Topology by Improved Cooling: Susceptibility and Size Distributions

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We use a cooling algorithm based on an improved action with scale invariant instanton solutions, which needs no monitoring or calibration and has an inherent cut off for dislocations. In an application to the \textit{SU}(2) theory the method provides good susceptibility data and physical size distributions of instantons.

1. THE IMPROVED COOLING

In [1] we presented an improved cooling algorithm which fulfills requirements qualifying it as a method to eliminate UV noise and study topological excitations on the lattice:

- it smoothes out the short range fluctuations, including “dislocations”
- it ensures stability of instantons (including their size) above a certain threshold \( \rho_0 \simeq 2.3a \) and needs thus no monitoring or calibration.

We apply this method to \textit{SU}(2) to study susceptibility, size distribution, distances and shape of instantons and we report here results concerning scaling and boundary condition effects.

Since the Wilson action decreases with the instanton size, it leads to an unphysical abundance of small instantons in a Monte Carlo simulation. Under Wilson cooling even wide instantons shrink and decay. This can be corrected for by modifying the lattice action. Starting from [2] one can construct a one parameter set of actions with no \( \mathcal{O}(a^2) \) and \( \mathcal{O}(a^4) \) corrections using five fundamental, planar loops of size \( m \times n \):

\[
S_{m,n} = \sum_{x,\mu,\nu} \text{Tr} \left( 1 - \frac{1}{2} \left[ \begin{array}{c} \nu \\ x \\ \mu \end{array} \right] + \nu \left[ \begin{array}{c} \mu \\ x \\ \mu \end{array} \right] \right) \quad (1)
\]

\[
S = \sum_{i=1}^{5} c_i \frac{1}{m_i n_i} S_{m_i,n_i} \quad (2)
\]

\( c_i \) are coefficients given by:

\[
c_1 = (19 - 55 c_5)/9, \quad c_2 = (1 - 64 c_5)/9
\]
\[
c_3 = (-64 + 640 c_5)/45, \quad c_4 = 1/5 - 2 c_5
\]
\[
\text{with } c_5 = 1/20.
\]

The cooling algorithm exactly minimizes the local action at each step and involves no further calibration or engineering. The size dependence of the action for the instantons is \( \theta \)–function like, giving \( 8 \pi^2 \) to better than 0.1% for \( \rho > \rho_0 \) and a steep descent (slightly configuration dependent) in the interval \( (0.8 - 1.0) \rho_0 \). Since the threshold is fixed in lattice units, it should not affect physical sizes if \( a \) is small enough. For definiteness we use everywhere as radius in lattice units:

\[
\rho_4 \equiv \rho_{\text{peak}}^4 = 6/\left( \pi^2 Q_{\text{peak}} \right) \quad (4)
\]

with \( Q_{\text{peak}} \) the topological charge density at the center of the instanton (using \( S_{\text{peak}} \) instead gives no significant difference). Notice that the \( Q(5Li) \) charge operator produces reasonable density data already on still rough configurations and can be used to observe instanton-anti-instanton (IA) pairs (which annihilate later in the cooling since they are not minima of the action).

The question of stability of instantons cannot be
completely disentangled from finite size effects. It is proven that \( Q = \pm 1 \) solutions do not exist on finite periodic volumes \( \mathbb{R}^3 \). Practically this means that isolated instantons above \( \approx 1/4 \)th of the lattice size are unstable. To avoid this affecting the physical distances one can use twisted boundary conditions or make at least one of the lattice sizes very large — both these conditions allow to stabilize wider instantons \( \mathbb{R}^3 \). The results of \( \mathbb{R}^3 \) were obtained using twisted b.c. with \( k = (1,1,1) \), \( m = (0,0,0) \) (t.b.c. in the following).

The analysis undertaken here concerns the determination of physical quantities like topological susceptibility and size distribution in view of the two problems raised above, namely (A) threshold effects (for rough lattices \( \rho_0 \) may already represent a physically relevant distance) and (B) finite size effects (small lattices may not accommodate large, physical instantons). Correspondingly we analyze here the scaling behavior and the dependence on size and boundary conditions.

