STRICT POLYNOMIAL SUPERFUNCTORS AND UNIVERSAL EXTENSION CLASSES FOR ALGEBRAIC SUPERGROUPS

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Abstract. The category of strict polynomial superfunctors, recently defined by Axtell, generalizes the category of strict polynomial functors. In this paper we compute extension groups in the category of strict polynomial superfunctors, and exhibit certain universal extension classes for the general linear supergroup. Some of these classes restrict to the classes exhibited by Friedlander and Suslin for the general linear group, while others arise from “super” phenomena. We use these extension classes to show that the cohomology ring of a finite supergroup scheme (equivalently, of a finite-dimensional (super-)cocommutative Hopf superalgebra) over a field is a finitely-generated algebra. Implications for the rational cohomology of the general linear supergroup are also discussed.

1. Introduction

Let $k$ be a field of positive characteristic $p$, and let $d$ be a positive integer. Friedlander and Suslin introduced the category $\mathcal{P}$ of strict polynomial functors over $k$ as part of their investigation into the cohomology of finite $k$-group schemes [12]. They calculated the Yoneda algebra in $\mathcal{P}$ of the $r$-th Frobenius twist of the identity functor, exhibiting certain universal extension classes for the general linear group. These extension classes enabled them to show that the cohomology ring of a finite $k$-group scheme, or equivalently of a finite-dimensional cocommutative Hopf algebra over $k$, is a finitely-generated $k$-algebra. The purpose of this article is to extend all of these results to the world of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces and group schemes. In particular, we prove:

Theorem (5.4.1, 5.4.2, 5.6.3). Let $k$ be a field, and let $G$ be a finite $k$-supergroup scheme, or, equivalently, a finite-dimensional (super-)cocommutative Hopf superalgebra over $k$. Then the cohomology ring $H^\bullet(G, k)$ is a finitely-generated $k$-superalgebra, and for each finite-dimensional $G$-superspace $M$, the cohomology group $H^\bullet(G, M)$ is a finitely-generated $H^\bullet(G, k)$-superspace.

Recall that a superspace is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space. The category $\text{succ}$ of $k$-superspaces is a symmetric monoidal category with braiding $T : V \otimes W \to W \otimes V$ defined by

$$T(v \otimes w) = (-1)^{\overline{v}\cdot\overline{w}} w \otimes v.$$ 

Here $\overline{v}, \overline{w} \in \mathbb{Z}/2\mathbb{Z} := \mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$ denote the degrees of homogeneous elements $v \in V$ and $w \in W$. Then a Hopf superalgebra over $k$ is a Hopf algebra object in the symmetric monoidal category $\text{succ}$. Similarly, an affine $k$-supergroup scheme is an affine group scheme object in $\text{succ}$. Each ordinary Hopf algebra over $k$ can be viewed as a Hopf superalgebra concentrated in degree $\overline{0}$. Conversely, if $H$ is a $k$-Hopf superalgebra, then $H$ is a subalgebra of the not-necessarily-cocommutative ordinary Hopf algebra $H \# k\mathbb{Z}_2$, the Radford biproduct (or Majid bosonization) of $H$ with the group algebra $k\mathbb{Z}_2$, and the cohomology ring of $H \# k\mathbb{Z}_2$ identifies with the degree-$\overline{0}$ component of $H^\bullet(H, k)$. Thus, the above theorem can be viewed as a generalization of the Friedlander–Suslin finite-generation result to a class of braided Hopf algebras, or to a larger class of ordinary Hopf algebras. Reducing the gradings modulo 2, our theorem also applies to $\mathbb{Z}$-graded Hopf algebras and group schemes.

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The main focus of this article is the calculation of extension groups in the category $\mathcal{P}$ of strict polynomial superfunctors, recently defined by Axtell [1], and the application of those results to the cohomology of the general linear supergroup $GL(m|n)$. We write $\mathcal{P}_d$ for Axtell’s category of degree-$d$ “strict polynomial functors of type I.” Axtell shows for $m, n \geq d$ that $\mathcal{P}_d$ is equivalent to the category of finite-dimensional left supermodules for the Schur superalgebra $S(m|n,d)$, and hence also equivalent to the category of degree-$d$ polynomial representations for the supergroup $GL(m|n)$. Axtell also defines a category of “strict polynomial functors of type II,” which for $n \geq d$ is equivalent to the category of finite-dimensional left supermodules for the Schur superalgebra $Q(n,d)$ of type $Q$ defined by Brundan and Kleshchev [3] §4. We don’t consider the cohomology of these “type II” functors at this time, but we expect that to do so would yield some interesting new connections to the representation theory of the supergroup $Q(n)$.

A degree-$d$ homogeneous strict polynomial functor $F$ can be restricted to the category $\mathcal{V}_T$ of purely even finite-dimensional $k$-superspaces, or to the category $\mathcal{V}_T$ of purely odd finite-dimensional $k$-superspaces. These two restrictions are each naturally degree-$d$ homogeneous strict polynomial functors in the sense of [2], though they are rarely isomorphic. Now suppose $k$ is a (perfect) field of characteristic $p > 2$. Given a $k$-superspace $V$ and a positive integer $r$, write $V^{(r)}$ for the $k$-superspace obtained by twisting the $k$-module structure on $V$ by the Frobenius morphism $\lambda \mapsto \lambda^p$. Since the decomposition $V = V_0 \oplus V_T$ of a superspace $V$ into its even and odd subspaces is not functorial (it is not compatible with the composition of odd linear maps), there is no corresponding direct sum decomposition of the identity functor $I : V \mapsto V$. On the other hand, for each $r \geq 1$, the $r$-th Frobenius twist functor $I^{(r)} : V \mapsto V^{(r)}$ decomposes as a direct sum $I^{(r)} = I_0^{(r)} \oplus I_1^{(r)}$ in $\mathcal{P}_p$ with $I_0^{(r)}(V) = V_0^{(r)}$ and $I_1^{(r)}(V) = V_T^{(r)}$. This can be thought of as a functor analogue of the fact the Frobenius morphism for $GL(m|n)$ has image in the underlying purely even subgroup $GL_m \times GL_n$ of $GL(m|n)$. In general, there does not appear to be a natural way to extend ordinary strict polynomial functors to the structure of strict polynomial superfunctors, though Frobenius twists of strict polynomial functors can be lifted to $\mathcal{P}$ in several ways; see Section 2.7.

The decomposition $I^{(r)} = I_0^{(r)} \oplus I_1^{(r)}$ leads to a matrix ring decomposition

\begin{equation}
\begin{pmatrix}
\Ext^*_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}) & \Ext^*_{\mathcal{P}}(I_1^{(r)}, I_0^{(r)}) \\
\Ext^*_{\mathcal{P}}(I_0^{(r)}, I_1^{(r)}) & \Ext^*_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)})
\end{pmatrix}
\end{equation}

of the Yoneda algebra $\Ext^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$. One of our main results (Theorem 4.7.1) is the calculation, up to certain constants that are equal to either 1 or $-1$, of the structure of this algebra in terms of distinguished extension classes

$$\bar{e}_i \in \Ext^{2p^{i-1}}_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)}) \quad \text{for } 1 \leq i \leq r,$$

$$\bar{c}_r \in \Ext^{r}_{\mathcal{P}}(I_1^{(r)}, I_0^{(r)}),$$

$$\bar{c}_r^\Pi \in \Ext^{r}_{\mathcal{P}}(I_1^{(r)}, I_1^{(r)}).$$

For example, we show that $\Ext^*_{\mathcal{P}}(I_0^{(r)}, I_0^{(r)})$ is a commutative algebra generated by $\bar{e}_1, \ldots, \bar{e}_r$ and $\bar{c}_r , \bar{c}_r^\Pi$ subject only to the relations $\bar{e}_1^2 = \cdots = \bar{e}_r^2 = 0$, and the restriction functor $F \mapsto F|_{\mathcal{V}_T}$ maps the subalgebra generated by $\bar{e}_1, \ldots, \bar{e}_r$ isomorphically onto $\Ext^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$. Some notable differences from the classical case include the fact that $\Ext^*_{\mathcal{P}}(I^{(r)}, I^{(r)})$ is highly non-commutative and contains non-nilpotent elements. We also observe by degree consideration that precomposition with $I^{(r)}$ does not induce an injective map on Ext-groups in $\mathcal{P}$; cf. [9] Corollary 1.3.

Our main applications to the cohomology of the general linear supergroup and the cohomology of finite group schemes are presented in Section 5. We show for each $r \geq 1$ that the restrictions to $GL(m|n)$ of the extension classes $e_r = \bar{e}_r$, $e_r^\Pi$, $c_r$, and $e_r^\Pi$ provide in a natural way the universal extension classes for $GL(m|n)$ that were conjectured in [2] Conjecture 5.4.1. Combined with our previous results in [7], this proves cohomological finite-generation for finite supergroup schemes over fields of characteristic $p > 2$. The case $p = 2$ reduces to the case of ordinary finite $k$-group schemes...
(since then every finite $k$-supergroup scheme is a finite $k$-group scheme), while the case $p = 0$ follows swiftly (though not trivially) from a structure theorem of Kostant; see Section 5.6. We also show in Section 5.5 that the restriction of $e_r$ to $GL(m|n)$ naturally produces a nonzero class in the rational cohomology group $H^2(GL(m|n),k)$. This stands in contrast to the well-known fact that if $G$ is a reductive algebraic group, then $H^i(G,k) = 0$ for all $i > 0$. It also demonstrates, in contrast to the classical case, that the embedding of the category of polynomial representations for $GL(m|n)$ into the category of all rational representations for $GL(m|n)$ does not induce isomorphisms on extension groups. We are not aware of any other nontrivial calculations of $H^i(GL(m|n),k)$ in the literature, though Brundan and Kleshchev have shown that $H^1(Q(n),k)$ is a one-dimensional odd superspace $[5, Corollary 7.8]$. An obvious open problem is to calculate the complete structure of the rational cohomology ring $H^*(GL(m|n),k)$.

The paper is organized as follows: In Sections 2 and 3 we give definitions and basic results that are needed for doing Ext-group calculations in the category $\mathcal{P}$. Readers acquainted with ordinary strict polynomial functors may find much of this material quite familiar, though some attention must be paid to make the transition from ordinary to $\mathbb{Z}_2$-graded objects. In particular, $\mathcal{P}$ is not an abelian category, so some care must be taken in defining the Ext-groups we will study (Sections 3.1–3.2), in defining operations on extension groups (Sections 3.2–3.4), and in interpreting cohomology classes in terms of $n$-extensions (Section 3.5). In Section 4 we compute the Yoneda algebra $\text{Ext}_\mathcal{P}^*(I^{(r)}, I^{(r)})$. Our strategy parallels the inductive approach in [12] using hypercohomology spectral sequences. For the base case of the induction, we consider a super analogue $\Omega$ of the de Rham complex, which satisfies a super Cartier isomorphism: $H^*(\Omega_{pq}) \cong \Omega_n(1)$. This isomorphism does not, however, preserve the cohomological degree, and this failure directly leads to the extension class $c_1$; see Remark 4.1.3 and Section 5.2. We remark that our proof of Lemma 4.1.1 makes critical use of the specific spectral sequence calculations in [12, §4]. Finally, in Section 5 we present the previously-mentioned applications to the cohomology of $GL(m|n)$ and to the cohomology of finite supergroup schemes. Our calculations in Section 5.3 relating for each $r \geq 1$ the extension class $c_r$ to the class $c_{r+1}$, are strongly influenced by the calculations of Franjou et al. [9] computing extension groups between classical exponential functors.

Conventions. Except when indicated, $k$ will denote a perfect field of positive characteristic $p > 2$, and we will follow the notation, terminology, and conventions laid out in [7, §2]. In particular, we assume that the reader is familiar with the standard sign conventions of “super” linear algebra. All vector spaces are $k$-vector spaces, and all unadorned tensor products denote tensor products over $k$. Set $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, and write $V = V_0 \oplus V_1$ for the decomposition of a superspace $V$ into its even and odd subspaces. Given a homogeneous element $v \in V$, write $\bar{v} \in \mathbb{Z}_2$ for the $\mathbb{Z}_2$-degree of $v$. Except when indicated, all isomorphisms will arise from even linear maps; we typically reserve the symbol “∼” for isomorphisms arising from odd linear maps. Set $\mathbb{N} = \{0, 1, 2, \ldots\}$.}

2. Preliminaries

2.1. Strict polynomial superfunctors. Let $\mathbf{svec}$ be the category whose objects are the $k$-superspaces and whose morphisms are the $k$-linear maps between superspaces. Let $\mathcal{V}$ be the full subcategory of $\mathbf{svec}$ whose objects are the finite-dimensional $k$-superspaces, and let $\mathbf{svec}_e$ and $\mathcal{V}_e$ be the underlying even subcategories having the same objects as $\mathbf{svec}$ and $\mathcal{V}$, respectively, but only the even linear maps as morphisms.

Then $\mathbf{svec}_e$ and $\mathcal{V}_e$ are abelian categories. Given $V, W \in \mathcal{V}$, let $T : V \otimes W \to W \otimes V$ be the supertwist map, which is defined on homogeneous simple tensors by $T(v \otimes w) = (-1)^{\bar{v} \bar{w}} w \otimes v$. For each $n \in \mathbb{N}$, there exists a right action of the symmetric group $\mathfrak{S}_n$ on $V^\otimes n$ such that the transposition $(i, i + 1) \in \mathfrak{S}_n$ acts via the linear map

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1. In $[7]$, $\mathbf{svec}$ is denoted $\mathbf{svec}_e$, while $\mathbf{svec}_e$ is denoted $\mathbf{svec}$.

2. From now, whenever we state a formula in which a homogeneous degree has been specified, we mean that the formula is true as written for homogeneous elements, and that it extends linearly to non-homogeneous elements.
(1_V)^{(i-1)} \otimes T \otimes (1_V)^{(n-i-1)}. \) Set \( \Gamma^n(V) = (V^\otimes n)^{\mathcal{S}_n}. \) Then \( \Gamma^n : V \mapsto \Gamma^n(V) \) is an endofunctor on \( \mathcal{V}_{ev}. \) If \( V = V^\mathfrak{g}, \) then \( \Gamma^n(V) = \Gamma^n(V), \) where \( \Gamma^n \) denotes the ordinary \( n \)-th divided power (i.e., symmetric tensor) functor.

For each \( A, B \in \mathcal{V}, \) the supertwist map induces an isomorphism \( A^\otimes n \otimes B^\otimes n \cong (A \otimes B)^{\otimes n}, \) which is an isomorphism of \( \mathcal{S}_n \)-modules if we consider \( A^\otimes n \otimes B^\otimes n \) as a right \( \mathcal{S}_n \)-module via the diagonal map \( \mathcal{S}_n \to \mathcal{S}_n \times \mathcal{S}_n. \) This means that \( \Gamma^n(A) \otimes \Gamma^n(B) \) is naturally a subspace of \( \Gamma^n(A \otimes B). \) In particular, if \( \phi : A \otimes B \to C \) is an even linear map, then there exists an induced even linear map

\[
\Gamma^n(\phi) : \Gamma^n(A) \otimes \Gamma^n(B) \to \Gamma^n(C).
\]

Now given \( n \in \mathbb{N}, \) define \( \Gamma^n \mathcal{V} \) to be the category whose objects are the same as those in \( \mathcal{V}, \) whose morphisms are defined by \( \text{Hom}_{\Gamma^n \mathcal{V}}(V, W) = \Gamma^n \text{Hom}_k(V, W), \) and in which the composition of morphisms is induced as in (2.1.1) by the composition of linear maps in \( \mathcal{V}. \) Alternatively, there exists by [1] Lemma 3.1 a natural isomorphism \( \Gamma^n \text{Hom}_k(V, W) \cong \text{Hom}_k(\mathcal{S}_n, V^\otimes n, W^\otimes n). \) Then composition in \( \Gamma^n \mathcal{V} \) can be viewed as the composition of \( k\mathcal{S}_n \)-module homomorphisms.

**Definition 2.1.1.** Let \( n \in \mathbb{N}. \) A **homogeneous strict polynomial superfunctor of degree \( n \)** is an even linear functor \( F : \Gamma^n \mathcal{V} \to \mathcal{V}, \) i.e., a covariant functor \( F : \Gamma^n \mathcal{V} \to \mathcal{V} \) such that for each \( V, W \in \mathcal{V}, \) the function \( F_{V, W} : \Gamma^n \text{Hom}_k(V, W) \to \text{Hom}_k(F(V), F(W)) \) is an even linear map. Given degree-\( n \) homogeneous strict polynomial superfunctors \( F \) and \( G, \) a homomorphism \( \eta : F \to G \) consists for each \( V \in \mathcal{V} \) of a map \( \eta(V) \in \text{Hom}_k(F(V), G(V)) \) such that for each \( \phi \in \text{Hom}_f \mathcal{V}(V, W) \) one has

\[
\eta(W) \circ F(\phi) = (-1)^{\text{deg} \eta} G(\phi) \circ \eta(V).
\]

We denote by \( \mathcal{P}_n \) the category whose objects are the homogeneous strict polynomial superfunctors of degree \( n \) and whose morphisms are the homomorphisms between those functors. The category \( \mathcal{P} \) of arbitrary strict polynomial superfunctors is defined to be the category \( \prod_{n \in \mathbb{N}} \mathcal{P}_n. \)

**Remark 2.1.2.** Let \( \Gamma \mathcal{V} \) be the category whose objects are the same as those in \( \mathcal{V}, \) whose morphisms are defined by \( \text{Hom}_{\Gamma \mathcal{V}}(V, W) = \prod_{n \in \mathbb{N}} \text{Hom}_\mathcal{V}(V, W), \) and in which composition of morphisms is defined componentwise using the composition laws in each \( \Gamma^n \mathcal{V}. \) Then each \( F \in \mathcal{P} \) is naturally an even linear functor \( F : \Gamma \mathcal{V} \to \mathcal{V}_{ev} \) follows: Let \( F = \bigoplus_{n \in \mathbb{N}} F^n \) be the polynomial decomposition of \( F, \) i.e., the decomposition of \( F \) into the sum of its homogeneous components \( F^n \in \mathcal{P}_n. \) Similarly, write \( \phi = \prod_{n \in \mathbb{N}} \phi_n \) for the decomposition of \( \phi \in \text{Hom}_{\Gamma \mathcal{V}}(V, W). \) Now \( F : \Gamma \mathcal{V} \to \mathcal{V}_{ev} \) is defined on objects by \( F(V) = \bigoplus_{n \in \mathbb{N}} F^n(V), \) and on morphisms by \( F(\phi) = \prod_{n \in \mathbb{N}} F^n(\phi_n), \) i.e., \( F(\phi) \) acts on the summand \( F^n(V) \) of \( F(V) \) by the linear map \( F^n(\phi_n) : F^n(V) \to F^n(W). \)

Conversely, let \( F : \Gamma \mathcal{V} \to \mathcal{V}_{ev} \) be an even linear functor. Then for each \( V \in \Gamma \mathcal{V}, \) the function \( F_{V, V} : \text{Hom}_{\Gamma \mathcal{V}}(V, V) \to \text{Hom}_\mathcal{V}(F(V), F(V)) \) makes \( F(V) \) into a left \( \text{Hom}_{\Gamma \mathcal{V}}(V, V) \)-module, and the decomposition \( \text{Hom}_{\Gamma \mathcal{V}}(V, V) = \prod_{n \in \mathbb{N}} \text{Hom}_\mathcal{V}(V, V) \) leads to an expression \( \text{id}_{F} = \prod_{n \in \mathbb{N}} \text{id}_{F^n} \) of the identity morphism as an infinite sum of orthogonal idempotents. Set \( F^n(V) = \text{id}_{F^n}(F(V)), \) and given \( \phi \in \text{Hom}_{\Gamma \mathcal{V}}(V, W), \) define \( F^n(\phi) : F^n(V) \to F^n(W) \) to be the map \( \text{id}_{F^n} \circ F(\phi) \circ \text{id}_{F^n}. \) Then \( F : \Gamma \mathcal{V} \to \mathcal{V}_{ev} \) is an even linear functor. If \( F^n(V) \in \mathcal{V} \) for each \( V \in \mathcal{V}, \) then \( F^n \in \mathcal{P}_n. \) If also \( F(\mathcal{V}) = \bigoplus_{n \in \mathbb{N}} F^n(\mathcal{V}) \) for each \( V \in \mathcal{V}, \) then \( F \in \mathcal{P} \).

**Remark 2.1.3.** Let \( V, W \in \mathcal{V}, \) and let \( \phi \in \text{Hom}_k(V, W)_\mathfrak{g}. \) Then for \( n \in \mathbb{N}, \phi^\otimes n \in \Gamma^n \text{Hom}_k(V, W). \) Thus, it follows that a homogeneous strict polynomial superfunctor \( F \in \mathcal{P}_n \) defines an ordinary functor \( \mathcal{V}_{ev} \to \mathcal{V}_{ev} \) that acts on objects by \( V \mapsto F(V) \) and on morphisms by \( \phi \mapsto F(\phi^\otimes n). \) Throughout the paper, we will often state remarks (such as this one) in the context of homogeneous superfunctors, and then leave it to the reader to consider how those remarks can be extended to the non-homogeneous case, and vice versa.

The category \( \mathcal{P} \) is not an abelian category, though the underlying even subcategory \( \mathcal{P}_{ev} \) of the same objects as \( \mathcal{P} \) but only the even homomorphisms, is an abelian category in which kernels and cokernels are computed “pointwise” in the category \( \mathcal{V}. \) More generally, if \( \eta \) is a homogeneous
homomorphism in $\mathcal{P}$, then the kernel, cokernel, and image of $\eta$ are again objects in $\mathcal{P}$. Only the even homomorphisms in $\mathcal{P}$ are genuine natural transformations between functors.

Observe that $\mathcal{V}_{ev} = \mathcal{V}_0 \oplus \mathcal{V}_1$, where $\mathcal{V}_0$ is the full subcategory of $\mathcal{V}_{ev}$ having as objects just the purely even superspaces, i.e., the superspaces $V$ with $V = V_0$, and $\mathcal{V}_1$ is the full subcategory of $\mathcal{V}_{ev}$ having as objects just the purely odd superspaces (and as morphisms just the even linear maps between them). In turn, $\mathcal{V}_0$ and $\mathcal{V}_1$ are each isomorphic to the category $\mathcal{V}$ of arbitrary finite-dimensional $k$-vector spaces. Replacing $\mathcal{V}$ by $\mathcal{V}_0$ or $\mathcal{V}_1$ in the definition of $\Gamma^n\mathcal{V}$, one obtains categories $\Gamma^n(\mathcal{V}_0)$ and $\Gamma^n(\mathcal{V}_1)$, that are each isomorphic to the ordinary analogue $\Gamma^n\mathcal{V}$ of $\Gamma^n\mathcal{V}$.

Let $V, W \in \mathcal{V}_0$. Since $\text{Hom}_k(V, W)$ is a purely even space, there exists a natural identification $\text{Hom}_{\Gamma^n\mathcal{V}}(V, W) = \text{Hom}_{\Gamma^n\mathcal{V}}(V, W)$. Thus, it makes sense to consider the restriction of $F \in \mathcal{P}_n$ to the category $\Gamma^n(\mathcal{V}_0) \cong \Gamma^n\mathcal{V}$. Similarly, we can consider the restriction of $F$ to $\Gamma^n(\mathcal{V}_1) \cong \Gamma^n\mathcal{V}$. By abuse of notation, we denote these restrictions by $F|_{\mathcal{V}_0}$ and $F|_{\mathcal{V}_1}$, respectively. In either case, we can then consider $F : \Gamma^n\mathcal{V} \to \mathcal{V}$ as a functor to $\mathcal{V}$ by forgetting the $\mathbb{Z}/2\mathbb{Z}$-gradings on objects in $\mathcal{V}$. Then $F|_{\mathcal{V}_0}$ and $F|_{\mathcal{V}_1}$ are purely even strict polynomial functors of degree $n$ in the sense of [12]; cf. the comments following [11 §3.7]. In particular, the restriction map $F \mapsto F|_{\mathcal{V}_0}$ defines an exact linear functor from $\mathcal{P}$ to the category $\mathcal{P}$ of ordinary strict polynomial functors, which we call restriction from $\mathcal{P}$ to $\mathcal{P}$.

In this paper we will consider strict polynomial superfunctors that restrict to well-known ordinary strict polynomial functors. In these situations we generally use a boldface symbol for the strict polynomial superfunctor and a non-boldface version of the same symbol for the functor’s restriction to $\mathcal{V}$. In particular, we consider superfunctors that restrict to the divided power algebra functor $\Gamma$ (i.e., the symmetric tensor functor), the symmetric algebra functor $S$, and the exterior algebra functor $\Lambda$ (which is also isomorphic to the anti-symmetric tensor functor). We write $\mathcal{P}_n$ and $\mathcal{P}$ for the ordinary analogues of $\mathcal{P}_n$ and $\mathcal{P}$, respectively.

2.2. Constructions. In this section we describe the details behind some standard constructions involving strict polynomial superfunctors. We omit the details for the direct sum $F \oplus G \in \mathcal{P}_n$ of two superfunctors $F, G \in \mathcal{P}_n$.

2.2.1. Composition of functors. Let $F \in \mathcal{P}_m$ and $G \in \mathcal{P}_n$. The composite functor $F \circ G \in \mathcal{P}_{mn}$ is defined on objects by $(F \circ G)(V) = F(G(V))$. To describe the action on morphisms, observe that the inclusion of the Young subgroup $(\mathcal{S}_n)^\times m$ into $\mathcal{S}_{mn}$ induces for each $U \in \mathcal{V}$ an inclusion $\Gamma^{mn}(U) \hookrightarrow \Gamma^m(\Gamma^n(U))$. Then $(F \circ G)|_{\mathcal{V}_{GW}}$ is the composite linear map $\Gamma^{mn} \text{Hom}_k(V, W) \xrightarrow{\Gamma^m(G|_{GW})} \Gamma^m(\text{Hom}_k(G(V), G(W))) \xrightarrow{F_{GW}|_{\mathcal{V}_{GW}}} \text{Hom}_k(F(G(V)), F(G(W)))$.

2.2.2. Tensor products. Let $F \in \mathcal{P}_m$ and $G \in \mathcal{P}_n$. The tensor product $F \otimes G \in \mathcal{P}_{m+n}$ is defined on objects by $(F \otimes G)(V) = F(V) \otimes G(V)$. The inclusion of the Young subgroup $\mathcal{S}_m \times \mathcal{S}_n$ into $\mathcal{S}_{m+n}$ induces for each $U \in \mathcal{V}$ an inclusion $\Gamma^{m+n}(U) \hookrightarrow \Gamma^m(\Gamma^n(U))$. Then $(F \otimes G)|_{\mathcal{V}_{GW}}$ is the composite linear map $\Gamma^{m+n} \text{Hom}_k(V, W) \hookrightarrow \Gamma^m \text{Hom}_k(V, W) \otimes \Gamma^n \text{Hom}_k(V, W) \xrightarrow{F_{GW}|_{\mathcal{V}_{GW}}} \text{Hom}_k((F \otimes G)(V), (F \otimes G)(W))$.

2.2.3. Dual functors. Given $U \in \mathcal{V}$, set $U^* = \text{Hom}_k(U, k)$. Then the dual $F^\# \in \mathcal{P}_n$ of $F \in \mathcal{P}_n$ is defined on objects by $F^\#(V) = F(V^*)^*$. On morphisms, $(F^\#)|_{\mathcal{V}_{GW}}$ is the composite map $\Gamma^n \text{Hom}_k(V, W) \cong \Gamma^n \text{Hom}_k(W^*, V^*) \xrightarrow{F_{GW}|_{\mathcal{V}_{GW}}} \text{Hom}_k(F(W^*), F(V^*)) \cong \text{Hom}_k(F^\#(V), F^\#(W))$, where the first and last isomorphisms are induced by sending a linear map $\psi$ to its transpose $\psi^*$. 
Recall that $V$ is naturally isomorphic to its double dual $(V^*)^*$ via the map $\Phi(V) : V \to (V^*)^*$ that is defined for $v \in V$ and $g \in V^*$ by $\Phi(V)(v)(g) = (-1)^{v \cdot g} g(v)$. Let $\Phi : I \to I^#$ be the natural transformation lifting $\Phi(V)$. Then $F$ identifies with $F^{##}$ via the composite isomorphism

$$F = F \circ I \xrightarrow{F \circ \Phi} I \circ I^# \xrightarrow{\Phi \circ (F \circ I^#)} I^# \circ F \circ I^#.$$ 

Now let $F,G \in \mathcal{P}_n$, and let $\eta \in \text{Hom}_{\mathcal{P}_n}(F,G)$. Then $\eta^# \in \text{Hom}_{\mathcal{P}_n}(G^#,F^#)$ is defined by $\eta^#(V) = \eta(V)^*$. If $\sigma \in \text{Hom}_{\mathcal{P}_n}(G,H)$, then $(\sigma \circ \eta)^# = (-1)^{\sigma \cdot \eta^#} \sigma^#$. Thus, $(-)^#$ defines an equivalence of categories $\mathcal{P}_n \simeq \mathcal{P}_n^{op-}$, where $\mathcal{P}_n^{op-}$ denotes the category with the same objects and morphisms as the opposite category of $\mathcal{P}_n$, but in which the composition law has been modified so that $\eta \circ_{op} \sigma$ is now equal to $(-1)^{\sigma \cdot \eta} \circ_{op} \sigma$; cf. [1, §3.4].

2.3. Examples. Throughout this section, let $m,n \in \mathbb{N}$, let $V,W \in \mathcal{V}$, and let $\phi \in \mathfrak{g}^n$ Hom$_k(V,W)$. We freely identify $\phi$ with an element of Hom$_{k \mathfrak{g}_n}(V^\otimes n, W^\otimes n)$. Let $\{x_1, \ldots, x_s\}$ be a basis for $V_T$, and let $\{y_1, \ldots, y_t\}$ be a basis for $V^*_T$.

2.3.1. Tensor products. Let $U \in \mathcal{V}$. Then the functors $V \mapsto V \otimes U$ and $V \mapsto U \otimes V$ are objects in $\mathcal{P}_1$. These functors act on morphisms by sending $\phi \in \text{Hom}_k(V,W)$ to the tensor products of maps $\phi \otimes \text{id}_U$ and $\text{id}_T \otimes \phi$, respectively. Then the supertwist map $T : V \otimes U \to U \otimes V$ lifts to an isomorphism $(- \otimes U) \cong (U \otimes -)$. Taking $U = k$, we get the identity functor $I : \mathcal{V} \to \mathcal{V}$.

2.3.2. Parity change. Let $k^0[1]$ be a one-dimensional purely odd superspace. Then the parity change functor $\Pi$ is the functor $\Pi : \mathcal{V} \to \mathcal{V}$ given by $\Pi(V) = k[1]^0 \otimes V$, i.e., $\Pi$ is defined by reversing the $\mathbb{Z}_2$-grading, and on morphisms by $\Pi(\phi) = (-1)^{\phi} \phi$, i.e., if $\phi : V \to W$ is an even linear map, then $\Pi(\phi) : \Pi(V) \to \Pi(W)$ is equal to $\phi$ if $\phi$ is a map between the underlying spaces, while if $\phi$ is odd, then $\Pi(\phi) = -\phi$. In contradiction to our stated convention on the use of boldface and non-boldface versions of the same symbol, set $\Pi = - \otimes k^0[1]$. Then $\Pi$ acts on objects by $\Pi(V) = \Pi(V)$, and acts on morphisms by $\Pi(\phi) = \phi$, i.e., $\Pi(\phi)$ is (always) equal to $\phi$ as a map between the underlying spaces.

Write $\text{id}_{V \to \Pi(V)}$ for the identity map on $V$ considered as an odd linear map $V \to \Pi(V)$. Then $\text{id}_{V \to \Pi(V)}$ lifts for each $F \in \mathcal{P}$ to an odd isomorphism $\text{id}_{F \to \Pi\circ F} : F \to \Pi \circ F$. More generally, for each $F,G \in \mathcal{P}$ there exist odd isomorphisms

$$\text{Hom}_{\mathcal{P}}(F,G) \cong \text{Hom}_{\mathcal{P}}(F,\Pi \circ G), \quad \eta \mapsto \text{id}_{G \to \Pi \circ G} \circ \eta,$$

that are natural with respect to even homomorphisms in either variable. Similar comments apply to the odd isomorphisms $F \simeq \Pi \circ F$ induced by the superspace map $V \to V$, $v \mapsto (-1)^{\overline{v}} v$.

2.3.3. Tensor powers. The $n$-th tensor power functor $\otimes^n \in \mathcal{P}_n$ is defined on objects by $(\otimes^n)(V) = V^\otimes n$, and on morphisms by $(\otimes^n)(\phi) = \phi$. Equivalently, $\otimes^n = I^\otimes n$. Set $T = \bigoplus_{m \in \mathbb{N}} \otimes^m$. Then $T(V)$ is the tensor superalgebra on $V$. Since $(\otimes^m) \otimes (\otimes^n) = \otimes^{m+n}$, it follows that the multiplication map on $T(V)$ lifts to a natural transformation $m_T : T \otimes T \to T$. Also, given $\sigma \in \mathfrak{g}_n$, it follows that the right action of $\sigma$ on $V^\otimes n$ lifts to a natural transformation $\otimes^n \to \otimes^n$, which we also denote $\sigma$.

2.3.4. Symmetric powers. The $n$-th symmetric power functor $S^n \in \mathcal{P}_n$ is defined on objects by $S^n(V) = (V^\otimes n)_{\mathfrak{g}_n} = (V^\otimes n)/(z - (z,\sigma) : z \in V^\otimes n, \sigma \in \mathfrak{g}_n)$. On morphisms, $S^n(\phi) : S^n(V) \to S^n(W)$ is the linear map that is naturally induced by $\phi$.

