COMPACT 9-POINT FINITE DIFFERENCE METHODS WITH HIGH ACCURACY ORDER AND/OR M-MATRIX PROPERTY FOR ELLIPTIC CROSS-INTERFACE PROBLEMS

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Abstract. In this paper we develop finite difference schemes for elliptic problems with piecewise continuous coefficients that have (possibly huge) jumps across fixed internal interfaces. In contrast with such problems involving one smooth non-intersecting interface, that have been extensively studied, there are very few papers addressing elliptic interface problems with intersecting interfaces of coefficient jumps. It is well known that if the values of the permeability in the four subregions around a point of intersection of two such internal interfaces are all different, the solution has a point singularity that significantly affects the accuracy of the approximation in the vicinity of the intersection point. In the present paper we propose a fourth-order 9-point finite difference scheme on uniform Cartesian meshes for an elliptic problem whose coefficient is piecewise constant in four rectangular subdomains of the overall two-dimensional rectangular domain. Moreover, for the special case when the intersecting point of the two lines of coefficient jumps is a grid point, such a compact scheme, involving relatively simple formulas for computation of the stencil coefficients, can even reach sixth order of accuracy. Furthermore, we show that the resulting linear system for the special case has an M-matrix, and prove the theoretical sixth order convergence rate using the discrete maximum principle. Our numerical experiments demonstrate the sixth (for the special case) and at least fourth (for the general case) accuracy orders of the proposed schemes. In the general case, we derive a compact third-order finite difference scheme, also yielding a linear system with an M-matrix. In addition, using the discrete maximum principle, we prove the third order convergence rate of the scheme for the general elliptic cross-interface problem.

1. Introduction and problem formulation

Interface problems arise in many applications such as modeling of underground waste disposal, oil reservoirs, composite materials, and many others. Some approaches to the solution of the elliptic interface problem with a smooth non-intersecting interface of coefficient jumps were provided by the immersed interface methods (IIM, see [7–9, 14, 16, 19, 20, 24, 29, 32] and references therein) and matched interface and boundary methods (MIB, see [13, 30, 31, 33, 34]). Both methods belong to the class of the finite difference methods (FDM). Elliptic interface problems with intersecting interfaces appear in many applications, but perhaps the most notorious example is the modeling of geological porous media flows (see e.g. [1, 3–6, 15, 17, 18, 23, 25, 27]). A classical problem of this type is formulated by the Society of Petroleum Engineers, the so-called SPE10 problem (see https://www.spe.org/web/csp/datasets/set02.htm). Here we consider a 2D simplification of this problem that involves interface intersections of vertical straight lines and horizontal straight lines, so that the permeability coefficient in the four subregions in the vicinity of an intersection point has different values. Even for this relatively simple cross-interface problem, the only compact 9-point finite difference method in the literature, that we are aware of, is the scheme in [2], that is third-order consistent, and uses special non-uniform meshes. To our knowledge, the convergence rate of this scheme has never been proven. We should also note here that the difficulty of the problem is usually
exacerbated if the jumps of the permeability coefficient across the different interfaces are very large (of several orders of magnitude). Details of the physical background of the elliptic interface problem can be found in [28].

In practice, usually the permeability variation occurs on scales that are very small as compared to the size of the medium, and therefore the solution of the interface problem is highly oscillatory. This causes the appearance of the so-called pollution effect in the error of its numerical approximation. In order to obtain a reasonable low-order numerical solution to such problems one needs to employ a very fine, possibly nonuniform grid, that captures the small scale features. Therefore, the development of higher-order compact approximations can help to reduce the computational costs. Compared to the finite element or finite volume methods, the FDM does not require the integration of highly-oscillatory or discontinuous functions. Furthermore, a compact 9-point scheme (in 2D) yields a sparse linear system that can be solved very efficiently.

Finally we should mention, that even in the relatively simple case of elliptic interface problems involving a smooth non-intersecting interface of coefficient jumps, the theoretical proof of convergence of the various proposed finite difference schemes is usually missing. The only exceptions are presented in [21] using discrete maximum principle for a second order scheme, and [12] using numerically verified discrete maximum principle for a fourth order scheme. The compact 9-point schemes considered in the present paper possess the M-matrix property, that guarantees the discrete maximum principle for the numerical solution. In turn, this property greatly facilitates the proof of their convergence rate.

In this paper we develop numerical approximations to the following elliptic cross-interface problem: Given the domain \( \Omega := (l_1, l_2) \times (l_3, l_4) \) with \( l_1, l_2, l_3, l_4 \in \mathbb{R} \), then consider:

\[
\begin{align*}
- \nabla \cdot (a \nabla u) &= f \quad \text{in } \Omega \setminus \Gamma, \\
[u] &= \phi_p \quad \text{on } \Gamma_p \text{ for } p = 1, 2, 3, 4, \\
[a \nabla u \cdot \vec{n}] &= \psi_p \quad \text{on } \Gamma_p \text{ for } p = 1, 2, 3, 4, \\
u &= g \quad \text{on } \partial \Omega,
\end{align*}
\]

where the cross-interface \( \Gamma \) is given by \( \Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \{(\xi, \zeta)\} \) with

\[
\begin{align*}
\Gamma_1 &= \{\xi\} \times (\zeta, l_4), \\
\Gamma_2 &= \{\xi\} \times (l_3, \zeta), \\
\Gamma_3 &= (\xi, l_2) \times \{\zeta\}, \\
\Gamma_4 &= (l_1, \xi) \times \{\zeta\}, \\
(\xi, \zeta) &\in \Omega.
\end{align*}
\]

As usual, that the square brackets here denote the jump of the corresponding function, i.e. for \((\xi, \zeta) \in \Gamma_p\) with \( p = 1, 2 \) (on the vertical line of the cross-interface \( \Gamma \)),

\[
[u](\xi, \zeta) := \lim_{x \to \xi^+} u(x, y) - \lim_{x \to \xi^-} u(x, y), \quad [a \nabla u \cdot \vec{n}](\xi, \zeta) := \lim_{x \to \xi^+} a(x, y) \frac{\partial u}{\partial x}(x, y) - \lim_{x \to \xi^-} a(x, y) \frac{\partial u}{\partial x}(x, y);
\]

while for \((x, \zeta) \in \Gamma_p\) with \( p = 3, 4 \) (i.e., on the horizontal line of the cross-interface \( \Gamma \)),

\[
[u](x, \zeta) := \lim_{y \to \zeta^+} u(x, y) - \lim_{y \to \zeta^-} u(x, y), \quad [a \nabla u \cdot \vec{n}](x, \zeta) := \lim_{y \to \zeta^+} a(x, y) \frac{\partial u}{\partial y}(x, y) - \lim_{y \to \zeta^-} a(x, y) \frac{\partial u}{\partial y}(x, y).
\]

Note that the interface curve \( \Gamma \) divides the domain \( \Omega \) into 4 subdomains:

\[
\Omega_1 := (l_1, \xi) \times (\zeta, l_4), \quad \Omega_2 := (\xi, l_2) \times (\zeta, l_4), \quad \Omega_3 := (\xi, l_2) \times (l_3, \zeta), \quad \Omega_4 := (l_1, \xi) \times (l_3, \zeta).
\]

See Fig. [1] for an illustration, where \( a_p := a \chi_{\Omega_p} \), \( f_p := f \chi_{\Omega_p} \), and \( u_p := u \chi_{\Omega_p} \) for \( p = 1, 2, 3, 4 \).

To derive a compact 9-point scheme that approximates the cross-interface problem (1.1), we assume that:

\[
\begin{align*}
(A1) \quad & a_p \text{ is a positive constant in } \Omega_p, \\
(A2) \quad & \text{The restriction of the solution } u_p \text{ and of the source term } f_p \text{ has uniformly continuous partial derivatives of (total) orders up to seven and five, respectively, in each } \Omega_p \text{ for } p = 1, 2, 3, 4, \\
(A3) \quad & \text{The essentially one-dimensional functions } \phi_p \text{ and } \psi_p \text{ in (1.1) on the interface } \Gamma_p \text{ have uniformly continuous derivatives of orders up to seven and six respectively for } p = 1, 2, 3, 4.
\end{align*}
\]
The remainder of the paper is organized as follows. In Section 2.1, we derive a compact 9-point scheme with sixth order of consistency for interior grid points in Theorem 2.1. For grid points near the interface, as illustrated by Fig. 2, we have two cases:

**Case 1:** If the point \((\xi, \zeta)\) of intersection of the interfaces is a grid point (see the left panel of Fig. 2), we derive in Section 2.2 a compact 9-point scheme that has a seventh order of consistency at every grid point lying on the cross-interface. The stencil coefficients are given in Theorems 2.2 and 2.3. In Section 3.1 we prove that this scheme is sixth-order accurate, using the discrete maximum principle satisfied by it. The results of some numerical experiments, demonstrating the sixth-order convergence rate of the scheme, are presented in Section 4.1.

**Case 2:** If \((\xi, \zeta)\) is not a grid point (see the right panel in Fig. 2), we derive in Section 2.2 a compact 9-point scheme with fourth order of consistency for every grid point neighboring the interface. Next we show in Section 3.2 that this scheme does not satisfy the M-matrix property. Subsequently, we obtain a compact scheme satisfying the M-matrix property, with a consistency order three at grid points neighboring the interface points except for the vicinity of the intersection point \((\xi, \zeta)\), and order two at grid points neighboring \((\xi, \zeta)\). Since the M-matrix property immediately implies that the scheme satisfies a discrete maximum principle, this allows us to prove that the overall convergence rate of the scheme is of order three. In Section 4.2 we provide some numerical results that seem to suggest that the scheme given in Theorems 2.1, 2.4 and 2.5, that does not satisfy a discrete maximum principle, is fifth-order accurate.

In Section 5, we summarize the main contributions of this paper. Finally, in Section 6 we present the proofs for the results stated in Sections 2 and 3.

2. **High order compact 9-point schemes using uniform Cartesian grids**

In this section, we present some compact finite difference schemes on uniform Cartesian grids for the elliptic cross-interface problem in (1.1). To improve readability, the technical proofs of the results stated in this section are deferred to Section 6.
We start by introducing a uniform Cartesian mesh on the domain:

\[ \Omega := (l_1, l_2) \times (l_3, l_4), \quad \text{with} \quad l_4 - l_3 = N_0(l_2 - l_1) \quad \text{for some positive integer } N_0, \]

containing the grid points:

\[ x_i := l_1 + ih, \quad i = 0, \ldots, N_1, \quad \text{and} \quad y_j := l_3 + jh, \quad j = 0, \ldots, N_2, \quad h := \frac{l_2 - l_1}{N_1} = \frac{l_4 - l_3}{N_2}, \]

where \( N_1 \) is a positive integer and \( N_2 := N_0N_1 \). We also define \((u_h)_{i,j}\) to be the value of the numerical approximation \(u_h\) of the exact solution \(u\) of the elliptic cross-interface problem \([1]\), at the grid point \((x_i, y_j)\). For stencil coefficients \(\{C_{k,\ell}\}_{k,\ell=-1,0,1}\) with \(C_{k,\ell} \in \mathbb{R}\) in the compact 9-point stencil centered at a grid point \((x_i, y_j)\), the discrete operator \(L_h\) acting on \(u_h\) is defined to be:

\[
L_h u_h := \sum_{k=-1}^{1} \sum_{\ell=-1}^{1} C_{k,\ell} (u_h)_{i+k,j+\ell}. \tag{2.1}
\]

Similarly, the action of the discrete operator \(L_h\) on the exact solution \(u\) is given by:

\[
L_h u := \sum_{k=-1}^{1} \sum_{\ell=-1}^{1} C_{k,\ell} u(x_i + kh, y_j + \ell h). \tag{2.2}
\]

2.1. Compact 9-point stencils at interior points. The following compact FDM with a consistency order six for \([1.1]\) at interior points is well known in the literature (e.g., see [11, 26]).

**Theorem 2.1.** Consider \((x_i, y_j) \in \Omega\) with all 9 points \((x_i \pm h, y_j \pm h) \in \overline{\Omega}_p\) for some \(p \in \{1, 2, 3, 4\}\). Assume that \(u_p := u\chi_{\Omega_p}\) and \(f_p := f\chi_{\Omega_p}\) have uniformly continuous partial derivatives of (total) orders up to seven and five, respectively, in \(\Omega_p\). Let the discrete operator \(L_h\) be defined in \([2.1]\) with the stencil coefficients

\[
C_{0,0} = 20, \quad C_{0,-1} = C_{1,0} = C_{1,0} = -4, \quad C_{1,1} = C_{-1,1} = C_{-1,1} = C_{-1,-1} = -1.
\]

Then the compact 9-point finite difference scheme \(h^{-2}L_h u_h = \frac{1}{a(x_i, y_j)} F\) with

\[
F := 6f(x_i, y_j) + \frac{\eta^2}{2} \left[ \frac{\partial f}{\partial x}(x_i, y_j) + \frac{\partial f}{\partial y}(x_i, y_j) \right] + \frac{h^4}{60} \left[ \frac{\partial^4 f}{\partial x^4}(x_i, y_j) + \frac{\partial^4 f}{\partial y^4}(x_i, y_j) \right] + \frac{h^4}{10} \frac{\partial^4 f}{\partial x^2 \partial y^2}(x_i, y_j),
\]

has a sixth order of consistency at the interior grid point \((x_i, y_j)\) for \(-\nabla \cdot (a \nabla u) = f\).

2.2. Compact 9-point stencils at grid points on the interface. In this subsection, we now discuss how to find a compact FDM of a consistency order seven at grid points \((x_i, y_j)\) lying on the cross-interface (i.e., Case 1 in Section [1] see the left panel of Fig. [2]). Note that all interfaces \(\Gamma_1, \ldots, \Gamma_4\) are open intervals and hence they do not contain the intersection point \((\xi, \zeta)\).

In order to devise the explicit formulas for the coefficients of the compact scheme, we need some notations and definitions. Recall that \(a_p := a\chi_{\Omega_p}\) is a positive constant, \(f_p := f\chi_{\Omega_p}\), and \(u_p := u\chi_{\Omega_p}\) for \(p = 1, 2, 3, 4\). For \((x_i^*, y_j^*) \in \overline{\Omega}_p\) with \(p = 1, 2, 3, 4\), using only the information of \(u_p\) in \(\overline{\Omega}_p\), we can define

\[
u^{(m,n)} := \frac{\partial^{m+n} u_p}{\partial x^m \partial y^n}(x_i^*, y_j^*), \quad f^{(m,n)} := \frac{\partial^{m+n} f_p}{\partial x^m \partial y^n}(x_i^*, y_j^*).
\]

For \((x_i^*, y_j^*) = (\xi, y_j^*) \in \Gamma_p\) with \(p = 1, 2\), we define \(\phi^{(n)} := \frac{d^n \phi_p(\xi, y_j^*)}{d y^n}|_{y=y_j^*}\) and \(\psi^{(n)} := \frac{d^n \psi_p(\xi, y_j^*)}{d y^n}|_{y=y_j^*}\), while for \((x_i^*, y_j^*) = (x_i^*, \zeta) \in \Gamma_p\) with \(p = 3, 4\), we similarly define \(\phi^{(n)} := \frac{d^n \phi_p(x_i^*, \zeta)}{d x^n}|_{x=x_i^*}\) and \(\psi^{(n)} := \frac{d^n \psi_p(x_i^*, \zeta)}{d x^n}|_{x=x_i^*}\). Note that \(\phi, \psi\) in \([1.1]\) are essentially 1D functions defined on the line segment \(\Gamma_p\).

Define \(N_0 := \mathbb{N} \cup \{0\}\) and for \(M \in \mathbb{N}_0\), we define the following index subsets of \(\mathbb{N}_0^2\) as follows:

\[
\Lambda_M := \{(m, n) \in \mathbb{N}_0^2 : m + n \leq M\}, \quad \Lambda_M^1 := \{(m, n) \in \Lambda_M : m = 0, 1\}, \quad \Lambda_M^2 := \Lambda_M \backslash \Lambda_M^1. \tag{2.3}
\]
The illustrations for $\Lambda^1$ and $\Lambda^2$ are shown in Fig. 12 in Section 6. We shall also define the following bivariate polynomials (which will be used in our compact FDMs later):

$$G_{M,m,n}(x,y) := \sum_{\ell=0}^{\lfloor \frac{7}{2} \rfloor} \frac{(-1)^{\ell} x^{m+2\ell} y^{n-2\ell}}{(m+2\ell)!(n-2\ell)!}, \quad H_{M,m,n}(x,y) := \sum_{\ell=1}^{1+\lfloor \frac{3}{2} \rfloor} \frac{(-1)^{\ell} x^{m+2\ell} y^{n-2\ell+2}}{(m+2\ell)!(n-2\ell+2)!},$$

where $\lfloor x \rfloor$ is the floor function representing the largest integer less than or equal to $x \in \mathbb{R}$.

