Vector and Tensor Contributions to the Luminosity Distance

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We compute the vector and tensor contributions to the luminosity distance fluctuations in first order perturbation theory, and we expand them in spherical harmonics. This work presents the formalism with a first application to a stochastic background of primordial gravitational waves.

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I. INTRODUCTION

The distance–redshift relation for far away objects plays an important role in cosmology. It has led Hubble, or rather Lemaitre [1], to discover the expansion of the Universe; and the distance–redshift relation to far away Supernovae type Ia is at the origin of last year’s Nobel Prize in physics for the discovery of the accelerated expansion of the Universe [2, 3].

A next step that has been initiated recently considers the angular and redshift fluctuations of the luminosity distance, which may also contain important information about our Universe [4, 7]. One important unsolved problem is the question how strongly the distance–redshift relation may be affected by the fact that the actual Universe is not homogeneous and isotropic, but the matter distribution and also the geometry have fluctuations. To first order in perturbation theory these fluctuations can average out in the mean and are therefore expected to be small.

However, it has been found that they are significantly larger than the naïvely expected value that would be of the order of the gravitational potential, namely, \( \sim 10^{-5} \). An analysis in first order gave fluctuations of the order of \( 10^{-3} \), hence 100 times larger than the naïve estimate [5]. Recently, Ben-Dayan et al. [8] have calculated a second order contribution to the distance–redshift relation of the order of \( \sim 10^{-3} \). Evidently, if the second order term is as large as the first order, this means that perturbation theory cannot be trusted. On the other hand, fully nonlinear toy models, which have been studied in the past, always gave relatively small modifications of the luminosity distance if the size of the fluctuations, spherical voids [9] or parallel walls [10], is small compared to the Hubble scale. Hence the problem remains open.

So far, the perturbative analyses of the distance–redshift relation have concentrated on scalar perturbations. In this work, we want to study the contributions from vector and tensor perturbations on a Friedmann–Lemaitre (FL) universe. This is interesting for several reasons. First of all, tensor perturbations are generically produced during inflation, and hence their contribution has to be added for completeness. Second, a passing gravitational wave from some arbitrary source does generate a tensor perturbation in the distance–redshift relation to any far away object and could, at least in principle, be detected in this way. For single binary sources we have found that this effect is very small [11]; however, a stochastic background might lead to a detectable effect. Even though vector perturbations are usually not generated during inflation (and if they are they decay during the subsequent radiation dominated phase), they are relevant in many models with sources like, e.g., cosmic strings or primordial magnetic fields. A third important motivation to study vector and tensor contributions comes from the fact that at second order in perturbation theory, scalars also generate vector and tensor perturbations [12, 13]. In a complete second order treatment these have to be included. With the formalism developed in this work, such an inclusion is straightforward. We plan to report on the result of these second order contributions in a forthcoming paper [14]. A similar program is carried out in Refs. [15, 16]. There the authors discuss scalar, vector, and tensor perturbations and split them into \( E \) and \( B \) modes. The treatment of these papers is, however, more adapted to describe distortions of surveys and weak lensing, but the convergence calculated there is related to our distance fluctuations.

The paper is organized as follows. In the next section we discuss the luminosity-redshift relation perturbatively at first order. In Sec. III we apply these results to tensor perturbations. We first derive the general first order expressions, which we then expand in spherical harmonics. We also give a numerical example for the gravitational wave background from inflation. In Sec. IV we treat vector perturbations and in Sec. V we conclude. Some lengthy calculations and some details are deferred to four Appendices.

Notation: We use the metric signature \((-++,+,+\)). We denote the derivative w.r.t. the conformal time \( \eta \) with a dot.

II. THE DISTANCE–REDSHIFT RELATION

For an arbitrary geometry, defined through the metric \( g \), a distance measure \( D \) from a source moving with 4-velocity \( u_S = (x_S) \) and an observer moving with 4-velocity \( u_O = (x_O) \) can be obtained as a solution of the
Sachs focusing equation \([17]\):
\[
\frac{d^2 D}{d\lambda^2} = - \left( R + |\Sigma|^2 \right) D. \tag{1}
\]

Here \(\lambda\) is the affine parameter of a lightlike geodesic \(x^\mu(\lambda)\) from the source to the observer, \(x^\mu(\lambda_S) = x^\mu_S\), \(x^\mu(\lambda_O) = x^\mu_O\), and
\[
R = \frac{1}{2} R_{\mu\nu} k^\mu k^\nu \quad \text{with} \quad k^\mu = \frac{dx^\mu}{d\lambda}, \tag{2}
\]
\(k^\mu\) is the 4-velocity of the lightlike geodesic, and \(\Sigma\) is the complex shear of the 'screen' defined below. The source and observer are made out of baryons; hence we identify the 4-velocity field \(u^\mu(x)\) with the (baryonic and dark) matter velocity field.

Considering a thin light bundle with vertex at the source, the luminosity distance is given by
\[
D_L = (1 + z) D, \tag{3}
\]
where \(z\) denotes the source redshift, defined by
\[
1 + z = \frac{g_{\mu\nu} k^\mu u^\nu|_S}{g_{\mu\nu} k^\mu u^\nu|_O} = \frac{\omega_S}{\omega_O}. \tag{4}
\]

We are considering past light cones without caustics between the observer and source positions. This is well justified as we are treating small perturbations on a Friedmann background. See Ref. [18] for more details on the effect of caustics in the past light cone.

The complex shear of the light ray bundle, \(\Sigma\), is defined as follows \([19]\): Consider two spatial orthonormal vectors \(e_1\) and \(e_2\), which are normal to both the 4-velocity \(u_O\) and \(k\) at the observer position and which are parallel transported along \(k\), such that \(\nabla_k e_a = 0\) for \(a = 1, 2\). The vectors \(e_1\), \(e_2\) are a basis of the so-called screen. Note that we do not require that \(u\) be parallel transported along \(k\); hence \(e_1\), \(e_2\) are in general not normal to \(u\) elsewhere than at the observer. The complex shear is defined by
\[
\Sigma = \frac{1}{2} g(e, \nabla_k e), \quad \text{with} \quad e = e_1 + ie_2. \tag{5}
\]

We consider a light bundle with vertex at the source\(^1\). This leads to the following initial conditions (more details are found in Appendix A) for the Sachs focusing equation \([11]\)
\[
D(\lambda_S) = 0, \quad D'(\lambda_S) = \omega_S = -g_{\mu\nu} k^\mu u^\nu|_S. \tag{6}
\]

In a perturbed FL metric the Sachs focusing equation \([11]\) reduces, at first order, to
\[
\frac{d^2 D}{d\lambda^2} = -R D. \tag{7}
\]

Since the complex scalar shear \(\Sigma\) vanishes for a conformally flat spacetime, \(|\Sigma|^2\) contributes only at second order. To first order in \(R\), Eq. \([7]\) with initial conditions \([6]\) is solved by
\[
\frac{D(\lambda_O)}{\omega_S} = (\lambda_O - \lambda_S) - \int_{\lambda_S}^{\lambda_O} d\lambda' \int_{\lambda_S}^{\lambda'} d\lambda'' R(\lambda - \lambda_S) = (\lambda_O - \lambda_S) - \int_{\lambda_S}^{\lambda_O} d\lambda (\lambda - \lambda_S)(\lambda_O - \lambda)R, \tag{8}
\]
where we have used the identity
\[
\int_{\eta_S}^{\eta_O} d\eta' \int_{\eta_S}^{\eta'} d\eta'' f(\eta'') = \int_{\eta_S}^{\eta_O} d\eta (\eta_O - \eta) f(\eta) \tag{9}
\]
for the second equal sign.

Of course, in a perturbed FL universe \(R\) is not first order; it also has a zeroth order contribution. But a perturbed FL universe is conformally related by the scale factor \(a\) to a perturbed Minkowski spacetime and lightlike geodesics are invariant under conformal transformations. Two conformally related metrics,
\[
\tilde{g}_{\mu\nu} = a^2 g_{\mu\nu}
\]
have the same lightlike geodesic curves, and only the affine parameter changes, \(d\bar{x} = a^2 d\lambda\), such that \(k^\mu = a^{-2} k^\mu\). Also the (normalized) matter 4-velocity changes, \(\bar{u}^\mu = a^{-1} u^\mu\) so that the redshifts are related by
\[
\bar{z} + 1 = \frac{a_O}{a_S} (\delta z + 1), \quad \text{where} \quad \frac{a_O}{a_S} \equiv \bar{z} + 1 \tag{10}
\]
is the background redshift, i.e., the redshift in an unperturbed Friedmann–Lemaître universe, and \(\delta z\) is the source redshift according to definition \([4]\) w.r.t. to the perturbed Minkowski metric \(g\), while \(\bar{z}\) is the one w.r.t. to the perturbed FL metric \(\tilde{g}\). We remark that \(\bar{z}\) is the true (observed) redshift. In what follows we shall normalize the scale factor at the observer to one, \(a_O = 1\).

