Integrals of Motion in the Two Killing Vector Reduction of General Relativity

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Abstract

We apply the inverse scattering method to the midi-superspace models that are characterized by a two-parameter Abelian group of motions with two spacelike Killing vectors. We present a formulation that simplifies the construction of the soliton solutions of Belinskiĭ and Zakharov. Furthermore, it enables us to obtain the zero curvature formulation for these models. Using this, and imposing periodic boundary conditions corresponding to the Gowdy models when the spatial topology is a three torus $T^3$, we show that the equation of motion for the monodromy matrix is an evolution equation of the Heisenberg type. Consequently, the eigenvalues of the monodromy matrix are the generating functionals for the integrals of motion. Furthermore, we utilise a suitable formulation of the transition matrix to obtain explicit expressions for the integrals of motion. This involves recursion relations which arise in solving an equation of Riccati type. In the case when the two Killing vectors are hypersurface orthogonal the integrals of motion have a particularly simple form.
1. Introduction

The Einstein field equations for space-times admitting a two-dimensional Abelian group of isometries acting orthogonally and transitively on non-null orbits are non-linear partial differential equations in two variables. For timelike orbits the equations are elliptic, whereas for spacelike orbits the equations are hyperbolic [1]. Although the space-times admitting two commuting Killing vectors are not the most general, they can represent interesting physical situations with stationary axial symmetry, planar symmetry or cylindrical symmetry [2].

Since the pioneering work by Geroch [3], it has been known that the field equations in the stationary axisymmetric case admit an infinite dimensional group of symmetry transformations. This result has encouraged research in solution-generating methods, the idea being that the complete class of solutions can be generated from a particular solution, such as flat space [4]. Subsequently, several solution-generating techniques have been developed, such as the Kinnersley-Chitre transformations [5], the Hauser-Ernst formalism [6], Harrison’s Bäcklund transformations [7] and the Belinskiĭ and Zakharov inverse scattering method [8]. The relations between the different approaches were discussed by Cosgrove [4], with Kitchingham [9] adapting these methods to the hyperbolic case.

Our paper uses the framework of the Belinskiĭ and Zakharov inverse scattering method, for the case when the field equations are hyperbolic. The inverse scattering method is a powerful method for solving certain systems of non-linear partial differential equations. The main step in this formalism is to write down linear eigenvalue equations whose integrability conditions are the given non-linear system. The methods of functional analysis can be applied to generate new solutions of the linear system from old, and hence new solutions of the original system from old [10]. The particular solutions that can be generated are the soliton solutions. The soliton solutions share a number of common properties with classical particles, namely, they are localized solutions that propagate energy, have a particular velocity of propagation and some persistence of shape which is maintained even when two solitons collide [2]. As shown by Belinskiĭ and Zakharov, the soliton transformation needs to be generalized when applied in the context of the Einstein equations with a two-parameter Abelian group.
of motions. The generalization is that the stationary poles are substituted by the pole trajectories. We will present an equivalent formulation of the linearized system. Similarly to Belinskiǐ and Zakharov [8], our linearized system is defined with the use of two differential operators which involve derivatives with respect to the complex parameter $\lambda$. We then define a new complex parameter $\omega$, and show that this simplifies the linearized system. We also point out the properties of the map between the two complex parameters $\omega(t, z, \lambda)$ and $\lambda(t, z, \omega)$. We then show that our formulation yields soliton solutions equivalent to the original solitons of Belinskiǐ and Zakharov.

We next construct the zero curvature formulation for the system. The zero curvature formulation is an important characteristic of integrable systems. A direct consequence of the zero curvature formulation, for a given system, is the fact that, when periodic boundary conditions are imposed, the equation of motion for the monodromy matrix is an evolution equation of Heisenberg type [11]. Hence, the eigenvalues of the monodromy matrix are conserved. In the context of the two spacelike Killing vector reduction of general relativity, periodic boundary conditions amount to the compactification of the $z$ direction into a circle. This then corresponds to the Gowdy models when the spatial topology is a three torus $T^3$ [12].

The next step is to obtain a suitable parameterization for the transition matrix. To achieve this we have to solve a system of four partial differential equations. This problem reduces to one of solving two equations of Riccati type. We then look for solutions to the Riccati equations in the form of power series in $(\lambda \pm 1)$. The solutions are given through recursion relations. For integrable systems with fixed poles, the integrals of motion are then given by the coefficients in the Laurent expansion of the eigenvalues of the monodromy matrix around the poles $(\lambda \pm 1)$. However, in the case of the Einstein field equations for space-times admitting a two-dimensional Abelian group of isometries acting orthogonally and transitively, as we have remarked earlier, the fixed poles are substituted by the pole trajectories, i.e., $(\lambda(\omega, t, z) \pm 1)$. Due to the fact that $\lambda$ is time-dependent, i.e., $\partial_t \lambda(\omega, t, z) \neq 0$, we are not able to identify the "local" integrals of motion as the coefficients in the expansion of the eigenvalues of the transition matrix. Instead, "local" integrals of motion are given as the eigenvalues of the transition matrix for fixed values of the complex parameter $\omega$ in the domain in which all the relevant algebraic series converge uniformly. In the case when the Killing
vectors are hypersurface orthogonal the integrals of motion have a particularly simple form.

This paper is organized as follows. In section two we formulate the inverse scattering method as applied to the two Killing vector reduction of general relativity. This involves the feature that the derivatives defining the first-order formulation of the equations of motion also involve derivatives with respect to the spectral parameter. We show how this may be dealt with by defining a new complex parameter, and we discuss some properties of this map. We then show how the approach of Belinskiĭ and Zakharov may be adapted to our case, and thus we obtain the soliton solutions. In section three, we turn to the zero curvature representation of the equations of motion, using this to define a transition matrix in the usual way. The integrals of motion are then seen to be given in terms of the related monodromy matrix. In section four, we utilise a suitable formulation of the transition matrix to obtain explicit expressions for the integrals of motion. This involves recursion relations which arise in solving an equation of Riccati type. Finally, in section five we present our conclusions.

