A logical theory for strong and weak ontic necessities in branching time

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Abstract
Ontic necessities are those modalities universally quantifying over domains of ontic possibilities, whose “existence” is independent of our knowledge. An ontic necessity, called the weak ontic necessity, causes interesting questions. An example for it is “I should be dead by now”. A feature of this necessity is whether it holds at a state has nothing to do with whether its prejacent holds at the state. Is there a weak epistemic necessity expressed by “should”? Is there a strong ontic necessity expressed by “must”? How do we make sense of the strong and weak ontic necessities formally? In this paper, we do the following work. Firstly, we recognize strong/weak ontic/epistemic necessities and give our general ideas about them. Secondly, we present a complete logical theory for the strong and weak ontic necessities in branching time. This theory is based on the following approach. The weak ontic necessity quantifies over a domain of expected timelines, determined by the agent’s system of ontic rules. The strong ontic necessity quantifies over a domain of accepted timelines, determined by undefeated ontic rules.

Keywords: Strong and weak ontic necessities; Strong and weak epistemic necessities; Branching time; Ordered ontic rules; Undefeatable ontic rules

1 Introduction

1.1 Modalities

Modalities locate their prejackets in spaces of possibilities ([FV06]). Different modalities locate their prejackets in different kinds of spaces: epistemic ones, deontic ones, and so on, that is, different modalities have different flavors ([Mat16]). Different modalities locate their prejackets in different ways: universally, existentially, and so on, that is, different modalities have different forces ([Mat16]). Those modalities related to universal quantifications are called necessities and those related to existential quantifications are called possibilities. We look at two examples.

(1) The man talking aloud might be drunk.
(2) Bob must go to school.

The first sentence says that “The man talking aloud is drunk” is true at some epistemic possibility, and the second one says that “Bob goes to school” is true at all deontic possibilities.

1Note that there are two different senses of “possibility” in this paragraph.
Modalities are complex and there are many theories about them in the literature. We refer to [vF06] and [Mat16] for some general discussion. The most influential theory on modalities in linguistics is the one proposed by Kratzer in [Kra91], among her other works. The framework of this theory consists of a set of possible worlds and two functions, respectively called a modal base and an ordering source. The modal base determines which possible worlds are accessible to a possible world and the ordering source orders possible worlds. Many modalities can be interpreted in this framework.

1.2 Strong/weak ontic/epistemic necessities

Ontic possibilities are possible states in which our world is or could be if things had gone differently in the past. Ontic possibilities can be easily confused with epistemic ones, which are possible states that we consider possible to be the actual state. Ontic possibilities are independent of our knowledge about the actual state but epistemic ones are due to our ignorance about it. Epistemic possibilities are ontic too: If a possibility is not possible in the ontic sense, then it is not possible in the epistemic sense. Here is an example. Suppose we know that Adam bought a lottery yesterday, the winning number is just revealed and Adam loses. The possibility that Adam wins is ontic but not epistemic.

An ontic necessity, identified by Copley [Cop06], causes interesting problems. Look at the following examples.

(3) Suppose Jones is in a building when an earthquake hits. The building collapses. Luckily, nothing falls upon Jones and he emerges from the rubble as the only survivor. Talking to the media, Jones says the following:

*I ought to be dead right now.* (From [Yal16])

(4) Consider Rasputin. He was hard to kill. First his assassins poisoned him, then they shot him, then they finally drowned him. Let us imagine that we were there. Let us suppose that the assassins fed him pastries dosed with a powerful, fast-acting poison, and then left him alone for a while, telling him they would be back in half an hour. Half an hour later, one of the assassins said to the others, confidently, “He ought to be dead by now.” The others agreed, and they went to look. Rasputin opened his eyes and glared at them. “He ought to be dead by now!” they said, astonished. (From [Tho08])

The sentence “I ought to be dead right now” is felicitous in the scenario of 3 but its prejacent is clearly false. This implies that the necessity in it is not epistemic but ontic. This is the same for the second occurrence of “He ought to be dead by now” in 4. In English, this necessity can also be expressed by “should”.

Non-deontic “should” has been commonly treated as epistemic. Is there really an epistemic “should”? If so, how do we deal with it and the ontic “should”?

Many works on deontic logic and deontic modalities, including [Wer72], [McN96], [vFI08] and [PK16], have made it clear that deontic “must” and “ought to” have different logical properties. For example, “must” is stronger than “ought to”, and “ought to” admits exceptions but “must” does not.

Non-deontic “must” and “should” have differences as well. The following examples show that non-deontic “must” is stronger than “should”.

2This is where the notions of “strong necessity” and “weak necessity” are from.
(5) The beer should be cold; in fact, it must be. (From Cop06)
(6) #The beer must be cold; in fact, it should be. (From Cop06)

We already see that “should φ” and “not φ” coexist. However, “must φ” and “not φ” do not, as shown by the following examples.

(7) #Our guests must be home by now, but they are not. (Adapted from Cop06)
(8) #The beer must be cold by now, but it isn’t. (From Cop06)

Is there an ontic “must”? If so, how do we deal with it and the epistemic “must”? Chinese “yiding” corresponds to “must”, and “yinggai” corresponds to “should”. Similar examples with the given ones can be established with “yiding” and “yinggai”. We refer to [WW21] for a semantic analysis of “yiding” and “yinggai”.

1.3 Related work

There have been some studies about ontic “should” in the literature, which, besides Cop06, include Swa08, Tho08, Pin09 and Yal16. There have been some works on epistemic “should”, including Sto94 and Man19. However, they do not distinguish between epistemic “should” and ontic “should”. As far as we read, no works in the literature explicitly discuss ontic “must”, except Yalcin Yal16, who does not think there is an ontic “must”. There have been many works on epistemic “must”, including Kar72, Sto94, Yal07, vFG10 and Man19.

1.4 Our work

We focus on strong and weak ontic necessities in this work. Firstly, we recognize strong/weak ontic/epistemic necessities and give our general ideas about them. The ideas are as follows. There is an agent. The domains for strong/weak ontic/epistemic necessities respectively consist of accepted ontic possibilities, which are determined by facts and undefeatable ontic rules, expected ontic possibilities, which are determined by facts and ontic rules, accepted epistemic possibilities, which are determined by facts, undefeatable ontic rules and indicative rules, and expected epistemic possibilities, which are determined by facts, ontic rules and indicative rules. Secondly, we present a complete logic for strong and weak ontic necessities in branching time following these ideas.

The rest of this paper is structured as follows. In Section 2, we discuss some present work about strong/weak ontic/epistemic necessities. In Section 3, we first show that there is an epistemic “should” and an ontic “must”. Then we present our general ideas about strong/weak epistemic/ontic necessities. The logical theory for strong and weak ontic necessities in time flow is presented in Section 4. In Section 5, we look at the logical theory from some general perspectives. This paper is concluded in Section 6, where we point out some future work as well. We put proofs for some results to Section B in the appendix.
2 Strong/weak ontic/epistemic necessities in the literature

2.1 Weak ontic and epistemic necessities

Here we just discuss some works studying ontic “should” and do not discuss those works which only study epistemic “should”, since we focus on ontic necessities in this work.

Copley [Cop06] thinks that “should” can involve either epistemic or ontic necessity. The two readings are determined by the kind of evidence and the direction of reasoning. Epistemic “should” is concerned with the evidence of what is known and the reasoning from present evidence to past or present events. Ontic “should” is concerned with the evidence of facts and the reasoning from earlier causing facts to later caused facts. We look at two examples adapted from [Cop06]:

(9) Yesterday evening, you saw that the clouds had been building up and that a thunderstorm had been approaching. This morning, before opening the curtain, you say the following: 

It should have rained.

(10) I don’t understand it. The ground is wet, even the leaves on the trees are wet, as far as the eye can see. It SHOULD have rained. But you’re telling me it’s just a very sophisticated sprinkler system.

According to Copley [Cop06], “should” is ontic in the first sentence and epistemic in the second one.

Copley [Cop06] hints that epistemic “should” also admits future-oriented evidence. For example, she thinks that “should” is epistemic in the following sentence:

(11) She raised less money than the other candidate, she had a lousy campaign manager, and what’s more, she’s actually a convicted felon. Our candidate SHOULD lose the election. 

Unless somehow all the other candidate’s supporters stay home.

This point seems incompatible with her ideas about epistemic “should”, as the reasoning is from earlier facts to later facts in this kind of example.

Generally speaking, we think Copley’s thoughts make very good sense, although they seem under development in some sense. It will be seen later that these thoughts influence our ideas about strong/weak ontic/epistemic necessities much.

Thomson [Tho08] thinks that the weak ontic necessity can be handled using time and probability. It should be the case that φ if and only if it was probably the case that φ. Yalcin [Yal16] considers this idea not good. Here is an example from [Yal16]:

(12) An urn has one black and four white marbles. A marble is selected at random. We observe it is black. Then we can say: It was probable that the marble selected would be white. However, it would be odd to say: The marble selected should be white.

Yalcin [Yal16] does not think that “should” can have an epistemic reading, although he does not show this. He thinks the weak ontic necessity is closely related to normality: It quantifies over a domain of normal worlds. We will compare our theory to this work later.
2.2 Strong ontic and epistemic necessities

As mentioned previously, many works in the literature study epistemic “must”. Again, since we focus on ontic necessities in this work, we just consider [Yal16], which seems the only work discussing the issue of ontic “must”.

Yalcin [Yal16] tentatively thinks that “must” cannot express a strong ontic necessity. He has two arguments.

