Kinematic self-similar plane symmetric solutions

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Abstract
This paper is devoted to classifying the most general plane symmetric spacetimes according to kinematic self-similar perfect fluid and dust solutions. We provide a classification of the kinematic self-similarity of the first, second, zeroth and infinite kinds with different equations of state, where the self-similar vector is not only tilted but also orthogonal and parallel to the fluid flow. This scheme of classification yields 24 plane symmetric kinematic self-similar solutions. Some of these solutions turn out to be vacuum. These solutions can be matched with the already classified plane symmetric solutions under particular coordinate transformations. As a result, these reduce to 15 independent plane symmetric kinematic self-similar solutions.

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1. Introduction
Einstein field equations (EFEs) are highly nonlinear, second-order coupled partial differential equations and hence could not be solved unless certain symmetry assumptions are taken on the spacetime metric. There has been recent literature [1–7, and references therein] which shows a significant interest in the study of various symmetries. Self-similarity leads to ordinary differential equations (ODEs) and their mathematical treatment is relatively simple. Invariance of the field equations under a scale transformation indicates that there exist scale invariant solutions to the EFEs. These solutions are known as self-similar solutions. Although self-similar solutions are only special solutions, they often play an important role in cosmological situations and gravitational collapse.

There exist several preferred geometric structures in self-similar models and a number of natural approaches may be used in studying them. The three most common ones are the co-moving, homothetic and Schwarzschild approaches. Each of the approaches has its
individual physical interpretational advantages and they are all complementary. In the co-moving approach, pioneered by Cahill and Taub [8], the coordinates are adopted to the fluid 4-velocity vector. This probably affords the best physical insight and is the most convenient one. In general relativity (GR), a self-similarity defined by the existence of a homothetic vector (HV) field is called self-similarity of the first kind (or homothety or continuous self-similarity). There exists a natural generalization of homothety called kinematic self-similarity, which is defined by the existence of a kinematic self-similar (KSS) vector field.

Cahill and Taub [8] gave the concept of self-similarity corresponding to Newtonian self-similarity of the homothetic class. Carter and Henriksen [9, 10] defined self-similarity of the second, zeroth and infinite kinds. The only compatible barotropic equation of state with self-similarity of the first kind is \( p = k \rho \). The classification of the self-similar perfect fluid solutions of the first kind in the dust case \( (k = 0) \) has been provided by Carr [1]. The case \( 0 < k < 1 \) has been studied by Carr and Coley [2]. Coley [11] has shown that the FRW solution is the only spherically symmetric homothetic perfect fluid solution in the parallel case. McIntosh [12] has discussed that a stiff fluid \( (k = 1) \) is the only compatible perfect fluid with the homothety in the orthogonal case. Benoit and Coley [13] have studied analytic spherically symmetric solutions of the EFEs coupled with a perfect fluid and admitting a KSS vector of the first, second and zeroth kinds. Sintes et al [14] have considered spherically, plane and hyperbolic symmetric spacetimes which admit a KSS vector of the infinite kind with perfect fluid. Carr et al [15, 16] have explored the KSS vector associated with the critical behaviour observed in the gravitational collapse of spherically symmetric perfect fluid with equation of state \( p = k \rho \). Further, they have investigated solution space of self-similar spherically symmetric perfect fluid models and physical aspects of these solutions. Coley and Goliath [17] have investigated self-similar spherically symmetric cosmological models with a perfect fluid and a scalar field with an exponential potential.

The assumption of self-similarity is very powerful in finding analytical solutions. The group \( G_3 \) contains two special cases of physical interest, i.e., spherical and plane symmetry. Most of the literature is available on spherical symmetric spacetimes. Maeda et al [3, 4] have studied the KSS vector of the second, zeroth and infinite kinds in the tilted, parallel and orthogonal cases. The same authors [5] have also discussed the classification of the spherically symmetric KSS perfect fluid and dust solutions. This analysis has provided some interesting solutions. Recently, Sharif and Sehar [7] have investigated the KSS solutions for the cylindrically symmetric spacetimes. This analysis has been extensively given for the perfect fluid and dust cases with tilted, parallel and orthogonal vectors by using different equations of state. Some interesting consequences have been developed. The same authors have also studied the properties of such solutions for spherically symmetric [18], cylindrically symmetric [19] and plane symmetric spacetimes [20].

In a recent paper, Sharif and Sehar [21] have explored the KSS solutions for the plane symmetric spacetimes under certain assumption. The investigation is incomplete due to this restriction on plane symmetric spacetimes. In this paper, we drop this restriction and deal with the most general plane symmetric spacetimes. This analysis provides many more interesting self-similar solutions. The paper has been organized as follows. In section 2, we briefly review KSS vectors of different kinds corresponding to the plane symmetric spacetimes. Sections 3 and 4 are devoted to the titled perfect fluid and dust solutions, respectively. The orthogonal perfect fluid and dust solutions are investigated in section 5. Sections 6 and 7 are used to explore the parallel perfect fluid and dust cases, respectively. In the last section, we present a summary of the results and their discussion.
2. Plane symmetric spacetime and kinematic self-similarity

A plane symmetric spacetime is a Lorentzian manifold possessing a physical stress–energy tensor. This admits $SO(2) \times \mathbb{R}^2$ as the minimal isometry group in such a way that the group orbits are spacelike surfaces of constant curvature. The most general plane symmetric metric is given in the form \[ d^2 = e^{2\nu(t,x)} \, dt^2 - e^{2\mu(t,x)} \, dx^2 - e^{2\lambda(t,x)} (dy^2 + dz^2), \] (1)

where $\nu, \mu$ and $\lambda$ are arbitrary functions of $t$ and $x$. The energy–momentum tensor for a perfect fluid can be written as

\[ T_{ab} = \left[ \rho(t,x) + p(t,x) \right] u_a u_b - p(t,x) g_{ab} \quad (a, b = 0, 1, 2, 3), \] (2)

where $\rho$ is the density, $p$ is the pressure and $u_a$ is the 4-velocity of the fluid element. In the co-moving coordinate system, the 4-velocity can be written as $u_a = (e^{\nu(t,x)}, 0, 0, 0)$. Using equations (1) and (2), we can write the EFEs as

\[ \kappa \rho = e^{-2\mu} \left( 2\lambda_{xx} \mu_x - 3\lambda_x^2 - 2\lambda_{xx} + 2\lambda_{xt} + \lambda_{tx} \right) + e^{-2\nu} \left( 2\lambda_{tt} \mu_t + \lambda_{tx} \right), \] (3)

\[ 0 = \lambda_{tx} - \lambda_t v_x + \lambda_x \lambda_t - \lambda_{tx} \mu_t, \] (4)

\[ \kappa p = e^{-2\mu} \left( \lambda_x^2 + 2\lambda_x v_x - e^{-2\nu} \left( 2\lambda_{tt} - 2\lambda_t v_t + 3\lambda_t^2 \right) \right), \] (5)

\[ \kappa p = e^{-2\mu} \left( v_{xx} + v_x^2 + v_t \lambda_x + \lambda_x^2 + \lambda_{xx} - \lambda_x \mu_x - v_t \mu_x \right) - e^{-2\nu} \left( \lambda_{tt} - \lambda_t v_t + \lambda_t^2 + \mu_{tt} + \mu_t^2 + \lambda_t \mu_t - v_t \mu_t \right). \] (6)

It follows from the conservation of energy–momentum tensor, $T^{ab}_{\phantom{ab};b} = 0$, that

\[ \mu_t = - \frac{\rho \delta}{(\rho + p)} - 2\lambda_t, \] (7)

and

\[ v_t = - \frac{p \alpha}{(\rho + p)}. \] (8)

For a plane symmetric spacetime, the general form of a vector field $\xi$ can be written as

\[ \xi_a \frac{\partial}{\partial x^a} = h_1(t,x) \frac{\partial}{\partial t} + h_2(t,x) \frac{\partial}{\partial x}, \] (9)

where $h_1$ and $h_2$ are arbitrary functions. The vector field $\xi$ can have three cases, i.e., parallel, orthogonal and tilted. They are distinguished by the relation between the generator and a timelike vector field, which is identified as the fluid flow, if it exists. When $\xi$ is parallel to the fluid flow $h_2 = 0$ and when $\xi$ is orthogonal to the fluid flow $h_1 = 0$. When both $h_1$ and $h_2$ are non-zero, $\xi$ is tilted to the fluid flow. The tilted case is the most general among them.

