RESOLUTIONS OF MODULI SPACES
AND HOMOLOGICAL STABILITY

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Abstract. We describe partial semi-simplicial resolutions of several moduli spaces of interest in topology, geometry and algebra. These are: moduli spaces of finite sets, labelled configuration spaces of points in an open manifold, and, our primary motivation, moduli spaces of surfaces with tangential structure.

This allows us to prove homology stability results for all of these moduli spaces, which often improve the known stability ranges and give explicit stability ranges in many new cases. In each of these cases the stable homology can also be identified.

1. Introduction

Let \( \Sigma_g \) be an oriented surface of genus \( g \) with \( r \) boundary components, and \( \Gamma_{g,r} \) denote its mapping class group: the group of isotopy classes of diffeomorphisms. Over the last thirty years there has been an intense interest in the homological aspects of this family of groups, stemming principally from the rational homology equivalence \( B\Gamma_g \simeq \mathbb{Q} \text{M}_g \) from the classifying space of \( \Gamma_g \) to Riemann’s moduli space of complex curves of genus \( g \).

A fundamental contribution in this direction is due to Harer [11], in which, inspired by the formal similarity between the family of mapping class groups and families of arithmetic groups, he shows that these groups exhibit homological stability: each of the natural maps between the \( \Gamma_{g,r} \) induces a homology isomorphism in some range of degrees which tends to infinity with \( g \). This stability range was later improved by Ivanov [14], and extended to also deal with certain coefficient modules.

More recently, there has been a great deal of activity generalising and extending these results. Wahl [25] has extended the techniques of Harer and Ivanov to prove homological stability for the mapping class groups \( \mathcal{N}_{n,r} \) of non-orientable surfaces, and hence a Madsen–Weiss theorem for these. Cohen and Madsen [5, 6] defined certain moduli spaces \( S_{g,r}(X) \) of “surfaces \( \Sigma_g \) in a background space \( X \)” (which specialises to \( B\Gamma_{g,r} \) when \( X \) is a point) and used techniques of Ivanov to prove homological stability for these. Boldsen [3] has recently given a much improved stability range for the mapping class groups of oriente surfaces (and also the spaces of \( S_{g,r}(X) \)), following to some degree an unpublished manuscript of Harer [12] (although he is unable to verify every claim made in that manuscript and so his range is slightly worse than that claimed by [12]).
The purpose of this paper is to generalise the above results by working in the context of moduli spaces of surfaces with tangential structure, which we shortly explain, which encompasses all of the known spaces of interest (as well as the mapping class groups, which occur as fundamental groups of certain moduli spaces). We remark that studying these moduli spaces is akin to studying the full diffeomorphism group, not just the mapping class group, and that while some of our arguments one may translate and prove using only mapping class groups and discrete techniques, there are some (cf. [12]) that seem to require the full diffeomorphism group.

1.1. Moduli spaces of surfaces. A convention on notation for surfaces: we write $\Sigma^r_g$ for the orientable surface of genus $g$ with $r$ boundary components, and $\Sigma$ for an arbitrary orientable surface, possibly with boundary. We write $S^r_n$ for the non-orientable surface of genus $n$ with $r$ boundary components, and $S$ for an arbitrary non-orientable surface, possibly with boundary. We write $F$ for an arbitrary surface.

Let us give a model of the moduli spaces we have in mind. Let $\theta : X \to BO(2)$ be a Serre fibration, $\gamma_2 \to BO(2)$ be the canonical bundle, and $\theta^*\gamma_2 \to X$ be the bundle classified by $\theta$. Define $\text{Bun}(TF; \theta^*\gamma_2)$ to be the space of bundle maps $TF \to \theta^*\gamma_2$, i.e. fibrewise linear isomorphisms, in the compact-open topology. Given a bundle map $\delta : TF|_{\partial F} \to \theta^*\gamma_2$, which we call a boundary condition, define $\text{Bun}_\delta(TF; \theta^*\gamma_2; \delta)$ to be the space of bundle maps $TF \to \theta^*\gamma_2$ that restrict to the map $\delta$ on the boundary. Let $\text{Diff}_\delta(F)$ denote the group of diffeomorphisms of $F$ which restrict to the identity diffeomorphism on some neighbourhood of the boundary, and equip it with the $C^\infty$ topology.

**Definition 1.1.** The moduli space of surfaces of topological type $F$ with $\theta$-structure and boundary condition $\delta$ is the homotopy quotient

$$M^\theta(F; \delta) := \text{Bun}_\delta(TF; \theta^*\gamma_2; \delta) \sslash \text{Diff}_\delta(F).$$

If we define $E^\theta(F; \delta) := \text{Bun}_\delta(TF; \theta^*\gamma_2; \delta) \times F \sslash \text{Diff}_\delta(F)$, where the group acts diagonally, the moduli space carries a universal smooth $F$-bundle

$$F \to E^\theta(F; \delta) \to M^\theta(F; \delta)$$

equipped with a bundle map $T^eE^\theta(F; \delta) \to \theta^*\gamma_2$ from the vertical tangent bundle, satisfying appropriate boundary conditions. If we do not wish to introduce notation for a boundary condition, we may write $M^\theta(F)$ to denote $M^\theta(F; \delta)$ with an unspecified boundary condition $\delta$.

The examples of tangential structure we have in mind are: no structure at all $BO(2) \to BO(2)$, orientations $BSO(2) \to BO(2)$, Spin structures $BSpin(2) \to BO(2)$ and any of these together with a map to a background space, e.g. $BSO(2) \times Y \to BO(2)$. These have all been studied in the literature, but in a companion paper [23] we investigate some more exotic tangential structures that have not been studied before.
Gluing together surfaces with \( \theta \)-structure defines various stabilisation maps, which are generated by the stabilisation maps

\[
\alpha(g) : M^0(\Sigma_g^r) \to M^0(\Sigma_{g+1}^{r-1}) \\
\beta(g) : M^0(\Sigma_g^r) \to M^0(\Sigma_{g+1}^{r+1}) \\
\gamma(g) : M^0(\Sigma_g^r) \to M^0(\Sigma_{g+1}^{r+1})
\]

which glue on a pair of pants by the legs, a pair of pants by the waist, and a disc, respectively. A qualitative statement of our main theorem is:

**Theorem.** The homology groups \( H_*(M^0(\Sigma)) \) stabilise in every degree if and only if the zeroth homology groups \( H_0(M^0(\Sigma)) \) stabilise.

Our main results, Theorems 9.3 and 9.4, are quantitative and give homological stability ranges for the stabilisation maps, both for orientable and non-orientable surfaces. The statement of the full stability theorem is in \[3\] and is quite complicated, but here we give some of its most interesting corollaries for particular tangential structures.

### 1.2. Oriented surfaces.

Consider the tangential structure \( \theta : BSO(2) \to BO(2) \), and let us write \( M^+(\Sigma_g^r; \delta) \) for the corresponding moduli space of oriented surfaces. Note that there is a homotopy equivalence \( M^+(\Sigma_g^r; \delta) \simeq B\Gamma_{g,r} \) to the classifying space of the oriented mapping class group of \( \Sigma_g^r \). Then

(i) \( \alpha(g)_* : H_*(M^+(\Sigma_g^r)) \to H_*(M^+(\Sigma_g^{r-1})) \) is an epimorphism for \( 3* \leq 2g+1 \) and an isomorphism for \( 3* \leq 2g-2 \).

(ii) \( \beta(g)_* : H_*(M^+(\Sigma_g^r)) \to H_*(M^+(\Sigma_g^{r+1})) \) is an isomorphism for \( 3* \leq 2g \) and an epimorphism in all degrees.

(iii) \( \gamma(g)_* : H_*(M^+(\Sigma_g^{r+1})) \to H_*(M^+(\Sigma_g^r)) \) is an epimorphism for \( r \geq 0 \) it is an epimorphism in all degrees; for \( r = 0 \) it is an epimorphism for \( 3* \geq 2g+3 \).

This coincides with the stability range recently obtained by Boldsen [3], except that our range for closing the last boundary component is slightly better.

Cohen and Madsen [5] introduced certain moduli spaces of surfaces with maps to a background space \( X \), denoted \( S_{g,r}(X) \), and studied their homology stability when \( X \) is simply-connected. In our notation these are simply the spaces \( M^0(\Sigma_g^r; \delta) \) for \( \theta : BSO(2) \times X \to BO(2) \), with boundary condition that \( \partial \Sigma_g^r \) is mapped constantly to a basepoint in \( X \). When \( X \) is simply-connected our methods also show

(i) \( \alpha(g)_* : H_*(M^0(\Sigma_g^r)) \to H_*(M^0(\Sigma_{g+1}^{r-1})) \) is an epimorphism for \( 3* \leq 2g \) and an isomorphism for \( 3* \leq 2g-3 \).

(ii) \( \beta(g)_* : H_*(M^0(\Sigma_g^r)) \to H_*(M^0(\Sigma_{g+1}^{r+1})) \) is an epimorphism for \( 3* \leq 2g-1 \) and an isomorphism for \( 3* \leq 2g-4 \).

(iii) \( \gamma(g)_* : H_*(M^0(\Sigma_g^{r+1})) \to H_*(M^0(\Sigma_g^r)) \) is an isomorphism for \( 3* \leq 2g-1 \).

For \( r > 0 \) it is an isomorphism in all degrees; for \( r = 0 \) it is an epimorphism in degrees \( 3* \leq 2g+2 \).

As is now well known [16, 8, 5, 9], in both cases the stable homology coincides with that of the infinite loop space of a certain Thom spectrum. We will not discuss the stable homology in this paper.
1.3. **Faber’s conjecture.** Let $\Gamma_g$ be the mapping class group of $\Sigma_g$ and $M_g$ be the moduli space of smooth projective curves of genus $g$. Faber [7] has made a detailed conjecture concerning the structure of the tautological algebra $R^*(M_g) \subset A^*(M_g)$ inside the rational Chow algebra, which is the subalgebra generated by the Mumford–Morita–Miller classes $\kappa_i \in A^i(M_g)$. Part b) of his conjecture states that the classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate the tautological algebra, and that there are no relations between them in degrees $* \leq \lfloor g/3 \rfloor$.

One may also make this conjecture in cohomology instead of in the Chow algebra. There is an isomorphism $H^*(M_g; \mathbb{Q}) \cong H^*(\Gamma_g; \mathbb{Q})$ as long as $g \geq 2$, and these are topologically defined classes $\kappa_i \in H^{2i}(\Gamma_g; \mathbb{Q})$ which correspond under the above isomorphism to the projections of the $\kappa_i$ to the cohomology of $M_g$. Let us denote $R^*(\Gamma_g) \subset H^*(\Gamma_g; \mathbb{Q})$ the subalgebra generated by these classes.

Morita [19] has proved the generation part of the conjecture in cohomology: the algebra $R^*(\Gamma_g)$ is generated by the classes $\kappa_1, \ldots, \kappa_{[g/3]}$. Later, Ionel [13] proved the generation part of the conjecture in the Chow algebra. Below we prove the "no relations" part of the conjecture, which also follows from an unpublished stability range of Harer [12] and the work of Boldsen [3]. Because of the homotopy equivalences $M^+(\Sigma_g) \simeq B\Gamma_{g,r}$, our stability theorem can equally well be applied to the mapping class groups.

**Corollary 1.2.** There are no relations between the $\kappa_i$ in $H^{2*}(M_g; \mathbb{Q})$, and hence in $A^*(M_g)$, in degrees $* \leq \lfloor g/3 \rfloor$.

**Proof.** We work in $H^*(\Gamma_g; \mathbb{Q})$, and so double degrees. Note that all stabilisation maps between the $\Gamma_{g,r}$ induce epimorphisms between the tautological algebras $R^*(\Gamma_{g,r})$, as all stabilisation maps are natural for the $\kappa_i$. The homology epimorphism ranges given in the stability theorem dualise to cohomology monomorphism ranges. These are also ranges for monomorphisms, hence isomorphisms, between tautological algebras.

A relation between the $\kappa_i$ in $R^*(\Gamma_g)$ would also give such a relation in $R^*(\Gamma_{g,1})$. The stabilisation map $\Gamma_{g,1} \to \Gamma_{\infty}$ induces an epimorphism in homology, so an isomorphism on tautological algebras, in degrees $* \leq \lfloor 2g/3 \rfloor$. By the affirmed Mumford conjecture [20] [16], there are no relations between the $\kappa_i$ in $H^*(\Gamma_{\infty}; \mathbb{Q})$ so neither are there any in $H^*(\Gamma_{g,1}; \mathbb{Q})$ in degrees $* \leq \lfloor 2g/3 \rfloor$. The tautological algebra is concentrated in even degrees, so this range is equivalent to $* \leq 2\lfloor g/3 \rfloor$. \hfill $\Box$

1.4. **Unoriented surfaces.** Consider the tangential structure $\text{Id} : BO(2) \to BO(2)$, and let us write $M(S^r_n; \delta)$ for the corresponding moduli space of non-orientable surfaces. Note that there is a homotopy equivalence $M(S^r_n; \delta) \simeq BN_{n,r}$ to the classifying space of the unoriented mapping class group of $S^r_n$. In this case there is an extra stabilisation map

$$\mu(n) : M(S^r_n) \longrightarrow M(S^r_{n+1})$$

given by gluing on a Möbius band with a disc removed. Then

(i) $\alpha(n)_* : H_*(M(S^r_n)) \to H_*(M(S^r_{n-1}))$ is an epimorphism for $3* \leq n$ and an isomorphism for $3* \leq n - 3$.

(ii) $\beta(n)_* : H_*(M(S^r_n)) \to H_*(M(S^r_{n+1}))$ is an isomorphism for $3* \leq n - 1$ and a monomorphism in all degrees.
(iii) \( \gamma(n)_*: H_*(\mathcal{M}(S^r_n)) \to H_*(\mathcal{M}(S^r_{n+1})) \) is an isomorphism for \( 3* \leq n - 1 \). For \( r > 0 \) it is an epimorphism in all degrees; for \( r = 0 \) it is an epimorphism for \( 3* \leq n + 2 \).

(iv) \( \mu(n)_*: H_*(\mathcal{M}(S^r_n)) \to H_*(\mathcal{M}(S^r_{n+1})) \) is an epimorphism for \( 3* \leq n \) and an isomorphism for \( 3* \leq n - 3 \).

This stability range for non-orientable surfaces improves on the previously best known range, due to Wahl [25], which was of slope 1/4. In fact, there is a slightly better range, but it does not admit a pleasant description: it improves the range given here by one degree for certain values of \( n \) modulo 6; we discuss it in [9,3].

As in the oriented case we can also consider the tangential structure \( \theta: BO(2) \times X \to BO(2) \), so the spaces \( \mathcal{M}^\theta(S^r_n; \delta) \) are moduli spaces of non-orientable surfaces equipped with a map to the background space \( X \). When \( X \) is simply connected our methods also show

(i) \( \alpha(n)_*: H_*(\mathcal{M}^\theta(S^r_n)) \to H_*(\mathcal{M}^\theta(S^r_{n+1})) \) is an epimorphism for \( 3* \leq n - 1 \) and an isomorphism for \( 3* \leq n - 4 \).

(ii) \( \beta(n)_*: H_*(\mathcal{M}^\theta(S^r_n)) \to H_*(\mathcal{M}^\theta(S^r_{n+1})) \) is an epimorphism for \( 3* \leq n - 2 \) and an isomorphism for \( 3* \leq n - 5 \).

(iii) \( \gamma(n)_*: H_*(\mathcal{M}^\theta(S^r_{n+1})) \to H_*(\mathcal{M}^\theta(S^r_n)) \) is an isomorphism for \( 3* \leq n - 2 \).

For \( r > 0 \) it is an epimorphism in all degrees; for \( r = 0 \) it is an epimorphism for \( 3* \leq n + 1 \).

(iv) \( \mu(n)_*: H_*(\mathcal{M}^\theta(S^r_n)) \to H_*(\mathcal{M}^\theta(S^r_{n+1})) \) is an epimorphism for \( 3* \leq n - 1 \) and an isomorphism for \( 3* \leq n - 4 \).

1.5. Configuration spaces. As a warm up to our principal examples of moduli spaces of surfaces, we explain how the same methods may be used to obtain homology stability results for configuration spaces.

Let \( Y \) be a connected manifold that is the interior of a manifold with boundary \( \bar{Y} \), and \( X \) be a path connected topological space. Define \( C_n(Y; X) \) to be the configuration space of \( n \) unordered points in \( Y \) with labels in \( X \), topologised as \( (Y^n \setminus \Delta) \times \Sigma_n X^n \) where \( \Delta \) denotes the fat diagonal. For each point \( b \in \partial \bar{Y} \) there is a stabilisation map

\[ s_b : C_n(Y; X) \to C_{n+1}(Y; X) \]

which adds a new point near \( b \).

Our principal result concerning labelled configuration spaces is then as follows, which is proved as Theorem 4.3.

**Theorem.** Let \( Y \) be a connected smooth manifold which is the interior of a manifold with boundary \( \bar{Y} \), and has dimension greater than 1. The map \( s_b : C_n(Y; X) \to C_{n+1}(Y; X) \) is a homology isomorphism in degrees \( 2* \leq n \).

Many cases of this theorem have already appeared in the literature. In particular the work of McDuff [18] and Segal [24 Proposition A.1] treat the case of arbitrary \( Y \) and \( X = * \), and the work of Lehrer–Segal [15] Theorem 3.2] treats the case \( Y = \mathbb{R}^d \) and arbitrary \( X \), although their methods are not able to give an explicit range. Earlier work of ArnoI’d [1] and F. Cohen [4] is related to the case when \( Y = \mathbb{R}^d \).

1.6. The resolution point of view. In many moduli-theoretic situations the following general scheme applies, which we illustrate with an example. Let \( \mathcal{M}_n \)
be the space of configurations of \( n \) unordered points in \( \mathbb{R}^\infty \), and let \( \mathcal{M}_n^1 \) be the space of \( n \) unordered points in \( \mathbb{R}^\infty \) \textit{with a distinguished point}. One can ask how far the map \( \mathcal{M}_n^1 \to \mathcal{M}_n \), which forgets that a point is distinguished, is from being a homotopy equivalence, and study the problem from the following point of view: map a \( k \)-dimensional manifold \( N^k \) into \( \mathcal{M}_n \), and try to lift this map to \( \mathcal{M}_n^1 \).

Certainly we can solve this lifting problem locally in \( N \): a map \( f : N \to \mathcal{M}_n \) classifies a \( n \)-fold covering space \( \tilde{N} \to N \) (along with an embedding \( \tilde{N} \to N \times \mathbb{R}^\infty \), but this contains no homotopical information and we will ignore it), and a map \( \bar{f} : N \to \mathcal{M}_n^1 \) classifies a \( n \)-fold covering space with section, and sections of \( \tilde{N} \to N \) certainly exist locally in \( N \). Thus we may find an (ordered, good) open cover \( U_\alpha \) of \( N \) and sections \( s_\alpha \) of \( \tilde{N}|_{U_\alpha} \to U_\alpha \). Now we may try to fit these sections together to form a global section over \( N \). Over intersections \( U_\alpha \cap U_\beta \) the sections \( s_\alpha, s_\beta \) may agree, but most probably do not. Thus over this intersection we have \( \text{two} \) ordered distinguished points in each fibre, so most naturally obtain a map into \( \mathcal{M}_n^2 \), the space of \( n \) unordered points in \( \mathbb{R}^\infty \) with two ordered distinguished points.

Continuing in the obvious way, we obtain a lift not to \( \mathcal{M}_n^1 \) but to the realisation of a semi-simplicial space

\[
\| \cdots \mathcal{M}_n^{2 \cdot} \| = \mathcal{M}_n^1 \to \mathcal{M}_n.
\]

If we try to solve the \textit{relative} lifting problem for this map, we find that we can do so for \( k \leq n - 2 \) (c.f. Proposition 3.2, where we show that the homotopy fibre of this map is in fact a wedge of \( (n - 1) \)-spheres). Thus within a certain range of dimensions the realisation of this semi-simplicial space is an adequate substitute for the space \( \mathcal{M}_n \), but it has the benefit of being constructed from the simpler spaces \( \mathcal{M}_n^{i+1} \simeq \mathcal{M}_{n-i-1}, i \geq 0 \). In particular the homology of the semi-simplicial space \( \| \mathcal{M}_n^{**} \| \) is obtained by means of a spectral sequence from the homology of the spaces \( \mathcal{M}_{n-i-1} \): this allows one to prove certain homological results about the spaces \( \mathcal{M}_n \) by induction on \( n \). The purpose of this article is to put this scheme into practice in the situations described above.

1.7. Outline. In §2 we give some standard notions concerning semi-simplicial spaces and the spectral sequence coming from their skeletal filtration. In §3 we give the full details of the above example, and in §4 we extend it to the spaces \( C_n(Y; X) \), configuration spaces of \( n \) points in an open manifold \( Y \) with labels in a path connected space \( X \). These two sections are logically independent of the remaining sections, but we feel they motivate many of the later techniques. In §5–§6 we study our main example, the moduli spaces \( \mathcal{M}^\theta(F) \) of surfaces of the topological type of \( F \), equipped with a \( \theta \)-structure. In the first few sections we give definitions of the resolutions we shall use, and in §9 we give the full statements of main homological stability theorems for these spaces, when the underlying surface has non-empty boundary, and discuss examples. In §10 and §11 we provide the proofs of the claims made in §9. In §12 we discuss stability for closing the last boundary component, which is slightly more subtle than the previous case, and discuss examples. In Appendix A we deduce the connectivities of certain complexes of arcs in surfaces, starting from results of Harer [11] and Wahl [25], which are necessary to prove the results of §9. This is included as an appendix as it may be of interest independent of the body of the article. In Appendix B we introduce an elementary construction in semi-simplicial spaces.
which occurs in several places in this theory, and give a criterion for it to have a highly connected realisation.

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2. Semi-simplicial spaces and resolutions

Let $\Delta^{op}$ denote the simplicial category, that is, the opposite of the category $\Delta$ having the finite ordered sets $[n] = \{0, 1, ..., n\}$ and morphisms the weakly monotone maps. A simplicial object in a category $C$ is a functor $X_\cdot : \Delta^{op} \to C$. Let $\Lambda \subset \Delta$ be the subcategory having all objects but only the strictly monotone maps. Call $\Lambda^{op}$ the semi-simplicial category and a functor $X_\cdot : \Lambda^{op} \to C$ a semi-simplicial object in $C$. A (semi-)simplicial map $f : X_\cdot \to Y_\cdot$ is a natural transformation of functors: in particular, it has components $f_n : X_n \to Y_n$.

The geometric realisation of a semi-simplicial space $X_\cdot$ is

$$\|X_\cdot\| = \bigsqcup_{n \geq 0} X_n \times \Delta^n / \sim$$

where the equivalence relation is $(d_i(x), y) \sim (x, d^i(y))$, for $d^i : \Delta^0 \to \Delta^{0+1}$ the inclusion of the $i$-th face. Note that there is a homeomorphism

$$\|X_\cdot\| \cong \hocolim_{\Lambda^{op}} X_\cdot$$

where the homotopy colimit is taken in the category of unpointed topological spaces.

If $X_\cdot$ is a semi-simplicial pointed space, its realisation as a pointed space is

$$\|X_\cdot\|_\ast = \bigvee_{n \geq 0} X_n \times \Delta^n / \sim$$

where $d_i(x) \times y \sim x \times d^i(y)$. Recall that the half smash product of a space $Y$ and a pointed space $C$ is the pointed space $C \times Y := C \times Y / \ast \times Y$. There is again a homeomorphism

$$\|X_\cdot\|_\ast \cong \hocolim_{\Lambda^{op}} X_\cdot$$

where the homotopy colimit is taken in the category of pointed topological spaces.

If $X_\cdot$ denotes levelwise addition of a disjoint basepoint, then there is a homeomorphism $\|X_\cdot\|_\ast \cong \|X_\cdot\|_\ast^+$. The skeletal filtration of $\|X_\cdot\|$ gives a strongly convergent first quadrant spectral sequence

$$(\text{sSS}) \quad E_{s,t}^1 = h_t(X_s) \Rightarrow h_{s+t}(\|X_\cdot\|)$$

for any connective generalised homology theory $h_*$. The $d^1$ differential is given by the alternating sum of the face maps, $d^1 = \sum (-1)^i(d_i)_\ast$. This spectral sequence coincides with the Bousfield–Kan spectral sequence for the homology of a homotopy colimit, and is natural for simplicial maps. There is also a pointed analogue, using reduced homology.

