Convergence Analysis for Rectangular Matrix Completion
Using Burer-Monteiro Factorization and Gradient Descent

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Abstract

We address the rectangular matrix completion problem by lifting the unknown matrix to a
positive semidefinite matrix in higher dimension, and optimizing a nonconvex objective over
the semidefinite factor using a simple gradient descent scheme. With $O(\mu r^2 \kappa^2 n \max(\mu, \log n))$
random observations of an $n_1 \times n_2$ $\mu$-incoherent matrix of rank $r$ and condition number $\kappa$, where
$n = \max(n_1, n_2)$, the algorithm linearly converges to the global optimum with high probabil-
ity.

1 Introduction

A growing body of recent research is shedding new light on the role of nonconvex optimization for
tackling large scale problems in machine learning, signal processing, and convex programming. This work is developing techniques that help to explain the surprising effectiveness of relatively
simple first-order algorithms for certain nonconvex optimizations.

When applied to problems that can be formulated as semidefinite programs, these techniques
can often be viewed as part of a framework proposed by Burer and Monteiro [2]. The Burer-
Monteiro technique is based on factoring the semidefinite variable, and applying classical opti-
mization techniques to the resulting nonconvex objective over the factor. While worst-case com-
plexity considerations imply that such an approach cannot succeed in general, a series of recent
papers [9, 31, 28, 11, 1] has shown the strategy to be remarkably effective for a number of problems
of practical interest, with analytical convergence guarantees and strong empirical performance.

In this paper, we enlarge the collection of problems to which the Burer-Monteiro technique can
be successfully applied, by analyzing the convergence properties of gradient descent applied to the
problem of rectangular matrix completion from incomplete measurements. The standard matrix
completion problem asks for the recovery of a low rank matrix $X^* \in \mathbb{R}^{n_1 \times n_2}$ given only a small
fraction of observed entries. Let $\Omega$ be the set of $m$ indices of the observed entries. Fixing a target
rank $r \ll \min(n_1, n_2)$, the natural, but nonconvex objective is

$$
\min_{X \in \mathbb{R}^{n_1 \times n_2}} \quad \text{rank}(X)
$$

subject to $X_{ij} = X^*_{ij}, (i, j) \in \Omega.$

(1)
In order for this problem to be well-posed, it is important to understand when \(X^*\) is identifiable and, in particular, the unique minimizer of (1). Moreover, because the problem is in general NP-hard, it is essential to identify tractable families of instances, together with efficient algorithms having global convergence guarantees.

In the current work, we apply the factorization technique by “lifting” the matrix \(X^*\) to a positive semidefinite matrix \(Y^* \in \mathbb{R}^{(n_1+n_2) \times (n_1+n_2)}\) in higher dimension. Lifting is an established method that recasts vector or matrix estimation problems in terms of positive semidefinite matrices with special structure. It has been applied to sparse eigenvector approximation [12] and phase retrieval [8], where the lifted matrix is of rank one. As explained in detail below, we can construct \(Y^*\) to be of the same rank as \(X^*\), thus obtaining a factorization \(Y^* = Z^*Z^{*\top}\) for some \(Z^* \in \mathbb{R}^{(n_1+n_2) \times r}\), and transforming the original matrix completion problem into the problem of recovering the semidefinite factor \(Z^*\). We formulate this as minimizing a nonconvex objective \(f(Z)\), to which we apply a gradient descent scheme, using a particular spectral initialization. Our analysis of this algorithm establishes a lower bound on the number of matrix measurements that are sufficient to guarantee identifiability of the true matrix and geometric convergence of the gradient descent algorithm, with explicit bounds on the rate.

In the following section we give a full description of our approach. Our theoretical results are presented in Section 3, with detailed proofs contained in the appendix. Our analysis subsumes the case where \(X^*\) is positive semidefinite. In Section 4 we briefly review related work. The experimental results are presented in Section 5, and we conclude with a brief discussion of future work in Section 6.

2 Semidefinite Lifting, Factorization, and Gradient Descent

For any \((n_1+n_2) \times r\) matrix \(Z\), we will use \(Z_{(i)}\) to denote its \(i\)th row, and \(Z_U\) and \(Z_V\) to denote the top \(n_1\) and bottom \(n_2\) rows. The operator, Frobenius and \(\ell_\infty\) norm of matrices are denoted by \(||\cdot||\), \(||\cdot||_F\) and \(||\cdot||_\infty\), respectively. We define \(\|Z\|_{2,\infty} = \max_i \|Z_{(i)}\|_2\) as the largest \(\ell_2\) norm of its rows, and similarly \(\|Z\|_{\infty,2} = \max \{\|Z\|_{2,\infty}, \|Z^\top\|_{2,\infty}\}\). Let \(P_\Omega : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^{n_1 \times n_2}\) be the operator

\[
P_\Omega(X)_{ij} = \begin{cases} 
X_{ij} & \text{if } (i,j) \in \Omega, \\
0 & \text{otherwise}.
\end{cases}
\]

(2)

In this paper, we focus on completing an incoherent or “non-spiky” matrix \(X^*\). With \(U^*\Sigma^*V^*\) denoting the rank-\(r\) SVD of \(X^*\), we assume \(X^*\) is \(\mu\)-incoherent, as defined below.

**Definition 1.** The matrix \(X^*\) is \(\mu\)-incoherent with respect to the canonical basis if its singular vectors satisfy

\[
\|U^*\|_{2,\infty} \leq \sqrt{\frac{\mu r}{n_1}}, \quad \|V^*\|_{2,\infty} \leq \sqrt{\frac{\mu r}{n_2}},
\]

(3)

where \(\mu\) is a constant.\(^1\)

\(^1\)Note that \(\mu \geq 1\), since \(r = \|U^*\|_F^2 = \sum_{i \in [n_1]} \|U^*_{(i)}\|_2^2 \leq \mu r\).
Our main interest is the uniform model where \( m \) entries of \( X^\star \) are observed uniformly at random, though we shall analyze a Bernoulli sampling model, where each entry of \( X^\star \) is observed with probability \( p = m/n_1 n_2 \). One can transfer the results back to the uniform model, as the probability of failure under the uniform model is at most twice that under the Bernoulli model; see [6, 7].

Using the rank-\( r \) SVD of \( X^\star \), we can lift \( X^\star \) to

\[
Y^\star = \begin{bmatrix} U^\star \Sigma^\star U^\star \top & X^\star \\ X^\star \top & V^\star \Sigma^\star V^\star \top \end{bmatrix} = Z^\star Z^\star \top, \quad \text{where } Z^\star = \begin{bmatrix} U^\star \\ V^\star \end{bmatrix} \Sigma^{1/2}.
\]  

(4)

The symmetric decomposition of \( Y^\star \) is not unique; our goal is to find a matrix in the set

\[
S = \left\{ \tilde{Z} \in \mathbb{R}^{(n_1+n_2) \times r} \mid \tilde{Z} = Z^\star R \text{ for some } R \text{ with } RR^\top = R^\top R = I \right\},
\]

(5)

since for any \( \tilde{Z} \in S \) we have \( X^\star = \tilde{Z} U^\top \tilde{Z} V^\top \). Let \( \Omega \) denote the corresponding observed entries of \( Y^\star \), and consider minimization of the squared error

\[
\min_Z \frac{1}{2p} \sum_{(i,j) \in \Omega} (ZZ^\top - Y^\star_{ij})^2 = \min_Z \frac{1}{2p} \| \mathcal{P}_\Omega(ZZ^\top - Y^\star) \|_F^2.
\]

(6)

Note that \( Y^\star \) is not the unique minimizer of (6), nor is it the only possible positive semidefinite lifting of \( X^\star \). For example, let \( P \) be an \( r \times r \) nonsingular matrix, and form the matrices

\[
Z' = \begin{bmatrix} U^\star \Sigma^{1/2} P \\ V^\star \Sigma^{1/2} P^{-1} \end{bmatrix}, \quad Y' = \begin{bmatrix} U^\star \Sigma^{1/2} P^2 \Sigma^{1/2} U^\star \top & X^\star \\ X^\star \top & V^\star \Sigma^{1/2} P^{-2} \Sigma^{1/2} V^\star \top \end{bmatrix}.
\]

(7)

Since \( \Omega \) does not contain any entry in the top-left or bottom-right block, \( Y' \) is also a minimizer of (6). Thus, the solution set of the lifted problem is much larger than the set \( S \) of actual interest. For the sake of simple analysis, we shall focus on exact recovery of \( Y^\star \) only, and thus impose an additional regularizer to align the column spaces of \( Z_U \) and \( Z_V \), as in [28]. The regularized loss is

\[
f(Z) = \frac{1}{2p} \| \mathcal{P}_\Omega(ZZ^\top - Y^\star) \|_F^2 + \frac{\lambda}{4} \| Z^\top DZ \|_F^2, \quad \text{where } D = \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix}.
\]

(8)

While this apparently introduces an extra tuning parameter, our analysis shows that to establish linear convergence, \( \lambda \) should be smaller than the absolute constant \( \frac{2}{9} \), and thus one may treat \( \lambda \) as a fixed number.

It is discussed in [11] that one needs to ensure the iterates stay incoherent. Let \( \mathcal{C} \) be the set of incoherent matrices

\[
\mathcal{C} = \left\{ Z : \| Z \|_{2,\infty} \leq \sqrt{\frac{2\mu^2}{n}} \| Z^0 \| \right\}
\]

(9)

where we assume \( \mu \) is known and \( Z^0 \) will be determined.
Our algorithm is simply gradient descent on \( f(Z) \), with projection onto \( C \). Let \( M = p^{-1} \mathcal{P}_\Omega(UV^T - X^*) \). Then the gradient of \( f \) is given by

\[
\nabla f(Z) = \begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix} Z + \lambda DZZ^T DZ.
\] (10)

The projection \( \mathcal{P}_C \) to the feasible set \( C \) has closed form solution, given by row-wise clipping:

\[
\mathcal{P}_C(Z)_{(i)} = \begin{cases} 
Z_{(i)} & \text{if } \|Z_{(i)}\| \leq \sqrt{\frac{2\mu r}{n}} \|Z^0\|, \\
\frac{Z_{(i)}}{\|Z_{(i)}\|} \cdot \sqrt{\frac{2\mu r}{n}} \|Z^0\| & \text{otherwise.}
\end{cases}
\] (11)

Note that \( X^0 \equiv p^{-1} \mathcal{P}_\Omega(X^*) \) is an unbiased estimator of \( X^* \) under the Bernoulli model. To initialize, we thus construct \( Z^0 \) from the top rank-\( r \) factors of \( X^0 \).

