A map between Galilean relativity and special relativity

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A map is discussed that connects, in 1+1 dimensions, Galilei’s relativity to Einstein’s special relativity. By means of this map it is possible to derive special-relativistic formulas from the corresponding Galilean ones. Beyond being interesting on its own, this map is also significant with respect to a recent debate on the extension of relativistic symmetries to the Planck scale (especially in the framework of the so-called doubly special relativity). The map in fact provides an explicit example of how can be misleading to interpret a mathematical correspondence between two relativity schemes as an argument in favor of their physical equivalence.

I. INTRODUCTION

Special relativity has been proposed in [1, 2] by Einstein with the aim to extend the relativity principle from the mechanical to the electromagnetic phenomena. The hypothesis of the invariance of the speed of light \( c \) has played a central role in Einstein’s original derivation. However, over the last century, special relativity has been re-derived starting from many different viewpoints (see e.g. [3–8]). From the special-relativistic formulas the corresponding Galilean ones are recovered in the limit in which all the speeds involved are small when compared to the speed of light, i.e. in the \( c^{-1} \to 0 \) limit. Whereas, from a mathematical point of view, special relativity can be seen as a deformation of Galilei’s relativity, being \( c^{-1} \) the deformation parameter, from a physical point of view the observables (and their relations) in the two frameworks remain well distinct. The fact that we are dealing with different theories and with different physical predictions, unless the \( c^{-1} \to 0 \) limit is not taken, is widely accepted. More recently extensions of special relativity have been considered in literature [9–12], motivated by the aim to extend special relativity including a second invariant scale, \( \lambda^{-1} \), eventually connected with the Planck length/energy. This scale \( \lambda^{-1} \) is expected to play a role in the new transformation formulas between inertial observers analogous to the role played by \( c^{-1} \) in the transition from Galilean relativity to special relativity. The resulting relativity scheme, also known as doubly-special relativity, should represent an extension of special relativity in the same sense in which special relativity has represented an extension of Galilei’s relativity. However reasons of concern have been raised from certain authors [13–15] ultimately motivated by the fact that a mathematical map can be found [16] that allows to describe doubly-special-relativistic formulas in terms of the corresponding special-relativistic ones. The existence of the mentioned map [16] would prove, according to these authors (see especially [14]), that we are dealing with the same fundamental theory but written in terms of different variables. Various objections have already been raised against this type of arguments [17, 18]. The aim of this paper is to provide a counterexample by showing that a map between Galilean relativity and special relativity can be found as well. The map allows us to rewrite special-relativistic formulas directly from the corresponding Galilean ones, and vice versa, without recurring to the \( c^{-1} \to 0 \) limit. The existence of such a map represents in our opinion a strong argument against the deduction of the physical equivalence of two relativity frameworks motivated by a mathematical correspondence between them. The paper is organized as follows. In Section II we construct the Galilei-Einstein map in the coordinate space, in Section III we discuss the map in the energy-momentum sector, in Section IV we discuss the possibility to extend the map to 3+1 dimensions, finally, in Section V, we present our final remarks.

II. THE MAP IN THE COORDINATE SPACE

We start our derivation of the Galilei-Einstein map remembering that the infinitesimal actions of the boosts on the coordinate space can be written, in 1+1 space-time dimensions, in the form

\[
\Delta x' \simeq \Delta x + \delta_N \Delta x,
\]

\[
\Delta t' \simeq \Delta t + \delta_N \Delta t,
\]
where \( \delta N \Delta x = \xi \{ N, \Delta x \} \) and \( \delta N \Delta t = \xi \{ N, \Delta t \} \) are the infinitesimal actions, \( \xi \) is the boost parameter and \( N \) is the boost generator. The actions of the boosts on the space-time coordinates can be expressed in terms of the Poisson brackets

\[
\{ N, \Delta x_E \} = \Delta t_E, \\
\{ N, \Delta t_E \} = \Delta x_E/c^2,
\]

for the special-relativistic case, and in terms of the brackets

\[
\{ N, \Delta x_G \} = \Delta t_G, \\
\{ N, \Delta t_G \} = 0,
\]

for the Galilean transformations. Here we look for a map between the Lorentz-Einstein algebra \([3] - [4]\) and the Galilei algebra \([5] - [6]\) of the form \( \Delta x_E = \Delta x_E(\Delta x_G, \Delta t_G) \), \( \Delta t_E = \Delta x_E(\Delta x_G, \Delta t_G) \). Substituting in Eqs.\([3] - [4]\), and using Eqs.\([5] - [6]\), we find the following system of differential equations:

\[
\Delta t_G \frac{\partial \Delta x_E}{\partial x_G} = \Delta t_E, \\
\Delta t_G ^2 \frac{\partial \Delta t_E}{\partial x_G} = \Delta x_E/c^2,
\]

whose solutions can be written as

\[
\Delta x_E = F_1 (c \Delta t_G) \sinh \left( \frac{\Delta x_G}{c \Delta t_G} \right) + F_2 (c \Delta t_G) \cosh \left( \frac{\Delta x_G}{c \Delta t_G} \right), \\
\Delta t_E = F_1 (c \Delta t_G) \cosh \left( \frac{\Delta x_G}{c \Delta t_G} \right) + F_2 (c \Delta t_G) \sinh \left( \frac{\Delta x_G}{c \Delta t_G} \right),
\]

where \( F_1 \) and \( F_2 \) depend on the Galilei time \( c \Delta t_G \) alone. For dimensional reasons we can assume that

\[
F_1 (c \Delta t_G) = \alpha c \Delta t_G, \\
F_2 (c \Delta t_G) = \beta c \Delta t_G,
\]

with \( \alpha \) and \( \beta \) dimensionless constants. Substituting Eqs.\([11] - [12]\) into Eqs.\([9] - [10]\), and defining \( v_G = \Delta x_G/\Delta t_G \), we get the final form of the Galilei-Einstein map in the space-time sector

\[
\Delta x_E = \alpha \sinh \left( \frac{v_G}{c} \right) + \beta \cosh \left( \frac{v_G}{c} \right), \\
\Delta t_E = \Delta t_G \left[ \alpha \cosh \left( \frac{v_G}{c} \right) + \beta \sinh \left( \frac{v_G}{c} \right) \right].
\]

The freedom in the choice of the parameters \( \alpha \) and \( \beta \) can be better understood by looking at the way in which the map acts on the velocity space. The speed-transformation rule can be obtained by taking the ratio between both sides of \([13]\) and \([14]\) getting

\[
v_E = c \frac{\alpha \sinh \left( \frac{v_G}{c} \right) + \beta \cosh \left( \frac{v_G}{c} \right)}{\alpha \cosh \left( \frac{v_G}{c} \right) + \beta \sinh \left( \frac{v_G}{c} \right)}.
\]

We notice that if we set \( \alpha = \beta \) we get \( v_E = c \) for every value of \( v_G \). In this case the whole velocity space of the Galilean relativity is mapped into the light-cone of the special relativity. If instead we choose \( \beta = 0 \) we get

\[
v_E = c \tanh \left( \frac{v_G}{c} \right),
\]

that maps the Galilean speeds into subluminal Einstein speeds, \( v_E < c \). Finally, if we choose \( \alpha = 0 \), we obtain

\[
v_E = c \coth \left( \frac{v_G}{c} \right),
\]

that corresponds to the case of superluminal propagation in Einstein relativity, \( v_E > c \).

To gain further insight we can consider the action of the map on the Galilei and the Lorentz-Einstein space-time invariant. Using Eqs.\([13] - [14]\) is easily found that the Lorentz-Einstein (space-time) invariant is mapped into the Galilei (time) invariant as follows

\[
\Delta x_E^2 - c \Delta t_E^2 = c^2 \Delta t_G^2 \left( \alpha^2 - \beta^2 \right).
\]
Thus we see that in general if $|\alpha| > |\beta|$ two Galilei events whose space and time distances are respectively $\Delta x_G$ and $\Delta t_G$ are mapped into space-like distant Lorentz-Einstein events. If $|\alpha| = |\beta|$ the couple of Galilean events is mapped into events that belong to the same light cone. Finally if $|\alpha| > |\beta|$ every couple of Galilean event is mapped into time-like distant space-time events (see FIG. 1). Equations (24)-(25) can be easily inverted providing the inverse map in space-time coordinates:

$$\Delta t_E = \Delta t_G \cosh \left( \frac{v_G}{c} \right),$$

$$\Delta x_E = c \Delta t_G \sinh \left( \frac{v_G}{c} \right),$$

that are the coordinates of a Rindler observer.
III. THE MAP IN THE ENERGY-MOMENTUM SPACE

