Characterization of deferred type statistical convergence and $P$-summability method for operators: Applications to $q$-Lagrange–Hermite operator

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The present work considers two important convergence techniques, namely, deferred type statistical convergence and $P$-summability method in respect of linear positive operators. With regard to these techniques, following closely the ideas developed in the articles (Appl. Math. Lett. 18, 2005, 1339-1344, and Sain Paulo J. Math. Sci. 13, 2019, 696-707), we state and prove two general non-trivial Korovkin-type approximation results for positive linear operators. Further, we define an operator based on multivariate $q$-Lagrange–Hermite polynomials and exhibit the applicability of the above theorems to these operators.

KEYWORDS
deferred type statistical convergence, modulus of continuity, multivariate Lagrange–Hermite polynomials, $P$-summability, $q$-Lagrange–Hermite polynomials

MSC CLASSIFICATION
41A25, 41A36, 33C45, 26A15

1 INTRODUCTION: DEFERRED WEIGHTED $A$-STATISTICAL CONVERGENCE AND $P$-SUMMABILITY METHOD

In Pure and Applied Mathematics, the idea of convergence of a sequence in a given space $X$ provides a lot of insight and applications. Once a sequence is convergent, we are free to use the analogy into continuity of a function, and hence, it is easy to dive in the areas of Topological Spaces, Measure Theory, Functional Analysis, Numerical Analysis, Mathematical Modeling, and so on, but what would be the situation if the sequence is not convergent, can we still say something about it? The answers are indeed affirmative and occur in the form of Summability Theory, particularly Statistical Convergence. In the past few decades, these notions of convergence have been deeply studied and generalized. The methods not only catch the fine details of some special class of sequences but also provide the impression of convergence to such sequences (non-convergent in usual sense). In the present article, we discuss two such methods: a generalized form of statistical convergence and $P$-summability method and their applications to the theory of approximation. In this row, we first, briefly, introduce the concept of statistical convergence and its deferred type generalization. After that, we shall discuss the $P$-summability method.

Let $M$ be a subset of the set of natural numbers $\mathbb{N}$, and for each $n \in \mathbb{N}$, we define $M_n = \{ m \in M : m \leq n \}$. The natural density $d(M)$ of the set $M$ is defined by the limit(if exists) of the sequence $\langle \frac{|M_n|}{n} \rangle$. More precisely,

$$d(M) = \lim_{n \to \infty} \frac{|M_n|}{n}. $$
Any sequence \( (x_n) \) is called statistically convergent\(^1 \) to \( l \) if, for each \( \epsilon > 0 \), the set \( \{ k \in \mathbb{N} : k \leq n \text{ and } |x_k - l| \geq \epsilon \} \) has density zero, that is,

\[
\lim_{n \to \infty} \frac{|\{ k \in \mathbb{N} : k \leq n \text{ and } |x_k - l| \geq \epsilon \}|}{n} = 0.
\]

This definition clearly indicates that every convergent sequence is always statistically convergent, while the converse need not to be true in general. For a counter example, we can opt the sequence \( (x_n) = \begin{cases} 1, & \text{if } n = m^2, m \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases} \). Clearly, \( (x_n) \) is not convergent in the usual sense while one can easily verify that it is statistically convergent to 0. Fridy\(^2 \) introduced the concept of statistically Cauchy sequence and established its equivalence with the statistical convergence of a sequence. Braha et al\(^3 \) proved a Korovkin-type theorem for the \( 2\pi \)-periodic continuous functions on \( \mathbb{R} \) by using the statistical summability of the de la Vallée Poussin mean. Karakaya and Chishti\(^4 \) derived the concept of weighted statistical convergence, and the idea was later modified by Mursaleen et al\(^5 \).

Let us assume that \( (s_k) \) be a sequence such that \( s_k \geq 0 \) and

\[
S_n = \sum_{k=1}^{n} s_k, \quad s_1 > 0,
\]

denotes its partial sum. Now, for a sequence \( (x_n) \), set

\[
u_n = \frac{1}{S_n} \sum_{k=1}^{n} s_k x_k, \quad n \in \mathbb{N}.
\]

Then, the sequence \( (x_n) \) is called weighted statistically convergent to a number \( l \) if, for any given \( \epsilon > 0 \), the following holds:

\[
\lim_{n \to \infty} \frac{|\{ k \in \mathbb{N} : k \leq S_n \text{ and } s_k |x_k - l| \geq \epsilon \}|}{S_n} = 0.
\]

Through de la Vallée Poussin mean, Belen and Mohiuddine\(^6 \) introduced the notion of weighted \( \lambda \)-statistical convergence which includes the weighted statistical convergence for \( \lambda_n = n \). If \( X_1 \) and \( X_2 \) are sequence spaces such that for every infinite matrix \( A = (a_{n,k}) : X_1 \to X_2 \), we have the transformation \( (Ax)_n = \sum_{k=1}^{\infty} a_{n,k} x_k \). Then, the matrix \( A \) is called regular if \( \lim_{n \to \infty} (Ax)_n = l \) whenever \( \lim x_k = l \). For a non-negative regular matrix \( A = (a_{n,k}) \), Freedman and Sember\(^7 \) proposed the concept of \( A \)-statistical convergence. The sequence \( (x_n) \) is called \( A \)-statistically convergent to a number \( l \) (denoted by \( \text{stat}_A - \lim x_n = l \)), if

\[
\lim_{n \to \infty} \sum_{k:|x_k-l|\geq\epsilon} a_{n,k} = 0, \quad \text{for every } \epsilon > 0.
\]

Mohiuddine\(^8 \) defined statistical weighted \( A \)-summability of a sequence and discussed its relationship with the weighted \( A \)-statistical convergence. Mohiuddine and Alami\(^9 \) proposed weighted Lacunary equistatistical convergence and derived the Korovkin and Voronovskaya-type approximation theorems. Kadak and Mohiuddine\(^10 \) extended the concept of almost convergence and its statistical versions by using the difference operator involving \( (p,q) \)-gamma function and proved the Korovkin-type theorem for functions of two variables and some other approximation results. Srivastava et al\(^11 \) derived a more general idea of \( A \)-statistical convergence and coined the word deferred weighted \( A \)-statistical convergence. Suppose \( (b_n) \) and \( (c_n) \) are the sequences of non-negative integers satisfying the regularity conditions \( b_n < c_n; \lim_{n \to \infty} c_n = \infty \). Now, if we set

\[
S_n = \sum_{m=b_n+1}^{c_n} s_m,
\]

for any given sequence \( (s_n) \) of non-negative real numbers and for any given sequence \( (x_n) \), its respective deferred weighted mean by \( \rho_n = \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} s_m x_m \). Then, the sequence \( (x_n) \) is called deferred weighted summable (denoted by \( c^{\text{DWS}}_A - \lim x_n = l \)) to \( l \) if \( \lim \rho_n = l \). Further, we call \( (x_n) \) to be deferred weighted \( A \)-summable to (denoted by \( c^{\text{DWS}}_A - \lim x_n = l \)) a number \( l \) if

\[
\lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k=1}^{\infty} s_m a_{m,k} x_k = l.
\]
Let $c^{DWS}$ be the space of all deferred weighted summable sequences; then, an infinite matrix $A = (a_{n,k})$ is called deferred weighted regular matrix if

$$(Ax)_n = \sum_{k=1}^{\infty} a_{n,k}x_k \in c^{DWS} \text{ for every convergent sequence } x = (x_n),$$

with

$$c^{DWS} - \lim_{n \to \infty} (Ax)_n = \text{stat}_A - \lim_{n \to \infty} x_n.$$  

