MULTIMOMENTUM HAMILTONIAN FORMALISM
IN FIELD THEORY

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Abstract

The standard Hamiltonian machinery, being applied to field theory, leads to infinite-dimensional phase spaces. It is not covariant. In this article, we present covariant finite-dimensional multimomentum Hamiltonian formalism for field theory. This formalism has been developed from 70th as multisymplectic generalization of the Hamiltonian formalism in mechanics. In field theory, multimomentum canonical variables are field functions and momenta corresponding to derivatives of fields with respect all world coordinates, not only the time. In case of regular Lagrangian densities, the multimomentum Hamiltonian formalism is equivalent to the Lagrangian formalism, otherwise for degenerate Lagrangian densities. In this case, the Euler-Lagrange equations become undetermined and require additional conditions. In gauge theory, they are gauge conditions. In general case, these supplementary conditions remain elusive. In the framework of the multimomentum Hamiltonian machinery, one obtaines them automatically as a part of Hamilton equations. In case of semiregular and almost regular Lagrangian densities, we get comprehensive relations between Lagrangian and multimomentum Hamiltonian formalisms. The key point consists in the fact that, given a degenerate Lagrangian density, one must consider a family of associated multimomentum Hamiltonian forms in order to exhaust solutions of the Euler-Lagrange equations. We spell out degenerate quadratic and affine Lagrangian densities. The most of field models are of these types. As a result, we get the general procedure of describing constraint field systems.

1 Introduction

We follow the generally accepted geometric description of classical fields by sections of fibred manifolds \( Y \to X \). Lagrangian and Hamiltonian formalisms on fibred manifolds are phrased in terms of jet spaces. Given a fibred manifold \( Y \to X \), the \( k \)-order jet space \( J^kY \) of \( Y \) comprises the equivalence classes \( j^k_x s \), \( x \in X \), of sections \( s \) of \( Y \) identified by the first \((k+1)\) terms of their Taylor series at a point \( x \). One utilizes the well-known facts that (i) the jet space \( J^kY \) is a finite-dimensional smooth manifold and (ii) a \( k \)-order differential operator on sections of \( Y \) can be represented by a morphism of \( J^kY \) to a vector bundle over \( X \). As a consequence, the dynamics of field systems is formulated in terms of finite-dimensional configuration and phase spaces. Moreover, we get the differential geometric
description of this dynamics, for there is the 1:1 correspondence between sections of the jet bundle $J^1Y \to Y$ and connections on $Y \to X$.

In field theory, we can restrict ourselves to the first order Lagrangian formalism when the configuration space is $J^1Y$. Given fibred coordinates $(x^\mu, y^i)$ of $Y$, the jet space $J^1Y$ is endowed with the adapted coordinates $(x^\mu, y^i, y^i_\mu)$ which bring it into a finite-dimensional smooth manifold. A first order Lagrangian density is represented by a horizontal exterior density

$$L = \mathcal{L}(x^\mu, y^i, y^i_\mu) \omega,$$

on $J^1Y \to X$. The associated Euler-Lagrange operator $\mathcal{E}_L$ on the second order jet manifold $J^2Y$ sets up the system of second order Euler-Lagrange equations for sections of the fibred manifold $Y$. Its canonical extension $\mathcal{E}'_L$ to the sesquiholonomic subbundle $\hat{J}^2Y$ of the repeated jet manifold $J^1J^1Y$ yields the equivalent system of first order Euler-Lagrange equations for sections $s$ of the fibred jet manifold $J^1Y \to X$. The Poincaré-Cartan form

$$\Xi_L = \pi_i^\lambda dy^i \wedge \omega_{\lambda} - \pi_i^\lambda y^i_\lambda \omega + \mathcal{L} \omega, \quad \pi_i^\lambda = \partial_{\lambda} \mathcal{L},$$

associated with a first order Lagrangian density $L$ is uniquely defined. With $\Xi_L$, we have the Cartan equations

$$\bar{s}^* (u | d\Xi_L) = 0$$

for sections $\bar{s}$ of $J^1Y \to X$ where $u$ is an arbitrary vertical vector field on $J^1Y \to X$. The equations (2) are equivalent to the first order Euler-Lagrange equations on sections $s$ which are jet prolongations $\bar{s} = J^1s$ of sections $s$ of $Y \to X$.

At present, we observe the following main Hamiltonian approaches to field theory:

(i) the standard Hamiltonian formalism;

(ii) the Hamilton-De Donder formalism which has been developed as the counterpart of the higher order Lagrangian formalism in the framework of the calculus of variations [1-6];

(iii) the multimomentum Hamiltonian formalism which is the multisymplectic generalization of the conventional Hamiltonian formalism to fibred manifolds over an $n$-dimensional base $X$, not only $\mathbb{R}$ [7-12].

In the straightforward manner when evolution is governed by the Poisson bracket, the standard Hamiltonian formalism leads to infinite-dimensional symplectic spaces [13, 15]. Its application to constraint field systems follows the Dirac’s procedure generalized to the infinite-dimensional case [14, 15]. In the naive Hamiltonian approach to field theory, the Hamilton equations are replaced with the equations of the Hamilton-De Donder type with respect to only a time coordinate $x^0$ by means of substituting solutions $y^i_0$ of the equations $p^i_0 = \pi^i_0(x^\mu, y^i, y^i_\mu)$ into the Cartan equations (2).

In the calculus of variations, the phase space is the manifold

$$Z = \bigwedge^{n-1} T^*X \wedge T^*Y$$

(3)
into which the Poincaré-Cartan form $\Xi_L$ takes its values \[6, 9, 16\]. This manifold is endowed with the coordinates $(x^\lambda, y^i, p_\lambda^i, p)$ such that

$$(x^\mu, y^i, p_\mu^i, p) \circ \Xi_L = (x^\mu, y^i, \pi_\mu^i, \pi_\mu^i y^i - \mathcal{L}).$$

It carries the canonical form

$$\Xi = p\omega + p_\lambda^i dy^i \wedge \omega_\lambda, \quad \omega_\lambda = \partial_\lambda \omega,$$  \(4\)

and the corresponding polisymplectic form $d\Xi$. In case of $n = 1$, the form $\Xi$ reduces to the Liouville form $\Xi = Edt + p_i dy^i$ for the homogeneous formalism of mechanics where $E$ is the energy variable.

In the Hamilton-De Donder approach, a Hamiltonian form fails to be introduced intrinsically. Given a Lagrangian density $L$, it is defined to be the pullback of the canonical form \(4\) by the natural injection $i_L$ of a submanifold $Z_L = \Xi_L(J^1Y)$ into $Z$. The corresponding equations are the Hamilton-De Donder equations

$$\tau^*(u \circ dH_L) = 0$$  \(5\)

for sections $\tau$ of the fibred manifold $Z_L \rightarrow X$ where $u$ is an arbitrary vertical vector field on $Z_L \rightarrow X$. When the Poincaré-Cartan morphism $\Xi_L : J^1Y \rightarrow Z$ is almost regular, the Hamilton-De Donder equations \(5\) are equivalent to the Cartan equations \(2\) \[3\]. Consequently, in comparison with the Lagrangian machinery, the Hamilton-De Donder formalism takes no advantage of describing constraint systems.

In the multimomentum Hamiltonian formalism, the phase space is the Legendre manifold

$$\Pi = \wedge^n T^*X \otimes Y T X \otimes V^*Y$$  \(6\)

into which the Legendre morphism $\hat{L}$ associated with a Lagrangian density $L$ on $J^1Y$ takes its values. This manifold is provided with the fibred coordinates $(x^\lambda, y^i, p_\lambda^i)$ such that

$$(x^\mu, y^i, p_\mu^i) \circ \hat{L} = (x^\mu, y^i, \pi_\mu^i).$$

The Legendre manifold \(6\) carries the generalized Liouville form

$$\theta = -p_\lambda^i dy^i \wedge \omega \otimes \partial_\lambda$$  \(7\)

corresponding to the canonical bundle monomorphism

$$\theta : \Pi \rightarrow \wedge^{n+1} Y T^*Y \otimes T X.$$

and the associated multisymplectic form

$$\Omega = dp_\lambda^i \wedge dy^i \wedge \omega \otimes \partial_\lambda.$$  \(8\)
If \( X = \mathbb{R} \), these reproduce respectively the Liouville form and the symplectic form in mechanics.

