Application of MKSOR iteration with trapezoidal approach for system of Fredholm integral equations of second kind

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Abstract. This paper focuses on solving system of Fredholm integral equations of second kind numerically by using the first order quadrature scheme with Modified Kaudd Successive Over Relaxation (MKSOR) iterative method. The discretization process has been done by using the first order quadrature scheme, trapezoidal rule in order to construct trapezoidal approximation equation. A system of linear equations can be generated from this trapezoidal approximation equation. Three iterative methods were used to solve the system of linear equations which are Gauss-Seidel (GS), Successive Over Relaxation (SOR) and Modified Kaudd Successive Over Relaxation (MKSOR). The efficiency of these methods was analysed in solving three considered problems. Based on the numerical results, it can be pointed out that MKSOR method is superior to the SOR and GS methods in terms of execution time and complexity.

1. Introduction
Mathematical modelling in various fields such as science and engineering often leads to integral equation or system of integral equations. Integral equations involved in numerous applications such as queuing theory, population genetics, continuum mechanics, geophysics, quantum mechanics, fracture mechanics, fluid mechanics, potential theory, radiation, optimization, mathematical economics, medicine, acoustics and radiative heat transfer problems [1]. There are many types of integral equation. Consequently, a system of Fredholm integral equations of second kind which is one of the significant integral equations is considered in this paper. Generally, a system of Fredholm integral equations of second kind be defined as

\[ g_p(x) = Z_p(x) + \sum_{q=1}^{m} \int_{a}^{b} R_{pq}(x,s) g_q(s) ds \]

where \( g_p(x) \) is a known function, \( R_{pq}(x,s) \) is a Kernel function, \( Z_p \) and \( R_{pq} \) are continuous functions and \( g_q(x) \) is an unknown function [2].

There are many numerical methods have been studied by many researchers in solving problem of system of integral equations. Majeed [3] proposed modified midpoint method meanwhile Rasulov et al. [4] developed a new algorithm for solving a system of integral equations. Maleknejad et al. also applied some numerical methods for solving the problem (1) such as Block-Pulse functions, Rationalized Haar functions method, Taylor series expansion method and the method of collocation with Legendre polynomial [5-8]. Other methods that have been applied to solve the problem of system of integral equations are Open Newton-Cotes formula [2], the Adomian decomposition method [9-10], Bernstein polynomial [11], modified trapezoidal method [12] and Orthogonal triangular functions method [13].
However, the objective of this paper is to solve the system of Fredholm integral equations of second kind by using trapezoidal rule with MKSOR iteration.

2. Trapezoidal Approximation Equation

In this section, the discretization of problem (1) is discussed by using first order quadrature scheme namely trapezoidal rule. This discretization process produces the trapezoidal approximation equation. Prior to that, consider the general form of trapezoidal rule as follows

\[ \int_a^b g(s) \, ds = \frac{\Delta h}{2} (Z_a + Z_b) \]  

(2)

Let the interval \([a, b]\) be divided into several sets \(\{x_0, x_1, x_2, ..., x_n\}\) with the number of \((n)\) subintervals. Then, the integral function \(Z(x)\) on the interval \([a, b]\) can be expressed as

\[ \int_a^b Z(x) \, dx = \int_{x_0}^{x_n} Z(x) \, dx \]  

(3)

For illustration purpose, consider \(a = 0, b = n\) and \(i = 0, 1, 2, ..., n\), then equation (3) can be rewritten as

\[ \int_a^b Z(x) \, dx = \int_{x_0}^{x_1} Z(x) \, dx + \int_{x_1}^{x_2} Z(x) \, dx + \int_{x_2}^{x_3} Z(x) \, dx + ... + \int_{x_{n-1}}^{x_n} Z(x) \, dx + \int_{x_n}^{b} Z(x) \, dx \]  

(4)

By applying the trapezoidal rule over equation (4), it can be expressed as

\[ \int_{x_0}^{x_n} Z(x) \, dx = \frac{\Delta h}{2} (Z_0 + 2Z_1 + 2Z_2 + ... + 2Z_{n-1} + Z_n) \]  

(5)

From equation (5), the weights quadrature coefficient, \(A_j\) in general form of quadrature scheme which is

\[ \int_a^b g(s) \, ds = \sum_{j=0}^{n} A_j g(s_j) + \epsilon_n(g) \]  

(6)

can be expressed as \(A_j = \begin{cases} \frac{1}{2} \Delta h, & j = 0, n \\ \Delta h, & j = 1, 2, ..., n-1 \end{cases}\) where \(n\) is subintervals with equal width, \(\Delta h\) which can be defined as \(\Delta h = \frac{b-a}{(n)}\) and the error, \(\epsilon_n(g)\) be ignored.

