Non linear stability of an expanding universe 
with $S^1$ isometry group

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Abstract

We prove the existence for an infinite proper time in the expanding 
direction of spacetimes satisfying the vacuum Einstein equations on a 
manifold of the form $\Sigma \times S^1 \times R$ where $\Sigma$ is a compact surface of genus $G > 1$. The Cauchy data are supposed to be invariant with respect to 
the group $S^1$ and sufficiently small, but we do not impose a restrictive 
hypothesis made in the previous work [1].

1 Introduction.

An einsteinian universe is a pair $(V, (4)g)$, with $V$ a smooth 4 dimensional 
manifold and $(4)g$ a lorentzian metric on $V$ which satisfies the Einstein equa-
tions. Such a universe satisfies the classical causality requirements if it is 
globally hyperbolic, equivalently if $V$ is a product $V = M \times R$ with each 
$M_t \equiv M \times \{t\}$ space like and a Cauchy surface, i.e. intersected once by each 
causal curve (timelike or null). It is well known that given a 3 dimensional 
manifold endowed with a properly riemannian metric $\bar{g}$ and a symmetric 2 
tensor $K$ satisfying the Einstein constraint equations, there exists (modulo 
appropriate functional hypotheses on the data) a globally hyperbolic vac-
um (the theorem extends to classical sources which admit of a well posed 
Cauchy problem) einsteinian universe such that $M_{t_0}$ is a Cauchy surface, and 
the spacetime metric $(4)g$ induces on $M_{t_0}$ the metric $\bar{g}$ while $K$ is the extrin-
sic curvature of $M_{t_0}$. This solution is unique, up to isometry, in the class of 
maximal spacetimes (i.e. which cannot be embedded in a larger one). In
spite of its formulation using $R$ this solution is a local one: $t$ is just a coordinate, it has no intrinsic meaning: the physically meaningful quantity is the proper time, determined by the metric $(^{(4)} g)$. The main problems which remain open in this field are the infinite proper time existence, or the formation of singularities and their nature.

In this article we will prove the existence for an infinite proper time, in the expanding direction, of vacuum einsteinian universes with Cauchy data which are in a neighbourhood of a vacuum einsteinian universe defined as follows.

- $V = M \times \mathbb{R}$ is such that $M$ is a compact manifold of the form $M = S^1 \times \Sigma$, with $\Sigma$ a smooth, orientable surface.

- The spacetime metric is invariant under the group $S^1$. It is given by:

$$(^{(4)} g) = -4dt^2 + 2t^2 \sigma + \theta^2$$

with $\sigma$ a metric on $\Sigma$ independent of $t$ and of scalar curvature $-1$, and $\theta$ a 1-form on $S^1$. In local coordinates $x^a$ on $\Sigma$ and $dx^3$ on $S^1$ we have:

$$\sigma = \sigma_{ab}dx^a dx^b, \quad a, b = 1, 2 \quad \theta = dx^3.$$ 

The property

$$R(\sigma) = -1$$

implies, by the Gauss Bonnet theorem, that the surface $\Sigma$ has genus $G > 1$.

Remark (important fact for the Thurston classification). The lorentzian 3 metric $-4dt^2 + 2t^2 \sigma$ is homogeneous in $t$, but not the 4 metric.

The above universe is a particular case of the ones described in the next section.

## 2 $S^1$ invariant einsteinian universes.

The spacetime manifold is a product $S^1 \times \Sigma \times \mathbb{R}$, where $\Sigma$ is a smooth, compact, orientable 2 - manifold of genus $G > 1$. The spacetime 4 metric is invariant under the action of the group $S^1$, with spacelike orbits $S^1 \times \{x\} \times \{t\}$. We restrict here our study to the so called polarized case where the
orbits are orthogonal to 3 dimensional lorentzian sections. The metric can be written, without loss of generality and for later convenience, in the form:

\[ (4)\ g = e^{-2\gamma(3)}g + e^{2\gamma(\theta)}^2, \]  

with \(\gamma\) a scalar function and \(\gamma\) an arbitrary lorentzian 3-metric on \(\Sigma \times R\) which we write under the usual form adapted to the Cauchy 2+1 splitting:

\[ (3)\ g = -N^2dt^2 + g_{ab}(dx^a + \nu^a dt)(dx^b + \nu^b dt) \]  

\(N, \nu, g = g_{ab}dx^a dx^b\) are \(t\) dependent scalar (lapse), vector (shift), metric on \(\Sigma\) and \(\theta = dx^3\) a 1-form on \(S^1, x^3\) a periodic coordinate. We denote by \(\partial_\alpha\) the Pfaff derivatives in the coframe \(\theta^0 = dt, \theta^a = dx^a + \nu^a dt,\) greek indices taking the values 0, 1 or 2. It holds that:

\[ \partial_0 = \frac{\partial}{\partial t} - \nu^a \frac{\partial}{\partial x^a} \]  

We introduce the extrinsic curvature \(k_t\) of \(\Sigma_t\) i.e. we set

\[ k_{ab} = -\frac{1}{2N} \hat{\partial}_0 g_{ab}, \]  

with \(\hat{\partial}_0\) the operator on time dependent tensors on \(\Sigma\) given by:

\[ \hat{\partial}_0 = \frac{\partial}{\partial t} - L_{\nu^t}. \]  

with \(L_{\nu^t}\) the Lie derivative with respect to \(\nu_t\). The mean curvature \(\tau\) of \(\Sigma_t\), which will play a fundamental role in our later estimates, is:

\[ \tau \equiv g^{ab}k_{ab}. \]  

3 Equations.

The metric \((4)\ g\) is supposed to satisfy the vacuum Einstein equations,

\[ Ricci((4)\ g) = 0. \]

The equations \((4)\ R_{\alpha3} = 0\) are identically satisfied by the metric 2.1.
3.1 Wave equation for $\gamma$.

