Segre varieties, CR geometry and Lie symmetries of second order PDE systems.

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Abstract. We establish a link between the CR geometry of real analytic submanifolds in \( \mathbb{C}^n \) and the geometric PDE theory. The main idea of our approach is to consider biholomorphisms of a Levi-nondegenerate real analytic Cauchy-Riemann manifold \( M \) as pointwise symmetries of a second order holomorphic PDE system defining the Segre family of \( M \). This allows to employ the well-elaborated PDE tools in order to study the biholomorphism group of \( M \). We give several examples and applications to the CR geometry: the results on the finite dimensionality of the biholomorphism group and precise estimates of its dimension, explicit parametrization of the Lie algebra of infinitesimal automorphisms etc. We deduce these results as a special case of more general statements concerning related properties of symmetries of second order PDE systems.

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1 Introduction

The main goal of the present paper is to establish the relationship between the CR geometry of a real analytic generic submanifold of \( \mathbb{C}^{n+m} \) and the geometric (or formal) theory of PDE. We apply a general method which is due to S.Lie in order to study infinitesimal symmetries of a holomorphic completely overdetermined involutive second order PDE system with first order relations and \( n \) independent and \( m \) dependent variables. For any given system of this class this method allows to determine whether the dimension of the Lie algebra of infinitesimal symmetries if finite; if this is the case, the Lie method leads to explicit recursive formulae which permit to compute terms of any order in the Taylor expansion of coefficients of an infinitesimal symmetry of such a system and to show that these expansions (and so any symmetry) are uniquely determined by their terms of finite order. This gives a precise upper estimate of the dimension of the symmetry group for such a system and an explicit parametrization of the symmetry group.

From the complex analysis point of view our interest in these questions is explained by the fact that the Segre family of a real analytic generic Levi nondegenerate subvariety \( M \) in \( \mathbb{C}^{n+m} \) (introduced to the modern theory by S.S.Chern and S.Webster) is a family of (graphs of) solutions of a holomorphic completely overdetermined involutive PDE system with \( m \) dependent and \( n \) independent variables and some additional first order equations if the real codimension of \( M \) is \( > 1 \), i.e. if \( M \) is not a hypersurface. Systems without first order relations were studied in our previous paper \[29\] so in the present paper we consider the more complicated higher codimensional case. The biholomorphic invariance of the Segre family means precisely that every biholomorphic
automorphism of \( M \) is a Lie symmetry of the PDE system defining its Segre family i.e. maps the graph of a solution to the graph of another solution. So we show how PDE symmetries techniques can be used in order to study the complex geometry of real analytic submanifolds in \( \mathbb{C}^n \) and to obtain precise upper estimates of the dimension and explicit parametrization of their automorphism groups etc.; various results of this type have been obtained by several authors using different methods (see a more detailed discussion below). But it is worth to emphasize that systems describing the Segre families of real analytic submanifolds form a very special subclass in the class of holomorphic completely overdetermined involutive systems with first order relations. So we consider a much more general situation and generalize some known results on automorphisms of CR manifolds.

In the present paper we pay more attention to the development of basic tools of the proposed PDE approach to the CR geometry and do not consider the most general classes of CR manifolds in order to avoid technical complications and long computations. However, the proposed method allows to obtain much more general and precise results not only for CR manifolds, but for symmetries of wide classes of PDE as well. Our main conclusion is that the very intensively developing theory of CR maps can be naturally viewed as a part of the geometric PDE theory and actually studies special pointwise symmetries of special holomorphic PDE systems. From our point of view, the further progress in the study of CR maps between real analytic submanifolds in \( \mathbb{C}^n \) may be achieved by application of advanced tools of the formal PDE theory combining with complex algebraic and differential geometry methods. This provides the natural framework for the CR geometry of real analytic manifolds and links it with the classical complex geometry.

2 Generatities of the Lie theory

In this section we recall certain basic tools of the Lie method of study of infinitesimal symmetries of differential equations. They are very well known to the experts in the geometric PDE theory and the differential geometry; for reader’s convenient we give a brief exposition. A more detailed information and the proofs of all statements of this section can be found in [7], [21], [23], [24].

2.1. Local transformation groups and symmetry groups.

Let \( \Omega \) be a domain in \( \mathbb{C}^n \). A local group of biholomorphic transformations acting on \( \Omega \) is given by a (local) connected complex Lie group \( G \), a domain \( D \) such that \( \{e\} \times \Omega \subset D \subset G \times \Omega \), and a holomorphic map \( \varphi : D \to \Omega \) with the following properties: (i) if \((h, x) \in D\), \((g, \varphi(h, x)) \in D\), and also \((gh, x) \in D\), then \( \varphi(g, \varphi(h, x)) = \varphi(gh, x)\); (ii) for all \( x \in \Omega \), \( \varphi(e, x) = x \); (iii) if \((g, x) \in D\), then \((g^{-1}, \varphi(g, x)) \in D\) and \( \varphi(g^{-1}, \varphi(g, x)) = x\).

Historically the notion of a group of transformations was introduced by S.Lie in connection with a study of transformations preserving a given PDE system (or more precisely, the space of its solutions). Such transformations are called symmetries (sometimes, the Lie symmetries, pointwise symmetries, classical symmetries). In the present paper we apply the Lie method of studying of PDE symmetries to a special but geometrically important class of holomorphic completely overdetermined second order PDE systems with first order relations, i.e. systems of the form

\[
(S) : u_{x_1}^1 = F_{ij}(x, u, u_x^1), \quad i, j = 1, \ldots, n, \quad u_x^k = H^k(x, u, u_x^1), \quad k = 2, \ldots, m
\]

where \( x = (x_1, \ldots, x_n) \) are independent variables, \( u(x) = (u^1(x), \ldots, u^m(x)) \) are unknown functions
(dependent variables), \( u^j = (u^j_1, ..., u^j_n) \) and \( F_{ij}, H^k \) are holomorphic functions (of course, we will always assume that \( F_{ij} = F_{ji} \)). Since this system is highly overdetermined, it is natural to assume that it satisfies some compatibility conditions. We will assume that such a system satisfies some integrability conditions of the Frobenius type (see below). This class of systems naturally arises in various areas of the geometry and PDE.

The solutions of such a system are holomorphic vector valued functions \( u = u(x) \); denote by \( \Gamma_u \) the graph of a solution \( u \).

**Definition 2.1** A symmetry group \( \text{Sym}(S) \) of a system \( (S) \) is a local complex transformation group acting on a domain in the space \( \mathbb{C}^n \times \mathbb{C}_u^m \) of independent and dependent variables with the following property: for every solution \( u(x') \) of \( (S) \) and every \( g \in G \) such that the image \( g(\Gamma_u) \) is defined, it is a graph of a solution of \( (S) \).

Often the largest symmetry group is of main interest (and so we write the symmetry group); for us this is not very essential since our methods give a description of any symmetry group for given system. In order to fix the terminology, everywhere below by the symmetry group we mean the largest one.

The definition of a symmetry group given above is not very well working in practice in the sense that it does not give an efficient tool to find the Lie symmetries. The main idea of the Lie method is to study the Lie algebra of a symmetry group instead of the group itself.

### 2.2. Vector fields and one-parameter transformation groups.

Consider a one parameter local complex Lie group of transformations (LTG) \( x^* = X(x, t) \) with the identity \( t = 0 \) acting on a complex manifold with local coordinates \( x = (x_1, ..., x_n) \). Let \( \theta(x) = \frac{\partial X}{\partial t} \big|_{t=0} \). The vector field \( X = X(x) = \sum_{j=1}^n \theta_j(x) \frac{\partial}{\partial x_j} \) is called the infinitesimal generator of our (LTG); we use the vector notation: if \( f = (f_1, ..., f_k) \) is a holomorphic vector function, then \( Xf = (Xf_1, ..., Xf_k) \). In particular, \( Xx = \theta(x) \). Recall that there exists a parametrization \( \tau(t) \) such that the above (LTG) is equivalent to the solution of the initial value problem for the first order ODE system \( \frac{dx^*}{dt} = \theta(x^*) \) (the First Fundamental Lie Theorem). A one-parameter (LTG) can be found from its infinitesimal generator by means of the Lie series (the exponential map): \( x^* = e^{tX}x = x + tXx + (t^2/2)X^2x + ... \), where \( X^k := XX^{k-1}, k = 1, 2, ..., X^0f(x) := f(x), t \in \mathbb{C} \). In the general case of a \( d \)-dimensional Lie transformation group \( G \) any group element in a neighborhood of the identity can be obtained by the exponential map for a suitable vector field from the Lie algebra of \( G \). So every local Lie group is completely determined by a vector field basis \( \{X_1, ..., X_d \} \) of its Lie algebra and can be explicitly parametrized via the exponential map \( e^{\sum t_jX_j} = \Pi e^{t_jX_j} \); the parameters \( t_1, ..., t_d \) are local coordinates on \( G \). The exponential map can be used as a definition of a symmetry group; this group is a finite dimensional Lie group if and only its Lie algebra is finite dimensional.

### 2.3. Jet bundles and prolongations of group actions.

The second key tool of the Lie theory is the notion of prolongation of an LTG action to a jet bundle. Recall this construction. Let \( f \) and \( g \) be two holomorphic maps in a neighborhood of the origin in \( \mathbb{C}^n \) to \( \mathbb{C}^m \) taking the origin to the origin. As usual, we say that they have the same \( r \)-jet at the origin if \( \partial^\alpha f(0) = \partial^\alpha g(0) \) for every \( \alpha : |\alpha| \leq r \) where we use the following notation (which we will keep everywhere throughout this paper): \( \partial^\alpha \varphi = \frac{\partial^\alpha \varphi}{\partial x_{\alpha_1} ... \partial x_{\alpha_r}} \) for \( \alpha = (\alpha_1, ..., \alpha_r), \alpha_1 \leq ... \leq \alpha_r, \) and \( |\alpha| := r \).

More generally, let \( M \) and \( N \) be two complex manifolds and \( f : M \rightarrow N, g : M \rightarrow N \) be two holomorphic maps. Let \( x \) and \( u \) be local holomorphic coordinates near \( p \in M \) and \( q \in N \) respectively such that \( x(p) = 0, u(q) = 0 \). We say that \( f \) and \( g \) have the same \( r \)-jet at \( p \), if \( u \circ f \circ x^{-1} \)
and \( u \circ g \circ x^{-1} \) have the same \( r \)-jet. It is easy to see that the definition is correct, i.e. does not depend on the choice of the coordinates. The relation that two maps have the same \( r \)-jet at \( p \) is an equivalence relation and the equivalence class with the representative \( f \) is denoted by \( j^r_p(f) \); it is called the \( r \)-jet of \( f \) at \( p \). The point \( p \) is called the source and the point \( q \) the target of \( j^r_p(f) \).

Denote by \( J^r_p(M,N) \) the set of all \( r \)-jets of maps from \( M \) to \( N \) with the source \( p \) and the target \( q \) and consider the set \( J^r(M,N) = U_{p \in M, q \in N} J^r_p(M,N) \). Consider also the natural projections \( \pi_M : J^r(M,N) \to M \) and \( \pi_N : J^r(M,N) \to N \) defined by \( \pi_M(j^r_p(f)) = p \) and \( \pi_N(j^r_p(f)) = f(p) \).

Declaring the pullbacks of open sets in \( M \) and \( N \) lifts canonically to a fiber preserving biholomorphism \( g \). Using local coordinates \( f \) in \( M,N \) induces a biholomorphic change of local coordinates in \( J^r(M,N) \) as follows: Set \( u^{(1)}(\alpha) = (u^1_\alpha, ..., u^m_\alpha, ..., u^n_\alpha) \), \( u^{(s)}(\alpha) = (u^s_\alpha) \) with \( j = 1, ..., m \), \( \alpha = (\alpha_1, ..., \alpha_s) \), \( \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_s \). The chart \( h \) is defined by

\[
h : j^r_p(f) \mapsto (x_j, u^k, u^{(1)}, ..., u^{(r)})
\]

\[
x_j = x_j(p'), u^k = u^k(f(p')), u^{(1)}_{\alpha_1...\alpha_s} = \partial^\alpha (u^j \circ f \circ x^{-1})(x(p')) \leq s \leq r
\]

for \( p' \) close enough to \( p \). These coordinates are called the natural coordinates on \( J^r(M,N) \). The Leibnitz formula and the chain rule imply that biholomorphic changes of local coordinates on \( M \) and \( N \) induce a biholomorphic change of local coordinates in \( J^r(M,N) \). This defines the natural structure of a complex manifold on the space \( J^r(M,N) \) and equips it with the structure of a holomorphic fiber bundle over \( M \times N \) with the natural projection \( \pi_{M \times N} : J^r(M,N) \to M \times N \).

Let \( G \) be a local group of biholomorphic transformations acting on \( M \times N \). Every biholomorphism \( g \in G \), \( g : M \times N \to M \times N \), \( g : (x,u) \mapsto (x^*, u^*) \) close enough to the identity lifts canonically to a fiber preserving biholomorphism \( g^{(r)} : J^r(M,N) \to J^r(M,N) \) as follows: if \( u = f(x) \) is a holomorphic function near \( p \), \( q = f(p) \) and \( u^* = f^*(x^*) \) is its image under \( g \) (that is the graph of \( f^* \) is the image of the graph of \( f \) under \( g \) near the point \( (p^*, q^*) = g(p, q) \)), then the jet \( j^r_p(f^*) \) is by the definition the image of \( j^r_p(f) \) under \( g^{(r)} \). In particular, a one-parameter local Lie group of transformations \( G \) canonically lifts to \( J^r(M,N) \) as a one-parameter Lie group of transformations \( G^{(r)} \) which is called the r-prolongation of \( G \). The infinitesimal generator \( X^{(r)} \) of \( G \) is called the r-prolongation of the infinitesimal generator \( X \) of \( G \).

Our considerations will be purely local so \( M \) and \( N \) will be open subsets in \( \mathbb{C}^m \) and \( \mathbb{C}^m \) respectively. In this case we write \( J^1(n,m) \) instead of \( J^1(M,N) \).