Now, for a non-negative deferred weighted regular matrix $A = (a_{n,k})$ and $K_c \subset \mathbb{N} = \{ k \in \mathbb{N} : |x_k - l| \geq \epsilon \}$, a sequence $(x_n)$ is said to be deferred weighted $A$-statistically convergent to $l$ (denoted by $\text{stat}_A^{DWS} - \lim_{n \to \infty} x_n = l$) if, for each $\epsilon > 0$, the deferred weighted $A$-density of $K_c$, that is, $d^A_{DW}(K_c)$ is zero. It means that

$$d^A_{DW}(K_c) = \lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in K_c} s_m a_{m,k} = 0.$$  

We call the sequence $(x_n)$ to be deferred weighted $A$-statistically convergent to the number $l$ with the rate $o(\gamma_n)$ (please see Duman et al.\textsuperscript{12,13}) if

$$\lim_{n \to \infty} \frac{1}{\gamma_n} \left\{ \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in K_c} s_m a_{m,k} \right\} = 0.$$  

In this case, we write $x_n - l = \text{stat}_A^{DWS} - o(\gamma_n)$. Having discussed the study of statistical convergence and its various types of generalizations by researchers, we now turn our attention to the next equally important convergence technique, namely, $P$-summability method.

Suppose that $p(u) = \sum_{n=1}^{\infty} p_n u^{n-1}$ is a power series of non-negative coefficients with radius of convergence $R \in (0, \infty]$. Any sequence $(x_n)$ is called convergent to a number $l$ by means of $P$-summability method (or $P$-summable/power series summability)\textsuperscript{14} if

$$\lim_{u \to R^-} \frac{1}{p(u)} \sum_{n=1}^{\infty} x_n p_n u^{n-1} = l. \quad (1.1)$$

Additionally, the $P$-summability method is said to be regular\textsuperscript{15} iff

$$\lim_{u \to R^-} \frac{p_n u^{n-1}}{p(u)} = 0, \text{ for all } n \in \mathbb{N}. \quad (1.2)$$

In particular, if $p_n = 1, \forall n \in \mathbb{N}$, then $p(u) = \frac{1}{1-u}$ and $R = 1$; hence, the power series method includes the Abel’s method, and if we choose $p_n = \frac{1}{(n-1)!}, \forall n \in \mathbb{N}$, then we obtain $p(u) = e^u$ and $R = \infty$, and so the power series method turns into Borel method.

The idea of convergence by the $P$-summability method is more general in comparison to the classical convergence. One can see this by taking an example of the sequence $(x_n) = \begin{cases} 1, & \text{n is even} \\ 0, & \text{n is odd} \end{cases}$. Clearly, it is not convergent in the usual sense, but if we take $p_n = 1, \forall n \in \mathbb{N}$, we have $p(u) = \frac{1}{1-u}, |u| < 1$, which implies that $R = 1$, and hence from (1.1), we have

$$\lim_{u \to 1^-} (1-u) \sum_{n=1}^{\infty} x_n u^{n-1} = \lim_{u \to 1^-} \frac{(1-u)}{u} \sum_{n=1}^{\infty} u^{2n} = \frac{1}{2},$$

therefore, $(x_n)$ converges to $\frac{1}{2}$ in the sense of $P$-summability method.

Due to this generality of convergence by power series method over the classical one, many authors have made significant contributions in this direction. The interested reader may refer to previous studies.\textsuperscript{16–20}
2 | GENERAL THEOREMS

Let \((B[a, b], \| \cdot \|)\) be the normed space of all bounded functions on \([a, b]\) with the sup-norm and \(C[a, b]\) denote the subspace of \(B[a, b]\) consisting of all continuous functions. We consider

\[
m = \sup_{x \in [a, b]} |x| \quad \text{and} \quad \| \Omega_n((s - x)^2) \| = \mu_n^{(2)}.
\]

First of all, Gadjiév and Orhan\(^{21}\) established a Korovkin-type approximation theorem via statistical convergence. Duman and Orhan\(^{13}\) derived the Korovkin-type result using the concept of \(A\)-statistical convergence. Erkuş and Duman\(^{22}\) extended it to the \(m\)-dimensional case. Duman and Orhan\(^{23}\) established a Korovkin-type theorem via \(A\)-statistical convergence for functions in a weighted space. Karakaya and Chishti\(^{4}\) introduced the concept of weighted statistical convergence for different sequences of positive linear operators; see, for instance, previous studies.\(^{24–29}\) Srivastava et al\(^{11}\) defined the notion of deferred weighted \(A\)-statistical convergence for linear positive operators \((LPO)\) constructed by means of \((p, q)\)-Lagrange polynomials. Following the proofs given in the papers,\(^{13,21}\) in the following theorem, we obtain an analogous result for the deferred weighted \(A\)-statistical convergence of a sequence of positive linear operators.

**Theorem 1.** Let \(\Omega_n : C[a, b] \to B[a, b]\) be a sequence of positive linear operators and \(A = (\alpha_{nk})\) be a non-negative deferred weighted regular matrix. Suppose \((a_n)\) and \((b_n)\) be sequences of non-negative integers satisfying the regularity conditions. If \(g_i(s) = s^i, \text{ for } i = 0, 1, 2\), then

\[
\text{stat}^{DW}_A - \lim_n \| \Omega_n(g_i) - g_i \| = 0, \quad \text{for } i = 0, 1, 2;
\]

if and only if

\[
\text{stat}^{DW}_A - \lim_n \| \Omega_n(g) - g \| = 0, \quad \forall g \in C[a, b].
\]

**Proof.** Let us assume that, for all \(g \in C[a, b]\), we have

\[
\text{stat}^{DW}_A - \lim_n \| \Omega_n(g) - g \| = 0,
\]

then

\[
\text{stat}^{DW}_A - \lim_n \| \Omega_n(g_i) - g_i \| = 0, \quad \text{for } i = 0, 1, 2
\]

is obvious. Conversely, assume that

\[
\text{stat}^{DW}_A - \lim_n \| \Omega_n(g_i) - g_i \| = 0, \quad \text{for } i = 0, 1, 2.
\]

Since \(g \in C[a, b]\), we have

\[
|g(s) - g(x)| \leq 2 \|g\| \cdot \forall s, x \in [a, b].
\]

As \(g\) is uniformly continuous on \([a, b]\), for a given \(c > 0, \exists \delta > 0\) such that \(|g(s) - g(x)| < c\), whenever \(|s - x| < \delta, s, x \in [a, b]\). If \(|s - x| \geq \delta\), then

\[
|g(s) - g(x)| \leq 2 \|g\| \frac{(s - x)^2}{\delta^2}.
\]