Once we have established the procedure in the coordinate space it is straightforward to extend it to the energy-momentum space. As we did in the previous section we write the action of the boosts using the Poisson-bracket notation $\delta_N E_E = \xi \{ N, E_E \}$ and $\delta_N P_E = \xi \{ N, P_E \}$, being the infinitesimal actions given by

$$\{ N, P_E \} = E_E/c^2,$$
$$\{ N, E_E \} = P_E,$$

in the special-relativity case, and by

$$\{ N, P_G \} = m,$$
$$\{ N, E_G \} = P_G,$$

in the case of Galilean relativity. Again we look for a map between the two algebras of the general form $E_E = E_E(E_G, P_G), P_E = P_E(E_G, P_G).$ From Eqs. (28)- (29) it follows that the map satisfies the system of equations

$$P_G \frac{\partial E_E}{\partial E_G} + m \frac{\partial E_E}{\partial P_G} = P_E,$$
$$P_G \frac{\partial P_E}{\partial E_G} + m \frac{\partial P_E}{\partial P_G} = E_E/c^2.$$

The solutions of the above system are

$$P_E = mc \left[ \hat{\alpha} \sinh \left( \frac{P_G}{mc} \right) + \hat{\beta} \cosh \left( \frac{P_G}{mc} \right) \right],$$
$$E_E = mc^2 \left[ \hat{\alpha} \cosh \left( \frac{P_G}{mc} \right) + \hat{\beta} \sinh \left( \frac{P_G}{mc} \right) \right],$$

where $\hat{\alpha}$ and $\hat{\beta}$ are dimensionless constants. From Eqs. (34)- (35) the invariant mass Casimir can be written as

$$E_E^2 - c^2 P_E^2 = m^2 c^4 \left( \hat{\alpha}^2 - \hat{\beta}^2 \right).$$

If $|\hat{\alpha}| > |\hat{\beta}|$ we get the energy-momentum dispersion relation of a particle (bradion) propagating slower than light ($v_E = dE_E/dP_E < c$). If $|\hat{\alpha}| < |\hat{\beta}|$ we get the energy-momentum dispersion relation of a particle (tachyon) propagating faster than light ($v_E > c$). Finally if $|\hat{\alpha}| = |\hat{\beta}|$ we get the energy-momentum dispersion relation of a massless particle propagating exactly at the speed of light ($v_E = c$). As in the coordinate space, if we request that the map reduces to the identity (i.e. $E_E = E_G, P_E = P_G$) in the limit $c^{-1} \to 0$, then we find that it must be $\hat{\alpha} = 1$ and $\hat{\beta} = 0$. Under this assumption the map reduces to the simpler

$$P_E = mc \sinh \left( \frac{P_G}{mc} \right) = mc^2 \sinh \left( \sqrt{\frac{2E_G}{mc}} \right),$$
$$E_E = mc^2 \cosh \left( \frac{P_G}{mc} \right) = mc^2 \cosh \left( \sqrt{\frac{2E_G}{mc}} \right),$$

the inverse map being

$$P_G = mc \ln \left[ \frac{P_E}{mc} + \sqrt{\left( \frac{P_E}{mc} \right)^2 + 1} \right],$$
$$E_G = \frac{mc^2}{2} \ln^2 \left[ \frac{E_E}{mc^2} + \sqrt{\left( \frac{E_E}{mc^2} \right)^2 - 1} \right].$$

We have already derived, in space-time coordinates, the transformation that maps the particle velocity. Here we notice that using the standard definition of particle speed, $v = dE/dp$, we can derive the same formula in the momentum space. In fact we have that

$$v_E = \frac{dE_E}{dP_E} = \frac{dE_E}{dE_G} \frac{dE_G}{dP_G} \frac{dP_G}{dP_E} = v_G \frac{dE_E}{dE_G} \left( \frac{dP_E}{dP_G} \right)^{-1},$$

and applying the map of Eqs.\([34]-[35]\) to the expression \([40]\) we get the desired formula of Eq.\([15]\) in the energy-momentum space. The map of Eqs.\([38]-[39]\) connects Galilei algebra to Lorentz-Poincaré algebra. This allows us to rewrite physical quantities, expressed in the energy-momentum space and transforming according to Galilei relativity, in terms of quantities transforming according to special relativity. It is easy to show that Galilean transformation rules between inertial observers are mapped into the corresponding special-relativistic ones:

\[
\begin{align*}
\{ p_G &= p_G - v_G m \\
E_G &= E_G + v_G p
\end{align*}
\quad \leftrightarrow \quad
\begin{align*}
\{ p_E &= \gamma(p_E - v_E E_E) \\
E_E &= \gamma(E_E - v_E / c^2 p_E).
\end{align*}
\]