To state our compact FDMs later, we shall use some auxiliary polynomials of $h$. For $w \in \mathbb{R}$, $M \in \mathbb{N}_0$ and $m,n \in \mathbb{N}_0$, we define the univariate polynomials of $h$ with the parameter $w$ for the vertical interface $\Gamma_1$ or $\Gamma_2$ to be

$$G_{M,m,n}^w := \sum_{\ell=-1}^{1} C_{-1,\ell} G_{M,m,n}(wh - h, \ell h),$$

$$H_{M,m,n}^{-w} := \sum_{\ell=-1}^{1} C_{-1,\ell} H_{M,m,n}(wh - h, \ell h), \quad H_{M,m,n}^{+w} := \sum_{k=0}^{1} \sum_{\ell=-1}^{1} C_{k,\ell} H_{M,m,n}(wh + kh, \ell h).$$

![Figure 3](image-url)  

**Figure 3.** First panel: compact 9-point stencils in Theorem 2.2 with $(x_i, y_j) = (x^*_i, y^*_j) \in \Gamma_1 \cup \Gamma_2$. Second panel: compact 9-point stencils with $(x_i, y_j) = (x^*_i, y^*_j) \in \Gamma_3 \cup \Gamma_4$. Third panel: compact 9-point stencil in Theorem 2.3 with $(x_i, y_j) = (x^*_i, y^*_j) = (\xi, \zeta)$. The grid point $(x_i, y_j)$ is indicated by the red color.

We first consider the compact discretization at grid points lying on the vertical interface line $\Gamma_1$ or $\Gamma_2$. The modification of the scheme corresponding to the horizontal interfaces $\Gamma_3$ or $\Gamma_4$ is straightforward and we will only briefly mention it afterwards.

**Theorem 2.2.** Consider a grid point $(x_i, y_j)$ such that $(x_i, y_j) \in \Gamma_1$ (see the first panel of Fig. 3). Assume that $u_p := u \chi_{\Omega_p}$ and $f_p := f \chi_{\Omega_p}$ have uniformly continuous partial derivatives of (total) orders up to seven and five, respectively, in each $\Omega_p$ for $p = 1, 2$. Also assume that the essentially one-dimensional functions $\phi_1$ and $\psi_1$ on the interface $\Gamma_1$ have uniformly continuous derivatives of orders up to seven and six, respectively. Let $(x^*_i, y^*_j) := (x_i, y_j)$ and $\mathcal{L}_h$ be the discrete operator in (2.1) with the stencil coefficients

$$C_{1,0} = -4, \quad C_{1,-1} = C_{1,1} = -1, \quad C_{-1,-1} = C_{-1,1} = -\alpha, \quad C_{0,-1} = C_{0,1} = -2(1 + \alpha),$$

$$C_{-1,0} = -4\alpha, \quad C_{0,0} = 10(1 + \alpha), \quad \alpha := a_1/a_2 > 0. \quad (2.6)$$

Then the compact 9-point finite difference scheme $h^{-1} \mathcal{L}_h u_h = h^{-1} F$ has a seventh order of consistency at the grid point $(x_i, y_j) \in \Gamma_1$, where

$$F := \frac{1}{a_1} \sum_{(m,n) \in \Lambda_5} f_1^{(m,n)} H_{t,m,n}^{-0} + \frac{1}{a_2} \sum_{(m,n) \in \Lambda_5} f_2^{(m,n)} H_{t,m,n}^{+0} - \sum_{n=0}^{7} \phi_1^{(n)} G_0^0g_{t,0,n} - \frac{1}{a_1} \sum_{n=0}^{6} \psi_1^{(n)} G_0^0g_{t,1,n}, \quad (2.7)$$

and $H_{t,m,n}^{-0}, H_{t,m,n}^{+0}, G_0^0g_{t,0,n}, G_0^0g_{t,1,n}$ are defined in (2.5).
For \((x_i, y_j) \in \Gamma_2\) (see the first panel of Fig. [3]), we can obtain the compact 9-point finite difference scheme with a consistency order seven by using \(\alpha := \frac{a_4}{a_3}\), replacing the subscript 1 by 2 for \(\Gamma_1\), \(\phi_1\), and \(\psi_1\), and replacing all the subscripts 1 and 2 by 4 and 3 for \(f_1\) and \(f_2\), respectively, in Theorem [2.2].

Similarly to Theorem [2.2], we can specify the scheme for grid points lying on the horizontal line \(\xi, \zeta\) (see the second panel of Fig. [3]) by modifying the stencil coefficients to:

\[
C_{0,1} = -4, \quad C_{1,1} = 1, \quad C_{-1,-1} = -\alpha, \quad C_{-1,0} = C_{1,0} = -2(1 + \alpha), \quad C_{0,-1} = -4\alpha, \quad C_{0,0} = 10(1 + \alpha), \quad \alpha := \frac{a_3}{a_2} > 0.
\]  

(2.8)

The right hand side vector in this case is given by:

\[
F := \frac{1}{a_3} \sum_{(m,n) \in \Lambda_2} f_3^{(m,n)} H_{m,n}^{-,0} + \frac{1}{a_2} \sum_{(m,n) \in \Lambda_2} f_2^{(m,n)} H_{m,n}^{+,0} - \sum_{m=0}^{7} \phi_3^{(m)} G_{m,0}^{-0} - \frac{6}{a_3} \sum_{m=0}^{6} \psi_3^{(m)} G_{m,1}^{-0} 
\]

(2.9)

where the polynomials for the horizontal interface are defined as:

\[
\tilde{G}_{M,m,n}^{-w} := \sum_{k=-1}^{1} C_{k,1} G_{M,m,n}(wh - h, kh),
\]

\[
\tilde{H}_{M,m,n}^{-w} := \sum_{k=-1}^{1} C_{k,1} H_{M,m,n}(wh - h, kh),
\]

\[
\tilde{H}_{M,m,n}^{+w} := \sum_{k=-1}^{1} \sum_{\ell=0}^{1} C_{k,\ell} H_{M,m,n}(wh + \ell h, kh).
\]

(2.10)

For \((x_i, y_j) \in \Gamma_4\) (see the second panel of Fig. [3]), we can obtain the scheme by using \(\alpha = \frac{a_4}{a_1}\), replacing the subscript 3 by 4 for \(\Gamma_3\), \(\phi_3\), and \(\psi_3\), and replacing all the subscripts 2 and 3 by 1 and 4 for \(f_2\) and \(f_3\), respectively in (2.8) - (2.9).

Finally, we handle the case when the intersecting interface point \((\xi, \zeta)\) is a grid point.

**Theorem 2.3.** Consider the grid point \((x_i, y_j) = (\xi, \zeta)\) (see the third panel of Fig. [3]). Assume that \(u_p := u\chi_p\) and \(f_p := f\chi_p\) have uniformly continuous partial derivatives of (total) orders up to seven and five, respectively, in each \(\Omega_p\) for \(p = 1, 2, 3, 4\). Also assume that the essentially one-dimensional functions \(\phi_p\) and \(\psi_p\) on the interface \(\Gamma_p\) have uniformly continuous derivatives of orders up to seven and six, respectively, for \(p = 1, 2, 3, 4\). Let \((x_i, y_j) := (\xi, \zeta)\) and \(L_h\) be the discrete operator in (2.1) with the stencil coefficients

\[
C_{-1,1} := \frac{-a_3^2(a_2 + a_3)}{a_2^2(a_1 + a_4)}, \quad C_{0,1} := \frac{-2(a_1 + a_2)}{a_2}, \quad C_{1,1} := -1,
\]

\[
C_{-1,0} := \frac{-2a_1(a_2 + a_3)}{a_2^2}, \quad C_{0,0} := \frac{5(a_2 + a_3)(a_1 + a_2)}{a_2^2}, \quad C_{1,0} := \frac{-2(a_2 + a_3)}{a_2},
\]

\[
C_{-1,-1} := \frac{-a_1a_4(a_2 + a_3)}{a_2^2(a_1 + a_4)}, \quad C_{0,-1} := \frac{2a_3(a_1 + a_2)}{a_2^2}, \quad C_{1,-1} := \frac{-a_3}{a_2}.
\]

(2.11)

Then the compact 9-point finite difference scheme \(h^{-1}L_h u_h = h^{-1}F\) has a seventh order of consistency at the grid point \((x_i, y_j)\) (i.e., \((\xi, \zeta)\)), where

\[
F := \sum_{(m,n) \in \Lambda_2^1} F_{m,n}^{1,0,0} + \sum_{(m,n) \in \Lambda_2^3} F_{m,n}^{0,0,0} + \sum_{(m,n) \in \Lambda_2^1} \Phi_{m,n}^{1,0,0} + \sum_{m=0}^{7} \phi_{m}^{0,0} + \sum_{(m,n) \in \Lambda_2^1} \Psi_{m,n}^{1,0,0} + \sum_{m=0}^{6} \psi_{m}^{0,0}
\]

(2.12)

and \(F_{m,n}^{1,0,0}, F_{m,n}^{0,0,0}, \Phi_{m,n}^{1,0,0}, \phi_{m}^{0,0}, \Phi_{m,n}^{1,0,0}, \phi_{m}^{0,0}, \Psi_{m,n}^{1,0,0}, \psi_{m}^{0,0}\) are defined in (6.26) - (6.29).

2.3. **Compact 9-point stencils at grid points neighboring the interface.** In this subsection, we derive a compact scheme with a consistency order four for every grid point neighboring the interface \(\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \{(\xi, \zeta)\}\) (i.e., Case 2 in Section [1]), see the right panel of Fig. [2].

If the grid point \((x_i, y_j) \notin \Gamma\) is in the vicinity of the interface line \(\Gamma_1\) (see the first panel of Fig. [4]), the compact scheme with a consistency order four is given in the theorem below.
Theorem 2.4. Consider a grid point \((x_i, y_j) \notin \Gamma\) such that \((x_i^*, y_j^*) := (x_i - wh, y_j) \in \Gamma_1\) with \(0 < w < 1\), \((\xi, \zeta) \notin (x_i, y_j) + (-h, h)^2\) (see the first panel of Fig. 4). Assume that \(u_p := u_{\chi_{\Omega_p}}\) and \(f_p := f_{\chi_{\Omega_p}}\) have uniformly continuous partial derivatives of (total) orders up to four and two, respectively, in each \(\Omega_p\) for \(p = 1, 2\). Also assume that the essentially one-dimensional functions \(\phi_1\) and \(\psi_1\) on the interface \(\Gamma_1\) have uniformly continuous derivatives of orders up to four and three, respectively. Let \(L_h\) be the discrete operator in (2.1) with the stencil coefficients

\[
\begin{align*}
C_{-1,1} &:= -(r_4 \alpha^2 + r_5 \alpha)/\beta, & C_{0,1} &:= -1, & C_{1,1} &:= -(r_6 + t_1 \alpha^2 + r_7 \alpha)/\beta, \\
C_{-1,0} &:= -(r_8 \alpha^2 + r_9 \alpha)/\beta, & C_{0,0} &:= -(s_1 + s_2 \alpha^2 + s_3 \alpha)/\beta, & C_{1,0} &:= -(r_{10} + r_{11} \alpha^2 + r_{12} \alpha)/\beta, \\
C_{-1,-1} &:= C_{1,1}, & C_{0,-1} &:= C_{0,1}, & C_{1,-1} &:= C_{1,1},
\end{align*}
\]

where \(\alpha := a_1/a_2 > 0\), \(\beta := r_1 + r_2 \alpha^2 + r_3 \alpha > 0\) and

\[
\begin{align*}
\begin{array}{l}
r_1 := (2w + 1)^2(w + 2)(w - 1)^2, \\
r_2 := 4w^3 - 4w^4 + 5w^2 + 6w^2 - 8w^2 + 2, \\
r_3 := -8w^5 + 6w^3 - 2w^2 + 4, \\
r_4 := 4w^4 + 4w^3 - w^2 + 1, \\
r_5 := -(2w + 1)^2(w - 1)^3, \\
r_7 := 8w^5 - 20w^4 + 14w^3 - 3w^2 + 1, \\
r_8 := -8w^4 + 8w^3 + 10w^2 - 6w + 4, \\
r_9 := 8w^4 - 8w^3 - 10w^2 + 6w + 4, \\
r_{10} := 8w^5 - 16w^4 + 14w^3 - 8w^2 - 2w + 4, \\
r_{11} := 8w^5 - 24w^4 + 38w^3 - 22w^2 + 8w, \\
r_{12} := -16w^5 + 40w^4 - 52w^3 + 30w^2 - 6w + 4, \\
t_1 := -4w^5 + 12w^4 - 13w^3 + 8w^2 - w, \\
t_2 := -8w^5 + 8w^4 + 22w^3 - 18w^2 + 10w - 10, \\
t_3 := 16w^5 + 12w^4 - 8w^2 - 20,
\end{array}
\end{align*}
\]

Then the compact 9-point finite difference scheme \(h^{-1}L_h u_h = h^{-1}F\) has a fourth order of consistency at the grid point \((x_i, y_j) \notin \Gamma\) with \((\xi, \zeta) \notin (x_i, y_j) + (-h, h)^2\), where

\[
F := \frac{1}{a_1} \sum_{(m,n) \in \Lambda_2} f_1^{(m,n)} H_{4,m,n}^{-w} + \frac{1}{a_2} \sum_{(m,n) \in \Lambda_2} f_2^{(m,n)} H_{4,m,n}^{+w} - \sum_{n=0}^{4} \frac{\phi_1^{(n)}}{\alpha_1} C_{4,0,n}^{w} - \frac{3}{a_1} \sum_{n=0}^{3} \psi_1^{(n)} C_{4,1,n}^{w}
\]

(2.15)

and \(H_{4,m,n}^{+w}, H_{4,m,n}^{-w}, C_{4,m,n}^{w}, C_{4,m,n}^{-w}\) are defined in (2.5). Moreover, up to a multiplicative constant for normalization, the stencil coefficients \(\{C_{k,\ell}\}_{k,\ell=-1,0,1}\) in (2.13) are unique.

![Figure 4](image_url)

**Figure 4.** Left: compact 9-point stencil in Theorem 2.4 with \((x_i^*, y_j^*) := (x_i - wh, y_j) \in \Gamma_1\) and \(0 < w < 1\). Right: compact 9-point stencil with \((x_i^*, y_j^*) := (x_i + wh, y_j) \in \Gamma_1\) and \(0 < w < 1\). The grid point \((x_i, y_j)\) is indicated by the red color, while the interface point \((x_i^*, y_j^*) \in \Gamma_1\) is indicated by the black color and is not a grid point.

If a grid point \((x_i, y_j)\) satisfies \((x_i^*, y_j^*) := (x_i + wh, y_j) \in \Gamma_1\) (see the second panel of Fig. 4) or \((x_i^*, y_j^*) := (x_i \pm wh, y_j) \in \Gamma_2\) or \((x_i^*, y_j^*) := (x_i, y_j \pm wh) \in \Gamma_3, \Gamma_4\) with \(0 < w < 1\), then the compact 9-point scheme with a consistency order four at the grid point \((x_i, y_j) \notin \Gamma\) can be obtained similarly.

If the grid point \((x_i, y_j) \notin \Gamma\) is situated as shown in the first panel of Fig. 5, a compact scheme of a consistency order four can be specified as in the following theorem.

