho\sigma}$ that could have contributed to Eq. \[63\] via Eq. \[64\]. However, this decomposed, identically, as

\[ D^\lambda D^\rho \mathcal{R}_{\mu\lambda\rho\sigma} = D^2 \mathcal{R}_{\mu\nu} - D^\lambda D_\mu \mathcal{R}_{\lambda\nu}, \quad (70) \]

after employing the Bianchi identity. In addition, the missing of the $O(5)$ term is concluded by the fact that covariant objects of $O(5)$ do not appear as matrices.

 Arbitrariness left in the counterterms of field redefinition is referred to as permissible schemes for the quantisation. The corresponding ambiguity is reduced by means of the minimal setting of the anomaly \[21\] which amounts to possibly weakening the anomaly conditions. Meanwhile, this ensures the faithfulness of the resultant essential symmetry requirement. Subject to this principle, schemes other than $k = 2$ may be adopted in Eqs. \[19 - 21\] instead. As the consequence, the anomaly terms will be displaced, moving, to and fro, between $\tilde{T}_a^a$ and $\tilde{T}_{01} - \tilde{T}_{10}$. However, the resultant condition must remain unaltered. The situation also resembles the treatment of Weyl anomaly pedagogically reviewed in Ref. \[24\] where all the anomalies can be shifted to the trace part of energy-momentum tensor.

**IV. THE CONFORMAL ALGEBRAS**

After clarifying the status of energy-momentum tensor we turn to the direct study of the structure of generators of transformations. For the two dimensions, the conformal generators are known to be provided in terms of holomorphic
and anti-holomorphic expressions $L_0$ and $L_0^\bar{c}$. According to the convention of \cite{41}, the forms of conformal generators $L_m$ are given as

$$L_m = \oint dz \zeta^{m-1} J(z) = \int dx^1 e^{m \tilde{z}} \tilde{T}^1(\tilde{z}),$$  
(71)

$$L_m = \oint d\bar{z} \bar{\zeta}^{m-1} J(\bar{z}) = \int dx^1 e^{m \tilde{z}} \tilde{T}(\tilde{z}),$$  
(72)

where

$$z = x^1 + ix^0,$$
(73)

$$\bar{z} = x^1 - ix^0,$$
(74)

$$\zeta = \exp z,$$
(75)

$$\bar{\zeta} = \exp \bar{z}.$$  
(76)

and the contour integrations in Eqs. (71) and (72) make sense of the radial quantisation discussed in the context of conformal field theory.

On the other hand, the treatment of Eqs. (42 - 45) plays the crucial role of conservation relation for the conformal currents, namely,

$$\left[ (\tilde{T}_{00})^0, (\tilde{T}_{00})^n \right] = \left[ (\tilde{T}_{00})^0, (\tilde{T}_{00})^n \right] = n \left( \tilde{T}_{01} \right)^n,$$  
(77)

$$\left[ (\tilde{T}_{01})^0, (\tilde{T}_{00})^n \right] = \left[ (\tilde{T}_{01})^0, (\tilde{T}_{00})^n \right] = n \left( \tilde{T}_{01} \right)^n,$$  
(78)

$$\left[ (\tilde{T}_{01})^m, (\tilde{T}_{00})^0 \right] = \left[ (\tilde{T}_{01})^m, (\tilde{T}_{00})^0 \right] = -m \left( \tilde{T}_{00} \right)^m,$$  
(79)

$$\left[ (\tilde{T}_{01})^0, (\tilde{T}_{01})^m \right] = \left[ (\tilde{T}_{01})^m, (\tilde{T}_{01})^0 \right] = -m \left( \tilde{T}_{01} \right)^m,$$  
(80)

once the conformal anomaly given by Eq. (69) is put to zero.

