Weighted $L^p$ estimates on the infinite rooted $k$-ary tree

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Abstract

In this paper sufficient conditions for weighted weak and strong type $(p, p)$ estimates with $p > 1$ for the centered maximal function on the infinite rooted $k$-ary tree are provided. Here the line initiated by the authors and Safe in Ombrosi et al. (Int Math Res Not IMRN 4:2736–2762, 2021) and providing a further extension of the use of techniques due to Naor and Tao in (J Funct Anal 259(3):731–779, 2010) is continued. The fact that the class of weights from the sufficient conditions is wider for $L^p$ estimates than the one obtained in [16] is established as well. Some results highlighting the pathological nature of the weighted $L^p$ theory in this setting are settled. It is shown that the $A_p$ condition is no precise in this setting, since there exist weights such that the $L^p$ boundedness holds but the $A_p$ condition is not satisfied. It is also shown that the Sawyer type testing condition is not sufficient either for the strong type to hold and also that strong and weak type estimates are not equivalent in this setting. It will be shown as well that the one weight results can be extended to the two weight setting.

1 Introduction and main results

In the seventies Muckenhoupt [12] established the following result. If $1 < p < \infty$ and $M$ stands for the Hardy-Littlewood the following statements are equivalent

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• \( w \in A_p \), namely
\[
[w]_{A_p} = \sup_B \frac{1}{|B|} \int_B w \left( \frac{1}{|B|} \int_B w^{-\frac{1}{p-1}} \right)^{p-1} < \infty \tag{1.1}
\]

where each \( B \) is a ball of \( \mathbb{R}^n \).

• The strong type estimate
\[
\|Mf\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \tag{1.2}
\]
holds.

• The weak type estimate
\[
\|Mf\|_{L^{p,\infty}(w)} \lesssim \|f\|_{L^p(w)} \tag{1.3}
\]
holds.

Throughout the next decades this seminal work led to the development of the theory of weights that has been a very fruitful area since then. Also a number of generalizations for a number of settings has been developed since then, for spaces of homogeneous type [1,9–11,19] , spaces of non-homogeneous type [6,17,23], or even in the discrete setting [2,7,15,20,21]. This note can be framed in that last group. Let us provide our setting.

Given \( k \geq 2 \) we denote by \( T^k \) the infinite rooted \( k \)-ary tree, namely, the infinite rooted tree such that each vertex has \( k \) children. We shall write just \( T \) in case there is no place to confusion. Abusing of notation, we will also use \( T \) to denote the set of the vertices of the tree. We can define the metric measure space \((T,d,\mu)\) where \( d \) is the usual tree metric, namely \( d(x,y) \) is the number of edges of the unique path between \( x \) and \( y \), and \( \mu \) is the counting measure defined on parts of the set of vertices. Given \( A \subset T \) we denote \( |A| = \mu(A) \) and
\[
\int_A f(x)dx = \sum_{x \in A} f(x).
\]

We will also denote
\[
Mf(x) = \sup_{r \in \mathbb{N} \cup \{0\}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy,
\]
\[
M^o f(x) = \sup_{r \in \mathbb{N} \cup \{0\}} A^o_r f(x)
\]
where \( A^o_r f(x) = \frac{1}{|S(x,r)|} \int_{S(x,r)} |f(y)|dy \) and
\[
B(x, r) = \{ y \in T : d(x,y) \leq r \},
\]
\[
S(x, r) = \{ y \in T : d(x,y) = r \}.
\]
In our case, in contrast with the standard Euclidean setting, spheres are not sets of measure 0. Actually $|S(x, r)| \approx k^r$. Furthermore, for $k \geq 2$, we have that $M^{\circ} \simeq M$ with constant independent of $k$ (see [16, Proposition 2.1]). At this point it is important to note that the measure on $T$ is far from being doubling, even the upper doubling condition of [8] fails. That fact disables the possibility of using the classical machinery available to deal the maximal function on $T$. However Naor and Tao [13] managed to overcome those difficulties using a combinatorial and expander argument to show that $M$ is of weak type $(1,1)$ with constant independent of $k$. Note that such a result on the homogenous tree was obtained independently by Cowling, Meda and Setti in [4] and that the strong type $(p, p)$ on free groups had already been obtained by Nevo and Stein in [14].

Pushing forward techniques in [13], the study of the theory of weights for the infinite rooted $k$-ary tree was initiated in [16]. The main results there were the following Fefferman-Stein inequality

$$w \left( \{ x \in T : Mf(x) > t \} \right) \lesssim \frac{1}{t} \int_T |f(x)|M(w^s)(x)^{\frac{1}{s}} dx \quad s > 1$$

and the quite surprising fact that there exists a weight $w$ such that for any positive integer $n$, if $M^n w = (M \circ \cdots \circ M)^n(w)$, then $M^n w(x) \lesssim w(x)$ is not sufficient for the weighted weak type $(1,1)$ nor even the weighted $(p, p)$ type for every $p > 1$ to hold. This shows that the choice for the maximal function in the right hand side of the Fefferman-Stein inequality is sharp, in the sense that $M(w^s)(x)^{\frac{1}{s}}$ cannot be replaced for any number of iterations of the maximal function.

We would like to note as well that in [16] it was showed that there exist non trivial weights $w$ such that $M(w^s)(x)^{\frac{1}{s}} \lesssim w(x)$ and therefore there are weights such that the weighted weak type $(1,1)$ estimate holds and hence the weighted strong type $(p, p)$ estimate for $p > 1$ holds as well. A natural question is whether it is possible to obtain more general classes of weights for $p > 1$. In this work we will pursue that direction providing novel results in the range $p > 1$. We will show that the class of suitable weights for this range is wider than the one obtained for the endpoint estimate in [16].

