Many-body entanglement in a topological chiral ladder

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We find that the topological phase transition in a chiral ladder is characterized by dramatic signatures in many body entanglement entropy between the legs, close to half-filling. The value of entanglement entropy for various fillings close to half-filling is identical, at the critical point, but splays out on either side, thus showing a sharp signature at the transition point. A second signature is provided by the change in entanglement entropy when a particle is added (or subtracted) from half-filling which turns out to be exactly −log 2 in the trivial phase, but zero in the topological phase. A microscopic understanding of tendencies to form singlets along the rungs in the trivial phase, and along the diagonals in the topological phase, is afforded by a study of concurrence. At the topological phase transition the magnitude of the derivative of the average concurrence of all the rungs shows a sharp peak. Also, at the critical point, the average concurrence is the same for various fillings close to half-filling, but splays out on either side, just like entanglement entropy.

Topological states of matter [1, 2] have been at the centre of physics research in the last decade or so. One of the reasons for excitement has been the apparent simplicity of the models involved underneath which rich physics lies, and continues to be unearthed. Topologically significant states are often accompanied by the presence of ‘edge states’ with metallic properties while the bulk is gapped and insulating. It has been long realized that entanglement in the many-body ground state can be a useful diagnostic for topological order. While the scaling to leading order of entanglement entropy is governed by the famous ‘area law’ [3, 4], it is the subleading part that is linked with topological order and has now come to be known as topological entanglement entropy [5–7]. A finer tool, namely the entanglement spectrum has also been widely used [8–14].

In this Letter, we point out that entanglement in the many body ground state, when considered in a comparative study of various fillings close to half-filling may show a dramatic signature at a topological phase transition. We choose a specific system, namely a two leg chiral ladder, that has received a lot of attention in recent times [15–17], but not from an entanglement perspective. This simple system turns out to be rich with a Meissner to vortex phase transition, and in the presence of diagonal hopping, a trivial-to-topological phase transition. The study of entanglement in the many body and single particle ground states in this system offers useful fresh insights for not only the topological phase transition, but for the Meissner to vortex phase transition as well. In order to compute entanglement entropy in the many body ground states of these systems, we exploit the clever techniques of Peschel and co-workers [18, 19].

A further feature in our work is an investigation into the role of concurrence [20, 21], a measure of two-site entanglement, whose study allows us to track microscopic quantum correlations, and how they alter at the topological phase transition. Any study of entanglement involves the specification of subsystem and its complement; in the ladder system, a natural subsystem to work with is one of the legs. With such a choice, it is of interest to understand not only the entanglement content between the two full legs, but also to consider the entanglement in each rung separately, where concurrence proves to be handy.

The system consists of a two-leg ladder of non-interacting fermions subjected to a uniform magnetic flux ϕ per plaquette. The horizontal division for the computation of entanglement entropy, is delineated.

FIG. 1: The ladder system consists of two legs a and b with uniform magnetic flux ϕ per plaquette. The horizontal division for the computation of entanglement entropy, is delineated.

The Hamiltonian can be written as [15, 16, 22]

\[
H = -t \sum_\ell (e^{i\varphi} a_{\ell+1} a_\ell + e^{-i\varphi} b_{\ell+1} b_\ell) - t' \sum_\ell a_\ell^{\dagger} b_\ell
- t_d \sum_\ell (a_\ell^{\dagger} b_{\ell+1} + b_\ell^{\dagger} a_{\ell+1}) + H.c.,
\]

(1)

where the operator \( a_\ell(b_\ell) \) is the annihilation operator at site \( \ell \) in the right(left) leg of the ladder. The parameters \( t, t', t_d \) are the hopping amplitudes along the legs of the ladder, along the rungs of the ladder without magnetic field and along the diagonals of each plaquette, respectively, and \( L \) is the length of the ladder. It is useful to define \( \xi = \ell \sqrt{t/t'} \), \( \xi_d = \ell t / t_d \). Using an appropriate gauge, the magnetic field is absorbed into the hopping term \( (t \rightarrow te^{i\varphi}, \varphi = \phi/2) \) by Peierls substitution.