(41)

Also the energy-momentum dispersion relations, connected to the Casimir invariant of the corresponding algebras, are mapped into each other

\[E_G = \frac{p_G^2}{2m} \leftrightarrow E_E^2 = c^2 p_E^2 + m^2 c^4.\]

(42)

One can also recover the relation between the energy-momentum and the speed of a particle. In fact using Eq.\([15]\) and Eq.\([21]\) one finds

\[
p_G = m v_G \quad \leftrightarrow \quad p_E = \frac{m v_E}{\sqrt{1 - (v_E/c)^2}},
\]

(43)

\[E_G = \frac{1}{2} m v_G^2 \quad \leftrightarrow \quad E_E = \frac{m c^2}{\sqrt{1 - (v_E/c)^2}}.
\]

(44)

Now we are ready to analyze another key issue of the correspondence between Galilei relativity and Einstein special relativity: the energy-momentum conservation rule. Energy-momentum conservation rule in special relativity can be written as

\[E_1E + E_2E = E_{3E}, \quad p_1E + p_2E = p_{3E}.
\]

(45)

(46)

Substituting expressions \([34]-[35]\) we find that the corresponding Galilean conservation rules read

\[
m_{1L} \cosh \left( \frac{P_{1G}}{m_{1G}c} \right) + m_{2L} \cosh \left( \frac{P_{2G}}{m_{2G}c} \right) = m_{3L} \cosh \left( \frac{P_{3G}}{m_{3G}c} \right),
\]

(47)

\[
m_{1L} \sinh \left( \frac{P_{1G}}{m_{1G}c} \right) + m_{2L} \sinh \left( \frac{P_{2G}}{m_{2G}c} \right) = m_{3L} \sinh \left( \frac{P_{3G}}{m_{3G}c} \right).
\]

(48)

Formulas \([47]-[48]\) represent fully covariant expressions in 1+1-dimensional Galilei relativity, for every value of the parameter \(c\), as can be easily verified observing that

\[
\{ N, m_{iG} \cosh \left( \frac{P_{iG}}{m_{iG}c} \right) \} = \frac{m_{iG}}{c} \sinh \left( \frac{P_{iG}}{m_{iG}c} \right), \quad \{ N, m_{iG} \sinh \left( \frac{P_{iG}}{m_{iG}c} \right) \} = \frac{m_{iG}}{c} \cosh \left( \frac{P_{iG}}{m_{iG}c} \right).
\]

(49)

However Eqs.\([47]-[48]\) are not the right energy-momentum conservation laws. The real ones can be obtained eliminating the invariant parameter, i.e. taking the \(c^{-1} \to 0\) limit. In this way one finds

\[m_1 + m_2 = m_3, \quad p_{1G} + p_{2G} = p_{3G},
\]

(50)

(51)

that are the right conservation laws of Galilei relativity. Instead, if we had started from \([50]-[51]\) we would have obtained for the energy-momentum conservation laws of special relativity

\[m_1 c \ln \left( \frac{c P_{1E} + E_{1E}}{m_1 c} \right) + m_2 c \ln \left( \frac{c P_{2E} + E_{2E}}{m_2 c} \right) = m_3 c \ln \left( \frac{c P_{3E} + E_{3E}}{m_3 c} \right),
\]

(52)

whose covariance can be easily proved noticing that

\[
\{ N, m_i \} = 0, \quad \{ N, \ln \left( \frac{c P_{iE} + E_{iE}}{m_i c} \right) \} = \frac{E_{iE}/c + p_{Ei}}{c P_{iE} + E_{iE}} = c^{-1}.
\]

(54)

Again one does not obtain the real special-relativistic energy-momentum conservation laws starting from the real Galilean ones, but again the energy-momentum conservation laws are compatible with the covariance. To obtain the real energy-momentum conservation rules of special relativity one has to start from the deformed, but still covariant, Galilean mass-momentum conservation rules of Eqs.\([47]-[48]\).
IV. ON THE POSSIBILITY TO EXTEND THE MAP TO 3+1 SPACE-TIME DIMENSIONS

