Thermofield Double States in Group Field Theory

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Abstract

Group field theories are higher-rank generalizations of matrix/tensor models, and encode the simplicial geometries of quantum gravity. In this paper, we study the thermofield double states in group field theories. The starting point is the equilibrium Gibbs states in group field theory recently found by Kotecha and Oriti, based on which we construct the thermofield double state as a “thermal” vacuum respecting the Kubo-Martin-Schwinger condition. We work with the Weyl $C^*$-algebra of group fields, where the group fields and their Hermitian conjugations respectively annihilate and create quantum polyhedra in the sense of second quantization. A particular type of thermofield double states with single type of symmetry are then obtained from the squeezed states on this Weyl algebra. In particular, the “tilde” system is obtained from the original system via modular conjugations, and we interpret the “tilde” system as an emergent referential system. The thermofield double states, when viewed as states on the group field theory Fock vacuum, are condensate states at finite flow parameter $\beta$. We suggest that the equilibrium flow parameters $\beta$ of this type of thermofield double states in the group field theory condensate pictures of black hole horizon and quantum cosmology are related to the inverse temperatures in gravitational thermodynamics.

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1 Introduction

Searching for the correct theory of quantum gravity is a recurring theme in theoretical physics. Many candidate theories have been proposed over the years, but no consensus is reached. In this respect, it is worthwhile to find some common features from different theories of quantum gravity. The group field theory (GFT) approach to quantum gravity can be related to many different approaches to quantum gravity, and hence is a natural place to find the common features of different theories.

GFTs are originally proposed to generate the simplicial quantum gravity such as the Ponzano-Regge model by topological lattice field theories with fields defined on the gauge group [7, 36]. It is realized in [14] that the GFTs can also be utilized to generate the spin foam models and to assure the triangulation independence of spin foams. Since then GFT provides an alternative way of viewing the spin foam models. In essence, the group fields are functions over the gauge group and hence incorporate the internal gauge symmetry quantum numbers into the fields, which is also the case for matrix models or tensor models. The Feynman diagrams in matrix or tensor models are simplicial graphs, and in GFT these simplicial Feynman graphs can be related to the triangulations of simplicial quantum gravity models including spin foam models. Consequently the GFTs are natural background-independent field-theoretic models of quantum gravity based on discrete simplicial structures. See [30] for an introduction to the above aspects.

The field-theoretic structure allows us to relate GFTs to many other approaches to quantum gravity in addition to simplicial quantum gravity: (i) when the gauge group is non-Abelian the group fields are noncommutative and can be formulated as a noncommutative field theory or noncommutative geometry [17, 5]; (ii) the simplicial Feynman graphs generated by GFTs can be interpreted as quantum geometric excitations, and the GFTs can be thus formulated as a second quantization of loop quantum gravity (LQG) built on the Ashtekar-Lewandowski vacuum where the excitations are the spin networks [38]; (iii) the tensor-model structure and the spin-network structure can be combined into a tensor network representation of GFTs [11, 9] where a holographic duality can be built in analogy to random tensor networks; (iv) the algebra of creation and annihilation operators in the second quantization leads to an algebraic formulation of GFT [27], thereby allowing the utilization of many techniques from algebraic quantum field theory.

The second quantization formulation of GFT inspires many recent developments in GFT in relation to LQG. Basically, the second quantization formalism in quantum mechanics is designated to study the quantum many-body systems, and similarly the second quantization formalism of GFT or LQG is expected to describe the many-body physics, such as condensate states, of quantum spacetime “atoms”. See [37] for some early intuitions. Indeed, the Schwinger-Dyson equation in GFT takes the form of a mean field equation for Bose-Einstein condensation (BEC) and describes the effective hydrodynamics of the emergent quantum spacetime. Remarkably, such a GFT condensate picture has been successfully applied to obtain a modified Friedmann equation in cosmology [19] and to explain the entropy of a quantum black hole [41]. A moment of reflection shows
an obvious missing point in the GFT condensate picture of quantum gravity: there is no well-defined statistical mechanics for quantum gravitational states, since the Hamiltonian in canonical pure gravity is zero (and the Hamiltonian for GFT is undefined due to the lack of time variable), not to mention the lack of definitions of many thermodynamical quantities in quantum gravity. Nevertheless, recently in [29] the equilibrium Gibbs states for GFT are constructed by the methods that do not explicitly use the pure-gravity Hamiltonian. One of the methods is Jaynes’ maximum entropy principle for constrained systems, which is further used in [10, 28] to study the generalized Gibbs states and background-independent statistical mechanics of quantum tetrahedra. The other method is based on the operator-algebraic Kubo-Martin-Schwinger (KMS) condition [22] and the algebraic formulation of GFT mentioned above. The Gibbs states in GFT are important for understanding the statistical mechanical aspects of quantum gravity in general, and they have advantages over other approaches to the equilibrium states in quantum gravity, such as [2], since in GFT the collective many-body physics can be studied by many familiar field-theoretic techniques, which in turn helps the study of classical limit for quantum gravity.

In this paper, we study the thermofield double (TFD) states in GFT based on the obtained Gibbs states of GFT, as a first step towards the extension from equilibrium states to equilibrium field theories “at finite temperature”. We take the operator-algebraic approach to Gibbs states, so that the algebraic TFD can be formulated as in [33]. In particular, this algebraic approach to TFD uses Tomita-Takesaki modular theory to get the “tilde” algebra \( \hat{\mathcal{M}} \) of TFD as the modular conjugate \( \mathcal{M}' \) of the von Neumann algebra \( \mathcal{M} \) of the original system. For a factor von Neumann algebra \( \mathcal{M}, \mathcal{M} \neq \mathcal{M}' \) in general, but their GNS representation Hilbert spaces are the same. In this generalized sense, the algebraic TFD states of GFT can not simply describe some doubled quantum geometries, but indicate the emergence of new fields when flows or transformations are present in GFT.

The TFD states in GFT, when viewed as states on the Fock vacuum, are in effect GFT condensate states. But the TFD states carry the equilibrium parameter \( \beta \), the “inverse temperature”, so they are very suitable for studying black holes (cf. [26]). Algebraically, the “inverse temperature” \( \beta \) also parametrizes the algebraic symmetry of the GFT algebraic states, which allows us to relate the “thermal” behavior to the symmetry of quantum gravitational states, a link deeply implied by black hole thermodynamics.

We begin in Sec. 2 with the basic definitions of GFT, its algebraic formulation and the GFT Gibbs states. For the completeness of presentation, we include the essential proofs from [27, 29]. In Sec. 3 we define and study the TFD states of GFT using Tomita-Takesaki modular theory. By taking inspiration from the second-quantization interpretation of GFT, we first construct entangled squeezed states on the Weyl \( C^* \)-algebra of GFT and show that such algebraic squeezed states can be expressed in the standard form of TFD states (3.9) with a single set of generators. A quantum geometric interpretation of the obtained TFD states as the gravitational plus the emergent referencial states is also given. In Sec 4 we show some algebraic properties of the TFD states when viewed
as GFT condensate states on the Fock vacuum, and study the example of TFD shell condensate as a black hole horizon as well the sphere condensate for quantum cosmology to qualitatively discuss the meaning of $\beta$ in this context. We summarizes this paper in Sec. 5.

In Appendix A we include another algebraic approach to the TFD extension of GFT based on the quantum deformation of the Hopf algebra of group fields.

2 Group field theories and Gibbs states

Group fields are fields (or complex-valued functions) defined on $n$ copies of the gauge group of interest, where the $n$ copies of gauge group is designated in such a way that the interaction term of GFT can be mapped to some simplicial complexes. Formally GFTs are similar to tensor models, but the group fields have continuous indices, i.e. with infinite-dimensional tensor fields. The physical input is the choice of the form of GFT action in which the simplicial structure of some models of quantum gravity can be encoded.

Let us work with the recent definitions given in [9]. Suppose the gauge group $G$ of interest is a locally compact Lie group with unimodular Haar measure $\mu$.

Definition 2.1. A group field $\phi$ is a $\mu^{\times n}$-integrable complex-valued function over the $n$-fold direct product of $G$,

$$\phi : G^{\times n} \to \mathbb{C}; \quad (g_1, g_2, ..., g_n) \mapsto \phi(g_1, g_2, ..., g_n)$$

such that

$$\int_{G^{\times n}} \phi(g_1, g_2, ..., g_n)\overline{\phi(g_1, g_2, ..., g_n)}d\mu^{\times n} < \infty.$$  \hspace{1cm} (2.2)

The group fields carry a representation of a Hilbert space as follows. Each group field $\phi$ corresponds to a vector $|\phi\rangle$. These vectors have inner products as in (2.2)

$$\langle \phi|\phi' \rangle = \int_{G^{\times n}} \phi(g_1, g_2, ..., g_n)\overline{\phi'(g_1, g_2, ..., g_n)}d\mu^{\times n}.$$  \hspace{1cm} (2.3)

By the Cauchy-Schwarz inequality and the condition (2.2), this inner product $\langle \phi|\phi' \rangle$ is finite, so that the vectors $|\phi\rangle$ are in a Hilbert space $\mathcal{H} = L^2(G^{\times n}, \mu^{\times n})$. The group fields $\phi$ as complex-valued functions can be recovered by assigning a linear functional $\langle g_1, ..., g_n |$ in the dual space $\mathcal{H}^*$ to $|\phi\rangle$ such that

$$\phi(g_1, ..., g_n) = \langle g_1, ..., g_n | \phi \rangle.$$  \hspace{1cm} (2.4)

The linear functionals $\langle g_1, ..., g_n |$ can be chosen to be multi-linear, that is, $\langle g_1, ..., g_n | = \langle g_1 | \otimes ... \otimes \langle g_n |$. In this case, the Hilbert space $\mathcal{H}$ can be factorized as

$$\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i, \quad \mathcal{H}_i = L^2(G, \mu)_i$$  \hspace{1cm} (2.5)
where the vectors \( |\phi_i\rangle \in \mathcal{H}_i \) satisfy \( \langle g_i | \phi \rangle = \langle g_i | \phi_i \rangle \). Notice that the factorization of \( |\phi\rangle \in \mathcal{H} \) does not entail the factorization of the group field \( \phi(g_1, ..., g_n) \). In the special case where the group field \( \phi \) is completely factorized, \( \phi \) becomes the product \( \prod_{i=1}^n \langle g_i | \phi_i \rangle \) of \( n \) independent one-fold group fields. To avoid ambiguity, we assume in the following that the group fields \( \phi \) cannot be factorized anymore.

To construct the action for GFT, consider first the change of group elements in a group field \( \phi(g_1, ..., g_n) \rightarrow \phi(g'_1, ..., g'_n) \) by the following transformation

\[
\phi(g'_1, ..., g'_n) = \int_{G^{\times n}} \prod_{i=1}^n d\mu(g_i) C(g_1, ..., g_n, g'_1, ..., g'_n) \phi(g_1, ..., g_n)
\]

where the integration kernel \( C(g_1, ..., g_n, g'_1, ..., g'_n) \) is called the covariance in analogy to the free tensor models \([21]\). When expressed in terms of the Hilbert space representation \( (2.4) \), the covariance \( C \) is an endomorphism of the linear functionals,

\[
C : \langle g_1, ..., g_n | \mapsto \int_{G^{\times n}} \prod_{i=1}^n d\mu(g_i) C(g_1, ..., g_n, g'_1, ..., g'_n) \langle g_1, ..., g_n |.
\]

The integration kernel \( C(g_1, ..., g_n, g'_1, ..., g'_n) \) thus encodes how the \( g_i \) are transformed to \( g'_i \). Then the kinematical term in the action can be formulated in analogy to the free tensor field theories.

**Definition 2.2.** Let \( C \) be a covariance endomorphism on \( \mathcal{H}^* \). A covariance \( K \) is inverse to \( C \) if \( C \circ K = 1_\mathcal{H} \). The kinematical action of GFT is

\[
S_0[\phi] = \frac{1}{2} \int_{G^{\times n}} \prod_{i=1}^n d\mu(g_i) \prod_{j=1}^n d\mu(g'_j) \phi(g_1, ..., g_n) K(g_1, ..., g_n, g'_1, ..., g'_n) \phi(g'_1, ..., g'_n).
\]

With \( (2.8) \), the probability measure \( D\phi \exp\{-S_0[\phi]\} \) becomes a multivariate Gaussian if \( D\phi \) is chosen as the Lebesgue measure on \( \mathcal{C} \). As in usual quantum field theories, an interaction term can be added to the free action, i.e. \( S_0 + \lambda S_{\text{int}} \) where \( \lambda \) is a coupling constant. The form of \( S_{\text{int}} \) in GFT should be chosen in such a way that the Feynman diagrams obtained by perturbatively expanding the probability measure \( D\phi \exp\{-S_0[\phi] - \lambda S_{\text{int}}[\phi]\} \) are dual to simplicial complexes.

