Blowing up generalized Kähler 4-manifolds

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Abstract

We show that the blow-up of a generalized Kähler 4-manifold in a non-degenerate complex point admits a generalized Kähler metric. As with the blow-up of complex surfaces, this metric may be chosen to coincide with the original outside a tubular neighbourhood of the exceptional divisor. To accomplish this, we develop a blow-up operation for bi-Hermitian manifolds.

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1 Introduction

Let \((M, \mathbb{J}_+, \mathbb{J}_-, \mathbb{J}_+^+, \mathbb{J}_-^-)\) be a generalized Kähler 4-manifold such that both generalized complex structures \(\mathbb{J}_+, \mathbb{J}_-\) have even type, meaning that they are equivalent to either a complex or symplectic structure at every point. In other words, their underlying real Poisson structures \(P_+, P_-\) have either rank 0 (at complex points) or 4 (at symplectic points). The structure \(\mathbb{J}_\pm\) is equipped with a canonical section \(s_\pm\) of its anticanonical line bundle, vanishing on the locus \(D_\pm\) of complex points, where \(P_\pm\) has rank zero. From [6], it follows that the symplectic leaves of \(P_+\) and \(P_-\) must be everywhere transverse, so that \(D_+, D_-\) are disjoint.

It was shown in [2] that in a neighbourhood of a complex point \(p \in D_+\) which is nondegenerate, in the sense of being a nondegenerate zero of \(s_+\), there are complex coordinates \((w, z)\) such that the generalized complex structure \(\mathbb{J}_+\) is equivalent to that defined by the differential form

\[
\rho_+ = w + dw \wedge dz.
\]  (1.1)

Note that \(D_+ = w^{-1}(0)\), along which \(\rho_+|_{D_+} = dw \wedge dz\) defines a complex structure, whereas for \(w \neq 0\), we have \(\rho_+ = w \exp(B + i\omega)\), for \(B + i\omega = d \log w \wedge dz\), defining a symplectic form \(\omega\) away from \(D_+\), as required.

It was then shown [2, Theorem 3.3] that the complex blow-up at \(p\) using the coordinates \((z, w)\) inherits a generalized complex structure. We detail in Section 2 why this structure is independent of the chosen coordinates. Thus we obtain a canonical blow-up \((\tilde{M}, \tilde{\mathbb{J}}_+)\) of \((M, \mathbb{J}_+)\) at \(p\), equipped with a generalized holomorphic map \(\pi : \tilde{M} \rightarrow M\) which is an isomorphism outside the exceptional divisor \(E = \pi^{-1}(p)\). The complex locus \(\tilde{D}_+\) of the blow-up is the proper transform of \(D_+\), and the exceptional divisor \(E\) is a 2-sphere which intersects \(\tilde{D}_+\) transversely at one point and is Lagrangian with respect to \(\omega\) elsewhere; this makes \(E\) a generalized complex brane [2].
In Section 5.1, we use the bi-Hermitian tools developed in Section 3 to construct a degenerate generalized Kähler structure on the blow-up, in the sense that the metric degenerates along the exceptional divisor $E$. Finally, in Section 5.2, we use a deformation procedure detailed in Section 4 to obtain a positive-definite metric, defining a generalized Kähler structure such that $\pi$ is an isomorphism away from a tubular neighbourhood of the exceptional divisor $E$. The generalized complex structure $J$ does not lift uniquely to the blow-up, as there is no preferred choice of symplectic area for $E$; this degree of freedom inherent in the generalized Kähler blow-up is familiar from the usual Kähler blow-up operation.

2 Generalized complex blow-up

Let $(w, z)$ be standard coordinates for $M = \mathbb{C}^2$, and consider the generalized complex structure $\mathcal{J}$ defined by the form $\rho_+$ given in (1.1). This structure extends uniquely to a generalized complex structure $\tilde{\mathcal{J}}$ the blow-up $\tilde{M} = [\mathbb{C}^2 : 0]$ of the plane in the origin, simply because the anticanonical section $\sigma = w\partial_w \wedge \partial_z$ does. That is, the line generated by $\rho_+$ may be written

$$\langle \rho_+ \rangle = e^\sigma \Omega^{2,0}(M),$$

and in the two blow-up charts $(w_0, z_0) = (w/z, z)$ and $(w_1, z_1) = (w, z/w)$, this pulls back to the line $e^\tilde{\sigma} \Omega^{2,0}(\tilde{M})$, where

$$\tilde{\sigma} = w_0 \partial_{w_0} \wedge \partial_{z_0} = \partial_{w_1} \wedge \partial_{z_1}.$$ 

Clearly, $\tilde{\sigma}$ drops rank along the proper transform of $w^{-1}(0)$, namely $w_0^{-1}(0)$.

The above construction of $\tilde{\mathcal{J}}$ uses the complex structure defined by $(w, z)$, but this complex structure is not determined canonically by $\mathcal{J}$. That is, there are automorphisms $\Phi = (\varphi, B) \in \text{Diff}(M) \ltimes \Omega^{2,\text{cl}}(M, \mathbb{R})$ of $\mathcal{J}$ for which $\varphi$ is not a holomorphic automorphism of $\mathbb{C}^2$. To show that $\tilde{\mathcal{J}}$ is independent of the particular complex structure used to perform the blow-up, we must show that any such automorphism $\Phi \in \text{Aut}(\mathcal{J})$ with $\varphi(0) = 0$ lifts to the blow-up $[\mathbb{C}^2 : 0]$.

