QUANTIZATIONS OF LOCAL SURFACES AND REBEL INSTANTONS

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ABSTRACT. We construct explicit deformation quantizations of the noncompact complex surfaces $Z_k := \text{Tot}(O_{\mathbb{P}^1}(-k))$ and describe their effect on moduli spaces of vector bundles and instanton moduli spaces. We introduce the concept of rebel instantons, as being those which react badly to some quantizations, misbehaving by shooting off extra families of noncommutative instantons. We then show that the quantum instanton moduli space can be viewed as the étale space of a constructible sheaf over the classical instanton moduli space with support on rebel instantons.

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1. Motivation and results

In this work we clarify a specific aspect of the quantization of SU(2) instantons, by describing explicitly the effect that deformation quantization has on moduli spaces of instantons over noncompact complex surfaces. Our main contribution is to identify those classical instantons which react badly to certain choices of quantization, by shooting off extra families of noncommutative instantons. We call them rebel instantons. Thus, rebel instantons cause the quantum moduli spaces to become larger than the corresponding moduli on the classical limit.

Our complex surfaces of choice are total spaces of negative line bundles on the complex projective line $\mathbb{P}^1$, namely the surfaces $Z_k := \text{Tot}(O_{\mathbb{P}^1}(-k))$ for $k \geq 1$. Our interest in them originates in the wish to understand how the mathematical physics on them reacts to contraction of the complex curve $\mathbb{P}^1$ to a point. This can only be done in the case of negative normal bundle, hence the choice of $-k$ neighbourhoods. Contracting this
curve produces a singular surface whenever $k > 1$. In fact, the case $k = 1$ — where the contraction produces the smooth surface $\mathbb{C}^2$ — allows for quantization without any rebel instantons, in contrast to what happens when $k > 1$.

For each surface $Z_k$ we compute all possible holomorphic Poisson structures, and then calculate explicitly corresponding deformation quantizations, writing out star products.

We study instantons and their moduli via the Kobayashi–Hitchin correspondence, that is, by constructing holomorphic vector bundles over these surfaces. We describe vector bundles concretely using matrices, in the spirit of the ADHM construction, but in a somewhat further simplified manner which is made possible by the filtrability of bundles on $Z_k$. Accordingly, an instanton or vector bundle on $Z_k$ can be described by a single upper-triangular matrix with polynomial entries. A similar use of filtrability allows us to describe vector bundles over noncommutative deformations and to describe how their moduli spaces change after deforming the surface.

Our main result is the following:

**Theorem (Thm. 9.6).** The quantum instanton moduli space $\mathbb{Q}_I^{[1]}(Z_k(\sigma))$ can be viewed as the étale space of a constructible sheaf over the classical instanton moduli space $\mathbb{M}_I(Z_k)$, which is supported on a closed subvariety, being trivial over

\[ S_0 := \{ P \in \mathbb{M}_I(Z_k) \mid p_{1,k-j+1} \neq 0 \} \]

and having stalk of dimension $i$ over

\[ S_i := \{ P \in \mathbb{M}_I(Z_k) \mid p_{1,k-j+1} = \cdots = p_{1,k-j+i-1} = 0, \ p_{1,k-j+i+1} \neq 0 \} . \]

Since we are dealing with moduli space of bundles on non-compact varieties, the topology of moduli spaces of bundles is rather subtle, already for the classical moduli space $[BGK2]$. A detailed analysis of the topology of the quantum moduli space would certainly be important and might provide insight into further interesting phenomena, such as studying the effect of evaluating the formal deformation parameter $\hbar$ to a non-zero constant where possible. However, these considerations would deserve a separate treatment and we thus choose to sideline these issues for the present article by simply viewing the quantum moduli space as the étale space (or sheaf space) of a constructible sheaf, constant over each stratum, making the map to the classical moduli space continuous.

Nonetheless Thm. 9.6 already shows that the effect of noncommutative deformations on instantons is radically different from the effect of commutative ones. In fact, [BaG, Thm. 7.3] showed that (nontrivial) classical deformations of $Z_k$ admit no instantons, although the surfaces $Z_k$ have rich instanton moduli spaces. (See Not. 9.2 for the representation of instantons in canonical coordinates.)

We now describe some of the literature on the subject and next the structure of this paper. Instantons on noncommutative spaces have been considered from various points of view, starting with instantons on noncommutative $\mathbb{R}^4$ and noncommutative tori [NS] and their relations with string theory [SW]. Nekrasov–Schwarz [NS] proposed a modification of the ADHM equations, which describe instantons on a noncommutative $\mathbb{R}^4$. Kapustin–Kuznetsov–Orlov [KKO] showed that the complex point of view can also be generalized to the noncommutative setting, identifying these noncommutative instantons with algebraic vector bundles on a noncommutative projective plane $\mathbb{P}^2_\hbar$ framed at a line at infinity. The space of solutions to these modified ADHM equations turns out to yield a smooth compactification of the moduli space of instantons on the (commutative) $\mathbb{R}^4$, so that the
noncommutative viewpoint also sheds light onto classical instantons. (Other interesting features and generalizations are as follows. This compactification can be viewed as the moduli space of torsion-free sheaves on $\mathbb{P}^2$ framed at a fixed line at infinity, see Nakajima [N]. Furthermore, from the point of view of instanton counting, moduli of torsion-free sheaves were extensively used as partial compactifications of moduli of instantons, in the very successful instanton partition function defined by Nekrasov, and explored in [NO, NY, GL]. There are also approaches to noncommutative instantons from the point of view of noncommutative geometry as for example in [BvS, CLS]. We shall not pursue these aspects here.)

Classical $\text{SU}(2)$ instantons on $\mathbb{C}P^2$ can be identified with holomorphic rank 2 bundles on $\mathbb{C}P^2$ via the Kobayashi–Hitchin correspondence, which for $k = 1$ was proven by King [Ki] and for $k \geq 2$ in [GKM]. Considering deformation quantizations $A$ of the sheaf $\mathcal{O}_{\mathbb{C}P^2}$ we obtain instantons on noncommutative deformations of $\mathbb{C}P^2$ as locally free sheaves of $A$-modules, generalizing the concept of holomorphic vector bundle to these noncommutative spaces. In general the surfaces $Z_k$ do not admit locally constant holomorphic Poisson structures, and instead of the Moyal product we shall thus consider the Kontsevich star product extended to these surfaces.

The paper is organized as follows. In §§2–4 we describe the noncommutative deformation theory of $\mathbb{C}P^2$. In §5 we review the theory of vector bundles on $\mathbb{C}P^2$ and their commutative deformations and discuss vector bundles on noncommutative deformations of $\mathbb{C}P^2$ in §6. We use the explicit expressions of star products obtained in §4 to show in §7 that purely noncommutative deformations have nontrivial moduli of vector bundles. Applications to (noncommutative) instantons are described in §§8–9.

2. Poisson geometry

In this section we study the holomorphic Poisson geometry of the surfaces $Z_k = \text{Tot} \mathcal{O}_{\mathbb{P}^1}(-k)$ for $k \geq 1$, which will be the starting point for quantizations in subsequent sections.

**Definition 2.1.** [LPV] A holomorphic Poisson bracket on a complex manifold or smooth complex algebraic variety $X$ is a $\mathbb{C}$-bilinear map

$$\{ \cdot, \cdot \} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$$

satisfying

- $\{ f, g \} = -\{ g, f \}$ \hspace{1cm} (skew-symmetry)
- $\{ fg, h \} = f \{ g, h \} + \{ f, h \} g$ \hspace{1cm} (Leibniz rule)
- $\{ \{ f, g \}, h \} = \{ f, \{ g, h \} \} + \{ \{ f, h \}, g \}$ \hspace{1cm} (Jacobi identity)

for all $f, g, h \in \mathcal{O}_X$.

Equivalently, a holomorphic Poisson structure may be described by a holomorphic bivector field $\sigma \in \mathcal{H}^0(X, \Lambda^2T_X)$ whose Schouten–Nijenhuis bracket $[\sigma, \sigma] \in \mathcal{H}^0(X, \Lambda^3T_X)$

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1Here we write $f, g, h \in \mathcal{O}_X$ for sections of $\mathcal{O}_X$ over some open set. More precisely, a holomorphic Poisson bracket on $\mathcal{O}_X$ is a family of Poisson brackets indexed by the open sets of $X$ and compatible with the restriction morphisms of the sheaf $\mathcal{O}_X$. 

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is zero. The associated Poisson bracket is then given by the pairing \( \langle \cdot , \cdot \rangle \) between vector fields and forms

\[
\{ f , g \} = \langle \sigma , df \wedge dg \rangle.
\]

**Remark 2.2.** On a smooth surface \( X \) the condition \([\sigma , \sigma ] = 0\) is satisfied for any bivector field \( \sigma \in H^0(\Lambda^2 T_X) \) since \( \Lambda^3 T_X = 0 \), and thus \( H^0(\Lambda^2 T_X) \) may be identified with the space of holomorphic Poisson structures.

We now focus on the surfaces \( Z_k \) and their Poisson structures, which we shall describe explicitly in canonical coordinates.

**Notation 2.3.** We fix coordinate charts \( U, V \) on \( Z_k \), which we will refer to as *canonical coordinates*, where

\[
U = \mathbb{C}^2_{z,u} = \{(z, u) \in \mathbb{C}^2\} \quad \text{and} \quad V = \mathbb{C}^2_{\xi,\nu} = \{ (\xi, \nu) \in \mathbb{C}^2 \}
\]

such that on \( U \cap V = \mathbb{C}^+ \times \mathbb{C} \) we identify

\[
(\xi, \nu) = (z^{-1}, z^k u).
\]

We denote by \( \ell \) the ideal of the zero section of \( Z_k \) regarded as a divisor. Hence \( \ell \) is generated by \( u \) on the \( U \)-chart, and by \( v \) on the \( V \)-chart.

In canonical coordinates, a holomorphic bivector field \( \sigma \in H^0(\Lambda^2 \mathcal{T}_{Z_k}) \) is of the form \( \sigma_U \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u} \) on \( U \) and \( \sigma_V \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial \nu} \) on \( V \), where \( \sigma_U \) (resp. \( \sigma_V \)) is a holomorphic function in \( z \) and \( u \) (resp. \( \xi \) and \( \nu \)) whose precise form will be given in Lem. 2.6.

Given a bivector field \( \sigma \) on \( Z_k \) the corresponding *Poisson bracket* \( \{ f , g \}_\sigma \) of two global functions \( f, g \in H^0(Z_k, \mathcal{O}) \) may be written in canonical coordinates as

\[
\{ f , g \}_\sigma|_U = \left( \sigma_U \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u} , df \wedge dg \right) = \sigma_U \left( \frac{\partial f_U}{\partial z} \frac{\partial g_U}{\partial u} - \frac{\partial f_U}{\partial u} \frac{\partial g_U}{\partial z} \right),
\]

\[
\{ f , g \}_\sigma|_V = \left( \sigma_V \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial \nu} , df \wedge dg \right) = \sigma_V \left( \frac{\partial f_V}{\partial \xi} \frac{\partial g_V}{\partial \nu} - \frac{\partial f_V}{\partial \nu} \frac{\partial g_V}{\partial \xi} \right),
\]

where \( d \) is the exterior derivative and \( f_U, g_U \) denote the restrictions of \( f, g \) to \( U \) written in \((z, u)\)-coordinates, and similarly for \( f_V, g_V \).

**Notation 2.5.** When referring to a bivector field \( \sigma \in H^0(\Lambda^2 \mathcal{T}_{Z_k}) \) or its corresponding Poisson structure, we will work in canonical coordinates in the basis \( \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial u} \) and \( \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial \nu} \) and only write its coefficient functions as a pair \( (\sigma_U, \sigma_V) \) — in fact, we often just write \( \sigma_U \) in \((z, u)\)-coordinates, as \( \sigma_V \) can be recovered by writing \(-z^{k-2} \sigma_U \) as a function of \( \xi \) and \( \nu \) via the change of variables \((2.5)\).

An application of the exponential sheaf sequence

\[
0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0
\]

shows that any line bundle on \( Z_k \) can be identified with the pullback \( \pi^* \mathcal{O}_\mathbb{P}^1(n) \) of the projection \( \pi : Z_k \rightarrow \mathbb{P}^1 \) to the zero section of \( Z_k \). We write \( \mathcal{O}_{Z_k}(n) \) or \( \mathcal{O}(n) \) for \( \pi^* \mathcal{O}_\mathbb{P}^1(n) \), whose change of coordinates from \( U \) to \( V \) can be given by the transition function \( z^{-n} \).

Here \( n \in \mathbb{Z} \) is the first Chern class of \( \mathcal{O}(n) \).
To calculate Poisson structures, note that $\mathcal{N}^2 T_{Z_k}$ is the anticanonical line bundle. The transition matrix for the tangent bundle of $Z_k$ is given in canonical coordinates by the Jacobian matrix of the change of coordinates $(z, u) \mapsto (z^{-1}, z^k u)$ of the manifold. For $Z_k$

$$\text{Jac}_{UV} = \begin{pmatrix} \frac{\partial z^{-1}}{\partial u} & \frac{\partial z^{-1}}{\partial v} \\ \frac{\partial z^k u}{\partial u} & \frac{\partial z^k u}{\partial v} \end{pmatrix} \equiv \begin{pmatrix} -z^{-2} & 0 \\ k z^{-1} u & z^k \end{pmatrix}.$$ (2.7)

The transition function for the anticanonical line bundle is then given by the determinant of this Jacobian, which is $-z^{k-2}$. We can thus identify $H^0(Z_k, \mathcal{N}^2 T_{Z_k}) \cong H^0(Z_k, \mathcal{O}(-k+2))$ as the space of Poisson structures.

**Lemma 2.8.** A general Poisson structure $\sigma \in H^0(Z_k, \mathcal{N}^2 T_{Z_k})$ on $Z_k$ is given in canonical coordinates by

$$\begin{pmatrix} \sigma_U \cdot \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}, \sigma_V \cdot \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} \end{pmatrix}$$

where $(\sigma_U, \sigma_V)$ is of the form

1. $(f_U + z g_U, -\xi f_V - g_V)$ for $k = 1$
2. $(f_U, -f_V)$ for $k = 2$
3. $(u f_U + z u g_U + z^2 u h_U, -\xi^2 v f_V - \xi v g_V - v h_V)$ for $k \geq 3$

for any global functions $(f_U, f_V), (g_U, g_V), (h_U, h_V) \in H^0(Z_k, \mathcal{O}_{Z_k})$.

In the basis $\left( \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} \right)$, the space of Poisson structures $H^0(Z_k, \mathcal{N}^2 T_{Z_k})$ is thus generated by

1. $(1, -\xi), (z, -1)$ for $k = 1$
2. $(1, -1)$ for $k = 2$
3. $(u, -\xi^2 v), (zu, -\xi v), (z^2 u, -v)$ for $k \geq 3$

as a module over global functions.