**Definition 2.3.** Let \( \phi_i(g_1, ..., g_n) \equiv \phi(\{g_i^{(j)}\}), i = 1, ..., n; j = 1, ..., n + 1 \), be \( n + 1 \) group fields defined on \( G^{\times n} \). The interaction term in the action of GFT is

\[
S_{\text{int}}[\phi] = \frac{1}{n + 1} \int_{G^{\times (n+1)}} \prod_{i,j=1, i \neq j}^{n+1} d\mu(g_i^{(j)}) V(\{g_i^{(1)}\}, ..., \{g_i^{(n+1)}\}) \phi(g_i^{(1)}) ... \phi(g_i^{(n+1)})
\]

where \( V(\{g_i^{(1)}\}, ..., \{g_i^{(n+1)}\}) \) is an integration kernel satisfying the closure constraints

\[
V(\{g_i^{(1)}\}, ..., \{g_i^{(n+1)}\}) = \int_{G^{\times (n+1)}} \prod_{j=1}^{n+1} d\mu(h_j) V(\{h_1 g_i^{(1)}\}, ..., \{h_{n+1} g_i^{(1)}\}), \quad \forall h_j \in G.
\]

\[(2.10)\]
The Feynman diagrams in the leading order of the \( \lambda \)-expansion thus consist of the \( n \)-stranded graphs representing the free propagation, defined by \( K \), of the \( n \) group elements in the argument of \( \phi(g_1, \ldots, g_n) \). \((n + 1)\) \( n \)-stranded graphs can meet at a vertex and the group fields get convoluted by \( V \). The higher order terms in the \( \lambda \)-expansion then represent more complicated simplicial complexes.

Notice that the closure constraint (2.10) imposes gauge symmetry \((G)\) at a given interaction vertex, which means only the gauge-invariant group fields contribute to the interactions in GFT. We can therefore impose the global gauge invariance condition on group fields (2.1) and the kinematical kernel (2.8),

\[
\phi(g_1, \ldots, g_n) = \phi(hg_1, \ldots, hg_n),
\]

\[
K(\{g_i\}, \{g'_i\}) = \int_{G \times 2} d\mu(h) d\mu(h') K(\{hg_i\}, \{h'g'_i\}),
\]

for \( h, h' \in G \). The closure constraint (2.11) also means that a single group field is dual to a polyhedron. To see this, let us take recourse from the noncommutative metric representation [5]. A group field in the noncommutative metric representation is obtained by the group Fourier transformation of the group fields, i.e.

\[
\hat{\phi}(\{x_i\}) = \int \prod_i d\mu(g_i) \phi(\{g_i\}) \prod_i e_{g_i}(x_i), \quad e_{g_i}(x_i) = e^{\text{Tr}g_i x_i}, \quad g_i \in G, x_i \in g
\]

where the trace in the plane-wavefunction \( e_{g}(x) \) is the trace on the Lie algebra \( g \) of \( G \) such that \( \text{Tr}\tau_i \tau_j = -\delta_{ij} \) for generators \( \tau_i \) of \( g \). The Lie algebra elements \( x_i \) can be associated to the vectors normal to the faces of a polyhedron in that \( |x_i| = \sqrt{\text{Tr}_g(x_i, x_j)} \) defines the area and \( x_i/|x_i| \) defines the unit normal vector of a face. In this representation the gauge-invariance of \( \phi(\{g_i\}) \) translates to the closure condition \( \sum_i x_i = 0 \) of a polyhedron. Thus, the group fields convoluted by \( V \) represent a 2-complexes containing these polyhedra.

The GFT thus defined can generate the amplitude of simplicial quantum gravity.

**Example 2.4.** In the Boulatov-Ooguri model [7, 36] over \( G \), the choices of \( K \) and \( V \) are respectively

\[
K(\{g_i\}, \{g'_i\}) = \int_G d\mu(h) \prod_{i=1}^n \delta(hg_i g'_i),
\]

\[
V(\{g^{(1)}_i\}, \ldots, \{g^{(n+1)}_i\}) = \int_G \prod_{i=1}^n d\mu(h_i) \prod_{i<j} \delta(h_i g^{(j)}_i, h_j g^{(j)}_i).
\]

When \( n = 4 \) and \( G = SO(4) \), the convolution of 4-stranded graphs at a vertex is dual to a 4-dimensional simplicial complex, i.e. a polyhedron. The Feynman amplitude gives us the \( 15j \) symbol in the spin basis, which is the vertex amplitude in spin foam models.
Example 2.5. Pithis et al. [43] choose the $K$ and $V$ as

\[ K(\{g_i\}, \{g'_i\}) = \prod_{i=1}^{n} \delta(g_i g'_i) \left[ -\sum_i \Delta_{g_i} + m^2 \right], \]  
\[ V(\{g^{(1)}_i\}, ..., \{g^{(n+1)}_i\}) = \sum_{m=2}^{n+1} \frac{1}{n-1} \phi^\dagger(g^{(1)}_i) ... \phi^\dagger(g^{(m)}_i) \prod_{m+1}^{n+1} \delta(\phi(g^{(m)}_i), 1) \]  

where $\Delta_{g_i}$ is the Laplacian on the group manifold and the constant $m$ can be related to the spin foam edge weight. The $V$ in (2.17) give a tensorial nonlinear interaction term. The numerical analysis in [43] shows that such nonlinear interaction term calls for a non-Fock representation of GFT condensates, which we shall study in the following.

The way in which $S_{\text{int}}$ enter the partition function, or the probability measure, implies that the interaction term is an observable in the free theory, i.e. $D\phi e^{-S_0 - \lambda S_{\text{int}}}$. Therefore general observables in GFT can be defined similarly as specific convolutions of group fields [42]:

Definition 2.6. A trace observable of GFT is a functional of the group fields with all the group elements are traced over, that is,

\[ O[\phi] = \int_{G^j} \prod_{i,j} d\mu(g^{(j)}_i) B(\{g^{(j)}_i\}) \prod_{i,j} \phi(g^{(j)}_i) \]  

where $j \in J$ not necessarily bounded by $(n + 1)$ and $B$ is an integration kernel encoding the ways of convolution or tracing. A partial trace observable is a functional of the group fields with parts of the group elements are traced over, thereby being a functional $O_p[\phi, g]$ of both group fields and the untraced group elements.

Example 2.7. Consider the spin network states in LQG as GFT observables [38, 39]. The streamlined structure of a spin network graph $\gamma$ in LQG consists of a set $E$ of edges colored with $SU(2)$ spins $j$ and a set $V$ of $n$-valent vertices to each of which is assigned an intertwiner operator $I$. Then $n$ colored edges are contracted with an $n$-valent vertex to form a spin network graph. Each vertex $v \in V$ carries a quantum state $|I_v\rangle$ that represents a quantum polyhedron in the dual quantum geometry, and on each edge the spin-$j$ representation space can be chosen as a Hilbert space $H_j$. Because $v$ is $n$-valent, the intertwiner states $|I_v\rangle$ live in the Hilbert space

\[ H_v \equiv \bigotimes_{i=1}^{n} H_{j_i, v} \]  

where $H_{j_i, v}$ is the Hilbert space of the $i$-th spin-$j_i$ edge attached to the vertex $v$. Comparing (2.5) and (2.19), we see that the intertwiner state $|I_v\rangle$ corresponds to a GFT state $|\phi\rangle$. The difference is that $|I_v\rangle$ will be projected on a spin basis $\{|j\rangle, j \in \mathbb{N}/2\}$ to obtain a spin network wavefunction, while $|\phi\rangle$ will be projected on a group basis $\langle\{g_i\}, g_i \in G$ to...
obtain a group field. By writing \( |I_v⟩ = \otimes_i |j_i, v⟩ \) for all untraced vertices in \( \mathcal{V} \). Next, a complete contraction of open edges can be made by matching the spins on a pair of open edges, which can be simply imposed on the spin basis as \( \delta(j, j') \langle j | \otimes \langle j' | \). Therefore a spin network wavefunction on a closed graph \( \gamma \) is

\[
\Psi_{\gamma,j,I_v} = \sum_{i=1}^{n} \sum_{\{j_i\}} M(j_i, j'_i) \delta(j_i, j'_i) \prod_{v \neq v' \in \mathcal{V}} \langle j'_i, v' | j_i, v⟩ \quad (2.20)
\]

where we have labelled the contracted open edges with the same \( i \) and the \( M \)'s are elements of the representation matrix. Lifting the spin basis and intertwiner states respectively to the group basis and GFT states we obtain the spin network observable in GFT in the form of (2.18) with the gauge-invariant gluing kernel

\[
B(\{g_i^{(j)}\}) = \int_G \prod_{i=1}^{n} d\mu(h_i) \prod_{i,j \neq j'} \delta(h_i g_i^{(j)} h_i, h_i g_i^{(j')}) M(h_i g_i^{(j)}, h_i g_i^{(j')}) \quad (2.21)
\]

with some coefficients \( M \). When the spin network graph is not closed, we have after contraction spin network states on the remaining open edges. These “boundary” spin network states have a tensor network representation [11],

\[
|\Psi_{\partial \gamma}⟩ = \bigotimes_{e \in E - \partial \gamma} \langle M_e | \bigotimes_{v \in \mathcal{V}} | I_v⟩ \quad (2.22)
\]

where \( \langle M_e | \) represents the contraction, possibly more general than (2.20), along the edge \( e \).

### 2.1 Algebraic formulation of group fields

From Example [27] we see that the observables in GFT are built from multiple group fields, so generally speaking GFT is a many-body theory of group fields. The second quantization formalism for quantum many-body systems then becomes a natural language for GFT and the closely related LQG [38, 27].

It is instructive to adopt the quantum geometric interpretation of a group field, namely a gauge-invariant group field, similar to a vertex in a spin network graph, corresponds to a dual quantum polyhedron. One can therefore define a GFT vacuum state \( |\Omega⟩ \) without any quantum geometric excitation. Then a (gauge-invariant) group field \( \phi \) and its conjugate \( \phi^\dagger \) respectively annihilates and creates a quantum polyhedron,

\[
\phi^\dagger(\{g_i\}) |\Omega⟩ = |\{g_i\}⟩, \quad \phi(\{g_i\}) |\Omega⟩ = 0. \quad (2.23)
\]

\footnote{Note that the group basis replaces the spin basis in the dual loop quantization based on Dittrich-Geiller vacuum with additional flatness constraints imposed on each closed loop or on the intertwiner states on each vertices [13]. In the case of GFT, these flatness constraints are not imposed, thereby covering also the intermediate “squeezed” cases.}

\footnote{A traditional way of lifting is using the Peter-Weyl theorem to relate the group representation \( \rho(g) \) to the discrete indices of the spin-\( j \) representations.}
These creation and annihilation operators satisfy the *bosonic* canonical commutation relations (CCR),
\[
[\phi(\{g_i\}), \phi(\{g_j\})^\dagger] = \delta(\{g_i\}, \{g_j\}).
\] (2.24)

These operators define a bosonic Fock space
\[
\mathcal{H}_{\text{Fock}} = \bigoplus_{N>0} \text{sym} \mathcal{H}^\otimes N
\] (2.25)
where \(\mathcal{H}\) is given in (2.5). The occupation number basis, the field operators and in this Fock space and the many-body operators for the action can be formulated in the usual manner [38].

To introduce the algebraic formulation of GFT, let us write down the field operators in the group basis
\[
\Psi(\psi) = \int_{G^n} \prod_{i=1}^n d\mu(g_i) \psi(\{g_i\}) \phi(\{g_i\}), \quad \Psi(\psi)^\dagger = \int_{G^n} \prod_{i=1}^n d\mu(g_i) \psi(\{g_i\}) \phi(\{g_i\})^\dagger
\] (2.26)
where the \(\psi(\{g_i\})\) are single-polyhedron wavefunctions. Namely, the group fields are field-operator-valued distributions on the space of “single-particle” wavefunctions. These field operators \(\Psi\) inherits the CCR algebra of \(\phi\) with the \(\delta\)-function replaced by the \(L^2\) inner product of the \(\psi\)’s
\[
[\Psi(\psi), \Psi(\psi)^\dagger] = \int_{G^n} \prod_{i=1}^n d\mu(g_i) \psi(\{g_i\}) \psi(\{g_i'\}) \equiv (\psi, \psi').
\] (2.27)
Notice that the above CCR algebra is defined with respect to the single-body space \(\mathcal{H}^{(1)}\), but it is the same as the CCR algebra obtained from the full Fock space [16]. The inner product \((\psi, \psi')\) in (2.27) induces a symplectic form \(\text{Im}(\psi, \psi')\) on the space of test functions \(\psi\), and hence a Weyl algebra for GFT can be defined on the thus obtained phase space [27]. Here we recollect a more direct description:

**Proposition 2.8.** Let \(\Psi(\psi), \Psi(\psi)^\dagger\) be the GFT field operators as in (2.26). Then the exponentiated operators \(W(\psi) = e^{\sqrt{2}i\int (\Psi(\psi) + \Psi(\psi)^\dagger)}\) form a Weyl \(C^*\)-algebra.