Consider in local coordinates a vector field \( X(x,u) = \sum_{j=1}^n \theta^j(x,u) \frac{\partial}{\partial x^j} + \sum_{k=1}^m \eta^k(x,u) \frac{\partial}{\partial u^k} \). In the natural coordinates its r-prolongation has the form

\[
X^{(r)} = X + \sum_{j,\mu} \eta^j_\mu \frac{\partial}{\partial u^j} + ... + \sum_{i_1,\ldots,i_r,\mu} \eta^{i_1i_2i_3\ldots i_r}_\mu \frac{\partial}{\partial u^{i_1i_2i_3\ldots i_r}}
\]

In order to compute the coefficients of this prolongation, define the operator of total derivative:

\[
D_i = \frac{\partial}{\partial x^i} + \sum_k u^k_i \frac{\partial}{\partial u^k} + \sum_{\mu,j} u^k_{ij} \frac{\partial}{\partial u^{\mu_j}} + ...
\]

The following elementary statement gives an explicit recursive formula for the coefficients of a prolongation and is the main computational tool in the Lie theory.
Proposition 2.2 One has

\[ \eta^\mu_i = D_i \eta^\mu_i - \sum_j (D_i \theta^j) u^\mu_j, \eta^\mu_{i_1\ldots i_r} = D_i \eta^\mu_{i_1\ldots i_r} - \sum_j (D_i \theta^j) u^\mu_{i_1\ldots i_r} - \sum_j (D_i \theta^j) u^\mu_{i_1\ldots i_r} \]

In particular the second prolongation \( X(2) \) is given by \( X(2) = X(1) + \sum_{\mu;i_1 \neq i_2} \eta^\mu_{i_1 i_2} \frac{\partial}{\partial u_{i_1 i_2}} \) with \( X(1) = X + \sum_{\mu,i} \eta^\mu_i \frac{\partial}{\partial u_i} \)

Proposition 2.2 implies the following formula giving an explicit expression for the coefficients of \( X(2) \):

\[
\begin{align*}
\eta^\mu_{i_1 i_2} &= \eta^\mu_{i_2 x_{i_1}} + u^\mu_{i_1} \left[ \eta^\mu_{i_2 u_{i_1}} - \theta^i_{x_{i_1} x_{i_2}} \right] + u^\mu_{i_2} \left[ \eta^\mu_{i_1 u_{i_2}} - \theta^i_{x_{i_1} x_{i_2}} \right] + \sum_{k \neq \mu} u^k_{i_2} \eta^\mu_{x_{i_1} u_{i_2}} \\
&+ \sum_{k \neq \mu} u^k_{i_2} \eta^\mu_{x_{i_1} u_{i_2}} - \sum_{k \neq \mu} u^k_{i_2} \eta^\mu_{x_{i_1} u_{i_2}} - \sum_{k \neq \mu} u^k_{i_2} \eta^\mu_{x_{i_1} u_{i_2}} - \sum_{k \neq \mu} u^k_{i_2} \eta^\mu_{x_{i_1} u_{i_2}} \\
&+ \sum_{r \neq \mu, p \neq \mu} u^r_{i_2} u^p_{i_1} \eta^\mu_{u_{i_1} u_{i_2}} + \sum_{t \neq \mu} u^t_{i_1} u^t_{i_2} \left[ -\theta^i_{x_{i_1} u_{i_2} + \eta^\mu_{u_{i_1} u_{i_2}}} \right] + \sum_{q \neq \mu} u^q_{i_2} u^q_{i_1} \left[ -\theta^i_{x_{i_1} u_{i_2} + \eta^\mu_{u_{i_1} u_{i_2}}} \right] \\
&+ \left[ \eta^\mu_{i_1 u_{i_2}} - \theta^i_{x_{i_1} u_{i_2}} \right] u^\mu_{i_1 i_2} + \sum_{a,b,s} u^a_{i_2} u^b_{i_1} \theta^s_{u_{i_1} u_{i_2}} + \Lambda^\mu_{i_1 i_2}
\end{align*}
\]

for \( i_1 \neq i_2 \) and

\[
\begin{align*}
\eta^\mu_{i_1 i_2} &= \eta^\mu_{i_2 x_{i_1}} + u^\mu_{i_1} \left[ 2\eta^\mu_{i_1 u_{i_1}} - \theta^i_{x_{i_1}} \right] + 2 \sum_{k \neq \mu} u^k_{i_1} \eta^\mu_{x_{i_1} u_{i_2}} - \sum_{k \neq \mu} u^k_{i_1} \theta^k_{x_{i_1} u_{i_2}} - 2 \sum_{k \neq \mu} u^k_{i_1} \theta^k_{x_{i_1} u_{i_2}} \\
&+ \sum_{r \neq \mu, p \neq \mu} u^r_{i_2} u^p_{i_1} \eta^\mu_{u_{i_1} u_{i_2}} + \sum_{t \neq \mu} u^t_{i_1} u^t_{i_2} \left[ -\theta^i_{x_{i_1} u_{i_2} + \eta^\mu_{u_{i_1} u_{i_2}}} \right] + \sum_{q \neq \mu} u^q_{i_2} u^q_{i_1} \left[ -\theta^i_{x_{i_1} u_{i_2} + \eta^\mu_{u_{i_1} u_{i_2}}} \right] \\
&+ \left[ \eta^\mu_{i_1 u_{i_2}} - 2\theta^i_{x_{i_1} u_{i_2}} \right] (u^\mu_{i_1})^2 - \sum_{a,b,s} u^a_{i_2} u^b_{i_1} \theta^s_{u_{i_1} u_{i_2}} + \Lambda^\mu_{i_1 i_2}
\end{align*}
\]

with

\[
\Lambda^\mu_{i_1 i_2} = \sum_{s} u^s_{i_2} \eta^\mu_{i_1 s} - \sum_{p} u^p_{i_2 p} \theta^p_{x_{i_1}} - \sum_{j} u^\mu_{i_1 j} \theta^j_{x_{i_2}} - \sum_{p,q} u^p_{i_2 p} u^q_{i_1 q} \theta^p_{u_{i_1}} - \sum_{p,q} u^p_{i_2 p} u^q_{i_1 q} \theta^p_{u_{i_1}} - \sum_{s,j} u^s_{i_1 j} u^s_{i_2 j} \theta^s_{u_{i_1}} \]

2.4. Infinitesimal symmetries of differential equations. An infinitesimal generator of a one-parameter group of symmetries of a system of PDE (S) is called an infinitesimal symmetry of this system. They form a Lie algebra with respect to the Lie bracket which is denoted by \( \text{Lie}(S) \).

Let (S) be a holomorphic PDE system of order \( r \); we consider its solutions on \( M \) with values in \( N \). Then it defines naturally a complex subvariety \( (S_r) \) in the jet space \( J^r(M,N) \) obtained by the replacing of the derivatives of dependent variables by the corresponding natural coordinates in the jet space.
Example 1. Let $M \subset \mathbb{C}^2$, $N \subset \mathbb{C}$ be domains, $(S)$ be a holomorphic second order ODE $u_{xx} = F(x,u,u_x)$, $(x,u) \in M \times N$. Let $(x,u_1,u_1)$ be the natural coordinates on the jet space $J^2(2,1)$. Then $(S_2)$ is a complex 3-dimensional submanifold in $J^2(M \times N)$ defined by the equation $u_{11} = F(x,u,u_1)$.

Example 2. More generally, let $M \subset \mathbb{C}^n$, $N \subset \mathbb{C}^m$ be domains, $(S)$ be a holomorphic completely overdetermined second order system: $(S) : u_{x_ix_j}^k = F_{ij}^k(x,u,u_x)$, $k = 1,\ldots,m$, $i,j = 1,\ldots,n$, $(x,u) \in M \times N$. Denote by $(x,u,u^k_i,u^k_j)$ the natural coordinates on $J^2(n,m)$. Then $(S_2)$ is a complex submanifold in $J^2(M \times N)$ defined by the equations $u^k_{ij} = F_{ij}^k(x,u,u^{(1)})$ where $u^{(1)} = (u^1_i)$.

Since $\pi_M : J^r(M \times N) \to M$ also is a fiber bundle over $M$, every holomorphic map $u : M \to N$ defines a section of this bundle by $p \mapsto j_p^r(u)$. So $u$ is a holomorphic solution of the system $(S)$ if and only if the section $p \mapsto j_p^r(u)$ is contained in the variety $(S_r)$.

If $(S_r)$ is a regular submanifold of $J^r(M,N)$, the system $(S)$ is called of maximal rang. Thus every system $(S)$ of maximal rang can be identified with a complex submanifold of the holomorphic fiber bundle $\pi_{M \times N} : J^r(M \times N) \to M \times N$ and its solutions can be identified with sections of the holomorphic fiber bundle $\pi_M : J^r(M \times N) \to M$. As we have seen in the above examples, completely overdetermined systems always are of maximal rang.

Definition 2.3 A system $(S)$ is called locally regular, if for every point $P \in J^r(M,N)$ with the natural projection $\pi_{M \times N}(P) = (p,q) \in M \times N$ there exists a solution $u(x)$ of $(S)$ holomorphic near $p$ such that $j_p^r(u) = P$.

A holomorphic function $F$ is called an invariant function for a one-parameter LTG with an infinitesimal generator $X$ if $F(x^s) \equiv F(x)$. It is easy to see that $F(x^s) = e^{sX}F(x)$; this implies that $F$ is invariant if and only if $XF(x) = 0$. A complex subvariety $V = \{ F(x) = 0 \}$, where $F$ is a vector valued holomorphic function of maximal rang, is called an invariant variety for a one-parameter LTG if $F(x^s) = 0$ when $F(x) = 0$. Clearly, $V$ is an invariant variety if and only if $XF(x) = 0$ for every $x \in V$ that is $X$ is a tangent field to $V$.

The importance of these notions explains by the following simple but fundamental statement (see for instance [23, 24]):

Proposition 2.4 (The Lie criterion) A vector field $X$ is an infinitesimal symmetry of a locally regular system $(S)$ of order $r$ and of maximal rang if and only if the variety $(S_r)$ is invariant for the $r$-prolongation $X^{(r)}$.

It follows by the Cauchy existence theorem that every system of ordinary differential equations (solved with respect to the highest order derivatives) is locally regular. In the case of several independent variables we need the Frobenius theorem which imposes integrability conditions.

A holomorphic completely overdetermined second order PDE system of the form

$$(S) : u^k_{x_ix_j} = F_{ij}^k(x,u,u_x), i,j = 1,\ldots,n, k = 1,\ldots,m$$

is always of maximal rang, but in general it is locally regular. So we need to assume that it satisfies the integrability condition in the following sense: the distribution on the tangent bundle of the jet space $J^1(n,m)$ defined by the differential forms
ω_i^k = du_i^k - \sum_j F_{ij}^k(x, u, u(1))dx_j, \phi_i^k = du_i^k - \sum_i u_i^kdx_i

is completely integrable. We call such systems completely integrable or involutive. It follows by the Frobenius theorem that every involutive system is locally regular. Thus, the last proposition is applicable for this class of systems.

In the next section we will see that this proposition gives an efficient tool for the computation of infinitesimal symmetries of holomorphic completely overdetermined second order involutive systems with additional first order relations.

3 Segre varieties, holomorphic maps and PDE symmetries

Denote by $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^m$ the standard coordinates in $\mathbb{C}^{n+m}$. All our considerations will be purely local, so all neighborhoods, domains etc. (which we usually even do not mention) always are supposed to be as small as we need (the most rigorous way is to use the language of germs; following the classical tradition we do not employ it in order to avoid useless formalizations). By a real analytic submanifold $\mathcal{M}$ of codimension $m$ in $\mathbb{C}^{n+m}$ we mean the zero set $\mathcal{M} = \{Z : r(Z, \overline{Z}) = 0\}$ of a real analytic $\mathbb{R}^m$-valued map $r = (r^1, ..., r^m)$ of maximal rank. Such a manifold is called generic if $\partial r^1 \wedge ... \wedge \partial r_m \neq 0$. In this paper we consider generic manifolds only. The holomorphic tangent space $H_p(\mathcal{M})$ at a point $p \in \mathcal{M}$ is the maximal complex subspace of the tangent space of $\mathcal{M}$ at $p$. $\mathcal{M}$ is called Levi nondegenerate at $p$ if the two following conditions hold:

(i) there exists a linear combination of the Levi forms $L^j_p(u, v) = \sum_{j,k} r^j_z \overline{r}^k_z (p) u_j v_k, u, v \in H_p(\mathcal{M})$ which is a nondegenerate hermitian form on $H_p(\mathcal{M})$

(ii) the forms $L^j_p(u, v)$ are $\mathbb{C}$-linearly independent.

We say that $\mathcal{M}$ is Levi nondegenerate if it is Levi nondegenerate at every point. Often some authors call $\mathcal{M}$ Levi-nondegenerate if a slightly weaker condition holds instead of (i): the Levi form of $\mathcal{M}$ (considered as a vector valued hermitian form) has the trivial kernel. Our methods can be easily carried to this case (and even to a much more general situation). In the present paper we restrict ourselves by the consideration of the above class of varieties in order to avoid supplementary computations and complications of the notations.

A map $f : \mathcal{M} \rightarrow \mathcal{M}$ defined and biholomorphic in a neighborhood of $\mathcal{M}$ is called a biholomorphism or a (biholomorphic) automorphism of $\mathcal{M}$. These maps form a group with respect to the composition which is called the group of biholomorphisms or the automorphism group of $\mathcal{M}$ and is denoted by $\text{Aut}(\mathcal{M})$.

The study of automorphism groups of real submanifolds in $\mathbb{C}^{n+m}$ is a traditional problem of the geometric complex analysis and the complex differential geometry. An important fact here is that such a group (in the Levi nondegenerate case) is always a real finite dimensional Lie group. This phenomenon is due to the intrinsic geometry of a real submanifold induced by the complex structure of the ambient space. It has been studied in the founder works of E.Cartan [9], N.Tanaka [32], S.S.Chern - J.Moser [10] for the case of real hypersurfaces. Cartan, Tanaka and Chern study the equivalence problem for a $G$-structure corresponding to the natural CR-structure sitting on a real hypersurface in $\mathbb{C}^{n+1}$ and solve the equivalence problem for this structure using Cartan’s equivalence
method for general G-structures. In particular, this gives a complete list of biholomorphic invariants of a hypersurface. Moser solves the equivalence problem via his theory of a normal form of a real analytic hypersurface with respect to the action of local biholomorphisms. This theory gives many additional useful information about biholomorphic maps of real hypersurfaces. In particular, it leads to an explicit parametrization of the automorphism group. The approaches of Cartan - Tanaka - Chern and Moser have been developed for the case of submanifolds of higher codimension in the works of V.Beloshapka, A.Loboda, V.Ezhov - A.Isaev - G.Schmalz and other authors.

Another natural approach is to study the Lie algebra $\mathfrak{Lie}(M)$ of the automorphism group of a real analytic manifold $M$. Vector fields in $\mathfrak{Lie}(M)$ are called infinitesimal automorphisms of $M$. The knowledge of the Lie algebra allows to refine a neighborhood of the identity in the automorphism group via the exponential map i.e. essentially to describe completely the group in the local situation. The results in this direction have been obtained by E.Bedford - S.Pinchuk, V.Beloshapka, N.Tanaka, A.Tumanov, N.Stanton, and other authors.