2. RESULTS AND DISCUSSION

We base our analysis on the following \( SU(2) \) Monte Carlo simulations (heat–bath, Wilson):

(1): \( 12^3 \times N_t, \beta = 2.4, (a \approx 0.12 \text{fm}) \)

(a): \( N_t = 12, \) t.b.c., (b): \( N_t = 12, \) p.b.c., and (c): \( N_t = 36, \) p.b.c.;

(2): \( 12^4, \beta = 2.5, (a \approx 0.085 \text{fm}), \) t.b.c.;

(3): \( 24^4, \beta = 2.6, (a \approx 0.06 \text{fm}), \) t.b.c.,

Here (1) and (3) are new data from 200, respectively 84 configurations (100, respectively 200 sweeps apart, after 20000 sweeps for thermalization), (2) are older data \( \mathbb{R}^3 \) from 160 configurations (250 sweeps apart).

**Topological Susceptibility:** Since the topological charge \( Q(5 Li) \) stabilizes very fast to an integer within less than 1% (see Fig. 1), we expect deviations from the physical values in the susceptibility to be only of type (A) or (B) above. The results are presented in Fig. 2 and Table 1. The susceptibility settles early in the cooling and seems to scale very well both with the cut-off (compare the (1) and (3) data) and with the volume (compare (1) and (2)), with small deviations showing up in (1) (beta=2.4) which will be argued to be mostly of type (A). Assuming the Witten–Veneziano formula to hold the result agrees excellently with the phenomenological expectation. Recent results for \( SU(2) \) in \( \mathbb{R}^3 \) are about 15% higher than the ones we quote.

![Figure 1. Evolution of action and charge of a typical configuration (t.b.c) under 5Li cooling.](image1)

![Figure 2. Topological Susceptibility: t.b.c. (1a), (2) and (3) (squares), and p.b.c. (1b) (triangles) and (1c) (circles) – see text. The bars in (1) give an indication of the errors which are not included in the data to make the figure clearer.](image2)

| sw | (1a) | (1b) | (1c) | (3) | sw | (2) |
|----|------|------|------|-----|----|-----|
| 5  | 199(3)| 200(5)| 201(5)| 205(16)| 0  | 219(8)|
| 20 | 200(3)| 200(6)| 199(5)| 199(15)| 20 | 212(7)|
| 50 | 198(3)| 198(6)| 195(5)| 197(15)| 60 | 212(7)|
| 150| 196(4)| 195(6)| 192(6)| 197(15)| 140| 212(7)|
| 300| 195(4)| 193(6)| 189(6)| 194(14)| 300| 211(7)|

**Instanton Size Distribution:** The size of the instantons is calculated following the continuum
ansatz. We verified that a discretized form of this ansatz with periodicity corrections and generalized to an ellipsoid fits very well the 5Li-densities (charge and action) of isolated instantons already after few cooling sweeps and that it can also handle overlapping structures. The fitted radius averaged over directions agrees quite well with \( \rho_{\text{peak}} \) Eq.(4). Wide instantons above \( 1/2N_s \) show considerable anisotropy and for such configurations different definitions of \( \rho \) may lead to different values. We have restricted to the \( \rho_{\text{peak}} \) definition except for the old data in [1] where the size was extracted differently. Consequently the upper part of the size distribution for (2) cannot be directly compared with (1) and (3).

\[ fig3 \quad \text{Size distributions. Crosses, squares, triangles, circles correspond respectively to lattices (3), (1a), (1b), (1c), the dotted vertical line is } \rho_0. \]

The new results are presented in Fig. 3. The comparison of the (1) and (3) data shows that at \( \beta = 2.4 \) the size distribution already begins to get out of the threshold region, indicating a reduced effect of the latter on physical instantons. The slight decay of the susceptibility observed in (1) can be completely understood as due to the disappearance of few, small but still physically relevant instantons close to the threshold.

Finite size effects can be estimated by comparing (1a,b,c). The results agree perfectly for low cooling sweeps but for (1b) (12, p.b.c) a shift towards smaller sizes at 300 sweeps is observed. We believe it to be mostly due to the instability of isolated instantons with p.b.c. This conclusion is further supported by comparing the distributions (1a,b) and (1a’,b’) obtained from (1a,b) by leaving out configurations of \( S = 8\pi^2 \) (see Fig. 4).

\[ fig4 \quad \text{Size distribution for (1a) and (1b) once subtracted the configurations with } S = 8\pi^2. \]

More detailed results, including larger statistics for \( \beta = 2.6 \) and an analysis for quantities like distance distributions, cross – correlations of size and distances and charge distributions will be presented elsewhere.

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