**Theorem 2.1.** Any automorphism of $\mathcal{J}$ on $M = \mathbb{C}^2$ fixing the origin lifts to the blow-up $\tilde{M}$ of $M$ in the origin.
Proof. Let \( \Phi = (\varphi, B) \in \text{Aut}(\mathcal{J}) \), meaning that

\[
e^B \varphi^* (w + dw \wedge dz) = e^\lambda (w + dw \wedge dz),
\]  

(2.1)

for some \( \lambda \in C^\infty(M, \mathbb{C}) \). Also, assume \( \varphi(0) = 0 \). Let \( p : \tilde{M} \to M \) be the blow-down map. We will show that \( \varphi \) lifts to \( \tilde{\varphi} \in \text{Diff}(\tilde{M}) \) such that \( p \circ \tilde{\varphi} = \varphi \circ p \), and then \( (\tilde{\varphi}, p^* B) \in \text{Aut}(\tilde{\mathcal{J}}) \) is the required lift of the automorphism. The lift \( \tilde{\varphi} \) exists if and only if the functions \( \tilde{w} = \varphi^* w, \tilde{z} = \varphi^* z \) are in the ideal generated by \( w \) and \( z \) in \( C^\infty(M, \mathbb{C}) \). By a theorem of Malgrange [10], this is equivalent to the following constraints:

\[
\frac{\partial^{p+q} \tilde{w}}{\partial^p w \partial^q \tilde{z}} \bigg|_{(0,0)} = 0 \quad \text{and} \quad \frac{\partial^{p+q} \tilde{z}}{\partial^p \tilde{w} \partial^q \tilde{z}} \bigg|_{(0,0)} = 0, \quad \text{for all } p, q \in \mathbb{N}. \tag{2.2}
\]

To verify (2.2), we rewrite (2.1) as follows:

\[
\tilde{w} + d\tilde{w} \wedge d\tilde{z} = e^\lambda e^{-B} (w + dw \wedge dz) = e^\lambda (w + dw \wedge dz - wB), \tag{2.3}
\]

where the summand of degree four is omitted from the last term since it vanishes. From this we immediately conclude that \( \tilde{w} = e^\lambda w \), so that \( \tilde{w} \) satisfies (2.2). But then

\[
d\tilde{w} \wedge d\tilde{z} = d(e^\lambda w) \wedge d\tilde{z} = e^\lambda (dw + wd\lambda) \wedge (\frac{\partial \tilde{z}}{\partial w} dw + \frac{\partial \tilde{z}}{\partial w} dw + \frac{\partial \tilde{z}}{\partial z} dz + \frac{\partial \tilde{z}}{\partial z} dz).
\]

By (2.3), this coincides with \( e^\lambda (dw \wedge dz - wB) \), and equating \( dw \wedge dz \) components we obtain

\[
(1 + w \frac{\partial \lambda}{\partial w}) \frac{\partial \tilde{z}}{\partial w} - w \frac{\partial \lambda}{\partial w} \frac{\partial \tilde{z}}{\partial w} = -wB_{w\tilde{z}}.
\]

Solving for \( \frac{\partial \tilde{z}}{\partial w} \) we obtain, near \((0,0)\),

\[
\frac{\partial \tilde{z}}{\partial w} = \frac{w(\frac{\partial \lambda}{\partial w} \frac{\partial \tilde{z}}{\partial w} - B_{w\tilde{z}})}{1 + w \frac{\partial \lambda}{\partial w}}. \tag{2.4}
\]

Similarly, equating \( dw \wedge d\tilde{w} \) components yields, near \((0,0)\),

\[
\frac{\partial \tilde{z}}{\partial \tilde{w}} = \frac{w(\frac{\partial \lambda}{\partial \tilde{w}} \frac{\partial \tilde{z}}{\partial \tilde{w}} - B_{w\tilde{w}})}{1 + w \frac{\partial \lambda}{\partial \tilde{w}}}. \tag{2.5}
\]

Finally, (2.4), (2.5) imply that (2.2) holds for \( \tilde{z} \), as required. \( \square \)
3 Bi-Hermitian approach

Our main tool for describing the geometry of the blow-up will be the bi-Hermitian approach to generalized Kähler geometry [6], which we describe briefly here. Since we are interested in a neighbourhood of a point, we may assume that the torsion 3-form $H$ of our generalized Kähler structure is cohomologically trivial. Such a generalized Kähler structure determines and is determined by a Riemannian metric $g$, a 2-form $b$, and a pair of complex structures $I_+, I_-$ which are compatible with $g$ and satisfy the condition

$$\pm d_c^\pm \omega_\pm = db,$$

(3.1)

where $\omega_\pm = gI_\pm$ are the usual Hermitian 2-forms and $d_c^\pm = [d, I_\pm^*]$ are the real Dolbeault operators associated to $I_\pm$. The correspondence between the generalized Kähler pair $\mathcal{J}_+, J_-$ and the above bi-Hermitian data is as follows:

$$J_\pm = \frac{1}{2} \begin{pmatrix} 1 & -b & 1 \end{pmatrix} \begin{pmatrix} I_+ \pm I_- & -\omega^1_\pm \mp \omega^{-1}_\pm \omega_+ & \omega_+ \mp \omega_- \mp \omega_+ \pm I_+^* \pm I_-^* \end{pmatrix} \begin{pmatrix} 1 \\ b \\ 1 \end{pmatrix}.$$  