The common feature of all these works is a direct study of a mixed "real-complex" structure of a hypersurface embedded to $\mathbb{C}^{n+1}$. This leads to computations with power series containing "mixed" terms of the type $Z^k \bar{Z}^l$ in order to find biholomorphic invariants. There is another way to find biholomorphic invariants of a real analytic submanifold in $\mathbb{C}^{n+m}$. For a fixed point $\zeta \in \mathbb{C}^{n+m}$ close enough to $M$ consider the complex submanifold $Q(\zeta) = \{Z : r(Z,\zeta) = 0\}$. It is called the Segre variety for B.Segre who introduced these objects. The basic property of the Segre varieties is their biholomorphic invariance: for every automorphism $f \in \text{Aut}(M)$ and any $\zeta$ one has $f(Q(\zeta)) = Q(f(\zeta))$. For the approach developed in the present paper, the utilisation of the complex conjugation in the definition of the Segre surface is technically inconvenient. So we consider the complex hypersurface $Q^*(\zeta) = Q(\overline{\zeta})$. Then for every $f \in \text{Aut}(M)$ one has $f(Q^*(\zeta)) = Q^*(f(\overline{\zeta}))$. Thus, $f$ maps any element of the family $\{Q^*(\zeta)\}_{\zeta}$ to another one. This property is crucial for our paper since it can be viewed from the geometric PDE point of view. Of course, we still call $Q^*(\zeta)$ the Segre variety and omit the star.

The Segre varieties were reintroduced to the modern theory in the important works of S.S.Chern and S.Webster and turned out to be a very useful tool for a study of holomorphic maps. The theory of Segre varieties has been applied to the study of analytic and algebraic extension of holomorphic maps by M.S.Baouendi - P.Ebenfelt - L.P.Rothschild, K.Diederich - S.Webster, K.Diederich - J.E.Fornaess, K.Diederich - S.Pinchuk, S.Webster, J.Faran and S.Webster also studied related geometric invariants.

M.S.Baouendi - P.Ebenfelt - L.P.Rothschild and D.Zaitsev used the Segre varieties geometry in order to obtain results concerning estimates of dimension and parametrization of automorphism groups for various classes of higher codimensional manifolds.

Our approach also makes use of the Segre varieties but the important difference is that we consider the subject from a more general PDE point of view. It is necessary to stress that the basic idea goes back to the founders works of B.Segre, E.Cartan and S.Lie’s school.

B.Segre observed that in $\mathbb{C}^2$ the set of Segre varieties of a Levi nondegenerate real analytic hypersurface $M$ (which is called the Segre family of $M$) is a regular two parameter family of holomorphic curves and so represents the trajectories of solutions of a holomorphic second order ordinary differential equation. The invariance of the Segre family with respect to $\text{Aut}(M)$ means that every biholomorphism of $M$ can be considered as a symmetry of the differential equation defining its Segre family.
Segre’s observation is of fundamental importance since it links the CR geometry with the PDE theory.

The study of symmetries of a second order ordinary differential equation (in some sense, completed) has been proceeded by S.Lie and his student A.Tresse \cite{31} (see also \cite{12, 23, 22}). In particular, such group is always a complex Lie group of dimension \(\leq 8\); this important fact allowed to B.Segre to conclude that \(\text{Aut}(\mathcal{M})\) is a real dimensional Lie group.

The idea of Segre can be naturally generalized to higher dimension as follows.

First of all, we consider the case where \(\mathcal{M}\) is a real analytic Levi nondegenerate hypersurface in \(\mathbb{C}^{n+1}\) through the origin.

After a biholomorphic change of coordinates in a neighborhood of the origin by the equation
\[
\{ w + \overline{w} + \sum_{j=1}^{n} \varepsilon_j z_j \overline{z_j} + R(Z, \overline{Z}) = 0 \}
\]
where \(\varepsilon_j = 1\) or \(-1\) and \(R = o(|Z|^2)\). For every point \(\zeta = (\eta_1, \ldots, \eta_n, \omega)\) the corresponding Segre variety is given by
\[
 w + \omega + \sum_{j=1}^{n} \varepsilon_j z_j \eta_j + R_z(\zeta) = 0.
\]
If we consider the variables \(x_j = z_j\) as independent ones and the variable \(w = u(x)\) as dependent one this equation can be rewritten in the form
\[
u + \omega + \sum_{j=1}^{n} \varepsilon_j x_j \eta_j + R(x, \zeta) = 0 \tag{1}
\]
(after an application of the implicit function theorem in order to remove \(u\) from \(R\)). Taking the derivatives in \(x_k\) we obtain the equations
\[
u_{x_k} + \varepsilon_k \eta_k + R_{x_k}(x, \zeta) = 0, \quad k = 1, \ldots, n \tag{2}
\]
The equations (1), (2) and the implicit function theorem imply that \(\zeta = \zeta(x, u, u_{x_1}, \ldots, u_{x_n})\) is a holomorphic function; taking again the partial derivatives in \(x_j\) in (2), we obtain the following completely overdetermined second order holomorphic PDE system:
\[
(\mathcal{S}_\mathcal{M}): \nu_{x_j, x_k} = F_{jk}(x, u, u_x), j, k = 1, \ldots, n
\]
with \(u_x = (u_{x_1}, \ldots, u_{x_n})\). It is very important to point out that this system necessarily satisfies the integrability condition of the Frobenius type. More precisely, with such a system one can associate the differential forms
\[
\omega_i = du_i - \sum_j F_{ij}(x, u, u^{(1)}) dx_j, \phi = du - \sum_i u_i dx_i
\]
defined on the jet space \(J^1(n, 1)\). It follows directly from the representation (1) of its integral manifolds that the distribution defined by these forms on the tangent bundle of \(J^1(n, 1)\) is completely integrable and so satisfies the Frobenius condition. The property of biholomorphic invariance of the Segre varieties means that any biholomorphism of \(\Gamma\) transforms the graph of a solution of \((\mathcal{S}_\mathcal{M})\) to the graph of another solution, i.e. is a Lie symmetry of \((\mathcal{S}_\mathcal{M})\). This naturally leads to a general consideration of a holomorphic involutive PDE system of the form
Thus, the study of biholomorphisms of real analytic Levi nondegenerate hypersurfaces can be reduced to the study of symmetries of holomorphic involutive PDE systems (with one dependent variable). However, the systems corresponding to Segre families form a very special subclass between involutive systems since the coefficients of (1) satisfy additional conjugation relations due to the fact that the defining function \( r \) is real valued. We point out here that the importance of the study of this class of PDE systems has been realized by S.S.Chern [11] who solved the equivalence problem for this class of systems with one dependent variable (see also the work of J.Faran [17]).

Now consider the higher codimensional case. First of all, we introduce the class of PDE systems which plays the major role in the present paper.

Let \( \mathcal{S} \) be a holomorphic second order PDE system with additional first order relations of the form

\[
\begin{align*}
\frac{\partial^2 u^1}{\partial x_i \partial x_j} &= F_{ij}(x, u, \frac{\partial u^1}{\partial x}), i, j = 1, \ldots, n \\
\frac{\partial u^k}{\partial x} &= G^k(x, u, \frac{\partial u^1}{\partial x}), k = 2, \ldots, m
\end{align*}
\]

In order to simplify the notations we introduce the dependent variables \( w := u^1 \) and \( v = (u^2, \ldots, u^m) \) so \( u = (w, v) \). Then our system can be rewritten in the form

\[
\begin{align*}
\frac{\partial w}{\partial x_i} &= F_{ij}(x, v, w, \frac{\partial w}{\partial x}), \\
\frac{\partial v^k}{\partial x} &= G^k(x, v, w, \frac{\partial w}{\partial x})
\end{align*}
\]

where we use the notation \( w_x = (w_{x_1}, \ldots, w_{x_n}) \), \( v^k_x = (v^k_{x_1}, \ldots, v^k_{x_n}) \), \( G^k = (G^k_1, \ldots, G^k_n) \). We will also use the notation \( v_x = (v^2_x, \ldots, v^m_x) \), \( G = (G^2, \ldots, G^m) \).

Consider a complex subvariety \( \Gamma \) in the jet space \( J^1(n, m) \) defined by \( (x, u, \frac{\partial u^1}{\partial x}) : \frac{\partial u^1}{\partial x} = G(x, u, \frac{\partial u^1}{\partial x}) \) in the natural coordinates. Then \( (x, u, \frac{\partial u^1}{\partial x}) \) are holomorphic local coordinates on \( \Gamma \) and we may consider the 1-forms defined on \( \Gamma \) as follows:

\[
\begin{align*}
\omega_i &= dw_i - \sum_j F_{ij}(x, u, \frac{\partial u^1}{\partial x}) dx_j, \\
\phi^1 &= dw - \sum_j w_j dx_j, \\
\phi^k &= dv^k - \sum_j G^k_j(x, u, \frac{\partial u^1}{\partial x}) dx_j, \quad k > 1
\end{align*}
\]

We say that the system \( \mathcal{S} \) is \textit{completely integrable} or \textit{involutive} if the distribution defined by these forms on the tangent bundle of \( \Gamma \) is completely integrable that is satisfies the Frobenius condition. It follows by the Frobenius theorem that if \( \mathcal{S} \) is involutive then it is locally regular i.e. for every point of the complex submanifold

\[
\mathcal{S}_2 : w_{ij} = F(x, u, \frac{\partial u^1}{\partial x}), v^{(1)} = G(x, u, \frac{\partial u^1}{\partial x})
\]

of \( J^2(n, m) \) there exists a solution of \( \mathcal{S} \) whose jet coincides with this point. In view of the Frobenius criterion the graphs of solutions of \( \mathcal{S} \) form a holomorphic foliation of \( \Gamma \) with \( n \)-dimensional leafs and depending on \( (n+m) \)-parameters if and only if \( \mathcal{S} \) is involutive.
Let now $\mathcal{M}$ be a Levi nondegenerate quadric in $\mathbb{C}^{n+m}$ given by $w_k + \overline{w}_k = L^k(z), \overline{z} >$, $k = 1, \ldots, m$ where every $L^k$ is a hermitian operator on $\mathbb{C}^n$ and $<z, \zeta> = \sum_{j=1}^{n} z_j \zeta_j$. We can assume that the hermitian form $<L^1(z), \overline{z}>$ is nondegenerate. For $(z, \omega) \in \mathbb{C}^n \times \mathbb{C}^m$ the corresponding Segre variety is $Q(\zeta, \omega) = \{(z, w) : w_k + \omega_k = L^k(z), \zeta >\}$. If we consider $x := z$ as independent variables and $u := w$ as dependent, then $Q(\zeta, \omega)$ is a graph of $u$: $Q(\zeta, \omega) = \{(x, u) : u^k + \omega_k = L^k(x), \zeta >\}$.

Let us construct a PDE system with a general solution given by the above family. First of all, considering the first partial derivatives we obtain the following system:

$$u_{x_i x_j}^1 = 0, u_{x_i}^k = A^k u_{x_i}^1, k = 2, \ldots, m$$

whose sets of solutions coincides with the Segre family of $\mathcal{M}$.

This construction can be immediately generalized to any Levi nondegenerate real analytic submanifold. Indeed, let $\mathcal{M}$ be a real analytic Levi nondegenerate submanifold in $\mathbb{C}^{n+m}$ through the origin. Then in a neighborhood of the origin it can be represented in the form $w_k + \overline{w}_k = L^k(z), \overline{z} > + o(|Z|^2)$, $k = 1, \ldots, m$. For $(z, \omega) \in \mathbb{C}^n \times \mathbb{C}^m$ the corresponding Segre variety is $Q(\zeta, \omega) = \{(z, w) : w_k + \omega_k = L^k(z), \zeta > + R^k(z, \zeta, \omega)\}$ where $R^k$ contains no term of order $\leq 2$ (after an application of the implicit function theorem if it is necessary). Consider $x := z$ as independent variables and $u := w$ as dependent, then $Q(\zeta, \omega)$ is a graph of $u$:

$$Q(\zeta, \omega) = \{(x, u) : u^k + \omega_k = L^k(x), \zeta > + R^k(x, \zeta, \omega)\} \tag{3}$$

Considering the first partial derivatives we obtain the following system:

$$u_{x_i}^k = L^k(x)x_i, \zeta > + R_{x_i}^k(x, \zeta, \omega) \tag{4}$$

Applying the implicit function theorem to (3), (4) we get that $(\zeta, \omega) = \varphi(x, u, w_x)$, where $\varphi$ is a holomorphic function. It is worth to point out that the implicit function theorem allows to compute by recursion a term of any order in the expansion of $\varphi$, so our method is totally constructive. Using $\varphi$ in order to exclude the parameters $\zeta, \omega$ from those equations of (4) which are not used yet, we obtain holomorphic equations of the form $u_{x_i}^k = A^k u_{x_i}^1 + \psi(x, u, u_{x_i}^1), k = 2, \ldots, m$ with holomorphic function $\psi$ without terms of order $\leq 1$.

Next, we consider the second order partial derivatives $u_{x_i x_j}^1 = R_{x_i x_j}^1(x, \zeta, \omega)$ and replace $\zeta, \omega$ by $\varphi$. We obtain the holomorphic equations $w_{x_i x_j} = F_{ij}(x, u, w_x)$. Thus, finally we obtain that $u(x)$ satisfy the following holomorphic PDE system:

$$(S_{\mathcal{M}}) : w_{x_i x_j} = F_{ij}(x, u, u_x), i \leq j, u_{x_i}^k = A^k w_x + G^k(x, u, w_x)$$

Since the solutions of this system (given by (3)) depend on $(n+m)$ parameter, it follows by the Frobenius theorem that this system is involutive (in particular, (3) represents all solutions of this system).
The biholomorphic invariance of the Segre family of $\mathcal{M}$ means that every biholomorphism of $\mathcal{M}$ is a symmetry of the constructed PDE system.

Therefore, in the case where $\text{Sym}(\mathcal{S}_M)$ is a finite dimensional Lie group, $\text{Aut}(\mathcal{M})$ is its finite dimensional real Lie subgroup (since it is obviously closed). In order to obtain a precise estimate of its dimension, we recall the following useful observation due to E.Cartan \[9\]. Let a holomorphic vector field $X$ be an infinitesimal generator of $\text{Aut}(\mathcal{M})$ (this means that we consider the real time $t$ in the corresponding Lie series). This is equivalent to the fact that $ReX$ is a tangent vector field to $\mathcal{M}$. On the other hand, $X$ is an infinitesimal symmetry of $(\mathcal{S}_M)$. Indeed, every biholomorphism from the corresponding real one-parameter group takes an element of the Segre family to another one, so $X$ is tangent to $\text{Sym}(\mathcal{S}_M)$ considered as a real Lie group; but since $X$ is a holomorphic vector field, it is necessarily in $\text{Lie}(\mathcal{S}_M)$. This is equivalent to the fact that $ReX$ is a tangent vector field to $\mathcal{M}$. Therefore, the real dimension of $\text{Aut}(\mathcal{M})$ is majorated by the complex dimension of $\text{Lie}(\mathcal{S}_M)$.