The multimomentum Hamiltonian formalism is phrased intrinsically in terms of Hamiltonian connections which play the role similar Hamiltonian vector fields in the symplectic geometry [8, 11, 12]. We say that a connection \( \gamma \) on the fibred Legendre manifold \( \Pi \to X \) is a Hamiltonian connection if the form \( \gamma \rfloor \Omega \) is closed. Then, a Hamiltonian form \( H \) on \( \Pi \) is defined to be an exterior form such that

\[
dH = \gamma \rfloor \Omega
\]

for some Hamiltonian connection \( \gamma \). Note that the manifold \( Z \) is the 1-dimensional affine bundle over the Legendre manifold \( \Pi \). There is the 1:1 correspondence between the Hamiltonian forms \( H \) on \( \Pi \) and the sections \( h \) of the bundle \( Z \to \Pi \) so that \( H \) are the pullbacks of the canonical form \( \Xi \) on \( Z \) by the sections \( h \).

The key point consists in the fact that every Hamiltonian form admits splitting

\[
H = p^i_\lambda dy^i \wedge \omega_\lambda - p^i_\lambda \Gamma^j_\lambda \omega - \tilde{\Gamma} \omega = p^i_\lambda dy^i \wedge \omega_\lambda - \mathcal{H} \omega
\]

where \( \Gamma \) is a connection on the fibred manifold \( Y \). In physical applications, one can consider this splitting as the workable definition of Hamiltonian forms on \( \Pi \). Given the Hamiltonian form \( H \), the equality (9) comes to the Hamilton equations

\[
\partial_\lambda r^i(x) = \partial^i_\lambda \mathcal{H}, \quad \partial_\lambda r^\lambda_i(x) = -\partial_i \mathcal{H}
\]

for sections \( r \) of the Legendre manifold \( \Pi \to X \).

If a Lagrangian density \( L \) is hyperregular (i.e. \( \hat{L} \) is a diffeomorphism), there exists the unique Hamiltonian form \( H \) such that the first order Euler-Lagrange equations and the Hamilton equations are equivalent.

If \( X = \mathbb{R} \), the multimomentum Hamiltonian formalism reproduces the familiar Hamiltonian formalism of mechanics. In this case, we have the well-known bijection between the Hamiltonian forms and the Hamiltonian connections. If one deals with constraints, this bijection makes necessary the Dirac-Bergman procedure of constructing the chain of \( k \)-class primary constraints spaces [14].

In the multimomentum Hamiltonian formalism when \( n > 1 \), we have a set of Hamiltonian connections for the same Hamiltonian form. Therefore, one can choose solutions of the Hamilton equations living on a constraint space. Moreover, in case of a degenerate Lagrangian system when the constraint space is \( Q = \hat{L}(J^1Y) \), we observe a family of different Hamiltonian forms \( H \) associated with the same Lagrangian density \( L \).

If a Lagrangian density is degenerate, the system of the Euler-Lagrange equations is underdetermined and require additional conditions. In gauge theory, they are gauge conditions. In general case, these gauge-type conditions remain elusive. In the framework of the multimomentum Hamiltonian formalism, we get them as a part of the Hamilton equations.
Different gauge-type conditions correspond to different associated multimomentum Hamiltonian forms.

We shall restrict our consideration to (i) semiregular Lagrangian densities \( L \) when the preimage \( \hat{L}^{-1}(p) \) of any point \( p \in Q \) is a connected submanifold of \( J^1Y \) and (ii) almost regular Lagrangian densities when \( J^1Y \) is a fibred manifold over \( Q \) which is an imbedded submanifold of \( \Pi \). These notions of degeneracy seem most appropriate. Lagrangian densities of fields are almost always both semiregular and almost regular. In this case, we get the comprehensive relation between solutions of the Euler-Lagrange equations and the Hamilton equations [11, 18].

If a Lagrangian density \( L \) is semiregular, all associated Hamiltonian forms \( H \) consist with each other on the constraint space \( Q \). For an associated Hamiltonian form \( H \), every solution of the corresponding Hamilton equations which lives on the constraint space \( Q \) yields a solution of the Euler-Lagrange equations. One may hope that, conversely, for any solution of the Euler-Lagrange equations, there exists the corresponding solution of the Hamilton equations for some associated Hamiltonian form. In case of an almost regular Lagrangian density which is semiregular, such a complete family of local Hamiltonian forms always exists.

In Section 6, we spell out models with affine and almost regular quadratic Lagrangian densities. In this case, a complete family of associated Hamiltonian forms always exists. The corresponding Hamilton equations are separated in the dynamic equations and the gauge-type conditions independent of momenta \( p^\lambda \). As a result, we get the universal procedure of describing constraint field theories. We apply it to the gauge theory of principal connections and the gravitation theory in Palatini variables.

## 2 Technical preliminary

All maps throughout are of class \( C^\infty \) and manifolds are real, Hausdorff, finite-dimensional, second-countable and connected.

A fibred manifold is defined to be the surjective submersion \( \pi : Y \to X \). A locally trivial fibred manifold is called a bundle. By \( VY \) and \( V^*Y \), we denote vertical tangent and vertical cotangent bundles of \( Y \) respectively.

Given fibred manifolds \( Y \to X \) and \( Y' \to X' \), let \( \Phi : Y \to Y' \) be a fibred manifold morphism over \( f : X \to X' \). If \( f = \text{Id}_X \), we call \( \Phi \) the fibred morphism \( Y \to_x Y' \) over \( X \).

Given a fibred manifold \( Y \to X \) and a manifold morphism \( f : X' \to X \), by \( f^*Y \) is meant the pullback of \( Y \) by \( f \) over \( X' \). For the sake of simplicity, the pullbacks \( \pi^*(TX) \) and \( \pi^*(T^*X) \) are denoted by \( TX \) and \( T^*X \) respectively.

On fibred manifolds, we consider the following special types of differential forms:

(i) exterior horizontal forms \( Y \to \wedge T^*X \);
(ii) tangent-valued horizontal forms

\[ \phi : Y \rightarrow \overset{\rightarrow}{\bigwedge} T^*X \otimes TY, \]
\[ \phi = dx^{\lambda_1} \wedge \ldots \wedge dx^{\lambda_r} \otimes (\phi_{\lambda_1 \ldots \lambda_r}^\mu \partial_\mu + \phi_{\lambda_1 \ldots \lambda_r}^i \partial_i); \]

and, in particular, soldering forms \( \sigma_i^\lambda dx^\lambda \otimes \partial_i; \)

(iii) pullback-valued forms

\[ Y \rightarrow \overset{\rightarrow}{\bigwedge} T^*Y \otimes TX, \quad Y \rightarrow \overset{\rightarrow}{\bigwedge} T^*Y \otimes V^*Y. \]

Horizontal \( n \)-forms are called horizontal densities.

Given a fibred manifold \( Y \rightarrow X \), the first order jet manifold \( J^1Y \) of \( Y \) is both the fibred manifold \( J^1Y \rightarrow X \) and the affine bundle \( J^1Y \rightarrow Y \) modelled on the vector bundle \( T^*X \otimes_Y VY \). The adapted coordinates \( (x^\lambda, y^i, y^i_\lambda) \) of \( J^1Y \) are compatible with these fibrations:

\[ x^\lambda \rightarrow x'^\lambda(x^\mu), \quad y^i \rightarrow y'^i(x^\mu, y^j), \quad y^i_\lambda = \left( \frac{\partial y'^i}{\partial y^j} y^j_\mu + \frac{\partial y'^i}{\partial x^\mu} \right) \frac{\partial x^\lambda}{\partial x'^\mu}. \]

There is the canonical bundle monomorphism (the contact map)

\[ \lambda : J^1Y \rightarrow T^*X \otimes TY, \quad \lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i). \]

Let \( \Phi \) be a fibred morphism of \( Y \rightarrow X \) to \( Y' \rightarrow X \) over a diffeomorphism of \( X \). Its jet prolongation reads

\[ J^1\Phi : J^1Y \rightarrow J^1Y', \quad y'^i_\mu \circ J^1\Phi = (\partial_\lambda \Phi^j + \partial_j \Phi^i y^j_\mu) \frac{\partial x^\lambda}{\partial x'^\mu}. \]

The jet prolongation of a section \( s \) of \( Y \rightarrow X \) is the section \( y'^i_\mu \circ J^1s = \partial_\mu s^i \) of \( J^1Y \rightarrow X \).