We consider the first one. In Kratzerian framework, we do the following to evaluate “must $\phi$” at a possible world $w$. By a set of propositions, we get a set of possible worlds accessible to $w$. By another set of propositions, we get an ordering among possible worlds. Then we check whether $\phi$ is true at all minimal accessible worlds to $w$. Suppose that “must” can express a strong ontic necessity. Note “must $\phi$” implies $\phi$. It is implied that for every possible world $w$, $w$ is a minimal accessible world to itself. Why? Assume not. Then we can make the following happen: “Must $p$” is true at $w$ but $p$ is false there. Then there is a contradiction. Yalcin thinks there are no conceptual reasons that the evaluation world needs to be minimal among its accessible worlds. Then “must” cannot express a strong ontic necessity.

We consider this argument not very convincing. Firstly, it is unclear whether there are conceptual reasons that the evaluation world needs to be minimal among its accessible worlds. Secondly, we do not have to suppose Kratzerian framework in dealing with modalities all the time.

Now we consider the second argument. Look at the following example:

(13) a. I must sneeze.
   b. I should sneeze.

Yalcin thinks that the first sentence does not imply the second one. Suppose that “must” can express a strong ontic necessity. Then “must” in the first sentence should express a strong ontic necessity and “should” in the second sentence should express a weak one. Then there is a problem.

We do not consider this argument very convincing either. There is some weirdness with the implication from the first sentence to the second one. However, it is not clear whether the weirdness is logical or not.

3 Recognitions of strong/weak ontic/epistemic necessities and our general ideas about them

3.1 Recognitions of strong/weak ontic/epistemic necessities

We think that “should” can express an epistemic necessity. Look at the following example.

(14) a. Jones is at home alone in the evening. A severe earthquake hits and the house topples. The next morning we come close to the house. We say the following:
   
   Jones should be dead by now but we have no idea whether he is.

b. During digging, we hear a weak voice. We say the following:
Jones should still be alive.

c. Finally, we find Jones alive. We say the following:

Jones should be dead by now but he is not.

It is clear that “should” in the sentence in [14c] is ontic. The sentence “Jones should be dead by now” is true in the first and third situations for the same reason. Then “should” in the sentence in [14a] is ontic as well. Then it is reasonable to think that “Jones should be dead by now” is true in the second situation too. Then “should” in the sentence in [14b] should be epistemic.

We think that “must” can also express an ontic necessity. Here is an example.

(15) Adam and Bob bought lotteries yesterday. Due to some reason, they exchanged their lotteries. The winning number is just generated randomly and Adam wins. Here is a sentence:

Adam must be the winner.

There is a clear sense in which this sentence is true, which means that “must” in it can be epistemic. However, there is also a sense in which this sentence is false. This means that “must” in it can be ontic as well.

We think that “could” is the dual of ontic “must” but not the dual of ontic “should”. The reason is that “could not” is synonymous with “must not” but not with “should not”. We also think that “might” is the dual of epistemic “must” but not the dual of epistemic “should”. “Should” does not seem to have a dual in English, no matter whether it is ontic or epistemic.

3.2 Our general ideas about strong/weak ontic/epistemic necessities

Fix an agent. There is a space of logical possibilities, which are possible states in which the world can be in the logical sense. The world is in a state, that is, the actual one. The agent knows some facts, which can be about the world in the past or the present. The possibilities in the space are both ontic and epistemic.

The agent accepts two kinds of rules about the world: ontic ones and indicative ones. Ontic rules concern which ontic possibilities are expected. Ontic rules also concern which epistemic possibilities are expected: If a possibility is not ontically expected, then it is not epistemically expected either. Indicative rules just concern which epistemic possibilities are expected.

Ontic rules can be of many kinds: natural laws (Light is faster than sound), common natural phenomena (It is cold in Beijing during the winter), daily experiences (Lack of engine oil causes damage to engines), and so on. Indicative rules can also be of many kinds: common observations (Fast windmills indicate big wind), daily experiences (Blinking engine lights indicate damaged engines), customs (Jones wears a pink cap on sunny days), and so on.

Facts and ontic rules determine expected ontic possibilities. For example, by the fact it is winter now and the rule it is cold in Beijing during the winter, those possibilities where it is cold in Beijing are expected.

Conflicts may arise among ontic rules in specific situations. Here is an example. Suppose that it is winter and the El Niño condition has occurred. Then the following ontic rules are conflicting: It is cold in Beijing in winter and the El Niño condition causes warm winter in Beijing.
It is complex how facts and ontic rules determine expected ontic possibilities. Here we just mention two points. Firstly, some kinds of rules tend to override some other kinds of rules. For example, natural laws tend to override daily experiences. Secondly, special rules tend to override general rules. For example, the rule the El Niño condition causes warm winter in Beijing tends to override the rule it is cold in Beijing in winter.

Ontic “should” quantifies over the domain of expected ontic possibilities. It is fine for the agent that the actual world is not expected. As a result, the agent can accept “should φ” and “not φ” at the same time.

The agent might learn new ontic rules in the future and she is ready for some new ones to defeat some old ones. However, she might take some rules as undefeatable. For example, she might take the following rule as undefeatable: The sun rises tomorrow. The possibilities violating any undefeatable rules are unaccepted and she just does not consider them.

Ontic “must” quantifies over the domain of accepted ontic possibilities. It is not fine for the agent that the actual world is unaccepted, as this would mean that some undefeatable rules are defeated. As a result, the agent cannot accept “must φ” and “not φ” at the same time.

Facts, ontic rules and indicative rules determine expected epistemic possibilities. Here is an example. By the fact the engine light is blinking and the indicative rule blinking engine lights indicate damaged engines, those possibilities where there is a problem with the engine are expected.

Conflicts among ontic or indicative rules may arise. In addition, conflicts among present facts and ontic or indicative rules may also occur. For example, if the engine light is blinking, then the current fact the engine is just fine conflicts with the indicative rule blinking engine lights indicate damaged engines.

About how facts, ontic rules and indicative rules determine expected epistemic possibilities, we just want to mention one thing: Present facts override rules. For example, in a situation where the engine light is blinking, the present fact the engine is just fine overrides the rule blinking engine lights indicate damaged engines.

Epistemic “should” quantifies over the domain of expected epistemic possibilities.

The agent might take some ontic or indicative rules as undefeatable, which determine accepted epistemic possibilities. Epistemic “must” quantifies over the domain of accepted epistemic possibilities.

**Remarks** We often vary in which rules are accepted, which rules override which rules, and which rules are undefeatable. Consequently, we often disagree on whether it must/should be the case that φ.

3.3 Analyses of some examples

In the above, we use the following example to show an epistemic “should”:

(14) a. Jones is at home alone in the evening. A severe earthquake hits and the house topples. The next morning we come close to the house. We say the following:

Jones should be dead by now but we have no idea whether he is.

b. During digging, we hear a weak voice. We say the following:
Jones should still be alive.

c. Finally, we find Jones alive. We say the following:

Jones should be dead by now but he is not.

“Should” in the sentence in [14a] is both ontic and epistemic. Why? There is a fact about the past: Jones has been buried in the rubble for some time. There is an implicit ontic rule: Being buried in rubble causes death. Then Jones should be dead by now in the ontic sense. There are no present facts or indicative rules. Then Jones should be dead by now in the epistemic sense. “Should” in the sentence in [14b] is epistemic. There is a fact: A weak voice comes. There is an implicit indicative rule: Voices indicate lives. This rule overrides the rule being buried in rubble causes death. Then Jones should still be alive in the epistemic sense. “Should” in the sentence in [14c] is ontic and the reason is similar with why “should” in the sentence in [14a] is ontic.

The following examples are from [Cop06]:

[9] Yesterday evening, you saw that the clouds had been building up and that a thunderstorm had been approaching. This morning, before opening the curtain, you say the following sentence:

It should have rained.

[10] I don’t understand it. The ground is wet, even the leaves on the trees are wet, as far as the eye can see. It SHOULD have rained. But you’re telling me it’s just a very sophisticated sprinkler system.

Copley uses the first example to explain the ontic “should” and the second one to show that epistemic “should” exists. We think that “should” in the first example is both ontic and epistemic. Why? By the two mentioned facts and an implicit ontic rule clouds and thunderstorms cause rain, we can get it should have rained in the ontic sense. There are no present facts or indicative rules. Then we can get it should have rained in the epistemic sense too. We think that “should” in the second example is epistemic. Why? By the two mentioned present facts and an implicit ontic rule wet ground and leaves indicate rain, we can get it should have rained in the epistemic sense.

Copley [Cop06] thinks that “should” in the following example is epistemic.

[11] She raised less money than the other candidate, she had a lousy campaign manager, and what’s more, she’s actually a convicted felon. Our candidate SHOULD lose the election. Unless somehow all the other candidate’s supporters stay home.

We think that “should” in this example is both epistemic and ontic. By the three facts mentioned in the example and an implicit ontic rule less money, being lousy and bad reputation cause loss, we can get our candidate should lose the election in the ontic sense. There are no indicative rules. Then we can get our candidate should lose the election in the epistemic sense.

Yalcin [Yal16] uses the following example to show that ontic “should” cannot be defined by time and “probable”.

[12] An urn has one black and four white marbles. A marble is selected at random. We observe it is black. Then we can say: It was probable that the marble selected would be white. However, it would be odd to say: The marble selected should be white.
Why is “The marble selected should be white” odd in this example? The reason is that no ontic rule here guarantees that the selected marble is expected to be white.

