KPZ universality in a classical single random walker under continuous measurement

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We introduce and study a new model consisting of a single classical random walker undergoing continuous monitoring at rate $\gamma$ on a discrete lattice. Although such a continuous measurement cannot affect physical observables, it has a non-trivial effect on the probability distribution of the random walker. When $\gamma$ is small, we show analytically that the time-evolution of the latter can be mapped to the Stochastic Heat Equation (SHE). In this limit, the width of the log probability thus follows a Family-Vicsek scaling law, $N^\alpha f(t/N^{\alpha/\beta})$, with growth exponent $\beta = 1/3$ and roughness exponent $\alpha = 0.5$ corresponding to the Kardar-Parisi-Zhang (KPZ) universality class. When $\gamma$ is further increased outside this regime, we find numerically that, while $\beta$ remains constant, $\alpha$ monotonously grows before saturating at a value $\alpha = 1$ at high monitoring rates. Furthermore, beyond the initial power-law characterized by $\beta$ and whenever the measurement rate exceeds a finite threshold, we observe the emergence of a second growth regime at intermediate times which could be the landmark of a measurement-induced transition.

In this paper, we unveil a connection between the KPZ universality class and classical information theory by studying a single classical random walker undergoing continuous monitoring.

Continuous or weak measurement has enjoyed considerable interest in the previous decades within the quantum community as it provides a non-destructive way to obtain information about a given quantum system \(17\) \(18\). Its advent led to many interesting applications such as quantum Zeno effects \(19\), quantum trajectories \(20\), quantum Maxwell demons \(21\), or direct observation of quantum jumps \(22\). Recently, a number of studies investigated the consequences of repeated projections or continuous monitoring on the evolution of quantum many-body systems. For systems undergoing both a random unitary evolution and measurements, a result that has aroused considerable interests lately is the existence of a measurement-induced phase transition in the entanglement entropy \(23\) \(24\). In particular, these contributions focus on entanglement or Rényi entropies, i.e. information-related quantities which are likely salients in classical systems as well. As such, it is natural to wonder whether the same phenomenology of measurement-induced phase separation also features in classical physics. This study constitutes a first attempt at answering this question.

We first present the framework that we use to model weak, continuous measurements on a generic stochastic dynamics. We then focus on the specific case of a single random walker diffusing on a lattice with the occupancy at each site being continuously monitored. When the measurement rate $\gamma$ is small, we find that the standard deviation of the log probability follows a Family-Vicsek scaling law with growth exponent $\beta \approx 1/3$ and roughness exponent $\alpha \approx 0.5$, which corresponds to the KPZ universality class \(3\). By performing a perturbative analysis around $\gamma = 0$, we then show analytically that this KPZ-like behavior is due to a direct mapping of the dynamics onto the Stochastic Heat Equation (SHE). In the last part, we explore the evolution of $\alpha$ and $\beta$ upon increasing $\gamma$ beyond the KPZ regime. We find numerically that $\alpha$ monotonously increases with $\gamma$ before reaching a plateau at $\alpha = 1$ for high measurement rates. Finally, beyond the initial growth characterized by $\beta$, we further point out the emergence of a second power-law regime at intermediate times whenever $\gamma$ exceeds a critical threshold. This last feature could be the landmark of a measurement-induced phase transition.

Continuous monitoring We begin by introducing the formalism of continuous monitoring. It is directly inspired from weak measurement and trajectories frameworks of quantum mechanics \(17\) \(36\) \(39\) and can be thought as a simple hidden Markov process \(40\).

In the absence of monitoring, the system undergoes
a stochastic dynamics of generator $\mathcal{L}$ on a classical configuration space $\mathcal{M}(\mathcal{C})$ with total number of configurations $\Omega$. The time-evolution of the probability distribution $P_t$ is given by the master equation

$$\frac{d}{dt} P_t(C) = \mathcal{L}(P_t(C)).$$

We assume that the stationary state is unique and is further given by the maximally entropic state $P_\infty = \Omega^{-1}$.

