Porosities of the sets of attractors

By

Paweł Klinga and Adam Kwela

Abstract

This paper is another attempt to measure the difference between the family \( A[0, 1] \) of attractors for iterated function systems acting on \([0, 1]\) and a broader family, the set \( A_w[0, 1] \) of attractors for weak iterated function systems acting on \([0, 1]\).

It is known that both \( A[0, 1] \) and \( A_w[0, 1] \) are meager subsets of the hyperspace \( K([0,1]) \) (of all compact subsets of \([0,1]\) equipped in the Hausdorff metric). Actually, \( A[0,1] \) is even \( \sigma \)-lower porous while the question about \( \sigma \)-lower porosity of \( A_w[0,1] \) is still open.

We prove that \( A[0, 1] \) is not \( \sigma \)-strongly porous in \( K([0,1]) \). Moreover, we show that \( A_w[0, 1] \setminus A[0, 1] \) is dense in \( K([0,1]) \). 

1 Introduction

We are mainly interested in two families of compact subsets of the space \([0, 1]^d\), where \( d \in \mathbb{N} \):

- the set \( A[0, 1]^d \) of attractors for iterated function systems acting on \([0, 1]^d\),
- the set \( A_w[0, 1]^d \) of attractors for weak iterated function systems acting on \([0, 1]^d\).

We will consider both \( A[0, 1]^d \) and \( A_w[0, 1]^d \) as subsets of the hyperspace \( K([0,1]^d) \) of all compact subsets of \([0, 1]^d\) equipped in the Hausdorff metric.

This approach is present in literature. In [11] Theorem 3.9 (see also [2]) E. D’Aniello and T.H. Steele proved that the family \( A[0, 1]^d \) is a meager subset of

\begin{itemize}
\item 2010 Mathematics Subject Classification: Primary: 28A80. Secondary: 26A18.
\item Key words and phrases: attractors, iterated function systems, weak iterated function systems, Banach fractals, upper porosity, \( \sigma \)-upper porosity, lower porosity, \( \sigma \)-lower porosity, strong porosity, \( \sigma \)-strong porosity
\end{itemize}
2

Basic notions and definitions

2.1 Hyperspace of compact subsets

Let \((X, d)\) be a compact metric space. By \(K(X)\) we denote the family of all compact subsets of \(X\). We will consider \(K(X)\) as a topological space equipped with the Hausdorff metric \(d_H\), which is given by

\[
d_H(A, B) = \min \left\{ r \geq 0 : A \subseteq \bar{B}_r \text{ and } B \subseteq \bar{A}_r \right\},
\]

where for \(C \in K(X)\) and \(r \geq 0\) we write

\[
\tilde{C}_r = \bigcup_{c \in C} \left\{ x \in X : d(x, c) \leq r \right\}.
\]

For \(C \in K(X)\) and \(r > 0\) by \(B_H(C, r)\) we will denote the ball in \(K(X)\) of radius \(r\) centered at \(C\). Throughout the paper we will assume that \(X = [0, 1]^d\). For convenience in this case we will write \(K[0, 1]^d\) instead of \(K([0, 1]^d)\). Moreover, if \(d = 1\) then we will write \(K[0, 1]\) instead of \(K[0, 1]_1\).
For $r \in \mathbb{R}$, $x = (x_1, \ldots, x_d) \in [0, 1]^d$ and $A \in K[0, 1]^d$ we will write
\[
    r \cdot A = \{(ry_1, \ldots, ry_d) : (y_1, \ldots, y_d) \in A\},
\]
\[
    x + A = \{(x_1 + y_1, \ldots, x_d + y_d) : (y_1, \ldots, y_d) \in A\}.
\]
Note that both $rA$ and $x + A$ are compact subsets of $\mathbb{R}^d$. Thus, $rA \subseteq [0, 1]^d$ and $x + A \subseteq [0, 1]^d$ provided that $rA \subseteq [0, 1]^d$ and $x + A \subseteq [0, 1]^d$.

### 2.2 Iterated function systems

By contraction on $X$ we understand a function $f : X \to X$ such that:
\[
    \exists L \in (0, 1) \quad \forall x, y \in X \quad d(f(x), f(y)) \leq L \cdot d(x, y).
\]
The smallest $L$ for which this inequality holds is called a Lipschitz constant of $f$ and denoted $Lip(f)$.

