Wavefront retrieval through random pupil plane phase probes: Gerchberg-Saxton approach

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A pupil plane wavefront reconstruction procedure is proposed based on analysis of a sequence of focal plane images corresponding to a sequence of random pupil plane phase probes. The developed method provides the unique nontrivial solution of wavefront retrieval problem and shows global convergence to this solution demonstrated using a Gerchberg-Saxton implementation. The method is general and can be used in any optical system that includes deformable mirrors for active/adaptive wavefront correction. The presented numerical simulation and lab experimental results show low noise sensitivity, high reliability and robustness of the proposed approach for high quality optical wavefront restoration. Laboratory experiments have shown $\lambda/14$ rms accuracy in retrieval of a poked DM actuator fiducial pattern with spatial resolution of 20-30 $\mu$m that is comparable with accuracy of direct high-resolution interferometric measurements.

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1. INTRODUCTION

Optical applications often require knowledge of both the phase and amplitude of an optical wavefront. Such problems (or similar) arise in various fields such as electron microscopy \cite{1}, X-ray crystallography \cite{2}, astronomy \cite{3} and optical imaging \cite{4}, quantum state tomography \cite{5}, etc. The particular interest of the authors of this paper is inspired by the requirements of stellar coronagraphy for high-contrast imaging of exoplanets where the wavefront reconstruction at different planes across the coronagraph instrument is needed to create a model of the instrument used for wavefront control algorithms such as Electric Field Conjugation (EFC) \cite{6,7} Additionally, wavefront control methods such as EFC require accurate wavefront reconstruction to cancel out aberration-induced speckles in the science focal plane and create regions of high contrast.

Since it is not always possible to measure the desired wavefront directly, different methods have been developed that allow its reconstruction based on intensity measurements only.

It is clear that phase information is absent from intensity-only measurements \cite{8} and cannot be reconstructed, unless some technique is used that encodes phase information in a sequence of intensity images in some controlled fashion. Otherwise, phase restoration can only be performed under certain assumptions about the reconstructed wavefront. It was found in earlier works \cite{9–12} that for almost every pupil plane wavefront a unique solution exists if the focal plane intensities are known together with some constraints applied to the pupil aperture function. In this case the wavefront reconstruction problem can be formulated as the problem of a complex-valued signal reconstruction from the modulus of its Fourier transform. However, even if the unique solution exists, it is not always possible to find this solution, due to convergence issues associated with the wavefront reconstruction algorithms \cite{13–15}. Note that the above mentioned solution uniqueness is not absolute, because a few trivial ambiguities still remain unsolved and can affect an algorithm convergence \cite{16} beside the widely discussed non-convexity of Fourier magnitude constraint \cite{13,17}.

In this paper we propose a pupil plane wavefront recon-
2. METHOD

Historically, the first successful phase retrieval method was proposed by Gerchberg and Saxton [18]. The algorithm was originally developed to restore the pupil plane wavefront assuming that both the pupil plane intensities $I(u) = |E(u)|^2$ and the focal plane intensities $i(r) = |e(r)|^2$ are known. The pupil plane (complex-valued) electric field $E(u)$ and the focal plane electric field $e(r)$ are related by the Fourier transform

\[ e(r) = |e(r)| \exp[i\varphi(r)] = \int E(u) \exp[2\pi i ur] du, \]

\[ E(u) = |E(u)| \exp[i\Phi(u)] = \int e(r) \exp[-2\pi i ur] dr, \]  

where $|E(u)|$ and $|e(r)|$ are the wavefront amplitudes, $\Phi(u)$ and $\varphi(r)$ are the wavefront phases, and $u$ and $r$ are radius-vectors in the pupil plane and the focal plane respectively. A constant phase factor as well as the normalizing/scaling factor $1/\lambda f$, where $\lambda$ is the wavelength and $f$ is the system focal length, are not included in Eq. 1 for simplicity.

The Gerchberg-Saxton iterative loop bounces between pupil and focal plane and performs the following sequence of steps in the $(k + 1)$-th iteration (Fig. 1):

1. Fourier transform of the current focal plane estimate $E_k(u)$
\[ e_k(r) = |e_k(r)| \exp[i\varphi_k(r)] = \int E_k(u) \exp[2\pi i ur] du. \]  

2. Replace the amplitude $|e_k(r)|$ of the resulting Fourier transform with the measured focal plane amplitude $|e(r)|$
\[ g_k(r) = |e(r)| \exp[i\varphi_k(r)]. \]

3. Inverse Fourier transform of the current focal plane wavefront estimation $g_k(r)$
\[ G_k(u) = |G_k(u)| \exp[i\Phi_k(u)] = \int g_k(r) \exp[-2\pi i ur] dr. \]

4. The next pupil plane wavefront estimate could be obtained by replacing the amplitude $|G_k(u)|$ with the measured pupil plane amplitude $|E(u)|$
\[ E_{k+1}(u) = |E(u)| \exp[i\Phi_k(u)]. \]

Though the pupil and focal plane intensity distributions used in the loop as constraints provide an efficient locally converging algorithm [13, 18, 19], they cannot guarantee the global convergence to the unique solution even in the case where such a solution exists [13, 20]. As a result the algorithm stagnates near the closest local minimum that could be very different from the real solution that should be found (see examples in [21, 22]). The main reasons for the stagnation are existence of ambiguous (non-unique) pupil plane wavefronts that are close to the measuring wavefront [20], on one hand, and non-convexity of the same Fourier magnitude sets [13, 17], on another hand. Even trivial ambiguities such as the global tip/tilt or conjugate inversion of the wavefront in combination with loose or symmetric wavefront support could be a reason for the algorithm stagnation [14, 16, 22, 23] near a solution where the shifted or twin wavefronts appear to be combined with the recovering wavefront. Modifications of the Gerchberg-Saxton algorithm such as basic input-output and hybrid input-output algorithms by Fienup [19, 24] improve convergence to the global solution in some cases but still demonstrate stagnation in other cases [14, 20, 22].

