GLOBAL ANALYSIS OF A STOCHASTIC TB MODEL WITH VACCINATION AND TREATMENT

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ABSTRACT. In this paper, a stochastic model is formulated to describe the transmission dynamics of tuberculosis. The model incorporates vaccination and treatment in the intervention strategies. Firstly, sufficient conditions for persistence in mean and extinction of tuberculosis are provided. In addition, sufficient conditions are obtained for the existence of stationary distribution and ergodicity. Moreover, numerical simulations are given to illustrate these analytical results. The theoretical and numerical results show that large environmental disturbances can suppress the spread of tuberculosis.

1. Introduction. Tuberculosis (TB) is a chronic epidemic disease caused by the bacterium *Mycobacterium tuberculosis* (MTB). The WHO estimates that 10.4 million people fell ill with TB and 1.8 million died of the disease in 2015 (WHO, 2017) and reports that TB is one of the top 10 causes of death from an infectious disease worldwide. Due to the great harm of TB on public health, it is important to increase our understanding of TB transmission dynamics. Mathematical models are powerful tools to gain insights into the transmission of infectious diseases [7, 8, 9, 15, 16, 21, 28, 29]. Over the past two decades, there have been a lot of mathematical models to investigate the transmission dynamics of TB [4, 10, 22, 26].

It is widely recognized that most infections with MTB do not cause TB disease and 90 – 95% of infections remain asymptomatic, though roughly one-third of the world’s population is infected with MTB. Thus reasonable TB models should incorporate latent tuberculosis cases, such as [1, 18, 27]. These mathematical models have provided useful information for studying transmission mechanisms of TB. However, most of the models do not pay more attention to the influence of intervention strategies on the transmission dynamics of TB.

Vaccination and treatment (of infectious individuals) are two of the most common implemented intervention strategies for infectious diseases, especially in the case of TB. In 2017, Gao and Huang [6] formulated an ordinary differential equation (ODE)
They also incorporated vaccination and treatment into the TB model. In this model the population is divided into five classes based on disease status, namely the uninfected individuals, the vaccinated individuals, the treated individuals, the infected individuals (individuals infected and shedding virus) and the recovered individuals (individuals previously infected with the virus but not currently shedding virus). At time $t$, the numbers in each of these classes are denoted by $S(t)$, $V(t)$, $T(t)$, $L(t)$ and $I(t)$, respectively. The model also involves the following parameters:

- $\Lambda$ the constant recruitment rate of the population,
- $\beta$ the scaled transmission rate between $S(t)$ and $I(t)$,
- $\rho_1 \beta$ the scaled transmission rate between $S(t)$ and $T(t)$,
- $p$ the proportional coefficient of vaccinated susceptible $S(t)$,
- $u$ the per-capita natural mortality rate,
- $\alpha$ the disease-induced death rate of $I(t)$,
- $l$ the proportion coefficient from $S(t)$ to $L(t)$,
- $\delta$ the rate at which $L(t)$ becomes $I(t)$,
- $\rho$ the rate at which $T(t)$ becomes $L(t)$,
- $\gamma$ the rate at which $I(t)$ becomes $T(t)$.

In terms of those parameters the ODE TB model takes the form:

$$
\begin{align*}
\frac{dS}{dt} &= \Lambda - \beta S(I + \rho_1 T) - (u + p)S, \\
\frac{dV}{dt} &= pS - uV, \\
\frac{dL}{dt} &= l\beta S(I + \rho_1 T) - (u + \delta)L + \rho T, \\
\frac{dI}{dt} &= (1 - l)\beta S(I + \rho_1 T) + \delta L - (u + \alpha + \gamma)I, \\
\frac{dT}{dt} &= \gamma I - (u + \rho)T.
\end{align*}
$$

(1)

One drawback of the aforementioned model is that it does not account for random factors that may influence the transmission dynamics of TB, because environmental fluctuations such as heavy rains, droughts, high winds and earthquakes can also affect the spread of diseases [5, 12, 13, 19, 23, 24, 25, 31, 33]. Mathematically, stochastic processes are usually used to describe the environmental noise. By using stochastic differential equations (SDE), several recent studies have taken into

Figure 1. Transfer diagram of the ODE TB model
account the effects of environmental fluctuations in the study of infectious disease dynamics [2, 3, 11, 20, 32, 30]. For instance, Chang et al. [3] formulated a stochastic SIRS system with two different nonlinear incidence rates and double epidemic asymmetrical hypothesis. They obtained thresholds for the persistence in mean and extinction of epidemic. By considering the role of the vaccine, Zhao et al. [32] proposed a stochastic SIVS system and found that large environmental noise could suppress the outbreak of diseases. Keeping all the above work in mind, in this paper, we assume that environmental fluctuations are directly proportional to the population in model (1). This is a standard technique in stochastic modeling (see e.g. [23]). Then we obtain the following stochastic TB model:

\[
\begin{aligned}
    dS &= [\Lambda - \beta S(I + \rho S) - (u + p)S] \, dt + \sigma_1 S \, dW_1(t), \\
    dV &= (\rho S - uV) \, dt + \sigma_2 V \, dW_2(t), \\
    dL &= (\beta S(I + \rho S) - (u + \delta)L + \rho T) \, dt + \sigma_3 L \, dW_3(t), \\
    dI &= [(1 - l)\beta S(I + \rho T) + \delta L - (u + \alpha + \gamma)I] \, dt + \sigma_4 I \, dW_4(t), \\
    dT &= [\gamma I - (u + \rho)T] \, dt + \sigma_5 T \, dW_5(t),
\end{aligned}
\]

where \(W_i(t) \ (i = 1, \cdots, 5)\) is an independent Brownian motion defined on the complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})\), and \(\sigma_i^2\) is the intensity of \(W_i(t) \ (i = 1, \cdots, 5)\).

The main purpose of the paper is to study how environmental fluctuations affect the spread of tuberculosis, and to further investigate the existence of stationary distribution and ergodicity of the stochastic TB system. The rest of the paper is organized as follows: In Section 2, we introduce some preliminary lemmas for further work. In Section 3, we explore sufficient conditions ensuring persistence in mean and extinction of tuberculosis. In Section 4, by using Lyapunov method we obtain sufficient conditions for the existence of stationary distribution and ergodicity. In Section 5, some numerical simulations are conducted to verify the analytical results. Finally, some concluding remarks are presented.

2. Preliminaries. Throughout this paper, let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions (i.e. it is increasing and right continuous while \(\mathcal{F}_0\) contains all \(\mathcal{P}\)-null sets). \(\mathbb{R}_n^+ = \{(x_1, \cdots, x_n) : x_i > 0, i = 1, \cdots, n\}\), \(\mathbb{R}^l\) is an Euclidean \(l\)-space and a.s. means almost surely.

