Singularities and full convergence of the Möbius-invariant Willmore flow in the 3-sphere

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Abstract

In this article we continue our investigation of the Möbius-invariant Willmore flow, starting to move in arbitrary $C^\infty$-smooth and umbilic-free initial immersions $F_0$, which map some fixed compact torus $\Sigma$ into $\mathbb{R}^n$ respectively $S^n$. Here, we investigate the behaviour of flow lines $\{F_t\}$ of the Möbius-invariant Willmore flow in $S^3$ starting with Willmore energy below $8\pi$, as the time $t$ approaches the maximal time of existence $T_{\text{max}}(F_0)$ of $\{F_t\}$. We particularly investigate the formation of singularities of flow lines of the Möbius-invariant Willmore flow in finite time. We will firstly consider general flow lines in $S^3$, and afterwards we will focus on flow lines, which start moving in smooth parametrizations of Hopf-tori in $S^3$. We will see that any “singularity” of a flow line $\{F_t\}$ of the MIWF can be interpreted as the support of a 2-rectifiable, integral varifold $\mu$ in $\mathbb{R}^4$, which is the measure-theoretic limit of the sequence of varifolds $\{H^2|_{F_{t_j}}\}$, for an appropriate sequence of times $t_{j_l} \nearrow T_{\text{max}}(F_0)$, and that the support $\text{spt}(\mu)$ has either degenerated to a point or is homeomorphic to either a 2-sphere or a compact torus in $S^3$. In the “non-degenerate” case in which the “singularity” $\text{spt}(\mu)$ is a compact surface of genus 1, it can be parametrized by a bi-Lipschitz homeomorphism at least of class $(W^{2,2} \cap W^{1,\infty})(\Sigma)$, being uniformly conformal w.r.t. some appropriate smooth metric of vanishing scalar curvature on the torus $\Sigma$. Under certain additional conditions on the considered flow line $\{F_t\}$, a “non-degenerate singularity” of $\{F_t\}$ can be parametrized by a uniformly conformal diffeomorphism of class $W^{4,2}(\Sigma, \mathbb{R}^4)$. Finally, if the initial immersion $F_0$ of a flow line $\{F_t\}$ is assumed to parametrize a smooth Hopf-torus in $S^3$ with Willmore energy smaller than $8\pi$, then we obtain stronger types of convergence of particular subsequences of $\{F_{t_{j_l}}\}$ to uniformly conformal $W^{4,2}$-parametrizations of certain limit Hopf-tori, and this insight yields in the third main theorem a surprisingly simple criterion for full $C^m$-convergence of such a flow line $\{F_t\}$ of the MIWF to a smooth diffeomorphism parametrizing a Clifford-torus, as $t \nearrow \infty$, for every fixed $m \in \mathbb{N}$.

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1 Introduction and main results

In this article, the author continues his investigation of the Möbius-invariant Willmore flow (MIWF) - evolution equation (1) - in $S^3$ respectively in $\mathbb{R}^3$. In the author’s article

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The author proved short-time existence and uniqueness of the flow
\[ \partial_t f_t = -\frac{1}{2} \frac{1}{|A_{f_t}|^4} \left( \triangle f_t \bar{H}_{f_t} + Q(A_{f_t})(\bar{H}_{f_t}) \right) \equiv -\frac{1}{|A_{f_t}|^4} \nabla_{L^2} \mathcal{W}(f_t), \]
which is well-defined on differentiable families of $C^4$-immersions $f_t$ mapping some arbitrarily fixed smooth compact torus $\Sigma$ either into $\mathbb{R}^n$ or into $\mathbb{S}^n$, for $n \geq 3$, without any umbilic points. As already pointed out in the author’s article [17], the “umbilic free condition” $|A_{f_t}|^2 > 0$ on $\Sigma$, implies $\chi(\Sigma) = 0$ for the Euler-characteristic of $\Sigma$, which forces the MIWF to be only well-defined on families of sufficiently smooth umbilic-free tori, immersed into $\mathbb{R}^n$ or $\mathbb{S}^n$. In equation (1), $\mathcal{W}$ denotes the Willmore-functional
\[ \mathcal{W}(f) := \int_{\Sigma} K_f^M + \frac{1}{2} |\bar{H}_f|^2 \, d\mu_{f^*(\text{g_euc})}, \]
which can more generally be considered on $C^4$-immersions $f : \Sigma \rightarrow M$, from any closed smooth Riemannian orientable surface $\Sigma$ into an arbitrary smooth Riemannian manifold $M$, where $K_f^M(x)$ denotes the sectional curvature of $M$ w.r.t. the “immersed tangent plane” $Df_x(T_x\Sigma)$ in $T_f(x)M$. In those cases being relevant in this article, obviously $K_f \equiv 0$ for $M = \mathbb{R}^n$ or $K_f \equiv 1$ for $M = \mathbb{S}^n$. For ease of exposition, we will only consider the case $M = \mathbb{S}^n$, for $n \geq 3$, in the sequel. Given some immersion $f : \Sigma \rightarrow \mathbb{S}^n$, we endow the torus $\Sigma$ with the pullback $g_f := f^*\text{g_euc}$ of the Euclidean metric on $\mathbb{S}^n$, i.e. with coefficients $g_{ij} := \langle \partial_i f, \partial_j f \rangle$, and we let $A_f$ denote the second fundamental form of the immersion $f$, defined on pairs of tangent vector fields $X, Y$ on $\Sigma$ by:
\[ A_f(X, Y) := D_X(D_Y(f)) - P^{\text{Tan}(f)}(D_X(D_Y(f))) \equiv (D_X(D_Y(f)))^{\perp f}, \]
where $D_X(V)|_x$ denotes the projection of the usual derivative of a vector field $V : \Sigma \rightarrow \mathbb{R}^{n+1}$ in direction of the vector field $X$ into the respective fiber $T_{f(x)}\mathbb{S}^n$ of $T\mathbb{S}^n$, $P^{\text{Tan}(f)} : \bigcup_{x \in \Sigma} \{x\} \times T_{f(x)}\mathbb{S}^n \rightarrow \bigcup_{x \in \Sigma} \{x\} \times T_{f(x)}(f(\Sigma)) =: \text{Tan}(f)$ denotes the bundle morphism which projects the entire tangent space $T_{f(x)}\mathbb{S}^n$ orthogonally onto its subspace $T_{f(x)}(f(\Sigma))$, the tangent space of the immersion $f$ in $f(x)$, for every $x \in \Sigma$, and where $^{\perp f}$ abbreviates the bundle morphism $\text{Id}_{T_{f(x)}\mathbb{S}^n} - P^{\text{Tan}(f)}$. Furthermore, $A_f^0$ denotes the tracefree part of $A_f$, i.e.
\[ A_f^0(X, Y) := A_f(X, Y) - \frac{1}{2} g_f(X, Y) \bar{H}_f \]
and $\bar{H}_f := \text{Trace}(A_f) \equiv A_f(e_i, e_i)$ (“Einstein’s summation convention”) denotes the mean curvature vector of $f$, where $\{e_i\}$ is a local orthonormal frame of the tangent bundle $T\Sigma$. Finally, $Q(A_f)$ respectively $Q(A_f^0)$ operates on vector fields $\phi$ which are sections into the normal bundle of $f$, i.e. which are normal along $f$, by assigning $Q(A_f)(\phi) := A_f(e_i, e_j)(A_f(e_i, e_j), \phi)$, which is by definition again a section into the normal bundle of $f$. Weiner computed in [49], Section 2, that the first variation of the Willmore functional $\nabla_{L^2} \mathcal{W}$ in some smooth immersion $f$, in direction of a smooth section $\phi$ into the normal bundle of $f$, is in both cases $M = \mathbb{S}^n$ and $M = \mathbb{R}^n$, $n \geq 3$, given by:
\[ \nabla_{L^2} \mathcal{W}(f, \phi) = \frac{1}{2} \int_{\Sigma} \langle \triangle f \bar{H}_f + Q(A_f^0)(\bar{H}_f), \phi \rangle \, d\mu_{f^*(\text{g_euc})} =: \int_{\Sigma} \langle \nabla_{L^2} \mathcal{W}(f), \phi \rangle \, d\mu_{f^*(\text{g_euc})}. \]
The decisive difference between the flow (1) and the classical $L^2$-Willmore-gradient-flow, i.e. the $L^2$-gradient-flow of the functional in (2), is the factor $\frac{1}{|A_{f_t}^0|}(|x)$ which is finite in $x \in \Sigma$, if and only if $x$ is not a umbilic point of the immersion $f_t$. Just as in the case $M = \mathbb{R}^n$, there holds the following important lemma for the case $M = S^n$, $n \geq 3$, see also Lemma 1 in the author’s article [17]:

**Lemma 1.1.** 1) Let $\Sigma$ be a smooth, compact, orientable surface without boundary and $n \geq 3$. The Willmore-functional $\mathcal{W}$, applied to immersions $f \in C^4(\Sigma, S^n)$, is invariant w.r.t. composition with any conformal transformation $\Phi$ of $S^n$, i.e. there holds

$$\mathcal{W}(\Phi(f)) = \mathcal{W}(f) \quad \forall \Phi \in \text{M"obs}(S^n).$$ (5)

2) Let $\Sigma$ be a smooth torus, $\Phi$ an arbitrary M"obius transformation of $S^n$, $n \geq 3$, and $f : \Sigma \rightarrow S^n$ a $C^4$-immersion without any umbilic points, i.e. satisfying $|A_{f_t}^0|^2 > 0$ on $\Sigma$. Then there holds the following rule for the transformation of the differential operator $f \mapsto |A_{f_t}^0|^{-4} \nabla_{L^2} \mathcal{W}(f)$ (of fourth order) w.r.t. the change from $f$ to the composition $\Phi \circ f$:

$$|A_{\Phi(f)}^0|^{-4} \nabla_{L^2} \mathcal{W}(\Phi(f)) = D\Phi(f) \cdot \left(|A_{f_t}^0|^{-4} \nabla_{L^2} \mathcal{W}(f)\right)$$ (6)

on $\Sigma$.

Since for the differential operator $\partial_t$ the chain rule applied to continuously differentiable families $\{f_t\}$ of $C^4$-immersions yields the same transformation formula as in (6), i.e. $\partial_t(\Phi(f_t)) = D\Phi(f_t) \cdot \partial_t(f_t)$, we achieve the following corollary of Lemma 1.1, analogously to Corollary 1 in the author’s article [17]:

**Corollary 1.1.** Any family $\{f_t\}$ of $C^4$-immersions $f_t : \Sigma \rightarrow S^n$ without any umbilic points, i.e. with $|A_{f_t}^0|^2 > 0$ on $\Sigma$ $\forall t \in [0, T]$, solves the flow equation

$$\partial_t f_t = -\frac{1}{2} |A_{f_t}^0|^{-4} \left(\Delta f_t \vec{H}_{f_t} + Q(A_{f_t}^0)(\vec{H}_{f_t})\right)$$

$$\equiv - |A_{f_t}^0|^{-4} \nabla_{L^2} \mathcal{W}(f_t)$$ (7)

if and only if its composition $\Phi(f_t)$ with an arbitrary M"obius transformation $\Phi$ of $S^n$ solves the same flow equation again, thus, if and only if

$$\partial_t(\Phi(f_t)) = - |A_{\Phi(f_t)}^0|^{-4} \nabla_{L^2} \mathcal{W}(\Phi(f_t))$$

holds $\forall t \in [0, T]$ and $\forall \Phi \in \text{M"obs}(S^n)$.

This corollary precisely points out the conformal invariance of the flow (1) and explains its name: the “M"obius-invariant Willmore flow”, abbreviated “MIWF”. The conformal invariance of the MIWF obviously distinguishes it geometrically from the classical Willmore flow. First of all, one might think that this difference was extremely helpful in view of Theorem 4.2 in [27] respectively in view of Theorem 4.2 in [43], which seem to guarantee us here, that induced metrics $g_{\text{pull}_j}(F_t, \Sigma)$ along any fixed flow line $\{F_t\}$ of the MIWF would be conformally equivalent to smooth metrics $g_{\text{point}, j}$ of vanishing scalar curvature, such that the resulting conform factors $u_{t_j}$ can be uniformly estimated in $L^\infty(\Sigma)$, for
some sequence of times $t_j \nearrow T_{\max}(F_0)$. But since the conformal invariance of the MIWF lets us apply only finitely many conformal transformations to a fixed flow line \{\(F_t\)} of the MIWF - in order to either estimate its life span, or in order to determine its behaviour as \(t \nearrow T_{\max}(F_0)\) - neither Theorem 4.2 in [27], nor Proposition 2.2 in [43], nor Theorems 3.2 and 4.2 in [43], nor Theorem 4.1 in [23] can be applied here, in order to obtain any valuable information about the limiting behaviour of a fixed flow line \{\(F_t\)} of the MIWF, as \(t \nearrow T_{\max}(F_0)\). The only accessible general information in this situation appears to be given by Theorem 5.2 in [23] respectively by Theorem 1.1 in [39] telling us, that the complex structures \(g_{\text{eucl}}(F_t(\Sigma))\) corresponding to the conformal classes of the induced metrics \(g_{\text{eucl}}\) are contained in some compact subset of the moduli space \(\mathcal{M}_1\), i.e. cannot diverge to the boundary of \(\mathcal{M}_1 \cong \mathbb{H}/\text{PSL}_2(\mathbb{Z})\); see here also Theorem 5.1 in [27] for the earliest reference concerning this type of result. But this information does not suffice, neither in order to exclude “loss of topology” nor “degeneration to a constant map” for an arbitrarily chosen sequence of immersions \{\(F_{t_j}\)} belonging to some fixed flow line \{\(F_t\)} of the MIWF, as \(t_j \nearrow T_{\max}(F_0)\); see here Section 5.1 in [41] for some illustrative examples and explanations, and compare also to the statements of Theorems 1.4, 3.2 and 4.2 in [43] and of Theorem 5.1 in [23].

The second important difference between the MIWF and the classical Willmore flow is their “different scaling behaviour”. One can easily compute, that for any smooth immersion \(f: \Sigma \rightarrow \mathbb{R}^n\) and for any \(\rho > 0\) there holds:

\[
A_{\rho f} = \rho A_f, \quad A_{\rho f}^0 = \rho A_f^0, \quad \tilde{H}_{\rho f} = \rho^{-1} \tilde{H}_f \quad \text{on} \ \Sigma, \quad \triangle_{\rho f}(\tilde{H}_{\rho f}) = \rho^{-3} \triangle_f(\tilde{H}_f) \quad \text{and} \quad Q(A_{\rho f}^0).\tilde{H}_{\rho f} = \rho^{-3} Q(A_f^0).\tilde{H}_f \quad \text{on} \ \Sigma,
\]

which implies that for any flow line \{\(f_t\)\}_{t \geq 0} of the classical Willmore flow in \(\mathbb{R}^n\) and for any \(\rho > 0\) the family \{\(\rho f_{\rho^{-1}t}\)\}_{t \geq 0} is again a flow line of the Willmore flow in \(\mathbb{R}^n\), whereas any flow line \{\(f_t\)\} of the MIWF in \(\mathbb{R}^n\) can be scaled with any factor \(\rho > 0\) in the ambient space \(\mathbb{R}^n\), without losing its property of being a flow line of the MIWF, and no time-gauge is necessary here just by construction of the MIWF. In particular, we are not able here, to adopt the “blow-up construction” of Section 4 in [24] and its powerful combination with the lower bound on the life span of any flow line of the classical Willmore flow from Theorem 1.2 of [25], leading to the first and famous “full convergence results” for the classical Willmore flow in Theorem 5.1 of [24] and later to its improvement in Theorem 5.2 of [26]. Interestingly, any attempt to obtain - at least - only an estimate on the life span of a general flow line of the MIWF, following the lines of the fundamental paper [25] or of its adaption to the “inverse Willmore flow” in [31], \(^1\) runs into the following serious problem: Any reasonable estimate of the life span of a geometric flow - only depending on geometric data at time \(t = 0\) - follows from a criterion of the “singular time” \(T_{\max}\) in terms of “blow up” of appropriately chosen geometric tensor fields along an arbitrarily fixed flow line as \(t \nearrow T_{\max}\), and the technical tool behind such a characterization is a suitable substitute of “Bernstein-Bando-Shi-estimates” - estimating covariant derivatives of any order of the mentioned tensor fields in \(L^\infty\) both w.r.t. space- and time-variables - which can only be proven by induction over the order of covariant differentiation; see Theorem 8.1

\(^1\)The most classical example for the investigation of long-time behaviour of a geometric flow is certainly Hamilton’s Ricci flow on compact, closed manifolds, for which the singular time \(T_{\max}(g_0)\) of a flow line \{\(g_t\)\} is characterized by the time of “blow up” of the Riemannian curvature tensor \(R_{ij}(t)\) along \(\{g_t\}\) as \(t \nearrow T_{\max}(g_0)\); see Chapter 8 of [2] for an elegant proof, being based on “Bernstein-Bando-Shi-estimates”.

in [2], or Theorem 3.5 in [24] and Section 4 in [25]. But the factor $|A_0^f|^{-4}$ in front of the $L^2$-gradient of the Willmore energy in (7) would produce “too many covariant derivatives” of $|A_0^f|^2$ in each step of such an attempted induction. Reducing the MIWF in $S^3$ to the degenerate version (105) of the classical elastic energy flow (106) - investigated in [8] and [9] - by means of the Hopf-fibration $\pi : S^3 \to S^2$, then one will observe exactly the same problematic phenomenon, although (105) is a geometric flow of fourth order moving closed curves in $S^2$ and thus especially “subcritical”; see here also Remark 6.2 below. Therefore, one should reduce one’s optimistic expectation to find simple criteria for flow lines of the MIWF to be singular in terms of the behaviours of well-known geometric quantities along the respective flow lines.

On the other hand, there are two remarkable, technical simplifications of the “reduced MIWF” starting in Hopf-tori in $S^3$: We can prove in Proposition 5.6 below in a fairly elementary way, that the conformal structures of parametrizations $F_t^j$ of Hopf-tori along any fixed flow line $\{F_t\}$ of the MIWF have to stay in some compact subset of the moduli space $\mathcal{M}_1 \cong \mathbb{H}/PSL_2(\mathbb{Z})$, confirming again the general information from Theorem 5.2 in [23] respectively from Theorem 1.1 in [39], which we had already quoted above. Moreover, in Theorem 1.2 (i) below we can also prove elementarily, that any flow line of the reduced MIWF cannot run neither into a single point nor into a limit surface which is homeomorphic to a 2-sphere, which means more precisely that at least for particular subsequences of arbitrarily fixed sequences $t_j \nearrow T_{\max}(F_0)$ the constant genus 1 of the embedded tori $F_{t_j}(\Sigma)$ is preserved in the limit; and this fact will enable us to directly apply here the result of Theorem 1.1 (ii) below - whence the strong Proposition 2.4 in [43] together with the proven Willmore conjecture [30] - estimating the conformal factors of the induced metrics $g_{\text{eucl}}(F_t^j(\Sigma))$ of the Hopf-tori $F_t^j(\Sigma)$ via the deep Theorem 3.1 in [27] without any further condition on the considered flow line $\{F_t\}$.

In accordance to the qualitative difficulties described above, the first two main theorems of this article, Theorems 1.1 and 1.2, show that it is rather unlikely to be able to rule out singularities of flow lines of the MIWF in terms of a “no curvature concentration”-condition along their flowing immersions, as the time $t$ approaches the respective “maximal time of existence” from the past - see here statement (15) and Remark 6.1 below - indicating a stark contrast to the behaviour of the classical Willmore flow in any $\mathbb{R}^n, n \geq 3$, on account of Theorem 1.2 in [25].

Surprisingly, the mathematical situation improves greatly, if one assumes additionally global existence of a flow line of the MIWF, starting in a Hopf-torus in $S^3$ with Willmore energy smaller than $8\pi$: In Theorem 1.3 we can combine the results of Theorem 1.2 below with the first “full convergence result” in [20], i.e. Theorem 1.1 in [20], and with Proposition 5.1 in [18], in order to prove that any such global flow line $\{F_t\}$ of the MIWF converges - up to smooth reparametrization - fully in the $C^m(\Sigma, \mathbb{R}^4)$-norm, for any fixed $m \in \mathbb{N}$, to a smooth and diffeomorphic parametrization of a Clifford-torus, if and only if the mean curvature vectors $\vec{H}_{F_t, S^3}$ along $F_t$ stay uniformly bounded in $L^\infty(\Sigma)$, for all $t \geq 0$.

In order to finally obtain Theorem 1.3 below, we will firstly aim at a better understanding of “how either singular flow lines of the MIWF behave as $t \nearrow T_{\max}(F_0) < \infty$”, i.e. which types of “singularities” the MIWF can only produce in finite time and how certain geometric and analytical data along singular flow lines influence the regularity properties of the “singularities” as $t \nearrow T_{\max}(F_0) < \infty$, see here Definition 1.1 (b) and (c) below, or how global flow lines of the MIWF behave as $t \nearrow \infty$, provided they start moving with
Willmore energy smaller than \(8\pi\). Our techniques of examination will consist of a combination of Kuwert’s and Schätzle’s [27], [28], [43] and Rivière’s [39], [40], [41] investigation of sequences of immersions of a compact Riemann surface \(\Sigma\) into some \(\mathbb{R}^n\) of fixed genus \(p > 0\), which either have “sufficiently small” Willmore energy and whose conformal classes cannot approach the boundary of the moduli space \(\mathcal{M}_p\) or which minimize the Willmore energy under fixed conformal class, with Rivière’s [37], [38], [41] and Bernard’s [4] discovery and examination of certain conservation laws originating from the conformal invariance of the Willmore functional, and with Palmurella’s and Rivière’s [34] resulting “first step” towards a new, powerful theory of “weak flow lines” of the classical Willmore flow. All mentioned papers by Kuwert, Schätzle, Rivière and Palmurella have one topic in common, namely “applied Gauge theory” - following Müller and Sverak [32] and Hélein [14] - in the sense that they either directly investigate the possibility to parametrize certain sets of embedded surfaces in \(\mathbb{R}^n\) - up to application of appropriate Möbius transformations of the ambient space \(\mathbb{R}^n\) - by means of uniformly conformal immersions, whose conformal factors can be a-priori bounded in \(L^\infty(\Sigma)\) via appropriate, modern refinements of the fundamental “Wente-estimates”, or in the sense that they apply the results of this fundamental research, in order to answer several prominent questions arising in the study of solutions to the (constrained) Willmore equation and of the classical Willmore flow.

First of all, we should introduce the following fundamental definitions, where parts (b) and (c) of the following definition are motivated by the classical terminology of Section 8.2 in [2], examining flow lines of the Ricci flow.

**Definition 1.1.** Let \(\Sigma\) be a smooth compact torus and \(n \geq 3\) an integer.

a) A “flow line” of the MIWF (1) in the ambient manifold \(M = \mathbb{R}^n\) or \(M = S^n\) is a smooth family \(\{f_t\}_{t \in [0,T]}\) of smooth immersions of \(\Sigma\) into \(M\), such that the resulting smooth function \(f : \Sigma \times [0,T) \rightarrow M\) satisfies equation (1) classically on \(\Sigma \times [0,T)\), i.e. such that

\[
\partial_t f_t(x) = -\frac{1}{2}\frac{1}{|A_{f_t}(x)|^4} \left( \Delta_{f_t} \vec{H}_{f_t}(x) + Q(A_{f_t}) \right) \text{ holds pointwise in every } (x,t) \in \Sigma \times [0,T).
\]

b) Let \(F_0 : \Sigma \rightarrow M\) be a smooth and umbilic-free immersion and \(\{F_t\}_{t \in [0,T]}\) a smooth flow line of the MIWF starting in \(F_0\). We call \([0,T)\) the “interval of maximal existence” of the MIWF starting in \(F_0\), if either \(T = \infty\), or if there holds \(T < \infty\) and there is not: an \(\epsilon > 0\) and a smooth solution \(\{\tilde{F}_t\}_{t \in [0,T + \epsilon]}\) of the MIWF with \(\tilde{F}_t = F_t\) on \(\Sigma\) for \(t \in [0,T)\). In both cases the element \(T \in \mathbb{R} \cup \{\infty\}\) is uniquely determined by the initial immersion \(F_0\), and we call it the “maximal time of existence” of the MIWF starting in \(F_0\), in symbols “\(T_{\text{max}}(F_0)\)”, respectively we call \(\{F_t\}_{t \in [0,T_{\text{max}}(F_0)]}\) the “maximal solution” of flow equation (1).

c) If \(T_{\text{max}}(F_0)\) is finite, then we also call \(T_{\text{max}}(F_0)\) “the singular time” of the flow line \(\{F_t\}\) of the MIWF starting in \(F_0\). In this case we also say, that the respective flow line of the MIWF “forms a singularity” as \(t \nearrow T_{\text{max}}(F_0)\), and we call such a flow line of the MIWF “singular”.

d) We call a family \(\{f_t\}_{t \in [0,T]}\) of immersions of \(\Sigma\) into \(\mathbb{R}^n\) a “generalized flow line of the MIWF in \(\mathbb{R}^n\)”, if the resulting function \(f : \Sigma \times [0,T] \rightarrow \mathbb{R}^n\) is of class...
$W^{1,p}([0,T], L^p(\Sigma, \mathbb{R}^n)) \cap L^p([0,T], W^{4,p}(\Sigma, \mathbb{R}^n))$, for some $T \in (0, \infty)$ and some $p \in (3, \infty)$, and satisfies the “relaxed MIWF-equation”:

$$(\partial_t f_t(x))^{1-t} = -\frac{1}{2} \frac{1}{|A_{f_t}(x)|^4} \left( \Delta_{f_t} H_{f_t}(x) + Q(A_{f_t}(x))(\mathcal{H}_{f_t}(x)) \right)$$

in a.e. $(x,t) \in \Sigma \times [0,T]$. Here, $(1-t)$ abbreviates the projection of the velocity vector $\partial_t f_t(x)$ into the normal space of the immersion $f_t$ within $\mathbb{R}^n$, over any fixed $x \in \Sigma$, as in (3).

We should note here, that Definition 1.1, (a)–(c), makes sense because of Theorem 1 in [17] respectively Theorems 2 and 3 in [19], proving existence and uniqueness of smooth short-time solutions of the MIWF with $C^\infty$-smooth, umbilic-free initial immersion of a smooth torus into $\mathbb{R}^n$ respectively $\mathbb{S}^n$. The choice of parabolic $L^p$-spaces in Definition 1.1 (d) with $p > 3$ has been motivated by Theorems 1–5 in [19]. Moreover, it follows immediately from Definition 1.1 and from flow equation (1), that every immersion $F_t$ of a smooth flow line of the MIWF has to be a parametrization of a umbilic-free torus. Concerning the question, how either “limit objects” as $t \nearrow T_{\text{max}}(F_0) = \infty$ of global flow lines of the MIWF or “singularities” of singular flow lines of the MIWF - in the sense of Definition 1.1 (c) - can only look like, we obtain a first orientation by means of the following general result.

**Theorem 1.1.** Let $\Sigma$ be an arbitrary smooth compact torus, and let $\{F_t\}$ be some flow line of the MIWF starting in a smooth and umbilic-free immersion $F_0 : \Sigma \rightarrow \mathbb{S}^3$ with $W(F_0) < 8\pi$, and let $t_j \nearrow T_{\text{max}}(F_0)$ be an arbitrary sequence.

1) There is a subsequence $\{F_{t_{j_l}}\}$ and some integral, 2-rectifiable varifold $\mu$ with unit Hausdorff-2-density, whose support is either a point in $\mathbb{S}^3$ - iff $\mu = 0$ - or a closed, embedded and orientable Lipschitz-surface in $\mathbb{S}^3$ of genus either 0 or 1, and such that

$$\mathcal{H}^2|_{F_{t_{j_l}}(\Sigma)} \rightarrow \mu \quad \text{weakly as Radon measures on } \mathbb{R}^4$$

and $F_{t_{j_l}}(\Sigma) \rightarrow \text{spt}(\mu)$ as subsets of $\mathbb{R}^4$ in Hausdorff distance,

as $l \nearrow \infty$.

2) In the “non-degenerate case”, in which there holds $\mu \neq 0$ and genus($\text{spt}(\mu)$) = 1 for a limit varifold $\mu$ in (9), appropriate reparametrizations of possibly another subsequence of the immersions $\{F_{t_{j_l}}\}$ in (9) converge weakly in $W^{2,2}(\Sigma, \mathbb{R}^4)$ and weakly* in $W^{1,\infty}(\Sigma, \mathbb{R}^4)$ to a $(W^{2,2} \cap W^{1,\infty})(\Sigma)$-parametrization $f$ of $\text{spt}(\mu)$, which is also a bi-Lipschitz homeomorphism

$$f : \Sigma \overset{\cong}{\rightarrow} \text{spt}(\mu),$$

and $f$ is “uniformly conformal w.r.t. $g_{\text{poin}}$ on $\Sigma$” in the sense that $f^*g_{\text{euc}} = e^{2u}g_{\text{poin}}$ holds on $\Sigma$, for some smooth zero scalar curvature metric $g_{\text{poin}}$ on $\Sigma$ and for some real-valued function $u \in L^\infty(\Sigma)$ with $\|u\|_{L^\infty(\Sigma)} \leq \Lambda = \Lambda(\{F_{t_{j_l}}\}, \mu) < \infty$. Moreover, in this case we can reinterpret the integral varifold $\mu$ in (9) in two ways, namely there holds:

$$\mu_f = \mu = \mathcal{H}^2|_{\text{spt}(\mu)} \quad \text{on } \mathbb{R}^4,$$
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where $\mu_f := f(\mu_{f^*\text{euc}})$ is the canonical surface measure of $f(\Sigma) = \text{spt}(\mu)$ in $\mathbb{R}^4$. The coinciding varifolds $\mu$ and $\mu_f$ have weak mean curvature vectors $\vec{H}_\mu, \vec{H}_{\mu_f}$ in $L^2(\Sigma, \mu)$, and they satisfy exactly:

$$4W(\mu) := \int_{\mathbb{R}^4} |\vec{H}_\mu|^2 d\mu = \int_{\mathbb{R}^4} |\vec{H}_{\mu_f}|^2 d\mu_f = \int_\Sigma |\vec{H}_{f,\mathbb{R}^4}|^2 d\mu_{f^*\text{euc}} \equiv 4W(f).$$

Finally, the projection $[\Sigma, f^*\text{euc}] = [\Sigma, \text{gpo} \text{n}]$ of the conformal structure $(\Sigma, \text{gpo} \text{n})$ into the moduli space $M_1$ does not depend on the choice of $\Sigma$ and $\text{gpo} \text{n}$ respectively of $f$, thus it only depends on the integral varifold $\mu$ in (9). Therefore, every subsequence of the sequence $\{F_{t_\jmath l}\}$ in (9) contains another subsequence $\{F_{t_\jmath l_p}\}$, such that after appropriate reparametrization:

$$[\Sigma, \bar{F}_{t_\jmath l_p}^* (\text{euc})] \rightarrow [\text{spt}(\mu)] \text{ in } M_1 \text{ as } p \rightarrow \infty.$$  

3) If a limit varifold $\mu$ in (9) satisfies $\mu \neq 0$ and genus$(\text{spt}(\mu)) = 1$ - as above in the second part of the theorem - and additionally $W(\mu) = \lim_{k \rightarrow \infty} W(F_{t_\jmath k})$ for some subsequence $\{F_{t_\jmath k}\}$ of the weakly convergent sequence $\{F_{t_\jmath l}\}$ from (9), then there is another subsequence of $\{F_{t_\jmath k}\}$ - which we term again $\{F_{t_\jmath l}\}$ - and some appropriate family of smooth diffeomorphisms $\Theta_k : \Sigma \rightarrow \Sigma$, such that the reparametrizations $\bar{F}_{t_\jmath k} := F_{t_\jmath k} \circ \Theta_k$ converge strongly:

$$\bar{F}_{t_\jmath k} \rightarrow f \quad \text{in } W^{2,2}(\Sigma, \text{gpo} \text{n}) \quad \text{as } k \rightarrow \infty,$$

where $f$ is the uniformly conformal bi-Lipschitz-parametrisation of $\text{spt}(\mu)$ from (11), and moreover every immersion $\bar{F}_{t_\jmath k}$ in (14) is a uniformly bi-Lipschitz homeomorphism of $(\Sigma, \text{gpo} \text{n})$ upon its image in $(S^3, \text{gpo} \text{n})$. Furthermore, in this case there is for every fixed $x \in S^3$ some further subsequence $\{\bar{F}_{t_\jmath k_m}\}$ of the sequence $\{\bar{F}_{t_\jmath k}\}$, such that for any $\varepsilon > 0$ there is some sufficiently small $\eta > 0$ satisfying:

$$\int_\Omega |A_{\bar{F}_{t_\jmath k_m}}^{-1}(B^l_m(x) \cap S^3)|^2 d\mu_{\bar{F}_{t_\jmath k_m}^* \text{euc}} < \varepsilon \quad \forall m \in \mathbb{N}. \quad (15)$$

In particular, the measures

$$M_l(\Omega) := \inf \left\{ \int_{F_{t_\jmath l}^{-1}(B^l_m \cap S^3)} |A_{F_{t_\jmath l}}|^2 d\mu_{F_{t_\jmath l}^* \text{euc}} \mid B \supseteq \Omega \text{ and } B \text{ is a Borel subset of } \mathbb{R}^4 \right\}$$

on $\mathbb{R}^4$ do not concentrate in any point of the ambient space $\mathbb{R}^4$ as $l \rightarrow \infty$.