**Proof.** For (1) we need to calculate $H^0(Z_1, \mathcal{O}(1))$, which is Lem. A.1. For (2) and (3) we need to calculate $H^0(Z_k, \mathcal{O}(-k+2))$, which is Lem. A.2. \(\square\)

**Remark 2.9.** The space of Poisson structures on $Z_k$ is of infinite dimension over $\mathbb{C}$, but restricted to the $n$th infinitesimal neighbourhood $\ell^{(n)}$, the space of Poisson structures is of dimension

$$\left\lfloor \frac{n(n+4)}{2} \right\rfloor$$

for $k = 1$

$$n^2$$

for $k = 2$

$$\frac{n((n-1)k+4)}{2}$$

for $k \geq 3$.

**Notation 2.10.** Write $Z_{\geq m}$ for $Z_k$ with $k \geq m$.

To describe explicit quantizations of Poisson structures on non-affine varieties, the following property will be useful.

**Definition 2.11.** A bivector field $\sigma \in H^0(X, \mathcal{N}^2 T_X)$ is **tangent** to a divisor $D \subset X$ if

$$\{\mathcal{O}_X, \mathcal{I}_D\} \subset \mathcal{I}_D$$

i.e. if the ideal sheaf $\mathcal{I}_D$ of $D$ is a Poisson ideal. Geometrically, $\sigma$ is tangent to $D$ if for every function $f \in \mathcal{O}_X$, the restriction of the Hamiltonian vector field $X_f = \{f, \cdot \}_\sigma$ to the divisor $D$ is tangent to $D$. 
In §4 we shall quantize Poisson structures tangent to the complement of an affine coordinate chart, so we record their particular form.

**Proposition 2.12.** Consider the open immersion \( \mathbb{C}^2 \cong U \subset Z_k \) with complementary divisor \( D = Z_k \setminus U = \{(0,v) \in V\} \cong \mathbb{C} \). Then the space of Poisson structures tangent to \( D \) is generated by

\[
\begin{align*}
(1) & \quad (1, -\xi) \quad \text{for } k = 1 \\
(2) & \quad (u, -\xi^2 v), (zu, -\xi v) \quad \text{for } k \geq 2
\end{align*}
\]

as a module over global functions.

**Proof.** This follows from Lem. 2.8 and the observation that since \( D \setminus V = (\xi) \), the coefficient function \( \sigma_V \) of a Poisson structure \( (\sigma_U, \sigma_V) \) which is tangent to \( D \) should be a multiple of \( \xi \) in \( V \)-coordinates. \( \square \)

Poisson structures on \( Z_k \) are depicted in Fig. 1, where dots represent the monomials (in \( U \)-coordinates) which appear in the expression of a general Poisson structure on \( Z_k \) and circled dots those of a Poisson structure tangent to \( D = Z_k \setminus U \).

Recall that the \( r \)th degeneracy locus of a holomorphic Poisson structure on a complex manifold or algebraic variety \( X \) is defined as

\[ D_{2r}(\sigma) = \{ x \in X \mid \text{rank } \sigma(x) \leq 2r \}, \]

where \( \sigma \) is viewed as a map \( T_X^* \rightarrow TX \) by contracting a 1-form with the bivector field \( \sigma \). At a given point on a complex surface a holomorphic Poisson structure has either full rank, or rank 0. Thus, for the surfaces \( Z_k \) we call \( \text{dgn}(\sigma) := D_0(\sigma) \) the degeneracy locus of \( \sigma \) and this degeneracy locus is given by the zeros of the coefficient functions \( \sigma_U \) and \( \sigma_V \).

A non-degenerate holomorphic Poisson structure \( \sigma \) is called a **holomorphic symplectic structure** as \( \sigma \) determines a non-degenerate closed holomorphic 2-form \( \omega \) by

\[ \omega(X_f, X_g) = \{ f, g \}_\sigma \]

where \( X_f \) denotes the Hamiltonian vector field associated to a function \( f \).

As we shall see in Prop. 2.16, \( Z_2 \) is the only one of the surfaces \( Z_k \) that admits a holomorphic symplectic structure and when this structure is algebraic, it is unique up to scaling.
Example 2.13. If $\sigma$ is as in Lem. 2.8 with global functions $(f_U, f_V), (g_U, g_V), (h_U, h_V)$ simply (nonzero) constants, we have

$$d\text{gn}(\sigma) = \begin{cases} 
\pi^{-1}(x) & \text{for } k = 1 \\
\emptyset & \text{for } k = 2 \\
\pi^{-1}(x) \cup \pi^{-1}(y) \cup \ell & \text{for } k \geq 3 
\end{cases}$$

where $\ell \subset \mathcal{Z}_k$ is the zero section and $x, y \in \mathbb{P}^1$ are two points, possibly equal. These degeneracy loci are depicted in Fig. 2.

For $k = 2$ the Poisson structure is non-degenerate and defines a holomorphic symplectic form.

If $\sigma$ is a Poisson structure which is tangent to a fibre $D = \pi^{-1}(x) \simeq \mathbb{C}$ of the bundle projection $\pi : \mathcal{Z}_k \to \mathbb{P}^1$, the picture of Fig. 2 reduces to two cases:

$$d\text{gn}(\sigma) = \begin{cases} 
\pi^{-1}(x) & \text{for } k = 1 \\
\pi^{-1}(x) \cup \ell & \text{for } k \geq 2. 
\end{cases} \quad (2.14)$$

We can set $U = \mathcal{Z}_k \setminus D$ so that $x \in \mathbb{P}^1$ is given in canonical coordinates by $\xi = 0$ and in these coordinates the form of $\sigma$ is given in Prop. 2.12.

Equation (2.14) describes the degeneracy locus for a non-zero linear combination of generators of those Poisson structures tangent to a fibre $D = \pi^{-1}(x)$. The degeneracy locus of a general Poisson structure $\sigma$ tangent to $D$ may be more complicated. However, to construct quantizations of $\sigma$, it is only necessary to require that $d\text{gn}(\sigma)$ contain a fibre of the projection $\pi$. The effect of quantizations on moduli then only depends on whether $d\text{gn}(\sigma)$ also contains all of $\ell$ or not and so we introduce the following notation.

Notation 2.15. Write $\sigma_0$, respectively $\sigma$, for Poisson structures on $\mathcal{Z}_k$ such that

$$\ell \not\subset d\text{gn}(\sigma_0) \supset \pi^{-1}(x)$$

$$\ell \subset d\text{gn}(\sigma) \supset \pi^{-1}(x)$$

respectively.

Note that $\sigma_0$ is a “minimally degenerate” Poisson structure on $\mathcal{Z}_1$, whereas $\sigma$ denotes Poisson structures which are degenerate on all of $\ell$, which can occur on $\mathcal{Z}_k$ for all $k \geq 1$.

Proposition 2.16. $\mathcal{Z}_k$ admits a holomorphic symplectic structure if and only if $k = 2$. Moreover, if the holomorphic symplectic structure on $\mathcal{Z}_2$ is algebraic, then it is a constant multiple of the canonical symplectic structure on $\mathcal{Z}_2 \simeq \mathbb{T}^*\mathbb{P}^1$.

Proof. A holomorphic symplectic structure corresponds to a nowhere vanishing global holomorphic section of $\Lambda^2 \mathcal{T}_{\mathcal{Z}_k} \simeq \mathcal{O}(-k+2)$, which exists only when this bundle is trivial, i.e. when $k = 2$.

For $\mathcal{Z}_2$, the canonical non-degenerate Poisson structure is given by $(1, -1)$, so that in coordinates the bracket of two functions $f, g$ is given by

$$\{f, g\}|_U = \left\langle \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial \eta}, df \wedge dg \right\rangle$$

$$\{f, g\}|_V = -\left\langle \frac{\partial}{\partial \xi} \wedge \frac{\partial}{\partial \eta}, df \wedge dg \right\rangle.$$
If a non-degenerate Poisson structure on $Z_2$ is algebraic, i.e. an algebraic section of $\Lambda^2 T_{Z_2}$, it is a constant multiple of the canonical non-degenerate Poisson structure, as every non-constant polynomial in $z, u$ has at least one zero. (This does not hold in the analytic category, as there are non-constant non-vanishing complex analytic sections.) □

**Alternative proof.** We shall see in §9 that $Z_k$ is the minimal resolution of the $(1,1)$ surface singularity $X_k = \mathbb{C}^2/\Gamma$, where $\Gamma$ is generated by $\gamma = \left( \begin{smallmatrix} \omega & 0 \\ 0 & \omega \end{smallmatrix} \right)$ for $\omega$ a primitive $k$th root of unity. A resolution of $\mathbb{C}^2/\Gamma$ admits a holomorphic symplectic form if and only if $\Gamma \subset SL(2, \mathbb{C})$. However, $\det \gamma = \omega^2 = 1$ if and only if $k = 2$. □

**Remark 2.17.** Here we discuss the case of degenerate Poisson structures on $Z_k$, leaving the holomorphic symplectic case on $Z_2$ for future work. More precisely, our construction works for Poisson structures tangent to a fibre of $\pi : Z_k \to \mathbb{P}^1$ (see Def. 2.11 and Prop. 4.1), which the holomorphic symplectic structure on $Z_2$ does not satisfy (cf. Rem. 4.9).

### 3. Deformation quantization and star products

Let $\hbar$ be a formal variable and denote by $\mathcal{O}_X[\hbar]$ the completed tensor product $\mathcal{O}_X \hat{\otimes} \mathbb{C}[\hbar]$, viewed as a sheaf of $\mathbb{C}[\hbar]$-vector spaces on $X$, where a “section” over an open set $U$ is given by a formal power series $f = \sum_0^\infty f_n \hbar^n$ with each $f_n \in \mathcal{O}_X(U)$. We shall turn $\mathcal{O}_X[\hbar]$ into a sheaf of associative $\mathbb{C}[\hbar]$-algebras by formally deforming the usual commutative product on functions. The augmentation $\mathbb{C}[\hbar] \to \mathbb{C}$ induces an augmentation $\mathcal{O}_X[\hbar] \to \mathcal{O}_X$.

**Definition 3.1.** A star product on a complex manifold (or smooth complex algebraic variety) $X$ is a $\mathbb{C}[\hbar]$-bilinear associative product

$$\star : \mathcal{O}_X[\hbar] \times \mathcal{O}_X[\hbar] \to \mathcal{O}_X[\hbar]$$

which is of the form

$$(f, g) \mapsto fg + \sum_{n=1}^\infty B_n(f, g) \hbar^n$$

where the $B_n$ are bidifferential operators, i.e. bilinear operators which are differential operators in both arguments.
Definition 3.2. [Y, §0.1] Two star products $\star, \star'$ on $X$ are said to be gauge equivalent, if there exists an isomorphism $(\mathcal{O}_X [\hbar], \star) \cong (\mathcal{O}_X [\hbar], \star')$ which commutes with the augmentations $\mathcal{O}_X [\hbar] \to \mathcal{O}_X$.

Definition 3.3. Let $(X, \sigma)$ be a holomorphic Poisson manifold with associated Poisson bracket $\{\cdot, \cdot\}_\sigma$. A deformation quantization of $(X, \sigma)$ is a pair $(X, \star_\sigma)$, where $\star_\sigma$ is a star product on $X$ with $B_1(f, g) = \{f, g\}_\sigma$, that is

$$f \star_\sigma g = fg + \{f, g\}_\sigma \hbar + \cdots$$

We set $\mathcal{A}^\sigma := (\mathcal{O}[\hbar], \star_\sigma)$ the sheaf of formal functions with holomorphic coefficients on the quantization $(X, \star_\sigma)$.

We call $Z_k(\sigma) = (Z_k, \mathcal{A}^\sigma)$ a noncommutative deformation of $Z_k$. As we usually work with a specified fixed Poisson structure, we usually use the abbreviated notations $\mathcal{A}, \{\cdot, \cdot\}$ and $\star$.

The existence of star products was first proved in the $C^\infty$ setting in Kontsevich’s seminal paper [Ko1] and later generalized to the algebro-geometric setting [Ko2, Y]. We quote this generalization in a simplified form.

Theorem 3.4. [Y, Cor. 11.2] Let $X$ be a complex algebraic variety with structure sheaf $\mathcal{O}_X$ and assume that $H^1(X, \mathcal{O}_X)$ and $H^2(X, \mathcal{O}_X)$ vanish. Then there is a bijection

$$\{\text{Poisson deformations of } \mathcal{O}_X\}/\sim \leftrightarrow \{\text{associative deformations of } \mathcal{O}_X\}/\sim$$

where $\sim$ denotes gauge equivalence.

Note that the surfaces $Z_k$ satisfy the hypothesis of Thm. 3.4.

Remark 3.5. Working with the sheaf of algebras $(\mathcal{O}_X [\hbar], \star)$ we are restricting ourselves to deformations which are in some sense “purely noncommutative”. More generally [Ko2, KS, Y], in the non-affine setting one may consider formal deformations of $\mathcal{O}_X$, which do not necessarily have $\mathcal{O}_X [\hbar]$ as underlying sheaf of $\mathbb{C}[\hbar]$-vector spaces, but could simultaneously deform $\mathcal{O}_X$ in some commutative direction, corresponding to simultaneously deforming the restriction morphisms of the sheaf $\mathcal{O}_X$. Although $Z_k$ does admit commutative deformations, we are restricting ourselves to these purely noncommutative directions for two reasons. On the one hand, we expect that turning on a commutative direction of deformation would imply that the moduli spaces of vector bundles become very small or even trivial, as is the case for the purely commutative deformations [BaG]. On the other hand, it seems difficult to obtain explicit simultaneous deformations in the generality we achieve in this article, as it was shown in [BaF, §5.3] that simultaneous commutative and noncommutative deformations of $Z_k$ may be obstructed.

3.1. The Kontsevich star product on $\mathbb{C}^d$. As part of an explicit quasi-isomorphism of $L_\infty$ algebras in the proof of his formality theorem, Kontsevich [Ko1] gave an explicit construction for a star product on $(\mathbb{R}^d, \pi)$ for some (real) Poisson structure $\pi$. The formula applies without change to the complex setting for a (holomorphic) Poisson structure on $\mathbb{C}^d$, which we will use in §4 to give explicit star products on $Z_k$.

We briefly recall the definition of this star product and refer to [Ko1, Ko2] for details.
Definition 3.6. Let $\sigma$ be a Poisson structure on $\mathbb{C}^d$. The Kontsevich star product $\star^K_{\sigma}$ on $\mathbb{C}^d$ is given by

$$ f \star^K_{\sigma} g = fg + \sum_{n=1}^{\infty} \hbar^n \sum_{\Gamma \in \mathcal{G}_{n,2}} w_{\Gamma}(f, g) $$  \hspace{1cm} (3.7)  

where

- $\mathcal{G}_{n,2}$ is the set of admissible graphs with $n$ unfilled ("first type") and 2 filled ("second type") vertices,
- $B_\Gamma$ is the bidifferential operator for $\sigma$ associated to the graph $\Gamma$, and
- $w_\Gamma$ is the weight of the graph $\Gamma$ obtained as the integral over a certain configuration space.