While Eqs. (71) and (72) carry the consequence of explicitly employing the Bessel-Hagen form of Eq. (4) in two dimensions, it turns out that the Fourier modes of the light-cone component of improved energy-momentum tensor can be identified with the conformal generators at the instant $\tau = 0$. To be concrete, regard that

$$\tilde{T}_{++} = \frac{1}{2} \left( \tilde{T}_{00} + \tilde{T}_{01} \right) - \frac{1}{4} \left\{ (\tilde{T}_{01} - \tilde{T}_{10}) + (\tilde{T}_{00} - \tilde{T}_{11}) \right\},$$  
(81)

$$\tilde{T}_{--} = \frac{1}{2} \left( \tilde{T}_{00} - \tilde{T}_{01} \right) + \frac{1}{4} \left\{ (\tilde{T}_{01} - \tilde{T}_{10}) - (\tilde{T}_{00} - \tilde{T}_{11}) \right\}. $$  
(82)

Meanwhile, the light-cone components of the improved energy-momentum tensor deviate from $\frac{1}{2} \left( \tilde{T}_{00} + \tilde{T}_{01} \right)$ and $\frac{1}{2} \left( \tilde{T}_{00} - \tilde{T}_{01} \right)$ only by the anomalies. Hence, the light-cone derivatives of these light-cone components of the improved energy-momentum tensor appear to be

$$\left[ (\tilde{T}_{00})^0 - (\tilde{T}_{01})^0, (\tilde{T}_{++})^h \right] = \left[ (\tilde{T}_{00})^0 - (\tilde{T}_{01})^0, \frac{1}{2} \left( \tilde{T}_{00} + \tilde{T}_{01} \right)^h \right] = 0,$$  
(83)

$$\left[ (\tilde{T}_{00})^0 + (\tilde{T}_{01})^0, (\tilde{T}_{--})^h \right] = \left[ (\tilde{T}_{00})^0 + (\tilde{T}_{01})^0, \frac{1}{2} \left( \tilde{T}_{00} - \tilde{T}_{01} \right)^h \right] = 0,$$  
(84)

if the secondary constraints induced by the anomaly-free conditions given in Eqs. (69) are also invoked. Indeed, these respond to the conservation relations

$$\partial_+ \tilde{T}_{++} = \partial_- T(z) = 0,$$  
(85)

$$\partial_+ \tilde{T}_{--} = \partial_- \tilde{T}(\bar{z}) = 0,$$  
(86)

where Wick rotations are considered in accordance with Eqs. (73) and (74). Therefore, the identifications of the Fourier modes of the improved energy-momentum tensor with the generators of conformal transformations in Eqs.
can be explicitly given as
\[ e^{2m\tau} (\tilde{T}_{++})^m = \frac{1}{2} \int_0^\pi d\sigma e^{2m(\sigma + \gamma)} \tilde{T}_{++} = \int_0^{2\pi} dx^1 e^{m(x^1 + ix^0)} T(z) \equiv L_m, \]  
\[ e^{-2m\tau} (\tilde{T}_{--})^m = \frac{1}{2} \int_0^\pi d\sigma e^{2m(\sigma - \gamma)} \tilde{T}_{--} = \int_0^{2\pi} dx^1 e^{m(x^1 - ix^0)} \tilde{T}(\bar{z}) \equiv \bar{L}_m, \]
where the variable \( \sigma \) has been re-scaled in order to have the conventional range of integration.