The equation $(4)^\mathbb{R}33 = 0$ implies that the function $\gamma$ satisfies the wave equation on $(\Sigma \times R, (3)^\mathbb{g})$. This equation reads:

$$-N^{-1}\partial_0(N^{-1}\partial_0\gamma) + Ng^{ab}\nabla_a(N\partial_b\gamma) + N^{-1}\tau\partial_0\gamma = 0 \quad (3.1)$$

3.2 3 dimensional Einstein equations.

When $(4)^\mathbb{R}33 = 0$ and $(4)^\mathbb{R}_{33} = 0$ the equations $(4)^\mathbb{R}_{\alpha\beta} = 0$ are equivalent to Einstein’s equations on $\Sigma \times R$ for the metric $(3)^\mathbb{g}$ with source the stress energy tensor of the scalar field $\gamma$, namely:

$$(3)^\mathbb{R}_{\alpha\beta} = 2\partial_\alpha\gamma\partial_\beta\gamma \quad (3.2)$$

In dimension 3 the Einstein equations are non dynamical, except for the conformal class of $g$. We set on $\Sigma \times R$

$$g_{ab} = e^{2\lambda}\sigma_{ab},$$

We impose that, on each $\Sigma_t$, $\sigma_t$ has scalar curvature $R(\sigma_t) = -1$, i.e. $\sigma_t \in M_{-1}$, which is no restriction since any metric $\sigma$ on $\Sigma$, which is of genus greater than 1, is conformal to such a metric, and $e^{2\lambda}$ is to be determined.

As a gauge condition we suppose that the mean extrinsic curvature $\tau$ is constant on each $\Sigma_t$, i.e. depends only on $t$. We will construct an expanding space time, i.e. we take $\tau$ to be negative and increasing from a value $\tau_0 < 0$.

The moment of maximum expansion will be attained if $\tau$ reaches the value $\tau = 0$. For convenience we define the parameter $t$ by

$$t = -\tau^{-1}, \quad t \geq t_0. \quad (3.3)$$

The following equations (momentum constraint) hold on $\Sigma_t$:

$$-Ne^{-2\lambda(3)}R_{0a} \equiv D_bh^b_a = L_a \equiv -2D_a\gamma' \quad (3.4)$$

with $h_{ab} = k_{ab} - \frac{1}{2}g_{ab}\tau$ the traceless part of $k_t$, $D_a$ the covariant derivative in the metric $\sigma_t$ and indices raised with $\sigma^{ab}$, and we have set:

$$\dot{\gamma} = e^{2\lambda}\gamma' \quad \text{with} \quad \gamma' = N^{-1}\partial_0\gamma.$$

Given $\sigma, \gamma$ and $\dot{\gamma}$ this is a linear equation for $h$ on $\Sigma_t$, independent of $\lambda$. The general solution is the sum of a solution $q$ of the homogeneous equation, a
trace and divergence free tensor called a TT tensor, and a conformal Lie derivative \( r \). Such tensors are \( L^2(\sigma) \) orthogonal.

The so called hamiltonian constraint on \( \Sigma_t \), on the other hand, is given by

\[
2N^{-2(3)}S_{00} \equiv R^{(3)}g - g^{ac}g^{bd}h_{bc}h_{ad} + \frac{1}{2}\tau^2 = 2(N^{-2}\partial_0\gamma\partial_0\gamma + g^{ab}\partial_a\gamma\partial_b\gamma) \quad (3.5)
\]

When \( g_{ab} = e^{2\lambda}\sigma_{ab} \) it becomes a semilinear elliptic equation in \( \lambda \):

\[
\Delta \lambda = f(x, \lambda) \equiv p_1 e^{2\lambda} - p_2 e^{-2\lambda} + p_3, \quad (3.6)
\]

with

\[
p_1 \equiv \frac{1}{4}\tau^2, p_2 \equiv |\gamma|^2 + \frac{1}{2}|h|^2, p_3 \equiv \frac{1}{2}R(\sigma) - |D\gamma|^2
\]

The equation \( ^{(3)}R_{00} = \rho_{00} \) gives for \( N \) the linear elliptic equation

\[
\Delta N - \alpha N = -e^{2\lambda}\partial_\tau \tau \quad \text{with} \quad \alpha \equiv e^{-2\lambda}(|h|^2 + |\dot{u}|^2) + \frac{1}{2}e^{2\lambda}\tau^2 \quad (3.7)
\]