4) As in the second and third part of the theorem we consider a limit varifold $\mu$ in (9) satisfying $\mu \neq 0$ and genus$(\text{spt}(\mu)) = 1$, and we assume here again the existence of a subsequence $\{F_{t_\jmath k}\}$ of the weakly convergent sequence $\{F_{t_\jmath l}\}$ in (9) with the property $\lim_{k \rightarrow \infty} W(F_{t_\jmath k})$. Suppose that $\{F_{t_\jmath k}\}$ satisfies also $\|A_{F_{t_\jmath k}}^0\|_{L^\infty(\Sigma)} \leq K$ and $\frac{dW(F_{t_\jmath l})}{dt}|_{t=t_k} \leq K$ for all $k \in \mathbb{N}$ and for some large number $K > 1$, then the limit parametrisation $f$ of $\text{spt}(\mu)$ in (11) is of class $W^{4,2}((\Sigma, \text{gpo} \text{n}), \mathbb{R}^4)$. \qed
Remark 1.1. We should remark here, that the most degenerate case \( \text{spt}(\mu) = \{\text{point}\} \) in the first part of Theorem 1.1 corresponds to the two special cases, in which certain embeddings \( F_{t_j} \) belonging to a fixed flow line \( \{F_t\} \) of the MIWF (1) in the ambient manifold \( M = \mathbb{R}^3 \) either diverge uniformly to \( \infty \), i.e. leave any compact subset of \( \mathbb{R}^3 \), as \( t_j \nearrow T_{\text{max}}(F_0) \), or converge in Hausdorff-distance to a single point in \( \mathbb{R}^3 \). This follows immediately from convergence (10), from the conformal invariance of the MIWF and via stereographic projection of \( S^3 \setminus \{\text{point}\} \) onto \( \mathbb{R}^3 \).

Moreover, if we let the MIWF start moving in a smooth parametrization of a Hopf-torus, then we can derive more precise information from the flow equation (1). In particular, for such flow lines of the MIWF we can rule out both degenerate cases \( \mu = 0 \) and \( \mu \neq 0 \wedge \text{spt}(\mu) \cong S^2 \) for singularities respectively for limit surfaces, appearing in the first part of Theorem 1.1. Precisely, we prove the following theorem.

Theorem 1.2. Let \( \{F_t\} \) be some flow line of the MIWF starting in a smooth and simple parametrization \( F_0 : \Sigma \rightarrow S^3 \) of a smooth Hopf-torus in \( S^3 \) with \( \mathcal{W}(F_0) < 8\pi \). Moreover, we fix an arbitrary sequence of times \( t_j \nearrow T_{\text{max}}(F_0) \).

1) Any weakly convergent subsequence \( \mathcal{H}^2[F_{t_j}] \) of corresponding varifolds as in (9) converges weakly to a non-trivial, integral 2-rectifiable varifold \( \mu \) with unit Hausdorff-2-density, whose support is an embedded Hopf-torus in \( S^3 \). In particular, for any such subsequence \( \{F_{t_j}\} \) all statements of the second part of Theorem 1.1 about \( \{F_{t_j}\} \) itself and about its limit varifold \( \mu \) hold here. Hence, such an embedded limit Hopf-torus \( \text{spt}(\mu) \) possesses a uniformly conformal bi-Lipschitz and \( W^{2,2} \)-parametrization \( f : (\Sigma, gpoin) \xrightarrow{\cong} \text{spt}(\mu) \) from statement (11), with \( f^*(g_{\text{euc}}) = e^{2u} gpoin \) and \( \|u\|_{L^\infty(\Sigma)} \leq \Lambda = \Lambda(\{F_{t_j}\}, \mu) < \infty \), and with \( \mathcal{W}(f) \leq \mathcal{W}(F_0) < 8\pi \), where \( gpoin \) is some appropriate smooth zero scalar curvature metric on \( \Sigma \).

2) We suppose that there is some large constant \( K > 0 \), such that \( \|\bar{H}_{F_{t_j}}\|_{L^\infty(\Sigma)} \) remains uniformly bounded by \( K \) for all \( j \in \mathbb{N} \), and we consider a subsequence \( \{F_{t_{j_k}}\} \) of \( \{F_{t_j}\} \) as in the first part of this theorem. Then any weakly/weakly* convergent sequence \( \{\tilde{F}_{t_{j_k}}\} \) as in (58) and (60), which we had obtained from \( \{F_{t_{j_k}}\} \) in the second part of Theorem 1.1, can be uniformly estimated in the way:

\[
\|\nabla gpoin(\tilde{F}_{t_{j_k}})\|^2_{W^{2,2}(\Sigma, gpoin)} \leq C(\mathcal{W}(F_0), K, \Sigma, gpoin, \Lambda) \cdot \left( \int_{\Sigma} |\nabla L^2 \mathcal{W}(\tilde{F}_{t_{j_k}})|^2 \, d\mu_{\tilde{F}_{t_{j_k}}} + 1 \right)
\]

for every \( k \in \mathbb{N} \), where \( gpoin \) and \( \Lambda \) are as in the first part of this theorem.

3) We assume the same requirements as in part (2) of this theorem and additionally, that the speed of \( \text{"energy decrease" } \frac{d}{dt} \mathcal{W}(F_t) \) shall remain uniformly bounded at the prescribed points of time \( t = t_j \nearrow T_{\text{max}}(F_0) \). Then, any weakly/weakly* convergent sequence \( \{\tilde{F}_{t_{j_k}}\} \) as in (58) and (60), which we had already considered above in the second part of this theorem, converges also weakly in \( W^{4,2}(\Sigma, gpoin), \mathbb{R}^4 \) strongly in \( W^{3,2}(\Sigma, gpoin), \mathbb{R}^4 \) and also in \( C^{2,\alpha}(\Sigma, gpoin), \mathbb{R}^4 \), for any fixed \( \alpha \in (0, 1) \), to the uniformly conformal parametrization \( f : \Sigma \xrightarrow{\cong} \text{spt}(\mu) \) of the corresponding limit Hopf-torus \( \text{spt}(\mu) \) from the first part of this theorem, and statement (15) holds here.
for the sequence \(\{F_t\}_{t \geq 0}\) respectively for the original sequence of embeddings \(\{F_t\}_{t \geq 0}\). In particular, the parametrization \(f\) is a bi-Lipschitz homeomorphism between \((\Sigma, g_{\text{poin}})\) and \(\text{spt}(\mu)\) and additionally of regularity class \(W^{4,2}(\Sigma, g_{\text{poin}}, \mathbb{R}^4)\).

Finally we infer from Theorem 1.2 above and from Theorem 1.1 in [20] a “full convergence result” for global flow lines of the MIWF starting in Hopf-tori in \(S^3\), only under the two additional conditions that they shall start moving with Willmore energy smaller than \(8\pi\) and that the mean curvature vectors along those flow lines shall remain uniformly bounded in \(L^\infty\) for all times.

**Theorem 1.3.** Let \(\{F_t\}_{t \geq 0}\) be a global flow line of the MIWF starting in some smooth and simple parametrization \(F_0 : \Sigma \to S^3\) of a smooth Hopf-torus in \(S^3\) with \(W(F_0) < 8\pi\). If there is some large constant \(K > 0\), such that \(\|\tilde{H}_{F_t, S^3}\|_{L^\infty(\Sigma)}\) remains uniformly bounded by \(K\) for all \(t \in [0, \infty)\), then for each fixed \(m \in \mathbb{N}\) there is a smooth family of smooth diffeomorphisms \(\Theta_t \equiv \Theta_t^m : \Sigma \xrightarrow{\cong} \Sigma\), such that the reparametrized flow line \(\{F_t \circ \Theta_t\}_{t \geq 0}\) converges fully in \(C^m(\Sigma, \mathbb{R}^4)\) to a smooth and diffeomorphic parametrization of some torus in \(S^3\), which is conformally equivalent to the standard Clifford-torus in \(S^3\), and this convergence takes place at an exponential rate, as \(t \nearrow \infty\).

**Corollary 1.2.** Let \(\{\gamma_t\}_{t \geq 0}\) be a global flow line of the degenerate elastic energy flow (105) below, starting in some smooth, closed and simple curve \(\gamma_0 : S^1 \to S^2\) with elastic energy \(E(\gamma_0) : = \int_{S^1} 1 + |\tilde{k}_{\gamma_0}|^2 \, d\mu_{\gamma_0} < 8\); see here Proposition 5.2 below. If there is some large constant \(K > 0\), such that the maximal curvature \(\|\tilde{k}_{\gamma_t}\|_{L^\infty(S^1)}\) remains uniformly bounded by \(K\) for all \(t \in [0, \infty)\), then for each \(m \in \mathbb{N}\) there is a smooth family of smooth diffeomorphisms \(\Theta_t \equiv \Theta_t^m : S^1 \xrightarrow{\cong} S^1\), such that the reparametrized flow line \(\{\gamma_t \circ \Theta_t\}_{t \geq 0}\) converges fully in \(C^m(S^1, \mathbb{R}^3)\) to a smooth embedding of some great circle in \(S^2\), and this convergence takes place at an exponential rate, as \(t \nearrow \infty\).

## 2 Definitions and preparatory remarks

To start out, we recall some basic differential geometric terms in the following definition. As in the introduction, we endow the unit \(n\)-sphere with the Euclidean scalar product of \(\mathbb{R}^{n+1}\), i.e. we set \(g_{S^n} : = \langle \cdot, \cdot \rangle_{\mathbb{R}^{n+1}}\). In this section we shall closely follow the classical book [21] about “compact Riemann surfaces”, Rivière’s lecture notes [41] and Tromba’s introduction [48] into “Teichmüller Theory”, in order to obtain an overview about the basic terms of Riemann surface theory, i.e. about “complex structures” and “conformal classes of smooth metrics” on compact and orientable surfaces, and about their “isomorphism classes”, which will play a fundamental role in the statements and proofs of our three main theorems below.

**Definition 2.1.** Let \(\Sigma\) be a smooth, compact and orientable manifold of real dimension 2 and of arbitrary genus \(p \geq 0\).

1) We call a finite atlas \(\{\psi_i : \Omega_i \xrightarrow{\cong} B^2_\rho(0)\}_{i=1,\ldots,n}\) of \(\Sigma\) “conformal” if all its transition maps

\[
\psi_j \circ \psi_k^{-1} : \psi_k(\Omega_k \cap \Omega_j) \xrightarrow{\cong} \psi_j(\Omega_k \cap \Omega_j)
\]

are holomorphic in the classical sense of complex analysis, and thus also conformal.
2) We term two conformal atlases equivalent, if their union is again a conformal atlas of $\Sigma$.

3) Any equivalence class of conformal atlases on $\Sigma$ is called a “complex structure” on $\Sigma$. Having endowed $\Sigma$ with one of its complex structures, we call $\Sigma$ a “compact complex manifold” of complex dimension 1 or shorter a “compact Riemann surface”.

4) We term the complex structures $S_1(\Sigma_1)$ and $S_2(\Sigma_2)$ of two compact Riemann surfaces of the same genus “isomorphic”, if there is a “conformal diffeomorphism” $f : \Sigma_1 \xrightarrow{\sim} \Sigma_2$, i.e. a bijective holomorphic map $f$ between $\Sigma_1$ and $\Sigma_2$, whose differential $D_xf : T_x\Sigma_1 \xrightarrow{\sim} T_{f(x)}\Sigma_2$ is an invertible complex linear map between corresponding tangent spaces.

5) The set of all isomorphism classes of complex structures on compact Riemann surfaces of genus $p \geq 0$ is called the “moduli space of compact Riemann surfaces of genus $p$”, in symbols: $\mathcal{M}_p$.

6) Strengthening the equivalence relation in (4) by requiring the isomorphism $f$ between $S_1(\Sigma_1)$ and $S_2(\Sigma_2)$ to be isotopic to some fixed conformal diffeomorphism between $\Sigma_1$ and $\Sigma_2$, then the resulting set of equivalence classes is called the “Teichmüller space of compact Riemann surfaces of genus $p$”, in symbols: $\mathcal{T}_p$.

7) A “conformal class” respectively “conformal structure” on $\Sigma$ is a set $[g]$ of Riemannian metrics $g = e^{2\upsilon}g_0$ on $\Sigma$, where $g_0$ is a fixed smooth metric and $\upsilon \in L^\infty(\Sigma)$ an arbitrary bounded function defined on $\Sigma$, called the “conformal factor” of $g$ w.r.t. $g_0$.

**Remark 2.1.** 1) Any complex structure $S(\Sigma)$ on some compact and orientable surface $\Sigma$ automatically yields a conformal class $[g]$ of smooth metrics on $\Sigma$, which are compatible with any atlas $\mathcal{A}$ of $\Sigma$ belonging to the given complex structure $S(\Sigma)$, in the sense that there holds $\psi_i^* (g_{\text{eucl}}) = e^{-2\upsilon_i} g$ on $\Omega_i$, $\upsilon_i \in L^\infty(\Omega_i)$, for any chart $\psi_i : \Omega_i \xrightarrow{\sim} B_1^2(0)$ of the atlas $\mathcal{A}$ as in the first part of Definition 2.1; see here Lemma 2.3.3 in [21]. Hence, a complex structure on any fixed compact orientable surface $\Sigma$ automatically yields a conformal structure on $\Sigma$ in a canonical way, and actually also the converse holds on account of Theorem 3.11.1 in [21] respectively Theorems 2.9 and 2.13 in [41], i.e. every Riemannian metric $g$ on a smooth, compact and orientable surface $\Sigma$ yields a certain conformal atlas $\mathcal{B}$ on $\Sigma$ - as in the first part of Definition 2.1 - with the additional property, to only consist of isothermal charts $\psi_i : \Omega_i \xrightarrow{\sim} B_1^2(0)$ w.r.t. the prescribed metric $g$. Therefore, the two seemingly unrelated concepts of a “compact and orientable surface endowed with a complex structure” and of a “smooth, compact and orientable surface endowed with a conformal class of Riemannian metrics” coincide and thus will be used synonymously henceforth. See here also Section 1.3 in [48].

2) By the “Uniformization Theorem”, Theorem 4.4.1 in [21], the moduli space $\mathcal{M}_1$ of a compact torus, i.e. of a compact Riemann surface of genus 1, is the upper half plane $\mathbb{H}$ modulo the action of the modular group $\text{PSL}_2(\mathbb{Z})$, whereas the Teichmüller space $\mathcal{T}_1$ of a compact torus is the entire upper half plane $\mathbb{H}$. See here also Theorems 2.7.1 and 2.7.2 in [21].
3) It moreover follows from “Poincaré’s Theorem” - see Section 1.5 in [48] for an elegant proof treating the case “genus(Σ) > 1” - that every prescribed conformal class [g₀] of Riemannian metrics respectively every complex structure $S(Σ)$ on a compact surface $Σ$ of genus $p ≥ 1$ contains a unique smooth metric, called $g_{\text{poin}} = g_{\text{poin}}(S(Σ))$, of constant scalar curvature $K_{g_{\text{poin}}}$ and unit volume, i.e. such that $K_{g_{\text{poin}}} ≡ \text{const}(p) ∈ ℝ$ on $Σ$ and with $\mu_{g_{\text{poin}}}(Σ) = 1$. Choosing some $g$ from the prescribed conformal class $[g₀]$, one considers the canonical “Ansatz” $g_{\text{poin}} := e^{-2u}g$ for the unknown “conformal factor” $u$ on $Σ$, and Poincaré’s Theorem follows from the fact, that Liouville’s elliptic PDE

$$-\Delta_g(u) + K_{g_{\text{poin}}} e^{-2u} = K_{g} \quad \text{on} \quad Σ$$

possesses for every prescribed negative number $K_{g_{\text{poin}}}$ a unique smooth solution $u$ and especially for $K_{g_{\text{poin}}} ≡ 0$ a unique one-parameter family $\{u + r\}_{r ∈ ℝ}$ of smooth solutions. Integrating equation (17) over $Σ$ w.r.t. $μ_g$, one infers from Gauss-Bonnet’s theorem, that the constant $K_{g_{\text{poin}}}$ is determined by:

$$K_{g_{\text{poin}}} = \frac{2π \chi(Σ)}{μ_{g_{\text{poin}}}(Σ)} = \frac{4π (1 - p)}{μ_{g_{\text{poin}}}(Σ)},$$

see also formula (3.5) in [27]. Hence, for every prescribed genus $p ≥ 1$ of $Σ$ a constant scalar curvature metric $g_{\text{poin}}$ within any prescribed conformal class $[g₀]$ is indeed uniquely determined by the additional requirement that “$μ_{g_{\text{poin}}}(Σ) = 1$”, and in this case there has to hold exactly $K_{g_{\text{poin}}} = 4π (1 - p)$ by (18).

On account of this last remark we know in particular, that any smooth immersion $f$ of a compact surface $Σ$ into some $ℝ^n$, $n ≥ 3$, yields a Riemannian metric $g_f := f^*g_{\text{euc}}$, which is conformally equivalent to a uniquely determined smooth metric $g_{\text{poin}}$ on $Σ$ of constant scalar curvature and unit volume, i.e. such that $f^*g_{\text{euc}} = e^{2u}g_{\text{poin}}$ holds for function $u ∈ C^∞(Σ)$. This basic result will play a central role in the proofs of our first two main results, Theorems 1.1 and 1.2 below. Now by Theorem 4.3 and Corollary 4.4 in [41] respectively by Theorem VII.12 in [38] - building on work by Müller and Sverak [32] and Hélein [14] about a geometric application of “compensated compactness”-estimates and quasiconformal mapping theory - this result also holds more generally for immersions $f : Σ → ℝ^n$ of class $W^{1,∞}$ with second fundamental form $A_f$ in $L^2(Σ)$ in the precise sense, that the induced metric $f^*g_{\text{euc}}$ gives rise to a unique complex structure, to a unique smooth metric $g_{\text{poin}}$ of constant scalar curvature and unit volume and to a unique function $u ∈ L^∞(Σ)$ such that $f^*g_{\text{euc}} = e^{2u}g_{\text{poin}}$ on $Σ$. We shall therefore introduce the concept of “uniformly conformal Lipschitz immersions” as in Section 2 of [28].

**Definition 2.2.** Let $g_{\text{poin}}$ be a smooth constant scalar curvature metric on some smooth compact, orientable surface $Σ$.

1) We call a Lipschitz map $f : Σ → S^3$ a “Lipschitz immersion uniformly conformal w.r.t. $g_{\text{poin}}$”, if there is some function $u ∈ L^∞(Σ)$ such that $f^*g_{\text{euc}} = e^{2u}g_{\text{poin}}$.

2) If $f$ is additionally of class $W^{2,2}(Σ, ℝ^4)$, then we call $f$ a $(W^{1,∞} ∩ W^{2,2})$-immersion uniformly conformal w.r.t. $g_{\text{poin}}$ on $Σ$.

3) Similarly, we define these two notions for Lipschitz maps $f : Σ → ℝ^3$, replacing $(S^3, g_{\text{euc}})$ by $(ℝ^3, g_{\text{euc}})$ and $ℝ^4$ by $ℝ^3$ in parts (1) and (2) of this definition.
In order to further proceed, we have to collect some terminology from classical Differential Geometry.

**Definition 2.3.** Let $\Sigma$ be a smooth surface, $F: \Sigma \rightarrow S^3$ a fixed $C^2$-immersion and $\psi: \Omega \subset \Sigma \rightarrow B^2_1(0)$ an arbitrary smooth chart of an atlas of $\Sigma$.

1) We will denote the resulting partial derivatives on $\Omega$ by $\partial_i$, $i = 1, 2$, the coefficients $(g_F)_{ij} := (\partial_i F, \partial_j F)_{g_{\text{euc}}}$ of the first fundamental form of $F$ w.r.t. $\psi$ and the associated Christoffel-symbols $(\Gamma_F)_{kl}^m := (\partial_k F, \partial_l F)_{g_{\text{euc}}}$ of $(\Omega, F^* g_{\text{euc}})$.

2) Moreover, for any vector field $V \in C^2(\Sigma, \mathbb{R}^4)$ we define the first covariant derivatives $\nabla_i^F(V) \equiv \nabla_i^F(\partial_i F)(V)$, $i = 1, 2$, w.r.t. $F$ as the projections of the usual partial derivatives $\partial_i(V)(x)$ of $V: \Sigma \rightarrow \mathbb{R}^4$ into the respective tangent spaces $T_{F(x)} S^3$ of the 3-sphere, $\forall x \in \Sigma$, and the second covariant derivatives by:

$$\nabla_{kl}^F(V) \equiv \nabla_k^F \nabla_l^F(V) := \nabla_k^F(\nabla_l^F(V)) - (\Gamma_F)_{kl}^m \nabla_m^F(V).$$ (19)

Moreover, for any $C^2$-vector field $V: \Sigma \rightarrow \mathbb{R}^4$ we define the projections of its first derivatives into the normal bundle of the prescribed immersion $F$ within $T S^3$ by:

$$\nabla_i^{\perp F}(V) \equiv (\nabla_i^F(V))^{\perp F} := \nabla_i^F(V) - P_{\text{Tan}(F)}(\nabla_i^F(V))$$

and the “normal second covariant derivatives” of $V$ w.r.t. the immersion $F$ by

$$\nabla_k^{\perp F} \nabla_l^{\perp F}(V) := \nabla_k^{\perp F}(\nabla_l^{\perp F}(V)) - (\Gamma_F)_{kl}^m \nabla_m^{\perp F}(V).$$

3) We define the Beltrami-Laplacian w.r.t. $F: \Sigma \rightarrow S^3$ by

$$\Delta_F(V) \equiv \Delta_{F,S^3}(V) := g_F^{kl} \nabla_k^F \nabla_l^F(V), \quad \text{for any } V \in C^2(\Sigma, \mathbb{R}^4),$$

its projection $\Delta_F^{\perp F} := (g_F^{kl} \nabla_k^F \nabla_l^F(V))^{\perp F}$ into the normal bundle of the surface $F(\Sigma)$ within $T S^3$ and the “normal Beltrami-Laplacian” by $\Delta_F^{\perp F}(V) := g_F^{kl} \nabla_k^{\perp F} \nabla_l^{\perp F}(V)$. \hfill \Box

**Remark 2.2.** For any fixed $C^2$-immersion $F: \Sigma \rightarrow \mathbb{R}^n$, for $n = 3$ or $n = 4$, the notions in parts (1)–(3) of Definition 2.3 are defined analogously, up to exchanging the tangent space $T_{F(x)} S^3$ by the constant tangent space $T_{F(x)} \mathbb{R}^3 \cong \mathbb{R}^3$ respectively $T_{F(x)} \mathbb{R}^4 \cong \mathbb{R}^4$, in any fixed $x \in \Sigma$. We will sometimes simply write $\Delta_F$ instead of $\Delta_{F,\mathbb{R}^3}$. \hfill \Box

Strongly motivated by Rivière’s [37], [38] and Bernard’s [4] work on the divergence form of the Euler-Lagrange-operator $\nabla L_2 W$ of the Willmore functional, we shall use the terminology of Definition 2.3 and of Remark 2.2 in a slightly generalized manner, and we shall follow here exactly Section 7 in [41], Sections VII.5.2 and VII.6.5 in [38], or Section 1.5 in [34] defining for any fixed Lipschitz-immersion $F: \Sigma \rightarrow \mathbb{R}^3$ with second fundamental form $A_F$ in $L^2(\Sigma)$ its “weak Willmore operator” below in (20) as a distribution of second order on $\Sigma$, thus generalizing its classical meaning in (4) for smooth immersions $f$ of $\Sigma$ into $\mathbb{R}^3$. The theory of “distributions” $T \in \mathcal{D}'(\Sigma)$ - to be used in Definition 2.4 below - is introduced in Chapters II-VI of [15] or also in Sections 5.15–5.23 of [1].
**Definition 2.4.** Let $\Sigma$ be a compact surface as in Definition 2.1, and let $F : \Sigma \to \mathbb{R}^3$ be a Lipschitz immersion with second fundamental form $A_F$ in $L^2(\Sigma)$ and with induced metric $g_F := F^*g_{\text{euc}}$.

1) We define the “weak Willmore operator” $\nabla_{L^2}W(F)$ as the following distribution on $\Sigma$ of second order:

$$\langle \nabla_{L^2}W(F), \varphi \rangle_{D'(\Sigma)} := \frac{1}{2} \int_\Sigma (\bar{H}_F, \triangle_F \varphi)_{g_{\text{euc}}} - g_F^{\nu\alpha} g_F^\mu \langle (A_F)_{\xi\nu}, \bar{H}_F \rangle_{g_{\text{euc}}} \langle \partial_\mu F, \partial_\alpha \varphi \rangle_{g_{\text{euc}}}$$

$$- g_F^{\nu\alpha} g_F^\mu \langle (A_F^0)_{\xi\nu}, \bar{H}_F \rangle_{g_{\text{euc}}} \langle \partial_\mu F, \partial_\alpha \varphi \rangle_{g_{\text{euc}}},$$

for $\forall \varphi \in C^\infty(\Sigma, \mathbb{R}^3)$.  

2) Moreover, we shall define the differential 1-form

$$w_F : \partial_\nu \mapsto \frac{1}{2} \left( \nabla_F^\mu (\bar{H}_F) + g_F^{\nu\alpha} \langle (A_F)_{\xi\nu}, \bar{H}_F \rangle_{g_{\text{euc}}} \partial_\mu F + g_F^{\nu\alpha} \langle (A_F^0)_{\xi\nu}, \bar{H}_F \rangle_{g_{\text{euc}}} \partial_\mu F \right)$$

mapping smooth vector fields on $\Sigma$ into sections of $F^*T\mathbb{R}^3$ as a distribution on $\Sigma$ of first order:

$$\langle w_F(\partial_\nu), \varphi \rangle_{D'(\Sigma)} := \frac{1}{2} \int_\Sigma -\langle \bar{H}_F, \nabla_F^\nu \varphi \rangle_{g_{\text{euc}}} + g_F^{\nu\alpha} \langle (A_F)_{\xi\nu}, \bar{H}_F \rangle_{g_{\text{euc}}} \langle \partial_\mu F, \varphi \rangle_{g_{\text{euc}}}$$

$$+ g_F^{\nu\alpha} \langle (A_F^0)_{\xi\nu}, \bar{H}_F \rangle_{g_{\text{euc}}} \langle \partial_\mu F, \varphi \rangle_{g_{\text{euc}}},$$

for $\forall \varphi \in C^\infty(\Sigma, \mathbb{R}^3)$, for $\nu = 1, 2$. As in [34], formula (2.5), and in [38], Corollary VII.3, we will also use the short notation:

$$w_F = \frac{1}{2} \left( \nabla_F^\mu (\bar{H}_F) + \langle (A_F)_{\xi\nu}, \bar{H}_F \rangle_{g_{\text{euc}}} + \langle (A_F^0)_{\xi\nu}, \bar{H}_F \rangle_{g_{\text{euc}}}, \right),$$

for the distributional 1-form in (21). \[ \square \]

**Remark 2.3.** Concerning the definition of the “weak Willmore operator” in the first part of Definition 2.4 we should mention here, that the definition in line (20) becomes an identity for smooth immersions $F : \Sigma \to \mathbb{R}^3$ on account of Theorem VII.7 in [38], respectively on account of the main theorem in [4], originating from Theorem 1.1 in [37]. We only have to recall the fact, that for smooth $F$ the $L^2$-gradient $\nabla_{L^2}W(F)$ of $W$ can be directly computed, yielding the differential operator in (24), which coincides with the expression in (4) respectively (20) via integration by parts and the main theorem in [4]:

$$\langle \nabla_{L^2}W(F), \varphi \rangle_{L^2(\Sigma, C^\infty)} = \langle \nabla_{L^2}W(F), \varphi \rangle_{D'(\Sigma)} \quad \forall \varphi \in C^\infty(\Sigma, \mathbb{R}^3).$$

(23)

For smooth immersions into $\mathbb{S}^3$ this is no longer true. In order to see this, but also in view of the proof of Theorem 1.1, we should point out here some precise, basic facts from Differential Geometry:

On account of Lemma 2.1 in [33] the Willmore functional of a $C^\infty$-smooth immersion $F : \Sigma \to \mathbb{S}^n$, $n \geq 3$, the pullback metric induced by $F$, the tracefree part $A_F^0$ of the second fundamental form of $F$, its squared length $|A_F^0|^2$, and the classical Willmore Lagrange operator

$$\nabla_{L^2}W(F) = \frac{1}{2} \left( \triangle_F^\perp \bar{H}_F + Q(A_F^0)(\bar{H}_F) \right)$$

(24)
remain unchanged, if the immersion \( F : \Sigma \rightarrow \mathbb{S}^n \subset \mathbb{R}^{n+1} \) is interpreted as an immersion of \( \Sigma \) into \( \mathbb{R}^{n+1} \), although this distinction has a certain effect on its entire second fundamental form and on its mean curvature vector. Denoting by \( A_{F,\mathbb{R}^{n+1}} \) and \( \tilde{H}_{F,\mathbb{R}^{n+1}} \) the second fundamental form and the mean curvature vector of \( F : \Sigma \rightarrow \mathbb{S}^n \) interpreted as an immersion into \( \mathbb{R}^{n+1} \) and similarly by \( A_{F,\mathbb{S}^n} \) and \( \tilde{H}_{F,\mathbb{S}^n} \) the second fundamental form and the mean curvature vector of \( F : \Sigma \rightarrow \mathbb{S}^n \) interpreted as an immersion into \( \mathbb{S}^n \), we have by formulae (2.1)–(2.4) in [33]:

\[
\begin{align*}
    g_{F,\mathbb{R}^{n+1}} &= g_{F,\mathbb{S}^n} \quad (25) \\
    A_{F,\mathbb{R}^{n+1}} &= A_{F,\mathbb{S}^n} - F g_F \quad (26) \\
    A_{F,\mathbb{R}^{n+1}}^0 &= A_{F,\mathbb{S}^n}^0 \quad (27) \\
    \tilde{H}_{F,\mathbb{R}^{n+1}} &= \tilde{H}_{F,\mathbb{S}^n} - 2F \quad (28) \\
    \Delta_{F,\mathbb{R}^{n+1}}^{\perp} \tilde{H}_{F,\mathbb{R}^{n+1}} + Q(A_{F,\mathbb{R}^{n+1}}^0)(\tilde{H}_{F,\mathbb{R}^{n+1}}) &= \Delta_{F,\mathbb{S}^n}^{\perp} \tilde{H}_{F,\mathbb{S}^n} + Q(A_{F,\mathbb{S}^n}^0)(\tilde{H}_{F,\mathbb{S}^n}). \quad (29)
\end{align*}
\]

Finally, we will also need:

**Definition 2.5.** 1) Let \( \Sigma \) be a compact smooth torus and \( n \geq 3 \) an integer. We denote by \( \text{Imm}_{\text{uf}}(\Sigma, \mathbb{R}^n) \) the subset of \( C^2(\Sigma, \mathbb{R}^n) \) consisting of umbilic-free immersions, i.e.:

\[
\text{Imm}_{\text{uf}}(\Sigma, \mathbb{R}^n) := \{ f \in C^2(\Sigma, \mathbb{R}^n) \mid f \text{ is an immersion satisfying } |A_f|^2 > 0 \text{ on } \Sigma \}.
\]

2) For a smooth regular curve \( \gamma : [a, b] \rightarrow \mathbb{S}^2 \) and a smooth tangent vector field \( W \) on \( \mathbb{S}^2 \) along \( \gamma \) we will denote by \( \nabla_{\gamma, w}(W) \) the classical covariant derivative of the vector field \( W \) w.r.t. the tangent vector field \( \partial_{\gamma, w} \) along \( \gamma \) - being projected into \( T_{\mathbb{S}^2} \) - moreover by \( \nabla_{\gamma, [w]}(W) \) the covariant derivative of the tangent vector field \( W \) along \( \gamma \) w.r.t. the unit tangent vector field \( \frac{\partial_{\gamma, w}}{|\partial_{\gamma, w}|} \) along \( \gamma \), and finally by \( \nabla_{\gamma, [w]}(W) \) the orthogonal projection of the tangent vector field \( \nabla_{\gamma, [w]}(W) \) into the normal bundle of the curve \( \gamma \) within \( T_{\mathbb{S}^2} \).