The following example is mentioned in the introduction.

(4) Consider Rasputin. He was hard to kill. First his assassins poisoned him, then they shot him, then they finally drowned him. Let us imagine that we were there. Let us suppose that the assassins fed him pastries dosed with a powerful, fast-acting poison, and then left him alone for a while, telling him they would be back in half an hour. Half an hour later, one of the assassins said to the others, confidently, “He ought to be dead by now.” The others agreed, and they went to look. Rasputin opened his eyes and glared at them. “He ought to be dead by now!” they said, astonished. (From [Tho08])

The first “ought to” in this example is both ontic and epistemic. There is a fact: Rasputin was fed with a powerful and fast-acting poison half an hour ago. There is only one implicit ontic rule: Powerful and fast-acting poison causes quick death. Then we can get he ought to be dead by now in both ontic and epistemic senses. The second “ought to” in this example is just ontic. The reason is that the present fact Rasputin is still alive overrides the abovementioned rule in determining expected epistemic possibilities.

In the above, we use the following example to show an ontic “must”:

(15) Adam and Bob bought lotteries yesterday. Due to some reason, they exchanged their lotteries. The winning number is just generated randomly and Adam wins. Here is a sentence:

Adam must be the winner.

There is a fact: Adam wins. Then there is only one epistemic possibility, the actual one. Then the sentence is true in the epistemic sense. There are no ontic rules. Then all possibilities are ontically possible. Then the sentence is false in the ontic sense.

Here are two more examples for “must”:

(16) Our guests must be home by now. They left half-an-hour ago, have a fast car, and live only a few miles away. (From [Lee71])

(17) Xander must be there. His car is outside and his lights are on. (Adapted from [Cop06])

“Must” in the first sentence is both ontic and epistemic and the domain for it is determined by the three mentioned facts and an implicit ontic law. “Must” in the second sentence is epistemic and the domain for it is determined by the two mentioned present facts and an implicit indicative law.

What follows is an example where both “must” and “should” occur.

(18) Alice must not be home yet; she hasn’t called, and she always calls right away to let us know she got back safely. She ought to be home already. I hope there wasn’t an accident. (From [Sil])

“Must” in this example is epistemic and “should” is ontic. There is an implicit undefeatable indicative rule: She always calls right away after getting home. Then Alice must not be home yet. By some ontic rule, which is not given in the example, she ought to be home already.
4 A logical theory for strong and weak ontic necessities in branching time

4.1 Our approach

For simplicity, we assume the agent knows everything about the world in the past and the present. The world can evolve in different ways and there are different timelines. The agent holds some ontic rules concerning which timelines are expected. The ontic rules may conflict with each other and there is a priority order among them. The ordered ontic rules as a whole determine which timelines are expected. The agent takes some ontic rules as undefeatable. The timelines violating any undefeatable ontic rules are unaccepted. Fix a moment. It may happen that no expected timelines pass through the moment. However, there are always accepted timelines passing through it.

The sentence “It should be the case now that $\phi$” is true at a moment if and only if $\phi$ is true with respect to all expected timelines, no matter whether they pass through the moment or not. It may happen that “It should be the case now that $\phi$” is true at a moment but “It is the case now that $\phi$” is false with respect to all timelines passing through the moment.

The sentence “It must be the case now that $\phi$” is true at a moment if and only if $\phi$ is true with respect to all accepted timelines, no matter whether they pass through the moment or not. It never happens that “It must be the case now that $\phi$” is true at a moment but “It is the case now that $\phi$” is false with respect to all timelines passing through the moment.

We look at an example explicitly involving time by our approach, which will be our running example.

Example 1. People of a tribe capture some animals and put them in isolated cages in a room. Every evening they randomly open two cages and leave the room with its door locked. The next morning they come and release the survivor. One day, three animals remain: a tiger, a dog, and a goat. Here are three sentences:

(19) a. The tiger must be alive tomorrow.
    b. The tiger should be alive tomorrow.
    c. The dog must be alive tomorrow.

The next morning they come outside the room. Here are three sentences:

(20) a. The tiger must be alive right now.
    b. The tiger should be alive right now.
    c. The dog must be alive right now.

They enter the room and find that the tiger is killed by the dog. Here are three sentences:

(21) a. The tiger must be alive right now.
    b. The tiger should be alive right now.
    c. The dog must be alive right now.
There are three ontic rules: *Tigers kill dogs*, *Tigers kill goats*, and *Dogs kill goats*. They are not comparable. None of them is undefeatable. For example, a strong goat could kill a weak tiger with its cavel. Since the first and second ontic rules are not undefeatable, the sentence in 19a cannot be uttered in the first situation. Then we should reject the sentences in 20a and 21a. According to these ontic rules, it is unexpected that the tiger will be killed. Then we can utter the sentence in 19b in the first situation. Then we should accept the sentences in 20b and 21b. The sentence in 19c cannot be uttered in the first situation. Then we should reject the sentences in 20c and 21c.

Remarks “Must” and “should” in these sentences can have epistemic readings too. In the first situation, there are just three ontic rules. Then ontic and epistemic possibilities are identical there. Then the ontic and epistemic readings of the sentences in 19a, 19b, and 19c are respectively equivalent. Due to similar reasons, the ontic and epistemic readings of the sentences in 20a, 20b, and 20c are respectively equivalent in the second situation. In the third situation, besides the three ontic rules, there are three present facts: *The tiger is dead, the dog is alive and the goat is alive*. Then the three ontic rules are overridden in determining epistemic possibilities. Then there is only one epistemic possibility. As a result, the epistemic readings of the sentences in 21a, 21b, and 21c are respectively false, false, and true. Note that their ontic readings are respectively false, true, and false.

4.2 Language

**Definition 1** (The language $\Phi_{SWONBT}$). Let $AP$ be a countable set of atomic propositions and $p$ range over it. The language $\Phi_{SWONBT}$ of the Logic for Strong and Weak Ontic Necessities in Branching Time (SWONBT) is defined as follows:

$$\phi ::= p \mid \bot \mid \neg \phi \mid (\phi \land \phi) \mid X\phi \mid Y\phi \mid [S]\phi \mid [W]\phi$$

The intuitive reading of featured formulas of this language:

- $X\phi$: $\phi$ will be true at the next moment.
- $Y\phi$: $\phi$ was true at the last moment.
- $[S]\phi$: $\phi$ must be true at the present moment.
- $[W]\phi$: $\phi$ should be true at the present moment.

The propositional connectives $\top, \lor, \rightarrow$ and $\leftrightarrow$ are defined as usual. The dual $[S]\phi$ of $[S]\phi$ is defined as $\neg[S]\neg\phi$, meaning $\phi$ could be true at the present moment. The dual $[W]\phi$ of $[W]\phi$ is defined as $\neg[W]\neg\phi$. As mentioned, $[W]\phi$ does not seem to have an intuitive reading. We introduce it due to technical reasons.

4.3 Semantic settings

4.3.1 Models

**Definition 2** (Models for $\Phi_{SWONBT}$). A tuple $M = (W, <, V)$ is a model for $\Phi_{SWONBT}$ if the following conditions are satisfied:
• $W$ is a nonempty set of states;
• $<$ is a serial relation on $W$ meeting the following condition: There is a $r \in W$, called the root, such that for every $w \in W$, there is a unique finite sequence $x_0, \ldots, x_n$ of states such that $x_0 = r$, $x_n = w$ and $x_0 < \cdots < x_n$;
• $V : \text{AP} \to \mathcal{P}(W)$ is a valuation.

Models for $\Phi_{\text{SWONBT}}$ are based on the so-called serial discrete rooted trees. Intuitively, models indicate how the world could evolve in time flow. There is a starting point but no ending point. The past is determined but the future is open.

What follow are some notions related to models which will be used later. An infinite sequence $x_0, x_1, \ldots$ of states is called a timeline if $x_0 = r$ and $x_0 < x_1 < \ldots$. We use $\text{TL}(M)$ to indicate the set of all timelines of $M$. For any timeline $\pi$ and natural number $i$, we use $\pi[i]$ to indicate the $i + 1$-th element of $\pi$. We call the set of natural numbers the clock.

For every model $M$, timeline $\pi$ and natural number $i$, $(M, \pi, i)$ is called a pointed model.

**Example 2** (Pointed models). Figure 1 indicates a pointed model $(M, \pi_2, 1)$. The intuitive reading of $(M, \pi_2, 1)$ is as follows. The present state is $\pi_2[1]$, that is, $w^1_1$, which has an alternative $w^1_2$. Both $w^1_1$ and $w^1_2$ evolve from the root $w^0_0$. Both of them have two next moments. The timeline $\pi_2$ is in consideration.

![Figure 1: A pointed model](image)

### 4.3.2 Contexts

**Definition 3** (Contexts). Let $M$ be a model. A tuple $C = (\mathcal{L}, \succ, U)$ is called a context for $M$ if the following conditions are met:

• $\mathcal{L}$ is a finite (possibly empty) set of (possibly empty) sets of timelines of $M$;
• $\succ$ is an irreflexive and transitive relation on $\mathcal{L}$;
• \(U\) is a (possibly empty) subset of the set of maximal elements of \(L\) with respect to \(\succ\).

The elements of \(L\) are called ontic rules and the elements of \(U\) are called undefeatable ones.

Intuitively, two ontic rules \(L_1\) and \(L_2\) in the relation \(\succ\) means that \(L_1\) has higher priority than \(L_2\). We use \(\epsilon\) to indicate the special context \((\emptyset, \emptyset, \emptyset)\).