We define a KSS vector $\xi$ such that

\[ \mathcal{L}_\xi h_{ab} = 2\delta h_{ab}, \] (10)

\[ \mathcal{L}_\xi u_a = \alpha u_a, \] (11)

where $h_{ab} = g_{ab} - u_a u_b$ is the projection tensor, $\alpha$ and $\delta$ are constants. The ratio, $\alpha/\delta$, is called the similarity index which gives rise to the following two cases:

\[ \begin{align*}
(1) \quad & \delta \neq 0; \\
(2) \quad & \delta = 0.
\end{align*} \]
Case 1. When $\delta \neq 0$, we can choose it as unity and the KSS vector for the titled case can take the following form:

$$\xi^a \frac{\partial}{\partial x^a} = (\alpha t + \beta) \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}.$$  \hfill (12)

In this case, the similarity index, $\alpha/\delta$, further implies the following three possibilities:

(i) $\delta \neq 0$, $\alpha = 1$ ($\beta$ can be taken to be zero),
(ii) $\delta \neq 0$, $\alpha = 0$ ($\beta$ can be taken to be unity),
(iii) $\delta \neq 0$, $\alpha \neq 0$, 1 ($\beta$ can be taken to be zero).

The first case 1(i) is referred to the self-similarity of the first kind. In this case, $\xi$ is a homothetic vector and the self-similar variable $\xi$ turns out to be $x/t$. For the second case 1(ii), it is called the self-similarity of the zeroth kind and the self-similar variable becomes $\xi = x e^{-t}$. The last case 1(iii) is known as the self-similarity of the second kind and the self-similar variable turns out to be

$$\xi = \frac{x}{(at)^{\beta}}.$$

For the case (1), the metric functions take the following form:

$$v = v(\xi), \quad \mu = \mu(\xi), \quad e^{\lambda} = x e^{\lambda(\xi)}.$$  \hfill (13)

Case 2. In the second case (2), when $\delta = 0$ and $\alpha \neq 0$ ($\alpha$ can be unity and $\beta$ can be re-scaled to zero), the self-similarity is referred to the infinite kind. Here, the KSS vector $\xi$ becomes

$$\xi^a \frac{\partial}{\partial x^a} = t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}.$$  \hfill (14)

and the self-similar variable takes the form $\xi = x/t$. Consequently, the metric functions will become

$$v = v(\xi), \quad \mu = -\ln(x) + \mu(\xi), \quad \lambda = \lambda(\xi).$$  \hfill (15)

It is mentioned here that, for $\delta = 0 = \alpha$, the KSS vector $\xi$ reduces to $KV$.

When the KSS vector $\xi$ is parallel to the fluid flow, we obtain

$$\xi^a \frac{\partial}{\partial x^a} = f(t) \frac{\partial}{\partial t},$$  \hfill (16)

where $f(t)$ is an arbitrary function. It is worth mentioning point here that we obtained [21] contradictory results in the first, second and zeroth kinds while for the infinite kind the self-similar variable was $x$. As a result, there was no solution when $\xi$ was parallel to the fluid flow in the first, second and zeroth kinds except for the infinite kind. However, this analysis of the most general plane symmetric spacetimes yields self-similar variable $x$ in each kind and hence we can expect solution. The metric functions for the first, second, zeroth and infinite kinds, respectively, will be

$$v = v(x), \quad \mu = \ln(t) + \mu(x), \quad \lambda = \ln(t) + \lambda(x),$$
$$v = (\alpha - 1) \ln(t) + v(x), \quad \mu = \ln(t) + \mu(x), \quad \lambda = \ln(t) + \lambda(x),$$
$$v = -\ln(t) + v(x), \quad \mu = \ln(t) + \mu(x), \quad \lambda = \ln(t) + \lambda(x),$$
$$v = v(x), \quad \mu = \mu(x), \quad \lambda = \lambda(x).$$  \hfill (17)

If the KSS vector $\xi$ is orthogonal to the fluid flow, it follows that

$$\xi^a \frac{\partial}{\partial x^a} = g(x) \frac{\partial}{\partial x}.$$  \hfill (18)
where \( g(x) \) is an arbitrary function and the self-similar variable is \( t \). The corresponding metric functions for the first, second, zeroth and infinite kinds, respectively, will take the following form:

\[
\begin{align*}
\nu &= \ln(x) + \nu(t), \quad \mu = \mu(t), \quad \lambda = \ln(x) + \lambda(t), \\
\nu &= \alpha \ln(x) + \nu(t), \quad \mu = \mu(t), \quad \lambda = \ln(x) + \lambda(t), \\
\nu &= \nu(t), \quad \mu = \mu(t), \quad \lambda = \ln(x) + \lambda(t), \\
\nu &= \ln(x) + \nu(t), \quad \mu = -\ln(x) + \mu(t), \quad \lambda = \lambda(t).
\end{align*}
\] (19)

The following two types of polytropic equations of state (EOS) will be assumed. The first equation of state, denoted by EOS(1), is

\[
p = k\rho^\gamma,
\]

where \( k \) and \( \gamma \) are constants. Another EOS is the following [17]:

\[
p = k \rho \gamma, \quad \rho = \frac{m \rho + p}{\gamma - 1},
\]

where \( m \rho \) is a constant which corresponds to the baryon mass and \( n(t, r) \) corresponds to baryon number density. We call this equation as the second equation of state EOS(2). Note that we take \( k \neq 0 \) and \( \gamma \neq 0 \), for EOS(1) and EOS(2). EOS(3) is given by

\[
p = k\rho, \quad -1 \leq k \leq 1, \quad k \neq 0.
\]

3. Tilted perfect fluid case

3.1. Self-similarity of the first kind

It follows from the EFEs that the energy density \( \rho \) and pressure \( p \) must take the following form:

\[
k\rho = \frac{1}{x^2} \rho(\xi),
\]

\[
k p = \frac{1}{x^2} p(\xi),
\] (20) (21)

where the self-similar variable is \( \xi = x/t \). When the EFEs and the equations of motion for the matter field are satisfied, it yields a set of ODEs and hence equations (3)–(8) reduce to

\[
\dot{\rho} = -(\dot{\mu} + 2\dot{\lambda})(\rho + p),
\]

\[
2\dot{p} - \dot{p} = \dot{\nu}(\rho + p),
\]

\[
e^{2\nu} \rho = 2\mu + 2\lambda \dot{\mu} - 4\dot{\lambda} - 3\dot{\lambda}^2 - 2\dot{\lambda} - 1,
\]

\[
0 = 2\lambda \dot{\mu} + \dot{\lambda}^2,
\]

\[
0 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\mu} - \dot{\mu} - \dot{\mu}^2 - \dot{\nu},
\]

\[
e^{2\nu} p = 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\dot{\nu} + 2\dot{\nu},
\]

\[
0 = 2\dot{\lambda} \dot{\mu} - 2\dot{\lambda}^2 - 2\dot{\lambda},
\]

\[
e^{2\nu} p = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\mu} + \dot{\mu}^2 + \dot{\nu}^2 - \dot{\mu} - \dot{\mu}^2 - \dot{\nu},
\]

\[
0 = -\dot{\lambda} - \dot{\lambda}^2 - \dot{\mu} - \dot{\mu}^2 - \dot{\nu} - \dot{\mu} - \dot{\mu} - \dot{\nu}.
\] (22) (23) (24) (25) (26) (27) (28) (29) (30)

Here, dot means derivative with respect to \( \ln(\xi) \).
3.1.1. Equation of state (3). If a perfect fluid satisfies EOS(3), equations (20) and (21) yield that
\[ p = k\rho \quad \text{(case I).} \]  
(31)

From equation (25), we have two possibilities either \( \dot{\lambda} = 0 \) or \( \dot{\lambda} = -2\dot{\mu} \). For the first possibility, we obtain the following vacuum solution:
\[ \nu = \ln \left( c_0 \xi^{(1+\sqrt{2})} \right), \quad \mu = c_1, \quad \lambda = c_2, \quad \rho = \text{constant}, \quad k = -3 \pm \sqrt{2}. \]  
(32)

The corresponding metric is
\[ ds^2 = \left( \frac{1}{t} \right)^{(2+2\sqrt{2})} dt^2 - dx^2 - x^2 (dy^2 + dz^2). \]  
(33)

The second possibility leads to contradiction.