Set $S = \bigoplus_{n \in \mathbb{N}} S^n$. Then $S(V)$ is the symmetric superalgebra on $V$. It is the free commutative superalgebra generated by the superspace $V$. As an algebra, $S(V)$ is isomorphic to the ordinary tensor product of algebras $S(V_0) \otimes \Lambda(V_T)$, and identifies with the algebra denoted $S_s(V)$ in [7, §2.3].
Since the product in $S(V)$ is induced by the product in $T(V)$, the multiplication map in $S(V)$ lifts to a natural transformation $m_S : S \otimes S \to S$. A basis for $S(V)$ is given by the set
\begin{equation}
\{x_1^{a_1} \cdots x_s^{a_s}y_1^{b_1} \cdots y_t^{b_t} : a_i, b_j \in \mathbb{N}, 0 \leq b_j \leq 1\}
\end{equation}
of monomials in $S(V)$.

### 2.3.5. Exterior powers

Write $\sigma \mapsto (-1)^{\sigma}$ for the one-dimensional sign representation of $\mathfrak{S}_n$. The $n$-th exterior power functor $\Lambda^n \in \mathcal{P}_n$ is defined on objects by

$$\Lambda^n(V) = (V^{\otimes n})/(z - (-1)^{\sigma}(z)) : z \in V^{\otimes n}, \sigma \in \mathfrak{S}_n).$$

On morphisms, $\Lambda^n(\phi) : \Lambda^n(V) \to \Lambda^n(W)$ is the linear map that is naturally induced by $\phi$.

Set $\Lambda = \bigoplus_{n \in \mathbb{N}} \Lambda^n$. Then $\Lambda(V)$ is the exterior superalgebra on $V$. In the terminology of [7, §2.1], $\Lambda(V)$ is the free graded-commutative graded superalgebra generated by $V$ when $V$ is considered as a Z-graded superspace concentrated in degree 1. As an algebra, $\Lambda(V)$ is isomorphic to the graded tensor product of superalgebras $\Lambda(V_0) \otimes S(V_1)$. As for $S(V)$, the multiplication map in $\Lambda(V)$ lifts to a natural transformation $m_{\Lambda} : \Lambda \otimes \Lambda \to \Lambda$. A basis for $\Lambda(V)$ is given by the set
\begin{equation}
\{a_1^{a_1} \cdots a_s^{a_s}y_1^{b_1} \cdots y_t^{b_t} : a_i, b_j \in \mathbb{N}, 0 \leq a_i \leq 1\}
\end{equation}
of monomials in $\Lambda(V)$. The algebra $\Lambda(V)$ identifies with the algebra denoted $\Lambda_s(V)$ in [7, §2.3].

### 2.3.6. Divided powers

The $n$-th divided power functor $\Gamma^n \in \mathcal{P}_n$ is defined on objects by

$$\Gamma^n(V) = (V^{\otimes n})_{\mathfrak{S}_n} = \{z \in V^{\otimes n} : z.\sigma = z \text{ for all } \sigma \in \mathfrak{S}_n\}$$

and on morphisms by $\Gamma^n(\phi) = \phi|_{\Gamma^n(V)}$. Set $\Gamma = \bigoplus_{n \in \mathbb{N}} \Gamma^n$.

Let $J \subseteq \mathfrak{S}_{m+n}$ be a set of right coset representatives for the Young subgroup $\mathfrak{S}_m \times \mathfrak{S}_n$ of $\mathfrak{S}_{m+n}$.

Then $\sum_{\sigma \in J} \sigma$ defines a natural transformation $(\otimes^m) \otimes (\otimes^n) \to \otimes^{m+n}$. This natural transformation restricts to a natural transformation $\Gamma^m \otimes \Gamma^n \to \Gamma^{m+n}$ that does not depend on the choice of $J$. Summing over all $m, n \in \mathbb{N}$, there exists a natural transformation $m_{\Gamma} : \Gamma \otimes \Gamma \to \Gamma$. Given $v \in V_0^\gamma$ and $n \in \mathbb{N}$, set $\gamma_n(v) = v \otimes \cdots \otimes v \in \Gamma^n(V)$. The definition of $\gamma_n(v)$ also makes sense if $v \in V_{\gamma}$ and $n = 1$, but $\gamma_n(v) \notin \Gamma^n(V)$ if $v \in V_{\gamma}$ and $n \geq 2$. By an adaptation of the arguments in [2, IV.5], one can show that the product $m_{\Gamma}(\Gamma(V) \otimes \Gamma(V) \to \Gamma(V)$ makes $\Gamma(V)$ into a commutative superalgebra, and that the set
\begin{equation}
\{\gamma_{a_1}(x_1) \cdots \gamma_{a_s}(x_s) y_1^{b_1} \cdots y_t^{b_t} : a_i, b_j \in \mathbb{N}, 0 \leq b_j \leq 1\}
\end{equation}
of monomials in $\Gamma(V)$ is a basis for $\Gamma(V)$. In particular, $\Gamma(V)$ is isomorphic to the tensor product of superalgebras $\Gamma(V_0) \otimes \Lambda(V_1)$ (cf. [2 (4) and Lemma 4.1]), and is generated as an algebra by the subspace $V_{\gamma} \subseteq \Gamma^1(V)$ together with the elements of the form $\gamma_{e^r}(v)$ for $v \in V_{\gamma}$ and $e \geq 0$.

### 2.3.7. Alternating powers

The $n$-th alternating power functor $A^n \in \mathcal{P}_n$ is defined on objects by

$$A^n(V) = \{z \in V^{\otimes n} : z.\sigma = (-1)^{\sigma} z \text{ for all } \sigma \in \mathfrak{S}_n\}$$

and on morphisms by $A^n(\phi) = \phi|_{A^n(V)}$. Set $A = \bigoplus_{n \in \mathbb{N}} A^n$.

Again let $J$ be a set of right coset representatives for $\mathfrak{S}_m \times \mathfrak{S}_n$ in $\mathfrak{S}_{m+n}$. Then $\sum_{\sigma \in J} (-1)^{\sigma} \sigma$ restricts to a natural transformation $A^m \otimes A^n \to A^{m+n}$ that does not depend on the choice of $J$. Summing over all $m, n \in \mathbb{N}$, there exists a natural transformation $m_{A} : A \otimes A \to A$. Now arguing as for $\Gamma$, one can show that the product $m_{A}(V)$ makes $A(V)$ into a graded-commutative graded superalgebra, with the grading induced by considering $V$ as a graded superspace concentrated in degree 1. One can also show that the set
\begin{equation}
\{a_1^{a_1} \cdots a_s^{a_s} \gamma_{b_1}(y_1) \cdots \gamma_{b_t}(y_t) : a_i, b_j \in \mathbb{N}, 0 \leq a_i \leq b_j \leq 1\}.
\end{equation}

\footnote{If $A$ and $B$ are graded superalgebras, then multiplication of homogeneous simple tensors in $A \otimes B$ is defined by $(a \otimes b)(c \otimes d) = (-1)^{\deg(a) \deg(b)} (ac \otimes bd)$, where $\deg(x)$ denotes the Z-degree of $x$. The Z-degree of $\alpha \otimes b$ is defined by $\deg(a \otimes b) = \deg(a) + \deg(b)$.}
of monomials in $A(V)$ is a basis for $A(V)$, and hence that $A(V)$ is isomorphic as an algebra to the graded tensor product of algebras $A(V^n) \otimes \Gamma(V^n)$. In particular, $A(V)$ is generated as an algebra by the subspace $V_0 \subseteq A^1(V)$ together with the elements of the form $\gamma_v e(v)$ for $v \in V_T$ and $e \geq 0$.

2.4. Bisuperfunctors. Let $\mathcal{V} \times \mathcal{V}$ denote the direct product of the category $\mathcal{V}$ with itself. Thus, the objects of $\mathcal{V} \times \mathcal{V}$ are pairs $(V, W)$ with $V, W \in \mathcal{V}$, while a morphism $\phi : (V, W) \rightarrow (V', W')$ in $\mathcal{V} \times \mathcal{V}$ is a linear map $\phi : V \oplus W \rightarrow V' \oplus W'$ such that $\phi(V) \subseteq V'$ and $\phi(W) \subseteq W'$. Now given $d, e \in \mathbb{N}$, define $\Gamma_d^e(\mathcal{V} \times \mathcal{V})$ to be the category with the same objects as $\mathcal{V} \times \mathcal{V}$, but in which all compositions are $\otimes$, the external direct sum $\oplus$, and the composite $F \circ (\psi, \phi) : (V, W) \rightarrow (V', W')$ of $\mathcal{V}$.

Definition 2.4.1. Let $d, e \in \mathbb{N}$. A strict polynomial bisuperfunctor of bidegree $(d, e)$ is an even linear superfunctor $F : \Gamma_d^e(\mathcal{V} \times \mathcal{V}) \rightarrow \mathcal{V}$. Given two such functors $F$ and $G$, a homomorphism $\eta : F \rightarrow G$ consists for each $(V, W) \in \mathcal{V}$ of a linear map $\eta(V, W) : F(V, W) \rightarrow G(V, W)$ such that for each $\phi \in \text{Hom}_{\Gamma_d^e(\mathcal{V} \times \mathcal{V})}((V, W), (V', W'))$, one has

\[ \eta(V', W') \circ F(\psi) = (-1)^{\psi e} G(\phi) \circ \eta(V, W). \]

We denote by $\mathcal{P}_d^e$ the category whose objects are the strict polynomial bisuperfunctors of bidegree $(d, e)$ and whose morphisms are the homomorphisms between those functors. Given $n \in \mathbb{N}$, the category $\mathcal{P}(n)$ of strict polynomial bisuperfunctors of total degree $n$ is the coequalizer $\prod_{d+e=n} \mathcal{P}_d^e$, and the category $\text{bi-}\mathcal{P}$ of arbitrary strict polynomial bisuperfunctors is the category $\prod_{n \in \mathbb{N}} \mathcal{P}(n)$.

Given $F \in \mathcal{P}_d^e$, $F' \in \mathcal{P}_r^e$, $G \in \mathcal{P}_m$, and $H \in \mathcal{P}_n$, one can construct:

(2.4.3) the internal direct sum $F \oplus F' \in \text{bi-}\mathcal{P}_m((V, W) \mapsto F(V, W) \oplus F'(V, W));$

(2.4.4) the external direct sum $G \boxplus H \in \text{bi-}\mathcal{P}_n((V, W) \mapsto G(V) \oplus H(W));$

(2.4.5) the composite $G \circ F \in \text{bi-}\mathcal{P}_m((V, W) \mapsto G(F(V, W));$

(2.4.6) the composite $F \circ (G, H) \in \mathcal{P}_m \times \mathcal{P}_n((V, W) \mapsto F(G(V, W), H(W));$

(2.4.7) the internal tensor product $F \otimes F' \in \mathcal{P}_{d+s}^e((V, W) \mapsto F(V, W) \otimes F'(V, W));$

(2.4.8) the external tensor product $G \boxtimes H \in \mathcal{P}_{m+n}(V, W) \mapsto G(V) \otimes H(W));$ and

(2.4.9) the dual $F^* \in \mathcal{P}_d^e(V, W) \mapsto F(V, W)^*.$

We leave the details of these constructions to the reader.

Define $\iota^0_m : \mathcal{P}_m \rightarrow \mathcal{P}_m^0$ and $\iota^0_n : \mathcal{P}_n \rightarrow \mathcal{P}_n^0$ by $\iota^0_m(G)(V, W) = G(V)$ and $\iota^0_n(H)(V, W) = H(W)$. Then $\iota^0_m$ and $\iota^0_n$ induce isomorphisms $\mathcal{P}_m \cong \mathcal{P}_m^0$ and $\mathcal{P}_n \cong \mathcal{P}_n^0$, and one can immediately check for $G \in \mathcal{P}_m$ and $H \in \mathcal{P}_n$ that $G \boxplus H = \iota^0_m(G) \boxplus \iota^0_n(H)$ and $G \boxtimes H = \iota^0_m(G) \otimes \iota^0_n(H)$. Set $I_{1,0} = \iota^0_n(I)$ and $I_{0,1} = \iota^0_n(I)$. Then $\iota^0_m(G) = G \circ I_{1,0}$, $\iota^0_n(H) = H \circ I_{0,1}$, and the direct sum functor

\[ \Sigma : (V, W) \mapsto V \oplus W \]

is equal to the internal direct sum $I_{1,0} \oplus I_{0,1}$.

Remark 2.4.2. Given $n \in \mathbb{N}$, define $\Gamma^n(\mathcal{V} \times \mathcal{V})$ to be the category that is obtained by replacing $\mathcal{V}$ with $\mathcal{V} \times \mathcal{V}$ in the definition of $\Gamma^n \mathcal{V}$. In particular, if $(V, W), (V', W') \in \mathcal{V} \times \mathcal{V}$, then

\[ \text{Hom}_{\Gamma^n(\mathcal{V} \times \mathcal{V})}((V, W), (V', W')) = \Gamma^n[\text{Hom}_k(V, V') \otimes \text{Hom}_k(W, W')]. \]

The exponential isomorphism for $\Gamma$ (discussed next in Section 2.5) induces an isomorphism

\[ \text{Hom}_{\Gamma^n(\mathcal{V} \times \mathcal{V})}((V, W), (V', W')) \cong \prod_{d+e=n} \text{Hom}_{\Gamma_d^e(\mathcal{V} \times \mathcal{V})}((V, W), (V', W')). \]
that is compatible with the composition of morphisms. Taking \((V,W) = (V',W')\), this yields a decomposition in \(\Gamma^n(V \times V)\) of the identity morphism \(\text{id}_{(V,W)}\) as a sum of orthogonal idempotents. Now let \(F : \Gamma^n(V \times V) \to V\) be an even linear functor. Then by reasoning similar to that described in the second paragraph of Remark 2.1.2, it follows that the idempotent decomposition of \(\text{id}_{(V,W)}\) for each \((V,W) \in V \times V\) induces a direct sum decomposition \(\bigoplus_{d+e=n} F^e_d\) of \(F\) with \(F^e_d \in \mathcal{P}_d\). In other words, \(F \in \mathcal{P}(n)\). Conversely, each \(F \in \mathcal{P}(n)\) defines a natural even linear functor \(F : \Gamma^n(V \times V) \to V\); cf. the first paragraph of Remark 2.1.2. Thus, we can identify \(\mathcal{P}(n)\) with the category of even linear functors \(F : \Gamma^n(V \times V) \to V\) whose morphisms satisfy \(2.4.2\).

2.5. \(\mathcal{P}\)-algebras and exponential superfunctors.

**Definition 2.5.1.** (cf. [19], §3) A functor \(A \in \mathcal{P}\) is a \(\mathcal{P}\)-algebra if there exist even homomorphisms \(k \to A\) and \(m_A : A \otimes A \to A\) such that for each \(V \in \mathcal{V}\), the induced maps make \(A(V)\) into a \(k\)-superalgebra. We say that \(A\) is commutative if each \(A(V)\) is then a commutative superalgebra. We say that \(A\) is graded if there exists a decomposition \(A = \bigoplus_{n \in \mathbb{Z}} A^n\) such that \(m_A\) restricts for each \(m,n \in \mathbb{Z}\) to a homomorphism \(A^m \otimes A^n \to A^{m+n}\), and we say that \(A\) is graded-commutative if each \(A(V)\) is then a graded-commutative graded superalgebra with respect to the grading \(A(V) = \bigoplus_{n \in \mathbb{Z}} A^n(V)\). If \(A\) and \(B\) are graded \(\mathcal{P}\)-algebras, then \(\eta \in \text{Hom}_\mathcal{P}(A,B)\) is a (graded) \(\mathcal{P}\)-algebra homomorphism if each \(\eta(V)\) is a homomorphism of (graded) superalgebras.

The functors \(S\) and \(\Gamma\) are examples of commutative \(\mathcal{P}\)-algebras. Any \(\mathcal{P}\)-algebra can be made into a graded \(\mathcal{P}\)-algebra via its polynomial grading, i.e., the grading defined by the functor’s polynomial decomposition. The polynomial gradings make \(\Lambda\) and \(A\) into graded-commutative \(\mathcal{P}\)-algebras. Now let \(A\) and \(B\) be graded \(\mathcal{P}\)-algebras. We denote by \(A \otimes B\) the graded \(\mathcal{P}\)-algebra such that for each \(V \in \mathcal{V}\), \((A \otimes B)(V)\) is equal to the graded tensor product of algebras \(A(V) \otimes B(V)\). Similarly, if \(A\) and \(B\) are graded \(\mathcal{P}\)-coalgebras, then the graded tensor product \(A \otimes B\) is naturally a graded \(\mathcal{P}\)-coalgebra (the reader can fill in the coalgebra analogue of Definition 2.5.1). If the \(\mathbb{Z}\)-grading on either \(A\) or \(B\) is trivial (or purely even), then we may denote \(A \otimes B\) simply as \(A \times B\).

**Definition 2.5.2.** Let \(A \in \mathcal{P}\). Then \(A\) is a graded \(\mathcal{P}\)-bialgebra if \((\mathcal{P})\) is both a graded \(\mathcal{P}\)-algebra and a graded \(\mathcal{P}\)-coalgebra, and if the coproduct \(\Delta_A : A \to A \otimes A\) is a homomorphism of graded \(\mathcal{P}\)-algebras. (If the grading on \(A\) is purely even, then we may drop the adjective graded.)

Given a (graded) \(\mathcal{P}\)-algebra \(A\), we can consider the composite bisuperfunctor homomorphism
\[
\mu_A : A \boxtimes A = (A \circ I_{1,0}) \otimes (A \circ I_{0,1}) \to (A \circ \Sigma) \otimes (A \circ \Sigma) = (A \circ A) \circ \Sigma \to A \circ \Sigma
\]
induced by the inclusions \(I_{1,0} \hookrightarrow \Sigma, I_{0,1} \hookrightarrow \Sigma\) and the product \(m_A : A \otimes A \to A\).

**Definition 2.5.3.** A graded \(\mathcal{P}\)-algebra \(A\) is an exponential superfunctor if \(\mu_A\) is an isomorphism of strict polynomial bisuperfunctors, i.e., if for each \(V,W \in \mathcal{V}\), the composite map
\[
A(V) \otimes A(W) \to A(V \oplus W) \otimes A(V \oplus W) \to A(V \oplus W)
\]
induced by the inclusions \(V \hookrightarrow V \oplus W\) and \(W \hookrightarrow V \oplus W\) and by multiplication in \(A(V \oplus W)\), lifts to an isomorphism of strict polynomial bisuperfunctors \(A \boxtimes A \cong A \circ \Sigma\).

**Example 2.5.4.** The functors \(S, \Gamma, \Lambda, \) and \(A\) are exponential superfunctors. More generally, if \(A\) and \(B\) are exponential superfunctors, then so is \(A \otimes B\).

Let \(D : \mathcal{V} \to \mathcal{V} \times \mathcal{V}\) be the diagonal functor \(V \mapsto (V,V)\), and let \(\Delta : I \to D\) be the natural transformation that lifts the usual diagonal map on vector spaces. Let \(A\) be an exponential superfunctor. Composing the isomorphism \(A \boxtimes A \cong A \circ \Sigma\) with \(D\), we obtain an isomorphism \(A \otimes A \cong A \circ D\) of polynomial superfunctors. Define \(\Delta_A\) to be the composite homomorphism
\[
\Delta_A : A = A \circ I \overset{\Delta}{\rightarrow} A \circ D \overset{(\mu_A \otimes D)^{-1}}{\rightarrow} A \otimes A.
\]
Then \(\Delta_A\) defines a coassociative coproduct on \(A\), making \(A\) into a (graded) \(\mathcal{P}\)-coalgebra.
Now suppose in addition for each \( V, W \in \mathcal{V} \) that \( \mu_A \) induces an isomorphism of graded superalgebras \( A(V) \otimes A(W) \cong A(V \oplus W) \). Then \( \Delta_A \) is a homomorphism of graded \( \mathcal{P} \)-algebras (because it is the composition of two algebra homomorphisms), and hence endows \( A \) with the structure of a graded \( \mathcal{P} \)-bialgebra. In particular, \( S \) and \( \Gamma \) are (trivially graded) commutative cocommutative \( \mathcal{P} \)-bialgebras, and \( \Delta \) and \( A \) are graded-commutative graded-cocommutative graded \( \mathcal{P} \)-bialgebras. Using the descriptions in Section\ref{graded} for \( \Gamma, S, \Lambda \) and \( A \) in terms of \( S, \Gamma, \) and \( \Lambda \), one can check that the coproducts restrict to the usual coproducts for \( S, \Gamma, \) and \( \Lambda \).

**Remark 2.5.5.** Given a graded \( \mathcal{P} \)-coalgebra \( A \), there exists a bisuperfunctor homomorphism

\[
\lambda_A : A \circ \Sigma \to (A \otimes A) \circ \Sigma = (A \circ \Sigma) \otimes (A \circ \Sigma) \to (A \circ I_{1,0})(A \circ I_{0,1}) = A \boxtimes A
\]

induced by the coproduct \( \Delta_A : A \to A \otimes A \) and the projections \( \Sigma \to I_{1,0} \) and \( \Sigma \to I_{0,1} \). If \( A \) is an exponential superfunctor with coproduct defined by \((2.5.2)\), then \( \lambda_A = \mu_A^{-1} \). Conversely, suppose \( A \) is a graded \( \mathcal{P} \)-coalgebra and \( \lambda_A \) is an isomorphism. Let \( + : D \to I \) be the natural transformation defined by \( +(v_1, v_2) = v_1 + v_2 \). Then the composite

\[
A \otimes A \xrightarrow{(\lambda_A \circ D)^{-1}} A \circ D \xrightarrow{A(+)} A,
\]

defines an associative product \( m_A \) making \( A \) into a graded \( \mathcal{P} \)-algebra, and the homomorphism \( \mu_A \) defined in terms of \((2.5.4)\) is equal to \( \lambda_A^{-1} \).

### 2.6. Duality isomorphisms.

As described in Section\ref{graded}, \( \Gamma \) is a commutative cocommutative \( \mathcal{P} \)-bialgebra. Then by duality so is \( \Gamma^\# \). Since \( \Gamma^1 = I \), we have \( (\Gamma^\#)^1 = I^\# \). Then the isomorphism \( I \cong I^\# \) together with the multiplication map on \( \Gamma^\# \) defines a natural transformation

\[
\otimes^n = I^\# \otimes^n \xrightarrow{n} (\Gamma^\#)^n
\]

that factors through \( S^n \). Summing over all \( n \in \mathbb{N} \), it follows that there exists a unique algebra homomorphism \( \theta : S \to \Gamma^\# \) that extends the identification \( S^1 = I \cong I^\# = (\Gamma^\#)^1 \). Now by an adaptation of the arguments in \cite[IV.5.11]{2}, one can show that \( \theta \) is an isomorphism. Explicitly, let \( \{x_1, \ldots, x_s\} \) and \( \{y_1, \ldots, y_t\} \) be bases for \( V_0 \) and \( V_1 \), and let \( \{x_1^s, \ldots, x_s^s\} \) and \( \{y_1^t, \ldots, y_t^t\} \) be the corresponding dual bases. Then \( \theta(V)(x_1^s \cdots x_s^s y_1^t \cdots y_t^t) \) evaluates to \( \pm 1 \) on the basis monomial \( \gamma_0(x_1^s) \cdots \gamma_n(x_n^s)(y_1^t)^h \cdots (y_t^t)^h \) and evaluates to \( 0 \) on all other basis monomials in \( \Gamma(V)^s \) of the form \((2.3.4)\). The isomorphism \( S \cong \Gamma^\# \) is compatible with the coproducts on \( S \) and \( \Gamma^\# \), so \( S \cong \Gamma^\# \) as \( \mathcal{P} \)-bialgebras. By duality, \( \Gamma^\# \cong \Lambda \) as \( \mathcal{P} \)-bialgebras as well. An entirely parallel argument shows that \( \Lambda \cong A^\# \) and \( A \cong A^\# \) as graded \( \mathcal{P} \)-bialgebras.

**Remark 2.6.1.** There exist unique algebra homomorphisms \( S \to \Gamma \) and \( \Lambda \to A \) extending the identifications \( S^1 = I = \Gamma^1 \) and \( \Lambda^1 = I = A^1 \). Given \( V \in \mathcal{V} \), the induced maps \( S(V) \xrightarrow{\Gamma} A(V) \) and \( \Lambda(V_0) \to A(V_0) \) are surjective, hence isomorphisms by dimension comparison. If \( k \) is a field of characteristic zero, then the maps \( S \to \Gamma \) and \( \Lambda \to A \) are isomorphisms.

### 2.7. Frobenius twists.

Given \( r \geq 1 \) and \( V \in \mathcal{V} \), write \( \varphi : k \to k \) for the \( p^r \)-power map \( \lambda \mapsto \lambda^{p^r} \), and set \( V^{(r)} = k \otimes_k V \). In other words, \( V^{(r)} \) is equal to \( V \) as an additive group, but a scalar \( \lambda \in k \) acts on \( V^{(r)} \) by \( \lambda^{p^r} \) acts on \( V \). The \( r \)-th Frobenius twist functor \( I^{(r)} \in \mathbb{P}_{p^r} \) is defined on objects by \( I^{(r)}(V) = V^{(r)} \). To describe the action of \( I^{(r)} \) on morphisms, first observe for each \( V \in \mathcal{V} \) that the \( p^r \)-power map defines an algebra homomorphism \( S(V)^{(r)} \to S(V) \) that is natural with respect to even linear maps on \( V \). By abuse of notation, we also denote this homomorphism by \( \varphi \). If \( z = z_0 + z_1 \) is the decomposition of \( z \) into its even and odd components, then \( z^p = (z_0^p) \), so \( \varphi \) has image in the subalgebra \( S(V_0) \) of \( S(V) \). One can check that if \( A, B \in \mathcal{V} \), then there exists
a commutative diagram

\[
\begin{array}{ccc}
S(A \otimes B)^{(r)} & \rightarrow & S(A)^{(r)} \otimes S(B)^{(r)} \\
\varphi \downarrow & & \varphi \otimes \varphi \\
S(A \otimes B) & \rightarrow & S(A) \otimes S(B)
\end{array}
\]

(2.7.1)

in which the horizontal arrows are the natural algebra maps.

By duality, there exists for each \(V \in \mathcal{V}\) an algebra homomorphism \(\varphi^d : \Gamma(V) \rightarrow \Gamma(V)^{(r)}\), which we refer to as the dual Frobenius morphism, that is also natural with respect to even linear maps on \(V\). Explicitly, \(\varphi^d\) acts on the generators for \(\Gamma(V)\) described in Section 2.3.6 by

\[
\varphi^d(z) = \begin{cases} 
0 & \text{if } z \in V^*_T \subseteq \Gamma^1(V), \\
\gamma_{p^e}^{-1}(v) & \text{if } z = \gamma_{p^e}(v) \text{ for some } e \in \mathbb{N}, v \in V^0, \text{ and } e \geq r, \\
0 & \text{if } z = \gamma_{p^e}(v) \text{ for some } e \in \mathbb{N}, v \in V^0, \text{ and } e < r.
\end{cases}
\]

(2.7.2)

Then \(\varphi^d\) is determined by its restriction to the subalgebra \(\Gamma(V^0)\) of \(\Gamma(V)\), and has image in the subalgebra \(\Gamma(V^0)^{(r)}\) of \(\Gamma(V)^{(r)}\). Now given \(V, W \in \mathcal{V}\), \(I_{V,W}^{(r)}\) is the linear map

\[
\Gamma^r \text{Hom}_k(V, W) \xrightarrow{\varphi^d} [\Gamma^1 \text{Hom}_k(V, W)^{(r)}] = \text{Hom}_k(V^{(r)}, W^{(r)})^e.
\]

Dualizing (2.7.1), and using the fact that \(\varphi^d\) is natural with respect to even linear maps, it follows that \(I_{V,W}^{(r)}\) is compatible with the composition of morphisms. From now on, for \(r \geq 1\) and \(F \in \mathcal{P}_n\), set \(F^{(r)} = F \circ I^{(r)} \in \mathcal{P}_{p^r n}\). Since (2.7.3) has image in the space of even linear maps, it follows for \(r \geq 1\) that there exist subfunctors \(I_0^{(r)}\) and \(I_1^{(r)}\) of \(I^{(r)}\) such that \(I_0^{(r)}(V) = V^0, I_1^{(r)}(V) = V^r,\) and \(I^{(r)} = I_0^{(r)} \oplus I_1^{(r)}\). Additionally,

\[
I_1^{(r)} = \Pi \circ I_0^{(r)} \circ \Pi.
\]

Lemma 2.7.1. Let \(F \in \mathcal{P}_n\), and let \(r \geq 1\). Then \(F \circ I_0^{(r)}\) and \(F \circ I_1^{(r)}\) are summands of \(F \circ I^{(r)}\).

Proof. We prove that \(F \circ I_0^{(r)}\) is a summand of \(F \circ I^{(r)}\); the proof for \(F \circ I_1^{(r)}\) is entirely analogous.

Given \(V \in \mathcal{V}\), let \(\iota_V : V^0 \rightarrow V\) and \(\pi_V : V \rightarrow V^0\) be the natural (but non-functorial) inclusion and projection maps, respectively. These are even linear maps, and \(\pi_V \circ \iota_V\) is the identity on \(V^0\). Set \(\iota_V = (\iota_V)^{\otimes p^n} \in \Gamma^{p^n} \text{Hom}_k(V^0, V),\) and set \(\pi_V = (\pi_V)^{\otimes p^n} \in \Gamma^{p^n} \text{Hom}_k(V, V^0)\). Then the composite \(\pi_V \circ \iota_V \in \Gamma^{p^n} \text{Hom}_k(V^0, V^0)\) is the identity map on \(V^0\) in the category \(\mathcal{P}_{p^n} \mathcal{V}\). Now applying the functor \(F \circ I^{(r)}\), we get maps

\[
(F \circ I^{(r)})(V^0) \xrightarrow{(F \circ I^{(r)})(\iota_V)} (F \circ I^{(r)})(V) \xrightarrow{(F \circ I^{(r)})(\pi_V)} (F \circ I^{(r)})(V^0)
\]

whose composite is the identity. But \((F \circ I^{(r)})(V^0) = (F \circ I_0^{(r)})(V),\) so to prove that \(F \circ I_0^{(r)}\) is a summand of \(F \circ I^{(r)}\), it suffices to show that

\[
(F \circ I^{(r)})(\iota_V) : (F \circ I_0^{(r)})(V) \rightarrow (F \circ I^{(r)})(V) \quad \text{and} \quad (F \circ I^{(r)})(\pi_V) : (F \circ I^{(r)})(V) \rightarrow (F \circ I_0^{(r)})(V)
\]

lift to natural transformations. We verify this for \((F \circ I^{(r)})(\iota_V),\) the argument for \((F \circ I^{(r)})(\pi_V)\) being entirely analogous. Let \(W \in \mathcal{V}\), and let \(\phi \in \Gamma^{p^n} \text{Hom}_k(V, W)\). We must show that

\[
(F \circ I^{(r)})(\phi) \circ (F \circ I^{(r)})(\iota_V) = (F \circ I^{(r)})(\iota_W) \circ (F \circ I_0^{(r)})(\phi).
\]

(2.7.5)
We may assume that \( \phi \in \mathfrak{p}^{p^{n}}[\text{Hom}_{k}(V,W)_{\underline{n}}] \), since otherwise (2.7.2) implies that \((F \circ I^{(r)})(\phi) = 0\) and \((F \circ I^{(r)})(\phi) = 0\). By definition, if \( \psi \in \mathfrak{p}^{p^{r}} \text{Hom}_{k}(V,W) \), then
\[
I^{(r)}(\psi) = I^{(r)}((\pi_{W})^{\otimes p^{r}} \circ \phi \circ (I_{V})^{\otimes p^{r}}).
\]
Then it follows that \((F \circ I^{(r)})(\phi) = (F \circ I^{(r)})(\pi_{W} \circ \phi \circ I_{V})\). Now since \( \phi \in \mathfrak{p}^{p^{n}}[\text{Hom}_{k}(V,W)_{\underline{n}}] \), it follows that \( \phi \circ I_{V} = I_{W} \circ (\pi_{W} \circ \phi \circ I_{V}) \), and hence that (2.7.5) is true.

**Remark 2.7.2.** In general, if \( F \in \mathcal{P}_{n} \) and \( r \geq 1 \), then \( F \circ I^{(r)} \neq (F \circ I^{(r)})(r) \) or \((F \circ I^{(r)})(r) \neq (F \circ I^{(r)})(r) \).

Now let \( A \in \mathcal{P} \) be an exponential superfunctor. Then

\[
A^{(r)} = A \circ (I^{(r)}_{0} \oplus I^{(r)}_{1}) \cong (A \circ I^{(r)}_{0}) \otimes (A \circ I^{(r)}_{1})
\]
as \( \mathcal{P} \)-algebras. This observation applies in particular to \( S, \Gamma, \Lambda, \) and \( A \). For \( n \in \mathbb{N} \), set

\[
\begin{align*}
S^{0(r)} &= S^{0} \circ I^{(r)}_{0}, & \Lambda^{0(r)} &= \Lambda^{0} \circ I^{(r)}_{0}, & \Gamma^{0(r)} &= \Gamma^{0} \circ I^{(r)}_{0}, \\
\Lambda^{1(r)} &= S^{1} \circ I^{(r)}_{1}, & \Gamma^{1(r)} &= \Lambda^{1} \circ I^{(r)}_{1}, & S^{1(r)} &= \Lambda^{1} \circ I^{(r)}_{1}.
\end{align*}
\]
(2.7.6)

For each \( V \in \mathcal{V} \), one has \( S^{0(r)}(V) = S^{0}(V_{\underline{0}}^{(r)}) \) and \( \Lambda^{1(r)}(V) = \Lambda^{1}(V_{\underline{1}}^{(r)}) \) as abstract vector spaces. As a superspace, \( S^{0(r)}(V) \) is always a purely even superspace, while \( \Lambda^{1(r)}(V) \) is a purely even (resp. odd) superspace if \( n \) is even (resp. odd). Similar identifications and comments apply to the other composite functors defined in (2.7.6).