**Definition 4** (Hierarchy of ontic rules in contexts). Let \(C = (L, \succ, U)\) be a context for a model \(M\). Define \(HI(C)\), the hierarchy of ontic rules in \(C\), as the sequence \((L_0, \ldots, L_n)\), which is constructed in the following way:

- Let \(L_0 = \{L \in L \mid L\) is a maximal element of \(L\}\};
- If \(L_0 \cup \cdots \cup L_k \neq L\), let \(L_{k+1} = \{L \in L \mid L\) is a maximal element of \(L \setminus (L_0 \cup \cdots \cup L_k)\}\), or else stop.

Here are some observations about \(HI(C) = (L_0, \ldots, L_n)\). Firstly, if \(L_0 = \emptyset\), then \(n = 0\). Secondly, \(L_0, \ldots, L_n\) are pairwise disjoint and their union is \(L\).

Similar ways of defining hierarchies can also be found in the literature on social choice theory, such as [JZP18].

**Example 3** (Hierarchy of ontic rules in contexts).

- \(HI(\epsilon) = \emptyset\). Note \(\emptyset\) is not the empty sequence but the sequence with the empty set as its only element.
- Let \(C = (L, \succ, U)\) be a context for a model \(M\), where \(L = \{L_1, L_2, L_3\}\), \(L_1 \succ L_2\) and \(L_1 \succ L_3\). Then \(HI(C) = (\{L_1\}, \{L_2, L_3\})\).

**Definition 5** (Accepted and expected timelines by contexts). Let \(M = (W, <, V)\) be a model, \(C = (L, \succ, U)\) be a context for \(M\), and \(HI(C) = (L_0, \ldots, L_n)\) be the hierarchy of ontic rules in \(C\).

Define the set \(A(C)\) of accepted timelines by \(C\) as \(\bigcap U\). Note specially, \(\bigcap \emptyset = TL(M)\).

Define the set \(E(C)\) of expected timelines by \(C\) as follows:

- Suppose \(\bigcap L_0 = \emptyset\). Then \(E(C) := \emptyset\).
- Suppose \(\bigcap L_0 \neq \emptyset\). Then \(E(C) := \bigcap L_0 \cap \cdots \cap \bigcap L_k\), where \((L_0, \ldots, L_k)\) is the longest initial segment of \((L_0, \ldots, L_n)\) such that \(\bigcap L_0 \cap \cdots \cap L_k \neq \emptyset\).

It can be seen \(E(C) \subseteq A(C)\).

**Example 4** (Expected timelines by contexts).

- Let \(C = (L, \succ, U)\) be a context for a model \(M\) such that \(HI(C) = \emptyset\). Then \(E(C) = TL(M)\).
- Let \(C = (L, \succ, U)\) be a context for a model \(M\) such that \(HI(C) = (\{\emptyset\}, \{\{\pi_1, \pi_2\}\})\). Then \(E(C) = \bigcap \{\emptyset\} = \emptyset\).
The reason for requiring $\pi$ to be in $A(C)$ is that the timelines outside $A(C)$ are unaccepted for the agent and there is no point for her to consider them.

**Definition 6** (Contextualized models and contextualized pointed models). For every model $M$, context $C$, timeline $\pi$ in $A(C)$, and natural number $i$, $(M, C, \pi, i)$ is called a contextualized model and $(M, C, \pi, i)$ is called a contextualized pointed model.

The reason for requiring $\pi$ to be in $A(C)$ is that the timelines outside $A(C)$ are unaccepted for the agent and there is no point for her to consider them.

**Example 5** (Contextualized pointed models). Figure 2 indicates a contextualized pointed model $(M, C, \pi_3, 1)$, where $C = ([, >, U)$ is a context such that $L = \{L_1, L_2, L_3\}$, $L_1 > L_2 > L_3$, and $U = \{L_1\}$, where $L_1 = \{\pi_1, \pi_2, \pi_3\}$, $L_2 = \{\pi_1, \pi_2\}$, and $L_3 = \{\pi_4\}$. It can be verified that $HI(C) = \{\{L_1\}, \{L_2\}, \{L_3\}\}$, $A(C) = \{\pi_1, \pi_2, \pi_3\}$ and $E(C) = \{\pi_1, \pi_2\}$. This means that $\pi_4$ is unacceptable, and $\pi_3$ and $\pi_4$ are unexpected.

![Figure 2: A contextualized pointed model](image)

### 4.4 Semantics

**Definition 7** (Semantics for $\Phi_{SWONBT}$). $M, C, \pi, i \vdash \phi$, formulas $\phi$ in $\Phi_{SWONBT}$ being true at contextualized pointed models $(M, C, \pi, i)$, is defined as follows:

- $M, C, \pi, i \vdash p$ $\iff$ $\pi[i] \in V(p)$
- $M, C, \pi, i \not\vdash \bot$
- $M, C, \pi, i \vdash \neg \phi$ $\iff$ $M, C, \pi, i \not\vdash \phi$
- $M, C, \pi, i \vdash \phi \land \psi$ $\iff$ $M, C, \pi, i \vdash \phi$ and $M, C, \pi, i \vdash \psi$
- $M, C, \pi, i \vdash X\phi$ $\iff$ $M, C, \pi, i + 1 \vdash \phi$
- $M, C, \pi, i \vdash Y\phi$ $\iff$ $M, C, \pi, i - 1 \vdash \phi$, if $i > 0$
- $M, C, \pi, i \vdash [S]\phi$ $\iff$ $M, C, \pi', i \vdash \phi$ for every $\pi' \in A(C)$
- $M, C, \pi, i \vdash [W]\phi$ $\iff$ $M, C, \pi', i \vdash \phi$ for every $\pi' \in E(C)$
It can be verified that
\[ M, C, \pi, i \Vdash \langle S \rangle \phi \iff M, C, \pi', u \Vdash \phi \text{ for some } \pi' \in A(C) \]
\[ M, C, \pi, i \Vdash \langle W \rangle \phi \iff M, C, \pi', u \Vdash \phi \text{ for some } \pi' \in E(C) \]

We say that a formula \( \phi \) is valid \((\models_{SWONBT} \phi)\) if \( M, C, \pi, i \Vdash \phi \) for all contextualized pointed models \((M, C, \pi, i)\).

**Example 6.** We show how Example 7 is analyzed in the formalization. We use \( f_x \) to indicate \( x \) is free and use \( a_x \) to indicate \( x \) is alive, where \( x \) can be \( t \) (the tiger), \( d \) (the dog) or \( g \) (the goat).

*Figure 3* denotes a model \( M \). Let \( C = (L_0, \succ, U) \) be context, where

- \( L_0 \) has three ontic rules: \( L_1 = \{\pi_1, \pi_3, \pi_4, \pi_5, \pi_6\} \), \( L_2 = \{\pi_1, \pi_2, \pi_3, \pi_5, \pi_6\} \) and \( L_3 = \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5\} \);
- \( \succ = \emptyset \);
- \( U = \emptyset \).

The ontic rule \( L_1 \) says that if the tiger and the dog are free, the tiger is alive. The ontic rules \( L_2 \) and \( L_3 \) are read similarly.

The following can be verified: \( H_I(C) = (\{L_1, L_2, L_3\}) \), \( A(C) = TL(M) \), and \( E(C) = \{\pi_1, \pi_3, \pi_5\} \).

The formulas \( [S]Xa_t \) and \( [S]A_t \) respectively mean the tiger must be alive tomorrow and the tiger must be alive right now. It can be verified that \( [S]Xa_t \) is false at \((M, C, \pi_2, 0)\) and \( [S]A_t \) is false at \((M, C, \pi_2, 1)\).

The formulas \([W]Xa_t \) and \([W]A_t \) respectively mean the tiger should be alive tomorrow and the tiger should be alive right now. It can be verified that \([W]Xa_t \) is true at \((M, C, \pi_2, 0)\) and \([W]A_t \) is true at \((M, C, \pi_2, 1)\).

The formulas \( [S]Xa_d \) and \( [S]A_d \) respectively mean the dog must be alive tomorrow and the dog must be alive right now. It can be verified that \( [S]Xa_d \) is false at \((M, C, \pi_2, 0)\) and \( [S]A_d \) is false at \((M, C, \pi_2, 1)\).

*Figure 3: A model*
4.5 Remarks

The function of contexts in the semantics is to determine a set of accepted timelines and a set of expected timelines. The operators \([S]\) and \([W]\) are universal operators over the two sets, respectively. However, how the two sets are determined is not reflected in the truth conditions of \([S]\phi\) and \([W]\phi\). This makes it possible that we can directly give the two sets in the semantics without changing the set of valid formulas.

**Definition 8 (AE pointed models).** A tuple \((M, AT, ET, \pi, i)\) is a AE pointed model if \(M\) is a model, \(AT\) is a nonempty set of timelines of \(M\), \(ET\) is a (possibly empty) subset of \(AT\), \(\pi\) is a timeline in \(AT\), and \(i\) is a natural number.

Define truth conditions of formulas in \(\Phi_{SWONBT}\) at AE pointed models in an imagined way.

**Theorem 1.** For every \(\phi \in \Phi_{SWONBT}\), \(\phi\) is true at all AE pointed models if and only if \(\phi\) is true at all contextualized pointed models.

We skip the proof for this result.

However, this does not mean that the formalization given above is not meaningful. It offers ways to determine accepted timelines and expected timelines. These ways will make a difference when how accepted/expected timelines are determined matters. Based on this work, we plan to deal with conditional weak ontic necessity, where the antecedent changes expected timelines and how it changes expected timelines is dependent on how expected timelines are determined.