3.2. Self-similarity of the second kind

It follows from the EFEs that the energy density \( \rho \) and pressure \( p \) can be written as
\[ \kappa \rho = \frac{1}{x^2} \left[ \rho_1(\xi) + \frac{x^2}{t^2} \rho_2(\xi) \right], \]  
(34)
\[ \kappa p = \frac{1}{x^2} \left[ p_1(\xi) + \frac{x^2}{t^2} p_2(\xi) \right], \]  
(35)

where the self-similar variable is \( \xi = x/(at)^{\frac{1}{2}} \). When the EFEs and the equations of motion for the matter field are satisfied for \( O[\left( \dot{\xi} \right)^0] \) and \( O[\left( \dot{\xi} \right)^2] \) terms separately, we obtain the following ODEs:
\[ \dot{\rho}_1 = -\left( \dot{\mu} + 2\dot{\lambda} \right)(\rho_1 + p_1), \]  
(36)
\[ \dot{\rho}_2 + 2\alpha \rho_2 = -\left( \dot{\mu} + 2\dot{\lambda} \right)(\rho_2 + p_2), \]  
(37)
\[ -\dot{p}_1 + 2\dot{p}_1 = \dot{\nu}(\rho_1 + p_1), \]  
(38)
\[ -\dot{p}_2 = \dot{\nu}(\rho_2 + p_2), \]  
(39)
\[ e^{2\mu} \rho_1 = 2\dot{\mu} + 2\dot{\mu} \dot{\lambda} - 4\dot{\lambda} - 3\dot{\lambda}^2 - 2\ddot{\lambda} - 1, \]  
(40)
\[ \alpha^2 e^{2\nu} \rho_2 = 2\ddot{\lambda} + \dot{\lambda}^2, \]  
(41)
\[ 0 = \dot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} - \mu - \dot{\nu} - \dot{\lambda} \dot{\mu}, \]  
(42)
\[ e^{2\mu} p_1 = 1 + 2\dot{\lambda} + \dot{\lambda}^2 + 2\dot{\nu} + 2\dot{\lambda} \dot{\nu}, \]  
(43)
\[ \alpha^2 e^{2\nu} p_2 = -2\ddot{\lambda} - 3\dot{\lambda}^2 - 2\alpha \dot{\lambda} + 2\alpha \dot{\nu}, \]  
(44)
\[ e^{2\mu} p_1 = \dot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} + \dot{\lambda} \dot{\nu} + \dot{\nu}^2 - \dot{\mu} - \dot{\mu} \dot{\lambda} - \dot{\nu} \dot{\mu}, \]  
(45)
\[ \alpha^2 e^{2\nu} p_2 = -\ddot{\lambda} - \dot{\lambda}^2 - \alpha \dot{\lambda} - \mu - \dot{\mu}^2 - \alpha \dot{\mu} + \dot{\lambda} \dot{\nu} + \dot{\mu} \dot{\nu} - \dot{\lambda} \dot{\mu}. \]  
(46)
3.2.1. Equations of state (1) and (2). When a perfect fluid satisfies EOS(1) for \( k \neq 0 \) and \( \gamma \neq 0, 1 \), equations (34) and (35) become

\[
\alpha = \gamma, \quad p_1 = 0 = p_2, \quad p_2 = \frac{k}{(8\pi G)^{(\gamma-1)/2}} \xi^{-2\gamma} \rho_1 \gamma \quad \text{(case I)},
\]

or

\[
\alpha = \frac{1}{\gamma}, \quad p_2 = 0 = \rho_1, \quad p_1 = \frac{k}{(8\pi G)^{(\gamma-1)/2}} \xi^{2\gamma} \rho_2 \gamma \quad \text{(case II)}.
\]

For a perfect fluid with EOS(2) and \( k \neq 0, \gamma \neq 0, 1 \), it follows from equations (34) and (35) that

\[
\alpha = \gamma, \quad p_1 = 0, \quad p_2 = \frac{k}{m_b \gamma (8\pi G)^{(\gamma-1)/2}} \xi^{-2\gamma} \rho_1 \gamma = (\gamma - 1) \rho_2 \quad \text{(case III),}
\]

or

\[
\alpha = \frac{1}{\gamma}, \quad p_2 = 0, \quad p_1 = \frac{k}{m_b \gamma (8\pi G)^{(\gamma-1)/2}} \xi^{2\gamma} \rho_2 \gamma = (\gamma - 1) \rho_1 \quad \text{(case IV)}.
\]

In case I, equation (37) gives rise to two possibilities, i.e. either \( \dot{\mu} = -2 \dot{\lambda} \) or \( p_2 = 0 \). For the first possibility we meet a contradiction. In the second option, we obtain the following vacuum solution:

\[
\nu = c_1, \quad \mu = \frac{1}{7} \ln \xi + c_2, \quad \lambda = -\ln \xi + c_3, \\
\rho_1 = 0 = p_1, \quad \rho_2 = 0 = p_2, \quad \alpha = \frac{3}{2}.
\]

The corresponding metric is

\[
ds^2 = dt^2 - \frac{2^{2/3} x}{(3t)^{2/3}} \, dx^2 - \left( \frac{3t}{2} \right)^{4/3} (dy^2 + dz^2).
\]

For case II, equation (36) shows that either \( \ddot{\mu} = -2 \ddot{\lambda} \) or \( p_1 = 0 \). The first possibility leads to contradiction and the second possibility yields the same solution as given by equation (51).

In case III, equation (38) implies that either \( \rho_1 = 0 \) or \( \dot{\nu} = 0 \). For the first option, equation (41) implies that either \( \dot{\lambda} = 0 \) or \( \ddot{\lambda} = -2 \dot{\mu} \). The case when \( \dot{\lambda} = 0 \) gives a contradiction and the option \( \ddot{\lambda} = -2 \dot{\mu} \) implies the same solution as given by equation (51). The second case \( \dot{\nu} = 0 \) and the case IV also lead to the same solution as equation (51).

