Four neutrino oscillation analysis of the Superkamiokande atmospheric neutrino data

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(June, 2000)

Abstract

Superkamiokande atmospheric neutrino data (contained and upward going through $\mu$ events) for 990 days are analyzed in the framework of four neutrinos (three active and one sterile neutrinos) without imposing constraints of Big Bang Nucleosynthesis. It is shown that the wide range of the oscillation parameters is allowed at 90% confidence level ($0.1 \lesssim |U_{s1}|^2 + |U_{s2}|^2 \leq 1$), where the best fit point has some contribution of $\nu_{\mu} \leftrightarrow \nu_s$ and contribution of $\Delta m_{\text{LSND}}^2$. The case of pure $\nu_{\mu} \leftrightarrow \nu_s$ oscillation is excluded at 99.7%CL (3.0$\sigma$) which is consistent with the recent analysis by the Superkamiokande group. Combining this result with the analysis by Giunti, Gonzalez-Garcia and Peña-Garay, it is found that the Large Mixing Angle and Vacuum Oscillation solutions of the solar neutrino problem are also allowed.

14.60.P, 26.65, 28.41, 96.60.J

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I. INTRODUCTION

There have been a few experiments which suggest neutrino oscillations: the solar neutrino deficit [1–6], the atmospheric neutrino anomaly [7–16], and the LSND data [17]. If we assume that all these three are caused by neutrino oscillations then we need at least four species of neutrinos and schemes with sterile neutrinos have been studied by many people [18–25]. By generalizing the discussion on Big Bang Nucleosynthesis (BBN) from the two neutrino scheme [26] to four neutrino case, it has been shown [20] that the neutrino mixing angles are strongly constrained not only by the reactor data [19] but also by BBN if one demands that the number $N_\nu$ of effective neutrinos be less than four. In this case the $4 \times 4$ MNS matrix [27] splits approximately into two $2 \times 2$ block diagonal matrices, and the solar neutrino deficit is explained by $\nu_e \leftrightarrow \nu_s$ oscillations with the Small Mixing Angle (SMA) MSW solution [28] and the atmospheric neutrino anomaly is accounted for by $\nu_\mu \leftrightarrow \nu_\tau$. On the other hand, some people have given conservative estimate for $N_\nu$ [29] and if their estimate is correct then we no longer have strong constraints on the mixing angles of sterile neutrinos. Recently Giunti, Gonzalez-Garcia and Peña-Garay [30] have analyzed the solar neutrino data in the four neutrino scheme without BBN constraints. They have shown that the scheme is reduced to the two neutrino framework in which only one free parameter $c_s \equiv |U_{s1}|^2 + |U_{s2}|^2$ appears in the analysis. Their conclusion is that the SMA MSW solution exists for the entire region of $0 \leq c_s \leq 1$, while the Large Mixing Angle (LMA) and Vacuum Oscillation (VO) solutions survive only for $0 \leq c_s \lesssim 0.2$ and $0 \leq c_s \lesssim 0.4$, respectively. In this paper the Superkamiokande atmospheric neutrino data (contained and upward going through $\mu$ events) are analyzed in the same scheme as in [30], i.e., in the four neutrino scheme with all the constraints of accelerators and reactors and without BBN constraints. Zenith angle dependence of atmospheric neutrinos has been analyzed by many theorists [31,23–25] as well as by experimentalists [7,8,10,13–15].

II. FOUR NEUTRINO SCHEME

Here we adopt the notation in [20] for the $4 \times 4$ MNS matrix:

$$
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau \\
\nu_s
\end{pmatrix} = U
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3 \\
\nu_4
\end{pmatrix},
$$

$$
U \equiv
\begin{pmatrix}
U_{e1} & U_{e2} & U_{e3} & U_{e4} \\
U_{\mu1} & U_{\mu2} & U_{\mu3} & U_{\mu4} \\
U_{\tau1} & U_{\tau2} & U_{\tau3} & U_{\tau4} \\
U_{s1} & U_{s2} & U_{s3} & U_{s4}
\end{pmatrix}

\equiv R_{34}(\pi/2 - \theta_{34}) R_{24}(\theta_{24}) R_{23}(\pi/2) e^{2i\delta_1 \lambda_3} R_{23}(\theta_{23}) e^{-2i\delta_1 \lambda_3} e^{\sqrt{6}i\delta_3 \lambda_{15}/2}
\times R_{14}(\theta_{14}) e^{-\sqrt{6}i\delta_3 \lambda_{15}/2} e^{2i\delta_2 \lambda_5/\sqrt{3}} R_{13}(\theta_{13}) e^{-2i\delta_2 \lambda_5/\sqrt{3}} R_{12}(\theta_{12}),
$$

(1)

1The result in [20] was refined later in [22] with more careful treatment.
where \(c_{ij} \equiv \cos \theta_{ij}\), \(s_{ij} \equiv \sin \theta_{ij}\) and

\[
R_{jk}(\theta) \equiv \exp (i T_{jk} \theta),
\]

is a \(4 \times 4\) orthogonal matrix with

\[
(T_{jk})_{lm} = i (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}),
\]

and \(2 \lambda_3 \equiv \text{diag}(1, -1, 0, 0), 2 \sqrt{3} \lambda_8 \equiv \text{diag}(1, 1, -2, 0), 2 \sqrt{6} \lambda_{15} \equiv \text{diag}(1, 1, 1, -3)\) are diagonal elements of the \(SU(4)\) generators.