Studying this kind of weighted estimates may have as well connections with ergodic theory. Observe that when $k$ is odd $T$ is almost identifiable with the free group on $\frac{k+1}{2}$ generators. Naor and Tao settled the weak type $(1,1)$ estimate for the maximal function on the $k$-ary tree, but the non-amenability of the free group disabled the possibility of using standard arguments to transfer that result to that setting. In fact, later on Tao [22] disproved weak type $(1, 1)$ estimate for the free group. His argument relied upon showing that $f \in L^1$ is not sufficient for the convergence of averaging operators in combination with arguments in [14] that showed that such pointwise convergence follows from the weak type $(1,1)$ estimate. On the other hand an earlier positive result due to Bufetov [3] showed that strengthening conditions in Tao’s result, namely, assuming that $f \in L \log L$ then those averaging operators converge.

Coming back to weighted estimates, a first natural question would be studying the relationship between some analogue of the $A_p$ condition as stated in (1.1) above, that...
in this case would read as \( w \in A_p \) if
\[
\sup_{x \in T, r \geq 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} w \left( \frac{1}{|B(x, r)|} \int_{B(x, r)} w^{-\frac{1}{p-1}} \right)^{p-1} < \infty, \tag{1.4}
\]
and strong and weak type estimates. We shall prove that there exists weights \( w \) such that the weak type \( (1, 1) \) holds, and consequently \( M \) is bounded on \( L^p(w) \) for every \( p > 1 \), but also \( w \notin A_p \) for any \( p > 1 \) (See Ex 2 in Theorem 1.3). Consequently the set of the balls \( \{ B(x, r) \}_{x \in T, r \geq 0} \) is not a Muckenhoupt basis as the ones considered, for instance, in [5], since \( A_p \) does not characterize the strong type \( (p, p) \) of the maximal function, in contrast with the classical setting that we presented above in which (1.1) and (1.2) were equivalent conditions. Note that our results differ from other results in the literature. For instance in Badr and Martell [2], a doubling measure is considered, endowing the graph with a structure of space of homogeneous type and hence enabling them to use covering lemmas. In the case of Hebisch, Steger [7], the setting considered allows them to have a Calderón-Zygmund type decomposition, which is not available in our case either.

Another pathological situation in our setting is that there exist weights such that (1.3) holds but (1.2) does not (See Ex 3 in Theorem 1.3). Due to the aforementioned results it seems clear that \( A_p \) type conditions do not seem neither precise nor suitable in this setting. Observe that if we assume that the strong type \( (p, p) \) with respect to a weight \( w \) holds and \( E, F \subset T \) then we have that
\[
1 \otimes w \left( \{ (x, y) \in E \times F : d(x, y) = r \} \right) \sim k^r \int_E A_r^\circ (\chi_F w) = k^r \int_F A_r^\circ (\chi_E) w \\
\leq k^r w(F)^{1-\frac{1}{p}} \left( \int_T A_r^\circ (\chi_E)^p w \right)^{\frac{1}{p}} \lesssim k^r w(F)^{1-\frac{1}{p}} w(E)^{\frac{1}{p}}, \tag{1.5}
\]
where \( 1 \otimes w \) stands for the product measure, namely
\[
1 \otimes w(A \times B) = |A| w(B).
\]
It is not difficult to see that in general the necessary condition (1.5) will not be a sufficient condition, nor even for the weighted weak type \( (p, p) \) estimate. However assuming a slightly better decay on \( k \) we manage to obtain a sufficient condition which is one of the main results of this paper.

**Theorem 1.1** Let \( k \geq 2 \) be an integer. Let \( p > 1 \) and let \( w \) be a weight. Assume that there exist \( 0 < \beta < 1 \) and \( \beta \leq \alpha < p \) such that
\[
1 \otimes w \left( \{ (x, y) \in E \times F : d(x, y) = r \} \right) \lesssim k^{r\beta} w(E)^{\frac{\alpha}{p}} w(F)^{1-\frac{\alpha}{p}}. \tag{1.6}
\]
Then
\[
\|Mf\|_{L^p_{\sigma, \infty}(w)} \lesssim \|f\|_{L^p_{\sigma, p}(w)}. \tag{1.7}
\]
Furthermore, if $\beta < \alpha$ then

$$\|Mf\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \quad (1.8)$$

and also

$$\|Mf\|_{L^{p'}(\sigma_p)} \lesssim \|f\|_{L^{p'}(\sigma_p)} \quad (1.9)$$

where $\sigma_p = w^{-\frac{1}{p-1}}$.

At this point some remarks are in order. First observe that if $0 < \frac{\beta}{\sigma} < 1$, then (1.8) follows from (1.7) by interpolation, however that is not the case for (1.9). We will settle both estimates relying upon the fact that under that assumption on $\beta$ and $\alpha$, actually we can show something slightly stronger, namely that

$$\sum_{r=0}^{\infty} \|A_r^0 f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}.$$  

Our condition is also sharp in the sense that if $\beta = \alpha$ we can provide a weight for which the weak type $(p, p)$ holds but the strong type $(p, p)$ does not and consequently the weak type $(q, q)$ fails as well for every $q < p$ (see Ex 3 in Theorem 1.3).

Observe also that in the classical setting $w \in A_p$ implies that $\sigma \in A_{p'}$ and hence no additional argument is required to show that (1.9) holds. In our case, even though (1.6) stated for $w$ does not imply its dual analogue for $\sigma_p$, we are still able to use it to prove (1.9). This, however may not be the case for the weak type estimates (see Ex 3 in Theorem 1.3).

Checking the sufficient condition in Theorem 1.1 directly could be quite involved. The following Corollary reduces the question to a condition that is easier to check than (1.6). Before presenting that result it is convenient to introduce a bit of notation.

In what follows we shall call $T_0$ the set given by the root of the tree, $T_j$ the set of the children of the vertices in $T_{j-1}$ and so on. Observe that the sets considered in the condition in (1.6) may have non-empty intersection with several different levels. With the condition in the following Corollary it suffices to study the behavior of the weight in each level instead.

