Entanglement in simple spin networks with a boundary

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Abstract: We investigate the bipartite entanglement for the boundary states in a simple type of spin networks with dangling edges, in which the two complementary parts are linked by two or more edges. Firstly, the spin entanglement is considered in the absence of the intertwiner entanglement. By virtue of numerical simulations, we find that the entanglement entropy usually depends on the group elements. More importantly, when the intertwiner entanglement is taken into account, we find that it is in general impossible to separate the total entanglement entropy into the contribution from spins on edges and the contribution from intertwiners at vertices. These situations are in contrast to the case when the two vertices are linked by a single edge.

Key words: entanglement, spin network, loop quantum gravity

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1 Introduction

Entanglement is the prominent phenomenon in quantum physics. Recently, it has been discovered that it also plays a key role in understanding the emergence of spacetime in the framework of holographic gravity[1]. On the one hand, the Ryu-Takayanagi (RT) formula provides a geometric description for the entanglement entropy of a subsystem on a boundary, which is measured by the area of the minimal surface in the bulk[2]. Such an area law is analogous to the Bekenstein-Hawking entropy for black holes. On the other hand, the behavior of quantum entanglement reflects the structure of the spacetime such that the background information can be extracted from the correlations of quantum states in a many-body system[3]. In particular, it turns out that the holographic properties of AdS spacetime can be captured by various types of tensor network states, such as multiscale entanglement renormalization ansatz (MERA)[4–8], perfect tensor networks[9], as well as hyperinvariant tensor networks[10–12].

Above attempts of investigating the structure of spacetime by entanglement are background dependent. In particular, the RT formula is proposed in the large N limit such that the perturbations in the bulk are controlled by the classical Einstein equations. It is quite intriguing to explore the role of quantum entanglement in the emergence of spacetime in a background independent manner, because the holographic nature of gravity is believed to be at the core of the quantum theory of gravity, which is beyond the large N limit of the gauge theory in standard AdS/CFT correspondence, where the bulk geometry is fixed and higher order corrections to gravity are greatly suppressed. When the gravity is strong enough, the dynamics of the bulk geometry can not be treated in a perturbative manner. One has to face the quantum nature of the background when building the geometry of the spacetime from the microscopic point of view by virtue of entanglement. In loop quantum gravity, it is well known that the geometry of spacetime itself can be quantized and the quantum states of the gravitational field are described by spin network states, which are SU(2) gauge invariant in four dimensional spacetime[13, 14]. Thus spin networks provide a very clear description of the atomic structure of the quantum geometry. In the traditional treatment, spin network states are mainly considered for closed graphs with fixed spins and intertwiners, such that they form a set of basis states in the Hilbert space of the gravitational field. It is clear that for a closed graph, a spin network is just a basis state without carrying any entanglement. Thus, in the past the entanglement structure of spin networks has rarely been addressed. Recently, the role of entanglement in building the geometry of spacetime has been revealed[3], and several publications on the relationship between quantum entanglement and spin networks have appeared[15–20].

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consider the superpositions of intertwiners and spins, or many spin network states, while the second is to consider the spin networks for an open graph with dangling edges. In ref. [19], the notion of spin networks has been extended to the non-closed graphs with dangling edges to describe the quantum geometry with a boundary, and the RT formula is understood in the coarse graining process. In this context, the SU(2) gauge invariance is only imposed on the internal vertex, while the uni-valent vertices linked to dangling edges are not gauge invariant. The associated degrees of freedom become physical on the boundary and are described by the boundary spin states. In ref. [19], the entanglement structure is investigated for a specific type of spin networks in which two neighboring vertices are linked by a single edge, and the notion of intertwiner entanglement is proposed. Moreover, the contribution from intertwiner entanglement at vertices and spin entanglement on edges are separated. Interestingly, one finds in this case that the spin entanglement from the edge, with irreducible representation \( j \), always contributes to the entanglement entropy with the term \( \ln(2j+1) \), which is independent of the group elements.