An admissible graph $\Gamma \in \mathcal{G}_{n,2}$ has two filled vertices representing the two entries of $B_\Gamma$ and $n$ unfilled vertices. Arrows which start at unfilled vertices represent derivatives of the target of the arrow. We shall use the notation $\partial_i$ for the derivative with respect to the $i$th coordinate of $\mathbb{C}^d$ and write $\sigma^{ij}$ for the coefficient function of $\partial_i \land \partial_j$. The $\sigma^{ij}$ are holomorphic functions on $\mathbb{C}^d$ and define a skew-symmetric $d \times d$ matrix.

Example 3.8. The bidifferential operator for the graph $\Gamma$ of Fig. 3 is given by

$$ B_\Gamma(f, g) = \sum_{1 \leq i, j, k, l, m, n \leq d} \sigma^{ij} \partial_j(\sigma^{kl}) \partial_l(\sigma^{mn}) \partial_k \partial_m(f) \partial_n(g). $$

The graphs with arrows ending in unfilled vertices represent bidifferential operators which take derivatives of $\sigma^{ij}$. When the Poisson structure $\sigma$ is constant and non-degenerate, i.e.

$$ \sigma = \sum_{i, j} \sigma^{ij} \partial_i \land \partial_j $$ \hspace{1cm} $\sigma^{ij} = -\sigma^{ji} \in \mathbb{C}$

graphs with arrows ending in unfilled vertices represent the zero operator and thus, for each $n$, there is only one graph in $\mathcal{G}_{n,2}$ which contributes to $B_n$ (see Fig. 4). For $\sigma$ constant, the Kontsevich star product $\star^K_{\sigma}$ then coincides with the Moyal product $\star^M_{\sigma}$ given
The graphs in $G_{n,2}$ for $n = 1, 2, 3, 4$ contributing to $B_n$ in the Moyal product.

by

$$f \star_M g = fg + \hbar \sum_{i,j} \sigma^{ij} \partial_i (f) \partial_j (g) + \frac{\hbar^2}{2} \sum_{i,j,k,l} \sigma^{ij} \sigma^{kl} \partial_i \partial_k (f) \partial_j \partial_l (g) + \cdots$$

$$= \sum_{n=0}^{\infty} \hbar^n \sum \prod_{k=1}^{n} \sigma^{i_k j_k} \times \prod_{k=1}^{n} \partial_{i_k} (f) \times \prod_{k=1}^{n} \partial_{j_k} (g), \quad (3.9)$$

where the symbol $\times$ denotes the usual product.

In general a holomorphic Poisson structure $\sigma$ will not be constant, not even locally. (For the surfaces $Z_k$ see Lem. 2.8 and Prop. 2.16.) In this case the formula for the Kontsevich star product also involves derivatives of the coefficient functions $\sigma^{ij}$.

Lemma 3.10. [Di] Up to second order in $\hbar$ the Kontsevich star product for $\sigma$ on $\mathbb{C}^d$ is given by

$$f \star^K g = fg + \hbar \sum_{i,j} \sigma^{ij} \partial_i (f) \partial_j (g) + \frac{\hbar^2}{2} \sum_{i,j,k,l} \sigma^{ij} \sigma^{kl} \partial_i \partial_k (f) \partial_j \partial_l (g) + \cdots$$

To give the Kontsevich star product explicitly to higher orders one would have to compute the weights of all admissible graphs. Already for a single graph this computation is non-trivial, yet necessary even if one were restrict oneself to, say, linear Poisson structures. For example, Felder–Willwacher [FW] computed the weight of the graph shown in Fig. 5 to be $\zeta (3) \hbar^2 / 182$ up to rationals, omitting 50 terms known to be rational. (The precise weight was recently given as $\frac{13}{3000} \pi^2 + \frac{\zeta (3)}{260} \pi^4$ in [BPP], which relates the

Figure 4. The graphs in $G_{n,2}$ for $n = 1, 2, 3, 4$ contributing to $B_n$ in the Moyal product.
The graph of $[FW]$ contributing to $B_7$ with weight $\zeta(3)^2/\pi^6$ up to rationals.

weights to multiple zeta values with integer coefficients.) As each unfilled vertex in this graph has only one ingoing arrow, the associated bidifferential operator is non-zero even for linear Poisson structures, and thus it will contribute to the expression of $B_7$ for any non-constant Poisson structure.

For the computations of cohomology and Ext$^i$ groups in §6.2 we will not need the precise weights in the Kontsevich star product as it suffices to know the possible bidifferential operators that appear in its expression.

3.2. **Star products on $\mathbb{Z}_k$.** The cohomology of $\mathbb{Z}_k$ can be described by the cohomology of the commutative diagram

$$
\begin{array}{ccc}
Z_k & \longrightarrow & V \\
\downarrow & & \downarrow \\
U \cap V & \longrightarrow & V \\
U & \longrightarrow & V
\end{array}
$$

(3.11)

where the arrows are inclusions of open sets.

Dual to (3.11), we have a commutative diagram of (commutative) algebras

$$
\begin{array}{ccc}
\mathcal{O}(Z_k) & \longrightarrow & \mathcal{O}(V) \\
\downarrow & & \downarrow \\
\mathcal{O}(U) \cap \mathcal{O}(V) & \longrightarrow & \mathcal{O}(V)
\end{array}
$$

where the arrows are the restriction morphisms.
To define a star product on $Z_k$, one has to find star products on $\mathcal{O}[\hbar](W)$, where $W \in \{Z_k, U, V, U \cap V\}$ making

$$
\mathcal{O}[\hbar](Z_k) \xrightarrow{\star} \mathcal{O}[\hbar](U) \xrightarrow{\star} \mathcal{O}[\hbar](V) \xrightarrow{\star} \mathcal{O}[\hbar](U \cap V)
$$

(3.12)

a commutative diagram of associative algebras.

We now describe quantizations of Poisson structures explicitly.

4. Quantizing degenerate Poisson structures

To obtain explicit examples of deformation quantizations of Poisson structures on $Z_k$, we adapt the construction given in Kontsevich [Ko2] of a quantizable compactification of a smooth affine Poisson variety. We show that the open immersion of a coordinate chart $U \subset Z_k$ is quantizable in case the Poisson structure is tangent to the divisor $D = Z_k \setminus U$, see Def. 2.11.

**Proposition 4.1.** Let $\sigma$ be a Poisson structure on $Z_k$ such that is tangent to $D = Z_k \setminus U$. Then $U \subset Z_k$ is a quantizable open immersion, i.e. the Kontsevich star product on $U$ extends to a global star product on $Z_k$.

**Proof.** Let $D = Z_k \setminus U$ and write the algebras $\mathcal{O}(U \cap V), \mathcal{O}(U), \mathcal{O}(V), \mathcal{O}(Z_k)$ in the $\mathbb{C}$-vector space basis $\{z^j u^l\}$ where $i \geq 0$ and

$$
\begin{align*}
-\infty < l < \infty & \quad \text{on } U \cap V \\
0 \leq l < \infty & \quad \text{on } U \\
-\infty < l \leq ki & \quad \text{on } V \\
0 \leq l \leq ki & \quad \text{on } U \cup V = Z_k.
\end{align*}
$$

(4.2)

The case $k = 1$. As seen in Fig. 1, a Poisson structure on $Z_1$ tangent to $D$ is generated by the monomial 1 in $U$-coordinates, i.e. such a Poisson structure is of the form $\sigma_U = f$ for some global function $f \in \mathcal{H}^0(Z_1, \mathcal{O})$.

First assume that $f$ is constant. The Kontsevich star product on $U \cong \mathbb{C}^2$ thus coincides with the Moyal product (3.9). Denoting by $\star$ the restriction of this star product to $U \cap V$, the star product of two arbitrary monomials $z^{i_1} u^{l_1}, z^{i_2} u^{l_2} \in \mathcal{O}(U \cap V)$ is then given by

$$
z^{i_1} u^{l_1} \star z^{i_2} u^{l_2} = \sum_{n \geq 0} a_n z^{i_1 + l_1 - n} u^{i_2 + l_2 - n} \hbar^n
$$

(4.3)

for some constant coefficients $a_n \in \mathbb{Q}$ depending only on $l_1, i_2, i_1, i_2$. In particular,

$$
a_0 = 1 \\
a_1 = l_1 i_2 - l_2 i_1 \\
a_2 = \frac{1}{2} (l_1 (l_1 - 1) i_2 (i_2 - 1) - 2 l_1 l_2 i_1 i_2 + l_2 (l_2 - 1) i_1 (i_1 - 1)).
$$
If both \((l_1, i_1)\) and \((l_2, i_2)\) satisfy one and the same bound of (4.2), then so does \((\max\{0, l_1 + l_2 - n\}, i_1 + i_2 - n)\). Hence, \(\ast\) preserves the subalgebras \(\mathcal{O}(Z_i), \mathcal{O}(U),\) and \(\mathcal{O}(V)\) of \(\mathcal{O}(U \cap V)\) and thus defines a global star product.

If \(f\) is not constant, its power series expansion in \(U\)-coordinates is of the form 
\[
 f(x) = \sum_{i=0}^{\infty} \sum_{l=0}^{d_i} f_{i,l} x^l \frac{\partial^l}{\partial x^l}.
\] (Note that \(f\) being a global function means that the monomials appearing in the power series expansion of \(f\) satisfy all of the bounds (4.2).

Now the Kontsevich star product also has contributions coming from graphs \(\Gamma \in \mathcal{G}_{n,2}\) with unfilled vertices representing a copy of the Poisson structure \(\sigma_U = f\). However, an unfilled vertex lowers the powers of \(z\) and \(u\) each by 1, but simultaneously multiplies by \(\sigma_U = f\), so that the exponents still satisfy the bounds (4.2).

The case \(k \geq 2\). Poisson structures tangent to \(D\) are generated by the monomials \(u, zu\) over global functions (see Prop. 2.12 and Fig. 1). The proof is now the same as for \(k = 1\) with \(\sigma_U\) having vanishing constant term.

\[\square\]

**Remark 4.4.** The open immersion \(U \subset Z_k\) satisfies some of the hypotheses of a quantizable compactification in the sense of [Ko2, Def. 4], except for \(Z_k\) not being compact and the divisor \(D = Z_k \setminus U\) not being ample. However, the complement of \(D\) is affine and even isomorphic to \(\mathbb{C}^2\), which is the deciding factor for the construction. So, even though we do not have a compactification, we still obtain a quantization.

Given a quasi-projective variety \(X\) and an affine subvariety \(U \subset X\), one could of course consider a quantizable smooth compactification \(U \subset X \subset \overline{X}\) and consider quantizations of Poisson structures which are tangent to \(\overline{X} \setminus U\). However, if one is interested in quantizations of non-compact \(X\), this strategy is more restrictive. For the case at hand, we have a smooth compactification \(U \subset Z_k \subset F_k\) to the \(k\)th Hirzebruch surface, but the space of Poisson structures which are tangent to the divisor \(F_k \setminus U\) is of dimension 3 for \(k = 1\) and of dimension 2 for \(k \geq 2\), whereas the space of quantizable Poisson structures for the open immersion \(U \subset Z_k\) is infinite-dimensional.

**Example 4.5.** Let \(\sigma\) be the Poisson structure \((1, -\xi)\) on \(Z_1\) and consider the quantizable open immersion \(U \subset Z_1\). Prop. 2.12 shows that \((1, -\xi)\) is tangent to \(D = Z_1 \setminus U = \{\xi = 0\}\) and Prop. 4.1 shows that there exists a global star product on \(Z_1\), which can be computed in canonical coordinates by the Moyal product on \(U\).

The corresponding star product on the \(V\) chart is then given, up to second order, by:

\[
f \ast g = fg - \hbar \left( \frac{\xi}{2} \partial_\xi(f) \partial_v(g) - \xi \partial_v(f) \partial_\xi(g) \right)
+ \hbar^2 \left( \frac{\xi^2}{2} \partial_\xi^2(f) \partial_v^2(g) - \xi^2 \partial_v \partial_v(f) \partial_\xi \partial_v(g) + \frac{\xi^2}{2} \partial_v^2(f) \partial_\xi^2(g)
+ \xi \partial_\xi(f) \partial_\xi^2(g) + \xi \partial_v(f) \partial_\xi \partial_v(g)
+ \xi \partial_v \partial_v(f) \partial_v(g) + \xi \partial_v^2(f) \partial_v(g)
- \partial_v(f) \partial_v(g)
- \nu \partial_v(f) \partial_v^2(g) - \nu \partial_v^2(f) \partial_v(g) \right).
\]
To verify this, rewrite the Moyal product on $U$ using the identities
\[
\partial_\xi = -\xi^2 \partial_\xi + \xi v \partial_v \\
\partial_v = \xi^{-1} \partial_v
\]
obtained by the change of coordinates and the commutation relations
\[
[\partial_\xi, \xi] = \text{id}, \quad [\partial_v, v] = \text{id}
\]
where $\xi, v$ in (4.6) are thought of as differential operators of order 0.

The following proposition shows that certain Poisson structures which in canonical coordinates do not satisfy the hypotheses of Prop. 4.1 may still be quantized by the same method after performing a linear change of coordinates.

**Proposition 4.7.** Let $\sigma = (\sigma_U, \sigma_V)$ be a Poisson structure on $Z_k$ which on $U$ is of the form $\sigma_U = u^d P(z)$ for $d \geq 1$ and $P(z)$ a polynomial in $z$. Then there exists a linear change of coordinates $U \cong U'$ such that $U' \subset Z_k$ is a quantizable open immersion.

**Proof.** First note that by Lem. 2.8, $P(z)$ is a polynomial of degree $n = (d-1)k + 2$. We can thus write
\[
\sigma_U = u^d \prod_{1 \leq i \leq n} (z - \lambda_i).
\]
Now, choose any $\lambda_i$ and consider the linear change of coordinates $U \cong U'$ given by $(z, u) \mapsto (z', u)$ for $z' = z + \lambda_i$. In these coordinates we have that
\[
\sigma_U = z'u^d \prod_{i \neq j} (z' - \lambda_j).
\]
Then $\sigma$ is tangent to the divisor $\{z' = 0\}$ and $U' \subset Z_k$ is a quantizable open immersion. \(\square\)

**Corollary 4.8.** For $k \neq 2$, any linear combination of the generators of Poisson structures on $Z_k$ can be quantized via an open immersion $C^2 \cong U' \subset Z_k$.

For the holomorphic symplectic structure on $Z_2$ the same method does not apply (see Kontsevich [Ko1, §3.5] for more details).

**Remark 4.9.** The Moyal product on $U \cong C^2$, does not extend naively to a star product on $Z_2$. In fact, the Moyal product on $\mathcal{O}[\hbar](U)$ does not preserve the subalgebra $\mathcal{O}[\hbar](Z_k)$. Recall from (3.9) that the bidifferential operators $B_n$ in the expression of the Moyal product on $C^2_{z,u}$ are given by
\[
B_n = \frac{1}{n!} \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} \partial_z^{n-i} \partial_u f \partial_z^i \partial_u^{n-i} g.
\]
Now consider the star product of the functions $zu, z^2 u \in \mathcal{O}(Z_2) \subset \mathcal{O}(U)$. Then
\[
z^2 u \star zu = z^3 u^2 + B_1(z^2 u, zu) \hbar + B_2(z^2 u, zu) \hbar^2 + \cdots
\]
\[
= z^3 u^2 + 2z^2 u \hbar + 2z \hbar^2 + \cdots
\]
But $2z \notin \mathcal{O}(Z_2)$, so $z^2 u \star zu \notin \mathcal{O}[\hbar](Z_k)$. 


