Research on solving strategy of structural reliability index based on probabilistic interval hybrid model

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Abstract. In this paper, a hybrid model is established in which the randomness of probabilistic parameters coexists with the uncertainty of interval parameters. Based on the reliability theory, the evaluation criteria of structural reliability under the hybrid model were proposed, and the two-layer cycle method and direct iterative algorithm were derived to calculate the structural reliability index. The correctness and effectiveness of the model and algorithm are verified by examples of a cantilever beam and a three-story frame structure. The hybrid model combines the probabilistic model with the interval model effectively and has practical significance for the engineering application of reliability theory.

1. Introduction

In practical engineering, the following situations are often encountered: some uncertain parameters can be described by probability model due to sufficient information, while other uncertain parameters can only be described by interval model due to lack of sufficient sample data support. Therefore, it is of great engineering significance to study the hybrid model with probabilistic random parameters and interval parameters coexisting.

In recent years, Ben-Haim and Yakov[1] preliminarily studied the concept of structural non-probabilistic reliability based on the convex model, and proposed that if the system can tolerate large uncertainties without failure, then the system is reliable; otherwise, if the system can only tolerate small uncertainties, then the system is unreliable. Y S Cheng and G W Zeng[2] used parameters to characterize the measurement of structural non-probabilistic reliability under the interval model, and then proposed an optimization model of structural non-probabilistic reliability under the condition of structural quality limitation. Based on interval analysis, S X Guo and Z Z Lv[3] proposed to use non-probabilistic reliability metrics to measure the reliability of the system when uncertain parameters are interval variables. Starks et al. [4], Qiu et al. [5], S X Guo and Z Z Lv[6] studied the structural reliability evaluation problem with the coexistence of random parameters and interval parameters by using the improved stochastic algorithm, interval analysis method and two-level functional function method. Du et al. [7] proposed a reliability optimization method dealing with the mixed case of interval and probability, and designed the structure by looking for the most unfavorable combination of interval variables.

This paper established a probability-interval hybrid model, taking into account both the randomness of probabilistic parameters and the uncertainty of interval parameters. Based on the reliability theory, the evaluation criteria of structural reliability under the hybrid model are proposed. The two-layer cycle method and direct iteration algorithm were derived to calculate the structural reliability index, and the advantages and disadvantages of the two methods were compared. Finally, numerical examples are given to verify the correctness and effectiveness of the proposed model and algorithm.
2. Probabilistic - interval hybrid model description

In practical engineering, some uncertain variables have sufficient sample information to obtain their random distribution characteristics. Other uncertain quantities are unknown but bounded variables which are independent of each other, these are interval variables. At this point, we use probability model to describe random variables \( x = \{x_1, x_2, L, x_m\}^T \). The interval model is used to describe the interval variables \( y = \{y_1, y_2, L, y_n\}^T \).

\[
x \in \{x: p_j(x), \ j=1,2,L,m\}
\]

\[
y \in \{y: y_L \leq y \leq y_U, \ i=1,2,L,n\}
\]

In the formula, \( p_j(x) \) is the probability density function of random variables.

The random variables were converted to approximate standard normal distribution variables \( u \) by numerical methods, and the interval variables \( y \) were converted to standardized variables \( q \) by the following standardized methods

\[
q_i = \frac{y_i - \bar{y}_i}{\Delta y_i}
\]

In the formula, \( \bar{y}_i \) is the mean value, and \( \Delta y_i \) is the variable range length. After standardization conversion, the uncertain variables \( u \) and \( q \) are described as

\[
u \in \{u: u \sim N(0,1), \ j=1,2,L,m\}
\]

\[
q \in \{q: ||q|| \leq 1, \ i=1,2,L,n\}
\]

The original limit state function \( G(x,y) \) is mapped to its standardized form \( g(u,q) \).

3. Structural reliability evaluation under hybrid model

In the case of mixed uncertainty, there is a family of limit state surfaces in the standard space in the limit state \( g(u,q) = 0 \). The whole space is divided into three regions: the safe region \( \Omega_s = \{u|g(u,q) > 0\} \), the critical boundary region \( \Omega_c = \{u|g(u,q) = 0\} \) and the failure region \( \Omega_f = \{u|g(u,q) < 0\} \), the interface between the safe region and the critical boundary region is called the most probable surface. Figure 1 shows the two-dimensional space situation of \( u_1 \) and \( u_2 \).