Another group of algorithms for the wavefront retrieval are “diversity” algorithms, that analyze intensity changes in the focal (or some intermediate) plane caused by a few predetermined changes of the pupil plane to determine the measured pupil plane wavefront. The desired “diversity” can be produced by either the pupil plane phase [25–28] or amplitude [29–31] changes that provide the unique solution for the wavefront retrieval. It has been shown [32] that for a special kind of the amplitude diversity (exponential pupil illumination) using three different pupil illuminations provides the analytically unique (up to the global phase) 2-dimensional pupil wavefront retrieval. This conclusion seems to be correct for any kind of the wavefront
diversity (either phase [16, 33] or amplitude or combine) unless the pupil wavefront modulation functions appear to have special symmetry or cross-symmetry. So, the pupil wavefront diversity removes the main reason of the wavefront retrieval stagnation, namely non-uniqueness of the solution. Though the non-convexity of the used constraints still can be relevant (theoretically) to some open-loop phase wavefront diversity applications, the related numerical problems can be easily solved by changing the predetermined diversities. As a result, the diversity algorithms are much faster and show much better global convergence in comparison with the algorithms from the first group. The simplest phase diversity may be produced by defocusing (like in the classical Hartmann optical test) that introduces a quadratic phase variation across the pupil. Such diversity, however, is not very sensitive to non-symmetric phase aberrations [34] At the same time, more complicated diversity methods often need additional optics that not always can be easily implemented in the optical system and is the source of non-common path errors. The used wavefront diversities also need to be well calibrated [35] that may not always be available. Additionally, perfect knowledge of the optical system parameters is often needed to use the phase diversities [36].

The question is could algorithms be developed that have advantages of both methods without their disadvantages? To answer this question let's use the Gerchberg-Saxton algorithm to restore the pupil plane wavefront that is temporarily changing under the influence of atmospheric turbulence, assuming that the pupil plane amplitude is constant (is equal to 1).

Consider the following modification of the Gerchberg-Saxton algorithm:

1. Fourier transform of the current wavefront estimate obtained in the time moment $t_k$

   $$e_k(r, t_k) = |e_k(r, t)| \exp[i\phi_k(r, t_k)] =$$
   $$= \int E_k(u, t_k) \exp[i2\pi ur]du.$$  \hspace{1cm} (6)

2. Replace the amplitude $|e_k(r, t_k)|$ of the resulting Fourier transform with the new focal plane amplitude $|e(r, t_{k+1})|$ that is changing with time

   $$g_k(r, t_{k+1}) = |e(r, t_{k+1})| \exp[i\phi_k(r, t_k)].$$ \hspace{1cm} (7)

**Fig. 2.** Gerchberg-Saxton phase retrieval for a sequence of correlated pupil plane aberrations produced by Kolmogorov turbulence ($D/r_0=30, c_{\Delta \phi}=\lambda/46, r_f=0.99$) in absence of focal plane image noise. The defocus $a$ is equal to $\lambda/20$. The sequence of correlated pupil plane aberrations, the pupil wavefront estimates and the focal plane images that match the presented pupil plane phases are shown (from top to bottom) together with corrected PSFs obtained after 1, 2, 21, 101 and 121 iterations.
number iterations needed for convergence

\[ \begin{array}{cccc}
\text{Number iterations needed for convergence} & 0 & 50 & 100 \\
\hline
\text{Defocus parameter (a)} & 0.54 & 0.44 & 0.34 & 0.24 & 0.14 \\
\end{array} \]

Fig. 3. The algorithm convergence speed vs. the focal plane image defocus (Kolmogorov turbulence, \( D/r_0 = 30 \), \( \sigma_{\Delta \phi} = \lambda / 46 \), \( r_I = 0.99 \)).

3. Inverse Fourier transform of the current focal plane wavefront estimation \( g_k(r) \)

\[
G_k(u, t_{k+1}) = |G_k(u, t_{k+1})| \exp[i\Phi_k(u, t_{k+1})] = \int g_k(r, t_{k+1}) \exp[-i2\pi ur] d\mathbf{u}. \tag{8}
\]

4. The next pupil plane wavefront estimate could be obtained by replacing the amplitude \( |G_k(u, t_{k+1})| \) with the measured pupil plane amplitude that is equal to 1

\[
F_{k+1}(u, t_{k+1}) = \exp[i\Phi_k(u, t_{k+1})]. \tag{9}
\]

It is obvious that the algorithm will never converge if the sequential wavefronts are completely uncorrelated. However, if the closest wavefronts are highly correlated and the sequential wavefront change is small enough to provide continuous transition between sequential pupil plane phases, one can expect the algorithm to converge to the “common” part of those wavefronts (note that this “common” part is changing with time). A few main advantages of the proposed approach in comparison with the original Gerchberg-Saxton algorithm can be seen. The first advantage is the existence of the unique solution for the sequence of continuously changing wavefronts. Though, it is difficult to give a pure mathematical proof for this conclusion it could be explained as follows. Lets assume that a continuous transition exists that transforms one random pupil phase distribution in another. During this transition both focal plane wavefronts and intensities are also transformed continuously. Taking into account that the probability of the unique solution of the phase retrieval is almost equal to 1, it is always possible to trace the actual solution along the transition path by using the continuity of the transition and assuming that the actual solution is known in some point of the transition.

The second advantage is the absence of local minima that can capture the iterative solution and cause stagnation. The presence of stagnation is associated with ambiguous solutions in the vicinity of the stagnation point [20]. The ambiguous phases are not keeping the same along the continuous phase transition and can trap the iterative solution that appears in their vicinity. However, they evolve differently from each other and from the global solution. As soon as the focal plane intensities related to the global and the ambiguous solutions become incompatible (because of the pupil plane phase change) the captured iterative solution will be released and continue to converge until it will be captured by the global solution (Fig. 2). The iterative solution captured by the global solution could be also influenced by the close ambiguous solutions that could cause a temporary iteration divergence (see spikes in Fig. 5, top) that terminates as soon as the global and the ambiguous focal plane intensities become incompatible. The aforementioned conclusions have been demonstrated through the numerical simulations (Fig. 2) and lab experiments that will be discussed in more detail later. The most surprising finding is fast algorithm convergence to the global solution even in the presence of strong photon noise and background noise as described in Sections 3B and 3E.

Since the Gerchberg-Saxton algorithm could produce high quality dynamical wavefront retrieval, the same approach can be generalized to recover the pupil plane wavefront in any optical system. In many cases correlated random pupil plane phase aberrations can be created, for example, with a deformable mirror (DM) and the Gerchberg-Saxton loop can be used to recover these aberrations combined with the static aberrations of the optical system. As soon as the iterative algorithm has found the global “common” solution, gradual reduction of the random DM shapes amplitude can force convergence to the real static aberration of the optical system.

