Optical Ray Transfer Matrix Model of the Turbulent Cells Cascade

Alzbeta KOVALOVA 1, Lucie HUDCOVA 1, Rastislav ROKA 2

1 Dept. of Radio Electronics, Brno University of Technology,Technicka 3082/12, 616 00 Brno, Czech Republic
2 Institute of Telecommunications, Slovak University of Technology, Ilkovicova 3, 812 19 Bratislava, Slovakia

Alzbeta.Kovalova@vut.cz, hudcova@vut.cz, rastislav.roka@stuba.sk
Submitted November 18, 2021 / Accepted August 24, 2022 / Online first October 17, 2022

Abstract. The paper offers a new approach to modeling atmospheric turbulence consisting of turbulent cells whose size is larger than the optical beam width. Particular turbulent cells are approximated by an optical element matrix. The ray transfer matrix method is presented, through which the optical elements can be described in the matrix form. A deflection simulation was performed that indicated the behavior of the optical beam by passing through the optical element. Furthermore, the calculation of the deflection vector is described together with a cascade model of turbulent cells. The matrix calculation for the cascade of optical elements is also expressed.

Keywords
Atmospheric turbulence, matrix optics, cascade of turbulent cells, ABCD matrix, reciprocity

1. Introduction

Optical Wireless Communication (OWC) refers to transmission in unguided propagation media in the visible, infrared (IR) and ultraviolet (UV) bands [1]. When implementing the OWC link in an optical channel, the serious effects of atmospheric phenomena should be considered and eliminated. Significant dependence of the transmission properties of the Free Space Optical (FSO) system on the state of the atmosphere, causing deterioration of the FSO system’s availability [2], [3], is one of the system drawbacks. Despite the stochastic nature of the atmosphere, the FSO system has advantages such as unlicensed bandwidth, more flexibility in designing optical network architecture, easy installation, insensitivity to electromagnetic interference and data security [4].

Constituent particles of the atmosphere, mainly water molecules, carbon dioxide, and ozone, cause absorption, scattering, and atmospheric attenuation [5]. Furthermore, there is a beam divergence loss in FSO communication, which depends on the free space loss (wavelength dependent) [3]. Another atmospheric effect that has a significant impact on the performance of the FSO system is atmospheric turbulence. Atmospheric turbulence leads to irradiance fluctuations, beam spreading, beam wandering and loss of spatial coherence of a laser beam [2], [6]. There are several works focused on modeling and analysis of FSO systems. Some of them focus on describing the various limitations of FSO systems and the various ways to improve the performance of such an atmospheric channel [5]. Further works consists of analyzes of modulation techniques in the presence of strong or weak turbulence in the communication channel or modeling of optical wave diffraction [7–11].

Several mathematical methods [12] are used to quantify atmospheric turbulence and its effect on the optical system. Each of these methods works with turbulence through the parameters of the transmission medium, whether it is air temperature, air pressure, or time and space changes in refractive index [12–14].

This paper offers a new perspective on turbulence as a cascade of the turbulent cells. The turbulent cells of the cascade are approximated by optical elements defined by ray transfer matrices. Due to the model are observed the deflection parameters, which together form the deflection vector.

The turbulent matrix model can be linked to instantaneous quantification methods working with deflection parameters, especially in a dynamically changing communication channels [15]. By extending the model to the influence of other physical phenomena (e.g. absorption, diffraction, scattering, etc.), the model can be applied for prediction of the optical beam propagation in turbulent medium. In such a case, it would be possible to determine the matrix defining the turbulence and, based on its parameters, determine the deflection vector of the input beam.

2. Introduction to Ray Transfer Matrix

Turbulence consists of turbulent cells of various scale sizes [12]. The ray transfer matrix method, which is based on the principles of geometric optics, is used for our work. It deals with large-scale turbulent cells that cause the optical
beam deflection in the optical wireless links. This method offers a way to perceive a turbulent cell as an optical element. The following technique works with the beam description of optical radiation, which is part of geometric optics. The presence of paraxial rays is a requirement for using this technique [16], for which the following applies

$$\tan(\theta) \approx \theta$$

where $\theta$ represents the angle subtended by the optical beam with the auxiliary axis parallel to the optical system axis.

The propagation of paraxial rays through various optical elements is described by ray transfer matrices [17] also known as ABCD matrices. Optical elements can be optical prisms, thick or thin lenses, mirrors or interfaces where the refraction of an optical beam occurs [17].

