Spectral multipliers, Bochner-Riesz means and uniform Sobolev inequalities for elliptic operators

Adam Sikora  
Macquarie University, Sydney  
sikora@maths.mq.edu.au

Lixin Yan  
Sun Yat-sen University, Guangzhou  
mcsylx@mail.sysu.edu.cn

Xiaohua Yao  
Central China Normal University, Wuhan  
yaoxiaohua@mail.ccnu.edu.cn

June 17, 2015

Abstract

This paper comprises two parts. In the first, we study $L^p$ to $L^q$ bounds for spectral multipliers and Bochner-Riesz means with negative index in the general setting of abstract self-adjoint operators. In the second we obtain the uniform Sobolev estimates for constant coefficients higher order elliptic operators $P(D) - z$ and all $z \in \mathbb{C} \setminus [0, \infty)$, which give an extension of the second order results of Kenig-Ruiz-Sogge [39]. Next we use perturbation techniques to prove the uniform Sobolev estimates for Schrödinger operators $P(D) + V$ with small integrable potentials $V$. Finally we deduce spectral multiplier estimates for all these operators, including sharp Bochner-Riesz summability results.

Contents

1 Introduction 2

2 Preliminaries 7
   2.1 Gaussian estimates and Davies-Gaffney estimates 8
   2.2 Stein-Tomas restriction type condition 9

3 Spectral multipliers and Bochner-Riesz means 10
   3.1 Bochner-Riesz means with negative index 10
   3.2 Bochner-Riesz means imply spectral multiplier estimates 16
   3.3 Estimates for the Bochner-Riesz means with negative index 21

4 Uniform Sobolev inequalities for elliptic operators with constant coefficients 23

\*2010 Mathematics Subject Classification: Primary: 58J50; Secondary: 42B15, 42B20, 35P15, 47F05.
\*Key words and phrase: Spectral multiplier, Bochner-Riesz mean, uniform Sobolev inequality, spectral measure, restriction type estimate.
1 Introduction

In this paper we investigate $L^p$ to $L^q$ estimates for spectral multiplier operators including Bochner-Riesz means with negative index in the general setting of abstract self-adjoint operators as well as elliptic differential operators. We also study closely related issue of the uniform Sobolev estimates. In this section we review these ideas, present our results, and put them in context.

Suppose that $X$ is a metric measure space and that $L$ is a nonnegative self-adjoint operator acting on the space $L^2(X)$. Such an operator admits a spectral resolution $E_L(\lambda)$. If $F$ is a real-valued Borel function $F$ on $[0, \infty)$, we can define the operator $F(L)$ by the formula

\begin{equation}
F(L) = \int_0^\infty F(\lambda) \, dE_L(\lambda).
\end{equation}

By the spectral theory the norm $\|F(L)\|_{L^2 \to L^2}$ is bounded by $L^\infty$ norm of the function $F$ (on the spectrum of $L$). We call $dE_L(\lambda)$ the spectral measure associated with the operator $L$. A significant problem often considered in the spectral multiplier theory is to describe sufficient conditions on $F$ to ensure the boundedness of extension of multiplier $F(L)$ from the operator defined on $L^2(X)$ to one acting between some $L^p(X)$ spaces or even more general functional spaces defined on $X$. Since the fundamental works of Mikhlin and Hörmander on Fourier multipliers [43, 37], one usually looks for conditions formulated in terms of differentiability of the function $F$. In addition the special instance of the Bochner-Riesz mean described below is also often investigated.

In the last fifty or so years spectral multipliers theory and the Bochner-Riesz means have attracted a lot of attention and have been studied extensively by many authors. The existing literature is too broad to list all significant contributions to the subject. Therefore here we mention only some examples of papers devoted or related to this research area such as [2, 9, 15, 23, 29, 32, 35, 36, 39, 41, 44, 51, 50, 53]. We wish to point out papers, which investigate sharp spectral multiplier results, the main focus of our study and quote in addition [14, 16, 19, 27, 33, 46, 47]. We refer the reader for references in all works cited above for more comprehensive list of relevant literature.

One of the most significant and more often considered instance of spectral multipliers is the Bochner-Riesz mean of the operator $L$. To define it, we put

\begin{equation}
S_R^\alpha(\lambda) = \frac{1}{\Gamma(\alpha + 1)} \left(1 - \frac{\lambda}{R}\right)^\alpha = \frac{1}{\Gamma(\alpha + 1)} \begin{cases} 
(1 - \frac{\lambda}{R})^\alpha & \text{for } \lambda \leq R \\
0 & \text{for } \lambda > R.
\end{cases}
\end{equation}

Then, we call the operator $S_R^\alpha(L)$ defined by (1.1) the Bochner-Riesz mean of order $\alpha$. The additional factor $\frac{1}{\Gamma(\alpha + 1)}$ is just reparametrization for positive $\alpha$ which is convenient to use if one consider negative range of $\alpha$, see [3, 7]. The case $\alpha = 0$ corresponds to the spectral projector $E_L([0, R])$, while for $\alpha > 0$ one can think of (1.2) as a smoothed version of the spectral projector, where the magnitude of $\alpha$ increases the order of smoothness. In perspective the Bochner-Riesz means of the operator $L$ which we develop here play a special role because they are not only the aim of our study but also a crucial tool in this paper.
To be able to describe and discuss our results we have to introduce some standard notation. Throughout this paper we assume that \((X, d, \mu)\) is a metric measure space with a Borel measure \(\mu\). We denote by \(B(x, \rho) = \{ y \in X, d(x, y) < \rho \}\) the open ball with centre \(x \in X\) and radius \(\rho > 0\). We often just use \(B\) instead of \(B(x, \rho)\). Given \(\lambda > 0\), we write \(\lambda B\) for the \(\lambda\)-dilated ball which is the ball with the same centre as \(B\) and radius \(\lambda \rho\). We set \(V(x, \rho) = \mu(B(x, \rho))\) the volume of \(B(x, \rho)\).

We say that \((X, d, \mu)\) satisfies the doubling condition, see [17, Chapter 3], if there exists a constant \(C > 0\) such that

\[
V(x, 2\rho) \leq CV(x, \rho) \quad \forall \rho > 0, \ x \in X.
\]

If this is the case, then there exist constants \(n\) and \(C\) such that for all \(\lambda \geq 1\) and \(x \in X\)

\[
V(x, \lambda \rho) \leq C \lambda^n V(x, \rho).
\]

In the sequel we want to consider \(n\) as small as possible and we always assume that condition (D) and \((D_n)\) are valid. In the standard Euclidean space with the Lebesgue measure \(n\) coincides with its dimension.

Next we describe the notion of Davies-Gaffney estimates, see [10, 11, 18]. Given a subset \(E \subseteq X\), we denote by \(\chi_E\) the characteristic function of \(E\) and set

\[
P_E f(x) = \chi_E(x) f(x).
\]

Consider again a non-negative self-adjoint operator \(L\) and an exponent \(m \geq 2\). We say that the semigroup \(e^{-tL}\) generated by \(L\) satisfies \(m\)-th order Davies-Gaffney estimates, if there exist constants \(C, \ c > 0\) such that

\[
\|P_{B(x,t^{1/m})} e^{-tL} P_{B(y,t^{1/m})}\|_{2 \to 2} \leq C \exp\left(- c \left(\frac{d(x,y)}{t^{1/m}}\right)^m\right)
\]

for all \(t > 0\) and \(x, y \in X\).

Another condition which we usually impose on the semigroup generated by \(L\) can be described in the following way. We assume that for some \(1 \leq p < 2\),

\[
\|e^{-tP_{B(x,s)}}\|_{p \to 2} \leq CV(x,s)^{\frac{1}{p} - \frac{1}{2}} \left(\frac{s}{t}\right)^{n\left(\frac{1}{p} - \frac{1}{2}\right)}
\]

holds for all \(x \in X\) and \(s \geq t > 0\).

Recall that the semigroup \(e^{-tL}\) generated by \(L\) is said to satisfy \(m\)-th order (pointwise) Gaussian estimate \(GE_m\), see for instance [10, Proposition 2.9], if semigroup \(e^{-tL}\) has integral kernels \(p_t(x,y)\) and there exist constants \(C, \ c > 0\)

\[
|p_t(x,y)| \leq \frac{C}{V(x,t^{1/m})} \exp\left(- c \left(\frac{d^m(x,y)}{t}\right)^{\frac{1}{m}}\right)
\]

for all \(t > 0\) and \(x, y \in X\). It is not difficult to note that both conditions \((DG_m)\) and \((G_{p,2,m})\) for any \(1 \leq p < 2\) follow from Gaussian estimates \((GE_m)\). On the other hand, there are many operators which satisfy Davies-Gaffney estimates \((DG_m)\) for which the standard pointwise Gaussian estimates \((GE_m)\) fail. For example, Schrödinger operators with inverse-square potential see [18, 45], second order elliptic operators with rough lower order terms, see [42], or higher order elliptic operators with bounded measurable coefficients, see [21].
For semigroups generated by differential operators the parameter \( m \geq 2 \) above usually corresponds to their order. The above estimates especially in the case \( m = 2 \) are the main focus of heat kernel theory. It is a well-established area of mathematics, which provides a deep understanding of Gaussian estimates \( (\text{GE}_m) \) and a broad class of examples (operators and ambient spaces), for which such estimates hold, see e.g. Davies [20], Ouhabaz [44] and Grigor’yan [31] and literature therein.

It is known that conditions \((\text{DG}_m)\) and \((\text{G}_{p,2,m})\) for some \( 1 \leq p < 2 \) imply that the spectral operator \( F(L) \) is bounded on \( L^r(X) \) for all \( p < r < p' \) for any bounded Borel function \( F: \mathbb{R}_+ \to \mathbb{C} \) such that

\[
(1.3) \quad \sup_{r>0} \|\eta F(t \cdot)\|_{L^r} < \infty
\]

for some \( k > n(1/p - 1/2) \), see for example [2, 9, 27] and Proposition 2.2 below. Here \( \eta \in C_c^\infty(0, \infty) \) is an arbitrary non-zero auxiliary function. In particular, if \( p = 1 \) then this corresponds to a spectral multiplier version of the classical Mikhlin theorem.

In general spectral multiplier theorems based on norm in (1.3) do not lead to critical exponent for Bochner-Riesz summability. To obtain such sharp results weaker Sobolev norms \( W^{a,q}(\mathbb{R}) \) \((1 \leq q \leq \infty)\) are considered, see [14, 27, 46]. The general perspective of these papers is that condition \((\text{ST}_{p,2,m}^q)\) below, imply sharp \( W^{a,q}(\mathbb{R}) \) version of spectral multipliers. See also [33, 41]. Condition \((\text{ST}_{p,2,m}^q)\) is motivated by classical Stein-Tomas restriction theorem, see (1.4) below. We point out that considering different values of \( q \in [1, \infty] \) are often essential for applications. If \( q = \infty \), then conditions \((\text{ST}_{p,2,m}^{\infty})\) and \((\text{G}_{p,2,m})\) are equivalent for every \( 1 \leq p < 2 \). The case \( q = 2 \), corresponding to classical Hörmander theorem, has another characterization in terms of the spectral measure \( dE_L(\lambda) \).

Namely, given any \( 0 \leq p < 2 \) condition \((\text{ST}_{p,2,m}^{2})\) is equivalent to the following estimate

\[
(1.4) \quad \|dE_L(\lambda)\|_{p \to p'} \leq C \lambda^{p/(p-1)} \lambda^{(n+3)/2} R^*_R \lambda^\frac{3}{2}, \quad \lambda > 0.
\]

It is a remarkable fact that estimate (1.4) not only play a crucial role in spectral multiplier theory and Bochner-Riesz analysis but can be also regarded as a significant example of restriction type results in harmonic analysis. Indeed, if \( \Delta \) is the standard Laplace operator in \( \mathbb{R}^n \), then a \( T^*T \) argument yields

\[
dE_\Delta(\lambda) = (2\pi)^{-n} \lambda^{(n-1)/2} R^*_R \lambda
\]

where \( R_\lambda \) is the restriction operator defined by relation \( R_\lambda (f)(\omega) = \hat{f}(\sqrt{\lambda} \omega) \), where \( \hat{f} \) is the Fourier transform of \( f \) and \( \omega \in S^{n-1} \) (the unit sphere). Thus it follows from the celebrated Stein-Tomas theorem, see [52] that the spectral projection measure \( dE_{-\Delta}(\lambda) \) is bounded as an operator acting from \( L^p(\mathbb{R}^n) \) to \( L^q(\mathbb{R}^n) \) for any \( 1 \leq p \leq 2(n+1)/(n+3) \). Let us also mention that in [33] spectral estimate (1.4) was obtained in the setting of Laplace type operator acting on asymptotically conic manifolds.

As we said in Abstract this paper comprises two parts. In first we will study the \( L^p \to L^q \) mapping properties of spectral multipliers and Bochner-Riesz means with negative index in the general setting of abstract self-adjoint operators. In the case of standard Laplace operator and Fourier transform such negative index means were studied in [3, 7, 13, 34]. In our discussion we consider condition that Bochner-Riesz mean \( S_R^q(\sqrt{L}) \) satisfies the \((p,q)\)-estimate, if there exists a constant \( C > 0 \) such that for all \( R > 0 \)

\[
(\text{BR}_{p,q,m}^q) \quad \|S_R^q(\sqrt{L}) P_{B(x, \rho)}\|_{p \to q} \leq CV(x, \rho)^{\frac{1}{q} - \frac{1}{p}} (R\rho)^{n(\frac{1}{p} - \frac{1}{q})}
\]
for all \(x \in X\) and all \(\rho \geq 1/R\), see Section 3 below. In this context, we will show that, on an abstract level, Bochner-Riesz means with negative index can be used to study spectral multipliers. Roughly speaking, under the assumption that \(V(x, \rho) \geq C\rho^n\) for all \(x \in X\) and \(\rho > 0\), if \(L\) satisfies Davies-Gaffney estimates \((DG_m)\) and \((G_{p_0,2,m})\) for some \(1 \leq p_0 < 2\), and \((BR^{\alpha}_{p,q,m})\) for \(\alpha \geq -1\) and \(p_0 < p < q < p'_0\), then for any \(F \in W^{\beta,1}(\mathbb{R})\) such that \(\text{supp} F \subseteq [1/4, 4]\), the operator \(F(r \sqrt[n]{L})\) is bounded from \(L'(X)\) to \(L'(X)\) for all \(p \leq r \leq s \leq q\) and \(\beta > n(1/p - 1/r) + n(1/s - 1/q) + \alpha + 1\).

As an application we establish the \(L^p \to L^q\) mapping properties of Bochner-Riesz means of the operator \(L\) with negative indexes. To be able to provide more detail description of our results we introduce additional notation, which partially coincides with one considered in [4, 7]. Given some \(1 \leq p < 2\), we set

\[
A = \left(1, \frac{n + 1 + 2\alpha}{2n}\right), \quad A' = \left(\frac{n - 1 - 2\alpha}{2n}, 0\right),
\]

\[
B(p) = \left(\frac{n + 1 + 2\alpha}{2n} + \alpha - \frac{2\alpha}{p}, \frac{n + 1 + 2\alpha}{2n}\right), \quad B'(p) = \left(\frac{n - 1 - 2\alpha}{2n}, \frac{2\alpha}{p} - \alpha + \frac{n - 1 - 2\alpha}{2n}\right),
\]

\[
C(p) = \left(\frac{1}{p}, \frac{n + 1 + 2\alpha}{2n}\right), \quad C'(p) = \left(\frac{n - 1 - 2\alpha}{2n}, 1 - \frac{1}{p}\right),
\]

\[
D(p) = \left(\frac{1}{2} + \alpha - \frac{2\alpha}{p}, \frac{1}{2}\right), \quad D'(p) = \left(\frac{1}{2}, \frac{1}{2} - \alpha + \frac{2\alpha}{p}\right).
\]

Denote by \(\Delta_{\alpha}(p,n)\) the open pentagon with vertices \(A, B(p), B'(p), A', (1,0)\). In Section 3 we will show that if we assume that \(C^{-1}L^{\alpha} \leq V(x, r) \leq C L^{\alpha}\) for all \(x \in X\) and \(r > 0\), and that \(L\) satisfies estimates \((DG_m), (G_{p_0,2,m})\) for some \(1 \leq p_0 < 2\) then for any \(p_0 < p < 2\),

\[
(BR^{-1}_{p,p',m}) \Rightarrow (BR^{\alpha}_{p,s,m})
\]

if each of the following conditions holds:

1. \(\alpha > n(1/p - 1/2) - 1/2, p_0 < r \leq s < p'_0, r < q_a\) and \(q'_a < s\) where \(q_a = \max\{1, \frac{2\alpha}{n + 1 + 2\alpha}\}\).

2. \(n(1/p - 1/2) - 1/2 \geq \alpha > 0, p_0 < r \leq s < p'_0, (1/r, 1/s) \in \Delta_{\alpha}(p,n)\) and \((1/r, 1/s)\) is strictly below the lines joining the point \((1/2, 1/2)\) to \((C(p))\) and \((C'(p))\).

3. \(-1/2 < \alpha \leq 0, p_0 < r \leq s < p'_0, (1/r, 1/s) \in \Delta_{\alpha}(p,n)\) and \((1/r, 1/s)\) is strictly below the lines joining \(D(p)\) to \((C(p))\); \(D(p)\) to \((D'(p))\) and \((D'(p))\) to \((C'(p))\).

4. \(-1 < \alpha \leq -1/2, p_0 < r \leq s < p'_0, \alpha - \frac{2\alpha}{p} < \frac{1}{r} - \frac{1}{s}, r < q'_a\) and \(q_a < s\), where \(1/q_a = 1 + \alpha - (2\alpha + 1)/p\).

In our paper we do not investigate the endpoint type results. The perspective developed in [14] suggests that such endpoint estimates can only be obtained in the second order case \(m = 2\).

Next consider \(D = -i(\partial_1, \ldots, \partial_n)\) and operator \(P(D)\) where \(P\) is a real elliptic polynomial of order \(m \geq 2\). The second part of this paper is devoted to restriction type estimates and Bochner-Riesz means of negative order of differential operators \(P(D) + V\), where \(V(x)\) are nonnegative potentials, see Sections 4-6 below. In the sequel, we write \(H_0 = P(D)\) and \(H = P(D) + V\). If \(0 \leq V \in L^1_{\text{log}}(\mathbb{R}^n)\), then it is well known that \(H_0\) and \(H\) can be defined as nonnegative self-adjoint operators on \(L^2(\mathbb{R}^n)\). Our approach to investigation of spectral multiplier operators \(F(H)\) is to obtain the restriction type
estimates (1.4) for $H$. We are able to do this under the standard non-degenerate condition of the homogeneous elliptic polynomial $P(\xi)$ on $\mathbb{R}^n$:

\begin{equation}
\det \left( \frac{\partial^2 P(\xi)}{\partial \xi_i \partial \xi_j} \right)_{i,j=1} \neq 0, \quad \xi \neq 0.
\end{equation}

The above condition is equivalent to the fact that the compact smooth hypersurface $\Sigma = \{ \xi \in \mathbb{R}^n; \ |P(\xi)| = 1 \}$ has nonzero Gaussian curvature everywhere. In terms of the Fourier transform we can express the spectral decomposition of $H_0$ by the following formula

$$dE_{H_0}(\lambda)f = (\delta(P - \lambda)^\sim)^{-1} \int_{\Sigma} e^{i\xi \hat{f}(\xi)} \frac{d\sigma_\lambda(\xi)}{\nabla P}.$$ 

Hence based on the non-degenerate assumption (1.5), for any $1 \leq p \leq 2(n+1)/(n+3)$, the spectral measure estimates (1.4) for $H_0$ follow from restriction theorem on the general surface $\Sigma$, see e.g. [30] and Stein [50, P. 364].