The repeated jet manifold \( J^1J^1Y \), by definition, is the first order jet manifold of \( J^1Y \rightarrow X \). It is provided with the adapted coordinates \( (x^\lambda, y^i, y^i_\lambda, y^i_{\mu\lambda}) \). Its subbundle \( \overset{\wedge}{J}^2Y \) with \( y^i_{(\lambda)} = y^i_\lambda \) is called the sesquiholonomic jet manifold. The second order jet manifold \( J^2Y \) of \( Y \) is the subbundle of \( \overset{\wedge}{J}^2Y \) with \( y^i_{\lambda\mu} = y^i_{\mu\lambda} \).

Given a fibred manifold \( Y \rightarrow X \), a jet field \( \Gamma \) on \( Y \) is defined to be a section of the jet bundle \( J^1Y \rightarrow Y \). A global jet field is a connection on \( Y \). By means of the contact map \( \lambda \), every connection \( \Gamma \) on \( Y \) can be represented by the tangent-valued form \( \lambda \circ \Gamma \) on \( Y \). For the sake of simplicity, we denote this form by the same symbol

\[ \Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i). \]

Let \( \Gamma \) be a connection and \( \sigma \) be a soldering form on a fibred manifold \( Y \). Then, their affine sum \( \Gamma + \sigma \) is a connection on \( Y \). Let \( \Gamma \) and \( \Gamma' \) be connections on \( Y \). Then, their affine difference \( \Gamma' - \Gamma \) is a soldering form on \( Y \).
The Legendre manifold $\Pi$ of a fibred manifold $Y$ is the composite manifold

$$\pi_{\Pi X} = \pi \circ \pi_{\Pi Y} : \Pi \to Y \to X$$

endowed with the fibred coordinates $(x^\lambda, y^i, p^\lambda_i)$:

$$p^\lambda_i = J \frac{\partial y^j}{\partial y^\lambda} \frac{\partial x^\lambda}{\partial x^\mu} p^\mu_j, \quad J^{-1} = \det \left( \frac{\partial x^\lambda}{\partial x^\mu} \right).$$

By $J^1\Pi$ is meant the first order jet manifold of $\Pi \to X$. It is provided with the adapted fibred coordinates $(x^\lambda, y^i, p^\lambda_i, y^{i(\mu)}, p^\lambda_i\mu)$.

By a momentum morphism, we call a fibred morphism $\Phi : \Pi \to J^1Y$.

Given a momentum morphism $\Phi$, its composition with the contact map $\lambda$ is represented by the horizontal pullback-valued 1-form

$$\Phi = dx^\lambda \otimes (\partial_\lambda + \Phi^\lambda_i(p) \partial_i)$$

on $\Pi \to X$. For instance, let $\Gamma$ be a connection on $Y \to X$. Then, $\widetilde{\Gamma} = \Gamma \circ \pi_{\Pi Y}$ is a momentum morphism. Conversely, every momentum morphism $\Phi$ of the Legendre manifold $\Pi$ of $Y$ defines the associated connection $\Gamma_\Phi = \Phi \circ \tilde{0}$ on $Y \to X$ where $\tilde{0}$ is the global zero section of the Legendre bundle $\Pi \to Y$.

## 3 Lagrangian formalism

Given a Lagrangian density $L$, the jet manifold $J^1Y$ is provided with the Lagrangian multisymplectic form

$$\Omega_L = \left( \partial_j \pi^\lambda_i dy^j + \partial^\mu_i \pi^\lambda_i dy^j_\mu \right) \wedge dy^i \wedge \omega \otimes \partial_\lambda.$$

Using the pullback of this form and the Poincaré-Cartan form $\Xi_L$ to the repeated jet manifold $J^1J^1Y$, one can construct the exterior form

$$\Lambda_L = d\Xi_L - \lambda \Omega_L = (y^i_\lambda - y^i_\lambda^\lambda) d\pi^\lambda_i \wedge \omega + (\partial_i - \hat{\partial}_\lambda \partial^\lambda_i) L dy^i \wedge \omega,$$

on $J^1J^1Y$. Its restriction to the second order jet manifold $J^2Y$ reproduces the familiar variational Euler-Lagrange operator

$$\mathcal{E}_L : J^2Y \to T^*Y,$$

$$\mathcal{E}_L = (\partial_i - \hat{\partial}_\lambda \partial^\lambda_i) L dy^i \wedge \omega, \quad \hat{\partial}_\lambda = \partial_\lambda + y^i_\lambda \partial_i + y^i_\mu \partial^\mu_i.$$
The restriction of the form (13) to the sesquiholonomic jet manifold $\tilde{J}^2Y$ of $Y$ defines the sesquiholonomic extension

$$\mathcal{E}'_L : \tilde{J}^2Y \to \mathbb{T}^*Y$$

(15)

of the Euler-Lagrange operator (14). It has the form (14) with nonsymmetric coordinates $y^i_{\mu\lambda}$.

Let $\pi$ be a section of the fibred jet manifold $J^1Y \to X$ such that its first order jet prolongation $J^1\pi$ takes its values into $\text{Ker} \mathcal{E}'_L$. Then, it satisfies the system of first order Euler-Lagrange equations

$$\partial_\lambda \pi^i = \pi^i, \quad \partial_\lambda \mathcal{L} - (\partial_\lambda \pi^i_\lambda \partial_j + \partial_\lambda \pi^i_\mu \partial_j) \partial_j \mathcal{L} = 0.$$  

(16)

They are equivalent to the familiar second order Euler-Lagrange equations

$$\mathcal{E}_L \circ J^2s = 0$$

(17)

for sections $s$ of $Y \to X$. We have $\pi = J^1s$.

4 Multimomentum Hamiltonian formalism

Let $\Pi$ be the Legendre manifold (6) provided with the generalized Liouville form $\theta$ (7) and the multisymplectic form $\Omega$ (8).

**Definition 1:** We say that a jet field (resp. a connection)

$$\gamma = dx^\lambda \otimes (\partial_\lambda + \gamma^i_\lambda \partial_i + \gamma^i_\mu \partial_i)$$

on the Legendre manifold $\Pi \to X$ is a Hamiltonian jet field (resp. a Hamiltonian connection) if the exterior form

$$\gamma \mid \Omega = dp^i_\lambda \wedge dy^i \wedge \omega + \gamma^i_\lambda dy^i \wedge \omega - \gamma^i_\lambda dp_i^\lambda \wedge \omega$$

is closed.

Hamiltonian connections constitute an affine subspace of connections on $\Pi \to X$. The following construction shows that this subspace is not empty.

Every connection $\Gamma$ on $Y \to X$ is lifted to the connection

$$\gamma = \tilde{\Gamma} = dx^\lambda \otimes [\partial_\lambda + \Gamma^i_\lambda(y) \partial_i + (-\partial_j \Gamma^i_\lambda(y)p^\mu_i - K^\mu_\nu \lambda(x)p^\nu_j + K^\alpha \omega_\lambda(x)p^\mu_j) \partial_j]$$

on $\Pi \to X$ where $K$ is a linear symmetric connection on the bundles $TX$ and $T^*X$. We have the equality

$$\tilde{\Gamma} \mid \Omega = d(\tilde{\Gamma} \mid \theta)$$

(18)

which shows that $\tilde{\Gamma}$ is a Hamiltonian connection.

**Definition 2:** An exterior $n$-form $H$ on the Legendre manifold $\Pi$ is called a Hamiltonian form if, on an open neighborhood of each point of $\Pi$, there exists a Hamiltonian jet field
satisfying the equation $\gamma \lceil \Omega = dH$. This jet field $\gamma$ is termed the Hamiltonian jet field for $H$.

Note that Hamiltonian forms throughout are considered modulo closed forms since closed forms do not make any contribution in the Hamilton equations.

