A 3-D Meshless Method for Solving Maxwell’s Equations by Using the Steger-Warming Flux Vector Splitting Approach

Yukun Gao\textsuperscript{1,2}, Hongquan Chen\textsuperscript{2}, Shengguan Xu\textsuperscript{2}, Jiale Zhang\textsuperscript{2}, Cheng Cao\textsuperscript{2} and Huangqi Gao\textsuperscript{2}

\textsuperscript{1} School of Mechanical Engineering, Anhui University of Technology, Maanshan, 243002, China  
\textsuperscript{2} College of Aerospace Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, China  
Email: gaoyk1984@126.com

Abstract. In order to solve Maxwell’s equations so as to study the electromagnetic stealth characteristics of targets, a 3-D meshless method is developed. According to the meshless method, the spatial derivatives on clouds of points are computed by using the weighted least square approach. After that, the Steger-Warming flux vector splitting approach is used to calculate the physical flux of Maxwell's equations. By using the developed method, the calculated bistatic radar cross sections (RCS) of a 3-D sphere are obtained, which are agree with the series solutions. Finally, the electromagnetic scattering characteristics for a 3-D stealth aircraft model with different situations is given, which shows the developed method has the ability to accommodate complicated 3-D configurations with multi-element.

1. Introduction
The research on solving Maxwell’s equations and analyzing the electromagnetic stealth characteristics of targets has been a hot topic for a long time. In order to compute the electromagnetic scattering fields and the RCS for a given body, various numerical methods have been developed, including finite-difference time-domain (FDTD) methods [1], finite-volume time-domain (FVTD) methods [2] as well as finite-element time-domain (FETD) methods [3]. Some researchers have noticed that these traditional mesh methods need to generate mesh cells in the computational domain, so the mesh methods are constrained by meshing. However, meshless methods only use points and do not require mesh cells, which are flexible to treat complicated configurations [4]. For this reason, meshless methods have been proposed and applied to calculate the electromagnetic fields [5-7]. Meshless methods by using basis functions, element free Galerkin methods and meshless methods referring to Computational Fluid Dynamics (CFD) are three kinds of typical meshless methods. The former two kinds need to select basis functions, which will affect the companion matrix as well as the boundary condition [8]. The latter kind is from CFD for solving Euler equations, which is different from the former two kinds in spatial discretization [4].

For this reason, the author has developed the 2-D meshless method to solve Maxwell's equations [7]. This paper intends to improve and develop the meshless method in dealing with 3-D problems. The spatial derivatives are calculated by the weighted least square technique. The introduce of weighted factor can relax the requirement of the method for the distribution of points to a certain extent. The physical flux is calculated by the Steger-Warming flux vector splitting approach on each cloud of points. Then, a typical 3-D example is used to validate the method. Finally, an aircraft model
example is presented to show the ability of the developed method in dealing with complicated 3-D configurations with multi-element to a certain extent.

2. 3-D Meshless Method

2.1. Maxwell’s Equations

3-D dimensionless Maxwell's equations may be written as:

\[
\frac{\partial W}{\partial t} + \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) = S
\]  

where

\[
W = \begin{bmatrix}
\varepsilon E_x & \varepsilon E_y & \varepsilon E_z & \mu H_x & \mu H_y & \mu H_z
\end{bmatrix}^T, \quad
F_1 = \begin{bmatrix}
0 & -H_y & 0 & E_z & -E_y & 0
\end{bmatrix}^T, \quad
F_2 = \begin{bmatrix}
-H_z & 0 & H_x & E_z & 0 & -E_x
\end{bmatrix}^T, \quad
F_3 = \begin{bmatrix}
0 & H_y & -H_z & 0 & -E_y & E_x
\end{bmatrix}^T, \quad
S = \begin{bmatrix}
-\sigma E_x & -\sigma E_y & -\sigma E_z & -\sigma_m H_x & -\sigma_m H_y & -\sigma_m H_z
\end{bmatrix}^T, \quad
E = (E_x, E_y, E_z)
\]

is the electric field vector, \( \mathbf{H} = (H_x, H_y, H_z) \) is the magnetic field vector, \( \varepsilon \) is the electric permittivity, \( \sigma \) is the conductivity, \( \mu \) is the magnetic permeability, \( \sigma_m \) is the magnetic resistivity.

