Fully Nonlinear Elliptic Equations on Hermitian Manifolds for Symmetric Functions of Partial Laplacians

Mathew George¹ · Bo Guan¹ · Chunhui Qiu²

Received: 13 December 2021 / Accepted: 16 March 2022 / Published online: 18 April 2022 © Mathematica Josephina, Inc. 2022

Abstract
We consider a class of fully nonlinear second-order elliptic equations on Hermitian manifolds closely related to the general notion of $\mathcal{G}$-plurisubharmonicity of Harvey–Lawson and an equation treated by Székelyhidi–Tosatti–Weinkove in the proof of Gauduchon conjecture. Under fairly general assumptions, we derive interior estimates and establish the existence of smooth solutions for the Dirichlet problem as well as for equations on closed manifolds.

Keywords Fully nonlinear elliptic equations · Hermitian manifolds · Partial Laplacians · Tangent cone at infinity · Rank · A priori estimates · Dirichlet problem

Mathematics Subject Classification 35J15 · 35J60 · 58J05

1 Introduction

In a remarkable paper where the Gauduchon conjecture [24] which extends Calabi–Yau Theorem to non-Kähler metrics was proved, Székelyhidi–Tosatti–Weinkove [2]
solved the Monge–Ampère type equation

\[ (\Delta u \omega - \sqrt{-1} \partial \bar{\partial} u + \chi(\partial u, \bar{\partial} u))^n = \psi \omega^n \]  \hspace{1cm} (1.1)

on a compact Hermitian manifold \((M^n, \omega)\), where for a function \(u \in C^2(M)\), 
\(\chi(\partial u, \bar{\partial} u)\) is a real \((1,1)\) form which depends on \(\partial u, \bar{\partial} u\) linearly and satisfies some additional assumptions (which for convenience we shall refer as STW conditions). This equation also has close connections to the form-type Monge–Ampère equations studied by Fu-Wang-Wu [3, 4], and the notion of \(n-1\) plurisubharmonic functions in the sense of Harvey–Lawson [5–7].

In the same paper [2], Székelyhidi–Tosatti–Weinkove also treated more general equations of the form

\[ f(\lambda(\Delta u \omega - \sqrt{-1} \partial \bar{\partial} u + \chi(\partial u, \bar{\partial} u))) = \psi \]  \hspace{1cm} (1.2)

under the same (STW) conditions on \(\chi\), where \(\lambda(U)\) denotes the eigenvalues of a real \((1,1)\) form \(U\) with respect to \(\omega\), and \(f\) is a symmetric function of \(n\) variables satisfying the structure conditions of Caffarelli–Nirenberg–Spruck [8], i.e., \(f\) is assumed to be defined in a symmetric open and convex cone \(\Gamma \subset \mathbb{R}^n\) with vertex at the origin,

\[ \Gamma_n = \{ \lambda \in \mathbb{R}^n : \lambda_i > 0 \} \subset \Gamma, \]  \hspace{1cm} (1.3)

and satisfies the conditions

\[ f_i = \frac{\partial f}{\partial \lambda_i} > 0 \text{ in } \Gamma, \quad 1 \leq i \leq n, \]  \hspace{1cm} (1.4)

\[ f \text{ is a concave function in } \Gamma, \]  \hspace{1cm} (1.5)

\[ \sup_{\partial \Gamma} f < \inf M \psi \]  \hspace{1cm} (1.6)

and, in addition

\[ \lim_{t \to \infty} f(t\lambda) = \sup_{\partial \Gamma} f, \quad \forall \lambda \in \Gamma. \]  \hspace{1cm} (1.7)

There are other important fully nonlinear equations of second order in complex geometry with explicit dependence on the solution and its gradient, such as the Fu-Yau equation [9, 10] arising from the Strominger system in superstring theory and its higher dimension extensions studied by Phong–Picard–Zhang [11–13] and Chu–Huang–Zhu [14]. In general, the presence of these terms and the gradients especially causes substantial difficulties in solving the equations, even in the case of linear dependence for the Monge–Ampère type equation (1.1).

It is clear that Székelyhidi–Tosatti–Weinkove [2] primarily focused on equation (1.1) in order to prove the Gauduchon conjecture. It seems worthwhile to take a closer look at equation (1.2) from different angles and from a more PDE point of

\(\copyright\) Springer
view. This is one of our main purposes in the current paper. We shall assume in place of (1.4) that $f$ satisfies the weaker condition

$$f_i \equiv \frac{\partial f}{\partial \lambda_i} \geq 0 \text{ in } \Gamma, \quad 1 \leq i \leq n. \quad (1.8)$$

In order to describe our results, we recall some notions from [15] and [16]. For a fixed real number $\sigma \in (\sup_{\partial \Gamma} f, \sup_{\Gamma} f)$ define

$$\Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) > \sigma \}.$$

**Lemma 1.1** Under conditions (1.8) and (1.5), the level hypersurface of $f$

$$\partial \Gamma^\sigma = \{ \lambda \in \Gamma : f(\lambda) = \sigma \},$$

which is the boundary of $\Gamma^\sigma$, is smooth and convex.

**Proof** If $f$ satisfies both (1.4) and (1.5), then $\partial \Gamma^\sigma$ is clearly smooth and convex. This is still true under the weaker condition (1.8) in place of (1.4). For if $Df(\lambda_0) = 0$ for some $\lambda_0 \in \Gamma$ then by (1.8) and (1.5), we see that $Df = 0$ in the cone $\lambda_0 + \Gamma_n$ with vertex at $\lambda_0$. Therefore, $f(\lambda_0) = \sup_\Gamma f > \sigma$, showing that $Df \neq 0$ on $\partial \Gamma^\sigma$. So $\partial \Gamma^\sigma$ is smooth. \hfill \Box

For $\lambda \in \partial \Gamma^\sigma$ let

$$\nu_\lambda = \frac{Df(\lambda)}{|Df(\lambda)|}$$

denote the unit normal vector to $\partial \Gamma^\sigma$ at $\lambda$.

**Definition 1.2** ([15]). For $\mu \in \mathbb{R}^n$ let

$$S^\sigma_\mu = \{ \lambda \in \partial \Gamma^\sigma : \nu_\lambda \cdot (\mu - \lambda) \leq 0 \}.$$

The tangent cone at infinity to $\Gamma^\sigma$ is defined as

$$C^+_\sigma = \{ \mu \in \mathbb{R}^n : S^\sigma_\mu \text{ is compact} \}.$$

Clearly $C^+_\sigma$ is a symmetric convex cone. As in [15], one can show that $C^+_\sigma$ is open. Note that the unit normal vector of any supporting plane to $C^+_\sigma$ lies in $\Gamma_n$. For a vector $\nu \in \Gamma_n$, let $r(\nu)$ denotes the number of nonzero components of $\nu$.

**Definition 1.3** ([16]). The rank of $C^+_\sigma$ is defined to be

$$\min\{r(\nu) : \nu \text{ is the unit normal vector of a supporting plane to } C^+_\sigma \}.$$
Our first result concerns interior estimates for second-order derivatives. Let \( u \in C^4(M) \) be an admissible solution of the equation

\[
f(\lambda(\Delta u \omega - \sqrt{-1} \partial \bar{\partial} u + \chi[u])) = \psi[u]
\]

(1.9)

where \( \chi[u] = \chi(z, u, \partial u, \bar{\partial} u) \) and \( \psi[u] = \psi(z, u, \partial u, \bar{\partial} u) \) are \( C^2 \) in \( (z, u, \partial u, \bar{\partial} u) \).

We assume

\[
\lim_{t \to +\infty} f(t \mathbf{1}) - \sup_M \psi[u] \geq c_0 > 0
\]

(1.10)

where \( \mathbf{1} = (1, \ldots, 1) \in \Gamma \). So \( c_0 \) may depend on \( |u|_{C^4(M)} \). This is clearly a necessary and fairly mild condition for the following result to hold.

**Theorem 1.4** *In addition to (1.5), (1.6), (1.8), and (1.10) assume that \( f \) satisfies*

\[
\sum f_i \lambda_i \geq -C_0 \sum f_i \text{ in } \Gamma
\]

(1.11)

*for some constant \( C_0 \geq 0 \), and*

\[
\text{rank of } C_0^+ \geq 2, \quad \forall \inf_M \psi[u] \leq \sigma \leq \sup_M \psi[u].
\]

(1.12)

*Let \( B_r \) be a geodesic ball of radius \( r \) in \( M \). Then*

\[
\sup_{B_{\frac{r}{2}}} |\partial \bar{\partial} u| \leq \frac{C}{r^2}
\]

(1.13)

*where \( C \) depends on \( c_0, C_0, |u|_{C^4(B_r)} \) and other known data.*

Geometrically, condition (1.11) means that at any point \( \lambda_0 \in \Gamma \), the distance from the origin to the tangent plane at \( \lambda_0 \) of the level hypersurface \( \partial \Gamma f(\lambda_0) \) has a uniform bound \( C_0 \). It is satisfied in most applications and is weaker than assumption (1.7) which implies

\[
\sum f_i \lambda_i \geq 0 \text{ in } \Gamma.
\]

(1.14)

By an inequality of Lin–Trudinger [17], the rank of \( C_0^+ \) is \( n - k + 1 \) for \( f = \sigma_k^{\frac{1}{k}} \) and \( \sigma > 0 \), where

\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}
\]

is the \( k \)-th elementary symmetric function defined on the Garding cone

\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, \ for \ 1 \leq j \leq k \}.
\]
So Theorem 1.4 applies to \( f = \sigma^\frac{1}{k} \) for \( k \leq n - 1 \) but excludes equation (1.1) which corresponds to \( f = \sigma^\frac{1}{n} \).

Theorem 1.4 heavily relies on a crucial lemma in Sect. 2. From the proof, we shall see that there exists \( \beta_0 > 1 \) such that Theorem 1.4 extends to the equation

\[
f(\lambda(\Delta u \omega - \beta \sqrt{-1} \partial \bar{\partial} u + \chi[u])) = \psi[u],
\]

for all \( \beta < \beta_0 \). It would be interesting to decide the supremum value, \( \sup \beta_0 \), of such \( \beta_0 \) and study the equation in the limiting case.

The second purpose of the current paper is to extend Theorem 1.4 to a more general setting.