Suppose for the moment that \( F \in \mathcal{P}_{n} \) is an ordinary homogeneous strict polynomial functor. Let \( r \geq 1 \), and let \( F^{(r)} \in \mathcal{P}_{p^{r}n} \) be the ordinary \( r \)-th Frobenius twist of \( F \). We can lift \( F^{(r)} \) to the structure of a homogeneous strict polynomial superfunctor in several different ways. First, since \( (2.7.3) \) has image in the space of even linear maps, it follows that \( I^{(r)}_{0} \) and \( I^{(r)}_{1} \) define even linear functors \( \Gamma^{p^{r}} \mathcal{V} \to \Gamma^{p^{r}}(\mathcal{V}_{\underline{0}}) \) and \( \Gamma^{p^{r}} \mathcal{V} \to \Gamma^{p^{r}}(\mathcal{V}_{\underline{1}}) \), respectively. Then making the natural identifications \( \mathcal{V}_{\underline{0}} \cong \mathcal{V} \) and \( \mathcal{V}_{\underline{1}} \cong \mathcal{V} \), we can consider \( F \) as an even linear functor \( \Gamma^{p^{r}}(\mathcal{V}_{\underline{0}}) \to \mathcal{V}_{\underline{0}} \) or as an even linear functor \( \Gamma^{p^{r}}(\mathcal{V}_{\underline{1}}) \to \mathcal{V}_{\underline{1}} \). Next, the inclusions \( \mathcal{V}_{\underline{0}} \hookrightarrow \mathcal{V} \) and \( \mathcal{V}_{\underline{1}} \hookrightarrow \mathcal{V} \) are even linear functors. We denote the composite even linear functors by

\[
F \circ I^{(r)}_{0} : \Gamma^{p^{r}} \mathcal{V} \to \Gamma^{p^{r}}(\mathcal{V}_{\underline{0}}) \xrightarrow{F} \mathcal{V}_{\underline{0}} \hookrightarrow \mathcal{V}, \quad \text{and}
\]
\[
F \circ I^{(r)}_{1} : \Gamma^{p^{r}} \mathcal{V} \to \Gamma^{p^{r}}(\mathcal{V}_{\underline{1}}) \xrightarrow{F} \mathcal{V}_{\underline{1}} \hookrightarrow \mathcal{V}.
\]
(2.7.7)
(2.7.8)

Finally, we can optionally compose (2.7.7) or (2.7.8) with the parity change functor \( \Pi \). (Composing with \( \Pi \) has the same result as composing with \( \Pi \).) This gives four different ways we can lift \( F^{(r)} \in \mathcal{P}_{p^{r}n} \) to an element of \( \mathcal{P}_{p^{r}n} \). For example, the four liftings of \( I^{(r)} \) obtained in this fashion are \( I^{(r)}_{0}, I^{(r)}_{1}, \Pi \circ I^{(r)}_{0}, \) and \( \Pi \circ I^{(r)}_{1} \). More generally, if \( F^{(r)} \) denotes one of the four liftings of \( F^{(r)} \) to \( \mathcal{P} \) described above, then the other three liftings are \( F^{(r)} \circ \Pi, \Pi \circ F^{(r)}, \) and \( \Pi \circ F^{(r)} \circ \Pi \).

One can now show that each of the functors defined in (2.7.6) is a lifting of either \( S^{0(r)}, \Lambda^{0(r)} \), or \( \Gamma^{0(r)} \) to the category \( \mathcal{P} \). In particular, one can use this observation to see that

\[
\begin{align*}
S^{0(r)} &\cong S^{0} \circ \Pi \\
\Lambda^{0(r)} &\cong \Lambda^{0} \circ \Pi \\
\Gamma^{0(r)} &\cong \Gamma^{0} \circ \Pi
\end{align*}
\]
if \( n \) is even, and
\[
\begin{align*}
S^{1(r)} &\cong S^{1} \circ (S^{0(r)} \circ \Pi) \\
\Lambda^{1(r)} &\cong \Lambda^{1} \circ (\Lambda^{0(r)} \circ \Pi) \\
\Gamma^{1(r)} &\cong \Gamma^{1} \circ (\Gamma^{0(r)} \circ \Pi)
\end{align*}
\]
if \( n \) is odd.

From now on, if \( F \in \mathcal{P}_{n} \) and \( r \geq 1 \), then unless stated otherwise we will use (2.7.7) to consider \( F^{(r)} \) as a homogeneous strict polynomial superfunctor. Then for all \( F \in \mathcal{P}_{n} \), one has \( F^{(r)} \mid \mathcal{V}_{\underline{0}} \cong F^{(r)} \) as ordinary strict polynomial functors. On the other hand, if \( F \in \mathcal{P}_{n} \), then using (2.7.7) to lift \( F \mid \mathcal{V}_{\underline{0}} \in \mathcal{P}_{n} \) to \( \mathcal{P}_{n} \) does not in general result in a strict polynomial superfunctor that is isomorphic to the original functor \( F \).
3. Cohomology of strict polynomial superfunctors

3.1. Projectives and injectives. Recall that \( \mathcal{P}_{ev} \) is an abelian category. We say that \( P \in \mathcal{P} \) is projective if the functor \( \text{Hom}_\mathcal{P}(P, -) : \mathcal{P}_{ev} \to \text{svcc}_{ev} \) is exact, and that \( Q \in \mathcal{P} \) is injective if \( \text{Hom}_\mathcal{P}(-, Q) : \mathcal{P}_{ev} \to \text{svcc}_{ev} \) is exact. Given \( F, G \in \mathcal{P} \), one has \( \text{Hom}_\mathcal{P}(F, G)_\Sigma = \text{Hom}_\mathcal{P}_{ev}(F, G) \), while the odd isomorphisms in (2.3.1) restrict to odd isomorphisms

\[
\text{Hom}_\mathcal{P}(F, G)_\Sigma \simeq \text{Hom}_\mathcal{P}(F, \Pi \circ G)_\Sigma = \text{Hom}_\mathcal{P}_{ev}(F, \Pi \circ G),
\]

(3.1.1)

that are natural with respect to even homomorphisms in either variable. Then it follows that a functor is projective (resp. injective) in the above-defined sense if and only if it is projective (resp. injective) as an object in the abelian category \( \mathcal{P}_{ev} \).

Given \( d \in \mathbb{N} \) and \( V \in \mathcal{V} \), set \( \Gamma^{d,V} = \Gamma^d \text{Hom}_\mathcal{P}_{ev}(V, -) \) and set \( S^{d,V} = S^d(V \otimes -) \simeq (\Gamma^{d,V})^\# \). Then by Yoneda's Lemma, there exist for each \( F \in \mathcal{P}_d \) natural isomorphisms

\[
\text{Hom}_\mathcal{P}_d(\Gamma^{d,V}, F) \cong F(V) \quad \text{and} \quad \text{Hom}_\mathcal{P}_d(F, S^{d,V}) \cong F^\#(V).
\]

(3.1.2)

It follows that \( \Gamma^{d,V} \) is projective in \( \mathcal{P}_d \) and \( S^{d,V} \) is injective in \( \mathcal{P}_d \). The next theorem follows from \([1]\) Proposition A.1 and the proof of \([1]\) Theorem 4.2.

**Theorem 3.1.1 (Axtell).** Let \( m, n, d \in \mathbb{N} \) such that \( m, n \geq d \), and set \( V = k^{m|n} = k^m \oplus \Pi(k^n) \). Then the functor \( \Gamma^{d,V} \oplus (\Pi \circ \Gamma^{d,V}) \) is a projective generator in \( (\mathcal{P}_d)_{ev} \), and evaluation on \( V \) induces an equivalence of categories between \( \mathcal{P}_d \) and the category of finite-dimensional left supermodules for the Schur superalgebra \( S(m|n, d) := \text{End}_{k\mathcal{E}_d}(k^{m|n})^\odot(d) \cong \text{End} \Gamma_\mathcal{P}_{ev}(V) \).

3.2. Cohomology groups and Yoneda products. Theorem 3.1.1 implies that the category \( \mathcal{P}_{ev} \) contains enough projectives and enough injectives. Then for each \( F, G \in \mathcal{P} \) we can define the extension group \( \text{Ext}^n_{\mathcal{P}}(F, G) \). Specifically, define \( \text{Ext}^n_{\mathcal{P}}(-, G) \) to be the \( n \)-th right derived functor of \( \text{Hom}_\mathcal{P}(-, G) : \mathcal{P}_{ev} \to \text{svcc}_{ev} \). Equivalently, \( \text{Ext}^n_{\mathcal{P}}(F, G) \) is the value of the \( n \)-th right derived functor of \( \text{Hom}_\mathcal{P}(F, -) : \mathcal{P}_{ev} \to \text{svcc}_{ev} \) when applied to \( G \). With these definitions, the isomorphisms in (3.1.1) extend to odd isomorphisms

\[
\text{Ext}^n_{\mathcal{P}}(F, G)_\Sigma \simeq \text{Ext}^n_{\mathcal{P}}(F, \Pi \circ G)_\Sigma = \text{Ext}^n_{\mathcal{P}_{ev}}(F, \Pi \circ G),
\]

(3.2.1)

Let \( (C, d^C) \) and \( (D, d^D) \) be chain complexes in the category \( \mathcal{P}_{ev} \), and let \( n \in \mathbb{Z} \). Define an even (resp. odd) chain map \( \varphi : C \to D[n] \) to consist for each \( i \in \mathbb{Z} \) of an even (resp. odd) homomorphism \( \varphi_i : C_{i+n} \to D_i \) such that \( d^D_i \circ \varphi_i = \varphi_{i-1} \circ d^C_{i+n} \). Then an even chain map \( \varphi : C \to D[n] \) is precisely a chain map of degree \( -n \) in the category \( \mathcal{P}_{ev} \), while an odd chain map \( \varphi : C \to D[n] \) is equivalent by (3.1.1) to an even chain map \( \varphi' : (\Pi \circ D)[n] \) of degree \( n \). Say that two even (resp. odd) chain maps \( \varphi, \psi : C \to D[n] \) are even (resp. odd) homotopic, and write \( \varphi \simeq \psi \), if there exists for each \( i \in \mathbb{Z} \) an even (resp. odd) homomorphism \( \Sigma_i : C_{i+n} \to D_{i+1} \) such that \( \varphi_i - \psi_i = d^D_{i+1} \circ \Sigma_i + \Sigma_{i-1} \circ d^C_{i+n} \).
As usual, the property of being even (resp. odd) homotopic is an equivalence relation on the set of even (resp. odd) chain maps. Clearly, the composition of two homogeneous chain maps is again a homogeneous chain map. The reader can immediately verify that if \( \varphi, \varphi' : C \to D[n] \) are homotopic chain maps of the same parity, and if \( \psi : D \to E[m] \) is another homogeneous chain map, then \( \psi \circ \varphi \simeq \psi \circ \varphi' \). Similarly, if \( \psi : E \to C[m] \) is a homogeneous chain map, then \( \varphi \circ \psi \simeq \varphi' \circ \psi \). The reader can also check for example that if \( C \) is a projective resolution in \( \mathcal{P}_e \) and if \( D \) is a (positively graded) acyclic chain complex in \( \mathcal{P}_e \), then for each homogeneous homomorphism \( \varphi : H_0(C) \to H_0(D) \), there exists a chain map \( \tilde{\varphi} : C \to D \) of the same parity that induces \( \varphi \), and any two such chain maps inducing \( \varphi \) are homotopic.

Now let \( F, G, H \in \mathcal{P} \), and let \( C, D, E \) be projective resolutions in the category \( \mathcal{P}_e \) of \( F, G, \) and \( H \), respectively. It is well-known that \( \text{Ext}^n_{\mathcal{P}_e}(G, H) \) and \( \text{Ext}^n_{\mathcal{P}_e}(F, G) \) identify with the sets of homotopy classes of odd chain maps \( \psi : D \to E[m] \) and \( \varphi : C \to D[n] \), respectively. Moreover, under these identifications, the Yoneda product

\[
\text{Ext}^n_{\mathcal{P}_e}(G, H) \otimes \text{Ext}^n_{\mathcal{P}_e}(F, G) \to \text{Ext}^{n+n}_{\mathcal{P}_e}(F, H)
\]

is induced by the composition of chain maps. Using (3.1.1) and (3.2.1), it follows that \( \text{Ext}^n_{\mathcal{P}}(G, H) \) and \( \text{Ext}^n_{\mathcal{P}}(F, G) \) identify with the sets of homotopy classes of odd chain maps \( \psi : D \to E[m] \) and \( \varphi : C \to D[n] \). Making these identifications for the even and odd subspaces of \( \text{Ext}^n_{\mathcal{P}}(G, H) \) and \( \text{Ext}^n_{\mathcal{P}}(F, G) \), the previous paragraph implies that we can use the composition of homotopy classes of homogeneous chain maps to lift (3.2.2) to a well-defined even bilinear map

\[
\text{Ext}^n_{\mathcal{P}}(G, H) \otimes \text{Ext}^n_{\mathcal{P}}(F, G) \to \text{Ext}^{m+n}_{\mathcal{P}}(F, H),
\]

which we call the Yoneda product of extensions in \( \mathcal{P} \). Associativity of (3.2.3) follows from the fact that composition of chain maps is associative. In particular, if \( F \in \mathcal{P} \), then \( \text{Ext}^n_{\mathcal{P}}(F, F) \) has the structure of a graded superalgebra, which we call the Yoneda algebra of \( F \).

We leave it to the reader to formulate bisuperfunctor analogues of the definitions in this section.

**Remark 3.2.1.** It follows from the above definition for the Yoneda product (3.2.3) that products of homogeneous elements can be expressed using the isomorphisms (3.2.1) in terms of Yoneda products in the category \( \mathcal{P}_e \). For example, the product \( \text{Ext}^n_{\mathcal{P}}(G, H) \otimes \text{Ext}^n_{\mathcal{P}}(F, G) \to \text{Ext}^{m+n}_{\mathcal{P}}(F, H) \) can be computed via the composite map

\[
\text{Ext}^n_{\mathcal{P}}(G, H) \otimes \text{Ext}^n_{\mathcal{P}}(F, G) \to \text{Ext}^n_{\mathcal{P}_e}(G, H) \otimes \text{Ext}^n_{\mathcal{P}_e}(F, G) \to \text{Ext}^{m+n}_{\mathcal{P}_e}(F, H) \to \text{Ext}^{m+n}_{\mathcal{P}}(F, H),
\]

while \( \text{Ext}^n_{\mathcal{P}}(G, H) \otimes \text{Ext}^{n}_{\mathcal{P}}(F, G) \to \text{Ext}^{m+n}_{\mathcal{P}}(F, H) \) can be computed via the composite map

\[
\text{Ext}^n_{\mathcal{P}}(G, H) \otimes \text{Ext}^n_{\mathcal{P}}(F, G) \to \text{Ext}^n_{\mathcal{P}_e}(G, H) \otimes \text{Ext}^n_{\mathcal{P}_e}(F, G) \to \text{Ext}^{m+n}_{\mathcal{P}_e}(F, H) \to \text{Ext}^{m+n}_{\mathcal{P}}(F, H).
\]

**Remark 3.2.2.** If \( F \in \mathcal{P}_m, G \in \mathcal{P}_n, \) and \( m \neq n \), then \( \text{Hom}_{\mathcal{P}}(F, G) = 0 \). From this it immediately follows that \( \text{Ext}^n_{\mathcal{P}}(F, G) = 0 \) whenever \( F \) and \( G \) are homogeneous of different degrees. In the future we will often apply this observation without further comment.

### 3.3. Operations on cohomology groups.

#### 3.3.1. Duality.

Let \( Q \) and \( R \) be injective resolutions in the category \( \mathcal{P}_e \) of \( F \) and \( G \), respectively. By duality, the even (resp. odd) subspace of \( \text{Ext}^n_{\mathcal{P}}(F, G) \) identifies with the set of homotopy classes of even (resp. odd) chain maps \( \psi : Q[n] \to R \). Since the duality functor \( F \mapsto F^\# \) sends projective objects to injective objects and vice versa, it then follows that the operation of sending a homogeneous chain map \( \varphi : C \to D[n] \) to the corresponding dual map \( \varphi^\# : D[n]^\# \to C^\# \) induces an isomorphism \( \text{Ext}^n_{\mathcal{P}}(F, G) \cong \text{Ext}^n_{\mathcal{P}}(G^\#, F^\#) \), which we denote by \( z \mapsto z^\# \). Moreover, if \( z \) and \( w \) are homogeneous elements such that the Yoneda product \( z \cdot w \) makes sense, then it follows that
(z · w)# = (−1)^{n_1+n_2} w# · z#, since this holds when composing homogeneous morphisms in \( P \). Finally, since \( F \cong F^{##} \), it follows that \( \text{Ext}^*_{\mathcal{P}}(F^{##}, G^{##}) \cong \text{Ext}^*_{\mathcal{P}}(F, G) \). Thus, we can consider the duality functor as inducing an anti-involution on extension groups in \( \mathcal{P} \).

### 3.3.2 Precomposition

Let \( H \in \mathcal{P} \). Precomposition with \( H \), \( F \to F \circ H \), defines an exact even linear endofunctor on the category \( \mathcal{P}_{ev} \). Then for each \( F,G \in \mathcal{P} \), there exists an induced even linear map \( \text{Ext}^*_{\mathcal{P}_{ev}}(F,G) \to \text{Ext}^*_{\mathcal{P}_{ev}}(F \circ H, G \circ H) \) that is compatible with the Yoneda product in \( \mathcal{P}_{ev} \). Since precomposition by \( H \) commutes with the isomorphisms in (3.4.1), we can use the isomorphisms in (3.2.1), together with Remark 3.2.1, to deduce that the operation of precomposition by \( H \) lifts to an even linear map \( \text{Ext}^*_{\mathcal{P}}(F,G) \to \text{Ext}^*_{\mathcal{P}}(F \circ H, G \circ H) \) that is compatible with the Yoneda product.

### 3.3.3 Conjugation by \( \Pi \)

The operation \( F \mapsto \Pi \circ F \) of postcomposition with \( \Pi \) preserves projective resolutions in \( \mathcal{P}_{ev} \), and it sends homogeneous chain maps to homogeneous chain maps of the same parity, and is compatible with the composition of homomorphisms. Then it follows from the interpretation of \( \text{Ext}^*_{\mathcal{P}}(F,G) \) in terms of homotopy classes of homogeneous chain maps that this operation extends for each \( F,G \in \mathcal{P} \) to an even isomorphism \( \text{Ext}^*_{\mathcal{P}}(F,G) \cong \text{Ext}^*_{\mathcal{P}}(\Pi \circ F, \Pi \circ G) \) that is compatible with the Yoneda product (3.2.3). Now given \( F \in \mathcal{P} \), set \( F^{\Pi} = \Pi \circ F \circ \Pi \). We refer to the operation \( F \mapsto F^{\Pi} \) as conjugation by \( \Pi \). Combining the comments in this section with those of Section 3.3.2, it follows for each \( F,G \in \mathcal{P} \) that there exists an even isomorphism \( \text{Ext}^*_{\mathcal{P}}(F,G) \cong \text{Ext}^*_{\mathcal{P}}(F^{\Pi}, G^{\Pi}) \) that is compatible with the Yoneda product (3.2.3). We denote this map by \( z \mapsto z^{\Pi} \), and refer to it as the conjugation action of \( \Pi \) on extension groups in \( \mathcal{P} \). Since \( \Pi \circ \Pi = I \), then \( (z^{\Pi})^{\Pi} = z \).

### 3.4 Cup products and coproducts

Let \( V,W \in \mathcal{V} \), and let \( d, e \in \mathbb{N} \). The exponential property for \( \Gamma \) implies that \( \Gamma^{d \cdot V} \otimes \Gamma^e \cdot W \) is isomorphic to a direct summand of \( \Gamma^{d + e, V \otimes W} \in \mathcal{P}_{d + e} \). Then it follows from Theorem 3.1.1 and the Künneth theorem that if \( P \) and \( Q \) are projective resolutions in \( \mathcal{P}_{ev} \) of \( A \) and \( C \), respectively, then the tensor product of complexes \( P \otimes Q \) is a projective resolution in \( \mathcal{P}_{ev} \) of \( A \otimes C \). Similarly, the external tensor product \( P \boxtimes Q \) is a projective resolution in \( (\text{bi-\text{P}})_{ev} \) of \( A \boxtimes C \). Now given \( B,D \in \mathcal{P} \), there exist well-defined even linear maps

\[
\text{Ext}^n_{\mathcal{P}}(A,B) \otimes \text{Ext}^m_{\mathcal{P}}(C,D) \to \text{Ext}^{m+n}_{\mathcal{P}}(A \otimes C, B \otimes D) \quad \text{and}
\]

\[
\text{Ext}^n_{\mathcal{P}}(A,B) \otimes \text{Ext}^m_{\mathcal{P}}(C,D) \to \text{Ext}^{m+n}_{\mathcal{P}}(A \boxtimes C, B \boxtimes D)
\]

induced by sending cocycles \( \varphi : P \to B \) and \( \psi : Q \to D \) to the tensor product of maps \( \varphi \otimes \psi : P \otimes Q \to B \otimes D \) or to the external tensor product of maps \( \varphi \boxtimes \psi : P \boxtimes Q \to B \boxtimes D \), respectively. The same argument as for Proposition 3.6 shows that (3.4.2) induces an isomorphism

\[
\kappa : \text{Ext}^n_{\mathcal{P}}(A,B) \otimes \text{Ext}^m_{\mathcal{P}}(C,D) \cong \text{Ext}^{m+n}_{\mathcal{P}}(A \boxtimes C, B \boxtimes D).
\]

Suppose \( A \) is a \( \mathcal{P} \)-coalgebra and \( B \) is a \( \mathcal{P} \)-algebra. Then there exist even bilinear maps

\[
\text{Ext}^n_{\mathcal{P}}(A,C) \otimes \text{Ext}^m_{\mathcal{P}}(A,D) \to \text{Ext}^{m+n}_{\mathcal{P}}(A \otimes C, D),
\]

\[
\text{Ext}^m_{\mathcal{P}}(C,B) \otimes \text{Ext}^n_{\mathcal{P}}(D,B) \to \text{Ext}^{m+n}_{\mathcal{P}}(C \otimes D, B),
\]

\[
\text{Ext}^n_{\mathcal{P}}(A,B) \otimes \text{Ext}^m_{\mathcal{P}}(A,B) \to \text{Ext}^{m+n}_{\mathcal{P}}(A,B)
\]

that arise in the usual fashion from (3.4.1) by composing with the maps in cohomology induced by the coproduct \( \Delta_A : A \to A \otimes A \), the product \( m_A : B \otimes B \to B \), or both. We refer to these maps as the cup products of the corresponding Ext-groups.

Given a graded \( \mathcal{P} \)-algebra \( B = \bigoplus_{n \in \mathbb{Z}} B^n \), consider the diagram

\[
\begin{array}{ccc}
B^i \otimes B^j & \xrightarrow{m_B} & B^{i+j} \\
\downarrow & & \downarrow \\
B^j \otimes B^i & \xrightarrow{m_B} & B^{i+j}
\end{array}
\]

for \( \tau : B^i \otimes B^j \to B^j \otimes B^i \).
Lemma 3.4.1. Let essentially the same “straightforward (but tiresome)” reasoning as [9, Lemma 1.11].

If either of these conditions is satisfied, say that is the supertwist map. If in which the left-hand vertical arrow is induced by the supertwist map. Set \( \varepsilon(B) = 0 \) if for each \( i, j \in \mathbb{Z} \) the above diagram commutes, and set \( \varepsilon(B) = 1 \) if for each \( i, j \in \mathbb{Z} \) the above diagram commutes up to the sign \((-1)^{ij}\), i.e., if the grading makes \( B \) into a graded-commutative \( \mathcal{P} \)-algebra. If either of these conditions is satisfied, say that \( B \) is \( \varepsilon(B) \)-commutative. Similarly, one defines the notion of \( \varepsilon(A) \)-cocommutativity for a graded \( \mathcal{P} \)-coalgebra \( A \). Now the next lemma follows from essentially the same “straightforward (but tiresome)” reasoning as [9, Lemma 1.11].

\[ \text{Lemma 3.4.1. Let } A = \bigoplus_{n \in \mathbb{Z}} A^n \text{ be a graded } \mathcal{P} \text{-coalgebra, and let } B = \bigoplus_{n \in \mathbb{Z}} B^n \text{ be a graded } \mathcal{P} \text{-algebra. Consider the diagram} \]

\[
\begin{array}{c}
\text{Ext}_{\mathcal{P}}(A^i, B^j) \otimes \text{Ext}_{\mathcal{P}}(A^j, B^i) \twoheadrightarrow \text{Ext}_{\mathcal{P}}^s(A^{i+j}, B^{i+j}) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
\text{Ext}_{\mathcal{P}}(A^i, B^j) \otimes \text{Ext}_{\mathcal{P}}(A^i, B^i) \twoheadrightarrow \text{Ext}_{\mathcal{P}}^s(A^{i+j}, B^{i+j})
\end{array}
\]

in which the horizontal arrow, and the left-hand vertical arrow is the supertwist map. If \( A \) is \( \varepsilon(A) \)-cocommutative and if \( B \) is \( \varepsilon(B) \)-commutative, then the diagram commutes up to the sign \((-1)^{s+t+\varepsilon(A)ij+\varepsilon(B)ij}\).

Recall that the diagonal functor \( D : \mathcal{V} \rightarrow \mathcal{V} \times \mathcal{V} \) and the direct sum functor \( \Sigma : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{V} \) are adjoint functors (on both sides). Using the definition (2.4.10) of homomorphisms in the category \( \Gamma^n(\mathcal{V} \times \mathcal{V}) \), it follows that \( D \) and \( \Sigma \) extend to a pair of adjoint functors \( \Gamma^n(\mathcal{V}) \rightarrow \Gamma^n(\mathcal{V} \times \mathcal{V}) \) and \( \Gamma^n(\mathcal{V} \times \mathcal{V}) \rightarrow \Gamma^n(\mathcal{V}) \), which we also denote \( D \) and \( \Sigma \). Then precomposition by \( D \) and \( \Sigma \) defines a pair of adjoint functors \( \mathcal{P}_n \rightarrow \mathcal{P}(n) \), \( F \mapsto F \circ \Sigma \), and \( \mathcal{P}(n) \rightarrow \mathcal{P}_n \), \( F \mapsto F \circ D \). Extending componentwise, we get a pair of exact adjoint functors \( \mathcal{P} \rightarrow \text{bi-} \mathcal{P} \), \( F \mapsto F \circ \Sigma \), and \( \text{bi-} \mathcal{P} \rightarrow \mathcal{P} \), \( G \mapsto G \circ D \). Then for each \( F \in \mathcal{P} \) and \( G \in \text{bi-} \mathcal{P} \), one gets an isomorphism

\[
\alpha : \text{Ext}_{\text{bi-} \mathcal{P}}(F \circ \Sigma, G) \cong \text{Ext}_{\mathcal{P}}(F, G \circ D)
\]

that is natural with respect to even homomorphisms in either \( F \) or \( G \). Now the next theorem follows by the same reasoning as its classical analogue; cf. [9, §1.7] and [20, §§5.3–5.4].

Theorem 3.4.2. Let \( A \in \mathcal{P} \) be an exponential superfunctor, let \( C \in \mathcal{P}_m \), and let \( D \in \mathcal{P}_n \). Write \( \bigoplus_{n \in \mathbb{N}} A^n \) for the polynomial decomposition of \( A \). Then for each \( m, n \in \mathbb{N} \), the cup products (3.4.4) and (3.4.5) induce isomorphisms

\[
\text{Ext}_{\mathcal{P}}(A^m, C) \otimes \text{Ext}_{\mathcal{P}}(A^n, D) \cong \text{Ext}_{\mathcal{P}}(A^{m+n}, C \otimes D), \quad \text{and}
\]

\[
\text{Ext}_{\mathcal{P}}(C, A^m) \otimes \text{Ext}_{\mathcal{P}}(A^n, D) \cong \text{Ext}_{\mathcal{P}}(C \otimes D, A^{m+n}).
\]

Let \( F \in \mathcal{P} \). We say that \( F \) is additive if the external direct sum \( F \oplus F \in \text{bi-} \mathcal{P} \) is isomorphic as a strict polynomial bisuperfunctor to \( F \circ \Sigma \), i.e., if for each \( V, W \in \mathcal{V} \), there exists a bifunctorial isomorphism \( F(V) \oplus F(W) \cong F(V \oplus W) \). For example, if \( A \) is an exponential superfunctor with polynomial decomposition \( \bigoplus_{n \in \mathbb{N}} A^n \), \( A^0 = k \), and \( n \) is the least positive integer such that \( A^n \neq 0 \), then \( A^n \) is additive. The next theorem is thus closely related to Theorem 3.4.2. Its proof follows from a repetition of the proof of [12, Theorem 2.13], after first applying (3.2.1) to reduce to extension groups in the abelian category \( \mathcal{P}_{ev} \).

Theorem 3.4.3. Let \( T \) and \( T' \) be homogeneous strict polynomial superfunctors of positive degrees, and let \( F \in \mathcal{P} \) be an additive functor. Then \( \text{Ext}_{\mathcal{P}}(F, T \otimes T') = 0 \).

Finally, let \( A \) be a \( \mathcal{P} \)-algebra and let \( B \) be a \( \mathcal{P} \)-coalgebra. Then the coproduct

\[
\text{Ext}_{\mathcal{P}}(A, B) \rightarrow \text{Ext}_{\mathcal{P}}(A, B) \otimes \text{Ext}_{\mathcal{P}}(A, B)
\]
is defined in terms of the coproduct \( \Delta_B \), the isomorphism \( \alpha \) of (3.4.8), the homomorphism \( \mu_A \) of (2.5.1), and the isomorphism \( \kappa \) of (3.4.3) as the composite linear map

\[
\Ext^\bullet(A,B) \xrightarrow{\Delta_B} \Ext^\bullet(A,B \otimes B) \xrightarrow{\alpha^{-1}} \Ext^\bullet_{k\Sigma}(A \circ \Sigma, B \boxtimes B) \\
\xrightarrow{\mu_A^*} \Ext^\bullet_{k\Sigma}(A \boxtimes A, B \boxtimes B) \xrightarrow{\kappa^{-1}} \Ext^\bullet(A,B) \otimes \Ext^\bullet(A,B).
\]

(3.4.12)

The reader can formulate the coproduct analogue of Lemma 3.4.1 by replacing the horizontal arrows in (3.5.1) with the corresponding (left-facing) coproducts. If \( A \) is an exponential superfunctor, then the composition \( \kappa^{-1} \circ \mu_A^* \circ \alpha^{-1} \) in (3.4.12) is the inverse of the cup product isomorphism (3.4.9); cf. the first paragraph of Remark 2.5.5.

3.5. \( \Ext^n \) and \( n \)-extensions. Recall that, while \( \mathcal{P} \) is not an abelian category, it is closed under kernels and cokernels of homogeneous morphisms. Then it makes sense to consider exact sequences in \( \mathcal{P} \) in which each morphism is homogeneous. Given \( F,G \in \mathcal{P} \), define a homogeneous \( n \)-extension of \( F \) by \( G \) in \( \mathcal{P} \) to be an exact sequence

\[
E : 0 \to G = E_{n+1} \to E_n \to \cdots \to E_1 \to E_0 = F \to 0
\]

(3.5.1)

in \( \mathcal{P} \) in which each morphism is homogeneous. Define the parity of \( E \) to be the sum of the parities of the morphisms appearing in \( E \). Say that two \( n \)-extensions \( E \) and \( E' \) of \( F \) by \( G \) in \( \mathcal{P} \) satisfy the relation \( E \sim E' \) if there exists a commutative diagram

\[
\begin{array}{ccc}
0 & \to & G \\
\downarrow & & \downarrow \\
0 & \to & E_n \\
\downarrow & & \downarrow \\
0 & \to & E'_n \\
\downarrow & & \downarrow \\
0 & \to & G \\
\downarrow & & \downarrow \\
0 & \to & E_1 \\
\downarrow & & \downarrow \\
0 & \to & F \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

(3.5.2)

in which each vertical arrow is a homogeneous morphism in \( \mathcal{P} \). Then the relation \( E \sim E' \) generates an equivalence relation \( E \sim E' \) on the set of homogeneous \( n \)-extensions of \( F \) by \( G \).

One can show that if \( E \sim E' \), and hence if \( E \sim E' \), then \( E \) and \( E' \) must be of the same parity. Define \( \Yext^\bullet_\mathcal{P}(F,G)_\Pi \) (resp. \( \Yext^\bullet_\mathcal{P}(F,G)_\bar{\Pi} \)) to be the set of equivalence classes of even (resp. odd) homogeneous \( n \)-extensions of \( F \) by \( G \) in \( \mathcal{P} \). One can also show that the operation of composing (i.e., splicing) homogeneous extensions induces a well-defined product between equivalence classes. We call this product the composition of homogeneous extensions in \( \mathcal{P} \).

**Proposition 3.5.1.** For each \( F,G \in \mathcal{P} \) and each \( n \in \mathbb{N} \), there exist bijections

\[
\theta_0 : \Ext^n_\mathcal{P}(F,G)_\Pi \to \Yext^n_\mathcal{P}(F,G)_\Pi, \quad \text{and} \quad \theta_1 : \Ext^n_\mathcal{P}(F,G)_\bar{\Pi} \to \Yext^n_\mathcal{P}(F,G)_\bar{\Pi}
\]

under which the Yoneda product (3.2.3) of homogeneous elements corresponds to the composition of homogeneous extensions in \( \mathcal{P} \).