The shift \( \nu \) satisfies the equation (resulting from the expression for \( h \))

\[
(L_\sigma n)_{ab} \equiv D_an_b + D_bn_a - \sigma_{ab}D_c\nu^c = f_{ab} \quad \text{with} \quad n_a \equiv \nu_a e^{-2\lambda} \quad (3.8)
\]

\[
f_{ab} \equiv 2Ne^{-2\lambda}h_{ab} + \partial_t\sigma_{ab} - \frac{1}{2}\sigma_{ab}\sigma^{cd}\partial_t\sigma_{cd}
\]

We require the metric \( \sigma_t \) to lie in some chosen cross section \( Q \rightarrow \psi(Q) \) of the fiber bundle \( M_{-1} \rightarrow T_{\text{Teich}} \) with \( T_{\text{Teich}} \) (Teichmüller) the space of classes of conformally inequivalent riemannian metrics, identified with \( R^{6G-6} \). The solvability condition for the shift equation determines \( dQ/dt \) in terms of \( h \).

One obtains an ordinary differential system for the evolution of \( Q \) by requiring that the \( t \)-dependent spatial tensor \( ^{(3)}R_{ab} = 2\partial_a u \partial_b u \), which is TT by the previously solved equations, be \( L^2 \) orthogonal to TT tensors.
4 Local existence theorem.

The Cauchy data on $\Sigma_{t_0}$ are:

1. A $C^\infty$ riemannian metric $\sigma_0$ which projects onto a point $Q(t_0)$ of $Teich$ and a $C^\infty$ tensor $q_0$ which is TT in the metric $\sigma_0$. The spaces $W^p_s$ and $H_s \equiv W^2_s$ are the usual Sobolev spaces of tensor fields on the riemannian manifold $(\Sigma, \sigma_0)$.

2. Cauchy data for $\gamma$ and $\dot{\gamma}$ on $\Sigma_0$, i.e.

$$\gamma(t_0, .) = \gamma_0 \in H_2, \dot{\gamma}(t_0, .) = \dot{\gamma}_0 \in H_1$$

**Theorem 4.1** The Cauchy problem with the above data for the Einstein equations with $S^1$ isometry group (polarized case) has, if $T - t_0$ is small enough, a solution with $\gamma \in C^0([t_0, T], H_2) \cap C^1([t_0, T], H_1)$; $\lambda, N, \nu \in C^0([t_0, T], W^p_3, 1 < p < 2$ and $N > 0$ while $\sigma \in C^1([t_0, T], C^\infty)$ with $\sigma_t$ uniformly equivalent to $\sigma_0$. This solution is unique up to $t$ parametrization of $\tau$ and choice of a cross section of $M_{-1}$ over $Teich$.

The proof is by solving alternately elliptic systems for $h, \lambda, N, \nu$ on each $\Sigma_t$, the wave equation for $u$ and the differential system satisfied by $Q$. The iteration converges if $T - t_0$ is small enough.

We will prove that the solution exists for all $t \geq t_0 > 0$, and for an infinite proper time, by obtaining a priori estimates of the various norms, and of a strictly positive lower bound independent of $T$ for $N$.

5 Energy estimates.

We omit the index $t$ when the context is clear. We denote by $\| \cdot \|$ and $\| \cdot \|_g$ the $L^2$ norms in the $\sigma$ and $g$ metric. We denote by $C_\sigma$ numbers depending only on $(\Sigma, \sigma)$. A lower case index $m$ or $M$ denotes respectively the lower or upper bound of a scalar function on $\Sigma_t$. It follows from the equations satisfied by $N$ and $\lambda$, by the maximum principle, that

$$0 \leq N_m \leq N_M \leq 2 \frac{\partial \tau}{\tau^2} \quad \text{and} \quad e^{-2\lambda_m} \leq \frac{1}{2} \tau^2 \quad (5.1)$$
5.1 First energy estimate.

Inspired by the Hamiltonian constraint, we define the energy by

$$ E(t) \equiv \|\gamma'\|^2_g + \|D\gamma\|^2_g + \frac{1}{2} \|h\|^2_g $$

(5.2)

By integrating the Hamiltonian constraint over \((\Sigma_t, g)\) and using the constancy of \(\tau\) and the Gauss-Bonnet theorem, we find, with \(\chi\) the Euler characteristic of \(\Sigma\), that

$$ E(t) = \frac{\tau^2}{4} \text{Vol}_g(\Sigma_t) + 4\pi \chi $$

(5.3)

and after some manipulations, that

$$ \frac{dE(t)}{dt} = \tau \int \left( \frac{1}{2} |h|^2_g + |\gamma'|^2 \right) N\mu_g. $$

(5.4)

We see that \(E(t)\) is a non-increasing function of \(t\) if \(\tau\) is negative.