2.1. Pairs of spaces. If $f : A \to X$ is a continuous map, write $C_f$ for its homotopy cofibre. Write $(X, A)$ for the pair of spaces, where we silently consider $f$ to be an inclusion. We may sometimes confuse $(X, A)$ with $C_f$, and in particular treat $(X, A)$ as a pointed space.
2.2. Relative semi-simplicial spaces. Let \( f_\bullet : X_\bullet \to Y_\bullet \) be a map of semi-simplicial spaces. Then the levelwise homotopy cofibres form a semi-simplicial pointed space \( C_{f_\bullet} \), and

\[
\|C_{f_\bullet}\|_s \cong C_{\|f_s\|}
\]

as homotopy colimits commute. In particular, the spectral sequence (ResS) for this semi-simplicial pointed space is

\[
E^1_{s,t} = \tilde{h}_t(C_{f_s}) \cong h_t(Y_s, X_s) \Rightarrow \tilde{h}_{s+t}(C_{\|f_s\|}) \cong h_{s+t}(\|Y_s\|, \|X_s\|).
\]

2.3. Augmented semi-simplicial spaces. An augmentation of a (semi-)simplicial space \( X_\bullet \) is a space \( X_{-1} \) and a map \( \epsilon : X_0 \to X_{-1} \) such that \( \epsilon d_0 = \epsilon d_1 : X_1 \to X_{-1} \). An augmentation induces a map \( \|\epsilon\| : \|X_\bullet\| \to X_{-1} \). In this case there is a spectral sequence defined for \( s \geq -1 \),

\[
E^1_{s,t} = h_t(X_s) \Rightarrow h_{s+t+1}(\|\epsilon\|, X_s),
\]

for any connective generalised homology theory \( h_s \). The \( d^1 \) differentials are as above for \( s > 0 \), and \( d^1 : E^1_{0,t} \to E^1_{-1,t} \) is given by \( \epsilon_s \).

There is also a relative version of this construction. Let \( f : (\epsilon_X : X_\bullet \to X_{-1}) \Rightarrow (\epsilon_Y : Y_\bullet \to Y_{-1}) \) be a map of augmented semi-simplicial spaces. There is a spectral sequence defined for \( s \geq -1 \),

\[
E^1_{s,t} = h_t(X_s, Y_s) \Rightarrow h_{s+t+1}(\|\epsilon_{X_s}\|, C_{\|\epsilon_Y\|}).
\]

2.4. Resolutions. For our purposes, a resolution of a space \( X \) is an augmented semi-simplicial space \( X_\bullet \to X \) such that the map \( \|X_\bullet\| \to X \) is a weak homotopy equivalence. An \( n \)-resolution of a space \( X \) is an augmented semi-simplicial space \( X_\bullet \to X \) such that the map \( \|X_\bullet\| \to X \) is \( n \)-connected.

2.5. Homotopy fibres. We will typically show that an augmented semi-simplicial space \( X_\bullet \to X \) is an \( n \)-resolution by showing that the homotopy fibre of \( \|X_\bullet\| \to X \) is \((n-1)\)-connected. For this it is useful to know that the homotopy fibre may be computed levelwise. Let us write \( f_n : X_n \to X \) for the unique composition of face maps.

Lemma 2.1. The square

\[
\begin{array}{c}
\|\text{hofib}_x f_\bullet\| \\
\downarrow \\
\{x\} \\
\downarrow \\
X
\end{array}
\]

is weakly homotopy cartesian.

Proof. Without loss of generality we may suppose that the \( f_n \) are fibrations and that \( X \) is a CW complex (so locally contractible), by passing to a homotopy equivalent diagram. As \( f_{n-1}^{-1}(x) \to \text{hofib}_x f_n \) is a weak homotopy equivalence, it is enough to show that

\[
\|f_{n-1}^{-1}(x)\| \to \|X_\bullet\| \to X
\]

is a homotopy fibre sequence. Certainly \( \|f_{n-1}^{-1}(x)\| \) is the geometric fibre over \( x \), so we will show that \( \|X_\bullet\| \to X \) is a local quasi-fibration, hence quasi-fibration, and the result follows.
For any point \( a \in X \) let \( U_a \) be a contractible neighbourhood, and \( b \in U_a \). The fibre over \( b \) is \( \|f_n^{-1}(b)\| \) and the preimage of \( U_a \) is \( \|f_n^{-1}(U_a)\| \). As each \( f_n \) is a fibration, the map
\[
f_n^{-1}(b) \rightarrow f_n^{-1}(U_a)
\]
is a levelwise weak homotopy equivalence, so a weak homotopy equivalence on realisation, as required. \( \square \)

3. Finite sets

We will start gently with the moduli spaces of finite sets, which is the example discussed in [10.6]. Our point-set model for this moduli space is
\[
B\Sigma_n = \text{Emb}(\{1, \ldots, n\}, \mathbb{R}^\infty)/\Sigma_n,
\]
the unordered configuration space of \( n \) points in \( \mathbb{R}^\infty \). There is a stabilisation map \( s : B\Sigma_n \rightarrow B\Sigma_{n+1} \), uniquely defined up to homotopy, given by adding a new point "near infinity". More precisely, we choose an embedding \( \mathbb{R}^\infty \xrightarrow{\varepsilon} [0, 1]^\infty \subset \mathbb{R}^\infty \) isotopic to the identity, and define \( s \) to send a configuration \( C \) to \( \varepsilon(C) \cup (2, 0, 0, \ldots) \).

**Theorem 3.1** (Nakaoka [21]). The map \( H_*(B\Sigma_n) \rightarrow H_*(B\Sigma_{n+1}) \) is an isomorphism in degrees \( 2* \leq n \).

3.1. **Resolution.** Let \( E = \{(C, p) \in B\Sigma_n \times \mathbb{R}^\infty \mid p \in C\} \). Forgetting the point \( p \) defines a covering space
\[
\pi : E \rightarrow B\Sigma_n.
\]
Let \( B\Sigma_n^i = E \times_{B\Sigma_n} E \times_{B\Sigma_n} \cdots \times_{B\Sigma_n} E \) consist of those tuples \((C, p_0, \ldots, p_i)\) with the \( p_j \) all distinct. There are maps \( d_j : B\Sigma_n \rightarrow B\Sigma_{n+1}^i \) for \( j = 0, \ldots, i \) given by forgetting the \( j \)-th point. Furthermore, there is a unique map \( B\Sigma_n^i \rightarrow B\Sigma_n \) given by forgetting all points.

**Proposition 3.2.** \( B\Sigma_n^* \rightarrow B\Sigma_n \) is an augmented semi-simplicial space. The fibre \( \|F(C)^*\| \) of \( \|B\Sigma_n^*\| \rightarrow B\Sigma_n \) over a configuration \( C \in B\Sigma_n \) is homotopy equivalent to a wedge of \((n - 1)\)-spheres. In particular this gives a \((n - 1)\)-resolution of \( B\Sigma_n \).

**Proof.** That \( B\Sigma_n^* \rightarrow B\Sigma_n \) is an augmented semi-simplicial space is clear: all that is required is to observe that the maps \( d_j \) respect the simplicial identities.

The fibre over a configuration \( C \) is the realisation of the semi-simplicial space \( F(C)^* \) with \( i \)-simplices the set of ordered subsets of \( C \) of cardinality \((i + 1)\). Certainly \( \|F(C)^*\| \) has dimension \((n - 1)\), so it is enough to show it is \((n - 2)\)-connected. We prove this by induction on \( n \), and note that it is clear for \( n < 2 \). Choose a \( p \in C \) and consider the inclusion \( F(C-p)^* \rightarrow F(C)^* \). Note that \( \|F(C-p)^*\| \rightarrow \|F(C)^*\| \) is nullhomotopic, as the point \( p \) can be added, and furthermore \( \|F(C-p)^*\| \simeq \vee S^{n-2} \) by inductive assumption. The levelwise homotopy cofibre is \( \bigvee_{j=0}^{n} F(C-p)^{i-1}_j \) which can be alternatively written as \( F(C-p)^{i-1}_j \times \{0, \ldots, \bullet\} \). This is the construction of Appendix B applied to \( F(C-p)^* \).

Consider the augmented semi-simplicial space
\[
\| \cdots F(C-p)^{i+1} \Rightarrow F(C-p)^i \| \xrightarrow{d_i} F(C-p)^{i-1}.
\]
This has fibre \( \|F(C - \{p, p_0, \ldots, p_{i-1}\})^*\| \) over \( \{p_0, \ldots, p_{i-1}\} \in F(C-p)^{i-1} \), which by inductive hypothesis is a wedge of \((n - i - 2)\)-spheres, so the map \( d_i \) is \((n - i - 2)\)-connected. By the discussion in Appendix B the map \( \vee S^{n-1} \simeq \Sigma \|F(C-p)^*\| \rightarrow \)
integral homology applied to the semi-simplicial space \( B \) and we wish to deduce something about the range \( 2^* \leq 3.2 \).

**Proof of Theorem 3.1.** transitively on \( F \) and by a permutation on the second. The second factor is contractible, so \( Y \) and hence also on \( B_{\Sigma_n}^i \).

Note \( B_{\Sigma_n}^i \) is homotopy equivalent to its subspace \( Y \) where the distinguished points have first coordinate 1, and the undistinguished points have first coordinate negative. There is a homeomorphism

\[
\pi: Y \to \text{Emb}\{1, ..., n - i - 1, 1 \times \mathbb{R}^\infty \}
\]

and the automorphism \( \sigma \) preserves the subspace \( Y \), acts trivially on the first factor and by a permutation on the second. The second factor is contractible, so \( \sigma \) is homotopic to the identity on \( Y \), and hence also on \( B_{\Sigma_n}^i \).

**Remark 3.4.** There is an alternative (equivalent) slightly more abstract point of view on this resolution. Let us define \( B_{\Sigma_n} := * / \Sigma_n \) to be the homotopy quotient. The semi-simplicial set \( F(\{1, ..., n\})^* \) has an action of \( \Sigma_n \), and we can define \( B_{\Sigma_n}^i := F(\{1, ..., n\})^i / \Sigma_n \) to be the homotopy quotient.

Then there is a homotopy equivalence \( \|F(\{1, ..., n\})^*\| / \Sigma_n \simeq \|B_{\Sigma_n}^i\| \), so a fibration sequence

\[
\|F(\{1, ..., n\})^*\| \to \|B_{\Sigma_n}^i\| \to B_{\Sigma_n}.
\]

It then remains to identify \( B_{\Sigma_n}^i \), which may be seen to be \( B_{\Sigma_n-1} \) as \( \Sigma_n \) acts transitively on \( F(\{1, ..., n\})^i \) with stabiliser \( \Sigma_n-1 \).

3.2. **Proof of Theorem 3.1.** Note that Theorem 3.1 is trivially true for \( n = 0 \), as the spaces \( B_{\Sigma_n} \) are all connected. Consider the spectral sequence (SSS) in integral homology applied to the semi-simplicial space \( B_{\Sigma_n}^{i+1} \). It is

\[
E^1_{s,t} = H_t(B_{\Sigma_n-s}) \Rightarrow H_{s+t}(\|B_{\Sigma_n}^i\|) \cong H_{s+t}(B_{\Sigma_n+1}) \quad \text{for} \quad s + t \leq n - 1
\]

and we wish to deduce something about the range \( 2s \leq n \), so need \( \lfloor \frac{n}{2} \rfloor \leq n - 1 \), which holds for all \( n \geq 1 \). In this spectral sequence the \( d^1 \) differential is given by the alternating sum of the face maps. However, these are all freely homotopic, so induce the same map on homology. Thus \( d^1 : E^1_{odd,s} \to E^1_{even,s} \) is zero and \( d^1 : E^1_{even,s} \rightarrow E^1_{odd,s} \) is given by the stabilisation map,

\[
s_s : H_t(B_{\Sigma_n-2s}) \to H_t(B_{\Sigma_n-2s+1}).
\]

Applying Theorem 3.1 for \( n' < n \) points, we see that this stabilisation map is an isomorphism for \( 2t \leq n - 2s \), so that \( E^2_{s,s} \) is trivial for bidegrees \( (2s, t) \) with \( 2(t+2s) \leq n+2s \) and \( s > 0 \), and bidegrees \( (2s+1, t) \) with \( 2(t+2s+1) \leq n+2s+2 \) and \( s > 0 \). Observing Figure 1 in total degrees \( 2s \leq n \) the spectral sequence collapses at \( E^2 \) and is concentrated along \( s = 0 \). Thus Theorem 3.1 holds.
Figure 1. $E^1$ page of the spectral sequence converging to $H_*(|B\Sigma_{n+1}|)$.

4. Configuration spaces and labels

For $Y$ a connected manifold and $X$ a path connected space, let us define

$$C_n(Y; X) := \text{Emb}([1, \ldots, n], Y) \times_{\Sigma_n} X^n$$

to be the moduli space of $n$ unordered points in $Y$ labelled by $X$. Let us write $Y_k$ for the manifold $Y$ with $k$ points removed (as $Y$ is assumed to be connected, the homeomorphism type of $Y_k$ is independent of which points are removed). In this section we will prove a generalisation of Theorem 3.1 to the spaces $C_n(Y; X)$. We also give a different spectral sequence argument, which is closer to that we will use in the following sections.

Let $Y$ be the interior of a manifold with boundary $ar{Y}$, and $b \in \partial \bar{Y}$. We may choose an embedding $e : \bar{Y} \hookrightarrow Y$ that is the identity outside a small neighbourhood of $b$, misses the point $b$ (and so also misses a contractible neighbourhood $U \subset \bar{Y}$ of it), and is isotopic to the identity. Given a configuration of $n$ points $C \subset Y$, $e(C) \subset Y$ is a configuration of $n$ points in $Y$ that do not lie in $U \cap Y$. We may then add a point in this open set (labelled by some $\ell \in X$) to the configuration to obtain a configuration $s_b(C)$ of $(n+1)$ points in $Y$. This defines a continuous map

$$s_b : C_n(Y; X) \rightarrow C_{n+1}(Y; X)$$

whose homotopy class only depends on the boundary component $b \in \partial \bar{Y}$ lies on (in particular, it is independent of $\ell$, as $X$ is assumed to be path connected).

**Theorem 4.1.** Let $Y$ be a connected manifold which is the interior of a manifold with boundary $\bar{Y}$, and has dimension greater than 1. The map $s_b : C_n(Y; X) \rightarrow C_{n+1}(Y; X)$ is a homology epimorphism in degrees $2* \leq n$ and a homology isomorphism in degrees $2* \leq n - 2$.

**Remark 4.2.** The condition that $Y$ is the interior of a manifold with boundary can probably be removed by instead working with the ends of $Y$ as in [24, Appendix to §5]. We have not pursued this approach as we are primarily interested in fairly innocuous manifolds which do admit a boundary.

**Remark 4.3.** Of course, stability for integral homology implies stability for any other connective homology theory.

In the case $X = \ast$, the fact that these spaces exhibit homological stability at all is due to McDuff [18] and a stability range similar to the above was later obtained by Segal [24, Proposition A.1]. For $Y = \mathbb{R}^d$ and $X$ arbitrary, the fact that these
spaces exhibit homological stability is due to Lehrer and Segal [15], though their methods are indirect and so unable to provide a stability range.

**Remark 4.4.** Although they only discuss configurations in Euclidean space, the transfer argument of Lehrer–Segal [15 Section 3] can be applied verbatim to show that the maps $C_n(Y;X) \to C_{n+1}(Y;X)$ always give homology monomorphisms for $n \geq 0$. This improves Theorem 4.1 to say that the stabilisation maps are isomorphisms in degrees $2s \leq n$.

Define

$$C_n(Y;X)^i := \{(C,p_0,...,p_i) \in C_n(Y;X) \times Y^{i+1} \mid p_j \in C, p_j \neq p_k\}.$$ 

There are maps $d_j : C_n(Y;X)^i \to C_n(Y;X)^{i-1}$ given by forgetting the $j$-th point, and $C_n(Y;X)^i \to C_n(Y;X)$ given by forgetting all points.

**Proposition 4.5.** $C_n(Y;X)^\ast \to C_n(Y;X)$ is an augmented semi-simplicial space, and an $(n-1)$-resolution. Furthermore, there are homotopy fibre sequences

$$C_{n-i-1}(Y_{i+1};X) \to C_n(Y;X)^i \xrightarrow{\pi} \text{Emb}(\{1,...,i+1\}, Y) \times X^{i+1} =: A_i(Y;X).$$

**Proof.** The first statement requiring proof is that the fibration $\|C_n(Y;X)^\ast\| \to S_n(Y;X)$ has $(n-1)$-connected fibre. However, its homotopy fibre over a labelled configuration $C$ in $Y$ is $\|F(C)^\ast\|$, which we showed in Proposition 3.2 to be a wedge of $(n-1)$-spheres.

The map $\pi$ sends a tuple $(C,p_0,...,p_i)$ to $(p_0,...,p_i,\ell(p_0),...,\ell(p_i))$ where $\ell(p_j)$ denotes the label in $X$ of $p_j \in C$. This is a fibration (in fact a fibre bundle) and the fibre over $(p_0,...,p_i,x_0,...,x_i)$ is $C_{n-i-1}(Y \setminus \{p_0,...,p_i\}; X)$. □

Consider the stabilisation map $s_b : C_n(Y;X) \to C_{n+1}(Y;X)$ for $b$ a point of $\partial Y$. Let us write $R_n(Y,b)$ for the pair $(C_{n+1}(Y;X), C_n(Y;X))$, where $C_n(Y;X)$ is thought of as a subspace via the map $s_b$. If the reader prefers they can consider $R_n(Y,b)$ to be the mapping cone of $s_b$: we will be interested only in its connectivity. The map $s_b$ induces a simplicial map on resolutions $s_b^\ast : C_n(Y;X)^\ast \to C_{n+1}(Y;X)^\ast$ and we write $R_n(Y,b)^\ast$ for the simplicial pair $(C_{n+1}(Y;X)^\ast, C_n(Y;X)^\ast)$. There is then an augmented semi-simplicial object in the category of pairs of spaces

$$R_n(Y,b)^\ast \to R_n(Y,b).$$

4.1. **Proof of Theorem 4.1** We proceed by induction on $n$. Note that the statement of Theorem 4.1 is equivalent to the statement that $R_n(Y,b)$ is $\frac{1}{2}$-connected. As $Y$ is connected and of dimension at least two, all the spaces involved are connected and so the statement is trivially true for $n \leq 1$. For $n \geq 2$, we can apply [RASS] to the augmented semi-simplicial pair of spaces $R_n(Y,b)^\ast \to R_n(Y,b)$. It has the form

$$E_{s,t}^1 = H_t(C_{n+1}(Y;X)^\ast, C_n(Y;X)^s) \Rightarrow 0 \quad \text{for} \ s + t \leq n$$

and $n \geq n/2$, so the range we wish to study is within the range that the spectral sequence converges to zero.

By Proposition 4.5, there is a relative Serre spectral sequence

$$\tilde{E}_{s,t}^2 = H_t(A_i(Y;X); H_s(C_{n-i}(Y_{i+1};X), C_{n-i-1}(Y_{i+1};X))) \Rightarrow H_{s+t}(C_{n+1}(Y;X)^s, C_n(Y;X)^t),$$
and applying Theorem 4.4 to $Y_{i+1}$ and $n - i - 1$ points we see that $E_{s,t}^2 = 0$ for $2s \leq n - i - 1$. Thus $H_*(C_{n+1}(Y;X), C_n(Y;X))$ is trivial in degrees $2s \leq n - i - 1$, and the inclusion of the fibre to the total space gives a homology epimorphism in degrees $2s \leq n - i + 1$.

This implies that $E_{s,t}^1 = 0$ for $2t \leq n - s - 1$, so for $2(s + t) \leq n + s - 1$. Thus the augmentation

$$d_1 : H_*(C_{n+1}(Y;X), C_n(Y;X)) \to H_*(C_{n+1}(Y;X), C_n(Y;X))$$

is an epimorphism in degrees $2s \leq n$.

Consider the inclusion of the relative fibre

$$(C_n(Y_1;X), C_{n-1}(Y_1;X)) \to (C_{n+1}(Y;X), C_n(Y;X))$$

over a point $(1,t) \in A_0(Y;X)$ with $1 \in Y$ near $b \in \partial \overline{Y}$. By the Serre spectral sequence calculation above, this gives a homology epimorphism in degrees $2s \leq n + 1$. Proposition 4.10 (given with its proof in 4.3 as the method of proof is different to the methods used in this section) says that the map

$$p_b : (C_n(Y;X), C_{n-1}(Y;X)) \to (C_n(Y_1;X), C_{n-1}(Y_1;X))$$

which adds a puncture near $b$ also induces a homology epimorphism in degrees $2s \leq n$. Thus the composition

$$(C_n(Y;X), C_{n-1}(Y;X)) \xrightarrow{p_b} (C_n(Y_1;X), C_{n-1}(Y_1;X)) \to (C_{n+1}(Y;X), C_n(Y;X))$$

is a homology epimorphism in degrees $2s \leq n$. However, this map is homotopic to the stabilisation map $s_b$ on pairs, and so is relatively nullhomotopic, which means the homology of the pair $(C_{n+1}(Y;X), C_n(Y;X))$ vanishes in degrees $2s \leq n$ as required.

4.2. Corollaries. Theorem 4.4 has several immediate corollaries in group homology, only using Euclidean spaces for $Y$. For any group $G$ there is a homotopy equivalence $C_n(\mathbb{R}^\infty; BG) \simeq B(\Sigma_n \wr G)$ coming from the homotopy fibre sequence

$$\text{map}(\{1, \ldots, n\}, BG) = BG^n \to C_n(\mathbb{R}^\infty; BG) \to C_n(\mathbb{R}^\infty; *) = B\Sigma_n.$$

**Corollary 4.6.** The map $\Sigma_n \wr G \to \Sigma_{n+1} \wr G$ is a homology epimorphism in degrees $2s \leq n$, and a homology isomorphism in degrees $2s \leq n - 2$.

Recall that $C_n(\mathbb{R}^2; *) = B\beta_n$ is a classifying space for Artin’s braid group on $n$ strands.

**Corollary 4.7.** The map $\beta_n \to \beta_{n+1}$, given by adding a strand, is a homology epimorphism in degrees $2s \leq n$, and a homology isomorphism in degrees $2s \leq n - 2$. Similarly for the groups $\beta_n(G)$, where the wreath product is formed using the natural homomorphism $\beta_n \to \Sigma_n$ sending a braid to the permutation of its ends.

**Corollary 4.8.** For any path connected space $X$, the map between homotopy quotients $X^n/\beta_n \to X^{n+1}/\beta_{n+1}$ is a homology epimorphism in degrees $2s \leq n$ and a homology isomorphism in degrees $2s \leq n - 2$.

**Remark 4.9.** Note that $C_n(\mathbb{R}^2; S^1) \simeq BR\beta_n$ is a classifying space for the ribbon braid group on $n$ ribbons (as $R\beta_n = \beta_n \wr \mathbb{Z}$), so in particular these groups have the above homology stability range.
These three corollaries are not new: in particular, they all follow directly from the work of F. Cohen [3], which in fact computes the homology of all the spaces involved: one may then simply observe the claimed stability ranges. Stability of the braid groups $\beta_n$ is originally due to Arnol’d [1].

4.3. Adding and removing punctures. In order to finish the proof of Theorem 4.1, it is necessary to study the map $p_y : C_n(Y; X) \to C_n(Y_1; X)$ that adds a puncture to the manifold $Y$ near the point $y \in Y$, and its natural partner, the map $f_1 : C_n(Y_1; X) \to C_n(Y; X)$ that fills in a puncture. The composition

$$C_n(Y; X) \xrightarrow{p_y} C_n(Y_1; X) \xrightarrow{f_1} C_n(Y; X)$$

is homotopic to the identity, so studying the homological effect of one map is equivalent to studying the homological effect of the other. The main result of this section is that the relative homology groups $H_*(C_n+1(Y; X), C_n(Y; X))$ exhibit homological stability for adding and removing punctures, that is, the maps $R_n(Y, b) \to R_n(Y_1, b) \to R_n(Y, b)$ are both homology equivalences in a range.

**Proposition 4.10.** The relative homology groups $H_*(C_{n+1}(Y; X), C_n(Y; X))$ exhibit homological stability for adding and removing punctures; more precisely, the maps $R_n(Y, b) \to R_n(Y_1, b) \to R_n(Y, b)$ are both homology equivalences in degrees $2s \leq n + 1$.

