SIMULATING A POSITRON CONVERTER IN BMAD

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Abstract

A model to describe the output particle distribution generated by particles impinging on a planar target has been developed. This model was developed to simulate positron production in the Cornell CESR Linac but can be applied to targets using different types of particles.

To model a specific converter target, the output particle distribution is first simulated by tracking particles using the Geant4 toolkit which models the fundamental physics of the conversion process. The Monte Carlo distribution from Geant4 is fitted to a set of functions and the function coefficients are saved for use in simulations. Using the fit functions in a simulation is not only more portable but is also more than an order of magnitude faster than running Geant4. This model has been successfully incorporated into the BMAD simulation toolkit.

The output position, angular orientation, and momentum distribution is modeled. Preliminary support for modeling the polarization distribution is also discussed.

INTRODUCTION

A common method of producing positrons for use in an accelerator is through the use of a positron converter[1,2]. This is typically a slab of heavy metal, such as tungsten, located in a Linac which is bombarded with electrons with energies of order 100 MeV. The electrons emit photons via bremsstrahlung which in turn decay to $e^+e^-$ pairs via the reaction

\[e^- + Z \rightarrow e^- + Z + \gamma \rightarrow e^- + Z + e^- + e^+.\]

After the converter, the electrons and any other particles produced can be filtered off with a dipole magnet effectively “converting” a beam of incoming electrons into a beam of outgoing positrons.

Both bremsstrahlung from electrons traveling through the field of a nucleus and the subsequent $e^+e^-$ pair production have been discussed at length in the literature[3,9]. Many of the theoretical treatments of bremsstrahlung and $e^+e^-$ pair production are reviewed by Tsai[9]. Analytic expressions for both the bremsstrahlung radiation spectrum and secondary electron energy spectrum as functions of target thickness have been derived[3], and the problem has been treated across a wide range of incoming electron energies[3].

The bremsstrahlung cross section has been tabulated for a wide range of electron energies[5,6]. The bremsstrahlung cross section has been tabulated for a wide range of electron energies and target materials[5,6]. Recently, computer codes have been developed to calculate the spectra and angular distributions of bremsstrahlung[7,8], although these are designed to work at low energies ($< 3$ MeV).

Despite the extensive study of bremsstrahlung and pair production that exists in the literature, there is no closed form description of the kinematics of positrons produced by bremsstrahlung from electrons impinging on a target. This problem has been treated numerically[10], but there is no existing machinery to incorporate these results into a full accelerator simulation. These prior efforts also do not address the production of polarized positrons from a beam of polarized electrons, a topic which is of current interest[11].

In response, the authors have developed a converter model and this model has been incorporated in the BMAD [12] accelerator toolkit. The model is designed to be flexible, accommodating any converter material and thickness as well as supporting arbitrary incoming and outgoing particle species. The model is also fast to use since the expensive physics calculations are performed only once when the converter model coefficients are being calculated. This is especially important when designing a machine with a converter as optimization simulations are typically time intensive.

One challenge in constructing the model was how to handle the complexity coming from the fact that the outgoing distribution has five dimensions (two spatial and three momentum) along with the incoming particle momentum, and orientation. Additionally, the target thickness and particle spin are potential degrees of freedom.

THE CONVERTER MODEL

The first step is to simulate the conversion process and calculate conversion probability coefficients. The coefficients can then be used to generate particles as part of a simulation of the machine the converter is a part of.

The coefficient calculation has two parts. The first part is to simulate the quantum electrodynamic[13] interactions of the incoming particles with the converter to produce bremsstrahlung photons and subsequent positrons. The distribution of the outgoing particles calculated in this first step is then used to calculate the coefficients for the expressions used to model the distribution.

Coordinate System

For concreteness, it will be assumed that the impinging beam is composed of electrons and the outgoing beam composed of positrons. The model itself is species agnostic.