**Proof.** First we construct \( \theta_0 \). Let \( \Yext^n_{\mathcal{P}_{ev}}(F,G) \) be the set of equivalence classes of \( n \)-extensions of \( F \) by \( G \) in the category \( \mathcal{P}_{ev} \). Since \( \mathcal{P}_{ev} \) is an abelian category, it is well-known that there exists a bijection \( \theta' : \Ext^n_{\mathcal{P}_{ev}}(F,G) \to \Yext^n_{\mathcal{P}_{ev}}(F,G) \) under which the Yoneda product for \( \Ext^n_{\mathcal{P}_{ev}}(F,G) \) corresponds to the composition of extensions for \( \Yext^n_{\mathcal{P}_{ev}}(F,G) \). Next, the inclusion of categories \( \mathcal{P}_{ev} \hookrightarrow \mathcal{P} \) induces a function \( \theta'' : \Yext^n_{\mathcal{P}_{ev}}(F,G) \to \Yext^n_{\mathcal{P}}(F,G)_\Pi \) that is compatible with the composition of extensions. We claim that \( \theta'' \) is a bijection. Assuming this, the composite

\[
\Ext^\bullet_\mathcal{P}(F,G)_\Pi = \Ext^\bullet_{\mathcal{P}_{ev}}(F,G) \xrightarrow{\theta'} \Yext^\bullet_{\mathcal{P}_{ev}}(F,G) \xrightarrow{\theta''} \Yext^\bullet_{\mathcal{P}}(F,G)_\Pi
\]

then provides the desired bijection \( \theta_0 \).

To see that \( \theta'' \) is a surjection, let \([E] \in \Yext^n_{\mathcal{P}}(F,G)_\Pi\), and write \( E \) as in (3.5.1). If each morphism appearing in \( E \) is even, then \([E]\) is in the image of \( \theta'' \), and we are done. Otherwise, let \( i \) be the least index such that the morphism \( E_{i+1} \to E_i \) is odd. Then one can show that \( E \sim E' \), where \( E' \)}
is obtained from $E$ by replacing $E_{i+1}$ by $\Pi \circ E_{i+1}$, and replacing the morphisms $E_{i+1} \to E_i$ and $E_{i+2} \to E_{i+1}$ by the composites $\Pi \circ E_{i+1} \simeq E_{i+1} \to E_i$ and $E_{i+2} \to E_{i+1} \simeq \Pi \circ E_{i+1}$, respectively. Now arguing by induction on $i$, and using the assumption that $E$ was even, it follows that $E$ is equivalent to an $n$-extension in which each morphism is even, and hence that $[E] \in \im(\theta'')$. To see that $\theta''$ is an injection, let $\theta'' : \Ext^b_P(F,G)_{\Pi} \to \Ext^b_P(F,G)_{\Pi} = \Ext^b_{Pev}(F,G)$ be defined by precisely the same procedure as in the first half of the proof of [13, Theorem IV.9.1]. Then the composite $\theta'' \circ \theta'' \circ \theta'$ is the identity map, which proves that $\theta''$ must be one-to-one.

Define $\theta_1$ to be the composite function

$$\Ext^b_P(F,G)_{\Pi} \simeq \Ext^b_P(F,\Pi \circ G)_{\Pi} \xrightarrow{\theta_0} \Yext^b_P(F,\Pi \circ G)_{\Pi} \simeq \Yext^b_P(F,G)_{\Pi},$$

where the last isomorphism is the obvious parity-reversing bijection induced by the odd isomorphism $G \simeq \Pi \circ G$. Then $\theta_1$ is a bijection, and using Remark 3.2.1 one can then check that $\theta_0$ and $\theta_1$ satisfy the compatible with the respective products. □

3.6. Hypercohomology. Since $\Hom_P(-,-)$ defines a bifunctor $P_{ev} \to \text{svcc}_{ev}$, contravariant in the first variable and covariant in the second, we can consider for each chain complex $A$ in $P_{ev}$ and each cochain complex $C$ in $P_{ev}$ the $n$-th hypercohomology group $\Ext^n_P(A,C)$ of $\Hom_P(-,-)$, as defined for example in [6, XVII.2]. In particular, we will make extensive use of the fact that if $A$ is an object in $P$ considered as a chain complex concentrated in degree zero, then the two hypercohomology spectral sequences converging to $\Ext^*_P(A,C)$ are each right modules over the Yoneda algebra $\Ext^*_P(A,A)$. For the reader’s convenience we briefly describe some of the details behind this construction in this special case.

Let $A \in P$, and let $C$ be a (non-negative) cochain complex in $P_{ev}$. Let $P = P_*$ be a projective resolution in $P_{ev}$ of $A$, and let $Q = Q^{*,*}$ be an injective Cartan-Eilenberg resolution in $P_{ev}$ of $C$. For $i,j \in \mathbb{N}$, set $\Hom_P(P,Q)^{i,j} = \bigoplus_{r+s = j} \Hom_P(P_r,Q^{i,s})$. This indexing induces on $\Hom_P(P,Q)$ the structure of a first quadrant double complex in which the horizontal differential is induced by the horizontal differential from $Q$, and in which the vertical differential is induced by the differential from $P$ and the vertical differential from $Q$. Now $\Ext^n_P(A,C)$ is the $n$-th cohomology group of the double complex $\Hom_P(P,Q)$. Interpreting homogeneous elements of $\Ext^m_P(A,A)$ as homotopy classes of homogeneous chain maps $P \to P[m]$, the right action of $\Ext^m_P(A,A)$ on $\Ext^n_P(A,C)$,

$$\Ext^m_P(A,C) \otimes \Ext^m_P(A,A) \to \Ext^{m+n}_P(A,C),$$

is then induced by the composition of homomorphisms

$$\Hom_P(P,Q) \otimes \Hom_P(P,P) \to \Hom_P(P,Q).$$

The first quadrant double complex $\Hom_P(P,Q)$ gives rise to two spectral sequences that each converge to $\Ext^*_P(A,C)$. The first hypercohomology spectral sequence arises from computing the cohomology of the double complex first along columns, and takes the form

$$I^{E_1}_{s,t} = \Ext^s_P(A,C^t) \Rightarrow \Ext^{s+t}_P(A,C),$$

Then the differential $d_1 : I^{E_1}_{s,t} \to I^{E_1}_{s+1,t}$ identifies with the map in cohomology induced by the differential $C^s \to C^{s+1}$ of the complex $C$. The second hypercohomology spectral sequence arises from computing the cohomology of the double complex first along rows, and takes the form

$$II^{E_2}_{s,t} = \Ext^s_P(A,\Pi^t(C)) \Rightarrow \Ext^{s+t}_P(A,C).$$

The two spectral sequences are related by the composite map

$$II^{E_2}_{s,0} \to II^{E_0}_{s,0} \hookrightarrow \Ext^s_P(A,C) \to I^{E_0}_{s,0} \Rightarrow I^{E_0}_{s,0},$$

which identifies with the map in cohomology $\Ext^*_P(A,\Pi^0(C)) \to \Ext^*_P(A,C^0)$ induced by the inclusion $\Pi^0(C) \hookrightarrow C^0$; cf. [6, XVII.3], though what we index as the first spectral sequence is indexed there as the second, and vice versa. The reader can check that the filtrations on $\Hom_P(P,Q)$ that
give rise to (3.6.1) and (3.6.2) are compatible with the right action of \( \text{Hom}_P(P, P) \), and thus (3.6.1) and (3.6.2) become spectral sequences of right \( \text{Ext}^*_P(A, A) \)-modules. In particular, the right action on the \( E_1 \)-page of (3.6.1) and the right action on the \( E_2 \)-page of (3.6.2) identify with the corresponding Yoneda products defined in Section 3.2.

**Remark 3.6.1.** By general abstract nonsense, restriction from \( P \) to \( P \) extends for each \( F, G \in P \) to a linear map \( \text{Ext}^*_P(F, G) \to \text{Ext}^*_P(F|_{\mathcal{V}^r}, G|_{\mathcal{V}^r}) \) that is compatible with Yoneda products. More generally, if \( A \in P \) and if \( C \) is a cochain complex in \( P_{ev} \), then restriction from \( P \) to \( P \) induces a linear map \( \text{Ext}^*_P(A, C) \to \text{Ext}^*_P(A|_{\mathcal{V}^r}, C|_{\mathcal{V}^r}) \) on hypercohomology groups.

4. The Yoneda algebra \( \text{Ext}^*_P(I^{(r)}, I^{(r)}) \)

Our goal in this section is to describe the structure of the Yoneda algebra \( \text{Ext}^*_P(I^{(r)}, I^{(r)}) \). Since \( I^{(r)} = I_0^{(r)} \oplus I_1^{(r)} \), it follows that \( \text{Ext}^*_P(I^{(r)}, I^{(r)}) \) is isomorphic to the matrix ring \([1.0.1] \), it suffices to describe each of the components of the matrix ring, and to describe the possible products between them. Our strategy is based on the approach in [12] using hypercohomology spectral sequences.

4.1. The super de Rham complex. Set \( \Omega = S \otimes A \), and recall from Section 2.3 that \( \Omega \) inherits from \( S \) and \( A \) the structure of a \( P \)-algebra. Given \( i, n \in \mathbb{N} \) with \( i \leq n \), define \( \Omega^n_i \) to be the subfunctor \( S^{n-i} \otimes A^i \) of \( \Omega \). Following [12 §4], we call \( i \) the cohomological degree and \( n \) the total degree of \( \Omega^n_i \). Now set \( \Omega_n = \bigoplus_{i=0}^{n} \Omega^n_i \) and \( \Omega^i = \bigoplus_{n=0}^{i} \Omega^n_i \). Then the cohomological grading \( \Omega = \bigoplus_{i \in \mathbb{N}} \Omega^i \) makes \( \Omega \) into a graded-commutative graded \( P \)-algebra.

The product and coproduct maps on \( S \) and \( A \) induce natural transformations

\[
d : \Omega^n_i = S^{n-i} \otimes A^i \to S^{n-i-1} \otimes I \otimes A^i \to S^{n-i-1} \otimes A^{i+1} = \Omega^{n+1}_i, \quad \text{and} \\
\kappa : \Omega^n_i = S^{n-i} \otimes A^i \to S^{n-i} \otimes I \otimes A^{i-1} \to S^{n-i+1} \otimes A^{i-1} = \Omega^{n-1}_i.
\]

One can check for each \( V \in \mathcal{V} \) that \( d(V) \) and \( \kappa(V) \) each make \( \Omega(V) \) into a differential graded superalgebra, and hence that \( d \) and \( \kappa \) are differentials. We refer to the resulting complexes \((\Omega, d)\) and \((\Omega, \kappa)\) as the *super de Rham complex* and the *super Koszul complex*. On \( \mathcal{V}_r \), \((\Omega, d)\) and \((\Omega, \kappa)\) restrict to the ordinary de Rham complex \((\Omega, d)\) and the ordinary Koszul complex \( K^z := (\Omega, \kappa) \), respectively; cf. [10, 12]. On \( \mathcal{V}_T \), \((\Omega, d)\) and \((\Omega, \kappa)\) restrict to the dual de Rham complex \((\Omega^\#, d^\#)\) and the dual Koszul complex \( K^z^\# := (\Omega^\#, \kappa^\#) \). Specifically, \( \Omega^n_i|_{\mathcal{V}_T} \cong (\Omega^{n-i})^\# \).

If \( V \) is an abelian Lie superalgebra, then \((\Omega(V), \kappa(V))\) identifies with the Koszul resolution of \( V \) as studied in [7, §3.1]. In particular, \((\Omega, \kappa)\) is acyclic.

**Lemma 4.1.1.** On \( \Omega_n \), the transformation \( d \circ \kappa + \kappa \circ d \) acts as multiplication by \( n \).

**Proof.** Let \( V \in \mathcal{V} \), and set \( f = d(V) \circ \kappa(V) + \kappa(V) \circ d(V) \). Since \( d(V) \) and \( \kappa(V) \) each make \( \Omega(V) \) into a differential graded superalgebra, it follows that \( f \) acts as an ordinary algebra derivation on \( \Omega(V) \), i.e., \( f(ab) = f(a) \cdot b + a \cdot f(b) \) for all \( a, b \in \Omega(V) \). Then it suffices to consider the action of \( f \) on a set of algebra generators for \( \Omega(V) = S(V) \otimes A(V) \), where the conclusion is easily verified. \( \square \)

Since the de Rham differential \( d \) is a derivation, it follows that the cohomology \( H^*(\Omega) \) of \( \Omega \) with respect to \( d \) inherits the structure of a graded-commutative graded \( P \)-algebra. Since \( d \) restricts the total degree, one gets \( H^*(\Omega) = \bigoplus_{n \in \mathbb{N}} H^*(\Omega^n) \), and Lemma 4.1.1 implies that \( H^*(\Omega^n) \neq 0 \) only if \( p \mid n \). The next theorem is a super analogue of the ordinary Cartier isomorphism. Unlike its ordinary analgoue, the super Cartier isomorphism does not preserve the cohomological degree.

**Theorem 4.1.2.** There exists a \( P \)-algebra isomorphism \( \Omega^{(1)} \cong H^*(\Omega) \) that restricts for each \( n \in \mathbb{N} \) to an isomorphism \( \Omega^{(1)} \cong H^*(\Omega^n) \).

**Proof.** First we show for each \( V \in \mathcal{V} \) that \( \Omega(V^{(1)}) \cong H^*(\Omega(V)) \) as superalgebras. Then we show that this family of isomorphisms lifts to an isomorphism of strict polynomial superfunctors. Recall
from Section 2.5 that $\Omega$ is an exponential superfunctor. For $V, W \in \mathcal{Y}$, the exponential isomorphism $\Omega(V \oplus W) \cong \Omega(V) \otimes \Omega(W)$ defines an isomorphism of complexes between $\Omega(V \oplus W)$ and the tensor product of complexes $\Omega(V) \otimes \Omega(W)$. Then to prove that $\Omega(V^{(1)}) \cong H^\bullet(\Omega(V))$ as superspaces, it suffices by the Künneth theorem to assume that $V$ is one-dimensional.

First suppose $V = k$, and let $v \in V$ be nonzero. Then $\Omega^0_n(V) = S^n(V) \otimes A^0(V)$ is spanned by $v^n \otimes 1$, $\Omega^1_n(V) = S^{n-1}(V) \otimes A^1(V)$ is spanned by $v^{n-1} \otimes v$, and $\Omega^2_n(V) = 0$ otherwise. Now $d(v^n \otimes 1) = n \cdot (v^{n-1} \otimes v)$, so $H^1(\Omega_n(V))$ is one-dimensional and spanned by the class of $v^n \otimes 1$ if $i = 0$ and $p \mid n$, is one-dimensional and spanned by the class of $v^{n-1} \otimes v$ if $i = 1$ and $p \mid n$, and is zero otherwise. Then $H^\bullet(\Omega_n(V)) \cong \Omega_n(V^{(1)})$ as superspaces.

Next suppose $V = \Pi(k)$, and let $v \in V$ be nonzero. Then $\Omega^0_n(V) = S^0(V) \otimes A^n(V)$ is spanned by $1 \otimes \gamma_n(v)$, $\Omega^1_n(V) = S^1(V) \otimes A^{n-1}(V)$ is spanned by $v \otimes \gamma_{n-1}(v)$, and $\Omega^2_n(V) = 0$ otherwise. Now $d(v \otimes \gamma_{n-1}(v)) = n \cdot (1 \otimes \gamma_{n}(v))$, so $H^1(\Omega(V))$ is one-dimensional and spanned by the class of $v \otimes \gamma_{n-1}(v)$ if $i = n - 1$ and $p \mid n$, is one-dimensional and spanned by the class of $1 \otimes \gamma_{n}(v)$ if $i = n$ and $p \mid n$, and is zero otherwise. Then $H^\bullet(\Omega_n(V)) \cong \Omega_n(V^{(1)})$ as superspaces.

Now let $V \in \mathcal{Y}$ be arbitrary. Let $\{v_1, \ldots, v_k\}$ be a homogeneous basis for $V$, and let $\{v'_1, \ldots, v'_\ell\}$ be the same set but considered as a basis for $V^{(1)}$. From the previous two paragraphs, it follows not only that $\Omega(V^{(1)}) \cong H^\bullet(\Omega(V))$ as superspaces, but that $H^\bullet(\Omega(V))$ is generated as a superalgebra by the cohomology classes of $v'_i \otimes 1$ and $v'_i \otimes v_i$ for $\overline{v_i} = \overline{0}$, and the classes of $v_i \otimes \gamma_{n-1}(v_i)$ and $1 \otimes \gamma_{pe}(v'_i)$ for $\overline{v_i} = \overline{1}$ and $e \geq 1$. These elements generate a subalgebra of $\Omega(V)$ isomorphic to $\Omega(V^{(1)})$. Explicitly, there exists an injective algebra homomorphism $\theta(\cdot) : \Omega(V^{(1)}) \hookrightarrow \Omega(V)$ satisfying

$$ (4.1.1) \quad \begin{align*} v'_i \otimes 1 & \mapsto v_i^0 \otimes 1, \\ 1 \otimes v'_i & \mapsto v_i^{-1} \otimes v_i \end{align*} \quad \text{for } \overline{v_i} = \overline{0}, \quad \begin{align*} v'_i \otimes 1 & \mapsto v_i \otimes \gamma_{pe-1}(v_i), \\ 1 \otimes \gamma_{pe}(v'_i) & \mapsto 1 \otimes \gamma_{pe+1}(v_i) \end{align*} \quad \text{for } \overline{v_i} = \overline{1} \text{ and } e \geq 0. $$

Then $\theta(V)$ induces an isomorphism of superalgebras $\Omega(V^{(1)}) \cong H^\bullet(\Omega(V))$, which by abuse of notation we also denote by $\theta(V)$. Using the relations in the divided power algebra and the fact that $A^\bullet = \lambda$ for all $\lambda \in \mathbb{F}_p \subseteq k$, one can check that $\theta(V)(1 \otimes \gamma_{n}(v'_i)) = 1 \otimes \gamma_{pe}(v_i)$ for all $n \in \mathbb{N}$.

Set $F = H^\bullet(\Omega) \in \mathcal{P}$. Now we show that the isomorphism $\theta(V) : \Omega(V^{(1)}) \rightarrow F(V)$ lifts to an isomorphism of strict polynomial superfunctors. To do this, we must show for each $V, W \in \mathcal{Y}$, each $\phi \in \Gamma^m \text{Hom}_k(V, W)$, and each $z \in \Omega_n(V^{(1)})$ that

$$ (4.1.2) \quad [\theta(V) \circ \Omega^{(1)}(\phi)](z) = [F(\phi) \circ \theta(V)](z). $$

Since the multiplication morphisms for $\Omega^{(1)}$ and $H^\bullet(\Omega)$ are natural transformations, it suffices to verify (4.1.2) as $z$ ranges over a set of generators of the algebra $\Omega(V^{(1)})$. By linearity, it also suffices to verify (4.1.2) as $\phi$ ranges over a basis for $\Gamma^m \text{Hom}_k(V, W)$. We will also find it convenient not to consider $F(\phi)$ directly, but to consider the function $\Omega(\phi)$ that induces $F(\phi)$.

Let $\{w_1, \ldots, w_m\}$ be a homogeneous basis for $W$, and let $\{w'_1, \ldots, w'_m\}$ be the same set but considered as a basis for $W^{(1)}$. For each $1 \leq i \leq \ell$ and $1 \leq j \leq m$, let $e_{i,j} \in \text{Hom}_k(V, W)$ be the “matrix unit” satisfying $e_{i,j}(v_a) = \delta_{i,a} w_i$, and let $e'_{i,j}$ be the corresponding element of $\text{Hom}_k(V^{(1)}, W^{(1)})$. Then the $e_{i,j}$ form a homogeneous basis for $\text{Hom}_k(V, W)$. Consequently, $\Gamma(\text{Hom}_k(V, W))$ admits a basis consisting of all monomials $\prod_{i,j} \gamma_{a_{i,j}}(e_{i,j})$ (the products taken, say, in the lexicographic order) with $a_{i,j} \in \mathbb{N}$ and $a_{i,j} \leq 1$ if $e_{i,j} = \overline{1}$. Of course, a similar statement holds for $\Gamma(\text{Hom}_k(V^{(1)}, W^{(1)}))$.

Now to verify (4.1.2), we can assume that $\phi = \prod_{i,j} \gamma_{a_{i,j}}(e_{i,j})$ for some $a_{i,j} \in \mathbb{N}$ with $a_{i,j} \leq 1$ if $e_{i,j} = \overline{1}$, and that $z$ is one of the generators for $\Omega(V^{(1)})$ appearing in (4.1.1).

First we consider the expression $[\theta(W) \circ \Omega^{(1)}(\phi)](z)$. Since $(\Omega^{(1)}(\phi) = \Omega(\varphi^\#(\phi))$, where $\varphi^\#$ is the dual Frobenius morphism described in (2.7.2), it follows that $\varphi^\#(\phi) = 0$ if $p \nmid a_{i,j}$ for some $a_{i,j}$, and $\varphi^\#(\phi) = \prod_{i,j} \gamma_{a_{i,j} \cdot p}(e'_{i,j})$ otherwise. So suppose $p \mid a_{i,j}$ for each $a_{i,j}$; say $a_{i,j} = p b_{i,j}$. If $\tau' = \overline{1}$
and $z = v'_n \otimes 1 \in \Omega_1(V^{(1)})$, then by assumption $\sum i,j a_{i,j} = p$, so $\phi = \gamma_p(e_{i,j})$ for some $i,j$. Then

$$[\theta(W) \circ \Omega^{(1)}(\gamma_p(e_{i,j}))](v'_n \otimes 1) = \delta_{j,n} \cdot \theta(W)(w'_i \otimes 1) = \delta_{j,n} (w'_i \otimes 1),$$

and similarly,

$$[\theta(W) \circ \Omega^{(1)}(\gamma_p(e_{i,j}))](1 \otimes v'_n) = \delta_{j,n} (w'_i \otimes w_i)$$

if $\tau'_n = \overline{0}$, and

$$[\theta(W) \circ \Omega^{(1)}(\gamma_p(e_{i,j}))](v'_n \otimes 1) = \delta_{j,n} (w_i \otimes \gamma_{p-1}(w_i))$$

if $\tau'_n = \overline{1}$.

Now suppose $\tau'_n = \overline{1}$ and $z = 1 \otimes \gamma_{p'}(v'_n) \in \Omega_{p'}(V^{(1)})$. Then $\sum i,j a_{i,j} = p^{p'+1}$, and

$$[\theta(W) \circ \Omega^{(1)}(\phi)](z) = [\theta(W) \circ \Omega^{(1)}(\phi)](1 \otimes \gamma_{p'}(v'_n))$$

$$= \theta(W) \left( \prod_{i,j} a_{i,j} \neq 0 \right) \left( \delta_{j,n} \cdot (1 \otimes \gamma_{a_{i,j}}(w_i)) \right)$$

$$= \prod_{a_{i,j} \neq 0} \delta_{j,n} \cdot (1 \otimes \gamma_{a_{i,j}}(w_i)).$$

Next we consider the expression $[\Omega^{(1)}(\phi) \circ \theta(V)](z)$. Set $c = \sum i,j a_{i,j}$, let $H$ be the Young subgroup $\prod_{i,j} S_{a_{i,j}}$ of $G_2$, and let $J \subset G_2$ be a set of right coset representatives for $H$. Then as in [2 IV.5.3], we can write $\phi = \prod_{i,j} \gamma_{a_{i,j}}(e_{i,j})$ in the form

$$\phi = \sum_{\sigma \in H} \left( \otimes_{i,j} (e_{i,j})^{a_{i,j}} \right) \cdot \sigma,$$

where the factors in the tensor product are taken in the lexicographic order. Now suppose $\tau'_n = \overline{0}$ and $z = v'_n \otimes 1$. Then $\sum i,j a_{i,j} = p$, and $[\Omega^{(1)}(\phi) \circ \theta(V)](v'_n \otimes 1) = \gamma_{p}(v'_n) \otimes 1$. From (4.1.3) it follows that $S^p(\phi)(v'_n) = 0$ unless $a_{i,j} = 0$ for all $j \neq n$. So suppose $a_{i,j} = 0$ for all $j \neq n$. Then

$$S^p(\phi)(v'_n) = \left( \prod_{i=1}^{p} w_i^{a_{i,n}} \right) \in S^p(W).$$

Since $k$ is a field of characteristic $p$, the multinomial coefficient in this expression is equal to zero if $a_{i,n} < p$ for some $i$. Then $[\Omega^{(1)}(\phi) \circ \theta(V)](v'_n \otimes 1) = 0$ unless $\phi = \gamma_p(e_{i,j})$ for some $i$ and $j$, and in this case one has $[\Omega^{(1)}(\phi) \circ \theta(V)](v'_n \otimes 1) = \delta_{j,n}(w'_i \otimes 1)$. A similar analysis shows that $[\Omega^{(1)}(\phi) \circ \theta(V)](1 \otimes v'_n) = \delta_{j,n}(w'_i \otimes 1)$, but that $\phi = \prod_{i,j} \gamma_{a_{i,j}}(e_{i,j})$ and $a_{i,j} \neq 0$ for $j \neq n$, or if $a_{i,j} \neq 0$ for more than one factor in the product, then $[\Omega^{(1)}(\phi) \circ \theta(V)](1 \otimes v'_n)$ is either equal to zero or ($\sigma_{i,j} = 0$ for $j \neq n$) is equal to the coboundary of $\left( \prod_{1 \leq i \leq m} w_i^{a_{i,n}} \right) \otimes 1$.

Now suppose $\tau'_n = \overline{1}$, $z = 1 \otimes \gamma_{p'}(v'_n)$, and $\phi = \prod_{i,j} \gamma_{a_{i,j}}(e_{i,j})$. Then

$$[\Omega^{(1)}(\phi) \circ \theta(V)](1 \otimes \gamma_{p'}(v'_n)) = [\Omega^{(1)}(\phi)](1 \otimes \gamma_{p'+1}(v'_n)) = \prod_{a_{i,j} \neq 0} \delta_{j,n} \cdot (1 \otimes \gamma_{a_{i,j}}(w_i)).$$

The analysis of $\Omega(V)$ from the one-dimensional case shows that the last expression in this equation is a coboundary unless $p \mid a_{i,j}$ for all $a_{i,j}$. Finally, suppose $\tau'_n = \overline{1}$ and $z = v'_n \otimes 1$. Then

$$[\Omega^{(1)}(\phi) \circ \theta(V)](v'_n \otimes 1) = [\Omega^{(1)}(\phi)](v'_n \otimes \gamma_{p-1}(v'_n))$$

$$= [\Omega^{(1)}(\phi)](1 \otimes \gamma_{p}(v'_n))$$

$$= [\Omega^{(1)}(\phi)](1 \otimes \gamma_{p}(v'_n)).$$

Now $\Omega^{(1)}(\phi)(1 \otimes \gamma_{p}(v'_n)) = \prod_{a_{i,j} \neq 0} \delta_{j,n} \cdot (1 \otimes \gamma_{a_{i,j}}(w_i))$, and this expression is a coboundary unless $p \mid a_{i,j}$ for all $a_{i,j}$. But $\kappa(V) \circ d(V) = -d(V) \circ \kappa(V)$ on $\Omega_{p}(V)$ by Lemma 4.1.1, so it follows that $[\Omega^{(1)}(\phi) \circ \theta(V)](v'_n \otimes 1)$ is a coboundary unless $p \mid a_{i,j}$ for all $a_{i,j}$, i.e., unless $\phi = \gamma_p(e_{i,j})$ for some $i$ and $j$. On the other hand, if $\phi = \gamma_p(e_{i,j})$, then $[\Omega^{(1)}(\phi) \circ \theta(V)](v'_n \otimes 1) = \delta_{j,n}(w_i \otimes \gamma_{p-1}(w_i))$.

Combining the observations of the previous four paragraphs, (4.1.2) then follows. \qed

**Remark 4.1.3.** As a $\mathcal{P}$-algebra, $\Omega^{(1)} ≅ S^{(1)} \otimes A^{(1)}$. Identifying $H^*(\Omega_{p,n})$ with $\Omega_{p,n}^{(1)}$ via the super Cartier isomorphism, one gets, in the notation of (2.7.6),

$$H^*(\Omega_{p,n}) = \bigoplus_{a+b+c+d=n} \left( S^{a(1)} \otimes A^{b(1)} \otimes (A^{c(1)} \otimes g \otimes \Gamma^{d(1)}) \right).$$
4.2. The super Koszul kernel subcomplex. Given \( i, n \in \mathbb{N} \) with \( i \leq n \), set

\[
K^i_n = \ker \{ \kappa : \Omega^i_n \to \Omega^{i-1}_n \}.
\]

It follows from Lemma 4.1.1 that \( (Kpn, d) \) is a subcomplex of \( (\Omega_{pn}, d) \). We call \( (Kpn, d) \) the Koszul kernel subcomplex of \( \Omega_{pn} \). On \( V_P \), \( K^i_n \) restricts to the Koszul kernel subfunctor \( \bar{K}^i_n \) defined in [10,12]. In particular, \( (Kpn, d) \) restricts to the ordinary Koszul kernel subcomplex \( (Kpn, d) \). On \( V_P \), \( K^i_n \) restricts to \( (K^{n-i}_n)^\# \), and \( (Kpn, d) \) restricts to \( (K^\#_{pn}, d^\#) \).

Since \( (\Omega, \kappa) \) is acyclic, there exist for each \( 0 \leq i \leq n \) and \( r \geq 1 \) short exact sequences in \( P_{ev} \)

\[
0 \to K^i_n \to \Omega^i_n \xrightarrow{\kappa^i_n} K^{i-1}_n \to 0,
\]

\[
(4.2.1)
\]

\[
0 \to K^i_0 \circ I_0^r \to \Omega^i_n \circ I_0^r \xrightarrow{\kappa^i_n} K^{i-1}_n \circ I_0^r \to 0,
\]

\[
0 \to K^i_1 \circ I_1^r \to \Omega^i_n \circ I_1^r \xrightarrow{\kappa^i_n} K^{i-1}_n \circ I_1^r \to 0.
\]

Since \( K^0_n = \Omega^0_n \cong S^n \), \( K^i_n = 0 \), and \( \Omega^n_n \cong A^n \), the associated long exact sequences in cohomology together with Theorem 3.4.3 then imply:

**Lemma 4.2.1.** Suppose \( F \in \mathcal{P} \) is additive. Then for all \( i, n, r \in \mathbb{N} \) with \( i < n \) and \( r \geq 1 \),

\[
\text{Ext}^s_{\mathcal{P}}(F, S^n) \cong \text{Ext}^{s+i}_{\mathcal{P}}(F, K^i_n) \cong \text{Ext}^{s+n-1}_{\mathcal{P}}(F, A^n),
\]

\[
(4.2.2)
\]

\[
\text{Ext}^s_{\mathcal{P}}(F, S_0^{n(r)}) \cong \text{Ext}^{s+i}_{\mathcal{P}}(F, K^i_n \circ I_0^r) \cong \text{Ext}^{s+n-1}_{\mathcal{P}}(F, A_0^{n(r)}),
\]

\[
(4.2.3)
\]

\[
\text{Ext}^s_{\mathcal{P}}(F, A_1^{n(r)}) \cong \text{Ext}^{s+i}_{\mathcal{P}}(F, K^i_n \circ I_1^r) \cong \text{Ext}^{s+n-1}_{\mathcal{P}}(F, \Gamma_1^{n(r)}).
\]

\[
(4.2.4)
\]

**Corollary 4.2.2.** Suppose \( F \in \mathcal{P} \) is additive. Then for all \( n, r \in \mathbb{N} \) with \( r \geq 1 \),

\[
\text{Ext}^s_{\mathcal{P}}(F, S_0^{n(r)}) \cong \text{Ext}^{s+n-1}_{\mathcal{P}}(F, A_0^{n(r)}) \quad \text{and}
\]

\[
\text{Ext}^s_{\mathcal{P}}(F, A_1^{n(r)}) \cong \text{Ext}^{s+n-1}_{\mathcal{P}}(F, \Gamma_1^{n(r)}).
\]

**Proof.** The reader can check that \( F \in \mathcal{P} \) is additive if and only if \( F \circ \Pi \) and \( \Pi \circ F \) are additive. Then the isomorphisms follow from \( (4.2.3), (4.2.4) \), and \( (2.7.9) \) by considering the conjugation action of \( \Pi \) on extension groups in \( \mathcal{P} \). \( \square \)

**Remark 4.2.3.** The connecting homomorphisms that induce the isomorphisms in Lemma 4.2.1 can be realized as left Yoneda products by the extension classes of the short exact sequences in \( (4.2.1) \); cf. [13, IV.9]. Thus, the isomorphism \( \text{Ext}^s_{\mathcal{P}}(F, S^n) \cong \text{Ext}^{s+n-1}_{\mathcal{P}}(F, A^n) \) identifies with left multiplication by the extension class of the super Koszul complex \( Kz_n \). Using \( (2.7.7) \) to consider the \( r \)-th Frobenius twists of the ordinary Koszul complex \( Kz_n \) and its dual \( K^\#_n \) as exact sequences in \( \mathcal{P} \), the composite isomorphism in \( (4.2.3) \) identifies with left multiplication by the extension class in \( \text{Ext}^{n-1}_{\mathcal{P}}(S_0^{n(r)}, A_0^{n(r)}) \) of \( Kz_n^{n(r)} \), and the composite isomorphism in \( (4.2.4) \) identifies with left multiplication by the extension class in \( \text{Ext}^{n-1}_{\mathcal{P}}(A_1^{n(r)}, \Gamma_1^{n(r)}) \) of

\[
\begin{cases}
Kz_n^{\#(r)} \circ \Pi & \text{if } n \text{ is even}, \\
\Pi \circ Kz_n^{\#(r)} & \text{if } n \text{ is odd}.
\end{cases}
\]

Conjugating by \( \Pi \) (if \( n \) is odd), or pre-composing with \( \Pi \) (if \( n \) is even), we see that similar remarks also apply to the isomorphisms in Corollary 4.2.2. Since the Yoneda product is associative, it follows that the isomorphisms in Lemma 4.2.1 and Corollary 4.2.2 commute with right multiplication by elements of the Yoneda algebra \( \text{Ext}^s_{\mathcal{P}}(F, F) \).

