Partial Supersymmetry Breaking
and
$\mathcal{N} = 2 \ U(N_c)$ Gauge Model with Hypermultiplets in Harmonic Superspace

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Abstract

We provide a manifestly $\mathcal{N} = 2$ supersymmetric formulation of the $\mathcal{N} = 2$ $U(N_c)$ gauge model constructed in terms of $\mathcal{N} = 1$ superfields in hep-th/0409060. The model is composed of $\mathcal{N} = 2$ vector multiplets in harmonic superspace and can be viewed as the $\mathcal{N} = 2 \ U(N_c)$ Yang-Mills effective action equipped with the electric and magnetic Fayet-Iliopoulos terms. We generalize this gauge model to an $\mathcal{N} = 2 \ U(N_c)$ QCD model by introducing $\mathcal{N} = 2$ hypermultiplets in harmonic superspace which include both the fundamental representation of $U(N_c)$ and the adjoint representation of $U(N_c)$. The effect of the magnetic Fayet-Iliopoulos term is to shift the auxiliary field by an imaginary constant. Examining vacua of the model, we show that $\mathcal{N} = 2$ supersymmetry is spontaneously broken down to $\mathcal{N} = 1$.

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1 Introduction

It is widely appreciated that $\mathcal{N} = 2$ supersymmetry imposes a strong constraint on the four-dimensional theory but yet leaves rich physical ingredients. For example, $\mathcal{N} = 2$ supersymmetric field theories develop controllable quantum effects \[1\]. Unconstrained $\mathcal{N} = 2$ superfields provide a manifestly $\mathcal{N} = 2$ supersymmetric formulation of them.

There are two types of unconstrained $\mathcal{N} = 2$ vector superfields. First type, developed in \[2\] for the abelian case and in \[3\] \[4\] for the non-abelian case, is constructed on the usual $\mathcal{N} = 2$ superspace $\mathbb{R}^{4|8}$ parametrized by $(x^m, \theta^i, \bar{\theta}^i)$. The second type is constructed on harmonic superspace $\mathbb{R}^{4|8} \times S^2$ developed in \[5\] \[6\] \[7\] \[8\] (see \[9\] for introduction), and parametrized by

$$
(x^m_A, \theta^\pm, \bar{\theta}^\pm, u^\pm_i) = (x^m - 2i\theta^i\sigma^m\bar{\theta}^j u^+_i u^-_j, \theta^i u^+_i, \bar{\theta}^i u^-_i, u^\pm_i)
$$ (1.1)

in the analytic basis. The $S^2 = SU(2)/U(1)$ is parametrized by harmonic variables $u^\pm_i$

$$
(u^+_i, u^-_i) \in SU(2), \quad u^+_i = \varepsilon_{ij} u^{+j}, \quad u^{+i} u^{-i} = 1, \quad \overline{u^{+i}} = u^{-i}. \quad (1.2)
$$

For $\mathcal{N} = 2$ hypermultiplets, harmonic superspace makes it possible to construct the off-shell $\mathcal{N} = 2$ unconstrained hypermultiplets, the $q^+$- and $\omega$-hypermultiplets. Thus harmonic superspace provides a manifestly $\mathcal{N} = 2$ supersymmetric formulation of $\mathcal{N} = 2$ supersymmetric theories in terms of off-shell $\mathcal{N} = 2$ unconstrained superfields. This property is very useful in the quantum calculations \[10\] of supersymmetric models. In the present paper, $\mathcal{N} = 2$ superfields in harmonic superspace are utilized in the construction of $\mathcal{N} = 2 \ U(N_c)$ gauge model coupled with $\mathcal{N} = 2$ hypermultiplets.

In \[11\], the $\mathcal{N} = 2 \ U(N_c)$ gauge model is constructed in terms of $\mathcal{N} = 1$ vector and chiral superfields in the adjoint representation of the gauge group $U(N_c)$. This model is a non-abelian generalization of the $U(1)$ gauge model with abelian constrained $\mathcal{N} = 2$ vector superfields \[13\] constructed by Antoniadis, Partouche and Taylor (APT) in \[12\], and can be regarded as $\mathcal{N} = 2 \ U(N_c)$ Yang-Mills (YM) low-energy effective action (LEEA) specified by a holomorphic function $\mathcal{F}$ and equipped with the electric and magnetic Fayet-Iliopoulos (FI) terms. In \[11\] \[14\] \[15\], the vacua of the model are examined. The $\mathcal{N} = 2$ and $U(N_c)$ symmetry are spontaneously broken to $\mathcal{N} = 1$ and $\prod^n_{n=1} U(N_n)$ symmetry with \[ \sum^n_{n=1} N_n = N_c, \] respectively. The associated Nambu-Goldstone (NG) fermion is shown to be provided by the overall $U(1)$ part in $U(N_c)$. In addition, the spectrum on the vacua is
completely clarified in \cite{15}.

The key for the partial supersymmetry breaking is the modification of the local version of the extended supersymmetry algebra by an additional spacetime independent term \cite{18, 19}. This modification is caused by the magnetic FI term. An important observation made in the study of the APT model \cite{12, 19, 20, 21} is that the effect of the magnetic FI term is to shift the auxiliary field in the $\mathcal{N} = 2$ vector superfield by an imaginary constant (while that of the electric FI term is to shift the symplectic dual auxiliary field by an imaginary constant). This observation is shown to be useful in this paper in the construction of the magnetic FI term for $\mathcal{N} = 2$ $U(N_c)$ gauge model with/without hypermultiplets.

In this paper, we provide a manifestly $\mathcal{N} = 2$ supersymmetric formulation of the $\mathcal{N} = 2$ $U(N_c)$ gauge model given in \cite{11}. For this purpose we employ $\mathcal{N} = 2$ vector multiplets $V^{++}$ in harmonic superspace. The magnetic FI term is introduced so as to shift the auxiliary field in $V^{++}$ by an imaginary constant. This causes $\mathcal{N} = 2$ supersymmetry to be broken spontaneously to $\mathcal{N} = 1$. In addition, we generalize this gauge model to an $\mathcal{N} = 2$ $U(N_c)$ QCD model coupled with $\mathcal{N} = 2$ hypermultiplets, $q^+$ and $\omega$, in harmonic superspace which include both the fundamental representation of $U(N_c)$ and the adjoint representation of $U(N_c)$. We determine the form of the magnetic FI term such that it shifts the auxiliary field in $V^{++}$ by an imaginary constant. Examining vacua of the model, we show that this model also describes partial supersymmetry breaking. It should be noted that the magnetic FI term of the $U(N_c)$ gauge model coupled with hypermultiplets in the adjoint representation is the same as that of the $U(N_c)$ gauge model without hypermultiplets. However in presence of hypermultiplets in the fundamental representation the magnetic FI term develops an additional term. This additional term overcomes the difficulty \cite{20, 21} in coupling fundamental hypermultiplets to the APT model.

This paper is organized as follows. In section 2, we introduce $\mathcal{N} = 2$ vector multiplets $V^{++}$ in harmonic superspace and construct an $\mathcal{N} = 2$ $U(N_c)$ gauge model equipped with the electric and magnetic FI terms. The vacua of the model are examined in section 3. We show that the model describes spontaneous partial supersymmetry breaking. Introducing $\mathcal{N} = 2$ hypermultiplets, $q^+$ and $\omega$, in harmonic superspace we generalize the model to the $\mathcal{N} = 2$ $U(N_c)$ QCD model equipped with the electric and magnetic FI terms in section 4. We introduce the magnetic FI term so as to shift the auxiliary field by an imaginary constant. In section 5, we show that owing to this property, $\mathcal{N} = 2$ supersymmetry is

$^3$This series of works \cite{11, 14, 15} is based on $\mathcal{N} = 1$ superspace and construction of most general $\mathcal{N} = 2$ Lagrangian based on special Kähler geometry \cite{16} which were developed after tensor calculus \cite{17}.
broken down to $\mathcal{N} = 1$ spontaneously. The supersymmetry transformation law of the components in the vector multiplet $V^{++}$ and the hypermultiplets, $q^+$ and $\omega$, is found in appendix B. In this paper, we follow the notation for harmonic superspace given in [9] (see appendix A) and one for spinors given in [23].

2 $\mathcal{N} = 2 \ U(N_c)$ gauge model

We introduce a set of $\mathcal{N} = 2$ vector superfields $V^{++} = V^{++} a t_a$ where $N_c \times N_c$ hermitian matrices $t_a$ ($a = 0, 1, \ldots, N_c^2 - 1$) generate $u(N_c)$, $[t_a, t_b] = i f^a_{bc} t_a$, and $t_0$ represents the overall $u(1)$ generator. $V^{++}$ is the analytic superfield satisfying $D^+ V^{++} = D^+ V^{++} = 0$. In the analytic basis $(x_A, \theta^\pm, \bar{\theta}^\pm, u_i^\pm)$ in [11], $D^\pm$ and $\bar{D}^\pm$ are given as

$$D_\alpha^+ = \frac{\partial}{\partial \theta^- \alpha}, \quad D_\alpha^- = -\frac{\partial}{\partial \theta^+ \alpha} + 2i (\sigma^m \bar{\theta}^-)_\alpha \frac{\partial}{\partial x^m A},$$

$$\bar{D}_\dot{\alpha}^+ = \frac{\partial}{\partial \bar{\theta}^- \dot{\alpha}}, \quad \bar{D}_\dot{\alpha}^- = -\frac{\partial}{\partial \bar{\theta}^+ \dot{\alpha}} - 2i (\bar{\theta}^\pm \sigma^m)_\dot{\alpha} \frac{\partial}{\partial x^m A}, \quad (2.1)$$

and thus $V^{++}$ is a superfield in analytic subspace $(x_A, \theta^+, \bar{\theta}^+, u_i^\pm)$. In the Wess-Zumino (WZ) gauge $V^{++}$ is given as

$$V^{++} = -2i \bar{\theta}^+ \sigma^m \bar{\theta}^+ v_m (x_A) - i \sqrt{2} (\theta^+)^2 \bar{\phi}(x_A) + i \sqrt{2} (\bar{\theta}^+)^2 \phi(x_A) + 4 (\bar{\theta}^+)^2 \theta^+ \lambda^i (x_A) u_i^- - 4 (\theta^+)^2 \bar{\theta}^+ \bar{\lambda}^i (x_A) u_i^- + 3 (\theta^+)^2 (\bar{\theta}^+)^2 D^{ij} (x_A) u_i^- u_j^- , \quad (2.2)$$

where $v_m$, $\phi$, $\lambda^i$ and $D^{ij}$ are vector, complex scalar, $SU(2)$ doublet Weyl spinor and auxiliary field, respectively. $D^{ij}$ is symmetric with respect to $(i, j)$ so that $D^i_{ij} = \varepsilon_{ijk} D^{jk}$ is an $SU(2)$ matrix, $D^i_{ij} = i D^A (\tau_A^i)$ $ij$. The reality $V^{++} = \bar{V}^{++}$, where the tilde “$\sim$” means the analyticity preserving conjugation $9$ (see appendix A), implies that $D^{ij} = \bar{D}^{ij}$ because $\bar{D}^{ij} u_i^- u_j^- = \bar{D}^{ij} u_i^- u_j^- = \bar{D}^{ij} u_i^- u_j^-$, and that the $D^A$ is a real three vector $\bar{D}^A = D^A$.