In our theory, contexts are not dependent on states. As a result, the truth values of \([S]\phi\) and \([W]\phi\) at a contextualized pointed model \((M, C, \pi, i)\) are independent of \(\pi[i]\). This means that our theory is not local. This is coincident with an essential feature of the strong and weak ontic necessities: Whether they hold at a state is determined by the system of ontic rules and has nothing to do with whether their prejacents are true at this state.

4.6 Expressivity

The language \(\Phi_{SWONBT}\) is as expressive as a fragment of it where \([S]\) and \([W]\) are not nested in any way.

**Lemma 1.** The following two groups of formulas are valid:

1. (a) \(X\neg \phi \leftrightarrow \neg X\phi\)
   (b) \(X(\phi \land \psi) \leftrightarrow (X\phi \land X\psi)\)
   (c) \(XY\phi \leftrightarrow \phi\)
   (d) \(X[S]\phi \leftrightarrow [S]X\phi\)
   (e) \(X[W]\phi \leftrightarrow [W]X\phi\)

2. (a) \(Y\neg \phi \leftrightarrow (Y \bot \lor \neg Y\phi)\)
   (b) \(Y(\phi \land \psi) \leftrightarrow (Y\phi \land Y\psi)\)
   (c) \(YX\phi \leftrightarrow (Y \bot \lor \phi)\)
   (d) \(Y[S]\phi \leftrightarrow [S]Y\phi\)
   (e) \(Y[W]\phi \leftrightarrow [W]Y\phi\)
This result can be easily shown.

For any natural number $n$ (possibly 0), we use $X^n$ and $Y^n$ to respectively denote the sequences of $n$ Xs and $n$ Ys.

**Definition 9 (The language $\Phi_{SWXXYY}$).** Define a language $\Phi_{SWXXYY}$ as follows, where $n$ ranges over the set of natural numbers:

$$\phi ::= X^n p \mid X^n \bot \mid Y^n p \mid Y^n \bot \mid \neg \phi \mid (\phi \land \phi) \mid [S] \phi \mid [W] \phi$$

The following result can be easily shown by Lemma 1.

**Theorem 2.** There is an effective function $\delta$ from $\Phi_{SWONBT}$ to $\Phi_{SWXXYY}$ such that for every $\phi \in \Phi_{SWONBT}$, $\vdash_{SWONBT} \phi \iff \delta(\phi)$.

**Definition 10 (Closed formulas).** Closed formulas of $\Phi_{SWONBT}$ are defined as follows, where $\phi$ is in $\Phi_{SWONBT}$:

$$\chi ::= [S] \phi \mid [W] \phi \mid \neg \chi \mid (\chi \land \chi)$$

By the following result, which is easy to verify, the truth value of a closed formula at a contextualized pointed model $(M, C, \pi, i)$ is independent of $\pi$.

**Theorem 3.** Fix a closed formula $\chi$. Let $M$ be a model, $C$ be a context for $M$, and $i$ be a natural number. Then $M, C, \pi, i \models \chi$ if and only if $M, C, \pi', i \models \chi$ for all $\pi$ and $\pi'$ in $A(C)$.

**Lemma 2.** The following formulas are valid:

1. $[S](\phi \land \psi) \leftrightarrow ([S] \phi \land [S] \psi)$
2. $[S](\chi \lor \phi) \leftrightarrow (\chi \lor [S] \phi)$, where $\chi$ is a closed formula
3. $[W](\phi \land \psi) \leftrightarrow ([W] \phi \land [W] \psi)$
4. $[W](\chi \lor \phi) \leftrightarrow (\chi \lor [W] \phi)$, where $\chi$ is a closed formula

This result is easy to show.

**Definition 11 (The language $\Phi_{XXYY}$).** Define a language $\Phi_{XXYY}$ as follows:

$$\alpha ::= X^n p \mid X^n \bot \mid Y^n p \mid Y^n \bot \mid \neg \alpha \mid (\alpha \land \alpha)$$

**Definition 12 (The language $\Phi_{SW-1}$).** Define a language $\Phi_{SW-1}$ as follows, where $\alpha$ is in $\Phi_{XXYY}$:

$$\phi ::= \alpha \mid \neg \phi \mid (\phi \land \phi) \mid [S] \alpha \mid [W] \alpha$$

Note $[S]$ and $[W]$ are not allowed to be nested in $\Phi_{SW-1}$.

**Theorem 4.** There is an effective function $\gamma$ from $\Phi_{SWXXYY}$ to $\Phi_{SW-1}$ such that for every $\phi \in \Phi_{SWXXYY}$, $\vdash_{SWONBT} \phi \iff \gamma(\phi)$.

The proof for this result is put to Section A in the appendix.

By Theorem 2 and this theorem, $\Phi_{SWONBT}$ is as expressive as $\Phi_{SW-1}$.
4.7 Axiomatization

We use $\Phi_{PL}$ to indicate the language of the Propositional Logic.

**Definition 13** (The axiomatic system SWONBT). Define an axiomatic system SWONBT as follows:

**Axioms:**

1. **Axioms for the Propositional Logic**
2. **Axioms for $X$:**
   - (a) $X\neg\phi \leftrightarrow \neg X\phi$
   - (b) $X(\phi \wedge \psi) \leftrightarrow (X\phi \wedge X\psi)$
   - (c) $XY\phi \leftrightarrow \phi$
   - (d) $X[S]\phi \leftrightarrow [S]X\phi$
   - (e) $X[W]\phi \leftrightarrow [W]X\phi$
   - (f) $\neg X\neg T$
3. **Axioms for $Y$:**
   - (a) $Y\neg\phi \leftrightarrow (Y\bot \lor \neg Y\phi)$
   - (b) $Y(\phi \wedge \psi) \leftrightarrow (Y\phi \wedge Y\psi)$
   - (c) $YX\phi \leftrightarrow (Y\bot \lor \phi)$
   - (d) $Y[S]\phi \leftrightarrow [S]Y\phi$
   - (e) $Y[W]\phi \leftrightarrow [W]Y\phi$
   - (f) $(S)Y\bot \rightarrow ((S)\alpha \rightarrow \alpha)$, where $\alpha$ is in $\Phi_{PL}$
4. **Axioms for $[S]$:**
   - (a) $[S](\phi \rightarrow \psi) \rightarrow ([S]\phi \rightarrow [S]\psi)$
   - (b) $[S](\chi \lor \phi) \leftrightarrow (\chi \lor [S]\phi)$, where $\chi$ is a closed formula
   - (c) $[S]\phi \rightarrow \phi$
5. **Axioms for $[W]$:**
   - (a) $[W](\phi \rightarrow \psi) \rightarrow ([W]\phi \rightarrow [W]\psi)$
   - (b) $[W](\chi \lor \phi) \leftrightarrow (\chi \lor [W]\phi)$, where $\chi$ is a closed formula
6. **Axioms for that $[S]$ is stronger than $[W]$:** $[S]\phi \rightarrow [W]\phi$

**Inference rules:**

1. Modus ponens: From $\phi$ and $\phi \rightarrow \psi$, we can get $\psi$;
2. Generalization of $X$: From $\phi$, we can get $X\phi$;
3. Generalization of $Y$: From $\phi$, we can get $Y\phi$;
4. Generalization of $[S]$: From $\phi$, we can get $[S]\phi$;

5. Generalization of $[W]$: From $\phi$, we can get $[W]\phi$;

6. Replacement of equivalent subformulas: From $\psi \leftrightarrow \psi'$, we can get $\phi \leftrightarrow \phi'$, where $\psi$ is a subformula of $\phi$ and $\phi'$ is the result of replacing $\psi$ by $\psi'$ in $\phi$.

We use $\vdash_{\text{SWONBT}} \phi$ to indicate $\phi$ is derivable in SWONBT.

**Theorem 5.** The axiomatic system SWONBT is sound and complete with respect to the set of valid formulas of $\Phi_{\text{SWONBT}}$.

The proof for this result is put to Section B in the appendix.

## 5 Zooming out

We look at the logical theory presented above from some general perspectives.

### 5.1 Comparisons to Yalcin’s theory for weak ontic necessity

The theory for weak ontic necessity presented by Yalcin [Yal16] is influenced by Veltman [Vel96] and Yalcin [Yal07]. Its evaluation context is a tuple $(W, S, N, w)$. Here $W$ is the set of all possible worlds, called the logical truth. $S$ is a subset of $W$, called an information state. $S$ represents a collection of facts. $N$ is a finite set of subsets of $W$ meeting the following condition: $W \in N$ and $\bigcap N \neq \emptyset$. Every element of $N$ indicates a normality proposition. The elements of $\bigcap N$ are called max normal worlds. “Should $\phi$” is true at $(W, S, N, w)$ if and only if for all max normal worlds $u$, $\phi$ is true at $(W, S, N, u)$.

Let us put time aside and just consider the weak ontic necessity.

Our theory is similar to Yalcin’s theory in the following aspects. Firstly, a set of expected worlds, which is a domain for the weak ontic necessity in our theory, share similar intuitions with a set of max normal worlds, which is a domain for the weak ontic necessity in Yalcin’s theory. Secondly, ontic rules in our theory and normality propositions in Yalcin’s theory play a similar role in the following sense: Ontic rules are used to determine expected worlds, while normality propositions are used to determine max normal worlds.