**Corollary 1.2** Let $1 < p < \infty$, and let $w$ be a weight such that there exists a real number $\delta < 1$ such that if $x \in T_j$ and $|i - j| \leq r$ we have that

$$w(T_i \cap S(x, r)) \lesssim k^{\frac{r + i - j}{2}(p-\delta)} k^r \delta w(x). \quad (1.10)$$

Then

$$\|Mf\|_{L^{p,\infty}(w)} \lesssim \|f\|_{L^p(w)}.$$
Furthermore, if \( p < q \) then

\[
\| Mf \|_{L^q(w)} \lesssim \| f \|_{L^q(w)}
\]

and also

\[
\| Mf \|_{L^q(\sigma_q)} \lesssim \| f \|_{L^q(\sigma_q)}
\]

where \( \sigma_q = w^{-\frac{1}{q-1}} \).

It is worth observing that if (1.10) holds for a certain \( p > 1 \), then it holds as well for every \( q > p \). This fact will be crucial for us to settle the strong type estimates in Corollary 1.2.

We shall observe that, as in the case of the condition in Theorem 1.1, Ex 3 from Theorem 1.3 also shows that (1.10) in Corollary 1.2 is sharp.

In the following Theorem we gather some relevant examples and features of the maximal function in the setting of the \( k \)-ary tree.

**Theorem 1.3** In the context of the \( k \)-ary tree with \( k \in \mathbb{N} \setminus \{1\} \) the following statements are true.

Ex 1 Let \( p > 1 \). If \( w(x) = \sum_{j=0}^{\infty} k^{j(p-1)} \chi_{T_j}(x) \) we have that

\[
\| Mf \|_{L^{p,\infty}(w)} \lesssim \| f \|_{L^{p}(w)}
\]

and also that for every \( q > p \)

\[
\| Mf \|_{L^q(w)} \lesssim \| f \|_{L^q(w)} \quad \text{and} \quad \| Mf \|_{L^q(\sigma_q)} \lesssim \| f \|_{L^q(\sigma_q)}
\]

but \( M(w^s)^{\frac{1}{q}} \nless w \) for any \( s \geq 1 \).

Ex 2 For \( \delta \in \left( \frac{1}{2}, 1 \right) \) let \( w(x) = \sum_{j=0}^{\infty} \frac{1}{k^j} \chi_{T_j}(x) \). Then, for every \( q > 1 \),

\[
\| Mf \|_{L^q(w)} \lesssim \| f \|_{L^q(w)}
\]

but also for every \( p > 1 \)

\[
\sup_{x \in T, r > 0} \frac{1}{|S(x, r)|} \int_{S(x, r)} \frac{1}{w} \left( \frac{1}{|S(x, r)|} \int_{S(x, r)} w^{-\frac{1}{p-1}} \right)^{p-1} = \infty.
\]

Ex 3 Let \( p > 1 \) and

\[
w(x) = \sum_{j=0}^{\infty} k^{j(p-1)} \chi_{T_j}(x).
\]

Then \( \| M \|_{L^{p,\infty}(w)} \lesssim \| f \|_{L^{p}(w)} \) holds but \( \| Mf \|_{L^{p}(w)} \lesssim \| f \|_{L^{p}(w)} \) does not.
We provide now some remarks regarding the statements contained in the theorem above. We first note that in Ex 1 we provide weights for which the endpoint estimate does not hold but weak and strong type estimates do. Ex 2 shows that we can find weights for which the maximal function is bounded for every $q > 1$ but $w \notin A_p$ for every $p > 1$. Observe that the result that we provide seems stronger than the one announced a few lines above, since we are going to settle it with balls $B(x, r)$ replaced by spheres $S(x, r)$ which yields a more restrictive condition since for $h \geq 0$

$$\frac{1}{|S(x, r)|} \int_{S(x, r)} h \lesssim \frac{1}{|B(x, r)|} \int_{B(x, r)} h$$

as it was shown in [16, Proposition 2.1]. In Ex 3 another fundamental difference between the classical theory and the theory on $k$-ary trees is portrayed. There exist weights such that the weighted weak type inequality holds but the strong type does not.

The remainder of the paper is organized as follows. Section 2 is devoted to provide a proof for Theorem 1.3 and some further remarks. In Section 3 we give detailed proofs of Theorem 1.1 and Corollary 1.2. Finally, in Section 4 we give two weight results and a remark concerning the Sawyer testing condition.

## 2 Proof of Theorem 1.3 and some further remarks

First we are going to give our proof of Theorem 1.3.

We begin settling Ex 1. To settle all the estimates it suffices to show that (1.10) holds and to apply (1.2).

Let $x \in T_j$ and assume that $|i - j| \leq r$ and that $m \in \{0, \ldots, r\}$ is the only integer such that $i = j + r - 2m$. Then

$$w(T_i \cap S(x, r)) = |T_i \cap S(x, r)| k^{i(p-1)} \leq k^{i(p-1)} k^{r-m} = k^{(r-m)(2p-1)} k^{(1-p)r} w(x).$$

Hence (1.10) holds with $\delta = 1 - p$, since $\frac{i-j+r}{2} = r - m$.

To show that $M(w^s)^\frac{1}{s} \not\lesssim w$ observe that for every $r \geq 0$

$$\left( \frac{1}{|S(0, r)|} \int_{S(0, r)} w^s \right)^\frac{1}{s} \geq \left( \frac{k^{r(p-1)s+rs}}{kr} \right)^\frac{1}{s} = k^{r(p-1)+r\frac{1}{s}}$$

This shows that $M(w^s)^\frac{1}{s} (0) = \infty$ for every $s \geq 1$ but $w(0) < \infty$ and we are done.

We continue settling Ex 2. Note that the boundedness follows from Corollary 2.1 (which will be settled below) with $s = \frac{1}{2}$. Now we shall show that $w$ does not satisfy the classical $A_p$ condition for any $p > 1$. Let us fix $p > 1$. Let $x \in T_j$ and let us consider $S(x, j)$. Then

$$\frac{1}{|S(x, j)|} \int_{S(x, j)} w \gtrsim \frac{1}{k^j}.$$
On the other hand
\[
\left( \frac{1}{|S(x, j)|} \int_{S(x, j)} w^{-\frac{1}{p-1}} \right)^{p-1} \gtrsim \left( \frac{1}{k^j} \int_{S(x, j) \cap T_{2j}} w^{-\frac{1}{p-1}} \right)^{p-1} \\
\gtrsim \left( \frac{k^j \delta}{k^j} \right)^{p-1} = k^{2j \delta}.
\]
Hence
\[
\frac{1}{|S(x, j)|} \int_{S(x, j)} w^{-\frac{1}{p-1}} \gtrsim k^{2j \delta} \frac{1}{k^j} = k^{j(2 \delta - 1)}
\]
and letting \( j \to \infty \) the desired conclusion follows.