2.2. Distribution of Points and Construction of Clouds of Points

Firstly, the meshless method need to distribute points in the computational domain. For convenience, mesh points can be used directly sometimes, other approach of distribution of points can also be used if needed. According to the meshless method, each cloud of points \( C_i \) then should be constructed (see figure 1).

![Figure 1. Schematic diagram of cloud of points \( C_i \).](image)

For 2-D situations, firstly, the central point \( i \) is selected as a point on \( C_i \) naturally. Then, several points around it should be selected properly as satellite points on \( C_i \). For example, a circle can be determined by setting the point \( i \) as its center and the length of \( r_i \) as its radius. Then, points 1–6 in the circle are selected as satellite points on \( C_i \). The total number of satellite points selected on \( C_i \) can be controlled by \( r_i \), which is related to the distance between any two points in the local area. For 3-D situations, a sphere can be determined similarly, and all the points in the sphere are selected naturally as points on \( C_i \).

2.3. Calculation of Spatial Derivatives

The function \( f = f(x, y, z) \) near the point \( i \) on \( C_i \) can be expressed as:
\[ f = f_i + a_i h + a_i l + a_i m + \cdots + \frac{1}{n!}\left( h \frac{\partial f}{\partial x} + l \frac{\partial f}{\partial y} + m \frac{\partial f}{\partial z} \right)^n + O\left(h^{n+1}, l^{n+1}, m^{n+1}\right) \]  

(2)

where \( f_i = f(x_i, y_i, z_i) \), \( h = x - x_i \), \( l = y - y_i \), \( m = z - z_i \), \( a_i (i = 1, 2, 3) \) are the spatial derivatives at the point \( i \). Then, equation (2) can be written for linear approximation as \( \tilde{f} = f_i + a_i h + a_i l + a_i m \). At each satellite point \( k (k = 1, \ldots, M) \), the function value \( f_k \) is available. Different from Ref. [7], the total error of \( f_k \) can be estimated by \( G = \frac{1}{2} \sum_{i=1}^{M} \omega_k (f_k - \tilde{f_k})^2 \), where the coefficient \( \omega_k \) is the weighted factor. In this paper, referring to Ref. [9], set \( \omega_k = 1/\overline{r}_k^2 \) (set the distance between the central point \( i \) and all satellite points \( k \) as \( r_1, r_2, \ldots, r_M \), let \( r_{\max} = \max \{r_1, r_2, \ldots, r_M\} \). \( \overline{r}_k = r_k / r_{\max} \). In general, the introduce of the weighted factor can weaken the affection which makes the matrix ill-conditioned caused by points distributed too near, and can help to improve the flexibility of distributing points. Then, \( a_i (i = 1, 2, 3) \) can be determined by the following equations so as to minimize \( G \):

\[
\begin{bmatrix}
\sum \omega_k^2 h_k^2 \\
\sum \omega_k^2 h_k l_k \\
\sum \omega_k^2 h_k m_k
\end{bmatrix}
\begin{bmatrix}
\alpha_k \\
\beta_k \\
\gamma_k
\end{bmatrix}
= \begin{bmatrix}
\sum \omega_k^2 h_k (f_k - f_i) \\
\sum \omega_k^2 l_k (f_k - f_i) \\
\sum \omega_k^2 m_k (f_k - f_i)
\end{bmatrix}
\]  

(3)

The linear function can be expressed as

\[ \tilde{f} = f_i + \sum \alpha_k (f_k - f_i) h + \sum \beta_k (f_k - f_i) l + \sum \gamma_k (f_k - f_i) m \], where coefficients \( \alpha_k \), \( \beta_k \) as well as \( \gamma_k \) are only determined by the coordinates of points on \( C_i \), which can be calculated before iterative computation. Then, the spatial derivatives \( a_i (i = 1, 2, 3) \) can be expressed as:

\[ a_1 = \sum \alpha_k (f_k - f_i), \quad a_2 = \sum \beta_k (f_k - f_i), \quad a_3 = \sum \gamma_k (f_k - f_i) \]  

(4)

\( a_i (i = 1, 2, 3) \) can also be expressed as:

\[ a_1 = \sum \alpha_{ik} (f_k - f_i), \quad a_2 = \sum \beta_{ik} (f_k - f_i), \quad a_3 = \sum \gamma_{ik} (f_k - f_i) \]  

(5)

where coefficients \( \alpha_{ik} \), \( \beta_{ik} \) as well as \( \gamma_{ik} \) can be calculated before iterative computation.