The distance \(D\) is not affected by a conformal factor, so that the effect of the expansion on the distance simply leads to a rescaling \([5]\)
\[
\bar{D}_L = (1 + \bar{z}) D_L. \tag{11}
\]

We now compute the luminosity distance in a perturbed Minkowski spacetime, \(D_L\), and then relate it to the one in a FL spacetime, \(\bar{D}_L\), by the above rescaling. Let \((1, n^i)\) be the 0-order term of the lightlike velocity vector \(k^\mu\) (in the nonexpanding Minkowski spacetime).

The lightlike condition implies \(|n|^2 = 1\). We normalize the affine parameter \(\lambda\) such that \(\omega_S = \kappa_S^2 = 1\) at 0th order. To determine the redshift \(\delta z\), we have to solve the perturbed geodesic equation for \(\mu = 0\) only (in order to determine \(\kappa_O^0\) to first order), since the peculiar velocities are already first order. The Christoffel symbols of Minkowski space vanish, so that the geodesic equation for \(\mu = 0\) to first order is simply
\[
\frac{dk^0}{d\lambda} + \Gamma_{00}^0 + 2\Gamma_{10}^0 n^1 + \Gamma_{ij}^0 n^i n^j = 0. \tag{12}
\]
We normalize the affine parameter $\lambda$ such that $k^0_O = 1$, and Eq. (12) is solved by

$$k^0_O = 1 - \int_{\lambda_S}^{\lambda_0} d\lambda \left( \Gamma^0_{00} + 2 \Gamma^0_{0i} n^i + \Gamma^0_{ij} n^i n^j \right).$$

(13)

The geodesic equation (12) will be useful also in order to express the distance $D$ in terms of the conformal time $\eta$ instead of the affine parameter $\lambda$. For this we use

$$k^0 = \frac{d\eta}{d\lambda} = 1 \lambda - \int_{\lambda_S}^{\lambda} d\lambda' \left( \Gamma^0_{00} + 2 \Gamma^0_{0i} n^i + \Gamma^0_{ij} n^i n^j \right),$$

(14)

which, in first order, leads to

$$\lambda_0 - \lambda_S = \eta_0 - \eta_S + \int_{\eta_S}^{\eta_0} d\eta \int_{\eta_S}^{\eta_0} d\eta' \left( \Gamma^0_{00} + 2 \Gamma^0_{0i} n^i + \Gamma^0_{ij} n^i n^j \right).$$

(15)

The conformal time and the background redshift are not observable. We want to write the distance as a function of the true (observed) redshift $\tilde{z} = z + \delta z$, where $\delta z \equiv (1 + z) \delta z$ according to Eq. (10). Following the approach presented in [3] we compute

$$D_L(\eta_S, n) = \tilde{D}_L(\eta(\tilde{z}), n) \equiv \tilde{D}_L(\tilde{z}, n),$$

(16)

with

$$\frac{d\tilde{z}}{d\tilde{z}} \tilde{D}_L(\tilde{z}, n) \bigg|_{\tilde{z} = z} = \frac{d\tilde{z}}{d\tilde{z}} D_L(\tilde{z}, n) + \text{first order}$$

$$= (1 + \tilde{z})^{-1} \tilde{D}_L + \mathcal{H}_S^{-1} \text{first order},$$

where $\mathcal{H}_S = \frac{\dot{a}}{a} |_{S}$. (17)

In other words, we evaluate the distance at the true (observed) redshift $\tilde{D}_L(\tilde{z}, n)$ by using Eqs. (10, 11, 15) in order to relate $\tilde{D}_L(\tilde{z}, n)$ to $D_L(\eta_S, n)$.

From Sec. III on, to simplify the notation, we denote the true (observed) redshift with $z$ instead of $\tilde{z}$. We shall not use $\tilde{z}$ anymore.

III. THE DISTANCE-REDSHIFT RELATION FROM TENSOR PERTURBATIONS

We first consider a perturbed Minkowski metric with tensor perturbations only, defined by

$$ds^2 = -d\eta^2 + (\delta_{ij} + 2H_{ij}) dx^i dx^j,$$

(18)

where the tensor perturbations are divergence-free $H^i_{kj} = 0$, traceless $H^i_i = 0$, symmetric $H_{ij} = H_{ji}$, and spatial $H_{00} = 0$. By definition, a spin-2 perturbation is gauge-invariant. To use a notation consistent with the next section, we introduce the gauge invariant shear on the $\{ t = \text{constant} \}$ hypersurfaces $\sigma_{ij} = H_{ij}$

(see, e.g., [20]).

A. The perturbation equations

From the Ricci tensor calculated in Appendix B we obtain

$$\mathcal{R} = -\frac{1}{2} n^i n^j \Box H_{ij}, \quad \text{where } \Box = \partial^\mu \partial_\mu.$$ (19)

Note that this is the Minkowski space d’Alembertian, without expansion. The geodesic equation (13) for $\mu = 0$ leads to (see Appendix B for details)

$$k^0_O = 1 - \int_{\lambda_S}^{\lambda_0} d\lambda \sigma_{ij} n^i n^j.$$ (20)

We consider the 4-velocity $(u^\mu) = (1, 0)$ because the spin-2 perturbations can not source peculiar velocities at linear order, so that we obtain the redshift to first order

$$1 + \delta z = \frac{1}{k^0_O} = 1 + \int_{\lambda_S}^{\lambda_0} d\lambda \sigma_{ij} n^i n^j.$$ (21)

With Eqs. (8, 11, 15) we find the luminosity distance in a perturbed FL universe with $d\tilde{s}^2 = a^2 d\tilde{s}^2$, as a function of the background redshift

$$\tilde{D}_L(\tilde{z}, n) = (1 + \tilde{z}) (\eta_0 - \eta_S) \times \left( 1 + \int_{\eta_S}^{\eta_0} d\eta \sigma_{ij} n^i n^j \right.$$ \left. + \int_{\eta_S}^{\eta_0} d\eta \frac{\eta_S - \eta}{\eta_0 - \eta_S} \sigma_{ij} n^i n^j \right.$$ \left. - \int_{\eta_S}^{\eta_0} d\eta \frac{\eta - \eta_S}{\eta_0 - \eta_S} \mathcal{R} \right). (22)

We have again used (9) to reduce the double integral. We finally express the luminosity distance in terms of the true, observed redshift $z$. Using Eqs. (16, 17), we obtain

$$D_L(z, n) = (1 + z) (\eta_0 - \eta_S) \times \left( 1 - \frac{\mathcal{H}_S^{-1}}{\eta_0 - \eta_S} \int_{\eta_S}^{\eta_0} d\eta \sigma_{ij} n^i n^j \right.$$ \left. + \int_{\eta_S}^{\eta_0} d\eta \frac{\eta_0 - \eta}{\eta_0 - \eta_S} \sigma_{ij} n^i n^j \right.$$ \left. - \int_{\eta_S}^{\eta_0} d\eta \frac{\eta - \eta_S}{\eta_0 - \eta_S} \mathcal{R} \right). (23)

The origin of the different terms in the redshift–distance relation is as follows: the first line is the unperturbed expression for the luminosity distance in a FL universe at the observed redshift $z$, the term on the second line derives from the redshift correction, the one on the third line from the relation between the conformal time $\eta$ and the affine parameter $\lambda$, and the one on the last line from the Sachs focusing equation. We can interpret this last term as a lensing effect. The first two terms come from the perturbation of the redshift.
For a fluid with a vanishing anisotropic stress the redshift–distance relation becomes
\[
\hat{D}_L(z, n) = (1 + z) (\eta_O - \eta_S) \\
\times \left( 1 - \frac{\mathcal{H}_S^{-1}}{\eta_O - \eta_S} \int_{\eta_S}^{\eta_O} d\eta \, \sigma_{ij} n^i n^j + \int_{\eta_S}^{\eta_O} d\eta \, \frac{\eta_O - \eta}{\eta_O - \eta_S} \sigma_{ij} n^i n^j + \int_{\eta_S}^{\eta_O} d\eta \, \frac{(\eta - \eta_S)(\eta_O - \eta)}{\eta_O - \eta_S} \mathcal{H} n^i n^j \sigma_{ij} \right),
\]
where we used the Einstein equation \[20\] and Eq. \[19\] to replace \( \mathcal{R} \). If the cosmic fluid is not ideal, but has anisotropic stresses, these add to the right-hand side of Eq. \[25\] (see \[20\]) and correspondingly to the last line in Eq. \[24\], the lensing term.

