THE MODIFIED CALABI-YAU PROBLEMS FOR CR-MANIFOLDS

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Dedicated to the memory of Xiao-Song Lin.

Abstract. In this paper, we derive a partial result related to a question of Professor Yau: “Does a simply-connected complete Kähler manifold \( M \) with negative sectional curvature admit a bounded non-constant holomorphic function?”

Main Theorem. Let \( M^{2n} \) be a simply-connected complete Kähler manifold \( M \) with negative sectional curvature \( \leq -1 \) and \( S_\infty(M) \) be the sphere at infinity of \( M \). Then there is an explicit bounded contact form \( \beta \) defined on the entire manifold \( M^{2n} \).

Consequently, if \( M^{2n} \) is a simply-connected Kähler manifold with negative sectional curvature \( -a^2 \leq \text{sec}_M \leq -1 \), then the sphere \( S_\infty(M) \) at infinity of \( M \) admits a bounded contact structure and a bounded pseudo-Hermitian metric in the sense of Tanaka-Webster.

We also discuss several open modified problems of Calabi and Yau for Alexandrov spaces and CR-manifolds.

0. Introduction

In this paper, we will provide a detailed construction of bounded contact structures on a simply-connected complete Kähler manifold \( M \) with negative sectional curvature \( \leq -1 \). Afterwards, we will discuss the related open problems inspired by Calabi and Yau.

In 1979, Professor S. T. Yau [Y1] asked the following question.

Problem 0.1. (Yau [Y1]) Let \( M^{2n} \) be a simply-connected complete Kähler manifold \( M \) with negative sectional curvature \( \leq -1 \). Does \( M^{2n} \) admit a bounded non-constant holomorphic function?

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In fact, an even more attractive problem in complex analytic differential geometry is to characterize bounded domains in \( C^n \) within noncompact manifolds.

**Problem 0.2.** (Yau [Y]) Let \( M^{2n} \) be a simply-connected complete Kähler manifold \( M \) with negative sectional curvature \( \leq -1 \). Is \( M \) bihomeomorphic to a bounded domain in \( C^n \)?

There were some partial progress made by Bland [Bl] and Nakano-Ohsawa [NO]. Under extra assumptions, they proved the existence of CR functions on the ideal boundary \( S_\infty(M) \). In [Bl], two sufficient conditions were given for a complete Kähler manifold \( M \) of non-positive sectional curvature to admit nonconstant bounded holomorphic functions, which seems also to guarantee that \( M \) is a relatively compact domain with smooth boundary.

The precise definition of ideal boundary \( S_\infty(M) \) can be found in [BGS].

**Theorem 0.3.** Let \( M^{2n} \) be a simply-connected complete Kähler manifold \( M \) with negative sectional curvature \( \leq -1 \) and \( S_\infty(M) \) be the sphere at infinity of \( M \). Then there is an explicit bounded contact form \( \beta \) defined on the entire manifold \( M^{2n} \).

Consequently, if \( M^{2n} \) is a simply-connected Kähler manifold with negative sectional curvature \( -a^2 \leq \sec M \leq -1 \), then the sphere \( S_\infty(M) \) at infinity of \( M \) admits a bounded contact structure and a bounded pseudo-Hermitian metric in the sense of Tanaka-Webster.

Our proof of Theorem 0.3 was inspired by Gromov’s bounded cohomology [Gro-2] and calculations in [CaX].