3.2.2. Equation of state (3). For a perfect fluid satisfying EOS(3), equations (34) and (35) yield that

\[
p_1 = k \rho_1, \quad p_2 = k \rho_2 \quad \text{(case V)}.
\]

This implies two options either \( k = -1 \) or \( k \neq -1 \). For \( k = -1 \), equations (36)–(46) lead to the same solution as for EOS(1) and EOS(2) given by equation (51). For \( k \neq -1 \), the case \( \rho_1 \neq 0, \rho_2 \neq 0 \) leads to a contradiction. The case, when \( \rho_1 = 0 \) and \( \rho_2 \) is arbitrary, implies that

\[
\nu = c_1, \quad \mu = \frac{3}{7} \ln \xi + c_2, \quad \lambda = -\ln \xi + c_3, \\
\rho_1 = 0 = p_1, \quad \rho_2 = \text{constant} = p_2, \quad \alpha = \frac{1}{2}.
\]

The resulting plane symmetric metric becomes

\[
ds^2 = dt^2 - \frac{64x^3}{t^6} \, dx^2 - \frac{t^4}{16} (dy^2 + dz^2).
\]
For the case when $\rho_2 = 0$ and $\rho_1$ is arbitrary, equation (41) implies that either $\lambda = 0$ or $\dot{\lambda} = -2\dot{\mu}$. For the first possibility, it follows that

$$
\nu = \frac{2k}{k + 1} \ln \xi + c_1, \quad \mu = c_2, \quad \lambda = c_3, \quad p_1 = \text{constant},
$$

and hence the plane symmetric spacetime will take the following form:

$$
ds^2 = \left(\frac{x}{(\alpha t)^{1/\alpha}}\right)^4 k_{k+1} \ln \xi + c_1 - dx^2 - x^2(dy^2 + dz^2). \quad (57)
$$

For the second possibility, equations (42) and (44) further imply two possibilities either $\dot{\mu} = 0$ or $\alpha = \frac{3}{2}$. When $\dot{\mu} = 0$ we obtain the same solution as equation (56). For $\alpha = \frac{3}{2}$, we can solve the system of equations by assuming either $\dot{\mu} = 0$ or $\dot{\nu} = 0$. Assuming $\dot{\mu} = 0$, we obtain the same solution as given by equation (56). If we take $\dot{\nu} = 0$, we have a contradiction.

### 3.3. Self-similarity of the zeroth kind

For this case, the EFEs show that the quantities $\rho$ and $p$ must be of the form

$$
\kappa \rho = \frac{1}{x^2} [\rho_1(\xi) + x^2 \rho_2(\xi)],
$$

$$
\kappa p = \frac{1}{x^2} [p_1(\xi) + x^2 \rho_2(\xi)], \quad (58)
$$

where the self-similar variable is $\xi = \frac{t}{x}$. Assuming that the EFEs and the equations of motion for the matter field are satisfied for $O[1(x)^0]$ and $O[1(x)^2]$ terms separately, it follows that

$$
\rho_1 = -(\dot{\mu} + 2\dot{\lambda})(\rho_1 + p_1),
$$

$$
\rho_2 = -(\dot{\mu} + 2\dot{\lambda})(\rho_2 + p_2),
$$

$$
-\dot{p}_1 + 2p_1 = \dot{\nu}(\rho_1 + p_1),
$$

$$
-\dot{p}_2 = \dot{\nu}(\rho_2 + p_2),
$$

$$
e^{2\nu} \rho_1 = 2\mu - 4\lambda - 3\dot{\lambda}^2 - 2\dot{\lambda} + 2\lambda \dot{\mu} - 1,
$$

$$
e^{2\nu} \rho_2 = 2\lambda \dot{\mu} + \dot{\lambda}^2,
$$

$$
0 = \dot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} - \dot{\mu} - \lambda \dot{\mu} - \lambda \dot{\nu},
$$

$$
e^{2\nu} p_1 = 1 + 2\lambda + \dot{\lambda}^2 + 2\nu + 2\dot{\lambda} \dot{\nu},
$$

$$
e^{2\nu} p_2 = 2\lambda \dot{\nu} - 2\dot{\lambda} - 3\dot{\lambda}^2,
$$

$$
e^{2\nu} p_1 = \lambda + \lambda^2 + \dot{\lambda} \dot{\nu} + \dot{\nu}^2 - \mu - \dot{\lambda} \dot{\mu} - \mu \dot{\nu},
$$

$$
e^{2\nu} p_2 = -\dot{\lambda} - \dot{\lambda}^2 + \dot{\lambda} \dot{\nu} - \dot{\lambda} \dot{\mu} + \mu \dot{\nu} - \mu \dot{\mu} - \mu \dot{\nu}. \quad (70)
$$

### 3.3.1. EOS(1) and EOS(2)

Here, equations (58) and (59) imply that

$$
p_1 = 0 = \rho_1, \quad p_2 = \left(\frac{k}{(8\pi G)^{\nu-1}}\right)^{\nu} \rho_2^{\nu} \quad \text{(case I)}. \quad (71)
$$
For EOS(2), it turns out that
\[ p_1 = 0 = \rho_1, \quad p_2 = \frac{k}{m_2 \gamma (8 \pi G)^{\gamma-1}} \left[ \rho_2 - \frac{p_2}{(\gamma - 1)} \right]^\gamma \] (case II). (72)

In both cases, we get the same set of equations which, when solved, yield the following solution for both EOS(1) and EOS(2):

\[
\begin{align*}
\nu &= c_1, \\
\mu &= -\ln \xi \pm \ln (\xi^3 - c_3) + c_2, \\
\lambda &= -\ln \xi + c_4, \\
\rho_1 &= 0 = p_1, \\
\rho_2 &= -\frac{3(\xi^3 + c_3)}{e^{2t}(\xi^3 - c_3)}, \quad p_2 = \text{constant}.
\end{align*}
\] (73)

The corresponding metric is
\[ ds^2 = dt^2 - \left(\frac{x^3 - c_3 e^{2t}}{x e^{2t}}\right)^2 dx^2 - e^{2t}(dy^2 + dz^2). \] (74)

### 3.3.2. EOS(3)

Here, it follows from equations (58) and (59) that
\[ p_1 = k \rho_1, \quad p_2 = k \rho_2. \] (75)

Proceeding in a similar fashion as in the case of self-similarity of the second kind with EOS(3), we obtain, for \( k = -1 \), the following solution:

\[
\begin{align*}
\nu &= c_1, \\
\mu &= -\ln \xi + c_2, \\
\lambda &= -\ln \xi + c_3, \\
\rho_1 &= 0 = p_1, \quad \rho_2 = \text{constant} = -p_2.
\end{align*}
\] (76)

The corresponding plane symmetric metric is
\[ ds^2 = dr^2 - \left(\frac{x^3 - e^{-4t}}{x e^{-4t}}\right)^2 dx^2 - e^{2t}(dy^2 + dz^2). \] (77)

The case \( k \neq -1 \) again leads to three options either \( \rho_1 \neq 0 \neq \rho_2 \), \( \rho_1 = 0 \) or \( \rho_2 = 0 \). The first case gives a contradiction. For the second option, we obtain the following solution:

\[
\begin{align*}
\nu &= c_1, \\
\mu &= 2 \ln \xi + c_2, \\
\lambda &= -\ln \xi + c_3, \\
\rho_1 &= 0 = p_1, \quad \rho_2 = \text{constant} = p_2.
\end{align*}
\] (78)

The plane symmetric metric for this solution becomes
\[ ds^2 = dr^2 - x^4 e^{-4t} dx^2 - e^{2t}(dy^2 + dz^2). \] (79)

In the case, when \( \rho_2 = 0 \), equation (65) yields two possibilities either \( \lambda = 0 \) or \( \lambda = -2 \mu \). For both possibilities, we obtain the same solution as in the case of the second kind with EOS(3) given by equation (56) (\( \alpha = 0 \)). The corresponding metric is
\[ ds^2 = (x e^{-t})^{\lambda} dr^2 - \lambda dx^2 - e^{2t}(dy^2 + dz^2). \] (80)

