Semi-modular forms from Fibonacci–Eisenstein series

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Abstract
In a 2021 paper, M. Just and the second author defined a class of “semi-modular forms” on \( \mathbb{C} \setminus \mathbb{R} \), in analogy with classical modular forms, that are “half modular” in a particular sense; and constructed families of such functions as Eisenstein-like series using symmetries related to integer partitions. Looking for further natural examples of semi-modular behavior, here we construct a family of Eisenstein-like series to produce semi-modular forms, using symmetries related to Fibonacci numbers instead of partitions. We then consider other Lucas sequences that yield semi-modular forms.

Keywords Eisenstein series · Lucas sequence · Fibonacci sequence · Modular form · Semimodular form · General linear group

1 Introduction and statement of results
In a 2021 paper [5], M. Just and the second author defined a class of “semi-modular forms” on \( \mathbb{C} \setminus \mathbb{R} \), in analogy with classical modular forms, that are “half modular” in a particular sense, and produced examples of such functions using special constructions related to integer partitions that give a combinatorial interpretation to the transfor-
Quite apart from the theory of partitions, here we construct new examples of semi-modular forms using symmetries from Fibonacci numbers and Lucas sequences.

We give a quick overview of modular forms theory in order to describe semi-modularity. Let us recall the canonical generators of the general linear group \( \text{GL}_2(\mathbb{Z}) \) are

\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

such that \( \text{GL}_2(\mathbb{Z}) = \langle T, U, V \rangle \). An important subgroup of \( \text{GL}_2(\mathbb{Z}) \) is the special linear group \( \text{SL}_2(\mathbb{Z}) \), which is well known to be generated by \( T \) together with the matrix

\[
S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{GL}_2(\mathbb{Z}).
\]

Functions of \( z \) in the upper half-plane \( \mathbb{H} \) invariant under \( \langle S, T \rangle = \text{SL}_2(\mathbb{Z}) \) up to a simple multiplier in \( z \) are modular forms, a class of functions central to number theory \([1, 8]\).

For a canonical example, recall the weight-\( m \) Eisenstein series, the prototype of an integer weight holomorphic modular form, convergent for \( m > 2 \), \( z \in \mathbb{H} \):

\[
G_m(z) = \sum_{j,k \in \mathbb{Z}} (jz + k)^{-m}.
\]

(1)

Note that if \( m \) is odd, every term cancels its negative, and \( G_m(z) = 0 \). Taking \( m \mapsto 2m \), for \( m > 1 \), the function \( G_{2m} : \mathbb{H} \to \mathbb{C} \) satisfies the defining properties of a modular form:

(i) \( G_{2m}(\frac{-1}{z}) = z^{2m} G_{2m}(z) \) (weighted invariance under inversion matrix \( S \)),

(ii) \( G_{2m}(z + 1) = G_{2m}(z) \) (invariance under translation matrix \( T \)).

As defined in \([5]\), a semi-modular form is a function of a complex variable that is invariant under one of the two matrices \( S, T \), together with being invariant under a second matrix \( M \) that is complementary in a natural sense to \( S, T \) —thus, as we mentioned above, it is “half modular.” To give more detail about the sense in which matrix \( M \) complements \( \text{SL}_2(\mathbb{Z}) \), we want \( M \in \text{GL}_2(\mathbb{Z}) \) to be a “nice” transformation matrix such that one can partition the general linear group, in terms of its generators, into the modular group \( \text{SL}_2(\mathbb{Z}) \) together with \( M \); i.e., such that one can write

\[
\text{GL}_2(\mathbb{Z}) = \langle M, S, T \rangle.
\]

In principle, the matrices \( M, S, T \) yield three families of functions invariant on \( \langle S, T \rangle \), \( \langle M, S \rangle \) and \( \langle M, T \rangle \), respectively. We count modular forms as semi-modular as well, but the existence of a complementary matrix \( M \) is implicit in the terminology.