**Proof.** Let $D$ be the closed $\dim(Y)$-dimensional disc of unit radius, and choose an open embedding $e : D \hookrightarrow Y$ away from the point $b$. We will decompose the space $C_n(Y; X)$ into a pair of open sets: let $U \subset C_n(Y; X)$ be the subspace consisting of those configurations with a unique closest point in $D$ to 0; let $V \subset C_n(Y; X)$ be the subspace consisting of those configurations with no point in $D$ at 0. The sets $U$ and $V$ give an open cover of $C_n(Y; X)$. We identify the homotopy types of $U$, $V$ and $U \cap V$ as follows.

(i) There is a fibre sequence $C_{n-1}(Y_1; X) \to U \to D \times X$, where the second map picks out the unique closest point in $D$ to 0 and its label, and this fibration is trivial.

(ii) There is a homotopy equivalence $V \cong C_n(Y \setminus \{0\}; X) \cong C_n(Y_1; X)$.

(iii) There is a fibre sequence $C_{n-1}(Y_1; X) \to U \cap V \to (D \setminus \{0\}) \times X$, which is the restriction of the fibration in (i) and hence trivial.

By excision, the homology of the pair

$$(U \cup V, V) \cong (C_n(Y; X), C_n(Y_1; X))$$

is canonically isomorphic to that of the pair

$$(U, U \cap V) \cong C_{n-1}(Y_1; X) \times X_+ \times (D, D \setminus \{0\}).$$

Then as the homotopy cofibre of $f_1 : R_n(Y_1; b) \to R_n(Y; b)$ can be identified with that of

$$s_b : (C_n(Y; X), C_n(Y_1; X)) \to (C_{n+1}(Y; X), C_{n+1}(Y_1; X))$$

its homology can be identified with that of the homotopy cofibre of

$$C_{n-1}(Y_1; X) \times X_+ \times S^{\dim(Y)} \xrightarrow{s_b \wedge \Id} C_n(Y_1; X) \times X_+ \times S^{\dim(Y)}$$

which is $\left[\frac{n-1}{2}\right] + \dim(Y) \geq \left[\frac{n+1}{2}\right]$ connected, by applying Theorem for $n-1$. □
Remark 4.11. This method of proving Proposition 4.10 is somewhat ad hoc, and the reader may have expected an approach where one resolves the map $R_n(Y) \to R_n(Y)$ and proceeds by induction. This would certainly be a more coherent approach, but the author could not get it to work.

5. Moduli spaces of surfaces

We now move on to the main example of interest, moduli spaces of surfaces equipped with some tangential structure. Particular examples of these recover the mapping class groups of oriented or unoriented surfaces, Cohen–Madsen’s moduli spaces $\mathcal{S}_{g,n}(X)$ of surfaces in a background space, the Spin mapping class groups introduced by Masbaum [17, 2], as well as many other examples, some of which we will study in a future article [23].

Definition 5.1. We write $\Sigma^r_g$ for the orientable surface of genus $g$ with $r$ boundary components, and $S^r_n$ for the non-orientable surface of genus $n$ with $r$ boundary components (i.e. $\#^n\mathbb{RP}^2 \coprod^r D^2$). We write $F$ for an arbitrary surface, $\Sigma$ for an arbitrary orientable surface and $S$ for an arbitrary non-orientable surface.

In the remainder of this section we give a precise definition of the moduli spaces $\mathcal{M}^\theta(F,\delta)$ that we will be using, and describe models for stabilisation maps between them. In §6 we describe two semi-simplicial resolutions of the moduli spaces $\mathcal{M}^\theta(\Sigma^r_g,\delta)$, which will both need to be used in order to prove homological for these moduli spaces. In §7 we describe three semi-simplicial resolutions of the moduli spaces $\mathcal{M}^\theta(S^r_n,\delta)$, which will all need to be used in order to prove homological for these moduli spaces. In §8 we describe the notion of $k$-triviality for a tangential structure $\theta$. This is the main condition we will require on $\theta$ in order to prove that the moduli spaces of $\theta$-surfaces exhibit homological stability. §9 contains the statement of the general stability theorem, and gives a recurrence relation which computes the stability range. We also give solutions to this recurrence relation in the main cases of interest. In §10–11 we give the proofs of the stability theorems, which is essentially a formal consequence of the definition of $k$-triviality. In §12 we discuss homological stability for closing off the last boundary component of a surface. This must be treated by different methods, as the resolutions constructed use the boundary of a surface in an essential way. Instead, we show how to resolve moduli spaces of closed manifolds (of any dimension) by moduli spaces of manifolds with boundary.

In order to precisely define the moduli spaces of manifolds we need to introduce the notion of a tangential structure $\theta$, which is simply a Serre fibration $\theta : X \to BO(d)$. We write $\gamma_d \to BO(d)$ for the universal rank $d$ vector bundle, and $\theta^*\gamma_d \to X$ for the vector bundle classified by $\theta$. Given a $d$-dimensional vector bundle $V \to B$, let $\text{Bun}(V, \theta^*\gamma_d)$ denote the space of bundle maps (that is, fibrewise linear isomorphisms) in the compact-open topology, and call it the space of $\theta$-structures on $V$.

Remark 5.2. The space $\text{Bun}(V, \theta^*\gamma_d)$ is homotopy equivalent to any of the more usual definitions of spaces of $\theta$-structures. In particular it is homotopy equivalent to the space of pairs $(f : B \to X, \varphi : f^*\theta^*\gamma_d \simeq V)$ of a continuous map from $B$ to $X$ and a vector bundle isomorphism between the bundle $\theta^*\gamma_d$ pulled back to $B$ and $V$. 
If $F$ is a $d$-manifold, possibly with boundary $\partial F$ (in which case we choose once and for all a collar $[0, 1) \times \partial F \rightarrow F$), and $\delta : TF|_{\partial F} \rightarrow \theta^* \gamma_d$ is a fixed $\theta$-structure on the tangent bundle over the boundary of $F$, let

$$\text{Bun}_\partial(TF, \theta^* \gamma_d; \delta)$$
denote the space of those bundle maps which agree with $\delta$ when restricted to the boundary.

Define the topological group $\text{Diff}_\partial^\prime(F)$ to be those diffeomorphisms of $F$ which fix pointwise the collar $[0, \epsilon) \times \partial F \hookrightarrow F$, equipped with the $\mathcal{C}^\infty$ topology. Let $\text{Diff}_\partial^\prime(F)$ denote $\colim_{\epsilon \rightarrow 0} \text{Diff}_\partial^\prime(F)$, and define the moduli space of $\theta$-manifolds of topological type $F$ with boundary condition $\delta$ to be the homotopy quotient

$$\mathcal{M}^\theta(F; \delta) := \text{Bun}_\partial(TF, \theta^* \gamma_d; \delta)/\text{Diff}_\partial(F).$$

This space carries a smooth $F$-bundle

$$\mathcal{E}^\theta(F; \delta) := \text{Bun}_\partial(TF, \theta^* \gamma_d; \delta) \times F/\text{Diff}_\partial(F),$$

where the group acts diagonally. The bundle $F \rightarrow \mathcal{E}^\theta(F; \delta) \rightarrow \mathcal{M}^\theta(F; \delta)$ is the universal smooth $F$ bundle equipped with a $\theta$-structure on its vertical tangent bundle satisfying the boundary condition $\delta$ on each fibre.

It will be convenient to sometimes omit the notation $\delta$ and write simply $\mathcal{M}^\theta(F)$ to mean $\mathcal{M}^\theta(F; \delta)$ for some unspecified $\delta$.

A point-set model for this moduli space may be described as follows. Given an embedding $e : \partial F \rightarrow [0) \times \mathbb{R}^\infty$, let

$$\text{Emb}_\partial^\prime(F, [0, 1] \times \mathbb{R}^\infty; e)$$
denote the space of embeddings (given the Whitney $\mathcal{C}^\infty$-topology) which agree with the cylindrical embedding $\text{Id} \times e : [0, \epsilon) \times \partial F \hookrightarrow [0, \epsilon) \times \mathbb{R}^\infty$ on the thin collar $[0, \epsilon) \times \partial F \hookrightarrow F$. Let

$$\text{Emb}_\partial(F, [0, 1] \times \mathbb{R}^\infty; e) := \colim_{\epsilon \rightarrow 0} \text{Emb}_\partial^\prime(F, [0, 1] \times \mathbb{R}^\infty; e).$$

The space $\text{Emb}_\partial^\prime$ has a continuous action by the topological group $\text{Diff}_\partial^\prime(F)$, and $\text{Diff}_\partial(F)$ acts continuously on $\text{Emb}_\partial$ and we may define

$$\mathcal{M}^\theta(F; \delta) := \text{Emb}_\partial(F, [0, 1] \times \mathbb{R}^\infty; e) \times_{\text{Diff}_\partial(F)} \text{Bun}_\partial(TF, \theta^* \gamma_d; \delta)$$
as a particular model for the moduli space. Note that $e$ is chosen from the contractible space $\text{Emb}(\partial F, \mathbb{R}^\infty)$, so changing it does not affect the homotopy type of this space and we may ignore it.

**Definition 5.3.** Let us say boundary conditions $\delta$, $\delta' : TF|_{\partial F} \rightarrow \theta^* \gamma_d$ for $F$ are *isomorphic* if they lie in the same path component of $\text{Bun}(TF|_{\partial F}, \theta^* \gamma_d)$. In this case a choice of path $\delta_t$ between them is called an *isomorphism* and determines a homotopy equivalence

$$\varphi(\delta_t) : \mathcal{M}^\theta(F; \delta) \simeq \mathcal{M}^\theta(F; \delta')$$
by gluing on to $F$ a copy of $\partial F \times [0, 1]$ with $\theta$-structure given by the path $\delta_t$. 
5.1. Gluing. We are interested in manifolds of dimension 2. Suppose we have a surface with boundary condition \((F, \delta : TF|_{\partial F} \to \theta^*\gamma_d)\), and suppose furthermore that there are chosen boundary components \(\partial_1 F, \partial_2 F\); a diffeomorphism \(\psi : \partial_1 F \cong \partial_2 F\) and an isomorphism of boundary conditions \(\varphi : (\delta|_{\partial_1 F}) \circ \tau \cong \psi^*(\delta|_{\partial_2 F})\), where \(\tau : TF|_{\partial_1 F} \to TF|_{\partial_2 F}\) is the involution that fixes vectors along \(\partial_1 F\) and inverts vectors orthogonal to it.

Let \(F'\) be the surface obtained from \(F\) by attaching a cylinder \([0, 1] \times S^1\) to \(F\) via \(\{0\} \times S^1 \cong \partial_1 F\) and \(\{1\} \times S^1 \cong \partial_2 F\). There is then an injection \(\text{Diff}_\theta(F) \to \text{Diff}_\theta(F')\) given by extending diffeomorphisms by the identity over the cylinder. The cylinder admits a \(\theta\)-structure compatible with \(\delta\) under this identification, using \(\varphi\). Let \(\delta'\) be the boundary condition on \(F'\) given by \(\delta\) on all the boundaries except the \(i\)-th and \(j\)-th. Then there is a \(\text{Diff}_\theta(F)\)-equivariant map

\[
\text{Bun}_\theta(F; \delta) \longrightarrow \text{Bun}_\theta(F'; \delta')
\]

given by extending a bundle map over the cylinder using \(\varphi\). Taking the homotopy quotient gives a continuous map

\[
\Gamma(\psi, \varphi) : \mathcal{M}^\theta(F; \delta) \longrightarrow \mathcal{M}^\theta(F'; \delta').
\]

In particular, if we have two surfaces with boundary conditions \((F, \delta : TF|_{\partial F} \to \theta^*\gamma_d)\) and \((F', \delta' : TF'|_{\partial F'} \to \theta^*\gamma_d)\), along with the data \(\psi : \partial_i F \cong \partial_j F\) and \(\varphi : (\delta|_{\partial_1 F}) \circ \tau \cong \psi^*(\delta|_{\partial_2 F})\) then there is a gluing map

\[
\mathcal{M}^\theta(F; \delta) \times \mathcal{M}^\theta(F'; \delta') \overset{\cong}{\longrightarrow} \mathcal{M}^\theta(F \coprod F', \delta \coprod \delta') \overset{\Gamma(\psi, \varphi)}{\longrightarrow} \mathcal{M}^\theta(F \cup F'; \delta \cup \delta')
\]

where \(F \cup F'\) is the union of the surfaces along \(\psi\), and \(\delta \cup \delta'\) is the induced boundary condition. Similarly, we can glue more than one pair of boundaries together at a time.

5.2. Stabilisation maps. Suppose we have two surfaces with boundary conditions \((F, \delta : TF|_{\partial F} \to \theta^*\gamma_d)\) and \((F', \delta' : TF'|_{\partial F'} \to \theta^*\gamma_d)\), and a some identified boundary components \(\partial F \leftrightarrow \partial_0 F \leftrightarrow \partial F'\). This determines a diffeomorphism between the images of these embeddings, \(\psi : \partial_0 F \cong \partial_0 F'\). Suppose also that \((\delta|_{\partial_0 F}) \circ \tau = \psi^*(\delta'|_{\partial_0 F'})\), so we may take \(\varphi = \text{Id}\). Then choosing a \(\theta\)-structure \(\ell : TF' \to \theta^*\gamma\) extending \(\delta'\) determines a map

\[
\mathcal{M}^\theta(F; \delta) \longrightarrow \mathcal{M}^\theta(F \cup F'; \delta \cup \delta')
\]

which we call a stabilisation map.

It will be convenient to have a second model for stabilisation maps. Suppose that \((F, \delta)\) is a surface with boundary condition, and that we have identified a pair of embedded intervals \(e : \{0, 1\} \times [0, 1] \hookrightarrow \partial F\) in the boundary of \(F\). Choosing a \(\theta\)-structure \(\ell\) on \([0, 1] \times [0, 1]\) extending that defined over \(\{0, 1\} \times [0, 1]\) by \(e^*(\delta)\) determines a map

\[
\sigma(e, \ell) : \mathcal{M}^\theta(F; \delta) \longrightarrow \mathcal{M}^\theta(F' = F \cup_e [0, 1] \times [0, 1]; \delta').
\]

Note that \(F'\) will have corners, and these need to be smoothed. We remark that all stabilisation maps in the first model arise (up to isomorphism of boundary conditions) via iterating this construction.

(i) If the intervals lie on different boundaries of \(F\) (and if \(F\) is orientable, the intervals are coherently oriented in the sense that there is an orientation of \(F\) which restricts to the standard orientations of each interval) we denote
this stabilisation map $\alpha_{ij}$, where the relevant boundaries are the $i$-th and $j$-th. This has the effect of gluing on a pair of pants to the $i$-th and $j$-th boundaries of $F$.

(ii) If the intervals both lie on the $i$-th boundary and are coherently oriented we denote it $\beta^i$. This has the effect of gluing on a pair of pants to the $i$-th boundary of $F$.

(iii) If the intervals both lie on the $i$-th boundary and are oppositely oriented we denote it $\mu^i$. This has the effect of gluing on a Möbius band with a disc removed to the $i$-th boundary of $F$.

There is a final stabilisation map: if the $i$-th boundary component of $F$ has a $\theta$-structure which bounds a disc, we may choose such a $\theta$-disc and glue it to $F$ to produce a $F'$ with one fewer boundary components. We denote this map by $\gamma^i$.

**Definition 5.4.** If the $\theta$-structure $e^*\delta$ is equal on $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$, there is an $\ell$ which is constant along the first coordinate direction. Call such a stabilisation map a trivial stabilisation.

Let $\theta(F) : TF|_{\partial F} \to \theta^*\gamma$ be a surface with boundary condition. Let $(a : [0, 1] \hookrightarrow F, u : a^*TF \to \theta^*\gamma)$ be an arc in $F$ between points $b_0, b_1 \in \partial F$, with a $\theta$-structure $u$ on the arc which restricts to $a^*(\delta|_{b_i})$ at each $i$. Let $F'$ be the surface obtained by cutting $F$ along the arc $a([0, 1])$, and $\delta_a$ be the new boundary condition on $\partial F'$ induced by $\delta$ on the old part and $u$ on the new part of the boundary. Then there is a trivial stabilisation

$$\sigma_u : M^\theta(F'; \delta_a) \to M^\theta(F; \delta)$$

given by identifying the two intervals in $\partial F'$ on which the $\theta$-structure is given by $u$. Up to isomorphism of boundary conditions, all trivial stabilisation maps occur in this way. We write $\alpha_u, \beta_u$ and so on, when the stabilisation map is of this type.

### 6. Resolutions for orientable surfaces

6.1. **The handle resolution.** Let $b_0, b_1$ be a pair of fixed points on a single boundary component of $\Sigma$, and fix a choice of orientation of the boundary component containing the $b_i$, which induces an orientation of $\Sigma$. Write $A = A_{b_0,b_1} = \text{Emb}_0([0,1],\Sigma)$ for the space of embeddings of arcs in $\Sigma$ from $b_0$ to $b_1$. Let $H(\Sigma)^i \subset A^{i+1}$ consist of tuples $(a_0, ..., a_i)$ such that

(i) the $a_j$ are disjoint except at their end points,

(ii) cutting along all the arcs $a_j$ does not disconnect $\Sigma$,

(iii) the ordering $a_0, ..., a_i$, the clockwise ordering of the arcs at $b_0$ and the anticlockwise ordering at $b_1$ all agree. Note that the notions of clockwise and anticlockwise orderings make sense as we have an orientation of $\Sigma$ coming from the orientation of a boundary.

Define

$$\mathcal{H}^\theta(\Sigma; \delta)^i \colonequals H(\Sigma)^i \times \text{Bun}_\theta(T\Sigma, \theta^*\gamma_2; \delta) \parallel \text{Diff}_\partial(\Sigma)$$

where the group acts diagonally. The maps $d_j : H(\Sigma)^i \to H(\Sigma)^{i-1}$ given by forgetting the $j$-th arc induce maps $d_j : \mathcal{H}^\theta(\Sigma; \delta)^i \to \mathcal{H}^\theta(\Sigma; \delta)^{i-1}$, and there is also a map $\mathcal{H}^\theta(\Sigma; \delta)^i \to M^\theta(\Sigma; \delta)^i$ given by forgetting all arcs.
Proposition 6.1. $\mathcal{H}^\theta(\Sigma; \delta)^\bullet \to \mathcal{M}^\theta(\Sigma; \delta)$ is an augmented semi-simplicial space. The homotopy fibre $\|H(\Sigma)^\bullet\|$ of $\|\mathcal{H}^\theta(\Sigma; \delta)^\bullet\| \to \mathcal{M}^\theta(\Sigma; \delta)$ over a $\theta$-surface $(M, \ell)$ is $(g(\Sigma)-2)$-connected. In particular this gives a $(g(\Sigma)-1)$-resolution of $\mathcal{M}^\theta(\Sigma; \delta)$.

Proof. That it is an augmented semi-simplicial space is immediate. That $\|H(\Sigma)^\bullet\|$ is the homotopy fibre of $\|\mathcal{H}^\theta(\Sigma; \delta)^\bullet\| \to \mathcal{M}^\theta(\Sigma; \delta)$ follows by the discussion in §2.5.

We may form the semi-simplicial set $\pi_0H(\Sigma)^\bullet$ by taking levelwise sets of path components. This is precisely the semi-simplicial set associated to the complex $B_0(\Sigma)$ of Ivanov [14], which he has shown to be homotopy equivalent to a wedge of $(g(\Sigma)-1)$-spheres. We include a proof of this fact as Theorem A.1 in Appendix A, where we deduce it from a theorem of Harer. Given this, it is enough to show that $\|H(\Sigma)^\bullet\| \to \|\pi_0H(\Sigma)^\bullet\|$ is a homotopy equivalence. We will do this by showing it is a levelwise homotopy equivalence, that is, that each $H(\Sigma)^i$ has contractible path components.

This relies on a theorem of Gramain [10, Théorème 5] which we rephrase here: let $F$ be a compact surface with boundary, and $x_0, x_1$ be distinct points on $\partial F$. Let $P(((0, 1], 0, 1), (F, x_0, x_1))$ denote the space of smooth embeddings $f : [0, 1] \to F$ sending 0, 1 to $x_0, x_1$ respectively and being disjoint from the boundary otherwise, equipped with the $C^\infty$ topology. Then this space has contractible components.

Note that our $H(\Sigma)^0$ is a union of path components of Gramain’s space of arcs $P(((0, 1], 0, 1), (\Sigma, b_0, b_1))$, and hence has contractible components. The maps $d_0 : H(\Sigma)^i \to H(\Sigma)^{i-1}$ are fibrations by [22], and the fibres can be identified with a union of path components of Gramain’s space $P(((0, 1], 0, 1), (\Sigma', b_0, b_1))$ where $\Sigma'$ is a surface obtained from $\Sigma$ by cutting along $i$ arcs. As inductive hypothesis we may suppose $H(\Sigma)^{i-1}$ has contractible components, and by Gramain’s theorem so does the fibre, so $H(\Sigma)^i$ does also.

Let $\tilde{H}(\Sigma)^i$ be the spaces defined similarly to $H(\Sigma)^i$, except by disjoint embeddings of $(i + 1)$ “lenses” $L_j, j = 0, \ldots, i$,

![A lens](image.png)

with their corners at $b_0$ and $b_1$ respectively, such that the core arcs satisfy the conditions (i) - (iii) above. There is a continuous map $\tilde{H}(\Sigma)^i \to H(\Sigma)^i$ given by restricting to the core arcs, and this is a homotopy equivalence as each arc in $\Sigma$ has a contractible space of thickenings to a lens. There is a map

$$\tilde{H}(\Sigma)^i \times \text{Bun}_\theta(T\Sigma, \theta^*\gamma_2; \delta) \overset{\tilde{\tau}}{\longrightarrow} \prod_{j=0}^i \text{Bun}_\theta(TL_j, \theta^*\gamma_2; \delta|_{b_0}, \delta|_{b_1}) =: A_i^\theta$$

that restricts the bundle map of the tangent bundle of $\Sigma$ to the $(i + 1)$ lenses defined by a point in $\tilde{H}(\Sigma)^i$. This map is $\text{Diff}_\theta(\Sigma)$-invariant, and hence descends to a map

$$\mathcal{H}^\theta(\Sigma; \delta)^i \overset{\pi}{\longrightarrow} A_i^\theta$$
where $\mathcal{H}^\theta(\Sigma; \delta)^i$ is defined in the same way as $\mathcal{H}^\theta(\Sigma; \delta)^i$ but using $\bar{H}(\Sigma)^i$ instead of $H(\Sigma)^i$.

Let us define $X_i := \coprod_{j=0}^i L_j$, and fix a point $u : TX_i \to \theta^*\gamma_2$ in $A^\theta_i$. Points in the fibre of the map $\bar{\pi}$ over $u$ consist of $(i+1)$ lenses in $\Sigma$ along with a $\theta$-structure on $\Sigma$ that is fixed on these lenses to be $u$. The group Diff$_\theta(\Sigma)$ acts transitively on the space $\bar{H}(\Sigma)^i$ of lenses in $\Sigma$, so we can identify the homotopy quotient of the fibre by this group as

$$\text{Bun}_{\mathcal{D},X_i}(T\Sigma, \theta^*\gamma_2; \delta \cup u) / \text{Stab}_{\text{Diff}_\theta(\Sigma)}(X_i \hookrightarrow \Sigma).$$

This stabiliser subgroup is equivalent to Diff$_\theta(\Sigma \setminus X_i)$, and $\text{Bun}_{\mathcal{D},X_i}(T\Sigma, \theta^*\gamma_2; \delta \cup u)$ is equivariantly homeomorphic to $\text{Bun}_{\mathcal{D}}(T\Sigma \setminus X_i, \theta^*\gamma_2; \delta \cup \partial u)$, where $\partial u$ is the boundary condition induced by $u$ on the boundary of $X_i$. Thus this space is by definition $\mathcal{M}^\theta(\Sigma \setminus X_i; \delta \cup \partial u)$.

**Proposition 6.2.** There is a homotopy equivalence $\mathcal{H}^\theta(\Sigma_{g+1}^\theta; \delta)^i \simeq \bar{\mathcal{H}}(\Sigma_{g+1}^\theta; \delta)^i$ and a homotopy fibre sequence

$$\mathcal{M}^\theta(\Sigma_{g+1}^\theta; \delta)^i \to \bar{\mathcal{H}}(\Sigma_{g+1}^\theta; \delta)^i \to A^\theta_i,$$

meaning that the homotopy fibre over each point $u$ of the base is of the homotopy type claimed, for some $\delta'$ that depends on $u$.