For the sake of simplicity, we define:

(i) \(a \wedge b = \min\{a, b\}\), \(a \vee b = \max\{a, b\}\) and \(\sigma = \sigma_1 \vee \sigma_2 \vee \sigma_3 \vee \sigma_4 \vee \sigma_5\);

(ii) \(\langle X \rangle = \frac{1}{t} \int_0^t X(s) \, ds\), where \(X(t) \in [0, +\infty)\) is an integrable function;

(iii) \(R_0 = \frac{6\beta(1+\rho_1)\Delta - u(u+p)}{(u+p+1/2\sigma_1^2)(u+\delta+1/2\sigma_2^2)(u+\alpha+\gamma+1/2\sigma_4^2)(u+\rho+1/2\sigma_5^2)}\).

**Lemma 2.1.** The solution of system (2) has the following properties:

\[
\begin{aligned}
    \lim_{t \to \infty} \frac{S(t)}{t} &= 0, & \lim_{t \to \infty} \frac{V(t)}{t} &= 0, & \lim_{t \to \infty} \frac{L(t)}{t} &= 0, & \lim_{t \to \infty} \frac{I(t)}{t} &= 0 \ a.s., \\
    \lim_{t \to \infty} \frac{T(t)}{t} &= 0, & \lim_{t \to \infty} \frac{\ln S(t)}{t} &\leq 0, & \lim_{t \to \infty} \frac{\ln V(t)}{t} &\leq 0, & \lim_{t \to \infty} \frac{\ln L(t)}{t} &\leq 0 \ a.s., \\
    \lim_{t \to \infty} \frac{\ln I(t)}{t} &\leq 0, & \lim_{t \to \infty} \frac{\ln T(t)}{t} &\leq 0 \ a.s.
\end{aligned}
\]

In addition, if \(u \geq \frac{1}{2}\sigma_1^2\), we have
From the first equation of system (2) we have

\[ \lim_{t \to \infty} \frac{1}{t} \left( \int_0^t S(s)dW_1(s) + \int_0^t V(s)dW_2(s) + \int_0^t L(s)dW_3(s) + \int_0^t I(s)dW_4(s) + \int_0^t T(s)dW_5(s) \right) = 0 \text{ a.s.} \]

Lemma 2.1 can be proved by using the similar approach in [17], and hence is omitted.

**Lemma 2.2.** For any given initial value \((S(0), V(0), L(0), I(0), T(0)) \in \mathbb{R}^5_+\), system (2) admits a unique positive solution \((S(t), V(t), L(t), I(t), T(t)) \in \mathbb{R}^5_+\) on \(t \geq 0\) with probability 1.

The proof of Lemma 2.2 is similar to Mao et al. [23], hence we omit it.

Let \(X(t)\) be a regular time-homogeneous Markov process in \(\mathbb{R}^l\) given by the SDE

\[ dX(t) = b(X)dt + \sum_{m=1}^k \sigma_m(X)dW_m(t). \]

The diffusion matrix of \(X(t)\) is described by

\[ A(x) = (a_{ij}(x)), \quad a_{ij}(x) = \sum_{m=1}^k \sigma_m^i(x)\sigma_m^j(x). \]

**Assumption 2.1 ([14, 22]).** There exists a bounded open domain \(U \subset \mathbb{R}^l\) with regular boundary \(\Gamma\), having the following properties:

(B.1): there is a positive number \(M > 0\) such that \(\sum_{i,j=1}^l a_{ij}(x)\xi_i\xi_j \geq M|\xi|^2, x \in U, \xi \in \mathbb{R}^l\).

(B.2): there exists a nonnegative \(C^2\)-function \(Q\) such that the differential operator \(LQ\) is negative for any \(\mathbb{R}^l \setminus U\).

**Lemma 2.3 ([14]).** If Assumption 2.1 holds, then the Markov process \(X(t)\) has a unique ergodic stationary distribution \(\mu(\cdot)\).

3. **Stochastic endemic dynamics.** In this section, we focus on the persistence in mean and extinction of TB under stochastic perturbations.

**Theorem 3.1.** For any given initial value \((S(0), V(0), L(0), I(0), T(0)) \in \mathbb{R}^5_+\), the TB of system (2) will die out if \(u \geq \frac{1}{2}\sigma^2\) and \(R_0 < 1\), i.e.,

\[ \lim_{t \to \infty} I(t) = 0, \quad \lim_{t \to \infty} L(t) = 0, \quad \lim_{t \to \infty} T(t) = 0 \text{ a.s.} \]

**Proof.** From the first equation of system (2) we have

\[ S(t) - S(0) = \Delta t - \int_0^t \beta S(s)[I(s) + \rho_1 T(s)]ds - (u + p) \int_0^t S(s)ds + \int_0^t \sigma_1 S(s)dW_1(s). \]  

(3)

It then follows that

\[ \lim_{t \to \infty} \langle S \rangle = \frac{1}{u + p} \lim_{t \to \infty} \left[ \Lambda - \frac{S(t) - S(0)}{t} - \beta(S(I + \rho_1 T)) + \frac{1}{t} \int_0^t \sigma_1 S(s)dW_1(s) \right] \]

\[ \leq \frac{1}{u + p} \lim_{t \to \infty} \left[ \Lambda + \frac{1}{t} \int_0^t \sigma_1 S(s)dW_1(s) \right]. \]  

(4)
By Lemma 2.1, we obtain that
\[
\lim_{t \to \infty} \langle S \rangle \leq \frac{\Lambda}{u + p} \quad \text{a.s.}
\] (5)
Applying Itô’s formula to \( Q = \ln(L + I + T) \), we have
\[
dQ(t) = \mathcal{L}Q dt + \frac{1}{L + I + T} (\sigma_3 L dW_3(t) + \sigma_4 I dW_4(t) + \sigma_5 T dW_5(t)),
\] (6)
where
\[
\mathcal{L}Q = \frac{1}{L + I + T} \left[ \beta S(I + \rho_1 T) - u(L + I + T) - \alpha I \right]
\]
\[
- \frac{1}{2(L + I + T)^2} (\sigma_3^2 L^2 + \sigma_4^2 I^2 + \sigma_5^2 T^2)
\]
\[
\leq \beta S + \rho_1 \beta S - u - \frac{1}{2(L + I + T)^2} (\sigma_3^2 L^2 + \sigma_4^2 I^2 + \sigma_5^2 T^2)
\]
\[
\leq \beta(1 + \rho_1) S - u - \frac{1}{6} (\sigma_3^2 \wedge \sigma_4^2 \wedge \sigma_5^2).
\]
Integrating both sides of (6) from 0 to \( t \), we see that
\[
\ln(L(t) + I(t) + T(t)) \leq \ln(L(0) + I(0) + T(0)) + \beta(1 + \rho_1) \int_0^t S(s) ds
\]
\[
- \frac{ut}{6} (\sigma_3^2 \wedge \sigma_4^2 \wedge \sigma_5^2)
\]
\[
+ \int_0^t [\sigma_3 dW_3(s) + \sigma_4 dW_4(s) + \sigma_5 dW_5(s)].
\] (7)
Dividing both sides of (7) by \( t \) and then taking the limit superior leads to
\[
\limsup_{t \to \infty} \frac{\ln(L + I + T)}{t} \leq \frac{\beta(1 + \rho_1) \Lambda}{u + p} - u - \frac{1}{6} (\sigma_3^2 \wedge \sigma_4^2 \wedge \sigma_5^2)
\]
\[
\leq \frac{1}{6} (\sigma_3^2 \wedge \sigma_4^2 \wedge \sigma_5^2) \left[ \frac{6[\beta(1 + \rho_1) \Lambda - u(u + p)]}{(u + p)(\sigma_3^2 \wedge \sigma_4^2 \wedge \sigma_5^2)} - 1 \right]
\] (8)
Inequality (8) implies directly the assertion of the theorem. \( \Box \)