5.2 Second energy estimate.

We define the energy of gradient \(\gamma\) by

$$ E^{(1)}(t) \equiv \int_{\Sigma_t} (J_0 + J_1) \mu_g, \quad J_1 = |\Delta_g \gamma|^2, \quad J_0 = |D\gamma'|^2 $$

(5.5)

After lengthy but straightforward calculation, we have found (see [1], indices raised with \(g\))

$$ \frac{dE^{(1)}}{dt} - 2\tau E^{(1)} = \tau \int_{\Sigma_t} \left\{ NJ_0 + (N - 2)(J_0 + J_1) \right\} \mu_g + Z $$

(5.6)

$$ Z \equiv 2 \int_{\Sigma_t} \left\{ Nh^{ac} \partial_a \gamma' \partial_b \gamma' + 2Nh^{ab} \nabla_a \partial_b \gamma \Delta_g \gamma + (\nabla_b (\partial^a N \partial_a \gamma) + \tau \partial_b N \gamma')(\partial^b \gamma') + [(2\partial_a Nh^{ac} - 4N \partial^c \gamma \gamma') \partial_c \gamma + 2\partial^a N \partial_a \gamma' + \gamma' \Delta_g N] \Delta_g \gamma \right\} \mu_g $$

(5.7)

We set

$$ E(t) = \varepsilon^2, \quad E^{(1)}(t) = \tau^{-2} \varepsilon^2. $$

(5.8)
6 Elliptic estimates.

We have shown in [1] that if the sum $\varepsilon^2 + \varepsilon_1^2$ is bounded by a number $c$ - hypothesis $H_c$ - the quantities $N, h, \lambda, \nu$ can be bounded on $\Sigma_t$ by using the elliptic equations they satisfy. In particular, denoting by $\lambda_M$ the supremum of $\lambda$ on $\Sigma_t$,}

\[
\frac{1}{\sqrt{2}} |\tau| e^{\lambda_M} \leq 1 + CC_{\sigma_t}(\varepsilon^2 + \varepsilon_1^2) \tag{6.1}
\]

where $C$ denotes a number depending on $c$, and $C_{\sigma_t}$ a Sobolev constant of $(\Sigma_t, \sigma_t)$.

We have also obtained estimates for $h$, and for $N$, namely

\[
\| h \|_{L^\infty(g_t)} \leq CC_{\sigma_t}|\tau|\{\varepsilon + (\varepsilon + \varepsilon_1)^2\} \tag{6.2}
\]

\[
0 \leq 2 - N_m \leq CC_{\sigma_t}(\varepsilon^2 + \varepsilon_1), \quad ||DN||_{L^\infty(g_t)} \leq CC_{\sigma_t}|\tau|(\varepsilon^2 + \varepsilon_1). \tag{6.3}
\]

All these estimates would be sufficient to prove that the energies remain uniformly bounded for all $t \geq t_0$, if small enough initially, if we knew an uniform (in $t$) bound of the Sobolev constants $C_{\sigma_t}$. We can obtain such a bound only if the total energy, $\varepsilon^2 + \varepsilon_1^2$ decays when $t$ increases.

7 Corrected energy estimates.

The decay we are looking for will be obtained through the introduction of "corrected energies" whose $t$ derivatives take advantage of the negative part of the derivative of the corresponding original energy to give a negative definite contribution.

7.1 First corrected energy.

One defines a corrected first energy by the formula, where $\alpha$ is a positive number:

\[
E_\alpha(t) = E(t) - 2\alpha \tau \int_{\Sigma_t} (\gamma - \gamma') \gamma' \mu_g, \quad \text{with} \quad \gamma' = \frac{1}{Vol_\sigma \Sigma_t} \int_{\Sigma_t} \gamma \mu_\sigma \tag{7.1}
\]
We estimate the complementary term using the Poincaré inequality which gives
\[ ||\gamma - \bar{\gamma}||_g \leq e^{\lambda_M} ||\gamma - \bar{\gamma}||_{\sigma_t} \leq e^{\lambda_M} \Lambda_{\sigma_t}^{-1/2} ||D\gamma||_{\sigma_t}. \] (7.2)
therefore, by the Cauchy-Schwarz inequality, since \( ||D\gamma||_{\sigma_t} = ||D\gamma||_g \)
\[ |\tau\int_{\Sigma_t} \gamma'(\gamma - \bar{\gamma}) \mu_g| \leq |\tau| e^{\lambda_M} \Lambda_{\sigma_t}^{-1/2} ||\gamma'||_g ||D\gamma||_g. \] (7.3)
We deduce from this inequality that:
\[ E(t) \leq \frac{1}{1 - a_t} E_\alpha(t) \quad \text{with} \quad a_t \equiv \frac{\alpha \tau |e^{\lambda_M}|}{\Lambda_{\sigma_t}^{1/2}} \]
We will have \( a_t < 1 \) if
\[ \alpha < \frac{\Lambda_{\sigma_t}^{1/2}}{|\tau| e^{\lambda_M}} \] (7.4)
We have seen that there exists numbers \( C \) and \( C_\sigma \) such that
\[ |\tau| e^{\lambda_M} \leq \sqrt{2}(1 + CC_\sigma (\varepsilon^2 + \varepsilon_1^2)) \] (7.5)
We suppose that there exist numbers \( \Lambda > 0 \) and \( \delta > 0 \), independent of \( t \), such that for all \( t \) it holds that
\[ \Lambda_{\sigma_t}^{1/2} \geq \Lambda^{1/2} (1 + \delta) \] (7.6)
then it holds that
\[ \frac{\Lambda_{\sigma_t}^{1/2}}{|\tau| e^{\lambda_M}} \geq \frac{\Lambda^{1/2} (1 + \delta)}{\sqrt{2}(1 + CC_\sigma (\varepsilon^2 + \varepsilon_1^2))} > \frac{1}{\sqrt{2}} \Lambda^{1/2} \] (7.7)
as soon as
\[ CC_\sigma (\varepsilon^2 + \varepsilon_1^2) < \delta \] (7.8)
When this inequality is satisfied we can choose any number \( \alpha \) such that
\[ \alpha \leq \frac{1}{\sqrt{2}} \Lambda^{1/2} \] (7.9)
and so insure that \( a_t < 1 \). For instance if
\[ CC_\sigma (\varepsilon^2 + \varepsilon_1^2) < \frac{\delta}{2} \] (7.10)
then
\[ 1 - a_t \equiv 1 - \frac{\alpha \tau |e^{\lambda_M}|}{\Lambda_{\sigma_t}^{1/2}} \geq \frac{\delta}{2(1 + \delta)} \] (7.11)
7.2 Decay of the corrected energy.

