Solutions of the Bogoliubov-de Gennes equation with position dependent Fermi–velocity and gap profiles

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It is shown that bound state solutions of the one dimensional Bogoliubov-de Gennes (BdG) equation may exist when the Fermi velocity becomes dependent on the space coordinate. The existence of bound states in continuum (BIC) like solutions has also been confirmed both in the normal phase as well as in the superconducting phase. We also show that a combination of Fermi velocity and gap parameter step-like profiles provides scattering solutions with normal reflection and transmission.

Introduction
The Bogoliubov-de Gennes (BdG) equation plays a particularly important role in the context of superconductivity [1, 2]. This equation is also essential in the study of Andreev refelction [3–5]. Recently the BdG equation with a linear potential [6] has been studied in the presence of modified uncertainty principle or a minimal length formalism [7, 8]. Since its inception the Minimal length formalism [9, 10] has been studied in various contexts. In particular various quantum mechanical models have been studied within the minimal length formalism to understand the effect of the minimal length parameter on observables like energy [11–20]. These phenomena actually can be used to obtain a bound on the minimal length/momentum parameter. In ref. [6] it was shown that in the BdG equation bound states do not exist in the absence of minimal length (maximal momentum) while in the presence of minimal length (maximal momentum) bound states do exist with energy depending on the relevant parameter introduced by the model.

In this article our objective is to suggest an alternative scenario to create bound states or bound states in continuum (BIC) in the BdG equation. Here our approach would be to allow space modulation of the Fermi velocity $v_F$. Such a scenario has been considered earlier in various contexts e.g, in graphene [21–24], one dimensional hetero-structures [25] etc. It was also shown that in one dimensional hetero-structures space modulation of Fermi velocity helps creation of bound states as well as BIC [26]. Here it will be shown that such a scenario may be replicated in the case of the BdG equation also. Finally we also address the question of normal reflectance and transmission within the BdG equation. It will be shown that a step-like order parameter ($\Delta$) and Fermi velocity ($v_F$) profiles entails scattering solutions with non zero reflection.

Formalism
To begin with we note that the BdG Hamiltonian in the Andreev approximation is given by [2, 3, 6]

$$H = \begin{pmatrix} v_F p_x & \Delta \\ -\Delta & -v_F p_x \end{pmatrix}$$

where $v_F, \Delta$ are respectively the Fermi velocity and the superconductor order (gap) parameter. However it may be noted that when the Fermi velocity depends on the space coordinate the operator $H$ no longer remains Hermitian and in order to maintain Hermiticity the term $v_F p_x$ has to be replaced by $\sqrt{v_F(x)} p_x \sqrt{v_F(x)}$ [25]. With this replacement and allowing for a position dependent $\Delta$ the Hamiltonian in Eq. (1) becomes:

$$H = \begin{pmatrix} \sqrt{v_F(x)} p_x \sqrt{v_F(x)} & \Delta(x) \\ -\Delta(x) & -\sqrt{v_F(x)} p_x \sqrt{v_F(x)} \end{pmatrix}$$

In the following we will discuss the solutions of the corresponding component equations associated to the BdG Hamiltonian of Eq. (2) upon introducing a two-component spinor $\psi^T = (\psi_1, \psi_2)$:

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

dealing in particular with stationary state solutions of the associated eigenvalue equation $H \psi = E \psi$:

$$H \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = E \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$  

First of all we easily verified that the time-dependent Dirac equation, c.f. Eq. (3), associated to the Hamiltonian given in Eq.(1) admits a conserved probability current. The continuity equation:

$$\frac{\partial}{\partial x} j(x, t) + \frac{\partial}{\partial t} \rho(x, t) = 0,$$

can be derived by using straightforward procedures, i.e. combining appropriately the two components of Eq. (3). The probability current $j(x, t)$ is found to be given by:

$$j(x, t) = (\sqrt{v_F} \psi_1)^* (\sqrt{v_F} \psi_1) - (\sqrt{v_F} \psi_2)^* (\sqrt{v_F} \psi_2)$$

and the time dependent probability density is as usual $\rho(x, t) = \langle \psi | \psi \rangle = \psi_1^* \psi_1 + \psi_2^* \psi_2$. 