A function $f : X \to X$ is a weak contraction if for every pair of distinct points $x, y \in X$ it is true that $d(f(x), f(y)) < d(x, y)$.

Every finite collection of contractions on $X$ will be called an iterated function system (IFS, in short) acting on $X$. Similarly, a finite collection of weak contractions on $X$ will be called a weak iterated function system (wIFS, in short) acting on $X$.

### 2.3 Attractors

Let $\{s_1, \ldots, s_k\}$ be an IFS acting on $X$. By $S : K(X) \to K(X)$ we denote the Hutchinson operator for $\{s_1, \ldots, s_k\}$, i.e.
\[
    S(A) = \bigcup_{i=1}^k s_i(A).
\]
If $(X, d)$ is complete, then so is $(K(X), d_H)$. Therefore, by applying the Banach fixed-point theorem, the equation $S(A) = A$ has a unique compact solution. Such set is called the attractor for the iterated function system $\{s_1, \ldots, s_k\}$.

Analogously, for a wIFS $\{s_1, \ldots, s_k\}$ acting on $X$, a compact set $A \in K(X)$ satisfying $\bigcup_{i=1}^k s_i(A) = A$ is called a weak IFS attractor for the system $\{s_1, \ldots, s_k\}$. It is shown in [4] that for every weak iterated function system there exists a weak IFS attractor provided that $X$ is a compact space.
The form of the Hutchinson operator imposes that attractors are self-similar (at least in some sense), so they are often used to describe fractals. Clearly, an IFS attractor is a weak IFS attractor. However, the reverse inclusion is not true by [11].

By $A[0,1]^d$ and $A_w[0,1]^d$ we will denote the sets of all attractors for IFS and wIFS, respectively, acting on $[0,1]^d$. If $d = 1$ then we will write $A[0,1]$ and $A_w[0,1]$ instead of $A[0,1]^1$ and $A_w[0,1]^1$, respectively.

2.4 Porosities

Fix a metric space $(X,d)$ and denote by $B(z,\delta)$ a ball in $(X,d)$ of radius $\delta > 0$ centered at $z \in X$. For any subset $A$ of $X$ and any point $x \in X$ we define

$$p(A,x) = \limsup_{R \to 0^+} \frac{\sup\{r \geq 0 : B(y,r) \subseteq B(x,R) \setminus A \text{ for some } y \in X\}}{R},$$

$$p(A,x) = \liminf_{R \to 0^+} \frac{\sup\{r \geq 0 : B(y,r) \subseteq B(x,R) \setminus A \text{ for some } y \in X\}}{R}.$$ Observe that $p(A,x) = 1$ whenever $x \not\in A$ and $0 \leq p(A,x) \leq \frac{1}{2}$ in the opposite case (see [11, Remark 1.2.(i)]).

We say that $A$ is:

- lower porous if $p(A,x) > 0$ for every $x \in X$,
- upper porous if $p(A,x) > 0$ for every $x \in X$,
- strongly porous if $p(A,x) \geq \frac{1}{2}$ for every $x \in X$ (equivalently, $p(A,x) = \frac{1}{2}$ for every $x \in \overline{X}$),
- $\sigma$-lower porous ($\sigma$-upper porous, $\sigma$-strongly porous) if it is a countable union of lower porous (upper porous, strongly porous) sets.

Clearly, each lower porous set as well as each strongly porous set is upper porous. What is more, since the condition defining upper porosity is a strengthening of nowhere density, each $\sigma$-upper porous set is meager.

In this known that on the real line there are upper porous sets that are not $\sigma$-lower porous. In fact, there is even a closed strongly porous set which is not $\sigma$-lower porous ([9, Corollary 4.1.(ii)]). On the other hand, there exist
countable upper porous sets which are not strongly porous (see \cite{10} Remark 1.2(iv) and Example 4.6).

We will use a different, but equivalent (thanks to \cite{10} Remark 1.2(ii)),
definition of strong porosity: \( A \) is strongly porous, if for each \( x \in X \) there
are two sequences \((x_n) \subseteq X \) and \((r_n) \subseteq (0, +\infty) \) such that:

- \( \lim_{n} x_n = x \),
- \( \lim_{n} \frac{r_n}{d(x,x_n)} = 1 \),
- \( B(x_n, r_n) \cap A = \emptyset \) for all \( n \).