(3.2)

It was observed in [7] that the bi-Hermitian condition endows the complex structure $I_\pm$ with a holomorphic Poisson structure $\sigma_\pm$ with real part

$$Q = \text{Re}(\sigma_+) = \text{Re}(\sigma_-) = \frac{1}{8}[I_+, I_-]g^{-1}.$$  

(3.3)

Indeed, $\sigma_\pm$ derives from a pair of transverse holomorphic Dirac structures as described in [6], though we shall not make use of this here.

Any pair of complex structures satisfies the following identity for the commutator:

$$[I_+, I_-] = (I_+ - I_-)(I_+ - I_-).$$  

(3.4)

Therefore, the zeros of $Q$ coincide with the loci where $I_+ = I_-$ or $I_+ = -I_-$. From (3.2), we see that the real Poisson structures $P_\pm$ underlying $\mathcal{J}_\pm$ are given by

$$P_\pm = -\frac{1}{2}(\omega^1_\pm \mp \omega^{-1}_\pm) = \frac{1}{8}(I_+ \mp I_-)g^{-1}.$$  

(3.5)

Therefore, we conclude that the zero locus of $Q$, and hence $\sigma_\pm$, is the union of the zero loci for $P_+, P_-$, namely the subsets $D_+, D_-$ discussed in section 1.

The holomorphic Poisson structure $(I_\pm, \sigma_\pm)$ provides an economical means to describe the full generalized Kähler structure, as observed in [5].

Theorem 3.1 ([5], Theorem 6.2). Let $(I_0, \sigma_0)$ be a holomorphic Poisson structure with $\text{Re}(\sigma_0) = Q$. Any closed 2-form $F$ satisfying the equation

$$FI_0 + I_0^* F + F Q F = 0$$  

(3.6)

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defines an integrable complex structure \( I_1 = I_0 + QF \), a symmetric tensor \( g = -\frac{1}{2}F(I_0 + I_1) \), and a 2-form \( b = -\frac{1}{2}F(-I_0 + I_1) \) such that

\[
dc^c_0 \omega_0 = -dc^c_1 \omega_1 = db.
\]

If \( g \) is positive-definite, then \((g, I_0, I_1)\) defines a bi-Hermitian structure satisfying (3.1), and hence a generalized Kähler structure, where \( J_- \) is the symplectic structure \( F \).

As is hinted at in Theorem 3.1, in which \( g \) need not be positive-definite, it will be useful in studying the blowup for us to relax the generalized Kähler condition, allowing degenerations of the Riemannian metric while maintaining the remaining constraints.

**Definition 3.2.** A degenerate bi-Hermitian structure \((g, b, I_+, I_-)\) consists of a possibly degenerate tensor \( g \in \Gamma^\infty(\text{Sym}^2 T^*) \), a 2-form \( b \in \Omega^2 \), and two integrable complex structures \( I_+, I_- \), such that \( gI^\pm + I^\pm g = 0 \) and

\[
dc^c_+ \omega_+ = -dc^c_- \omega_- = db,
\]

where \( \omega_\pm = gI_\pm \). Informally, it is a generalized Kähler structure where \( g \) may be degenerate.

Degenerate bi-Hermitian structures arising from the construction in Theorem 3.1 as solutions to (3.6) enjoy a composition operation which we now review (see [5] for details).

If \( F_{01} \) is a closed 2-form solving

\[
F_{01}I_0 + I_0^*F_{01} + F_{01}QF_{01} = 0,
\]

for a holomorphic Poisson structure \((I_0, \sigma_0)\) with \( \text{Re}(\sigma_0) = Q \), then it determines a second holomorphic Poisson structure \((I_1, \sigma_1)\) with \( \text{Re}(\sigma_1) = Q \), via \( I_1 = I_0 + QF_{01} \). If we then have another closed 2-form \( F_{12} \), such that

\[
F_{12}I_1 + I_1^*F_{12} + F_{12}QF_{12} = 0,
\]

then it determines a third holomorphic Poisson structure \((I_2, \sigma_2)\) with \( \text{Re}(\sigma_2) = Q \), via \( I_2 = I_1 + QF_{12} \). Rewriting (3.7) and (3.8) as the pair

\[
F_{01}I_0 + I_0^*F_{01} = 0, \quad F_{12}I_1 + I_1^*F_{12} = 0,
\]

we see that the closed 2-form \( F_{02} = F_{01} + F_{12} \) satisfies

\[
F_{02}I_0 + I_0^*F_{02} + F_{02}QF_{02} = F_{02}I_0 + I_0^*F_{02} = F_{01}(I_2 - I_1) - (I_1^* - I_0^*)F_{12} = F_{01}QF_{12} - F_{01}QF_{12} = 0.
\]
We may interpret this in the following way: a solution to (3.7) defines a degenerate bi-Hermitian structure with constituent complex structures \((I_0, I_1)\), and a solution to (3.8) does the same, but with complex structures \((I_1, I_2)\). These two degenerate bi-Hermitian structures may be composed in the sense that the sum \(F_{02} = F_{01} + F_{12}\) defines a new degenerate bi-Hermitian structure with constituent complex structures \((I_0, I_2)\). This composition may be viewed as a groupoid (see Figure 1).