For a non-trivial potential $V$ one can not use the Fourier transform to obtain description of spectral resolution of the operator $H = H_0 + V$. Therefore we have to develop another perspective to analyse the spectral properties of $H$, which is based on perturbation techniques and some of ideas developed in Section 3. In our approach we use Stone’s formula:

\begin{equation}
dE_H(\lambda)f = (2\pi i)^{-1} \left( R_H(\lambda + i0) - R_H(\lambda - i0) \right) f, \quad \lambda > 0,
\end{equation}

where $R_H(\lambda \pm i0)$ are defined as boundaries of the resolvent $(H - z)^{-1}$ of $H$ with $z \in \mathbb{C}/[0, \infty)$. To obtain the required bound of $R_H(\lambda \pm i0)$, we establish the following uniform Sobolev type estimate for the operator $P(D)$

\begin{equation}
\left\| u \right\|_{L^p(\mathbb{R}^n)} \leq C \left| z \right|^{\frac{n}{2} - \frac{1}{2} - \frac{1}{q} - 1} \left\| \left( P(D) - z \right) u \right\|_{L^p(\mathbb{R}^n)}, \quad z \neq 0,
\end{equation}

where $n > m \geq 2$ and the pairs $(p, q)$ satisfy the following conditions:

\begin{equation}
\min \left( \frac{1}{p} - 1, \frac{1}{2} - \frac{1}{q} \right) > \frac{1}{2n}, \quad \frac{2}{n+1} < \left( \frac{1}{p} - \frac{1}{q} \right) \leq \frac{m}{n},
\end{equation}

see Corollary 4.2 below. Note that on the Sobolev embedding line $1/p - 1/q = m/n$ estimate (1.7) does not contain the term which depends on $|z|$, which means that it is uniform for all $z \in \mathbb{C}$ as its name suggests. The proof of (1.7) is based on analysis of oscillatory integral operator related to restriction theorem, see e.g. [48], which essentially relies on the non-degenerate curvature condition on the hypersurface $\Sigma$ above.

In the case $P(D) = -\Delta$, estimate (1.7) and their more general non-elliptic variants were obtained by Kenig, Ruiz and Sogge and motivated by certain unique continuation theorems for the operators $P(D)$, see [39]. Here using (1.7) and the following perturbed resolvent identity

\begin{equation}
R_H(\lambda \pm i0) = R_{H_0}(\lambda \pm i0) \left( I + VR_{H_0}(\lambda \pm i0) \right)^{-1}, \quad \lambda > 0,
\end{equation}

we will verify $L^p$-version of the limiting absorption principle (1.6) for $H$. Some versions of (1.9) and the limiting absorption principle were used by Agmon in his celebrated scattering work [1] on different weighted subspaces of $L^2(\mathbb{R}^n)$. 


In Theorem 5.8 below, based on the limiting absorption principle and uniform Sobolev estimate we prove that there exists a constant $c_0 > 0$ such that if
\begin{equation}
\|V\|_\mathbb{R}^+ + \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(x)|}{|x-y|^{n-m}} dx \leq c_0,
\end{equation}
then the spectral measure estimates (1.4) hold for $H = P(D) + V$ for all $1 \leq p < \min\left(\frac{2(n+1)}{n+3}, \frac{n}{m}\right)$. Note that when $m = 2$ and $n \geq 3$ then the range of $p$ is the same as for the standard Laplace operator, see Remark 5.9 below. Note also that if $V \in L^{\frac{n}{n-2}} \cap L^{\frac{n}{n+2}}$ then the expression described in (1.10) is finite. This provides a large class of rough potentials $V$ to which our result can be applied. It is an interesting question whether it is enough to assume that the expression defined by (1.10) is finite instead of being small enough. It is plausible to expect that further sophistication of our approach can lead to result of this type but we are not going to study this issue here.

In our approach we need to assume that the semigroup $e^{-tH}$ generated by $H = P(D) + V$, satisfies estimates $(DG_m)$ or $(GE_m)$. Now, since $V(x)$ is nonnegative, it is comparably easy to show the Davies-Gaffney estimates $(DG_m)$ see Lemma 6.1 below. In addition if $m > n$ or $m = 2$, then it is well-known that the Gaussian estimates $(GE_m)$ for $e^{-tH}$ always hold for all $0 \leq V \in L^1_{\text{loc}}$, see e.g [5]. On the other hand, if $4 \leq m \leq n$, then generally, the Gaussian bound of $e^{-tH}$ may fail to hold. We describe some results of this type in Section 6 but do not discuss here all relevant details, instead we refer the reader to [21, 24].

The layout of the paper is as follows. In Section 2 we recall some basic properties of heat kernels and state some known spectral multiplier results. In Section 3 we will show that, at an abstract level, Bochner-Riesz means with negative index implies spectral multiplier estimates corresponding to functions supported in dyadic intervals, which can be used to study $L^p \rightarrow L^q$ mapping properties of Bochner-Riesz means with negative index. In Section 4 we prove the uniform Sobolev estimate (1.7) for constant coefficient higher order elliptic operators on $\mathbb{R}^n$. We then use the standard perturbation technique to obtain estimates of spectral projectors for elliptic operators $P(D) + V$ with certain potentials $V$ on $\mathbb{R}^n$ in Section 5. From this, we can deduce spectral multiplier estimates of these elliptic operators, including Bochner-Riesz summability results in Section 6.

Throughout, the symbols “$c$” and “$C$” will denote (possibly different) constants that are independent of the essential variables.

## 2 Preliminaries

In this section we discuss some basic properties of Gaussian, Davies-Gaffney and Stein-Tomas type estimates. For $1 \leq p \leq +\infty$, we denote the norm of a function $f \in L^p(X, d\mu)$ by $\|f\|_p$, by $\langle ., . \rangle$ the scalar product of $L^2(X, d\mu)$, and if $T$ is a bounded linear operator from $L^p(X, d\mu)$ to $L^q(X, d\mu)$, $1 \leq p, q \leq +\infty$, we write $\|T\|_{p \rightarrow q}$ for the operator norm of $T$. For a given function $F : \mathbb{R} \rightarrow \mathbb{C}$ and $R > 0$, we define the function $\delta_R F : \mathbb{R} \rightarrow \mathbb{C}$ by putting $\delta_R F(x) = F(Rx)$. Given $p \in [1, \infty]$, the conjugate exponent $p'$ is defined by $1/p + 1/p' = 1$.

For a function $W : M \rightarrow \mathbb{R}$, let $M_W$ the operator of multiplication by $W$, that is
\[ (M_W f)(x) = W(x) f(x). \]

In the sequel, we shall identify the function $W$ and the operator $M_W$. That is, if $T$ is a linear operator, we shall denote by $W_1 T W_2$ the operator $M_{W_1} T M_{W_2}$. We also set $V^\alpha(x) = V(x, t)^\alpha$. 


2.1 Gaussian estimates and Davies-Gaffney estimates

**Proposition 2.1.** Let $m \geq 2$ and $1 \leq p < 2$. Let $L$ be a non-negative self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates $(DG_m)$ and condition $(G_{p,2,2})$. Then for all $p < r \leq q < p'$ and for all $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1/r - 1/q$

\begin{equation}
\|V_t^\alpha e^{-r^mL}V_t^\beta\|_{r \to q} \leq C,
\end{equation}

and

\begin{equation}
\|V_t^\alpha (I + t^mL)^{N/m}V_t^\beta\|_{r \to q} \leq C
\end{equation}

for every $N > n(1/r - 1/q)$.

**Proof.** From condition $(G_{p,2,2})$

\begin{equation}
\|P_{B(x,t)}e^{-r^mL}P_{B(y,t)}\|_{p \to p'} \leq CV(x,t)^{1/p}V(y,t)^{1/p'}.
\end{equation}

Let $\tilde{p} \in (p, 2)$. Note that it follows from the doubling condition $(D_n)$ that

\[ V(y,\rho) \leq C \left(1 + \frac{d(x,y)}{\rho}\right)^{n} V(x,\rho) \quad \forall \rho > 0, \, x, y \in X \]

From the above estimate, (2.3) and Davies-Gaffney estimates $(DG_m)$, the Riesz-Thorin interpolation theorem give the following

\begin{equation}
\|P_{B(x,t)}e^{-r^mL}P_{B(y,t)}\|_{n \to s} \leq CV(x,t)^{1/p}V(y,t)^{1/p'} \exp \left( -c't\left(\frac{d(x,y)}{t}\right)^{n}\right).
\end{equation}

By (ii) of [11, Proposition 2.1], we obtain that for $\tilde{p} \leq r \leq q \leq \tilde{p}'$ and for all $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1/r - 1/q$

\begin{equation}
\|V_t^\alpha e^{-r^mL}V_t^\beta\|_{r \to q} \leq C,
\end{equation}

which proves (2.1).

Next, for $t > 0$,

\[(I + t^mL)^{-N/m} = C_N \int_0^\infty e^{-s} s^{N/m-1} e^{-s^mL} ds\]

for some $C_N$. It then follows that for every $p < r \leq q < p'$,

\begin{equation}
\|V_t^\alpha (I + t^mL)^{-N/m}V_t^\beta\|_{r \to q} \leq C_N \int_0^\infty e^{-s} s^{N/m-1} \|V_t^\alpha e^{-s^mL}V_t^\beta\|_{r \to q} ds.
\end{equation}

Observe that for every $z \in X$, if $s < 1$, then

\[ V(z, t) \leq C s^{-n/m} V(z, s^{1/m}t) \]

and if $s > 1$, then $V(z, t) \leq CV(z, s^{1/m}t)$. Estimate (2.6) yields (2.2) for $N > n(1/r - 1/q)$. This ends the proof.
2.2 Stein-Tomas restriction type condition

Let us recall the restriction type estimates \((ST^q_{p,2,m})\), which were originally introduced in \([27]\) for \(p = 1\), and then in \([14]\) for general \(1 < p < 2\). Consider a non-negative self-adjoint operator \(L\) and exponents \(p\) and \(q\) such that \(1 \leq p < 2\) and \(1 \leq q \leq \infty\). Following \([46]\), we say that \(L\) satisfies the Stein-Tomas restriction type condition if for any \(R > 0\) and all Borel functions \(F\) such that \(\text{supp} \ F \subset [0, R]\),

\[
(ST^q_{p,2,m}) \quad \left\| F(\sqrt[n]{L})P_{B(x,\rho)} \right\|_{p \to 2} \leq CV(x, \rho)^{\frac{1}{2} - \frac{1}{q}}(R\rho)^{\left(\frac{q}{2} - \frac{1}{q}\right)} \left\| \delta_R F \right\|_q
\]

for all \(x \in X\) and all \(\rho \geq 1/R\).

As we mentioned in Introduction this condition is motivated by analysis of the standard Laplace operator \(\Delta = -\sum_{i=1}^n \delta_{\eta_i}^2\) on \(\mathbb{R}^n\). It is not difficult to observe, see \([14, \text{Proposition 2.4}]\), that for \(q = 2\) the condition \((ST^2_{p,2,2})\) is equivalent to the \((p, 2)\) Stein-Tomas restriction estimate

\[
\left\| dE_{\sqrt[n]{\lambda}}(\lambda) \right\|_{p \to p'} \leq \mathcal{C} \lambda^{-\left(1/p - 1/p'\right) - 1}
\]

for all \(1 \leq p \leq \frac{2(n+1)}{n+3}\).

Note that if condition \((ST^q_{p,2,m})\) holds for some \(q \in [1, \infty)\), then \((ST^q_{p,2,m})\) is automatically valid for all \(q \geq \tilde{q}\) including the case \(\tilde{q} = \infty\). It is known that if \(q = \infty\), then the condition \((ST^\infty_{p,2,m})\) follows from the standard elliptic estimates, that is, to be more precise the conditions \((ST^\infty_{p,2,m})\) and \((G_{p,2,m})\) are equivalent, see for instance \([46, \text{Proposition 2.2}]\).

We start with stating very general spectral multiplier result. Point \((i)\) of the following proposition can be easily applied in a wide range of situations but usually does not give the sharp result and the differentiability assumption can often be relaxed. However, this general statement helps to avoid nonessential technicalities while discussing sharp spectral multiplier results. Point \((ii)\) usually leads to optimal results but verifying condition \((ST^q_{p,2,m})\) is quite difficult. For the proof, we refer the reader to \([9]\) (for point \((i)\)) and \([46, \text{Theorem 5.1}]\) (for both parts). Recall that \(n\) is the doubling dimension from condition \((D_n)\) and \(\eta \in C^{\omega}(0, \infty)\) is a non-zero auxiliary function and \(C_k\) is a space of \(k\) times continuously differentiable functions on the real line.

**Proposition 2.2.** Let \(L\) be a non-negative self-adjoint operator on \(L^2(X)\) satisfying Davies-Gaffney estimates \((DG_m)\). Then

\((i)\) Assume that the condition \((G_{p,2,m})\) holds for some \(p\) satisfying \(1 \leq p < 2\). Then for any bounded Borel function \(F\) such that

\[
\sup_{\rho > 0} \left\| \eta \delta_r F \right\|_{C^k} < \infty
\]

for some integer \(k > n(1/p - 1/2)\), the operator \(F(L)\) is bounded on \(L'(X)\) for all \(p < r < p'\).

\((ii)\) Assume that the condition \((ST^q_{p,2,m})\) holds for some \(p, q\) satisfying \(1 \leq p < 2\) and \(1 \leq q \leq \infty\). Then for any bounded Borel function \(F\) such that \(\sup_{\rho > 0} \left\| \eta \delta_r F \right\|_{W^\alpha_q} < \infty\) for some \(\alpha > \max\{n(1/p - 1/2), 1/q\}\), the operator \(F(L)\) is bounded on \(L'(X)\) for all \(p < r < p'\).

A significant example of spectral multipliers are Bochner-Riesz means. Let us recall that Bochner-Riesz operators of index \(\alpha\) for a non-negative self-adjoint operator \(L\) are defined by the formula

\[
S^\alpha_R(L) = \frac{1}{\Gamma(\alpha + 1)} \left( \frac{I - L}{R} \right)^\alpha, \quad R > 0.
\]
Bochner-Riesz analysis studies the range of $\alpha$ for which the operators $S^\alpha_R(L)$ are uniformly bounded on $L^p$. Applying spectral multiplier theorems to study boundedness of Bochner-Riesz means is often an efficient test to check if the considered result is sharp or not.

**Corollary 2.3.** Suppose that the operator $L$ satisfies Davies-Gaffney estimate ($\text{DG}_m$) and condition ($ST^q_{p,2,m}$) with some $1 \leq p < 2$ and $1 \leq q \leq \infty$. Then for all $p < r < p'$ and $\alpha > n(1/p - 1/2) - 1/q$, (3.2)

$$
\sup_{R > 0} \left\| S^\alpha_R(L) \right\|_{r \to r} \leq C.
$$

**Proof.** For the proof, we refer the reader to [46, Corollary 4.4].

## 3 Spectral multipliers and Bochner-Riesz means

Assume that $(X,d,\mu)$ satisfies the doubling condition, that is (D). Suppose $L$ is a nonnegative self-adjoint operator acting on the space $L^2(X)$. Such an operator admits a spectral resolution $E_L(\lambda)$. If $F$ is a real-valued Borel function $F$ on $[0, \infty)$, then one can define the operator $F(L)$ by the formula (3.1)

$$
F(L) = \int_0^\infty F(\lambda) \, dE_L(\lambda).
$$

By spectral theory if the function $F$ is bounded, then the operator $F(L)$ is bounded as an operator acting on $L^2(X)$. Many authors study necessary conditions on function $F$ to ensure that $F(L)$ is bounded as operator action on $L^p$ spaces for some range of $p$. However we are interested here in estimates of $L^p \to L^q$ norm of $F(L)$ for some $1 \leq p < q \leq \infty$ especial in situation when $F$ is potentially unbounded.

Observe that for $\text{Im} \lambda \neq 0$, the resolvent family $(L - \lambda)^{-1}$ is a holomorphic family of bounded operators on $L^2(X)$. Throughout this article, we assume that:

*The resolvent family of the operator $L$ extends continuously to the real axis as a bounded operator in a weaker sense, e.g., between weighted $L^2$-spaces.*

It is then differentiable in $\lambda$ up to the real axis. This property is satisfied by many operators, e.g., constant coefficients higher order elliptic operators $P(D)$ described in Section 4 below. Under this assumption, we find that $E_L(\lambda)$ is differentiable in $\lambda$ and Stone’s formula for operator $L$ is valid

$$
\frac{d}{d\lambda} E_L(\lambda) = \frac{1}{2\pi i} \left( (L - (\lambda + i0))^{-1} - (L - (\lambda - i0))^{-1} \right).
$$

In this case we write (abusing notation somewhat) $dE_L(\lambda)$ for the derivative of $E_L(\lambda)$ with respect to $\lambda$. Stone’s formula gives a mechanism for analysing the spectral measure, namely we need to analyse the limit of the resolvent $(L - \lambda)^{-1}$ on the real axis, see Sections 4-6 below.

### 3.1 Bochner-Riesz means with negative index

In the same way in which we defined the Bochner-Riesz means of the operator $L$ one can also define Bochner-Riesz means of its root of order $m$, that is $\sqrt[m]{L}$. Similarly as before, for every $R > 0$ the Bochner-Riesz means of index $\alpha$ for the operator $\sqrt[m]{L}$ are defined by the formula (3.2)

$$
S^\alpha_R(\sqrt[m]{L}) = \frac{1}{\Gamma(\alpha + 1)} \left( I - \frac{\sqrt[m]{L}}{R} \right)^\alpha_+, \quad \alpha > -1.
$$
When $\alpha = -1$, we set $S_R^{-1}(\sqrt{L}) = R^{-1}dE_{\sqrt{L}}(R)$. Given some $\alpha \geq -1$ and $1 \leq p < q \leq \infty$, we say that the Bochner-Riesz mean $S_R^\alpha(\sqrt{L})$ satisfies the $(p,q)$-estimate, if there exists a constant $C > 0$ such that for all $R > 0$,

$$(\text{BR}^\alpha_{p,q,m}) \quad \|S_R^\alpha(\sqrt{L})P_{B(x,\rho)}\|_{p \to q} \leq CV(x,\rho)^\frac{1}{\frac{1}{p} - \frac{1}{q}} (R\rho)^m (\frac{1}{p} - \frac{1}{q})$$

for all $x \in X$ and all $\rho \geq 1/R$.