3 A regularity theorem

**Theorem 3.1.** Let \( \Sigma \) be a compact Riemann surface, \( g_{\text{poin}} \) a smooth metric of constant scalar curvature belonging to the conformal class of \( \Sigma \). Then any uniformly conformal \( (W^{2,2} \cap W^{1,\infty})\)-immersion \( F : \Sigma \rightarrow \mathbb{R}^3 \), whose distributional Willmore operator \( \nabla L^2 W(F) \) from (20) can be identified with a function of class \( L^2((\Sigma, g_{\text{poin}}), \mathbb{R}^3) \), is of class \( W^{4,2}((\Sigma, g_{\text{poin}}), \mathbb{R}^3) \).

**Proof.** In order to prove this theorem, we will introduce and apply a machinery developed by Rivière [38], [41], Bernard [4] and Palmurella [34], based on the formulation (20) of the Euler-Lagrange equations of the Willmore functional in divergence form, on Hodge decompositions of differential forms and on \( L^p \)-respectively \( L^2 \)-regularity theory. Since we have to show this regularity result only locally and since we aim to apply Proposition 8.1 below, which only holds for uniformly conformal immersions w.r.t. the Euclidean metric.
Singularities and full convergence of the Möbius-invariant Willmore flow on an open domain in $\mathbb{R}^2$, we choose some coordinate patch $\Omega \subset \Sigma$, some isothermal chart $\psi : B_2^1(0) \rightarrow \Omega$, i.e. satisfying $\psi^*(g_{\text{poin}}) = e^{2u} g_{\text{euc}}$ on the open unit disc $B_2^1(0)$, for some smooth, bounded function $v$ on $B_2^1(0)$; see here the first part of Remark 2.1. Now, since there holds $F^*(g_{\text{euc}}) = e^{2u} g_{\text{poin}}$ on $\Sigma$, for some function $u \in L^\infty(\Sigma)$, the composition $F \circ \psi : B_2^1(0) \rightarrow \mathbb{R}^3$ is of class $(W^{2,2} \cap W^{1,\infty})(B_2^1(0))$ and satisfies:

$$
(F \circ \psi)^* (g_{\text{euc}}) = e^{2u} g_{\text{poin}} = e^{2u_0 + 2v} g_{\text{euc}} \quad \text{on } B_2^1(0).
$$

Hence, we are going to prove $W^{1,2}_{\text{loc}}(B_2^1(0))$-regularity of the uniformly conformal $(W^{2,2} \cap W^{1,\infty})$-immersion $f := F \circ \psi : B_2^1(0) \rightarrow \mathbb{R}^3$. First of all, we infer from the basic requirement of the theorem, namely that $\nabla_{L^2} W(F)$ from (20) be a function of class $L^2((\Omega, g_{\text{poin}}), \mathbb{R}^3)$ for any coordinate patch $\Omega$, that its pullback $\nabla_{L^2} W(F) \circ \psi = \nabla_{L^2} W(f)$ via the above isothermal chart $\psi$ has to be a function of class $L^2(B_2^1(0), \mathbb{R}^3)$. Hence, we infer from Proposition 4.3 in [34], that we would actually obtain our desired result - locally on $B_2^1(0)$ - if we knew that the mean curvature vector $\vec{H}_f$ of $f$ was of class $L^p_{\text{loc}}(B_2^1(0), \mathbb{R}^3)$, for some $p > 2$. We will now follow the lines of Proposition 4.6 in [34], in order to prove that actually $\vec{H}_f \in L^p_{\text{loc}}(B_2^1(0), \mathbb{R}^3)$, for every $r \in (1, \infty)$. Secondly, we see by formulæ (20) and (21) and by Section 5.17 in [1], that the distributional covariant divergence of the distributional 1-form $\omega_F$ in (21) and (22) is exactly the distributional Willmore operator $\nabla_{L^2} W(F)$ in (20), i.e.:

$$
2 \langle \text{div}_F (\omega_F), \varphi \rangle_{\mathcal{D}'(\Sigma)} = 2 \int_{\Sigma} \langle g_F^{\mu\alpha} \nabla_\alpha (w_F(\partial_\nu)), \varphi \rangle_{\Omega} \, d\mu_{g_F} = \int_{\Sigma} \langle \vec{H}_F, g_F^{\mu\alpha} \nabla_\alpha (\nabla_\mu \varphi) \rangle_{g_{\text{euc}}} - g_F^{\mu\alpha} g_F^{\nu\beta} (\langle A_F \xi_\nu, \vec{H}_F \rangle_{g_{\text{euc}}} (\partial_\mu F, \nabla_\alpha \varphi)_{g_{\text{euc}}} - g_F^{\nu\beta} g_F^{\mu\alpha} (\langle A_F \xi_\nu, \vec{H}_F \rangle_{g_{\text{euc}}} (\partial_\mu F, \nabla_\alpha \varphi)_{g_{\text{euc}}}) \, d\mu_{g_F} \equiv 2 \langle \nabla_{L^2} W(F), \varphi \rangle_{\mathcal{D}'(\Sigma)}
$$

$\forall \varphi \in C^\infty(\Sigma, \mathbb{R}^3)$. We write therefore simply

$$
\text{div}_F (\omega_F) = \nabla_{L^2} W(F) \quad \text{in } \mathcal{D}'(\Omega),
$$

(31)
as if these were smooth functions respectively smooth differential forms. Equation (31) translates via our isothermal chart $\psi$ and equation (30) into the equation

$$
d^* (\omega_f) = e^{2u_0 + 2v} d^* (\omega_f) = e^{2u_0 + 2v} \nabla_{L^2} W(f) \quad \text{in } \mathcal{D}'(B_2^1(0)),
$$

(32)
where we have used equation (30) and the adjoint operators

$$
d^* = - \ast d^* : \Omega^1(B_2^1(0)) \rightarrow \Omega^0(B_2^1(0)) \quad \text{and } d^* : \Omega^1(B_2^1(0)) \rightarrow \Omega^0(B_2^1(0))
$$
of the exterior derivative $d : \Omega^0(B_2^1(0)) \rightarrow \Omega^1(B_2^1(0))$ w.r.t. the Euclidean metric and w.r.t. the pullback metric $g_f := F^*(g_{\text{euc}}) = e^{2u_0 + 2v} g_{\text{euc}}$ on the open unit disc $B_2^1(0)$ respectively. As usual, $\Omega^1(B_2^1(0))$ and $\Omega^0(B_2^1(0))$ denote the vector spaces of differential forms of degrees 1 and 0, using the standard notation $\Omega^n(B_2^1(0)) := \Gamma(\Lambda^n(B_2^1(0)))$ for $n \in \mathbb{N}_0$; see here Section 3.3 in [22]. Now on account of Theorem 3.3 in [41], there is a unique weak $W^{1,1}$-solution $\mathcal{L}$ of the elliptic Dirichlet-problem

$$
(d^* d + dd^*) (Z) \equiv \Delta_{g_{\text{euc}}}^3 (Z) = e^{2u_0 + 2v} \nabla_{L^2} W(f) \quad \text{on } B_2^1(0), \quad Z = 0 \quad \text{on } \partial B_2^1(0),
$$

(33)
and we infer from $L^2$-theory that here $\mathcal{L} \in W^{2,2}(B^2_1(0), \mathbb{R}^3)$, taking our requirements into account that $\nabla_{L^2} W(F) \in L^2(\Omega, \mathbb{R}^3)$ and that $u$ and $v$ are bounded functions. Now we have by (32) and (33):

$$d^\ast (w_f - d\mathcal{L}) = 0 \quad \text{in} \quad \mathcal{D}'(B^2_1(0)).$$

Hence, since $B^2_1(0)$ is simply connected, we obtain from the weak version of Poincaré’s Lemma a distribution $L$ on $B^2_1(0)$ satisfying

$$\ast dL = w_f - d\mathcal{L} \quad \text{in} \quad \mathcal{D}'(B^2_1(0)), \quad (34)$$

which is a Hodge-decomposition of $w_f$ in the sense of formula (7.15) in [41], see also Section 10.5 in [16]. Applying again $-\ast d$ to equation (34), and employing then the explicit formulation (22) of the 1-form $w_f$, we obtain:

$$\Delta^\mathbb{R}^3_{g_{euc}}(L) = (d^\ast d + dd^\ast)(L) = -\ast d \ast d(L) = -d(\ast d(L) = d^\ast(\ast dL - \ast w_f)$$

$$= d^\ast \left( \ast d\mathcal{L} - \frac{1}{2} (\nabla^f(\bar{H}_f) + \langle (A_f), \bar{H}_f \rangle_{g_{euc}} + \langle (A^0_f), \bar{H}_f \rangle_{g_{euc}}) \right)$$

$$= -d^\ast \left( \frac{1}{2} \left( \langle (A_f), \bar{H}_f \rangle_{g_{euc}} + \langle (A^0_f), \bar{H}_f \rangle_{g_{euc}} \right) \right) \quad \text{in} \quad \mathcal{D}'(B^2_1(0)). \quad (35)$$

Since the embedding $f = F \circ \psi$ is of class $(W^{2,2} \cap W^{1,\infty})(B^2_1(0))$, the expression in parantheses on the right hand side in (35) $\ast \frac{1}{2} \left( \langle (A_f), \bar{H}_f \rangle_{g_{euc}} + \langle (A^0_f), \bar{H}_f \rangle_{g_{euc}} \right)$ is of class $L^1(B^2_1(0))$. Hence, as in the proof of Theorem 3.3.6 in [14] we can extend this $L^1$-function trivially to entire $\mathbb{R}^2$, then consider its convolution $\Psi_f$ with the singular kernel $K(\zeta) := \frac{1}{2\pi} \log(|\zeta|)$ over entire $\mathbb{R}^2$, and see that the weak derivative $\nabla \Psi_f$ of $\Psi_f$ is given by another convolution, namely:

$$\nabla \Psi_f(z) = \int_{\mathbb{R}^2} \nabla K(z - \zeta) \left( \frac{1}{2} \left( \langle (A_f), \bar{H}_f \rangle_{g_{euc}} + \langle (A^0_f), \bar{H}_f \rangle_{g_{euc}} \right)(\zeta) \right) d\mathcal{L}^2(\zeta), \quad \forall \ z \in \mathbb{R}^2,$$

which is contained in the Lorentz space $L^{2,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ by estimate (3.39) in [14]. On the other hand, on account of formula (35) and just by construction of $\Psi_f$ and $\nabla \Psi_f$ we have $\Delta^\mathbb{R}^3_{g_{euc}}(d\Psi_f) = \Delta^\mathbb{R}^3_{g_{euc}}(L)$ in $\mathcal{D}'(B^2_1(0))$, implying especially the smoothness of $d\Psi_f - L$ locally in $B^2_1(0)$ on account of “Weyl’s Lemma”. Therefore, we finally infer that $L$ is of class $L^{2,\infty}_{\text{loc}}(B^2_1(0), \mathbb{R}^3)$; see here also the application of Theorem 3.5 in [41] within the proof of Theorem 7.4 in [41]. Moreover on account of Theorem 3.3 in [41], there are unique weak $W^{1,1}$-solutions $\mathcal{R}$ and $\mathcal{S}$ of the elliptic Dirichlet-problems

$$(d^\ast d + dd^\ast)(Z) \equiv \Delta^\mathbb{R}^3_{g_{euc}}(Z) = -\langle df, d\mathcal{L} \rangle \quad \text{on} \quad B^2_1(0), \quad Z = 0 \quad \text{on} \partial B^2_1(0), \quad (36)$$

and

$$(d^\ast d + dd^\ast)(Z) \equiv \Delta^\mathbb{R}^3_{g_{euc}}(Z) = -\langle df, d\mathcal{L} \rangle \quad \text{on} \quad B^2_1(0), \quad Z = 0 \quad \text{on} \partial B^2_1(0), \quad (37)$$

where the terminology “$df \times d\mathcal{L}$” and “$\langle df, d\mathcal{L} \rangle$” had been introduced and explained in Section 2.1 of [41]. Using $L^p$-theory, we infer from $f \in W^{1,\infty}(B^2_1(0), \mathbb{R}^3)$ and from $d\mathcal{L} \in W^{1,2}(B^2_1(0), \mathbb{R}^3)$ to $L^p(B^2_1(0))$, for every $p \in (1, \infty)$, that the weak solutions $\mathcal{R}$ and $\mathcal{S}$ of equations (36) and (37) are here of class $W^{2,p}(B^2_1(0))$, for every $p \in (1, \infty)$. Moreover, since $f$ is uniformly conformal with $f^\ast(g_{euc}) = e^{2u\psi+2\nu}g_{euc}$, we infer from (19):

$$\Delta^\mathbb{R}^3_{g_{euc}}(f) = e^{2u\psi+2\nu} \bar{H}_f \quad \text{on} \quad B^2_1(0). \quad (38)$$
Hence, we infer from formulae (34) and (36)–(38) and from formula (2.60) in [41] as in the proof of Theorem 7.4 in [41] that:

\[ d^*(-df \times \vec{H}_f - \langle *df \rangle \times L - dR) = 0 \text{ in } \mathcal{D}'(B_1^2(0)), \]

and that

\[ d^*(-\langle *df \rangle, L) - dS = 0 \text{ in } \mathcal{D}'(B_1^2(0)). \]

Hence, as in Theorem 7.5 of [41] we obtain again from the weak version of Poincaré’s Lemma two distributions \( R \) and \( S \) satisfying:

\[
\begin{align*}
*dr &= - df \times \vec{H}_f - \langle *df \rangle \times L - dR \text{ in } \mathcal{D}'(B_1^2(0)), \\
*ds &= - \langle *df \rangle, L - dS \text{ in } \mathcal{D}'(B_1^2(0)).
\end{align*}
\]

(39) \hspace{1cm} (40)

On account of \( L \in L^2_{2,\infty}(B_1^2(0), \mathbb{R}^3), R, S \in W^{2,p}(B_1^2(0)), \) for any \( p \in (1, \infty), \) and \( \vec{H}_f \in L^2(B_1^2(0), \mathbb{R}^3) \hookrightarrow L^2_{2,\infty}(B_1^2(0), \mathbb{R}^3), \) we infer from (39) and (40), that \( dR \) and \( dS \) are 1-forms of regularity class \( L^2_{2,\infty}(B_1^2(0)) \) in particular, i.e. that \( R \) and \( S \) are actually Sobolev-functions of class \( W^{1,2}_{2,\infty}(B_1^2(0)) \), again as in Theorem 7.5 of [41]. Now, combining this again with equations (34), (39) and (40), we see that all conditions of Proposition 8.1 below are satisfied, and we thus obtain from the first two equations of that proposition the following system of weak partial differential equations of second order for the pair \((R, S) \in W^{1,2}_{2,\infty}(B_1^2(0), \mathbb{R}^4):\)

\[
\Delta_{\text{g}_{\text{eucl}}}^\mathbb{R}^3(R) = dN_f \times (dR + *dR) - \langle dN_f, dS + *dS \rangle + \langle *df \rangle \times dL
\]

and

\[
\Delta_{\text{g}_{\text{eucl}}}^\mathbb{R}^3(S) = \langle dN_f, *dR \rangle + \langle dN_f, dR \rangle - \langle dN_f, dS \rangle + \langle *df \rangle \times dL
\]

in \( \mathcal{D}'(B_1^2(0)) \),

which should here be interpreted as an elliptic system of equations of the form:

\[
\begin{align*}
\Delta_{\text{g}_{\text{eucl}}}^\mathbb{R}^3(R) &= \left( dN_f \times (dR + *dR) - \langle dN_f, dS + *dS \rangle \right) + \left[ dN_f \times dR - \langle dN_f, dS \rangle + \langle *df \rangle \times dL \right] \\
\Delta_{\text{g}_{\text{eucl}}}^\mathbb{R}^3(S) &= \left( \langle dN_f, *dR \rangle \right) + \left[ \langle dN_f, dR \rangle - \langle dN_f, dS \rangle + \langle *df \rangle \times dL \right]
\end{align*}
\]

(41)

whose leading term in round brackets consists of 4 components containing sums of \((2 \times 2)\)-Jacobians as in Theorem 8.1 below and whose remainder term consists of 4 components containing at least functions of class \( L^2(B_1^2(0)) \), because the immersion \( f \) is of class \( W^{1,\infty}(B_1^2(0), \mathbb{R}^3) \) and its Gauss map \( N_f : \frac{f_{x_1} \times f_{x_2}}{|f_{x_1} \times f_{x_2}|} \) satisfies \( dN_f \in L^2(B_1^2(0)), \) moreover because \( dR \) and \( dS \) are of class \( W^{1,p}(B_1^2(0)) \subseteq L^\infty(B_1^2(0)) \), and because \( dL \in W^{1,2}(B_1^2(0)) \hookrightarrow L^r(B_1^2(0)), \) for any \( r \in (1, \infty). \) We therefore infer from Theorem 8.1 below, that the pair of solutions \((R, S)\) of system (41) is even of class \( W^{2,2}_{2,\infty}(B_1^2(0), \mathbb{R}^4) \), for every \( p \in (1, 2), \) and thus that \( dR \) and \( dS \) are 1-forms of class \( W^{1,p}_{2,\infty}(B_1^2(0)) \), which embeds into \( L^r_{2,\infty}(B_1^2(0)) \) for every \( r \in (1, \infty), \) because \( p \in (1, 2) \) has been arbitrary. We can therefore conclude by means of the third equation of Proposition 8.1, that \( \Delta_{\text{g}_{\text{eucl}}}^\mathbb{R}^3(f) \) is of class \( L^r_{2,\infty}(B_1^2(0), \mathbb{R}^3) \) for every \( r \in (1, \infty), \) and thus that also \( \vec{H}_f \in L^r_{2,\infty}(B_1^2(0), \mathbb{R}^3) \) for every \( r \in (1, \infty), \) on account of equation (38). As mentioned in the beginning of the proof, we now only have to apply Proposition 4.3 in [34], and the assertion of the theorem follows immediately. \( \square \)

### 4 Proof of Theorem 1.1

In parts (1)–(3) of Theorem 1.1 any immersion \( F \) of \( \Sigma \) into \( S^3 \) is interpreted as an immersion of \( \Sigma \) into \( \mathbb{R}^4. \) The corresponding effect of this choice on geometric tensors and scalars of
such immersions is summarized in Remark 2.3. Hence, we shall simply write $F^* g_{\text{euc}}$ instead of $F^* \langle \cdot , \cdot \rangle_{\mathbb{R}^4}$, $A_F$ instead of $A_{F,\mathbb{R}^4}$, $A_F^0$ instead of $A_{F,\mathbb{R}^4}^0$, $\tilde{H}_F$ instead of $\tilde{H}_{F,\mathbb{R}^4}$, etc.

1) First of all we infer from $W(F_0) < 8\pi$ and from the monotonicity of $W(F_t)$ along the flow line $\{F_t\}$ of the MIWF, that

$$W(F_{t_{j+1}}) < W(F_{t_j}) < 8\pi \quad \forall j \in \mathbb{N},$$

which together with formula (28) yields inequality A.20 in [26] for each integral 2-varifold $\mu_j := \mathcal{H}^2|_{F_{t_j}(\Sigma)}$, and hence formulae A.17 and A.21 in [26] prove, that each immersion $F_{t_j}$ is a smooth embedding of $\Sigma$ into $\mathbb{S}^3$. Moreover, we infer from the compactness of each embedded surface $F_{t_j}(\Sigma)$ and from formulae A.6 and A.16 in [26] applied to each integral 2-varifold $\mu_j := \mathcal{H}^2|_{F_{t_j}(\Sigma)}$, that

$$\rho^{-2} \mathcal{H}^2(F_{t_j}(\Sigma) \cap B_\rho(x)) \leq \text{const.} W(F_{t_j}) \leq \text{const.} \quad \forall j \in \mathbb{N},$$

for every fixed $x \in \mathbb{R}^4$ and every $\rho > 0$. Hence, combining this again with (42) and with formula (28), we can infer from Allard’s Compactness Theorem, that there is some subsequence $\{F_{t_{j_l}}\}$ of $\{F_{t_j}\}$ such that

$$\mathcal{H}^2|_{F_{t_{j_l}}(\Sigma)} \rightharpoonup \mu \quad \text{weakly as Radon measures on } \mathbb{R}^4,$$

as $l \to \infty$, for some integral, 2-rectifiable varifold $\mu$ in $\mathbb{R}^4$, which is just the asserted convergence (9). Combining (42) and (43) with the fact that there holds $F_{t_{j_l}}(\Sigma) \subset \mathbb{S}^3$, $\forall l \in \mathbb{N}$, we can argue as in the beginning of the proof of Proposition 2.1 in [43], respectively as in the proof of Theorem 4.2 in [27], following exactly the lines of the proof of Theorem 3.1 in [45], pp. 310–311, via Simon’s monotonicity formula (1.2) in [45] for 2-rectifiable integral varifolds in $\mathbb{R}^4$:

$$F_{t_{j_l}}(\Sigma) \to \text{spt}(\mu) \quad \text{as subsets of } \mathbb{R}^4 \text{ in Hausdorff distance, as } l \to \infty,$$

which is just the asserted convergence (10) and shows particularly, that $\text{spt}(\mu)$ has to be contained in $\mathbb{S}^3$. Moreover, as explained below Remark 2.1 we obtain unique smooth metrics $g_{\text{poin},j}$ of zero scalar curvature and unit volume on $\Sigma$, such that each immersion $F_{t_j}$ is uniformly conformal w.r.t. $g_{\text{poin},j}$ in the sense of Definition 2.2, i.e. such that there holds:

$$(F_{t_j})^*(g_{\text{euc}}) = e^{2u_j} g_{\text{poin},j} \quad \text{on } \Sigma, \quad \forall j \in \mathbb{N},$$

for unique functions $u_j \in C^\infty(\Sigma)$. On account of statement (42) we can apply here Theorem 5.2 in [23] respectively Theorem 1.1 in [39] and thus infer, that the complex structures $S(F_{t_j}(\Sigma))$ corresponding to the conformal classes of the metrics $g_{\text{poin},j}$ in (45) are contained in some compact subset of the moduli space $\mathcal{M}_1$. Hence, again up to extraction of a subsequence - which we shall relabel again - there have to exist diffeomorphisms $\Phi_j : \Sigma \xrightarrow{\sim} \Sigma$, such that

$$\Phi_j^* g_{\text{poin},j} \to g_{\text{poin}} \quad \text{smoothly as } j \to \infty$$

for some zero scalar curvature and unit volume metric $g_{\text{poin}}$ on $\Sigma$. Now if $\mu = 0$, then $\text{spt}(\mu)$ can only be a single point in $\mathbb{S}^3$. This follows from Theorem 5.1 and Corollary
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5.1 in [23] and from the proof of Theorem 5.2 in [23], taking here convergences (43), (44) and (46), and equation (45) into account; see here also Theorem 1 in [5]. Provided we have \( \mu \neq 0 \), then we can apply here Proposition 2.1 in [43] and infer, that \( \text{spt}(\mu) \) is an orientable and embedded Lipschitz-surface, either of genus 0 or of genus 1, and that \( \mu \) has constant Hausdorff-2-density 1, and this has already proved the first part of the theorem.

2) Now, we suppose that the limit varifold \( \mu \) in (9) respectively in (43) is non-trivial and that its support \( \text{spt}(\mu) \) is an embedded torus in \( S^3 \). For ease of notation and in view of the aims of this part of the theorem we may relabel henceforth the subsequence \( \{F_{\epsilon_j}\} \) in (9) and (10) into \( \{F_{\epsilon_j}\} \). Now, on account of the assumptions in this part of the theorem - just having been mentioned above - we may apply Propositions 2.1 and 7.2 in [43], in order to see that there is a smooth compact torus \( \bar{\Sigma} \) and a homeomorphic \((W^{2,2} \cap W^{1,\infty})\)-parametrization \( F : \bar{\Sigma} \overset{\pi}{\to} \text{spt}(\mu) \) of \( \text{spt}(\mu) \), being uniformly conformal w.r.t. some smooth metric \( \bar{g} \) on \( \bar{\Sigma} \) and whose “pushforward”-measure \( F(\text{meas}(\mu_{\text{euc}})) =: \mu_F \) coincides with \( \mu \) on \( \mathbb{R}^4 \) by formula (2.5) in [43] and has square integrable weak mean curvature vector \( \bar{H}_\mu \), with \( W(\mu_F) = W(F) \in [4\pi, \infty) \) by formula (7.8) in [43]. Moreover, on account of the proven Willmore conjecture, Theorem A in [30], combined with the fundamental Theorem 1.7 in [40], and by means of stereographic projection \( \mathcal{P} \) from \( S^3 \setminus \{(0,0,0,1)\} \) into \( \mathbb{R}^3 \) - assuming here that \( \text{spt}(\mu) \) is contained in \( S^3 \setminus \{(0,0,0,1)\} \) without loss of generality on account of convergence (10) - we conclude from the above properties of \( F \):

\[
W(\bar{\Sigma}) = W(\mathcal{P} \circ F) \geq \min\{W(f) \mid f \in (W^{2,2} \cap W^{1,\infty})(\bar{\Sigma}, \mathbb{R}^3), \ f^*(\text{meas}(\mu_{\text{euc}})) \geq \varepsilon \bar{g}, \text{ for some } \varepsilon > 0\} \geq 2\pi^2.
\]

Combining this with \( W(\mu_F) = W(\mu_{\|F\|}) = W(F) \) and with statement (42), we obtain:

\[
W(\mu) + \varepsilon_4 \equiv W(\mu) + \frac{8\pi}{3} = W(F) + \frac{8\pi}{3} \geq 2\pi^2 + \frac{8\pi}{3} > 8\pi > W(\mu_0) \geq \limsup_{j \to \infty} W(F_{\epsilon_j}),
\]

obtaining exactly condition (2.91) in [43]. We may therefore apply here Proposition 2.4 in [43], building on the proof of Proposition 2.1 in [43], i.e. on Lemma 2.1 in [27], on Theorem 3.1 in [45], and on Theorems 6.1 and 3.1 in [27], respectively on its variants, Theorems 5.1 and 5.2, in [43]. Hence, considering again the conformality relation (45) of the embeddings \( F_{\epsilon_j} \) w.r.t. certain smooth metrics \( g_{\text{poin},j} \) of zero constant curvature, we conclude from Proposition 2.4 in [43], that the resulting conformal factors \( u_j \in C^\infty(\Sigma) \) of the pullback metrics \( (F_{\epsilon_j})^*(\text{meas}(\mu_{\text{euc}})) \) in (45) are uniformly bounded on \( \Sigma \), i.e. that there holds here:

\[
(F_{\epsilon_j})^*(\text{meas}(\mu_{\text{euc}})) = e^{2u_j} g_{\text{poin},j} \quad \text{with} \quad \|u_j\|_{L^\infty(\Sigma)} \leq \Lambda, \quad \forall j \in \mathbb{N}.
\]

Here, the large constant \( \Lambda \) only depends on the value \( \delta := 8\pi - W(\mu_0) > 0 \) and on certain local measure theoretic properties of \( \mu \) and on certain geometric data of its support \( \text{spt}(\mu) \) and of the embedded surfaces \( F_{\epsilon_j}(\Sigma) \subset S^3 \). The exact derivation of the bound \( \Lambda \) in (48) is explained in the proof of Proposition 2.1 in [43], where one can trace back the precise applications of the technical results in [27], i.e. of Lemma
2.1 and Theorems 6.1 and 3.1 in [27], to the considered sequence of embeddings $F_t$ into $\mathbb{S}^3$ and to their induced metrics $g_{\text{euc}}[F_t(\Sigma)]$ in (45). Now we can take the new information in (48) and dive into the proof of Proposition 6.1 in [43]. As in that proof we infer from (42) and (48) and from Lemma 5.2 in [27], that the conformal structures corresponding to the conformal classes of the metrics $(F_t)^*(g_{\text{euc}})$ respectively of the metrics $g_{\text{poin},j}$ - by (48) - are compactly contained in the moduli space $\mathcal{M}_1$. Hence, again up to extracting a subsequence - which we shall relabel again - there have to exist diffeomorphisms $\Phi_j : \Sigma \xrightarrow{\phi} \Sigma$, such that

$$\Phi_j^* g_{\text{poin},j} \to g_{\text{poin}}$$

smoothly as $j \to \infty$ (49)

for some zero scalar curvature and unit volume metric $g_{\text{poin}}$ on $\Sigma$, similarly to convergence (46). In view of statements (48) and (49), in view of the desired assertion of this part of the theorem and since the Willmore functional is invariant w.r.t. smooth reparametrization, we shall replace the embeddings $F_t$ by their reparametrizations $\tilde{F}_t := F_t \circ \Phi_j$, for every $j \in \mathbb{N}$, so that we may continue our proof with the modified equations

$$(\tilde{F}_t)^*(g_{\text{euc}}) = e^{2u_j \circ \Phi_j} \Phi_j^* g_{\text{poin},j} \quad \text{with} \quad \| u_j \circ \Phi_j \|_{L^\infty(\Sigma)} \leq \Lambda, \quad \forall j \in \mathbb{N}, \quad (50)$$

instead of equations (48). For ease of notation we shall rename $\Phi_j^* g_{\text{poin},j}$ again into $g_{\text{poin},j}$ and $u_j \circ \Phi_j$ into $u_j$, $\forall j \in \mathbb{N}$, so that we can assume the convergence

$$g_{\text{poin},j} \to g_{\text{poin}}$$

smoothly as $j \to \infty$, (51)

instead of the convergence in (49). Moreover, from equations (50), together with formulae (17) and (18) above we infer, just as in formula (6.4) of [43]:

$$-\Delta_{\tilde{F}_t}(g_{\text{euc}})(u_j) = K_{\tilde{F}_t}(g_{\text{euc}}) \quad \text{on} \quad \Sigma,$$

or expressed equivalently:

$$-\Delta_{g_{\text{poin},j}}(u_j) = e^{2u_j} K_{\tilde{F}_t}(g_{\text{euc}}) \quad \text{on} \quad \Sigma.$$ (52)

