Meson bound states in multiflavour massive Schwinger model*

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Abstract
The problem of meson bound states with $N_f$ massive fermions in two dimensional quantum electrodynamics is discussed. We speculate about the spectrum of the lightest particles by means of the effective semiclassical description. In particular, the systems of fundamental fermions with $SU(2)$ and $SU(3)$ flavour symmetries broken by massive terms are investigated.

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1 Introduction

The literature on the Schwinger model [1] and its generalizations should refer to at least a hundred of interesting and important papers. This theoretical model is usually used to demonstrate some important phenomena known from more realistic models. It is easy to see and investigate here screening and confinement, $U(1)$ problem, chiral symmetry breaking and topological vacua, formation of bound states and finite temperature effects. On the other hand, known non-perturbative techniques tested here, like lattice calculations or sum rules, reveal many pitfalls, what makes their use doubtful. Surprisingly, the two-dimensional Schwinger model shares many phenomenological features in common with four-dimensional QCD. While one can state that the problem of massless charged fermions in two dimensions has been elaborated in all details [2]–[11], the situation when fermion masses are finite [12]–[21] still requires further investigations. In particular, it would be interesting to understand in the theory with the confinement the mechanism of formation of the lightest physical particles from the fundamental fermions. This subject is discussed in this paper.

Our paper is organized as follows: in the Section 2, we introduce briefly the multiflavour massive Schwinger model in its bosonized version. Classical equations of motion and ground states are described. In the Section 3, we discuss the lightest meson bound states in the case when fundamental fermions ('quarks') are heavy (or in other words the coupling constant is weak). To make our paper self-contained, in the first part of this section we remind the most important points of the semiclassical quantization procedure applied to the 'particle-like' solutions of the sine–Gordon theory. This knowledge is used in subsequent developments. In the Section 4, the analysis of the lightest meson states is extended to the case of light quarks (the strong coupling regime). We discuss the model for $SU(2)$ and $SU(3)$ flavour groups separately, both in unbroken and broken cases. In the Appendix, we advertise the method useful to derive approximately particle-like solutions to the classical field equations. This method is explained on the simplest example of the sine–Gordon theory, but its advantage is that it can be used to more involved problems.
2 Lagrangian

We will consider $N_f$ fundamental charged and massive Dirac fermions in (1+1)-dimensional Minkowski spacetime, with the following Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{a=1}^{N_f} \bar{\psi}^a (i\gamma^\mu \partial_\mu - e\gamma^\mu A_\mu - m_a) \psi^a, \quad (1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (2)$$

e is the coupling constant (dimension of mass). We allow for masses of fermions $m_a$ to be different for each flavour $a$. In the exactly solvable massless case $m_a = 0$, the classical system possesses a global symmetry group $U(N)_L \times U(N)_R = U(1)_V \times U(1)_A \times SU(N)_L \times SU(N)_R$. At the quantum level, the axial symmetry $U(1)_A$ is broken down by the anomaly. In the massive case, the flavour symmetry $SU(N)_L \times SU(N)_R$ is broken explicitly to $U(1)^{N_f-1}$, leaving only $(N_f-1)$ conserved flavour numbers. We are going to make use of the standard (Abelian) bosonization rules,

$$N_{m_a} [\bar{\psi}^a \gamma^\mu \psi^a] = \frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \Phi^a, \quad (3)$$

$$N_{m_a} [\bar{\psi}^a \psi^a] = -c m_a N_{m_a} [\cos \sqrt{4\pi} \Phi^a], \quad (4)$$

where $N_{m_a}$ denotes normal-ordering with respect to the fermion mass $m_a$, $\Phi^a$ is the family of canonical pseudoscalar fields. The appearance of the constant $c = e\gamma / 2\pi$ ($\gamma$ is the Euler constant) is due to the use of the specific renormalization scheme. Obviously, physical quantities do not depend on this choice. Then, one can derive the bosonized version of (1):

$$\mathcal{L} = \frac{1}{2} F_{01}^2 + \sum_{a=1}^{N_f} \frac{1}{2} (\partial_\mu \Phi^a)^2 + \frac{e}{\sqrt{\pi}} F_{01} \left( \frac{N_f}{\sum_{a=1}^{N_f} \Phi^a + \theta / \sqrt{4\pi}} \right)$$

$$+ \sum_{a=1}^{N_f} c m_a^2 N_{m_a} [\cos \sqrt{4\pi} \Phi^a] + \text{const.} \quad (5)$$