In our first statement of this section we note that considering the Bochner-Riesz means of the operators $L$ and $\sqrt{L}$ are essentially equivalent under some assumptions of the operator $L$.

**Lemma 3.1.** Suppose that $(X,d,\mu)$ satisfies the doubling condition (D) and that the semigroup corresponding to a non-negative self-adjoint operator $L$ satisfies estimates $(\text{DG}_m)$ and $(\text{G}_{p_0,2,m})$ for some $1 \leq p_0 < 2$. Then for every $\alpha \geq -1$ and $1 \leq p_0 < p < q < p_0'$, $(\text{BR}^\alpha_{p,q,m})$ is equivalent to

$$(3.3) \quad \|S_R^\alpha(L)P_{B(x,\rho)}\|_{p \to q} \leq CV(x,\rho)^\frac{1}{\frac{1}{p} - \frac{1}{q}} (R\rho)^m (\frac{1}{p} - \frac{1}{q})$$

for all $x \in X$ and all $\rho \geq 1/R$.

**Proof.** Let $\varphi$ be a non-zero $C_0^\infty$ function on $\mathbb{R}$ such that $\varphi(s) = 1$ if $s \in [-1/2, 3/2]$ and $\varphi(s) = 0$ if $|s| \geq 2$. For $s > 0$, we write

$$(3.4) \quad (1 - s)_+^\alpha = (1 - s^{1/m})_+^\alpha (1 + \sum_{k=1}^{m-1} s^{k/m})^\alpha \varphi(s)$$

and

$$(3.5) \quad (1 - s^{1/m})_+^\alpha = (1 - s)_+^\alpha (1 + \sum_{k=1}^{m-1} s^{k/m})^{-\alpha} \varphi(s).$$

We apply Proposition 2.2 to obtain that for $p_0 < r < p_0'$, there exists a constant $C > 0$ independent of $R$ such that

$$(3.6) \quad \left\| \left( 1 + \sum_{k=1}^{m-1} \left( \frac{\sqrt{L}}{R} \right)^k \right)^{\frac{\alpha}{m}} \varphi \left( \frac{L}{R^m} \right) \right\|_{r \to r} \leq C.$$ 

This, together with (3.4) and (3.5), proves Lemma 3.1. \qed

It is easy to note that for the standard Bochner-Riesz means corresponding to the Fourier transform and the standard Laplace operator, condition $(\text{BR}^\alpha_{p,q,m})$ implies the same estimates for all exponents $r, s$ such that $1 < r \leq p < q \leq s < \infty$. In our next statement we show that this is a quite general situation limited only by a range of $L^p$ spaces on which the semigroup generated by $L$ acts and enjoys generalised Gaussian estimates.

**Lemma 3.2.** Suppose that there exists a constant $C > 0$ such that $C^{-1} \rho^n \leq V(x,\rho) \leq C \rho^n$ for all $x \in X$ and $\rho > 0$. Next assume that $L$ is a non-negative self-adjoint operator acting on $L^2(X)$ satisfying estimates $(\text{DG}_m)$ and $(\text{G}_{p_0,2,m})$ for some $1 \leq p_0 < 2$ and that $\alpha \geq -1$. Then $(\text{BR}^\alpha_{p,q,m})$ with $p_0 < p < q < p_0'$ implies $(\text{BR}^\alpha_{r,s,m})$ for all $p_0 < r \leq p < q \leq s < p_0'$.

In particular, if the operator $L$ satisfies the Gaussian estimate $(\text{GE}_m)$, then $(\text{BR}^\alpha_{p,q,m})$ implies $(\text{BR}^\alpha_{r,s,m})$ for all $1 \leq r \leq p < q \leq s \leq \infty$. 

11
Proof. We first show that $(BR_{p,q,m}^α)$ implies $(BR_{p,s,m}^α)$ for $1 \leq p_0 \leq p < q < s < p_0'$. We choose a function $φ \in C_0^∞(-2,2)$ such that $φ(s) = 1$ if $s < 1$; $0$ if $s > 2$. Let $N > n(1/q - 1/s)$. Note that $V(x, ρ) \geq C^{-1}ρ^n$ for all $x \in X$ and $ρ > 0$. For $p_0 < q < s < p_0'$, it follows by Proposition 2.1 that

$$\left\Vert \left(1 + \frac{L}{λ^m}\right)^{-N/m}\right\Vert_{q→s} \leq Cλ^{(\frac{1}{q} - \frac{1}{s})} \left\Vert V(x, λ^{-1})^{\frac{1}{q} - \frac{1}{s}} \left(1 + \frac{L}{λ^m}\right)^{-N/m}\right\Vert_{q→s} \leq Cλ^{(\frac{1}{q} - \frac{1}{s})}. \quad (3.7)$$

Hence by Proposition 2.2

$$\left\Vert φ\left(\frac{\sqrt{L}}{λ}\right)\right\Vert_{q→s} = \left\Vert φ\left(\frac{\sqrt{L}}{λ}\right)\left(1 + \frac{L}{λ^m}\right)^{N/m}\right\Vert_{q→q} \left\Vert \left(1 + \frac{L}{λ^m}\right)^{-N/m}\right\Vert_{q→s} \leq Cλ^{(\frac{1}{q} - \frac{1}{s})}. \quad (3.8)$$

Note that $(1 - s/λ)_+^α = φ(s/λ)(1 - s/λ)_+^α$ and $V(x, ρ) \leq Cρ^n$ for all $x \in X$ and $ρ > 0$. It follows that

$$\|S_{α}(\sqrt{L})P_{B(x,ρ)}\|_{p→s} = \left\| φ\left(\frac{\sqrt{L}}{λ}\right)S_{α}(\sqrt{L})P_{B(x,ρ)}\right\|_{p→s} \leq \left\| φ\left(\frac{\sqrt{L}}{λ}\right)\right\|_{q→q} \|S_{α}(\sqrt{L})P_{B(x,ρ)}\|_{p→q} \leq Cλ^{(\frac{1}{q} - \frac{1}{s})}V(x, ρ)^{\frac{1}{q} - \frac{1}{p}}(λρ)^{n(\frac{1}{q} - \frac{1}{s})} \leq CV(x, ρ)^{\frac{1}{q} - \frac{1}{p}}(λρ)^{n(\frac{1}{q} - \frac{1}{s})} \quad (3.9)$$

since $V(x, ρ) \leq Cρ^n$ for all $x \in X$ and $ρ > 0$. Hence, $(BR_{p,q,m}^α) \Rightarrow (BR_{p,s,m}^α)$ for $1 \leq p_0 \leq p < q < s < p_0'$. A similar argument as above shows that $(BR_{p,q,m}^α)$ implies $(BR_{r,q,m}^α)$ for $1 \leq p_0 < r \leq p < q < p_0'$.

As we notice before, condition $(G_{1,2,m})$ follows from $(GE_m)$, so the second part of the Lemma 3.2 follows from the first part. This ends the proof of Lemma 3.2. \[\Box\]

Our next result is a version of Lemma 3.2 corresponding to the case $α = -1$. In this situation the proof simplifies and we can omit the Gaussian bounds assumptions from the statement.

**Lemma 3.3.** Let a nonnegative self-adjoint operator $H$ satisfy the $(p_0, p_0')$-restriction estimate for some $1 < p_0 < 2$ such that

$$\|dE_H(λ)\|_{p_0→p_0'} \leq Cλ^{-\frac{1}{p} + \frac{1}{p_0} - \frac{1}{p_0'}}, \quad (3.10)$$

In addition, if there exists some $k > 0$ such that

$$\|(1 + tH)^{-1}\|_{p→p_0} \leq C_k t^{-\frac{1}{p} + \frac{1}{p_0}}, \quad t > 0, \quad (3.11)$$

for some $1 \leq p \leq p_0 < 2$. Then the estimate

$$\|dE_H(λ)\|_{q→q'} \leq Cλ^{-\frac{1}{p} + \frac{1}{q'}}, \quad (3.12)$$

holds for all $p \leq q \leq p_0$.  

12
Proof. By interpolation, it suffices to prove the endpoint case $q = p$. In fact, we observe that

$$(1 + H/\lambda)^{-2k}dE_H(\lambda) = 2^{-2k}dE_H(\lambda).$$

Then by duality it follows that

$$\parallel 2^{-2k}dE_H(\lambda) \parallel_{p \to p'} = \parallel (1 + H/\lambda)^{-k}dE_H(\lambda)(1 + H/\lambda)^{-k} \parallel_{p \to p'} \leq \parallel (1 + H/\lambda)^{-k}dE_H(\lambda) \parallel_{p_0 \to p_0'}(1 + H/\lambda)^{-k} \parallel_{p_0' \to p'} \leq C\lambda^{\frac{1}{2}(\frac{1}{p} - \frac{1}{p_0})}.$$  

This ends the proof. □

Remark 3.4. The resolvent power $(1 + tH)^{-k}$ in condition (3.11) can be replaced by the semigroup $e^{-tH}$, which actually are equivalent by some standard arguments.

Next we describe a useful notation of one-dimensional homogeneous distributions $\chi_+^\alpha$ and $\chi_-^\alpha$ coming from [36] and defined by

$$(3.13) \quad \chi_\pm^\alpha = \frac{x_\pm^\alpha}{\Gamma(\alpha + 1)}, \quad \text{Re} \alpha > -1,$$

where $\Gamma$ is the Gamma function and

$$x_+^\alpha = x^\alpha \quad \text{if} \quad x \geq 0 \quad \text{and} \quad x_-^\alpha = 0 \quad \text{if} \quad x < 0;$$

$$x_-^\alpha = |x|^\alpha \quad \text{if} \quad x \leq 0 \quad \text{and} \quad x_-^\alpha = 0 \quad \text{if} \quad x > 0.$$  

It easy to note that $\chi_\pm^\alpha$ are well defined distributions for Re $\alpha > -1$. From a straightforward observation

$$\frac{d}{dx} \chi_\pm^\alpha = \pm \alpha x_\pm^{\alpha - 1},$$

it follows that

$$\frac{d}{dx} \chi_\pm^\alpha = \pm \alpha x_\pm^{\alpha - 1}$$

for all Re $\alpha > 0$. One can use the above relation to extend the family of functions $\chi_\pm^\alpha$ to a family of distributions on $\mathbb{R}$ defined for all $\alpha \in \mathbb{C}$, see [36, Ch III, Section 3.2] for details. Since $1 - \chi_-^0(x)$ is the Heaviside function, it follows that

$$\chi_-^{-k} = (\pm 1)^k \delta_0^{(k-1)}, \quad k = 1, 2, \ldots,$$

where $\delta_0$ is the $\delta$-Dirac measure.

A straightforward computation shows that for all $w, z \in \mathbb{C}$

$$(3.14) \quad \chi_-^w * \chi_-^z = \chi_-^{w+z+1},$$

where $\chi_-^w * \chi_-^z$ is the convolution of the distributions $\chi_-^w$ and $\chi_-^z$, see [36, (3.4.10)]. If supp $F \subset [0, \infty)$, we then define the Weyl fractional derivative of $F$ of order $\nu$ by the formula

$$(3.15) \quad F^{(\nu)} = F * \chi_-^{-\nu-1}, \quad \nu \in \mathbb{C}$$

and we note that for every $\nu \in \mathbb{C}$,

$$F^{(\nu)} * \chi_-^{-1} = F * \chi_-^{-\nu-1} * \chi_-^{-1} = F,$$
Proposition 3.6. in terms of estimate (BR and for fixed \( \xi \). Here \( \xi \) finishes the proof. □

Fourier multiplier and is bounded on all \( L^p(\mathbb{R}) \) spaces. Namely, if \( \text{supp} \, F \subset [0, \infty) \), then for any \( 1 \leq p \leq \infty \) and \( \nu \in \mathbb{R} \) we define the Weyl-Sobolev norm of \( F \) by the formula

\[
\|F\|_{WS^{\nu,p}} = \|F\|_p + \|F^{(\nu)}\|_p.
\]

Remark 3.5. Note that for \( 1 < p < \infty \) the Weyl-Sobolev norm is equivalent to the standard Sobolev norm, that is

\[
c\|F\|_{W^{\nu,p}} \leq \|F\|_{WS^{\nu,p}} \leq C\|F\|_{W^{\nu,p}}
\]

whereas for \( p = 1 \)

\[
\|F\|_{WS^{\nu,1}} \leq C_{\varepsilon}\|F\|_{W^{\nu,1,\varepsilon}}
\]

for any \( \varepsilon > 0 \).

Proof. Note that \( (1 - d^2/dx^2)^{-(\alpha+1/2)}I_\alpha \) is an example of classical (one dimensional) Hörmander type Fourier multiplier and is bounded on all \( L^p(\mathbb{R}) \) spaces for \( 1 < p < \infty \). Next set

\[
I_\alpha f = \chi^{-\alpha-2}_- f
\]

and for fixed \( \varepsilon > 0 \) consider the operator \( (1 - d^2/dx^2)^{-(\alpha+1/2)}I_\alpha \). An argument as in [36, Example 7.1.17, p. 167 and (3.2.9) p.72] shows that \( (1 - d^2/dx^2)^{-(\alpha+1/2)}I_\alpha f = f \ast \eta \) where \( \eta \) is the locally integrable function

\[
\hat{\eta}(\xi) = \frac{-ie^{i\alpha/2}\xi_-(\alpha+1) + ie^{-i\alpha/2}\xi_-(\alpha+1)}{(1 + \xi^2)^{(\alpha+1/2)/2}}.
\]

Here \( \xi_- = \max(0, \xi) \) and \( \xi_- = -\max(0, \xi) \). A standard argument shows that \( \eta \in L^1(\mathbb{R}) \). Hence

\[
\|I_{12} \| \leq C\|\delta R F \ast \chi^{-\alpha-2}_-\|_1 \leq C\|(1 - d^2/dx^2)^{-(\alpha+1/2)}\delta R F\|_1 = C\|\delta R F\|_{W^{\nu+1,1,\varepsilon}(\mathbb{R})}.
\]

This finishes the proof. □

In our next results we will explain how to estimate the \( L^p \rightarrow L^q \) norm of general multiplier \( F(L) \) in terms of estimate \( (BR_{p,q,m}^\alpha) \).

Proposition 3.6. Suppose that \((X, d, \mu)\) satisfies the doubling property \((D)\) and a non-negative self-adjoint operator \( L \) acting on \( L^2(X) \) satisfying condition \((BR_{p,q,m}^\alpha)\) for \( \alpha \geq -1 \) and \( 1 \leq p < q \leq \infty \). Then for every \( \varepsilon > 0 \) there exists a constant \( C_\varepsilon \) such that for any \( R > 0 \) and all Borel functions \( F \) for which \( \text{supp} \, F \subset [R/2, R] \)

\[
\|F(\sqrt{L})P_{B(x,\rho)}\|_{p \rightarrow q} \leq C \nu(x, \rho)^{(\frac{1}{2} - \frac{1}{p})(\frac{1}{2} - \frac{1}{q})}\|\delta R F\|_{WS^{\nu+1,1,\varepsilon}(\mathbb{R})}
\]

for all \( x \in X \) and all \( \rho \geq 1/R \).
Proof. We use the formula (3.16) to obtain that for \( \alpha \geq -1 \),

\[
F(\sqrt[n]{L}) = \frac{1}{\Gamma(\alpha + 1)} \int_0^\infty F * \chi_{-2}^{-\alpha}(\lambda)(\lambda - \sqrt[n]{L})_+^\alpha d\lambda.
\]

Since \( \text{supp } F \subseteq [R/2, R] \), one can rewrite

\[
F(\sqrt[n]{L}) = \int_0^\infty \lambda^\alpha S^\alpha_{\lambda R}(\sqrt[n]{L})(\delta_R F * \chi_{-2}^{-\alpha})(\lambda)d\lambda
\]

\[
= \left( \int_0^2 + \int_2^\infty \right) \lambda^\alpha S^\alpha_{\lambda R}(\sqrt[n]{L})(\delta_R F * \chi_{-2}^{-\alpha})(\lambda)d\lambda
\]

\[
= I_1 + I_2.
\]

We observe that if \( \lambda \in (2, \infty) \) and \( \tau \in (1/2, 1) \), then \( \chi_{-2}^{-\alpha}(\lambda - \tau) = 0 \), and so \( (\delta_R F * \chi_{-2}^{-\alpha})(\lambda) = 0 \). Hence \( I_2 = 0 \). To estimate the term \( I_1 \) we use condition \((BR_{p,q,m}^\alpha)\) to obtain

\[
\|F(\sqrt[n]{L})P_{\lambda \rho \omega}\|_{p \rightarrow q} \leq \int_0^2 \lambda^\alpha \|S^\alpha_{\lambda R}(\sqrt[n]{L})P_{\lambda \rho \omega}\|_{p \rightarrow q} (\delta_R F * \chi_{-2}^{-\alpha})(\lambda)d\lambda
\]

\[
\leq CV(x, \rho)^{\gamma \beta \alpha}(R\rho)^{\gamma \beta \alpha} \int_0^2 \lambda^{\gamma \beta \alpha} \|\delta_R F * \chi_{-2}^{-\alpha}(\lambda)\|d\lambda
\]

\[
= CV(x, \rho)^{\gamma \beta \alpha}(R\rho)^{\gamma \beta \alpha} \left( \int_0^{1/4} + \int_{1/4}^2 \right) \lambda^{\gamma \beta \alpha} \|\delta_R F * \chi_{-2}^{-\alpha}(\lambda)\|d\lambda
\]

\[
= CV(x, \rho)^{\gamma \beta \alpha}(R\rho)^{\gamma \beta \alpha} (I_{11} + I_{12}).
\]

For the term \( I_{11} \) we use the fact that \( \text{supp } \delta_R F \subseteq [1/2, 1] \) to obtain that if \( \lambda \in (0, 1/4) \) and \( \tau \in (1/2, 2) \), then \( \|\chi_{-2}^{-\alpha}(\lambda - \tau)\| \leq C \). This shows

\[
I_{11} = C \int_0^{1/4} \lambda^{\gamma \beta \alpha} \|\delta_R F * \chi_{-2}^{-\alpha}(\lambda)\|d\lambda
\]

\[
\leq C \int_0^{1/4} \int_0^{1/4} \lambda^{\gamma \beta \alpha} \|\chi_{-2}^{-\alpha}(\lambda - \tau)\|d\tau d\lambda
\]

\[
\leq C \|\delta_R F\|_1 \int_0^{1/4} \lambda^{\gamma \beta \alpha} d\lambda
\]

\[
\leq C \|\delta_R F\|_1,
\]

where we used the fact that \( n(1/p - 1/q) + \alpha > -1 \). Now we estimate the term \( I_{12} \), and note that

\[
I_{12} \leq C \int_{1/4}^2 \lambda^{\gamma \beta \alpha} \|\delta_R F * \chi_{-2}^{-\alpha}(\lambda)\|d\lambda
\]

\[
\leq C \int_{1/32}^\infty \|\delta_R F * \chi_{-2}^{-\alpha}(\lambda)\|d\lambda \leq \|\delta_R F\|_{WS^{\alpha + 1}}
\]

This ends the proof. \( \square \)
3.2 Bochner-Riesz means imply spectral multiplier estimates

In this section we will show that Bochner-Riesz means can be used to study spectral multipliers corresponding to functions supported in dyadic intervals. We assume that \((X, d, \mu)\) is a metric measure space satisfying the doubling property and \(n\) is the doubling dimension from condition \((D_n)\).

