Tunneling Between a Pair of Parallel Hall Droplets

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Abstract

In this paper, we examine interwell tunneling between a pair of fractional quantum Hall liquids in a double quantum well system in a tilted magnetic field. Using a variational Monte Carlo method, we calculate moments of the intra-Landau level tunneling spectrum as a function of in-plane field component \( B_\parallel \) and interwell spacing \( d \). This is done for variety of incompressible states including a pair of \( \nu = 1/3 \) layers ([330]), pair of \( \nu = 1/5 \) layers ([550]), and Halperin’s [331] state. The results suggest a technique to extract interwell correlations from the tunneling spectral data.

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I. INTRODUCTION

Traditionally experimental studies of the quantum Hall effect have been restricted to magnetotransport, surface acoustic wave, and capacitance spectroscopy studies. However, during the last several years new spectroscopic techniques have been developed. These techniques include photoluminescence, inelastic light scattering, and tunneling spectroscopy. The latter has just begun to be used to investigate the tunneling spectra between a pair of weakly coupled two-dimensional electron liquids and between a two-dimensional liquid and a three-dimensional doped substrate.

This paper will be primarily concerned with tunneling between a pair of parallel quantum wells in the high field regime. Experiments of this sort have been performed by Eisenstein, Pfeiffer, and West. For filling fractions ranging from \( \nu = 0.48 \) to \( \nu = 0.83 \) per layer, these authors find the following features in the I-V characteristics. At low voltages, the I-V characteristics exhibit a pseudogap with activated temperature dependence. (The activation temperature is about 5-10K.) Above this pseudo-gap, Eisenstein et al. find a featureless band of intra-Landau level excitations which peaks at a voltage \( 2\Delta_2 \sim 0.45e^2/\epsilon l_m \). At still higher voltages, they find a second gap which separates the intra-Landau excitations from the inter-Landau level and inter-subband excitations. In view of the above experimental results, it is clear that a detailed theoretical understanding of inter-well tunneling in the high field regime would be desirable.

To date, the theory of inter-well tunneling in the high field regime has focussed on dynamical issues like the size and origin of the pseudo-gap. For instance, Johannson and Kinaret found a tunneling pseudogap in the non-linear I-V characteristic using a Wigner crystal model. Above the pseudo-gap, they find a featureless band similar to that found in the experiment. In addition to this work, there are a number of exact diagonalization calculations of the single electron spectral functions which may be used to calculate the tunneling conductance. Because of finite size effects, the diagonalization calculations do not obtain continuous I-V curves. Nevertheless, these calculations and the Wigner crystal...
model all seem to obtain a peak in the tunnel current at a voltage which is consistent with experiment. In addition Yang and MacDonald have calculated the tunneling density of states of a disordered 2D electron gas in a strong magnetic field. They find a suppression of the tunneling conductance at small voltages but no pseudogap. He, Platzman, and Halperin also find pseudo-gap behavior in a pair of $\nu = 1/2$ Halperin Lee Read Fermi liquids. Finally, Efros and Pikus have studied a lattice-gas model of a classical electron liquid using a Monte-Carlo methods.

In contrast to those references which focus on dynamical issues, this paper will focus on features of the equilibrium interwell correlations which may be extracted from the tunneling conductance. We will argue that a significant understanding of static correlations may be achieved by studying the tilt angle dependence of a few spectral moments extracted from experiments using a tilted field geometry. The case for this will be made through an examination of tunneling spectral moments obtained from a variational Monte Carlo (vMC) calculation. The vMC method described here, we feel, gives results which are complementary to those obtained from exact diagonalization and independent boson model calculations. Of course the vMC method does suffer from certain limitations. Most notably vMC requires the use of reasonably manageable wavefunctions. This requirement will restrict our investigations to the simplest incompressible states which can occur in double well systems. One might be concerned that a comparison of these results with the experimental data would be meaningless since the data is restricted to compressible states whose filling fraction ranges from .48 to .83 per layer. We would argue however that this is not a serious problem for two reasons: First, the experimental data of Eisenstein et al is found to be rather insensitive to the filling fraction. Secondly, the results reported below involve the first few spectral moments which we will argue are primarily sensitive to short range interwell correlations and are less sensitive to to slight changes in filling fraction.

The need to work with simple trial wavefunctions gives rise to a second problem which we will now describe: Consider the tunneling between a pair of $\nu = 1/3$ states. A question one might like to ask is what sort of inter-well correlations will be induced and how might
one detect them in a tunneling experiment. The difficulty which is encountered is that the wavefunction which includes the correlations is not simply the product of two $\nu = 1/3$ Laughlin wavefunction, it is some perturbation of the product wavefunction. Unfortunately, identifying the form of the perturbed wavefunction and using it in a calculation would add significantly to the complexity of the discussion. To avoid this problem we proceed as follows: 

(i.) First, we will calculate the relevant spectral moments using unperturbed wavefunctions like a pair of $\nu = 1/3$ and a pair of $\nu = 1/5$. The results of this calculation may then be compared with experiment in order to address issues regarding the quality of data and the quality of the theoretical model.