**Theorem 2.5.** Consider a grid point \((x_i, y_j) \notin \Gamma\) such that \((x_i^*, y_j^*) := (\xi, \zeta) = (x_i - w_1 h, y_j - w_2 h)\) with \(0 < w_1, w_2 < 1\) (see the first panel of Fig. 5). Assume that \(u_p := u_{\chi_{\Omega_p}}\) and \(f_p := f_{\chi_{\Omega_p}}\) have uniformly continuous partial derivatives of (total) orders up to four and two, respectively, in each \(\Omega_p\) for \(p = 1, 2, 3, 4\). Also assume that the essentially one-dimensional functions \(\phi_p\) and \(\psi_p\) on the interface \(\Gamma_p\) have uniformly continuous derivatives of orders up to four and three, respectively, for \(p = 1, 2, 3, 4\). Let \(L_h\) be the discrete operator in (2.1) with the stencil coefficients \(\{C_{k,\ell}\}_{k,\ell=-1,0,1}\)
assume that the essentially one-dimensional functions \( \phi \) and \( \psi \) have partial derivatives of (total) orders up to seven and five, respectively, in each \( \Omega \). It allows us to prove their convergence rate. For the sake of readability, the proofs of the results stated in this section are provided in Section 6.

For a scheme with stencil coefficients \( \{C_{k,\ell}\}_{k,\ell=-1,0,1} \), let us introduce the following sign condition:

\[
\begin{align*}
C_{k,\ell} &> 0, & \text{if} & \quad (k, \ell) = (0, 0), \\
C_{k,\ell} &\leq 0, & \text{if} & \quad (k, \ell) \neq (0, 0),
\end{align*}
\]

and summation condition:

\[
\sum_{k=-1}^{1} \sum_{\ell=-1}^{1} C_{k,\ell} = 0. \tag{3.2}
\]

Under suitable boundary conditions (such as Dirichlet boundary conditions), it is well known that the sign condition (3.1) and summation condition (3.2) together guarantee the resulting coefficient matrix to be an M-matrix [22].

### 3.1. Case 1: The interfaces intersection point \((\xi, \zeta)\) is a grid point.

**Theorem 3.1.** Consider the elliptic cross-interface problem in (1.1) with the interfaces intersection point \((\xi, \zeta)\) being a grid point. Assume that \( w_p := u_{\chi_{\Omega_p}} \) and \( f_p := f_{\chi_{\Omega_p}} \) have uniformly continuous partial derivatives of (total) orders up to seven and five, respectively, in each \( \Omega_p \) for \( p = 1, 2, 3, 4 \). Also assume that the essentially one-dimensional functions \( \phi_p \) and \( \psi_p \) on the interface \( \Gamma_p \) have uniformly
continuous derivatives of orders up to seven and six, respectively, for \( p = 1, 2, 3, 4 \). Then the matrix of the linear system resulting from the compact 9-point scheme given by Theorems 2.1 to 2.3, including the corresponding modifications for the other parts of the interface \( \Gamma \), is an M-matrix. Consequently, under the above assumptions, the scheme is sixth-order accurate, i.e. there exists a positive constant \( C \), independent of \( h \), such that

\[
\|u - u_h\|_\infty \leq Ch^6,
\]

where \( u \) is the exact solution of (1.1), and \( u_h \) is its numerical approximation.

### 3.2. Case 2: The interfaces intersection point \((\xi, \zeta)\) is not a grid point.

We first consider the compact 9-point scheme, with a consistency order four at a grid point \((x_i, y_j) \notin \Gamma\), in Theorem 2.4 where \((x^*_i, y^*_j) := (x_i - wh, y_j) \in \Gamma_1\) with \(0 < w < 1\) (see the first panel of Fig. 4). The summation condition \((3.2)\) for \([C_{k,\ell}]_{k,\ell=-1,0,1}\) in \((2.13)\) can be directly verified. From \((2.14)\), we can check that \(r_p > 0\) for all \(p = 1, 2, \ldots, 12\) and \(w \in (0, 1)\), while \(s_p < 0\) for all \(p = 1, 2, 3\) and \(w \in (0, 1)\). Therefore, the stencil coefficients \([C_{k,\ell}]_{k,\ell=-1,0,1}\) in \((2.13)\) satisfy:

\[
\begin{cases}
C_{k,\ell} > 0, & \text{if} \quad (k, \ell) = (0, 0), \\
C_{k,\ell} < 0, & \text{if} \quad (k, \ell) \notin \{(0, 0), (1, -1), (1, 1)\},
\end{cases}
\]

for any positive \(a_1, a_2\), and \(w \in (0, 1)\). Since we know from Theorem 2.4 that all the coefficients \([C_{k,\ell}]_{k,\ell=-1,0,1}\) in \((2.13)\) are unique after normalization, we observe from the above inequalities that the FDM in Theorem 2.4 satisfies the sign condition in \((3.1)\) if and only if

\[
C_{1,-1} \leq 0 \quad \text{and} \quad C_{1,1} \leq 0.
\]

By \((2.13)\) and \((2.14)\), we have

\[
C_{1,-1} = C_{1,1} = \frac{-(r_6 + r_7\alpha + t_1\alpha^2)}{r_1 + r_2\alpha^2 + r_3\alpha}, \quad \alpha = \frac{a_1}{a_2} > 0,
\]

\[
t_1 = -4w^5 + 12w^4 - 13w^3 + 8w^2 - w,
\]

and \(r_p > 0\) for all \(p = 1, 2, 3, 6, 7\) and \(w \in (0, 1)\). We can easily check that \(t_1 > 0\) for \(w \in [0.162, 1)\) and \(t_1 < 0\) for \(w \in (0, 0.161]\). So \(w \in [0.162, 1)\) is required to achieve \((3.4)\) for all positive \(a_1, a_2\). When \((x_i - h, y_j)\) (see the left panel in Fig. 4) or \((x_i + h, y_j)\) (see the right panel in Fig. 4) is the center point, \((1 - w) \in [0.162, 1)\) is also necessary. Thus the conditions in \((3.4)\) only hold for all positive \(a_1, a_2\) if \(w \in [0.162, 0.838]\). However, for \(w \in (0, 1)\) outside \([0.162, 0.838]\), one can always find particular positive \(a_1, a_2\) so that \((3.4)\) fails. In order to achieve the sign condition for proving convergence, it is necessary to lower the consistency order.

Now we propose the following compact scheme with a consistency order three such that \([C_{k,\ell}]_{k,\ell=-1,0,1}\) satisfies the sign condition \((3.1)\) and the summation condition \((3.2)\) for all positive \(a_1, a_2\) and \(w \in (0, 1)\).

**Theorem 3.2.** Consider a grid point \((x_i, y_j) \notin \Gamma\) such that \((x^*_i, y^*_j) := (x_i - wh, y_j) \in \Gamma_1\) with \(0 < w < 1\), \((\xi, \zeta) \notin (x_i, y_j) + (-h, h)^2\) (see the first panel of Fig. 4). Assume that \(u_p := u_{\chi_{\Omega_p}}\) and \(f_p := f_{\chi_{\Omega_p}}\) have uniformly continuous partial derivatives of (total) orders up to three and one, respectively, in each \(\Omega_p\) for \(p = 1, 2\). Also assume that the essentially one-dimensional functions \(\phi_1\) and \(\psi_1\) on the interface \(\Gamma_1\) have uniformly continuous derivatives of orders up to three and two, respectively. Let \(L_h\) be the discrete operator in \((2.1)\) with the stencil coefficients

\[
\begin{align*}
C_{0,0} := 1, & \quad C_{-1,1} := C_{-1,1} := \frac{[r_1\alpha^2 + r_5\alpha + t_1\alpha^2 + t_2\alpha]}{\beta}, \\
C_{1,1} := C_{1,-1} := \rho, & \quad C_{0,1} := C_{0,-1} := \frac{[s_1 + s_2\alpha^2 + s_3\alpha + t_3 + t_4\alpha^2 + t_5\alpha]}{\beta}, \\
C_{-1,0} := \frac{[s_4\alpha^2 + s_5\alpha + t_6\alpha + t_7\alpha + t_8]}{\beta}, & \quad C_{1,0} := \frac{[s_6 + s_7\alpha^2 + s_8\alpha + t_8 + t_9\alpha^2 + t_{10}\alpha]}{\beta},
\end{align*}
\]

\((3.5)\)
where \( \alpha := a_1/a_2 > 0, \beta := r_1 + r_2\alpha^2 + r_3\alpha > 0, \rho \in \mathbb{R} \) is a free parameter, and
\[
\begin{align*}
r_1 &:= 12(w^3 - w^2 - w + 1), \\
r_2 &:= 4(w^3 + 3w^2 + 2w), \\
r_3 &:= 4(-w^3 + w + 3), \\
r_4 &:= 4(2w^3 + w + 3), \\
r_5 &:= 4(-2w^3 - w + 3), \\
s_1 &:= 6(-w^2 + 2w - 1), \\
s_2 &:= 6(-w^2 + 2w + 1), \\
s_3 &:= 12(w^2 - 1), \\
s_4 &:= 4(-w^3 + w - 3), \\
s_5 &:= 4(4w^3 - w - 3), \\
s_6 &:= 12(-2w^3 + 3w^2 - 1), \\
s_7 &:= 4(2w^3 - 3w^2 - w), \\
s_8 &:= 4(8w^3 - 6w^2 + w - 3), \\
t_1 &:= -4w^3 + 6w^2 - 2w, \\
t_2 &:= 4w^3 - 6w^2 + 2w, \\
t_3 &:= -6w^3 + 9w^2 - 3, \\
t_4 &:= -2w^3 - 3w^2 - w, \\
t_5 &:= 8w^3 - 6w^2 + w - 3, \\
t_6 &:= 8w^3 - 12w^2 - 2w, \\
t_7 &:= -8w^3 + 12w^2 + 2w - 6, \\
t_8 &:= 6(-w^2 + 2w - 1), \\
t_9 &:= -6w^2, \\
t_{10} &:= 12(w^2 - w).
\end{align*}
\] (3.6)

Then the compact 9-point finite difference scheme \( h^{-1}L_h u_h = h^{-1}F \) has a third order of consistency at the grid point \((x_i, y_j) \not\in \Gamma \) with \((\xi, \zeta) \not\in (x_i, y_j) + (-h, h)^2\), where
\[
F := \frac{1}{a_1} \sum_{(m,n) \in \Lambda_1} f_1^{(m,n)} H_{3,m,n} - \frac{1}{a_2} \sum_{(m,n) \in \Lambda_1} f_2^{(m,n)} H_{3,m,n} - 3 \sum_{n=0}^3 \phi_1^{(n)} G_{3,0,n} - \frac{1}{a_1} \sum_{n=0}^2 \psi_1^{(n)} G_{3,1,n}
\] (3.7)
and \( H_{3,m,n}^{-w} H_{3,m,n}^{w}, G_{3,0,n}^{w}, G_{3,1,n}^{w} \) are defined in (2.5). Furthermore, for all positive \( a_1, a_2 \) and \( w \in (0, 1) \), \( \{C_{k,l}\}_{k,l=-1,0,1} \) in (3.3) always satisfies the summation condition (3.2) for all \( \rho \in \mathbb{R} \), while \( \{C_{k,l}\}_{k,l=-1,0,1} \) in (3.5) satisfies the sign condition (3.1) if and only if \( \rho \) belongs to the following nonempty interval:
\[
\left[ \frac{-t_3 + t_9\alpha^2 + t_5\alpha}{s_1 + s_9\alpha^2 + s_3\alpha}, \frac{-t_6\alpha^2 + t_7\alpha}{s_4 + s_7\alpha^2 + s_5\alpha}, \frac{-t_8 + t_9\alpha^2 + t_{10}\alpha}{s_6 + s_7\alpha^2 + s_8\alpha} \right], \quad \min \left[ 0, -\frac{t_1\alpha^2 + t_2\alpha}{r_1\alpha^2 + r_5\alpha} \right].
\] (3.8)

For the range of the parameter \( \rho \) to achieve the M-matrix property, in fact, the nonempty interval in (3.8) always contains the following subintervals:

\[
\begin{cases}
[-0.2, -0.018], & \text{if } \alpha \in (0, 1) \text{ and } w \in (0, 1/2), \\
[-0.04, 0], & \text{if } \alpha \in (0, 1) \text{ and } w \in [1/2, 1), \\
0, & \text{if } \alpha \in [1, +\infty) \text{ and } w \in (0, 1/2), \\
[-0.1, -0.012], & \text{if } \alpha \in (1, +\infty) \text{ and } w \in [1/2, 1).
\end{cases}
\] (3.9)

In order to produce an overall scheme satisfying the M-matrix property, it is also necessary to lower the order of consistency for grid points near the interfaces intersection point \((\xi, \zeta)\). Such a scheme is given in the next theorem.

**Theorem 3.3.** Consider a grid point \((x_i, y_j) \not\in \Gamma \) such that \((x_i^*, y_j^*) := (\xi, \zeta) = (x_i - w_1 h, y_j - w_2 h)\) with \(0 < w_1, w_2 < 1\) (see the first panel of Fig. 7). Assume that \( u_p := u_{X_{\Omega_p}} \) and \( f_p := f_{X_{\Omega_p}} \) have uniformly continuous partial derivatives of (total) orders up to two and zero, respectively, in each \( \Omega_p \) for \( p = 1, 2, 3, 4 \). Also assume that the essentially one-dimensional functions \( \phi_p \) and \( \psi_p \) on the interface \( \Gamma_p \) have uniformly continuous derivatives of orders up to two and one, respectively, for \( p = 1, 2, 3, 4 \). Let \( L_h \) be the discrete operator in (2.1) with the stencil coefficients
\[
C_{-1,0} := -[a_1 a_2 r_2 + a_1 a_3 r_1]/\beta, \\
C_{0,-1} := -[a_1 a_4 r_4 + a_2 a_3 r_3]/\beta, \\
C_{0,1} := -[a_1 a_2 r_8 + a_1 a_3 r_7 + a_2^2 r_6 + a_2 a_3 r_5]/\beta, \\
C_{1,0} := -[a_1 a_2 r_{12} + a_1 a_3 r_{11} + a_2^2 r_{10} + a_2 a_3 r_9]/\beta, \\
C_{0,0} := 1, \\
C_{-1,-1} = C_{-1,1} = C_{1,0} = C_{1,1} := 0,
\]
where \( \beta := -(a_1 a_2 s_4 + a_1 a_3 s_3 + a_2^2 s_2 + a_2 a_3 s_1) > 0 \) and
\[
\begin{align*}
r_1 &:= 2w_2^2 - w_2 + 1, \\
r_2 &:= -2w_2^2 + w_2 + 1, \\
r_3 &:= -2w_1^2 + w_1 + 1, \\
r_4 &:= 2w_1^2 - w_1 + 1, \\
r_5 &:= -w_2(2w_2^2 - w_1 - 1), \\
r_6 &:= (w_2 - 1)(2w_2^2 - w_1 - 1), \\
r_7 &:= w_2(2w_2^2 - w_1 + 1), \\
r_8 &:= -(w_2 - 1)(2w_2^2 - w_1 + 1), \\
r_9 &:= -(2w_2^2 - w_2 - 1)(w_1 - 1), \\
r_{10} &:= (2w_2^2 - w_2 - 1)(w_1 - 1), \\
r_{11} &:= (2w_2^2 - w_2 + 1)w_1, \\
r_{12} &:= -(2w_2^2 - w_2 - 1)w_1, \\
s_1 &:= 2(w_2 - 1)w_1 w_2 + w_2^2 - w_2 w_1 + 2w_2^2 - 2, \\
s_2 &:= -2(w_1 w_2)(w_1 + w_2 + 1), \\
s_3 &:= -2(w_2 - 1)w_1^2 - 2(w_2 - w_2)w_1 - 2w_2^2 - 2, \\
s_4 &:= 2(w_2 - 1)(w_1^2 + w_1 w_2 + w_1 + 1).
\end{align*}
\] (3.11)

Then the compact 9-point finite difference scheme \( h^{-1}L_h u_h = h^{-1}F \) has a second order of consistency at the grid point \((x_i, y_j) \not\in \Gamma \) with \((\xi, \zeta) \in (x_i, y_j) + (-h, h)^2\), where
\[
F := \sum_{(m,n) \in \Lambda_2} F_2^{w_1,w_2} + \sum_{(m,n) \in \Lambda_0} F_2^{w_1,w_2} + \sum_{(m,n) \in \Lambda_2} \phi_1^{w_1,w_2} + \sum_{m=0}^2 \phi_1^{w_1,w_2} + \sum_{(m,n) \in \Lambda_2} \psi_2^{w_1,w_2} + \sum_{m=0}^1 \psi_2^{w_1,w_2}
\]
and \( F_{2,m,n}^{1,w_1,w_2}, F_{2,m,n}^{w_1,w_2}, \Phi_{2,m,n}^{1,w_1,w_2}, \Phi_{2,m,n}^{w_1,w_2}, \Psi_{2,m,n}^{1,w_1,w_2}, \Psi_{2,m,n}^{w_1,w_2} \) are defined in (6.26)–(6.29). Furthermore, \( \{C_{k,l}\}_{k,l=-1,0,1} \) in (3.10) satisfy both, the sign condition (3.1), and the summation condition (3.2) for all positive \( a_1, a_2, a_3, a_4 \) and \((w_1, w_2) \in (0, 1)^2\).

