Insights from Simple Stochastic Models of Earth’s Long-Term Carbon Cycle and Climate

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Key Points:

• We explore models of randomness in the long-term climate and carbon cycle
• Symmetric noise in the CO₂ concentration is instructive, but incompatible with the snowball record
• Modeling noise in the CO₂ outgassing rate, instead, is more flexible and can be qualitatively reconciled with the snowballs

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Abstract
Over its multibillion-year history, Earth has exhibited a wide range of climates. Its history ranges from snowball episodes where the surface was mostly or entirely covered by ice to periods much warmer than today, where the cryosphere was virtually absent. Our understanding of greenhouse gas evolution over this long history, specifically carbon dioxide, is mainly informed by deterministic and mechanistic models. However, the complexity of the carbon cycle and its uncertainty over time motivates study of non-deterministic models, where key elements of the cycle are represented by inherently stochastic processes. By doing so, we can learn what models of variability are compatible with Earth’s climate record instead of how exactly this variability is produced, working backward. Here we discuss two simple stochastic models of variability in the carbon system and how they relate to Earth’s snowball record in particular. The first, which is the most simple and represents CO₂ concentration directly as a stochastic process, is instructive and perhaps intuitive, but is incompatible with this record. The second, which separates carbon source from sink and represents CO₂ outgassing as a stochastic process instead of concentration, is more flexible. When outgassing fluctuates over longer periods, as opposed to brief and rapid excursions from a mean state, this model is more compatible with the snowball record. The contrast between these models illustrates what kind of variability may have characterized the long-term carbon cycle.

Plain Language Summary
The Earth’s long-term climate record is complicated and variable. At some points, our planet has been extremely cold and almost entirely covered in ice (snowball periods). At others, the climate was much warmer than today and the surface was ice-free. These changes may have many overlapping causes and are quite complex. In light of this, we use simple models to understand how randomness in the carbon cycle affects the long-term climate. In one model, the atmospheric carbon dioxide concentration is treated as a randomly fluctuating quantity. We can learn from this model, but it is at odds with the timing of Earth’s snowball periods. In a second model, the CO₂ outgassing rate is represented by a randomly fluctuating quantity instead of the concentration itself. These models offer some qualitative lessons about the kind of randomness that is compatible with Earth’s snowball episodes.

1 Introduction
Geological and geochemical evidence indicates that Earth’s climate has varied dramatically over at least the past 2.5 billion years. Three “Snowball Earth” periods have been identified, when most or all of the planet’s surface was covered in ice (Hoffman & Schrag, 2002; R. Pierrehumbert et al., 2011). The first of these episodes occurred at the beginning of the Paleoproterozoic era, approximately 2.5 Gya (Evans et al., 1997; Kirschvink et al., 2000). The next two occurred during the Cryogenian period of the Neoproterozoic era, about 700 Mya and with inception times roughly 50 Ma apart (Hoffman et al., 1998; Rooney et al., 2015; Prave et al., 2016). At other points in time, like the Cretaceous and the early Eocene “equable” climates, the Earth was warm and ice-free at high latitudes (Berner, 1990; Greenwood & Wing, 1995).

What caused snowball climates at some points, but warm climates at others? Over geologic time, the atmospheric carbon dioxide (CO₂) concentration, and therefore the climate, is thought to be regulated by volcanic outgassing and the silicate weathering feedback (Urey, 1952; Walker et al., 1981a; Marshall et al., 1988; Berner & Lasaga, 1989). In this picture, when atmospheric carbon dioxide levels change, the resulting change in temperature and runoff modifies the global cation flux to the ocean and the rate of marine carbonate burial, counteracting the initial CO₂ perturbation. Together, silicate weathering and marine carbonate burial sequester CO₂ from the ocean and atmosphere un-
Earth’s temperature and CO₂ concentration are not well known deep into the planet’s history. Although the Sun’s luminosity has slowly increased since the birth of the solar system, there is no evidence for secular warming of the climate (Feulner, 2012). To compensate for the increase in absorbed solar radiation, the mean CO₂ concentration has very likely decreased over time. This decrease has not been smooth, but rather accompanied by significant variability, the cause of which may be multifactorial and is still an active area of research (Franks et al., 2014; Montañez et al., 2016; Lenardic et al., 2016; Macdonald et al., 2019; Park et al., 2021; Baum et al., 2022).

