Theoretical Physics Institute
University of Minnesota

UMN-TH-1232/93
TPI-MINN-93/61-T
December 1993

e^+ e^- \rightarrow \tau^+ \tau^- \text{ at the threshold and beyond}

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Abstract

The excitation curve for the $\tau^+ \tau^-$ production in electron positron annihilation near the threshold is revisited with the aim of updating and extending a previous work. We find that the corrections of the relative magnitude $O(\alpha)$ near the threshold are significantly contributed by the radiative modification of the Coulomb interaction between the $\tau$ leptons. The interpolation between the Coulomb effects at the threshold and the relativistic effects at higher energies is considered and the resulting formula is argued to have relative accuracy up to, but not including, terms of the order of $\alpha^2$ at any velocity of the $\tau$ leptons.
The dramatic improvement in the accuracy of the measured mass of the $\tau$ lepton, achieved in the BES experiment\cite{1}, has demonstrated one of the advantages of studying the production of the $\tau^+ \tau^-$ pair in $e^+ e^-$ annihilation in the immediate vicinity of the threshold. In view of the special kinematical and background advantages of the threshold region\cite{2} one may believe that experiments at the $\tau^+ \tau^-$ threshold will be continued at the existing machine and, hopefully, at the much anticipated tau-charm factory\cite{3}. In view of the probable improvement in the precision of the experimental data, it is worthwhile to have sufficiently accurate theoretical description of the $e^+ e^- \rightarrow \tau^+ \tau^-$ cross section at the threshold and beyond. In addition, this process is of a theoretical interest of its own, since it provides a ‘test ground’ for QED almost as good as the muon or electron magnetic anomaly.

The well-known lowest-order QED formula for the cross section

$$\sigma_0(e^+ e^- \rightarrow \tau^+ \tau^-) = \frac{2\pi\alpha^2}{3s} v (3 - v^2),$$

where $v = \sqrt{1 - 4m_\tau^2/s}$ is the velocity of each of the $\tau$ leptons in the c.m. frame, is subject to corrections, arising from various sources:

- $i$ – radiation from the initial electron and positron,
- $ii$ – vacuum polarization in the time-like photon,
- $iii$ – corrections to the spectral density $\rho(q^2) = -\frac{1}{3} \sum_X \langle 0 | j_\mu(-q) | X \rangle \langle X | j_\mu(q) | 0 \rangle$ of the electromagnetic current $j_\mu = (\bar{\tau} \gamma_\mu \tau)$ of the tau leptons. The latter corrections generally include both the electromagnetic interaction between the $\tau$ leptons and the final state radiation.

In higher orders there appears interference between the effects $i - iii$, however its relative magnitude is suppressed by at least the factor $\alpha^2$, which makes it beyond the scope of the present paper.

Because of the radiative effects the actually measured cross section is given by the formula

$$\sigma(W) = \int_W^W r(W, w) |1 - \Pi(w)|^{-2} \sigma(w) \, dw,$$

where $W = \sqrt{s}$ is the total energy in the c.m. system. The weight function $r(W, w)$ describes the radiation from the initial state\cite{4} and $|1 - \Pi(w)|^{-2}$ is the vacuum polarization factor\cite{5}. These two effects are standard and are routinely accounted for in the data analyses, while the dynamics of the final state is encoded in the cross section $\sigma(w) =$...
In this letter we concentrate on the behavior of $\bar{\sigma}(w)$ in the threshold region, where the velocity $v$ of the produced $\tau$'s is a small parameter. In this region the QED effects in the spectral density $\rho(w^2)$ are determined by the Coulomb interaction between the $\tau$ leptons at distances $r \sim (m_\tau v)^{-1}$, the parameter for which effects is given by $\pi \alpha / v$, while the relativistic effects and the radiation of transversal photons by the $\tau$ leptons are suppressed by at least the factor $v^2$. In the non-relativistic region one can effectively separate the dynamics at short distances $r \sim m_\tau^{-1}$, at which the leptons are produced by the electromagnetic current and that at the distances $r \sim (m_\tau v)^{-1}$, which determine the Coulomb effects, by replacing the current $j_\mu$ by a local non-relativistic operator. In the lowest order in the loop corrections, arising at short distances, this replacement can be written as (c.f. Ref. [6])