Algorithm 1: Projected gradient descent for matrix completion

**Input:** \( \Omega, \{X^*_{ij} : (i, j) \in \Omega \}, m, n_1, n_2, r, \lambda, \eta \)

**Initialization**
- \( p = m/n_1 n_2 \)
- \( U^0 \Sigma^0 V^0^T = \text{rank-}r \text{ SVD of } p^{-1} \mathcal{P}_\Omega(X^*) \)
- \( Z^0 = [U^0 \Sigma^0 ; V^0 \Sigma^0] \)
- \( k \leftarrow 0 \)

**Repeat**
- \( M^k = p^{-1} \mathcal{P}_\Omega(Z^k U Z_k^T V - X^*) \)
- \( \nabla f(Z^k) = \begin{bmatrix} 0 & M^k \\ M^k^T & 0 \end{bmatrix} Z^k + \lambda DZZ^T DZ^k. \)
- \( Z^{k+1} = \mathcal{P}_C\left(Z^k - \frac{\eta}{\|Z^0\|^2} \nabla f(Z^k)\right) \)
- \( k \leftarrow k + 1 \)

**Until convergence;**

**Output:** \( \hat{Z} = Z^k, \hat{X} = Z^k U Z_k^T. \)

**Remarks.** (i) The step size \( \eta \) is normalized by \( \|Z^0\|^2 \). Our analysis will establish linear convergence when taking step sizes of the form \( \eta/\sigma_1^* \), where \( \eta \) is a sufficiently small constant. We replace \( \sigma_1^* \) by \( \|Z^0\|^2 \) in the actual algorithm since it is unkown in practice. (ii) The feasible set (9) depends on \( Z^0 \) as well. Under above spectral initialization, our analysis shows that when \( p \geq O(\mu r^2 \log n/n) \), the term \( \sqrt{\frac{2\mu r}{n}} \|Z^0\| \) is an upper bound of \( \|Z^*\|_{2,\infty} \) with high probability (see Corollary 1 below). This means \( S \) is a subset of \( C \). Note that this does not change the global optimality of \( Z^* \) and its equivalent elements, since \( f(Z^*) = 0 \). In practice, we find that the iterates of our algorithm remain incoherent, so that one may drop the projection step. (iii) The column
space regularizer (8) is also imposed for analysis only. When \( \lambda = 0 \), our algorithm typically converges to another PSD lifted matrix of \( X^* \), with minor difference from \( Y^* \) in the top-left and bottom-right blocks.

In the following section we state and sketch a proof of our main convergence result for this algorithm.

## 3 Main Result: Convergence Analysis

**Theorem 1.** Suppose that \( X^* \) is of rank \( r \), with condition number \( \kappa = \sigma_1^*/\sigma_r^* \), and \( \mu \)-incoherent as defined in Definition 1. Suppose further that we observe \( m \) entries of \( X^* \) chosen uniformly at random. Let \( Y^* = Z^*Z^*^\top \) be the lifted matrix as in (4) and write \( n = \max(n_1, n_2) \). Then there exist universal constants \( c_0, c_1, c_2, c_3 \) such that if
\[
m \geq c_0 \mu r^2 \kappa^2 \max(\mu, \log n) n,
\]
then with probability at least \( 1 - c_1 n^{-c_2} \) the iterates of Algorithm 1 converge to \( Z^* \) geometrically, when using regularization parameter \( \lambda = 1/9 \), correctly specified input rank \( r \), and constant step size \( \eta/\sigma_1^* \) with \( \eta \leq c_3/\mu^2 r^2 \kappa \).

We shall analyze the Bernoulli sampling model, as justified in Section 2. For simplicity, we assume \( n = n_1 = n_2 \) throughout the analysis. For the case \( n_1 > n_2 \), the convergence claim still holds with high probability for \( p \geq O\left(\mu \kappa^2 r^2 \max(\mu, \log n_1)/n_2\right) \).

Let us define the distance to \( Z^* \) in terms of the solution set \( S \).

**Definition 2.** Define the distance between \( Z \) and \( Z^* \) as
\[
d(Z, Z^*) = \min_{\widetilde{Z} \in S} \| Z - \widetilde{Z} \|_F = \min_{R \in \mathbb{R}^{n \times n}, \text{rank}(R) = r, R^\top R = I} \| Z - Z^* R \|_F.
\]

The next theorem establishes the global convergence of Algorithm 1, assuming that the input rank is correctly specified. The proof sketch is given in the next subsection.

**Theorem 2.** There exist universal constants \( c_0, c_1, c_2 \) such that if \( p \geq c_0 \mu r^2 \kappa^2 \max(\mu, \log n) \), with probability at least \( 1 - c_1 n^{-c_2} \), the initialization \( Z^0 \) satisfies
\[
d(Z^0, Z^*) \leq \frac{1}{4} \sqrt{\sigma_r^*}.
\]

Moreover, there exists a universal constant \( c_3 \) such that when using constant step size \( \eta/\sigma_1^* \) with \( \eta \leq \frac{c_3}{\mu^2 r^2 \kappa} \) and initial value \( Z^0 \) obeying (13), the \( k \)th step of Algorithm 1 with \( \lambda = 1/9 \) satisfies
\[
d(Z^k, Z^*) \leq \frac{1}{4} \left(1 - \frac{27}{256} \frac{\eta}{\kappa}\right)^{k/2} \sqrt{\sigma_r^*},
\]
with probability at least \( 1 - 6n^{-3} \).
Remarks.

(i) After each update, the distance of our iterates to $Z^*$ is reduced by at least a factor of $1 - O(1/\mu^2 r^2 \kappa^2)$.

(ii) Hence, the output $\hat{Z}$ satisfies $d(\hat{Z}, Z^*) \leq \varepsilon$ after at most $\lceil 2 \log^{-1} \left( 1/(1 - \frac{27}{256} \cdot \frac{\eta}{\kappa}) \right) \log (\sqrt{\sigma_r^2}/4\varepsilon) \rceil$ iterations.

3.1 Proof Sketch

Our proof idea is of the same nature as the analysis in [9, 31]. We show two appealing properties when sufficient entries are observed. First, our spectral initialization produces a starting point within the $O(\sigma_r^2)$ neighbourhood of the solution set.

**Theorem 3.** There exist universal constants $c, c_1, c_2$, such that if $p \geq c \mu r^2 \kappa^2 \log n$ then with probability at least $1 - c_1 n^{-c_2}$,

$$d(Z^0, Z^*) \leq \frac{1}{4} \sqrt{\sigma_r^2}.$$  

To demonstrate this, we exploit the concentration around the mean of $p^{-1} P_\Omega(X^*)$. See Appendix B for the proof. Using this theorem, we can immediately show that $Z^*$ and all other elements of $S$ are contained in the feasible set (9).

**Corollary 1.** With probability at least $1 - c_1 n^{-c_2}$, $\sqrt{2\mu/n} \|Z^0\| \geq \|Z^*\|_{2,\infty}$.

The second crucial property is that $f(Z)$ is well-behaved within the $O(\sigma_r^2)$ neighbourhood, so that the iterates moves closer to the optima in every iteration. The key step is to set up a local regularity condition [9] similar to Nesterov’s conditions [22].

**Definition 3.** Let $Z = \arg \min_{\tilde{Z} \in S} \|Z - \tilde{Z}\|_F$ denote the matrix closest to $Z$ in the solution set. We say that $f$ satisfies the regularity condition $RC(\varepsilon, \alpha, \beta)$ if there exist constants $\alpha, \beta$ such that for any $Z$ satisfying $d(Z, Z^*) \leq \varepsilon$, we have

$$\langle \nabla f(Z), Z - Z^* \rangle \geq \frac{1}{\alpha} \sigma_r^2 \|Z - Z^*\|^2_F + \frac{1}{\beta \sigma_r^2} \|\nabla f(Z)\|^2_F.$$

Using this condition, one can show the iterates converges linearly to the optima if we start close enough to $Z^*$.

**Theorem 4.** Consider the update $Z^{k+1} = P_C \left( Z^k - \frac{\mu}{\sigma_r^2} \nabla f(Z^k) \right)$. If $f$ satisfies $RC(\varepsilon, \alpha, \beta)$, $d(Z^k, Z^*) \leq \varepsilon$, and $0 < \mu \leq \min(\alpha/2, 2/\beta)$, then

$$d(Z^{k+1}, Z^*) \leq \sqrt{1 - \frac{2\mu}{\alpha \kappa} d(Z^k, Z^*)}.$$  

6
The following theorem illustrates the local regularity of \( f(Z) \). Nesterov’s criterion is established upon strong convexity and strong smoothness of the objective. Here we show analogous curvature and smoothness conditions holds for \( f(Z) \) locally – within the \( O(\sqrt{\sigma^*}) \) neighbourhood – with high probability. Interestingly, we found that to show the local curvature, it suffices to set \( \lambda < \frac{2}{9} \). The proof can be found in Appendix C, for which we have generalized some technical lemmas of [11].

**Theorem 5.** Let the regularization constant be set to \( \lambda = \frac{1}{9} \). There exists a universal constant \( c \), such that if \( p \geq c \max(\mu^2 r^2 \kappa^2, \mu r \log n) \), then \( f \) satisfies \( RC(\frac{1}{4} \sqrt{\sigma^*}, 512/27, 6514\mu^2 r^2 \kappa) \) with probability at least \( 1 - 6n^{-3} \).

### 4 Related Work

Matrix completion is one instance of the general low rank linear inverse problem

\[
\text{find } X \text{ of minimum rank such that } \mathcal{A}(X) = b, \tag{14}
\]

where \( \mathcal{A} \) is an affine transformation and \( b = \mathcal{A}(X^*) \) is the measurement of the ground truth \( X^* \). Considerable progress has been made towards algorithms for recovering \( X^* \) including both convex and nonconvex approaches. One of the most popular methods is nuclear norm minimization, a convenient convex relaxation of rank minimization. It was first proposed in [13, 23], and analyzed under a certain restricted isometry property (RIP). Subsequent work clarified the conditions for reconstruction, and studied recovery guarantees for both exact and approximately low rank matrices, with or without noise [6, 7, 21, 10]. One significant advantage for this approach is its near-optimal sample complexity. Under the same incoherence assumption as ours, Chen [10] establishes the currently best-known lower bound of \( O(\mu r n \log^2 n) \) samples. Using a closely related notion of incoherence, Negahban and Wainwright [21] show that if \( X^* \) is “\( \alpha \)-nonspiky” with \( \frac{\|X^*\|_\infty}{\|X^*\|_F} \leq \frac{\alpha}{\sqrt{mn}} \), then \( O(\alpha^2 r n \log n) \) samples are sufficient for exact recovery. However, convexity and low sample complexity aside, in practice the power of nuclear norm relaxation is limited due to high computational cost. The state-of-art algorithms for nuclear norm minimization are proximal methods that perform iterative singular value thresholding [3, 27]. However, such algorithms don’t scale to large instances because the per-iteration the SVD is expensive.