The field strength $\bar{W}$ is given by

$$\bar{W} = -\frac{1}{4} (D^+)^2 \sum_{n=1}^{\infty} \int dv_1 \cdots dv_n (-i)^{n+1} \frac{V^{++}(v_1) \cdots V^{++}(v_n)}{(u^1 v_1^+)(v_1^+ v_2^+)(v_2^+ v_3^+)(v_3^+ u_+^n)} \quad . \quad (2.3)$$

It is straightforward to derive the following expression by using formulas given in [9] (see also [6]):

$$\bar{W} = -i \sqrt{2} \phi - 2 \bar{\theta}^i \lambda_i + \bar{\theta}^i \bar{\theta}^j D_{ij} + \bar{\theta}^i \sigma^{mn} \bar{\theta}^m v_{mn} + \frac{4}{3} i (\bar{\theta}^i \bar{\theta}^j) D_m \lambda_i \sigma^m \bar{\theta}_j - \frac{2}{3} \sqrt{2} (\bar{\theta}^i \bar{\theta}^j) [\phi, \bar{\phi} \lambda_j] + i \sqrt{2} (\bar{\theta}^i)^4 \eta^{mn} D_m \phi + i (\bar{\theta}^i)^4 \varepsilon_{ij} \lambda^i \lambda^j + i \sqrt{2} (\bar{\theta}^i)^4 [\phi, [\psi, \bar{\phi}]] - 2 i \bar{\theta}^+ \bar{\theta}^- [\phi, \bar{\phi}] + \cdots \quad (2.4)$$
where

\[ v_{mn} = \partial_m v_n - \partial_n v_m + i[v_m, v_n], \]
\[ D_m \phi = \partial_m \phi + i[v_m, \phi], \quad D_m \lambda = \partial_m \lambda + i[v_m, \lambda] \]

(2.5)

and ellipsis represents terms which do not contribute to the action.

We consider the action

\[ S_V = -\frac{i}{4} \int d^4x \left[ (D)^4 F(W) - (\bar{D})^4 \bar{F}(\bar{W}) \right], \]

(2.6)

where \( F \) is an analytic trace function of \( W = W^a t_a \) and \( (D)^4 = \frac{1}{16} (D^+)^2 (D^-)^2 \). The action \( S_V \)

reduces in component fields to

\[
S_V = \int d^4x \left[ -g_{ab} D^m \phi^a D^m \phi^b - \frac{1}{2} f_{ab} \lambda^i \sigma^m D_m \lambda^i + \frac{1}{2} F_{ab} |\lambda^i \sigma^m D_m \bar{\lambda}^b - \frac{i}{4} F_{abc} \lambda^i \lambda^c \bar{\lambda}^b + \frac{i}{4} F_{abc} |\lambda^i \sigma^mn \bar{\lambda}^b + \frac{i}{4} F_{abc} |\lambda^i \sigma^mn \bar{\lambda}^b - \frac{i}{4} F_{abc} |\lambda^i \sigma^mn \bar{\lambda}^b + \frac{i}{4} F_{abc} |\lambda^i \sigma^mn \bar{\lambda}^b - \frac{i}{4} F_{abc} |\lambda^i \sigma^mn \bar{\lambda}^b + \frac{i}{4} F_{abc} |\lambda^i \sigma^mn \bar{\lambda}^b \right. 

(2.7)

where \( F_{a1\ldots an} \equiv \partial^n F/\partial W^a_1 \ldots \partial W^a_n \) and \( g_{ab} \equiv \text{Im} F_{ab} \). \( F_{a1\ldots an} \) represents \( F_{ab\ldots} \) evaluated at \( \theta^\pm = \bar{\theta}^\pm = 0 \). We have used the relation

\[ f_{ba}^c F_c = -f_{bd}^c (i\sqrt{2}\phi^d) F_{ca} \]

(2.8)

which follows from the fact that \( \phi^a \) and \( F_{a|} \) transform as adjoint under \( U(N_c) \).

### 2.1 electric & magnetic FI terms

We introduce the electric FI term

\[
S_e = \int dud\zeta^{(-4)} \text{tr} (\Xi^{+} + V^{++}) + \text{c.c.} = \int d^4x \xi \epsilon_{ij} D_{ij}^0 + \text{c.c.},
\]

(2.9)

where \( d\zeta^{(-4)} = d^4x d^4t^+ \) and \( \Xi^{++} = \xi \epsilon_{ij} u^+_i u^+_j \) is the electric FI parameter. This term develops a constant imaginary part in the dual auxiliary field \( D_D^{aij} \) of \( W^a_D \equiv F_a \). To see
this we derive the equation of motion for $V^{++}$ from $S_V + S_e$. Rewriting $S_V$ as an integral over the analytic subspace as was done in [23] for the bare theory in which $F$ is quadratic, and then varying with respect to $V^{++}$, we obtain

$$\frac{i}{16}(D^+)^2 F_a + \Xi^{++} \delta_a^0 + c.c. = 0 .$$

(2.10)

This may be rewritten as

$$(D^+)^2 (F_a + 8i \xi^{ij} \theta_i \theta_j \delta_a^0) - (D^+)^2 (\bar{F}_a - 8i \xi^{ij} \bar{\theta}_i \bar{\theta}_j \delta_a^0) = 0 .$$

(2.11)

Because $F_a = W_a^D = \theta^i \theta^j D_{ij}^a + \cdots$, the effect of the electric FI term is to shift the dual auxiliary field by an imaginary constant, $D_{ij}^a \to D_{ij}^a + 8i \xi^{ij} \delta_a^0$.

Next we introduce the magnetic FI term of the form

$$S_{YM}^m = \int d^4x (D^4)^4 \xi^{ij} \theta_i \theta_j \left( F_0 + \frac{1}{2} F_{00} 4i \xi^{kl} \theta_k \theta_l \right) + c.c.$$

$$= \int d^4x \left[ \xi^{ij} \theta_i \theta_j (F_{0a} D_{ij}^a - F_{0ab} |\lambda^a_i \lambda^b_j|) + 2i F_{00} [\xi^{ij} \theta_i \theta_j D_{ij}^a] \right] + c.c. .$$

(2.12)

We note that this term reduces to $S_e + \text{const.}$ for the bare theory. The effect of this term is to shift the auxiliary field $D_{ij}^a$ by an imaginary constant. This can be seen as follows.

Observe that $S_{YM}^m$ and the $D$-dependent terms in $S_V$ may be rewritten as

$$- \frac{i}{4} \int d^4x (D^4) \left[ \frac{1}{2} F_a |\theta^i \theta^j D_{ij}^a| + \frac{1}{2} F_{ab} |\theta^i \theta^j D_{ij}^a - 2\theta^k \lambda^b_k + \theta^k \sigma^mn \theta_{km} \theta_{bn}| \right]$$

$$+ \frac{1}{6} F_{abc} [3\theta^i \theta^j D_{ij}^a (-2\theta^l \lambda^b_l) (-2\theta^l \lambda^c_l)] + c.c. .$$

(2.13)

where

$$D_{ij}^a = D_{ij}^a + 4i \xi^{ij} \delta^a_0 , \quad \bar{D}_{ij}^a = D_{ij}^a - 4i \xi^{ij} \delta^a_0 .$$

(2.14)

This implies that

$$S_V + S_{YM}^m = - \frac{i}{4} \int d^4x (D^4) F(\hat{W}) + c.c. .$$

(2.15)

where $\hat{W}$ is $W$ with the replacement $D_{ij}^a \to \bar{D}_{ij}^a$ (similarly $\bar{W}$ is $\bar{W}$ with the replacement $D_{ij}^a \to \bar{D}_{ij}^a$). We note that due to this effect the supersymmetry transformation law

$$\delta_\eta \lambda^a_{ai} = (D_a)^i \eta^i + \cdots$$

(see appendix B) changes to

$$\delta_\eta \lambda^a_{ai} = (D_a)^i \eta^i + \cdots , \quad \delta_\eta \bar{\lambda}^{ai} = -(\bar{D}_a)^i \bar{\eta}^i + \cdots .$$

(2.16)
Gathering all together, the total action for the $U(N_c)$ gauge model is

$$S_{YM} = S_V + S_e + S_m^Y .$$  

(2.17)

The terms including the auxiliary field $D_{ij}^a$

$$\int d^4x \left[ \frac{1}{4} g_{ab} D^{a\ij} D_{ij}^b + (\xi^{ij} + \bar{\xi}^{ij}) D_{ij}^0 + (\xi_D^{ij} F_{0a} + \bar{\xi}_D^{ij} \bar{F}_{0a}) D_{ij}^c 
+ \frac{i}{4} F_{abc} \lambda^a \lambda_{bc} D_{ij}^c
- \frac{i}{4} F_{abc} |\bar{\lambda}^a \bar{\lambda}^b D_{ij}^c| \right]$$

(2.18)

lead to

$$D^{aij} = -2g^{ab} \left[ (\xi^{ij} + \bar{\xi}^{ij}) \delta^0_{ai} + \xi_D^{ij} F_{0b} + \bar{\xi}_D^{ij} \bar{F}_{0b} + \frac{i}{4} F_{bcd} \lambda^c \lambda^d_j - \frac{i}{4} \bar{F}_{bcd} |\bar{\lambda}^c \bar{\lambda}^d_j \right].$$  

(2.19)