We stress again that quite similarly to the hypersurface case, systems defining the Segre families form a very special subclass of the class of second order holomorphic involutive systems with first order relations.

We have proved the following

**Proposition 3.1** The Segre family of a real analytic Levi nondegenerate submanifold $\mathcal{M}$ of $\mathbb{C}^{n+m}$ is a general solution of a holomorphic second order completely overdetermined involutive PDE system with $n$ independent and one dependent variables and first order relations. This system is canonically associated with $\mathcal{M}$ and is denoted by $(\mathcal{S}_M)$.

If $\text{Sym}(\mathcal{S}_M)$ is a finite dimensional complex Lie group, then $\text{Aut}(\mathcal{M})$ is its real Lie subgroup embedded to $\text{Sym}(\mathcal{S}_M)$ as a totally real submanifold.

We conclude this section by some examples. It is easy to show (see \[24\]) that every 6-dimensional quadric in $\mathbb{C}^4$ is linearly equivalent to one of the following quadrics:

$$\mathcal{M}_1: w_1 + \overline{w}_1 = z_1 \overline{z}_1 + z_2 \overline{z}_2, \quad w_2 + \overline{w}_2 = z_1 \overline{z}_1 - z_2 \overline{z}_2,$$

$$\mathcal{M}_2: w_1 + \overline{w}_1 = z_1 \overline{z}_1 - z_2 \overline{z}_2, \quad w_2 + \overline{w}_2 = z_1 \overline{z}_2 + z_2 \overline{z}_1,$$

$$\mathcal{M}_3: w_1 + \overline{w}_1 = z_1 \overline{z}_2 + z_2 \overline{z}_1, \quad w_2 + \overline{w}_2 = z_1 \overline{z}_1.$$

Considering independent variables $x = z$ and dependent variables $u = w$ we get that the systems defining the corresponding Segre families are

$$(S)_1^1: u_{x,i,j}^1 = 0, \quad i,j = 1,2, \quad u_{x,1}^1 = u_{x,1}^2 = u_{x,2}^1 = u_{x,2}^2 = -u_{x,2},$$

$$(S)_2^1: u_{x,i,j}^1 = 0, \quad i,j = 1,2, \quad u_{x,1}^2 = -u_{x,2}^1, \quad u_{x,2}^2 = u_{x,1}^1,$$

$$(S)_3^1: u_{x,i,j}^1 = 0, \quad i,j = 1,2, \quad u_{x,1}^1 = u_{x,2}^1, \quad u_{x,2}^2 = 0.$$

In the next two sections we develop a general approach in order to study infinitesimal symmetries of second order holomorphic involutive systems with first order relations. Much more advanced tools can be found in \[24\] \[25\]; we only adapt for our case a very elementary part of the general theory.
4 Completely integrable systems, their deformations and infinitesimal symmetries

Consider a holomorphic second order involutive PDE system \((S_0)\):

\[
(S_0) : w_{x_1} x_{i_2} = F_{i_1 i_2} (x, u, w_x), i_1, i_2 = 1, ..., n, \mu = 1, ..., m, \\
v_{x_j}^k = G_j^k (x, u, w_x), k = 2, ..., m, j = 1, ..., n
\]

with \(n\) independent variables \(x\) and \(m\) dependent variables \(u = (w, v) \in \mathbb{C} \times \mathbb{C}^{m-1}\).

By a completely integrable holomorphic deformation of the system \((S_0)\) we mean a PDE system of the form

\[
(S_\varepsilon) : w_{x_1} x_{i_2} = F_{i_1 i_2} (\varepsilon, x, u, w_x), v_x = G (\varepsilon, x, u, w_x)
\]

where \(F_{i_1 i_2}^\varepsilon, G\) are holomorphic functions in \((x, u, w_x)\) and real analytic with respect to a (vector-valued) parameter \(\varepsilon\); they satisfy \(F_{i_1 i_2}^{\varepsilon = 0} \equiv F_{ij}, G^{\varepsilon = 0} = G\) and are such that this system is completely integrable for every fixed \(\varepsilon\).

For every \(\varepsilon\) we can consider all first order partial derivatives of the equations \(v_x = G(\varepsilon, x, u, w_x)\) and then substitute \(w_{x_1} x_{i_2} = F_{ij} (x, u, w_x)\) in order to remove the second order derivatives of \(w\) in the right sides. The obtained PDE system has the form

\[
(S_\varepsilon) : u_{x_1}^{i_2} = F_{i_1 i_2}^\varepsilon (\varepsilon, x, u, u^{(1)}), v^{(1)} = G (\varepsilon, x, u, w^{(1)})
\]

and obviously has the same space of solutions as the initial system, so has the same symmetry group. We will work with this system.

In order to study \(\text{Lie}(S_\varepsilon)\) we apply the general Lie method to the deformed system \((S_\varepsilon)\). This system defines a complex subvariety \((S_\varepsilon^2)\) of the jet space \(J^2(n, m)\) given by the equations

\[
u_{x_1}^{i_2} = F_{i_1 i_2}^\varepsilon (\varepsilon, x, u, u^{(1)}), v^{(1)} = G (\varepsilon, x, u, w^{(1)})
\]

and in view of the integrability condition this system is locally regular. Therefore the Lie criterion implies that \(X = \sum \theta^j \frac{\partial}{\partial x_j} + \sum \eta^\mu \frac{\partial}{\partial w^\mu}\) is in \(\text{Lie}(S_\varepsilon)\) if and only if \(X^{(2)}\) is tangent to \((S_\varepsilon^2)\). This is equivalent to the following equations:

\[
X^{(2)} u^\mu = X^{(2)} (F_{i_1 i_2}^\varepsilon (\varepsilon, x, u, u^{(1)})) = X^{(1)} (F_{i_1 i_2}^\varepsilon (\varepsilon, x, u, u^{(1)})) \\
X^{(2)} (v^{(1)} - G (\varepsilon, x, u, w^{(1)})) = 0, (x, u, u^{(1)}, u^{(2)}) \in (S_\varepsilon^2)
\]

Clearly, this is a linear condition on the coefficients \(\theta, \eta\) of \(X\) and their partial derivatives up to the second order. We explain now how to construct explicitly the corresponding linear second order PDE system with holomorphic coefficients for \(\theta, \eta\) equivalent to this condition.
Set \( \hat{\eta}_{i1i2}^\mu = \eta_{i1i2}^\mu - \Lambda_{i1i2}^\mu \). Then we have

\[
\hat{\eta}_{i1i2}^\mu = -\Lambda_{i1i2}^\mu + X^{(1)}(F_{i1i2}^\mu(\varepsilon, x, u, u^{(1)}), (x, u, u^{(1)}, u^{(2)})) \in (S_2)^\varepsilon
\]

Set \( L_2 = \{(x, u, u^{(1)}, u^{(2)}): u_{i1i2}^\mu = F_{i1i2}^\mu(\varepsilon, x, u, u^{(1)}) \} \) and \( L_1 = \{(x, u, u^{(1)}, u^{(2)}): v^{(1)} = G(\varepsilon, x, u, u^{(1)}) \} \), so \((S_2^\varepsilon) = L_1 \cap L_2\).

Using the equalities \( u_{i1i2}^\mu = F_{i1i2}^\mu(\varepsilon, x, u, u^{(1)}) \) we replace \( u_{i1i2}^\mu \) by \( F_{i1i2}^\mu \) in \( \Lambda_{i1i2}^\mu \) and denote obtained expressions by \( \hat{\Lambda}_{i1i2}^\mu \). We point out that they are linear in \( \partial \theta, \partial \eta \) (the vector functions formed by all first order partial derivatives of \( \theta^j, \eta^\mu \)). We get the equations

\[
\hat{\eta}_{i1i2}^\mu|_{L_2} = -\hat{\Lambda}_{i1i2}^\mu(\varepsilon, x, u, u^{(1)}, \theta, \eta, \partial \theta, \partial \eta) + \phi_{i1i2}^\mu(\varepsilon, x, u, u^{(1)}, \theta, \eta, \partial \theta, \partial \eta)
\]

where holomorphic functions \( \phi_{i1i2}^\mu(\varepsilon, x, u, u^{(1)}, \theta, \eta, \partial \theta, \partial \eta) = X^{(1)}(F_{i1i2}^\mu(\varepsilon, x, u, u^{(1)})) \) are linear with respect to \( \theta, \eta, \partial \theta, \partial \eta \). On the other hand, \( \hat{\eta}_{i1i2}^\mu|_{L_2} = \sum_{|\alpha| \leq 3} A_{i1i2}^\mu(u^{(1)})^\alpha \) where the coefficients \( A_{i1i2}^\mu \) are integer linear combinations with constant coefficients of second order partial derivatives of \( \theta, \eta \) (of course, we suppose that \( A_{i1i2}^\mu \) are defined for every \( \alpha \) allowing them to vanish identically).

Next we need to restrict our expressions on \( \eta \): \( \hat{\eta}_{i1i2}^\mu|_{(S_2)_{\varepsilon}} = \sum_{|\beta| \leq 2} B_{i1i2}^\mu(u^{(1)})^\beta \) where \( B_{i1i2}^\mu = \sum_{\gamma \leq 2} c_{\gamma}^\mu \partial \gamma \theta j = \sum_{\gamma \leq 2} d_{\alpha}^\mu \partial \alpha \eta^k \) where we write \( A = (\mu, i_1, i_2, \beta) \) for simplicity of notations and the coefficients are holomorphic functions in \( (\varepsilon, x, u) \).

Developing the right sides of (5) into power series with respect to \( u^{(1)} \) we obtain the series of the form \( \sum_{\alpha} f_{i1i2}^\mu(\varepsilon, x, u, \theta, \eta, \partial \theta, \partial \eta)(u^{(1)})^\alpha \) where the holomorphic coefficients \( f_{i1i2}^\mu(\varepsilon, x, u, \theta, \eta, \partial \theta, \partial \eta) \) are linear with respect to \( \theta, \eta, \partial \theta, \partial \eta \).

Replacing here \( u^{(1)} \) by \( G(\varepsilon, x, u, w^{(1)}) \) and developing the obtained expressions in power series in \( w^{(1)} \), we obtain that (5) implies \( \sum_{|\beta| \leq 2} B_{i1i2}^\mu(u^{(1)})^\beta = \sum_{|\beta| \leq 2} B_{i1i2}^\mu(\varepsilon, x, u, \theta, \eta, \partial \theta, \partial \eta)(u^{(1)})^\beta \) that is \( B_{i1i2}^\mu = p_{i1i2}^\mu(\varepsilon, x, u, \theta, \eta, \partial \theta, \partial \eta) \) for any \( \mu, i_1 \leq i_2, |\beta| \) where the right sides are linear with respect to \( \theta, \eta, \partial \theta, \partial \eta \). By the Noetherian property, there exists a finite number \( N \) (independent of \( \varepsilon \)) such that this equivalent to \( B_{i1i2}^\mu = p_{i1i2}^\mu(\varepsilon, x, u, \theta, \eta, \partial \theta, \partial \eta), |\beta| \leq N \).

We get a linear PDE system of the form (using the notation \( z = (x, u) \)):

\[
\sum_{|\alpha| = 2} (a_{1,\alpha}^t(\varepsilon, z) \partial^\alpha \theta j + \sum_{|\beta| = 2} b_{1,\beta}^t(\varepsilon, z) \partial^\beta \eta^k = c_t^t(\varepsilon, z, \theta, \eta, \partial \theta, \partial \eta), t = 1, ..., N_1
\]

\[
\sum_{j,k} d_{j,k}^i(\varepsilon, z) \frac{\partial \theta j}{\partial z k} + \sum_{i,l} e_{i,l}^i(\varepsilon, z) \frac{\partial \eta^i}{\partial z l} = f^i(\varepsilon, z, \theta, \eta), t = 1, ..., N_2
\]

\[
\sum_p g_p^t(\varepsilon, z) \theta p + \sum_q h_q^t(\varepsilon, z) \eta^q = l^t(\varepsilon, z), t = 1, ..., N_3
\]

where the right sides are linear functions in \( \theta, \eta, \partial \theta, \partial \eta \) (recall that our initial tangency conditions are linear with respect to \( \theta, \eta \). Therefore, the right sides do not contain terms without \( \theta, \eta \) and their derivatives; in particular, \( l^t \) vanishes identically).
Now we proceed quite similarly with the equations

\[ X^{(1)}v^{(1)} = X^{(1)}G(\varepsilon, x, u, w^{(1)}) \]  

(9)

We have \( \eta^\mu_i = \sum_{|\alpha| \leq 2} Q^\mu_{i\alpha} [u^{(1)}]^\alpha \) where \( Q^\mu_{i\alpha} \) are linear combinations of second order partial derivatives of \( \theta, \eta \) with constant coefficients. The equations (9) can be rewritten in the form

\[ \eta^\mu_i = X^{(1)} G^\mu_i (\varepsilon, x, u, w^{(1)}) = \sum_{p=1}^n \eta^1_p \psi^\mu_{ip}(\varepsilon, x, u, w^{(1)}) + \phi^\mu_i (\varepsilon, x, u, w^{(1)}, \theta, \eta), \]

\[ \mu = 2, ..., m, i = 1, ..., n \]

where

\[ \psi^\mu_{ip}(\varepsilon, x, u, w^{(1)}) = \frac{\partial G^\mu_i}{\partial u_p} = \sum_\delta \Psi^\mu_\delta [w^{(1)}]^\delta, \]

\[ \phi^\mu_i (\varepsilon, x, u, w^{(1)}, \theta, \eta) = \sum_j \theta_j \frac{\partial G^\mu_i}{\partial x_j} + \sum_k \eta^k \frac{\partial G^\mu_i}{\partial u^k} = \sum_\delta \Phi^\mu_\delta [w^{(1)}]^\delta \]

In particular, the functions \( \phi^\mu_i (\varepsilon, x, u, w^{(1)}, \theta, \eta) \) are linear with respect to \( \theta, \eta \). This is equivalent to the equalities

\[ \sum_{|\alpha| \leq 2} Q^\mu_{i\alpha} [u^{(1)}]^\alpha = \sum_{|\alpha| \leq 2, p=1, ..., n} Q^1_{ip} \psi^\mu_{ip}(\varepsilon, x, u, w^{(1)})[u^{(1)}]^\alpha + \phi^\mu_i (\varepsilon, x, u, w^{(1)}, \theta, \eta) \]  

(10)

under the condition \( v^{(1)} = G(\varepsilon, x, u, w^{(1)}) = \sum_\gamma g_\gamma(\varepsilon, x, u)[w^{(1)}]^\gamma \).