5. Geometry of commutative instantons

In this section we summarize the known results about vector bundles and moduli for commutative deformations of $Z_k$ which will be used to construct the noncommutative counterparts in §6.

As seen in §2, holomorphic line bundles on $Z_k$ are classified by their first Chern class and we denote by $O_{Z_k}(n)$ the line bundle with first Chern class $n$, omitting the subscript when it is clear from the context. Recall that a rank $r$ bundle $E$ on $X$ is called filtrable if there exists an increasing filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{r-1} \subset E_r = E$ of subbundles such that $E_i/E_{i-1} \in \text{Pic} \, X$, where $1 \leq i \leq r$.

**Theorem 5.1.** [G1, Lem. 3.1, Thm. 3.2] Holomorphic vector bundles on $Z_k$ are algebraic and filtrable.

**Definition 5.2.** [B] Let $E$ be a rank $r$ holomorphic vector bundle on $Z_k$. The restriction of $E$ to the zero section $\ell \simeq \mathbb{P}^1$ is a rank $r$ bundle on $\mathbb{P}^1$, which by Grothendieck’s lemma splits as a direct sum of line bundles. Thus, $E|_{\ell} \simeq O_{\mathbb{P}^1}(j_1) \oplus \cdots \oplus O_{\mathbb{P}^1}(j_r)$. We call $(j_1, \ldots, j_r)$ the splitting type of $E$. When $E$ is a rank 2 bundle with first Chern class 0, then the splitting type is $(j, -j)$ for some $j \geq 0$ and we say for short that $E$ has splitting type $j$.

**Remark 5.3.** Filtrability of vector bundles on $Z_k$ implies that moduli of rank 2 vector bundles are parametrized by classes in $\text{Ext}^1(O(j_2), O(j_1))$ and algebraicity implies that such spaces of extensions are finite dimensional. For suitable numerical invariants or a suitable notion of stability, one may extract finite-dimensional moduli spaces from the naive quotient of the vector spaces $\text{Ext}^1(O(j_2), O(j_1))$ modulo bundle isomorphisms.

For applications to SU$(2)$ instantons, one considers bundles with vanishing first Chern class. Thus, we study the quotient $\text{Ext}^1(O(j), O(-j))/\sim$, where $\sim$ denotes bundle isomorphism. However, this quotient has an extremely complicated structure, in particular it is non-Hausdorff in the analytic topology. A stratification into Hausdorff components was presented in [BGK2, Thm. 4.15] by using two numerical invariants, whose sum makes up the local second Chern class (see Rem. 5.7).

To define the local Chern class, consider the (affine) surface $X_k$ obtained by contracting the zero section $\ell \subset Z_k$ to a point. Then $X_k \simeq \mathbb{C}^2$ and for $k \geq 2$ one obtains the $\frac{1}{k}(1, 1)$ surface singularity and $\tau: Z_k \rightarrow X_k$ is its (toric) resolution, which is given by inclusion of fans as shown in Fig. 6.

**Remark 5.4.** Observe that $X_k \simeq \mathbb{C}^2/\Gamma$, where $\Gamma \subset \text{GL}(2, \mathbb{C})$ is a cyclic group of order $k$ with a generator acting on $\mathbb{C}^2$ via multiplication by $\gamma = \left( \begin{smallmatrix} \omega & 0 \\ 0 & \omega^k \end{smallmatrix} \right)$ for $\omega$ a primitive $k$th root of unity. Thus $X_k$ is the $A_1$ surface singularity, but for $k \geq 3$ we have that det $\gamma = \omega^2 \neq 1$. In particular, $\Gamma \not\subset \text{SU}(2)$ so that $Z_{\geq 3}$ is not an ALE space, see [Kr, Thm. 1.2].

**Definition 5.5.** Let $E$ be a holomorphic rank 2 bundle on $Z_k$ and let $\tau: Z_k \rightarrow X_k$ be the contraction map. The local second Chern class of $E$ is the local holomorphic Euler characteristic of $E$ around $\ell$, that is,

$$\chi(\ell, E) = h^0(X_k, (\tau_* E)^{\vee \vee} / \tau_* E) + h^0(X_k, R^1 \tau_* E).$$ (5.6)

**Remark 5.7.** The terms on the right-hand side of (5.6) define two independent holomorphic invariants of the vector bundle $E$. In [BGK2] these invariants were called the width of $E$,
since $h^0(X_k, (\tau_* E)\otimes /_h /_E)$ measures the default of the direct image from being locally free, and the height of $E$, since $h^0(X_k, R^1 \tau_* E)$ measures how far $E$ is from being a split extension. These are two independent numerical invariants, and the pair stratifies the moduli spaces $M(Z_k)$ into Hausdorff components [BGK2, Thm. 4.15].

To describe actual moduli spaces, [GKM, Def. 5.2] give an ad-hoc definition of stability, calling a rank 2 vector bundle on $Z_k$ (framed) stable when it is holomorphically trivial (and framed) on $Z_k \setminus \ell$.

**Notation 5.8.** Denote by $M_j(Z_k)$ the subspace of $\text{Ext}^1(\mathcal{O}_{Z_k}(j), \mathcal{O}_{Z_k}(-j))/\sim$ consisting of those classes corresponding to stable vector bundles, where $\sim$ denotes bundle isomorphism.

Moduli spaces of rank 2 bundles on $Z_k$ were studied in [G2, Thm. 3.5] for the case $k = 1$ and in [BGK2, Thm. 4.11] for the cases $k \geq 1$. The moduli spaces of stable bundles with splitting type $j$ on $Z_k$, turn out to be smooth quasi-projective varieties of dimension $2j - k - 2$ [BGK2, Thm. 4.11]. In fact, we have:

**Theorem 5.9.** [BGK2, Thm. 4.11] The moduli space of rank 2 holomorphic bundles on $Z_k$ with vanishing first Chern class and splitting type $j$ contains an open dense subset isomorphic to $\mathbb{P}^{2j-k-2}$ minus a closed subvariety of codimension at least $k + 1$.

6. **Geometry of noncommutative deformations**

We now study vector bundles over noncommutative deformations $Z_k(\sigma) = (Z_k, A^\sigma)$ of $Z_k$ (see Def. 3.3).

**Definition 6.1.** We call a locally free sheaf of $A^\sigma$-modules of rank $r$ over $Z_k(\sigma)$ a vector bundle of rank $r$ over $Z_k(\sigma)$. A vector bundle of rank 1 over $Z_k(\sigma)$ is called a line bundle.

We recall some properties of deformation quantizations and refer to Kashiwara–Schapira [KS] for more details.

**Proposition 6.2.** [KS] Let $(X, \sigma)$ be a holomorphic Poisson manifold and let $A = A^\sigma$ be a deformation quantization of $\mathcal{O}_X$.

(i) Let $\mathcal{E}$ be a coherent sheaf of $A$-modules without $h$-torsion. If $\mathcal{E}/h\mathcal{E}$ is locally free of rank $r$ as a sheaf of $\mathcal{O}_X$-modules, then $\mathcal{E}$ is locally free of rank $r$ as a sheaf of $A$-modules.
implies that a rank 7 noncommutative associative deformation of $A$ which can be phrased using the coordinate charts of $\mathcal{O}_X$.

Definition 6.4. Let $\mathcal{O}_X$ over $A$. It gives, for any $k > 0$ and for any coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, then $H^k(U, \mathcal{G}) = 0$ for any coherent sheaf $\mathcal{G}$ of $A$-modules.

Here (ii) says that a Leray cover for $(X, \mathcal{O}_X)$ is also a Leray cover for $(X, A)$. In particular, we may use the canonical coordinate charts on $Z_k$ to calculate cohomology or extension groups of coherent (or locally free) sheaves of $A$-modules. After showing that rank 2 bundles are extensions of line bundles (Thm. 6.9), we will calculate these extension groups and use them to obtain moduli of vector bundles over noncommutative deformations in §7.

In terms of canonical coordinate charts for $Z_k$ Def. 6.1 implies that a rank $r$ vector bundle over a noncommutative $Z_k(\sigma)$ is given by two free rank $r$ modules over $U$ and $V$ with a global structure defined by a transition matrix, i.e. a $\star$-invertible $r \times r$ matrix with entries in $A^\sigma(U \cap V)$, determining the vector bundle uniquely up to isomorphism, where an isomorphism of vector bundles on $Z_k(\sigma)$ is an isomorphism of $A^\sigma$-modules, which can be phrased using the coordinate charts of $Z_k$ as follows.

Definition 6.3. Let $E$ and $E'$ be vector bundles over $Z_k(\sigma)$ defined by transition matrices $T$ and $T'$ respectively. An isomorphism between $E$ and $E'$ is given by a pair of matrices $A_U$ and $A_V$ with entries in $A^\sigma(U)$ and $A^\sigma(V)$, respectively, which are invertible with respect to $\star$ and such that

$$T' = A_V \star T \star A_U.$$

Definition 6.4. Let $X$ be a complex manifold (resp. smooth algebraic variety) and let $\mathcal{A}$ be a noncommutative associative deformation of $\mathcal{O}_X$ over $\mathbb{C}[\hbar]$. The augmentation $\mathcal{A} \to \mathbb{C} \otimes_{\mathbb{C}[\hbar]} \mathcal{A} \simeq \mathcal{O}_X$ is given by $\mathcal{A} \to \mathcal{A}/\hbar \mathcal{A}$. Augmentation induces a map on quasi-coherent sheaves of $\mathcal{A}$-modules and the image of a sheaf $\mathcal{F}$ of $\mathcal{A}$-modules is called the classical limit of $\mathcal{F}$. The image of any cocycle or cohomology class $\alpha \in H^1(X, \mathcal{F}(U))$, for $U \subset X$ open, is called the classical limit of $\alpha$.

6.1. Line bundles.

Lemma 6.5. Let $\mathcal{A}$ be a deformation quantization of $\mathcal{O}$. Then an $\mathcal{A}$-module $S$ is acyclic if and only $S = S/\hbar S$ is acyclic.

Proof. Consider the short exact sequence

$$0 \to S \xrightarrow{\hbar} S \to S \to 0.$$

It gives, for $j > 0$ surjections

$$H^j(X, S) \xrightarrow{\hbar} H^j(X, S) \to 0.$$

This immediately implies that $H^j(X, S) = 0$ for $j > 0$. The converse is immediate.

Definition 6.6. Let $Z_k(\sigma)$ be a noncommutative deformation of $Z_k$. Denote by $\mathcal{A}(j)$ the line bundle over $Z_k(\sigma)$ with transition function $z^{-j}$.

Proposition 6.7. Any line bundle on $Z_k(\sigma)$ is isomorphic to $\mathcal{A}(j)$ for some $j \in \mathbb{Z}$, i.e. $\text{Pic}(Z_k(\sigma)) \simeq \mathbb{Z}$.

Proof. Let $f = f_0 + \sum_{n=1}^\infty \tilde{f}_n \hbar^n \in \mathcal{A}^\star(U \cap V)$ be the transition function for $\mathcal{L}$. Then there exist functions $a_0 \in \mathcal{O}^\star(U)$ and $a_0 \in \mathcal{O}^\star(V)$ such that $a_0 f_0 a_0 = z^{-j}$ and viewing $a_0$ resp.
\( \alpha_0 \) as elements in \( A^*(U) \) resp. \( A^*(V) \) one has \( \alpha_0 \star f \star \alpha_0 = z^{-j} + \sum_{n=1}^{\infty} f_n h^n \) for some \( f_n \in \mathcal{O}(U \cap V) \). We may thus assume that the transition function of \( \mathcal{L} \) is \( \sum_{n=1}^{\infty} f_n h^n \).

To give an isomorphism \( \mathcal{L} \simeq A(f) \) it suffices to define functions \( a_n \in \mathcal{O}(U) \) and \( \alpha_n \in \mathcal{O}(V) \) satisfying
\[
(1 + \sum_{n=1}^{\infty} a_n h^n) \star (z^{-j} + \sum_{n=1}^{\infty} f_n h^n) \star (1 + \sum_{n=1}^{\infty} \alpha_n h^n) = z^{-j}.
\] (6.8)

Collecting terms by powers of \( \hbar \), (6.8) is equivalent to the system of equations
\[
S_n + z^{-j} a_n + z^{-j} \alpha_n = 0 \quad n = 1, 2, \ldots
\]
where \( S_n \) is a finite sum involving \( f_i, B_i \) for \( i \leq n \), but only \( a_i, \alpha_i \) for \( i < n \). The first terms are
\[
S_1 = f_1
S_2 = f_2 + a_1 f_1 + a_1 f_1 + B_1(a_1, z^{-j}) + B_1(z^{-j}, a_1) = a_1 z^{-j} \alpha_1
S_3 = f_3 + B_2(a_1, z^{-j}) + B_2(z^{-j}, a_1) + B_1(a_2, z^{-j})
\]
\[
+ B_1(z^{-j}, a_2) + B_1(a_1, f_1) + B_1(a_1, z^{-j} a_1) + B_1(z^{-j}, a_1)
+ a_2 f_1 + a_2 z^{-j} \alpha_1 + a_1 f_2 + a_1 f_1 a_1 + a_1 z^{-j} \alpha_2 + f_2 a_1 + f_1 a_2
\]

Since \( H^1(Z_k, \mathcal{O}) = 0 \) we can solve these equations recursively, for example by defining \( a_n \) to cancel all terms of \( z^j S_n \) having positive powers of \( z \) and setting \( a_n = z^j S_n - a_n \). \( \square \)

6.2. Vector bundles. Generalizing Thm. 5.1 to the noncommutative setting, we prove filtrability and formal algebraicity for bundles over deformation quantizations of \( Z_k \). Let \( \mathcal{Z}_{k}(\sigma) \) be a noncommutative deformation of \( Z_k \).

**Theorem 6.9.** Vector bundles over \( \mathcal{Z}_{k}(\sigma) \) are filtrable.

**First proof.** Let \( E \) be the rank 2 bundle given by transition matrix \( T = T_0 + \sum_{n=1}^{\infty} T_n h^n \).

Let \( (A_U, A_V) \) be an isomorphism of the classical limit \( E/\hbar E \) (with transition matrix \( T_0 \)) with a filtered bundle, i.e.
\[
A_V T_0 A_U = \begin{pmatrix} z^{j_1} & b_0 + O(\hbar) \\ 0 & z^{j_2} + O(\hbar) \end{pmatrix}
\]

with \( j_1 \geq j_2 \). Then \( (A_U, A_V) \) gives an isomorphism with the bundle given by
\[
A_V \star T \star A_U = \begin{pmatrix} z^{j_1} + O(\hbar) & b_0 + O(\hbar) \\ O(\hbar) & z^{j_2} + O(\hbar) \end{pmatrix} = T'.
\] (6.10)

Now choose \( \alpha_n, \delta_n \) and \( d_n, e_n \) as in the proof of Prop. 6.7 such that the \( (1,1) \) and \( (2,2) \) entries of (6.10) are taken to the transition functions of \( A(-j_1) \) and \( A(-j_2) \), respectively.