The main difference between our theory and Yalcin’s theory is that the domain for the weak ontic necessity is computed in different ways: Our work uses a priority order among ontic rules, while Yalcin’s theory just uses a set of normality propositions.

### 5.2 The strong and weak ontic necessities are S5 and K45 operators respectively

The logic S5 is a normal modal logic with the special axioms $\Box \phi \to \phi$, $\Box \phi \to \Box \Box \phi$ and $\phi \to \Box \Diamond \phi$. Models of S5 are based on relational structures $(W, R)$, where $R$ is reflexive, transitive and symmetric. The logic K45 is a normal modal logic with the special axioms $\Box \phi \to \Box \Box \phi$ and $\Diamond \phi \to \Box \Diamond \phi$. Models of K45 are based on relational structures $(W, R)$, where $R$ is transitive and euclidean.
The following result can be easily shown.

**Lemma 3.**

1. \(\vdash_{\text{SWONBT}} [S]\phi \leftrightarrow [S][S]\phi.\)
2. \(\vdash_{\text{SWONBT}} \phi \leftrightarrow [S][S]\phi.\)
3. \(\vdash_{\text{SWONBT}} [W]\phi \leftrightarrow [W][W]\phi.\)
4. \(\vdash_{\text{SWONBT}} (S)\phi \leftrightarrow [S][S]\phi.\)

From this lemma and the definition of the system SWONBT, we can see that S5 is a sublogic of SWONBT under the correspondence between \(\square\) and \([S]\), and K45 is a sublogic of SWONBT under the correspondence between \(\square\) and \([W]\).

Define uniform substitutions of formulas of S5 in \(\Phi_{\text{SWONBT}}\) under the correspondence between \(\square\) and \([S]\) in the usual way. Define uniform substitutions of formulas of K45 in \(\Phi_{\text{SWONBT}}\) under the correspondence between \(\square\) and \([W]\) in the usual way.

**Theorem 6.** Let \(\phi\) be a formula of S5. Then \(\phi\) is valid with respect to S5 if and only if all of the uniform substitutions of \(\phi\) in \(\Phi_{\text{SWONBT}}\) under the correspondence between \(\square\) and \([S]\) are valid with respect to SWONBT.

**Proof.** Assume \(\phi\) is valid with respect to S5 and \(\phi'\) is a uniform substitution of \(\phi\) in \(\Phi_{\text{SWONBT}}\). By the completeness of S5, there is a derivation \(\phi_1, \ldots, \phi_n\) for \(\phi\) in S5. Let \(\phi'_1, \ldots, \phi'_n\) be the result of performing uniform substitutions to \(\phi_1, \ldots, \phi_n\) in a similar way with how \(\phi'\) is gotten from \(\phi\). Note \(\phi'_n\) equals to \(\phi'\). As S5 is a sublogic of SWONBT, \(\phi'_1, \ldots, \phi'_n\) is a derivation of \(\phi'\) in SWONBT. By the soundness of SWONBT, \(\phi'\) is valid with respect to SWONBT.

Assume \(\phi\) is not valid with respect to S5. Then there is a pointed model for S5 where \(\phi\) is false. Let \(\phi'\) be the result of replacing \(\square\) in \(\phi\) by \([S]\). Then \(\phi'\) is a uniform substitution of \(\phi\) in \(\Phi_{\text{SWONBT}}\). Then we can get a contextualized pointed model for SWONBT where \(\phi'\) is false. We skip the details. Then \(\phi'\) is not valid with respect to SWONBT.

**Theorem 7.** Let \(\phi\) be a formula of K45. Then \(\phi\) is valid with respect to K45 if and only if all of the uniform substitutions of \(\phi\) in \(\Phi_{\text{SWONBT}}\) under the correspondence between \(\square\) and \([W]\) are valid with respect to SWONBT.

**Proof.** Assume \(\phi\) is valid with respect to K45 and \(\phi'\) is a uniform substitution of \(\phi\) in \(\Phi_{\text{SWONBT}}\). By a similar argument as the one given in the proof for Theorem 6 we can show that \(\phi'\) is valid with respect to SWONBT.

Assume \(\phi\) is not valid with respect to K45. Then there is a pointed model for K45 where \(\phi\) is false. Let \(\phi'\) be the result of replacing \(\square\) in \(\phi\) by \([W]\). Then \(\phi'\) is a uniform substitution of \(\phi\) in \(\Phi_{\text{SWONBT}}\). Then we can get a contextualized pointed model for SWONBT where \(\phi'\) is false. We skip the details. Then \(\phi'\) is not valid with respect to SWONBT.

### 5.3 Comparisons to PCTL* 

Full Computation Tree Logic (CTL*), introduced in [EH86], is a branching-time temporal logic studied well in computer science. PCTL* is an extension of CTL* with two past operators [Rey05]. The featured formulas of PCTL* are as follows:
• $X\phi$: $\phi$ will be true at the next moment.

• $\phi U \psi$: $\phi$ will be true until $\psi$ is true.

• $Y\phi$: $\phi$ was true at the last moment.

• $\phi S \psi$: $\phi$ has been true since $\psi$ was true.

• $A\phi$: $\phi$ is necessarily true at the present moment.

It is natural to compare PCTL* and SWONBT from the perspective of how the operators $X$, $Y$, $A$ and $[S]$ are dealt with.

The pointed models for PCTL* are tuples $(M, \pi, i)$ given in Section 4.3. The truth conditions of $X\phi$, $Y\phi$ and $A\phi$ are as follows:

\[
M, \pi, i \models X\phi \iff M, \pi, i + 1 \models \phi
\]

\[
M, \pi, i \models Y\phi \iff M, \pi, i - 1 \models \phi, \text{ if } i > 0
\]

\[
M, \pi, i \models A\phi \iff M, \pi', i \models \phi \text{ for every } \pi' \text{ such that } \pi'[i] = \pi[i]
\]

The next moment operator and the last moment operator are handled in the same way in PCTL* and SWONBT. However, the operator $A$ in PCTL* is different from the operator $[S]$ in SWONBT. The main difference is that $A$ is an epistemic necessity while $[S]$ is an ontic one. This can be seen from the fact that $p \rightarrow A p$ is valid in PCTL* but $p \rightarrow [S]p$ is not valid in SWONBT.

6 Looking backward and forward

In this work, we first provide general ideas to make distinctions between strong/weak ontic/epistemic necessities. Then we present a logical theory for strong and weak ontic necessities in branching time, adopting the following approach. The agent has ordered ontic rules concerning how the world evolves in time flow, which determine expected timelines. The agent takes some ontic rules as undefeatable, which determine accepted timelines. The strong ontic necessity quantifies over the domain of accepted timelines and the weak ontic necessity quantifies over the domain of expected timelines.

There is some interesting work worth doing in the future.

Dealing with strong and weak epistemic necessities In this work, based on our general ideas about strong/weak ontic/epistemic necessities, we present a logical theory for strong/weak ontic necessities. It is natural to present a logical theory for strong and weak epistemic necessities next.

Handling conditional weak ontic necessity Based on this work, we plan to propose a formal theory for conditional weak ontic necessity in the future. Here are two examples of conditional weak ontic necessity.

(22) If Obama wins and his policies are as dreadful as expected, things should be much worse in two or three years and Dems will get the blame.

(23) If Adam had worked in his office yesterday, he should be dead by now.
Adding two temporal operators to SWONBT

The logic SWONBT is not very expressive: it just contains two simple temporal operators: the next moment operator and the last moment operator. It is natural to introduce the operators until and since to it.

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A Proofs for expressivity of SWONBT

Theorem 4. There is an effective function $\gamma$ from $\Phi_{SWXXYY}$ to $\Phi_{SW-1}$ such that for every $\phi \in \Phi_{SWXXYY}$, $\models_{SWONBT} \phi \leftrightarrow \gamma(\phi)$.

Proof. Respectively define the depth of formulas of $\Phi_{SWXXYY}$ with respect to $[S]$ and $[W]$ in the usual way. Pick a formula $\phi$ in $\Phi_{SWXXYY}$. Define $\gamma(\phi)$ as follows.

- Fix a sub-formula $[S]\psi$ of $\phi$ whose depth with respect to $[S]$ is two if $\phi$ has such a sub-formula. Then the depth of $\psi$ with respect to $[S]$ is one.
- There is a formula $\psi'$ in $\Phi_{SWXXYY}$ such that $\models_{SWONBT} \psi \leftrightarrow \psi'$ and $\psi'$ is in the form of $\chi_1 \land \cdots \land \chi_n$, where every $\chi_x$ is in the form of $[S]H_1 \lor \cdots \lor [S]H_n \lor (S)I_1 \lor \cdots \lor (S)I_n \lor [W]J_1 \lor \cdots \lor [W]J_n \lor (W)K_1 \lor \cdots \lor (W)K_n \lor L_1 \lor \cdots \lor L_n$, where all $H_x, I_x, J_x, K_x$ and $L_x$ are in $\Phi_{XXYY}$. Then $\models_{SWONBT} [S]\psi \leftrightarrow [S]\psi'$.
- By Lemma 2 $\models_{SWONBT} [S]\psi' \leftrightarrow ([S]\chi_1 \land \cdots \land [S]\chi_n)$.
- Let $[S]_x = [S]([S]H_1 \lor \cdots \lor [S]H_n \lor (S)I_1 \lor \cdots \lor (S)I_n \lor [W]J_1 \lor \cdots \lor [W]J_n \lor (W)K_1 \lor \cdots \lor (W)K_n \lor L_1 \lor \cdots \lor L_n)$). By Lemma 2 $\models_{SWONBT} [S]_x \leftrightarrow \xi_x$, where $\xi_x = [S]H_1 \lor \cdots \lor [S]H_n \lor (S)I_1 \lor \cdots \lor (S)I_n \lor [W]J_1 \lor \cdots \lor [W]J_n \lor (W)K_1 \lor \cdots \lor (W)K_n \lor [S]L_1 \lor \cdots \lor [S]L_n$).
- Then $\models_{SWONBT} ([S]\chi_1 \land \cdots \land [S]\chi_n) \leftrightarrow (\xi_1 \land \cdots \land \xi_n)$. We replace $[S]\psi$ in $\phi$ by $\xi_1 \land \cdots \land \xi_n$.
- By repeating the previous steps, we can get a formula $\phi_1$ in $\Phi_{SWXXYY}$ such that $\models_{SWONBT} \phi \leftrightarrow \phi_1$ and $\phi_1$ has the depth at most one with respect to $[S]$.
Lemma 6.