The separation of spin entanglement and intertwiner entanglement in a network looks peculiar if one recalls the nonlinear nature of entanglement entropy. One may speculate if it is always possible to separate the entropy into these two contributions in a general spin network. This clarification would improve our understanding of the structure of entanglement in spin networks. Therefore, in this paper we further develop the results of [19] by considering a more situation of spin network with dangling edges where two neighboring vertices are linked by two or more edges, in either direct or indirect manner. We investigate the bipartite entanglement entropy associated with the boundary degrees of freedom on dangling edges. Moreover, for simplicity, we perform numerical analysis for a simple type of spin networks containing two multi-valent vertices or several tri-valent vertices. We believe that the results are general enough and could be applicable to more complicated spin networks. We first consider the spin entanglement from edges in the absence of intertwiner entanglement. By virtue of numerical evaluations we demonstrate that, in general, the entanglement entropy depends on the group elements on edges, which has previously been pointed out in ref. [19]. Our numerical results imply that once the spins on edges are defined, bounds for the spin entanglement should exist. Secondly, we consider the bipartite entanglement entropy in the presence of intertwiner entanglement. In this case, we find that, in general, it is not possible to separate the total entropy into spin entanglement and intertwiner entanglement. Mathematically, it can not be written as a sum of two distinct parts any more. Our conclusions and outlook are given in the last section.

2 Entanglement in the absence of intertwiner entanglement

In this section, we evaluate the bipartite entanglement entropy for a few spin networks in the absence of intertwiner entanglement. First, we consider the case when the two neighboring vertices are linked by two edges directly. In general, a spin network is a graph \( \Gamma \) composed of edges and vertices, which could be closed or non-closed. The spin network state for a non-closed graph \( \Gamma \) with dangling edges \( o \) is denoted by \( | \Gamma, \{ j_e, j_o \}, \{ I_e \}, \{ M_o \} \rangle \), where \( j_e \) denotes the spin on the internal edge \( e \) and \( I_e \) denotes the intertwiner at internal vertex \( e \), while spin \( j_o \) and magnetic quantum number \( M_o \) are assigned to each dangling edge \( o \). The corresponding spin network function can be written as

\[
\langle \{ h_e \}, \{ h_o \} | \Gamma, \{ j_e, j_o \}, \{ I_e \}, \{ M_o \} \rangle = \sum_{m_e, n_e, m_o} \prod_e U^e_{m_e,n_e}(h_e) \prod_v (I_v)^{\{ j_v \}| m_v; n_v, m_o} \prod_o U^o_{m_o,M_o}(h_o),
\]

where \( h_e \) and \( h_o \) are holonomies along the internal edge \( e \) and dangling edge \( o \), respectively, and \( U^j \) is the matrix representation of SU(2) group with spin \( j \). This kind of spin networks is constructed for a spatial region with a boundary. The total Hilbert space is composed of the Hilbert space associated with the bulk \( H \) and the Hilbert space associated with the boundary \( \bar{H} \). Thus, a spin network state can be written as the direct product of two parts, namely \( | \Gamma, \{ j_e, j_o \}, \{ I_e \}, \{ M_o \} \rangle = \otimes | \Gamma^{(j_e;j_o)} | \otimes | j_o, M_o \rangle \). We define a boundary state \( | \bar{\Psi} \rangle | \Gamma, \{ j_e, j_o \}, \{ I_e \}, \{ h_o \} \rangle \) as \( (| \Gamma, \{ j_e, j_o \}, \{ I_e \}, \{ M_o \} | h_e \rangle, | h_o \rangle) \). Next, we study the bipartite entanglement in the boundary spin state \( | \bar{\Psi} \rangle | \Gamma, \{ j_e, j_o \}, \{ I_e \}, \{ h_o \} \rangle \).