By eliminating the auxiliary field using this equation, the action $S_{YM}$ becomes

$$S'_{YM} = \int d^4x \left[ \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{pot}} + \mathcal{L}_{\text{Pauli}} + \mathcal{L}_{\text{mass}} + \mathcal{L}_{4\text{fermi}} \right]$$

(2.20)

where

$$\mathcal{L}_{\text{kin}} = -g_{ab} D^a \phi^a D_m \phi^b - \frac{1}{2} F_{ab} |\bar{\lambda}^a \sigma^m D_m \lambda^b - \frac{1}{2} F_{ab} |\lambda^a \sigma^m D_m \bar{\lambda}^b - \frac{1}{4} g_{ab} v_{mn} v^{bmn} - \frac{1}{8} \text{Re} F_{ab} |\varepsilon^m n p q v_{mn} v^{bp} ,$$

(2.21)

$$\mathcal{L}_{\text{pot}} = -g_{ab} P^a P^b - \frac{1}{4} g^{ab} D^{aij} |\bar{D}_{ij}^a| + 2i \xi_D^{ij} F_{0b} - 2i \bar{\xi}_D^{ij} \bar{F}_{0b}$$

$$= -g_{ab} P^a P^b - \frac{1}{4} g^{ab} D^{aij} |\bar{D}_{ij}^a| - 2i (\xi^{ij} + \bar{\xi}^{ij}) (\xi_{Dij} - \bar{\xi}_{Dij}) ,$$

(2.22)

$$\mathcal{L}_{\text{Pauli}} = \frac{i}{4} (F_{abc} |\lambda^a \sigma^m \lambda^b - F_{abc} |\bar{\lambda}^a \sigma^m \bar{\lambda}^b) v_{mn}^c ,$$

(2.23)

$$\mathcal{L}_{\text{mass}} = \frac{1}{2} g_{ab} \left[ \lambda^a f_{cd} (i \sqrt{2} \phi^c) \lambda_d^b + \lambda^a f_{cd} (-i \sqrt{2} \bar{\phi}^c) \lambda_d^b - \frac{1}{2} g^{ab} (\xi^{ij} + \bar{\xi}^{ij}) \delta^0_{ai} + \xi_D^{ij} F_{0b} + \bar{\xi}_D^{ij} \bar{F}_{0b} ) (F_{bed} |\lambda^e \lambda_j^d - \bar{F}_{bed} |\lambda^e \bar{\lambda}_j^d$$

$$- \frac{1}{4} D^{aij} F_{abc} |\lambda_j^b - \xi_D^{ij} F_{0ab} |\lambda_j^b ,$$

$$= \frac{1}{2} g_{ab} \left[ \lambda^a f_{cd} (i \sqrt{2} \phi^c) \lambda_d^b + \lambda^a f_{cd} (-i \sqrt{2} \bar{\phi}^c) \lambda_d^b$$

$$+ \frac{i}{4} D^{aij} F_{abc} |\lambda_j^b - \xi_D^{ij} F_{0ab} |\lambda_j^b ,$$

(2.24)

$$\mathcal{L}_{4\text{fermi}} = -\frac{i}{12} F_{abc} |\lambda^a \lambda^b \lambda^c \lambda^d_j + \frac{i}{12} F_{abcd} |\lambda^a \lambda^b \lambda^c \lambda^d_j (F_{bej} |\lambda^e \lambda_j^f - \bar{F}_{bej} |\lambda^e \bar{\lambda}_j^f$$

(2.25)

and

$$\sqrt{2} P^a = -if^a_{be} \phi^b \phi^c = -ik_b^a \phi^b = +ik_a^b \phi^b ,$$

(2.26)
\[ D^{aij} = -2g^{ab} \left[ (\xi^{ij} + \bar{\xi}^{ij})\delta_b^0 + \xi^{ij}_D\mathcal{F}_{0b} + \bar{\xi}^{ij}_D\mathcal{F}_{0b} \right] \]  
(2.27)

\[ D^{aij} = D^{aij} + 4i\xi^{ij}_D\delta^0_b = -2g^{ab} \left[ (\xi^{ij} + \bar{\xi}^{ij})\delta_b^0 + (\xi^{ij}_D + \bar{\xi}^{ij}_D)\mathcal{F}_{0b} \right] \]  
(2.28)

\( P_a \) is the Killing potential for the Killing vector \( k_a = k^b\partial_b \) which generates \( U(N_c) \) isometry of the special Kähler geometry.

### 3 Vacua of \( \mathcal{N} = 2 \) \( U(N_c) \) gauge model

We examine vacua of the model defined by \( S_{YM} = S_V + S_e + S_{YM}^m \). The scalar potential \( \mathcal{V} = -\mathcal{L}_{pot} \) is

\[ \mathcal{V} = \frac{1}{4}g_{ab}D^{aij}\bar{D}_{ij} + g_{ab}P^aP^b + 2i(\xi^{ij} + \bar{\xi}^{ij})(\xi_{Di} - \bar{\xi}_{Di}) \]  
(3.1)

In the three-vector notation,

\[ \xi_j^i = i\xi^A(\tau_A)_j^i, \quad \xi_D^i = i\xi^A_D(\tau_A)_j^i, \quad D^{aij} = iD^{aA}(\tau_A)_j^i, \quad A = 1, 2, 3, \]

\[ D^{aij}\bar{D}_{ij} = -D^{aij}\bar{D}_{bj} = D^{aA}\bar{D}^{bA} \]

this is written as

\[ \mathcal{V} = \frac{1}{2}g_{ab}D^{aA}\bar{D}^{bA} + \frac{1}{2}g_{ab}P^aP^b + 4i(\xi^A + \bar{\xi}^A)(\xi^A_D - \bar{\xi}^A_D) \]

\[ = 2g^{ab} \left[ (\xi^A + \bar{\xi}^A)\delta^0_a + (\xi^A_D + \bar{\xi}^A_D)\mathcal{F}_{0a} \right]^2 + \frac{1}{2}g_{ab}P^aP^b + 4i(\xi^A + \bar{\xi}^A)(\xi^A_D - \bar{\xi}^A_D). \]  
(3.3)

The last term is a constant. We demand positive definiteness of \( g_{ab} \). In order to determine the vacuum, we examine \( \partial\mathcal{V}/\partial(W^a) = 0 \). The second term \( g_{ab}P^aP^b \) in \( \mathcal{V} \) tells us that \( \langle \phi^r \rangle = 0 \) where \( t_r \) represent non-Cartan generators and \( \langle \ast \rangle \) means the vacuum expectation value (vev) of \( \ast \). The vacuum condition is

\[ \langle \partial_a \mathcal{V} \rangle = \frac{i}{4}\langle \mathcal{F}_{abc} | D^{bA}D^{cA} \rangle = 0 \]  
(3.4)

where we note \( \langle D^{cA} \rangle = \langle D^{cA} \rangle \) because fermions do not acquire vevs. Let us examine the case with

\[ \mathcal{F} = \sum_n \frac{c_n}{n!} \text{tr}(W)^n \]

(3.5)

for concreteness. Let \( E_{\bar{i}i}, \bar{i} = 1, \ldots, N_c \) be the fundamental matrix which has 1 at the \((i, \bar{j})\)-component and 0 otherwise. Cartan generators may be written as \( t_i = E_{\bar{i}i} \). We
have $\langle \partial_i V \rangle = 0$, because $F_{\bar{r}i\bar{m}} = \langle D^r \rangle = 0$. Noting that the points specified by $\langle F_{ri} \rangle = 0$ correspond to unstable vacua, we derive the vacuum condition

$$
\sum_A \langle D^{iA} D^{\dagger A} \rangle = 0, \quad i = 1, \ldots, N_c.
$$

(3.6)

This is solved by

$$
\langle \phi \rangle = \text{diag}(a_1, \cdots, a_1, a_2, \cdots, a_2, \cdots),
$$

(3.7)

which means that the gauge symmetry $U(N_c)$ is broken down to $\prod_n U(N_n)$ with $\sum N_n = N_c$.

On the other hand, the supersymmetry transformation of $\lambda^{ai}$ reduces on the vacuum to

$$
\langle \delta_\eta \lambda^{\bar{i}\bar{j}} \rangle = \langle D^{\dagger j} \rangle \eta^j = i \langle D^{iA} \rangle (\tau_A)^{ij} \eta^j, \quad \langle \delta_\eta \lambda^r \rangle = \langle D^{ri} \rangle \eta^i = 0.
$$

(3.8)

A necessary condition for the partial supersymmetry breaking is

$$
\langle \det D^{\dagger j} \rangle = 2 \sum_A \langle D^{iA} D^{\dagger A} \rangle = 0
$$

for some $i$. This is obviously satisfied by the vacua (3.6). Let us look at the mass term for $\lambda^{ij}$

$$
- \frac{i}{4} \langle F_{\bar{m}I} \rangle \lambda^{\bar{i}} \langle D^{\dagger j} \rangle \lambda^{\bar{j}} = - \frac{i}{4} \langle F_{\bar{m}I} \rangle \langle D^{iA} \rangle \lambda^{\bar{i}} (\tau_2 \tau_A)^{ij} \lambda^{\bar{j}}.
$$

(3.10)

Because of (3.9), a half of the fermions $\lambda^{\bar{i}}$, $i = 1, 2$, is massless. In fact, diagonalizing the mass matrix by a matrix $U^{ij}$, we see that the supersymmetry transformation of the massless combination of $\lambda^{\bar{i}}$, say $U^1 \lambda^{\bar{i}}$, is non-trivial. In the ordinary hermitian matrices $t_a$, $a = 0, \cdots, N_c^2 - 1$, of $u(N_c)$, this means that $U^1 \lambda^{0j}$ is massless and $\delta_\eta (U^1 \lambda^{0j}) \neq 0$. Thus the Nambu-Goldstone fermion for partial supersymmetry breaking lies in the overall $U(1)$ part of the massless combination of $\lambda^{\bar{i}}$, $U^1 \lambda^{0j}$.