Research generally attempts to understand variability in atmospheric CO₂ concentration in a deterministic fashion, by investigating specific components of the carbon cycle and their potential effect on climate. For example, we can attempt to constrain the independent effects of continental configuration, topography, biochemistry, ocean circulation, and a wide range of other processes. Here we take a different approach, investigating the effects of explicitly random or stochastic processes in the long term carbon cycle. This approach views the climate system as an inherently stochastic, complex system and attempts to understand what models of randomness agree with our sparse and incomplete record of Earth’s climate on the longest timescales. Specifically, we anchor our modeling and discussion to the aforementioned snowball earth episodes, as they represent very important excursions from the mean state. Our conception of long-term climate variability must be compatible with the snowball record.

That is not to say that stochastic models are absent from the climate literature. Not at all. There is a long history of stochastic climate modeling touching virtually all components of the earth system (Hasselmann, 1976; Imkeller & Von Storch, 2001; Imkeller & Monahan, 2002; Franzke et al., 2015). However, they generally focus on shorter-term dynamics and specific phenomena like the El Niño-Southern Oscillation. In this paper, we set the complexity of the three-dimensional, chaotic climate system aside and examine the consequences of simple stochastic models of the carbon cycle over the past ~2 Gyr. We do not suggest that our modeling is a precise representation of Earth’s long-term carbon record, but use it to understand what kinds of variability (what models) are compatible with the snowball record.

At the beginning of Section 2, we explain the primary physical equations and assumptions governing both of these models. In Section 2.1, we interrogate the stochastic model recently published by Wordsworth (2021) and discuss its compatibility with Earth’s record of snowball periods. In 2.2 we describe an alternative model with a deterministic weathering sink but stochastic CO₂ outgassing. In Section 3 we explain how we solve the model equations numerically and produce a large ensemble of these solutions. In Section 4 we present the results of these simulations, compare with the model of Section 2.1, and discuss compatibility with Earth’s snowball record. Finally, in Section 5, we offer some concluding remarks and discuss possible avenues of future research.
where \( C \) is the system’s heat capacity, \( T \) is the global mean surface temperature, \( t \) is time, \( F \) is the time-dependent incident solar flux, \( \alpha \) is the planetary albedo, OLR is the temperature-dependent and \( \text{CO}_2 \)-dependent outgoing longwave radiation, and \( f \) is the atmospheric \( \text{CO}_2 \) concentration in ppm. We set the albedo to a constant 30%.

In reality, due do the response of clouds and ice to surface temperature, albedo is a complex function of temperature. However, the goal of this model is to simulate the likelihood of different temperatures over time, not represent the full-complexity of the ice-albedo feedback. Instead, we assume a hypothetical snowball threshold temperature of \( T_{\text{snow}} = 280 \text{ K} \) (R. Pierrehumbert et al., 2011) where the ice-albedo feedback runs away. The precise value of this threshold does not affect our conclusions.

The solar flux increases over time according to the standard approximation (Gough, 1981),

\[
F(t) = F_0 \left[ 1 + \frac{2}{5} \left( 1 - t/t_0 \right) \right]^{-1},
\]

where \( F_0 \) is the modern value of 1366 W/m\(^2\) and \( t_0 \) is 4.5 Gyr.

The heat capacity of the ocean-atmosphere system is such that the temperature approaches equilibrium over \( \sim 10^3 \text{ yr} \), much more rapidly than the \( 10^6-10^9 \text{ yr} \) timescales of interest in our simulations. Accordingly, we assume instantaneous temperature equilibrium and set \( \dot{T} \) to zero. We also linearize OLR around Earth’s preindustrial surface temperature and \( \text{CO}_2 \) concentration of \( T_e = 288 \text{ K} \) and \( f_0 = 285 \text{ ppm} \), respectively.

\[
\text{OLR} = \text{OLR}_0 + a (T - T_e) - b \log \left( \frac{f}{f_0} \right),
\]

where \( a = 2 \text{ W/m}^2/\text{K} \) (Abbot, 2016) and \( b = 5.35 \text{ W/m}^2 \) (Myhre et al., 1998). To balance Equation 1 at preindustrial conditions and \( t = 4.5 \text{ Gyr} \), \( \text{OLR}_0 \) must be equal to \( F_0 (1 - A)/4 \). These assumptions, plugged into Equation 1, yield

\[
0 = \frac{F(t)}{4} \left[ 1 - a \right] - \text{OLR}_0 - a (T - T_e) + b \log \left( \frac{f}{f_0} \right)
\]

which can be solved directly for temperature as a function of time and \( \text{CO}_2 \) concentration,

\[
T(t, f) = T_e + \frac{1}{a} \left[ \frac{F(t)}{4} \left( 1 - \alpha \right) - \text{OLR}_0 + b \log \left( \frac{f}{f_0} \right) \right].
\]