*Proposition 3:* Let $H$ be a Hamiltonian form. For any exterior horizontal density $\widetilde{H} = \mathcal{H}\omega$ on the fibred Legendre manifold $\Pi \rightarrow X$, the form $H - \widetilde{H}$ is a Hamiltonian form. Conversely, if $H$ and $H'$ are Hamiltonian forms, their difference $H - H'$ is an exterior horizontal density on $\Pi \rightarrow X$.

In virtue of Proposition 3, Hamiltonian forms constitute an affine space modelled on a linear space of the exterior horizontal densities on $\Pi \rightarrow X$. A glance at the equality (18) shows that this affine space is not empty. Given a connection $\Gamma$ on a $Y \rightarrow X$, its lift $\widetilde{\Gamma}$ on $\Pi \rightarrow X$ is a Hamiltonian connection for the Hamiltonian form

$$H_\Gamma = \mathcal{H}\lceil \theta = p^\lambda_i dy^i \wedge \omega_\lambda - p^\lambda_i \Gamma^i_\lambda(y)\omega.$$  \hfill (19)

It follows that every Hamiltonian form on the Legendre manifold $\Pi$ can be given by the expression (11).

Moreover, a Hamiltonian form has the canonical splitting (10) as follows. Every momentum morphism $\Phi$ represented by the pullback-valued form (12) on $\Pi$ yields the associated Hamiltonian form

$$H_\Phi = \Phi\lceil \theta = p^\lambda_i dy^i \wedge \omega_\lambda - p^\lambda_i \Phi^i_\lambda \omega.$$  \hfill (20)

Conversely, every Hamiltonian form $H$ defines the associated momentum morphism

$$\widetilde{H} : \Pi \rightarrow J^1Y, \quad y^i_\lambda \circ \widetilde{H} = \partial^i_\lambda \mathcal{H},$$

and the associated connection $\Gamma_H = \widetilde{H} \circ \widetilde{0}$ on $Y \rightarrow X$. As a consequence, we have the canonical splitting

$$H = H_{\Gamma_H} - \widetilde{H}. \hfill (21)$$

*Definition 4:* The Hamilton operator $\mathcal{E}_H$ for a Hamiltonian form $H$ is defined to be the first order differential operator

$$\mathcal{E}_H : J^1\Pi \rightarrow \wedge^{n+1} T^*\Pi,$$

$$\mathcal{E}_H = dH - \tilde{\Omega} = [(y^i_\lambda - \partial^i_\lambda \mathcal{H})dp^\lambda_i - (p^\lambda_i dy^i + \partial^i_\lambda \mathcal{H} dy^i) \wedge \omega] \wedge \omega \hfill (22)$$

where

$$\tilde{\Omega} = dp^\lambda_i \wedge dy^i \wedge \omega_\lambda + p^\lambda_i dy^i \wedge \omega - y^i_\lambda dp^\lambda_i \wedge \omega$$

is the pullback of the multisymplectic form $\Omega$ onto $J^1\Pi$.

For any jet field $\gamma$ on the Legendre manifold $\Pi$, we have

$$\mathcal{E}_H \circ \gamma = dH - \gamma \lceil \Omega.$$
It follows that $\gamma$ is a Hamiltonian jet field for a Hamiltonian form $H$ if and only if it takes its values into $\text{Ker} \, E_H$, that is, satisfies the algebraic Hamilton equations

$$
\gamma^i_\lambda = \partial^i_\lambda \mathcal{H}, \quad \gamma^\lambda_\iota = -\partial^\lambda_\iota \mathcal{H}.
$$

(23)

In particular, Hamiltonian jet fields associated with the same Hamiltonian form differ from each other in soldering forms $\tilde{\sigma}$ on $\Pi \to X$ which obey the relation

$$
\tilde{\sigma}^i \Omega = 0, \quad \tilde{\sigma}^\lambda_\iota = 0, \quad \tilde{\sigma}^\lambda_\iota = 0.
$$

(24)

The Hamilton operator (22) is an affine morphism and $\text{Ker} \, E_H \to \Pi$ is an affine subbundle of the jet bundle $J^1\Pi \to \Pi$. Moreover, a glance at the condition (24) shows that, for all Hamilton operators $E_H$, this affine subbundle is modelled on the same vector subbundle of the bundle $T^*X \otimes_{\Pi} V \Pi$. As a consequence, we have the following assertion.

**Proposition 5:** A Hamiltonian connection for every Hamiltonian form always exists.

Let $r$ be a section of the fibred Legendre manifold $\Pi \to X$ such that its jet prolongation $J^1r$ takes its values into $\text{Ker} \, E_H$. Then, the Hamilton equations (23) are brought to the first order differential Hamilton equations (11a) and (11b) Conversely, if $r$ is a solution (resp. a global solution) of the Hamilton equations (11a) and (11b) for a Hamiltonian form $H$, there exists an extension of this solution to a Hamiltonian jet field (resp. a Hamiltonian connection) $\gamma$ which has an integral section $r$, that is, $\gamma \circ r = J^1r$.

## 5 Constraint systems

This Section is devoted to relations between the Lagrangian formalism on fibred manifolds and the multimomentum Hamiltonian formalism in case of degenerate Lagrangian densities.

Given a fibred manifold $Y \to X$, let $L$ be a first order Lagrangian density. One observes that, when the Legendre morphism $\hat{L}$ is a diffeomorphism, the corresponding Lagrangian system meets the unique equivalent Hamiltonian system. It follows that, if a Legendre morphism is regular at a point, the corresponding Lagrangian system reduced to an open neighborhood of this point has the equivalent local Hamiltonian system. In order to keep this equivalence in case of degenerate systems, we require that the image of a configuration space by the Legendre morphism contains all points where the momentum morphism is regular.

**Definition 6:** We shall say that a Hamiltonian form $H$ is associated with a Lagrangian density $L$ if $H$ satisfies the relations

$$
\hat{L} \circ \hat{H} \big|_Q = \text{Id}_Q, \quad Q = \hat{L}(J^1Y),
$$

(25a)

$$
H = H_{\hat{H}} + L \circ \hat{H}.
$$

(25b)
The relation (25b) results in the condition

\[(p_i^\mu - \partial_i^\mu L \circ \hat{H})\partial_i^\mu \partial_\alpha^a \mathcal{H} = 0.\]

A glance at this condition shows that (i) the condition (25a) is the corollary of the condition (25b) if the momentum morphism \(\hat{H}\) is regular at all points of \(Q\) and (ii) \(\hat{H}\) is not regular outside the constraint space \(Q\).

Let us emphasize that different Hamiltonian forms can be associated with the same Lagrangian density.

**Proposition 7:** Given a Lagrangian density \(L\), let \(\Phi\) be a momentum morphism associated with the Legendre morphism \(\hat{L}\), that is,

\[\hat{L} \circ \Phi \bigg|_Q = \text{Id}_Q.\]  

Let us consider the Hamiltonian form

\[H_{LA} = H_\Phi + L \circ \Phi\]  

where \(H_\Phi\) is the Hamiltonian form (24). If

\[\hat{H}_{LA} = \Phi,\]  

then the Hamiltonian form (27) is associated with \(L\).

If a Lagrangian density \(L\) is hyperregular, there always exists the unique Hamiltonian form

\[H = H_{\hat{L}^{-1}} + L \circ \hat{L}^{-1}\]

associated with \(L\).

Contemporary field theories are almost never regular, but semiregular and almost regular. Also in this case, one can get the workable relations between Lagrangian and multimomentum Hamiltonian formalisms \([11, 12, 18]\).

**Proposition 8:** All Hamiltonian forms associated with a semiregular Lagrangian density \(L\) consists with each other on the constraint space \(Q\), and the Poincaré-Cartan form \(\Xi_L\) for \(L\) is the pullback

\[\Xi_L = H \circ \hat{L}, \quad \pi_i^\lambda y^i_\lambda - \mathcal{L} = \mathcal{H}(x^\mu, y^i, \pi_i^\lambda),\]

of any associated Hamiltonian form.