Hence, we can write

\[
|g(s) - g(x)| < c + 2 \|g\| \frac{(s - x)^2}{\delta^2}, \quad \forall s, x \in [a, b]. \tag{2.1}
\]

In view of the monotonicity of \(\Omega_n\), we get

\[
|\Omega_n(g(s); x) - g(x)| \leq \Omega_n(|g(s) - g(x)|; x) + |g(x)| \Omega_n(1; x) - 1,
\]
hence from (2.1), we obtain
\[
|\mathcal{G}_n(g(x); x) - g(x)| \leq \mathcal{G}_n \left( \epsilon + 2 \| g \| \frac{(s - x)^2}{\delta^2}; x \right) + |g(x)| |\mathcal{G}_n(1; x) - 1|.
\]

As, we see that
\[
\mathcal{G}_n(c; x) + \frac{2 \| g \|}{\delta^2} \mathcal{G}_n \left( (s - x)^2; x \right) = \epsilon + \epsilon (\mathcal{G}_n(1; x) - 1) + \frac{2 \| g \|}{\delta^2} \left[ (\mathcal{G}_n(s^2; x) - x^2) - 2x (\mathcal{G}_n(s; x) - x) + x^2 (\mathcal{G}_n(1; x) - 1) \right],
\]

hence
\[
\|\mathcal{G}_n(g) - g\| \leq \epsilon + \left( \| g \| + \frac{2 \| g \| m^2}{\delta^2} + \epsilon \right) \|\mathcal{G}_n(1) - 1\| + \frac{2 \| g \|}{\delta^2} \|\mathcal{G}_n(s^2) - x^2\| + \frac{4 \| g \| m}{\delta^2} \|\mathcal{G}_n(s) - x\|. \tag{2.2}
\]

Now, for any \( \epsilon' > 0 \), we define the followings sets:
\[
\begin{align*}
U_1 &= \{ n \in \mathbb{N} : \|\mathcal{G}_n(g) - g\| \geq \epsilon' \}; \\
U_2 &= \left\{ n \in \mathbb{N} : \left( \| g \| + \frac{2 \| g \| m^2}{\delta^2} + \epsilon \right) \|\mathcal{G}_n(1) - 1\| \geq \frac{\epsilon' - \epsilon}{3} \right\}; \\
U_3 &= \left\{ n \in \mathbb{N} : \frac{2 \| g \|}{\delta^2} \|\mathcal{G}_n(s^2) - x^2\| \geq \frac{\epsilon' - \epsilon}{3} \right\}; \\
U_4 &= \left\{ n \in \mathbb{N} : \frac{4 \| g \| m}{\delta^2} \|\mathcal{G}_n(s) - x\| \geq \frac{\epsilon' - \epsilon}{3} \right\},
\end{align*}
\]

then it is obvious that \( U_1 \subset U_2 \cup U_3 \cup U_4 \), and hence,
\[
\frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in U_1} s_m a_{m,k} \leq \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in U_2} s_m a_{m,k} + \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in U_3} s_m a_{m,k} + \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in U_4} s_m a_{m,k}.
\]

In view of our hypothesis, we have
\[
\lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in U_i} s_m a_{m,k} = 0,
\]
for \( i = 2, 3, \) and \( 4 \). Therefore,
\[
\lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in U_i} s_m a_{m,k} = \text{stat}^{DW}_A - \lim_n \|\mathcal{G}_n(g) - g\| = 0.
\]

This completes the proof of the theorem. □

Remark 1. If \( b_n = 0, c_n = n, \) and \( s_n = 1, \forall n \in \mathbb{N} \), then the deferred weighted \( A \)-statistical convergence reduces to the \( A \)-statistical convergence; therefore, our Theorem 1 reduces to the Korovkin-type result given in Duman and Orhan. If \( b_n = 0, c_n = n, \forall n \in \mathbb{N} \), and \( A = (C, 1) \), the Cesàro matrix, then deferred weighted \( A \)-statistical convergence turns into weighted statistical convergence; hence, Theorem 1 yields the Korovkin-type theorem proved in Mursaleen et al. In the case where \( b_n = 0, c_n = n, \) and \( s_n = 1, \forall n \in \mathbb{N} \), and \( A = (C, 1) \), we obtain statistical convergence, and so our Theorem 1 includes the Korovkin-type result of Gadjev and Orhan.

Duman et al. initiated the study of the rate of \( A \)-statistical convergence of positive linear operators. Duman determined the rate of \( A \)-statistical convergence of positive linear convolution operators by using the modulus of continuity. For some other studies in this direction, we refer to previous studies. Following the techniques developed in the papers, in our next theorem, we determine the rate of deferred weighted \( A \)-statistical convergence by the operators \( \mathcal{G}_n(g) \) for all \( g \in C[a, b] \).
Theorem 2. Let the matrix \( A \) and the sequences \( \langle a_n \rangle, \langle b_n \rangle \) be as defined in Theorem 1. Further, let \( \langle \gamma_n \rangle \) be a positive non-increasing sequence of real numbers. For \( g \in C[a, b] \), if

\[
\left\{ g \| + \omega \left( g; \sqrt{\mu_n^{(2)}} \right) \right\} \left\| \mathfrak{G}_n(1) - 1 \right\| + 2 \omega \left( g; \sqrt{\mu_n^{(2)}} \right) = \text{statDW}_A - o(\gamma_n), \quad \text{as } n \to \infty,
\]

then

\[
\| \mathfrak{G}_n(g) - g \| = \text{statDW}_A - o(\gamma_n), \quad \text{as } n \to \infty.
\]

Proof. For any \( g \in C[a, b] \), it is well known that

\[
|g(s) - g(x)| \leq \left\{ 1 + \frac{(s-x)^2}{\delta^2} \right\} \omega(g; \delta) \delta > 0.
\]

Hence, by the monotonocity and linearity of operators \( \mathfrak{G}_n \), we get

\[
|\mathfrak{G}_n(g; x) - g(x)| \leq |g(x)| |\mathfrak{G}_n(1; x) - 1| + \omega(g; \delta) + \omega(g; \delta) |\mathfrak{G}_n(1; x) - 1| + \frac{\omega(g; \delta)}{\delta^2} |\mathfrak{G}_n((s-x)^2); x|
\]

which implies that

\[
\| \mathfrak{G}_n(g) - g \| \leq \left\{ g \| + \omega(g; \delta) \right\} \| \mathfrak{G}_n(1) - 1 \| + \omega(g; \delta) + \frac{\omega(g; \delta)}{\delta^2} \left\| \mathfrak{G}_n((s-x)^2) \right\|.
\]

Taking \( \delta = \sqrt{\| \mathfrak{G}_n((s-x)^2) \|} = \sqrt{\mu_n^{(2)}} \), we get

\[
\| \mathfrak{G}_n(g) - g \| \leq \left\{ g \| + \omega \left( g; \sqrt{\mu_n^{(2)}} \right) \right\} \| \mathfrak{G}_n(1) - 1 \| + 2 \omega \left( g; \sqrt{\mu_n^{(2)}} \right).
\]

Taking into account our hypothesis (2.3), the proof of the theorem is completed. \( \square \)

Özgüç and Taş\textsuperscript{34} established a general Korovkin-type theorem by using power series method for a sequence of positive linear operators defined on the space of continuous real-valued functions on a Hausdorff space \( X \) with at least two points. Taş and Atlihan\textsuperscript{35} investigated the Korovkin-type theorem for the sequence of positive linear operators defined on \( C[a, b] \) by using power series method. For further studies in this direction, one can see previous studies.\textsuperscript{16,27,36,37} Proceeding in a manner similar to the proof of Theorem 1 proved in Taş and Atlihan,\textsuperscript{35} in the following result, we obtain a Korovkin-type approximation theorem for the operators \( \mathfrak{J}_n \) using \( P \)-summability method.