Let \( K \leq n \) be a fixed positive integer. Set

\[
\mathcal{I}_K = \{(i_1, \ldots, i_K) : 1 \leq i_1 < \cdots < i_K \leq n, \ i_j \in \mathbb{N}\}
\]

and for \( I = (i_1, \ldots, i_K) \in \mathcal{I}_K \) define

\[
\Lambda_I(\lambda) = \sum_{i \in I} \lambda_i = \lambda_{i_1} + \cdots + \lambda_{i_K}, \ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n.
\]

(1.16)

For convenience we fix an order for the elements in \( \mathcal{I}_K \):

\[
I_1, \ldots, I_N, \text{ where } N = \frac{n!}{K!(n-K)!}
\]

and write

\[
\Lambda(\lambda) = (\Lambda_{I_1}(\lambda), \ldots, \Lambda_{I_N}(\lambda))
\]

or simply,

\[
\Lambda(\lambda) = (\Lambda_1(\lambda), \ldots, \Lambda_N(\lambda)).
\]

We consider the equation

\[
f(\Lambda(\sqrt{-1} \partial \bar{\partial} u + X[u])) = \psi[u],
\]

(1.17)

where

\[
\Lambda(\sqrt{-1} \partial \bar{\partial} u + X[u]) = \Lambda(\lambda(\sqrt{-1} \partial \bar{\partial} u + X[u]))
\]

and

\[
\lambda(\sqrt{-1} \partial \bar{\partial} u + X[u]) = (\lambda_1, \ldots, \lambda_n)
\]
denotes the eigenvalues of \( \sqrt{-1} \partial \bar{\partial} u + X[u] \) with respect to \( \omega \), \( X[u] = X(z, u, \partial u, \bar{\partial} u) \) is a real \((1, 1)\) form depending on \( u, \partial u, \) and \( \bar{\partial} u \), and \( f \) is a symmetric concave function defined in an open symmetric convex cone \( \Gamma \subset \mathbb{R}^N \) with vertex at the origin, \( \Gamma_N \subset \Gamma \), and satisfies

\[
f_i = f_{\Lambda_i} = \frac{\partial f}{\partial \Lambda_i} \geq 0 \text{ in } \Gamma, \quad 1 \leq i \leq N. \tag{1.18}
\]

We may similarly define admissible functions by requiring \( \Lambda(\sqrt{-1} \partial \bar{\partial} u + X[u]) \in \Gamma \). Clearly, equations (1.2) and (1.17) coincide when \( K = n - 1 \).

Equation (1.17) is closely related to the notion of \( G \)-plurisubharmonicity Harvey–Lawson introduced and studied in a long series of papers (see e.g. [5–7]). As in [16] we denote

\[
\mathcal{P}_k := \{ \lambda \in \mathbb{R}^n : \lambda_{i_1} + \cdots + \lambda_{i_k} > 0, \ \forall \ 1 \leq i_1 < \cdots < i_k \leq n \}.
\]

A function \( u \) is \( k \)-plurisubharmonic in the sense of Harvey–Lawson [6] if

\[
\lambda(\sqrt{-1} \partial \bar{\partial} u + X) \in \mathcal{P}_k.
\]

Define

\[
\rho_k(\lambda) := \prod_{1 \leq i_1 < \cdots < i_k \leq n} (\lambda_{i_1} + \cdots + \lambda_{i_k}), \ 1 \leq k \leq n.
\]

Then \( f(\lambda) = \log \rho_k(\lambda) \) satisfies (1.4) and (1.5) in \( \mathcal{P}_k \), and the rank of \( C_\sigma^+ \) is \( k \). Clearly,

\[
\rho_1(\lambda) = \sigma_n(\lambda), \ \rho_n(\lambda) = \sigma_1(\lambda).
\]

Note that \( \rho_K(\lambda) = \sigma_N(\Lambda(\lambda)) \). So the equation

\[
\rho_K(\lambda(\sqrt{-1} \partial \bar{\partial} u + \chi)) = \psi
\]

for \( 1 \leq K \leq n \) is included in the examples of equation (1.17) given by

\[
f(\Lambda) = \sigma_k^{1/k}(\Lambda), \ 1 \leq k \leq N.
\]

In particular, equation (1.1) is equivalent to

\[
\rho_{n-1}(\lambda(\sqrt{-1} \partial \bar{\partial} u + X)) = \psi.
\]

Theorem 1.4 holds for equation (1.17) with the modification of replacing the rank condition (1.12) by

\[
\text{rank of } C_\sigma^+ \geq \frac{N(n - K)}{n} + 1, \ \forall \sup_{\partial \Gamma} f < \sigma < \sup f. \tag{1.19}
\]
Indeed, it extends to the equation

\[ f(\Lambda(\sqrt{-1}\bar{\partial}u + X[u])) - \beta \Lambda'(\sqrt{-1}\bar{\partial}u + X[u])) = \psi[u], \tag{1.20} \]

for some \( \beta > 0 \), where \( \Lambda' = (\Lambda_{I_1'}, \ldots, \Lambda_{I_n'}) \) and \( I_j' \) denotes the complement of \( I_j \) in \( \{1, \ldots, n\} \); we state a more complete result for equation (1.20) to include global and boundary estimates on second derivatives.

**Theorem 1.5** Under conditions (1.5), (1.18), (1.11), and (1.19), there exists constant \( \beta_0 > 1 \) such that for any \( \beta < \beta_0 \) and admissible solution \( u \in C^4(M) \) of equation (1.20) satisfying (1.10), the interior estimate (1.13) holds for any geodesic ball \( B_r \) in \( M \). Suppose in addition that \( (M^n, \omega) \) is a Hermitian manifold with smooth boundary \( \partial M \) and compact closure \( \bar{M} = M \cup \partial M \), and \( u \in C^4(\bar{M}) \). Then

\[
\max_{\bar{M}} |\partial \bar{\partial} u| \leq C \left( 1 + \max_{\partial M} |\partial \bar{\partial} u| \right) \tag{1.21}
\]

and

\[
\max_{\partial M} |\nabla^2 u| \leq C. \tag{1.22}
\]

One may also derive interior gradient estimates under suitable growth conditions of \( X \) and \( \psi \) on \( u \) and \( \nabla u \); see Theorem 4.1 for the precise statement.

We now turn to existence of admissible solutions for equation (1.17). As we shall see from the proofs the results extend to equation (1.20) for \( \beta < \beta_0 \) (as in Theorem 1.5) without difficulty. For simplicity, we only state our results here for \( X \) and \( \psi \) both independent of \( u \) and \( \nabla u \); more general results are stated in Sect. 5. We first consider the Dirichlet problem.

**Theorem 1.6** Let \( (M^n, \omega) \) be a Hermitian manifold with smooth boundary \( \partial M \) and compact closure \( \bar{M} = M \cup \partial M \), \( X \) a smooth real \((1,1)\) form on \( \bar{M} \), \( \psi \in C^\infty(\bar{M}) \) and \( \varphi \in C^0(\partial M) \). Suppose (1.5), (1.6), (1.18), (1.11), and (1.19) hold, and that there exists an admissible viscosity subsolution \( u \in C^0(\bar{M}) \) satisfying

\[
\begin{cases}
  f(\Lambda(\sqrt{-1}\bar{\partial}u + X)) \geq \psi \text{ in } \bar{M}, \\
  u = \varphi \text{ on } \partial M.
\end{cases} \tag{1.23}
\]

Equation (1.17) admits a unique admissible solution \( u \in C^\infty(M) \cap C^0(\bar{M}) \) with \( u = \varphi \) on \( \partial M \) and, for any subdomain \( \Omega \subset \bar{\Omega} \subset M \),

\[
|u|_{C^{2,\alpha}(\Omega)} \leq C, \quad 0 < \alpha \leq 1 \tag{1.24}
\]

where \( C \) depends on the distance from \( \Omega \) to \( \partial M \).

If in addition \( \varphi \in C^\infty(\partial M) \) then \( u \in C^\infty(\bar{M}) \) and satisfies the global estimate

\[
|u|_{C^{2,\alpha}(\bar{M})} \leq C, \quad 0 < \alpha \leq 1. \tag{1.25}
\]
Remark 1.7 More precisely, for a domain \( \Omega \) in \( M \), we have
\[
|\nabla u|^2 + |\nabla^2 u| \leq \frac{C}{d^2} \left\{ 1 + \sup_M u - u \right\} \quad \text{in} \ \Omega
\] (1.26)
and
\[
|\nabla^2 u|_{C^\alpha(\Omega)} \leq \frac{C}{d^{2+\alpha}} \left\{ 1 + \sup_M u - \inf_{\Omega} u \right\}, \quad 0 < \alpha \leq 1
\] (1.27)
where \( d \) is the distance from \( \Omega \) to \( \partial M \), and \( C \) is independent of \( d \).

Remark 1.8 In order to derive (1.25), it is enough to assume \( \varphi \in C^{3,1}(\partial M) \) while the optimal condition is expected to be \( \varphi \in C^{2,\alpha}(\partial M) \); this will be treated in details elsewhere.

An interesting question would be when there exist admissible subsolutions. Under suitable conditions on \( f \), one should be able to follow the construction of Caffarelli–Nirenberg–Spruck [8] with appropriate modifications, for instance, on manifolds with \( K \)-plurisubharmonic exhausting functions.

Equation (1.17) belongs to the general class of equations of the form
\[
f(\lambda(\sqrt{-1}\bar{\partial}\partial u + X[u])) = \psi[u]
\] (1.28)
which corresponds to (1.17) for \( K = 1 \). Consequently, when both \( X \) and \( \psi \) are independent of \( u \) and \( \nabla u \), results of Szekelyhidi [18] apply to equation (1.17) on compact Hermitian manifold (without boundary). Under assumption (1.19), we obtain the following existence result for equation (1.17) without the key \( C \)-subsolution condition in [18] (which is equivalent to the tangent cone at infinity condition introduced in [15] for a type I cone \( \Gamma \) that there exists a function \( u \in C^2(M) \) with \( \lambda(\sqrt{-1}\bar{\partial}\partial u + X) \in C^+_{\psi} \) on \( M \) [19]; recall from [8] that \( \Gamma \) is of type I if each positive \( \lambda_i \) axis is contained in \( \partial\Gamma \)).