The ladder model shows a trivial-to-topological phase transition for any general \( \varphi \), which is signaled by a change
in winding number. As shown in Fig. 2, there is a change in winding number from 0 to 1 on increasing $\xi_d$ while keeping $\varphi$ constant. The Fourier transform of Eq. 1 can be cast into the general form of a $2 \times 2$ matrix in terms of Pauli matrices as $\mathcal{H}(k) = d_0 I + d_x \sigma_x + d_y \sigma_y + d_z \sigma_z$, where $d_0 = -\cos \varphi \cos k$, $d_x = -\xi - \xi_d \cos k$, $d_y = 0$, and $d_z = -\sin \varphi \sin k$. Here, only $d_x$ and $d_z$ contribute to the winding number calculation. Eliminating the Fermi sea, we have the trajectory

$$\left(\frac{d_x + \xi}{\xi_d}\right)^2 + \left(\frac{d_z}{\sin \varphi}\right)^2 = 1,$$

from which the winding number around the origin is computed (Fig. 2). So, there is a trivial-to-topological phase transition at $\xi = \xi_d$ for any $\varphi$; the only role of $\varphi$ is to alter the minor axis length in the winding number calculation.

The most widely used quantity to measure entanglement in a pure state of a bipartite system is entanglement entropy, which is nothing but the von Neumann entropy of the subsystem with respect to its complement. The entanglement entropy of the subsystem with respect to its complement is given in terms of the eigenvalues $C_i$ of the subsystem correlation matrix by

$$S = -\sum_i (C_i \ln C_i + (1 - C_i) \ln(1 - C_i)).$$

FIG. 2: The change in winding number with respect to change in $\xi_d$: (a) Trivial phase $\xi_d < \xi$, (b) the critical point $\xi_d = \xi$, and (c) the topological phase $\xi_d \geq \xi$.

FIG. 3: The subsystem entanglement entropy ($S_a$) for the horizontal cut (Fig. 1) as a function of $\xi_d$, close to half-filling. Number of rungs $L = 1000$, $\varphi = \frac{\pi}{2}$ and $\xi = 1$ with open boundary conditions (OBC) imposed. $S_a$ takes the same value for various fillings close to half-filling at the critical point. $S_a$ attains a minimum right after the topological phase transition ($\xi_d = 1$). The inset shows only the data for half-filling in an extended region.

Here we focus on the entanglement between the two legs of the ladder as shown in Fig. 1. The ladder model lends itself naturally to the horizontal division. Most studies of entanglement in chains (of the SSH model [23], for example) have looked at the vertical division, where a horizontal division does not exist. The supplementary section does contain a discussion of entanglement in the ladder model but with the other natural division, namely the vertical division. In the absence of diagonal hopping, when $\xi$ dominates, the rungs of the system tend to form singlets. Furthermore, in the limit where the legs hopping $t$ goes to zero, entanglement entropy goes to $N_p \log 2$, where $N_p$ is the number of particles, and $\log 2$ is the contribution from each singlet. As the legs hopping and the magnetic flux contribution along the legs of the ladder are turned on, a deviation from the value $N_p \log 2$ is seen, although it continues to be of this order of magnitude. Fig. 3 shows the variation of entanglement entropy as a function of diagonal hopping, in the vicinity...
of half-filling. In the trivial phase \((\xi_d < \xi)\), as the diagonal hopping is increased the singlets along the rungs are systematically weakened, and therefore the entanglement entropy decreases steadily. However, for \((\xi_d > \xi)\) edge states appear, and form singlets (evidence for this comes from a study of concurrence which appears later).

This causes the entanglement entropy to increase when \(\xi_d\) is increased in the topological phase. In addition, the large values of \(\xi_d\) cause singlets to be formed along the diagonals, which once again, in the limit of very large \(\xi_d\) yield a total entanglement entropy of \(N_p \log 2\), although from a different mechanism here. The topological phase transition is thus signalled by the entanglement entropy attaining a value independent of filling in the vicinity of half-filling. The entanglement entropy also attains a minimum soon after the topological phase is entered; this minimum seems to be directly correlated with the gap in the spectrum closing.

