Representations of Two-Colour BWM Algebras and Solvable Lattice Models

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Abstract

Many of the known solutions of the Yang-Baxter equation, which are related to solvable lattice models of vertex- and IRF-type, yield representations of the Birman-Wenzl-Murakami algebra. From these, representations of a two-colour generalization of the Birman-Wenzl-Murakami algebra can be constructed, which in turn are used to derive trigonometric solutions to the Yang-Baxter equation. In spirit, this construction resembles the fusion procedure, in the sense that starting from known solutions of the Yang-Baxter equation new solutions can be obtained.

1 Introduction

Over the past fifteen years, since Baxter’s famous book first appeared in print [1], the full relevance of the Yang-Baxter equation (YBE) in the theory of solvable two-dimensional lattice models has been realized. Moreover, many fruitful and partly unexpected connections to other branches of mathematics and physics, including for example quantum groups, knot and link invariants, and integrable quantum field theories with purely elastic (factorized) scattering, have been discovered.

Among the several algebraic techniques used to construct solutions to the YBE, a particularly interesting approach is based on braid-monoid algebras (BMA) [2]. By a procedure called Baxterization [3], one reduces the problem of finding representations of the Yang-Baxter algebra (YBA) to the simpler task of constructing representations of a certain BMA. Examples of BMA for which a Baxterization is known are the Temperley-Lieb algebra [4], the Birman-Wenzl-Murakami (BWM) algebra [5,6] and their recently introduced [7] ‘dilute’ [8–13] and ‘two-colour’ generalizations [14,15].

In this short note, we present explicit expressions for ‘vertex-type’ representations of the two-colour BWM (TCBWM) algebra, by which we understand representations acting on a tensor product space. This is complementary to the content of Ref. [15] where exclusively RSOS (restricted solid-on-solid) type representations were considered, for which the representation space is spanned by the set of all paths on certain graphs. Here, the representations are constructed by suitably combining representations of the ordinary BWM algebra underlying the solvable vertex models of Bazhanov [16] and Jimbo [17]. By the known Baxterization of the TCBWM algebra [15], this yields solutions of the YBE and hence solvable lattice models of vertex type.

In contrast to the case of RSOS type representations discussed in Ref. [15], we do not expect that the corresponding solvable vertex models are new. Instead, we expect a similar scenario as in the dilute case, where the $R$-matrices constructed...
from representations of the dilute BWM algebra are related by a suitable gauge transformation to other members of the lists of well-known $R$-matrices \[16,17\]. For one series of $R$-matrices, this has been explicitly demonstrated in Ref. \[10\].

2 The TCBWM algebra

For lack of space, we cannot repeat the complete definition of the TCBWM algebra here, thus we only give a rough sketch of the idea, see Refs. \[7,15\] for details.

An ordinary BMA \[2\] is generated by two types of generators, the braids (with their inverses) and the monoids or Temperley-Lieb (TL) operators. Besides the usual braid relations and the defining relations of the TL algebra, a number of relations involving both types of generators are imposed. All relations have a natural interpretation in terms of continuous moves in a diagrammatic presentation of the generators acting on arrays of strings \[2\], hinting at the connection to the theory of knot and link invariants. The defining relations also involve two central elements customarily denoted by $\sqrt{Q}$ and $\omega$, which in the graphical interpretation are associated to closed loops and so-called ‘twists’, respectively. The BWM algebra is a certain quotient of a BMA, in which the braid generators satisfy a cubic reduction relation \[5,6\].

The TCBWM algebra is a two-colour generalization of the BWM algebra, with two species (colours) of strings. The generators now carry colour indices

\[
\begin{align*}
p_j^{(c,c')} & \quad \text{(projectors)} \\
b_j^{(c,c)} , & \quad b_j^{(c,c')} \quad \text{(braids)} \\
e_j^{(c,c')} & \quad \text{(Temperley-Lieb operators)}
\end{align*}
\]

and one has additional generators $p_j^{(c,c')}$ which act as projection operators on certain colours. Here, $c, c' \in \{1, 2\}$ label the two colours, and $\bar{c} = 3 - c$ denotes the complementary (or opposite) colour. Furthermore, the generators carry a label $j \in \{1, 2, \ldots, N-1\}$. In the graphical interpretation, this means that the corresponding generator only acts on the two of the $N$ strings, namely on those labeled by $j$ and $j+1$. The central elements mentioned above now also carry colour indices, thus we have factors of $\sqrt{Q_c}$ and $\omega_c$ associated to closed loops and to ‘twists’ in a string of colour $c$, respectively.