The convergence rate of the overall scheme, that satisfies the M-matrix property, is claimed in the next theorem.

**Theorem 3.4.** Consider the elliptic cross-interface problem in (1.1) and assume that the intersection point \((\xi, \zeta)\) is not a grid point. Assume that \( u_p := u\chi_{\Omega_p} \) and \( f_p := f\chi_{\Omega_p} \) have uniformly continuous partial derivatives of (total) orders up to four and two, respectively, in each \( \Omega_p \) for \( p = 1, 2, 3, 4 \). Also assume that the essentially one-dimensional functions \( \phi_p \) and \( \psi_p \) on the interface \( \Gamma_p \) have uniformly continuous derivatives of orders up to four and three respectively for \( p = 1, 2, 3, 4 \). Then the matrix of the linear system resulting from the compact 9-point scheme given by Theorems 2.1, 3.2 and 3.3 with \( \rho \) inside the nonempty interval in (3.8), including the corresponding modifications for the other parts of the interface \( \Gamma \), is an M-matrix. Consequently, under above assumptions, the scheme is third-order accurate, i.e., there exists a positive constant \( C \), independent of \( h \), such that:

\[
\|u - u_h\|_\infty \leq Ch^3,
\]  

where \( u \) is the exact solution of (1.1), and \( u_h \) is its numerical approximation.

### 4. Numerical experiments

Let us now choose \( \Omega = (0, 1)^2 \) and \( N_1 = N_2 := 2^J \) for some \( J \in \mathbb{N}_0 \). To quantify the accuracy of the various schemes in the numerical examples presented below we use the relative error in the \( l_2 \) norm: \( \epsilon_h := \frac{\|u_h - u\|_2}{\|u\|_2} \), where:

\[
\|u_h - u\|_2 := h^2 \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} ((u_h)_{i,j} - u(x_i, y_j))^2, \quad \|u\|_2 := h^2 \sum_{i=0}^{N_1} \sum_{j=0}^{N_2} (u(x_i, y_j))^2,
\]

as well as its infinity norm:

\[
\|u_h - u\|_\infty := \max_{0 \leq i \leq N_1, 0 \leq j \leq N_2} |(u_h)_{i,j} - u(x_i, y_j)|.
\]

The theoretical rate of convergence of the various schemes presented above is verified on a set of numerical examples.

**Table 1.** Performance in Examples 1 to 3 of the scheme given in Theorems 2.1 to 2.3 on uniform Cartesian meshes with \( h = 2^{-J} \). Note that \( \epsilon_h := \frac{\|u_h - u\|_2}{\|u\|_2} \).




| \( J \) | \( \epsilon_h \) | \( \|u_h - u\|_\infty \) | \( \|u_h - u\|_2 \) | \( \|u_h - u\|_\infty \) | \( \|u_h - u\|_2 \) | \( \|u_h - u\|_\infty \) | \( \|u_h - u\|_2 \) |
|---|---|---|---|---|---|---|---|
| 2 | 1.046E-03 | 1.537E-03 | 1.236E-05 | 2.146E-06 | 2.185E-02 | 2.185E-02 |
| 3 | 1.303E-05 | 6.3 | 2.225E-05 | 6.1 | 2.093E-07 | 5.9 | 3.930E-08 | 5.9 |
| 4 | 1.887E-07 | 6.1 | 3.396E-07 | 6.0 | 3.488E-09 | 5.9 | 6.340E-10 | 6.0 |
| 5 | 2.852E-09 | 6.0 | 5.286E-09 | 6.0 | 5.673E-11 | 5.9 | 1.027E-11 | 5.9 |
| 6 | 4.380E-11 | 6.0 | 8.216E-11 | 6.0 | 9.043E-13 | 6.0 | 1.598E-13 | 6.0 |
| 7 | 7.888E-13 | 5.8 | 1.424E-12 | 5.9 | 6.283E-08 | 6.0 | 6.283E-04 | 6.0 |
| 8 | 9.783E-10 | 6.0 | 1.598E-13 | 6.0 | 9.783E-06 | 6.0 |

4.1. Compact 9-point scheme in Theorems 2.1 to 2.3 (if \((\xi, \zeta)\) is a grid point):
Example 1. Let \( \Omega = (0, 1)^2 \) and \( (\xi, \zeta) = (1/2, 1/2) \). The functions in (1.1) are given by
\[
\begin{align*}
a_1 &= a\chi_{\Omega_1} = 10^{-5}, & a_2 &= a\chi_{\Omega_2} = 10^5, & a_3 &= a\chi_{\Omega_3} = 10^{-5}, & a_4 &= a\chi_{\Omega_4} = 10^5, \\
u_1 &= u\chi_{\Omega_1} = -\sin(2\pi x) \exp(-y) - \sin(2\pi (-y + 1)) \exp(-y), \\
u_2 &= u\chi_{\Omega_2} = -\sin(2\pi (-x + 1)) \exp(-y) - \sin(2\pi (-y + 1)) \exp(-y), \\
u_3 &= u\chi_{\Omega_3} = -\sin(2\pi (-x + 1)) \exp(-y) - \sin(2\pi y) \exp(-y), \\
u_4 &= u\chi_{\Omega_4} = -\sin(2\pi x) \exp(-y) - \sin(2\pi y) \exp(-y),
\end{align*}
\]
and \( g, f, \phi_p, \psi_p \) for \( p = 1, 2, 3, 4 \) in (1.1) are obtained by plugging the above functions into (1.1). Note that \( \phi_p = 0 \) for \( p = 1, 2, 3, 4 \). The numerical results are presented in Table 1 and Fig. 6.

|       |       |       |
|-------|-------|-------|
| \( a_1 \) | \( a_2 \) |       |
| \( 10^{-5} \) | \( 10^5 \) |       |
| \( a_3 \) | \( a_4 \) |       |
| \( 10^{-5} \) | \( 10^5 \) |       |

**Figure 6.** Example 1: the coefficient \( a(x, y) \) (left), the numerical solution \( (u_h)_{i,j} \) (middle), and \( |(u_h)_{i,j} - u(x_i, y_j)| \) (right), at all grid points \( (x_i, y_j) \) on \( \Omega \) with \( h = 2^{-7} \), for the scheme in Theorems 2.1 to 2.3.

Example 2. Let \( \Omega = (0, 1)^2 \) and \( (\xi, \zeta) = (1/2, 1/2) \). The functions in (1.1) are given by
\[
\begin{align*}
a_1 &= a\chi_{\Omega_1} = 10^7, & a_2 &= a\chi_{\Omega_2} = 10^{-3}, & a_3 &= a\chi_{\Omega_3} = 10^4, & a_4 &= a\chi_{\Omega_4} = 10^{-6}, \\
u_1 &= u\chi_{\Omega_1} = (x^3 + (1 - y)^3) \exp(-x + y), & u_2 &= u\chi_{\Omega_2} = ((1 - x)^3 + (1 - y)^3) \exp(-x + y), \\
u_3 &= u\chi_{\Omega_3} = ((1 - x)^3 + y^3) \exp(-x + y), & u_4 &= u\chi_{\Omega_4} = (x^3 + y^3) \exp(-x + y),
\end{align*}
\]
and \( g, f, \phi_p, \psi_p \) for \( p = 1, 2, 3, 4 \) in (1.1) are obtained by plugging the above functions into (1.1). Again, \( \phi_p = 0 \) for \( p = 1, 2, 3, 4 \). The numerical results are presented in Table 1 and Fig. 7.

|       |       |       |
|-------|-------|-------|
| \( a_1 \) | \( a_2 \) |       |
| \( 10^7 \) | \( 10^{-3} \) |       |
| \( a_3 \) | \( a_4 \) |       |
| \( 10^{-6} \) | \( 10^4 \) |       |

**Figure 7.** Example 2: the coefficient \( a(x, y) \) (left), the numerical solution \( (u_h)_{i,j} \) (middle), and \( |(u_h)_{i,j} - u(x_i, y_j)| \) (right), at all grid points \( (x_i, y_j) \) on \( \Omega \) with \( h = 2^{-6} \), for the scheme in Theorems 2.1 to 2.3.
Example 3. Let $\Omega = (0, 1)^2$ and $(\xi, \zeta) = (1/4, 1/8)$. The functions in (1.1) are given by

$$\begin{align*}
a_1 &= a\chi_{\Omega_1} = 10^{-4}, & a_2 &= a\chi_{\Omega_2} = 10^{-5}, & a_3 &= a\chi_{\Omega_3} = 2 \times 10^{-4}, & a_4 &= a\chi_{\Omega_4} = 10^6, \\
u_1 &= u\chi_{\Omega_1} = \sin(8\pi x)\sin(8\pi y)/a_1, & u_2 &= u\chi_{\Omega_2} = \sin(8\pi x)\sin(8\pi y)/a_2, \\
u_3 &= u\chi_{\Omega_3} = \sin(8\pi x)\sin(8\pi y)/a_3, & u_4 &= u\chi_{\Omega_4} = \sin(8\pi x)\sin(8\pi y)/a_4,
\end{align*}$$

and functions $g, f_p, \phi_p, \psi_p$ for $p = 1, 2, 3, 4$ in (1.1) are obtained by plugging the above functions into (1.1). Note that $\phi_p = 0$ for $p = 1, 2, 3, 4$. The numerical results are presented in Table 1 and Fig. 8.

![Figure 8](image.png)

**Figure 8.** Example 3: the coefficient $a(x, y)$ (left), $-(u_h)_{i,j}$ (middle), and $|u_h - u(x, y)|$ (right), at all grid points $(x_i, y_j)$ on $\Omega$ with $h = 2^{-8}$, for the scheme in Theorems 2.1 to 2.3.

**Table 2.** Performance in Examples 4 to 6 of the scheme in Theorems 2.1, 2.4 and 2.5 on uniform Cartesian meshes with $h = 2^{-J}$. Note that $\epsilon_h := \frac{\|u_h - u\|_2}{\|u\|_2}$.

| J  | $\epsilon_h$ | order $\|u_h - u\|_\infty$ order | $\epsilon_h$ | order $\|u_h - u\|_\infty$ order | $\epsilon_h$ | order $\|u_h - u\|_\infty$ order |
|----|--------------|-----------------------------|--------------|-----------------------------|--------------|-----------------------------|
| 3  | 4.1236E-02   | 4.022E-02                   | 5.236E-01    | 9.973E-01                   | 6.873E-05    | 5.4 1.943E+00               |
| 4  | 5.4082E-04   | 4.9 1.317E-03               | 2.057E-02    | 4.7 3.724E-02               | 1.960E-06    | 5.4 5.972E-02               |
| 5  | 6.9528E-06   | 5.4 3.036E-05               | 1.261E-03    | 4.0 2.335E-03               | 1.199E-07    | 3.7 3.84E-03                |
| 6  | 7.3116E-07   | 4.9 9.905E-07               | 5.273E-05    | 4.6 1.060E-04               | 3.058E-09    | 5.3 1.242E-04               |
| 7  | 8.1293E-08   | 4.6 3.660E-08               | 1.674E-06    | 5.0 3.207E-06               | 6.623E-11    | 5.5 4.067E-06               |

4.2. Compact 9-point scheme given in Theorems 2.1, 2.4 and 2.5 (if $(\xi, \zeta)$ is not a grid point):

**Example 4.** Let $\Omega = (0, 1)^2$ and $(\xi, \zeta) = (\pi/5, \pi/8)$. The functions in (1.1) are given by

$$\begin{align*}
a_1 &= a\chi_{\Omega_1} = 10^4, & a_2 &= a\chi_{\Omega_2} = 10^{-4}, & a_3 &= a\chi_{\Omega_3} = 10^4, & a_4 &= a\chi_{\Omega_4} = 10^{-4}, \\
u_1 &= u\chi_{\Omega_1} = \cos(5x)\cos(5y) = u_2 = u\chi_{\Omega_2} = \cos(12x)\exp(y - x), \\
u_3 &= u\chi_{\Omega_3} = \sin(5x)\cos(5y), & u_4 &= u\chi_{\Omega_4} = \sin(12y)\exp(-x - y),
\end{align*}$$

and $g, f_p, \phi_p, \psi_p$ for $p = 1, 2, 3, 4$ in (1.1) are obtained by plugging the above functions into (1.1). Note that $\phi_p \neq 0$ for $p = 1, 2, 3, 4$. The numerical results are presented in Table 2 and Fig. 9.

**Example 5.** Let $\Omega = (0, 1)^2$ and $(\xi, \zeta) = (\pi/4, \pi/10)$. The functions in (1.1) are given by

$$\begin{align*}
a_1 &= a\chi_{\Omega_1} = 10^4, & a_2 &= a\chi_{\Omega_2} = 10^{-6}, & a_3 &= a\chi_{\Omega_3} = 10^3, & a_4 &= a\chi_{\Omega_4} = 10^{-5}, \\
u_1 &= u\chi_{\Omega_1} = \sin(16y), & u_2 &= u\chi_{\Omega_2} = \cos(16x), & u_3 &= u\chi_{\Omega_3} = \sin(16y), & u_4 &= u\chi_{\Omega_4} = \cos(16x),
\end{align*}$$

and $g, f_p, \phi_p, \psi_p$ for $p = 1, 2, 3, 4$ in (1.1) are obtained by plugging the above functions into (1.1). Note that $\phi_p \neq 0$ for $p = 1, 2, 3, 4$. The numerical results are presented in Table 2 and Fig. 10.
Example 4. Let \( \Omega = (0,1)^2 \) and \((\xi, \zeta) = (\pi/6, \pi/8)\). The functions in (1.1) are given by
\[
\begin{align*}
a_1 &= a\chi_{\Omega_1} = 10^{-4}, \quad a_2 = a\chi_{\Omega_2} = 10^5, \quad a_3 = a\chi_{\Omega_3} = 10^{-4}, \quad a_4 = a\chi_{\Omega_4} = 10^6, \\
u_1 &= u\chi_{\Omega_1} = \sin(4(x+y))/a_1, \quad u_2 = u\chi_{\Omega_2} = \cos(2(x-y))/a_2, \\
u_3 &= u\chi_{\Omega_3} = \sin(4(x-y))/a_3, \quad u_4 = u\chi_{\Omega_4} = \cos(2(x+y))/a_4,
\end{align*}
\]
and \(g, f_p, \phi_p, \psi_p\) for \(p = 1, 2, 3, 4\) in (1.1) are obtained by plugging the above functions into (1.1). Note that \(\phi_p \neq 0\) for \(p = 1, 2, 3, 4\). The numerical results are presented in Table 2 and Fig. 11.
5. Conclusion

This paper is aimed at the development of compact finite difference schemes of high orders for the cross-interface problems in (1.1), under the assumptions (A1)-(A3). The main contributions of this paper can be summarized as follows:

(1) We derive compact 9-point finite difference schemes that are sixth-order accurate, if the internal interfaces coincide with some grid lines, and fourth-order accurate otherwise. The schemes use uniform meshes and all formulas of the schemes are constructed explicitly for all grid points which can be easily implemented.

(2) In case that the internal interfaces of coefficient jumps are matched by grid lines, the resulting scheme satisfies the M-matrix property and the discrete maximum principle, that allows us to prove that it has the sixth-order convergence rate. If the interfaces are not matched by grid lines, the resulting linear system does not satisfy the M-matrix property, and we were unable to theoretically prove its fourth/fifth-order convergence rate. The fourth/fifth-order convergence rate was verified only numerically.

(3) In the latter case, we derive a compact, 9-point, third-order accurate scheme satisfying the M-matrix property, and we prove its third-order convergence rate using the discrete maximum principle.