Another popular convex surrogate for the rank function is the max-norm [25, 15], given by \( \|X\| = \min_{X=UV^T} \|U\|_{2,\infty} \|V\|_{2,\infty} \). For certain types of problems, the max-norm offers better generalization error bounds than the nuclear norm [24]. But practically solving large scale problems that incorporate the max-norm is also non-trivial. In 2010, Lee et al. [20] rephrased the max-norm constrained problem as an SDP, and applied Burer-Monteiro factorization. Although this ends up with an \( \ell_{2,\infty} \) constraint similar to ours (9), we emphasize that the constraint plays a different role in our setting. While [25, 20] use it to promote low rank solutions, our purpose is to enforce incoherent solutions; and experimental results suggest that one can drop it. Moreover, the convergence of projected gradient descent for this problem was not previously understood.
In a parallel line of work, the problem of developing techniques that exactly solve nonconvex formulations has attracted significant recent research attention. In chronological order, Keshavan et al. [19] proposed a manifold gradient method for matrix completion. They factorize $X^* = U^*\Sigma^*V^*$, where $U^* \in \mathbb{R}^{n_1 \times r}, U^\top U = n_1 I_{n_1}$ and $V^* \in \mathbb{R}^{n_2 \times r}, V^\top V = n_2 I_{n_2}$. Similar to our definition of $S$, the equivalent classes of $U^*$ and $V^*$ are Grassmann manifolds of $r$ dimensional subspaces. The authors then minimize the nonconvex objective $F(U,V) = \min_{S \in \mathbb{R}^{r \times r}} \left\| P_{\Omega}(USV^\top - X^*) \right\|_F^2$ over the manifolds. In each iteration, $U$ and $V$ are updated along their manifold gradients, followed by the update of the optimal scaling matrix $S$. This algorithm enjoys a linear rate of global convergence. However, its per-iteration update also has high computational complexity, see Section 5 for details. In the same year, Jain et al. [17] suggested minimizing the squared residual $\|A(X) - b\|^2$ under a rank constraint $\text{rank}(X) \leq r$. While this constraint is nonconvex, projection onto the feasible set can be computed using low rank SVD. Under certain RIP assumption on $A$, Jain et al. establish the global convergence of projected gradient descent for (14). This algorithm is named Singular Value Projection (SVP). Yet in the setting of completion, only experimental support for the effectiveness of SVP is provided. More importantly, SVP also suffers from expensive per-iteration SVD for large scale problems. Jain et al. [18] further analysed an alternating least squares algorithm (ALS) for (14) in 2011. ALS factorizes $X = UV^\top$ where $U \in \mathbb{R}^{n_1 \times r}$ and $V \in \mathbb{R}^{n_2 \times r}$, and alternately minimizes $\|A(UV^\top) - b\|_2^2$ over $U$ and $V$, while fixing the other factor. Although analytical results are obtained, in practice the least square problems are often ill-posed.

In 2014, Candès et al. [9] proposed Wirtinger flow for phase retrieval. Wirtinger flow is a fast first-order algorithm that minimizes a fourth order (nonconvex) objective, geometrically converging to the global optimum. While previous work [8, 4, 5] lifts the phase retrieval problem into an SDP where the solution is rank one, this works bridges SDP and first-order algorithms via the Burer-Monteiro technique. It has inspired further research on related topics; last year, the authors of [31, 28, 11] considered factorizations for (14), assuming $X^*$ is semidefinite, and proved global optimality of first-order algorithms under appropriate initializations. Tu et al. [28] have extended this algorithm to handle rectangular matrix via asymmetric factorization, and have shown exact recovery of $X^*$, assuming $A$ satisfies a certain RIP. They use lifting implicitly, factorizing $X = Z_UZ_V^\top$ and applying gradient updates on both factors $Z_U$ and $Z_V$ simultaneously, with the nonconvex objective function

$$g(Z_U, Z_V) = \frac{1}{2p}\left\| P_{\Omega}(Z_UZ_V^\top - X^*) \right\|_F^2 + \frac{\lambda}{4} \left\| Z_U^\top Z_U - Z_V^\top Z_V \right\|_F^2. \quad (15)$$

Their proof strategy also shows convergence of $Z$ in the lifted space. For the specific case of matrix completion, Chen and Wainwright [11] obtained guarantees when $X^*$ is semidefinite. Our work generalizes the results obtained in [28, 11], extending the recent literature on first-order algorithms for factorized models.

After completing this work we learned of independent research of Sun and Luo [26], who also analysed a gradient descent for rectangular matrix completion. Their algorithm is similar to ours, with additional Frobenius norm constraints on the factors. The authors established a sample complexity of $O(r^7\kappa^6)$ observations; in comparison our bound scales as $O(r^2\kappa^2)$. In
terms of convergence rate, they show that to obtain an $\varepsilon$-optimal solution, their algorithm requires $O(\text{poly}(n) \log(1/\varepsilon))$ iterations, while our time complexity is independent of $n$.

In other related work, [30, 29] also studied nonconvex optimization methods to address matrix completion problem. These algorithms still require low rank SVD in each iteration.

5 Experiments

We conduct experiments on synthetic datasets to support our analytical results. As the column space regularizer and incoherence constraint of our gradient method (GD) are merely for analytical purpose, we drop them in all the experiments; simply optimize the $\ell_2$ loss
\[
\min_X \frac{1}{2} \| P_{\Omega}(X - X^*) \|_F^2 + \| X \|_*,
\]
where $\lambda = 0$ will enforce the minimizer fitting the observed values exactly. We use ADMM to solve (16). It is based on the algorithm for the matrix approach in [27], and can neatly handle the case $\lambda = 0$. We emphasize there is no computational difference between cases whether $\lambda$ is zero or not. All methods are implemented in MATLAB and the experiments were run on a Linux machine with a 3.4GHz Intel Core i7 processor and 8 GB memory.

Computational Complexity Table 1 summarizes the per-iteration complexity of all the methods for completing a $n \times n$ matrix. Since $M^k$ is a sparse matrix with $m$ nonzero entries, and we have dropped the regularizer and constraint, our method GD only needs $2mr + m + n^2r$ operations to compute the gradient, and $4nr$ operations to update the iterate. The computation of nuclear is dominated by singular value thresholding and updating the objective value, which require the $O(n^3)$ cost full SVD. Similarly, SVP needs $O(n^2r)$ operations to compute the rank-$r$ SVD for low rank projection. For OptSpace, $O(mr + n^2r + nr^2)$ operations are needed to compute the manifold gradient and line search. The most expensive part is to determine the optimal scaling matrix $S \in \mathbb{R}^{r \times r}$, which boils down to solving a $r^2$ by $r^2$ dense linear system. In total $O(mr^3 + n^2r^2 + nr^4 + r^6)$ operations are used to construct and solve this system. One can see that GD significantly reduces the computation than the others. Though the dominating terms for SVP and GD are in the same order, in practice the partial SVD are more expensive than the gradient update, especially on large instances.

Runtime Comparison We randomly generated a true matrix $X^*$ of size $4000 \times 2000$ and rank 3. It is constructed from the rank-3 SVD of a random $4000 \times 2000$ matrix with i.i.d normal entries. We sampled $m = 199057$ entries of $X^*$ uniformly at random, where $m$ is roughly equal to $2nr \log n$ with $n = 4000$ and $r = 3$. For simplicity, we feed SVP, OptSpace and GD with the true rank. For all these methods, we use the randomized algorithm of Halko et al. [16] to compute the low rank SVD, which is approximately 15 times faster than MATLAB built-in SVD on instances of such
| Method  | Complexity                  |
|---------|-----------------------------|
| GD      | $2mr + m + nr + 4nr$        |
| SVP     | $O(n^2r)$                   |
| OptSpace| $O(mr^3 + n^2r^2 + nr^4 + nr^6)$ |
| nuclear | $O(n^3)$                    |

Table 1: Per-iteration complexities.

size. We report relative error measured in the Frobenius norm, defined as $\|\hat{X} - X^*\|_F / \|X^*\|_F$. For nuclear, we set $\lambda = 0$ to enforce exact fitting. The convergence speed of ADMM mildly depends on the choice of penalty parameter. We tested 5 values 0.1, 0.2, 0.5, 1, 1.5 and selected 0.2, which leads to fastest convergence. Similarly, for SVP, we would like to choose the largest step size for which the algorithm is converging. We evaluated 15, 20, 30, 35, 40 and selected 30. The step size is chosen for GD in the same way. Five values 20, 50, 70, 75, 80 are tested for $\eta$ and we picked 70. For OptSpace, we compared fixed step sizes 0.50, 0.10, 0.050, 0.010, 0.005 with line search, and found the algorithm converged fastest under line search. Figure 1a shows the results. Our method outperforms all the competing approaches. To further illustrate the speed improvement of GD, we run larger instances of size $10000 \times 5000$ and $20000 \times 5000$, whether the true rank is 40. The parameters are selected in the same manner, and we terminate the computation once the relative error is below $1e^{-9}$. We report the results of GD and SVP in Figure 1b; nuclear and OptSpace do not scale to such sizes. It is clear that the runtime of GD increases much slower than SVP.

Sample Complexity We evaluate the number of observations required by GD for exact recovery. For simplicity, we consider square but asymmetric $X^*$. We conducted experiments in 4 cases, where the randomly generated $X^*$ is of size $500 \times 500$ or $1000 \times 1000$, and of rank 10 or 20. In each case, we compute the solutions of GD given $m$ random observations, and a solution with relative error below $1e^{-6}$ is considered to be successful. We run 20 trials and compute the empirical probability of successful recovery. The results are shown in Figure 1c. For all four cases, the phase transitions occur around $m \approx 3.5nr$. This suggests that the actual sample complexity of GD may scale linearly with both the dimension $n$ and the rank $r$.

6 Conclusion

We propose a lifting procedure together with Burer-Monteiro factorization and a first-order algorithm to carry out rectangular matrix completion. While optimizing a nonconvex objective, we establish linear convergence of our method to the global optimum with $O(\mu r^2 n \max(\mu, \log n))$ random observations. We conjecture that $O(nr)$ observations are sufficient for exact recovery, and that the column space regularizer can be dropped. We provide empirical evidence showing this algorithm is fast and scalable, with performance that is favorable to existing procedures, suggesting that lifting techniques may be promising for much more general classes of problems.
Figure 1: (a) Runtime comparison where $X^*$ is $4000 \times 2000$ and of rank 3. 199057 entries are observed. (b) Runtime growth of GD and SVP. (c) Sample complexity of gradient scheme.

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A Technical Lemmas

Another way of writing the objective function is

$$f(Z) = \frac{1}{2p} \sum_{l=1}^{2m} (\langle A_l, ZZ^\top \rangle - b_l)^2 + \frac{\lambda}{4} \|Z^\top DZ\|^2_F,$$

where $l$ is an index of $\Omega$, $A_l$ is a matrix with 1 at the corresponding observed entry and 0 elsewhere.