Theorem 3. For \( g \in C[a, b] \), the sequence \( \langle \mathfrak{J}_n \rangle \) satisfies

\[
\lim_{u \to R} \frac{1}{p(u)} \sum_{n=1}^{\infty} \| \mathfrak{J}_n(g) - g \| p_u u^{n-1} = 0,
\]

iff

\[
\lim_{u \to R} \frac{1}{p(u)} \sum_{n=1}^{\infty} \| \mathfrak{J}_n(s^i) - x^i \| p_u u^{n-1} = 0,
\]

for \( i = 0, 1, 2 \).
Proof. It is easy to see that Equation (2.5) ⇒ (2.6); thus, we shall prove the converse. Assume that the condition (2.6) is true. Using (2.2), we can write

\[
\lim_{u \to R} \frac{1}{p(u)} \sum_{n=1}^{\infty} \| \mathcal{Q}_n(g) - g \| p_n u^{n-1} \leq e + \left( \|g\| + \frac{2 \|g\| m^2}{\delta^2} + e \right) \lim_{u \to R} \frac{1}{p(u)} \sum_{n=1}^{\infty} \| \mathcal{Q}_n(1) - 1 \| c_n u^{n-1} \\
+ \frac{2 \|g\|}{\delta^2} \lim_{u \to R} \frac{1}{p(u)} \sum_{n=1}^{\infty} \left\| \mathcal{Q}_n(s^2) - s^2 \right\| p_n u^{n-1} \\
+ \frac{4 \|g\| m}{\delta^2} \lim_{u \to R} \frac{1}{p(u)} \sum_{n=1}^{\infty} \| \mathcal{Q}_n(s) - s \| c_n u^{n-1}.
\]

Hence, in view of the hypothesis and the arbitrariness of \( \epsilon \), we have the desired result. \( \square \)

Now, we furnish an example to show that Theorem 3 is a non-trivial generalization of the classical Korovkin-type theorem given by Korovkin.\(^{38}\) For this purpose, we define a sequence of positive linear operators for which Theorem 3 holds true while the Korovkin theorem does not work.

Example 1. For \( f \in C[0, 1] \), let us define a sequence of linear operators as follows:

\[
T_n(f; x) = (1 + \gamma_n)B_n(f; x),
\]

where

\[
\gamma_n = \begin{cases} 
1, & n = m^3, \ m \in \mathbb{N} \\
0, & \text{otherwise},
\end{cases}
\]

and \( B_n(f; x) \) denotes the sequence of Bernstein polynomials. Then, clearly, \( T_n \) is a positive linear operator, and the sequence \( (\gamma_n) \) is not convergent (in the usual sense), but it is convergent in the sense of power series method as shown below: Let \( p_n = 1 \), then \( p(u) = \frac{1}{1-u} \) and \( R = 1 \). We have to prove that

\[
\lim_{u \to 1^-} (1 - u) \sum_{m=1}^{\infty} u^{m^3} = 0.
\]

From Polya and Szegö\(^{39}\) (Ex. 35, p54), we have

\[
\lim_{u \to 1^-} (1 - u) \sum_{m=1}^{\infty} u^{m^3} = \lim_{u \to 1^-} (1 - u)^{2/3} \lim_{u \to 1^-} \sqrt[3]{1-u} \sum_{m=1}^{\infty} u^{m^3} \\
= \lim_{u \to 1^-} (1 - u)^{2/3} \left( \frac{4}{3} \right) = 0.
\]

From (2.7), we obtain

\[
T_n(e_i; x) = (1 + \gamma_n)B_n(e_i; x), \ i = 0, 1, 2.
\]

It is known\(^{38}\) that \( B_n(e_i; x) \to e_i(x) \), as \( n \to \infty \), uniformly in \( x \in [0, 1] \), for \( i = 0, 1, 2 \); hence, it follows that \( T_n(e_i; x) \to e_i(x) \), for \( i = 0, 1, 2 \) in the sense of power series method, and thus by Theorem 3, the operator \( T_n(f; x) \) converges to \( f(x) \) in the sense of power series method, but the classical Korovkin theorem does not work here as the sequence \( (\gamma_n) \) is not convergent in the usual sense.

Following Taş and Atlıhan\(^{35}\) (Theorem 2), in the next result, we determine the rate of convergence for the operators \( \mathcal{Q}_n \) by using power series method with the aid of the modulus of continuity.

Theorem 4. If \( g \in C[a, b] \) and \( \phi(u) \) is some positive function on \( (0, R) \) such that

\[
\frac{1}{p(u)} \sum_{n=1}^{\infty} \left\| g \right\| + o \left( g; \sqrt[p]{\mu_n^{(2)}} \right) \| \mathcal{Q}_n(1) - 1 \| p_n u^{n-1} + \frac{2}{p(u)} \sum_{n=1}^{\infty} o \left( g; \sqrt[p]{\mu_n^{(2)}} \right) p_n u^{n-1} = O(\phi(u)), \text{ as } u \to R^-,
\]

where
then
\[
\frac{1}{p(u)} \sum_{n=1}^{\infty} \| \mathcal{Q}_n(g) - g \|_p u^{n-1} = O(\phi(u)), \text{ as } u \to R^+.
\]

**Proof.** From the inequality (2.4), we see that
\[
\frac{1}{p(u)} \sum_{n=1}^{\infty} \| \mathcal{Q}_n(g) - g \|_p u^{n-1} \leq \frac{1}{p(u)} \sum_{n=1}^{\infty} \| g \|_p + o(g; \delta) \| \mathcal{Q}_n(1) - 1 \|_p u^{n-1}
\]
\[
+ \frac{1}{p(u)} \sum_{n=1}^{\infty} o(g; \delta) \left( 1 + \frac{1}{\delta^2} \mu_n^{(2)} \right) p_n u^{n-1}.
\]

Now, choosing \( \delta = \sqrt{\mu_n^{(2)}} \) and considering the hypothesis, we reach the required result. \( \square \)

**Remark 2.** Since Abel method of convergence is a particular case of the power series method, our Theorems 3 and 4 are more general than the corresponding results, namely, Theorems 2.2 and 2.3 given in Söylemez and Ünver. 40

### 3 CONSTRUCTION OF THE Q-LAGRANGE–HERMITE OPERATORS

The past few decades have been witness to the multivariate extension of many well-known orthogonal polynomials, for some notably mentioned works one may refer to previous studies. 41–43 Very recently, researchers have started to make efforts to construct sequences of LPO using the multivariate polynomials. In this row, for \( g \in C[0, 1] \), Erkuş et al 44 have defined a sequence of LPO using multivariate Lagrange polynomials and studied its approximation behavior by means of statistical convergence method. Mursaleen et al 45 attempted to define a \( q \)-(basic) analogue of the operator defined in Erkuş et al, 44 while Baxhaku et al 46 proposed a slight modification in the attempted operator and extended the study to the bivariate and Generalized Boolean Sum (GBS) cases.