**Proof.** Denoting \(\Phi(\psi) = \frac{1}{\sqrt{2}}(\Psi(\psi) + \Psi(\psi)^\dagger)\), we have \([\Phi(\psi_1), \Phi(\psi_2)] = i\text{Im}(\psi_1, \psi_2)\). Then by the Baker-Hausdorff formula, we see that \(W(\psi)\) satisfy the defining relation for a Weyl algebra,
\[
W(\psi_1)W(\psi_2) = e^{-\frac{i}{2}\text{Im}(\psi_1, \psi_2)}W(\psi_2 + \psi_1).
\] (2.28)
\(W(\psi)\) is unitary, since \(\Phi(\psi)\) is Hermitian. Hence \(W(\psi)\)’s form a Weyl algebra \(W\). The Hermitian conjugation defines the involution. The \(W\)’s as bounded linear functionals can be represented on some Hilbert space \(\mathcal{K}\), so we can assign a \(C^*\)-norm defined on the irreducible representations of \(\mathcal{K}\),
\[
\|W\|_{C^*} = \sup_{\pi_{\mathcal{K}}} \|\pi_{\mathcal{K}}(W)\|_{\mathcal{K}} = \sup_{\pi} \sqrt{\text{Tr}(\pi_{\mathcal{K}}(W), \pi_{\mathcal{K}}(W)^\dagger)},
\] (2.29)
to \(\mathcal{W}\), thereby making it a Weyl \(C^*\)-algebra.
The Fock space structure of GFT can be recovered from the GNS representation \((\mathcal{H}_{\text{Fock}}, \pi_F, |\Omega\rangle)\) of \(\mathcal{W}\). Here the GNS Hilbert space and the vacuum state are the same as the Fock space \((2.25)\) and respectively the vacuum in \((2.23)\) if the algebraic states on \(\mathcal{W}\) are given by the quasi-free states

\[
\omega_F(W(\psi)) = \langle \Omega | \pi_F(W(\psi)) | \Omega \rangle = e^{i \langle \Phi(\psi) | \Omega \rangle} = e^{-\frac{\langle \psi, \psi \rangle}{4}} \tag{2.30}
\]

where we have used \(\| \Phi(\psi) | \Omega \rangle \|^2_{H_{\text{Fock}}} = \langle \psi, \psi \rangle\). The important point is that the representation \(\pi_F(W(\psi)) = e^{i \Phi(\psi)}\) only represents the pure states in \(H_{\text{Fock}}\) created by the field operator \(\Phi\), thereby making the representation irreducible. The bicommutant of \(\pi_F(W(\psi))\) is then the whole space \(\pi''_F(W(\psi)) = \mathcal{B}(H_{\text{Fock}})\) of bounded linear functionals, which is a von Neumann algebra. We thus have, alternative to the Weyl \(C^*\)-algebra \(\mathcal{W}\), an algebraic description of GFT with the von Neumann algebra \(\mathcal{B}(H_{\text{Fock}})\).

We remark that working with the von Neumann algebra is advantageous in that the interactions and dynamics can be studied algebraically through the Tomita-Takesaki modular theory \([47]\) without explicitly using the Hamiltonian. Now that at a finite number \(N\) of excitations in a subspace \(\mathcal{H}_N\), \(\mathcal{B}(\mathcal{H}_N) = \bigoplus_{n=1}^{N} \mathcal{B}(\mathcal{H}_n)\) is a finite factor, there exist regular normal tracial states on \(\mathcal{B}(\mathcal{H}_N)\), whereby we have the statistical mixture of the Fock states:

\[
\omega_\rho[W] = \text{Tr}(\rho \pi_F(W)) \quad \text{where } \rho \text{ is a trace class density operator.}
\]

For the infinite \(N\) case, we can consider the Weyl algebra \(\mathcal{W}\) as a quasi-local algebra, i.e. the weak closure \(\bigcup_{N} \mathcal{B}(\mathcal{H}_N)\), then the normal states in \(\mathcal{H}_{\text{Fock}}\) can be obtained by the \(C^*\)-inductive limit of the local normal states in the subspace \(\mathcal{H}_N\) \([16]\), i.e. the local normal folium of states. In the following, the tracial states on \(\mathcal{B}(\mathcal{H}_{\text{Fock}})\) should be understood in this sense.

### 2.2 KMS condition and Gibbs states

In the algebraic formulation of a physical system in terms of a \(C^*\)-algebra or a von Neumann algebra, the equilibrium states are the states satisfying the algebraic KMS condition \([22]\):

**Definition 2.9.** Let \(\alpha_t\) be a one-parameter group of automorphisms of a \(C^*\)-algebra \(\mathcal{A}\) (or a von Neumann algebra \(\mathcal{M}\)). Let \(\omega[A], A \in \mathcal{A}\) be the algebraic states on \(\mathcal{A}\). The KMS condition is

\[
\omega[A \alpha_t+i\beta(B)] = \omega[A] \alpha_t(B), \quad t, \beta \in \mathbb{R}. \tag{2.31}
\]

The algebraic states satisfying the KMS condition are KMS states. The KMS states are stationary with respect to the transformations or flows from a one-parameter group of algebraic automorphisms, i.e. \(\omega[\alpha_t(A)] = \omega[A]\). Here \(\beta\) is a flow parameter which is not necessarily the inverse temperature.

In GFT, an existing one-parameter group of algebraic automorphisms consists of the left (or right) translations on \(G\). Consider the collective left translations on \(G^{\times n}\),

\[
L_{(g^t)} : G \to G; \quad (g_1, ..., g_n) \mapsto (g_1^tg_1, ..., g_n^tg_n) \tag{2.32}
\]
Lemma 2.11. Let \( \exp(\cdot) \) be the exponential map, where \( \rho \) is a density operator \( \in \mathcal{W} \) of the space of “single-polyhedron” wavefunctions \( \psi(\{g_i\}) \) as \( L_{\{g_i\}}^\ast \psi(\{g_i\}) = \psi(\{L_{\{g_i\}^{-1}}^\ast \{g_i\}\}) \). This thereby induces a flow on the Weyl elements:

\[
\alpha_{\{g_i\}} : \mathcal{W} \to \mathcal{W}; \quad W(\psi) \mapsto \alpha_{\{g_i\}}(W(\psi)) = W(L_{\{g_i\}}^\ast \psi(\{g_i\})) \tag{2.33}
\]

which is a \(*\)-automorphism on \( \mathcal{W} \). Because the von Neumann algebra of interest is the whole \( \mathcal{B}(\mathcal{H}_\text{Fock}) \), the \(*\)-automorphisms \( \alpha_{\{g_i\}} \) can be represented as unitary transformations \( U(g), g \in G \), on \( \mathcal{H}_\text{Fock} \) by the bounded linear transformation theorem,

\[
U(\{g_i\}) \Psi(\{g_i\}) U(\{g_i\})^{-1} = \Psi(\{g'_i g_i\}). \tag{2.34}
\]

Such a unitary operator \( U \) admits a Hermitian generator \( \mathcal{G} \) such that \( U(t) = e^{it\mathcal{G}} \) for some parameter \( t \in \mathbb{R} \). Not surprisingly, this \( \mathcal{G} \) gives rise to the canonical Gibbs form of KMS states \([29]\):

**Lemma 2.10.** Let \( \pi_F(W) \) be the Fock GNS representation map for \( W \in \mathcal{W} \) and \( \omega_\beta[W] = Tr(\rho \pi_F(W)) \) be a mixed algebraic state on \( \mathcal{W} \) with a density operator \( \rho \). If the \( \omega_\beta[W] \)'s are KMS states with respect to an one-parameter group of automorphisms \( \alpha_{t} \), then \( \rho \propto e^{-\beta \mathcal{G}} \) for some \( \beta \in \mathbb{R} \).

**Proof.** The KMS condition for \( \omega_\beta[W] \) is \( \omega_\beta[W \alpha_{t+\beta}(W')] = \omega_\beta[\alpha_{t}(W')W] \). When \( t = 0 \), this becomes

\[
\text{Tr}(\rho \pi_F(W') \pi_F(W)) = \text{Tr}(\rho \pi_F(W) \pi_F(\alpha_{t+\beta}(W'))).
\]

Using \( W = e^{i\Phi}, U = e^{it\mathcal{G}} \) and (2.34), we have

\[
\text{Tr}(\rho \pi_F(W') \pi_F(W)) = \text{Tr}(\rho \pi_F(W) e^{-\beta \mathcal{G}} \pi_F(W') e^{\beta \mathcal{G}}) = \text{Tr}(e^{-\beta \mathcal{G}} \pi_F(W') e^{\beta \mathcal{G}} \pi_F(W)).
\]

Therefore \( \rho \pi_F(W') = e^{-\beta \mathcal{G}} \pi_F(W') e^{\beta \mathcal{G}} \rho \), which entails \( [e^{\beta \mathcal{G}} \rho, \pi_F(W')] = 0 \) or \( e^{\beta \mathcal{G}} \rho \in \pi_F(W) \). Since \( \pi_F(W) = \mathcal{B}(\mathcal{H}_\text{Fock}) \) and \( \pi_F \) is irreducible, we have that \( \pi_F(W) \propto 1 \). Hence \( \rho \propto e^{-\beta \mathcal{G}} \).

Next, to apply the above result to the collective left translations (2.33), we suppose that \( G \) is connected, which is indeed the case for the gauge group \( SU(2) \) of LQG \([29]\):

**Lemma 2.11.** Let \( G \) be a connected Lie group and \( \mathfrak{g} \) be its Lie algebra. The KMS states (i.e. density operators) \( \rho_X \) with respect to the collective left translations \( \alpha_{\{g_i\}}, g_i' \in G \) have the canonical Gibbs form

\[
\rho_X = \frac{1}{Z} e^{-\beta \mathcal{G}_X}, \quad Z = \text{Tr} e^{-\beta \mathcal{G}_X}, \mathcal{G}_X = iU_\ast(X), \quad \beta \in \mathbb{R}, X \in \mathfrak{g} \tag{2.35}
\]

where \( U_\ast \) is the anti-Hermitian representation of \( \mathfrak{g} \).

**Proof.** Consider \( X \in \mathfrak{g} \), then \( X \) can be mapped to a \( g_X(t) = e^{tX} \in G \) through the exponential map, where \( t \in \mathbb{R} \) is a parameter such that \( g_X(0) = 1_G, (dg_X/dt)_{t=0} = X \). The left translations on \( G \) become \( L_{g_X} = L_{e^{tX}} = g_X(\cdot) \). Let \( U : G \to U_\mathcal{H} \) be a
strongly continuous unitary representation of \( G \) by the unitary operators on some Hilbert space \( \mathcal{H}' \), then the composition \( U_X = U \circ g_X \) is a strongly continuous one-parameter \((t)\) group of unitary operators in \( \mathcal{U}_{\mathcal{H}'} \) (see \[29\] for the proof). Hence \( U_X \) is a strongly continuous unitary representation of \( \alpha_g \). By Stone’s theorem, we have \( U_X = e^{-i\mathcal{G}_X t} \) for some Hermitian generator \( \mathcal{G}_X \) from \( \mathcal{H}' \). Applying Lemma \[2.10\] then gives the Gibbs states. Finally, the anti-Hermitian \( U^*_X(X) \) comes from \( U_X(t) = U(e^{tX}) = e^{tU^*_X(X)} = e^{-it\mathcal{G}_X} \).

Finally, in the case of GFT, we have \( \mathcal{H}' = \mathcal{H}_{\text{Fock}} \) and the connected group is \( G^{\times n} \). Notice that the collective left translations on \( G \), when written as \( g \cdot \cdot = e^{tX_i} \cdot \cdot \), have right-invariant generators \( X \), otherwise the (global) gauge invariance will be violated when acting different \( g_i \). We therefore have

**Theorem 2.12** (Kotecha-Oriti \[29\]). Let \( \mathcal{H}_{\text{Fock}} \) be the Fock space for a GFT on \( G^{\times n} \), and let \( \alpha_{\{g_i\}} \) be the collective left translations on \( G^{\times n} \), then the KMS states \( \rho_g \) with respect to \( \alpha_{\{g_i\}} \) take the canonical Gibbs form

\[
\rho_g = e^{-i\mathcal{G}_X t}/Z, \beta \in \mathbb{R}
\]

where

\[
U_X(g) = \int_{G^{\times n}} \prod_{i=1}^n d\mu(g_i) \phi(g_i) L_{R_{g^*}X} \phi(g_i)
\]

(2.36)

with \( L_{R_{g^*}X} \) being the Lie derivative along the right-invariant generators \( R_{g^*}X \) of \( L_gX \) for \( X \in \mathfrak{g} \).