In \cite{8} Theorem 3.1 it is shown that the set \( A[0,1]^d \) is \( \sigma \)-lower porous. The
question whether \( A_{w}[0,1]^d \) is \( \sigma \)-lower porous is still open (see \cite{8} Question 4.4).

For more on porosities see for instance \cite{9}, \cite{10} or \cite{13}.

3 Preliminary results

We start this section with a result enabling to transfer some results about
attractors in \( K[0,1] \) to arbitrary dimension \( d \in \mathbb{N} \). However the proof can
be found in \cite{7}, we repeat it for the sake of completeness.

\begin{lemma} \label{lem:transfer}
(\cite{7} Lemma 2.2). For each \( d \in \mathbb{N} \) the following hold:

- \( A \in A[0,1] \) if and only if \( A \times \{0\}^{d-1} \subseteq A[0,1]^d \),
- \( A \in A_{w}[0,1] \) if and only if \( A \times \{0\}^{d-1} \subseteq A_{w}[0,1]^d \).
\end{lemma}

\begin{proof}
Observe that if \( f : [0,1]^d \rightarrow [0,1]^d \) is a (weak) contraction then so is
\( f | [0,1] \times \{0\}^{d-1} \). On the other hand, for each (weak) contraction \( f : [0,1] \rightarrow
[0,1] \) the map \( g : [0,1]^d \rightarrow [0,1] \) given by \( g(x_1, \ldots, x_d) = (f(x_1), 0, \ldots, 0) \) is
a (weak) contraction as well.
\end{proof}

Recall that \( A[0,1]^d \) is dense in \( K[0,1]^d \), for every \( d \in \mathbb{N} \), as it contains
all nonempty finite sets. In particular, \( A[0,1]^d \) cannot be itself nowhere
dense. The following observation implies that the same is true for the set
\( A_{w}[0,1]^d \setminus A[0,1]^d \).

\begin{proposition}
The set \( A_{w}[0,1]^d \setminus A[0,1]^d \) is dense in \( K[0,1]^d \), for every
\( d \in \mathbb{N} \).
\end{proposition}
Proof. Before the main part of the proof, we need to perform a construction in \([0, 1]\), which will be needed later. This part is a simple modification of [7, Proposition 3.1], however we repeat the reasoning for the sake of completeness.

Let \(f : [0, 1] \to [0, 1]\) be the weak contraction given by \(f(x) = x - x^2\) for all \(x \in [0, 1]\). Define a sequence of intervals by \(I_1 = [0, \frac{1}{2}]\) and \(I_{n+1} = [0, f(\max I_n)]\). Inductively pick points \(x_n\) for \(n \in \mathbb{N}\) in such a way that for each \(n \in \mathbb{N}\) we have:

(a) \(x_n \in I_1\),

(b) \(x_{n+1} < x_n\),

(c) \(x_n - x_{n+1} > x_{n+1} - x_{n+2}\),

(d) \(\{x_i : i \in \mathbb{N}\} \cap (I_n \setminus I_{n+1})\) is finite,

(e) \(k_n > n + n \cdot (\sum_{i=1}^{n-1} k_i)\), where \(k_n = |\{x_i : i \in \mathbb{N}\} \cap (I_n \setminus I_{n+1})|\).

Note that item (d) guarantees that \(\lim_n x_n = 0\). Put \(X = \{0\} \cup \{x_n : n \in \mathbb{N}\}\).

Observe that \(X \in A_w[0, 1]\). Indeed, let \(g : [0, 1] \to [0, 1]\) be the weak contraction such that \(g(x_n) = x_{n+1}\) for all \(n \in \mathbb{N}\) (such \(g\) exists by items (b) and (c)). As \(\lim_n \max I_n = 0\), the unique fixed point of \(g\) is 0. Thus, if \(h : [0, 1] \to [0, 1]\) is the function constantly equal to \(x_1\) then \(X = g[X] \cup h[X]\).

We are ready for the main part of the proof. As the family of all finite nonempty sets is dense in \(K[0, 1]^d\), it suffices to show that for every finite nonempty set \(F \subseteq [0, 1]^d\) and every \(\delta > 0\) there is an attractor in \(B_H(F, \delta) \cap (A_w[0, 1]^d \setminus \mathcal{A}[0, 1]^d)\).