(ii.) Secondly we address the issue of the identification and characterization of inter-well correlations by studying the Halperin \cite{331} state. This wavefunction displays significant inter-well correlations and the required calculations are straightforward to implement. The results of the calculation on the [331] state suggests a general approach to analysing tunneling data. We believe the approach should be able to identify inter-well correlations independent of whether they are present in the zeroth order wavefunction (as is the case with the [331]) or are present only in the exact wavefunction (as is the case with a pair of $\nu = 1/3$ liquids). Of course the calculation of spectral moments obtained for the [331] state could also be compared to experimental data, should such data ever become available.

Consider, therefore, the device shown in fig. 1 on which a tilted magnetic field $\vec{B} = (B_\parallel, 0, B_\perp)$ is applied. We will model the double well system using the Hamiltonian $H = H_0 + H_t$ where $H_0$ is defined to be

$$H_0 \equiv -\frac{\hbar^2}{2m_{eff}} \sum_\alpha \int dr c_\alpha^\dagger (\hat{\nabla} - i\frac{e}{\hbar c} A_z(\vec{r})) c_\alpha + \frac{1}{2} \sum_{\alpha\beta} \int dr_1 dr_2 V_{\alpha\beta}(r_1 - r_2)c_\alpha^\dagger(r_1)c_\beta^\dagger(r_2)c_\beta(r_2)c_\alpha(r_1)$$

(1)

where $H_t \equiv \int d^2r [t_0 \exp(i\frac{e}{\hbar c} \oint dz A_z(\vec{r})) c_\uparrow^\dagger c_\downarrow + h.c.]$ is the tunneling term. The electron-electron interaction is taken to be $V_{\uparrow\uparrow}(r) = V_{\downarrow\downarrow}(r) = e^2/\epsilon r$ and $V_{\uparrow\downarrow}(r) = e^2/\epsilon (r^2 + d^2)^{1/2}$ where $d$ is the interwell spacing and $\epsilon$ is the dielectric constant.

Our discussions will focus on pairs of $\nu = 1/m$ states (also denoted [mm0]), and the
Halperin [331] state recently observed by Eisenstein et al \(13\) in double well systems and by Suen et al \(14\) in wide single quantum wells. The Halperin \([mmm]\) states \(15-17\) are described by the wavefunction

\[
\Psi_{mmm} \equiv \prod_{i<j}(z_i - z_j)^m(\sigma_i, \sigma_j) \prod_i e^{-|z_i|^2/4l_m^2} \tag{2}
\]

where

\[
m(\sigma_i, \sigma_j) \equiv \begin{cases} 
  m & \sigma_i = \sigma_j \\
  n & \sigma_i \neq \sigma_j 
\end{cases} \tag{3}
\]

and where \(\sigma\) is a pseudospin index which labels the two wells. For simplicity, we will assume that the actual electron spin is frozen out by the magnetic field. The \([mmm]\) wavefunction describes an incompressible state with a total filling factor \(\nu = \nu_\uparrow + \nu_\downarrow = 2\pi l_m^2 n_0 = 2/(m+n)\) where \(n_0\) is the total electron density on the two layers and where \(l_m \equiv (\hbar c/eB_\perp)^{1/2}\).

As a function of the voltage bias \(V\), between the two wells, we wish to calculate the inter-well tunneling current \(I_t\). To do this we first define the tunneling operator

\[
S_+ (\mathbf{q}) = \int d^2r \ c_\uparrow^+ c_\downarrow \ exp \left( -\frac{ie}{\hbar c} dB_{||} y \right) \tag{4}
\]

where

\[
\mathbf{q} = 2\pi dB_{||}/\Phi_0 \hat{y}
\]

Expanded in powers of \(t_0\), the tunneling current is

\[
I_t = e < j_z >_V + 2e |t_o|^2 \ Im[X_{ret} (eV_o)] + ... \tag{5}
\]

where

\[
X_{ret} (\omega) \equiv \int_{-\infty}^{\infty} dt \ e^{i\omega t} \langle [S_+ (q, t) , S_+ (q, 0)] \rangle \tag{6}
\]

In eqn. 5, the first term would describe a Josephson effect, if such exists, the second term describes incoherent tunneling. In the following discussion, we will only consider unpolarized states \(m \neq n\) where the first term in eqn. 5 vanishes. In this case, no Josephson effect effect
will occur. (We refer the reader to appendix B for a more detailed discussion regarding the absence of a Josephson effect.)

Now using variational Monte Carlo, we will calculate various spectral moments of the form

\[ W_k(B_\parallel) = \int d(eV) \ I_t(V, B_\parallel)(eV)^k \]  