Let \( \omega \) be the Kähler metric on \( M^{2n} \). It is clear that \( d\omega = 0 \). When \( M^{2n} \) is a simply-connected complete Kähler manifold with negative sectional curvature \( \leq -1 \), Gromov observed that there must be a bounded 1-form \( \beta \) with

\[
  d\beta = \omega. 
\]  

(0.1)

The proof of Gromov’s assertion was outlined in [Pa] and [JZ]. In this paper, we provide a detailed proof of Gromov’s assertion in §1. A similar sub-linear estimates for equation (0.1) on manifolds with non-positive curvature was given by the first author and Xavier in [CaX].
1. Bounded solutions to $d\beta = \alpha$ on manifolds with negative curvature

In this section, we prove Theorem 0.3. In addition, we present a new direct proof of Gromov's bounded cohomology theorem of negative curvature, see Theorem 1.4 and its proof below. Gromov's original approach to Theorem 1.4 below was based a volume estimate of $k$-dimensional cone over a $(k-1)$-dimensional chain, and then use a dual space argument to complete the proof. Our new method is to work on $k$-chains directly with a controlled Poincaré lemma for negative curvature. Our approach might have some potential independent applications.

Throughout this section $(M^m, g)$ will be a complete simply-connected manifold of negative sectional curvature $\leq -1$. Let also $\alpha$ be a bounded smooth closed $k$-form on $M$ with $k \geq 1$. Since $M^m$ is diffeomorphic to $\mathbb{R}^m$ there exists a form $\beta$ such that $d\beta = \alpha$. The purpose of this section is to show that $\beta$ can be chosen to be bounded. The proof will follow from the Poincaré lemma by a comparison argument.

Fix $p \in M$ and denote by $\exp_p : T_p M \to M$ the exponential map based at $p$.

**Lemma 1.1.** Consider the maps $\tau_t : M \to M$, given by $x \mapsto \exp_p(t \exp_p^{-1}(x))$, where $0 \leq t \leq 1$. Then

$$|(\tau_t)_* \xi| \leq \frac{\sinh tr}{\sinh r} |\xi|$$

(1.1)

for every tangent vector $\xi$, where $r = d(x, p)$.

**Proof.** Let $\sigma : [0, 1] \to M^n$ be the geodesic segment joining $p$ to $x$, $\xi \in T_x M^n$ and $y = (\exp_p)^{-1}(x) \in T_p M^n$. By a straightforward computation one has

$$(\tau_t)_* \xi = (d \exp_p)_t((\exp_p)^{-1}(x))[td(\exp_p)^{-1}(x) \xi]$$

$$= (d \exp_p)_t y \{t[d(\exp_p)_y]^{-1} \xi\}.$$ 

Recall that $\sigma(t) = \exp_p(ty)$. It is now manifest from the above formula that

$$J(tr) := (\tau_t)_* \xi$$

(1.2)

is the Jacobi field along $\sigma$ satisfying $J(0) = 0$, $J(r) = \xi$. On the other hand, since the sectional curvatures are $\leq -1$, we estimate the function $f(s) := |J(s)|$ by a method inspired by Gromov. It is sufficient to verify

$$\frac{|J(s)|}{\sinh s} \leq \frac{|J(r)|}{\sinh r},$$

(1.3)

for all $0 \leq s \leq r$. 


We may assume that \( r > 0 \), otherwise the inequality (1.1) holds trivially. To do this, we consider the function
\[
\eta(s) = \frac{f(s)}{\sinh s}.
\]
It is sufficient to verify
\[
\frac{f(s)}{\sinh s} \leq \frac{f(r)}{\sinh r} \quad \text{or} \quad \eta'(s) \geq 0.
\tag{1.4}
\]
Since we have
\[
\eta'(s) = f'(s) \sinh s - f(s) \cosh s \quad \text{or} \quad \frac{f'(s)}{[\sinh s]^2} = \frac{f''(s)}{|J|^2} \geq f(s),
\tag{1.6}
\]
we have \( f''(s) = |J(s)|'' \) and
\[
[f'(s) \sinh s - f(s) \cosh s]' = f''(s) \sinh s - f(s) \sinh s \geq 0.
\tag{1.5}
\]
Recall that the curvature tensor \( R \) is given by
\[
R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]}Z
\]
where \([X,Y] = XY - YX\) is the Lie bracket of \( X \) and \( Y \).