**Theorem 3.7.** Suppose that there exists a constant \(C > 0\) such that \(V(x, \rho) \geq C\rho^\alpha\) for all \(x \in X\) and \(\rho > 0\). Let \(L\) be a non-negative self-adjoint operator \(L\) acting on \(L^2(X)\) satisfying Davies-Gaffney estimates \((DG_m)\) and condition \((G_{p,2,m})\) for some \(1 \leq p_0 < 2\). Next assume that condition \((BR_{p,q,m}^\alpha)\) holds for \(\alpha \geq -1\) and \(p_0 < p < q < p_0'\). Let \(p \leq r \leq s \leq q\), and \(\beta > n(1/p - 1/r) + n(1/s - 1/q) + \alpha + 1\). Then for a Borel function \(F\) such that \(\text{supp}\, F \subseteq [1/4, 4]\) and \(F \in W^{\beta,1}(\mathbb{R})\), the operator \(F(t \sqrt{L})\) is bounded from \(L'(X)\) to \(L^1(X)\). In addition,

\[(3.17) \sup_{t > 0} t^{n(\frac{1}{r} - \frac{1}{s})} \|F(t \sqrt{L})\|_{r \to s} \leq C \|F\|_{W^{\beta,1}(\mathbb{R})}.\]

**Proof.** Let \(\phi \in C_c^\infty(\mathbb{R})\) be a function such that \(\text{supp}\, \phi \subseteq \{\xi : 1/4 \leq |\xi| \leq 1\}\) and \(\sum_{\ell \in \mathbb{Z}} \phi(2^{-\ell} \lambda) = 1\) for all \(\lambda > 0\). Set \(\phi_0(\lambda) = 1 - \sum_{\ell=1}^\infty \phi(2^{-\ell}\lambda),\)

\[(3.18) G^{(0)}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi_0(\tau) \hat{G}(\tau) e^{i\lambda \tau} \, d\tau\]

and

\[(3.19) G^{(\ell)}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(2^{-\ell} \tau) \hat{G}(\tau) e^{i\lambda \tau} \, d\tau,\]

where \(G(\lambda) = F(\sqrt{\lambda}) e^{i\lambda}\). Note that by the Fourier inversion formula

\[G(\lambda) = \sum_{\ell=0}^{\infty} G^{(\ell)}(\lambda).\]

Then

\[(3.20) F(t \sqrt{\lambda}) = G(\lambda) e^{-t^2} = \sum_{\ell=0}^{\infty} G^{(\ell)}(\lambda) e^{-t^2} = \sum_{\ell=0}^{\infty} F^{(\ell)}(t \sqrt{\lambda})\]

so for any \(f \in L'(X)\)

\[(3.21) \|F(t \sqrt{L})f\|_s \leq \sum_{\ell=0}^{\infty} \|F^{(\ell)}(t \sqrt{L})f\|_s, \quad p \leq r \leq s \leq q.\]

Next we fix \(\varepsilon > 0\) such that

\[(3.22) 2n\varepsilon[(1/s - 1/r) + (1/p - 1/q)] \leq \beta - n(1/s - 1/r) - n(1/p - 1/q) - \alpha - 1.\]

For every \(t > 0\) and every \(\ell\) set \(\rho_\ell = 2^{(1+\varepsilon)t}\). Then we choose a sequence \((x_n) \in X\) such that \(d(x_i, x_j) > \rho_\ell/10\) for \(i \neq j\) and \(\sup_{x \in X} \inf_i d(x, x_i) \leq \rho_\ell/10\). Such sequence exists because \(X\) is separable. Now set \(B_i = B(x_i, \rho_\ell)\) and define \(\overline{B_i}\) by the formula

\[\overline{B_i} = \overline{B}\left(x_i, \frac{\rho_\ell}{10}\right) \setminus \bigcup_{j < i} \overline{B}\left(x_j, \frac{\rho_\ell}{10}\right),\]
where \( \mathcal{B}(x, \rho) = \{ y \in X : d(x, y) \leq \rho \} \). Note that for \( i \neq j \), \( \mathcal{B}(x_i, \frac{\rho}{20}) \cap \mathcal{B}(x_j, \frac{\rho}{20}) = \emptyset \).

Observe that for every \( k \in \mathbb{N} \),
\[
\sup_i \# \{ j : d(x_i, x_j) \leq 2^k \rho \} \leq \frac{V(x, 2^{k+1} \rho)}{V(x, \frac{\rho}{20})} \leq \frac{V(y, 2^{k+2} \rho)}{V(y, \frac{\rho}{20})} \leq C 2^{kn}.
\]

(3.23)

Set \( D_\rho = \{(x, y) \in X \times X : d(x, y) \leq \rho \} \). It is not difficult to see that
\[
D_\rho \subseteq \bigcup_{\{i,j : d(x_i, x_j) < 2\rho \}} \mathcal{B}_i \times \mathcal{B}_j \subseteq D_{4\rho}.
\]

(3.24)

Now let \( \psi \in C_c^\infty(1/16, 4) \) be a function such that \( \psi(\lambda) = 1 \) for \( \lambda \in (1/8, 3) \), and we decompose
\[
F^{(\ell)}(t \sqrt{n}L) f = \sum_{i, j : d(x_i, x_j) < 2\rho} P_{B_i} [\psi F^{(\ell)}(t \sqrt{n}L)] P_{B_j} f + \sum_{i, j : d(x_i, x_j) < 2\rho} P_{B_i} [(1 - \psi) F^{(\ell)}(t \sqrt{n}L)] P_{B_j} f + \sum_{i, j : d(x_i, x_j) \geq 2\rho} P_{B_i} F^{(\ell)}(t \sqrt{n}L) P_{B_j} f = I + II + III.
\]

(3.25)

**Estimate for I.** Note that \( p \leq r \leq s \leq q \). By Hölder’s inequality
\[
\| \sum_{i, j : d(x_i, x_j) < 2\rho} P_{B_i} (\psi F^{(\ell)}(t \sqrt{n}L) P_{B_j} f \|^s_s \leq \sum_i \| P_{B_i} (\psi F^{(\ell)}(t \sqrt{n}L) P_{B_j} f \|^s_s \leq C \sum_i \| P_{B_i} (\psi F^{(\ell)}(t \sqrt{n}L) P_{B_j} f \|^s_s \\
\leq C \sum_i \| P_{B_i} (\psi F^{(\ell)}(t \sqrt{n}L) P_{B_j} f \|^s_s \\
\leq C \sum_j V(B_i)^s \| P_{B_j} f \|^s_s \\
\leq C \sum_{x \in X} \{ V(x, \rho) \}^{s(1 - s)} V(x, \rho \}^{s(1 - s)} \| \psi F^{(\ell)}(t \sqrt{n}L) P_{B_j} f \|^s_s \\
\leq C \sum_{x \in X} \{ V(x, \rho) \}^{s(1 - s)} V(x, \rho \}^{s(1 - s)} \| \psi F^{(\ell)}(t \sqrt{n}L) P_{B_j} f \|^s_s
\]

(3.26)

In Proposition 3.6 we assume \( \text{supp} F \subseteq [R/2, R] \). To adjust the multipliers which we consider here to this requirement we write \( (\psi F^{(\ell)}(t \sqrt{n}L) = \sum_{k=0}^7 (x_{2^{-k} - 2^{-3} \psi} F^{(\ell)}(t \sqrt{n}L) Now by Proposition 3.6 for every \( \ell \geq 4 \),
\[
\| (\psi F^{(\ell)}(t \sqrt{n}L) P_{B(x, \rho)} f \|_{p \rightarrow q} \leq C \sum_{k=0}^7 \| V(x, \rho) \|^{-\frac{1}{p} + \frac{1}{q}} 2^{k(1 + \epsilon)(\frac{1}{p} - \frac{1}{q})} \| \delta_{2^{-k} - 2^{-3} \psi} (\psi F^{(\ell)}(t \cdot) \|_{W^{s+1,1}}
\]

17
Next we note that it follows from (3.19) that

\[ \|G^{(\ell)}\|_{WS^{\alpha+1,1}} \leq C2^{(\alpha+1)\ell}\|G^{(\ell)}\|_1 \]

Hence for \( k = 0, 1, \ldots, 7, \)

\[ \|\delta^{2k-3}_{23}((\psi F^{(\ell)})(t))\|_{WS^{\alpha+1,1}} \leq C\|G^{(\ell)}\|_{WS^{\alpha+1,1}} \leq C2^{(\alpha+1)\ell}\|G^{(\ell)}\|_1 \]

This gives

\[ \|(\psi F^{(\ell)})(t\sqrt{\ell})P_{B(x,\rho)}\|_p \leq V(x,\rho)^{\frac{1}{\rho} - \frac{1}{q}}2^{(1+\varepsilon)n(\frac{1}{\rho} - \frac{1}{q})}2^{(\alpha+1)\ell}\|G^{(\ell)}\|_1 \]

for every \( \ell \geq 4. \) On the other hand, for \( \ell = 0, 1, 2, 3, \) we note that by Proposition 2.3 of [46], \((G_{p_0,2^m}) \Rightarrow (ST_{p_0,2^m}^\infty),\) and thus \( \|(\psi F^{(\ell)})(t\sqrt{\ell})P_{B(x,\rho)}\|_p \leq CV(x,\rho)^{\frac{1}{\rho} - \frac{1}{q}}\|F\|_1. \) Since \( V(x,\rho) \geq C\rho^\alpha \) for all \( x \in X \) and \( \rho > 0, \) we have that

\[
\sum_{\ell=1}^{\infty} \left( \sum_{i,j,d(x_{i,j})<2\rho} P_{B_i}((\psi F^{(\ell)})(t\sqrt{\ell})P_{B_j}f) \right) \\
\leq 3 \sup_{\ell=1} \left\{ V(x,\rho)^{\frac{1}{\rho} - \frac{1}{q}} \left\|(\psi F^{(\ell)})(t\sqrt{\ell})P_{B(x,\rho)}\right\|_p \right\} \\
+ \sup_{\ell=4} \left\{ V(x,\rho)^{\frac{1}{\rho} - \frac{1}{q}} \left\|(\psi F^{(\ell)})(t\sqrt{\ell})P_{B(x,\rho)}\right\|_p \right\} \\
\leq C\rho^{n(\frac{1}{\rho} - \frac{1}{q})}(\|F\|_1 + \sum_{\ell=4}^{\infty} 2^{(1+\varepsilon)n(\frac{1}{\rho} - \frac{1}{q})}2^{(\alpha+1)\ell}\|G^{(\ell)}\|_1) \\
\leq C\rho^{n(\frac{1}{\rho} - \frac{1}{q})}\|G\|_{B^\alpha_{1,1}},
\]

(3.27)

where \( \gamma = n(1/p - 1/r) + n(1/s - 1/q) + \alpha + 1 + \delta \) and \( \delta = \varepsilon n(1/p - 1/r) + \varepsilon n(1/s - 1/q). \) The last inequality follows from definition of Besov space. See e.g. [6, Chap. VI]. By (3.22)

\[ W^\beta_{1,1} \subseteq B^\gamma_{1,1} \quad \text{with} \quad \|G\|_{B^\gamma_{1,1}} \leq C\|G\|_{W^\beta_{1,1}} \]

where \( \gamma = n(1/p - 1/r) + n(1/s - 1/q) + \alpha + 1 + \delta, \) see again [6]. However, \( \text{supp } F \subseteq [1/4, 4] \) so \( \|G\|_{W^\beta_{1,1}} \leq \|F\|_{W^\beta_{1,1}}. \) Hence the foregoing estimates give

(3.28)

LHS of (3.27) \( \leq C\rho^{n(\frac{1}{\rho} - \frac{1}{q})}\|F\|_{W^\beta_{1,1}}. \)

**Estimate of II.** Repeat an argument leading up to (3.26), it is easy to see that

\[ \| \sum_{i,j,d(x_{i,j})<2\rho} P_{B_i}((1-\psi)F^{(\ell)})(t\sqrt{\ell})P_{B_j}f \|_s \leq C \sup_{x \in X} \|((1-\psi)F^{(\ell)})(t\sqrt{\ell})P_{B(x,\rho)}\|_{r-s} \|f\|_r \]

\[ \leq C\|((1-\psi)F^{(\ell)})(t\sqrt{\ell})\|_{r-s} \|f\|_r, \]

where, for a fixed \( N, \) one has the uniform estimates

\[ \left| \left( \frac{d}{d\lambda} \right)^N ((1-\psi)F^{(\ell)})(\lambda) \right| \leq C\lambda 2^{-\ell N}(1 + |\lambda|)^{-N} \|F\|_{L^1(\mathbb{R})}. \]
But Proposition 2.1 then implies that for every \( p \leq r \leq s \leq q \),
\[
\|(1 - \psi)F^{(t)}(t \sqrt[p]{L})\|_{r \rightarrow s} \leq \|(1 - \psi)F^{(t)}(t \sqrt[p]{L}) (1 + t \sqrt[p]{L})^M\|_{s \rightarrow s} \|(1 + t \sqrt[p]{L})^{-M}\|_{r \rightarrow s} 
\leq C2^{-\ell N} t^{\mu(t, 1 - \frac{1}{2})} \|F\|_{L^1(\mathbb{R})},
\]
which gives
\[
(3.29) \quad \sum_{\ell = 0}^{\infty} \|(1 - \psi)F^{(t)}(t \sqrt[p]{L})\|_{r \rightarrow s} \leq C t^{\mu(t, 1 - \frac{1}{2})} \|F\|_{L^1(\mathbb{R})}.
\]

**Estimate of III.** Note that
\[
\left\| \sum_{i, j : d(x_i, x_j) \geq 2^{(1+\varepsilon)t}} P_{B_i} F^{(t)}(t \sqrt[p]{L}) P_{B_j} f \right\|_s \leq \sum_{i} \left\| \sum_{j : d(x_i, x_j) \geq 2^{(1+\varepsilon)t}} P_{B_i} F^{(t)}(t \sqrt[p]{L}) P_{B_j} f \right\|_s \leq \sum_{i} \left( \sum_{j : d(x_i, x_j) \geq 2^{(1+\varepsilon)t}} \left\| P_{B_i} F^{(t)}(t \sqrt[p]{L}) P_{B_j} f \right\|_s \right)^s.
\]
Recall that \( G(\lambda) = F(\sqrt[p]{L}) e^{t} \). By the formula (3.19), it follows from an argument as in [46, Lemma 4.3] that For all \( \ell = 0, 1, 2, \ldots \) and all \( x_i, x_j \) with \( d(x_i, x_j) \geq 2^{(1+\varepsilon)t} \), there exist some positive constants \( C, c_1, c_2 > 0 \) such that for \( p_0 < r \leq s < p' \),
\[
\left\| P_{B_i} F^{(t)}(t \sqrt[p]{L}) P_{B_j} f \right\|_s \leq C \left\| P_{B_i} f \right\|_r \int_{-\infty}^{\infty} \phi(2^{\ell - \varepsilon} \tau) \hat{G}(\tau) \left\| P_{B_i} e^{(t\tau - 1)e^{it}L} P_{B_j} \right\|_{r \rightarrow s} d\tau \leq C t^{\mu(t, 1 - \frac{1}{2})} e^{-c_2 \frac{d(x_i, x_j)}{2^t}} \exp \left( -c_2 \frac{d(x_i, x_j)}{2^t} \right) \left\| F \right\|_r \left\| P_{B_i} f \right\|_r.
\]
which, together with the Cauchy-Schwarz inequality, yields
\[
\left\| \sum_{i, j : d(x_i, x_j) \geq 2^{(1+\varepsilon)t}} P_{B_i} F^{(t)}(t \sqrt[p]{L}) P_{B_j} f \right\|_s \leq C t^{\mu(t, 1 - \frac{1}{2})} e^{-c_1 \frac{d(x_i, x_j)}{2^{(1+\varepsilon)t}}} \left\| F \right\|_r \left\{ \sum_{j : d(x_i, x_j) \geq 2^{(1+\varepsilon)t}} \exp \left( -c_2 \frac{d(x_i, x_j)}{2^t} \right) \left\| P_{B_j} f \right\|_r \right\}^{s/r} \leq C t^{\mu(t, 1 - \frac{1}{2})} e^{-c_1 \frac{d(x_i, x_j)}{2^{(1+\varepsilon)t}}} \left\| F \right\|_r \left\| f \right\|_r.
\]
Therefore,
\[
\sum_{\ell = 0}^{\infty} \left\| \sum_{i, j : d(x_i, x_j) \geq 2^{(1+\varepsilon)t}} P_{B_i} F^{(t)}(t \sqrt[p]{L}) P_{B_j} f \right\|_s \leq C t^{\mu(t, 1 - \frac{1}{2})} \sum_{\ell = 0}^{\infty} e^{-c_1 \frac{d(x_i, x_j)}{2^{(1+\varepsilon)t}}} \left\| F \right\|_r \left\| f \right\|_r \leq C t^{\mu(t, 1 - \frac{1}{2})} \left\| F \right\|_r \left\| f \right\|_r.
\]
Estimate (3.17) then follows from (3.21), (3.25), (3.27), (3.28), (3.29) and (3.31). This completes the proof of Theorem 3.7. □

19
Remark 3.8. From the proof of Theorem 3.7, we can see that the result of

$$\sup_{r>0} \|F(t \sqrt[n]{L})\|_{r \rightarrow r} \leq C$$

in Theorem 3.7 (i.e., $r = s$ in (3.17)) holds under the assumption that $(X,d,\mu)$ satisfies the doubling condition (D) only. In this case, we do not need the assumption that $V(x,\rho) \geq C\rho^n$ for all $x \in X$ and $\rho > 0$. See also Theorem 4.2, [46].

The following corollary is a consequence of Theorem 3.7.

Corollary 3.9. Suppose that there exists a constant $C > 0$ such that $V(x,\rho) \geq C\rho^n$ for all $x \in X$ and $\rho > 0$. Next assume that a non-negative self-adjoint operator $L$ acting on $L^2(X)$ satisfies estimates (DG$_m$) and (G$_{p,\infty}$) for some $1 \leq p_0 < 2$. Then $(BR^{\alpha}_{p,q})$ for $\alpha \geq -1$ and $p_0 < p < q < p'_0$, implies

$$\left\|S^\delta_{R}(L)\right\|_{r \rightarrow s} \leq C R^{n\left(\frac{1}{2} - \frac{1}{r}\right)}$$

for all $p \leq r \leq s \leq q$ and $\Re \delta > \alpha + n(1/p - 1/r) + n(1/s - 1/q)$.

In particular, if $(BR^{\alpha}_{p_0,q_0})$ holds for

$$(3.32) \quad \left(\frac{p_0}{q_0}, \frac{1}{q_0}\right) = \left(\frac{n + 1 + 2\alpha}{2n} - \frac{2\alpha}{n + 1}, \frac{n + 1 + 2\alpha}{2n}\right),$$

then

$$\left\|S^\delta_{R}(L)\right\|_{r \rightarrow r} \leq C$$

for all $p_0 \leq r \leq p'_0$ and $\delta > n(1/p_0 - 1/2) - 1/2$.