For simplicity, we first consider a spin network with only two vertices \( A \) and \( B \). When the spins on edges are defined, the Hilbert space of the bulk is given by the products of two intertwiners,

\[
H_{AB} = H_A \otimes H_B
\]

where \( H_A \) and \( H_B \) are the spaces of intertwiners attached to two vertices \( A \) and \( B \), respectively,

\[
H_A = \text{Inv}_{SU(2)}[V_j^1 \otimes ... \otimes V_j^p \otimes V^{J_1} \otimes ... \otimes V^{J_n}]
\]

\[
H_B = \text{Inv}_{SU(2)}[V^{K_1} \otimes ... \otimes V^{K_q} \otimes V^{J_1} \otimes ... \otimes V^{J_n}],
\]

(3)
The numerical results for various spins are shown in Fig. 1. Firstly, we note that the entanglement entropy is not independent of the group elements any more; it is a function of the parameter \(\theta\). Secondly, we find that the entropy satisfies the bounds 
\[
|\ln(2j_1 + 1) - \ln(2j_2 + 1)| \leq E_{\rho_A}(\theta) \leq \ln(2j_1 + 1) + \ln(2j_2 + 1).
\]
In fact, in this special case, since there is only one dangling edge at each vertex, a stronger upper bound holds 
\[
E_{\rho_A}(\theta) \leq \min\{\ln(2j_1 + 1), \ln(2j_2 + 1)\}.
\]
networks. As an example, we consider the spin network shown in Fig. 2. The corresponding boundary state is given as
\[ |\Psi_{J_1,J_4,J_2,J_3}\rangle = \sum_{M_1} NC^{J_1 j_1 j_4}_{M_1 m_1 k_4} D^{m_1 n_1} U_{n_1}^{k_1*}(g(\theta)) C^{J_2 j_2 j_1}_{M_2 m_2 k_1} D^{m_2 n_2} U_{n_2}^{k_2*}(g(0)) C^{J_3 j_3 j_2}_{M_3 m_3 k_2} D^{m_3 n_3} U_{n_3}^{k_3*}(g(0)) C^{J_4 j_4 j_3}_{M_4 m_4 k_3} D^{m_4 n_4} U_{n_4}^{k_4*}(g(0)) |M_1 M_2 M_3 M_4\rangle, \tag{8} \]
where, for a bipartite system, we have chosen \( A = \{J_1, J_4\} \) and \( B = \{J_2, J_3\} \) and \( l = 1, \cdots 4 \). The reduced density matrix for the bipartite entanglement entropy is given as \( \rho_A = Tr_B(|\Psi_{J_1,J_4,J_2,J_3}\rangle \langle \Psi_{J_1,J_4,J_2,J_3}|) \). Numerical results for a few specific spins are shown in Fig. 2. We note that the entanglement entropy is generally a function of the parameter \( \theta \). In particular, when two parts are linked by two edges with spins \( J_1 \) and \( J_3 \), respectively, we find that the entropy is bounded as \( |\ln(2j_1+1) - \ln(2j_3+1)| \leq E_{\rho_A}(\theta) \leq \ln(2j_1+1) + \ln(2j_3+1) \).

3 Entanglement in the presence of intertwiner entanglement

In this section, we take the intertwiner entanglement into account. In ref. [19], it was shown that when two neighboring vertices \( A \) and \( B \) are linked by a single edge carrying a spin \( j \), as shown in Fig. 1, then the total entanglement entropy of the boundary states can be separated into two parts, one from intertwiner entanglement at vertices and the other from spin entanglement, which is nothing but \( \ln(2j+1) \), the maximal entropy allowed by the spin on the edge and independent of the group elements of the holonomy. We point out that the following relation plays a crucial role in the separation of spin entanglement and intertwiner entanglement; it is the orthogonal relation between two intertwiners
\[ \sum_{N_1 \cdots N_q} \langle I^{k_2} | N_1 \cdots N_q m \rangle \langle N_1 \cdots N_q m' | I^{k_2}\rangle = \frac{1}{2j+1} \delta_{k_2 k_2'} \delta_{m m'}, \tag{9} \]
where \( |I^{k_2}\rangle \) represents the \( k_2 \)-th component of the intertwiner state, and \( N_i \) \( (i = 1, \cdots , q) \) is the magnetic quantum number of the spin on the \( i \)-th dangling edge, while \( m \) is the magnetic quantum number of the spin \( j \) on the single
Fig. 3. (1) Two neighboring vertices $A$ and $B$ are linked by a single edge carrying a spin $j$. The dashed line denotes the entanglement of two interwiners. (2) The sketch of the orthogonal relation described by Eq.(9).