2. The Inverse Scattering Method and the Soliton Solutions

We will consider the midi-superspace models that are characterized by the existence of a two-parameter Abelian group of motions with two spacelike Killing vectors (the case when one Killing vector is timelike and the other spacelike is similar and we will not consider it separately). Let us choose coordinates adapted to the action of the symmetry group so that the metric assumes the following form [8]

\[
ds^2 = -f dt^2 + f dz^2 + g_{ab} dx^a dx^b,
\]

where \(a, b = 1, 2\), \(\{x^0, x^1, x^2, x^3\} = \{t, x, y, z\}\), \(f\) is a positive function and \(g_{ab}\) is a symmetric two-by-two matrix. The function \(f\) and the matrix \(g_{ab}\) depend only on the co-ordinates \(\{t, z\}\), or equivalently on the null co-ordinates \(\{\xi, \eta\} = \{\frac{1}{2}(z+t), \frac{1}{2}(z-t)\}\). There is a freedom to perform the co-ordinate transformations

\[
\{\xi, \eta\} \rightarrow \{\tilde{\xi}(\xi), \tilde{\eta}(\eta)\}.
\]

It is easy to see that the transformations (2.2) preserve both the conformally flat two-metric \(f(-dt^2 + dz^2)\) and the positivity of the function \(f\) if \(\partial_{\xi} \tilde{\xi} \partial_{\eta} \tilde{\eta} > 0\).
The complete set of vacuum Einstein equations for the metric (2.1) decomposes into two groups of equations [8]. The first group determines the matrix $g_{ab}$ and can be written as a single matrix equation

$$
\partial_\eta (\alpha \partial_\xi g^{-1}) + \partial_\xi (\alpha \partial_\eta g^{-1}) = 0,
$$

where $\alpha^2 = \det g$ and $\{\xi, \eta\}$ are the null co-ordinates. The second group of equations determines the function $f(\xi, \eta)$ in terms of a given solution of (2.3):

$$
\begin{align*}
\partial_\xi (\ln f) &= \frac{\partial^2 (\ln \alpha)}{\partial \xi (\ln \alpha)} + \frac{1}{4 \alpha \alpha_\xi} Tr A^2, \\
\partial_\eta (\ln f) &= \frac{\partial^2 (\ln \alpha)}{\partial \eta (\ln \alpha)} + \frac{1}{4 \alpha \alpha_\eta} Tr B^2,
\end{align*}
$$

where $\alpha_\xi = \partial_\xi \alpha$, $\alpha_\eta = \partial_\eta \alpha$ and the matrices $A$ and $B$ are defined by

$$
A = -\alpha \partial_\xi g^{-1}, \quad B = \alpha \partial_\eta g^{-1}.
$$

The dynamics of the system is thus essentially determined by eqn. (2.3) and for this reason we will concentrate on it in the following.

Taking the trace of eqn. (2.3) and using the definition for $\alpha$, we obtain

$$
\alpha_\xi \eta = 0.
$$

The two independent solutions of this equation are

$$
\begin{align*}
\alpha &= a(\xi) + b(\eta), \\
\beta &= a(\xi) - b(\eta).
\end{align*}
$$

Using the transformations (2.2), one can bring the functions $a(\xi)$ and $b(\eta)$ to a prescribed form. However, we will consider the general form without specifying the functions $a(\xi)$ and $b(\eta)$ in advance.

Let us now consider the system of equations

$$
\begin{align*}
\nabla_\eta A + \nabla_\xi B &= 0, \\
\partial_\eta A - \partial_\xi B + [A, B] &= 0,
\end{align*}
$$

where $A$ and $B$ are the matrices previously defined.
where $\nabla_\xi = \partial_\xi + \alpha \alpha^{-1}$, $\nabla_\eta = \partial_\eta + \alpha_\eta \alpha^{-1}$ and $[\ , \ ]$ denotes the commutator in the Lie algebra of the group GL(2,R). The general solution of the equation (2.8b) is of the form

$$A = \partial_\xi ll^{-1}, \quad B = \partial_\eta ll^{-1},$$

(2.9)

where $l$ is an element of the group $GL(2, R)$. In addition we impose the constraint $\alpha^2 = \det l$. Eqn. (2.6) follows from the trace of eqn. (2.8a), once we substitute eqn. (2.9) into (2.8a). However, we still have more degrees of freedom in eqn. (2.8a) than in (2.3). We thus need to impose some additional conditions in order to recover the correct number of degrees of freedom. For this reason we impose the constraint $l = l^\tau$, where $l^\tau$ is the transpose of the matrix $l$. It then follows, once the additional constraints are imposed, that eqns. (2.8) are equivalent to eqn. (2.3). Due to the constraint $l = l^\tau$, the matrices $A, B$ in eqn. (2.9) must be taken to be only those which can be written in this form using a symmetric matrix $l$. The sets of such matrices $A, B$ form subsets (but not subgroups) of $GL(2, R)$.

The crucial step in the inverse scattering method is to define the linearized system whose integrability conditions are the equations of interest, in our case eqns. (2.8). Following ref. [8], we first define the two differential operators

$$D_\xi = \partial_\xi - \frac{\alpha_\xi}{\alpha} \frac{\lambda + 1}{\lambda - 1} \lambda \partial_\lambda,$$  \hspace{1cm} (2.10a)

$$D_\eta = \partial_\eta - \frac{\alpha_\eta}{\alpha} \frac{\lambda - 1}{\lambda + 1} \lambda \partial_\lambda,$$  \hspace{1cm} (2.10b)

where $\lambda$ is a complex parameter independent of the co-ordinates $\{\xi, \eta\}$. Notice that $D_\xi$ and $D_\eta$ are invariant under the coordinate transformation $\{\lambda, \xi, \eta\} \rightarrow \{\frac{1}{\lambda}, \xi, \eta\}$. It is straightforward to see that the differential operators $D_\xi$ and $D_\eta$ commute since $\alpha$ satisfies the wave equation (2.6)

$$[D_\xi, D_\eta] = \frac{\alpha_\xi \eta}{\alpha} \left( \frac{\lambda + 1}{\lambda - 1} - \frac{\lambda - 1}{\lambda + 1} \right) \lambda \partial_\lambda = 0.$$  \hspace{1cm} (2.11)