Substituting these power series into (10) we get the following equality (using the vector notation): \( \sum_\beta T^\beta [w^{(1)}]^\beta = \sum_\beta S^\beta [w^{(1)}]^\beta + \sum_\beta P^\beta [w^{(1)}]^\beta \) of power series with vector valued coefficients \( T_\beta, S_\beta \) which are linear combinations of first order partial derivatives of \( \theta, \eta \) with coefficients holomorphic in \( (\varepsilon, x, u) \) and \( P_\beta(\varepsilon, x, u, \theta, \eta) \) being linear in \( \theta, \eta \). So we obtain the following system of the equations: \( T_\beta - S_\beta - P_\beta = 0 \) which in view of the Noetherian condition is equivalent to \( T_\beta - S_\beta - P_\beta = 0, |\beta| \leq N \) for a finite \( N \).

So we have a first order linear system of equations:

\[ \sum_{j,k} \hat{a}_{j,k}(\varepsilon, z) \frac{\partial \theta_j}{\partial z_k} + \sum_{i,l} \hat{b}_{i,l}(\varepsilon, z) \frac{\partial \eta^l}{\partial z_i} = \hat{c}(\varepsilon, z, \theta, \eta), t = 1, ..., N_4, \]  

(11)

\[ \sum_p \hat{d}(\varepsilon, z) \theta_p + \sum_q \hat{e}(\varepsilon, z) \eta^q = \hat{f}(\varepsilon), t = 1, ..., N_5 \]  

(12)

As above, the right sides does not contain terms without \( \theta, \eta \) (for instance, \( \hat{f} \equiv 0 \)).

We have proved the following
Theorem 4.1 The vector field $X$ defines an infinitesimal symmetry of $(S^c)$ if and only if its coefficients satisfy the united system (1), (7), (8), (11), (12). The Lie algebra $\text{Lie}(S^c)$ is finite dimensional if and only if the linear space of holomorphic solutions of this united system is finite dimensional.

The constructed linear holomorphic PDE system is called the (infinitesimal) Lie equations associated with $(S^c)$. As an important example, let us construct the Lie equations for a PDE system of the form

$$u^k_{x_ix_j} = 0, i, j = 1, ..., n, k = 1, ..., m$$

$$v^k_x = M^k w_x, k = 2, ..., m$$

We call such a system by a flat system with relations $(S_{flat})$. Obviously, such a system is involutive.

The variety $(S_{flat})_2$ defined by $(S_{flat})$ is given by the equations

$$u^k_{ij} = 0, k = 1, ..., m, i, j = 1, ..., n$$

$$v^{(1)} = Mw^{(1)}$$

where the matrix $M$ is formed by the matrices $M^k$ as vertical blocks. Let a vector field $X = \sum_{j=1}^n \theta^j \frac{\partial}{\partial x_j} + \sum_{\mu=1}^m \eta^\mu \frac{\partial}{\partial u^\mu}$ be in $\text{Lie}(S_{flat})$ i.e. an infinitesimal symmetry of $(S_{flat})$.

Since our system is locally regular and of maximal rank, $X \in \text{Lie}(S_{flat})$ if and only if $X^{(2)}$ is tangent to $(S_{flat})_2$ i.e.

$$X^{(2)} u^\mu_{i_1i_2} = 0, i_1, i_2 = 1, ..., n, \mu = 1, ..., m$$

$$X^{(2)} (v^{(1)} - Mw^{(1)}) = X^{(1)} (v^{(1)} - Mw^{(1)}) = 0$$

$$u^\mu_{i_1i_2} = 0, v^{(1)} = Mw^{(1)}$$

The first line equations imply that

$$\eta^\mu_{i_1i_2} = 0, (x, u, u^{(1)}, u^{(2)}) \in (S_{flat})_2$$

for any $\mu$ and any $i_1 \leq i_2$. We point out also that the equations $u^\mu_{i_1i_2} = 0$ imply

$$\Lambda^\mu_{i_1i_2} = 0$$

Set $L_2 = \{(x, u, u^{(1)}, u^{(2)}) : u^\mu_{i_1i_2} = 0\}$ and $L_1 = \{(x, u, u^{(1)}, u^{(2)} : v^{(1)} = Mw^{(1)}\}$, so $(S_{flat})_2 = L_1 \cap L_2$.

In view of (16)
where the coefficients \( A^\mu_{i_1 i_2 \alpha} \) are integer linear combinations of second order partial derivatives of \( \theta, \eta \).

Next we need to restrict the polynomials (17) on \( L_1 \). Replacing \( v(1) \) by \( Mw(1) \) in (17) we obtain

\[
\eta^\mu_{i_1 i_2} | L_2 = \sum_{|\alpha| \leq 3} A^\mu_{i_1 i_2 \alpha} [u^{(1)}]^\alpha
\]

(17)

for all \( \mu, i_1 \leq i_2, \beta \). This is a linear second order PDE system with constant coefficients which represents the Lie equations for \((S_{flat})\). We emphasize the very important property of this system: every equation of second (resp. first) order contains only the second (resp. first) order partial derivatives.

In the next section we recall some general properties of linear PDE systems with holomorphic coefficients useful for a study of the Lie equations.

5 Symbols, prolongations and solutions of linear systems

In this section we adapt general methods of the formal PDE theory for our case. Much more general methods and tools can be found in \([23, 24]\).

As usual, by a holomorphic linear PDE system of order \( q \) with \( n \) independent variables \( y \) and \( m \) dependent variables \( \tau \) we mean a system of the form

\[
(R_q) : \sum_{j=1, \ldots, m; |\alpha| \leq q} a^{\alpha}_{ji}(y) \partial^\alpha \tau^j = 0, i = 1, \ldots, s
\]

where \( a^{\alpha}_{ji} \) are holomorphic functions. We use the same notation for the subvariety in the jet space \( J^q(n, m) \) corresponding to this system:

\[
(R_q) : \sum_{j=1, \ldots, m; |\alpha| \leq q} a^{\alpha}_{ji}(y) \tau^j = 0, i = 1, \ldots, s
\]
A (holomorphic) solution of such a system is a function \( \tau(y) \) holomorphic on a domain \( D \) of definition of the coefficients such that \( j^x_i(\tau) \in (\mathcal{R}_q) \) for every \( x \in D \). We denote by \( \text{Sol}(\mathcal{R}_q) \) the vector space of the solutions of \( (\mathcal{R}_q) \).

The symbol \( G_q(y^0) \) of \( (\mathcal{R}_q) \) at a point \( y^0 \) is a linear subspace of the complex affine space with coordinates \( v^j_{\alpha} \), \( j = 1, ..., m \); \( |\alpha| = q \), \( \alpha_1 \leq ... \leq \alpha_q \); \( \alpha \in \{1, ..., n\} \), defined by

\[
(G_q) : \sum_{j=1, ..., m; |\alpha|=q} a^i_{\alpha j}(y^0)v^j_{\alpha} = 0, i = 1, ..., s
\]

The \( r \)-prolongation \( (\mathcal{R}_{q+r}) \) of \( (\mathcal{R}_q) \) is a linear system which we get if we add to \( (\mathcal{R}_q) \) the equations obtained by taking all the partial derivatives of order \( \leq r \) in every equation of \( (\mathcal{R}_q) \), that is

\[
(\mathcal{R}_{q+r}) : \sum_{j=1, ..., m; |\alpha|=q} \partial^\beta(a^i_{\alpha j}(y)\partial^\alpha \tau^j) = 0, i = 1, ..., s, |\beta| \leq r
\]

Obviously, it has the same space of solutions. The symbol of \( (\mathcal{R}_{q+r}) \) is denoted by \( G_{q+r}(y^0) \).

The system \( (\mathcal{R}_q) \) is called of finite type at \( y^0 \) if \( G_{q+r}(y^0) = \{0\} \) for some \( r \). If a system is of finite type at every point, we say simply that it is of finite type. The smallest \( r \) with this property is called the type of \( (\mathcal{R}_q) \) and is denoted by \( \text{type}(\mathcal{R}_q) \).

**Theorem 5.1** Suppose that \( (\mathcal{R}_q) \) is of finite type at some point \( y^0 \). Then the dimension of the space of solutions of \( (\mathcal{R}_q) \) holomorphic in a neighborhood of \( y^0 \) is finite.

**Proof:** The fact that \( G_{q+r}(y^0) = \{0\} \) for some \( r \) implies that \( (\mathcal{R}_{q+r}) \) contains a subsystem which can be solved with respect to all partial derivatives of order \( q + r \) and so can be represented in the form (in a neighborhood of \( y^0 \)):

\[
\partial^\alpha \tau^j = \sum_{k=1, ..., m; |\beta| \leq q+r-1} (b^j_{k\beta}(y)\partial^\beta \tau^k), j = 1, ..., m, |\alpha| = q + r
\]

This implies by the chain rule and recurrence that all derivatives of \( \tau^j \) of order \( \geq q + r \) at \( y^0 \) are determined by derivatives of order \( \leq q + r - 1 \), which means that the dimension of \( \text{Sol}(\mathcal{R}_q) \) is finite.

This proof is quite constructive and allows to obtain explicit recursive formulae for the Taylor expansions at \( y^0 \) of solutions of \( (\mathcal{R}_q) \). This also means that the dimension of \( \text{Sol}(\mathcal{R}_q) \) is majorated by \( \dim J^d(n, m) \) where \( d = \text{type}(\mathcal{R}_q) - 1 \). Of course this estimate is not precise since the partial derivatives at \( y^0 \) of \( \tau \) of order \( \leq d \) satisfy a system of linear algebraic equations \( (L) \) arising from the equations of \( (\mathcal{R}_{q+r}) \) of order \( < (q + r) \). Solving this system we can precisely determine the dimension of the space \( \text{Sol}(\mathcal{R}_q) \) for any concrete system \( (\mathcal{R}_q) \). More precisely, applying the Cramer rule to \( (L) \) we can represent some partial derivatives of \( \tau \) at \( y^0 \) of order \( \leq d \) (principal derivatives) as linear combinations of others (parametric derivatives). The number of parametric derivatives is equal to the dimension of \( \text{Sol}(\mathcal{R}_q) \) and they form a set of natural parameters on \( \text{Sol}(\mathcal{R}_q) \).

Let \( (\mathcal{R}_q^\varepsilon) \) be an analytic family of linear systems given by
ordinary differential equation. algebra tools.

study of a finite number of prolongations and their symbols, i.e. by means of the elementary linear 

jet space Char denoted by \( S \) considered a holomorphic equation (\( \lambda \) set of such \( \lambda \) of \( I \subset \mathbb{C} \). dimLie characteristic variety finite type. On of the possibilities here is to consider its

linear map \( \sigma \) in involutive system (\( R \). Suppose that the completed Lie equations for \( R \). Theorem 5.3 Suppose that the system \( R \) is of finite type. Then for every \( \varepsilon \) close enough to the origin the system (\( R \) is of finite type and type(\( R \) \( R \)). Furthermore, \( \dim Sol(R_0) \leq \dim Sol(R_0) \). The proof is immediate since the rank of a linear algebraic system defining the symbol of the prolonged system does not decrease with respect to small perturbations of the coefficients so type(\( R \) \( R \)). Similarly, if (\( L \) is a linear algebraic system for the partial derivatives of order \( \leq \) type(\( R \) arising from the equations of the lower orders, then rank(\( L \) \( L \)) and the number of the parametric derivatives decreases so \( \dim Sol(R_0) \leq \dim Sol(R_0) \).

In general a linear system of order \( q \) may contain some equations of order \( < q \). However, if we add to such a system all the equations of order \( \leq q \) obtained from the equations of lower order by taking all the partial derivatives of a suitable order, we obtain a system with the same space of solutions. We call such a system the completion of (\( R \) or the completed system (\( R \). We also point out that every linear system can be reduced to a system of the first order by introducing the supplementary dependent variables; so one may work with these systems only.

Applying these results to the completed Lie equations deduced in the previous section for an involutive system (\( S^0 \) and its holomorphic involutive deformation, we obtain the following

Theorem 5.3 Suppose that the completed Lie equations for (\( S^0 \) form a system of finite type \( d \) at some point \( (x^0, u^0) \). Then \( \dim Lie(S^0) \) is finite and for any \( \varepsilon \) close enough to the origin \( \dim Lie(S_\varepsilon) \leq \dim Lie(S^0) \).

In view of this result it is of clear interest the question how to check up if a given system is of finite type. On of the possibilities here is to consider its characteristic variety. Let \( \lambda \) be a vector of \( \mathbb{C}^m \). We use the notation \( \lambda^\alpha = \lambda^{\alpha_1}...\lambda^{\alpha_s} \). A vector \( \lambda \) is called a characteristic (co)vector at \( y \) if the linear map \( \sigma_\lambda(y) : \mathbb{C}^m \rightarrow \mathbb{C}^n \) given by the matrix \( \sigma_\lambda(y) : \sum_{|\alpha|\leq q} a^i_{\alpha}(y)\lambda^\alpha \) is not injective. The set of such \( \lambda \) is an algebraic variety in \( \mathbb{C}^m \) which is called the characteristic variety at \( y \) and is denoted by \( \operatorname{Char}_y(R_q) \).

The following criterion is useful (see [24], p.195): a system (\( R_q \) is of finite type if and only if \( \operatorname{Char}_y(R_q) \) is zero for every \( y \) (we do not use it in the present paper).

Of course, this statement says nothing about a value of the type of (\( R_q \). However, if the system (\( R_q \) is known to be of finite type, its type can be determined by direct computations using the study of a finite number of prolongations and their symbols, i.e. by means of the elementary linear algebra tools.

As an example we study the Lie equations in the simplest classical case of a second order ordinary differential equation. We denote by \( x \in \mathbb{C} \) and \( u \in \mathbb{C} \) the independent and dependent variables respectively and consider a holomorphic equation (\( S \) : \( u_{xx} = F(x, u, u_x) \). This equation define a hypersurface in the jet space \( J^2(1, 1) : (S_2) : u_{11} = F(x, u, u_1) \).
A holomorphic vector field \( X = \theta \frac{\partial}{\partial x} + u \frac{\partial}{\partial y} \) is an infinitesimal symmetry of \((S)\) if and only if its 2-prolongation \( X^{(2)} = X + \eta_1 \frac{\partial}{\partial u_1} + \eta_{11} \frac{\partial}{\partial u_{11}} \) is tangent to \((S)\) that is \( X^{(2)}(u_1 - F(x, u, u_1)) = 0, (x, u, u_1, u_{11}) \in (S_2) \).