Then, substitute \(x = x_i\) on the interval \([a, b]\) and construct a system of Fredholm integral equations of second kind by considering equation (1) with \(P = 1, 2\) and \(m = 2\). A system that consists two equations can be formed as follows

\[ g_1(x_i) = Z_1(x_i) + \int_a^b R_{11}(x_i, s)g_1(s) \, ds + \int_a^b R_{12}(x_i, s)g_2(s) \, ds \]  

(7)

\[ g_2(x_i) = Z_2(x_i) + \int_a^b R_{21}(x_i, s)g_1(s) \, ds + \int_a^b R_{22}(x_i, s)g_2(s) \, ds \]  

(8)

where \(i = 0, 1, 2, ..., n\). By imposing equation (5) into equations (7) and (8), the trapezoidal approximation equations for a system of Fredholm integral equations can be expressed as

\[ g_1(x_0) = \left( \frac{\Delta h}{2} \right) R_{11}(x_0, s_0)g_{1,0} + \Delta h R_{11}(x_0, s_1)g_{1,1} + \Delta h R_{11}(x_0, s_2)g_{1,2} + ... + \Delta h R_{11}(x_0, s_{n-1})g_{1,n-1} \]  

(9)

\[ g_2(x_0) = \left( \frac{\Delta h}{2} \right) R_{21}(x_0, s_0)g_{2,0} + \Delta h R_{21}(x_0, s_1)g_{2,1} + \Delta h R_{21}(x_0, s_2)g_{2,2} + ... + \Delta h R_{21}(x_0, s_{n-1})g_{2,n-1} \]  

(10)

For the simplicity, consider \(R_{pq}(x_i, s_j) = R_{pq}(i, j)\), \(g_p(s_j) = g_{pj}\) and \(Z_p(x_i) = Z_{pi}\). Furthermore, a linear system in matrix form can be generated by manipulating the approximate equations (9) and (10) and can be defined as

\[ R \bar{g} = \bar{Z} \]  

(11)

where
R = \begin{bmatrix} R_A & R_B \\ R_C & R_D \end{bmatrix}
\begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{2n} \end{bmatrix}
= Z, where
Z = \begin{bmatrix} Z_{1,0} & Z_{1,1} & \cdots & Z_{1,n} \\ Z_{2,0} & Z_{2,1} & \cdots & Z_{2,n} \end{bmatrix}.

\begin{align*}
\begin{bmatrix} R_A \\ R_B \\ R_C \\ R_D \end{bmatrix} \text{ are submatrices which are defined as follows:}

R_A &= \begin{bmatrix}
1 - \frac{\Delta h}{2} R_{11}(0,0) & -\Delta h R_{11}(0,1) & \cdots & -\frac{\Delta h}{2} R_{11}(0,n) \\
-\frac{\Delta h}{2} R_{11}(1,0) & 1 - \Delta h R_{11}(1,1) & \cdots & -\frac{\Delta h}{2} R_{11}(1,n) \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{\Delta h}{2} R_{11}(n,0) & -\Delta h R_{11}(n,1) & \cdots & 1 - \frac{\Delta h}{2} R_{11}(n,n) 
\end{bmatrix} \\
R_B &= \begin{bmatrix}
-\frac{\Delta h}{2} R_{12}(0,0) & -\Delta h R_{12}(0,1) & \cdots & -\frac{\Delta h}{2} R_{12}(0,n) \\
-\frac{\Delta h}{2} R_{12}(1,0) & -\Delta h R_{12}(1,1) & \cdots & -\frac{\Delta h}{2} R_{12}(1,n) \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{\Delta h}{2} R_{12}(n,0) & -\Delta h R_{12}(n,1) & \cdots & -\frac{\Delta h}{2} R_{12}(n,n) 
\end{bmatrix} \\
R_C &= \begin{bmatrix}
-\frac{\Delta h}{2} R_{21}(0,0) & -\Delta h R_{21}(0,1) & \cdots & -\frac{\Delta h}{2} R_{21}(0,n) \\
-\frac{\Delta h}{2} R_{21}(1,0) & -\Delta h R_{21}(1,1) & \cdots & -\frac{\Delta h}{2} R_{21}(1,n) \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{\Delta h}{2} R_{21}(n,0) & -\Delta h R_{21}(n,1) & \cdots & -\frac{\Delta h}{2} R_{21}(n,n) 
\end{bmatrix} \\
R_D &= \begin{bmatrix}
1 - \frac{\Delta h}{2} R_{22}(0,0) & -\Delta h R_{22}(0,1) & \cdots & -\frac{\Delta h}{2} R_{22}(0,n) \\
-\frac{\Delta h}{2} R_{22}(1,0) & 1 - \Delta h R_{22}(1,1) & \cdots & -\frac{\Delta h}{2} R_{22}(1,n) \\
\vdots & \vdots & \ddots & \vdots \\
1 & -\frac{\Delta h}{2} R_{22}(n,1) & \cdots & 1 - \frac{\Delta h}{2} R_{22}(n,n) 
\end{bmatrix}
\end{align*}