**Remark 4.2.4.** Since the restriction functor \( \mathcal{P} \to \mathcal{P} \) maps \( Kz_n \) to the ordinary Koszul complex \( Kz_n \), and since it forgets the fact that \( Kz_n^{(r)} \) had been lifted to \( \mathcal{P} \), it follows that the isomorphisms...
in \((4.2.2)\) and \((4.2.3)\) fit into commutative diagrams
\[
\begin{array}{ccc}
\text{Ext}_*^t(F, S^n) & \xrightarrow{\sim} & \text{Ext}_*^{t+n-1}(F, A^n) \\
\downarrow & & \downarrow \\
\text{Ext}_*^t(F|_{\mathcal{V}_n}, S^n) & \xrightarrow{\sim} & \text{Ext}_*^{t+n-1}(F|_{\mathcal{V}_n}, \Lambda^n)
\end{array}
\]
and
\[
\begin{array}{ccc}
\text{Ext}_*^t(F, S^{n(r)}) & \xrightarrow{\sim} & \text{Ext}_*^{t+p'-1}(F, \Lambda_0^{n(r)}) \\
\downarrow & & \downarrow \\
\text{Ext}_*^t(F|_{\mathcal{V}_{n(r)}}, S^{n(r)}) & \xrightarrow{\sim} & \text{Ext}_*^{t+p'-1}(F|_{\mathcal{V}_{n(r)}}, \Lambda^{n(r)})
\end{array}
\]
in which the vertical arrows are the corresponding restriction homomorphisms. The isomorphisms in the bottom rows of each diagram are the isomorphisms in \([12, \text{Proposition 4.4}]\), which also admit descriptions as left multiplication by the extension classes of \(K_{zn}\) and \(K_{zn}^{(r)}\), respectively.

It is straightforward to check that the Koszul differential is compatible with the super Cartier isomorphism. For example, if \(x_0 = 1\) and \(e \geq 0\), then
\[
[\theta(V) \circ \kappa(V^{(1)}))((1 \otimes \gamma_{p'_{-1}}(x_0^n))) = \theta(V) (x_0^n \otimes 1) \cdot \theta(V) (1 \otimes \gamma_{p'_{-1}}(x_0^n)) = (v_0 \otimes \gamma_{p_{-1}}(x_0^n)) \cdot (1 \otimes \gamma_{p+1_{-1}}(x_0^n)) = v_0 \otimes \gamma_{p+1_{-1}}(x_0^n).
\]

Then the next result follows from a word-for-word repetition of the proof of \([10, \text{Proposition 3.5}]\).

**Proposition 4.2.5.** The super Cartier isomorphism restricts for each \(n \in \mathbb{N}\) to an isomorphism
\[
K_n^{(1)} \cong \mathcal{H}_n^{*}(K_{pn}).
\]
In particular, the inclusion of complexes \(K_{pn} \hookrightarrow \Omega_{pn}\) induces an inclusion \(\mathcal{H}_n^{*}(K_{pn}) \hookrightarrow \mathcal{H}_n^{*}(\Omega_{pn})\).

Replacing \(K_n\) by \(\mathcal{H}_n^{*}(K_{pn})\), there is the following analogue of Lemma \(4.2.1\).

**Lemma 4.2.6.** Let \(F \in \mathcal{P}\) be an additive functor and let \(n \in \mathbb{N}\). Then for \(0 \leq t \leq n - 1\),
\[
\begin{align*}
(4.2.5) \quad & \text{Ext}_*^t(F, S^{n(1)}) \cong \text{Ext}_*^t(F, \mathcal{H}^{t}(K_{pn})) \cong \text{Ext}_*^{t+t}(F, \mathcal{H}^{t}(K_{pn})), \\
(4.2.6) \quad & \text{Ext}_*^t(F, \Lambda_1^{n(1)}) \cong \text{Ext}_*^t(F, \mathcal{H}^{n(p-1)}(K_{pn})) \cong \text{Ext}_*^{t+t}(F, \mathcal{H}^{n+p(p-1)}(K_{pn})),
\end{align*}
\]
and \(\text{Ext}_*^t(F, \mathcal{H}^{t}(K_{pn})) = 0\) if \(n \leq t < n(p-1)\) or \(t \geq pn\).

**Proof.** Consider the family of short exact sequences obtained by replacing \(n\) by \(pn\) and \(\kappa\) by \((-1)^{t} \kappa\) in \((4.2.1)\). It follows from Lemma \(4.1.3\) that this family fits together into a short exact sequence of complexes whose associated long exact sequence in cohomology takes the form
\[
\cdots \rightarrow \mathcal{H}^{t-1}(K_{pn}^{*}) \rightarrow \mathcal{H}^{t}(K_{pn}^{*}) \rightarrow \mathcal{H}^{t}(\Omega_{pn}^{*}) \rightarrow \mathcal{H}^{t+1}(K_{pn}^{*}) \rightarrow \cdots.
\]
Since \(\mathcal{H}^{t}(K_{pn}^{*}) = \mathcal{H}^{t+1}(K_{pn}^{*})\), and since by Proposition \(4.2.5\) the inclusion of complexes \(K_{pn}^{*} \hookrightarrow \Omega_{pn}^{*}\) induces an inclusion at the level of cohomology groups, the long exact sequence decomposes into a family of short exact sequences of the form
\[
(4.2.7) \quad 0 \rightarrow \mathcal{H}^{t}(K_{pn}) \rightarrow \mathcal{H}^{t}(\Omega_{pn}) \rightarrow \mathcal{H}^{t-1}(K_{pn}) \rightarrow 0.
\]
Identifying \(\mathcal{H}^{t}(\Omega_{pn})\) with a subfunctor of \(\Omega_{n}^{(1)}\) as in Remark \(4.1.3\) and identifying \(\mathcal{H}^{t-1}(K_{pn})\) with a subfunctor of \(K_{n}^{(1)}\), the map \(\mathcal{H}^{t}(\Omega_{pn}) \rightarrow \mathcal{H}^{t-1}(K_{pn})\) in \((4.2.7)\) identifies with \((-1)^{t} \kappa^{(1)}\).

Now consider the long exact sequence in cohomology obtained by applying \(\text{Hom}(F, -)\) to \((4.2.7)\). It follows from Theorem \(3.4.3\) and Remark \(4.1.3\) that \(\text{Ext}_*^t(F, \mathcal{H}^{t}(\Omega_{pn})) = 0\), and hence that \(\text{Ext}_*^t(F, \mathcal{H}^{t-1}(K_{pn})) \cong \text{Ext}_*^{t+t}(F, \mathcal{H}^{t}(K_{pn}))\), except perhaps if \(t\) equals 0, \(n\), \(n(p-1)\), or \(pn\). Since \(\mathcal{H}^{0}(K_{pn}) = \mathcal{H}^{0}(\Omega_{pn}) \cong \mathcal{S}_0^{n(1)}\) and \(\mathcal{H}^{pn}(K_{pn}) = 0\), this establishes \((4.2.5)\), the second isomorphism in \((4.2.6)\), and the equality \(\text{Ext}_*^t(F, \mathcal{H}^{t}(K_{pn})) = 0\) for \(t \geq pn\).
Next recall that the Koszul differential induces an isomorphism $\Omega_n^{p} \cong K_n^{-1}$. Since the second map in (4.2.7) identifies with $(-1)^{2r}K^{(1)}$, it follows that the collection of maps $\Omega^{t}(\Omega_{pn}) \to \Omega^{t-1}(K_{pn})$ from (4.2.7) also restricts to an isomorphism from the subfunctor $\Omega^{n}(1)$ of $H^{*}(\Omega_{pn})$ to the subfunctor $K_{n}^{-1}(1)$ of $H^{*}(K_{pn})$. Applying Remark 4.1.3 and Theorem 3.4.3, one gets

$$\Ext^s_P(F, \Omega^{n+1}(1)) \cong \Ext^s_P(F, \Omega^{n}(\Omega_{pn}) \oplus H^{m}(\Omega_{pn})) \cong \Ext^s_P(F, \Lambda^{1} \oplus \Gamma^{n}(1)).$$

Then taking $t = n$ in (4.2.7), it follows in the associated long exact sequence in cohomology that

$$\Ext^s_P(F, \Omega^{n}(\Omega_{pn})) \cong \Ext^s_P(F, \Omega^{n-1}(K_{pn}))$$

and hence that $\Ext^s_P(F, H^{n}(K_{pn})) = 0$. Combined with the observations of the previous paragraph, this shows that $\Ext^s_P(F, H^{t}(K_{pn})) = 0$ for $n \leq t < n(p-1)$. Finally, taking $t = n(p-1)$ in (4.2.7), it now follows from the exact long sequence in cohomology, Remark 4.1.3 and Theorem 3.4.3 that the inclusion $K_{pn} \hookrightarrow \Omega_{pn}$ induces isomorphisms

$$\Ext^s_P(F, H^{n(p-1)}(K_{pn})) \cong \Ext^s_P(F, H^{n(p-1)}(\Omega_{pn})) \cong \Ext^s_P(F, \Lambda^{1}(1)).$$

\[\square\]

4.3. Vector space structure of $\Ext^s_P(I^{(r)}, S_0^{p^r-1}(1))$. Our goal in this section is to describe for $r \geq 1$ the $E_2$-pages of the spectral sequence

$$\Ext_2^{s,t} = \Ext_2^{s,t}(I^{(r)}, H^{t}(\Omega_{p^r})) \Rightarrow \Ext^{s+t}(I^{(r)}, \Omega_{p^r}),$$

and

$$\Ext_2^{s,t} = \Ext_2^{s,t}(I^{(r)}, H^{t}(K_{p^r})) \Rightarrow \Ext^{s+t}(I^{(r)}, K_{p^r})$$

obtained by taking $A = I^{(r)}$ and $C = \Omega_{p^r}$ or $C = K_{p^r}$ in (3.6.2). Since $I^{(r)} = I^{(r)}_{0} \oplus I^{(r)}_{1}$, it follows that (4.3.1) and (4.3.2) each decompose into a direct sum of hypercohomology spectral sequences. Specifically, given $l \in \{0, 1\}$, let

$$\Ext_2^{s,t} = \Ext_2^{s,t}(I^{(r)}_{l}, H^{t}(\Omega_{p^r})) \Rightarrow \Ext^{s+t}(I^{(r)}_{l}, \Omega_{p^r}),$$

and

$$\Ext_2^{s,t} = \Ext_2^{s,t}(I^{(r)}_{l}, H^{t}(K_{p^r})) \Rightarrow \Ext^{s+t}(I^{(r)}_{l}, K_{p^r})$$

be the hypercohomology spectral sequences obtained by taking $A = I^{(r)}_{l}$ and $C = \Omega_{p^r}$ or $C = K_{p^r}$ in (3.6.2), respectively. Then (4.3.1) is the direct sum of the spectral sequences obtained by taking $l = 0$ and $l = 1$ in (4.3.3), and similarly for (4.3.2) and (4.3.4). We will exploit these direct sum decompositions to make explicit calculations. As a byproduct of our work, we will obtain the vector space structure of the extension group $\Ext^s_P(I^{(r)}, S_0^{p^r-1}(1))$. First we compute the abutments of the spectral sequences (4.3.3) and (4.3.4).

Lemma 4.3.1. Let $r \geq 1$, and let $l \in \{0, 1\}$. Then:

$$\Ext^s_P(I^{(r)}_{l}, \Omega_{p^r}) = \begin{cases} k & \text{if } l = 0 \text{ and either } s = 0 \text{ or } s = 2p^r - 1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Ext^s_P(I^{(r)}_{l}, K_{p^r}) = \begin{cases} k & \text{if } l = 0 \text{ and } s \text{ is even with } 0 \leq s < 2p^r, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First take $A = I^{(r)}_{l}$ and $C = \Omega_{p^r}$ in (3.6.1). Then it follows from Theorem 3.4.3 and (4.2.2), the injectivity of $S^n$, and (3.1.2) that $E_{1}^{s,t} = k$ if $l = 0$ and $s = t = 0$, or if $l = 0$, $s = p^r$, and $t = p^r - 1$, but that $E_{s,t} = 0$ otherwise. Since the nonzero terms in the spectral sequence are of non-adjacent total degrees, it follows that the spectral sequence collapses at the $E_1$-page and that $\Ext^s_P(I^{(r)}_{l}, \Omega_{p^r})$ is described as claimed. Now take $A = I^{(r)}_{l}$ and $C = K_{p^r}$ in (3.6.1). Then as above, it follows from (4.2.2) that $E_{1}^{s,t} = k$ if $l = 0$ and $s = t$ with $0 \leq s < p^r$, but that $E_{s,t} = 0$ otherwise. Again, the nonzero terms in the spectral are of non-adjacent total degrees, so it follows that $\Ext^s_P(I^{(r)}_{l}, K_{p^r})$ is described as in the statement of the lemma. \[\square\]
Our focus for the rest of this section is on analyzing the spectral sequences (4.3.3) and (4.3.4).

For the rest of this section, set \( q = p^{r-1} \). Theorem [4.3.3] and Remark [4.1.3] imply that

\[
E^{s,0}_{2,\ell} \cong \text{Ext}^s_P(I^{(r)}_\ell, s q^{(1)}_0), \quad E^{\ell,0}_{2,\ell} \cong \text{Ext}^\ell_P(I^{(r)}_\ell, A^{q^{(1)}}_0),
\]

(4.3.5)

and \( E^{s,t}_{2,\ell} = 0 \) if \( t \notin \{0, q, (p-1)q, pq\} \). Then Lemma [4.2.1], (2.7.4), and (2.7.9) imply

\[
E^{s,0}_{2,0} = E^{s+q-1,q}_{2,0} = E^{s+q-1,(p-1)q}_{2,0} \cong E^{s+2(q-1),pq}_{2,0}, \quad \text{and}
\]

(4.3.6)

Similarly, Lemma [4.2.6] implies that

\[
E^{s,0}_{2,\ell} = E^{s+t,t}_{2,\ell} \quad \text{and} \quad E^{s,(p-1)q}_{2,\ell} = E^{s+t,(p-1)q+\ell}_{2,\ell} \quad \text{for } 0 \leq t \leq q - 1, \quad \text{but}
\]

(4.3.7)

\[
E^{s,t}_{2,\ell} = 0 \quad \text{if } q \leq t < (p-1)q \text{ or if } t \geq pq.
\]

The proof of Lemma [4.2.6] shows that the inclusion \( K_{p^r} \hookrightarrow \Omega_{p^r} \) induces isomorphisms

(4.3.8)

\[
E^{0,0}_{2,\ell} \cong E^{s,0}_{2,\ell} \quad \text{and} \quad E^{s,(p-1)q}_{2,\ell} \cong E^{s,(p-1)q}_{2,\ell}.
\]

Finally, Lemma [4.3.1] implies that

\[
\bigoplus_{i+j=s} E^{i,j}_{\infty,\ell} = \begin{cases} k & \text{if } \ell = 0 \text{ and } s = 0 \text{ or } s = 2p^r - 1, \\ 0 & \text{otherwise}, \end{cases}
\]

(4.3.9)

\[
\bigoplus_{i+j=s} E^{i,j}_{\infty,\ell} = \begin{cases} k & \text{if } \ell = 0 \text{ and } s \text{ is even with } 0 \leq s < 2p^r, \\ 0 & \text{otherwise}. \end{cases}
\]

**Theorem 4.3.2.** Set \( q = p^{r-1} \). In the spectral sequences \( E_{2,0} \) and \( E_{2,1} \), one has

\[
E^{s,0}_{2,0} \cong E^{s+q-1,q}_{2,0} \cong E^{s+q-1,(p-1)q}_{2,0} \cong E^{s+2(q-1),pq}_{2,0} \cong \begin{cases} k & \text{if } s \equiv 0 \mod 2q \text{ and } s \geq 0, \\ 0 & \text{otherwise}, \end{cases}
\]

(4.3.10)

\[
E^{s,0}_{2,1} \cong E^{s+q-1,q}_{2,1} \cong E^{s+q-1,(p-1)q}_{2,1} \cong E^{s+2(q-1),pq}_{2,1} \cong \begin{cases} k & \text{if } s \equiv p^r \mod 2q \text{ and } s \geq p^r, \\ 0 & \text{otherwise}. \end{cases}
\]

(4.3.11)

**Proof.** The proof is by induction on \( s \). To help avoid getting lost in a sea of spectral sequence notation, we break the proof into four steps that we illustrate in Figures [1][4]. The figures are drawn for the case \( p = 5 \) and \( r = 2 \), but are representative (via an appropriate rescaling) of the general situation. The figures illustrate the information about each spectral sequence that has been explicitly determined, or can be deduced using the isomorphisms and equalities preceding the theorem, at the completion of that step of the proof. In each figure, a solid horizontal line indicates a row in which terms may be nonzero. On those lines, an open dot at the point \((s, t)\) means that the corresponding term is zero, a closed dot at \((s, t)\) means that the corresponding term is isomorphic to \( k \), and a lack of any dot indicates that no information about that term has yet been determined. Arrows indicate that the corresponding differential has been determined to be an isomorphism. In the diagrams for \( E_{2,0} \), a dashed line passes through terms of total degree \( 2p^r - 1 \).

**Step 1.** Since (4.3.6) is valid for all \( s \in \mathbb{Z} \), and since (4.3.3) and (4.3.4) are first quadrant spectral sequences, (4.3.10) and (4.3.11) are true for \( s < 0 \). Next, it follows from (4.3.9) that \( E^{0,0}_{2,0} = k \) and \( E^{0,0}_{2,1} = 0 \), and hence that (4.3.10) and (4.3.11) are true if \( s = 0 \). See Figure 1.

**Step 2.** Since \( E^{0,0}_{\infty,\ell} = 0 \), it follows that \( E^{0,0}_{2,1} \neq 0 \) only if there exist integers \( i, j \) with \( i + j = s - 1 \) and \( i \leq s - 2 \) such that \( E^{i,j}_{2,1} \neq 0 \). Now an induction argument using the information from Step 1
and the isomorphism $\overline{E}_{2,1}^{s,0} \cong \overline{E}_{2,1}^{s+q-1,q}$ shows that $\overline{E}_{2,1}^{s,0} = 0$ for $s \leq p^r - 1$. Then (4.3.11) is true for $s \leq p^r - 1$. See Figure 2.

Step 3. Since $\overline{E}_{\infty,0}^p = 0$ if $s + t \notin \{0, 2p^r - 1\}$, and since $\overline{E}_{2,0}^{s,(p-1)q} = \overline{E}_{2,0}^{s,pq} = 0$ for $s \leq p^r + q - 2$ by Step 2, it follows for $0 < s < 2p^r - 1$ that the differential on the $(q + 1)$-th page of (4.3.3) induces an isomorphism $\overline{E}_{q+1}^p : \overline{E}_{q+1}^{s-q-1,q} \to \overline{E}_{2,0}^q$. Then an induction argument like that in the proof of (12) (4.5.5)) shows that (4.3.10) is true for $s < 2p^r - 1$. Now combining the validity of (4.3.10) for $s < 2p^r - 1$ with (4.3.7), (4.3.8), and the result of Step 2, it follows that all terms of total degree $2p^r - 1$ in $E_{2,0}$ are zero except perhaps $E_{2,0}^{2p^r-1,1}$, and that $E_{2,0}^{2p^r-1,q-1} \cong k$ is the only nonzero term of total degree $2p^r - 2$ in $E_{2,0}$. Since there is no differential between these terms, and since $E_{\infty,0}^p = 0$ in all odd total degrees, it follows that $E_{2,0}^{2p^r-1,0} = 0$. Then by (4.3.8), (4.3.10) is true for $s = 2p^r - 1$. See Figure 3 in which we also show the data so far for $E_{2,0}$ and $E_{2,1}$.

Step 4. We have established so far that (4.3.10) is true for $s < 2p^r$, and that (4.3.11) is true for $s < p^r$. Given $m \in \mathbb{N}$, set $u = p^r + 2qm$ and set $v = u + p^r = 2p^r + 2qm$. We now proceed by induction on $m$ to show that if (4.3.10) is true for $s < v$ and if (4.3.11) is true for $s < u$, then (4.3.10) is true for $s < v + 2q$ and (4.3.11) is true for $s < u + 2q$.

First, it follows from the induction hypothesis and (4.3.7) and (4.3.8) that the nonzero terms $E_{2,1}^{s,t}$ (resp. $E_{2,0}^{s,t}$) with $(p-1)q \leq t < pq$ that have been explicitly determined so far are in distinct even (resp. odd) total degrees. Similarly, the nonzero terms $E_{2,1}^{s,t}$ (resp. $E_{2,0}^{s,t}$) with $0 \leq t < q$ that have been explicitly determined so far are in distinct odd (resp. even) total degrees. Then since $E_{\infty,1} = 0$, there must be a nontrivial differential originating at the term $E_{2,1}^{u-(p-1)q-1,(p-1)q} \cong k$. By the induction hypothesis, $E_{2,1}^{u,q} = 0$ for all $s, t$ with $0 < t < q$ and $s + t = u$. Then the differential

\[(4.3.12)\quad d_{(p-1)q+1} : E_{2,1}^{u-(p-1)q-1,(p-1)q} \to E_{2,1}^{u,0}\]

must be nontrivial. By the induction hypothesis there are no other nontrivial terms of total degree $u - 1$ in $E_{2,1}$, so since $E_{\infty,1} = 0$, this differential must be an isomorphism. Then $k \cong E_{2,1}^{u,0} \cong \overline{E}_{2,1}^{u,0}$, and by a similar argument as for Step 2, we deduce from the induction hypothesis that $\overline{E}_{2,1}^{u,0} = 0$ for $u < s < u + 2q$. Then (4.3.11) is true for $s < u + 2q$.

Applying (4.3.6), (4.3.7), and (4.3.8), we now get that $E_{2,0}^{u+q-1,(p-1)q} \cong k$. Reasoning as in the previous paragraph, we also see that this term cannot be the image of any nontrivial differential. But $E_{2,0}$ is zero in all total degrees $\geq 2p^r$, so we deduce that there must be a nontrivial differential originating at $E_{2,0}^{u+q-1,(p-1)q}$. By the induction hypothesis, $E_{2,0}^{u,q} = 0$ for all $s, t$ with $0 < t < q$ and $s + t = u$. Then as in the previous paragraph, we deduce that the differential

\[(4.3.13)\quad d_{(p-1)q+1} : E_{2,0}^{u+q-1,(p-1)q} \to E_{2,0}^{u+pq,0} = E_{2,0}^{v,0}\]

is an isomorphism. So $k \cong E_{2,0}^{v,0} \cong \overline{E}_{2,0}^{v,0}$. Finally, again reasoning as for Step 2, it follows from the induction hypothesis that $\overline{E}_{2,0}^{v,0} = 0$ for $v < s < v + 2q$ and hence that (4.3.10) is true for $s < v + 2q$; see Figure 4. This completes the induction argument laid out at the beginning of Step 4, and hence completes the inductive proof that (4.3.10) and (4.3.11) are true for all $s$. \qed

4.4. Module structure of $\Ext_P^q(I^{(r)}, S_0^{p^r-1})$. In this section we continue our investigation of the hypercohomology spectral sequences (4.3.3) and (4.3.4), with the goal of describing the structure of $\Ext_P^q(I^{(r)}, S_0^{p^r-1})$ as a right module over the Yoneda algebra $\Ext_P(I^{(r)}, I^{(r)})$. The next lemma, which sharpens an observation from Step 3 of the proof of Theorem 4.3.2 is a crucial ingredient in our calculation of the right module structure of $\Ext_P^q(I^{(r)}, S_0^{p^r-1})$. Its proof makes critical use of the explicit cohomology calculations in [12, §4]. In this section, set $q = p^r - 1$. 

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Lemma 4.4.1. In the spectral sequence $E_{2,0}^s \Rightarrow \text{Ext}_{\mathcal{P}}^s(I_0^r, \Omega_{p^r})$, the differential
\[
d_{q+1} : E_{2,0}^{s-q-1,q} \to E_{2,0}^{s,0}
\]
is an isomorphism if $0 < s < 2p^r$, but is trivial if $s = 2p^r$.

Proof. As observed in Step 3 of the proof of Theorem 4.3.2, the indicated differential is an isomorphism if $0 < s < 2p^r - 1$. It is also an isomorphism if $s = 2p^r - 1$, since then both terms are zero. So it suffices to prove the case $s = 2p^r$ of the lemma.

Set $E_{2,0} = E_{2,0}^s$, and for the next two paragraphs let $E_{2,0}^s = \text{Ext}_{\mathcal{P}}^s(I^r, \Omega_{p^r}^{(1)}) \Rightarrow \text{Ext}_{\mathcal{P}}^{s+t}(I^r, \Omega_{p^r})$ be the second hypercohomology spectral sequence in the category $\mathcal{P}$ for the ordinary de Rham complex $\Omega_{p^r}$; this is one of the spectral sequences that was the focus of attention in the proof of the case $j = 1$ of [12, Theorem 4.5]. By general abstract nonsense, the restriction functor $\mathcal{P} \to \mathcal{P}$ induces a homomorphism of spectral sequences $E \to E$ such that the induced maps $E_2 \to E_2$ and $E_\infty \to E_\infty$ identify with the restriction homomorphisms described in Remark 3.6.1.

We claim that the restriction map $\text{Ext}_{\mathcal{P}}^s(I_0^r, \Omega_{p^r}) \to \text{Ext}_{\mathcal{P}}^s(I^r, \Omega_{p^r})$ is an isomorphism, and hence that the morphism $E \to E$ induces an isomorphism of bigraded vector spaces $E_\infty \sim E_\infty$.

To see why this claim is sufficient to prove the lemma, recall from [12] that
\[
k \cong E_{2,0}^{2p^r-q-1,q} \cong E_{2,0}^{2p^r-q-1,q} \cong \text{Ext}_{\mathcal{P}}^{2p^r-1}(I^r, \Omega_{p^r}).
\]
Also, $\text{Ext}_{\mathcal{P}}^{2p^r-1}(I_0^r, \Omega_{p^r}) \cong k$ by Lemma 4.3.1 and the only nonzero terms of total degree $2p^r - 1$ in $E_2$ are $E_{2,0}^{2p^r-q-1,q} \cong k$ and $E_{2,0}^{2p^r+1-1,q} \cong k$. Then for the induced map $E_\infty \to E_\infty$ to be an isomorphism, the restriction map $E_2^{2p^r-q-1,q} \to E_2^{2p^r-q-1,q}$ must be an isomorphism, and $E_2^{2p^r-q-1,q}$ must consist of permanent cycles. In particular, $d_{q+1} : E_2^{2p^r-q-1,q} \to E_2^{2p^r,0}$ must be trivial.

Now we prove the claim that the restriction functor $\mathcal{P} \to \mathcal{P}$ induces an isomorphism
\[
\text{Ext}_{\mathcal{P}}^s(I_0^r, \Omega_{p^r}) \sim \text{Ext}_{\mathcal{P}}^s(I^r, \Omega_{p^r}).
\]
For the rest of this proof, let
\[
E_1^s = \text{Ext}_{\mathcal{P}}^s(I_0^r, \Omega_{p^r}), \quad E_1^{s+t} = \text{Ext}_{\mathcal{P}}^{s+t}(I_0^r, \Omega_{p^r}), \quad \text{and}
\]
be the first hypercohomology spectral sequences for $\Omega_{p^r}$ and $\Omega_{p^r}$ in the categories $\mathcal{P}$ and $\mathcal{P}$, respectively. Then the restriction functor induces a homomorphism of spectral sequences $E \to E$.

As shown in the case $j = 0$ of [12, (4.5.2)] and in the proof of Lemma 4.3.1 these spectral sequences collapse at the $E_1$-page. Then it suffices to show that the induced map $E_1 \to E_1$ is an isomorphism.

The only nonzero terms in the two $E_1$-pages are $E_1^{0,0} \cong k \cong E_1^{p^r,0} - 1$ and $E_1^{0,0} \cong k \cong E_1^{p^r,0} - 1$. The $p^r$-power map $I_0^r \to S^p$ spans $\text{Hom}_{\mathcal{P}}(I_0^r, S^p)$, and is mapped by the restriction functor to the $p^r$-power map $I^r \to S^p$, which spans $\text{Hom}_{\mathcal{P}}(I^r, S^p)$. Then the map $E_1^{0,0} \to E_1^{0,0}$ is an isomorphism. Now taking $n = p^r$ and $F = I_0^r$ in the first commutative diagram in Remark 4.2.4 this implies that the restriction map $E_1^{p^r-1,0} \to E_1^{p^r-1,0}$ is an isomorphism as well. \(\square\)

Remark 4.4.2. Let $E \to E$ be the homomorphism of spectral sequences from the second and third paragraphs of the proof of Lemma 4.4.1. It follows from Remark 4.2.4 that this homomorphism gives rise to a commutative diagram
\[
\begin{array}{ccc}
E_2^{0,0} & \sim & E_2^{q-1,q} \xrightarrow{d_{q+1}} E_2^{2q,0} \\
\downarrow & & \downarrow \\
E_2^{0,0} & \sim & E_2^{q-1,q} \xrightarrow{d_{q+1}} E_2^{2q,0}
\end{array}
\]
in which the vertical arrows are the corresponding restriction homomorphisms. By essentially the same reasoning as in the last paragraph of the proof of Lemma 4.4.1, the restriction map $E_2^{0,0} \rightarrow E_2^{0,0}$ is an isomorphism. Then the lemma together with [12 (4.5.6)] implies that the restriction map

$$\text{Ext}_P^{2p^{r}-1}(I_0^{(r)}, S_0^{q(r-1)(1)}) \rightarrow \text{Ext}_P^{2p^{r}-1}(I^{(r)}, S_0^{q(r-1)(1)})$$

is also an isomorphism.

**Lemma 4.4.3.** Let $r, s \in \mathbb{N}$ with $r \geq 1$. There exist isomorphisms of right $\text{Ext}_P^{\bullet}(I^{(r)}, I^{(r)})$-modules

(4.4.1) $\text{Ext}_P^{\bullet}(I^{(r)}, S_1^{q(1)}) \cong \text{Ext}_P^{s+q-1}(I^{(r)}, \Lambda_{1}^{q(1)}) \cong \text{Ext}_P^{s+p^{r}}(I^{(r)}, S_0^{q(1)})$, and

(4.4.2) $\text{Ext}_P^{\bullet}(I^{(r)}, S_0^{q(1)}) \cong \text{Ext}_P^{s+q-1}(I^{(r)}, \Lambda_{0}^{q(1)}) \cong \text{Ext}_P^{s+p^{r}}(I^{(r)}, S_0^{q(1)})$.

In particular, there exist isomorphisms of right $\text{Ext}_P^{\bullet}(I^{(r)}, I^{(r)})$-modules

(4.4.3) $\text{Ext}_P^{\bullet}(I^{(r)}, S_1^{q(1)}) \cong \text{Ext}_P^{s+2p^{r}}(I^{(r)}, S_1^{q(1)})$, and

(4.4.4) $\text{Ext}_P^{\bullet}(I^{(r)}, S_0^{q(1)}) \cong \text{Ext}_P^{s+2p^{r}}(I^{(r)}, S_0^{q(1)})$.

**Proof.** It suffices to prove (4.4.1), since then (4.4.2) follows from conjugating by the parity change functor II, and then (4.4.3) and (4.4.4) follow from composing (4.4.1) and (4.4.2). The first isomorphism in (4.4.1) is an isomorphism of right $\text{Ext}_P^{\bullet}(I^{(r)}, I^{(r)})$-modules by Corollary 4.2.2 and Remark 4.2.3, so it suffices to establish the second isomorphism in (4.4.1).

Theorem 4.3.2 implies that the nonzero terms in the $E_2$-page of (4.3.4) are in distinct total degrees, and the total degree of a nonzero term satisfies a parity condition depending on the term’s row; cf. Step 4 of the proof of Theorem 4.3.2. Then since $E_{\infty,0}^{s,t} = 0$ if $s + t \geq 2p^{r}$, and since $E_{\infty,1}^{0,0} = 0$, it follows that each differential in (4.3.4) that can be nontrivial is nontrivial. Specifically, if $0 \leq t < q$ and $s \equiv 0 \mod 2q$ with $s \geq 2p^{r}$, then

$$d_{(p-1)q+1} : E_{2,0}^{s-(p-1)q-1+t,(p-1)q+t} \rightarrow E_{2,0}^{s+t,t}$$

is nontrivial, hence an isomorphism. Similarly, if $s \equiv p^{r} \mod 2q$ and $s \geq p^{r}$, then

$$d_{(p-1)q+1} : E_{2,1}^{s-(p-1)q-1+t,(p-1)q+t} \rightarrow E_{2,1}^{s+t,t}$$

is an isomorphism; we illustrate the nontrivial differentials in (4.3.4) in Figure 5. Combining the $t = 0$ cases of these two observations, it follows for all $s \geq 0$ that the differential

$$d_{(p-1)q+1} : E_{2}^{s+q-1,(p-1)q} \rightarrow E_{2}^{s+p^{r},0}$$

in (4.3.2) defines an isomorphism $\text{Ext}_P^{s+q-1}(I^{(r)}, \Lambda_{1}^{q(1)}) \cong \text{Ext}_P^{s+p^{r}}(I^{(r)}, S_0^{q(1)})$. This isomorphism is moreover compatible with the right action of $\text{Ext}_P^{\bullet}(I^{(r)}, I^{(r)})$, since (4.3.2) is a spectral sequence of right $\text{Ext}_P^{\bullet}(I^{(r)}, I^{(r)})$-modules.

In this rest of this section we will apply Lemmas 4.4.1 and 4.4.3 and the results of Section 4.3 to describe $\text{Ext}_P^{\bullet}(I^{(r)}, S_0^{q(1)})$ as a right module over the Yoneda algebra $\text{Ext}_P^{\bullet}(I^{(r)}, I^{(r)})$. We will also make use of Corollaries 4.5.5 and 4.6.2, which do not rely on the results of this section.

Observe that $\text{Ext}_P^{\bullet}(I^{(r)}, S_0^{q(1)})$ identifies with the bottom row of the spectral sequence (4.3.1). Moreover, applying (4.2.3), we can identify the differential $d_{q+1} : E_{2}^{s-q-1,q} \rightarrow E_{2}^{s,0}$ in (4.3.1) with an $\text{Ext}_P^{\bullet}(I^{(r)}, I^{(r)})$-equivariant map

(4.4.5) $d_{q+1} : \text{Ext}_P^{s-2q}(I^{(r)}, S_0^{q(1)}) \rightarrow \text{Ext}_P^{*}(I^{(r)}, S_0^{q(1)})$.