The beam transmission technique shown in Fig. 1 operates with two reference planes, input and output planes. Both planes are perpendicular to the optical axis of the system. At each point in the optical system, an optical axis corresponding to the central beam is defined. The optical beam enters the inhomogeneous area through the input plane at a distance $x_1$ from the optical axis and propagates in the direction that makes an angle $\theta_1$ with an auxiliary optical axis. The inhomogeneous area is defined by the refractive index $n_2$, where the refractive index $n_1$ represents the surrounding medium. After propagation to the output plane, this beam is at a distance $x_2$ from the optical axis and propagates further at an angle $\theta_2$ [17].

The beam deflection on the output plane is evaluated by the equation

$$x_2 = A \cdot x_1 + B \cdot \theta_1$$

and the angle of deflection of the beam from the optical axis is evaluated from the following relation

$$\theta_2 = C \cdot x_1 + D \cdot \theta_1.$$  

These expressions can be written in the matrix form [17]

$$\begin{bmatrix} x_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1 \\ \theta_1 \end{bmatrix}$$

where the ABCD matrix represents the area between input and output planes where the optical element is situated. This matrix combines the deflection of the outgoing beam with the deflection of the incident beam.

2.1 Approximation of Turbulent Cells and Inter-Turbulent Spaces

A turbulent atmosphere is considered to be a non-stationary medium where the refractive index $n$ changes. Thick lens is a proper optical element for approximating a turbulent cell. Defined matrix of thick lens assumes a change in the refractive index as the optical beam passes through the optical structure [18].

The ABCD matrix defined for a thick lens (see Fig. 2) is given by multiplying the three matrices containing the descriptive parameters of the lens [16]

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} n_2/n_1 & 0 \\ 1/n_2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ l \end{bmatrix}$$

where $n_1$ is refractive index outside of lens, $n_2$ is refractive index of the lens itself, $R_1$ is radius of curvature of the first surface, $R_2$ is radius of curvature of the second surface and $l$ represents the center thickness of lens.

The ABCD matrix defined for medium of constant refractive index (see Fig. 3), which represents the inter-turbulent space in the turbulent cell cascade model, is given by [16]

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 \\ l \end{bmatrix}$$

where $l$ denotes the width of the area with constant refractive index. The matrix is derived using a trigonometric function and paraxial approximation. A more detailed description of the derivation and calculation can be found in Sec. 3.
2.2 ABCD Turbulence Simulation

In the following text, \( L \) will denote the ray transfer matrix (ABCD matrix) for medium of constant refractive index (inter-turbulent space) and \( T \) will denote the ray transfer matrix for the thick lens (turbulent cell). Figure 4 shows two variations of the optical beam transition. One of them represents a reference model consisting of a medium with a constant refractive index. The refractive index of the optical element has the same value as the refractive index of the surrounding medium. The second variation is a model of turbulent cell approximated by a thick lens. The simulation based on Fig. 4 points to five cases of beam propagation:

- The first case is the reference model \( L_0 \) with a medium with a constant refractive index
- The remaining four cases \( T_1 - T_4 \) are formed by a turbulent cell model with different sizes and refractive indices.

Geometric parameters (radii of curvature, central thickness of the lens) and refractive indices for specific simulation models are given in the Tab. 1. According to relationship (10), the refractive index \( n(\lambda) \) depends on the wavelength of the used laser (\( \lambda = 0.6328 \mu m \)), the atmospheric pressure (\( P = 102850 \text{ Pa} \)), and the thermodynamic temperature \( T \) varying according to Tab. 1.

| Model | \( R \) [cm] | \( t \) [cm] | \( T \) [K] | \( n [-] \) |
|-------|--------------|--------------|------------|-----------|
| \( L_0 \) | 1            | 2            | 295.65     | 1.027502  |
| \( T_1 \) | 1.5          | 3            | 296.15     | 1.027436  |
| \( T_2 \) | 2            | 4            | 296.65     | 1.027410  |
| \( T_3 \) | 2.5          | 5            | 295.15     | 1.027549  |
| \( T_4 \) | 3            | 6            | 294.65     | 1.027596  |

Tab. 1. Parameters for each case of the model. \( R \) stands for radius of curvature \((R = R_1 = R_2)\), \( t \) is central thickness of lenses, \( T \) presents thermodynamic temperature, and \( n \) is the refractive index calculated according to (10).