3. NUMERICAL SIMULATIONS

Though the above idea looks promising, a few more things need to be established. First of all, the algorithm convergence should be proved/checked and the conditions should be found when the algorithm converges to the exact solution. However, even in the case that such solution exists the algorithm must demonstrate an appropriate convergence speed and noise sensitivity.
The numerical simulations described in this section address these questions.

Two different data sets have been used to simulate random pupil plane phase distributions. The first one (Model I) is based on a “boiling” turbulence model [37]. In this model the spatial variations are described by Kolmogorov turbulence model, and the temporal variations are determined by inhomogeneities disintegration speed that depends on the inhomogeneities size (scale). The chosen turbulence strength \( D \) is the entrance pupil size, and \( r_0 \) is Fried parameter) creates the wavefront aberrations that are similar to those introduced by the atmosphere across the 3-meter aperture under 1” seeing. The temporal sampling for the Model I is chosen such that the disintegration time of the smallest spatial inhomogeneities is equal to the sampling step (it means that these inhomogeneities are uncorrelated for sequential wavefronts). The selected simulation parameters provides sequential focal plane image correlation 0.4-0.5 consistent with results of astronomical speckle-interferometric measurements performed at times comparable with the lifetime of stellar interferograms under seeing conditions that are close to the simulated seeing conditions [38, 39]. The second data set (Model II) is the sequence of uncorrelated Gaussian phase screens with the spatial correlation radius \( \sigma = D/12 \) and the rms phase value of about \( \lambda/5 \).

Unfortunately, both data sets do not provide the sequential wavefront correlation needed for the algorithm convergence. The linear wavefront interpolation has been used to produce a data set with larger sequential correlation. In this procedure the pupil phases \( \varphi_{i+2} \) with the sequence numbers \( i \times N \) are generated in accordance with Model I or II and the gaps between them are filled with phases \( \varphi_{k+i} = k \times \varphi_{i+2} \) for sequential pupil plane phase difference rms \( \sigma_{\Delta \varphi} \). After each iteration, the pupil wavefront can be corrected with the obtained wavefront estimate. The optical quality of the corrected focal plane image is used in this paper to evaluate the algorithm performance. The quality of the focal plane images is characterized by the related Strehl ratio \( S \).

The algorithm convergence speed vs. correlation of sequential focal plane images \( r_1 \). The number of iterations needed to reach Strehl ratio \( S \) larger than \( S_{\text{thr}} \) is presented (Kolmogorov turbulence, \( D/r_0 = 30 \)). Note, that for \( r_1 \) less than 0.94 the algorithm does not converge.

Fig. 5. The algorithm convergence: the dependence of the phase retrieval quality \( S \) on the iteration number is presented for the correlation coefficient \( r_1 \) equal to 0.91 (top), 0.94 (bottom) and 0.99 (top) (Kolmogorov turbulence, \( D/r_0 = 30 \)).

Fig. 6. The algorithm convergence speed vs. correlation of sequential focal plane images \( r_1 \). The number of iterations needed to reach Strehl ratio \( S \) larger than \( S_{\text{thr}} \) is presented (Kolmogorov turbulence, \( D/r_0 = 30 \)). Note, that for \( r_1 \) less than 0.94 the algorithm does not converge.

A. Algorithm convergence

Numerical simulations show that in the case of large phase aberrations (like in Model I) it is difficult to achieve algorithm convergence unless a small defocus \( a(8 + x^2 + y^2 - D^2)/D^2 \) term, where \( a \) is defocus parameter, added to the pupil plane phase. The convergence absence can be easily explained by the wavefront degeneracy presented in the focal plane where the sign of the optical beam convergence changes. Because of this singularity two conjugately inverted pupil plane solutions \( \varphi(r) \) and \( -\varphi(-r) \) appear to be close to each other during each iteration step resulting in the convergence failure. Sufficient defocus

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Fig. 7. Gerchberg-Saxton phase retrieval for a sequence of correlated pupil plane aberrations in a presence of photon noise (Kolmogorov turbulence, $D/r_0=30$, $r_1=0.99$, $n_{ph}=2$ photons/speckle, $a = \lambda/20$). The actual pupil wavefront, its estimate after 105 iterations are shown together with the related focal plane photon limited image and the corrected PSF ($St=0.526$). The focal plane region highlighted by the rectangle is shown with full resolution in the related image corner.

(it does not need to be known) removes the focal plane wavefront degeneracy (it does not solve the phase ambiguity caused by the conjugate inversion though) that recover the convergence to one of the conjugately inverted phase solutions (Fig. 3). Finally, the defocus sign could be used to distinguish between two possible solutions.

It should be noted that for smaller phase aberrations the iterative process converges even without defocusing. However, the convergence speed in this case is very low and can be improved by a factor of 20 using focal plane image defocusing. For example, the global solution for the Model I with $D/r_0 = 10$ can be obtained after 800-1000 Gerchberg-Saxton iterations without any image defocusing. The same solution could be reached only after 40-60 iterations if defocus is introduced.

Analysis performed for Model I with the defocus parameter $a = \lambda/20$ shows (Fig. 4–6) that the algorithm does not converge as long as the correlation coefficient $r_I$ is less than 0.9 ($c_{\Delta \varphi} > \lambda/18$) though it shows negligible $St$ improvement for smaller $r_I$ values. The algorithm only converges (meaning that it forms the Airy-like focal plane image structure) if $r_I$ is larger than 0.94 ($c_{\Delta \varphi} = \lambda/18$). Once $r_I$ reaches 0.97 ($c_{\Delta \varphi} = \lambda/25$) the Strehl ratio becomes larger than 0.95 for more than 50% of the corrected focal plane images (the $St_{im}$ definition). For $c_{\Delta \varphi} = \lambda/46$ more than 90% of the corrected images have $St$ larger than 0.95, though 5% of them still can show $St$ as low as 0.2 (Fig. 5, top) being affected by the ambiguous phase solutions. The temporary algorithm divergence caused by the phase ambiguity issue is quickly changed by the algorithm convergence that occurs once the pupil wavefront change.

The image defocus and high correlation of sequential images provide fast and guaranteed algorithm convergence to the global solution (Fig. 3–6). The convergence speed can be increased even more. An appropriate method will be presented in the next subsection.

