1. Introduction

Parshin’s conjecture states that $K_i(X)\mathbb{Q} = 0$ for $i > 0$ and $X$ smooth and projective over a finite field $\mathbb{F}_q$. The purpose of this paper is to break up Parshin’s conjecture into several independent statements, in the hope that each of them is easier to attack individually. If $CH_n(X, i)$ is Bloch’s higher Chow group of cycles of relative dimension $n$, then in view of $K_i(X)\mathbb{Q} \cong \oplus_n CH_n(X, i)\mathbb{Q}$, Parshin’s conjecture is equivalent to Conjecture $P(n)$ for all $n$, stating that $CH_n(X, i)\mathbb{Q} = 0$ for $i > 0$, and all smooth and projective $X$. We show assuming resolution of singularities that Conjecture $P(n)$ is equivalent to the conjunction of three conjectures $A(n)$, $B(n)$, and $C(n)$, and give several equivalent versions of these conjectures. This is most conveniently formulated in terms of weight homology. We define $H_{W}^*(X, \mathbb{Q})$ to be the homology of the complex $CH_n(W(X))\mathbb{Q}$, where $W(X)$ is the weight complex defined by Gillet-Soulé shifted by $2n$. Then, in a nutshell, Conjecture $A(n)$ states that for all schemes $X$ over $\mathbb{F}_q$, the niveau spectral sequence of $CH_n(X, *)\mathbb{Q}$ degenerates to one line, Conjecture $C(n)$ states that for all schemes $X$ over $\mathbb{F}_q$ the niveau spectral sequence of $H^W_n(X, \mathbb{Q})(n)$ degenerates to one line, and Conjecture $B(n)$ states that, for all $X$ over $\mathbb{F}_q$, the two lines are isomorphic. The conjunction of $A(n)$, $B(n)$, and $C(n)$ clearly implies $P(n)$, because $H^W_n(X, \mathbb{Q})(n) = 0$ for $i \neq 2n$ and $X$ smooth and projective over $\mathbb{F}_q$, and we show the converse. Note that in the above formulation, Parshin’s conjecture implies statements on higher Chow groups not only for smooth and projective schemes, but gives a way to calculate $CH_n(X, i)\mathbb{Q}$ for all $X$.

A more concrete version of $A(n)$ is that, for every smooth and projective scheme $X$ of dimension $d$ over $\mathbb{F}_q$, $CH_n(X, i)\mathbb{Q} = 0$ for $i > d - n$. A reformulation of $B(n)$ is that for every smooth and projective scheme $X$ of dimension $d > n + 1$, the following sequence is exact

$$0 \rightarrow K^M_{d-n}(k(X))\mathbb{Q} \rightarrow \oplus_{x \in X^{(1)}} K^M_{d-n-1}(k(x))\mathbb{Q} \rightarrow \oplus_{x \in X^{(2)}} K^M_{d-n-2}(k(x))\mathbb{Q}$$

and that this sequence is exact at $K^M_{1}(k(X))\mathbb{Q}$ if $d = \text{dim } X = n + 1$.

In the second half of the paper, we focus on the case $n = 0$, because of its applications in [5]. A different version of weight homology has been studied by

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Jannsen [11], and he proved that Conjecture C(0) holds under resolution of singularities. We use this to give two more versions of Conjecture P(0). The first is that there is an isomorphism from higher Chow groups with \( \mathbb{Q}_l \)-coefficients to \( l \)-adic cohomology, for all \( X \) and \( i \),

\[
CH_0(X, i) \mathbb{Q}_l \oplus CH_0(X, i + 1) \mathbb{Q}_l \rightarrow H_i(X_{et}, \hat{\mathbb{Q}}_l).
\]

Finally, conjecture P(0) can also be recovered from, and implies a structure theorem for higher Chow groups of smooth affine schemes: For all smooth and affine schemes \( U \) of dimension \( d \) over \( \mathbb{F}_q \), the groups \( CH_0(U, i) \) are torsion for \( i \neq d \), and the canonical map \( CH_0(U, d) \mathbb{Q}_l \rightarrow H_d(U_{et}, \hat{\mathbb{Q}}_l) \mathbb{Gal}(\mathbb{F}_q) \) is an isomorphism.

Finally, we reproduce an argument of Levine showing that if \( F \) is the absolute Frobenius, the push-forward \( F_* \) acts like \( q^n \) on \( CH_n(X, i) \), and the pull-back \( F^* \) acts on motivic cohomology \( H^1_{et}(X, \mathbb{Z}(n)) \) like \( q^n \) for all \( n \). As a Corollary, Conjecture P(0) follows from finite dimensionality of smooth and projective schemes over finite fields in the sense of Kimura [13].