Consider an electron incident on the upstream face of a positron converter of thickness $T$, with momentum $p_-$ perpendicular to the face of the converter as depicted in Figure 1. The $(u, v, s)$ coordinate system (Figs. 1and 2) is defined with $s$ perpendicular to the converter surface and the coordinate origin at the point where the electron would emerge if it was undeflected through the converter. Positrons produced in the converter emerge from the downstream face with some radial displacement $r$ from the origin and at an angle $\theta$ relative to the $u$ axis as depicted in Figure 2. Define
the rotated \((x, y, s)\) coordinate system so that \(x\) is in the direction of \(r\).

By symmetry, \(\theta\) must be uniformly distributed from 0 to \(2\pi\). The kinematic properties of the produced positrons which must be modeled are its radial displacement \(r\) and outgoing momentum \(p_+\). The outgoing momentum is described in terms of its magnitude, \(p_+\), and the slopes along the \(x\) and \(y\) directions:

\[
\frac{dx}{ds} = \frac{p_x}{p_+}, \quad \frac{dy}{ds} = \frac{p_y}{p_+} \tag{1}
\]

**Probability Distributions**

At the downstream surface, positrons produced in the converter are described by the distribution

\[
P\left(p_+, r, \frac{dx}{ds}, \frac{dy}{ds}, p_-, \theta\right); \tag{2}
\]

\(P\) is the probability, per incoming electron, of producing an outgoing positron with momentum magnitude \(p_+\), radial displacement \(r\), and momentum orientation \((dx/ds, dy/ds)\). \(P\) will have a dependence on the converter thickness \(t\) and incoming momentum \(p_-\) (non-normal incidence is discussed below). The converter thickness dependence is important if the converter thickness is allowed to vary in a simulation where the positron yield is to be maximized. To model the thickness and electron momentum dependence, the distribution \(P\) is calculated at a number of user defined thicknesses \(t\) and electron momenta \(p_-\). Interpolation is then used when generating particles with the model.

For a given \(t\) and \(p_-\), the integral of \(P\) over the space \((p_+, r, dx/ds, dy/ds)\) will be the number of positrons \(n_+\) produced per electron

\[
\int P\left(p_+, r, \frac{dx}{ds}, \frac{dy}{ds}\right) dp_+ dr d\left(\frac{dx}{ds}\right) d\left(\frac{dy}{ds}\right) = n_+ \tag{3}
\]

To simplify the computations, \(P\) is decomposed into two subdistributions,

\[
P\left(p_+, r, \frac{dx}{ds}, \frac{dy}{ds}\right) = P_1(p_+, r) P_2\left(\frac{dx}{ds}, \frac{dy}{ds}; p_+, r\right) \tag{4}
\]

where \(P_1\) is the probability, per incident electron, of producing a positron with momentum magnitude \(p_+\) and radial displacement \(r\), and \(P_2\) is the probability, for a given \(p_+\) and \(r\), of producing a positron with momentum orientation \((dx/ds, dy/ds)\). \(P_1\) is normalized to \(n_+\) and \(P_2\) is normalized to one.

The \(P_1\) probability distribution is characterized by a two dimensional lookup table that gives \(P_1\) at specific values of \(p_+\) and \(r\). \(P_2\) is approximated using a heuristically derived skewed Lorentzian distribution

\[
P_2\left(\frac{dx}{ds}, \frac{dy}{ds}; p_+, r\right) = \frac{1 + \beta \frac{dx}{ds}}{1 + \alpha_1^2 \left(\frac{dx}{ds} - c_x\right)^2 + \alpha_2^2 \left(\frac{dy}{ds}\right)^2} \tag{5}
\]

where, as explained below, \(\beta, \alpha_1, \alpha_2\) and \(c_x\) will be characterized as functions of \(p_+\) and \(r\), and \(A\) is calculated from the normalization of \(P_2\).