**Theorem 3.2.** For any given initial value \((S(0), V(0), L(0), I(0), T(0)) \in \mathbb{R}_+^5\), the TB of system (2) will persistence in mean if \( u \geq \frac{1}{2} \sigma^2 \) and \( R_0 > 1 \), i.e.,
\[
\liminf_{t \to \infty} (I) \geq \frac{4(u + p)}{\gamma(c_4(u + p) + \beta \rho_1)} (u + p + \frac{1}{2} \sigma_1^2) (R_0^{\frac{1}{2}} - 1) > 0 \quad \text{a.s.}
\]
\[
\liminf_{t \to \infty} (L) \geq \frac{4p}{(u + \delta)(c_4(u + p) + \beta \rho_1)} (u + p + \frac{1}{2} \sigma_1^2) (R_0^{\frac{1}{2}} - 1) > 0 \quad \text{a.s.}
\]
\[
\liminf_{t \to \infty} (T) \geq \frac{4}{c_4(u + p) + \beta \rho_1} (u + p + \frac{1}{2} \sigma_1^2) (R_0^{\frac{1}{2}} - 1) > 0 \quad \text{a.s.}
\]

**Proof.** Consider the Lyapunov function
\[
Q = \ln S + c_1 \ln L + c_2 \ln I + c_3 \ln T + c_4 T,
\]
where
\[
c_1 = \frac{u + p + \frac{1}{2} \delta}{u + \delta + \frac{1}{2} \sigma_3^2}, \quad c_2 = \frac{u + p + \frac{1}{2} \sigma_1^2}{u + \beta + \frac{1}{2} \sigma_2^2}, \quad c_3 = \frac{u + p + \frac{1}{2} \sigma_4^2}{u + \rho + \frac{1}{2} \sigma_5^2}, \quad c_4 = \frac{\beta}{\gamma}.
\]
Applying Itô's formula to $Q$, we have
\[ dQ(t) = \mathcal{L}Q dt + \sigma_1 dW_1(t) + c_1 \sigma_3 dW_3(t) + c_2 \sigma_4 dW_4(t) + (c_3 + c_4 T) \sigma_5 dW_5(t), \]
where
\[ \mathcal{L}Q = \frac{1}{S} [\Lambda - \beta S(I + \rho_1 T) - (u + p) S] + \frac{c_1}{L} [\beta S(I + \rho_1 T) - (u + \delta) L + \rho T] \\
+ \frac{c_2}{T} [(1 - l) \beta S(I + \rho_1 T) + \delta L - (u + \alpha + \gamma) I] + \frac{c_3}{T} [\gamma I - (u + p) T] \\
+ c_4 [\gamma I - (u + \rho) T] - \frac{1}{2} \left( \sigma_1^2 + c_1 \sigma_3^2 + c_2 \sigma_4^2 + c_3 \sigma_5^2 \right) \\
= \frac{\Lambda}{S} - \beta(I + \rho_1 T) - (u + p) + \frac{c_1}{L} \beta S(I + \rho_1 T) - c_1(u + \delta) + \frac{c_1}{L} \rho T \\
+ \frac{c_2}{T} (1 - l) \beta S(I + \rho_1 T) + \frac{c_2}{T} \delta L - c_2(u + \alpha + \gamma) + \frac{c_3}{T} \gamma I - c_3(u + \rho) \\
+ c_4 [\gamma I - (u + \rho) T] - \frac{1}{2} \left( \sigma_1^2 + c_1 \sigma_3^2 + c_2 \sigma_4^2 + c_3 \sigma_5^2 \right) \\
\geq \frac{\Lambda}{S} + \frac{c_1}{L} \beta \rho_1 S T + \frac{c_2}{T} \delta L + \frac{c_3}{T} \gamma I + (c_4 \gamma - \beta) I - [c_4(u + \rho) + \beta \rho_1] T \\
- (u + p + \frac{1}{2} \sigma_1^2) - c_1(u + \delta + \frac{1}{2} \sigma_3^2) - c_2(u + \alpha + \gamma + \frac{1}{2} \sigma_4^2) \\
- c_3(u + \rho + \frac{1}{2} \sigma_5^2) \\
\geq 4 \sqrt{c_1 c_2 c_3 \Lambda \beta \rho_1 \gamma + (c_4 \gamma - \beta) I - [c_4(u + \rho) + \beta \rho_1] T - (u + p + \frac{1}{2} \sigma_1^2)} \\
- c_1(u + \delta + \frac{1}{2} \sigma_3^2) - c_2(u + \alpha + \gamma + \frac{1}{2} \sigma_4^2) - c_3(u + \rho + \frac{1}{2} \sigma_5^2).
\]
Direction calculation yields that
\[ \mathcal{L}Q \geq - [c_4(u + \rho) + \beta \rho_1] T + 4 \sqrt{c_1 c_2 c_3 \Lambda \beta \rho_1 \gamma - 4(u + p + \frac{1}{2} \sigma_1^2)} \\
\geq - [c_4(u + \rho) + \beta \rho_1] T + 4 \left[ \Lambda \beta \rho_1 \gamma (u + p + \frac{1}{2} \sigma_1^2) \right]^{\frac{3}{2}} \\
- 4(u + p + \frac{1}{2} \sigma_1^2) \\
= - [c_4(u + \rho) + \beta \rho_1] T + 4(u + p + \frac{1}{2} \sigma_1^2)(R_0^\frac{1}{2} - 1).
\]
Integrating both sides of (9) from 0 to $t$, we have
\[ (c_4(u + \rho) + \beta \rho_1) \langle T \rangle \geq 4(u + p + \frac{1}{2} \sigma_1^2)(R_0^\frac{1}{2} - 1) - \frac{Q(t) - Q(0)}{t} + \frac{M(t)}{t}, \]
where
\[ M(t) = \int_0^t \sigma_1 dW_1(\theta) + c_1 \int_0^t \sigma_3 dW_3(\theta) + c_2 \int_0^t \sigma_4 dW_4(\theta) + \int_0^t (c_3 + c_4 T(\theta)) \sigma_5 dW_5(\theta). \]
Taking the limit inferior of (10), it then follows from Lemma 2.1 that
\[ \liminf_{t \to \infty} \langle T \rangle \geq \frac{4}{c_4(u + \rho) + \beta \rho_1}(u + p + \frac{1}{2} \sigma_1^2)(R_0^\frac{1}{2} - 1) > 0 \ a.s. \]
Integrating both sides of the last equation of system (2) from 0 to \( t \) leads to
\[
\langle I \rangle = \frac{1}{\gamma} \left[ (u + p) \langle T \rangle - \frac{T(t) - T(0)}{t} - \frac{1}{t} \int_0^t T(s) \sigma_5 dW_5(t) \right].
\] (11)