It is also insightful to study the entropy difference when a particle is either added or removed from the half-filled state:

\[
\Delta S = S_{hf+1} - S_{hf}. \quad (5)
\]

Fig. 4 shows that in the topological phase \(\Delta S\) goes to zero, whereas in the trivial phase, \(\Delta S\) is \(-\log 2\). The \(-\log 2\) difference would be expected in the limit of the legs hopping going to zero, because the half-filled state can then be thought of as \(N_p\) singlets. The removal or addition of one particle would then result in the destruction of entropy equal to that of one singlet. But, it is remarkable that this difference remains exactly \(-\log 2\), even in the presence of legs hopping and flux. In the topological phase, the limit of large but finite \(\xi_d\) is a useful reference. At half-filling, exactly one of the edge states is occupied, and this contributes zero to the entanglement, while the remaining electrons form singlets along the diagonals. When one particle is added to this state, it lands in the other edge, which also contributes nothing to the entanglement, and thus \(\Delta S\) would be zero. We see from Fig. 4 that this feature is exact throughout the topological phase for \(\varphi = \pi/2\). For other values of \(\varphi\) though, we see that as one approaches the critical point within the topological phase, the edge states do contribute to the entanglement, thus causing \(\Delta S\) to overshoot zero. This seems to be related to the edge states being not completely localized at the edges, when \(\varphi\) is decreased.

In order to acquire a finer understanding of the nature of the many body ground state wavefunctions, in various phases of the system, it is useful to study two-site entanglement. An excellent measure for this purpose is concurrence \([21, 24–28]\). One nice feature of concurrence is that for a number conserving Hamiltonian, the two-site concurrence is readily obtained, regardless of whether the density matrix is pure or mixed. This follows from the structure of the reduced density matrix for two sites \(i\) and \(j\) (with \(i < j\)), which can be written as

\[
\rho_{ij} = \begin{pmatrix} u_{ij} & 0 & 0 & 0 \\ 0 & w_{1ij} & z_{ij} & 0 \\ 0 & z_{ij} & w_{2ij} & 0 \\ 0 & 0 & 0 & v_{ij} \end{pmatrix}, \quad (6)
\]

where, \(u_{ij} = \langle (1-n_i)(1-n_j) \rangle, w_{1ij} = \langle (1-n_i)n_j \rangle, w_{2ij} = \langle n_i(1-n_j) \rangle, v_{ij} = \langle n_in_j \rangle\) and \(z_{ij} = \langle c_i^\dagger c_j \rangle\). The concurrence is then given by

\[
C = 2 \max(0, |z| - \sqrt{uv}). \quad (7)
\]

However, in the noninteracting framework, \(C\) can be directly calculated from the subsystem correlation matrix of the two sites. Employing Wick’s theorem one can decompose the four point correlators into two point correlators; the zero non zero elements of the reduced density matrix \(\rho_{ij}\) are then simplified in terms of the correlation matrix.

The study of concurrence between two sites is maximum and equal to unity when they form a singlet. For the ladder model, we can expect that the system at half-filling, has a tendency to form singlets in each rung when the rungs hopping is high and when the diagonal hopping is small. As the diagonal hopping is increased, the tendency to form singlets along the diagonals would be enhanced. The study of concurrence reinforces these expectations. Fig. 5(a) shows the concurrence between the two sites on a rung, averaged over all rungs, as a function of \(\xi_d\). Also included in the same figure is the concurrence between the sites on a diagonal, averaged over all diagonals of the ladder. It is seen that for large \(\xi_d\), the rungs concurrence drops to zero, whereas for small \(\xi_d\), the diagonal concurrence is zero. This type of a feature has been reported in the literature in the context of the SSH model \([24]\) and has been called ‘sudden-death of concurrence’ - it is a quantum information effect, and the value of \(\xi_d\) at which this change happens does not seem to have any significance for the phases of the model. Moreover the
A study of the concurrence between the edge sites is profitable for an investigation into the role of quantum correlations within the edge states in the topological phase. We notice that this shows a sharp change at the topological phase transition point $\xi_d = \xi$. For $\xi_d \geq \xi$, when the topological phase has just been entered, although the overall concurrence between rungs continues to decrease, we observe that the concurrence in the edge states increases in a brief range. This suggests that the edge states when they have just formed have singlet-like nature; however as the diagonal rungs are cranked up, this character steadily decreases as the diagonals become more and more singlet-like. The point at which the edge-state concurrence begins once again to decrease seems to be connected to the appearance of an enhanced density of states in the topological region, which is discussed in the supplementary section.