B. Photon noise

Although the numerical simulations above show good performance there are fundamental limits on wavefront reconstruction errors set by photon noise that could completely break the algorithm convergence. It is expected that the algorithm performance including the quality of wavefront reconstruction and the convergence speed should fall down in presence of photon noise. However, those expectations are not completely right. First of all, the presented approach shows decent convergence even in the case of fluxes as low as 1-2 photons per speckle (Fig. 7). The cost is in the form of maximal reachable Strehl ratio which decreases with source brightness if the correlation between sequential images is high enough (Fig. 8). At the same time, the algorithm restores high Strehl ratio even for fluxes as low as 1-2 photons per speckle when the correlation is not high enough for the algorithm convergence in the “no photon noise” case (Fig. 9).

Secondly, the unexpected benefit is that the convergence speed in the low photon rate mode is noticeably (by a factor of 2-3) faster in comparison with “no photon noise” case (Fig. 10).

The faster convergence of the low photon case can be explained by a simpler solution with fewer degrees of freedom, but obtained at a cost in surface quality in the form of an ultimately lower Strehl ratio. Taking into account this faster convergence, the following algorithm modification can be applied to improve convergence speed in the “no photon noise” case. This modification assumes that the phase retrieval procedure should be applied to the low photon rate images first and switch to the “no photon noise” case in the following iterations. Such artificial “photonization” of the focal plane images could be done programmatically with an appropriate photon noise generator.
As a result, the convergence speed of “no photon noise” case can be additionally improved by a factor about 2 (Fig. 10, 11). It is interesting to note, that the flux of 1-2 photons per speckle means that 1-2 photons fall within a sub-aperture of size $r_0$ allowing reasonable wavefront sensing with classical Hartmann method. It explains why the considered approach converges in the “photon noise” case and makes both approaches in some sense equivalent to each other.

C. Phase retrieval with DM introduced probes

In the previous sections we have shown how Gerchberg-Saxton approach can be used to restore random, dynamically changing pupil plane phases that satisfy Kolomogorov statistics. Particular statistical properties of those phases, however, were not used. It means that the same approach can be used for pupil plane phases with arbitrary statistics including the case that statistics are slowly changing between iterations. The key is maintaining a high enough sequential wavefront correlation for the algorithm convergence. So, if the rms of the random pupil plane phases asymptotically approaches 0, the Gerchberg-Saxton algorithm should also converge to zero phase or to the static aberration of the optical system if such an aberration is present. Since a particular wavefront statistics is not important, Model II will be used in all subsequent numerical simulations and laboratory experiments. We also will not apply anymore the procedure improvements using defocusing and “photonization” approaches discussed above. It results in lower converging speed of the algorithm and appearance of two possible conjugately inverted phase solutions $\phi(r)$ and $-\phi(-r)$ that will be transformed to the same orientation for comparison.

Numerical simulation results related restoration of static pupil plane wavefronts are presented in Fig. 12 and Fig. 13. In these runs the first sample (uncorrelated with other samples) of the used Model II sequence plays role of a static aberration of the optical system. This static aberration has been restored by using remaining correlated Model II samples. The phase amplitude of probes (rms of $\lambda/5$) was not changing during the initial iteration stage needed for the iterative solution to be captured by the dynamically changing global solution. After 500 iterations, as soon as the global solution was found, we start to reduce the amplitude gradually from iteration to iteration until it becomes negligible. It took 100 iterations to decrease the amplitude of the phase probes from the rms of $\lambda/5$ to the rms of $\lambda/500$ while keeping the phase restoration rms error at $\lambda/150$ level. As soon as the phase probe amplitude is reduced to the level of $\lambda/150$, the stagnation caused by the conjugately inverted solution becomes the main factor that restricts the algorithm convergence.

The achieved phase retrieval accuracy is appropriate for most of the optical applications, even taken into account that the stagnation cannot be overcome unless some focal plane defocus is introduced (Section 3A). This approach could be used to determine static aberrations for any optical system where random, correlated wavefront aberrations could be created with a DM, for example. Since particular random pupil plane phases do not need to be known during the wavefront retrieval, the procedure does not need any accurate calibration either of the DM or the rest of the optical system. All geometrical calibrations needed to match the pupil plane and focal plane wavefronts can be performed by only analysing the plane intensity measurements. As a result the procedure itself could be used as a powerful tool that provides the optical system calibration needed for other applications [6, 7, 40].

D. Pupil plane amplitude restoration

Apart from the pupil phase retrieval the considered random phase probe approach can be also used the determine the amplitude of the pupil wavefront. The simplest way that can provide both the phase and the amplitude estimates is to change pupil plane constrains in the fourth step of the phase restoration procedure (Section 3) as follows [23]
This local solution can be found by solving a linear optimization problem where the pupil plane amplitudes are varying to satisfy the phase distribution. There is the unique local pupil plane amplitude solution of the complex wavefront restoration problem. Faster phase convergence in comparison with amplitude convergence (meaning that the phase variability in iterations should prevail) should be provided to escape from the local amplitude minima. To decrease the amplitude variation a projective algorithm [41] with a feedback parameter (relaxation constant) $\alpha$ ($0 \leq \alpha \leq 1$) could be used instead of step 4a:

4b. The next pupil plane wavefront estimate is equal

$$E_{k+1}(u, t_{k+1}) = \begin{cases} G_k(u, t_{k+1}) = & |A_{k+1}(u, t_{k+1})| \exp[i\Phi_k(u, t_{k+1})] \\ & \text{inside of the pupil,} \\ 0 & \text{outside of the pupil,} \end{cases}$$

where

$$|A_{k+1}(u, t_{k+1})| = a|A_{k}(u, t_k)| + (1-a)|G_k(u, t_{k+1})|.$$
Fig. 13. The pupil plane amplitude retrieval for 4 different restoration scenarios:
a) the procedure 4a (Eq. 10) is used to recover the amplitude of dynamic wavefront (no amplitude and phase convergence);
b) after the initial iteration stage needed to capture the global phase solution, the projective algorithm 4b ($\alpha = 0.8$) was used to recover the amplitude of dynamic wavefront (phase convergence only can be achieved);
c) the projective algorithm 4b ($\alpha = 0.8$) is used to recover the amplitude of static wavefront with the preliminary estimated pupil plane phase (negligible errors of the phase solution result in an ambiguous amplitude solution);
d) switch to the projective algorithm 4b took place immediately after the initial iteration stage. Within next $\sim 1000$ iterations normalization $P$ of random phase probes was not changed. At the end of this step $P$ started gradually decrease to 0 providing convergence to the actual static pupil plane wavefront. On the contrary with the previous case (c), this scenario provides both phase and amplitude global convergence.