Proof. Let $F(\lambda) = (1 - \lambda^m)^\phi$ and $\delta = \sigma + i\tau$. We set

$$F(\lambda) = F(\lambda)\phi(\lambda^m) + F(\lambda)(1 - \phi(\lambda^m)) =: F_1(\lambda^m) + F_2(\lambda^m),$$

where $\phi \in C^{\infty}(\mathbb{R})$ is supported in $\{\lambda : |\lambda| \geq 1/4\}$ and $\phi = 1$ for all $|\lambda| \geq 1/2$. It is known that if $0 < s < \sigma + 1$, then $(1 - |\lambda|^m)^\phi \in W^{s,1}(\mathbb{R})$ with $\left\|(1 - |\lambda|^m)^\phi \right\|_{W^{s,1}(\mathbb{R})} \leq C e^{\sigma r}$ for constants $C, c > 0$ independent of $s$, see for example [8, Lemma 4.4]. This, in combination with Theorem 3.7, shows that

$$\sup_{R>0} R^{n(1 - \frac{1}{r})} \left\|F_2 \left(\frac{L}{R^m}\right)\right\|_{r \rightarrow s} \leq C$$

for all for $p \leq r \leq s \leq q$ and $\sigma > \alpha + n(1/p - 1/r) + n(1/s - 1/q)$. On the other hand, we note that $V(x,\rho) \geq C\rho^n$ for all $x \in X$ and $\rho > 0$. by Propositions 2.1 and 2.2 that for $p_0 < r < s < p'_0$ and for $N > m(1/p - 1/2)$,

$$\left\|F_1 \left(\frac{L}{R^m}\right)\right\|_{r \rightarrow s} = \left\|F_1 \left(\frac{L}{R^m}\right) \left(1 + \frac{L}{R^m}\right)^{N/m}\right\|_{r \rightarrow r} \left\|\left(1 + \frac{L}{R^m}\right)^{N/m}\right\|_{r \rightarrow s} \leq C R^{n\left(\frac{1}{2} - \frac{1}{r}\right)}.$$

This proves $(BR^{\delta}_{r,s})$.

Now assume (3.32). It follows that $\left\|S^\delta_{R}(\sqrt[n]{L})\right\|_{p_0 \rightarrow p_0} \leq C$ for $\Re \delta > \alpha + n(1/p_0 - 1/q_0) = n(1/p_0 - 1/2) - 1/2$. By duality and interpolation, $(BR^{\delta}_{r,s})$ holds for all $p_0 \leq r \leq p'_0$ and $\delta > n(1/p_0 - 1/2) - 1/2$. The proof is complete.
3.3 Estimates for the Bochner-Riesz means with negative index

In previous section in Corollary 3.9 we prove that one can narrow the gap between $p$ and $q$ in condition $(\text{BR}_{p,q,m}^\alpha)$ by increasing the order of Bochner-Riesz means $\alpha$. In this section we describe relation between various $(\text{BR}_{p,q,m}^\alpha)$ of a different nature. This time the argument is based on $T^*T$ type argument and Stein’s complex interpolation. The heart of the matter in our discussion is the fact that Stein-Tomas restriction estimate is essentially equivalent with the full description of $L^p \to L^q$ mapping properties of Bochner-Riesz means of order $1/2$.

Given some $1 \leq p < 2$, we set

$$A = \left(1, \frac{n+1+2\alpha}{2n}\right), \quad A' = \left(\frac{n-1-2\alpha}{2n}, 0\right),$$

$$B(p) = \left(\frac{n+1+2\alpha}{2n} + \alpha - \frac{2\alpha}{p}, \frac{n+1+2\alpha}{2n}\right), \quad B'(p) = \left(\frac{n-1-2\alpha}{2n}, \frac{2\alpha - \alpha + n-1-2\alpha}{p}\right),$$

$$C(p) = \left(\frac{1}{p}, \frac{n+1+2\alpha}{2n}\right), \quad C'(p) = \left(\frac{n-1-2\alpha}{2n}, 1 - \frac{1}{p}\right),$$

$$D(p) = \left(\frac{1}{2} + \alpha - \frac{2\alpha}{p}, \frac{1}{2}\right), \quad D'(p) = \left(\frac{1}{2}, 1 - \alpha + \frac{2\alpha}{p}\right).$$

Denote by $\Delta_\alpha(p, n)$ the open pentagon with vertices $A, B(p), B'(p), A'$ and $(1, 0)$. Namely,

$$\Delta_\alpha(p, n) = \left\{(\frac{1}{r}, \frac{1}{s}) \in (0, 1) \times (0, 1) : \min\left(\frac{1}{r} - \frac{1}{2}, \frac{1}{2} - \frac{1}{s}\right) > -\frac{2\alpha + 1}{2n}, \ \alpha - \frac{2\alpha}{p} < \frac{1}{r} - \frac{1}{s}\right\}.$$

We are now in position to state our next result.

**Theorem 3.10.** Suppose that there exists a constant $C > 0$ such that $C^{-1} \rho^\alpha \leq V(x, \rho) \leq C \rho^\alpha$ for all $x \in X$ and $\rho > 0$. Next assume that a non-negative self-adjoint operator $L$ acting on $L^2(X)$ satisfies estimates $(\text{DG}_m)$ and $(\text{G}_{p,0,2,m})$ for some $1 \leq p_0 < 2$ and $(\text{BR}^{-1}_{p,0,m}^\alpha)$ holds for some $p_0 < p < 2$.

$$\|S_{R_m}^\alpha(L)\|_{\ell_r \to \ell_s} \leq CR^{\alpha(\frac{1}{r} - \frac{1}{s})}$$

if each of the following conditions holds:

1. $\alpha > n(1/p - 1/2) - 1/2, p_0 < r \leq s < p_0', r < q_{\alpha'} < s$ where $q_{\alpha} = \max\{1, \frac{2\alpha}{n+1+2\alpha}\}$.

2. $n(1/p - 1/2) - 1/2 \geq \alpha > 0, p_0 < r \leq s < p_0', (1/r, 1/s) \in \Delta_\alpha(p, n)$ and $(1/r, 1/s)$ is strictly below the lines joining the point $(1/2, 1/2)$ to $C(p)$ and $C'(p)$.

3. $-1/2 < \alpha \leq 0, p_0 < r \leq s < p_0', (1/r, 1/s) \in \Delta_\alpha(p, n)$ and $(1/r, 1/s)$ is strictly below the lines joining $D(p)$ to $C(p); D(p)$ to $D'(p)$ and $D'(p)$ to $C'(p)$.

4. $-1 < \alpha \leq -1/2, p_0 < r \leq s < p_0', \alpha - \frac{2\alpha}{p} < \frac{1}{r} - \frac{1}{s}, r < q_{\alpha'}$ and $q_{\alpha} < s$, where $1/q_{\alpha} = 1 + \alpha - (2\alpha + 1)/p$.

The proof of Theorem 3.10 is based on the following interpolation result.

**Lemma 3.11.** Suppose that there exists a constant $C > 0$ such that $V(x, r) \geq Cr^\alpha$ for all $x \in X$ and $r > 0$. Next assume that a non-negative self-adjoint operator $L$ acting on $L^2(X)$ satisfies estimates
(DGₘ) and (Gₚ₀,₂ₘ) for some 1 ≤ p₀ < 2 and (BRₜᵢₗ,₁) holds for some δᵢ, pᵢ, qᵢ, i = 1, 2 such that p₀ < pᵢ ≤ qᵢ < p₀' and −\frac{2δᵢ}{n+1} ≤ \left(\frac{1}{pᵢ} - \frac{1}{qᵢ}\right). Then for every θ ∈ (0, 1),

\[ \|S_{\lambda}^{α}(\sqrt[p₀]{L})\|_{p₀\rightarrow q₀} ≤ Cλ^{n\left(\frac{1}{p₀} - \frac{1}{q₀}\right)}, \quad λ > 0 \]

holds for α > δ₀ = θδ₁ + (1 - θ)δ₂ and

\[ \frac{1}{p₀} = \frac{θ}{p₁} + \frac{1 - θ}{p₂}, \quad \frac{1}{q₀} = \frac{θ}{q₁} + \frac{1 - θ}{q₂}. \]

Proof. By Lemma 3.2 and Corollary 3.9 for any Re δ > δ₁

\[ \|S_{\lambda}^{α}(\sqrt[pᵢ]{L})\|_{pᵢ\rightarrow qᵢ} ≤ Cλ^{n\left(\frac{1}{pᵢ} - \frac{1}{qᵢ}\right)}, \quad λ > 0 \]

for i = 1, 2. The proof then follows from Stein’s classical complex interpolation theorem [49] for analytic families of operators. □

Proof of Theorem 3.10. We first show that for all n(1/p - 1/2) - 1/2 ≥ α > -1/2 and every ε > 0, (BRₜᵢₗ,⁺,ε) holds for (1/r, 1/s) = C(p) = \left(\frac{1}{p}, \frac{n+1+2α}{2n}\right).

Indeed we assume that L satisfies condition (BR⁻¹,p₀,m) for some p₀ < p < 2. By Lemma 3.2 estimate (BR⁻¹,r,+,m) holds for all p₀ < r ≤ p ≤ p' ≤ s ≤ p₀'. Next by Corollary 3.9

\[ \|S_{\lambda}^{α+2ε}(\sqrt[pᵢ]{L})\|_{pᵢ\rightarrow pᵢ'} ≤ Cλ^{n\left(\frac{1}{pᵢ} - \frac{1}{pᵢ'}\right)}, \quad λ > 0 \]

for all ε > 0. By TT⁺ argument,

\[ \|S_{\lambda}^{α+ε}(\sqrt[p]{L})\|_{p\rightarrow 2} = \|S_{\lambda}^{α+2ε}(\sqrt[p]{L})\|_{pᵢ\rightarrow pᵢ'} ≤ Cλ^{n\left(\frac{1}{pᵢ} - \frac{1}{pᵢ'}\right)}, \quad λ > 0 \]

where q = \frac{n+1+2α}{n+1}. This proves estimate (BR⁺⁺,ε) for (1/r, 1/s) = C(p) = \left(\frac{1}{p}, \frac{n+1+2α}{2n}\right).

Now point (2) follows from the above observation, straightforward L² estimates for α = 0 (that is \|S_{\lambda}^{α}(\sqrt[p]{L})\|₂ ≤ 1), duality and Lemmas 3.2 and 3.11. The proof of point (1) is simple adjustment of the above argument based on the fact that in virtue of Lemma 3.3 for any p₀ < p₁ < p₂ < 2 condition (BR⁻¹,p₀',m) implies (BR⁻¹,p₀,p⁺',m).

Interpolation using Lemma 3.11 between (3.33) and L² estimates \|S_{\lambda}^{α}(\sqrt[p]{L})\|₂ ≤ 1 yields

\[ \|S_{\lambda}^{α+ε}(\sqrt[p]{L})f\|₂ ≤ Cλ^{n\left(\frac{1}{p} - \frac{ε}{2}\right)} ||f||ₚ \]

for \( \frac{1}{r} = \frac{1}{p} + \frac{2α}{p} - \frac{2α}{2n} \) and \( -\frac{1}{r} ≤ α ≤ 0 \), which means that (BR⁺⁺,ε) holds for (1/r, 1/s) = D(p) = \left(\frac{1}{2} + \frac{2α}{2n} - \frac{2α}{p}, \frac{1}{2}\right).

Now point (3) is a consequence of estimates (BR⁺⁺,ε) for (1/r, 1/s) = D(p) and (1/r, 1/s) = C(p), the Riesz-Thorin theorem (or Lemma 3.11), duality and Lemma 3.2.

Since (BR⁺⁺,ε) holds for (1/r, 1/s) = C(p) and α = -1/2, we can apply the argument similar to the discussion described above and assumption (BR⁻¹,p₀,p⁺',m) to obtain point (4). The proof of Theorem 3.10 is end. □
Remark 3.12. It follows from Lemma 3.2 that if the operator $L$ satisfies the Gaussian upper bounds (GE$_m$) and condition (BR$_{p,p',m}^{-1}$) for some $1 \leq p < 2$, then restriction $p_0 < r \leq s < p'_0$ can be removed from all points (1)-(4) and set $\Delta_0(p,n)$ can be replaced by

$$\tilde{\Delta}_0(p,n) = \left\{ \left( \frac{1}{r}, \frac{1}{s} \right) \in [0, 1] \times [0, 1] : \min \left( \frac{1}{r} - \frac{1}{2}, \frac{1}{2} - \frac{1}{s} \right) > -\frac{2\alpha + 1}{2n}, \quad \alpha - \frac{2\alpha}{p} < \frac{1}{r} - \frac{1}{s} \right\}.$$

Remark 3.13. Note also that if we know that

$$\|S_{R^n}^{(-1/2)}(L)\|_{r,s} \leq CR^{(\frac{1}{s} - \frac{1}{r})},$$

then by interpolation, we can further extend the range of $r$ and $s$ in point (4) to obtain essentially the same optimal results as in the case of the standard Laplace operator.

4 Uniform Sobolev inequalities for elliptic operators with constant coefficients

In this section we will consider $L^p \to L^q$ uniform boundedness of the resolvent of the higher order elliptic differential operators. Let $n \geq 2$ and $P(\xi)$ be the real homogeneous elliptic polynomial of order $m \geq 2$ on $\mathbb{R}^n$ satisfying the following non-degenerate condition:

$$\det \left( \partial^2 P(\xi) \over \partial \xi_i \partial \xi_j \right)_{i,j=1} \neq 0, \quad \xi \neq 0,$$

which is equivalent to the fact that hypersurface

$$\Sigma = \{ \xi \in \mathbb{R}^n; \quad |P(\xi)| = 1 \},$$

has nonzero Gaussian curvature everywhere, see [12, 50]. Without loss of generality, we may assume that $P(\xi) > 0$ for all $\xi \neq 0$.

Throughout this section we always assume that $H_0 := P(D)$, where $D = -i(\partial_1, \ldots, \partial_n)$ and $P(D)$ is the nonnegative self-adjoint operator associated with the elliptic polynomial $P(\xi)$ on $L^2(\mathbb{R}^n)$ and that $P$ satisfies conditions (4.1) and (4.2).

In our next results we will show that the operator $(H_0 - z)^{-\alpha}$ can be defined for all values of $z \in C$ including $z \geq 0$ by taking limits from upper or lower half-plane gives different operators if $\alpha > 0$, see (4.18) below. Hence it is convenient to introduce notation $C^\pm$. If $z$ is not a positive real, then this coincides with the standard complex numbers. For $z = \lambda > 0$ we consider two possibilities $\lambda + i0$ or $\lambda - i0$. The topology of $\mathbb{C}^\pm$ again coincides with topology of $\mathbb{C}$ except of set consisting of $\lambda + i0$ or $\lambda - i0$ where the limit can be only taken from the corresponding upper and lower half-planes.

The following statement is our main result in this section.

Theorem 4.1. Let $n \geq 2$, $m \geq 2$ and $z \in \mathbb{C}$. Consider arbitrary auxiliary cutoff function $\psi$ such that $\psi \in C_0^\infty(\mathbb{R}), \psi(s) \equiv 1$ if $s \in [-2, 2]$ and $\psi$ is supported in the interval $[-4, 4]$. Assume that $1/2 \leq \alpha < (n+1)/2$ for $n \geq 3$ and $0 < \alpha < 3/2$ for $n = 2$. Suppose also that exponents $(p, q)$ satisfy the following conditions:

$$\min \left( \frac{1}{p} - \frac{1}{2}, \frac{1}{2} - \frac{1}{q} \right) > \frac{2\alpha - 1}{2n}.$$
\begin{equation}
\frac{2\alpha}{n+1} < \left( \frac{1}{p} - \frac{1}{q} \right).
\end{equation}

Then there exists positive constants \( C_{p,q} \) independent of \(|z|\) such that
\begin{equation}
\| (H_0 - z)^{-\alpha} \psi(H_0/|z|) \|_{p \to q} \leq C_{p,q} \|z\|^{\frac{\alpha}{\min\{\frac{n}{p}, \frac{1}{q}\}}}, \quad \forall z \in \mathbb{C}^\pm \setminus \{0\}.
\end{equation}

Moreover, for the same range of \( \alpha \) and exponents \((p, q)\) the corresponding Bochner-Riesz means of order \(-\alpha\), \( S_\lambda^{-\alpha}(H_0) = \frac{1}{(1-\alpha)} \left( 1 - \frac{H_0}{\lambda} \right)_z^{-\alpha} \) is well defined for all \( \lambda > 0 \) and satisfies similar estimates
\begin{equation}
\| S_\lambda^{-\alpha}(H_0) \|_{p \to q} \leq C_{p,q} \lambda^{\frac{\alpha}{\min\{\frac{n}{p}, \frac{1}{q}\}}}, \quad \lambda > 0.
\end{equation}

Next assume in addition that \( m\alpha > n \) or that \( \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{n} \), \( p \neq 1 \) and \( q \neq \infty \) for \( m\alpha \leq n \).

Then
\begin{equation}
\| (H_0 - z)^{-\alpha} \|_{p \to q} \leq C_{p,q} \|z\|^{\frac{\alpha}{\min\{\frac{n}{p}, \frac{1}{q}\}}}
\end{equation}

for all \( z \in \mathbb{C}^\pm \setminus \{0\} \).

\textbf{Proof.} We only discuss the case \( m\alpha < n \). The other cases are similar or simpler. We begin our discussion with verifying estimates (4.5) and (4.7) and postpone considering the operator \( S_\lambda^{-\alpha}(H_0) \) to the end of proof. Set \( z = re^{i\theta} \) with \( r > 0 \). If \( \delta < |\theta| \leq \pi \) for some \( \delta > 0 \), then the operator \((H_0 - e^{i\theta})^{-\alpha}\) is a standard constant coefficient pseudo-differential operator of order \(-am\) with a symbol \((P(\xi) - e^{i\theta})^{-\alpha}\). Hence resolvent estimate (4.7) follows from the standard Sobolev estimates and a scaling argument in \( r \). A similar argument shows that for any \( p \leq q \) the multiplier \((H_0 - e^{i\theta})^{-\alpha}\psi(H_0)\) is bounded as operator from \( L^p \) to \( L^q \). Thus we can assume that \( 0 < |\theta| \leq \delta \) and by symmetry it is enough to consider only the case \( \Im z > 0 \).