By using Lemma 2.1 and taking the limit inferior of (11) again, we obtain that
\[
\liminf_{t \to \infty} \langle I \rangle = \frac{u + p}{\gamma} \liminf_{t \to \infty} \langle T \rangle 
\geq \frac{4(u + p)}{\gamma(c_4(u + p) + \beta \rho_1)} (u + p + \frac{1}{2} \sigma_1^2) (R_0^\delta - 1) > 0 \text{ a.s.}
\]
Similarly, integrating both sides of the third equation of system (2) from 0 to \( t \), we have
\[
(u + \delta) \langle L \rangle = \beta(SI + \rho_1 ST) + \rho \langle T \rangle - \frac{L(t) - L(0)}{t} + \frac{\sigma_3}{t} \int_0^t L(s) dW_3(t) 
\geq \rho \langle T \rangle - \frac{L(t) - L(0)}{t} + \frac{\sigma_3}{t} \int_0^t L(s) dW_3(t).
\] (12)
This, together with Lemma 2.1 show immediately that
\[
\liminf_{t \to \infty} \langle L \rangle \geq \frac{\rho}{u + \delta} \liminf_{t \to \infty} \langle T \rangle 
\geq \frac{4\rho}{(u + \delta)(c_4(u + p) + \beta \rho_1)} (u + p + \frac{1}{2} \sigma_1^2) (R_0^\delta - 1) > 0 \text{ a.s.}
\]
This completes the proof of Theorem 3.2.

4. Stationary distribution and ergodicity. In the absence of environmental disturbances, i.e., \( \sigma_i = 0, \ i = 1, \cdots, 5 \), the full system (2) is reduced to the deterministic system (1). Gao and Huang [6] obtained that if \( R_0^\delta > 1 \) system (1) admits a unique endemic equilibrium \( E^* = (S^*, V^*, L^*, I^*, T^*) \) which is globally asymptotically stable. However, there is no endemic equilibrium in the stochastic system (2). In this section, by constructing suitable Lyapunov functions, we carry out the stationary distribution and ergodicity of system (2) to illustrate the cycling phenomena of TB.

**Theorem 4.1.** The solution of system (2) admits a unique ergodic stationary distribution if \( R_0^\delta > 1 \).

**Proof.** Define a \( C^2 \)-function
\[
Q_1 = M(-n_1 \ln S - n_2 \ln L - n_3 \ln I - n_4 \ln T - (1 + \frac{1}{M}) \frac{\beta T}{\gamma}) 
+ \frac{1}{\theta + 1} (S + V + L + I + T)^{\theta+1} - \ln S - \ln V - \ln L 
= MQ_1 + Q_2 - \ln S - \ln V - \ln L,
\] (13)
where \( n_1, n_2, n_3, n_4, \theta \) and \( M \) are positive constants, which will be defined later. Clearly, we have
\[
\liminf_{k \to \infty, (S, V, L, I, T) \in R_0^\delta \setminus U_k} \tilde{Q}(S, V, L, I, T) = +\infty,
\]
where \( U_k = \prod_{i=1}^{k} \left( \frac{5}{k} \right) \). According to the continuity of \( \tilde{Q}(S, V, L, I, T) \), it must have a minimum point \((S_0, V_0, L_0, I_0, T_0) \in \mathbb{R}_+^5 \). It follows that the \( C^2 \) function

\[
Q(S, V, L, I, T) = \tilde{Q}(S, V, L, I, T) - \tilde{Q}(S_0, V_0, L_0, I_0, T_0)
\]

is a positive definite function.

Applying Itô's formula to \( Q_1 \), one obtains the differential operator \( \mathcal{L} \) of \( Q_1 \) as follows:

\[
\mathcal{L}Q_1 = -\frac{n_1}{S} [\Lambda - \beta S(I + \rho_1 T) - (u + p)S] - \frac{n_2}{L} [\beta S(I + \rho_1 T) - (u + \delta)L + \rho T]
- \frac{n_3}{I} [(1 - l)\beta S(I + \rho_1 T) + \delta L - (u + \alpha + \gamma)I] - \frac{n_4}{T} [\gamma I - (u + \rho)T]
- (1 + \frac{1}{M})\beta(I - (u + \rho)T) + \frac{1}{2}n_1\sigma_1^2 + \frac{1}{2}n_2\sigma_2^2 + \frac{1}{2}n_3\sigma_3^2 + \frac{1}{2}n_4\sigma_4^2
\]

\[
= -\frac{n_1I}{S} + n_1\beta(I + \rho_1 T) - \frac{n_2\beta S(I + \rho_1 T)}{L} - \frac{n_2\rho T}{L}
- \frac{n_3}{I}[(1 - l)\beta S(I + \rho_1 T) - \frac{n_3\delta L}{I} - \frac{n_4\gamma I}{T} - (1 + \frac{1}{M})\beta I]
+ (1 + \frac{1}{M})\beta(u + \rho) + n_1(u + p + \frac{1}{2}\sigma_1^2)
+ n_2(u + \delta + \frac{1}{2}\sigma_2^2) + n_3(u + \alpha + \gamma + \frac{1}{2}\sigma_3^2) + n_4(u + \rho + \frac{1}{2}\sigma_4^2)
\]

\[
\leq -\frac{n_1I}{S} + (n_1 - 1 - \frac{1}{M})\beta I - \frac{n_2\beta \rho_1 ST}{L} - \frac{n_3\delta L}{I} - \frac{n_4\gamma I}{T}
+ \left[ n_1 \rho_1 + (1 + \frac{1}{M})\frac{(u + \rho)}{\gamma} \right] \beta T + n_1(u + p + \frac{1}{2}\sigma_1^2)
+ n_2(u + \delta + \frac{1}{2}\sigma_2^2) + n_3(u + \alpha + \gamma + \frac{1}{2}\sigma_3^2) + n_4(u + \rho + \frac{1}{2}\sigma_4^2)
\]

\[
\leq -4\sqrt{n_1n_2n_3n_4\Lambda \beta \rho_1 \delta \gamma} + \left[ n_1 \rho_1 + (1 + \frac{1}{M})\frac{(u + \rho)}{\gamma} \right] \beta T
+ \left[ n_1 - 1 - \frac{1}{M} \right] \beta I + n_1(u + p + \frac{1}{2}\sigma_1^2) + n_2(u + \delta + \frac{1}{2}\sigma_2^2)
+ n_3(u + \alpha + \gamma + \frac{1}{2}\sigma_3^2) + n_4(u + \rho + \frac{1}{2}\sigma_4^2).
\]

Let

\[
n_1 = 1, \ n_2 = \frac{n_1(u + p + \frac{1}{2}\sigma_1^2)}{u + \delta + \frac{1}{2}\sigma_2^2}, \ n_3 = \frac{n_1(u + p + \frac{1}{2}\sigma_2^2)}{u + \alpha + \gamma + \frac{1}{2}\sigma_3^2}, \ n_4 = \frac{n_1(u + p + \frac{1}{2}\sigma_3^2)}{u + \rho + \frac{1}{2}\sigma_4^2}.
\]

