Abstract

We establish a Lagrangian variational framework for general relativistic continuum theories that permits the development of the process of Lagrangian reduction by symmetry in the relativistic context. Starting with a continuum version of the Hamilton principle for the relativistic particle, we deduce two classes of reduced variational principles that are associated to either spacetime covariance, which is an axiom of the continuum theory, or material covariance, which is related to particular properties of the system such as isotropy. The covariance hypotheses and the Lagrangian reduction process are efficiently formulated by making explicit the dependence of the theory on given material and spacetime tensor fields that are transported by the world-tube of the continuum via the push-forward and pull-back operations. It is shown that the variational formulation, when augmented with the Gibbons-Hawking-York (GHY) boundary terms, also yields the Israel-Darmois junction conditions between the solution at the interior of the relativistic continua and the solution describing the gravity field produced outside from it. The expression of the first variation of the GHY term with respect to the hypersurface involves some extensions of previous results that we also derive in the paper. We consider in details the application of the variational framework to relativistic fluids and relativistic elasticity. For the latter case, our setting also allows to clarify the relation between formulations of relativistic elasticity based on the relativistic right Cauchy-Green tensor or on the relativistic Cauchy deformation tensor. The setting developed here will be further exploited for modelling purpose in subsequent parts of the paper.

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1 Introduction

General relativistic continuum theories are essential in the understanding of the structure and evolution of astrophysical systems. Phenomena of high interest which request these theories include gravitational collapse, supernova explosion, galaxy formation, accretion onto a neutron star or a black hole, as well as the emission of gravitational waves. While the use of relativistic fluid models is well-established, relativistic elasticity models are also believed to be of high relevance for instance for the physics of neutron star crusts.

As we will review below, variational principles have played a main role in the derivation of relativistic continuum models, and still form today an essential modelling tool in this area on the theoretical side. In general, for mechanical systems in absence of irreversible process, should they be relativistic or not, the most elegant and physically justified variational principle is the Hamilton principle. This principle is free from any constraints on the field variations, and can be constructed directly from the knowledge of the Lagrangian density of the system. It however requires to work in the material (or Lagrangian) representation, which corresponds to the description of each individual material particle. Depending on the problem at hands, this may not be the most practical description, one reason being that the equations in material coordinates often take a complicated form compared to their Eulerian (spatial) or convective (body) versions. It is thus desirable to formulate the variational principle directly in terms of the variables associated to those representations. Several types of Eulerian variational formulations have been developed.
for relativistic fluids and solids, with various types of constraints and from various point of views, see below. This makes hard the study of the relation between them and their derivation from a unified point of view for both general relativistic fluids and solids.

The most efficient and justified approach to obtain in full generality the variational formulation for relativistic continua in such representations would be to systematically deduce it from a continuum version of the Hamilton principle for the relativistic particle by making use of the symmetries of the system. These symmetries are associated to the spacetime covariance, which is a requested physical condition, and also to particular properties of the system like isotropy or homogeneity. Such a systematic approach to rigorously derive relevant variational formulations from the classical Hamilton principle by symmetries is well-known in Newtonian continuum mechanics and fits with the general process of Lagrangian reduction by symmetry that we will review later. It is therefore highly desirable to develop an analogue process for the general relativistic case, that also yields a reliable approach for the derivation of useful extensions of those theories that will be explored in subsequent parts of this work.

**Goal of the paper.** The first purpose of this paper is to establish a Lagrangian variational framework for general relativistic continuum theories that systematically parallels the Lagrangian reduction approach mentioned above for the Newtonian case. In particular, our framework allows to rigorously deduce several useful variational formulations by starting from the most natural and physically justified principle, namely, a continuum version of the Hamilton principle for the relativistic particle, as illustrated below:

\[
\delta \int_{\lambda_0}^{\lambda_1} \sqrt{-g(\dot{x}, \dot{x})} \ m_0 c^2 \ d\lambda = 0 \quad \Rightarrow \quad \delta \int_{\lambda_0}^{\lambda_1} \int_B \sqrt{-g(\dot{\Phi}, \dot{\Phi})} \ (c^2 + W) \ d\lambda \wedge R_0 = 0.
\]

The left hand side represents the Hamilton principle for a relativistic particle of mass \(m_0\) with world-line \(x(\lambda)\) in the spacetime \((M, g)\), where \(\dot{x} = dx/d\lambda\). On the right hand side we write its continuum version which plays a fundamental role in this paper: the map \(\Phi(\lambda, X) \in M\), with \(X \in B\) and \(\dot{\Phi} = d\Phi/d\lambda\), denotes the world-tube of a relativistic continuum media parameterized with the material manifold \(B\), the volume form \(R_0\) on \(B\) is the mass form, and \(W\) is an internal energy expression which depends on the material and spacetime tensor fields occurring in the theory.

From this principle, and exactly as in the nonrelativistic case, symmetries of the Lagrangian density can be used to systematically deduce two main types of variational formulations yielding the relativistic version of the convective (body) and spatial (Eulerian) variational formulations. The two classes of symmetries that we consider are spacetime covariance, which is an axiom of the continuum theory, and material covariance, which is related to specific properties of the continuum.

In order to develop a systematic relativistic analogue to the “reduction by symmetry” approach of Newtonian continua, we make explicit the dependence of the theory on tensor fields given on the parametrization manifold and on the spacetime.
This implies, for instance, the inclusion of reference tensor fields associated to the mass density and to the world-velocity.

The second purpose of this paper concerns the junction or matching conditions between the solution at the interior of the relativistic continua and the solution describing the gravity field produced outside from it. The matching conditions between two spacetimes is a fundamental issue in general relativity which has been considered in classic works by Darmois [1927], Lichnerowicz [1955], O’Brien and Synge [1952], Israel [1966], and Bonnor and Vickers [1981]. The Israel-Darmois junction conditions state that the interior and exterior Lorentzian metrics induce the same metric on the boundary hypersurface (assumed in this work to consist of spacelike and timelike pieces) and that, in absence of singular matter distribution, define on it the same extrinsic curvature (second fundamental form). These conditions have important applications in several gravitational topics such as wormholes (Visser [1989a,b], Lobo, Simpson, and Visser [2020]), interior structures of black-holes (Barrabés and Frolov [1996]), gravitational collapse (Fayos, Jean, Llanta, and Senovilla [1992], Fayos, Senovilla, and Torres [1996], Münch [2021]), or bubble dynamics in the early Universe (Berezin, Kuzmin, and Tkachev [1987], Blau, Guendelman, and Guth [1987], Deng and Vilenkin [2017], Deng [2020]), to name only a few works.

We shall show how these junction conditions, as well as their implication on the boundary conditions on the stress-energy-momentum tensor, can be directly obtained from a natural extension of the variational formulation by including the Gibbons-Hawking-York (GHY) term associated to the boundary of the relativistic continuum. This boundary term, given by the integral of the trace of the extrinsic curvature, has been introduced in York [1972], Gibbons and Hawking [1977] and makes the gravitational variational principle well-defined on spacetimes with boundaries. While the variation of the GHY term with respect to the metric is well-known, its variation with respect to the hypersurface involves some extensions of previous results on the first variation of integrated mean curvature that we also derive in the paper.

### Variational principles for relativistic continua.

Variational principles for general relativistic continua were first given by Taub [1954] for the case of the perfect fluid. Taub’s variational principle is based on varying the particles’ world-lines and the metric and gives both the field equations for the gravitational field created by the fluid and the equations of motion of the fluid in this gravitational field. This variational approach was then further developed and applied to several models occurring in special relativistic elasticity (Grot [1971], Maugin and Eringen [1972a]; Maugin [1972a]; Maugin and Eringen [1972b]), general relativistic elasticity (Maugin [1971], Carter [1973]), and relativistic fluids (Carter and Khalatnikov [1992], Carter and Langlois [1995, 1998]). This type of variational formulations forms today an essential tool for modelling in general relativistic continuum theories, as shown for instance in the recent works Gavassino, Antonelli, and Haskell [2020], Bau and Wasserman [2020], Feng and Carloni [2020], Anderson and Comer [2021], Anderson [2021].
While these approaches are based on varying the world-lines of the continuum, variational principles have also been proposed based on varying the projection map which assigns to each point of the continuum in spacetime its material label (Kijowski and Magli [1992], Beig and Schmidt [2003]). Other types of variational principles involve arbitrary Eulerian variations, rather than arbitrary world-lines variations, but require the introduction of Clebsch-type auxiliary fields or Lagrange multiplier terms in the action functional (Schutz [1970], Maugin [1972b], Khalatnikov and Lebedev [1982], Lebedev and Khalatnikov [1982]). Approaches with auxiliary fields are however harder to use for modelling purpose and lack physical justification.

The general Lagrangian variational framework that we develop in this paper encompasses the approach of Taub [1954] and subsequent works mentioned above on relativistic fluid and elasticity by recasting them in a systematic and unified Lagrangian covariant reduction approach. It contains however essential differences from these approaches. In order to make a clear distinction between the material, spacetime, and body descriptions, and to rigorously relate the processes of covariant reduction to the symmetries of Lagrangian density, we introduce the appropriate material tensor fields on the reference manifolds, so that the corresponding Eulerian quantities are given by the push-forward with the world-tube $\Phi$. For instance, we introduce the material mass density $R$, the material entropy density $S$, as well as a material vector $W$, so that the corresponding Eulerian mass density, entropy density, and world-velocity are written as $\rho = \Phi_* R$, $\varsigma = \Phi_* S$, $w = \Phi_* W$. Making explicit the dependence of the Lagrangian density on such tensor fields, rather than quantities such as the proper mass density or the matter current that are constructed from them and from the Lorentzian metric, is crucial for the development of the Lagrangian covariant reduction process. It allows for a general definition of material covariance in relation with the isotropy of the continuum, which permits the construction of a fully Eulerian variational formulation by covariant reduction. This variational formulation is shown to be also available for anisotropic continua at the expense of the inclusion of additional material tensor fields. Finally, the present approach also makes the variational formulation free from any Lagrange multiplier terms such as the constant magnitude constraint for the world-velocity in Taub [1954], and allows to deduce all the variational formulations from the continuum version of the Hamilton principle for the relativistic particle. In addition, since we are not only considering the variations of the metric, but also of the continuum configuration, our variational approach is directly applicable to continuum mechanics in a fixed gravitational field and to special relativity. The approach developed here can be seen as the covariant relativistic version of the process of Lagrangian reduction by symmetry in free boundary Newtonian continuum mechanics developed in Gay-Balmaz, Marsden, and Ratiu [2012] that we will review in §2. For Newtonian fluids on fixed domains, such techniques reduce to the Euler-Poincaré reduction on diffeomorphism groups Holm, Marsden, and Ratiu [1998] which itself has its roots in the Lie group geometric formulation of the ideal fluid due to Arnold [1966]. We refer to Tables 1 and 2 for a schematic correspondence between the relativistic and nonrelativistic Lagrangian reduction processes.
2 Reduced variational formulations of Newtonian continuum theories

Let us consider a non relativistic continuum body (fluid or solid) whose configuration at each time can be described by an embedding \( \varphi : \mathcal{B} \to \mathcal{S} \). Here \( \mathcal{S} = \mathbb{R}^n \) is the ambient space in which the body evolves and \( \mathcal{B} \) is the reference configuration of the body, given by a compact \( n \)-dimensional submanifold of \( \mathbb{R}^n \) with smooth boundary. A motion of the body is described by a curve \( t \in [t_0, t_1] \mapsto \varphi_t \in \text{Emb}(\mathcal{B}, \mathcal{S}) \) in the manifold of all embeddings of \( \mathcal{B} \) into \( \mathcal{S} \). Given a motion \( \varphi_t \), the location at time \( t \) of the point of the continuum with label \( X \in \mathcal{B} \) is given by \( x = \varphi_t(X) = \varphi(t, X) \in \mathcal{S} \).

We review below the symmetry reduced variational formulations for free boundary continuum theories by following Gay-Balmaz, Marsden, and Ratiu [2012]. These variational formulations are systematically deduced from the Hamilton principle by exploiting the symmetries of the systems. These symmetries are best explained by explicitly emphasize the dependence of the theory on given fixed tensor fields on the reference configuration \( \mathcal{B} \) and on the ambient space \( \mathcal{S} \), see Marsden and Hughes [1983]; Simo, Marsden, and Krishnaprasad [1988]; Gay-Balmaz, Marsden, and Ratiu [2012]. We will restrict our approach to continuum theories depending on two volume forms \( R, S \in \Omega^n(\mathcal{B}) \) representing the mass and entropy densities in the reference configuration, and on two Riemannian metrics \( G \in S^2_+(\mathcal{B}) \) and \( g \in S^2_+(\mathcal{S}) \) on \( \mathcal{B} \) and \( \mathcal{S} \), respectively. By adopting this setting, the relevant dynamic fields for the description of the continuum in the symmetry reduced formulations, besides the velocity, are obtained by applying the pull-back or push-forward operations associated with the configuration map \( \varphi \). For instance the Eulerian mass density \( \varrho \), the Eulerian entropy density \( \varsigma \), and the Cauchy deformation tensor \( c \) are obtained as

\[
\varrho = \varphi^*R, \quad \varsigma = \varphi^*S, \quad c = \varphi^*G, \tag{2.1}
\]

while the right Cauchy-Green tensor is

\[
C = \varphi^*g. \tag{2.2}
\]

We will also describe the mass and entropy density by using the functions \( \rho \) and \( s \) such that

\[
\rho \mu(g) = \varrho \quad \text{and} \quad s \mu(g) = \varsigma, \tag{2.3}
\]

with \( \mu(g) \) the volume form associated to \( g \) on \( \mathcal{S} \). The specific entropy defined by

\[
\eta = s / \rho \tag{2.4}
\]

will also be used. For simplicity, we shall consider only zero traction boundary conditions and ignore surface tension effects. The variational treatment can be extended to this case, see Gay-Balmaz, Marsden, and Ratiu [2012].

2.1 Material description and the Euler-Lagrange equations

Given the fields \( R, S, G, g \) as above, we consider a class of continuum bodies described by Lagrangian functions \( L : T\text{Emb}(\mathcal{B}, \mathcal{S}) \times \Omega^n(\mathcal{B}) \times \Omega^n(\mathcal{B}) \times S^2_+(\mathcal{B}) \times S^2_+(\mathcal{S}) \to \mathbb{R} \).
of the form

\[ L(\varphi, \dot{\varphi}, R, S, G, g \circ \varphi) = \int_B \mathcal{L}(\dot{\varphi}, T\varphi, R, S, G, g \circ \varphi), \tag{2.5} \]

with Lagrangian density

\[ \mathcal{L}(\dot{\varphi}, T\varphi, R, S, G, g \circ \varphi) = \frac{1}{2} g(\dot{\varphi}, \dot{\varphi}) R - \mathcal{W}(T\varphi, R, S, G, g \circ \varphi) R. \tag{2.6} \]

In (2.5), \( T \varphi : T\mathcal{B} \rightarrow TS \) denotes the tangent map (derivative) to the configuration map \( \varphi \in \text{Emb}(\mathcal{B}, S) \), also known as the deformation gradient, and \( \dot{\varphi} \in T_{\varphi} \text{Emb}(\mathcal{B}, S) \) is the material velocity of the continua with \( T_{\varphi} \text{Emb}(\mathcal{B}, S) := \{ \dot{\varphi} \in C^\infty(\mathcal{B}, TS) \mid \dot{\varphi}(X) \in T_{\varphi(X)}S \} \) the tangent space to \( \text{Emb}(\mathcal{B}, S) \) at \( \varphi \). The first term in (2.6) represents the kinetic energy density and the second term is (minus) the total stored energy density, with \( \mathcal{W} \) an arbitrary function of the point values of \( T\varphi, R, S, G, g \circ \varphi \). Expressions (2.5) and (2.6) are referred to as the Lagrangian function and Lagrangian densities in the material representation.

In this representation, the equations of motion are obtained by the Hamilton principle

\[ \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_{t_0}^{t_1} L(\varphi_\varepsilon, \dot{\varphi}_\varepsilon, R, S, G, g \circ \varphi_\varepsilon) dt = 0 \tag{2.7} \]

for arbitrary variations \( \varphi_\varepsilon \) of the curve \( \varphi(t) \in \text{Emb}(\mathcal{B}, S) \) with fixed endpoints, exactly as in classical (finite dimensional) Lagrangian mechanics. Note that \( R, S, G, g \) are held fixed in this variational principle, i.e, they appear as parameters in this formulation.

**Material frame indifference.** From the axiom of material frame indifference, the function \( \mathcal{W} \) in (2.6) must satisfy

\[ \mathcal{W}(T(\psi \circ \varphi), R, S, G, \psi^* g \circ \psi \circ \varphi) = \mathcal{W}(T\varphi, R, S, G, g \circ \varphi), \tag{2.8} \]

for all spatial diffeomorphisms \( \psi \in \text{Diff}(S) \). This implies that \( \mathcal{W} \) depends on \( T\varphi \) and \( g \) only through the right Cauchy-Green tensor \( C = \varphi^* g \), i.e., \( \mathcal{W} \) depends only on the point values of \( R, S, G, C \). As a consequence, there is a function \( W \), the convective stored energy function, such that

\[ \mathcal{W}(T\varphi, R, S, G, g \circ \varphi) = W \left( \frac{R}{\mu(C)}, \frac{S}{R}, G, C \right). \]

Note that we decided to choose the first two arguments of \( W \) as the combinations \( \frac{R}{\mu(C)} \) and \( \frac{S}{R} \) to ease the comparison with the relativistic case later, and because it is in this form that the concrete expression for \( W \) are usually given. Of course other forms of the function \( W \), such as \( W(R, S, G, C) \), are possible as long as there is a bijective correspondence between the representations.
Material covariance. By definition, the function $\mathcal{W}$ satisfies the property of material covariance if the equality

$$\mathcal{W}(T(\varphi \circ \psi), \psi^* R, \psi^* S, \psi^* G, g \circ \varphi \circ \psi) = \mathcal{W}(T\varphi, R, S, G, g \circ \varphi) \circ \psi$$

(2.9)

holds for all body diffeomorphisms $\psi \in \text{Diff}(B)$. Assuming that material frame indifference in (2.8) is satisfied, condition (2.9) can be equivalently written in terms of $\mathcal{W}$ as

$$\mathcal{W}(\rho \circ \varphi, \eta \circ \varphi, \psi^* G, \psi^* C) = \mathcal{W}(\rho, \eta, C) \circ \psi,$$

for all $\psi \in \text{Diff}(B)$. This condition on $\mathcal{W}$ means that the continuum is homogeneous isotropic, see Marsden and Hughes [1983, §3.5]. It allows the definition of a function $\varpi$, the Eulerian stored energy function, such that

$$\varpi(\rho, \eta, c, g) = \mathcal{W}(\rho \circ \varphi, \eta \circ \varphi, \varphi^* c, \varphi^* g) \circ \varphi^{-1},$$

with $c = \varphi^* G$ the Cauchy deformation tensor.

### 2.2 Reduced variational principle in the convective frame

By assuming the condition (2.8) on $\mathcal{W}$, the Lagrangian density $\mathcal{L}$ can be written in terms of $R$, $S$, $G$, and $C = \varphi^* g$ as

$$\mathcal{L}(\dot{\varphi}, T\varphi, R, S, G, g \circ \varphi) = \frac{1}{2} C(\mathcal{V}, \mathcal{V}) R - \mathcal{W} \left( \frac{R}{\mu(C)}, \frac{S}{R}, G, C \right) =: \mathcal{L}(\mathcal{V}, R, S, G, C),$$

where we defined the convective velocity $\mathcal{V} = T\varphi^{-1} \circ \dot{\varphi} \in \mathfrak{X}(B)$. Therefore, $\mathcal{L}$ has an associated convective Lagrangian density $\mathcal{L}$ as defined above.

In this case, the Hamilton principle (2.7) can be shown to induce the following convective variational formulation in terms of $\mathcal{L}$:

$$\delta \int_{t_0}^{t_1} \mathcal{L}(V, R, S, G, C) dt = 0, \text{ for variations } \delta V = \partial_t \zeta - [V, \zeta], \quad \delta C = \mathcal{L} \zeta C,$$

where $\zeta = T\varphi^{-1} \circ \delta \varphi$ is an arbitrary time dependent vector field on $B$, vanishing at $t = t_0, t_1$, and $[V, \zeta]$ the Lie bracket of vector fields, see Fig. 2.1 on the left. A direct application of this principle yields the equations in convective representation as

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \mathcal{V}} - \mathcal{L}_V \frac{\partial \mathcal{L}}{\partial \mathcal{V}} = \frac{\partial \mathcal{L}}{\partial \mathcal{C}} : \nabla C - 2 \text{div} \mathcal{V} \left( C \cdot \frac{\partial \mathcal{L}}{\partial \mathcal{C}} \right),$$

(2.10)

together with the boundary condition

$$\left( \mathcal{V} \otimes \frac{\partial \mathcal{L}}{\partial \mathcal{V}} - 2 C \cdot \frac{\partial \mathcal{L}}{\partial \mathcal{C}} \right) (\cdot, N^\flat) = 0 \text{ on } \partial B,$$

(2.11)

see Gay-Balmaz, Marsden, and Ratiu [2012] for a detailed derivation. In (2.10), $\mathcal{L}_V \frac{\partial}{\partial \mathcal{V}}$ denotes the Lie derivative of the one-form density $\frac{\partial \mathcal{L}}{\partial \mathcal{V}}$ in the direction $V$ and $\text{div} \mathcal{V}$ is the divergence operator associated to a given torsion free covariant derivative $\nabla$. In (2.11) the outward pointing normal vector field $N$ and the flat operator are computed relative to the metric $C$. 