It is also illuminating to study concurrence in the rungs, for a range of fillings close to half-filling. The inset of Fig. 5(b) shows that at the topological phase transition, the average rungs concurrence, similar to entanglement entropy, also attains the same value for various fillings close to half-filling, and nicely splays out on either side of the topological phase transition. The derivative of concurrence shows a sharp feature at the topological phase transition. Concurrence as a diagnostic as a phase transition has been used in the SSH model [24], the Harper model [28], and in spin chains [29]. Fig. 5(c) looks at the dependence of rungs concurrence, as a function of filling, both in the trivial and the topological phases. We see that in the trivial phase, concurrence has a peak at half-filling, whereas this dramatically becomes a dip, as soon as the topological phase is entered. Furthermore, in the limit of very large diagonal hopping, this dip becomes a broad basin, close to half-filling. At the topological phase transition, it is an almost entirely smooth curve, except for a tiny peak at half-filling which comes from the edge states - we have verified that the corresponding model with periodic boundary conditions shows a completely smooth curve, indicating that the tiny peak must indeed be a consequence of the edge states.

To summarize, many body entanglement close to half-filling can provide dramatic signatures at a topological phase transition. We show this by considering the specific system of a chiral ladder. The entanglement entropy between the legs of the ladder is independent of filling, close to half-filling, if one is exactly at the topological critical point, whereas this independence is lost on either side of the transition. A similar feature is also shown by average concurrence in all the rungs of the ladder. The magnitude of the derivative of this concurrence has a dramatic peak at the transition point. Addition or subtraction of a particle at half-filling can lead to either a precise change of $-\log 2$ in entanglement entropy in the trivial phase, or no change in the topological phase, due to the presence of edge states. In this Letter, we have emphasized the usefulness of considering the entanglement entropy between the two legs of the ladder in the many-body eigenstates. A study of single-particle entanglement with the same division captures the Meissner to vortex phase transition; these details can be found in the supplementary section. The study of entanglement entropy in the many-body ground state, but with a vertical division of the subsystem provides further insights into the topological phase transition - this too can be found in the supplementary section. Our work opens up the question of how general these features of many-body entanglement are for topological phase transitions. Recent work [17] shows that an electric field in this system can lead to chiral Bloch oscillations. Whether this can give rise to special entanglement effects is worth investigating.

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I. LADDER HAMILTONIAN

Since the Hamiltonian is translationally invariant along the legs, its momentum space version reads

\[ H = 2t \sum_k c_k^\dagger \mathcal{H}(k) c_k, \]

where

\[ \mathcal{H}(k) = \begin{bmatrix} -\cos(\varphi - k) & -\xi - \xi_d \cos k \\ -\xi - \xi_d \cos k & -\cos(\varphi + k) \end{bmatrix}, \]

with \( c_k = [a_k^\dagger b_k^\dagger] \), \( c_k = [a_k b_k^\dagger] \), \( \xi = \frac{t_1}{2\pi}, \xi_d = \frac{t_d}{\pi} \). The dispersion consists of two bands with \( E_+(E_-) \) being the energies of higher(lower) energy band:

\[ E_{\pm}(k) = -\cos \varphi \cos k \pm \sqrt{(\xi + \xi_d \cos k)^2 + \sin^2 \varphi \sin^2 k}. \]

The corresponding eigenvectors are

\[ \Gamma_k^\dagger |0\rangle = \frac{1}{N} \left( (\xi + \xi_d \cos k)a_k^\dagger + Y b_k^\dagger \right) |0\rangle \]

\[ \Omega_k^\dagger |0\rangle = \frac{1}{N} \left( -Ya_k^\dagger + (\xi + \xi_d \cos k)b_k^\dagger \right) |0\rangle, \]

where \( \Gamma_k^\dagger, \Omega_k^\dagger \) are new creation operators for the lower\((E_-)\) and higher\((E_+)\) energy bands respectively, with \( Y = \sqrt{(\xi + \xi_d \cos k)^2 + \sin^2 \varphi \sin^2 k} \) and \( N = \sqrt{(\xi + \xi_d \cos k)^2 + Y^2} \) is the normalization constant.