**Definition 3.3** ([5]). Fix a real manifold \(M\) with real Poisson structure \(Q\). Then we may define a groupoid whose objects are holomorphic Poisson structures \((I_i, \sigma_i)\) on \(M\) with \(\text{Re}(\sigma_i) = Q\) and whose morphisms \(\text{Hom}(i, j)\) are real closed 2-forms \(F_{ij}\) such that the following two equations hold.

\[
I_j - I_i = QF_{ij} \\
F_{ij}I_j + I_j^*F_{ij} = 0.
\]

The composition of morphisms is then simply addition of 2-forms \(F_{ij} + F_{jk}\).

**Remark 3.4.** Combined with Theorem 3.1, this definition provides a composition operation for the degenerate bi-Hermitian structures determined by the 2-forms \(F_{ij}\).

### 4 Flow construction

We now review a method, introduced in [8] and developed in [5], for modifying a bi-Hermitian structure of the kind studied in the previous section using a smooth real-valued function. The method proceeds essentially by solving (3.8) using the flow of a suitably-chosen vector field, and then composing this solution with the given bi-Hermitian structure viewed as a solution to (3.7). This is a direct analog of the well-known modification of a Kähler form by adding \(f\) to the Kähler potential.

**Theorem 4.1** ([8, 5]). Let \((I_0, \sigma_0)\) be a holomorphic Poisson structure with \(Q = \text{Re}(\sigma_0)\), and let \(f\) be a smooth real-valued function. Let \(\phi_t\) be the time-\(t\) flow of the
Hamiltonian vector field $X = Q(df)$. Then, so far as the flow is well-defined, the closed 2-form

$$F_t = \int_0^t \varphi_s^*(dd^c_{I_0} f) ds \quad (4.1)$$

satisfies Equation 3.6, i.e.

$$F_t I_0 + I_0^* F_t + F_t QF_t = 0.$$

**Remark 4.2.** The above flow generates a family of integrable complex structures $I_t = I_0 + QF_t$, which are all equivalent, since $I_t = \varphi_t(I_0)$. If $f$ is strictly plurisubharmonic for $I_0$, i.e. defines a Riemannian metric $h = -(dd^c_{I_0} f)$, then from (4.1) we have

$$\lim_{t \to 0} t^{-1} F_t = dd^c_{I_0} f,$$

implying that the symmetric tensor

$$g_t = -\frac{1}{2} F_t (I_0 + I_t)$$

satisfies $\lim_{t \to 0} t^{-1} g_t = h$, so that $g_t$ defines a Riemannian metric for sufficiently small $t \neq 0$, and so by Theorem 4.1, we obtain a generalized Kähler structure $(g_t, I_0, I_t, b_t)$.

## 5 Generalized Kähler blow-up

We now apply the machinery of the preceding sections to the problem of blowing up the generalized Kähler 4-manifold $(M, J_+, J_-)$ introduced in Section 1 at a nondegenerate point $p \in D_+$ in the complex locus of $J_+$. The first step (§ 5.1) is to blow up the generalized complex structure $J_+$ and obtain a degenerate bi-Hermitian structure. In the second step (§ 5.2) we deform the degenerate bi-Hermitian structure by composing it with another degenerate bi-Hermitian structure obtained from the flow construction (§ 4). Finally (§ 5.3), we prove that the resulting deformation is positive-definite, defining a generalized Kähler structure on the blow-up.

### 5.1 Simultaneous blow-up

**Lemma 5.1.** In a neighbourhood of the nondegenerate point $p \in D_+$, there exist complex coordinates $(u_\pm, v_\pm)$ such that the holomorphic Poisson structure $(I_\pm, \sigma_\pm)$ is given by $u_\pm \partial_{u_\pm} \wedge \partial_{v_\pm}$.

**Proof.** From the normal form for $J_+$ near $p$ given by Equation 1.1, it follows that $P_+$ is isomorphic to $\text{Im}(w \partial_w \wedge \partial_z)$. In particular, $P_+$ vanishes linearly.
along $D_+$. By Equations 3.3, 3.4, and 3.5, and since $D_-$ is disjoint from $D_+$, it follows that $Q = -\frac{1}{2}[I_+, I_-]g^{-1}$ has linear vanishing along $D_-$ as well. This means that the holomorphic Poisson structure $\sigma_{\pm}$ is a section of a holomorphic line bundle $\Lambda^2 T_{1,0}$ with a nondegenerate zero at $p$. Hence we may choose $I_\pm$-complex coordinates $(u_\pm, v_\pm)$ near $p$ such that $\sigma_{\pm} = u_\pm \partial_{u_\pm} \wedge \partial_{v_\pm}$, as required.

We now demonstrate that the coordinates $(u_\pm, v_\pm)$ placing $\sigma_{\pm}$ into standard form are closely related to the coordinates $(w, z)$ placing $\mathbb{J}_+$ into the standard form 1.1.