The \( k = 0 \) moment is

\[ W_0(B_\parallel) = 2\pi e|t_0|^2 NS_{++}(2\pi B_\parallel d/\phi_0) \]

where the pseudospin correlation function is defined by

\[ S_{\pm\pm}(k) \equiv \frac{1}{N} \langle \tilde{S}_\pm(k) \tilde{S}_\pm(k) \rangle \]  

where \( \tilde{S}_\pm(q) \) is the Fourier transform of the interwell tunneling operator, \( c_\uparrow^\dagger c_\downarrow \), projected onto the lowest Landau level. The \( k = 1 \) moment is

\[ W_1(B_\parallel) = 2\pi e|t_0|^2 Nf_{++}(2\pi B_\parallel d/\phi_0) \]

where the oscillator strength is

\[ f_{++}(q) = \frac{1}{2N} \langle [\tilde{S}_+^\dagger(q), [\tilde{H}, \tilde{S}_+(q)] \rangle \rangle \]

and where \( N \) is the total number of electrons in either well. Results for \( W_1(B_\parallel) \), will be presented in terms of \( \langle eV \rangle \) the mean voltage bias which is defined by \( \langle eV \rangle \equiv W_1(B_\parallel)/W_0(B_\parallel) = f_{++}(q)/s_{++}(q) \). This expression is exact to the extent that the \( |mmn> \) wavefunction is the exact ground state. \( \langle eV \rangle \) may be written in the more explicit form

\[ \langle eV \rangle = \frac{\langle [\tilde{S}_+^\dagger(q), [\tilde{H}, \tilde{S}_+(q)] \rangle \rangle}{\langle \tilde{S}_+^\dagger(q) \tilde{S}_+(q) \rangle} \]

This expression is, of course, the basis of the single mode approximation. We do not, however, refer to the variational Monte Carlo calculation as a single mode approximation since it does not assume a single collective mode.

The oscillator strength \( f_{++}(q) \) may be calculated using the following expression:

\[ f_{++}(k) = \frac{1}{4} \int \frac{d^2k}{(2\pi)^2} \left\{ a_{\alpha\beta}(k, q) S_{\alpha\beta}(k) + b(k - q, q) [S_{++}(k) + S_{--}(k)] \right\} \]

where the various quantities will be defined as follows: \( S_{\alpha\beta}(q) \) is a structure factor matrix
\[ S_{\alpha\beta}(k) = \frac{1}{N} \langle \vec{n}_\alpha (-k) \vec{n}_\beta (k) \rangle \] (12)

and \( b(q,k) \) and \( a_{\alpha\beta}(k,q) \) are

\[ b(k,q) = 2V_{\uparrow\uparrow}(k)e^{(kq + k\vec{q})/2} - V_{\uparrow\downarrow}(k) (e^{\vec{q} \cdot k} + e^{k \cdot \vec{q}}) \] (13)

and

\[ a_{\alpha\beta}(k,q) = V_{\alpha\gamma}(k) m_{\gamma\beta}(k,q) + m_{\alpha\gamma}(k,q) V_{\gamma\beta}(k) \] (14)

where

\[ m(k,q) = - \begin{bmatrix} 1 - \exp(\vec{r}q - \vec{q}k)/2 \cr c.c. \cr 1 \end{bmatrix} \exp(-|q|^2/2). \] (15)

This expression is obtained from the definition [equation (9)] using manipulations of the sort described in reference [18].

The structure factor matrix is been calculated for the [330] and [550] states in ref. [18]. It has also been calculated for the [331] state in ref. [19]. So only the calculation of \( S_{++}(k) = S_{--}(k) \) will need to be done. This is the task of the next section. Readers uninterested in the technical details of this may simply skip that section.

**II. MONTE CARLO CALCULATION OF THE PSEUDO-SPIN CORRELATOR**

This section describes the calculation of \( S_{++}(k) \) using the variational Monte Carlo method. To do this we first introduce the pair spin correlation function:

\[ g_{++}(r) \equiv \Omega^2 \frac{\langle N_1, N_2 | \vec{S}_-(z') \vec{S}_+(z) | N_1 N_2 \rangle}{\langle N_1 N_2 | N_1 N_2 \rangle} \] (16)

where \( \vec{S}_\pm(r) \) denotes the spin of the i-th electron projected to the lowest Landau level, i.e. \( \vec{S}_\pm = \sigma_\pm \vec{\delta}(r - r_i) \) where \( \vec{\delta}(r - r_i) \) is the Fourier transform of \( e^{-ik \cdot \delta_j} e^{-i\vec{k} \cdot z_j} \). The pair spin correlation function is related to \( S_{++}(k) \) as follows:

\[ S_{++}(k) = x_\downarrow e^{-|k|^2/2} + n_0 x_\downarrow x_\uparrow \int d^2 r \ e^{-i \vec{k} \cdot \vec{r}} g_{++}(\vec{r}) \] (17)
In equation (16) \( \Omega \) denotes the total area of the two dimensional electron gas, and \( x_\sigma \) is the fraction of electrons on the \( \sigma \) well.

Using the explicit form of \( \Psi_{mmn} \) and assuming balanced quantum wells, i.e. \( N^\uparrow = N^\downarrow \), one can write the pair spin correlation function in the form

\[
g_{++}(z - z') = \frac{\Omega^2(-1)^{n-m}}{|z' - z|^{2(m-n)}} < \delta_{\uparrow\sigma_1}\delta_{\downarrow\sigma_2}\delta(z' - z_1)\delta(z - z_2)A_{z_1z_2}[z_i] >_N \tag{18}
\]

where the weight factor \( A_{z_1z_2}[z_k] \) is defined by

\[
A_{z_1z_2}[z_k] = \left\lfloor \frac{N^\uparrow + N^\downarrow}{\Pi_{k=N^\uparrow+2}^{k=N^\uparrow+1}} \left( z_1 - z_k \right) \left( z_2 - z_k \right) \right\rfloor^{m-n} \tag{19}
\]

and where \( < (...) >_N \) denotes

\[
\frac{\langle N^\uparrow + 1, N^\downarrow - 1 | (...) | N^\uparrow + 1, N^\downarrow - 1 \rangle}{\langle N^\uparrow + 1, N^\downarrow - 1 | N^\uparrow + 1, N^\downarrow - 1 \rangle} \tag{20}
\]