The different vacua are labelled by the angle parameter $\theta$ and the relevant constant in (5) should be adjusted to ensure zero vacuum energy. After
integrating out the electric field $F_{01}$, we arrive at the effective Lagrangian describing the system of interacting pseudoscalar fields defined by

$$L_{\text{eff}} = \sum_{a=1}^{N_f} \frac{1}{2} (\partial_\mu \Phi^a)^2 - V_{\text{eff}}$$

$$V_{\text{eff}} = \frac{e^2}{2\pi} \left( \sum_{a=1}^{N_f} \Phi^a + \frac{\theta}{\sqrt{4\pi}} \right)^2 - \sum_{a=1}^{N_f} c m_a^2 N_m [\cos\sqrt{4\pi} \Phi^a] + \text{const} .$$  \hspace{1cm} (6)

The topological charges of pseudoscalar fields $\Phi^a$ are related to flavour quantum numbers of fundamental fermions through the relations:

$$Q^a = \frac{1}{\sqrt{\pi}} \Phi^a \bigg|_{-\infty}^{+\infty} .$$  \hspace{1cm} (7)

Classical Euler–Lagrange field equations can be derived,

$$\partial_t^2 \Phi^a - \partial_x^2 \Phi^a + \frac{e^2}{\pi} \left( \sum_{b=1}^{N_f} \Phi^b + \frac{\theta}{\sqrt{4\pi}} \right) + c \sqrt{4\pi} m_a^2 \sin \left( \sqrt{4\pi} \Phi^a \right) = 0 .$$ \hspace{1cm} (8)

Classical vacua are easy to determine from the requirements $\partial V_{\text{eff}}/\partial \Phi^a = 0$, namely:

$$\Phi^a = \frac{1}{\sqrt{4\pi}} \arcsin \frac{A}{m_a^2} + \sqrt{\pi} n^a , \quad n^a \in \mathbb{Z} , \quad \sum_{a=1}^{N_f} n^a = 0 ,$$ \hspace{1cm} (9)

where the constant $A$ is subject to the equation

$$\sum_{a=1}^{N_f} \arcsin \frac{A}{m_a^2} + \theta + c \left( \frac{2\pi}{e} \right)^2 A = 0 .$$ \hspace{1cm} (10)

All finite-energy (localized) solutions should approach asymptotically the vacuum:

$$\Phi^a \xrightarrow{x \to \pm\infty} \frac{1}{\sqrt{4\pi}} \arcsin \frac{A}{m_a^2} + \sqrt{\pi} n^a_{\pm} ,$$ \hspace{1cm} (11)

$$\partial_\mu \Phi^a \xrightarrow{x \to \pm\infty} 0 ,$$ \hspace{1cm} (12)

$$\sum_{a} n^a_{\pm} = 0 .$$ \hspace{1cm} (13)
There are no restrictions on our considerations if we set all \( n^a \) equal zero. Thus, we have \( n^a = Q^a \), and Eq. (13) means that all finite-energy solutions are chargeless (charge screening).

For \( \theta = 0 \) (no CP breaking) we find

\[
\Phi^a_{\text{vac}} = \sqrt{\pi} n^a , \quad n^a \in \mathbb{Z} , \quad \sum_{a=1}^{N_f} n^a = 0 ,
\]

and we fix the constant in (6) to obtain the potential

\[
V_{\text{eff}} = \sum_{a=1}^{N_f} c m_a^2 N_m \{ 1 - \cos \sqrt{4\pi} \Phi^a \} + \frac{e^2}{2\pi} \left( \sum_{a=1}^{N_f} \Phi^a \right)^2 ,
\]

For \( \theta \neq 0 \), the equation (10) has still a solution corresponding to a classical vacuum. However, the case of \( \theta = \pm \pi \) is special, and the degeneracy of the vacuum structure appears.