Theorem 1.9 Let \((M^n, \omega)\) be a compact Hermitian manifold, \( X \) a smooth positive \((1, 1)\) form and \( \psi \in C^{\infty}(M) \). In addition to (1.5), (1.18), (1.11), and (1.19), assume in place of (1.6) that
\[
\sup_{\partial\Gamma} f \leq 0 < \psi < \sup_{\Gamma} f = \lim_{t \to +\infty} f(t\mathbf{1}).
\] (1.29)

Then there exists a unique constant \( b \) such that the equation
\[
f(\Lambda(\sqrt{-1}\bar{\partial}\partial u + X)) = e^b \psi
\] (1.30)
has a unique admissible solution \( u \in C^{\infty}(M) \) with
\[
\sup_M u = 0.
\]
Without the assumption $X > 0$, equation (1.30) may become degenerate. Note that when $X > 0$ constants are admissible functions. On the other hand, if $X \leq 0$, there are no admissible functions on compact manifolds (without boundary).

Comparing to equation (1.17), the general equation (1.28) is much more difficult, and most of the results in this paper no longer hold without additional assumptions even in the case $X$ and $\psi$ are independent of $u$ and $\nabla u$. We also note that, among others, equation (1.28) will only be degenerate elliptic if condition (1.4) is replaced by (1.8). For $X[u] = X(\partial u, \bar{\partial} u)$ and $\psi[u] = \psi(\partial u, \bar{\partial} u)$, Guan–Nie [16] derived the second-order estimates for Eq. (1.28) on closed Hermitian manifolds under rather strong conditions; we refer the reader to [16] for more references.

For $f = (\sigma_n / \sigma_k)^{\frac{1}{n-k}}$, the Dirichlet problem for Eq. (1.2) was recently studied by Feng–Ge–Zheng [20]. In [21], Guan–Qiu–Yuan treated equations of the form

$$F(\gamma \Delta u \omega - \beta \sqrt{-1} \partial \bar{\partial} u + \chi[u]) = \psi[u]$$

(1.31)

for $\gamma > 0, \beta < \gamma$. One should be able to extend most of results described in the current paper to the limiting case $\beta = \gamma$ for equation (1.31). It also seems interesting to study equation (1.17) with $\Lambda_f$ defined in (1.16) replaced by other symmetric functions of $K$ variables.

The rest of this paper is organized as follows. In Sect. 2, we prove some key properties of the tangent cone at infinity related to its rank. The second derivative and gradient estimates are derived in Sects. 3 and 4, respectively. We also establish the Harnack inequality using the gradient estimates for positive solutions when $X$ and $\psi$ are independent of $u$ and $\nabla u$. Section 5 concerns the existence of admissible solutions, where we prove Theorems 1.6, 1.9 and their extensions.

The second author wishes to thank Duong Phong for stimulating communications, especially for his comments on results described in [21], which motivated us to carry out the current work. We also thank the referee for very helpful comments.

## 2 Rank of Tangent Cone at Infinity

In this section, $f$ is assumed to satisfy (1.5), (1.8), and (1.11). We shall prove the following key lemmas.

**Lemma 2.1** Let $P = \{ \mu \in \mathbb{R}^n : \nu \cdot \mu = c \}$ be a hyperplane, where $\nu$ is a unit vector. Suppose that there exists a sequence $\{ \lambda_k \}$ in $\partial \Gamma^{\sigma}$ with

$$\lim_{k \to +\infty} \nu_{\lambda_k} = \nu, \quad \lim_{k \to +\infty} \nu_{\lambda_k} \cdot \lambda_k = c, \quad \lim_{k \to +\infty} |\lambda_k| = +\infty. \quad (2.1)$$

Then $P$ is a supporting hyperplane to $C^+_\sigma$ at a nonvertex point.

**Proof** Assume $\mu_0 \in \mathbb{R}^n$ with $\nu \cdot \mu_0 \leq c - \epsilon$ for some $\epsilon > 0$. Then

$$\lim_{k \to +\infty} \nu_{\lambda_k} \cdot (\mu_0 - \lambda_k) = \nu \cdot \mu_0 - c \leq -\epsilon.$$
This means that $S_{\mu_0}^\sigma$ is noncompact and therefore $\mu_0 \notin C_\sigma^+$. As $\epsilon > 0$ is arbitrary and $C_\sigma^+ \subset P^+$, we see that $C_\sigma^+ \subset P^+ = \{ \mu \in \mathbb{R}^n : v \cdot \mu > c \}$.

By symmetry, the vertex of $C_\sigma^+$ is at a point $a = (a, \ldots, a)$ for some real number $a$. It is easy to see that $a \in P$, i.e.,

$$a = \frac{c}{\sum v^i}$$

where $v = (v^1, \ldots, v^n)$. Indeed, clearly $a \geq c(\sum v^i)^{-1}$ for otherwise $a \notin P^+$. Let $R_k$ be the ray from $a$ passing $\lambda_k$ so $R_k \subset C_\sigma^+ \subset P^+$. On the other hand, $R_k$ would not completely lie in $P^+$ for $k$ large if $a > c(\sum v^i)^{-1}$, a contradiction.

Let $\mu_k$ be the point of intersection of $R_k$ and the unit sphere centered at $a$. Passing to a subsequence, we may assume $\mu_k \rightarrow \mu \in \overline{C_\sigma^+}$ as $k \rightarrow +\infty$. Clearly,

$$0 \geq v_k \cdot (\mu_k - \lambda_k) \geq v_k \cdot (a - \lambda_k).$$

Taking limit as $k \rightarrow +\infty$ we obtain $v \cdot \mu = c$. This show that $\mu \in P \cap \partial C_\sigma^+$. □

**Lemma 2.2** Let $r \geq 1$ be the rank of $C_\sigma^+$. There exists $c_1 > 0$ such that at any point $\lambda \in \partial \Gamma^\sigma$ where without loss of generality we assume $f_1 \leq \cdots \leq f_n$,

$$\sum_{i \leq n-r+1} f_i \geq c_1 \sum f_i.$$

**Proof** We prove by contradiction. Suppose that for each $k = 1, 2, \ldots$ there exists $\lambda_k \in \partial \Gamma^\sigma$ with $f_1(\lambda_k) \leq \cdots \leq f_n(\lambda_k)$ and

$$\sum_{i \leq n-r+1} f_i(\lambda_k) \leq \frac{1}{k} \sum f_i(\lambda_k). \tag{2.2}$$

Note that there is $t > 0$ such that $t 1 \in \partial \Gamma^\sigma$, that is, $f(t 1) = \sigma$. By the concavity of $f$,

$$\sum f_i(t - \lambda_i) \geq f(t 1) - f(\lambda) = 0.$$

Therefore,

$$\sum f_i \lambda_i \leq t \sum f_i.$$

By (1.11), we can pass to a subsequence and assume that

$$\lim_{k \rightarrow +\infty} v_{\lambda_k} = v, \quad \lim_{k \rightarrow +\infty} v_{\lambda_k} \cdot \lambda_k = c, \quad \lim_{k \rightarrow +\infty} |\lambda_k| = +\infty.$$

According to Lemma 2.1, the hyperplane $P = \{ \mu \in \mathbb{R}^n : v \cdot \mu = c \}$ is a supporting hyperplane to $C_\sigma^+$. However, $v$ has at most $r - 1$ nonzero components, which is a contradiction. □
Lemma 2.3 Let $P = \{ \mu \in \mathbb{R}^n : \nu \cdot \mu = c \}$, where $\nu$ is a unit vector, be a tangent hyperplane to $\partial C^+_\sigma$. Then there exists a sequence $\{ \lambda_k \}$ in $\partial \Gamma$ with

$$\lim_{k \to +\infty} \nu_{\lambda_k} = \nu, \quad \lim_{k \to +\infty} \nu_{\lambda_k} \cdot \lambda_k = c, \quad \lim_{k \to +\infty} |\lambda_k| = +\infty. \quad (2.3)$$

Proof We may assume $\partial \Gamma \cap \partial C^+_\sigma \cap P = \emptyset$ for otherwise it contains a ray (see [15]) and (2.3) holds for some $\lambda_k \in \partial \Gamma \cap \partial C^+_\sigma \cap P$.

Suppose that $P$ is the tangent hyperplane to $\partial C^+_\sigma$ at $\mu \in P \cap \partial C^+_\sigma$. We have $\mu \notin \partial \Gamma$ and $\mu_k = \mu + \nu_k \in C^+_\sigma$ for all $k \geq 1$ as $\nu \in \Gamma_n$ and $\mu + \Gamma_n \in C^+_\sigma$. Let

$$S_{\mu_k} = \{ \lambda \in \partial \Gamma : \nu \cdot (\mu_k - \lambda) = 0 \}$$

and choose $\lambda_k \in S_{\mu_k}^0$ satisfying

$$\nu \cdot (\lambda_k - \mu) = \min_{\lambda \in S_{\mu_k}^0} \nu \cdot (\lambda - \mu) \geq 0;$$

such $\lambda_k$ exists as $S_{\mu_k}^0$ is compact. As $\partial \Gamma \cap \partial C^+_\sigma \cap P = \emptyset$, we have

$$\lim_{k \to +\infty} |\lambda_k| = +\infty.$$ 

So the last limit in (2.3) holds. Next,

$$\nu_{\lambda_k} \cdot (\lambda_k - \mu) = \nu_{\lambda_k} \cdot (\lambda_k - \mu_k) + \nu_{\lambda_k} \cdot (\mu_k - \mu) = \frac{v_k \cdot \nu}{k}.$$

Therefore,

$$\lim_{k \to +\infty} \nu_{\lambda_k} \cdot (\lambda_k - \mu) = 0. \quad (2.4)$$

Passing to a subsequence, we may assume

$$\lim_{k \to +\infty} \nu_{\lambda_k} = \nu'.$$

By (2.4) we have

$$\lim_{k \to +\infty} \nu_{\lambda_k} \cdot \lambda_k = \nu' \cdot \mu \equiv c'.$$

By Lemma 2.1, we see that $P' = \{ \eta \in \mathbb{R}^n : \nu' \cdot \eta = c' \}$ is a supporting plane to $C^+_\sigma$ at $\mu$. By uniqueness of the tangent plane, $P' = P$ and therefore $\nu' = \nu$. \qed
3 The Second-Order Estimates

In this section, we establish the second-order estimates in Theorems 1.4 and 1.5 for admissible solutions. We first fix some notation. In local coordinates \( z = (z_1, \ldots, z_n) \), we shall write

\[
\omega = \sqrt{-1} g_{ij} dz_i \wedge d\bar{z}_j
\]  

(3.1)

and \( \{g^{i\bar{j}}\} = \{g_{i\bar{j}}\}^{-1} \). The Christoffel symbols \( \Gamma^k_{ij} \), torsion, and curvature tensors are given, respectively, by

\[
\nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j} = \Gamma^k_{ij} \frac{\partial}{\partial z_k}, \quad T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}
\]

and

\[
R_{i\bar{j}k\bar{l}} = -g_{mi} \frac{\partial \Gamma^n_{ik}}{\partial z_j} = -\frac{\partial g_{ki}}{\partial z_i \partial \bar{z}_j} + g_{pq} \frac{\partial g_{k\bar{q}}}{\partial z_i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}_j}
\]

where \( \nabla \) is the Chern connection on \((M^n, \omega)\).