Fig. 4. (1) Two neighboring vertices $A$ and $B$ are linked by two edges carrying spin $j_1$ and $j_2$, respectively. (2) The extension of the orthogonal relation does not hold.

This orthogonal relation can be represented as a diagram, as shown in Fig.3(2). Obviously, this identity is applied during the evaluation of the reduced density matrix such that the final result can be written as a product of the spin contribution and the intertwiner contribution, as shown in Eq.(24) in [19]. However, when two vertices are linked by two or more edges, we find that this situation does not hold any more. In general, the bipartite entanglement entropy can not be separated into a spin part and an intertwiner part. For explicitness, we consider two vertices $A$ and $B$ linked by two edges carrying spin $j_1$ and $j_2$, respectively, as shown in Fig.4(1). For the evaluation of the reduced density matrix, we need to simplify the contractions of tensors. Unfortunately, we find that the following identity, need to separate the intertwiner entanglement from spin entanglement, does not hold,

$$\sum_{N_1 \cdots N_q} \left< t^{k_2} | N_1 \cdots N_q m_1 m_2 \right> \left< N_1 \cdots N_q m_1' m_2' | t^{k_2'} \right> \neq \frac{1}{(2j_1 + 1)(2j_2 + 1)} \delta_{k_2 k_2'} \delta_{m_1 m_1'} \delta_{m_2 m_2'}.$$  \hspace{1cm} (10)

This is diagrammatically sketched in Fig.4(2). We provide the proof for this statement in the Appendix. Similarly, one can show that such relations are also absent when two vertices are linked by more than two edges indirectly. Therefore, for a general spin network with dangling edges, it is not possible to separate the total entropy into the contributions from spins on the edges and from intertwiners at vertices.

The above orthogonal relation is not a necessary condition for separating the intertwiner indices and spin indices. However, we remark that, in a general case, they can not be separated if two of vertices are linked by more than one path. To support this statement, we evaluate the total entanglement entropy and the intertwiner entanglement entropy numerically for a few specific spin networks. An example is shown in Fig.5 and the boundary spin state...
the entanglement entropy, which is the reduced density matrix is

\[ \rho = C_{m_1 m_2}^{j_1 j_2} \rho_{n_1 n_2}^{j_1 j_2} D_{m_1 m_2}^{n_1 n_2} \]

where \( i = 1, 2 \) and \( k_1, k_2 \) are possible spins on virtual edges inside intertwiner A and B, respectively.

If we take all spins on the dangling edges to be \( \frac{1}{2} \), then the spins on virtual edges inside an intertwiner can be 0 and 1. With this assumption, the boundary state takes the following general form,

\[ |\Psi\rangle = \sum_{M, N} (\varphi_{00} C_{00}^{00} C_{01}^{11} + \frac{1}{3} \varphi_{01} C_{01}^{01} C_{11}^{11} + \frac{1}{3} \varphi_{11} C_{01}^{01} C_{11}^{11} + \frac{1}{3} \varphi_{10} C_{01}^{01} C_{11}^{11}) |M_1 M_2 N_1 N_2\rangle, \tag{11} \]

where \( \varphi_{00} \) and \( \varphi_{11} \) are two components in the intertwiner space. It should be noted that the other two components \( \varphi_{01} \) and \( \varphi_{10} \) do not appear in the above equation simply because the contraction of the corresponding CG coefficients in these terms vanishes.