### 3 Heavy quarks.

At first, we consider the case when the Lagrangian (bare) masses of fundamental fermions are much larger than the scale of electromagnetic interactions \( e \).

Let us divide the Lagrangian (3) into two parts,

\[
L_{0\text{eff}} = \sum_{a=1}^{N_f} \frac{1}{2} (\partial_{\mu} \Phi^a)^2 + \sum_{a=1}^{N_f} c m_a^2 N_m \{ \cos \sqrt{4\pi} \Phi^a \} ,
\]

\[
L_{\text{int}} = \frac{e^2}{2\pi} \left( \sum_{a=1}^{N_f} \Phi^a + \frac{\theta}{\sqrt{4\pi}} \right)^2 .
\]

The first part defines the system of \( N_f \) sine–Gordon fields. The weak interactions between them (17) are important when fields are close to their vacuum values.

Since the theory is superrenormalizable, in order to subtract all ultraviolet divergencies it is enough to replace the unordered functions of fields in the Lagrangian by their normal-ordered counterparts. Moreover, we do not need to take much care of the renormalization of quadratic forms of fields.
(kinetic terms and Coulomb interactions), because it effects only in the addition of an infinite constant to the Lagrangian. Thus, everything we want to know about the ultraviolet renormalization is contained in the prominent formula derived by Coleman [22]:

$$\cos (\beta \Phi) = \left( \frac{m}{\Lambda} \right)^{\beta^2/4\pi} N_m [\cos (\beta \Phi)] ,$$  \hspace{1cm} (18)

where $\Lambda$ is the ultraviolet cutoff. The above formula allows us to compare two different renormalization scales,

$$N_m [\cos (\beta \Phi)] = \left( \frac{\mu}{m} \right)^{\beta^2/4\pi} N_\mu [\cos (\beta \Phi)] .$$  \hspace{1cm} (19)

Thus, the effect of finite renormalization is a multiplication by the power of mass ratios. For free fermions (16), the anomalous dimension $\beta^2/4\pi$ is equal one, thus the operator (18) acts just like a free fermion mass operator.

If we want to have the standard form of sine–Gordon Lagrangian, we need to renormal-order $L^0_{\text{eff}}$ (16),

$$L^0_{\text{eff}} = \sum_{a=1}^{N_f} N_{M_a} \left[ \frac{1}{2} (\partial_\mu \Phi^a)^2 + \frac{M^2}{\beta^2} \cos (\beta \Phi^a) \right] ,$$  \hspace{1cm} (20)

where $\beta = \sqrt{4\pi}$ and the renormalized mass is:

$$M_a = \beta^2 c m_a = 2 e^\gamma m_a .$$  \hspace{1cm} (21)

As the system of classical field equations derived from $L^0_{\text{eff}} + L^\text{int}_{\text{eff}}$ (20, 17) admits localized (particle-like) solutions, the semiclassical quantization can be used to find, at least the lowest, bound states.

There exist two different types of particle-like solutions for the sine–Gordon equation. The first one refers to solutions which are time–independent in their rest frames (static, solitonic), called usually solitons ($Q^a = +1$) and antisolitons ($Q^a = -1$),

$$\Phi^a = Q^a \frac{4}{\beta} \arctan \left( e^{-M^a(x-x_a)} \right) .$$  \hspace{1cm} (22)

The energy density for such a solution is localized around some point $x_a$. The classical masses are $M^0_{\text{cl}} = 8 M^a/\beta^2$. As the quantum (finite) corrections do not introduce any new mass scale, we can denote the quantum mass of
solitary solitons as $M_{qu}^a = 8M^a/\beta'^2$, and quantum effects are reduced to the change of the coupling constant. The semiclassical WKB quantization around solitons or antisolitons gives the following effective coupling constant \[ \beta'^2 = \frac{\beta^2}{1 - \frac{\beta^2}{8\pi}} \approx \frac{\sqrt{4\pi}}{8\pi} . \] (23)

For $\beta = \sqrt{4\pi}$, the same result has been obtained by the calculation based on Feynman integrals [25] and it is very close to the numerical value obtained with variational methods [26]. Presumably, for $\beta = \sqrt{4\pi}$ the result of WKB approximation can be expected to be exact.