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In terms of the components the eigenvalue equation given in Eq.(4) reads

\[
\sqrt{v_F(x)}p_x \sqrt{v_F(x)} \psi_1 + \Delta(x) \psi_2 = E \psi_1
\]

\[
- \sqrt{v_F(x)}p_x \sqrt{v_F(x)} \psi_2 + \Delta(x) \psi_1 = E \psi_2
\]

In order to find the components \(\psi_{1,2}\) we now multiply the above equations from the left by \(\sqrt{v_F(x)}\) and writing

\[
\sqrt{v_F(x)}\psi_{1,2} = \phi_{1,2}
\]

we obtain:

\[
v_F(x)p_x \phi_1 + \Delta(x) \phi_2 = E \phi_1 \quad (9a)
\]

\[
-v_F(x)p_x \phi_2 + \Delta(x) \phi_1 = E \phi_2 \quad (9b)
\]

The components can now be easily decoupled. For example, it can be easily shown that the component \(\phi_1\) satisfies the equation

\[
-v_F^2(x)\frac{d^2 \phi_1}{dx^2} - v_F(x)v_F'(x)\frac{d \phi_1}{dx} + \frac{\Delta(x)^2}{\hbar^2} \phi_1 = \frac{E^2}{\hbar^2} \phi_1 \quad (10)
\]

The lower component \(\phi_2\) is then computed from Eq. (9a). We shall now choose specific velocity and gap profiles to examine creation of bound states, BIC and scattering solutions.

Constant gap and hyperbolic Fermi velocity profile. First we consider a step-like velocity profile along with constant gap:

\[
v_F(x) = v_0 \cosh^2(\alpha x), \quad v_0 > 0, \quad (11a)
\]

\[
\Delta(x) = \Delta = \text{const.} \quad (11b)
\]

Eq. (10) can be further reduced and brought to the standard Schrödinger form:

\[
\frac{d^2 \phi_1}{dq^2} + \epsilon^2 \phi_1 = 0, \quad (12)
\]

where \(q\) and \(\epsilon\) are given by

\[
q = \int \frac{dx}{v_F(x)}, \quad \epsilon^2 = \frac{E^2 - |\Delta|^2}{\hbar^2} \quad (13)
\]

In this case we find from Eq. (13)

\[
q = \frac{\tanh(\alpha x)}{\alpha v_0}, \quad -\frac{1}{\alpha v_0} < q < \frac{1}{\alpha v_0} \quad (14)
\]

We now consider \(E^2 > |\Delta|^2\) i.e, \(\epsilon > 0\). In this case the solution of Eq.(12) is given by

\[
\phi_1(q) = c_1 \sin(\epsilon q) + c_2 \cos(\epsilon q) \quad (15)
\]

where \(c_{1,2}\) are arbitrary constants. For bound states the wave function \(\phi_1(q)\) should vanish at the boundary values i.e,

\[
\phi_1(\pm \frac{1}{\alpha v_0}) = 0 \quad (16)
\]

Then from Eq. (15) it follows that

\[
\phi_{n1}(q) = N_n \sin \left( \sqrt{\frac{E_n^2 - |\Delta|^2}{\hbar^2}} q \right) \quad (17a)
\]

\[
E_n^2 = n^2\pi^2\alpha^2 v_0^2 \hbar^2 + |\Delta|^2, \quad n = 1, 2, 3, \ldots \quad (17b)
\]

where \(N_n\) is a normalization constant. We now go back to the \(x\) space to obtain \(\psi_1(x)\) via Eq.(8). Subsequently one has to use (9) to obtain the component \(\psi_2(x)\). Thus the complete solution of the BdG equation is given by:

\[
\psi_n = \begin{pmatrix} \psi_{n1} \\ \psi_{n2} \end{pmatrix} = N_n \begin{pmatrix} \sech(\alpha x) \sin(n\pi \tanh(\alpha x)) \\ \frac{\alpha v_0}{\Delta} \cos(n\pi \tanh(\alpha x)) \end{pmatrix} \quad (18)
\]

The normalization constant can be determined from the relation

\[
\int_{-\infty}^{\infty} \psi_n^\dagger \psi_n dx = \int_{-\infty}^{\infty} \left( \psi_{n1}^* \psi_{n1} + \psi_{n2}^* \psi_{n2} \right) dx = 1 \quad (19)
\]

and is given by

\[
N_n = \sqrt{\frac{\alpha}{2}} \left[ 1 + (n\pi\gamma)^2 \right]^{-\frac{1}{2}} \quad (20)
\]

From the wave-function given in Eq. (18) one can easily deduce the probability density \(\rho_n(x) = \psi_{n1}^* \psi_{n1} + \psi_{n2}^* \psi_{n2}\).