### 3.4. Self-similarity of the infinite kind

In this case, the EFEs indicate that the quantities \( \rho \) and \( p \) must be of the following form:

\[
\begin{align*}
\kappa \rho &= \rho_1(\xi) + \frac{1}{r^2} \rho_2(\xi), \\
\kappa p &= p_1(\xi) + \frac{1}{r^2} p_2(\xi).
\end{align*}
\] (81) (82)
where $\xi = \frac{t}{x}$. The requirement that the EFEs and the equations of motion for the matter field are satisfied for $O[(\ell)^0]$ and $O[(\ell)^{-2}]$ terms separately leads to the following equations:

\[
\rho_1 = - (\dot{\mu} + 2\dot{\lambda})(\rho_1 + p_1), \quad (83)
\]
\[
\rho_2 + 2\rho_2 = - (\dot{\mu} + 2\dot{\lambda})(\rho_2 + p_2), \quad (84)
\]
\[
-\dot{p}_1 = \dot{v} (\rho_1 + p_1), \quad (85)
\]
\[
-\dot{p}_2 = \dot{v} (\rho_2 + p_2), \quad (86)
\]
\[
e^{2\nu} \rho_1 = 2\dot{\lambda} \dot{\mu} - 3\dot{\lambda}^2 - 2\dot{\lambda}, \quad (87)
\]
\[
e^{2\nu} \rho_2 = 2\dot{\lambda} \dot{\mu} + \dot{\lambda}^2, \quad (88)
\]
\[
0 = \ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda} \dot{\mu} - \dot{\mu} \dot{\lambda}, \quad (89)
\]
\[
e^{2\nu} p_1 = \dot{\lambda}^2 + 2\dot{\lambda} \dot{\nu}, \quad (90)
\]
\[
e^{2\nu} p_2 = - 2\dot{\lambda} - 3\dot{\lambda}^2 - 2\dot{\lambda} + 2\dot{\lambda} \dot{\nu}, \quad (91)
\]
\[
e^{2\nu} p_1 = \ddot{\lambda} + \dot{\lambda}^2 + \dot{\lambda} \dot{\nu} + \dot{\nu}^2 + \dot{\lambda} \dot{\mu} - \dot{\mu} \dot{\nu}, \quad (92)
\]
\[
e^{2\nu} p_2 = - \ddot{\lambda} - \dot{\lambda}^2 - \dot{\lambda} \dot{\nu} - \dot{\mu} \dot{\lambda} - \dot{\mu}^2 - \dot{\mu} \dot{\nu} - \dot{\lambda} \dot{\mu}. \quad (93)
\]

### 3.4.1. EOS(1) and EOS(2).

For EOS(1), equations (81) and (82) imply that
\[
p_2 = 0 = p_2, \quad p_1 = k(8\pi G)^{\gamma-1} \rho_1^{\gamma} \quad \text{(case I)}.
\]

For EOS(2), it implies that
\[
p_2 = 0 = p_2, \quad p_1 = \frac{k}{m_0^{\gamma}(8\pi G)^{\gamma-1}} \left(\rho_1 - \frac{p_1}{(\gamma - 1)}\right)^\gamma \quad \text{(case II)}.
\]

In both cases, equation (88) shows that either $\dot{\lambda} = \text{constant}$ or $\dot{\lambda} = -2\dot{\mu}$. If $\dot{\lambda} = \text{constant}$, equations (86) and (90), respectively, imply that $\rho_1 = 0 = p_1$ and we are left with equations (92) and (93). Solving these two equations lead to $\ddot{\nu} + \ddot{\nu}^2 - \ddot{\mu} - \ddot{\mu}^2 - \ddot{\mu} = 0$ which satisfies for four different possibilities. For $\ddot{\mu} = 0 = \ddot{\nu}$, we trivially get Minkowski spacetime. For $\ddot{\mu} = 0$, the solution turns out to be
\[
\nu = \ln(\ln \xi - \ln c_1) + c_2, \quad \mu = c_3, \quad \dot{\lambda} = c_4.
\]
\[
\rho_1 = 0 = p_1, \quad \rho_2 = 0 = p_2.
\]

The metric will be
\[
dx^2 = \left[\ln \left(\frac{x}{c_1 t}\right)\right]^2 dt^2 - \frac{1}{x^2} dx^2 - (dy^2 + dz^2) \quad (c_1 \neq 0).
\]

In the case $\ddot{v} = 0$, we obtain
\[
\nu = c_1, \quad \mu = \ln(\ln \xi - \ln c_2) + c_3, \quad \dot{\lambda} = c_4.
\]
\[
\rho_1 = 0 = p_1, \quad \rho_2 = 0 = p_2.
\]

The corresponding metric is
\[
dx^2 = dt^2 - \frac{1}{x^2} \left[\ln \left(\frac{x}{c_2 t}\right)\right]^2 dx^2 - (dy^2 + dz^2) \quad (c_2 \neq 0).
\]

Finally, for the last possibility $\ddot{\nu} + \ddot{\nu}^2 = 0$ and $-\ddot{\mu} - \ddot{\mu}^2 - \ddot{\mu} = 0$, equations (92) and (93) imply $\ddot{\mu} \ddot{\nu} = 0$ and again we have the above possibilities. The second case, when $\dot{\lambda} = -2\dot{\mu}$, gives the same solution as given by equation (97).
3.4.2. EOS(3). Equations (81) and (82) imply that
\[ p_1 = k \rho_1, \quad p_2 = k \rho_2 \] (case III).

When \( k = -1 \), this gives rise to the same solution as for EOS(1) and EOS(2). The second case, i.e., \( k \neq -1 \), also leads to the same results as in EOS(1) and EOS(2).

4. Tilted dust case

4.1. Self-similarity of the first kind

When we take \( \rho = 0 \) in equations (22)–(30) for the tilted perfect fluid case with self-similarity of the first kind, equation (23) gives either \( \dot{\nu} = 0 \) or \( \rho = 0 \). Both cases yield contradiction.

4.2. Self-similarity of the second kind

Here, for \( p_1 = 0 = p_2 \), equations (38) and (39) imply that either \( \nu = \text{constant} \) or \( \rho_1 = 0 = \rho_2 \). For the first possibility, we obtain the following solution:
\[ \nu = c_1, \quad \mu = \ln \left( c_3 \xi^{-1/2} \left( \xi^{3/2} + 2c_2^{3/2} \right) \right), \quad \lambda = -\ln \xi + c_4, \]
\[ \rho_1 = 0 = p_1, \quad \rho_2 = \frac{2}{3c_5} \left( 2 - 3 \left( \frac{\xi^{3/2}}{\xi^{3/2} + 2c_2^{3/2}} \right) \right), \quad p_2 = 0, \]
\[ \alpha = \frac{3}{2}. \]
The corresponding metric is
\[ ds^2 = dt^2 - \left( \frac{x^2 + c_2^2}{\frac{3c_5}{2}} \right) \frac{2x^{3/2}}{\frac{3c_5}{2}} \left( \frac{\xi^{3/2}}{\xi^{3/2} + 2c_2^{3/2}} \right) dx^2 - \left( \frac{3t^2}{2} \right)^{4/3} (dy^2 + dz^2). \]

4.3. Self-similarity of the zeroth kind

This case gives contradiction and hence there is no solution.

4.4. Self-similarity of the infinite kind

In this case, equations (85) and (86) imply that either \( \nu = \text{constant} \) or \( \rho_1 = 0 = \rho_2 \). In the first case, we obtain the following solution:
\[ \nu = c_1, \quad \mu = -\ln \xi + \ln(\xi - c_2) + c_3, \quad \lambda = c_4, \]
\[ \rho_1 = 0 = \rho_2. \]
The corresponding metric is
\[ ds^2 = dx^2 - \frac{(x - c_2t)^2}{x^4} dx^2 - (dy^2 + dz^2). \]

For the second case, when \( \rho_1 = 0 = \rho_2 \), equations (88) and (90) imply that either \( \dot{\lambda} = 0 \) or \( \dot{\mu} = \dot{\nu}, \lambda = -2\mu \). The first option yields exactly the same result as for the tilted perfect fluid with self-similarity of the infinite kind using EOS(1) and EOS(2) and are given by equations (96), (98) and Minkowski spacetime. The other possibility implies a Minkowski spacetime.
5. Orthogonal perfect fluid and dust cases