### 3 Thermo field dynamics extension via Tomita-Takesaki theory

TFD is a reformulation of the real time formalism, i.e. the closed time path formalism, of quantum field theories at finite temperature \[12\]. Basically, TFD intends to rewrite the partition function of a thermal quantum system in the form of a vacuum-vacuum transition amplitude as in a zero-temperature field theory. For a state \( \rho \) in equilibrium at inverse temperature \( \beta_T = 1/T \), the thermal partition function is \( Z = \text{Tr}(e^{-\beta_T H}) \), the trace of the Gibbs state with some Hamiltonian \( H \). Then the thermal average of an operator \( A \) can be written as

\[
\langle A \rangle_{\beta_T} = \frac{1}{Z} \text{Tr}(A e^{-\beta_T H}) \equiv \langle 0, \beta_T | A | 0, \beta_T \rangle
\]

(3.1)

where the thermal vacuum \( | 0, \beta_T \rangle \), when written in the energy eigenbasis \( \{ |n \rangle \} \), is

\[
| 0, \beta_T \rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\frac{1}{2\beta_T E_n}} |n \rangle \otimes |\tilde{n} \rangle.
\]

(3.2)

The tensor product basis states \( |n \rangle \otimes |\tilde{n} \rangle \) is in the doubled Hilbert space \( \mathcal{H} \otimes \tilde{\mathcal{H}} \) where \( \mathcal{H} \) is the quantum system of interest and \( \tilde{\mathcal{H}} \) is a copy of \( \mathcal{H} \) so as to produce the same \( \delta \)-functions, i.e. \( \delta_{mn} = \langle m |\tilde{n} \rangle = \langle m | n \rangle \). In the formalism of second quantization, the
coefficients \( e^{-\frac{1}{2} \beta T E_n} \) comes from the Bogoliubov transformation between the two sets of annihilation and creation operators: \( (a, a^\dagger) \) for \( \mathcal{H} \) and \( (\tilde{a}, \tilde{a}^\dagger) \) for \( \tilde{\mathcal{H}} \). The time evolutions in \( \mathcal{H} \otimes \tilde{\mathcal{H}} \) are then generated by the free total Hamiltonian \( H - \tilde{H} \), and this total Hamiltonian annihilates the thermal vacuum, \( (H - \tilde{H}) |0, \beta\rangle = 0 \). Interaction terms \( \lambda (L_I - \tilde{L}_I) \) can be added to the free total Lagrangian \( L_0 - \tilde{L}_0 \), so that the perturbative Feynman diagrams can be calculated for the doubled system.

The \( \tilde{\mathcal{H}} \) in TFD is usually considered as a fictitious tool to express the thermal mixed state in \( \mathcal{H} \) as a pure entangled state in \( \mathcal{H} \otimes \tilde{\mathcal{H}} \). However, since the thermal vacuum is entangled, \( \tilde{\mathcal{H}} \) surely contains the physical information of \( \mathcal{H} \). Indeed, the \( \tilde{\mathcal{H}} \) in a conformal field theory has a physical interpretation as one side of the emergent dual holographic geometry in AdS/CFT \[32\]. It is therefore meaningful to study the physical meaning of the TFD in other contexts. In the following we shall investigate the quantum geometric meaning of \( \tilde{\mathcal{H}} \) in the TFD extension of GFT.

### 3.1 Algebraic thermo field dynamics

The thermal vacuum \([3,2]\) of TFD is based on a well-defined Hamiltonian \( H \) of a thermal system generating the time evolutions. For GFT and the related theories of quantum gravity, the Hamiltonian might not exist, but the operator-algebraic formulation is still valid.

Let \( \mathcal{A} \) be a \( C^* \)-algebra of physical observables with faithful algebraic states \( \omega \), then we have the GNS representation \((\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)\) of \( \mathcal{A} \) constructed from \( \omega \). Let \( \mathcal{M} = \pi''_\omega(\mathcal{A}) \) be the von Neumann algebra generated by \( \pi_\omega(\mathcal{A}) \), then for \( M \in \mathcal{M} \) it allows an antilinear operator \( S \) on \( \mathcal{H}_\omega \) such that \( SM\Omega_\omega = M^\dagger \Omega_\omega \), or equivalently \( S|A\rangle = |A^\dagger\rangle \) for \( |A\rangle \in \mathcal{H}_\omega \). The polar decomposition \( S = J\Delta^{1/2} \) defines the modular conjugation operator \( J = J^\dagger \) and the modular operator \( \Delta = S^\dagger S \). Then the Tomita-Takesaki modular theory of von Neumann algebra \([17]\) tells us that

\[
J\mathcal{M}J = \mathcal{M}', \quad \Delta^t \mathcal{M} \Delta^{-t} = \mathcal{M}, \quad t \in \mathbb{R}
\]

where the former is the commutation relation and the latter dictates the “time evolution”. The modular operator \( \Delta \) thus generates a flow \( \Delta^t(\cdot) \Delta^{-t} \equiv \sigma_t \) of the algebra without referring to an explicit Hamiltonian. The following algebraic formulation of TFD also does not explicitly use the Hamiltonian of the system, and hence have wider range of applicability.

**Theorem 3.1 (Ojima [35]).** For a thermal quantum system described by a von Neumann algebra \( \mathcal{M} \) constructed from the KMS states \( \omega_{KMS} \) with respect to the flow generated by the modular operator \( \sigma_t \), the TFD of \( \mathcal{M} \) consists of the “tilde” system described by \( \tilde{\mathcal{M}} = J\mathcal{M}J = \mathcal{M}' \) with \( J \) being the modular conjugate operator and the thermal vacuum \( \Omega_{\omega_{KMS}} \) which is the GNS vacuum of \( \omega_{KMS} \).

**Proof.** We first note that the flow \( \sigma_t \) can be considered as a flow with generator \( \tilde{H} \) such that \( \Delta^t(\cdot) \Delta^{-t} = e^{i\tilde{H}t}(\cdot)e^{-i\tilde{H}t} \) for \( \Delta = e^{-\beta \tilde{H}} \) and \( \beta = -1 \). Given the GNS vacuum
\( \Omega_{\text{KMS}} \), one has \( \Delta^\omega \Omega_{\text{KMS}} = 0 \), which implies \( \tilde{H} \Omega_{\text{KMS}} = 0 \). Since \( J\Delta J = \Delta^{-1} \), one has furthermore \( J\tilde{H}J = -\tilde{H} \). If the system described by \( \mathcal{M} \) has a well-defined Hamiltonian \( H \), then \( \tilde{H} \) admits the expression \( \tilde{H} = H - JHJ \), which indicates that the “tilde” space should be obtained by the modular conjugation. To proceed without a Hamiltonian, we suppose \( \tilde{H} \) is merely a generator of the flow \( \sigma_t, t \in \mathbb{R} \). Then we have
\[
M^\dagger \Omega_{\text{KMS}} = SM \Omega_{\text{KMS}} = J\Delta^{-1/2} M \Omega_{\text{KMS}} = Je^{-\beta H/2} M \Omega_{\text{KMS}},
\]
which corresponds to the thermal condition underlying the KMS condition in TFD \cite{12} with only one additional \( J \). From the Tomita-Takesaki theory, we also have the properties that \( J\Omega_{\text{KMS}} = \Omega_{\text{KMS}} \) and \( JMJ \in \mathcal{M}' \). So by identifying \( JM = \tilde{M} \in \mathcal{M} \), we obtain \( \langle A \rangle_\beta = (\Omega_{\text{KMS}}, A \Omega_{\text{KMS}}) \) satisfying the KMS condition. 

Notice that in general \( \mathcal{M}' \) is not the same as \( \mathcal{M} \) (because we want \( \mathcal{M} \) to be a factor), but on the level of elements \( M \in \mathcal{M} \) and \( JMJ \in \mathcal{M}' \) the corresponding Fock-space structures are the same. The reason is that for the GNS representation \((\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)\) of \( \mathcal{A} \) constructed from a state \( \omega \) and the von Neumann algebra \( \mathcal{M} = \pi'_\omega(\mathcal{A}) \), since the vacuum \( \Omega_\omega \) is cyclic and separating, we have \( \mathcal{H}_\omega = \pi_\omega \Omega_\omega = \mathcal{M}' \Omega_\omega = \mathcal{M}' \Omega_\omega \) where the overline denotes the norm closure \cite{16}. In this sense, the above algebraic TFD is more general than the Hilbert-space version.

### 3.2 Thermofield double states in group field theories

Let us first recall the thermal vacuum of a free bosonic oscillator with Hamiltonian \( H_b = ca^\dagger a \) \cite{12},
\[
|0, \beta \rangle_b = \sqrt{1 - e^{-\beta \epsilon}} \sum_{n=0}^{\infty} e^{-n \beta \epsilon/2} |n \rangle |\tilde{n} \rangle = e^{\theta(\beta)(a^\dagger \tilde{a} - \tilde{a} a)} |0 \rangle |\tilde{0} \rangle \tag{3.4}\]
where \( (a, a^\dagger) \) are the annihilation and creation operators on \( \tilde{\mathcal{H}} \), and the parameter \( \theta(\beta) \) is defined by the Bogoliubov transformation coefficients
\[
cosh \theta = (1 - e^{-\beta \epsilon})^{-1/2}, \quad \sinh \theta = e^{-\beta \epsilon/2}(1 - e^{-\beta \epsilon})^{-1/2}. \tag{3.5}\]
The thermal vacuum \eqref{3.4} is a squeezed state with the temperature-dependent squeezing parameter \( \theta(\beta) \). Now to formulate the TFD of GFT, we first need to perform the GNS construction from the Weyl algebra \( \mathcal{W} \) and a new vacuum \( |\Omega_S \rangle \) different from the Fock vacuum \( |\Omega_F \rangle = |\Omega \rangle \). From \eqref{3.4}, we expect the new vacuum \( |\Omega_S \rangle \) to define a \( \beta \)-dependent squeezed state representation similar to the coherent state representation \cite{27}. The squeezed states on a Weyl \( C^* \)-algebra have been studied, for example, in \cite{24,25}. We similarly define the GFT squeezed states on \( \mathcal{W} \):

**Definition 3.2.** Let \( \mathcal{W} \) be the Weyl \( C^* \)-algebra for GFT with Weyl elements \( W(\psi) \). A squeezed state on \( \mathcal{W} \) is
\[
\omega_S(W(\psi)) = \omega_F(W(\psi)) e^{-\frac{\epsilon}{2} \Re \{c(\psi, \bar{\psi})\}}, \quad c \in \mathbb{C} \tag{3.6}\]
where \( \omega_F(W(\psi)) \) is the Fock state \cite{22,30}. 

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The extra term added to the Fock state can be understood as coming from the variances (or fluctuations) of the self-adjoint field operators $\Phi, \Phi^\dagger$ of $(\Psi, \Psi^\dagger)$ in the expectation values of a squeezed vacuum $|\Omega_S\rangle$. By requiring the special relations

$$
\langle \Omega_S | \Psi(\psi) \Psi(\psi') | \Omega_S \rangle = c(\psi, \psi'),
$$

$$
\langle \Omega_S | \Psi^\dagger(\psi) \Psi^\dagger(\psi') | \Omega_S \rangle = \bar{c}(\psi, \psi'),
$$

$$
\langle \Omega_S | \Psi(\psi) | \Omega_S \rangle = 0,
$$

$$
\langle \Omega_S | \Psi^\dagger(\psi) | \Omega_S \rangle = 0,
$$

where $c \in \mathbb{C}$, we see that the the terms in (3.7) contribute to the covariance as $\text{Re}\{c(\psi, \bar{\psi})\}$. We can write the squeezed vacuum $|\Omega_S\rangle$ in a Fock space in the following way. Consider the Fock space representation of pure states $\omega$ on $\mathcal{W}$. Let us consider instead the subalgebra $\omega$ of convex combination of product Fock states. On the other hand, if $|\psi\rangle$ is not Fock states, so $|\psi\rangle$ can be written as convex combination values of a squeezed vacuum $|\Omega_S\rangle$. Hence the definition of squeezed vacuum state still holds.