The algebras of interest amount to the explicit derivation, on shell of Eq. (69), as follows,
\[
\left[ \frac{\tilde{\xi}_{01} \pm \tilde{\xi}_{00}}{2} \right]^m, \left( \frac{\tilde{\xi}_{01} \pm \tilde{\xi}_{00}}{2} \right)^n = (n - m) \left( \frac{\tilde{T}_{01} \pm \tilde{T}_{00}}{2} \right)^{n+m} + \frac{D}{12} \left( \frac{(m^3 - m) - (n^3 - n)}{2} \right) \delta_{m+n,0} \\
\frac{m(1 - \delta_{n,0}) - n(1 - \delta_{m,0})}{16\pi^2 T} \int_{n+m} \left( \frac{g_{\alpha} \pm g_{\bar{\alpha}}}{2} \right) \\
\frac{m(1 - \delta_{n,0}) - n(1 - \delta_{m,0})}{12\pi^3 T^2} \int_{n+m} \left( \frac{g_{\alpha} \pm g_{\bar{\alpha}}}{2} \right) \\
\frac{(m^3 - m)(1 - \delta_{n,0}) - (n^3 - n)(1 - \delta_{m,0})}{48\pi^3 T^2} \int_{n+m} \left( \frac{g_{\alpha} \pm g_{\bar{\alpha}}}{2} \right) \\
\frac{(m^3 - m)(1 - \delta_{n,0}) - (n^3 - n)(1 - \delta_{m,0})}{24\pi^3 T^2} \int_{n+m} \left( \frac{g_{\alpha} \pm g_{\bar{\alpha}}}{2} \right) + O(5) + \cdots, \tag{89}
\]
where
\[
g_{\alpha} = \frac{1}{T^2} \Pi G(1 - \alpha_3) \bar{E}_2 G \Pi + \bar{X}(1 - \alpha_4) \bar{E}_2 \bar{X} + \frac{2}{T} \Pi G ((1 - \alpha_1) E_1 + (1 - \alpha_2) E_3) \bar{X}, \tag{90}
\]
\[
g_{\bar{\alpha}} = \frac{1}{T^2} \Pi G(1 - \bar{\alpha}_3) \bar{E}_2 \Pi \bar{G} + \bar{X}(1 - \bar{\alpha}_4) \bar{E}_2 \bar{X} + \frac{2}{T} \Pi G ((1 - \bar{\alpha}_1) E_1 + (1 - \bar{\alpha}_2) \bar{E}_3) \bar{X}. \tag{91}
\]

Remarkably, the dependence of algebras of generators on the improvement terms is observed in Eqs. (89). Thus, the closure of algebras for the light-components of energy-momentum tensor could not be a generic feature. In addition, both the two algebras mentioned above can not be simultaneously closed up to c-numbers in view of the particular structure given in terms of the ± signs. Therefore, it may be of interest whether there exist some particular choices of field variables for which one of the algebras of Eqs. (89) may reproduce the desired closed Virasoro version. This situation seems to be realised by requiring either \( \bar{E}_i \)'s = \( E_i \)'s or \( \bar{E}_i \)'s = \( -E_i \)'s, each of which is accompanied with the conditions \( \alpha_i \)'s = \( \bar{\alpha}_i \)'s, but none of these solutions are permissible to the consistency equations given in Eqs. (49) + (50). However, the possibility for further compacting the form of Eqs. (89) remains. For example, considerable simplifications can be achieved by means of putting the quantity \( DY \) zero and/or discarding some of the traceless parts of the six matrices for the improvement terms. The traceless part of matrices are reducible because they are restrained only by four of the consistency conditions in Eqs. (49) + (50).

For consistency check, it is worthwhile to note that all the coefficients for the central extension in Eqs. (89) tend to vanish when one of the generators are given in the zeroth mode. The corresponding structure may also be formally derived from Eqs. (89), namely,
\[
\left[ \frac{\tilde{T}_{01} \pm \tilde{T}_{00}}{2} \right]^0, \left( \frac{\tilde{T}_{01} \pm \tilde{T}_{00}}{2} \right)^n = \left[ \frac{\tilde{T}_{00}}{2} \right]^0, \left( \frac{\tilde{T}_{01} \pm \tilde{T}_{00}}{2} \right)^n = \left[ \frac{\tilde{T}_{00}}{2} \right]^0, \left( \frac{\tilde{T}_{01} \pm \tilde{T}_{00}}{2} \right)^n = n \left( \frac{\tilde{T}_{01} \pm \tilde{T}_{00}}{2} \right)^n, \tag{92}
\]
based on Eqs. (77-80). Certainly, the conservation relations above coincides with the ones suggested by the Virasoro version.