In the following analysis we will consider the situation where non-negligible contributions from the largest mass squared difference \(\Delta m^2_{\text{LSND}}\) appear in the oscillation probability \(P(\nu_\mu \to \nu_\mu)\). To avoid contradiction with the negative result of the CDHSW disappearing experiment on \(\nu_\mu \to \nu_\mu\), we will take \(\Delta m^2_{\text{LSND}} = 0.3 \text{eV}^2\) as a reference value.

Having assumed \(U_{e3} = U_{e4} = 0\) and \(\Delta m^2_{21} = 0\), we have only mixings among \(\nu_\mu, \nu_\tau, \nu_s\) in the analysis of atmospheric neutrinos, and the Schrödinger equation we have to consider is

\[
i \frac{d}{dx} \begin{pmatrix} \nu_\mu(x) \\ \nu_\tau(x) \\ \nu_s(x) \end{pmatrix} = \left[ \bar{U} \text{diag}( -\Delta E_{32}, 0, \Delta E_{43}) \bar{U}^{-1} + \text{diag}(0, 0, A(x)) \right] \begin{pmatrix} \nu_\mu(x) \\ \nu_\tau(x) \\ \nu_s(x) \end{pmatrix},
\]

where
\[ \Delta E_{ij} \equiv \Delta m_{ij}^2 / 2E \text{ and } A(x) \equiv G_F N_n(x)/\sqrt{2} \] stands for the effect due to the neutral current interactions between \( \nu_\mu \), \( \nu_\tau \) and matter in the Earth after adding the unit matrix diag \((A(x), A(x), A(x))\) to the right hand side of \((3)\). Since \( \nu_e \) does not oscillate with any other neutrinos, the only oscillation probability which is required in the analysis of atmospheric neutrinos is \( P(\nu_\mu \rightarrow \nu_\mu) \). In the present case, we have mass hierarchy \( |\Delta E_{32}| \gg |\Delta E_{43}|, |A(x)| \) and to the leading order in \( |\Delta E_{43}|/|\Delta E_{32}| \) and \( |A(x)|/|\Delta E_{32}| \) we can obtain the analytical expression for the oscillation probability \( P(\nu_\mu \rightarrow \nu_\mu) \) in adiabatic approximation, i.e., assuming that the derivative \( dA(x)/dx \) is not large compared to \( \Delta E_{jk} \).

As we will see in the appendix, with such approximation we can show that the oscillation probability \( P(\nu_\mu \rightarrow \nu_\mu) \) is invariant under \( \theta_{34} \rightarrow -\theta_{34} \) and \( \delta_1 \rightarrow \pi - \delta_1 \) to the leading order in \( |\Delta E_{43}|/|\Delta E_{32}| \) and \( |A(x)|/|\Delta E_{32}| \). So we will study our scheme for the range \( 0 \leq \theta_{24} \leq \pi/2, 0 \leq \theta_{23} \leq \pi/2, -\pi/2 \leq \theta_{34} \leq \pi/2, 0 \leq \delta_1 \leq \pi/2 \). Note that all the three mixing angles would lie in the first quadrant if there were no matter effects.

III. ANALYSIS OF THE ATMOSPHERIC NEUTRINO DATA

We calculate the disappearance probability \( P(\nu_\mu \rightarrow \nu_\mu) \) by solving \((3)\) numerically, and evaluate the number of events:

\[
N(\mu) = (1 - r_\mu) N_0(\mu) + r_\mu N_0(e),
\]

\[
N(e) = (1 - r_e) N_0(e) + r_e N_0(\mu),
\]

\[
N_0(\mu) = n_T \int dE \int dq \int d\cos \Theta \int d\cos \theta \int d\varphi \ \epsilon_\mu(q)
\times \frac{\partial^3 F_\mu(E, \theta)}{\partial E \partial d\cos \theta \partial d\varphi} \cdot \frac{\partial^2 \sigma_\mu(E, q)}{\partial dq \partial d\cos \psi} \cdot \frac{d\cos \psi}{d\cos \theta} \ P(\nu_\mu \rightarrow \nu_\mu; E, \theta),
\]

\[
N_0(e) = n_T \int dE \int dq \int d\cos \Theta \int d\cos \theta \int d\varphi \ \epsilon_e(q)
\times \frac{\partial^3 F_e(E, \theta)}{\partial E \partial d\cos \theta \partial d\varphi} \cdot \frac{\partial^2 \sigma_e(E, q)}{\partial dq \partial d\cos \psi} \cdot \frac{d\cos \psi}{d\cos \theta},
\]

where \( d^3 F_\alpha / dE d\cos \theta d\varphi \) \((\alpha = \mu, e)\) is the flux of atmospheric neutrinos \( \nu_\alpha \) \((\alpha = \mu, e)\) with energy \( E \) from the zenith angle \( \theta \) \([37]\), \( n_T \) is the effective number of target nucleons, \( \epsilon_\alpha(q) \) is the detection efficiency function for charged leptons \( \alpha \) \((\alpha = \mu, e)\) \([38]\), \( d\sigma_\alpha/dq d\cos \psi \) \((\alpha = \mu, e)\) is the differential cross section of the interaction \( \nu_\alpha N \rightarrow \alpha X \) \((\alpha = e \text{ or } \mu)\); for sub-GeV events quasi-elastic scatterings \( \nu_\alpha N \rightarrow \alpha N' \) are dominant and the cross-section given in \([39]\) is used, while for multi-GeV events the inclusive cross-section for \( \nu_\alpha N \rightarrow \alpha X \) given in \([40]\) is used), and \( \Theta \) is the zenith angle of the direction from which the charged lepton \( \alpha \) comes \((\text{See Fig. 1 of [23]}\)). \( r_\mu \) and \( r_e \) stand for ratios of contamination due to misidentification.
of $\mu$-like and $e$-like events and we have put $r_\mu=0.03$, $r_e=0.06$ for sub-GeV events, $r_\mu=0.007$, $r_e=0.12$ for fully contained multi-GeV events, $r_\mu=0$ for partially contained multi-GeV and upward going through $\mu$ events.