The second type of particle-like solutions for the sine–Gordon equation corresponds to periodic in time solutions, called breather modes:

$$\Phi^a = \frac{4}{\beta} \arctan \left( \frac{\gamma \sin \omega t}{\omega \cosh \gamma x} \right) ,$$

(24)

where $T = 2\pi/\omega$ is a period and $\gamma = \sqrt{(M^a)^2 - \omega^2}$. These solutions are localized and ‘topologically chargeless’ $Q^a = 0$. Their quantization via WKB methods [23, 24] is analogous to the quantization of the Bohr orbits of the hydrogen atom. The Bohr–Sommerfeld quantization condition is here:

$$S + ET = 2\pi n ,$$

(25)

where $n$ is positive integer, $S$ is the action per one period, and $E$ is the energy. Again, it can be checked that all quantum effects can be acknowledged in the change of coupling constant according to (23). The masses of quantum states produced on the basis of classical breather modes are:

$$M_n^a = \frac{16M^a}{\beta'^2} \sin \frac{n\beta'^2}{16} ,$$

(26)

where $n = 1, 2, 3, ... < 8\pi/\beta'^2$. For $\beta = \sqrt{4\pi}$ there are no such states.

After this brief description of the spectrum of the quantized sine–Gordon system, we can pass to the discussion of the quantum states for the Schwinger model (16, 17). The main result of interactions (17) is to impose the zero total charge condition (13). The lightest bound states is composed of a pair of a soliton and an antisoliton. Its mass (up to $e/m_a$ corrections) is given by

$$M = M_{qu}^a + M_{qu}^b = \frac{2e\gamma}{\pi} (m_a + m_b) .$$

(27)
Let us specify $\theta = 0$ and $M^a = M^b = M$, and $Q^a = -Q^b = 1$. Then, the corresponding pair of wave packets (22) is an exact solution of equations of motion provided that $x_a = x_b$, i.e. the soliton and the antisoliton are localized at the same point. It can be immediately checked that the energy of Coulomb interactions (17) is zero, so that both particle-like waves do not interact together. The state is composed of a free soliton and a free antisoliton, living together around the same point $x_a = x_b$. But if we try to separate them for a small distance $\Delta x = x_a - x_b$, they start to interact and the energy increase due to Coulomb interactions (17) can be easily calculated,

$$\Delta E = \frac{e^2 M}{\pi^2} (\Delta x)^2.$$  

(28)

We have observed that the asymptotic freedom and the confinement in the Schwinger model can be already seen at the classical level, provided that the bosonization has been performed. In this way, the bosonization provides us with some kind of the dual description of the model.

The meson state described above were composed of a pair of wave packets (22) corresponding to two particle-like solutions of different flavours. There exist also states where ‘quark’ and ‘antiquark’ wave packets are of the same flavour. However, these solutions are of different type, being breather modes for the system defined by the equations (8). Their masses (close to $2M^a$) and their shapes can be found using the approximate method described in the Appendix.

## 4 Light quarks

In the strong coupling limit, where $e \gg m_a$ it is much more convenient to perform appropriate change of field variables:

$$\chi^a = O_5^a \Phi^b + \frac{\theta}{\sqrt{4\pi N_f}} \delta_i^a$$  

(29)

Using the orthogonal matrix $O$ defined below:

\[
O_1^a = \frac{1}{\sqrt{N_f}}(1,1,\ldots,1) \\
O_2^a = \frac{1}{\sqrt{N_f(N_f-1)}}(1,1,\ldots,1, -N_f + 1)
\]
\[ O_a^3 = \frac{1}{\sqrt{(N_f - 1)(N_f - 2)}}(1, 1, \ldots 1, -N_f + 2, 0) \]
\[
\ldots
\]
\[ O_a^{N_f} = \frac{1}{\sqrt{2}}(1, -1, 0, \ldots, 0) \] (30)