In order to explain the construction of \( q \)-Lagrange–Hermite operator, we shall need to recall some important definitions from the quantum calculus (widely known as \( q \)-calculus). Let us assume that \( |q| < 1 \); then, we recall the \( q \)-(basic) analogues of a natural number and Pochhammer symbol as
\[
[n]_q := \frac{1 - q^n}{1 - q}; \text{ and } (\rho; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1 - \rho)(1 - \rho q) \cdots (1 - \rho q^{n-1}), & \text{if } n \in \mathbb{N}, \end{cases}
\]
respectively, where \( \rho \) is some arbitrary parameter.

Inspired by the multivariate extension of Lagrange polynomials, 47 Altin and Erkuş 48 introduced the multivariate Lagrange–Hermite polynomials \( h_{p}^{(\beta_{1}; \beta_{2}; \ldots; \beta_{m})}(z_{1}, z_{2}, \ldots, z_{n}) \) generated by the expression
\[
\prod_{i=1}^{r} (1 - t_i z_i)^{-\beta_i} = \sum_{p=0}^{\infty} h_{p}^{(\beta_{1}; \beta_{2}; \ldots; \beta_{m})}(z_{1}, z_{2}, \ldots, z_{n}) t^{p},
\]
with \( |t| < \min(|z_{1}|^{-1}, |z_{2}|^{-1/2}, \ldots, |z_{r}|^{-1/r}) \). Recently, Erkuş 49 proposed a \( q \)-(basic) analogue of the above Lagrange–Hermite polynomial generated by
\[
\prod_{i=1}^{r} \frac{1}{(z_i t_{i}; q)_{p_{i}}^{\rho_{i}}} = \sum_{p=0}^{\infty} h_{p,q}^{(\beta_{1}; \beta_{2}; \ldots; \beta_{m})}(z_{1}, z_{2}, \ldots, z_{r}) t^{p}. \tag{3.1}
\]

From (3.1), it is easy to obtain the explicit form of the polynomial \( h_{p,q}^{(\beta_{1}; \beta_{2}; \ldots; \beta_{m})}(z_{1}, z_{2}, \ldots, z_{r}) \) as
\[
h_{p,q}^{(\beta_{1}; \beta_{2}; \ldots; \beta_{m})}(z_{1}, z_{2}, \ldots, z_{r}) = \sum_{l_{1}+2l_{2}+\ldots+rl_{r}=p} \left\{ \prod_{k=1}^{r} \left( q^{\beta_{k}}; q \right)_{l_{k}} \frac{(z_{k})_{l_{k}}}{(q, q)_{l_{k}}} \right\}.
\]
Hence, using (3.1), we have the identity

\[
\prod_{i=1}^{r} (z_i^q, q)_{\rho_i} \sum_{p=0}^{\infty} \left( \sum_{l_i+2l_{i-1}+\ldots+l_1=p} \prod_{k=1}^{r} \left( q^{\rho_k}, q \right)_{l_k} (z_k^q)_{\rho_k} \right) t^p = 1.
\] (3.2)

Now, motivated by the above studies, for \( g \in C[0, 1] \), we propose the following sequence of Lagrange–Hermite positive linear operators:

\[
\mathfrak{R}_{n,q}^{a(1), \ldots, a(r)} (g; x) = \left\{ \prod_{i=1}^{r} \left( a_n^{(i)} x_i q \right)_n \right\}_{p=0}^\infty \sum_{l_i+2l_{i-1}+\ldots+l_1=p} \left\{ \prod_{k=1}^{r} \left( q^{n_k}, q \right)_{l_k} \left( a_n^{(k)} \right)_{l_k} \right\} g \left( \frac{[l_1]}{[n+l_1-1]} \right) x^p,
\] (3.3)

where \( a^{(i)} = \langle a_n^{(i)} \in (0, 1) \rangle_{n \in \mathbb{N}} \) are sequences of real numbers.

In the following lemmas, we obtain estimates for some raw moments of the operators defined by (3.3).

**Lemma 1.** The operators \( \mathfrak{R}_{n,q}^{a(1), \ldots, a(r)} (.; x) \) satisfy

\[
\mathfrak{R}_{n,q}^{a(1), \ldots, a(r)} (1; x) = 1, \text{ for all } x \in [0, 1].
\]

**Proof.** Using (3.2), proof of the lemma is straightforward; hence, the details are omitted. \(\square\)

**Lemma 2.** For the operators \( \mathfrak{R}_{n,q}^{a(1), \ldots, a(r)} (.; x) \), we have

\[
\mathfrak{R}_{n,q}^{a(1), \ldots, a(r)} (s; x) = x a_n^{(1)}.
\]

**Proof.** From the definition (3.3) of the Lagrange–Hermite operator, we can write

\[
\mathfrak{R}_{n,q}^{a(1), \ldots, a(r)} (s; x) = \left\{ \prod_{i=1}^{r} \left( a_n^{(i)} x_i q \right)_n \right\}_{p=0}^\infty \sum_{l_i+2l_{i-1}+\ldots+l_1=p} \left\{ \prod_{k=1}^{r} \left( q^{n_k}, q \right)_{l_k} \left( a_n^{(k)} \right)_{l_k} \right\} \left( \frac{[l_1]}{[n+l_1-1]} \right) x^p.
\]