Then for
\[
A_V = \begin{pmatrix} 1 + \sum_{n=1}^{\infty} \alpha_n h^n \\ 0 \\ 1 + \sum_{n=1}^{\infty} \delta_n h^n \end{pmatrix}
A_U = \begin{pmatrix} 1 + \sum_{n=1}^{\infty} \alpha_n h^n \\ 0 \\ 1 + \sum_{n=1}^{\infty} d_n h^n \end{pmatrix}
\]
we have that
\[
A_V \star T' \star A_U = \begin{pmatrix} z^{j_1} & b_0 + O(\hbar) \\ O(\hbar) & z^{j_2} + O(\hbar) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} \alpha_n h^n \\ 0 \\ \sum_{n=1}^{\infty} \delta_n h^n \end{pmatrix} = T''
\]
for some integer $N \geq 1$. Now choose $c'_N$ and $\gamma'_N$ such that
\[ z^{j_1} \gamma'_N + c_N + z^{j_2} c'_N = 0 \]
which is possible since $H^1(Z_k, O(j_1-j_2)) = 0$, as $j_1 \geq j_2$. Then
\[ \left( \begin{array}{c} 1 \\ \gamma'_N h^N \\ 1 \end{array} \right) \ast T'' \ast \left( \begin{array}{c} 1 \\ c'_N h^N \\ 1 \end{array} \right) = \left( \begin{array}{cc} z^{j_1} + O(h) & b_0 + O(h) \\ O(h^{N+1}) & z^{j_2} + O(h) \end{array} \right) = T''' \ast .
\]

As for the isomorphism between the bundles defined by $T'$ and $T''$, find matrices taking the $(1, 1)$ and $(2, 2)$ entries of $T''$ to $z^{j_1}$ and $z^{j_2}$ as above. We thus get that an isomorphic bundle may be given by a transition matrix of the form
\[ \left( \begin{array}{cc} z^{j_1} & b_0 + O(h) \\ O(h^{N+1}) & z^{j_2} \end{array} \right).
\]

Applying the principle of (strong) induction, we conclude that $E$ is isomorphic to the bundle given by a transition matrix
\[ \left( \begin{array}{cc} z^{j_1} & b_0 + \sum_{n=1}^{\infty} b_n h^n \\ 0 & z^{j_2} \end{array} \right) .
\]

In particular, any rank 2 bundle is an extension of line bundles. The calculation for rank $r$ is similar. \hfill \Box

**Second proof.** This is a generalization of Ballico–Gasparim–Köppe [BGK, Thm. 3.2] to the noncommutative case. Let $E$ be a sheaf of $A$-modules. Lem. 6.5 gives that the classical limit $E_0 = E/hE$ is acyclic as a sheaf of $A$-modules (and equivalently as a sheaf of $O$-modules) if and only if $E$ is acyclic as a sheaf of $A$-modules.

Filtrability for a bundle $E$ over $Z_k$ is obtained due to the vanishing of cohomology groups $H^i(Z_k, E \otimes S^n N^*)$ for $i = 1, 2$, where $N^*$ is the conormal bundle of $\ell \subset Z_k$ and $n > 0$ are integers, the proof proceeds by induction on $n$. In the noncommutative case, let $S$ denote the kernel of the projection $A^{(n)} \to A^{(n-1)}$. By construction we have that $S/hS = S^n N^*$ and the required vanishing of cohomologies is guaranteed by Lem. 6.5. \hfill \Box

As for $Z_k$, rank 2 vector bundles on $Z_k(\sigma)$ are thus extensions of line bundles. Hence, moduli spaces of rank 2 bundles may be built out of quotients of Ext$^1$'s.

**Definition 6.11.** [G1, Thm. 3.3] showed that a rank 2 bundle $E$ on $Z_k$ with first Chern class $c_1(E) = 0$ can be given by a canonical transition matrix
\[ T_0 = \left( \begin{array}{cc} z^j & p \\ 0 & z^{-j} \end{array} \right) \quad \text{with} \quad p = \sum_{i=0}^{j-1} \sum_{l=k-i+1}^{j+2} p_{il} z^i u^j \in \text{Ext}^1(O(j), O(-j)).
\]

Accordingly, for a noncommutative deformation $Z_k(\sigma)$ we define the corresponding notion of canonical transition matrix as:
\[ T = \left( \begin{array}{cc} z^j & p \\ 0 & z^{-j} \end{array} \right) \quad \text{with} \quad p = \sum_{n=0}^{\infty} p_n h^n \in \text{Ext}^1(A(j), A(-j)).
\]

We will now see that each $p_n$ can be given the canonical form of the classical case.
Lemma 6.12. Let $\mathcal{A}$ be a deformation quantization of $\mathcal{O}_{\mathbb{C}^n}$. There is an injective map of $\mathbb{C}$-vector spaces

$$\text{Ext}^1_{\mathcal{A}}(\mathcal{A}(j), \mathcal{A}(-j)) \rightarrow \bigoplus_{n=0}^{\infty} \text{Ext}^1_{\mathcal{O}}(\mathcal{O}(j), \mathcal{O}(-j))[\hbar] \quad p = p_0 + \sum_{n=1}^{\infty} p_n \hbar^n \mapsto (p_0, p_1 \hbar, p_2 \hbar^2, \ldots)$$

where $p_i \in \text{Ext}^1(\mathcal{O}(j), \mathcal{O}(-j))$.

Proof. Two extension classes $p = p_0 + \sum_{n=1}^{\infty} p_n \hbar^n$ and $p' = p'_0 + \sum_{n=1}^{\infty} p'_n \hbar^n$ are equivalent if there exist two functions $b = b_0 + \sum_{n=1}^{\infty} b_n \hbar^n \in \mathcal{A}(U)$ and $\beta = \beta_0 + \sum_{n=1}^{\infty} \beta_n \hbar^n \in \mathcal{A}(V)$ such that

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^j \\ 0 \end{pmatrix} = \begin{pmatrix} p' \\ z^j \end{pmatrix} = \begin{pmatrix} p \\ 0 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

that is

$$p + \beta \star z^{-j} = p' + z^j \star b$$

(6.14)
or equivalently

$$p_n + \beta_n z^{-j} + \sum_{i=1}^{n} B_i(\beta_{n-i}, z^{-j}) = p'_n + z^j b_n + \sum_{i=1}^{n} B_i(z^j, b_{n-i}) \quad \text{for } n \in \mathbb{N}.$$

For each $n$ we may choose $b_n$ and $\beta_n$ reducing $p_n$ to the canonical form

$$p_n = \sum_{i=0}^{\frac{n+1}{2}} \sum_{i=k-j+1}^{j-1} p^n_{ij} z^i u^j$$

as in (6.11).

Lemma 6.16. Let $\mathcal{Z}(\sigma)$ be the deformation quantization of $\mathcal{O}_{\mathbb{C}^n}$ with Poisson structure $\sigma = (1, -\xi)$. Then any $p \in \text{Ext}^1_{\mathcal{A}}(\mathcal{A}(j), \mathcal{A}(-j))$ is represented in the canonical form $p = \sum_{n=0}^{\infty} p_n \hbar^n$ where $p_n$ is of the form $p_n = \sum_{i=0}^{2j-2} \sum_{i=-j+1}^{j-1} p^n_{ij} z^i u^j$.

Proof. On the $U$ chart, the holomorphic Poisson structure $\sigma$ is induced by the bivector $\frac{\partial}{\partial t} \wedge \frac{\partial}{\partial s}$, and the corresponding star product is given by the Moyal product (3.9).

As in the proof of Lem. 6.12, elements $p = p_0 + \sum_{n=1}^{\infty} p_n \hbar^n$ and $p' = p'_0 + \sum_{n=1}^{\infty} p'_n \hbar^n$ in $\text{Ext}^1(\mathcal{A}(j), \mathcal{A}(-j))$ are equivalent if there exist two functions $b \in \mathcal{A}(U)$ and $\beta \in \mathcal{A}(V)$ of the form (6.13) such that

$$p' = p + \beta \star z^{-j} - z^j \star b.$$
We carry out calculations on the $U$ chart, hence using the Moyal product. For $n = 1, 2$ this gives:

$$p'_1 = p_1 + \beta_1 z^{-j} - \left( \frac{\partial}{\partial u^i} \beta_0 \right) \left( \frac{\partial}{\partial z^j} \right) z^i b_1 + \left( \frac{\partial}{\partial z^j} \right) \left( \frac{\partial}{\partial u^i} \beta_0 \right)$$

$$p'_2 = p_2 + \beta_2 z^{-j} - \left( \frac{\partial}{\partial u^i} \beta_1 \right) \left( \frac{\partial}{\partial z^j} \right) z^i b_2 + \left( \frac{\partial}{\partial z^j} \right) \left( \frac{\partial}{\partial u^i} \beta_1 \right)$$

etc. But we then find out that ultimately the calculations repeat at each step the same type of calculation done on neighbourhood zero, thus at each step the shape of the polynomial is the same as what we have in the classical limit.

\[ \square \]

**Theorem 6.17.** Let $\sigma$ be a Poisson structure on $Z_k$ tangent to the divisor $Z_k \setminus U$ and let $A$ be the quantization of the open immersion $U \subset Z_k$. Then

$$\text{Ext}^1_A(A(f), A(-j)) \cong \text{Ext}^1_O(O(f), O(-j))[\hbar].$$

**Proof.** Recall from the proof of Lem. 6.12 that an extension class $p \in \text{Ext}^1_A(A(f), A(-j))$ may be reduced to the form $p = p_0 + \sum_{n=1}^{\infty} p_n \hbar^n$, where $p_i \in \text{Ext}^1_O(O(f), O(-j))$. It remains to show that this form cannot be reduced further.

For $Z_1$, this is the content of Lem. 6.16. For $Z_{\geq 2}$, Prop. 2.12 shows that for $\sigma$ tangent to $D$, $\sigma_U = u f_U + z g_U$ for some global functions $f_U, g_U$. Since $\sigma_U$ is a multiple of $u$, it follows that the bilinear operators $B_{u, v}$ in the expression of the star product never lower the exponents of $u$, cf. the proof of Prop. 4.1. Thus the star products in (6.14) never contain terms with exponents of $u$ low enough to reduce the general formal series of $p_n$ any further.

\[ \square \]

**Definition 6.18.** We say that $p = \sum p_n \hbar^n \in O[\hbar]$ is formally algebraic if $p_n$ is a polynomial for every $n$.

We say that a vector bundle over $Z_k(\sigma)$ is formally algebraic if it is isomorphic to a vector bundle given by formally algebraic transition functions. In addition, if there exists $N$ such that $p_n = 0$ for all $n > N$, we then say that $p$ is algebraic.

**Corollary 6.19.** Vector bundles on noncommutative deformations of $Z_k$ are formally algebraic.

**Proof.** Lemma 6.12 shows that rank 2 vector bundles over $Z_k(\sigma)$ are formally algebraic, and in light of Thm. 6.9, we obtain the result for all ranks.

\[ \square \]

**Remark 6.20.** A quantization of $Z_k$ will also give a quantization of the singular affine surface $X_k = \text{Spec}(\mathcal{H}^0(Z_k, O))$, obtained from $Z_k$ by contracting the zero section $\ell$ to a point. ($X_k$ contains an isolated $\frac{1}{2}(1, 1)$ singularity, see §9 and Fig. 6.) Indeed, a quantization of $Z_k$ gives a star product on $O_Z_k(U)$ for all open sets $U$, and thus in particular a star product on the algebra of global functions $\mathcal{H}^0(Z_k, O) \cong H^0(X_k, O)$.

Note that Poisson structures on singular affine toric varieties (of which $X_k$ is a particular case) are known to always admit quantizations (see Filip [F]), but in general there are also counterexamples to the quantization of singular Poisson algebras (see Mathieu [M] and also Schedler [Sch]).

Since $\text{Ext}^1_Z(O(f), O(-j))$ is finite dimensional, we have that if $Z_k(\sigma)$ is a noncommutative deformation having finite order in $\hbar$, then the space of extensions $\text{Ext}^1_A(A(f), A(-j))$...
is also finite dimensional. For moduli of vector bundles on \( Z_k(\sigma) \) up to all orders of \( h \) see Rem. 7.20.

7. **Moduli of bundles on noncommutative deformations**

We now calculate moduli spaces for vector bundles on deformation quantizations of 
\((Z_k, \sigma)\) for \( \sigma \) a holomorphic Poisson structure.

**Remark 7.1.** Analogous to the situation of Rem. 5.3, one has filtrability of vector bundles also in the noncommutative case (Lem. 6.9) so that moduli of rank 2 vector bundles can be described as quotients of \( \text{Ext}^1 \) of line bundles. Furthermore, we have formal algebraicity (Cor. 6.19) as the extension groups are \( \mathbb{N} \)-graded by powers of \( h \) with finite-dimensional graded components, or indeed finite-dimensional if we take \( h \) only up to a fixed finite power. We then obtain finite-dimensional quotients taking \( \text{Ext}^1 \) modulo bundle isomorphisms, where here isomorphisms are defined in Def. 6.3 using the (noncommutative) star product.

We thus may proceed as in the classical (commutative) setting and extract moduli spaces from extension groups of line bundles, by considering extension classes up to bundle isomorphism.

**Notation 7.2.** We denote by \( \mathfrak{M}_j(Z_k(\sigma)) \) the subspace of the quotient \( \text{Ext}^1_j(A(j), A(-j))/\sim \) consisting of those classes of formally algebraic vector bundles, whose classical limit is a stable vector bundle of charge \( j \). Here \( \sim \) denotes bundle isomorphism as in Def. 6.3. We denote by \( \mathfrak{M}_j(n)(Z_k(\sigma)) \) the moduli of bundles obtained by imposing the cut-off \( h^n+1 = 0 \), that is, the superscript \( (n) \) means quantized to level \( n \). Accordingly \( \mathfrak{M}_j^{(1)}(Z_k(\sigma)) \) stands for first-order quantization and \( \mathfrak{M}_j^{(0)}(Z_k(\sigma)) = \mathfrak{M}_j(Z_k) \) recovers the classical moduli space of Def. 5.8 obtained when \( h = 0 \).

**Definition 7.3.** The **splitting type** of a vector bundle \( E \) on \( Z_k(\sigma) \) is defined to be the splitting type of its classical limit as in Def. 5.2. Hence, when the classical limit is an \( \text{SL}(2, \mathbb{C}) \) bundle, the splitting type of \( E \) is the smallest integer \( j \) such that \( E \) can be written as an extension of \( A(j) \) by \( A(-j) \).