Having constructed the map in 1+1 dimensions one could wonder if a similar map can be found in 3+1 space-time dimensions as well. We focus again in the energy-momentum sector but the same procedure can be easily reproduced in the space-time sector, leading to the same result. We recall that, in the energy-momentum sector, the relevant Lorentz-Poincaré algebra in 3+1 dimension can be written as

\[
\{N_{Ei}, P_{Ej}\} = E_{Ec}^{-2} \delta_{ij} \quad \{N_{Ei}, E_E\} = P_{Ei} \tag{55}
\]

\[
\{N_{Ei}, N_{Ej}\} = -c^{-1} e_{ijk} M_{Ek} \quad \{M_{Ei}, M_{Ej}\} = \epsilon_{ijk} M_{Ek} \tag{56}
\]

\[
\{M_{Ei}, P_{Gj}\} = \epsilon_{ijk} P_{Ek} \quad \{M_{Ei}, N_{Ej}\} = \epsilon_{ijk} N_{Ek} \tag{57}
\]

\[
\{N_{Ei}, E_E\} = P_{Ei} \quad \{M_{Ei}, E_E\} = 0. \tag{58}
\]

whereas the Galilei algebra reads

\[
\{N_{Ei}, P_{Ej}\} = m \delta_{ij} \quad \{N_{Ei}, E_E\} = P_{Ei} \tag{59}
\]

\[
\{N_{Ei}, N_{Ej}\} = 0 \quad \{M_{Ei}, M_{Ej}\} = \epsilon_{ijk} M_{Ek} \tag{60}
\]

\[
\{M_{Ei}, P_{Gj}\} = \epsilon_{ijk} P_{Ek} \quad \{M_{Ei}, N_{Ej}\} = \epsilon_{ijk} N_{Ek} \tag{61}
\]

\[
\{N_{Ei}, E_E\} = P_{Ei} \quad \{M_{Ei}, E_E\} = 0. \tag{62}
\]

As in the 1+1 dimensional case we look for transformations of the type

\[
E_E = E_E(E_G, \vec{P}_G), \quad \vec{P}_E = \vec{P}_E(E_G, \vec{P}_G),
\]

that substituted in (55) with the help of (59) furnish the following system of differential equations

\[
\frac{\partial P_{Ej}}{\partial P_{Gi}} m + \frac{\partial P_{Ej}}{\partial E_G} P_{Gi} = \delta_{ij} E_{Ec}^{-2}, \quad \frac{\partial E_E}{\partial P_{Gi}} m + \frac{\partial E_E}{\partial E_G} P_{Gi} = P_{Ei}. \tag{63}
\]

We notice here that due to the rotational symmetry one can write the Lorentz-Einstein momentum as a function of the Galilei energy and momentum in the form

\[
\vec{P}_E = \vec{P}_G F(E_G, P_{G}^2),
\]

where \( F \) is a rotationally-invariant function. Substituting \( \vec{P}_E \) in (63) we get

\[
\delta_{ij} (Fm - E_{Ec}^{-2}) + P_{Gj} P_{Gi} \left( 2 \frac{\partial F}{\partial P_{G}^2} m + \frac{\partial F}{\partial E_G} \right) = 0. \tag{65}
\]

Considering the rotational properties of this last equation, and that it must hold also for \( i \neq j \), it follows that it must be

\[
\left( 2 \frac{\partial F}{\partial P_{G}^2} m + \frac{\partial F}{\partial E_G} \right) = 0, \tag{66}
\]

that means that \( F \) must have the form

\[
F(E_G, P_{G}^2) = F \left( E_G - \frac{P_{G}^2}{2m} \right). \tag{67}
\]

From (65) and (66) also follows that

\[
E_E = mc^2 F \left( E_G - \frac{P_{G}^2}{2m} \right). \tag{68}
\]

Substituting \( E_E \) in Eq.(64) one finds \( P_E = 0 \) which implies that also \( F(E_G, P_{G}^2) = 0 \) and \( E_G = 0 \). Thus we cannot find a extension of the map (34)-(35) for the 3+1-dimensional case.