Using some elementary transformations, \( \frac{[l_1]}{[q(q)],_l} = \frac{1}{(1-q(q); q),_l} \) and \( \frac{[q(q); q),_l}{[n+l_1-1]} = (1-q(q); q),_l-1 \), we have

\[
\mathfrak{R}_{n,q}^{a(1), \ldots, a(r)} (s; x) = x a_n^{(1)} \left\{ \prod_{i=1}^{r} \left( a_n^{(i)} x_i q \right)_n \right\}_{p=0}^\infty \sum_{l_i+2l_{i-1}+\ldots+l_1=p} \left\{ \prod_{k=1}^{r} \left( q^{n_k}, q \right)_{l_k-1} \cdots \left( q^{n_k}, q \right)_{l_k-1} \right\} \left( \frac{a_n^{(1)} \cdots a_n^{(r)}}{[q(q); q)_l-1} \right) x^p
\]

\[
= x a_n^{(1)} \left\{ \prod_{i=1}^{r} \left( a_n^{(i)} x_i q \right)_n \right\}_{p=0}^\infty \sum_{k=1}^{q^{n_k}, q)_{l_k} \cdots \left( q^{n_k}, q \right)_{l_k-1} \right\} \left( \frac{[l_1]}{[n+l_1-1]} \right) x^p
\]

\[
= x a_n^{(1)} \left\{ \prod_{i=1}^{r} \left( a_n^{(i)} x_i q \right)_n \right\}_{p=0}^\infty \sum_{k=1}^{q^{n_k}, q)_{l_k} \cdots \left( q^{n_k}, q \right)_{l_k-1} \right\} \left( \frac{[l_1]}{[n+l_1-1]} \right) x^p
\]

\[
= x a_n^{(1)} \left\{ \prod_{i=1}^{r} \left( a_n^{(i)} x_i q \right)_n \right\}_{p=0}^\infty \sum_{k=1}^{q^{n_k}, q)_{l_k} \cdots \left( q^{n_k}, q \right)_{l_k-1} \right\} \left( \frac{[l_1]}{[n+l_1-1]} \right) x^p
\]

\[
= x a_n^{(1)}, \text{ in view of (3.2)}.
\]

\(\square\)

**Lemma 3.** The operators \( \mathfrak{R}_{n,q}^{a(1), \ldots, a(r)} (.; x) \) satisfy

\[
\mathfrak{R}_{n,q}^{a(1), \ldots, a(r)} (s^2; x) \leq q \left( x a_n^{(1)} \right)^2 + \frac{x a_n^{(1)}}{[n]};
\]
and

\[ |\mathfrak{R}_{n,q}^{a_1^{(1)},\ldots,a_r^{(r)}}(s^2 - x^2; x)| \leq 2x^2(1 - a_n^{(1)}) + \frac{x\alpha_n^{(1)}}{[n]_q}. \]

**Proof.** From the definition of the operator \( \mathfrak{R}_{n,q}^{a_1^{(1)},\ldots,a_r^{(r)}}(\cdot; x) \), we have

\[
\mathfrak{R}_{n,q}^{a_1^{(1)},\ldots,a_r^{(r)}}(s^2; x) = \left\{ \prod_{i=1}^{r} \left( a_n^{(i)} x^i; q \right)_n \right\} \sum_{p=1}^{\infty} \sum_{l_i \geq 1} \left\{ \prod_{x=1}^{r} \left( q^n; q \right)(\frac{a_n^{(i)} x^i}{q^n; q})_l \right\} \left( \frac{[l_1]_q}{[n + l_1 - 1]_q} \right)^2 x^{p-1}
\]

\[
= x\alpha_n^{(1)} \left\{ \prod_{i=1}^{r} \left( a_n^{(i)} x^i; q \right)_n \right\} \sum_{p=1}^{\infty} \sum_{l_i \geq 1} \frac{[l_1]_q}{[n + l_1 - 1]_q} \left( q^n; q \right)(\frac{a_n^{(i)} x^i}{q^n; q})_l x^{p-1}
\]

\[
= x\alpha_n^{(1)} \left\{ \prod_{i=1}^{r} \left( a_n^{(i)} x^i; q \right)_n \right\} \sum_{p=1}^{\infty} \sum_{l_i \geq 1} \left( q^n; q \right)(\frac{a_n^{(i)} x^i}{q^n; q})_l x^{p-1} = \sum_1^{1} + \sum_2^{2}, \text{ say.}
\]

Here,

\[
\sum_1 = x\alpha_n^{(1)} \left\{ \prod_{i=1}^{r} \left( a_n^{(i)} x^i; q \right)_n \right\} \sum_{p=1}^{\infty} \sum_{l_i \geq 1} \left( q^n; q \right)(\frac{a_n^{(i)} x^i}{q^n; q})_l x^{p-1}.
\]

Since \( \frac{1}{[n+l_1]} \leq \frac{1}{[n]_q} \), using (3.2), we have

\[
\sum_1 \leq \frac{x\alpha_n^{(1)}}{[n]_q}.
\]
Now,

\[
\sum_2 = q(n^{(1)}) \left\{ \prod_{i=1}^{r} \left( a_n^{(i)} x^2; q \right)_n \right\} \sum_{p=1}^{\infty} \left\{ \sum_{l_i \geq 1} (q^n; q)_{l_i-1} \cdots (q^n; q)_l \right\} \\
\times \left( \frac{[l_1 - 1] q}{[n + l_1 - 1] q} \right) \left( a_n^{(1)} x^2; q \right)_n \right\} \sum_{p=2}^{\infty} \left\{ \sum_{l_i \geq 2} (q^n; q)_{l_i-2} \cdots (q^n; q)_l \right\} \chi^{p-2}
\]

\[
= q(n^{(1)}) \left\{ \prod_{i=1}^{r} \left( a_n^{(i)} x^2; q \right)_n \right\} \sum_{p=2}^{\infty} \left\{ \sum_{l_i \geq 2} (q^n; q)_{l_i-2} \cdots (q^n; q)_l \right\} \chi^{p-2}
\]

Since \(\frac{[n + l_i - 2] q}{[n + l_i - 1] q} < 1\), in view of (3.2), we get

\[
\sum_2 \leq q(n^{(1)})^2.
\]

Finally, using the estimates of \(\sum_1\) and \(\sum_2\) in (3.4), we obtain

\[
\Re_{n,q}^{\alpha^{(1)},\cdots,\alpha^{(r)}; x} \leq q(n^{(1)})^2 + \frac{x a_n^{(1)}}{[n]_q}. \tag{3.5}
\]

Hence, we can write

\[
\Re_{n,q}^{\alpha^{(1)},\cdots,\alpha^{(r)}; x} - x^2 \leq x^2 (q(n^{(1)})^2 - 1) + \frac{x a_n^{(1)}}{[n]_q} = -x^2 (1 - q(n^{(1)})^2) + \frac{x a_n^{(1)}}{[n]_q}.
\]

As \(q, a_n^{(1)} \in (0,1)\), we have

\[
\Re_{n,q}^{\alpha^{(1)},\cdots,\alpha^{(r)}; x} - x^2 \leq \frac{x a_n^{(1)}}{[n]_q}. \tag{3.6}
\]
Since the operator $\mathfrak{R}_{n,q}^{(\ldots)}(\cdot;x)$ is positive and linear, using Lemmas 1 and 2, we may write

$$0 \leq \mathfrak{R}_{n,q}^{(\ldots)}((s-x)^2;x) = \mathfrak{R}_{n,q}^{(\ldots)}(s^2;x) - 2sx\mathfrak{R}_{n,q}^{(\ldots)}(s;x) + x^2;$$

$$\Rightarrow -2x^2(1 - \alpha_n^{(1)}) \leq \mathfrak{R}_{n,q}^{(\ldots)}(s^2;x) - x^2,$$

or, $-2x^2(1 - \alpha_n^{(1)}) - \frac{x\alpha_n^{(1)}}{[n]_q} \leq \mathfrak{R}_{n,q}^{(\ldots)}(s^2;x) - x^2.$

Hence, in view of (3.6), we obtain

$$\left| \mathfrak{R}_{n,q}^{(\ldots)}(s^2;x) - x^2 \right| \leq 2x^2(1 - \alpha_n^{(1)}) + \frac{x\alpha_n^{(1)}}{[n]_q}.$$

This completes the proof. \[\square\]

4 | APPLICATIONS

In this section, we show the convergence of our $q$-Lagrange–Hermite operators $\mathfrak{R}_{n,q}^{(\ldots)}$ by applying the earlier explained convergence techniques. For this purpose, we shall need the following important assumptions:

**Assumption 1.** We consider the sequence $\alpha_n^{(1)} \in (0, 1)$, convergent (in classical sense) to 1 as $n$ tends to infinity.