The defining relations of the TCBWM algebra can most easily be understood directly from the diagrammatic interpretation: they are given by considering all possible colourings of the diagrams corresponding to the defining relations of the ordinary BWM algebra. In addition, we demand that any products leading to diagrams with colour mismatches vanish in the algebra. This statement basically reduces to the requirement that the generators $p_j^{(c,c')}$ are indeed orthogonal projectors.

Before we move on to the representations, let us briefly present the relations defining the BWM quotient. For the coloured braids, one has the cubic reduction relations

\[
\left( b_j^{+(c,c)} - q_c^{-1} p_j^{(c,c)} \right) \left( b_j^{+(c,c)} + q_c p_j^{(c,c)} \right) \left( b_j^{+(c,c)} - \omega_c p_j^{(c,c)} \right) = 0; \quad (2)
\]

and the Temperley-Lieb generators $e_j^{(c,c)}$ are given by quadratic expressions in the
braid as follows
\[ e_j^{(c,c)} = p_j^{(c,c)} + \frac{b_j^{+(c,c)} - b_j^{-(c,c)}}{q_c - q_c^{-1}}, \]
where \( q_c \) is related to \( \sqrt{Q_c} \) and \( \omega_c \) by
\[ \sqrt{Q_c} = 1 + \frac{\omega_c - \omega_c^{-1}}{q_c - q_c^{-1}} \]
which is a consequence of Eqs. (2) and (3) and the defining relations of the TL algebra.

3 Representations of the TCBWM algebra

We now present explicit expressions for representations of the TCBWM algebra. We build these representations on a pair of representations of the ordinary BWM algebra, which are labeled by affine Lie algebras \( B^{(1)}_n \), \( C^{(1)}_n \), and \( D^{(1)}_n \) and underlie the corresponding series of \( R \)-matrices of Refs. [16,17]. Thus the TCBWM representations are labeled by pairs \( (G^{(1)}_{c_1}, n_1, G^{(1)}_{c_2}, n_2) \), where \( G_c \in \{B, C, D\} \) and the index \( c \) refers to the colour. Given \( G_c \) and \( n_c \), the representation still contains two parameters \( q_c \), which determine \( \omega_c \) through
\[ \omega_c = \begin{cases} \frac{q_c^{2n_c}}{q_c^{2n_c+1}} & \text{if } G_c = B \\ \frac{-q_c^{2n_c+1}}{q_c^{2n_c-1}} & \text{if } G_c = C \\ q_c^{2n_c-1} & \text{if } G_c = D \end{cases} \]
and thereby \( \sqrt{Q_c} \) by Eq. (4).

The two-colour BWM representations we are interested in act on the tensor product space
\[ V_c = \bigotimes_{j=1}^{N} C^{d_c}, \]
where \( d = d_1 + d_2 \) is given by
\[ d_c = \begin{cases} 2n_c + 1 & \text{if } G_c = B \\ 2n_c & \text{if } G_c \in \{C, D\} \end{cases} \]
and \( V_c = \bigotimes_{j=1}^{N} C^{d_c} \) is the representation space for the single-colour representations.
In these representations, the generators (1) act non-trivially only in two factors (labeled by \( j \) and \( j + 1 \)) of the \( N \)-fold tensor product, wherefor it suffices to give the expressions for two factors which thus are \( d^2 \times d^2 \) matrices which we denote by the same symbols (1) without the site index \( j \).