**Lemma 5.2.** In a sufficiently small neighbourhood $U$ of $p$ where the following coordinates are defined, the functions $u_\pm, v_\pm$ lie in the ideal of $C^\infty(U, \mathbb{C})$ generated by $w, z$.

**Proof.** Let $\rho_+$ be the generator (1.1) defined by $\mathbb{J}_+$ in $U$, and let $\rho_- = e^\beta$ be the generator defined by $\mathbb{J}_-$, which has symplectic type in $U$, so that $\beta = B + i\omega$ is a complex 2-form such that $\omega$ is symplectic.

The holomorphic Poisson structures $\sigma_{\pm} = u_\pm \partial_{u_\pm} \wedge \partial_{v_\pm}$ define generalized complex structures in $U$ via the differential forms

$$u_\pm + du_\pm \wedge dv_\pm \in e^{\sigma_{\pm}} \Omega^{2,0}_\pm.$$

In [6], it is shown that these holomorphic Poisson structures may be expressed as a certain “wedge product” of the underlying generalized complex structures $(\mathbb{J}_+, \mathbb{J}_-)$. Explicitly, this provides the following identities$^1$:

$$e^{\bar{\beta}}(w - dw \wedge dz) = e^{\lambda_-}(u_- + du_- \wedge dv_-)$$

$$e^\beta(w - dw \wedge dz) = e^{\lambda_+}(u_+ + du_+ \wedge dv_+),$$

for smooth functions $\lambda_+, \lambda_- \in C^\infty(U, \mathbb{C})$. Comparing these equations to (2.3), we see that the argument in the proof of Theorem 2.1 implies the required constraint on $u_\pm, v_\pm$. \hfill $\square$

**Theorem 5.3.** The complex structures $I_-, I_+$ underlying a generalized Kähler 4-manifold $(M, \mathbb{J}_+, \mathbb{J}_-)$ both lift to the blow-up of $(M, \mathbb{J}_+)$ at a nondegenerate complex point $p \in D_+$.

**Proof.** Let $\psi : U \to \mathbb{C}^2$ be the chart defined by $(w, z)$ in the normal form (1.1) and let $\varphi_\pm : U \to \mathbb{C}^2$ be the chart defined by $(u_\pm, v_\pm)$ in the normal form given by Lemma 5.1. Then $\chi_\pm = \psi \circ \varphi_\pm^{-1}$ is a diffeomorphism and $\chi_\pm(0) = 0$.

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$^1$In general, if $\rho_{\pm}$ generate the canonical line bundles of $\mathbb{J}_{\pm}$, then $\rho_+^T \wedge \rho_-^T$ generates $e^{\sigma_{\pm}} \Omega^{n,0}(M, I_+)$ and $\rho_+^T \wedge \varphi_-^T$ generates $e^{\sigma_-} \Omega^{n,0}(M, I_-)$. Here $\rho^T$ is the reversal anti-automorphism of forms.
The complex structure $I_\pm$ lifts to the blow-up $\tilde{M}$ precisely when the diffeomorphism $\chi_\pm$ lifts to a diffeomorphism of blow-ups $\tilde{\chi}_\pm : [\varphi_\pm(U) : 0] \to [\psi(U) : 0]$. This occurs if and only if $u_\pm$ and $v_\pm$ are contained in the ideal generated by $w, z$, which is itself guaranteed by Lemma 5.2.

**Remark 5.4.** It follows from the theorem that the complex structure $\tilde{I}_\pm$ we obtain on the blow-up of $(M, J_\pm)$ may be identified with the usual complex blow-up of $(M, I_\pm)$ at $p$. Furthermore, since the holomorphic Poisson structure $\sigma_\pm$ vanishes at $p$, it follows that $\sigma_\pm$ lifts to a holomorphic Poisson structure on the blow-up.

We now apply Theorem 5.3 to obtain a degenerate bi-Hermitian structure on the blow-up of $(M, J_\pm, J_-)$ at $p \in D_\pm$. Let $(g, I_+, I_-, b)$ be the bi-Hermitian structure on $M$ defined by the generalized Kähler structure.

**Corollary 5.5.** Let $(\tilde{M}, \tilde{J}_\pm)$ be the blow-up of the generalized complex 4-manifold $(M, J_\pm)$ at the nondegenerate point $p \in D_\pm$, with blow-down map $\pi$. Then $\tilde{M}$ inherits a degenerate bi-Hermitian structure $(\tilde{g}, \tilde{b}, \tilde{I}_+, \tilde{I}_-)$ such that $\pi : (\tilde{M}, \tilde{I}_\pm) \to (M, I_\pm)$ is a usual holomorphic blow-down and $\tilde{g} + \tilde{b} = \pi^* (g + b)$.