We set:

\[ \frac{dE_\alpha}{dt} = \frac{dE}{dt} - R_\alpha \]

with (the terms explicitly containing the shift \( \nu \) give an exact divergence which integrates to zero)

\[ R_\alpha = 2\alpha \tau \int_{\Sigma_t} \left\{ \partial_0 \gamma' (\gamma - \tilde{\gamma}) + \gamma' \partial_0 (\gamma - \tilde{\gamma}) - N\tau \gamma' (\gamma - \tilde{\gamma}) \right\} \mu_g \]

\[ + 2\alpha \frac{d\tau}{dt} \int_{\Sigma_t} \gamma' (\gamma - \tilde{\gamma}) \mu_g \] (7.12)

To simplify the writing we suppose that \( \int_{\Sigma_t} \gamma' \mu_g = 0 \), this quantity is conserved in time if \( \gamma \) satisfies the wave equation 3.1. Some elementary computations using 3.1 and integration by parts show that, using also \( \frac{d\tau}{dt} = \tau^2 \):

\[ R_\alpha = 2\alpha \tau \int_{\Sigma_t} \left\{ [1/2 |h|^2 + (1 - 2\alpha)|\gamma'|^2 + 2\alpha|\nabla \gamma|^2] - 2\alpha \tau \gamma' (\gamma - \tilde{\gamma}) \right\} \mu_g + \tau A \] (7.13)

We write \( \frac{dE_\alpha}{dt} \) in the form

\[ \frac{dE_\alpha}{dt} = \tau \int_{\Sigma_t} \left\{ \frac{1}{2} |h|^2 + (1 - 2\alpha)|\gamma'|^2 + 2\alpha |\nabla \gamma|^2 \right\} \mu_g + \tau A \] (7.14)

where \( A \) can be estimated with higher order terms in the energies, using the inequality 6.3 satisfied by \( N - 2 \), since \( A \) reads

\[ A \equiv \int_{\Sigma_t} (N - 2)[\frac{1}{2} |h|^2 + (1 - 2\alpha)|\gamma'|^2 + 2\alpha |\nabla \gamma|^2] \mu_g \] (7.15)

We look for a positive number \( k \) such that the difference \( \frac{dE_\alpha}{dt} - k\tau E_\alpha \) can be estimated with higher order terms in the energies. We have:

\[ \frac{dE_\alpha}{dt} - k\tau E_\alpha = 2\tau \int_{\Sigma_t} \left\{ [\frac{1}{2} |h|^2 + (1 - 2\alpha - \frac{k}{2})|\gamma'|^2 + (2\alpha - \frac{k}{2}) |\nabla \gamma|^2] - \alpha(1 - k)\tau \gamma' (\gamma - \tilde{\gamma}) \right\} \mu_g + \tau A \] (7.16)

\[ = 2\tau \int_{\Sigma_t} \left\{ [\frac{1}{2} |h|^2 + (1 - 2\alpha - \frac{k}{2})|\gamma'|^2 + (2\alpha - \frac{k}{2}) |\nabla \gamma|^2] - \alpha(1 - k)\tau \gamma' (\gamma - \tilde{\gamma}) \right\} \mu_g + \tau A \] (7.17)
We have treated in \[1\] the case where \( \Lambda \geq \frac{1}{8}, \alpha = \frac{1}{4} \). In this case it is possible to take \( k = 1 \) and obtain immediately

\[
\frac{dE_{\frac{1}{4}}}{dt} - \tau E_{\frac{1}{4}} \leq |\tau A|.
\]

(7.18)

In the general case we have

\[
\frac{dE_\alpha}{dt} - k\tau E_\alpha \leq 2\tau \int_{\Sigma_t} \left\{ [(1 - 2\alpha - \frac{k}{2})|\gamma'|^2 + (2\alpha - \frac{k}{2})|D\gamma|^2_g] - \alpha(1 - k)\tau\gamma'(\gamma - \bar{\gamma}) \right\} \mu_g + |\tau A|
\]

(7.19)

The estimate 7.3 together with the inequality 7.7 gives

\[
|\tau \int_{\Sigma_t} \gamma'(\gamma - \bar{\gamma}) \mu_g| \leq \sqrt{2}\Lambda^{-1/2}|\gamma'|_g||D\gamma||_g
\]

(7.20)