In each case the initial amplitude approximation is equal to constant across the pupil. The actual pupil plane amplitude (Eq. 12), the amplitude estimates, the feedback parameter ($\alpha$) and the normalization coefficient $P$ are shown for each convergence stage of considered scenarios. Each presented amplitude is accompanied with corresponding actual (top, left corner) and recovered (top, right corner) pupil plane phases. The same visualization parameters are applied for all the amplitudes presented in the same row.
absence of the convergence to the global phase/amplitude solution for small $a$, only local convergence could be observed;

- the convergence to the global solution when $a$ is close to 1;

- the pupil plane phase diversity is essential for the amplitude restoration. In absence of the diversity the algorithm converges to one of the local solution with negligible phase error, but and large amplitude error;

- the amplitude restoration loop converges very slow in comparison with pure phase restoration.

E. Additive background noise

In the case of a static aberration and in absence of noise the modification (4b) of the Gerchber-Saxton procedure converges to a solution $E(u) = |E(u)| \exp[i\Phi(u)]$ that includes not only pupil plane phase $\Phi(u)$ but also the pupil plane amplitude estimate $|E(u)|$. Unfortunately, both $\Phi(u)$ and $|E(u)|$ estimates are sensitive to the background image noise produced, for example, by the focal plane detector (Fig. 14).

The numerical wavefront retrieval simulations with noisy focal plane images (the noise level is about 1% measured relative to the average intensity of the brightest speckles) are presented in Fig. 15. A few observations related to the wavefront retrieval in presence of the background image noise should be noticed. First of all, the algorithm converges more slowly in comparison with the “no noise” case. Secondly, the recovered pupil phases are degraded by clearly visible white noise caused by the background image noise (Fig. 15). Finally, even low level background noise (as low as 1%) completely brakes the amplitude restoration (Fig. 15). At the same time the pupil phase estimates look reasonable even in the case “both amplitude and phase” restoration. The pupil phase estimate seems to be insensitive to the applied pupil amplitude.

An averaging method

$$ E(u) = |E(u)| \exp[i\Phi(u)] = \langle E_i(u) \rangle, \quad (13) $$

allows to improve the wavefront estimates degraded by the background noise. In Eq. 13 $E_i(u)$ is the i-th pupil plane wavefront estimate, $\Phi(u) = \arg\langle E_i(u) \rangle$, $|E(u)| = |\langle E_i(u) \rangle|$ and $\langle \ldots \rangle$ means averaging of wavefront estimates (the wavefront tip/tilts should be corrected before the averaging). The averaging can be applied as soon as the amplitude of random phase probes becomes negligible. It does not provide, however any wavefront improvement unless a simple image filtration procedure is applied (Fig. 15). In this procedure each of the used images should be filtered in the frequency domain by cutting all the frequencies beyond the pupil cut-off frequency (Fig. 14). In the case of the “phase only” restoration the filtering results in significant growth of the Strehl ratio even without wavefront averaging. In our simulations with the implemented filtering the Strehl ratio has been improved from 0.68 to 0.91 and reached 0.998 after averaging of 1000 pupil plane wavefronts. The similar improvement has been obtained for the pupil plane phase in the case of “both amplitude and phase” restoration. The noise filtering also improves the amplitude estimates (Fig. 15). However, it does not provide unbiased global amplitude convergence, that can be only
obtained by applying additional amplitude constraints. So, the background noise makes the pupil amplitude retrieval difficult and unpractical, especially taken into account low convergence speed of considered algorithms. It is interesting to note, that in presence of the background noise, the amplitude estimates sometimes show correlation with the most distinguished pupil plane phase features. This effect was noticed in the numerical simulations and was confirmed in the laboratory demonstration discussed in the next Section.

4. EXPERIMENTAL RESULTS

The main results of the numerical simulations have been confirmed in a few laboratory experiments that were performed at the NASA Ames Coronagraph Testbed [42]. During these experiments the phase and amplitude in the entrance pupil of the PIAA (Phase Induced Amplitude Apodization) coronagraph have been restored by using a set of random phase probes created by the deformable mirror located in the entrance pupil of the coronagraph.

A. Experiment description

With a few small differences the PIAA coronagraph setup at NASA Ames is similar to the setup used in the EXCEDE demonstration [43]. The front-end of this setup that feeds into the PIAA coronagraph was used in the wavefront retrieval experiments. The experiment optical layout is the following (Fig 16).

The spherical wave formed by the point source S1 (655nm laser) passes through the diffractive pupil [44] DP and is focused in the front-end focus F1 at the distance of 531 mm from the diffractive pupil. The distance between the point source and the diffractive pupil is equal to 473 mm. The diffractive pupil is a spherical mirror with the curvature radius of 500 mm that has a low frequency diffractive grating on its surface. The beam is collimated farther by an off-axis parabolic mirror OAP1, and is reflected by the fold mirror M1 and the deformable mirror DM. Finally, the collimated beam is focused by the second off-axis parabola OAP2 in the first focus of the PIAA coronagraph. The focal distances of the OAP1 and OAP2 are 305 mm and 127 mm respectively. The diffractive pupil is optically conjugated with the DM and they both are optically conjugated with the first PIAA mirror. A circular pupil stop PA with the diameter of 9.3 mm is used to restrict the beam size. The pupil stop is located up-stream of the DM at the minimal distance that prevents the beam vignetting.

The focal plane images are focused on the Basler acA3800-14um camera with the pixel size of 1.67 µm that is small enough to provide an appropriate image sampling (the system \( \lambda_f/D = 9.2 \) µm). The camera was operated in the video mode that provides 14 twelve-bit frames/sec. After averaging 100 frames the 600 × 600 px raw focal plane images clipped down to 512 × 512 px were used in the wavefront retrieval procedure. The averaging was needed to reduce the camera background noise and increase the camera dynamic range.

In our experiments the 32 × 32 Boston Micromachines MEMS DM was used. The image sampling provided a spatial resolution of 86 pixels across the pupil (≈2.5 pixels/DM actuator) as measured from the image power spectrum.

The fiducial pattern applied to the DM was the flat DM surface with three actuators poked in the same “positive” direction and one actuator poked in the opposite direction. The related DM voltage map [45] and focal plane image are presented in Fig. 17. The maximal amplitude of poked actuators reaches 130–140 nm that results in not more than 0.02 degradation in Strehl ratio (in comparison with the flat DM case).