Using new fields the Lagrangian density (6) takes the form:

\[ L_{\text{eff}} = \frac{1}{2} \sum_{a=1}^{N_f} (\partial_\mu \chi^a)^2 - \frac{1}{2} \mu^2 (\chi^1)^2 + \sum_{a=1}^{N_f} c_m^2 N_{ma} \left[ \cos \sqrt{4\pi}(O^T \chi)^a - \frac{\theta}{N_f} \right] + \text{const}, \] (31)

where \( \mu^2 = N_f e^2 / \pi \). The topological charges are defined now with respect to \( \chi^a \) fields, and they can be interpreted as the electromagnetic charge

\[ Q = \frac{1}{\sqrt{2\pi}} \chi^1 \bigg|_{+\infty}^{-\infty}, \] (32)

and \( SU(N_f) \) hypercharges for other fields. For instance, in the case of two flavours \( N_f = 2 \)

\[ I^3 = \frac{1}{\sqrt{2\pi}} \chi^2 \bigg|_{+\infty}^{-\infty} \] (33)

is a third component of isospin. For \( N_f = 3 \) we have two conserved hypercharges,

\[ Y = \frac{2}{\sqrt{3}} \frac{1}{\sqrt{2\pi}} \chi^2 \bigg|_{+\infty}^{-\infty} \] (34)
\[ I^3 = \frac{1}{\sqrt{2\pi}} \chi^3 \bigg|_{+\infty}^{-\infty}. \] (35)

Following Coleman [13], we can decouple the only heavy field \( \chi^1 \). It is straightforward to perform necessary renormal-ordering,

\[ N_{ma} \left[ \cos \sqrt{\frac{4\pi}{N_f}}(\chi^1 - \frac{\theta}{\sqrt{4\pi N_f}}) \right] = \left( \frac{\mu}{m_a} \right)^{1/2} N_{mu} \left[ \cos \sqrt{\frac{4\pi}{N_f}}(\chi^1 - \frac{\theta}{\sqrt{4\pi N_f}}) \right] \]

and remove \( \chi^1 \) from (31),

\[ L_{\text{eff}}^{\text{light}} = \frac{1}{2} \sum_{a=2}^{N_f} (\partial_\mu \chi^a)^2 \]
The simplest case \( N_f = 2 \) was already elaborated by Coleman \[13\]. He noticed that the Lagrangian density (37) is equivalent (for \( N_f = 2 \)) to the sine-Gordon theory,

\[
\mathcal{L}^{\text{light}}_{N_f=2} = \frac{1}{2} (\partial_\mu \chi^2)^2 + \frac{1}{2} M^2 N_M \left[ \cos \left( \sqrt{2\pi} \chi^2 \right) \right],
\]

where

\[
M = \left( e^{\gamma/2} \sqrt{m_1^2 + m_2^2 + 2m_1 m_2 \cos \theta} \right)^{2/3}.
\]

The quantum states of the sine–Gordon theory were described briefly in the previous section. Here we have \( \beta = \sqrt{2\pi} \), what implies that there are two breathers (24) with masses \( \mathcal{M} \) and \( \mathcal{M}\sqrt{3} \) (\( \mathcal{M} \) is here the quantum mass of the soliton). Coleman identified three solutions: the soliton \( (Q = 0, I_3 = +1) \), the antisoliton \( (Q = 0, I_3 = -1) \) and the lighter breather \( (Q = 0, I_3 = 0) \) as the components of the isotriplet. These are the lightest physical states. Coleman noticed that they form the degenerate isotriplet even if the \( SU(2) \) flavour symmetry is apparently broken (i.e. \( m_1/m_2 \) is very large or very small). However, it is not true that the \( SU(2) \) symmetry is restored here exactly. The differences in the masses of the isotriplet states arise as we take into account the corrections of the order \( m_a/\mu \). Note that the origin of \( I_3 = 0 \) state (the analogue of \( \pi^0 \)) is different that the origin of \( I_3 = \pm 1 \) states (the analogues of \( \pi^\pm \)). Another interesting thing is that the dependence on the vacuum parameter \( \theta \) is only via the sine–Gordon mass value. As far as the heavier breather is concerned (that of mass \( \mathcal{M}\sqrt{3} \)), Coleman wrongly \[19\] identified that as the isosinglet state (the analogue of \( \eta \) particle). In fact, the isosinglet state should be matched with the non-trivial configuration for \( \chi_1 \), so that its mass is of the order \( \mu \). Then, the heavier breather mode should be rather interpreted as a bound state of pions (or \( \pi^0 \) excitation).