We write \( z = (\lambda + i\epsilon)^m \) for \( \lambda > 0 \) and \( 0 < \epsilon << 1 \). Since \(|z| \sim \lambda^m\), by homogeneity, it suffices to estimate \((H_0 - (1 + i\epsilon))^m\) and \((H_0 - (1 + i\epsilon)^m)\psi(H_0)\) for \( 0 < \epsilon << 1 \). Let \( K^\alpha \) be the convolution kernel of \((H_0 - (1 + i\epsilon))^m\). By the inverse Fourier transform
\begin{equation}
K^\alpha = F^{-1}\left\{ (P(\xi) - (1 + i\epsilon)^m)^{-\alpha} \right\}.
\end{equation}

Note that \( K^\alpha = K_1 + K_2 \), where
\begin{equation}
K_1 = F^{-1}\left( \frac{\psi(P^{1/m}(\xi))}{(P(\xi) - (1 + i\epsilon)^m)^\alpha} \right)
\end{equation}

and
\begin{equation}
K_2 = F^{-1}\left( \frac{1 - \psi(P^{1/m}(\xi))}{(P(\xi) - (1 + i\epsilon)^m)^\alpha} \right).
\end{equation}

It is clear that to show (4.5) and (4.7) it is enough to verify that \( K_1 \) satisfies (4.5), whereas (4.7) holds for \( K_2 \).

\textit{Estimate (4.7) for \( K_2 \).} To estimate \( K_2 \) we note for any \( \alpha > 0 \) it is symbol of order \(-m\alpha\) that is
\begin{equation}
\left| D^{\beta}\left( \frac{1 - \psi(P^{1/m}(\xi))}{(P(\xi) - (1 + i\epsilon)^m)^\alpha} \right) \right| \leq C_{\alpha} \left( \alpha + |\xi| \right)^{-m\alpha - |\beta|}.
\end{equation}

Hence \(|K_2(\xi)| \leq C_{N} |\xi|^{m\alpha - n - N} \) for any \( N \in \mathbb{N} \) and by Young’s inequality and interpolation
\begin{equation}
\| K_2 \ast f \|_q \leq C_{p,q} \| f \|_p.
\end{equation}
for all \((p, q)\) satisfying \(0 \leq \frac{1}{p} - \frac{1}{q} \leq \frac{m}{n}\) and \((p, q) \neq \left(\frac{n}{m}, \infty\right), \left(1, \frac{n}{m-n}\right).

Estimate (4.5) for \(K_1\). To estimate \(K_1\) we use the stationary phase principle. We write

\( \text{(4.9)} \)

\[ K_1(x) = \int_{\mathbb{R}^n} \frac{e^{i\langle x, \xi \rangle}}{(P^{1/m}(\xi) - 1 - i\varepsilon)\alpha} d\xi = \int_0^\infty s^{n-1} \frac{\tilde{\psi}(s)}{(s - 1 - i\varepsilon)^\alpha} \left( \int_{\Sigma} e^{is\omega d\omega} \right) ds, \]

where \(\tilde{\psi}(s) = \psi(s)(s^{m-1} + s^{m-2}(1 + i\varepsilon) + \ldots + (1 + i\varepsilon)^{m-1})^{-\alpha}.\)

Note that \(K_1\) is the Fourier transform of compactly supported distribution including taking limits with \(\varepsilon\) goes to \(0\) so \(|K_1(x)| \leq C\) for all \(|x| \leq 1\). To handle the remaining case \(|x| > 1\), we recall the following stationary phase formula for the Fourier transform of a smooth measure on hypersurface \(\Sigma\)

\( \text{(4.10)} \)

\[ \int_{\Sigma} e^{is\omega d\omega} = |\tilde{\psi}(s)| + |\tilde{\psi}(s)| e^{-i\alpha(y)}, \]

where for say \(|y| \geq 1/4\), the coefficients satisfy

\( \text{(4.11)} \)

\[ \left| \frac{\partial^\beta}{\partial y^\beta} c_+(y) \right| + \left| \frac{\partial^\beta}{\partial y^\beta} c_-(y) \right| \leq C_\alpha |y|^{-\beta}, \quad \beta \in \mathbb{N}_0. \]

and \(\phi_\pm(y) = (\alpha, \omega_\pm(y))\) are smooth homogeneous function of degree one. Here \(\omega_\pm(y)\) are the two points of \(\Sigma\) such that \(\pm \frac{y}{|y|}\) are the positive normal direction of \(\Sigma\) at these points. Thus by (4.9) and (4.10)

\( \text{(4.12)} \)

\[ K_1(x) = \sum_{\pm} \int_0^\infty \frac{s^{n-1} \tilde{\psi}(s)}{(s - 1 - i\varepsilon)^\alpha} \left( |x| \frac{\tilde{\psi}(s)}{s} c_\pm(s) e^{i\alpha y_\pm(x)} \right) ds \]

where

\[ b_\pm(x) = \int_{-\infty}^\infty \frac{(s + 1)^{n-1}}{(s - i\varepsilon)^\alpha} c_\pm((s + 1)x) e^{i\alpha y_\pm(x)} ds. \]

Note that the function \(s \mapsto (s + 1)^{n-1} \tilde{\psi}(s) + c_\pm((s + 1)x)\) is smooth and compactly supported so it is easy to check that

\[ |\partial^\beta b_\pm(x)| \leq C_\beta |x|^{\alpha+\beta-1}, \quad |x| > 1/4 \]

uniformly in \(\varepsilon > 0\).

Hence in view of (4.12), we can further smoothly decompose \(K_1(x) = K'(x) + K''(x)\) in such a way that \(\text{supp} K' \subset B(0, 1)\) (the unit ball of \(\mathbb{R}^n\)), \(|K'(x)| \leq C\) for all \(x\) and \(K''\) can be expressed as

\[ K''(x) = \sum_{\pm} |x|^{\alpha+\beta} a_\pm(x) e^{i\alpha y_\pm(x)}, \]

where \(a_\pm \in C^\infty(\mathbb{R}^n)\) satisfy \(a_\pm(x) = 0\) for \(|x| \leq 1/2\) and \(|\partial^\beta a_\pm(x)| \leq C_\beta |x|^{-\beta}\) for any \(\beta \in \mathbb{N}_0.\) By Young's inequality

\( \text{(4.13)} \)

\[ ||K' * f||_q \leq C||f||_p \]

for all \(1 \leq p \leq q \leq \infty.\)
To estimate $K''$, we note that by the assumption $\alpha < (n + 1)/2$ and $|K''(x)| \leq (1 + |x|)^{(n+1-2\alpha)/2}$. Hence

\begin{equation}
(4.14) \quad \|K'' * f\|_q \leq C\|f\|_p
\end{equation}

for all $(p, q)$ satisfying that $\frac{n-1+2\alpha}{2n} \leq \frac{1}{p} - \frac{1}{q} \leq 1$ but $(p, q) \neq (1, \frac{2\alpha}{n+1-2\alpha}), (\frac{2\alpha}{n+1-2\alpha}, \infty)$. However this argument does not give the whole range of pairs $(p, q)$ for which (4.14) holds. It is possible to extend it by making use of the oscillatory factor $e^{\alpha |y|}$ in the integral operator

$$K'' * f(x) = \sum_{\pm} \int_{\mathbb{R}^n} |x-y|^{-\frac{n+1+\alpha}{2}} a_\pm(x-y) e^{\pm i\phi_\pm(x-y)} f(y) dy.$$ 

In fact, under the assumption that $\Sigma$ has nonzero Gaussian curvature everywhere, the phase function $\phi_\pm(x-y)$ satisfies the so-called $n \times n$-Carleson-Sjölin conditions, see [48, p.69] or [50, p.392]. Hence the celebrated Carleson-Sjölin argument can be used to estimate $K'' * f$.

Let $\beta(s) \in C^\infty_c(\mathbb{R})$ be a such function that supp $\beta \in [\frac{1}{2}, 2]$ and \(\sum_0^\infty \beta(2^{-\ell}s) = 1\) for $s \geq 1/2$. Set $K''_\ell(x) = \beta(2^{-\ell}|x|) K''(x)$ for all $\ell = 0, 1, 2, \ldots$ so

$$K'' * f(x) = \sum_{\ell=0}^\infty (K''_\ell * f)(x),$$

where

$$K''_\ell * f(x) := \int_{\mathbb{R}^n} |x-y|^{-\frac{n+1+\alpha}{2}} \beta(2^{-\ell}|x-y|) a_\pm((x-y)) e^{\pm i\phi_\pm(x-y)} f(y) dy.$$ 

Put $\lambda = 2^\ell$. By homogeneity

$$(K''_\ell * f)(\lambda x) = \lambda^{-\frac{n+1+\alpha}{2}} \int_{\mathbb{R}^n} w(x-y) e^{\pm i\phi_\pm(x-y)} f(\lambda y) dy,$$

where $w(x) = |x|^{-\frac{n+1+\alpha}{2}} \beta(|x|) a_\pm(\lambda x) \in C^\infty_c(\mathbb{R}^n \setminus 0)$ satisfying $|\partial^\alpha w(x)| \leq C_\alpha$ for any $\alpha$. Now we can apply Carleson-Sjölin argument, see [48, p.69] or [50, p.392], to conclude that

\begin{equation}
(4.15) \quad \|K''_\ell * f\|_q \leq C\lambda^{-n/p + (n-1+2\alpha)/2} \|f\|_p, \quad \lambda = 2^\ell, \ell = 0, 1, \ldots,
\end{equation}

which $q = \frac{2n}{n+1} p', 1 \leq p < 2n/(n-1+2\alpha)$ for all $\alpha \geq 1/2$ if $n \geq 3$ and $\alpha > 0$ if $n = 2$. By interpolation between (4.14) and (4.15)

\begin{equation}
(4.16) \quad \|K'' * f\|_q \leq C\|f\|_p
\end{equation}

for all $(p, q)$ such that $\frac{2\alpha}{n+1} < \frac{1}{p} - \frac{1}{q} \leq 1$ and

$$\min\left(\frac{1}{p} - \frac{1}{2}, \frac{1}{2} - \frac{1}{q}\right) > \frac{2\alpha - 1}{2n}.$$ 

Therefore (4.8), (4.13) together with (4.16) yield estimate (4.7).
Next we consider Bochner-Riesz mean operator $S_{\lambda}^{-\alpha}(H_0) = \frac{1}{\Gamma(1-\alpha)} (1 - \frac{H_0}{\lambda})^{\alpha}$. By homogeneity we can set $\lambda = 1$. Note also that

$$(x \pm i0)^{-\alpha} = \chi_+^{\alpha} + e^{\pm i\alpha} \chi_-^{\alpha},$$

so

$$e^{i\pi\alpha}(x - i0)^{-\alpha} - e^{-i\pi\alpha}(x + i0)^{-\alpha} = 2i \sin(\pi\alpha)\chi_+^{\alpha} = 2i \sin(\pi\alpha)\Gamma(1-\alpha)\chi_-^{\alpha} = 2i\pi \frac{\chi_+^{\alpha}}{\Gamma(\alpha)}.$$

Employing analytic continuation shows that (4.17) is valid for all $\alpha \in \mathbb{C}$, see also [36, (3.2.11)]. By (4.17)

$$2i\pi \frac{S_{1}^{-\alpha}(H_0)}{\Gamma(\alpha)} = 2i\pi \frac{\chi_+^{\alpha}(1 - H_0)}{\Gamma(\alpha)} = e^{i\pi\alpha}(H_0 - 1 - i0)^{-\alpha} - e^{-i\pi\alpha}(H_0 - 1 + i0)^{-\alpha}$$

$$= e^{i\pi\alpha}(H_0 - 1 - i0)^{-\alpha} \psi(H_0) - e^{-i\pi\alpha}(H_0 - 1 + i0)^{-\alpha} \psi(H_0).$$

Hence we obtain estimate (4.6) as a direct consequence of (4.5). This ends the proof of Theorem 4.1.

For $\alpha = 1$ the formula (4.17) simplifies to following relation

$$\frac{1}{2\pi i}(\lambda + i0)^{-1} - (\lambda - i0)^{-1} = \delta_0,$$

see [36, Example 3.1.13]. Then the above relation in turn implies the well-known absorption principle which connects the spectral projections $dE_{H_0}(\lambda)$ and the resolvent $R_0(z) = (H_0 - z)^{-1}$

$$dE_{H_0}(\lambda)f = \frac{1}{2\pi i}(R_0(\lambda + i0) - R_0(\lambda - i0))f.$$  

We will use the case $\alpha = 1$ of Theorem 4.1 and (4.18) to investigate the spectral resolution of Schrödinger type operators $H = P(D) + V$ with integrable potentials $V$, see Section 5 below. Therefore we summarise this particular case of Theorem 4.1 in the following corollary.

**Corollary 4.2.** Suppose that $n \geq 2$, $m \geq 2$ and the operator $H_0$ satisfies the assumptions of Theorem 4.1. Then

$$||dE_{H_0}(\lambda)||_{p \to q} \leq C \lambda^{\frac{m}{2}(\frac{1}{p} - \frac{1}{q})^{-1}}, \lambda > 0$$

for all exponents $(p, q)$ such that

$$\min\left(\frac{1}{p} - \frac{1}{2}, \frac{1}{2} - \frac{1}{q}\right) > \frac{1}{2n}$$

and

$$\frac{2}{n + 1} < \left(\frac{1}{p} - \frac{1}{q}\right).$$

Next assume in addition that $m > n$ or that $\frac{1}{p} - \frac{1}{q} \leq \frac{m}{n}$, $p \neq 1$ and $q \neq \infty$ for $m \leq n$. Then

$$||R_0(z)||_{p \to q} \leq C |z|^{\frac{m}{2}(\frac{1}{p} - \frac{1}{q})^{-1}}$$

for all $z \in \mathbb{C}^\pm \setminus \{0\}$. 

---

27
In the next step we will study the boundary resolvent manifolds estimates (not necessarily elliptic). In the setting of the Laplace operator on asymptotically conic non-trapping manifolds estimates (4.20) from Corollary 4.2 was obtained by Kenig, Ruiz and Sogge in [39]. (In fact, they were able to prove such a uniform estimate for a larger class of operators, where the standard Laplace operator $\Delta$ was replaced by a homogeneous second order constant coefficient differential operator, non-degenerate but not necessarily elliptic). In the setting of the Laplace operator on asymptotically conic non-trapping manifolds estimates (4.20) were obtained by Guillarmou and Hassell in [32]. For other results of this type see also [34] and the references within.

The classical Bochner-Riesz means operators $S^{\alpha}_{1}(\Delta)$ with a negative index $-\alpha$ corresponding to the standard Laplace operator have been studied by many authors, see for example [3, 7, 13, 34] and references therein.

## 5 Restriction type estimates for Schrödinger operators $P(D) + V$

In this section we will establish $L^{p} \to L^{q}$ estimates for the perturbed resolvent $R_{H}(z) = (z - H)^{-1}$ for any $z \neq 0$, where $H := H_{0} + V = P(D) + V$ is a self-adjoint operator with the real valued potential $V$. For simplicity we assume that $V \geq 0$ belong to $L^{1}_{\text{loc}}(\mathbb{R}^{n})$. Then it is well-known that the operator $H$ can be defined as a self-adjoint extension by the following non-negative closed form

$$Q_{V}(f) := \int_{\mathbb{R}^{n}} P(\xi) |\hat{f}(\xi)|^{2} d\xi + \int_{\mathbb{R}^{n}} V|f|^2 dx$$

for all $f \in W^{m,2}(\mathbb{R}^{n})$ such that $\int V|f|^2 dx < \infty$.

In order to obtain the estimates for the resolvent $R_{H}(z) = (z - H)^{-1}$, a crucial step will be to pass from (4.20) to a similar estimate for $H$ by writing the standard perturbation formula:

$$R_{H}(z)f = R_{0}(z)(I + VR_{0}(z))^{-1}f, \quad \text{Im}z \neq 0.$$  

In the next step we will study the boundary resolvent $R_{H}(\lambda \pm i0)$ and by Stone’s formula

$$dE_{H}(\lambda)f = \frac{1}{2\pi i}(R_{H}(\lambda + i0) - R_{H}(\lambda - i0))f$$

deduce restriction type estimates for the spectral projection measure $dE_{H}(\lambda)$. Using the notation $\mathbb{C}^{\pm}$ introduced at the beginning of section 4 for $\lambda > 0$ and $z = \lambda \in \mathbb{C}^{\pm}$ we always assume that $R_{H}(z) = R_{H}(\lambda \pm i0)$. We first verify the following lemma.

**Lemma 5.1.** Suppose $n \geq 2$, $m \geq 2$ and that $H_{0}$ satisfies assumptions of Theorem 4.1. Assume also that exponents $(p, q)$ satisfy all conditions listed in Corollary 4.2, $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ and that $0 \leq V \in L^{r}(\mathbb{R}^{n})$. Then

$$\|VR_{0}(z)\|_{p \to p} \leq C\|V\|_{r} |z|^{rac{2(n+1)}{n+2q} - 1}, \quad \forall z \in \mathbb{C}^{\pm} \setminus \{0\}.$$  

**Proof.** Let $M_{V}$ be the multiplication operator defined by $M_{V}f = V(x)f(x)$. Then by Hölder’s inequality $\|M_{V}\|_{q \to p} \leq \|V\|_{r}$ and by Corollary 4.2

$$\|VR_{0}(z)\|_{p \to p} \leq \|M_{V}\|_{q \to p}\|R_{0}(z)\|_{p \to q} \leq C\|V\|_{r} |z|^{rac{2(n+1)}{n+2q} - 1}, \quad \forall z \in \mathbb{C}^{\pm} \setminus \{0\}.$$  

\[\Box\]
Assume now that the exponent \( p \) satisfies the relation \( \max\left(\frac{2n}{n+m}, 1\right) < p < \frac{2(n+1)}{n+3} \). Note that then the pair \( (p, p') \) satisfies all conditions from Corollary 4.2. This yields the following corollary.

**Corollary 5.2.** Suppose again that \( n \geq 2, m \geq 2 \), \( H_0 \) satisfies assumptions of Theorem 4.1 and that \( 0 \leq V \in L^{\frac{4n}{n+1}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) where \( s = \max\left(\frac{n}{m}, 1\right) \). Then there exists a constant \( C > 0 \) such that

\[
\|VR_0(z)\|_{p \to p} \leq C \|z\|^{\frac{2(n+1) - 3p}{2(n+3) - 3p}}, \quad \forall z \in \mathbb{C}^+ \setminus \{0\}
\]

for all \( \max\left(\frac{2n}{n+m}, 1\right) < p < \frac{2(n+1)}{n+3} \).

In particular, there exists a constant \( \delta > 0 \) such that the operator \( I + VR_0(z) \) is invertible on \( L^p(\mathbb{R}^n) \) and

\[
\sup_{|z| > \delta} \|(I + VR_0(z))^{-1}\|_{p \to p} \leq C.
\]

for all \( z \in \mathbb{C}^+ \cap \{|z| \geq \delta\} \).

**Proof.** We only discuss the case \( m > n \) because the proof for the case \( m \leq n \) is similar. In the considered situation \( \frac{2n}{n+m} < p < \frac{2(n+1)}{n+3} \) so if we set \( \frac{1}{r} = \frac{1}{p} - \frac{1}{p'} \), then \( \frac{m}{n} < r < \frac{n}{2} \). Hence by Lemma 5.1

\[
\|VR_0(z)\|_{p \to p} \leq C\|V\|_{r} \cdot |z|^{\frac{2(n+1) - 3p}{2(n+3) - 3p}} \leq C\|V\|_{n}^{\frac{1}{2(n+1)/2}} \cdot |z|^{\frac{2(n+1) - 3p}{2(n+3) - 3p}} \cdot \|z|^{\frac{3p - 1}{2(n+3) - 3p}}, \quad \forall z \in \mathbb{C}^+ \setminus \{0\},
\]

where \( \theta = \left(\frac{1}{p} - \frac{1}{p'} - \frac{m}{n}\right)/(\frac{2}{n+1} - \frac{m}{n}) \). Note that

\[
\|V\|_{n}^{\frac{1}{2(n+1)/2}} \cdot |z|^{\frac{3p - 1}{2(n+3) - 3p}} \leq (1 + |V|_{n})\cdot (1 + |V|_{n}),
\]

hence there exists a constant \( C \) depending on \( n, m, V \) such that estimate (5.5) holds.