Let us make a comment on the relation between the present construction and that of [11]. The vacuum condition means that

$$
(2(\xi^A + \bar{\xi}^A) + (\xi^A_{\bar{D}} + \bar{\xi}^A_{\bar{D}}) \langle \text{Re} F_{\bar{m}I} \rangle)^2 = (\xi^A_{\bar{D}} + \bar{\xi}^A_{\bar{D}})^2 \langle g_{\bar{m}} \rangle^2,
$$

(3.11)

$$
(2(\xi^A + \bar{\xi}^A) + (\xi^A_{\bar{D}} + \bar{\xi}^A_{\bar{D}}) \langle \text{Re} F_{\bar{m}I} \rangle) (\xi^A_{\bar{D}} + \bar{\xi}^A_{\bar{D}}) \langle g_{\bar{m}} \rangle = 0.
$$

(3.12)

By using $SU(2)$, we may choose $(\xi^A_{\bar{D}} + \bar{\xi}^A_{\bar{D}}) = (0, -m, 0)$ with a real constant $m$ without loss of generality. Then (3.12) implies

$$
\langle \text{Re} F_{\bar{m}} \rangle = \frac{2}{m} (\xi^2 + \bar{\xi}^2) \equiv -2 \frac{c}{m}
$$

(3.13)
with a real constant \( e \). Substituting this into (3.11) we find

\[
\langle g_\mu \rangle = \pm \frac{2}{m} \sqrt{(\xi^1 + \bar{\xi}^1)^2 + (\xi^3 + \bar{\xi}^3)^2}.
\]  

(3.14)

By using \( U(1) \), residual rotational symmetry along 2-axis, we may choose \((\xi^1 + \bar{\xi}^1) = 0 \) and \((\xi^3 + \bar{\xi}^3) = \xi \) with a real constant \( \xi \) without loss of generality, so that

\[
\langle g_\mu \rangle = \pm 2 \frac{\xi}{m}.
\]  

(3.15)

Thus we solve the vacuum condition by \( \langle \mathcal{F}_\mu \rangle = -2(\frac{\xi}{m} \pm i \frac{\bar{\xi}}{m}) \). By fixing \( SU(2) \) appropriately we have managed to reproduce the conclusion of \([11]\).

4 \( \mathcal{N} = 2 U(N_c) \) Gauge model with Hypermultiplets

We generalize the \( \mathcal{N} = 2 U(N_c) \) gauge model to the \( \mathcal{N} = 2 U(N_c) \) QCD model coupled with the \( q^+ \)- and \( \omega \)-hypermultiplets. We consider both the fundamental representation of \( U(N_c) \) and the adjoint representation of \( U(N_c) \).

4.1 \( q^+ \) hypermultiplets

The \( q^+ \) hypermultiplet is an analytic superfield satisfying \( D^+ q^+ = \bar{D}^+ q^+ = 0 \), and can be expanded as

\[
q^+ = F^+(x_A, u) + \theta^+ \psi(x_A, u) + \bar{\theta}^+ \bar{\kappa}(x_A, u) + (\theta^+)^2 M^-(x_A, u) + (\bar{\theta}^+)^2 N^-(x_A, u)
\]

\[
+ i \theta^+ \sigma^m \bar{\theta}^+ A^m_\mu (x_A, u) + (\theta^+)^2 \bar{\theta}^+ \gamma^{(-2)}(x_A, u) + (\bar{\theta}^+)^2 \theta^+ \chi^{(-2)}(x_A, u)
\]

\[
+ (\theta^+)^2 (\bar{\theta}^+)^2 P^{(-3)}(x_A, u)
\]

(4.1)

in the analytic basis. The physical fields are \( SU(2)_A \) doublet complex scalars (to be denoted as \( f^i \)) contained in \( F^+ \) and a pair of \( SU(2)_A \) isosinglet spinors, \( \psi \) and \( \kappa \), where \( SU(2)_A \) is the automorphism of \( \mathcal{N} = 2 \).

We begin with a set of hypermultiplets \( q^{+\mu} \) where \( \mu = u = 1, \ldots, N_c \) for fundamental \( q^+ \) while \( \mu \equiv a = 0, 1, \ldots, N_c^2 - 1 \) for adjoint \( q^+ \). The \( U(N_c) \) gauged matter action for \( q^+ \) is (see appendix C for the symplectic covariant form)

\[
S_q^{\text{gauged}} = - \int du d\zeta \bar{\zeta}^{(-4)} \bar{q}^{+\mu} D^{++} q^{+\mu}, \quad D^{++} q^{+\mu} = D^{++} q^{+\mu} + i V^{++a} (T_a)^\mu_{\nu} q^{+\nu}
\]

(4.2)

where \( D^{++} = \partial^{++} - 2 i \theta^+ \sigma^m \bar{\theta}^+ \frac{\partial}{\partial \theta^+} + \theta^+ \alpha \frac{\partial}{\partial \theta^+} + \bar{\theta}^+ \bar{\alpha} \frac{\partial}{\partial \bar{\theta}^+} \) and \( \partial^{++} = u^i \frac{\partial}{\partial u^i} \). \( T_a \) is understood as

\[
(T_a)^\mu_{\nu} = \begin{cases} (t_a)^u_v & \text{for fundamental } q^+ \\ \text{ad}(t_a)^b_c = if_{abc} & \text{for adjoint } q^+ \end{cases}
\]

(4.3)
This action is invariant under the $U(N_c)$ gauge transformation

$$\delta q^{+\mu} = i\epsilon^a (T_a)^{\mu} \nu q^{+\nu}, \quad \delta q^+ = -i\epsilon^a q^+_a (T_a)^{\nu}. \quad (4.4)$$

In other words, the $U(N_c)$ isometry generated by the Killing vector $\lambda^+ = i(T_a)^{\nu} q^{+\nu} \partial_{+\mu}$ has been gauged [22]. The Killing potential for $\lambda^+ = \bar{q}^+_a \lambda^+ = -i\epsilon^a q^+_a (T_a)^{\nu} q^{+\nu}.$

The equation of motion $D^{++}q^{+\mu} = 0$ is expanded with respect to the order of $\theta$ as

$$\theta^{++} F^+ = 0, \quad (4.6)$$
$$\theta^{++} \psi = \theta^{++} \bar{\kappa} = 0, \quad (4.7)$$
$$\theta^{++} M^- + \sqrt{2} \bar{\phi} F^+ = 0, \quad (4.8)$$
$$\theta^{++} N^- - \sqrt{2} \bar{\phi} F^+ = 0, \quad (4.9)$$
$$\theta^{++} A_m^- - 2 D_m F^+ = 0, \quad (4.10)$$
$$\theta^{++} \bar{\gamma}^{(-2)} - i\sigma D_m \phi + \sqrt{2} \bar{\phi} \bar{\kappa} - 4i\bar{\lambda} F^+ u_i^- = 0, \quad (4.11)$$
$$\theta^{++} \chi^{(-2)} + i\sigma D_m \phi - \sqrt{2} \phi \psi + 4i\lambda F^+ u_i^- = 0, \quad (4.12)$$
$$\theta^{++} P^{(-3)} - D^m A_m^- + \sqrt{2} \phi N^- - \sqrt{2} \phi M^- - 2i\bar{\lambda} \psi u_i^- + 2i\bar{\lambda} \kappa u_i^- + 3iD^i u_i^- u_j^- F^+ = 0, \quad (4.13)$$

where $D_m = \partial_m + i v_m$. Here, we have omitted $U(N_c)$ index understanding SW-NE contraction for the $U(N_c)$ index. For example, $\phi F^+$ means $\phi^u \nu F^{+\nu} = \phi^a (t_a)^{\nu} F^{+\nu}$ for fundamental $q^+$, while $\phi^a F^{+b} = \phi^a \text{ad}(t_c)^a b F^{+b} = \phi^a f^a \phi F^{+b}$ for adjoint $q^+$. We find that these equations can be solved by

$$(4.6) \rightarrow F^+ = f^i (x_A) u_i^+, \quad (4.14)$$
$$(4.7) \rightarrow \psi = \psi (x_A), \quad \bar{\kappa} = \bar{\kappa} (x_A), \quad (4.15)$$
$$(4.8) \rightarrow M^- = -\sqrt{2} \phi f^i u_i^-, \quad (4.16)$$
$$(4.9) \rightarrow N^- = \sqrt{2} \phi f^i u_i^-, \quad (4.17)$$
$$(4.10) \rightarrow A_m^- = 2 D_m f^i u_i^-, \quad (4.18)$$
$$(4.11) \rightarrow \bar{\gamma}^{(-2)} = 2i\bar{\lambda} f^j u_j^- u_i^-, \quad (4.19)$$
$$i\sigma D_m \phi - 2i\bar{\lambda} f^i u_i^- = \sqrt{2} \bar{\phi} \bar{\kappa} = 0, \quad (4.20)$$
$$(4.12) \rightarrow \chi^{(-2)} = -2i\lambda f^j u_j^- u_i^-, \quad (4.21)$$
$$i\sigma D_m \phi - 2i\bar{\lambda} f^i u_i^- = \sqrt{2} \phi \psi = 0, \quad (4.22)$$
$$(4.13) \rightarrow P^{(-3)} = -iD^i f^k u_i^- u_j^- u_k^-, \quad (4.23)$$
$$D^m D_m f^i - (\bar{\phi} \phi + \bar{\phi} \phi) f^i + i\lambda^i \psi - i\bar{\lambda} \kappa + iD^i f_j = 0. \quad (4.24)$$
The infinitely many auxiliary fields contained in \( q^+ \) can be eliminated by using these equations except for (4.20), (4.22) and (4.24) which are dynamical. As a result, the action \( S^\text{gauged} \) reduces to

\[
S^\text{gauged} = \int d^4 x \left[ -\bar{f}^i_{\mu} D^m D_m f^i_{\mu} - \bar{f}_{\mu}(\phi \bar{\phi} + \bar{\phi} \phi)^{\mu}_\nu f^i_{\mu} + if_{\mu} D^{ij\mu} f^j_{\nu} + \frac{i}{2} \bar{\psi}_\mu \sigma^m D_m \psi^\mu - \frac{i}{2} K^I \sigma^m D_m \bar{\kappa}^I + \frac{i}{\sqrt{2}} \bar{\psi}_\mu \tilde{\lambda}^{ij\mu} f^i_{\mu} - i\bar{f}_\mu \lambda^{ij\mu}_\nu \psi^\nu \right. \\
+ \frac{i}{\sqrt{2}} K^I \phi^\mu \psi^\nu + \frac{1}{2} \sqrt{2} \bar{\psi}_\mu \bar{\phi}^\mu \bar{\kappa}^I \right]. \tag{4.25}
\]

The gauge transformation and the Killing potential reduce respectively to \( \delta f^i_{\mu} = \epsilon^a (\ell_a)_i^\mu \) and \( \Lambda^+ = Q^a_{ij} u^+_i u^+_j \) where

\[
(\ell_a)_i^\mu = i(T_a)^{\mu}_{\nu} f^i_{\nu}, \quad (\ell_a)_i^\mu = i f^i_{\nu}(T_a)^{\mu}_{\nu}, \quad Q^a_{ij} = \frac{i}{2} (\bar{f}_\nu(T_a)^{\mu}_{\nu} f^i_{\mu} + \bar{f}_\nu(T_a)^{\mu}_{\nu} f^j_{\mu}) \tag{4.26}
\]

Hence,

\[
- \bar{f}_i(\phi \bar{\phi} + \bar{\phi} \phi)f^i = (\bar{\ell}_a)_{ij} (\ell_b)^j (\phi^a \bar{\phi}^b + \bar{\phi}^a \phi^b), \quad i \bar{f}_i D^{ij} f_j = Q^{ij}_a D^a_{ij}. \tag{4.27}
\]

We introduce flavors to the action \( S^\text{gauged} \). We just regard the field \( q^{+\mu} \) as an \( N_f \)-dimensional vector. The action is simply

\[
S^\text{gauged} = - \int d^4 x q^{+\mu} D^{+\mu} q^{+\mu I}. \tag{4.28}
\]

where \( I = 1, \ldots, N_f \). This implies that component fields \( f, \psi \) and \( \kappa \) are understood as \( N_f \)-dimensional vectors. We suppress the flavor index in the followings.