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1 Inversion \( z \mapsto -1/z \) is related to conjugation of Ferrers-Young diagrams of partitions in \([5]\).
We note that finding a matrix $M$ to complement $S$, $T$ is all but trivial; the details are in identifying matrices $M$ that are canonical in some sense, and that also admit reasonably natural functions invariant on $\langle M, S \rangle$ and $\langle M, T \rangle$. In [5], the “even function” matrix

$$M' = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

was used to produce such families: the family $\langle M', T \rangle$ includes functions like $\cos z$, $z \in \mathbb{C}$. Using detailed constructions in partition theory, the authors define “partition Eisenstein series” invariant on $\langle M', S \rangle$, that are analogous to (1) but summed over integer partitions.

Now, partitions and modular forms are closely connected classically, so it seems natural to look to partitions for generalizations of modularity. In [5], which uses symmetries in the Ferrers-Young diagrams of partitions in a very specific way to induce semi-modular behavior, the authors ask if other classes of semi-modular forms arise from natural symmetries in areas of mathematics beyond the confluence of partitions and modular forms.

We will now show that semi-modularity does arise outside of the usual partitions-modular forms universe; we construct a family of semi-modular forms using symmetries related to classical Fibonacci numbers instead of partitions. Recall for $n \geq 0$, the $n$th Fibonacci number $F_n$ is defined by $F_0 := 0$, $F_1 := 1$, and for $n > 1$,

$$F_n := F_{n-1} + F_{n-2}. \quad (3)$$

This recursion extends to negative indices by defining, for $n \geq 1$, $F_{-n} := (-1)^{n-1} F_n$.

Much like partitions, Fibonacci numbers are additive objects that appear throughout the mathematical sciences, particularly due to their close connection to the golden ratio $\phi := \frac{1 + \sqrt{5}}{2} = \lim_{n \to \infty} F_n/F_{n-1}$. Via evaluations of the Rogers–Ramanujan continued fraction [11], the golden ratio plays a role in the theory of modular forms—giving a second-hand link to modular forms for Fibonacci numbers. More recent works directly connect Fibonacci numbers and modular forms (see e.g., [2, 7, 9]).

Seeking new examples of semi-modular forms, we will define a Fibonacci variant of (1).

**Definition 1** For fixed $m > 1$, $z \in \mathbb{C}$, $z \neq \phi, -\frac{1}{\phi}$ or $F_n/F_{n-1}$ for any $n \in \mathbb{Z}$, let

$$\mathcal{F}_m(z) := \sum_{j=\infty}^{\infty} (F_j z + F_{j-1})^{-m},$$

where the $F_n$ are Fibonacci numbers, $n \in \mathbb{Z}$. We call $m > 1$ the “weight” of the series.

We note this is a single bilateral summation, as opposed to a double sum as in (1), and is convergent on its domain when $m > 1$ by comparison with $\zeta(m)$. To justify the

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2 See e.g., papers on the $q$-bracket of Bloch and Okounkov such as [3, 4, 10, 12, 13].
domain conditions, note that for any \( n \in \mathbb{Z}, n \neq 1 \), the term \( F_{-n+1} z + F_n \) vanishes in the denominator when \( z = F_n/F_{n-1} \) due to the minus sign attached to exactly one of the negatively indexed terms, giving a pole of order \( m \). Since \( z = \phi, -1/\phi \) are the respective limits of the infinite sequence of poles \( F_n/F_{n-1} \) and \( F_{-n}/F_{-n-1} \) as \( n \to \infty \), these limiting values themselves represent essential singularities.

This “Fibonacci–Eisenstein” series \( \mathcal{F}_k \) has nice transformation properties at even weights.

**Theorem 2** Let \( z \in \mathbb{C}, z \neq \phi, -1/\phi, F_n/F_{n-1} \) for any \( n \in \mathbb{Z} \). For \( k \geq 1 \) we have

(i) \( \mathcal{F}_2(k(-1/z)) = z^{2k} \mathcal{F}_2(k(z)) \);

(ii) \( \mathcal{F}_2(k(1-z)) = \mathcal{F}_2(k(z)) \).