We find:

\[
\rho_n(x) = \frac{\alpha}{2} \frac{\sech^2 \alpha x}{\left[ 1 + (n\pi\gamma)^2 \right]^{\frac{3}{2}}} \left[ 2 \sin^2(n\pi \tanh(\alpha x)) + (n\pi\gamma)^2 \right] \quad (21)
\]

and we observe that for large values of the quantum number \(n\) the probability density becomes independent of \(n\):

\[
\lim_{n \to \infty} \rho_n(x) = \frac{\alpha}{2} \sech^2 \alpha x
\]
Eq. (18) actually develop nodes (zeros of the probability density) only if $\gamma = 0$. Such behaviour is depicted in Fig. 1 where we plot the probability density for the first excited states and for different choices of the parameter $\gamma$. For a stationary state the probability density is time independent and thus the continuity equation dictates that the current satisfies $\partial j/\partial x = 0$, or that $j(x,t)$ be a constant in space. Indeed for the solutions of the discrete levels given by Eq. (18) the current can be easily derived, using the shorthand notation $A_n(x) = n\pi \tanh(\alpha x)$, as:

$$j_n(x,t) = \frac{N^2}{\alpha v_0} \sin^2[A_n] - \frac{(n\pi\gamma)^2}{2} \cos^2[A_n]$$

Eq. (18) actually develop nodes (zeros of the probability density) only if $\gamma = 0$. Such behaviour is depicted in Fig. 1 where we plot the probability density for the first excited states and for different values of the parameter $\gamma = 0.25, 0.5, 0.75, 1$.

Figure 1. Probability density for the discrete levels given in Eq. (18), $\rho_n(x)\alpha^{-1}$. Shown are the first few excited levels ($n = 1, 2, 3$), as function of the position $x$ (measured in units of $\alpha^{-1}$) for different values of the parameter $\gamma := 0.25, 0.5, 0.75, 1$.
Note that the explicit limiting form of $\rho_{\psi}(x)$ as $x \to \pm \infty$ ensures that they decay to 0 probability densities for various values of the dimensionless parameter $\mu$. The probability density is found to be given as:

$$\psi^> = \begin{pmatrix} \psi_1^>(x) \\ \psi_2^>(x) \end{pmatrix} = N^> \sech(\alpha x) \left( \sin \left[ \frac{E}{\Delta} \tanh(\alpha x) \right] + i \frac{\Delta}{\alpha} \mu \cos \left[ \frac{E}{\Delta} \tanh(\alpha x) \right] \right)$$

Eq. (28) is easily checked to be exactly normalised to unity.

The current $j^>$ is computed using the wave functions in Eq. (25) into the general expression given in Eq. (6). A straightforward computation gives:

$$j^> = -\frac{\alpha v_0}{2} \frac{\mu^2}{1 + \mu^2 - \frac{\gamma}{2\mu} \sin \left( \frac{2\mu}{\gamma} \right)}.$$  

In Fig. 2 we plot the probability density $\rho_{\psi}(x)$ for different values of the parameter $\gamma$ as function of the position $x$ (measured in units of $\alpha^{-1}$) for different values of the parameter $\gamma := 0.5$ (left) and $\gamma := 0.75$ (right).

From the above expressions it may be observed that since $\sin$ and the $\cos$ can not exceed unity the $\sech(\alpha x)$ term in the wave functions ensures that they decay to 0 as $x \to \pm \infty$.