Following a calculation in [BGS], by our assumption of \( \sec_M \leq -1 \) we have
\[
f''(s) = |J(s)|'' \geq f(s),
\]
where we used the assumption that \( \langle J'', J \rangle = -\langle R(\sigma', J)\sigma', J \rangle \geq |J|^2 \).

It follows from (1.5)-(1.6) that (1.4) holds. This completes the proof of (1.3) as well as Lemma 1.1. \( \square \)

Recall that if \( \alpha \) is a \( k \)-form and \( Z \) is a vector field, then \( (\alpha|Z) \) is the \((k-1)\)-form given by
\[
(\alpha|Z)(\xi_1, \cdots, \xi_{k-1}) = \alpha(Z, \xi_1, \cdots, \xi_{k-1}).
\]

For the sake of completeness we give a proof of the following elementary result.

\textbf{Lemma 1.2.} Let \( \Psi \) be a closed \( k \)-form in \( \mathbb{R}^m \). Then the \((k-1)\)-form \( \Phi \) defined by
\[
\Phi(x) = r\int_0^1 [(\tau_t)^*(\Psi \frac{\partial}{\partial r})](x)dt
\]
satisfies \( d\Phi = \Psi \); here \( \frac{\partial}{\partial r} = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} r = (\sum_{i=1}^{m} x_i^2)^{1/2} \) and \( \tau_t(x) = tx \).

**Proof.** By the standard proof of the Poincaré lemma ([SiT], p.130), \( \Phi \) can be taken to be

\[
\Phi(x) = \sum_{i_1 < \ldots < i_k} (-1)^{j-1} x_{i_j} \left( \int_0^1 t^{k-1} \Psi_{i_1 \ldots i_k} (tx) dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k},
\]

where \( \Psi = \sum_{i_1 < \ldots < i_k} \Psi_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \).

In particular, one has

\[
\Phi(x) = \sum_{i_1 < \ldots < i_k} \sum_{j=1}^{k} x_{i_j} \left( \int_0^1 t^{k-1} \Psi_{i_1 \ldots i_k} (tx) dt \right) \frac{\partial}{\partial x_{i_j}} \left( \int_0^1 t^{k-1} \Psi_{i_1 \ldots i_k} (tx) dt \right) dx_{i_1} \wedge \cdots \wedge dx_{i_k},
\]

as desired. \( \square \)

We would also like to borrow another elementary but useful observation of Gromov, in order to prove our main theorem

**Lemma 1.3.** (Gromov, [Cha, page 124]) Suppose that \( f \) and \( h \) are positive integrable functions, of real variable \( r \), for which

\[
\frac{f}{g}
\]

is an increasing with respect to \( r \). Then the function

\[
\frac{\int_0^r f}{\int_0^r g}
\]

is also increasing with respect to \( r \geq 0 \).

Let us now provide a new detailed proof of a theorem of Gromov.

**Theorem 1.4.** (Gromov) Let \( M^m \) be a simply-connected complete Riemannian manifold with negative sectional curvature \( \leq -1 \). Suppose that
α is bounded closed k-form with \( k \geq 2 \). There is a bounded \((k - 1)\)-form \( \beta \) with \( d\beta = \alpha \) satisfying
\[
\|\beta\|_{L^\infty} \leq \frac{1}{k - 1} \|\alpha\|_{L^\infty}.
\] (1.7)

**Proof.** Let \((x_1, \ldots, x_n)\) be Euclidean coordinates in \(T_pM\) and consider the pull-back metric \( h \) of the metric \( g \) under \( \exp_p : T_pM \to M \). Observe that there are now two ways to interpret the map \( \tau_t \). The first interpretation comes from Lemma 1.1 with \((M, g)\) being replaced by \((T_pM, h)\); alternatively, one can think of \( \tau_t \) as the self-map of \( T_pM \), \((x_1, \ldots, x_n) \mapsto t(x_1, \ldots, x_n)\), that appears in the Poincaré lemma (Lemma 1.2). It is an easy and yet basic observation that these two ways of thinking about \( \tau_t \) give rise to the same map.