We now suppose that an external observer wishes to extract information from the system by measuring it but in a non-totally efficient manner, for instance by taking blurry snapshots of it. Explicitly, we consider that an ancilla couples to the system for a short amount of time $\delta t$ such that the generated correlation is of order $\delta t$. Instead of directly monitoring the system, the observer measures the state of the ancilla, which provides indirect information about the system. Repeating this procedure with $M$ ancillae and taking the limit $M \rightarrow 0$, $M \rightarrow \infty$ while keeping $M\delta t := t$ fixed allows to express the continuous monitoring as a stochastic differential equation (SDE).

In what follows, the configuration space will be given by the set $\{X_1, \ldots, X_N\} := \bar{X}$ where the $X_j$’s can take values $\pm 1$: $+1$ corresponds to an occupied site while $-1$ to an empty one. We suppose that all sites will be independently monitored. The ancilla monitoring site $j$ is also described by a random variable $A_j$ which can take binary values $a_j \in \{-1,1\}$. We denote by $P(\bar{X}, A_j)$ the joint probability of the union system + ancillae to be in a given configuration. We fix this probability distribution to

$$P(\bar{X}, A_j) = P(\bar{X}) \frac{1 + \sqrt{\gamma} a_j X_j}{2},$$

where $P(\bar{X})$ is the reduced probability of the system only. The interpretation of $\bar{X}$ is the following: the state of the ancilla and the sites are positively correlated (e.g., if $A_j = 1$, it’s more likely to find $X_j$ in $1$). Once a measurement of the ancilla’s state has been made with outcome $a_j$, the probability distribution is updated with probability $1 + \frac{\sqrt{\gamma}}{2} a_j \langle X_j \rangle$ to

$$P(\bar{X}) \rightarrow P(\bar{X} | A_j = a_j) = P(\bar{X}) \frac{1 + \sqrt{\gamma} a_j X_j}{1 + \frac{\sqrt{\gamma}}{2} a_j \langle X_j \rangle},$$

where $\langle X_j \rangle := \sum_{i} X_j P_i(\bar{X})$. In the SM [11], we show that repeating this procedure $M$ times and taking the limit $M \rightarrow \infty$, $\delta t \rightarrow 0$ while keeping $M\delta t = t$ fixed leads, in the Itô prescription, to the following evolution for the probability distribution

$$dP_t(\bar{X}) = \frac{\sqrt{\gamma}}{2} P_t(\bar{X})(X_j - \langle X_j \rangle_t) dB^j_t,$$

where $dB^j_t$ are site-independent Brownian processes with variance $dt$ and Itô rules $dB^i_t dB^k_t = \delta_{i,k}dt$. Because of the measurement, note that $P_t$ is both a probability distribution and a stochastic variable. Consequently, there are two types of averages in the problem: $\langle \rangle$ denotes average with respect to $P_t$, while $\mathbb{E}[\cdot]$ denotes average with respect to the Brownian processes $\{B^j_t\}$.

As measurements occur independently on every site, we obtain the stochastic evolution of the monitored system as the sum of (4) and (4):

$$dP_t = L(P_t) dt + \int dP_t(\bar{X})(X_j - \langle X_j \rangle_t) dB^j_t.$$  \hspace{1cm} (5)

Note that, since $\mathcal{L}$ preserves the total probability and $\sum_j P_t(\bar{X})(X_j - \langle X_j \rangle_t) = 0$, the probability distribution $P_t$ remains normalized at every time $t$ for each realization of the process. In addition, continuous monitoring can not affect the expectation of any physical observable $\mathbb{E}[\langle O(\bar{X}) \rangle]$ since this quantity is linear in $P_t$ and $d\mathbb{E}[P_t] = d\mathbb{E}[\mathcal{L}(P_t)]dt$. This is expected since, contrary to the quantum case, classical measurements can’t affect the actual physical state of the system. Note, however, that quantities depending non-linearly on $P_t$ such as the Gibbs-Shannon entropy will be affected by monitoring, even after the average $\mathbb{E}[\cdot]$ has been taken.