This is the governing equation of all our subsequent modeling. We note that the linearization in Equation 3 is a reasonable approximation when \( f \) is within \( 10-10^5 \text{ ppm} \) and \( T \) is within about 280-290 K. At higher temperatures, the value of \( b \) increases. However, we are primarily interested in temperatures below the equilibrium value of 288 K, so moderate error at high-temperatures is acceptable.

### 2.1 Stochastic \( \text{CO}_2 \) Concentration

Equation 5 defines temperature in terms of time \( t \) (capturing the brightening sun) and the carbon dioxide concentration \( f \). To introduce randomness into this system, one possibility is to model \( f \) as a stochastic process. We analyze and discuss this approach here, along with its primary assumptions and implications, before continuing with the development of our new model equations.

Recently, Wordsworth (2021) used an Ornstein-Uhlenbeck process (Jacobs, 2010; Dobrow, 2016) to represent randomness in \( f \). This approach models \( f \) with Gaussian noise that relaxes to a prescribed value \( \chi \). Previous modeling uses a nondimensional representation, \( y = f/f_0 \), and the model equation is

\[
dy = \frac{1}{\tau} \left[ y - \frac{\chi(t)}{f_0} \right] dt + \sigma dB,
\]
Figure 1. A single realization of the stochastic CO$_2$ concentration model described by Equations 5 and 6, over 250 Myr, with $\tau = 3$ Myr and $\sigma = 10$. Panels (a) and (b) show the CO$_2$ concentration over time, with the equilibrium value $\chi$ drawn in gray. Panel (a) shows the entire time series and panel (b) shows the same simulation, but zoomed into the $\sim$5 Myr surrounding the temperature minimum. Panels (c) and (d) show temperature evolution for the same simulation, again zoomed in for panel (d), with temperatures below 280 K highlighted in blue. Panels (b) and (d), in particular, highlight the exquisite temperature sensitivity when CO$_2$ concentrations are low. A relatively small change in $f$ near the minimum between 13 and 11 Mya produces a very dramatic drop in temperature.

where $\tau$ is the relaxation timescale, $\chi$ is the time-dependent value of $f$ that would achieve perfect temperature equilibrium (at $T_e$), $\sigma$ scales the noise, and $dB$ is a Weiner process (Gaussian or Brownian noise). In this case, $\tau$ is analogous to the timescale required for chemical weathering to restore temperature equilibrium.

In their model, randomness occurs directly in the CO$_2$ concentration and $f$ is the sole prognostic variable. The stochastic differential equation above is integrated numerically over $10^8$-$10^9$ yr and the resulting sequence of discrete $f$ values determines the simulated temperature history via Equation 5. As time proceeds, $f$ varies randomly but is restored to equilibrium over a timescale of $\tau \approx 3$ Ma, which is short compared to the integration timescale. Figure 1 shows one example simulation using this equation, where $\sigma$ was chosen for a reasonable chance that the temperature would reach $T_{\text{snow}}$.

Relaxation of $f$ toward $\chi$ happens considerably faster than the slow, insolation driven drift in $\chi$ itself. As a result, over many random realizations of this model, the CO$_2$ concentration is approximately normally distributed around $\chi$ at all points in time.

$$f \sim \mathcal{N}(\chi(t), \sigma^2),$$

where $\mathcal{N}$ is the normal distribution with mean of $\chi(t)$ and variance of $\sigma^2$. It is relatively straightforward to examine the resulting temperature probabilities over time, using Equation 5.