Theorem 4.3.2 and Corollary 4.6.2 imply that $\text{Ext}_P^{*}(I^{(r)}, S_0^{q(1)})$ is generated over $\text{Ext}_P^{*}(I^{(r)}, I^{(r)})$ by the $p^{r-1}$-power map $\varphi_{r-1} : I_0^{(r)} \rightarrow S_0^{q(1)}$, which spans $\text{Hom}_P(I^{(r)}, S_0^{q(1)})$. More specifically, we
get that each \( z' \in \text{Ext}^s_\mathcal{P}(I^{(r)}, S^{q(1)}_0) \) can be uniquely expressed in the form \( z' = \varphi_{r-1} \cdot z \) for some \( z \in \text{Ext}^s_\mathcal{P}(I^{(r)}, I_0^{(r)}) \) or \( z \in \text{Ext}^s_\mathcal{P}(I_1^{(r)}, I_0^{(r)}) \). Now fix \( e_r \in \text{Ext}^{2q_1-1}(I_1^{(r)}, I_0^{(r)}) \) such that

\[
\partial_{q+1}(\varphi_{r-1}) = \varphi_{r-1} \cdot e_r.
\]

Then \((4.4.5)\) takes the form \( \partial_{q+1}(\varphi_{r-1} \cdot z) = \varphi_{r-1} \cdot e_r \cdot z \). In particular, applying Lemma \(4.4.1\) we get for \( 0 \leq \ell < p \) that \( \text{Ext}^{2q_\ell}(I_0^{(r)}, S^{q(1)}_0) \) is spanned by \( \varphi_{r-1} \cdot (e_r)^\ell \), and hence that \( \text{Ext}^{2q_\ell}(I_0^{(r)}, I_0^{(r)}) \) is spanned by \( (e_r)^\ell \), but that \( (e_r)^p = 0 \).

Now define \( c_r \in \text{Ext}^{p}(I_1^{(r)}, I_0^{(r)}) \) such that \( \varphi_{r-1} \cdot c_r \) is the image of \( \varphi_{r-1}^{\Pi} = \Pi \circ \varphi_{r-1} \circ \Pi \) under the composite isomorphism \((4.4.1)\). In other words,

\[
d_{(p-1)q+1}((K z_q^{(1)})^{\Pi} : \varphi_{r-1}^{\Pi}) = \varphi_{r-1} \cdot c_r.
\]

Then \( \varphi_{r-1}^{\Pi} \cdot c_r^{\Pi} = (\varphi_{r-1} \cdot c_r)^{\Pi} \) is the image of \( \varphi_{r-1} \) under \((4.4.2)\), and \( \varphi_{r-1} \cdot (c_r \cdot c_r^{\Pi}) \) is the image of \( \varphi_{r-1} \) under \((4.4.4)\). In particular, \( c_r \cdot c_r^{\Pi} \) spans \( \text{Ext}^{2q_\ell}(I_0^{(r)}, I_0^{(r)}) \). Now using \((4.4.4)\), it follows by induction on \( \ell \) that \( \text{Ext}^{2q_\ell}(I_0^{(r)}, S^{q(1)}_0) \) is spanned by \( \varphi_{r-1} \cdot (c_r \cdot c_r^{\Pi}) \). Since \((4.4.4)\) is a map of right \( \text{Ext}^{p}(I^{(r)}, I^{(r)}) \)-modules, the observations in the last sentence of the previous paragraph imply by induction on the cohomological degree that

\[
\{ \varphi_{r-1} \cdot (c_r \cdot c_r^{\Pi}) \ell : (e_r)^{\ell_1} : \ell, \ell_1 \in \mathbb{N}, \ell_1 < p \}
\]

is a basis for \( \text{Ext}^{p}(I_0^{(r)}, S^{q(1)}_0) \), and hence that \( \{(c_r \cdot c_r^{\Pi}) \ell : (e_r)^{\ell_1} : \ell, \ell_1 \in \mathbb{N}, \ell_1 < p \} \) is a linearly independent subset of \( \text{Ext}^{p}(I_0^{(r)}, I_0^{(r)}) \). Conjugating by \( \Pi \), we get that \( \{(c_r^{\Pi} \cdot c_r) \ell : (e_r^{\Pi})^{\ell_1} : \ell, \ell_1 \in \mathbb{N}, \ell_1 < p \} \) is linearly independent in \( \text{Ext}^{p}(I_1^{(r)}, I_1^{(r)}) \), and hence that \( \{(c_r \cdot c_r^{\Pi}) \ell : (e_r^{\Pi})^{\ell_1} : \ell, \ell_1 \in \mathbb{N}, \ell_1 < p \} \) is linearly independent in \( \text{Ext}^{p}(I_1^{(r)}, I_0^{(r)}) \), since left multiplication by \( c_r^{\Pi} \) transforms any element in the latter set into an element in the former set. Then by Corollary \(4.6.2\) and dimension comparison, we conclude that

\[
\{ \varphi_{r-1} \cdot c_r \cdot (c_r \cdot c_r^{\Pi}) \ell : (e_r)^{\ell_1} : \ell, \ell_1 \in \mathbb{N}, \ell_1 < p \}
\]

is a basis for \( \text{Ext}^p(I_1^{(r)}, S^{q(1)}_0) \).

**Proposition 4.4.4.** Let \( e_r \) be as defined in \((4.4.6)\), and let \( c_r \) be as defined in \((4.4.7)\). Then

\[
\{ \varphi_{r-1} \cdot (e_r)^{\ell_1} : (c_r \cdot c_r^{\Pi})^{\ell_1} : \ell, \ell_1 \in \mathbb{N}, \ell_1 < p \}
\]

is a basis for \( \text{Ext}^{p}(I_0^{(r)}, S^{q^{r-1}(1)}_0) \), and

\[
\{ \varphi_{r-1} \cdot (e_r)^{\ell_1} : (c_r \cdot c_r^{\Pi})^{\ell_1} \cdot c_r : \ell, \ell_1 \in \mathbb{N}, \ell_1 < p \}
\]

is a basis for \( \text{Ext}^p(I_1^{(r)}, S^{q^{r-1}(1)}_0) \).

**Proof.** Recall from Section \(3.3.1\) that the duality functor \( F \mapsto F^\# \) extends to an anti-involution on extension groups. This anti-involution restricts for each \( s \) to an involution on the vector space \( \text{Ext}^{p}(I_0^{(r)}, I_0^{(r)}) \), which is always at most one-dimensional by Corollary \(4.5.5\). Then it follows that \( (e_r)^\# \) is equal to either \( e_r \) or \( -e_r \), and similarly that \( (c_r \cdot c_r^{\Pi})^\# = \pm (c_r \cdot c_r^{\Pi}) \). Now applying the anti-involution \( z \mapsto z^\# \) to the linearly independent set \( \{(c_r \cdot c_r^{\Pi})^{\ell_1} \cdot (e_r)^{\ell_1} : \ell, \ell_1 \in \mathbb{N}, \ell_1 < p \} \), we deduce that \((4.4.8)\) is a basis for \( \text{Ext}^{p}(I_0^{(r)}, S^{q^{r-1}(1)}_0) \). Since each monomial in \((4.4.9)\) can be transformed via right multiplication by \( c_r^{\Pi} \) into a monomial in \((4.4.8)\), it follows by dimension comparison that \((4.4.9)\) must be a basis for \( \text{Ext}^{p}(I_1^{(r)}, S^{q^{r-1}(1)}_0) \). \( \square \)
The restriction functor $\mathcal{P} \to \mathcal{P}$ gives rise to a commutative diagram

$$
\begin{align*}
\Ext_P^r(I_0^{(r)}, I_0^{(r)}) & \to \Ext_P^r(I_0^{(r)}, S_0^{p^r-1}(1)) \\
\downarrow & \downarrow \\
\Ext_P^r(I^{(r)}, I^{(r)}) & \to \Ext_P^r(I^{(r)}, S_0^{p^r-1}(1))
\end{align*}
$$

in which the horizontal arrows are induced by the corresponding $p^r-1$-power maps. The diagram in Remark 4.4.2 implies that the restriction functor maps the extension class $\varphi_{r-1} \cdot e_r$ to the extension class denoted $\overline{e}_r$ in [12, p. 244]. Then the commutativity of the above diagram implies that the restriction functor sends $e_r$ to the distinguished extension class denoted $e_r$ in [12, p. 244]. In other words, $e_r|_{\mathcal{V}_\sigma} = e_r$. On the other hand, $c_r|_{\mathcal{V}_\sigma} = 0$ because $I_1^{(r)}|_{\mathcal{V}_\sigma} = 0$.

Let $j \geq 1$. Recall from Section 3.3.2 that precomposition with $I_0^{(j)}$ extends to an even linear map on extension groups in $\mathcal{P}$. We denote this map by $z \mapsto z^{(j)}$. Since for each $F \in \mathcal{P}$ one has $(F \circ I_0^{(j)})|_{\mathcal{V}_\sigma} = (F|_{\mathcal{V}_\sigma}) \circ I^{(j)}$, it follows that the map $z \mapsto z^{(j)}$ is compatible with the restriction functor $\mathcal{P} \to \mathcal{P}$. Then taking $j = r - i$, there exists for each $i \geq 1$ a commutative diagram

$$
\begin{align*}
\Ext_P^r(I_0^{(i)}, I_0^{(i)}) & \to \Ext_P^r(I_0^{(i)}, I_0^{(r)}) \\
\downarrow & \downarrow \\
\Ext_P^r(I^{(i)}, I^{(i)}) & \to \Ext_P^r(I^{(i)}, I^{(r)})
\end{align*}
$$

Moreover, the bottom arrow of this diagram is an injection by [12, Corollary 4.9]. Since for $i \geq 1$ the extension class $e_i \in \Ext_P^r(I_0^{(i)}, I_0^{(i)})$ restricts to the nonzero class $e_i \in \Ext_P^r(I_0^{(r)}, I_0^{(i)})$, it follows from the commutativity of the diagram that $e_i^{(r-i)} \neq 0$. In Section 4.6 we will show that $\Ext_P^r(I_0^{(r)}, I_0^{(r)})$ is generated as an algebra by the extension classes $e_1^{(r-1)}, e_2^{(r-2)}, \ldots, e_{r-1}^{(1)}, e_r, (e_r \cdot c_r^{\Pi})$.

**Remark 4.4.5.** Precomposition by $I^{(1)}$ induces for each $r \geq 0$ an injective algebra homomorphism $\Ext_P^r(I^{(r)}, I^{(r)}) \to \Ext_P^r(I^{(r+1)}, I^{(r+1)})$; see [12, Corollary 4.9]. On the other hand, for $r \geq 1$ the homomorphism $\Ext_P^r(I^{(r)}, I^{(r)}) \to \Ext_P^r(I^{(r+1)}, I^{(r+1)})$ induced by precomposition with $I^{(1)}$ sends $c_r$ and $c_r^{\Pi}$ to zero, because $\Ext_P^r(I^{(r+1)}, I^{(r+1)})$ is zero in degree $p^r$; see Corollary 4.5.5.

**Remark 4.4.6.** Let $\alpha : S \to \Gamma$ be the unique $\mathcal{P}$-algebra homomorphism extending the identification $S^1 = I = \Gamma^1$. The restriction of $\alpha$ to $S^p$ fits into an exact sequence

$$
0 \to I_0^{(1)} \to S^p \to \Gamma^p \to I_0^{(1)} \to 0
$$

in which the second arrow is the $p$-power map, and the fourth arrow is the dual Frobenius map. By [12, Lemma 4.12], the restriction of (4.4.10) to $\mathcal{V}_\sigma$ represents the element $e_1 \in \Ext_P^2(I^{(1)}, I^{(1)})$. Since $e_1|_{\mathcal{V}_\sigma} = e_1$, it follows that (4.4.10) is a representative extension for the cohomology class $e_1$ under the bijection $\theta_0$ of Proposition 3.5.1.

4.5. **Vector space structure of $\Ext_P^r(I^{(r)}, S_0^{p^r-1}(j))$, $j \geq 1$.** Recall for $n \in \mathbb{N}$ that $\Omega_n$ denotes the component of total degree $n$ of the ordinary de Rham complex functor $\Omega$, and $K_n$ denotes the Koszul kernel subfunctor of $\Omega_n$ defined in [12, §4]. Given $j \geq 1$, we use (2.7.7) to consider $\Omega_n^{(j)}$ and $K_n^{(j)}$ as strict polynomial superfunctors. Then for $1 \leq i \leq n$ one has

$$
\Omega_n^{ij} = (S^{n-i} \otimes \Lambda^i) \circ I_0^{(j)} = S_0^{n-i(j)} \otimes \Lambda_0^{ij}.
$$

Now let $j, r \in \mathbb{N}$ with $1 \leq j \leq r$. Our goal in this section is to describe the spectral sequences

$$
E_2^{s,t} = \Ext_P^s(I^{(r)}, H^t(\Omega_n^{(j)}_{p^{r-s}})) \Rightarrow \Ext_P^{s+t}(I^{(r)}, \Omega_n^{(j)}_{p^{r-s}}),
$$
Lemma 4.5.2. Assume by way of induction that \( 1 \leq \ell \) obtained by taking \( A = I^{(r)} \) and \( C = C_{p^r-j}^{(j)} \) in (3.6.2). As in Section 4.3, we will exploit the fact that for \( \ell \in \{0, 1\} \), there exist spectral sequences

\[
E_{2, \ell}^{s,t} = \text{Ext}_P^s(I^{(r)}, \text{H}^t(K^{(j)}_{p^r-j})) \Rightarrow \text{Ext}_P^{s+t}(I^{(r)}, K^{(j)}_{p^r-j})
\]

(4.5.3) that are direct summands of (4.5.2) and (4.5.3), respectively. One of the results of our investigation will be the following analogue of [12, Theorem 4.5], which computes the row \( t = 0 \) of (4.5.2):

**Theorem 4.5.1.** Let \( j, r \in \mathbb{N} \) with \( 1 \leq j \leq r \). Then

\[
\text{Ext}_P^s(I^{(r)}_0, S_0^{p^r-j}(j)) \cong \text{Ext}_P^{s+p^r-j-1}(I^{(r)}_0, \Lambda_0^{r-j}(j)) \cong \begin{cases} k & \text{if } s \equiv 0 \mod 2p^r-j \text{ and } s \geq 0, \\ 0 & \text{otherwise.} \end{cases}
\]

\[
\text{Ext}_P^s(I^{(r)}_1, S_0^{p^r-j}(j)) \cong \text{Ext}_P^{s+p^r-j-1}(I^{(r)}_1, \Lambda_0^{r-j}(j)) \cong \begin{cases} k & \text{if } s \equiv p^r \mod 2p^r-j \text{ and } s \geq p^r, \\ 0 & \text{otherwise.} \end{cases}
\]

Theorem 4.5.1 is true if \( j = 1 \) by Theorem 4.3.2 and (4.3.5), so for the rest of this section let us assume by way of induction that \( 1 \leq j < r \) and that Theorem 4.5.1 is true for the given values of \( j \) and \( r \). Using this assumption, we can compute the abutments of (4.5.2) and (4.5.3):

**Lemma 4.5.2.** Let \( j, r \in \mathbb{N} \) with \( 1 \leq j < r \). Then:

\[
\text{Ext}_P^s(I^{(r)}_0, K^{(j)}_{p^r-j}) = \begin{cases} k & \text{if } s \geq 0 \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}
\]

\[
\text{Ext}_P^s(I^{(r)}_0, \Omega^{(j)}_{p^r-j}) = \begin{cases} k & \text{if } s \geq 0 \text{ and } s \equiv 0 \mod 2p^r-j \text{ or } s \equiv -1 \mod 2p^r-j, \\ 0 & \text{otherwise.} \end{cases}
\]

\[
\text{Ext}_P^s(I^{(r)}_1, K^{(j)}_{p^r-j}) = \begin{cases} k & \text{if } s \geq p^r \text{ and } s \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}
\]

\[
\text{Ext}_P^s(I^{(r)}_1, \Omega^{(j)}_{p^r-j}) = \begin{cases} k & \text{if } s \equiv p^r \mod 2p^r-j \text{ and } s \geq p^r, \\ k & \text{if } s \equiv p^r - 1 \mod 2p^r-j \text{ and } s \geq p^r - 1 + 2p^r-j, \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** In this proof set \( q = p^r-j \) and let \( \ell \in \{0, 1\} \). First take \( A = I^{(r)}_\ell \) and \( C = K^{(j)}_{q} \) in (3.6.1). If \( s \geq q \), then \( K^q = 0 \), and hence \( E_1^{s,t} = 0 \). On the other hand, if \( s < q \), then (4.2.3) implies that

\[
E_1^{s,t} \cong E_0^{s-t,s} \cong \text{Ext}_P^{s-t,s}(I^{(r)}_\ell, S^q_0).
\]

By assumption, Theorem 4.5.1 is true for the given values of \( r \) and \( j \), so we can apply it via the above isomorphism to explicitly describe the vector space \( E_1^{s,t} \). Specifically, if \( 0 \leq s < q \), then

\[
E_1^{s,t} = \begin{cases} k & \text{if } \ell = 0, t \geq 0, \text{ and } t \equiv 0 \mod 2q, \\ k & \text{if } \ell = 1, t \geq p^r, \text{ and } t \equiv p^r \mod 2q, \\ 0 & \text{otherwise.} \end{cases}
\]

In particular, the total degrees of any two nonzero terms in the \( E_1 \)-page of the spectral sequence must be of the same parity. Then it follows that all differentials in the spectral sequence are zero, and hence that \( E_1 \cong E_\infty \). But \( E_\infty \cong \text{Ext}_P^*(I^{(r)}_\ell, K^{(j)}_q) \), so the calculation of \( \text{Ext}_P^*(I^{(r)}_\ell, K^{(j)}_q) \) follows from the explicit calculation of the \( E_1 \)-page as a total vector space.
Now take $A = I_r^{(r)}$ and $C = \Omega^q$ in (3.6.1). Then applying Theorem 3.4.3 and Theorem 4.5.1 for the given values of $r$ and $j$, the calculation of $\text{Ext}_P^s(I_r^{(r)}, \Omega^q)$ follows from a repetition of the proof of [12] (4.5.2). Specifically, Theorem 3.4.3 implies that $E_{s,t}^1 = 0$ unless $s = 0$ or $s = q$, and the differential between these two columns fits into an exact sequence

$$0 \to E_{s,t}^0 \to \text{Ext}_P^s(I_r^{(r)}, \Omega^q) \to \text{Ext}_P^{s+1-q}(I_r^{(r)}, \Lambda^q_0) \to E_{s+1,t-q}^\infty \to 0.$$  

From Theorem 4.5.1 it follows that the second and third terms in this exact sequence are never simultaneously nonzero. Then the differential between the columns $s = 0$ and $s = q$ vanishes, which implies that $E_1 \cong E_\infty$. Now the calculation of $E_\infty \cong \text{Ext}_P^s(I_r^{(r)}, \Omega^q)$ follows from Theorem 4.5.1 and the above four-term exact sequence. In particular, if $\ell = 1$ and $s \equiv 0 \mod 2q$ with $s \geq 0$, or if $\ell = 1$ and $s \equiv p^r \mod 2q$ with $s \geq p^r$, then the edge maps

$$\text{Ext}_P^s(I_r^{(r)}, \Omega^q) \to E_{0,s}^0 \cong \text{Ext}_P^s(I_r^{(r)}, \Omega^q_0)$$

are isomorphisms of one-dimensional spaces.

Our focus for the rest of this section is on analyzing the spectral sequences (4.5.4) and (4.5.5). Whereas the super Cartier isomorphism does not preserve the cohomological degree, the ordinary Cartier isomorphism [10, 3.3] induces isomorphisms of strict polynomial superfunctors

$$\text{H}^*(\Omega^q) \cong \text{H}^*(\Omega^{q+1}) \quad \text{and} \quad \text{H}^*(K_{p^r-j}) \cong \text{H}^*(K_{p^r-j+1}).$$

that preserve the cohomological degree. For the rest of this section, set $q = p^r-j-1$. Then

$$E_{2,\ell}^s \cong \text{Ext}_P^s(I_r^{(r)}, \Omega^{q+1}) \quad \text{and} \quad E_{2,\ell}^s \cong \text{Ext}_P^s(I_r^{(r)}, K^{q+1}).$$

Now (4.5.1) and Theorem 3.4.3 imply that $E_{2,\ell}^s = 0$ unless $t = 0$ or $t = q$. Next, the inclusion of complexes $K_{p^r-j} \hookrightarrow \Omega^{q+1}$ induces a map of spectral sequences $E \to \tilde{E}$ that on the $E_2$-page identifies with the map in cohomology induced by the inclusion $K_q \hookrightarrow \Omega^{q+1}$. In particular, since $K_q \cong \Omega_q$, the induced map $E_{2,\ell}^0 \to \tilde{E}_{2,\ell}^0$ is an isomorphism. The following lemma now follows from a word-for-word repetition of the proof of [12] (4.5.4).

**Proposition 4.5.3.** In (4.5.5), all differentials to terms in the row $t = 0$ are zero. Hence,

$$E_{\infty,\ell}^s \cong E_{2,\ell}^s \cong \text{Ext}_P^s(I_r^{(r)}, \Omega^{q+1}).$$

**Corollary 4.5.4.** On the $E_2$-page of (4.5.5), the total degrees of any two nonzero terms must be of the same parity. Consequently, all differentials in (4.5.5) are zero, and $E_{2,\ell} \cong E_{\infty,\ell}$.

**Proof.** As in the proof of Lemma 4.5.2, we have $E_{2,\ell}^s = 0$ if $t \geq q$, and $E_{2,\ell}^s \cong E_{2,\ell}^{s-t}$ if $0 \leq t < q$. Now $E_{2,\ell}^{s,0} \cong E_{\infty,\ell}^{s,0}$ by Proposition 4.5.3 and the total vector space structure of $E_{\infty,\ell}$ is given by Lemma 4.5.2. In particular, the total degrees of any two nonzero terms in $E_{\infty,\ell}$ must be of the same parity. Then the same conclusion also follows for $E_{2,\ell}$, whence the conclusion of the corollary.

We can now complete the proof of Theorem 4.5.1.

**Proof of Theorem 4.5.1.** By assumption, the theorem is true for the given values of $r$ and $j$. Then by induction, it suffices to show that the theorem is also true for $j+1$. By Corollary 4.5.4 and its proof, we have $E_{2,\ell} \equiv E_{\infty,\ell}$, $E_{2,\ell}^{s,0} \equiv E_{2,\ell}^{s+t}$ for $0 \leq t < q$, and $E_{2,\ell}^{s,t} = 0$ for $t \geq q$. Also, $E_{2,\ell}^{s,0} = 0$ for $s < 0$. Then by Lemma 4.5.2 and induction on $s$, we must have

$$E_{2,\ell}^{s,0} \cong \text{Ext}_P^s(I_r^{(r)}, \Omega^{q+1}) \cong \begin{cases} k & \text{if } \ell = 0, s \geq 0, \text{ and } s \equiv 0 \mod 2q, \\ k & \text{if } \ell = 1, s \geq p^r, \text{ and } s \equiv p^r \mod 2q, \\ 0 & \text{otherwise}, \end{cases}$$

This uses the assumption $r > j$ and its consequence $2q = 2p^r-j > 2$.\[\square\]
in order for $\bigoplus_{u+v=s} E_{2,\ell}^{u,v}$ to have the same dimension as $\bigoplus_{u+v=s} E_{n,\ell}^{u,v} \cong \text{Ext}^s_\mathcal{P}(I^{(r)}_0, K_{p^{r-j}(j)})$. □

**Corollary 4.5.5.** Let $r \geq 1$. Then

$$\text{Ext}^s_\mathcal{P}(I^{(r)}_1, I^{(r)}_1) \cong \text{Ext}^s_\mathcal{P}(I^{(r)}_0, I^{(r)}_0) \cong \begin{cases} k & \text{if } s \geq 0 \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

$$\text{Ext}^s_\mathcal{P}(I^{(r)}_1, I^{(r)}_0) \cong \text{Ext}^s_\mathcal{P}(I^{(r)}_1, I^{(r)}_0) \cong \begin{cases} k & \text{if } s \geq p^r \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** The first isomorphism in each line follows from conjugating by $\Pi$, while the second in each line is true by the case $j = r$ of Theorem 4.5.1 □

### 4.6. Module structure of $\text{Ext}^*_\mathcal{P}(I^{(r)}, S_0, p^{r-j}(j))$, $j \geq 1$.

Again let $j, r \in \mathbb{N}$ with $1 \leq j < r$, and set $q = p^{r-j-1}$. In this section we continue our investigation of the spectral sequence (4.5.4), with the goal of describing $\text{Ext}^*_\mathcal{P}(I^{(r)}, S_0, p^{r-j}(j))$ as a right module over the Yoneda algebra $\text{Ext}^*_\mathcal{P}(I^{(r)}, I^{(r)})$.

The spectral sequence (4.5.4) has only two nonzero rows,

$$E^{s,0}_{2,\ell} \cong \text{Ext}^s_\mathcal{P}(I^{(r)}_\ell, S_0^{(j+1)}) \quad \text{and} \quad E^{s,q}_{2,\ell} \cong \text{Ext}^s_\mathcal{P}(I^{(r)}_\ell, \Lambda_0^{(j+1)}) \cong \text{Ext}^s_\mathcal{P}(I^{(r)}_\ell, S_0^{(j+1)}),$$

and hence only one nontrivial differential, which identifies with a map

$$(4.6.2) \quad \partial_{q+1} : \text{Ext}^{s-2q}_\mathcal{P}(I^{(r)}_\ell, S_0^{(j+1)}) \rightarrow \text{Ext}^s_\mathcal{P}(I^{(r)}_\ell, S_0^{(j+1)})$$

that fits into a four-term exact sequence

$$(4.6.3) \quad 0 \rightarrow E^{s-q-1,q}_{\infty,\ell} \rightarrow \text{Ext}^{s-2q}_\mathcal{P}(I^{(r)}_\ell, S_0^{(j+1)}) \rightarrow \text{Ext}^s_\mathcal{P}(I^{(r)}_\ell, S_0^{(j+1)}) \rightarrow E^{s,0}_{\infty,\ell} \rightarrow 0.$$

Suppose $\ell = 0$. By Theorem 4.5.1 the second and third terms of this exact sequence are both isomorphic to $k$ if $s - 2q \geq 0$ and $s \equiv 0 \mod 2q$; the second (and hence also the first) term is zero, but the third (and hence also the fourth) term is isomorphic to $k$ if $s = 0$; and the second and third terms (and hence the end terms as well) are zero for all other values of $s$. So suppose $s - 2q \geq 0$ and $s \equiv 0 \mod 2q$. If $s \not\equiv 0 \mod 2p^{r-j}$, then Lemma 4.5.2 implies that the end terms of (4.6.3) are both zero, and hence that the differential is an isomorphism. Now suppose $s \equiv 0 \mod 2p^{r-j}$. Since $E^{s-1,0}_{2,0} = 0 = E^{s,0}_{2,0}$, it follows from Lemma 4.5.2 that the end terms of (4.6.3) must both be isomorphic to $k$, and hence the differential must be trivial. In particular, if $s \equiv 0 \mod 2p^{r-j}$, then the maps $E^{s,0}_{2,0} \rightarrow E^{s,0}_{\infty,0} \rightarrow \text{Ext}^s_\mathcal{P}(I^{(r)}_0, \Omega_{p^{r-j}(j)})$ are isomorphisms. Combined with the observations from the end of the proof of Lemma 4.5.2, this implies for $s \equiv 0 \mod 2p^{r-j}$ that the composite map (3.6.3) is an isomorphism. A similar analysis can also be applied if $\ell = 1$. We summarize the results of both analyses in the following theorem (cf. [12] (4.5.6)).

**Theorem 4.6.1.** Let $j, r \in \mathbb{N}$ with $1 \leq j < r$, and identify the nonzero rows of (4.5.4) as in (4.6.1).

1. Suppose $\ell = 0$. If $s \not\equiv 0 \mod 2p^{r-j}$, then (4.6.2) is an isomorphism. If $s \equiv 0 \mod 2p^{r-j}$, then (4.6.2) is trivial, and the $p$-power map $\varphi : S_0^{p^{r-j}(j+1)} \rightarrow S_0^{p^{r-j}(j)}$ induces an isomorphism

$$\text{Ext}^s_\mathcal{P}(I^{(r)}_0, S_0^{p^{r-j}(j+1)}) \cong \text{Ext}^s_\mathcal{P}(I^{(r)}_0, S_0^{p^{r-j}(j)}).$$

2. Suppose $\ell = 1$. If $s \not\equiv p^r \mod 2p^{r-j}$, then (4.6.2) is an isomorphism. If $s \equiv p^r \mod 2p^{r-j}$, then (4.6.2) is trivial, and $\varphi : S_0^{p^{r-j}(j+1)} \rightarrow S_0^{p^{r-j}(j)}$ induces an isomorphism

$$\text{Ext}^s_\mathcal{P}(I^{(r)}_1, S_0^{p^{r-j}(j+1)}) \cong \text{Ext}^s_\mathcal{P}(I^{(r)}_1, S_0^{p^{r-j}(j)}).$$

Repeated application of Theorem 4.6.1 yields:
Corollary 4.6.3. Let $1 \leq j < r$. The $p^{r-j}$-power map $\varphi_{r-j} : I_0^{(r)} \hookrightarrow S_0^{p^{r-j}(j)}$ induces isomorphisms

\[
\text{Ext}_p^s(I_0^{(r)}, I_0^{(r)}) \cong \text{Ext}_p^s(I_0^{(r)}, S_0^{p^{r-j}(j)}) \quad \text{if } s \equiv 0 \mod 2p^{r-j}, \text{ and}
\]

\[
\text{Ext}_p^s(I_1^{(r)}, I_0^{(r)}) \cong \text{Ext}_p^s(I_1^{(r)}, S_0^{p^{r-j}(j)}) \quad \text{if } s \equiv p^r \mod 2p^{r-j} \text{ and } s \geq p^r.
\]

Theorem 4.6.1 also implies the following direct analogue of [12, Corollary 4.6]:

Corollary 4.6.3. Let $j, r \in \mathbb{N}$ with $1 \leq j < r$. Let $\ell \in \{0, 1\}$, and let

\[V \subseteq \text{Ext}_p^{\bullet}(I_\ell^{(r)}, S_0^{p^{r-j-1}(j+1)})\]

be a graded subspace such that the $p$-power map $\varphi : S_0^{p^{r-j-1}(j+1)} \to S_0^{p^{r-j}(j)}$ induces a surjection from $V$ onto $\text{Ext}_p^{\bullet}(I_\ell^{(r)}, S_0^{p^{r-j}(j)})$, and such that $V$ is stable with respect to the endomorphism

\[d_{p^{r-j-1}(j+1)} : \text{Ext}_p^{\bullet-2p^{r-j-1}(j+1)}(I_\ell^{(r)}, S_0^{p^{r-j-1}(j+1)}) \to \text{Ext}_p^{\bullet}(I_\ell^{(r)}, S_0^{p^{r-j-1}(j+1)}).
\]

Then $V = \text{Ext}_p^{\bullet}(I_\ell^{(r)}, S_0^{p^{r-j-1}(j+1)})$.

Proof. We give the proof in the case $\ell = 1$, the proof for $\ell = 0$ being entirely analogous. The first assumption on $V$ implies that $\text{Ext}_p^{\bullet}(I_1^{(r)}, S_0^{p^{r-j-1}(j+1)}) \subseteq V$ for all $s \geq p^r$ with $s \equiv p^r \mod 2p^{r-j}$. Then the second assumption on $V$ implies that $\text{Ext}_p^{s}(I_1^{(r)}, S_0^{p^{r-j-1}(j+1)}) \subseteq V$ for all $s \geq p^r$ with $s \equiv p^r \mod 2p^{r-j-1}$, and hence that $V = \text{Ext}_p^{\bullet}(I_1^{(r)}, S_0^{p^{r-j-1}(j+1)})$.

Recall the cohomology classes $e_1^{(r-1)}, e_2^{(r-2)}, \ldots, e_{r-1}^{(1)}, e_r, c_r$ that were defined in Section 4.4. We can now use these classes to describe a basis for $\text{Ext}_p^{\bullet}(I_\ell^{(r)}, S_0^{p^{r-j}(j)})$.

Proposition 4.6.4. Let $j, r \in \mathbb{N}$ with $1 \leq j \leq r$. Then the set of monomials

(4.6.4) $\{\varphi_{r-j} \cdot (e_{r-j+1}^{(j-1)})^{\ell_1} \cdots (e_{r-1}^{(1)})^{\ell_j} (e_r)^{\ell_1} (c_r \cdot c_r^\Pi)^{\ell} : 0 \leq \ell_1, \ell_2, \ldots, \ell_j < p, \ell \geq 0\}$

is a basis for $\text{Ext}_p^{\bullet}(I_0^{(r)}, S_0^{p^{r-j}(j)})$, and the set of monomials

(4.6.5) $\{\varphi_{r-j} \cdot (e_{r-j+1}^{(j-1)})^{\ell_1} \cdots (e_{r-1}^{(1)})^{\ell_j} (e_r)^{\ell_1} (c_r \cdot c_r^\Pi)^{\ell} \cdot c_r : 0 \leq \ell_1, \ell_2, \ldots, \ell_j < p, \ell \geq 0\}$

is a basis for $\text{Ext}_p^{\bullet}(I_1^{(r)}, S_0^{p^{r-j}(j)})$.

Proof. The proof is by induction on $j$ in a direct generalization of the argument used for the proof of [12, Corollary 4.7]. The base case of the induction argument is handled using Proposition 4.4.4. Then the induction step is handled using Theorem 4.6.1 Corollary 4.6.3 and the twisting functor $F \mapsto F \circ I_0^{(j)}$ in the same way that the proof of [12, Corollary 4.7] uses [12, Corollary 4.6] and the twisting functor $F \mapsto F \circ I^{(j)}$.