In this form, the gradient can be written as

$$\nabla f(Z) = \frac{1}{p} \sum_{l=1}^{2m} \left( \langle A_l, ZZ^\top \rangle - b_l \right) (A_l + A_l^\top)Z + \lambda DZ \left( Z^\top DZ \right)^{\Gamma}$$

$$= \frac{1}{p} \sum_{l=1}^{2m} \left( \langle A_l, HZ^\top + ZH^\top + HH^\top \rangle \right) (A_l + A_l^\top)(Z + H) + \lambda \Gamma.$$
We will use the following facts throughout the proof:

\[ \|Z\|_{2,\infty} = \|Z^*\|_{2,\infty} \leq \sqrt{\frac{\mu r}{n}} \sigma^*_1, \tag{17} \]

\[ \|H\|_{2,\infty} \leq 3 \sqrt{\frac{\mu r}{n}} \sigma^*_1, \tag{18} \]

\[ \langle (A_l + A_l^T)B, C \rangle = \langle A, BC^T + CB^T \rangle, \tag{19} \]

\[ Z^T \bar{Z} \text{ is positive semidefinite, } H^T \bar{Z} \text{ is symmetric.} \tag{20} \]

Inequality (17) is a direct result of Definition 1. To see (18), note that
\[ \|H\|_{2,\infty} \leq \|Z\|_{2,\infty} + \|Z\|_{2,\infty} \leq \sqrt{\frac{2\mu r}{n}} \sigma_1 + \sqrt{\frac{\mu r}{n}} \sigma^*_1, \]
and \[ |\sigma_1 - \sigma^*_1| \leq \frac{1}{10} \sigma^*_1 \] by the discussion of initialization in Appendix B. For (20), it holds that
\[ \arg\min_{RR^T=R^TR=I} \|Z - Z^*R\|^2_F = AB^T, \]

where \( \Lambda \Lambda B^T \) is the SVD of \( Z^* \). Clearly, \( Z^\top \bar{Z} \) is positive semidefinite, and \( H^\top \bar{Z} = Z^\top \bar{Z} - Z^\top Z = B \Lambda \Lambda B^T - Z^\top Z \) is symmetric.

Next, we list several technical lemmas that are utilized later. We will use \( c \) to denote a numerical constant, whose value may vary from line to line.

**Lemma 1.** For any \( Z \) of the form \( Z = \begin{bmatrix} Z_U \\ Z_V \end{bmatrix} = \begin{bmatrix} U \Sigma \frac{1}{2} R \\ V \Sigma \frac{1}{2} R \end{bmatrix}, \) where \( U, V, R \) are unitary matrices and \( \Sigma \succeq 0 \) is a diagonal matrix, we have
\[ \left\| ZZ^\top - Z^* Z^*\right\|_F \leq 2 \left\| U \Sigma V^\top - U^* \Sigma^* V^*\right\|_F. \]

**Proof.** Recall that
\[ Z^* = \begin{bmatrix} Z^*_U \\ Z^*_V \end{bmatrix} = \begin{bmatrix} U^* \Sigma^* \frac{1}{2} \\ V^* \Sigma^* \frac{1}{2} \end{bmatrix} \]
where \( X^* = U^* \Sigma^* V^*\top \). We have
\[ \left\| ZZ^\top - Z^* Z^*\right\|_F^2 = \left\| U \Sigma U^\top - U^* \Sigma^* U^*\right\|_F^2 + \left\| V \Sigma V^\top - V^* \Sigma^* V^*\right\|_F^2 + 2 \left\| U \Sigma V^\top - U^* \Sigma^* V^*\right\|_F^2, \tag{21} \]
and
\[ \left\| U \Sigma U^\top - U^* \Sigma^* U^*\right\|_F^2 + \left\| V \Sigma V^\top - V^* \Sigma^* V^*\right\|_F^2 = 2 \left( \left\| \Sigma \right\|_F^2 + \left\| \Sigma^* \right\|_F^2 - \langle \Sigma, U^* U^\top \Sigma^* U^*\top U + V^\top V^*\top \Sigma^* V^*\top V \rangle \right). \tag{22} \]
We can lower bound
\[
\langle \Sigma, U^\top U^* \Sigma^* U^* U + V^\top V^* \Sigma^* V^* V \rangle
\]
\[
= \sum_{i=1}^{r} \sigma_i \left( U^\top U^* \Sigma^* U^* U + V^\top V^* \Sigma^* V^* V \right)_{ii}
\]
\[
= \sum_{i=1}^{r} \sum_{k=1}^{r} \sigma_i \sum_{k=1}^{r} \sigma_k^* \left( (U^\top U^*)^2_{ik} + (V^\top V^*)^2_{ik} \right)
\]
\[
\geq \sum_{i=1}^{r} \sum_{k=1}^{r} \sigma_i \sigma_k^* \cdot 2 (U^\top U^*)_{ik} (V^\top V^*)_{ik}
\]
\[
= 2 \langle \Sigma, U^\top U^* \Sigma^* V^* V \rangle.
\]
Combining (22) and (23), we obtain
\[
\left\| U \Sigma U^\top - U^* \Sigma^* U^* \right\|_F^2 + \left\| V \Sigma V^\top - V^* \Sigma^* V^* \right\|_F^2
\]
\[
\leq 2 \left( \left\| \Sigma \right\|_F^2 + \left\| \Sigma^* \right\|_F^2 - 2 \langle \Sigma, U^\top U^* \Sigma^* V^* V \rangle \right)
\]
\[
= 2 \left( \left\| U \Sigma V^\top \right\|_F^2 + \left\| U^* \Sigma^* V^* \right\|_F^2 - 2 \langle U \Sigma V^\top, U^* \Sigma^* V^* \rangle \right)
\]
\[
= 2 \left\| U \Sigma V^\top - U^* \Sigma^* V^* \right\|_F^2.
\]
(24)
Plugging (24) back into (21), we obtain the lemma.

We will exploit the following two known concentration results.

**Lemma 2** (Chen [10], Lemma 2). There exist universal constants $c, c_1, c_2$, such that for any fixed matrix $X^* \in \mathbb{R}^{n \times n}$, with probability at least $1 - c_1 n^{-c_2}$ it holds that
\[
\left\| p^{-1} P_\Omega (X^*) - X^* \right\| \leq c \left( \frac{\log n}{p} \left\| X^* \right\|_{\infty} + \sqrt{\frac{\log n}{p} \left\| X^* \right\|_{\infty,2}} \right).
\]

**Lemma 3** (Candès and Recht [6], Theorem 4.1). Define subspace
\[
T = \left\{ M \in \mathbb{R}^{n_1 \times n_2} : M = U^* X^T + Y V^* V^T \text{ for some } X \text{ and } Y \right\}.
\]
Let $P_T$ be the Euclidean projection onto $T$. There is a numerical constant $c$ such that for any $\delta \in (0, 1)$, if $p \geq c \mu_0 \log n \delta^2 n$, then with probability $1 - 3n^{-3}$, we have
\[
p^{-1} \left\| P_T P_\Omega P_T - p P_T \right\| \leq \delta.
\]

**Lemma 4** upper bounds the spectral norm of the adjacency matrix of a random Erdős-Rényi graph. It is a variant of Lemma 7.1 of Keshavan et al. [19], which uses known results of Feige and Ofek [14].
Lemma 4 (Chen and Wainwright [11], Lemma 9). Suppose that $\bar{\Omega} \subset [n] \times [n]$ is the set of edges of a random Erdős-Rényi graph with $n$ nodes, where any pair of nodes is connected with probability $p$. There exists two numerical constants $c_1, c_2$ such that, for any $\delta \in (0, 1)$, if $p \geq \frac{c_1 \log n}{\delta^2 n}$, then with probability at least $1 - \frac{1}{2} n^{-4}$, uniformly for all $x, y \in \mathbb{R}^n$ it holds that

$$p^{-1} \sum_{(i,j) \in \bar{\Omega}} x_i y_j \leq (1 + \delta) \|x\|_1 \|y\|_1 + c_2 \sqrt{\frac{n}{p}} \|x\|_2 \|y\|_2.$$  \hspace{1cm} (26)

We refer readers to [19] for a complete proof, in particular noticing that one can choose $p$ large enough so that the constant factor in the first term in (26) is only $1 + \delta$.

Lemma 5, 6 and 7 are direct generalizations of Lemma 4 and 5 of [11].

Lemma 5. There exists a constant $c$ such that, for any $\delta \in (0, 1)$, if $p \geq \frac{c}{\delta^2} \max \left( \frac{\log n}{n}, \frac{\mu^2 r^2 \kappa^2}{n} \right)$, then with probability at least $1 - \frac{1}{32} n^{-4}$, uniformly for all $H$ such that $\|H\|_{2,\infty} \leq 3 \sqrt{\frac{\mu r}{n}} \sigma_1^*$, we have

$$p^{-1} \left\| \mathcal{P}_\Omega (HH^\top) \right\|_F^2 \leq (1 + \delta) \|H\|_F^4 + \delta \sigma_r^* \|H\|_F^2.$$  \hspace{1cm} (27)

Proof. It holds that

$$p^{-1} \left\| \mathcal{P}_\Omega (HH^\top) \right\|_F^2 = p^{-1} \sum_{(i,j) \in \bar{\Omega}} \langle H(i), H(j) \rangle^2 \leq p^{-1} \sum_{(i,j) \in \bar{\Omega}} \left\| H(i) \right\|_2^2 \left\| H(j) \right\|_2^2.$$  \hspace{1cm} (28)

Since $\bar{\Omega}$ is a reduced sampling of $Y \in \mathbb{R}^{2n \times 2n}$ under a Bernoulli model, Lemma 4 is applicable here. Assume $p \geq \frac{c_1 \log (2n)}{\delta^2 (2n)}$, we then have with probability at least $1 - \frac{1}{32} n^{-4}$, for all $H$ such that $\|H\|_{2,\infty} \leq 3 \sqrt{\frac{\mu r}{n}} \sigma_1^*$,

$$p^{-1} \left\| \mathcal{P}_\Omega (HH^\top) \right\|_F^2 \leq p^{-1} \sum_{(i,j) \in \bar{\Omega}} \left\| H(i) \right\|_2^2 \left\| H(j) \right\|_2^2 \leq (1 + \delta) \left( \sum_{i \in [2n]} \left\| H(i) \right\|_2^2 \right)^2 + c_2 \sqrt{\frac{2n}{p}} \sum_{i \in [2n]} \left\| H(i) \right\|_2^4 \leq (1 + \delta) \|H\|_F^4 + c_2 \sqrt{\frac{2n}{p}} \|H\|_F^2 \|H\|_{2,\infty}^2 \leq \|H\|_F^2 \left( 1 + \delta \right) \|H\|_F^2 + \sqrt{\frac{16c_2^2 \mu^2 r^2 \sigma_1^4}{pn}}.$$
where the second line follows from Lemma 4 and the last line follows from \( \|H\|_{2,\infty} <= 3 \frac{\mu r}{n}\sigma_1^* \).