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2. Parshin’s conjecture

We fix a perfect field \( k \) of characteristic \( p \), and consider the category of separated schemes of finite type over \( k \). We recall some facts on Bloch’s higher Chow groups [1], see [3] for a survey. Let \( z_n(X, i) \) be the free abelian group generated by cycles of dimension \( n+i \) on \( X \times_k \Delta^i \) which meet all faces properly, and let \( z_n(X, *) \) be the complex of abelian groups obtained by taking the alternating sum of intersection with face maps as differential. We define \( CH_n(X, i) \) as the \( i \)th homology of this complex and motivic Borel-Moore homology to be

\[
H^i_c(X, \mathbb{Z}(n)) = CH_n(X, i - 2n).
\]

For a proper map \( f : X \rightarrow Y \) we have a push-forward \( z_n(X, *) \rightarrow z_n(Y, *) \), for a flat, quasi-finite map \( f : X \rightarrow Y \), we have a pull-back \( z_n(X, *) \rightarrow z_n(Y, *) \), and a closed embedding \( i : Z \rightarrow X \) with open complement \( j : U \rightarrow X \) induces a localization sequence

\[
\cdots \rightarrow H^i_c(Z, \mathbb{Z}(n)) \xrightarrow{i_*} H^i_c(X, \mathbb{Z}(n)) \xrightarrow{j^*} H^i_c(U, \mathbb{Z}(n)) \rightarrow \cdots.
\]

If \( X \) is smooth of pure dimension \( d \), then \( H^i_c(X, \mathbb{Z}(n)) \cong H^{2d-i}(X, \mathbb{Z}(d-n)) \), where the right hand side is Voevodsky’s motivic cohomology [13]. For a finitely generated field \( F \) over \( k \), we define \( H^i_c(F, \mathbb{Z}(n)) = \text{colim}_U H^i_c(U, \mathbb{Z}(n)) \), where the colimit runs through \( U \) of finite type over \( k \) with field of functions \( F \). For the reader who is more familiar with motivic cohomology, we mention that Voevodsky’s theorem implies
that for a field $F$ of transcendence degree $d$ over $k$, we have

$$H_i^c(F, \mathbb{Z}(n)) \cong H^{2d-i}(F, \mathbb{Z}(d-n)) \cong \begin{cases} 0 & i < d + n \\ K^M_{d-n}(F) & i = d + n. \end{cases}$$

The latter isomorphism is due to Nesterenko-Suslin and Totaro. It follows formally from localization that there are spectral sequences

$$E^1_{s,t}(X) = \bigoplus_{x \in X_{(s)}} H^c_{s+t}(k(x), \mathbb{Z}(n)) \Rightarrow H^c_{s+t}(X, \mathbb{Z}(n)). \quad (1)$$

Here $X_{(s)}$ denotes points of $X$ of dimension $s$. Since $H^c_i(F, \mathbb{Z}(n)) = 0$ for $i < n + \text{trdeg } F$, the spectral sequence is concentrated in the area $0 \leq s \leq \dim X$ and $t \geq n$. If we let

$$\tilde{H}^c_i(X, \mathbb{Z}(n)) = E^2_{i+n,n}(X)$$

be the $i$th homology of the complex

$$0 \leftarrow \bigoplus_{x \in X_{(s)}} H^c_{2s}(k(x), \mathbb{Z}(n)) \leftarrow \cdots \leftarrow \bigoplus_{x \in X_{(s)}} H^c_{s+n}(k(x), \mathbb{Z}(n)) \leftarrow \cdots, \quad (2)$$

with $\bigoplus_{x \in X_{(s)}} H^c_{s+n}(k(x), \mathbb{Z}(n))$ in degree $s + n$, then we obtain a canonical and functorial map

$$\alpha : H^c_i(X, \mathbb{Z}(n)) \rightarrow \tilde{H}^c_i(X, \mathbb{Z}(n)).$$

Note that the groups in $\tilde{2}$ are Milnor-K-groups.

Parshin’s conjecture states that for all smooth and projective $X$ over $\mathbb{F}_q$, the groups $K_i(X)_{\mathbb{Q}}$ are torsion for $i > 0$. If Tate’s conjecture holds and rational equivalence and homological equivalence agree up to torsion for all $X$, then Parshin’s conjecture holds by [2]. Since $K_i(X)_{\mathbb{Q}} = \oplus_n CH_n(X, i)_{\mathbb{Q}}$, it follows that Parshin’s conjecture is equivalent to the following conjecture for all $n$.

Conjecture $P(n)$: For all smooth and projective schemes $X$ over the finite field $\mathbb{F}_q$, the groups $H^c_i(X, \mathbb{Q}(n))$ vanish for $i \neq 2n$.

We will refer to the following equivalent statements as Conjecture $A(n)$:

**Proposition 2.1.** For a fixed finite field $\mathbb{F}_q$ and integer $n$, the following statements are equivalent:

a) For all schemes $X/\mathbb{F}_q$ and all $i$, $\alpha$ induces an isomorphism $H^c_i(X, \mathbb{Q}(n)) \cong H^c_i(X, \mathbb{Q}(n))$.

b) For all finitely generated fields $k/\mathbb{F}_q$ with $d := \text{trdeg } k/\mathbb{F}_q$, and all $i \neq d + n$, we have $H^c_i(k, \mathbb{Q}(n)) = 0$.

c) For all smooth and projective $X$ over $\mathbb{F}_q$ and all $i > \dim X + n$, we have $H^c_i(X, \mathbb{Q}(n)) = 0$.

d) For all smooth and affine schemes $U$ over $\mathbb{F}_q$ and all $i > \dim U + n$, we have $H^c_i(U, \mathbb{Q}(n)) = 0$. 
Proof. a) ⇒ c), d): The complex (2) is concentrated in degrees \([2n, d + n]\).

c) ⇒ b): This is proved by induction on the transcendence degree. By de Jong’s theorem, we find a smooth and proper model \(X\) of a finite extension of \(k\). Looking at the niveau spectral sequence (1), we see that the induction hypothesis implies \(H^c_i(X, \mathbb{Q}(n)) = 0\) for \(i > d + n\) (see [2] for details).

d) ⇒ b): follows by writing \(k\) as a colimit of smooth affine scheme schemes of dimension \(d\).

b) ⇒ a): The niveau spectral sequence collapses to the complex (2).