Following now exactly the lines of the proof of Proposition 6.1 in [43], the equations in (52) yield together with the differential equations

$$\Delta^{\mathbb{R}^4}_{g_{\text{poin},j}}(\tilde{F}_t) = e^{2u_j} H_{\tilde{F}_t,\mathbb{R}^4} \quad \text{on} \quad \Sigma,$$

and together with statements (42), (50) and (51), and also using the fact that each $\tilde{F}_t$ maps $\Sigma$ into the compact 3-sphere, the estimates:

$$\| \nabla u_j \|_{L^2(\Sigma,g_{\text{poin}})} \leq C(\Lambda) \quad \text{and} \quad \| \tilde{F}_t \|_{W^{2,2}(\Sigma,g_{\text{poin}})} \leq C(\Lambda) \quad (54)$$

for every $j \in \mathbb{N}$. Hence, we obtain convergent subsequences $\{u_{jk}\}$ and $\{\tilde{F}_{tjk}\}$ of $\{u_j\}$ and $\{\tilde{F}_t\}$:

$$u_{jk} \to u \quad \text{weakly in} \quad W^{1,2}(\Sigma,g_{\text{poin}}) \quad (55)$$

$$u_{jk} \to u \quad \text{weakly* in} \quad L^\infty(\Sigma,g_{\text{poin}}) \quad (56)$$

$$u_{jk} \to u \quad \text{pointwise a.e. in} \quad \Sigma \quad (57)$$

and

$$\tilde{F}_{tjk} \to f \quad \text{weakly in} \quad W^{2,2}(\Sigma,g_{\text{poin}}) \quad (58)$$
as $k \to \infty$, for appropriate functions $u \in W^{1,2}(\Sigma, g_{\text{poin}}) \cap L^\infty(\Sigma, g_{\text{poin}})$ and $f \in W^{2,2}(\Sigma, g_{\text{poin}})$. Moreover, we infer from (50) and (51) that $\nabla g_{\text{poin}} \tilde{F}_{t_j}$ is uniformly bounded in $L^\infty(\Sigma, g_{\text{poin}})$. Hence, again recalling the fact that $\tilde{F}_{t_j}$ map $\Sigma$ into the 3-sphere, we obtain:

$$\| \tilde{F}_{t_j} \|_{W^{1,\infty}(\Sigma, g_{\text{poin}})} \leq \text{Const}(\Lambda).$$

(59)

We thus infer from Theorem 8.5 in [1] and from Theorem 4.12 in [1], i.e. from Arzela’s and Ascoli’s Theorem, that the convergent subsequence $\{\tilde{F}_{t_{jk}}\}$ in (58) also converges in the senses:

$$\tilde{F}_{t_{jk}} \rightharpoonup f \quad \text{weakly* in} \quad W^{1,\infty}(\Sigma, g_{\text{poin}})$$

(60)

and

$$\tilde{F}_{t_{jk}} \to f \quad \text{in} \quad C^0(\Sigma, g_{\text{poin}})$$

(61)

as $k \to \infty$. Now we can conclude from (50) and from the above convergences (51), (56), (57) and (60), that in the limit there holds actually:

$$f^*(g_{\text{euc}}) = e^{2u} g_{\text{poin}},$$

(62)

showing that $f$ is a uniformly conformal ($W^{2,2} \cap W^{1,\infty}$)-immersion w.r.t. $g_{\text{poin}}$ on $\Sigma$, with $\| u \|_{L^\infty(\Sigma)} \leq \Lambda$ for the same constant $\Lambda$ as in (48), depending on the sequence $\{F_{t_{lj}}\}$ from (9) and on the limit varifold $\mu$. Moreover, we obtain as in the proof of Proposition 2.1 in [43], line (2.30), that $\mu_f := f(\mu_f^*(g_{\text{euc}}))$ coincides with the varifold $\mu$ in (43). Similarly to the argument above before estimate (47), we now continue as in the proof of Proposition 2.1 in [43] and conclude from the facts, that $f \in W^{2,2}(\Sigma, \mathbb{R}^4)$ and that $\mathcal{W}(f) \leq \liminf_{k \to \infty} \mathcal{W}(\tilde{F}_{t_{jk}}) < 8\pi$ and from equation (62) via Proposition 7.2 in [43] that $f$ is injective, and that not only $\mu_f = \mu$ holds, but also $\mu_f = \mathcal{H}^2(f(\Sigma))$, and moreover that

$$\text{spt}(\mu_f) = f(\Sigma), \quad \text{thus also that} \quad \text{spt}(\mu) = f(\Sigma),$$

and that therefore

$$f : \Sigma \xrightarrow{\simeq} \text{spt}(\mu)$$

(63)

is a bi-Lipschitz continuous homeomorphism. Hence, assertions (11) and (12) are already proven. Continuing here as in the proof of Proposition 7.2 of [43], we infer that the coinciding integral varifolds $\mu_f$ have weak mean curvature vectors $\tilde{H}_{\mu_f}$ in $L^2(\mu_f)$ and that formula (13) holds here.

Moreover, we can argue as in Remark 2 on p. 1354 in [43], that any further conformal ($W^{2,2} \cap W^{1,\infty}$)-parametrization $\hat{f} : \hat{\Sigma} \to \text{spt}(\mu)$, with $\hat{f}^*(g_{\text{euc}}) = e^{2\hat{u}} \hat{g}_{\text{poin}}$, for another metric $\hat{g}_{\text{poin}}$ of zero scalar curvature and for another $\hat{u} \in L^\infty(\Sigma)$, has to be a bi-Lipschitz homeomorphism between $\hat{\Sigma}$ and $\text{spt}(\mu)$ on account of Proposition 7.2 in [43] and that the composition

$$\phi := f^{-1} \circ \hat{f} : (\hat{\Sigma}, \hat{g}_{\text{poin}}) \xrightarrow{\simeq} (\Sigma, g_{\text{poin}})$$

has to be either holomorphic or antiholomorphic, which additionally has a non-vanishing Jacobian in every point of $\hat{\Sigma}$. Hence, according to Definition 2.1 (iv) the map $\phi$ yields an isomorphism between the complex structures $S(\Sigma)$ and $\hat{S}(\Sigma)$.
induced by the conformal structures \((\Sigma, g_{\text{poin}})\) and \((\hat{\Sigma}, \hat{g}_{\text{poin}})\), and we may define unambiguously:

\[
[spt(\mu)] := [(\Sigma, g_{\text{poin}})] \in \mathcal{M}_1.
\]

Hence, there is a unique representative of \(spt(\mu)\) in the moduli space \(\mathcal{M}_1\), being obtained via any conformal \((W^{2,2} \cap W^{1,\infty})\)-parametrization \(\hat{f}\) of \(spt(\mu)\). Combining this fact with statements (49), (50) and (62) above, we infer that every subsequence of the original convergent sequence \(\{F_{t_j}\}\) in (9), which we had considered at the beginning of this part of the proof, contains another subsequence \(\{F_{t_{j_p}}\}\) such that after appropriate reparametrization:

\[
[(\Sigma, \hat{F}_{t_{j_p}}^* (g_{\text{euc}}))] = [(\Sigma, \Phi_{j_p}^* g_{\text{poin}, j_p})] \longrightarrow [(\Sigma, \hat{g}_{\text{poin}})] = [spt(\mu)] \quad \text{in } \mathcal{M}_1
\]
as \(p \to \infty\), which proves all assertions of the second part of this theorem.

3) If we suppose again that the support of the limit varifold \(\mu\) is an embedded torus and that also its Willmore energy \(\mathcal{W}(\mu)\) from (13) equals the limit of Willmore energies \(\mathcal{W}(F_{t_{j_k}})\), for some subsequence \(\{F_{t_{j_k}}\}\) of the sequence \(F_{t_j} : \Sigma \xrightarrow{\cong} F_{t_j}(\Sigma) \subset S^3\) from (9), then we can follow the argument at the end of the proof of Proposition 5.3 in [28]. First of all, similarly to the proof of the second part of the theorem, we ease our notation and relabel the given subsequence \(\{F_{t_{j_k}}\}\) again into \(\{F_{t_j}\}\), and then we replace them by appropriate reparametrizations \(\hat{F}_{t_j} := F_{t_j} \circ \Phi_j\), in order to have statements (50)–(54) and (59) at our disposal, and we obtain in the weak limits (58) and (60) a homeomorphic parametrization \(f \in (W^{2,2} \cap W^{1,\infty})(\Sigma, g_{\text{poin}})\) of \(spt(\mu)\) which is a uniformly conformal \((W^{2,2} \cap W^{1,\infty})\)-immersion w.r.t. \(g_{\text{poin}}\) on account of formula (62). Formula (62) implies particularly:

\[
\Delta_{g_{\text{poin}}}^\mathbb{R}^4 (f) = e^{2u} \mathcal{H}_f,_{\mathbb{R}^4} \quad \text{on } \Sigma,
\]
similarly to equations (53). Now, from our additional requirement that \(\mathcal{W}(\mu) = \lim_{j \to \infty} \mathcal{W}(F_{t_j})\), and from statements (13), (50), (62) and (64) we infer that

\[
\int_\Sigma |\Delta_{g_{\text{poin}, j}}^\mathbb{R}^4 (\hat{F}_{t_j})|^2 e^{-2u} \, d\mu_{g_{\text{poin}, j}} = \int_\Sigma |\mathcal{H}_{\hat{F}_{t_j}}|^2 e^{2u} \, d\mu_{g_{\text{poin}, j}} = \int_\Sigma |\mathcal{H}_{\hat{F}_{t_j}}|^2 e^{2u} \, d\mu_{\hat{F}_{t_j} g_{\text{euc}}} \quad (65)
\]

\[
\longrightarrow 4 \mathcal{W}(\mu) = \int_\Sigma |\mathcal{H}_f|^2 2 e^{2u} \, d\mu_{g_{\text{poin}}} = \int_\Sigma |\mathcal{H}_f|^2 2 e^{2u} \, d\mu_{g_{\text{poin}}} \quad (66)
\]
as \(j \to \infty\). Now we observe, that the bounds (50) and (54) imply the uniform bound

\[
\| \Delta_{g_{\text{poin}, j}}^\mathbb{R}^4 (\hat{F}_{t_j}) e^{-u_j} \|_{L^2(\Sigma, g_{\text{poin}})} \leq C(\Lambda)
\]

for every \(j \in \mathbb{N}\). Combining estimate (66) with convergence (65) and with the smooth convergence (51), we infer:

\[
\int_\Sigma |\Delta_{g_{\text{poin}, j}}^\mathbb{R}^4 (\hat{F}_{t_j})|^2 e^{-2u_j} \, d\mu_{g_{\text{poin}}} \longrightarrow \int_\Sigma |\Delta_{g_{\text{poin}}}^\mathbb{R}^4 (f)|^2 e^{-2u} \, d\mu_{g_{\text{poin}}}, \quad (67)
\]
as \(j \to \infty\). In order to finally prove our assertion in (14), we shall rather continue working with the subsequence \(\{t_{j_k}\}\) of \(\{t_j\}\) appearing in convergences (56)–(58).
instead of \{t_j\} itself. From estimate (66) we also infer - possibly for another subsequence, which we relabel again into \{t_{jk}\}:

\[
\Delta_{g_{\text{poin}},jk}^\mathbb{R}^4 (\tilde{F}_{t_{jk}}) e^{-u_{jk}} \rightharpoonup q \quad \text{weakly in } L^2(\Sigma, g_{\text{poin}})
\]  

(68)
as \(k \to \infty\), for some limit function \(q \in L^2(\Sigma)\). Moreover, on account of the convergences (57) and (58) and on account of (50) we can apply the result of exercise E8.3 in [1] and obtain for the same subsequence as in (68) that

\[
\Delta_{g_{\text{poin}},jk}^\mathbb{R}^4 (\tilde{F}_{t_{jk}}) e^{-u_{jk}} \rightharpoonup \Delta_{g_{\text{poin}}}^\mathbb{R}^4 (f) e^{-u} \quad \text{weakly in } L^1(\Sigma, g_{\text{poin}})
\]  

(69)
as \(k \to \infty\). Since weak convergence in \(L^2(\Sigma, g_{\text{poin}})\) implies weak convergence in \(L^1(\Sigma, g_{\text{poin}})\) and since weak limits in Banach spaces are unique, convergences (68) and (69) imply that

\[
\Delta_{g_{\text{poin}},jk}^\mathbb{R}^4 (\tilde{F}_{t_{jk}}) e^{-u_{jk}} \rightharpoonup \Delta_{g_{\text{poin}}}^\mathbb{R}^4 (f) e^{-u} \quad \text{weakly in } L^2(\Sigma, g_{\text{poin}})
\]  

(70)
as \(k \to \infty\). Combining this with convergence (67) we finally obtain strong \(L^2\)-convergence of the subsequence in (69) and (70):

\[
\Delta_{g_{\text{poin}},jk}^\mathbb{R}^4 (\tilde{F}_{t_{jk}}) e^{-u_{jk}} \rightharpoonup \Delta_{g_{\text{poin}}}^\mathbb{R}^4 (f) e^{-u} \quad \text{strongly in } L^2(\Sigma, g_{\text{poin}})
\]  

as \(k \to \infty\). Combining this again with estimate (50), with convergence (57) and with Vitali’s convergence theorem, we finally obtain:

\[
\Delta_{g_{\text{poin}},jk}^\mathbb{R}^4 (\tilde{F}_{t_{jk}}) \rightharpoonup \Delta_{g_{\text{poin}}}^\mathbb{R}^4 (f) \quad \text{strongly in } L^2(\Sigma, g_{\text{poin}})
\]  

(71)
as \(k \to \infty\). Just as in the end of the proof of Proposition 5.3 in [28], p. 506, we can infer now from convergence (71) and again from (51) and (54), that

\[
\partial_u \left( g_{\text{poin}}^u \sqrt{\det g_{\text{poin}}} \partial_v (\tilde{F}_{t_{jk}} - f) \right) = \\
= \partial_u \left( (g_{\text{poin}}^u \sqrt{\det g_{\text{poin}}} - g_{\text{poin},jk}^u \sqrt{\det g_{\text{poin},jk}}) \partial_v (\tilde{F}_{t_{jk}}) \right) \\
+ \sqrt{\det g_{\text{poin},jk}} \Delta_{g_{\text{poin}},jk}^\mathbb{R}^4 (\tilde{F}_{t_{jk}}) - \sqrt{\det g_{\text{poin}}} \Delta_{g_{\text{poin}}}^\mathbb{R}^4 (f) \\
\to 0 \quad \text{strongly in } L^2(\Sigma, g_{\text{poin}}),
\]
as \(k \to \infty\). Hence, by standard \(L^2\)-estimates for uniformly elliptic partial differential equations on \((\Sigma, g_{\text{poin}})\) together with convergence (61), we conclude that the subsequence \{\tilde{F}_{t_{jk}}\} in (71) of the reparametrized sequence of embeddings \{\tilde{F}_{t_j}\} converges strongly to \(f\) in \(W^{2,2}(\Sigma, g_{\text{poin}})\), just as asserted in (14).

Now we finally prove, that the subsequence \{\tilde{F}_{t_{jk}}\} found in (14) is uniformly bi-Lipschitz continuous. On account of the bound in (59) the sequence \{\tilde{F}_{t_{jk}}\} is uniformly Lipschitz continuous. Hence, if the asserted bi-Lipschitz property of \{\tilde{F}_{t_{jk}}\} would not hold, then there existed another subsequence \(f_m := \tilde{F}_{t_{jkm}}\) of \(\tilde{F}_{t_{jk}}\) and points \(p_m, q_m \in \Sigma\) such that

\[
|f_m(p_m) - f_m(q_m)| < \frac{1}{m} \text{dist}_{(\Sigma, g_{\text{poin}})}(p_m, q_m) \quad \forall m \in \mathbb{N}
\]  

(72)
holds. On account of the compactness of \((\Sigma, g_{\text{poin}})\), we first extract convergent subsequences \(p_{m'} \rightarrow p\) and \(q_{m'} \rightarrow q\) in \((\Sigma, g_{\text{poin}})\), which we relabel again. Inserting this into (72) and using the uniform convergence of \(\{f_m\}\) to \(f\) from (61), we obtain in the limit as \(m \rightarrow \infty\): \(|f(p) - f(q)| = 0\) and thus that \(p = q\) by injectivity of \(f\). Hence, similarly to the proof of Proposition 7.2 in [43] we now introduce local conformal coordinates about this limit point \(p\) w.r.t. the zero scalar curvature metrics \(g_{\text{poin}, j_{km}}\). By Definitions 2.3.1, 2.3.3 and 2.3.4 in [21] this means, that we consider open neighbourhoods \(U_m(p)\) of \(p\) mapping \(0\) to \(p\), such that

\[
\varphi_m g_{\text{poin}, j_{km}} = e^{2u_m} g_{\text{euc}} \quad \text{with} \quad \Delta_{g_{\text{euc}}}(u_m) = 0 \quad \text{on} \quad B_1^2(0),
\]

and with \(v_m \in L^\infty(B_1^2(0))\) and \(\|v_m\|_{L^\infty(B_1^2(0))} \leq C(g_{\text{poin}}) \quad \forall m \in \mathbb{N}\),

(73)

where we have used convergence (51). Using Cauchy estimates we thus also have:

\[
\|\nabla^s(v_m)\|_{L^\infty(B_{\frac{3}{4}}(0))} \leq C(g_{\text{poin}}, s) \quad \forall m \in \mathbb{N}
\]

(74)

and for each fixed \(s \in \mathbb{N}\). Hence, we obtain:

\[
(f_m \circ \varphi_m)^*(g_{\text{euc}}) = \varphi_m^*(e^{2u_{j_{km}}} g_{\text{poin}, j_{km}}) = e^{2u_{j_{km}}(\varphi_m g_{\text{poin}, j_{km}})} = e^{2u_{j_{km}}(\varphi_m + 2v_m)} g_{\text{euc}}
\]

showing that \(f_m \circ \varphi_m : B_{1}^2(0) \rightarrow S^3\) are smooth conformal embeddings w.r.t. the Euclidean metric on \(B_{1}^2(0)\). We set

\[
M := \sup_{m \in \mathbb{N}} \left( \|u_{j_{km}}\|_{L^\infty(\Sigma)} + \|v_m\|_{L^\infty(B_1^2(0))} \right) < \infty,
\]

and on account of the strong \(W^{2,2}\)-convergence (14) of the embeddings \(f_m\) and on account of estimates (73) and (74) we may choose \(\varrho \in (0,1)\) that small, such that

\[
\int_{B_{\varrho}^2(0)} |D^2(f_m \circ \varphi_m)|^2 d\mathcal{L}^2 < \frac{\pi}{2} \tan(\pi) e^{-6M}, \quad \forall m \in \mathbb{N}.
\]

(75)

On account of \(p_m \rightarrow p = q \leftarrow q_m\) in \((\Sigma, g_{\text{poin}})\) and due to estimate (73) we know that \(p_m, q_m \in \varphi_m(B_{\varrho}^2(0)) \subset \subset U_m(p)\) for sufficiently large \(m\), and thus we obtain from equation (50), convergence (51), estimate (75) and from Lemmata 4.2.7 and 4.2.8 in [32], similarly to the end of the proof of Proposition 7.2 in [43]:

\[
\text{dist}_{(\Sigma, g_{\text{poin}})}(p_m, q_m) \leq 2 e^{2M} \text{dist}_{(\Sigma, f_m g_{\text{euc}})}(p_m, q_m) \leq 2 \sqrt{2} e^{2M} |f_m(p_m) - f_m(q_m)|
\]

for large \(m\), which contradicts (72) for sufficiently large \(m\). Hence, the sequence of embeddings \(\tilde{F}_{t_{jk}} : \Sigma \xrightarrow{\tilde{\iota}} \tilde{F}_{t_{jk}}(\Sigma) \subset S^3\) is indeed uniformly bi-Lipschitz continuous. Now we choose some point \(x \in S^3\) arbitrarily. On account of the uniform bi-Lipschitz property of the embeddings \(\{\tilde{F}_{t_{jk}}\}\), we can find for any small \(r > 0\) some sufficiently small \(\eta > 0\), depending on \(x\) and \(r\), such that the preimages \(\tilde{F}_{t_{jk}}^{-1}(B_{\eta}^2(x) \cap S^3)\) are contained in open geodesic discs \(B_{\eta}^2(p_{k})\) about certain points \(p_k \in \Sigma\) w.r.t. the metric \(g_{\text{poin}, k}\), for every \(k \in \mathbb{N}\). On account of the compactness of \((\Sigma, g_{\text{poin}})\) we can
extract some convergent subsequence $p_{k_m} \to p$ in $\Sigma$, depending on $x \in S^3$. Setting now again $f_m := F_{t_{jkm}}$, we thus obtain the existence of some large $K = K(x) \in \mathbb{N}$, such that $f_m^{-1}(B^4_\eta(x) \cap S^3)$ is contained in $B^3_{2r}(p)$ for every $m \geq K$. Combining this again with the strong $W^{2,2}$-convergence (14) of the embeddings $f_m$, we infer that for every $\varepsilon > 0$ there is some sufficiently small $\eta > 0$, depending on $x$ and $\varepsilon$, such that:

$$
\int (f_m)^{-1}(B^4_\eta(x) \cap S^3) |A_{f_m}|^2 d\mu_{f_m,\text{euc}} \leq \int B^3_{2r}(p) |D^2 f_m|^2 d\mu_{f_m,\text{euc}} < \varepsilon, \quad \forall m \geq K,
$$

which has already proved the last assertion (15) of the third part of this theorem.

4) As in the third part of this theorem we consider a limit varifold $\mu$ in (9), whose support is a compact and embedded torus in $S^3$, and we consider again the same subsequence $\{F_{t_{jk}}\}$ of the weakly convergent sequence $\{F_{t_{jk}}\}$ in (9). Here we assume additionally, that this subsequence $\{F_{t_{jk}}\}$ satisfies:

$$
\|A^0_{F_{t_{jk}} S^3}\|_{L^\infty(\Sigma)} \leq K \quad \text{and} \quad \left| \frac{d}{dt} W(F_t) \right|_{t = t_{jk}} \leq K \quad \forall k \in \mathbb{N}, \quad (76)
$$

for some sufficiently large number $K > 1$. We know already from (14), (60) and (61), that some particular subsequence of $\{F_{t_{jk}}\}$ - which we relabel again - can be reparametrized into embeddings $\tilde{F}_{t_{jk}}$ which converge strongly in $W^{2,2}(\Sigma, g_{\text{poin}})$, weakly* in $W^{1,\infty}(\Sigma, g_{\text{poin}})$ and also uniformly:

$$
\tilde{F}_{t_{jk}} \rightarrow f \quad \text{in} \quad C^0(\Sigma, g_{\text{poin}}), \quad (77)
$$

as $k \to \infty$, to the uniformly conformal ($W^{2,2} \cap W^{1,\infty}$)-parametrization $f$ of spt($\mu$) from (63). Also recalling that the MIWF (1) is conformally invariant, we can therefore assume that the images $F_{t_{jk}}(\Sigma)$ of the entire subsequence $\{F_{t_{jk}}\}$ satisfy:

$$
F_{t_{jk}}(\Sigma) \subset S^3 \setminus B^3_\delta((0,0,0,1)) \quad \forall k \in \mathbb{N}, \quad (78)
$$

for some small $\delta > 0$. We therefore project the entire flow line $\{F_t\}$ of the MIWF stereographically from $S^3 \setminus \{(0,0,0,1)\}$ into $\mathbb{R}^3$, and we shall consider henceforth the sequence of stereographically projected embeddings $\mathcal{P} \circ F_{t_{jk}} : \Sigma \longrightarrow \mathbb{R}^3$ instead of the original sequence $\{F_{t_{jk}}\}$ in the formulation of this last part of the theorem. Using the flow equation (1) and its conformal invariance we compute by means of the chain rule and the additional requirements for the considered subsequence $\{F_{t_{jk}}\}$:

$$
\int_\Sigma \frac{1}{|A^0_{\mathcal{P} \circ F_{t_{jk}}} S^3|^4} |\nabla_{L^2} W(\mathcal{P} \circ F_{t_{jk}})|^2 d\mu_{(\mathcal{P} \circ F_{t_{jk}})^*(\text{euc})} =

= 2 \left| \frac{d}{dt} W(\mathcal{P} \circ F_t) \right|_{t = t_{jk}} \left| t = t_{jk} \right. = 2 \left| \frac{d}{dt} W(F_t) \right|_{t = t_{jk}} \leq 2K \quad \forall k \in \mathbb{N}.
$$

Moreover, we note that by (78) our condition $\|A^0_{F_{t_{jk}} S^3}\|^2 \leq K$ on $\Sigma$ also implies that

$$
\|A^0_{\mathcal{P} \circ F_{t_{jk}}} S^3\|^2 \leq K'(K, \delta) < \infty \quad \forall k \in \mathbb{N},
$$
and we therefore obtain the estimate:

\[
\int_{\Sigma} |\nabla_{L^2} \mathcal{W}(\mathcal{P} \circ F_{\tilde{t}_j})|^2 d\mu_{(\mathcal{P} \circ F_{\tilde{t}_j})^* (g_{\text{euc}})} \leq 2K (K')^2 \quad \text{for every } k \in \mathbb{N}.
\]  

(79)

Now, we shall again replace the embeddings \( \mathcal{P} \circ F_{\tilde{t}_j} \) by the embeddings \( \tilde{f}_k := \mathcal{P} \circ F_{\tilde{t}_j} \circ \Phi_{\tilde{t}_j} \equiv \mathcal{P} \circ \tilde{F}_{\tilde{t}_j} \) in view of (50) and (51), and we obtain from (79), (50) and (51) together with the area formula:

\[
\| \nabla_{L^2} \mathcal{W}(\tilde{f}_k) \|^2_{L^2(\Sigma, g_{\text{poin}})} \equiv \int_{\Sigma} |\nabla_{L^2} \mathcal{W}(\tilde{f}_k)|^2 d\mu_{g_{\text{poin}}} \leq 2e^{2\Lambda} \int_{\Sigma} |\nabla_{L^2} \mathcal{W}(\tilde{f}_k)|^2 d\mu_{\tilde{f}_k} (g_{\text{euc}})
\]

\[
= 2e^{2\Lambda} \int_{\Sigma} |\nabla_{L^2} \mathcal{W}(\mathcal{P} \circ F_{\tilde{t}_j})|^2 d\mu_{(\mathcal{P} \circ F_{\tilde{t}_j})^* (g_{\text{euc}})} \leq 4K (K')^2 e^{2\Lambda}
\]  

(80)

for every \( k \in \mathbb{N} \), where \( \Lambda \) denotes an upper bound for the new conformal factor \( \tilde{u}_{j_k} \) in \( L^\infty(\Sigma) \), appearing in:

\[
(\tilde{f}_k)^* (g_{\text{euc}}) = e^{2\tilde{u}_k} g_{\text{poin},j_k} \quad \text{on } \Sigma
\]  

(81)

for every \( k \in \mathbb{N} \), which holds on account of (50) and (78) and due to the conformality of the stereographic projection. On account of (80) there is some subsequence of the sequence \( \{\tilde{f}_k\} \), which we relabel into \( \{f_k\} \) again, and some function \( q \in L^2(\Sigma, g_{\text{poin}}) \) with values in \( \mathbb{R}^3 \), such that

\[
\nabla_{L^2} \mathcal{W}(f_k) \rightarrow q \quad \text{weakly in } L^2(\Sigma, g_{\text{poin}})
\]  

(82)

as \( k \rightarrow \infty \). Considering the homeomorphic parametrization \( \tilde{f} := \mathcal{P} \circ f \) of the limit torus \( \mathcal{P} (\text{spt}(\mu)) \subset \mathbb{R}^3 \) from (63) projected stereographically from \( \mathbb{S}^3 \setminus \{(0,0,0,1)\} \) into \( \mathbb{R}^3 \), we derive from (62), (77) and (78) immediately:

\[
(\tilde{f})^* (g_{\text{euc}}) = (\mathcal{P} \circ f)^* (g_{\text{euc}}) = e^{2\tilde{u}} g_{\text{poin}} \quad \text{on } \Sigma,
\]  

(83)

for some function \( \tilde{u} \in L^\infty(\Sigma) \), which has to additionally satisfy:

\[
\tilde{u}_k \rightarrow \tilde{u} \quad \text{pointwise a.e. in } \Sigma,
\]  

(84)

on account of (51), (81) and on account of the strong \( W^{2,2}(\Sigma, g_{\text{poin}}) \)-convergence of the sequence \( \{f_k\} \) to \( \tilde{f} \) due to (14). Obviously, \( \tilde{f} \) is a uniformly conformal \( (W^{2,2} \cap W^{1,\infty}) \)-immersion w.r.t. \( g_{\text{poin}} \) on account of (83) - just as \( f \) is by (62) - and statements (82) and (83) also imply, that

\[
\nabla_{L^2} \mathcal{W}(\tilde{f}_k) \rightarrow q \quad \text{weakly in } L^2(\Sigma, \tilde{f}^* g_{\text{euc}}),
\]  