**Single-particle problem** We now consider the specific case of a single random walker on a square lattice of $N$ sites with periodic boundary conditions. Let $p_j(t)$ be the probability for the particle to be at site $j$ at time $t$. We choose $\mathcal{L}$ to be the discrete Laplacian weighted by a diffusion constant $D$, i.e $\mathcal{L} = D \Delta_d$ with $\Delta_d p_j := p_{j-1} - 2p_j + p_{j+1}$. Starting from (3), we show in the SM [11] that the evolution of $p_j$ in the presence of monitoring is given by

$$dp_j = D \Delta_d p_j dt + \sqrt{\gamma} p_j dW^j_t,$$

with $dW^j_t := dB^j_t - \sum_m p_m dB^m_t$.

The stationary states corresponding to the first and second term on the right hand side of Eq. (5) are of quite different nature. The diffusive term favors the flat, maximally entropic distribution $p_j = 1/N$ while the measurement term favors the $N$ pointer states $p_j = \delta_{j,k}$ for fixed $k \in [1, N]$. For finite $D$ and $\gamma$, the stationary distribution of this model is non-trivial and, to the best of our knowledge, not known with a notable exception for $N = 2$. In the latter case, it turns out that the dynamics is equivalent to the one of a single qubit undergoing both thermal relaxation and quantum measurements and was treated in [12, 13].

Eq. (6) is reminiscent of the stochastic heat equation (SHE) with multiplicative noise [14] except that the noise $dW^j_t$ is not a single independent Brownian process. Indeed, $dW^j_t$ has 0 average and correlations $dW^j_t dW^k_t = (\delta_{j,k} - (p_j + p_k)) + \sum_m p_m^2 dt$. However, as we will now see, this correspondence becomes invalid in the limit of small $\gamma$. 


Small $\gamma$ regime We now perform a perturbative analysis around $\gamma = 0$ in the infinite system size limit $N \to \infty$ and use it to map the dynamics of the random walker [6] onto the SHE. Our starting point is the following small $\gamma$ expansion

$$p = p^{(0)} + \sqrt{\gamma} p^{(1)} + \gamma p^{(2)} + \cdots,$$

where $p^{(0)}$ is the stationary flat profile of the maximally entropic state, i.e., $p_j^{(0)}(t) = 1/N, \forall j, t$. Inserting (7) into (6), we obtain the evolution of $p^{(1)}$ as

$$dp_j^{(1)} = D\Delta d p_j^{(1)} dt + \frac{1}{N} (dB_t^j - \sum m \frac{1}{N} dB_t^m).$$

The term $\sum m \frac{dB_m^j}{N}$ has mean 0 and variance $1/N$ so it is subleading in the limit $N \to \infty$. In this regime, we thus get

$$dp_j^{(1)} \approx D\Delta d p_j^{(1)} dt + p_j^{(0)} dB_t^j.$$  

As $p_j^{(0)}$ is constant, (9) is nothing but the Edwards-Wilkinson [45] equation. The evolution of $p^{(2)}$ is obtained in a similar manner and yields

$$dp_j^{(2)} = D\Delta d p_j^{(2)} dt + p_j^{(1)} dB_t^j + p_j^{(0)} dB_t^j.$$  

As explained above, the variance of $I$ scales as $1/N$. The variance of $II$ is given by $\mathbb{E}[(\frac{1}{\sqrt{N}} \sum_j (p_j^{(1)})^2)]$. Using translational invariance, we have on the other hand that $\mathbb{E}[(p_j^{(0)})^2] = \mathbb{E}[(\frac{1}{\sqrt{N}} \sum_j (p_j^{(1)})^2)]$ so there is a factor of $N$ between the variance of the multiplicative noise term $p_j^{(1)} dB_t^j$ and $II$. Thus, in the limit of large $N$, we can neglect $I$ and $II$ to obtain

$$dp_j^{(2)} \approx D\Delta d p_j^{(2)} + p_j^{(1)} dB_t^j.$$  

This equation is structurally equivalent to Eq. (9). Thus, to order $\gamma$, the discrete SHE with multiplicative noise

$$dp_j = D\Delta d p_j + \sqrt{\gamma} p_j dB_t^j$$

is a good approximation of (6):

$$D\Delta d p_j dt + \sqrt{\gamma} p_j dW_t^j = D\Delta d p_j + \sqrt{\gamma} p_j dB_t^j + O(\gamma).$$  