The origin of the factor \( A_{z_1z_2}[z_k] \) in equation (18) may be understood as follows: First \( S_+(z) | N^\uparrow, N^\downarrow \rangle \neq | N^\uparrow + 1, N^\downarrow - 1 \rangle \) because the electron transfer from bottom to top well leaves \( m - n \) more zeros than the \( n \) required for a \( |mmn> \) state. Moreover, the new electron on the top well is bound to \( n \) zeros rather than the \( m \) zeros characteristic of a \( |mmn> \) state. As a result one must multiply, \( | N^\uparrow + 1, N^\downarrow - 1 > \) by an additional Jastrow factor in order to obtain \( S_+(z)|N^\uparrow, N^\downarrow> \). This extra Jastrow factor leads immediately to factor \( A_{z_1z_2}[z_k] \). The manipulations which lead to the complete expression presented in equation (18) may be found in appendix A.

The expectation on the right hand side of equation (18) may be obtained from a simulation of a pair of mobile impurities in a two component background plasma. To do this we rewrite the equation as

\[
g_{++}(z - z') = \Omega^2(-1)^{n-m} \delta_{\uparrow\sigma_1}\delta_{\downarrow\sigma_2}\delta(z' - z_1)\delta(z - z_2)A_{z_1z_2}[z_i] \frac{\langle z_i >_N}{|z_1 - z_2|^{2(m-n)}} \tag{21}
\]

Next, we split \( A_{z_1z_2}[z_i] \) into its modulus and a part with modulus 1:
Then, using importance sampling, we absorb the second factor in equation (22) as well as
the factor of $1/|z_1 - z_2|^2(m-n)$ into a new analogue plasma Hamiltonian for the Monte Carlo,
leaving only $\tilde{A}_{z_1z_2}[z_k]$ to average over. The classical Hamiltonian of the (modified)
analogue plasma is

$$\beta H_{\text{plasma}} = \sum_{i<j} Q_{ij} \ln r_{ij} + \sum_i r_i^2/2 \quad (23)$$

The interaction strength $Q_{ij}$ between a pair of background ions is $2m(\sigma_i, \sigma_j)$. For an impurity
and a background ion $Q_{1i} = Q_{2i} = m + n$. Between the two impurities $Q_{12} = 2n$.

During the simulation we calculate the pair spin correlation function using

$$g_{++}(r_n) \equiv \frac{1}{\Delta^2 \pi [n^2 - (n - 1)^2]x_i^2 \eta_0 N_s} \cdot B_{++}(r_n) \quad (24)$$

where the new notation is defined as follows:

$$N_s \equiv \sum_{k=1}^{N_{MC}} \sum_{i=1}^{2} 1 \quad (25)$$

where $i = 1, 2$ labels the impurity ions and $k$ labels the sampled configurations. The prime
indicates that only configurations where the impurity ions lie in a circle of radius $R_s$ centered
at the origin are to be included. $B_{++}(r_n)$ is a bin counter which keep track of the contribution
to the sum of $\tilde{A}_{z_1z_2}[z_i(k)]$ associated with impurity pairs with separation $z_1 - z_2$ such that
$n\Delta < |z_1 - z_2| < (n + 1)\Delta$ where $\Delta$ is the bin width. More precisely, $B_{++}(r_n)$ is defined by

$$B_{++}(r_n) \equiv \sum_{k=1}^{N_{MC}} \tilde{A}_{z_1z_2}[z_i(k)] \theta(|z_1 - z_2| - n\Delta) \theta((n + 1)\Delta - |z_1 - z_2|) \quad (26)$$

where $z_i(k)$ denotes the position of the i-th ion in the k-th sampled Monte Carlo cycle.

We used the Metropolis algorithm to calculate $B_{++}(r_n)$. During each cycle, we attempt
as many Monte Carlo moves as there are particles. We began our simulation with an initial
equilibration period of $10^3$ cycles. During this equilibration period, the ion step size was
adjusted until an average acceptance ratio of 0.5 was reached. The step size was kept fixed after the equilibration period. We then ran for another $2 \times 10^6$ cycles, sampling one out of every ten cycles. This gave a total of $N_{MC} = 2 \times 10^5$. Usually in simulations of this sort, one counts only those pairs where one of the impurities lies in a circle of radius $R_*$. In this way one can reduce finite size corrections to the pair spin correlation function. Various choices for $R_*$, system sizes, run lengths were tried. Ultimately, we concluded that $R_* \to \infty$ on a system of 200 particles per well gave an acceptable balance of systematic finite size errors and statistical error. Specifically, we concluded that the finite size error is smaller than the noise due to the Monte Carlo procedure. The statistical error in $g_{++}(r)$ for the run of $2 \times 10^6$ cycles was 0.003 for $0 < r/l_m < 4.5$ except near $r/l_m = 1.7$ where it increases to 0.008. These represent percentage errors of 1-3%.