Let \( u \in C^4(M) \) be an admissible solution of equation (1.17). We denote

\[
g[u] = \sqrt{-1} \partial \bar{\partial} u + X[u]
\]

and define \( G \) by

\[
G(g[u]) = f(\Lambda(g[u])).
\]

Consequently, equation (1.17) takes the form

\[
G(g[u]) = \psi[u].
\]  

(3.2)

Note that for any \((1, 1)\) forms \( A \) and \( B \),

\[
f(\Lambda(aA + bB)) = f(a\Lambda(A) + b\Lambda(B)).
\]

It follows from [8] that \( G(A) \) is concave in \( A \).

When \( K = n - 1 \), we have \( N = n \) and

\[
\Lambda(g[u]) = \lambda((\text{tr}g[u])\omega - g[u]).
\]

We see that equation (1.9) is covered by equation (3.2) for

\[
X[u] = \frac{\text{tr} \chi[u]}{n - 1} \omega - \chi[u].
\]
In local coordinates, we write
\[ g[u] = \sqrt{-1} g_{ij} dz_i \wedge dz_j = \sqrt{-1}(\partial_i \bar{\partial}_j u + X_{ij}) dz_i \wedge dz_j \]
and differentiate equation (3.2) twice to obtain
\[ G^{i\bar{j}} \nabla_p g_{i\bar{j}} = \nabla_p \psi[u] \tag{3.3} \]
and
\[ G^{i\bar{j}} \nabla_q \nabla_p g_{i\bar{j}} = \nabla_q \nabla_p \psi[u] - G^{i\bar{k}l} \nabla_p g_{i\bar{j}} \nabla_q g_{k\bar{l}}, \tag{3.4} \]
where
\[ G^{i\bar{j}} = \frac{\partial G}{\partial g_{i\bar{j}}}(g[u]), \quad G^{i\bar{j}k\bar{l}} = \frac{\partial^2 G}{\partial g_{i\bar{j}} \partial g_{k\bar{l}}}(g[u]). \]

To estimate \(|\partial \bar{\partial} u|\), as in [16] we follow ideas of Tosatti–Weinkove [22] and consider the quantity which is given in local coordinates by
\[ A := \sup_{z \in M} \max_{\xi \in T^{1,0}_z M} e^{(1+\gamma)\phi} e_{pq} \xi_p \xi_q (g^{k\bar{l}} g_{i\bar{j}} g_{k\bar{j}} \xi_i \xi_j)^{2+\gamma} / |\xi|^{2+\gamma} \tag{3.5} \]
where \(\phi\) is a function depending on \(u\) and \(|\nabla u|\), and \(\gamma > 0\) is a small constant to be determined.

Assume that \(A\) is achieved at an interior point \(z_0 \in M\) for some \(\xi \in T^{1,0}_z M\). We choose local coordinates around \(z_0\) such that \(g_{i\bar{j}} = \delta_{ij}\) and \(T^k_{ij} = 2 \Gamma^k_{ij}\) using the lemma of Streets and Tian [23], and that \(g_{i\bar{j}}\) is diagonal at \(z_0\) with
\[ g_{1\bar{1}} \geq g_{2\bar{2}} \geq \cdots \geq g_{n\bar{n}}. \]
As noted in [22], the maximum \(A\) is achieved for \(\xi = \partial_1\) at \(z_0\) when \(\gamma > 0\) is sufficiently small; see also [16]. We assume \(g_{1\bar{1}} \geq 1\); otherwise, we are done.

Let \(W = g_{i\bar{i}}^{-1} g^{k\bar{l}} g_{i\bar{j}} g_{k\bar{l}}\). We see that the function \(e^{(1+\gamma)\phi} g_{1\bar{1}}^{-1} W^\gamma\) which is locally well defined attains a maximum \(A = (e^\phi g_{1\bar{1}})^{1+\gamma}\) at \(z_0\) where \(W = g_{1\bar{1}}^2\),
\[ \frac{\partial_i (g_{1\bar{1}}^{-1} g_{1\bar{i}})}{g_{1\bar{i}}} + \frac{\gamma \partial_i W}{2W} + (1 + \gamma) \partial_i \phi = 0, \quad \frac{\partial_{\bar{i}} (g_{1\bar{1}}^{-1} g_{1\bar{i}})}{g_{1\bar{i}}} + \frac{\gamma \partial_{\bar{i}} W}{2W} + (1 + \gamma) \partial_{\bar{i}} \phi = 0 \tag{3.6} \]
for each $1 \leq i \leq n$, and
\[
0 \geq \frac{1}{\partial_i} G^{ij} \tilde{\partial}_i (g^{-1}_{i1} g_{1i}) - \frac{1}{\partial_i} G^{ij} \tilde{\partial}_i (g^{-1}_{i1} g_{1i}) \tilde{\partial}_j (g^{-1}_{i1} g_{1i}) + \frac{\gamma}{2W} G^{ij} \tilde{\partial}_i \tilde{\partial}_j W - \frac{\gamma}{2W^2} G^{ij} \tilde{\partial}_i \tilde{\partial}_j W + (1 + \gamma) G^{ij} \tilde{\partial}_i \tilde{\partial}_j \phi. \tag{3.7}
\]

In what follows we shall make use of calculations in [16]; for the reader’s convenience, we give a brief outline. First,
\[
\partial_i (g^{-1}_{i1} g_{1i}) = \nabla_i g_{1i}, \quad \partial_i W = 2g_{i1} \nabla_i g_{1i}, \tag{3.8}
\]
\[
\tilde{\partial}_j \partial_i (g^{-1}_{i1} g_{1i}) = \nabla_j \nabla_i g_{1i} + (\Gamma^m_{j1} \nabla_i g_{1m} - \Gamma^m_{j1} \nabla_i g_{1i}) + (\Gamma^m_{i1} \nabla_j g_{1i} - \Gamma^m_{i1} \nabla_j g_{1i}) + (\Gamma^m_{i1} \Gamma^m_{j1} - \Gamma^m_{i1} \Gamma^m_{j1}) g_{1i}. \tag{3.9}
\]

and
\[
\tilde{\partial}_j \partial_i W = 2g_{i1} \nabla_j \nabla_i g_{1i} + 2\nabla_i g_{1i} \nabla_j g_{1i} + \sum_{l>1} \nabla_i g_{li} \nabla_j g_{li} + \sum_{l>1} (\nabla_i g_{li} + \Gamma^l_{ij} g_{li})(\nabla_j g_{li} + \Gamma^l_{ij} g_{li}) + g_{i1} \sum_{l>1} (\Gamma^l_{ij} \nabla_i g_{1i} + \Gamma^l_{ij} \nabla_j g_{li}) - g_{i1} \sum_{l>1} \Gamma^m_{li} \Gamma^m_{j1} (g_{1i} + g_{ii}). \tag{3.10}
\]

It follows that
\[
G^{ij} \partial_i W \tilde{\partial}_j W = 4g_{i1}^2 G^{ij} \nabla_i g_{1i} \nabla_j g_{1i}, \tag{3.11}
\]
\[
G^{ij} \tilde{\partial}_i (g^{-1}_{i1} g_{1i}) \tilde{\partial}_j (g^{-1}_{i1} g_{1i}) = G^{ij} \nabla_i g_{1i} \nabla_j g_{1i} \tag{3.12}
\]

and, by Cauchy–Schwarz inequality,
\[
G^{ij} \tilde{\partial}_i \partial_j W \geq 2g_{i1} G^{ij} \nabla_i \nabla_j g_{1i} + 2G^{ij} \nabla_i g_{1i} \nabla_j g_{1i} - C g_{i1}^2 \sum G^{ij}, \tag{3.13}
\]
\[
G^{ij} \tilde{\partial}_i (g^{-1}_{i1} g_{1i}) \tilde{\partial}_j (g^{-1}_{i1} g_{1i}) \geq G^{ij} \nabla_i \nabla_j g_{1i} - C g_{i1} \sum G^{ij}, \tag{3.14}
\]

We derive from (3.6), (3.7) and (3.11)-(3.14) that
\[
\nabla_i g_{1i} + g_{i1} \partial_i \phi = 0, \quad \nabla_i g_{1i} + g_{i1} \tilde{\partial}_i \phi = 0 \tag{3.15}
\]
and

\[ 0 \geq \frac{1}{\varrho_{i\bar{j}}} G^{i\bar{j}} \nabla_i \nabla_i g_{i\bar{j}} - \frac{1}{\varrho_{i\bar{j}}} G^{i\bar{j}} \nabla_i g_{i\bar{j}} \nabla_i g_{i\bar{j}} + G^{i\bar{j}} \partial_i \partial_{\bar{j}} \phi \]

\[ + \frac{\gamma}{\varrho_{i\bar{j}}} \sum_{l>1} G^{i\bar{j}} \nabla_i g_{i\bar{j}} \nabla_i g_{i\bar{j}} + \frac{\gamma}{16 \varrho_{i\bar{j}}} \sum_{l>1} G^{i\bar{j}} \nabla_i g_{i\bar{j}} \nabla_i g_{i\bar{j}} - C \sum G^{i\bar{j}}. \tag{3.16} \]