Because of the well-known connection between the SHE and the KPZ equation via the Cole-Hopf transform [3], equivalence [13] suggests that the logarithm of $p_j$ might be a quantity of interest. Indeed, the discrete Cole-Hopf transform is defined as

$$h_j := \frac{1}{\sqrt{\gamma}} \log p_j.$$  

Note that since $p_j \in [0,1], h_j \in [-\infty,0]$. Using standard Itô calculus on (12), the stochastic dynamics for $h_j$ in the continuum limit is readily computed as

$$dh_j = D\Delta_d h_j + D\sqrt{\gamma} (\nabla_d h_j)^2 - \sqrt{\gamma} dt + dB_t^j,$$

which is nothing else than a discretization of the celebrated KPZ equation up to a linear shift in time $h_j \to h_j + \sqrt{\gamma}t$. Due to the mapping (13), the dynamics of the monitored random walker should thus share common features with the physics of interface growth, at least in the limit of small $\gamma$. One of the interesting quantities arising in the study of such interfaces is the so-called width $w$. It is defined as the average deviation from the spatial mean

$$w := \frac{1}{N} \sum_j (h_j - \bar{h})^2,$$

where $\bar{h} := \frac{1}{N} \sum_j h_j$. Starting from a flat initial profile, the Family-Vicsek (F-V) scaling relation [11, 10] conjectures that, for scale-invariant interfaces, the width should behave as

$$w \propto N^\alpha f \left( \frac{t}{N^{\alpha/\beta}} \right)$$

with $f(u) \propto u^\beta$ for $u \ll 1$ and $f(u) \propto \text{const}$ for $u \gg 1$. $\beta$ is called the growth exponent while $\alpha$ is called the roughening exponent. For models within the KPZ universality class, it has been shown that $\beta = 1/3$ and $\alpha = 0.5$ in one dimension [3]. Due to the mapping (13), we expect the width of the monitored random walker to follow (17) with KPZ exponents when $\gamma$ is small (see Fig. 1-top). However, beyond this regime, the scaling exponents could a priori differ and we tackle this question by performing numerical simulations of (6) using a standard Euler-Maruyama method.

**Numerical results** We now present numerical results obtained for the single-particle problem. We started our simulations with a flat initial profile $p_j(t=0) = 1/N$, i.e. $h_j(t=0) = -\frac{1}{\sqrt{\gamma}} \log N$. We simulated (6) and took the logarithm for every single realization to obtain the evolution of the process $h_j$.

We plot on Fig. 1-top the rescaled width $\hat{w} = w/N^\alpha$ as a function of the rescaled time $t = t/N^{\alpha/\beta}$ for different system sizes when $\gamma \sim 0$. We observe a collapse of the data at short and long $t$ to a F-V scaling relation with $\alpha = 0.45$ and $\beta = 0.38$. At intermediate $t$, the collapse is not perfect but note that such a deviation also features in the SHE (see inset): this confirms the validity of the mapping (13). As reported in e.g. [17, 18], such deviation from the F-V scaling could be attributed to the fact that a strong non-linearity is necessary to drive (15) to the KPZ fixed point.

We plot on Fig. 1-bottom $\hat{w}$ as a function of $\hat{t}$ away from the regime of weak monitoring. At short and long
saturation time \( t^* \approx N^{\alpha/\beta} \). When this second growth regime is absent from the simulations, there are two possibilities: either the system size \( N \) considered is too small or \( \tilde{t} \to \infty \), which would be an indication of a measurement-induced transition. Distinguishing between these two possibilities would require a careful analysis and extrapolation at \( N \to \infty \) of the behavior of \( \tilde{t} \) as a function of \( \gamma \) which is beyond the scope of this work and we defer it to subsequent studies.

