Chiral Quantization on a Group Manifold

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Abstract

The phase space of a particle on a group manifold can be split in left and right sectors, in close analogy with the chiral sectors in Wess Zumino Witten models. We perform a classical analysis of the sectors, and the geometric quantization in the case of $SU(2)$. The quadratic relation, classically identifying $SU(2)$ as the sphere $S^3$, is replaced quantum mechanically by a similar condition on non-commutative operators (‘quantum sphere’). The resulting quantum exchange algebra of the chiral group variables is quartic, not quadratic. The fusion of the sectors leads to a Hilbert space that is subtly different from the one obtained by a more direct (un-split) quantization.

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1 Introduction

The theory of Wess–Zumino–Witten (WZW) models has received a lot of attention since its inception. One of the reasons is that also a quantum theory corresponding to it is well-known, viz. the theory of unitarizable representations of affine Kac–Moody algebras. In recent years the canonical structure of the classical model has been unraveled. It is fair to say that the transition between these two was not established by some standard quantization procedure, like deformation quantisation (DQ) or geometric quantization (GQu), but rather guessed. A simplified model, analogous to WZW models but with a finite dimensional phase space, has been presented as a toy model for conformal field theory. This model is based on the free motion of a particle on a group manifold. The classical theory is straightforward, and was worked out to some extent in for SU(2), whereas the proposals of for other groups were derived in . A quantum theory with the appropriate ‘classical limit’ was proposed in, again without systematic derivation.

In this paper we start from the classical model of a free particle moving on a compact group manifold. One may take it as an approximation of the kinematics of the WZW model itself, when one leaves out the fluctuations and keeps only the zero modes. We will reformulate this model in such a way that one of the main properties of the WZW model, viz. chiral splitting, is mimicked. In the WZW model it refers to the presence of left and right conserved currents, for the particle there are left and right conserved momenta. It is desirable to use both conserved quantities to parametrize the phase space. To this end, we split the phase space, and the original model arises from a symplectic reduction. After this splitting, the symplectic structures of both sectors are identical (up to a sign). Passing to the quantum theory, each sector can be quantized separately, and afterwards both sectors should be fused. For the quantization in each sector, we will use the geometric quantization procedure. In this way the quantum theory emerges as the result of a systematic procedure. The result after the fusion is close, but not identical, to the result announced in . For the chiral sectors separately, it was proposed that the exchange algebra of the group matrix elements is quadratic, but the relations we find are quartic instead. Also, we obtain in some sense a quantum sphere: The classical $S^3$ relation $a^a a + b^b b = 1$ remains valid in the quantum case, but only with the specific operator ordering
The second section treats the classical model. Its subdivisions treat subsequently the canonical phase space in the usual form, the (classically) equivalent viewpoint as a symplectic reduction of a chirally split extended phase space, the classical treatment of one of the chiral sectors, and finally, the detailed treatment of SU(2) as an example and a preparation for the quantum theory. Whereas the classical treatment can be given for any compact semi-simple Lie group, we restrict ourselves to SU(2) to carry the quantization program to completion. This is done in the third section, following geometric quantization methods. Here we obtain the quartic exchange algebra for the group elements. In section four, we show how to fuse the chiral sectors, thus obtaining the quantum version of the classical reduction. This necessitates the construction of the inverse matrix of the group element with operator entries (‘antipode’). The fifth section contains some discussion. Finally, in the appendix, we present, from a slightly different viewpoint, a more detailed exposition of the structures used in splitting and fusing.

2 The classical model

2.1 Canonical Phase Space

We start with a compact group $G$ as configuration space, and let $\mathcal{G}$ denote the Lie algebra of $G$. The phase space is the tangent bundle of $G$, with elements $(g, p)$, and the natural choice for the Lagrangian is the square of the velocity, leading to the equations of motion

$$\frac{d}{dt}(g^{-1}\frac{d}{dt}g) = 0.$$  \hspace{1cm} (1)

Defining a momentum corresponding to the trivialization of the bundle $TG$ by the lift of the left action of $G$ on itself,

$$p_t = g^{-1}\frac{d}{dt}g,$$  \hspace{1cm} (2)

one sees that it is conserved, since that action is a symmetry of the system. In terms of this momentum, the symplectic form is the differential of the following Liouville one-form

$$\alpha = K(p_t, g^{-1}dg).$$  \hspace{1cm} (3)
We can use the conserved momentum to parametrize the solutions:

\[ g(t, g_o) = g_o \exp(pt). \] (4)