V. FINAL REMARKS AND IMPLICATION FOR THE TRIVIALITY OF DOUBLY SPECIAL RELATIVITY

We have argued that Galilei transformations in 1+1 dimensions can be mapped into Lorentz-Einstein transformations, and vice versa, for any (non-zero) value of the deformation parameter \( c \). We have also found that the map
depends on two further parameters whose values tune the mapping of Galilei events into space-like, time-like or light-like Lorentz-Einstein events. A similar result can be found for the energy-momentum sector. In this case, depending on the values of the parameters involved, we can have Galilei particles mapped into bradion-like, photon-like or tachyons-like Lorentz-Einstein particles. The map also acts on the velocity space, on the space-time invariant, on the energy-momentum dispersion relation, on the energy-momentum conservation rules and on all the other relevant physical quantities. With respect to the energy-momentum conservation rules we have shown that the Galilei mass-momentum conservation laws are not immediately mapped into the special-relativistic energy-momentum conservation laws. Rather they are mapped into Eqs. (32)-(33) that are still covariant expressions in special relativity but are not right ones. In order to get the right ones it is necessary to consider a proper combinations of the Galilei mass-momentum conservation laws (Eqs. (47)-(48)). We have derived the map at a classical level, however it is rather easy, by means of the substitutions \( E \to i\hbar \partial / \partial t, \) \( p \to i\hbar \partial / \partial x \) and Eqs. (42), to find a quantum version of the map changing a Schrödinger equation into a Klein-Gordon equation, or vice versa. The existence of this 1+1-dimensional map does not mean that 1+1-d Galilei relativity is equivalent to 1+1-d special relativity. The observables in the two schemes are different and so are the relations between them (the physical laws) and, as a consequence, the physical predictions, at least in the regime in which the map does not reduce to the identity. The existence of the map only allows us to write physical laws in one scheme in terms of the physical laws of the image theory. As we have already said, within the proposals of considering extensions of special relativity at the Planck scale [9,12] also similar maps have been found [16] connecting these models to special relativity. The existence of such maps, analogous to the map we have analyzed here, has led various authors to consider doubly special relativity as a simple rewriting of special relativity in term of nonstandard variables. Our result, that special relativity can be obtained by a nonlinear map from Galilei relativity, shows that special relativity could be considered as a deformation of Galilei relativity in the same way. Thus if one accepts, as we do, that Galilei relativity and special relativity have profound physical differences, one has to conclude that these maps can account for the richness of the target model rather than showing its triviality.

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APPENDIX A

Here we assume \( \alpha = 1 \) and \( \beta = 0 \), but the same calculation can be carried out for \( \alpha = 0 \) and \( \beta = 1 \), and then easily generalized to arbitrary values of \( \alpha \) and \( \beta \). Starting from the Lorentz boosts

\[
\begin{align*}
\Delta x'_E &= \gamma (\Delta x_E - v_E \Delta t_E) \\
\Delta t'_E &= \gamma (\Delta t_E - v_E / c^2 \Delta x_E)
\end{align*}
\]

we substitute the expressions (13)-(14) obtaining

\[
\begin{align*}
\frac{c \Delta t'_G}{\Delta x'_G} \sinh \left( \frac{\Delta x'_G}{c \Delta t'_G} \right) &= \frac{1}{\sqrt{1 - \tanh^2(v_G/c)}} c \Delta t_G \left[ \sinh \left( \frac{\Delta x_G}{c \Delta t_G} \right) - \tanh(v_G/c) \cosh \left( \frac{\Delta x_G}{c \Delta t_G} \right) \right] \\
\Delta t'_G \cosh \left( \frac{\Delta x'_G}{c \Delta t'_G} \right) &= \frac{1}{\sqrt{1 - \tanh^2(v_G/c)}} \Delta t_G \left[ \cosh \left( \frac{\Delta x_G}{c \Delta t_G} \right) - \tanh(v_G/c) \sinh \left( \frac{\Delta x_G}{c \Delta t_G} \right) \right].
\end{align*}
\]

Taking the square of both sides of (70) and subtracting from the first equation the second, after some algebraic manipulations we find

\[
\begin{align*}
\Delta t'_G = \Delta t_G \\
\cosh \left( \frac{\Delta x'_G}{c \Delta t'_G} \right) &= \cosh \left( \frac{\Delta x_G}{c \Delta t_G} - v_G/c \right) \Rightarrow \begin{cases} \Delta t'_G = \Delta t_G \\
\Delta x'_G = \Delta x_G - v_G \Delta t_G, \end{cases}
\end{align*}
\]

that are just the finite Galilei boosts. Repeating the passages in the reverse order we get Lorentz boosts starting from the Galilei ones.

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