In a similar way, we can get a formula \( \phi_2 \) in \( \Phi_{\text{SWXXX}} \) such that \( \models_{\text{SWONBT}} \phi_1 \leftrightarrow \phi_2 \) and \( \phi_2 \) has the depth at most one with respect to \( [w] \). Note \( \phi_2 \) is in \( \Phi_{\text{SW}-1} \).

Let \( \gamma(\phi) = \phi_2 \).

We can see that \( \models_{\text{SWONBT}} \phi \leftrightarrow \gamma(\phi) \) and \( \gamma \) is effective. \( \square \)

B Proofs for soundness and completeness of SWONBT

The following result can be easily shown:

Lemma 4.

1. \( \models_{\text{SWONBT}} [S](\phi \land \psi) \leftrightarrow ([S]\phi \land [S]\psi) \).
2. \( \models_{\text{SWONBT}} [w](\phi \land \psi) \leftrightarrow ([w]\phi \land [w]\psi) \).

Let \( \delta \) be the function from \( \Phi_{\text{SWONBT}} \) to \( \Phi_{\text{SWXXX}} \) mentioned in Theorem 2. The following result can be easily shown:

Lemma 5. For every \( \phi \in \Phi_{\text{SWONBT}} \), \( \models_{\text{SWONBT}} \phi \leftrightarrow \delta(\phi) \).

Let \( \gamma \) be the function from \( \Phi_{\text{SWXXX}} \) to \( \Phi_{\text{SW}-1} \) mentioned in Theorem 3. In a similar way as the proof for Theorem 4, the following result can be shown:

Lemma 6. For every \( \phi \in \Phi_{\text{SWXXX}} \), \( \models_{\text{SWONBT}} \phi \leftrightarrow \gamma(\phi) \).

We call \( p, \neg p, \bot \) and \( \top \) literals. We introduce some auxiliary notations:

- For every \( \phi \in \Phi_{\text{XXX}} \), we use \( \text{DJ}(\phi) \) to indicate the set of disjuncts of the disjunctive normal form of \( \phi \). Note the elements of \( \text{DJ}(\phi) \) are in the form of \( x^h \land l_1 \land \cdots \land l_m \land \neg x^{h'} \land l'_1 \land \cdots \land l'_n \), where all \( l_x \) and \( l'_x \) are literals.

- A formula \( \text{FCF} \) is called a full core formula if it is in the form of \( [S]H \land [S]\l_1 \land \cdots \land [S]\l_i \land [W]J \land [W]K_1 \land \cdots \land [W]K_k \land L \), where \( H, l_1, \ldots, l_i, J, K_1, \ldots, K_k, \) and \( L \) are all in \( \Phi_{\text{XXX}} \).

- Fix a full core formula \( \text{FCF} = [S]H \land [S]\l_1 \land \cdots \land [S]\l_i \land [W]J \land [W]K_1 \land \cdots \land [W]K_k \land L \). The following sequence \( \text{BS} \) is called the basic sequence for \( \text{FCF} \): \( (H \land \l_1, \ldots, H \land l_i, H \land J \land K_1, \ldots, H \land J \land K_k, H \land L) \). The following sequence \( \text{AS} \) is called an atomic sequence for \( \text{FCF} \): \( (H_1 \land l'_1, \ldots, H_i \land l'_i, H_{i+1} \land J_1 \land K_{i+1}, \ldots, H_{i+n} \land J_k \land K_k, H'' \land L') \), where \( H_1, \ldots, H_i, H_{i+1} \), \ldots, \( H_{i+n}, H'' \), \( H'' \), \( L' \) are all in \( \text{DJ}(H) \), \( l'_1 \in \text{DJ}(l_1), \ldots, l'_i \in \text{DJ}(l_i), J'_1, \ldots, J'_k \in \text{DJ}(J), K'_1, \ldots, K'_k \in \text{DJ}(K) \), and \( L' \in \text{DJ}(L) \).

- A formula \( \text{PCF} \) is called a partial core formula if it is in the form of \( [S]H \land [S]\l_1 \land \cdots \land [S]\l_i \land [W]J \land L \), where \( H, l_1, \ldots, l_i, J, \) and \( L \) are all in \( \Phi_{\text{XXX}} \).

- Fix a partial core formula \( \text{PCF} = [S]H \land [S]\l_1 \land \cdots \land [S]\l_i \land [W]J \land L \). The following sequence \( \text{BS} \) is called the basic sequence for \( \text{PCF} \): \( (H \land \l_1, \ldots, H \land l_i, H \land J \land L) \). The following sequence \( \text{AS} \) is called an atomic sequence for \( \text{PCF} \): \( (H'_1 \land l'_1, \ldots, H'_i \land l'_i, H'' \land L') \), where \( H'_1, \ldots, H'_i, H'' \) are all in \( \text{DJ}(H) \), \( l'_1 \in \text{DJ}(l_1), \ldots, l'_i \in \text{DJ}(l_i) \), and \( L' \in \text{DJ}(L) \).
• Let $S = (\phi_1, \ldots, \phi_n)$ be a finite nonempty sequence of formulas. We use $\bigwedge(S)S$ to indicate the formula $\langle S \rangle \phi_1 \land \cdots \land \langle S \rangle \phi_n$.

• Let $\Theta$ be a finite set of finite nonempty sequences of formulas. We use $\bigvee \bigwedge(S) \Theta$ to indicate the formula $\bigvee \{ \bigwedge(S)S \mid S \in \Theta \}$.

Lemma 7.

1. Let $FCF = [S]H \land \langle S \rangle I_1 \land \cdots \land \langle S \rangle I_n \land [\pi]J \land \langle \pi \rangle K_1 \land \cdots \land \langle \pi \rangle K_m \land L$ be a full core formula. Then if $FCF$ is consistent, there is an atomic sequence $AS = (H'_1 \land I'_1, \ldots, H'_n \land I'_n, H'' \land J' \land K'_1, \ldots, H'' \land J' \land K'_m, H'' \land L')$ for $FCF$ such that (1) all the elements of it are consistent and (2) if some element of it implies $\gamma \underline{\psi}$, then $l_1 \land \cdots \land l_n$ is consistent, where $l_1, \ldots, l_n$ are all the literals that are a conjunct of some element of this sequence.

2. Let $PCF = [S]H \land \langle S \rangle I_1 \land \cdots \land \langle S \rangle I_n \land [\pi]J \land L$ be a partial core formula. Then if $PCF$ is consistent, there is an atomic sequence $AS = (H'_1 \land I'_1, \ldots, H'_n \land I'_n, H'' \land J' \land K'_1, \ldots, H'' \land J' \land K'_m, H'' \land L')$ for $PCF$ such that (1) all the elements of it are consistent and (2) if some element of it implies $\gamma \underline{\psi}$, then $l_1 \land \cdots \land l_n$ is consistent, where $l_1, \ldots, l_n$ are all the literals that are a conjunct of some element of this sequence.

Proof. We just show the first item. The proof for the second one is similar. Assume $FCF$ is consistent. Let $BS = (H \land I_1, \ldots, H \land I_n, H \land J \land K_1, \ldots, H \land I_n \land K_m, H \land L)$ be its basic sequence. We can easily show $\vdash_{SWONBT} FCF \rightarrow \bigwedge(S)BS$. Then $\bigwedge(S)BS$ is consistent. Note $\vdash_{SWONBT} \bigwedge(S)BS \leftrightarrow \bigvee \bigwedge(S)AS \mid AS$ is an atomic sequence for $FCF$. Then there is an atomic sequence $AS = (H'_1 \land I'_1, \ldots, H'_n \land I'_n, H'' \land J' \land K'_1, \ldots, H'' \land J' \land K'_m, H'' \land L')$ for $FCF$ such that $\bigwedge(S)AS$ is consistent. Then all the elements of $AS$ are consistent. Assume some element of $AS$ implies $\gamma \underline{\psi}$. Let $l_1, \ldots, l_n$ be all the literals that are a conjunct of some element of $AS$. By Axiom 1, we can get $\vdash_{SWONBT} \bigwedge(S)AS \rightarrow (l_1 \land \cdots \land l_n)$. Then $l_1 \land \cdots \land l_n$ is consistent.

We say that a contextualized pointed model $(M, C, \pi, i)$ is linear if the domain of $M$ just consists of the elements of $\pi$.

Lemma 8. Let $\phi = X^{a_1}l_1 \land \cdots \land X^{a_i}l_n \land Y^{b_1}l'_1 \land \cdots \land Y^{b_i}l'_m$, where all $l_z$ and $l'_z$ are literals. Then if $\phi$ is consistent, it is satisfiable at a linear contextualized pointed model.