**Remark** We note that in these functions, the odd-weight cases do not vanish like (1).

The property (i) is, of course, weighted invariance with respect to matrix \( S \). Property (ii), symmetry around \( \text{Re}(z) = 1/2 \), is encoded in the matrix

\[
P = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

The Riemann zeta function \( \zeta(z) \) in the critical strip is an example of a function with (weighted) invariance around \( P \), as are the trigonometric functions \( \sin(\pi z), \cos(2\pi z) \).

Now, noting that, in fact, the generators of \( \text{GL}_2(\mathbb{Z}) \) can be written

\[
U = PTS, \quad V = SPTS^3,
\]

then one can alternatively view the general linear group as

\[
\text{GL}_2(\mathbb{Z}) = \langle P, S, T \rangle.
\]

From this perspective, modular forms invariant on \( \langle S, T \rangle \), and periodic functions of period 1 symmetric around the 1/2-line invariant on \( \langle P, T \rangle \) are members of a larger class of semi-modular forms invariant on two of the three generators \( P, S, T \) of \( \text{GL}_2(\mathbb{Z}) \). By Theorem 2, \( \mathcal{F}_2(k(z)) \) is a semi-modular form in this class, too, invariant on \( \langle P, S \rangle \).

**Remark** We note \( PS \) is a so-named Fibonacci matrix (see e.g., [9]) such that, for \( n \geq 1 \),

\[
(PS)^n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.
\]

In Sect. 2, we will establish the semi-modularity of \( \mathcal{F}_2 \). In Sect. 3, we discuss connections to other Lucas sequences, and produce infinite families of semi-modular forms.
2 Proof of the main result

Our proof of Theorem 2 depends on the following decomposition of the bilateral series \( \mathcal{F}_m(z) \) into two unilateral parts:

\[
\mathcal{F}_m(z) = \mathcal{F}_m^-(z) + \mathcal{F}_m^+(z),
\]

where

\[
\mathcal{F}_m^-(z) := \sum_{-\infty < n \leq 0} (F_n z + F_{n-1})^{-m}, \quad \mathcal{F}_m^+(z) := \sum_{1 \leq n < \infty} (F_n z + F_{n-1})^{-m}.
\]

Proof of Theorem 2 Recall \( F_0 = 0 \) and for negative indices, we define \( F_{-n} = (-1)^{n-1} F_n \). We only use even values \( m = 2k \) with \( k > 1 \) (even powers regularize the \( \pm \) sign behavior in our proof below). We will employ standard Eisenstein series techniques, together with the recursion (3). Note for the positively indexed terms of \( \mathcal{F}_{2k} \), we have

\[
\mathcal{F}_{2k}^+(z + 1) = \sum_{n \geq 1} (F_n (z + 1) + F_{n-1})^{-2k} = \sum_{n \geq 1} (F_n z + (F_n + F_{n-1}))^{-2k} \\
= \sum_{n \geq 1} (F_n z + F_{n+1})^{-2k} = z^{-2k} \sum_{n \geq 2} (F_n (1/z) + F_{n-1})^{-2k} \\
= z^{-2k} \mathcal{F}_{2k}^+(1/z) - 1.
\]

Similarly, for the non-positively indexed terms of \( \mathcal{F}_{2k} \), we have

\[
\mathcal{F}_{2k}^-(z + 1) = \sum_{n \leq 0} (F_n (z + 1) + F_{n-1})^{-2k} = \sum_{n \leq 0} (F_n z + (F_n + F_{n-1}))^{-2k} \\
= \sum_{n \leq 0} (F_n z + F_{n+1})^{-2k} = z^{-2k} \sum_{n \leq 1} (F_n (1/z) + F_{n-1})^{-2k} \\
= z^{-2k} \mathcal{F}_{2k}^-(1/z) + 1.
\]

Then using (5) to add the corresponding left- and right-hand sides of (6) and (7) yields

\[
\mathcal{F}_{2k} (z + 1) = z^{-2k} \mathcal{F}_{2k} (1/z),
\]

the first identity in the theorem. Along similar lines, we also have

\[
\mathcal{F}_{2k}^-(z) = \sum_{n \leq 0} (F_n (-z) + F_{n-1})^{-2k} = z^{-2k} \sum_{n \leq 0} (-F_n + F_{n-1} (1/z))^{-2k} \\
= z^{-2k} \sum_{n \geq 1} (F_n (1/z) + F_{n-1})^{-2k} = z^{-2k} \mathcal{F}_{2k}^+(1/z),
\]