Now, let us consider \( N_f = 3 \) case. We restrict ourselves to the case when the vacuum angle \( \theta \) is zero (\( C, P \) symmetries are not broken). Our effective Lagrangian contains now two light fields,

\[
\mathcal{L}^{\text{light}}_{N_f=3} = \frac{1}{2} (\partial_\mu \chi^2)^2 + \frac{1}{2} (\partial_\mu \chi^3)^2 - V_{\text{eff}},
\]

\[
V_{\text{eff}} = c \mu^{1/3} \left( m_1^{5/3} N_{m_1} \left[ 1 - \cos \left( \sqrt{2\pi} \chi^3 + \sqrt{2\pi/3} \chi^2 \right) \right] + \right.
\]
\begin{equation}
m_2^{5/3} N_{m_2} \left[ 1 - \cos \left( \sqrt{2\pi} \chi^3 - \sqrt{2/3} \chi^2 \right) \right] + m_3^{5/3} N_{m_3} \left[ 1 - \cos \frac{8\pi}{3} \chi^2 \right].
\end{equation}

It gives the following field equations of motion \((a = 2, 3)\),
\begin{equation}
\partial_t^2 \chi^a - \partial_x^2 \chi^a + \frac{\partial V_{\text{eff}}}{\partial \chi^a} = 0.
\end{equation}

At first, we discuss the case when all fermion masses are equal, namely \(m_1 = m_2 = m_3 = m\), so that \(SU(3)\) flavour symmetry remains unbroken. We list several exact classical solutions, which correspond to the lightest bound states. At the beginning, note that if the field \(\chi^2\) takes its vacuum value, say \(\chi^2 = 0\), the equations (41) reduce to the sine–Gordon equation for the field \(\chi \equiv \chi^3\),
\begin{equation}
\partial_t^2 \chi - \partial_x^2 \chi + \frac{M^2}{\sqrt{2\pi}} \sin \left( \sqrt{2\pi} \chi \right) = 0,
\end{equation}
where
\begin{equation}
M = \left( 2e^7 \mu^{1/3} m^{5/3} \right)^{1/2}.
\end{equation}

Therefore, a soliton, an antisoliton and the lighter one of breathers represent here three solutions with equal masses, and quantum numbers \(Y = 0\) and \(I_3 = +1, -1, 0\) respectively ('pions'). Because of the exact \(SU(3)\) flavour symmetry, there are more solutions of the same mass as pions. In order to obtain them, we need to put \(\chi^2/\sqrt{3} \pm \chi^3 = 0\) and observe that \(\chi \equiv \chi^2/\sqrt{3} \pm \chi^3\) satisfies the equation (42) (the sign corresponds to two alternatives). Solitons and antisolitons constructed in these two cases describe four bound states with quantum numbers \(Y = \pm 1\) and \(I_3 = \pm 1/2\) ('kaons'). On the other hand, periodic solutions (breathers) for both alternative cases can be identified, so that the lighter breather gives us the eighth state of the pion mass ('\(\eta_8\)-particle'). In this way, we have completed the whole meson octet, being the family of the lightest physical particles. Let us remark that this symmetric case allows to construct further exact solutions. Heavier breathers give next three particles (mass/octet mass = \(3/2 + \sqrt{3} \log (5 + \sqrt{24})\), \(Y = \pm 2\) and \(I = 0\)).