**Proof.** The homotopy equivalence is clear, as $\bar{H}(\Sigma)^i \to H(\Sigma)^i$ is a Diff$_\theta(\Sigma)$-equivariant map and a homotopy equivalence. We have shown in the discussion above that this homotopy fibre of the map is $\mathcal{M}^\theta(\Sigma_{g+1}^\theta \setminus X_i; \delta \cup \partial u)$. An Euler characteristic calculation using the conditions (i) – (iii) that define elements of $\bar{H}(\Sigma_{g+1}^\theta)^i$ shows that $\Sigma_{g+1}^\theta \setminus X_i \cong \Sigma_{g+1}^{\prime \prime \prime \prime \prime \prime}$, and the result follows. \qed

We will occasionally be slightly imprecise and freely interchange $\mathcal{H}^\theta(\Sigma_{g+1}^\theta; \delta)^i$ and $\bar{\mathcal{H}}(\Sigma_{g+1}^\theta; \delta)^i$, so for example say that there is a homotopy fibre sequence

$$\mathcal{M}^\theta(\Sigma_{g+1}^\theta; \delta)^i \to \mathcal{H}^\theta(\Sigma_{g+1}^\theta; \delta)^i \to A^\theta_i.$$

This should not lead to confusion.

### 6.2. The boundary resolution.

Let $b_0, b_1$ be a pair of fixed points on different boundary components of $\Sigma$, and fix a choice of orientation of $\Sigma$. As before, write $A = A_{b_0,b_1} = \text{Emb}_0([0,1], \Sigma)$ for the space of embeddings of arcs in $\Sigma$ from $b_0$ to $b_1$. Let $B(\Sigma)^i \subset A^{i+1}$ consist of tuples $(a_0, ..., a_i)$ such that

- (i) the $a_j$ are disjoint except at their end points,
- (ii) cutting along all the $a_j$ does not disconnect the surface,
- (iii) the ordering $a_0, ..., a_i$, the clockwise ordering of the arcs at $b_0$ and the anti-clockwise ordering at $b_1$ all agree.

Define

$$B^\theta(\Sigma; \delta)^i := B(\Sigma)^i \times \text{Bun}_{\mathcal{D}}(T\Sigma, \theta^*\gamma_2; \delta) / \text{Diff}_\theta(\Sigma)$$

where the group acts diagonally. The maps $d_j : B(\Sigma)^i \to B(\Sigma)^{i-1}$ given by forgetting the $j$-th arc induce maps $d_j : B^\theta(\Sigma; \delta)^i \to B^\theta(\Sigma; \delta)^{i-1}$, and there is also a map $B^\theta(\Sigma; \delta)^i \to \mathcal{M}^\theta(\Sigma; \delta)$ given by forgetting all arcs.

**Proposition 6.3.** $B^\theta(\Sigma; \delta)^* \to \mathcal{M}^\theta(\Sigma; \delta)$ is an augmented semi-simplicial space. The homotopy fibre $\lVert B(\Sigma)^i \rVert$ of $\lVert B^\theta(\Sigma; \delta)^i \rVert \to \mathcal{M}^\theta(\Sigma; \delta)$ over a $\theta$-surface $(M, \ell)$ is $(g(\Sigma) - 2)$-connected. In particular this gives a $(g(\Sigma) - 1)$-resolution of $\mathcal{M}^\theta(\Sigma; \delta)$. 


Proof. We again use the theorem of Gramain [10, Théorème 5] to identify the fibre with the complex $B_0(\Sigma)$ of Theorem A.1.

As in the last section, there are analogous spaces $\bar{B}(\Sigma)_i$ and $\bar{B}(\Sigma; \delta)_i$ defined using lenses in $\Sigma$ instead of arcs.

Proposition 6.4. There is a homotopy equivalence $B(\Sigma; \delta)_i \simeq \bar{B}(\Sigma; \delta)_i$ and a homotopy fibre sequence
\[ M(\Sigma^r_{g-i}; \delta') \to \bar{B}(\Sigma^r_{g+1}; \delta') \to \pi \to A^i_\theta, \]
meaning that the homotopy fibre over each point $u$ of the base is of the homotopy type claimed, for some $\delta'$ depending on $u$.

Proof. As in Proposition 6.2. To compute the topological type of surfaces the moduli space given by the fibre parametrises, note that after cutting out $(i+1)$ arcs of the type described above, we obtain $\Sigma^r_{g-i}$: cutting a single arc reduces Euler characteristic by 1, and decreases number of boundaries by one. The ordering condition implies that the remaining arcs now start and end at the same boundary component, so we are now in the situation of Proposition 6.2, so lose one genus and gain one boundary per remaining arc.

\[\square\]

7. Resolutions for non-orientable surfaces

The case of non-orientable surfaces is slightly more complicated than the orientable case, due to the following fact: cutting a non-orientable surface $S^r_n$ along an arc starting and ending on the same boundary component can produce either $S^r_{n-1}$ or $S^r_{n+1}$ (or even an orientable surface), depending on whether the arc is orientation reversing or not. Thus slightly more care is needed. Let us give some definitions which will be useful to describe the various possibilities.

Definition 7.1. An oriented point $b$ on the boundary of a non-orientable surface $S$ is a point along with a choice of unit tangent vector along the boundary of $S$.

If $b_0$, $b_1$ are oriented points on the same boundary component $\partial S$ of $S$, say they are coherently oriented if the orientations of $\partial b S$ induced by the tangent vectors at $b_0$ and $b_1$ agree. Say they are oppositely oriented otherwise.

If $a$ is an arc in $S$ between oriented points $b_0$, $b_1$, a normal orientation of $a$ is a choice of unit normal vector field. Say $a$ admits a normal orientation coherent with its endpoints if there is a normal orientation of $a$ which restricts at the $b_i$ to their orientations.

Disjoint arcs $a_j$ between oriented points $b_0$, $b_1$ can be ordered at each point $b_i$ in the following way: the tangent space of $S$ at $b_i$ has a canonical orientation coming from the inwards normal vector and the chosen orientation of the boundary at $b_i$. We can thus discuss the clockwise and anticlockwise orderings of the $a_j$ at each $b_i$.

7.1. The Möbius band resolution. Let $b_0$, $b_1$ be a pair of fixed coherently oriented points on a single boundary component of $S$, and write $A = A_{b_0,b_1} = \text{Emb}_b([0,1],S)$ for the space of embeddings of arcs from $b_0$ to $b_1$. Let $M(S)^i \subset A^{i+1}$ consist of tuples $(a_0, ..., a_i)$ such that

(i) the arcs $a_j$ admit a normal orientation coherent with their endpoints,
(ii) the $a_j$ are disjoint except at their end points,
where the group acts diagonally. The maps \( d \) for the space of embeddings of arcs from tuples \((a, b)\) forgeting the \( j \)-th arc induce maps \( d_j : M(S)^i \to M(S)^{i-1} \) given by forgetting the \( j \)-th arc and there is also a map \( M^\theta(S; \delta)^i \to M^\theta(S; \delta)^{i-1} \) given by forgetting all arcs.

**Proposition 7.2.** \( M^\theta(S; \delta)^* \to M^\theta(S; \delta) \) is an augmented semi-simplicial space. The homotopy fibre \( \| M(S)^* \| \to \| M^\theta(S; \delta)^* \| \to M^\theta(S; \delta) \) over a \( \theta \)-surface \((M, \ell)\) is \( (\lfloor \frac{n(S)}{3} \rfloor - 1) \)-connected. In particular this gives a \( [\frac{n(S)}{3}] \)-resolution of \( M^\theta(S; \delta) \).

**Proof.** We use the theorem of Gramain [10, Théorème 5] to identify the fibre with the complex \( C_0(S) \) of Theorem [A.2] which is shown there to have the correct connectivity.

As before, there are analogous spaces \( \tilde{M}(S)^i \) and \( \bar{M}^\theta(S; \delta)^i \) defined using lenses in \( S \) instead of arcs.

**Proposition 7.3.** There is a homotopy equivalence \( M^\theta(S; \delta)^i \simeq \tilde{M}^\theta(S; \delta)^i \) and a homotopy fibre sequence 
\[
M^\theta(S_n^i; \delta) \to \tilde{M}^\theta(S_n^i; \delta) \to A^\theta_i,
\]
meaning that the homotopy fibre over each point \( u \) of the base is of the homotopy type claimed, for some \( \delta' \) depending on \( u \).

**Proof.** We must just check what topological type of surface can be obtained by cutting the surface \( S_n^i \) along \((i+1)\) arcs of the type described above. Cutting along the first arc reduces the Euler characteristic by one, but preserves the number of boundary components as the ends of the arc are compatibly oriented. Thus it reduces the genus of the surface by one. Cutting along subsequent arcs does the same, by the ordering criterion, so we obtain \( S_{n-1} \).

### 7.2. The handle resolution.

Let \( b_0, b_1 \) be a pair of fixed oppositely oriented points on a single boundary component of \( S \), and write \( A = A_{b_0, b_1} = \text{Emb}_\partial([0, 1], S) \) for the space of embeddings of arcs from \( b_0 \) to \( b_1 \). Let \( H(S)^i \subset A^{i+1} \) consist of tuples \((a_0, ..., a_i)\) such that

(i) the arcs \( a_j \) admit a normal orientation coherent with their endpoints,
(ii) the \( a_j \) are disjoint except at their end points,
(iii) cutting along all the \( a_j \) does not disconnect the surface,
(iv) the ordering \( a_0, ..., a_i \), the clockwise ordering of the arcs at \( b_0 \) and the clockwise ordering at \( b_1 \) all agree.

Define 
\[
\mathcal{H}^\theta(S; \delta) := H(S)^i \times \text{Bun}_\partial(TS, \theta^* \gamma_2; \delta) / \text{Diff}_\partial(S)
\]
where the group acts diagonally. The maps \( d_j : H(S)^i \to H(S)^{i-1} \) given by forgetting the \( j \)-th arc induce maps \( d_j : \mathcal{H}^\theta(S; \delta)^i \to \mathcal{H}^\theta(S; \delta)^{i-1} \), and there is also a map \( \mathcal{H}^\theta(S; \delta)^i \to M^\theta(S; \delta) \) given by forgetting all arcs.
Proposition 7.4. \( \mathcal{H}^\theta(S; \delta)^{\bullet} \to \mathcal{M}^\theta(S; \delta) \) is an augmented semi-simplicial space. The homotopy fibre \( \| \mathcal{H}(S)^{\bullet} \| \) of \( \| \mathcal{H}^\theta(S; \delta)^{\bullet} \| \to \mathcal{M}^\theta(S; \delta) \) over a \( \theta \)-surface \((M, \ell)\) is homotopy equivalent to a wedge of \( (\lfloor \frac{n(S)}{2} \rfloor - 1) \)-spheres. In particular this gives a \( (\lfloor \frac{n(S)}{2} \rfloor - 1) \)-resolution of \( \mathcal{M}^\theta(S; \delta) \).

Proof. We use the theorem of Gramain [10] to identify the fibre with the complex \( D_0(S) \) of Theorem [A.3] which is shown there to have the correct connectivity. □

As before, there are analogous spaces \( \bar{H}(S)^{i} \) and \( \bar{H}^\theta(S; \delta)^{i} \) defined using lenses in \( S \) instead of arcs.

Proposition 7.5. There is a homotopy equivalence \( \mathcal{H}^\theta(S; \delta)^{i} \simeq \bar{H}^\theta(S; \delta)^{i} \) and a homotopy fibre sequence

\[
\mathcal{M}^\theta(S_n^{r+i+1}; \delta') \longrightarrow \bar{H}^\theta(S_n^{r}; \delta)^{i} \longrightarrow A^\theta_n,
\]

meaning that the homotopy fibre over each point \( u \) of the base is of the homotopy type claimed, for some \( \delta' \) depending on \( u \).

Proof. We must just check what topological type of surface can be obtained by cutting the surface \( S_n^{r} \) along \((i+1)\) arcs of the type described above. Cutting along the first arc reduces the Euler characteristic by one, and increases the number of boundary components by one, as the ends of the arc are oppositely oriented. Thus it reduces the genus of the surface by two and we obtain \( S_n^{r+i+1} \). Cutting along subsequent arcs does the same, by the ordering criterion, so we obtain \( S_n^{r+1} \) as claimed, for some \( \delta' \) depending on \( u \).

□

7.3. The boundary resolution. Let \( b_0, b_1 \) be a pair of fixed oriented points on different boundary components of \( S \), and write \( A = A_{b_0,b_1} = \text{Emb}_\partial([0,1], S) \) for the space of embeddings of arcs from \( b_0 \) to \( b_1 \). Let \( B(S)^i \subset A^{i+1} \) consist of tuples \((a_0, \ldots, a_i)\) such that

(i) the arcs \( a_j \) admit a normal orientation coherent with their endpoints,

(ii) the \( a_j \) are disjoint except at their end points,

(iii) cutting along all the \( a_j \) does not disconnect the surface,

(iv) the ordering \( a_0, \ldots, a_i \), the clockwise ordering of the arcs at \( b_0 \) and the anti-clockwise ordering at \( b_1 \) all agree.

Define

\[
B^\theta(S; \delta)^{i} := B(S)^i \times \text{Bun}_\partial(TS, \theta^* \gamma_2; \delta) / \text{Diff}_\partial(S)
\]

where the group acts diagonally. The maps \( d_j : B(S)^{i} \to B(S)^{i-1} \) given by forgetting the \( j \)-th arc induce maps \( d_j : B^\theta(S; \delta)^{i} \to B^\theta(S; \delta)^{i-1} \), and there is also a map \( B^\theta(S; \delta)^{i} \to \mathcal{M}^\theta(S; \delta) \) given by forgetting all arcs.

Proposition 7.6. \( B^\theta(S; \delta)^{\bullet} \to \mathcal{M}^\theta(S; \delta) \) is an augmented restricted simplicial space. The homotopy fibre \( \| B(S)^{\bullet} \| \) of \( \| B^\theta(S; \delta)^{\bullet} \| \to \mathcal{M}^\theta(S; \delta) \) over a \( \theta \)-surface \((M, \ell)\) is \( (\lfloor \frac{n(S)}{2} \rfloor - 2) \)-connected. In particular this gives a \( (\lfloor \frac{n(S)}{2} \rfloor - 1) \)-resolution of \( \mathcal{M}^\theta(S; \delta) \).

Proof. We use the theorem of Gramain [10] Théorème 5] to identify the fibre with the complex \( E_0(S) \) of Theorem [A.3] which is shown there to have the correct connectivity. □
As before, there are analogous spaces $\overline{B}(S)^i$ and $B^\theta(S; \delta)^i$ defined using lenses in $S$ instead of arcs.

**Proposition 7.7.** There is a homotopy equivalence $B^\theta(S; \delta)^i \simeq \overline{B}(S; \delta)^i$ and a homotopy fibre sequence

$$\mathcal{M}^\theta(S^{r+i}_{n-2i}; \delta') \to B^\theta(S^{r+1}_{n+1}; \delta)^i \xrightarrow{\pi} A^\theta_i,$$

meaning that the homotopy fibre over each point $b$ of the base is of the homotopy type claimed, for some $\delta'$ depending on $b$.

**Proof.** We must just check what topological type of surface can be obtained by cutting the surface $S^n_r$ along $(i+1)$ arcs of the type described above. Cutting along the first arc reduces the Euler characteristic by one, and reduced the number of boundary components by one, as the ends of the arc are on different boundary components. Thus it does not change the genus of the surface, and we obtain $S^n_r$. The remaining $i$ arcs now have ends on the same boundary component and give an element of $H(S^{r-1}_n)^{i-1}$, by the ordering criterion. Thus each subsequent arc reduces the genus by two and we obtain $S^{r+i}_{n-2i}$. \hfill \Box

8. $k$-TRIVIALITY OF $\theta$-STRUCTURES

Suppose we resolve $\mathcal{M}^\theta(\Sigma^r_g; \delta)$ with the boundary resolution where $b_0$ is on the first boundary and $b_1$ is on the second boundary. After gluing a strip between these two boundaries using a pair of intervals $e : \{0, 1\} \times [0, 1] \to \partial S^r_g$ disjoint from the $b_i$, this gives the handle resolution of $\mathcal{M}^\theta(\Sigma^r_{g+1}; \delta')$ with $b_0, b_1$ both on the first boundary, so we get a semi-simplicial map

$$\alpha^* : B^\theta(\Sigma^r_g; \delta)^* \to \mathcal{H}^\theta(\Sigma^r_{g+1}; \delta')^*$$

which is a resolution of the stabilisation map $\alpha^{12} : \mathcal{M}^\theta(\Sigma^r_g; \delta) \to \mathcal{M}^\theta(\Sigma^r_{g+1}; \delta')$.

Let us write $\alpha^*_g$ for the pair $(\mathcal{M}^\theta(\Sigma^r_{g+1}; \delta'), \mathcal{M}^\theta(\Sigma^r_g; \delta))$, and $(\alpha^*)_g$ for the resolution of pairs.

Similarly, suppose we resolve $\mathcal{M}^\theta(\Sigma^r_g; \delta)$ with the handle resolution where $b_0$ and $b_1$ are on the first boundary. After gluing a strip to the first boundary using a pair of intervals $e : \{0, 1\} \times [0, 1] \to \partial S^r_g$ in such a way as to separate $b_0$ and $b_1$, this gives the boundary resolution of $\mathcal{M}^\theta(\Sigma^r_{g+1}; \delta')$ with $b_0$ on the first boundary and $b_1$ on the second, so we get a semi-simplicial map

$$\beta^* : \mathcal{H}^\theta(\Sigma^r_g; \delta)^* \to B^\theta(\Sigma^r_{g+1}; \delta')^*$$

which is a resolution of the stabilisation map $\beta^1 : \mathcal{M}^\theta(\Sigma^r_g; \delta) \to \mathcal{M}^\theta(\Sigma^r_{g+1}; \delta')$. Let us write $\beta^*_g$ for the pair $(\mathcal{M}^\theta(\Sigma^r_{g+1}; \delta'), \mathcal{M}^\theta(\Sigma^r_g; \delta))$, and $(\beta^*)_g$ for the resolution of pairs.

Note that by Propositions 6.2 and 6.4 there are fibrations of pairs

\begin{equation}
\beta^r_{g-i-1} \to (\alpha^i)^r_g \to A^\theta_i,
\end{equation}

and

\begin{equation}
\alpha^r_{g-i-1} \to (\beta^i)^r_g \to A^\theta_i.
\end{equation}

When $i = 0$, let us write $\beta^r_{g-1}/u$ and $\alpha^r_{g-1}/u$ for the respective homotopy fibres over a point $u \in A^\theta_0$. 

The notion of \( k \)-triviality which we are about to introduce concerns only the very bottom of these resolutions. In each of the diagrams below, we call the top composition an \textit{elementary stabilisation map}.

\[
\begin{aligned}
\beta_g^{r-1}/u &\to (\alpha_0^r)_g \\
\{u\} &\to A_0^g
\end{aligned}
\quad
\begin{aligned}
\alpha_{g-1}^{r+1}/u &\to (\beta_0^r)_g \\
\{u\} &\to A_0^g
\end{aligned}
\]

Taking the disjoint union of the elementary stabilisation maps for one \( u \) in each path component of \( A_0^g \) gives

\[
\bigoplus_{[u] \in \pi_0 A_0^g} \beta_g^{r-1}/u \to (\alpha_0^r)_g \to \alpha_g^r \tag{8.3}
\]

and

\[
\bigoplus_{[u] \in \pi_0 A_0^g} \alpha_{g-1}^{r+1}/u \to (\beta_0^r)_g \to \beta_g^r \tag{8.4}
\]

\textbf{Example 8.1.} Consider the tangential structures given by the maps \( BO(2) \to BO(2) \) and \( BSO(2) \to BO(2) \), so either no tangential structure at all or orientations. These structures have the property that giving a boundary condition on a single boundary component of a surface induces a unique (up to isomorphism) boundary condition on the remaining boundaries and the interior of the surface.

The elementary stabilisation maps are the vertical maps in the following diagrams,

\[
\begin{array}{c}
\mathcal{M}^\theta(\Sigma_{g}^r ; \delta'_u) \xrightarrow{\alpha_{u}^{12}} \mathcal{M}^\theta(\Sigma_{g}^{r-1} ; \delta_u) \xrightarrow{\beta_u^1} \mathcal{M}^\theta(\Sigma_{g}^{r-1} ; \delta'_u) \\
\mathcal{M}^\theta(\Sigma_{g}^{r-1} ; \delta'_u) \xrightarrow{\beta_u^1} \mathcal{M}^\theta(\Sigma_{g}^{r} ; \delta) \xrightarrow{\alpha_{u}^{12}} \mathcal{M}^\theta(\Sigma_{g}^{r+1} ; \delta_u) \xrightarrow{\beta_u^1} \mathcal{M}^\theta(\Sigma_{g}^{r+1} ; \delta'_u) \\
\end{array}
\]

By the above remark, the boundary conditions \( \delta \) and \( \delta'_u \) must be isomorphic in each case, so there exist dotted isomorphisms of boundary conditions. Furthermore, for these tangential structures a stabilisation map is determined (up to homotopy) by which boundary components it glues a pair of pants to. Hence the dotted maps make all the triangles commute up to homotopy, and so give a relative nullhomotopy of the vertical maps.

We may then make the following definition.

\textbf{Definition 8.2.} The tangential structure \( \theta \) on orientable surfaces is \textit{\( k \)-trivial} if all \( k \)-fold compositions of elementary stabilisation maps are nullhomotopic. Such compositions simply correspond to stacking the diagrams of Example 8.1 upon each other \( k \) times.

Thus the example above shows that \( BO(2) \to BO(2) \) and \( BSO(2) \to BO(2) \) are both 1-trivial.

\textbf{8.1. \( k \)-triviality for non-orientable surfaces.} Suppose we resolve \( \mathcal{M}^\theta(S^r_n ; \delta) \) with the boundary resolution where \( b_0 \) is on the first boundary and \( b_1 \) is on the second boundary. After gluing a strip between these two boundaries using a pair of intervals \( e : \{0,1\} \times [0,1] \to \partial S^r_n \) disjoint from the \( b_i \), this gives the handle.
resolution of $\mathcal{M}^{\theta}(S^{r-1}_{n+2}; \delta')$ with $b_0, b_1$ both on the first boundary, so we get a semi-simplicial map
\[
\alpha^* : \mathcal{B}^{\theta}(S^n; \delta)^* \rightarrow \mathcal{H}^{\theta}(S^{r-1}_{n+2}; \delta')^*
\]
which is a resolution of the stabilisation map $\alpha^{12} : \mathcal{M}^{\theta}(S^n_{1}; \delta) \rightarrow \mathcal{M}^{\theta}(S^{r}_{n+2}; \delta')$. Let us write $\alpha_r^*$ for the pair $(\mathcal{M}^{\theta}(S^{r-1}_{n+2}; \delta'), \mathcal{M}^{\theta}(S^n_{1}; \delta))$, and $(\alpha^*)_{\mu}^*$ for the resolution of pairs.

Similarly, suppose we resolve $\mathcal{M}^{\theta}(S^n_0; \delta)$ with the handle resolution where $b_0$ and $b_1$ are on the first boundary. After gluing a strip to the first boundary using a pair of intervals $e : \{0, 1\} \times [0, 1] \rightarrow \partial S^n_0$ in such a way as to separate $b_0$ and $b_1$, this gives the boundary resolution of $\mathcal{M}^{\theta}(S^{r+1}_{n+1}; \delta')$ with $b_0$ on the first boundary and $b_1$ on the second, so we get a semi-simplicial map
\[
\beta^* : \mathcal{H}^{\theta}(S^n_0; \delta)^* \rightarrow \mathcal{B}^{\theta}(S^{r+1}_{n+1}; \delta')^*
\]
which is a resolution of the stabilisation map $\beta^1 : \mathcal{M}^{\theta}(S^n_0; \delta) \rightarrow \mathcal{M}^{\theta}(S^{r+1}_{n+1}; \delta')$. Let us write $\beta_r^*$ for the pair $(\mathcal{M}^{\theta}(S^{r+1}_{n+1}; \delta'), \mathcal{M}^{\theta}(S^n_0; \delta))$, and $(\beta^*)_{\mu}^*$ for the resolution of pairs.