$\chi^2$ is defined as

$$
\chi^2 = \chi^2_{\text{sub-GeV}} + \chi^2_{\text{multi-GeV}} + \chi^2_{\text{through}}
$$

where

$$
\chi^2_{\text{sub-GeV}} = \frac{\beta_s^2}{\sigma_{\beta_s}} + \sum_{j=1}^{10} \left\{ \frac{\alpha(1 - \beta_s^2)N^s_j(e) - n^s_j(e)}{n^s_j(e)} \right\}
$$

$$
\chi^2_{\text{multi-GeV}} = \frac{\beta_m^2}{\sigma_{\beta_m}} + \sum_{j=1}^{10} \left\{ \frac{\alpha(1 - \beta_m^2)N^m_j(e) - n^m_j(e)}{n^m_j(e)} \right\}
$$

$$
\chi^2_{\text{through}} = \frac{\alpha_{\text{th}}^2}{\sigma_{\alpha_{\text{th}}}} + \sum_{j=1}^{10} \left\{ \frac{\alpha_{\text{th}}N^1_j(\mu) - n^1_j(\mu)}{n^1_j(\mu)} \right\}
$$

are $\chi^2$ for sub-GeV, multi-GeV, and upward going through $\mu$ events, respectively, the summation on $j$ runs over the ten bins for each $\chi^2$, $N^a_j(\alpha)$ and $n^a_j(\alpha)$ ($a=s, m, \text{th}$) stand for theoretical predictions and data for the numbers of sub-GeV, multi-GeV, and upward going through $\mu$ events, and it is understood that $\chi^2$ is minimized with respect to all the normalization factors $\alpha$, $\beta_s$, $\beta_m$, $\alpha_{\text{th}}$. We have put $\sigma_s=0.08$, $\sigma_m=0.12$, $\sigma_{\alpha_{\text{th}}}=0.22$ and we have assumed that the overall flux normalization $\alpha$ in the contained events is a free parameter as in [10], and we have omitted the uncertainties of $E_\nu$ spectral index, relative normalization between PC and FC and up-down correlation for simplicity.

We have evaluated $\chi^2$ for $\theta_{24} = (25 + 5j)\degree$ ($j = 0, \ldots, 7$), $\theta_{34} = 15j\degree$ ($j = -6, \ldots, 6$), $\theta_{23} = 10j\degree$ ($j = 0, \ldots, 4$), $\delta_1 = 0\degree, 45\degree, 90\degree$, $\Delta m^2_{43} = 10^{-4+j/10}$eV$^2$ ($j = 5, \ldots, 20$) and it is found that $\chi^2$ has the minimum value

$$
\chi^2_{\text{min}} = 43.1 \quad (\chi^2_{\text{sub-GeV}} = 19.0, \chi^2_{\text{multi-GeV}} = 13.2, \chi^2_{\text{through}} = 11.6)
$$

for

$$
\Delta m^2_{43} = 10^{-2.9}$eV$^2 = 1.3 \times 10^{-3}$eV$^2, \quad (\theta_{24}, \theta_{34}, \theta_{23}) = (35\degree, 15\degree, 20\degree)
$$

for 45 degrees of freedom. For pure $\nu_\mu \leftrightarrow \nu_\tau$ ($\theta_{34} = \theta_{23} = 0$), the best fit is obtained

$$
\chi^2_{\text{min}}(\nu_\mu \leftrightarrow \nu_\tau) = 48.3 \quad (\chi^2_{\text{sub-GeV}} = 19.8, \chi^2_{\text{multi-GeV}} = 17.0, \chi^2_{\text{through}} = 10.6)
$$

for

$$
\Delta m^2_{43} = 2.0 \times 10^{-3}$eV$^2, \quad (\theta_{24}, \theta_{34}, \theta_{23}) = (40\degree, 0\degree, 0\degree).
$$

The allowed regions at $1\sigma$CL, $90\%$CL, $99\%$CL are obtained by $\chi^2 \leq \chi^2_{\text{min}} + \Delta \chi^2$, where $\Delta \chi^2=5.9, 9.2, 15.1$ for five degrees of freedom, respectively, and they are depicted in Fig. 1 for various values of $\theta_{24}$ and for $\delta_1 = 0$ (1(a)), $\delta_1 = \pi/4$ (1(b)) and $\delta_1 = \pi/2$ (1(c)). At $99\%$
confidence level we find $27^\circ \lesssim \theta_{24} \lesssim 58^\circ$. The difference of the best fit point among the all parameter space and the best fit for pure $\nu_\mu \leftrightarrow \nu_\tau$ ($\Delta m_{43}^2 = 10^{-27/10}$eV$^2$, $\theta_{24} = 40^\circ$, $\theta_{34} = 0^\circ$, $\theta_{23} = 0^\circ$) is $\Delta \chi^2 = 5.1$ which corresponds to 0.84\sigma, so this difference is not significant. Zenith angle dependence for contained events and upward going through muon events for three sets of the parameters including the case (12) and (14) is given in Fig. 2(a) and 2(b). Zenith angle dependence for no oscillation case that we obtained is reasonably in good agreement with the Superkamiokande result [12] and this puts confidence to the present analysis. In general, the reason that the best fit point is slightly away from pure $\nu_\mu \leftrightarrow \nu_\tau$ case is because a better fit to the multi-GeV contained events compensates a worse fit to the upward going through $\mu$ events, and in total the case of hybrid oscillations of $\nu_\mu \leftrightarrow \nu_\tau$ and $\nu_\mu \leftrightarrow \nu_s$ fits better to the data (See Figs. 2(a) and 2(b)).