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as well as
\[ 
\mathcal{F}_{2k}^+ (-z) = \sum_{n \geq 1} (F_n (-z) + F_{n-1})^{-2k} = z^{-2k} \sum_{n \geq 1} (-F_n + F_{n-1}(1/z))^{-2k} 
\]
\[ 
= z^{-2k} \sum_{n \leq 0} (F_n(1/z) + F_{n-1})^{-2k} = z^{-2k} \mathcal{F}_{2k}^- (1/z). 
\] (10)

We note here that the terms of \( \mathcal{F}_{2k}^- \) and \( \mathcal{F}_{2k}^+ \) swap places to produce the inversion \( -z \mapsto 1/z \). As above, adding the corresponding left- and right-hand sides of (9) and (10) gives
\[ 
\mathcal{F}_{2k}^- (-z) = z^{-2k} \mathcal{F}_{2k}^+ (1/z). 
\] (11)

Compare Eqs. (8) and (11) to yield the identity
\[ 
\mathcal{F}_{2k} (z + 1) = \mathcal{F}_{2k} (-z). 
\] (12)

Making the substitution \( z \mapsto -z \) in (11) and (12) completes the proof of the theorem. The domain restrictions on \( z \) were justified below Definition 1.

\[ \square \]

3 Extension to other Lucas sequences

At this stage, seeing how the proofs above follow easily from the recursion (3), a natural next question is are other Fibonacci-like sequences such as Lucas sequences subject to the same treatment, leading to further families of semi-modular functions? Recall for \( a, b \in \mathbb{Z} \) that a classical Lucas sequence \(^3\) \( \{L_n(a, b)\} \) is defined for \( n \geq 2 \) by the recursion
\[ 
L_n(a, b) := aL_{n-1}(a, b) - bL_{n-2}(a, b). 
\] (13)

With the initial conditions \( L_0(a, b) = 0, \ L_1(a, b) = 1 \), it is called a Lucas sequence of the first kind. When \( L_0(a, b) = 2, \ L_1(a, b) = a \), it is called a Lucas sequence of the second kind. Lucas sequences generalize a number of classical Fibonacci-like sequences; for instance, the Lucas sequence \( L_n(1, -1) \) of the first kind is the Fibonacci sequence \( F_n \), and the same form \( L_n(1, -1) \) of the second kind defines the classical Lucas numbers \( L_n \).

However, for our proof above to extend to other sequences, we require more than just the recursion, an even weight, and standard Eisenstein series manipulations. We need the initial terms defined such that \( L_n(a, b) \) has “alternating sign symmetry” \( L_{-n}(a, b) = (-1)^{n-1} L_n(a, b) \) around the term \( L_0(a, b) \) (or a similar near symmetry), so the positively and negatively indexed terms will make a swap as noted below Eqs. (9) and (10).

\[^3\] For a comprehensive survey of Fibonacci and Lucas sequences, see [6].
The Lucas numbers $L_n = L_n(1, -1)$ with initial conditions $L_0 := 2$, $L_1 := 1$, and recursion $L_n := L_{n-1} + L_{n-2}$ admit the similar extension $L_{-n} = (-1)^n L_n$.

**Theorem 3** Let $m > 1$, $z \in \mathbb{C}$, $z \neq \phi$, $-1/\phi$, $\frac{L_n}{L_{n-1}}$ for any $n \in \mathbb{Z}$, and for $m > 1$ define

$$\mathcal{L}_m(z) := \sum_{j=-\infty}^{\infty} (L_j z + L_{j-1})^{-m},$$

where the $L_n$ are Lucas numbers, $n \in \mathbb{Z}$. Then for $k \geq 1$ we have

(i) $\mathcal{L}_{2k}(-\frac{1}{z}) = z^{2k} \mathcal{L}_{2k}(z)$;
(ii) $\mathcal{L}_{2k}(1 - z) = \mathcal{L}_{2k}(z)$.