These turbulent models show how the direction and the size of the optical beam deflection can change after passing through a turbulent cell compared to passing through a stationary medium. Final results of the simulation are evaluated for three different beams (\( \alpha, \beta, \gamma \)) passing through each model. Each of the rays has a different input deflection \( x_1 \) according to Tab. 2, and its subsequent transition is realized for different large lenses with different values of refractive indices. With increasing matrix numbering (\( T_1 - T_4 \)), cell size and refractive index increase.

The results of simulation based on these five models are shown in the Fig. 5. The absolute difference of the output deflections from the reference model \( \Delta x_i \) is plotted on the y-axis and the values of the input deflection \( x_1 \) of the optical beam is plotted on the x-axis. The x-axis on Fig. 5 is formed by three points, which indicate the deflections of the beam on the input plane \( x_1 (\alpha, \beta, \gamma) \). For each model (\( L_0 - T_4 \)), the beam deflections on the output plane \( x_2 L_0, x_2 T_1 - 4 \) were determined according to the model type. Since the relatively small dimensions of the turbulent cells and the small changes in the refractive index are taken into account, the output deflections \( x_2 \) are slightly different. Therefore, it is more efficient to monitor the absolute difference of output deflections \( \Delta x_i \) from the reference model given by relation

\[
\Delta x_i = |x_{2L_0} - x_{2T_1 - 4}|
\]

where values of \( x_{2L_0}, x_{2T_1 - 4} \) for individual models are obtained from the matrix (4). Their dependence on the input deflection \( x_1 \) is evident. It is clear that the value of \( \Delta x_i \) will be zero at all points of the reference model.

| Beams | \( x_1 \) [cm] | \( \theta_1 \) [rad] |
|-------|---------------|---------------------|
| \( \alpha \) | 0.5           | 1.75 e-04           |
| \( \beta \)  | 1             | 3.49 e-04           |
| \( \gamma \) | 1.5           | 5.24 e-04           |

Tab. 2. Values of input deflections \( x_1 \) and angles of deflection \( \theta_1 \) for three different deflected optical beams denoted as \( \alpha, \beta, \gamma \). The input deflections \( x_1 \) are determined based on the transmitter distance, turbulent cell size, and paraxial condition.

Fig. 4. Three different deflected optical beams propagate
a) in a medium with a constant refractive index \( L_0 \) and
b) in a medium with a thick lens \( T_1, T_2, \alpha, \beta, \) and \( \gamma \) indicate specific optical beam deflections in each model.

Fig. 5. Dependence of the absolute difference of the output deflections from the reference model on the values of the input deflection of the optical beam. The dependence is realized for five models \( L_0 - T_4 \).
As turbulent cell sizes and refractive indices increase, so does the overall difference in output deflections from the reference model. By passing through a larger turbulent cell, a more significant deflection will occur.

3. Cascade System

The idea of a cascade system is to imagine atmospheric turbulence as several consecutive turbulent cells. This cascade represents part of the optical transmission path. Each turbulent cell has its own defined geometric parameters that best reflect the size of the actual turbulent cell. By modeling the turbulent cascade, it is possible to monitor the deflection of the beam not only on the detector of the receiver after passing through the entire cascade but also within it.

The following section offers a mathematical procedure for calculating the optical beam deflection with a matrix solution of a cascade of turbulent cells.

3.1 Calculation of Beam Deflection by Passing Through a Cascade of Two Turbulent Cells

To better illustrate the procedure for determining the individual deflections of the paraxial optical beam, the following calculation procedure will be based on the turbulent cascade model from the Fig. 6.

In the first place, it is necessary to determine the distance from the laser radiation source to the turbulent area \( l_{TX} \) and the deflection of the beam \( x_1 \). Using the trigonometric tangent function, it is possible to calculate the parameters of the beam deflection on the input plane of the lens

\[
x_1 = l_{TX} \cdot \tan(\theta_1)
\]

where \( x_1 \) is the deflection on the input plane, \( l_{TX} \) indicates the distance from the laser source to the turbulent cell. The value of the initial deflection angle \( \theta_1 \) of the laser beam must be greater than zero degrees (\( \tan(0) = 0 \)), which is a condition for the proper use of ray transfer matrix calculation. The beam deflection parameters at the input plane of the first turbulent cell of the cascade form the input vector denoted as

\[
X_{IN} = \begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix}.
\]  

In order to calculate the deflection vectors using matrix (5), the refractive indices in the given areas are determined based on the values of surrounding pressure, thermodynamic temperature, and laser wavelength. The refractive index \( n(\lambda) \) is calculated from the relation [13]

\[
n(\lambda) = 1 + 77.6 \cdot 10^{-6} \left(1 + 7.52 \cdot 10^{-3} \cdot \lambda^{-2} \right) \frac{P}{T} \]

where \( \lambda \) is optical wavelength in \( \mu m \), \( P \) is air pressure in \( Pa \) and \( T \) is thermodynamic temperature at the location in K. Spectral dependence of the refractive index of the atmosphere must be taken into account, so that its value can be evaluated for specific optical wavelengths. Therefore, the used equation depends not only on the state quantities, but also on the wavelength of the laser.