2.4. Calculation of Physical Flux

Based on \( C_i \), apply equation (5) to treat the flux of Maxwell’s equations:

\[
\left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) = \sum \alpha_{ik} (F_{1ik} - F_i) + \sum \beta_{ik} (F_{2ik} - F_i) + \sum \gamma_{ik} (F_{3ik} - F_i)
\]

\[
= \sum (\alpha_{ik} F_{1ik} + \beta_{ik} F_{2ik} + \gamma_{ik} F_{3ik}) - \sum (\alpha_{ik} F_i + \beta_{ik} F_{2i} + \gamma_{ik} F_{3i})
\]  

(6)

For \( \sum (\alpha_{ik} F_{1ik} + \beta_{ik} F_{2ik} + \gamma_{ik} F_{3ik}) \) being available, then how to accommodate \( \sum (\alpha_{ik} F_{1ik} + \beta_{ik} F_{2ik} + \gamma_{ik} F_{3ik}) \) will be introduced in the following text. Create a virtual interface at the midpoint between point \( i \) and points \( k \) (see figure 2) and define a numerical flux term
\[ Q_{ik} = \xi_{ik} F_1(W_{ik}) + \eta_{ik} F_2(W_{ik}) + \zeta_{ik} F_3(W_{ik}) \]
where \( \xi_{ik} = \alpha_{ik} / \sqrt{\alpha_{ik}^2 + \beta_{ik}^2 + \gamma_{ik}^2} \),
\( \eta_{ik} = \beta_{ik} / \sqrt{\alpha_{ik}^2 + \beta_{ik}^2 + \gamma_{ik}^2} \), and \( \zeta_{ik} = \gamma_{ik} / \sqrt{\alpha_{ik}^2 + \beta_{ik}^2 + \gamma_{ik}^2} \). Define \( d_{ik} = (\alpha_{ik}, \beta_{ik}, \gamma_{ik}) \), so equation (6) can be expressed as:

\[
\left( \frac{\partial F_{1j}}{\partial x} + \frac{\partial F_{2j}}{\partial y} + \frac{\partial F_{3j}}{\partial z} \right) = \sum_{k=1}^{M} Q_{ik} \left| d_{ik} \right| - \sum_{k=1}^{M} \left( \alpha_{ik} F_{1j} + \beta_{ik} F_{2j} + \gamma_{ik} F_{3j} \right)
\]

(7)

Figure 2. Schematic diagram of virtual interface on \( C_i \).

Use Steger-Warming flux vector splitting approach to split the Jacobian matrix of \( Q_{ik} \), the split matrices \( M^+ \) and \( M^- \) are obtained. Then, \( Q_{ik} \) can be written as
\[ Q_{ik} = Q^+_{ik} + Q^-_{ik} = M^+_{ik} W^+_{ik} + M^-_{ik} W^-_{ik}, \]
where \( W^+_{ik} \) and \( W^-_{ik} \) can be obtained by interpolation [7].

2.5. Temporal Discretization and Boundary Conditions
Then, the four-stage Runge-Kutta scheme [7] is carried out in temporal discretization, and the perfectly matched layer boundary condition as well as the perfect conductor boundary condition [2] are used so as to solve Maxwell’s equations.

3. Numerical Examples and Discussion
In this section, a 3-D sphere example is simulated to validate the developed method. Then, the electromagnetic scattering fields and the bistatic RCS for a 3-D stealth aircraft model with different situations are solved. In figure 3, the incident wave [9] is propagating along \( k' \) axis, \( k'' \) axis is the projection of \( k' \) axis in \( xoy \) plane, \( \varphi \) is the angle between the direction of \( k'' \) axis and \( x \) axis, \( \theta \) is the angle between the direction of \( k' \) axis and \( z \) axis.