II. THE VORTEX TO MEISSNER PHASE TRANSITION

Entanglement in the single-particle states provides useful signatures of the vortex-to-Meissner phase transition [1, 2], which will be described in this section.

A. No diagonal hopping

In the absence of diagonal hopping \((t_d = 0)\), there are two parameters: one is the uniform magnetic field and the other is \( \xi \). For a constant \( \xi \) and increasing \( \varphi \), the dispersion curve shows a phase transition from the Meissner to the vortex phase at a certain critical value of \( \varphi \). From the dispersion curves in Fig. 1, it can be seen that the lower energy band of the system possesses two minima at \( \pm k_y \) which become one minimum at \( k = 0 \), on decreasing \( \varphi \) below a critical value \( \varphi_c \). The critical value \( \varphi_c \) is obtained by minimizing the lower band energy \( E_-(k) \) with respect to \( k \), and demanding that \( k_y \neq 0 \). This yields

\[ \varphi_c = \cos^{-1} \left( -\frac{\xi + \sqrt{\xi^2 + 4}}{2} \right) \]

\[ \sin k_y = \pm \sqrt{\sin^2 \varphi_c - \cot^2 \varphi_c \xi^2}. \]

This phase transition is captured by other quantities like the chiral current [1]. The currents in the two legs are

\[ J_a = \sum_l J_{ai} = \frac{i}{2} \sum_l (e^{i\varphi} a_{l+1}^\dagger a_l - e^{-i\varphi} a_{l-1}^\dagger a_l) \]

\[ J_b = \sum_l J_{bi} = \frac{i}{2} \sum_l (e^{-i\varphi} b_{l+1}^\dagger b_l - e^{i\varphi} b_{l-1}^\dagger b_l). \]
(ξ) of the ladder for a constant magnetic field, as was reported earlier [1]. The chiral current initially increases with increase in ξ and beyond the critical point (ξc) it saturates indicating the vortex-to-Meissner phase transition.

Next we study the entanglement between the two legs of the ladder; in other words, we consider the horizontal division as shown in Fig.1 of the main paper. The vortex-to-Meissner transition is nicely captured at the single particle level. The two-point correlations restricted to the subsystem lattice sites yield the subsystem correlation matrix, from which the entanglement entropy can be computed. The subsystem correlation matrix pertaining to a is

$$ C_{a} = \left[ \frac{1}{L^2} \sum_{k} e^{-i(k(m-n))} (\xi + \xi d \cos k)^2 \right]_{L \times L}, \quad (10) $$

where the summation over k is made over all the occupied levels. For single particle ground states, it would just be one number. Fig. 3 shows subsystem entropy behaviour with change in ϕ and ξ.

Here, for small ϕ with constant ξ the system is in the maximally entangled state i.e. an entropy of log 2. This is because for small ϕ the system has a unique minimum for the lower energy band, which indicates that the particle could be on either of the legs of the ladder with equal probability (Eq.(4) with ξd = 0), which in turn means that the entropy is maximal. On increasing ϕ beyond ϕc, there are two degenerate minima for the lower energy band i.e. the particle is either on the upper leg (a) for positive momentum or on the lower leg (b) for negative momentum. Since the lack of information is minimal, this results in low entanglement entropy of the subsystem. Similar behaviour is shown with variation of ξ keeping ϕ constant. For ξ = 0 the two legs are disconnected, and therefore the entanglement entropy for subsystem a is zero. It increases on increasing ξ till ξc. Thereafter, it saturates because the minimum is unique and at k = 0 i.e. the maximally entangled state.

B. Diagonal hopping

When diagonal hopping $t_d$ is turned on, the ladder system is rich with multiple phases. First, there is still the vortex to Meissner phase transition. Once again, signatures for this transition are seen both in chiral current shown in Fig. 2(b), and in entanglement entropy for the same conditions as shown in Fig. 3(b). Diagonal hopping favours the Meissner phase as indicated by Eq. 4. It also contributes to making the probability of the particle being in either leg equiprobable in the Meissner phase, and thus making the ground state maximally entangled.