Therefore it holds that:

\[
\int_{\Sigma_t} \left\{ [(1 - 2\alpha - \frac{k}{2})|\gamma'|^2 + (2\alpha - \frac{k}{2})|D\gamma|^2_g] - \alpha(1 - k)\tau\gamma'(\gamma - \bar{\gamma}) \right\} \mu_g \geq
\]

\[
(1 - 2\alpha - \frac{k}{2})||\gamma'||^2_g + (2\alpha - \frac{k}{2})||D\gamma||^2_g - \alpha(1 - k)\sqrt{2}\Lambda^{-1/2}|\gamma'|_g||D\gamma||_g
\]

(7.21)

The above quadratic form in \( ||\gamma'||_g \) and \( ||D\gamma||_g \) is non-negative if

\( k \leq 4\alpha, \quad k \leq 2(1 - 2\alpha) \)

(7.22)

and \( k \) is such that

\[ 2\alpha^2\Lambda^{-1}(1 - k)^2 - 4(2\alpha - \frac{k}{2})(1 - 2\alpha - \frac{k}{2}) \leq 0 \]

(7.23)

the inequalities 7.23 imply

\[ k \leq 1, \]

(7.24)

The inequality 7.24 reads

\[ (1 - 2\Lambda^{-1}\alpha^2)k^2 - (1 - 2\Lambda^{-1}\alpha^2)2k - 2\Lambda^{-1}\alpha^2 + 8\alpha(1 - 2\alpha) > 0 \]

(7.25)

\[ k \leq 1, \]

(7.26)
That is, since \( 1 - 2\Lambda^{-1}\alpha^2 > 0 \),

\[
k^2 - 2k + 1 - \frac{(1 - 4\alpha)^2}{(1 - 2\Lambda^{-1}\alpha^2)} > 0
\]

equivalently

\[
k < 1 - \frac{1 - 4\alpha}{(1 - 2\Lambda^{-1}\alpha^2)^\frac{1}{2}}
\]

(7.27)

There will exist such a \( k > 0 \) if

\[
1 - 4\alpha < (1 - 2\Lambda^{-1}\alpha^2)^\frac{1}{2}
\]

(7.28)

that is

\[-2\Lambda^{-1}\alpha - 16\alpha + 8 > 0.\]

(7.29)

i.e.

\[
\alpha < \frac{4}{8 + \Lambda^{-1}}
\]

(7.30)

We have

\[
\frac{4}{8 + \Lambda^{-1}} \leq \frac{1}{4}, \text{ if } \Lambda \leq \frac{1}{8}
\]

(7.31)

An elementary computation shows that

\[
\frac{4}{8 + \Lambda^{-1}} \leq \frac{\Lambda^{\frac{1}{2}}}{\sqrt{2}}
\]

(7.32)

with the equality satisfied only when \( \Lambda = \frac{1}{8} \).

We choose \( \alpha \) such that it satisfies the inequality 7.31, and then \( k > 0 \) such that it satisfies 7.28.

### 7.3 Corrected second energy.

We define a **corrected second energy** \( E^{(1)}_\alpha \) by the formula

\[
E^{(1)}_\alpha(t) = E^{(1)}(t) + 2\alpha^{(1)}\tau \int_{\Sigma_t} \Delta_g \gamma' \mu_g
\]

Using again the Cauchy Schwarz inequality, and the Poincaré inequality to estimate \( ||\gamma'||_{g_t} \) in terms of \( ||D\gamma'||_{g_t} \) (the hypothesis \( \gamma' = 0 \) is not necessary here because on a compact manifold \( \int_{\Sigma_t} \Delta_g \gamma \mu_g = 0.\) we find, (with the same \( a_t \) as in the previous subsection)

\[
E^{(1)}(t) \leq \frac{1}{1 - a_t} E^{(1)}_\alpha(t)
\]

(7.34)
7.4 Decay of the second corrected energy.

We have found in [1], by straightforward but lengthy computations with the use of the wave equation for $\gamma$ and $^{(3)}R_{0}^{c} \equiv -N\nabla_{a}h^{ac} = 2\partial_{0}\gamma\partial^{\nu}\gamma$ together with Cauchy Schwarz and Sobolev theorems, an equality of the form:

$$\frac{dE_{\alpha}^{(1)}}{dt} \equiv \frac{dE^{(1)}_{\alpha}}{dt} + R_{\alpha}^{(1)}$$

where, with the choice $\tau = \frac{1}{t}$,

$$R_{\alpha}^{(1)} \equiv 2\alpha^{(1)}\tau \int_{\Sigma_{t}} \{-N|D\gamma'|^{2} + N|\Delta_{g}\gamma|^{2} + (N + 1)\tau\Delta_{g}\gamma\gamma'\} \mu_{g} + \tau Z_{\alpha}$$

where $Z_{\alpha}$, given by:

$$Z_{\alpha} \equiv 2\alpha \int_{\Sigma_{t}} \partial^{a}N(\partial_{a}\gamma\Delta_{g}\gamma + \gamma'\partial_{a}\gamma') + 2Nh^{ab}\nabla_{a}\partial_{b}\gamma\gamma' + 2\gamma'\partial_{c}\gamma(\partial_{a}Nh^{ac} - \gamma'\partial^{a}\gamma)) \mu_{g}$$