(85)

as \( k \rightarrow \infty \), with the same \( q \in L^2((\Sigma, g_{\text{poin}}), \mathbb{R}^3) \) as in (82). On the other hand, we can immediately infer from (14), (60), (61) respectively from (77), combined with (78), that the reparametrized embeddings \( \tilde{f}_k = \mathcal{P} \circ F_{\tilde{t}_j} \circ \Phi_{\tilde{t}_j} \) converge strongly in \( W^{2,2}(\Sigma, g_{\text{poin}}) \), weakly* in \( W^{1,\infty}(\Sigma, g_{\text{poin}}) \) and in \( C^0(\Sigma, g_{\text{poin}}) \) to the parametrization \( \tilde{f} = \mathcal{P} \circ f \) of the projected torus \( \mathcal{P} (\text{spt}(\mu)) \subset \mathbb{R}^3 \). Using particularly the fact that the \( \tilde{f}_k \) are uniformly bounded in \( W^{1,\infty}(\Sigma, g_{\text{poin}}) \) by (59) and (78), we can combine again Theorem 8.5 in [1] with the above mentioned convergences of the sequence
\{ \tilde{f}_k \}, with Remark 8.3(6) in [1] for \( X = L^1(\Sigma, g_{\text{poin}}) \), and with formulae (20) and (23), in order to conclude that

\[
\langle \nabla_{L^2} W(\tilde{f}_k), \varphi \rangle_{L^2(\Sigma, \tilde{f}_k^*(g_{\text{euc}}))} = \langle \nabla_{L^2} W(\tilde{f}_k), \varphi \rangle_{D'(\Sigma)} = \\
= \int_\Sigma (\tilde{H}_{\tilde{f}_k} \triangle \varphi)_{g_{\text{euc}}} - g_{\tilde{f}_k}^{\mu \kappa} \langle (A_{\tilde{f}_k})_{\zeta \nu}, \tilde{H}_{\tilde{f}_k} \rangle_{g_{\text{euc}}} \langle \partial_\mu \tilde{f}_k, \partial_\alpha \varphi \rangle_{g_{\text{euc}}} \\
- g_{\tilde{f}_k}^{\rho \alpha} g_{\tilde{f}_k}^{\mu \kappa} \langle (A_{\tilde{f}_k})_{\xi \nu}, \tilde{H}_{\tilde{f}_k} \rangle_{g_{\text{euc}}} \langle \partial_\mu \tilde{f}_k, \partial_\alpha \varphi \rangle_{g_{\text{euc}}} \\
= \int_\Sigma (\tilde{H}_{\tilde{f}}, \triangle \varphi)_{g_{\text{euc}}} - g_{\tilde{f}}^{\rho \alpha} g_{\tilde{f}}^{\mu \kappa} \langle (A_{\tilde{f}})_{\xi \nu}, \tilde{H}_{\tilde{f}} \rangle_{g_{\text{euc}}} \langle \partial_\mu \tilde{f}, \partial_\alpha \varphi \rangle_{g_{\text{euc}}} \\
- g_{\tilde{f}}^{\rho \alpha} g_{\tilde{f}}^{\mu \kappa} \langle (A_{\tilde{f}})_{\xi \nu}, \tilde{H}_{\tilde{f}} \rangle_{g_{\text{euc}}} \langle \partial_\mu \tilde{f}, \partial_\alpha \varphi \rangle_{g_{\text{euc}}} = \\
= \langle \nabla_{L^2} W(\tilde{f}), \varphi \rangle_{D'(\Sigma)} \quad (86)
\]

as \( k \to \infty \), for every fixed \( \varphi \in C^\infty(\Sigma, \mathbb{R}^3) \). In the above argument one has to recall, that the \((W^{2,2} \cap W^{1,\infty})\)-immersions \( \tilde{f}_k : \Sigma \to \mathbb{R}^3 \) and \( \tilde{f} : \Sigma \to \mathbb{R}^3 \) map into \( \mathbb{R}^3 \), but not into \( S^3 \), such that we can actually apply our special version in (20) of the distributional Willmore operator. Combining (86) with convergences (51), (84), (82) and (85), with equations (81) and (83) and with the uniform bound (50), then we obtain again via E8.3 in [1] in the limit as \( k \to \infty \):

\[
\langle \nabla_{L^2} W(\tilde{f}), \varphi \rangle_{D'(\Sigma)} = \lim_{k \to \infty} \langle \nabla_{L^2} W(\tilde{f}_k), \varphi \rangle_{L^2(\Sigma, \tilde{f}_k^*(g_{\text{euc}}))} = \\
= \lim_{k \to \infty} \langle \nabla_{L^2} W(\tilde{f}_k), \varphi \rangle_{L^2(\Sigma, \tilde{f}_k^*(g_{\text{euc}}))} = \langle q, \varphi \rangle_{L^2(\Sigma, \tilde{f}^*(g_{\text{euc}}))}
\]

\( \forall \varphi \in C^\infty(\Sigma, \mathbb{R}^3) \), where the function \( q \) is of class \( L^2((\Sigma, g_{\text{poin}}), \mathbb{R}^3) \) by (85). This implies that \( \nabla_{L^2} W(\tilde{f}) \) is here not only a distribution of second order acting on \( C^\infty(\Sigma, \mathbb{R}^3) \), but it can be identified with an \( \mathbb{R}^3 \)-valued function of class \( L^2((\Sigma, g_{\text{poin}})) \). We can therefore apply here Theorem 3.1 to the uniformly conformal \((W^{2,2} \cap W^{1,\infty})\)-immersion \( \tilde{f} \) and conclude, that \( \tilde{f} \) is actually of class \( W^{4,2}((\Sigma, g_{\text{poin}}), \mathbb{R}^3) \), hence that \( f = \mathcal{P}^{-1} \circ \tilde{f} \) is of class \( W^{4,2}((\Sigma, g_{\text{poin}}), \mathbb{R}^4) \).

\[ \square \]

5 Dimension-reduction of the Möbius-invariant Willmore flow and proof of Theorem 1.2

The basic ingredient of the approach to Theorem 1.2 is obviously the Hopf-fibration

\[ S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2 \]

and its equivariance w.r.t. rotations on \( S^3 \) and \( S^2 \) (see formula (88) below) and also w.r.t. the first variation of the Willmore-energy along Hopf-tori in \( S^3 \) and closed curves in \( S^2 \) (see formula (102) below). In order to work with the most effective formulation of the Hopf-fibration, we consider \( S^3 \) as the subset of the four-dimensional \( \mathbb{R} \)-vector space \( \mathbb{H} \) of quaternions, whose elements have length \( 1 \), i.e.

\[ S^3 := \{ q \in \mathbb{H} | \bar{q} \cdot q = 1 \}. \]

We shall use the usual notation for the generators of the division algebra \( \mathbb{H} \), i.e. \( 1, i, j, k \). We therefore decompose every quaternion in the way

\[ q = q_1 + iq_2 + jq_3 + kq_4, \]
for unique “coordinates" $q_1, q_2, q_3, q_4 \in \mathbb{R}$, such that in particular there holds
\[ \bar{q} = q_1 - i q_2 - j q_3 - k q_4. \]
Moreover, we identify
\[ S^2 = \{ q \in \text{span}\{1, j, k\} | \bar{q} \cdot q = 1 \} = S^3 \cap \text{span}\{1, j, k\}, \]
and use the particular involution $q \mapsto \bar{q}$ on $\mathbb{H}$, which fixes the generators $1, j$ and $k$, but sends $i$ to $-i$. Following [35], we employ this involution to write the Hopf-fibration in the elegant way
\[ \pi : \mathbb{H} \rightarrow \mathbb{H}, \quad q \mapsto \bar{q} \cdot q, \tag{87} \]
for $q \in \mathbb{H}$. We shall remember here its most important, easily proved properties from Lemma 2.1 in [18] in the following lemma, without proof.

**Lemma 5.1.**
1) $\pi(S^3) = S^2$.

2) $\pi(e^{i \varphi} q) = \pi(q), \ \forall \varphi \in \mathbb{R} \text{ and } \forall q \in S^3$.

3) Moreover, there holds
\[ \pi(q \cdot r) = \bar{r} \cdot \pi(q) \cdot r \quad \forall q, r \in S^3. \tag{88} \]

Formula (88) means that right multiplication on $S^3$ translates equivariantly via the Hopf-fibration to rotation in $S^2$, since the Lie-group $S^3$ acts isometrically on $S^2$ in the following way: every $r \in S^3$ induces the rotation
\[ q \mapsto \bar{r} \cdot q \cdot r, \quad \text{for } q \in S^2. \tag{89} \]

4) The differential of $\pi$ in any $q \in \mathbb{H}$, applied to some $v \in \mathbb{H}$, reads
\[ D\pi_q(v) = \bar{v} \cdot q + \bar{q} \cdot v. \]

By means of the Hopf-fibration we introduce Hopf-tori in the following definition (see also [35] for further explanations):

**Definition 5.1.**
1) Let $\gamma : [a, b] \rightarrow S^2$ be a regular, closed $C^\infty$-curve in $S^2$, and let $\eta : [a, b] \rightarrow S^3$ be a smooth lift of $\gamma$ w.r.t. $\pi$ into $S^3$, i.e. a smooth map from $[a, b]$ into $S^3$ satisfying $\pi \circ \eta = \gamma$. We define
\[ X(s, \varphi) := e^{i \varphi} \cdot \eta(s), \quad \forall (s, \varphi) \in [a, b] \times [0, 2\pi], \tag{90} \]
and note that $(\pi \circ X)(s, \varphi) = \gamma(s), \ \forall (s, \varphi) \in [a, b] \times [0, 2\pi]$.

2) We call this map $X$ the “Hopf-torus-immersion" and its image respectively $\pi^{-1}(\text{trace}(\gamma))$ the “smooth Hopf-torus" in $S^3$ w.r.t. the smooth curve $\gamma$.

In order to compute the position of the projection of the conformal structure of a given Hopf-torus into the moduli space in terms of its profile curve $\gamma$, we introduce “abstract Hopf-tori":
Definition 5.2. Let $\gamma : [0, L/2] \to S^2$ be a path with constant speed 2 which traverses a simple closed smooth curve in $S^2$ of length $L > 0$ and encloses the area $A$ in the sense of Definition 5.2. Its associated Hopf-torus $\pi^{-1}(\text{trace}(\gamma)) \subset S^3$ endowed with the Euclidean metric of the ambient space $\mathbb{R}^4$ is conformally equivalent to its corresponding “abstract Hopf-torus” $C/\Gamma_{\gamma}$ in the sense of Definition 5.2. In particular, the projection of the point $(A/4\pi, L/4\pi) \in \mathbb{H}$ into the moduli space $\mathbb{H}/\text{PSL}_2(\mathbb{Z})$ yields exactly the conformal class of $(\pi^{-1}(\text{trace}(\gamma)), \text{euc})$ interpreted as a Riemann surface.

Proposition 5.1. Let $\gamma : S^1 \to S^2$ be a simple closed smooth curve of length $L$, which encloses the area $A$ in the sense of Definition 5.2. Its associated Hopf-torus $\pi^{-1}(\text{trace}(\gamma)) \subset S^3$ endowed with the Euclidean metric of the ambient space $\mathbb{R}^4$ is conformally equivalent to its corresponding “abstract Hopf-torus” $C/\Gamma_{\gamma}$ in the sense of Definition 5.2. In particular, the projection of the point $(A/4\pi, L/4\pi) \in \mathbb{H}$ into the moduli space $\mathbb{H}/\text{PSL}_2(\mathbb{Z})$ yields exactly the conformal class of $(\pi^{-1}(\text{trace}(\gamma)), \text{euc})$ interpreted as a Riemann surface.

Remark 5.1. On account of Proposition 5.1 the conformal structures $[\pi^{-1}(\text{trace}(\gamma))]$ and $[M\gamma]$ induced by $\pi^{-1}(\text{trace}(\gamma)) \subset S^3$ and $M\gamma \subset C$ coincide, and they lie in some prescribed compact subset $K$ of the moduli space $\mathcal{M}_1$, if and only if the pair of “moduli” $(A/2, L/2)$ of $M\gamma$ is situated at sufficiently large distance to the boundary of $\mathbb{H}$, i.e. if and only if the length of $\gamma$ is bounded from above and from below by two appropriate positive numbers $R_1(K), R_2(K)$. We will return to this fact in Proposition 5.6 below.

In order to rule out inappropriate parametrizations of Hopf-tori along the flow lines of the MIWF, we shall introduce “topologically simple” maps between tori in the following definition, see also Definition 1.3 and Remark 1.3 in [18].

Definition 5.3. 1) Let $\Sigma_1$ and $\Sigma_2$ be two compact tori. We term a continuous map $F : \Sigma_1 \to \Sigma_2$ simple, if it has mapping degree $\pm 1$, i.e. if the induced map

$$(F \circ \delta) : H_2(\Sigma_1, \mathbb{Z}) \xrightarrow{\cong} H_2(\Sigma_2, \mathbb{Z})$$

is an isomorphism between these two singular homology groups in degree 2.

2) Moreover, we denote by

$$\mathcal{H}(\Sigma, S^3) := \{ F \in C^\infty(\text{Imm}(\Sigma, S^3)) \mid F \text{ maps the torus } \Sigma \text{ simply onto } \pi^{-1}(\text{trace}(\gamma)), \text{ for some } \gamma \in C^\infty_{\text{reg}}(S^1, S^2) \}$$

the set of $C^\infty$-smooth “Hopf-torus immersions” mapping some compact smooth torus $\Sigma$ simply onto a smooth Hopf-torus $\pi^{-1}(\text{trace}(\gamma))$ in $S^3$. This set corresponds via stereographic projection of $S^3 \setminus \{(0,0,0,1)\}$ onto $\mathbb{R}^3$ to the set $\mathcal{H}(\Sigma, \mathbb{R}^3)$ of $C^\infty$-smooth “Hopf-torus-immersions”, mapping some compact torus $\Sigma$ simply onto a smooth “Hopf-torus in $\mathbb{R}^3$.”

In the next two propositions we recall some basic differential geometric formulae from Propositions 3.1 and 3.2 in [18], which particularly yield the useful correspondence between the MIWF and the variant (105) of the elastic energy flow in Proposition 5.4 below. The proofs are either straightforward or can be found in [18].
Proposition 5.2. The $L^2$-gradient of the elastic energy $\mathcal{E}(\gamma) := \int_{\gamma} 1 + |\vec{\kappa}|^2 \, d\mu_{\gamma}$, with $\vec{\kappa}$ as in (94) below, evaluated in an arbitrary closed curve $\gamma \in C^\infty_{\text{reg}}(S^1, S^2)$, reads exactly:

$$\nabla_{L^2} \mathcal{E}(\gamma)(x) = 2 \left( \nabla_{\gamma'} \right)^2 (\vec{\kappa}(\gamma))(x) + |\vec{\kappa}(\gamma)|^2 (x) \vec{\kappa}(\gamma)(x) + \vec{\kappa}(\gamma)(x), \quad \text{for } x \in S^1. \quad (91)$$

Using the abbreviation $\partial_s \gamma := \frac{\partial \gamma}{|\partial \gamma|}$ for the partial derivative of $\gamma$ normalized by arc-length, the leading term on the right hand side of equation (91) reads:

$$\left( \nabla_{\gamma'} \right)^2 (\vec{\kappa}(\gamma))(x) = \left( \nabla_{\delta \gamma} \right)^2 ((\partial_s \gamma)(x) - \langle \gamma(x), \partial_s \gamma(x) \rangle \gamma(x)) = \left( \partial_s \right)^4 \langle \gamma(x) \rangle - \langle \partial_s \rangle^2 \langle \gamma(x) \rangle \partial_s \gamma(x) + \langle |\nabla_{\delta \gamma}|^2 \rangle \langle \gamma(x) \rangle. \quad (92)$$

The fourth normalized derivative $(\partial_s)^4 \langle \gamma \rangle$ is non-linear w.r.t. $\gamma$, and at least its leading term can be computed in terms of ordinary partial derivatives of $\gamma$:

$$(\partial_s)^4 \langle \gamma \rangle = \frac{(\partial_s)^4 \langle \gamma \rangle}{|\partial_s \gamma|^4} - \frac{1}{|\partial_s \gamma|^4} \left( \langle \partial_s \rangle^4 \langle \gamma \rangle, \frac{\partial_s \gamma}{|\partial_s \gamma|} \right) \frac{\partial_s \gamma}{|\partial_s \gamma|} + \frac{1}{|\partial_s \gamma|^4} \langle \partial_s \rangle^2 \langle \gamma \rangle \cdot \partial_s \gamma \cdot \partial_s \gamma$$

+ rational expressions which only involve $(\partial_s)^2 \langle \gamma \rangle$ and $\partial_s \gamma$, (93)

where $C : \mathbb{R}^6 \rightarrow \text{Mat}_{3,3}(\mathbb{R})$ is a Mat$_{3,3}(\mathbb{R})$-valued function, whose components are rational functions in $(y_1, \ldots, y_6) \in \mathbb{R}^6$. □

Proposition 5.3. Let $F : \Sigma \rightarrow S^3$ be an immersion which maps the compact torus $\Sigma$ simply onto some Hopf-torus in $S^3$, and let $\gamma : S^1 \rightarrow S^2$ be a smooth regular parametrization of the closed curve $\text{trace}(\pi \circ F)$, which performs exactly one loop through its trace. Let moreover

$$\vec{\kappa} := \vec{\kappa} := \vec{\kappa} \equiv \vec{\kappa} := := \frac{1}{|\gamma|^2} (\gamma \cdot \nu_{\gamma} \cdot \nu_{\gamma})$$

be the curvature vector along the curve $\gamma$, for a unit normal field $\nu_{\gamma} \in \Gamma(TS^2)$ along the trace of $\gamma$, and $\kappa_{\gamma} := (\vec{\kappa}, \nu_{\gamma})_{\mathbb{R}^3}$ the signed curvature along $\gamma$. Then there is some $\varepsilon = \varepsilon(F, \gamma) > 0$ such that for an arbitrarily fixed point $x^* \in S^1$ the following differential-geometrical formulae hold for the immersion $F$:

$$A_{F,S^3}(\eta_F(x)) = N_F(\eta_F(x)) \begin{pmatrix} 2\kappa_{\gamma}(x) & 1 \\ 1 & 0 \end{pmatrix} \quad (95)$$

where $\eta_F : S^1 \cap B_{\varepsilon}(x^*) \rightarrow \Sigma$ denotes an arbitrary horizontal smooth lift of $\gamma|_{S^1 \cap B_{\varepsilon}(x^*)}$ w.r.t. the fibration $\pi \circ F$, as introduced in Lemma 7.1, and $N_F$ denotes a fixed unit normal field along the immersion $F$. This implies

$$\tilde{H}_{F,S^3}(\eta_F(x)) = \text{trace} A_{F,S^3}(\eta_F(x)) = 2\kappa_{\gamma}(x) N_F(\eta_F(x))$$

$\forall x \in S^1 \cap B_{\varepsilon}(x^*)$, for the mean curvature vector of $F$ and also

$$A_{F,S^3}^0(\eta_F(x)) = N_F(\eta_F(x)) \begin{pmatrix} \kappa_{\gamma}(x) & 1 \\ 1 & -\kappa_{\gamma}(x) \end{pmatrix} \quad (96)$$

$\forall x \in S^1 \cap B_{\varepsilon}(x^*)$, for the mean curvature vector of $F$ and also
and \( |A^{0}_{F,S^3}|^2(\eta_F(x)) = 2(\kappa_\gamma(x))^2 + 1 \),

\[
Q(A^{0}_{F,S^3})(\tilde{H}_{F,S^3})(\eta_F(x)) = (A^{0}_{F,S^3})_{ij}(\eta_F(x))(A^{0}_{F,S^3})_{ij}(\eta_F(x)), \tilde{H}_{F,S^3}(\eta_F(x))
= 4(\kappa_\gamma^3(x) + \kappa_\gamma(x)) N_F(\eta_F(x))
\]  

and finally

\[
\Delta \frac{1}{F}(\tilde{H}_{F,S^3})(\eta_F(x)) = 8 \left( \nabla_{\gamma'} \right)^2 (\kappa_\gamma)(x) N_F(\eta_F(x))
\]

and for the traced sum of all derivatives of \( A_F \) of order \( k \in \mathbb{N} \):

\[
||\nabla_{L^2}\mathcal{W}(F)(\eta_F(x))||^k = 2^{2k} \left| \left( \nabla_{\gamma'} \right)^{k}(\kappa_\gamma) \right|^2
\]  

\( \forall \, x \in S^1 \cap B_{\varepsilon}(x^\star) \). In particular, we derive

\[
\nabla_{L^2}\mathcal{W}(F)(\eta_F(x)) = 2 \left( 2 \left( \nabla_{\gamma'} \right)^2 (\kappa_\gamma)(x) + \kappa_\gamma^3(x) + \kappa_\gamma(x) \right) N_F(\eta_F(x)),
\]

and the “Hopf-Willmore-identity”:

\[
D\pi_F(\eta_F(x)) \cdot (\nabla_{L^2}\mathcal{W}(F)(\eta_F(x))) = 4 \left( 2 \left( \nabla_{\gamma'} \right)^2 (\kappa_\gamma) + |\kappa_\gamma| |\tilde{\kappa}_\gamma| + \tilde{\kappa}_\gamma \right)(x)
\equiv 4 \nabla_{L^2}\mathcal{E}(\gamma)(x)
\]

\( \forall \, x \in S^1 \cap B_{\varepsilon}(x^\star) \), where there holds \( \pi \circ F \circ \eta_F = \gamma \) on \( S^1 \cap B_{\varepsilon}(x^\star) \), as in Lemma 7.1 below. Finally, we have

\[
\mathcal{W}(F) = \int_{\Sigma} 1 + \frac{1}{4} |\tilde{H}_{F,S^3}|^2 \, d\mu_F = \pi \int_{S^1} 1 + |\kappa_\gamma|^2 \, d\mu_\gamma = \pi \mathcal{E}(\gamma),
\]

and

\[
\int_{\Sigma} \frac{1}{|A^{0}_{F,S^3}|^2} |\nabla_{L^2}\mathcal{W}(F)|^2 \, d\mu_F = \frac{1}{4} \int_{\Sigma} \frac{1}{|A^{0}_{F,S^3}|^4} \left| \Delta \frac{1}{F}(\tilde{H}_{F,S^3}) + Q(A^{0}_{F,S^3})(\tilde{H}_{F,S^3}) \right|^2 \, d\mu_F =
\]

\[
= \pi \int_{S^1} \frac{1}{(\kappa_\gamma^2 + 1)^2} 2 \left( \nabla_{\gamma'} \right)^2 (\kappa_\gamma) + |\kappa_\gamma|^2 |\tilde{\kappa}_\gamma| + \tilde{\kappa}_\gamma \right|^2 \, d\mu_\gamma
\]

\[
\equiv \pi \int_{S^1} \frac{1}{(\kappa_\gamma^2 + 1)^2} |\nabla_{L^2}\mathcal{E}(\gamma)|^2 \, d\mu_.
\]

Now we arrive at the main result of this section, a reduction of the MIWF to some “degenerate variant” of the classical elastic energy flow on \( S^2 \) by means of the Hopf-fibration:

**Proposition 5.4.** Let \([0,T] \subset \mathbb{R} \) be a non-void compact interval, and let \( \gamma_t : S^1 \rightarrow S^2 \) be a smooth family of closed smooth regular curves, \( t \in [0,T] \), which traverse their traces in \( S^2 \) exactly once, for every \( t \in [0,T] \). Moreover, let \( F_t : \Sigma \rightarrow S^3 \) be an arbitrary smooth family of smooth immersions, which map some compact smooth torus \( \Sigma \) simply onto the Hopf-tori \( \pi^{-1}(\text{trace}(\gamma_t)) \subset S^3 \), for every \( t \in [0,T] \). Then the following statement holds:

The family of immersions \( \{ F_t \} \) moves according to the MIWF-equation (1) on \([0,T] \times \Sigma \)
- up to smooth, time-dependent reparametrizations $\Phi_t$ with $\Phi_0 = \text{id}_\Sigma$ - if and only if there is a smooth family $\sigma_t: S^1 \to S^1$ of reparametrizations with $\sigma_0 = \text{id}_{S^1}$, such that the family $\{\gamma_t \circ \sigma_t\}$ satisfies the “elastic energy evolution equation”

$$\partial_t \tilde{\gamma}_t = -\frac{1}{(K^2_{\gamma_t} + 1)^2} \left( 2 \left( \nabla \frac{\gamma_t}{|\gamma_t|} \right)^2 (\tilde{K}_{\gamma_t}) + |\tilde{K}_{\gamma_t}|^2 \tilde{K}_{\gamma_t} + \tilde{K}_{\gamma_t} \right) \equiv -\frac{1}{(K^2_{\gamma_t} + 1)^2} \nabla_{L^2} E(\tilde{\gamma}_t) \quad (105)$$

on $[0, T] \times S^1$, where $\nabla_{L^2} E$ denotes the $L^2$-gradient of the elastic energy $E$, as above in Proposition 5.2.

**Proof.** The proof is built on Propositions 5.2 and 5.3 and is essentially a copy of the proof of the corresponding Proposition 3.3 in [18], modulo only minor modifications.

Moreover, we will need the following simple short-time existence and uniqueness result for smooth flow lines of the elastic energy flow (105):

**Proposition 5.5.** Let $\gamma_0: S^1 \to S^2$ be a $C^\infty$-smooth, closed and regular curve. Then there is some small $T > 0$ and a $C^\infty$-smooth solution $\{\gamma_t\}$ of the elastic energy flow (105) on $S^1 \times [0, T]$.

**Proof.** The proof works exactly as the proof of Theorem 3.1 in [9], where short time existence and uniqueness of the classical elastic energy flow

$$\partial_t \gamma_t = -\left( 2 \left( \nabla \frac{\gamma_t}{|\gamma_t|} \right)^2 (\tilde{K}_{\gamma_t}) + |\tilde{K}_{\gamma_t}|^2 \tilde{K}_{\gamma_t} + \tilde{K}_{\gamma_t} \right) \equiv -\nabla_{L^2} E(\gamma_t) \quad (106)$$

for smooth curves $\gamma_t: S^1 \to S^2$ is proved by means of a concrete, stereographic chart from $\mathbb{R}^2$ onto $S^2 \setminus \{(0, 0, 1)\}$ and by means of normal representation $\tilde{\gamma}_t := \gamma_0 + u_t N_{\gamma_0}$ of the projected plane curves $\tilde{\gamma}_t$ w.r.t. the projected plane initial curve $\gamma_0: S^1 \to \mathbb{R}^2$. The only difference here is the additional factor $\frac{1}{(K^2_{\gamma_t} + 1)^2}$ arising in (105) in front of the right hand side of (106). Since the curvature vector $\tilde{K}_{\gamma_0}$ of the projected curve $\tilde{\gamma}_t = \gamma_0 + u_t N_{\gamma_0}$ is explicitly computed in the proof of Theorem 3.1 in [9] and since the factor $\frac{1}{(K^2_{\gamma_0} + u_t N_{\gamma_0})^2 + 1)^2}$ - now to be multiplied with the right hand side of equation (3.1) in [9] - is bounded from above by 1 and from below by $\frac{1}{2}$ on $S^1 \times [0, T]$ for every “perturbation” $\{u_t\}$ which is sufficiently small in $C^{2+\alpha/2}(S^1 \times [0, T], \mathbb{R}^2)$, the final argument of the proof of Theorem 3.1 in [9] - employing linearization of the quasilinear parabolic differential equation (3.1) in [9] and parabolic Schauder theory - remains here exactly the same.

In the following proposition we collect the most fundamental information about flow lines of evolution equation (105), of our variant of the elastic energy flow.

**Proposition 5.6.** Let $\{\gamma_t\}_{t \in (0, T)}$, with either $T > 0$ or $T = \infty$, be a flow line of evolution equation (105), starting in a smooth closed curve $\gamma_0: S^1 \to S^2$ with elastic energy $E_0 := E(\gamma_0) < 8$. Then the following statements hold:

1) The elastic energy $E(\gamma_t)$ along the flow line $\{\gamma_t\}$ is monotonically decreasing and stays strictly smaller than 8,

2) the curves $\gamma_t$ are smooth embeddings of $S^1$ into $S^2$,
3) the lengths of the curves $\gamma_t$ are uniformly bounded from above by $E_0$ and from below by $\pi$, 

4) the areas of the domains $\Omega_t$ lying on the left hand sides of the embedded curves $\gamma_t$ in $S^2$ - see Definition 5.2 - are bounded from below by $2(\pi - 2)$ and from above by $2(\pi + 2)$, 

5) the conformal structures induced by the Euclidean metric of $\mathbb{S}^3$ restricted to the Hopf-tori $\pi^{-1}(\text{trace}(\gamma_t))$ lie in a compact subset of the moduli space $\mathcal{M}_1 \cong \mathbb{H}/\text{PSL}_2(\mathbb{Z})$, for every $t \in [0, T)$.