$$\bar{\sigma}(w) = \frac{2\pi^2 \alpha^2}{m_\tau^4} \text{Im} G(0, 0; m_\tau v^2), \quad (3)$$

where $G(x, y; E)$ is the non-relativistic Green's function at energy $E = w - 2m_\tau$. For free particles the Green's function is given by

$$G_0(x, y; \frac{p^2}{m}) = \frac{m \exp(i p |x - y|)}{4\pi |x - y|} \quad (4)$$

so that $\text{Im} G_0(0, 0; m_\tau v^2) = m^2 v / (4\pi)$ and the non-relativistic limit of the 'bare' cross section (1) is reproduced. The solution for the Green's function in the Coulomb potential $V(r) = -\alpha / r$ is also well known and at the origin it reads as $\text{Im} G_c(0, 0; m_\tau v^2) = F_c \text{Im} G_0(0, 0; m_\tau v^2)$ with the Coulomb factor $F_c$ given by

$$F_c = \frac{\pi \alpha / v}{1 - \exp(-\pi \alpha / v)}. \quad (5)$$

The $v^{-1}$ behavior of this factor at $v \to 0$ makes the cross section at $v = 0$ finite and of a measurable magnitude:

$$\bar{\sigma}(v = 0) = \frac{\pi^2 \alpha^3}{2m_\tau^2} (1 + O(\alpha)) = 0.236 \text{ nb} (1 + O(\alpha)). \quad (6)$$

The fact that the spectral density of the current at small velocity is determined by the Coulomb interaction calls for the idea that the radiative corrections to this interaction should show up at the level of $O(\alpha)$ rather than the usual $O(\alpha^2)$. Since the higher order effects, involving exchange and radiation of transversal photons at large distances, are
suppressed by the factor $v^2$, the only radiative effect at these distances is due to running of the coupling $\alpha$, which is also known as the Uehling-Serber radiative correction to the potential (see e.g. in the textbook [7]). For a preliminary estimate of the magnitude of this effect one can replace $\alpha$ in the Coulomb factor (5) by the effective coupling at the momentum $p = m_\tau v$:

$$\alpha \rightarrow \alpha \left(1 + \frac{2\alpha}{3\pi} \ln \frac{m_\tau v}{m_e}\right).$$

(7)

One can readily see that the ratio $m_\tau v/m_e$ is large even for small $v$, so that the effect can be of the order of one percent. This estimate however requires some elaboration, which is the main subject of the present work. Namely, non-logarithmic terms can be essential and also the estimate of the characteristic distances, essential for the Coulomb factor, is modified by the Coulomb interaction at $v \sim O(\alpha)$. To find the effect quantitatively we use the full form of the Uehling-Serber correction to the Coulomb potential due to the electron-positron vacuum polarization [7]:

$$\delta V(r) = -\frac{2\alpha^2}{3\pi} r \int_1^\infty e^{-2m_\tau x} \left(1 + \frac{1}{2x^2}\right) \frac{\sqrt{x^2 - 1}}{x^2} dx ,$$

(8)

and consider this correction as a perturbation, thus finding the modification of the wave function at the origin in the form

$$\delta \psi(0) = -\int G_c(0, r; m_\tau v^2) \delta V(r) \psi_c(r) d^3r ,$$

(9)

where $\psi_c(r)$ is the S-wave wave function at energy $E = m_\tau v^2$ in the Coulomb field $-\alpha/r$:

$$\psi_c(r) = Ce^{-ipr} \frac{1}{\Gamma(1 + i\lambda)} F_1(1 + i\lambda, 2, 2ipr)$$

(10)

with $p = m_\tau v$, $\lambda = m_\tau \alpha/(2p) = \alpha/(2v)$ and $C$ being normalization constant. Using the representation of the Coulomb Green’s function in the form

$$G_c(0, r; p^2/m_\tau) = -\frac{1}{2\pi} \frac{m_p}{e^{ipr}} \int_0^\infty e^{2ipt} \left(1 + \frac{t}{t}\right)^{i\lambda} dt ,$$

(11)

we find that the corrected expression for the square of the wave function at the origin can be written as:

$$|\psi_c(0) + \delta \psi(0)|^2 = |\psi_c(0)|^2 (1 + h) ,$$

(12)

where the correction $h$ after integration over $r$ in eq.(10) is given by
\[ h = \frac{2\alpha}{3\pi} \left[ -2\lambda \Im \int_0^\infty dt \int_1^\infty dx \left( \frac{1+t}{t} \right)^{i\lambda} \frac{(t + z x v^{-1})^{i\lambda-1}}{(t + 1 + z x v^{-1})^{i\lambda+1}} \left( 1 + \frac{1}{2x^2} \right) \frac{\sqrt{x^2 - 1}}{x^2} \right] \] (13)

with \( z = m_e/m_\tau \).