V. DISCUSSIONS

To summarise, the quantisation should not be restricted by the field content prescribed a priori; field contents themselves pertains to the properties of the quantised theory itself. The incorporation with field redefinition exploits the possibility to end up the quantisation with permissible configurations against the conventional starting point.

Considerably, the method of canonical quantisation further arms the traditional field theoretical treatments with extra interesting information. The corresponding intuitive implementation for the field redefinition appears more viable, in contrast to those within the path integral approaches, by means of its relevance to the canonical transformation among dynamically equivalent systems [24-26], even though further generalisation remains possible [50]. The ambiguity contribution for various operators and their algebras thus responds to reformulating the theory with alternative field variables. Specifically, this leaves freedom appropriate for the historical Schwinger terms [1,2] in the current algebra. Moreover, as it was shown in section [11] nor can be the form for the charge algebra heuristically predicted by its classical version since the quantisation entails the field redefinition that exceeds beyond the scope of the appending total divergence to the current operators.

In practise, speaking of the results for the nonlinear sigma model in Section [11] the field redefinition in the Hamiltonian scheme first retains the embedding flexibility for quantisation and then regains the consistency by deviating field properties from the convention scalar prescription after a simple covariant requirement [25]. Noteworthy is only that the use of the term energy-momentum tensor stress energy tensor is prevented in our formulation. Derived according to the variation with respect to the space-time metric, the meaning of such term is less concrete in the Hamiltonian dynamics of flat space-time, even though possible linkages of this to the Belinfante tensor could be imagined [24, 29]. Furthermore, the conformal anomaly conditions indeed appear to be less strict than the conventional Weyl anomaly conditions [21, 22, 23, 24]. Remarkably, the additional restrictions, e.g. \( D = 26 \), are originated from the reparametrisation invariance that imposes the equivalence between the Weyl and conformal symmetry on the two dimensional field theories [24, 27].

The calculation in Section [1V] explicitly probes the quantised algebras at the charge level. We find it impossible to recover the the closed conformal algebras, i.e., the Virasoro version, at the two loop level for the problem at hand. However, the conformal currents conserve whenever the symmetry is recovered. Subsequently, the compatibility with the conventional closure expectation based on the earlier lower-order treatments [10] remains undisturbed. This is because the discrepancy for the conformal algebra is only up to \( O(4) \) which corresponds to conventional two-loop level, where the symmetric and traceless properties for the canonical energy-momentum tensor have been lost.

To conclude, exploiting the formulation in terms of alternative sets of field variables should of fundamental interests for any models. While the field redefinition responds to generalising the field content, the present study the nonlinear sigma model in fact offers consistent issues: In the historical treatment of Yang-Mill models [11, 29, 54], it was known that the gauge invariant conformal charge cannot form the closed algebras. This does not actually bother the conformal symmetry but exhibits the non-integrable structure [34, 35] in response to the path-dependent finite transformation of gauge fields. Hence, vector potential could be important ingredient for the conformal symmetry equipped with algebra of nontrivial form. On the other hand, the field variables of nontrivial spin structure may enter [29] into the nonlinear sigma model, and respond to the vector fields, by means of the failure of canonical energy-momentum tensor to be manifestly symmetric and traceless at the two loop order. Moreover, gauge potentials are essential components for the study of the fermionization of nonlinear sigma model [48, 49]; the presence of the such degree of freedom need not come as surprise. Our work attempts to address itself to the concrete picture and to the further applicability for the canonical quantisation. Of course, more open questions are left, in a straightforward manner. These may concern formulating on the curved space-time, explicit incorporating gauge fields or even the super-fields. However, we close the present work by further claiming the following fundamental point. The employment of the Hamiltonian field redefinition subject to preservation of canonical commutators needs not be the most generic handling of field theory [50]. We hope that through all the efforts made on the nonlinear sigma model more insight for the field theory in general can be revealed.
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