**B. Spherical harmonic analysis**

We want to determine the power spectrum of the luminosity distance. In the unperturbed FL background the luminosity distance to the redshift \( z \) is given by
\[
\hat{D}_L(z) = (1 + z) (\eta_O - \eta_S).
\]
We define the relative difference in the luminosity distance as
\[
\Delta_L(z, n) = \frac{\hat{D}_L(z, n) - \hat{D}_L(z)}{\hat{D}_L(z)} = \frac{1}{\eta_O - \eta_S} \int_{\eta_S}^{\eta_O} d\eta \left[ - \mathcal{H}_S^{-1} (\eta_O - \eta) + (\eta - \eta_S)(\eta_O - \eta) \mathcal{H} \right] \sigma_{ij} n^i n^j.
\]
Note that we evaluate the unperturbed distance at the true, observable redshift.

We are interested in the angular power spectrum of this observable, \( c_\ell (z, z') \), which depends on the redshift of the two sources and is defined by the two point correlation function
\[
\langle \Delta_L(\mathbf{z}, \mathbf{n}) \Delta_L(\mathbf{z}', \mathbf{n}') \rangle = \frac{1}{4\pi} \sum_\ell (2\ell + 1) c_\ell(z, z') P_\ell(\mathbf{n} \cdot \mathbf{n}').
\]

In the distance–redshift relation \[24\] [and, in particular, in Eq. \[24\] for an ideal fluid] we have several times the term \( n^i n^j \sigma_{ij} (\eta, \mathbf{x}(\eta)) \) where \( \mathbf{x}(\eta) = \mathbf{x}_O - \mathbf{n} (\eta_O - \eta) \). In terms of its Fourier transform this is
\[
n^i n^j \sigma_{ij} (\eta, \mathbf{x}(\eta)) = \int \frac{d^3k}{(2\pi)^3} \hat{\sigma}_{ij} (\eta, \mathbf{k}) n^i n^j e^{-i \mathbf{k} \cdot \mathbf{x}(\eta)}.
\]

Without loss of generality we choose \( \mathbf{x}_O = \mathbf{0} \). Setting \( \mathbf{k} = \hat{\mathbf{k}} |\mathbf{k}| = \hat{\mathbf{k}} k, \; \mu = \hat{\mathbf{k}} \cdot \mathbf{n}, \; \Delta \eta = \eta_O - \eta \),
\[
(30)
\]
we obtain
\[
n^i n^j \sigma_{ij} (\eta, \mathbf{x}(\eta)) = \int \frac{d^3k}{(2\pi)^3} \hat{\sigma}_{ij} (\eta, \mathbf{k}) n^i n^j e^{i \mu k \Delta \eta}.
\]
Writing the exponential in terms of spherical Bessel functions
\[
(31)
\]
we find
\[
n^i n^j \sigma_{ij} = \int \frac{d^3k}{(2\pi)^3} n^i n^j \hat{\mathcal{J}}_\ell (k \Delta \eta) P_\ell (\mu),
\]
\[32\]
With respect to a helicity basis in Fourier space
\[
\{ \mathbf{e}^{(+)}(\mathbf{e}^{(-)}), \hat{\mathbf{k}} \}, \; \mathbf{e}^{(\pm)} = \sqrt{\frac{1}{2}} (\mathbf{e}_1 \pm i \mathbf{e}_2),
\]
such that
\[
(33)
\]
we have
\[
\hat{\mathcal{J}}_\ell (k \Delta \eta) = \hat{\mathcal{J}}^{(+)} (k \Delta \eta) \mathbf{e}^{(+)} \hat{\mathbf{k}} \pm \hat{\mathcal{J}}^{(-)} (k \Delta \eta) \mathbf{e}^{(-)} \hat{\mathbf{k}} \hat{\mathbf{k}}.
\]
\[34\]
We introduce the spherical harmonics with respect to some arbitrary \( \mathbf{z} \) direction given by a unit vector \( \mathbf{e} \) as \( Y_{\ell m}(\mathbf{n}, \mathbf{e}) \), since we shall use them w.r.t. different \( \mathbf{z} \) axes. The addition theorem of spherical harmonics is
\[
P_\ell (\mu) = \frac{4\pi}{2\ell + 1} \sum_m Y_{\ell m}^* (\hat{\mathbf{k}}, \mathbf{e}) Y_{\ell m} (\mathbf{n}, \mathbf{e}).
\]
\[35\]
Using the following spherical harmonics definition
\[
Y_{2\pm \phi}(\mathbf{n}, \hat{\mathbf{k}}) = \sqrt{\frac{15}{8\pi}} \sin^2 \theta e^{\pm 2i \phi} = \sqrt{\frac{15}{2\pi}} \frac{1 - \mu^2}{2} e^{\pm 2i \phi},
\]
we can rewrite Eq. \[33\] as
\[
n^i n^j \sigma_{ij} = \int \frac{d^3k}{(2\pi)^3} \hat{\mathcal{J}}_\ell (k \Delta \eta) P_\ell (\mu) \sum_{\ell, m} \sqrt{\frac{2\pi}{15}} \frac{4\eta \mathcal{J}_\ell^m (k \Delta \eta)}{\mathcal{Y}_{\ell m}^* (\hat{\mathbf{k}}, \mathbf{e})} Y_{\ell m} (\mathbf{n}, \mathbf{e}).
\]
\[36\]
We now introduce the initial tensor power spectrum \( P_H (k) \) through
\[
\langle \hat{\mathcal{H}}^{(\pm)} (\eta_S, \mathbf{k}) \hat{\mathcal{H}}^{(\pm)} (\eta_S', \mathbf{k}') \rangle = (2\pi)^3 \delta^{(3)} (\mathbf{k} - \mathbf{k}') P_H (k) T_k (\eta_S) T_k (\eta_S'),
\]
\[37\]
where $T_k(\eta)$ is the transfer function with the initial condition $T_k(\eta) \to (k\eta - \alpha)$ 1. The $\delta^{(3)}(k-k')$ is a consequence of stochastic homogeneity. For the shear we have then

$$\langle \hat{\sigma}^\pm(\eta_S, k) \hat{\sigma}^{\pm*}(\eta_S', k') \rangle = (2\pi)^3 \delta^{(3)}(k-k') P_H(k) \tilde{T}_k(\eta_S)\tilde{T}_k(\eta_S').$$ \hspace{1cm} (41)

Next, we express the terms in the distance–redshift relation with the help of the power spectrum of the integrand,

$$\langle n' n'\sigma_{ij}n'^i n'^k\sigma_{lk} \rangle = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1) \tilde{c}_\ell(\eta, \eta') P_\ell(n \cdot n').$$

A lengthly but straightforward calculation yields \[21\] \[2\]

Using the Limber approximation (see Appendix D) for the time integrals in Eq. (24), and, in particular, Eqs. (15), (16) and (17), we can simplify the time integrals, and we find the coefficients (under the ideal fluid assumption)

$$\tilde{c}_\ell(z, z') \simeq \frac{1}{\pi} \frac{(\ell + 2)!}{(\ell - 2)!} k^2 P_H(k) \tilde{T}_k(\eta') (A + B\tilde{H}(\eta, k) + C\tilde{H}'^2(\eta, k)), \hspace{1cm} (43)

$$

$$\tilde{T}_k(\eta) + 2\dot{H}\tilde{T}_k(\eta) + k^2\tilde{T}_k(\eta) = 0,$$ \hspace{1cm} (52)

with initial condition $T_k(\eta_{in}) = 1$ and $\dot{T}_k(\eta_{in}) = 0$ for $k\eta_{in} \ll 1$. In a matter (or radiation) dominated universe this differential equation can be solved analytically in terms of Bessel functions. The growing (not decaying) mode is given by

$$T_k(\eta) = (k\eta)^{1/2-q} Y_{1/2-q}(k\eta), \hspace{1cm} (53)$$

and $Y_\nu$ is the Bessel function of the second kind of order $\nu$. At late times, when the cosmological constant dominates, we cannot write the scale factor $a(\eta)$ as a power law and we have no analytic solution to (52). To determine the $c_\ell$ coefficients, we have solved the differential Eq. (52) numerically.