Because the function $\mathcal{L}_{2k}(z)$ enjoys (weighted) invariance with respect to $\langle P, S \rangle$ just like $\mathcal{F}_{2k}(z)$, it is a semi-modular form in the same class.

**Proof** Replace $F_n$ with $L_n$ in the proof of Theorem 2, noting the arguments about the poles and singularities at $\phi$, $-1/\phi$ also hold here since $\lim_{n \to \infty} L_n/L_{n-1} = \phi$. □

The “alternating sign symmetry” required for these proofs is not a general property of Lucas sequences, but can be obtained in more general cases.

**Definition 4** For fixed $a, b \neq 0$, $m > 1$, $z \in \mathbb{C}$, $z \neq \frac{L_n(a,b)}{L_{n-1}(a,b)}$, for any $n \in \mathbb{Z}$, $z \neq \lim_{n \to \infty} \frac{L_n(a,b)}{L_{n-1}(a,b)}$ or $\lim_{n \to \infty} -\frac{L_{n-1}(a,b)}{L_n(a,b)}$ if the limits exist, let

$$\mathcal{L}_{a,b,m}(z) := \sum_{j=-\infty}^{\infty} (L_j (a, b) z + L_{j-1}(a, b))^{-m},$$

where $L_n(a, b)$ is a Lucas sequence of either the first or the second kind (to be specified).

The analytic conditions generalize those in Definition 1 with identical justifications, noting by (13) that $\mathcal{L}_{0,b,m}(z)$ clearly diverges by consideration of geometric series.

Taking $b = -1$ in Lucas sequences $L_n(a, b)$ of the first kind, one sets $L_0(a, -1) := 0$, $L_1(a, -1) := 1$, and for $a \in \mathbb{Z}$, $a \neq 0$, one has $L_n(a, -1) := a L_{n-1}(a, -1) + L_{n-2}(a, -1)$. These $L_n(a, -1)$ are referred to as $a$-Fibonacci numbers, and generalize the properties of $F_n$. It is easy to check that we can extend the indices using $L_{-n}(a, -1) := (-1)^{n-1} L_n(a, -1)$ for this Lucas sequence of the first kind. Moreover, a similar relation $L_{-n}(a, -1) := (-1)^n L_n(a, -1)$ also holds when $a \neq 0$ if it is taken as a Lucas sequence of the second kind, setting $L_0(a, -1) := 2$, $L_1(a, -1) := a$.

**Theorem 5** For $a \in \mathbb{Z}\setminus\{0\}$, let $L_n(a, -1)$ be a Lucas sequence of either the first or the second kind. Take $z \in \mathbb{C}$, $z \neq \frac{L_n(a,b)}{L_{n-1}(a,b)}$ for any $n \in \mathbb{Z}$, $z \neq \lim_{n \to \infty} \frac{L_n(a,b)}{L_{n-1}(a,b)}$ or $\lim_{n \to \infty} -\frac{L_{n-1}(a,b)}{L_n(a,b)}$. Then for $k \geq 1$ we have

(i) $\mathcal{L}_{a,-1,2k}(-\frac{1}{z}) = z^{2k} \mathcal{L}_{a,-1,2k}(z)$;
(ii) $\mathcal{L}_{a, -1, 2k}(a - z) = \mathcal{L}_{a, -1, 2k}(z)$.

**Remark** Setting $a = 1$, then Theorem 5 subsumes both Theorems 2 and 3.

The property (i) above is weighted invariance with respect to matrix $S$. Property (ii), symmetry around Re$(z) = a/2$, is encoded in the matrix

$$P_a = \begin{pmatrix} -1 & a \\ 0 & 1 \end{pmatrix}.$$  

Noting that $P_a T^a = PT$, then by (4), we also have

$$U = P_a T^a S, \quad V = S P_a T^a S^3.$$  

Thus, one can write

$$\text{GL}_2(\mathbb{Z}) = \langle P_a, S, T \rangle,$$

and $\mathcal{L}_{a, -1, 2k}(z)$ satisfies our definition of a semi-modular form.