The next step is to consider the following linear system

$$D_\xi \psi = - \frac{A}{\lambda - 1} \psi,$$  \hspace{1cm} (2.12a)

$$D_\eta \psi = \frac{B}{\lambda + 1} \psi,$$  \hspace{1cm} (2.12b)
where $\psi(\lambda, \xi, \eta)$ is a complex matrix function, and the real matrices $A$, $B$ and the real function $\alpha$ do not depend on the complex parameter $\lambda$. The integrability conditions for the system (2.12) are eqns. (2.8). To prove this we first apply the operator $D_\eta$ to eqn. (2.12a) and subtract from this the result of applying the operator $D_\xi$ to eqn. (2.12b). The left-hand side vanishes using eqn. (2.11) and the right-hand side is a rational function of $\lambda$ which vanishes if eqns. (2.8) are satisfied. In order to take into account the additional constraints that we have imposed we require that

$$\bar{\psi}(\bar{\lambda}) = \psi(\lambda), \quad g = \psi\left(\frac{1}{\lambda}\right)\psi^\tau(\lambda), \quad (2.13a)$$

where $\bar{\lambda}$ is the complex conjugate to $\lambda$, $g(\xi, \eta)$ is a symmetric two-by-two matrix of functions, whose determinant is $\alpha^2$, and $\psi(\lambda)$ satisfies eqns. (2.12). The condition (2.13b) implies that $A$ and $B$ have the correct number of degrees of freedom. To see this, we apply the $D_\xi$ operator to equation (2.13b). Using the fact that $\psi$ satisfies eqn. (2.12a) we obtain

$$\partial_\xi g = \frac{1}{\lambda - 1}\left(\lambda A g - g A^\tau\right). \quad (2.14a)$$

Similarly, using (2.12b), we obtain

$$\partial_\eta g = \frac{1}{\lambda + 1}\left(\lambda B g + g B^\tau\right). \quad (2.14b)$$

Taking the transposes of the right-hand sides of equations (2.14), we deduce that $Ag = gA^\tau$ and $Bg = gB^\tau$. Consequently, $A$ and $B$ have the form

$$A = \partial_\xi g g^{-1}, \quad B = \partial_\eta g g^{-1}. \quad (2.15)$$

The standard application of the inverse scattering method to field theories in (1+1)-dimensions differs from the present situation in that the linearized system does not usually involve differentiation with respect to the spectral parameter $\lambda$, as occurs here. The present situation can be improved by defining a new complex parameter $\omega$ by

$$\omega = \frac{1}{2} \left(\frac{\alpha}{\lambda} + 2\beta + \alpha \lambda\right). \quad (2.16)$$
A straightforward calculation shows that $D_\xi \omega = D_\eta \omega = 0$. Furthermore, by making a co-ordinate transformation $\{\lambda, \xi, \eta\} \to \{\omega, \xi, \eta\}$ we reduce $D_\xi$ and $D_\eta$ to $\partial_\xi$ and $\partial_\eta$ respectively. In order to write the linearized system (2.12) in the new co-ordinates we have to invert the relation (2.16), i.e., we have to know $\lambda$ as a function of $(\omega, \xi, \eta)$. The inverse of (2.16) is not unique - we encounter the following two solutions for $\lambda$:

$$
\lambda_\pm = \left(\frac{\omega - \beta}{\alpha}\right) \pm \left(\left(\frac{\omega - \beta}{\alpha}\right)^2 - 1\right)^{\frac{1}{2}}.
$$

At this point, we have a choice of two different co-ordinate systems, the first corresponding to $\lambda_+$ and the second to $\lambda_-$. We can now write the linearized system in the new co-ordinates as

$$
\partial_\xi \psi = -\frac{A}{\lambda_\pm - 1} \psi,
$$

$$
\partial_\eta \psi = \frac{B}{\lambda_\pm + 1} \psi,
$$

where $\psi(\omega, \xi, \eta)$ and $\lambda_\pm$ is given by (2.17). It is straightforward to check that the integrability conditions for this system are equations (2.8). To show this it is necessary to use equations (2.7) and (2.17).

Let us point out some properties of the map (2.16). The transformation $\lambda \to \frac{1}{\lambda}$ leaves (2.16) invariant. The $\lambda$ plane is mapped into the two-sheeted Riemann surface which covers the entire $\lambda$ plane with the branch points at $\omega = \beta \pm \alpha$. The map (2.16) takes the circle $|\lambda| = \rho$, into the ellipse $C_\rho$ given by the parametric equation

$$
u = \frac{\alpha}{2} \left(\rho + \frac{1}{\rho}\right) \cos \phi + \beta,$$

$$
u = \frac{\alpha}{2} \left(\rho - \frac{1}{\rho}\right) \sin \phi,$$

where $\omega = u + iv$ and $\phi$ is the phase of $\lambda$, i.e., $\lambda = re^{i\phi}$. In particular, the image of the circle $|\lambda| = 1$ is a closed interval on the real axis in the $\omega$ plane. From our previous discussion it follows that the two formulations, the first defined by $(\lambda, \xi, \eta)$ and the linearized system (2.12), and the second defined by $(\omega, \xi, \eta)$ and the linearized system (2.18) are completely equivalent. In what follows we will use both formulations.

It is straightforward to obtain the soliton solutions following the inverse scattering method [8]. We begin by setting $\psi|_{\lambda=\pm \infty} = 1$. Then, from eqn. (2.13) it follows that
\( g(\xi, \eta) = \psi(0, \xi, \eta) \). In the inverse scattering approach we start with a given solution \( g_0(\xi, \eta) \) of eqns. (2.3). From eqn. (2.15) we then determine \( A_0(\xi, \eta) \) and \( B_0(\xi, \eta) \). Substituting \( A_0(\xi, \eta) \) and \( B_0(\xi, \eta) \) into eqn. (2.12) and solving the linearized system we obtain \( \psi_0(\lambda, \xi, \eta) \), with \( g_0(\xi, \eta) = \psi_0(0, \xi, \eta) \). We now define \( \chi \) as

\[
\psi(\lambda, \xi, \eta) = \chi(\lambda, \xi, \eta) \psi_0(\lambda, \xi, \eta).
\]  