The reduced density matrix for bipartition is given as \( \rho_{M_1 M_2} = Tr_{N_1 N_2}(|\Psi\rangle \langle \Psi|) \). It is straightforward to obtain the entanglement entropy, which is

\[ E = -Tr(\rho_{M_1 M_2} \ln \rho_{M_1 M_2}) \]

\[ = \frac{3|\varphi_{00}|^2}{3|\varphi_{00}|^2 + |\varphi_{11}|^2} \ln(|\varphi_{00}|^2) - \frac{|\varphi_{11}|^2}{3|\varphi_{00}|^2 + |\varphi_{11}|^2} \ln(|\varphi_{11}|^2) \]

\[ + \frac{|\varphi_{11}|^2 - 3|\varphi_{00}|^2}{3|\varphi_{00}|^2 + |\varphi_{11}|^2} \ln 3 \ln(3|\varphi_{00}|^2 + |\varphi_{11}|^2). \tag{13} \]

On the other hand, the intertwiner entanglement entropy is determined by the matrix \( \varphi_{k_1 k_2} \),

\[ \varphi_{k_1 k_2} = \begin{pmatrix} \varphi_{00} & \varphi_{01} \\ \varphi_{10} & \varphi_{11} \end{pmatrix}. \tag{14} \]

The reduced density matrix is \( \rho_{k_1} = \frac{\varphi_{k_1 k_2} \varphi_{k_1 k_2}^\dagger}{Tr(\varphi_{k_1 k_2} \varphi_{k_1 k_2}^\dagger)} \). The entanglement entropy between intertwiners is

\[ E_I = -Tr(\rho_{k_1} \ln \rho_{k_1}) = - (a_+ \ln a_+ + a_- \ln a_-), \tag{15} \]

where \( a_+ = (\frac{1}{3} + \frac{1}{2})^2 \frac{4|\varphi_{00}|^2 - |\varphi_{01}|^2 - |\varphi_{10}|^2 - |\varphi_{11}|^2)}{|\varphi_{00}|^2 + |\varphi_{01}|^2 + |\varphi_{10}|^2 + |\varphi_{11}|^2} \). In Tab.1, we evaluate the entanglement entropy \( E \) of the boundary spin state and the entanglement entropy \( E_I \) of intertwiners for a few specific values of intertwiner parameters. It manifestly indicates that the total entanglement entropy measured in boundary states can not be written as the sum of the spin contribution and the intertwiner contribution. For instance, in the fifth column of the table, the entanglement entropy between intertwiners is even larger than the entanglement entropy for the boundary state. In the last column, the total entanglement entropy of the boundary state is zero, but the entanglement of intertwiners is not. In the next-to-last column, “meaningless” means that the boundary state \( |\Psi\rangle \) vanishes. Finally, we remark that the total entanglement entropy is not larger than \( \ln 4 \) in all cases considered, simply because all dangling edges carry spin 1/2. In general, the bounds we found in the previous section do not hold any more when the intertwiner entanglement is involved.
4 Conclusions and outlook

In this paper, we have investigated the bipartite entanglement for the boundary spin states in spin networks with dangling edges. In particular, we have constructed a simple type of spin network in which two complementary parts are linked by two paths, either in a direct or indirect manner. The numerical evaluation of entanglement entropy leads to the following two main results. Firstly, in the absence of the intertwiner entanglement, the entanglement entropy for the boundary state depends on the group elements of the holonomy, which can not be simply determined by the spins $j_1$ and $j_2$ on the edges connecting the complementary parts. Nevertheless, we have proposed a bound for the entanglement entropy, which is $|\ln(2j_1+1) - \ln(2j_2+1)| \leq E \leq \ln(2j_1+1) + \ln(2j_2+1)$. It would be very important to prove or test this bound in a general case. Secondly, when the intertwiner entanglement is taken into account, the total entanglement can not be written, in general, as the sum of intertwiner entanglement and spin entanglement, but as a mixture of these two contributions.

Although we have only considered the simple case with two paths connecting two vertices, we believe that the above statements could be applicable to more complicated cases in which two vertices are linked by more than two edges directly, or by indirect paths.