The algorithm was run as follows:

1. The set of independent Gaussian phase screens with the correlation radius of 12 DM actuators was generated to produce the correlated random phase probes. The related DM voltage was calculated based on the simplest quadratic model of the DM deflection curve. The amplitude normalization of the probes was chosen experimentally such that the main lobe of the focal plane image (Fig. 17) had become completely destructed by created aberrations (Fig. 14). Note, that the random phase probes do not need any DM calibration to be produced.

2. The recovery procedure included three main steps.

In the initial iteration stage needed to capture the global phase solution number of partially correlated phase screens
between independent phase screens $N$ was equal to 200. During this stage the amplitude (normalization) of the random phase probes was not changed and the pupil amplitude was set to be a constant. As soon as the global phase solution was captured (after 500-700 iterations), the second step starts. To obtain the general phase and amplitude solution the amplitude/normalization of the random phase probes was gradually reduced (by factor 0.9 between two independent phase probes) until it became negligible (after 2000 iterations approximately). During this step $N$ was subsequently reduced to 100, 70 and 50 to speed-up convergence and the amplitude feedback coefficient $\alpha$ was varying between 0.9 and 0.99. In the third step the final phase/amplitude solution was obtained by averaging 1000 iterations.

**Phase retrieval**

The restored pupil plane phase/DM surface and pupil amplitude are presented in Fig. 17. The pupil amplitude estimate is very noisy (the rms of 15-20%) that is consistent the numerical simulation results. Although it looks homogeneous, some correlation of the estimate with the strongest pupil phase features (dead actuators) can be noticed. The recovered phase map has much more details. First of all, the poked actuators are clearly recognizable in the correct location and polarity. The recovered actuator poked amplitudes match the commanded signal (see discussion in Section 4C). Ignoring small tip/tilt and defocus terms, the pupil phase looks flat except for the right edge of the pupil where the phase growth of 0.7\(\lambda\) is observed and two dead actuators marked as “5” and “6” in Fig. 17 and 18. There are also a few DM areas with the phase depression possibly caused by the DM flattening errors or optical aberrations produced by other optical elements. Finally, the poorly recognizable vertical and horizontal phase structures are likely caused by the quilting pattern formed by DM actuators [46].

All the above features, except the DM quilting pattern, are highly repeatable and consistent between different wavefront retrieval runs. The periodic structure of the DM surface cannot be firmly recognized because the spatial resolution of the obtained pupil phase distributions is not high enough to resolve it. The pupil phase map resolution is limited by two following factors:

1. the applied pupil plane and focal plane sampling;
2. the dynamic range of the focal plane images that is limited by the camera.

To allow restoration of the amplitude and location the periodic DM grating the dynamic range of the camera should be large enough to linearly detect the weak focal plane speckles with contrast of a few times $10^{-5}$ near the DM diffractive orders (contrast of $10^{-3}$). These speckles are the most sensitive to the spatial shift of the of the DM grating. To expand the dynamic range of the images a high dynamic imaging tool was implemented that allows to combine a set of images taken with different exposure times (each image is an average of 100 frames) in one image with the dynamic range of 5 orders of magnitude. We also doubled the pupil plane spatial resolution by increasing the size of collected images to $1200 \times 1200$ and clipping them down to $1024 \times 1024$. The final pupil plane phase estimate with resolution about 5 pixels/actuator (172 pixels across the pupil) is shown in Fig. 18 (two independently restored maps

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**Fig. 17.** Results of the laboratory demonstration. The restored amplitude and phase are shown together with a focal plane image and the DM applied voltage. The fiducial poked actuators (1, 2, 3, and 4) and the dead DM actuators (5 and 6) are marked. The restored flat DM phase (unaveraged) and the poked actuators – flat DM difference are also shown.

**Fig. 18.** Phase retrieval with implemented high dynamic range imaging. The periodic DM actuators grid is clearly visible both in the obtained phase maps and in the DM interferometric map measured with ZYGO interferometer [45]. The restored phase maps are presented for two independent retrieval runs. The difference between runs (tip/tilt is corrected) shows the systematic pattern in the restored maps. The periodic component of the systematic error can be reduced by shifting the second restored phase by (1,3) pixels. The following DM structures are marked in shown maps: poked actuators (1, 2, 3 and 4), dead actuators (5 and 6) the DM diffractive orders/periodic structure (a), OAPs produced vertical structures (b), the DM boundary and the pupil aperture. Marks "4", "5" and "6" are black because of negative phase, and marks "1", "2" and "3" are white because of positive phase.
are presented). In comparison with the obtained low resolution map, the DM surface grating with amplitude of about $\lambda/30$ nm ($\sim 20$ nm) is clearly visible in high resolution maps. The DM features are well-matched in both maps. The phase maps show noticeable tip/tilt between low and high resolution wavefronts caused by the long term instability of the optical setup. Note, that we did not use an image re-centering procedure, so the observed tip/tilt is simply caused by the changed location of the source.

C. Phase retrieval reliability

To determine reliability of the obtained solution two independent wavefront retrieval runs with the identical settings were performed (Fig. 18). Though both phase maps demonstrate good consistence, more detailed consideration shows the presence of systematic errors between them. Following is the analysis of the difference between these independent phase solutions:

1. Low order modes of two obtained phase maps are practically identical, except for tip/tilt differences between different trials.

2. The fiducial actuator pokes are well-matched both in location and amplitude. They are completely subtracted in the phase difference map (Fig. 18). The relative amplitudes of poked actuators (1) are consistent with the amplitude ratios obtained from the DM deflection curve measured with ZYGO interferometer [45]. This comparison assumes that deflection of the poked actuators can be described by the same quadratic deflection curve measured under reasonable suggestion that inter-actuator stroke effect [47] is the same for the poked actuators. Unfortunately, it was impossible to determine the actual amplitude of pokes. The poke amplitudes calculated as average of the DM deflection curve based values and the inter-actuator stroke estimates [47] are presented in Table 1.

3. Systematic pattern is seen in the phase difference that consists of two different components. The first component is a regular grid-like structure produced by the DM surface grating. Setting in one of the phase maps 1 pixel of horizontal shift and 3 pixel of vertical shift results in the DM grid artifact disappearing that suggest an ambiguity in the spatial location of the grid pattern.