(2.20)

The linearized eqns. for \( \chi \), from eqns. (2.12), are

\[
D_\xi \chi = \frac{1}{\lambda - 1} \left(-A \chi + \chi A_0\right),
\]

(2.21a)

\[
D_\eta \chi = \frac{1}{\lambda + 1} \left(B \chi - \chi B_0\right).
\]

(2.21b)

We also have the constraint on \( \chi \), from eqns. (2.13),

\[
g(\xi, \eta) = \chi(\frac{1}{\lambda}, \xi, \eta) g_0(\xi, \eta) \chi^\tau(\lambda, \xi, \eta).
\]

(2.22)

Setting \( \chi |_{\lambda = \pm \infty} = I \) in (2.22) it follows that

\[
g(\xi, \eta) = \chi(0, \xi, \eta) g_0(\xi, \eta).
\]

(2.23)

The soliton solutions are characterized by the points in the \( \lambda \) plane at which the determinant of \( \chi \) is equal to zero and \( \chi^{-1} \) has a simple pole, and similarly, the points at which the determinant of \( \chi^{-1} \) is equal to zero and \( \chi \) has a simple pole [8]. Thus, \( \chi \) and \( \chi^{-1} \) are rational matrix functions of \( \lambda \) with a finite number of simple poles

\[
\chi = I + \sum_{k=1}^{N} \left( \frac{R_k}{\lambda - \mu_k} + \frac{\bar{R}_k}{\lambda - \bar{\mu}_k} \right),
\]

(2.24a)

\[
\chi^{-1} = I + \sum_{k=1}^{N} \left( \frac{S_k}{\lambda - \nu_k} + \frac{\bar{S}_k}{\lambda - \bar{\nu}_k} \right),
\]

(2.24b)

where the matrices \( R_k \) and \( S_k \) do not depend on \( \lambda \), and \( \bar{\mu}_k \) is the complex conjugate of \( \mu_k \), and \( \bar{\nu}_k \) of \( \nu_k \). From eqn. (2.24a), it is easy to check that the condition \( \chi |_{\lambda = \pm \infty} = I \) is satisfied. Also, from eqn. (2.22) it follows that the positions of the poles of \( \chi \) and

\footnote{Integration of the linearized system is straightforward in the case when the metric \( g_0(\xi, \eta) \) is diagonal [13]. However, when \( g_0(\xi, \eta) \) is non-diagonal this step is non-trivial, an example of this being the case of the Bianchi type II models [14,15].}
the poles of $\chi^{-1}$ are related by $\mu_k \nu_k = 1$, $k = 1, ..., N$. This implies, using eqn. (2.16),
that $\omega_k = \frac{1}{2}\left(\frac{\alpha}{\mu_k} + 2\beta + \alpha \mu_k\right) = \frac{1}{2}\left(\frac{\alpha}{\nu_k} + 2\beta + \alpha \nu_k\right)$, $k = 1, ..., N$.

The next step is to rewrite eqns. (2.21) in the more convenient form

$$\begin{align*}
-\frac{A}{\lambda - 1} &= \left(D_\xi \chi\right) \chi^{-1} - \chi \frac{A_0}{\lambda - 1} \chi^{-1}, \\
\frac{B}{\lambda + 1} &= \left(D_\eta \chi\right) \chi^{-1} + \chi \frac{B_0}{\lambda + 1} \chi^{-1}.
\end{align*}
$$

(2.25a)  (2.25b)

Substituting eqn. (2.24) into (2.25) and setting the residuals at the poles $\lambda = \mu_k$ of the right-hand-side equal to zero, we obtain a set of $N$ first order equations for the matrices $\mathcal{R}_k$, $k = 1, ..., N$. Similarly, from the identity $\chi \chi^{-1} = I$ we obtain the system of $N$ algebraic equations

$$\mathcal{R}_k \chi^{-1}(\mu_k) = 0,$n

(2.26a)

and from eqn. (2.22) we obtain another system of $N$ algebraic equations

$$\mathcal{R}_k g_0 \left(1 + \sum_{l=1}^{N} \left(\frac{\mathcal{R}_l^\tau}{\nu_k - \mu_l} + \frac{\bar{\mathcal{R}}_l^\tau}{\nu_k - \bar{\mu}_l}\right)\right) = 0,$n

(2.26b)

where we have used the identity $\mu_k \nu_k = 1$. We also have similar equations for the matrices $\bar{\mathcal{R}}_k$ and the poles $\lambda = \bar{\mu}_k$, $k = 1, ..., N$. This yields the $n$-soliton solutions equivalent to those of Zakharov and Belinskiĭ, [8]. In addition, equations (2.4), which determine the function $f$, can then be integrated explicitly following ref. [8,16]. We will not present these results here. Instead, we will turn to the study of the zero curvature formulation and the integrals of motion.

3. The Zero Curvature Formulation and The Integrals of Motion

We now construct the zero curvature formulation for equations (2.8) and, using the techniques of ref. [11] in the case when the $z$ direction is compactified into a circle $S^1$, obtain the generating functional for the integrals of motion. Our first step is to perform a co-ordinate transformation $\{\xi, \eta\} \rightarrow \{t, z\}$ and define $U(t, z, \lambda)$ and $V(t, z, \lambda)$ by

$$\begin{align*}
U(t, z, \lambda) &= \frac{1}{2} \left(-\frac{A}{\lambda - 1} + \frac{B}{\lambda + 1}\right), \\
V(t, z, \lambda) &= \frac{1}{2} \left(-\frac{A}{\lambda - 1} - \frac{B}{\lambda + 1}\right).
\end{align*}
$$

(3.1a)  (3.1b)
The equation of motion for $U$ and $V$ is

$$D_t U - D_z V + [U, V] = 0, \quad (3.2)$$

where $D_t = \frac{1}{2} \left( D_\xi - D_\eta \right)$, $D_z = \frac{1}{2} \left( D_\xi + D_\eta \right)$, with $D_\xi$ and $D_\eta$ given by eqn. (2.10). Substituting eqn. (3.1) into (3.2), it is straightforward to show that eqn. (3.2) is equivalent to eqns. (2.8).