We are interested in the von Neumann algebra $\mathcal{M} \otimes \mathcal{M}' \equiv \mathcal{M}$ of the TFD type. So let $\mathcal{W} = \mathcal{W} \otimes \mathcal{W}$ be the Weyl $C^*$-algebra corresponding to $\mathcal{M}$ and $\omega_\mathcal{W} : \mathcal{W} \rightarrow \mathbb{C}$ be the algebraic states on $\mathcal{W}$. By the same reasoning of Sec. 2.1, we can construct from $\omega_\mathcal{W}$ the GNS triple $(\mathcal{H}_\mathcal{W}, \pi_\mathcal{W}, |\Omega_\mathcal{W}\rangle)$. If this GNS representation restricted to $\mathcal{W}$ is Fock, we have $\mathcal{W} = J\mathcal{W}J = \mathcal{W}' = \eta \mathbf{1}$ with $\eta$ a constant. Let us denote $\mathcal{W}_0 = \mathcal{W} \otimes \mathbf{1}$. Likewise, when the total GNS representation on $\mathcal{W}$ is Fock, we still have the pure states $\omega_{\mathcal{W},F}$. If furthermore $\omega_{\mathcal{W},F}$ is separable and satisfy $\omega_{\mathcal{W},F} |\omega = \omega_F$, then $\omega_{\mathcal{W},F} \in \text{conv}(\omega_F \otimes \mathbf{1})$, the norm closure of convex combination of product Fock states. On the other hand, if $\omega_\mathcal{W} = \omega_{\mathcal{W},\text{KMS}}$ is the KMS states on $\mathcal{W}$ with respect to the left translations on the group-element arguments (which is not affected by $J$) in $\psi$, then $\mathcal{W} \neq \eta \mathbf{1}$. In this case, $\mathcal{W}_0$ becomes a subalgebra of $\mathcal{W}$. We can also define the (two-mode) squeezed states $\omega_{\mathcal{W},S}$ on $\mathcal{W}$ with the replacement $\Psi(\psi') \rightarrow \tilde{\Psi}(\psi')$. In this case, the relations in (3.7) can be violated, but the definition of squeezed vacuum state still holds.

For $\mathcal{M}$ to describe an algebraic TFD, it is required that $|\Omega_\mathcal{W}\rangle = |\Omega_S\rangle = |\Omega_{\text{KMS}}\rangle$. The following result shows that $\omega_{\mathcal{W},S}$ is qualified to be a TFD state, i.e. an entangled squeezed states.

**Proposition 3.3.** Let $\mathcal{W} = \mathcal{W} \otimes \mathcal{W}'$ be a Weyl $C^*$-algebra. Then the squeezed states $\omega_{\mathcal{W},S}$ on $\mathcal{W}$ is an entangled state.

**Proof.** Consider the Fock space representation of pure states $\omega_{\mathcal{W},F}$. Then any state on $\mathcal{W}$ can be written as convex combination $\omega_\mathcal{W} = \sum_i \lambda_i \omega_{\mathcal{W},F,(i)}$ with $\sum_i \lambda_i = 1$. If the squeezed states $\omega_{\mathcal{W},S}$ on $\mathcal{W}$ is separable, then we must have $\omega_{\mathcal{W},S} \in \text{conv}(\omega_F \otimes \mathbf{1})$. Since the $\omega_{\mathcal{W},S}$ are not Fock states, so $\mathcal{W}' \neq \eta \mathbf{1}$ and $\omega_{\mathcal{W},S} \notin \text{conv}(\omega_F \otimes \mathbf{1})$. Hence the $\omega_{\mathcal{W},S}$ are entangled.

Alternatively, we would like to avoid using $\omega_{\mathcal{W},F}$ which has not been explicitly constructed. Let us consider instead the subalgebra $\mathcal{W}_0$ of $\mathcal{W}$. Then the restriction of the domain of $\omega_{\mathcal{W},S}$ to $\mathcal{W}_0$ is the squeezed states $\omega_\mathcal{S}$ on $\mathcal{W}_F$, and it has the expression $\omega_\mathcal{S}(W) = \text{Tr}(\rho_\mathcal{S} \pi_F(W))$ for $W \in \mathcal{W}$ in the sense of local normal folium. By the definition
of $\omega_S$ (3.6), we have that $\rho_S$ is reducible and has the convex decomposition $\rho_S = \sum_n \lambda_n \rho_F^{(n)}$ where $n$ labels the particle number in the Fock space. By the algebraic approach [3], the von Neumann entropy between $\omega_{W_0,S}$ and $\omega_{W-W_0,S}$ is $S = -\sum_n \lambda_n \ln \lambda_n \neq 0$. Therefore $\omega_{W,S}$ is entangled. \hfill $\square$

Next, we need to find the Bogoliubov transformation that relates the squeezed vacuum states to the KMS states and find a way to return to the Fock space representation. Let us first consider the simple case of $|\Omega_S\rangle$:

**Theorem 3.4.** Let $W$ be the Weyl C$^*$-algebra for GFT. For the squeezed vacuum state $|\Omega_S\rangle$ on $W = W \otimes W'$, there exist a Bogoliubov transformation such that $|\Omega_S\rangle$ is transformed into the form of a bosonic TFD state with respect to the KMS states $\omega_{KMS}$ on $W$:

$$|\Omega_S\rangle = \sqrt{1 - e^{-i\beta u_*}} \sum_{n=0}^{\infty} e^{-i\beta u_*/2} |\{g_i\}, n\rangle |\{g'_i\}, \tilde{n}\rangle$$

(3.9)

where $u_*$ is an eigenvalue of $U_*$ (2.36), and the Fock states are the GNS Fock states.

**Proof.** In the simpler case where $b \in \mathbb{R}$, the thermal vacuum (3.8) has the same form as in (3.4), and hence the Bogoliubov transformation similar to (3.5) gives the TFD states in the standard form.

In the more general algebraic case, we need to find the *-automorphism $\alpha_T$ on $W$ such that $\alpha_T(W(\psi)) = W(T\psi)$ for $W(\psi) \in W$ and a symplectic transformation $T$ on the space of $\psi$'s. Notice that in (3.8) $b < \infty$, so by Theorem 2 of [25] there exists a unitary $U$ on the GNS space $H_W$ such that $\pi_W(\alpha_T(W')) = U\pi_W(W')U^*$. Denote $\Phi = \frac{b}{2} \Psi(\psi)\Psi(\psi') + \frac{i}{2} \Psi(\psi')\Psi(\psi)$ in (3.8) and consider $\pi_W = \pi_{W,F}$ as the Fock representation, then we have $U = e^{-it\Phi}$. Therefore, we can choose

$$\alpha_T(W(\psi)) = UW(\psi)U^* = e^{-it\Phi}W(\psi)e^{it\Phi} = W(T\psi),$$

with the symplectic transformation

$$T = \cosh(\varphi) + j \sinh(\varphi'), \quad \varphi = \delta(\psi(\{g_i\}, |b|), \varphi' = \delta(\psi'(\{g_i\}), |b|)$$

where $\varphi$ is positive and self-adjoint, and $j = -1$ for the bosonic case. When acting on the field operators, $\alpha_T$ gives the Bogoliubov transformations of the field operators on the Fock space $H_{W,F}$,

$$\alpha_T(\Psi(\psi)) = \Psi(\cosh \varphi \psi) + \tilde{\Psi}(\sinh \varphi' \psi') = \phi \cosh |b| - \tilde{\phi} \sinh |b|. \quad (3.10)$$

By choosing $b$ as in (3.5) and using the Gibbs states $\rho_g$ on $W$, we can obtain the TFD states.

$$|\Omega_S\rangle = \sqrt{1 - e^{-i\beta u_*}} \sum_{n=0}^{\infty} e^{-i\beta u_*/2} |\{g_i\}, n\rangle |\{g'_i\}, \tilde{n}\rangle.$$ 

(3.11)
This is because by setting the Bogoliubov operator $\chi$ as $\cosh 2\chi = \coth(i\beta u_*/2)$, we can rewrite the KMS state $\omega_{\text{KMS}}$ on $\mathcal{W}$ in the Fock representation as $\omega_{\text{KMS}}(W(\psi)) = e^{-\cosh 2\chi(\psi,\psi)/4}$. Since $\cosh 2\chi = \cosh^2 \chi + \sinh^2 \chi$, we have

$$\omega_{\text{KMS}}(W(\psi)) = e^{-\frac{1}{4}(\cosh^2 \chi + \sinh^2 \chi)(\psi,\psi)} = \omega_F(W(\cosh \chi \psi)) \otimes \omega_F(W(\sinh \chi \psi)).$$

Hence, by choosing $\chi \psi = |b\psi|$, we can obtain the TFD form (3.11). Since in equilibrium the Gibbs state $\rho_g$ is invariant under the group-translation flow $\alpha_{(g)}$, it is safe for the purpose of proving existence to take a constant eigenvalue $u_*$ of $U_*$ in $\rho_g$. In this way, the partition function of $\rho_g$ becomes $Z = \sum_n e^{-\beta g_n} = (1 - e^{-i\beta u_*})^{-1}$, and $|\Omega_S\rangle$ takes the standard form.

**Corollary 3.5.** Let $|\Omega_S\rangle$ be as above. Then both $\omega_S$ on $\mathcal{W}$ and the reduced state to $\mathcal{W}$ are KMS with respect to a group-translation flow $\alpha_{(b)}$ on the group fields.

**Proof.** Let $\rho_S = |\Omega_S\rangle \langle \Omega_S|$. By using $\langle \{g_i\}, \tilde{n}\{g'_i\}, \tilde{n}\rangle = \delta_{nm}$, we retain the Gibbs states on $\mathcal{W}$,

$$\rho_{\mathcal{W}} = \text{Tr}_{\mathcal{W}} \rho_S = \frac{1}{Z} \sum_n e^{-i\beta u_*} \langle \{g_i\}, n \rangle \langle \{g_i\}, n \rangle.$$ 

Consider the group-translation flow $\alpha_{(b)}$ on $\mathcal{W}$ with the same generator $G = iU_*$ as in $|\Omega_S\rangle$. The KMS condition is satisfied for Gibbs states following the standard arguments [12], with the Hamiltonian replaced by the generator $G$. 

We remark that the single-mode expression (3.9) can be changed to the multi-mode expression by considering the multi-mode expansion of the test functions $\psi$. For instance, consider the expansion of $\psi(g)$ with respect to the $\text{SU}(2)$ group basis, then

$$\Psi(\psi) = \int_G \prod_{i=1}^n d\mu(g) \sum_j \text{Tr}[\psi^j D^j(g)] \phi(\{g_i\}), \quad j \in \frac{N}{2},$$

where $D^j$ is the Wigner matrix. Since in (3.10) we have identified $|b\psi|$ with $\psi$, now the squeeze parameter $b$ also has an expansion into $b_j$, and as a consequence, the eigenvalue $u_*$ has a corresponding expansion into $u_{*,j}$. Then

$$|\Omega_S\rangle \sim \sum_{n=0}^\infty \sum_{j \in \mathbb{N}/2} e^{-i\beta u_{*,j}/2} \langle \{g_i\}, n \rangle \langle \{g'_i\}, \tilde{n}\rangle.$$ 

Moreover, since the CCR algebra structure is the same for all finite or infinite $N$, the $|\Omega_S\rangle$ can be extended to the full Fock space. Written in terms of the multi-particle occupation number basis, it becomes

$$|\Omega_S\rangle = \sum_{N_i} \sum_{n_i} \sum_{j \in \mathbb{N}/2} M(N_i, n_i, j) e^{-i\beta u_{*,j}/2} \langle \{g_i\}; \{n_i\}\rangle \langle \{g'_i\}; \tilde{n}\rangle.$$ 

where $M(N_i, n_i, j)$ is a normalization factor and $N_i$ is the number of particles in the subspace $\mathcal{H}_N^\perp$.

To recapitulate, the algebraic TFD extension of GFT results in the extended Weyl algebra $\mathcal{W} \otimes \mathcal{W}'$ and the squeezed vacuum $|\Omega_S\rangle$ such that the algebraic states $\omega_S[W] = \langle \Omega_S|W|\Omega_S\rangle$ are KMS with respect to a flow on the group fields.
3.3 Quantum geometric interpretation

In constructing the algebraic formulation of GFT, one has adopted the quantum geometric interpretation from LQG. It is therefore mandatory to consider the quantum geometric interpretation of the TFD states in GFT. At first sight, the “tilde” Hilbert space will simply copy the original Hilbert space. However, in the algebraic formulation of TFD \( \tilde{M} = M' \neq M \) in general. In other words, the “tilde” system has an observable algebra different from that of the original GFT system (of LQG), and they should represent different but correlated physical systems.