6. Proofs of Results Stated in Sections 2 and 3

\begin{align*}
(0, 0) & \quad (1, 0) \\
(0, 1) & \quad (1, 1) \\
(0, 2) & \quad (1, 2) \\
(0, 3) & \quad (1, 3) \\
(0, 4) & \quad (1, 4) \\
(0, 5) & \quad (1, 5) \\
(0, 6) & \quad (1, 6) \\
(0, 7) & \quad (2, 0) \quad (3, 0) \quad (4, 0) \quad (5, 0) \quad (6, 0) \quad (7, 0) \\
(0, 1) & \quad (1, 1) \\
(0, 2) & \quad (1, 2) \\
(0, 3) & \quad (1, 3) \\
(0, 4) & \quad (1, 4) \\
(0, 5) & \quad (1, 5) \\
(0, 6) & \quad (1, 6) \\
(0, 7) & \quad (2, 0) \quad (3, 0) \quad (4, 0) \quad (5, 0) \quad (6, 0) \quad (7, 0)
\end{align*}

\textbf{Figure 12.} Red trapezoid: \( \{(m, n) : (m, n) \in \Lambda_1^2 \} \). Blue trapezoid: \( \{(m, n) : (m, n) \in \Lambda_2^2 \} \). Note that \( \Lambda_1^2 \cup \Lambda_2^2 = \Lambda_7 := \{(m, n) \in \mathbb{N}_0^2 : m + n \leq 7 \} \).

For the proofs of the theorems in this paper we first need to establish some auxiliary identities about the solution \( u \) of the elliptic cross-interface problem in (1.1). Let \( (x_i^*, y_j^*) \in \Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \{(\xi, \zeta)\} \) such that \( x_i^* \in (x_i - h, x_i + h) \) and \( y_j^* \in (y_j - h, y_j + h) \). Recall that for \( p = 1, \ldots, 4 \),

\[
\begin{align*}
 u_p & := u_{\chi_{\Omega_p}} , \\
 f_p & := f_{\chi_{\Omega_p}} , \\
 u_p^{(m,n)} & := \frac{\partial^{m+n} u_p}{\partial x^m \partial y^n}(x_i^*, y_j^*) , \\
f_p^{(m,n)} & := \frac{\partial^{m+n} f_p}{\partial x^m \partial y^n}(x_i^*, y_j^*) .
\end{align*}
\]
As in [10,11], we can derive from \(-\nabla \cdot (a \nabla u) = f\) that

\[
u_p^{(m,n)} = (-1)^{\left\lfloor \frac{m}{2} \right\rfloor} u_p^{(\text{odd}(m),n-m-\text{odd}(m))} + \frac{1}{a_p} \sum_{\ell=1}^{[m/2]} (-1)^{\ell} f_p^{(m-2\ell,n+2\ell-2)}, \quad \forall (m,n) \in \Lambda^2_M, \tag{6.1}\]

\[
u_p^{(m,n)} = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} u_p^{(n+m-\text{odd}(n),\text{odd}(n))} + \frac{1}{a_p} \sum_{\ell=1}^{[n/2]} (-1)^{\ell} f_p^{(m+2\ell-2,n-2\ell)}, \quad \forall (n,m) \in \Lambda^2_M, \tag{6.2}\]

where the floor function \(|x|\) is defined to be the largest integer less than or equal to \(x \in \mathbb{R}\), and

\[
\text{odd}(m) := \begin{cases} 
0, & \text{if } m \text{ is even,} \\
1, & \text{if } m \text{ is odd.} 
\end{cases} \tag{6.3}\]

Similarly to [11, (2.5)-(2.10)] and [10, (2.9)-(2.13)], using the above identities, and using Taylor expansions at the point \((x_i^*, y_j^*)\), we obtain

\[
u_p(x + x_i^*, y + y_j^*) = \sum_{(m,n) \in \Lambda^1_M} u_p^{(m,n)} G_{M,m,n}(x,y) + \frac{1}{a_p} \sum_{(m,n) \in \Lambda^1_M} f_p^{(m,n)} H_{M,m,n}(x,y) + O(h^{M+1}), \tag{6.4}\]

\[
u_p(x + x_i^*, y + y_j^*) = \sum_{(n,m) \in \Lambda^1_M} u_p^{(m,n)} G_{M,m,n}(y,x) + \frac{1}{a_p} \sum_{(n,m) \in \Lambda^1_M} f_p^{(m,n)} H_{M,m,n}(y,x) + O(h^{M+1}), \tag{6.5}\]

for \((x,y) \in (-2h,2h)^2\), where the bivariate polynomials \(G\) and \(H\) are defined in (2.4).

From \([u] = \phi_1\) and \([a \nabla u \cdot \nabla] = \psi_1\) on \(\Gamma_1\) in [1,1], we obtain

\[
u^{(0,n)}_1 = \nu^{(0,n)}_2 = \phi_1^{(n)}, \quad \nu^{(1,n)}_1 = \frac{a_2}{a_1} \nu^{(1,n)}_2 - \frac{1}{a_1} \psi_1^{(n)}, \quad n = 0,1,2,\ldots, M. \tag{6.6}\]

The above identities will be frequently used in the following proofs.

**Proof of Theorem 2.2.** Note that \((x_i, y_j) = (x_i^*, y_j^*) \in \Gamma_1\). Since the discrete operator \(L_h u\) in (2.2) involves both \(u_1\) and \(u_2\) (see the first panel of Fig. 3), using jump conditions in (1.1) across the interface \(\Gamma_1\), we can replace all \(\{u_1^{(m,n)} : (m,n) \in \Lambda^1_M\}\) on the right-hand side of (6.3) with \(p = 1\) by \(\{u_2^{(m,n)} : (m,n) \in \Lambda^1_M\}\). More precisely, using identities in (6.5) to replace \(\{u_1^{(m,n)} : (m,n) \in \Lambda^1_M\}\)
in (6.3) with \( p = 1 \), we obtain
\[
u_1(x + x_i^*, y + y_j^*) = \sum_{n=0}^{M} u_1^{(0,n)} G_{M,0,n}(x, y) + \sum_{n=0}^{M-1} u_1^{(1,n)} G_{M,1,n}(x, y) + \sum_{(m,n) \in \Lambda_{M-2}} f_1^{(m,n)} H_{M,m,n}(x, y),
\]
\[
= \sum_{n=0}^{M} \left( u_2^{(0,n)} - \phi_1^{(n)} \right) G_{M,0,n}(x, y) + \sum_{n=0}^{M-1} \left( \frac{a_2}{a_1} u_2^{(1,n)} - \psi_1^{(n)} \right) G_{M,1,n}(x, y)
\]
\[
+ \frac{1}{a_1} \sum_{(m,n) \in \Lambda_{M-2}} f_1^{(m,n)} H_{M,m,n}(x, y), \tag{6.6}
\]
for \((x, y) \in (-2h, 2h)^2\). Using (6.3) with \( p = 2 \), we deduce from the definition of \( \mathcal{L}_h u \) in (2.2) that
\[
C_{-1,1} u_1(x_i - h, y_j + h) + C_{0,1} u_2(x_i, y_j + h) + C_{1,1} u_2(x_i + h, y_j + h)
\]
\[
\mathcal{L}_h u = + C_{-1,0} u_1(x_i - h, y_j) + C_{0,0} u_2(x_i, y_j) + C_{1,0} u_2(x_i + h, y_j)
\]
\[
+ C_{-1,1} u_1(x_i - h, y_j - h) + C_{0,1} u_2(x_i, y_j - h) + C_{1,1} u_2(x_i + h, y_j - h)
\]
\[
= \sum_{(m,n) \in \Lambda_M^1} u_2^{(m,n)} I_{m,n} + \frac{1}{a_1} \sum_{(m,n) \in \Lambda_{M-2}} f_1^{(m,n)} H_{M,m,n}^0 + \frac{1}{a_2} \sum_{(m,n) \in \Lambda_{M-2}} f_2^{(m,n)} H_{M,m,n}^0 \tag{6.7}
\]
\[
- \sum_{n=0}^{M-1} \phi_1^{(n)} G_{M,0,n} + \frac{1}{a_1} \sum_{n=0}^{M-1} \psi_1^{(n)} G_{M,1,n}^0 + \mathcal{O}(h^{M+1}), \quad \text{as} \quad h \to 0,
\]
where \( H_{M,m,n}^0, H_{M,m,n}^+, G_{M,m,n}^0 \) are defined in (2.5), and
\[
I_{m,n} := \left( \frac{a_2}{a_1} \right)^m \sum_{\ell=-1}^{1} C_{-1,\ell} G_{M,m,n}(-h, \ell h) + \sum_{k=0}^{1} \sum_{\ell=-1}^{1} C_{k,\ell} G_{M,m,n}(kh, \ell h). \tag{6.8}
\]
Then the conditions
\[
I_{m,n} = \mathcal{O}(h^{M+1}), \quad \text{for all} \quad (m,n) \in \Lambda_M^1, \tag{6.9}
\]
can be equivalently rewritten as a system of linear equations on the unknowns \( \{ C_{k,\ell} \}_{k,\ell=-1,0,1} \). By calculation, we observe that (6.9) has a nontrivial solution \( \{ C_{k,\ell} \}_{k,\ell=-1,0,1} \) if and only if \( M \leq 7 \). Moreover, for \( M = 7 \), up to a multiplicative constant for normalization, (2.6) is the unique solution to (6.9). Therefore, for the solution \( \{ C_{k,\ell} \}_{k,\ell=-1,0,1} \) in (2.6) with \( M = 7 \), we conclude that
\[
h^{-1} \mathcal{L}_h(u - u_h) = h^{-1} \sum_{(m,n) \in \Lambda_M^1} u_2^{(m,n)} I_{m,n} = \mathcal{O}(h^7), \quad h \to 0,
\]
which proves the seventh order of consistency of the compact finite difference scheme in Theorem 2.2.

Proof of Theorem 2.3. Note that \((x_i, y_j) = (x_i^*, y_j^*) = (\xi, \zeta)\). Since the discrete operator \( \mathcal{L}_h u \) in (2.2) involves \( u_p \) with \( p = 1, 2, 3, 4 \) (see the third panel of Fig. 3), using jump conditions in (1.1) across interfaces \( \Gamma_1, \Gamma_3 \) and \( \Gamma_4 \) (see the first panel of Fig. 13), we can replace all \( \{ u_3^{(m,n)} : (n,m) \in \Lambda_M^1 \} \) and \( \{ u_4^{(m,n)} : (n,m) \in \Lambda_M^1 \} \) on the right-hand side of (6.4) with \( p = 3, 4 \) by \( \{ u_2^{(m,n)} : (m,n) \in \Lambda_M^1 \} \) from the following (6.10)–(6.23). Then the compact 9-point scheme with a consistency order seven is derived from (6.24)–(6.33).
Similarly to (6.6), \([u] = \phi_p\) and \([a \nabla u \cdot \bar{n}] = \psi_p\) on \(\Gamma_p\) with \(p = 3, 4\) imply that:

\[
\begin{align*}
\tag{6.10}
u_3(x + x_1^*, y + y_2^*) &= \sum_{(m,n) \in \Lambda^1_M} u_2^{(m,n)} \left( \frac{a_2}{a_3} \right)^n G_{M,n,m}(y,x) - \sum_{m=0}^M \phi_3^{(m)} G_{M,0,m}(y,x) \\
&\quad - \frac{1}{a_3} \sum_{m=0}^{M-1} \psi_3^{(m)} G_{M,1,m}(y,x) + \frac{1}{a_3} \sum_{(m,n) \in \Lambda_{M-2}} f_3^{(m,n)} H_{M,n,m}(y,x) + O(h^{M+1}),
\end{align*}
\]

\[
\begin{align*}
\tag{6.11}
u_4(x + x_1^*, y + y_2^*) &= \sum_{(m,n) \in \Lambda^1_M} u_1^{(m,n)} \left( \frac{a_1}{a_4} \right)^n G_{M,n,m}(y,x) - \sum_{m=0}^M \phi_4^{(m)} G_{M,0,m}(y,x) \\
&\quad - \frac{1}{a_4} \sum_{m=0}^{M-1} \psi_4^{(m)} G_{M,1,m}(y,x) + \frac{1}{a_4} \sum_{(m,n) \in \Lambda_{M-2}} f_4^{(m,n)} H_{M,n,m}(y,x) + O(h^{M+1}),
\end{align*}
\]

for \(x, y \in (-2h, 2h)\). On the other hand, (6.2) and (6.5) lead to:

\[
\begin{align*}
\tag{6.12}
u_1^{(\text{odd}(m), n+\text{odd}(m))} &= \begin{cases} 
\frac{a_2}{a_1} u_2^{(0,m+n)} - \phi_1^{(m+n)}, & \text{if } m \text{ is even}, \\
\frac{a_2}{a_1} u_2^{(1,m+n-1)} - \frac{1}{a_1} \psi_1^{(m+n-1)}, & \text{if } m \text{ is odd},
\end{cases} & \text{for all } (m, n) \in \Lambda^2_M,
\end{align*}
\]

i.e.,

\[
\begin{align*}
\tag{6.13}
u_1^{(\text{odd}(m), n+\text{odd}(m))} &= \left( \frac{a_2}{a_1} \right)^{\text{odd}(m)} u_2^{(\text{odd}(m), n+\text{odd}(m))} - \text{odd}(m+1) \phi_1^{(m+n)} \\
&\quad - \text{odd}(m) \psi_1^{(m+n-1)}, & \text{for all } (m, n) \in \Lambda^2_M.
\end{align*}
\]

(6.13) implies that:

\[
\begin{align*}
\tag{6.14}
u_p^{(m,n)} &= (-1)^{\frac{m}{2}} \nu_p^{(\text{odd}(m), n+\text{odd}(m))} + \sum_{\ell=1}^{\lfloor m/2 \rfloor} (-1)^\ell \frac{1}{a_p} f_p^{(m-2\ell,n+2\ell-2)}, & \forall (m, n) \in \Lambda^2_M & \text{and } p = 1, 2.
\end{align*}
\]

From (6.12) and (6.13) with \(p = 1\), we observe that:

\[
\begin{align*}
\tag{6.15}
u_1^{(m,n)} &= (-1)^{\frac{m}{2}} \frac{a_2}{a_1} u_2^{(\text{odd}(m), n+\text{odd}(m))} - (-1)^{\frac{m}{2}} \nu_1^{(\text{odd}(m), n+\text{odd}(m))} \\
&\quad - (-1)^{\frac{m}{2}} \frac{\text{odd}(m)}{a_1} \psi_1^{(m+n-1)} + \frac{1}{a_1} \sum_{\ell=1}^{\lfloor m/2 \rfloor} (-1)^\ell f_1^{(m-2\ell,n+2\ell-2)}, & \text{for all } (m, n) \in \Lambda^2_M.
\end{align*}
\]

By (2.3), we have:

\[
\{ (m, n) : (m, n) \in \Lambda^1_M \} \setminus \{ (m, n) : (m, n) \in \Lambda^2_M \} = \{ (0, 0), (0, 1), (1, 0), (1, 1) \}. \tag{6.15}
\]

Note that for \(m = 0, 1\), the summation \(\sum_{\ell=1}^{\lfloor m/2 \rfloor} \) in (6.13) and (6.14) is empty. So (6.13) with \(p = 2\) and (6.15) result in:

\[
\begin{align*}
\tag{6.16}
u_2^{(m,n)} &= (-1)^{\frac{m}{2}} u_2^{(\text{odd}(m), n+\text{odd}(m))} + \frac{1}{a_2} \sum_{\ell=1}^{\lfloor m/2 \rfloor} (-1)^\ell f_2^{(m-2\ell,n+2\ell-2)}, & \forall (n, m) \in \Lambda^1_M.
\end{align*}
\]