![Figure 1. mixed reliability index in U-space](attachment:image.png)

The purpose of reliability assessment is to reasonably quantify the safety degree of the structure. In the mixed case, the distribution characteristics of the interval uncertainty variables are unknown, and
any improper distribution assumption may lead to dangerous assessment. Therefore, the reliability of the structure under the probabilistic and interval hybrid model is defined as: the probability that the structure can at least complete the predetermined function of the structure within a specified period of time for any possible realization of the interval uncertain variables in the interval set. As an analogy of the probabilistic reliability index, the reliability index of the mixed model can be further defined as the following constraint minimization problem:

\[
\beta_m = \text{sgn}\left( g(0,q^*)^T \left[ u^{MPP} \right] \right) \quad (6)
\]

\( u^{MPP} \) is the most probable point of the surface in the limit state (MPP), \( q^* \) is called the worst case point (WCP), and they are the solution of the following optimization problem respectively:

\[
\begin{align*}
\min_{u} & \quad u^T u \\
\text{s.t.} & \quad g(u,q) = 0 \\
& \quad |q| \leq 1 \quad (i = 1, 2, \ldots, n)
\end{align*}
\]  

(7)

In the formula

\[
\begin{align*}
g(u,q^*) &= \min_{q} g(u,q) \\
\text{s.t.} & \quad |q| \leq 1 \quad (i = 1, 2, \ldots, n)
\end{align*}
\]  

(8)

4. Solving strategy of reliability index under mixed model

4.1. Mathematical programming method (MPM)

The above optimization problem is a nested optimization problem. Equation (7) seeks MPP of random variables, while equation (8) seeks WCP of interval uncertain variables. Combining the above two expressions, it is equivalent to the following single-layer cycle minimization problem:

\[
\begin{align*}
\min_{u,q} & \quad u^T u \\
\text{s.t.} & \quad g(u,q) = 0 \\
& \quad |q| \leq 1 \quad (i = 1, 2, \ldots, n)
\end{align*}
\]  

(9)

In the above formula, both random variables \( u \) and interval uncertain variables \( q \) are used as design variables of the optimization problem, so the optimization problem can be solved by mathematical programming algorithm based on gradient. Here, the sequential quadratic programming algorithm is adopted\(^8\).

4.2. Direct iteration method (DIM)

For nested optimization equation (7) and (8), the inner and outer layer optimization problems can be solved alternatively according to the idea of two-layer cycle method.

Assume that iteration in step \( k \) obtains \( u^{(k)} \) and \( q^{(k)} \). In practical engineering, due to the relatively small variation range of the interval uncertainty parameters, it can be considered that the limit state function changes monotonically within its interval boundary range, and the constraints in equation (8) will all reach the tight constraints. The first-order Taylor series expansion of the limit state function is carried out at the place \( q^{(k)} \), and the inner optimization (8) can be rewritten as

\[
\begin{align*}
\min_{q} & \quad g\left(u^{(k)},q^{(k)}\right) + \sum_{i=1}^{n} \left( G_{q_i}^{(k)} \right)^T \left( q_i^{(k)} - q_i^{(k)} \right) \\
\text{s.t.} & \quad |q| = 1 \quad (i = 1, 2, \ldots, n)
\end{align*}
\]  

(10)

In the formula, \( G_{q_i}^{(k)} = \frac{\partial g\left(u,q\right)}{\partial q_i} \bigg|_{\left(u^{(k)},q^{(k)}\right)} \)

Using the KKT optimality condition, we can get
\[
\begin{align*}
\left\{ \begin{array}{l}
G_{q_i}^{(i)} + \text{sgn} \left( q_i^* \right) \lambda_i = 0 \\
\left| q_i^* \right| = 1
\end{array} \right. \\
\left( i = 1, 2, L, n \right) 
\end{align*}
\]  \quad (11)

\( \lambda > 0 \) is the Lagrangian multiplier. Solve the above equation
\[
\left\{ \begin{array}{l}
\lambda_i = \left| G_{q_i}^{(i)} \right| \\
q_i^* = -\text{sgn} \left( G_{q_i}^{(i)} \right) \\
\end{array} \right. \\
\left( i = 1, 2, L, n \right) 
\]  \quad (12)

Therefore, the update iteration formula of WCP is
\[
q_{i+1}^{(k)} = \left[ (q_1^{(k+1)})^T, (q_2^{(k+1)})^T, \ldots, (q_n^{(k+1)})^T \right]^T \\
= \left[ \text{sgn} \left( G_{q_1}^{(k)} \right), \text{sgn} \left( G_{q_2}^{(k)} \right), \ldots, \text{sgn} \left( G_{q_n}^{(k)} \right) \right]^T 
\]  \quad (13)

Then, using HL-RF iterative algorithm, MPP in the standard U-space can be updated as
\[
u_{i+1}^{(k)} = \frac{G_u^{(k)}^T \nu_i^{(k)} - g(\nu_i^{(k)}, q_i^{(k+1)})}{G_u^{(k)}^T G_u^{(k)}} 
\]  \quad (14)

In the formula, \( G_u^{(k)} = \frac{\partial g(\nu, q)}{\partial \nu} \), The iterative formula updates the uncertain variables in turn until the convergence criterion is satisfied, and then the mixed reliability index \( \beta_n \) is calculated.