We now turn to the discussion of the case when \(m \equiv m_1 = m_2\) and \(\mu \gg m_3 \gg m\), i.e. the 'strange' quark is much heavier but still below the
The states of the lowest isotriplet satisfy the equation (42) with the same assignment of the solutions as before. The lowest isodoublet states can be constructed in the similar way as in the case of equal masses. The only difference is that the combinations of the fields \(\chi^2/\sqrt{3} \pm \chi^3\) do not take vacuum values, but they remain still small with respect to \(\chi \equiv \chi^2/\sqrt{3} \mp \chi^3\). Solving the static equations of motion (41) (using methods given in the Appendix) for these variables we found that the mass ratio between isotriplet states (pions) and isodublet states (kaons) is approximately \(\pi^2\sqrt{r}/4\sqrt{3}\) where \(r = (m_3/m)^{5/3}\).

Finally, we discuss the case when \(m \equiv m_1 = m_2\) and \(m_3 \gg \mu \gg m\), i.e. one of the flavours refers to heavy quarks. In this case we rotate both of the light fields \(\Phi^a (a = 1, 2)\) through the transformation (29) into the new field variables \(\chi^a\), and heavy field \(\Phi^3\) remains here as the third field variable. Using the new fields, our effective lagrangian (6) (with \(N_f = 3\) and \(\theta = 0\)) takes the form:

\[
\mathcal{L}_{eff} = \frac{1}{2} (\partial_\mu \chi^1)^2 + \frac{1}{2} (\partial_\mu \chi^2)^2 + \frac{1}{2} (\partial_\mu \Phi^3)^2 - V_{eff}
\]

\[
V_{eff} = \frac{e^2}{\pi} \left( \chi^1 + \frac{\Phi^3}{\sqrt{2}} \right)^2 - cm^2N_m [\cos \sqrt{2\pi}(\chi^1 + \chi^2) + \cos \sqrt{2\pi}(\chi^1 - \chi^2)] - cm^2M_{m_3} \cos \sqrt{4\pi} \Phi^3 + const. \tag{44}
\]

To consider the lowest states, let us put the heavy field in its vacuum state \(\Phi^3 = 0\). Then, we derive from the equations of motion that field \(\chi^1\) is in its vacuum state as well, and field \(\chi^2\) satisfies the sine–Gordon equation with the mass \(M = \sqrt{4\pi cm}\) and \(\beta = \sqrt{2\pi}\). The three solutions of the equal mass to this equation, a soliton, an antisoliton and one of the breathers represent the three partners of the isotriplet (as it was described after Eq.(42)). The other light states with a nontrivial contribution from the heavy field can also be constructed. These are isodoublets \(I^3 = \pm 1/2\) with nonzero flavour number (7) \(Q^3 = \pm 1\), built on the static solutions of the equation of motion. Since the flavour symmetry is broken, the mass of this state differs from that of the isotriplet. We found the approximate value of the mass ratio isotriplet mass/isodoublet mass = \(\pi^2m_3/(12m)\). We also calculated the mass of the \(\eta\)-particle (the appropriate solution is based on the breather), and its value calculated up to the first order of the method described in the Appendix, is consistent with the value predicted by Gell-Mann–Okubo mass formula \(m_{\pi}^2 + 3m_\eta^2 = 4m_K^2\) with the accuracy of 10 per cent.
5 Summary

In this paper, we have described the lightest physical meson states formed from the fundamental fermion fields in the Schwinger model. Of course, the picture is dependent on the hierarchy between the quark masses and the scale of interactions. For heavy quarks, we have noticed that both confinement and asymptotical freedom can be anticipated at the classical level. Physical quantities are analytical both in the coupling constant and in the inverses of quark masses, so that both parameters can be used to define perturbative expansions. To describe the physics of light quarks governed by the Schwinger model, we find it convenient to use a different set of bosonic field variables. If all light quark masses are equal (i.e. SU\(N\) flavour symmetry occurs), then we have checked for SU\(2\) and SU\(3\) examples that the lightest physical states belong to the mesonic SU\(N\) multiplets, excluding always the singlet state. The mass of the singlet state lies near the scale of interactions (i.e. the Schwinger mass). It is important to note here that the members of the same multiplet correspond usually to several different types of classical solutions of the bosonized model. In the case of SU\(2\) symmetry, the breaking of flavour symmetry can be noticed only as a 'hyperfine' splitting of the lightest multiplet ('hyperfine' means here being of the order of the inverse of the Schwinger mass). In the case of SU\(3\) symmetry (and higher groups), the situation is different. When the flavour symmetry is broken, we notice immediately within the lightest multiplet the splitting of states corresponding to different isospins.