4. The second component of the systematic pattern appears itself as irregular mainly vertical stripes across the pupil. Their presence can be explained by existence of some irregular vertical phase structure introduced by the optics and assuming some variability in the structure position caused, for example, by slow drift in the optical setup. The source of this structure seems to be the OAPs surface errors produced during diamond turning process. The diamond turning OAP errors are detectable in the reflected light at large incidence angles. They are also responsible for the higher brightness tilted horizontal band that is visible in the focal plane PSF (marked with “b” in Fig. 18).

5. The phase difference rms is around $\lambda/14$ and is dominated by the discussed systematic pattern.

The first identified systematic component of the phase error clear shows an ambiguity in the phase solution that should be discussed. This special kind ambiguity is observed when the focal plane image can be decomposed in two or more components that either do not interfere with each other or the interference terms are negligible. It happens, for example, in a practically interesting case when one of the components is produced by a multiplicative pupil plane phase grating. The diffractive orders formed by the grating interfere with background speckles produced by another component. However, in many cases, the brightness of the background speckles is negligible in comparison with the diffractive orders brightness, so the interference term can be considered as equal to zero. As a result, all phase solutions where the phase grating is arbitrarily shifted in the pupil plane produce undistinguished focal plane intensity distributions meaning that the phase retrieval have an infinite number of ambiguous solutions. The interferometric term, of course, is not equal to zero precisely. However, any optical instability or registration noise affect the algorithm convergence. As a result, the iteration converges to one of the solutions with an arbitrary location of the pupil plane phase grating. The problem is additionally complicated by factors such as:

- the conjugately inverted phase solutions;
- small amplitude of the DM diffractive peaks. The relative contrast of the DM diffraction peaks is less that 10$^{-3}$. All the peaks can reduce the Strehl ratio of the focal plane PSF not more than by 0.01-0.02. It means that higher than used correlation rate between sequential random phase probes and larger number of iterations is required to recover phase features responsible for such small Strehl degradation.

Despite high thermal and mechanical stability of the used optical setup [48], some long term relative drift of the optical elements is observed. This drift produces beam-walk that results in the pupil plane phase temporal variations affecting the phase retrieval algorithm. The effect is mainly responsible for the irregular vertical structure detected in the phase difference map.

Optical beams that form the PSF central lobe and the DM diffractive orders reflect from different parts of the OAP2. The OAP2 produces aberrations that are different for collimated beams that form the PSF central lobe and the DM diffractive orders. Together with the thermal and mechanical temporal instabilities these non-common path errors can affect the parameters of the recovered DM surface grid.

Finally, an inaccurate estimate of the pupil size can be responsible for slightly different location of the wavefronts recovered in different wavefront retrieval runs.

| Poked actuator | Actuator amplitude | Relative amplitude |
|----------------|--------------------|--------------------|
|                | restored           | applied             |
| 1              | 135 nm             | 173 nm             | 1.0 | 1.0 |
| 2              | 75 nm              | 102 nm             | 0.56 | 0.59 |
| 3              | 94 nm              | 133 nm             | 0.70 | 0.77 |
| 4              | -115 nm            | -153 nm            | -0.85 | -0.88 |

Table 1. Amplitude of poked actuators.
Fig. 19. Experimental comparison of the random phase probe and the standard Gerchberg-Saxton approaches. The pupil plane phase restored by using random phase probes (left) and Gerchberg-Saxton method with the constant (middle) and random (right) pupil plane initial phase are shown.

All these factors contribute in the final phase difference map, but it is very difficult to estimate any particular factor contribution. Though the amplitude of the DM produced phase grating matches the amplitude measured with ZYGO the relative location of the grating remains uncertain. This uncertainty can be solved with the discussed defocusing method.

It is also interesting to compare the obtained solution with the standard Gerchberg-Saxton phase retrieval solution based on the same laboratory data. In Fig. 19 results of such comparison are shown. The focal plane image shown in Fig. 18 has been used to estimate the pupil plane phase through a few independent Gerchberg-Saxton phase retrieval runs. Each run started with different initial pupil plane phase satisfied Model II statistics with rms varying from 0 (constant pupil plane phase) to 2\(\lambda\). In each case, the standard Gerchberg-Saxton algorithm demonstrate fast local convergence (after 40-50 first iterations, where the most significant changes occur, the algorithm stagnates near a local solution). None of obtained wavefronts is well consistent with poked actuators geometry and the solution obtained with the random phase probes approach. However, some similarity of results obtained with this two methods should be noted though.

5. DISCUSSION

The discussed random phase probes approach can be used to test any optical system, where random correlated phase aberrations can be produced. The location of these aberrations in the system and their shape seems to be not very important if they are located far enough from the focal plane of the system. Such aberrations can be also caused differently. For example, they can be introduced by the DM, slow air turbulence or convection with the amplitude that can be controlled, the temperature gradients in the system, random relative motion of the different optical elements, etc. As soon as the amplitude of the aberration creation process is approaching to 0, the convergence of the algorithm to the real system wavefront is expected. The method can be successful in the case both large (as large as a few wavelength) and small optical aberrations. The method is especially promising in the case when the optical system performance is limited by non-common path propagation errors. It could provide accurate, high resolution wavefront sensing that is not influenced by any additional optics aberrations and does not need any well calibrated camera travel along the focus direction needed for most phase diversity methods.

A short list of possible method applications includes:

1. **Optics testing and calibration.** The method can be used for any optical system where direct interferometric measurements are difficult or impossible. It could particularly interesting for testing either very large optics (like large telescope mirrors) where the random phase diversity is easily introduced by thermal gradients existing in the mirror or small optical elements with the random phase probes produced by the DM. The method can be also applied to measure the DM surface itself providing a powerful and accurate the DM calibration tool.

2. **Coronagraphy.** In the case of complex optical setups, both the system wavefront and the contribution of different parts of the system can be independently measured by putting additional imaging cameras in an appropriate system focus. As a result the complete, high-accuracy system model could be produced. Such a model creation is very important for direct imaging of Earth-like planets around nearby stellar systems that would allow the planets spectral measurements and characterization of planetary atmospheres. The direct imaging of exoplanets requires an extraordinary capability for the imaging system. This system should be able to detect a planet that is ten orders of magnitude fainter in comparison with the star and is completely indistinguishable from the stellar PSF diffraction wings. Current starlight suppression systems based on a combination of coronagraphy and wavefront control are able to reach the \(10^{-9} - 10^{-10}\) broadband imaging contrast needed for direct exoplanet studies. However, the used wavefront control algorithms, such as EFC [6], provide good local convergence only, assuming that the highly accurate model of the starlight suppression system is required to reach the necessary system performance. The random phase probes based algorithms could provide such a model. These algorithms are fast, very accurate and free from non-common path errors that can affect the system performance. They do not need any system reconfiguration to be used and can be running remotely to perform the system calibration with the existing optical setup during space missions.