Finally we settle Ex 3. Note that the weak type \((p, p)\) estimate with respect to \(w\) follows from Ex 1. Now let \( f = \chi_{T_0} \). Then we clearly have that \( \| f \|_{L^p(w)} = 1 \). On the other hand
\[
\| Mf \|_{L^p(w)} = \int_{T} (Mf)^p w \geq \sum_{i>0} \int_{T_i} (Mf)^p w \\
\gtrsim \sum_{i>0} \int_{T_i} \frac{1}{k^{pi}} k^i (p-1) = \sum_{i>j} \frac{1}{k^{pi}} k^i \\
= \sum_{i>0} 1 = +\infty.
\]
Thus, assuming \( \| Mf \|_{L^p(w)} \lesssim \| f \|_{L^p(w)} \), we clearly reach a contradiction.

This ends the proof of Theorem 1.3.

Now we present the following corollary of Theorem 1.1.

**Corollary 2.1** If there exists some \( s > 1 \) such that
\[
M_s w \lesssim w,
\]
where \( M_s (w) = M(w^s)^{\frac{1}{s}} \) then we have that for every \( p > 1 \)
\[
\| Mf \|_{L^p(w)} \lesssim \| f \|_{L^p(w)}
\]
and also that
\[
\| Mf \|_{L^p(\sigma_p)} \lesssim \| f \|_{L^p(\sigma_p)}.
\]
This corollary shows that the sufficient condition obtained in [16], namely \( M_s w \lesssim w \) also allows to provide strong type estimates. Observe that in that even though the strong type \((p, p)\) estimate follows by interpolation, that is not the case for the strong \((p', p')\) estimate in terms of the dual weight, and then Theorem 1.1 yields new estimates.
Proof of Corollary 2.1 Recall that in [16, Lemma 2.2] it was shown that for $1 < s < \infty$

$$1 \otimes w (\{(x, y) \in E \times F : d(x, y) = r\}) \lesssim k^{r^{\frac{1}{s'}}} M_s w(E)^{\frac{1}{s'+1}} w(F)^{\frac{1}{s'+1}}.$$ 

Observe that then

$$1 \otimes w (\{(x, y) \in E \times F : d(x, y) = r\}) \lesssim k^{r^{\beta}} w(E)^{\alpha} w(F)^{1 - \frac{\alpha}{p}}$$

and consequently the result follows from Theorem 1.1 with

$$\beta = \frac{s'}{s' + 1}, \quad \alpha = \frac{s'}{s' + 1} p,$$

since $\alpha > \beta$. \qed

In light of the results we have obtained we wonder whether, as in the classical setting, given a weight $w$ and $p > 1$

$$\|Mf\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \iff \|Mf\|_{L^p(\sigma_p)} \lesssim \|f\|_{L^p(\sigma_p)}.$$

3 Proof of Theorem 1.1 and Corollary 1.2

3.1 Proof of Theorem 1.1

We will follow the scheme devised in Naor and Tao [13] and further exploited in [16]. We begin with the following Lemma.

Lemma 3.1 Let $k \geq 2$ be an integer. Let $p > 1$ and let $w$ be a weight. Assume that there exist $0 < \beta < 1$ and $\beta \leq \alpha < p$ such that

$$1 \otimes w (\{(x, y) \in E \times F : d(x, y) = r\}) \lesssim k^{r^{\beta}} w(E)^{\frac{\alpha}{p}} w(F)^{1 - \frac{\alpha}{p}}.$$ 

Let $r > 0$ and $\lambda > 0$. Then

$$w \left( \left\{ A_r^\circ (|f|) \geq \lambda \right\} \right) \lesssim \sum_{n \in \mathbb{N} \cup \{0\}} \sum_{1 \leq 2^n \leq 2^{kr}} \left( \frac{2^n}{k^r} \right)^{\frac{1 - \beta}{\alpha}} \frac{1}{2^{\frac{\alpha}{p}}} 2^{\beta \frac{\alpha}{p}} w \left( \left\{ |f| \geq 2^n \lambda \right\} \right).$$

Proof We can assume without loss of generality $f$ to be non-negative, and by homogeneity that $\lambda = 1$. We bound

$$f \leq \frac{1}{2} + \sum_{n \in \mathbb{N} \cup \{0\}} \sum_{1 \leq 2^n \leq 2^{kr}} 2^n \chi_{E_n} + f \chi_{\{f \geq \frac{1}{2} k^r\}}, \quad (3.1)$$
where $E_n$ is the sublevel set

$$E_n = \left\{ 2^{n-1} \leq f < 2^n \right\}. \tag{3.2}$$

Hence

$$A^o_r(f) \leq \frac{1}{2} + \sum_{n \in \mathbb{N} \cup \{0\}} 2^n A^o_r(x_{E_n}) + A^o_r(f x_{\{ f \geq \frac{1}{2} k r \}}). \tag{3.3}$$

First we observe that

$$w \left( A^o_r \left( f x_{\{ f \geq \frac{1}{2} k r \}} \right) \neq 0 \right) \leq w \left( \bigcup_{y \in \{ f \geq \frac{1}{2} k r \}} S(y, r) \right) \leq \sum_{y \in \{ f \geq \frac{1}{2} k r \}} w(S(y, r)). \tag{3.4}$$