Here, the self-similar variable is $\xi = t$ in each kind. The EFEs and the equations of motion for the perfect fluid of the first kind give the following set of equations:

$$\dot{\mu} = 0, \quad \text{(105)}$$

$$e^{2\nu}(e^{-2\mu} + \rho) = \lambda'^2, \quad \text{(106)}$$

$$e^{2\nu}(3e^{-2\mu} - p) = 3\lambda'' + 2\lambda' - 2\lambda'\nu', \quad \text{(107)}$$

$$e^{2\nu}(e^{-2\mu} - p) = \lambda'' + \lambda'^2 - \lambda'\nu', \quad \text{(108)}$$

$$2\lambda(\rho + p) = -\rho', \quad \text{(109)}$$

$$\rho = p, \quad \text{(110)}$$

where prime indicates derivative with respect to $\xi = t$. Equation (110) gives an equation of state for this system of equations. Solving these equations simultaneously, we arrive at the following solution:

$$\nu = \ln \left( \frac{p'}{4p\sqrt{(c_0 + p)}} \right), \quad \mu = c_1, \quad \lambda = -\frac{1}{4} \ln(p) + \ln(c_2), \quad \text{(111)}$$

$$\rho = p, \quad p^2 p - 2(1 + p)(p'' p - p'^2) = 0. \quad \text{(112)}$$

For the perfect fluid case of the second and zeroth kinds, we obtain contradiction. The perfect fluid case of the infinite kind gives Minkowski spacetime.

For the dust case, we take $p = 0$ in the equations for the perfect fluid case. In the self-similarity of the first kind, equation (110) shows that the resulting spacetime must be vacuum. Equation (106) gives $e^{2\nu} e^{2\mu} = \lambda'^2$ and we obtain

$$\nu = \nu(\xi), \quad \lambda = c_0 \int e^{2\mu(\xi)} \, d\xi, \quad \mu = c_1, \quad \rho = 0 = p. \quad \text{(113)}$$

The metric becomes

$$dx^2 = x^2 e^{2\nu(\xi)} \, dt^2 - dx^2 - x^2 \exp \left( 2c_0 \int e^{2\mu(\xi)} \, dt \right) (dy^2 + dz^2). \quad \text{(114)}$$

For the self-similarity of the second, zeroth and infinite kinds, we arrive at a contradiction due to one or the other reason and hence there is no solution.

6. Parallel perfect fluid case

6.1. Self-similarity of the first kind

Here, the self-similar variable is $\xi = x$ and the metric functions are given by equation (17). A set of ODEs in terms of $\xi$ are

$$\nu' = 0, \quad \text{(115)}$$

$$\rho = 3 e^{-2\nu} + e^{-2\mu}(2\lambda_\nu \mu' - 3\lambda'^2 - 2\lambda''), \quad \text{(116)}$$

$$p = e^{-2\mu}(\lambda'' + 2\lambda_\nu v') - e^{-2\nu}, \quad \text{(117)}$$

$$0 = \rho + 3p. \quad \text{(118)}$$
Here, prime denotes derivative with respect to $\xi = x$. Equation (118) indicates an equation of state. Using equation (114) in rest of the equations, we get $p' = 0$. Solving the remaining equations, we obtain

$$
\nu = c_1, \quad \mu = c_2, \quad \lambda = c_3 \xi + c_4, \quad \rho = 0 = p
$$

(120)

and the corresponding spacetime is

$$
d s^2 = d t^2 - t^2 d x^2 - t^2 e^{2\lambda(x)} (d y^2 + d z^2).
$$

(121)

### 6.2. Self-similarity of the second kind

For this kind, the self-similar variable is also $\xi = x$ and the metric functions are given by equation (17). The EFEs imply that the quantities $\rho$ and $p$ must be of the form

$$
\begin{align*}
\kappa \rho &= t^{-2} \rho_1(\xi) + t^{-2a} \rho_2(\xi), \\
\kappa p &= t^{-2} p_1(\xi) + t^{-2a} p_2(\xi).
\end{align*}
$$

(122, 123)

A set of ODEs in terms of $\xi$ will be

$$
\nu' = 0, \\
e^{2\mu} \rho_1 = 2\lambda' \mu' - 3\lambda'^2 - 2\lambda'', \\
\rho_2 = 3 e^{-2\nu}, \\
e^{2\mu} p_1 = \lambda'^2, \\
e^{2\nu} p_2 = 2\alpha - 3, \\
e^{2\mu} p_1 = \lambda'' + \lambda'^2 - \lambda''', \\
e^{2\nu} p_2 = 2\alpha - 3, \\
0 = \rho_1 + 3p_1, \\
0 = (3 - 2\alpha)\rho_2 + 3p_2, \\
-p_1' = 0, \\
-p_2' = 0.
$$

(124-134)

#### 6.2.1. EOS(1) and EOS(2)

When a perfect fluid satisfies EOS(1), equations (122) and (123) imply that

$$
p_2 = 0 = \rho_1, \quad \alpha = \frac{1}{\gamma}, \quad p_1 = \frac{k}{\gamma} (8\pi G)^{(1-\gamma)} \rho_2^\gamma 
$$

(135)

For EOS(2), it turns out that

$$
p_2 = 0, \quad \alpha = \frac{1}{\gamma}, \quad p_1 = \frac{k}{m_b (8\pi G)^{(1-\gamma)}} \rho_2^\gamma = (\gamma - 1)\rho_1
$$

(136)

Case I gives a contradiction and case II yields the following solution:

$$
\nu = c_1, \quad \mu = \ln \lambda' + c_2, \quad \lambda = \lambda(\xi), \\
\rho_1 = -3p_1 = \text{constant}, \quad \rho_2 = \frac{3}{c_0^2}, \quad p_2 = 0, \quad \alpha = \frac{3}{2}
$$

(137)

The spacetime becomes

$$
d s^2 = d t^2 - t^2 d x^2 - t^2 e^{2\lambda(x)} (d y^2 + d z^2).
$$

(138)
6.2.2. EOS(3). For EOS(3), equations (122) and (123) show that
\[ p_1 = k \rho_1, \quad p_2 = k \rho_2. \] (139)
Here, equations (131) and (132) imply that \( \rho_1 = 0 \) and equation (124) gives \( \dot{\nu} = 0 \) while equation (127) implies that \( \dot{\lambda} = 0 \). Solving the remaining equations, it turns out that
\[ \nu = c_1, \quad \mu = \mu(\xi), \quad \lambda = c_2, \quad \rho_1 = 0 = p_1, \]
\[ \rho_2 = \frac{3}{c_0}, \quad p_2 = -\frac{(3 - 2\alpha)}{c_0}, \quad k = -\frac{(3 - 2\alpha)}{3}. \] (140)
This gives the following spacetime:
\[ ds^2 = t^{2(\alpha - 1)} dt^2 - t^2 e^{2\mu(x)} dx^2 - t^2 (dy^2 + dz^2). \] (141)

6.3. Self-similarity of the zeroth kind

For this kind, the self-similar variable is again \( \xi = x \) and the plane symmetric metric functions are given by equation (17). The EFEs imply that the quantities \( \rho \) and \( p \) must be of the form
\[ \kappa \rho = t^{-2} p_1(\xi) + p_2(\xi), \] (142)
\[ \kappa p = t^{-2} p_1(\xi) + p_2(\xi). \] (143)
ODEs are
\[ \nu' = 0, \] (144)
\[ e^{2\mu} p_1 = 2\lambda' \mu' - 3\lambda'^2 - 2\lambda'', \] (145)
\[ \rho_2 = 3 e^{-2\nu}, \] (146)
\[ e^{2\mu} p_1 = \lambda'^2, \] (147)
\[ e^{2\nu} p_2 = -3, \] (148)
\[ e^{2\nu} p_1 = \lambda'' + \lambda'^2 - \lambda' \mu', \] (149)
\[ e^{2\nu} p_2 = -3, \] (150)
\[ 0 = \rho_1 + 3 p_1, \] (151)
\[ 0 = \rho_2 + p_2, \] (152)
\[ -p_1' = 0, \] (153)
\[ -p_2' = 0. \] (154)