In summary, the \( \mathcal{N} = 2 U(N_c) \) gauged action for \( N_f \) hypermultiplets \( q^{+\mu} \) in the fundamental representation of \( U(N_c) \) is

\[
S^\text{gauged}_{\text{fund}} = \int d^4 x \left[ -\bar{f}^i_{\mu} D^m D_m f^i_{\mu} - \bar{f}_{\mu}(\phi \bar{\phi} + \bar{\phi} \phi)^{\mu}_\nu f^i_{\mu} + if_{\mu} D^{ij\mu} f^j_{\nu} + \frac{i}{2} \bar{\psi}^a_{\mu} \sigma^{mn} D_m \bar{\kappa}^a - \frac{i}{2} K^I \sigma^m D_m \bar{\kappa}^I + \frac{i}{\sqrt{2}} \bar{\psi}^a_{\mu} \tilde{\lambda}^{ij\mu} f^i_{\mu} - i\bar{f}_\mu \lambda^{ij\mu}_\nu \psi^\nu \right. \\
+ \frac{i}{\sqrt{2}} K^I \phi^\mu \psi^\nu + \frac{1}{2} \sqrt{2} \bar{\psi}^a_{\mu} \bar{\phi}^\mu \bar{\kappa}^I \right] \tag{4.29}
\]

where \( \hat{Q}^{ij}_a \equiv Q^{ij}_a \big|_{T_a = \text{fund}} \). On the other hand, the \( \mathcal{N} = 2 U(N_c) \) gauged action for \( N_a \) hypermultiplets \( q^{+a} \) in the adjoint representation of \( U(N_c) \) is

\[
S^\text{gauged}_{\text{adj}} = \int d^4 x \left[ -\bar{f}^a_{\mu} D^m D_m f^a_{\mu} - \bar{f}_{\mu}(\phi \bar{\phi} + \bar{\phi} \phi)^{\mu}_\nu f^a_{\mu} + if_{\mu} D^{ij\mu} f^j_{\nu} + \frac{i}{2} \bar{\psi}^a_{\mu} \sigma^{mn} D_m \bar{\kappa}^a - \frac{i}{2} K^I \sigma^m D_m \bar{\kappa}^a + \frac{i}{\sqrt{2}} \bar{\psi}^a_{\mu} \tilde{\lambda}^{ij\mu} f^i_{\mu} - i\bar{f}_\mu \lambda^{ij\mu}_\nu \psi^\nu \right. \\
+ \frac{i}{\sqrt{2}} K^I \phi^\mu \psi^\nu + \frac{1}{2} \sqrt{2} \bar{\psi}^a_{\mu} \bar{\phi}^\mu \bar{\kappa}^I \right]. \tag{4.30}
\]

where \( \hat{Q}^{ij}_a \equiv Q^{ij}_a \big|_{T_a = \text{adj}} \).
4.2 \( \omega \) hypermultiplets

The \( \omega \) hypermultiplet is known as a real hypermultiplet, \( \omega = \bar{\omega} \). Here we combine two of them to a complex superfield, or equivalently we do not impose the reality condition. Such a complex \( \omega \) hypermultiplet is expanded as

\[
\omega(\zeta, u) = \omega(x_A, u) + \theta^+ \psi^-(x_A, u) + \bar{\theta}^+ \bar{\kappa}^- (x_A, u) + (\theta^+)^2 M^- (x_A, u) + (\bar{\theta}^+)^2 N^- (x_A, u) + i\theta^+ \sigma^m \bar{\theta}^+ A_m^- (x_A, u) + (\theta^+)^2 \bar{\theta}^+ \gamma^{(-3)} (x_A, u) + (\bar{\theta}^+)^2 \theta^+ \chi^{(-3)} (x_A, u)
\]

\[+(\theta^+)^4 P^{(-4)} (x_A, u). \quad (4.31)\]

We consider the \( U(N_c) \) gauged action

\[
S^\text{gauged}_\omega = \frac{1}{2} \int dud\zeta^{(-4)} \bar{\omega}(D^{++})^2 \omega, \quad D^{++} = D^{++} + iV^{++}, \quad V^{++} = V^{++} T_a, \quad (4.32)
\]

where \( T_a = (t_a)^u_v, u, v = 1, \cdots, N_c \) for \( \omega^u \) in the fundamental representation of \( U(N_c) \), while \( T_a = \text{ad}(t_a)^b_c = if_{abc} a, b = 0, 1, \cdots, N_c^2 - 1 \) for \( \omega^a \) in the adjoint representation of \( U(N_c) \). The equation of motion, \( (D^{++})^2 \omega = 0 \), is expanded with respect to the order of \( \theta \) as

\[
(\partial^{++})^2 \omega = 0, \quad (4.33)
\]

\[
(\partial^{++})^2 \psi^- = (\partial^{++})^2 \bar{\kappa}^- = 0, \quad (4.34)
\]

\[
(\partial^{++})^2 M^- + 2\sqrt{2} \bar{\phi} \partial^{++} \omega = 0, \quad (4.35)
\]

\[
(\partial^{++})^2 N^- - 2\sqrt{2} \bar{\phi} \partial^{++} \omega = 0, \quad (4.36)
\]

\[
(\partial^{++})^2 A_m^- - 4\bar{\partial}^{++} D_m \omega = 0, \quad (4.37)
\]

\[
(\partial^{++})^2 \gamma^{(-3)} - 2i\sigma^m D_m \partial^{++} \psi^- + 2\sqrt{2} \bar{\phi} \partial^{++} \bar{\kappa}^- - 8i\bar{\lambda} \bar{u}_i \partial^{++} \omega - 4i\bar{\lambda} \bar{u}_i^+ \omega = 0, \quad (4.38)
\]

\[
(\partial^{++})^2 \chi^{(-3)} + 2i\sigma^m D_m \partial^{++} \bar{\kappa}^- - 2\sqrt{2} \bar{\phi} \partial^{++} \psi^- + 8i\bar{\lambda} \bar{u}_i \partial^{++} \bar{\kappa}^- + 4i\bar{\lambda} \bar{u}_i^+ \omega = 0, \quad (4.39)
\]

\[
(\partial^{++})^2 P^{(-4)} - 2D^m \partial^{++} A_m^- + 2D^m D_m \omega + 2\sqrt{2} \bar{\phi} \partial^{++} N^- - 2\sqrt{2} \bar{\phi} \partial^{++} M^- - 2i\lambda \bar{u}_i \psi^- - 4i\bar{\lambda} \bar{u}_i \partial^{++} \psi^- + 2i\bar{\lambda} \bar{u}_i^+ \bar{\kappa}^- + 4i\bar{\lambda} \bar{u}_i^+ \partial^{++} \bar{\kappa}^- + 6iD^{ij} \bar{u}_i^+ \bar{u}_j^- \partial^{++} \omega - 2(\bar{\phi} \partial\phi + \phi \bar{\partial} \overline{\phi}) \omega = 0. \quad (4.40)
\]

We find that these are solved by

\[
\omega(x_A, u) = \frac{1}{\sqrt{2}} \omega(x_A) + \omega^{ij}(x_A) u_i^+ u_j^- , \quad (4.41)
\]

\[
\psi^-(x_A, u) = \psi^j(x_A) u_i^- , \quad \bar{\kappa}^-(x_A, u) = \bar{\kappa}^i(x_A) u_i^- , \quad (4.42)
\]

\[
M^-(x_A, u) = -\sqrt{2} \bar{\phi} \omega^{ij} u_i^- u_j^- , \quad (4.43)
\]

\[
N^-(x_A, u) = \sqrt{2} \phi \omega^{ij} u_i^- u_j^- , \quad (4.44)
\]
where (4.49), (4.51) and (4.52) which are dynamical, the action reduces to

\[ i\sigma^m D_m \bar{\psi}^i - \sqrt{2} \phi \bar{\psi}^i + 2i \lambda_j \bar{\psi}^i + \sqrt{2} i \bar{\lambda}^i \omega = 0, \]

Eliminating infinitely many auxiliary fields by using these equations except for (4.47), (4.49), (4.51) and (4.52) which are dynamical, the action reduces to

\[ S^\text{gauged}_\omega = \int dx^4 \left[ + \frac{1}{2} \bar{\omega}^{ij} D_m \bar{\omega}_{ij} + \frac{1}{2} \bar{\omega} D^m D_m \omega - \frac{i}{4} \bar{k}^i \sigma^m D_m \bar{k}_i + \frac{i}{4} \bar{\psi}^i \sigma^m D_m \psi_i \right. \]

\[ \left. - \frac{\sqrt{2}}{4} \bar{k}^i \phi \psi_i + \frac{i}{2} \bar{k}^i \bar{\lambda}^j \omega_{ij} - \frac{\sqrt{2}}{4} ik^i \lambda_i \omega + \frac{i}{2} \bar{k}^i \bar{\lambda}^j \psi_j + \frac{\sqrt{2}}{4} i \bar{k} \bar{\lambda}^i \psi_i \right. \]

\[ \left. - \frac{\sqrt{2}}{4} \bar{\psi}^i \bar{\bar{k}}^i - \frac{i}{2} \bar{\omega} \bar{\lambda}^i \bar{\bar{k}}_i - \frac{\sqrt{2}}{4} i \bar{\psi}^i \bar{\lambda}^j \omega_{ij} + \frac{\sqrt{2}}{4} i \bar{\psi}^i \bar{\lambda} \omega \right] \]

where

\[ S^i_a = \frac{i}{2} \bar{\omega}^i k_a \right. \]

It is easy to introduce flavors to the action by regarding \( \omega \) as a vector \( \omega^I \)

\[ S^\text{gauged}_\omega = \frac{1}{2} \int du d\omega (-4) \bar{\omega} I (\mathcal{D}^+)^2 \omega^I, \]

where \( I = 1, \cdots, N^\omega_f \) for fundamental \( \omega \) and \( I = 1, \cdots, N^\omega_a \) for adjoint \( \omega \). We omit the flavor index below.