Finally, we may resolve $\mathcal{M}^{\theta}(S^n_0; \delta)$ with the Möbius band resolution where $b_0$ and $b_1$ are on the first boundary. After gluing a Möbius band in so as to not invert the relative orientations of $b_0$ and $b_1$, we obtain the Möbius band resolution of $\mathcal{M}^{\theta}(S^{r+1}_{n+1}; \delta')$, so get a semi-simplicial map
\[
\mu^* : \mathcal{M}^{\theta}(S^n_0; \delta)^* \rightarrow \mathcal{M}^{\theta}(S^{r+1}_{n+1}; \delta')^*
\]
which is a resolution of the stabilisation map $\mu^1 : \mathcal{M}^{\theta}(S^n_0; \delta) \rightarrow \mathcal{M}^{\theta}(S^{r+1}_{n+1}; \delta')$. Let us write $\mu_r^*$ for the pair $(\mathcal{M}^{\theta}(S^{r+1}_{n+1}; \delta'), \mathcal{M}^{\theta}(S^n_0; \delta))$, and $(\mu^*)_{\mu}^*$ for the resolution of pairs.

Note that by Propositions 7.3, 7.5 and 7.7 there are fibrations of pairs
\begin{align}
\beta^{r+1-2i}_{n-2i} & \rightarrow (\alpha^i)_r^n \rightarrow A^\theta_i, \\
\alpha^{r+i+1}_{n-2(i+1)} & \rightarrow (\beta^i)_r^n \rightarrow A^\theta_i,
\end{align}
and
\[
\mu^{r-1-i}_{n-i} \rightarrow (\mu^i)_r^n \rightarrow A^\theta_i.
\]
When $i = 0$, let us write $\beta^{r-1}_{n-2}/u$, $\alpha^{r+1}_{n-2}/u$ and $\mu^{r-1}_{n-1}/u$ for the respective homotopy fibres over a point $u \in A^\theta_0$.

For non-orientable surfaces, as well as elementary stabilisation maps coming from $(\alpha^0)_r^n$ and $(\beta^0)_r^n$ as in the orientable case there are also elementary stabilisation maps coming from $(\mu^0)_r^n$
\[
\mu^{r-1}_n/u \rightarrow (\mu^0)_r^n \rightarrow \mu^n_r
\]
which assemble to give
\[
\prod_{[u] \in \pi_0 A^\theta_0} \mu^{r-1}_{n-1}/u \rightarrow (\mu^0)_r^n \rightarrow \mu^n_r.
\]
**Definition 8.3.** The tangential structure $\theta$ on non-orientable surfaces is $k$-trivial if all the $k$-fold compositions of elementary stabilisation maps coming from $\alpha$ and $\beta$ are nullhomotopic.

The tangential structure $\theta$ on non-orientable surfaces is $k$-trivial for projective planes if all the $k$-fold compositions of elementary stabilisation maps coming from $\mu$ are nullhomotopic.

**Example 8.4.** It is easy to see that the tangential structure $\theta : BO(2) \to BO(2)$ on non-orientable surfaces is also 1-trivial and 1-trivial for projective planes, by an argument similar to that of Example 8.1.

**8.2. Proving $k$-triviality.** Let us describe briefly what is involved in proving that a tangential structure is $k$-trivial. For simplicity let us suppose that $k = 2$ is even and that we are resolving a map of type $\alpha$ between orientable surfaces. Then we are required to produce the dotted map in the following diagram, making each triangle commute up to homotopy.

We have added on the right a schematic picture of the stabilisation maps and boundary conditions involved. Although it is not required, it is often easiest to produce the dotted map as a stabilisation map, corresponding to gluing on a surface $\Sigma_{g-K+1}$ equipped with some $\theta$-structure (satisfying the boundary conditions $(D, A + B)$). Let us call such a $\theta$-surface $\Sigma_{\Delta}$. Then commutativity of the diagram of stabilisation maps simply corresponds to the equations

$$\Sigma_{\Delta} \circ \Sigma_T \cong \Sigma_R$$

$$\Sigma_B \circ \Sigma_{\Delta} \cong \Sigma_L$$

holding, where $\cong$ means that the $\theta$-surfaces are in the same path component of the relevant moduli space.

**Definition 8.5.** When $k > 1$, we say $\theta$ is strongly $k$-trivial if there are always diagonal stabilisation maps making both triangles homotopy commute. Of course, strong $k$-triviality implies $k$-triviality.

**Remark 8.6.** Any 1-trivial $\theta$-structure is strongly 2-trivial, so all the examples we have seen so far are strongly 2-trivial.

Of course, the left and right surfaces in our schematic picture do not have independent $\theta$-structures on them, but it is not easy to describe the relation between their $\theta$-structures in any generality. In practice (see the companion paper [23] in
which we prove 4-triviality for several tangential structures) it is often easier to forget any relation between the $\theta$-structures of the left and right surfaces except that $\Sigma_T \circ \Sigma_L \cong \Sigma_R \circ \Sigma_B$, and show:

**Proposition 8.7** $(k = 2K)$. Let us be given any $\theta$-surfaces $\Sigma_L$, $\Sigma_R$, $\Sigma_B$, $\Sigma_T$ with boundary conditions $(A + B, C, D, E + F)$ as depicted in the figure above. Suppose that the $\theta$-structures on $\Sigma_T \circ \Sigma_L$ and $\Sigma_R \circ \Sigma_B$ are isomorphic (i.e. in the same path component of the relevant moduli space).

If there is always a $\theta$-surface $\Sigma_\Delta$ with underlying surface $\Sigma_{K-1}^{1+2}$ and boundary conditions $(D, A + B)$ such that $\Sigma_\Delta \circ \Sigma_T \cong \Sigma_R$ and $\Sigma_B \circ \Sigma_\Delta \cong \Sigma_L$, then $\theta$ is strongly $k$-trivial for maps of type $\alpha$. There is a similar statement for maps of type $\beta$.

**Proposition 8.8** $(k = 2K + 1)$. Let us be given any $\theta$-surfaces $\Sigma_L$, $\Sigma_R$, $\Sigma_B$, $\Sigma_T$ with boundary conditions $(A + B, C, D + E, F)$ as depicted in the figure below. Suppose that the $\theta$-structures on $\Sigma_T \circ \Sigma_L$ and $\Sigma_R \circ \Sigma_B$ are isomorphic (i.e. in the same path component of the relevant moduli space).

If there is always a $\theta$-surface $\Sigma_\Delta$ with underlying surface $\Sigma_{K-1}^{1+2}$ and boundary conditions $(D + E, A + B)$ such that $\Sigma_\Delta \circ \Sigma_T \cong \Sigma_R$ and $\Sigma_B \circ \Sigma_\Delta \cong \Sigma_L$, then $\theta$ is strongly $k$-trivial for maps of type $\alpha$. There is a similar statement for maps of type $\beta$.

These propositions imply strong $k$-triviality of maps of type $\alpha$ and $\beta$ for both orientable and non-orientable surfaces. For non-orientable surfaces there is also a condition for maps of type $\mu$ to be strongly $k$-trivial, which the reader may easily formulate.

**8.3. Formal $k$-triviality.** There is a condition on the path components of the moduli spaces that implies strong $k$-triviality, but which usually delivers a far from optimal $k$. As the slope of the stability range we will produce depends on $k$, this usually gives a far from optimal stability range. However, for some purposes it is enough to know merely the existence of a stability range (for example, to apply the methods of [8] to identify the stable homology of $M^\theta(F)$, as we demonstrate in [23]), and in those cases it is easy to check this condition. We formulate the
condition for orientable surfaces: the reader will immediately see the changes that
must be made for the non-orientable case.

**Definition 8.9.** Let us say a \( \theta \)-structure becomes \textit{constant at genus} \( \ell \) if the maps
\( \alpha(g) \) and \( \beta(g) \) are isomorphisms on \( \pi_0 \) for all \( g \geq \ell \), and epimorphisms for all
\( g \geq \ell - 1 \).

**Proposition 8.10.** Suppose \( \theta \) becomes constant at genus \( \ell \). Then it is strongly
\( k \)-trivial for \( k \leq \max(2,2\ell +1) \).

**Proof.** Let us suppose \( \ell > 0 \), as the remaining case is similar. We will verify the
conditions of Proposition 8.8. Let us be given the \( \theta \)-surfaces \( \Sigma_L, \Sigma_R, \Sigma_B, \Sigma_T \) with
boundary conditions \( (A + B, C, D + E, F) \), and such that \( \Sigma_T \circ \Sigma_L \cong \Sigma_R \circ \Sigma_B \).

The proposed \( \Sigma_\Delta \) is an element of \( \mathcal{M}^\theta(\Sigma_{\ell-1}^{2+2}; D + E, A + B) \), whose set of path
components fits into the commutative diagram

\[
\begin{array}{ccc}
\pi_0(\mathcal{M}^\theta(\Sigma_{\ell-1}^{2+2}; D + E, A + B)) & \xrightarrow{\Sigma_T} & \pi_0(\mathcal{M}^\theta(\Sigma_{\ell}^{2+1}; D + E, C)) \\
\Sigma_T & \downarrow & \Sigma_T \\
\pi_0(\mathcal{M}^\theta(\Sigma_{\ell}^{1+2}; F, A + B)) & \xrightarrow{\Sigma_B} & \pi_0(\mathcal{M}^\theta(\Sigma_{\ell+1}^{1+1}; F, C)).
\end{array}
\]

The only conditions on \( [\Sigma_\Delta] \) is that it should map to \( [\Sigma_R] \) under the horizontal
map and \( [\Sigma_L] \) under the vertical map, so it is enough to show that such an
element exists. But \( [\Sigma_R] \) and \( [\Sigma_L] \) map via isomorphisms to the same element in
\( \pi_0(\mathcal{M}^\theta(\Sigma_{\ell+1}^{1+1}; F, C)) \), so choosing any \( [\Sigma_\Delta] \) mapping to \( [\Sigma_R] \) by the horizontal
map (which we can do as the top map is surjective) works, because the bottom map is
bijective. The case of maps of type \( \beta \) is similar. \( \square \)

9. The stability theorems

9.1. **Caveat.** Whatever tangential structure \( \theta \) we have in mind, there are important
differences between orientable and non-orientable surfaces. One should \textit{not}
attempt to prove a stability theorem which mixes the two, and one should be wary
of trying to prove stability for closing the last boundary, as the following example
indicates.

**Proposition 9.1.** Let \( \theta = \text{Id} : BO(2) \to BO(2) \). The maps

\[
\mu^1 : \mathcal{M}^\theta(\Sigma_g^1) \to \mathcal{M}^\theta(S_{2g+1}^1)
\]

\[
\gamma^1 : \mathcal{M}^\theta(\Sigma_g^2) \to \mathcal{M}^\theta(\Sigma_g)
\]

are not monomorphisms on \( H_2(-; \mathbb{Q}) \) for \( g \gg 0 \).

**Proof.** There is a homotopy equivalence \( \mathcal{M}^\theta(\Sigma_g^1) \simeq B\Gamma_{g,1} \) with the classifying
space of the (orientation preserving!) mapping class group of \( \Sigma_g^1 \). It is known that
for large genus \( H^2(\Gamma_{g,1}; \mathbb{Q}) \) has rank one, generated by the first Mumford–Morita–
Miller class \( \kappa_1 \). There is a homotopy equivalence \( \mathcal{M}^\theta(S_{2g+1}^1) \simeq BN_{2g+1,1} \) with the
classifying space of the unoriented mapping class group of \( S_{2g+1}^1 \). It is known [25]
that for large genus this group only has rational cohomology in degrees divisible
by 4. This shows the first map is not a monomorphism on second homology.

There is a homotopy equivalence \( \mathcal{M}^\theta(\Sigma_g) \simeq B(\Gamma_g \rtimes \mathbb{Z}/2) = B(\Gamma_g) \) with the
classifying space of the unoriented mapping class group of \( \Sigma_g \). It is easy to check that
the change of orientation map \( \mathbb{Z}/2 \) acts as \( \{\pm 1\} \) on \( H^2(\Gamma_g; \mathbb{Q}) = \mathbb{Q} \langle \kappa_1 \rangle \), so
$H^2(\Gamma(\Sigma_g); \mathbb{Q}) = 0$. This shows the second map is not a monomorphism on second homology. \hfill \Box

9.2. Stability of $H_0$. A necessary condition for a family of spaces to exhibit homological stability is that their sets of path components (or alternatively, their zeroth homology) stabilises. This will be one of the two requirements of our stability theorem: the other is that the tangential structure should be $k$-trivial for some $k$.

Definition 9.2. In the context of orientable surfaces, say a tangential structure $\theta$ stabilises at genus $h$ if all the maps

\[
\begin{align*}
\alpha(g) &: H_0(\mathcal{M}^\theta(\Sigma_g^{r+1})) \to H_0(\mathcal{M}^\theta(\Sigma_g^{r+1})) \\
\beta(g) &: H_0(\mathcal{M}^\theta(\Sigma_g^r)) \to H_0(\mathcal{M}^\theta(\Sigma_g^{r+1}))
\end{align*}
\]

are epimorphisms for $g \geq h$.

In the context of non-orientable surfaces, say a tangential structure $\theta$ stabilises at genus $h$ if all the maps

\[
\begin{align*}
\alpha(n) &: H_0(\mathcal{M}^\theta(S_n^{r+1})) \to H_0(\mathcal{M}^\theta(S_{n+2}^r)) \\
\beta(n) &: H_0(\mathcal{M}^\theta(S_n^r)) \to H_0(\mathcal{M}^\theta(S_{n+1}^{r+1}))
\end{align*}
\]

are epimorphisms for $n \geq h$. Say a tangential structure $\theta$ stabilises at genus $h$ for projective planes if all the maps

\[
\mu(n) : H_0(\mathcal{M}^\theta(S_n^r)) \to H_0(\mathcal{M}^\theta(S_{n+1}^{r+1}))
\]

are epimorphisms for $n \geq h$.

9.3. Statement of the stability theorems. The functions $F$, $G$ and $H$ in the statement of the following theorems are described in the following section, where we give recurrence relations for computing them. A simple analysis of these recurrence relations shows that the functions $F$, $G$ and $H$ all tend to infinity: they are non-decreasing, and it is easy to see there cannot be a point after which all the functions must remain constant.

Theorem 9.3. Let $\theta : X \to BO(2)$ be a tangential structure that stabilises at genus $h$ and is $k$-trivial. Then\[(i) \alpha(g)_* : H_*(\mathcal{M}^\theta(\Sigma_g^r)) \to H_*(\mathcal{M}^\theta(\Sigma_g^{r+1})) \text{ is an epimorphism for } * \leq F(g) \text{ and an isomorphism for } * \leq F(g) - 1.
(ii) \beta(g)_* : H_*(\mathcal{M}^\theta(\Sigma_g^r)) \to H_*(\mathcal{M}^\theta(\Sigma_g^{r+1})) \text{ is an epimorphism for } * \leq G(g) \text{ and an isomorphism for } * \leq G(g) - 1.
(iii) \gamma(g)_* : H_*(\mathcal{M}^\theta(\Sigma_g^r)) \to H_*(\mathcal{M}^\theta(\Sigma_g^{r+1})) \text{ is an isomorphism for } * \leq G(g) \text{ and an epimorphism in all degrees, as long as } r \geq 2.

Theorem 9.4. Let $\theta : X \to BO(2)$ be a tangential structure that stabilises at genus $h$ and is $k$-trivial. Then\[(i) \alpha(n)_* : H_*(\mathcal{M}^\theta(S_n^r)) \to H_*(\mathcal{M}^\theta(S_{n+2}^{r+1})) \text{ is an epimorphism for } * \leq F(n) \text{ and an isomorphism for } * \leq F(n) - 1.
(ii) \beta(n)_* : H_*(\mathcal{M}^\theta(S_n^r)) \to H_*(\mathcal{M}^\theta(S_{n+1}^{r+1})) \text{ is an epimorphism for } * \leq G(n) \text{ and an isomorphism for } * \leq G(n) - 1.
(iii) \gamma(n)_* : H_*(\mathcal{M}^\theta(S_n^r)) \to H_*(\mathcal{M}^\theta(S_{n+1}^{r+1})) \text{ is an isomorphism for } * \leq G(n) \text{ and an epimorphism in all degrees, as long as } r \geq 2.
If \( \theta \) stabilises at genus \( h' \) for projective planes and is \( k' \)-trivial for projective planes, then

(iv) \( \mu(n)_* : H_*(\mathcal{M}^\theta(S^r_n)) \to H_*(\mathcal{M}^\theta(S^r_{n+1})) \) is an epimorphism for \( * \leq H(n) \) and an isomorphism for \( * \leq H(n) - 1 \).

Stability for closing the last boundary component off is quite subtle, and requires some extra notation that we shall not introduce yet. We give the statement and a discussion of examples in §12 typically when there is stability for closing the last boundary component, it is in the same range as the penultimate boundary.

Remark 9.5. Both of the above theorems hold in slightly more generality: integral homology \( H_* \) can be replaced by any connective homology theory \( h_* \). This follows from the above by the Atiyah–Hirzebruch spectral sequence.

Remark 9.6. Note that if a stabilisation map \( \beta \) creates a new boundary component whose boundary condition bounds a disc, then it has a right inverse by gluing in that disc, and hence is injective in all degrees on homology. This increases the range in which it is an isomorphism by 1. For tangential structures such as \( BO(2) \to BO(2) \), \( BSO(2) \to BO(2) \), and these along with maps to a simply-connected background space, all boundary conditions bound a disc, and so all \( \beta \) are injective in homology.

Similarly, whenever a stabilisation map \( \gamma \) is not closing the last boundary component, it has a left inverse \( \beta \), and so is surjective in all degrees on homology.

9.4. Recurrence relations for orientable surfaces. In the case of orientable surfaces, we wish to find functions \( F, G : \mathbb{N} \to \mathbb{N} \) such that

(F) \[ H_*(\alpha^r_g) = 0 \quad * \leq F(g), \]

(G) \[ H_*(\beta^r_g) = 0 \quad * \leq G(g), \]

and it will be convenient to simultaneously find functions \( X, Y : \mathbb{N} \to \mathbb{N} \) such that

(X) \[ H_* \left( \prod_{[u] \in \pi_0 A^r_{n-1}} \beta^r_{g-1}/u \right) \to H_*(\alpha^r_g) \text{ is epi} \quad * \leq X(g), \]

(Y) \[ H_* \left( \prod_{[u] \in \pi_0 A^r_{n-1}} \alpha^r_{g-1}/u \right) \to H_*(\beta^r_g) \text{ is epi} \quad * \leq Y(g). \]

We suppose that all these functions are non-decreasing, and increase at most 1 at each step. We will show in §10 that any functions satisfying the following three conditions also satisfy the above conditions.

9.4.1. Conditions from stabilisation of \( H_0 \). If \( \theta \) stabilises at genus \( h \) then we obtain the inequalities

\[ F(g), G(g), X(g), Y(g) \geq 0 \quad \text{for} \quad g \geq h. \]

As a convention we define all the functions to be \(-1\) for \( g < h \).
9.4.2. **Conditions from connectivity of the arc complex.** We require that

If \( X(g) \geq 1 \) then \( X(g) \leq g \),

If \( Y(g) \geq 1 \) then \( Y(g) \leq g - 1 \).

This is only a condition for surfaces of very low genus, as \( X \) and \( Y \) will be essentially linear of slope less than 1.

9.4.3. **Conditions from the inductive step.** We require the functions to satisfy

\[
X(g) \leq \min(Y(g - 1) + 1, F(g - 1) + 1, G(g) + 1),
\]

\[
Y(g) \leq \min(X(g - 2) + 1, F(g - 1) + 1, G(g - 1) + 1).
\]

Furthermore for \( k = 1 \),

\[
F(g) \leq X(g),
\]

\[
G(g) \leq Y(g),
\]

and for \( k > 1 \)

\[
F(g) \leq \min(X(g + 1 - \left\lfloor \frac{k}{2} \right\rfloor), Y(g + 1 - \left\lfloor \frac{k}{2} \right\rfloor)),
\]

\[
G(g) \leq \min(X(g - \left\lfloor \frac{k}{2} \right\rfloor), Y(g + 1 - \left\lceil \frac{k}{2} \right\rceil)).
\]

9.5. **Recurrence relations for non-orientable surfaces.** In the case of non-orientable surfaces, we wish to find functions \( F, G, H : \mathbb{N} \to \mathbb{N} \) such that

(F) \( H_*(\alpha_n^*) = 0 \) * \( \leq F(n) \),

(G) \( H_*(\beta_n^*) = 0 \) * \( \leq G(n) \),

(H) \( H_*(\mu_n^*) = 0 \) * \( \leq H(n) \),

and it will be convenient to simultaneously find functions \( X, Y, Z : \mathbb{N} \to \mathbb{N} \) such that

(X) \( H_*(\prod_{[u] \in \pi_0 \mathcal{A}_0^g} \beta_{n-1/u}^*) \to H_*(\alpha_n^*) \) is epi * \( \leq X(n) \),

(Y) \( H_*(\prod_{[u] \in \pi_0 \mathcal{A}_0^g} \alpha_{n-2/u}^*) \to H_*(\beta_n^*) \) is epi * \( \leq Y(n) \),

(Z) \( H_*(\prod_{[u] \in \pi_0 \mathcal{A}_0^g} \mu_{n-1/u}^*) \to H_*(\mu_n^*) \) is epi * \( \leq Z(n) \).

We suppose that all these functions are non-decreasing, and increase at most 1 at each step. We will show in \([\text{III}]\) that any functions satisfying the following three conditions also satisfy the above conditions.
9.5.1. Conditions from stabilisation of $H_0$. If $\theta$ stabilises at genus $h$ then we obtain the inequalities

$$F(n), G(n), X(n), Y(n) \geq 0 \quad \text{for} \quad n \geq h.$$  

If $\theta$ stabilises at genus $h'$ for projective planes then we obtain the inequalities

$$H(n), Z(n) \geq 0 \quad \text{for} \quad n \geq h'.$$

As a convention we define all the functions to be $-1$ for $n < h$ (or $h'$ respectively).

9.5.2. Conditions from connectivity of the arc complex. We require that

If $X(n) \geq 1$ then $X(n) \leq \lfloor \frac{n}{2} \rfloor$,

If $Y(n) \geq 1$ then $Y(n) \leq \lfloor \frac{n}{2} \rfloor - 1$,

If $Z(n) \geq 1$ then $Z(n) \leq \lfloor \frac{n}{3} \rfloor$.

For $X$ and $Y$ this is only a condition for surfaces of very low genus, as these functions will be essentially linear of slope less than 1. For $Z$ this is an important condition.

9.5.3. Conditions from the inductive step. We require the functions to satisfy

$$X(n) \leq \min(Y(n - 2) + 1, F(n - 2) + 1, G(n) + 1),$$

$$Y(n) \leq \min(X(n - 4) + 1, F(n - 2) + 1, G(n - 2) + 1),$$

$$Z(n) \leq H(n - 2) + 1.$$  

Furthermore if $\theta$ is $k$-trivial for $k = 1$,

$$F(n) \leq X(n),$$

$$G(n) \leq Y(n),$$

and for $k > 1$

$$F(n) \leq \min(X(n + 2 - 2\lfloor \frac{k}{2} \rfloor), Y(n + 2 - 2\lfloor \frac{k}{2} \rfloor)),$$

$$G(n) \leq \min(X(n - 2\lfloor \frac{k}{2} \rfloor), Y(n + 2 - 2\lfloor \frac{k}{2} \rfloor)).$$

Finally, if $\theta$ is $k'$-trivial for projective planes

$$H(n) \leq Z(n - k' + 1).$$

9.6. Oriented surfaces. We consider the tangential structure $\theta : BSO(2) \rightarrow BO(2)$. The moduli spaces of oriented surfaces are always path connected, and so stabilise at genus 0. By Example 8.1 this tangential structure is 1-trivial.

It is easy to verify that $F(g) = X(g) = \lfloor \frac{2g+1}{3} \rfloor$ and $G(g) = Y(g) = \lfloor \frac{2g}{3} \rfloor$ give an optimal solution to the recurrence relations of §9.4 with $h = 0$ and $k = 1$. Furthermore, all boundary conditions on orientable surfaces bound a disc, so $\beta(g)_*$ is always injective. Thus: $\alpha(g)_*$ is an epimorphism for $3* \leq 2g + 1$ and an isomorphism for $3* \leq 2g - 2$; $\beta(g)_*$ is an isomorphism for $3* \leq 2g$ and always a monomorphism; $\gamma(g)_*$ is an isomorphism for $3* \leq 2g$ and always an epimorphism, as long as one is not closing the last boundary component. We will see in §12 that in fact $\gamma(g)_*$ is always an isomorphism for $3* \leq 2g$, even for the last boundary component.