As in [16] and [21], we have

\[ \nabla_i \nabla_i g_{i\bar{j}} - \nabla_i \nabla_i g_{i\bar{j}} = R_{i\bar{j}i\bar{j}} - R_{i\bar{j}i\bar{j}} - T_{i\bar{j}}^l \nabla_i g_{i\bar{j}} - \overline{T_{i\bar{j}l}} \nabla_i g_{i\bar{j}} \]

\[ - T_{i\bar{j}}^l \overline{T_{i\bar{j}l}} + H_{i\bar{j}}. \tag{3.17} \]

where

\[ H_{i\bar{j}} = \nabla_i \nabla_i X_{i\bar{j}} - \nabla_i \nabla_i X_{i\bar{j}} - 2 \Re \{ T_{i\bar{j}}^l \nabla_i X_{i\bar{j}} \} + R_{i\bar{j}i\bar{j}} X_{i\bar{j}} - R_{i\bar{j}i\bar{j}} X_{i\bar{j}} - T_{i\bar{j}}^l \overline{T_{i\bar{j}l}} X_{i\bar{j}}. \]

It follows from Schwarz inequality that

\[ G^{i\bar{j}} \nabla_i \nabla_i g_{i\bar{j}} \geq G^{i\bar{j}} \nabla_i \nabla_i g_{i\bar{j}} - \frac{\gamma}{32 \varrho_{i\bar{j}}} G^{i\bar{j}} \nabla_i g_{i\bar{j}} \nabla_i g_{i\bar{j}} \]

\[ - C \varrho_{i\bar{j}} \sum G^{i\bar{j}} + G^{i\bar{j}} H_{i\bar{j}}. \tag{3.18} \]

From (3.4), (3.15), (3.16), and (3.18), we derive

\[ \varrho_{i\bar{j}} G^{i\bar{j}} \partial_i \partial_{\bar{j}} \phi \leq - \nabla_i \nabla_1 \psi + \frac{1 + \gamma}{\varrho_{i\bar{j}}} G^{i\bar{j}} \nabla_i g_{i\bar{j}} \nabla_i g_{i\bar{j}} + C \varrho_{i\bar{j}} \sum G^{i\bar{j}} \]

\[ - \frac{\gamma}{32 \varrho_{i\bar{j}}} \sum_{l>1} G^{i\bar{j}} (\nabla_i g_{i\bar{j}} \nabla_i g_{i\bar{j}} + \nabla_i g_{i\bar{j}} \nabla_i g_{i\bar{j}}) - G^{i\bar{j}} H_{i\bar{j}} \]

\[ \leq - \nabla_i \nabla_1 \psi + (1 + \gamma) \varrho_{i\bar{j}} G^{i\bar{j}} \nabla_i \nabla_i \phi \]

\[ + C \varrho_{i\bar{j}} \sum G^{i\bar{j}} - G^{i\bar{j}} H_{i\bar{j}}. \tag{3.19} \]

By straightforward calculations (see, e.g., [16, 21]),

\[ G^{i\bar{j}} H_{i\bar{j}} \geq 2 G^{i\bar{j}} \Re \{ X_{i1, \xi} \nabla_{\alpha} \nabla_i \nabla_i u \} - 2 G^{i\bar{j}} \Re \{ X_{i1, \xi} \nabla_{\alpha} \nabla_i \nabla_i u \}
\]

\[ - C \varrho_{i\bar{j}}^2 \sum G^{i\bar{j}} - C \sum G^{i\bar{j}} |\nabla_i \nabla_k u|^2 \]

\[ - C \sum_k |\nabla_i \nabla_k u|^2 \sum G^{i\bar{j}} \]

\[ \geq 2 \Re \{ X_{i1, \xi} \nabla_{\alpha} \psi \} + 2 \varrho_{i\bar{j}} G^{i\bar{j}} \Re \{ X_{i1, \xi} \nabla_{\alpha} \phi \}
\]

\[ - C |A|^2 \sum G^{i\bar{j}}. \]
Henceforth for convenience, we denote
\[ |A_i|^2 = g_{ii}^2 + \sum_k |\nabla_i \nabla_k u|^2, \quad |A|^2 = \sum |A_i|^2. \]

Next,
\[
\nabla_\alpha \psi = \psi_\alpha + \psi_u \nabla_\alpha u + \psi_{\xi_\beta} \partial_\alpha \partial_\beta u + \psi_{\xi_\beta} \partial_\alpha \bar{\partial}_\beta u
\] (3.21)
and
\[
\nabla_1 \nabla_1 \psi \geq \psi_{\xi_\alpha} \nabla_\alpha \nabla_1 u + \psi_{\xi_\alpha} \nabla_\alpha \nabla_1 u - C|A_1|^2
\] (3.22)

Finally, from (3.19)-(3.22), we derive
\[
g_{11} G^{i\bar{j}} (\bar{\partial}_i \partial_\ell \phi - (1 + \gamma) \bar{\partial}_i \phi \partial_\ell \phi) \leq C (g_{11} |\nabla \phi| + |A|^2) \left(1 + \sum G^{i\bar{j}}\right). \] (3.23)

Let \( \phi = \log(\eta/h) \) where \( h = (1 - \gamma |\nabla u|^2) \), \( \gamma \) as before, and \( \eta \) is a smooth function to be chosen; we require \( \gamma \) small to satisfy \( 2\gamma |\nabla u|^2 \leq 1 \). By straightforward calculations,
\[
\partial_i |\nabla u|^2 = \nabla_k u \nabla_i \nabla_k u + \nabla_k u \nabla_i \nabla_k u
\] (3.24)
and
\[
\bar{\partial}_i \partial_\ell |\nabla u|^2 = \nabla_i \nabla_\ell u \nabla_k \nabla_i \nabla_k u + \nabla_i \nabla_\ell u \nabla_k \nabla_i \nabla_k u + 2\Re \{\nabla_\ell u \nabla_\ell \nabla_i \nabla_k u\}
\] (3.25)
\[
= \nabla_i \nabla_\ell u \nabla_k \nabla_i \nabla_k u + \nabla_i \nabla_\ell u \nabla_k \nabla_i \nabla_k u + 2\Re \{\nabla_\ell u \nabla_k \nabla_i \nabla_k u\}
\] (3.25)
\[
+ R_{ik\bar{l}} \nabla_i u \nabla_\ell u \nabla_k \nabla_k u - T^i_{ik} \nabla_\ell u \nabla_k \nabla_k u - T^i_{ik} \nabla_k u \nabla_\ell u \nabla_k \nabla_k u
\] (3.25)
\[
\geq (1 - \gamma) |A_i|^2 + 2\Re \{\nabla_\ell u \nabla_k \nabla_i \nabla_k u\} - C|\nabla u|^2.
\] (3.25)

It follows that
\[ G^{i\bar{j}} \bar{\partial}_i h \partial_\ell h \leq C \gamma^2 G^{i\bar{j}} |A_i|^2 \]
and
\[ -G^{i\bar{j}} \bar{\partial}_i h \partial_\ell h \geq \gamma (1 - \gamma) \sum G^{i\bar{j}} |A_i|^2 - C|A| \sum G^{i\bar{j}} - C|A|. \] (3.26)

Clearly |\nabla \phi| \leq |\nabla h|/h + |\nabla \eta|/\eta and
\[
G^{i\bar{j}} \bar{\partial}_i \phi \partial_\ell \phi \leq \frac{1 + \gamma}{h^2} G^{i\bar{j}} \bar{\partial}_i h \partial_\ell h + \frac{1 + \gamma}{\gamma \eta^2} G^{i\bar{j}} \bar{\partial}_i \eta \partial_\ell \eta. \] (3.27)
We see that
\[ G^{ii}(\bar{\partial}_i \bar{\partial}_i \varphi) - (1 + \gamma) \bar{\partial}_i \varphi \bar{\partial}_i \varphi \geq - \frac{\bar{\partial}_i \bar{\partial}_i \varphi}{h} + \frac{3\gamma \bar{\partial}_i \bar{\partial}_i \varphi}{h^2} - \frac{C}{\gamma} G^{ii} \bar{\partial}_i \eta \bar{\partial}_i \eta + G^{ii} \bar{\partial}_i \eta \bar{\partial}_i \eta \]
\[ \geq \gamma(1 - \gamma - C\gamma^2) \sum G^{ii} |A_i|^2 - \frac{C}{\gamma \eta^2} G^{ii} \bar{\partial}_i \eta \bar{\partial}_i \eta \]
\[ + \frac{1}{\eta} G^{ii} \bar{\partial}_i \eta \bar{\partial}_i \eta - C|A| \sum G^{ii} - C|A|. \]

We can now fix \( \gamma \) sufficiently small, further requiring that \( 2\gamma + C\gamma^2 \leq 1 \), to derive
\[ \gamma^2 g_{1\bar{1}} \sum G^{ii} |A_i|^2 + g_{1\bar{1}} G^{ii} \left( \frac{\bar{\partial}_i \partial_i \eta}{\eta} - \frac{C \bar{\partial}_i \eta \bar{\partial}_i \eta}{\eta^2} \right) \leq C|A|^2 \sum G^{ii}. \]  
(3.28)

From Lemma 2.2, we have the following key inequality for each \( i \geq 1 \),
\[ G^{ii} = \sum_{l=1}^{N} f_{\lambda_l} \frac{\partial \lambda_l}{\partial \lambda_i} = \sum_{i \in I} f_{\lambda_l} \geq c_1 \sum_{l=1}^{N} f_{\lambda_l}, \]  
(3.29)
where the sum \( \sum_{i \in I} f_{\lambda_l} \) is taken over all \( I \in \mathcal{I}_K \) with \( i \in I \), since \( \lambda(g) = (g_{1\bar{1}}, \ldots, g_{n\bar{n}}) \). Note that when \( g_{i\bar{j}} \) is diagonal, so is \( G^{i\bar{j}} \).

Consequently, by (3.28), we obtain
\[ \frac{c_1 \gamma^2}{2} g_{1\bar{1}} |A|^2 \sum f_{\lambda_l} + g_{1\bar{1}} G^{ii} \left( \frac{\bar{\partial}_i \partial_i \eta}{\eta} - \frac{C \bar{\partial}_i \eta \bar{\partial}_i \eta}{\eta^2} \right) \leq C|A|^2 \]  
(3.30)
provided that \( g_{1\bar{1}} \) is large enough.