6.3.1. EOS(1) and EOS(2). In the case of EOS(1), equations (142) and (143) yield
\[ \rho_1 = 0 = p_1, \quad p_2 = k (8\pi G)^{(1-\gamma)}/\rho_2^\gamma \] (case I). (155)
For EOS(2), it turns out that
\[ p_1 = 0 = p_1, \quad p_2 = \frac{k}{m_\gamma (8\pi G)^{(1-\gamma)}/\rho_2^\gamma} \left[ \rho_2 - \frac{p_2}{(\gamma - 1)} \right]^\gamma \] (case II). (156)
In both the cases, we obtain the same solution as
\[ \nu = c_1, \quad \mu = \mu(\xi), \quad \lambda = c_2, \]
\[ \rho_1 = 0 = p_1, \quad \rho_2 = -p_2 = \text{constant} \] (157)
and the resulting metric is
\[ ds^2 = \frac{1}{t^2} dt^2 - t^2 e^{2\mu(\xi)} dx^2 - t^2 (dy^2 + dz^2). \] (158)
6.3.2. EOS(3). Equations (142) and (143) show that
\[ p_1 = k\rho_1, \quad p_2 = k\rho_2. \] (159)
Here, equations (151) and (152) imply that either \( \rho_1 = 0 \) or \( \rho_2 = 0 \). Equation (146) gives a contradiction for \( \rho_2 = 0 \) and hence \( \rho_1 = 0 \). Also, equation (151) shows that \( k = -1 \) hence this gives the same solution as in EOS(1) and EOS(2) given by equation (158).

6.4. Self-similarity of the infinite kind

Again we have the self-similar variable \( \xi = x \) and the spacetime metric coefficients are given by equation (17). A set of ODEs will be
\[-e^{2\mu}\rho = 3\lambda'^2 + 2\kappa'' - 2\lambda'\mu',\] (160)
\[e^{2\mu}\rho = \lambda'^2 + 2\lambda'v',\] (161)
\[e^{2\mu}\rho = \lambda'' + \lambda^2 + \lambda'v' + v'' - \lambda'\mu' - v'\mu',\] (162)
\[-p' = v'(\rho + p).\] (163)

We consider the following four possibilities to solve the above set of equations:
1. \( v' = \mu' \)
2. \( v' = \lambda' \)
3. \( \lambda' = \mu' \)
4. \( v' = \lambda' = \mu' \).

The first case gives the following solution:
\[ v = \mu = c_1, \quad \lambda = c_2\xi + c_3, \]
\[ \rho = -3p = \text{constant} \] (164)
and the corresponding metric is
\[ ds^2 = dt^2 - dx^2 - e^{2\xi}(dy^2 + dz^2). \] (165)
The second case corresponds to Minkowski spacetime. For the case (iii), we obtain the following solution:
\[ v = c_1, \quad \lambda = \mu = c_3 - \ln(\xi - c_2), \]
\[ \rho = 3p, \quad p = -1 \] (166)
and the metric is given by
\[ ds^2 = dt^2 - \frac{1}{\xi^2}(dx^2 + dy^2 + dz^2). \] (167)
The last case yields Minkowski spacetime.

7. Parallel dust case

7.1. Self-similarity of the first kind

Setting \( p = 0 \) in the equations for the parallel perfect fluid case with self-similarity of the first kind, we finally have a contradiction and hence we do not have any self-similar solution.
7.2. Self-similarity of the second kind

For \( p_1 = 0 = p_2 \), equations (124) and (134) show that \( \nu = \) constant = \( \lambda \), respectively, and we get the same solution as given by equation (157) with \( \rho_2 = 0 = p_2 \) and \( \alpha = \frac{1}{2} \) but the corresponding metric is

\[
\text{d}s^2 = t \, \text{d}t^2 - t^2 \, e^{2\mu(x)} \, \text{d}x^2 - t^2 (\text{d}y^2 + \text{d}z^2). \tag{168}
\]

7.3. Self-similarity of the zeroth kind

When we take \( p_1 = 0 = p_2 \), equations (148) and (150) lead to contradiction.

7.4. Self-similarity of the infinite kind

For \( p = 0 \), equation (163) shows that either \( \nu = \) constant or \( \rho = 0 \). In the first case, the resulting spacetime is Minkowski. For \( \rho = 0 \), equation (161) implies that either \( \lambda' = 0 \) or \( \lambda' = -2\nu' \). When \( \lambda' = 0 \), we obtain \( \nu'' + \nu^2 - \mu' \nu' = 0 \) which implies that either \( \nu' = 0 \) or \( \mu' = 0 \). For the first possibility, we obtain Minkowski spacetime. For the second possibility, we get the following vacuum solution:

\[
\nu = \ln(c_2(x - c_1)), \quad \mu = c_3, \quad \lambda = c_4, \quad \rho = 0 = p. \tag{169}
\]

The metric for this spacetime is

\[
\text{d}s^2 = (c_2(x - c_1))^2 \, \text{d}t^2 - (\text{d}x^2 + \text{d}y^2 + \text{d}z^2). \tag{170}
\]

For \( \lambda' = -2\nu' \), equations (160) and (162) imply that \( 2\nu'' - 3\nu^2 - \mu' \nu' = 0 \) which gives either \( \nu' = 0 \) or \( \mu' = 0 \). The first possibility leads to the Minkowski spacetime and the second possibility gives the following vacuum solution:

\[
\nu = \ln \left( \frac{c_2}{(3x - c_1)^{1/3}} \right), \quad \mu = c_3, \quad \lambda = c_4, \quad \rho = 0 = p \tag{171}
\]

and the corresponding metric is

\[
\text{d}s^2 = \left( \frac{c_2}{(3x - c_1)^{2/3}} \right) \, \text{d}t^2 - (\text{d}x^2 + \text{d}y^2 + \text{d}z^2). \tag{172}
\]

8. Summary and discussion

Recent literature [3–5, 7, 18–21] indicates keen interest in the self-similar solutions and their physical features. Maeda et al [3–5] have classified the spherically symmetric KSS perfect fluid and dust solutions. Sharif and Sehar [7] have extended this analysis for the classification of the KSS cylindrically symmetric solutions. Recently, the same authors [21] have explored the KSS solutions for the plane symmetric spacetimes under a certain restriction, i.e., \( \mu = 0 \) for the sake of simplicity. Consequently, the classification was incomplete in the sense that we were missing many such cases where a solution could be possible. This paper deals with the most general plane symmetric spacetimes and provides self-similar solutions even in those cases where we obtain null results [21]. We have classified KSS perfect fluid and dust solutions for the cases when the KSS vector is tilted, orthogonal and parallel to the fluid flow by using EOS(1), EOS(2) and EOS(3). This gives rise to 24 plane symmetric self-similar solutions out of which we obtain 15 independent solutions.

It is found that EOS(1) and EOS(2) are incompatible with the self-similarity of the first kind in the tilted perfect fluid case. For EOS(3), we obtain solution with constant density. For
the self-similarity of the second kind with EOS(1) and EOS(2), we obtain a vacuum solution. For EOS(3) with \( k = -1 \), it follows the same solution as for EOS(1) and EOS(2). The case \( k \neq -1 \) leads to two self-similar solutions, one of these \( (\rho_1 = 0) \) represents a stiff fluid. The zeroth kind with EOS(1) and EOS(2) yields a solution. EOS(3) gives three solutions, one is a vacuum solution and other is a stiff fluid solution. In the case of the infinite kind for EOS(1) and EOS(2), we find three vacuum solutions while EOS(3) also leads to vacuum solutions both for \( k = -1 \) and \( k \neq -1 \).

For the tilted dust case with self-similarity of the second kind, we obtain the same solution as for the tilted perfect fluid with EOS(1) and EOS(2) and a dust solution for \( \alpha = 3/2 \). The self-similarity of the infinite kind leads to four vacuum solutions, one of them is Minkowski spacetime. There is no solution in any other kind.