The results for [330] and [331] are presented in figure 2. First consider [330] data. In the absence of interlayer correlations one can calculate $\bar{g}_{++}(r)$ exactly. To do this, one notes that for any pair of uncorrelated liquid

$$< \bar{S}_-(z)\bar{S}_+(z') > = < [\bar{c}_+^+(z)\bar{c}_+(z)][\bar{c}_-^+(z')\bar{c}_-(z')] >$$

$$= \nu_4(1 - \nu_t)(2\pi)^{-2}\exp-|z-z'|^2/2 \quad (27)$$

This is obtained by Wick factorizing the left hand side into products of the single particle matrix. Such a manipulation is exact in the absence of interwell correlations. One then uses this result together with eqn. [7] to find that

$$\bar{g}_{++}(r) = -e^{-r^2/2} \quad (28)$$

This result is the solid line through the [330] data presented in fig. 2. The agreement between Monte Carlo data and the analytical theory is quite satisfactory.

Next consider the [331] Monte Carlo data. As is the case with [330], the pair spin correlation function for [331] is short ranged. It peaks at $r = 1.7l_m$ and is quite small for $r > 5.0l_m$. This behavior is quite different than what would be obtained from an
[mmm] state. In particular if \( n \to m \), then \( A_{zz'}[z_k] \to 1 \) and \( \lim_{r \to \infty} g_{++}(r) \to x_\uparrow^2 \). This demonstrates that [mmm] exhibits ODLRO. The presence of ODLRO in the [mmm] state has been discussed by Zee and Wen in the context of a possible Josephson effect for this system. Given the absence of the ODLRO in the [331], we conclude that a Wen-Zee type Josephson effect will not occur for this state.

Instead of working directly with the numerical \( g_{++}(r) \), it is convenient to work with an analytic fit. A convenient choice to fit \( g_{++}(r) \) is

\[
g_{++}(r) = a(r^2 + br^4) \exp \left[ -\frac{1}{2} \left( \frac{r}{s} \right)^2 \right].
\]

To within the accuracy of the Monte Carlo data, one can fit \( g_{++}(r) \) from the simulation with 200 particles per well with \( a = 0.130 \), \( b = 0.0 \) and \( s = 1.189 \). The fit for \( g_{++}(r) \) is the solid line passing through the [331] data in fig. 2.

Having obtained the analytic fit for \( g_{++}(r) \), we insert this into equation (17) to obtain the spin correlation function \( S_{++}(k) \) and the integrated spectral weight \( W_0(B_{||}) \). See fig. 3. Of course, for the [mm0] state \( W_0 \) may be obtained analytically using eqn. 28. In fact, one can use eqn. 28 to obtain \( W_0(B_{||}) \) for any pair of parallel liquid states with filling fractions \((\nu_\uparrow, \nu_\downarrow)\) provided that interwell correlations can be ignored. The un-correlated limit of \( W_0 \) will be denoted by \( W_{nc} \) where

\[
\frac{W_{nc}(B_{||})}{\Omega} = \frac{e |t_0|^2}{\hbar l_m^2} \nu_\uparrow (1 - \nu_\downarrow) \exp \left[ -\frac{1}{2} \left( \frac{d}{l_m B_{||}} \right)^2 \right]
\]

\( W_{nc} \) is also plotted in fig. 3. The results for \( W_0(B_{||}) \) will be discussed in detail in the next section.

**III. IN-PLANE FIELD DEPENDENCE OF THE TUNNELING SPECTRUM**

In figure 3, we present the integrated spectral weight \( W_0(B_{||}) \) which was obtained from the Monte Carlo procedure described in the previous section. We observe that interwell tunneling is more suppressed for the [330] than for the [331]. This is due to the strong
interlayer correlations in the [331] state: The correlation hole which occurs in the opposite layer increases the number of unblocked final states. To understand this, consider the following argument: Momentum conservation requires that electrons tunnel along the tilted magnetic field lines. Hence even in the absence of manybody correlations, tunneling will be suppressed because the relevant matrix element involves the overlap of a pair of Gaussian wavefunctions displaced by $d_* \equiv d(B_\perp/B_\parallel)$. See fig. 4. Thus eqn. 30 is telling us that for a pair $\nu = 1/m$ states, the integrated tunneling conductance is determine by this single electron effect. Intrawell correlations are irrelevant to $W_0$ since the only approximation which went into the derivation of $W_{nc}$ was the Wick factorization of the spin correlator in eqn. 30.

Contrast this state of affairs with what happens in the [331] state (or any other [mmn] state with $n \neq 0$). In this case, tunneling will be enhanced when $B_\parallel = 0$, because the electron will tunnel directly into its correlation hole on the opposite quantum well. However, if the field is tilted, the electron will miss the correlation hole. Hence an in-plane magnetic field will reduce $W_0$. See fig. 5. To separate correlation effects from the suppression of single electron tunneling, we define the ‘correlation enhancement ratio’ $R \equiv W_0(B_\parallel)/W_{nc}(B_\parallel)$. In fig. 4, we plot $R$ vs $B_\parallel$. Also plotted in fig. 5 is the radial distribution function $g_{\uparrow\downarrow}(d_*)$ which is defined by

$$x_\alpha x_\beta n_0 g_{\alpha\beta}(\vec{r}) \equiv \frac{1}{N} \left\langle \sum_{\sigma_i=\alpha, \sigma_j=\beta} \delta(\vec{r} + \vec{r}_i - \vec{r}_j) \right\rangle$$

(31)

where $\alpha, \beta = \uparrow, \downarrow$. This result for $g_{\uparrow\downarrow}(d_*)$ was previously obtained in ref. 19. We see from fig. 5 that the correlation enhancement is maximum when $d_*$ lies within the correlation hole and has a minimum when $d_*$ is near $3.2l_m$, the radius of the first coordination shell.