By the concavity of \( f \) and Schwarz inequality, we derive
\[ \sqrt{g_{1\bar{1}}} \sum f_{\lambda_l} = \sqrt{g_{1\bar{1}}} \sum f_{\lambda_l} - \sum f_{\lambda_l} \Lambda_l(g) + \sum f_{\lambda_l} \Lambda_l(g) \]
\[ \geq f(\sqrt{g_{1\bar{1}}}) - f(\lambda(g)) - \epsilon \sum f_{\lambda_l} \Lambda_l^2 - \frac{1}{4\epsilon} \sum f_{\lambda_l} \]
\[ \geq \frac{c_0}{2} - \epsilon \sum f_{\lambda_l} \Lambda_l^2 - \frac{1}{4\epsilon} \sum f_{\lambda_l} \]
by assumption (1.10), provided that \( g_{1\bar{1}} \) is sufficiently large. So from (3.30) we obtain
\[ g_{1\bar{1}} |A|^2 \sum f_{\lambda_l} + \sqrt{g_{1\bar{1}}}|A|^2 + C g_{1\bar{1}} G^{ii} \left( \frac{\bar{\partial}_i \partial_i \eta}{\eta} - \frac{C \bar{\partial}_i \eta \bar{\partial}_i \eta}{\eta^2} \right) \leq C|A|^2. \]  
(3.31)

To derive the interior estimate, following [24], we take \( \eta \) to be a smooth function with compact support in \( B_r \subset M \) satisfying
\[ 0 \leq \eta \leq 1, \quad \eta |_{B_r} \equiv 1, \quad |\nabla \eta| \leq \frac{C}{r}, \quad |\nabla^2 \eta| \leq \frac{C}{r^2}, \]  
(3.32)
so that
\[ G^{ij} \partial_i \partial_j \eta - \frac{C}{\eta} G^{ij} \partial_i \eta \partial_j \eta \leq \frac{C}{r^2} \sum f_{\lambda_i}. \] (3.33)

We derive a bound \( \eta g_{11} \leq \frac{C}{r^2} \) at \( z_0 \) which gives (1.13).

Suppose that \((M^n, \omega)\) is a Hermitian manifold with smooth boundary \( \partial M \). Taking \( \eta = 1 \), we obtain the global estimate (1.21). The boundary estimate (1.22) may be derived as in [21] with some minor modifications; we shall omit the details here.

**Remark 3.1** When \( X \) and \( \psi \) are independent of \( u \) and \( \nabla u \), we obtain
\[
\sup_{B_r^2} |\bar{\partial} \partial u| \leq \frac{C}{r^2} \left\{ 1 + \sup_{B_r} |\partial u|^2 \right\}, \] (3.34)

**4 Gradient Estimates**

In this section, we derive gradient estimates. As usual, it requires suitable growth conditions of \( X \) and \( \psi \) on \( u \) and its gradient. Concerning \( X \), we assume the following sublinear growth assumption for \( D\zeta X \) when \( |\zeta| \) is sufficiently large,
\[
|D\zeta X(z, u, \zeta, \bar{\zeta})| \leq \rho_0 |\zeta|, \] (4.1)
and
\[
X \leq (\rho_0 |\zeta|^2 + \rho_1) \omega, \quad D_u X \leq (\rho_0 |\zeta|^2 + \rho_1) \omega, \] (4.2)
where \( \rho_1 = \rho_1(z, u) \) and \( \rho_0 = \rho_0(z, u, |\zeta|) \rightarrow 0^+ \) as \( |\zeta| \rightarrow \infty \); clearly we may assume \( \rho_0(z, u, t) \) to be increasing in \( t > 0 \). The first assumption in (4.2) will also be crucial to the existence, more precisely to the upper bound, of admissible solutions of the Dirichlet problem in Theorem 5.2 in Sect. 5. We impose a similar condition on \( \psi \) but, as in [21], associate it with the growth of \( f \):
\[
|D\zeta \psi(z, u, \zeta, \bar{\zeta})| \leq \rho_0 f(|\zeta|^2 1)/|\zeta|, \] (4.3)
and
\[
\psi \leq \rho_0 f(|\zeta|^2 1) + \rho_1(z, u), \quad -D_u \psi \leq \rho_0 f(|\zeta|^2 1) + \rho_1(z, u). \] (4.4)

**Theorem 4.1** Let \( u \in C^3(\bar{B}_r) \) be an admissible solution of equation (1.17). Under conditions (1.5), (1.6), (1.18), (1.11), (1.19), and (4.1)-(4.4),
\[
|\partial u|^2 \leq \frac{C}{r^2} \left\{ 1 + \sup_{B_r} u - u \right\} \text{ in } B_r^2 \] (4.5)
for some uniform constant \( C > 0 \).
Remark 4.2 We may replace (4.1) and (4.3) by the weaker linear-like growth conditions

\[ |D_\xi X(z, u, \xi, \bar{\xi})| \leq \varrho(z, u)|\xi|, \]  
\[ |D_\xi \psi(z, u, \xi, \bar{\xi})| \leq \varrho(z, u)f(|\xi|^2)/|\xi|, \]

where \( \varrho > 0 \). In this case we can derive the bound

\[ |\partial u|^2 \leq \frac{C}{r^2} \left( 1 + u - \inf_{B_r} u \right)^N \text{ in } B_{r} \]

for sufficiently large \( N > 0 \); see, e.g., [21].

Remark 4.3 When \( \omega \) is Kähler, (4.6) and (4.7) can be replaced by the assumptions

\[ D_{p_i p_j}^2 X(z, u, p)\xi_i \xi_j \leq \varrho(z, u)|\xi|^2, \quad \text{for } \xi \in T_z M, \]

that is, \( X \) is semi-concave in \( \nabla u \), the real gradient of \( u \), and

\[ D_{p_i p_j}^2 \psi(z, u, p)\xi_i \xi_j \geq -\varrho(z, u)f(|p|^2)|\xi|^2/|p|^2, \quad \xi \in T_z M. \]

Proof of Theorem 4.1 Consider

\[ \max_M e^\phi |\nabla u|^2 \]

where \( \phi \) is a function to be chosen. Suppose it is attained at an interior point \( z_0 \in M \). We choose local coordinates \( (z_1, \ldots, z_n) \) such that \( g_{i\bar{j}} = \delta_{ij}, \) \( T_{ij} = 2\Gamma^k_{ij} \) and \( g_{ij} \) are diagonal at \( z_0 \). The function \( \log |\nabla u|^2 + \phi \) achieves a maximum at \( z_0 \) where we assume \( |\nabla u| \geq 1 \). Therefore, at \( z_0 \),

\[ \partial_i |\nabla u|^2 + |\nabla u|^2 \partial_i \phi = 0, \quad \bar{\partial}_i |\nabla u|^2 + |\nabla u|^2 \bar{\partial}_i \phi = 0 \]

and

\[ G^{i\bar{i}} \bar{\partial}_i \partial_i |\nabla u|^2 - G^{i\bar{i}} \bar{\partial}_i |\nabla u|^2 \partial_i |\nabla u|^2 |\nabla u|^4 + G^{i\bar{i}} \bar{\partial}_i \partial_i \phi \leq 0. \]

By (3.24), (3.25) and Schwarz inequality,

\[ \bar{\partial}_i |\nabla u|^2 \partial_i |\nabla u|^2 \leq 2|\nabla u|^2 |Q_i|^2 \]

and

\[ \bar{\partial}_i \partial_i |\nabla u|^2 \geq (1 - \gamma)|Q_i|^2 + 2\Re \{ \nabla_k u \nabla_{\bar{k}} \nabla_i \nabla_j u \} - C|\nabla u|^2 \]
where $0 < \gamma < 1/6$ and

$$|Q_i|^2 = \nabla_i \nabla_k u \nabla_i \nabla_k u + \nabla_i \nabla_k u \nabla_i \nabla_k u = \sum_k (|\nabla_i \nabla_k u|^2 + |\nabla_i \nabla_k u|^2).$$

It follows from (3.3) that

$$G^{ij} \nabla_k \nabla_i \nabla_j u = G^{ij} (\nabla_k g_{ij} - \nabla_k X_{ij}) = \nabla_k \psi - G^{ij} \nabla_k X_{ij}. \quad (4.15)$$

So

$$G^{ij} \bar{\partial}_i \bar{\partial}_j |\nabla u|^2 \geq G^{ij} (1 - \gamma) |Q_i|^2 - C |\nabla u|^2 \sum G^{ii} + R \quad (4.16)$$

where

$$R = 2 \Re \{ (\nabla_k \psi - G^{ij} \nabla_k X_{ij}) \nabla_k u \}.$$

We derive

$$G^{ij} \bar{\partial}_i \bar{\partial}_j \phi \frac{1 + \gamma}{2} G^{ij} \bar{\partial}_i \phi \bar{\partial}_i \phi \leq - \frac{R}{|\nabla u|^2} + C \sum G^{ii}. \quad (4.17)$$

Let $\phi = \log(\eta/h)$ where $h = 1 + \sup_M u - u$. So

$$G^{ij} \bar{\partial}_i \bar{\partial}_j \phi = \frac{1}{h} G^{ij} \bar{\partial}_i \bar{\partial}_j u + \frac{1}{h^2} G^{ij} \bar{\partial}_i \bar{\partial}_j u \bar{\partial}_i u + \frac{1}{\eta} G^{ij} \bar{\partial}_i \partial_i \eta - \frac{1}{\eta^2} G^{ij} \bar{\partial}_i \eta \bar{\partial}_i \eta. \quad (4.18)$$

As before, $|\nabla \phi| \leq |\nabla u|/h + |\nabla \eta|/\eta$ and

$$G^{ij} \bar{\partial}_i \phi \bar{\partial}_i \phi \leq \frac{1 + \gamma}{h^2} G^{ij} \bar{\partial}_i \bar{\partial}_j u \bar{\partial}_i u + \frac{1 + \gamma}{\eta^2} G^{ij} \bar{\partial}_i \eta \bar{\partial}_i \eta.$$