Clearly, R is the coefficient matrix, g and Z represent unknown vector and known vector respectively.

3. Derivation of Proposed Iterative Method

As mentioned above, the linear system in equation (11) will be solved iteratively by using GS, SOR and MKSOR iterative methods in order to get the value of the unknown function, \( g_q(x) \). The GS iterative method is known as a modification of the Jacobi iterative method in the sense of using the most recent calculated values [14]. Meanwhile, SOR iterative method is similar as GS method but involves the relaxation parameters \( \omega \in (0, 2) \) which allows for increased accuracy [14-15].

3.1. MKSOR Iterative Method

Youssef has introduced KSOR method [15]. The advantage of this method is that all calculations are done by updating the first component in the first equation of the first step. However, this method has been modified to form a new method known as a method MKSOR [16] as follows:

\begin{align*}
\begin{bmatrix} g_{i}^{(k+1)}(0,0) \\ g_{i}^{(k+1)}(0,1) \\ \vdots \\ g_{i}^{(k+1)}(n,0) \\
\end{bmatrix} &= \frac{1}{(1 + \omega^2)} \begin{bmatrix} 1 + \frac{\omega^2}{R_{ii}} \sum_{j=1}^{n} R_{ij} g_{j}^{(k+1)} - \sum_{j=1}^{n} R_{ij} g_{j}^{(k)} - \sum_{j=1}^{n} R_{ij} g_{j}^{(k+1)} \\
\end{bmatrix} \quad (13)
\end{align*}

where \( i = 0, 2, 4, ..., n - 2 \).
where \( i = 1, 3, 5, \ldots, n - 1 \).

4. Numerical Experiment

In order to evaluate the performance of GS, SOR and MKSOR iterative methods, there are three considered examples of system of Fredholm integral equations of second kind. Three parameters recorded in numerical comparison which are number of iterations (Iter), computational time in seconds (Time) and maximum error (Error). In addition to that, the considered tolerance error is \( \varepsilon = 10^{-16} \) and constant in various size grids.

Problem 1 [10]

\[
g_1(x) = \frac{2}{3} e^x - \frac{1}{4} + \int_0^1 \left( \frac{1}{3} e^s g_1(s) + s^2 g_2(s) \right) ds \tag{14}
g_2(x) = \frac{3}{2} x - x^2 + \int_0^1 (x^2 e^{-s} g_1(s) - x g_2(s)) ds \tag{15}
\]

The exact solutions for the system of Fredholm integral equations (11) are \( g_1(x) = e^x \) and \( g_2(x) = x \).

Problem 2 [10]

\[
g_1(x) = \frac{x}{16} + \frac{17}{36} + \int_0^1 \left( g_1(s) + g_2(s) \right) ds \tag{16}
g_2(x) = x^2 - \frac{19}{12} x + 1 + \int_0^1 x s (g_1(s) + g_2(s)) ds \tag{17}
\]

The exact solutions for the system of Fredholm integral equations (12) are \( g_1(x) = x + 1 \) and \( g_2(x) = x^2 + 1 \).

Problem 3 [2]

\[
g_1(x) = \frac{5}{6} x^2 - \frac{25}{12} x + 1 + \int_0^1 x (1 + s) g_1(s) ds + \int_0^1 x^2 s g_2(s) ds \tag{18}
g_2(x) = x^4 - \frac{1}{5} x^2 - \frac{7}{12} x + \int_0^1 x s g_1(s) ds + \int_0^1 (x^2 - x s) g_2(s) ds \tag{19}
\]

The exact solutions for the system of Fredholm integral equations (13) are \( g_1(x) = x^2 + 1 \) and \( g_2(x) = x^4 \).