**Proposition 9:** Let \(H\) be a Hamiltonian form associated with a semiregular Lagrangian density \(L\). The Hamilton operator \(\mathcal{E}_H\) for \(H\) satisfies the relation

\[\Lambda_L = \mathcal{E}_H \circ J^1 \hat{L}.\]

**Proposition 10:** (i) Let a section \(r\) of \(\Pi \to X\) be a solution of the Hamilton equations (11a) and (11b) for a Hamiltonian form \(H\) associated with a semiregular Lagrangian
density $L$. If $r$ lives on the constraint space $Q$, the section $\overline{s} = \widehat{H} \circ r$ of $J^1Y \to X$ satisfies the first order Euler-Lagrange equations (16). (ii) Given a semiregular Lagrangian density $L$, let $\overline{s}$ be a solution of the first order Euler-Lagrange equations (16). Let $H$ be a Hamiltonian form associated with $L$ so that

$$\widehat{H} \circ \widehat{L} \circ \overline{s} = \overline{s}. \quad (30)$$

Then, the section $r = \widehat{L} \circ \overline{s}$ of $\Pi \to X$ is a solution of the Hamilton equations (11a) and (11b) for $H$. It lives on the constraint space $Q$. (iii) For every sections $\overline{s}$ and $r$ satisfying (i) or (ii), we have the relations

$$\overline{s} = J^1 s, \quad s = s_{\Pi Y} \circ r$$

where $s$ is a solution of the second order Euler-Lagrange equations (17).

We shall say that a family of Hamiltonian forms $H$ associated with a semiregular Lagrangian density $L$ is complete if, for each solution $\overline{s}$ of the first order Euler-Lagrange equations (16), there exists a solution $r$ of the Hamilton equations (11a) and (11b) for some Hamiltonian form $H$ from this family so that

$$r = \widehat{L} \circ \overline{s}, \quad \overline{s} = \widehat{H} \circ r, \quad \overline{s} = J^1(s_{\Pi Y} \circ r).$$

In virtue of Proposition 10, such a complete family exists if and only if, for each solution $\overline{s}$ of the Euler-Lagrange equations for $L$, there exists a Hamiltonian form $H$ from this family so that the condition (30) holds.

We establish existence of a complete family of Hamiltonian forms associated with a Lagrangian density which is both semiregular and almost regular [12, 18].

**Proposition 11:** Let $L$ be an almost regular Lagrangian density. (i) On an open neighborhood of each point $q \in Q$, there exists a momentum morphism $\Phi$ which obeys the conditions (26) and (28) and so, the local Hamiltonian form (27) is associated with $L$. (ii) If $L$ is a semiregular Lagrangian density, there exists a complete family of local associated Hamiltonian forms on an open neighborhood of each point $p \in Q$.

Note that, if the imbedded constraint space $Q$ is not an open subbundle of $\Pi$, it can not be provided with the multisymplectic structure. Therefore, to consider solutions of the Hamilton equations on $Q$, one must introduce a Hamiltonian form and the corresponding Hamilton equations at least on some open neighborhood of $Q$. Another way consists in constructing the Hamilton-De Donder type equations on the imbedded constraint space $Q$. Let an almost regular Lagrangian density $L$ be semiregular and $H_Q$ the restriction of the associated Hamiltonian forms to $Q$. For sections $r$ of the fibred manifold $Q \to X$, we can write the equations

$$r^*(u|dH_Q) = 0 \quad (31)$$

where $u$ is an arbitrary vertical vector field on $Q \to X$. Since $dH_Q \neq dH \mid Q$, these equations fail to be equivalent to the Hamilton equations restricted to $Q$. At the same time, we have $Z_L = H_Q(Q)$ and the equations (31) are equivalent to the Hamilton-De Donder equations (3).
6 Quadratic and affine Lagrangian densities

In this Section, we construct complete families of Hamiltonian forms associated with almost regular quadratic and affine Lagrangian densities. They are semiregular.

Let us consider a quadratic Lagrangian density given by the coordinate expression

\[ L = L_\omega, \quad L = \frac{1}{2}a_{ij}^\lambda(y)y_i^\lambda y_j^\mu + b_i^\lambda(y)y_i^\lambda + c(y), \] (32)

where \( a, b \) and \( c \) are local functions on \( Y \) with the corresponding transformation laws. It is semiregular. The associated Legendre morphism reads

\[ p_i^\lambda \circ \hat{L} = a_{ij}^\lambda y_j^\mu + b_i^\lambda. \] (33)

It is an affine morphism over \( Y \). We have the corresponding linear morphism

\[ \overline{L} : T^*X \otimes VY \to \Pi, \quad p_i^\lambda \circ \overline{L} = a_{ij}^\lambda y_j^\mu. \]

Note that almost all quadratic Lagrangian densities of field models take the form

\[ \mathcal{L} = \frac{1}{2}a_{ij}^\lambda \overline{y}_i^\lambda \overline{y}_j^\mu + c(y), \quad \overline{y}_\mu^i = y_\mu^i - \Gamma_\mu^i, \] (34)

where \( \Gamma \) is a connection on \( Y \). It is equivalent to the fact that the constraint space \( Q \) given by the Legendre morphism (33) contains the zero section \( 0(Y) \) of \( \Pi \to Y \). Then, \( \text{Ker} \hat{L} \) is an affine subbundle of \( J^1Y \to Y \) and there exists a connection \( \Gamma \) on \( Y \) such that

\[ \Gamma : Y \to \text{Ker} \hat{L}, \quad a_{ij}^\lambda \Gamma_j^i + b_i^\lambda = 0. \] (35)

With this connection, the Lagrangian density (32) can be brought into the form (34). If the Lagrangian density (32) is regular, the connection (35) is unique.

Let \( L \) be an almost regular quadratic Lagrangian density such that \( 0(Y) \subset Q \). Then, there exists a linear pullback-valued horizontal 1-form

\[ \sigma : \Pi \to T^*X \otimes VY, \quad \overline{y}_\lambda^i \circ \sigma = \sigma_{ij}^\lambda p_j^\mu, \] (36)

on \( \Pi \to X \) which satisfies the condition

\[ \overline{L} \circ \sigma \circ i_Q = i_Q \] (37)

where \( i_Q \) denotes the imbedding of \( Q \) into \( \Pi \).

The connection (35) and the form (36) are the key objects in our construction.

Since \( \overline{L} \) and \( \sigma \) are linear morphisms, \( \overline{L} \circ \sigma \) is a surjective submersion of \( \Pi \) onto \( Q \). It follows that

\[ \sigma = \sigma \circ \overline{L} \circ \sigma, \] (38)
and we have the splitting
\[ J^1Y = \text{Ker} \hat{L} \oplus \text{Im} \sigma. \] (39)
Moreover, there exists \( \sigma \) such that
\[ \Pi = \text{Ker} \sigma \oplus Q. \] (40)

Given a form \( \sigma \) and a connection (35), let us consider the affine momentum morphism
\[ \Phi = \Gamma + \sigma, \quad \Phi^i = \Gamma^i_\lambda(y) + \sigma^{ij}_\lambda p^\mu_j. \] (41)
It is associated with the Legendre morphism (33). Conversely, every affine momentum morphism associated with (33) is of the type (41). The Hamiltonian form \( H_{L\Phi} \) corresponding to \( \Phi \) is associated with the Lagrangian density (32) due to the condition (38). It reads
\[ H = p^i_\lambda dy^i \wedge \omega^\lambda - (\Gamma^i_\lambda(p^\lambda_i - \frac{1}{2}b^\lambda_i) + \frac{1}{2}\sigma^{ij}_\lambda p^\mu_i p^\mu_j - c)\omega. \] (42)
The canonical splitting (21) of this Hamiltonian form shows that \( \Gamma_H = \Gamma \) and the Hamiltonian density \( \tilde{H} \) is quadratic, but it becomes affine on \( \text{Ker} \sigma \).

We aim to show that the Hamiltonian forms (42) for different connections (35) constitute the complete family.