In the same way, one can also introduce a (conserved) right momentum
\[ p_r = -\frac{d}{dt}g g^{-1}, \] and a second parametrization of the solution
\[ g(t, g_o) = \exp(-p_r t)g_o. \] (5)

### 2.2 Split Phase Space

Having a large set of global conserved quantities at one’s disposal one naturally wants to use them all. In the WZW models, in a similar way, conserved left and right currents are both present. In [3], this was used to parametrize the solutions (their equations (2) and (3) ) as

\[ g = P g_0 \tilde{P} \] (6)

where \( P \) and \( \tilde{P} \) are path–ordered integrals of the chiral currents, and \( g_0 \) fixes the initial condition. In this formula, the values of the left and right currents are not completely independent: this corresponds to the well–known fact that the representation of left and right zero modes, or the monodromy, should be the same. The exponentials should in fact be in the same conjugacy class. This fact complicates the description of phase space using both left and right conserved currents. This feature is reproduced in the particle model, when we try to use both left and right momenta the particle phase space. From eqs. (4) and (5) it is clear that, when \( p_l \) and \( p_r \) parametrize the same solution, they are in the same conjugacy class (up to a sign)

\[ p_r = -Ad_{g_0}p_l. \] (7)

Let us try to describe this interdependence. The conjugacy classes (adjoint orbits) are parametrized by the elements of an arbitrarily chosen (but fixed) Weyl chamber \( W \). Now take some values for \( p_l \) and \( p_r \), and let \( w \in W \) be the element characterizing their orbit. Then, to reconstruct \( p_l \) from \( w \), we need in addition some group element \( \sigma_l \) such that \( p_l = Ad_{\sigma_l} w \). This group element is determined up to multiplication (from the right) by the stabilizer of \( w \). For generic (regular) \( w \), this stabilizer is isomorphic to the
maximal torus $T$ (fixed by the choice of Weyl chamber) of $G$. So instead of this group element, it is sufficient to give its right coset, i.e. an element of $G/T$. To restore $p_r$, we need another independent element of the coset space, $p_r = -Ad_{\sigma_r}w$. We can conclude that the space of momenta is isomorphic to $W \times G/T \times G/T$. In order to reconstruct $g$, we have to give some element $h$ of the torus stabilizing $W$. then $g = \sigma_r h \sigma_r^{-1}$. Consequently for the whole phase space, we have locally

$$TG \approx (W \times G/T \times G/T) \times T. \quad (8)$$

The details of this construction can be found in the appendix.

In this picture, the symplectic form induced by eq. (3) on $G/T$ is the symplectic structure of [12], while $W$ and $T$ are canonically conjugated. This suggests that, by doubling the Weyl chamber $W$ and the torus $T$ in the description eq. (8), one can realize this system as a quotient of two symplectic manifolds, each being a copy of $W \times G$. This is not quite trivial, since one has to show first that the local description as $G/T \times T$ fits together to $G$ (which is demonstrated in the appendix), and second that a suitable symplectic reduction can be found. We now describe this product structure ('splitting') and the symplectic reduction ('fusion').

The extended phase space $P$ is a product $P_l \times P_r$, where $P_{l,r} = W_{l,r} \times G_{l,r}$ are two copies the same manifold. They are equipped with exact symplectic forms, which are exterior derivatives, $\omega_{l,r} = d\theta_{l,r}$ of the one-forms

$$\theta_{l,r} := K(w_{l,r}, g_{l,r}^{-1} dg_{l,r}). \quad (9)$$

The symplectic structure on $P$ is given as the difference $\omega_l - \omega_r$. From this expression it is evident that the chiral sectors are canonically independent, i.e. the functions on the right sector Poisson-commute with the functions on the left one. The original phase space $TG$ is restored by symplectic reduction induced by the (first class) constraints identifying the Weyl chambers, $w_l - w_r = 0$. The coordinates on the product torus $T_l T_r$ are canonically conjugated to these constraints, and consequently the two copies of $T$ are identified as well. Explicitly, the original variables $g$ and $p$ are given in terms of the extended phase space variables $\{w_l = w, g_l; w_r = w, g_r\}$ by

$$g = g_t g_r^{-1}, p = Ad_{g_r}w \quad (10)$$

This explains the mechanism of this reduction.
2.3 Classical Chiral Sector

Here we describe in more detail the structure of a single sector (and will omit the subscript). We will express the symplectic form in terms of Lie algebraic data.