The resulting power spectrum $c_\ell(z, z)$ for different source redshifts is shown in Fig. 1.

**FIG. 1:** We show the tensor power spectrum rescaled by $\ell^4$ for the fluctuations in the luminosity distance for different values of the source redshift ($z = 0.5$, dotted pink line; $z = 1$, dotted blue line; $z = 2$, dashed green line; $z = 3$, long-dashed orange line; $z = 4$, solid red line). In the figure we have set $\eta = 1$.

Clearly, for sufficiently large $\ell$, $c_\ell(z, z) \propto \ell^{-4}$. The simplest way to understand this scaling is to note that once a mode enters the horizon, the tensor fluctuations scale like $\int \sigma d\eta \sim H \propto a_\eta/a \propto (k\eta)^{-q}$, where $a_\eta = a(\eta = 1/k)$ denotes the value of the scale factor at horizon entry. For modes that enter during the radiation era $q = 1$, while for modes that enter during the matter era $q = 2$. Hence $\int \sigma d\eta \propto H \propto H_{in}/k^3$ is acquiring a factor $k^{-q}$ with respect to the scale invariant initial spectrum. This leads to a red spectrum, $k^3 \int \sigma^2 \propto k^{-2q}$ and $\ell^4 c_\ell(z, z) \propto \ell^{-2q}$. This spectrum turns from
$c_{\ell} \propto \ell^{-4}$ for scales that enter the horizon in the radiation era to $c_{\ell} \propto \ell^{-6}$ for scales that enter the horizon in the matter era. For $z = 4$ this happens roughly at $\ell \sim 20$. Of course, the transition is quite gradual.

Comparing Fig. 1 with the results from scalar perturbations [3], we see first that the tensor contribution is much smaller, nearly 8 orders of magnitude. We obtain $\ell^2 c_{\ell}(z) \sim 5 \times 10^{-13}$ for $z = 4$ and $\ell \sim 40$ while scalar perturbations yield $\ell^2 c_{\ell}(z) \sim 10^{-5}$ for $z = 4$ and $\ell \sim 100$. Furthermore, despite also being proportional to the lensing term, it scales differently with $\ell$. This comes from the fact that the scalar lensing term is determined by the spectrum of $k^2 \Psi$, where $\Psi$ is the scale invariant Bardeen potential, while for scales that enter during radiation dominated expansion, $\sigma_{ij} n^i n^j$ is suppressed by a factor of $1/k$.

Interestingly the tensor signal is not monotonic in redshift up to $z \simeq 2$. It has a sharp minimum at $z \simeq 1.65$. To illustrate this, we also plot $c_{\ell}(z, z)$ as a function of the source redshift for different values of $\ell$ in Fig. 2.

The signal drops to 0 at $z_c = 1.65$. This comes from the fact that it is dominated by two terms with opposite sign. To see this, we also show the contributions from the three terms in the square bracket of (27) individually in Fig. 3.

If the source redshift is small, $\eta_S \sim \eta_0$, the first term $\propto -H_S^{-1} \sim -\eta_S/2$ dominates, while for large redshifts, $\eta_S \ll \eta_0$, the second term $\propto (\eta_0 - \eta)$ dominates. If $\sigma_{ij} n^i n^j$ has a definite sign, the result inherits this sign for small redshifts and the opposite sign for large redshifts. The sign changes happens around $\eta_S = \eta_0/2$ corresponding to a redshift $z_c \sim 3$. This is not expected to be very precise; in particular, we have neglected the time dependence of the transfer function in this argument. The more precise numerical evaluation gives $z_c \simeq 1.65$. Interestingly this redshift is close to the maximum of the angular diameter distance $D_A(z) = (\eta_0 - \eta_S)/(1 + z)$.

The results shown in Figs. 1 to 3 have been calculated with the following cosmological parameters: $h = 0.7$, $H_O^{-1} = 2997.9h^{-1}$Mpc, $\Omega_m h^2 = 0.13$, $\Omega_r h^2 = 4.17 \times 10^{-5}$, and $\Omega_\Lambda = 1 - \Omega_m - \Omega_r$.

IV. THE DISTANCE–REDSHIFT RELATION FROM VECTOR PERTURBATIONS

We now consider vector perturbations. As for tensor perturbations, we can divide out the cosmic expansion for lightlike geodesics. Hence we can consider Minkowski space with purely vector perturbations. The metric is then given by

$$ds^2 = -d\eta^2 - 2B_i dx^i d\eta + (\delta_{ij} + H_{ij} + \dot{H}_{ij}) dx^i dx^j, \quad (54)$$

where the perturbations are divergence-free, $B^i = H^i = 0$. Using $B_{ij} = B_{(ij)}$ and $H_{ij} = H_{(ij)}$, where ( ) denotes symmetrization, the shear on the constant time hypersurface is given by $\sigma_{ij} = B_{ij} + \dot{H}_{ij}$ or in 3-vector notation $\sigma_i = B_i + \dot{H}_i$, and $\sigma_{ij} = \sigma_{(i,j)}$. This quantity is gauge invariant [20].

A. The perturbation equations

With the Ricci tensor calculated in Appendix C we obtain for vector perturbations

$$R = \frac{1}{2} \left( \nabla^2 (\sigma n^i) + \dot{\sigma}_{ij} n^i n^j \right). \quad (55)$$
To determine the redshift we first evaluate the geodesic solution \[ (13) \] with the Christoffel symbols derived in Appendix \[ (21) \].

\[ k_0^2 = 1 - \int_{\lambda_s}^{\lambda_o} d\lambda \sigma_{ij}n^i n^j. \]  

(Vector perturbations can have a nonvanishing peculiar velocity term. We define the observer 4-velocity \((u'^{\mu}) = (1, B^i + v^i)\). The peculiar velocity \(v^i\) defined in this way is gauge invariant. It is the vorticity of the matter flow. We now obtain

\[ 1 + \delta z = -\frac{1 + n_i v^i_O}{k_O^2 + n_i v^i_O} = 1 + n_i \left( v^i_O - v^i_S \right) + \int_{\lambda_s}^{\lambda_o} d\lambda \sigma_{ij} n^i n^j. \]  

After a short calculation, using the results of Sec. II, we find the luminosity distance

\[ D_L(z,n) = (1+z)(\eta_O - \eta_S) \times \left( 1 + n_i v^i_O - 2 n_i v^i_S + \int_{\eta_S}^{\eta_O} d\eta \sigma_{ij} n^i n^j \right) + \frac{1}{\eta_O - \eta_S} \int_{\eta_S}^{\eta_O} d\eta \int_{\eta_S}^{\eta_O} d\eta' (\eta - \eta')(\eta - \eta') \eta - \eta - \int_{\eta_S}^{\eta_O} d\eta (\eta - \eta)(\eta - \eta) \frac{R}{\eta - \eta - \eta}. \]  

Since we are interested in expressing the luminosity distance as a function of the true redshift, we have to evaluate Eq. \( (55) \) at \( z \) and subtract the correction term defined through Eqs. \( (10) \) and \( (17) \),

\[ \Delta L(z,n) = -\frac{2}{\eta_O - \eta_S} \int_{\eta_S}^{\eta_O} d\eta' \sigma_{ij} n^i n^j \left( 1 - \frac{H^{-1}}{\eta_O - \eta_S} \right) \]

\[ \times \left( 1 - \frac{H^{-1}}{\eta_O - \eta_S} n_i v^i_O - n_i v^i_S \right) \]

\[ \left( 1 - \frac{H^{-1}}{\eta_O - \eta_S} n_i v^i_O - n_i v^i_S \right) \]

The expression depends only on the gauge-invariant quantities \( v^i \) and \( \sigma_{ij} \) as it should. We note also that we did not assume any gravitational theory yet. Indeed the procedure used so far is completely geometrical. If one is interested in general relativity (GR), then the two gauge-invariant variables \( v^i \) and \( \sigma_{ij} \) are not independent but related via Einstein’s equations \( (21) \),

\[ \nabla^2 \sigma_{ij} = -16\pi G \rho^2 c^2 (\tilde{\rho} + \tilde{p}), \]  

where \( \tilde{\rho} \) and \( \tilde{p} \) are the background density and pressure, respectively.