Let us consider the Hamilton equations (11a) for the Hamiltonian form (42). For sections \( r \) of \( \Pi \to X \), they read
\[ J^1s = (\Gamma + \sigma) \circ r, \quad s = \pi_{HY}. \] (43)
With splitting (39), we have the following two surjections
\[ S : J^1Y \to \text{Ker} \hat{L}, \quad S : y^i_\lambda \to y^i_\lambda - \sigma^{ik}_\lambda (a^{\mu\mu}_{kj} y^j_\mu + b^k_i), \]
\[ F : J^1Y \to \text{Im} \sigma, \quad F = \sigma \circ \hat{L} : y^i_\lambda \to \sigma^{ik}_\lambda (a^{\mu\mu}_{kj} y^j_\mu + b^k_i). \]
With respect to these projections, the Hamilton equations (43) are brought into the pair of equations
\[ S \circ J^1s = \Gamma \circ s, \] (44)
\[ F \circ J^1s = \sigma \circ r. \] (45)

On an analogy with gauge theory, by a gauge-type class is meant the preimage \( F^{-1}(\mathbf{y}) \) of every point \( \mathbf{y} \in \text{Im} \sigma \). Then, the equations (44) make the sense of gauge-type conditions. We say that these conditions are universal if they single out one representative of every gauge-type class. It is readily observed that the conditions (44) are universal gauge-type conditions on sections of the jet bundle \( J^1\Pi \to \Pi \) when the algebraic Hamilton equations (23) are considered, otherwise on sections \( r \) of \( \Pi \to X \). At the same time, they show that it is condition on \( S \circ J^1s \) which can supplement the underdetermined Euler-Lagrange
equations for the degenerate Lagrangian density (32). Moreover, for every section \( s \) of \( Y \), there exists a connection \( \Gamma \) (35) such that the gauge-type conditions (44) hold. In this case, the momentum morphism (41) satisfies the condition
\[
\Phi \circ \hat{L} \circ J^1_s = J^1_s.
\]
Hence, the Hamiltonian forms (42) constitute a complete family. Note that Hamiltonian forms from this family differ only in connections \( \Gamma \) which are responsible for the gauge-type condition (44).

Let us consider now an affine Lagrangian density
\[
L = \mathcal{L}\omega, \quad \mathcal{L} = b^i(y) y^i_\lambda + c(y).
\]
(46)
It is almost regular and semiregular. The corresponding Legendre morphism reads
\[
\hat{p}^i_\lambda \circ \hat{L} = b^i_i(y).
\]
(47)
It follows that, in case of an affine Lagrangian density, the corresponding constraint space is given by the section \( b \) of \( \Pi \to Y \).

It is easy to see that the Hamiltonian form
\[
H = H_\Gamma + L \circ \Gamma = \hat{p}^i_\lambda dy^i \wedge \omega_\lambda - (\hat{p}^i_\lambda - b^i_i) \Gamma^i_\lambda \omega + c\omega,
\]
(48)
for any connections \( \Gamma \) on \( Y \) is associated with \( L \) (46). The corresponding momentum morphism reads
\[
y^i_\lambda \circ \hat{H} = \Gamma^i_\lambda.
\]
(49)
We thus observe that, in case of the affine Hamiltonian density, the Hamilton equations (11a) reduce to the gauge-type conditions (19).

For any section \( s \) of \( Y \), we can choose the connection \( \Gamma \) which has the integral section \( s \). Then, the corresponding momentum morphism (49) obeys the condition
\[
\hat{H} \circ \hat{L} \circ J^1_s = J^1_s.
\]
It follows that the Hamiltonian forms (48) constitute the complete family.

7 Gauge theory of principal connections

In this Section, the manifold \( X \) is assumed to be oriented. It is provided with a nondegenerate fibre metric \( g_{\mu\nu} \) and \( g^{\mu\nu} \) in the tangent and cotangent bundles of \( X \). We denote \( g = \det(g_{\mu\nu}) \).

Gauge theory of principal connections is described by the degenerate quadratic Lagrangian density, and its multimomentum Hamiltonian formulation exemplifies the common attributes of the degenerate quadratic models. The feature of gauge theory consists in the fact that splittings (39) and (40) are canonical.
Let $P \to X$ be a principal bundle with a structure Lie group $G$ which acts on $P$ on the right by the law

$$r_g : P \to Pg, \quad g \in G.$$ 

A principal connection is defined to be a $G$-equivariant global jet field $A$ on $P$:

$$A \circ r_g = J^1r_g \circ A, \quad g \in G.$$ 

There is the 1:1 correspondence between the principal connections $A$ on $P$ and the global sections of the bundle $C = J^1P/G$ called the bundle of principal connections. It is the affine bundle modelled on the vector bundle

$$\overline{C} = T^*X \otimes V^G P, \quad V^G P = VP/G.$$ 

Given a bundle atlas $\Psi^P$ of $P$, the bundle $C$ is provided with the fibred coordinates $(x^\mu, k^m_\mu)$ so that

$$(k^m_\mu \circ A)(x) = A^m_\mu(x)$$

are coefficients of the local connection 1-form of a principal connection $A$ with respect to the atlas $\Psi^P$.

The first order jet manifold $J^1C$ of the bundle $C$ provided with the adapted coordinates $(x^\mu, k^m_\mu, k^m_{\mu \lambda})$. There exists the canonical splitting (50)

$$J^1C = C_+ \oplus C_- = (J^2P/G) \oplus (\wedge T^*X \otimes V^G P),$$

$$k^m_{\mu \lambda} = \frac{1}{2}(k^m_{\mu \lambda} + k^m_{\lambda \mu} + c^m_{nl}k^n_{\lambda}k^l_{\mu}) + \frac{1}{2}(k^m_{\mu \lambda} - k^m_{\lambda \mu} - c^m_{nl}k^n_{\lambda}k^l_{\mu}),$$

over $C$ where $C_+ \to C$ is the affine bundle modelled on the vector bundle

$$\overline{C}_+ = 2T^*X \otimes V^G P.$$ 

There are the corresponding canonical surjections:

(i) $S : J^1C \to C_+.$

(ii) $F : J^1C \to C_-$ where

$$F = \frac{1}{2}F^m_{\lambda \mu}dx^\lambda \wedge dx^\mu \otimes e_m, \quad F^m_{\lambda \mu} = k^m_{\mu \lambda} - k^m_{\lambda \mu} - c^m_{nl}k^n_{\lambda}k^l_{\mu},$$

The Legendre manifold of the bundle $C$ of principal connections reads

$$\Pi = \wedge T^*X \otimes TX \otimes [C \times \overline{C}]^*.$$ 

It is provided with the fibred coordinates $(x^\mu, k^m_\mu, p^m_{\mu \lambda})$ and has the canonical splitting

$$\Pi = \wedge T^*X \otimes [\overline{C}_+ \oplus C_-]^* = \Pi_+ \oplus \Pi_-.$$(51)
\[(k^m_\mu, p^m_\mu) = (k^m_\mu, p^{(\mu\lambda)}_m) = \frac{1}{2}[p^m_\mu + p^{(\lambda\mu)}_m] + (k^m_\mu, p^m_\mu) = \frac{1}{2}[p^m_\mu - p^{(\lambda\mu)}_m].\]

On the configuration space (50), the conventional Yang-Mills Lagrangian density \(L_{YM}\) is given by the expression

\[L_{YM} = \frac{1}{4\varepsilon^2}a^G_{mn}g^{\lambda\mu}g^{\beta\nu}\mathcal{F}^m_\lambda\mathcal{F}^n_\nu\sqrt{|g|}\omega(52)\]

where \(a^G\) is a nondegenerate \(G\)-invariant metric in the Lie algebra of \(G\). It is almost regular and semiregular. The Legendre morphism associated with the Lagrangian density (52) takes the form

\[\hat{L}_{YM} : J^1C \to Q = \Pi_- \subset \Pi,\]

\[\hat{p}^{(\mu\lambda)}_m \circ \hat{L}_{YM} = 0, \quad (53a)\]

\[\hat{p}^{[\mu\lambda]}_m \circ \hat{L}_{YM} = \varepsilon^{-2}a^G_{mn}g^{\lambda\alpha}g^{\beta\nu}\mathcal{F}^m_\alpha\mathcal{F}^n_\beta\sqrt{|g|}. \quad (53b)\]

A glance at these expressions shows that \(C_+ = \text{Ker} \hat{L}\) and \(\Pi_- = Q\). It follows that the splittings (50) and (51) are similar the splittings (39) and (40). Therefore, to construct the complete family of multimomentum Hamiltonian forms associated with the Yang-Mills Lagrangian density (52), we can follow the general procedure from previous Section.