(7.35)

can be estimated with higher order terms in the energies. Using 5.6 we obtain that:

$$\frac{dE_{\alpha}^{(1)}}{dt} - (2 + k)\tau E_{\alpha}^{(1)} = \tau \int_{\Sigma_{t}} \{(2N - 2 - k - 2\alpha N)J_{0} + (2\alpha N + N - 2 - k)J_{1}\} \mu_{g}$$

$$+ 2\alpha\tau^{2} \int_{\Sigma_{t}} \{(N + 1 - 2 - k)\Delta_{g}\gamma\gamma'\} \mu_{g} + Z + \tau Z_{\alpha}$$

(7.36)

which we write:

$$\frac{dE_{\alpha}^{(1)}}{dt} - (2 + k)\tau E_{\alpha}^{(1)} = \tau \int_{\Sigma_{t}} \{(2 - k - 4\alpha)J_{0} + (4\alpha - k)J_{1}\} \mu_{g}$$

$$+ 2\alpha\tau^{2} \int_{\Sigma_{t}} \{(1 - k)\Delta_{g}uu'\} \mu_{g} + Z + \tau Z_{\alpha} + Z_{N}$$

(7.37)

where $Z_{N}$ can be estimated with higher order terms in the energies through the estimate of $N - 2$.

The same estimates as those done for the first corrected energy show that, under the hypothesis made previously, it holds that the term linear in the energies on the right hand side of the above equality is always non negative for the following choices of $\alpha$ and $k$:
• 1. \( \Lambda \geq \frac{1}{8} \). We can choose \( \alpha = \frac{1}{4} \) and \( k = 1 \).

• 2. \( \Lambda < \frac{1}{8} \). We must then choose \( \alpha \) and \( k \) satisfying the inequalities 7.31 and 7.28.

In all cases the estimate of the higher order terms in the energies are the same ones as obtained in [1], and the following equality holds:

\[
\frac{dE^{(1)}_\alpha}{dt} = (2 + k)\tau E^{(1)}_\alpha + |\tau|^3 B
\]

where \( B \) is a polynomial in first and second derivatives of \( \gamma, h, Dh, DN, D^2N \) whose many terms can all be bounded using previous estimates by a polynomial in \( \varepsilon \) and \( \varepsilon_1 \) whose terms are at least of degree 3 and the coefficients bounded by \( CC_{\sigma_t} \), under the \( H_\varepsilon \) hypothesis, with \( c > 0 \) a given appropriate number.

8 Decay of the total energy.

We define \( y(t) \) to be the **total corrected energy** namely:

\[ y(t) \equiv E_\alpha(t) + \tau^{-2}E^{(1)}_\alpha \]

It bounds the total energy \( x(t) \equiv E_{tot}(t) \equiv \varepsilon^2 + \varepsilon_1^2 \) by

\[ x(t) \equiv \varepsilon^2 + \varepsilon_1^2 \leq \frac{1}{1 - a_t} y(t) \quad \text{with} \quad a_t \equiv \frac{\alpha |\tau| e^\lambda M}{\Lambda_{\sigma_t}^{\frac{3}{2}}} \]

We make the following a priori hypothesis, for all \( t \geq t_0 \) for which the considered quantities exist

- **Hypothesis \( H_\sigma \)**: 1. The numbers \( C_{\sigma_t} \) are uniformly bounded by a constant \( M_\sigma \).

2. There exist \( \Lambda > 0 \) and \( \delta > 0 \) such that the inequality 7.6 is satisfied.

- **Hypothesis \( H_E \)**: The energies \( \varepsilon_t^2 \) and \( \varepsilon_{1,t}^2 \) satisfy the inequality 7.8.
We choose $\alpha$ such that (the case $\Lambda \geq \frac{1}{8}$, $\alpha = \frac{1}{4}$ was considered in [1]).

\[
\alpha < \frac{4}{8 + \Lambda^{-1}} < \frac{1}{4} \quad \text{with} \quad \Lambda < \frac{1}{8} \tag{8.1}
\]

Under the hypotheses $H_c, H_E$ and $H_\sigma$ there exists a number $M > 0$ such that, for all $t$:

\[
1 - a_t \geq M > 0. \tag{8.2}
\]

Under the hypothesis $H_E$ all powers of $E_{tot}$ greater than 3/2 are bounded by the product of $E_{tot}^{3/2}$ by a constant.

We denote by $M_t$ any given number dependent on the bounds of these $H$’s hypothesis but independent of $t$.

Under the hypotheses $H_c$, $H_\sigma$ and $H_E$ the function $y$ satisfies a differential inequality of the form

\[
\frac{dy}{dt} \leq -\frac{k}{t}(y - M_1 y^{3/2}) \tag{8.3}
\]

We suppose that $y_0 \equiv y(t_0)$ satisfies

\[
y_0^{1/2} < \frac{1}{2M_1} \tag{8.4}
\]

Then $y$ starts decreasing, continues to decrease as long as it exists and satisfies an inequality which gives by integration, after setting $y = z^2$,

\[
\log\left\{\frac{z(1 - M_1 z_0)}{(1 - M_1 z)z_0}\right\} + \frac{1}{2}k\log\frac{t}{t_0} \leq 0 \quad \text{a fortiori} \quad t^k y \leq \frac{t_0^k y_0}{(1 - M_1 z_0)^2}
\]

hence, using the hypotheses and previous bounds, the decay estimate

\[
t^k x(t) \leq M_2 x_0 \quad \text{with} \quad M_2 \leq \frac{4t_0^k}{(1 - a_t)(1 - a_0)} \leq \frac{4t_0^k}{M^2}
\]

9 Teichmüller parameters.

We require the metric $\sigma_t$ to remain, when $t$ varies, in some cross section of $M_{-1}$ over the Teichmüller space $\mathcal{T}$. 