### B. Spherical harmonic analysis

As for the tensor perturbations, we are interested in the term \( n^i n^j \sigma_{ij} \). The main difference is that in the vector case we have \( \sigma_{ij} = \sigma_{(ij)} \) in real space and \( \tilde{\sigma}_{ij} = -ik(i \tilde{\sigma}_j) \) in Fourier space. This leads to

\[ n^i n^j \tilde{\sigma}_{ij} e^{ik \Delta \eta} = -kn^j \tilde{\sigma}_j \frac{\partial e^{ik \Delta \eta}}{\partial (k \Delta \eta)} \]  

With this we can write Eq. \( (59) \) as

\[ n^i n^j \sigma_{ij} = -\int \frac{d^3 k}{(2\pi)^3} k n^j \tilde{\sigma}_j \sum_{t=0}^{\infty} (2\ell + 1) i^\ell j_\ell^+(k \Delta \eta) P_\ell(\mu). \]  

With the helicity basis defined in Sec. III, the addition theorem of the spherical harmonics \( (57) \), and

\[ Y_{1,\pm1}(\mathbf{n},\mathbf{\hat{k}}) = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi} = \mp \sqrt{\frac{3}{(2\pi)^3}} \frac{1 - \mu^2}{2} \frac{e^{i\phi}}{3}, \]  

we obtain

\[ n^i n^j \sigma_{ij} = \int \frac{d^3 k}{(2\pi)^3} k \left( Y_{1,1}(\mathbf{n},\mathbf{\hat{k}}) \tilde{\sigma}^+ - Y_{1,-1}(\mathbf{n},\mathbf{\hat{k}}) \tilde{\sigma}^- \right) \]

\[ \times 4\pi \int \sum_{\ell,m} i^\ell j_\ell^k(kr) Y_{\ell m}^*(\mathbf{\hat{k}}, \mathbf{e}) Y_{\ell m}(\mathbf{n}, \mathbf{e}). \]  

If we assume that vector perturbations have been generated at some time in the past, we can define the vector power spectrum as for the tensor case as

\[ \langle \tilde{\sigma}^\pm (\mathbf{n}, \mathbf{\hat{k}}) \tilde{\sigma}^{\pm*} (\mathbf{n}', \mathbf{\hat{k}}') \rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}') P_{\sigma}(k) T_{\ell}(\eta_S) T_{\ell}(\eta_S). \]  

If we do not want to consider the case of early generation, we simply have to replace \( P_{\sigma}(k) T_{\ell}(\eta_S) T_{\ell}(\eta_S) \) by a time-dependent power spectrum, \( P_{\sigma}(k, \eta_S, \eta_S) \). The model under consideration (e.g., cosmic strings) then has to be used to determine this time-dependent power spectrum. If, however, vector perturbations evolve freely, we can then compute the shear power spectrum as for tensors,

\[ \langle n^i n^j \sigma_{ij} n^i n^j \sigma_{ik} \rangle = \frac{1}{4\pi} \sum_{\ell} (2\ell + 1) \bar{c}_\ell(\eta, \eta') P_\ell(\mathbf{n} \cdot \mathbf{n}'), \]  

\[ (67) \]
and a lensing part. At redshift $z$, we obtain a shear power spectrum of the form \[ \propto \sigma \left( \frac{\eta}{\sqrt{D}} \right) (1+z)^{6.5}, \]

which cannot be factorized into a random initial spectrum and a deterministic transfer function.

As mentioned above, in more realistic scenarios, where vector perturbations are generated, e.g., via anisotropic stresses from topological defects or by second order perturbations, we obtain a shear power spectrum of the form $P_\sigma (k, \eta, \eta')$, which cannot be factorized into a random initial spectrum and a deterministic transfer function.

In Appendix C 2 we nevertheless, for sake of completeness, continue with expression (68) to derive the vector angular power spectrum for the luminosity distance fluctuations. We do not repeat the lengthy, complicated, and not very illuminating formulas here.

V. CONCLUSIONS AND OUTLOOK

In this paper we have calculated the angular power spectrum of the linear vector and tensor fluctuations in the distance–redshift relation. For vector perturbations we have simply derived the formulas and for tensor fluctuations we have applied them to an initial spectrum of scalar perturbations at second order generated from scalar perturbations at second order. We therefore expect that the tensor signal is not monotonically increasing with redshift as we would expect it if the affine parameter $\lambda$ is normalized such that $\omega_S = 1$. In general one can use the Sachs focusing equation also with a different affine parameter normalization. Indeed from the distance definition $D = \sqrt{|\det D|}$, where $D$ is the Jacobi matrix that satisfies the differential equation
\[ \frac{d^2D}{d\lambda^2} = \begin{pmatrix} -R & \text{Re} (F) \\ \text{Im} (F) & -R + \text{Re} (F) \end{pmatrix} D, \]

with $F = \frac{i}{2} R_{\mu\nu\omega\gamma} \epsilon^{\mu\nu\delta} k^\omega k^\gamma$. The determinant of the Jacobi matrix $D$ describes the area of the thin light beam and its square root is therefore a distance. (A1), if the affine parameter $\lambda$ is normalized such that $\omega_S = 1$. In general one can use the Sachs focusing equation also with a different affine parameter normalization. Indeed from the distance definition
\[ D = \sqrt{\frac{dA_0}{d\Omega_S}}, \]

we find, setting $\omega = 1$,
\[ \bar{D} = \sqrt{\frac{dA_0}{d\Omega_S}} = \bar{\omega} \sqrt{\frac{dA_0}{d\Omega_S}} = \bar{\omega} D. \]

Since we are considering a light beam with a vertex at the source position, the initial conditions of the Sachs focusing equation are
\[ D (\lambda_S) = 0, \quad \left. \frac{dD (\lambda)}{d\lambda} \right|_{\lambda = \lambda_S} = 1, \]

if we normalize $\lambda$ such that $\omega_S = 1$. The general initial conditions for an arbitrary affine parameter $\lambda$, are given by (B6). Choosing $\omega_S = 1 + z$ one obtains the luminosity distance while $\omega_S = (1+z)^{-1}$ gives the angular diameter distance.

Appendix B: Details for tensor perturbations

Here we write down the nonvanishing Christoffel symbols and the Ricci tensor for the metric (15),
\[ \Gamma^0_{ij} = \eta^{0l} (H_{dl,i} + H_{i,j,l} - H_{ij,l}) = H_{ij}, \]
\[ \Rightarrow \sum_i \Gamma^0_{ii} = 0, \]
\[ \Gamma^i_{j0} = \eta^{ik} (H_{ko,j} + H_{jk,0} - H_{i0,k}) = \hat{H}_{ji}, \]
\[ \Rightarrow \Gamma^i_{j0} = 0, \]
\[ \Gamma^i_{jl} = \eta^{ik} (H_{jk,l} + H_{kl,j} - H_{il,j}) = H_{ij,l} + H_{i,j,l} - H_{ji,l}, \]
\[ \Rightarrow \Gamma^i_{ij} = 0. \]

Acknowledgments

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Appendix A: Sachs focusing equation

Much of this work is based on the Sachs focusing equation (17). It has been shown (23) that the distance can be defined as
\[ D = \sqrt{|\det D|}, \]

where $D$ is the Jacobi matrix that satisfies the differential equation
\[ \frac{d^2D}{d\lambda^2} = \begin{pmatrix} -R & \text{Re} (F) \\ \text{Im} (F) & -R + \text{Re} (F) \end{pmatrix} D, \]

with $F = \frac{i}{2} R_{\mu\nu\omega\gamma} \epsilon^{\mu\nu\delta} k^\omega k^\gamma$. The determinant of the Jacobi matrix $D$ describes the area of the thin light beam and its square root is therefore a distance. (A1), if the affine parameter $\lambda$ is normalized such that $\omega_S = 1$. In general one can use the Sachs focusing equation also with a different affine parameter normalization. Indeed from the distance definition
\[ D = \sqrt{\frac{dA_0}{d\Omega_S}}, \]

and the solid angle aberration (17)
\[ \frac{d\Omega}{d\Omega_S} = (k_{\mu} u^\mu)^2 = \frac{\omega^2}{\bar{\omega}^2}, \]

we find, setting $\omega = 1$,
\[ \bar{D} = \sqrt{\frac{dA_0}{d\Omega_S}} = \bar{\omega} \sqrt{\frac{dA_0}{d\Omega_S}} = \bar{\omega} D. \]

Since we are considering a light beam with a vertex at the source position, the initial conditions of the Sachs focusing equation are
\[ D (\lambda_S) = 0, \quad \left. \frac{dD (\lambda)}{d\lambda} \right|_{\lambda = \lambda_S} = 1, \]

if we normalize $\lambda$ such that $\omega_S = 1$. The general initial conditions for an arbitrary affine parameter $\lambda$, are given by (B6). Choosing $\omega_S = 1 + z$ one obtains the luminosity distance while $\omega_S = (1+z)^{-1}$ gives the angular diameter distance.