The double integral in the latter equation was calculated numerically\[1\]. The plot of the magnitude of the relative correction \( h \) as a function of velocity \( v \) is shown in Fig.1 along with the result of the simple logarithmic estimate from the equations (5) and (7). One can see that this estimate becomes invalid in the region \( v < O(\alpha) \), while it fits the exact result quite well at higher velocity. Also in Fig.1 is shown the result of calculation of the contribution to \( h \) of the muon vacuum polarization, which, as expected, is quite small. We also find that in the limit \( v \to 0 \) and \( m_e/m_\tau \to 0 \) the quantity in the square brackets in eq.(13) is equal to 4, so that \( h(v = 0, m_e = 0) = 8\alpha/(3\pi) \).

In order to have a full description of the \( O(\alpha) \) correction to the excitation of the \( \tau^+ \tau^- \) channel near the threshold one should also take into account the finite renormalization of the electromagnetic current, which comes from short distances \( r \sim m^{-1} \), which is well known\[8\] to result in the overall factor \( (1 - 4\alpha/\pi) \), which, unlike the term \( h \), does not depend on the velocity in the threshold region, i.e. unless the terms \( O(v^2) \) are taken into account.

Summarizing this discussion we find that the cross section for production of non-relativistic \( \tau \) leptons with velocity \( v \) including the terms of the relative magnitude \( O(\alpha) \) is given by

\[ \bar{\sigma} = \frac{\pi^2 \alpha^3}{2m_\tau^2} \frac{1}{1 - \exp(-\pi \alpha/v)} \left( 1 - \frac{4\alpha}{\pi} + h \right) \] (14)

where \( h \) is determined by the running of the QED coupling \( \alpha \) and the dominant contribution to it due to the electron-positron vacuum polarization is given by eq.(13) and which is plotted on Fig.1.

At the present accuracy of experimental measurement of \( m_\tau \), which is about 0.3 MeV\[9\], the \( O(\alpha) \) corrections discussed here can be completely ignored. A simple analysis shows that these terms become essential for measurements with accuracy of the order of \( 10^{-2} \) MeV, at which level one should also take into account the contribution of the

\[ ^{1}\text{In particular we found that the numerical integration routines built into Mathematica provide a reliable, though not the fastest, platform for this calculation.} \]
Coulomb bound states\footnote{We take this opportunity to correct some misprints and other minor flaws of the discussion of the interpolating formula in Ref.\cite{2}. These flaws however do not affect the results of the data analysis in the BES experiment\cite{1}, which used the formulas from Ref.\cite{2}.}. For completeness we cite here that the binding energy of the \(n\)-th level is

\[
|E| = \frac{m_\tau \alpha^2}{4 n^2} \approx \frac{24 \text{KeV}}{n^2}
\]

and its \(e^+e^-\) decay rate is given by

\[
\Gamma_{ee} = \frac{m_\tau \alpha^5}{6 n^3} \approx \frac{6.1 \text{eV}}{n^3} .
\]

Equation (14) is applicable in the non-relativistic limit, i.e. the relative magnitude of the corrections to it is determined by \(v^2\). On the other hand the exact in \(v\) formula of the first order in \(\alpha\) is known (see e.g. in Schwinger’s textbook \cite{8}): \(\bar{\sigma}(e^+e^{-} \rightarrow \tau^+\tau^-) = \sigma_0 \left(1 + \frac{\alpha}{\pi} S(v) \right)\), where \(\sigma_0\) is the bare cross section, eq.\((1)\), and