In summary, the \( U(N_c) \) gauged action for \( N^\omega_f \) hypermultiplets \( \omega^u \) in the fundamental representation of \( U(N_c) \) is

\[ S^\text{gauged}_{\text{fund}} = \int dx^4 \left[ + \frac{1}{2} \bar{\omega}^{ij} D^m D_m \omega_{ij} + \frac{1}{2} \bar{\omega} D^m D_m \omega + \frac{i}{4} \bar{k}^i \sigma^m D_m \bar{k}_i + \frac{i}{4} \bar{\psi}^i \sigma^m D_m \psi_i \right. \]

\[ \left. - \frac{1}{2} \bar{\omega}^{ij} (\bar{\phi} + \phi \bar{\phi}) v_{\omega^v} - \frac{\sqrt{2}}{4} i k^i u \lambda^j v_{\omega^v} + \bar{S}^ij a^i_{\bar{a}} \right. \]

\[ \left. + \left( \frac{\sqrt{2}}{4} k^i \phi^u \psi_i^v + \frac{i}{2} k^i \lambda^j u_{\omega^v} - \frac{\sqrt{2}}{4} i k^i u \lambda^j v_{\omega^v} + \frac{i}{2} \bar{k}^i \bar{\lambda}^j u \psi^v_j + \frac{\sqrt{2}}{4} i \bar{k} \bar{\lambda}^i u v_{\psi^v_i} \right) \right] \]
with \( S_{\text{adj}}^{\omega} = S_{\text{adj}}^{\omega}|_{T_a = t_a} \), while that for \( N_a^{\omega} \) hypermultiplets \( \omega^a \) in the adjoint representation of \( U(N_c) \) is

\[
S_{\text{adj}}^{\omega} = \\
\int dx^4 \left[ \frac{1}{2} \bar{\omega}_a^{ij} \mathcal{D}^m \mathcal{D}_m \omega_{ij} + \frac{1}{2} \bar{\omega}_a \mathcal{D}^m \mathcal{D}_m \omega_{ij} - i \kappa_a^i \sigma^m \mathcal{D}_m \bar{\kappa}_i + i \frac{\psi^a}{4} \bar{\sigma}^m \mathcal{D}_m \psi^a_{ij} \right. \\
- \frac{1}{2} \bar{\omega}^{ij}_a (\bar{\phi} \phi + \phi \bar{\phi})_b \omega_{ij}^b - \frac{1}{2} \bar{\omega}_a (\bar{\phi} \phi + \phi \bar{\phi})^a_b \omega^b + \hat{S}_{\text{adj}}^{ij} D^a_{ij} \\
+ \left( \frac{\sqrt{2}}{4} \kappa_a^i \tilde{\phi}^a \tilde{b} \psi_{ij}^b + i \frac{\sqrt{2}}{4} i \kappa_a^i \tilde{\lambda}^a \tilde{b} \omega_{ij}^b - \frac{\sqrt{2}}{4} i \kappa_a^i \tilde{\lambda}^a \tilde{b} \omega^b + i \frac{\sqrt{2}}{4} i \kappa_a^i \tilde{\lambda}^a \tilde{b} \psi_{ij}^b \right) \\
\left. + \text{c.c.} \right] \tag{4.57}
\]

with \( \hat{S}_{\text{adj}}^{ij} = S_{\text{adj}}^{ij}|_{T_a = \text{adj}(t_a)} \).

The \( \mathcal{N} = 2 \) \( U(N_c) \) QCD action coupled with \( N_f \) fundamental \( q^{+u} \), \( N_a \) adjoint \( q^{+a} \), \( N_f^{\omega} \) fundamental \( \omega^u \) and \( N_a^{\omega} \) adjoint \( \omega^a \) is given as

\[
S_{V,q,\omega} = S_V + S_{\text{fund}}^{\omega} + S_{\text{adj}}^{q} + S_{\text{fund}}^{q} + S_{\text{adj}}^{\omega} \tag{4.58}
\]

with (2.7), (4.29), (4.30), (4.56) and (4.57).

### 4.3 electric & magnetic FI terms

Next, we add the electric and magnetic FI terms to the action \( S_{V,q,\omega} \) in (4.58). The electric FI term \( S_e \) is the same as (2.9). As was seen in subsection 2.1, the effect of the magnetic FI term is to shift the auxiliary field \( D \) by an imaginary constant. Thanks to this property the magnetic FI term does not affect the superinvariance of the total action which now includes \( S_e \). This observation leads to the magnetic FI term of the form

\[
S_m^{\text{QCD}} = S_m^{\text{YM}} + \int d^4 x \ 2i (\hat{Q}_{0}^{ij} + \hat{S}_{0}^{ij})(\xi_D^{ij} - \bar{\xi}_D^{ij}) \tag{4.59}
\]

where \( S_m^{\text{YM}} \) is given in (2.12). The last term in (4.59) shifts the terms, \( \hat{Q}_{0}^{ij} D_{ij}^0 \) in \( S_{\text{fund}}^{\omega} \) (4.29) and \( \hat{S}_{0}^{ij} D_{ij}^0 \) in \( S_{\text{fund}}^{q} \) (4.56), as

\[
\hat{Q}_{0}^{ij} D_{ij}^a + \hat{S}_{0}^{ij} D_{ij}^a + 2i (\hat{Q}_{0}^{ij} + \hat{S}_{0}^{ij})(\xi_D^{ij} - \bar{\xi}_D^{ij}) = \frac{1}{2} (\hat{Q}_{0}^{ij} + \hat{S}_{0}^{ij}) D_{ij}^a + \text{c.c.} \tag{4.60}
\]

where \( D \) is given in (2.14). The terms, \( \hat{Q}_{0}^{ij} D_{ij}^0 \) in \( S_{\text{fund}}^{\omega} \) (4.30) and \( \hat{S}_{0}^{ij} D_{ij}^0 \) in \( S_{\text{adj}}^{\omega} \) (4.57), vanish because \( \text{ad}(t_0) = 0 \), and so the adjoint matters do not appear in (4.59).

Thus, we observe that

\[
S_{V,q,\omega} + S_m^{\text{QCD}} = S_{V,q,\omega}|_{D \rightarrow D} \tag{4.61}
\]
where $|D\to D|$ means the replacement $D^{\alpha ij} \to D^{\alpha ij} (D^{\alpha ij} \to \bar{D}^{\alpha ij})$.

Gathering all together, the total action is

$$S_{QCD} = S_{V,q,\omega} + S_c + S_{m}^{QCD},$$

(4.62)

with (4.59), (2.5) and (4.60). The auxiliary field $D^{\alpha ij}$ is solved by

$$D^{\alpha ij} = -2g^{ab}
\left[
(\xi^{ij} + \bar{\xi}^{ij})\delta_b^0 + \xi_D^{ij}\mathcal{F}_b^0 + \bar{\xi}_D^{ij}\mathcal{F}_b^0
\right]
+ \mathcal{F}_{bcd}^i \lambda_i^c \lambda_j^d
- \mathcal{F}_{bcd}^i \lambda_i^c \lambda_j^d
$$

(4.63)

while the $D^{\alpha ij}$ is given as

$$D^{\alpha ij} = -2g^{ab}
\left[
(\xi^{ij} + \bar{\xi}^{ij})\delta_b^0 + (\xi_D^{ij} + \bar{\xi}_D^{ij})\mathcal{F}_b^0 + \bar{Q}_b + \bar{Q}_b + \bar{S}_b + \bar{S}_b
\right]
+ \mathcal{F}_{bcd}^i \lambda_i^c \lambda_j^d
- \mathcal{F}_{bcd}^i \lambda_i^c \lambda_j^d
$$

(4.64)

After eliminating auxiliary field $D$ by using (4.63), the action of the $N = 2$ $U(N_c)$ QCD model becomes

$$S_{QCD} = \int d^4x \left[ \mathcal{L}_{kin} + \mathcal{L}_{pot} + \mathcal{L}_{Pauli} + \mathcal{L}_{mass} + \mathcal{L}_{4}\text{fermi} \right]$$

(4.65)

where

$$\mathcal{L}_{kin} = -g_{ab}D^mD^aD^bD_m\bar{\psi}^b - \frac{1}{2}\mathcal{F}_{ab}^i \lambda_i^a \sigma^m D_m \lambda_i^b
- \frac{1}{4}g_{ab}v_m^a v_m^b \bar{v}^a v^b
- \frac{i}{2}v_a \sigma^m D_m \bar{\psi}^a - \frac{i}{2}\kappa_a \sigma^m D_m \bar{\psi}^a
- \frac{i}{2}D^a D_m \bar{\psi}^a - \frac{i}{2}\kappa_a \sigma^m D_m \bar{\psi}^a
+ \frac{1}{2}\mathcal{F}_{ab}^i \lambda_i^a \sigma^m D_m \bar{\psi}^a
- \frac{1}{2}v_a \sigma^m D_m \bar{\psi}^a
$$