We will look at rank 2 bundles of a fixed splitting type \( j \) on the first formal neighbourhood \( \ell^{(1)} \) of \( \ell \subset Z_k \). In order for the relevant space of extensions to be non-zero, one should assume \( k \leq 2j - 2 \). Note that whenever \( j \leq k \leq 2j - 2 \) the full moduli of splitting type \( j \) bundles on \( Z_k \) is supported on \( \ell^{(1)} \). For general \( k \), the moduli spaces of bundles on \( \ell^{(1)} \) are dense open subspaces of the full moduli spaces of bundles on \( Z_k \), so even by restricting to bundles on \( \ell^{(1)} \) one obtains a partial description of the moduli space of bundles on all of \( Z_k \). We thus refrain from introducing new notation, keeping the same notation as in Not. 7.2.

Moreover, we present the calculation only up to first order in \( h \), although we note that the explicit formulae of the star products given in §3.2 enable one to determine the moduli also to higher orders in \( h \) (see Rem. 7.20).

Let \( p + p'h \) and \( q + q'h \) be two extension classes in \( \text{Ext}^1_j(A(j), A(-j)) \) which are of splitting type \( j \), i.e. in canonical \( U \)-coordinates \( p, p', q, q' \) are multiples of \( u \).

The bundles defined by \( p + p'h \) and \( q + q'h \) are isomorphic, if there exist invertible matrices

\[
\begin{pmatrix}
a + a'h & b + b'h \\
c + c'h & d + d'h
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
a + a'h & \beta + \beta'h \\
\gamma + \gamma'h & \delta + \delta'h
\end{pmatrix}
\]
whose entries are holomorphic on $U$ and $V$, respectively, such that

$$
\begin{pmatrix}
\alpha + a'\hbar & \beta + b'\hbar \\
\gamma + \gamma'\hbar & \delta + \delta'\hbar
\end{pmatrix} \star \begin{pmatrix}
\frac{z^j}{\hbar} & q + q'\hbar \\
0 & z^{-j}
\end{pmatrix} = \begin{pmatrix}
\frac{z^j}{\hbar} & p + p'\hbar \\
0 & z^{-j}
\end{pmatrix} \star \begin{pmatrix}
\alpha + a'\hbar & b + b'\hbar \\
\gamma + \gamma'\hbar & d + d'\hbar
\end{pmatrix}.
$$

(7.4)

We wish to determine the constraints such an isomorphism imposes on the coefficients of $q$ and $q'$. This is more convenient if rewritten by right-multiplying (7.4) with the inverse of \( \begin{pmatrix} z^j & q+q'\hbar \\ 0 & z^{-j} \end{pmatrix} \), or more precisely by the right inverse with respect to \( \star \), which (modulo $\hbar^2$) is

$$
\begin{pmatrix}
\frac{z^{-j}}{\hbar} & -q - q'\hbar + 2z^{-j}\{z^j, q\} \\
0 & z^j
\end{pmatrix}.
$$

On $\ell^{(1)}$ we have that $u^2 = 0$ and therefore $a = a_0 + a_1u$, $\alpha = \alpha_0 + \alpha_1u$, etc., where $a_1$, $\alpha_1$, etc. are holomorphic functions in $z$.

We first observe that for the classical limit the calculations are given in [G1, §3.1]. In particular, following the details of the proof of [ibid., Prop. 3.3], we may assume that $a_0 = \alpha_0 = d_0 = \beta_0$ are constant, and $b = \beta = 0$. Since we already know that on the classical limit the only equivalence on $\ell^{(1)}$ is given projectivization, we may assume that $p = q$ keeping in mind that there is a projectivization to be done in the end. We may also assume that the determinants of the changes of coordinates on the classical limit are 1. Accordingly, we may simplify (7.4) to:

$$
\begin{pmatrix}
\alpha + a'\hbar & \beta'\hbar \\
\gamma + \gamma'\hbar & \delta + \delta'\hbar
\end{pmatrix} = \begin{pmatrix}
\frac{z^j}{\hbar} & p + p'\hbar \\
0 & z^{-j}
\end{pmatrix} \star \begin{pmatrix}
\alpha + a'\hbar & b'\hbar \\
\gamma + \gamma'\hbar & d + d'\hbar
\end{pmatrix} \star \begin{pmatrix}
\frac{z^{-j}}{\hbar} & -p - (q' - 2\{z^j, p\}z^{-j})\hbar \\
0 & z^j
\end{pmatrix}
$$

(7.5)

where $a_0 = d_0 = \alpha_0 = \delta_0 = 1$.

Since we already know the moduli on the classical limit, we only need to study the terms containing $\hbar$, which after multiplying are:

$$
(1, 1) = a' + \{z^j, a, z^{-j}\} + \{z^j, a\}z^{-j} + \{pc, z^{-j}\} + \{p, c\}z^{-j} + \{pc' + p'c\}z^{-j}
$$

$$
(2, 1) = z^{-2j}c'.
$$

$$
(1, 2) = -(a, p)z^j - \{z^j, a\}p + \{z^j, p\}a + \{pd, z^j\} + \{p, d\} + 2z^{-j}\{z^j, p\}pc + z^j b' - (pd' + q'a)z^j + (pd' + p'd)z^j - (pc' + p'c + q'c)p
$$

$$
(2, 2) = d' + \{z^{-j}d, z^j\} + \{z^{-j}, d\}z^j - \{z^{-j}, c, p\} - \{z^{-j}, c\}p - (pc' + q'c)z^{-j} + 2\{z^j, p\}z^{-2j}c.
$$

All four terms must be adjusted by using free variables to only contain expressions which are holomorphic on $V$ in order to satisfy (7.5). For example, the (2, 1) term shows that this condition is satisfied precisely when $c'$ is a section of $\mathcal{O}(2j)$. A simple verification by computing the Poisson brackets shows that the (1, 1) and (2, 2) terms can always be adjusted by choosing, say, $c$ and $d'$ appropriately, leaving the coefficients of $a'$ free.

It remains to analyze the term (1, 2). Because we are working on the first formal neighbourhood of $\ell$, we may drop terms in $u^2$ (recall that we assume that $p, p', q'$ are
Similarly, the vanishing of the coefficient of $z^j$ imposes:

$$p_{10}' - q_{10}' = (a_{00}' - d_{01}' + 4a_{12})p_{10} - (-a_{00}' + d_{01}' - a_{11})p_{11}$$

$$- 4(c_{02}p_{11}^2 + 2c_{03}p_{10}p_{11} + c_{04}p_{10}^2).$$

Similarly, the vanishing of the coefficient of $z^j$ imposes:

$$p_{11}' - q_{11}' = (a_{01}' - d_{01}' + 4a_{12})p_{10} - (-a_{01}' + d_{01}' - a_{11})p_{11}$$

$$- 4(c_{02}p_{11}^2 + 2c_{03}p_{10}p_{11} + c_{04}p_{10}^2).$$

Since for $P \in \mathcal{M}_2(Z_1)$ its coefficients $p_{10}$ and $p_{11}$ do not vanish simultaneously, we can choose coefficients of $a, d$ or $c$ to solve both (7.10) and (7.11). We observe that two out of $a, d, c$ will already have been fixed on a previous step, when canceling coefficients in the $(1, 1)$ and $(2, 2)$ terms, but there remains always one of them free to be chosen. Hence, we can solve both equations for any values of $q_{10}'$ and $q_{11}'$, so that for Poisson structure $\sigma_0 = (1, -\xi)$ two bundles over $Z_1(\sigma_0)$ are isomorphic whenever their classical limits are, giving an isomorphism of moduli spaces

$$\mathcal{M}_2^{(1)}(Z_1(\sigma_0)) = \mathcal{M}_2(Z_1).$$
Example 7.12 \((k = 1\) and \(j = 2\) and \(\sigma = u\)). For \(\sigma = (u, -\xi^2 v)\), which by Prop. 4.1 defines a global star product on \(Z_1\), equations (7.10) and (7.11) simplify to
\[
\begin{align*}
    p_1' - q_1' &= (a'_{00} - d'_{00})p_0 \\
    p_1' - q_1' &= (a'_{01} - d'_{01})p_0 + (a'_{02} - d'_{02})p_1
\end{align*}
\]
which can be written as the Toeplitz system
\[
\begin{pmatrix}
    p_{11} - q_{11} \\
    p_{10} - q_{10}
\end{pmatrix} = \begin{pmatrix}
    p_{10} & p_{11} \\
    0 & p_{10}
\end{pmatrix} \begin{pmatrix}
    a'_{11} - d'_{10} \\
    a'_{01} - d'_{00}
\end{pmatrix}.
\]
(7.13)
It then follows that the moduli space behaviour is quite different from the case studied in Ex. 7.9. Here, if \(p_{10} \neq 0\) the matrix \(\begin{pmatrix}
    p_{10} & p_{11} \\
    0 & p_{10}
\end{pmatrix}\) is invertible, so we can solve (7.13) for any \(q'\), giving the equivalence relation
\[
(p_{10}, p_{11}, p_{10}', p_{11}') \sim (\lambda p_{10}, \lambda p_{11}, q_{10}', q_{11}') \quad \text{if } p_{10} \neq 0.
\]
So that the fibre of the projection
\[
\mathbb{M}^{(1)}_2(Z_1(\sigma)) \overset{\pi}{\longrightarrow} \mathbb{M}_2(Z_1).
\]
to the classical limit is just a point provided \(p_{10} \neq 0\).

However, if \(p_{10} = 0\), that is, over the single point \(P = [0 : 1]\) in the classical moduli space \(\mathbb{M}_2(Z_1)\), (7.13) imposes the additional constraint that \(q_{11}' = p_{10}'\). (The coefficient \(q_{11}'\) remains arbitrary because \(p_{11}\) must be nonzero in this case.)

We thus obtain the following equivalence relation:
\[
(p_{10}, p_{11}, p_{10}', p_{11}') \sim (\lambda p_{10}, \lambda p_{11}, p_{10}', q_{11}') \quad \text{if } p_{10} = 0.
\]
Here we observe that both \(p_{11}'\) and \(q_{11}'\) are arbitrary, so that the resulting equivalence relation on the fibre over \(P = [0 : 1] \in \mathbb{M}_2(Z_1)\) can equivalently be represented by
\[
(0, p_{11}, p_{10}', *) \sim (0, \lambda p_{11}, p_{10}', *)
\]
(where * denotes an arbitrary complex number) is parametrized by the different values of \(p_{10}' \in \mathbb{C}\). Therefore, we have obtained that the fibre \(L = \pi^{-1}([0 : 1]) = [0 : 1] \times \mathbb{C}\) is an affine line. Equivalently, the collection of points \(L \subset \mathbb{M}^{(1)}_2(Z_1(\sigma))\) that have \(P\) as its classical limit is \(L = [0 : 1] \times \mathbb{C}\). We can thus view
\[
\begin{tikzcd}
\mathbb{M}^{(1)}_2(Z_1(\sigma)) \arrow{d}
\\
\mathbb{M}_2(Z_1)
\end{tikzcd}
\]
as the étale space of a skyscraper sheaf supported at \(P\).

Example 7.14 \((k = 1\) and \(j = 3\) and \(\sigma_0 = 1\)). Here the imposed constraints are that the coefficients of \(z^2u, z^3u, z^4u, z^5u\) must vanish. To illustrate the calculation, we list the first two. The analogues of equations (7.10) and (7.11), corresponding to the vanishing of the
coefficients of $z^2 u$ and $z^3 u$ are

\[
p'_{1-1} - q'_{1-1} = (a'_{00} - d'_{00} + 2a_{11} + 8d_{11})p_{1-1} + 6d_{10}p_{10} - 6\left(c_{00}(p^2_{11} + 2p_{10}p_{12}) + 2c_{01}(p_{10}p_{11} + p_{1-1}p_{12}) + c_{02}(p^2_{10} + 2p_{1-1}p_{11}) + 2c_{03}p_{1-1}p_{10} + c_{04}p^2_{1-1}\right)
\]

\[
p'_{10} - q'_{10} = (a'_{00} - d'_{00} + 3a_{12} + 9d_{12})p_{1-1} - 6\left(2c_{00}p_{11}p_{12} + c_{01}(p^2_{11} + 2p_{10}p_{12}) + 2c_{02}(p_{10}p_{11} + p_{1-1} + p_{12}) + c_{03}(p^2_{10} + 2p_{1-1}p_{11}) + 2c_{04}p_{1-1}p_{10} + c_{05}p^2_{1-1}\right)
\]

and similarly for the coefficients of $z^4 u$ and $z^5 u$. It then turns out that over the points belonging to the moduli space $\mathcal{M}_3(Z_1)$ we can solve all the constraint equations, so that $q$ is arbitrary, and once again as in Ex. 7.9 we obtain an isomorphism between the quantum and the classical moduli:

\[
\mathcal{M}^{(1)}_3(\mathbb{Z}_1(\sigma_0)) \cong \mathcal{M}_3(Z_1).
\]

Generalizing Exs. 7.9 and 7.14 we obtain the following theorem.

**Theorem 7.15.** Let $\sigma_0$ be the Poisson structure which in canonical coordinates is $\sigma_0 = (1, -\xi)$. Then for each $j$ we obtain an isomorphism

\[
\mathcal{M}^{(1)}_j(\mathbb{Z}_1(\sigma_0)) \cong \mathcal{M}_j(Z_1).
\]

**Example 7.16** $(k = 1$ and $j = 3$ and $\sigma = u$). For $j = 3$, the relevant monomials of (7.6) are $z^2 u, \ldots, z^5 u$ and the vanishing condition can be written as the Toeplitz system

\[
\begin{pmatrix}
  p'_{12} - q'_{12} \\
  p'_{11} - q'_{11} \\
  p'_{10} - q'_{10} \\
  p'_{1-1} - q'_{1-1}
\end{pmatrix} = \begin{pmatrix}
  p_{1-1} & p_{10} & p_{11} & p_{12} \\
  0 & p_{1-1} & p_{10} & p_{11} \\
  0 & 0 & p_{1-1} & p_{10} \\
  0 & 0 & 0 & p_{1-1}
\end{pmatrix} \begin{pmatrix}
  a'_{03} - d'_{03} \\
  a_{02} - d_{02} \\
  a'_{01} - d'_{01} \\
  a_{00} - d_{00}
\end{pmatrix}.
\]

(7.17)

As in Ex. 7.12 the fibres of the projection

\[
\mathcal{M}^{(1)}_3(\mathbb{Z}_1(\sigma)) \xrightarrow{\pi} \mathcal{M}_3(Z_1)
\]

vary depending on the coordinates of the point $[p_{1-1} : p_{10} : p_{11} : p_{12}] \in \mathcal{M}_3(Z_1) \subset \mathbb{P}^3$. We have:

- If $p_{1-1} \neq 0$ we can solve all the equations by choosing $a'$ and $d'$ appropriately. Hence, there exists an isomorphism for any value of $q'$ and consequently the fibre in this case is only a point. Thus

\[
S_0 := \{ P \in \mathcal{M}_3(Z_1) \mid p_{1-1} \neq 0 \}
\]

is an open set of the classical moduli space over which the map $\pi$ is an isomorphism.
Now assume \( p_{1-1} = 0 \) but \( p_{10} \neq 0 \). Then the fourth equation imposes the condition \( q'_{1-1} = p'_{1-1} \) but the other equations can be solved for any values of \( q'_{10}, q'_{11}, q'_{12} \). So that isomorphism gives the equivalence relation

\[
[0 : 1 : p_{11} : p_{12}](p'_{1-1}, p'_{10}, p'_{11}, p'_{12}) \sim [0 : 1 : p_{11} : p_{12}](p'_{1-1}, *, *, *)
\]

where * stands for an arbitrary complex value. Thus, the fibre of \( \pi \) over a point \([0 : 1 : p_{11} : p_{12}]\) is a copy of \( \mathbb{C} \) parametrized by the different values of \( p'_{1-1} \). In other words, setting

\[
S_1 := \{ P \in \mathcal{M}_3(Z_1) \mid p_{1-1} = 0, \ p_{10} \neq 0 \}
\]

we have that if \( P \in S_1 \) then \( \pi^{-1}(P) \cong \mathbb{C} \).