Proof. First of all, we know that any flow line $\{\gamma_t\}$ of equation (105) satisfies:

$$\frac{d}{dt}E(\gamma_t) = \langle \nabla_{L^2}E(\gamma_t), \partial_t\gamma_t \rangle_{L^2(\mu_{\gamma_t})} = -\int_{\mathbb{S}^1} \frac{1}{(\kappa^2_{\gamma_t} + 1)^2} \left| 2 \left( \nabla^\perp_{\gamma_t/|\gamma_t|} \kappa_{\gamma_t} \right)^2 (\kappa_{\gamma_t} + |\kappa_{\gamma_t}|) + |\kappa_{\gamma_t}|^2 \kappa_{\gamma_t}^2 \right| d\mu_{\gamma_t} \leq 0 \quad (107)$$

for every $t \in [0, T]$, which proves already the first assertion, taking here also $E(\gamma_0) < \pi$ into account. Combining this with formula (103) and with the Li-Yau-inequality, we obtain that the corresponding Hopf-tori $\pi^{-1}(\text{trace}(\gamma_t))$ are embedded surfaces in $\mathbb{S}^3$, implying that their profile curves $\gamma_t$ have to map $\mathbb{S}^1$ injectively into $\mathbb{S}^2$, thus implying that $\gamma_t$ have to be smooth embeddings, as well. As for the third assertion, we infer from (107) in particular the two inequalities:

$$\text{length}(\gamma_t) \leq E(\gamma_0), \quad (108)$$
$$\int_{\mathbb{S}^1} |\kappa_{\gamma_t}|^2 d\mu_{\gamma_t} \leq E(\gamma_0), \quad (109)$$

for every $t \in [0, T]$. Applying the elementary inequality

$$\left( \int_{\mathbb{S}^1} |\kappa| d\mu_{\gamma_t} \right)^2 \geq 4\pi^2 - \text{length}(\gamma)^2,$$

which holds for closed smooth regular paths $\gamma : \mathbb{S}^1 \to \mathbb{S}^2$, proved by Teufel [46], one can easily derive the lower bound

$$\text{length}(\gamma) \geq \min \left\{ \pi, \frac{3\pi^2}{E(\gamma)} \right\}, \quad (110)$$

see Lemma 2.9 in [9], for any closed smooth regular path $\gamma : \mathbb{S}^1 \to \mathbb{S}^2$. In combination with the monotonicity of $E(\gamma_t)$ in (107) and with the requirement that $E(\gamma_0) < \pi$ we obtain:

$$\text{length}(\gamma_t) \geq \min \left\{ \pi, \frac{3\pi^2}{E(\gamma_0)} \right\} = \pi, \quad (111)$$

for every $t \in [0, T]$, which proves the third assertion. The fourth assertion of the proposition now follows from Gauss-Bonnet’s Theorem for simply connected subdomains $\Omega$ of $\mathbb{S}^2$ with smooth boundary $\partial \Omega$:

$$\mathcal{H}^2(\Omega) + \int_{\partial \Omega} \kappa_{\partial \Omega} d\mathcal{H}^1 = 2\pi,$$
yielding here together with Cauchy-Schwarz' inequality, with the first statement of the proposition and with $\mathcal{E}_0 < 8$:

$$\mathcal{H}^2(\Omega_t) \geq 2\pi - \int_{\mathbb{S}^1} |\kappa_{\gamma_t}| \, d\mu_{\gamma_t} \geq 2\pi - \frac{\mathcal{E}_0}{2} > 2(\pi - 2),$$

and similarly also:

$$\mathcal{H}^2(\Omega_t) \leq 2\pi + \int_{\mathbb{S}^1} |\kappa_{\gamma_t}| \, d\mu_{\gamma_t} \leq 2\pi + \frac{\mathcal{E}_0}{2} < 2(\pi + 2),$$

for every $t \in [0, T]$. Now, the last assertion of the proposition about the conformal structures induced by the Euclidean metric of $\mathbb{S}^3$ restricted to Hopf-tori $\pi^{-1}({\text{trace}}(\gamma_t))$ follows immediately from the third statement of the proposition combined with Remark 5.1. $\square$

**Proof of Theorem 1.2:**

1) By means of Proposition 5.4 and on account of the uniqueness of classical flow lines both of the MIWF in $\mathbb{S}^3$ and of the elastic energy flow (105) in $\mathbb{S}^2$ one can easily prove, e.g. as in the proof of the first part of Theorem 2.1 in [18] treating the classical Willmore flow, that the unique classical flow line $\{F_t\}$ of the MIWF has to consist of smooth parametrizations of Hopf-tori in $\mathbb{S}^3$, whenever it starts moving in a smooth and simple parametrization $F_0$ of a Hopf-torus in $\mathbb{S}^3$, and that - via projection by the Hopf-fibration - the flow line $\{F_t\}$ yields a smooth flow line $\{\gamma_t\}_{t \in [0, T_{\text{max}}(F_0))]}$ of the elastic energy flow (105), starting to move in a smooth closed curve $\gamma_0 : \mathbb{S}^1 \rightarrow \mathbb{S}^2$, whose traces satisfy: $\text{trace}(\gamma_t) = \pi(F_t(\Sigma))$ for every $t \in [0, T_{\text{max}}(F_0))]$. Now, in order to prove the first statement of Theorem 1.2, we consider the arbitrarily chosen sequence $t_j \nearrow T_{\text{max}}(F_0)$ of the statement of the theorem, and we recall that by statement (9) there is at least a subsequence $\{F_{t_{j_t}}\}$ of $\{F_{t_j}\}$ and some corresponding integral, 2-rectifiable varifold $\mu$ in $\mathbb{R}^4$, such that

$$\mathcal{H}^2[F_{t_{j_t}}(\Sigma)] \rightharpoonup \mu \quad \text{weakly as Radon measures on } \mathbb{R}^4,$$  \hspace{1cm} (112)

as $l \nearrow \infty$. Moreover, as mentioned above, for every $t \in [0, T_{\text{max}}(F_0))$ the image $F_t(\Sigma)$ in $\mathbb{S}^3$ is a Hopf-torus, and the projection of these tori via the Hopf-fibration yields closed and smooth curves in $\mathbb{S}^2$, which can be appropriately parametrized by smooth paths $\gamma_t : \mathbb{S}^1 \rightarrow \mathbb{S}^2$, such that the family $\{\gamma_t\}_{t \in [0, T_{\text{max}}(F_0))]$ is a classical flow line of the elastic energy flow (105), starting with initial elastic energy $\mathcal{E}_0 = \mathcal{E}(\gamma_0) < 8$ on account of formula (103) and due to $\mathcal{W}(F_0) < 8\pi$. We may therefore apply here Proposition 5.6 and infer, that the curves $\gamma_t : \mathbb{S}^1 \rightarrow \mathbb{S}^2$ are smooth embeddings, whose lengths are uniformly bounded from below and from above, and that they enclose areas on $\mathbb{S}^2$ in the sense of Definition 5.2, whose $\mathcal{H}^2$-measures are bounded from below by the positive number $2(\pi - 2)$, for every $t \in [0, T_{\text{max}}(F_0))$. This implies first of all, that the diameters of the Hopf-tori $F_{t_{j_t}}(\Sigma) = \pi^{-1}(\text{trace}(\gamma_{t_{j_t}}))$ - interpreted as subsets of $\mathbb{R}^4$ - are bounded from below for every $t \in \mathbb{N}$. Hence, it follows as in the proof of Proposition 2.2 in [43], that the integral limit varifold $\mu$ in (112) satisfies $\mu \neq 0$. Therefore, we can infer from the first part of Theorem 1.1, that $\text{spt}(\mu)$ is an embedded, closed and orientable Lipschitz-surface in $\mathbb{S}^3$ either of genus 0 or of
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genus 1. Moreover, we know by weak lower semicontinuity of the Willmore energy \( W \) w.r.t. the convergence in (112) - see [42] - that
\[
W(\mu) := \frac{1}{4} \int_{\mathbb{R}^4} |\bar{H}_\mu|^2 \, d\mu \leq \liminf_{l \to \infty} \frac{1}{4} \int_{\mathbb{R}^4} |H_{F_{l\mu},\mathbb{R}^4}|^2 \, d\mu_{g^{F_{l\mu}}} \leq W(F_0) < 8\pi,
\]
and we can also infer from the fact that \( F_{l\mu}(\Sigma) \subset \mathbb{S}^3, \forall l \in \mathbb{N} \), and from (112) - as in the proof of the first part of Theorem 1.1 - that
\[
F_{l\mu} \rightarrow \text{spt}(\mu) \quad \text{as subsets of } \mathbb{R}^4 \text{ in Hausdorff distance, as } l \to \infty.
\]
Because of \( F_{l\mu}(\Sigma) = \pi^{-1}(\text{trace}(\gamma_{l\mu})) \), we can apply the Hopf-fibration to the convergence in (114) and infer that
\[
\text{trace}(\gamma_{l\mu}) \rightarrow \pi(\text{spt}(\mu)) \quad \text{as subsets of } \mathbb{R}^3 \text{ in Hausdorff distance, as } l \to \infty.
\]
Moreover, we note that for any smooth curve \( c : (-\varepsilon, \varepsilon) \rightarrow \mathbb{S}^2 \) we have the formula
\[
\langle N_{\mathbb{S}^2}(c(t)), c''(t) \rangle = I F_{\mathbb{S}^2}(c(t)), \forall t \in (-\varepsilon, \varepsilon),
\]
from elementary Differential Geometry, where \( N_{\mathbb{S}^2} \) and \( I F_{\mathbb{S}^2} \) denote the Gauss-map and the second fundamental form of the standard embedding \( \mathbb{S}^2 \hookrightarrow \mathbb{R}^3 \), respectively. Hence, requiring also that \( c'(t) \) has length one for \( t \in (-\varepsilon, \varepsilon) \), we see that the “normal component” \( \langle N_{\mathbb{S}^2}(c(t)), c''(t) \rangle \) of the curvature \( \kappa_{c'}^{\mathbb{S}^2}(t) \equiv c''(t) \) of the curve \( c \) - when considered as a path in \( \mathbb{R}^3 \) - is exactly given by \( I F_{\mathbb{S}^2}(c(t)) \) and thus equals 1, the only possible principle curvature of the standard unit sphere \( \mathbb{S}^2 \). We can therefore reformulate the elastic energy \( \mathcal{E} \) of any smooth closed curve \( \gamma : \mathbb{S}^1 \rightarrow \mathbb{S}^2 \) in the way:
\[
\mathcal{E}(\gamma) \equiv \int_{\mathbb{S}^1} 1 + |\kappa_{\gamma}^{\mathbb{S}^2}|^2 \, d\mu_\gamma = \int_{\mathbb{S}^1} |\kappa_{\gamma}^{\mathbb{R}^3}|^2 \, d\mu_\gamma,
\]
which is simply the standard elastic energy of a smooth closed curve in \( \mathbb{R}^3 \). Combining now formula (116) with the bounds (108) and (109), again Allard’s compactness theorem implies that the integral 1-varifolds \( \nu_l := \mathcal{H}^1|_{\text{trace}(\gamma_{l\mu})} \) converge weakly - up to extraction of another subsequence - to an integral 1-varifold \( \nu \) in \( \mathbb{R}^3 \):
\[
\mathcal{H}^1|_{\text{trace}(\gamma_{l\mu})} \rightharpoonup \nu \quad \text{weakly as Radon measures on } \mathbb{R}^3,
\]
as \( l \to \infty \). Now we also know, that the 1-dimensional Hausdorff-densities \( \theta^1(\nu_l) \) exist in every point of \( \mathbb{R}^3 \) and satisfy \( \theta^1(\nu_l) \geq 1 \) pointwise on \( \text{spt}(\nu_l) = \gamma_{l\mu} \), since \( \gamma_{l\mu} \) are closed and smooth curves. Hence, combining this with convergence (117), formula (116), and with estimates (108), (109), we see that the conditions of Proposition 9.2 below are satisfied by the integral 1-varifolds \( \nu_l = \mathcal{H}^1|_{\text{trace}(\gamma_{l\mu})} \) with \( n = 1, m = 2, \alpha = \frac{1}{2} \) and \( \beta = \frac{1}{4} \), and we infer first of all together with convergence (115):
\[
\pi(\text{spt}(\mu)) \leftarrow \text{trace}(\gamma_{l\mu}) = \text{spt}(\mathcal{H}^1|_{\text{trace}(\gamma_{l\mu})}) \rightarrow \text{spt}(\nu)
\]
as subsets of \( \mathbb{R}^3 \) in Hausdorff distance, as \( l \to \infty \), which implies in particular: \( \pi(\text{spt}(\mu)) = \text{spt}(\nu) \), and we infer furthermore:
\[
\text{spt}(\nu) = \{ x \in \mathbb{R}^3 \mid \exists x_l \in \text{spt}(\nu_l) \forall l \in \mathbb{N} \text{ such that } x_l \to x \}.
\]
Since \( \text{spt}(\mu) \) is already known to be either an embedded 2-sphere or an embedded compact torus, the obtained equation \( \pi(\text{spt}(\mu)) = \text{spt}(\nu) \) particularly shows, that \( \text{spt}(\nu) \) is a compact and path-connected subset of \( S^2 \). Moreover, since \( \nu \) is a 1-rectifiable varifold, one can easily derive from Theorem 3.2 in [44], that \( \nu \) coincides with the measure \( \theta^1(\nu) \cdot \mathcal{H}^1_{[\theta^1(\nu)>0]} \) on entire \( \mathbb{R}^3 \) and that \( [\theta^1(\nu) > 0] \) is a countably 1-rectifiable subset of \( \mathbb{R}^3 \), where \( \theta^1(\nu) \) denotes the upper 1-dimensional Hausdorff-density of \( \nu \), compare to Paragraph 3 in [44]. Since \( \nu \) is here additionally integral, we therefore infer especially:

\[
\mathcal{H}^1(A) = \int_A (\theta^1(\nu))^{-1} \cdot \theta^1(\nu) \, d\mathcal{H}^1 = \int_A (\theta^1(\nu))^{-1} \, d\nu \leq \nu(A) < \infty \tag{120}
\]

for all \( \mathcal{H}^1 \)-measurable subsets \( A \) of \( [\theta^1(\nu) > 0] \),

recalling that \( \nu \) is especially a Radon measure on \( \mathbb{R}^3 \), and that here the upper 1-dimensional density \( \theta^1(\nu) \) has to satisfy \( \theta^1(\nu) \geq 1 \) \( \nu \)-almost everywhere on \( \mathbb{R}^3 \). Moreover, again on account of convergence (117), formula (116), and estimates (108), (109), the conditions of Proposition 9.1 are satisfied by \( \nu_l = \mathcal{H}^1_{[\text{trace}(\gamma_l)]} \) with \( n = 1 \), \( m = 2 \), \( \alpha = \frac{1}{2} \) and \( \beta = \frac{1}{4} \), and we can conclude together with equation (119):

\[
\theta^1(\nu, x) \text{ exists and } \theta^1(\nu, x) \geq \limsup_{l \to \infty} \theta^1(\nu_l, x_l) \geq 1 \text{ for every } x \in \text{spt}(\nu), \tag{121}
\]

where we have chosen an arbitrary point \( x \in \text{spt}(\nu) \) and an appropriate sequence \( x_l \in \text{spt}(\nu_l) \) with \( x_l \to x \) in \( \mathbb{R}^3 \), according to equation (119), and where we have again used the obvious fact that \( \theta^1(\nu_l) \geq 1 \) pointwise on \( \text{spt}(\nu_l) = \gamma_{l_j} \) for every \( l \in \mathbb{N} \). Combining the obvious, general fact that \( [\theta^1(\xi) > 0] \) is contained in \( \text{spt}(\xi) \), for any rectifiable 1-varifold \( \xi \), with (121), we finally obtain:

\[
[\theta^1(\nu) \geq 1] \subset [\theta^1(\nu) > 0] \subset \text{spt}(\nu) \subset [\theta^1(\nu) \geq 1], \tag{122}
\]

proving that these three subsets of \( S^2 \) coincide with each other. Since the set \( [\theta^1(\nu) > 0] \) is already known to be countably 1-rectifiable, statement (122) proves in particular, that the compact set \( \pi(\text{spt}(\mu)) = \text{spt}(\nu) \) is actually a countably 1-rectifiable subset of \( S^2 \), which additionally has to have finite \( \mathcal{H}^1 \)-measure on account of (120), simply taking here \( A = \text{spt}(\nu) = [\theta^1(\nu) > 0] \). This shows especially that \( \text{spt}(\nu) \) cannot be a dense subset of \( S^2 \), because otherwise there would hold \( \text{spt}(\nu) = \text{spt}(\nu) = S^2 \) and therefore \( \mathcal{H}^2(\text{spt}(\nu)) = 4\pi \), which obviously contradicts the fact that \( \mathcal{H}^1(\text{spt}(\nu)) < \infty \). Hence, there has to be some point \( x_0 \in S^2 \) and some radius \( \rho > 0 \), such that \( \text{spt}(\nu) \subset S^2 \setminus B^3_\rho(x_0) \). Without loss of generality we may assume, that \( x_0 \) is exactly the north pole \((0,0,1)\). On account of convergence (118) we can thus infer, that the converging sets \( \text{trace}(\gamma_{l_j}) \) still have to be contained in \( S^2 \setminus B^3_\rho(x_0) \) for sufficiently large \( l \in \mathbb{N} \), say for every \( l \in \mathbb{N} \) without loss of generality. Hence, we can apply here stereographic projection \( P : S^2 \setminus \{x_0\} \to \mathbb{R}^2 \), \( (x,y,z) \mapsto \frac{1}{1-z}(x,y) \), and thus map all the sets \( \text{trace}(\gamma_{l_j}) \) stereographically onto closed planar curves with smooth and regular parametrizations \( \gamma_{l_j} := P(\gamma_{l_j}) \), which have to be contained in some compact subset \( K = K(\rho) \) of \( \mathbb{R}^2 \). Now, given any smooth closed curve \( \gamma : S^1 \to S^2 \setminus B^3_\rho(x_0) \) we can compare the pointwise values of
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its Euclidean curvature $|\kappa_\gamma^E|$ with the corresponding values of the Euclidean curvature $|\kappa^E_{\mathcal{P}(\gamma)}|$ of the stereographically projected curve $\mathcal{P}(\gamma) : \mathbb{S}^1 \to \mathbb{R}^2$. Hence, there is a constant $C = C(\varrho) > 0$, being independent of $\gamma$, such that:

$$|\kappa^E_{\mathcal{P}(\gamma)}| \leq C(\varrho) |\kappa_\gamma^E| \quad \text{pointwise on } \mathbb{S}^1,$$

provided there holds $\gamma : \mathbb{S}^1 \to \mathbb{S}^2 \setminus B_\varrho^3(x_0)$. Recalling now formula (116) and noting that there holds trivially also

$$|\partial_\nu(\mathcal{P}(\gamma))| \leq \tilde{C}(\varrho) |\partial_\nu\gamma| \quad \text{pointwise on } \mathbb{S}^1,$$

for any smooth closed curve $\gamma : \mathbb{S}^1 \to \mathbb{S}^2 \setminus B_\varrho^3(x_0)$, we finally arrive at the estimates:

$$\int_{\mathbb{S}^1} |\kappa^E_{\mathcal{P}(\gamma)}|^2 \, d\mu_{\mathcal{P}(\gamma)} \leq C^2(\varrho) \tilde{C}(\varrho) \int_{\mathbb{S}^1} |\kappa_\gamma^E|^2 \, d\mu_\gamma = C^2(\varrho) \tilde{C}(\varrho) \mathcal{E}(\gamma)$$

and also

$$\text{length}(\mathcal{P}(\gamma)) = \int_{\mathbb{S}^1} \text{length}(\gamma) \leq \tilde{C}(\varrho) \text{length}(\gamma) \leq \tilde{C}(\varrho) \mathcal{E}(\gamma),$$

for any smooth closed curve $\gamma : \mathbb{S}^1 \to \mathbb{S}^2 \setminus B_\varrho^3(x_0)$. Moreover, on account of the first point of Proposition 5.6, we know that the curves $\gamma_t$ - moving along the elastic energy flow (105) - satisfy:

$$\mathcal{E}(\gamma_t) = \int_{\mathbb{S}^1} \frac{1}{2} |\kappa_{\gamma_t}|^2 \, d\mu_{\gamma_t} \leq \mathcal{E}(\gamma_0) < \infty,$$

for every $t \in [0, T_{\max}(F_0))$. Recalling now that we proved above, that trace($\gamma_t$) is contained in $\mathbb{S}^2 \setminus B_\varrho^3(x_0)$ holds for every $l \in \mathbb{N}$, we can combine estimates (125), (126) and (127) and arrive at the estimates:

$$\int_{\mathbb{S}^1} |\kappa^E_{\mathcal{P}(\gamma_t)}|^2 \, d\mu_{\mathcal{P}(\gamma_t)} \leq 8 C^2(\varrho) \tilde{C}(\varrho)$$

and

$$\text{length}(\mathcal{P}(\gamma_t)) \leq 8 \tilde{C}(\varrho),$$

for every $l \in \mathbb{N}$. Since we also know that the traces of the projected curves $\mathcal{P}(\gamma_t)$ are contained in a compact subset $K(\varrho)$ of $\mathbb{R}^2$, we can apply Theorem 3.1 in [3] on account of the estimates in (128) and infer, that at least some subsequence of the projected paths $\mathcal{P}(\gamma_t)$ converges weakly in $W^{2,2}(\mathbb{S}^1, \mathbb{R}^2)$ and thus strongly in $C^1(\mathbb{S}^1, \mathbb{R}^2)$ to some closed limit curve $\gamma^* : \mathbb{S}^1 \to \mathbb{R}^2$, whose trace has to be contained again in the compact subset $K(\varrho)$ of $\mathbb{R}^2$. In combination with convergence (118), the trace of the closed path $\mathcal{P}^{-1}(\gamma^*)$ has to coincide with spt$(\nu)$, which proves that spt$(\nu)$ and thus $\pi(\text{spt}(\mu))$ is not only a compact and path-connected, countably 1-rectifiable subset of $\mathbb{S}^2$, but even the trace of a closed $C^1$-curve in $\mathbb{S}^2$. However, spt$(\mu)$ is already known to be either an embedded 2-sphere or an embedded compact torus. Hence, only the latter alternative can hold here, and therefore the compact torus spt$(\mu)$ is both an open and a closed subset of the $C^1$-Hopf-torus $\pi^{-1}(\pi(\text{spt}(\mu)))$. Hence, they have to exactly coincide: spt$(\mu) = \pi^{-1}(\pi(\text{spt}(\mu)))$, and spt$(\mu)$ has therefore turned out to be an embedded $C^1$-Hopf-torus, whose profile curve can be parametrized by
the closed $C^1$-path $\mathcal{P}^{-1}(\gamma^*)$. This shows in particular, that all requirements of the second part of Theorem 1.1 are satisfied here, so that all statements of the second part of Theorem 1.1 have to hold here for the considered sequence of embeddings $\{F_{t_j}\}$ and for their non-trivial limit varifold $\mu$ from (112). We can therefore especially infer from statement (11), that the embedded Hopf-torus $\text{spt}(\mu)$ possesses a uniformly conformal bi-Lipschitz parametrization $f : (\Sigma, g_{\text{poin}}) \overset{\cong}{\rightarrow} \text{spt}(\mu)$, for some zero scalar curvature metric $g_{\text{poin}}$ on $\Sigma$, with conformal factor $u \in L^\infty(\Sigma)$ bounded by some constant $\Lambda$ depending on the sequence $\{F_{t_j}\}$ from (112) and on $\mu$ - as explained in the proof of the second part of Theorem 1.1 - and with $\mathcal{W}(f) = \mathcal{W}(\mu) < 8\pi$ on account of formulae (13) and (113).

2) As in the first part of this proof of this theorem we are going to fix some arbitrarily chosen subsequence $\{t_{j_l}\}$ of $t_j \not\nearrow T_{\max}$, for which $\{F_{t_{j_l}}\}$ has the property in (112) with some 2-varifold $\mu$ in the limit. We know from the first part of this theorem, that $\mu$ has to be a non-trivial integral 2-varifold, whose support is an embedded Hopf-torus in $\mathbb{S}^3$, and that therefore all statements of the second part of Theorem 1.1 hold here for the appropriately chosen subsequence $\{F_{t_{j_l}}\}$. In particular, we know that the immersions $F_{t_{j_l}}$ can be reparametrized by smooth diffeomorphisms $\Phi_{j_l} : \Sigma \overset{\cong}{\rightarrow} \Sigma$ in such a way, that the reparametrizations $\tilde{F}_{t_{j_l}} := F_{t_{j_l}} \circ \Phi_{j_l}$ are uniformly conformal w.r.t. certain metrics $g_{\text{poin},j_l}$ of vanishing scalar curvature and with smooth conformal factors $u_{j_l}$, which are uniformly bounded in $L^\infty(\Sigma, g_{\text{poin}})$ and in $W^{1,2}(\Sigma, g_{\text{poin}})$, i.e. with:

$$\|u_{j_l}\|_{L^\infty(\Sigma, g_{\text{poin}})} \leq \Lambda, \quad \text{for every } l \in \mathbb{N},$$

(129)

and

$$\|u_{j_l}\|_{W^{1,2}(\Sigma, g_{\text{poin}})} \leq C(\Lambda), \quad \text{for every } l \in \mathbb{N}.$$  

(130)

Here the bound $\Lambda$ depends on the difference $8\pi - \mathcal{W}(F_0) > 0$ and on certain GMT-properties of $\mu$ and of the embedded surfaces $F_{t_{j_l}}(\Sigma) \subset \mathbb{S}^3$ following from the proofs of Proposition 2.1 in [43] and of Theorem 3.1 in [27] - as already mentioned in the proof of the second part of Theorem 1.1 - and $g_{\text{poin}}$ is a certain zero scalar curvature metric which satisfies on account of (51) - again up to extraction of a subsequence:

$g_{\text{poin},j_l} \rightarrow g_{\text{poin}}$ smoothly as $l \rightarrow \infty$, 

(131)

as explained below formula (49). Hence, we also infer the estimates

$$\|\tilde{F}_{t_{j_l}}\|_{W^{1,\infty}(\Sigma, g_{\text{poin}})} + \|\tilde{F}_{t_{j_l}}\|_{W^{2,2}(\Sigma, g_{\text{poin}})} \leq \text{Const}(\Lambda), \quad \text{for every } l \in \mathbb{N},$$

(132)

from the proof of the second part of Theorem 1.1, implying the existence of a particular subsequence $\{\tilde{F}_{t_{j_l}}\}$ of $\{\tilde{F}_{t_{j}}\}$, which converges in the three senses (58), (60) and (61) to the uniformly conformal bi-Lipschitz homeomorphism $f : (\Sigma, g_{\text{poin}}) \overset{\cong}{\rightarrow} \text{spt}(\mu)$ from the first part of this theorem. Since we aim to prove the desired estimate (16) for this particular subsequence $\{\tilde{F}_{t_{j_l}}\}$ of $\{\tilde{F}_{t_{j}}\}$, we should relabel here this subsequence $\{\tilde{F}_{t_{j_l}}\}$ again into $\{\tilde{F}_{t_{j}}\}$, just for ease of notation. Now combining estimates (129), (130) and (132) with convergence (131), we can proceed exactly as in the proof of Theorem 4.1 and of Proposition 5.2 in [34], in order to prove estimate (155) below, yielding quickly the desired estimate (16), for every $j \in \mathbb{N}$. First of all, comparing the assumptions of Theorem 4.1 in [34] with our knowledge in formulae
(129)–(132) it is worth mentioning, that we can exchange our estimate (130) by the slightly weaker estimate (134) below, which also follows from the equations

\[-\triangle_{g_{\text{poin}},j}(u_j) = e^{2u_j} K_{\tilde{F}^*_{tj}(g_{\text{euc}})} \text{ on } \Sigma\]  

in (52), together with the elementary estimates

\[\int_{\Sigma} |K_{\tilde{F}^*_{tj}(g_{\text{euc}})}| d\mu_{\tilde{F}^*_{tj}(g_{\text{euc}})} \leq \frac{1}{2} \int_{\Sigma} |A_{\tilde{F}^*_{tj}(g_{\text{euc}})}|^2 d\mu_{\tilde{F}^*_{tj}(g_{\text{euc}})} = 2 \mathcal{W}(F_{tj}) < 16 \pi, \; \forall \; j \in \mathbb{N},\]  

and with the statement of the fifth part of Proposition 5.6, as exactly pointed out in the proof of Theorem 5.4 in [41]:

\[\|\nabla^{\text{poin}}(u_j)\|_{L^2,\infty(\Sigma)} \leq C(g_{\text{poin}}) \mathcal{W}(F_{tj}) \leq C(g_{\text{poin}}) \mathcal{W}(F_0), \; \forall \; j \in \mathbb{N}.\]  

(134)

Here, \(L^{2,\infty}(\Sigma)\) denotes the Lorentz space as in Proposition 8.1 below, satisfying especially \(L^2(\Sigma, g_{\text{poin}}) \hookrightarrow L^{2,\infty}(\Sigma, g_{\text{poin}})\), see for example [47]. Now we use our hypothesis, that the mean curvature vectors \(\vec{H}_{\tilde{F}_{tj},S^3}\) - which we shall abbreviate here simply by \(\vec{H}_{\tilde{F}_{tj}}\) - of the embeddings \(F_{tj} : \Sigma \rightarrow S^3\) remain uniformly bounded for all \(j \in \mathbb{N}\), i.e. that there is some large number \(K\), such that \(\|\vec{H}_{\tilde{F}_{tj}}\|_{L^\infty(\Sigma)} \leq K\) holds for every \(j \in \mathbb{N}\). Since in our considered situation every immersion \(F_t\) parametrizes some Hopf-torus, for \(t \in [0, T_{\text{max}}(F_0))\), we can combine our uniform bound on \(\|\vec{H}_{\tilde{F}_{tj}}\|_{L^\infty(\Sigma)}\) with formulae (95) and (96) in Proposition 5.3 and infer, that the entire second fundamental forms \(A_{\tilde{F}_{tj},S^3}\) - which we shall abbreviate here simply by \(A_{F_{tj}}\) - of the embeddings \(F_{tj} : \Sigma \rightarrow S^3\) can be uniformly bounded:

\[\|A_{F_{tj}}\|_{L^2(\Sigma)}^2 \equiv \|g^{kl}_{\tilde{F}_{tj}} g_{\tilde{F}_{tj}}^{ik} \langle (A_{F_{tj}})_{ik}, (A_{F_{tj}})_{hl} \rangle_{\mathbb{R}^4} \|_{L^\infty(\Sigma)} \leq (K^2 + 2)\]  

for every \(j \in \mathbb{N}\). Since these scalars are invariant w.r.t. smooth reparametrizations of the embeddings \(F_{tj}\), statement (135) implies that also

\[\|A_{F_{tj}}\|_{L^2(\Sigma)}^2 \equiv \|g^{kl}_{\tilde{F}_{tj}} g_{\tilde{F}_{tj}}^{ik} \langle (A_{F_{tj}})_{ik}, (A_{F_{tj}})_{hl} \rangle_{\mathbb{R}^4} \|_{L^\infty(\Sigma)} \leq (K^2 + 2),\]  

(136)

for every \(j \in \mathbb{N}\). Now, on account of the uniform convergence (61) and on account of the conformal invariance of the flow (1), we can assume - as in (78) - that the images of the sequence \(\{\tilde{F}_{tj}\}\) are contained in \(S^3 \setminus B^4_{\delta}((0,0,0,1))\) for some \(\delta > 0\). We may therefore apply the stereographic projection \(\mathcal{P}\) to the entire sequence \(\{\tilde{F}_{tj}\}\) and obtain new embeddings \(f_j := \mathcal{P} \circ \tilde{F}_{tj} : \Sigma \rightarrow \mathbb{R}^3\), which are again uniformly conformal, i.e. satisfy as in (81):

\[f^*_j(g_{\text{euc}}) = e^{2u_j} g_{\text{poin},j} \text{ with } \|\tilde{u}_{ij}\|_{L^\infty(\Sigma)} < \check{\Lambda} \text{ and }\]  

\[\|\nabla^{g_{\text{poin}}}(\tilde{u}_{ij})\|_{L^2,\infty(\Sigma,g_{\text{poin}})} \leq \check{\Lambda} \text{ for every } j \in \mathbb{N},\]  

(137)

for every \(j \in \mathbb{N}\) and for some large constant \(\check{\Lambda}\), which depends on certain GMT-properties of the sequence \(\{F_{tj}\}\) in (112) and of the limit varifold \(\mu\) - as the constant \(\Lambda\) in (129) did - on \(g_{\text{poin}}\) and additionally on \(\delta\). Moreover, we also obtain the estimates in (132) for the new embeddings \(f_j : \Sigma \rightarrow \mathbb{R}^3\):

\[\|f_j\|_{W^{1,\infty}(\Sigma,g_{\text{poin}})} + \|f_j\|_{W^{2,2}(\Sigma,g_{\text{poin}})} \leq \text{Const}(\Lambda,\delta), \; \forall \; j \in \mathbb{N},\]  

(139)
and also estimate (136) for the new embeddings $f_j : \Sigma \rightarrow \mathbb{R}^3$:

$$\|A_{f_j}\|_{L^\infty(\Sigma)} \equiv \|g^{ik}_{f_j} (A_{f_j})_{ik}, (A_{f_j})_{kl}\|_{L^\infty(\Sigma)} \leq C(K, \delta)$$

for every $j \in \mathbb{N}$. Combining now (139) and (140), we finally infer, that for every small $\varepsilon_0 > 0$ there is some small $R_0 > 0$, depending on $\varepsilon_0$, $K$ and $\delta$, such that

$$\int_{B^\text{gpoint}_{R_0}(x_0)} |A_{f_j}|^2 \, d\mu_{f_j^*(g_{\text{euc}})} \leq \varepsilon_0, \quad \forall j \in \mathbb{N}$$

independently of $x_0 \in \Sigma$, where “$B^\text{gpoint}_{R_0}(x_0)$” denotes the open geodesic disc of radius $r$ about the center point $x_0$ in $\Sigma$ w.r.t. the fixed zero scalar curvature metric $g_{\text{point}}$. Now, we fix some $x_0 \in \Sigma$ arbitrarily, we consider some small $\varepsilon_0 > 0$ and some small $R_0 > 0$ as in (141), and we introduce isothermal charts $\psi_j : B^2_1(0) \xrightarrow{\cong} U_j(x_0) \subset B^\text{gpoint}_{R_0}(x_0)$ w.r.t. $g_{\text{point},j}$ on open neighborhoods $U_j(x_0)$ of $x_0$, for each $j \in \mathbb{N}$, as already above in (73). Hence, we consider here again harmonic and bounded functions $v_j$ on $B^2_1(0)$, such that

$$\psi_j^*g_{\text{point},j} = e^{2v_j} g_{\text{euc}} \text{ on } B^2_1(0), \quad \text{with } \|v_j\|_{L^\infty(B^2_1(0))} \leq C(\text{gpoint}) \quad \forall j \in \mathbb{N},$$

and with

$$\|\nabla^s(v_j)\|_{L^\infty(B^2_{1/s}(0))} \leq C(\text{gpoint}, s) \quad \forall j \in \mathbb{N},$$

and for each fixed $s \in \mathbb{N}$, we have again used convergence (131). In particular, the compositions $f_j \circ \psi_j : B^2_1(0) \rightarrow \mathbb{R}^3$ are uniformly conformal w.r.t. the Euclidean metric on $B^2_1(0)$:

$$(f_j \circ \psi_j)^*(g_{\text{euc}}) = \psi_j^*(e^{2\tilde{u}_j} g_{\text{point},j}) = e^{2\tilde{u}_j \circ \psi_j} \psi_j^*g_{\text{point},j} = e^{2\tilde{u}_j \circ \psi_j + 2v_j} g_{\text{euc}} \text{ on } B^2_1(0).$$