Let us further assume \( p \geq \frac{162c^2\mu^2r^2\kappa^2}{\delta^2n} \), then we can bound

\[
p^{-1} \left\| P_\Omega( H H^\top) \right\|_F^2 \leq \|H\|_F^2 \left( (1 + \delta) \|H\|_F^2 + \delta\sigma_1^* \right).
\]

The final threshold we obtain is thus \( p \geq \frac{c}{\delta^2} \max \left( \frac{\log n}{n}, \frac{\mu^2r^2\kappa^2}{n} \right) \) for some constant \( c \). \( \square \)

**Lemma 6.** There exists a constant \( c \) such that, if \( p \geq \frac{c \log n}{n} \), then with probability at least \( 1 - 2n^{-4} \), uniformly for all matrices \( A, B \) of proper sizes,

\[
p^{-1} \left\| P_\Omega(AB^\top) \right\|_F^2 \leq 2n \min \left\{ \left\| A \right\|_F^2 \left\| B \right\|_{2,\infty}^2, \left\| B \right\|_F^2 \left\| A \right\|_{2,\infty}^2 \right\}
\]

**Proof.** Let \( \Omega_{Y_i} = \{ j : (i,j) \in \Omega \} \) denote the set of entries sampled in the \( i \)th row of \( AB^\top \). Note that only \( n \) entries of each row are sampled with probability \( p \). Using a binomial tail bound, if \( p \geq \frac{c \log n}{n} \) for sufficiently large \( c \), the event \( \max_{i \in [2n]} |\Omega_{Y_i}| \leq 2pn \) holds with probability at least \( 1 - n^{-4} \). Conditioning on this event, we have for all \( A, B \) of proper size,

\[
p^{-1} \left\| P_\Omega(AB^\top) \right\|_F^2 = p^{-1} \sum_{i=1}^n \sum_{j \in \Omega_{Y_i}} (A_{(i)}, B_{(j)})^2
\]

\[
\leq p^{-1} \sum_{i=1}^n \left\| A_{(i)} \right\|_2^2 \sum_{j \in \Omega_{Y_i}} \left\| B_{(j)} \right\|_2^2
\]

\[
\leq p^{-1} \sum_{i=1}^n \left\| A_{(i)} \right\|_2^2 \max_{i \in [2n]} |\Omega_{Y_i}| \left\| B \right\|_{2,\infty}^2
\]

\[
\leq 2n \left\| A \right\|_F^2 \left\| B \right\|_{2,\infty}^2.
\]

Similarly we can prove with probability at least \( 1 - n^{-4} \),

\[
p^{-1} \left\| P_\Omega(AB^\top) \right\|_F^2 \leq 2n \left\| B \right\|_F^2 \left\| A \right\|_{2,\infty}^2.
\]

\( \square \)

The following lemma establishes restricted strong convexity and smoothness of the observation operator for matrices in \( T \).

**Lemma 7.** Let \( T \) be the subspace defined in (25). There exists a universal constant \( c \) such that, if \( p \geq \frac{c \mu r \log n}{\delta^2 n} \), with probability at least \( 1 - 3n^{-3} \), uniformly for all \( A \in T \), we have

\[
p(1 - \delta) \left\| A \right\|_F^2 \leq \left\| P_\Omega(A) \right\|_F^2 \leq p(1 + \delta) \left\| A \right\|_F^2.
\]

Consequently, uniformly for all \( A, B \in T \),

\[
|p^{-1} \langle P_\Omega(A), P_\Omega(B) \rangle - \langle A, B \rangle| \leq \delta \left\| A \right\|_F \left\| B \right\|_F.
\]

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Proof. By Lemma 3, with probability at least $1 - 3n^{-3}$, for any $X \in \mathbb{R}^{n \times n}$ it holds that
\begin{equation}
  p(1 - \delta) \|X\|_F \leq \|P_{\Omega}P_{\Omega}P_{T}(X)\|_F \leq p(1 + \delta) \|X\|_F.
\end{equation}
Let $A$ be a matrix in $T$. Rewriting $\|P_{\Omega}(A)\|_F^2 = \langle P_{\Omega}P_{T}(A), P_{\Omega}P_{T}(A) \rangle = \langle A, P_{T}P_{\Omega}P_{T}(A) \rangle$, and using the Cauchy-Schwarz inequality and (32) we can bound
\begin{equation}
  \|P_{\Omega}(A)\|_F^2 \leq p(1 + \delta) \|A\|_F^2.
\end{equation}
In addition, we have
\begin{equation}
  \|P_{\Omega}(A)\|_F^2 = \langle A, P_{T}P_{\Omega}P_{T}(A) \rangle = \langle A, P_{T}P_{\Omega}P_{T}(A) - pP_{T}(A) + pP_{T}(A) \rangle \geq - \|A\|_F \|P_{T}P_{\Omega}P_{T} - pP_{T}\|_F + p \|A\|_F^2 \geq p(1 - \delta) \|A\|_F^2,
\end{equation}
where (a) follows from Lemma 3. Combining (33) and (34) proves (30). To show (31), let $A' = \frac{A}{\|A\|_F}$ and $B' = \frac{B}{\|B\|_F}$. Both $A' + B'$ and $A' - B'$ are in $T$. We have
\begin{align}
  \langle P_{\Omega}(A'), P_{\Omega}(B') \rangle &= \frac{1}{4} \left\{ \|P_{\Omega}(A' + B')\|_F^2 - \|P_{\Omega}(A' - B')\|_F^2 \right\} \\
  &\leq \frac{1}{4} \left\{ (1 + \delta)p \|A' + B'\|_F^2 - (1 - \delta)p \|A' - B'\|_F^2 \right\} \\
  &= \frac{1}{4} \left\{ 2\delta p \|A'\|_F^2 + \|B'\|_F^2 \right\} + 4p \langle A', B' \rangle \\
  &= p\delta + p\langle A', B' \rangle,
\end{align}
where (b) follows from (30). Thus, we have
\begin{equation}
  p^{-1} \langle P_{\Omega}(A), P_{\Omega}(B) \rangle = p^{-1} \|A\|_F \|B\|_F \langle P_{\Omega}(A'), P_{\Omega}(B') \rangle \leq \delta \|A\|_F \|B\|_F + \langle A, B \rangle.
\end{equation}
Similarly, we can show
\begin{equation}
  p^{-1} \langle P_{\Omega}(A), P_{\Omega}(B) \rangle \geq -\delta \|A\|_F \|B\|_F + \langle A, B \rangle.
\end{equation}
\hfill \Box

B Initialization

B.1 Proof of Theorem 3

Let $\delta$ denote the upper bound of $\|p^{-1}P_{\Omega}(X^*) - X^*\|$ as in Lemma 2, and let $\sigma_1 \geq \ldots \geq \sigma_n$ denote the singular values of $p^{-1}P_{\Omega}(X^*)$. By Weyl’s theorem, we have
\begin{equation}
  |\sigma_i - \sigma_i^*| \leq \delta, \quad i \in [n].
\end{equation}
Note this implies $\sigma_{t+1} \leq \delta$, as $\sigma_{t+1}^* = 0$.

By definition, $Z^0 = [U; V]\Sigma^2$, where $U\Sigma V^T$ is the rank-$r$ SVD of $p^{-1}\mathcal{P}_\Omega(X^*)$. According to Lemma 1, one has

$$
\left\| Z^0 Z^0^T - Z^* Z^*^T \right\|_F \leq 2 \left\| U\Sigma V^T - X^* \right\|_F
$$

$$
\leq 2\sqrt{2r} \left\| U\Sigma V^T - X^* \right\|
$$

$$
\leq 2\sqrt{2r} (\left\| U\Sigma V^T - p^{-1}\mathcal{P}_\Omega(X^*) \right\| + \left\| p^{-1}\mathcal{P}_\Omega(X^*) - X^* \right\|)
$$

$$
= 4\sqrt{2r} \delta,
$$

where (a) holds because rank$(U\Sigma V^T - X^*) \leq 2r$, (b) holds since $\left\| U\Sigma V^T - p^{-1}\mathcal{P}_\Omega(X^*) \right\| = \sigma_{t+1} \leq \delta$.

Let $H = Z^0 - \overline{Z}^0$. We want to bound $d(Z^0, Z^*)^2 = \|H\|_F^2$. According to (20), $H^T \overline{Z}^0$ is symmetric and $Z^0^T \overline{Z}^0$ is positive semidefinite. Hence we can write

$$
\left\| Z^0 Z^0^T - Z^* Z^*^T \right\|_F^2
$$

$$
= \left\| H\overline{Z}^0^T + \overline{Z}^0 H^T + HH^T \right\|_F^2
$$

$$
= \text{tr} \left( (H^T H)^2 + 2(H^T \overline{Z}^0)^2 + 2(H^T H)(\overline{Z}^0^T \overline{Z}^0) + 4(H^T H)(H^T \overline{Z}^0) \right)
$$

$$
\geq \text{tr} \left( (4 - 2\sqrt{2})(H^T H)(H^T \overline{Z}^0) + 2(H^T H)(\overline{Z}^0^T \overline{Z}^0) \right)
$$

$$
= (4 - 2\sqrt{2}) \text{tr} ((H^T H)(Z^0^T \overline{Z}^0)) + (2\sqrt{2} - 2) \left\| H\overline{Z}^0^T \right\|_F^2,
$$

where in the second line we used that $H^T \overline{Z}^0$ is symmetric. Besides, as $Z^0^T \overline{Z}^0$ is positive semidefinite, $(4 - \sqrt{2}) \text{tr}((H^T H)(Z^0^T \overline{Z}^0))$ is nonnegative. Therefore,

$$
\left\| Z^0 Z^0^T - Z^* Z^*^T \right\|_F^2 \geq (2\sqrt{2} - 2) \left\| H\overline{Z}^0^T \right\|_F^2 \geq 4(\sqrt{2} - 1)\sigma^*_r \|H\|_F^2.
$$

Combining (39) and (41), it follows that

$$
d(Z^0, Z^*)^2 \leq \frac{\left\| Z^0 Z^0^T - Z^* Z^*^T \right\|_F^2}{4(\sqrt{2} - 1)\sigma^*_r} \leq \frac{8r}{(\sqrt{2} - 1)\sigma^*_r} \delta^2.
$$
Therefore, it suffices to show

\[ d(Z^0, Z^*)^2 \leq \frac{8r}{(\sqrt{2} - 1)\sigma_r^*} \delta^2 \]

\[ \overset{(a)}{=} cr \left( \frac{\log n}{p} \|X^*\|_\infty + \sqrt{\frac{\log n}{p} \|X^*\|_{\infty, 2}} \right)^2 \]

\[ \overset{(b)}{\leq} c r^2 \left( \frac{\mu r \log n \|X^*\|_\infty}{pm} + \sqrt{\frac{\mu r \log n \|X^*\|_{\infty, 2}}{pm}} \right)^2 \]

\[ \leq \frac{1}{16} \sigma_r^* , \]

where in \((a)\) we replaced \(\delta\) using Lemma 2, and \((b)\) holds since by our incoherence assumption \(\|X^*\|_{\infty, 2} \leq 8r(\sqrt{2} - 1)\sigma_r^*\delta^2 \leq \frac{1}{16} \sigma_r^* \). As a result, \(2\sigma_1 \geq \sigma_r^* \).