Using the Gersten resolution, the statement in the proposition implies that on a smooth \(X\) of dimension \(d\), the motivic complex \(\mathbb{Q}(d - n)\) is concentrated in degree \(d - n\), and if \(C_n := \mathcal{H}^{d-n}(\mathbb{Q}(d - n)) = CH_n(\cdot, d - n)\), then \(CH_n(X, i) = H^{d-n-i}(X, C_n)\).

Statement (2.1) is part of the following affine analog of \(P(n)\):

**Conjecture** \(L(n)\): For all smooth and affine schemes \(U\) of dimension \(d\) over the finite field \(\mathbb{F}_q\), the group \(H^c_r(U, \mathbb{Q}(n))\) vanishes unless \(d \leq i \leq d + n\).

Since \(H^c_r(U, \mathbb{Q}(n)) \cong H^{2d-i}(U, \mathbb{Q}(d - n))\), Conjecture \(L(n)\) can be thought of as an analog of the affine Lefschetz theorem.

### 3. Weight homology

This section is inspired by Jannsen [11]. Throughout this section we assume resolution of singularities over the field \(k\). Let \(\mathcal{C}\) be the category with objects smooth projective varieties over a field \(k\) of characteristic 0, and \(\text{Hom}_\mathcal{C}(X, Y) = \bigoplus_i CH^{\dim Y_i}(X \times Y_i)\), where \(Y_i\) runs through the connected components of \(Y\). Let \(\mathcal{H}\) be the homotopy category of bounded complexes over \(\mathcal{C}\). Gillet and Soulé define in [8], for every separated scheme of finite type, a weight complex \(W(X) \in \mathcal{H}\) satisfying the following properties [8, Thm. 2] (our notation differs from loc.cit. in variance):

a) \(W(X)\) is represented by a bounded complex

\[
M(X_0) \leftarrow M(X_1) \leftarrow \cdots \leftarrow M(X_k)
\]

with \(\dim X_i \leq \dim X - i\), where \(M(X_i)\) placed in degree \(i\).

b) \(W(\cdot)\) is covariant functorial for proper maps.

c) \(W(\cdot)\) is contravariant functorial for open embeddings.

d) If \(T \to X\) is a closed embedding with open complement \(U\), then there is a distinguished triangle

\[
W(T) \xrightarrow{i_*} W(X) \xrightarrow{j^*} W(U).
\]
e) If $D$ is a divisor with normal crossings in a scheme $X$, with irreducible components $Y_i$, and if $Y^{(r)} = \coprod_{i \neq r} \cap_{i \in I} Y_i$, then $W(X - D)$ is represented by

$$M(X) \leftarrow M(Y^{(1)}) \leftarrow \cdots \leftarrow M(Y^{(\dim X)}).$$

The argument in loc.cit. only uses that resolution of singularities exists over $k$, and we assume from now on that $k$ is such a field.

Given an additive covariant functor $F$ from $C$ to an abelian category, we define weight (Borel-Moore) homology $H^W_i(X, F)$ as the $i$th homology of the complex $F(W(X))$. Weight homology has the functorialities inherited from b) and c), and satisfies a localization sequence deduced from d). If $K$ is a finitely generated field over $k$, then we define $H^W_i(K, F)$ to be $\colim_i H^W(U, F)$, where the (filtered) limit runs through integral varieties having $K$ as their function field. Similarly, a contravariant functor $G$ from $C$ to an abelian category gives rise to weight cohomology (with compact support) $H^W_i(X, G)$.

As a special case, we define the weight homology group $H^W_i(X, \mathbb{Z}(n))$ as the $i - 2n$th homology of the homological complex of abelian groups $CH_n(W(X))$.

**Lemma 3.1.** We have $H^W_i(X, \mathbb{Z}(n)) = 0$ for $i > \dim X + n$. In particular, $H^W_i(K, \mathbb{Z}(n)) = 0$ for every finitely generated field $K/k$ and every $i > \trdeg K/k + n$.

**Proof.** This follows from the first property of weight complexes together with $CH_n(T) = 0$ for $n > \dim T$. \hfill $\square$

It follows from Lemma 3.1 that the niveau spectral sequence

$$E^2_{s,t}(X) = \oplus_{x \in X(\alpha)} H^W_{s+t}(k(x), \mathbb{Z}(n)) \Rightarrow H^W_{s+t}(X, \mathbb{Z}(n))$$

is concentrated on and below the line $t = n$. Let

$$\tilde{H}^W_i(X, \mathbb{Z}(n)) = E^2_{i+n,n}(X)$$

be the $i$th homology of the complex

$$0 \leftarrow \oplus_{x \in X(\alpha)} H^W_{2n}(k(x), \mathbb{Z}(n)) \leftarrow \cdots \leftarrow \oplus_{x \in X(\alpha)} H^W_{s+n}(k(x), \mathbb{Z}(n)) \leftarrow \cdots,$$

where $\oplus_{x \in X(\alpha)} H^W_{s+n}(k(x), \mathbb{Z}(n))$ is placed in degree $s + n$. Then we obtain a canonical and natural map