The case $j = r$ of Proposition 4.6.4 immediately gives:

Corollary 4.6.5. Let $r \geq 1$. Then the set of monomials

(4.6.6) $\{(e_1^{(r-1)})^{\ell_1} \cdots (e_{r-1}^{(1)})^{\ell_j} (e_r)^{\ell_1} (c_r \cdot c_r^\Pi)^{\ell} : 0 \leq \ell_1, \ell_2, \ldots, \ell_r < p, \ell \geq 0\}$

is a basis for $\text{Ext}_p^{\bullet}(I_0^{(r)}, I_0^{(r)})$, and the set of monomials

(4.6.7) $\{(e_1^{(r-1)})^{\ell_1} \cdots (e_{r-1}^{(1)})^{\ell_j} (e_r)^{\ell_1} (c_r \cdot c_r^\Pi)^{\ell} \cdot c_r : 0 \leq \ell_1, \ell_2, \ldots, \ell_r < p, \ell \geq 0\}$

is a basis for $\text{Ext}_p^{\bullet}(I_1^{(r)}, I_0^{(r)})$. 
4.7. Algebra structure of $\text{Ext}^*_p(I^r, I^r)$. Up to certain structure coefficients that are equal to ±1, we can now describe the multiplicative structure of the Yoneda algebra $\text{Ext}^*_p(I^r, I^r)$.

Theorem 4.7.1. Let $r \geq 1$. For $1 \leq i \leq r$, set $\tilde{e}_i = e_i^{(r-i)}$. Then $\text{Ext}^*_p(I^r, I^r)$ is generated as a $k$-algebra by $\tilde{e}_1, \ldots, \tilde{e}_r$, and $\tilde{e}_1^\Pi, \ldots, \tilde{e}_r^\Pi$, subject only to the following relations:

1. For each $1 \leq i \leq r$, $\tilde{e}_i \cdot \tilde{e}_r = \tilde{e}_i^\Pi \cdot \tilde{e}_r = \tilde{e}_i \cdot \tilde{e}_r^\Pi = 0$.
2. For each $1 \leq i, j \leq r$, $\tilde{e}_j^\Pi \cdot \tilde{e}_i = 0$ and $\tilde{e}_i^\Pi = (\tilde{e}_i^\Pi)^p = 0$.
3. For each $1 \leq i \leq r$, there exists $\lambda_i \in \{±1\}$ such that

$$\tilde{e}_i \cdot \tilde{e}_r = \lambda_i \cdot \tilde{e}_r \cdot \tilde{e}_i^\Pi$$

and

$$\tilde{e}_i \cdot \tilde{e}_r^\Pi = \lambda_i \cdot \tilde{e}_r^\Pi \cdot \tilde{e}_i.$$ 

4. The subalgebra generated by $\tilde{e}_1, \ldots, \tilde{e}_r, (\tilde{e}_r \cdot \tilde{e}_r^\Pi), \tilde{e}_1^\Pi, \ldots, \tilde{e}_r^\Pi, (\tilde{e}_r^\Pi \cdot \tilde{e}_r)$ is commutative. In particular, the even restriction functor $\mathcal{P} \to \mathcal{P}$, $F \mapsto F|_{\mathcal{V}_m}$, induces an isomorphism from the subalgebra of $\text{Ext}^*_p(I^r, I^r)$ generated by $\tilde{e}_1, \ldots, \tilde{e}_r$ onto the Yoneda algebra $\text{Ext}^*_p(I^r, I^r)$.

Proof. First, the fact that $\text{Ext}^*_p(I^r, I^r)$ is generated as an algebra by the given list of elements follows from Corollary [4.6.5] on the matrix ring decomposition (1.0.1) of $\text{Ext}^*_p(I^r, I^r)$, and the conjugation action of $\Pi$. The matrix ring decomposition of $\text{Ext}^*_p(I^r, I^r)$ also implies (1) and the first string of equalities in (2), while $\tilde{e}_i^p = (\tilde{e}_i^\Pi)^p = 0$ for each $1 \leq i \leq r$ by the observations following (4.4.6). Next, consider the pair of commuting endomorphisms $z \mapsto z^\Pi$ and $z \mapsto z^\Pi$ on $\text{Ext}^*_p(I^r, I^r)$, and the composite anti-involution $z \mapsto z^\Pi$. It follows as in the proof of Proposition 4.4.4 that $(\tilde{e}_i)^\Pi = ±(\tilde{e}_i)$ for each $1 \leq i \leq r$. Similarly, $(\tilde{e}_r \cdot \tilde{e}_r^\Pi)(\tilde{e}_i)^\Pi = ±(\tilde{e}_i \cdot \tilde{e}_r)$ for each $1 \leq i \leq r$. But $(\tilde{e}_i \cdot \tilde{e}_r^\Pi)(\tilde{e}_i)^\Pi = (\tilde{e}_i)(\tilde{e}_r^\Pi)(\tilde{e}_i)^\Pi = ±(\tilde{e}_i \cdot \tilde{e}_r^\Pi)$ for some $\lambda_i \in \{±1\}$. Conjugating by $\Pi$, we get $\tilde{e}_i \cdot \tilde{e}_r^\Pi = \lambda_i \cdot \tilde{e}_r^\Pi \cdot \tilde{e}_i$ as well. This proves (3). Now to prove (4), it suffices to show that the subalgebra generated by $\tilde{e}_1, \ldots, \tilde{e}_r$ is commutative.

Let $A$ be the subalgebra of $\text{Ext}^*_p(I^r, I^r)$ generated by $\tilde{e}_1, \ldots, \tilde{e}_r$. Since $\tilde{e}_i$ restricts to the extension class denoted $e_i^{(r-i)}$ in (12) (as observed after Proposition 4.4.4), and since the classes $e_1^{(r-1)}, \ldots, e_r$ generate $\text{Ext}^*_p(I^r, I^r)$ as an algebra, we deduce that the restriction functor $\mathcal{P} \to \mathcal{P}$ induces a surjective algebra homomorphism $A \to \text{Ext}^*_p(I^r, I^r)$. Corollary [4.6.5] implies that the two algebras are of the same dimension, and hence that the homomorphism is an isomorphism. Then since $\text{Ext}^*_p(I^r, I^r)$ is a commutative algebra, $A$ is as well. 

\[ \square \]

Problem 4.7.2. Determine for each $1 \leq i \leq r$ the structure constant $\lambda_i$.

5. Applications of the universal extension classes

In this section we present our main application of the extension classes $e_r$ and $c_r$ exhibited in Section 4.4 that the cohomology ring of a finite $k$-supergroup scheme is a finitely-generated $k$-algebra. We also discuss some implications for the cohomology of $GL(m|n)$. We begin in Section 5.1 by recalling some of our previous work on the cohomology of finite supergroup schemes.

5.1. Recollections on the cohomology of finite supergroup schemes. An affine $k$-supergroup scheme $G$ is equivalent to the data of its coordinate superalgebra $k[G]$, a commutative Hopf superalgebra over $k$. The commutativity of $k[G]$ implies for each $r \in \mathbb{N}$ that the $p^r$-power map defines a Hopf superalgebra homomorphism $k[G]^{(r)} \to k[G]$. The $r$-th Frobenius kernel $G_r$ of $G$ is then the scheme-theoretic kernel of the corresponding comorphism $P_G^r : G \to G^{(r)}$. In other words, $G_r$ is the affine $k$-supergroup scheme with coordinate superalgebra $k[G_r] = k[G]/(\sum_{j \in I_r} k[G] f^p)$. Here $I_r$ denotes the augmentation ideal of $k[G]$. An affine $k$-supergroup scheme is algebraic if $k[G]$ is a finitely-generated $k$-algebra, is finite if $k[G]$ is finite-dimensional, and is infinitesimal if it is finite and if $I_r$ is nilpotent. If $G$ is infinitesimal, then the minimal non-negative integer $r$ such that $f^{p^r} = 0$ for all $f \in I_r$ is the height of $G$. For example, if $G$ is an affine $k$-supergroup scheme, then the Frobenius kernel $G_r$ is infinitesimal of height $r$. 


Let $G$ be a finite $k$-supergroup scheme. Then the dual superalgebra $k[G]^*$ is a finite-dimensional cocommutative Hopf superalgebra, and the category of left $G$-supermodules is isomorphic to the category of left $k[G]^*$-superalgebras. Using the fact that the supertwist map makes $\mathbf{V}_{ev}$ into an abelian braided monoidal category, one gets from [16, Theorem 3.12] that the cohomology ring $H^r(G, k) \cong H^r(k[G]^*, k)$ is a graded-commutative superalgebra, i.e., if $w \in H^r(G, k)$ and $z \in H^s(G, k)$, then $z \cdot w = (-1)^{ij}(w \cdot z)$. More generally, let $M$ be a trivial $G$-supermodule. Then there exist natural even isomorphisms

\[(5.1.6)\quad H^i(G, M) \cong H^i(G, k) \otimes M \cong \text{Hom}_k(M^*, H^i(G, k)).\]

Considering a homogeneous element $z \in H^i(G, M)$ as a linear map $z : M^* \to H^i(G, k)$ of the same parity, the graded-commutativity of $H^* (G, k)$ then implies that $z$ extends uniquely to an even homomorphism of graded superalgebras

\[(5.1.7)\quad \text{Hom}_k(M^*, H^i(G, k)) \cong \text{H}^i(G, M) \cong H^i(G, k) \otimes M \cong \text{Hom}_k(M^*, H^i(G, k)).\]

Here $S(V(i))$ and $\Lambda(V(i))$ denote the “$i$-refractions” of the graded superspaces $S(V)$ and $\Lambda(V)$, i.e., $S^j(V(i)) = S^j(V)$ for each $j \in \mathbb{N}$, while $S^j(V(i)) = 0$ if $j \notin i\mathbb{Z}$, and similarly for $\Lambda(V(i))$.

In [7, §5.3–5.4], we considered for each finite $k$-supergroup scheme $G$ and each finite-dimensional $G$-supermodule $M$ the following questions:

(5.1.4) Is $H^* (G, k)$ a finitely-generated $k$-algebra?

(5.1.5) Is $H^* (G, M)$ finitely-generated under the cup product action of $H^* (G, k)$?

In [7, Theorem 5.3.3], we showed that if the answers to these questions are yes whenever $G$ is an infinitesimal $k$-supergroup scheme, then the answers are yes for every finite $k$-supergroup scheme. In [7, Theorem 5.4.2], we showed further that if $G$ is infinitesimal of height $r$, then the answers to the above two questions are yes provided that there exists a closed embedding $G \to GL(m|n)_r$ for some $m, n \in \mathbb{N}$ with $(m+n) \not\equiv 0 \mod p$, and provided that, for these values of $m$ and $n$, there exist certain conjectured cohomology classes $e_r^{m,n}$ and $c_r^{m,n}$ for $GL(m|n)$ whose restrictions to $GL(m|n)_r$ admit particular descriptions. To precisely state the required conditions for $e_r^{m,n}|_{G_1}$ and $c_r^{m,n}|_{G_1}$, we first recall the May spectral sequence constructed in [7, §5.2].

Let $G$ be an algebraic $k$-supergroup scheme, and let $\mathfrak{g} = \text{Lie}(G)$ be the Lie superalgebra of $G$. Then by [7, Corollary 5.2.3], there exists a spectral sequence of rational $G$-supermodules

\[(5.1.6)\quad E_0^{i,j} = \Lambda^j(\mathfrak{g}^*) \otimes S^i(\mathfrak{g}_0^2)^{(1)} \Rightarrow H^{i+j}(G_1, k).\]

Here $S(\mathfrak{g}_0^2)^{(1)}$ denotes the 2-refraction of $S(\mathfrak{g}_0^2)$, considered as a rational $G$-supermodule via the Frobenius morphism $F_G : G \to G^{(1)}$. Identifying $\Lambda(\mathfrak{g}^*)$ with the graded tensor product of algebras $\Lambda(\mathfrak{g}_0^2)^g \otimes S(\mathfrak{g}_0^2)$, we get from [7, Proposition 3.5.3 and Lemma 5.2.2] that the subalgebra $S(\mathfrak{g}_0^2)^p \Lambda(\mathfrak{g}^*)$ consisting of the $p$-th powers in $S(\mathfrak{g}_0^2)$ is a subalgebra of permanent cycles in $E_0$, and that $S(\mathfrak{g}_0^2)^{p} \cong S(\mathfrak{g}_0^{2p})^{(1)}$ as graded $G$-supermodules.

Now let $m$ and $n$ be positive integers, let $G = GL(m|n)$ be the corresponding general linear supergroup, and let $\mathfrak{g} = \mathfrak{gl}(m|n) = \text{Hom}_k(k^{m|n}, k^{m|n})$ be the Lie superalgebra of $G$. The conjectured classes $e_r^{m,n}$ and $c_r^{m,n}$ mentioned above are cohomology classes

\[e_r^{m,n} \in H^{2p^r-1}(GL(m|n), \mathfrak{gl}(m|n)^{(r)})\quad \text{and} \quad c_r^{m,n} \in H^{p^r}(GL(m|n), \mathfrak{gl}(m|n)^{(r)})\]

whose restrictions $e_r^{m,n}|_{G_1}$ and $c_r^{m,n}|_{G_1}$ admit the following descriptions:

\[(5.1.7)\quad \text{The $G$-supermodule homomorphism } e_r^{m,n}|_{G_1} : S^*(\mathfrak{g}_0^2(2p^r-1))^{(r)} \to H^*(G_1, k) \text{ induced by (5.1.2)} \text{ is equal to the composition of the $p^{r-1}$-power map } S^*(\mathfrak{g}_0^2(2p^r-1))^{(r)} \to S^*(\mathfrak{g}_0^2(2p^r))^{(1)} \text{ with the horizontal edge map } S^*(\mathfrak{g}_0^2(2p^r))^{(1)} \to H^*(G_1, k) \text{ of (5.1.6).}\]
Our goal is to show that, up to rescaling, \((e, \varepsilon)\) with the vertical edge map of \((e, \varepsilon)\) has image equal to the subalgebra of \(\Lambda(g^*)\) generated by all \(p^r\)-th powers in \(S(g^*) \subset \Lambda(g^*)\).

Our goal is to show that the extension classes \(e_r, e^p_r, c_r, \text{ and } c^p_r\) exhibited in Section 4.4 provide in a natural way the conjectured classes \(e_{r,m,n}\) and \(c_{r,m,n}\). Specifically, evaluation on the superspace \(k^{m|n}\) defines an exact functor from \(\mathcal{P}\) to the category of rational \(G\)-supermodules. This functor then induces for each \(T, T' \in \mathcal{P}\) a natural even linear map \(\text{Ext}^*_\mathcal{P}(T, T') \to \text{Ext}_G^*(T(k^{m|n}), T'(k^{m|n})), \; \; z \mapsto z|_G\). Observe that

\[
\begin{align*}
\mathfrak{g}_m &= \text{Hom}_k(k^{m|0}, k^{m|0}), \quad \mathfrak{g}_{m+1} = \text{Hom}_k(k^{0|n}, k^{m|0}), \quad \text{and} \\
\mathfrak{g}_n &= \text{Hom}_k(k^{0|n}, k^{0|n}), \quad \mathfrak{g}_{-1} = \text{Hom}_k(k^{m|0}, k^{0|n}).
\end{align*}
\]

are each naturally subspaces of \(\mathfrak{g} = \mathfrak{g}(m|n)\), with \(\mathfrak{g}_n \oplus \mathfrak{g}_m = \mathfrak{g}_T^n\), and \(\mathfrak{g}_{-1} \oplus \mathfrak{g}_{m+1} = \mathfrak{g}_T^m\). Restricting \(e_r, e^p_r, c_r, \text{ and } c^p_r\) to \(G\), we get cohomology classes

\[
\begin{align*}
e_r|_G &\in \text{Ext}_G^{2p-1}(k^{m|0}(r), k^{m|0}(r)), \quad \cong \; \text{H}^{2p-1}(G, \mathfrak{g}_m^r), \\
e^p_r|_G &\in \text{Ext}_G^{2p-1}(k^{0|n}(r), k^{0|n}(r)), \quad \cong \; \text{H}^{2p-1}(G, \mathfrak{g}_n^r), \\
c_r|_G &\in \text{Ext}_G^{p+1}(k^{0|n}(r), k^{m|0}(r)), \quad \cong \; \text{H}^{p+1}(G, \mathfrak{g}_T^m), \\
c^p_r|_G &\in \text{Ext}_G^{p+1}(k^{m|0}(r), k^{0|n}(r)), \quad \cong \; \text{H}^{p+1}(G, \mathfrak{g}_T^n).
\end{align*}
\]

Our goal is to show that, up to rescaling, \((e_r + e^p_r)|_G \in \text{H}^{2p-1}(G, \mathfrak{g}_T^n)\) provides the conjectured class \(e_{r,m,n}\) and \((c_r + c^p_r)|_G \in \text{H}^p(G, \mathfrak{g}_T^n)\) provides the conjectured class \(c_{r,m,n}\). We begin in Sections 5.2 and 5.3 by showing for each \(r \geq 1\) that \(e_r\) and \(c^p_r\) restrict nontrivially to \(G_1\). Then in Section 5.4 we verify the conditions \((5.1.7)\) and \((5.1.8)\).

5.2. Restriction of \(e_1.\) In this section, fix positive integers \(m\) and \(n\), let \(G = GL(m|n)\) be the corresponding general linear supergroup, and let \(G_1\) be its first Frobenius kernel.

**Lemma 5.2.1.** \(c_1|_{G_1} \neq 0\) and \(c^p_1|_{G_1} \neq 0.\)

**Proof.** Given \(1 \leq i, j \leq m + n\), write \(e_{i,j}\) for the corresponding \((m + n) \times (m + n)\) matrix unit whose \((i,j)\)-entry is equal to 1 and whose other entries are 0. Let \(U\) be the one-dimensional odd subsupergroup scheme of \(G\) such that for each commutative \(k\)-superalgebra \(A\),

\[
U(A) = \{I_{m+n} + a \cdot e_{m,m+1} : a \in A_T\},
\]

where \(I_{m+n}\) denotes the \((m + n) \times (m + n)\) identity matrix. Then \(U\) is a subsupergroup scheme of \(G_1\). We will show that \(c_1|_{G_1} \neq 0\) by showing that \(c_1|_U \neq 0.\) First, consider the super Koszul kernel complex \(K_p\). Proposition 4.25 implies that \(\text{H}^0(K_p) \cong I_0^{(1)}, \; \text{H}^{0-1}(K_p) \cong I_1^{(1)}, \text{and } \text{H}^i(K_p) = 0\) for \(i \notin \{0, p-1\}\). Then by Proposition 3.5.1 the augmented complex

\[
K' : 0 \to I_0^{(1)} \to K_p^0 \to K_p^1 \to \cdots \to K_p^{p-2} \to K_p^{p-1} \to I_1^{(1)} \to 0
\]

represents an element \(\widehat{c}_1\) of the one-dimensional space \(\text{Ext}_\mathcal{P}^1(I_1^{(1)}, I_0^{(1)})\). In particular, \(\widehat{c}_1\) is a scalar multiple of \(c_1.\) We will show that \(\widehat{c}_1|_{U} \neq 0\) and hence that \(c_1|_{U} \neq 0.\)

Let \(x_1, \ldots, x_m\) be the standard basis for \(k^m,\) and let \(y_1, \ldots, y_n\) be the standard basis for \(k^n.\) Then we consider \(x_1, \ldots, x_m, y_1, \ldots, y_n\) as a homogeneous basis for \(k^{m|n},\) with \(\mathfrak{f}_1 = \mathfrak{0}\) and \(\mathfrak{g}_j = \mathfrak{1}\) for each \(i\) and \(j.\) Let \(\Lambda(z)\) be the ordinary exterior algebra over a one-dimensional odd superspace

\[\text{for each } d \in \mathbb{N}, \text{evaluation on } k^{m|n} \text{ defines an exact functor from } \mathcal{P}_d \text{ to the category of finite-dimensional supermodules for the Schur superalgebra } S(m|n, d), \text{ which is a quotient of the superalgebra } \text{Dist}(G) \text{ of distributions for the supergroup } G = GL(m|n); \text{ cf. } [5, \text{ Theorem 3.2}] \text{ and } [8, \text{ §2}]. \text{ Since the category of integrable } \text{Dist}(G)\text{-supermodules is isomorphic to the category of rational } G\text{-supermodules} [5, \text{ Corollary 3.5}], \text{ we thus get an exact functor from } \mathcal{P}_d \text{ to the category of rational } G\text{-supermodules}.\]
spanned by the vector $z$. The category of left $U$-supermodules is isomorphic to the category of left supermodules for $k[U]^* \cong \Lambda(z)$. Under this isomorphism, the action of $\Lambda(z)$ on $k^{m|n}$ is defined by $z.x_i = 0$ and $z.y_j = \delta_{j,1} \cdot x_m$. This equivalence also induces an isomorphism
\begin{equation}
\text{Ext}^*_U(I_1^{(1)}(k^{m|n}), I_0^{(1)}(k^{m|n})) \cong \text{Ext}^*_\Lambda(z)(k^{0|n(1)}, k^{m|0(1)}),
\end{equation}
where $k^{0|n(1)}$ and $k^{m|0(1)}$ are considered as trivial $\Lambda(z)$-supermodules. A projective resolution of the trivial $\Lambda(z)$-module $k$ is given by the complex $P_*$, where $P_i = \Lambda(z)$ for each $i \geq 0$, and for $i \geq 1$ the differential $P_i \to P_{i-1}$ is multiplication by $z$. Then $P^i := P_* \otimes k^{0|n(1)}$ is a projective resolution of $k^{0|n(1)}$, and the image of $\overline{c}_i|U$ under the isomorphism (5.2.2) is computed by lifting the identity function $k^{0|n(1)} \to k^{0|n(1)}$ to a chain map $\varphi : P' \to K'(k^{m|n})$, and then taking the cohomology class in $\text{Hom}_{\Lambda(z)}(P', k^{m|0(1)})$ of the cocycle $\varphi_p : P'_p \to k^{m|0(1)}$.

For each $i \geq 0$, $P'_i$ is a free $\Lambda(z)$-supermodule of rank $n$ with a homogeneous $\Lambda(z)$-basis by the vectors $1 \otimes y_j^{(1)}$ for $1 \leq j \leq n$. By definition, $K'_i(k^{m|n})$ is a subspace of $S^{p-i}(k^{m|n}) \otimes A^i(k^{m|n})$. We express elements of $S(k^{m|n})$ and $A(k^{m|n})$ in terms of the bases (2.3.2) and (2.3.5). Then the action of $z$ on $S(k^{m|n}) \otimes A(k^{m|n})$ is given by
\begin{align*}
z, &\left[ (x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_n}) \otimes (x_1^{c_1} \cdots x_m^{c_m} \gamma_d_1(y_1) \cdots \gamma_d_r(y_n)) \right] \\
&= b_1 \cdot \left[ (x_1^{a_1} \cdots x_m^{a_m+1} y_1^{b_1-1} \cdots y_n^{b_n}) \otimes (x_1^{c_1} \cdots x_m^{c_m} \gamma_d_1(y_1) \cdots \gamma_d_r(y_n)) \right] \\
&\quad + (-1)^{b_1+\cdots+b_n} \left[ (x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_n}) \otimes (x_1^{c_1} \cdots x_m^{c_m+1} \gamma_d_1-1(y_1) \cdots \gamma_d_r(y_n)) \right].
\end{align*}

We now define a family of $\Lambda(z)$-supermodule homomorphisms $\varphi_i : P'_i \to K_p^{p-i}(k^{m|n})$ as follows: For $0 \leq i \leq p - 1$, define $\varphi_i$ such that $\varphi_i(1 \otimes y_j^{(1)}) = \delta_{j,1} \cdot \frac{1}{(p-1)!} \cdot [x_m y_1 \otimes \gamma_{p-i-1}(y_1)]$. Define $\varphi_p$ such that $\varphi_p(1 \otimes y_j^{(1)}) = \delta_{j,1} \cdot \frac{1}{(p-1)!} \cdot x_m = -\delta_{j,1} \cdot x_m^{(1)} \in k^{m|0(1)}$, and set $\varphi_i = 0$ for $i > p$. Then one can check that this family of homomorphisms defines a chain map $\varphi : P' \to K'(k^{m|n})$ lifting the identity function $k^{0|n(1)} \to k^{0|n(1)}$. Since $\varphi_p$ is nonzero, and since the differential on the complex $\text{Hom}_{\Lambda(z)}(P'_*, k^{m|0(1)})$ is trivial, it follows that $\overline{c}_i|U \neq 0$, and hence $c_i|G_1 \neq 0$. The proof that $c_r^{\Pi}|G_1 \neq 0$ is entirely analogous, replacing $U$ by the transpose sub-supergroup $U'$ of $G_1$ defined by $U'(A) = \{ I_{m+n} + a \cdot e_{m+1,m} : a \in A_T \}$. 

5.3. **Restriction of $c_r$.** Again, fix positive integers $m$ and $n$, and set $G = GL(m|n)$. Let $U$ be be as defined in (5.2.1), and let $H^*(U, k)$ denote the cohomology ring $\text{Ext}^*_U(k, k) \cong \text{Ext}^*_\Lambda(z)(k, k)$, which is known to be a polynomial algebra generated by an odd generator $\beta$ of cohomological degree 1. For each $r \geq 1$ and for each pair of trivial $U$-supermodules $V^{(r)}$ and $W^{(r)}$, there exists a natural identification
\begin{equation}
\text{Ext}^*_U(V^{(r)}, W^{(r)}) \cong H^*(U, k) \otimes \text{Hom}_k(V, W)^{(r)}.
\end{equation}
Taking $r = 1$, the proof of Lemma 5.2.1 shows that, up to a nonzero scalar factor, $c_1|U$ identifies with $\beta \cdot \alpha^{(1)}$, where $\alpha : k^m \to k^m$ is the linear map defined with respect to the standard bases for $k^n$ and $k^m$ by the $m \times n$ unit matrix $e_{m,1}$. Similarly, replacing $U$ by its transpose $U'$, $c_1^{\Pi}|U'$ identifies with a nonzero scalar multiple of $\beta \cdot \alpha^{(1)}$, where $\alpha^t : k^m \to k^n$ is the transpose of $\alpha$. Our goal in this section is to extend these identifications to all $r \geq 1$. Specifically, we will show by induction on $r$ that, under the identification (5.3.1), $c_r|U$ identifies with a nonzero scalar multiple of $\beta \cdot \alpha^{(r)}$, and by symmetry that $c_r^{\Pi}|U'$ identifies with a nonzero scalar multiple of $\beta \cdot \alpha^{(r)}$, and hence that $c_r|G_1 \neq 0$ and $c_r^{\Pi}|G_1 \neq 0$.

Before delving into details, let us first indicate the general strategy of our induction argument. First, consider $I_1^{(r)}$ and $I_0^{(r)}$ as subfunctors of $\Gamma_1^{(r)} := A \circ I_1^{(r)}$ and $S_0^{(r)} := S \circ I_0^{(r)}$, respectively. Then for each $i \geq 1$, we get the $i$-fold cup product $c_i^{\Pi j} \in \text{Ext}^p_{\Pi}(\Gamma_1^{(r)}, S_0^{(r)})$. Assuming by way of
induction that $c_r|_U$ identifies with a nonzero scalar multiple of $\beta^{p^r} \cdot \alpha^{(r)}$, it follows that $(c_{r,k})|_U$ identifies with a nonzero scalar multiple of $\beta^{p^r} \cdot \alpha^{(r)}$, where $\alpha_i$ denotes the composite map

$$\Gamma^i(k^n) = ((k^n) \otimes i)^\circ i \hookrightarrow (k^n)^{\otimes i} \xrightarrow{\alpha^{\otimes i}} (k^m)^{\otimes i} \to S^i(k^m).$$

In particular, $(c_{r,i})|_U \neq 0$, hence $c_{r,i} \neq 0$. One immediately checks that $\alpha_p$ equals the composite

$$\Gamma^p(k^n) \xrightarrow{\varphi^p} (k^n)^{(1)} \xrightarrow{(\alpha^{(1)})} (k^m)^{(1)} \xrightarrow{\varphi} S^p(k^m),$$

where $\varphi^#$ and $\varphi$ denote the dual Frobenius map and the $p$-power map, respectively. Then $\beta^{p^{r+1}} \cdot \alpha_p^{(r+1)}$ is the image of $\beta^{p^{r+1}} \cdot \alpha_p^{(r+1)}$ under the map in cohomology

$$(\varphi^#, \varphi) : \text{Ext}_U^{pr+1}(k^{n(r+1)}, k^{m(r+1)}) \to \text{Ext}_U^{pr+1}(\Gamma^p(k^n), S^p(k^m))$$

induced in the obvious way by $\varphi^#$ and $\varphi$. Our plan is to show that the corresponding homomorphism between extension groups in $P$,

$$(\varphi^#, \varphi) : \text{Ext}_P^{pr+1}(I_1^{(r+1)}, I_0^{(r+1)}) \to \text{Ext}_P^{pr+1}(\Gamma_p^{(r)}, S_0^{(r)}),$$

maps $c_{r+1}$ to a nonzero scalar multiple of $c_{r+1}^p$; cf. [9 Corollary 5.9]. Then, using the fact that $(\varphi, \varphi)$ is an injection, which can be seen from (3.5.3) by the fact that $\varphi^# : \Gamma^p(k^n) \to k^{n(1)}$ is a surjection and $\varphi : k^{m(1)} \to S^p(k^m)$ is an injection, and using the compatibility of $(\varphi^#, \varphi)$ with the restriction map $z \mapsto z|_U$, it follows that $c_{r+1}|_U$ identifies with a nonzero multiple of $\beta^{p^{r+1}} \cdot e_{m,1}^{(r+1)}$.

The homomorphism (5.3.3) factors as a composite of homomorphisms

$$(\varphi^#, \varphi) : \text{Ext}_P^{pr+1}(I_1^{(r+1)}, I_0^{(r+1)}) \xrightarrow{(\varphi^#)\ast} \text{Ext}_P^{pr+1}(\Gamma_1^{(r)}, I_0^{(r+1)}) \xrightarrow{\varphi\ast} \text{Ext}_P^{pr+1}(\Gamma_1^{(r)}, S_0^{(r)}),$$

induced first by $\varphi^#$ and then by $\varphi$. Corollary 4.6.2 implies via conjugation by $P$ and via duality that $(\varphi^\ast)$ is an isomorphism in cohomological degree $p^{r+1}$. Then to complete the induction argument, it suffices to show that the image of $\varphi_\ast$ is the subspace spanned by $c_{r+1}^p$. First, recall the map (3.6.3) that relates the first and second hypercohomology spectral sequences; taking $A = \Gamma_1^{(r)}$ and $C = \Omega_p^{(r)}$, this map identifies with $\varphi_\ast$. We will prove that $c_{r+1}^p \in \text{im}(\varphi_\ast)$ by analyzing (3.6.3).

**Proposition 5.3.1.** In (3.6.3), the image of $\varphi_\ast$ is the subspace spanned by $c_{r+1}^p$.

**Proof.** Set $V = \text{Ext}_P^\bullet(I_1^{(r)}, I_0^{(r)})$. Taking $A = \Gamma_1^{(r)} := A \circ I_1^{(r)}$, and taking $B = S_0^{(r)} := S \circ I_0^{(r)}$ or $B = \Lambda_0^{(r)} := A \circ I_0^{(r)}$, respectively, the cup product (3.4.4) induces linear maps

$$V \otimes d \to \text{Ext}_P^\bullet(I_1^{(d)}, S_0^{(d)}), \text{ and}$$

$$V \otimes d \to \text{Ext}_P^\bullet(I_1^{(d)}, \Lambda_0^{(d)}),$$

By Lemma 3.4.1 these maps factor, respectively, through linear maps

$$\Theta_S^d : S^{d}(V) \to \text{Ext}_P^\bullet(I_1^{(d)}, S_0^{(d)}), \text{ and}$$

$$\Theta_A^d : \Lambda^d(V) \to \text{Ext}_P^\bullet(I_1^{(d)}, \Lambda_0^{(d)}).$$

As an inductive tool for analyzing the terms appearing in (3.6.3), we will first prove by induction on $d$ that $\Theta_S^d$ and $\Theta_A^d$ are isomorphisms for $1 \leq d < p$. The case $d = 1$ is a tautology, so suppose that $1 < d < p$. Then the composite homomorphism

$$S_0^{d(r)} \to (S_0^{1(r)}) \otimes d = (I_0^{(r)}) \otimes d \to S_0^{d(r)}$$

induced by the coproduct and product morphisms for $S_0^{(r)}$ is equal to multiplication by the nonzero scalar $d!$. This implies that the map in cohomology

$$\text{Ext}_P^\bullet(I_1^{(d)}, (I_0^{(r)}) \otimes d) \to \text{Ext}_P^\bullet(I_1^{(d)}, S_0^{(d)}(r))$$

induced by the product in $S_0^{(r)}$ is a surjection, and hence by (3.4.9) that $\Theta^d_S$ is a surjection as well. Replacing $S_0^{(r)}$ with $A_0^{(r)}$, we get by the same reasoning that $\Theta^d_\Lambda$ is a surjection. Then to prove that $\Theta^d_S$ and $\Theta^d_\Lambda$ are isomorphisms, it suffices to show that they are injections.

Given a cochain complex $C$ in $P_{ev}$, write $E(d, C)$ for the corresponding first hypercohomology spectral sequence obtained by taking $A = \Gamma_1^{(r)}$ in (3.6.1). As in [9] Proposition 4.1, if $C'$ and $C''$ are cochain complexes in $P_{ev}$, then the cup product isomorphism (3.4.9) induces an isomorphism of spectral sequences $E(d', C') \otimes E(d'', C'') \cong E(d' + d'', C' \otimes C'')$. In particular, using (2.7.7) to consider the $r$-th Frobenius twist of the ordinary de Rham complex $\Omega$ as a complex of strict polynomial superfunctions, the cup product induces an isomorphism $E(1, \Omega_1^{(r)}) \otimes \Omega_1^{(r)} \cong E(d, (\Omega_1^{(r)})^{\otimes d})$. We then get a homomorphism of spectral sequences $E(1, \Omega_1^{(r)}) \otimes \Omega_1^{(r)} \to E(d, \Omega_d^{(r)})$ by composing with the map of chain complexes $(\Omega_1^{(r)})^{\otimes d} \to \Omega_d^{(r)}$ induced by the multiplication morphism for $\Omega$.