We use the following notation (13). The set of simple roots dual to the chosen Weyl chamber (see page 3 and the appendix), is $\Delta$, the set of roots of the Lie algebra is $\Phi$, and the set of positive roots is $\Phi^+$. The element of the Cartan subalgebra $K$-dual to the root $\beta$ is $t_\beta = i[e_\beta, e_{-\beta}]$. In addition we introduce $\theta^{\alpha_i}$, the one–form dual to the simple root $t_{\alpha_i}$, and $\omega^\beta$, the left invariant one–form dual to the root vector $e_\beta$. Finally, $w_i$ is the coordinate in the Weyl chamber in the basis dual to the one formed by $t_{\alpha_i}$.

The left–invariant one–form $g^{(-1)}dg$ is

$$i \sum_{\alpha_i \in \Delta} \theta^{\alpha_i} t_{\alpha_i} + \sum_{\beta \in \Phi^+} \omega^\beta e_\beta.$$  

(11)

Taking into account the commutation relations and the orthogonality properties of the root subspaces in the Lie algebra, one can rewrite the symplectic form eq.(93) as

$$\omega = \sum_{\alpha_i \in \Delta} dw_i \wedge \theta^{\alpha_i} + i \sum_{\beta \in \Phi^+} K(w, t_\beta) \omega^\beta \wedge \omega^{-\beta}.$$  

(12)

From this, the Poisson brackets follow immediately. The functions on the group are generated by the matrix elements of all representations [14]. Let $M_i(g) (i = 1, 2)$ be the matrix corresponding to some chosen representation 'i', and $m_i$ the corresponding matrix representation of the Lie algebra. The general Poisson brackets of these generators is given by

$$\{M_1, \otimes M_2\}(g) = M_1(g) \otimes M_2(g) r_{12}$$  

(13)

where one takes the tensor product of matrices, and the Poisson bracket of the entries. In eq.(13), we used the exchange operator

$$r_{12} = \sum_{\beta \in \Phi^+} \frac{i}{K(w, t_\beta)} [m_1(e_{-\beta}) \otimes m_2(e_\beta) - m_1(e_\beta) \otimes m_2(e_{-\beta})].$$  

(14)

This formula generalizes the one presented in [11] (3.24), where it is derived for the specific case where both '1' and '2' are in the same representation. The following completes the Poisson bracket relations:

$$\{w_i, M\}(g) = M(g)m(t_{\alpha_i}).$$  

(15)
The derivation of the above Poisson brackets becomes easy if one expresses the exterior derivative of $M$ in terms of the differentiations along the left invariant vector fields $t^L_{\alpha}, e^L_{\alpha}$ and $e^L_{-\alpha}$ corresponding to the chosen basis elements of $G$:

$$dM(g) = M(g) \left[ \sum_{\alpha_i \in \Delta} m(t_{\alpha_i}) \theta^{\alpha_i} + \sum_{\alpha > 0} m(e_{\alpha}) \omega^\alpha + m(e_{-\alpha}) \omega^{-\alpha} \right].$$

(16)

This formula follows from the definition of the left invariant vector field $x^L$ corresponding to $x \in G$, $(x^L f)(g) = \frac{d}{dt} f(ge^{tx})|_0$. In particular, for the matrix function we have $(x^L M)(g) = M(g) m(x)$. The equation for the hamiltonian vector field $x^M$ of the function $M$, $i(x^M) \omega + dM = 0$, takes a very simple form if $x^M$ is expanded in terms of the left invariant basis. The solution is given by

$$x^M(g) = M(g) \left[ \sum_{\alpha > 0} \frac{i}{K(w, t_{\alpha})} [m(e_{-\alpha}) e^L_{\alpha} - m(e_{\alpha}) e^L_{-\alpha}] - \sum_{\alpha_i \in \Delta} m(t_{\alpha_i}) \frac{\partial}{\partial w^i} \right]$$

and eqs. (13,15) follow from the definition of the Poisson bracket.

As demonstrated, the classical treatment can be given quite generally. For the quantisation, we will not be able to work it out completely, but will restrict ourselves to $SU(2)$. For that reason, and to provide an explicit example for the above, we treat this case in detail in the next subsection.