IV. DISCUSSIONS

(1) Pure $\nu_\mu \leftrightarrow \nu_s$ oscillation is obtained only for $\theta_{23} = 0$, $\theta_{34} = \pm 90^\circ$, and in this case $\chi^2$ satisfies $\chi^2 - \chi^2_{\min} \geq 17.9$ for any value of $\Delta m_{43}^2$, $\theta_{34}$ and $\delta_1$ and it implies that pure $\nu_\mu \leftrightarrow \nu_s$ oscillation is excluded at 99.7\%CL (3.0\sigma CL). This is consistent with the recent claim by Superkamiokande group [11,12].

(2) For some $\Delta m_{43}^2$, $\theta_{24} \delta_1$ (e.g., $\Delta m_{43}^2 = 1.3 \times 10^{-3}$eV$^2$, $\theta_{24} = 50^\circ$, $\theta_{23} = 30^\circ$, $\delta_1 = 0$), $\theta_{34} = 90^\circ$ is allowed at 90\%CL, and one might wonder how such a solution can give a reasonable fit to the data. For this set of the oscillation parameters, we find that the oscillation probabilities without matter effects are given by

\[ P(\nu_\mu \rightarrow \nu_\tau; \text{without matter effects}) = \frac{3}{8} \cos^2 \left( \frac{5\pi}{18} \right) \]
\[ P(\nu_\mu \rightarrow \nu_\nu; \text{without matter effects}) = \sin^2 \left( \frac{5\pi}{9} \right) \left[ \frac{1}{32} + \frac{3}{4} \sin^2 \left( \frac{\Delta m_{43}^2 L}{4E} \right) \right], \quad (15) \]

where $(3/8) \cos^2 (5\pi/18) \simeq 0.15$, $\sin^2 (5\pi/9) \simeq 0.97$, and we have averaged over rapid oscillations which come from $\Delta m_{32}^2$. From (15) we observe that zenith angle dependence with a non-maximal coefficient comes solely from $\nu_\mu \leftrightarrow \nu_s$ oscillation and both $\nu_\mu \leftrightarrow \nu_\mu$ and $\nu_\mu \leftrightarrow \nu_s$ oscillations have constant contribution in the oscillation probabilities. As was shown in [23], the zenith angle dependence of contained events is explained well also by $\nu_\mu \leftrightarrow \nu_s$ oscillation, but the fit of $\nu_\mu \leftrightarrow \nu_s$ to upward going $\mu$ events is poor [11,12]. In the present case, we have $\chi^2_{\nu_{sub-GeV}} = 19.7$, $\chi^2_{\nu_{multi-GeV}} = 15.5$, $\chi^2_{\nu_{through}} = 15.0$ and we observe that this solution is allowed at 90\%CL because a poor fit to the upward going through $\mu$ events is compensated by a good fit to the multi-GeV contained events.

(3) One of the remarkable features of our result is that there is a region where relatively small value of $\theta_{24}(\sim 30^\circ)$ is allowed at 90\%CL for $\theta_{23} \simeq 20^\circ$, i.e., there exists a solution in which all the mixing angles are relatively small $|\theta_{jk}| \lesssim \pi/6$, and such situation does not occur for $\theta_{23} = 0$. This phenomena is reminiscent of the work [11] on three flavor analysis of the atmospheric neutrino data, in which it was argued that relatively small mixing angles could account for the data because of matter effects if there were no CHOOZ constraint [12]. However, in the present case it turns out that the matter effect is not so important, i.e., even if we put the density of the matter to zero the fit to the data is still reasonable.
The reason that the fit to the data is good for $\theta_{24} \sim 30^\circ$ is because the disappearance probability behaves as $1 - P(\nu_\mu \leftrightarrow \nu_\mu) = A + B \sin^2(\Delta m_{43}^2 L/4E)$ ($A, B$ are constant) and even if the coefficient $B$ is relatively small the presence of the constant term $A$ compensates the goodness of the fit to a certain extent. This argument applies both to the multi-GeV $\mu$-like contained events and the upward going through muon events.

To combine the present result with the analysis of the solar neutrinos in [30], it is necessary to obtain the value of $c_s \equiv |U_{s1}|^2 + |U_{s2}|^2$ for each point. In our parametrization (1) we have

$$U_{s1} = -s_{12}(c_{23}c_{34} + s_{23}s_{24}s_{34}e^{i\delta_1})$$
$$U_{s2} = c_{12}(c_{23}c_{34} + s_{23}s_{24}s_{34}e^{i\delta_1})$$  \hspace{1cm} (16)

so that

$$c_s \equiv |U_{s1}|^2 + |U_{s2}|^2 = |c_{23}c_{34} + s_{23}s_{24}s_{34}e^{i\delta_1}|^2.$$  \hspace{1cm} (17)

The contours of $c_s = 0.2, 0.4, 0.6, 0.8$ are plotted together with the allowed region for various $\Delta m_{43}^2$ in Fig. 3(a) ($\delta_1 = 0$), 3(b) ($\delta_1 = \pi/4$) and 3(c) ($\delta_1 = \pi/2$). For each value of $\delta_1$ some point in the allowed region satisfies $c_s \lesssim 0.2$ for $40^\circ \lesssim \theta_{24} \lesssim 50^\circ$. We find that the allowed region at 90\% confidence level satisfies $0.15 \lesssim c_s \lesssim 1$ for $\delta_1 = 0$, $0.10 \lesssim c_s \lesssim 1$ for $\delta_1 = \pi/4$, $0.05 \lesssim c_s \lesssim 1$ for $\delta_1 = \pi/2$, respectively. Hence combination of the present result with the analysis in [30] suggests that the LMA and VO solutions as well as SMA solution of the solar neutrino problem are possible for some region in the parameter space. Recently Superkamiokande group has announced [43] that the SMA and VO solutions of the solar neutrino problem are disfavored at 95\% confidence level. If Nature is described by a four neutrino scenario, therefore, the present scheme with $c_s \lesssim 0.2$ may be the right solution for all the oscillation data.