Now, using the isothermal charts $\psi_j : B^2_1(0) \xrightarrow{\cong} U_j(x_0)$ statement (141) implies:

$$\int_{B^2_1(0)} |A_{f_j \circ \psi_j}|^2 \, d\mu_{(f_j \circ \psi_j)^*(g_{\text{euc}})} \leq \varepsilon_0, \quad \text{for every } j \in \mathbb{N}.$$  

(144)

Since we use several different references on gauge theory in this proof, we mention here Section 5.1 in [14], explaining that condition (144) can also be formulated in terms of the usual Euclidean metric and the $L^2$-measure on $B^2_1(0) \subset \mathbb{R}^2$, taking equation (143) and the conformal invariance of the Dirichlet functional into account:

$$\int_{B^2_1(0)} |A_{f_j \circ \psi_j}|^2 \, d\mathcal{L}^2 \leq \varepsilon_0, \quad \text{for every } j \in \mathbb{N}.$$  

(145)

Now we aim at estimating $\|\nabla(\tilde{u}_j \circ \psi_j + v_j)\|_{L^2(B^2_1(0))}$, for sufficiently small $r \in (0, R_0)$, in terms of the controllable quantity $\varepsilon_0$ and the given upper bounds $K$ and $\Lambda$, for every $j \in \mathbb{N}$. To this end, we follow the proof of Theorem 5.5 in [41], and then we will combine it with estimates (137) and (145) and with Theorem 2.1 in [13], p. 78, on sufficiently small discs about 0 in $\mathbb{R}^2$. First of all, on account of estimate (145) we may apply “Hélein’s lifting theorem”, Theorem 4.2 in [41], in order to obtain pairs of functions $e^1_j, e^2_j \in W^{1,2}(B^2_1(0), S^2)$ which satisfy both:

$$N_j = e^1_j \times e^2_j \quad \text{on } B^2_1(0)$$

and

$$\int_{B^2_1(0)} |\nabla e^1_j|^2 + |\nabla e^2_j|^2 \, d\mathcal{L}^2 \leq C \int_{B^2_1(0)} |A_{f_j \circ \psi_j}|^2 \, d\mathcal{L}^2 \leq C \varepsilon_0, \quad \text{for every } j \in \mathbb{N}.$$  

(146)
having used here already estimate (145), where $N_j : \mathbb{B}_2^3(0) \rightarrow \mathbb{S}^2$ denote unit normals along the conformal embeddings $f_j \circ \psi_j$, and where $C$ is an absolute constant. Now we use Theorem 3.8 in [41] and estimate (146) and infer, that the unique weak solutions $\mu_j$ of the Dirichlet boundary value problems:

$$-\triangle_{euc}(\mu_j) = \sum_{k=1}^{3} \det(\nabla(e^1_j)_k, \nabla(e^2_j)_k) \equiv \star N_j^* \text{vol}_{\mathbb{S}^2} = e^{2\delta_j \circ \psi_j + 2\nu_j} K(f_j \circ \psi_j)^* \text{geuc} \quad \text{on} \ B_2^3(0)$$

and $\mu_j = 0$ on $\partial B_2^3(0)$, can be estimated in $W^{1,2}(\mathbb{B}_2^3(0)) \cap L^{\infty}(\mathbb{B}_2^3(0))$:

$$\| \mu_j \|_{L^{\infty}(\mathbb{B}_2^3(0))} + \| \mu_j \|_{W^{1,2}(\mathbb{B}_2^3(0))} \leq C \varepsilon_0, \quad \text{for every} \ j \in \mathbb{N}, \quad (148)$$

similarly to estimates (129) and (130), but estimates (148) are local and therefore more precise. See here also Theorem 6.1 in [27] and its variant in Proposition 5.1 of [43]. Now, on account of formulae (2.48), (2.51), (4.10) and (5.21) in [41] - compare here also to equations (17), (52) and (133) above - the conformal factors $\lambda_j = \tilde{u}_j \circ \psi_j + v_j$ of the conformal embeddings $f_j \circ \psi_j$ satisfy exactly equation (147) on $B_1^3(0)$, implying that the differences $\lambda_j - \mu_j$ are real-valued, harmonic functions on $B_1^3(0)$, for every $j \in \mathbb{N}$. Hence, as in the proof of Theorem 5.5 in [41] we can estimate by means of Cauchy-estimates on every disc $B_2^3(0)$, for $r \in (0, \frac{1}{8})$:

$$\| \nabla(\lambda_j - \mu_j) \|_{L^{\infty}(B_r^2(0))} = \| \nabla(\lambda_j - \mu_j - (\lambda_j - \mu_j)_{B_2^3(0)}) \|_{L^{\infty}(B_r^2(0))} \leq \frac{C}{r^3} \int_{B_r^2(0)} \| \lambda_j - \mu_j - (\lambda_j - \mu_j)_{B_2^3(0)} \| d\mathcal{L}^2,$$

for every $j \in \mathbb{N}$, where $(\lambda_j - \mu_j)_{B_2^3(0)}$ denotes the mean value of $\lambda_j - \mu_j$ over $B_2^3(0)$ w.r.t. the Lebesgue measure $\mathcal{L}^2$. Now, we fix some $p \in (1,2)$ and combine (149) with Hölder’s inequality and with Poincaré’s inequality, in order to obtain:

$$\| \nabla(\lambda_j - \mu_j) \|_{L^{\infty}(B_r^2(0))} \leq \frac{C}{r^3} r^{-\frac{1}{p} - \frac{2}{p}} (2r)^{2-\frac{4}{p}} \left( \int_{B_r^2(0)} |\lambda_j - \mu_j - (\lambda_j - \mu_j)_{B_2^3(0)}|^p d\mathcal{L}^2 \right)^{1/p} \leq C_p r^{-\frac{3+2-\frac{4}{p}}{p} + 1} \left( \int_{B_r^2(0)} |\nabla(\lambda_j - \mu_j)|^p d\mathcal{L}^2 \right)^{1/p}$$

$$= C_p r^{-\frac{2}{p}} \left( \int_{B_r^2(0)} |\nabla(\lambda_j - \mu_j)|^p d\mathcal{L}^2 \right)^{1/p} \quad (150)$$

for every $r \in (0, \frac{1}{8})$, taking the exact scaling behaviour of Poincaré’s inequality in line (150) into account. Using again the harmonicity of $\lambda_j - \mu_j$ respectively of its gradient $\nabla(\lambda_j - \mu_j)$, we easily obtain from estimate (150) and from Theorem 2.1 and Remark 2.2 in [13], p. 78 - but exchanging here the interior $L^2$-estimates in Remark 2.2 by interior $L^p$-estimates:

$$\| \nabla(\lambda_j - \mu_j) \|_{L^{\infty}(B_r^2(0))} \leq \tilde{C}_p r^{-\frac{2}{p}} \left( \int_{B_r^2(0)} |\nabla(\lambda_j - \mu_j)|^p d\mathcal{L}^2 \right)^{1/p}$$

$$= \tilde{C}_p \left( \int_{B_r^2(0)} |\nabla(\lambda_j - \mu_j)|^p d\mathcal{L}^2 \right)^{1/p} \quad (151)$$
for every $j \in \mathbb{N}$, for every $r \in \left(0, \frac{1}{8}\right)$ and for the fixed $p \in (1, 2)$. Hence, we obtain from estimates (138), (142), (148) and (151), that

$$
\left( \int_{B_{r}^{2}(0)} |\nabla(\lambda_j - \mu_j)|^2 d\mathcal{L}^2 \right)^{1/2} \leq \sqrt{\pi} r \left\| \nabla(\lambda_j - \mu_j) \right\|_{L^\infty(B_{r}^{2}(0))} \leq \sqrt{\pi} C \tilde{C}_p r \left( \int_{B_{r}^{2}(0)} |\nabla(\lambda_j - \mu_j)|^p d\mathcal{L}^2 \right)^{1/p} \leq \text{Const}(\tilde{A}, \varepsilon_0, p) r
$$

(152)

for every $j \in \mathbb{N}$ and for every $r \in \left(0, \frac{1}{8}\right)$, where we have combined estimates (138), (142) and (148) with the continuity of the embedding $L^{2,\infty}(B_{r}^{2}(0)) \hookrightarrow L^{p}(B_{r}^{2}(0))$, for the fixed $p \in (1, 2)$, in order to obtain the last inequality in (152). See here Section 3.2 in [41] and the literature mentioned there. Hence, on account of (152) we can determine some small radius $r_0 \in \left(0, \frac{1}{8}\right)$, depending only on $\Lambda, \varepsilon_0, p, g_{\text{poin}}$ and on $\delta$, such that the integral $\int_{B_{r_0}^{2}(0)} |\nabla(\lambda_j - \mu_j)|^2 d\mathcal{L}^2$ is smaller than $\varepsilon_0^2$. Combining this with estimate (148) we finally infer, that the conformal factors $\lambda_j = \tilde{u}_j \circ \psi_j + v_j$ of the conformal embeddings $f_j \circ \psi_j$ satisfy:

$$
\left( \int_{B_{r_0}^{2}(0)} |\nabla(\tilde{u}_j \circ \psi_j + v_j)|^2 d\mathcal{L}^2 \right)^{1/2} \leq (C + 1) \varepsilon_0, \quad \text{for every } j \in \mathbb{N},
$$

(153)

where $C > 1$ is the same absolute constant as in (148), where $r_0$ depends only on $\Lambda, \varepsilon_0, p, g_{\text{poin}}$ and on $\delta$, and where $\varepsilon_0$ had to be chosen sufficiently small in (141). On account of convergence (131) and estimate (142), and on account of the conformal invariance of the Dirichlet-integral estimate (153) implies immediately:

$$
\left( \int_{B_{r_0}^{2}(0)} |\nabla(\tilde{u}_j)|^2 d\mu_{g_{\text{poin}}(x_0)} \right)^{1/2} \leq (C + 2) \varepsilon_0, \quad \text{for every } j \in \mathbb{N},
$$

(154)

for the same absolute constant $C$ as in (148) and (153), where $B_{r_0}^{2}(x_0)$ denotes an open geodesic disc of radius $r_0$ about the fixed center point $\psi_j(0) = x_0 \in \Sigma$ w.r.t. the metric $g_{\text{poin}}$ from (131), and where $g_0$ is a sufficiently small positive number, which only depends on $\Lambda, \varepsilon_0, p, g_{\text{poin}}$ and on $\delta$, just as $r_0$ does. Estimates (153) and (154) are also asserted in Theorem 2.2 in [34], where the reader is advised to check our basic reference [41] as well. Gathering all estimates in (137)–(145) and in (153)–(154), we can actually apply the entire reasoning of the proof of Theorem 4.1 in [34], in particular estimates (4.10) and (4.13)–(4.16) in Proposition 4.7 and Lemma 4.9 of [34], in order to obtain here the estimate

$$
\| \nabla_{g_{\text{poin}}(f_j)} \|_{W^{3,2}(B_{r_0}^{2}(x_0))}^2 \leq \text{Const} \cdot \left( \int_{B_{r_0}^{2}(x_0)} |\nabla L^2 \mathcal{W}(f_j)|^2 d\mu_{f_j^*(g_{\text{aux}})} + 1 \right)
$$

(155)

for every $j \in \mathbb{N}$, provided $\varepsilon_0$ had been chosen sufficiently small in (141), where the small radius $g_0$ had been determined in (154) and where the constant in (155) depends on $g_{\text{poin}}, \mathcal{W}(F_0), K, \Lambda, \delta$ and on the choice of $\varepsilon_0$. Now, the final step of the proof works as in the end of the proof of Proposition 5.2 in [34]. Since the center $x_0 \in \Sigma$ of the open geodesic disc $B_{g_0}(x_0)$ in $(\Sigma, g_{\text{poin}, j})$ had been chosen arbitrarily on $\Sigma$, since the metrics $g_{\text{poin}, j}$ converge smoothly to the metric $g_{\text{poin}}$ by (131) and
since Σ is compact, we infer from (155) by means of covering Σ by the images of finitely many appropriate coordinate patches \( \psi^j : B^2_{r_0}(0) \rightarrow \psi^j(B^2_{r_0}(0)) \subset \Sigma \), \( i = 1, \ldots, N = N(r_0, g_{\text{poin}}, \Sigma) \), independently of \( j \in \mathbb{N} \), that estimate (16) actually holds globally on Σ for the sequence of embeddings \( f_j = \mathcal{P} \circ \tilde{F}_{t_j} : \Sigma \rightarrow \mathbb{R}^3 \), with a large constant depending only on \( g_{\text{poin}}, \mathcal{W}(F_0), K, \Lambda, \Sigma \) and on \( \delta \), similarly to the constant in (155). In order to obtain estimate (16) also for the original embeddings \( \tilde{F}_{t_j} : \Sigma \rightarrow \mathbb{S}^3 \), we only have to apply now the inverse stereographic projection \( \mathcal{P}^{-1} : \mathbb{R}^3 \rightarrow \mathbb{S}^3 \setminus \{(0,0,0,1)\} \) to \( f_j \), explicitly given by
\[
(x, y, z) \mapsto \frac{1}{x^2 + y^2 + z^2 + 1}(2x, 2y, 2z, x^2 + y^2 + z^2 - 1).
\]
Here, we have to recall that the operator
\[
\text{Imm}_{\text{ul}}(\Sigma, \mathbb{R}^n) \rightarrow \mathbb{R}, \quad f \mapsto \int_{\Sigma} \frac{1}{|A_j^0|^4} |\nabla L^2 \mathcal{W}(f)|^2 \, d\mu_{f_j^*(\text{g_{euc}})}
\]
is a conformal invariant for every \( n \geq 3 \), and use that here \( |A_j^0|^2 \equiv |A_{\tilde{F}_{t_j}, \mathbb{S}^3}|^2 \) remains bounded on \( \Sigma \) - for all \( j \in \mathbb{N} \) - from above in terms of the uniform upper bound in (136), i.e. by the required upper bound \( K \) for \( \sup_{j \in \mathbb{N}} \sup_{\Sigma} |\tilde{F}_{t_j, \mathbb{S}^3}| \), and also from below by the number 2 by formula (97) in Proposition 5.3 combined with the fact, that here the considered flow line \( \{F_t\} \) of the MIWF consists of smooth parametrizations of Hopf-tori in \( \mathbb{S}^3 \) - as already derived from Proposition 5.4 in the beginning of the proof of the first part of Theorem 1.2. Obviously, this yields also upper and lower bounds for the traces \( |A_j^0|^2 \) on \( \Sigma \) in terms of \( K \) and \( \delta \), as in (140).

3) Now we additionally presume, that the speed \( \frac{d}{dt} \mathcal{W}(F_t) \) of “energy decrease” remains uniformly bounded by some large constant \( W \) at every time \( t = t_j \), and we consider a weakly/weakly* convergent subsequence \( \{\tilde{F}_{t_{j_k}}\} \) in \( W^{2,2}(\Sigma, g_{\text{poin}}, \mathbb{R}^4) \) and in \( W^{1,\infty}(\Sigma, g_{\text{poin}}, \mathbb{R}^4) \) as in (58) and (60), which we had obtained from the original sequence \( \{F_{t_j}\} \) in part (2) of Theorem 1.1 by extraction of an appropriate subsequence \( \{F_{t_{j_k}}\} \) and by appropriate reparametrization of each embedding \( F_{t_{j_k}} \). Hence, we have here additionally:
\[
\int_{\Sigma} \frac{1}{|A_{\tilde{F}_{t_{j_k}}}^0|^4} |\nabla L^2 \mathcal{W}(F_{t_{j_k}})|^2 \, d\mu_{\tilde{F}_{t_{j_k}}^*(\text{g_{euc}})} = 2 \left| \frac{d}{dt} \mathcal{W}(F_{t_{j_k}}) \right| \leq 2W \quad \text{for every } k \in \mathbb{N}.
\]
(156)

On account of the uniform upper bound (135), we obtain from (156) that
\[
\int_{\Sigma} |\nabla L^2 \mathcal{W}(F_{t_{j_k}})|^2 \, d\mu_{\tilde{F}_{t_{j_k}}^*(\text{g_{euc}})} \leq (K^2 + 2)^2 2W, \quad \text{for every } k \in \mathbb{N},
\]
and thus also
\[
\int_{\Sigma} |\nabla L^2 \mathcal{W}(\tilde{F}_{t_{j_k}})|^2 \, d\mu_{\tilde{F}_{t_{j_k}}^*(\text{g_{euc}})} \leq (K^2 + 2)^2 2W, \quad \text{for every } k \in \mathbb{N},
\]
(157)
on account of the invariance of the differential operator \( F \mapsto \nabla L^2 \mathcal{W}(F) \) w.r.t. smooth reparametrization and on account of the definition of \( \tilde{F}_{t_j} \) below formula
Combining now estimates (16) and (157) estimating the considered sequence \( \{\tilde{F}_{ijk}\} \), and recalling that the embeddings \( \tilde{F}_{ijk} \) converge in the senses (58), (60) and (61), then the “principle of subsequences” yields that \( \{\tilde{F}_{ijk}\} \) also converges weakly in \( W^{4,2}((\Sigma, g_{\text{poin}}), \mathbb{R}^4) \) to the conformal bi-Lipschitz parametrization 
\[ f : (\Sigma, g_{\text{poin}}) \rightarrow \text{spt}(\mu) \]
of the corresponding limit Hopf-torus \( \text{spt}(\mu) \) from the first part of this theorem, i.e. to the parametrization of the support of the limit varifold \( \mu \) of the weakly convergent subsequence \( \{F_{ijk}\} \) from line (112). Obviously, the limit parametrization \( f \) has to be of class \( W^{4,2}((\Sigma, g_{\text{poin}}), \mathbb{R}^4) \) in this case. From Rellich’s embedding theorem, A 8.4 in [1], we immediately infer also strong convergence of the sequence \( \{\tilde{F}_{ijk}\} \) to \( f \) in \( W^{3,2}((\Sigma, g_{\text{poin}}), \mathbb{R}^4) \), as \( k \to \infty \). Since this implies together with formula (13) and with convergence (60) in particular:
\[ W(\mu) = W(f) = \lim_{k \to \infty} W(\tilde{F}_{ijk}) = \lim_{k \to \infty} W(F_{ijk}), \]
all conditions of the third part of Theorem 1.1 are satisfied by the sequence \( \{\tilde{F}_{ijk}\} \), and therefore statement (15) has to hold here, just as asserted, for the reparametrized sequence \( \{\tilde{F}_{ijk}\} \) or equivalently for the original sequence \( \{F_{ijk}\} \) itself - taking the fact into account that \( A_{\tilde{F}_{ijk}} = (A_{F_{ijk}}) \circ \Phi_{ij} \) for every \( k \in \mathbb{N} \) and that the functional in (15) is invariant w.r.t. smooth reparametrization. Finally, we infer from estimates (16) and (157) together with the compactness of the embedding \( W^{4,2}((\Sigma, g_{\text{poin}}), \mathbb{R}^4) \hookrightarrow C^{2,\alpha}((\Sigma, g_{\text{poin}}), \mathbb{R}^4) \), for any \( \alpha \in (0,1) \), and again from the “principle of subsequences”, that \( \{F_{ijk}\} \) converges to \( f \) in \( C^{2,\alpha}((\Sigma, g_{\text{poin}}), \mathbb{R}^4) \), as \( k \to \infty \).}

For the proof of Theorem 1.3 we firstly recall here Theorem 1.1 of [20].

**Proposition 5.7.** Let \( \Sigma \) be a smooth compact torus, and let \( F^* : \Sigma \rightarrow M(\Sigma^{1 \times S^1}) \) be a smooth diffeomorphic parametrization of a compact torus in \( S^3 \), which is conformally equivalent to the standard Clifford-torus \( C \) via some conformal transformation \( M \in \text{Mob}(S^3) \), and let some \( \beta \in (0,1) \) and \( m \in \mathbb{N} \) be fixed. Then, there is some small neighborhood \( W = W(\Sigma, F^*, m) \) about \( F^* \) in \( h^{2+\beta}(\Sigma, \mathbb{R}^4) \), such that for every \( C^\infty \) smooth initial immersion \( F_1 : \Sigma \rightarrow S^3 \), which is contained in \( W \), the unique flow line \( \{P(t,0,F_1)\}_{t \geq 0} \) of the MIWF exists globally and converges - up to smooth reparametrization - fully to a smooth and diffeomorphic parametrization of a torus in \( S^3 \), which is again conformally equivalent to the standard Clifford-torus \( C \). This full convergence takes place w.r.t. the \( C^m(\Sigma, \mathbb{R}^4) \)-norm and at an exponential rate, as \( t \nearrow \infty \).

**Proof of Theorem 1.3:**

We assume the considered flow line \( \{F_t\} \) of the MIWF to start moving in a smooth and simple parametrization \( F_0 : \Sigma \rightarrow S^3 \) of a Hopf-torus in \( S^3 \) with \( W(F_0) < 8\pi \) and to be global, i.e. with \( T_{\text{max}}(F_0) = \infty \). First of all, we conclude from this condition and from the flow equation (1), similarly as in (156):

\[
\int_0^\infty \int_\Sigma \frac{1}{|A_{F_t}|^3} |\nabla L^2 W(F_t)|^2 d\mu_{F_t(g_{\text{vac}})} dt = -2 \lim_{T \to \infty} \int_0^T \frac{d}{dt} W(F_t) dt = 2 \lim_{T \to \infty} (W(F_0) - W(F_T)) < 2 W(F_0).
\]
Hence, there is some sequence $t_j \nearrow \infty$, such that
\[
2 \frac{d}{dt} \mathcal{W}(F_{t_j}) = \int_{\Sigma} \frac{1}{|A_{F_{t_j}}|^4} \left| \nabla L^2 \mathcal{W}(F_{t_j}) \right|^2 d\mu_{F_{t_j}^*(\text{guc})} \rightarrow 0, \tag{158}
\]
as $j \to \infty$. Moreover, we have assumed that there is some large constant $K > 0$, such that $\| \tilde{H}_{F_t} \|_{L^\infty(\Sigma)}$ remains uniformly bounded by $K$ for all $t \in [0, \infty)$, implying here again estimates (136) and (140). Now, on account of statement (158) we see, that the sequence $\{F_{t_j}\}$ satisfies all requirements of the third part of Theorem 1.2, i.e. we can use exactly the diverging sequence of times $t_j \nearrow T_{\text{max}}(F_0) = \infty$ satisfying (158) in the third part of Theorem 1.2. Hence, any reparametrized subsequence $\{\tilde{F}_{t_{j_k}}\}$ converging weakly/weakly* as in (58) and (60) - which we had considered in the second and third part of Theorem 1.2 - converges even weakly in $W^{4,2}(\Sigma, g_{\text{poin}}), \mathbb{R}^4$ and strongly in $W^{3,2}(\Sigma, g_{\text{poin}}), \mathbb{R}^4$ to a uniformly conformal bi-Lipschitz homeomorphism $\tilde{f}$ between $(\Sigma, g_{\text{poin}})$ and a Hopf-torus $\text{spt}(\mu)$ in $\mathbb{S}^3$, where $g_{\text{poin}}$ is some appropriate smooth metric on $\Sigma$ of vanishing scalar curvature. Now we argue as in (78) and apply the stereographic projection $\mathcal{P} : \mathbb{S}^3 \setminus \{(0,0,0,1)\} \to \mathbb{R}^3$. Hence, we can conclude that also the smooth embeddings $\tilde{f}_k := \mathcal{P} \circ \tilde{F}_{t_{j_k}}$ converge weakly in $W^{4,2}(\Sigma, g_{\text{poin}}), \mathbb{R}^3$ and strongly in $W^{3,2}(\Sigma, g_{\text{poin}}), \mathbb{R}^3$ to the uniformly conformal bi-Lipschitz homeomorphism $\tilde{f} := \mathcal{P} \circ f$ between $(\Sigma, g_{\text{poin}})$ and $\mathcal{P}(\text{spt}(\mu)) \subset \mathbb{R}^3$. Now using the conformal invariance of the MIWF, convergence (158) also implies that:
\[
\int_{\Sigma} \frac{1}{|A_{\tilde{f}_k}|^4} \left| \nabla L^2 \mathcal{W}(\tilde{f}_k) \right|^2 d\mu_{\tilde{f}_k^*(\text{guc})} = 2 \frac{d}{dt} \mathcal{W}(\tilde{f}_k) = 2 \frac{d}{dt} \mathcal{W}(\mathcal{P} \circ \tilde{F}_{t_{j_k}}) = 2 \frac{d}{dt} \mathcal{W}(\tilde{F}_{t_{j_k}}) \rightarrow 0,
\]
as $k \to \infty$. Combining this with estimate (140), we obtain:
\[
\int_{\Sigma} \left| \nabla L^2 \mathcal{W}(\tilde{f}_k) \right|^2 d\mu_{\tilde{f}_k^*(\text{guc})} \rightarrow 0, \quad \text{as } k \to \infty. \tag{159}
\]
Now similarly to the argument in line (86), we can compute here by means of formulae (20), (23) and (159) and by means of the above mentioned strong convergence of $\{\tilde{f}_k\}$ in $W^{3,2}(\Sigma, g_{\text{poin}}), \mathbb{R}^3$ to $\tilde{f}$:
\[
0 \leftarrow \langle \nabla L^2 \mathcal{W}(\tilde{f}_k), \varphi \rangle_{L^2(\Sigma, \tilde{f}_k^*(\text{guc}))} = \langle \nabla L^2 \mathcal{W}(\tilde{f}_k), \varphi \rangle_{\mathcal{D}'(\Sigma)} \rightarrow \langle \nabla L^2 \mathcal{W}(\tilde{f}), \varphi \rangle_{\mathcal{D}'(\Sigma)}, \tag{160}
\]
as $k \to \infty$, for every fixed $\varphi \in C^\infty(\Sigma, \mathbb{R}^3)$. Hence, we infer from (160) that $\nabla L^2 \mathcal{W}(\tilde{f}) \equiv 0$ in the distributional sense of (20), i.e. that the uniformly conformal bi-Lipschitz homeomorphism $\tilde{f}$ is “weakly Willmore” on $\Sigma$ exactly in the sense of Corollary 7.3 in [41] respectively Definition VII.3 in [38]. We can therefore immediately infer from Theorem 7.11 in [41] respectively from Corollary VII.6 in [38], that $\tilde{f}$ is actually a smooth diffeomorphism between $(\Sigma, g_{\text{poin}})$ and an embedded, classical Willmore surface in $\mathbb{R}^3$, where we have strongly relied on the fact, that $\tilde{f}$ has already been known to be a uniformly conformal bi-Lipschitz homeomorphism onto $\mathcal{P}(\text{spt}(\mu))$. Hence applying now inverse stereographic projection from $\mathbb{R}^3$ to $\mathbb{S}^3 \setminus \{(0,0,0,1)\}$ and recalling the statement of the third part of Theorem 1.2, the original limit embedding $\tilde{f} : \Sigma \xrightarrow{\cong} \text{spt}(\mu)$ turns out to parametrize a smooth Willmore-Hopf-torus in $\mathbb{S}^3$. Since we had assumed that $\mathcal{W}(F_0) < 8\pi$, we must have - on account of the proven Willmore conjecture, Theorem A in [30] - in the limit
as \( t_{jk} \nearrow \infty \): \( W(f) \in [2\pi^2, 8\pi) \), and thus by formula (103): \( E(\gamma) \in [2\pi, 8) \) for the elastic energy of any fixed smooth profile curve \( \gamma \) of the Willmore-Hopf-torus spt(\( \mu \)). Moreover, we obtain here immediately from formula (104), that any smooth profile curve \( \gamma \) of the Willmore-Hopf-torus spt(\( \mu \)) is an "elastic curve in \( S^2 \), i.e. solves the equation:

\[
2 \left( \nabla_{\gamma} \right)^2 (\kappa_{\gamma}) + |\kappa_{\gamma}|^2 \kappa_{\gamma} + \kappa_{\gamma} \equiv 0 \quad \text{on} \quad S^1.
\]

Hence, on account of the second part of Proposition 5.1 in [18] we can infer from \( E(\gamma) \in [2\pi, 8) \), that here actually \( E(\gamma) = 2\pi \) has to hold, implying that \( \gamma \) parametrizes a great circle in \( S^2 \). Hence, spt(\( \mu \)) = \( \pi^{-1}(\text{trace}(\gamma)) \) has to be the Clifford-torus in \( S^3 \) up to a particular conformal transformation of \( S^3 \). We can therefore conclude here, that \( f : (\Sigma, g_{\text{poin}}) \xrightarrow{\cong} M(\frac{1}{\sqrt{2}}(S^1 \times S^1)) \) smoothly and diffeomorphically, where \( M \) is an appropriate Möbius-transformation of \( S^3 \). Now we can again conclude from the statement of Theorem 1.2, that the reparametrized embeddings \( \tilde{F}_{jk} \) converge also in \( C^{2,\alpha}(\Sigma, g_{\text{poin}}), \mathbb{R}^4 \) to the limit embedding \( f \), as \( k \to \infty \), for any fixed \( \alpha \in (0,1) \). Hence, we can apply here Proposition 5.7 to \( F^* := f \), any \( \beta < \alpha \) and to \( F_1 = \tilde{F}_{jk} \) for some sufficiently large \( k^* >> 1 \), such that \( \tilde{F}_{jk} \in W(\Sigma, f, m) \) in the terminology of Proposition 5.7 above, where we have used here that

\[
C^{2,\alpha}(\Sigma, g_{\text{poin}}), \mathbb{R}^4 \hookrightarrow h^{2+\beta}((\Sigma, g_{\text{poin}}), \mathbb{R}^4)
\]

is a continuous embedding, provided \( \beta < \alpha \). Hence, we obtain from Proposition 5.7 the existence of some smooth family of smooth diffeomorphisms \( \theta_s : \Sigma \xrightarrow{\cong} \Sigma \), for \( s \geq 0 \) and with \( \theta_0 = \text{id}_\Sigma \), depending on the choice of \( m \in \mathbb{N} \) in the formulation of this theorem, such that the reparametrized flow line \( \{P(s,0,\tilde{F}_{jk}) \circ \theta_s\}_{s \geq 0} \) of the MIWF - i.e. being a smooth solution to the "relaxed MIWF-equation" (8) - satisfies:

\[
P(s,0,\tilde{F}_{jk}) \circ \theta_s \to G \quad \text{in} \quad C^m(\Sigma, \mathbb{R}^4)
\]

fully as \( s \to \infty \) and at an exponential rate, where \( G : \Sigma \xrightarrow{\cong} M^*(\frac{1}{\sqrt{2}}(S^1 \times S^1)) \) is a smooth diffeomorphism and where \( M^* \) is another appropriate Möbius transformation of \( S^3 \). Hence, using the fact that \( \tilde{F}_{jk} = F_{jk} \circ \Phi_{jk} \), as defined below (49) - is a smooth reparametrization of the embedding \( F_{jk} \), i.e. of the element of the flow line \( \{F_t\}_{t \geq 0} \) at time \( t = t_{jk} \), we can finally conclude from (161) and from the invariance of the MIWF w.r.t. time-independent and smooth reparametrization, that the smooth family of diffeomorphisms

\[
\Theta_t \equiv \Theta_{t^m} := \begin{cases} 
\Phi_{jk} : & 0 \leq t \leq t_{jk}, \\
\Phi_{jk} \circ \theta_{t-t_{jk}} : & t \geq t_{jk},
\end{cases}
\]

has all desired properties of the theorem, and Theorem 1.3 is proved.

\[
\square
\]

6 Appendix A

**Remark 6.1.** We should remark here first of all, that the \( W^{4,2} \)-regularity which we have achieved for the conformal parametrization \( f : (\Sigma, g_{\text{poin}}) \xrightarrow{\cong} \text{spt}(\mu) \) of a limit torus spt(\( \mu \)) in the fourth part of Theorem 1.1 and in the third part of Theorem 1.2, is even stronger than
the minimally required regularity of umbilic-free initial immersions of the torus \( \Sigma \) into any \( \mathbb{R}^n, n \geq 3 \), in which the “relaxed MIWF-equation” from line (8) can be uniquely started; see the exact short-time existence statement in Proposition 6.1 below. But unfortunately, still the conformal parametrization \( f \) of the limit torus \( \text{spt}(\mu) \) in the fourth part of Theorem 1.1 might fail to be umbilic-free, or the limit Hopf-torus \( \text{spt}(\mu) \) in the third part of Theorem 1.2 - which is automatically umbilic-free - might depend on the choice of the weakly convergent subsequence \( \{F_{t_j}\} \) in lines (9)–(10) of the originally considered sequence of embeddings \( \{F_t\} \) - for any fixed sequence of times \( t_j \cap T_{\text{max}}(F_0) \). Therefore, neither the fourth part of Theorem 1.1 nor the third part of Theorem 1.2 of this article can be combined with Proposition 6.1 below, in order to obtain any sufficient conditions, under which the “relaxed MIWF” (8) can be uniquely restarted at a singular time \( T_{\text{max}}(F_0) < \infty \), which would actually rule out the existence of singular times of the MIWF under the stated conditions in these two limit-regularity results. Vice versa this insight means, that we can neither conclude from the fourth part of Theorem 1.1 nor from the third part of Theorem 1.2, that either the supremum of the mean curvature, i.e. of \( \| \vec{H}_{F_t} \|_{L^\infty(\Sigma)} \), or the supremum of \( |A_{F_t}^0|^2 \) over \( \Sigma \), or the speed of Willmore-energy-decrease, i.e. \( \frac{4}{\pi} W(F_t) \), have to “blow up” along a singular flow line \( \{F_t\} \) of the MIWF in general, as \( t \) approaches the singular time \( T_{\text{max}}(F_0) < \infty \) from the past. Moreover, for the same reason the third part of Theorem 1.1 and also the third part of Theorem 1.2 show us on account of statement (15), that the phenomenon of “concentration of curvature in the ambient space \( \mathbb{R}^n \)” of embeddings \( \{F_t\} \) moving along the MIWF in \( \mathbb{S}^3 \) is probably not a criterion for the respective flow line \( \{F_t\} \) to be singular, i.e. to develop a singularity in finite time \( T_{\text{max}}(F_0) \), in contrast to the famous statement of Theorem 1.2 in [25] about the classical Willmore flow in \( \mathbb{R}^n, n \geq 3 \), and even the initial energy bound “\( W(F_0) < 8\pi \)” does not improve this picture. Hence, we can conclude from Theorems 1.1 and 1.2, that there are no obvious geometric criteria for a flow line of the MIWF in \( \mathbb{S}^3 \) to develop a singularity in finite time - not even in a low Willmore energy regime - indicating a stark contrast to the behaviour of the classical Willmore flow in \( \mathbb{R}^n \); see Theorem 1.2 in [25] and Theorem 5.2 in [26].