**Proof** This is very similar to the proof of Theorem 2; one could decompose $L_n(a, -1)$ into positively and non-positively indexed terms and carry out the same manipulations. Here, we summarize those steps without decomposing into two halves. First of all, we have

$$\mathcal{L}_{a, -1, 2k}(z + a) = \sum_{j=-\infty}^{\infty} \left[ L_j(a, -1)(z + a) + L_{j-1}(a, -1) \right]^{-2k} \quad (14)$$

$$= \sum_{j=-\infty}^{\infty} \left[ L_j(a, -1)z + (a L_j(a, -1) + L_{j-1}(a, -1)) \right]^{-2k}$$

$$= z^{-2k} \sum_{j=-\infty}^{\infty} \left[ -L_j(a, -1) + L_{j+1}(a, -1)(1/z) \right]^{-2k}$$

$$= z^{-2k} \mathcal{L}_{a, -1, 2k}(1/z),$$

where we reverse indices $\sum_{n=-\infty}^{\infty} \mapsto \sum_{n=\infty}^{-\infty}$ in the final summation to indicate the positively and non-positively indexed terms make a swap just as we noted in Eqs. (9) and (10). Similarly, we have

$$\mathcal{L}_{a, -1, 2k}(-z) = \sum_{j=-\infty}^{\infty} \left[ L_j(a, -1)(-z) + L_{j-1}(a, -1) \right]^{-2k} \quad (15)$$

$$= z^{-2k} \sum_{j=-\infty}^{\infty} \left[ -L_j(a, -1) + L_{j-1}(a, -1)(1/z) \right]^{-2k}$$

$$= z^{-2k} \mathcal{L}_{a, -1, 2k}(1/z).$$

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4 The matrix $P_a$ plays a role in work in the literature relating Fibonacci matrices to modular forms and Poincaré series; and $P_3S$ is noted in [9] to be “almost a Fibonacci matrix.”
Compare (14) and (15) to yield

\[ L_{2k}(z + a) = L_{2k}(-z). \] (16)

Making the substitution \( z \mapsto -z \) in (15) and (16) completes the proof. The domain restrictions on \( z \) delete the infinite sequences of poles at those values, as well as their respective limits that represent essential singularities, as indicated below Definition 4.

It follows from the recursion (13) that, in general, one can define

\[ L_{-n}(a, b) := (-1]^\ell L_n(a, b)/b^n, \] (17)

where \( \ell = 1 \) if \( L_n(a, b) \) is a Lucas sequence of the first kind, and \( \ell = 2 \) if it is a Lucas sequence of the second kind. From (17), it is clear \( b = -1 \) is the only substitution that produces the desired “alternating sign symmetry” for sequences of either kind. Perhaps other substitutions, like setting \( b \) equal to other roots of unity, would yield further relations for \( L_{a, b, 2k}(z) \).

4 Further questions

We note that the \( a = 0 \) case of \( P_a \) is the complementary matrix (2) used in [5] to produce semi-modular forms, and the \( a \neq 0 \) cases yield semi-modularity in Theorem 5. Then, the set \( \{P_a : a \in \mathbb{Z}\} \) of mirror symmetries around \( \text{Re}(z) = a/2 \) is the complete collection of complementary matrices, such that \( \text{GL}_2(\mathbb{Z}) = \langle P_a, S, T \rangle \), which have thus far produced examples of semi-modular forms.

Do there exist matrices that are not of the form \( P_a \), that both complement \( S, T \) as generators of \( \text{GL}_2(\mathbb{Z}) \) and produce natural semi-modular forms? Based on connections such as those proved in [9] between \( P_a \), Fibonacci matrices, and Poincaré series, are there further links between semi-modularity and classical modular forms? Moreover, since we now find semi-modularity to arise from symmetries in Lucas sequences, as well as from partition theory as in [5], one wonders: what other symmetric, additive or recursive structures in mathematics may be used to construct semi-modular Eisenstein-like series?\(^5\)

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