2.3 Reduced variational principle in the Eulerian frame

Assuming material covariance (2.9), we obtain that the material Lagrangian density $\mathcal{L}$ is invariant under the right action of the group Diff($\mathcal{B}$) and thus induces the spatial Lagrangian density $\ell$ as

$$\mathcal{L}(\dot{\varphi}, T\varphi, R, S, G, g \circ \varphi) = \varphi^*[\ell(u, \rho, \varsigma, c, g)]$$

with

$$\ell(u, \rho, \varsigma, c, g) = \frac{1}{2} g(u, u) \rho - \varpi(\rho, \eta, c, g) \rho,$$

where $u = \dot{\varphi} \circ \varphi^{-1}$ is the Eulerian velocity and we recall the relations (2.1), (2.3), and (2.4).

The Hamilton principle (2.7) induces the following Eulerian variational formulation

$$\frac{d}{ds} \bigg|_{s=0} \int_{t_0}^{t_1} \int_{\varphi(s)(\mathcal{B})} \ell(u_s, \rho_s, \varsigma_s, c_s, g) dt = 0 \quad \text{for variations}$$

$$\delta u = \partial_t \xi + [u, \xi], \quad \delta \rho = -\mathcal{L}_{\xi \rho}, \quad \delta \varsigma = -\mathcal{L}_{\xi \varsigma}, \quad \delta c = -\mathcal{L}_{\xi c},$$

where $\xi = \delta \varphi \circ \varphi^{-1}$ is an arbitrary time dependent vector field on $\varphi(\partial \mathcal{B})$ vanishing at $t = t_0, t_1$, see Fig. 2.1 on the right. A direct application of this principle yields the equations in Eulerian representation as

$$\frac{\partial}{\partial t} \frac{\partial \ell}{\partial u} + \mathcal{L}_u \frac{\partial \ell}{\partial u} = \rho \frac{\partial \ell}{\partial \rho} + \varsigma \frac{\partial \ell}{\partial \varsigma} - \frac{\partial \ell}{\partial c} : \nabla c + 2 \text{div}_{\nabla} \left( \frac{\partial \ell}{\partial c} : c \right)$$

(2.12)

together with the boundary conditions

$$\left( \left( \ell - \rho \frac{\partial \ell}{\partial \rho} - \varsigma \frac{\partial \ell}{\partial \varsigma} \right) \delta - 2 \frac{\partial \ell}{\partial c} : c \right) (\cdot, n^b) = 0 \quad \text{on} \quad \varphi(\partial \mathcal{B}),$$

with $n$ the outward pointing unit normal vector field to $\varphi(\partial \mathcal{B})$ with respect to $g$ and where $b$ is the flat operator associated to $g$. In (2.12) $\nabla$ is a torsion free covariant derivative and $\text{div}_{\nabla}$ the associated divergence operator. For instance, one can choose the Levi-Civita covariant derivative associated to $g$ or the one associated to $c$. In the latter case, the second to last term in (2.12) vanishes. The continuity equations $\partial_t \rho + \mathcal{L}_u \rho = 0, \partial_t \varsigma + \mathcal{L}_u \varsigma = 0,$ and $\partial_t c + \mathcal{L}_u c = 0$ follow from (2.1). One also notices that (2.12) can be equivalently written as

$$\partial_t \frac{\partial \ell}{\partial u} + \text{div}_{\nabla} \left( \left( \ell - \rho \frac{\partial \ell}{\partial \rho} - \varsigma \frac{\partial \ell}{\partial \varsigma} \right) \delta + u \otimes \frac{\partial \ell}{\partial u} - 2 \frac{\partial \ell}{\partial c} : c \right) = \frac{\partial \ell}{\partial g} : \nabla g,$$

(2.13)

where the right hand side vanishes if $\nabla$ is chosen as the Levi-Civita covariant derivative associated to $g$. The $(1,1)$ tensor field density appearing on the left hand side takes the coordinate expression

$$\left( \ell - \rho \frac{\partial \ell}{\partial \rho} - \varsigma \frac{\partial \ell}{\partial \varsigma} \right) \delta^\mu_{\nu} + u^\mu \otimes \frac{\partial \ell}{\partial u^\nu} - 2 \frac{\partial \ell}{\partial c_{\mu \alpha}} c_{\nu \alpha}. $$
We refer to Gay-Balmaz, Marsden, and Ratiu [2012] for a detailed derivation of these equations. For motion in a fixed domain and in absence of $c$, this formulation reduces to the Euler-Poincaré reduction on diffeomorphism groups, see Holm, Marsden, and Ratiu [1998].

\[
\delta \int_{t_0}^{t_1} \int_B L \, dt = 0, \quad \delta \varphi \text{ free}
\]

Figure 2.1: Illustration of the variational principles in the three representations of Newtonian continuum mechanics.

**Remark 2.1** (Lagrangian reduction by symmetry). In this review we have put the emphasize on the Lagrangian densities, rather than on the Lagrangian functions which are obtained by integrating the densities on the body as in Gay-Balmaz, Marsden, and Ratiu [2012]. The reason being that we shall work with Lagrangian densities in the relativistic case. Using the integrated Lagrangian makes, however, more transparent the link with the general process of Lagrangian reduction by symmetry. This general process can be described as follows. Consider a Lagrangian function $L : TQ \to \mathbb{R}$ defined on the tangent bundle $TQ$ of the configuration manifold $Q$ of some mechanical system. Consider a free and proper action of a Lie group $G$ on $Q$ and assume that $L$ is invariant under the action of $G$ naturally induced on $TQ$ (the tangent lifted action). Consider the associated reduced Lagrangian $\ell : (TQ)/G \to \mathbb{R}$ defined on the quotient space. Then, the Lagrangian reduction approach yields a very systematic way to derive the reduced Euler-Lagrange equations for $\ell$ on $(TQ)/G$, by considering the variational principle on $(TQ)/G$ induced from the Hamilton principle on $TQ$, see Marsden and Scheurle [1993], Cendra, Marsden, and Ratiu [2001]. In the case of free boundary continuum mechanics reviewed above, we have $Q = \text{Emb} (\mathcal{B}, S)$ and the symmetry group $G$ is a subgroup of $\text{Diff} (\mathcal{B})$ or $\text{Diff} (S)$. Continuum mechanics has also been studied from the point of view of Hamiltonian reduction by symmetry, with the aim of systematically deriving the
Poisson brackets governing the dynamics in the Eulerian and convective representations. We refer to Marsden and Weinstein [1983], Marsden, Ratiu, and Weinstein [1984], Lewis, Marsden, Montgomery, and Ratiu [1986], Mazer and Ratiu [1989] for the Eulerian description and to Holm, Marsden, and Ratiu [1986], Simo, Marsden, and Krishnaprasad [1988] for the convective description.

3 Covariance properties and reduced Lagrangian densities

In this section we describe a geometric setting for relativistic continua that allows to emphasize two notions of covariance: the usual notion of covariance with respect to spacetime diffeomorphisms as well as the notion of covariance with respect to diffeomorphisms of the reference configuration. These are referred to as space-time covariance and material covariance. This setting is a natural extension of the geometric approach to Newtonian continuum mechanics reviewed above in §2.

Fundamental for our description are the notions of material (or reference) tensor fields and spacetime tensor fields which are given tensors fields on the reference configuration and on the spacetime. Their identification is crucial for the two definitions of covariance.

Using this setting and under the covariance assumptions, it is possible to systematically define three related Lagrangian densities for a given relativistic continuum theory. Each of these three Lagrangian densities depends on dynamical variables that are subject to variations in the critical action principle, as well as on parametric variables that are held fixed. We use the following terminology for these densities:

(1) The material Lagrangian density, denoted $\mathcal{L}$, which depends on the configuration map of the continua (the world-tube) and its first derivatives;

(2) The spacetime (or Eulerian) Lagrangian density, denoted $\ell$, defined from $\mathcal{L}$ under the assumption of material covariance;

(3) The convective Lagrangian density, denoted $\mathcal{L}$, defined from $\mathcal{L}$ under the assumption of spacetime covariance.

We shall consider the material Lagrangian density as the primary object, from which the expressions of $\mathcal{L}$ and $\ell$ are deduced under the corresponding covariance assumption. This is justified by the fact that the critical action principle for the material Lagrangian density is given by the Hamilton principle applied to the configuration map of the continuum (the world-tube). This principle doesn’t involve any constraints in the variations of the variables and is the field theoretic analogue to the Hamilton principle $\delta \int L(q, \dot{q}) dt = 0$ of classical mechanics. The critical action principles for the densities $\mathcal{L}$ and $\ell$ take however a more involved form, with constrained variations, that is systematically deduced from the Hamilton principle for $\mathcal{L}$. 
3.1 Geometric setting and Lagrangian density in material description

Here we first review the definition of the world-tube and the associated generalized velocity and world-velocity for relativistic continua. Then, we introduce the notion of reference tensor fields and spacetime tensor fields, and we give the general geometric setting for the definition of the Lagrangian density of a relativistic continuum in the material description. In this description, the equation of evolution are obtained by the Hamilton principle applied to the world-tube.

**World-tube and world-velocity.** Consider a \((n+1)\)-dimensional spacetime \(M\) endowed with a Lorentzian metric \(g\). The relativistic motion of a continuum media in \(M\) is described by an embedding \(\Phi : D = [a, b] \times B \to M\), called the *world-tube*, where \(B\) is a \(n\)-dimensional compact orientable manifold with smooth boundary describing the material continuum, and where \(g(\partial_\lambda \Phi, \partial_\lambda \Phi) < 0\). For each \(X \in B\), the curve \(\lambda \in I \mapsto \Phi(\lambda, X) \in M\) is the world-line of the particle with label \(X \in B\).

The *generalized velocity* of the media is the vector field on \(\Phi(D)\) defined as
\[
w = \partial_\lambda \Phi \circ \Phi^{-1} \in \mathcal{X}(\Phi(D)),
\]
and its normalized version
\[
u = \frac{c}{\sqrt{-g(w,w)}}w \in \mathcal{X}(\Phi(D)), \quad g(u,u) = -c^2,
\]
is the *world-velocity*. Note that we can write the generalized velocity as \(w = \Phi_* \partial_\lambda\), where \(\Phi_*\) denotes the push-forward of the vector field \(\partial_\lambda\) by \(\Phi\).

For our subsequent development it is advantageous to consider the general setting in which \(D\) is an arbitrary \((n+1)\)-dimensional manifold with piecewise smooth boundary and \(W \in \mathcal{X}(D)\) is a given nowhere vanishing vector field on \(D\). This setting is helpful both for conceptual and notational reasons. In this context, the generalized velocity of the world-tube is defined as
\[
w = \Phi_*W \in \mathcal{X}(\Phi(D)).
\]
The relevant case is \(D = [a, b] \times B\) and \(W = \partial_\lambda\) in which case (3.2) recovers (3.1).

**Reference and spacetime tensor fields.** The dynamics of relativistic continua depends on given tensor fields on \(D\) and \(M\), called *reference tensor fields* and *spacetime tensor fields*, respectively. For simplicity of the exposition we assume that,

---

1^Note the different notation used for the Riemannian metric \(g\) on the ambient space \(S\) in §2 and the Lorentzian metric \(g\) on spacetime \(M\) used here.
besides the reference vector field $W$ on $\mathcal{D}$, the dynamics depends on a given $(p,q)$-tensor field $K$ on $\mathcal{D}$, and a given $(r,s)$-tensor field $\gamma$ on $\mathcal{M}$. We use the notations $K \in \mathcal{T}_p^q(\mathcal{D})$ and $\gamma \in \mathcal{T}_r^s(\mathcal{M})$. It is assumed that

$$\mathcal{L}_W K = 0,$$

where $\mathcal{L}_W$ denotes the Lie derivative of a tensor field in the direction $W$. Thus, $K$ is assumed to be constant on the flow $\phi_\tau$ of $W$, i.e.

$$\phi_\tau^* K = K.$$

**Remark 3.1** (Generalization and examples). We present the theory for general tensor fields, but it directly also applies to tensor field densities, differential forms, and pseudo-Riemannian metrics, for instance. Also the extension to the case of the dependence on a collection $\{K_i\}$ and $\{\gamma_j\}$ of reference and spacetime tensor fields is straightforward. The collection of spacetime tensor fields necessarily includes the Lorentzian metric $g$. An important reference tensor field for our development is the reference volume form $R \in \Omega^{n+1}(\mathcal{D})$, $\mathcal{D} = [a,b] \times \mathcal{B}$, defined by

$$R = d\lambda \wedge \pi_\mathcal{B}^* R_0,$$

where $R_0 \in \Omega^n(\mathcal{B})$ is a volume form on $\mathcal{B}$, the mass form, and $\pi_\mathcal{B} : [a,b] \times \mathcal{B} \to \mathcal{B}$ is the projection. The assumption (3.3) is satisfied with the vector field $W = \partial_\lambda$ since

$$\mathcal{L}_{\partial_\lambda} R = \mathcal{L}_{\partial_\lambda} d\lambda \wedge \pi_\mathcal{B}^* R_0 + d\lambda \wedge \mathcal{L}_{\partial_\lambda} \pi_\mathcal{B}^* R_0 = 0.$$

Given a world-tube $\Phi : \mathcal{D} \to \mathcal{M}$, the Eulerian expression of a given reference tensor field $K$ is defined by the push-forward of $K$ onto $\Phi(\mathcal{D}) \subset \mathcal{M}$ by the world-tube:

$$\kappa = \Phi_* K \in \mathcal{T}_p^q(\Phi(\mathcal{D})).$$

From the formula $\Phi_* (\mathcal{L}_W K_i) = \mathcal{L}_{\Phi_* W} \Phi_* K_i$, we get the important relation

$$\mathcal{L}_{w} \kappa = 0,$$

where $w$ is the generalized velocity. Similarly, the material expression of a given spacetime tensor field $\gamma$ is defined by the pull-back of $\gamma$ to $\mathcal{D}$ by the world-tube:

$$\Gamma = \Phi^* \gamma \in \mathcal{T}_r^s(\mathcal{D}).$$

Note that although $\Phi$ is not a diffeomorphism $\mathcal{D} \to \mathcal{M}$, but only an embedding, the pull-back of an arbitrary tensor field can be defined since $\mathcal{D}$ and $\mathcal{M}$ have the same dimension\(^2\).

\(^2\)Explicitly, for $\alpha^1, ..., \alpha^r \in T_X^* \mathcal{D}$ and $u_1, ..., u_s \in T_X \mathcal{D}$, we have

$$\Gamma(X)(\alpha^1, ..., \alpha^r, u_1, ..., u_s) = \gamma(\Phi(X))((T_X^* \Phi)^{-1}(\alpha^1), ..., (T_X^* \Phi)^{-1}(\alpha^r), T_X \Phi(u_1), ..., T_X \Phi(u_s))$$

with $T_X \Phi : T_X \mathcal{D} \to T_{\Phi(X)} \mathcal{M}$ and $T_X^* \Phi : T_{\Phi(X)}^* \mathcal{M} \to T_X^* \mathcal{D}$ isomorphisms and $T_X \mathcal{M} = T_{\Phi(X)}[\Phi(\mathcal{D})]$ for all $X \in \mathcal{D}$. 
3.1 Geometric setting and Lagrangian density in material description. The material description of a relativistic continuum uses the world-tube as a primary variable. This description involves both the configuration manifold $\mathcal{D}$ and the spacetime $\mathcal{M}$, as opposed to the Eulerian representation and the convective representation, which only use one of them. It is in the material description that the critical action principle takes the simplest form. It is given by the Hamilton principle for the variations of the world-tube.

In general, the Lagrangian density $\mathcal{L}$ of a relativistic continuum depends on the material point $X \in \mathcal{D}$ and on the value of the world-tube and of its first derivative at this point. It also depends parametrically on the point values of the given reference and spacetime tensor fields $W$, $K$ and $\gamma$. It is thus a bundle map

$$\mathcal{L} : J^1(\mathcal{D} \times \mathcal{M}) \times T^p\mathcal{D} \times T_q^n\mathcal{M} \to \wedge^{n+1}\mathcal{D} \tag{3.6}$$

covering the identity on $\mathcal{D}$. Here $J^1(\mathcal{D} \times \mathcal{M}) \to \mathcal{D} \times \mathcal{M}$ denotes the first jet bundle of the trivial fiber bundle $\mathcal{D} \times \mathcal{M} \to \mathcal{D}$. The vector fiber of $J^1(\mathcal{D} \times \mathcal{M})$ at $(X, x) \in \mathcal{D} \times \mathcal{M}$ is $L(T_X\mathcal{D}, T_x\mathcal{M})$, the space of linear maps $T_X\mathcal{D} \to T_x\mathcal{M}$. Also, $T^p_q\mathcal{D} \to \mathcal{D}$ denotes the vector bundle of $(p, q)$ tensors on $\mathcal{D}$, similarly for $T^n_q\mathcal{M} \to \mathcal{M}$. Note that $J^1(\mathcal{D} \times \mathcal{M}) \times T^p\mathcal{D} \times T^n_q\mathcal{M}$ is a bundle over $\mathcal{D} \times \mathcal{M}$, which can also be regarded as a bundle over $\mathcal{D}$. It is the latter interpretation that is understood in (3.6).

The case when $\mathcal{L}$ depends on differential forms or pseudo-Riemannian metrics, see Remark 3.1, is treated similarly, by considering appropriate subbundles of $T^p_q\mathcal{D}$ and $T^n_q\mathcal{M}$.

Sometimes it is required to appropriately restrict the bundles to open subbundles in order to have the Lagrangian density well-defined. This is typically the case to ensure the condition $\langle \partial_\lambda \Phi, \partial_\lambda \Phi \rangle < 0$. We will not indicate such restrictions as they are obvious from the context.

In local coordinates the Lagrangian density $\mathcal{L}$ reads

$$\mathcal{L} = \mathcal{L}(X^a, x^\mu, v_\alpha^\mu, W^b, K_{b_1...b_q}^{a_1...a_p}, \gamma_{\mu_1...\mu_n}) \, d^{n+1}X. \tag{3.7}$$

Here $X^a, a = 1, ..., n + 1$ and $x^\mu, \mu = 1, ..., n + 1$ are local coordinates on $\mathcal{D}$ and $\mathcal{M}$, and $(X^a, x^\mu, v_\alpha^\mu)$ are the local coordinates induced on $J^1(\mathcal{D} \times \mathcal{M})$. We also use the notation $d^{n+1}X = dX^1 \wedge ... \wedge dX^{n+1}$. The evaluation of the Lagrangian density on a world tube $\Phi$, a reference vector field $W \in \mathfrak{X}(\mathcal{D})$, a reference tensor field $K \in T^p_q(\mathcal{D})$, and a spacetime tensor field $\gamma \in T^n_q(\mathcal{M})$, is written as

$$\mathcal{L}(j^1\Phi, W, K, \gamma \circ \Phi), \tag{3.7}$$

where $j^1\Phi : \mathcal{D} \to J^1(\mathcal{D} \times \mathcal{M})$ denotes the first jet extension of the world-tube $\Phi$, given locally as $X^a \mapsto (X^a, \Phi^a(X^a), \Phi^\mu_b(X^a))$.

For given $W, K, and \gamma$, the Hamilton principle for this Lagrangian density reads

$$\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_{\mathcal{D}} \mathcal{L}(j^1\Phi_\varepsilon, W, K, \gamma \circ \Phi_\varepsilon) = 0 \tag{3.8}$$

for arbitrary variations $\Phi_\varepsilon$ of the world tube $\Phi$. It is important to note that $W, K, \gamma$ are held fixed during the process of taking variations. This principle yields the
Euler-Lagrange equations and boundary conditions

\[
\partial_a \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^a} - \frac{\partial \mathcal{L}}{\partial \Phi} = \partial_a \mu_{\mu_1 \ldots \mu_r} \partial_{\mu_1} \cdots \partial_{\mu_r} \mathcal{L},
\]

where \( d^n X_a = i_{\partial_a} d^{n+1} X \). This boundary condition arises from letting the variations \( \delta \Phi = \frac{\delta}{\delta \epsilon} \big|_{\epsilon=0} \Phi_{\epsilon} \) be arbitrary on \( \partial D \). Other boundary conditions can be obtained by imposing restriction on the boundary variations or adding a boundary term in the action functional.

### 3.2 Spacetime covariance and the convective Lagrangian density

In this section we state the definition of spacetime covariance for a Lagrangian density of the type (3.7), which is a general assumption on relativistic theories. We show that when a Lagrangian density is spacetime covariant, it has an associated convective Lagrangian density.

We say that the Lagrangian density (3.7) is \textit{spacetime covariant} if it satisfies

\[
\mathcal{L}(j^1(\psi \circ \Phi), W, K, \psi \circ \psi \circ \Phi) = \mathcal{L}(j^1 \Phi, W, K, \gamma \circ \Phi), \tag{3.9}
\]

for all \( \psi \in \text{Diff}(M) \) and for all \( \Phi, W, K, \gamma \).

A simple example of a spacetime covariant Lagrangian density is \( \mathcal{L} : J^1(D \times M) \times S^2 L M \to \wedge^{n+1} D \) given by

\[
\mathcal{L}(j^1 \Phi, g \circ \Phi) = \Phi^* [\mu(g)], \tag{3.10}
\]

where \( S^2 L M \subset T^0_2 M \to M \) denotes the bundle of Lorentzian metrics on \( M \) and \( \mu(g) \) is the volume form associated to the Lorentzian metric \( g \). Spacetime covariance is checked as

\[
\mathcal{L}(j^1(\psi \circ \Phi), \psi \circ g \circ \psi \circ \Phi) = (\psi \circ \Phi)^* [\mu(\psi \circ g)] = \Phi^* [\mu(g)] = \mathcal{L}(j^1 \Phi, g \circ \Phi).
\]

More generally, \( \mathcal{L} : J^1(D \times M) \times V D \times S^2 L M \to \wedge^{n+1} D \) given by

\[
\mathcal{L}(j^1 \Phi, R, g \circ \Phi) = e \left( \frac{R}{\Phi^* [\mu(g)]} \right) R \tag{3.11}
\]

with \( V D \subset \wedge^{n+1} D \to D \) the bundle of volume forms and \( e : \mathbb{R} \to \mathbb{R} \) a strictly positive function, is spacetime covariant.