### B.2 Proof of Corollary 1

By the incoherence assumption, we have \(\|Z^*\|_{2, \infty} \leq \sqrt{\frac{\mu r}{n} \sigma_1^*} \), see \((17)\). It suffices to show \(2\sigma_1 \geq \sigma_r^* \). From the above discussion, we can see that

\[
\frac{8r}{(\sqrt{2} - 1)\sigma_r^*} \delta^2 \leq \frac{1}{16} \sigma_r^* \Rightarrow \delta \leq \frac{1}{16} \sigma_r^* .
\]

By Wely’s theorem, we have \(|\sigma_1 - \sigma_1^*| \leq \frac{1}{16} \sigma_1^*.\) As a result, \(2\sigma_1 \geq \sigma_r^* \).

### C Regularity Condition

Analogous to the restricted strong convexity (RSC) and restricted strong smoothness (RSS), we show that with high probability our objective function \(f\) satisfies the local curvature and local smoothness conditions defined below.
• **Local Curvature Condition**

There exists constant $c_1, c_2$ such that for any $Z$ satisfying $d(Z, Z^*) \leq \frac{1}{4} \sqrt{\sigma^*}$,

$$\langle \nabla f(Z), H \rangle \geq c_1 \| H \|_F^2 + c_2 \| H^T DZ \|_F^2.$$ 

• **Local Smoothness Condition**

There exist constants $c_3, c_4$ such that for any $Z$ satisfying $d(Z, Z^*) \leq \frac{1}{4} \sqrt{\sigma^*}$,

$$\| \nabla f(Z) \|_F^2 \leq c_3 \| H \|_F^2 + c_4 \| H^T DZ \|_F^2.$$ 

C.1 **Proof of the Local Curvature Condition**

$$\langle \nabla f(Z), H \rangle$$

$$= \frac{1}{p} \left( \sum_{l=1}^{2m} \langle A_l, HZ^T + ZH^T + HH^T \rangle \cdot \langle (A_l + A_l^T)(Z + H), H \rangle \right) + \lambda \text{tr}(H^T \Gamma)$$

$$= \frac{1}{p} \left( \sum_{l=1}^{2m} \langle A_l, HZ^T + ZH^T \rangle \cdot \langle A_l, HH^T + 2HH^T \rangle \right) + \lambda \text{tr}(H^T \Gamma)$$

$$= \frac{1}{p} \left\{ \sum_{l=1}^{2m} \langle A_l, HZ^T + ZH^T \rangle^2 + \sum_{l=1}^{2m} 2\langle A_l, HH^T \rangle^2 + \sum_{l=1}^{2m} 3\langle A_l, HZ^T + ZH^T \rangle \langle A_l, HH^T \rangle \right\}$$

$$+ \lambda \text{tr}(H^T \Gamma)$$

$$(ii) \geq \frac{1}{p} \left\{ \frac{a^2 + b^2}{2} - \frac{3}{\sqrt{2}} \left( \sum_{l=1}^{2m} \langle A_l, HZ^T + ZH^T \rangle \right)^2 \right\} + \lambda \text{tr}(H^T \Gamma)$$

$$= \frac{1}{p} \left\{ \left( a - \frac{3}{2\sqrt{2}} b \right)^2 - \frac{1}{8} b^2 \right\} + \lambda \text{tr}(H^T \Gamma)$$

$$(iii) \geq \frac{1}{p} \left( \frac{a^2}{2} - \frac{5}{4} b^2 \right) + \lambda \text{tr}(H^T \Gamma)$$

$$= \frac{1}{2} p^{-1} \| P_\Omega(HZ^T + ZH^T) \|_F^2 - \frac{5}{2} p^{-1} \| P_\Omega(HH^T) \|_F^2 + \lambda \text{tr}(H^T \Gamma).$$  \hspace{1cm} (47)$$

where we used equation (19) for $(i)$, the Cauchy-Schwarz inequality for $(ii)$, inequality $(a-b)^2 \geq \frac{a^2}{2} - b^2$ for $(iii)$. Finally, in the last line we used $\sum_{l=1}^{2m} \langle A_l, M \rangle^2 = \| P_\Omega(M) \|_F^2$.  

We first lower bound $\frac{1}{2} p^{-1} \| P_\Omega(HZ^T + ZH^T) \|_F^2$. By the symmetry of $\Omega$, it is equal to
\[ p^{-1} \left\| P_\Omega(H_U Z_V^T + Z_U H_V^T) \right\|_F^2, \text{ which expands to} \]
\[ p^{-1} \left\| P_\Omega(H_U Z_V^T) \right\|_F^2 + p^{-1} \left\| P_\Omega(Z_U H_V^T) \right\|_F^2 + 2p^{-1} \langle P_\Omega(H_U Z_V^T), P_\Omega(Z_U H_V^T) \rangle. \tag{48} \]

As both \( H_U Z_V^T \) and \( Z_U H_V^T \) belong to \( T \), we use Lemma 7 to lower bound above three terms, respectively. This gives us
\[
\frac{1}{2} p^{-1} \left\| P_\Omega(HZ^T + Z H^T) \right\|_F^2 \\
\geq (1 - \delta) \left( \left\| H_U Z_V^T \right\|_F^2 + \left\| Z_U H_V^T \right\|_F^2 \right) + 2 \langle H_U Z_V^T, Z_U H_V^T \rangle - 2\delta \left\| H_U Z_V^T \right\|_F \left\| Z_U H_V^T \right\|_F \\
\geq (1 - \delta) \left( \left\| H_U Z_V^T \right\|_F^2 + \left\| Z_U H_V^T \right\|_F^2 \right) + 2 \langle H_U Z_V^T, Z_U H_V^T \rangle - \delta \left( \left\| H_U Z_V^T \right\|_F^2 + \left\| Z_U H_V^T \right\|_F^2 \right) \\
\geq (1 - 2\delta) \sigma_r^* \left( \left\| H_U \right\|_F^2 + \left\| H_V \right\|_F^2 \right) + 2 \langle H_U Z_V^T, Z_U H_V^T \rangle \\
= (1 - 2\delta) \sigma_r^* \left\| H \right\|_F^2 + 2 \langle H_U Z_V^T, Z_U H_V^T \rangle. \tag{49} \]

where we used \( \left\| H_U Z_V^T \right\|_F^2 \geq \sigma_r^* \left\| H_U \right\|_F^2 \) and \( \left\| Z_U H_V^T \right\|_F^2 \geq \sigma_r^* \left\| H_V \right\|_F^2 \) for \((iv)\).

Until now, we obtain
\[
\langle \nabla f(Z), H \rangle \geq (1 - 2\delta) \sigma_r^* \left\| H \right\|_F^2 + 2 \langle H_U Z_V^T, Z_U H_V^T \rangle + \lambda \text{tr}(H^T \Gamma) - \frac{5}{2} p^{-1} \left\| P_\Omega(HH^T) \right\|_F^2. \tag{50} \]

Next, we lower bound \( 2 \langle H_U Z_V^T, Z_U H_V^T \rangle + \lambda \text{tr}(H^T \Gamma) \) together. Rewriting
\[
2 \langle H_U Z_V^T, Z_U H_V^T \rangle = \langle H, \begin{bmatrix} 0 & Z_V^T H_U^T \\ Z_V H_U^T & 0 \end{bmatrix} Z \rangle = \langle H, \frac{1}{2} (ZH^T - DZ^T D) \rangle, \\
ZZ^T - ZZ^T = HH^T + ZZ^T + HZ^T, \tag{51} \]

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and plugging in $\Gamma = DZZ^TDZ$, we then have

$$
2\langle H_UZ_V^T, Z_UH_V^T \rangle + \lambda \text{tr}(H^T \Gamma)
= \langle H, \frac{1}{2}(ZH^T - DZH^T D)Z \rangle + \lambda \langle H, D(ZZ^T - ZZ^T D)Z \rangle + \lambda \langle H, D(ZZ^T)DZ \rangle
+ \lambda \langle H, D(\overline{Z}\overline{Z}^T)DH \rangle
\leq \langle H, \frac{1}{2}(ZH^T - DZH^T D)Z \rangle + \lambda \langle H, D(ZZ^T - \overline{Z}\overline{Z}^T)DZ \rangle + \lambda \|Z^T \|_F^2 + (a)
\leq \lambda \|Z^T \|_F^2 + \lambda \|H^T \|_F^2 + \lambda \|H^T \|_F^2 + 3\lambda \text{tr}(H^T DHDH^T DZ)
+ \left( \lambda - \frac{1}{2} \right) \text{tr}(H^T DZH^T DZ)
\leq \lambda \|Z^T \|_F^2 + \lambda \|H^T \|_F^2 + 3\lambda \|H^T \|_F^2 + \left( \lambda - \frac{1}{2} \right) \|Z^T \|_F^2
\leq \lambda \|Z^T \|_F^2 + \left( \lambda - \frac{9}{4} \lambda \right) \|Z^T \|_F^2 + \left( \frac{3}{2} \|Z^T \|_F^2 \right).
$$

(52)

Equality (a) holds because $\overline{Z}^T \overline{Z} = 0$. We plug in (51) in (b). For (c), we use $\overline{Z}^T \overline{Z} = 0$ and that $H^T \overline{Z}$ is symmetric. Inequality (d) follows from Cauchy-Schwarz. Hence, for any $\lambda < \frac{9}{4}$, the term $2\langle H_UZ_V^T, Z_UH_V^T \rangle + \lambda \text{tr}(H^T \Gamma)$ is positive.