**Conclusion and perspectives** In this paper we introduced and studied a model for a single random walker undergoing continuous measurement. In the regime of weak monitoring, we mapped the time-evolution of its probability distribution onto the SHE. We deduced that, in this regime, the width of the log probability follows a F-V scaling relation with growth and scaling exponents \( \beta \approx 1/3 \) and \( \alpha = 0.5 \) (KPZ universality class). Beyond weak monitoring, we numerically find that increasing \( \gamma \) affects the scale-invariant properties of the interface. While the growth exponent remains constant \( \beta \approx 1/3 \), the roughness exponent \( \alpha \) monotonously ramps up from \( 0.5 \) to \( 1 \) when \( \gamma \) varies from \( 0 \) to \( 8 \). Finally, for strong measurement rate, we pointed out the existence of a second growth regime, characterized by a different exponent \( \beta' \).

Our study opens the door to several interesting directions. The emergence of this new growth regime at high \( \gamma \) calls for our immediate attention as it could indicate a measurement-induced phase transition. This phenomena might also feature in higher dimension. For instance, it is documented that the KPZ equation indicates a measurement-induced transition. This would for a quantum version called the QSSEP \([51, 52]\). The study of both SSEP and QSSEP would thus provide a unified framework to disentangle the properties specific to quantum and classical systems under continuous monitoring.

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\end{align*} \]
FIG. 2. **Left:** Log-Log plot of the width $w$ as a function of time for different system sizes $N$ at fixed $\gamma = 4$. For sufficiently large $N$, a second growth regime emerges after a finite time $\tilde{t}$. **Middle:** Log-Log plot of the width $w$ as a function of time for different measurement rates $\gamma$ at fixed $N = 128$. A second growth regime appears for sufficiently high $\gamma$. We see on this figure that $\alpha$, $\tilde{t}$ and $\beta'$ are all $\gamma$-dependent quantities. **Right:** roughening exponent $\alpha$ as a function of $\gamma$.

Medenjak and Lorenzo Piroli for useful discussions. T.J acknowledges support from the Swiss National Science Foundation under Division II. During the writing of the manuscript, it came to our knowledge that two works with a similar objective of studying measurement effects on chaotic, classical systems but with a focus on phase transition were put as preprints [53, 54].

**References**

[1] A.-L. Barabási and H. E. Stanley, *Fractal Concepts in Surface Growth* (Cambridge University Press, 1995).

[2] G. Ódor, Rev. Mod. Phys. 76, 663 (2004).

[3] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. 56, 889 (1986).

[4] I. Corwin, arXiv e-prints, arXiv:1106.1596 (2011), arXiv:1106.1596 [math.PR].

[5] P. Meakin, P. Ramanlal, L. M. Sander, and R. C. Ball, Phys. Rev. A 34, 5091 (1986).

[6] E. Murray, in Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability: Held at the Statistical Laboratory, University of California, June 20-July 30, 1960, Vol. 2 (Univ of California Press, 1960) p. 223.

[7] M. Kolb, R. Botet, and R. Jullien, Phys. Rev. Lett. 51, 1123 (1983).

[8] J. M. Kim and J. M. Kosterlitz, Phys. Rev. Lett. 62, 2280 (1989).

[9] A. Nahum, J. Ruhman, S. Vijay, and J. Haah, Phys. Rev. X 7, 031016 (2017).

[10] D. Bernard and P. L. Doussal, EPL (Europhysics Letters) 131, 10007 (2020).

[11] T. Jin, A. Krajenbrink, and D. Bernard, Phys. Rev. Lett. 125, 040603 (2020).

[12] C. Zu, F. Machado, B. Ye, S. Choi, B. Kobrin, T. Mittiga, S. Hsieh, P. Bhattacharyya, M. Markham, D. Twitchen, A. Jarmola, D. Budker, C. R. Laumann, J. E. Moore, and N. Y. Yao, Nature 597, 45 (2021).

[13] J. De Nardis, M. Medenjak, C. Karrasch, and E. Ilievski, Phys. Rev. Lett. 124, 210605 (2020).

[14] E. Ilievski, J. De Nardis, S. Gopalakrishnan, R. Vasseur, and B. Ware, Phys. Rev. X 11, 031023 (2021).