The top row of Figure 2 shows distributions in the CO$_2$ concentration $f$ at three different times over the most recent 500 Mya. Earlier in time, the sun is fainter so a higher $\chi$ value is required for a mean temperature of 288 K. At later times, the brighter sun results in lower $\chi$ values, shifting the distribution down. The bottom row of this figure shows the same distributions, but plotted against the corresponding temperatures. Because of
In the top row, normal distributions in the atmospheric CO\textsubscript{2} concentration ($f$), for several $\sigma$ values, at three different snapshots in time. The $\chi$ values are the distribution means required to produce a mean temperature of $T_e = 288$ K using Equation 5. The bottom row shows the exact same distributions, but plotted in temperature space instead. These plots show how, even though the mean temperature does not change (by design), the space of probable temperatures widens over time and cold temperatures become strongly favored. The logarithmic relationship between $T$ and $f$ produces this effect. The hypothetical snowball temperature of 280 K becomes dramatically more likely at later times.

Figure 3 shows exactly how much more likely cold excursions become at later times in this model: orders of magnitude. The left panel shows the cumulative probability of the CO\textsubscript{2} concentration required to achieve a hypothetical snowball temperature of 280 K, for several values of $\sigma$, over time. Note the logarithmic vertical axis. Beyond the most recent periods of time, a snowball is effectively impossible. The right panel shows the temperature corresponding to the first percentile of the $f$ distributions over time, showing a precipitous drop in this temperature as time proceeds.

To summarize, this model represents randomness in the climate system by making the CO\textsubscript{2} concentration approximately equivalent to a normally distributed random variable with a mean value that drifts over time to maintain stable temperature. Outside the most recent period, stochastically induced cold temperatures are extremely low probability—effectively prohibited. For higher values of $\sigma$, as Wordsworth (2021) showed, the same transition occurs as time proceeds, but it simply occurs at earlier times.

However, it is worth considering whether randomness in the climate system is well represented by Gaussian noise in the CO\textsubscript{2} concentration directly. First, the principle results of this model strongly contradict Earth’s geologic record, which indicates snowball
Figure 3. On the left, the cumulative probability of snowball temperatures for different values of $\sigma$ over time. Each line represents the probability that random variation in $f$, as defined by Equation 6, would produce $T \leq 280$ K. As expected from Figure 2, the snowball probability plummets at earlier times. On the right, the temperatures corresponding to the first percentile of the $f$ distributions. Although the first percentile is an arbitrary metric, it illustrates the very strong bias against cold temperatures at earlier times.

Episodes occurred roughly 2.5 Gya and 700 Mya. To be fair, Wordsworth (2021) focuses mainly on application to exoplanets, but in that case assuming Earth-like weathering sinks and Sun-like stellar properties considerably limits the model’s applicability.

Second, it is questionable to assume that CO$_2$ concentrations are well represented by symmetric noise. To first order, atmospheric CO$_2$ is controlled over long periods by volcanic outgassing (source) and chemical weathering (sink). It is intuitive to assume that volcanic outgassing is noisy, as it is comprised of eruptions of various magnitudes and timescales. This justifies a noisy, non-negative, source of carbon dioxide. However, it is not clear why there should be a proportionally noisy sink, as well.

When CO$_2$ concentration exceeds the equilibrium value ($\chi$) it is relaxed down using a weathering timescale $\tau$. This relaxation corresponds to the physical process of chemical weathering and carbon burial. However, because the noise in $f$ is symmetric, when the concentration drops below $\chi$, it is relaxed up over the same timescale $\tau$. It is not clear what physical process this corresponds to, awkwardly implying that the volcanic outgassing is a simple function of the CO$_2$ concentration.

This model yields conceptual insights. It demonstrates the basic point that any change in atmospheric CO$_2$ concentration produces a larger temperature response when the initial concentration is lower. This response is strongly skewed toward cold temperatures because of the temperature’s logarithmic dependence on CO$_2$. Figure 2 demonstrates these effects. However, this model contradicts Earth’s snowball record and may be of limited relevance to exoplanets because of its strong assumptions about Earth-like weathering behavior and boundary conditions. It also makes no explicit distinction between weathering and outgassing, limiting its physical interpretability.
2.2 Stochastic CO$_2$ Outgassing

We develop a model with a more physically consistent representation of long-term CO$_2$ outgassing and removal. In our model, the temperature is still driven by CO$_2$ concentration and the secular brightening of the sun defined by Equation 5. However, in this case the CO$_2$ concentration is not a stochastic variable itself, but is governed by the balance between deterministic weathering (sink) and stochastic volcanic outgassing (source).

\[ \dot{C}(t) = V(t) - W(C), \]  