$$\gamma : \tilde{H}^W_i(X, \mathbb{Z}(n)) \to H^W_i(X, \mathbb{Z}(n)).$$

Consider the canonical map of covariant functors $\pi' : z_n(-, *) \to CH_n(-)$ on the category of smooth projective schemes over $k$, sending the cycle complex to its highest cohomology. Then by [11 Thm.5.13, Rem.5.15], the set of associated homology functors extends to a homology theory on the category of allvarieties over $k$. The argument of [11 Prop. 5.16] show that the extension of the associated homology functors for $z_n(-, *)$ are higher Chow groups $CH_n(-, i)$. The extension $CH_n(-)$ are by definition the functors $H^W_i(-, \mathbb{Z}(n))$. We obtain a functorial map

$$\pi : H^W_i(X, \mathbb{Z}(n)) \to H^W_i(X, \mathbb{Z}(n)).$$
Lemma 3.2. For \( i = 2n \), and all schemes \( X \), the map \( \pi \) is an isomorphism \( H^2_{2n}(X, \mathbb{Z}(n)) \cong H^W_{2n}(X, \mathbb{Z}(n)) \). In particular, \( H^W_{2d}(K, \mathbb{Z}(d)) \cong \mathbb{Z} \) for all fields \( K \) of transcendence degree \( d \) over \( k \).

Proof. The statement is clear for \( X \) smooth and projective. We proceed by induction on the dimension of \( X \). Using the localization sequence for both theories, we can assume that \( X \) is proper. Let \( f : X' \to X \) be a resolution of singularities of \( X \), \( Z \) be the closed subscheme (of lower dimension) where \( f \) is not an isomorphism, and \( Z' = Z \times_X X' \). Then we conclude by comparing localization sequences

\[
\begin{array}{c}
H^2_{2n}(Z', \mathbb{Z}(n)) \longrightarrow H^2_{2n}(Z, \mathbb{Z}(n)) \oplus H^2_{2n}(X', \mathbb{Z}(n)) \longrightarrow H^2_{2n}(X, \mathbb{Z}(n)) \longrightarrow 0 \\
\| \quad \| \quad \downarrow \\
H^W_{2n}(Z', \mathbb{Z}(n)) \longrightarrow H^W_{2n}(Z, \mathbb{Z}(n)) \oplus H^W_{2n}(X', \mathbb{Z}(n)) \longrightarrow H^W_{2n}(X, \mathbb{Z}(n)) \longrightarrow 0.
\end{array}
\]

\( \square \)

The map \( \pi \) for fields induces a map \( \beta : \tilde{H}^i_1(X, \mathbb{Q}(n)) \to \tilde{H}^W_i(X, \mathbb{Q}(n)) \), which fits into the (non-commutative) diagram

\[
\begin{array}{ccc}
H^i(X, \mathbb{Z}(n)) & \xrightarrow{\pi} & H^W_i(X, \mathbb{Z}(n)) \\
\alpha \downarrow & & \uparrow \gamma \\
\tilde{H}^i(X, \mathbb{Z}(n)) & \xrightarrow{\beta} & \tilde{H}^W_i(X, \mathbb{Z}(n)).
\end{array}
\]

We now return to the situation \( k = \mathbb{F}_q \), and compare weight homology to higher Chow groups using their niveau spectral sequences. We saw that the niveau spectral sequence for higher Chow groups is concentrated above the line \( t = n \), and that the niveau spectral sequence for weight homology is concentrated below the line \( t = n \). Our aim is to show that Parshin’s conjecture is equivalent to both being rationally concentrated on this line, and that the resulting complexes are isomorphic.

The following statements will be referred to as Conjecture \( B(n) \):

Proposition 3.3. For a fixed integer \( n \), the following statements are equivalent:

a) The map \( \beta \) induces an isomorphism \( \tilde{H}^i(X, \mathbb{Q}(n)) \cong \tilde{H}^W_i(X, \mathbb{Q}(n)) \) for all schemes \( X \) and all \( i \).

b) The map \( \pi \) induces an isomorphism \( H^i_{d+n}(k, \mathbb{Q}(n)) \cong H^W_{d+n}(k, \mathbb{Q}(n)) \) for all finitely generated fields \( k/\mathbb{F}_q \), where \( d = \text{trdeg} k/\mathbb{F}_q \).

c) For every smooth and projective \( X \) over \( \mathbb{F}_q \) we have \( \tilde{H}^i_{d+n}(X, \mathbb{Q}(n)) = \tilde{H}^W_{d+n-1}(X, \mathbb{Q}(n)) = 0 \) if \( d = \dim X > n + 1 \), and \( \tilde{H}^i_{2n+1}(X, \mathbb{Q}(n)) = 0 \) if \( \dim X = n + 1 \). Note that assuming \( A(n), c) \) is equivalent to \( H^i_{d+n}(X, \mathbb{Q}(n)) = \tilde{H}^W_{d+n-1}(X, \mathbb{Q}(n)) = 0 \) for all smooth and projective \( X \) of dimension \( d > n + 1 \) and \( d = n + 1 \), respectively, hence are part of Conjecture \( P(n) \).
Proof. \( b) \Rightarrow a) \) is trivial, and \( a) \Rightarrow b) \) follows by a colimit argument because

\[
\colim_{U \subseteq X} \tilde{H}_i^c(X, \mathbb{Q}(n)) \cong \begin{cases} 
H_{d+n}^c(k(X), \mathbb{Q}(n)) & i = d + n; \\
0 & \text{otherwise}.
\end{cases}
\]

\( a) \Rightarrow c) \) follows because for \( X \) smooth and projective of dimension \( d \), the cohomology of the complex \( [2] \) tensored with \( \mathbb{Q} \) equals \( \tilde{H}_i^c(X, \mathbb{Q}(n)) = H_1^W(X, \mathbb{Q}(n)) \) for \( i = d + n \) and \( i = d + n - 1 \) (or \( i = 2n + 1 \) in case \( d = n + 1 \)). An inspection of the niveau spectral sequence \( [3] \) shows that this is a subgroup of \( H_1^W(X, \mathbb{Q}(n)) = 0 \).