It then follows that

\[
\mathcal{L}Q_1 \leq -4\sqrt{n_1n_2n_3n_4\Lambda \beta \rho_1 \delta \gamma} - \frac{\beta}{M} I + \left[ \rho_1 + (1 + \frac{1}{M})\frac{(u + \rho)}{\gamma} \right] \beta T
+ 4(u + p + \frac{1}{2}\sigma_1^2).
\]
By Itô’s formula, we obtain the differential operator $\mathcal{L}$ of $Q_2$ as follows:

\[
\mathcal{L}Q_2 = (S + V + L + I + T)^\theta \left[ \alpha - u(S + V + L + I + T) - \alpha I \right]
\]
\[
+ \frac{\theta}{2} (S + V + L + I + T)^{\theta - 1} (\sigma_1^2 S^2 + \sigma_2^2 V^2 + \sigma_3^2 L^2 + \sigma_4^2 T^2)
\]
\[
\leq \Lambda (S + V + L + I + T)^\theta - u(S + V + L + I + T)^{\theta + 1}
\]
\[
+ \frac{\theta}{2} (\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2) (S + V + L + I + T)^{\theta + 1}
\]
\[
= \Lambda (S + V + L + I + T)^\theta - \phi(S + V + L + I + T)^{\theta + 1}
\]
\[
\leq C_0 - \frac{1}{2} \phi(S + V + L + I + T)^{\theta + 1}
\]
\[
\leq C_0 - \frac{1}{2} \phi(S^{\theta + 1} + V^{\theta + 1} + L^{\theta + 1} + I^{\theta + 1} + T^{\theta + 1}),
\]

where we choose $\theta$ sufficiently small such that $\phi = u - \frac{\theta}{2} \sigma^2 > 0$ and

\[
C_0 = \sup_{(S,V,L,I,T) \in \mathbb{R}^4_+} \left\{ \Lambda (S + V + L + I + T)^\theta - \frac{\phi}{2} (S + V + L + I + T)^{\theta + 1} \right\} < \infty.
\]

Similarly, one obtains

\[
\mathcal{L}(-\ln S) = -\frac{\Lambda}{S} + \beta (I + \rho_1 T) + u + p + \frac{1}{2} \sigma_1^2,
\]
\[
\mathcal{L}(-\ln V) = -\frac{p S}{V} + u + \frac{1}{2} \sigma_2^2,
\]
\[
\mathcal{L}(-\ln L) \leq u + \delta - \frac{\beta T}{L} + \frac{1}{2} \sigma_3^2,
\]
\[
\mathcal{L}(-\ln I) \leq -\frac{\delta L}{I} + u + \alpha + \gamma + \frac{1}{2} \sigma_4^2.
\]

Making use of (15)-(17), we then derive that

\[
\mathcal{L}Q \leq -4M(u + p + \frac{1}{2} \sigma_1^2)(R_0^\frac{1}{4} - 1) + (M + 1)(\rho_1 + \frac{u + \rho}{\gamma}) \beta T
\]
\[
+ C_0 - \frac{1}{2} \phi(S^{\theta + 1} + V^{\theta + 1} + L^{\theta + 1} + I^{\theta + 1} + T^{\theta + 1})
\]
\[
- \frac{\Lambda}{S} - \frac{p S}{V} - \frac{\beta T}{L} - \frac{\delta L}{I} + p + 4u + \delta + \alpha + \gamma + \frac{1}{2} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2).
\]
For convenience, we define
\[
F_1 = \sup_{T \in \mathbb{R}_+} \left\{ -4M(u + p + \frac{1}{2} \sigma_1^2)(R_0^{\frac{1}{2}} - 1) \\
+ (M + 1)(\rho_1 + \frac{u + p}{\gamma})\beta T - \frac{1}{4} \phi T^{\theta + 1} \right\} < \infty,
\]
(19)

\[
F_2 = C_0 + p + 4\epsilon + \delta + \alpha + \gamma + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2) < \infty.
\]

Constructing the compact subset \( D \) as follows:
\[
D = \{ \epsilon \leq S \leq \epsilon^{-1}, \epsilon^2 \leq V \leq \epsilon^{-2}, \epsilon^2 \leq L \leq \epsilon^{-2}, \epsilon^3 \leq I \leq \epsilon^{-3}, \epsilon \leq T \leq \epsilon^{-1} \},
\]
where \( \epsilon > 0 \) is sufficiently small such that
\[
- \left( \frac{\Lambda}{\epsilon} \wedge \frac{p}{\epsilon} \wedge \frac{\rho}{\epsilon} \wedge \frac{\phi}{\epsilon^{2\theta+1}} \wedge \frac{1}{\epsilon^{4\theta+1}} \right) + F_1 + F_2 < -1,
\]
(20)

Dividing the complementary set of \( D \) into ten parts as follows:
\[
\mathbb{R}_+^5 \setminus D = \bigcup_{i=1}^{10} D_i^c,
\]
where
\[
D_1^c = \{ (S, V, L, I, T) \in \mathbb{R}_+^5 | 0 < S < \epsilon \},
D_2^c = \{ (S, V, L, I, T) \in \mathbb{R}_+^5 | \epsilon \leq S, 0 < V < \epsilon^2 \},
D_3^c = \{ (S, V, L, I, T) \in \mathbb{R}_+^5 | 0 < T < \epsilon \},
D_4^c = \{ (S, V, L, I, T) \in \mathbb{R}_+^5 | \epsilon \leq T, 0 < L < \epsilon^2 \},
D_5^c = \{ (S, V, L, I, T) \in \mathbb{R}_+^5 | \epsilon^2 \leq L, 0 < I < \epsilon^3 \},
D_6^c = \{ (S, V, L, I, T) \in \mathbb{R}_+^5 | S > \epsilon^{-1} \},
D_7^c = \{ (S, V, L, I, T) \in \mathbb{R}_+^5 | V > \epsilon^{-2} \},
D_8^c = \{ (S, V, L, I, T) \in \mathbb{R}_+^5 | L > \epsilon^{-2} \},
D_9^c = \{ (S, V, L, I, T) \in \mathbb{R}_+^5 | I > \epsilon^{-1} \},
D_{10}^c = \{ (S, V, L, I, T) \in \mathbb{R}_+^5 | T > \epsilon^{-3} \}.
\]

Now we analyze the range of differential operators \( \mathcal{L}Q(S, V, L, I, T) \) on each domain.