This stability range coincides with the range recently obtained by Boldsen [8] for surfaces with boundary, and improves it slightly for closing the last boundary.
9.7. A remark on sharpness. Let us discuss how sharp this stability range is for oriented surfaces, where the moduli space $\mathcal{M}^+(\Sigma_g)$ is homotopy equivalent to the classifying space of the mapping class group $B\Gamma_{g,r}$. We assume familiarity with the Mumford–Morita–Miller classes

$$\kappa_i \in H^2(\mathcal{M}^+(\Sigma_g^n); \mathbb{Q}).$$

Morita [19] has shown the classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate the tautological algebra (that is, the algebra generated by the $\kappa_i$) in $H^*(\mathcal{M}^+(\Sigma_g); \mathbb{Q})$. It is also known by the Madsen–Weiss theorem [16] that stably $H^*(\mathcal{M}^+(\Sigma_1^n); \mathbb{Q}) \cong H^*(\mathcal{M}^+(\Sigma_\infty); \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \ldots]$. We have mapping

$$\mathcal{M}^+(\Sigma_g) \xrightarrow{\kappa_1} \mathcal{M}^+(\Sigma_g^1) \xrightarrow{s_\infty} \mathcal{M}^+(\Sigma_\infty^1)$$

which are natural for Miller–Morita–Mumford classes. By Morita’s theorem $\kappa_{[g/3]+1}$ is algebraically dependent on the lower $\kappa_i$ in the cohomology of $\mathcal{M}^+(\Sigma_g)$, and so also in the cohomology of $\mathcal{M}^+(\Sigma_g^1)$. It is not dependent on the lower $\kappa_i$ in the cohomology of $\mathcal{M}^+(\Sigma_\infty^1)$, and so the stabilisation map $s_\infty$ cannot be surjective on rational (hence integral) homology in degree $2([g/3]) + 1$. Thus any stability range (for oriented surfaces) given by functions $F(g)$, $G(g)$ must satisfy

$$\min(F(g), G(g)) \leq 2\left\lfloor \frac{g}{3} \right\rfloor + 1.$$

The range we obtain for oriented surfaces satisfies $\min(F(g), G(g)) = \left\lfloor \frac{2g}{3} \right\rfloor$, so is optimal when $g \equiv 2 \mod 3$.

9.8. Unoriented surfaces. We consider the tangential structure $\theta : BO(2) \to BO(2)$. The moduli spaces of unoriented surfaces are always path connected, and so stabilise at genus 0. By Example [8,1], this tangential structure is both 1-trivial and 1-projective for projective planes.

It is easy to verify that $F(n) = X(n) = \left\lfloor \frac{n}{3} \right\rfloor + e_2(n)$, $G(n) = Y(n) = \left\lfloor \frac{n-1}{3} \right\rfloor + e_6(n)$ and $H(n) = Z(n) = \left\lfloor \frac{n}{3} \right\rfloor$ solve the recurrence relations for $k = k' = 1$ and $h = h' = 0$, where $e_i(n)$ is 1 if $n = i + 6k$, $k \geq 0$, and 0 otherwise. Furthermore, all boundary conditions for this tangential structure bound a disc so $\beta(g)$, is always injective. Thus: $\mu(n)_*$ is an epimorphism for $3* \leq n$ and an isomorphism for $3* \leq n - 3$; $\alpha(n)_*$ is an epimorphism for $3* \leq n + 3e_2(n)$ and an isomorphism for $3* \leq n + 3(e_2(n) - 1)$; $\beta(n)_*$ is an epimorphism for $3* \leq n - 1 + 3e_6(n)$ and a monomorphism in all degrees; $\gamma(n)_*$ is an isomorphism for $3* \leq n - 1 + 3e_6(n)$ and an epimorphism in all degrees, as long as one is not closing the last boundary component. We will see in [12] that in fact $\gamma(n)_*$ is always an isomorphism for $3* \leq n - 1 + 3e_6(n)$, even when closing the last boundary component.

This stability range improves on the previously best known range, due to Wahl [25], which is of slope $1/4$.

9.9. Background spaces. Given a tangential structure $\theta : X \to BO(2)$, let us consider the new tangential structure $\theta \times Y = \theta \circ \pi_1 : X \times Y \to BO(2)$ for a path connected space $Y$. If we denote the corresponding moduli spaces by $\mathcal{M}^\theta \times Y(F; \delta)$, then up to isomorphism of boundary conditions we may always assume that after picking a point $c_i$ on each boundary component $\partial_i F$, $\delta$ sends each $c_i$ to a basepoint $* \in Y$.

**Proposition 9.7.** Let $\theta : X \to BO(2)$ be a tangential structure which is strongly $k$-trivial and stabilises (for orientable or non-orientable surfaces) at genus $h$. Let
Let \( Y \) be a simply connected space. Then \( \theta \times Y := \theta \circ \pi_X : X \times Y \to BO(2) \) is strongly \( \max(2, k) \)-trivial and stabilises at genus \( h \).

Similarly, suppose \( \theta \) is strongly \( k' \)-trivial for projective planes and stabilises at genus \( h' \) for projective planes. Then \( \theta \times Y \) is strongly \( \max(2, k') \)-trivial and stabilises at genus \( h' \).

Proof. Let us first show that the path components of \( \mathcal{M}^{\theta \times Y}(F) \) stabilise at the same genus as those of \( \mathcal{M}^\theta(F) \). Without loss of generality we may assume that the “maps to \( Y \)” part of a boundary condition for \( \theta \times Y \) is the constant map to a basepoint \( \ast \in Y \), so that boundary conditions are of the form \( \delta \times \{\ast\} \) with \( \delta \) a boundary condition for \( \theta \). There is a natural fibration sequence with section

\[
(9.1) \quad \text{map}(F, \partial F; Y, \ast) \to \mathcal{M}^{\theta \times Y}(F; \delta \times \{\ast\}) \to \mathcal{M}^\theta(F; \delta)
\]

and so an exact sequence of sets

\[
\ast \to \pi_0(\text{map}(F, \partial F; Y, \ast)) \to \pi_0(\mathcal{M}^{\theta \times Y}(F)) \to \pi_0(\mathcal{M}^\theta(F)) \to \ast.
\]

If \( F = S \) is non-orientable then \( \pi_0(\text{map}(F, \partial F; Y, \ast)) = H_2(Y)/2H_2(Y) = H_2(Y; \mathbb{F}_2) \), and we have the exact sequence of sets

\[
\ast \to H_2(Y; \mathbb{F}_2) \to \pi_0(\mathcal{M}^{\theta \times Y}(F)) \to \pi_0(\mathcal{M}^\theta(F)) \to \ast,
\]

so if \( \pi_0(\mathcal{M}^\theta(F)) \) stabilises at genus \( k \), then so does \( \pi_0(\mathcal{M}^{\theta \times Y}(F)) \).

If \( F = \Sigma \) is orientable then \( \pi_0(\text{map}(F, \partial F; Y, \ast)) = \pi_2(Y) = H_2(Y) \), and we have the exact sequence of sets

\[
\ast \to H_2(Y) \to \pi_0(\mathcal{M}^{\theta \times Y}(F)) \to \pi_0(\mathcal{M}^\theta(F)) \to \ast.
\]

So if \( \pi_0(\mathcal{M}^\theta(F)) \) stabilises at some genus then so does \( \pi_0(\mathcal{M}^{\theta \times Y}(F)) \), as the maps on bases of these exact sequences of sets are epimorphisms and maps on fibres are isomorphisms (unless we are stabilising a disc by attaching a Möbius band, which gives \( H_2(Y) \to H_2(Y; \mathbb{F}_2) \) on fibres which is not an isomorphism, but is in any case an epimorphism, which is enough).

Let us now show that if \( \theta \) is strongly \( k \)-trivial for \( k \geq 2 \) then so is \( \theta \times Y \). Given diagrams as in [S2] we have to produce a diagonal stabilisation map, or what is the same thing, a \( \theta \times Y \)-surface \( \Sigma_\Delta \) satisfying the necessary compatibility conditions.

Such a diagram has an underlying diagram of \( \theta \)-surfaces, and by assumption we can find a \( \theta \)-surface \( \Sigma'_\Delta \) satisfying the necessary compatibility conditions. Thus we are only required to produce a map \( \Sigma'_\Delta \to Y \) making the “maps to \( Y \)” part of the diagram commute. The “maps to \( Y \)” part of a \( \theta \times Y \)-structure on a surface is determined by an element of \( H_2(Y) \) (or \( H_2(Y; \mathbb{F}_2) \) if the surface is non-orientable, but let us suppose we are dealing with orientable surfaces in the remainder of the proof). Thus there are classes \( [\Sigma_L], [\Sigma_R], [\Sigma_B], [\Sigma_T] \in H_2(Y) \), such that

\[
[\Sigma_T] + [\Sigma_L] = [\Sigma_R] + [\Sigma_B].
\]

Define \( [\Sigma_\Delta] := [\Sigma_L] - [\Sigma_B] \), and let \( \Sigma_\Delta \) denote the \( \theta \times Y \)-surface with underlying \( \theta \)-surface \( \Sigma'_\Delta \) and map to \( Y \) given by \( [\Sigma_\Delta] \in H_2(Y) \). It is simple to verify that this \( \Sigma_\Delta \) gives the required diagonal stabilisation map.

As a concrete example, the tangential structure \( \theta : BSO(2) \times Y \to BO(2) \) and a boundary condition \( \delta \) on \( \Sigma'_g \) where all the boundary is sent to a basepoint in \( Y \) gives \( \mathcal{M}^\theta(\Sigma'_g; \delta) \simeq S_{g,r}(Y) \). The tangential structure \( BSO(2) \to BO(2) \) stabilises at genus 0 and is strongly 1-trivial, so \( \theta \) stabilises at genus 0 and is
From the fibration sequence (9.1) we see that the simplest non-simply connected background space we could use, where $\Gamma(\Sigma) = \pi_1(\Sigma)$, is always an isomorphism for $3^p \leq 2g - 3$; $\beta(g)_r$ is an epimorphism for $3^p \leq 2g - 1$ and an isomorphism for $3^p \leq 2g - 4$; $\gamma(g)_r$ is an isomorphism for $3^p \leq 2g - 1$ and always an epimorphism, as long as one is not closing the last boundary component. We will see in §10.1 that in fact $\gamma(g)_r$ is always an isomorphism for $3^p \leq 2g - 1$, even when closing the last boundary component.

This stability range coincides with the range recently obtained by Boldsen [3] for surfaces with boundary.

9.10. A remark on non-simply connected background spaces. Let us consider the simplest non-simply connected background space we could use, $Y = S^1$. From the fibration sequence (9.1) we see that $\pi_0(S_{g,r}(S^1)) = H^1(\Sigma_g^r, \partial \Sigma_g^r; \mathbb{Z})/\Gamma(\Sigma_g^r)$ where $\Gamma(\Sigma) = \pi_0(\text{Diff}_\partial(\Sigma))$ is the mapping class group of $\Sigma$. Furthermore, the exact sequence of abelian groups

$$
0 \rightarrow H^0(\Sigma_g^r) \rightarrow H^0(\partial \Sigma_g^r) \rightarrow H^1(\Sigma_g^r, \partial \Sigma_g^r; \mathbb{Z}) \rightarrow \mathbb{Z}^{2g} \rightarrow 0
$$

defines a canonical trivial $\Gamma(\Sigma_g^r)$-submodule $\mathbb{Z}^{r-1}$ of $H^1(\Sigma_g^r, \partial \Sigma_g^r; \mathbb{Z})$ with quotient module $\mathbb{Z}^{2g}$, on which $\Gamma(\Sigma_g^r)$ acts through its surjection onto the symplectic group $Sp_{2g}(\mathbb{Z})$. One can check that the set of orbits $\mathbb{Z}^{2g}/Sp_{2g}(\mathbb{Z})$ is in natural bijection with the natural numbers, via the $Sp_{2g}(\mathbb{Z})$-invariant map $\text{gcd} : \mathbb{Z}^{2g} \rightarrow \mathbb{N}$. Thus there is a natural exact sequence of pointed sets

$$
\mathbb{Z}^{r-1} \rightarrow \pi_0(S_{g,r}(S^1)) \rightarrow \mathbb{N}
$$

so as a set the components of $S_{g,r}(S^1)$ are naturally independent of $g$ but not of $r$. One might hope, as the author does, that the spaces $S_{g,r}(S^1)$ still exhibit homological stability for the map $S_{g,r}(S^1) \rightarrow S_{g+1,r}(S^1)$ which attaches a torus with trivial map to $S^1$, as it can be shown that this induces a bijection on $H_0$. Furthermore, attaching a torus with non-trivial map to $S^1$ can be shown to not induce a bijection on $H_0$. In the current framework the natural hypothesis is stability of path components for all stabilisation maps, no matter what tangential structure is chosen on the new torus. Galatius and the author [9] have been able to identify the homology of the direct limit $S_{\infty,1}(S^1)$, where one stabilises by attaching the torus $1 \in \mathbb{N} = \pi_0(S_{1,1}(S^1))$, without making use of homological stability. It would be most surprising if there were not some underlying homological stability statement; the author intends to investigate this matter in a future paper.

10. Proof of stability for orientable surfaces with boundary

Definition 10.1. Let us write $A_g$ to denote the statement “Equation (A) of [3,4] holds for all $g \leq g^*$”.

Suppose that the conditions of Theorem [3,4] hold: the moduli spaces of $\theta$-surfaces stabilise at genus $h$ and are $k$-trivial. Then $H_0(\alpha_g^r) = H_0(\gamma_g^r) = 0$ for all $g \geq h$, so certainly the statements $F_h$ and $G_h$ hold, and consequently $X_h$ and
\(Y_k\) hold too. This starts an induction and the following theorem provides the inductive step, and hence implies Theorem 9.3. We leave it to the sceptical reader to convince themselves that these implication do indeed give an inductive proof.

**Theorem 10.2.** Suppose the hypotheses of Theorem 9.3 hold, then there are implications

(i) \(X_g\) and \(Y_g \Rightarrow F_g\),
(ii) \(Y_g\) and \(X_{g-1} \Rightarrow G_g\),
(iii) \(F_{g-1}, G_g\) and \(Y_{g-1} \Rightarrow X_g\),
(iv) \(F_{g-1}, G_{g-1}\) and \(X_{g-2} \Rightarrow Y_g\).

Recall that we have resolutions of pairs

\[
(\alpha^*)^r \rightarrow \alpha^r_g \quad (\beta^*)^r \rightarrow \beta^r_g
\]
and by Propositions 6.2 and 6.4 there are fibrations of pairs

\[
(10.1) \quad \beta^r_{g+i-1} \rightarrow (\alpha^i)^r_g \rightarrow A^0_i
\]
and

\[
(10.2) \quad \alpha^r_{g+i+1} \rightarrow (\beta^i)^r_g \rightarrow A^0_i.
\]

When \(i = 0\), let us write \(\beta^r_{g-1}/u\) and \(\alpha^r_{g+1}/u\) for the respective fibres over \(u \in A^0_0\). It is also convenient to define

\[a_0^0 := \pi_0(A^0_0)\).

**Proof of Theorem 10.2** (i) and (ii). We have that \(\theta\) is \(k\)-trivial: suppose for simplicity that \(k = 2K\) is even. For all \(g' \leq g\), the maps

\[
\prod_{[u] \in a_0^0} \beta^r_{g'/u} \rightarrow \alpha^r_{g'} \quad \prod_{[u] \in a_0^0} \alpha^r_{g'/u} \rightarrow \beta^r_{g'}
\]
are homology epimorphisms in degrees \(* \leq X(g')\) and \(* \leq Y(g')\) respectively (when \(X_{g'}\) and \(Y_{g'}\) hold respectively). Composing them with each other \(k = 2K\) times we obtain a nullhomotopic map

\[
\prod_{[u_1], \ldots, [u_k] \in a_0^0} \alpha^r_{g-K/u_1, \ldots, u_k} \rightarrow \prod_{[u_1], \ldots, [u_k] \in a_0^0} \beta^r_{g-K+1/u_1, \ldots, u_k-1} \rightarrow \cdots \rightarrow \alpha^r_g
\]
which is a homology epimorphism in a range of degrees. Supposing \(X_g\) and \(Y_{g-1}\) hold, this range is given by the range in which the first two maps are homology epimorphisms (as it increases for each subsequent pair of maps), which is

\[
\min(Y(g - K + 1), X(g - K + 1))
\]
which by construction is at least \(F(g)\). The cases of \(k\) odd and the resolution of \(\beta^r_g\) are identical. \(\square\)

**Lemma 10.3.** Suppose \(G_g\). Then

\[
H_0(A^\theta_s; H_*(\beta^r_{g-s})) \rightarrow H_*(((\alpha^s)^r_g)
\]
is a homology epimorphism in degrees \(* \leq G(g-s) + 1\), and both groups are trivial in degrees \(* \leq G(g - s)\).

Suppose \(F_{g-1}\). Then

\[
H_0(A^\theta_s; H_*(\alpha^r_{g-s-1})) \rightarrow H_*(((\beta^s)^r_g)
\]
is a homology epimorphism in degrees $* \leq F(g - s - 1) + 1$, and both groups are trivial in degrees $* \leq F(g - s - 1)$.

**Proof.** This is immediate from the relative Serre spectral sequence for the fibrations (10.1) and (10.2), and the vanishing range for the homology of the fibres implied by the statements $G_g$ and $F_{g-1}$. □

**Proof of Theorem 10.2 (iii).** We study the spectral sequence (RAsSS) for the augmented semi-simplicial space $(\alpha^r)_{g} \to \alpha^r_{g-1}$, for which we write $E^1_{s,t}$. The spectral sequence converges to zero in total degrees $s + t \leq g - 1$, by the connectivity of the resolutions, and we wish to draw a conclusion about the column $s = -1$, $t \leq X(g)$, so $s + t \leq X(g) - 1$. Thus we require that $X(g) \leq g$, but this is guaranteed by one of the conditions on the function $X$.

The previous lemma identifies $E^1_{s,t}$ as zero for $t \leq G(g - s)$ and a quotient of $H_t(\bigoplus \beta_{g-s}^{-s-1})$ for $t \leq G(g - s) + 1$. Recall that $X(g) \leq G(g) + 1$, and as $G(g) \leq G(g - s) + s$, (as the function $G$ increases by at most 1 at each step) it also follows that $X(g) \leq G(g - s) + s + 1$ for all $s \geq 0$. As $G_g$ holds, a chart of the $E^1$-page of the spectral sequence is given on the left of Figure 3.

Figure 3.

In Figure 3 a solid dot denotes an unknown group, the absence of a dot denotes the zero group, and a grey dot denotes elements in the image

$$\bigoplus_{[u_1],...,[u_{s+1}],[u] \in a_0} H_t(\alpha_{g-s-1}^{r+s-1}/u_1,\ldots,u_{s+1},u) \to \bigoplus_{[u_1],...,[u_{s+1}]} \in a_0 E^1_{s,t},$$

which as $Y_{g-1}$ holds certainly occurs for $t \leq Y(g - s) \leq Y(g - 1) - s + 1$, hence $t + s \leq Y(g - 1) + 1$, so in particular $t + s \leq X(g) \leq Y(g - 1) + 1$ as indicated in the figure.

Thus the augmentation map is surjective in degrees $* \leq X(g)$ modulo the images of the higher differentials coming from the grey dots. We will show that these differentials are zero (as they factor through the augmentation map), so that the augmentation map is indeed surjective in this range of degrees.

Consider the map $f = (\alpha_u, \beta_u) : \coprod_{[u] \in a_0^e} \beta_{g-1}^{r-1}/u \to \alpha_g^r$ and the induced map on resolutions. We have given on the right of Figure 3 a chart of the spectral
sequence of the resolution of a single $\beta_g^{r-1}/u$, and it looks the same for a union of them. Note that $X(g) \leq F(g - 1) + 1$ and $F(g - 1 - s) \geq F(g - 1) - s$, so as $F_{g-1}$ holds this gives the vanishing line in the second chart. Note that the map $f$ lifts to the space of 0-simplices in the resolution of $\alpha_g^r$ (because it is the space of 0-simplices). Let us write $\tilde{E}_{s,t}$ for the spectral sequence of the first space, so the vanishing line implies that the group $\tilde{E}_{s,X(g)}^{s+1}$ consists of cycles until $\tilde{E}_{s,X(g)}^{s+1}$. The fact that $f$ lifts to the 0-simplices means that the induced map of spectral sequences is zero on the $s = -1$ column for all pages $r \geq 2$. We then have the following diagram, where the vertical maps are induced by $f$.

\[
\begin{array}{c c c}
E_{s,t}^{s+1} & d_{s+1} & E_{s,t}^{s+1} \\
\downarrow 0 & \downarrow \circ & \downarrow 1 \\
\tilde{E}_{s,t}^{s+1} & d_{s+1} & \tilde{E}_{s,t}^{s+1}
\end{array}
\]

Note 1 is an epimorphism as $Y_{g-1}$ holds, and 2 is an epimorphism as the right hand square commutes, so it follows that the top $d_{s+1}$ differential is zero, as required. □

Proof of Theorem 10.2 (iv). We study the spectral sequence (RAsSS) for the augmented semi-simplicial space $(\beta^r)_g \to \beta_g^r$, for which we write $E_{s,t}^1$. The spectral sequence converges to zero in total degrees $s + t \leq g - 2$, by the connectivity of the resolutions, and we wish to draw a conclusion about the column $s = -1$, $t \leq Y(g)$, so $s + t \leq Y(g) - 1$. Thus we require that $Y(g) \leq g - 1$, but this is guaranteed by one of the conditions on the function $Y$.

The previous lemma identifies $E_{s,t}^1$ as zero for $t \leq F(g - s - 1)$ and a quotient of $H_t(\prod \beta_{g,s-1}^{r+s-1})$ for $t \leq F(g - s - 1) + 1$. Recall that $Y(g) \leq F(g - 1) + 1$, and as $F(g - s) \leq F(g - 1) - s + 1$, (as the function $F$ increases by at most 1 at each step) it also follows that $Y(g) \leq F(g - s) + s$ for all $s \geq 0$. As $F_{g-1}$ holds, a chart of the $E^1$-page of the spectral sequence is given on the left of Figure 4.

\[
\begin{array}{c c c c c c c c}
\beta_g & \alpha_g^{s+1} & \alpha_g^{s+2} & \alpha_g^{s+3} & \alpha_g^{s+4} & \beta_{g-1} & \beta_{g-2} & \beta_{g-3} & \beta_{g-4}
\end{array}
\]

\[
\begin{array}{c c c c c c c c}
Y(g) & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}
\]

\[
\begin{array}{c c c c c c c c}
-1 & 0 & 1 & 2 & 3
\end{array}
\]

\[
\begin{array}{c c c c c c c c}
\beta_g & \alpha_g^{s+1} & \alpha_g^{s+2} & \alpha_g^{s+3} & \alpha_g^{s+4} & \beta_{g-1} & \beta_{g-2} & \beta_{g-3} & \beta_{g-4}
\end{array}
\]

\[
\begin{array}{c c c c c c c c}
Y(g) & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}
\]

\[
\begin{array}{c c c c c c c c}
-1 & 0 & 1 & 2 & 3
\end{array}
\]

**Figure 4.**
In Figure 4, a solid dot denotes an unknown group, the absence of a dot denotes the zero group, and a grey dot denotes elements in the image of stabilisation maps $\alpha$ with respect to $\text{M"obius}$ bands, which is in any case only slightly different, and in fact slightly easier: the stabilisation maps $\alpha$ are in a sense “dual”, whereas the stabilisation map $\mu$ is “self-dual”.