Let \(F\) be a finite nonempty subset of \([0, 1]^d\) and \(\delta > 0\) be such that \(2\delta < \min\{d(x, y) : x, y \in F, x \neq y\}\). Fix any \(w \in F\) and consider the set \(Y = F \cup (w + \delta \cdot (X \times \{0\}^{d-1}))\). Clearly, \(Y \in B_H(F, \delta)\). By Lemma 3.1, \(w + \delta \cdot (X \times \{0\}^{d-1}) \in A_w[0, 1]^d\). Hence, \(Y \in A_w[0, 1]^d\) as \(Y \setminus (w + \delta \cdot (X \times \{0\}^{d-1}))\) is finite.

To finish the proof, we will show that \(Y \notin A[0, 1]^d\). Assume towards contradiction that \(f_1, \ldots, f_k : [0, 1]^d \to [0, 1]^d\) are standard contractions such that \(Y = \bigcup_{i=1}^k f_i[Y]\). Denote \(X' = w + \delta \cdot (X \times \{0\}^{d-1})\) and \(I'_n = w + \delta \cdot (I_n \times \{0\}^{d-1})\) for all \(n \in \mathbb{N}\). Find \(n \in \mathbb{N}\) such that:

(i) \(n > k \cdot |F|\) (here \(|F|\) denotes the cardinality of the finite set \(F\)),
(ii) if \( f_i(w) \neq w \) then \( f_i[X'] \cap X' \cap I_n = \emptyset \) (notice that \( f_i(w) \neq w \) implies \( f_i[X'] \cap X' \) being finite, by \( \lim_n x_n = 0 \)),

(iii) \( \max_{i \leq k} \text{Lip}(f_i) \cdot |I_n| < |I_{n+1}| \) (recall that for \( f = x(1 - x) \) and any \( L \in (0, 1) \) we have \( |f(x) - f(0)| = |x| \cdot |1 - x| > L|x - 0| \) for all \( x \in (0, 1 - L) \)).

We will justify that \( X' \cap I'_n \setminus I'_{n+1} \) cannot be covered by \( \bigcup_{i \leq k} f_i[Y] \). If \( f_i(w) \neq w \) then \( f_i[X'] \cap X' \cap I_n = \emptyset \) by item (ii). If \( f_i(w) = w \) then item (iii) implies that \( f_i[I_n'] \subseteq I_{n+1}' \). Therefore,

\[
(X' \cap I'_n \setminus I'_{n+1}) \cap \bigcup_{i \leq k} f_i[Y] = (X' \cap I'_n \setminus I'_{n+1}) \cap \bigcup_{i \leq k} f_i[F \cup (X' \cap I'_1 \setminus I_n)].
\]

Using \( |X' \cap I'_1 \setminus I_n'| = \sum_{i=1}^{n-1} k_i \), we see that

\[
\left| \bigcup_{i \leq k} f_i[F \cup (X' \cap I'_1 \setminus I_n)] \right| \leq k \cdot \left( |F| + \sum_{i=1}^{n-1} k_i \right) < n + n \cdot \sum_{i=1}^{n-1} k_i < k_n = |X' \cap (I_n' \setminus I_{n+1})|.
\]

Thus, the proof is finished. \( \square \)

We end with showing that \( \sigma \)-strong-porosity is an appropriate notion for our purposes – settling whether \( A[0, 1]^d \) is \( \sigma \)-strongly-porous will require full depth of \( A[0, 1]^d \) (not only the information that all finite nonempty sets are in \( A[0, 1]^d \)).

**Proposition 3.3.** The collection of all finite nonempty subsets of \( [0, 1]^d \) is \( \sigma \)-strongly-porous in \( K[0, 1]^d \), for every \( d \in \mathbb{N} \).

**Proof.** Observe that the family of all finite nonempty subsets of \( [0, 1]^d \) is equal to \( \mathcal{P}(D) \cup \bigcup_{k \in \omega} \mathcal{F}_k \), where

\[
D = \{(x_1, \ldots, x_d) \in [0, 1]^d : \forall i \leq d x_i = 0 \text{ or } x_i = 1 \},
\]

\[
\mathcal{F}_k = \{F \subseteq [0, 1]^d : |F| = k, F \setminus D \neq \emptyset \}.
\]

Clearly, \( \mathcal{P}(D) \) is strongly porous as a finite set. Thus it suffices to show that \( \mathcal{F}_k \) is strongly porous for each \( k \in \mathbb{N} \).