From (3.29) it follows that

$$\frac{1}{h^2} G^{ij} \bar{\partial}_i \bar{\partial}_j u \bar{\partial}_i u - \frac{1 + \gamma}{2} G^{ij} \bar{\partial}_i \phi \bar{\partial}_i \phi \geq \frac{1 - 3\gamma}{2h^2} G^{ij} \bar{\partial}_i \bar{\partial}_j u \bar{\partial}_i u - \frac{C}{\eta^2} G^{ij} \bar{\partial}_i \eta \bar{\partial}_i \eta \geq \frac{c_1 |\nabla u|^2}{4h^2} \sum G^{ii} - \frac{C}{\eta^2} G^{ii} \bar{\partial}_i \eta \bar{\partial}_i \eta. \quad (4.18)$$

By the concavity of $f$ and assumption (1.11),

$$|\nabla u|^2 \sum G^{ii} \geq f(|\nabla u|^2) - f(\Lambda) + G^{ii} g_{ii} \geq f(|\nabla u|^2) - \psi - C \sum G^{ii},$$

Similarly,

$$G^{ij} \bar{\partial}_i \bar{\partial}_j \phi = G^{ij} \bar{\partial}_i \phi - G^{ij} \bar{\partial}_j u \bar{\partial}_j u \geq -G^{ij} X_{ij} - C \sum G^{ii}. \quad (4.19)$$
Combining the above inequalities, and taking $\eta$ as in the previous section, we derive

\[
\frac{c_1}{8h^2} \| \nabla u \|^2 + \frac{c_1}{8h^2} f \| \nabla u \|^2 \cdot 1 \\
\leq -\frac{1}{h} G^{ii} \delta_i u - \frac{1}{h} \eta G^{ii} \delta_i \eta + \frac{C}{\eta^2} G^{ii} \delta_i \eta \delta_i \eta \\
+ \frac{c_1 \psi}{8h^2} + C \sum_i G^{ii} - \frac{R}{|\nabla u|^2} \\
\leq \frac{1}{h} G^{ii} X_{ii} + \frac{c_1 \psi}{8h^2} - \frac{R}{|\nabla u|^2} + C \sum_i G^{ii}. \tag{4.19}
\]

Consequently, we obtain a bound $\eta \| \nabla u \|^2 \leq C/r^2$ if $\psi$ and $X$ are independent of $u$ and $\partial u$. In the general case, by (3.21) and (4.11),

\[
\operatorname{Re} \{ \nabla_k \psi \nabla k u \} = \psi u \| \nabla u \|^2 + \operatorname{Re} \{ \psi \alpha \partial_\alpha \| \nabla u \|^2 \} \\
+ \operatorname{Re} \{ \psi_k \nabla k u + \psi \alpha \Gamma_k^l \nabla l u \nabla k u \} \\
= \| \nabla u \|^2 (\psi u - \operatorname{Re} \{ \psi \alpha \partial_\alpha \psi \}) \\
+ \operatorname{Re} \{ \psi_k \nabla k u + \psi \alpha \Gamma_k^l \nabla l u \nabla k u \} \tag{4.20}
\]

and similarly,

\[
G^{ii} \operatorname{Re} \{ \nabla_k \nabla k X_{ii} \} = \| \nabla u \|^2 B - \frac{\| \nabla u \|^2}{\eta} G^{ii} \operatorname{Re} \{ X_{ii, \alpha} \partial_\alpha \eta \} \\
\leq \| \nabla u \|^2 B + \frac{C \| \nabla u \|^2 |D \xi \psi|}{\sqrt{\eta}} \sum_i G^{ii} \tag{4.21}
\]

where

\[
A = \psi u - \frac{1}{h} \operatorname{Re} \{ \psi \alpha \partial_\alpha u \} + \frac{1}{\| \nabla u \|^2} \operatorname{Re} \{ \psi_k \nabla k u + \psi \alpha \Gamma_k^l \nabla l u \nabla k u \}, \\
B = G^{ii} X_{ii, u} - \frac{1}{h} G^{ii} \operatorname{Re} \{ X_{ii, \alpha} \partial_\alpha u \} + \frac{1}{\| \nabla u \|^2} G^{ii} \operatorname{Re} \{ (X_{ik} + X_{ii, \alpha} \Gamma_k^l \nabla l u) \nabla k u \}.
\]

By assumptions (4.1) and (4.3),

\[
\frac{1}{h} G^{ii} X_{ii} + \frac{c_1 \psi}{8h^2} - \frac{R}{\| \nabla u \|^2} \leq C H \sum_i G^{ii} + CE \\
+ \frac{C_\Omega \| \nabla u \|}{\sqrt{\eta}} \sum_i G^{ii} + \frac{C_\Omega f \| \nabla u \|^2 \cdot 1}{|\nabla u| \sqrt{\eta}} \quad \text{(4.22)}
\]
where
\[ E = |\nabla \zeta \psi||\nabla u|^{-1} + (\psi_u)^+ + \psi^+ + |D \zeta \psi||\nabla u| \leq \varrho_0 f(|\nabla u|^2 \mathbf{1}) \]
by (4.3) and (4.4), and
\[ H = |\nabla \zeta X||\nabla u|^{-1} + \text{tr} X^+ + \text{tr}(D_u X)^+ + |D \zeta X||\nabla u| \leq \varrho_0|\nabla u|^2 \]
by (4.1) and (4.2). Plugging these back to (4.22), we obtain a bound \( \eta|\nabla u|^2 \leq C/r^2 \) at \( z_0 \) from which (4.5) follows.

\[ \square \]

**Remark 4.4** Assume both \( X \) and \( \psi \) are independent of \( u \). Taking \( \eta = 1 \) we obtain
\[ |\nabla u|^2 \leq C \left( 1 + \sup_M u - u \right) \]
on \( M \)
for any admissible solution \( u \in C^3(M) \) of equation (1.17). It follows that
\[ \left| \left( 1 + \sup_M u - u(x) \right)^{1/2} - \left( 1 + \sup_M u - u(y) \right)^{1/2} \right| \leq C \text{dist}(x, y), \forall x, y \in M \]
where \( \text{dist}(x, y) \) denotes the distance between \( x \) and \( y \) in \( (M, \omega) \). In particular,
\[ \sup_M u - \inf_M u \leq Cd^2 \quad (4.23) \]
where \( d \) is the diameter of \( M \).

**Remark 4.5** Assume \( u > 0 \) on \( B_r \) and take \( h = u^2 \) in the previous proof. We can derive a bound \( \eta|\nabla u|^2 \leq C/r^2 \) at \( z_0 \) if \( \psi \) and \( X \) are independent of \( u \) and \( \partial u \). This gives
\[ \frac{|\nabla u|^2}{u^2} \leq C \text{ in } B_{\zeta}. \]
Consequently, we obtain the Harnack inequality
\[ \sup_{B_{\zeta}} u \leq C \inf_{B_{\zeta}} u. \]

**Remark 4.6** For a Hermitian manifold \((M^n, \omega)\) with boundary \( \partial M \), taking \( \eta = 1 \), we derive the global gradient estimate
\[ \max_M |\partial u| \leq C \max_{\partial M} |\partial u| + C. \]
5 Existence

In this section, we outline proofs of Theorems 1.6 and 1.9. With the aid of the a priori estimates in the specific forms derived in the previous sections, we can extend these theorems to cover more general cases where \( X \) and \( \psi \) are allowed to depend on \( u \) and \( \nabla u \).

The following result extends Theorem 1.9.

**Theorem 5.1** Let \((M^n, \omega)\) be a compact Hermitian manifold. Assume that (1.5), (1.18), (1.11), (1.19), (1.29) and (4.1)–(4.4) hold. In addition, assume that \( X = X(z, \nabla u) \) and \( \psi = \psi(z, \nabla u) \) are both independent of \( u \), and \( X(z, 0) > 0 \) on \( M \). Then there exists a unique constant \( b \) such that the equation

\[
f(\Lambda(\sqrt{-1} \bar{\partial} \partial u + X[u])) = e^b \psi[u]
\]

(5.1)

has a unique admissible solution \( u \in C^\infty(M) \) up to a constant.

**Proof** To begin with we assume \( X[u] = X(z, u, \nabla u) \) and \( \psi[u] = \psi(z, u, \nabla u) \) satisfying

\[- D_u X, \ D_u \psi \geq 0.
\]

(5.2)

First we choose \( A > 0 \) sufficiently large such that \( H \equiv f(\Lambda(A\omega)) > 0 \). For \( t \in [0, 1] \), consider the equation

\[
f(\Lambda(\sqrt{-1} \bar{\partial} \partial u + X[u] + tA\omega)) = e^{tu} \psi'[u]
\]

(5.3)

where \( \psi'[u] = tH + (1 - t)\psi[u] \). Note that for \( t > 0 \), by (5.2)

\[D_u(e^{tu} \psi'[u]) \geq Ht^2e^{tu} > 0.
\]

So the solution of equation (5.3) is unique and its linearized operator is nonsingular for \( t > 0 \). When \( t = 1 \) the solution is \( u^1 = 0 \).

Suppose \( u \in C^\infty(M) \) is the admissible solution of equation (5.3) for fixed \( t > 0 \), and assume \( \sup_M u \geq 0 \). At a point where \( u \) attains its maximum value, by (5.2), we have

\[
e^{tu} \psi^1(z, 0, 0) \leq e^{tu} \psi^1(z, u, 0) \leq f(\Lambda(X(z, u, 0) + tA\omega)) \leq f(\Lambda(X(z, 0, 0) + tA\omega).
\]

Therefore,

\[
t \sup_M u \leq \sup_{z \in M} \frac{f(\Lambda(X(z, 0, 0) + A\omega))}{\psi'[z, 0, 0]} \leq C.
\]

(5.4)
Similarly, suppose \( \inf_M u \leq 0 \). Then
\[
\inf_M u \geq \inf_{z \in M} \log \frac{f(\Lambda(X(z, 0, 0)))}{\psi^t(z, 0, 0)} \geq -C. \tag{5.5}
\]

Consequently, we may apply the continuity method to obtain a unique admissible solution \( u^t \in C^\infty(M) \) of equation (5.3) for all \( t \in (0, 1] \) with the bound
\[
-C \leq t \inf_M u^t \leq t \sup_M u^t \leq C, \quad 0 < t \leq 1 \tag{5.6}
\]
where \( C \) is independent of \( t \).