Observe that choosing

$$E = \{ y \} \quad F = S(y, s)$$

in the hypothesis

$$w(S(y, s)) \lesssim k^{\frac{p}{\alpha}} \rho_s w(y)$$

and consequently

$$w \left( A^o_r \left( f x_{\{ f \geq \frac{1}{2} k r \}} \right) \neq 0 \right) \leq \sum_{y \in \{ f \geq \frac{1}{2} k r \}} w(S(y, r)) \lesssim k^{\frac{p}{\alpha}} \rho_s \sum_{y \in \{ f \geq \frac{1}{2} k r \}} w(y) = k^{\frac{p}{\alpha}} \rho_s w \left( \{ f \geq \frac{1}{2} k r \} \right).$$
Thus we have that combining the estimates above

\[
\begin{align*}
    w(A_r^o f \geq 1) & \leq w \left( \sum_{n \in \mathbb{N} \cup \{0\}, 1 \leq 2^n \leq k^r} 2^n A_r^o \left( \chi_{E_n} \right) \geq \frac{1}{2} \right) + w \left( A_r^o \left( f \chi_{\{ f \geq 1/2 k^r \}} \right) \neq 0 \right) \\
    & \leq w \left( \sum_{n \in \mathbb{N} \cup \{0\}, 1 \leq 2^n \leq k^r} 2^n A_r^o \left( \chi_{E_n} \right) \geq \frac{1}{2} \right) + k^r \beta^p r w \left( \{ f \geq 1/2 \} \right).
\end{align*}
\]

Let \( \gamma > 0 \) to be chosen. Note that if

\[
\sum_{n \in \mathbb{N} \cup \{0\}, 1 \leq 2^n \leq k^r} 2^n A_r^o \left( \chi_{E_n} \right) \geq \frac{1}{2}
\]

then we necessarily have for some \( n \in \mathbb{N} \), such that \( 1 \leq 2^n \leq k^r \),

\[
A_r^o \left( \chi_{E_n} \right) \geq \frac{1}{2n+4} \left( \frac{2^n}{k^r} \right) \gamma \frac{8(2^\gamma - 1)}{2^\gamma}.
\]

Indeed, otherwise we have that

\[
\begin{align*}
    \frac{1}{2} & \leq 8(2^\gamma - 1) \sum_{n \in \mathbb{N} \cup \{0\}, 1 \leq 2^n \leq k^r} 2^n A_r^o \left( \chi_{E_n} \right) \leq \frac{8(2^\gamma - 1)}{2^\gamma 16 k^r \gamma} \sum_{n \in \mathbb{N} \cup \{0\}, 1 \leq 2^n \leq k^r} 2^{\gamma n} \\
    & \leq \frac{8(2^\gamma - 1)}{2^\gamma 16 k^r \gamma} \left( \frac{2^\gamma k^r \gamma - 1}{2^\gamma - 1} \right) < \frac{1}{2}
\end{align*}
\]

which is a contradiction. Thus

\[
\begin{align*}
    w(A_r^o f \geq 1) & \leq \sum_{n \in \mathbb{N} \cup \{0\}, 1 \leq 2^n \leq k^r} w(F_n) + k^r \beta^p r w \left( \{ f \geq 1/2 \} \right)
\end{align*}
\]

where

\[
F_n = \left\{ A_r^o \left( \chi_{E_n} \right) \geq \frac{1}{2n+4} \left( \frac{2^n}{k^r} \right) \gamma \frac{8(2^\gamma - 1)}{2^\gamma} \right\}.
\]
Note that $F_n$ is finite and observe that since $A_r^\circ$ is a selfadjoint operator,

\[
\frac{1}{Kr} \mathbb{1} \otimes w \left( \{(x, y) \in E_n \times F_n : d(x, y) = r\} \right) = \frac{1}{Kr} \sum_{x \in E_n} \sum_{y \in F_n \atop d(x, y) = r} w(y) \simeq \int_T \chi_{E_n} A_r^\circ (w \chi_{F_n})(y) = \int_{F_n} w A_r^\circ (\chi_{E_n})(y)
\]

\[
\gtrsim w(F_n) \frac{1}{2n+4} \left( \frac{2n}{Kr} \right)^\gamma.
\]

Now, using the hypothesis

\[
\frac{1}{Kr} \mathbb{1} \otimes w \left( \{(x, y) \in E_n \times F_n : d(x, y) = r\} \right) \lesssim k^{r(\beta - 1)} w(E_n) \frac{\alpha}{\beta} w(F_n)^{1 - \frac{\alpha}{\beta}}.
\]

Hence

\[
w(F_n) \frac{1}{2n+4} \left( \frac{2n}{Kr} \right)^\gamma \lesssim \frac{1}{8 (2^\gamma - 1)} k^{r(1 - \beta) - \gamma} w(E_n) \frac{\alpha}{\beta} w(F_n)^{1 - \frac{\alpha}{\beta}}.
\]

\[
\iff w(F_n) \frac{\beta}{2^\gamma - 1} \lesssim k^{-r(1 - \beta - \gamma) - \gamma n} w(E_n) \frac{\alpha}{\beta}.
\]

\[
\iff w(F_n) \lesssim \left( \frac{1}{2^\gamma - 1} \right) \frac{\beta}{\alpha} k^{-r(1 - \beta - \gamma) - \gamma n} w(E_n).
\]

Choosing $\gamma = 1 - t\beta > 0$ with $t > 1$ we have that

\[
1 - \beta - \gamma = 1 - \beta - (1 - t\beta) = 1 - \beta - 1 + t\beta = \beta(t - 1)
\]

and

\[
1 - \gamma = 1 - (1 - t\beta) = \beta t = \beta(t - 1) + \beta.
\]

Hence

\[
w(F_n) \lesssim \left( \frac{1}{2^{1-t\beta} - 1} \right) \frac{\beta}{\alpha} \left( \frac{2n}{Kr} \right)^{(t-1)\frac{\beta}{\alpha}} 2^{\beta \frac{\beta}{\alpha} n} w(E_n)
\]

and we may choose $t = \frac{1 + \frac{1}{\beta}}{2}$. Then

\[
w(F_n) \lesssim \left( \frac{2n}{Kr} \right)^{\frac{1-\beta}{2\beta} \frac{\beta}{\alpha}} 2^{\beta \frac{\beta}{\alpha} n} w(E_n)
\]

\[
= \left( \frac{2n}{Kr} \right)^{\frac{1-\beta}{2} \frac{\beta}{\alpha}} 2^{\beta \frac{\beta}{\alpha} n} w(E_n)
\]
and this yields the desired conclusion.  