**Proposition 3.2 (The convective Lagrangian density).** Let \( \mathcal{L} \) be a Lagrangian density (3.7) with the spacetime covariance property (3.9). Then, there is an associated \textit{convective Lagrangian density}, given by a bundle map \( \mathcal{L} : T D \times T^0_4 D \times T^*_s D \to \wedge^{n+1} D \) covering the identity on \( D \), such that

\[
\mathcal{L}(j^1 \Phi, W, K, \gamma \circ \Phi) = \mathcal{L}(W, K, \Gamma), \tag{3.12}
\]

where

\[
\Gamma := \Phi^* \gamma \in T^*_s (D).
\]
Proof. We fix an embedding $\Phi_0 : D \to N_0 \subset M$. Then it suffices to choose $\psi \in \text{Diff}(M)$ to be an arbitrary extension of $\Phi_0 \circ \Phi^{-1} : \Phi(D) \to N_0$ and we have

$$\mathcal{L}(j^1\Phi, W, K, \gamma \circ \Phi) = \mathcal{L}(j^1\Phi_0, W, K, (\Phi_0)\ast \Gamma \circ \Phi_0).$$

We note that $(\Phi_0)\ast \Gamma \circ \Phi_0 = T^*\Phi_0 \circ \Gamma$ depends only on the values $\Gamma(X)$ of $\Gamma$, not on $\Gamma$ as a field. We define

$$\mathcal{L}(W, K, \Gamma) := \mathcal{L}(j^1\Phi_0, W, K, (\Phi_0)\ast \Gamma \circ \Phi_0).$$

One then check, using again spacetime covariance, that $\mathcal{L}$ does not depend on the embedding $\Phi_0$. ■

For the examples (3.10) and (3.11), the corresponding convective Lagrangian densities are the bundle maps $\mathcal{L} : S^2_L D \to \wedge^{n+1} D$ and $\mathcal{L} : V D \times S^2_L D \to \wedge^{n+1} D$ given by

$$\mathcal{L}(\Gamma) = \mu(\Gamma) \quad \text{and} \quad \mathcal{L}(R, \Gamma) = e\left(\frac{R}{\mu(\Gamma)}\right) R,$$

where $\Gamma = \Phi^*g$.

Note that in local coordinates the convective Lagrangian density reads

$$\mathcal{L} = \mathcal{L}(X^a, W^b, K_{b_1...b_q}^{a_1...a_p}, \Gamma_{b_1...b_s}^{a_1...a_r}) d^{n+1}X.$$  \hfill (3.13)

In particular, $\mathcal{L}$ can depend explicitly on $X \in D$ in general. We refer to $\mathcal{L}$ as the convective (or body) Lagrangian density associated to $\mathcal{L}$, since it depends only on material (or body) tensor fields, without reference to $M$.

Remark 3.3 (Choice of spacetime tensors and isotropy subgroup). In some situations one particular choice for the spatial tensor field $\gamma$ is enough for the description of the continuum theory under study. This is the case for instance for the Minkowski metric in the case of special relativity. In such situations, for the definition (3.12) to hold, it is enough to assume that $\mathcal{L}$ satisfies the spacetime covariance (3.9) for $\psi$ in the isotropy subgroup of the chosen field. For $M = \mathbb{R}^4$ and $\gamma$ the Minkowski metric, this corresponds to covariance with respect to the Poincaré group. The convective Lagrangian is then defined only for $\Gamma$ which can be written in terms of the given $\gamma$ as $\Gamma = \Phi^*\gamma$ for some world-tube $\Phi$.

3.3 Material covariance and the spacetime Lagrangian density

In this section we state the definition of material covariance for a Lagrangian density of the type (3.7). We show that when a Lagrangian density is materially covariant, it has an associated spacetime (or Eulerian) Lagrangian density. As opposed to spacetime covariance, material covariance is related to the properties of the material, such as isotropy. However, material covariance can also be achieved in the anisotropic case by suitably extending the collection of material tensor fields of the theory, see Remark 6.5.
We say that the Lagrangian density (3.7) is \textit{materially covariant} if it satisfies

\[ \mathcal{L}(j^1(\Phi \circ \varphi), \varphi^*W, \varphi^*K, \gamma \circ \Phi \circ \varphi) = \varphi^* \left[ \mathcal{L}(j^1\Phi, W, K, \gamma \circ \Phi) \right], \quad (3.14) \]

for all \( \varphi \in \text{Diff}(\mathcal{D}) \) and for all \( \Phi, W, K, \gamma \). While the material covariance is stated in (3.14) in terms of the world-tube and tensor fields, it is really a property of the Lagrangian density as a bundle map (3.7), as a direct check in local coordinates shows.

Examples of materially covariant Lagrangian densities are given by (3.10) and (3.11).

\textbf{Proposition 3.4 (The spacetime Lagrangian density).} \textit{Let} \( \mathcal{L} \) \textit{be a Lagrangian density (3.7) with the material covariance property (3.14)}. \textit{Then, for any domain} \( N \subset \mathcal{M} \) \textit{diffeomorphic to} \( \mathcal{D} \), \textit{there exists an associated \textit{spacetime Lagrangian density}, given by a bundle map} \( \ell_N : TN \times T_q^p N \times T_q^s N \rightarrow \Lambda^{n+1} N \) \textit{covering the identity on} \( N \) \textit{such that}

\[ \mathcal{L}(j^1\Phi, W, K, \gamma \circ \Phi) = \Phi^*[\ell_N(w, \kappa, \gamma)], \quad (3.15) \]

where

\[ N = \Phi(\mathcal{D}) \subset \mathcal{M}, \quad w = \Phi_*W \in \mathfrak{X}(\Phi(\mathcal{D})), \quad \text{and} \quad \kappa = \Phi_*K \in T_q^p(\Phi(\mathcal{D})). \]

\textbf{Proof.} Given \( N \subset \mathcal{M} \) \textit{diffeomorphic to} \( \mathcal{D} \), \textit{we define} \( \ell_N : TN \times T_q^p N \times T_q^s N \rightarrow \Lambda^{n+1} N \) \\
\textit{as}

\[ \ell_N(w, \kappa, \gamma) := \Phi^*[\mathcal{L}(j^1\Phi, \Phi^*w, \Phi^*K, \gamma \circ \Phi)], \quad (3.16) \]

where \( \Phi : \mathcal{D} \rightarrow \mathcal{M} \) \textit{is an embedding with} \( \Phi(\mathcal{D}) = N \). \textit{By using the material covariance (3.14)} \textit{one checks that this definition does not depend on the chosen embedding with} \( \Phi(\mathcal{D}) = N \). \textit{Even though the relation (3.16) uses the fields} \( w, \kappa, \gamma \), \textit{this equality really defines} \( \ell_N \) \textit{on the point values} \( w(x), \kappa(x), \gamma(x) \) \textit{as a direct check in local coordinates shows, i.e., it gives} \( \ell_N \) \textit{as a map defined on the bundle} \( TN \times T_q^p N \times T_q^s N \). \( \blacksquare \)

We refer to \( \ell_N \) as the spacetime Lagrangian associated to \( \mathcal{L} \) on \( N \), since it only depends on spacetime tensor fields, \textit{without reference to} \( \mathcal{D} \). \textit{Gluing together these expressions, we get a spacetime Lagrangian density defined over the whole spacetime, denoted simply} \( \ell : TM \times T_q^p M \times T_q^s M \rightarrow \Lambda^{n+1} M \).

\textit{For the examples} (3.10) and (3.11), \textit{the associated spacetime Lagrangians are the bundle maps} \( \ell : S^2 L^2 M \rightarrow \Lambda^{n+1} M \) \textit{and} \( \ell : V M \times S^2 L^2 M \rightarrow \Lambda^{n+1} M \)

\[ \ell(g) = \mu(g) \quad \text{and} \quad \ell(\varrho, g) = e \left( \frac{\varrho}{\mu(g)} \right) g. \]

\textit{Given a world-tube} \( \Phi : \mathcal{D} \rightarrow \mathcal{M} \), \textit{the volume form} \( \varrho = \Phi_*R \) \textit{is Eulerian expression of} \( R \) \textit{on} \( \Phi(\mathcal{D}) \), \textit{see (3.4).}
Remark 3.5 (Choice of material tensors and isotropy subgroup). In many situations, see the examples later, the description of the continuum relies only on one particular choice for the fields $W$ or $K$, such as $W = \partial_\lambda$ for $D = [a, b] \times B$. In this case, for the definition (3.15) to hold, it is enough to assume that $L$ satisfies the material covariance (3.14) for $\varphi$ in the isotropy subgroup of the chosen fields. For instance, if we focus on $W = \partial_\lambda$, the corresponding isotropy subgroup is

$$\text{Diff}_{\partial_\lambda}(D) = \{\varphi \in \text{Diff}(D) \mid \varphi^* \partial_\lambda = \partial_\lambda\} \simeq \text{Diff}(B) \otimes F(B, \mathbb{R}) \ni (\psi, f)$$

(3.17)

with $\varphi(\lambda, X) = (\lambda + f(X), \psi(X))$. In this case, $\ell_N$ in (3.16) is defined only for $w$ which can be written in terms of the given $W$ as $w = \Phi_* W$ for some world-tube $\Phi$, similarly for other fields $K$. This remark is analogous to Remark 3.3 about spacetime covariance.

4 Eulerian and convective covariant reductions

In this section we derive the Eulerian form, resp. the convective form of the Hamilton principle (3.8) for relativistic continua under the assumption of material covariance, resp. spacetime covariance. To obtain the Eulerian form, we use the relation (3.15) between the material Lagrangian density and its spacetime version as well as the definition of the Eulerian form of the given material tensor fields. Similarly, to obtain the convective form, we use the relation (3.12) between the material Lagrangian density and its convective version, as well as the definition of the convective form of the given spacetime tensor fields. These derivations use some technical results on Lie derivatives that we give below.

4.1 Preliminaries on Lie derivatives

Recall that the local expression of the Lie derivative of a $(p, q)$-tensor field $\kappa$ along a vector field $\zeta$ is

$$(L_\zeta \kappa)_{\beta_1 \cdots \beta_p}^{\alpha_1 \cdots \alpha_q} = \partial_r K_{\beta_1 \cdots \beta_q \gamma}^{\alpha_1 \cdots \alpha_p} \xi^\gamma - K_{\beta_1 \cdots \beta_q}^{\alpha_1 \cdots \alpha_p \gamma} \partial_r \xi^\gamma + K_{\beta_1 \cdots \beta_q \rho}^{\alpha_1 \cdots \alpha_p \nu} \partial_\rho \xi^\nu$$

(4.1)

where $K$ is the $(p + 1, q + 1)$ tensor field defined by

$$K_{\beta_1 \cdots \beta_q \gamma}^{\alpha_1 \cdots \alpha_p} = \sum_r (K_{\beta_1 \cdots \beta_q \gamma \beta_r}^{\alpha_1 \cdots \alpha_p \rho} \delta^\rho_{\beta_r} - K_{\beta_1 \cdots \beta_q}^{\alpha_1 \cdots \alpha_{r-1} \gamma \beta_r \beta_{r+1} \cdots \beta_q \rho} \delta^\rho_{\beta_r}).$$

(4.2)

Formulas (4.1) can also be written in terms of a given torsion free covariant derivative $\nabla$ on $M$ by replacing $\partial_r$ with $\nabla_r$. With such choice, we have the global formula

$$L_\zeta \kappa = \nabla_\zeta \kappa + \tilde{K} : \nabla \zeta.$$

(4.3)
For future use, we recall these formulas in the case where $\kappa$ is, respectively, a function $f$, a vector field $w = w^\mu \partial_\mu$, a 2 covariant symmetric tensor field $c = c_{\mu\nu} dx^\mu \otimes dx^\nu$, and a $(n + 1)$-form $\varrho = \overline{\gamma} dx^{n+1}$:

\[
\begin{align*}
\mathcal{L}_\zeta f &= \partial_\gamma f \zeta^\gamma \\
\mathcal{L}_\zeta w^\mu &= \partial_\gamma w^\mu \zeta^\gamma - w^\gamma \partial_\gamma \zeta^\mu \\
\mathcal{L}_\zeta c_{\mu\nu} &= \partial_\gamma c_{\mu\nu} \zeta^\gamma + c_{\gamma\nu} \partial_\gamma \zeta^\mu + c_{\mu\gamma} \partial_\gamma \zeta^\nu \\
\mathcal{L}_\zeta \varrho &= \partial_\gamma (\overline{\gamma} \zeta^\gamma).
\end{align*}
\]

The derivation of the reduced Euler-Lagrange equations uses the following technical result.

**Lemma 4.1.** Let $\kappa$ be a $(p,q)$-tensor field and $\pi$ a $(q,p)$-tensor field density. Then, we have locally

\[
(L_\zeta \kappa)_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q}^{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q} = \zeta^\mu \left( \partial_\mu \kappa_{\beta_1 \ldots \beta_q}^{\alpha_1 \ldots \alpha_p} - \partial_\mu (\overline{\kappa} \beta_1 \ldots \beta_q^{\mu \alpha_1 \ldots \alpha_p}) \right) + \partial_\mu (\overline{\kappa} \beta_1 \ldots \beta_q^{\mu \alpha_1 \ldots \alpha_p} \zeta^\mu). \tag{4.4}
\]

The same formula holds with $\partial_\gamma$ replaced by $\nabla_\gamma$, with $\nabla$ a torsion free covariant derivative. In this case, we have the global formula

\[
L_\zeta \kappa \cdot \pi = \nabla_\kappa \kappa \cdot \pi - \text{div} \overline{\nabla} (\pi : \overline{\kappa}) \cdot \zeta + \text{div} ((\pi : \overline{\kappa}) \cdot \zeta), \tag{4.5}
\]

where the dot symbol “.” in $L_\zeta \kappa \cdot \pi$ and $\nabla_\zeta \kappa \cdot \pi$ denotes the full contraction and where $\pi : \overline{\kappa}$ is the $(1,1)$ tensor field density obtained by contracting all the respective indices of $\overline{\kappa}$ and $\pi$ except the last covariant and contravariant indices of $\overline{\kappa}$.

**Proof.** This follows directly from a computation in local coordinates using (4.1) and (4.2). ■

**Remark 4.2** (Divergences). Note that the first divergence operator is associated to the torsion free covariant derivative $\nabla$, while the last one is canonically defined since it acts on a vector field density, hence the notations $\text{div} \nabla$ and $\text{div}$ in (4.5). Given a $(1,1)$ tensor field density $T = T^\mu_\nu \partial_\mu \otimes dx^\nu \otimes dx^{n+1}$ and a vector field density $X = X^\mu \partial_\mu \otimes dx^{n+1}$, these divergences are defined as

\[
\begin{align*}
\text{div} \nabla T(u) &= \nabla_\partial_\mu T(u, dx^\mu) = \nabla_\mu T^\mu_\nu u^\nu dx^{n+1}, \quad \forall u \in TM \\
\text{div} X &= d(\partial_\mu \langle dx^\mu, X \rangle) = d(X^\mu dx^\mu_\mu) = \partial_\mu X^\mu dx^{n+1}.
\end{align*}
\]

where $\partial_\mu \langle dx^\mu, X \rangle$ denotes the insertion of the vector $\partial_\mu$ in the $(n+1)$-form $\langle dx^\mu, X \rangle$. We note that for $\ell$ a $(n+1)$-form and $\zeta$ a vector field, we have

\[
\text{div}(\ell \zeta) = d(\partial_\mu \langle dx^\mu, X \rangle) = L_\zeta \ell. \tag{4.7}
\]
For a function $f$, a vector field $w = w^\mu \partial_\mu$, a 2 covariant symmetric tensor field $c = c_{\mu\nu} dx^\mu \otimes dx^\nu$, and a $(n + 1)$-form $\varrho = \bar{\omega}^n dx$, formula (4.4) gives

$$
\mathcal{L}_\zeta f \bar{\pi} = \zeta^i \partial_i f \bar{\pi} \\
\mathcal{L}_\zeta w^\mu \pi_\mu = \zeta^i (\partial_i w^\nu \pi_\nu + \partial_\nu (w^\nu \pi_\gamma)) - \partial_\nu (w^\nu \pi_\mu \zeta^\mu) \\
\mathcal{L}_\zeta c_{\mu\nu} \pi^{\mu\nu} = \zeta^i (\partial_i c_{\mu\nu} \pi^{\mu\nu} - 2 \partial_\mu (c_{\gamma\nu} \pi^{\mu\nu})) + 2 \partial_\gamma (c_{\mu\nu} \pi^{\gamma\nu} \zeta^\mu) \\
\mathcal{L}_\zeta \bar{\omega} = -\zeta^i \partial_i \pi + \partial_\gamma (\bar{\omega} \pi^{\gamma}).
$$

### 4.2 Reduced Euler-Lagrange equations on spacetime

We now present the variational principle induced in the Eulerian description by the Hamilton principle (4.2) and consider the associated spacetime Lagrangian density $\mathcal{L} = \Phi \circ \partial \pi + \partial_\gamma (\pi \gamma)$.

Given a $(1, 1)$ tensor field density $T$, written locally as $T = T^{\mu}_\nu dx^\mu \otimes dx^\nu$, we use the notation $\operatorname{tr}(T) = T^{\mu}_\nu dx^\mu \otimes dx^\nu$ for the contraction of the contravariant index and the first density index. We denote by $i_D : \partial N \to N$ the inclusion.

**Theorem 4.3** (Covariant Eulerian reduction). Let $\mathcal{L} : J^1(\mathcal{D} \times \mathcal{M}) \times T^\mathcal{D} \times T^T_\mathcal{D} \times T^T_\mathcal{M} \to \wedge^{n+1} \mathcal{D}$ be a Lagrangian density with the material covariant property (3.14) and consider the associated spacetime Lagrangian density $\ell : TM \times T^\mathcal{M} \times T^T_\mathcal{M} \to \wedge^{n+1} \mathcal{M}$.

Fix the reference tensor fields $W \in \mathcal{X}(\mathcal{D})$, $K \in \mathcal{T}^p_\mathcal{D}(\mathcal{D})$, and the spacetime tensor field $\gamma \in \mathcal{T}^p_\mathcal{M}(\mathcal{M})$. For each world-tube $\Phi : \mathcal{D} \to \mathcal{M}$, define $N = \Phi(\mathcal{D})$, $w = \Phi_* W$, and $\kappa = \Phi_* K$. Then, the following statements are equivalent:

(i) $\Phi$ is a critical point of the **Hamilton principle**

$$
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_{\mathcal{D}} \mathcal{L}(J^1 \Phi_\varepsilon, W, K, \gamma \circ \Phi_\varepsilon) = 0
$$

for arbitrary variations $\Phi_\varepsilon$.

(ii) $\Phi$ is a solution of the **Euler-Lagrange equations**

$$
\partial_\alpha \frac{\partial \mathcal{L}}{\partial \Phi_\alpha^\mu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \Phi_\gamma \gamma} = 0, \quad \frac{\partial \mathcal{L}}{\partial \Phi_\alpha^\mu} d^n X_\alpha = 0.
$$

(iii) $w \in \mathcal{X}(N)$ and $\kappa \in \mathcal{T}^p_\mathcal{M}(N)$ are critical points of the **Eulerian variational principle**

$$
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_{N_\varepsilon} \ell(w_\varepsilon, \kappa_\varepsilon, \gamma) = 0,
$$

(4.9)
4.2 Reduced Euler-Lagrange equations on spacetime

for arbitrary variations \( \Phi_\varepsilon \), where \( N_\varepsilon = \Phi_\varepsilon(\mathcal{D}) \), \( w_\varepsilon = (\Phi_\varepsilon)_* W \), \( \kappa_\varepsilon = (\Phi_\varepsilon)_* K \). In particular the variations of \( w \) and \( \kappa \) are constrained of the form \( \delta w = -L_\zeta w \) and \( \delta \kappa = -L_\zeta \kappa \) where \( \zeta = \delta \Phi \circ \Phi^{-1} \) is an arbitrary vector field on \( N \).