The coefficients have the following expressions:

\[
\eta_1 = \eta_x + (\eta_u - \theta_x) u_1 - \theta_u(u_1)^2, \\
\eta_{11} = \eta_{xx} + (2\eta_{ux} - \theta_{xx}) u_1 + (\eta_{uu} - 2\theta_{xu})(u_1)^2 - \theta_{uu}(u_1)^3 + (\eta_u - 2\theta_x) u_{11} - 3\theta_u u_1 u_{11}
\]

Consider the expansion \( F(x, u, u_1) = \sum_{\nu \geq 0} f^\nu(x, u)(u_1)^\nu \); after elementary computations following the described above general method we obtain the following system \((R_2)\) of infinitesimal Lie equations:

\[
\eta_{xx} = 2f^0 \theta_x + f^1 \eta_x - f^0 \eta_u + f^0 \theta_x + f^0 \eta, \\
2\eta_{ux} - \theta_{xx} = f^1 \theta_x - 3f^0 \theta_u + f^1 \theta + f^1 \eta, \\
\eta_{uu} - 2\theta_{xu} = 2f^1 \theta_u + 3f^3 \eta_x + f^2 \theta + f^2 \eta, \\
-\theta_{uu} = -f^3 \theta_x + f^2 \theta_u + 4f^4 \eta_x + f^3 \theta + f^3 \eta, \\
(2 - \nu) f^\nu \theta_x + (4 - \nu) f^{\nu-1} \theta_u + (\nu + 1) f^{\nu+1} \eta_x + f^\nu \theta + f^\nu \eta = 0, \nu \geq 4
\]

Actually only a finite number of these equations are independent. But we show that the first 4 second order equations form a finite type system. Thus, we consider a system \((R'_2)\):

\[
\eta_{xx} = 2f^0 \theta_x + f^1 \eta_x - f^0 \eta_u + f^0 \theta + f^0 \eta \\
2\eta_{ux} - \theta_{xx} = f^1 \theta_x - 3f^0 \theta_u + f^1 \theta + f^1 \eta \\
\eta_{uu} - 2\theta_{xu} = 2f^1 \theta_u + 3f^3 \eta_x + f^2 \theta + f^2 \eta \\
-\theta_{uu} = -f^3 \theta_x + f^2 \theta_u + 4f^4 \eta_x + f^3 \theta + f^3 \eta
\]

The symbol \(G'_2\) of this system is a linear 2- dimensional subspace of the space \(Q^6\) with coordinates \(v^1_{11}, v^1_{12}, v^2_{11}, v^2_{12}, v^2_{22}\) defined by the equations

\[
v^2_{11} = 0, 2v^2_{12} - v^1_{11} = 0, v^2_{22} - 2v^1_{12} = 0, v^1_{22} = 0
\]

A vector \(\lambda \in Q^2\) will be characteristic if and only if the matrix with the lines \((0, \lambda_1^2), (\lambda_2^2, 2\lambda_1 \lambda_2), (-2\lambda_1 \lambda_2, \lambda_3^2), (\lambda_2^2, 0)\) has the rank \(\leq 1\); this implies the the characteristic variety is equal to zero and so our system is of finite type.

Its 1-prolongation \(G'_3\) is a subspace of \(Q^8\) with the coordinates \(v^1_{111}, v^1_{112}, v^1_{122}, v^1_{222}, v^2_{111}, v^2_{112}, v^2_{122}, v^2_{222}\) given by the equations
we still have additional first order equations in the Lie equations (dimLie that parametric derivatives. Then the values of all second order derivatives of \( \theta \) in involutive PDE symmetries these equations imposes additional analytic restrictions on the parameters determined by (19) - (22) and the values of all derivatives at \((a,0)\) so we see immediately that \( G_3' = \{0\} \), i.e. \((R_2')\) is of type 1. Solving its 1-prolongation \((R_3')\) with respect to the partial derivatives of the third order, we obtain the following explicit representations:

\[
\begin{align*}
\theta_{xxx} &= -f^1\theta_{xx} + 7 f_0\theta_{xu} + 2 f^1\eta_{xu} - 2 f^0\eta_{uu} + 4 (f^0_u - f^1_x)\theta_x + 5 f^0_u\theta_u + f^1_u \eta_x + (2 f^1_{xu} - f^1_{xu}) \theta + (2 f^0_{xu} - f^1_{xu}) \eta, \\
\theta_{xxu} &= -f^0\theta_{uu} - f^3\eta_{xx} + (1/3) (f^1_u - f^2_x) \theta_x - (f^0_u + f^1_u) \theta_u - (1/3) (5 f^3_x + 2 f^2_u) \eta_x \\
&+ (1/3) f^0_u \eta_u + (1/3) (f^1_{xu} - 2 f^2_{xx}) \theta + (1/6) (2 f^1_{xu} - f^2_{xx}) \eta, \\
\theta_{xuu} &= f^3 \theta_{xx} - 4 f^4 \eta_{xx} - f^2_{xu} \theta - (4 f^4_x + f^3_u) \eta_x - f^3_x \theta - f^3 \eta, \\
\eta_{xxx} &= 3 f^0_x \theta_x + (f^1_x + f^0_u) \eta_x - f^0 \eta_u + 2 f^0 \eta_{xx} + f^1 \eta_{xx} - f^0 \eta_{xu} + f^2_{xx} \theta + f^3 \eta_u, \\
\eta_{xxu} &= 2 f^0 \theta_{xu} + f^1 \eta_{xu} - f^0 \eta_{uu} + 2 f^0 \theta_{xu} + f^1 \theta_{xu} + f^2_{xu} \theta + f^3 \eta_{xu}, \\
\eta_{xuu} &= -2 f^0 \theta_{uu} - f^3 \eta_{xx} + (1/3) (2 f^1_x - f^2_x) \theta_x - 2 f^0 \theta_u - (1/3) (f^1_x + f^2_u) \eta_x + (2/3) f^1_u \eta_u + (1/3) (2 f^1_{xu} - f^2_{xu}) \theta, \\
\eta_{uuu} &= 2 f^3 \theta_{xx} + 2 f^4 \theta_{xu} - 8 f^4 \eta_{xx} + 3 f^3 \eta_{xx} - (2 f^1_u - f^2_x) \theta_u + (f^3_u - 8 f^4_x) \eta_x + f^2 \eta_u, \\
&+ (f^2_{xu} - 2 f^3_x) \theta + (f^2_u - 2 f^2_{xu}) \eta
\end{align*}
\]

Fix a point \((x_0, u_0)\) and attach the values \(a_1 := \theta(x_0, u_0), a_2 := \eta(x_0, u_0), a_3 := \theta_x(x_0, u_0), a_4 := \theta_u(x_0, u_0), a_5 := \eta_x(x_0, u_0), a_6 := \eta_u(x_0, u_0), a_7 = \theta_{xx}(x_0, u_0), a_8 = \theta_{xu}(x_0, u_0)\) to the parametric derivatives. Then the values of all second order derivatives of \(\theta, \eta\) at \((x_0, u_0)\) are determined by (13) - (22) and the values of all derivatives at \((x_0, u_0)\) of order \(\geq 3\) are determined by the former expressions for the third order partial derivatives via the chain rule. This means that \(\text{dim} \text{Lie}(S) \leq 8\) and this estimate is precise since in the flat case where \(F \equiv 0\) one has \(\text{dim} \text{Lie}(S) = 8\).

Of course, the constructed vector fields are in general just the candidates to be in \(\text{Lie}(S)\) since we still have additional first order equations in the Lie equations \((R_q)\). The fact that \(\theta, \eta\) satisfy these equations imposes additional analytic restrictions on the parameters \(a_j\) so actually \(\text{Lie}(S)\) is parametrized by a some analytic subvariety in the space \(\mathbb{C}^8\) of the parameters \(a_j\).

The present description of symmetries of a second order ordinary differential equation has been obtained by L.Dickson [12]. Since the Segre family of a Levi nondegenerate hypersurface in \(\mathbb{C}^2\) is a set of solutions of such equation, the present method allows to obtain an explicit parametrization of its automorphism group. This argument can be directly generalized to second order holomorphic involutive PDE symmetries

\[u^k_{x_1 x_2} = F^k_{ij}(x, u, u_x), k = 1, ..., m, i, j = 1, ..., n\]

Using this method and the explicit formulae for the 2-prolongation of a vector field on \(\mathbb{C}^n \times \mathbb{C}^m\), the author proved in [23] that the Lie algebra of infinitesimal symmetries of such a system has a
dimension \( \leq (n+m+2)(n+m) \) and every infinitesimal symmetry is determined by a second order Taylor expansion at a given point (the Lie equations are of type 1). In the special case where \( n = 1 \) i.e. for a system of ordinary differential equations this result was established by F.Gonzales-Gascon and A.Gonzales-Lopez [18] (see also [22]). In particular, this implies the results of Tanaka [32] and Chern - Moser [10] on the majoration of the dimension of the automorphism group of a real analytic Levi nondegenerate hypersurface in \( \mathbb{C}^{n+1} \), its parametrization etc.

It is important to emphasize that such an explicit parametrization of the Lie algebra of infinitesimal symmetries can be obtained for every system with the Lie equations of finite type. In what follows we restrict ourselves just by the study of symbols of the Lie equations in order to avoid complicated formulae.

We conclude this section by a statement concerning the special case of linear PDE systems with constant coefficients. The main example of these systems is given by the Lie equations for a flat manifold derived in the previous section.

Consider a linear PDE system with constant coefficients of the form

\[
(\mathcal{R}_q) : \sum_{i,|\alpha|=q_k} a^k_{i\alpha} \partial^\alpha u^i = 0, k = 1, ..., K
\]

where \( q_k = \max_k q_k \). We emphasize that every equation of this system of order \( q_k \) contains the partial derivatives of the same order \( q_k \) only. In particular, the Lie equations for a flat system deduced in the previous section are of this class.

A holomorphic in a neighborhood of the origin map \( u = (u^1, ..., u^m) \) is a solution of \((\mathcal{R}_q)\) if and only if

\[
\partial^\beta (\sum a^k_{i\alpha} \partial^\alpha u^i)|_{x=0} = \sum a^k_{i\alpha} \partial^{\beta+\alpha} u^i|_{x=0} = 0, k = 1, ..., K
\]

for every \( \beta \).

This is equivalent to

\[
\sum_{i:|\alpha|=q_k,|\beta|=s-q_k} a^k_{i\alpha} (\partial^{\beta+\alpha} u^i)|_{x=0} = 0, k = 1, ..., K, s = q, q + 1, ...
\]

In the complex affine space with the coordinates \((v^i_{i_1, ..., i_s})\), \( i \in \{1, ..., m\}, i_1 \leq ... \leq i_s, \) \( i_j \in \{1, ..., n\} \) consider a subspace \( V_s \) defined by the linear algebraic system

\[
\sum_{i:|\alpha|=q_k,|\beta|=s-q_k} a^k_{i\alpha} v^i_{\beta+\alpha} = 0, k = 1, ..., K
\]

for \( s = q, q + 1, ... \).

**Proposition 5.4** The dimension of the space \( \text{Sol}(\mathcal{R}_q) \) is finite if and only if there exists an \( s \) such that \( V_s = \{0\} \). In this case the completion of \((\mathcal{R}_q)\) is a system of finite type and every solution is a polynomial of degree \( < s \).
Proof: Suppose that there exists an \( s \) such that \( V_s = \{0\} \). In view of (23) this means that the completion of \( (R_q) \) is a system of finite type majorated by \( s \). Moreover, (23) shows that in this case all partial derivatives of \( u \) of order \( s \) vanish identically.

Let now the dimension of \( \text{Sol}(R_q) \) is finite. Suppose by contradiction that there exists an increasing sequence \( (s_t) \) such that every \( V_{s_t} \) is non-trivial. Let \( (v_{i_1,...,i_{s_t}}) \) be a non-zero vector in \( V_{s_t} \).

Consider the map \( u_t = (u_1^t, ..., u_m^t) \) whose components are the homogeneous polynomials of degree \( s_t \) satisfying \( \frac{\partial^{s_t} u_j}{\partial x_{i_1}...\partial x_{i_{s_t}}}(0) = v_{i_1,...,i_{s_t}} \). Then for every \( t \) the function \( u_t \) satisfies (23) for \( s = s_t \); but since it is homogeneous polynomial of degree \( s_t \), clearly it satisfies (23) for all other \( s \). Therefore, every \( u_t \) is a solution of \( (R_q) \): a contradiction.

In particular, we have the following

**Corollary 5.5** Suppose that \( (R_q) \) has a finite dimensional solution space and let \( (R_{\varepsilon q}) \) be its holomorphic deformation. Then for every \( \varepsilon \) small enough \( \dim \text{Sol}(R_{\varepsilon q}) \leq \dim \text{Sol}(R_q) \).

### 6 General flat systems with first order linear relations

In this section we consider a flat system \((S)\) of the form

\[
\begin{align*}
\frac{\partial u_i^1}{\partial x_j} &= 0, i, j = 1, ..., n \\
\frac{\partial u_i^k}{\partial x} &= A_k^1 u_i^1, k = 2, ..., n
\end{align*}
\]

with \( n \) independent and \( m \) dependent variables. We apply a geometric method in order to describe the symmetries of this system without computations. The basic idea goes back to S.Lie - G.Scheffers [19] (see also [18]); a related result also was obtained by B.Shipman [27]. The present proof is a direct generalization of author’s argument about the rationality of holomorphic maps between quadrics in \( \mathbb{C}^n [10] \).

**Theorem 6.1** Suppose that the matrices \( A^1 := \text{Id}_n, A^2, ..., A^m \) are linearly independent. Then \( \text{Lie}(S) \) is finite dimensional.

Proof: Fix an infinitesimal symmetry \( X \in \text{Lie}(S) \) and for \( t \in \mathbb{C} \) close enough to the origin consider the flow \( f(t, x, u) = e^{tX} \) generated by \( X \).

The set \( \text{Sol}(S) \) of solutions of \((S)\) is an \( (n+m) \)- parameter family of affine subspaces of \( \mathbb{C}^n \times \mathbb{C}^m \) of the form \( Q(\zeta, \omega) = \{(x, u) : u = \omega + <x, A\zeta>\} \) where \( \omega + <x, A\zeta> = \omega^j + <x, A^j> \), \( j = 1, ..., m \). The parameters \( (\zeta, \omega) \in \mathbb{C}^{n+m} \) give a natural holomorphic coordinate system on \( \text{Sol}(S) \) which is an \( (n+m) \)-dimensional complex manifold.

The fact that \( f_t \) takes any solution to another solution means that for any \( (\zeta, \omega) \) there exists a point \( (\zeta^*, \omega^*) \) such that \( f_t(Q(\zeta, \omega)) = Q(\zeta^*, \omega^*) \) that is

\[
\begin{align*}
h_t(x, \omega^j + <x, A\zeta^*>) = \omega^j_t + <g_t(x, \omega^j + <x, A\zeta^*>), A\zeta^*_t>
\end{align*}
\]

where \( f_t = (g_t, h_t) \).

Thus, \( f_t \) induces a map
\[ f_t^*: \text{Sol}(S) \rightarrow \text{Sol}(S), \]
\[ f_t^* (\zeta, \omega) \mapsto (\zeta_t^*, \omega_t^*) \]

**Lemma 6.2** The family \( \{ f_t \} \) is a family of biholomorphisms holomorphically depending on the parameter \( t \).