V. CONCLUSIONS

We have shown in the framework of four neutrino oscillations without assuming the BBN constraints that the Superkamiokande atmospheric neutrino data are explained by wide range of the oscillation parameters which implies hybrid oscillations of $\nu_\mu \leftrightarrow \nu_\tau$ and $\nu_\mu \leftrightarrow \nu_\tau$ as well as hybrid oscillations with $\Delta m_{\text{atm}}^2$ and $\Delta m_{\text{SND}}^2$. The case of pure $\nu_\mu \leftrightarrow \nu_\mu$ is excluded at 3.0$\sigma$CL in good agreement with the Superkamiokande analysis. It is found by combining the analysis on the solar neutrino data by Giunti, Gonzalez-Garcia and Peña-Garay that the LMA and VO solutions as well as SMA solution of the solar neutrino problem are allowed. This gives us another possibility in phenomenology of neutrino oscillations and such scenarios deserve further study.

ACKNOWLEDGMENTS

I would like to thank Alex Friedland for showing me how to obtain interpolation in 3 dimensions by Mathematica. This research was supported in part by a Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture, #12047222, #10640280.
APPENDIX

With hierarchy $|\Delta E_{32}| \gg |\Delta E_{43}|, |A(x)|$ we can obtain the analytical expression for the oscillation probability $P(\nu_\mu \to \nu_\mu)$ to the leading order in $|\Delta E_{43}|/|\Delta E_{32}|$ and $|A(x)|/|\Delta E_{32}|$ in adiabatic approximation. The purpose of this Appendix is to show that the oscillation probability $P(\nu_\mu \leftrightarrow \nu_\mu)$ is invariant under $\theta_{34} \to -\theta_{34}$ and $\delta_1 \to \pi - \delta_1$ to the leading order in $|\Delta E_{43}/\Delta E_{32}|$ and $|A(x)/\Delta E_{32}|$. To do that, it is convenient to use different parametrization:

\[
V \equiv R_{13} \left( \frac{\pi}{2} \right) D R_{23}(\varphi_{23}) R_{13}(\varphi_{13}) D^{-1} R_{12}(\varphi_{12}), \tag{18}
\]

where $D \equiv \text{diag}(e^{i\delta},1,1)$. To relate this parametrization with (7), we have to multiply diagonal phase matrices

\[
V = \text{diag} \left( e^{i\omega_1}, e^{i\omega_2}, e^{-i(\omega_1 + \omega_2)} \right) \tilde{U} \text{diag} \left( e^{i\gamma_1}, e^{i\gamma_2}, e^{-i(\gamma_1 + \gamma_2)} \right), \tag{19}
\]

where

\[
\omega_1 = \frac{1}{3} \arg U_{s2} + \frac{1}{3} \arg U_{s3} - \frac{1}{3} \arg U_{\mu 4}, \quad \omega_2 = \frac{1}{3} \arg U_{s2} + \frac{1}{3} \arg U_{s3} + \frac{2}{3} \arg U_{\mu 4},
\]

\[
\gamma_1 = -\frac{1}{3} \arg U_{s2} + \frac{2}{3} \arg U_{s3} + \frac{1}{3} \arg U_{\mu 4}, \quad \gamma_2 = \frac{2}{3} \arg U_{s2} - \frac{1}{3} \arg U_{s3} + \frac{1}{3} \arg U_{\mu 4}. \tag{20}
\]

Comparing the both hand sides of (19), we obtain

\[
\tan \varphi_{12} = \frac{|U_{s3}|}{|U_{s2}|}, \quad \tan \varphi_{13} = \frac{c_{24}s_{34}}{\sqrt{c_{24}^2 s_{34}^2 + s_{24}^2}},
\]

\[
\tan \varphi_{23} = \frac{c_{24} s_{34}}{s_{24}}, \quad \arg (-V_{e3}) = \pi - \delta' = -\arg U_{\mu 4} - \arg U_{s2} - \arg U_{s3}. \tag{21}
\]

Using this parametrization, we have

\[
V^{-1} \text{diag} (0, 0, A) V + (0, 0, \Delta E_{43})
\]

\[
= R_{12}(\varphi_{12})^{-1} D R_{13}(\varphi_{13} - \varphi_{13}^M)^{-1} \text{diag} (t_-, 0, t_+) R_{13}(\varphi_{13} - \varphi_{13}^M) D^{-1} R_{12}(\varphi_{12})
\]

\[
= \text{diag} \left( C_{12}^2(t_- \tilde{C}^2 + t_+ \tilde{S}^2), S_{12}^2(t_- \tilde{C}^2 + t_+ \tilde{S}^2), t_+ \tilde{C}^2 + t_- \tilde{S}^2 \right)
\]

\[
+ \frac{1}{2} \sin 2\varphi_{12}(t_- \tilde{C}^2 + t_+ \tilde{S}^2) \lambda_1 + \frac{1}{2} C_{12}(t_- - t_+) \sin 2\tilde{\varphi}(\lambda_4 \cos \delta' - \lambda_5 \sin \delta')
\]