So far the discussion of identifying the correlation hole by using $W_0(B_\parallel)$ has focused on the [331] state. However, the discussion is in fact a bit more general: Indeed, even though the [mm0] wavefunction gives $R = 1$, the exact ground state would deviate from the [mm0] state. In this case$^{24}$, $g_{\uparrow\downarrow}$ will develop a weak correlation hole which deepens as $d/l_m$ decreases. Presumably $R$ would then reflect the existence of the induced correlation hole.
Next we wish to consider $< eV >$. To calculate $< eV >$ we first evaluate $f_{++}(q)$ using eqn. 11 and then we use eqn. 10 to obtain $< eV >$. Numerical quadratures were verified by comparing them with analytical results valid in the $d = 0$ limit. The sensitivity of $< eV >$ to the choice of fit parameters $(a, b, s)$ (see eqn. 29) was also studied. Typically the error in $< eV >$ due to different choices of fit parameters was less than 5% for changes of up to 10% in the fitting parameters. The best fit results are presented in figs. 7 and 8a-c for various well spacings for the [330], [550] and [331] states.

The first thing that one notices in fig. 8 is the strong dependence of $< eV >$ on the layer spacing: As $d \to 0$, $< eV >$ collapses when $B_{\parallel} = 0$. The reason for this is simple. For $d = 0$ and $t_0 = 0$ the Hamiltonian is $SU(2)$ invariant but the [331] wavefunction is not. As a result $S_{++}(q)|331>$ would be a Goldstone mode which implies that $< eV >$ vanishes at $q = 0$. Note however, that numerical diagonalizations\textsuperscript{25,26} indicate that the [331] state becomes unstable for $d/l_m$ less than some critical value around 0.5. The likely explanation is that the band edge of $I(V)$, i.e. $2\Delta_1$ collapses before $< eV >$ does.

Next we observe that the $B_{\parallel}$ dependence of $< eV >$ is more rapid for the [331] state than for the [330] state. This rapid dependence of $< eV >$ on $B_{\parallel}$ coincides with $d_*$ moving out of the correlation hole. Once $d_*$ moves beyond the first coordination shell interwell correlations become irrelevant and the dependence of $< eV >$ on $B_{\parallel}$ will be similar to that obtained for a pair of uncorrelated fluids i.e. like [330]. This then explains the similarity of the $< eV >$ vs. $B_{\parallel}$ curves obtained for [331] and [330] (see fig. 8) when $q l_m > 3$.

To summarize, we have computed tunneling spectral moments $W_0$ and $W_1$ of the [330], [550], and [331] states as a function of the in-plane magnetic field in a tilted field geometry. We argue that the ratio $R = W_0(B_{\parallel})/W_{nc}(B_{\parallel})$ provides a qualitative method for imaging the correlation hole in $g_{\uparrow\downarrow}(d_*)$. We have also studied the behavior of $< eV > = W_1/W_0$. The results presented in fig. 8 show that the mean of the intra-Landau level spectrum will rapidly increase as $d_* = dB_{\parallel}/B_{\perp}$ moves through the first configuration shell. Finally, arguments presented in appendix B demonstrate that no Josephson effect of the sort proposed by Wen and Zee can occur if $m \neq n$. 
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APPENDIX: A. DERIVATION OF PAIR SPIN CORRELATION FUNCTION

In this appendix, we will derive the explicit form of the pair spin correlation function, i.e. equation (18) from its definition given in eqn. (16). To recast eqn. (16) into the desired form, we will need the identity

\[ S_+ (z) | N_\uparrow, N_\downarrow > = (-1)^{N_\downarrow} \Pi' \left[ \prod_j (z - z_j)^{\Delta(\sigma_j)} \right] \rho_\uparrow (z') | N_\uparrow + 1, N_\downarrow - 1 > \]  

where \( \Delta_j = m(\downarrow, \sigma_j) - m(\uparrow, \sigma_j) \) and where \( \Pi'_j \) means to omit the \( j = i \) factor. The systematic derivation which gives the above result is straightforward but will not be given, since a few minutes reflection should convince the reader that the above expression is correct. In particular, the Jastrow factor on the left side of equation (A1) has already been discussed, so one only needs to consider the factor \( \rho_\uparrow (z') \). This factor ensures that the state on the right hand side has the \( i \)-th particle located in the top well at position \( z \) which, of course, is the case for \( S_+ (z) | N_\uparrow, N_\downarrow > \).