Diagonal hopping also induces a topological phase transition, which is the focus of the main paper. With open boundary conditions (OBC), the energy spectra for a constant ξ and a constant magnetic flux with varying ξd is shown in Fig. 4(a). A pair of zero energy states i.e. edge states appear for ξd ≥ ξ signalling a trivial-to-topological phase transition in the system. We observe that for $\phi = \frac{\pi}{2}$, the spectrum is symmetric about the zero energy modes. For other ϕ, the symmetry is broken; however, the topological phase transition always happens at the same point $\xi_a = \xi$, independent of ϕ. The case $\phi = \frac{\pi}{2}$ is special. Here, the unitary transformation $U_c = (\sigma_x + \sigma_y)/\sqrt{2}$ applied to eq. 2 yields a structure similar to that of the SSH model. One way to visualize the ladder model as a generalization of the SSH model is as follows. Consider a series of unit cells each consisting of two-sites (one of type ‘a’ and the other of type ‘b’) as shown in Fig. 5(a). If the neighboring unit cells are connected in such a way that only type ‘b’ site of one unit cell couples with type ‘a’ of the neighbouring cell, then the SSH model is obtained (Fig. 5(b)). On the other hand, if every site of a unit cell couples with every site of the neighboring cell, then the ladder model is obtained. The momentum space Hamiltonian under a suitable unitary transformation yielding a structure identical to that of the SSH model is a further consequence of the special case of $\phi = \frac{\pi}{2}$. In the SSH model, the wave functions for the two edge states are localized on the ends of the chain. Analogously, in the ladder model, edge states (Figs. 4(b) and 4(c)) are localized on the ends (first and last unit cells) i.e. ($a_1, b_1$) and ($a_L, b_L$) and decay exponentially in the bulk. Unlike edge states, any other typical wave
function of the system has some random distribution over all the sites as shown in Fig. 4(d).

Apart from the topological phase transition $\xi_d$ can also trigger the vortex to Meissner phase transition. Below a critical value $\xi_d \leq \xi_{dc}$ in the vortex phase [1], a dense region in the energy spectrum can be discerned. This is a signature of enhanced degeneracy which in turn is a consequence of the presence of two minima (maxima) in the lower (higher) energy band. Beyond $\xi_{dc}$, in the Meissner phase, energy density is diminished. The critical point $\xi_{dc}$ is the point at which the two minima merge into one, and can be analytically computed for general $\varphi$:

$$\xi_{dc} = \begin{cases} \frac{-(\xi + \cos \varphi) + \sqrt{(\xi - \cos \varphi)^2 + 4 \sin^2 \varphi}}{2}, & \xi \leq \frac{\sin^2 \varphi}{\cos \varphi} \\
0, & \xi \geq \frac{\sin^2 \varphi}{\cos \varphi} \end{cases}$$

Conversely, in terms of $\xi_d$ the critical point $\xi_c$ is given by

$$\xi_c = \begin{cases} \frac{\sin^2 \varphi - \xi_d^2 - \xi_d \cos \varphi}{\xi_d + \cos \varphi}, & \xi_d \leq -\frac{\cos \varphi + \sqrt{1 + 3 \sin^2 \varphi}}{2} \\
0, & \xi_d \geq -\frac{\cos \varphi + \sqrt{1 + 3 \sin^2 \varphi}}{2} \end{cases}$$

Eq. (12) shows that the critical point $\xi_c$ decreases on increasing diagonal hopping up to $\xi_d = 1$ as already suggested by a study of chiral current in Fig. 2(b). For $\xi_d \geq 1$, $\xi_c = 0$ because the relative critical value is never negative. A further look at Fig. 4(a) reveals that, there is a re-appearance of dense states in the topological region. This is a consequence of degeneracy due to the appearance of two maxima (minima) in the lower (higher) energy band. This is a new phase phase transition, different from vortex-Meissner, and appears to have not been reported earlier and has features of a van Hove singularity [3]. This new critical point is given by

$$\xi_{dc}' = \left(\frac{\xi + \sqrt{\xi^2 + 4}}{2}\right)_{\varphi = \frac{\pi}{2}}.$$  