The turbulent cell size defined by the central thickness \( t \) and radii of curvature \( R_{1,2} \), and the refractive index \( n \) are used to calculate the matrix \( T_1 \) (Fig. 6). The \( T_1 \) label represents the ABCD matrix for the first turbulent cell. By the parameters of deflection on the input plane \( (x_1, \theta_1) \) and the matrix \( T_1 \) the deflection on the output plane are obtained

\[
\begin{pmatrix} x_2 \\ \theta_2 \end{pmatrix} = T_1 \begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix}.
\]  

After finding the value of the parameters \( x_2 \) and \( \theta_2 \), the value of the deflection parameters on the input plane of the second lens \( (x_3, \theta_3) \) are calculated.

The trigonometric tangent function determines the deflection in the input plane of the second lens due to the known angle of deflection and inter-turbulent distance \( l \). In this simplified model, the optical beam always propagates toward the optical axis of the system. The deflection of the laser beam in the inter-turbulent area is defined for four potential cases, which are shown in Fig. 7:

(a) The beam deflects from the area above the optical axis to above it with a positive deflection value, but a negative value of deflection angle (Fig. 6 - L1).

(b) The beam deflects from the area above the optical axis to below it. The deflection angle is negative, and deflection on the output plane acquires a negative value (Fig. 6 - L1).

(c) The beam deflects from the area below the optical axis to below it. The deflection angle is positive, but deflection acquires a negative value (Fig. 6 - L2).

(d) The beam deflects above the optical axis with a positive value of both deflection parameters (Fig. 6 - L2).
where the negative sign in the deflection parameter for the inter-turbulent space

\[ \theta_2 = \frac{x_3}{l_{\text{RX}}} \] (12)

where the matrix in this formula represents the matrix \( L_2 \) on the Fig. 6

\[ X_{\text{OUT}} = \begin{pmatrix} x_5 \\ \theta_5 \end{pmatrix} = L_2 \begin{pmatrix} x_4 \\ \theta_4 \end{pmatrix}. \] (16)

For all cases of transmission of an optical beam in the inter-turbulent space, depending on the inter-turbulent distance and the size of the parameters of the input deflection vector, a positive or negative beam deflection can be detected.

### 3.2 Matrix Cascade Calculation

Imagine a cascade of \( N \) turbulent cells according to Fig. 8. Each turbulent cell is defined by its ray transfer matrix \( T_1 \sim T_n \) and each inter-turbulent space is defined by matrix \( L_1 \sim L_n \). Vector \( X_{\text{IN}} \) points to a primary deflection of the optical beam at the input of the turbulent cascade and \( X_{\text{OUT}} \) points to the vector of the resulting beam deflection by passing through the whole cascade.

Through the paraxial approximation [16] according to the equation (1), it is possible to write the ABCD matrix of this form for the inter-turbulent space

\[ L_n = \begin{pmatrix} 1 & l_n \\ 0 & 1 \end{pmatrix}, \] (17)

as was shown in the previous subsection. As a result of this simplification, neither the matrices defined for turbulent cells nor the matrices for inter-turbulent space contain evolving deflection parameters. Therefore, it is possible to perform matrix multiplication of individual optical elements of the turbulent cascade.

Figure 9 shows a diagram pointing to a matrix cascade calculation of the output deflection vector \( X_{\text{OUT}} \). The technique of calculating the output deflection of the beam incident on the detector is evident from the given diagram. Multiplication of matrices is performed by notation

\[ X_{\text{OUT}} = X_{\text{IN}} \cdot T_n \cdot L_n \cdot T_{n-1} \cdot L_{n-1} \cdots T_1 \cdot X_{\text{IN}} \] (18)

where \( X_{\text{IN}} \) denotes the vector of cascade deflection input parameters and \( X_{\text{OUT}} \) denotes the vector of cascade deflection output parameters. By reversing the order of the matrices, it is possible to calculate for the given channel the deflection of the optical beam propagating in the opposite direction.