Set $E = E(1, \Omega_1^{(r)})$, and set $E = E(d, \Omega_d^{(r)})$. The only nonzero columns in $E_1$ are $E_1^{0, \bullet} \cong V$ and $E_1^{1, \bullet} \cong V$. With these identifications, the differential $d_1 : E_1^{0, \bullet} \to E_1^{1, \bullet}$ is the identity. Next, using (3.4.9) we can write

$$E_1^{s, \bullet} \cong \text{Ext}^s_P(\Gamma_1^{(r)}, S_0^{d-s}(r) \otimes \Lambda_s^{(r)}) \cong \text{Ext}^s_P(\Gamma_1^{r-d}(s), S_0^{d-s}(r)) \otimes \text{Ext}^s_P(\Gamma_1^{(r)}, \Lambda_s^{(r)}).$$

Thus, if $0 < s < d$, then $E_1^{s, \bullet} \cong S^{d-s}(V) \otimes \Lambda^s(V)$ by induction on $d$. It now follows that the homomorphism of spectral sequences $(E_1)^{\otimes d} \to E$ factors on the $E_1$-page as the composition of two maps, $\rho : (E_1)^{\otimes d} \to \Omega_d(V)$ and $\sigma : \Omega_d(V) \to E_1$, such that:

- $\rho$ is induced by the identifications $E_1^{0, \bullet} = S^1(V)$ and $E_1^{1, \bullet} = \Lambda(V)$, and by multiplication in $\Omega(V)$; and
- $\sigma$ restricts for each $s$ to a map $\sigma_s : \Omega_d^{(r)} \to E_1^{s, \bullet}$ such that $\sigma_0 = \Theta^d_S$, $\sigma_d = \Theta^d_\Lambda$, and for $0 < s < d$, $\sigma_s$ is the isomorphism $S^{d-s}(V) \otimes \Lambda^s(V) \cong E_1^{s, \bullet}$.

Using the above factorization, it follows that $\sigma$ fits into a commutative diagram

$$\begin{array}{cccccccc}
S^d(V) & \longrightarrow & S^{d-1}(V) \otimes \Lambda(V) & \longrightarrow & \cdots & \longrightarrow & S^1(V) \otimes \Lambda^{d-1}(V) & \longrightarrow & \Lambda^d(V) \\
\Theta^d_S & \downarrow & \Theta^d_\Lambda \\
E_1^{0, \bullet} & \xrightarrow{d_1} & E_1^{1, \bullet} & \xrightarrow{d_1} & \cdots & \xrightarrow{d_1} & E_1^{d-1, \bullet} & \xrightarrow{d_1} & E_1^{d, \bullet}
\end{array}$$

in which the top row is the de Rham complex $\Omega_d(V)$. In other words, $\sigma$ defines a map of cochain complexes. Since $\Omega_d(V)$ is an exact complex unless $d \equiv 0 \mod p$, this implies, by considering the left-most square in the diagram, that $\Theta^d_S$ is an injection, hence an isomorphism. The proof that $\Theta^d_\Lambda$ is an isomorphism is completely parallel to the argument given for $\Theta^d_S$: replace the de Rham complex $\Omega$ with the Koszul complex $K^z$, and interchange the roles of $S$ and $\Lambda$.

Now suppose $d = p$. Consider momentarily the second hypercohomology spectral sequence

$$H^2 E^{s,t}_2 = \text{Ext}^s_P(\Gamma_1^{(r)}, H^t(\Omega_p^{(r)})) \Rightarrow \text{Ext}^{s+t}(P_1^{(r)}, \Omega_p^{(r)}).$$

Applying the ordinary Cartier isomorphism, Theorem 4.5.1, conjugation by $H$, and duality,

$$H^2 E^{s,t}_2 \cong \text{Ext}^s_P(\Gamma_1^{(r)}, I_0^{(r+1)}) \cong \begin{cases} k & \text{if } s \equiv p^{-1} \mod 2p, \ s \geq p^{r+1} \text{, and } t \in \{0, 1\}, \\ 0 & \text{otherwise}. \end{cases}$$

This implies that all differentials in the spectral sequence are trivial, and that the composite

$$H^2 E^{p^{r+1},0}_2 \Rightarrow H^2 E^{p^{r+1},0}_\infty \Rightarrow \text{Ext}^{p^{r+1}}_{P_1}((\Gamma_1^{(r)}, \Omega_p^{(r)}))$$

is an isomorphism of one-dimensional spaces. Next consider the first hypercohomology spectral sequences $E := E(p, \Omega_p^{(r)})$ and $E := E(p, K^z_p^{(r)})$. Then $c_{\nu p}^{(r)} \in E_1^{0,p^{r+1}}$. By playing the spectral
sequences $E$ and $\mathbf{E}$ off each other, we will show that $c_i^p$ spans the space of permanent cycles in $E_1^{0,p^{r+1}}$, and hence that the composite
\[
\text{Ext}_P^{p^{r+1}}(\Gamma_p^{(r)}, \Omega_p^{(r)}) \to E_\infty^{0,p^{r+1}} \hookrightarrow E_1^{0,p^{r+1}}
\]
is an isomorphism of one-dimensional spaces, finishing the proof of the theorem.

As in the case $1 \leq d < p$, we get by induction on $d$ the existence of chain maps $\sigma : \Omega_p(V) \to E_1$ and $\tau : K_zp(V) \to E_1$ such that

- $\sigma$ satisfies the same properties as in the case $1 \leq d < p$,
- the restriction $\tau_0 : N_p(V) = K_z^{(p)}(V) \to E_1^{0,p}$ is equal to $\Theta_p$,
- the restriction $\tau_p : S_p(V) = K_z^{(p)}(V) \to E_1^{p,p}$ is equal to $\Theta_p^p$, and
- for $0 < s < p$, the restriction $\tau_s : K_z^{s}(V) \to E_1^{s,s}$ is an isomorphism.

The exactness of $K_zp(V)$ then implies that $E_2^{2s,0} = 0$ for $1 \leq s \leq p - 2$, and that $\Theta_p$ is an injection.

We also get $E_2^{p-1,0} \cong \ker(\Theta_p^p)$ and $E_2^{p,0} \cong \ker(\Theta_p^p)$. Since $\Pi(V)(\Omega_p(V)) = 0$ for $i > 1$ by the Cartier isomorphism, it follows that $E_2^{p,0} \cong \ker(\Theta_p^p)$ and $E_2^{p-1,0} \cong \ker(\Theta_p^p)$, and hence $E_2^{s,0} = 0$ for $2 \leq s \leq p - 1$. Then $E_2^{0,0} \cong \ker(\Theta_p^p) \cong E_2^{p,0}$ by the commutativity of the diagram

\[
\Lambda^d(V) \xrightarrow{n^d(V)} K_z^p(V) \\
\Theta_p \downarrow \quad \tau_1 \downarrow \\
E_1^{0,0} \xrightarrow{d_1} E_1^{1,0}
\]

Now the only nonzero columns in the two spectral sequences are $E_2^{0,0} \cong E_2^{p,0}, E_2^{p-1,0}, E_2^{p,0},$ and $E_2^{1,0}$. Since $V$ is concentrated in cohomological degrees $m \equiv p^r \mod 2$, it follows that
\[
E_2^{p,m} \cong \text{Ext}_P^m(\Gamma_p^{(r)}, S^{p^{r}}) \cong E_2^{0,m} \quad \text{if } m \not\equiv p^{r+1} \mod 2p,
\]
and hence that
\[
E_2^{0,i} \cong E_2^{p,i} \cong E_2^{p,i-p+1} \cong E_2^{p,i-p+1} \quad \text{for all } i \leq p^{r+1} - 2,
\]
with the middle isomorphism induced by the differential $d_p : E_2^{0,i} \to E_2^{p,i-p+1}$. Similarly, using (5.3.5), the fact that $E_2^{s,0} = 0$ for $1 \leq s \leq p - 2$, and the fact that $E \Rightarrow 0$ by the exactness of the complex $K_z^{p}(\cdot)$, it follows that there exist isomorphisms
\[
E_2^{0,i} \cong E_2^{p,i} \cong E_2^{p,i-p+1} \cong E_2^{p,i-p+1} \quad \text{for all } i \leq p^{r+1} + p - 3.
\]
In particular, (5.3.6), (5.3.7), and (5.3.8) imply that
\[
E_2^{0,i} \cong E_2^{p,i-p+1} \cong E_2^{p,i-p+1} \cong E_2^{p,i-2p+2} \quad \text{for all } i \leq p^{r+1} + p - 3.
\]
Trivially, $E_2^{0,i} = 0 = E_2^{0,i}$ if $i < 0$. Then an induction argument using (5.3.9) and the isomorphism $E_2^{0,0} \cong E_2^{p,0}$ shows that $E_2^{p,i} = 0$ for all $i \leq p^{r+1} + p - 3$. In particular, $E_2^{p,i} = 0$ for $i = p^{r+1} - p$ and $i = p^{r+1} - p + 1$. This implies that $E_2^{0,p^{r+1}}$ is the only nonzero term of total degree $p^{r+1}$ in the $E_2$-page of its spectral sequence, and that $E_2^{0,p^{r+1}}$ consists of permanent cycles. Since $[E^{0,p^{r+1}}_\infty]^{p^{r+1}} \cong k$ by the calculations in the previous paragraph, we conclude then that $E_2^{0,p^{r+1}} \cong k$. Finally, write $(c_i^p)^p$ for the $p$-th power of $c_i$ in $S^p(V)$. Then $(c_i^p)^p$ is in the kernel of the de Rham differential $S^p(V) \to S^{p-1}(V) \otimes \Lambda^1(V)$. Since $\sigma : \Omega_p(V) \to E_1$ is a chain map, and since $\sigma_0 = \Theta_S^p$, this implies
that $c_r^{(p)}$ is a nonzero element of the one-dimensional space $E_2^{0,p+1}$. Thus, $c_r^{(p)}$ spans the space of permanent cycles in $E_1^{0,p+1}$. This completes the proof. □

We have now completed the proof outlined at the start of this section. Thus:

**Theorem 5.3.2.** For each $r \geq 1$, the classes $c_r$ and $c_r^{(p)}$ restrict nontrivially to $GL(m|n)_1$.

**Remark 5.3.3.** A version of Proposition 5.3.1 also holds for the cohomology classes $e_{r+1}$ and $e_r^{(p)}$, i.e., the image of $e_{r+1}$ in $\text{Ext}^{2p'}_{\mathcal{G}}(\Gamma_0^{(r)}, \mathcal{S}_0^{(p)})$ is a nonzero scalar multiple of the cup product $e_r^{(p)}$. This can be proved through an argument similar to that given for Proposition 5.3.1 or it can be deduced from [6, Corollary 5.9] using the fact that $e_r|_{\mathcal{V}_0} = e_r$ for each $r \geq 1$. More generally, it is natural to expect that analogues of the Ext-group calculations in [6, §§4–5] should hold for the exponential superfunctors defined in (2.7.6).

### 5.4. Cohomological finite generation in positive characteristic

Our main theorem is:

**Theorem 5.4.1.** Let $k$ be a field of characteristic $p \geq 3$. Let $G$ be a finite $k$-supergroup scheme, and let $M$ be a finite-dimensional $G$-supermodule. Then $H^*(G, k)$ is a finitely-generated $k$-algebra, and $H^*(G, M)$ is finitely-generated under the cup product action of $H^*(G, k)$.

**Proof.** By [7, Remark 5.4.3], we may assume that the field $k$ is perfect, and by [7, Theorem 5.3.3], we may assume that $G$ is infinitesimal. Then by [7, Theorem 5.4.2], it suffices to show for $m, n, r \geq 1$ that, up to rescaling, $e_r^{m,n} := (e_r + e_{r}^{(p)})|_{GL(m|n)}$ and $c_r^{m,n} := (c_r + e_{r}^{(p)})|_{GL(m|n)}$ satisfy (5.1.7) and (5.1.8)\(^6\). Since (5.1.7) and (5.1.8) are conditions in terms of superalgebra homomorphisms, it suffices by multiplicativity and linearity to consider $e_r$, $e_{r}^{(p)}$, $c_r$, and $c_{r}^{(p)}$ separately.

Set $G = GL(m|n)$, and let $g_{+1}$ and $g_{-1}$ be the subalgebras of $g = gl(m|n)$ defined in (5.1.9). As in [5, §2], let $T$ be the maximal torus in $G$ consisting of diagonal matrices, and let $\epsilon_i : T \to G_m$ be the homomorphism that picks out the $i$-th entry of a diagonal matrix. Then the character group $X(T)$ of $T$ is the free abelian group generated by $\epsilon_1, \ldots, \epsilon_{m+n}$, and each $G$-supermodule $M$ decomposes into a direct sum of $T$-weight spaces. In particular, the matrix unit $e_{ij} \in gl(m|n)$ is of weight $\epsilon_i - \epsilon_j$. Now $(g_{+1}^{(p)})^{(r)}$ is the irreducible $G$-supermodule of highest weight $p^r(\epsilon_{m+1} - \epsilon_m)$, and $(g_{-1}^{(p)})^{(r)}$ is the irreducible $G$-supermodule of highest weight $p^r(\epsilon_1 - \epsilon_{m+n})$. Since $c_r$ and $c_{r}^{(p)}$ each restrict nontrivially to $G_1$, it follows that the induced $G$-supermodule homomorphisms $c_r|_{G_1} : (g_{+1}^{(p)})^{(r)} \to H^p(G_1, k)$ and $c_{r}^{(p)}|_{G_1} : (g_{-1}^{(p)})^{(r)} \to H^p(G_1, k)$ are injections.

The root system of $G$ with respect to $T$ is the set $\Phi = \{\epsilon_i - \epsilon_j : 1 \leq i, j \leq m + n, i \neq j\}$. Write $\Phi_{\text{odd}}$ for the subset of odd roots, i.e., for the set of weights of $T$ in $g_{+1}$. We claim that the only term of total degree $p^r$ in the $E_0$-page of (5.1.6) that contains vectors of weight $p^r\alpha$ for $\alpha \in \Phi_{\text{odd}}$ is $E_0^{0,p^r}$. To see this, observe that any weight vector in the $E_0$-page of (5.1.6) can be written as a product of homogeneous monomials from $\Lambda(g_{+1}^{(p)})$, $S(g_{+1}^{(p)})$, and $S(g_{-1}^{(p)}(2))^{(1)}$. If $v$ is a vector of weight $\sum_{i=1}^{m+n} a_i \epsilon_i$ in $\Lambda(g_{+1}^{(p)})$ or $S(g_{-1}^{(p)}(2))^{(1)}$, then $\sum_{i=1}^{m} a_i = 0$ and $\sum_{i=1}^{m+n} a_{m+i} = 0$. This implies that any monomial of weight $p^r\alpha$ in the $E_0$-page of (5.1.6) with $\alpha \in \Phi_{\text{odd}}$ must be divisible by a monomial in $S^{p^r}(g_{+1}^{(p)})$. More specifically, if $v$ is a monomial of total degree $p^r$ and weight $p^r\alpha$ for some $\alpha \in \Phi_{\text{odd}}$, then $v$ must be in the image of the $G$-supermodule homomorphism

\[
(g_{+1}^{(p)})^{(r)} \hookrightarrow S(g_{+1}^{(p)}) \hookrightarrow \Lambda^{p^r}(g_{+1}^{(p)}) \cong E_0^{0,p^r}
\]

in which the first arrow raises elements to the $p^r$-power. The image of (5.4.1) consists of permanent cycles by [7, Proposition 3.5.3], and hence is in the image of the vertical edge map of (5.1.6). This implies that the composition of the injective map $(c_r + e_{r}^{(p)})|_{G_1} : (g_{+1}^{(p)})^{(r)} \to H^p(G_1, k)$ with the

\(^6\)The assumption $m \neq n \neq 0 \mod p$ in [7, Conjecture 5.4.1], and hence also in [7, Theorem 5.4.2], turns out to be unnecessary, though it should be assumed that $m, n \geq 1$. 

vertical edge map of (5.1.6) has the same image as (5.4.1). Then extending multiplicatively, we conclude that $e^{m,n}_r := (e_r + e^{\Pi}_r)|_G$ satisfies (5.1.8).

Next we show by induction on $r$ that, up to rescaling, (5.1.7) is satisfied by $e^{m,n}_r := (e_r + e^{\Pi}_r)|_G$. The inductive step is handled by an argument like that in the second paragraph of Section 5.3 using Remark 5.3.3 instead of Proposition 5.3.1 so it suffices to consider the case $r = 1$. First recall that $H^\bullet(G_1,k)$ identifies with the cohomology ring $H^\bullet(V(g),k)$ for the restricted enveloping superalgebra $V(g)$ of $g$ (cf. [7] Remark 4.4.3), and that $H^\bullet(V(g),k)$ can be computed using the projective resolution $X(g)$ of $k$ defined in [7] §3.3 (see also [17] §6). As a graded superspace,

$$X(g) = V(g) \otimes A(g) \otimes \Gamma(g_\mathfrak{g}(2))(1),$$

i.e., $X_1(g) = \bigoplus_{s+2t=1} V(g) \otimes A^s(g) \otimes \Gamma^t(g_\mathfrak{g}(2))(1)$. The differential on $X(g)$ depends on the choice of a fixed homogeneous basis for $g$; we work with the basis $\{e_{ij} : 1 \leq i,j \leq m+n\}$ of matrix units. Then $d_1 : X_1(g) \to X_0(g)$ satisfies $d_1(1 \otimes e_{ij} \otimes 1) = e_{ij} \otimes 1 \otimes 1$, and $d_2 : X_2(g) \to X_1(g)$ satisfies

$$d_2(1 \otimes (e_{ij} \cdot e_{st}) \otimes 1) = e_{ij} \otimes e_{st} \otimes 1 - (-1)^{\bar{e}_{ij} \cdot \bar{e}_{st}} e_{st} \otimes e_{ij} \otimes 1 - 1 \otimes |e_{ij}, e_{st}| \otimes 1, \quad \text{and}$$

$$d_2(1 \otimes 1 \otimes e^{(1)}_{ij}) = (e_{ij})^{p-1} \otimes e_{ij} \otimes 1 - 1 \otimes |e^{(1)}_{ij}| \otimes 1 \quad \text{for} \, \bar{e}_{ij} = 0.$$ Here $z \mapsto z^{[p]}$ is the $p$-map making $g_\mathfrak{g}$ into a restricted Lie algebra, which sends $z \in g_\mathfrak{g}$ to its ordinary $p$-th matrix power. Given $e_{ij} \in g$, let $e^*_{ij}$ be the corresponding dual basis element in $g^*$, and given $e_{ij} \in g_\mathfrak{g}$, let $g_{ij} \in \text{Hom}_{V(g)}(X_2(g),k)$ be the homomorphism that is dual to the basis element $1 \otimes 1 \otimes e^{(1)}_{ij}$ of $X_2(g)$. Then the $g_{ij}$ are cocycles in $\text{Hom}_{V(g)}(X_2(g),k)$ by [7] Lemma 3.5.2, and the proof of [7] Proposition 3.5.3 shows (modulo a reindexing of the spectral sequence) that the horizontal edge map of (5.1.6) sends $e^{(1)}_{ij} \in (g^\mathfrak{g})_1(1) \cong E_0^{2,0}$ to the cohomology class of $g_{ij}$. Then to finish the proof, it suffices to show that $(e_1 + e^{\Pi}_1)|_{G_1} : (g^\mathfrak{g})_1(1) \to H^0(G_1,k)$ also sends $e^{(1)}_{ij}$ to the cohomology class of $g_{ij}$.

Recall that $e_1$ is the extension class in $\mathcal{P}$ of the exact sequence (4.4.10). Then $e_1(k^{m|n})$ is made into an exact sequence of restricted $g$-supermodules by having $g$ act trivially on $k^{m|0}(1)$, and by giving $S^p(k^{m|n})$ and $\Gamma^p(k^{m|n})$ the $g$-supermodule structures induced by the natural action of $g$ on $(k^{m|n})^{\otimes p}$. As in the proof of Lemma 5.2.1, the cohomology class $e_1|_{G_1} \in \text{Ext}^2_{G_1}(k^{m|0}(1), k^{m|0}(1))$ can be described by lifting the identity function $k^{m|0}(1) \to k^{m|0}(1)$ to a chain homomorphism

$$\cdots \longrightarrow X_2(g) \otimes k^{m|0}(1) \longrightarrow X_1(g) \otimes k^{m|0}(1) \longrightarrow X_0(g) \otimes k^{m|0}(1) \longrightarrow k^{m|0}(1)$$

and then taking the cohomology class of the map $\varphi_2$.

As in the proof of Lemma 5.2.1, let $x_1, \ldots, x_m, y_1, \ldots, y_n$ be the standard homogeneous basis for $k^{m|n}$. Then the reader can check that the following formulas uniquely define a chain homomorphism $\varphi : X(g) \otimes k^{m|0}(1) \to e_1(k^{m|n})$ lifting the identity map on $k^{m|0}(1)$:

- $\varphi_0((1 \otimes 1 \otimes 1) \otimes x^{(1)}_s) = \gamma_p(x_s)$,
- $\varphi_1((1 \otimes e_{ij} \otimes 1) \otimes x^{(1)}_s) = \delta_{j,s} \cdot \frac{1}{(p-1)!} \cdot x^{p-1}_1 \cdot x^{p-1}_s$ for $e_{ij} \in g$,
- $\varphi_2((1 \otimes 1 \otimes e^{(1)}_{ij}) \otimes x^{(1)}_s) = \delta_{j,s} \cdot x^{(1)}_s$ for $e_{ij} \in g^\mathfrak{g}$, and
- $\varphi_2((1 \otimes z \otimes 1) \otimes x^{(1)}_s) = 0$ for all $z \in A^2(g)$.

This implies that if $e^{*}_{ij} \in g^*_m$, then $e^{*}_{ij}|_{G_1} : (g^\mathfrak{g})^{*}_1(1) \to H^2(G_1,k)$ sends $e^{*}_{ij}$ to the cohomology class of $g_{ij}$. Similarly, if $e^{*}_{ij} \in g^*_n$, then $e^{*}_{ij}|_{G_1} : (g^\mathfrak{g})^{*}_n(1) \to H^2(G_1,k)$ sends $e^{*}_{ij}$ to the class of $g_{ij}$. Thus,
\[(e_1 + e_n)|_{G_1} : \Omega^1_{\mathbb{G}} \rightarrow H^2(G_1, k)\) is equal to the corresponding edge map of (5.1.6), and extending multiplicatively, we conclude for \(r = 1\) that (5.1.7) is satisfied by \(c_1^{m,n} := (e_1 + e_n)|_{G_1}\). \(\square\)

Applying the equivalence between finite supergroup schemes over \(k\) and finite-dimensional cocommutative Hopf superalgebras over \(k\), we immediately get:

**Corollary 5.4.2.** Let \(k\) be a field of characteristic \(p \geq 3\), let \(A\) be a finite-dimensional cocommutative Hopf superalgebra over \(k\), and let \(M\) be a finite-dimensional \(A\)-supermodule. Then the cohomology ring \(H^*(A, k)\) is a finitely-generated \(k\)-algebra, and \(H^*(A, M)\) is finitely-generated under the cup product action of \(H^*(A, k)\).

### 5.5. Implications for the cohomology of \(GL(m|n)\).

Recall from [12, Corollary 3.13] that if \(T\) and \(T'\) are homogeneous strict polynomial functors of degree \(d\), and if \(n \geq d\), then evaluation on the vector space \(k^n\) induces an isomorphism \(\text{Ext}^*_{\mathcal{P}_d}(T, T') \cong \text{Ext}^*_{\mathcal{P}_{d,n}}(T(k^n), T'(k^n))\). Taking \(d = 0\), the category \(\mathcal{P}_0\) identifies with the semisimple category of finite-dimensional \(k\)-vector spaces, and \(GL_n\) acts trivially on \(T(k^n)\) for each \(T \in \mathcal{P}_0\). Then a special case of the previous isomorphism is the well-known fact that \(\text{Hom}_{GL_n}(k, k) \cong k\), but \(\text{Ext}^i_{GL_n}(k, k) = 0\) for all \(i > 0\). The next proposition implies that there is no analogue of [12, Corollary 3.13] for \(\mathcal{P}_d\) and \(GL(m|n)\).

**Proposition 5.5.1.** Let \(m\) and \(n\) be positive integers. Then \(\text{Ext}^2_{GL(m|n)}(k, k) \neq 0\).

**Proof.** Set \(G = GL(m|n)\), and set \(\mathfrak{g} = \text{Lie}(G) = \mathfrak{gl}(m|n)\). Write \(\mathfrak{g}_\mathbb{C} = \mathfrak{g}_m \oplus \mathfrak{g}_n\) as in (5.1.9), and let \(tr_m : \mathfrak{g}_m \rightarrow k\) and \(tr_n : \mathfrak{g}_n \rightarrow k\) be the usual trace functions on \(\mathfrak{g}_m\) and \(\mathfrak{g}_n\), respectively. We consider \(tr_m\) and \(tr_n\) as elements of \(\mathfrak{g}^*\) in the obvious way. Then the supertrace function \(\text{str} : (\mathfrak{g}(m|n) \rightarrow k)\) is defined by \(\text{str} = tr_m - tr_n\). By [7, Corollary 5.2.3], the \(E_2\)-page of the spectral sequence (5.1.6) has the form \(E_2^{ij} = H^j(\mathfrak{g}, k) \otimes S^i(\mathfrak{g}_0^*(2))^{(1)}\), where \(H^*(\mathfrak{g}, k)\) denotes the ordinary Lie superalgebra cohomology of \(\mathfrak{g}\). Immediate calculation using the Koszul resolution for \(\mathfrak{g}\) [7, §3.1] shows that \(H^1(\mathfrak{g}, k) \cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*\) is spanned by the supertrace function. Now one can check that the differential \(d_2 : E_2^{0,1} \rightarrow E_2^{2,0}\) maps \(E_2^{0,1} \cong H^1(\mathfrak{g}, k)\) onto the subspace of \(E_2^{2,0} \cong (\mathfrak{g}_0^*)^{(1)}\) spanned by \(\text{str}^{(1)}\). This implies that the horizontal edge map of (5.1.6) induces an injection \((\mathfrak{g}_0^*)^{(1)} \hookrightarrow H^2(G_1, k)\).

In particular, the image of \(\text{tr}_m^{(1)} \in (\mathfrak{g}_0^*)^{(1)}\) under the edge map \((\mathfrak{g}_0^*)^{(1)} \rightarrow H^2(G_1, k)\) is nonzero.

Now as in the proof of [12, Lemma 1.4], consider the commutative diagram

\[
\begin{array}{cccc}
\text{Ext}^2_{GL}(k, \mathfrak{g}_m^{(1)}) & \rightarrow & \text{Ext}^2_{G_1}(k, \mathfrak{g}_m^{(1)}) & \rightarrow & \text{Hom}_k((\mathfrak{g}_m^{(1)}), H^2(G_1, k)) \\
\downarrow \text{tr}_m^{(1)}, & & \downarrow \circ \text{tr}_m^{(1)} & & \\
\text{Ext}^2_{G}(k, k) & \rightarrow & \text{Ext}^2_{G_1}(k, k). & &
\end{array}
\]

The proof of Theorem 5.4.1 and the observation at the end of the previous paragraph imply that the image of \(e_1|_G\) across the top row and right-hand column of this diagram is nonzero. Then the image of \(e_1|_G\) down the left-hand column and across the bottom row of the diagram must also be nonzero. In particular, \((\text{tr}_m^{(1)})_* (e_1|_G)\) must be a nonzero class in \(\text{Ext}^2_G(k, k)\). \(\square\)

**Problem 5.5.2.** Compute the rational cohomology ring \(H^*(GL(m|n), k) = \text{Ext}^*(GL(m|n), k, k)\).

**Remark 5.5.3.** It follows from a result of Kujawa [15, Lemma 3.6] that \(\text{Ext}^1_{GL(m|n)}(L, L) = 0\) for each irreducible rational \(GL(m|n)\)-supermodule \(L\). On the other hand, Brundan and Kleshchev have shown for the supergroup \(Q(n)\) that \(H^1(Q(n), k) \cong \Pi(k)\) [4, Corollary 7.8].

### 5.6. Cohomological finite generation in characteristic zero.

In this section we prove cohomological finite-generation for finite-dimensional cocommutative Hopf superalgebras over fields of characteristic zero, using a theorem of Kostant describing the structure of these algebras. The next result is Theorem 3.3 of [14] (it is evident from the context that the theorem as published contains
a typo: the word ‘commutative’ should be the word ‘cocommutative’). Given a group $G$, we write $kG$ for the group algebra of $G$ over $k$, i.e., the ring of $k$-linear combinations of elements of $G$, in which multiplication is induced by multiplication in $G$.

**Theorem 5.6.1** (Kostant). Let $k$ be an algebraically closed field of characteristic zero, and let $A$ be a cocommutative Hopf superalgebra over $k$. Let $G$ be the group of group-like elements in $A$, let $\mathfrak{g}$ be the Lie superalgebra of primitive elements in $A$, and let $U(\mathfrak{g})$ be the universal enveloping superalgebra of $\mathfrak{g}$. Then there exists an isomorphism of Hopf superalgebras $A \cong kG\#U(\mathfrak{g})$, where the smash product is taken with respect to the homomorphism $\pi: G \to GL(\mathfrak{g})$ defined by $\pi(g)(x) = g x g^{-1}$.

Since $U(\mathfrak{g})$ is infinite-dimensional whenever $\mathfrak{g} \neq 0$, and since any purely odd Lie superalgebra is automatically abelian, we immediately get:

**Corollary 5.6.2.** Let $k$ be an algebraically closed field of characteristic zero, and let $A$ be a finite-dimensional cocommutative Hopf superalgebra over $k$, then there exists a finite group $G$, a finite-dimensional odd superspace $V$, and a representation $\pi: G \to GL(V)$ such that $A$ is isomorphic as a Hopf superalgebra to the smash product $kG\#\Lambda(V)$ formed with respect to $\pi$.

Now we prove the characteristic zero analogue of Corollary 5.4.2.

**Theorem 5.6.3.** Let $k$ be a field of characteristic zero, let $A$ be a finite-dimensional cocommutative Hopf superalgebra over $k$, and let $M$ be a finite-dimensional $A$-supermodule. Then $H^\bullet(A,k)$ is a finitely-generated $k$-algebra, and $H^\bullet(A,M)$ is a finitely-generated $H^\bullet(A,k)$-module.

**Proof.** Since $H^\bullet(A,k) \otimes \overline{k} \cong H^\bullet(A \otimes \overline{k}, \overline{k})$ as $\overline{k}$-algebras, and since $H^\bullet(A,M) \otimes \overline{k} \cong H^\bullet(A \otimes \overline{k}, M \otimes \overline{k})$ as modules under the previous isomorphism, we may assume that $k$ is algebraically closed. Then by Corollary 5.6.2 there exists a finite group $G$, a finite-dimensional odd superspace $V$, and a representation $\pi: G \to GL(V)$ such that $A$ is isomorphic as a Hopf superalgebra to the smash product $kG\#\Lambda(V)$ formed with respect to $\pi$. Since the exterior algebra $\Lambda(V)$ is a normal subalgebra of $A$, and since the Hopf superalgebra quotient $A/\Lambda(V)$ is isomorphic to the group algebra $kG$, we get for each $A$-supermodule $M$ a Lyndon–Hochschild–Serre spectral sequence

\[ E_2^{ij}(M) = H^i(kG, H^j(\Lambda(V), M)) \Rightarrow H^{i+j}(A,M). \]

The group algebra $kG$ is semisimple by Maschke’s Theorem, so $E_2^{ij}(M) = 0$ for $i > 0$. Then the spectral sequence collapses at the $E_2$-page, yielding $H^\bullet(A,M) \cong H^\bullet(\Lambda(V), M)^G$. Next, the ring $H^\bullet(\Lambda(V), k)$ is isomorphic as a graded $k$-algebra to the symmetric algebra $S(V^*)$. This identification is an isomorphism of $kG$-modules, where the $kG$-module structure on $S(V^*)$ is induced by the contragredient representation of $G$ on $V^*$. Then by Noether’s finiteness theorem, the cohomology ring $H^\bullet(A,k) \cong S(V^*)^G$ is a finitely-generated $k$-algebra, and $S(V^*)$ is finitely-generated as a module over $S(V^*)^G$. Then to prove that $H^\bullet(A,M)$ is a finitely-generated $H^\bullet(A,k)$-module, it suffices to show that $H^\bullet(\Lambda(V), M)$ is a finitely-generated $H^\bullet(\Lambda(V), k)$-supermodule.

Consider $V$ as an abelian Lie superalgebra. Then $\Lambda(V)$ is the enveloping superalgebra of $V$, and $H^\bullet(\Lambda(V), M)$ is the Lie superalgebra cohomology group $H^\bullet(V,M)$ studied in §§3.1–3.2. Let $C^\bullet(V,M)$ be the cochain complex defined in §§3.2 whose cohomology is equal to $H^\bullet(V,M)$. Then the cup product makes $C^\bullet(V,M)$ into a finitely-generated differential graded (either left or right) $C^\bullet(V,k)$-supermodule; see the discussion after Remarks 3.2.2 and Footnote 2 in §§2.6. Since $V$ is abelian, the differential on $C^\bullet(V,k)$ is trivial, and we get $H^\bullet(V,k) \cong C^\bullet(V,k) \cong S(V^*)$. Passing to cohomology, this implies that $H^\bullet(V,M)$ is finitely-generated under the (left or right) cup product action of $H^\bullet(V,k) \cong C^\bullet(V,k)$. \(\square\)

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Figure 1. End of Step 1 of the proof of Theorem 4.3.2.

Figure 2. End of Step 2 of the proof of Theorem 4.3.2.
Figure 3. End of Step 3 of the proof of Theorem 4.3.2.
Figure 4. End of Step 4 of the proof of Theorem 4.3.2.
Figure 5. Differentials in the spectral sequence [4.3.4].