Appendix B: Details for tensor perturbations

Here we write down the nonvanishing Christoffel symbols and the Ricci tensor for the metric (15),
\[ \Gamma^0_{ij} = \eta^{0l} (H_{dl,j} + H_{i,j,l} - H_{ij,l}) = H_{ij}, \]
\[ \Rightarrow \sum_i \Gamma^0_{ii} = 0, \]
\[ \Gamma^i_{j0} = \eta^{ik} (H_{ko,j} + H_{jk,0} - H_{i0,k}) = \hat{H}_{ji}, \]
\[ \Rightarrow \Gamma^i_{j0} = 0, \]
\[ \Gamma^i_{jl} = \eta^{ik} (H_{jk,l} + H_{kl,j} - H_{il,j}) = H_{ij,l} + H_{i,j,l} - H_{ji,l}, \]
\[ \Rightarrow \Gamma^i_{ij} = 0. \]
These components lead to

\[ R_{00} = R_{0i} = 0, \quad R_{ij} = H_{ij} - H_{i,ji} = -\Box H_{ij}. \quad (B5) \]

\[ R = \frac{1}{2} (R_{00} + 2 R_{0i} n^i + R_{ij} n^i n^j) = -\frac{1}{2} n^i n^j \Box H_{ij}. \quad (B7) \]

**Appendix C: Details for vector perturbations**

1. Christoffel symbols and Ricci tensor

Here we write down the nonvanishing Christoffel symbols and the Ricci tensor for the metric \([43]\).

\[ \Gamma^i_{00} = -\dot{B}_i, \quad (C1) \]

\[ \Gamma^i_{j0} = \frac{1}{2} (\partial_h (H_{ij} + H_{ji}) - B_{i,j} + B_{j,i}), \quad (C2) \]

\[ \Gamma^i_{0i} = \dot{H}_{i,i} \quad \Rightarrow \quad \Gamma^i_{0i} = 0, \quad (C3) \]

\[ \Gamma^i_{ij} = \frac{1}{2} (B_{i,j} + B_{j,i} + \dot{H}_{ij} + \dot{H}_{ji}) = \sigma_{ij}, \quad (C4) \]

\[ \Gamma^0_{ii} = B_{i,i} + \dot{H}_{i,i} \quad \Rightarrow \quad \sum_i \Gamma^0_{ii} = 0, \quad (C5) \]

\[ \Gamma^i_{jk} = H_{i,jk}, \quad \Gamma^i_{ik} = H_{i,ik} \quad \Rightarrow \quad \Gamma^i_{ik} = 0, \quad (C6) \]

\[ R_{00} = 0, \quad (C9) \]

\[ R_{0i} = \frac{1}{2} (B_{i,j} + \dot{H}_{i,j}) = \frac{1}{2} \nabla^2 \sigma_i, \quad (C10) \]

\[ R_{ij} = \frac{1}{2} (B_{i,j} + B_{j,i} + \dot{H}_{ij} + \dot{H}_{ji}) = \hat{\sigma}_{ij}. \quad (C11) \]

With these Ricci tensor components we can easily compute

\[ R = \frac{1}{2} (R_{00} + 2 R_{0i} n^i + R_{ij} n^i n^j) = -\frac{1}{2} (\nabla^2 \sigma_i n^i + \hat{\sigma}_{ij} n^i n^j). \quad (C12) \]

2. The \(e_\ell\) coefficients

We first derive in detail Eq. \([65]\). We use the relation

\[ Y_{1,\pm 1}(n, \hat{k}) = \sqrt{\frac{4\pi}{3}} \sum_{m'=-1}^1 \frac{1}{Y_{1,m'}(n, e)} Y_{1,m'}(\hat{k}, e). \quad (C13) \]

Here \(Y_{1,m'}(\hat{k}, e)\) is the vector spherical harmonic. The general addition theorem for spin weighted spherical harmonics used in Eq. \([C13]\) above can be found, e.g., in \([20]\).

Applying this to Eq. \([65]\), we obtain

\[ n^i n^j \hat{\sigma}_{ij} = \left( \frac{4\pi}{3} \right) \sum_{\ell m} \frac{1}{Y_{\ell m}(n)} Y_{\ell m}(n) \times \int \frac{d^3k}{(2\pi)^3} k^j j^i_j(k\Delta \eta) Y_{\ell m}^*(\hat{k}) \begin{pmatrix} \hat{\sigma}^+ - \sigma^+_m(k\Delta \eta) - \hat{\sigma}^- \sigma^-_{-m}(k\Delta \eta) \end{pmatrix}. \quad (C14) \]

Here we omit the arbitrary unit vector \(e\) in the notation, and we have introduced the helicity basis,

\[ \hat{\sigma}_{\ell} = \hat{\sigma}^{+} \sigma^{e\ell}_{\ell} + \hat{\sigma}^{-} \sigma^{-e\ell}_{\ell} \]

defined in Eq. \([54]\). In the special case with \(n = e\) we obtain

\[ e^i e^j \sigma_{ij} = \frac{4\pi}{\sqrt{3}} \int \frac{d^3k}{(2\pi)^3} k^j j^i_j(k\Delta \eta) Y_{10}^*(\hat{k}) \begin{pmatrix} \hat{\sigma}^+ - \sigma^+_0(k\Delta \eta) - \hat{\sigma}^- \sigma^-_0(k\Delta \eta) \end{pmatrix}. \quad (C15) \]

where we have used

\[ Y_{10}(e, e) = \sqrt{\frac{2\ell + 1}{4\pi}} \delta_{m0}, \quad (C16) \]

and Eq. \([C13]\) for \(m = e\) which yields,

\[ Y_{1,\pm 1}(e, \hat{k}) = \sqrt{\frac{4\pi}{3}} \sum_{m'=-1}^1 \frac{1}{Y_{1,m'}(e, e)} Y_{1,m'}^*(\hat{k}, e) = \mp \in Y_{1,10}(\hat{k}, e). \quad (C17) \]

Since the two point correlation function \([67]\) depends on the angle \(n \cdot n'\) only we can set \(n' = e\) without loss of generality. With this we find

\[ \langle n^i n^j \hat{\sigma}_{ij} e^i e^j \rangle = \frac{8}{3\sqrt{3}} \left( \sum_{\ell m} \frac{1}{Y_{\ell m}(n)} \frac{1}{Y_{1,m'}(n)} \right) \times \begin{pmatrix} 1 & 1 \end{pmatrix} \times \int d\Omega \hat{k} f_m^*(\hat{k}) Y_{\ell m}^*(\hat{k}) Y_{10}^*(\hat{k}), \quad (C18) \]

where we have introduced

\[ f_m^*(\hat{k}) = -Y_{1,m'}^*(\hat{k}) - Y_{1,10}(\hat{k}) + Y_{1,m'}^*(\hat{k}) Y_{1,10}(\hat{k}) \quad (C19) \]

Since the spherical harmonics form an orthogonal basis on \(S^2\), we can expand the product of two of them again in terms of spherical harmonics using the Clebsch–Gordan coefficients \([24]\). In the case of Eq. \([C18]\) we use

\[ Y_{\ell m}^*(\hat{k}) Y_{10}^*(\hat{k}) = \sum_{L=|\ell - \tilde{\ell}|}^{\ell + \tilde{\ell}+1} \frac{\ell + \tilde{\ell} + 1}{4\pi (2L + 1)} \langle \ell, 0, \tilde{\ell}, 0 | L, m, 0, \tilde{\ell}, 0 | L, m \rangle Y_{\ell m}^*(\hat{k}). \quad (C20) \]
The dependence of the spherical harmonics on the azimuthal angle $\phi$,

$$Y_{LM}^* \propto e^{-im\phi} \quad \text{and} \quad \pm 1 Y_{Lm'}^* \propto e^{-im'\phi} \quad (C21)$$

implies that the integral over angles in Eq. (C18) only contributes for $m' = - m$. Therefore also $m \in \{-1,0,1\}$.