We introduce two index sets
\[ I_1 = \{1, 2, \ldots, d_1\} \]
\[ I_2 = \{d_1 + 1, d_1 + 2, \ldots, d_1 + d_2\} \]
to keep apart labels referring to different colours. For \( \alpha \in I_c \), we define
\[ \tilde{\alpha} = \alpha - \frac{d_c + 1}{2} - d_1 \delta_{c,2} \in \left\{ -\frac{d_c - 1}{2}, -\frac{d_c - 3}{2}, \ldots, -\frac{d_c - 1}{2} \right\} \]
\[ \alpha' = d_c + 1 - \alpha + 2d_1 \delta_{c,2} \in I_c \]
such that the possible values of $\tilde{\alpha}$ for both colours lie symmetrically around zero. Furthermore, we introduce symbols $\epsilon_{\alpha}$ and $\bar{\epsilon}_{\alpha}$ which are defined as follows:

For $\alpha \in I_c$ and $G_c \in \{B, D\}$:

$$\epsilon_{\alpha} = \begin{cases} \tilde{\alpha} + \frac{1}{2} & \alpha < \alpha' \\ \tilde{\alpha} & \alpha = \alpha' \\ \tilde{\alpha} - \frac{1}{2} & \alpha > \alpha' \end{cases} \quad (11)$$

$$\bar{\epsilon}_{\alpha} = \begin{cases} \tilde{\alpha} + 1/2 & \alpha < \alpha' \\ \tilde{\alpha} & \alpha = \alpha' \\ \tilde{\alpha} - 1/2 & \alpha > \alpha' \end{cases} \quad (12)$$

For $\alpha \in I_c$ and $G_c = C$:

$$\epsilon_{\alpha} = \begin{cases} 1 & \alpha < \alpha' \\ -1 & \alpha > \alpha' \end{cases} \quad (13)$$

$$\bar{\epsilon}_{\alpha} = \begin{cases} \tilde{\alpha} + 1/2 & \alpha < \alpha' \\ \tilde{\alpha} - 1/2 & \alpha > \alpha' \end{cases} \quad (14)$$

Note that $\alpha = \alpha' \in I_c$ can only occur for representations labeled by $B^{(1)}_{c,n}$ because these have an odd $d_c = 2n_c + 1$.

With these notations, we obtain a two-colour representation by setting

$$p^{(c,c')} = \sum_{\alpha \in I_c} \sum_{\beta \in I_{c'}} E_{\alpha,\alpha} \otimes E_{\beta,\beta} \quad (15)$$

$$b^{+, (c,c')} = \sum_{\alpha \in I_c} q_c^{-1} \left[1 + (q_c - 1) \delta_{\alpha,\alpha'}\right] E_{\alpha,\alpha} \otimes E_{\alpha,\alpha} + \sum_{\alpha \neq \beta \in I_c} \left[1 + (q_c - 1) \delta_{\alpha,\beta}\right] E_{\alpha,\beta} \otimes E_{\beta,\alpha} - (q_c - q_c^{-1}) \sum_{\alpha < \beta \in I_c} E_{\alpha,\alpha} \otimes E_{\beta,\beta} + (q_c - q_c^{-1}) \sum_{\alpha > \beta \in I_c} \epsilon_{\alpha} \epsilon_{\beta} q_c^{\tilde{\alpha}' - \tilde{\beta}'} E_{\alpha',\beta} \otimes E_{\alpha',\beta'} \quad (16)$$

$$b^{-, (c,c')} = \sum_{\alpha \in I_c} q_c \left[1 + (q_c^{-1} - 1) \delta_{\alpha,\alpha'}\right] E_{\alpha,\alpha} \otimes E_{\alpha,\alpha} + \sum_{\alpha \neq \beta \in I_c} \left[1 + (q_c^{-1} - 1) \delta_{\alpha,\beta}\right] E_{\alpha,\beta} \otimes E_{\beta,\alpha} - (q_c - q_c^{-1}) \sum_{\alpha > \beta \in I_c} E_{\alpha,\alpha} \otimes E_{\beta,\beta} - (q_c - q_c^{-1}) \sum_{\alpha < \beta \in I_c} \epsilon_{\alpha} \epsilon_{\beta} q_c^{\tilde{\alpha}' - \tilde{\beta}'} E_{\alpha',\beta} \otimes E_{\alpha',\beta'} \quad (17)$$

$$b^{(c,c')} = \sum_{\alpha \in I_c} \sum_{\beta \in I_c} E_{\beta,\alpha} \otimes E_{\alpha,\beta} \quad (18)$$