\[
+ \frac{1}{2} S_{12}(t_- - t_+) \sin 2\tilde{\varphi}(\lambda_6 \cos \delta' - \lambda_7 \sin \delta'), \tag{22}
\]

where $\lambda_j (j = 1, \ldots, 8)$ are the $3 \times 3$ Gell-Mann matrices with normalization $\text{tr}(\lambda_j \lambda_k) = 2\delta_{jk}$, $S_{12} \equiv \sin \varphi_{12}, C_{12} \equiv \cos \varphi_{12}, \tilde{S} \equiv \sin(\varphi_{13} - \varphi_{13}^M), \tilde{C} \equiv \cos(\varphi_{13} - \varphi_{13}^M)$ and $\varphi_{13}^M$ is the effective mixing angle in matter given by

\[
\tan 2\varphi_{13}^M = \frac{\Delta E_{43} \sin 2\varphi_{13}}{\Delta E_{43} \cos 2\varphi_{13} - A^i}, \tag{23}
\]

and $t_{\pm}$ are the eigenvalues defined by
Putting everything together, we obtain

\[ t_{\pm} \equiv \frac{1}{2} \left( A + \Delta E_{43} \pm \sqrt{\Delta E_{43} \cos 2\varphi_{13} - A}^2 + (\Delta E_{43} \sin 2\varphi_{13})^2 \right). \]  \tag{24}

Putting

\[ \Lambda \equiv \frac{1}{2\Delta E_{32}} \left[ \sin 2\varphi_{12}(t_- \tilde{C}^2 + t_+ \tilde{S}^2)\lambda_1 + C_{12}(t_- - t_+ \sin 2\tilde{\varphi}(\lambda_4 \cos \delta' - \lambda_5 \sin \delta') \right], \]  \tag{25}

we have

\[ e^{-i\Lambda} \left[ \begin{array}{c} \text{diag}(-\Delta E_{32}, 0, 0) + V^{-1} \text{diag}(0, 0, A) + \text{diag}(0, 0, \Delta E_{43}) \end{array} \right] e^{i\Lambda} \]

\[ = \text{diag}(-\Delta E_{32}, 0, 0) + (C_{12}(t_- \tilde{C}^2 + t_+ \tilde{S}^2)I_3 \]

\[ + \frac{1}{2} \left[ (t_- \tilde{C}^2 + t_+ \tilde{S}^2)(S^2 - 2C^2_{12}) + t_- \tilde{C}^2 + t_+ \tilde{S}^2 \right] \text{diag}(0, 1, 1) \]

\[ + \frac{1}{2} \left[ S^2_{\Delta E_{32}} + t_- \tilde{C}^2 - t_+ \tilde{S}^2 \right] \tilde{\lambda}_8 + \frac{1}{2} S_{12}(t_- - t_+) \sin 2\tilde{\varphi}(\lambda_6 \cos \delta' - \lambda_7 \sin \delta'), \]  \tag{26}

where \( I_3 \equiv \text{diag}(1, 1, 1), \tilde{\lambda}_8 \equiv \text{diag}(0, 1, -1). \) The last two lines in (26) can be diagonalized by noting

\[ \mathcal{F}\tilde{\lambda}_8 + \mathcal{G}(\lambda_6 \cos \delta' - \lambda_7 \sin \delta') \]

\[ = \mathcal{F}\tilde{\lambda}_8 + \mathcal{G} e^{\frac{i}{2} \delta' \tilde{\lambda}_8} \lambda_6 e^{-\frac{i}{2} \delta' \tilde{\lambda}_8} = e^{\frac{i}{2} \delta' \tilde{\lambda}_8} \left( \mathcal{F}\tilde{\lambda}_8 + \mathcal{G}\lambda_6 \right) e^{-\frac{i}{2} \delta' \tilde{\lambda}_8} \]

\[ = e^{\frac{i}{2} \delta' \tilde{\lambda}_8} e^{-i\psi \lambda_7} \sqrt{\mathcal{F}^2 + \mathcal{G}^2} \tilde{\lambda}_8 e^{i\psi \lambda_7} e^{-\frac{i}{2} \delta' \tilde{\lambda}_8}, \]  \tag{27}

where

\[ \mathcal{F} \equiv \frac{1}{2} \left[ S^2_{\Delta E_{32}}(t_- \tilde{C}^2 + t_+ \tilde{S}^2) - t_- \tilde{C}^2 + t_+ \tilde{S}^2 \right] \]

\[ \mathcal{G} \equiv \frac{1}{2} S_{12}(t_- - t_+) \sin 2\tilde{\varphi} \]  \tag{28}

and the effective mixing angle \( \psi \) in matter is given by

\[ \tan 2\psi \equiv \frac{\mathcal{G}}{\mathcal{F}}. \]  \tag{29}

Putting everything together, we obtain

\[ V \text{diag}(-\Delta E_{32}, 0, \Delta E_{43}) V^{-1} + \text{diag}(0, 0, A) \]

\[ = V e^{i\delta' \tilde{\lambda}_8} e^{-i\psi \lambda_7} e^{i\Lambda} \left\{ \begin{array}{c} \text{diag}(-\Delta E_{32}, 0, 0) + (C_{12}(t_- \tilde{C}^2 + t_+ \tilde{S}^2)I_3 \]

\[ + \frac{1}{2} \left[ (t_- \tilde{C}^2 + t_+ \tilde{S}^2)(S^2 - 2C^2_{12}) + t_- \tilde{C}^2 + t_+ \tilde{S}^2 \right] \text{diag}(0, 1, 1) + \sqrt{\mathcal{F}^2 + \mathcal{G}^2} \tilde{\lambda}_8 \right\} \]