[15] D. Wei, A. Rubio-Abadal, B. Ye, F. Machado, J. Kemp, K. Strakaew, S. Hollerith, J. Rui, S. Gopalakrishnan, N. Y. Yao, I. Bloch, and J. Zeiher, arXiv e-prints, arXiv:2107.00038 (2021), arXiv:2107.00038 [cond-mat.quant-gas].

[16] Q. Fontaine, D. Squizzato, F. Baboux, I. Melonio, A. Lemaître, M. Morassi, I. Sagnes, L. Le Gratiet, A. Harouri, M. Wouters, I. Carusotto, A. Amo, M. Richard, A. Minguzzi, L. Canet, S. Ravets, and J. Bloch, arXiv e-prints, arXiv:2112.09550 (2021), arXiv:2112.09550 [cond-mat.mes-hall].

[17] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. 60, 1351 (1988).

[18] J. M. Raimond, M. Brune, and S. Haroche, Rev. Mod. Phys. 73, 565 (2001).

[19] W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, Phys. Rev. A 41, 2295 (1990).

[20] S. Kocsis, B. Braverman, S. Ravets, M. J. Stevens, R. P. Mirin, L. K. Shalm, and A. M. Steinberg, Science 332, 1170 (2011).

https://www.science.org/doi/pdf/10.1126/science.1202218.

[21] N. Cottet, S. Jezouin, L. Bretheau, P. Campagne-Ibarcq, Q. Ficheux, J. Anders, A. Aufèves, R. Azouti, P. Rouchon, and B. Huard, Proceedings of the National Academy of Sciences 114, 7561 (2017).

https://www.pnas.org/doi/pdf/10.1073/pnas.1704827114.

[22] Z. K. Minev, S. O. Mundhada, S. Shankar, P. Reinhold, R. Gutiérrez-Jáuregui, R. J. Schoelkopf, M. Mirrahimi, H. J. Carmichael, and M. H. Devoret, Nature 570, 200 (2019).

[23] X. Cao, A. Tilloy, and A. De Luca, SciPost Phys. 7, 024 (2019).

[24] B. Skinner, J. Ruhman, and A. Nahum, Phys. Rev. X 9, 031009 (2019).