**Case 1.** \((S, V, L, I, T) \in D_1^c\). We derive from (18) that
\[
\mathcal{L}Q \leq -\frac{\Lambda}{S} + C_0 + p + 4\epsilon + \delta + \alpha + \gamma + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)
- 4M(u + p + \frac{1}{2} \sigma_1^2)(R_0^{\frac{1}{2}} - 1) + (M + 1)(\rho_1 + \frac{u + p}{\gamma})\beta T - \frac{1}{4} \phi T^{\theta + 1}
\]
(21)
\[
\leq -\frac{\Lambda}{S} + F_1 + F_2 \leq -\frac{\Lambda}{\epsilon} + F_1 + F_2.
\]

**Case 2.** \((S, V, L, I, T) \in D_2^c\). Using (18) one obtains
\[
\mathcal{L}Q \leq -\frac{pS}{V} + C_0 + p + 4\epsilon + \delta + \alpha + \gamma + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)
\]
\begin{equation}
-4M(u+p+ \frac{1}{2}\sigma_1^2)(R_0^{\theta+1} - 1) + (M+1)(\rho_1 + \frac{u+\rho}{\gamma})\beta T - \frac{1}{2}\phi T^{\theta+1} \\
\leq -\frac{pS}{V} + F_1 + F_2 \leq -\frac{p}{\varepsilon} + F_1 + F_2.
\end{equation}

**Case 3.** \((S,V,L,I,T) \in D_5^c\). It is easy to show from (18) that

\begin{equation}
LQ \leq -4M(u+p+ \frac{1}{2}\sigma_1^2)(R_0^{\theta+1} - 1) + (M+1)(\rho_1 + \frac{u+\rho}{\gamma})\beta T \\
+ C_0 + p + 4u + \delta + \alpha + \gamma + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2) \\
\leq -4M(u+p+ \frac{1}{2}\sigma_1^2)(R_0^{\theta+1} - 1) + (M+1)(\rho_1 + \frac{u+\rho}{\gamma})\beta \varepsilon + F_2.
\end{equation}

**Case 4.** \((S,V,L,I,T) \in D_4^c\). Note from (18) that

\begin{equation}
LQ \leq -\frac{\rho T}{L} + C_0 + p + 4u + \delta + \alpha + \gamma + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2) \\
- 4M(u+p+ \frac{1}{2}\sigma_1^2)(R_0^{\theta+1} - 1) + (M+1)(\rho_1 + \frac{u+\rho}{\gamma})\beta T - \frac{1}{2}\phi T^{\theta+1}
\end{equation}

\leq -\frac{\rho T}{L} + F_1 + F_2 \leq -\frac{\rho}{\varepsilon} + F_1 + F_2.

**Case 5.** \((S,V,L,I,T) \in D_5^c\). By (18) we can easily show that

\begin{equation}
LQ \leq -\frac{\delta L}{T} + C_0 + p + 4u + \delta + \alpha + \gamma + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2) \\
- 4M(u+p+ \frac{1}{2}\sigma_1^2)(R_0^{\theta+1} - 1) + (M+1)(\rho_1 + \frac{u+\rho}{\gamma})\beta T - \frac{1}{2}\phi T^{\theta+1}
\end{equation}

\leq -\frac{\delta L}{T} + F_1 + F_2 \leq -\frac{\delta}{\varepsilon} + F_1 + F_2.

**Case 6.** \((S,V,L,I,T) \in D_6^c\). Making use of (18) one obtains that

\begin{equation}
LQ \leq -\frac{1}{2}\phi S^{\theta+1} - 4M(u+p+ \frac{1}{2}\sigma_1^2)(R_0^{\theta+1} - 1) + (M+1)(\rho_1 + \frac{u+\rho}{\gamma})\beta T \\
- \frac{1}{2}\phi T^{\theta+1} + C_0 + p + 4u + \delta + \alpha + \gamma + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)
\end{equation}

\leq -\frac{1}{2}\phi \frac{1}{\varepsilon^{\theta+1}} + F_1 + F_2.

**Case 7.** \((S,V,L,I,T) \in D_7^c\). We know from (18) that

\begin{equation}
LQ \leq -\frac{1}{2}\phi V^{\theta+1} - 4M(u+p+ \frac{1}{2}\sigma_1^2)(R_0^{\theta+1} - 1) + (M+1)(\rho_1 + \frac{u+\rho}{\gamma})\beta T \\
- \frac{1}{2}\phi T^{\theta+1} + C_0 + p + 4u + \delta + \alpha + \gamma + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)
\end{equation}

\leq -\frac{1}{2}\phi \frac{1}{\varepsilon^{\theta+1}} + F_1 + F_2.

**Case 8.** \((S,V,L,I,T) \in D_8^c\). Using (18), we can show that

\begin{equation}
LQ \leq -\frac{1}{2}\phi L^{\theta+1} - 4M(u+p+ \frac{1}{2}\sigma_1^2)(R_0^{\theta+1} - 1) + (M+1)(\rho_1 + \frac{u+\rho}{\gamma})\beta T \\
- \frac{1}{2}\phi T^{\theta+1} + C_0 + p + 4u + \delta + \alpha + \gamma + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2)
\end{equation}

\leq -\frac{1}{2}\phi \frac{1}{\varepsilon^{2\theta+1}} + F_1 + F_2.
Case 9. \( (S,V,L,I,T) \in D_9^c \). In view of (18), one obtains that
\[
\mathcal{L}Q \leq -\frac{1}{4} \phi T^{\theta+1} - 4M(u + p + \frac{1}{2} \sigma_1^2)(R_0^{\frac{3}{4}} - 1) + (M + 1)(\rho_1 + \frac{u + \rho}{\gamma})T \\
- \frac{1}{4} \phi T^{\theta+1} + C_0 + p + 4u + \delta + \alpha + \gamma + \frac{1}{2} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2) \\
\leq -\frac{1}{4} \frac{\sigma_1^2}{\beta_{\theta+1}} + F_1 + F_2. 
\] (29)

Case 10. \( (S,V,L,I,T) \in D_{10}^c \). We see from (18) that
\[
\mathcal{L}Q \leq -\frac{1}{2} \phi T^{\theta+1} - 4M(u + p + \frac{1}{2} \sigma_1^2)(R_0^{\frac{3}{4}} - 1) + (M + 1)(\rho_1 + \frac{u + \rho}{\gamma})T \\
- \frac{1}{2} \phi T^{\theta+1} + C_0 + p + 4u + \delta + \alpha + \gamma + \frac{1}{2} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2) \\
\leq -\frac{1}{2} \frac{\sigma_1^2}{\beta_{\theta+1}} + F_1 + F_2. 
\] (30)

It follows from (20)-(30) that
\[
\mathcal{L}Q < -1 \text{ for all } (S,V,L,I,T) \in \mathbb{R}_5^+ \setminus D,
\]
which means condition (B.2) of Assumption 2.1 is satisfied.