Proof. Assume $\phi$ is consistent. It is easy to see that there is a formula $\phi'$ such that $\vdash_{SWONBT} \phi \leftrightarrow \phi'$ and $\phi' = X^{a_1} \phi_1 \land \cdots \land X^{a_i} \phi_i \land Y^{b_1} \psi_1 \land \cdots \land Y^{b_i} \psi_f$, where $0 \leq a_1 < \cdots < a_n$, $0 < c_1 < \cdots < c_d$, and $\phi_1, \ldots, \phi_i, \psi_1, \ldots, \psi_f$ are conjunctions of literals. By Axioms 1, $\phi_1, \ldots, \phi_i$ are all consistent. Then we can easily define a linear contextualized pointed model $(M, C, \pi, 0)$ such that $M, C, \pi, 0 \models X^{a_1} \phi_1 \land \cdots \land X^{a_i} \phi_i \land Y^{b_1} \psi_1 \land \cdots \land Y^{b_i} \psi_f$. We skip the details. Then $M, C, \pi, 0 \models \phi$.

Lemma 9.

1. Let $FCF = [S]H \land \langle S \rangle I_1 \land \cdots \land \langle S \rangle I_n \land [\pi]J \land \langle \pi \rangle K_1 \land \cdots \land \langle \pi \rangle K_m \land L$ be a full core formula. Then $FCF$ is satisfiable if there is an atomic sequence $AS = (H'_1 \land I'_1, \ldots, H'_n \land I'_n, H'' \land J' \land K'_1, \ldots, H'' \land J' \land K'_m, H'' \land L')$ for $FCF$ such that (1) all the elements of it are satisfiable and (2) if some element of it implies $\gamma \underline{\psi}$, then $l_1 \land \cdots \land l_n$ is satisfiable, where $l_1, \ldots, l_n$ are all the literals that are a conjunct of some element of this sequence.

$^3$This means $\vdash_{SWONBT} \psi \rightarrow \gamma \underline{\psi}$ for some element $\psi$ of $AS$.

$^4$This means $\models_{SWONBT} \psi \rightarrow \gamma \underline{\psi}$ for some element $\psi$ of $AS$. 

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2. Let $\textbf{PCF} = [S]H \land (S)l_1 \land \cdots \land (S)l_n \land [W]J \land L$ be a partial core formula. Then $\textbf{PCF}$ is satisfiable if there is an atomic sequence $\textbf{AS} = (H'_1 \land l'_1, \ldots, H'_n \land l'_n, H'' \land L')$ for $\textbf{PCF}$ such that (1) all the elements of it are satisfiable and (2) if some element of it implies $Y_\bot$, then $l_1 \land \cdots \land l_n$ is satisfiable, where $l_1, \ldots, l_n$ are all the literals that are a conjunct of some element of this sequence.

Proof.

1. Assume there is an atomic sequence $\textbf{AS} = (H'_1 \land l'_1, \ldots, H'_n \land l'_n, H'' \land L')$ for $\textbf{PCF}$ such that (1) all the elements of it are satisfiable and (2) some element of it implies $Y_\bot$, and $l_1 \land \cdots \land l_n$ is satisfiable, where $l_1, \ldots, l_n$ are all the literals that are a conjunct of some element of this sequence.

2. Note we use $\epsilon$ to denote the special context $(\emptyset, \emptyset, \emptyset)$. By Lemma 8 there are pairwise disjoint linear contextualized pointed models $(M_1, \epsilon, \pi_1, 0), \ldots, (M_i, \epsilon, \pi_i, 0), (M'_1, \epsilon, \pi'_1, 0), \ldots, (M'_k, \epsilon, \pi'_k, 0), (H'', \epsilon, \pi'', 0)$, and $(H''', \epsilon, \pi''', 0)$, which respectively satisfy the elements of $\textbf{AS}$ and $l_1 \land \cdots \land l_n$.

3. Let $w$ be the root of $H''$. Let $M$ be the model constructed in the following way:

   - Replace the roots of $M_1, \ldots, M_i, M'_1, \ldots, M'_k, H''$ by $w$. Let $M_1, \ldots, M_i, M'_1, \ldots, M'_k, H''$ be the resulted models and $\lambda_1, \ldots, \lambda_i, \lambda'_1, \ldots, \lambda'_k, \lambda''$ be the timelines of these models.

   - Merge the resulted models in an imagined way.

   Let $C$ be the context $(\{\{\lambda'_1, \ldots, \lambda'_k\}\}, \emptyset, \emptyset)$.

4. It can be verified that the elements of $\textbf{AS}$ are respectively true at the following contextualized pointed models: $(M, C, \lambda_1, 0), \ldots, (M, C, \lambda_i, 0), (M, C, \lambda'_1, 0), \ldots, (M, C, \lambda'_k, 0), (M, C, \lambda'', 0)$.

5. Let $\textbf{BS} = (H \land l_1, \ldots, H \land l_n, H \land J \land K_1, \ldots, H \land J \land K_k, H \land L)$ be the basic sequence for $\textbf{FCF}$. Then all the elements of it are respectively true at these contextualized pointed models. It can be verified $M, C, \lambda'', 0 \vDash \textbf{FCF}$.

6. Assume there is an atomic sequence $\textbf{AS} = (H'_1 \land l'_1, \ldots, H'_n \land l'_n, H'' \land L')$ for $\textbf{FCF}$ such that (1) all the elements of it are satisfiable and (2) no element of it implies $Y_\bot$.

7. By Lemma 8 there are pairwise disjoint linear contextualized pointed models $(M_1, \epsilon, \pi_1, 0), \ldots, (M_i, \epsilon, \pi_i, 0), (M'_1, \epsilon, \pi'_1, 0), \ldots, (M'_k, \epsilon, \pi'_k, 0), (H'', \epsilon, \pi'', 0)$, which respectively satisfy the elements of $\textbf{AS}$.

8. Let $w$ be a new state for these models. Let $M$ be the resulted model by merging $w$ and these models in an imagined way. Let $\lambda_1, \ldots, \lambda_i, \lambda'_1, \ldots, \lambda'_k, \lambda''$ be the results of prefixing $\pi_1, \ldots, \pi_i, \pi'_1, \ldots, \pi'_k, \pi''$ with $w$. Let $C$ be the context $(\{\{\lambda'_1, \ldots, \lambda'_k\}\}, \emptyset, \emptyset)$.

9. It can be verified that the elements of $\textbf{AS}$ are respectively true at the contextualized pointed models $(M, C, \lambda_1, 0), \ldots, (M, C, \lambda_i, 0), (M, C, \lambda'_1, 0), \ldots, (M, C, \lambda'_k, 0), (M, C, \lambda'', 0)$.

10. Let $\textbf{BS} = (H \land l_1, \ldots, H \land l_n, H \land J \land K_1, \ldots, H \land J \land K_k, H \land L)$ be the basic sequence for $\textbf{FCF}$. Then its elements are respectively true at these contextualized pointed models. It can be verified $M, C, \lambda'', 0 \vDash \textbf{FCF}$.
2. The proof for this item is similar. The main difference is that the needed context for satisfying PCF is \((\emptyset, \emptyset, \emptyset)\), according to which no timeline is expected.

**Theorem 5.** The axiomatic system SWONBT is sound and complete with respect to the set of valid formulas of \(\Phi_{\text{SWONBT}}\).

**Proof.** The soundness of SWONBT can be easily verified and we skip it. Let \(\phi\) be a consistent formula in \(\Phi_{\text{SWONBT}}\). By Lemmas 5 and 6, the formula \(\gamma(\delta(\phi))\) in \(\Phi_{\text{SW}^{-1}}\) is consistent. Let \(\phi'\) be a formula in \(\Phi_{\text{SW}^{-1}}\) such that \(\vdash_{\text{SWONBT}} \gamma(\delta(\phi)) \leftrightarrow \phi'\) and \(\phi'\) is in the disjunctive normal form. Then some disjunct \(\psi\) of \(\phi'\) is consistent. Note \(\psi\) is in the form of \([S]H_1 \land \cdots \land [S]H_h \land [S]I_1 \land \cdots \land [S]I_i \land [W]J_1 \land \cdots \land [W]J_j \land [W]K_1 \land \cdots \land [W]K_k \land \land L\), where all \(H_x, I_x, J_x, K_x\) and \(L\) are in \(\Phi_{\text{XXYY}}\). Also note \(\vdash_{\text{SWONBT}} [S]\top, \vdash_{\text{SWONBT}} [S]\top, \vdash_{\text{SWONBT}} [W]\top,\) and \(\vdash_{\text{SWONBT}} \top\). Then we can assume that some \(H_1, I_1, J_1\) and \(L\) exist. Note it is possible that no \(K_x\) exists. Let \(H = H_1 \land \cdots \land H_h\) and \(J = J_1 \land \cdots \land J_j\). Let \(\psi' = [S]H \land [S]I_1 \land \cdots \land [S]I_i \land [W]J_1 \land [W]J_2 \land \cdots \land [W]J_k \land \land L\). By Lemma 4 \(\vdash_{\text{SWONBT}} \psi \leftrightarrow \psi'\). Then \(\psi'\) is consistent. Note \(\psi'\) is a full core formula or a partial core formula. By Lemmas 2, 3 and 9, \(\psi'\) is satisfiable. By soundness of SWONBT, \(\phi\) is satisfiable. \(\square\)