We may also replace the form \( \alpha \) that appears in the statement of Lemma 1.2 by a closed form \( \Psi \) on \( T_pM \) which is bounded in the induced metric \( h \). Let \( \Phi \) be given by Lemma 1.2 and observe that, by Lemma 1.1,
\[
|(\tau_t)^* \varphi(x)|_h \leq \left( \frac{\sinh tr}{\sinh r} \right)^{k-1} |\varphi(\tau_t(x))|_h, \quad k \geq 2,
\] (1.8)
holds for any \((k - 1)\)-form \( \varphi \) on \( T_pM \); here \( |\cdot|_h \) is any one of the equivalent norms induced by \( h \). Since \( |\frac{\partial}{\partial r}| = 1 \), it follows from (1.3) and Lemma 1.2 that
\[
|\Phi(x)|_h \leq r \int_0^1 \left( \frac{\sinh tr}{\sinh r} \right)^{k-1} |\Psi(tx)|_{\frac{\partial}{\partial r}} |x|_h \, dt
\]
\[
= \int_0^r \left( \frac{\sinh s}{\sinh r} \right)^{k-1} |\Psi(t \cdot)|_{\frac{\partial}{\partial r}} |x|_h \, ds
\]
\[
\leq \int_0^r \left( \frac{\sinh s}{\sinh r} \right)^{k-1} ds \sup_{0 \leq s \leq r} |\Psi(s \frac{\partial}{\partial r})|_h
\]
(1.9)

Choosing \( f(r) = (\sinh r)^{k-1} \) and \( \hat{g}(r) = (k - 1)(\sinh r)^{k-2} \cosh r \) in Lemma 1.3, we see that \( \int_0^r \frac{f'(s)}{\hat{g}(s)} ds \to \frac{1}{(k - 1)(\sinh r)^2} > 0 \) and
\[
\int_0^r \frac{(\sinh s)^{k-1} ds}{(\sinh r)^{k-1}} \leq \frac{1}{k - 1}.
\] (1.10)

It follows from (1.9)-(1.10) that
\[
|\Phi(x)|_h \leq \frac{1}{k - 1} \sup |\Psi|_h.
\] (1.11)
Hence $\Phi$ is a bounded solution of $d\Phi = \Psi$ and the proof of Theorem 1.4 is completed. \hfill $\square$

**Proof of Main Theorem:**

Our main theorem (Theorem 0.3.) can be derived as follows. We fix a base point $p$ as above. There is a differential structure $\Xi_p$ imposed on $S_\infty(M)$ given by the map

$$ F_p : B_1(0) \to M \cup S_\infty(M) $$

$$ \vec{v} \to \text{Exp}_p[\vec{v} / (1 - |\vec{v}|)]. $$

For $p \neq q$, the transitive map $F_q^{-1} \circ F_p : \overline{B_1(0)} \to \overline{B_1(0)}$ is not necessarily smooth. However, we fix one differential structure $\Xi_p$ on $S_\infty(M)$ via the map $F_p$.

Let $J$ be the complex structure of our Kähler manifold $M$. Let $r(x) = d(x, p)$ and $\beta = J \circ dr$, i.e., $\beta(\vec{w}) = dr(J\vec{w})$ for all $\vec{w} \in T_x(M)$. When $-a^2 \leq \sec M - 1$, it is known that

$$ |X|^2 \leq |(\nabla_X dr)(X)| = |\text{Hess}(r)(X, X)| \leq a|X|^2 $$

for all $X \in T_x(\partial B_r(p))$ with $r >> 1$.

Since $M$ is Kähler, we have $\nabla_X J = 0$. It follows that $|\nabla_X \beta| \leq a|X|$ for $X \in T_x(\partial B_r(p))$ with $r >> 1$.