Next we verify estimate (5.6). Note that \( \frac{m}{n}(1 - \frac{1}{p'}) - 1 < 0 \) so there exists a constant \( \delta > 0 \) such that \( \|VR_0(z)\|_{p \to p} \leq \frac{1}{r} \) for all \( |z| > \delta \). By the standard Neumann series argument the last estimate yields \( \|(1 + VR_0(z))^{-1}\|_{p \to p} \leq 2 \).

In order to use Corollary 5.2 to establish the \( L^p \)-estimates of the spectral projections measure \( dE_H(\lambda) \), we need the following lemma essentially due to Hörmander [36, Chapter 14].

**Lemma 5.3.** Let \( 0 \leq V \in L^\infty(\mathbb{R}^n) \) with compact support. Then the equality

\[
\langle R_H(z)f, g \rangle = \langle R_0(z)(I + VR_0(z))^{-1}f, g \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^n)
\]

holds for all \( z \in \mathbb{C}^+ \setminus (\{0\} \cup \Lambda) \), where \( \Lambda \) is the set of positive discrete eigenvalues of \( H = P(D) + V \). In particular, the functions on the both sides of (5.7) are continuous on \( z \in \mathbb{C}^+ \setminus (\{0\} \cup \Lambda) \) and analytic in its interior.

**Proof.** The potential \( V \) is a bounded and compactly supported function so it is the short range perturbation of \( P(D) \), see Hörmander [36, page 246 of Chapter 14]. Therefore, the equality (5.7) immediately follows from Hörmander [36, Theorem 14.5.4 of Section 14.5].

**Remark 5.4.** The set \( \Lambda \) is the point spectrum of \( H \) with the finite multiplicity, which is discretely embed into positive real line. It would be interesting to show that \( \Lambda \) is empty for general higher order elliptic operator \( P(D) + V \). In the case of second order operators, the absence of positive eigenvalues has been studied in depth by many authors and confirmed for potential with decay of the order \( o(1/|x|) \) and some integrable class, see e.g. Hörmander [36, Chapter 14], Koch and Tataru [40] and references therein.
Proposition 5.5. Under the assumptions of Corollary 5.2 there exists a constant \( \delta > 0 \) such that

\[
\| R_H(z) \|_{p \to p'} \leq C |z|^\frac{n}{p} |\frac{1}{z} - 1|^{-1}, \quad z \in \mathbb{C}^\pm \cap \{ |z| \geq \delta \}
\]

and

\[
\| dE_H(\lambda) \|_{p \to p'} \leq C \lambda^{\frac{n}{p} - \frac{1}{2}}, \quad \lambda \geq \delta
\]

for all \( \max \left( \frac{2n}{n+m} \right) < p < \frac{2(n+1)}{n+3} \).

Proof. It suffices to prove estimate (5.8) for large \( |z| > \delta \) since estimates (5.9) immediately follow from (5.8) by Stone’s formula (5.3). Similarly, we only discuss the case \( m > n \). Given \( \frac{2n}{n+m} < p < \frac{2(n+1)}{n+3} \), then \( V \in L'(\mathbb{R}^n) \) with \( r = \frac{p}{p-1} \in (\frac{2n}{m}, \frac{n+1}{2}) \). In order to obtain (5.8), we need to establish equality (5.7) for \( |z| > \delta \) as \( 0 \leq V \in L'(\mathbb{R}^n) \). Firstly, we can take a monotonically increasing sequence of \( 0 \leq V_k \in L' \) with compact support such that \( V_k(x) \) converges to \( V(x) \) as \( k \to \infty \) in both pointwise and \( L' \) norm sense. By Lemma 5.3 for every \( k \)

\[
\langle R_H(z)f, g \rangle = \langle R_0(z)(I + V_k R_0(z))^{-1}f, g \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^n)
\]

for all \( z \in \mathbb{C}^\pm \setminus \{(0) \cup \Lambda_k \} \), where \( \Lambda_k \) is the point spectrum of the operator \( H_k = P(D) + V_k \). Note that \( ||V_k||_r \leq ||V||_r \) for every \( k \) so by Corollary 5.2 the estimates \( ||(I + V_k R_0(z))^{-1}||_{p \to p} \leq C \) hold uniformly for all \( |z| > \delta \), which implies that the set \( |z| > \delta \) has no eigenvalues of \( H_k \) (i.e. \( \Lambda_k \cap \{ |z| > \delta \} \) is an empty set). Indeed, if \( \lambda \in \Lambda_k \) and \( \lambda > \delta \), then there is a \( 0 \neq g_k \in L^2 \) such that

\[
(\lambda - H_k)g_k = (\lambda - P(D) - V_k)g_k = 0.
\]

Set \( f_k = V_k g_k \). Then \( g_k = R_0(\lambda + i0)f_k \) and \( (I + V_k R_0(\lambda + i0))f_k = 0 \). Note that \( f_k \neq 0 \) and it belongs to \( L'(\mathbb{R}^n) \) by Hölder’s inequality. This contradicts the existence of inverse \( (I + V_k R_0(\lambda + i0))^{-1} \) as bounded operator on \( L'(\mathbb{R}^n) \). Thus equality (5.10) actually holds on \( \mathbb{C}^\pm \cap \{ |z| > \delta \} \), the both sides of which are analytic on the interior of \( \mathbb{C}^\pm \cap \{ |z| > \delta \} \) and continuous up to their boundary.

Now we can extend the argument above to a general potential \( 0 \leq V \in L' \) by taking limit in equality (5.10). Note that for \( |z| \geq \delta \),

\[
(I + V_k R_0(z))^{-1} - (I + VR_0(z))^{-1} = (I + V_k R_0(z))^{-1}((V_k - VR_0(z))(I + VR_0(z))^{-1}.
\]

By Corollary 5.2

\[
||(I + V_k R_0(z))^{-1} - (I + VR_0(z))^{-1}||_{p \to p} \leq C||V_k - V||_r \to 0
\]

as \( k \to \infty \). Hence the left side of (5.10) converges uniformly to \( \langle R_0(z)(I + VR_0(z))^{-1}f, g \rangle \) on any compact subset \( K \) of \( \mathbb{C}^\pm \cap \{ |z| \geq \delta \} \) and the function \( z \to \langle R_0(z)(I + VR_0(z))^{-1}f, g \rangle \) is continuous on the set \( \mathbb{C}^\pm \cap \{ |z| \geq \delta \} \) and analytic in its interior. On the other hand, for any \( k \) we define the closed forms \( Q_{V_k} \) associated with \( H_k = P(D) + V_k \) by

\[
Q_{V_k}(f) := \int_{\mathbb{R}^n} P(\xi) |\hat{f}(\xi)|^2 d\xi + \int_{\mathbb{R}^n} V_k |f|^2 dx, \quad f \in W^{m,2}(\mathbb{R}^n).
\]

Then since the increasing sequence of nonnegative forms \( Q_{V_k}(f) \) monotonically converges to the form \( Q_V(f) \) defined in (5.1), so by Kato [38, Theorem 3.13a] it follows that \( \langle R_H(z)f, g \rangle \) converges to \( \langle R_H(z)f, g \rangle \) for each \( \text{Re } z < 0 \). Hence on the common domain \( \mathbb{C}^\pm \cap \{ \text{Re } z < -\delta \} \)

\[
\langle R_H(z)f, g \rangle = \langle R_0(z)(I + VR_0(z))^{-1}f, g \rangle, \quad f, g \in \mathcal{S}(\mathbb{R}^n).
\]

Note that both side of (5.12) extend analytically into the interior of \( \mathbb{C}^\pm \cap \{ |z| \geq \delta \} \), so by the uniqueness of analytic function extension equality (5.12) holds on \( \mathbb{C}^\pm \cap \{ |z| \geq \delta \} \). Thus estimate (5.8) follows from estimates (4.20) and (5.6). \( \square \)
If \( m < n \) and the \( L^\infty \) norm of potential \( V \) is small then we can extend Proposition 5.5 to a small values of frequency \( \lambda \). The proof is based on the uniform Sobolev \( L^p \to L^q \) estimates of the free resolvent \( R_0(z) \), which yields the required estimates for pairs \( (p,q) \) on the Sobolev line \( 1/p - 1/q = m/n \).

**Proposition 5.6.** Suppose that \( n \geq 2, m \geq 2 \), \( H_0 \) satisfies assumptions of Theorem 4.1 and that \( 0 \leq V \in L^\infty(\mathbb{R}^n) \). There exists a constant \( c_0 > 0 \) such that when \( \|V\|_m^\infty \leq c_0 \), then

\[
\|R_H(z)\|_{p \to p'} \leq C |z|^{\frac{m}{2} \left( \frac{1}{p} - \frac{1}{p'} \right)}^{-1}, \quad \forall z \in \mathbb{C}^\infty \setminus \{0\}
\]

and

\[
\|dE_H(\lambda)\|_{p \to p'} \leq C |z|^{\frac{m}{2} \left( \frac{1}{p} - \frac{1}{p'} \right)}^{-1}, \quad \lambda > 0
\]

for all

\[
\frac{2n}{n+m} \leq p < \min \left( \frac{2(n+1)}{n+3}, \frac{n}{m} \right).
\]

**Proof.** If \( p \) satisfies condition (5.15), then there exists \( q > 1 \) such that the pair \( (p,q) \) lies on the Sobolev line \( \frac{1}{p} - \frac{1}{q} = \frac{m}{n} \) and \( \frac{1}{p} - \frac{1}{2} > \frac{2n}{2n+1} \). This means that all conditions for exponents \( (p,q) \) listed in Corollary 4.2 hold. Hence by Lemma 5.1

\[
\|VR_0(z)\|_{p \to p} \leq C\|V\|_m^\infty, \quad \forall z \in \mathbb{C}^\infty \setminus \{0\}.
\]

Setting \( c_0 = C \) in the above estimate ensures that \( \|VR_0(z)\|_{p \to p} \leq \frac{1}{2} \) and

\[
\sup_{|z| > 0} \|(I + VR_0(z))^{-1}\|_{p \to p} \leq 2.
\]

Now the estimates (5.13) and (5.14) follows from (5.2) and Stone’s formula (5.3). \( \square \)

In Theorem 5.8 below we shall extend estimates (5.15) to the large range \( 1 \leq p < \min \left( \frac{2(n+1)}{n+3}, \frac{n}{m} \right) \). In particular, if \( m = 2, n \geq 3 \), this range is optimal and coincides with one described in Corollary 4.2. The argument we use is based on Lemmas 3.3 and 5.7 below.

**Lemma 5.7.** Assume that \( n > m \geq 2 \), \( H = H_0 + V \) for a potential \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n) \) and that \( H_0 = P(D) \) satisfies the assumptions of Theorem 4.1. There exists a constant \( c_0 > 0 \) such that if

\[
\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{V(x)}{|x-y|^{n-m}} dx \leq c_0,
\]

then estimate

\[
\|(I + tH)^{-1}\|_{p \to q} \leq C_{p,q} t^{-\frac{m}{2} \left( \frac{1}{p} - \frac{1}{q} \right)}, \quad t > 0
\]

holds for all pairs \( (p,q) \) such that \( 0 \leq 1/p - 1/q < m/n \). Moreover, for \( k \in \mathbb{N} \) large enough

\[
\|(I + tH)^{-k}\|_{p \to q} \leq C_{k,p,q} t^{-\frac{m}{2} \left( \frac{1}{p} - \frac{1}{q} \right)}, \quad t > 0
\]

hold for all \( 1 \leq p \leq q \leq \infty \).
Proof. We first prove that

\[ (5.18) \quad \|tV(I + tP(D))^{-1}\|_{1\rightarrow 1} \leq \frac{1}{2}. \]

Note that

\[ \|tV(I + tP(D))^{-1}\|_{1\rightarrow 1} \leq \|VP(D)^{-1}\|_{1\rightarrow 1} \|tP(D)(I + tP(D))^{-1}\|_{1\rightarrow 1} \leq C\|VP(D)^{-1}\|_{1\rightarrow 1}. \]

Since the fundamental solution of \( P(D) \) is bounded by \( O(|x|^{m-n}) \) so

\[ \|VP(D)^{-1}\|_{1\rightarrow 1} \leq C \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(x)|}{|x - y|^{n-m}} dx. \]

Thus when \( c_0 \) is enough small then (5.18) holds.

Next we prove estimates (5.16). Taking adjoint and using interpolation we reduce the proof to the case \( p = 1 \). By (5.18) and Neumann series argument

\[ \left\| \left( I + tV(I + tP(D)^{-1}) \right)^{-1} \right\|_{1\rightarrow 1} \leq 2. \]

Writing the standard perturbation formula in our notation yields

\[ (I + tH)^{-1} = (I + tP(D)^{-1})(I + tV(I + tP(D)^{-1}))^{-1} \]

for all \( t > 0 \). Hence if \( 0 \leq 1 - 1/q < m/n \), then it follows from the Sobolev embedding that

\[ \|(I + tH)^{-1}\|_{1\rightarrow q} \leq 2 \|I + tP(D)^{-1}\|_{1\rightarrow q} \leq Ct^{-\frac{m}{n}(1 - \frac{1}{q})}. \]

To verify estimate (5.17) we note that by (5.16) for any \( k \in \mathbb{N} \),

\[ \|(I + tH)^{-k}\|_{p\rightarrow p} \leq C \]

for all \( 1 \leq p \leq \infty \). It follows from an interpolation argument that it suffices to show that for \( k \in \mathbb{N} \) large enough

\[ \|(I + tH)^{-k}\|_{1\rightarrow \infty} \leq Ct^{-\frac{m}{n}}, \quad t > 0. \]

To prove the above relation we iterate \( k \)-times the resolvent \((I + tH)^{-1}\). Choose \( k \in \mathbb{N} \) such that

\[ 0 < \frac{1}{k} \leq m/n. \]

Let \( p_1 = 1 \) and for each \( 1 \leq i \leq k \) we define \( p_{i+1} \) by putting \( 1/p_i - 1/p_{i+1} = \frac{1}{k} \). Note that \( p_{k+1} = \infty \). By estimate (5.16)

\[ \|(I + tH)^{-k_0}\|_{1\rightarrow \infty} \leq \prod_{i=1}^{k_0} \|(I + tH)^{-1}\|_{p_i\rightarrow p_{i+1}} \leq Ct^{-\frac{m}{n}}, \quad t > 0. \]

This concludes the proof of Lemma 5.7. \( \square \)

The following result is a consequence of Lemma 5.7.
Theorem 5.8. Assume that $n > m \geq 2$, $H = H_0 + V$ for a potential $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ and that $H_0 = P(D)$ satisfies the assumptions of Theorem 4.1. There exists a small constant $c_0 > 0$ such that if
\begin{equation}
\|V\|_{\frac{n}{m}} + \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{V(x)}{|x - y|^{n-m}} dx \leq c_0,
\end{equation}
then the estimate
\begin{equation}
\|dE_H(\lambda)\|_{p \rightarrow p'} \leq C \lambda^{\frac{n}{m} \left(\frac{1}{p'} - \frac{1}{p}\right)} - 1, \quad \lambda > 0
\end{equation}
holds for all $1 \leq p < \min\left\{\frac{2(n+1)}{n+3}, \frac{n}{m}\right\}$. 

Proof. This is a consequence of Proposition 5.6, Lemma 3.3 and Lemma 5.7.

Remark 5.9. If $m = 2$ and $n \geq 3$ (for example $H = -\Delta + V$), then $\frac{2(n+1)}{n+3} \leq \frac{n}{2}$, hence we obtain almost the optimal range $1 \leq p < \frac{2(n+1)}{n+3}$ for estimate (5.20). The endpoint estimate, $p = \frac{2(n+1)}{n+3}$, can also be verified by the following resolvent estimate:
\begin{equation}
\left\|(Q(D) - z)^{-1}\right\|_{2(n+1)} \leq C |z|^{-\frac{n}{n+3}}, \quad \forall z \in \mathbb{C}^n \setminus \{0\},
\end{equation}
where $Q(D)$ is any second order homogeneous elliptic operator, see e.g. Stein [49, page 370].

6 Applications

As an illustration of our results we will discuss a class of possible applications, which include $m$-th order elliptic operators with some positive potentials and Schrödinger operator with the inverse-square potential.

6.1 Higher order elliptic operators with potentials

In this subsection, we show Hörmander-type spectral multiplier theorem for elliptic $m$-th order operators perturbed by potentials. Our discussion requires the following lemma.

Lemma 6.1. Let $P(D)$ be a positive elliptic $m$-th order homogeneous operator and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ be a potential. Then the semigroup $e^{-\lambda H}$ generated by $H = P(D) + V$ satisfies the $m$-th order Davies-Gaffney estimates, that is, there exist constants $c, C > 0$ such that for all $t > 0$ and all $x, y \in \mathbb{R}^n$,
\begin{equation}
\|P_{B(x,t^{1/m})}e^{-\lambda t}P_{B(y,t^{1/m})}\|_{2 \rightarrow 2} \leq C \exp (-c \left(\frac{|x - y|}{t^{1/m}}\right)^{m/m-1}).
\end{equation}

Proof. The proof of (6.1) is based on the ideas of Barbatis, Davies [5] and Dungey [26]. Consider the set of linear functions $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ of the form $\psi(x) = a \cdot x$, where $a = (a_1, \ldots, a_n) \in \mathbb{S}^{n-1}$. Then for $\lambda \in \mathbb{R}$ we consider the conjugated operator
\begin{equation}
H_{\lambda \psi} = e^{-\lambda \psi}He^{\lambda \psi} = P_{\lambda \psi}(D) + V,
\end{equation}
where $P_{\lambda \psi}(D) = e^{-\lambda \psi}P(D)e^{\lambda \psi} = P(D) - i\lambda a$. Note that $V \geq 0$, then there exists some constant $d_0 > 0$ such that
\begin{equation}
\text{Re}(H_{\lambda \psi}f,f) \geq \text{Re}(P_{\lambda \psi}(D)f,f) \geq -d_0\lambda \|f\|_2^2.
\end{equation}
Let \( f_i = e^{-tH}f \) for \( f \in L^2(\mathbb{R}^n) \). Then
\[
\frac{d}{dt} \|f_i\|_2^2 = -\langle H\phi, f_i \rangle - \langle f_i, H\phi \rangle = -2 \text{Re} \langle H\phi, f_i \rangle \leq 2d_0\lambda^m \|f_i\|_2^2,
\]
which implies that
\[
\|e^{-tH}f\|_2 \leq e^{2d_0\lambda^m t} \|f\|_2.
\]
Note that \( e^{-tH} = e^{-\lambda t} e^{-tH^{\phi}} \). We get that
\[
\|e^{-\lambda t} \exp(-tH) e^{tH} \|_{2 \rightarrow 2} \leq e^{2d_0\lambda^m t}.
\]
Now we consider \( a = (a_1, a_2, a_3) \in S^{n-1} \) such that \( \psi(x) - \psi(y) = |x - y| \). Then
\[
\left\| P_{B(x,t^{1/n})} e^{-tH} P_{B(y,t^{1/n})} \right\|_{2 \rightarrow 2} \leq C e^{\lambda t^2 - 4(1/2 - 1/m)}. \]
Taking infimum over \( \lambda \) in the above inequality, we obtain estimate (6.1).