(iv) \( w \in \mathfrak{X}(N) \) and \( \kappa \in \mathfrak{T}^p_q(N) \) are solution of the reduced Euler-Lagrange equations on spacetime

\[
\begin{align*}
\text{div} \nabla \left( \ell \delta + w \otimes \frac{\partial \ell}{\partial w} - \frac{\partial \ell}{\partial \kappa} : \hat{\kappa} \right) &= \frac{\partial \nabla \ell}{\partial x} + \frac{\partial \ell}{\partial \gamma} \cdot \nabla \gamma \\
\iota_{\partial N} \left( \text{tr} \left( \ell \delta + w \otimes \frac{\partial \ell}{\partial w} - \frac{\partial \ell}{\partial \kappa} : \hat{\kappa} \right) \cdot \zeta \right) &= 0, \ \forall \zeta \ \text{on} \ \partial N \\
\mathcal{L}_w \kappa &= 0,
\end{align*}
\]

written with the help of a torsion free covariant derivative \( \nabla \). In local coordinates, writing \( \ell = \bar{\ell} d^{n+1} x \), the boundary condition reads

\[
\left( \ell \delta + w \otimes \frac{\partial \bar{\ell}}{\partial w} - \frac{\partial \bar{\ell}}{\partial \kappa} : \hat{\kappa} \right)^\mu \nu d^n x^\mu = 0, \ \text{on} \ \partial N.
\]

**Proof.** For simplicity, we include only the material tensor \( K \), since the treatment of the vector field \( W \) is a particular case of it. Using Lemma 4.1 and (4.8), we have

\[
\frac{d}{d \varepsilon} \bigg|_{\varepsilon=0} \int \ell((\Phi_\varepsilon)_* K, \gamma)
= \int_N \frac{\partial \ell}{\partial \kappa} \cdot \delta \kappa + \int_{\partial N} \iota_{\delta \Phi \circ \Phi^{-1}} \ell
= -\int_N \left( \frac{\partial \ell}{\partial \kappa} \cdot \nabla \kappa - \text{div} \nabla \left( \frac{\partial \ell}{\partial \kappa} : \hat{\kappa} \right) \cdot \zeta \right) + \int_{\partial N} \text{tr} \left( \ell \delta - \frac{\partial \ell}{\partial \kappa} : \hat{\kappa} \right) \cdot \zeta
= -\int_N \left( \text{div} \nabla \left( \ell \delta \frac{\partial \ell}{\partial \kappa} : \hat{\kappa} \right) \cdot \zeta \right) + \frac{\partial \nabla \ell}{\partial x} \cdot \zeta + \frac{\partial \ell}{\partial \gamma} \cdot \nabla \gamma + \frac{\partial \ell}{\partial \gamma} \cdot \nabla \gamma \right)
\]

In the third equality we used \( \iota_{\zeta} \ell = \text{tr}(\ell \delta) \cdot \zeta \). In the fourth equality we used the following formula

\[
\nabla \zeta[\ell(\kappa, \gamma)] = \frac{\partial \nabla \ell}{\partial x} \cdot \zeta + \frac{\partial \ell}{\partial \kappa} \cdot \nabla \kappa + \frac{\partial \ell}{\partial \gamma} \cdot \nabla \gamma \quad (4.11)
\]

for the derivative of the bundle map \( \ell \) with respect to the given connection \( \nabla \), with \( \frac{\partial \nabla \ell}{\partial x} \) the derivative of \( \ell \) with respect to the base point defined with the help of \( \nabla \).

To include the reference vector field \( W \) we note that for \( w \) a vector field, we have \( \frac{\partial \ell}{\partial w} : \hat{w} = -w \otimes \frac{\partial \ell}{\partial w} \).

Since the vector field \( \zeta \) is arbitrary, we get the equations (4.10).

\[\blacksquare\]

The result stated in the above theorem still holds when material covariance is satisfied only with respect to the isotropy subgroup \( \text{Diff}_{W,K}(\mathcal{D}) \) of the material tensor fields, see Remark 3.5.
Remark 4.4 (On the form of the boundary condition). We chose to write the boundary condition in (4.10) without introducing a Lorentzian metric, since we didn’t assume the presence of a metric in the continuum theory so far. If such a metric is present then, under the hypothesis that the boundary consist of nondegenerate hypersurfaces, the unit normal vector field can be used to rewrite the boundary condition in a more concrete way. This will be considered later, see Lemma 4.7.

Spacetime covariance. We now examine the case in which the material Lagrangian density \( L \) in (3.7) is also spacetime covariant with respect to \( \text{Diff}(\mathcal{M}) \), see (3.9), in addition to the material covariance (3.14).

Lemma 4.5. Let \( \mathcal{L} : J^1(\mathcal{D} \times \mathcal{M}) \times \mathcal{T}^0 \mathcal{D} \times \mathcal{T}^1 \mathcal{M} \rightarrow \wedge^{n+1} \mathcal{D} \) be a material covariant Lagrangian density and consider the associated spacetime Lagrangian \( \ell : \mathcal{T} \mathcal{M} \times \mathcal{T}^0 \mathcal{M} \times \mathcal{T}^1 \mathcal{M} \rightarrow \wedge^{n+1} \mathcal{M} \). Then if \( \mathcal{L} \) is also spacetime covariant, we have

\[
\psi^* \left[ \ell(\psi_* w, \psi_* \kappa, \psi_* \gamma) \right] = \ell(w, \kappa, \gamma), \quad \forall \psi \in \text{Diff}(\mathcal{M}).
\]

This implies

\[
\frac{\partial \nabla \ell}{\partial x} = 0 \quad \text{and} \quad -w \otimes \frac{\partial \ell}{\partial w} + \frac{\partial \ell}{\partial \kappa} : \hat{\kappa} + \frac{\partial \ell}{\partial \gamma} : \hat{\gamma} = \ell \delta.
\]

Proof. The first assertion follows from the relation (3.15) combined with the spacetime covariance condition (3.9). Taking a path of diffeomorphisms \( \psi \in \text{Diff}(\mathcal{M}) \) passing through the identity at \( \varepsilon = 0 \) and taking the \( \varepsilon \)-derivative of the condition \( \ell(\psi_* w, \psi_* \kappa, \psi_* \gamma) = \psi^* \left[ \ell(\psi_* w, \kappa, \gamma) \right] \), we get \( \frac{\partial \ell}{\partial w} \cdot \mathcal{L}_{\varepsilon} \cdot \mathcal{L}_{\varepsilon} \cdot \mathcal{L}_{\varepsilon} = \text{div}(\mathcal{L}_{\varepsilon}) \), for the vector field \( \zeta = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \psi \), where \( \text{div}(\mathcal{L}_{\varepsilon}) \) is the divergence of the vector field density \( \mathcal{L}_{\varepsilon} \), see (4.6)–(4.7). Choosing a torsion free covariant derivative \( \nabla \) and using the formulas (4.3), \( \text{div}(\mathcal{L}) = \nabla \mathcal{L}[\mathcal{L}] + \ell \div \mathcal{L} = \nabla \mathcal{L}[\mathcal{L}] + \ell \delta : \mathcal{L} \), and (4.11), the requested identity follows since the vector field \( \zeta \) is arbitrary.

In the next corollary, we obtain that the reduced spacetime Euler-Lagrange equations can be written exclusively in terms of the partial derivative with respect to \( \gamma \), when \( \mathcal{L} \) satisfies both covariance properties. We recall that \( \gamma \) is not a variable in the spacetime description, but a given fixed parameter.

Corollary 4.6. Assume that the Lagrangian density (3.7) is material and spacetime covariant. Then the reduced Euler-Lagrange equations (4.10) are equivalently written as

\[
\begin{cases}
\text{div} \nabla \left( \frac{\partial \ell}{\partial \gamma} : \hat{\gamma} \right) = \frac{\partial \ell}{\partial \gamma} : \nabla \gamma \\
\n^*_{\partial N} \left( \text{tr} \left( \frac{\partial \ell}{\partial \gamma} : \hat{\gamma} \right) \cdot \zeta \right) = 0, \quad \forall \zeta \quad \text{on} \quad \partial N \\
\mathcal{L}_w \kappa = 0.
\end{cases}
\]

(4.12)

The local form of the boundary conditions is

\[
\left( \frac{\partial \ell}{\partial \gamma} \right)_\mu d^n x_\mu = 0.
\]
These results stated above still hold when material covariance is satisfied only with respect to the isotropy subgroup \( \text{Diff}_{W,K}(\mathcal{D}) \) of the material tensor fields, see Remark 3.5. Spacetime covariance with respect to the whole group \( \text{Diff}(\mathcal{M}) \) is however needed.

**The case of a Lorentzian metric on spacetime.** The equations obtained above hold for any spacetime tensor field \( \gamma \), not necessarily a metric. We shall consider in this paragraph the special case in which \( \gamma \) is a Lorentzian metric, denoted \( \gamma = g \), hence the Lagrangian density, is a bundle map of the form

\[
\mathcal{L} : J^1(\mathcal{D} \times \mathcal{M}) \times T\mathcal{D} \times T^0_\mathcal{D} \mathcal{D} \times S^2_L \mathcal{M} \to \wedge^{n+1} \mathcal{D},
\]

written as \( \mathcal{L}(j^1\Phi, W, K, g \circ \Phi) \).

We shall need the following result to write the boundary condition when a Lorentzian metric is given. Recall that an hypersurface \( \Sigma \) in \( \mathcal{M} \) is called nondegenerate if the induced metric \( i_\Sigma^* g \) is nondegenerate with \( i_\Sigma : \Sigma \to \mathcal{M} \) the inclusion. Given a normal vector field \( n \) to a nondegenerate hypersurface \( \Sigma \), we use the notation

\[
\epsilon = g(n, n) \in \{ \pm 1 \}.
\]

Such an hypersurface is said to be timelike, resp., spacelike, if \( i_\Sigma^* g \) is Lorentzian \((\epsilon = 1) \), resp. Riemannian \((\epsilon = -1) \).

**Lemma 4.7.** Let \( T \) be a \((1,1)\) tensor field density and assume that the piecewise smooth boundary \( \partial N \) consists of nondegenerate hypersurfaces with respect to the Lorentzian metric \( g \). Then we have

1. \( i^*_\partial N[\text{tr}(T) : \zeta] = \epsilon i^*_\partial N[i_n(T(\zeta, n^b))] \), for all \( \zeta \in \mathfrak{X}(N) \), with \( n \) a unit vector normal to \( \Sigma \) and \( \epsilon = g(n, n) \in \{ \pm 1 \} \).
2. \( i^*_\partial N[\text{tr}(T) : \zeta] = 0, \forall \zeta \in T\mathcal{N}|_{\partial N} \iff T(\zeta, n^b) = 0 \) on \( \partial N \), \( \forall \zeta \in T\mathcal{N}|_{\partial N} \).

**Proof.** We write \( T = T \otimes \mu(g) \), where \( \mu(g) \) is the volume form associated to \( g \) and \( T \) is a \((1,1)\) tensor field. To prove (1) we consider the vector field \( X = T(\zeta, \cdot) \) on \( \partial N \). We assume that the boundary consists of spacelike and timelike pieces. Let \( n \) be the outward pointing normal vector field to \( \partial N \). We decompose \( X \) as \( X = \epsilon g(X, n)n + X^\| \) with \( \epsilon = g(n, n) \). We compute

\[
i^*_\partial N[\text{tr}(T) : \zeta] = i^*_\partial N[i_{\partial \alpha} T(\zeta, dx^\alpha)] = i^*_\partial N[X(dx^\mu) i_{\partial \mu} \mu(g)] = \epsilon i^*_\partial N[i_n(T(\zeta, n^b))],
\]

for all \( \zeta \in \mathfrak{X}(N) \). To prove (2), we note that

\[
\epsilon i^*_\partial N[i_n(T(\zeta, n^b))] = \epsilon i^*_\partial N[T(\zeta, n^b)i_n \mu(g)] = \epsilon T(\zeta, n^b)|_{\partial N} i^*_\partial N(i_n \mu(g)).
\]

Since \( i^*_\partial N(i_n \mu(g)) \) is a volume form on the boundary \( i^*_\partial N[i_n(T(\zeta, n^b))] = 0, \forall \zeta \) is equivalent to \( T(\zeta, n^b)|_{\partial N} = 0, \forall \zeta \). The latter is in turn equivalent to \( T(\zeta, n^b) = 0 \) on \( \partial N \), \( \forall \zeta \in T\mathcal{N}|_{\partial N} \). \( \blacksquare \)
Corollary 4.8. Assume that the Lagrangian density $(4.13)$ is material covariant and consider the associated spacetime Lagrangian density $\ell(w, \kappa, g)$. Then the reduced Euler-Lagrange equations are

$$
\begin{align*}
\text{div} \nabla \left( \ell \delta + w \otimes \frac{\partial \ell}{\partial w} - \frac{\partial \ell}{\partial \kappa} : \mathring{\kappa} \right) &= \frac{\partial \nabla \ell}{\partial x} \\
\left( \ell \delta + w \otimes \frac{\partial \ell}{\partial w} - \frac{\partial \ell}{\partial \kappa} : \mathring{\kappa} \right)(\cdot, n^b) &= 0 \quad \text{on} \quad \partial N
\end{align*}
$$

where $\nabla$ is the Levi-Civita covariant derivative, $n$ is a unit normal vector field to $\partial N$ (assumed to consist of nondegenerate hypersurfaces), and $b$ is the flat operator, all associated of $g$.

If, in addition, the material Lagrangian density is also spacetime covariant, then these equations can be written as

$$
\text{div} \nabla \frac{\partial \ell}{\partial g} = 0, \quad \frac{\partial \ell}{\partial g}(\cdot, n^b) = 0 \quad \text{on} \quad \partial N, \quad \mathcal{L}_w \kappa = 0.
$$

Proof. When $\gamma = g$ is a Lorentzian metric, we can choose $\nabla$ as the Levi-Civita covariant derivative of $g$, so that $\nabla g = 0$. Hence the first equation in $(4.10)$ reduces to the first equation of $(4.15)$. The boundary condition is obtained by using Lemma 4.7.

In the spacetime equivariant case, we note that for $\gamma = g$, we have

$$
\left( \frac{\partial \ell}{\partial g} : \mathring{g} \right)_\mu^\nu = 2 g_{\nu \gamma} \frac{\partial \ell}{\partial g_{\gamma \mu}}
$$

from $(4.2)$. Hence, using also Lemma 4.7, $(4.12)$ becomes $(4.15)$. ■

4.3 Reduced convective Euler-Lagrange equations

In this section we derive the convective form of the Hamilton principle $(3.8)$ for relativistic continua under the assumption of spacetime covariance. We use the relation $(3.12)$ between the material Lagrangian density and its convective version as well as the definition of the material form of the given spacetime tensor field.

Theorem 4.9. Let $\mathcal{L} : J^1(\mathcal{D} \times \mathcal{M}) \times T\mathcal{D} \times T_q^p \mathcal{D} \times T_s^r \mathcal{M} \to \wedge^{n+1} \mathcal{D}$ be a spacetime covariant Lagrangian density and consider the associated covariantly reduced convective Lagrangian density $\mathcal{L} : T\mathcal{D} \times T_q^p \mathcal{D} \times T_s^r \mathcal{D} \to \wedge^{n+1} \mathcal{D}$.

Fix the reference tensor fields $W \in \mathfrak{X}(\mathcal{D})$, $K \in T_q^p (\mathcal{D})$, and the spacetime tensor field $\gamma \in T_s^r (\mathcal{M})$. For each world-tube $\Phi : \mathcal{D} \to \mathcal{M}$, define $\Gamma = \Phi^* \gamma$. Then, the following statements are equivalent:

(i) $\Phi$ is a critical point of the Hamilton principle

$$
\left. \frac{d}{d\varepsilon} \right|_{\varepsilon = 0} \int_{\mathcal{D}} \mathcal{L}(j^1 \Phi_\varepsilon, W, K, \gamma \circ \Phi_\varepsilon) = 0
$$

for arbitrary variations $\Phi_\varepsilon$. 

(ii) \( \Phi \) is a solution of the Euler-Lagrange equations
\[
\partial_a \frac{\partial \bar{L}}{\partial \Phi_a} - \partial_\mu \frac{\partial \bar{L}}{\partial \Phi_\mu} = \frac{\partial \bar{L}}{\partial \gamma} \partial_\mu \gamma, \quad \frac{\partial \bar{L}}{\partial \Phi_\mu} d^n X_a = 0.
\]

(iii) \( \Gamma \in T^r_s (\mathcal{D}) \) is a critical point of the convective variational principle
\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_\mathcal{D} \mathcal{L} (W, K, \Phi^* \gamma) = 0,
\]
for arbitrary variations \( \Phi_\varepsilon \). In particular, the variation of \( \Gamma \) is constrained of the form \( \delta \Gamma = Z \Gamma \), where \( Z = T \Phi^{-1} \circ \delta \Phi \) is an arbitrary vector field on \( \mathcal{D} \).

(iv) \( \Gamma \in T^r_s (\mathcal{D}) \) is a solution of the reduced convective Euler-Lagrange equations
\[
\begin{aligned}
\text{div} \left( \mathcal{L} \delta \mathcal{L} \right) = & \frac{\partial \mathcal{L}}{\partial \Gamma} \cdot \nabla K + \frac{\partial \mathcal{L}}{\partial K} \cdot \nabla \hat{K} + \frac{\partial \mathcal{L}}{\partial W} \cdot \nabla W + \frac{\partial \mathcal{L}}{\partial \Gamma} \cdot \nabla X \\
& \left( i_{\delta} \mathcal{L} \right) \left( \mathcal{L} \delta \mathcal{L} \right) = 0, \forall Z \text{ on } \partial \mathcal{D},
\end{aligned}
\]
written with the help of a torsion free covariant derivative \( \nabla \) on \( \mathcal{D} \). In local coordinates, denoting \( \mathcal{L} = \bar{L} d^{n+1} X \), the boundary condition reads
\[
\left( \frac{\partial \bar{L}}{\partial \Gamma} : \hat{\Gamma} \right) a^b d^n X_a = 0, \text{ on } \partial \mathcal{D}.
\]

Proof. The proof is left to the reader. It uses the results of §4.1 and is similar to that of Theorem 4.3.

The result stated in the above theorem still holds when spacetime covariance is satisfied only with respect to the isotropy subgroup \( \text{Diff}_\gamma (\mathcal{M}) \) of the spacetime tensor field, see Remark 3.3.

Material covariance. We now examine the case in which the material Lagrangian density \( \mathcal{L} \) in (3.7) is also materially covariant with respect to \( \text{Diff} (\mathcal{D}) \), see (3.14), in addition to the spacetime covariance (3.9). The proofs are left to the reader.

Lemma 4.10. Let \( \mathcal{L} : J^1 (\mathcal{D} \times \mathcal{M}) \times T^p \mathcal{D} \times T^q \mathcal{D} \times T^r \mathcal{M} \to \wedge^{n+1} \mathcal{D} \) be a spacetime covariant Lagrangian density and consider the associated convective Lagrangian density \( \mathcal{L} : T^p \mathcal{D} \times T^q \mathcal{D} \times T^r \mathcal{M} \to \wedge^{n+1} \mathcal{D} \). Then if \( \mathcal{L} \) is also material covariant, we have
\[
\varphi^* [\mathcal{L} (W, K, \Gamma)] = \mathcal{L} (\varphi^* W, \varphi^* K, \varphi^* \Gamma), \quad \forall \varphi \in \text{Diff} (\mathcal{D}).
\]

This implies
\[
\frac{\partial \mathcal{L}}{\partial X} = 0 \quad \text{and} \quad - W \otimes \frac{\partial \mathcal{L}}{\partial W} + \frac{\partial \mathcal{L}}{\partial K} \cdot \hat{K} + \frac{\partial \mathcal{L}}{\partial \Gamma} : \hat{\Gamma} = \mathcal{L} \delta.
\]
Corollary 4.11. Assume that the Lagrangian density \((3.7)\) is material and spacetime covariant. Then the reduced convective Euler-Lagrange equations \((4.16)\) are equivalently written as
\[
\begin{align*}
\text{div} \left( -W \otimes \frac{\partial L}{\partial W} + \frac{\partial L}{\partial K} : \hat{K} \right) &= \frac{\partial L}{\partial W} \cdot \nabla W + \frac{\partial L}{\partial K} \cdot \nabla K, \\
\iota_{\partial D}^* \left( \operatorname{tr} \left( L \delta + W \otimes \frac{\partial L}{\partial W} - \frac{\partial L}{\partial K} : \hat{K} \right) \cdot Z \right) &= 0, \quad \forall Z \text{ on } \partial D.
\end{align*}
\]
The local form of the boundary conditions is
\[
\left( \tilde{L} \delta + W \otimes \frac{\partial \tilde{L}}{\partial W} - \frac{\partial \tilde{L}}{\partial K} : \hat{K} \right)^{a} \frac{d^{b} X_{b}}{d^{a}} = 0.
\]
Note that, in a similar way with the case of Corollary 4.6, here \(W\) and \(K\) are not dynamic variables in the convective description, but given fixed parameters. Also, these results still hold when the spacetime covariance is satisfied only with respect to the isotropy subgroup \(\text{Diff}_{\gamma}(M)\) of the spacetime tensor fields, see Remark 3.3. Material covariance with respect to the whole group \(\text{Diff}(\mathcal{D})\) is however needed.

In the Figure below we give the analogue of Fig. 2.1 in the relativistic case.

![Image of variational principles](image)

Figure 4.1: Illustration of the variational principles in the three representations of relativistic continuum mechanics for general material and spacetime tensor fields \(K, W, \gamma\) with associated dynamic fields \(\kappa = \Phi K, w = \Phi W, \Gamma = \Phi^* \gamma\). Compare to Fig. 2.1.

5 Coupling with the Einstein equations and junction conditions

In this section, we shall show how the variational formulation for continua developed above can be coupled to the gravitation theory. In particular, the variational formulation that we develop in this section is able to produce
the field equations for the gravitational field created by the relativistic continuum, both at the interior and outside the continuum;

(2) the equations of motion of the continuum in this gravitational field;

(3) the junction conditions between the solution at the interior of the relativistic continuum and the solution describing the gravity field produced outside from it.