In fact the normalization constant can be exactly evaluated and is given by

$$N^> = \sqrt{\frac{2}{\alpha}} \left[ (1 + \mu^2) - \frac{\gamma}{2\mu} \sin \left( \frac{2\mu}{\gamma} \right) \right]$$

We note that for values of the energy close to $|\Delta|$ or $\mu \to 0$, the probability density approaches a limiting shape that depends only on the parameter $\gamma$:

$$\lim_{\mu \to 0^+} \rho^>(x) = \frac{\alpha \sech^2(\alpha x)}{2(1 + \frac{\gamma}{\alpha^2})} \left[ 1 + \frac{2}{\gamma^2} \tanh^2(\alpha x) \right].$$

Note that the explicit limiting form of $\rho_{\psi}(x)$ given in

$$\psi^< = \begin{pmatrix} \psi_1^<(x) \\ \psi_2^<(x) \end{pmatrix} = N^< \sech(\alpha x) \left( \sin \left[ \frac{E}{\Delta} \tanh(\alpha x) \right] + i \frac{\Delta}{\alpha} \nu \cosh \left[ \frac{E}{\Delta} \tanh(\alpha x) \right] \right)$$

It is not difficult to see that both $\psi_1^>$, go to zero as $x \to \pm \infty$. In fact the normalization constant can be
Figure 3. Probability density \( \rho_<(x) \alpha^{-1} \) for the BIC solutions in the region \( E > |\Delta| \) given in Eq. (32). Shown are the probability densities for various values of the dimensionless parameter \( \nu \), as function of the position \( x \) (measured in units of \( \alpha^{-1} \)) for different values of the parameter \( \gamma := 0.5 \) (left) and \( \gamma := 0.75 \) (right).

found to be

\[
N_<^{-1} = \sqrt{\frac{2}{\alpha}} \sqrt{\frac{\gamma}{2\nu} \sinh \left( \frac{2\nu}{\gamma} \right) - (1 - \nu^2)} \tag{33}
\]

From the wave-function given in Eq. (32) one can easily deduce the probability density \( \rho_<(x) = \psi_1^* \psi_1^< + \psi_2^* \psi_2^< \). We find:

\[
\rho_<(x) = \frac{\alpha^2}{2} \frac{\text{sech}^2 \alpha x}{\sinh \left( \frac{2\nu}{\gamma} \right) - (1 - \nu^2)} \tag{34}
\]

The current \( j_< \) is again computed using the wave functions in Eq. (32) into the general expression given in Eq. (6). A straightforward computation now gives:

\[
j_<= -\frac{\alpha v_0}{2} \frac{\nu^2}{\gamma \sinh \left( \frac{2\nu}{\gamma} \right) - (1 - \nu^2)}. \tag{35}\]

In Fig. 3 we have plotted the probability density \( \rho_<(x) \) which clearly exhibits the BIC nature of the wave functions (note that the parameter \( \nu \) can be changed continuously in the interval \([0, 1]\)). We observe that for vanishing values of the energy, \( E \to 0 \) or \( \nu \to 0 \), the probability density approaches a shape that depends only on the parameter \( \gamma \) and is therefore independent of the energy:

\[
\lim_{\nu \to 0} \rho_<(x) = \lim_{\nu \to 0^+} \rho_>(x) = \frac{\alpha \text{sech}^2(\alpha x)}{2(1 + \frac{2\nu}{\gamma})} \left( 1 + \frac{2}{\gamma^2} \tanh^2(\alpha x) \right). \tag{36}\]

Note that the explicit limiting form of \( \rho_<(x) \) given in Eq. (36) is easily seen to coincide with the limiting form obtained in Eq. (28) and thus again to be exactly normalised to unity. Thus the BdG equation possesses BIC solutions in both the normal phase, \( E > |\Delta| \) and the superconducting phase, \( E \leq |\Delta| \). In Fig. 4 we show the probability currents as function of the energy in units of the gap parameter \( |\Delta| \) (\( E/|\Delta| \)) where we combine the results found above about the probability currents, c.f., Eqs. (22,29,35).

We remark that with the specific choice in Eq. 11 of the velocity profile the solutions described above, discrete bound states and bound states in continuum (BIC) exhaust all solutions. There are no scattering states. This is not surprising since the velocity profile in Eq. 11 has been chosen so, just to be able to find bound states and BIC solutions. In order to have scattering solutions different velocity and/or gap profiles must be chosen. We will take up this point in the next paragraph.

Figure 4. Minus the probability current in units of \( \alpha v_0/2 \), \( -j/(\alpha v_0/2) \), for both discrete bound states and BIC solutions as function of the energy \( E \) in units of the gap parameter \(|\Delta|\). The BIC current extends to the region \( E < |\Delta| \) while in the region \( E \geq |\Delta| \) we show both the currents of the BIC-like solution (continuous line) as well as the currents of the discrete bound states (full dots).