Finally, we drop $\lambda \|\frac{1}{2}Z^T DHD - H^T D\|_F^2$ and use Lemma 5 to upper bound $p^{-1} \|P_\Omega(HH^T)\|_F^2$:

$$
\langle \nabla f(Z), H \rangle \geq (1 - 2\delta)\sigma_r^* \|H\|_F^2 + \left( \frac{1}{2} \frac{9}{4} \lambda \right) \|Z^T \|_F^2 - \frac{5}{2}(1 + \delta) \|H\|_F^2 - \frac{5}{2}\lambda \|Z^T \|_F^2
\geq \left( (1 - 2\delta)\sigma_r^* - \frac{5}{2}(1 + \delta) \|H\|_F^2 - \frac{5}{2}\lambda \|Z^T \|_F^2 \right) \|H\|_F^2 + \left( \frac{1}{2} \frac{9}{4} \lambda \right) \|Z^T \|_F^2.
$$

(53)

For simplicity, we take $\lambda = \frac{1}{9}$, $\delta = \frac{1}{16}$, and we have $\|H\|_F^2 \leq \frac{1}{16}\sigma_r^*$. This leads to

$$
\langle \nabla f(Z), H \rangle \geq \frac{283}{512} \sigma_r^* \|H\|_F^2 + \left( \frac{1}{4} \right) \|Z^T \|_F^2.
$$

(54)

Note that this lower bound holds with high probability uniformly for all $Z$ such that $d(Z, Z^*) \leq \frac{1}{3} \sqrt{\sigma_r^*}$, since Lemma 5 and 7 hold uniformly.

When the ground truth $X^*$ is positive semidefinite, we don’t need to do lifting nor impose the regularizer. Using Lemma 7, we can lower bound $\frac{1}{2}p^{-1} \|P_\Omega(HZ^T + ZH^T)\|_F^2 \geq (1 - \delta)\sigma_r^* \|H\|_F^2$. 

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directly. Taking proper constants, we can obtain the standard restricted strong convexity condition:

\[ \langle \nabla f(Z), H \rangle \gtrsim \sigma^*_r \|H\|_F^2. \]

C.2 Proof of the Local Smoothness Condition

To upper bound \( \|\nabla f(Z)\|_F^2 = \max_{\|W\|_F=1} |\langle \nabla f(Z), W \rangle|^2 \), it suffices to show that for any \( n \times r \) \( W \) of unit Frobenius norm, \( |\langle \nabla f(Z), W \rangle|^2 \) is upper bounded. We first write

\[
\langle \nabla f(Z), W \rangle \\
= \frac{1}{p} \sum_{l=1}^{2m} \left( \langle A_l, H\tilde{Z}^T + \tilde{Z}H^T \rangle + \langle A_l, HH^T \rangle \right) \cdot \langle (A_l + A_l^T)(\tilde{Z} + H), W \rangle + \lambda \text{tr}(W^T\Gamma) \\
\overset{(i)}{=} \frac{1}{p} \sum_{l=1}^{2m} \left( \langle A_l, H\tilde{Z}^T + \tilde{Z}H^T \rangle + \langle A_l, HH^T \rangle \right) \left( \langle A_l, WW^T + WW^T \rangle + \langle A_l, WH^T + HW^T \rangle \right) \\
+ \lambda \text{tr}(W^T\Gamma) \\
= \frac{1}{p} \left\{ \langle \mathcal{P}_\Omega(H\tilde{Z}^T + \tilde{Z}H^T), \mathcal{P}_\Omega(W\tilde{Z}^T + \tilde{Z}W^T) \rangle + \langle \mathcal{P}_\Omega(HH^T), \mathcal{P}_\Omega(W\tilde{Z}^T + \tilde{Z}W^T) \rangle \\
+ \langle \mathcal{P}_\Omega(H\tilde{Z}^T + \tilde{Z}H^T), \mathcal{P}_\Omega(WH^T + HW^T) \rangle + \langle \mathcal{P}_\Omega(HH^T), \mathcal{P}_\Omega(WH^T + HW^T) \rangle \right\} \\
+ \lambda \text{tr}(W^T\Gamma),
\]

(55)
where we used (19) for (i). Since \((a + b + c + d + e)^2 \leq 5(a^2 + b^2 + c^2 + d^2 + e^2)\), we have

\[
\langle \nabla f(Z), W \rangle^2 \\
\leq \frac{5}{p^2} \left\{ \langle P_\Omega(HZ^T + ZH^T), P_\Omega(WZ^T + ZW^T) \rangle^2 + \langle P_\Omega(HH^T), P_\Omega(WZ^T + ZW^T) \rangle^2 \\
+ \langle P_\Omega(HZ^T + ZH^T), P_\Omega(WH^T + HW^T) \rangle^2 + \langle P_\Omega(HH^T), P_\Omega(WH^T + HW^T) \rangle^2 \right\} \\
+ 5\lambda^2 \text{tr}(W^T \Gamma)^2 \\
\leq \frac{5}{p^2} \left( \| P_\Omega(HZ^T + ZH^T) \|_F^2 + \| P_\Omega(HH^T) \|_F^2 \right) \\
\cdot \left( \| P_\Omega(WZ^T + ZW^T) \|_F^2 + \| P_\Omega(WH^T + HW^T) \|_F^2 \right) + 5\lambda^2 \| \Gamma \|_F^2 \| W \|_F^2 \\
\leq \frac{5}{p} \left( 2 \| P_\Omega(HZ^T) \|_F^2 + 2 \| P_\Omega(ZH^T) \|_F^2 + 2 \| P_\Omega(HH^T) \|_F^2 \right) \\
\cdot \left( \frac{1}{p} \left( 2 \| P_\Omega(WZ^T) \|_F^2 + 2 \| P_\Omega(ZW^T) \|_F^2 + 2 \| P_\Omega(WH^T) \|_F^2 + 2 \| P_\Omega(HW^T) \|_F^2 \right) \\
+ 5\lambda^2 \| \Gamma \|_F^2, \\
\right) \\
(56)
\]

where we used the Cauchy-Schwarz inequality for (ii), and \((a + b)^2 \leq 2(a^2 + b^2)\) for (iii). We then use Lemma 6 to upper bound (1), (2), (4), (5), (6), (7), and Lemma 5 for (3). Also since \(\| W \|_F = 1\), one has

\[
\langle \nabla f(Z), W \rangle^2 \\
\leq 5 \left( 8n \| H \|_F^2 \| Z \|_{2,\infty}^2 + (1 + \delta) \| H \|_F^4 + \delta \sigma^*_H \| H \|_F^2 \right) \cdot \left( 8n \| Z \|_{2,\infty}^2 + 8n \| H \|_{2,\infty}^2 \right) \\
+ 5\lambda^2 \| \Gamma \|_F^2 \\
= 40n \left( 8n \| Z \|_{2,\infty}^2 + (1 + \delta) \| H \|_F^2 + \delta \sigma^*_H \right) \| H \|_F^2 \cdot \left( 8n \| Z \|_{2,\infty}^2 + 8n \| H \|_{2,\infty}^2 \right) + 5\lambda^2 \| \Gamma \|_F^2 \\
\leq 400\mu r \sigma^*_H \left( 8\mu r \sigma^*_H + (1 + \delta) \| H \|_F^2 + \delta \sigma^*_H \right) \| H \|_F^2 + 5\lambda^2 \| \Gamma \|_F^2, \\
(57)
\]

where in the last line we plugged in \(\| Z \|_{2,\infty} \leq \sqrt{\frac{\mu r}{n}} \sigma^*_H\) and \(\| H \|_{2,\infty} \leq 3\sqrt{\frac{\mu r}{n}} \sigma^*_H\), i.e. (17) and (18).
Next, we bound
\[
\|\Gamma\|_F^2 = \left\| D(ZZ^T - ZZ^T)DZ + DZZ^T DZ \right\|_F^2
\leq 2 \left\| D(ZZ^T - ZZ^T)DZ \right\|_F^2 + 2 \left\| DZZ^T DZ \right\|_F^2
\leq 2 \left\| ZZ^T - ZZ^T \right\|_F^2 \|Z\|^2 + 2 \left\| Z \right\|_F^2 \left\| Z^T DZ \right\|_F^2
\leq 2 \left( \|HH^T\|_F^2 + \|ZH^T\|_F^2 + \|ZH^T\|_F^2 \right) \|Z\|^2 + 2 \left\| Z \right\|_F^2 \left\| Z^T DZ \right\|_F^2
\leq 6 \left( \|H\|_F^2 + 2 \|Z\|_F^2 \right) \|H\|_F^2 \|Z\|^2 + 2 \left\| Z \right\|_F^2 \left\| Z^T DZ \right\|_F^2
\leq 6 \left( \|H\|_F^2 + 4\sigma_1^* \right) \|H\|_F^2 \|Z\|^2 + 4\sigma_1^* \left\| Z^T DZ \right\|_F^2
\] (58)

Inequality \((a)\) holds because \(\|AB\|_F \leq \|A\| \|B\|_F\) and \(\|D\| = 1\). To get \((b)\), for the first term in the 3rd line we expand \(ZZ^T - ZZ^T\), for the second term we expand \(Z = \bar{Z} + H\) and use \(Z^T DZ = 0\). For \((c)\), we use \(\|AB\|_F \leq \|A\| \|B\|_F \leq \|A\|_F \|B\|_F\). Last, \((d)\) holds because \(\|Z\|^2 = 2\sigma_1^*\).

Finally, we combine (57) and (58). As before, take \(\lambda = \frac{1}{9}\), \(\delta = \frac{1}{16}\), and \(\|H\|_F^2 \leq \frac{1}{16}\sigma_1^*\), we obtain
\[
\|\nabla f(Z)\|_F^2
\leq \left\{ 400\mu r\sigma_1^* \left( 8\mu r\sigma_1^* + (1 + \delta) \|H\|_F^2 + \delta\sigma_1^* \right) + 30\lambda^2 \left( \|H\|_F^2 + 4\sigma_1^* \right) \|Z\|^2 \right\} \|H\|_F^2
\leq \left\{ 400\mu r\sigma_1^* \left( 8\mu r\sigma_1^* + (1 + \delta) \|H\|_F^2 + \delta\sigma_1^* \right) + \frac{735}{8} \sigma_1^* \lambda^2 \left( \|H\|_F^2 + 2\sigma_1^* \right) \right\} \|H\|_F^2
\leq \left\{ 3257\mu^2 r^2 \sigma_1^2 \|H\|_F^2 + \frac{20}{81} \sigma_1^* \left\| Z^T DZ \right\|_F^2 \right\}
\] (59)

where for \((a)\) we used \(\|Z\| \leq \|H\| + \|Z\| \leq \frac{1}{3}\sqrt{\sigma_1^*} + \sqrt{2\sigma_1^*} \leq \frac{2}{3}\sqrt{\sigma_1^*}\), for \((b)\) we used \(\mu, r \geq 1\).

As before, this condition holds uniformly for all \(Z\) such that \(d(Z, Z^*) \leq \frac{1}{3}\sqrt{\sigma_1^*}\).