**Proposition 6.1.** Let \( \Sigma \) be a smooth compact torus, and let some real number \( p \in (3, \infty) \) and some integer \( n \geq 3 \) be arbitrarily fixed. For every umbilic-free immersion \( U_0 \) of regularity class \( W^{4,\frac{4}{p}}(\Sigma, \mathbb{R}^n) \) there is some sufficiently small time \( T > 0 \) and a short-time solution \( \{f_t\}_{t \in [0,T]} \) of the relaxed MIWF-equation from line (8), i.e. of

\[
(\partial_t f_t(x))^{_{\perp f_t}} = -\frac{1}{2} \frac{1}{|A_{f_t}^0(x)|^4} \left( \triangle_{f_t} \vec{H}_{f_t}(x) + Q(A_{f_t}^0)(\vec{H}_{f_t})(x) \right), \quad \forall (x,t) \in \Sigma \times [0,T],
\]

with \( f_0(x) = U_0(x), \quad \forall x \in \Sigma, \)

in the B-space \( W^{1,p}([0,T]; L^p(\Sigma, \mathbb{R}^n)) \cap L^p([0,T]; W^{4,p}(\Sigma, \mathbb{R}^n)) \) from Definition 1.1 (d).

**Proof.** This follows immediately from Theorem 2 (i) in [19].

**Remark 6.2.** As already mentioned in the introduction of this paper, the natural approach in [8] and [9] or also in [31] aiming at criteria for long-time existence and subconvergence of the classical elastic energy flow (106) and of the “inverse Willmore flow” suddenly break down when applied to our degenerate variant (105) of the classical elastic energy flow (106), i.e. when applied to the “reduction” (105) of the MIWF moving Hopf-tori in \( \mathbb{S}^3 \) via the Hopf-fibration from Section 5. In particular our final results, Theorem 1.3 and
Corollary 1.2, cannot be proved by means of such classical and fairly elementary methods, applied to the degenerate elastic energy flow (105) moving closed smooth curves on $S^2$. We shall take a closer look at this striking phenomenon in this remark. The canonical approach would be here, to combine formulæ (2.14), (2.18), (2.21) and (2.23) in Lemma 2.3 of [9] with our flow equation (105), in order to obtain evolution equations for the curvature vector $\vec{\kappa}_{\gamma_t}$ and for the arclength $d\mu_{\gamma_t}$ along a flow line $\{\gamma_t\}_{t\in[0,T]}$ of evolution equation (105), which yields here exactly:

\[
(\nabla_t)^{\perp}(\vec{\kappa}_{\gamma_t}) + \left(\nabla_{\gamma_t}^{\perp}\right)^2\left(\frac{1}{(\kappa_{\gamma_t}^2 + 1)^2} \left( 2 \left(\nabla_{\gamma_t}^{\perp}\right)^2(\vec{\kappa}_{\gamma_t}) + (1 + \kappa_{\gamma_t}^2) \vec{\kappa}_{\gamma_t} \right) \right) = 0
\]  

for any $t \in [0,T]$. Moreover, we have for any family $\{\Psi_t\}_{t\in[0,T]}$ of “normal” vector fields along the curves $\gamma_t$:

\[
(\nabla_t)^{\perp}(\vec{\kappa}_{\gamma_t}) + \left(\nabla_{\gamma_t}^{\perp}\right)^2\left(\frac{1}{(1 + \kappa_{\gamma_t}^2)^2} \left( 2 \left(\nabla_{\gamma_t}^{\perp}\right)^2(\vec{\kappa}_{\gamma_t}) + (1 + \kappa_{\gamma_t}^2) \vec{\kappa}_{\gamma_t} \right) \right) = 0
\]  

for any $t \in [0,T]$. In view of statement (165) below, we need some appropriate tool for a clean induction argument. To this end one can use the following proposition, which should be compared with Proposition 4.2 in [31].

**Proposition 6.2.** Let $\{\gamma_t\}_{t\in[0,T]}$ be a smooth flow line of evolution equation (105), and let $\{\Psi_t\}_{t\in[0,T]}$ be a family of smooth normal vector fields along $\{\gamma_t\}_{t\in[0,T]}$, satisfying

\[
(\nabla_t)^{\perp}(\vec{\kappa}_{\gamma_t}) + \left(\nabla_{\gamma_t}^{\perp}\right)^2\left(\frac{1}{(1 + \kappa_{\gamma_t}^2)^2} \left( 2 \left(\nabla_{\gamma_t}^{\perp}\right)^2(\vec{\kappa}_{\gamma_t}) + (1 + \kappa_{\gamma_t}^2) \vec{\kappa}_{\gamma_t} \right) \right) = Y_t
\]

for some smooth normal vector field $Y_t$ along $\{\gamma_t\}_{t\in[0,T]}$. Then its covariant derivative
\[ \Psi_t := \nabla_s \Phi_t \text{ satisfies the equation} \]
\[
\nabla_t^\perp (\Psi_t) + \frac{2}{(\kappa_t^2 + 1)^2} \nabla_s^4 (\Psi_t) = \nabla_s^\perp (Y_t) + \frac{8}{(\kappa_t^2 + 1)^3} (\kappa_t, \nabla_s^\perp (\kappa_t)) \nabla_s^4 (\Phi_t) \\
- \left[ \frac{2}{(\kappa_t^2 + 1)^2} (\nabla_s^\perp)^2 (\kappa_t), \nabla_s^\perp (\kappa_t) \right] + \frac{\kappa_t^2}{1 + \kappa_t^2} \endeck
\]
\[ \text{for every } t \in [0, T], \text{ where we abbreviated above } \nabla_s := \nabla_{\gamma'} |_{\gamma'} \text{ for ease of notation.} \]

Now, following Sections 3 and 4 in [8] and Sections 2.2 and 4.1 in [9] and regarding the uniform bounds (108) and (111) we introduce some technically useful notations.

**Definition 6.1.** 1) For a fixed smooth, closed and regular curve \( \gamma : S^1 \to S^2 \), and integers \( b \geq 2 \) and \( a \geq 0, c \geq 0 \), we call “\( P_b^{a,c}(\kappa_{\gamma}) \)” any finite linear combination of products
\[
(\nabla_s^\perp)^i (\kappa_{\gamma}) \ast \ldots \ast (\nabla_s^\perp)^i (\kappa_{\gamma})
\]
with \( i_1 + \ldots + i_b = a \) and \( \max i_j \leq c \).

2) Scale-invariant Sobolev-norms: For any smooth, closed, regular curve \( \gamma : S^1 \to S^2 \) we define:
\[
\| (\nabla_s^\perp)^i (\kappa_{\gamma}) \|_{p,a} := \text{length}(\gamma)^{i+1-1/p} \left( \int_{S^1} |(\nabla_s^\perp)^i (\kappa_{\gamma})|^p \mathrm{d}\mu_\gamma \right)^{1/p},
\]
and
\[
\| \kappa_{\gamma} \|_{k,p} := \sum_{i=0}^{k} \| (\nabla_s^\perp)^i (\kappa_{\gamma}) \|_p,
\]
for \( i \in \mathbb{N} \) and \( 1 \leq p < \infty \).

Combining now formulae (162)–(164) and Proposition 6.2, we obtain the following general evolution equation by induction, which compares to Proposition 4.3 in [31] or also to Lemma 3.1 in [8].

**Proposition 6.3.** Let \( \{ \gamma_t \}_{t \in [0, T]} \) be a flow line of evolution equation (105) and \( m \in \mathbb{N}_0 \). Then the vector field \( (\nabla_s^\perp)^m (\kappa_{\gamma}) \) satisfies the equation
\[
\nabla_t^\perp ((\nabla_s^\perp)^m (\kappa_{\gamma})) + \frac{2}{(\kappa_t^2 + 1)^2} (\nabla_s^\perp)^4 ((\nabla_s^\perp)^m (\kappa_{\gamma})) = \\
= \frac{1}{(\kappa_t^2 + 1)^3} P_3^{m+1,m+3} (\kappa_{\gamma}) + \sum_{(a,b,d) \in I(m)} \frac{1}{(\kappa_t^2 + 1)^d} P_b^{a,m+2} (\kappa_{\gamma}),
\]
(165)
for any $t \in [0,T]$, where we have used the notation of Definition 6.1 and where the set $I(m)$ consists of those triples $(a,b,d) \in \mathbb{N}_0^3$, such that $a \in \{m+4,m+2,m\}$, $b$ is odd and $\leq 2m+5$, and $1 \leq d \leq m+4$.

This result should be combined with the following basic interpolation inequality, which we quote from Lemma 4.3 in [8]:

**Proposition 6.4.** Let $\gamma: S^1 \rightarrow S^2$ be a closed, smooth, regular curve in $S^2$, and $k \in \mathbb{N}$, $a,c \in \mathbb{N}_0$, with $c \leq k-1$, $b \in \mathbb{N}$ with $b \geq 2$, there is some constant $C$, depending only on $k$ and $b$, such that there holds:

$$
\int_{S^1} |P_{p}^{n,c}(\kappa_{\gamma})| \mu_{\gamma} \leq C \text{length}(\gamma)^{1-a-b} \| \kappa_{\gamma} \|_{k,2} \| \kappa_{\gamma} \|_{2}^{b-\beta},
$$

where $\beta := \frac{a+b-1}{k}$.

Unfortunately, we cannot obtain any $L^\infty-L^\infty$-estimates for the covariant derivatives $(\nabla_{\gamma})^m(\kappa_{\gamma})$, $m \in \mathbb{N}$, from a combination of Propositions 6.3 and 6.4, because the usual trick “to multiply equation (165) with $(\nabla_{\gamma})^m(\kappa_{\gamma})$ and then to integrate by parts twice” does not elegantly work here on account of the factor $\frac{1}{(\kappa_{\gamma}^2+1)^{2}}$ in formula (165), and moreover because of too many terms of relatively high order on the right hand side of formula (165), in order to allow for a successful interpolation via formula (166). We are therefore even unable to rule out the existence of singularities of flow lines of the flow (105) under reasonable initial conditions, although (105) is a substantial simplification of the MIWF, from both the analytic and the geometric point of view.

**Remark 6.3.** In Theorems 1.1 and 1.2 we have considered only sequences $\{F_t\}$ with $t_j \nearrow T_{\text{max}}(F_0)$, whom we reparametrize in such a way that each embedding $\tilde{F}_t := F_t \circ \Phi_j$ is uniformly conformal w.r.t. some smooth metric of zero scalar curvature $g_{\text{poin},j}$ and such that these metrics $g_{\text{poin},j}$ - up to extraction of a subsequence - converge smoothly to some fixed smooth metric of zero scalar curvature $g_{\text{poin}}$ on the abstract compact torus $\Sigma$. Instead we could also choose $\Sigma$ to be simply the standard “Clifford-torus” in $\mathbb{R}^3$, and we could try to reparametrize every single embedding $F_t$ of the flow line $\{F_t\}$, i.e. for every single $t \in [0,T_{\text{max}}(F_0))$, in such a way, that the new, reparametrized family of embeddings $\{\tilde{F}_t\}_{t \in [0,T_{\text{max}}(F_0))}$ is still a flow line of the “relaxed MIWF-equation” (8) and has much better analytical properties, considered from the perspective of parabolic PDEs. As in Section 5 of [34] the motivation for this idea is obvious: We try to use a systematical and continuous gauge of all metrics $\{F_t^{*}(g_{\text{euc}})\}_{t \in [0,T_{\text{max}}(F_0))}$ into uniformly conformal metrics

$$
\{\tilde{F}_t^{*}(g_{\text{euc}})\}_{t \in [0,T_{\text{max}}(F_0))} = \{e^{2u_t} g_{\text{poin}}\}_{t \in [0,T_{\text{max}}(F_0))},
$$

where $g_{\text{poin}}$ should be some fixed smooth metric of zero scalar curvature on the Clifford-torus, in order to prove - under appropriate “mild” conditions on the considered flow line $\{F_t\}_{t \in [0,T_{\text{max}}(F_0))}$ - by a parabolic bootstrap argument $C^\infty$-smoothness of the “generalized” flow line $\{\tilde{F}_t\}$ up to $t = T_{\text{max}}(F_0)$ and actually including $t = T_{\text{max}}(F_0)$, provided $T_{\text{max}}(F_0)$ is here supposed to be finite. As in the classical example of the Ricci flow this would yield an immediate contradiction, and thus there would have to hold here $T_{\text{max}}(F_0) = \infty$, whence ruling out singularities along $\{F_t\}$ in finite time, at least under appropriate additional conditions on the original flow line $\{F_t\}$. However, comparing the “weak Willmore flow”
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for immersions of $S^2$ into $\mathbb{R}^3$ in [34] with the MIWF (1), we discover at least two basic problems when trying to perform such a program for the MIWF:

1) We neither have formula (2.2) in Theorem 2.4 of [34] nor the estimates of Theorem 2.7 of [34] at our disposal, because the main results of Müller’s and De Lellis’ papers [10] and [11] and of Kuwert’s and Scheuer’s article [29] only hold for the classical Willmore flow moving immersions of the 2-sphere $S^2$ into $\mathbb{R}^3$ respectively $\mathbb{R}^n$, whereas we have to consider immersions of a fixed torus $\Sigma$ - for example of the Clifford-torus - into $S^3$. We therefore do not have any tool, in order to either bound the deviation $\| e^{u_t} - 1 \|_{L^\infty(\Sigma)}$ of the conformal factors $e^{u_t}$ of $\tilde{F}_t^\ast(g_{\text{euc}})$ uniformly on entire $\Sigma$ or the areas and barycenters of the immersions $\tilde{F}_t$ in terms of their Willmore energies $W(F_t)$ respectively areas $A(F_t)$, or in terms of any other controllable geometric, tensorial quantity, uniformly for all $t \in [0, T_{\text{max}}(F_0))$. Compare here to the proof of Theorem 1.9 in [34].

2) Another fundamental problem lies in the fact, that first of all the embeddings $F_t$ are uniformly conformal w.r.t. varying metrics $g_{\text{point}}(t)$ of zero scalar curvature, and that the moduli space $\mathcal{M}_1$ is isomorphic to $\mathbb{H}/\text{PSL}_2(\mathbb{Z})$ by Theorem 2.7.2 in [21], whereas there is not up to conformal automorphisms - only one conformal class on $S^2$ by Corollary 5.4.1 in [21]. Therefore the reasoning yielding Lemma 1.1 in [34] breaks down, i.e. we cannot achieve an explicit and useful formula - see formula (1.8) in [34] - for the tangential vector field $\{U(F_t)\}$ along $\{F_t\}$, which yields the exact “modified flow equation”

$$\partial_t \tilde{F}_t = -\frac{1}{|A_{\tilde{F}_t}|^4} \nabla_{L^2} W(\tilde{F}_t) + U(\tilde{F}_t) \quad \text{on } \Sigma \times [0, T_{\text{max}}(F_0))$$

for the correctly gauged flow line $\{\tilde{F}_t\}$. However, such a computation is of fundamental importance, because the “relaxed MIWF-equation” (8) is obviously not uniformly parabolic and suffers even from “non-uniqueness”. Hence, not being able here to perform the “DeTurck-trick” explicitly, the proofs of Proposition 5.3 and Corollary 5.5 in [34] cannot be carried over to the present situation, which would show here how to extend the gauged flow line $\{\tilde{F}_t\}$ of the MIWF smoothly into the time $t = T_{\text{max}}(F_0)$ of maximal existence.

7 Appendix B

In this appendix we recall and quote Lemma 5.1 in [18], i.e. the existence of horizontal smooth lifts of some arbitrary closed path $\gamma : S^1 \to S^2$, w.r.t. fibrations of the type $\pi \circ F$ for “simple” parametrizations $F : \Sigma \to \pi^{-1}(\text{trace}(\gamma)) \subset S^3$, in the sense of Definition 5.3. The following lemma is an important tool in the proof of Propositions 5.3 and 5.4 of this article.

**Lemma 7.1.** [Lemma 5.1 in [18]] Let $\gamma : S^1 \to S^2$ be a regular smooth path, which traverses a closed smooth curve in $S^2$ exactly once, and let $F : \Sigma \to S^3$ be a smooth immersion, which maps a smooth compact torus $\Sigma$ simply and smoothly onto the Hopf-torus $\pi^{-1}(\text{trace}(\gamma)) \subset S^3$. 

1) For every fixed \( s^* \in S^1 \) and \( q^* \in \pi^{-1}(\gamma(s^*)) \subset S^3 \) there is a unique horizontal smooth lift \( \eta(s^*,q^*) : \text{dom}(\eta(s^*,q^*)) \rightarrow \pi^{-1}(\text{trace}(\gamma)) \subset S^3 \), defined on a non-empty, open and connected subset \( \text{dom}(\eta(s^*,q^*)) \subset S^1 \), of \( \gamma : S^1 \rightarrow S^2 \) w.r.t. the Hopf-fibration \( \pi \), such that \( \text{dom}(\eta(s^*,q^*)) \) contains the point \( s^* \) and such that \( \eta(s^*,q^*) \) attains the value \( q^* \) in \( s^* \); i.e. \( \eta(s^*,q^*) \) is a smooth path in the torus \( \pi^{-1}(\text{trace}(\gamma)) \), which intersects the fibers of \( \pi \) perpendicularly and satisfies:

\[
(\pi \circ \eta(s^*,q^*))(s) = \gamma(s) \quad \forall s \in \text{dom}(\eta(s^*,q^*)) \quad \text{and} \quad \eta(s^*,q^*)(s^*) = q^* ,
\]

and there is exactly one such function \( \eta(s^*,q^*) \) mapping the open and connected subset \( \text{dom}(\eta(s^*,q^*)) \subset S^1 \) into \( \pi^{-1}(\text{trace}(\gamma)) \subset S^3 \).

2) There is some \( \epsilon = \epsilon(F,\gamma) > 0 \) such that for every fixed \( s^* \in S^1 \) and every \( x^* \in (\pi \circ F)^{-1}(\gamma(s^*)) \) there is a horizontal smooth lift \( \eta_F(s^*,x^*) \) of \( \gamma|_{S^1 \cap B_\epsilon(s^*)} \) w.r.t. the fibration \( \pi \circ F : \Sigma \longrightarrow \text{trace}(\gamma) \subset S^2 \), attaining the value \( x^* \) in \( s^* \), i.e. \( \eta_F(s^*,x^*) \) is a smooth path in the torus \( \Sigma \) which intersects the fibers of \( \pi \circ F \) perpendicularly and satisfies:

\[
(\pi \circ F \circ \eta_F(s^*,x^*))(s) = \gamma(s) \quad \forall s \in S^1 \cap B_\epsilon(s^*) \quad \text{and} \quad \eta_F(s^*,x^*)(s^*) = x^* .
\]

This implies in particular, that for the above \( \epsilon = \epsilon(F,\gamma) > 0 \) the function \( \eta_F \mapsto F \circ \eta_F \) maps the set \( \mathcal{L}(\gamma|_{S^1 \cap B_\epsilon(s^*)},\pi \circ F) \) of horizontal smooth lifts of \( \gamma|_{S^1 \cap B_\epsilon(s^*)} \) w.r.t. \( \pi \circ F \) surjectively onto the set \( \mathcal{L}(\gamma|_{S^1 \cap B_\epsilon(s^*)},\pi) \) of horizontal smooth lifts of \( \gamma|_{S^1 \cap B_\epsilon(s^*)} \) w.r.t. \( \pi \).

**Proof.** See the proof of Lemma 5.1 in [18].

### 8 Appendix C

We shall now quote the important Proposition 4.5 of [34] respectively Corollary 7.6 in [41], which we needed in the proof of Theorem 3.1; see also formula (2.21) in [4] for the same statement, but formulated in the \( C^\infty \)-smooth setting. As in the proof of Theorem 3.1 we are going to employ here the “exterior derivative” \( \delta : \Omega^0(B^2_1(0)) \rightarrow \Omega^1(B^2_1(0)) \) and its adjoint \( \delta^* = - \star \delta : \Omega^1(B^2_1(0)) \rightarrow \Omega^0(B^2_1(0)) \) w.r.t. the Euclidean metric on the open unit disc \( B^2_1(0) \), connecting the vector spaces \( \Omega^0(B^2_1(0)) \) and \( \Omega^1(B^2_1(0)) \) of differential forms of degrees 1 and 0. Here we have denoted as usual: \( \Omega^\alpha(B^2_1(0)) := \Gamma(\Lambda^\alpha(B^2_1(0))) \), for \( n \in \mathbb{N}_0 \); see here also Section 10.5 in [16] and Section 3.3 in [22] for valuable explanations.

**Proposition 8.1.** Let \( f : B^2_1(0) \rightarrow \mathbb{R}^3 \) be a uniformly conformal \( (W^{1,\infty} \cap W^{2,2}) \)-immersion w.r.t. \( g_{\text{eucl}} \) on \( B^2_1(0) \) - see Definition 2.2 - and let the differential one-form \( w_f \) from line (21) be Hodge-decomposable into \( w_f = d\mathcal{L} + \star dL \) in \( \mathcal{D}'(B^2_1(0)) \), for some functions \( \mathcal{L} \in W^{2,2}(B^2_1(0) ; \mathbb{R}^3) \) and \( L \in L^{2,\infty}(B^2_1(0) ; \mathbb{R}^3) \). Moreover, suppose that there are functions \( R \) and \( S \) of class \( W^{2,p}(B^2_1(0)) \) for some \( p > 2 \), and functions \( R, S \) of class \( W^{1,2,\infty}(B^2_1(0)) \), such that

\[
-df \times \tilde{H}_f - ((\star df), L) = dR + \star dS \quad \text{in} \quad \mathcal{D}'(B^2_1(0)),
\]

and \( -((\star df), L) = dS + \star dS \) in \( \mathcal{D}'(B^2_1(0)) \).
Then the following equations hold for $L$, $R$, $S$, $L$, $R$ and $S$ in the sense of distributions:

\[
\begin{align*}
\triangle_{\text{euc}}(R) &= dN_f \times (dR + \star dR) - \langle dN_f, dS + \star dS \rangle + (\star df) \times dL, \\
\triangle_{\text{euc}}(S) &= \langle dN_f, dR + \star dR \rangle + \langle \star df, dL \rangle, \\
\triangle_{\text{euc}}(f) &= df \times (dR + \star dR) + \langle df, dS + \star dS \rangle \quad \text{in } D'(B^2_1(0)),
\end{align*}
\]

where $N_f := \frac{f_{x_1} \times f_{x_2}}{|f_{x_1} \times f_{x_2}|}$ denotes one of the two Gauss maps along the immersion $f$.

Furthermore, we employed the following regularity result in the proof of Theorem 3.1, which is an improvement of the famous “Wente-estimate” for weak $W^{1,2}$-solutions of the “constant mean curvature differential equation” on bounded domains in $\mathbb{R}^2$. Its origin can be traced back either to Wente’s, Brezis’ and Coron’s investigation of this equation in [50] and [6], or also to Theorems 3.2.1 and 3.3.1 in [32], which are built on the seminal papers [12] and [7] about “Hardy-BMO-duality” and “compensated compactness”. One can also find a similar result in Chapter IV of Rivière’s dissertation [36], pp. 100–103, and in Theorem 3.10 of his lecture notes [41]. Theorem 8.1 was employed in Palmurella’s dissertation [34], exactly in the stated generality; see Theorem 2.1 in [34].

**Theorem 8.1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $a \in W^{1,2}(\Omega, \mathbb{R})$, $f \in L^p(\Omega, \mathbb{R})$ for some $p \in (1, 2)$, and let $u \in W^{1,2}(\Omega, \mathbb{R})$ be a weak solution of the distributional elliptic equation:

\[-\triangle u = \langle \nabla \perp a, \nabla u \rangle_{\mathbb{R}^2} + f \quad \text{in } D'(\Omega).\]

Then there holds: $u \in W^{2,p}_{\text{loc}}(\Omega, \mathbb{R})$. \hfill \Box

### 9 Appendix D

In the proof of the first part of Theorem 1.2 we employed the following two GMT-results, which can be quickly derived from the general “monotonicity formula” for $n$-rectifiable varifolds in $\mathbb{R}^{n+m}$ with locally bounded first variations; see Paragraph 17 in [44].

**Proposition 9.1.** Let $\nu_j$ be $n$-rectifiable varifolds defined on an open subset $\Omega \subseteq \mathbb{R}^{n+m}$, $n, m \in \mathbb{N}$, with locally bounded first variations $\delta \nu_j$ and such that for every open ball $B_\varepsilon := B^{n+m}_\varepsilon(x_0) \subseteq \Omega$ there holds:

\[\nu_j(\Omega) + \varepsilon^{1-\alpha - \beta} \nu_j(B_\varepsilon)^{\alpha - 1} \parallel \delta \nu_j \parallel (B_\varepsilon) \leq \Lambda \quad \forall j \in \mathbb{N},\]

where $\alpha$ and $\beta$ are appropriate positive numbers and $\Lambda >> 1$ sufficiently large. If moreover $\nu_j \rightharpoonup \nu$ weakly as Radon measures on $\Omega$, then the $n$-dimensional Hausdorff-density $\theta^n(\nu)$ of $\nu$ exists in every point of $\Omega$, and for any convergent sequence $x_j \rightharpoonup x_0 \in \Omega$ there holds:

\[\theta^n(\nu, x_0) \geq \limsup_{j \to \infty} \theta^n(\nu_j, x_j).\] \hfill \Box
Proposition 9.2. Let $\nu_j$ be $n$-rectifiable varifolds defined on an open subset $\Omega \subseteq \mathbb{R}^{n+m}$, $n,m \in \mathbb{N}$, with locally bounded first variations $\delta \nu_j$ and such that for every open ball $B_\epsilon := B_{\epsilon}^{n+m}(x_0) \subseteq \Omega$ there holds:

$$\nu_j(\Omega) + \epsilon^{1-\alpha_n-\beta} \nu_j(B_\epsilon)^{\alpha_n-1} \| \delta \nu_j \| (B_\epsilon) \leq \Lambda \quad \forall j \in \mathbb{N},$$

where $\alpha$ and $\beta$ are appropriate positive numbers and $\Lambda \gg 1$ sufficiently large. If moreover the $n$-dimensional Hausdorff-densities $\theta^n(\nu_j)$ of $\nu_j$ satisfy $\theta^n(\nu_j) \geq 1$ on $\text{spt}(\nu_j)$, for each $j \in \mathbb{N}$, and if $\nu_j \rightharpoonup \nu$ weakly as Radon measures on $\Omega$, then there holds:

$$\text{spt}(\nu_j) \rightarrow \text{spt}(\nu) \text{ in Hausdorff distance locally in } \Omega,$$

as $j \to \infty$, and even more precisely:

$$\text{spt}(\nu) = \{ x \in \Omega | \exists x_j \in \text{spt}(\nu_j) \text{ for every } j \in \mathbb{N} \text{ such that } x_j \rightarrow x \}.$$ 

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