Let us consider connections (35) on the bundle \(C\) of principal connections which take their values into \(\text{Ker} \hat{L}\):

\[S : C \to C_+, \quad S^m_{\mu\lambda} - S^m_{\lambda\mu} - c^m_{nl}k^n_{\lambda\mu} = 0. \quad (54)\]

For all these connections, the Hamiltonian forms

\[H = \hat{p}^{(\mu\lambda)}_m dk^m_\mu \wedge \omega_\lambda - \hat{p}^{(\mu\lambda)}_m S^m_{\mu\lambda} \omega - \hat{H}_{YM}\omega, \quad (55)\]

\[\hat{H}_{YM} = \varepsilon^2/4a^G_{mn}g^{\lambda\alpha}g^{\beta\nu}[p^m_\alpha[p^{(\mu\beta)}_n]_\lambda]_\mu |g|^{-1/2}, \quad (56)\]

are associated with the Lagrangian density \(L_{YM}\) and constitute the complete family. Moreover, we can minimize this complete family if we restrict our consideration to connections (54) of the following type. Given a symmetric linear connection \(K\) on the cotangent bundle \(T^*X\) of \(X\), every principal connection \(B\) on \(P\) is lifted to the connection \(S_B\) (54) such that

\[S_B \circ B = S \circ J^1B, \quad (57)\]

\[S^m_{B\mu\lambda} = \frac{1}{2}[c^m_{nl}k^n_{\lambda\mu} + \partial_\lambda B^m_\mu + \partial_\mu B^m_\lambda - c^m_{nl}(k^n_{\mu\lambda}B^l_\nu + k^n_{\lambda\nu}B^l_\mu)] - K^\beta_{\mu\lambda}(B^m_{\beta} - k^m_{\beta}). \quad (58)\]

We denote the Hamiltonian form (53) for the connections \(S_B\) (54) by \(H_B\). The corresponding Hamilton equations for sections \(r\) of \(\Pi \to X\) read

\[\partial_\lambda p^\mu_\mu = -c^m_{nl}k^{[\mu\nu]}_n + c^m_{nl}B^{(\mu\nu)}_l - K^\beta_{\mu\lambda}(p^\lambda_\nu + p^\nu_\lambda), \quad (57)\]

\[\partial_\lambda k^m_\mu + \partial_\mu k^m_\lambda = 2S^m_{B(\mu\lambda)}. \quad (58)\]
plus the equation (53b).

The equations (58) and (53b) are similar to the equations (44) and (15) respectively. The equations (53b) and (47) restricted to the constraint space (53a) are the familiar Yang-Mills equations for \( A = \pi_{HC} \circ r \). These equations are the same for all Hamiltonian forms \( H_B \) and exemplify the Hamilton-De Donder equations (4). Different Hamiltonian forms \( H_B \) lead to different equations (58) which play the role of the gauge-type condition

\[ S_B \circ A = S \circ J^1 A. \]

At the same time, given a solution \( A \) of the Yang-Mills equations, there always exists a multimomentum Hamiltonian form \( H_B \) such that

\[ \tilde{H}_B \circ \tilde{L}_M \circ J^1 A = J^1 A. \]

For instance, this is \( H_{B=A} \). It follows that the Hamiltonian forms \( H_B \) constitute a complete family.

The gauge-type conditions (58) however differ from the familiar gauge conditions in gauge theory. Gauge-type classes introduced in previous Section are preimages \( F^{-1}(e) \) of points \( e \in C_- \), whereas the orbits of the gauge group acting on \( J^1 C \) belong to preimages \( (\text{pr}_1 \circ F)^{-1}(v) \) of points

\[ v \in \Lambda^2 T^* X \otimes V^G P. \]

The gauge-type conditions (58) are universal conditions on sections of the jet bundle \( J^1 C \to C \), whereas the familiar gauge conditions are locally universal conditions on sections of \( J^1 C \to X \) and single out a representative in each class of gauge conjugate potentials (with accuracy to the Gribov ambiguity). Namely, if \( A \) is a solution of the Yang-Mills equations, there exists a gauge conjugate \( A' \) which also is a solution of the same Yang-Mills equations and satisfies given gauge conditions. However, not every solution of the Yang-Mills equations obeys one or another gauge condition utilized in gauge theory. In this sense, the system of these gauge conditions is not complete. In gauge theories, this deficiency is not important since all conjugate gauge potentials are treated as physically equivalent, otherwise for other degenerate field systems, e.g. the Proca field. At the same time, we have a complete family of gauge-type conditions. For instance, in cases of electromagnetic fields and the Proca field, the gauge-type condition (58) takes the form

\[ \partial_\mu A_\nu + \partial_\nu A_\mu = \alpha_{\mu\nu}(x). \]

In particular, one can reproduce the standard Lorentz and \( \alpha \)-gauge conditions or introduce other conditions on \( \partial_\mu A_\nu + \partial_\nu A_\mu \).

8 Gravity in multimomentum variables

In this Section, \( X \) is a 4-dimensional world manifold which obeys the well-known topological conditions in order that a gravitational field exists on \( X^4 \).
In the gauge gravitation theory, classical gravity is described by pairs of tetrad fields and reducible Lorentz connections which play the role of gauge gravitational potentials [17]. In the absence of Dirac fermion fields, one can follow the affine-metric formulation of General Relativity when gravitational variables are both a pseudo-Riemannian metric \( g \) on a world manifold \( X^4 \) and a linear connection \( K \) on \( TX \). We call them a world metric and a world connection.

The world connections are associated with principal connections on the principal bundle \( LX \to X^4 \) of linear frames in \( TX \). The structure group of \( LX \) is \( GL_4 = GL_+^+(4, \mathbb{R}) \). Hence, there is the 1:1 correspondence between the world connections and the global sections of the bundle of principal connections

\[
C = J^1 LX/GL_4.
\]

There is the 1:1 correspondence between the world metrics \( g \) on \( X^4 \) and the global sections of the bundle \( \Sigma \) of pseudo-Euclidean bilinear forms in tangent spaces to \( X^4 \). This bundle is associated with \( LX \). The 2-fold covering of the bundle \( \Sigma \) is the quotient bundle \( LX/\text{SO}(3, 1) \).

Thus, the configuration space of the affine-metric gravitational variables is represented by the product of the corresponding jet manifolds

\[
J^1 C \times J^1 \Sigma.
\] (59)

It is provided with the adapted coordinates \((x^\mu, g^{\alpha\beta}, k^\alpha_{\beta\mu}, g^{\alpha\beta}_{\lambda}, k^{\alpha}_{\beta\mu\lambda})\) with respect to a holonomic bundle atlas of \( LX \).

Also the phase space \( \Pi \) is the product of the Legendre manifolds (3) of the bundles \( C \) and \( \Sigma \). The fibred coordinates of \( \Pi \) are \((x^\mu, g^{\alpha\beta}, k^\alpha_{\beta\mu}, p^{\alpha\beta}_{\lambda}, p^\alpha_{\beta\mu\lambda})\).