Lagrangian densities. In general, when the variational setting developed in §3-§4 is coupled with field theories, the spacetime tensor fields $\gamma$ that appear in the Lagrangian densities are not fixed and their dynamic is governed by Euler-Lagrange equations on spacetime associated to the variations $\delta \gamma$. We shall restrict here to the coupling with gravitation, i.e., take $\gamma = g$ a Lorentzian metric. We thus consider the Lagrangian density of the continuum as a bundle map

$$\mathcal{L} : J^1(D \times M) \times T^* D \times T^p \! \times \! S^2 L M \rightarrow \wedge^{n+1} D$$

written as $\mathcal{L}(j^1 \Phi, W, K, g \circ \Phi)$. For simplicity of the exposition we assume that $\mathcal{L}$ is materially covariant with respect to the isotropy subgroup of $(W, K)$ so that there is the associated spacetime Lagrangian density

$$\ell : TM \times T^p \! \times \! S^2 L M \rightarrow \wedge^{n+1} M \quad (5.1)$$

written as $\ell(w, \kappa, g)$, see §3.3.

Recall that the Einstein equations in the vacuum are the Euler-Lagrange equations for the Einstein-Hilbert Lagrangian

$$\ell_{EH} : J^2 S^2 L M \rightarrow \wedge^{n+1} M, \quad \ell_{EH}(j^2 g) = \frac{1}{2\chi} R(g) \mu(g), \quad (5.2)$$

defined on the second order jet bundle of $S^2 L M \rightarrow M$. Here $\chi = 8\pi G c^{-4}$ and $R(g)$ is the scalar curvature of $g$. Recall that it is defined as $R = g^{\alpha\beta} Ric_{\alpha\beta} = g^{\alpha\beta} R^{\lambda}_{\alpha\beta}$ with $R(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u, v]}$ the Riemann curvature tensor of $g$, with local coordinates $R^{\lambda}_{\alpha\beta} = \partial_\mu \Gamma^\lambda_{\alpha\beta} - \partial_\beta \Gamma^\lambda_{\alpha\mu} + \Gamma^\lambda_{\sigma\mu} \Gamma^\sigma_{\alpha\beta} - \Gamma^\lambda_{\sigma\beta} \Gamma^\sigma_{\alpha\mu}$. The spacetime covariance of $\ell_{EH}$ reads

$$\ell_{EH}(j^2 (\varphi^* g)) = \varphi^* [\ell_{EH}(j^2 g)], \quad \forall \varphi \in \text{Diff}(M). \quad (5.3)$$

We recall that the Einstein tensor field $G^{\alpha\beta} = Ric^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta}$ satisfies the Bianchi identities

$$G^{\alpha\beta ; \beta} = 0,$$

which can be seen as a direct consequence of (5.3).

Gibbons-Hawking-York boundary term. In order to couple continuum mechanics with gravitation we shall modify the Einstein-Hilbert action in order to take into account of the boundary of the spacetime $N$ occupied by the continuum. We
shall consider a well-known modification due to York [1972], Gibbons and Hawking [1977] for spacetimes with boundary, by adding the integral of the trace of the extrinsic curvature (or second fundamental form) on the boundary of the continuum.

We recall the definition and our conventions on extrinsic curvature. Consider a nondegenerate hypersurface $\Sigma \subset M$, choose a normal vector field $n$, and denote $\epsilon = g(n,n) \in \{\pm 1\}$. This choice together with the choice of an orientation of $M$ fixes the orientation of $\Sigma$. The orthogonal decomposition $TM|_\Sigma = T\Sigma \oplus T\partial\Sigma$ reads $u = u^\parallel + u^\perp$ with $u^\parallel = u - \epsilon g(u,n)n$ and $u^\perp = \epsilon g(u,n)n$. Given $u,v \in \mathfrak{X}(\Sigma)$, and an extension $\tilde{v}$ of $v$, to a neighborhood of $\Sigma$ in $M$, we have the orthogonal decomposition

$$\nabla_u \tilde{v} = (\nabla_u \tilde{v})^\parallel + (\nabla_u \tilde{v})^\perp = \nabla_u^\Sigma v - \epsilon K(u,v)n,$$

where $\nabla^\Sigma$ is the Levi-Civita covariant derivative of the (Riemannian or Lorentzian) metric $h = i_\Sigma^* g$ and $K(u,v) = -g(\nabla_u \tilde{v}, n) = g(u, \nabla_v n)$ is the extrinsic curvature of $\Sigma$. The trace of $K$ with respect to $h$ is denoted $k = \operatorname{Tr} K = h^{ab} K_{ab}$, where $h_{ab} dx^a dx^b$ is the local expression of $h$ on $\Sigma$. Note that $k$ is, up to a constant factor, the mean curvature of $\Sigma$. We shall use the notations $K(g)$ and $k(g)$ to emphasize the dependency on Lorentzian metric $g$. We shall also consider the trace $\operatorname{Tr}(K^2) = h^{ab} K_{ab} h^{cd} K_{cd}$. The Gibbons-Hawking-York (GHY) term associated to an oriented nondegenerate hypersurface $\Sigma$ is

$$\frac{1}{\chi} \int_\Sigma \epsilon k(g) \mu(i_\Sigma^* g), \quad \text{(5.4)}$$

with $\mu(i_\Sigma^* g)$ the volume form associated to $i_\Sigma^* g$ and to the orientation of $\Sigma$.

### Action functional for general relativistic continuum.

We shall denote by $N^- = \Phi(D) \subset M$ the portion of spacetime occupied by the continuum, denoted $N$ earlier, and we shall write $N^+ = M - \operatorname{int}(N^-)$. For simplicity we assume that the spacetime $M$ has no boundary and we have $\partial N^+ = \partial N^-$ assumed to be piecewise smooth. While $\partial N^+$ and $\partial N^-$ are equals as manifolds, they have opposite orientation as boundaries of $N^+$ and $N^-$, the latter having the orientation induced from that of $M$.

We denote by $g^−$ and $g^+$ the Lorentzian metrics on $N^-$ and $N^+$, assumed to be smooth. It is assumed that $g^+$ and $g^−$ induce the same metric on the boundary, i.e.,

$$i_{\partial N^−} g^− = i_{\partial N^+} g^+ =: h \quad \text{(5.5)}$$

with $i_{\partial N^±} : \partial N^{±} \to N^{±}$ the inclusions. We also assume that $h$ is nondegenerate, i.e., either Lorentzian or Riemannian, on each piece of the boundary. Condition (5.5), sometimes referred to as the preliminary junction condition, ensures that the boundary has a well-defined geometry. It is valid even in the presence of a singular matter distribution on the boundary. The remaining content of the Israel-Darmois conditions, which is sometimes referred to as the Lanczos-Israel condition, will be obtained from the variational formulation.

The total action functional is constructed by adding the action functional of the continuum with Lagrangian density (5.1) on $N^- = \Phi(D)$, the Einstein-Hilbert
action functionals on $N^-$ and $N^+$, as well as the corresponding GHY boundary terms, which yields

\[
\int_{N^-} \ell(w, \kappa, g^-) + \frac{1}{2\chi} \int_{N^-} R(g^-)\mu(g^-) + \frac{1}{2\chi} \int_{N^+} R(g^+)\mu(g^+)
\]
\[
+ \frac{1}{\chi} \int_{\partial N^-} \epsilon k(g^-)\mu^-(h) + \frac{1}{\chi} \int_{\partial N^+} \epsilon k(g^+)\mu^+(h).
\]

(5.6)

In the last two terms of (5.6), $\epsilon$ is 1, resp., $-1$, on the timelike, resp., spacelike piece of the boundary, and $\mu^\pm(h)$ is the volume form on $\partial N^\pm$ associated to $h$ and to the boundary orientation of $\partial N^\pm$.

The case when the spacetime $M$ itself has a boundary can be easily considered by adding the appropriate GHY term associated to $\partial M$.

Below we shall derive the equations as well as the boundary and junction conditions by extending the covariant reduced variational principle developed above to the action functional (5.6). The presence of the GHY is essential for this derivation, as we shall see below. While the variation of the GHY term with respect to the metric is well-known, we shall derive the variation with respect to the hypersurface, thereby extending previous results.

### 5.1 First variation of the Gibbons-Hawking-York term

In this section we consider a smooth nondegenerate hypersurface $\Sigma$ in the spacetime $(M, g)$ and denote $h = i^*_\Sigma g$ the induced metric on $\Sigma$. We assume that $\Sigma$ has a piecewise smooth boundary $\partial \Sigma$, consisting of nondegenerate pieces, with outward pointing unit normal vector field $\nu$. We write $\sigma = h(\nu, \nu) \in \{\pm 1\}$ along $\partial \Sigma$ and denote $\gamma = i^*\partial \Sigma h$ the induced metric on $\partial \Sigma$.

The orientation of $M$ and the choice of a normal vector field $n$ on $\Sigma$ determines the orientations of $\Sigma$ and of $\partial \Sigma$. We denote by $\mu(h)$ and $\mu(\gamma)$ the volume forms associated to the metrics and these orientations.

**First variation with respect to the metric.** We recall the variation of the GHY boundary term with respect to the Lorentzian metric in the Lemma below. We refer to Lehner, Myers, Poisson, and Sorkin [2016] for a detailed proof of this result, which is stated here in an intrinsic form.

**Lemma 5.1.** The first variation of the GHY term with respect to the spacetime metric $g$ is

\[
2 \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_{\Sigma} k(g)\mu(h) = \int_{\Sigma} (\mathcal{K} - kh)\delta h^{-1} - g(\delta V, n)\mu(h) - \int_{\partial \Sigma} \sigma h(\delta W, n)\mu(\gamma),
\]

where the vector fields $\delta V$ and $\delta W$ on $M$ and $\Sigma$ are defined from the variation $\delta g$ as

\[
\delta V^\mu = g^{\alpha\beta}\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\Gamma^\beta_{\alpha\beta}
\]

and

\[
\delta W = (i^*_\Sigma (i_n \delta g))^{\gamma h}.
\]

We note that in the last term $\sigma$ can take the values 1 or $-1$ on various parts of $\partial \Sigma$. 

First variation with respect to the hypersurface. We derive in this paragraph the variation of the GHY boundary terms with respect to the hypersurface. The resulting first variation formula extends earlier results obtained for the case of mean curvature of surfaces in the Euclidean space, see Dogan and Nochetto [2011] and reference therein, and in a space of constant sectional curvature, see Gruber, Toda, and Tran [2019]. Our formula shows the occurrence of the Ricci tensor of the ambient Lorentzian metric. As opposed to most of the earlier approaches, our proof is elementary, based on a direct computation.

**Theorem 5.2.** The first variation of the GHY term with respect to the hypersurface is

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_{\Sigma} k(g) \mu(h) = \int_{\Sigma} \delta \varphi^\perp (k^2 - \text{Tr}(K^2) - \text{Ric}(n,n)) \mu(h) + \int_{\partial \Sigma} \sigma h(k \delta \varphi^\parallel - \epsilon \text{grad}^\Sigma(\delta \varphi^\perp), \nu) \mu(\gamma),
\]

where \( \varphi \) is a path of diffeomorphisms of \( \mathcal{M} \) with \( \varphi_0 = \text{id} \) and we use the orthogonal decomposition

\( \delta \varphi|_\Sigma = \delta \varphi^\perp n + \delta \varphi^\parallel. \)

The proof proceeds in several lemmas. Some of the computations will be done in local charts \( \psi : U \subset \Sigma \to \psi_\Sigma(U) \subset \mathbb{R}^n \) of \( \Sigma \) and \( \psi_M : V \subset \mathcal{M} \to \psi_M(V) \subset \mathbb{R}^{n+1} \) of \( \mathcal{M} \). By appropriate restriction, we get the local representation of \( \varphi \) as

\[
\varphi_{\text{loc}} = \psi_M \circ \varphi \circ \psi_\Sigma^{-1} : \psi_\Sigma(U) \subset \mathbb{R}^n \to \psi_M(V) \subset \mathbb{R}^{n+1}.
\]

For brevity, we shall suppress the notation \((\cdot)_{\text{loc}}\) everywhere. Without loss of generality, we can assume that locally

\[
\varphi_\varepsilon(y) = \varphi(y) + \epsilon(f(y)n(\varphi(y)) + T_y \varphi(X(y))),
\]

where \( f = \delta \varphi^\perp \in C^\infty(\Sigma) \) and \( X = \delta \varphi^\parallel \in \mathfrak{X}(\Sigma) \).

**Lemma 5.3.** We have the following formulas:

1. \[
\frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} T \varphi_\varepsilon(v) = (df \cdot v)n + f \nabla_v n + \nabla_{\mathcal{D}} v X - \epsilon K(v, X)n, \text{ for all } v \in T\Sigma;
\]
2. \[
\frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} n_\varepsilon \circ \varphi_\varepsilon = -\epsilon \text{grad}^\Sigma f + (i_X K)^\sharp h \text{ on } \Sigma;
\]

where \( \nabla \) and \( \nabla^\Sigma \) are Levi-Civita covariant derivative associated to \( g \) on \( \mathcal{M} \) and to \( h \) on \( \Sigma \), and \( \frac{D}{D\varepsilon} \alpha(\varepsilon) \) denotes the covariant derivative of a curve \( \alpha(\varepsilon) \in T\mathcal{M} \) with respect to \( \nabla \).
Proof. For (1), we compute
\[
\frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} T_y\varphi_\varepsilon(v) = \frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} \frac{d}{dt} \bigg|_{t=0} \varphi_\varepsilon(y + tv) = \frac{D}{Dt} \bigg|_{t=0} \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \varphi_\varepsilon(y + tv)
\]
\[
= \frac{D}{Dt} \bigg|_{t=0} (f(y + tv)n(\varphi(y + tv)) + T_{y+tv}\varphi(X(y + tv)))
\]
\[
= (df \cdot v)n(\varphi(y)) + f(y)\nabla T_{y\varphi(v)}n + \nabla T_{y\varphi(v)} T\varphi(X)(y)
\]
\[
= (df \cdot v)n(\varphi(y)) + f(y)\nabla T_{y\varphi(v)}n + T_{y\varphi(\nabla^\Sigma X)} - \epsilon K(v, X)n(\varphi(y)).
\]
To prove (2) we first note that from the equality \(g(\varphi_\varepsilon(y))(n_\varepsilon(\varphi_\varepsilon(y)), T_y\varphi_\varepsilon(v)) = 0\), \(\forall v \in T\Sigma\), we get
\[
g(\varphi(y))\left( \frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} n_\varepsilon(\varphi_\varepsilon(y)), T_y\varphi_\varepsilon(v) \right) + g(\varphi(y))\left( n(\varphi(y)), \frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} T\varphi_\varepsilon(v) \right) = 0, \quad \forall v.
\]
By using (1) in the second term, we get
\[
g(\varphi(y))\left( \frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} n_\varepsilon(\varphi_\varepsilon(y)), T_y\varphi_\varepsilon(v) \right) = -\epsilon df \cdot v + K(v, X), \quad \forall v. \quad (5.7)
\]
From the equality \(g(\varphi_\varepsilon(y))(n_\varepsilon(\varphi_\varepsilon(y)), n_\varepsilon(\varphi_\varepsilon(y))) = \epsilon\), we get
\[
g(\varphi(y))\left( \frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} n_\varepsilon(\varphi_\varepsilon(y)), n(\varphi(y)) \right) = 0
\]
hence we can write \(\frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} n_\varepsilon(\varphi_\varepsilon(y)) = T_y\varphi(W(y))\) for some vector field \(W\) on \(\Sigma\). Inserting this equality in (5.7) and using \(g(\varphi(y))(T_y\varphi(W(y)), T_y\varphi(v)) = h(y)(W(y), v)\),
we get the equality claimed in (2).

Lemma 5.4. The variation of the induced metric \(h = \frac{i}{\varepsilon}g\) and associated volume \(\mu(h)\) with respect to the hypersurface are
\[
\begin{enumerate}
\item \(\delta h = 2fK + \mathcal{L}Xh;\)
\item \(\delta(\mu(h)) = (fk + \text{div}^\Sigma X) \mu(h).\)
\end{enumerate}
\]
Proof. To prove (1), we note that for all \(u, v \in T\Sigma\), we have
\[
\delta h(y)(u, v) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} g(\varphi_\varepsilon(y))(T_y\varphi_\varepsilon(u), T_y\varphi_\varepsilon(v))
\]
\[
= g(\varphi(y))\left( \frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} T_y\varphi_\varepsilon(u), T_y\varphi_\varepsilon(v) \right)
\]
\[
+ g(\varphi(y))\left( T_y\varphi(u), \frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} T_y\varphi_\varepsilon(v) \right).
\]
Using Lemma 5.3 (1), we get
\[
\delta h(y)(u, v) = 2f(y)K(y)(u, v) + h(y)(\nabla^\Sigma u, v) + h(y)(u, \nabla^\Sigma v)
\]
\[
= 2f(y)K(y)(u, v) + \mathcal{L}Xh(u, v).
\]
To prove (2), we first use (1) and \(h^{ab} \mathcal{L}_X h_{ab} = h^{ab} h_{cb} X^c_{fa} + h^{ab} h_{ac} X^c_{fb} = 2 \text{div}^\Sigma X\) to compute

\[
\delta \text{det} h = (\text{det} h) h^{ab} \delta h_{ab} = (\text{det} h) h^{ab} (2f K_{ab} + (\mathcal{L}_X h)_{ab}) = 2(\text{det} h)(f k + \text{div}^\Sigma X).
\]

Then, for \(\text{det}(h) > 0\), we have

\[
\delta \sqrt{\text{det}(h)} = \frac{1}{2\sqrt{\text{det}(h)}} 2(\text{det} h)(f k + \text{div}^\Sigma X) = \sqrt{\text{det}(h)}(f k + \text{div}^\Sigma X),
\]

from which (2) follows, similarly for \(\text{det}(h) < 0\).

**Lemma 5.5.** The variation, with respect to the hypersurface, of the extrinsic curvature \(K\) and its trace \(k\) is

1. \(\delta K = f K^2 - f R(n, n, n, n) - \epsilon \nabla^\Sigma d f + \mathcal{L}_X K;\)
2. \(\delta k = -f \text{Tr}_h(K^2) - f \text{Ric}(n, n) - \epsilon \Delta^\Sigma f + d k \cdot X;\)

where \(K^2_{ab} = K_{ac} h^{ef} K_{fb}\).

**Proof.** (1) From the definition of the extrinsic curvature

\[
K(y)(u, v) = g(\varphi(y)) \left( T_y \varphi(u), \nabla_{T_y \varphi(v)} n \right) = g(\varphi(y)) \left( T_y \varphi(u), \frac{D}{Dt} \bigg|_{t=0} n(\varphi(y + tv)) \right),
\]

we get

\[
\delta K(y)(u, v) = \frac{d}{d \varepsilon} \bigg|_{\varepsilon=0} g(\varphi_\varepsilon(y)) \left( T_y \varphi_\varepsilon(u), \frac{D}{Dt} \bigg|_{t=0} n_\varepsilon(\varphi_\varepsilon(y + tv)) \right)
\]

\[
= g(\varphi(y)) \left( T_y \varphi_0(u), \frac{D}{Dt} \bigg|_{t=0} n_\varepsilon(\varphi(\varepsilon + tv)) \right) \quad (5.8)
\]

1. We start by treating the first term in (5.8). Since the vector field in the second slot is parallel to \(\Sigma\), by Lemma 5.3 (1) the first term in (5.8) can be written as

\[
g(\varphi(y)) \left( f(y) \nabla_{T_y \varphi(u)} n + T_y \varphi(\nabla_u^\Sigma X), \nabla_{T_y \varphi(v)} n \right) = f(y)K^2(u, v) + K(\nabla_u^\Sigma X, v). \quad (5.9)
\]

The equality above is obtained from \(g(\varphi(y)) \left( \nabla_{T_y \varphi(\partial_a)} n, \nabla_{T_y \varphi(\partial_b)} n \right) = K_{ad} h^{df} K_{fb}\), which follows from the fact that the vector \(\nabla_{T_y \varphi(\partial_b)} n\) is parallel to \(\Sigma\) and can be written as \(\nabla_{T_y \varphi(\partial_b)} n^\alpha = \varphi_b^a h^{ab} K_{ca}\) by definition of \(K\).

1. To treat the second term in (5.8), we use

\[
\frac{D}{Dt} \frac{D}{D\varepsilon} \alpha(\varepsilon, t) - \frac{D}{Dt} \frac{D}{D\varepsilon} \alpha(\varepsilon, t) = R \left( \frac{d}{d\varepsilon} \gamma(\varepsilon, t), \frac{d}{Dt} \gamma(\varepsilon, t) \right) \alpha(\varepsilon, t)
\]
for $\alpha(\varepsilon, t) \in T_{\gamma(\varepsilon, t)}\mathcal{M}$ and we compute

\[
\frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} \frac{D}{Dt} \bigg|_{t=0} n_{\varepsilon}(\varphi_{\varepsilon}(y + tv))
= \frac{D}{Dt} \bigg|_{t=0} \frac{D}{D\varepsilon} \bigg|_{\varepsilon=0} n_{\varepsilon}(\varphi_{\varepsilon}(y + tv)) + R\left(\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \varphi_{\varepsilon}(y), \frac{d}{dt} \bigg|_{t=0} \varphi(y + tv)\right) n(\varphi(y))
= \frac{D}{Dt} \bigg|_{t=0} T_y \varphi \left(\left( -\varepsilon \nabla^\Sigma f + (i_X K)^{zh} \right)(y + tv) \right)
+ R(f(y)n(\varphi(y)) + T_y \varphi(X(y)), T_y \varphi(v)) n(\varphi(y)).
\]

From this equality, we can rewrite the second term in (5.8) as

\[
g(\varphi(y))\left( T_y \varphi(u), T_y \varphi \left( \frac{D}{Dt} \bigg|_{t=0} \left( -\varepsilon \nabla^\Sigma f + (i_X K)^{zh} \right)(y + tv) \right) \right)
+ R(\varphi(y))(T_y \varphi(u), n(\varphi(y)), f(y)n(\varphi(y)) + T_y \varphi(X(y)), T_y \varphi(v))
= h(y) \left( u, \nabla^\Sigma_v (-\varepsilon \nabla^\Sigma f + (i_X K)^{zh})(y) \right)
+ f(y)R(\varphi(y))(T_y \varphi(u), n(\varphi(y)), n(\varphi(y)), T_y \varphi(v))
+ R(\varphi(y))(T_y \varphi(u), n(\varphi(y)), T_y \varphi(X(y)), T_y \varphi(v)),
\]

where $R(u, v, w, z) = g(u, R(w, z)v)$. We note that the first term can be written as

\[-\varepsilon \left< \nabla^\Sigma_v df, u \right> + \nabla^\Sigma_v K(X, u) + K(\nabla^\Sigma_v X, u),\]

while from the Gauss-Codazzi equation, the last term can be written as $\nabla^\Sigma_X K(u, v) - \nabla^\Sigma_v K(u, X)$. From this, the second term of (5.8) can be written as

\[-\varepsilon \left< \nabla^\Sigma_v df, u \right> - f(y)R(\varphi(y))(n(\varphi(y)), T_y \varphi(u), n(\varphi(y)), T_y \varphi(v))
+ K(\nabla^\Sigma_v X, u) + \nabla^\Sigma_X K(u, v).\]

By summing the results obtained in (5.9) and (5.10), we get the statement claimed in (1).