### 2.4 The $SU(2)$ case

We shall identify $TSU(2)$ with $S^3 \times su(2)$, where $su(2)$ is the space of $2 \times 2$ antihermitian complex matrices. This space is spanned by $l_i := \frac{i}{2} \sigma_i$, where $\sigma_i$ are the Pauli matrices. The invariant form $K$ is simply proportional to a trace of a product:

$$K(p, \tilde{p}) := -2Tr(p\tilde{p}).$$

(18)

Now we parametrize the space of orbits. The Weyl chamber we choose is

$$W := \{ \tilde{w} : \tilde{w} = wl_3; w > 0 \}$$

(temporarily denoting the Weyl chamber element by $\tilde{w}$ to avoid confusion with the real value $w$). The remaining information about momenta is contained
in local sections $\sigma$ (86) of $G \rightarrow G/T$. Take $p$ and $\tilde{p}$ in the same conjugacy class. Then there exist sections $\sigma, \tilde{\sigma}$ such that

$$p = \text{Ad}\sigma wl_3, \quad \tilde{p} = \text{Ad}\tilde{\sigma} wl_3.$$ (20)

In accord with the approach presented in the previous section we will now focus our attention on the left sector only, and we will for convenience call this sector $(P, \omega)$.

Take the stereographic parametrization of $G/T \simeq S^2$. It is defined by two neighbourhoods $U_\pm$, which are identified with two complex planes with the transition $z_+ = z_-^{-1}$ on $U_+ := U_+ \cap U_-$. Take two local sections $\sigma_\pm : S^2 \rightarrow SU(2)$:

$$\sigma_+ = f_+ \begin{bmatrix} 1 & iz_- \\ iz_+ & 1 \end{bmatrix}, \quad \sigma_- = f_- \begin{bmatrix} -i\bar{z}_- & -1 \\ 1 & i\bar{z}_- \end{bmatrix},$$ (21)

where $f_\pm = f(z_\pm), f(z) := (1 + z\bar{z})^{-1/2}$. These cover the two neighbourhoods $O_\pm := \{p : w \pm p_3 \neq 0\} \subset \beta^{-1}(wl_3); \quad r \in \mathbb{R}_+$ (22)

of the adjoint orbit by $\sigma_\pm wl_3 \sigma^{-1}_\pm = p$. Any group element $g$ can be represented locally as

$$g_\pm = \sigma_\pm e^{i\sigma_3 q_\pm},$$ (23)

and on $U_+-$ we have

$$\sigma_- e^{iq_+ \sigma_3} = e^{iq_- \sigma_3}, \quad \sigma_+ = |z_+|^{-1} \begin{bmatrix} iz_+ & 0 \\ 0 & -iz_+ \end{bmatrix},$$ (24)

and consequently

$$g_+ = \sigma_+ e^{iq_+ \sigma_3} = \sigma_- e^{iq_- \sigma_3} = g_-,$$ (25)

so that the group element is well defined globally. Having displayed the above transition maps once, we will drop the $(\pm)$ indices, but emphasize that our calculations are globally true.

We represent an element $g \in SU(2)$ by a $2 \times 2$ matrix:

$$g \equiv \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix}, \quad a^* a + b^* b = 1.$$ (26)
From (23) it follows that \(a\) and \(b\) have the following local coordinate expressions:

\[
a = fe^{iq}; \quad b = i\bar{z}fe^{iq}
\]  

(27)

The ('left') Louville form \(\theta = K(\bar{w}, g^{-1}dg)\), where \(\bar{w} = wl_3\), is locally equal to:

\[
\theta = wdq + iw\frac{1}{2}f^2(\bar{z}dz - zd\bar{z}),
\]  

(28)

and the symplectic form reads:

\[
\omega := d\theta = dw \wedge dq + \frac{i}{2}f^2dw \wedge (\bar{z}dz - zd\bar{z}) + iw f^4(d\bar{z} \wedge dz).
\]  

(29)

Having a local expression for \(\omega\) we can find (local) expressions for the (global) hamiltonian vector fields

\[
X_a = e^{iq}(\frac{1}{2w} | z |^2 f \partial_q - if \partial_w + iz \frac{1}{fw} \partial z)
\]

\[
X_b = e^{iq}(-\frac{i}{2w} \bar{z}f \partial_q + \bar{z}f \partial_w + \frac{1}{fw} \partial_z)
\]

\[
X_w = \partial_q
\]  

(30)

and the Poisson brackets \(\{f, g\} := X_f g\):

\[
\{a^*, a\} = \frac{i}{w}bb^*, \quad \{a, b\} = 0 , \quad \{a, b^*\} = \frac{i}{w}aa^*, \quad \{b^*, b\} = -\frac{i}{w}aa^*,
\]  

(31)

\[
\{a, w\} = -ia , \quad \{a^*, w\} = ia^*, \quad \{b, w\} = -ib , \quad \{b^*, w\} = ib^*.
\]  