(8)

where $C$ is the total ocean atmosphere-ocean reservoir of CO$_2$, $V$ is the volcanic CO$_2$ outgassing rate, and $W$ is the rate of CO$_2$ sequestration via silicate weathering. Weathering is prescribed by the traditional WHAK formulation (Walker et al., 1981b), where the weathering rate is exponentially dependent on temperature changes,

\[ W(T) = k \exp \left( \frac{T - T_{e}}{T_{s}} \right). \]  

(9)

Here, $k$ is a calibration constant and $T_{s}$ is the exponential scaling factor. Note that we exclude direct dependence on carbon dioxide concentration because, with it, weathering can no longer be calibrated to the same equilibrium temperature at different solar forcing (R. T. Pierrehumbert, 2010).

To evaluate the weathering equation, we convert the total carbon $C$ to temperature $T$ by first converting to concentration $f$ in ppm. To do this, we use the simple ocean-atmosphere partitioning model of Mills et al. (2011),

\[ p = 0.78 \frac{C}{C + h}, \]  

(10)

where $p$ is the CO$_2$ partial pressure and $h$ is a constant $2.33 \times 10^8$ Tmole CO$_2$. From there, we assume a background partial pressure of approximately 1 bar to compute the CO$_2$ concentration $f$, which is used to evaluate Equation 5. Finally, the resulting temperature is used to compute the weathering and sequestration rate.

The outgassing rate $V$ fluctuates stochastically and is defined by an Ornstein-Uhlenbeck process,

\[ dV = \frac{1}{\tau} (\mu - V) \, dt + \sigma dB, \]  

(11)

where $\tau$ is a prescribed relaxation time scale, $\mu$ is a prescribed mean outgassing rate, $\sigma$ scales the noise term, and $dB$ is a Weiner process (Gaussian or Brownian noise). Note that this equation and its parameters are distinct from Equation 6. Here, $\tau$ no longer represents a weathering time scale because this stochastic process represents outgassing, not CO$_2$ concentration directly.

All together, our model consists of two differential equations. The first is a straightforward definition of how total ocean-atmosphere carbon changes in response to the balance of weathering and outgassing. The second defines stochastic variation in the CO$_2$ outgassing rate.

\[ \dot{C} = V(t) - W(C) \]  

(12)

\[ \dot{V} = \frac{1}{\tau} (\mu - V) + \sigma B(t) \]  

(13)

The difference between this model and the one discussed in Section 2.1 is that, here, neither the carbon reservoir $C$ nor the CO$_2$ concentration $f$ are themselves stochastic processes. The carbon sink and source terms are separate. Weathering is deterministic and depends implicitly on temperature, as defined in Equation 9. The outgassing rate $V$ is stochastic and modeled with an Ornstein-Uhlenbeck process. In contrast to the model in Section 2.1, these equations explicitly represent imbalances in the carbon system that drive long-term temperature change.
### Table 1. Static model parameters with values and units

| Parameter | Description                                | Value/Range | Units    |
|-----------|--------------------------------------------|-------------|----------|
| $f_0$     | preindustrial CO$_2$ concentration          | 285         | ppm      |
| $T_e$     | preindustrial temperature                   | 288         | K        |
| $\alpha$  | planetary albedo                            | 0.3         | dimensionless |
| $\text{OLR}_0$ | equilibrium outgoing longwave radiation    | 239.05      | W/m$^2$  |
| $h$       | ocean-atmosphere partitioning constant      | $2.33 \times 10^8$ | Tmole   |
| $\mu$     | mean volcanic outgassing rate               | 7           | Tmole/yr |
| $t_1$     | simulation start time                       | 2.5         | Gya      |
| $t_2$     | simulation end time                         | 0           | Gya      |
| $t_0$     | temporal scaling of insolation function     | 4.5         | Gya      |
| $t_{\text{spinup}}$ | model spinup duration                   | 0.5         | Gya      |
| $T_s$     | weathering temperature scaling              | 11.1        | K        |
| $T_{\text{snow}}$ | snowball threshold temperature             | 280         | K        |
| $k$       | weathering calibration constant             | 7           | Tmole/yr |
| $a$       | OLR temperature scaling                     | 2           | W/m$^2$/K |
| $b$       | OLR log $f$ scaling                        | 5           | W/m$^2$  |
| $F_0$     | modern solar insolation                     | 1366        | W/m$^2$  |
| $\tau$    | outgassing relaxation timescale             | $10^5 - 10^9$ | yr      |
| $\sigma$  | outgassing deviation/variability            | $10^{-5} - 10^{-2}$ | Tmole/yr |