Consider the map $f = (\beta_\ast, \alpha_\ast) : \prod_{[u] \in \alpha_0^e} \alpha_{g-1}^r / u \to \beta_g^r$ and the induced map on resolutions. We have given on the right of Figure 4 a chart of the spectral sequence of the resolution of a single $\alpha_{g-1}^r / u$, and it looks the same for a union of them. Note that $G(g - 1) + 1 \geq Y(g)$ and $G(g - 1 - s) \geq G(g - 1) - s$, so as $G_{g-1}$ holds this gives the vanishing line in the second chart. Note that the map $f$ lifts to the space of 0-simplices in the resolution of $\alpha_g^r$ (because it is the space of 0-simplices). Let us write $\tilde{E}_{s,t}^r$ for the spectral sequence of the first space, so the vanishing line implies that the group $\tilde{E}_{s,Y(g)-s}^r$ consists of cycles until $\tilde{E}_{s,Y(g)-s}^{r+1}$. The fact that $f$ lifts to the 0-simplices means that the induced map of spectral sequences is zero on the $s = -1$ column for all pages $r \geq 2$. We then have the following diagram, where the vertical maps are induced by $f$.

$$
\begin{array}{ccc}
E_{-1,Y(g)}^{s+1} & \xrightarrow{d_{s+1}} & E_{s,Y(g)-s}^{s+1} \\
\downarrow 0 & & \downarrow \circlearrowleft 1 \\
\tilde{E}_{-1,Y(g)}^{s+1} & \xleftarrow{d_{s+1}} & \tilde{E}_{s,Y(g)-s}^{s+1} \\
\end{array}
$$

Note $\circlearrowleft$ is an epimorphism as $X_{g-2}$ holds, and $\circlearrowright$ is an epimorphism as the right hand square commutes, so it follows that the top $d_{s+1}$ differential is zero, as required. \hfill $\square$

11. Proof of stability for non-orientable surfaces with boundary

The only difference in the non-orientable case is the proof for stabilisation with respect to $\text{M"obius}$ bands, which is in any case only slightly different, and in fact slightly easier: the stabilisation maps $\alpha$ and $\beta$ are in a sense “dual”, whereas the stabilisation map $\mu$ is “self-dual”.

Suppose that the conditions of Theorem 9.4 hold: the moduli spaces of 0-surfaces stabilise at genus $h$ and are $k$-trivial (or stabilise for projective planes at genus $h'$ and are $k'$-trivial for projective planes). Then $H_0(\alpha_n^r) = H_0(\beta_n^r) = 0$ for all $n \geq h$, (or $H_0(\mu_n^r) = 0$ for all $n \geq h'$), so certainly the statements $F_h$ and $G_h$ (or $H_{h'}$) hold, and consequently $X_h$ and $Y_h$ (or $Z_{h'}$) hold too. This starts an induction and the following theorem provides the inductive step, hence implies Theorem 9.4.
Theorem 11.1. Suppose the hypotheses of Theorem 9.4 hold, then there are implications

(i) \( X_n \) and \( Y_n \Rightarrow F_n \),
(ii) \( X_{n-2} \) and \( Y_n \Rightarrow G_n \),
(iii) \( F_{n-2}, G_n \) and \( Y_{n-2} \Rightarrow X_n \),
(iv) \( F_{n-2}, G_{n-2} \) and \( X_{n-4} \Rightarrow Y_n \).

There are also implications

(v) \( Z_n \Rightarrow H_n \),
(vi) \( Z_{n-2} \) and \( H_{n-1} \Rightarrow Z_n \).

The proofs of (i) – (iv) are identical to those in the last section and we omit them. Recall that we have a resolution of pairs

\[ (\mu^r)_{n} \Rightarrow \mu^r_n \]

and by Proposition 7.3 there are fibrations of pairs

\[ (11.1) \quad \mu^r_{n-i-1} \Rightarrow (\mu^r)_n \Rightarrow A^0_i. \]

When \( i = 0 \), let us write \( \mu^r_{n-1}/u \) for the fibre over \( u \in A^0_0 \).

Proof of Theorem 11.1 (v). We have that \( \theta \) is \( k' \)-trivial for projective planes. For all \( n' \leq n \), the maps

\[ \prod_{[u] \in a^0_0} \mu^r_{k-1}/u \Rightarrow \mu^r_{n'} \]

are homology epimorphisms in degrees \(* \leq Z(n')\). Composing them with each other \( k' \) times we obtain a nullhomotopic map

\[ \prod_{[u], ..., [u_k] \in a^0_0} \mu^r_{n-k}/u_1, ..., u_k \Rightarrow \prod_{[u], ..., [u_{k'-1}] \in a^0_0} \mu^r_{g-k'+1}/u_1, ..., u_{k'-1} \Rightarrow a^r_g \]

which is a homology epimorphism in a range of degrees. As \( Z_n \) holds, this range is given by the range in which the first map is a homology epimorphism (as it increases for each subsequent pair of maps), i.e. \( Z(n-k') \), which by construction is at least \( H(n) \).

Lemma 11.2. Suppose \( H_{n-1} \). Then

\[ H_0(A^0_0; H_s(\mu^r_{n-s-1})) \Rightarrow H_s((\mu^r)^r_n) \]

is a homology epimorphism in degrees \(* \leq H(n-s-1) + 1 \), and both groups are trivial in degrees \(* \leq H(n-s-1) \).

Proof. This is immediate from the relative Serre spectral sequence for the fibration \([11.1]\), and the vanishing range for the homology of the fibres implied by the statement \( H_{n-1} \).

Proof of Theorem 11.1 (vi). We study the spectral sequence for the augmented semi-simplicial space \( (\mu^r)^r_n \Rightarrow \mu^r_n \), for which we write \( E^1_{s,t} \). The spectral sequence converges to zero in total degrees \( s + t \leq \left[ \frac{n}{2} \right] - 1 \), by the connectivity of the Möbius band resolution, and we wish to draw a conclusion about the column \( s = -1, t \leq Z(n) \), so \( s + t \leq Z(n) - 1 \). Thus we require that \( Z(n) \leq \left[ \frac{n}{2} \right] \), but this is guaranteed by one of the conditions on the function \( Z \).

The previous lemma identifies \( E^1_{0,t} \) as zero for \( t \leq H(n-s-1) \) and a quotient of \( H_t(\prod \mu^r_{n-s-1}) \) for \( t \leq H(n-s-1) + 1 \). Recall that \( Z(n) \leq H(n-2) + 1 \) and
As $H(n) \leq H(n-s) + s$ (as the function $H$ increases by at most 1 at each step) it also follows that $Z(n) \leq H(n-s-1) + s + 1$ for all $s \geq 0$. As $H_{n-1}$ holds, a chart of the $E^1$-page of the spectral sequence is given as the figure below.

In the figure, a solid dot denotes an unknown group and the absence of a dot denotes the zero group. Thus the augmentation map is surjective in degrees $* \leq Z(n)$. □

12. Manifolds without boundary

In order to prove homology stability of $M^\theta(F)$ for surfaces $F$ without boundary, we cannot use the resolutions of the previous two sections which use arcs with ends on the boundary of a surface. Instead, we will define a new resolution using discs in the surface, and resolve $M^\theta(F)$ by moduli spaces of surfaces of the same genus but strictly more boundary components than $F$. The resolution is quite general, and exists for manifolds of any dimension and having any tangential structure.

12.1. The disc resolution. Fix a $d$-manifold $M$, possibly with boundary, and let $M^\theta(M; \delta)$ be the moduli space of $\theta$-manifolds with underlying manifold diffeomorphic to $M$ and boundary conditions $\delta$, as defined at the beginning of §5. Let $D(M)^i := \text{Emb}(\bigsqcup_{j=0}^i D^d_j, M)$ where $D^d_j$, $j = 0, \ldots, i$ denote standard closed unit discs in $\mathbb{R}^d$.

Define

$$D^\theta(M; \delta)^i := D(M)^i \times \text{Bun}_{\delta}(TM, \theta^* \gamma_d; \delta) / \text{Diff}(M)$$

where the group acts diagonally. The map $d_j : D(M)^i \to D(M)^{i-1}$ that forgets the $j$-th disc induces a map $d_j : D^\theta(M; \delta)^i \to D^\theta(M; \delta)^{i-1}$, and there are maps $D^\theta(M; \delta)^i \to M^\theta(M; \delta)$ that forget all the discs.

Proposition 12.1. $D^\theta(M; \delta)^\bullet \to M^\theta(M; \delta)$ is an augmented semi-simplicial space, and a resolution.

Proof. We must show that $\|D^\theta(M; \delta)^\bullet\| \to M^\theta(M; \delta)$ is a homotopy equivalence. Fix a point $\xi \in M^\theta(M; \delta)$. The maps $D^\theta(M; \delta)^i \to M^\theta(M; \delta)$ have homotopy fibre $D(M)^i$ over $\xi$, so the homotopy fibre after realisation is $\|D(M)^\bullet\|$, which we must show is contractible.

It is convenient to replace $\|D(M)^\bullet\|$ by an equivalent semi-simplicial space whose space of $i$-simplices is the space of $(i+1)$ distinct points of $M$ each equipped with a framing of the tangent space of $M$ at that point. One may see that this
is equivalent by the fibration sequence over the total space of the frame bundle of $M$,

$$\ast \simeq \text{Fib} \longrightarrow \text{Emb}(\tilde{D}^d, M^d) \xrightarrow{\pi} F_r(M)$$

where $\pi$ is the map sending an embedding $e$ to the image under $De$ of the standard frame at 0. That this map has contractible fibres follows by a scanning argument, similar to that which shows that the diffeomorphism group of an open disc is homotopy-equivalent to the orthogonal group. Call this semi-simplicial space $F(M)^\bullet$.

Suppose first that $M$ has a boundary component $\partial_b M$, and consider the spectral sequence [SSS] for this simplicial space

$$E^2_{s,t} = H_s(H_t(F(M)^\bullet, \partial_b)) \Rightarrow H_{s+t}(\|F(M)^\bullet\|).$$

Define a map $\bar{s} : F(M)^i \to F(M)^{i+1}$ by adding a new framed point near $\partial_0 M$ and declaring it to be the first point. This defines a chain contraction in positive simplicial degrees of the complex $(H_t(F(M)^\bullet, \partial_b), s)$, so $E^2_{s,t}$ is concentrated along $s = 0$ and

$$H_t(F(M)^1) \xrightarrow{d_0 - d_1} H_t(F(M)^0) \longrightarrow H_t(\|F(M)^\bullet\|) \longrightarrow 0$$

is exact. Thus it is enough to show that $d_0 - d_1$ is an epimorphism in positive degrees. Let $X \subset M$ be a $(d - 1)$-skeleton (so the inclusion is a homotopy equivalence, as $M$ is not closed), and $p \in M$ be a point not in $X$. Choosing a framing at $p$ defines a map

$$s : F(M)^0|_X \longrightarrow F(M)^1 \quad x \mapsto (x, p)$$

from the frame bundle of $M$ restricted to $X$ to $F(M)^1$. Then $d_0 \circ s : F(M)^0|_X \to F(M)^0$ is the inclusion, so a homology equivalence, and $d_1 \circ s : F(M)^0|_X \to F(M)^0$ is the constant map $p$. Thus $(d_0 - d_1) \circ s$ is surjective on the homology of $F(M)^0$ in positive degrees, so $d_0 - d_1$ is surjective on homology in positive degrees, and so $H_s(\|F(M)^\bullet\|)$ is trivial. It is easy to check that $\|F(M)^\bullet\|$ is also simply connected, so it is contractible (although all we really require for our applications is that it is acyclic).

Now suppose that $M$ is a $d$-manifold without boundary, let $D^d \hookrightarrow M$ be an embedded closed disc and define $\bar{M} = M - \bar{D}^d$ be the complement of the interior. Then $\bar{M} \to M$ induces an inclusion $F(\bar{M})^\bullet \to F(M)^\bullet$. In simplicial degree $i$ we can identify the homotopy cofibre as $\vee_{j=0}^i F(M)^{i-1} \ltimes (F_r(M)|_{D^d}, F_r(M)|_{\partial D^d})$, or equivalently $(F(M)^{i-1} \ltimes (F_r(M)|_{D^d}, F_r(M)|_{\partial D^d})) \ltimes \{0, \ldots, i\}$. The part coming from the frame bundle splits off as a constant factor, and we see that the levelwise cofibre is the semi-simplicial space

$$(F_r(M)|_{D^d}, F_r(M)|_{\partial D^d}) \ltimes (F(M)^{i-1} \ltimes \{0, \ldots, \bullet\}).$$

The second factor is just the construction of Appendix B applied to the semi-simplicial space $F(\bar{M})^\bullet$. The homotopy fibre of

$$\|\cdots F(\bar{M})^{i+1} \Rightarrow F(\bar{M})^i\| \xrightarrow{d_i} F(\bar{M})^{i-1}$$

over some configuration of $i$ framed points is simply $\|F(\bar{M} - \{i \text{ discs}\})^\bullet\|$ which is contractible by the case treated above, as $\bar{M}$ has boundary. Thus by the discussion
in Appendix [B]

\[ \|F(M)\|^{-1} \times \{0, \ldots, \bullet\}, \| \Sigma \| F(M) \| \simeq * \]

and so both \( F(M) \) and the cofibre of \( F(M) \rightarrow F(M^\bullet) \) have contractible realisation, so \( \|F(M^\bullet)\| \simeq * \) as required. \( \square \)

Relating the spaces \( D^\theta(M; \delta)^{i} \) to \( M^\theta(M - \{(i+1) \text{ discs}\}) \) is slightly complicated. Let \( X_i := \coprod_{j=0}^c D^d_j \) and define the space

\[ B^\theta_i := \text{Bun}(TX_i, \theta^\gamma_d) \]

so there is a continuous map \( \tilde{\pi} : D^\theta(M)^i \rightarrow B^\theta_i \) sending a tuple \((M^\ell, c_0, \ldots, c_i)\) to \( \coprod_{j=0}^c e^\ell_j \). There is a continuous map

\[ \pi : D^\theta(M; \delta)^i \rightarrow B^\theta_i \times \left( \text{Emb}(X_i, M)/\text{Diff}^\theta(M) \right) \]

given by \( \tilde{\pi} \) on the first factor, and the map which takes the ordinary quotient on the second factor.

**Lemma 12.2.** If \( M \) is orientable there is a natural bijection

\[ \text{Emb}(X_i, M)/\text{Diff}^\theta(M) = \begin{cases} \{ \pm 1 \}^{i+1} & \partial M \neq \emptyset \\ \{ \pm 1 \}^i & \partial M = \emptyset \end{cases} \]

If \( M \) is not orientable \( \text{Emb}(X_i, M)/\text{Diff}^\theta(M) = * \).

**Proof.** If \( M \) is orientable with boundary, given two embeddings \( c_0, c_1 : X_i \rightarrow M \) there is an orientation-preserving diffeomorphism \( \varphi : M \rightarrow M \) carrying \( c_0 \) to \( c_1 \), after perhaps changing \( c_0 \) by a standard orientation-reversing involution on some discs of \( X_i \). Thus the quotient is in bijection with the set of choices of orientation for each disc, so \( \{ \pm 1 \}^{i+1} \). If \( M \) has no boundary the same is true, except that now there is also an orientation-reversing diffeomorphism of \( M \) which acts as \(-1\) on \( \text{Emb}(X_i, M)/\text{Diff}^\theta(M) = \{ \pm 1 \}^{i+1} \), leaving quotient in bijection with \( \{ \pm 1 \}^i \).

If \( M \) is non-orientable any embedding can be carried to any other by a diffeomorphism: given the above discussion, to see this it is enough to show that \( c_0 \) can be changed by a standard orientation-reversing involution on one disc \( D^d \rightarrow M \) of \( X_i \). This can be done by choosing an orientation-reversing loop based at the centre of \( D^d \), and taking a diffeomorphism that drags \( D^d \) along this loop. \( \square \)

The map \( \pi \) does not hit every path component: its image on path components defines a subset

\[ \pi_0(C^\theta_i(M)) \subseteq \pi_0(B^\theta_i) \times \left( \text{Emb}(X_i, M)/\text{Diff}^\theta(M) \right) \]

consisting of those pairs \((\ell : TX_i \rightarrow \theta^\gamma_d, [c : X_i \rightarrow M])\) such that the diagram

\[
\begin{array}{ccc}
TX_i & \xrightarrow{\ell} & \theta^\gamma_d \\
\downarrow_{\delta} & & \uparrow_{\delta} \\
TM & \xleftarrow{T M|_{\partial M}} & \text{TM} \end{array}
\]

has a relative extension. Let \( C^\theta_i(M) \subseteq B^\theta_i \times \left( \text{Emb}(X_i, M)/\text{Diff}^\theta(M) \right) \) be the union of those path components in \( \pi_0(C^\theta_i(M)) \).
Proposition 12.3. There are homotopy fibre sequences

\[ (12.1) \quad \mathcal{M}^\theta(M - \{(i + 1) \text{ discs}\}; \delta') \rightarrow \mathcal{D}^\theta(M; \delta)^i \rightarrow C^\theta_i(M) \]

meaning that the homotopy fibre over each point \( b \) of the base is of the homotopy type claimed, for some \( \delta' \) depending on \( b \).

Proof. Before we take the homotopy quotient by \( \text{Diff}_\theta(M) \), there is a \( \text{Diff}_\theta(M) \)-invariant map

\[ D(M)^i \times \text{Bun}_\theta(TM; \theta^*\gamma_d; \delta) \rightarrow C^\theta_i(M) \]

which is a fibration. The fibre over a point \( b \) in the base consists of the data of

(i) \( (i + 1) \) embedded discs in \( M \) with their relative orientations fixed by \( b \),
(ii) a \( \theta \)-structure on \( M \) that is fixed on the discs by \( b \).

The group \( \text{Diff}_\theta(M) \) acts transitively on the set of embeddings of \( (i + 1) \) discs with fixed relative orientations, and the stabiliser group is \( \text{Diff}_\theta(M \setminus X_i) \). The space of \( \theta \)-structures on \( M \) fixed on the discs by \( b \) is homeomorphic to \( \text{Bun}_\theta(TM \setminus X_i, \theta^*\gamma_d; \delta \cup \partial b) \), so after taking the homotopy quotient of the fibres by \( \text{Diff}_\theta(M) \) we obtain

\[ \text{Bun}_\theta(TM \setminus X_i, \theta^*\gamma_d; \delta \cup \partial b)/\text{Diff}_\theta(M \setminus X_i) \simeq \mathcal{M}^\theta(M - \{(i + 1) \text{ discs}\}; \delta'). \]

\[ \square \]

Given an embedding \( \iota : M \hookrightarrow M' \) and a \( \theta \)-structure on \( M' \setminus M \) that agrees with \( \delta \) on \( \partial M \) and \( \delta' \) on \( \partial M' \), we obtain a simplicial map \( \iota^* : \mathcal{D}^\theta(M; \delta)^i \rightarrow \mathcal{D}^\theta(M'; \delta')^i \)

and for each \( i \) a map

\[ \pi_0(C^\theta_i(\iota)) : \pi_0(C^\theta_i(M)) \rightarrow \pi_0(C^\theta_i(M')). \]

We will be interested in conditions on \( \theta \) which ensure that these maps are always bijections. One simplification is that we only need to check the condition in simplicial degree zero.

Lemma 12.4. The maps \( \pi_0(C^\theta_i(\iota)) \) are bijections for all simplicial degrees \( i \) and all manifold inclusions \( \iota \) if and only if the maps \( \pi_0(C^\theta_0(\iota)) \) are bijections for all manifold inclusions \( \iota \).

Proof. There are natural epimorphisms

\[ \pi_0(C^\theta_i(M)) \rightarrow \pi_0(C^\theta_{i-1}(M)) \]

given by forgetting the last disc, with point preimages all naturally in bijection with \( \pi_0(C^\theta_0(M \setminus \bigsqcup_{i=0}^{i-1} D^d)) \). The claim now follows by induction on \( i \). \( \square \)

12.2. Closing the last boundary. Let us return to the case of surfaces. We wish to analyse the stabilisation maps \( \gamma : \mathcal{M}^\theta(F^1; \delta) \rightarrow \mathcal{M}^\theta(F) \) given by choosing a \( \theta \)-structure on the disc extending a given one on the boundary (where this is possible).

Definition 12.5. Say that a tangential structure \( \theta : X \rightarrow BO(2) \) is closeable if whenever there is a stabilisation map \( \gamma : \mathcal{M}^\theta(F^1; \delta) \rightarrow \mathcal{M}^\theta(F) \), the induced map

\[ \pi_0(C^\theta_0(F^1)) \rightarrow \pi_0(C^\theta_0(F)) \]

is a bijection.
Theorem 12.6. Let \( \theta \) be a tangential structure such that all the stabilisation maps \( \gamma^{r+1} : M^\theta(F^{r+1}) \to M^\theta(F^r) \) are homology isomorphisms for \( \ast \leq G(F) \) and homology epimorphisms in all degrees, as long as \( r > 0 \) (as in Theorem 9.3 or 9.4). Let \( \gamma : M^\theta(F^1 ; \delta) \to M^\theta(F) \) be a stabilisation map which caps off the last boundary component, and suppose \( \theta \) is closeable.

Then \( \gamma \) is a homology isomorphism in degrees \( \ast \leq G(F) \) and a homology epimorphism in degrees \( \ast \leq G(F) + 1 \).

Proof. The map \( \gamma \) induces a simplicial map
\[
\gamma^* : D^\theta(F^1 ; \delta^*) \to D^\theta(F)^*
\]
on disc resolutions, and we study the associated map of spectral sequences (RefSS). These are
\[
E^{1}_{s,t}(F^1) = H_t(D^\theta(F^1 ; \delta^*)) \Rightarrow H_{s+t}(M^\theta(F^1 ; \delta))
\]
and
\[
E^{1}_{s,t}(F) = H_t(D^\theta(F)^*) \Rightarrow H_{s+t}(M^\theta(F))
\]
On the \( E^1 \)-page the map of spectral sequences is \( H_t(D^\theta(F^1 ; \delta^*)) \to H_t(D^\theta(F)^*) \), which may be studied via the map of Serre spectral sequences for the fibrations [12.1] of Proposition [12.3]
\[
H_p(C_s(F^1); H_q(M^\theta(F^{s+2}))) \Rightarrow H_p(C_s(F); H_q(M^\theta(F^{s+1})))
\]
By the assumptions of closability \( C_s(F^1) \cong C_s(F) \), so the map on \( E^2 \)-pages is an isomorphism for \( q \leq G(F) \) and an epimorphism in all bidegrees. Thus the map on \( E^\infty \)-pages is an isomorphism in total degrees \( \ast \leq G(F) \) and an epimorphism in total degrees \( \ast \leq G(F) + 1 \).

This shows that the map \( E^{1}_{s,t}(F^1) \to E^{1}_{s,t}(F) \) is an isomorphism for \( t \leq G(F) \) and an epimorphism for \( t \leq G(F) + 1 \). By the same reason as above, this implies that the map on \( E^\infty \)-pages is an isomorphism in total degree \( \ast \leq G(F) \) and an epimorphism in total degree \( \ast \leq G(F) + 1 \) as required.

12.3. Examples and non-examples.

12.3.1. Oriented surfaces. Let us consider the tangential structure \( \theta : BSO(2) \to BO(2) \), so let \( \Sigma \) be an orientable surface. In this case \( \pi_0(B^0_{SO(2)}) = \{ \pm 1 \} \) as there are two orientations of \( D^2 \) and \( \pi_0(C^0_{SO(2)}(\Sigma)) \subseteq \{ \pm 1 \} \times (\Emb(X_0, \Sigma) / \Diff^0(\Sigma)) \) is
\[
\pi_0(C^0_{SO(2)}(\Sigma)) = \begin{cases} \{(1,1),(-1,-1)\} \subset \{\pm 1\}^2 & \Sigma \text{ has boundary} \\ \{(1,*),(-1,*)\} \subseteq \{1\} \times \{*\} & \Sigma \text{ has no boundary} \end{cases}
\]
as there is always a unique orientation of \( \Sigma \) extending one specified on a disc in \( \Sigma \). For \( \Sigma \) without boundary, the map \( C^0_{SO(2)}(\Sigma^1) \to C^0_{SO(2)}(\Sigma) \) is now easily seen to be a bijection: it sends \((1,1)\) to \((1,*)) \) and \((-1,-1)\) to \((-1,*)\). Thus Theorem 12.6 applies: the map \( \gamma : M^{SO(2)}(\Sigma^1) \to M^{SO(2)}(\Sigma) \) is a homology isomorphism in degrees \( 3\ast \leq 2g(\Sigma) \), and a homology epimorphism in degrees \( 3\ast \leq 2g(\Sigma) + 3 \).
12.3.2. **Unoriented surfaces.** Let us consider the tangential structure \( \theta = \text{Id} : BO(2) \to BO(2) \), and let \( S \) be a non-orientable surface. In this case the relevant set is \( \pi_0(B^0(2)) \times (\text{Emb}(X_0, S)/\text{Diff}^0(S)) = * \) so \( \pi_0(C^0(2)(S)) = * \) for any non-orientable surface \( S \), and so Theorem 12.6 applies: the map \( \gamma : \mathcal{M}^O(2)(S^1) \to \mathcal{M}^O(2)(S) \) is a homology isomorphism in degrees \( 3* \leq n(S) \) and a homology epimorphism in degrees \( 3* \leq n(S) + 3 \).