On the other hand, the diffusion matrix of model (2) is
\[
A = \begin{pmatrix}
\sigma_1^2 S^2 & 0 & 0 & 0 & 0 \\
0 & \sigma_2^2 V^2 & 0 & 0 & 0 \\
0 & 0 & \sigma_3^2 L^2 & 0 & 0 \\
0 & 0 & 0 & \sigma_4^2 I^2 & 0 \\
0 & 0 & 0 & 0 & \sigma_5^2 T^2
\end{pmatrix}. 
\] (31)

Choose
\[
G = \min_{(S,V,L,I,T) \in D} \{ \sigma_1^2 S^2, \sigma_2^2 V^2, \sigma_3^2 L^2, \sigma_4^2 I^2, \sigma_5^2 T^2 \} > 0.
\]

It then follows that
\[
\sum_{i,j=1}^5 A_{ij}(x)\xi_i\xi_j = \sigma_1^2 S^2 \xi_1^2 + \sigma_2^2 V^2 \xi_2^2 + \sigma_3^2 L^2 \xi_3^2 + \sigma_4^2 I^2 \xi_4^2 + \sigma_5^2 T^2 \xi_5^2 \\
\geq G\|\xi\|^2, \quad (S,V,L,I,T) \in D, \quad \xi \in \mathbb{R}^5,
\]
which implies condition (B.1) of Assumption 2.1 is satisfied.

Remark 1. From Theorem 4.1 we have that the property (B.2) of Assumption 2.1 is verified. Thus, the solution of system (2) is positive recurrent by the definition of positive recurrence and Lemma 4.1 in [14].

5. Numerical simulations. In this section, we employ the Milsteins Higher-Order Method [34] to confirm the main theoretical results of this paper. In order to distinguish carefully the difference between the stochastic model (2) and the corresponding deterministic model (1), we plot the solution curves of these systems. In this way, system (2) can be rewritten as the following discrete system:
A stochastic TB model with vaccination and treatment 2935

\begin{equation}
\begin{cases}
S_{n+1} = S_n + \Delta t(\Lambda - \beta S_n(\rho_1 I_n + \rho_1 T_n) - (u + p)S_n) + \sigma_1 S_n \sqrt{\Delta t} \xi_{1,n} \\
&+ \frac{1}{2} \sigma_1^2 S_n(\xi_{1,n}^2 - 1) \Delta t, \\
V_{n+1} = V_n + \Delta t(p S_n - u V_n) + \sigma_2 V_n \sqrt{\Delta t} \xi_{2,n} + \frac{1}{2} \sigma_2^2 V_n(\xi_{2,n}^2 - 1) \Delta t, \\
I_{n+1} = I_n + \Delta t(\beta S_n(\rho_1 I_n + \rho_1 T_n) - (u + \delta)L_n + \rho T_n) + \sigma_3 I_n \sqrt{\Delta t} \xi_{3,n} \\
&+ \frac{1}{2} \sigma_3^2 I_n(\xi_{3,n}^2 - 1) \Delta t, \\
T_{n+1} = T_n + \Delta t(\gamma I_n - (u + \rho)T_n) + \sigma_5 T_n \sqrt{\Delta t} \xi_{5,n} + \frac{1}{2} \sigma_5^2 T_n(\xi_{5,n}^2 - 1) \Delta t,
\end{cases}
\end{equation}

where $\Delta t = 0.01$, $\xi_{i,n}$ $i = 1, \ldots, 5$, $n = 1, 2, \ldots$ obeys the Gaussian distribution $N(0,1)$. To this end, we set $(S(0), V(0), L(0), I(0), T(0)) = (0.1, 0.1, 0.1, 0.1, 0.1)$.

**Example 5.1.** Choose the parameters: $\Lambda = 1, \beta = 0.8, \rho_1 = 0.01, u = 0.52, p = 0.7, l = 0.05, \rho = 1.5, \delta = 1.5, \alpha = 0.05, \gamma = 0.1, \sigma_1 = 0.95$ $i = 1, \ldots, 5$.

By simple calculation, we have $\bar{R}_0 = \frac{(u + \rho + \gamma \rho_1)(\beta + \beta(1-l)\Lambda)}{(u + p)[(u + \delta)[(u + p)[(u + \alpha) + w_1(u + p + \delta)]]]} = 1.0533 > 1$, thus the corresponding deterministic system (1) has a unique endemic equilibrium which is globally asymptotically stable (see Theorem 2.1 of [6] and Fig. 2(a)). However, for the stochastic system (2), the conditions $u = 0.52 > \frac{1}{2} \sigma^2 = 0.45125$ and $R_0 = 0.946 < 1$ of Theorem 3.1 are verified, which means that the diseases of system (2) will die out. This is aligned with the numerical results in Fig. 2(b). It should be noted that the only difference between system (1) and system (2) lies in the environmental noise, which suggests that environmental disturbance may help to curb the outbreak of diseases.

![Figure 2](image)

**Figure 2.** Trajectory of the solution of system (2) and its corresponding deterministic model (1)

**Example 5.2.** Choose the parameters: $\Lambda = 1, \beta = 0.8, \rho_1 = 0.2, u = 0.1, p = 0.1, l = 0.8, \rho = 0.2, \delta = 0.6, \alpha = 1, \gamma = 0.6, \sigma_1 = 0.02$ $i = 1, \ldots, 5$.

Theorem 3.2 indicates that the diseases of system (2) will persistence in mean if $u = 0.1 \geq \frac{1}{2} \sigma^2 = 0.0002$ and $\bar{R}_0 = 1.3684 > 1$. We obtain the corresponding
simulation results as shown in Fig. 3, which support the analytical results. In comparison with Example 5.1, we find that small environmental noise will not lead to the extinction of diseases. This suggests that diseases have a certain resistance to the environmental disturbance.

![Figure 3. Trajectory of the solution of system (2) and its corresponding deterministic model (1)](image)

**Example 5.3.** Choose the parameters: \( \Lambda = 1, \beta = 0.8, \rho_1 = 0.2, u = 0.1, p = 0.1, l = 0.8, \rho = 0.2, \delta = 0.6, \alpha = 1, \gamma = 0.6, \sigma_i = 0.02 \) for \( i = 1, \cdots, 5 \).

Straightforward calculation yields that \( R_0^* = 1.3684 > 1 \), which indicates that system (2) has a unique ergodic stationary distribution (See Theorem 4.1). This is consistent with the description depicted in Fig. 4.

6. **Concluding remarks.** In this paper, taking into account the vaccination and treatment of diseases, we proposed a stochastic model for the transmission dynamics of tuberculosis. Sufficient conditions for the persistence in mean and extinction of tuberculosis are obtained. In addition, the existence of a unique ergodic stationary distribution is confirmed by using the stochastic Lyapunov method and the Khasminskii’s theory. The stationary distribution indicates that the tuberculosis can become persistent in vivo. The theoretical work extended the results of the corresponding deterministic system. The results show that: (i) large stochastic noise can suppress the spread of tuberculosis, i.e., the tuberculosis will go to extinction if the intensity of noise is large enough; (ii) the tuberculosis has a certain resistance to the environmental noise, i.e., the tuberculosis will maintain its persistence if the environmental noise is sufficiently small, but the capacities of anti-noise are limited.

Some interesting questions deserve further investigation. One could investigate more realistic but more complex systems, for instance, multi-group tuberculosis systems with telephone noise and Lévy noise. We leave these investigations for future research.