\( c) \Rightarrow b) \): For \( n > d \), both sides vanish, whereas for \( d = n \), both sides are canonically isomorphic to \( \mathbb{Q} \). For \( n < d \), we proceed by induction on \( d \). Choose a smooth and projective model \( X \) for \( k \) and compare the exact sequences \( [2] \) and \( [4] \):

\[
\begin{array}{cccc}
A & \leftarrow & \bigoplus_{x \in X_{(d-1)}} H^c_{d+n-1}(k(x), \mathbb{Q}(n)) & \leftarrow & H^c_{d+n}(k, \mathbb{Q}(n)) \\
\| & & | & & |
B & \leftarrow & \bigoplus_{x \in X_{(d-1)}} H^W_{d+n-1}(k(x), \mathbb{Q}(n)) & \leftarrow & H^W_{d+n}(k, \mathbb{Q}(n))
\end{array}
\]

The terms on the left are \( A = H^c_{2n}(X, \mathbb{Q}(n)) \cong B = H^W_{2n}(X, \mathbb{Q}(n)) \) if \( d = n + 1 \), and \( A = \bigoplus_{x \in X_{(d-1)}} H^c_{d+n-2}(k(x), \mathbb{Q}(n)) \cong B = \bigoplus_{x \in X_{(d-2)}} H^W_{d+n-2}(k(x), \mathbb{Q}(n)) \) if \( d > n + 1 \). The upper sequence is exact by hypothesis, and an inspection of \( [3] \) shows that the lower sequence is exact because \( H^W_1(X, \mathbb{Q}(n)) = 0 \) for \( i > 2n \).

We refer to the following statements as Conjecture C(n):

**Proposition 3.4.** For a fixed integer \( n \), the following statements are equivalent:

a) For all schemes \( X \) over \( \mathbb{F}_q \) and all \( i \), the map \( \gamma \) induces an isomorphism \( H^c_1(X, \mathbb{Q}(n)) \cong H^W_1(X, \mathbb{Q}(n)) \).

b) For all finitely generated fields \( k/\mathbb{F}_q \) and \( i \neq \text{trdeg } k/\mathbb{F}_q + n \), we have \( H^W_1(k, \mathbb{Q}(n)) = 0 \).

c) For all smooth and projective \( X \), the map \( \gamma \) induces an isomorphism

\[
\tilde{H}_i^W(X, \mathbb{Q}(n)) = \begin{cases} 
0 & i > 0; \\
C_{H_n}(X) & i = 2n.
\end{cases}
\]

Proof. \( b) \Rightarrow a) \) is trivial and \( a) \Rightarrow b) \) follows by a colimit argument.

a) \( \Rightarrow c) \) is trivial for \( i > 2n \), and Lemma 3.2 for \( i = 2n \).

c) \( \Rightarrow b) \): This is proved like Proposition 2.1 by induction on the transcendence degree of \( k \). Let \( X \) be a smooth and projective model of \( k \). The induction hypothesis implies that the niveau spectral sequence \( [3] \) collapses to horizontal line \( t = n \) and the vertical line \( s = d \). Since it converges to \( H^W(X, \mathbb{Q}(n)) \), which is zero for \( i > 2n \), we obtain isomorphisms \( H^W_{d+n-i}(k(X), \mathbb{Q}(n)) \xrightarrow{d_1} \tilde{H}^W_{d+n-i}(X, \mathbb{Q}(n)) \) for \( 1 < i < d - n \), and an exact sequence

\[
H^W_{2n+1}(k(X), \mathbb{Q}(n)) \xrightarrow{d_{2n}} \tilde{H}^W_{2n}(X, \mathbb{Q}(n)) \rightarrow H^W_{2n}(X, \mathbb{Q}(n)) \rightarrow H^W_{2n}(k(X), \mathbb{Q}(n)).
\]
The claim follows because $\tilde{H}_W^i(X, \mathbb{Q}(n)) = H_W^i(X, \mathbb{Q}(n)) = 0$ for $i > 2n$, and because the maps $H_{2n}^i(X, \mathbb{Q}(n)) \xrightarrow{\sim} H_W^i(X, \mathbb{Q}(n)) \xleftarrow{\sim} \tilde{H}_W^i(X, \mathbb{Q}(n))$ are isomorphisms by Lemma 3.2 and hypothesis.

\section*{Theorem 3.5.} The conjunction of Conjectures $A(n)$, $B(n)$ and $C(n)$ is equivalent to $P(n)$.

\textbf{Proof.} Given Conjectures $A(n)$, $B(n)$, and $C(n)$, we get

$$H_c^i(X, \mathbb{Q}(n)) = \tilde{H}_c^i(X, \mathbb{Q}(n)) \cong \tilde{H}_W^i(X, \mathbb{Q}(n)) \cong H_W^i(X, \mathbb{Q}(n))$$

for all $X$, and the latter vanishes for smooth and projective $X$ and $i > 0$, hence Conjecture $P(n)$ follows. Conversely, Conjecture $P(n)$ implies Prop. 2.1(c), then 3.3(c), and finally 3.4(c) by using 2.1(a) and 3.3(a).