Combining the ingredients above we are in the position to settle Theorem 1.1.

**Proof of Theorem 1.1** We begin settling (1.7). Since as we noted in the introduction $M^0 f \simeq Mf$ it suffices to settle the result for $M^0 f$. By homogeneity we shall assume that $\lambda = 1$. Since $M^0 f = \sup_{r \geq 0} A^0_r f$, Lemma 3.1 implies that

$$w \left( M^0 f \geq 1 \right) \leq \sum_{r=0}^{\infty} w \left( A^0_r f \geq 1 \right)$$

$$\lesssim \sum_{r=0}^{\infty} \sum_{n \in \mathbb{N} \cup \{0\}} \left( \frac{2^n}{k^r} \right) \sum_{k \leq 2^{n-r}} 2^{\beta \frac{p}{\alpha}} w \left( \left\{ |f| \geq 2^{n-1} \right\} \right)$$

$$\lesssim \sum_{x \in T} \sum_{n=0}^{\infty} \left( \sum_{r \in \mathbb{N} \cup \{0\}} \frac{1}{k^r \left( 1 - \frac{\beta p}{2} + \frac{\beta p}{2} \right)} \right) \chi_{\{ |f(x)| \geq 2^{n-1} \}}(x) w(x)$$

$$\lesssim \sum_{x \in T} \sum_{n=0}^{\infty} 2^{\beta \frac{p}{\alpha} n} \chi_{\{ |f(x)| \geq 2^{n-1} \}}(x) w(x) \lesssim \sum_{x \in T} |f(x)|^{\frac{\beta p}{\alpha}} w(x).$$

Now we turn our attention to the case $\alpha > \beta$. We claim that

$$\sum_{r=0}^{\infty} \| A^0_r f \|_{L^p(w)} \lesssim \| f \|_{L^p(w)}. \quad (3.5)$$

Note that from this estimate we can derive both (1.8) and (1.9). In the case of (1.8) we have that

$$\| Mf \|_{L^p(w)} \leq \sum_{r=0}^{\infty} \| A^0_r f \|_{L^p(w)} \lesssim \| f \|_{L^p(w)}.$$

For (1.9), we may assume that $f \geq 0$. Note that since

$$\| Mf \|_{L^p(w)} = \sup_{\| g \|_{L^p(w)} = 1} \left| \int_T Mf g \right|$$
we can argue as follows

\[
\left| \int_T Mfg \right| \leq \int_T Mf |g| \leq \sum_{r=0}^{\infty} \int_T A_r^g(f) |g| = \sum_{r=0}^{\infty} \int_T f A_r^g(g)
\]

\[
\leq \|f\|_{L^p'(\sigma_p)} \sum_{r=0}^{\infty} \|A_r(g)\|_{L^p(w)}
\]

\[
\lesssim \|f\|_{L^p'(\sigma_p)} \|g\|_{L^p(w)}
\]

and then we are done. Hence we are left with settling (3.5). Note that

\[
\|A_r f\|_{L^p(w)}^p = p \int_0^\infty \lambda^{p-1} w(A_r f \geq \lambda) \, d\lambda
\]

\[
\leq \sum_{n \in \mathbb{N} \cup \{0\}} \left( \frac{2^n}{k^r} \right)^{\frac{1-\beta_p}{2}} 2^{\beta_p n} p \int_0^\infty \lambda^{p-1} w \left( \{|f| \geq 2^{n-1} \lambda\} \right) \, d\lambda
\]

\[
= \sum_{n \in \mathbb{N} \cup \{0\}} \left( \frac{2^n}{k^r} \right)^{\frac{1-\beta_p}{2}} 2^{\beta_p n} p \int_0^\infty \left( \frac{s}{2^{n-1}} \right)^{p-1} w \left( \{|f| \geq s\} \right) \frac{1}{2^{n-1}} ds
\]

\[
= \sum_{n \in \mathbb{N} \cup \{0\}} \left( \frac{2^n}{k^r} \right)^{\frac{1-\beta_p}{2}} 2^{\beta_p n} p \left( \frac{1}{2^{n-1}} \right)^{p} \int_0^\infty s^{p-1} w \left( \{|f| \geq s\} \right) ds
\]

\[
= \sum_{n \in \mathbb{N} \cup \{0\}} \left( \frac{2^n}{k^r} \right)^{\frac{1-\beta_p}{2}} 2^{\beta_p n} \left( \frac{1}{2^{n-1}} \right)^{p} \|f\|_{L^p(w)}^p.
\]

Hence

\[
\|A_r f\|_{L^p(w)} \lesssim \left( \sum_{n \in \mathbb{N} \cup \{0\}} \left( \frac{2^n}{k^r} \right)^{\frac{1-\beta_p}{2}} 2^{\beta_p n} \left( \frac{1}{2^{n-1}} \right)^{p} \right)^{\frac{1}{p}} \|f\|_{L^p(w)}.
\]
Using the estimate above we can control the left hand side of (3.5) by

$$\sum_{r=0}^{\infty} \left( \sum_{n \in \mathbb{N} \cup \{0\}} \frac{2^n}{k^r} \right)^{1/2 - \frac{\beta}{2}} \frac{2^\beta n^p}{2} \left( \frac{1}{2^{n-1}} \right)^p \leq \sum_{r=0}^{\infty} \frac{1}{k^r} \sum_{n \in \mathbb{N} \cup \{0\}} \frac{2^n}{k^r} \frac{1}{2^{n-1}} \leq 2^{(\beta - 1)n} \sum_{n=0}^{\infty} 2^{(\beta - 1)n} \lesssim \sum_{n=0}^{\infty} 2^{(\beta - 1)n} < \infty.$$  

This ends the proof of the Theorem. \(\square\)

### 3.2 Proof of Corollary 1.2

We begin settling the following Lemma.