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A STOCHASTIC TB MODEL WITH VACCINATION AND TREATMENT

Figure 4. The pictures on the left are trajectories of the solution of system (2). The pictures on the right are the distribution density functions of system (2)

REFERENCES

[1] S. Bowong and J. J. Tewa, Mathematical analysis of a tuberculosis model with differential infectivity, Communications in Nonlinear Science and Numerical Simulation, 14 (2009), 4010–4021.
[2] Y. Cai, Y. Kang, M. Banerjee and W. Wang, A stochastic sirs epidemic model with infectious force under intervention strategies, Journal of Differential Equations, 259 (2015), 7463–7502.
[3] Z. Chang, X. Meng and X. Lu, Analysis of a novel stochastic sirs epidemic model with two different saturated incidence rates, Physica A: Statistical Mechanics and its Applications, 472 (2017), 103–116.
[4] S. Choi, E. Jung and S.-M. Lee, Optimal intervention strategy for prevention tuberculosis using a smoking-tuberculosis model, Journal of Theoretical Biology, 380 (2015), 256–270.
[5] T. Feng, X. Meng, L. Liu and S. Gao, Application of inequalities technique to dynamics analysis of a stochastic eco-epidemiology model, Journal of Inequalities and Applications, 2016 (2016), Paper No. 327, 29 pp.
[6] D. Gao and N. Huang, A note on global stability for a tuberculosis model, Applied Mathematics Letters, 73 (2017), 163–168.
[7] S. Gao, L. Chen and Z. Teng, Pulse vaccination of an SEIR epidemic model with time delay, *Nonlinear Analysis: Real World Applications*, 9 (2008), 599–607.

[8] X. Han and P. E. Kloeden, *Immune System Virus Model*, Springer Singapore, Singapore, 2017.

[9] G. Huang, W. Ma and Y. Takeuchi, Global properties for virus dynamics model with Beddington-DeAngelis functional response, *Applied Mathematics Letters*, 22 (2009), 1690–1693.

[10] H. Huo and M. Zou, Modelling effects of treatment at home on tuberculosis transmission dynamics, *Applied Mathematical Modelling*, 40 (2016), 9474–9484.

[11] D. Jiang, C. Ji, N. Shi and J. Yu, The long time behavior of DI SIR epidemic model with stochastic perturbation, *Journal of Mathematical Analysis and Applications*, 372 (2010), 162–180.

[12] D. Jiang, Q. Liu, N. Shi, T. Hayat, A. Alsaedi and P. Xia, Dynamics of a stochastic HIV-1 infection model with logistic growth, *Physica A: Statistical Mechanics and its Applications*, 469 (2017), 706–717.

[13] D. Jiang and N. Shi, A note on nonautonomous logistic equation with random perturbation, *Journal of Mathematical Analysis and Applications*, 303 (2005), 164–172.

[14] R. Khasminskii, *Stochastic Stability of Differential Equations*, Stochastic Modelling and Applied Probability, 66. Springer, Heidelberg, 2012.

[15] P. E. Kloeden and C. Pötzche, Nonautonomous bifurcation scenarios in SIR models, *Mathematical Methods in the Applied Sciences*, 38 (2015), 3495–3518.

[16] A. Korobeinikov and G. C. Wake, Lyapunov functions and global stability for SIR, SIRS, and SIS epidemiological models, *Applied Mathematics Letters*, 15 (2002), 955–960.

[17] X. Leng, T. Feng and X. Meng, Stochastic inequalities and applications to dynamics analysis of a novel SIVS epidemic model with jumps, *Journal of Inequalities and Applications*, 2017 (2017), Paper No. 138, 25 pp.

[18] G. Li and Z. Jin, Global stability of a SEIR epidemic model with infectious force in latent, infected and immune period, *Chaos, Solitons & Fractals*, 25 (2005), 1177–1184.

[19] M. Liu, C. Bai and K. Wang, Asymptotic stability of a two-group stochastic SEIR model with infinite delays, *Communications in Nonlinear Science and Numerical Simulation*, 19 (2014), 3444–3453.

[20] Q. Liu, The threshold of a stochastic SIRS epidemic model under regime switching, *Nonlinear Analysis: Hybrid Systems*, 21 (2016), 49–58.

[21] Q. Liu and Q. Chen, Analysis of the deterministic and stochastic SIRS epidemic models with nonlinear incidence, *Physica A: Statistical Mechanics and its Applications*, 428 (2015), 140–153.

[22] Q. Liu, D. Jiang, N. Shi, T. Hayat and A. Alsaedi, Dynamics of a stochastic tuberculosis model with constant recruitment and varying total population size, *Physica A: Statistical Mechanics and its Applications*, 469 (2017), 518–530.

[23] X. Mao, G. Marion and E. Renshaw, Environmental brownian noise suppresses explosions in population dynamics, *Stochastic Processes and Their Applications*, 97 (2002), 95–110.

[24] X. Meng, S. Zhao, T. Feng and T. Zhang, Dynamics of a novel nonlinear stochastic SIS epidemic model with double epidemic hypothesis, *Journal of Mathematical Analysis and Applications*, 433 (2016), 227–242.

[25] A. Miao, J. Zhang, T. Zhang and B. Pradeep, Threshold dynamics of a stochastic model with vertical transmission and vaccination, *Computational and Mathematical Methods in Medicine*, 2017 (2017), Art. ID 4820183, 10 pp.

[26] D. Moualeu, A. N. Yakam, S. Bowong and A. Temgoua, Analysis of a tuberculosis model with undetected and lost-site cases, *Communications in Nonlinear Science and Numerical Simulation*, 41 (2016), 48–63.

[27] E. G. Nicholson, A. M. Geltemeyer and K. C. Smith, Practice guideline for treatment of latent tuberculosis infection in children, *Journal of Pediatric Health Care*, 29 (2015), 302–307.

[28] M. A. Nowak and R. M. May, Mathematical principles of immunology and virology, *Nature Medicine*, 410 (2001), 412–413.

[29] S. Ruan and W. Wang, Dynamical behavior of an epidemic model with a nonlinear incidence rate, *Journal of Differential Equations*, 188 (2003), 135–163.

[30] Q. Yang, D. Jiang, N. Shi and C. Ji, The ergodicity and extinction of stochastically perturbed SIR and SEIR epidemic models with saturated incidence, *Journal of Mathematical Analysis and Applications*, 388 (2012), 248–271.
[31] Q. Yang and X. Mao, Extinction and recurrence of multi-group seir epidemic models with stochastic perturbations, *Nonlinear Analysis: Real World Applications*, 14 (2013), 1434–1456.

[32] D. Zhao, T. Zhang and S. Yuan, The threshold of a stochastic sivs epidemic model with nonlinear saturated incidence, *Physica A: Statistical Mechanics and its Applications*, 443 (2016), 372–379.

[33] Y. Zhou, W. Zhang and S. Yuan, Survival and stationary distribution of a sir epidemic model with stochastic perturbations, *Applied Mathematics and Computation*, 244 (2014), 118–131.

[34] X. Zou and K. Wang, Numerical simulations and modeling for stochastic biological systems with jumps, *Communications in Nonlinear Science and Numerical Simulation*, 19 (2014), 1557–1568.

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