From (6.5),

\[
\begin{align*}
\tag{6.17}
u_1^{(0,0)} &= u_2^{(0,0)} - \phi_1^{(0)}, & u_1^{(0,1)} &= u_2^{(0,1)} - \phi_1^{(1)}, & u_1^{(1,0)} &= \frac{a_2}{a_1} u_2^{(1,0)} - \frac{\psi_1^{(0)}}{a_1}, & u_1^{(1,1)} &= \frac{a_2}{a_1} u_2^{(1,1)} - \frac{\psi_1^{(1)}}{a_1}.
\end{align*}
\]
So (6.14), (6.15) and (6.17) lead to:

\[ \begin{align*}
  u_1^{(m,n)} &= (-1)^{\frac{m}{2}} \left( \frac{a_2}{a_1} \right)^{\text{odd}(m)} u_2^{(\text{odd}(m),m+n-\text{odd}(m))} - (-1)^{\frac{m}{2}} \text{odd}(m+1) \phi_1^{(m+n)} \\
  &= -(-1)^{\frac{m}{2}} \text{odd}(m) \psi_1^{(m+n-1)} + \frac{1}{a_1} \sum_{\ell=1}^{\lfloor m/2 \rfloor} (-1)\ell f_1^{(m-2\ell,n+2\ell-2)}, \quad \forall (n, m) \in \Lambda^1_M.
\end{align*} \]  

(6.18)

(6.10) and (6.16) imply that:

\[ \begin{align*}
  u_3(x + x_i^*, y + y_j^*) + O(h^{M+1}) \\
  &= \sum_{(n,m) \in \Lambda_M^1} (-1)^{\frac{m}{2}} \left( \frac{a_2}{a_1} \right)^{\text{odd}(m)} u_2^{(\text{odd}(m),m+n-\text{odd}(m))} \left( \frac{a_1^{2n}}{a_4} \right) G_{M,n,m}(y,x) \\
  &\quad - \frac{1}{a_3} \sum_{n,m} 1^n G_{M,1,m}(y,x) \\
  &\quad + \frac{1}{a_1} \sum_{(n,m) \in \Lambda_M^1} \sum_{\ell=1}^{\lfloor m/2 \rfloor} (-1)\ell f_1^{(m-2\ell,n+2\ell-2)} \left( \frac{a_1^{2n}}{a_4} \right) G_{M,n,m}(y,x) + \frac{1}{a_3} \sum_{n,m} 1^n H_{M,n,m}(y,x).
\end{align*} \]

(6.19)

(6.11) and (6.18) imply that:

\[ \begin{align*}
  u_4(x + x_i^*, y + y_j^*) + O(h^{M+1}) \\
  &= \sum_{(n,m) \in \Lambda_M^1} (-1)^{\frac{m}{2}} \left( \frac{a_2}{a_1} \right)^{\text{odd}(m)} u_2^{(\text{odd}(m),m+n-\text{odd}(m))} \left( \frac{a_1}{a_4} \right) G_{M,n,m}(y,x) \\
  &\quad - \frac{1}{a_3} \sum_{n,m} 1^n G_{M,1,m}(y,x) \\
  &\quad + \frac{1}{a_1} \sum_{(n,m) \in \Lambda_M^1} \sum_{\ell=1}^{\lfloor m/2 \rfloor} (-1)\ell f_1^{(m-2\ell,n+2\ell-2)} \left( \frac{a_1}{a_4} \right) G_{M,n,m}(y,x) + \frac{1}{a_3} \sum_{n,m} 1^n H_{M,n,m}(y,x).
\end{align*} \]

(6.20)

By (6.2) and the definition of \( \Lambda_M^1 \) in (2.3), we have:

\[ \begin{align*}
  &\sum_{(n,m) \in \Lambda_M^1} (-1)^{\frac{m}{2}} \left( \frac{a_2}{a_1} \right)^{\text{odd}(m)} u_2^{(\text{odd}(m),m+n-\text{odd}(m))} G_{M,n,m}(y,x) \\
  &= \sum_{v=0}^{\lfloor M/2 \rfloor} (-1)^v u_2^{(0,2v)} G_{M,0,2v}(y,x) + \sum_{v=0}^{\lfloor (M-1)/2 \rfloor} (-1)^v u_2^{(0,2v+1)} G_{M,1,2v}(y,x) \\
  &\quad + \frac{a_2}{a_1} \sum_{v=0}^{\lfloor (M+1)/2 \rfloor - 1} (-1)^v u_2^{(1,2v)} G_{M,0,2v+1}(y,x) + \frac{a_2}{a_1} \sum_{v=0}^{\lfloor M/2 \rfloor - 1} (-1)^v u_2^{(1,2v+1)} G_{M,1,2v+1}(y,x),
\end{align*} \]

\[ \begin{align*}
  &= \sum_{n=0}^{M} (-1)^{\frac{n}{2}} u_2^{(0,n)} G_{M,\text{odd}(n),n-\text{odd}(n)}(y,x) + \frac{a_2}{a_1} \sum_{n=0}^{M-1} (-1)^{\frac{n}{2}} u_2^{(1,n)} G_{M,\text{odd}(n),n+1-\text{odd}(n)}(y,x),
\end{align*} \]

i.e.:

\[ \begin{align*}
  &\sum_{(n,m) \in \Lambda_M^1} (-1)^{\frac{m}{2}} \left( \frac{a_2}{a_1} \right)^{\text{odd}(m)} u_2^{(\text{odd}(m),m+n-\text{odd}(m))} G_{M,n,m}(y,x) = \sum_{(m,n) \in \Lambda_M^1} u_2^{(m,n)} (-1)^{\frac{m}{2}} \left( \frac{a_2}{a_1} \right)^{m} G_{M,\text{odd}(n),z}(y,x),
\end{align*} \]

(6.21)

where \( z := n + m - \text{odd}(n) \). Then (6.19) and (6.21) imply that:

\[ \begin{align*}
  u_3(x + x_i^*, y + y_j^*) + O(h^{M+1}) \\
  &= \sum_{(m,n) \in \Lambda_M^1} u_2^{(m,n)} (-1)^{\frac{n}{2}} \left( \frac{a_2}{a_3} \right)^{\text{odd}(n)} G_{M,\text{odd}(n),z}(y,x) + \frac{1}{a_3} \sum_{n,m} f_3^{(m,n)} H_{M,n,m}(y,x) \\
  &\quad + \sum_{(m,n) \in \Lambda_M^1} \sum_{\ell=1}^{\lfloor m/2 \rfloor} (-1)^\ell f_2^{(m-2\ell,n+2\ell-2)} \frac{a_2^n}{a_3^n} G_{M,n,m}(y,x) - \sum_{m=0}^{M} \frac{a_2^m}{a_3} G_{M,0,m}(y,x) - \frac{M-1}{a_3} \psi^{(m)} G_{M,1,m}(y,x),
\end{align*} \]

(6.22)
(6.20) and (6.21) imply that:

$$u_4(x + x^*_i, y + y^*_j) + \mathcal{O}(h^{M+1})$$

$$= \sum_{(m,n) \in \Lambda_M^1} u_{2}^{(m,n)} |a_1|^{(m)}/|a_4|^{(n)} G_{M,odd(n),z}(y, x) + \frac{1}{a_4} \sum_{(m,n) \in \Lambda_{M-2}} f_4^{(m,n)} H_{M,m,n}(y, x)$$

$$- \sum_{(m,n) \in \Lambda_M^1} \left( \frac{\text{odd}(m+1)}{(-1)^{\frac{m}{2}}} \phi_1^{(m+n)} + \frac{\text{odd}(m)}{(-1)^{\frac{m}{2}}} \psi_1^{(m+n-1)} \right) \left( \frac{a_1}{a_4} \right)^n G_{M,m,n}(y, x) - \sum_{m=0}^{M} \phi_4^{(m)} G_{M,0,m}(y, x)$$

(6.23)

$$+ \frac{1}{a_4} \sum_{(m,n) \in \Lambda_M^1} \sum_{s=1}^{[m/2]} (-1)^s f_1^{(m-2s,n+2s-2)} \left( \frac{a_1}{a_4} \right)^n G_{M,m,n}(y, x) - \frac{1}{a_4} \sum_{m=0}^{M-1} \psi_4^{(m)} G_{M,1,m}(y, x),$$

for \((x, y) \in (-2h, 2h)^2\) and \(z := n + m - \text{odd}(n)\).

Now, by (6.3) with \(p = 2\), (6.6), (6.22), and (6.23) with \((x^*_i, y^*_j) = (x_i, y_j)\) and (2.2), we have (note that \(w_1 = w_2 = 0\) due to \((x^*_i, y^*_j) = (x_i, y_j)\), and see the third panel of Fig. 3 for an illustration):

$$- \mathcal{L}_h u = + C_{-1,1} u_1(x_i - h, y_j + h) + C_{0,1} u_2(x_i, y_j + h) + C_{1,1} u_2(x_i + h, y_j + h)$$

$$+ C_{-1,0} u_1(x_i - h, y_j) + C_{0,0} u_2(x_i, y_j) + C_{1,0} u_2(x_i + h, y_j)$$

$$+ C_{-1,-1} u_4(x_i - h, y_j - h) + C_{0,-1} u_3(x_i, y_j - h) + C_{1,-1} u_3(x_i + h, y_j - h)$$

$$= \sum_{(m,n) \in \Lambda_M^1} \psi_2^{(m,n)} I_{m,n} + \sum_{(n,m) \in \Lambda_M^1} F_1^{1,0,0} M,m,n + \sum_{(n,m) \in \Lambda_{M-2}} F_1^{0,0,0} M,m,n + \sum_{(n,m) \in \Lambda_M^1} \Phi_1^{1,0,0} M,m,n$$

(6.24)

where:

$$I_{m,n} := \left( \frac{a_2}{a_4} \right)^m G_{M,m,n}^{1,u_1,w_2} + G_{M,m,n}^{2,w_1,w_2} + (-1)^{\frac{m}{2}} \left( \frac{a_2}{a_3} \right)^{\text{odd}(n)} G_{M,z,odd(n)}^{3,w_1,w_2} + (-1)^{\frac{m}{2}} \left( \frac{a_2}{a_1} \right)^{\text{odd}(n)} G_{M,z,odd(n)}^{4,w_1,w_2},$$

(6.25)

$$F_1^{1,u_1,w_2} : = \frac{1}{a_2} \sum_{s=1}^{\lfloor m/2 \rfloor} (-1)^s f_2^{(m-2s,n+2s-2)} G_{M,m,n}^{3,w_1,w_2} + \frac{1}{a_1} \left( \frac{a_1}{a_4} \right)^n \sum_{s=1}^{\lfloor m/2 \rfloor} (-1)^s f_1^{(m-2s,n+2s-2)} G_{M,m,n}^{4,w_1,w_2},$$

(6.26)

$$F_1^{0,u_1,w_2} := \sum_{p=1}^{4} \frac{f_p^{(m,n)}}{a_p} H_{M,m,n}^{p,u_1,w_2},$$

(6.27)

$$\Phi_1^{1,u_1,w_2} := \frac{\phi_4^{(m+n+1)}}{(-1)^{\frac{m+n}{2}}} \left( \frac{a_1}{a_4} \right)^n G_{M,m,n}^{4,w_1,w_2}, \quad \Phi_1^{1,u_1,w_2} := - \frac{\psi_4^{(m+1)}}{a_1} G_{M,1,m}^{1,u_1,w_2},$$

(6.28)

$$\psi_1^{(m+n-1)} := \frac{\psi_4^{(m)}}{(-1)^{\frac{m}{2}}} \left( \frac{a_1}{a_4} \right)^n G_{M,m,n}^{4,w_1,w_2}, \quad \Psi_1^{1,u_1,w_2} := - \frac{\phi_4^{(m)}}{a_1} G_{M,1,m}^{1,u_1,w_2},$$

(6.29)
Note that $I_{m,n}$ in (6.25) satisfies

$$I_{m,n} = O(h^{M+1}), \quad \forall (m, n) \in \Lambda^1_M. \quad (6.32)$$

Then the conditions in (6.32) can be equivalently rewritten as a system of linear equations for the unknowns $\{C_{k,\ell}\}_{k,\ell=-1,0,1}$. (6.32) has a nontrivial solution $\{C_{k,\ell}\}_{k,\ell=-1,0,1}$ if and only if $M \leq 7$. Moreover, for $M = 7$, up to a multiplicative constant for normalization, (2.11) is the unique solution to (6.32). Therefore, for the solution $\{C_{k,\ell}\}_{k,\ell=-1,0,1}$ in (2.11) with $M = 7$, we conclude that:

$$h^{-1}L_h(u - u_h) = h^{-1} \sum_{(m,n) \in \Lambda^1_M} u_{2(m,n)}I_{m,n} = O(h^7), \quad h \to 0, \quad (6.33)$$

which proves the seventh order of consistency of the FDM in Theorem 2.3. \hfill \Box

**Proof of Theorem 2.4.** Since $(x_i, y_j) = (x_i^* + wh, y_j^*)$, we only need to replace $(x_i, y_j)$ by $(x_i^* + wh, y_j^*)$ on the left-hand side of (6.7). Then the rest proof is same as the proof in Theorem 2.2. \hfill \Box

**Proof of Theorem 2.5.** Note that $(x_i - w_1h, y_j - w_2h) = (x_i^* + wh, y_j^*) = (\xi, \zeta)$, and the discrete operator $L_h u$ in (2.2) involves $u_p$ with $p = 1, 2, 3, 4$ (see the first panel of Fig. 5). Since Theorem 2.3 does not use jump conditions in (1.1) across the interface $\Gamma_2$ (see the first panel of Fig. 13), if we only extend the derivation of the special case in Theorem 2.3 to the general case in Theorem 2.5, the compact 9-point scheme with a consistency order four could not be derived. In order to use jump conditions across the interface $\Gamma_2$, we derive $\tilde{u}_4$ in (6.36). More precisely, by jump conditions across interfaces $\Gamma_2$ and $\Gamma_3$ (see the second panel of Fig. 13), we can replace all $\{u_4^{(m,n)} : (m, n) \in \Lambda^1_M\}$ on the right-hand side of (6.3) with $p = 4$ by $\{u_2^{(m,n)} : (m, n) \in \Lambda^1_M\}$ from the following (6.34)–(6.36).

Then the compact 9-point scheme with a consistency order four is derived in the rest of the proof. Firstly, similarly to (6.11)–(6.23), we have:

$$\tilde{u}_4(x + x_i^*, y + y_j^*) + O(h^{M+1})$$

$$= \sum_{(m,n) \in \Lambda^1_M} (-1)^{\lfloor \frac{m}{2} \rfloor} \frac{a_2}{a_3} u_2^{(m+n-\text{odd}(n),\text{odd}(n))} \frac{a_4}{a_3} G_{M,m,n}(x,y) + \sum_{(m,n) \in \Lambda^1_{M-2}} \frac{f_4^{(m,n)}}{a_4} H_{M,m,n}(x,y)$$

$$- \sum_{(m,n) \in \Lambda^1_M} (-1)^{\lfloor \frac{m}{2} \rfloor} \frac{a_2}{a_3} G_{M,m,n}(x,y) \bigg\{ \sum_{\ell=1}^{\lfloor n/2 \rfloor} \frac{(n-2\ell-2)}{a_3} f_3^{(m+2\ell-2,n-2\ell)} \bigg( \frac{a_3}{a_4} \bigg)^m G_{M,m,n}(x,y) - \sum_{n=0}^{M-1} \phi^{(n)}_{2} G_{M,0,n}(x,y) - \sum_{n=0}^{M-1} \psi^{(n)}_{2} G_{M,1,n}(x,y) \bigg\} (6.34)$$

On the other hand:

$$u_2^{(m+n-\text{odd}(n),\text{odd}(n))} = \begin{cases} u_2^{(n,0)}, & \text{if } n \text{ is even and } m = 0, \\ u_2^{(n+1,0)}, & \text{if } n \text{ is even and } m = 1, \\ u_2^{(n-1,1)}, & \text{if } n \text{ is odd and } m = 0, \\ u_2^{(n,1)}, & \text{if } n \text{ is odd and } m = 1, \\ \end{cases} \quad \forall (m, n) \in \Lambda^1_M.$$ 