Using this identity one then obtains

\[ < N_1, N_2 | \overline{S}_+ (z') \overline{S}_+ (z) | N_1 N_2 > = \]

\[ < N_\uparrow + 1, N_\downarrow - 1 | \Pi'_k (z_i - z_k)^{\Delta_i} \Pi_\uparrow (z') \Pi_\downarrow (z_i - z_i)^{\Delta_i} | N_\uparrow + 1, N_\downarrow - 1 > \]  

(A2)

In the above expression, one may replace \( \Pi_n^\sigma \) with \( \rho_n^\sigma \) for \( n = 1, 2 \). Then, after some straightforward manipulations of equation (A2), one may write \( g_{++}(r) \) in the form
\[
\Omega^{-2} g_{++} (z' - z) = \frac{(-1)^{n-m} Q[N']}{|z' - z|^{2(m-n)}} < \sigma_1^\uparrow \delta \sigma_2^\uparrow \delta (z' - z_1) \delta (z - z_2) A_{z'z_k} >_N \]  \hspace{1cm} (A3)

where \(Q[N_\sigma] \equiv< N_\sigma | N_\sigma >\) is the normalization of the Halperin wavefunction, \((N'_\sigma) \equiv (N_\uparrow + 1, N_\downarrow - 1)\), and where \(A_{z'z_k}\) was defined in equation (19).

According to the plasma analogy, one may interpret \(Q[N_\sigma]\) as the configuration integral of the two component impurity-free plasma. \(Q[N_\sigma]\) is related to the classical partition function by

\[
Z[N_\sigma] = \frac{Q[N_\sigma]}{N_\uparrow! N_\downarrow! \Lambda^{2(N_\uparrow + N_\downarrow)}} \]  \hspace{1cm} (A4)

where \(\Lambda\) is the (arbitrarily chosen) thermal wavelength of the analog plasma. Using this expression one can write the ratio of configuration integrals which appears in equation (A3) as

\[
\frac{Q[N_{\uparrow} + 1, N_\downarrow - 1]}{Q[N_\uparrow, N_\downarrow]} \approx \frac{N_\uparrow}{N_\downarrow} e^{-\beta(\mu_\uparrow - \mu_\downarrow)} \]  \hspace{1cm} (A5)

This is valid assuming that \(N_\sigma >> 1\).

Next we insert the configuration integral ratio into equation (A3) to get the general expression for the pair spin correlation function:

\[
\Omega^{-2} g_{++} (z' - z) = \frac{(-1)^{n-m}}{|z' - z|^{2(m-n)}} \frac{N_\uparrow}{N_\downarrow} e^{+\beta(\mu_\uparrow - \mu_\downarrow)} < \sigma_1^\uparrow \delta \sigma_2^\uparrow \delta (z' - z_1) \delta (z - z_2) A >_N \]  \hspace{1cm} (A6)

This is valid even if the wells are out of balance. For problems in which the wells are in balance, \(\mu_\uparrow = \mu_\downarrow\) and the above expression simplifies to equation (18).

**APPENDIX: B. ABSENCE OF THE JOSEPHSON EFFECT WHEN \(M \neq N\)**

Recently Wen and Zee have suggested that, if one could separately contact the two wells, then at \(T = 0\) the [mmm] states might exhibit a Josephson effect. Unfortunately, it seems unlikely that the Josephson effect exists at finite temperature since thermal fluctuations
would cause $< S_\pm (z) >$ to vanish and $< S_+ (z) S_- (z') >$ to decay algebraically. More likely would be some sort of fluctuation contribution to the tunneling current\textsuperscript{27}.

Nevertheless, it is interesting to consider whether a similar $T=0$ Josephson effect should exist in a more general $[mmn]$ state. Below we present a simple argument which strongly suggests that the $T=0$ Josephson effect does not exist unless $m=n$.

The argument begins by constructing a simple tunnelling Hamiltonian valid for the subspace of $|mmn >$ states of the form $|N_\uparrow, N_\downarrow >$. This ignores high energy bulk excitations and all edge excitations with non-vanishing wavevector. Next we define a new basis

$$|\theta_\uparrow, \theta_\downarrow > \equiv \sum_{N_1, N_2} a(N_1, N_2) \exp i(N_1 \theta_1 + N_2 \theta_2) |N_\uparrow, N_\downarrow >$$

(B1)

where $a(N_\uparrow, N_\downarrow)$ is peaked about $< N_\uparrow >$ and $< N_\downarrow >$ whose width obeys $N_\alpha >> \Delta N_\alpha >> 1$.

The tunneling Hamiltonian is

$$H_{\text{eff}} = 2t_J \cos(\theta_\uparrow - \theta_\downarrow) - \mu_\uparrow N_\uparrow - \mu_\downarrow N_\downarrow$$

(B2)

where $t_J = < N_\uparrow + 1, N_\downarrow - 1 | H_t | N_\uparrow, N_\downarrow >$ and where $\theta_\alpha$ and $N_\alpha$ obey canonical commutation relations.