$$e^{(c,c')} = \sum_{\alpha \in I_c} \sum_{\beta \in I_c} \epsilon_{\alpha} \epsilon_{\beta} q_c^{\tilde{\alpha}' - \tilde{\beta}'} E_{\alpha',\beta} \otimes E_{\alpha',\beta'} \quad (19)$$

where, as above, $c, c' \in \{1, 2\}$ and $\bar{c} = 3 - c$. The $d \times d$ matrices $E_{\alpha,\beta}$ have matrix elements $(E_{\alpha,\beta})_{i,j} = \delta_{i,\alpha} \delta_{j,\beta}$. The expressions for the generators $e^{(c,c')}$ follow from Eq. (3).
4 Solvable vertex models

The Boltzmann weights of solvable vertex models derived from representations of the TCBWM algebra are encoded in the matrix elements of the $R$-matrix. The Baxterization of Ref. [15] yields a general expression for a trigonometric $R$-matrix for any representation given in the previous section, with the restriction that the parameters $q_c$ have to be equal, i.e., $q_1 = q_2 = q$. Let us introduce two parameters $\lambda$ and $\eta$ by

$$ q = \exp(-i\lambda), \quad q^{2\eta} = \exp(-2i\eta\lambda) = \omega_1 \omega_2, \quad (20) $$

where of course the values of $\omega_c$ are determined by Eq. (5).

Denoting the spectral parameter by $u$, the $R$-matrix has the form [15]

$$ \hat{R}(u) = \sum_{c=1}^{2} \left\{ p^{(c,c)} + \frac{\sin(\eta\lambda-u)}{\sin(\eta\lambda)} p^{(c,\bar{c})} \right. $$

$$ \left. - \frac{\sin(u)}{2i \sin(\lambda) \sin(\eta\lambda)} \left( e^{i(\eta\lambda-u)} b^{+(c,c)} - e^{-i(\eta\lambda-u)} b^{-(c,c)} \right) \right. $$

$$ + \frac{\sin(u) \sin(\eta\lambda-u)}{\sin(\lambda) \sin(\eta\lambda)} b^{(c,\bar{c})} + \frac{\sin(u)}{\sin(\eta\lambda)} e^{i(c,\bar{c})} \right\} \quad (21) $$

which is completely symmetric in the two colours. It satisfies the quantum YBE

$$ (\hat{R}(u) \otimes I) (I \otimes \hat{R}(u+v)) (\hat{R}(v) \otimes I) = (I \otimes \hat{R}(v)) (\hat{R}(u+v) \otimes I) (I \otimes \hat{R}(u)) \quad (22) $$

where $I$ denotes the $d \times d$ unit matrix. These solutions of the YBE are crossing-symmetric [2] with crossing parameter $\eta\lambda$, and satisfy the inversion relation

$$ \hat{R}(u) \hat{R}(-u) = \varrho(u) \varrho(-u) I \otimes I \quad (23) $$

where the function $\varrho(u)$ is given by

$$ \varrho(u) = \frac{\sin(\lambda-u) \sin(\eta\lambda-u)}{\sin \lambda \sin \eta \lambda}. \quad (24) $$

5 Concluding remarks

We presented explicit representations of the TCBWM algebra acting on tensor product spaces. These representations are built on a pair of two representations of the ordinary BWM algebra. From any representation of this type, we construct an $R$-matrix via the Baxterization derived in Ref. [15], and hence a solvable lattice model of vertex type. Although the construction shows some similarity to the fusion procedure, these models are different from those constructed by fusion, compare e.g. Ref. [18] for the fusion of the dilute $A_L$ models. Note that the $R$-matrices related to the dilute BWM algebra [10,11] and to the dilute and two-colour TL algebras [14,7] can be recovered as special cases of the TCBWM result, see Ref. [15].

As already mentioned in the introduction, we expect that the $R$-matrices obtained in this way are not new, but related to other trigonometric $R$-matrices within the lists
given in Refs. [16,17] by gauge transformations. It would be interesting to know the precise relationship between these $R$-matrices and to understand their construction on the level of the underlying quantum groups.

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