**Lemma 3.2** Under the conditions of Corollary 1.2 we have that for every pair \(E, F\) of finite subsets of \(T\) and every non negative integer \(r\),

$$1 \otimes w \left( \{(x, y) \in E \times F : d(x, y) = r\} \right) \leq c_{p, \delta, k} \frac{p}{p-1} r w(F)^{1-\frac{1}{p}} w(E)^{\frac{1}{p}}.$$  

**Proof** We recall that we can split the tree \(T\) as \(T = \bigcup_{j=0}^{\infty} T_j\), where \(T_j\) is the generation of the tree at depth \(j\). We define \(E_j = E \cap T_j\) and \(F_j = F \cap T_j\). An element in \(E_j\) and an element in \(F_i\) can be at distance exactly \(r\), if and only if \(i = j + r - 2m\) for some \(m \in \{0, \ldots, r\}\). Hence we can write

$$1 \otimes w \left( \{(x, y) \in E \times F : d(x, y) = r\} \right) = \sum_{m=0}^{r} \sum_{i,j \in \mathbb{N} \cup \{0\}} \mathbb{1} \otimes w \left( \{(x, y) \in E_j \times F_i : d(x, y) = r\} \right). \quad (3.6)$$

Now we fix \(m \in \{0, \ldots, r\}\) and \(i, j \in \mathbb{N} \cup \{0\}\) such that \(i = j + r - 2m\). Note that if \(x \in T_j\) and \(y \in T_i\) are at distance \(r\) in \(T\), then the \(m^{th}\) parent of \(x\) coincides with the \((r - m)^{th}\) parent of \(y\). This leads to the fact that for each \(y \in T_i\) there exist at most \(k^m\) elements of \(x \in T_j\) with \(d(x, y) = r\). From this it readily follows that

$$1 \otimes w \left( \{(x, y) \in E_j \times F_i : d(x, y) = r\} \right) \leq k^m w(F_i).$$
On the other hand, by assumption, since \(\frac{r+i-j}{2} = r - m\),

\[
\mathbb{1} \otimes w\left(\{(x, y) \in E_j \times F_i : d(x, y) = r\}\right) \\
= \sum_{x \in E_j} w(F_i \cap S(x, r)) \lesssim \sum_{x \in E_j} k^{(r-m)(p-\delta)} kr^\delta w(x) \\
= k^{(r-m)(p-\delta)} kr^\delta w(E_j).
\]

Thus combining the ideas above

\[
\mathbb{1} \otimes w\left(\{(x, y) \in E_j \times F_i : d(x, y) = r\}\right) \\
\lesssim \min \left\{ k^{(r-m)(p-\delta)} kr^\delta w(E_j), k^m w(F_i) \right\}. \quad (3.7)
\]

Taking into account (3.6) and (3.7), to end the proof it suffices to show that

\[
\sum_{m=0}^r \sum_{i, j \in \mathbb{N} \cup \{0\}} \min \left\{ k^{(r-m)(p-\delta)} kr^\delta w(E_j), k^m w(F_i) \right\} \\
\leq c_{p, \delta} k^{\frac{p}{p-\delta+1}} r w(F)^{1-\frac{1}{p-\delta+1}} w(E)^{\frac{1}{p-\delta+1}}. \quad (3.8)
\]

Let us define \(c_j = \frac{w(E_j)}{k^{(p-\delta)j}}\) and \(d_j = \frac{w(F_j)}{k^j}\) for \(j \geq 0\) and \(c_j = d_j = 0\) for \(j < 0\) then,

\[
\sum_{j=0}^\infty k^{(p-\delta)j} c_j = w(E) \quad \text{and} \quad \sum_{j=0}^\infty k^j d_j = w(F), \quad (3.9)
\]

and we have that

\[
\sum_{m=0}^r \sum_{i, j \in \mathbb{N} \cup \{0\}} \min \left\{ k^{(r-m)(p-\delta)} kr^\delta w(E_j), k^m w(F_i) \right\} \\
= \sum_{m=0}^r \sum_{i, j \in \mathbb{N} \cup \{0\}} \min \left\{ k^{(p-\delta)(r-m+j)} k^\delta r c_j, k^m k^i d_i \right\} \\
= \sum_{m=0}^r \sum_{i, j \in \mathbb{N} \cup \{0\}} \min \left\{ k^{\delta r} k^{\frac{(i+j+r)(p-\delta)}{2}} c_j, k^{\frac{i+j+r}{2}} d_i \right\}.
\]
Taking the identity above into account, settling (3.8) reduces to show that

\[
\sum_{m=0}^r \sum_{i,j \in \mathbb{N} \cup \{0\}} \min \left\{ k^{\delta r} k^{(i+j+r)(p-\delta)/2} c_j, k^{i+j+r} d_i \right\}
\]

\[
\leq c_{p,\delta} k^{\frac{p-q}{p-\delta+1}} w(F)^{1-\frac{1}{p-\delta+1}} w(E)^{\frac{1}{p-\delta+1}}.
\]

To prove this inequality, we let \( \rho \) be a real parameter to be chosen later, and argue as follows

\[
\sum_{m=0}^r \sum_{i,j \in \mathbb{N} \cup \{0\}} \min \left\{ k^{\delta r} k^{(i+j+r)(p-\delta)/2} c_j, k^{i+j+r} d_i \right\}
\]

\[
\leq k^{\frac{p+\delta}{2}} r \sum_{i,j \in \mathbb{N} \cup \{0\}} k^{(i+j)(p-\delta)/2} c_j + k^{\frac{p}{2}} \sum_{i,j \in \mathbb{N} \cup \{0\}} k^{i+j+r} d_i
\]

\[
\leq k^{\frac{p+\delta}{2}} r \sum_{j=0}^\infty k^{j(p-\delta)/2} c_j + k^{\frac{p}{2}} \sum_{i=0}^\infty k^{i-r} d_i
\]

\[
= k^{\frac{p+\delta}{2}} r k^{\frac{p(p-\delta)}{2}} \sum_{j=0}^\infty k^{j(p-\delta)} c_j + k^{\frac{p}{2}} k^{\frac{p}{2} - \frac{\rho}{2}} \sum_{i=0}^\infty k^{i} d_i
\]