The fields $U(t, z, \omega)$ and $V(t, z, \omega)$ are expressed, in the co-ordinates $\{t, z, \omega\}$, by the formulae that can be obtained from (3.1) by the substitution $\lambda \rightarrow \lambda_+(t, z, \omega)$, where $\lambda_+(t, z, \omega)$ is defined in (2.17)\(^2\). Consequently, the equation of motion in this co-ordinate system is the zero curvature equation, i.e., eqn. (3.2) with the substitution $D_t \rightarrow \partial_t$ and $D_z \rightarrow \partial_z$. We will consider the case when periodic boundary conditions are imposed

$$U(t, z + 2L, \omega) = U(t, z, \omega), \quad (3.3a)$$
$$V(t, z + 2L, \omega) = V(t, z, \omega). \quad (3.3b)$$

Notice that we impose periodic boundary conditions on all fields, including $\alpha$ and $\beta$. These boundary conditions correspond to compactifying the $z$ direction into a circle $S^1$. For example, this corresponds to the Gowdy models when the spatial topology is a three torus $T^3$ [12]. We consider the transition matrix

$$T(t, z_1, z_0, \omega) = P \exp \int_{z_0}^{z_1} U(t, z, \omega) \, dz. \quad (3.4)$$

The transition matrix satisfies

$$\partial_{z_1} T(t, z_1, z_0, \omega) = U(t, z_1, \omega) T(t, z_1, z_0, \omega), \quad (3.5)$$

with the condition

$$T(t, z_1, z_0, \omega) \bigg|_{z_0 = z_1} = I. \quad (3.6)$$

We apply $\partial_t$ to eqn. (3.5), obtaining

$$\partial_t (\partial_z T) = \partial_t (U) T + U \partial_t (T). \quad (3.7)$$

\(^2\) From hereon we will restrict ourselves to the plus sign in eqn. (2.17).
Using the equations of motion for \( \{U, V\} \), the zero curvature equation (3.7) can be written

\[
\partial_z (\partial_t T - VT) = U (\partial_t T - VT) .
\] (3.8)

It follows from (3.5) that

\[
\partial_t T(t, z_1, z_0, \omega) = V(t, z_1, \omega) T(t, z_1, z_0, \omega) \\
+ T(t, z_1, z_0, \omega) C(t, z_0, \omega),
\] (3.9)

and, using the condition (3.6), we get \( C(t, z_0, \omega) = -V(t, z_0, \omega) \). Thus the equation of motion for the transition matrix is

\[
\partial_t T(t, z_1, z_0, \omega) = V(t, z_1, \omega) T(t, z_1, z_0, \omega) \\
- T(t, z_1, z_0, \omega) V(t, z_0, \omega),
\] (3.10)

We now define the monodromy matrix to be the transition matrix along the fundamental domain \(-L \leq z \leq L\), i.e.,

\[
T_L(t, \omega) = T(t, L, -L, \omega).
\] (3.11)

From eqn. (3.10), using the periodic boundary conditions (3.3), it follows that the equation of motion for the monodromy matrix \( T_L(t, \omega) \) is an evolution equation of Heisenberg type

\[
\partial_t T_L(t, \omega) = [V(t, L, \omega), T_L(t, \omega)].
\] (3.12)

This implies that the eigenvalues of the monodromy matrix \( T_L(t, \omega) \) are conserved, or equivalently

\[
\partial_t \text{Tr} T_L(t, \omega) = 0,
\] (3.13a)

\[
\partial_t \text{Tr} (T_L)^2(t, \omega) = 0.
\] (3.13b)

Our conclusion is that the functions

\[
E_L(\omega) = \text{Tr} T_L(t, \omega),
\] (3.14a)

\[
F_L(\omega) = \text{Tr} (T_L)^2(t, \omega),
\] (3.14b)
are the generating functions for the conservation laws. This result is a direct consequence of the zero curvature formulation and the periodic boundary conditions as shown above. Similar results are known for most if not all of the dynamical systems that admit a zero curvature formulation [11]. The standard method to obtain the explicit expressions for the integrals of motion involves solving the equations of Riccati type. The integrals of motion are then identified as the coefficients in the Laurent expansions of the generating functions. However, as we will show in the next section, in our case the generating functions (3.14) depend on the complex parameter \( \omega \) through \( \lambda_{\pm}(t, z, \omega) \). Consequently, we cannot identify the integrals of motion as the coefficients in the expansion of the generating functions. Instead, our integrals of motion are given by algebraic series, for those values of \( \omega \) for which the series converge uniformly.

4. The Integrals of Motion

We can now obtain explicit expressions for the integrals of motion. We first write the transition matrix in the form

\[
T(t, z_1, z_0, \omega) = \left( I + W(t, z_1, \omega) \right) e^{Z(t, z_1, z_0, \omega)} \left( I + W(t, z_0, \omega) \right)^{-1},
\]

where \( I \) is the identity two-by-two matrix, \( W(t, z, \omega) \) is an off-diagonal matrix and \( Z(t, z_1, z_0, \omega) \) is a diagonal matrix [11]. Substituting eqn. (4.1) into (3.5) we obtain the following system of equations

\[
\begin{align*}
\partial_{z_1} Z^{(1)} - \frac{1}{2} \left( U^{(3)} W^{(4)} + U^{(4)} W^{(3)} \right) - U^{(1)} &= 0, \\
\partial_{z_1} Z^{(2)} - \frac{1}{2} \left( U^{(3)} W^{(4)} - U^{(4)} W^{(3)} \right) - U^{(2)} &= 0,
\end{align*}
\]

\[
\begin{align*}
\partial_{z_1} W^{(3)} + W^{(3)} \left( \partial_{z_1} Z^{(1)} - \partial_{z_1} Z^{(2)} \right) - \left( U^{(1)} + U^{(2)} \right) W^{(3)} - U^{(3)} &= 0, \\
\partial_{z_1} W^{(4)} + W^{(4)} \left( \partial_{z_1} Z^{(1)} + \partial_{z_1} Z^{(2)} \right) - \left( U^{(1)} - U^{(2)} \right) W^{(4)} - U^{(4)} &= 0.
\end{align*}
\]

Here we have used the notation \( Z = Z^{(1)} \tau_{(1)} + Z^{(2)} \tau_{(2)} \), \( W = W^{(3)} \tau_{(3)} + W^{(4)} \tau_{(4)} \) and \( U = \sum_{i=1}^{4} U^{(i)} \tau_{(i)} \). Our choice for a basis in the Lie algebra of the group GL(2,R) is \( \tau_{(1)} = I, \tau_{(2)} = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \tau_{(3)} = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \) and \( \tau_{(4)} = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \).