**Proof:** The image \( f_t(Q(\zeta, \omega)) \) is given by

\[ \{(x^*, u^*): (x^*, u^*) = (g(t, x, \omega + < x, A\zeta >), h(t, x, \omega + < x, A\zeta >)), x \in \mathbb{C}^n \}. \]

For \( t = 0 \) one has \((g_0(\bullet), h_0(\bullet)) = (x, u)\) so for \( t \) small enough the implicit function theorem can be applied to \( x^* = g(t, x, \omega + < x, A\zeta >) \) and \( x = x(t, x^*, \zeta, \omega) \) is holomorphic. Substituting it to \( u^* = h(t, x, \omega + < x, A\zeta >) \) we obtain \( u^* = \varphi(t, x^*, \zeta, \omega) \) and \( \varphi \) is holomorphic. On the other hand, \( f_t(Q(\zeta, \omega)) = Q(\zeta_t^*, \omega_t^*) \) so \( \varphi(t, x^*, \zeta, \omega) = \omega_t^* + < x^*, A\zeta_t^* > \). In particular, \( \varphi_1(t, x^*, \zeta, \omega) = \omega_1^* + x_1^*(\zeta_1^*) + ... + x_n^*(\zeta_n^*) \) so every \( \zeta_j^* = \zeta_j^*(t, \zeta, \omega) \) is holomorphic and obviously \( \omega^* = \omega^*(t, \zeta, \omega) \) is holomorphic.

Consider the vector fields \( \mathcal{L}_\nu = \frac{\partial}{\partial \zeta_\nu} - \sum_{k=1}^{m} \left( \sum_{j=1}^{n} a_{ij}^k x_i \right) \frac{\partial}{\partial x^j} \) where \( A^k = (a_{ij}^k) \).

Applying them to (24) we get

\[ \mathcal{L}_\nu((\omega^*_j)_i) + < g_t(x, \omega + < x, A\zeta >), \mathcal{L}_\nu A^j \zeta_t^* >= 0 \]  

(25)

Consider (24), (25) as a linear system with respect to components of \( f_t \). Since \((\zeta_0^*, \omega_0^*) \equiv (\zeta, \omega)\), this system contains an \((n + m) \times (n + m)\) subsystem with the determinant \( \neq 0 \) for \( t \) small enough. Applying the Cramer rule we obtain that for any \((t, \zeta, \omega)\) fixed the map \( f_t(x, \omega + < x, A\zeta >) \) is a rational map in \( x \). Moreover, the degree of every such a map is uniformly bounded by \( n \).

The last step of the proof is to show the the space of solutions \( (S) \) is "large enough".

Set \((e_k = (0, ... , 1, ..., 0)) \in \mathbb{C}^n\) (1 on the \( k \)-position) and consider the vectors \( v_k(\zeta) = (e_k, < e_k, A^1 \zeta >, ..., < e_k, A^m \zeta >) \) (so \( v_k(\zeta) \in Q(\zeta, 0) \)).

**Lemma 6.3** The linear hull of \( \{ v_k(\zeta), \zeta \in \mathbb{C}^n \} \) coincides with \( \mathbb{C}^n \).

**Proof:** If the statement is false, there exists a \( \lambda \in \mathbb{C}^{n+m} \setminus \{0\} \) such that \( < \lambda, v_k(\zeta) >= 0 \) for any \( k, \zeta \) that is \( \lambda_k + \lambda_{n+} < e_k, A^1 \zeta > + ... + \lambda_{n+m} < e_k, A^m \zeta >= 0 \) for all \( \zeta, k \); therefore \( \lambda_k = 0 \) for every \( k = 1, ..., n \) and so \( < e_k, (\lambda_{n+1}A^1 + ... + \lambda_{n+m}A^m) \zeta >= 0 \) for every \( k, \zeta \), that is \( \lambda_{n+1}A^1 + ... + \lambda_{n+m}A^m = 0 \) : a contradiction which proves the lemma.

Fix now \((n + m)\) linearly independent complex lines \( l^1, ..., l^{n+m} \); every \( l^j \) is in some \( Q(\zeta, 0) \) through the origin. Every line generates a family of parallel lines and any line of such a family is in \( Q(\zeta, \omega) \) for some \( \omega \). After a linear change of variables in \( \mathbb{C}^{n+m} \) these families become the coordinate ones and the classical separate rationality theorem [3] implies that \( f_t \) is a rational map of degree \( \leq n \) for any \( t \) small enough that is
\[ f_t(x, u) = \frac{\sum_{|I| = 0}^{n} a_I(t)(x, u)^I}{\sum_{|I| = 0}^{n} b_J(t)(x, u)^J}. \]

Hence, \( X = \frac{df}{dt}\{t = 0\} \) is a vector field with rational coefficients of degree \( \leq n^2 \). Every such a coefficient is uniquely determined by a finite number \( d = d(n^2) \) of terms of its Taylor expansion at the origin. Therefore, the dimension of \( (S) \) is finite. This completes the proof of the theorem.

We say that a flat system \( (S) \) is nondegenerate if it satisfies the hypothesis of our proposition that the matrices \( A^j \) are linearly independent.

From Proposition 5.4 we obtain the following Corollary 6.4

**Corollary 6.4** The completed Lie equations of a nondegenerate flat system \( (S) \) form a PDE system of finite type and every infinitesimal symmetry \( X \in \text{Lie}(S) \) has polynomial coefficients of uniformly bounded degree.

Corollary 5.5 implies now one of our main results:

**Theorem 6.5** If \( (S^\epsilon) \) is an involutive holomorphic deformation of a nondegenerate flat system \( (S) \), then \( \text{dimLie}(S^\epsilon) \leq \text{dimLie}(S) \).

Now we can apply the obtained results in order to study biholomorphisms of Cauchy-Riemann manifolds.

Let \( M \) be a generic real analytic Levi nondegenerate submanifold in \( \mathbb{C}^{n+m} \) through the origin. After a biholomorphic change of coordinates it can be represented in the form \( w + \overline{w} = < L(z), \overline{z} > + o(|Z|^2) \). Denote by \( M_{\text{flat}} \) the corresponding quadric: \( w + \overline{w} = < L(z), \overline{z} > \). For real \( \epsilon \) close enough to the origin consider the following change of variables: \( z = \epsilon z', w = \epsilon^2 w' \).

In the new coordinates (we omit the primes) we get the manifold \( M^\epsilon : w + \overline{w} = < L(z), \overline{z} > + (1/\epsilon^2)R(\epsilon z, \epsilon \overline{z}, \epsilon^2 w, \epsilon^2 \overline{w}) \) biholomorphic to \( M \) for every \( \epsilon \). Since the function \((1/\epsilon^2)R(\epsilon z, \epsilon \overline{z}, \epsilon^2 w, \epsilon^2 \overline{w})\) extends to a function real analytic in \( \epsilon \) in a neighborhood of the origin and vanishing at the origin, the system \( S(M^\epsilon) \) defining the Segre family of \( M^\epsilon \) is a holomorphic involutive deformation of the flat system defining the Segre family of \( M_{\text{flat}} \).

It follows from the results of the previous sections that we have established the following result:

**Corollary 6.6** \( \text{Aut}(M) \) is a finite dimensional real Lie group. Moreover, \( \text{dimAut}(M) \) is majorated by the complex dimension of the flat PDE system defining the Segre family of \( M_{\text{flat}} \).

Various results of this type for this and more general classes of CR manifolds have been obtained by several authors [2, 3, 16, 21, 28, 33, 38] using different methods. We emphasize that our method can be adapted to a much more general situation and allows to obtain many additional information on the structure of the automorphism group.

Remark. We have introduced the small parameter \( \epsilon \) by analogy with the well-known scaling techniques (see for instance [1]). On the other hand, in our situation this argument can be considered as an application of the general PDE method of small parameter widely known in the classical mechanics.

The geometric method employed in this section allows to obtain only an imprecise estimate of the type of the Lie equations. In order to determine this type precisely, a direct linear algebra
computations can be used. In the next section we consider the special case of system with two dependent and two independent variables and show how the computations of the type can effectively be done.

7 Flat systems with linear relations, case \( n = 2, m = 2 \)

In the present section we consider the special case of study of (infinitesimal) symmetries of flat systems with first order relations.

Consider the following flat system \((S)\) given by

\[
\begin{align*}
  w^j_{x_1 x_1} &= 0, w^j_{x_1 x_2} = 0, w^j_{x_2 x_2} = 0, j = 1, 2 \\
  w^2_{x_1} &= a_{11} u^1_{x_1} + a_{12} u^1_{x_2}, \\
  w^2_{x_2} &= a_{21} u^1_{x_1} + a_{22} u^1_{x_2}
\end{align*}
\]

Our goal is to establish the following

**Proposition 7.1** Suppose that the matrices \(I d_2, A\) are linearly independent that is \((S)\) is nondegenerate. Then the corresponding Lie equations of \((S)\) form a PDE system of finite type 1.

Let a holomorphic vector field \(X = \theta^1 \frac{\partial}{\partial x_1} + \theta^2 \frac{\partial}{\partial x_2} + \eta^1 \frac{\partial}{\partial u^1} + \eta^2 \frac{\partial}{\partial u^2}\) be in \(\text{Lie}(S)\). First and second prolongations are

\[
X^{(1)} = X + \eta^1 \frac{\partial}{\partial u^1} + \eta^2 \frac{\partial}{\partial u^2} + \eta^1 \frac{\partial}{\partial u^1} + \eta^2 \frac{\partial}{\partial u^2}
\]

\[
X^{(2)} = X^{(1)} + \eta^1 \frac{\partial}{\partial u^1} + \eta^2 \frac{\partial}{\partial u^2} + \eta^1 \frac{\partial}{\partial u^1} + \eta^2 \frac{\partial}{\partial u^2} + \eta^1 \frac{\partial}{\partial u^1} + \eta^2 \frac{\partial}{\partial u^2}
\]

Following the general method described above, we have to consider the first order Lie equations:

\[
\begin{align*}
  \eta^2_2 ((S))^{(2)} &= a_{11} \eta^1_1 ((S))^{(2)} + a_{12} \eta^2_1 ((S))^{(2)} \\
  \eta^2_2 ((S))^{(2)} &= a_{21} \eta^1_1 ((S))^{(2)} + a_{22} \eta^2_1 ((S))^{(2)}
\end{align*}
\]

Computing the restrictions \(\eta^2_2 ((S))^{(2)}\) and comparing the coefficients near the powers of \(u^j_k\), we obtained the following linear first order PDE systems with constant coefficients for \(\theta, \eta\):

\[
\eta^2_{x_1} = a_{11} \eta^1_{x_1} + a_{12} \eta^1_{x_2}, \eta^2_{x_2} = a_{21} \eta^1_{x_1} + a_{22} \eta^1_{x_2}
\]

and

\[
\begin{align*}
  \eta^2_{u^1} + a_{11} \eta^2_{u^1} &= a_{11} \eta^2_{u^1} + \eta^2_{u^1} + \eta^2_{u^1} - a_{12} \theta^1_{x_2}, \\
  a_{12} \eta^2_{u^2} &= a_{12} \eta^1_{u^2} + a_{12} (a_{11} + a_{22}) \eta^1_{u^2} + a_{12} \theta^2_{x_1} + (a_{11} + a_{22}) \theta^2_{x_1} - a_{12} \theta^2_{x_2}, \\
  a_{21} \eta^2_{u^2} &= a_{21} \eta^1_{u^2} + (a_{21} + a_{22}) \eta^1_{u^2} - a_{21} \theta^1_{x_2} - a_{22} \theta^1_{x_2}, \\
  \eta^2_{u^1} + a_{22} \eta^2_{u^1} &= (a_{21} a_{12} + a_{22} \eta^1_{u^2} + a_{22} \eta^1_{u^2} + a_{21} \theta^1_{x_1} - a_{21} \theta^1_{x_1}
\end{align*}
\]
In view of our condition of linear independence of \( Id, \) \( A \) this last system implies that

\[
\eta_{u1}^2 = \phi_1(\eta_{ij}^1, \theta_{i}^k), \eta_{u2}^2 = \phi_1(\eta_{ij}^1, \theta_{i}^k)
\]

where \( \phi_s \) are linear functions.

Finally, we have two series of equations:

\[
a_{21} \left( \theta_{u1}^2 + a_{11} \theta_{u2}^2 - a_{12} \theta_{u2}^1 \right) = 0,
\]

\[
(a_{11} - a_{22}) \left( \theta_{u1}^2 - a_{12} \theta_{u2}^1 + a_{11} \theta_{u2}^2 \right) = 0,
\]

\[
a_{12} \left( \theta_{u1}^1 - a_{12} \theta_{u2}^1 + a_{11} \theta_{u2}^2 \right) = 0
\]

\[
a_{21} \left( \theta_{u1}^1 - a_{21} \theta_{u2}^2 - a_{22} \theta_{u2}^1 \right) = 0
\]

\[
(a_{11} - a_{22}) \left( \theta_{u1}^1 + a_{22} \theta_{u2}^1 - a_{21} \theta_{u2}^2 \right) = 0
\]

\[
a_{12} \left( \theta_{u1}^1 + a_{22} \theta_{u2}^1 - a_{21} \theta_{u2}^2 \right) = 0
\]

In view of the linear independence of the matrices \( Id_2, \) \( A \) this implies that

\[
\theta_{u1}^1 = -a_{22} \theta_{u2}^1 + a_{21} \theta_{u2}^2, \theta_{u1}^2 = a_{12} \theta_{u2}^1 - a_{11} \theta_{u2}^2
\]

It is useful to consider the differential consequences of these equalities:

\[
\theta_{u1}^1 = (a_{12}a_{21} + a_{22}^2)\theta_{u1}^1 - a_{22}a_{21}a_{11}\theta_{u2}^2,
\]

\[
\theta_{u1}^2 = -a_{21}a_{11}\theta_{u2}^2 + (a_{12}a_{21} + a_{22}^2)\theta_{u2}^2,
\]

\[
\theta_{u1}^1 = -(a_{12}a_{22} + a_{11}a_{12})\theta_{u2}^1 + (a_{12}a_{21} + a_{11}^2)\theta_{u2}^2
\]

Now we may similarly proceed the study of second order equations.
The second order Lie equations arise from the conditions

\[
\eta_{u1}^1|(S_2) = 0, \eta_{u1}^1|(S_2) = 0, \eta_{u1}^1|(S_2) = 0,
\]