Choosing \( \rho = \frac{2 \log k (w(F)/w(E))}{p-\delta+1} - \frac{(p+\delta-1)\rho}{p-\delta+1} \) it is not hard to check that

\[
k^{\frac{p+\delta}{2}} r k^{\frac{p(p-\delta)}{2}} w(E) + k^{\frac{p}{2}} k^{\frac{p}{2} - \frac{\rho}{2}} w(F) \leq c_{p,\delta} k^{\frac{p}{p-\delta+1}} w(F)^{1-\frac{1}{p-\delta+1}} w(E)^{\frac{1}{p-\delta+1}}
\]

for some constant \( c_{p,\delta} > 0 \) and we are done. \( \square \)

At this point we are in the position to settle Corollary 1.2.

**Proof of Corollary 1.2** First observe that for the weak type estimate, by Lemma 3.2, (1.6) is satisfied choosing \( \beta = \alpha = \frac{p}{p-\delta+1} \). For the strong type estimates note that also due to Lemma 3.2, then (1.6) stated for \( q \) in place of \( p \) is satisfied choosing \( \beta = \frac{p}{p-1+\delta} \) and \( \alpha = \frac{p}{p-1+\delta} \), which in turn yields the desired conclusion. \( \square \)

### 4 Two weight estimates

In this last section we gather our results regarding two weight estimates. The first of them is that with minor adjustments it is possible to prove the following two weighted version of Theorem 1.1.
Theorem 4.1 Let $k \geq 2$ be an integer. Let $p > 1$ and let $u$, $v$ be weights. Assume that there exist $0 < \beta < 1$ and $\beta \leq \alpha < p$ such that

$$1 \otimes u \left( \left\{ (x, y) \in E \times F : d(x, y) = r \right\} \right) \lesssim k^r \beta u(E)^\frac{\alpha}{p} u(F)^{1-\frac{\alpha}{p}}.$$

Then

$$\|Mf\|_{L^{\beta\alpha p, \infty}(u)} \lesssim \|f\|_{L^{\beta\alpha p}(v)}.$$

Furthermore, if $\beta < \alpha$ then

$$\|Mf\|_{L^p(u)} \lesssim \|f\|_{L^p(v)}$$

and also

$$\|Mf\|_{L^{p'}(\sigma v, p)} \lesssim \|f\|_{L^{p'}(\sigma u, p)}$$

where $\sigma_{\rho, p} = \rho^{-\frac{1}{p-1}}$.

Sketch of the proof With minor modifications, the argument supplied to settle Lemma 3.1 allows to show that for every $r > 0$ and $\lambda > 0$, we have that

$$u \left( \left\{ A^2_r(|f|) \geq \lambda \right\} \right) \lesssim \sum_{n \in \mathbb{N} \cup \{0\}} \left( \frac{2^n}{k^r} \right)^{\frac{1-p}{2} p \alpha} 2^\beta 2^n v \left( \left\{ |f| \geq 2^{n-1} \lambda \right\} \right).$$

To end the proof it suffices to mimic the argument in the proof of Theorem 1.1. \( \square \)

From this Theorem it is also possible to derive the following Corollary.

Corollary 4.2 Let $u$, $v$ be weights such that there exists $0 < \delta < 1$ such that if $x \in T_j$, $F \subset T_i$ and $|i - j| \leq r$,

$$u(F \cap S(x, r)) \lesssim k^{\frac{i-j+r}{2}(p-\delta)} k^{r \delta} v(x).$$

Then

$$\|Mf\|_{L^p(u)} \lesssim \|f\|_{L^p(v)}$$

Furthermore, if $p < q$ then

$$\|Mf\|_{L^q(u)} \lesssim \|f\|_{L^q(v)}$$

and also

$$\|Mf\|_{L^{q'}(\sigma v, q)} \lesssim \|f\|_{L^{q'}(\sigma u, q)}$$
where \( \sigma_{p,q} = \rho^{-\frac{1}{q-1}} \).

**Sketch of the proof** Arguing as in the proof of Corollary 1.2, we have that for every pair \( E, F \) of finite subsets of \( T \) and every non negative integer \( r \),

\[
1 \otimes u (\{(x, y) \in E \times F : d(x, y) = r\}) \leq c_{p,q} k^{\frac{p}{p-1} - \frac{1}{p-1}} v(F)^{\frac{1}{p-1} - \frac{1}{p-1}} u(E)^{\frac{1}{p-1}}.
\]

From this point the desired conclusion readily follows from Theorem 4.1.

At this point a natural question would be to consider whether testing conditions like the ones introduced by Sawyer in [18] are sufficient for strong type estimates to hold. Our next result shows that even in the one weight setting that is not the case.

**Theorem 4.3** Let \( p > 1 \) and

\[
w(x) = \sum_{j=0}^{\infty} k^{(p-1)j} \chi_{T_j}(x).
\]

Then we have that

\[
\| Mf \|_{L^p(w)} \lesssim \| f \|_{L^p(w)}
\]

does not hold but for every ball \( B \)

\[
\int_B M(\chi_B \sigma)^p w \lesssim \int_B \sigma
\]

where \( \sigma = w^{-\frac{1}{p-1}} \).

**Proof** By Ex 3 in Theorem 1.3 we know that the weighted strong type estimate does not hold for \( w \). Now we observe that

\[
\sigma(x) = \sum_{j=0}^{\infty} \frac{1}{k_j} \chi_{T_j}(x).
\]

For this weight it was shown in [16, Theorem 3.1] that

\[
M \sigma(x) \lesssim \sigma(x).
\]

Taking this into account

\[
\int_B M(\chi_B \sigma)^p w \lesssim \int_B \sigma^p w = \int_B \sigma
\]

and the testing condition holds.
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