Step-like Fermi velocity and gap profiles
It is well known that it is not entirely trivial to describe non zero reflectance within the massless Dirac equation.
For instance ref. [25] discusses a massless Dirac Hamiltonian with a position dependent Fermi velocity $v_F(x)$ finding that a simple step-like profile is not enough to get non zero reflectance. Here our aim is to show that with the Hamiltonian given in Eq. (2) introducing a step-like gap parameter allows scattering states with non vanishing reflectance. We take up the following profiles:

$$v_F(x) = v_F^+ \theta(-x) + v_F^- \theta(x),$$
$$\Delta(x) = \Delta^+ \theta(-x) + \Delta^- \theta(x).$$

We note that integrating the first order intertwining normal reflectance $\Delta$ vanishing phase difference between $\Delta^+$ and $\Delta^-$ to trigger normal reflectance. We implement the boundary conditions in Eqs. (38a&38b) in the above solutions $\phi_{1,2}(x)$ and find:

$$\phi_2(x) = \left\{ \begin{array}{ll}
A \left( \frac{E}{\Delta^+} - \frac{hv_F^+ k_-}{\Delta^+} \right) e^{+ik_- x} + B \left( \frac{E}{\Delta^+} + \frac{hv_F^+ k_+}{\Delta^+} \right) e^{-ik_+ x} & \text{if } x < 0 \\
C \left( \frac{E}{\Delta^-} - \frac{hv_F^- k_+}{\Delta^-} \right) e^{+ik_+ x} & \text{if } x > 0
\end{array} \right.$$  

We implement the boundary conditions in Eqs. (38a&38b) in the above solutions $\phi_{1,2}(x)$ and find:

$$B = A \frac{E}{\Delta^+} - \frac{hv_F^+ k_-}{\Delta^+} + \frac{hv_F^+ k_+}{\Delta^+}$$  

$$C = A + B$$

We note that if $\Delta(x) = \text{const.}$ (i.e. $\Delta^+ = \Delta^-$) there is no reflection because the coefficient $B$ vanishes identically even if $v_F \neq v_F^\pm$.

The probability current is then easily found from $j(x) = |\phi_1|^2 - |\phi_2|^2$ as:

$$j(x) = \begin{cases} 
\bar{j}_{\text{inc}} - j_{\text{refl.}} & \text{if } x < 0 \\
\bar{j}_{\text{trans.}} & \text{if } x > 0
\end{cases}$$

with:

$$\bar{j}_{\text{inc.}} = |A|^2 \left\{ \frac{1 - \left( \frac{E - hv_F^+ k_-}{\Delta^+} \right)^2}{|\Delta^+|^2} \right\}$$

$$\bar{j}_{\text{refl.}} = |B|^2 \left\{ \frac{\left( \frac{E + hv_F^+ k_-}{\Delta^+} \right)^2 - 1}{|\Delta^+|^2} \right\}$$

$$\bar{j}_{\text{trans.}} = |C|^2 \left\{ \frac{1 - \left( \frac{E - hv_F^+ k_+}{\Delta^-} \right)^2}{|\Delta^-|^2} \right\}$$

We find that Eq. (10) admits the solution:

$$\phi_1(x) = \begin{cases} 
A e^{ik_- x} & \text{if } x < 0 \\
B e^{ik_+ x} & \text{if } x > 0
\end{cases}$$

and the corresponding lower component $\phi_2(x)$ is found $x = 0$ one finds the following boundary conditions:

$$\phi_1(0^-) = \phi_1(0^+)$$
$$\phi_2(0^-) = \phi_2(0^+)$$

Then one can easily solve the corresponding second order equations in Eq. (10). Upon defining:

$$k_\pm = \frac{\sqrt{E^2 - |\Delta|^2}}{(hv_F^\pm)^2}$$

The following scattering solution is found:

$$\phi_1(x) = \begin{cases} 
A e^{ik_- x} + Be^{-ik_+ x} & \text{if } x < 0 \\
Ce^{ik_+ x} & \text{if } x > 0
\end{cases}$$

The lower component $\phi_2(x)$ is obtained using the intertwining relations in Eq. (9):