According to this rather simple formalism one expects the various Josephson effects if $t_J \neq 0$. To see if this is the case we compare the angular momentum of $|N_\uparrow, N_\downarrow >$ and $|N_\uparrow + 1, N_\downarrow - 1 >$. One readily finds that angular momentum difference between these two states is

$$\Delta L_z = (m - n)(N_\uparrow - N_\downarrow + 1)$$

(B3)

Because of the rotational invariance of $H_t$, $t_J = 0$ unless $m = n$. So a $T = 0$ Josephson effect can only occur for $[mmn]$ states. This is the result of Wen and Zee. Evidently such a Josephson effect does not occur for the $[331]$ state. Of course, the absence of the Josephson effect for the $[331]$ state was demonstrated by different methods in sec. II. See discussion below eqn. 28.

Several comments should be made about the above argument. The first is that one may readily include disorder into the argument. For example, suppose that the interwell
tunneling $t(z)$ is a random function of $z$ which fluctuates about a mean value $\langle t(z) \rangle$ and which is autocorrelated on some distance scale $\xi$. In this case a tunnelling event can change $L_z$ by an amount of order $R/\xi$ where $R$ is the radius of the electron droplet. So the no-Josephson argument fails if

$$(N_\uparrow - N_\downarrow)(m - n) < R/\xi \quad (B4)$$

An alternative way of writing this is to note that $N_\alpha = \pi R_\alpha^2 n_\alpha$ and that the difference in $k_F$ of the electrons in the two edge channels is $\Delta k_F = (R_\uparrow - R_\downarrow)/l_m$ so eqn. $[B4]$ becomes

$$(k_\uparrow - k_\downarrow) < 1/\xi \quad (B5)$$

The interpretation of this result is simple, all the tunnelling which occurs between the two wells for small voltage bias (i.e. $eV < 2\Delta_1$) occurs at the edges. Moreover, because of the requirement of momentum conservation, interedge tunneling can only occur because of disorder effects or because the external field is tilted.
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FIGURES

FIG. 1. Schematic drawing of the double quantum well. We ignore the finite well width $d_w$. The right and left wells are denoted $\sigma = \uparrow$ and $\sigma = \downarrow$, where $\sigma$ is the well index (or pseudospin index). The external components of the field $\vec{B}$ perpendicular and parallel to the wells are denoted by $B_\perp$ and $B_\parallel$.

FIG. 2. Pair spin correlation function data from the Monte Carlo runs. The solid line through the circles is the analytic result for the [330] state presented in eqn. (28). The solid line through the squares is the fit to the expansion given in eqn. (29) to the data. This data is for 200 particles per well and $2 \times 10^6$ Monte Carlo cycles.

FIG. 3. The integrated tunneling conductance spectral moment $W_0(B_\parallel)$ associated with inter-Landau level excitations of the [330] state (solid) and the [331] state (dashed).

FIG. 4. Schematic drawing illustrating single electron tunneling and the origin of $W_{nc}$. Because of momentum conservation, the electron tunnels along field lines. In a tilted field this reduces the matrix element associated with the overlap of the single-particle Gaussian wavefunctions. This figure based on an argument by J. Eisenstein.

FIG. 5. Schematic drawing of tunneling into the correlation hole of a [mmn] state with $n \neq 0$. Fewer final states are blocked by the correlation hole than would be blocked by an uncorrelated electron liquid. This effect tends to increase $W_0$ and decrease $<eV>$.

FIG. 6. (a.) A plot of the radial distribution function $g_{\uparrow\downarrow}(d_*)$ vs $d_* \equiv B_\parallel d / B_\perp l_m$, the lateral displacement experienced by an electron during a tunneling event. (b.) A plot of correlation enhancement factor $R$ vs $d_*$. Observe that $R$ has a minimum near the maximum of $g_{\uparrow\downarrow}(d_*)$ i.e. at the position of the first coordination shell. From tunneling data, one can measure $R(B_\parallel)$ which, in general, is expected to track $g_{\uparrow\downarrow}(d_*)$ as it does here for the [331] state.
FIG. 7. Comparison of mean voltage bias results obtained for the $[330]$ and $[331]$ states. The mean voltage bias $< eV >$ is the mean voltage of the intra-Landau level band which is observed in the tunneling conductance spectrum. Results are presented as a function of $(d/l_m)(B_{\parallel}/B_{\perp})$ where $d/l_m = 2.4$.

FIG. 8. Mean voltage bias vs. $B_{\parallel}d/B_{\perp}l_m$: (a.) Results for $[330]$ (i.e. a pair of $\nu = 1/3$) for various well spacing $d = 0, 0.5, 1.0, 1.5, 2.4$. (b.) Same as (a.) except for $[550]$ state. (c.) Same as (a.) except for $[331]$ state.
\[ W_0 \left( \frac{\Omega e t_0^2}{\hbar l_m^2} \right) \]
$\sigma = \uparrow$  \hspace{1cm}  $g_{\uparrow\downarrow}(r)$

$\sigma = \downarrow$

$B_{\parallel} = 0$

$B_{\parallel} \neq 0$
\begin{equation}
\langle eV \rangle \left( \frac{e^2}{\epsilon l_m} \right) = d = 2.4 l_m
\end{equation}
\[ \langle eV \rangle = \left( \frac{e^2}{\epsilon l_m} \right) \]

\[ B_{\parallel}d/B_{\perp}l_m \]

\[ d = 0 \quad d = 0.5 l_m \quad d = 1.0 l_m \quad d = 1.5 l_m \quad d = 2.4 l_m \]