(2) We have $\delta k = \delta(h^{ab} K_{ab}) = \delta h^{ab} K_{ab} + h^{ab} \delta K_{ab}$. By Lemma 5.4 (1) the first term is

\[
\delta h^{ab} K_{ab} = -h^{ad} \delta h_{dc} h^{cb} K_{ab} = -h^{ad} 2 f K_{dc} h^{cb} K_{ab} - h^{ad}(\mathcal{L}_X h_{dc}) h^{cb} K_{ab}
= -2 f Tr h(K^2) + \mathcal{L}_X h^{-1} : K,
\]

where we used $\mathcal{L}_X h^{-1} : h + h^{-1} : \mathcal{L}_X h = 0$. By Lemma 5.5 (1) the second term is

\[
h^{ab} \delta K_{ab} = f(y) \text{Tr} h(K^2) - f(y)Ric(n(\varphi(y)), n(\varphi(y))) - \varepsilon \Delta^\Sigma f + h^{-1} : \mathcal{L}_X K,
\]

where we used

\[
h^{ab} R(\varphi(y))(n(\varphi(y)), T_y \varphi(\partial_a), n(\varphi(y)), T_y \varphi(\partial_b)) = Ric(n(\varphi(y)), n(\varphi(y))).
\]

The final result follows from $h^{-1} : \mathcal{L}_X K + \mathcal{L}_X h^{-1} : \mathcal{L}_X K = \mathcal{L}_X (h^{-1} : K) = d k \cdot X.$
5.2 Variational formulation for relativistic continuum coupled to gravity

Proof of Theorem 5.2. From Lemma 5.5 (2) and Lemma 5.4 (2), we have
\[ \delta(k\mu(h)) = \delta k\mu(h) + k\delta(\mu(h)) \]
\[ = -f \operatorname{Tr}_h(K^2)\mu(h) - f \operatorname{Ric}(n(\varphi(y)), n(\varphi(y)))\mu(h) \]
\[ - \epsilon \Delta^\Sigma f\mu(h) + d\kappa \cdot X\mu(h) + k(fk + \operatorname{div}^\Sigma X)\mu(h) \]
\[ = f \left( k^2 - \operatorname{Tr}_h(K^2) - \operatorname{Ric}(n(\varphi(y)), n(\varphi(y))) \right)\mu(h) \]
\[ + \operatorname{div}^\Sigma(-\epsilon \operatorname{grad}^\Sigma f + kX)\mu(h). \]

The result then follows from the divergence theorem
\[ \int_{\Sigma} \operatorname{div}^\Sigma X\mu(h) = \int_{\partial\Sigma} \sigma h(X, \nu)\mu(\gamma), \]
for \( X \in \mathcal{X}(\Sigma) \), where we used that \( \partial\Sigma \) consists of nondegenerate pieces with outward pointing unit vector field \( \nu \), and where \( \sigma = h(\nu, \nu) \in \{ \pm 1 \} \) on \( \partial\Sigma \). □

5.2 Variational formulation for relativistic continuum coupled to gravity

From the developments of the previous section, we can now state and prove our main result concerning the variational formulation for relativistic continuum coupled to gravitation, based on the action functional given in (5.6). We shall consider \( D = [a, b] \times B \) where \( B \) has a smooth boundary and we assume that \( \Phi(a, B) \) and \( \Phi(b, B) \) are spacelike hypersurfaces, while \( \Phi([a, b], \partial B) \) is timelike since \( \Phi \) is a world-tube. In particular, the boundary does not contain null hypersurfaces and no contribution arises from the codimension-two surfaces at which the boundary pieces are joined together in our setting, see Remark 5.8 for these delicate issues.

As explained earlier, we assume that the Lorentzian metrics \( g^- \) and \( g^+ \) on which the action functional is evaluated satisfy the preliminary junction condition (5.5) and we derive the Lanczos-Israel junction condition as a critical point condition. It is also possible to obtain (5.5) from the variational formulation without assuming it a priori, by including the corresponding Lagrange multiplier term.

Theorem 5.6. Let \( \mathcal{L} : J^1(\mathcal{D} \times \mathcal{M}) \times T^D \times T_p^\mathcal{D} \times S^2_L \mathcal{M} \to \wedge^{n+1} \mathcal{D} \) be the Lagrangian density of the continuum assumed to be material and spacetime covariant, and consider the associated spacetime Lagrangian density \( \ell : TM \times T^p_D \mathcal{M} \times S^2_L \mathcal{M} \to \wedge^{n+1} \mathcal{M} \).

Fix the reference tensor fields \( W \in \mathcal{X}(\mathcal{D}) \) and \( K \in T_q^D(\mathcal{D}) \). For each world-tube \( \Phi : \mathcal{D} \to \mathcal{M} \), define \( N^- = \Phi(\mathcal{D}) \), \( N^+ = \mathcal{M} - \text{int}(N^-) \), \( w = \Phi_* W \), and \( \kappa = \Phi_* K \).

Consider smooth Lorentzian metrics \( g^\pm \in S^2_L(N^\pm) \) such that \( i^{\partial N^-}_* g^- = i^*_{\partial N^+} g^+ \).

Then the following statements are equivalent:

(i) \( w \in \mathcal{X}(N^-), \kappa \in T_q^D(N^-) \), \( g^\pm \in S^2_L(N^\pm) \) are critical points of the Eulerian variational principle
\[ \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \left[ \int_{N^-} \ell(w_\varepsilon, \kappa_\varepsilon, g^-_\varepsilon) + \frac{1}{2\chi} \int_{N^-} R(g^-_\varepsilon)\mu(g^-_\varepsilon) + \frac{1}{2\chi} \int_{N^+_\varepsilon} R(g^+_\varepsilon)\mu(g^+_\varepsilon) \right] \]
\[ + \frac{1}{\chi} \int_{\partial N^-_\varepsilon} \epsilon k(g^-_\varepsilon)\mu^-(h_\varepsilon) + \frac{1}{\chi} \int_{\partial N^+_\varepsilon} \epsilon k(g^+_\varepsilon)\mu^+(h_\varepsilon) \bigg] = 0, \] (5.11)
for arbitrary variations Φε and g± with \( i_{\partial N^\pm} g_\pm = i_{\partial N^\pm} g_\pm \), where \( N^- = \Phi_\varepsilon(\mathbb{D}) \), \( w_\varepsilon = (\Phi_\varepsilon), W \), \( \kappa_\varepsilon = (\Phi_\varepsilon), K \), such that \( \delta \Phi(a, \mathcal{B}) = \delta \Phi(b, \mathcal{B}) = 0 \) and \( \delta g|_{\Phi(a, \mathcal{B})} = \delta g|_{\Phi(b, \mathcal{B})} = 0 \). In particular the variations of \( w \) and \( \kappa \) are constrained of the form \( \delta w = -\mathcal{L}_\zeta w \) and \( \delta \kappa = -\mathcal{L}_\zeta \kappa \) where \( \zeta = \delta \Phi \circ \Phi^{-1} \) is an arbitrary vector field on \( N \) such that \( \zeta|_{\Phi(a, \mathcal{B})} = \zeta|_{\Phi(b, \mathcal{B})} = 0 \).

(ii) \( w \in \mathcal{X}(N^-) \), \( \kappa \in \mathcal{T}_q(N^-) \), \( g^\pm \in \mathcal{S}_L^2(N^\pm) \) satisfy

\[
\begin{align*}
\text{div} \left( \frac{\partial \ell}{\partial g^-} \right) &= 0, \quad \mathcal{L}_w \kappa = 0 \quad \text{on} \quad N^- \\
G(g^-) \mu(g^-) &= 2 \chi \frac{\partial \ell}{\partial g^-} \quad \text{on} \quad N^- \\
G(g^+) &= 0 \quad \text{on} \quad N^+ \\
[K] &= 0 \quad \text{on} \quad \partial N,
\end{align*}
\]

(5.12)

where \( 2 g^- \cdot \frac{\partial \ell}{\partial g^-} = \ell \delta + w \otimes \frac{\partial \ell}{\partial g^-} - \frac{\partial \ell}{\partial \kappa} : \hat{\kappa} \) and where \( [K] \) denotes the jump of the extrinsic curvature along \( \partial N \).

Proof. (1) Metric variation: We first fix the world-tube and take the variation with respect to the Lorentzian metrics \( g^\pm \). The variation of the Einstein-Hilbert terms is well-known and given by

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \left[ \frac{1}{2\chi} \int_{N^-} R(g^-) \mu(g^-) + \frac{1}{2\chi} \int_{N^+} R(g^+) \mu(g^+) \right] \\
= -\frac{1}{2\chi} \int_{N^-} G(g^-) : \delta g^- \mu(g^-) - \frac{1}{2\chi} \int_{N^+} G(g^+) : \delta g^+ \mu(g^+) \\
+ \frac{1}{2\chi} \int_{\partial N^-} \epsilon g^- (\delta V^-, n^-) \mu^-(h) + \frac{1}{2\chi} \int_{\partial N^+} \epsilon g^+ (\delta V^+, n^+) \mu^+(h).
\]

On the other hand, denoting \( \partial_p N^\pm \), \( p = 1, 2, \ldots \), the smooth pieces of the boundary \( \partial N^\pm \) and using Lemma 5.1 with \( \Sigma = \partial_p N^\pm \), the variation of the GHY terms is

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \left[ \frac{1}{\chi} \int_{\partial N^-} \epsilon k(g^-) \mu^-(h) + \frac{1}{\chi} \int_{\partial N^+} \epsilon k(g^+) \mu^+(h) \right] \\
= \frac{1}{2\chi} \int_{\partial N^-} ((K(g^-) - k(g^-) h) : \delta h^{-1} - g^- (\delta V^-, n^-) \mu^-(h) \\
+ \frac{1}{2\chi} \int_{\partial N^+} ((K(g^+) - k(g^+) h) : \delta h^{-1} - g^+ (\delta V^+, n^+) \mu^+(h) \\
- \frac{1}{2\chi} \sum_p \int_{\partial_p N^-} \sigma_p h(\delta W^-, \nu_p) \mu^-(\gamma) - \frac{1}{2\chi} \sum_p \int_{\partial_p N^+} \sigma_p h(\delta W^+, \nu_p) \mu^+(\gamma),
\]

where \( \nu_p \) is the outward pointing unit normal vector field to \( \partial_p N^\pm \) and \( \sigma_p = h(\nu_p, \nu_p) \in \{\pm\} \) on \( \partial(\partial_p N^\pm) \).
Note that the terms involving $\delta V$ cancel and hence, the critical conditions of (5.11) with respect to $\delta g^-, \delta g^+, \delta h$ are, respectively:

\[
\frac{\partial \ell}{\partial g^-} = \frac{1}{2\chi} G(g^-)\mu(g^-) \quad \text{on } N^-
\]
\[
0 = \frac{1}{2\chi} G(g^+)\mu(g^+) \quad \text{on } N^+
\]
\[
0 = K(g^-) - k(g^-)h + K(g^+) - k(g^+)h \quad \text{on } \partial N.
\]

The last condition is obtained by noting that $\mu^-(h) = -\mu^+(h)$ and $\partial N^-$ and $\partial N^+$ have opposite orientation. Since $K(g^+)$ and $k(g^+)$ are computed with respect to the orientation of $\partial N^+$, similarly for $K(g^-)$, $k(g^-)$ and $\partial N^-$, choosing a common orientation on $\partial N$, this condition reads $[K - kh] = 0$ or, equivalently, $[K] = 0$.

(2) World-tube variation: The world-tube cannot be a priori varied independently of the metric. Let us define $\zeta = \delta \Phi \circ \Phi^{-1}$ and write the orthogonal decomposition $\zeta = \zeta^\pm n^\pm + \zeta^\parallel$ along $\partial \Sigma$ with $\zeta^\pm = \epsilon g(\zeta, n^\pm)$. Taking the variation of the condition $i^\pm_{\partial N^-} g^- = i^\pm_{\partial N^+} g^+$ we get

\[
2\zeta^\pm K(g^-) + L_{\zeta^\parallel} i^\pm_{\partial N^-} g^- + i^\pm_{\partial N^-} \delta g^- = 2\zeta^\pm K(g^+) + L_{\zeta^\parallel} i^\pm_{\partial N^+} g^+ + i^\pm_{\partial N^+} \delta g^+ \quad \text{from Lemma 5.4.}
\]

From the junction condition already derived and from $i^\parallel_{\partial N^-} g^- = i^\parallel_{\partial N^+} g^+$, we obtain that at the critical condition already derived, $\delta \Phi$ can be varied independently of $g^\pm$.

We now take the variation of (5.11) with respect to the world-tube. As seen in the proof of Theorem 4.3, the variation of the term associated to the continuum is

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \int_{N^\varepsilon} \ell(w, \kappa, \delta, g^-) = -\int_{N^-} \left( \text{div}^\nabla \left( \ell \delta + w \otimes \frac{\partial \ell}{\partial w} - \frac{\partial \ell}{\partial \kappa} : \hat{\kappa} \right) \cdot \zeta - \frac{\partial \nabla \ell}{\partial x} \cdot \zeta \right)
\]

\[
+ \int_{\partial N^-} \text{tr} \left( \ell \delta + w \otimes \frac{\partial \ell}{\partial w} - \frac{\partial \ell}{\partial \kappa} : \hat{\kappa} \right) \cdot \zeta,
\]

with $\nabla$ the Levi-Civita covariant derivative associated to $g^-$ and $\zeta = \delta \Phi \circ \Phi^{-1}$.

The variation of the Einstein-Hilbert terms with respect to the world-tube is

\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \left[ \frac{1}{2\chi} \int_{N^\varepsilon} R(g^-)\mu(g^-) + \frac{1}{2\chi} \int_{N^\varepsilon} R(g^+)\mu(g^+) \right]
\]

\[
= \frac{1}{2\chi} \int_{\partial N^-} R(g^-) \zeta^\parallel \mu^- (h) + \frac{1}{2\chi} \int_{\partial N^+} R(g^+) \zeta^\parallel \mu^+ (h).
\]
Finally, using Theorem 5.2 with $\Sigma = \partial_p N^\pm$, the variation of the GHY terms yields
\[
\frac{d}{ds}\bigg|_{s=0} \left[ \frac{1}{\chi} \int_{\partial N^-} \epsilon k(g^-) \mu^- (h) + \frac{1}{\chi} \int_{\partial N^+} \epsilon k(g^+) \mu^+ (h) \right]
\]
\[
= \frac{1}{\chi} \int_{\partial N^-} \epsilon \zeta^- [k(g^-)^2 - Tr(K(g^-)^2) - Ric^-(n^-,n^-)] \mu^- (h)
\]
\[
+ \frac{1}{\chi} \sum_p \int_{\partial(\partial_p N^-)} \epsilon \sigma_p h(k(g^-))\zeta^- - \epsilon \text{grad}^{\partial N}(\zeta^-,\nu_p) \mu^- (\gamma)
\]
\[
+ \frac{1}{\chi} \int_{\partial N^+} \epsilon \zeta^+ [k(g^+)^2 - Tr(K(g^+)^2) - Ric^+(n^+,n^+)] \mu^+ (h)
\]
\[
+ \frac{1}{\chi} \sum_p \int_{\partial(\partial_p N^+)} \epsilon \sigma_p h(k(g^+))\zeta^+ - \epsilon \text{grad}^{\partial N}(\zeta^+,\nu_p) \mu^+ (\gamma),
\]
where we used the same notations $\partial_p N^\pm$, $\nu_p$, $\sigma_p$, as earlier.

By grouping these results, we get the following conditions
\[
\text{div}^N \left( \ell \delta + w \otimes \frac{\partial \ell}{\partial w} - \frac{\partial \ell}{\partial N} : \widehat{\kappa} \right) = \frac{\partial^N \ell}{\partial x} \quad \text{on} \quad N^-
\]
and, using Lemma 4.7 (1),
\[
\epsilon i_{n^-} \left( \ell \delta + w \otimes \frac{\partial \ell}{\partial w} - \frac{\partial \ell}{\partial N} : \widehat{\kappa} \right) (\zeta^-,n_-)
\]
\[
+ \frac{1}{2\chi} R(g^-)\zeta^- \mu^- (h) - \frac{1}{2\chi} R(g^+)\zeta^+ \mu^+ (h)
\]
\[
+ \frac{1}{\chi} \epsilon \zeta^+ [k(g^-)^2 - Tr(K(g^-)^2) - Ric^-(n^-,n^-)] \mu^- (h)
\]
\[
- \frac{1}{\chi} \epsilon \zeta^+ [k(g^+)^2 - Tr(K(g^+)^2) - Ric^+(n^+,n^+)] \mu^+ (h) = 0, \forall \zeta \quad \text{on} \quad \partial N^-.
\] (5.16)

By using $\zeta^+ = -\zeta^-, \mu^+(h) = -\mu^-(h)$, and the junction condition $[K] = 0$ already obtained above, see (5.15), condition (5.16) is equivalently written as
\[
\epsilon i_{n^-} \left( \ell \delta + w \otimes \frac{\partial \ell}{\partial w} - \frac{\partial \ell}{\partial N} : \widehat{\kappa} \right) (\zeta^-,n_-)
\]
\[
+ \frac{1}{\chi} \left( G(g^+)(n^+,n^+) - G(g^-)(n^-,n^-) \right) \epsilon \zeta^- \mu^- (h) = 0, \forall \zeta \quad \text{on} \quad \partial N.
\] (5.17)

From the spacetime equivariance of $\ell$ we have
\[
\ell \delta + w \otimes \frac{\partial \ell}{\partial w} - \frac{\partial \ell}{\partial N} : \widehat{\kappa} = 2g^- \cdot \frac{\partial \ell}{\partial g^-} \quad \text{and} \quad \frac{\partial^N \ell}{\partial x} = 0,
\]
see Lemma 4.5 with $\gamma = g^-$. Hence, using (5.13) and (5.14), the condition (5.17) reads
\[
G(g^-)(\zeta,n^-) \mu^- (h) - G(g^-)(n^-,n^-) \zeta^- \mu^- (h) = 0, \forall \zeta \quad \text{on} \quad \partial N.
\]
This is equivalent to
\[
i^*_{\partial N} (i_n^- G(g^-)) = 0.
\]
which can also be written as \( i_{\partial N} (i_{n} - [G]) = 0 \) since \( G(g^+) = 0 \). It turns out that this condition is a consequence of the junction condition already derived, as it is easily seen by using the Gauss-Codazzi equation, see also Remark 5.7.

So far we haven’t put any restriction on the variations and on the domain \( N^- \) except that its boundaries \( \partial p N^\pm \) are piecewise nondegenerate, with \( \partial (\partial p N^\pm) \) also nondegenerate, see Remark 5.8. We now use that \( D = [a,b] \times B \) where \( B \) has a smooth boundary, and also use \( \delta \Phi(a,B) = \delta \Phi(b,B) = 0 \) and \( \delta g|_{\Phi(a,B)} = \delta g|_{\Phi(b,B)} = 0 \). Hence the integrals over the codimension-two surfaces \( \partial (\partial p N^\pm) \) in (1) and (2) vanish.

\[ \square \]

Remark 5.7 (On the O’Brien-Synge junction conditions). From the Gauss-Codazzi equations on a nondegenerate hypersurface \( \Sigma \), one obtains the equalities \( G(n,n) = -\frac{1}{2}(\epsilon R^\Sigma + \text{Tr}(K^2) - k^2) \) and \( G(\zeta,n) = \nabla_{\partial_\lambda} K(\zeta,\partial_\lambda) h^{ab} - \nabla_\zeta k \) for all \( \zeta \in \mathfrak{X}(\Sigma) \). Hence the Darmois-Israel junction conditions \([h] = [K] = 0\) imply the conditions \([G(n,n)] = [G(\zeta,n)] = 0\) for all \( \zeta \in \mathfrak{X}(\Sigma) \), which are a generalized form of the O’Brien-Synge conditions, see Israel [1966] and O’Brien and Synge [1952].

Remark 5.8 (Null hypersurfaces and nonsmooth intersection). In our setting we have assumed that the boundaries include only nondegenerate hypersurfaces and that no contribution arises from the codimension-two surfaces at which the boundary pieces are joined together. On null hypersurfaces the GHY term is ill-defined and an appropriate replacement for the boundary contribution has to be considered in the Einstein-Hilbert action, Neiman [2012], Parattu, Chakraborty, Majhi, and Padmanabhan [2016]. Also, when the intersection between two pieces of the boundary is not smooth, the extrinsic curvature is singular and the boundary action acquires additional contributions from the intersection, Hartle and Sorkin [1981], Hayward [1993]. We refer to Lehner, Myers, Poisson, and Sorkin [2016] for a complete treatment of these situations.