We observe the following: (i) The \( M' \) is emergent in the sense that it is defined according to the algebraic TFD rules and the equilibrium GFT Gibbs states. This is similar to the inclusion of matter into quantum gravity by generalizing GFTs to those defined on the Drinfeld double \( D(G) \) of the gauge group \( G \), where the dual group of \( G \) encodes the matter content. (ii) To each GFT Fock state \( |\{g_i\}, n\rangle \) there corresponds a “tilde” state \( |\{g'_i\}, \tilde{n}\rangle \). In a sense, this means that every piece of quantum geometric data is coupled to some referential state, as is proposed for gravity coupled with scalar matter in [45]. (iii) The “tilde” states are nontrivial only when there are interactions, because for the free Fock states \( \mathcal{W}_F = \eta \mathbf{1} \) is trivial. Since these “tilde” states are in the commutant \( M' \) they do not affect the dynamics of \( M \), but they can be considered as reference frames or relational clocks. Summarizing the above observations, we propose to interpret the TFD states in GFT as quantum polyhedra dressed with the emergent referential “matter” states.

Given a gauge-invariant group field \( \phi(\{g_i\}) \), we can interpret it as creating a dual quantum polyhedron in the quantum geometry of LQG with each \( g_i \) on the edge dual to a face of the polyhedron. In the second quantization formalism, this gives rise to a single-polyhedron Fock state (in terms of field operators) \( \Psi(\{g_i\}) \equiv |\Delta^1\rangle \) and likewise for \( |\Delta^n\rangle \) of \( n \) independent polyhedra. These independent polyhedra can be glued according to the interaction term (2.9) in the GFT action or the tracial observables (2.18) (in the second quantized form). These group fields constitute the Weyl algebra \( \mathcal{W} \) or the von Neumann algebra \( M \). Now the TFD extension adds \( \tilde{M} = M' \) to form \( M \otimes M' \equiv M \). For reasons explained in Sec. 3.1, \( M' \) differs from \( M \) in general but with the same Fock space structure, which means the field operators \( \hat{\Psi}(\{g'_i\}) \) for \( M' \) and \( \hat{\Psi}(\{g_i\}) \) (or the wavefunctions \( \psi \)) are different functions on \( G^x \), and they satisfy \( [\hat{\Psi}(\{g_i\}), \hat{\Psi}(\{g'_i\})] = 0 \) and \( [\Psi(\{g_i\}), \hat{\Psi}(\{g'_i\})] = 0 \). The structure of \( M \) allows us to take \( \tilde{\phi} = \phi \), so that the differences between \( \hat{\Psi} \) and \( \hat{\Psi} \) are solely in the test wavefunctions \( \psi \neq \psi' \). The Fock states on \( M \) are then doubled to be \( |\Delta^n\rangle \), e.g.

\[
\Psi(\{g_i\}) \otimes \hat{\Psi}(\{g'_i\}) \otimes |\Omega\rangle \equiv |\Delta^1\rangle.
\]

The doubled states \( |\Delta^n\rangle \) are further transformed into the entangled squeezed vacuum

---

\(^3\)In the conventional formulation of second quantization, the field operators are independent of the choice of the complete set of single-body wavefunctions, if two different sets of creation/annihilation operators can be related, e.g. by \( b = \sum_i \psi_i^* \psi_i a_i \). Here we do not have such transformations as the original and “tilde” field operators are commutative to each other.
state

\[ |\Omega_S\rangle \equiv |\blacktriangle\rangle_{\text{TFD}} = \sum_n e^{-\beta G_n} |\blacktriangle^n\rangle, \quad (3.15) \]

which is the TFD state for a GFT in equilibrium with respect to a flow on group fields. Note that in the proof of (3.9) the differences in \( \psi, \psi' \) have been wiped out in the Bogoliubov transformation (3.10), whereby the differences in the observable algebras are invisible at the state level.

By definition, \( \mathcal{M} = \pi''(\mathcal{W}) \subset \mathcal{B}(\mathcal{H}_\omega) \) are constructed from the representation of the Weyl algebra of the gauge-invariant group fields, and hence both \( \mathcal{M} \) and \( \mathcal{M}' \) contain gauge-invariant operators in \( \mathcal{B}(\mathcal{H}_\omega) \). Consequently, we can envision each quantum polyhedron originally created by \( \Psi(\{g_i\}) \) to be decorated by an additional layer of polyhedral data created by \( \tilde{\Psi}(\{g_i\}) \) and then they are superposed into a TFD state. As an example, consider the a single-cube state \( |\triangle_\text{cube}\rangle \) created by \( \phi(\{g_i\}), i = 1, 2, \ldots, 8 \). Then the Fock states on \( M \) can be depicted in the dual quantum geometry as the “fat” graphs:

\[
|\triangle_\text{cube}\rangle = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{cube1.png}
\end{array} \quad \Rightarrow \quad |\blacktriangle_\text{cube}\rangle = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{cube2.png}
\end{array}.
\quad (3.16)
\]

And the TFD states are obtained by the superposition \( |\blacktriangle_\text{cube}\rangle_{\text{TFD}} \).

The reason for still putting the states of \( \mathcal{M}' \) on the original quantum geometry is that the \( \mathcal{M}' \) is obtained from \( \mathcal{M} \) and they are both subsets of \( \mathcal{B}(\mathcal{H}_\omega) \). For simplicity, we can identify \( g_i = g'_i \), so that they live on the same edge. In this way, a single-polyhedron “tilde” state \( \tilde{\Psi}(\{g_i\}) |\Omega\rangle \) is expected to register the local changes in \( \Psi(\{g_i\}) |\Omega\rangle \) under the considered flow. This is similar to using a quantum clock system, which is correlated with the main system of concern, to register the quantum time experienced by the main system [20]. In the current case, by interpreting the “tilde” states as reference states, we can treat the the flow on group fields as a relational flow defined with respect to these “tilde” states.

As for the “matter” degrees of freedom, we can only speculate the following: Suppose \( \mathcal{M} = \pi''(\mathcal{W}) = \pi'(K) \) where \( \pi(K) \) is a representation of a group \( K \), then \( \mathcal{M}' = \pi''(K) \) is also a von Neumann algebra, so that \( \pi''(K) = \pi(K) \). In other words, \( \mathcal{M}' \) is determined by the (irreducible) representations of the group \( K \), which is similar to how matter quantum fields are classified by the irreducible representation of the Lorentz group. However, we know nothing about the group \( K \) at present.

Generally speaking, the equilibrium Gibbs states and the TFD states in GFT are defined with respect to some flow on the group fields. The flow or transformation, which could be neither dynamical nor symmetric, relates different group fields. If we use an additional “tilde” system as a reference frame for such a flow, there will be exchanges of information or geometric data between the main system and the “tilde” referencial system, but in the equilibrium the total state takes the TFD form invariant under the flow. Now in the current case, the “tilde” system \( \mathcal{M}' \) is obtained (or emergent) from \( \mathcal{M} \), but if one treats \( \mathcal{M}' \) as encoding the quantum gravitational data copied from \( \mathcal{M} \), then
the total theory will go beyond a quantum theory of general relativity (e.g. bimetric theories of massive gravity). In view of the exchanges between $M$ and $M'$, a natural alternative is therefore to treat $M'$ as encoding quantum “matter” degrees of freedom emergent from the quantum gravitational $M$.

4 Group field theory condensate at finite $\beta$

The GFT condensates are proposed to approximate the continuum geometries as the BEC of the group fields that encode the fundamental building blocks of LQG. As typical collective excitations, the squeezed states created by group fields in the second quantization formalism give rise to the “dipole” GFT condensate states [16]. Based on the fact that the TFD states are two-mode squeezed states on the Fock vacuum, we show in this section that such TFD states are indeed a particular type of GFT condensate states on the Fock vacuum. The equilibrium parameter $\beta$ distinguishes such TFD condensates from other types of GFT condensates.

Let us first recall the homogeneous GFT condensates without a flow [40]. The GFT condensate states are basically the infinite superposition of the coherent excitations of group fields, e.g. $e^{\Psi\{g_i\}}|\Omega\rangle$ where $\Psi$ is the field operator. The homogeneity condition is imposed on the “single-polyhedron” wavefunctions $\psi\{g_i\}$ in the field operator $\Psi$'s: two field operators of GFT $\{\Psi_1\{g_i\}\},\{\Psi_2\{g_i\}\}$ are said to be homogeneously connected in $\psi$ if the “single-polyhedron” wavefunctions $\psi\{h_ig_i\}$ have an extra set of group-element arguments $\{h_i\}$ and satisfy the convolution condition for connecting a pair of $h$'s,

$$\Psi_1^\dagger\Psi_2^\dagger = \int_{G^n} \prod d\mu(g_i)d\mu(h_i) \prod \delta(h_i h_j^{-1}) \psi_1(\{...,h_i g_i,...\}) \psi_2(\{...,h_j g_j,...\}) \phi_1^\dagger \phi_2^\dagger.$$  (4.1)

A homogeneous dipole condensate states are then $e^{\Psi\{g_i\}\dagger\Psi\{g_i'\}\dagger}|\Omega\rangle$. This homogeneity of wavefunctions constrains the connectivity of the additional $\{h_i\}$ in the GFT condensate states, and we expect that there will be a similar homogeneity condition on the equilibrium states with respect to a flow on the group fields. On the other hand, when one identifies a homogeneous GFT condensate state as the seed state $|\text{seed}\rangle$ and generates other condensate states by acting the refinement operators on the seed state, e.g. $M|\text{seed}\rangle$, one needs to know more information about condensate states so as to distinguish different quantum geometries from the indistinguishable bosonic group fields. The “tilde” referential states in the TFD states then naturally serve this purpose for GFT with a flow. With these observations in mind, we consider the following

**Proposition 4.1.** For a GFT with a group-translation flow $\alpha_{\{g_i\}}$, the TFD vacuum $|\Omega_S\rangle$ is a GFT condensate state on the GFT Fock vacuum $|\Omega\rangle \otimes |\Omega\rangle$ without the flow. If the edge-homogeneity constraint $\delta(g_i g_i'^{-1})$ is imposed on $|\Omega_S\rangle$, then homogeneity of GFT condensates is equivalent to the homogeneity in the referential states.

**Proof.** That $|\Omega_S\rangle$ is formally a GFT condensate state on $|\Omega\rangle \otimes |\Omega\rangle$ is obvious from the expression [3.8]. The edge-homogeneity condition implemented by $\delta(g_i g_i'^{-1})$ in the test
wavefunctions constrains the two sets of group fields $\phi(\{g_i\})$ and $\tilde{\phi}(\{g'_i\})$ to live on the same edges with group elements $\{g_i\}$. This edge-homogeneity thus puts two layers of polyhedra on the same graph on which the group variables live. By constraining $g_i = g'_i$, the convolution condition (4.1) for $\psi(\{g_i\})$ can be transferred to that of $\psi'(\{g'_i\})$.

The TFD states add twofold new data about the flow to the GFT condensates: one is the “tilde” referencial states which register the responses of the quantum geometry under the flow, which have been put on the original quantum geometry; the other is the Gibbs states determining the squeezing induced by the flow. As a consequence, in addition to the homogeneity of referencial states, we also have the equilibrium condition:

**Definition 4.2.** In a GFT with field operators $\Psi(\{g_i\}), \Psi^\dagger(\{g_i\}), i \geq 4$ and a group-translation flow $\alpha_{\{g_i\}}$, the GFT states are said to be in equilibrium with respect to $\alpha_{\{g_i\}}$ at the equilibrium parameter $\beta$ if their TFD states have the same parameter $\beta$.

At a uniform $\beta$, one can choose a homogeneous seed state of GFT condensate and then refine it topologically as in [40]. Let us consider instead the TFD states with different $\beta$. Since the the parameters $\beta$ or the Gibbs states are new data added to the quantum geometry represented by GFT, we can specifically consider different condensate states without these new data related by the refinement operators, and then add to each layer $r$ of refinement the Gibbs states with a fixed $\beta_r$. Consequently, on each layer of some shell condensate states there is a distinct algebraic symmetry with equilibrium parameter $\beta$.

If $\beta$ can be interpreted as the inverse temperature, we then envisage a relation between $\beta$ and the refinement layer $r$ in analogy to the Tolman-Ehrenfest effect for equilibrium temperatures in a gravitational field.