To calculate coefficients, a program to simulate the passage of particles through matter is needed. This program was developed using the Geant4\[1\] particle physics library. A large number of electrons incident upon the converter are simulated and the produced positron distribution is recorded. From this, a fitting program is used to construct a two dimensional binned table of output probability \(P_1\) as a function of \(p_+\) and \(r\). The user may specify values for \(p_+\) and \(r\) to bin at or specify the desired number of bins in both dimensions to use. If the latter option is made, the fitting program will space the bins to give roughly equal number of particles in each bin. An example is shown in Figure 3. The probability of having a positron outside of the range of the table is taken to be zero. Inside the range of the table, the value of \(P_1(p_+, r)\) is found by linear interpolation of the table data.

For the \(P_2\) modeling, for each of the \((p_+, r)\) bin points for \(P_1\), a least squares fit is made using Eq. 5 and the distribution of \(dx/ds\) and \(dy/ds\) for particles in the bin. Since the distribution in Equation 5 is not normalizable over the entire \((dx/ds, dy/ds)\) plane, as part of the fit, the fitting program calculates values for \(|dx/ds|_{\text{min}}, |dx/ds|_{\text{max}}, |dy/ds|_{\text{min}}, |dy/ds|_{\text{max}}\), which are otherwise of which the value of \(P_2\) is taken to be zero. These cuts are made so that \(P_2\) covers 95% of the positrons generated by Geant4. The values of \(c_x, \alpha_1, \alpha_2, \beta, dx/ds_{\text{min}}, dx/ds_{\text{max}}, |dy/ds|_{\text{min}}, |dy/ds|_{\text{max}}\) are obtained by a fit of the Geant4

Figure 1: Positron converter coordinate system (side view).

Figure 2: Positron converter coordinate system (view of the downstream face).
data in each \((p_+, r)\) bin with the value of \(A\) fixed by the unit normalization of \(P_2\).

Once tables for the parameters \(c_x, \alpha_x, \) etc. are calculated at the bin points, the tables are used for each of these parameters to fit the parameters as a function of \(p_+\) and \(r\). The fit for a given parameter is has two pieces. Above a user definable threshold in \(p_+\), a two dimensional fit in \((p_+, r)\) is made of the form:

\[
f(p_+)g(r) \exp(k_1 p_+ + k_2 r) + C
\]

(6)

where \(f(p_+)\) and \(g(r)\) are third-order polynomials. Below the threshold, for each \(p_+\) in the \(P_1\) table, a 1D fourth-order polynomial fit in \(r\) is made. The 1D fits are performed for small values of \(p_+\) because equation 6 tends to fit the Geant4 data poorly when \(p_+ \leq 10\) MeV. Since most produced positrons are produced in this region, it is desirable for the \(P_2\) parameter fits to be more detailed when \(p_+\) is small.

Examples of the \(P_1\) and \(P_2\) distributions obtained from the Geant simulation are shown in Figures 3 and 4 respectively. The fit given in equation 5 models the data from Geant quite successfully, both when the parameters are obtained by direct fit to the data and when the parameters from the second order fits are used. The average \(\chi^2\) value for the direct fits is about 0.11, while the average \(\chi^2\) for the second order fits is about 0.14. The bins where the fit performs the worst have a \(\chi^2\) of about 1 for the direct fits, while the maximum \(\chi^2\) for the second order fits is about 2.2.

The Geant4 simulation and accompanying formalism assume that the incoming electrons are normally incident on the target. To adjust for a non-normal impinging electron when generating positrons with the converter model, the coordinate system is rotated so that the \(s\) axis is aligned along the incoming electron axis. The effective converter thickness used is adjusted to be \(T \sec \phi\) where \(\phi\) is the angle from the normal, as this is the distance from the electron beam’s entry point on the converter’s upstream face to it’s exit point on the downstream face. The distributions of positron momenta for \(\phi = 1\) and 5 degrees compared to the distribution for \(\phi = 0\).

Figure 3: \(P_1(p_+, r)\) for incoming electrons with \(p_+ c = 250\) MeV and a tungsten target of thickness \(T = 6.35\) mm. 