Thus, $\{\beta|_{\partial B_r(p)}\}$ defines an equi-continuous family of contact forms on $S_\infty(M)$. By Ascoli lemma, there is a subsequence to converge to a bounded contact form $\beta_\infty$ on $S_\infty(M)$. Since $\sec M \leq -1$, it is known that $d\beta(\vec{X}, \vec{X}) = \text{Hess}(r)(X, X) + \text{Hess}(r)(JX, JX) \geq 2|X|^2$ for all $X \in T_x(\partial B_r(p))$ and $X \perp \nabla r$, where $\vec{X} = \frac{\vec{X}}{\sqrt{2}}[X - \sqrt{-1}JX]$. Therefore, $\beta_\infty$ defines a non-trivial contact form on $S_\infty(M)$. Moreover, $\omega_\infty = d\beta_\infty$ gives rise to a pseudo-hermitian metric on $S_\infty(M)$.

Similarly, one can also choose $\beta^*$ satisfying $d\beta^* = \omega$, where $\omega$ is the Kähler form of $M$ and $\beta^*$ in the proof of Theorem 1.4. With extra efforts, one can show that $|\nabla \beta^*| \leq c_1$ for some constant $c_1$. Thus, $\{\beta^*|_{\partial B_r(p)}\}$ defines an equi-continuous family of contact forms on $S_\infty(M)$ as well.

This completes the proof of our main theorem.

2. The modified Calabi-Yau problems for singular spaces and CR-manifolds

In this section, we will discuss the generalized Calabi problems on Kähler manifolds with boundaries. In addition, we will comment on
the existence of positive sup-harmonic functions on (possibly singular) Alexandrov spaces with non-negative sectional curvature.

§A. Sup-harmonic functions on Alexandrov spaces with non-negative sectional curvature

Professor S. T. Yau also had earlier results on bounded harmonic functions on smooth complete Riemannian manifolds with non-negative Ricci curvature. We would like to extend this theorem of Yau to singular spaces.

In an important paper [Per1], Perelman provided an affirmative solution to the Cheeger-Gromoll soul conjecture. More precisely, he showed that “if a smooth complete non-compact Riemannian manifold $M^n$ of non-negative curvature has a point $p_0$ with strictly positive curvature $K|_{p_0} > 0$, then $M^n$ must be diffeomorphic to $\mathbb{R}^n$. In [Per1], Perelman also asked to what extent the conclusions of his paper [Per1] would hold for (possibly singular) Alexandrov spaces with non-negative curvature.

Recently, the first author together with Dai and Mei showed the following.

**Theorem A.1.** (Cao-Dai-Mei, 2007, [CaMD1]) Let $M^n$ be an $n$-dimensional complete, non-compact Alexandrov space with non-negative sectional curvature. Suppose that $M^n$ has no boundary and $M^n$ has positive sectional curvature on an non-empty open set. Then $M^n$ is contractible.

In 1976, Professor S. T. Yau proved the following Liouville type theorem.

**Theorem A.2.** (Yau, 1976 [Y3]) Let $M^n$ be an $n$-dimensional complete, non-compact smooth Riemannian space with non-negative Ricci curvature. Then any positive harmonic functions on $M^n$ must be a constant function.

On an (possibly singular) Alexandrov space, we introduce the following notion of sup-harmonic function.

**Definition 0.1.** Definition A.3 Let $M^n$ be an $n$-dimensional complete, non-compact Alexandrov space with non-negative sectional curvature. Suppose that $M^n$ has no boundary, $f : M^n \to \mathbb{R}$ is a Lipschitz continuous function and

$$f(x) \geq \frac{1}{\text{Area}(\partial B_{\varepsilon}(x))} \int_{\partial B_{\varepsilon}(x)} f dA$$  \hfill (A.1)

for any sufficiently small $\varepsilon > 0$. Then we say that $f$ is a sup-harmonic function on $M$. 
For example, \( f(x) = -[d(x, x_0)]^2 \) is a sup-harmonic function on \( M \), whenever \( M \) has non-negative sectional curvature in generalized sense.