Without loss of generality, let us work in the basic case of 4-fold group fields $\phi(\{g_i\}), i = 1, 2, 3, 4$, representing the 4-valent vertices dual to quantum tetrahedra. By Proposition 4.1, the topological connectivity between group fields in a TFD condensate state is transferred to the homogeneity of the referential states. So the white-black colored group fields are now colored by the “tilde” referential states:

\[
\left| \begin{array}{c} \text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array} \right> \equiv \left| \begin{array}{c} \text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array} \right>, \quad \left| \begin{array}{c} \text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array} \right> \equiv \left| \begin{array}{c} \text{1} \\
\text{2} \\
\text{3} \\
\text{4}
\end{array} \right>.
\]

(4.2)

Since the “tilde” states are only referencial, one can in principle refine the geometry by adding such colored vertices in the same way as [40]. The homogeneity condition constrains the connectivity of group fields (or their dual quantum polyhedra) in a GFT condensate, but not the parameter $\beta$ in a GFT condensate state of TFD form. For two group fields homogeneously connected in a TFD state at an equilibrium parameter $\beta$, we

\footnote{Note that when gluing different condensate states, the gluing edge generically breaks the closure constraint (closure defect) [2]. The problem of how to relate different algebraic symmetries defined with different $\beta$ does not arise in that the presence of closure defects breaks the gauge invariance of group fields, and hence the flow defined on the gauge-invariant group fields cannot be defined across the boundary.}
can depict the TFD superposition of two GFT Fock states by an additional dotted line, e.g.

\[ | \begin{array}{c} \text{15} \\ \text{14} \\ \text{13} \\ \text{12} \\ \text{8} \\ \text{9} \\ \text{10} \\ \text{11} \end{array} \rangle + | \begin{array}{c} \text{7} \\ \text{6} \\ \text{5} \\ \text{4} \\ \text{0} \\ \text{1} \\ \text{2} \\ \text{3} \end{array} \rangle \Rightarrow | \begin{array}{c} \text{15} \\ \text{14} \\ \text{13} \\ \text{12} \\ \text{8} \\ \text{9} \\ \text{10} \\ \text{11} \end{array} \rangle \beta \rangle \] (4.3)

where the gluing rules are obeyed such that only the edges on vertices of different colors (white \neq black) with the same referential index (e.g. 4 = 4) can be glued. In this way, the equilibrium condition constrains the algebraic symmetry of the group fields on top of the homogeneous condensate geometries and their refinements. Therefore, the TFD states with different \( \beta \) can still be homogeneously glued and refined, but they are distinguished by the different algebraic symmetries parameterized by \( \beta \).

**Example 4.3.** Consider the homogeneous shell condensate states in GFT, which can model the horizon of a black hole [41]. The seed state for a homogeneous shell condensate is chosen to be the following state

\[ | \text{seed} \rangle_b = | \begin{array}{c} \text{1} \\ \text{1} \\ \text{1} \\ \text{1} \\ \text{1} \\ \text{1} \\ \text{1} \end{array} \rangle , \] (4.4)

with two boundaries, i.e. two sets of open edges with indices 1 and 4 respectively. Since the boundaries consist of open edges, we do not consider any flow on the boundary edges. The refinement operators are

\[ \hat{r}_W : | \begin{array}{c} \text{1} \\ \text{1} \\ \text{1} \end{array} \rangle \mapsto | \begin{array}{c} \text{1} \\ \text{1} \\ \text{1} \end{array} \rangle , \quad \hat{r}_B : | \begin{array}{c} \text{1} \\ \text{1} \\ \text{1} \end{array} \rangle \mapsto | \begin{array}{c} \text{1} \\ \text{1} \\ \text{1} \end{array} \rangle \] (4.5)

and at the boundaries, for example a boundary vertex with index 4,

\[ \hat{r}_{W,\partial} : | \begin{array}{c} \text{4} \\ \text{4} \end{array} \rangle \mapsto | \begin{array}{c} \text{4} \\ \text{4} \end{array} \rangle \ , \quad \hat{r}_{B,\partial} : | \begin{array}{c} \text{4} \end{array} \rangle \mapsto | \begin{array}{c} \text{4} \end{array} \rangle \] (4.6)

where the open edges on the boundary are also refined. The refined shell condensate states are formally

\[ | r \rangle = \prod_r f_r(\hat{r}_B, \hat{r}_W, \hat{r}_{B,\partial}, \hat{r}_{W,\partial}) | \text{seed} \rangle_b \] (4.7)

where \( f_r \) is a function of the refinement operators.

Given an infinite number of possible refinements of a GFT condensate, one can choose a particular refinement level \( r_h \) of the shell condensate to model the horizon of a (quantum) black hole. In a generic shell condensate \( | r \rangle \), the boundary with index 4 is not only the outer boundary of \( | r \rangle \) but also the inner boundary of \( | r + 1 \rangle \) at the next level. But the connectivity between two shell condensate does not specify a black hole horizon. We notice that, with hindsight, the event horizon of a black hole possesses the surface gravity \( \kappa \) that relates the symmetry of the horizon to the temperature of Hawking radiation, and
in particular, the local surface gravity $\kappa = 1/\ell$, where $\ell$ is the proper distance from an observer (or an apparent horizon) to the horizon, and the local first law $\delta E = \kappa \delta A/8\pi$ [18] show the local (quantum) relation between gravity and matter. In the current case of TFD states of GFT, we have both the quantum gravitational states and the emergent “matter” states, which are correlated by the squeezing transformation. Therefore, for the shell condensate at a chosen refinement level to be able to model a black hole horizon, we expect the equilibrium parameter $\beta$ to play the role of $\kappa$, encoding the symmetry of the horizon as well as the local Hawking temperature.

For instance, consider the horizon shell $|r_h\rangle$ at equilibrium parameter $\beta_h$, and its next refinement $|r_h + 1\rangle = f_{r_h+1}(\hat{r}_B, \hat{r}_W, \hat{r}_{B,\beta}, \hat{r}_{W,\beta}) |r_h\rangle$ at $\beta_h'$. Parts of the boundary looks like the following

\[
\begin{array}{c}
| \beta \rangle + | \beta' \beta' \beta' \rangle \Rightarrow \quad (r_h + 1) \\
\end{array}
\]

(4.8)

In (4.8) we require $\beta' > \beta$, or equivalently $\beta$ is a monotonically increasing function $\beta(r)$ of the refinement level $r$. Then the $\beta'$ can correspond to the inverse of the local surface gravity $\kappa$, if the proper distance $\ell$ is also a monotonically increasing function $\ell(r)$ of the refinement level $r$. In other words, we can take

\[
\beta'(r_h + 1) \propto \ell(r_h + 1), \quad (4.9)
\]

so that $\beta'$ outside the horizon shell can be interpreted as the inverse local Hawking temperature. Meanwhile, in (4.8) the edge labeled by the group element $g$, is dual to a quantum area patch in the LQG quantum geometry, which gives rise to the constraint that the sum over the area eigenvalues in $|r_h\rangle$ is the area $A$ of the horizon. The proper distance $\ell$ should be defined with respect to this horizon, but this is not given in the quantum geometries. The TFD states of GFT, a posteriori, define the $\ell$ in terms of the equilibrium parameter $\beta$.

In short, for an equilibrium shell condensate state in GFT to model a black hole horizon, it is required that, in addition to the semiclassical conditions given in [41], the equilibrium parameter $\beta$ can be interpreted as the proper distance $\ell$ from an apparent horizon to the event horizon and satisfies the local first law of black hole thermodynamics.

We remark that the use of TFD states in Example 4.3 is different from [26, 32]; on the one hand, here a TFD state only defines single GFT shell condensate decorated by referencial “matter” states, instead of a double states; on the other hand, the TFD state entangles gravitational and “matter” states, unlike the trans-horizon entanglement. Nevertheless, the latter is consistent with the explanation of black hole entropy via gravity-matter entanglement [6].
Example 4.4. Let us turn to the GFT condensate quantum cosmology. For a GFT coupled with massless scalar fields $\varphi$, the group fields are defined on $G^{\times n} \times \mathbb{R}^n$. The scalar fields can be used as a relational clock if they are monotonic on $\mathbb{R}^n$, and then the GFT can be deparameterized to have a relational Hamiltonian generating the dynamics \cite{11}. This generic method, when applied to the cosmological case, has been illustrated in a toy model \cite{1} where the deparameterized Hamiltonian is the squeezing operator (in the spin basis). In GFT at equilibrium parameter $\beta$, the TFD state $|\Omega_S\rangle$ can be written in the form of a squeezed state as (3.8),

$$|\Omega_S\rangle = e^{i\varphi\{\bar{b}^2\tilde{\Psi}^\dagger(\psi)\tilde{\Psi}^\dagger(\psi')+\bar{b}^2\Psi(\psi)\tilde{\Psi}(\psi')\}}|\Omega\rangle \otimes |\Omega\rangle,$$

where we have written the parameter explicitly as the scalar field $\varphi$. By identifying the relational Hamiltonian (in the second-quantized form) $H_\varphi = -\frac{\bar{b}^2}{2}\tilde{\Psi}^\dagger(\psi)\tilde{\Psi}^\dagger(\psi') - \frac{\bar{b}^2}{2}\Psi(\psi)\tilde{\Psi}(\psi')$ with respect to $\varphi$, we see that the TFD squeezed state is the “time” evolution of the Fock vacuum generated by $H_\varphi$. In other words, $|\Omega_S\rangle$ is a solution of the relational Schrödinger equation $i\frac{d}{d\varphi}|\Omega_S\rangle = H_\varphi|\Omega_S\rangle$.

Notice that the scalar fields are not introduced in the setup of this paper, the $\varphi$ in (4.10) are in fact functions $\varphi(\beta)$ of the equilibrium parameter $\beta$. The referential “tilde” states are used to reflect the responses of the untilded states, and they can be created and annihilated as well, thereby not qualified as a monotonic relation clock. The obvious alternative is to consider the flow parameter $\beta(\varphi, b, n)$ as a monotonic function of $\varphi$ such that $\varphi(\beta)$ is also monotonic as an inverse function.

The relational evolution generated by $H_\varphi$ should be applicable to arbitrary initial states. Let us take the seed state for spatial 3-sphere as the initial state,

$$|\text{seed}\rangle_c = |\begin{array}{c} \varnothing \end{array}\rangle.$$

Then the squeezing $e^{i\varphi H_\varphi}|\text{seed}\rangle_c \equiv |\omega_S\rangle$ generates a series of new vertices. Recalling the quantum geometric interpretation of the “tilde” states, each $\tilde{\Psi}^\dagger(\psi)\tilde{\Psi}^\dagger(\psi')$ creates a colored vertex. So, in contrast to the model in \cite{1}, here the squeezing only refines $|\text{seed}\rangle_c$ in the same way as (4.5). Again, we impose the additional constraint that at each refinement level $r$ the GFT condensate state should be in equilibrium at $\beta_r$, which implies a equal-$\varphi(\beta_r)$ spatial hypersurface. Therefore, the refinement maps changes the equilibrium parameter $\beta$, e.g.

$$|\begin{array}{c} \varnothing \end{array}\rangle \beta \mapsto |\begin{array}{c} \varnothing \end{array}\rangle \beta'.$$

Next, suppose that each colored vertex occupies the minimal volume $v_0$ predicted by the LQG quantum geometry, then the total quantum volume of the refined 3-sphere is $V = v_0 n$ where $n$ is the number of vertices defined by the untilded number operator $N = \Psi^\dagger(\psi)\Psi(\psi)$. For example, $n = 4$ for $|\text{seed}\rangle_c$. The Friedman equation for GFT condensate quantum cosmology is obtained by calculating the evolution of $V$ as a scale.
factor with respect to $\varphi$. Without going into details, we observe that at a refinement level $r$, the volume $V_r$ of the GFT condensate not only sets the scale of the radius $l_A$ of the refined 3-sphere, but also the scale $\varphi$ of cosmic time. In this sense, we can interpret the parameter $\beta$ as the inverse “temperature” of the apparent horizon, $\beta(r) \propto l_A$, so that

$$\varphi(\beta, r) \propto \varphi(l_A, r)$$

(4.13)

depends monotonically on $V_r$.

Relating the $\beta$ to the scale or the number of vertices in a GFT condensate has another a posteriori advantage: a fixed $\beta$ means a fixed finite number of Fock excitations in the GFT condensate, so that the tracial states can be directly defined for this condensate state without worrying the types of von Neumann algebras. A point to note is that for a TFD state $| \Omega_S \rangle$, the number of excitation in the untilded system is $\langle \Omega_S | \phi^\dagger \phi | \Omega_S \rangle = \sinh^2 b(\beta) \equiv n$, which obviously has a monotonic dependence on $\beta$. Now consider the “off-diagonal” correlator $\langle \Omega_S | \Psi^\dagger(\psi) \Psi(\psi') | \Omega_S \rangle$. By the background-independent definition of the field operators and (3.10), we see that the $\psi'$ will be identified with $\psi$ after the Bogoliubov transformation to the unsmeared group fields. As a consequence, we have, by relating $\beta$ to $n$,

$$\langle \Omega_S | \Psi^\dagger(\psi) \Psi(\psi') | \Omega_S \rangle \sim n,$$

(4.14)

which shows the off-diagonal long-range order in $| \Omega_S \rangle$, an essential characterization of BEC [46].

5 Summary

In this paper, we have studied a particular type of TFD states of GFT in the operator-algebraic formulation with the second-quantization interpretation. In view of the equilibrium Gibbs states for GFT recently obtained in [29], this TFD states contribute to an equilibrium GFT “at finite temperature”, which in turn reveals the “thermal” aspects of quantum gravity.