\section*{Remark.} Propositions 2.1, 3.4, 3.3 as well as Theorem 3.5 remain true if we restrict ourselves to schemes of dimension at most $N$, and fields of transcendence degree at most $N$, for a fixed integer $N$.

\section*{Remark.} Gillet announced that one can obtain a rational version of weight complexes by using de Jong’s theorem on alterations instead of resolution of singularities. The same argument should then give the generalization [11, Thm.5.13]. In this case, all arguments of this section hold true rationally, except the proof of c) $\Rightarrow$ b) in Propositions 3.3 and 3.4 and the proof of $P(n) \Rightarrow A(n), B(n), C(n)$ in Theorem 3.5 which require that every finitely generated field over $\mathbb{F}_q$ has a smooth and projective model.

\section*{4. The case $n = 0$}

Since $H^0_c(X, \mathbb{Q}(0)) = CH_0(X, i)_{\mathbb{Q}}$, we use higher Chow groups in this section.

\textbf{Proposition 4.1.} We have $CH_0(X, i)_{\mathbb{Q}} \cong \tilde{H}_c^i(X, \mathbb{Q}(0))$ for $i \leq 2$, and the map $CH_0(X, 3)_{\mathbb{Q}} \rightarrow \tilde{H}_c^3(X, \mathbb{Q}(0))$ is surjective for all $X$. In particular, $A(0)$ holds in dimensions at most 2.

\textbf{Proof.} Since $H^i(k(x), \mathbb{Q}(1)) = 0$ for $i \neq 1$ and $H^i(k(x), \mathbb{Q}(0)) = 0$ for $i \neq 0$, this follows from an inspection of the niveau spectral sequence. \hfill $\square$

Jannsen [11] defines a variant of weight homology with coefficients $A$, $H_W^i(X, A)$, as the homology of the complex $\text{Hom}(CH^0(W(X)), A)$. Note that $H_W^i(X, \mathbb{Q}) = H^i(X, \mathbb{Q}(0))$ because $\text{Hom}(CH^0(X), \mathbb{Q}) \cong CH_0(X)_{\mathbb{Q}}$ for smooth and projective $X$, in a functorial way. Indeed, a map of connected, smooth and projective varieties induces the identity pull-back on $CH^0$ and the identity push-forward on $CH_0$. 

Theorem 4.2. (Jannsen) Under resolution of singularities, $H^W_a(k, A) = 0$ for $a \neq \text{trdeg } k/\mathbb{F}_q$, hence $H^W_i(X, A)$ is the homology of the complex

$$0 \leftarrow \bigoplus_{x \in X(\mathbb{F}_q)} H^0_a(k(x), A) \leftarrow \cdots \leftarrow \bigoplus_{x \in X(\mathbb{F}_q)} H^W_a(k(x), A) \leftarrow \cdots \tag{5}$$

for all schemes $X$. In particular, Conjecture $C(0)$ holds.

This is proved in [11] Prop. 5.4, Thm. 5.10. The proof only works for $n = 0$, because it uses the bijectivity of $CH^m(Y) \to CH^0(X)$ for a map of connected smooth and projective schemes $X \to Y$. The second statement follows using the niveau spectral sequence, which exists because $H^W_i(X, A)$ satisfies the localization property by property (4) of weight complexes.

Let $Z^c(0)$ be the complex of etale sheaves $z_0(-, \ast)$. For any prime $l$, consider $l$-adic cohomology

$$H_i(X_{\text{et}}, \hat{\mathbb{Q}}_l) := \mathbb{Q} \otimes_{\mathbb{Z}} \lim H_i(X_{\text{et}}, Z^c/l^U(0)).$$

In [4], we showed that for every positive integer $m$, and every scheme $f : X \to k$ over a perfect field, there is a quasi-isomorphism $Z^c/m(0) \cong Rf^! Z/m$. In particular, the above definition agrees with the usual definition of $l$-adic homology if $l \neq p = \text{char } \mathbb{F}_q$. If $\bar{X} = X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ and $\hat{G} = \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$, then there is a short exact sequence

$$0 \to H_{i+1}(\bar{X}_{\text{et}}, \hat{\mathbb{Q}}_l)^{\hat{G}} \to H_i(X_{\text{et}}, \hat{\mathbb{Q}}_l) \to H_i(\bar{X}_{\text{et}}, \hat{\mathbb{Q}}_l)^{\hat{G}} \to 0$$

and for $U$ affine and smooth, $H_i(U_{\text{et}}, \hat{\mathbb{Q}}_l)$ vanishes for $i \neq d, d - 1$ and $l \neq p$ by the affine Lefschetz theorem and a weight argument [12 Thm.3a]). The map from Zariski-hypercohomology of $Z^c/m(0)$ to etale-hypercohomology of $Z^c/m(0)$ induces a functorial map

$$CH_0(X, i)/m \to CH_0(X, i, Z/m) \to H_i(X_{\text{et}}, Z^c/m(0)),$$

hence in the limit a map

$$\omega : CH_0(X, i)_{\hat{\mathbb{Q}}_l} \to H_i(X_{\text{et}}, \hat{\mathbb{Q}}_l).$$

Similarly, the map $\bar{\omega} : CH_0(\bar{X}, i)_{\hat{\mathbb{Q}}_l} \to H_i(\bar{X}_{\text{et}}, \hat{\mathbb{Q}}_l)$ induces a map

$$\tau : CH_0(X, i + 1)_{\hat{\mathbb{Q}}_l} \leftarrow (CH_0(\bar{X}, i + 1)_{\hat{\mathbb{Q}}_l})^{\hat{G}} \to H_{i+1}(\bar{X}_{\text{et}}, \hat{\mathbb{Q}}_l)^{\hat{G}} \to H_i(X_{\text{et}}, \hat{\mathbb{Q}}_l)$$