### 5.2 Deformation of degenerate bi-Hermitian structure

The degenerate bi-Hermitian structure on $\tilde{M}$ obtained in Corollary 5.5 fails to define a generalized Kähler structure because $\tilde{g}$ is not positive-definite along the exceptional divisor $E$. We now apply Theorem 4.1 to obtain a second degenerate bi-Hermitian structure, which we use to modify $(\tilde{g}, \tilde{b}, \tilde{I}_+, \tilde{I}_-)$. The modification will leave the structures on $\tilde{M}$ unchanged outside a tubular neighbourhood $V_E$ of $E$ which blows down to a neighbourhood of $p$ in which $J_-$ has symplectic type and is given by a complex 2-form with imaginary part $\omega$. Let $\pi : \tilde{M} \to M$ denote the blow-down map, and write $\tilde{\omega} = \pi^* \omega$ for the pull-back of the symplectic form to $V_E$.

First we describe the degenerate bi-Hermitian structure using the formalism of Theorem 3.1. The complex structure $\tilde{I}_-$ and the 2-form $\tilde{\omega}$ satisfy (3.6), and so in $V_E$ we have

$$\tilde{I}_+ = \tilde{I}_- + \tilde{Q}\tilde{\omega},$$

where $\tilde{Q} = \text{Re}(\tilde{\sigma}_-) = \text{Re}(\tilde{\sigma}_+)$, as in (3.3), and $\tilde{\sigma}_\pm$ is the blown up holomorphic Poisson structure. In the following, we construct a closed 2-form $F_t$ in a possibly smaller tubular neighbourhood such that

$$\tilde{I}_+^t = \tilde{I}_+ + \tilde{Q}F_t$$

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2 The flow construction may be applied equally well to degenerate bi-Hermitian structures.

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defines a new complex structure $\overline{I}_+$. The final task, completed in Section 5.3, will be to show that the composition (5.1), in the sense of Definition 3.3, defines a generalized Kähler structure.

\[
\overline{I}_- \overline{\omega} \overline{I}_+ \xrightarrow{f_t} \overline{I}_+^t
\]

(5.1)

We now construct $f_t$. Let $(u, v)$ be $I_+$-holomorphic coordinates near $p$ such that $\sigma_+ = u \partial_u \wedge \partial_v$, and let $(u_0, v_0) = (u/v, v)$ and $(u_1, v_1) = (u, v/u)$ be the two affine charts covering a tubular neighbourhood $V_E$ of the exceptional divisor $E = u_0^{-1}(0) \cup v_0^{-1}(0)$. Using $u_0, v_1$ as affine coordinates on $E \cong \mathbb{CP}^1$, we may describe the Fubini-Study metric $\omega_E$ in terms of the Kähler potential

\[
f_0 = \log \left( \frac{u_0 \overline{u}_0}{1 + u_0 \overline{u}_0} \right) = \log \left( \frac{1}{1 + v_0 \overline{v}_0} \right),
\]

which is smooth away from $u_0 = 0$ and satisfies $i \partial \overline{\partial} f_0 = \omega_E$. Although $f_0$ is singular, we observe that its Hamiltonian vector field is smooth:

\[
Q(df_0) = \text{Re}(u_0 \partial_u \wedge \partial v_0) d \log \left( \frac{u_0 \overline{u}_0}{1 + u_0 \overline{u}_0} \right) = \frac{1}{1 + u_0 \overline{u}_0} \text{Re}(\partial v_0).
\]

Hence $Q(df_0)$ defines a smooth Poisson vector field on $V_E$.

Now choose a bump function $\epsilon \in C^\infty(V_E, [0, 1])$ which vanishes on a smaller tubular neighbourhood $U_E \subset V_E$ and is such that $1 - \epsilon$ has compact support in a closed disc bundle $K$ over $E$, with $U_E \subset K \subset V_E$. Consider the smooth function $f_\epsilon \in C^\infty(V_E, \mathbb{R})$ given by

\[
f_\epsilon = \epsilon \log(u \overline{u} + v \overline{v}) = \epsilon \log(v_0 \overline{v}_0(1 + u_0 \overline{u}_0)).
\]

Since $i \partial \overline{\partial} \log(v_0 \overline{v}_0(1 + u_0 \overline{u}_0)) = i \partial \overline{\partial} \log(1 + u_0 \overline{u}_0) = -i \partial \overline{\partial} f_0$, it follows that

\[
f = c(f_0 + f_\epsilon), \quad c \in \mathbb{R}_{>0}
\]

(5.2)

has the property that $X = Q(df)$ is a smooth Poisson vector field in $V_E$ and

\[
i \partial \overline{\partial} f = \begin{cases} c \omega_E & \text{in } U_E \\ 0 & \text{outside } K \end{cases}
\]

(5.3)

For sufficiently small $\delta > 0$, there exists an open neighbourhood $V_E'$, with $K \subset V_E' \subset V_E$, on which the flow $\varphi_t$ of $X$ is well-defined for all $t \in (-\delta, \delta)$. Also, choose $\delta$ small enough so that there is a neighbourhood $V_E''$ with $\overline{V}_E'' \subset V_E'$, with $\varphi_1(K) \subset V_E''$ for $t \in (-\delta, \delta)$. Using (5.3), we see that $\varphi_t^* (i \partial \overline{\partial} f)$ is smooth on $V_E''$, with compact support contained in $V_E''$, for all $t \in (-\delta, \delta)$.
We now apply Theorem 4.1 to the flow $\varphi_t$ on $V'_E$. This provides a solution

$$F_t = \int_0^t \varphi_s^* (dd^c_{I^+_1} f) ds$$

to Equation 3.6 for all $t \in (-\delta, \delta)$, with compact support in $V'_E$. Therefore, we obtain a family of complex structures on $V'_E$ given by

$$I^+_1 = I_+ + QF_t.$$  (5.4)

Since $F_t$ has compact support contained in $V'_E$, the complex structure $I^+_1$ may be extended to all of $\tilde{M}$ by setting it equal to $I_+$ outside $V'_E$. We summarize the above procedure in the following result.