Since $f_{mn}(k)$ contains only terms that either do not depend on $\theta$ or that are quadratic in $\sin(\theta)$ and $\cos(\theta)$ the only nonvanishing contributions are $L = 0$ or $L = 2$. Analogous to Eq. (C20) we write

$$Y_{\ell,m}(n) Y_{1,-m}(n) = \sum_{n} \frac{\sqrt{3(2\ell + 1)}}{4\pi(2n + 1)} \times (\ell, 0, 1) \langle n, 0 \rangle (\ell, m, 1, -m) \langle n, 0 \rangle Y_{n0}(n). \quad (C22)$$

The addition theorem for the spherical harmonics implies

$$P_n \langle n \cdot e \rangle = \sqrt{\frac{4\pi}{2n + 1}} Y_{n0}(n). \quad (C23)$$

Using these identities we can rewrite the correlation function $\langle n^i n^j \sigma_{ij} \rangle$ as

$$\langle n^i n^j \sigma_{ij} \rangle = \sum_{n} \sum_{\ell=0,2} \sum_{m=-1}^{1} \frac{i^{\ell-j}}{3\pi^{1/2}} \times (2\ell + 1) \left(2\ell + 1\right)(2L + 1)^{-1/2} \times (\ell, 0, 1) \langle L, 0 \rangle (\ell, m, 1, -m) \langle L, 0 \rangle Y_{Lm'0}(n). \quad (C24)$$

where we have introduced

$$B_{Lm} = \int d\Omega \frac{f_{m}(\hat{k}) Y_{Lm}^*(\hat{k})}{\langle \hat{k} \rangle}. \quad (C25)$$

The nonvanishing coefficients are given by

$$B_{00} = \frac{1}{\sqrt{\pi}} \quad B_{2\pm 1} = \frac{1}{2} \frac{3}{\sqrt{5\pi}} \quad B_{2,0} = -\frac{1}{\sqrt{5\pi}}. \quad (C26)$$

Computing the sum of the Clebsch–Gordan coefficients, we find

$$\langle n^i n^j \sigma_{ij} \rangle = \frac{\ell(\ell + 1)}{2\pi(2\ell + 1)} P_\ell \langle n \cdot e \rangle \times \int dk k^2 P_\ell(k) T_\ell(k) \langle \eta \rangle \times \left[ j_{l-1}(k(\Delta \eta)) j_{l-1}(k(\Delta \eta')) + j_{l+1}(k(\Delta \eta)) j_{l+1}(k(\Delta \eta')) \right]. \quad (C27)$$

Using this result for the $c_\ell$’s defined in Eq. (C28), we obtain directly Eq. (C28).

Now we compute the full $c_\ell$ coefficient defined in (C28). We use the Limber approximation (Appendix D) to do the time integrals $\int d\nu T_\nu(\eta) j_{l+1}(k(\eta_0 - \eta))$. We start with the Doppler terms, the first line of Eq. (C28), which contribute to the dipole term only

$$c_1^D = \frac{4\pi H_{S-1} H_{S-1}^*}{3 \Delta \eta_S \Delta \eta_{S'}} < |\bar{v}_O|^2 >. \quad (C28)$$

This dipole is the same as one from the scalar analysis [3]. We cannot, of course, decide which part of the observer velocity comes from scalar perturbations and which part from vector perturbations. Since this dipole term is highly nonlinear, we neglect it in the subsequent analysis and consider only $\ell \geq 2$. We now determine the other terms. From the peculiar velocity of the source we obtain

$$c_\ell^{(1)} = \frac{2\ell(\ell + 1)}{\pi(2\ell + 1)^2} \left(1 - \frac{H_{S-1}^* H_{S-1}}{\Delta \eta_S \Delta \eta_{S'}}\right) \int dk k^2 P_\ell(k) T_\ell^S(\eta) T_\ell^S(\eta_S) \sum_{l' = -1}^{\ell+1} j_{l'}(k\Delta \eta_S) j_{l'}(k\Delta \eta_{S'}) \quad (C29)$$

where we have introduced the velocity power spectrum defined by

$$\langle \tilde{v}^+ \langle \eta_S, k \rangle \rangle \langle \tilde{v}^+ \langle \eta_{S'}, k' \rangle \rangle = (2\pi)^3 \delta^{(3)}(k - k') P_\ell(k) T_\ell^S(\eta_S) T_\ell^S(\eta_{S'}) \quad (C30)$$

and $\tilde{v}^+$ is the peculiar velocity in terms of the helicity basis defined in Eq. (34).

The second line of the redshift–distance relation (30) leads to

$$c_\ell^{(2)} \approx \frac{\ell(\ell + 1)}{2\pi(1 + 2\ell)^2} \frac{H_{S-1}^* H_{S-1}^*}{\Delta \eta_S \Delta \eta_{S'}} \int dk k^2 P_S(k) \times \sum_{l' = -1}^{\ell+1} I_{l'1}^2 T_\ell^S(\eta_{l'1,k}) \Theta \left(\eta_{l'1,k} - \eta_S\right) \Theta \left(\eta_{l'1,k} - \eta_{S'}\right) + \left(\frac{\ell + 1}{\ell}\right) I_{l'1}^2 T_\ell^S(\eta_{l'1,k}) \Theta \left(\eta_{l'1,k} - \eta_{S'}\right) \Theta \left(\eta_{l'1,k} - \eta_{S'}\right)$$
The fourth line is composed of the two terms that contribute to \( R \) given in Eq. (33). Denoting them with superscripts (41), (42), and their correlation with (412) we obtain

\[
c_t^{(41)} \approx \frac{\ell (1 + \ell)}{2\pi (1 + 2\ell)^2} \frac{1}{\Delta \eta_s \Delta \eta_s'} \int dk P_\sigma(k) \sum_{\ell = 1, \ell + 1} \left[ \ell^2 (\eta_\ell - \eta_s)(\eta_\ell - \eta_s') \left( \frac{\ell}{\ell + 1} \right)^2 I_{\ell-k}^2 T_{\ell-k}^2 (\eta_\ell) \theta (\eta_\ell - \eta_s) \theta (\eta_\ell - \eta_s') \right] \\
+ \left( \frac{\ell}{\ell + 1} \right)^2 I_{\ell-k}^2 T_{\ell-k}^2 (\eta_\ell) \theta (\eta_\ell - \eta_s) \theta (\eta_\ell - \eta_s') \\
- \left( \frac{\ell}{\ell + 1} \right) I_{\ell-k} I_{\ell-k} T_{\ell-k} (\eta_\ell) \theta (\eta_\ell - \eta_s) \theta (\eta_\ell - \eta_s') + (S \leftrightarrow S') .
\]  

Next, we compute the cross terms between the different lines of the distance-redshift relation [50]. We start with the second and third lines,

\[
c_t^{(23)} \approx - \frac{\ell (1 + \ell)}{2\pi (1 + 2\ell)^2} \frac{1}{\Delta \eta_s \Delta \eta_s'} \int dk P_\sigma(k) \sum_{\ell = 1, \ell + 1} \left[ \ell^2 I_{\ell-k}^2 T_{\ell-k}^2 (\eta_\ell) \theta (\eta_\ell - \eta_s) \theta (\eta_\ell - \eta_s') \right] \\
+ \left( \frac{\ell}{\ell + 1} \right)^2 I_{\ell-k}^2 T_{\ell-k}^2 (\eta_\ell) \theta (\eta_\ell - \eta_s) \theta (\eta_\ell - \eta_s') \\
- \left( \frac{\ell}{\ell + 1} \right) I_{\ell-k} I_{\ell-k} T_{\ell-k} (\eta_\ell) \theta (\eta_\ell - \eta_s) \theta (\eta_\ell - \eta_s') + (S \leftrightarrow S') .
\]  

The second and fourth lines yield

\[
c_t^{(241)} \approx \frac{\ell (1 + \ell)}{4\pi (1 + 2\ell)^2} \frac{1}{\Delta \eta_s \Delta \eta_s'} \int dk k^2 P_\sigma(k)
\]
\[ c^{(42)}_\ell \approx \frac{\ell (\ell + 1)}{4\pi (\ell + 2)^2} \frac{1}{\Delta \eta_S} \frac{1}{\Delta \eta_S} \int dk k P_\sigma (k) \times \sum_{\ell' = 1, \ell' + 1} \left[ \left( \eta_{\ell', k} - \eta_{\ell', S} \right) (\ell' - 1) I_{\ell' - 1, \ell} T_k \left( \eta_{\ell', k} \right) \right] + (S \leftrightarrow S'), \tag{C37} \]

and, the third and fourth lines give

\[ c^{(34)}_\ell \approx - \frac{\ell (\ell + 1)}{4\pi (\ell + 2)^2} \frac{1}{\Delta \eta_S} \frac{1}{\Delta \eta_S} \int dk k P_\sigma (k) \times \sum_{\ell' = 1, \ell' + 1} \left[ (\eta_{\ell', k} - \eta_{\ell', S}) (\ell' - 1) I_{\ell' - 1, \ell} T_k \left( \eta_{\ell', k} \right) \right] + (S \leftrightarrow S'), \tag{C39} \]

\[ c^{(42)}_\ell \approx - \frac{\ell (\ell + 1)}{4\pi (\ell + 2)^2} \frac{1}{\Delta \eta_S} \frac{1}{\Delta \eta_S} \int dk k P_\sigma (k) \times \sum_{\ell' = 1, \ell' + 1} \left[ (\eta_{\ell', k} - \eta_{\ell', S}) (\ell' - 1) I_{\ell' - 1, \ell} T_k \left( \eta_{\ell', k} \right) \right] + (S \leftrightarrow S'). \tag{C40} \]