### 3 Simulations

We integrate the system of Equations 12 and 13 using the simple Euler-Maruyama method over a range of different parameters and analyze the statistics of the ensemble. We vary $\tau$ and $\sigma$ over discrete values between $10^5$-$10^9$ yr and $10^{-5}$-$10^{-2}$ Tmole/yr, respectively, performing many simulations with each available combination. We execute simulations in two stages. First, we perform a relatively small number of simulations over each combination of $\tau$ and $\sigma$. Second, we identify which combinations yield simulations with appreciable, but not unrealistically high, temperature variability and perform a much larger number of simulations using those combinations. This is a simple strategy to avoid wasting computation on inconsequential $(\tau, \sigma)$ pairs. All told, the ensemble is comprised of 27,551,040 unique simulations.

Each simulation begins at $t = 2$ Gya, when the equilibrium CO$_2$ concentration is slightly above $10^5$ ppm, and ends at 0 Gya (present day). Each simulation is also taken through a spin-up period of 0.5 Gya to avoid starting them at the exact equilibrium temperature, then integrated using 1 million time steps. To prevent evaluating the logarithm of a non-positive number, $V$ is restricted to be above zero and $C$ is restricted to be above a very small fraction of its equilibrium value at modern insolation. These restrictions have virtually no impact on our conclusions and only serve to prevent computation errors with parameter combinations that produce unrealistically high variability. For each simulation, we store the values of $C$, $V$, $T$, and $W$ at 17 equally spaced points in time. We also record the maximum and minimum value of each quantity along with the precise time of the temperature extrema. Table 1 consolidates all of the model parameters along with their values and units.

### 4 Results and Discussion

For orientation, Figure 4 shows the results of a single simulation with $\tau = 30 \times 10^6$ yr and $\sigma = 2 \times 10^{-4}$ Tmole/yr. The volcanic outgassing rate, which is a realization of the Ornstein-Uhlenbeck process defined in Equation 13, is shown on the top in
red. The total atmosphere-ocean carbon stock is shown on a log scale just below in blue. These are the prognostic variables of the model. The CO\textsubscript{2} concentration, temperature, and weathering rate are each derived from \( C \) as described in Section 2.

Figure 5 shows more realizations of the stochastic outgassing rate for different combinations of \( \tau \) and \( \sigma \), stacked on top of each other. When \( \tau \) is small, the outgassing rate quickly relaxes to the mean and the value of \( \sigma \) must be relatively high to produce random fluctuations of appreciable magnitude. In this regime, changes in the outgassing rate are dominated by short-term, rapid excursions. In an intermediate regime, with \( \tau = 30 \text{ Myr} \), longer-term fluctuations begin to appear because the relaxation time scale is longer, but rapid excursions are still evident. Finally, with \( \tau = 300 \text{ Myr} \), the outgassing rate is dominated by longer period drift with occasional and smaller magnitude short-term shocks.

Figure 6 shows the principle results of our ensemble. It shows the first temperature percentile, over time, for different values of \( \tau \). The first percentile is a somewhat arbitrary choice. However, the same results are evident for whatever small percentile is chosen. Smaller values of \( \tau \) produce more rapid fluctuations in the outgassing rate and the likelihood of cold excursions reaching the snowball temperature increases markedly with time. With higher values of \( \tau \), the outgassing history transitions to long-term variability, flattening the first temperature percentile curves. When \( \tau \) is small, snowballs are considerably more likely at later times. When \( \tau \) is large, the outgassing rate changes more slowly and snowball likelihood is more uniform over time.

Comparing the leftmost panel of Figure 6 to Figure 3 is revealing. Low values of \( \tau \) in Figure 6 recover similar results to Wordsworth (2021), who found that the likelihood of snowball events dramatically increases over time along with the secular brightening of the Sun (as explained in Section 2.1). For larger values of \( \tau \), which permit longer-duration excursions in \( V \) (Figure 5), the variation of the 1st percentile temperature over time is much smaller. This is more consistent with the Earth’s geological record of snowball episodes, which appear around 2.5 Ga, 720 Ma, and 560 Ma.