(the left map is an isomorphism by a trace argument). The sum

$$\varphi^i_X : CH_0(X, i)_{\hat{\mathbb{Q}}_l} \oplus CH_0(X, i + 1)_{\hat{\mathbb{Q}}_l} \to H_i(X_{\text{et}}, \hat{\mathbb{Q}}_l) \tag{6}$$

is compatible with localization sequences, because all maps involved in the definition are. The following proposition shows that Parshin’s conjecture can be recovered from and implies a structure theorem for higher Chow groups of smooth affine schemes; compare to Jannsen [11 Con. 12.4b]].
Proposition 4.3. The following statements are equivalent:

a) Conjecture $P(0)$.

b) For all schemes $X$, and all $i$, the map $\varphi^i_X$ is an isomorphism.

c) For all smooth and affine schemes $U$ of dimension $d$, the groups $CH_0(U, i)$ are torsion for $i \neq d$, and the composition $\omega : CH_0(U, d)_Q \to H_d(U_{et}, \hat{Q}_l) \to H_d(\bar{U}_{et}, \hat{Q}_l)^G$ is an isomorphism.

Proof. a) $\Rightarrow$ b): First consider the case that $X$ is smooth and proper. Then Conjecture $P(0)$ is equivalent to vanishing of the left hand sides of (6) for $i \neq 0, -1$, whereas the right hand side of (6) vanishes by the Weil-conjectures. On the other hand, $\varphi^0_X$ induces an isomorphism $CH_0(X) \otimes Q_l \cong H^{2d}(X_{et}, \hat{Q}_l(d))$ and $\varphi^{-1}_X$ induces an isomorphism $CH_0(X) \otimes Q_l \cong (CH_0(\bar{X}) \otimes Q_l)^G \cong H^{2d}(\bar{X}, \hat{Q}_l(d))^G$. Indeed, both sides are isomorphic to $Q_l$ if $X$ is connected. Using localization and alterations, the statement for smooth and proper $X$ implies the statement for all $X$.

b) $\Rightarrow$ a): The right-hand side of (6) is zero for $i \neq 0, -1$ by weight reasons for smooth and projective $X$, hence so is the left side.

c) $\Rightarrow$ b): We first assume that $X$ is smooth and affine. By hypothesis and the affine Lefschetz theorem, both sides of (6) vanish for $i \neq d, d - 1$, and are isomorphic for $i = d$. For $i = d - 1$, the vertical maps in the following diagram are isomorphisms by semi-simplicity,

\[
\begin{array}{ccc}
CH_0(X, d)_Q & \cong & CH_0(\bar{X}, d)^G_Q \\
\| & & \| \\
(CH_0(\bar{X}, d)_Q)^G & \longrightarrow & H_d(\bar{X}, \hat{Q}_l)^G \\
\longrightarrow & & \sim \longrightarrow \\
& & H_{d-1}(X, \hat{Q}_l).
\end{array}
\]

Hence the lower map $\varphi^{d-1}_X$ is an isomorphism because the upper map is. Using localization, the statement for smooth and affine $X$ implies the statement for all $X$.

Proposition 4.4. Under resolution of singularities, the following are equivalent:

a) Conjecture $P(0)$.

b) For every smooth affine $U$ of dimension $d$ over $F_q$, we have $CH_0(U, i)_Q \cong H^W_i(U, Q)$ for all $i$, and these group vanish for $i \neq d$.

c) For every smooth affine $U$ of dimension $d$ over $F_q$, the groups $CH_0(U, i)_Q$ vanish for $i > d$, and $CH_0(U, d)_Q \cong H^W_d(U, Q)$.
Proof. a) ⇒ b): It follows from the previous Proposition that \( CH_0(U, i) \) = 0 for \( i \neq d \). On the other hand, \( P(0) \) for all \( X \) implies that \( CH_0(X, i) \cong H^W_i(X, \mathbb{Q}) \) for all \( i \) and \( X \).

c) ⇒ a): The statement implies Conjecture A(0), and then Conjecture B(0), version b), for all \( X \). By Theorems 3.5 and 4.2, \( P(0) \) follows.

\[
\text{Proposition 4.5. Conjecture} \ P(0) \text{ for all smooth and projective } X \text{ implies the following statements:}
\]

a) (Affine Gersten) For every smooth affine \( U \) of dimension \( d \), the following sequence is exact:

\[
CH_0(U, d) \otimes \mathbb{Q} \rightarrow \bigoplus_{x \in U^{(0)}} H^d(k(x), \mathbb{Q}(d)) \rightarrow \bigoplus_{x \in U^{(1)}} H^{d-1}(k(x), \mathbb{Q}(d-1)) \rightarrow \cdots
\]

b) Let \( X = X_d \supseteq X_{d-1} \supseteq \cdots \supseteq X_0 \) be a filtration such that \( U_i = X_i - X_{i-1} \) is smooth and affine of dimension \( i \). Then \( CH_0(X, i) \mathbb{Q} \) is isomorphic to the \( i \)th homology of the complex

\[
0 \rightarrow CH_0(U_d, d) \mathbb{Q} \rightarrow CH_0(U_{d-1}, d-1) \mathbb{Q} \rightarrow \cdots \rightarrow CH_0(U_0, 0) \mathbb{Q} \rightarrow 0.
\]

The maps \( CH_0(U_i, i) \mathbb{Q} \rightarrow CH_0(U_{i-1}, i-1) \mathbb{Q} \rightarrow CH_0(U_{i-1}, i-1) \mathbb{Q} \) are from the localization sequence.