12.3.3. **Background spaces.** Let us consider the tangential structure \( \theta \times Y : X \times Y \to BO(2) \). If \( Y \) is connected then the forgetful map \( B^0_{\theta \times Y} \to B^0_{\theta} \) induces a bijection on path components and so \( \pi_0(C^0_{\theta \times Y}(\theta)) \) is stable if and only if \( \pi_0(C^0_{\theta}(-)) \) is stable. Thus adding “maps to a simply connected \( Y \)” to oriented or unoriented surfaces as above still has stability for closing the last boundary component.

12.3.4. **A non-example.** An important non-example is \( \theta = \text{Id} : BO(2) \to BO(2) \) for orientable surfaces, as remarked in Proposition 12.1. One may compute that \( \pi_0(C^0(2)(\Sigma)) = * \) but \( \pi_0(C^0(2)(\Sigma_g)) = \{ \pm 1 \} \).

More elaborately, note that if an orientable surface \( \Sigma \) has boundary then the forgetful map \( \mathcal{M}^SO(2)(\Sigma) \to \mathcal{M}^O(2)(\Sigma) \) is a homotopy equivalence, whereas if it does not have boundary there is a fibre sequence

\[
\mathcal{M}^SO(2)(\Sigma) \to \mathcal{M}^O(2)(\Sigma) \to BO\mathbb{Z}/2.
\]

This identifies \( H^*(\mathcal{M}^O(2)(\Sigma_g) ; \mathbb{Q}) \) with the invariant subring \( \mathbb{Q}[\kappa_1, \kappa_2, \ldots, \kappa_2]/(\text{involutions act as } \kappa_i \mapsto (-1)^i \kappa_i) \) up to degree \( 3* \leq 2g - 2 \). Thus the rational cohomology groups are independent of \( g \) in a range, but the isomorphism is not induced by the maps

\[
\mathcal{M}^O(2)(\Sigma_g) \to \mathcal{M}^O(2)(\Sigma_g^1) \to \mathcal{M}^O(2)(\Sigma_g+1).
\]

One can manufacture a moduli space that maps to both the spaces \( \mathcal{M}^O(2)(\Sigma_g) \) and \( \mathcal{M}^O(2)(\Sigma_g+1) \) and exhibits this stability, but it is not of the type discussed in this paper (it is essentially the classifying-space of the group of diffeomorphisms of \( \Sigma_g^1 \) that either fix the boundary pointwise or induce a standard orientation-reversing involution on it).

A. **On complexes of arcs in surfaces**

In the body of this paper we have required information on the connectivities of certain simplicial complexes which are slight modifications of those discussed in the literature. The purpose of this appendix is to deduce information about the complexes we need from that available in [11, 25]. It is included as an appendix as it may be of independent interest.

A.1. **Arcs in orientable surfaces.** Let \( \Sigma \) be an orientable surface with boundary, and let \( b_0, b_1 \) be distinct points on \( \partial \Sigma \). Let \( B\Sigma(\Sigma, \{b_0, b_1\}, \{b_0\}) \) be Harer’s simplicial complex [11], whose vertices are isotopy classes of properly embedded arcs in \( \Sigma \) from \( b_0 \) to \( b_1 \), and a collection of such span a simplex if they have representatives which are disjoint and do not disconnect \( \Sigma \). For any simplex \( \sigma \subset B\Sigma(\Sigma, \{b_0, b_1\}, \{b_0\}) \), one can order the arcs clockwise at \( b_0 \) and anticlockwise at \( b_1 \), and compare these orderings. Let \( B_0(\Sigma) \) denote the subcomplex consisting of those simplices where these two orderings agree. If \( b_0, b_1 \) lie on the same boundary component, this is the complex of the same name of Ivanov [14] and we
recover his theorem on its connectivity. We are grateful to N. Wahl for suggesting the following line of argument.

**Theorem A.1.** $B_0(\Sigma)$ is $(g - 2)$-connected.

*Proof.* Note that the theorem is clearly true for $g \leq 1$, as then the complex is either empty or discrete (so a wedge of copies of $S^0$). Thus we proceed by induction on $g$.

Recall that Harer has shown that $BX(\Sigma, \{b_0, b_1\}, \{b_0\})$ is homotopy equivalent to a wedge of copies of $S^{2g-2+\partial}$, where $\partial$ is the number of boundary components containing the $b_i$. His proof was slightly incomplete, but has been fixed in [25]. For $k \leq g - 2$ let $f : S^k \to B_0(\Sigma)$ be a continuous map, which we may assume to be simplicial for some triangulation of $S^k$, and $\hat{f} : D^{k+1} \to BX(\Sigma, \{b_0, b_1\}, \{b_0\})$ be a nullhomotopy in $BX(\Sigma, \{b_0, b_1\}, \{b_0\})$, which we may again suppose to be simplicial for some triangulation of $D^{k+1}$. We will show that $\hat{f}$ can be chosen to lie in $B_0(\Sigma)$.

Note that the vertices of a simplex $\sigma \subset BX(\Sigma, \{b_0, b_1\}, \{b_0\})$ are totally ordered, with the order given by the induced ordering at $b_0$. Thus it makes sense to talk of the front $p$-simplex or back $q$-simplex of $\sigma$. For each simplex $\sigma \subset D^{k+1}$, we may compute the maximal number $v(\sigma)$ such that the orderings at $b_0$ and $b_1$ agree on the first $v$ vertices of $\sigma$.

We may decompose a simplex $\sigma \subset D^{k+1}$ uniquely as $\sigma^g \ast \sigma^b$ where $\sigma^g$ is the face on the first $v(\sigma)$ vertices of $\sigma$, and $\sigma^b$ is the face on the remaining vertices. Note that $v(\sigma^b) = 0$ and $\hat{f}(\sigma^g)$ lies in $B_0(\Sigma)$. Thus it is enough to fix all simplices $\sigma^b$ with $v(\sigma^b) = 0$, which we call a fully bad simplex. Let $\sigma^b$ be a maximal (under inclusion of faces) fully bad simplex.

Let $\Sigma \setminus \sigma^b$ be the surface obtained by cutting along the vertices of $\sigma^b$. Note that $g(\Sigma \setminus \sigma^b) > g(\Sigma) - |\sigma^b|$ as

(i) if $b_0$ and $b_1$ lie on a single boundary component, the most genus that can be lost by cutting along $|\sigma^b|$ arcs is $|\sigma^b|$, and this happens *precisely* when the arcs are correctly ordered,

(ii) if $b_0$ and $b_1$ lie on different boundary components, then given $|\sigma^b|$ arcs, cutting along the first arc does not reduce the genus, but joins the two boundary components together, and cutting long the remaining arcs loses at most one genus per arc.

On the cut surface $\Sigma \setminus \sigma^b$ there are multiple copies of both $b_0$ and $b_1$, but we can single out a preferred copy of each as the copies that lie on the boundary of $\Sigma \setminus \sigma^b$ at the interface of $\partial \Sigma$ and the new boundary coming from the first arc of $\sigma^b$. We claim that there is a commutative diagram

$$\begin{align*}
\text{Link}(\sigma^b) \cong S^{k-|\sigma^b|+1} & \xrightarrow{\hat{f}} BX(\Sigma, \{b_0, b_1\}, \{b_0\}) \\
B_0(\Sigma \setminus \sigma^b) & \to BX(\Sigma \setminus \sigma^b, \{b_0, b_1\}, \{b_0\}).
\end{align*}$$

That $\hat{f}$ lifts canonically to the complex $BX(\Sigma \setminus \sigma^b, \{b_0, b_1\}, \{b_0\})$ follows as for any $\sigma \subset \text{Link}(\sigma^b)$, $v(\sigma \ast \sigma^b) = |\sigma|$ as otherwise there are more arcs that can be added.
to $\sigma^b$ to keep it fully bad. Thus the arcs of $\sigma$, when considered to lie in $\Sigma \setminus \sigma^b$, must start and end at the preferred copies of $b_0, b_1$. In fact this canonical lift lies in $B_0(\Sigma \setminus \sigma^b)$ as $\sigma \subset \text{Link}(\sigma^b)$ means that $v(\sigma) = |\sigma|$, so $\sigma$ consists of correctly ordered arcs.

The complex $B_0(\Sigma \setminus \sigma^b)$ is $(g(\Sigma \setminus \sigma^b) - 2)$-connected by hypothesis, and we may compute

$$g(\Sigma \setminus \sigma^b) > g(\Sigma) - |\sigma^b| \geq k - |\sigma^b| + 2$$

and so $k - |\sigma^b| + 1 \leq g(\Sigma \setminus \sigma^b) - 2$ and the downwards vertical map is nullhomotopic.

Write $F : D^{k - |\sigma^b| + 2} \to B_0(\Sigma \setminus \sigma^b)$ for a nullhomotopy.

We now define

$$\hat{f} \ast F : \partial \sigma^b \ast D^{k - |\sigma^b| + 2} \cong St(\sigma^b) \to BX(\Sigma, \{b_0, b_1\}, \{b_0\})$$

a modification of $\hat{f}$ on the star of $\sigma^b$. The new simplices are of the form $\alpha \ast \beta$ for $\alpha \subset \sigma^b$ a proper face, and $\beta$ mapping through $B_0(\Sigma \setminus \sigma^b)$ to $BX(\Sigma, \{b_0, b_1\}, \{b_0\})$.

Now $v(\alpha \ast \beta) \geq |\beta|$, so is positive unless $\beta = \emptyset$. But then $\alpha \subset \sigma^b$ has lower dimension, so we have replaced $St(\sigma^b)$ by simplices which are either not fully bad, or are fully bad but are of lower dimension than $\sigma^b$.

\[\Box\]

### A.2. Arcs in non-orientable surfaces

Let $S$ be a non-orientable surface and $b_0, b_1$ be oriented points on $\partial S$ i.e. points with a chosen orientation of the tangent space of $\partial S$. Let $C(S, b_0, b_1)$ denote the simplicial complex with vertices the isotopy classes of 1-sided arcs from $b_0$ to $b_1$, and where a collection of vertices span a simplex if they can be made disjoint, and have connected non-orientable (or genus 0) complement. This is related to the complexes $G(S, \Delta)$ of Wahl [25].

Using the orientation of the tangent space at each $b_i$, given by the chosen orientation of the boundary at $b_i$ and the inwards normal vector, we can order the arcs clockwise or anticlockwise at each of the $b_i$.

Let $b_0, b_1$ both lie on the same boundary component, and have coherent orientations. Let $C_0(S)$ denote the subcomplex of $C(S, b_0, b_1)$ where the clockwise ordering at $b_0$ coincides with the anticlockwise ordering at $b_1$. Note that for $\sigma^k$ a $k$-simplex of this complex, $S^r_\sigma \setminus \sigma^k \cong S^n_{n-k-1}$.

The argument in this case is slightly different than that of the other theorems in this appendix.

**Theorem A.2.** $C_0(S)$ is $\lfloor \frac{n-1}{2} \rfloor - 1$-connected.

**Proof.** Consider the subcomplex $G_0(S, \overrightarrow{b_0})$ of Wahl’s $G(S, \overrightarrow{b_0})$ consisting of those simplices which are ordered *palindromically*: the $k^{th}$ arc in the clockwise order is the $k^{th}$ arc in the anticlockwise order, for all $k$. Recall that $G(S, \overrightarrow{b_0})$ is $(n(S) - 3)$-connected [25, Theorem 3.3].

The complex $C_0(S, b_0, b_1)$ is homeomorphic to $G_0(S, \overrightarrow{b_0})$ as follows. Choose a path in the boundary from $b_1$ to $b_0$, then composing arcs with this path defines a map $C_0(S, b_0, b_1) \to G_0(S, \overrightarrow{b_0})$ which is easily seen to be simplicial and a bijection on sets of simplices.

Note first that the theorem is trivially true for $n(S) \leq 3$, so suppose it holds for all genera below $n(S)$. Let $k \leq \left\lfloor \frac{n(S)-1}{2} \right\rfloor - 1$ and take a continuous map $f : S^k \to G_0(S, \overrightarrow{b_0})$, which we may suppose is simplicial for some triangulation of the $k$-sphere. The composition $S^k \to G_0(S, \overrightarrow{b_0}) \to G(S, \overrightarrow{b_0})$ is nullhomotopic,
by the discussion above, so we may choose a nullhomotopy \( \hat{f} : D^{k+1} \to G(S, \overrightarrow{b_0}) \), which we may again suppose to be simplicial. We will modify this map relative to \( \partial D^{k+1} \) to have image in \( G_0(S, \overrightarrow{b_0}) \). Call a simplex \( \sigma^b \subseteq D^{k+1} \) fully bad if the first clockwise arc is not the first anticlockwise arc. For any simplex \( \sigma \), there is a maximal collection of vertices \( \{v_1, v_2, ..., v_k\} \) such that \( v_i \) is the \( i \)th arc of \( \sigma \) in both the clockwise and anticlockwise orderings. Call the face spanned by these \( \sigma^b \), and the face spanned by the remaining vertices \( \sigma^b \). Note that \( \sigma^b \) is fully bad, otherwise we can add its first arc to \( \sigma^b \). As \( \sigma = \sigma^b \circ \sigma^b \) and \( \hat{f}(\sigma^b) \subseteq G_0(S, \overrightarrow{b_0}) \), it is enough to remove the fully bad simplices in \( D^{k+1} \).

Let \( \sigma^b \subseteq D^{k+1} \) be a maximal (under face inclusion) fully bad simplex. If \( \sigma \subseteq \text{Link}(\sigma^b) \) then \( \hat{f}(\sigma) \) is in \( G_0(S, \overrightarrow{b_0}) \) as otherwise there are more arcs that can be added to \( \sigma_b \) keeping it fully bad. Furthermore, the arcs of \( \sigma \) occur first in \( \sigma \circ \sigma^b \).

Restricted to \( \text{Link}(\sigma^b) \cong S^{k+1-|\sigma^b|} \), the map \( \hat{f} \) lifts to \( G_0(S, \overrightarrow{b_0}) \) as in Theorem A.1. Note that \( n(S) > n(S \setminus \sigma^b) \geq n(S) - 2|\sigma^b| + 1 \) as removing the first arc loses a single genus, and removing subsequent arcs loses at most two genera per arc. Now \( G_0(S \setminus \sigma^b, \overrightarrow{b_0}) \) is \((n(S, \sigma^b) - 1)\)-connected by assumption, and

\[
3(k + 1 - |\sigma^b|) \leq n(S) - 1 - 3|\sigma^b| \leq n(S \setminus \sigma^b) - |\sigma^b| - 2 \leq n(S \setminus \sigma^b) - 4
\]

as \( |\sigma^b| \geq 2 \) for \( \sigma^b \) to be fully bad, so \( k + 1 - |\sigma^b| \leq \left\lfloor \frac{n(S, \sigma^b) - 1}{3} \right\rfloor - 1 \). Thus the lift of \( \hat{f} \) to \( G_0(S \setminus \sigma^b, \overrightarrow{b_0}) \) is nullhomotopic, say through a map \( F : D^{k+1-|\sigma^b|+1} \to G_0(S \setminus \sigma^b, \overrightarrow{b_0}) \). Then redefining \( \hat{f} \) on the star of \( \sigma^b \) by

\[
\hat{f} * F : \partial \sigma^b * D^{k+1-|\sigma^b|+1} \cong St(\sigma^b) \to G(S, \overrightarrow{b_0})
\]

gives a new \( \hat{f} \) which strictly reduces the dimension of fully bad simplices. \( \square \)

Let \( b_0, b_1 \) both lie on the same boundary component, and have opposite orientations. Let \( D_0(S, b_0, b_1) \) denote the subcomplex of \( C(S, b_0, b_1) \) where the clockwise ordering at \( b_0 \) coincides with the clockwise ordering at \( b_1 \). Note that for \( \sigma^k \) a \( k \)-simplex of this complex, \( S_n^r \setminus \sigma^k \cong S^{r+k+1}_{n-2(k+1)} \).

**Theorem A.3.** \( D_0(S) \) is homotopy equivalent to a wedge of \( \left\lfloor \frac{n}{2} \right\rfloor - 1 \)-spheres.

**Proof.** First note that this complex has dimension \( \left\lfloor \frac{n}{2} \right\rfloor - 2 \), so it is enough to show that it is \( \left\lfloor \frac{n}{2} \right\rfloor - 2 \)-connected.

Let \( D(S) \) be the complex with vertices isotomy classes of 1-sided arcs with ends in \( \{b_0, b_1\} \), where a collection of arcs span a simplex if they can be made disjoint and have connected non-orientable (or genus 0) complement. Note this complex is Wahl’s \( G(S, \{b_0, b_1\}) \) and so is \( n(S) - 3 \)-connected [25, Theorem 3.3]. Furthermore, \( D_0(S) \) is a subcomplex of \( D(S) \).

As usual, let \( k \leq \left\lfloor \frac{n}{2} \right\rfloor - 2 \) and \( f : S^k \to D_0(S) \) be a continuous map, which we assume to be simplicial for some triangulation of the \( k \)-sphere. The composition \( S^k \to D_0(S) \to D(S) \) is nullhomotopic, so there exists a nullhomotopy \( \hat{f} : D^{k+1} \to D(S) \), which may also suppose is simplicial. Say that \( \sigma^b \subseteq D^{k+1} \) is fully bad if the first arc at \( b_0 \) is not the first arc at \( b_1 \), and let \( \sigma^b \) be a maximal fully bad simplex. As usual \( \hat{f} \) restricted to \( \text{Link}(\sigma^b) \cong S^{k+1-|\sigma^b|} \) lifts to \( D_0(S \setminus \sigma^b) \), which by induction we may suppose to be a wedge of \( \left\lfloor \frac{n(S \setminus \sigma^b)}{2} \right\rfloor - 1 \)-spheres.

We must bound \( n(S \setminus \sigma^b) \) from below. As \( \sigma^b \) is fully bad, the first arc at \( b_0 \) is not the first arc at \( b_1 \). There are two possibilities. Firstly, the first arc at \( b_0 \) may...
end at \( b_0 \). Then, by parity, there must be another arc which starts and ends at the same point \( b_1 \). Cutting along both these arcs loses 2 genus, and there are then a remaining \(|\sigma^b| - 2\) arcs to cut along, which may remove at most 2 genus each. Then \( n(S \setminus \sigma^b) \geq n - 2 + 2(|\sigma^b| - 2) = n - 2|\sigma^b| + 2 \). The second possibility is that the first arc at \( b_0 \) may end at \( b_1 \), necessarily not first. If the first arc at \( b_1 \) ends at \( b_1 \) we are in the first case, with the roles of the \( b_i \) reversed. Thus we may assume that the first arc at \( b_1 \) ends at \( b_0 \). Cutting along the first arc at \( b_0 \) loses 2 genus and creates a new boundary, but cutting along the first arc at \( b_1 \) just joins the new boundary to the old. Thus \( n(S \setminus \sigma^b) \geq n - 2 + 2(|\sigma^b| - 2) = n - 2|\sigma^b| + 2 \) in this case also.

Now
\[
2(k + 1 - |\sigma^b|) \leq n - 2 - 2|\sigma^b| \leq n(S \setminus \sigma^b) - 4,
\]
and so \( k + 1 - |\sigma^b| \leq \left\lfloor \frac{n(S \setminus \sigma^b)}{2} \right\rfloor - 2 \). Thus the lift of \( \tilde{f} \) to \( D_0(S \setminus \sigma^b) \) is nullhomotopic, say through a map \( F : D^{k+1-|\sigma^b|+1} \rightarrow D_0(S \setminus \sigma^b) \). As usual we modify \( \tilde{f} \) on \( St(\sigma^b) \) by
\[
\tilde{f} \ast F : \partial \sigma^b \ast D^{k+1-|\sigma^b|+1} \cong St(\sigma^b) \rightarrow D(S)
\]
and this gives an improvement of \( \tilde{f} \), which strictly reduces dimension of fully bad simplices.

Let \( b_0, b_1 \) both lie on different boundary components. Let \( E_0(S, b_0, b_1) \) denote the subcomplex of \( C(S, b_0, b_1) \) where the clockwise ordering at \( b_0 \) coincides with the clockwise ordering at \( b_1 \). Note that for \( \sigma^k \) a \( k \)-simplex of this complex, \( S_n^r \setminus \sigma^k \cong S_{n-2k}^{r+k-1} \).

**Theorem A.4.** \( E_0(S) \) is \((\lceil \frac{n}{2} \rceil - 2)\)-connected.

**Proof.** Let \( E(S, b_0, b_1) \) denote the complex with vertices isotopy classes of 1-sided arcs with ends in \( \{b_0, b_1\} \), where a collection of arcs span a simplex if they can be made disjoint and have connected non-orientable (or genus 0) complement. Note this complex is Wahl’s \( G(S, \{\overrightarrow{b_0}, \overrightarrow{b_1}\}) \) and so is \((n(S) - 2)\)-connected [25, Theorem 3.3]. Furthermore \( E_0(S) \) is a subcomplex of \( E(S) \). The remainder is entirely analogous to Theorems A.3 and A.1. \( \square \)

**B. A CONSTRUCTION IN SEMI-SIMPLICIAL SPACES**

In this appendix we describe a simple construction in semi-simplicial spaces, and show how to determine the connectivity of its realisation in certain cases. Though this construction seems very natural, we are not aware of it being discussed in the literature. We refer to [22] for notation.

Let \( X_\bullet \) be a semi-simplicial space, and define a semi-simplicial pointed space \( X^+_{n-1} \times \bullet \) having \( X^+_{n-1} \times \{0, \ldots, n\} \) as its space of \( n \)-simplices, with face maps given by the pointed maps

\[
(B.1) \quad \partial_i(x \times j) = \begin{cases} 
  d_i(x) \times (j - 1) & \text{if } i < j, \\
  \ast & \text{if } i = j, \\
  d_{i-1}(x) \times j & \text{if } i > j.
\end{cases}
\]

**Proposition B.1.** Suppose \( X_\bullet \) has the property that the maps

\[
(*) \quad \| \cdots X_{i+1} \Rightarrow X_i \| \xrightarrow{d_i} X_{i-1}
\]

are \((k - i)\)-connected for all \(i\). Then the natural map

\[ \Sigma\|X_*\| \longrightarrow \|X_{-1}^+ \times \bullet\|_* \]

is \(k\)-connected. In particular, if the maps \(\Sigma\) are equivalences for all \(i\) then there is an equivalence \(\Sigma\|X_*\| \simeq \|X_{-1}^+ \times \bullet\|_*\).

**Proof.** Note that \(\text{hocofib}(\cdots X_{i+1}^+ \Rightarrow X_i^+), d \Rightarrow X_{i-1}^+ \) \(\cong\) \(\text{hocofib}(\cdots X_{i+1} \Rightarrow X_i), d \Rightarrow X_{i-1} \) which is \((k - i)\)-connected by hypothesis.

The semi-simplicial space \(X_{-1}^+ \times \bullet\) has a filtration by \(F^i(X_{n-1}^+ \times [n]) = \{x \times j : j \leq i\}\). The inclusion \(F^{i-1}(X_{n-1}^+ \times [n]) \hookrightarrow F^i(X_{n-1}^+ \times [n])\) is the inclusion of a collection of path components, so a cofibration, and the filtration quotient \(F^i(X_{i-1}^+ \times \bullet) / F^{i-1}(X_{i-1}^+ \times \bullet)\) is the semi-simplicial pointed space

\[
\begin{array}{ccccccc}
\cdots & X_{i+2} & \longrightarrow & X_{i+1} & \longrightarrow & X_i^+ & \longrightarrow & X_{i-1}^+ & \cdots & \ast & \cdots & \ast \\
& d_{i+2} & \longrightarrow & d_{i+1} & \longrightarrow & d_i & \longrightarrow & \end{array}
\]

where the grey arrows denote constant maps to the basepoint. The realisation of this semi-simplicial pointed space is

\[\|F^i(X_{i-1}^+ \times \bullet) / F^{i-1}(X_{i-1}^+ \times \bullet)\|_* \simeq \Sigma^i \text{hocofib}(\cdots X_{i+1}^+ \Rightarrow X_i^+), d \Rightarrow X_{i-1}^+ \]

which is \(k\)-connected by the remark above. Furthermore, \(F^0\|X_{-1}^+ \times \bullet\|_* \simeq \Sigma\|X_*\|\) by inspection, so the inclusion of the filtration zero part \(\Sigma\|X_*\| \hookrightarrow \|X_{-1}^+ \times \bullet\|_*\) is \(k\)-connected.

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