All the results of numerical experiments regarding to the three systems of Fredholm integral equations of second kind in equations (14-19) were recorded in Table 1. The numerical results showed that the SOR and MKSOR iterative methods has reduced the number of iterations approximately 26.32%-69.40% compared to GS method in solving three considered problems. Meanwhile in term of computational time, the SOR iteration has reduced approximately 46.15% -48.98% whereas the MKSOR iteration has reduced approximately 22.92%-71.43% compared to the GS method. Clearly, it seems that MKSOR method has the fastest time compared to GS and SOR methods at various size grids. Therefore, it can be pointed out that the MKSOR method is more efficient than GS and SOR methods in terms of computational time and complexity in solving the problem of system of second kind Fredholm integral equations.

5. Conclusions

In this paper, MKSOR method has been successfully applied in solving system of second kind Fredholm integral equations. Firstly, the discretization process has been done by using the trapezoidal rule to derive the corresponding trapezoidal approximation equation. Then it leads to construct a linear system to be solved iteratively via GS, SOR and MKSOR iterative methods. By referring Table 1, the numerical results show that implementation of the MKSOR method solved the three test problems with fastest computational time. However, MKSOR method has similar number of iteration as SOR but these two iterative methods have less number of iterations compared to GS. The accuracy of these three iterative methods are in a good agreement. Future study can be extended by using higher order discretization scheme and block iterative methods as discussed in EG [17-18], EDG [19] and AGE [20].
Table 1. Comparison of number of iterations (Iter), computational time in seconds (Time) and maximum absolute error (Error) on iterative methods for three considered problems.

| M  | Problem 1          | Problem 2          | Problem 3          |
|----|-------------------|-------------------|-------------------|
|    | GS    | SOR   | MKSOR | GS    | SOR   | MKSOR | GS    | SOR   | MKSOR |
| 800| 19     | 14    | 14    | 134   | 41    | 41    | 33    | 17    | 17    |
| 1200| 19    | 14    | 14    | 134   | 41    | 41    | 33    | 17    | 17    |
| 1600| 19    | 14    | 14    | 134   | 41    | 41    | 33    | 17    | 17    |
| 2000| 19    | 14    | 14    | 134   | 41    | 41    | 33    | 17    | 17    |
| 2400| 19    | 14    | 14    | 134   | 41    | 41    | 33    | 17    | 17    |

| Time (second) | 800  | 1200 | 1600 | 2000 | 2400 |
|--------------|-----|------|------|------|------|
| 800          |     | 0.16 | 0.37 | 0.65 | 1.05 |
| 1200         | 0.37| 0.46 | 0.81 | 0.65 | 2.00 |
| 1600         | 0.65| 0.81 | 0.48 | 0.65 | 2.00 |
| 2000         | 1.05| 1.27 | 0.75 | 1.05 | 1.44 |
| 2400         | 1.44| 1.83 | 1.11 | 1.44 | 1.83 |

| Error | 800    | 1200   | 1600   | 2000   | 2400   |
|-------|--------|--------|--------|--------|--------|
| 800   | 2.1358e-03 | 2.1358e-03 | 2.1358e-03 | 4.3589e-03 | 4.3589e-03 | 4.3589e-03 | 2.0349e-03 | 2.0349e-03 | 2.0349e-03 |
| 1200  | 1.4239e-03 | 1.4239e-03 | 1.4239e-03 | 2.9036e-03 | 2.9036e-03 | 2.9036e-03 | 1.3568e-03 | 1.3568e-03 | 1.3568e-03 |
| 1600  | 1.0680e-03 | 1.0680e-03 | 1.0680e-03 | 2.1768e-03 | 2.1768e-03 | 2.1768e-03 | 1.0177e-03 | 1.0177e-03 | 1.0177e-03 |
| 2000  | 8.5444e-04 | 8.5444e-04 | 8.5444e-04 | 1.7410e-03 | 1.7410e-03 | 1.7410e-03 | 8.1416e-04 | 8.1416e-04 | 8.1416e-04 |
| 2400  | 7.1204e-04 | 7.1204e-04 | 7.1204e-04 | 1.4506e-03 | 1.4506e-03 | 1.4506e-03 | 6.7848e-04 | 6.7848e-04 | 6.7848e-04 |
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