[25] Y. Li, X. Chen, and M. P. A. Fisher, Phys. Rev. B 100,
[26] M. Szyniszewski, A. Romito, and H. Schomerus, Phys. Rev. B 100, 064204 (2019).
[27] Y. Bao, S. Choi, and E. Altman, Phys. Rev. B 101, 104301 (2020).
[28] M. J. Gullans and D. A. Huse, Phys. Rev. Lett. 125, 070606 (2020).
[29] M. J. Gullans and D. A. Huse, Phys. Rev. X 10, 041020 (2020).
[30] A. Zabalo, M. J. Gullans, J. H. Wilson, S. Gopalakrishnan, D. A. Huse, and J. H. Pixley, Phys. Rev. B 101, 060301 (2020).
[31] A. Lavasani, Y. Alavirad, and M. Barkeshli, Nature Physics 17, 342 (2021).
[32] M. Buchhold, Y. Minoguchi, A. Altland, and S. Diehl, Phys. Rev. X 11, 041004 (2021).
[33] X. Turkeshi, A. Biella, R. Fazio, M. Dalmonte, and M. Schirò, Phys. Rev. B 103, 224210 (2021).
[34] P. Sierant and X. Turkeshi, Phys. Rev. Lett. 128, 130605 (2022).
[35] Z. Weinstein, Y. Bao, and E. Altman, Measurement-induced power law negativity in an open monitored quantum circuit (2022).
[36] J. Dalibard, Y. Castin, and K. Molmer, Phys. Rev. Lett. 68, 580 (1992).
[37] K. Jacobs and D. A. Steck, Contemporary Physics 47, 279 (2006) https://doi.org/10.1080/00107510601101934.
[38] M. Bauer, D. Bernard, and A. Tilloy, Journal of Statistical Mechanics: Theory and Experiment 2014 P09001 (2014).
[39] D. Bernard, T. Jin, and O. Shpielberg, EPL (Europhysics Letters) 121, 60006 (2018).
[40] L. Rabiner and B. Juang, IEEE ASSP Magazine 3, 4 (1986).
[41] Supplementary material.
[42] M. Bauer and D. Bernard, Letters in Mathematical Physics 104, 707 (2014).
[43] A. Tilloy, M. Bauer, and D. Bernard, Phys. Rev. A 92, 052111 (2015).
[44] L. Bertini and N. Cancrini, Journal of Statistical Physics 78, 1377 (1995).
[45] F. Edwards and D. R. Wilkinson, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences 381, 17 (1982).
[46] F. Family and T. Vicsek, Journal of Physics A: Mathematical and General 18, L75 (1985).
[47] C. Dasgupta, J. M. Kim, M. Dutta, and S. Das Sarma, Phys. Rev. E 55, 2235 (1997).
[48] C.-H. Lam and F. G. Shin, Phys. Rev. E 57, 6506 (1998).
[49] L. Canet, H. Chaté, B. Delamotte, and N. Wschebor, Phys. Rev. Lett. 104, 150601 (2010).
[50] G. Grinstein, M. A. Muñoz, and Y. Tu, Phys. Rev. Lett. 76, 4376 (1996).
[51] M. Bauer, D. Bernard, and T. Jin, SciPost Phys. 6, 45 (2019).
[52] D. Bernard and T. Jin, Phys. Rev. Lett. 123, 080601 (2019).
[53] J. Willsher, S.-W. Liu, R. Moessner, and J. Knolle, Measurement-induced phase transition in a classical, chaotic many-body system (2022).
[54] A. Pizzi, D. Malz, A. Nunnenkamp, and J. Knolle, Bridging the gap between classical and quantum many-body information dynamics (2022).
Appendix A: Weak measurements

In this appendix, we present the derivation of (5) and (6) of the main text.

The state of system is described by a set of random variables \( \{X_1, \cdots, X_N\} := \vec{X} \) which can take values \( \pm 1 \) where +1 corresponds to an occupied site and −1 to an empty site. The ancilla monitoring site \( j \) is also described by a random variable \( A_j \) which can take binary values \( \{-1, 1\} \). We denote by \( P(\vec{X}, A_j) \) the joint probability of the union system + ancilla to be in a given configuration. We fix this probability distribution to

\[
P(\vec{X}, A_j) = P(\vec{X}) \frac{1 + \varepsilon A_j X_j}{2}, \tag{A1}
\]

with \( \varepsilon = \sqrt{\frac{\gamma}{\delta t}} \) being a small parameter and \( P(\vec{X}) \) being the reduced probability of the system only. The joint distribution [A1] implies that the state of ancilla \( j \) and its corresponding site are positively correlated: if \( A_j = 1 \), it’s more likely to find \( X_j \) in state 1. The reduced probability \( P(A_j) \) for the \( j \)-th ancilla is given by

\[
P(A_j) = \sum_{\{\vec{X}\}} P(\vec{X}, A_j) = \frac{1}{2}(1 + \varepsilon A_j \langle X_j \rangle) . \tag{A2}
\]

Once a measurement of the state of the ancilla has been made with outcome \( A_j \), the probability distribution is updated with probability \( P(A_j) \) to

\[
P(\vec{X}) \rightarrow P(\vec{X}|A_j) = P(\vec{X}) \frac{1 + \varepsilon A_j X_j}{1 + \varepsilon A_j (X_j)} , \tag{A3}
\]

which is just Bayes law. We now repeat this procedure \( M \) times with a fresh ancilla \( A_j^{(n)} \) indexed by \( n \in [1, M] \). Expanding [A3] to order \( \varepsilon^2 \), we get:

\[
P_{n+1}(\vec{X}|A_j^{(n)}) = P_n(\vec{X}) \left( 1 + \varepsilon A_j^{(n)} X_j \right) \left( 1 - \varepsilon A_j^{(n)} \langle X_j \rangle_n + \varepsilon^2 \left( A_j^{(n)} \langle X_j \rangle_n \right)^2 \right) + O(\varepsilon^3) \tag{A4}
\]