By (5.6) we can find a sequence \( t_k \to 0 \) such that both of the following limits exist
\[
\lim_{k \to \infty} t_k \inf_M u^{t_k} = a, \quad \lim_{k \to \infty} t_k \sup_M u^{t_k} = b.
\]

Suppose now that \( X \) and \( \psi \) are both independent of \( u \). By (4.23) we see that \( a = b \) and therefore
\[
\lim_{k \to \infty} t_k u^{t_k} = b \quad \text{on } M.
\]

Moreover, by (4.23) and the estimates we have established, there exists a subsequence of
\[
\left\{ u^{t_k} - \sup_M u^{t_k} \right\}
\]
converging in \( C^{2,\alpha}(M) \) to a smooth admissible solution \( u \) of equation (5.1). Clearly,
\[
\inf_M u \geq -Cd^2, \quad \sup_M u = 0
\]
where \( d \) is the diameter of \( M \).

We next consider the Dirichlet problem
\[
\left\{ \begin{array}{l}
f(\Lambda(\sqrt{-1} \partial \bar{\partial} u + X[u])) = \psi[u] \text{ in } \tilde{M}, \\
u = \varphi \text{ on } \partial M.
\end{array} \right. \tag{5.7}
\]
Here \((M^n, \omega)\) is assumed as in Theorem 1.6 to be a Hermitian manifold with smooth boundary \( \partial M \) and compact closure \( \tilde{M} = M \cup \partial M \), and \( \varphi \in C^0(\partial M) \).

**Theorem 5.2** In addition to (1.5), (1.6), (1.18), (1.11), (1.19), (4.1)–(4.4) and (5.2), assume that there exists an admissible subsolution \( u \in C^0(M) \) satisfying
\[
\left\{ \begin{array}{l}
f(\Lambda(\sqrt{-1} \partial \bar{\partial} u + X[u])) \geq \psi[u] \text{ in } \tilde{M}, \\
u = \varphi \text{ on } \partial M
\end{array} \right. \tag{5.8}
\]
in the viscosity sense. The Dirichlet problem (5.7) admits a unique admissible solution $u \in C^\infty(M) \cap C^0(M)$. Moreover, $u \in C^\infty(M)$ if $\phi \in C^\infty(\partial M)$.

**Proof** We assume $u \in C^\infty(\bar{M})$ and $\phi \in C^\infty(\partial M)$; the general case follows from approximation. Let $u \in C^4(M) \cap C^0(\bar{M})$ be an admissible solution of problem (5.7). From the comparison principle and (5.2), we see that 

$$\frac{\partial u}{\partial v} \leq \frac{\partial u}{\partial v} \text{ on } \partial M \quad (5.9)$$

where $v$ denotes the exterior unit normal to $\partial M$. By assumptions (4.1) and (5.2),

$$\text{tr}X[u] \leq \text{tr}X(z, \inf_M u, \partial u, \bar{\partial} u) \leq \varrho_1(z, \inf_M u) + \varrho_0(z, \inf_M u, |\partial u|)|\partial u|^2.$$

Since $\Delta u + \text{tr}X[u] \geq 0$, we have

$$\Delta u + \text{tr}X(z, \inf_M u, \partial u, \bar{\partial} u) \geq 0. \quad (5.10)$$

It follows from the maximum principle that $u \leq h$ on $\bar{M}$ where $h \in C^\infty(\bar{M})$ is the unique solution of

$$\Delta h + \text{tr}X(z, \inf_M u, \partial h, \bar{\partial} h) = 0 \text{ in } M, \quad h = \phi \text{ on } \partial M. \quad (5.11)$$

By the assumption $\varrho_0 = \varrho_0(z, u, t) \to 0^+$ as $t \to \infty$ in (4.1) we see that problem (5.11) is solvable. Combining with the gradient estimate in Sect. 4 (see Remark 4.6), we have established the global $C^1$ bound

$$\sup_M |u| + \sup_M |\nabla u| \leq C. \quad (5.12)$$

By Theorem 1.5 and Evans–Krylov Theorem, we obtain the $C^{2,\alpha}$ estimate

$$|u|_{C^{2,\alpha}(\bar{M})} \leq C. \quad (5.13)$$

The existence and smoothness of a unique admissible solution now follow from the standard continuity method and the classical Schauder theory for linear uniformly elliptic equations.

**Remark 5.3** The assumption $\varrho_0 = \varrho_0(z, u, t) \to 0^+$ as $t \to \infty$ in (4.1) is critical to the existence of solution of problem (5.11). Consider the problem in a domain $\Omega \subset \mathbb{R}^n$

$$\Delta h + |\nabla h|^2 + b = 0 \text{ in } \Omega, \quad h = 0 \text{ on } \partial \Omega. \quad (5.14)$$
Clearly $h$ satisfies $\Delta e^h + be^h = 0$. Therefore, problem (5.14) has no solution if $b$ is the first eigenvalue of $\Omega$. Indeed, let $\phi_1$ be the first eigenfunction

$$\Delta \phi_1 + b\phi_1 = 0 \text{ in } \Omega, \quad \phi_1 = 0 \text{ on } \partial \Omega,$$

(5.15)

normalized so that

$$\sup_{\Omega} \phi_1 = 1.$$  

Note that $e^h > 1 \geq \phi_1$ in $\Omega$ since $\Delta e^h < 0$ in $\Omega$. We can find $t > 1$ such that

$$0 < \inf_{\Omega} (e^h - t\phi_1) < 1.$$  

Consequently, in the interior point $x_0$ where $e^h - t\phi_1$ attains the minimal value,

$$\Delta (e^h - t\phi_1) + b(e^h - t\phi_1) \geq b(e^h - t\phi_1) > 0$$

which is a contradiction.

When $u$ is a radial function, in spherical coordinates equation (5.14) reduces to the simple Riccati equation for $y = u'$

$$y' + \frac{n-1}{r} y + y^2 + b = 0.$$  

In the special case $n = 1$ and $b = 1$, the solution is $u(x) = -\log \cos x$ which is only defined for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

References

1. Gauduchon, P.: La 1-forme de torsion d’une variété hermitienne compacte. Math. Ann. 267, 495–518 (1984)
2. Székelyhidi, G., Tosatti, V., Weinkove, B.: Gauduchon metrics with prescribed volume form. Acta Math. 219, 181–211 (2017)
3. Fu, J.-X., Wang, Z.-Z., Wu, D.-M.: Form-type Calabi–Yau equations. Math. Res. Lett. 17, 887–903 (2010)
4. Fu, J.-X., Wang, Z.-Z., Wu, D.-M.: Form-type equations on Kähler manifolds of nonnegative orthogonal bisectional curvature. Calc. Var. PDE 52, 327–344 (2015)
5. Harvey, R., Lawson, H.B., Jr.: Plurisubharmonicity in a General Geometric Context, Geometry and Analysis No 1, Advanced Lectures in Mathematics, vol. 17, pp. 363–402. International Press, Somerville (2011)
6. Harvey, R., Lawson, H.B., Jr.: Geometric plurisubharmonicity and convexity: an introduction. Adv. Math. 230, 2428–2456 (2012)
7. Harvey, R., Lawson, H.B., Jr.: p-Convexity, p-plurisubharmonicity and the Levi problem. Indiana Univ. Math. J. 62, 149–169 (2013)
8. Caffarelli, L.A., Nirenberg, L., Spruck, J.: The Dirichlet problem for nonlinear second-order elliptic equations III: functions of eigenvalues of the Hessians. Acta Math. 155, 261–301 (1985)
9. Fu, J.-X., Yau, S.-T.: A Monge–Ampère-type equation motivated by string theory. Commun. Anal. Geom. 15, 29–75 (2007)
10. Fu, J.-X., Yau, S.-T.: The theory of superstring with flux on non-Kähler manifolds and the complex Monge–Ampère equation. J. Differ. Geom. 78, 369–428 (2008)
11. Phong, D.H., Picard, S., Zhang, X.-W.: On estimates for the Fu-Yau generalization of a Strominger system. J. Reine Angew. Math. 1, 1–32 (2016)
12. Phong, D.H., Picard, S., Zhang, X.-W.: The Fu-Yau equation with negative slope parameter. Invent. Math. 209, 541–576 (2017)
13. Phong, D.H., Picard, S., Zhang, X.-W.: Fu-Yau Hessian equations. J. Differ. Geom. 118, 147–187 (2021)
14. Chu, J.-C., Huang, L.-D., Zhu, X.-H.: The Fu-Yau equation in higher dimensions. Peking Math. J. 2, 71–97 (2019)
15. Guan, B.: Second order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds. Duke Math. J. 163, 1491–1524 (2014)
16. Guan, B., Nie, X.-L.: Second order estimates for fully nonlinear elliptic equations with gradient terms on Hermitian manifolds, arXiv:2108.03308
17. Lin, M., Trudinger, N.: On some inequalities for elementary symmetric functions. Bull. Austral. Math. Soc. 50, 317–326 (1994)
18. Székelyhidi, G.: Fully non-linear elliptic equations on compact Hermitian manifolds. J. Differ. Geom. 109, 337–378 (2018)
19. Guan, B.: The Dirichlet problem for fully nonlinear elliptic equations on Riemannian manifolds, arXiv:1403.2133v2
20. Feng, K., Ge, H.-B., Zheng, T.: The Dirichlet problem of fully nonlinear elliptic equations on Hermitian manifolds. arXiv:1905.02412
21. Guan, B., Qiu, C.-H., Yuan, R.-R.: Fully nonlinear elliptic equations for conformal deformations of Chern–Ricci forms. Adv. Math. 343, 538–566 (2019)
22. Tosatti, V., Weinkove, B.: The Monge–Ampère equation for (n-1)-plurisubharmonic functions on a compact Kähler manifold. J. Am. Math. Soc. 30, 311–346 (2017)
23. Streets, J., Tian, G.: Hermitian curvature flow. J. Eur. Math. Soc. 13, 601–634 (2011)
24. Guan, P.-F., Wang, G.-F.: Local estimates for a class of fully nonlinear equations arising from conformal geometry. Int. Math. Res. Not. 26, 1413–1432 (2003)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.