5. Examples

5.1. Cantilever beam

As shown in figure 2, the length of the cantilever beam is \( L = 10 \) m, and the material density is \( \rho = 2.551 \times 10^3 \text{ kg/m}^3 \), and its free end is subjected to the action of downward load \( F \). The elastic modulus of the material is \( E \), and the load \( F \) and the elastic modulus of the material \( E \) are assumed to be random variables. The beam diameter \( d \) is interval uncertain variable, and the specific uncertain properties are shown in Table 1. When the vertical displacement of the free end of the cantilever beam is greater than \( 1.5 \times 10^{-3} \) m, the structure is considered to be invalid, and the limit state equation is defined as
\[
g(d) = 1.5 \times 10^{-3} - \frac{64FL}{3\pi Ed^4} 
\]  \quad (15)

\[\text{Figure 2. Structure of cantilever beam}\]
Table 1. Uncertainty properties for the cantilever beam

| Uncertain variables | Type       | Mean value | COV  | Interval range |
|---------------------|------------|------------|------|----------------|
| $F \ (kN)$          | Normal     | 100        | 0.05 | --             |
| $E \ (GPa)$         | Normal     | 30         | 0.05 | --             |
| $d \ (m)$           | Interval   | 2.5        | --   | $d \in [2.4, 2.6]$ |

5.2. Three-storey frame structure

Figure 3 shows a three-storey frame structure with floor height of 4 meters. The structural columns and beams are made of reinforced concrete. The beam size is $300 \times 600 \text{mm}$. As shown in figure 4, the roof of each layer on the left side of the frame structure is subjected to rightward load $F$. It is assumed that the load $F$, material elastic modulus $E$ and material density $\rho$ are random variables, the column section is cube and the size is interval uncertain variable. Uncertainty properties are shown in Table 2.

When the horizontal displacement of the top layer of the frame structure is greater than $1.5 \times 10^5 \text{m}$, the structure is considered to be invalid. The limit state equation is defined as

$$g = 1.5 \times 10^3 - \delta \geq 0$$  \hspace{1cm} (16)

Table 2. Uncertainty properties for the Three-storey frame structure

| Uncertain variables | Type       | Mean value | COV  | Interval range |
|---------------------|------------|------------|------|----------------|
| $F \ (kN)$          | Normal     | 10000      | 0.05 | --             |
| $E \ (GPa)$         | Normal     | 30         | 0.05 | --             |
| $\rho \ (kg/m^3)$   | Normal     | $2.551 \times 10^3$ | 0.05 | --             |
| $b \ (m)$           | Interval   | 0.6        | --   | $b \in [0.59, 0.61]$ |

6. Results and discussion

The mathematical programming method and the direct iterative algorithm were used to solve the mixed reliability index respectively. The results of the two examples were listed in Table 3 and Table 4. In both methods, the initial uncertain variable value is selected as its mean value, and the convergence criterion is that the relative difference of the objective function of adjacent iteration steps is less than $10^{-4}$. The two methods get the same reliability index, but the iteration steps of the direct iteration algorithm are much less than those of the mathematical programming method. The iteration histories of the two methods are shown in figure 5 and figure 6.
Table 3. Summary of results for the cantilever beam

| Strategy | Reliability index $\beta_m$ | Iteration steps | MPP & WCP $F_{\text{MPP}}$ (kN) | $E_{\text{MPP}}$ (GPa) | $d^* (m)$ |
|-----------|----------------------------|-----------------|-------------------------------|------------------------|----------|
| MPM       | 2.42                       | 20              | 109.74                        | 27.28                  | 2.53     |
| DIM       | 2.42                       | 6               | 109.74                        | 27.28                  | 2.53     |

Table 4. Summary of results for the three-storey frame structure

| Strategy | Reliability index $\beta_m$ | Iteration steps | MPP & WCP $F_{\text{MPP}}$ (kN) | $E_{\text{MPP}}$ (GPa) | $b^* (m)$ |
|-----------|----------------------------|-----------------|-------------------------------|------------------------|----------|
| MPM       | 2.51                       | 87              | 10827                         | 27.43                  | 0.57     |
| DIM       | 2.51                       | 11              | 10827                         | 27.43                  | 0.57     |

Figure 5. Iteration processes of reliability evaluation for the cantilever beam

Figure 6. Iteration processes of reliability evaluation for the Three-storey frame structure

7. conclusion

This paper based on the structural reliability probability method and the interval model on the basis of the probability method, put forward the concept of reliability probability interval hybrid model with mixed uncertain parameters measurement. A reliability index solution strategy under the hybrid model is proposed. The examples validate the effectiveness of the proposed method. The solution strategy can be used to evaluate the structural safety performance of practical engineering, and it is of great significance to improve the safety performance of structural design.

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