15
Given a metric \( s \in M_{-1} \) the \textbf{Dirichlet energy} \( D_s(\sigma) \) of the metric \( \sigma \in M_{-1} \) is the energy of the (unique) harmonic diffeomorphism homotopic to the identity \( \phi : (\Sigma, \sigma) \to (\Sigma, s) \). It can be written by conformal invariance as

\[
D_s(\sigma) \equiv \int_{\Sigma} g^{ab} \partial_a \Phi^A \partial_b \Phi^B s_{AB}(\Phi) \mu_g
\]

Let \( \sigma_0 \) satisfy the hypothesis \( H_\sigma \), then there exists a number \( D \) such that if \( |D(\sigma) - D(\sigma_0)| \leq D \), called \textbf{Hypothesis} \( H_D \), then \( \sigma \) satisfies also the hypothesis \( H_\sigma \). We now estimate \( D(\sigma) \).

We have if \( \Phi \) is a harmonic map that

\[
\frac{d}{dt} D_s(\sigma) = \int_{\Sigma_t} \{ \tilde{\partial}_0 g^{ab} \partial_a \Phi^A \partial_b \Phi^B - N \tau g^{ab} \partial_a \Phi^A \partial_b \Phi^B \} s_{AB}(\Phi) \mu_g
\]

with

\[
\tilde{\partial}_0 g^{ab} = 2N e^{-4\lambda} h^{ab} + N e^{-2\lambda} h^{ab} \tau
\]

hence

\[
\frac{d}{dt} D_s(\sigma) = \int_{\Sigma_t} 2N e^{-2\lambda} h^{ab} \partial_a \Phi^A \partial_b \Phi^B s_{AB}(\Phi_\sigma) \mu_{\sigma}
\]

Using \( 0 < N \leq 2 \) and \( e^{-2\lambda} \leq \frac{\tau^2}{2} \) and the bound of \( \| h \|_{\infty} \) we find:

\[
\frac{d}{dt} D_s(\sigma) \leq |\tau| CC_\sigma [\varepsilon + (\varepsilon + \varepsilon_1)^2] D_s(\sigma)
\]

Under the hypotheses that we have made the Dirichlet energy satisfies the differential inequality

\[
\frac{1}{D_s(\sigma)} \frac{d}{dt} D_s(\sigma) \leq \frac{CM_\sigma M_{2\lambda}^{\frac{1}{2}} \tau^{\frac{1}{2}}}{t^{1+\frac{1}{4}}}
\]

By integration and elementary calculus we obtain the inequality, valid for all \( t \geq t_0 \) since \( k \) is a strictly positive number,

\[
|D(\sigma_t) - D(\sigma_0)| \leq M_3 x_0^{\frac{1}{2}}
\]
10 Global existence.

**Theorem 10.1** Let \((\sigma_0, q_0) \in C^\infty(\Sigma_0)\) and \((u_0, \dot{u}_0) \in H_2(\Sigma_0, \sigma_0) \times H_1(\Sigma_0, \sigma_0)\) be initial data for the polarized Einstein equations with \(U(1)\) isometry group on the initial manifold \(\Sigma_0 \times U(1)\); suppose that \(\sigma_0\) is such that \(R(\sigma_0) = -1\). Then there exists a number \(\eta > 0\) such that if \(x_0 \equiv E_{\text{tot}}(t_0) < \eta\) these Einstein equations have a solution on \(\Sigma \times S^1 \times [t_0, \infty)\), with initial values determined by \(\sigma_0, q_0, u_0, \dot{u}_0\). The solution has an infinite proper time extension since \(N_m > 0\). It is unique with \(\tau = -t^{-1}\) and \(\sigma_t\) in a chosen cross section of \(M_{-1}\) over \(T_{\text{eich}}\).

Proof. The same continuity argument as in [1].

It can be proved that when \(t\) tends to infinity the obtained solution tends to a metric of the type 1.1.

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References

[1] Y. Choquet-Bruhat and V. Moncrief, Future global einsteinian spacetimes with \(U(1)\) isometry group. C.R. Acad. Sci. Paris t. 332 série I (2001), 137-144. Detailed version in Ann. H. Poincaré 2 (2001) 1007-1064.

[2] V. Moncrief Reduction of Einstein equations for vacuum spacetimes with \(U(1)\) spacelike isometry group, Annals of Physics 167 (1986), 118-142

[3] Y. Choquet-Bruhat and V. Moncrief Existence theorem for solutions of Einstein equations with 1 parameter spacelike isometry group, Proc. Symposia in Pure Math, 59, 1996, H. Brezis and I.E. Segal ed. 67-80

[4] L. Andersson, V. Moncrief and A. Tromba On the global evolution problem in 2+1 gravity J. Geom. Phys. 23 1997 n°3 – 4,1991-205

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