\[ \times e^{-i\Lambda} e^{i\psi \lambda_7} e^{-\frac{i}{2} \delta' \tilde{\lambda}_8} V^{-1}, \]  \tag{30}

to first order in \( |\Delta E_{43}/\Delta E_{32}| \) and \( |A(x)/\Delta E_{32}|. \) The effective MNS matrix in matter is given by \( V e^{i\delta' \tilde{\lambda}_8/2} e^{-i\psi \lambda_7} \) to zeroth order, and the three eigenvalues in matter are \(-\Delta E_{32}, \xi_+, \xi_-\), where \( \xi_{\pm} \equiv (t_- \tilde{C}^2 + t_+ \tilde{S}^2)(S^2_{\Delta E_{32}} - 2C^2_{12}) + t_- \tilde{C}^2 + t_+ \tilde{S}^2 \pm \sqrt{\mathcal{F}^2 + \mathcal{G}^2}, \) to first order in
\[ |\Delta E_{43}/\Delta E_{32}| \text{ and } |A(x)/\Delta E_{32}|, \text{ after subtracting the contribution from } C^2_{12}(t_-\bar{C}^2 + t_+\bar{S}^2)I_3. \]

The oscillation probability \( P(\nu_\mu \rightarrow \nu_\mu) \) is given by

\[
P(\nu_\mu \rightarrow \nu_\mu) = 1 - 4|V_{\mu 1}|^2(1 - |V_{\mu 1}|^2) \sin^2 \left( \frac{\Delta E_{32}L}{2} \right) - 4|e^{i\theta'} C_\psi V_{\mu 2} + e^{-i\theta'} S_\psi V_{\mu 3}|^2 - e^{i\theta'} S_\psi V_{\mu 2} + e^{-i\theta'} C_\psi V_{\mu 3}|^2 \sin^2 \left( \frac{(\xi_+ - \xi_-)L}{2} \right) \]

(31)

to the leading order in \( |\Delta E_{43}|/|\Delta E_{32}| \) and \( |A(x)|/|\Delta E_{32}| \), where \( C_\psi \equiv \cos \psi, S_\psi \equiv \sin \psi \) and \( V_{\mu j} \) are the matrix elements given by (18):

\[
V_{\mu 1} \equiv -S_{12}C_{23} - C_{12}S_{23}S_{13}e^{i\theta'} \\
V_{\mu 2} \equiv C_{12}C_{23} - S_{12}S_{23}S_{13}e^{i\theta'} \\
V_{\mu 3} \equiv S_{23}C_{13}
\]

(32)

with \( S_{jk} \equiv \sin \varphi_{jk}, C_{jk} \equiv \cos \varphi_{jk} \). We observe that (31) is invariant under \( \theta_{34} \rightarrow -\theta_{34} \) and \( \delta_1 \rightarrow \pi - \delta_1 \) since under these transformation the mixing angles in \( V \) behave as \( \varphi_{12} \rightarrow \varphi_{12}, \varphi_{13} \rightarrow -\varphi_{13}, (\bar{\varphi}_{13} \rightarrow -\bar{\varphi}_{13}), \varphi_{23} \rightarrow \varphi_{23}, \arg U_{sj} \rightarrow -\arg U_{sj} (j=2,3), \arg U_{j4} \rightarrow \pi - \arg U_{j4}, \delta' \rightarrow \pi - \delta', \psi \rightarrow -\psi, V_{\mu j} \rightarrow V^*_{\mu j} (j=1,2), V_{j3} \rightarrow V_{j3}, |e^{i\theta'} C_\psi V_{\mu 2} + e^{-i\theta'} S_\psi V_{\mu 3}|^2 \rightarrow |e^{i\theta'} C_\psi V_{\mu 2} + e^{-i\theta'} S_\psi V_{\mu 3}|^2, | - e^{i\theta'} S_\psi V_{\mu 2} + e^{-i\theta'} C_\psi V_{\mu 3}|^2 \rightarrow | - e^{i\theta'} S_\psi V_{\mu 2} + e^{-i\theta'} C_\psi V_{\mu 3}|^2. \]

Therefore, the oscillation probability is invariant under \( \theta_{34} \rightarrow -\theta_{34}, \delta_1 \rightarrow \pi - \delta_1 \) to zeroth order in \( |\Delta E_{43}/\Delta E_{32}| \) and \( |A(x)/\Delta E_{32}| \).
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Figures

Fig.1 (a), (b), (c) The allowed region in the ($\theta_{34}$, $\theta_{23}$) plane for various values of $\theta_{24} = 27^\circ, 30^\circ, \ldots, 58^\circ$ and (a) $\delta_1 = 0$ ($\Delta m^2_{43} = 10^{-29/10}eV^2$), (b) $\delta_1 = \pi/4$ ($\Delta m^2_{43} = 10^{-27/10}eV^2$), (c) $\delta_1 = \pi/2$ ($\Delta m^2_{43} = 10^{-29/10}eV^2$), respectively. The solid, dashed and dotted lines represent 68%, 90%, 99% confidence level, respectively. The asterisk in Fig. 1 (a) stands for the best fit point. The value of $\Delta m^2_{43}$ for each set of figures is that of the best fit point with GIVEN value of $\delta_1$. The best fit with a fixed value of $\delta_1 = 0, \pi/4, \pi/2$ is obtained for $\Delta m^2_{43} = 10^{-29/10}eV^2$, $10^{-27/10}eV^2$, $10^{-29/10}eV^2$, respectively.