(4.66)

$$\mathcal{L}_{pot} = -\frac{1}{4}g_{ab}D^{\alpha ij} D^{\alpha ij} - g_{ab}P^a P^b + 2i\xi^{ij} D_{Dij} \mathcal{F}_{00} - 2i\xi^{ij} \xi^{ij} \mathcal{F}_{00}
+ 2i(\hat{Q}_0 + \hat{S}_0)(\xi^{ij} + \bar{\xi}^{ij}) - 2i(\xi^{ij} + \bar{\xi}^{ij})(\xi_{Dij} + \bar{\xi}_{Dij})
- \frac{1}{2}(\phi \phi + \phi \phi)_{v_i}^a v_i^a - \frac{1}{2}(\phi \phi + \phi \phi)_{v_i}^a v_i^a
- \frac{1}{2}(\phi \phi + \phi \phi)_{b_i}^a b_i^a
= -\frac{1}{4}g_{ab}D^{\alpha ij} D^{\alpha ij} - g_{ab}P^a P^b - 2i(\xi^{ij} + \bar{\xi}^{ij})(\xi_{Dij} - \bar{\xi}_{Dij})$$
\begin{align}
& - \bar{f}_u (\phi \bar{\phi} + \phi \bar{\phi})^u v f^v_i - \frac{1}{2} \bar{\omega}_u^i (\phi \bar{\phi} + \phi \bar{\phi})^u v \omega^v_i - \frac{1}{2} \bar{\omega}_u (\phi \bar{\phi} + \phi \bar{\phi})^u v \omega^v - i \bar{\omega}_u^i (\phi \bar{\phi} + \phi \bar{\phi})^u v \omega^v_i - \frac{1}{2} \bar{\omega}_a (\phi \bar{\phi} + \phi \bar{\phi})^a b \omega^b_i , \\
L_{\text{mass}} &= + \frac{1}{2} g_{ab} \lambda^a f^b_{cd} (\phi \bar{\phi})^u v f^v_i - \frac{1}{2} \sqrt{2} \delta_0 \bar{\phi} f^i | \mathcal{F}_{abc} | \chi^b \lambda^c_j , \\
& - i \bar{f}_u \lambda^i u v \psi^v_i + i \kappa_a \lambda^a b f^b_i + \frac{1}{2} \sqrt{2} \kappa_a \phi^a v \psi^v_i - i \bar{f}_a \lambda^i b v \psi^v_i + \frac{1}{2} \kappa_a \lambda^a b f^b_i + \frac{1}{2} \sqrt{2} \kappa_a \phi^a v \psi^v_i + \frac{\sqrt{2}}{4} \kappa_a \phi^a v \psi^v_i - \frac{\sqrt{2}}{4} i \bar{\omega}_u \lambda^a b v \psi^v_i + \frac{i \sqrt{2}}{4} \kappa_a \lambda^a b v \psi^v_i + \frac{i \sqrt{2}}{4} \kappa_a \phi^a b v \psi^v_i + \kappa_a \phi^a b v \psi^v_i + \kappa_a \lambda^a b v \psi^v_i + \kappa_a \phi^a b v \psi^v_i
& + c.c. ,
\end{align}

where

\begin{align}
D^{a ij} &= -2 g_{ab} (\xi^{ij} + \bar{\xi} \delta_0) \delta_0^b + \bar{\xi} \delta_0 \mathcal{F}_{0b} + \xi^{ij} \mathcal{F}_{0b} + \bar{\xi} \delta_0^b + \bar{\xi} \delta_0^b + \bar{\xi} \delta_0^b + \bar{\xi} \delta_0^b + \bar{\xi} \delta_0^b + \bar{\xi} \delta_0^b + \bar{\xi} \delta_0^b + \bar{\xi} \delta_0^b + \bar{\xi} \delta_0^b + \bar{\xi} \delta_0^b + \bar{\xi} \delta_0^b + \bar{\xi} \delta_0^b
\end{align}

\begin{align}
\mathcal{L}_{\text{Pauli}}, \mathcal{L}_{\text{4fermi}} \text{ and } \mathcal{P}^a \text{ are given in [2.28], [2.25] and [2.20], respectively. The action } S'_{\text{QCD}} \text{ without FI terms corresponds to the } \mathcal{N} = 2 \text{ action (see for example [16]) with flat four-dimensional space-time and with flat hyperkähler geometry of hypermultiplets.}

\section{Vacua of } \mathcal{N} = 2 \ U(N_c) \ QCD \ model

Let us examine vacua of the \mathcal{N} = 2 \ U(N_c) \ QCD model coupled with hypermultiplets and equipped with the electric and magnetic FI terms, \( S'_{\text{QCD}} \) in \[4.65\].

The vacua are determined by the scalar potential

\begin{align}
\mathcal{V} &= \frac{1}{4} g_{ab} D^{a ij} | \mathcal{D}^{b}_{ij} | + g_{ab} \mathcal{P}^a \mathcal{P}^b + 2i (\xi^{ij} + \bar{\xi} \delta_0) (\xi_{Dij} - \bar{\xi} D_{ij})
+ f_u (\phi \bar{\phi} + \phi \bar{\phi})^u v f^v_i + \frac{1}{2} \bar{\omega}_u^i (\phi \bar{\phi} + \phi \bar{\phi})^u v \omega^v_i + \frac{1}{2} \bar{\omega}_u (\phi \bar{\phi} + \phi \bar{\phi})^u v \omega^v
+ f^i_a (\phi \bar{\phi} + \phi \bar{\phi})^a b f^b_i + \frac{1}{2} \bar{\omega}_a^i (\phi \bar{\phi} + \phi \bar{\phi})^a b \omega^b_i + \frac{1}{2} \bar{\omega}_a (\phi \bar{\phi} + \phi \bar{\phi})^a b \omega^b_i
\end{align}

We demand positive definiteness of \( g_{ab} \). The second term in the first line tells us that \( \langle \phi^r \rangle = 0 \text{ where } r \in \text{ non-Cartan directions. The first term in the second line reduces to}

\begin{align}
\langle | \phi |^2 | f_i^a | f^a_i \rangle
\end{align}
because \( \langle \phi \bar{\phi} \rangle = \langle |\phi|^2 \rangle t_1 \). Let us examine the phase in which \( \langle \phi \bar{\phi} \rangle \neq 0 \) (so called the Coulomb phase). This leads to \( \langle f^i_i \rangle = 0 \), i.e., \( \langle f^i_i \rangle = 0 \), which implies that \( \langle \hat{Q}_{ij} \rangle = 0 \). Similarly one finds that \( \langle \omega_{ij}^u \rangle = \langle \omega_i^u \rangle = 0 \) and \( \langle \tilde{S}_{ij}^a \rangle = 0 \). On the other hand, the first term in the last line becomes

\[
\langle \bar{T}^i_{r}, \phi \bar{\phi} \hat{Q}_{ij} (f^a_{ij} f^a_{ij} + f^a_{ij} f^a_{ij}) f^i_i \rangle
\]

and thus we derive \( \langle f^i_i \rangle = 0 \) where \( r \in \text{non-Cartan directions} \), which implies that \( \langle \hat{Q}_{ij} \rangle = 0 \). Similarly we derive \( \langle \omega_{ij}^r \rangle = \langle \omega_i^r \rangle = 0 \) and \( \langle \tilde{S}_{ij}^a \rangle = 0 \). Summarizing we derived vevs at the vacua

\[
\langle f^i_i \rangle = \langle \omega_{ij}^u \rangle = \langle \omega_i^u \rangle = 0 \quad \text{and} \quad \langle f^i_i \rangle = \langle \omega_{ij}^r \rangle = \langle \omega_i^r \rangle = 0 .
\]

Finally we examine the first term in the first line in \( V \), say \( V_1 \). By using (5.4), we derive

\[
\langle \partial A V_1 \rangle = 0 , \quad A = \{ f^i_i, \omega_{ij}^i, \omega_i^a, f^i_i, \omega_{ij}^i, \omega_i^a \} .
\]

We note here that \( \langle f^i_i \rangle \), \( \langle \omega_{ij}^i \rangle \) and \( \langle \omega_i^a \rangle \) are not determined by the scalar potential \( V \). In addition, \( \partial A \partial A V = 0 \) for \( A = \{ f^i_i, \omega_{ij}^i, \omega_i^a \} \). This means that the adjoint matter scalars in the Cartan direction parametrize the flat directions in \( V \). The non-trivial vacuum condition is

\[
\langle \partial_a V_1 \rangle = \frac{i}{8} \langle \mathcal{F}_{abc} \rangle \mathcal{D}^{bij} \mathcal{D}_{ij}^a = \frac{i}{4} \langle \mathcal{F}_{abc} \rangle \mathcal{D}^{ba} \mathcal{D}^{ca}
\]

where \( \langle \mathcal{D}^{aA} \rangle \) is the same as one without hypermultiplets in (2.28) because \( \langle \hat{Q}_{ij} \rangle = \langle \hat{Q}_{ij} \rangle = \langle \tilde{S}_{ij}^a \rangle = \langle \tilde{S}_{ij}^a \rangle = 0 \). In this way, we arrive at the same vacuum condition as one for the model specified by \( S_{YM} \) in (2.20). As was explained in section 3, the supersymmetry transformation of \( \lambda^i \) on the vacua is

\[
\langle \delta_\eta (U^i_j \lambda^j) \rangle \neq 0 , \quad \langle \delta_\eta (U^2_j \lambda^j) \rangle = 0 , \quad \langle \delta_\eta \lambda_i^a \rangle = 0 .
\]

The supersymmetry transformation of matter fermions in \( \Psi \) and \( \omega \) is found in appendix B. Because the vevs of the supersymmetry transformation of matter fermions in the fundamental representation are proportional to those of matter bosons in the fundamental representation, they vanish. On the other hand, the vevs of the supersymmetry transformation of matter fermions in the adjoint representation are proportional to those of matter bosons in the adjoint representation which are in non-Cartan directions, and so they vanish. Thus we have for all matter fermions

\[
\langle \delta_\eta \Psi \rangle = 0 , \quad \Psi = \{ \kappa_i^a, \psi_i^a, \kappa_i^a, \psi_i^a, \kappa_i^a, \psi_i^a \} .
\]
Let us look at the mass terms of $\lambda \bar{\psi}$ in (4.68). Because of (5.4) they reduce at the vacua to (3.10), and thus $U^{1\bar{j}} \lambda \bar{\psi}$ is massless. This means in the ordinary basis spanned by hermitian matrices $t_a$ of $U(N_c)$, that the overall $U(1)$ fermion $U^{1\bar{j}} \lambda^0 \bar{\psi}$ is the Nambu-Goldstone fermion associated with the spontaneous partial supersymmetry breaking $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$.