By (6.16), for any $(m, n) \in \Lambda^1_M$ we have:

$$u_2^{(m+n-\text{odd}(n),\text{odd}(n))} = \begin{cases} (-1)^{\lfloor \frac{m}{2} \rfloor} u_2^{(0,n)} + \sum_{\ell=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-2\ell-2)}{a_3} f_2^{(n-2\ell,2\ell-2)}), & \text{if } n \text{ is even and } m = 0, \\ (-1)^{\lfloor \frac{m}{2} \rfloor} u_2^{(1,n)} + \sum_{\ell=1}^{\lfloor \frac{n}{2} \rfloor} \frac{a_2}{a_3} f_2^{(n+1-2\ell,2\ell-2)}, & \text{if } n \text{ is even and } m = 1, \\ (-1)^{\lfloor \frac{m}{2} \rfloor} u_2^{(0,n)} + \sum_{\ell=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(n-1-2\ell,1+2\ell-2)}{a_3} f_2^{(n-1-2\ell,1+2\ell-2)}, & \text{if } n \text{ is odd and } m = 0, \\ (-1)^{\lfloor \frac{m}{2} \rfloor} u_2^{(1,n)} + \sum_{\ell=1}^{\lfloor \frac{n}{2} \rfloor} \frac{a_2}{a_3} f_2^{(n-1-2\ell,1+2\ell-2)}, & \text{if } n \text{ is odd and } m = 1. \\ \end{cases}$$

Now we observe that:

$$u_2^{(m+n-\text{odd}(n),\text{odd}(n))} = (-1)^{\lfloor \frac{m}{2} \rfloor} u_2^{(m,n)} + \frac{1}{a_2} \sum_{\ell=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\ell} f_2^{(q-2\ell,\text{odd}(n)+2\ell-2)}, \quad \forall (m, n) \in \Lambda^1_M, \quad (6.35)$$
with $q := n - (-1)^m \text{odd}(n + m)$. So (6.34) and (6.35) imply that:

$$
\tilde{u}_4(x + x^*_i, y + y^*_j) + \mathcal{O}(h^{M+1}) = \sum_{(m,n) \in \Lambda^1_M} (-1)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor} \left( \frac{a_2}{a_3} \right)^{\text{odd}(n)} \left( \frac{a_3}{a_4} \right)^m u^{(m,n)}_{2,M,m,n}(x,y) + \frac{1}{a_4} \sum_{(m,n) \in \Lambda_{M-2}} f^{(m,n)}_4 H_{M,m,n}(x,y) \\
+ \frac{1}{a_2} \sum_{(m,n) \in \Lambda^1_M} \left( \frac{a_2}{a_3} \right)^{\text{odd}(n)} \left( \frac{a_3}{a_4} \right)^m \sum_{\ell=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^{\ell + \lfloor \frac{\ell}{2} \rfloor} f^{(q-2\ell, \text{odd}(n)+2\ell-2)}_3 G_{M,m,n}(x,y) \\
- \sum_{(m,n) \in \Lambda^1_M} \left( \frac{\text{odd}(n+1)}{-1} \right)^{\frac{m}{2}} \phi_3^{(m+n)} + \frac{\text{odd}(n)}{a_3(-1)} \psi_3^{(m+n-1)} \left( \frac{a_3}{a_4} \right)^m G_{M,m,n}(x,y) - \sum_{m=0}^{M-1} \frac{\psi_2^{(m)}}{a_4} G_{M,1,m}(x,y).
$$

(6.36)

with $q := n - (-1)^m \text{odd}(n + m)$.

Now, by (2.2), (6.3) with $p = 2$, (6.6), (6.22), (6.23) and (6.36) with $\tilde{u}_4(x, y) = (x^*_i + w_1 h, y^*_j + w_2 h)$, we have that:

$$
\mathcal{L}_h u = \sum_{\ell = 0}^{\infty} c_{-1,\ell} u_1(x_i - h, y_j + \ell h) + \sum_{k=0}^{\infty} c_{k,\ell} u_2(x_i + kh, y_j + \ell h) + \sum_{k=0}^{\infty} c_{-1,-1} u_3(x_i + kh, y_j - h) \\
+ \sum_{k=0}^{\infty} c_{-1,-1} u_4(x_i - h, y_j - h) + \sum_{k=0}^{\infty} c_{-1,-1} \tilde{u}_4(x_i - h, y_j - h) = \sum_{(m,n) \in \Lambda^1_M} u^{(m,n)}_2 \tilde{l}_{m,n} + \sum_{(m,n) \in \Lambda^1_M} \tilde{F}^{1,w_1,w_2}_{M,m,n} \\
+ \sum_{(m,n) \in \Lambda_{M-2}} \tilde{F}^{1,w_1,w_2}_{M,m,n} + \sum_{(m,n) \in \Lambda^1_M} \tilde{\phi}^{1,w_1,w_2}_{M,m,n} + \sum_{m=0}^{M} \tilde{\phi}^{1,w_1,w_2}_{M,m,n} + \sum_{m=0}^{M-1} \tilde{\phi}^{1,w_1,w_2}_{M,m,n} + \sum_{m=0}^{M-1} \tilde{\psi}^{1,w_1,w_2}_{M,m,n} + \sum_{m=0}^{M-1} \tilde{\psi}^{1,w_1,w_2}_{M,m,n} + \mathcal{O}(h^{M+1}),
$$

as $h \to 0$, where:

$$
\tilde{l}_{m,n} := \tilde{l}_{m,n} + (-1)^{\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor} \left( \frac{a_2}{a_3} \right)^{\text{odd}(n)} \left( \frac{a_3}{a_4} \right)^m G^{4,w_1,w_2}_{M,m,n}, \quad \tilde{F}^{1,w_1,w_2}_{M,m,n} := f^{(m,n)}_4 \tilde{F}^{1,w_1,w_2}_{M,m,n} + \frac{f^{(m,n)}}{a_4} H^{4,w_1,w_2}_{M,m,n},
$$

(6.37)

$$
\tilde{\phi}^{1,w_1,w_2}_{M,m,n} := \tilde{\phi}^{1,w_1,w_2}_{M,m,n} + \frac{\phi_3^{(m+n+1)}}{-1} \left( \frac{a_3}{a_4} \right)^m G^{4,w_1,w_2}_{M,m,n}, \quad \tilde{\phi}^{1,w_1,w_2}_{M,m,n} := \tilde{\phi}^{1,w_1,w_2}_{M,m,n} - \frac{\phi_2^{(m)}}{a_4} G^{4,w_1,w_2}_{M,m,n},
$$

(6.38)

$$
\tilde{\psi}^{1,w_1,w_2}_{M,m,n} := \tilde{\psi}^{1,w_1,w_2}_{M,m,n} + \frac{\psi_3^{(m+n+1)}}{-1} \left( \frac{a_3}{a_4} \right)^m G^{4,w_1,w_2}_{M,m,n}, \quad \tilde{\psi}^{1,w_1,w_2}_{M,m,n} := \tilde{\psi}^{1,w_1,w_2}_{M,m,n} - \frac{\psi_2^{(m)}}{a_4} G^{4,w_1,w_2}_{M,m,n},
$$

(6.39)

$$
\tilde{G}^{4,w_1,w_2}_{M,m,n} := \tilde{c}_{-1,-1} G^{4,w_1,w_2}_{M,m,n}(w_1 - h, w_2 - h), \quad \tilde{H}^{4,w_1,w_2}_{M,m,n} := \tilde{c}_{-1,-1} H^{4,w_1,w_2}_{M,m,n}(w_1 - h, w_2 - h),
$$

(6.40)

$$
\text{for every } c_{k,\ell}, \tilde{c}_{-1,-1} \in \mathbb{R}, \quad q := n - (-1)^m \text{odd}(n + m), \quad \tilde{I}_{m,n} := \tilde{I}_{m,n}^{(k,\ell)} \quad \tilde{F}^{1,w_1,w_2}_{M,m,n}, \quad \tilde{F}^{1,w_1,w_2}_{M,m,n}, \quad \tilde{\phi}^{1,w_1,w_2}_{M,m,n}, \quad \tilde{\phi}^{1,w_1,w_2}_{M,m,n}, \quad \tilde{\psi}^{1,w_1,w_2}_{M,m,n}, \quad \tilde{\psi}^{1,w_1,w_2}_{M,m,n},
$$

are obtained by replacing $C_{k,\ell}$ by $C_{k,\ell}$ in (6.25)–(6.29) for $k, \ell = -1, 0, 1$. We consider:

$$
C_{k,\ell} := \begin{cases} 
\tilde{c}_{-1,-1} + \tilde{c}_{-1,-1}, & \text{if } k = \ell = -1, \\
0, & \text{if } k = \ell = 0, \\
\tilde{c}_{k,\ell}, & \text{otherwise},
\end{cases}
$$

(6.42)

and

$$
\tilde{I}_{m,n} = \mathcal{O}(h^{M+1}), \quad \text{for all } (m,n) \in \Lambda^1_M.
$$

(6.43)

where $\tilde{I}_{m,n}$ is defined in (6.37). Then the conditions in (6.43) can be equivalently rewritten as a system of linear equations on the unknowns $\{c_{k,\ell}\}_{k,\ell=-1,0,1} \cup \{\tilde{c}_{-1,-1}\}$. By calculation, we observe that (6.43) has a nontrivial solution $\{C_{k,\ell}\}_{k,\ell=-1,0,1}$ defined in (6.42) if and only if $M \leq 4$. Furthermore, we
observe that \( \{C_{k,\ell}\}_{k,\ell=-1,0,1} \) defined in (6.42) is uniquely determined by solving (6.43) with \( M = 4 \). The rest of the proof is similar as the proof of Theorem 2.3. □

Proof of Theorem 3.1. Clearly, all the \( \{C_{k,\ell}\}_{k,\ell=-1,0,1} \) in Theorems 2.1 to 2.3 satisfy the sign condition (3.1) and the summation condition (3.2). For simplicity, we assume \( \Omega := (0, 1)^2 \) and \( h := 1/N \) with \( N \in \mathbb{N} \). We define \( \Omega_h := \Omega \cap (h\mathbb{Z}^2) \), \( \partial \Omega_h := \partial \Omega \cap (h\mathbb{Z}^2) \), \( \mathcal{O}_h := \mathcal{O} \cap (h\mathbb{Z}^2) \), and \( (x_i, y_j) := (ih, jh) \).

So \( \mathcal{O}_h := \{(x_i, y_j) : 0 \leq i, j \leq N\} \) and we also define \( V(\mathcal{O}_h) := \{(v_{i,j}) : 0 \leq i, j \leq N\} \) with \( (v)_{i,j} \in \mathbb{R} \) for any \( v \in V(\mathcal{O}_h) \), \( (v)_{i,j} \) represents the value of \( v \) at the point \( (x_i, y_j) \).

Recall that \( \Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \{(\xi, \zeta)\} \), so we define that:

\[
(\Delta_h u_{h})_{i,j} := \begin{cases} -h^{-2} L_h u_{h}, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Omega \setminus \Gamma, \\ -h^{-1} L_h u_{h}, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Gamma, \end{cases}
\]

(6.44)

where \( L_h \) is defined in Theorem 2.1 for \( (x_i, y_j) \in \Omega \setminus \Gamma \), and \( L_h \) is defined in Theorems 2.2 and 2.3 for \( (x_i, y_j) \in \Gamma \). Therefore, using FDMs in Theorems 2.1 to 2.3, we find \( u_h \in V(\mathcal{O}_h) \) satisfying:

\[
\Delta_h u_h = F := \begin{cases} \frac{-1}{a(x_i, y_j)} F, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Omega \setminus \Gamma, \\ \frac{-h}{k} F, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Gamma, \end{cases}
\]

(6.45)

on \( \Omega_h \) with \( u_h = g \) on \( \partial \Omega_h \),

where \( F \) is the right-hand side of FDM in Theorems 2.1 to 2.3.

Using (3.1) and (3.2), we now prove the discrete maximum principle: for any \( v \in V(\mathcal{O}_h) \) satisfying \( \Delta_h v \geq 0 \) on \( \Omega_h \), we must have \( \max_{(x_i, y_j) \in \Omega_h} v(x_i, y_j) \leq \max_{(x_i, y_j) \in \partial \Omega_h} v(x_i, y_j) \), where \( \Delta_h \) is defined in (6.44).

Suppose that \( \max_{(x_i, y_j) \in \Omega_h} v(x_i, y_j) > \max_{(x_i, y_j) \in \partial \Omega_h} v(x_i, y_j) \). Take \( (x_m, y_n) \in \Omega_h \) where \( v \) achieves its maximum in \( \Omega_h \). Because all the stencils satisfying (3.1) and (3.2), we have:

\[
\sum_{k,\ell \in \{-1,0,1\}, k \neq 0, \ell \neq 0} -C_{k,\ell}(v)_{m+k,n+\ell} \leq C_{0,0}(v)_{m,n},
\]

By

\[
0 \leq h^s (\Delta_h v)_{m,n} = -C_{0,0}(v)_{m,n} - \sum_{k,\ell \in \{-1,0,1\}, k \neq 0, \ell \neq 0} C_{k,\ell}(v)_{m+k,n+\ell},
\]

where \( s = 1, 2 \), we have

\[
C_{0,0}(v)_{m,n} \leq \sum_{k,\ell \in \{-1,0,1\}, k \neq 0, \ell \neq 0} -C_{k,\ell}(v)_{m+k,n+\ell} \leq C_{0,0}(v)_{m,n}.
\]

Thus, equality holds throughout and \( v \) achieves its maximum at all its nearest neighbors of \( (x_m, y_n) \). Applying the same argument to the neighbors in \( \Omega_h \) and repeat this argument, we conclude that \( v \) must be a constant contradicting our assumption. This proves the discrete maximum principle.

Let \( U_h := \{u(x_i, y_j)\}_{(x_i,y_j) \in \mathcal{O}_h} \). By Theorems 2.1 to 2.3, we have: \( \Delta h U_h = \tilde{F} + R \), where:

\[
|R(x_i, y_j)| \leq \begin{cases} Ch^6, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Omega \setminus \Gamma, \\ Ch^7, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Gamma, \end{cases}
\]

(6.46)

where \( C > 0 \) is independent of \( h \). Define \( E_h := U_h - u_h \) on \( \mathcal{O}_h \). By (6.45) and \( \Delta h U_h = \tilde{F} + R \),

\[
\Delta_h E_h = \Delta_h U_h - \Delta_h u_h = R \text{ on } \Omega_h \text{ with } E_h = 0 \text{ on } \partial \Omega_h.
\]

(6.47)

By (3.1), (3.2) and (6.44), we have:

\[
\Delta h 1 = 0 \text{ on } \Omega_h.
\]

(6.48)
We define the comparison function \( \theta := \frac{1}{24} (x - \xi)^2 + \frac{1}{24} (y - \zeta)^2 \) and \( \Theta := \{ \theta(x_i, y_j) : 0 \leq i, j \leq N \} \). For \( 0 < \xi, \zeta < 1 \), we have \( 0 \leq \Theta \leq \frac{1}{12} \) on \([0, 1]^2\). By \( \{ C_{k, \ell} \}_{k, \ell = -1, 0, 1} \) in Theorems \([2.1] \) to \([2.3] \) \([2.1] \), and \([2.2] \), we have

\[
(L_\Theta)_{i,j} = \sum_{k=-1}^{1} \sum_{\ell=-1}^{1} C_{k, \ell} \theta(x_i + kh, y_j + \ell h) = \begin{cases} -h^2, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Omega \setminus \Gamma, \\
-h^2 a_1 + a_2, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Gamma_1, \\
-h^2 a_1 + a_3, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Gamma_2, \\
-h^2 a_3 + a_2, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Gamma_3, \\
-h^2 (a_1 + a_3)(a_2 + a_3) / 4a_2 & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Gamma_4.
\end{cases}
\]

(6.44) leads to

\[
(D_h \Theta)_{i,j} = \begin{cases} -h^{-2}(L_\Theta)_{i,j} = 1, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Omega \setminus \Gamma, \\
h^{-1}(L_\Theta)_{i,j} = \frac{1}{h} (1 + a_2 a_3) > \frac{1}{h}, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Gamma_1, \\
h^{-1}(L_\Theta)_{i,j} = \frac{1}{h} (1 + a_3 a_2) > \frac{1}{h}, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Gamma_2, \\
h^{-1}(L_\Theta)_{i,j} = \frac{1}{h} (1 + a_1 a_2) > \frac{1}{h}, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Gamma_3, \\
h^{-1}(L_\Theta)_{i,j} = \frac{1}{h} (1 + a_1 a_3) > \frac{1}{h}, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Gamma_4.
\end{cases}
\]

(6.50)

From \( C > 0 \) in (6.46), we observe that

\[
4C h^6 (D_h \Theta)_{i,j} \geq \begin{cases} Ch^6, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Omega \setminus \Gamma, \\
Ch^7, & \text{if } (x_i, y_j) \in \Omega_h \text{ and } (x_i, y_j) \in \Gamma.
\end{cases}
\]

(6.51)

Note that \( E_h := U_h - u_h \). We deduce that (6.47) implies

\[
(D_h (E_h + 4Ch^6 \Theta))_{i,j} = (D_h E_h)_{i,j} + 4Ch^6 (D_h \Theta)_{i,j} = R(x_i, y_j) + 4Ch^6 (D_h \Theta)_{i,j}, \text{ for } (x_i, y_j) \in \Omega_h.
\]

(6.52)

By (6.46), (6.51), and (6.52),

\[
(D_h (E_h + 4Ch^6 \Theta))_{i,j} \geq 0 \quad \text{for } (x_i, y_j) \in \Omega_h.
\]

By the discrete maximum principle of \( \Delta_h \) on \( \Omega_h, C > 0 \) in (6.46), \( E_h = 0 \) on \( \partial \Omega_h \), and \( 0 \leq \Theta \leq \frac{1}{12} \) on \( \Omega \), we obtain that

\[
\max_{(x_i, y_j) \in \Omega_h} (E_h)_{i,j} \leq \max_{(x_i, y_j) \in \Omega_h} (E_h + 4Ch^6 \Theta)_{i,j} \leq \max_{(x_i, y_j) \in \partial \Omega_h} (E_h + 4Ch^6 \Theta)_{i,j} \leq \max_{(x_i, y_j) \in \partial \Omega_h} (E_h)_{i,j} + 4Ch^6 \times \max_{(x_i, y_j) \in \partial \Omega_h} (\Theta)_{i,j} = \frac{C}{3} h^6.
\]

(6.53)

A similar argument can be applied to \(-E_h \). Hence, \( \| E_h \|_\infty \leq \frac{C}{3} h^6 \). Thus (3.3) is proved. Finally, the sign condition (3.1), the summation condition (3.2), and the Dirichlet boundary condition of (4.1) together imply the M-matrix property.

\[\Box\]

**Proof of Theorem 3.2.** The derivation (3.5) and (3.7) is straightforward by the proof of Theorem 2.4 with \( M = 3 \). The summation condition (3.2) for \( \{ C_{k, \ell} \}_{k, \ell = -1, 0, 1} \) in (3.5) for any \( \rho \in \mathbb{R} \) can be verified easily.