So that the reflectance ($R$) and transmittance ($T$) coefficients can be deduced as:

$$R = \frac{j_{\text{refl.}}}{j_{\text{inc.}}} = \frac{|B|^2 - |\Delta|^2 + (E + hv_F^+ k_-)^2}{|A|^2 + |\Delta|^2 - (E - hv_F^+ k_-)^2}$$

$$T = \frac{j_{\text{trans.}}}{j_{\text{inc.}}} = \frac{|C|^2 |\Delta|^2 - |\Delta^+|^2 - (E - hv_F^+ k_+)^2}{|A|^2 |\Delta|^2 - (E - hv_F^+ k_-)^2}$$

With the help of Eqs. (40a&40b) the reflectance $R$ and transmittance $T$ can be explicitly computed as functions of the energy eigenvalue $E$. In Fig. 5 we show $R$ and $T$ for a particular model of steplike gap: $\Delta^+ = \Delta$ and $\Delta^- = \Delta e^{i\phi}$. We provide curves for $R$ and $T$ as functions of $E/|\Delta|$ for different values of the phase difference ($\phi = \pi, \pi/2, \pi/3$).

We now discuss the region $E^2 < \min \{|\Delta|^2, |\Delta^+|^2\}$ and we show that it admits a single bound state. Here we can define:

$$\kappa_- = \frac{\sqrt{|\Delta|^2 - E^2}}{(hv_F^+)^2}$$

$$\kappa_+ = \frac{\sqrt{|\Delta|^2 - E^2}}{(hv_F^-)^2}$$

We find that Eq. (10) admits the solution:

$$\phi_1(x) = \begin{cases} 
B e^{-ik_- x} & \text{if } x < 0 \\
C e^{ik_+ x} & \text{if } x > 0
\end{cases}$$

and the corresponding lower component $\phi_2(x)$ is found.
in the positive branch of the spectrum. Fig. 6 shows the merging of the two solutions of Eq. 45 and only if:

\[ E > \varepsilon > 0 \]

The boundary conditions in Eqs. (38a&38b) are easily found to admit a non vanishing solution for \( B \) and \( C \) if and only if:

\[ E + i h v_F \kappa_- \frac{\Delta}{\Delta^*} = E - i h v_F \kappa_+ \frac{\Delta}{\Delta^*} \]

which is a second order equation, see Eq. (43), for the eigenvalue \( E \) of the bound state. The two solutions are:

\[ E = \pm \sqrt{|\Delta^*| \Delta^*} \sin \phi \sqrt{2 \cos \phi - \frac{|\Delta^*| - |\Delta^2| \Delta^*}{|\Delta^*|^2}} \]

The boundary conditions in Eqs. (38a&38b) are easily found to admit a non vanishing solution for \( B \) and \( C \) if and only if:

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which is a second order equation, see Eq. (43), for the eigenvalue \( E \) of the bound state. The two solutions are:

\[ E = \pm \sqrt{|\Delta^*| \Delta^*} \sin \phi \sqrt{2 \cos \phi - \frac{|\Delta^*| - |\Delta^2| \Delta^*}{|\Delta^*|^2}} \]

Fig. 6 shows the merging of the two solutions of Eq. 45 in the positive branch of the spectrum.

We can then consider the simpler model \( \Delta^+ = \Delta \) and \( \Delta^- = \Delta e^{i\phi} \) and find that Eq. (45) reduces to:

\[ E = \pm \cos \frac{\phi}{2} \Delta \]

and in the positive branch of the spectrum we have then the single bound state at \( E = \cos (\phi /2) \Delta \) (blue line in Fig. 6). The corresponding spinor wave function \( \psi \) can be computed with the help of Eq. (8):

\[ \psi(x) = \begin{cases} 
B \frac{1}{\sqrt{v_F}} \left( e^{-i\phi/2} \right) e^{+\kappa_- x} & \text{if } x < 0, \\
B \frac{1}{\sqrt{v_F}} \left( e^{+i\phi/2} \right) e^{-\kappa_+ x} & \text{if } x > 0.
\end{cases} \]