In summary, we find that in the Coulomb phase $\langle \phi \rangle \neq 0$ the $\mathcal{N} = 2$ $U(N_c)$ QCD model $S_{\text{QCD}}$ (4.62) describes the spontaneous partial supersymmetry breaking $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$.

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\section*{A Notation}

The bar “\( \bar{\ } \)" means the complex conjugation
\begin{equation}
\bar{f}_i = \bar{f}_i, \tag{A.1}
\end{equation}
\begin{equation}
\bar{f}_i = \epsilon_{ij} \bar{f}_j = \epsilon_{ij} \epsilon_{jk} \bar{f}_k = -\bar{f}_i, \tag{A.2}
\end{equation}
\begin{equation}
\bar{\xi}^{ij} = \bar{\xi}^{ij}, \tag{A.3}
\end{equation}
\begin{equation}
\bar{\xi}_{ij} = \epsilon_{ik} \epsilon_{jl} \bar{\xi}^{kl} = \epsilon_{ik} \epsilon_{jl} \epsilon_{kp} \epsilon_{lq} \bar{\xi}^{pq} = \bar{\xi}^{ij}. \tag{A.4}
\end{equation}

The tilde “\( \sim \)" is the analyticity preserving conjugation, the product of the complex conjugation “\( \bar{\ } \)" and the antipodal map “\( \star \)"; \( (u^+)^* = u^- \) and \( (u^-)^* = -u^+ \)
\begin{equation}
\bar{u}^\pm = u^\mp, \quad \bar{\theta}^\pm = \bar{\theta}^\mp, \quad \bar{\theta}^{\pm \alpha} = \bar{\theta}^{\mp \alpha}. \tag{A.5}
\end{equation}
\begin{equation}
\bar{\theta}^{\pm} = \bar{\theta}^{\mp}, \quad \bar{\theta}^{\pm} = -\theta^\pm. \tag{A.6}
\end{equation}

\section*{B \( \mathcal{N} = 2 \) supersymmetry transformation}

Under \( \mathcal{N} = 2 \) supersymmetry, coordinates transform as
\begin{equation}
(x_A^m , \theta^\pm _\alpha , \bar{\theta}^{\pm \alpha} , u^\pm_i ) \rightarrow (x_A^m - 2i(\eta^i \sigma^m \bar{\theta}^+ + \theta^+ \sigma^m \bar{\eta}^i)u^-_i , \theta^\pm _\alpha + u^\pm_i \eta^i_\alpha , \bar{\theta}^{\pm \alpha} + u^\pm_i \bar{\eta}^i_\alpha , u^\pm_i ), \tag{B.1}
\end{equation}
which is generated by the differential operator
\begin{equation}
\eta Q + \bar{\eta} \bar{Q} \quad \text{with} \quad \begin{cases}
Q^\alpha_{ai} = -2i \sigma^m \bar{\theta}^+ u^-_i \partial_m + u^+_i \partial_{\theta^+ \alpha} + u^-_i \partial_{\theta^- \alpha}, \\
\bar{Q}^\alpha_{ai} = -2i \theta^+ \sigma^m u^-_i \partial_m + u^+_i \partial_{\bar{\theta}^+ \alpha} + u^-_i \partial_{\bar{\theta}^- \alpha}.
\end{cases} \tag{B.2}
\end{equation}

The supersymmetry transformation law of component fields in \( V^{++} \) is derived by matching the components with the appropriate power of \( \theta^\pm \), \( \bar{\theta}^\pm \) in \( \delta \eta V^{++} = (\eta Q + \bar{\eta} \bar{Q})V^{++} \) where the left hand side means transformations on the component fields. Because we are working in the WZ gauge, we gauge transform the resulting expression into the WZ gauge form. Thus we examine
\begin{equation}
\delta \eta V^{++} = (\eta Q + \bar{\eta} \bar{Q})V^{++} - \delta \eta V^{++}. \tag{B.3}
\end{equation}

Because the supersymmetry transformation is an infinitesimal transformation, it is enough to consider the infinitesimal gauge transformation \( \delta \eta V^{++} = -\mathcal{D}^{++} \lambda_g \). Observe that by choosing an analytic superfield \( \lambda_g \) as
\begin{equation}
\lambda_g = F_g + \theta^+ \lambda_g^- + \bar{\theta}^+ \bar{\lambda}_g^- + (\theta^+)^2 M_g^- + (\bar{\theta}^+)^2 N_g^- + i \theta^+ \sigma^m \bar{\theta}^+ A_g^- \\
+ (\theta^+)^2 \bar{\theta}^+ B_g^{(-3)} + (\bar{\theta}^+)^2 \theta^+ C_g^{(-3)} + (\theta^+)^4 D_g^{(-4)} \tag{B.4}
\end{equation}
with
\[ F_g = P_g^{(-3)} = 0, \]  
\[ \lambda_{g\alpha} = 2i(\sqrt{2}\eta^{\bar{\alpha}} \bar{\phi} + \sigma^m \eta^i v_m) u_i^-, \]  
\[ \bar{\lambda}_{g}^{-\dot{\alpha}} = -2i(\sqrt{2}\eta^{\alpha} \phi + \bar{\sigma}^m \eta^i v_m) u_i^-, \]  
\[ M_g^{-\dot{\alpha}} = 2\bar{\eta}^{\bar{\alpha}} \bar{\lambda}^i u_i^- u_j^-, \]  
\[ N_g^{-\dot{\alpha}} = -2\bar{\eta}^{\bar{\alpha}} \lambda^i u_i^- u_j^- , \]  
\[ A_{g}^{-m} = 2i(\eta^i \sigma_m \bar{\lambda}^j - \bar{\lambda}^i \sigma_m \eta^j) u_i^- u_j^- , \]  
\[ \lambda_g^{(-3)} = -2\eta^i D^{jk} u_i^- u_j^- u_k^-, \]  
\[ \bar{\lambda}_g^{(-3)} = -2\bar{\eta}^{\bar{j}} D^{jk} u_i^- u_j^- u_k^- , \]  
the \( N = 2 \) supersymmetry transformation law is obtained as
\[ \delta_\eta \phi = -i\sqrt{2}\epsilon_{ij} \eta^i \lambda^j , \]  
\[ \delta_{\bar{\eta}} \bar{\phi} = -i\sqrt{2}\epsilon_{ij} \bar{\eta}^i \bar{\lambda}^j , \]  
\[ \delta_\eta v_m = i\epsilon_{ij} (\eta^i \sigma_m \bar{\lambda}^j + \bar{\lambda}^i \sigma_m \eta^j) , \]  
\[ \delta_\eta \lambda^i_\alpha = \frac{1}{2} \sigma^m \bar{\sigma}^\alpha \eta^i v_m + \sqrt{2} \sigma^m \eta^i D_m \phi - i\eta^i [\phi, \bar{\phi}] + D^j \eta^j , \]  
\[ \delta_{\bar{\eta}} \bar{\lambda}^i_{\dot{\alpha}} = \frac{1}{2} \sigma^m \sigma^\dot{\alpha} \eta^i v_m - \sqrt{2} \sigma^m \eta^i D_m \bar{\phi} + i\eta^i [\bar{\phi}, \phi] - D^j \bar{\eta}^j , \]  
\[ \delta_\eta D^{ij} = -2i\eta^i \sigma^m D_m \bar{\lambda}^j + 2i D_m \lambda^i \sigma^m \bar{\eta}^j + 2\sqrt{2} \eta^i [\bar{\lambda}^j, \phi] + 2\sqrt{2} \bar{\eta}^j [\lambda^i, \bar{\phi}] . \]  
Because the \( U(N_c) \) gauged action for \( q^+ \) hypermultiplets is invariant under the infinitesimal gauge transformation, \( \delta_g q^+ = i\lambda q^+ \) and \( \delta_g V^{++} = -D^{++}\lambda \) with an analytic superfield \( \lambda \), the supersymmetry transformation of \( q^+ \) must be followed by the gauge transformation with \( \lambda \equiv \lambda_g \). Thus the supersymmetry transformation of the component fields in the \( q^+ \) hypermultiplet can be read off from
\[ \delta_\eta q^+ = (\eta Q + \bar{\eta} \bar{Q})q^+ - \delta_g q^+ \]  
with \( \delta_g q^+ = i\lambda_g q^+ \) as
\[ \delta_\eta f^i = \eta^i \psi + \bar{\eta}^i \bar{\kappa} , \]  
\[ \delta_{\bar{\eta}} \psi_\alpha = 2i(\sigma^m \eta^i)_{\alpha} D_m f_i - 2\sqrt{2} \eta_\alpha \bar{\phi} f_i , \]  
\[ \delta_\eta \bar{\kappa}_{\dot{\alpha}} = 2i(\bar{\eta}^i \sigma^m)_{\dot{\alpha}} D_m f_i + 2\sqrt{2} \bar{\eta}_\dot{\alpha} \phi f_i . \]  
Similarly by examining
\[ \delta_\eta \omega = (\eta Q + \bar{\eta} \bar{Q})\omega - \delta_g \omega , \]  
(B.19)
Here, we give the symplectic covariant form of \( S \). The invariant tensor of the symplectic group, \( U \), runs from 1 to 2 \( N \), where \( \tilde{\alpha} \) is related to \( \tilde{\alpha} \) by \( \tilde{\alpha} = \Omega_{\tilde{N} \tilde{M}} q_{\tilde{N}}^+ \), where \( \Omega_{\tilde{M} \tilde{N}} = \Omega_{\tilde{N} \tilde{M}} \) is the invariant tensor of the symplectic group, \( Sp(N) \) for fundamental \( q^+ \) or \( Sp(N^2) \) for adjoint \( q^+ \). The gauged action is

\[
S^\text{gauged}_q = \frac{1}{2} \int dud\epsilon^{-4} \left[ q^+_M D^{++} q^+_M \right],
\]

where \( D^{++} q^+_M = D^{++} q^+_M + i V^{++} q^+_N, \quad V^{++} q^+_N = V^{++} q^+_N T_a, \quad T_a = \begin{pmatrix} -T_a^T & 0 \\ 0 & T_a \end{pmatrix}. \]

The fundamental or adjoint representation of the gauge group is embedded in the symplectic matrix, \( Sp(N) \) or \( Sp(N^2) \), respectively. This action is invariant under \( U(N) \) gauge transformation \( \delta q^+_M = e^a \lambda^+_a M \) with \( \lambda^+_a M = i(T_a)^M q^+_N \).
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