Finally, based on our current work it is quite intriguing to further explore the relationship between quantum entanglement and quantum geometry, described by spin network states in loop quantum gravity. Our investigation is in progress and will be published in the near future[21].

5 Appendix

In this Appendix, we demonstrate the absence of the orthogonal relation for intertwiners when two vertices are linked by two edges, namely the inequality in Eq.(10), by applying the proof by contradiction. Assume that Eq.(10) is true. Let us consider the following contraction, which appears in the evaluation of the reduced density matrix

$$D = \sum_{N_1\cdots N_q} \langle T^{k_2}|N_1\cdots N_q m_1 m_2 \rangle \sum_{k_2'} \langle N_1\cdots N_q m_1' m_2'|T^{k_2'} \rangle \langle T^{k_2'}|M_1\cdots M_p m_1'' m_2'' \rangle.$$  \hspace{1cm} (16)
From Eq. (10), one can write Eq. (16) as,
\[
D = \frac{1}{(2j_1 + 1)(2j_2 + 1)} \delta_{m_1m_1'} \delta_{m_2m_2'} \langle I^{k_2}|M_1\cdots M_p m_1'' m_2'' \rangle.
\] (17)

For convenience, we define the operator \( \hat{P} = \sum |I^{k_2}\rangle \langle I^{k_2}| \), so that Eq. (16) can be rewritten as
\[
D = \sum_{N_1\cdots N_q} \langle I^{k_2}|N_1\cdots N_q m_1 m_2'\rangle \langle N_1\cdots N_q m_1' m_2'|\hat{P}|M_1\cdots M_p m_1'' m_2'' \rangle.
\] (18)

A diagrammatic sketch of Eqs. (16), (17) and (18) is shown in Fig. 6. We introduce the operator \( \hat{J}^2 = (\hat{J}_1 + \hat{J}_2)^2 \).

The action of operators \( \hat{J}_1 \) and \( \hat{J}_2 \) is defined as
\[
\langle m_1 m_2|\hat{J}_1|m_1' m_2'\rangle = \langle m_1|\hat{J}_1|m_1'\rangle \delta_{m_2 m_2'}
\]
\[
\langle m_1 m_2|\hat{J}_2|m_1' m_2'\rangle = \delta_{m_1 m_1'} \langle m_2|\hat{J}_2|m_2'\rangle,
\] (19)

where \( \hat{J}_i (j_i, m) = j_i (j_i + 1) |j_i, m\rangle \) \( (i = 1, 2) \). Next, we consider the following action of this operator on \( D \), which is denoted as \( F \) and shown in Fig. 7.

\[
F = \sum_{N_1\cdots N_q} \langle I^{k_2}|N_1\cdots N_q m_1 m_2'\rangle \sum_{m_1' m_2'} \langle m_1'' m_2''|\hat{J}^2|m_1' m_2'\rangle \langle N_1\cdots N_q m_1' m_2'|\hat{P}|M_1\cdots M_p m_1'' m_2'' \rangle.
\] (20)

On the one hand, by virtue of Eq. (17), Eq. (20) can be simplified as
\[
F = \frac{1}{(2j_1 + 1)(2j_2 + 1)} \langle m_1'' m_2''|\hat{J}^2|m_1 m_2'\rangle \langle I^{k_2}|M_1\cdots M_p m_1'' m_2'' \rangle.
\] (21)

On the other hand, from Eq. (18), we may rewrite Eq. (20) as
\[
F = \sum_{N_1\cdots N_q} \langle I^{k_2}|m_1 m_2 N_1\cdots N_q \rangle \sum_{m_1' m_2'} \langle m_1'' m_2''|\hat{J}^2|m_1' m_2'\rangle \langle N_1\cdots N_q m_1' m_2'|\hat{P}|M_1\cdots M_p m_1'' m_2'' \rangle.
\] (22)

Next, we prove that the operators \( \hat{J}^2 \) and \( \hat{P} \) commute with each other. For any \( |\Psi\rangle \in H_1 \otimes H_2 \otimes \cdots \otimes H_p \), we have \( \hat{P}|\Psi\rangle \in \text{Inv}_{SU(2)}[H_1 \otimes H_2 \otimes \cdots \otimes H_p] \). We also know that \( \hat{J}_1, \hat{J}_2, \hat{J}_3, \ldots, \hat{J}_p \) are commuting operators. Therefore, \( \hat{J}_1^2, \hat{J}_2^2, \ldots, \hat{J}_p^2 \) also commute with \( \hat{P} \).