After direct computations we obtain the following groups of equations:

\[
\eta_{x1}^1 = 0, \eta_{x2}^1 = 0, \eta_{x2}^2 = 0,
\]

\[
\theta_{u1}^1 + 2a_{11} \theta_{u1}^1 + a_{11}^2 \theta_{u2}^2 = 0, -a_{21} \left( \theta_{u1}^1 + a_{11} \theta_{u2}^2 \right) = 0, a_{21} \theta_{u2}^1 = 0,
\]

\[
\theta_{u1}^2 + 2a_{22} \theta_{u1}^2 + a_{22}^2 \theta_{u2}^2 = 0, -a_{12} \left( \theta_{u1}^2 + a_{22} \theta_{u2}^2 \right) = 0, a_{12} \theta_{u2}^2 = 0
\]
We have the following equations for $\eta^1$ and $\theta_1$:

\[
2\eta^1_{x_1 u^1} - \theta^1_{x_1 x_1} + 2a_{11} \eta^1_{x_1 u^2} = 0, \quad \eta^1_{u^1 u^1} - \theta^1_{x_1 x_2} + a_{11} \eta^1_{x_2 u^2} + a_{21} \eta^1_{x_1 u^2} = 0,
\]

\[
\eta^1_{u^1 u^1} - 2\theta^1_{x_1 u^1} + 2a_{11} \left( \eta^1_{u^1 u^2} - \theta^1_{x_1 x^2} \right) + a_{11} \eta^1_{u^1 u^2} = 0,
\]

\[
-\theta^1_{x_2 u^1} - a_{11} \theta^1_{x_2 x^2} + a_{21} \left( \eta^1_{u^1 u^2} - \theta^1_{x_1 x^2} \right) + a_{11} a_{21} \eta^1_{u^2 u^2} = 0,
\]

\[
-\theta^1_{x_2 x_2} + 2a_{21} \eta^1_{x_2 u^2} = 0, \quad a_{21} \left( -2\theta^1_{x_2 x^2} + 2a_{21} \eta^1_{u^2 u^2} \right) = 0
\]

We also have similar equations for $\eta^1$, $\theta_2$:

\[
2\eta^1_{x_2 u^1} - \theta^2_{x_2 x_2} + 2a_{22} \eta^1_{x_2 u^2} = 0, \quad \eta^1_{x_1 u^1} - \theta^2_{x_1 x_2} + a_{22} \eta^1_{x_1 u^2} + a_{12} \eta^1_{x_2 u^2} = 0,
\]

\[
\eta^1_{u^1 u^1} - 2\theta^2_{x_2 u^1} + 2a_{22} \left( \eta^1_{u^1 u^2} - \theta^2_{x_2 x^2} \right) + a_{22} \eta^1_{u^2 u^2} = 0,
\]

\[
-\theta^2_{x_1 u^1} - a_{22} \theta^2_{x_1 x^2} + a_{12} \left( \eta^1_{u^1 u^2} - \theta^2_{x_2 x^2} \right) + a_{22} a_{12} \eta^1_{u^2 u^2} = 0,
\]

\[
-\theta^2_{x_1 x_1} + 2a_{12} \eta^1_{x_1 u^2} = 0, \quad a_{12} \left( -2\theta^2_{x_1 x^2} + a_{12} \eta^1_{u^2 u^2} \right) = 0
\]

We have also the “mixed” equations containing $\eta^1$ and both of $\theta_1$, $\theta_2$:

\[
-2\theta^2_{x_1 u^1} + 2a_{12} \left( \eta^1_{u^1 u^2} - \theta^1_{x_1 x^2} \right) - 2a_{11} a_{12} \eta^1_{u^1 x^2} = 0,
\]

\[
\eta^1_{u^1 u^1} - \theta^1_{x_1 u^1} - \theta^2_{x_2 u^1} - a_{12} \theta^2_{x_2 x^2} + a_{12} \left( \eta^1_{u^1 u^2} - \theta^2_{x_2 x^2} \right)
\]

\[
+ a_{22} \left( \eta^1_{u^1 u^2} - \theta^2_{x_1 x^2} \right) - a_{21} \theta^2_{x_1 x^2} + (a_{12} a_{21} + a_{11} a_{22}) \eta^1_{u^2 u^2} = 0,
\]

\[
-\theta^2_{x_2 u^1} - a_{22} \theta^2_{x_2 x^2} + a_{21} \left( \eta^1_{u^1 u^2} - \theta^2_{x_2 x^2} \right) + a_{21} a_{22} \eta^1_{u^2 u^2} = 0
\]

Finally, we have the following series of equations:

\[
\theta^2_{u^1 u^1} + 2a_{12} \theta^2_{u^1 u^2} + 2a_{11} \theta^2_{u^2 u^2} + 2a_{11} a_{22} \theta^2_{u^1 u^2} + a_{11} \theta^2_{u^2 u^2} = 0,
\]

\[
a_{12} \left( 2\theta^2_{u^1 u^2} + 2a_{11} \theta^2_{u^2 u^2} + a_{12} \theta^2_{u^2 u^2} \right) = 0,
\]

\[
\theta^2_{u^1 u^1} + (a_{11} + a_{22}) \theta^2_{u^1 u^2} + a_{21} \theta^2_{u^2 u^2} + a_{11} a_{22} \theta^2_{u^2 u^2} + (a_{11} a_{22} + a_{12} a_{21}) \theta^2_{u^2 u^2} = 0,
\]

\[
\theta^2_{u^1 u^1} + (a_{11} + a_{22}) \theta^2_{u^1 u^2} + a_{12} \theta^1_{u^1 u^2} + a_{12} a_{22} \theta^1_{u^2 u^2} + (a_{11} a_{22} + a_{12} a_{21}) \theta^2_{u^2 u^2} = 0,
\]

\[
a_{21} \left( 2\theta^2_{u^1 u^2} + a_{21} \theta^2_{u^2 u^2} + 2a_{22} \theta^2_{u^2 u^2} \right) = 0,
\]

\[
\theta^2_{u^1 u^1} + a_{22} \theta^2_{u^1 u^2} + a_{21} \theta^2_{u^2 u^2} + 2a_{21} a_{22} \theta^2_{u^2 u^2} + a_{22} \theta^2_{u^2 u^2} = 0
\]

These equations together with earlier obtained first order ones form the system of Lie equations for $(S)$.

In order to show that the obtained second order linear PDE system is of finite type and the type is equal to 1 it is necessary to study the 1-prolongation of this system i.e. essentially the PDE system obtained by the consideration the first order partial derivatives of our equations.
Two cases can occur: the case where $a_{12} \neq 0$ or $a_{21} \neq 0$ and the case where $a_{12} = a_{21} = 0$ and $a_{11} \neq a_{22}$. In every case the direct elementary computation shows that the symbol of the 1-prolongation is trivial.

This completes the proof of the proposition.

As a corollary we obtain the following

**Corollary 7.2** Let $(S^c)$:

\[
\begin{align*}
u^j_{x_1} &= F^j_{11}(\varepsilon, x, u, u_x), \quad u^j_{x_2} = F^j_{12}(\varepsilon, x, u, u_x), \quad u^j_{x_2} = F^j_{22}(\varepsilon, x, u, u_x), \quad j = 1, 2 \\
u^2_{x_1} &= a_{11}u^1_{x_1} + a_{12}u^1_{x_2} + G_1(\varepsilon, x, u, u^1_x) \\
u^2_{x_2} &= a_{21}u^1_{x_1} + a_{22}u^1_{x_2} + G_2(\varepsilon, x, u, u^1_x)
\end{align*}
\]

be a holomorphic completely integrable deformation of the flat nondegenerate system $(S^0) = (S)$. Then for every $\varepsilon$ close to the origin enough one has $\dim\operatorname{Lie}(S^c) \leq \dim\operatorname{Lie}(S^0)$ and every infinitesimal symmetry of $(S^c)$ is determined by its second order Taylor expansion at the origin.

In particular, since the Segre family of a 6-dimensional real analytic Levi-nodegenerate manifold in $\mathbb{C}^4$ is described by a system of this class, the present method allows to obtain explicit recursive formulae for infinitesimal automorphisms of such a manifold.

In conclusion of this paper we emphasize again that our method can be used in order to obtain a very precise information on automorphisms of wide classes of CR manifolds and related PDE systems. For instance, if we replace the condition (i) in the definition of a Levi nondegenerate manifold by the slightly weaker condition of the triviality of the kernel of the Levi form, the Segre family will be given by a “mixed” PDE system containing second order partial derivatives of several dependent variables and first order equations with linear parts satisfying some independence conditions; our method works for this class of systems with minor modifications. The condition (ii) of the Levi nondegeneracy also can be replaced by a weaker assumptions on the highest Levi forms. This leads to systems where the terms of highest order (in the first order equations) satisfy some independence conditions. The most powerful algebraic tool for the study of the related Lie equations is given by the Spencer cohomology theory and the Cartan - Kahler theory of normal forms of analytic linear PDE systems (see [24]). Finally, the consideration of manifolds with the degenerate first Levi form leads to PDE systems which are not solved with respect to the highest partial derivatives. The study of their Lie symmetries needs more advanced tools of the local complex analytic geometry. Our approach also raises several other natural questions: equivalence problems and invariants of involutive second order PDE systems with first order relations, classifications of these systems with respect to the properties of symmetry group (non-compact, transitive, etc.) by analogy with very well known result of geometric complex analysis. But perhaps the most important problem is to develop in a systematic way the geometry of the Segre families of real analytic CR manifolds from the complex differential and algebraic geometry standpoint.

References

[1] M.S.Baouendi, P.Ebenfelt, L.P.Rothschild, *Algebraicity of holomorphic mappings between real algebraic sets in $\mathbb{C}^n$*, Acta Math. 177(1996), 225-273.
[2] M.S.Baouendi, P.Ebenfelt, L.P.Rothschild, *Rational dependence of smooth and analytic CR mappings on their jets*, Math. Ann. 315 (1999), 205-249.

[3] M.S.Baouendi, P.Ebenfelt, L.P.Rothschild, *CR automorphisms of real analytic CR manifolds in complex space*, Comm. Anal. Geom. 6 (1998), 291-315.

[4] E.Bedford, S.Pinchuk, *Convex domains with noncompact automorphism groups*, Mat. Sb. 185 (1994), 3-26.

[5] V.Beloshapka, *A uniqueness theorem for automorphisms of a nondegenerate surface in a complex space*, Math. Notes 47 (1990), 230-242.

[6] V.Beloshapka, *On holomorphic transformations of quadric*, Math. USSR Sb. 72 (1992), 189-205.

[7] G.W.Bluman, S.Kumei, *Symmetries and differential equations*, Springer-Verlag, 1989.

[8] S.Bohner, J.Martin, *Several complex variables*, Princeton Univ. Press, 1948.

[9] E.Cartan, *Sur la géométrie pseudoconforme des hypersurfaces de deux variables complexes*, Ann. Math. Pura Appl. 11 (1932)17-90.

[10] S.S.Chern, J.K.Moser, *Real hypersurfaces in complex manifolds*, Acta Math. 133 (1974), 219-271.

[11] S.S.Chern, *On the projective structure of a real hypersurface in \( \mathbb{C}^{n+1} \)*, Math. Scand. 36 (1975), 74-82.

[12] E.Dickson, *Differential equations from the group standpoint*, Ann. Math. 25(1924), 287-378.

[13] K.Diederich, S.Webster, *A reflection principle for degenerate real hypersurfaces*, Duke Math. J. 47 (1980), 835-845.

[14] K.Diederich, J.E.Fornaess, *Proper holomorphic mappings between real analytic pseudoconvex domains in \( \mathbb{C}^{n} \)*, Math. Ann. 282 (1988), 681- 700.

[15] K.Diederich, S.Pinchuk, *Proper holomorphic maps in dimension 2 extend*, Indiana Univ. Math. J. 44 (1995), 1089-1126.

[16] V.Ezhov, A.Isaev, G.Schmalz, *Invariants of elliptic and hyperbolic CR-structures of codimension 2*, Internat. J. Math. 10 (1999), 1-52.

[17] J.Faran, *Segre families and real hypersurfaces*, 60(1980), 135-172.

[18] F. Gonzalez-Gascon, A.Gonzalez-Lopez, *Symmetries of differential equations. IV*, J. Math. Phys. 24(1983), 2006-2021.

[19] S.Lie, G.Scheffers, *Vorlesungen uber Continuierliche Gruppen*, Chelsea, Bronx, NY. 1971.

[20] A.Loboda, *Real analytic generating manifolds of codimension 2 in \( \mathbb{C}^{4} \) and their biholomorphic mappings*, Math. USSR Izv. 33 (1989), 295-315.

[21] P.Olver, *Applications of Lie Groups to differential equations*, Springer-Verlag, 1986.

[22] P.Olver, *Equivalence, invariants and symmetry*, Cambridge Univ. Press. 1995.

[23] L.V.Ovsiannikov, *Group Analysis of Differential equations*, Academic Press, New York, 1982.

[24] J.-F.Pommaret, *Systems of partial differential equations and Lie pseudogroups*, Gordon and Breach Sci. Publ. 1978.

[25] J.-F.Pommaret, *Partial differential equations and group theory*, Kluwer, 1994.

[26] B.Segre, *Intorno al problem di Poincaré della representatione pseudo-conform*, Rend. Acc. Lincei, 13 (1931), 676-683.
[27] B.Shiffman, Projective geometry and Poincare’s theorem on automorphisms of the ball, Enseign. Math. 41 (1995), 201-216.
[28] N.Stanton, Infinitesimal CR automorphisms of real hypersurfaces, Amer. J. Math. 118(1996), 209-233.
[29] A.Sukhov, Segre varieties and Lie symmetries, Pub. IRMA, Lille, 1999, V.50.
[30] A.Sukhov, On CR mappings of real quadric manifolds, Mich. Math. J. 41(1999), 143-150.
[31] A.Tresse, Détermination des invariants ponctuels de l’équation différentielle du second ordre \( y'' = \omega(x, y, y') \), Hirzel, Leiptzig, 1896.
[32] N.Tanaka, On the pseudo-conformal geometry of hypersurfaces of the space of \( n \) complex variables, J. Math. Soc. Japan, 14 (1962), 397-429.
[33] A.Tumanov, Finite-dimensionality of the group of CR automorphisms of a standard CR manifold and proper holomorphic mappings of Siegel domains Funkts. Anal. i Pril. 17(1983), 49-61.
[34] S.M.Webster, On the Mapping Problem for algebraic real hypersurfaces Inv. Math. 43 (1977), 53-68.
[35] S.M.Webster, On the reflection principle in several complex variables, Proc. AMS 71(1978), 26-28.
[36] S.M.Webster, Double valued reflection in the complex plane, Enseign. Math. 42(1996), 25-48.
[37] S.Webster, Some birational invariants for algebraic real hypersurfaces, Duke Math. J. 45(1978), 39-46.
[38] D.Zaitsev, Germs of local automorphisms of real-analytic CR structures and analytic dependence of \( k \)-jets, Math. Res. Letters 41(1997), 823-842.

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