Figure 3. Schematic diagram of incident wave.

3.1. 3-D Sphere
Firstly, a 3-D sphere example is solved so as to validate the method. The sphere body is assumed to be perfect electric conductor with \( a = \lambda \), where \( a \) is the radius of the sphere. For the 3-D situation, considering the rapidly increasing in the amount of computation, a relatively small computational domain is used, which is \( 8\lambda \times 8\lambda \times 8\lambda \) with 463056 points.

In this section, the electromagnetic scattering characteristics of the sphere under beaming of different polarized electromagnetic waves are studied. The incident electromagnetic wave is
propagating along the $x$ axis with $\theta = 90^\circ$, $\varphi = 0^\circ$, and the polarization angle is set as $\alpha = 0^\circ$ and $\alpha = 90^\circ$ respectively. Contours of scattered field in $z = 0$ cross-section for the two different polarization situations are presented in figure 4. It could be observed the difference between the electromagnetic scattering characteristics for the two different polarization situations. Figure 5 shows that the bistatic RCS are agree with the series solutions [10].

![Figure 4](image1.png)

**Figure 4.** Contours of scattered field in $z = 0$ cross-section for 3-D sphere: (a) polarization angle $\alpha = 0^\circ$, (b) polarization angle $\alpha = 90^\circ$.

![Figure 5](image2.png)

**Figure 5.** RCS for 3-D sphere ($\theta = 90^\circ$): (a) polarization angle $\alpha = 0^\circ$, (b) polarization angle $\alpha = 90^\circ$.

### 3.2. 3-D Stealth Aircraft Model

Finally, the electromagnetic scattering characteristics for a 3-D stealth aircraft model with different situations is studied here. The aircraft model (see figure 6) is a low detectable delta-winged configuration with $4\lambda$ length from nose to tail and $4\lambda$ width between two wingtips, while the bomb is simulated with a smaller ellipsoid. The computational domain is also set as $8\lambda \times 8\lambda \times 8\lambda$ with 419433, 411827 and 424670 points respectively for three situations: (a) normal flight with the basic shape, (b) bomb bay doors open, (c) the bomb out of the bay.

![Figure 6](image3.png)

**Figure 6.** Points distribution on the surface of 3-D stealth aircraft model: (a) basic shape, (b) bomb bay doors open, (c) bomb out of bay.
The incident wave is propagating along the $z$ axis with the incident angle $\theta = 0^\circ$, $\varphi = 0^\circ$ and the polarization angle $\alpha = 0^\circ$. Contours of scattered field for the three situations are presented in figure 7, and the bistatic RCS are also given in figure 8. It could be observed the difference in the electromagnetic scattering characteristics and the stealth characteristics from comparison in figures 7 and 8. Figure 7 shows that scattered fields are strong in bistatic angle of $\theta = 0^\circ$ and $\theta = 180^\circ$ directions, which are agree with the distributions of the maximum values of the bistatic RCS.

![Figure 7. Contours of scattered field in $y = 0$ cross-section for 3-D stealth aircraft model: (a) basic shape, (b) bomb bay doors open, (c) bomb out of bay.](image)

![Figure 8. RCS for 3-D stealth aircraft model ($\varphi = 0^\circ$).](image)

4. Conclusions
By using the Steger-Warming flux vector splitting approach, a 3-D meshless method is developed. According to the meshless method, a weighted least square technique is employed to compute the spatial derivatives. Then, a Steger-Warming flux vector splitting approach is introduced to calculate the physical flux of Maxwell’s equations. The introduce of weighted factors can weaken the affection which makes the matrix ill-conditioned caused by points distributed too near, and can help to improve the flexibility of distributing points to a certain extent. The obtained bistatic RCS for a 3-D sphere are agree with the series solutions, which validates the developed method. Finally, the paper study the electromagnetic scattering characteristics and the stealth characteristics for a 3-D stealth aircraft model, which shows the developed method has the ability to accommodate complicated 3-D configurations to a certain extent.

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