Next assume that \( p_{1-1} = p_{10} = 0 \) but \( p_{11} \neq 0 \), then the third and fourth equations of (7.17) impose the conditions \( q'_{1-1} = p'_{1-1} \) and \( q'_{10} = p'_{10} \) but the remaining two equations can be solved for any values of \( q'_{11}, q'_{12} \). Hence in this case, isomorphism imposes the following equivalence relation

\[
[0 : 0 : 1 : p_{12}](p'_{1-1}, p'_{10}, p'_{11}, p'_{12}) \sim [0 : 0 : 1 : p_{12}](p'_{1-1}, p'_{10}, *, *)
\]

with the fibre of \( \pi \) over such a point being a copy of \( \mathbb{C}^2 \) parametrized by the values of \((p'_{1-1}, p'_{10})\). Thus, setting

\[
S_2 := \{ P \in \mathcal{M}_3(Z_1) \mid p_{1-1} = p_{10} = 0, \ p_{11} \neq 0 \}
\]

we have that if \( P \in S_2 \) then \( \pi^{-1}(P) \cong \mathbb{C}^2 \).

Lastly there would be the point \([0 : 0 : 1]\) to be considered, but it does not belong to \( \mathcal{M}_3(Z_1) \) because it corresponds to a vector bundle with charge 5 (see Def. 5.5 and Table 1), so we are already done.

In conclusion,

\[
\mathcal{M}_3^{(1)}(Z_1(\sigma)) \xrightarrow{\pi} \mathcal{M}_3(Z_1)
\]

can be viewed as the étale space of a constructible sheaf, with stalks of dimension \( i \) over the strata \( S_i \) for \( i = 0, 1, 2 \).

**Example 7.18** (\( \sigma_i \) multiple of \( u, k \) and \( j \) arbitrary). Let \( \sigma \) be a holomorphic Poisson structure on \( Z_k \) such that \( \sigma_i \) is a multiple of \( u \) and let \( Z_k(\sigma) \) be any deformation quantization of \( Z_k \). (Note that by Lem. 2.8 any Poisson structure on \( Z_{2,3} \) is of this form.)

Then the moduli space \( \mathcal{M}_3^{(1)}(Z_k(\sigma)) \) may be described by the Toeplitz system:

\[
\begin{pmatrix}
    p'_{1,j-1} - q'_{1,j-1} \\
    p'_{1,j-2} - q'_{1,j-2} \\
    \vdots \\
    p'_{1,k-j+2} - q'_{1,k-j+2} \\
    p'_{1,k-j+1} - q'_{1,k-j+1}
\end{pmatrix} = \begin{pmatrix}
    \cdots & \cdots & \cdots & \cdots & \cdots \\
    p_{1,k-j+1} & p_{1,k-j+2} & p_{1,k-j+3} & \cdots & p_{1,j-1} \\
    0 & p_{1,k-j+1} & p_{1,k-j+2} & \cdots & p_{1,j-3} \\
    0 & 0 & p_{1,k-j+1} & p_{1,k-j+2} & \cdots & p_{1,j-4} \\
    0 & 0 & 0 & p_{1,k-j+1} & \cdots & p_{1,j-5} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & 0 & \cdots & p_{1,k-j+1} \\
\end{pmatrix} \begin{pmatrix}
    a_{0,j-k-2} & a_{0,j-k-3} & \cdots & a_{0,j-k} \\
    a_{0,j-k-2} & a_{0,j-k-3} & \cdots & a_{0,j-k} \\
    \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
    a_{0,1} & a_{0,0} \\
\end{pmatrix}
\]
Exs. 7.16 and 7.18 readily generalize to give:

**Theorem 7.19.** The quantum moduli space \( \mathcal{M}_j^{(1)}(Z_k(\sigma)) \) can be viewed as the étale space of a constructible sheaf over the classical moduli space \( \mathcal{M}_j(Z_k) \), the sheaf being trivial over the open set

\[
S_0 := \{ P \in \mathcal{M}_j(Z_k) \mid p_{1,k-j+1} \neq 0 \}
\]

and with stalk of dimension \( i \) over the locally closed subvarieties

\[
S_i := \{ P \in \mathcal{M}_j(Z_k) \mid p_{1,k-j+1} = \cdots = p_{1,k-j+i} = 0, p_{1,k-j+i+1} \neq 0 \}.
\]

**Proof.** The classical moduli space gets stratified into disjoint subsets \( S_i \) over which the fibre of the projection

\[
\pi : \mathcal{M}_j^{(1)}(Z_k(\sigma)) \to \mathcal{M}_j(Z_k)
\]

has dimension \( i \). The strata are the dense open subset

\[
S_0 := \{ P \in \mathcal{M}_j(Z_k) \mid p_{1,k-j+1} \neq 0 \}
\]

and locally closed subsets of decreasing dimension

\[
S_i := \{ P \in \mathcal{M}_j(Z_k) \mid p_{1,k-j+1} = \cdots = p_{1,k-j+i} = 0, p_{1,k-j+i+1} \neq 0 \}.
\]

**Remark 7.20 (Higher powers of \( \hbar \)).** Repeating the calculation for higher powers of \( \hbar \), one finds that \( \mathcal{M}_j^{(2)}(Z_1(\sigma)) \simeq \mathcal{M}_j^{(1)}(Z_1(\sigma)) \) so that for splitting type 2, considering terms up to \( \hbar^2 \) does not give any more vector bundles. We thus expect that for arbitrary powers of \( \hbar \) one obtains an isomorphism \( \mathcal{M}_j^{(2)}(Z_1(\sigma)) \simeq \mathcal{M}_j^{(1)}(Z_1(\sigma)) \) with all splitting type 2 vector bundles supported on the first neighbourhood of \( \hbar \). In particular, this implies that bundles of splitting type 2 on \( Z_1(\sigma) \) are algebraic in the sense of Def. 6.18.

Similarly, Thm. 7.15 would imply

\[
\mathcal{M}_j(Z_1(\sigma_0)) \simeq \mathcal{M}_j(Z_1)
\]

for the minimally degenerate Poisson structure \( \sigma_0 \) and arbitrary \( j \), which corresponds to the statement that bundles of arbitrary splitting type on \( Z_1(\sigma_0) \) are algebraic.

On the other hand, the calculation of \( \mathcal{M}_j^{(2)}(Z_1(\sigma)) \) also suggests that for \( j \geq 3 \) and Poisson structures which are degenerate on all of \( \ell \), for example any Poisson structure on \( Z_{k \geq 3} \), the moduli space \( \mathcal{M}_j(Z_k(\sigma)) \) should also contain bundles whose presentation requires nontrivial coefficients of \( \hbar^n \) for \( n > 1 \). In particular, this implies that the moduli space \( \mathcal{M}_j^{(n)}(Z_k(\sigma)) \) is larger than \( \mathcal{M}_j^{(1)}(Z_k(\sigma)) \).

The calculation of \( \mathcal{M}_2^{(2)}(Z_1(\sigma)) \) is analogous to the calculations of Exs. 7.9 and 7.12 and can be reproduced by using the Kontsevich star product extended to \( Z_k \) as given in Prop. 4.1, whose terms up to second order can be obtained from Lem. 3.10. However, the calculation is considerably longer and we do not include it here, leaving a more general investigation for future work.
By definition an instanton on a 4-manifold $X$ is a connection $A$ minimizing the Yang–Mills functional

$$\text{YM}(A) = \int_X \text{tr} F \wedge F$$

where $F$ is the curvature of $A$. The Euler–Lagrange equations for the Yang–Mills functional

$$D(\ast F) = 0$$

(8.1)

are called the Yang–Mills equations (here $\ast$ denotes Hodge dual). Thus an instanton is a solution of the Yang–Mills equations. A linearized version of these equations is given by the anti-self-duality (ASD) equations

$$F^+ = \frac{1}{2} (F + \ast F) = 0.$$  
(8.2)

Equation (8.2) is sometimes called the instanton equation because its solutions also satisfy (8.1).

The translation from gauge theory to complex geometry is made via the so-called Kobayashi–Hitchin correspondence, which relates instantons and vector bundles as well as their moduli spaces. If $X$ is a complex Kähler surface and $E$ is an SU(2) bundle on $X$, then the moduli space $M_E$ of irreducible ASD connections on $E$ is a complex analytic space and each point in $M_E$ has a neighbourhood which is the base of a universal deformation of the corresponding stable vector bundle [DK, Prop. 6.4.4]. A version of such a correspondence for noncompact surfaces states:

**Lemma 8.3.** [GKM, Cor. 5.5] A holomorphic SL(2, $\mathbb{C}$) vector bundle on $Z_k$ corresponds to an SU(2) instanton if and only if its splitting type is a multiple of $k$.

The surfaces $Z_k$ have rich moduli spaces of instantons, which unfortunately disappear under any small commutative deformation of $Z_k$. In fact, for $j \equiv 0 \mod k$, we have:

**Theorem 8.4.** [BGK2, Thm. 4.11] The moduli space of irreducible SU(2) instantons on $Z_k$ with charge (and splitting type) $j$ is a quasi-projective variety of dimension $2j - k - 2$.

In contrast:

**Theorem 8.5.** [BaG, Thm. 7.3] Let $Z_k(\tau)$ be a nontrivial commutative deformation of $Z_k$. Then the moduli spaces of irreducible SU(2) instantons on $Z_k(\tau)$ are empty.

Disappearance of instantons under classical deformations gave us a strong motivation to explore noncommutative directions of deformations. From the point of view of algebraic deformation theory, both classical and noncommutative deformations can be regarded on equal footing — as components in the Hochschild–Kostant–Rosenberg decomposition of the Hochschild cohomology

$$\text{HH}^2(Z_k) \cong H^1(Z_k, \mathcal{T}_{Z_k}) \oplus H^0(Z_k, \Lambda^2 \mathcal{T}_{Z_k})$$

which parametrizes deformations of the Abelian category of (quasi)coherent sheaves in the sense of [LV]. (Note that simultaneous deformations of $Z_k$ in commutative and noncommutative directions may be obstructed. The obstruction calculus was studied in [BaF].) However, from a physics point of view we hoped, and intuitively expected, that moduli of instantons appear again on directions of noncommutative deformations. We
will see that this is indeed the case. Furthermore, we discover that some instantons react wildly to certain types of quantization, causing quantum moduli spaces to become larger than the classical ones.

The geometry underlying the disappearance of instantons on commutative deformations is the fact that if $Z_k(\tau)$ is any nontrivial commutative deformation of $Z_k$, then every holomorphic vector bundle on $Z_k(\tau)$ splits as a direct sum of line bundles [BaG, Thm. 6.10]. The key issue here is that for $\tau \neq 0$ the deformations $Z_k(\tau)$ are affine [BaG, Thm. 6.18]. In particular, by [BaG, Thm. 6.6] such a nontrivial deformation contains no compact complex curves. In physics language, we may say that there is no compact manifold which can hold the instanton charge.

In Yang–Mills theory instantons are well known to carry topological charges. Under the Kobayashi–Hitchin correspondence the charge of an SU(2) instanton on a complex surface translates into the second Chern class of its corresponding SL(2, C) holomorphic vector bundle. Here we use instead the concept of local charge, given that second Chern class of a bundle on $Z_k$ vanishes, since $H^4(Z_k, \mathbb{Z}) \cong H^4(S^2, \mathbb{Z}) = 0$. The terminology “local charge” is motivated by the fact that it provides a local contribution to the second Chern class when we consider a compact surface containing an embedded $Z_k$.

**Definition 8.6.** We define the normalized charge of an instanton on $Z_k$ to be the sum of the local second Chern class (Def. 5.5) of the bundle to which it corresponds and $\epsilon$, where $\epsilon = 1$ if $k \geq 2$ and 0 when $k = 1$. In what follows we will simply refer to this normalized charge as the charge of the instanton for brevity.

**Remark 8.7.** The reason for this normalization is that, once normalized, the minimal charge of an SU(2) instanton with splitting type $j$ equals $j$, and this allows us to express the theorems that follow in a much simpler way. We observe that the addition of $\epsilon = 1$ in the cases $k \geq 2$ corresponds to the fact that the surface $X_k$ obtained by contracting the $\mathbb{P}^1$ to a point has one singularity. In case we considered more general surfaces containing other contractible curves, the correct normalization should most likely be to count all singularities obtained by their contraction. For bounds on the values of instanton charges see [BGK2] and [GKM].

We note also that there exist more than one notion of Chern classes for sheaves on singular varieties. A particularly useful one, presented by Blache [Bl], is the concept of orbifold Chern class, defined using the familiar integration formula $\int_X \text{ch}(E) \cdot \text{td}(X)$ appearing in Hirzebruch–Riemann–Roch formulas, but which in the case of singular varieties differs from the Euler characteristic by a weighted counting of singularities. In fact, Blache proves the following formula:

$$\chi(X, E) = \int_X \text{ch}(E) \cdot \text{td}(X) + \sum_{x \in \text{Sing}(X)} \mu_{x, X}(E).$$

Here our $\epsilon$ may be regarded as keeping track of when the defect term $\mu$ is nonzero.

As an illustration we give instanton invariants from [BG] in Table 1.

9. Rebel instantons

We wish to investigate the effect that noncommutative deformations have on instantons — these effects can already be observed at first order in $\hbar$. We thus reinterpret of the results of §6 into the language of instantons.
Noncommutative versions of (8.1) and (8.2) and instanton solutions for the cases of noncommutative $\mathbb{R}^4$ are presented in [NS], with self-duality being given by the generalized ASD equations

$$F_{\mu\nu}^+ = 0$$

where the curvature of the connection is calculated by the generalized formula

$$F_{\mu\nu, j}^+ = \partial_{\mu}A_{\nu, j}^+ - \partial_{\nu}A_{\mu, j}^+ + A_{\mu, k}^+ \star A_{\nu, j}^k - A_{\nu, k}^+ \star A_{\mu, j}^k.$$

ASD connections are then automatically solutions to the deformed Yang–Mills equations:

$$\partial_{\mu}F_{\mu\nu}^- - A_{\mu} \star F_{\mu\nu}^- = 0.$$

Instantons on noncommutative $\mathbb{R}^4$ in this sense were further studied in [SW] for their relations with string theory, and in [KKO] on noncommutative projective planes. For our noncommutative deformations of the $Z_k$, we have obtained global star products and one could also approach the study of instantons using such deformed equations. However, it is more convenient to work directly in the language of vector bundles.