**Problem A.4. (Liouville-Yau type problem)** Let \( M^n \) be a \( n \)-dimensional complete, non-compact Alexandrov space with non-negative sectional curvature. Suppose that \( M^n \) has no boundary. Is it true that any positive sup-harmonic functions on \( M^n \) must be a constant function?

In [CaB], the first author and Benjamini studied a different Liouville-type problem of Schoen-Yau. One hopes to continue to work on Liouville-Yau type problem mentioned above.

§B. The generalized Calabi problems for Kähler domains with boundaries

The classical Calabi problems for Ricci curvatures on compact Kähler manifolds **without boundaries** have been successfully solved by Professor S. T. Yau.

**Theorem B.1. (Yau [Y4])** Let \( M^{2n} \) be a compact smooth Kähler manifold without boundary. Then the following is true: (1) For any Kähler form \( \omega_0 \in H^{(1,1)}(M^{2n}) \) and any \((1,1)\)-form \( \beta \) representing the first Chern class \( c_1(M^{2n}) \), there is a Kähler metric \( \tilde{\omega} = \omega_0 + i\partial\bar{\partial}f \) such that its Ricci curvature tensor satisfies

\[
\text{Ric}_{\tilde{\omega}} = \beta;
\]

(2) If the first Chern class \( c_1(M) \leq 0 \), then \( M^{2n} \) admits a Kähler-Einstein metric.

For a Kähler manifold \( \Omega \) with boundary \( M^{2n-1} = \partial \Omega \), we consider a similar problem. This problem is closely related to the existence problem of CR-Einstein metrics, or partially Einstein metrics.

**Definition B.2. (CR-Einstein metrics or partially Einstein metrics, [Lee2])** Let \( \Sigma^{2n-1} \) be a CR-hypersurface with CR-distribution \( \mathcal{H}_{\Sigma^{2n-1}} = \ker \theta \) for some contact 1-form \( \theta \) and let \( g_\theta(X, JY) = d\theta(X, JY) \) be a pseudo-hermitian metric as above. If the Ricci tensor of \( g_\theta \) satisfies

\[
\text{Ric}_{g_\theta}(X, Y) = cg_\theta(X, Y)
\]

for all \( X, Y \in \mathcal{H}_{\Sigma^{2n-1}} = \ker \theta \) where \( c \) is a constant, then \( g_\theta \) is called a CR-Einstein (partially Einstein) metric.

Inspired by Yau’s result, Lee proposed to study the CR-version of the Calabi problem.

**Problem B.3. (CR-Calabi Problems, [Lee2])** Let \( M^{2n-1} \) be a CR-manifold, \( \Phi \) be a closed form representing the first Chern class for the
bundle $T^{(1,0)}(M^{2n-1})$ and $\Phi_b(X, Y) = \Phi(X, Y)$ for $X, Y \in H_{\Sigma^{2n-1}} = \ker \theta$.

(1) Can we find a pseudo-metric $g_{\theta}$ such that its Ricci tensor satisfies

$$\text{Ric}_{g_{\theta}}(X, Y) = \Phi_b(X, Y)$$

for all $X, Y \in H_{\Sigma^{2n-1}} = \ker \theta$?

(2) Given a $(1, 1)$-form $\beta_b \in [c_1(M^{2n-2})]_b$, can we find a pseudo-metric $g_{\theta}$ such that its Ricci tensor satisfies

$$\text{Ric}_{g_{\theta}}(X, Y) = \beta(X, Y)$$

for all $X, Y \in H_{\Sigma^{2n-1}} = \ker \theta$?

The pseudo-Hermitian metric for general CR-manifolds was also discussed in [Ta1-2] and [Web]. Authors derived the following partial answer to Problem 3:

**Problem B.4.** ([CaCh]) Let $M^{2n-1}$ be the smooth boundary of a bounded strongly pseudo-convex domain $\Omega$ in a complete Stein manifold $V^{2n}$. Then for $n \geq 3$, $M^{2n-1}$ admits a CR-Einstein metric (or partially Einstein metric).