The explicit construction takes analogy from the the TFD states of bosonic oscillators, since the GFT state space in the second quantized formulation has the structure of the bosonic Fock space. We have defined the squeezed states (3.6) on the Weyl algebra of group fields and shown that these squeezed states can take the standard form of TFD states (3.9) on the doubled Hilbert space. We have worked in the generalized operator-algebraic approach to TFD where the “tilde” algebra of observables is the commutant of the original one, so that the original and the “tilde” system can be different in their algebras. This generalized formulation allows us to interpret the “tilde” states in the TFD states as emergent reference states, thereby constituting a consistent “fat” quantum geometric interpretation of the TFD states thus obtained for GFT.

The constructed TFD states are condensate GFT states “at finite temperature” when viewed from the original GFT vacuum. With this understanding, we have qualitatively
shown that the equilibrium flow parameter $\beta$ is related to the inverse temperature of gravitational thermodynamics in the GFT condensate description of black hole horizon and quantum cosmology.

The algebraic KMS Gibbs state is of course not the only equilibrium state in GFT. The maximal entropy principle gives the generalized Gibbs states of GFT [10, 28], which allows a formulation of statistical field theory of quantum polyhedra. In this respect, it is promising to investigate the thermal field theories in the context of GFT.

The algebraic TFD formulated via the Tomita-Takesaki modular theory is more general than the standard TFD at the algebraic level, but of course consistent with the the standard one at the Hilbert-space level. If one only focuses on the states without worrying the algebraic representations, one can, with hindsight, still interpret the TFD doubled states as physical plus referencial states. The question now is how far can this interpretation go, since simply copying the geometric data will lead us to gravity different from general relativity. An exception could be the general relativity with a cosmological constant, whose action in (2 + 1)-dimension can be written as a difference of two SU(2) Chern-Simons action or a TFD doubled action [34]. In (3 + 1)-dimension, the $SL(2, \mathbb{C})$ Chern-Simons theory (with two Chern-Simons term in the action) has been related to simplical quantum gravity with a cosmological constant in some recent works [23]. We thus hope to find in the future some deeper connections between TFD, $q$-deformation (cf. Appendix A) and quantum gravity with a cosmological constant.

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A Thermo field dynamics extension via Hopf algebra

In this appendix, we present another algebraic approach to the TFD extension of GFT via Hopf algebra. The starting point is the Hopf algebra of group fields or of $\mathbb{C}$-functions on group $G$, with which the GFT on the Drinfeld double $D(G)$ of $G$ has been studied in [31] (see also [4]). Our strategy here is, instead of considering the Drinfeld double, to apply the quantum-algebra interpretation of TFD doubling [8] to the Hopf algebra of group fields. However, we are unable to find a clear quantum geometric interpretation to the quantum algebra of TFD doubling for GFT.

A.1 Hopf algebra of group fields

The group fields, as functions on groups, can have the structure of a Hopf algebra. To see this, let us first recall that, in the noncommutative metric representation of GFT [5], the convolution of group fields is transformed into the noncommutative $\ast$-product of plane
wavefunctions $e_g(x)$, e.g.

$$\int_{\mathbb{R}} dx \hat{\phi}_1 \star \hat{\phi}_2(x) = \int_{\mathbb{R}} dx \int_G d\mu(g) \phi_1(g) \phi_2(g^{-1}) e_g \ast e_{g^{-1}}(x) = \int_{\mathbb{R}} dx \int_G d\mu(g) \phi_1(g) \phi_2(g^{-1}) e_{gg^{-1}}(x)$$

$$= \int_G d\mu(g) \phi_1(g) \phi_2(g^{-1}) \delta(gg^{-1}) = \int_G d\mu(g) \phi_1(g) \phi_2(g)$$ \hspace{1cm} (A.1)

where $\hat{\phi}(x) = \hat{\phi}(-x)$ and $e_g \ast e_{g'} = e_{gg'}$.

From (A.1) we can almost see the structure of a Hopf algebra. Indeed, we have exactly a Hopf algebra $A^*$ of group fields: The product or multiplication in $A^*$ is simply

$$\phi_1 \ast \phi_2(g) = \phi_1(1) \phi_2(g), \quad g \in G,$$ \hspace{1cm} (A.2)

and obviously the unit is the constant group field $\iota(g) = \phi_1(g) = 1$ which is the identity element of $A^*$. The coproduct in $A^*$ is

$$\Delta^* \phi(g, g') = \phi(gg'), \quad g, g' \in G.$$ \hspace{1cm} (A.3)

This is because the function algebra $\mathcal{F}(G \times G) \cong \mathcal{F}(G) \otimes \mathcal{F}(G)$, so that the tensor product in a Hopf algebra is transfered to a two-fold group field. The co-unit in $A^*$ is then $\varepsilon(\phi) = \phi(g_0)$ with $g_0$ being the identity element of $G$ such that $\iota \circ \varepsilon(\phi) = \phi_1(g_0) = 1$. The antipode is, as in (A.1), $s(\phi(g)) = \phi(g^{-1})$; the involution is simply the complex conjugation. It is easy to see that the defining relation for a Hopf algebra, after restoring the tensor product notation,

$$\ast \circ (1 \otimes s) \circ \Delta^* = \ast \circ (s \otimes 1) \circ \Delta^* = \iota \circ \varepsilon$$ \hspace{1cm} (A.4)

holds here.

The convolution of group fields (A.1) defines an inner product and hence a pairing of group fields,

$$\langle \phi_1, \phi_2 \rangle = \int_G d\mu(g) \phi_1(g) \phi_2(g).$$ \hspace{1cm} (A.5)

For reasons explained in [31], it is also useful to work with the dual algebra $A$ of $A^*$. $A$ is also a Hopf algebra with the product $\bullet$ given by the duality $\langle \phi_1 \bullet \phi_2, \phi \rangle = \langle \phi_1 \otimes \phi_2, \Delta^* \phi \rangle$, that is,

$$\phi_1 \bullet \phi_2(g) = \int_G d\mu(h) \phi_1(h) \phi_2(h^{-1}g), \quad g, h \in G.$$ \hspace{1cm} (A.6)

The identity of this $\bullet$-product is then the $\delta$-function $I(g) = \delta_{g_0}(g)$ such that $I \bullet \phi(g) = \phi(g)$. The coproduct and the co-unit in $A$ are respectively

$$\Delta \phi(g, g') = \phi(g) \delta_{g}(g'), \quad \varepsilon(\phi) = \int_G d\mu(g) \phi(g).$$ \hspace{1cm} (A.7)

The antipode $s$ is defined as in $A^*$, but the involution $\ast$ is now $\phi^*(g) = \overline{\phi(g^{-1})}$. In this case of $A$, we still have the relation $\bullet \circ (1 \otimes s) \circ \Delta = \bullet \circ (s \otimes 1) \circ \Delta = I \circ \varepsilon$, if we simply put $\Delta \phi = \phi(g) \otimes \phi(g)$ in the tensor product notation.
The •-product induces a presentation of the gauge invariance of group fields through the projector \( \vartheta = \vartheta \bullet \vartheta \) on \( A^\otimes n \) where

\[
\vartheta(\{g_i\}) = \int_G d\mu(h) \prod_{i=1}^n \delta_{g_i}(g_i^{-1}h).
\] (A.8)

It is easy to see that \( \vartheta \bullet \phi(\{hg_i\}) = \vartheta \bullet \phi(\{g_i\}) \) for \( h \in G \). The \( \vartheta \) can be considered as a differential, so that the \( \delta \)-functions in (A.8) are naturally the propagators of a single-fold group field. According to the quantum geometric interpretation of GFT, a gauge-invariant group field generates a quantum polyhedron. Then the Hopf algebra \( A \) dictates the way in which these quantum polyhedra are related. By recalling how the simplicial spin foams are generated by the perturbative Feynman diagrams of GFT, the \( A \) is furthermore related to the Hopf algebraic approach to the coarse-graining/renormalization in spin foam models [33]. For example, the •-product convolutes two group fields and hence glues two polyhedra; the coproduct \( \Delta \) unfolds a spin foam into two subfoams and trivializes or coarse grains one of them.

### A.2 Thermo field dynamics doubling

Now we show that the TFD can be obtained by the \( q \)-deformation of the Hopf algebra \( A \). To this end, let us rewrite the coproduct in \( A \) as

\[
\Delta : A \rightarrow A \otimes A; \quad \Delta \phi(g) = \phi(g)\delta_g(g') = \frac{1}{2}(\phi(g)\delta_g(g') \otimes 1 + 1 \otimes \phi(g)\delta_g(g')).
\] (A.9)

Adopting the second-quantization interpretation of GFT, we see that \( A \) can be made into a bosonic Heisenberg-Weyl algebra by adding a central operator \( h = \delta(\{g_i\}, \{g_j\})/2 \) and the number operator \( N = \phi^\dagger \phi \). For the purpose of describing TFD doubling, we can focus only on the \( q \)-deformation of the coproducts of \( \phi \):

\[
\Delta \phi_q(g) = \frac{1}{2}(\phi(g)\delta_g(g') \otimes q^{1/2} + q^{-1/2} \otimes \phi(g)\delta_g(g'))
\] (A.10)

where the \( q \) is required to satisfy \( |q| = 1 \).

The coproduct \( \Delta \) naturally doubles the degrees of freedom in the Fock space generated by the group fields \( (\phi, \phi^\dagger) \). The following result shows that the \( q \)-deformed coproduct can be related to the TFD doubling,

**Proposition A.1.** Let \( q = e^{2\theta} \) be the parameter of the \( q \)-deformation. If the \( \theta = \theta(\beta) \) is a squeezing parameter, then the TFD squeezed states in GFT are obtained by the \( q \)-deformed coproduct of group fields.

**Proof.** The following proof is an adaptation of [8] to GFT. Consider the “normalized”

\(^5\)Here the block transforms are encoded in the coproduct, so the antipode in \( A \) is much simpler.
$q$-deformed coproduct of group fields and their derivatives,

$$A_q \equiv \frac{\Delta \phi_q(g)}{\sqrt{|2|_q}} = \frac{1}{2 \sqrt{|2|_q}} (e^\theta \phi(g) \delta_q(g') + e^{-\theta} \tilde{\phi}(g) \delta_q(g')),$$

$$B_q \equiv \frac{1}{\sqrt{|2|_q}} \frac{\delta(\Delta \phi_q(g))}{\delta \theta} = \frac{1}{2 \sqrt{|2|_q}} (e^\theta \phi(g) \delta_q(g') - e^{-\theta} \tilde{\phi}(g) \delta_q(g')).$$

Then $A_q + B_q = 2e^\theta \phi(g')/\sqrt{|2|_q}$ and $A_q - B_q = 2e^{-\theta} \tilde{\phi}(g')/\sqrt{|2|_q}$. By introducing

$$A(\theta) = \frac{\sqrt{|2|_q}}{2 \sqrt{2}} (A_q(\theta) + A_q(-\theta) - B_q(\theta) + B_q(-\theta)),$$

$$B(\theta) = \frac{\sqrt{|2|_q}}{2 \sqrt{2}} (B_q(\theta) + B_q(-\theta) - A_q(\theta) + A_q(-\theta)),$$

we have for any $g' \in G$,

$$\phi(\theta) = \frac{1}{\sqrt{2}} (A(\theta) + B(\theta)) = \phi \cosh \theta - \tilde{\phi}^\dagger \sinh \theta \quad (A.11)$$

and likewise $\tilde{\phi}(\theta) = (A(\theta) - B(\theta))/\sqrt{2}$. Thus, (A.11) matches the Bogoliubov transformation for TFD squeezed vacuum $|\Omega_S\rangle$ such that $\phi(\theta) |\Omega_S\rangle = 0.$

Now that the TFD states can be obtained from the $q$-deformed Hopf algebra of group fields, it is tempting to relate the TFD states to the hyperbolic quantum geometry of $q$-deformed LQG [15]. However, in $q$-deformed LQG the gauge group is deformed into a quantum group, for example, $U_q(su(2))$, whereas in the TFD extension of GFT, what is deformed is the Hopf algebra of group fields which are still defined on the original gauge group. In other words, the $q$-deformed gauge group changes the algebraic data on, or the coloring of, the spin network states of the undeformed LQG, while in the TFD case the geometric coloring is preserved but the spin network states are squeezed into a particular condensate state. Physically, the $q$-deformation of the gauge group puts a “cutoff”, related to the cosmological constant, on the LQG vertex amplitudes, but the TFD vacuum state mixes the spin network states of LQG. In this sense, there does not seem to be a direct relation between these two quantum deformations.

From the point of view of coarse-graining spin foams, the $q$-deformed coproduct does not completely coarse grain the spin foams, but retains the mixing of all the subfoams.

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