For the \( \{ C_{k, \ell} \}_{k, \ell = -1, 0, 1} \) in (3.5), we can check that all \( r_p \) in (3.6) satisfy \( r_p > 0 \) for \( p = 1, 2, \ldots, 5 \) and \( w \in (0, 1) \), all \( s_p \) in (3.6) satisfy \( s_p < 0 \) for \( p = 1, 2, \ldots, 8 \) and \( w \in (0, 1) \). So (3.5) satisfies the sign condition (3.1), and if only if \( \rho \in \mathbb{R} \) satisfies:

\[
\max \{ b_1, b_2, b_3 \} \leq \rho \leq \min \{ \overline{b}_1, \overline{b}_2 \},
\]

(6.54)
which is just
\[
\begin{align*}
\bar{b}_1 & := \frac{-t_3 + t_4 \alpha^2 + t_5 \alpha}{s_1 + s_2 \alpha^2 + s_3 \alpha}, & \bar{b}_2 & := \frac{-t_6 \alpha^2 + t_7 \alpha}{s_4 \alpha^2 + s_5 \alpha}, & \bar{b}_3 & := \frac{-t_8 + t_9 \alpha^2 + t_{10} \alpha}{s_6 + s_7 \alpha^2 + s_8 \alpha}, \\
\bar{b}_1 & := 0, & \bar{b}_2 & := -\frac{t_1 \alpha^2 + t_2 \alpha}{r_4 \alpha^2 + r_5 \alpha}.
\end{align*}
\]
We now show that the interval in (3.8) (i.e., the interval in (6.54)) is nonempty. In particular, by a direct calculation, we obtain that:
\[
\begin{cases}
\max\{\bar{b}_1, \bar{b}_2, \bar{b}_3\} < -0.2 & \text{and} & -0.018 < \min\{\bar{b}_1, \bar{b}_2\}, & \text{if } \alpha \in (0, 1) \text{ and } w \in (0, 1/2], \\
\max\{\bar{b}_1, \bar{b}_2, \bar{b}_3\} < -0.04 & \text{and} & \min\{\bar{b}_1, \bar{b}_2\} = 0, & \text{if } \alpha \in (0, 1) \text{ and } w \in [1/2, 1), \\
\max\{\bar{b}_1, \bar{b}_2, \bar{b}_3\} < 0 & \text{and} & \min\{\bar{b}_1, \bar{b}_2\} = 0, & \text{if } \alpha \in [1, +\infty) \text{ and } w \in (0, 1/2], \\
\max\{\bar{b}_1, \bar{b}_2, \bar{b}_3\} < -0.1 & \text{and} & -0.012 < \min\{\bar{b}_1, \bar{b}_2\}, & \text{if } \alpha \in [1, +\infty) \text{ and } w \in [1/2, 1).
\end{cases}
\]
Thus, the interval in (3.8) is nonempty and there exists \( \rho \in \mathbb{R} \) such that \( \{C_{k,\ell}\}_{k,\ell=-1,0,1} \) in (3.5) satisfies the sign condition (3.1) for any positive \( a_1, a_2 \) and \( w \in (0, 1) \).

**Proof of Theorem 3.3.** The derivation of \( \mathcal{L}_h u_h = F \) is straightforward by the proof of Theorem 2.3 with \( M = 2 \) and \( (x_i, y_j) = (x_i^+, w_1 h, y_j^+ + w_2 h) \). The summation condition (3.2) for \( \{C_{k,\ell}\}_{k,\ell=-1,0,1} \) in (3.10) can be verified easily.

For the \( \{C_{k,\ell}\}_{k,\ell=-1,0,1} \) in (3.10), we can check that all \( r_p \) in (3.11) satisfy \( r_p > 0 \) for \( p = 1, 2, \ldots, 12 \) and \( (w_1, w_2) \in (0, 1)^2 \), all \( s_p \) in (3.11) satisfy \( s_p < 0 \) for \( p = 1, 2, \ldots, 4 \) and \( (w_1, w_2) \in (0, 1)^2 \). Thus, \( \{C_{k,\ell}\}_{k,\ell=-1,0,1} \) in (3.10) satisfies the sign condition (3.1) for any positive \( a_1, a_2, a_3, a_4 \) and \( (w_1, w_2) \in (0, 1)^2 \).

**Proof of Theorem 3.4.** Recall that \( \Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \{(\xi, \zeta)\} \). Let \( \Omega_{h,\Gamma} := \Omega_{h,\Gamma,1} \cup \Omega_{h,\Gamma,2} \cup \Omega_{h,\Gamma,3} \cup \Omega_{h,\Gamma,4} \cup \Omega_{h,\Gamma,\zeta} \), where \( \Omega_{h,\Gamma,1} := \left\{(x_i, y_j) : (x_i^+, w_1 h, y_j^+ + w_2 h) \right\}, \Omega_{h,\Gamma,2} := \left\{(x_i, y_j) : (x_i, y_j^+) \in \Gamma_3 \cup \Gamma_4, 0 < w < 1 \right\}, \Omega_{h,\Gamma,3} := \left\{(x_i, y_j) : (x_i^+, w_1 h, y_j^+ + w_2 h) = (\xi, \zeta), 0 < w_1, w_2 < 1 \right\} \) and \( \Omega_{h,\Gamma,\zeta} := \left\{(x_i, y_j) : (x_i, y_j) \right\} \). Then (3.12) is obtained similarly by the proof of Theorem 3.1 by the following replacements:

Replace (6.44) by:
\[
(\Delta_h u_h)_{i,j} := \begin{cases} 
- h^{-2} \mathcal{L}_h u_h, & \text{if } (x_i, y_j) \in \Omega_h \setminus \Omega_{h,\Gamma}, \\
- h^{-1} \mathcal{L}_h u_h, & \text{if } (x_i, y_j) \in \Omega_{h,\Gamma,1} \cup \Omega_{h,\Gamma,2} \cup \Omega_{h,\Gamma,3} \cup \Omega_{h,\Gamma,4}, \\
- h^{-1} \mathcal{L}_h u_h, & \text{if } (x_i, y_j) \in \Omega_{h,\Gamma,\zeta},
\end{cases}
\]
where \( \mathcal{L}_h \) is defined in Theorem 2.1 for \( (x_i, y_j) \in \Omega_{h,\Gamma,1} \cup \Omega_{h,\Gamma,2} \cup \Omega_{h,\Gamma,3} \cup \Omega_{h,\Gamma,4} \), and \( \mathcal{L}_h \) is defined in Theorem 3.3 for \( (x_i, y_j) \in \Omega_{h,\Gamma,\zeta} \).
Replace (6.46) by:
\[
|R(x_i, y_j)| \leq \begin{cases} 
C h^3, & \text{if } (x_i, y_j) \in \Omega_h \setminus \Omega_{h,\Gamma}, \\
C h^2, & \text{if } (x_i, y_j) \in \Omega_{h,\Gamma,1} \cup \Omega_{h,\Gamma,2} \cup \Omega_{h,\Gamma,3} \cup \Omega_{h,\Gamma,4}, \\
C h^2, & \text{if } (x_i, y_j) \in \Omega_{h,\Gamma,\zeta}.
\end{cases}
\]
We define the comparison function \( \theta_p := \theta_{\chi_{\Omega_h}} \) for \( p = 1, 2, 3, 4 \), and choose \( \theta_1 = \frac{1}{24}((x-\xi)^2 + (y-\zeta)^2)^2 + 10 \), \( \theta_2 = \frac{1}{24}((x-\xi)^2 + (y-\zeta)^2)^2 + 8 \), \( \theta_3 = \frac{1}{24}((x-\xi)^2 + (y-\zeta)^2)^2 + 9 \), \( \theta_4 = \frac{1}{24}((x-\xi)^2 + (y-\zeta)^2)^2 + 15 \). For \( 0 < \xi, \zeta < 1 \), we have \( 0 \leq \Theta \leq \frac{1}{12} + 15 = \frac{181}{12} \) on \( [0, 1]^2 \). Similar as in (6.49) and (6.50), by \( \{C_{k,\ell}\}_{k,\ell=-1,0,1} \) in Theorem 3.2 we have
\[
(\Delta_h \Theta)_{i,j} = - h^{-1}(\mathcal{L}_h \Theta)_{i,j} = h^{-1} C_1 + h C_2,
\]
where
\[
C_1 = 4 \frac{-6((w + 1)\alpha^2 + (1 - w)\alpha) \rho + 3w \alpha^2 - 3(w - 1)\alpha}{r_2 \alpha^2 + r_3 \alpha + r_1}, \quad C_2 = \frac{(p_1 \alpha^2 + p_2 \alpha + p_3) \rho + q_1 \alpha^2 + q_2 \alpha + q_3}{2(r_2 \alpha^2 + r_3 \alpha + r_1)}.
\]
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completes the proof of Theorem 3.4. □

For any choice of coefficients and and
\[ E \quad \text{and} \quad s \quad \text{all negative for } 0 < w < 1. \]

Similar as (6.49) and (6.50), by (6.60), we have
\[ (\Delta_h \Theta)_{i,j} = -h^{-1}(L_h \Theta)_{i,j} = h^{-1}C_3 + hC_4; \]

where
\[ C_3 = \frac{-a_1a_3e_1 - 2a_1a_2r_2 - a_2a_3r_3}{a_1a_2s_1 + a_1a_3s_3 + a_2^2s_2 + a_2a_3s_1}, \quad C_4 = \frac{1}{12} \frac{-(a_2r_2 + a_3r_1)(a_1r_4 + a_2r_3)}{a_1a_2s_1 + a_1a_3s_3 + a_2^2s_2 + a_2a_3s_1}, \]

and \( s_1, s_2, s_3, s_4, r_1, r_2, r_3, r_4 \) are defined in (3.11), \( e_1 = 2w_1^2 + 4w_2^2 - w_1 - 2w_2 + 3 \), \( (x_i, y_j) \in \Omega_h \), and
\[ (x_i - w_1 h, y_j - w_2 h) = (\xi, \zeta), 0 < w_1, w_2 < 1. \]

Since \( r_1, r_2, r_3, r_4, e_1 \) are all positive and \( s_1, s_2, s_3, s_4 \) are all negative for \( 0 < w_1, w_2 < 1 \), we have that the coefficients \( C_3 \) and \( C_4 \) of \( h^{-1} \) and \( h \) in \( (\Delta_h \Theta)_{i,j} \) in (6.59) are both positive for \( 0 < w_1, w_2 < 1 \) if and only if
\[ \rho < \min \left\{ \frac{w\alpha + (1 - w)}{2(w + 1)\alpha + 2(1 - w)}, -\frac{q_1\alpha^2 + q_2\alpha + q_3}{p_1\alpha^2 + p_2\alpha + p_3} \right\}. \]

For any choice of \( \rho \) satisfying \( \rho \leq 0 \), it is straightforward to observe that (6.58) holds for all \( w \in (0, 1) \) and for all \( \alpha \in (0, \infty) \).

Similar as (6.49) and (6.50), by \( \{C_{k,\ell}\}_{k,\ell=-1,0,1} \) in Theorem 3.3, we have
\[ (\Delta_h \Theta)_{i,j} = -h^{-1}(L_h \Theta)_{i,j} = h^{-1}C_3 + hC_4, \]

where
\[ C_1 = C_4 \]

and \( C_3, C_4 > 0 \). Choose \( C_5 := \max\{C, \frac{C_1}{C_4}, \frac{C}{C_3}\} \), where \( C \) is the positive constant in (6.55).

Then we can replace (6.51) by:
\[ C_5 h^3 (\Delta_h \Theta)_{i,j} = \begin{cases} C_5 h^3 \geq Ch^3, & \text{if } (x_i, y_j) \in \Omega_h \setminus \Omega_{h,\Gamma}, \\ C_5 h^3 (h^{-1}C_1 + hC_2) \geq C_5 C h^2 \geq Ch^2, & \text{if } (x_i, y_j) \in \Omega_h, \quad (x_i, y_j) \in \Omega_{h,\Gamma_1,\Gamma_2} \cup \Omega_{h,\Gamma_3,\Gamma_4}, \\ C_5 h^3 (h^{-1}C_3 + hC_4) \geq C_5 C h^2 \geq Ch^2, & \text{if } (x_i, y_j) \in \Omega_{h,\xi,\zeta}. \end{cases} \]

Note that \( E_h := U_h - u_h \). So we can replace (6.52) by:
\[ (\Delta_h (E_h + C_5 h^3 \Theta))_{i,j} = (\Delta_h E_h)_{i,j} + C_5 h^3 (\Delta_h \Theta)_{i,j} \geq \begin{cases} R(x_i, y_j) + Ch^3 \geq 0, & \text{if } (x_i, y_j) \in \Omega_h \setminus \Omega_{h,\Gamma}, \\ R(x_i, y_j) + Ch^2 \geq 0, & \text{if } (x_i, y_j) \in \Omega_{h,\Gamma_1,\Gamma_2} \cup \Omega_{h,\Gamma_3,\Gamma_4}, \\ R(x_i, y_j) + Ch^2 \geq 0, & \text{if } (x_i, y_j) \in \Omega_{h,\xi,\zeta}. \end{cases} \]

Finally, replace (6.53) by:
\[ \max_{(x_i, y_j) \in \Omega_h} (E_h)_{i,j} \leq \max_{(x_i, y_j) \in \Omega_h} (E_h + C_5 h^3 \Theta)_{i,j} \leq \max_{(x_i, y_j) \in \Omega_h} (E_h + C_5 h^3 \Theta)_{i,j} \leq \frac{181}{12} C_5 h^3. \]

A similar argument can be applied to \( -E_h \). Hence, \( \|E_h\|_{\infty} \leq \frac{181}{12} C_5 h^3 \). Thus (3.12) is proved. Finally, (3.1), (3.2), and the Dirichlet boundary condition of (1.1) together imply the M-matrix property. This completes the proof of Theorem 3.4.
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