One might be able to continue working on Problem B.3, using Kohn-Rossi’s $\bar{\partial}_b$-theory described below.

§C. The Calabi-Escobar type problem for Kähler domains with boundaries

The first author and Mei-Chi Shaw studied the CR-version of the Poincaré-Lelong equation $i\bar{\partial}_b\bar{\partial}_b u = \Psi_b$ in [CaS3]. The linearization equation for (B.2) is related to the CR-version of Poincaré-Lelong equation.

In fact, to solve the linear function

$$\bar{\partial}_b u = \beta_b \text{ on } b\Omega,$$  \hspace{1cm} (C.1)

Kohn and Rossi [KoRo] used the solutions to the $\bar{\partial}$-Cauchy problem to solve $\bar{\partial}_b u = \beta_b$ extrinsically as follows. Let us first choose an arbitrary smooth extension $\hat{\beta}$ on $\Omega$. If we can solve

$$\begin{cases}
\bar{\partial} v = \bar{\partial} \hat{\beta} \text{ on } \Omega \\
v_{\|X} = 0, \text{ for } X \in T_z^{(0,1)}(b\Omega)
\end{cases}$$

Clearly $\tilde{\beta} = \hat{\beta} - v$ is a $\bar{\partial}$-closed extension on $\Omega$ of $\beta$. If we solve

$$\bar{\partial} \tilde{u} = \tilde{\beta} - v \text{ on } (\Omega \cup b\Omega),$$  \hspace{1cm} (C.2)

then the restriction $u = \tilde{u}|_{\Omega}$ satisfies

$$\bar{\partial}_b [\tilde{u}]_{\Omega} = \beta_b \text{ on } b\Omega.$$

(Such a solution for $\bar{\partial}_b u = \beta_b$ on $b\Omega$ is unique up to $\bar{\partial}_b$-closed extensions.)
The details for solving the $\bar{\partial}$-Cauchy problem (C.2) was given in Chapter 9 of [ChSh].

In 1992, Escobar [Esc] was able to solve the non-linear curvature equation on manifolds with boundary.

**Theorem C.1.** (Escobar [Esc]) Let $\Omega \subset \mathbb{R}^n$ be a compact domain with smooth boundary $\partial \Omega$ and $n > 6$. Then there is a conformally flat metric $g$ on $\Omega$ such that the scalar curvature $\text{Scal}_g$ of $g$ is zero and the mean curvature $H_g$ of $(\partial \Omega, g)$ is constant:

$$\begin{cases}
\text{Scal}_g = 0 \text{ on } \Omega \\
H_g = c \text{ on } \partial \Omega,
\end{cases}$$

for some constant $c$.

Inspired by Theorem C.1 and the Kohn-Rossi’s solution to $\bar{\partial}$-Cauchy problem, we are interested in the following type.

**Problem C.2.** (Calabi-Escobar type problem) Let $\Omega$ be a compact domain in Stein manifold $M$ with smooth strongly pseudo convex boundary $b\Omega$, and let $H^{CR}_g$ be the partial sum of second fundamental form of $(b\Omega, g)$ over the CR-distribution $\ker \theta$ of $b\Omega$. Is there an Kähler-Einstein metric $g$ on $\Omega$ with constant CR-mean curvature on the boundary $b\Omega$? In another words, we would like to find the existence of solution to the following non-linear boundary problem:

$$\begin{cases}
\text{Ric}_g = c_1 g \text{ on } \Omega \\
H^{CR}_g = c_2 \text{ on } b\Omega
\end{cases}$$

for some constant numbers $c_1$ and $c_2$.

The linearalization of non-linear equation is the Poincaré-Lelong equation with boundary conditions. The first author and Mei-Chi Shaw [CaS] were able to solve

$$i\partial_b \bar{\partial}_b u = \Theta_b \text{ on } b\Omega$$

even for weakly pseudo-convex domains $\Omega$ in $\mathbb{C}P^n$.

One hopes to continue to work in direction, in order to investigate Problem C.2.

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