We note that given the continuity of the \( \phi_{1,2} \) components the corresponding \( \psi_{1,2} \) spinor components are in general not continuous (if \( v_F \neq v_F^+ \)). This may reflect in a discontinuity in the probability density. The spinor in Eq. (47) is straightforwardly normalised to unity and the corresponding probability density is computed as:

\[ x_0 \rho(x) = \frac{\phi}{2} \begin{cases} 
eq 0 & \text{if } x < 0 \\
\frac{1}{v_F^+} e^{-2 \frac{\phi}{v_F^+} \sin (\phi /2) \frac{x}{|\Delta|}} & \text{if } x > 0.
\end{cases} \]

Fig. 6. Positive branch of the energy eigenvalue obtained merging the two solutions of Eq. 44 given in Eq. (45) for different values of the parameter \( \epsilon = |\Delta_0|/|\Delta^*| \). Note that the solutions in Eq. (45) are symmetric under the exchange \( |\Delta^*| \leftrightarrow |\Delta| \) (or \( \epsilon \leftrightarrow 1/\epsilon \)).

Discussion and Conclusions

While the 1-dim BdG hamiltonian of Eq. (1) does not admit in general bound states, introducing a position dependent Fermi velocity and gap parameter as in Eq. (11) opens up the possibility of having bound states. Discrete bound states are found however only in the region \( E > |\Delta| \). Interestingly we find bound states in the continuum (BIC) like solutions. It is worth pointing out that BIC like solutions are found both in the region \( E > |\Delta| \) as well as in the region \( E \leq |\Delta| \). The two classes of BIC solutions merge continuously at \( E = |\Delta| \).
Figure 7. Probability density $x_0 \rho(x)$ as a function of $x/x_0$ for the step-like gap profile with a phase difference between $\Delta^-$ and $\Delta^+$. $\Delta^- = \Delta$ and $\Delta^+ = \Delta e^{i\phi}$. On the left we show the case $v_\rho = v_\rho^+$ (the probability density is continuous at $x = 0$), while on the right we have $v_\rho^+ / v_\rho^- = 0.9$ (the probability density shows a discontinuity at $x = 0$). In each plot the three curves are for three different values of the phase parameter $\phi$: $\phi = \pi$ (blue line), $\phi = \pi/2$ (orange line), and $\phi = \pi/3$ (green line).

can be seen explicitly by comparing the the limiting behavior of the probability density functions, respectively $\lim_{\nu \to 0^+} \rho_n(x)$ and $\lim_{\nu \to 0^+} \rho_\nu(x)$.

In particular we note how the BIC current $j_\nu$ in the region $E \leq |\Delta|$ (superconducting phase) merges with the BIC current $j_\nu$ in the region $E > |\Delta|$ (normal phase) ensuring continuity at $E = |\Delta|$. The same can be checked graphically by noticing that the curve $\mu = 0$ of Fig. 2 coincides with the curve $\nu = 0$ of Fig. 3.

Also we note how Fig. 4 shows explicitly that in the region $E \geq |\Delta|$, the probability currents of the discrete states (full dots), c.f. Eq. (22) exactly superimpose the the probability current of the BIC states (continuous line), c.f. Eq. (29).

In conclusion we have investigated the one dimensional BdG equation with position dependent Fermi velocity $v_F(x)$ and order parameter $\Delta(x)$. We have first considered a smooth velocity distribution $v_F(x) = v_0 \cosh(\alpha x)$ and constant order parameter $\Delta$ showing that it leads to the creation of bound states as well as bound states in continuum, but no scattering solutions. Interestingly enough we found that, with the introduction of a position dependent Fermi velocity profile, while in the normal phase ($E > |\Delta|$) are present both discrete bound states as well as BIC states, in the superconducting region ($E \leq |\Delta|$) there are only BIC states. Some features of quantities like the probability density have also been analysed. It has been shown that for large $n$ the probability density becomes independent of $n$. In the case of BIC the same feature is also observed as the probability density becomes independent of energy $E$ for large values of $E$.

We have then considered the case of combined step-like Fermi velocity and order parameter profiles. We find that a phase difference between $\Delta^+$ and $\Delta^-$ is sufficient to provide normal (non zero) reflectance in the region $E > \max \{|\Delta^+|, |\Delta^-|\}$. In the region $E < \min \{|\Delta^+|, |\Delta^-|\}$ a single bound state is found (positive branch of the spectrum). Other configurations could be the object of further work.

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