Fix \( k \in \mathbb{N}, F \in \mathcal{F}_k \) and any \( x \in F \setminus D \). Find three sequences \((y_n), (z_n) \subseteq [0, 1]^d \) and \((r_n) \subseteq (0, 1) \) such that:
• $r_n = d(y_n, x) = d(z_n, x)$ for all $n$,
• $y_n \neq z_n$ for all $n$,
• $B(y_n, r_n) \cap B(z_n, r_n) = \emptyset$ for all $n$,
• $\lim_n r_n = 0$.

This is possible as $x \notin D$.

Put $G_n = (F \setminus \{x\}) \cup \{y_n, z_n\}$. As $d_H(G_n, F) = r_n \to 0$, the sequence $(G_n)$ tends to $F$ (in $K[0, 1]^3$) and $\lim_n \frac{r_n}{d_H(G_n, F)} = 1$. To conclude the proof, we need to justify that $B_H(G_n, r_n) \cap F_k = \emptyset$.

Let $H \in B_H(G_n, r_n)$. Then $H$ has to intersect each of the balls $B(y_n, r_n)$, $B(z_n, r_n)$ and $B(w, r_n)$ for $w \in F \setminus \{x\}$. As those balls are pairwise disjoint, $H$ has to have at least $2 + k - 1 = k + 1$ elements. Hence, $H \notin F_k$ and the proof is finished. \qed

4 Attractors for IFSs

Theorem 4.1. The set $A[0, 1]$ is not $\sigma$-strongly porous in $K[0, 1]$.

Proof. We will prove the following.

$$\exists E \in A[0, 1] \exists \varepsilon > 0 \forall Y \in B_H(E, \varepsilon) \ B_H(Y, d_H(Y, E) \cdot 0.99) \cap A[0, 1] \neq \emptyset.$$

From this, the negation of the definition of $\sigma$-strong porosity will follow.

The idea of the proof is to find a correspondence between $A[0, 1]$ and the set $[-\frac{3}{2}, \frac{3}{2}]^N$ of all sequences with all terms belonging to the interval $[-\frac{3}{2}, \frac{3}{2}]$, then observe that in any decomposition of $[-\frac{3}{2}, \frac{3}{2}]^N$ into countably many pieces, one will be "large" (i.e. dense in some set $U \subseteq [-\frac{3}{2}, \frac{3}{2}]^N$ open in the product topology), pick any attractor $E$ corresponding to an element of $U$ and show that for any $Y \in B_H(E, \varepsilon)$ there is an open $V \subseteq U$ such that the attractor corresponding to any element of $V$ is in $B_H(Y, d_H(Y, E) \cdot 0.99)$.

The correspondence will be as follows: we will match $(x_n) \in [-\frac{3}{2}, \frac{3}{2}]^N$ (i.e. a sequence of reals) with a Cantor-like attractor. By a Cantor-like attractor we mean an attractor generated by two contractions: each of them will be "based" on the following functions:

$$f_1(x) = \frac{1}{10} x + \frac{2}{10}, \quad f_2(x) = \frac{1}{10} x + \frac{7}{10}.$$
Therefore, in each iteration a new image will be a "tenth" of a size of the previous image. However, also, every image of an interval can be moved by \( \frac{3}{2} \) of the tenth of the interval to the left or to the right (for instance, in the case of the first iteration: the left-hand image will be a subset of \([\frac{1}{20}, \frac{3}{20}]\) and the right-hand image will be a subset of \([\frac{14}{20}, \frac{16}{20}]\) – note that the distance between them is at least \( \frac{1}{10} \), so at the end we will still get an attractor for some IFS).

The endpoints of those images will be determined by the sequence \((x_n)\). For instance, a sequence constantly equal to 0 will "code" exactly the functions \(f_1, f_2\). A sequence constantly equal to \( -\frac{3}{2} \) will result in functions

\[
\frac{1}{10}x + \frac{1}{20}, \frac{1}{10}x + \frac{11}{20}
\]

and a sequence constantly equal to \( \frac{3}{2} \) will result in functions

\[
\frac{1}{10}x + \frac{7}{20}, \frac{1}{10}x + \frac{17}{20}
\]

And obviously, all the "in-between" shifts of intervals will be coded by terms between \( -\frac{3}{2} \) and \( \frac{3}{2} \). Below we present details.