\[
S(v) = \frac{1}{v} \left\{ (1 + v^2) \left[ \frac{\pi^2}{6} + \ln \left( \frac{1+v}{1-v} \right) \ln \left( \frac{1+v}{1-v} \right) + 2 \text{Li}_2 \left( \frac{1-v}{1+v} \right) + 2 \text{Li}_2 \left( \frac{1+v}{1-v} \right) \right] - 2 \text{Li}_2 \left( \frac{1-v}{2} \right) - 4 \text{Li}_2(v) + \text{Li}_2(v^2) \right\} + \left[ \frac{11}{8} (1 + v^2) - 3v + \frac{1}{2} \frac{v^4}{3-v^2} \right] \ln \left( \frac{1+v}{1-v} \right) + \\
6v \ln \left( \frac{1+v}{2} \right) - 4v \ln v + \frac{3}{4} v^2 \ln \left( \frac{1-v^2}{1-v^2} \right) 
\]

with \(\text{Li}_2(x) = -\int_0^x \ln(1-t) \, dt/t = \sum_{n=1}^{\infty} x^n/n^2\). The full relativistic formula of the first order in \(\alpha\) develops at low velocity the inaccuracy of the order of \((\alpha/v^2)^3\), which matches the inaccuracy \(O(v^2)\) of the non-relativistic formula (14) at \(v^2 \sim \alpha\), where the relative magnitude of the error in either of these formulas is \(O(v^2) \sim O(\alpha)\). One can however write an expression\footnote{We take this opportunity to correct some misprints and other minor flaws of the discussion of the interpolating formula in Ref.\cite{2}. These flaws however do not affect the results of the data analysis in the BES experiment\cite{1}, which used the formulas from Ref.\cite{2}.}, which interpolates between the non-relativistic formula (14), summing all the terms of the form \((\alpha/v)^k\) and \(\alpha (\alpha/v)^k\), and the full formula of the first order in \(\alpha\). The resulting interpolating formula can be argued to have relative accuracy \(O(\alpha^2)\) at any velocity\footnote{We take this opportunity to correct some misprints and other minor flaws of the discussion of the interpolating formula in Ref.\cite{2}. These flaws however do not affect the results of the data analysis in the BES experiment\cite{1}, which used the formulas from Ref.\cite{2}.}

The interpolating formula has the form

\[
\bar{\sigma}(e^+e^{-} \rightarrow \tau^+\tau^-) = \sigma_0 \left(1 + h \right) F_c \left( 1 + \frac{\alpha}{\pi} S(v) - \frac{\pi \alpha}{2v} \right) \]

where the lowest-order cross section \(\sigma_0\) is given by eq.\((1)\), the Coulomb factor \(F_c\) is in the equation \((3)\), the correction term \(h\), given by eq.\((13)\), accounts for running of \(\alpha\) in the Coulomb terms and the function \(S(v)\) is written in eq.\((17)\). One can notice that in eq.\((18)\) the first Coulomb term \(\pi \alpha/(2v)\) is subtracted from the full relativistic expression
of the first order in \( \alpha \), since this term is contained in the factor \( F_c \). The factor \((1 - 4\alpha/\pi)\), coming from finite renormalization of the current at the threshold, which is present in the equation (14), in the interpolating formula is contained in the factor with \( S(v) \). Equation (18) sums all terms of the form \((\alpha/v)^k\) and \(\alpha(\alpha/v)^k\) as well as all terms linear in \( \alpha \) with arbitrary dependence on \( v \). Therefore the corrections to the interpolating formula start with terms of the form \( \alpha^2(\alpha/v)^k \) with \( k \geq 0 \), which are at least as small as \( O(\alpha^2) \) in comparison with those included in eq.(18), independently of the velocity \( v \). Naturally, to also take into account the terms of the relative order \( \alpha^2 \) one needs a full calculation of the two-loop correction to the spectral density of the electromagnetic current at arbitrary \( v \) as well as a calculation of the interference between the mechanisms i – iii mentioned in the beginning of this paper.

This work was supported, in part, by the DOE grant DOE-AC02-83ER40105.

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Figure caption

Figure 1. The relative correction $h$ (eq.(13)) to the cross section of $\tau^+\tau^-$ production near the threshold, arising from modification of the Coulomb potential by the polarization of the electron-positron vacuum (solid) and of the muon one (dashed) vs. the velocity of the $\tau$ leptons in the c.m. system. Also is shown (dotted) the logarithmic approximation for the electron-positron contribution to $h$ according to eq.(7).