Because \( |\Psi\rangle \) is arbitrary, we get \( \hat{P}\hat{J}^2 \hat{P} = \hat{J}^2 \hat{P} \hat{P} \). If we take its transposed-conjugate \( \hat{P}^\dagger \hat{J}^2 \hat{P} = \hat{J}^2 \hat{P}^\dagger \hat{P} \), we get \( \hat{J}^2 \hat{P} = \hat{P} \hat{J}^2 \), i.e. \( [\hat{P}, \hat{J}^2] = 0 \). With this fact, Eq. (22) becomes
\[
F = \sum_{N_1\cdots N_q} \langle I^{k_2}|N_1\cdots N_q m_1 m_2'\rangle \sum_{m_1' m_2'} \langle N_1\cdots N_q m_1' m_2'|\hat{P}|M_1\cdots M_p m_1'' m_2'' \rangle \langle m_1' m_2'|\hat{J}^2|m_1'' m_2'' \rangle.
\] (23)

With the help of Eq. (17) and Eq. (18), the above equation can be further simplified as
\[
F = \frac{1}{(2j_1 + 1)(2j_2 + 1)} \delta_{m_1 m_1''} \delta_{m_2 m_2''} \sum_{m_1' m_2'} \langle I^{k_2}|m_1' m_2' M_1\cdots M_p |m_1'' m_2'' \rangle \langle m_1' m_2'|\hat{J}^2|m_1'' m_2'' \rangle.
\] (24)

A diagrammatic sketch of Eqs. (20)–(24) is shown in Fig. 7.

If we contract both Eq. (21) and Eq. (24) with \( \langle M_1\cdots M_p m_1'' m_2'' |I^{k_2}\rangle \), we get
\[
\sum_{k_2, m_2', m_1''} \langle I^{k_2}|M_1\cdots M_p m_1'' m_2'' \rangle \langle M_1\cdots M_p m_1'' m_2'' |I^{k_2}\rangle = \delta_{m_1 m_1''} \delta_{m_2 m_2''}

\sum \langle I^{k_2}|M_1\cdots M_p m_1'' m_2'' \rangle \langle m_1' m_2'|\hat{J}^2|m_1'' m_2'' \rangle \langle M_1\cdots M_p m_1'' m_2'' |I^{k_2}\rangle,
\] (25)
where \( i = 1, 2 \) and \( l = 1, \cdots, p \). Although this equation looks complicated, as there exist \( k_2, M_1, \cdots, M_p, m_1'', m_2'' \), such that \( \langle I^{k_2} | M_1 \cdots M_p m_1'' m_2'' \rangle \neq 0 \), Eq. (25) is nothing else but

\[
\langle m_1''' m_2''' | \hat{J}^2 | m_1 m_2 \rangle = K \delta_{m_1'''} \delta_{m_2'''}
\]

\[
= K \langle m_1''' m_2''' | m_1 m_2 \rangle,
\]

where \( K \in \mathbb{C} \) is a constant. This means that \( \hat{J}^2 \) has only one eigenvalue \( K \). However, when \( j_1 \geq \frac{1}{2} \) and \( j_2 \geq \frac{1}{2} \), \( \hat{J}^2 \) has at least two different eigenvalues \( (j_1 + j_2)(j_1 + j_2 + 1) \) and \( |j_1 - j_2|(|j_1 - j_2| + 1) \). Therefore, our starting assumption is not true and the orthogonal relation as shown in Eq. (10) does not exist.

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