On the configuration space (59), the Hilbert-Einstein Lagrangian density of General Relativity reads

\[
L_{HE} = -\frac{1}{2\kappa} g^{\beta\lambda} \mathcal{F}^{\alpha}_{\beta\alpha\lambda} \sqrt{-g}\omega, \quad \mathcal{F}^{\alpha}_{\beta\nu\lambda} = k^{\alpha}_{\beta\lambda\nu} - k^{\alpha}_{\beta\nu\lambda} + k^{\alpha}_{\nu\epsilon\lambda} k^{\epsilon}_{\beta\lambda} - k^{\alpha}_{\epsilon\lambda\nu} k^{\epsilon}_{\beta\lambda}.
\] (60)

It is affine, and we can follow the general procedure from Section 6. The corresponding Legendre morphism is given by the expressions

\[
p_{\alpha\beta\lambda} \circ \hat{L}_{HE} = 0, \quad p^\alpha_{\beta\nu\lambda} \circ \hat{L}_{HE} = \pi^\alpha_{\beta\nu\lambda} = \frac{1}{2\kappa} (\delta^{\nu}_{\alpha} g^{\beta\lambda} - \delta^{\lambda}_{\alpha} g^{\beta\nu}) \sqrt{-g}.
\] (61)

Let us construct complete family of Hamiltonian forms (48) associated with the Lagrangian density (60). To minimize it, we consider a certain subset of connections on the bundle \( C \times \Sigma \). Let \( K \) be a world connection and

\[
S_K^{\alpha}_{\beta\nu\lambda} = \frac{1}{2} [k^{\alpha}_{\epsilon\nu} k^{\epsilon}_{\beta\lambda} - k^{\alpha}_{\epsilon\lambda} k^{\epsilon}_{\beta\nu} + \partial_{\lambda} K^{\alpha}_{\beta\nu} + \partial_{\nu} K^{\alpha}_{\beta\lambda} - 2K^{\nu}_{\mu\lambda}(K^{\mu}_{\beta\epsilon} - k^{\mu}_{\beta\epsilon}) + K^{\epsilon}_{\beta\mu} k^{\alpha}_{\epsilon\nu} + K^{\epsilon}_{\beta\nu} k^{\alpha}_{\epsilon\lambda} - K^{\alpha}_{\epsilon\lambda} k^{\epsilon}_{\beta\nu} - K^{\alpha}_{\epsilon\nu} k^{\epsilon}_{\beta\lambda}]
\]
the corresponding connection (64) on the bundle $C$. Let $K'$ be another symmetric world connection which induces a connection on the bundle $\Sigma$. On the bundle $C \times \Sigma$, we consider the following connection

$$
\Gamma^\alpha_{\beta\lambda} = -K'^\alpha_{\epsilon\lambda}g^{\epsilon\beta} - K'^\beta_{\epsilon\lambda}g^{\alpha\epsilon}, \quad \Gamma^\alpha_{\beta\nu\lambda} = S_K^\alpha_{\beta\nu\lambda} - R^\alpha_{\beta\nu\lambda}
$$

(62)

where $R$ is the curvature of the connection $K$. The corresponding Hamiltonian form (68) is given by the expression

$$
H_{HE} = (p_{\alpha\beta} \lambda d\alpha^\beta + p_{\alpha} \beta\nu\lambda dK^\alpha_{\beta\nu}) \wedge \omega_\lambda - \mathcal{H}_{HE}\omega,
$$

$$
\mathcal{H}_{HE} = -p_{\alpha\beta} \lambda(K'^\alpha_{\epsilon\lambda}g^{\epsilon\beta} + K'^\beta_{\epsilon\lambda}g^{\alpha\epsilon}) + p_{\alpha} \beta\nu\lambda\Gamma^\alpha_{\beta\nu\lambda} - R^\alpha_{\beta\nu\lambda}(p_{\alpha} \beta\nu\lambda - \pi_{\alpha} \beta\nu\lambda).
$$

(63)

It is associated with the Lagrangian density $L_{HE}$. Given $H_{HE}$ (63), the corresponding covariant Hamiltonian forms for General Relativity read

$$
\partial_\gamma g^{\alpha\beta} + K'^\alpha_{\epsilon\lambda}g^{\epsilon\beta} + K'^\beta_{\epsilon\lambda}g^{\alpha\epsilon} = 0,
$$

$$
\partial_\gamma k^\alpha_{\beta\nu} = \Gamma^\alpha_{\beta\nu\lambda} - R^\alpha_{\beta\nu\lambda},
$$

(64a)

$$
\partial_\gamma p_{\alpha\beta} \lambda = p_{\epsilon\beta} R^\alpha_{\epsilon\beta\alpha} + p_{\epsilon\alpha} \beta R^\epsilon_{\alpha\beta\lambda} + \frac{1}{\kappa}(R_{\epsilon\alpha\beta} - \frac{1}{2}g_{\epsilon\alpha\beta}R)\sqrt{-g},
$$

(64b)

$$
\partial_\gamma p_{\alpha} \beta\nu\lambda = -p_{\epsilon\nu} \alpha (K'^\epsilon_{\alpha\gamma\lambda} + p_{\epsilon\nu} \beta (\nu\gamma)K'_{\alpha\gamma\lambda} - p_{\epsilon\nu} (\nu\gamma)K'^\epsilon_{\alpha\gamma\lambda} + p_{\epsilon} \beta (\nu\gamma)K'_{\alpha\gamma\lambda},
$$

(64c)

$$
\partial_\gamma p_{\alpha} \beta\nu\lambda = -p_{\epsilon\nu} \alpha (K'^\epsilon_{\alpha\gamma\lambda} + p_{\epsilon\nu} \beta (\nu\gamma)K'_{\alpha\gamma\lambda} - p_{\epsilon\nu} (\nu\gamma)K'^\epsilon_{\alpha\gamma\lambda} + p_{\epsilon} \beta (\nu\gamma)K'_{\alpha\gamma\lambda}. (64d)
$$

The equations (64a) and (64b) represent the gauge-type conditions (68). In accordance with the canonical splitting (58), the equations (64c) are brought into the pair of equations

$$
\mathcal{F}^\alpha_{\beta\nu\lambda} = R^\alpha_{\beta\nu\lambda},
$$

(65)

$$
\partial_\nu(K'^\alpha_{\beta\lambda} - k^\alpha_{\beta\lambda}) + \partial_\lambda(K'^\alpha_{\beta\nu} - k^\alpha_{\beta\nu}) - 2K^\epsilon_{(\nu\lambda)}(K'^\alpha_{\beta\epsilon} - k^\alpha_{\beta\epsilon}) + K^\epsilon_{\beta\nu\lambda}k^\alpha_{\epsilon\nu} + k^\epsilon_{\beta\nu\lambda}k^\alpha_{\epsilon\nu} - K^\alpha_{\epsilon\lambda}k^\epsilon_{\beta\nu} - K^\alpha_{\epsilon\nu}k^\epsilon_{\beta\lambda} = 0.
$$

(66)

For a given world metric $g$ and a world connection $k$, there always exist connections $K'$ and $K$ such that these gauge conditions hold. It follows that the Hamiltonian forms (63) constitute a complete family.

Being restricted to the constraint space (61), the equations (64c) and (64d) read

$$
\frac{1}{\kappa}(R_{\epsilon\alpha\beta} - \frac{1}{2}g_{\epsilon\alpha\beta}R)\sqrt{-g} = 0,
$$

(67)

$$
D_{\alpha}(\sqrt{-g}g^{\nu\beta}) - \delta_{\alpha} D_{\lambda}(\sqrt{-g}g^{\lambda\beta}) + \sqrt{-g}g^{\nu\beta}(k^\lambda_{\alpha\lambda} - k^\lambda_{\lambda\alpha}) + g^{\lambda\beta}(k^\nu_{\lambda\alpha} - k^\nu_{\alpha\lambda}) + \delta_{\alpha} g^{\lambda\beta}(k^\mu_{\nu\mu} + k^\mu_{\mu\lambda} - k^\mu_{\mu\lambda}) = 0,
$$

(68)

$$
D_{\lambda}g^{\alpha\beta} = \partial_\lambda g^{\alpha\beta} + k^\alpha_{\mu\lambda}g^{\mu\beta} + k^\beta_{\mu\lambda}g^{\alpha\mu}.
$$

(69)

Substituting the condition (65) into the equation (67), we obtain the Einstein equations

$$
\mathcal{F}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{F} = 0.
$$

(69)
The equations (68) and (69) are the familiar equations for a gravity in the nonsymmetric Palatini variables. The former is the equation for torsion and nonmetricity terms of the connection $k_{\beta\nu}$. In the absence of matter, it has the well-known solution

$$k_{\beta\nu} = \left\{_{\beta\nu}^{\alpha} \right\} - \frac{1}{2} \delta_{\nu}^{\alpha} V_{\beta}, \quad D_{\alpha}g^{\beta\gamma} = V_{\alpha}g_{\beta\gamma},$$

where $V_{\alpha}$ is an arbitrary covector field corresponding to the so-called projective freedom.

The multimomentum quantum field theory has been hampered by the lack of satisfactory commutation relations between multimomentum canonical variables. At the same time, the multimomentum Hamiltonian formalism can be extended to quantum field theory if one considers chronological forms, but not commutation relations \[21\]. Moreover, it may incorporate together the canonical and algebraic approaches to quantization of fields.

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