**Assumption 2.** We assume that the sequences $\langle q_n \rangle$ and $\langle q_n'' \rangle$ converge (again in classical sense) to $d$ and $d \in [0, 1)$, respectively, as $n$ tends to infinity.

4.1 | Convergence of $\mathfrak{R}_{n,q}^{(\ldots)}$ using deferred weighted $A$-statistical method

To prove the said convergence of our operator, we first recall the following important characteristics of a deferred weighted regular matrix given by Srivastava et al.\(^{11}\)

**Theorem 5.** Suppose $\langle b_n \rangle$ and $\langle c_n \rangle$ are the sequences of non-negative integers. Then, an infinite matrix $A = (a_{n,k})$ is said to be a deferred weighted regular matrix iff

$$\sup_n \sum_{k=1}^{\infty} \frac{1}{S_n} \left| \sum_{m=\max_{b_n+1}}^{c_n} a_{m,k} \right| < \infty;$$

$$\lim_{n \to \infty} \frac{1}{S_n} \sum_{m=\max_{b_n+1}}^{c_n} a_{m,k} = 0, \quad \forall k \in \mathbb{N}; \quad \text{(4.1)}$$

$$\lim_{n \to \infty} \frac{1}{S_n} \sum_{m=\max_{b_n+1}}^{c_n} \sum_{k=1}^{\infty} a_{m,k} = 1.$$

In view of Theorem 1, it follows that the sequence of operators $\langle \mathfrak{R}_{n,q}^{(\ldots)}(g) \rangle$ converges deferred weighted $A$-statistically to $g$ if the following holds:

$$\text{stat}_{A}^{DW} \lim_{n} \left\| \mathfrak{R}_{n,q}^{(\ldots)}(s') - x' \right\| = 0, \quad \text{for } i = 0, 1, 2.$$

Applying Lemma 1, the above condition is obvious for the case $i = 0$. Now, for $i = 1$, using Lemma 2, we can write

$$\left\| \mathfrak{R}_{n,q}^{(\ldots)}(s) - x \right\| \leq (1 - \alpha_n^{(1)}). \quad \text{(4.2)}$$
Let $\epsilon > 0$ be an arbitrary real number. If we consider the sets $V_1 = \{ n \in \mathbb{N} : \left\| \mathcal{R}_{a_n}^{(i), \cdots, a^{(i)}}(s) - x \right\| \geq \epsilon \}$ and $V_2 = \{ n \in \mathbb{N} : (1 - a^{(1)}_n) \geq \epsilon \}$; then from (4.2), we have $V_1 \subseteq V_2$, and hence,

$$\frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in V_1} s_m a_{m,k} \leq \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in V_2} s_m a_{m,k}$$

or,

$$\lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in V_1} s_m a_{m,k} \leq \lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in V_2} s_m a_{m,k}.$$

From Assumption 1, there exists a positive integer $n_0(\epsilon)$ such that $\lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \leq n_0} s_m a_{m,k} = \lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \leq n_0} s_m a_{m,k}$; thus,

$$\lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in V_1} s_m a_{m,k} \leq \lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in V_2} s_m a_{m,k}.$$

Now, using the fact given in (4.1) to the right-hand side of the above inequality, we have

$$\lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in V_2} s_m a_{m,k} = 0,$$

or $\text{stat}^{\text{DW}}_A \left\| \mathcal{R}_{a_n}^{(i), \cdots, a^{(i)}}(s) - x \right\| = 0$.

Now, for the case $i = 2$, using Lemma 3, we obtain

$$\left\| \mathcal{R}_{a_n}^{(1), \cdots, a^{(1)}}(s^2) - x^2 \right\| \leq 2(1 - a^{(1)}_n) + \frac{a^{(1)}_n}{[n]_{q_n}}. \quad (4.3)$$

Again, we construct the sets $W_1 = \{ n \in \mathbb{N} : \left\| \mathcal{R}_{a_n}^{(1), \cdots, a^{(1)}}(s^2) - x^2 \right\| \geq \epsilon \}$, $W_2 = \{ n \in \mathbb{N} : (1 - a^{(1)}_n) > \frac{\epsilon}{2} \}$, and $W_3 = \{ n \in \mathbb{N} : \frac{a^{(1)}_n}{[n]_{q_n}} > \frac{\epsilon}{2} \}$. From (4.3), it is easy to see that $W_1 \subseteq W_2 \cup W_3$, and therefore,

$$\lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in W_1} s_m a_{m,k} \leq \lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in W_2} s_m a_{m,k} + \lim_{n \to \infty} \frac{1}{S_n} \sum_{m=b_n+1}^{c_n} \sum_{k \in W_3} s_m a_{m,k}.$$

Finally, following the previous logic along with the assumptions and the fact given in (4.1), the required claim is established.

**Example 2.** Let $A$ be a Cesaro matrix defined as

$$a_{m,k} = \begin{cases} \frac{1}{m}, & 1 \leq k \leq m, \\ 0, & k > m. \end{cases}$$

Further, let $b_n = 3n$, $c_n = 5n$, $s_n = 1$, $a^{(1)}_n = \frac{n}{n+1}$, and $q_n = n^{-1/2}$, $\forall n \in \mathbb{N}$. Then, clearly, by Theorem 5, $A$ is a deferred weighted regular matrix, $a^{(1)}_n \in (0, 1)$ and tends to 1, as $n \to \infty$, and $\lim q_n = 1$, and $\lim q^n_n = 0$. Consequently, the sequence of operators $(\mathcal{R}_{a_n}^{(1), \cdots, a^{(1)}}(g))$ converges deferred weighted $A$-statistically to $g$, for any $g \in C[0, 1]$. 