6 Examples

We show in this section how the reduced variational framework proposed above applies to relativistic fluid and elasticity. It should be noted that this setting allows to obtain the complete system of field equations by a variational principle directly deduced from the Hamilton principle, by using the covariance assumptions and without the need to include any constraints or unphysical variables. In the case of relativistic elasticity, our setting also allows to clarify the relation between formulations based on the relativistic right Cauchy-Green tensor or on the relativistic Cauchy deformation tensor. For the examples below, we take \( D = [a,b] \times B \) and \( W = \partial_\lambda \).

6.1 General relativistic fluids

Material tensor fields. For the description of a relativistic fluid, besides the vector field \( \partial_\lambda \in \mathfrak{X}(D) \), two other material tensor fields are needed, given by volume
forms $R, S \in \Omega^{n+1}(\mathcal{D})$ with
\[
\mathcal{L}_{\partial_\lambda}^{} R = \mathcal{L}_{\partial_\lambda}^{} S = 0. \tag{6.1}
\]
These two forms correspond to the reference mass density and entropy density and are chosen as
\[
R = d\lambda \wedge \pi_0^* R_0 \quad \text{and} \quad S = d\lambda \wedge \pi_0^* S_0, \tag{6.2}
\]
where $R_0, S_0 \in \Omega^n(\mathcal{B})$ are volume forms on $\mathcal{B}$, called the mass form and entropy form. As shown in Remark 3.1, properties (6.1) hold for $R, S$ given in (6.2).

The corresponding Eulerian quantities are the generalized velocity $w$, generalized mass density $\varrho$, and generalized entropy density $\varsigma$ given by:
\[
w = \Phi_* W, \quad \varrho = \Phi_* R, \quad \varsigma = \Phi_* S.
\] (6.3)

From these quantities, the world-velocity, the proper mass and entropy densities, and the proper specific entropy are defined as
\[
u = \frac{cw}{\sqrt{-g(w,w)}}, \quad \rho = \frac{\sqrt{-g(w,w)} \varrho}{c \mu(g)}, \quad s = \frac{\sqrt{-g(w,w)} \varsigma}{c \mu(g)}, \quad \eta = \frac{s}{\rho} = \frac{\varsigma}{\varrho}. \tag{6.3}
\]

The distinction between the generalized and proper densities plays a central role in our approach and it arises since we are using world-tubes that do not necessarily satisfy the normalisation condition $\sqrt{-g(\dot{\Phi}, \dot{\Phi})} = c$. As we already commented earlier, this allows to formulate the variational principle in the material picture as a standard Hamilton principle, without any constraints.

Lagrangian densities for relativistic fluids. In the setting described above, assuming spacetime and material covariance, the corresponding Lagrangian densities are functions of the form
\[
\mathcal{L}^{}_{\Phi_* \partial_\lambda} R, S, g \circ \Phi), \quad \mathcal{L}^{}_{\partial_\lambda} R, S, \Gamma), \quad \ell^{}(w, \varrho, \varsigma, g).
\]

For a fluid with specific internal energy given by a function $e(\rho, \eta)$, the Lagrangian density is usually given in the Eulerian form, and is defined as (minus) the sum of the proper rest-mass energy density and proper internal energy density, namely
\[
\ell^{}(w, \varrho, \varsigma, g) = -\rho(c^2 + e(\rho, \eta))\mu(g)
\]
\[
= -\frac{1}{c} \sqrt{-g(w,w)} \left( c^2 + e\left( \frac{1}{c} \sqrt{-g(w,w)} \frac{\varrho}{\mu(g)}, \frac{\varsigma}{\varrho} \right) \right) \varrho, \tag{6.4}
\]

where in the second equality we used the expression of $\rho$ and $\eta$ in terms of $w, \varrho, \varsigma$ given in (6.3). From this, the material Lagrangian density is found from the relation
\[
\mathcal{L}^{}_{\Phi_* \partial_\lambda} R, S, g \circ \Phi) = \Phi^*[\ell^{}(w, \varrho, \varsigma, g)], \tag{6.5}
\]
with $w = \Phi_* \partial_\lambda, \varrho = \Phi_* R, \varsigma = \Phi_* S$ which gives
\[
\mathcal{L}^{}_{\Phi_* \partial_\lambda} R, S, g \circ \Phi) = -\frac{1}{c} \sqrt{-g(\dot{\Phi}, \dot{\Phi})} \left( c^2 + e\left( \frac{1}{c} \sqrt{-g(\dot{\Phi}, \dot{\Phi})} \frac{R}{\Phi^*[\mu(g)]}, \frac{S}{R} \right) \right) R. \tag{6.6}
\]
This Lagrangian is automatically materially covariant with respect to the isotropy subgroup of $\partial_\lambda$, see (3.17). The extension of $\mathcal{L}$ to arbitrary non vanishing vector fields $W$ with $\Gamma(W,W) < 0$ is found by writing $w = \Phi_*W$ in the relation (6.5) above, from which we deduce that $\mathcal{L}(j^1\Phi, W, R, S, g \circ \Phi)$ is given by the same expression (6.6) with $\Phi$ replaced by $T\Phi W$. Now, from its definition, this expression is automatically materially covariant under the whole group $\text{Diff}(\mathcal{D})$. Spacetime covariance is easily checked and leads to the convective Lagrangian density

$$\mathcal{L}(W, R, S, \Gamma) = -\frac{1}{c^2} \sqrt{-\Gamma(W,W)} \left( c^2 + e \left( \frac{1}{c^2} \sqrt{-\Gamma(W,W)} \frac{R}{\mu(\Gamma)}, S \right) \right).$$

**Remark 6.1 (Material Lagrangian, relativistic dust, and the relativistic particle).** The material Lagrangian for relativistic fluid given in (6.6) hasn’t been apparently considered earlier. This expression elegantly relates the Lagrangian for fluids to that of the relativistic particle. For instance, for the relativistic dust ($e = 0$), using (6.2), it takes the form

$$\mathcal{L}(\Phi_0, \partial_\lambda, R, g \circ \Phi) = -c \sqrt{-g(\Phi_0, \Phi_0)} \, d\lambda \wedge R_0.$$

This Lagrangian is nothing else than a continuum collection of the Lagrangian density for the relativistic particle

$$\mathcal{L}(x, \dot{x}) = -c \sqrt{-g(x, x)} \, d\lambda,$$

to which it reduces when $\mathcal{B}$ is just a point $\mathcal{B} = \{X\}$.

**Spacetime reduced Euler-Lagrange equations for fluids.** The results stated in Theorem 4.3 and 4.9 are directly applicable to the relativistic fluids. In particular, we get the following Eulerian variational principle and general forms of the spacetime reduced Euler-Lagrange equations by direct application of (4.9) and (4.14).

**Proposition 6.2.** The Eulerian variational formulation for relativistic fluid takes the form

$$\frac{d}{dz} \bigg|_{z=0} \int_{N_\epsilon} \ell(w_\epsilon, \varrho_\epsilon, \varsigma_\epsilon, g) = 0,$$

for arbitrary variations $\Phi_\epsilon$, where $N_\epsilon = \Phi_\epsilon(\mathcal{D})$, $w_\epsilon = (\Phi_\epsilon)_*W$, $\varrho_\epsilon = (\Phi_\epsilon)_*R$, $\varsigma_\epsilon = (\Phi_\epsilon)_*S$. In particular the variations of $w$, $\varrho$, and $\varsigma$ are constrained of the form

$$\delta w = -\mathcal{L}_\varsigma w, \quad \delta \varrho = -\mathcal{L}_\varsigma \varrho, \quad \delta \varsigma = -\mathcal{L}_\varsigma \varsigma,$$

where $\varsigma = \delta \Phi \circ \Phi^{-1}$ is an arbitrary vector field on $\mathcal{N}$.

The critical conditions associated to (6.7) are

$$\text{div} \left( \left( \ell - \varrho \frac{\partial \ell}{\partial \varrho} - \varsigma \frac{\partial \ell}{\partial \varsigma} \right) \delta + w \otimes \frac{\partial \ell}{\partial w} \right) = 0$$

$$\text{div} \left( \left( \ell - \varrho \frac{\partial \ell}{\partial \varrho} - \varsigma \frac{\partial \ell}{\partial \varsigma} \right) \delta + w \otimes \frac{\partial \ell}{\partial w} \right) (\cdot, n^\mathcal{N}) = 0 \text{ on } \partial \mathcal{N}$$

(6.8)
and the variables $w$, $\rho$, $\varsigma$ satisfy

$$L_w \rho = 0 \quad \text{and} \quad L_w \varsigma = 0.$$  \hfill (6.9)

We note that equations (6.9) are equivalently written in terms of the world-velocity, the proper mass and entropy density as

$$L_u (\rho \mu (g)) = 0 \quad \text{and} \quad L_u (s \mu (g)) = 0,$$

which are obtained by using the equality $L^1_{1w} (\varphi f) = L_w \varphi$.

**Relativistic fluid equations.** From the expression (6.4) of the Lagrangian density for fluids, one directly computes the partial derivatives

$$\frac{\partial \ell}{\partial w} = \frac{1}{c^2} u^b \left( c^2 + e + \rho \frac{\partial e}{\partial \rho} \right) \varrho, \quad \frac{\partial \ell}{\partial \varsigma} = - \frac{1}{c} \sqrt{-g(w, w)} \frac{\partial e}{\partial \eta},$$

which yields the stress-energy-momentum tensor for fluids

$$\mathfrak{T} \text{fluid} = \left( \ell - \frac{\partial \ell}{\partial \varrho} \varrho - \frac{\partial \ell}{\partial \varsigma} \varsigma \right) \delta + w \otimes \frac{\partial \ell}{\partial w} = \left( p \delta + \frac{1}{c^2} u \otimes u^b (\epsilon_{\text{tot}} + p) \right) \mu (g),$$  \hfill (6.10)

with $\epsilon_{\text{tot}} = \rho (c^2 + e (\rho, \eta))$ the total energy density and $p = \rho^2 \partial e / \partial \rho$ the pressure.

The first equation in (6.8) yields the relativistic Euler equation and energy equation

$$\frac{1}{c^2} (\epsilon_{\text{tot}} + p) \nabla_u u = - \mathcal{P} \nabla p \quad \text{and} \quad \text{div} (\epsilon_{\text{tot}} u) + p \text{div} u = 0$$  \hfill (6.11)

while the second one gives the vacuum boundary conditions

$$p |_{\partial N} = 0,$$  \hfill (6.12)

which follows from $\mathfrak{T} \text{fluid} (\cdot, n^b) = p n^b$ on $\partial N$. Note that we also have $g(u, u) = 0$ on the boundary, by definition of $u$ in terms of the world-tube. In (6.11) we introduced the orthogonal projector $\mathbb{P}$ onto $(\text{span} \ u)^\perp$, see (6.13) later.

**Coupling with the Einstein equations and junction conditions.** The extension of the variational formulation in Proposition 6.2 to the coupling with general relativity can be obtained by particularizing Theorem 5.6 to the fluid Lagrangian density (6.4). The resulting variational principle then yields the Einstein equations on $N^\pm$, the relativistic fluid equations (6.11) on $N^-$, as well as the Israel-Darmois junction conditions on $\partial N$. Recall from Remark 5.7 that the latter conditions imply the O’Brien-Synge boundary conditions. Using the Einstein equations on $N^\pm$ one directly gets from the O’Brien-Synge conditions and from the expression (6.10), the boundary conditions (6.12). To summarize, for the general relativistic fluid we have

$$[h] = [K] = 0 \implies p |_{\partial N} = 0.$$
6.2 General relativistic elasticity

As we have seen for the Newtonian case in §2, the description of relativistic elasticity requires the introduction of an additional material tensor field, namely, a Riemannian metric $G_0$ on $\mathcal{B}$. From it, we define the 2-covariant symmetric positive tensor $G = \pi^* G_0$ on $D = [a, b] \times \mathcal{B}$. It satisfies

$$\mathcal{L}_{\partial_\lambda} G = i_{\partial_\lambda} G = 0.$$ 

Given a Lorentzian metric $g$ on $\mathcal{M}$, a world-tube $\Phi : D \to \mathcal{M}$ and the associated world-velocity $u$, an important tensor on spacetime for the description of relativistic elasticity is the projection tensor $p$ defined by

$$p(v_x, w_x) = g(P(v_x), P(w_x)), \quad \text{for all } v_x, w_x \in T_x \mathcal{M},$$

where $v_x \mapsto P(v_x)$ denotes the orthogonal projector onto $(\text{span } u)^\perp$, see Carter and Quintana [1972]. This tensor plays somehow the same role as the Riemannian metric $g$ on $\mathcal{S}$ in the non relativistic case, see §2, while being now an unknown in the relativistic case. This point will be made precise below. Explicitly, $p$ and $P$ are given by

$$p = g + \frac{1}{c^2} u^b \otimes u^b = g - \frac{1}{g(w, w)} w^b \otimes w^b$$

$$P = \delta + \frac{1}{c^2} u \otimes u = \delta - \frac{1}{g(w, w)} w \otimes w^b.$$

**Relativistic deformation tensors.** Here we recall the definition of some relativistic deformation tensors, see, e.g., Grot and Eringen [1966]; Eringen and Maugin [1990]; Maugin [1978], then we clarify their intrinsic nature and their relation with the tensors $G$ and $p$ in simple geometric terms. In particular, the approach presented here also allows to clarify the relation between formulations of relativistic elasticity based on the relativistic right Cauchy-Green tensor and the metric $G$ on one hand, or based on the relativistic Cauchy deformation tensor and the projection tensor on the other hand. A main step for this clarification is the result stated in Lemma 6.3.

For $D = [a, b] \times \mathcal{B} \ni X = (\lambda, X)$, we denote by $X^A$, $A = 1, \ldots, n$ the coordinates on $\mathcal{B}$ and by $X^a$, $a = 0, \ldots, n$ the coordinates on $D$ with $X^a = \lambda$ for $a = 0$ and $X^a = X^a$, for $a = 1, \ldots, n$. Given a world-tube $\Phi$ and a Lorentzian metric $g$, the relativistic deformation gradient $F^A_B$ and the relativistic right Cauchy-Green tensor $C_{AB}$ are defined as

$$F^\mu_A = P^\mu_\lambda^A \Phi,^\lambda \quad \text{and} \quad C_{AB} = g_{\mu\nu} F^\mu_A F^\nu_B.$$ 

(6.14)

Note that, contrarily to what the notation may suggest, $C_{AB}$ is not a tensor field on $\mathcal{B}$ since it depends on $\lambda$. The inverse relativistic deformation gradient $(-1)^A_\mu$ and relativistic Cauchy deformation tensor $c_{\mu\nu}$ are defined by

$$(-1)^A_\mu = (\Phi^{-1})^A_\mu \quad \text{and} \quad c_{\mu\nu} = G_{AB}(-1)^A_\mu (-1)^B_\nu.$$ 

(6.15)
Lemma 6.3. The relativistic right Cauchy-Green tensor \( C \) and the relativistic Cauchy deformation tensor \( c \) are related to the tensors \( p \) and \( G = \pi^{*}_{B}G_0 \) via the world-tube \( \Phi \) as
\[
C = \Phi^{*}p \quad \text{and} \quad c = \Phi^{*}G.
\]
(6.16)

In particular \( C \) and \( c \) are tensor fields on \( D \) and \( M \), respectively. They satisfy
\[
i_{\lambda} C = 0 \quad \text{and} \quad i_{u} c = 0.
\]

For all \((\lambda, X) \in D \) and all \( x \in \Phi(D) \),
\[
C(\lambda, X) : T_X B \times T_X B \to \mathbb{R} \quad \text{and} \quad c(x) : \text{span}(u(x))^\perp \times \text{span}(u(x))^\perp \to \mathbb{R}
\]
are positive definite.

Proof. Consider the tensor field on \( D \) defined by \( C = \Phi^{*}p \). In coordinates, we have
\[
C_{ab} = p_{\mu \nu} \Phi^{\mu}_{,a} \Phi^{\nu}_{,b} = g_{\lambda \kappa} P^{\mu}_{\lambda \kappa \rho} P^{\nu}_{\rho \sigma} \Phi^{\mu}_{,a} \Phi^{\nu}_{,b}.
\]

On one hand we have \( C_{0b} = 0 \) for all \( b = 0, ..., n \) since \( P^{\mu}_{\lambda \kappa \rho} \Phi^{\mu}_{,0} = P^{\mu}_{\lambda \kappa \rho} w^{\lambda} \circ \Phi = 0 \), on the other hand \( C_{ab} \) coincides with the local expression given in (6.14) for \( a, b = 1, ..., n \).

Hence the tensor field \( C \) defined by \( C = \Phi^{*}p \) is canonically identified with the relativistic right Cauchy-Green tensor as defined in (6.14), and satisfies \( i_{\lambda} C = 0 \).

To prove the second equality we note that \( c = \Phi^{*}G = \Phi^{*} \pi^{*}_{B} G_0 = (\pi_{B} \circ \Phi^{-1})^{*} G_0 \) and it is easily seen that the local expression of \( (\pi_{B} \circ \Phi^{-1})^{*} G_0 \) coincides with that given in (6.15). We also have \( i_{u} c = i_{\Phi, \partial_{\lambda}} \Phi^{*} G = \Phi^{*} (i_{\partial_{\lambda}} G) = \Phi^{*} (i_{\partial_{\lambda}} \pi^{*}_{B} G_0) = 0. \)

Lagrangian densities for relativistic elasticity. In the setting described above, assuming spacetime covariance, the corresponding Lagrangian densities are functions of the form
\[
\mathcal{L}(j^{1}\Phi, \partial_{\lambda}, R, G, g \circ \Phi), \quad \mathcal{L}(\partial_{\lambda}, R, G, \Gamma).
\]

Under material covariance we can further define the Lagrangian density
\[
\ell(w, \varrho, c, g).
\]

We consider a general expression \( W(G_0, C) \) for the specific stored energy function for (possibly anisotropic) elasticity, written in terms of the given Riemannian metric \( G_0 \) on \( B \) and the relativistic right Cauchy-Green tensor \( C \). As a map it reads \( W : S^2_{\pm} B \times S^2_{\pm} B \to \mathbb{R} \), where \( S^2_{\pm} B \to B \) denotes the bundle of 2-covariant symmetric positive tensors, i.e., locally, we have \( W(X^A, G_0(X)_{AB}, C(\lambda, X)_{AB}) \). Note that an explicit dependence of \( W \) on \( X \in B \) is possible. Inspired by the expression (6.6) of the material Lagrangian density for fluids, we construct the following Lagrangian density for elasticity with stored energy \( W \):
\[
\mathcal{L}(j^{1}\Phi, \partial_{\lambda}, R, G, g \circ \Phi) = -\frac{1}{c} \sqrt{-g(\Phi, \Phi)} (c^2 + W(G_0, C)) R.
\]
(6.17)
6.2 General relativistic elasticity

It is important to note here that $C = \Phi^* p$ on the right hand side, is considered as a function of the material tensor $\partial_\lambda$, the spatial tensor $g$ and the first jet extension $j^1 \Phi$, namely, we have

$$C = C(j^1 \Phi, \partial_\lambda, g \circ \Phi) = \Phi^* p = \Phi^* \left( g - \frac{1}{g(w, w)} w^\flat \otimes w^\flat \right), \quad w = \Phi_* \partial_\lambda.$$ 

The Lagrangian density (6.17) is spacetime covariant, which follows from the fact that $\sqrt{-g(\dot{\Phi}, \dot{\Phi})}$ is spacetime covariant and from the identity

$$C(j^1(\psi \circ \Phi), \partial_\lambda, \psi_\ast g \circ \psi \circ \Phi) = C(j^1 \Phi, \partial_\lambda, g \circ \Phi),$$

for all $\psi \in \text{Diff}(M)$. The associated reduced convective Lagrangian density is

$$\mathcal{L}(\partial_\lambda, R, G, \Gamma) = -\frac{1}{c} \sqrt{-\Gamma(\partial_\lambda, \partial_\lambda)} \left( c^2 + W(G_0, C) \right) R,$$

where now $C$ appears as a function of $\partial_\lambda$ and $\Gamma$

$$C = C(\partial_\lambda, \Gamma) = \Gamma - \frac{1}{\Gamma(\partial_\lambda, \partial_\lambda)} \partial_\lambda^{\flat r} \otimes \partial_\lambda^{\flat r}.$$ 

Based on the expression above, one directly verifies that $\mathcal{L}$ is material covariant with respect to $\text{Diff}_{\partial_\lambda}(D)$, i.e. satisfies $\mathcal{L}(j^1(\Phi \circ \varphi), \partial_\lambda, \varphi^* R, \varphi^* G, g \circ \Phi \circ \varphi) = \varphi^* [\mathcal{L}(j^1 \Phi, \partial_\lambda, G, g \circ \Phi)]$ for all $\varphi \in \text{Diff}_{\partial_\lambda}(D)$, if and only if $W$ satisfies

$$W(\psi^* G_0, \psi^* C) = W(G_0, C) \circ \psi, \quad \text{for all } \psi \in \text{Diff}(\mathcal{B}),$$

see Remark 3.5, especially (3.17). This condition on $W$ means that the elastic material is homogeneous isotropic, see Marsden and Hughes [1983, §3.5].