Substituting eqns. (4.2a,b) into eqns. (4.2c,d), we obtain the system of equations that determines \( W(t, z_1, \omega) \)

\[
\partial_{z_1} W^{(3)} + U^{(4)} \left( W^{(3)} \right)^2 - 2U^{(2)} W^{(3)} - U^{(3)} = 0,
\]
\[ \partial_{z_1} W^{(4)} + U^{(3)} \left( W^{(4)} \right)^2 + 2U^{(2)} W^{(4)} - U^{(4)} = 0, \quad (4.3b) \]

with periodic boundary conditions on \( W(t, z_1, \omega) \), i.e., \( W(t, z_1, \omega) = W(t, z_1 + 2L, \omega) \).

Once we have solved eqns. \((4.3)\) for \( W(t, z_1, \omega) \), eqns. \((4.2a,b)\), together with the boundary condition \( Z(t, z_1, z_0, \omega) \big|_{z_1=z_0} = 0 \), determine \( Z(t, z_1, z_0, \omega) \):

\[
Z^{(1)}(t, z_1, z_0, \omega) = \int_{z_0}^{z_1} dz \left( U^{(1)} + \frac{1}{2} \left( U^{(3)} W^{(4)} + U^{(4)} W^{(3)} \right) \right), \quad (4.4a)
\]

\[
Z^{(2)}(t, z_1, z_0, \omega) = \int_{z_0}^{z_1} dz \left( U^{(2)} + \frac{1}{2} \left( U^{(3)} W^{(4)} - U^{(4)} W^{(3)} \right) \right). \quad (4.4b)
\]

The main problem is thus to obtain the solutions to the system \((4.3)\). These equations are of Riccati type. Given an equation of Riccati type with arbitrary coefficients together with a particular solution, it is possible to reduce the equation to a linear first order system \([17]\). However, we do not have particular solutions to equations \((4.3)\), so we need a different approach. We make a coordinate transformation \( \{t, z_1, \omega\} \rightarrow \{t, z_1, \lambda\} \) and as a result we obtain the equations \((4.3)\) in the form

\[
D_{z_1} W^{(3)} + U^{(4)} \left( W^{(3)} \right)^2 - 2U^{(2)} W^{(3)} - U^{(3)} = 0, \quad (4.5a)
\]

\[
D_{z_1} W^{(4)} + U^{(3)} \left( W^{(4)} \right)^2 + 2U^{(2)} W^{(4)} - U^{(4)} = 0, \quad (4.5b)
\]

We now expand the fields \( W^{(3)} \) and \( W^{(4)} \) as power series in \( \lambda - 1 \)

\[
W^{(3)} = \sum_{n=0}^{\infty} W^{(3)}_n (\lambda - 1)^n, \quad W^{(4)} = \sum_{n=0}^{\infty} W^{(4)}_n (\lambda - 1)^n. \quad (4.6)
\]

Substituting the first equation of \((4.6)\) and \((3.1a)\) into \((4.5a)\) and using the expansion

\[
\frac{1}{1+\lambda} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n (\lambda - 1)^n
\]

we obtain the recursion relation \((N = 0, 1, 2, \ldots)\)

\[
(N + 2) \frac{\alpha}{\alpha} W^{(3)}_{N+2} = \partial_{z_1} W^{(3)}_N - \frac{A^{(4)}}{2} \sum_{n=0}^{N+1} W^{(3)}_n W^{(3)}_{N+1-n} + \frac{B^{(4)}}{4} \sum_{m=0}^{N} \left(-\frac{1}{2}\right)^m \times
\]

\[
\sum_{n=0}^{N-m} W^{(3)}_n W^{(3)}_{N-m-n} - \left( \frac{3}{2} (N+1) \frac{\alpha}{\alpha} - A^{(2)} \right) W^{(3)}_{N+1} - \frac{N}{2} \frac{\alpha}{\alpha} W^{(3)}_N - \frac{1}{4} \sum_{n=0}^{N} \left(-\frac{1}{2}\right)^n \times
\]

\[
\left( \frac{\alpha}{\alpha} (N-n) + 2B^{(2)} \right) W^{(3)}_{N-n} - \frac{\alpha}{4\alpha} \sum_{n=1}^{N} \left(-\frac{1}{2}\right)^{n-1} (N-n) W^{(3)}_{N-n} + \frac{B^{(3)}}{2} \left(-\frac{1}{2}\right)^{N+1} \quad (4.7)
\]

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together with
\[
\frac{\alpha_5}{\alpha} W_1^{(3)} = -\frac{A^{(4)}}{2} \left( W_0^{(3)} \right)^2 + A^{(2)} W_0^{(3)} + \frac{A^{(3)}}{2}. \tag{4.8}
\]

Notice that \( W_0^{(3)} \) is arbitrary, and for every choice of \( W_0^{(3)} \) we have a different solution. From eqn. (4.5b), we obtain similar equations determining the \( W_n^{(4)} \), which may be obtained from eqns. (4.7) and (4.8) by the replacements \( W_n^{(3)} \rightarrow W_n^{(4)} \), \( A^{(4)} \leftrightarrow A^{(3)} \), \( B^{(4)} \leftrightarrow B^{(3)} \), \( A^{(2)} \rightarrow -A^{(2)} \) and \( B^{(2)} \rightarrow -B^{(2)} \).