\[
= P_n(\vec{X}) \left( 1 + \varepsilon A_j^{(n)} (X_j - \langle X_j \rangle_n) - \varepsilon^2 (X_j \langle X_j \rangle_n - \langle X_j \rangle_n^2) \right) + O(\varepsilon^3) , \tag{A5}
\]

where \( \langle \rangle_n \) indicates that the average has to be taken with \( P_n \) and we used that \( (A_j^{(n)})^2 = 1 \). The signal is defined as the sum of the measurement outputs on the ancilla, i.e \( S_{j,M} := \sum_{n=1}^{M} \varepsilon A_j^{(n)} \), from which we deduce its increment \( S_{j,M+1} - S_{j,M} = \varepsilon A_j^{(M+1)} \). Using [A2], we further obtain that

\[
\langle \varepsilon A_j^{(n)} \rangle = \frac{\varepsilon}{2} (1 + \varepsilon \langle X_j \rangle_n) - \frac{\varepsilon}{2} (1 - \varepsilon \langle X_j \rangle_n) = \varepsilon^2 \langle X_j \rangle_n , \tag{A6}
\]

\[\langle (\varepsilon A_j^{(n)})^2 \rangle = \varepsilon^2 .\]

These relations show that in the limit \( \varepsilon \to 0 \), \( M\delta t = t \), the signal converges in law towards a process described by the following stochastic differential equation

\[
dS_{j,t} = \frac{\gamma}{4} \langle X_j \rangle_t dt + \frac{\sqrt{7}}{2} dB^j_t , \tag{A7}
\]

where \( j \) is the site index and \( B^j_t \) is a 0-mean, site-independent Brownian process of variance \( \mathbb{E}[B^j_t B^j_t] = \delta_{ij} dt \). We can now replace the \( A_j \)'s in the evolution equation for the probability [A5] to get

\[
dP_t(\vec{X}) = P_t(\vec{X}) \left( \frac{\gamma}{4} \langle X_j \rangle_t dt + \frac{\sqrt{7}}{2} dB^j_t \right) (X_j - \langle X_j \rangle_t) - \frac{\gamma}{4} (X_j \langle X_j \rangle_t - \langle X_j \rangle_t^2) dt \right) = \frac{\sqrt{7}}{2} P_t(\vec{X}) (X_j - \langle X_j \rangle_t) dB^j_t . \tag{A8}
\]

If measurement processes occur independently on every site and we include the internal stochastic dynamics \( \mathcal{L} \) of the system, we obtain the following SDE for \( P_t \):

\[
dP_t = \mathcal{L}(P_t) dt + \frac{\sqrt{7}}{2} \sum_j P_t(\vec{X})(X_j - \langle X_j \rangle_t) dB^j_t , \tag{A9}
\]

which is [6] in the main text.
We now specify to the particular case of a single random walker on a discrete lattice of \( N \) sites with periodic boundary conditions. In this case, we define \( p_j(t) := \mathcal{P}_t(X_1, = -1, \cdots, X_j, = 1, \cdots X_N = -1) \) and we further have that

\[
\mathcal{L}(p_j) = D(p_{j-1} - 2p_j + p_{j+1}), \quad \langle X_j \rangle_t = \sum_{j' \neq j} -p_{j'} + p_j = -1 + 2p_j . \tag{A10}
\]

Inserting (A10) into (A9), we obtain the time-evolution of \( p_j \) (Eq. (6) in main text) as

\[
dp_j = D(p_{j-1} - 2p_j + p_{j+1}) + \frac{\sqrt{\gamma}}{2} \left( \sum_{m \neq j} p_m (-2p_j)dB^m_t + p_j (2 - 2p_j)dB^j_t \right),
\]

\[
= D(p_{j-1} - 2p_j + p_{j+1}) + \sqrt{\gamma}p_j \left( dB^j_t - \sum_m p_m dB^m_t \right) . \tag{A11}
\]