The results of this simple model suggest some constraints on the variability of volcanic outgassing over Earth’s history. For temperatures to generally remain above \( \sim 280 \text{ K} \), yet not to preclude early snowball events, \( \sigma \) may have been fairly small with a larger value of \( \tau \). Qualitatively speaking, variability in the carbon cycle and atmospheric \( \text{CO}_2 \) must have considerable long-term drift, not simply short-term excursions. This is because a model with only short-term variability exhibits a very dramatic preference for snowball events later in Earth’s history, contradicting the geologic record.

5 Conclusion

In this paper, we discuss simple models of carbon variability deep into Earth’s history. One model treats the atmospheric \( \text{CO}_2 \) concentration directly as a stochastic variable. While instructive, this model is limited in its flexibility and can only produce results that are quite clearly not compatible with the Earth’s history of snowball events. Another model, introduced here, treats carbon sources and sinks independently and considers the effect of randomly varying \( \text{CO}_2 \) outgassing with deterministic weathering. This model is slightly more complex, allowing for a range of different behavior that can be compared with the first model and with the snowball record. When outgassing varies over longer periods, specifically with a relaxation time scale of roughly \( \tau > 100 \text{ Myr} \), snowballs are possible early in Earth history, which is more compatible with observations.

Variability in the carbon cycle has often been characterized as either stochastic or deterministic. The usual conceptual division is between rapid, unpredictable volcanic forcing (stochastic) and a slow change in the background “weatherability” of the continental land masses. Although this is sometimes a useful construct, it’s important to remember that both of these processes can, and probably do, influence the climate simultane-
Figure 4. An example simulation of the system defined by Equations 12 and 13. In panel (a), the stochastic outgassing rate $V$. In panel (b), the total atmosphere-ocean CO$_2$ reservoir $C$. These are the prognostic model variables. Each of the remaining quantities is panels (c-e) is derived from $C$. The gray line in each panel shows the equilibrium value or the mean value appropriate for each quantity. For the outgassing and weathering in panels (a) and (e) it shows the mean value of 7 Tmole/yr. In panels (b) and (c) it shows the value required for the equilibrium temperature of $T_0 = 288$ K, over time. Finally, in panel (d), it shows the equilibrium temperature of 288 K.
Figure 5. Realizations of the Ornstein-Uhlenbeck process defined by Equation 13 for three exemplary pairs of $\tau$ and $\sigma$. Five independent realizations are stacked vertically on top of each other.

Figure 6. The value of the first temperature percentile over time in our ensemble. Moving to the right, each panel shows results for a higher/longer relaxation timescale $\tau$. Within each panel, results for different values of $\sigma$ are delineated by color. The hypothetical snowball temperature $T_{\text{snow}} = 280$ K is shown in each panel. When $\tau$ is small, the outgassing rate is characterized by rapid deviations (see Figure 5) and the likelihood of cold temperatures increases dramatically as time progresses. As $\tau$ becomes larger, outgassing rates vary over longer periods and the likelihood of cold temperatures becomes much more uniform over time. The first percentile is an arbitrary choice, but other low quantiles demonstrate the same basic relationships between $\tau$, $\sigma$, and the likelihood of low temperatures.
ously. For example, a snowball may only be triggered by the combination of slowly increasing weatherability and a relatively sudden change to volcanic outgassing. Our model results with high $\tau$ represent a simple demonstration of this scenario, where outgassing undergoes both long-term drift and occasional shorter-term shocks.

There is a wide range of opportunities to explore randomness in simple climate models. Our model has only considered one form of stochastic outgassing with a deterministic weathering mechanism that is dependent exclusively on temperature. This is, of course, an oversimplification (Macdonald et al., 2019; Park et al., 2021; Baum et al., 2022) and future work could investigate model equations that incorporate variability in the sink term. For example, it would be straightforward to include a randomly varying weatherability factor that is also represented by an Ornstein-Uhlenbeck process, but this term need not be stochastic at all.

6 Open Research

All of the code developed for this project was written in the Julia language (Bezanson et al., 2012, 2017). It is freely available on GitHub at [github.com/markbaum/random-volcanic-climate](https://github.com/markbaum/random-volcanic-climate) and publicly archived on Zenodo (Baum, 2022). Code for the WHAK weathering equation is borrowed from GEOCLIM.jl (Baum & Fu, 2022). Plots were created using the Julia wrapper of the open source Matplotlib library (Hunter, 2007).

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