**Remark 6.2.** Let \( H = P(D) + V \) with \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n) \). If \( m > n \) or \( m = 2 \), then it is well-known that the semigroup \( e^{-tH} \) satisfies the Gaussian estimates \( (\text{GE}_m) \)
\[
|p_t(x,y)| \leq C\tau^{-\frac{m}{2}} \exp \left( -c \left( \frac{|x-y|^m}{t} \right)^\frac{1}{m} \right)
\]
for some \( C, c > 0 \). On the other hand, if \( 4 \leq m \leq n \), then generally, the Gaussian bound of \( e^{-tH} \) may fail to hold. For these results and further details, see [21], [24] and therein references.

We are now able to state some results describing spectral multipliers for \( m \)-th order elliptic operators with positive potentials \( V \) on \( \mathbb{R}^n \). As above, let \( H = P(D) + V \) and \( 0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n) \). If \( n < m \), then by Remark 6.2, the Gaussian estimate (6.3) holds, which immediately implies Davies-Gaffney estimate (6.1) and condition \( (\text{DG}_m) \), see Section 2. Hence it follows from point (i) of Proposition 2.2 that for any \( 1 \leq p < 2 \), the spectral multiplier operator \( F(H) \) is bounded on \( L^q(\mathbb{R}^n) \) for all \( p < q < p' \) if a bounded Borel function \( F : [0, \infty) \rightarrow \mathbb{C} \) satisfies \( \sup_{r > 0} \| \eta \delta F \|_{C^k} < \infty \) for some \( k > n(1/p - 1/2) \) and some non-zero auxiliary function \( \eta \in C_0^\infty(0, \infty) \). In particular, as \( p = 1 \), this exactly corresponds to a spectral multiplier version of the classical Mikhlin theorem. However, for the cases \( n > m \), we need to impose a non-degenerate condition (4.1) on \( P(\xi) \). Now based on estimate (5.20) and Davies-Gaffney estimate (6.1), the following Hörmander type spectral multipliers result for \( H = P(D) + V \) holds.

**Theorem 6.3.** Suppose that \( n > m \geq 2 \), \( H_0 = P(D) \) satisfies the assumptions of Theorem 4.1 and that \( 0 \leq V \in L^\#(\mathbb{R}^n) \). There exists a small constant \( c_0 > 0 \) such that if
\[
\|V\|_m + \sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{V(x)}{|x-y|^{n-m}} dx \leq c_0,
\]
then for any \( 1 \leq p < \min \left( \frac{2(n+1)}{n+3}, \frac{n}{m} \right) \) and any bounded Borel function \( F : [0, \infty) \rightarrow \mathbb{C} \) satisfying \( \sup_{r > 0} \| \eta \delta F \|_{W^{0,2}} < \infty \) for \( \alpha > n(1/p - 1/2) \), the operator \( F(H) \) is bounded on \( L^q(\mathbb{R}^n) \) for all \( p < q < p' \). In addition,
\[
\|F(H)\|_{q \rightarrow q} \leq C_\alpha \sup_{r > 0} \| \eta \delta F \|_{W^{0,2}}.
\]
Proof. Note that \( n(1/p - 1/2) > 1/2 \) for all \( p \) above, hence by Theorem 5.8 and Lemma 6.1, Theorem 6.3 follows from point (ii) of Proposition 2.2. See also [46, Theorem 5.1].

Note that for any \( 1 \leq p \leq 2 \), the function \( (1 - \lambda)^{\delta} \in W^{m(1/p - 1/2),2} \) if \( \alpha > n(1/p - 1/2) - 1/2 \). Hence as a corollary, we can apply Theorem 6.3 to discuss the bounds of Bochner-Riesz means \( S_{R}^{\alpha}(H) \) where \( H = P(D) + V \).

**Corollary 6.4.** Let \( n, m, P(D) \) and \( V \) satisfy the same conditions as Theorem 6.3. Then it follows that for any \( 1 \leq p < \min\left(\frac{2(n+1)}{n+3}, \frac{n}{m}\right) \) and \( \alpha > n(1/p - 1/2) - 1/2 \), Bochner-Riesz means

\[
\sup_{R > 0} \left\| S_{R}^{\alpha}(H) \right\|_{r \rightarrow r} \leq C
\]

uniformly hold for any \( p < r < p' \). In particular, we can take \( r = p \) and \( 1 \leq p \leq \frac{2(n+1)}{n+3} \) if \( m = 2 \) and \( n \geq 3 \).

### 6.2 Schrödinger operator with the inverse-square potential

We consider the spectral estimates for Schrödinger operator \( H = -\Delta + V \) with the inverse square potential, that is \( V(x) = c/|x|^2 \). Fix \( n > 2 \) and assume that \( -(n-2)^2/4 < c \). Note that the potential \( V(x) \) does not satisfy with condition (6.4) even if \( c \) is very small. Hence in this subsection we will study this potentials case. First, define by quadratic form method \( H = -\Delta + V \) on \( L^{2}(\mathbb{R}^n, dx) \). The classical Hardy inequality

\[
-\Delta \geq \frac{(n-2)^2}{4}|x|^{-2},
\]

shows that for all \( c > -(n-2)^2/4 \), the self-adjoint operator \( H \) is non-negative. Set \( p_{c}^{*} = n/\sigma \), \( \sigma = \max((n-2)/2 - \sqrt{(n-2)^2/4 + c}, 0) \). If \( c \geq 0 \), then the semigroup \( \exp(-tH) \) is pointwise bounded by the Gaussian semigroup and hence acts on all \( L^{p} \) spaces with \( 1 \leq p \leq \infty \). If \( c < 0 \), then \( \exp(-tH) \) acts as a uniformly bounded semigroup on \( L^{p}(\mathbb{R}^n) \) for \( p \in (p_{c}^{*})', p_{c}^{*} \) and the range \( (p_{c}^{*})', p_{c}^{*} \) is optimal, see for example [42]. It was proved in [14, Section 10] that \( H \) satisfies restriction estimate

\[
\|dE_{H}(\lambda)\|_{p \rightarrow p'} \leq C\lambda^{\frac{n}{2} - \frac{2n}{n+3} - 1}, \quad \lambda > 0
\]

for all \( p \in (p_{c}^{*})', \frac{2n}{n+3} \). If \( c \geq 0 \), then (6.6) for \( p = (p_{c}^{*})' = 1 \) is included.

Assume that \( n = 3 \). Next we will use the standard perturbation techniques to prove the following result.

**Proposition 6.5.** Suppose that \( H = -\Delta + V \) on \( \mathbb{R}^3 \) and \( V(x) = c/|x|^2 \). Then there exists a constant \( c_0 > 0 \) such that if \( 0 \leq c \leq c_0 \), then estimate

\[
\|dE_{H}(\lambda)\|_{p \rightarrow p'} \leq C\lambda^{\frac{3}{2} - \frac{2}{3} - 1}, \quad \lambda > 0
\]

holds for all \( 1 \leq p \leq 4/3 \).
Proof. Because estimate (6.7) has known for all $p \in [1, 6/5]$ by (6.6) when $n = 3$, so it suffices to prove the spectral estimates for all $p \in [6/5, 4/3]$. We now start by recalling the well-known representation of the free resolvent $R_0(z) = (\Delta - z)^{-1}$

$$R_0(\zeta^2)g(x) = (-\Delta - \zeta^2)^{-1}g(x) = \begin{cases} \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{i|\zeta||x-y|}}{|x-y|} g(y) dy & \text{for } \text{Im} \zeta > 0, \\ \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{e^{-i|\zeta||x-y|}}{|x-y|} g(y) dy & \text{for } \text{Im} \zeta < 0, \end{cases}$$

see e.g. [22]. By elementary computations we obtain that for $z \neq 0$ and $1 < p < 3/2$, it follows from [22, Corollary 14] that

$$\|VR_0(z)g\|_p \leq c \int_{\mathbb{R}^3} \frac{|\Delta^{-1}(|g|)|^p}{|x|^{2p}} dx \leq cK(p) \int_{\mathbb{R}^3} |g|^p dx,$$

where $K(p) = \frac{p^2}{3(2-p)(p-1)}$. Then there exists a constant $c_0 > 0$ such that when $0 \leq c \leq c_0$, we have that $\|VR_0(z)\|_{p \to p} \leq 1/2$ and

$$\sup_{|z| > 0} \| (I + VR_0(z))^{-1} \|_{p \to p} \leq 2$$

for all $1 < p < 3/2$. Next

$$\|R_0(z)\|_{p \to p'} \leq C|z|^{\frac{1}{2} - \frac{1}{p'}}^{-1}, \quad z \in \mathbb{C}^+ \setminus \{0\}$$

for all $6/5 \leq p \leq 4/3$, see e.g. Stein[49, P. 370], so required estimate (6.7) for $6/5 \leq p \leq 4/3$ follows from the perturbation formula (5.2) and Stone’s formula (5.3).

Acknowledgements: We thank P. Chen and A. Hassell for useful discussions. A. Sikora was supported by Australian Research Council Discovery Grant DP 130101302. L. Yan was supported by NNSF of China (Grant No. 11371378). X. Yao was supported by NSFC (Grant No. 11371158), the program for Changjiang Scholars and Innovative Research Team in University (No. IRT13066).

References

[1] S. Agmon, Spectral properties of Schrödinger operators and scattering theory, *Ann. Sc. Norm. Super. Pisa, Cl. Sci.*, IV. Ser. 2 (1975) 151-218. 6

[2] G. Alexopoulos, Spectral multipliers on Lie groups of polynomial growth, *Proc. Amer. Math. Soc.* 120 (1994) 973-979. 2, 4

[3] J.-G. Bak, Sharp estimates for the Bochner-Riesz operator of negative order in $\mathbb{R}^2$, *Proc. Amer. Math. Soc.* 125 (1997) 1977–1986. 2, 4, 28

[4] J.-G. Bak, D. McMichael, D. Oberlin, $L^p - L^q$ estimates off the line of duality, *J. Austrl. Math. Soc. (Series A)* 58 (1995) 154–166. 5
[5] G. Barbatis and E. B. Davies, Sharp bounds on heat kernels of higher order uniformly elliptic operators, *J. Oper. Theory*. 36 (1996), 179-198. 7, 33

[6] J. Bergh and J. L"ofstr"om, *Interpolation spaces*. Springer-Verlag, Berlin- New York, 1976. 18

[7] L. B"orjeson, Estimates for the Bochner-Riesz operator with negative index, *Indiana U. Math. J.* 35 (1986) 225-233. 2, 4, 5, 28

[8] F. Bernicot, L. Grafakos, L. Song and L.X. Yan, The bilinear Bochner-Riesz problem, to appear in *J. Anal. Math*. 20

[9] S. Blunck, A H"ormander-type spectral multiplier theorem for operators without heat kernel. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5), (2003), 449–459. 2, 4, 9

[10] S. Blunck and P.C. Kunstmann, Weighted norm estimates and maximal regularity. *Adv. Differential Equations* 7 (2002), 1513–1532. 3

[11] S. Blunck and P.C. Kunstmann, Genealized Gaussian estimates and the Legendre transform. *J. Oper. Theory* 53 (2005), 351–365. 3, 8

[12] J. Bruna, A. Nagel and S. Wainger, Convex hypersurfaces and Fourier transforms, *Ann. of Math*. 127 (1988) 333–365. 23

[13] A. Carbery, F. Soria, Almost everywhere convergence of Fourier integrals for functions in Sobolev spaces, *Rev. Mat. Iberoamericana* 4 (1988) 319–337. 4, 28

[14] P. Chen, E.M. Ouhabaz, A. Sikora and L.X. Yan, Restriction estimates, sharp spectral multipliers and endpoint estimates for Bochner-Riesz means. To appear in *J. Anal. Math*. 2, 4, 5, 9, 35

[15] M. Christ, *L^p* bounds for spectral multipliers on nilpotent groups. *Trans. Amer. Math. Soc*. 328 (1991), 73–81. 2

[16] M. Christ and C.D. Sogge, The weak type *L^1* convergence of eigenfunction expansions for pseudo-differential operators. *Invent. Math.* 94 (1988), 421–453. 2

[17] R. Coifman and G. Weiss, Analyse harmonique non-commutative sur certains espaces homogènes. *Lecture Notes in Math*. 242. Springer, Berlin-New York, 1971. 3

[18] T. Coulhon and A. Sikora, Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem. *Proc. Lond. Math. Soc*. 96 (2008), 507–544. 3

[19] M. Cowling and A. Sikora, A spectral multiplier theorem for a sublaplacian on SU(2). *Math. Z*. 238 (2001), no. 1, 1–36. 2

[20] E.B. Davies, Heat Kernels and Spectral Theory, *Cambridge Tracts in Math*. 92, Cambridge Univ. Press, 1989. 4

[21] E.B. Davies, Limits on *L^p* regularity of self-adjoint elliptic operators. *J. Diff. Equa*. 135 (1997), 83–102. 3, 7, 34
[22] E.B. Davies and A. Hinz, Explicit constants for Rellich inequalities in $L^p(\Omega)$. *Math. Z.* **227** (1998), 511–523. 36

[23] L. De Cari, Unique continuation for a class of higher order elliptic operators. *Pacific J. Math.* **179** (1997), 1–10. 2

[24] Q. Deng, Y. Ding and X.H. Yao, Gaussian bound of heat kernel for elliptic differential operator with the potentials of Kato type. *J. Funct. Anal.* **266** (2014), 5377–5397. 7, 34

[25] Y. Ding and X.H. Yao, $H^p$-$H^q$ estimates for dispersive equations and related applications. *J. Funct. Anal.* **257** (2009), 2067–2087.

[26] N. Dungey, Sharp constants in higher-order heat kernel bounds. *Bull. Austral. Math. Soc.* **61** (2000), 189–200. 33

[27] X. T. Duong, E.M. Ouhabaz and A. Sikora, Plancherel-type estimates and sharp spectral multipliers. *J. Funct. Anal.* **196** (2002), 443–485. 2, 4, 9, 14

[28] J.E. Galé and T. Pytlik, Functional calculus for infinitesimal generators of holomorphic semigroups. *J. Funct. Anal.* **150** (1997), 307–355. 14

[29] M. Goldberg and W. Schlag, A limiting absorption principle for the three-dimensional Schrödinger equation with $L^p$ potentials. *Int. Math. Res. Not.* 2004, no. 75, 4049–4071. 2

[30] A. Greenleaf, Principal curvature and harmonic analysis. *Indiana Univ. Math. J.* **30** (1981), 519–537 6

[31] A. Grigor’ya, *Heat Kernel and Analysis on Manifolds*, AMS/IP Studies in Advanced Mathematics, **47**, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009. 4

[32] C. Guillarmou and A. Hassell, Uniform Sobolev estimates for non-trapping metric. *J. Inst. Math. Jussieu* **13** (2014), 599-632. 2, 28

[33] C. Guillarmou, A. Hassell and A. Sikora, Restriction and spectral multiplier theorems on asymptotically conic manifolds. *Anal. PDE.* (2013), 893–950. 2, 4

[34] S. Gutiérrez, Non trivial $L^q$ solutions to the Ginzburg-Landau equation. *Math. Ann.* **328** (2004), 1–25. 4, 28

[35] W. Hebisch, A multiplier theorem for Schrödinger operators. *Colloq. Math.* **60/61** (1990) 659–664. 2

[36] L. Hörmander, *The analysis of linear partial differential operators*, I, II. Springer-Verlag, Berlin, 1983. 2, 13, 14, 27, 29

[37] L. Hörmander, Estimates for translation invariant operators in $L^p$ spaces. *Acta Math.* **104** (1960), 93–140. 2

[38] T. Kato, *Perturbation Theory for Linear Operators*, 2nd edition, Springer-Verlag, 1980. 30
[39] C.E. Kenig, A. Ruiz and C.D. Sogge, Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators, *Duke Math. J.* 55 (1987), 329–347. 1, 2, 6, 28

[40] H. Koch and D. Tataru, Carleman estimates and absence of embedded eigenvalues. *Commun. Math. Phys.*, 267 (2006), 419-449. 29

[41] P.C. Kunstmann and M. Uhl, Spectral multiplier theorems of Hörmander type on Hardy and Lebesgue spaces, available at arXiv:1209.0358. 2, 4

[42] V. Liskevich, Z. Sobol and H. Vogt, On the $L^p$ theory of $C^0$-semigroups associated with second-order elliptic operators II. *J. Funct. Anal.* 193 (2002), 55–76. 3, 35

[43] S.G. Mikhlin, *Multidimensional singular integrals and integral equations*, Pergamon Press, Oxford, 1965 (translated from the Russian by W. J. A. Whyte. Translation edited by I.N. Sneddon). 2

[44] E.M. Ouhabaz, *Analysis of heat equations on domains*, London Math. Soc. Monographs, Vol. 31, Princeton Univ. Press (2005). 2, 4

[45] G. Schreieck and J. Voigt, Stability of the $L_p$-spectrum of generalized Schrödinger operators with form small negative part of the potential. In *Function Analysis (Essen, 1991)*, 95-105. Lecture Notes in Pure and Appl. Math., 150. Dekker, New York, 1994. 3

[46] A. Sikora, L.X. Yan and X.H. Yao, Sharp spectral multipliers for operators satisfying generalized Gaussian estimates. *J. Funct. Anal.* 266 (2014), 368–409. 2, 4, 9, 10, 18, 19, 20, 35

[47] A. Seeger and C.D. Sogge, On the boundedness of functions of (pseudo)-differential operators on compact manifolds. *Duke Math.* 59 (1989), 709–736. 2

[48] C.D. Sogge: *Fourier integrals in classical analysis*, Cambridge University Press, 1993. 6, 26

[49] E.M. Stein, Interpolation of linear operators, *Trans. Amer. Math. Soc.* 83 (1956), 482–492. 22, 33, 36

[50] E.M. Stein, *Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, (1993). 2, 6, 23, 26

[51] T. Tao, Some recent progress on the restriction conjecture. Fourier analysis and convexity, 217C243, *Appl. Numer. Harmon. Anal.*, Birkhäuser Boston, Boston, MA, 2004. 2

[52] P. Tomas, A restriction theorem for the Fourier transform. *Bull. Amer. Math. Soc.* 81 (1975), 477–478. 4

[53] Q. Zheng and X.H. Yao, Higher-order Kato class potential for Schrödinger operators, *Bull. London Math. Soc.*, 41 (2009), 293-301. 2