In the orthogonal perfect fluid with self-similarity of the first kind we obtain a solution in terms of pressure and with self-similarity of the infinite kind we obtain Minkowski spacetime. Any other kind does not provide a solution. The orthogonal dust case with self-similarity of the first kind yields a vacuum spacetime given by equation (112). All other kinds provide contradictory results.

In the parallel perfect fluid with the self-similarity of the first kind, we obtain a vacuum solution. The second kind leads to contradiction for EOS(1), but for EOS(2) we obtain a solution in which one fluid represents dust and the other vacuum. For EOS(3), we obtain a solution with arbitrary \( \mu \) and \( \alpha \neq \frac{3}{2} \). The zeroth kind yields a vacuum solution. There are three self-similar solutions with self-similarity of the infinite kind, one of which is Minkowski spacetime. For the parallel dust case, the first kind gives a contradiction. The second kind implies the same solution as in the parallel perfect fluid case with EOS(3) and \( p_2 = 0, \alpha = \frac{3}{2} \). We do not have any self-similar solution of zeroth kind in the dust case. However, we obtain three different vacuum solutions for the infinite kind.

It is interesting to note that all the self-similar solutions, except the solutions given by equations (80), (97), (99), (104), (111), (165), (167), (170), (172), found here correspond with the already classified solutions [23] under particular coordinate transformations. The metrics given by equations (52), (55), (74), (79) and (102) correspond to the class of metrics

\[
d_s^2 = dr^2 - e^{2\mu(t)} \, dx^2 - e^{2\lambda(t)} (dy^2 + dz^2). \tag{173}
\]

This metric has the four KVs admitting \( G_3 \otimes \mathfrak{g} \) with a spacelike \( \mathfrak{g} \) and can be matched with Kantowski–Sachs spacetimes [22]. The spacetime given by equations (33) and (57) corresponds to the class of metrics

\[
d_s^2 = e^{2\nu(x)} \, dr^2 - dx^2 - e^{2\lambda(x)} (dy^2 + dz^2) \tag{174}
\]

which has four KVs with the same Lie algebra and a timelike \( \mathfrak{g} \). The solution given by equation (113) can be matched with the solution

\[
d_s^2 = e^{2\nu(x)} [dr^2 - e^{2\mu/a} (dy^2 + dz^2)] - dx^2 \quad (\alpha \neq 0) \tag{175}
\]

which admits six KVs. The metrics (121) and (138) turn out to be equivalent to the metric

\[
d_s^2 = dr^2 - e^{2\nu(t)} [dx^2 + e^{2\mu/a} (dy^2 + dz^2)] \quad (\alpha \neq 0) \tag{176}
\]

which has six KVs with a Lie algebra identical to that of the metric (175) and belongs to the family of LRS metrics. Finally, the metrics given by equations (77), (141), (158) and (168) has the correspondence with the class of metrics given as

\[
d_s^2 = dr^2 - e^{2\nu(t)} (dx^2 + dy^2 + dz^2) \tag{177}
\]

admitting six KVs and represents FRW models. We also note that the solutions given by the metrics (99), (104) seem to have a similar nature while the solutions (165) and (167), and
Table 1. Tilted perfect fluid KSS solutions.

| Self-similarity         | Solution                                      |
|-------------------------|-----------------------------------------------|
| First kind (EOS(3))     | Solution given by equation (33)                |
| Second kind (EOS(1))    | None                                          |
| Second kind (EOS(2))    | Solution given by equation (52)                |
| Second kind (EOS(3)(i)) | Solution given by equation (52)                |
| Second kind (EOS(3)(ii))| Solution given by equation (55)                |
| Second kind (EOS(3)(iii))| Solution given by equation (57)               |
| Zeroth kind (EOS(1))    | Solution given by equation (74)                |
| Zeroth kind (EOS(2))    | Solution given by equation (74)                |
| Zeroth kind (EOS(3)(i)) | Solution given by equation (77)                |
| Zeroth kind (EOS(3)(ii))| Solution given by equation (79)                |
| Zeroth kind (EOS(3)(iii))| Solution given by equation (80)               |
| Infinite kind (EOS(1)(i))| Minkowski spacetime                            |
| Infinite kind (EOS(1)(ii))| Solution given by equation (97)               |
| Infinite kind (EOS(1)(iii))| Solution given by equation (99)              |
| Infinite kind (EOS(2))  | Same solutions as in EOS(1)                    |
| Infinite kind (EOS(3))  | Same solutions as in EOS(1)                    |

Table 2. Tilted dust KSS solutions.

| Self-similarity     | Solution                                      |
|---------------------|-----------------------------------------------|
| First kind          | None                                          |
| Second kind (i)     | Solution given by equation (102)              |
| Second kind (ii)    | Solution given by equation (52)               |
| Zeroth kind         | None                                          |
| Infinite kind (i)   | Solution given by equation (104)              |
| Infinite kind (ii)  | Minkowski spacetime                            |
| Infinite kind (iii) | Solution given by equation (97)               |
| Infinite kind (iv)  | Solution given by equation (99)               |

Table 3. Orthogonal perfect fluid KSS solutions.

| Self-similarity   | Solution                                      |
|-------------------|-----------------------------------------------|
| First kind        | Solution given by equation (111)             |
| Second kind       | None                                          |
| Zeroth kind       | None                                          |
| Infinite kind     | Minkowski spacetime                            |

solutions (170) and (172) can correspond to each other. It is worth mentioning that we obtain density either zero or positive in all the solutions except the one where it is not constant but can be positive. The physical properties of such solutions can be seen in [20]. Thus, we finally obtain 15 independent KSS plane symmetric solutions. The results can be summarized in the form of tables 1–6:

Finally, we would like to mention that Sintes et al [14] found solutions only for the infinite kind. However, we have studied KSS solutions of the most general plane symmetric spacetimes for all kinds. The KSS solutions of the infinite kind can be matched with those of [14]. The solutions given by equations (99) and (104) can be matched with equation (5.5) and
Table 4. Orthogonal dust KSS solutions.

| Self-similarity | Solution                                      |
|-----------------|-----------------------------------------------|
| First kind      | Solution given by equation (113)             |
| Second kind     | None                                          |
| Zeroth kind     | None                                          |
| Infinite kind   | None                                          |

Table 5. Parallel perfect fluid KSS solutions.

| Self-similarity    | Solution                                      |
|--------------------|-----------------------------------------------|
| First kind         | Solution given by equation (121)             |
| Second kind (EOS(1)) | None                                        |
| Second kind (EOS(2)) | Solution given by equation (138)             |
| Second kind (EOS(3)) | Solution given by equation (141)             |
| Zeroth kind (EOS(1)) | Solution given by equation (158)             |
| Zeroth kind (EOS(2)) | Solution given by equation (158)             |
| Zeroth kind (EOS(3)) | Solution given by equation (158)             |
| Infinite kind (i)  | Solution given by equation (165)             |
| Infinite kind (ii) | Minkowski spacetime                           |
| Infinite kind (iii) | Solution given by equation (167)             |
| Infinite kind (iv) | Minkowski spacetime                           |

Table 6. Parallel dust KSS solutions.

| Self-similarity | Solution                                      |
|-----------------|-----------------------------------------------|
| First kind      | None                                          |
| Second kind     | Solution given by equation (168)             |
| Zeroth kind     | None                                          |
| Infinite kind (i) | Minkowski spacetime                           |
| Infinite kind (ii) | Minkowski spacetime                           |
| Infinite kind (iii) | Solution given by equation (170)             |
| Infinite kind (iv) | Minkowski spacetime                           |
| Infinite kind (v) | Solution given by equation (172)             |

The solutions (97), (170) and (172) correspond to the solution (5.6) of [14]. The remaining solutions do not correspond to those solutions given in [14].

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