Figure 4: \(P_2\left(\frac{dx}{ds}, \frac{dy}{ds}; p_+, r\right)\) for incoming electrons with \(p_+ c = 250\) MeV and a tungsten target of thickness \(T = 6.35\) mm, and outgoing positrons with \(p_+ c = 5.15\) MeV and \(r = 0.37\) mm. The purple points indicate data obtained directly from the Geant simulation, while the green curve shows the fit to the data.

Figure 5: Relative discrepancy of the distributions of positron momenta for \(\phi = 1\) and 5 degrees compared to the distribution for \(\phi = 0\).

SPIN TRACKING

Polarization transfer from the incoming electrons to the outgoing positrons has also been modeled. Any incoming polarization \(S_\perp\) may be specified, and histograms describing \(S_x, S_y, \) and \(S_z\) of the produced positrons as functions of
Recently, the PEPPo collaboration\[11\] has published experimental results of polarized positron production from a polarized electron beam via bremsstrahlung, to which the spin tracking results from this simulation can be compared. This simulation yields polarization transfer efficiencies significantly higher than those observed experimentally for outgoing positrons with $p_+ c$ less than 5 MeV. In contrast, the simulation’s results for higher momentum positrons agrees closely with experiment. The discrepancy is likely due to approximations made by the Geant4 library in computing polarization transfer during the pair production step. While this disagreement shows that the spin tracking components of this simulation should not be relied on for accuracy at present, all of the infrastructure is in place to handle spin tracking. When more accurate spin tracking methods become available in Geant, this simulation, and the Bmad converter element, will be ready to use their results.

**THE BMAD CONVERTER ELEMENT**

Bmad is a software toolkit for the simulation of high energy charged particles and X-rays. To make use of the converter model, a converter element has been added to Bmad which uses the model coefficients as described above allowing for such things as a start to end simulation of the Cornell CESR Linac. The model coefficients are stored within a lattice input file which makes converter simulation portable to any Bmad based program.

For each incoming electron, Bmad randomly pulls values of $p_+, r_0$, $\frac{dx}{dr}$, and $\frac{d\phi}{dr}$ from the interpolated probability distributions. Outgoing positrons are assigned a weighting factor proportional to the likelihood of producing a positron from the given incoming electron. This weighting factor is then used with any statistical analysis.

Agreement between direct output from Geant4 and emulated output from Bmad is quite good, with the distribution of particles from Bmad coming within a few percent of the Geant distributions. This is illustrated in Figure 8. Bmad simulates the positron converter approximately 20 times faster than the direct Geant4 physics simulations.

As the figure shows, the relative discrepancy between the distributions of positrons produced by Geant and Bmad agree within a few percent under most circumstances. The two scenarios where this is not the case are for the lowest energy bins, and for bins with large $p_+$ and large $r$. The discrepancy in the first case is due to the fact that Bmad does not produce positrons with $p_+$ below the value of $p_+$ used as the center of the first bin. This causes Bmad to produce less low-energy positrons than Geant. This discrepancy is acceptable, as virtually all of these low-energy positrons would be lost within a short distance from the converter. The discrepancy for large $p_+$ and $r$ bins is also acceptable since very few positrons are produced in this region.

**CONCLUSION**

The converter model presented in this paper, when applied to positron production, provides simulation results that are in close agreement to Geant4’s direct simulation of bremsstrahlung radiation and pair production. A small sacrifice in accuracy is made in return for a significant increase in simulation speed, making this method appropriate for use in lattice simulation and optimization software such as Bmad.

One of the primary motivators for this work was the desire to fully model the CESR linac lattice. Work is currently underway to use the model developed here. A first goal in this endeavor is to assess the agreement between the Bmad model of the linac as a whole against the linac’s experimentally observed beam. Once good agreement is established, Bmad can be used to optimize the lattice design, including...
quadrupole and solenoid tuning. Other future applications could include the optimization of element positioning on the linac beam line.

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