Let the correspondence between \((x_n)\) and \((f_1, f_2)\) be the following: for each \(i \in \mathbb{N}\) let \(x_i\) be such that:

\[
\begin{align*}
x_1 & : f_1(0) = \frac{2}{10} + \frac{1}{10} \cdot x_1 \\
x_2 & : f_2(0) = \frac{7}{10} + \frac{1}{10} \cdot x_2 \\
x_3 & : f_1(f_1(0)) = f_1(0) + \frac{2}{100} + \frac{1}{100} \cdot x_3 \\
x_4 & : f_2(f_1(0)) = f_1(0) + \frac{7}{100} + \frac{1}{100} \cdot x_4 \\
x_5 & : f_1(f_2(0)) = f_2(0) + \frac{2}{100} + \frac{1}{100} \cdot x_5 \\
x_6 & : f_2(f_2(0)) = f_2(0) + \frac{7}{100} + \frac{1}{100} \cdot x_6 \\
x_7 & : f_1(f_1(f_1(0))) = f_1(f_1(0)) + \frac{2}{1000} + \frac{1}{1000} \cdot x_7
\end{align*}
\]

and so on.

Suppose to the contrary that \(A[0,1]\) is \(\sigma\)-strongly porous. Then so is the family \(\mathcal{F}\) of attractors corresponding (in the above way) to sequences in \([-\frac{3}{2}, \frac{3}{2}]^\mathbb{N}\). In particular, \(\mathcal{F} = \bigcup_{m \in \mathbb{N}} P_m\) where each \(P_m\) is a strongly porous subset of \(K[0,1]\). The sets \(P_m\) define a cover of \([-\frac{3}{2}, \frac{3}{2}]^\mathbb{N}\) by countably many
pieces. Since $[-\frac{3}{2}, \frac{3}{2}]^N$ is a complete metric space, by the Baire category theorem there is $m \in \mathbb{N}$ and an open set $I_1 \times I_2 \times \cdots \times I_k \times [-\frac{3}{2}, \frac{3}{2}]^N \subseteq [-\frac{3}{2}, \frac{3}{2}]^N$ (where $I_1, \ldots, I_k$ are open intervals in $[-\frac{3}{2}, \frac{3}{2}]$) such that the sequences corresponding to attractors from $P_m$ are dense in $I_1 \times I_2 \times \cdots \times I_k \times [-\frac{3}{2}, \frac{3}{2}]^N$.

For every $i \leq k$ put $x_i$ as a middle of an interval $I_i$. Then the sequence $(x_1, x_2, \ldots, x_k, 0, 0, \ldots) \in [-\frac{3}{2}, \frac{3}{2}]^N$ determines an attractor $E \in K[0, 1]$.

Let us take $\varepsilon$ such that it is half the length of an interval that is one level "below" $k$-th interval in the construction of the attractor. More formally, since $x_1$ and $x_2$ are responsible for the positions of intervals of length $\frac{1}{10}$, $x_3, x_4, x_5$ and $x_6$ are responsible for the positions of intervals of length $\frac{1}{100}$ etc., $\varepsilon = \frac{1}{2^{10^n}}$, where $n \in \mathbb{N}$ is such that $\sum_{i=1}^{n-2} 2^i < k \leq \sum_{i=1}^{n-1} 2^i$.

Now fix $Y \in B_H(E, \varepsilon)$ and to shorten the notation put $\delta = d_H(Y, E)$. Obviously, $\delta < \varepsilon = \frac{1}{2^{10^n}}$. Thus, there exists $j \geq n$ such that $\delta \in [\frac{1}{2} \cdot \frac{1}{10^{j+1}}, \frac{1}{2} \cdot \frac{1}{10^j}]$. Let $J_1, \ldots, J_{2j+2}$ be the intervals of length $\frac{1}{10^{j+2}}$ from the construction of the attractor $E$.

Observe that:

- $\tilde{Y}_\delta \supseteq E$ (since $\delta = d_H(Y, E)$),
- $\delta - \frac{99}{100} \delta = \frac{\delta}{100} < \frac{1}{2^{10^{j+2}}}$,
- the maximal possible shift of each $J_i$ is equal to
  \[
  \frac{3}{2} \cdot \frac{1}{10^{j+2}} = \frac{1}{10^{j+2}} + \frac{1}{2} \cdot \frac{1}{10^{j+2}} > |J_i| + \frac{\delta}{100},
  \]
- the set $\tilde{Y}_{\frac{99}{100}}$ is a union of closed intervals each of which has length at least $2 \cdot \frac{99}{100} \delta \geq \frac{99}{10^{j+2}} \geq \frac{1}{10^{j+2}} = |J_i|$.