3. **Adaptive Optics.** As soon as the pupil plane phase estimation is obtained, it could be used for adaptive wavefront correction in adaptive optics systems, particularly, to correct wavefront aberrations caused by atmosphere turbulence. It was shown above that even when the flux is as low as 1-2 photons/speckle the discussed approach allows the diffraction-limited wavefront correction with the Strehl ratio 0.2-0.5, assuming that the imaging frequency is high enough to provide 0.95 or higher correlation of the sequential focal plane images. Assuming the atmosphere coherence time of 20 ms (in visual spectral range) [38, 39, 49] the frequency imaging needed for the adaptive wavefront correction can be estimated as a few hundred - one thousand Hz. One more issue should be taken into account though. The Gerchberg-Saxton based wavefront estimates can be used only in the open loop applications. The phase discontinuities produced during the Gerchberg-Saxton loop break the algorithm convergence been applied for the close loop correction. To avoid such discontinuities the gradient descent approach (instead of the Gerchberg-Saxton) where the pupil phase continuity can be used to restrict possible phase solutions in the conditional optimization procedure.
Pupil phase (dynamical) Restored pupil phase Restoration errors

Pupil phase (static) Restored pupil phase Averaged pupil phase

Fig. 20. The pupil plane phase restoration for segmented apertures. The aperture consists of 15 different rectangular segments, each with random static piston (the piston rms of $\lambda/2$). The recovered wavefronts for the dynamically changing aberrations (a) and static aberration (b) are shown. In both cases, the correlated Model II sequence is used to simulate dynamically changing correlated pupil plane phases. In presence of 1% focal plane background noise (measured relative to the average intensity of the brightest speckles) the restored pupil phase rms ($\sigma$) is about $\lambda/20$ that allows the wavefront correction with Strehl ratio of about 0.92. The Strehl ratio reaches 0.993 after averaging 20 independent wavefront estimates (the static aberration case). To visualize phase restoration errors in the case of dynamically changing aberrations a bias of $3\sigma$ is added to the unwrapped error map meaning that the darkest and brightest pixels correspond to the $\pm 3\sigma$ levels.

4. Segmented apertures co-phasing. The random phase probe approach works equally well for both continuous and discontinuous pupil phase distributions. It makes it possible to use the discussed pupil plane wavefront reconstruction capability of the method to co-phase sub-apertures of big segmented mirrors [50]. In Fig. 20 random sub-aperture phases for a segmented pupil containing 9 segments are measured with the discussed approach. In the case when the segments are randomly moving relatively each other the procedure can provide the signal needed to dynamically correct the segment positions.

5. Dynamic processes study. The random phase probes approach can also be interesting in the case when the dynamical wavefront change is produced by the process itself. Additional advantages of this approach are the model independence of the method, the low method sensitivity to the photon noise and fast method convergence that could allow a reasonable solution even in the case when the available process statistics is very limited by the process duration for example. It could be applicable for wide class of applications ranged from the optical system state monitoring (important for newly developed space missions) to X-ray microscopy/tomography imaging.

6. CONCLUSION

In conclusion, we have proposed the random pupil phase probes approach that allows (in absence of noise) both pupil phase and amplitude retrieval assuming that the random probes provide sufficient sequential correlation of the focal plane images. The method is completely model independent excepting that an adequate propagation is considered. The method does not require particular probe shapes to be known and does not need any accurate calibration procedure to be used. Rather, it could provide good and reliable calibration of the optical system by itself.

In the case of phase retrieval considered approach shows low sensitivity to the photon noise and the registration noise. To increase accuracy the phase estimates an appropriate averaging procedure has been proposed. The considered algorithms provide fast and reliable pupil plane phase measurements for both large (of order a few $\lambda$) and small (optical surface structures as deep as 30-40 nm can be measured) phase aberrations. They work equally well in the case of both continuous and discontinuous pupil plane phase distributions and could be used to measure both static and dynamically changed pupil plane phases. Similarly to other focal plane wavefront sensing methods, the random phase probe estimates are not affected by non-common path errors and can be used by applications whose performance is limited by non-common path errors. The phase estimates appear to be insensitive to the pupil amplitude errors, that makes it possible to obtain a reliable phase solution even in the case when the related pupil amplitudes are unknown or very uncertain.

The phase retrieval solutions can be affected by systematic errors been applied to the periodical high frequency pupil plane structures which existence could not be efficiently broken by the applied phase probes. In this case an appropriate solution, probably, could be obtained by using slightly defocused focal plane images (it increases the related interferometric term) and switching to a gradient descent like optimization.

Due to phase discontinuities arising during the iterative loop, the Gerchberg-Saxton based phase retrieval algorithm could not be used in close loop applications. This limitation can be also relaxed by switching to a gradient descent like optimization where the continuity of the pupil phase can be used as a constraint for an appropriate phase solution.

On the contrary to the pupil phase solution, the related amplitude solution needs much more iterations to be reached. Even small registration noise is able to break the global convergence for the pupil amplitudes that makes the pupil amplitude measurements difficult and unpractical. However, similarly the pure phase solutions, a reliable amplitude solution could be probably obtained by switching to a constrained gradient descent like algorithm, where the appropriate amplitude constraints should be used to provide the algorithm convergence.

To be implemented the random phase probe method the following requirements should be met:

- the focal plane imaging procedure should provide an adequate image sampling;
- a possibility to create random pupil phase aberrations and to control their amplitudes with the DM, for example. Note, that the particular location of the DM is not very important. The DM can be placed anywhere in the optical system except the system focus.
- the imaging frequency should be high enough to keep high sequential correlation of the focal plane images in
the case when the random phase probes are creating by a rapid dynamical process like the atmosphere turbulence/convection.

– an appropriate propagation model should be used, that assumes that the procedure can be used in the case of Fresnel diffraction for example.

The described procedure can be implemented for a wide class of optical applications, including those that use the DM for adaptive wavefront correction.

Finally, the system noise not only limits the system performance, sometimes it helps to improve the performance.

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