Proof. a) follows because the spectral sequence (I) collapses, and b) by a diagram chase.

Remark. If we fix a smooth scheme \( X \) of dimension \( d \), and use cohomological notation, then by Proposition 2.1 and the Gersten resolution, we get that the rational motivic complex \( Q(d) \) is conjecturally concentrated in degree \( d \), say \( C_d = H^d(Q(d)) = CH_0(-, d) \mathbb{Q} = H^W_d(-, \mathbb{Q}) \). Then Conjecture L(0) says that \( H^i(U, C_d) = 0 \) for \( U \subseteq X \) affine and \( i > 0 \). This is analog to the mod \( p \) situation, where the motivic complex agrees with the logarithmic de Rham-Witt sheaf \( \mathbb{Z}/p(n) \cong \nu^d[-d] \), and \( H^i(U, \nu^d) = 0 \) for \( U \subseteq X \) affine and \( i > 0 \). The latter can be proved by writing \( \nu^d \) as the kernel of a map of coherent sheaves and using the vanishing of cohomology of coherent sheaves on affine schemes. This suggest that one might try to do the same for \( C_d \).

4.1. Frobenius action. Let \( F : X \rightarrow X \) be the Frobenius morphism induced by the \( q \)th power map on the structure sheaf.

Theorem 4.6. The push-forward \( F_* \) acts like \( q^n \) on \( CH_n(X, i) \), and the pull-back \( F^* \) acts on \( H^i(X, \mathbb{Z}(n)) \) as \( q^n \) for all \( n \).

The Theorem is well-known, but we could not find a proof in the literature. The proof of Soulé [14] Prop.2] for Chow groups does not carry over to higher Chow groups, because the Frobenius does not act on the simplices \( \Delta^n \), hence a cycle \( Z \subseteq \Delta^n \times X \) is not send to a multiple of itself by the Frobenius. We give an argument due to M. Levine.
Proof. Let $DM^-$ be Voevodsky’s derived category of bounded above complexes of Nisnevich sheaves with transfers with homotopy invariant cohomology sheaves. Then we have the isomorphisms

$$\begin{align*}
CH_n(X, i) &\cong \text{Hom}_{DM^-}(\mathbb{Z}(n)[2n+i], M_c(X)), \\
H^i(X, \mathbb{Z}(n)) &\cong \text{Hom}_{DM^-}(M(X), \mathbb{Z}(n)[i]).
\end{align*}$$

The action of the Frobenius is given by composition with $F$: $M_c(X) \to M_c(X)$ and $F: M(X) \to M(X)$, respectively.

The Frobenius acts on the category $DM^-$, i.e. for every $\alpha \in \text{Hom}_{DM^-}(X, Y)$ we have $F_Y \circ \alpha = \alpha \circ F_X$. This follows by considering composition of correspondences. Hence it suffices to calculate the action of the Frobenius on $\mathbb{Z}(n)$, i.e. show that $F = q^n \in \text{Hom}_{DM^-}(\mathbb{Z}(n), \mathbb{Z}(n)) \cong \mathbb{Z}$. But $\text{Hom}_{DM^-}(\mathbb{Z}(n), \mathbb{P}^n[-2n]) = CH_n(\mathbb{P}^n)$. The latter is the free abelian group generated by the generic point, and the Frobenius acts by $q^n$ on it. $\blacksquare$

Remark. 1) It would be interesting to write down an explicit chain homotopy between $F^*$ and $q^n$ on $\mathbb{Z}_n(X, *)$.

2) The proposition implies that the groups $CH_n(\mathbb{F}_q, i)$ are killed by $q^n - 1$, and that $CH_n(X, i)$ is $q$-divisible for $n < 0$.

Granted the Theorem, the standard argument gives the following Corollary, see also Jannsen [10, Thm. 12.5.7].

Corollary 4.7. Assume that for all be smooth and projective $X$ of dimension $d$, the kernel of the map $CH^d(X \times X) \to \text{End}_{\text{hom}}(M(X))$ is nilpotent. Then Conjecture $P(0)$ holds.

The hypothesis of the Corollary is satisfied if $X$ is finite dimensional in the sense of Kimura [13].

Proof. Using the existence of a zero-cycle $c$ of degree 1, we see that the projector $\pi_{2d} = [X \times c]$ is defined. Let $\tilde{X}$ be the motive ker $\pi_{2d} = X/\mathcal{L}^d$, where $\mathcal{L}$ is the Lefschetz motive. Consider the action of the geometric Frobenius $F \in \text{End}_{\text{rat}}(\tilde{X}) \subseteq CH^d(X \times X)$. Its image in the category of motives for homological equivalence is algebraic, and its minimal polynomial $P_{\tilde{X}}(T)$ has roots of absolute value $q^{\frac{1}{2}}$ for $0 \leq j < 2d$. By hypothesis, $P_{\tilde{X}}(F)^a = 0$ in $\text{End}_{\text{rat}}(\tilde{X})$ for some integer $a$, but by the Theorem, $F^*$ acts like $q^a$ on $CH^d(X, i)$. Hence $0 \neq P_{\tilde{X}}(q^a)^a = P_{\tilde{X}}(F^*)^a = 0$. $\blacksquare$

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