**Proposition 5.6.** The flow construction of Theorem 4.1, applied to the singular function $f$ given in (5.2), produces a smooth family of solutions $(F_t)_{t \in (-\delta, \delta)}$ to (3.6) with compact support in a tubular neighbourhood of the exceptional divisor, and hence we obtain a degenerate bi-Hermitian structure

$$(\tilde{g}'_t, \tilde{b}'_t, \tilde{I}^+_t, I_+)$$

on $\tilde{M}$, where $\tilde{I}^+_t$ is given by (5.4) and $\tilde{g}'_t, \tilde{b}'_t$ are as in Theorem 3.1, yielding

$$\tilde{g}'_t = -\frac{1}{2} F_t (I_+ + \tilde{I}^+_t).$$  (5.5)

In Section 5.3, we compose the above degenerate bi-Hermitian structure with that from Corollary 5.5 and show the resulting structure is positive-definite.

---

3The fact that $f$ is not smooth does not affect the validity of Theorem 4.1 in this case, as the vector field $X = Q(df)$ is a smooth Poisson vector field, and hence locally Hamiltonian.
Remark 5.7. The family of complex structures $\bar{I}_+$ on $\bar{M}$ constructed above defines a deformation of the blow-up complex structure $\bar{I}_+$ in the direction given by the class in $H^1(\mathcal{J})$ defined by the vector field $Z = Q(df)$, which is a holomorphic vector field on the annular neighbourhood of $E$ defined by $V_E \setminus K$. The $(1,0)$ part of $Z$ in this annular neighbourhood is (in the $(u_0,v_0)$ chart)

$$Z^{1,0} = c\sigma_*(\frac{\omega}{1+u_0\bar{u}_0} + \log(v_0\bar{v}_0(1 + u_0\bar{u}_0)))$$

$$= c(u_0\delta u_0 \wedge \delta v_0)(u_0^{-1} du_0 + v_0^{-1} dv_0)$$

$$= c(\delta v_0 - \frac{u_0}{v_0}\delta u_0).$$

This deformation class has a geometric interpretation: since $p \in D_+$ and $\sigma_+|_{D_+} = 0$, the contraction

$$\text{Tr}(d\sigma_+|_{D_+})$$

defines a holomorphic vector field $\chi$ on $D_+$. The flow of $c\chi$ then provides a path $p(t)$ of points on $D_+$. The family of blow-ups of $(M, I_+)$ at $p(t)$ provides a deformation of complex structure with derivative $[Z^{(1,0)}]$ at $t = 0$.

### 5.3 Positivity

Now that we have constructed the two degenerate bi-Hermitian structures on $\bar{M}$ occurring in (5.1), we must argue that their composition in the sense of Definition 3.3 is positive-definite. The composition is the (a priori degenerate) bi-Hermitian structure $(\bar{g}_t, \bar{b}_t, \bar{I}_-, \bar{I}_+)$, where

$$\bar{g}_t = -\frac{t}{2}((\bar{\omega} + F_t)(\bar{I}_- + \bar{I}_+))$$

$$\bar{b}_t = -\frac{t}{2}((\bar{\omega} + F_t)(-\bar{I}_- + \bar{I}_+)).$$

Rewriting this, we obtain

$$\bar{g}_t = -\frac{t}{2}\left((\bar{\omega}(\bar{I}_- + \bar{I}_+) + \bar{\omega}(\bar{I}_- + \bar{I}_+) + F_t(\bar{I}_- + \bar{I}_+) + F_t(\bar{I}_+ + \bar{I}_+)\right)$$

$$= \bar{g} + \bar{g}' - \frac{t}{2}(\bar{Q}QF_t - F_t\bar{Q}\bar{\omega}),$$

where we use the fact that $\bar{I}_- - \bar{I}_+ = \bar{Q}\bar{\omega}$ and $\bar{I}_+ - \bar{I}_- = \bar{Q}F_t$.

**Theorem 5.8.** Provided that $c$ in (5.2) is chosen small enough, the symmetric tensor $\bar{g}_t$ defined by (5.6) is positive-definite on $\bar{M}$ for sufficiently small $t \neq 0$, defining a generalized Kähler structure on the blow-up.

**Proof.** Since $F_t \to 0$ as $t \to 0$, it follows that $\bar{I}^t_+ \to \bar{I}_+$ as $t \to 0$. By Equation 5.5, therefore, we see that

$$\lim_{t \to 0} \frac{t}{2} \bar{g}'_t = -(dd^c f)(\bar{I}_+) = \begin{cases} c\omega_E & \text{in } U_E \\ 0 & \text{outside } K \end{cases}$$
where \( \omega_E \) is the Fubini-Study metric. This implies that \( \tilde{g}_t' \) is positive-definite when restricted to \( TE \) for sufficiently small nonzero \( t \), and hence \( \tilde{g} + \tilde{g}_t' \) is positive-definite in a neighbourhood of \( E \) for sufficiently small nonzero \( t \). Also, the third summand in (5.6) is proportional to \( \bar{\omega} \), which vanishes along \( E \).