We have computed this point, only for the case $\varphi = \frac{\pi}{2}$, where the bandstructure is symmetric (Fig. 4(a)). The maxima of the lower band appear at the zone boundary when $\xi \leq \xi_{dc}$. But for $\xi_d > \xi_{dc}'$ they start to move

![Fig. 4: (a) Energy spectra for ladder system with varying $\xi_d$. Squared amplitudes of the coefficients of wave function ($\psi_i^2$) for edge states are shown in (b) and (c), and for a typical bulk state in (d), with $\xi$ fixed at 1.5. The other parameters $\xi = 1$, number of rungs $L = 50$, $\varphi = \frac{\pi}{2}$ are common to all figures. The indices first run through 1 to $L$ among the ‘a’ sites, and then again 1 to $L$, through ‘b’ sites, hence the edge-states show signals at one edge and close to the centre of the figure, which is also an edge of the ladder.](image)

![Fig. 5: Visualizing the SSH model as being made up of two-site unit cells.](image)

![Fig. 6: Ladder system with vertical division of two subsystems.](image)

Another useful division for the purpose of studying entanglement entropy is to consider two subsystems obtained by cutting the ladder vertically as shown in Fig. 6. In the vertical division case, the entanglement entropy seems to be closely correlated with the band spectrum. For $\varphi = \frac{\pi}{2}$, the band spectrum is symmetric about the centre, as shown in Fig. 4(a). The band closes at $\xi_d = \xi$ and forms two degenerate edge modes when the boundary conditions are open.
When the boundary conditions are closed, the band gap closes at $\xi_d = \xi$, but the gap is non-zero on either side of the topological phase transition. From Fig. 7 and Fig. 8, we observe that at the topological phase transition point, the entanglement entropy takes on the same identical value for a range of fillings close to half-filling. Furthermore, with periodic boundary conditions, the entanglement entropy is maximum at the topological phase transition. The point where entanglement entropy attains a common value for a variety of fillings close to half-filling, and around which it splays out for different fillings, is thus a useful method to capture a topological phase transition. The magnitude of the entanglement here is much smaller than that obtained with horizontal division, because here it is only one (or two in the case of periodic boundary conditions) plaquette that contributes to the entanglement, whereas with horizontal division, every rung contributes, thus making the overall entanglement of the order of the number of particles $N_p$.

A further feature that entanglement entropy is sensitive to, is the enhanced degeneracy that appears in the band structure at the critical point described by Eq. 13. The sharp increase in density of states might be indicative of a van Hove singularity. A jump in entropy appears when the particle filling is half-filling minus three as shown in Fig. 7 and Fig. 8 both of which have this point at $\xi_d = 1.618$, in agreement with Eq. 13. The reason for this goes back to the argument we employed to reach Eq. 13. When particles are removed starting from half-filling (which results in only one of the two edge states being occupied), the first particle is removed from the edge state. The next two particles then come off from the maxima in the bands, and then there are four degenerate levels for the next particles. It is at this filling level, with the enhanced degeneracy, that entanglement entropy shows the sensitivity. Once again, calculating the entanglement entropy difference between half filled and one less than half filled captures the trivial to topological phase transition, as shown in Fig. 9. With open boundary conditions, for $(\xi_d < \xi)$, there is a finite value of $\Delta S = \log 2$ whereas after $(\xi_d = \xi)$ due to the degeneracy of the edge modes the entropy difference $\Delta S$ tends to zero. However, in the case of periodic boundary conditions, the gap is closed only at $\xi_d = 1$, hence the entropy difference goes to zero at that point alone. Once again, we see that this is a feature of the topological phase transition, and is thus completely independent of $\varphi$.

![Figure 7](image1.png)

**FIG. 7:** Subsystem entanglement entropy ($S_a$) with $\xi_d$ for $L = 1000$, $\xi = 1$, $\varphi = \frac{\pi}{2}$ with open boundary conditions.

![Figure 8](image2.png)

**FIG. 8:** Subsystem entanglement entropy ($S_a$) with $\xi_d$ for $L = 1000$, $\xi = 1$, $\varphi = \frac{\pi}{2}$ with periodic boundary conditions.

![Figure 9](image3.png)

**FIG. 9:** Subsystem entanglement entropy of ladder system ($\Delta S_a$) with $\xi_d$. (a)OBC, (b) PBC.

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