To determine the correlation between the peculiar velocities and the shear on the constant time hypersurface \( \sigma_{ij} \), we need to specify the gravitation theory. We choose GR by using the Einstein’s equations \([31]\). Correlating the term for the peculiar velocity of the source \( v_S \) with the others, we find

\[ c^{(12)}_\ell = \frac{-1}{16\pi G a_S^2 (\rho_S + p_S)} \left( \frac{H_S^{-1}}{\Delta \eta_S} \right) \frac{H_S^{-1}}{\Delta \eta_S} \frac{2\ell (\ell + 1)}{2(\ell + 1)^2} \int dk k^2 P_\sigma (k) T_k (\eta_S) \sum_{\ell' = 1, \ell' + 1} j^{ij}_\ell (k\Delta \eta) \int_{\eta_S}^{\eta_0} d\eta T_k (\eta) j^j_\ell (k\Delta \eta) \right] + (S \leftrightarrow S'), \tag{C41} \]

\[ c^{(13)}_\ell = \frac{1}{16\pi G a_S^2 (\rho_S + p_S)} \left( \frac{H_S^{-1}}{\Delta \eta_S} \right) \frac{H_S^{-1}}{\Delta \eta_S} \frac{2\ell (\ell + 1)}{2(\ell + 1)^2} \int dk k^2 P_\sigma (k) T_k (\eta_S) \sum_{\ell' = 1, \ell' + 1} j^{ij}_\ell (k\Delta \eta) \int_{\eta_S}^{\eta_0} d\eta (\eta_S - \eta) T_k (\eta) j^j_\ell (k\Delta \eta) \right] + (S \leftrightarrow S'). \tag{C41} \]
\[
\epsilon^\text{(141)}_\ell = \frac{1}{8\pi G a_5^2 (\rho_s + p_s)} \left( \frac{1 - H S^{-1}}{\Delta n_s} \right) \frac{1}{\Delta n_s^\prime \pi (2 \ell + 1)^2} \sum_{\ell = -\ell + 1}^{\ell} \left[ I_{\ell}^2 \frac{1}{\Delta n_s^\prime} P_{\ell} \left( \frac{\ell}{\Delta n_s} \right) T_{\ell/\Delta n_s}(\eta_s) \right. \\
\times \left( T_{\ell/\Delta n_s}(\eta_s + \frac{\Delta n_s}{\ell}) I_{\ell-1}(\ell - 1) \Theta \left( \eta_s - \eta_s' + \frac{\Delta n_s}{\ell} \right) - T_{\ell/\Delta n_s}(\eta_s) I_{\ell}(\ell + 1) \Theta (\eta_s - \eta_s') \right) \left( S \leftrightarrow S' \right),
\]

\[
\epsilon^\text{(142)}_\ell = \frac{1}{8\pi G a_5^2 (\rho_s + p_s)} \left( \frac{1 - H S^{-1}}{\Delta n_s} \right) \frac{1}{\Delta n_s^\prime \pi (2 \ell + 1)^2} \sum_{\ell = -\ell + 1}^{\ell} \left[ I_{\ell}^2 \frac{1}{\Delta n_s^\prime} P_{\ell} \left( \frac{\ell}{\Delta n_s} \right) T_{\ell/\Delta n_s}(\eta_s) \right. \\
\times \left( T_{\ell/\Delta n_s}(\eta_s + \frac{\Delta n_s}{\ell}) I_{\ell-1}(\ell - 1) \left( \eta_s - \eta_s' + \frac{\Delta n_s}{\ell} \right) \Theta (\eta_s - \eta_s') \right) + \left( S \leftrightarrow S' \right),
\]

Appendix D: Limber approximation

In this work we have used the Limber approximation \[2\] repeatedly. It approximates the integral of the product of a spherical Bessel function and a slowly varying function (e.g., a power law) by

\[
\int_{x_1}^{x_2} dx f(x) j_\ell(x) \approx I_{\ell} f(\ell - \ell) \Theta(x - x_1),
\]

for \( x_2 > x_1 \), where \( \Theta \) denotes the Heaviside function defined by

\[
\Theta(x) = \begin{cases} 
0, & x \leq 0 \\
1, & x > 0
\end{cases},
\]

and \( I_\ell^2 \approx 1.58/\ell \) describes the area under the first peak of the spherical Bessel function \( j_\ell(x) \). This rather crude approximation considers the contribution under the first peak only, and it usually gives an overestimation, but never by more than a factor of 2 \[2\]. Of course, if the function \( f \) varies heavily in the region of the first peak, \( \ell - 1 < x < \ell + 1 \), the approximation cannot be used.

We are, in particular, interested in (note that \( \Delta \eta = \eta_O - \eta \) and \( \Delta \eta' = \eta_O - \eta' \))

\[
\int_{\eta_s}^{\eta_o} d\eta T(\eta) j_\ell(k\Delta \eta) \approx \frac{1}{k} T(\eta_{k,k}) I_\ell \Theta(\eta_{k,k} - \eta),
\]

\[
\int_{\eta_s}^{\eta_o} d\eta T(\eta) j_\ell(k\Delta \eta') \approx \frac{1}{k} T(\eta_{k,k}) I_\ell \Theta(\eta_{k,k} - \eta),
\]

\[
\int_{\eta_s}^{\eta_o} d\eta \int_{\eta_s}^{\eta_o} d\eta' \frac{\hat{T}(\eta')}{(k\Delta \eta')^2} j_\ell(k\Delta \eta') 
\approx \frac{1}{k^2} \hat{T}(\eta_{k,k}) I_\ell \Theta(\eta_{k,k} - \eta),
\]

\[
\int_{\eta_s}^{\eta_o} d\eta (\eta - \eta_{s,c}) (\eta_{O,c} - \eta) \mathcal{H}(\eta) \hat{T}(\eta) j_\ell(k\Delta \eta) \frac{(k\Delta \eta)^2}{(k\Delta \eta')^2} 
\approx \frac{1}{k^2} \hat{T}(\eta_{k,k}) I_\ell \Theta(\eta_{k,k} - \eta) \mathcal{H}(\eta_{k,k}) \hat{T}(\eta_{k,k}) \Theta(\eta_{k,k} - \eta),
\]

\[
9 \]

\[
13
\]
\[
\int_{\eta_S}^{\eta_0} d\eta \int_{\eta_0}^{\eta_S} d\eta' T(\eta') j'_\ell(k \Delta \eta') \\
\approx \frac{\ell - 1}{k^2} T(\eta_{\ell-1,k}) I_{\ell-1} \Theta(\eta_{\ell-1,k} - \eta_S) \\
- \frac{\ell + 1}{k^2} T(\eta_{\ell,k}) I_{\ell} \Theta(\eta_{\ell,k} - \eta_S), 
\]
(D8)

\[
\int_{\eta_S}^{\eta_0} d\eta (\eta - \eta_S) (\eta_0 - \eta) T_k(\eta) j_\ell(k \Delta \eta) \\
\approx \frac{\ell}{k^2} T(\eta_{\ell,k}) I_{\ell} \Theta(\eta_{\ell,k} - \eta_S), 
\]
(D9)

\[
\int_{\eta_S}^{\eta_0} d\eta (\eta - \eta_S) (\eta_0 - \eta) \dot{T}(\eta) j'_\ell(k \Delta \eta) 
\]

We have used the following propriety of the spherical Bessel functions [24]:

\[
j'_\ell(k \Delta \eta) = j_{\ell-1}(k \Delta \eta) - \frac{\ell + 1}{k \Delta \eta} j_\ell(k \Delta \eta). 
\]
(D11)

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