Under material covariance, it is possible to define the corresponding spacetime Lagrangian density, see §3.3, which is found as

$$\ell(w, \varrho, c, g) = -\varrho \left( c^2 + \varpi(c, p) \right) \mu(g),$$

where the Eulerian version $\varpi(c, p)$ of the specific stored energy is defined by

$$\varpi(c, p) = W(\Phi^* c, \Phi^* p) \circ \Phi^{-1}$$

for a world-tube $\Phi : D \to N$ with $\Phi_* \partial_\lambda = w$. Note that $\varpi$ is defined only on symmetric covariant tensors $c, p$ such that their pull-back with respect to some word-tube are positive definite on $T \mathcal{B}$ and degenerate on $\partial_\lambda$, see (6.16). One uses (6.18) to check that this definition doesn’t depend on the world-tube with $\Phi_* \partial_\lambda = w$.

**Remark 6.4 (On the need of $c$ and $p$).** One can be surprised that only the projection tensor $p$, instead of the Lorentzian metric $g$, is enough to intrinsically determine the invariants of $c$ via $\varpi(c, p)$. This is due to the fact that both $c$ and $p$ are nondegenerate on the same subspace, namely $(\text{span } u)^\perp$. Note that from (6.20) it follows that $\varpi$ is well-defined only on such pair $(c, p)$ in general. Our approach thus includes the...
forms of energy considered in Grot and Eringen [1966] and Kijowski and Magli [1992]. Our approach also naturally shows that both tensor $c$ and $p$ are needed to express the stored energy of a relativistic material in the spacetime description, thereby confirming a remark of Gérard Maugin, see Maugin [1978b, Appendix I] about the unsoundness of some proposed formulations of relativistic elasticity based exclusively on $p$.

**Remark 6.5 (Material covariance for anisotropic elasticity).** Material covariance can also be obtained in the anisotropic case by making explicit the dependence of the stored energy function on additional material tensor fields, such as $n$ linearly independent one-forms $\{\alpha^K\}_{K=1}^n$ on $\mathcal{B}$. By writing the stored energy function as $W(G_0, \{\alpha^K\}, C)$, one notes that spacetime covariance trivially holds while material covariance now reads

$$W(\psi^* G_0, \{\psi^* \alpha^K\}, \psi^* C) = W(G_0, \{\alpha^K\}, C) \circ \psi, \quad \text{for all } \psi \in \text{Diff}(\mathcal{B}),$$

(6.21)

and hence does not preclude anisotropy. An Eulerian version of the stored energy function can thus be defined in the anisotropic case, similarly to (6.20), which now reads $\varpi(c, \{f^K\}, p)$, with the additional dependence on the one-forms $f^K = \Phi_* \alpha^K$.

From $\mathcal{L}\partial_t \alpha^K = 0$ and $i_{\partial_t} \alpha^K = 0$ one gets $\mathcal{L}_w f^K = 0$ and $i_{\mathcal{L}_w} f^K = 0$, from which the continuity equation $\mathcal{L}_u f^K = 0$ follows by using the formula $\mathcal{L}_fu f^K = df \wedge i_u f^K$.

**Spacetime reduced Euler-Lagrange equations.** By exploiting the setting presented above, the results stated in Theorem 4.3 and 4.9 are directly applicable to isotropic relativistic elasticity. In particular, we get the following Eulerian variational principle and spacetime reduced Euler-Lagrange equations by direct application of (4.9) and (4.14).

**Proposition 6.6.** The Eulerian variational formulation for relativistic isotropic elasticity takes the form

$$\frac{d}{de} \bigg|_{e=0} \int_{\mathcal{N}_e} \ell(w_e, \varrho_e, c_e, \mathfrak{g}) = 0,$$

(6.22)

for arbitrary variations $\Phi_e$, where $\mathcal{N}_e = \Phi_e(\mathcal{D})$, $w_e = (\Phi_e)_* W$, $\varrho_e = (\Phi_e)_* R$, $c_e = (\Phi_e)_* G$. In particular the variations of $w$, $\varrho$, and $c$ are constrained of the form

$$\delta w = -\mathcal{L}_\zeta w, \quad \delta \varrho = -\mathcal{L}_\zeta \varrho, \quad \delta c = -\mathcal{L}_\zeta c,$$

where $\zeta = \delta \Phi \circ \Phi^{-1}$ is an arbitrary vector field on $\mathcal{N}$.

The critical conditions associated to (6.22) are

$$\begin{cases} \text{div} \nabla \left( \left( \ell - \varrho \frac{\partial \ell}{\partial \varrho} \right) \delta + w \otimes \frac{\partial \ell}{\partial w} - 2 \frac{\partial \ell}{\partial c} \cdot c \right) = 0 \\ \left( \left( \ell - \varrho \frac{\partial \ell}{\partial \varrho} \right) \delta + w \otimes \frac{\partial \ell}{\partial w} - 2 \frac{\partial \ell}{\partial c} \cdot c \right) \cdot n = 0 \quad \text{on } \partial \mathcal{N}. \end{cases}$$

(6.23)

and the variables $w$, $\varrho$, $c$ satisfy

$$\mathcal{L}_w \varrho = 0 \quad \text{and} \quad \mathcal{L}_w c = 0.$$
We note that equations (6.24) are equivalently written in terms of the world-velocity as
\[ \mathcal{L}_u(\rho\mu) = 0 \quad \text{and} \quad \mathcal{L}_u c = 0. \]
Indeed, from the formula \( \mathcal{L}_f u c = f \mathcal{L}_u c + \nabla f \otimes i_u c + i_u c \otimes \nabla f \) and the property \( i_u c = 0 \), see Lemma 6.3, we obtain the equivalence \( \mathcal{L}_u c = 0 \iff \mathcal{L}_w c = 0 \).

**Remark 6.7 (Anisotropic elasticity).** In the anisotropic case, see Remark 6.5, the Lagrangian \( \ell \) also depends on the one-forms \( f^K \). From the general expression (4.14) of the reduced Euler-Lagrange equations, this results in the inclusion of the term
\[ -\sum_{K=1}^n \frac{\partial \ell}{\partial f^K} \otimes f^K \]
in both equations in (6.23). Also, the additional the continuity equations \( \mathcal{L}_w f^K = 0 \), or its equivalent formulation \( \mathcal{L}_u f^K = 0 \), appear in (6.24).

**Relativistic elasticity equations.** From the expression (6.19) of the Lagrangian density for relativistic elasticity, one directly computes the partial derivatives
\[
\frac{\partial \ell}{\partial w} = \frac{1}{c^2} w^\nu (c^2 + \varpi) \rho - \rho \frac{\partial \varpi}{\partial p_{\mu\nu}} \frac{\partial p_{\mu\nu}}{\partial w}(g), \quad \frac{\partial \ell}{\partial c} = -\rho \frac{\partial \varpi}{\partial c}(g),
\]
\[
\frac{\partial \ell}{\partial \varpi} = -\frac{\sqrt{-g(w,w)}}{c}(c^2 + \varpi).
\]
We note that
\[
\frac{\partial p_{\mu\nu}}{\partial w^\lambda} = \frac{2 w_{\mu} w_{\nu}}{g(w,w)^2} g_{\alpha\lambda} w^\alpha - \frac{1}{g_{\alpha\beta} w^\alpha w^\beta} (g_{\mu\lambda} w_{\nu} + g_{\nu\lambda} w_{\mu})
\]
which then gives
\[
\frac{\partial \varpi}{\partial p_{\mu\nu}} = -\frac{2}{g(w,w)} \frac{\partial \varpi}{\partial p_{\mu\nu}} p_{\mu\lambda} w_{\nu} = \frac{2}{g(w,w)} \frac{\partial \varpi}{\partial c_{\mu\nu}} c_{\mu\lambda} w_{\nu},
\]
where we used the spacetime covariance of \( \varpi \), i.e., \( \varpi(\varphi^* c, \varphi^* p) = \varpi(c, p) \circ \varphi \), for all \( \varphi \in \text{Diff}(\mathbb{M}) \), which yields
\[
\frac{\partial \varpi}{\partial c} \cdot c + \frac{\partial \varpi}{\partial p} \cdot p = 0.
\]
From this, one gets the stress-energy-momentum tensor for elasticity as
\[
\mathcal{T}_{el} = \left( \ell - \frac{\partial \ell}{\partial \varpi} \right) \delta + w \otimes \frac{\partial \ell}{\partial w} - 2 \frac{\partial \ell}{\partial c} \cdot c = (\epsilon_{\text{tot}} \frac{1}{c^2} u \otimes u - t_3) \mu(g),
\]
where \( \epsilon_{\text{tot}} = \rho(c^2 + \varpi(c, p)) \) is the total energy density and with the relativistic stress tensor
\[
t_{el} = -2 \rho \frac{\partial \varpi}{\partial c} \cdot c, \quad (t_{el})^\mu_\nu = -2 \rho \frac{\partial \varpi}{\partial c_{\lambda\alpha}} c_{\mu\nu}.
\]
The first equation in (6.23) yields the relativistic Euler-Cauchy equation and energy equation
\[
\frac{1}{c^2} (\epsilon_{\text{tot}} \nabla \cdot \mathbf{u} + u \mathbf{t}_{\text{el}} : \nabla \mathbf{u}) = \text{div} \mathbf{t}_{\text{el}} \quad \text{and} \quad \text{div}(\epsilon_{\text{tot}} \mathbf{u}) = \mathbf{t}_{\text{el}} : \nabla \mathbf{u} \quad (6.25)
\]
while the second one gives the zero traction boundary conditions
\[
\mathbf{t}_{\text{el}} (\cdot, n^\flat) = 0 \quad \text{on} \quad \partial N = 0, \quad (6.26)
\]
which follows from \( \mathbf{T}_{\text{el}} (\cdot, n^\flat) = \mathbf{t}_{\text{el}} (\cdot, n^\flat) \) on \( \partial N \). We have \( g(u, n) = 0 \) on \( \partial N \) as before.

**Coupling with the Einstein equations and junction conditions.** In a similar way with the fluid case, the extension of the variational formulation to the coupling with general relativity can be obtained by particularizing Theorem 5.6 to the Lagrangian density (6.19) of elasticity. One obtains the Einstein equations on \( N^\pm \), the equations (6.11) on \( N^- \), as well as the Israel-Darmois junction conditions on \( \partial N \). On the boundary one gets
\[
[h] = [K] = 0 \implies \mathbf{t}_{\text{el}} (\cdot, n^\flat) = 0 \quad \text{on} \quad \partial N = 0.
\]

### 6.3 General relativistic continua

The variational approach presented above for fluid and elasticity can be easily extended to more general continua with internal energy function \( W(\rho, \eta, G_0, C) \), which thus covers both the fluid and elasticity cases. We very briefly describe this situation below. For such continua the material Lagrangian takes the form
\[
\mathcal{L}(j^1 \Phi, \partial \lambda, R, S, G, g \circ \Phi)
\]
\[
= -\frac{1}{c}\sqrt{-g(\Phi, \Phi)} \left( c^2 + W\left(\frac{1}{c}\sqrt{-g(\Phi, \Phi)} \frac{R}{\Phi^*|\mu(g)|}, \frac{S}{R}, G_0, C\right)\right) R. \quad (6.27)
\]

This Lagrangian density is spacetime covariant. It is material covariant if and only if the continua is isotropic homogeneous,
\[
W(\rho \circ \psi, \eta \circ \psi, \psi^* G_0, \psi^* C) = W(\rho, \eta, G_0, C) \circ \psi, \quad \text{for all} \ \psi \in \text{Diff}(\mathcal{B}). \quad (6.28)
\]

Material covariance can be also achieved for the anisotropic case by following Remarks 6.5 and 6.7, and expressing the internal energy as a function of the form \( W(\rho, \eta, G_0, \{\alpha^K\}, C) \). A detailed study of this situation will be studied elsewhere.

Assuming isotropy for simplicity, the spacetime Lagrangian density reads
\[
\ell(w, g, \varsigma, c, g) = -\rho \left( c^2 + \varpi(\rho, \eta, c, p) \right) \mu(g)
\]
where we defined \( \varpi(\rho, \eta, c, p) = W(\rho \circ \Phi, \eta \circ \Phi, \Phi^* c, \Phi^* p) \circ \Phi^{-1} \), as in (6.20). Propositions 6.2 and 6.6 generalize to this case, yielding the spacetime reduced Euler-Lagrange equations in the general form
\[
\begin{cases}
\text{div} \nabla \left( \left( \ell - \frac{\partial \ell}{\partial \varphi} - \varsigma \frac{\partial \ell}{\partial \varsigma} \right) \delta + w \otimes \frac{\partial \ell}{\partial w} - 2 \frac{\partial \ell}{\partial c} \cdot c \right) = 0 \\
\left( \left( \ell - \frac{\partial \ell}{\partial \varphi} - \varsigma \frac{\partial \ell}{\partial \varsigma} \right) \delta + w \otimes \frac{\partial \ell}{\partial w} - 2 \frac{\partial \ell}{\partial c} \cdot c \right) (\cdot, n^\flat) = 0 \quad \text{on} \ \partial N
\end{cases} \quad (6.29)
\]
and $\mathcal{L}_w g = 0$, $\mathcal{L}_w \varsigma = 0$, and $\mathcal{L}_w c = 0$.

From this, one computes the stress-energy-momentum tensor as

$$\Sigma_{\text{cont}} = \left( \ell - \partial_\ell g - \varsigma \partial_\varsigma \right) \delta + w \otimes \partial_w - 2 \partial_\ell \cdot c = (\epsilon_{\text{tot}} \frac{1}{c^2} u \otimes u^b - t) \mu(g),$$

where $\epsilon_{\text{tot}} = \rho (c^2 + \varpi(\rho, \eta, c, p))$ is the total energy density and with the relativistic stress tensor

$$t = -P \left( \rho \frac{\partial \varpi}{\partial \rho} \delta + 2 \rho \frac{\partial \varpi}{\partial c} c \right) = -P p + t_{\text{el}}, \quad t^\mu_\nu = -P^{\mu}_\alpha \left( \rho \frac{\partial \varpi}{\partial \rho} \delta^\alpha_\nu + 2 \rho \frac{\partial \varpi}{\partial c_\lambda} c_{\lambda \nu} \right),$$

written in terms of the pressure $p$ and relativistic elastic stress tensor $t_{\text{el}}$.

The first equation in (6.29) yields the relativistic Euler-Cauchy equation and energy equation for the relativistic continuum as

$$\frac{1}{c^2} (\epsilon_{\text{tot}} + p) \nabla_u u + u \cdot t_{\text{el}} : \nabla u = -P \nabla p + \text{div} t_{\text{el}} \quad \text{and} \quad \text{div}(\epsilon_{\text{tot}} u) + p \text{div} u = t_{\text{el}} : \nabla u$$

while the second one gives the zero traction boundary conditions

$$t_{\text{el}}(\cdot, n^b) = p n^b \quad \text{on} \quad \partial N = 0. \quad (6.30)$$

The coupling with general relativity follows again from Theorem 5.6 and on the boundary one gets

$$[h] = [K] = 0 \implies t_{\text{el}}(\cdot, n^b) = p n^b \quad \text{on} \quad \partial N = 0.$$ 

We give below two tables that summarize the analogies between the variational framework for nonrelativistic and relativistic Lagrangian continuum theories. We focus on the spacetime reduced description. The convective reduced description can be described similarly.

Despite the apparent analogies in these tables, one should not forget crucial differences between the two situations. For instance (choosing $n = 3$) the push-forward and pull-back operations $\varphi_*$ and $\varphi^*$ act on tensor fields defined on the three dimensional manifolds $\mathcal{B}$ and $\mathcal{S}$, at each fixed time $t$, while $\Phi_*$ and $\Phi^*$ act on tensor fields defined on the four dimensional manifolds $\mathcal{D} = [a, b] \times \mathcal{B}$ and $\mathcal{M}$. Similarly, the divergence and Lie derivative operators act on objects defined on three dimensional manifolds, at each fixed time $t$ for the nonrelativistic case, while they act on objects defined on four dimensional manifolds in the relativistic case. Note that we have kept the first jet notation $j^1 \Phi$ to encode the first derivatives of the world-tube $\Phi : \mathcal{D} \to \mathcal{M}$ for the relativistic case. When $\mathcal{D} = [a, b] \times \mathcal{B} \ni X = (\lambda, X)$ we could also write it as the couple $(\partial_\lambda \Phi, T\Phi)$ with $T\Phi$ the tangent map to $\Phi(\lambda, \cdot) : \mathcal{B} \to \mathcal{M}$ for each fixed $\lambda \in [a, b]$, in analogy with the non-relativistic notation $(\dot{\varphi}, T\varphi)$.

7 Conclusion

We have established a Lagrangian variational framework for general relativistic continuum theories that systematically parallels the Lagrangian reduction approach
### Table 1: Schematic correspondence between relativistic and nonrelativistic Lagrangian continuum theories.

| | Nonrelativistic | Relativistic |
|---|---|---|
| **Material tensor fields** & associated spatial versions | $R \xrightarrow{\phi^*} \varrho$ | $\partial_\lambda \xrightarrow{\phi^*} w$ |
| | $S \xrightarrow{\phi^*} \varsigma$ | $\bar{R} \xrightarrow{\phi^*} \varrho$ |
| | $G \xrightarrow{\phi^*} c$ | $S \xrightarrow{\phi^*} \varsigma$ |
| Spatial tensor field & associated material version | $C \xleftarrow{\phi^*} g$ | $C \xleftarrow{\phi^*} p = p(g, w)$ |
| Spatial covariance | $\sim \mathcal{L}(\dot{\varphi}, T\varphi, R, S, G, g \circ \varphi)$ | $\sim \mathcal{L}(j^1\Phi, \partial_\lambda, R, S, G, g \circ \varphi)$ |
| | $= \frac{1}{2} g(\dot{\varphi}, \dot{\varphi}) R - W(\rho, \eta, G_0, C) R$ | $= -\sqrt{-g} \frac{\dot{\Phi}}{c} (c^2 + W(\rho, \eta, G_0, C)) R$ |
| Material covariance | $\sim \mathcal{L}(\dot{\varphi}, T\varphi, R, S, G, g \circ \varphi)$ | $\sim \mathcal{L}(j^1\Phi, \partial_\lambda, R, S, G, g \circ \varphi)$ |
| | $= \varphi^* [\ell(u, g, \varsigma, c, g)]$ | $= \Phi^* [\ell(w, g, \varsigma, c, p)]$ |
| | with | with |
| | $\ell(u, g, \varsigma, c, g) = \frac{1}{2} g(u, u) \varrho - \varpi(\rho, \eta, c, g) \varrho$ | $\ell(w, g, \varsigma, c, g) = -\sqrt{-g} \frac{w}{c} (c^2 + \varpi(\rho, \eta, c, p)) \varrho$ |

### Table 2: Relativistic and nonrelativistic spacetime reduced Euler-Lagrange equations for continuum theories.

| | Equations of motion for the continuum |
|---|---|
| **Nonrelativistic** | $\partial_t \frac{\partial \ell}{\partial \varrho} + \text{div} \nabla \left( \left( \ell - \varrho \frac{\partial \ell}{\partial \varrho} - \varsigma \frac{\partial \ell}{\partial \varsigma} \right) \delta + u \otimes \frac{\partial \ell}{\partial \delta} - 2 \frac{\partial \ell}{\partial \varsigma} \cdot c \right) = 0$ |
| | $\partial_t \varrho + \mathcal{L}_u \varrho = 0$, $\partial_t \varsigma + \mathcal{L}_u \varsigma = 0$, $\partial_t c + \mathcal{L}_u c = 0$ |
| **Relativistic** | $\text{div} \nabla \left( \left( \ell - \varrho \frac{\partial \ell}{\partial \varrho} - \varsigma \frac{\partial \ell}{\partial \varsigma} \right) \delta + w \otimes \frac{\partial \ell}{\partial \delta} - 2 \frac{\partial \ell}{\partial \varsigma} \cdot c \right) = 0$ |
| | $\mathcal{L}_w \varrho = 0$, $\mathcal{L}_w \varsigma = 0$, $\mathcal{L}_w c = 0$ |

Table 1: Schematic correspondence between relativistic and nonrelativistic Lagrangian continuum theories.

Table 2: Relativistic and nonrelativistic spacetime reduced Euler-Lagrange equations for continuum theories.

To nonrelativistic fluids and elasticity. Using spacetime and material covariance properties, our approach allows to rigorously deduce variational formulations in the convective and Eulerian descriptions by starting from the most natural and physically justified principle, namely, a continuum version of the Hamilton principle for the relativistic particle. In particular, our formulation does not need the inclusion of constraints or unphysical variables in the action functional.

We showed how this setting can be extended to the coupling of the continuum dynamics with gravitation theory by including the Gibbons-Hawking-York term as-
associated to the boundary of the relativistic continuum. In particular, the critical condition of the resulting variational principle gives the Einstein equations for the gravitational field created by the relativistic continuum, both at the interior and outside the continuum; the equations of motion of the continuum in this gravitational field; the junction conditions between the solution at the interior of the relativistic continuum and the solution describing the gravity field produced outside from it. The variation of the Gibbons-Hawking-York term term with respect to the boundary has necessitated some extensions of previous results on the first variation of mean curvature integrals.

The setting was then applied to relativistic fluids and relativistic elasticity by focusing on appropriate choices of reference tensor fields and Lagrangian densities. For elasticity, the variational setting also allowed to clarify the relation between formulations based on the relativistic right Cauchy-Green tensor and the metric $G$ on one hand, or based on the relativistic Cauchy deformation tensor and the projection tensor on the other hand. The resulting Israel-Darmois junction conditions were shown to imply the zero traction type boundary conditions for the continuum, via the O'Brien-Synge conditions. The setting developed here offers new tools for the modelling of general relativistic continuum theories by variational principles, which will be further exploited in subsequent parts of the paper.

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