**Definition 9.1.** Based on Def. 7.3 and the result of Lem. 8.3, a formally algebraic bundle on $Z_k(\sigma)$ of rank 2 and splitting type $nk$ is called a formal instanton. The charge of a formal instanton on $Z_k(\sigma)$ is defined as the charge of its classical limit (Def. 8.6), i.e. obtained by setting $\hbar = 0$.

**Definition 9.2.** According to Def. 6.11, instantons on $Z_k$ and on deformations of $Z_k$ can be represented in canonical coordinate charts by a pair $(j, p)$ of an integer and a formal expression $p = \sum p_n \hbar^n$ where $p_n$ are polynomials. For a fixed splitting type $j$, the instanton is thus determined by the coefficients of the corresponding polynomials, and in this case we denote the point on the corresponding moduli space simply by $[j]$ as in (7.8).

**Notation 9.3.** We will denote by $\mathcal{Q}_j(Z_k(\sigma))$ the moduli space of formal instantons of charge $j$ on $Z_k(\sigma)$, and by $\mathcal{Q}_j^{(m)}(Z_k(\sigma))$ the subspace obtained by imposing the cut

| monomial | width | height | charge |
|----------|-------|--------|--------|
| $z^{-1}u$ | 3     | 2      | 5      |
| $u$      | 1     | 2      | 3      |
| $zu$     | 1     | 2      | 3      |
| $z^2u$   | 3     | 2      | 5      |
| $u^2$    | 3     | 3      | 6      |
| $zu^2$   | 2     | 3      | 5      |
| $z^2u^2$ | 3     | 3      | 6      |
| $zu^3$   | 5     | 3      | 7      |
| $z^2u^3$ | 5     | 3      | 7      |
| $z^2u^4$ | 5     | 3      | 8      |
| zero     | 6     | 3      | 9      |

Table 1. Numerical invariants for rank 2 vector bundles of splitting type $j = 3$ on $Z_1$.
\[ \hbar = 0. \text{ Hence } \mathcal{Q}^{(0)}_{j_k}(Z_1(\sigma)) \text{ equals the classical moduli space } \mathcal{M}_{j_k}(Z_k) \text{ of instantons of charge } j. \]

Therefore, by definition we have:

**Lemma 9.4.** The classical limit of any formal instanton on \( Z_1(\sigma) \) is an instanton on \( Z_k \).

**Proof.** The correspondence between vector bundles and instantons may be applied to both quantum and classical moduli. The projection onto classical moduli translates as

\[
\begin{array}{ccc}
\mathcal{M}^{(1)}_j(Z_k(\sigma)) & \xleftarrow{\pi} & \mathcal{Q}^{(1)}_j(Z_k(\sigma)) \\
\mathcal{M}_j(Z_k) & \xrightarrow{\pi} & \mathcal{M}_{j_k}(Z_k)
\end{array}
\]

We now interpret the statements of Thms. 7.15 and 7.19 in terms of instantons. These get rephrased as:

**Theorem 9.5.** Let \( \sigma_0 \) be the Poisson structure on \( Z_1 \) which in canonical coordinates is described by \( \sigma_0 = (1, -\xi) \), hence \( \sigma_0 \) is degenerate at a single point of the line \( \ell \subset Z_1 \). Then for each value of the charge \( j \) we obtain an isomorphism between the quantum and the classical instanton moduli spaces

\[ \mathcal{Q}^{(1)}_j(Z_1(\sigma_0)) \cong \mathcal{M}_{j_k}(Z_k). \]

**Theorem 9.6.** The quantum instanton moduli space \( \mathcal{Q}^{(1)}_j(Z_k(\sigma)) \) can be viewed as the étale space of a constructible sheaf over the classical instanton moduli space \( \mathcal{M}_{j_k}(Z_k) \) which is supported on a closed subvariety, being trivial over

\[ S_0 := \{ P \in \mathcal{M}_j(Z_k) \mid p_{1,k-j+1} \neq 0 \} \]

and having stalk of dimension \( i \) over

\[ S_i := \{ P \in \mathcal{M}_j(Z_k) \mid p_{1,k-j+1} = \cdots = p_{1,k-j+i} = 0, \, p_{1,k-j+i+1} \neq 0 \}. \]

**Notation 9.7.** When regarding the quantum moduli space \( \mathcal{Q}^{(1)}_j(Z_k(\sigma)) \) as the étale space of a sheaf over the classical moduli space \( \mathcal{M}_{j_k}(Z_k) \), we will use the notation \( \mathcal{Q}^\sigma \) and call it the quantizing sheaf.

**Definition 9.8.** A (classical) instanton \( A \) is called a rebel instanton for \( \sigma \) if the stalk \( \mathcal{Q}^\sigma_A \) of the quantizing sheaf \( \mathcal{Q}^\sigma \) at \( A \) is nontrivial. The dimension of the stalk at \( A \) is called the level of rebelliousness of \( A \). Hence, if the stalk \( \mathcal{Q}^\sigma_A \) has rank \( n \) then \( A \) is said to present level \( n \) rebelliousness, or equivalently, \( A \) is called \( n \)-rebel instanton. For brevity the vocabulary rebel will be used to denote rebelliousness of any level \( n \geq 1 \).

An instanton that is not rebel for \( \sigma \) is called \( \sigma \)-tame.

In particular, in the language of Def. 9.8 (see also Not. 2.15) we can rephrase Thm. 9.5 as follows.

**Theorem 9.9.** Let \( \sigma_0 \) be a minimally degenerate Poisson structure on \( Z_1 \). Then all instantons on \( Z_1 \) are \( \sigma_0 \)-tame.
Note that the zero section \( \ell \) of \( Z_1 \) contracts to a smooth point and Thm. 9.9 stands in stark contrast to the case of \( k \geq 3 \) where the zero section \( \ell \) of \( Z_k \) contracts to a \( \frac{1}{k}(1, 1) \) singularity.

**Theorem 9.10.** Let \( k \geq 3 \). Then \( Z_k \) produces rebel instantons for any holomorphic Poisson structure \( \sigma \).

To be interesting from a physical perspective, it is important that the first-order deformations parametrized by a holomorphic Poisson structure can be continued to all higher orders, which was shown in §4, at least for Poisson structures tangent to a fibre of the projection to \( \mathbb{P}^1 \). Thms. 9.9 and 9.10 now exhibit a phenomenon that can already be observed at first order in \( \hbar \) and for higher orders in \( \hbar \) we expect Thms. 9.9 and 9.10 to generalize as follows.

Phrased in the language of (rebel) instantons, Rem. 7.20 suggests that for the minimally degenerate Poisson structure \( \sigma_0 \) on \( Z_1 \), we expect an isomorphism \( \mathbb{Q} \mathbb{P} \mathbb{V} (Z_j (\sigma_0)) \cong \mathbb{Z} / \ell \mathbb{V} (Z_j) \), i.e. the absence of rebel instantons implies that the moduli of noncommutative and classical instantons are isomorphic for arbitrary powers of \( \hbar \).

On the other hand we expect the existence of \( n \)-rebel instantons (as for example in Thm. 9.10) to imply the existence of noncommutative instantons whose minimal representatives \( p = \sum_{n \geq 0} p_n \hbar^n \) contain non-vanishing terms \( p_n \), i.e. \( n \)-rebel instantons produce noncommutative instantons up to \( \hbar^n \).

---

**APPENDIX A. COHOMOLOGY**

In this appendix we determine global sections of \( Z_k \) with line bundle coefficients. These are used in the proof of Lem. 2.8 to determine the space of Poisson structures on \( Z_k \).

The coordinate ring of \( Z_k \) is

\[
R = \mathbb{H}^0(Z_k, \mathcal{O}) = \mathbb{C}[x_0, x_1, \ldots, x_k] / (x_i x_{j+1} - x_{i+1} x_j)_{0 \leq i < j \leq k-1}.
\]

The contraction map \( Z_k \to \text{Spec} R \) is given in \( (z, u) \)-coordinates by \( z^l u \mapsto x_i \). We compute \( \mathbb{H}^0(Z_k, \mathcal{O}(j)) \).

**Lemma A.1.** The cohomology \( \mathbb{H}^0(Z_k, \mathcal{O}(j)) \) for \( j \geq 0 \) is generated as an \( R \)-module by the monomials \( \beta_i = z^l \) for \( 0 \leq i \leq j \) with relations

\[
\beta_i x_{i-1} - \beta_{i-1} x_i = 0
\]

where \( 1 \leq l \leq k \) and \( 1 \leq i \leq j \).

**Proof.** Let \( \mathcal{U} = \{ U, V \} \) be our canonical coordinates on \( Z_k \) and let \( \sigma \in \tilde{C}^0(\mathcal{U}, \mathcal{O}(j)) \) be a 0-cochain, thus \( \sigma \) consists of a pair of holomorphic sections \( (\sigma_U, \sigma_V) \). Writing out the general form of such a cochain, we have

\[
\sigma_U = \sum_{i=0}^k \sum_{l=0}^i \sigma_{ul} z^l u^l
\]

as this is just an arbitrary holomorphic map on \( U \), where the line bundle trivializes. As we know, \( \mathbb{H}^0 \) computes global holomorphic sections, so to find the cohomology class of \( \sigma \) we just need to find the most general such \( \sigma \) that extends holomorphically to \( V \). The transition function for the line bundle \( \mathcal{O}(j) \) is \( T = z^{-j} \). Changing coordinates then gives

\[
T \sigma = z^{-j} \sum_{i=0}^k \sum_{l=0}^i \sigma_{ul} z^l u^l
\]

which needs to be holomorphic on \( V \). This gives the condition that \( T \sigma \) must contain only terms \( z^r u^s \) with \( r \leq ks \). Terms that do not satisfy this condition are not cocycles, and must
be removed from the expression of $T\sigma$. We are thus left with: $T\sigma = \sum_{i=0}^{\infty} \sum_{l=0}^{ki+j} \sigma_{il}z^{l-j}u^i$ or equivalently $\sigma = \sum_{i=0}^{\infty} \sum_{l=0}^{ki+j} \sigma_{il}z^{l-k}u^{i}$ which can be rewritten as

$$\sigma = \sum_{l=0}^{j} \sigma_{il}z^l + \sum_{i=1}^{\infty} \sum_{l=0}^{ki+j} \sigma_{il}z^{l-k}u^{i}.$$ 

Now notice that on the right-hand side every term on the second sum can be obtained from a term on the first sum by multiplying by $(z^k u)^i$ which are global holomorphic functions on $Z_k$, thus $H^0(Z_k, O(j))$ with $j \geq 0$ is generated as an $R$-module by the monomials $\beta_i = z^i$ for $0 \leq i \leq j$. Since $j \geq 0$ there is at least one such $\beta_i$. These satisfy the equalities

$$\beta_0x_0 = \beta_0x_1 = z u$$

$$\beta_2x_0 = \beta_1x_1 = \beta_0x_2 = z^2 u,$$

$$\vdots$$

$$\beta_jx_0 = \beta_1x_1 \cdots \beta_jx_k = z^{j-1} u.$$

To compute $H^0(Z_k, O(-j))$, set $\nu = -j \mod k$, so that $-j = -qk + \nu$ with $0 \leq \nu < k$.

**Lemma A.2.** The cohomology group $H^0(Z_k, O(-j))$ for $j \geq 0$ is generated by the monomials $\alpha_i = z^i u^i$, for $0 \leq i \leq \nu$, with relations

$$\alpha_i x_{i-1} - \alpha_{i-1} x_i = 0$$

for $1 \leq i \leq \nu$ and $1 \leq l \leq k$.

**Proof.** As in the proof of Lem. A.1 we start with a 0-cochain $(\sigma_U, \sigma_V)$ having $\sigma_U = \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \sigma_{il}z^{l-k}u^i$, and we look for the corresponding $\sigma_V$ making this a 0-cocycle. Changing coordinates, we have

$$T\sigma = z^j \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \sigma_{il}z^{l-k}u^i.$$ 

Here, since $j > 0$, terms on $u^0$ are not holomorphic on $V$. To have holomorphicity, we need only terms $z^i u^i$ with $r \leq ks$, we are thus looking to satisfy the condition $j + l \leq ki$.

Thus, we arrive at the expression $T\sigma = \sum_{i=0}^{\infty} \sum_{l=0}^{\infty} \sigma_{il}z^{l-j}u^i$ which terms are nonzero only when $ki - j \geq 0$, that is, $i \geq \lfloor j/k \rfloor$. Using the notation set up just above for the Euclidean algorithm, we have $q = \lfloor j/k \rfloor$, and we are searching for holomorphic terms in the expression $T\sigma = \sum_{i=q}^{\infty} \sum_{l=0}^{\infty} \sigma_{il}z^{l-j}u^i$. Note that for $i = q$ we have $0 \leq l \leq kj - j = \nu$, thus, we may rewrite

$$T\sigma = \sum_{i=0}^{\nu} \sum_{l=0}^{ki+j} \sigma_{il}z^{l-j}u^i + \sum_{i=q}^{\infty} \sum_{l=0}^{\infty} \sigma_{il}z^{l-j}u^i,$$

or equivalently

$$T\sigma = \sum_{i=0}^{\nu} \sum_{l=q}^{\infty} \sigma_{il}z^{l-j}u^i + \sum_{i=1}^{\infty} \sum_{l=0}^{\infty} \sigma_{il}z^{l-j}u^{q+i}.$$
Thus
\[\sigma = \sum_{i=0}^{\nu} \sigma_{ij} \zeta^i u^q + \sum_{i=1}^{\infty} \sum_{j=0}^{\nu-k_i} \sigma_{ij} \zeta^i u^{q+i},\]
and the second sum on the right-hand side has only terms that can be obtained from the first sum via multiplying by \((z^{\pm k_i} u)^q\) which are global holomorphic functions on \(Z_k\). Hence \(H^0(Z_k, \mathcal{O}(-j))\) is generated by the monomials \(\alpha_i = z^i u^q\), for \(0 \leq i \leq \nu\). These satisfy the relations
\[
\begin{align*}
\alpha_1 x_0 &= \alpha_0 x_1 = z u^q \\
\alpha_2 x_0 &= \alpha_1 x_1 = \alpha_0 x_2 = z^2 u^q \\
& \vdots \\
\alpha_{\nu-k-1} x_{\nu-k} &= \alpha_{\nu-1} x_{\nu} = z^{\nu-1} u^q.
\end{align*}
\]
□

ACKNOWLEDGEMENTS

We thank Oren Ben-Bassat and Pushan Majumdar for helpful discussions and the referee for carefully reading our paper and making several useful suggestions.

S.B. acknowledges the generous support of the Studienstiftung des deutschen Volkes and the Max Planck Institute for Mathematics in Bonn, Germany. E.G. was partially supported by a Simons Associateship grant of ICTP and Network grant NT8 of the Office of External Activities at ICTP, Italy.

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