Thus, for each $i \leq 2^{j+2}$ the set $U_i \subseteq [-\frac{3}{2}, \frac{3}{2}]$ equal to the interior of

\[\left\{x \in \left[-\frac{3}{2}, \frac{3}{2}\right] : J_i + x \subseteq \tilde{Y}_{\frac{99}{100}} \right\}\]

is nonempty. Hence, there is an open subset of $I_1 \times I_2 \times \cdots \times I_k \times [-\frac{3}{2}, \frac{3}{2}]^N$ such that each attractor corresponding to an element of that open set is a subset of $\tilde{Y}_{\frac{99}{100}}$.

Actually, by the last item above, we can even conclude that the set $U_i \setminus [-10^i \delta, 10^i \delta]$ is nonempty for each $i \leq 2^{j+2}$. Indeed, the interval $[-10^i \delta, 10^i \delta]$
is responsible for shifts by at most $\frac{\delta}{100} < \frac{1}{2} \cdot \frac{1}{10^{j+2}}$ (as each $x \in [-\frac{3}{2}, \frac{3}{2}]$ corresponds to a shift of $J_i$ by $\frac{x}{10^{j+2}}$). Thus, it suffices to observe that $|J_i| = \frac{1}{10^{j+2}}$, 
$||[-\frac{1}{100} \delta, \frac{1}{100} \delta]| < \frac{1}{10^{j+2}}$ and $2 \frac{99}{100} \delta > 2 \frac{1}{10^{j+2}}$.

Now, for each $i \leq 2^{j+2}$ let $V_i \subseteq U_i$ be open and such that:

- if $i$ is odd (i.e., the closest interval $J_{i'}$ to $J_i$ is on the right-hand side of $J_i$) and $U_i \cap [-\frac{3}{2}, -10^j \delta) \neq \emptyset$ then $V_i = U_i \cap [-\frac{3}{2}, -10^j \delta)$,
- if $i$ is even (i.e., the closest interval $J_{i'}$ to $J_i$ is on the left-hand side of $J_i$) and $U_i \cap (10^j \delta, \frac{3}{2}] \neq \emptyset$ then $V_i = U_i \cap (10^j \delta, \frac{3}{2}]$,
- $V_i = U_i \setminus [-10^j \delta, 10^j \delta]$ in all other cases.

Let $H \in P_m$ be an attractor corresponding to any sequence in $I_1 \times I_2 \times \ldots \times I_k \times \left[\frac{3}{2}, \frac{3}{2}\right]^{\sum_{i=1}^{j+1} 2^i - k} \times V_1 \times \ldots \times V_{2^{j+2}} \times \left[\frac{3}{2}, \frac{3}{2}\right]^N$.

Then $H \subseteq \tilde{Y}_{\frac{99}{100} \delta}$. Moreover, $Y \subseteq \tilde{H}_{\frac{99}{100} \delta}$. Indeed, fix $y \in Y$ and let $e \in E$ be such that $d(y, e)$ is minimal possible (i.e., $d(y, e) = d(y, E)$). Without loss of generality we may assume that $e \in J_i$ for some odd $i \leq 2^{j+2}$ (the case of even $i$ is similar). Let $z$ be the point lying exactly in the middle between $J_i$ and $J_{i+1}$. Denote also by $e'$ the point corresponding to $e$ in $H$. Then $s = d(e, e')$ is the shift of $J_i$. There are three possibilities:

- $y \leq e'$ and $J_i$ was shifted to the left (i.e., towards $y$). In this case $d(y, e') = d(y, e) - s < \delta - \frac{\delta}{100} = \frac{99}{100} \delta$ as the shift was by more than $\frac{\delta}{100}$.
- $J_i$ was shifted to the left, but $y > e'$. In this case the most extreme possibility is $y = z$. However, we have:

$$d(z, e) \leq |J_i| + \frac{1}{2} \text{dist}(J_i, J_{i+1}) = \frac{1}{10^{j+2}} + \frac{2}{10^{j+2}} = \frac{3}{10^{j+2}}.$$ 