Fix \( c = c_0 \in \mathbb{R}_{>0} \) in the definition (5.2) of \( f \), and let \( U \subset U_E \) be a tubular neighbourhood of \( E \) where the third summand in (5.6) is so small that \( \tilde{g}_t \) is positive-definite in \( U \) for sufficiently small nonzero \( t \). Note that \( \tilde{g}_t \) is certainly positive-definite outside \( K \) (where it coincides with \( \tilde{g} \)), hence it remains to show that \( \tilde{g}_t \) is positive in the intermediate region \( K \setminus U \).

We have chosen \( U \) so that the third term in (5.6) is dominated there by the first two terms. This means that at each point in \( U \) and for each vector \( v \neq 0 \) (and for sufficiently small nonzero \( t \)), we have

\[
|\tilde{Q}(F_t v, \bar{\omega} v)| < \tilde{g}(v,v) + \tilde{g}_t'(v,v)
= \tilde{g}(v,v) - \frac{1}{2} F_t (\tilde{I}_+ + \tilde{I}_v) v, v
= \tilde{g}(v,v) - F_t (\tilde{I}_+ v, v) - \frac{1}{2} \tilde{Q}(F_t v, F_t v)
= \tilde{g}(v,v) - F_t (\tilde{I}_+ v, v) - \frac{1}{2} \tilde{Q}(F_t v, F_t v).
\]

Since (5.7) holds for \( c = c_0 \), it will also hold in \( U \) for \( c = \lambda c_0 \), for any \( \lambda \in (0,1) \), since for \( x, y \in \mathbb{R}_{\geq 0} \) and \( z \in \mathbb{R} \), we have the implication

\[
(x < y + z) \Rightarrow (\lambda x < \lambda(y + z) \leq y + \lambda z).
\]

Therefore we have shown positivity of \( \tilde{g}_t \) in \( U \) for any \( 0 < c \leq c_0 \), for sufficiently small nonzero \( t \).

Now observe that the first term of (5.6), i.e. \( \tilde{g} \), is positive-definite on \( K \setminus U \) and independent of \( c \), whereas the second and third terms are each proportional to \( c \). Hence by choosing \( c \neq 0 \) sufficiently small, we ensure that \( \tilde{g}_t \) is positive-definite on \( K \setminus U \), in addition to \( U \) and outside \( K \), for sufficiently small \( t \neq 0 \). This completes the proof.

\[\Box\]

### 6 Examples

By the work of Goto [4], we know that the choice of a holomorphic Poisson structure on a compact Kähler manifold gives rise to a family of generalized Kähler structures deforming the initial Kähler structure. In this way, one obtains nontrivial generalized Kähler structures on any compact Kähler surface with effective anti-canonical divisor \( D \). Performing a Kähler blow-up of such a surface at a point lying on \( D \), we obtain a new Kähler surface with effective anti-canonical divisor given by the proper transform of \( D \). Hence we
may apply the Goto deformation and obtain a generalized Kähler structure on the blow-up. We believe that our construction gives an explicit realization of Goto’s existence result in this case, as evidenced by Remark 5.7.

In the non-algebraic case, or for noncompact surfaces, our construction provides new generalized Kähler structures. For example, a result of Apostolov [1] states that for surfaces with odd first Betti number, a bi-Hermitian structure which is not strongly bi-Hermitian may only exist on blow-ups of minimal class VII surfaces with curves. If the minimal surface has a generalized Kähler structure, therefore, we may employ our result to obtain structures on the appropriate blow-ups.

Example 6.1 (Diagonal Hopf surfaces). $X = S^3 \times S^1$ admits a family of generalized Kähler structures with bi-Hermitian structure $(g, I_+, I_-)$ given by viewing $X$ as a Lie group, taking $g$ to be a bi-invariant metric, and $(I_+, I_-)$ to be left and right-invariant complex structures compatible with $g$ (see [6] for details). In these examples, $D_+$ and $D_-$ are nonempty disjoint curves which sum to the anti-canonical divisor. We may therefore blow up any number of points lying on $D_+ \cup D_-$ and obtain generalized Kähler structures on these manifolds, which are diffeomorphic to $(S^3 \times S^1)\# k \mathbb{C}P^2$. This provides another construction of bi-Hermitian structures on non-minimal Hopf surfaces, besides those discovered in [11, 9].

In a remarkable recent work [3], Fujiki and Pontecorvo obtained bi-Hermitian structures on hyperbolic and parabolic Inoue surfaces as well as Hopf surfaces, by carefully studying the twistor space of the underlying conformal 4-manifold. They then obtained bi-Hermitian structures when these surfaces are properly blown up, meaning that the surface is blown up at nodal singularities of the anti-canonical divisor. Finally, they obtained bi-Hermitian structures on a family of deformations of such blowups. We may of course blow up their minimal examples at smooth points of the anti-canonical divisor, using our procedure. It remains to determine how the various bi-Hermitian structures now known on $(S^3 \times S^1)\# k \mathbb{C}P^2$ are related.

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