In this way we have obtained the solutions to equations (4.5) in an open neighborhood of the point \( \lambda = 1 \). Our next step is to perform a coordinate transformation \( \{ t, z_1, \lambda \} \rightarrow \{ t, z_1, \omega \} \). Then \( W = W(t, z, \lambda_+(t, z, \omega)) \) becomes a function of \( t, z \) and \( \omega \), and eqn. (4.6) becomes an algebraic series in powers of \( (\lambda_+(t, z, \omega) - 1) \). Substituting this expansion and the expansion \( \frac{1}{1+\lambda_+} = \frac{1}{2} \sum_{n=0}^\infty (-\frac{1}{2})^n (\lambda_+ - 1)^n \) into (4.4) we obtain

\[
Z^{(1)}(t, z_1, z_0, \omega) = \int_{z_0}^{z_1} dz \left(-\frac{1}{2} \frac{1}{\lambda_+ - 1} \left(A^{(1)} + \frac{1}{2} \left(A^{(3)} W_0^{(4)} + A^{(4)} W_0^{(3)} \right)\right)\right) + \frac{1}{4} \sum_{n=0}^\infty \left( B^{(1)} \left(-\frac{1}{2} \right)^n - \left(A^{(3)} W_{n+1}^{(4)} + A^{(4)} W_{n+1}^{(3)} \right)\right) + \frac{1}{2} \sum_{m=0}^n \left( \frac{1}{2} \right)^m \left( B^{(3)} W_{n-m}^{(4)} + B^{(4)} W_{n-m}^{(3)} \right) (\lambda_+ - 1)^n. \tag{4.9a}
\]

\[
Z^{(2)}(t, z_1, z_0, \omega) = \int_{z_0}^{z_1} dz \left(-\frac{1}{2} \frac{1}{\lambda_+ - 1} \left(A^{(2)} + \frac{1}{2} \left(A^{(3)} W_0^{(4)} - A^{(4)} W_0^{(3)} \right)\right)\right) + \frac{1}{4} \sum_{n=0}^\infty \left( B^{(2)} \left(-\frac{1}{2} \right)^n - \left(A^{(3)} W_{n+1}^{(4)} - A^{(4)} W_{n+1}^{(3)} \right)\right) + \frac{1}{2} \sum_{m=0}^n \left( \frac{1}{2} \right)^m \left( B^{(3)} W_{n-m}^{(4)} - B^{(4)} W_{n-m}^{(3)} \right) (\lambda_+ - 1)^n. \tag{4.9b}
\]

where \( W_n^{(3)} \) and \( W_n^{(4)} \) are determined from eqns. (4.7) and (4.8). Thus we have obtained the functions \( Z^{(i)}(t, z_1, z_0, \omega) \), defined on open neighborhoods of the hypersurface \( \lambda_+(t, z, \omega) = 1 \) where the relevant algebraic series converge uniformly.

We now proceed to construct the functions \( Z^{(i)}(t, z_1, z_0, \omega) \), defined on open neighborhoods of the hypersurface \( \lambda_+(t, z, \omega) = -1 \). Our first step is to construct the solution to the equation (4.5) in an open neighborhood of the point \( \lambda = -1 \). We expand the fields \( W^{(3)} \) and \( W^{(4)} \) in powers of \( (\lambda + 1) \)

\[
W^{(3)} = \sum_{n=0}^\infty W_n^{(3)} (\lambda + 1)^n, \quad W^{(4)} = \sum_{n=0}^\infty W_n^{(4)} (\lambda + 1)^n. \tag{4.10}
\]
and the expansion \( \frac{1}{\lambda - 1} = -\frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{\lambda + 1}{2} \right)^n \) leads to the following recursion relation 
\((N = 0, 1, 2, \ldots)\)

\[
(N + 2) \frac{\alpha n}{\alpha} W_{N+2}^{(3)} = \partial_z W_N^{(3)} + \frac{B(4)}{2} \sum_{n=0}^{N+1} W_n^{(3)} W_{N+1-n} + \frac{A(4)}{4} \sum_{m=0}^{N} \frac{1}{2m} \times 
\sum_{n=0}^{N-m} W_n^{(3)} W_{N-m-n} + \left( \frac{3}{2}(N + 1) \frac{\alpha n}{\alpha} - B(2) \right) W_{N+1}^{(3)} - \frac{N}{2} \frac{\alpha n}{\alpha} W_N^{(3)} - \frac{1}{4} \sum_{n=0}^{N} \frac{1}{2n} \times 
\left( \frac{\alpha \xi}{\alpha} (N - n) + 2A(2) \right) W_{N-n}^{(3)} - \frac{A(3)}{2N+2}, \quad (4.11)
\]

together with

\[
\frac{\alpha n}{\alpha} W_1^{(3)} = \frac{B(4)}{2} \left( W_0^{(3)} \right)^2 - B(2) W_0^{(3)} - \frac{1}{2} B(3). \quad (4.12)
\]

As with eqn. (4.8), \(W_0^{(3)}\) is arbitrary, and for different choices of \(W_0^{(3)}\) we have different solutions. From (4.5b) and (4.10) we can obtain similar equations determining the \(W_n^{(4)}\), by the following replacements in those equations: \(W_n^{(3)} \rightarrow W_n^{(4)}\), \(A(4) \leftrightarrow A(3)\), \(B(4) \leftrightarrow B(3)\), \(A(2) \rightarrow -A(2)\) and \(B(2) \rightarrow -B(2)\).

We now perform the coordinate transformation \(\{t, z, \lambda\} \rightarrow \{t, z, \omega\}\). Then (4.10) becomes an algebraic series in powers of \((\lambda_+ + 1)\). Using the expansions (4.10,11) and \(\frac{1}{\lambda - 1} = -\frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{\lambda + 1}{2} \right)^n \) we obtain

\[
Z^{(1)}(t, z_1, z_0, \omega) = \int_{z_0}^{z_1} dz \left( \frac{1}{2} \left( \frac{1}{\lambda_+ + 1} \left( B(1) + \frac{1}{2} \left( B(3) W_0^{(4)} + B(4) W_0^{(3)} \right) \right) \right) 
+ \frac{1}{4} \sum_{n=0}^{\infty} \left( A(1) \frac{1}{2^n} + \left( B(3) W_{n+1}^{(4)} + B(4) W_{n+1}^{(3)} \right) \right) 
+ \frac{1}{2n+1} \left( A(3) W_{n-m}^{(4)} + A(4) W_{n-m}^{(3)} \right) (\lambda_+ + 1)^n \right), \quad (4.13a)
\]