Since $s \leq \frac{3}{2} \cdot \frac{1}{10^{j+2}}$, we get that:

$$d(z, e') = d(z, e) + s \leq \frac{3}{10^{j+2}} + \frac{3}{2} \cdot \frac{1}{10^{j+2}} < \frac{9}{2} \cdot \frac{1}{10^{j+2}} < \frac{99}{100} \cdot \frac{1}{2} \cdot \frac{1}{10^{j+1}} \leq \frac{99}{100} \delta.$$
• $J_i$ was shifted to the right. We will show that this implies that there is no $y \in Y$ such that $d(y, e) = d(y, E)$ for $e \in J_i$, i.e., this case is impossible. Indeed, if $y > z$ then $d(y, J_{i+1}) < d(y, J_i)$. On the other hand, if $y < \min J_i - \delta$ then $d(y, J_{i-1}) < d(y, J_i)$ (as $y \in \bigcup_{r \leq 2^{j+2}} (J_i^r)_{\delta}$).

Finally, if $\min J_i - \delta \leq y \leq z$ then it would be possible to shift $J_i$ by some $\frac{\delta}{100} < t < \frac{3}{2} \frac{1}{10^{j+2}}$ to the left in such a way that $t + J_i \subseteq B(y, \frac{99}{100} \delta)$ as $2 \frac{99}{100} \delta > |J_i|$ and

$$d(z, \min J_i) + \frac{\delta}{100} \leq \frac{3}{10^{j+2}} + \frac{1}{2} \frac{1}{100} \frac{1}{10} < \frac{99}{100} \frac{1}{2} \frac{1}{10^{j+1}} \leq \frac{99}{100} \delta,$$

$$d(\min J_i - \delta, \max J_i) - \frac{3}{2} \frac{1}{10^{j+2}} = \delta + |J_i| - \frac{3}{2} \frac{1}{10^{j+2}} = \delta - \frac{1}{2} \frac{1}{10^{j+2}} < \frac{99}{100} \delta,$$

(recall that $\delta < \frac{1}{2} \frac{1}{10^j}$).

This finishes the proof. \qed

Knowing the status of $\sigma$-strong porosity of $A[0,1]$, we ask the same question for the difference between this family and a broader one, namely the family of weak attractors.

**Question 4.2.** Is the set $A_w[0,1] \setminus A[0,1]$ $\sigma$-strongly-porous in $K[0,1]$?

**References**

[1] E. D’Aniello, T. H. Steele, Attractors for iterated function schemes on $[0,1]^N$ are exceptional, *Journal of Mathematical Analysis and Applications*, 424: 534-541, 2015.

[2] E. D’Aniello, T. H. Steele, Attractors for iterated function systems, *Journal of Fractal Geometry*, 3: 95-117, 2016.

[3] E. D’Aniello, T. H. Steele, Attractors for classes of iterated function systems, *European Journal of Mathematics*, 5: 116-137, 2019.

[4] M. Edelstein, On fixed and periodic points under contractive mappings, *Journal of London Mathematical Society s1-37*, 1: 74-79, 1962.

[5] A.S. Kechris, Classical Descriptive Set Theory, Springer, New York 1998.
[6] P. Klinga and A. Kwela, Borel complexity of the family of attractors for weak IFSs, submitted.

[7] P. Klinga and A. Kwela, IFSs and weak IFSs of size n and families of their attractors, submitted.

[8] P. Klinga, A. Kwela and M. Staniszewski, Size of the set of attractors for iterated function systems, *Chaos, Solitons and Fractals*, **128**: 104—107, 2019.

[9] M. Koc, Upper porous sets which are not $\sigma$-lower porous, *Real Analysis Exchange*, **35**: 21-30, 2009/2010.

[10] M. E. Mera, M. Morán, D. Preiss, and L. Zajíček, Porosity, $\sigma$-porosity and measures, *Nonlinearity*, **16**: 247-255, 2003.

[11] M. Nowak, M. Fernández-Martínez, Counterexamples for IFS-attractors, *Chaos Solitons & Fractals*, **89**: 316-321, 2016.

[12] L. L. Stachó and L. I. Szabó, A note on invariant sets of iterated function systems, *Acta Mathematica Hungarica*, **119**: 159-164, 2008.

[13] L. Zajíček, On $\sigma$-porous sets in abstract spaces, *Abstract and Applied Analysis*, vol. **2005**, no. **5**: 509-534, 2005.

Addresses:
Pawel Klinga
Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80 – 952 Gdańsk
Poland
e-mail: pawel.klinga@ug.edu.pl

Address:
Adam Kwela
Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80 – 952 Gdańsk
Poland
e-mail: adam.kwela@ug.edu.pl