Fig.2 (a), (b) The zenith angles dependence of (a) contained events and (b) upward going through muon events for $\Delta m^2_{43} = 10^{-29/10}eV^2$, $\theta_{24} = 50^\circ$, $\theta_{34} = 90^\circ$, $\theta_{23} = 30^\circ$ (fine dotted line), $\Delta m^2_{43} = 10^{-29/10}eV^2$, $\theta_{24} = 35^\circ$, $\theta_{34} = 15^\circ$, $\theta_{23} = 20^\circ$ (dashed line; best fit case) $\Delta m^2_{43} = 10^{-27/10}eV^2$, $\theta_{24} = 40^\circ$, $\theta_{34} = 0^\circ$, $\theta_{23} = 0^\circ$ (coarse dotted line; best fit case among pure $\nu_\mu \leftrightarrow \nu_\tau$ oscillations), no oscillation case (solid line), respectively. $\delta_1 = 0$ for all the three cases. The zenith angle dependence of the multi-GeV $\mu$-like events is different for the three sets of the oscillation parameters, but that of the upward going muon events is almost similar for the best fit case and the pure $\nu_\mu \leftrightarrow \nu_\tau$ case.

Fig.3 (a), (b), (c) The shadowed area is the allowed region projected on the ($\theta_{34}$, $\theta_{23}$) plane for various values of $\Delta m^2_{43}$ ($10^{-3.5}eV^2 \leq \Delta m^2_{43} \leq 10^{-2}eV^2$) for each value of $\theta_{24} = 27^\circ, 30^\circ, \ldots, 58^\circ$ and for (a) $\delta_1 = 0$, (b) $\delta_1 = \pi/4$, (c) $\delta_1 = \pi/2$, respectively, and the thin solid lines are boundary of the allowed region for various values of $\Delta m^2_{43}$. The solid, dashed, coarse dotted and fine dotted lines stand for the contours of $c_s^2 = |U_{s1}|^2 + |U_{s2}|^2 = |c_{23}c_{34} + s_{23}s_{24}s_{34}e^{i\delta_1}|^2 = 0.2, 0.4, 0.6, 0.8$, respectively. Solutions with $c_s < 0.2$ exist for $40^\circ \lesssim \theta_{24} \lesssim 50^\circ$ and for each value of $\delta_1$ and they can have Large Mixing Angle solutions of the solar neutrino problem.
$\theta_{24}=27^\circ$

$\theta_{24}=30^\circ$

$\theta_{24}=35^\circ$

$\theta_{24}=40^\circ$

$\theta_{24}=45^\circ$

$\theta_{24}=50^\circ$

$\theta_{24}=55^\circ$

$\theta_{24}=58.5^\circ$

Fig. 1 (a) $\delta_1=0$
Fig. 1 (b) \( \delta_1 = \pi/4 \)
$\theta_{24}=27^\circ$

$\theta_{24}=30^\circ$

$\theta_{24}=35^\circ$

$\theta_{24}=40^\circ$

$\theta_{24}=45^\circ$

$\theta_{24}=50^\circ$

$\theta_{24}=55^\circ$

$\theta_{24}=58.5^\circ$

Fig. 1 (c) $\delta_1=\pi/2$
no oscillation

\[ \Delta m^2_{43} = 10^{-29/10} \text{eV}^2, \theta_{24} = 50^\circ, \theta_{34} = 90^\circ, \theta_{23} = 30^\circ \]
\[ \Delta m^2_{43} = 10^{-29/10} \text{eV}^2, \theta_{24} = 35^\circ, \theta_{34} = 15^\circ, \theta_{23} = 20^\circ \]
\[ \Delta m^2_{43} = 10^{-27/10} \text{eV}^2, \theta_{24} = 40^\circ, \theta_{34} = 0^\circ, \theta_{23} = 0^\circ \]

**Fig. 2 (a)**
no oscillation

\[ \Delta m^2_{43} = 10^{-29/10} \text{eV}^2, \theta_{24} = 50^\circ, \theta_{34} = 90^\circ, \theta_{23} = 30^\circ \]

\[ \Delta m^2_{43} = 10^{-29/10} \text{eV}^2, \theta_{24} = 35^\circ, \theta_{34} = 15^\circ, \theta_{23} = 20^\circ \]

\[ \Delta m^2_{43} = 10^{-27/10} \text{eV}^2, \theta_{24} = 40^\circ, \theta_{34} = 0^\circ, \theta_{23} = 0^\circ \]

Fig.2 (b)
\( \theta_{24} = 30^\circ \)

\( \theta_{24} = 35^\circ \)

\( \theta_{24} = 40^\circ \)

\( \theta_{24} = 45^\circ \)

\( \theta_{24} = 50^\circ \)

\( \theta_{24} = 55^\circ \)

\( \delta_1 = 0 \)

\( C_s = 0.2 \)

\( C_s = 0.4 \)

\( C_s = 0.6 \)

\( C_s = 0.8 \)

Fig.3 (a)  \( \delta_1 = 0 \)
Fig. 3 (b) \( \delta_1 = \pi/4 \)
\[ \theta_{24} = 30^\circ \]

\[ \theta_{24} = 40^\circ \]

\[ \theta_{24} = 50^\circ \]

\[ \theta_{24} = 35^\circ \]

\[ \theta_{24} = 45^\circ \]

\[ \theta_{24} = 55^\circ \]

\[ \theta_{34} \]

\[ \theta_{34} \]

\[ C_s = 0.2 \]

\[ C_s = 0.4 \]

\[ C_s = 0.6 \]

\[ C_s = 0.8 \]

\[ \delta_1 = \pi/2 \]

**Fig. 3 (c)**