For the case \(X^*\) is positive semidefinite, as we don’t need to impose the regularizer, standard restricted strong smoothness condition follows:
\[
\|\nabla f(Z)\|_F^2 \lesssim \sigma_1^* \|H\|_F^2
\]
C.3 Proof of the Regularity Condition

Rearranging the terms in the smoothness condition (59), we can further bound

\[ \frac{\| \nabla f(Z) \|_F^2}{\mu^2 r^2 \kappa \sigma_1^*} \leq 3257 \sigma_1^* \| H \|_F^2 + \frac{1}{4} \| Z^\top D H \|_F^2. \]  

(60)

Combining equation (54) and (60), it follows that

\[ \langle \nabla f(Z), H \rangle \geq \frac{27}{512} \sigma_1^* \| H \|_F^2 + \frac{1}{6514 \mu^2 r^2 \kappa \sigma_1^*} \| \nabla f(Z) \|_F^2. \]  

(61)

Finally, by upper bounding the probability that Lemma 5, 6, or 7 fails, and the sample probability \( p \) these lemmas require, we conclude that once

\[ p \geq c \max \left( \frac{\mu r \log n}{n}, \frac{\mu^2 r^2 \kappa^2}{n} \right) \]  

(62)

for some absolute constant \( c \), regularity condition (61) holds with probability at least \( 1 - 6n^{-3} \).

D Linear Convergence

Let \( H^k = Z^k - Z^k \). Our iterate is \( Z^{k+1} = \mathcal{P}_C(Z^k - \eta \nabla f(Z^k)) \). Since \( \mathcal{P}_C \) is just row-wise clipping, by Lemma 8 we have

\[ \left\| \mathcal{P}_C \left( Z^k - \frac{\eta}{\sigma_1^*} \nabla f(Z^k) \right) - \tilde{Z}^k \right\|_F^2 \leq \left\| Z^k - \frac{\eta}{\sigma_1^*} \nabla f(Z^k) - \tilde{Z}^k \right\|_F^2. \]  

(63)

It follows that

\[
\begin{align*}
\left\| Z^{k+1} - \bar{Z}^k \right\|_F^2 & \leq \left\| Z^k - \frac{\eta}{\sigma_1^*} \nabla f(Z^k) - \bar{Z}^k \right\|_F^2 \\
& = \left\| H^k \right\|_F^2 + \frac{\eta^2}{\sigma_1^*} \left\| \nabla f(Z^k) \right\|_F^2 - \frac{2\eta}{\sigma_1^*} \langle \nabla f(Z^k), H^k \rangle \\
& \overset{(a)}{=} \left\| H^k \right\|_F^2 + \frac{\eta^2}{\sigma_1^*} \left\| \nabla f(Z^k) \right\|_F^2 - \frac{2\eta}{\sigma_1^*} \left( \frac{1}{\alpha \sigma_1^*} \left\| H^k \right\|_F^2 + \frac{1}{\beta \sigma_1^*} \left\| \nabla f(Z^k) \right\|_F^2 \right) \\
& = \left( 1 - \frac{2\eta}{\alpha \kappa} \right) \left\| H^k \right\|_F^2 + \frac{\eta(\eta - 2/\beta)}{\sigma_1^*} \left\| \nabla f(Z^k) \right\|_F^2 \\
& \overset{(b)}{\leq} \left( 1 - \frac{2\eta}{\alpha \kappa} \right) \left\| H^k \right\|_F^2,
\end{align*}
\]

(64)

where we use the definition of \( \text{RC}(\varepsilon, \alpha, \beta) \) for (a) and \( 0 < \eta \leq \min \left\{ \alpha/2, 2/\beta \right\} \) for (b). Therefore,

\[ d(Z^{k+1}, Z^*) = \min_{Z \in S} \left\| Z^{k+1} - \bar{Z}^k \right\|_F^2 \leq \sqrt{1 - \frac{2\eta}{\alpha \kappa}} d(Z^k, Z^*). \]  

(65)
Lemma 8. Let \( y \in \mathbb{R}^r \) be a vector such that \( \|y\|_2 \leq \theta \), for any \( x \in \mathbb{R}^r \). Then

\[
\| P_{\|\cdot\|_2 \leq \theta} (x) - y \|_2^2 \leq \| x - y \|_2^2 .
\]

Proof. If \( \|x\|_2 \leq \theta \), then \( P_{\|\cdot\|_2 \leq \theta} (x) = x \). Otherwise \( P_{\|\cdot\|_2 \leq \theta} (x) = \theta \bar{x} \), where \( \bar{x} = \frac{x}{\|x\|_2} \). Write

\[
y = (y^\top \bar{x}) \bar{x} + P_x^\perp (y),
\]

we have

\[
\| \theta \bar{x} - y \|_2^2 = \| \theta \bar{x} - (y^\top \bar{x}) \bar{x} \|_2^2 + \| P_x^\perp (y) \|_2^2 = (\theta - y^\top \bar{x})^2 + \| P_x^\perp (y) \|_2^2 .
\]

(66)

It suffices to show

\[
(\theta - y^\top \bar{x})^2 \leq (\|x\| - y^\top \bar{x})^2 .
\]

(67)

If \( y^\top \bar{x} \leq 0 \), then (67) holds because \( \|x\| > \theta \). If \( y^\top \bar{x} > 0 \), (67) still holds since \( \|x\| > \theta \geq \|y\| \geq y^\top \bar{x} \).

References

[1] Srinadh Bhojanapalli, Anastasios Kyrillidis, and Sujay Sanghavi. Dropping convexity for faster semi-definite optimization. arXiv:1509.03917, 2015.

[2] Samuel Burer and Renato D.C. Monteiro. A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization. Mathematical Programming, 95(2):329–357, 2003.

[3] Jian-Feng Cai, Emmanuel J Candès, and Zuowei Shen. A singular value thresholding algorithm for matrix completion. SIAM Journal on Optimization, 20(4):1956–1982, 2010.

[4] Emmanuel Candès, Thomas Strohmer, and Vladislav Voroninski. Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming. Communications on Pure and Applied Mathematics, 66(8):1241–1274, 2013.

[5] Emmanuel J Candès and Xiaodong Li. Solving quadratic equations via phaselift when there are about as many equations as unknowns. Foundations of Computational Mathematics, 14 (5):1017–1026, 2014.

[6] Emmanuel J Candès and Benjamin Recht. Exact matrix completion via convex optimization. Foundations of Computational mathematics, 9(6):717–772, 2009.

[7] Emmanuel J Candès and Terence Tao. The power of convex relaxation: Near-optimal matrix completion. Information Theory, IEEE Transactions on, 56(5):2053–2080, 2010.

[8] Emmanuel J Candès, Yonina C Eldar, Thomas Strohmer, and Vladislav Voroninski. Phase retrieval via matrix completion. SIAM Review, 57(2):225–251, 2015.

[9] Emmanuel J Candès, Xiaodong Li, and Mahdi Soltanolkotabi. Phase retrieval via Wirtinger flow: Theory and algorithms. Information Theory, IEEE Transactions on, 61(4):1985–2007, 2015.
[10] Yudong Chen. Incoherence-optimal matrix completion. *Information Theory, IEEE Transactions on*, 61(5):2909–2923, 2015.

[11] Yudong Chen and Martin J. Wainwright. Fast low-rank estimation by projected gradient descent: General statistical and algorithmic guarantees. arXiv:1509.03025, 2015.

[12] Alexandre d’Aspremont, Laurent El Ghaoui, Michael I Jordan, and Gert RG Lanckriet. A direct formulation for sparse pca using semidefinite programming. *SIAM Review*, 49(3):434–448, 2007.

[13] Maryam Fazel. Matrix rank minimization with applications. Technical report, Elec. Eng. Dept., Stanford University, 2002. PhD thesis.

[14] Uriel Feige and Eran Ofek. Spectral techniques applied to sparse random graphs. *Random Structures & Algorithms*, 27(2):251–275, 2005.

[15] Rina Foygel and Nathan Srebro. Concentration-based guarantees for low-rank matrix reconstruction. arXiv:1102.3923, 2011.

[16] Nathan Halko, Per-Gunnar Martinsson, and Joel A. Tropp. Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions. *SIAM Review*, 53(2):217–288, 2011.

[17] Prateek Jain, Raghu Meka, and Inderjit S Dhillon. Guaranteed rank minimization via singular value projection. In *Advances in Neural Information Processing Systems*, pages 937–945, 2010.

[18] Prateek Jain, Praneeth Netrapalli, and Sujay Sanghavi. Low-rank matrix completion using alternating minimization. In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*, pages 665–674. ACM, 2013.

[19] Raghunandan H Keshavan, Andrea Montanari, and Sewoong Oh. Matrix completion from a few entries. *Information Theory, IEEE Transactions on*, 56(6):2980–2998, 2010.

[20] Jason D Lee, Ben Recht, Nathan Srebro, Joel Tropp, and Ruslan R Salakhutdinov. Practical large-scale optimization for max-norm regularization. In *Advances in Neural Information Processing Systems*, pages 1297–1305, 2010.

[21] Sahand Negahban and Martin J. Wainwright. Restricted strong convexity and weighted matrix completion: Optimal bounds with noise. *The Journal of Machine Learning Research*, 13 (1):1665–1697, 2012.

[22] Yurii Nesterov. *Introductory lectures on convex optimization*, volume 87. Springer Science & Business Media, 2004.

[23] Benjamin Recht, Maryam Fazel, and Pablo A Parrilo. Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization. *SIAM Review*, 52(3):471–501, 2010.
[24] Nathan Srebro and Adi Shraibman. Rank, trace-norm and max-norm. In Learning Theory, pages 545–560. Springer, 2005.

[25] Nathan Srebro, Jason Rennie, and Tommi S Jaakkola. Maximum-margin matrix factorization. In Advances in neural information processing systems, pages 1329–1336, 2004.

[26] Ruoyu Sun and Zhi-Quan Luo. Guaranteed matrix completion via nonconvex factorization. In Foundations of Computer Science, IEEE 56th Annual Symposium on, pages 270–289, 2015.

[27] Ryota Tomioka, Kohei Hayashi, and Hisashi Kashima. Estimation of low-rank tensors via convex optimization. arXiv:1010.0789, 2010.

[28] Stephen Tu, Ross Boczar, Max Simchowitz, Mahdi Soltanolkotabi, and Benjamin Recht. Low-rank solutions of linear matrix equations via procrustes flow. In International Conference on Machine Learning, 2016.

[29] Ke Wei, Jian-Feng Cai, Tony F. Chan, and Shingyu Leung. Guarantees of riemannian optimization for low rank matrix completion. arXiv:1603.06610, 2016.

[30] Tuo Zhao, Zhaoran Wang, and Han Liu. A nonconvex optimization framework for low rank matrix estimation. In Advances in Neural Information Processing Systems, pages 559–567, 2015.

[31] Qingqing Zheng and John Lafferty. A convergent gradient descent algorithm for rank minimization and semidefinite programming from random linear measurements. In Advances in Neural Information Processing Systems, pages 109–117, 2015.