4.2 Convergence of $\mathfrak{R}_{n,q_n}^{a^{(1)},\ldots,a^{(r)}}$ using $P$-summability method

In order to show the uniform convergence of $(\mathfrak{R}_{n,q_n}^{b^{(1)},\ldots,b^{(r)}}(g))$ to $g$ on $[0, 1]$ by $P$-summability method, in view of Theorem 3, it is sufficient to establish that

$$
\lim_{u \to R} \frac{1}{p(u)} \sum_{n=1}^{\infty} \left\| \mathfrak{R}_{n,q_n}^{a^{(1)},\ldots,a^{(r)}}(s') - x \right\| p_n u^{n-1} = 0, \text{ for } i = 0, 1, 2.
$$

Using Lemma 1, it is easy to verify for the case when $i = 0$, that is, \( \lim_{u \to R} \frac{1}{p(u)} \sum_{n=1}^{\infty} \left\| \mathfrak{R}_{n,q_n}^{b^{(1)},\ldots,b^{(r)}}(1) - 1 \right\| p_n u^{n-1} = 0 \). Now, using Lemma 2, we have

$$
\frac{1}{p(u)} \sum_{n=1}^{\infty} \left\| \mathfrak{R}_{n,q_n}^{a^{(1)},\ldots,a^{(r)}}(s) - x \right\| p_n u^{n-1} \leq \frac{1}{p(u)} \sum_{n=1}^{\infty} \left( 1 - a_n^{(1)} \right) p_n u^{n-1}. \tag{4.4}
$$

From our Assumption 1, for a given $\epsilon > 0$, there exists a positive integer $n_0(\epsilon)$ such that $|a_n^{(1)} - 1| < \frac{\epsilon}{2}$, for all $n > n_0(\epsilon)$, thus

$$
\frac{1}{p(u)} \sum_{n=1}^{\infty} \left( 1 - a_n^{(1)} \right) p_n u^{n-1} \leq \frac{1}{p(u)} \sum_{n=1}^{n_0} \left( 1 - a_n^{(1)} \right) p_n u^{n-1} + \frac{\epsilon}{2} \sum_{n=n_0+1}^{\infty} p_n u^{n-1} < \frac{1}{p(u)} \sum_{n=1}^{n_0} \left( 1 - a_n^{(1)} \right) p_n u^{n-1} + \frac{\epsilon}{2} \sum_{n=1}^{\infty} p_n u^{n-1}.
$$

Since $(1 - a_n^{(1)})$ is a bounded sequence, \( \exists M_1 > 0 \) such that $M_1 = \max_{1 \leq n \leq n_0} (1 - a_n^{(1)})$; therefore,

$$
\frac{1}{p(u)} \sum_{n=1}^{\infty} \left( 1 - a_n^{(1)} \right) p_n u^{n-1} < \frac{M_1}{p(u)} \sum_{n=1}^{n_0} p_n u^{n-1} + \frac{\epsilon}{2}. \tag{4.5}
$$

Now, in view of the regularity condition given by (1.2), there exists $\delta_j(\epsilon) > 0$, such that \( \frac{p_j u^{n-1}}{p(u)} < \frac{\epsilon}{2M_1n_0} \), for all $R - \delta_j(u) < u < R$, and $j = 1, 2, \ldots, n_0(\epsilon)$. Consider $\delta(\epsilon) = \min \left( \delta_1(\epsilon), \delta_2(\epsilon), \ldots, \delta_{n_0}(\epsilon) \right)$, then \( \frac{p_j u^{n-1}}{p(u)} < \frac{\epsilon}{2M_1n_0} \), for all $R - \delta(\epsilon) < u < R$, and for every $n = 1, 2, \ldots, n_0$. Hence, from (4.5), we have

$$
\frac{1}{p(u)} \sum_{n=1}^{\infty} \left( 1 - a_n^{(1)} \right) p_n u^{n-1} < \frac{M_1}{2M_1n_0} + \frac{\epsilon}{2} = \epsilon.
$$

Consequently, from (4.4), we have

$$
\frac{1}{p(u)} \sum_{n=1}^{\infty} \left\| \mathfrak{R}_{n,q_n}^{a^{(1)},\ldots,a^{(r)}}(s) - x \right\| p_n u^{n-1} < \epsilon, \text{ } \forall u \in (R - \delta, R).
$$

By a similar reasoning and using Lemma 3, we can show that

$$
\frac{1}{p(u)} \sum_{n=1}^{\infty} \left\| \mathfrak{R}_{n,q_n}^{a^{(1)},\ldots,a^{(r)}}(s^2; x) - x^2 \right\| p_n u^{n-1} \leq \frac{1}{p(u)} \sum_{n=1}^{\infty} \left( 2(1 - a_n^{(1)}) + \frac{a_n^{(1)}}{\lfloor n \rfloor q_n} \right) p_n u^{n-1} < \epsilon, \text{ for some } \theta(\epsilon) > 0 \text{ and } \forall u \in (R - \theta, R),
$$

in view of the fact that the sequence \( \frac{a_n^{(1)}}{\lfloor n \rfloor q_n} \to 0 \), as $n \to \infty$ (keeping in mind the Assumption 2). This completes the requirement.
CONCLUSION AND COMMENTS

In this work, we visited two summability methods, namely, deferred type statistical convergence and $P$-summability, and demonstrated the applications of these methods to the constructive theory of approximation. With these two impressions of convergence, we established two non-trivial Korovkin-type convergence theorems for positive linear operators. In both the cases, we obtained stronger results than the classical Korovkin-type theorems for the corresponding operators; therefore, it may be useful to consider Theorems 1 and 3 when the classical conditions do not work. Using the multivariate $q$-Lagrange–Hermite polynomials, we constructed a new sequence of operators, proved some interesting inequalities concerning to the moments, and finally investigated the convergence through the said summability techniques.

The generalized sampling series introduced by Butzer and Stens\cite{50} is a powerful tool to investigate and prove the approximation problem on $\mathbb{R}$. The approximation of continuous and discontinuous functions by classical sampling operators was first initiated by Butzer et al.\cite{51} Costarelli et al.\cite{52} studied the Kantorovich sampling series for discontinuous signals (functions). Angamuthu et al.\cite{53} analyzed the behavior of the exponential sampling series and also investigated its rate of convergence. For some other significant studies in this direction, one can refer to previous studies.\cite{54-56} The study in fuzzy approximation theory began with the monograph by Anastassiou.\cite{57} The general approximation of a fuzzy system refers to whether the fuzzy system can approximate any continuous function defined on a compact set with arbitrary precision. Anastassiou and Duman\cite{58} proved a Korovkin-type approximation theorem for fuzzy positive linear operators by using the notion of $A$-statistical convergence. Sezer and Ç anak\cite{59} applied the power summability method to the space of fuzzy numbers $\mathbb{R}_F$ and gave some Tauberian theorems based on the said summability method. Yavuz\cite{60} established a fuzzy trigonometric Korovkin-type approximation theorem by using power series summability method and also derived another approximation result for fuzzy periodic continuous functions with the help of fuzzy modulus of continuity.

Thus, the present study is closely connected with the theories and applications of multivariate orthogonal polynomials, sampling approximation techniques, and fuzzy approximation theory. One can define the sampling operators based on the multivariate Lagrange polynomials and investigate their various convergence properties. The non-trivial Korovkin-type theorems (Theorems 1 and 3) may be extended in the fuzzy sense. Following the interesting paper by Ta¸san and Yurdakadım,\cite{20} the present study could also be discussed for the modular spaces. Finally, one may try to define and study the convergence of positive linear operators based on other multivariate polynomials.\cite{61}

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AUTHOR CONTRIBUTION

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