The Appendix

We describe here the general approximate method to derive particle-like solutions for the given system of non-linear field equations. We assume that we are dealing with two types of particle-like solutions: static solutions (solitons) and periodic solutions (breathers). We present the method looking at the simple example of the exactly solvable sine–Gordon equation, but there no obstacles if put it into use in more complicated cases.

The sine–Gordon equation reads,

\[
\partial_t^2 \Phi - \partial_x^2 \Phi + \frac{M^2}{\beta} \sin(\beta \Phi) = 0 .
\]

First, we are attempting to find a static solution localized around some point \(x_0\). We specify the asymptotic conditions \(\Phi(x = -\infty) = 0\) and \(\Phi(x = +\infty) = 2\pi/\beta\). Far away from the localization point \(x_0\) the field \(\Phi\) is close to its vacuum value, and the equation (45) can be linearized. We
write down the linear differential equations for \( x << x_0 \) and for \( x >> x_0 \) respectively. Having imposed proper asymptotic conditions, we glue both solutions together at the point \( x_0 \). The final result yields,

\[
\Phi = \begin{cases} 
\frac{\pi e^{M(x-x_0)}}{\beta} & \text{if } x \leq x_0 \\
\frac{2\pi}{\beta} - \frac{\pi e^{-M(x-x_0)}}{\beta} & \text{if } x \geq x_0
\end{cases}
\]  

(46)

It gives the mass \( \pi^2 M/\beta^2 \), being close to the exact result \( 8M/\beta^2 \). Using this procedure, one can calculate further corrections.

To find periodic solutions (breather modes) we need to use some more tricky procedure. Any periodic solution can be expanded in the Fourier series,

\[
\Phi = \sum_{n=1}^{\infty} \phi_n(x) \sin (\omega nt),
\]  

(47)

where \( \omega = 2\pi/T \). We restrict here to solutions antisymmetric in time, \( \Phi(-t, x) = -\Phi(t, x) \). It allows us also to write down the classical energy in the following way:

\[
E = \frac{1}{2} \int_{-\infty}^{+\infty} dx \left[ \partial_t \Phi \bigg|_{t=0} \right]^2.
\]  

(48)

The Bohr–Sommerfeld quantization condition (25) reads,

\[
\int_{-\infty}^{+\infty} dx \int_{0}^{T} dt \ |\partial_t \Phi|^2 = 2\pi N.
\]  

(49)

The quantum effects will come to drive the coupling constant (23). To find a classical solution being the starting point for the WKB quantization procedure (23), as a first approximation we take only the first term in the Fourier expansion (47). The lowest frequency \( \omega \) is derived from (49) (for \( N = 1 \)),

\[
\omega \int_{-\infty}^{+\infty} dx \phi_1^2(x) = 2,
\]  

(50)

where the Fourier coefficient \( \phi_1 \) is calculated from some ordinary differential equation, which is dependent of the parameter \( \omega \). The energy (mass) (48) corresponding to the lowest breather mode in this first approximation is given by the following simple formula,

\[
E = \omega.
\]  

(51)

This is nothing else but the famous de Broglie relation. Here, this approximate formula is as well as the harmonic approximation of the breather mode
solution is good enough. If we take the exact solution, we can verify that 
\[ E/\omega = 2\sqrt{3}/\pi \approx 1.102 \]
so that it is pretty accurate. But it is important to stress that the above method allows to calculate further corrections if one wishes.

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