\[
Z^{(2)}(t, z_1, z_0, \omega) = \int_{z_0}^{z_1} dz \left( \frac{1}{2} \left( \frac{1}{\lambda_+ + 1} \left( B(2) + \frac{1}{2} \left( B(3) W_0^{(4)} - B(4) W_0^{(3)} \right) \right) \right) 
+ \frac{1}{4} \sum_{n=0}^{\infty} \left( A(2) \frac{1}{2^n} + \left( B(3) W_{n+1}^{(4)} - B(4) W_{n+1}^{(3)} \right) \right) 
+ \frac{1}{2n+1} \left( A(3) W_{n-m}^{(4)} - A(4) W_{n-m}^{(3)} \right) (\lambda_+ + 1)^n \right), \quad (4.13b)
\]
where $W_n^{(3)}$ and $W_n^{(4)}$ are determined from the recursion relations derived for them above. We have thus determined the functions $Z^{(i)}(t, z_1, z_0, \omega)$ in an open neighbourhood of the hypersurface $\lambda_+(t, z, \omega) = -1$ where the relevant algebraic series converge uniformly.

Since we know (locally) the functions $Z^{(i)}(t, z_1, z_0, \omega)$, we can obtain expressions for the integrals of motion of our system. Using (4.1) and the fact that $W(t, z, \omega) = W(t, z + 2L, \omega)$ it follows that

\begin{align}
E_L(\omega) &= \text{Tr} \, T_L(t, \omega) = \text{Tr} \, e^{Z(L, -L, \omega)}, \\
F_L(\omega) &= \text{Tr} \, (T_L)^2(t, \omega) = \text{Tr} \, e^{2Z(L, -L, \omega)}.
\end{align}

From (4.14), after a straightforward calculation, it follows that

\begin{align}
Z^{(1)}(L, -L, \omega) &= \frac{1}{2} \ln \left( \frac{E_L^2(\omega) - F_L(\omega)}{2} \right), \quad (4.15a) \\
Z^{(2)}(L, -L, \omega) &= \frac{1}{2} \cosh^{-1} \left( \frac{F_L(\omega)}{E_L^2(\omega) - F_L(\omega)} \right). \quad (4.15b)
\end{align}

Consequently, from (3.13) and (4.15), we have

$$\partial_t Z^{(i)}(L, -L, \omega) = 0.$$ \quad (4.16)

The conclusion is that the $Z^{(i)}(L, -L, \omega)$ are the integrals of motion for every fixed value of $\omega$ which belongs to the domain in which the relevant series converge uniformly.

Our final remark is on the case when the two Killing vectors are hypersurface orthogonal. In our formulation, this case corresponds to the vanishing of the off-diagonal matrix $W$ in (4.1). The transition matrix $T$ is then diagonal. Consequently, our integrals of motion have a particularly simple form

\begin{align}
Z^{(1)}(L, -L, \omega) &= \int_{-L}^{L} dz \, U^{(1)}, \\
Z^{(2)}(L, -L, \omega) &= \int_{-L}^{L} dz \, U^{(2)},
\end{align}

with

\begin{align}
U^{(1)} &= \frac{1}{2} \left( -\frac{\alpha \xi}{\lambda_+ - 1} + \frac{\alpha \eta}{\lambda_+ + 1} \right), \quad (4.18a) \\
U^{(2)} &= \frac{1}{2} \left( -\frac{\gamma \xi}{\lambda_+ - 1} + \frac{\gamma \eta}{\lambda_+ + 1} \right). \quad (4.18b)
\end{align}
To obtain (4.18) we have used equations (2.15) together with the parametrization

\[ g = e^{\alpha \tau(1) + \gamma \tau(2)}. \quad (4.19) \]

Work on the geometrical interpretation of the integrals of motion \( Z^{(i)}(L, -L, \omega) \) and the boundary conditions that correspond to gravitational plane waves and gravitational cylindrical waves is in progress [18].

5. Conclusions

In this paper we have formulated the inverse scattering method as applied to the midisuperspace models characterized by a two-parameter Abelian group of motions with two spacelike Killing vectors. The application of the inverse scattering method to this model involved the feature that the derivatives defining the first-order formulation of the equations of motion also involved derivatives with respect to the spectral parameter \( \lambda \). We dealt with this by introducing a new spectral parameter \( \omega \). We also discussed the properties of the map between the two complex parameters \( \omega(t, z, \lambda) \) and \( \lambda(t, z, \omega) \). Then we showed how the approach of Belinskiĭ and Zakharov could be adapted to our case, and hence found the soliton solutions.

Our next step was to obtain the zero curvature representation of the Einstein field equations for space-times admitting a two-dimensional Abelian group of isometries, in the case when the orbits are spacelike. Then, using periodic boundary conditions which in the present context correspond to the three torus Gowdy models, we showed how the zero curvature formulation implies that the equations of motion for the transition matrix are of Heisenberg type. Consequently the eigenvalues of the transition matrix are conserved.

To obtain explicit expressions for the integrals of motion we had to solve a system of four partial differential equations. This problem was reduced to the problem of solving two equations of Riccati type. The solutions of these equations were given through recursion relations. The final results are the integrals of motion which are given as powers series in \( (\lambda_\pm(t, z, \omega) \pm 1) \). In the case when the two Killing vectors are hypersurface orthogonal the integrals of motion have a particularly simple form.
Future research will include the elaboration of the geometrical interpretation of the integrals of motion that we have obtained, as well as an investigation of different boundary conditions, such as those corresponding to planar symmetry and cylindrical symmetry [18]. The problem of quantization of the soliton solutions on the Bianchi type II background is under investigation and we will report our results shortly [15]. Finally, our main goal is to obtain the quantum theory for the two Killing vector reduction of general relativity [19].

6. Acknowledgments

We would like to thank A. Ashtekar, N. Kalogeropoulos, G. Mena-Marugán, R.S. Rajeev, L. Smolin, R. Sorkin, and G. Stephens for discussions. N.M. is very grateful to A. Ashtekar for financial support and the warm hospitality of the Relativity Group at Syracuse University. B.S. was supported by a QEII Fellowship from the Australian Government.

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