\[ X_{\text{IN}} \begin{pmatrix} n_1 \\ \vdots \\ n_{n_1} \end{pmatrix} L_1 \begin{pmatrix} n_2 \\ \vdots \\ n_{n_2} \end{pmatrix} T_1 \begin{pmatrix} n_3 \\ \vdots \\ n_{n_3} \end{pmatrix} \cdots L_{n-1} \begin{pmatrix} n_{n_{n-1}} \\ \vdots \\ n_{n_n} \end{pmatrix} T_{n-1} \begin{pmatrix} n_{n_{n-1}}+1 \\ \vdots \\ n_{n_1} \end{pmatrix} X_{\text{OUT}} \]

Fig. 8. Illustration model of the cascade of \( N \) turbulent cells.
derwater communications in the analysis of the propagation.

the results are applicable in other transmission media - un-
turbulent cells in the desert or planetary boundary layer. Also,
geneity, which is larger than the width of the optical beam -
FSO systems in the presence of local temperature inhomo-

go that the optical link works as efficiently as possible.

calculation will be able to specify the initial beam deflection

on initial and detected optical beams. The backward matrix

possibletodeterminethematrixofaturbulentmediumbased

medium. Via the matrix calculation technique, it would be

ing vector notation and matrix definition of the transmission

tors inside or outside the cascade, at the receiver.

a mathematical procedure for calculating the deflection vec-
turbulent cells via a matrix model. In addition, it presents

parameters that create the deflection vector.

The paper also includes a simulation aimed at comparing

development of the optical beam through different large
turbulent cells via a matrix model. In addition, it presents a

mathematical procedure for calculating the deflection vec-
tors inside or outside the cascade, at the receiver.

Turbulence in FSO systems can also be analyzed using

vector notation and matrix definition of the transmission

medium. Via the matrix calculation technique, it would be possible to determine the matrix of a turbulent medium based on initial and detected optical beams. The backward matrix calculation will be able to specify the initial beam deflection so that the optical link works as efficiently as possible.

Implementation of this method would be possible in

FSO systems in the presence of local temperature inhomogeneity, which is larger than the width of the optical beam -
turbulent cells in the desert or planetary boundary layer. Also,
the results are applicable in other transmission media - under-
water communications in the analysis of the propagation of
optical beams through layers with different temperatures,
i.e. different refractive indices. However, the work is in the
initial phase of the modeling for further processing in the
time and space domain.

4. Conclusion

In summary, this paper presents an introduction to the
ray transfer matrix method used for a mathematical description of atmospheric turbulence as an optical element. The designed turbulent model consists of a cascade of optical elements approximating turbulent cells. From the cascade of turbulent cells and in the presence of certain simplifications and requirements, there is the potential to work with mentioned mathematical technique. In this way, the model currently serves to observe the optical beam deflection parameters that create the deflection vector.

The paper also includes a simulation aimed at comparing the deflection of the optical beam through different large turbulent cells via a matrix model. In addition, it presents a mathematical procedure for calculating the deflection vectors inside or outside the cascade, at the receiver.

Implementation of this method would be possible in

FSO systems in the presence of local temperature inhomogeneity, which is larger than the width of the optical beam -
turbulent cells in the desert or planetary boundary layer. Also,
the results are applicable in other transmission media - under-
water communications in the analysis of the propagation of
optical beams through layers with different temperatures,
About the Authors . . .

Alzbeta KOVALOVA was born in 1997. She received her M.Sc. degree in 2021 in Electronics and Communication Technologies from Brno University of Technology (BUT), Czech Republic. She is currently a Ph.D. student at the Dept. of Radio Electronics at BUT. Her research interests include matrix optics, optical wireless communication, atmospheric turbulence and its impact on free space optics systems.

Lucie HUDCOVA was born in 1981. She graduated from the BUT in 2006, in 2010 she received a Ph.D. degree in Electronics and Communications at the same university. Since 2018, she is an Associate Professor at the Dept. of Radio Electronics. Her specialization is optoelectronics, transmission media in the optical spectrum, optical communications, and free-space optics. Her research work is focused on the modeling of the atmospheric and underwater effects in the optical domain.

Rastislav ROKA was born in 1972. He received his M.Sc. and Ph.D. degrees in Telecommunications from the Slovak University of Technology, Slovakia in 1995 and 2002. Since 2009, he is an Associated Professor at this institute. At present, his research activity is focused on the signal transmission through optical transport, metropolitan and access networks by means of new WDM and TDM technologies using advanced optical signal processing included various modulation and coding techniques.