(32)

There is an obvious generalization of the above algebra. Recall from the previous section that the definitions of Liouville and symplectic forms (eq.(93)) depend on an identification of the space of orbits with a fixed Weyl chamber. We are free to choose this Weyl chamber in an arbitrary way. If we take \(Ad_{g_0}l_3\) instead of \(l_3\) in eq.(92), the Liouville form \(\theta\) is

\[
\theta = K(wg_0l_3\bar{g}_0^{-1}, \omega),
\]  

(33)
where \( g_0 \equiv \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} \) is an arbitrary element of \( SU(2) \). The symplectic form can be computed from \( \omega := d\theta \) and the resulting Poisson algebra in general depends on \( g_0 \):

\[
\begin{align*}
\{ a, a^* \} &= -\frac{i}{w} c_1 b b^* - \frac{i}{2w} (a b \bar{c}_2 + a^* b^* c_2) \\
\{ b, b^* \} &= -\frac{i}{w} c_1 a a^* + \frac{i}{2w} (a b \bar{c}_2 + a^* b^* c_2) \\
\{ a, b \} &= \frac{i}{w} c_2 \\
\{ w, a \} &= ic_1 a - ic_2 b^* \\
\{ w, b \} &= ic_1 b + ic_2 a
\end{align*}
\]

(34)

where

\[
\begin{align*}
c_1 &= |\alpha|^2 - |\beta|^2; \\
c_2 &= 2\alpha^* \beta; \\
c_1^2 + |c_2|^2 &= 1.
\end{align*}
\]

(36)

This is a generalization of the algebra of eqs. (31) and (32), to which it is obviously equivalent. The former algebra is preferable to carry through the quantisation in the next section, because it is the only one for which the functions \( a \) and \( b \) on the group (Poisson) commute.

One can easily check that the change of Weyl chamber described in eq. (33) is equivalent to the right action of \( g_0 \) on \( G \). It is natural to ask whether there exists a Poisson structure on \( G \) such that the action

\[
G \times P \ni (g_0, (w, g)) \rightarrow (w, gg_0^{-1}) \in P
\]

preserves the relations of eqs. (31-32). The answer is affirmative for the quadratic relations eq. (31), if one imposes the following Poisson relations on the group parameters \( \alpha \) and \( \beta \):

\[
\begin{align*}
\{ \alpha, \beta \} &= -\frac{i}{w} \alpha \beta \\
\{ \alpha^*, \beta^* \} &= \frac{i}{w} \alpha^* \beta^*; \\
\{ \alpha, \beta^* \} &= -\frac{i}{w} \alpha \beta^* \\
\{ \alpha^*, \alpha \} &= \frac{2i}{w} \beta^* \beta; \\
\{ \beta^*, \beta \} &= 0.
\end{align*}
\]

(38)

These brackets define a one–parameter family of \( SU(2) \) Lie–Poisson groups, which was studied in [14]. It can be identified as a one-parameter family
of classical versions of the algebra of Woronowicz \[16, 17\]. As far as the
commutation rules with \( w \) are concerned, eq.(32), the above action of the
Lie–Poisson group eq.(38) does not preserve the Poisson brackets of \( w \) and
\( a, b \).

We close the classical treatment of \( SU(2) \) with a remark. We may think
about the commutative ring generated by \( a, b, a^*, b^* \) and the single relation
\( aa^* + bb^* = 1 \) as the algebra of functions on the manifold of \( SU(2) \). It is not
useful to think about this manifold as a group equipped with a multipli-
cation. The structure that is actually used, and that is compatible with the
symplectic structure, is that of a manifold with a simply transitive action
of the group on it. Only the left action of \( SU(2) \) on the manifold is consistent
with the Poisson algebra of the functions \( a, b, a^*, b^* \) and the Weyl chamber
variable \( w \).

3 Quantum Chiral Sector

Generally speaking, one may apply a variety of methods to quantize a given
classical system. Among the rather unambiguous methods, we mention de-
formation quantization \[6, 18\] and geometric quantization (GQu) \[7, 8\]. We
will use the second method (the simplest version \[19, 8\] is sufficient), because
it gives not only the quantum algebra of operators, but also the Hilbert space
representation.

First we have to build a complex line bundle over the phase space with
a connection, the curvature of which is proportional to the symplectic form
\( \omega \). This step is very simple because \( \omega \) is exact, and consequently the line
bundle is trivial. We can take a global section \( \lambda_0 \) and define the connection
by introducing the following covariant differential operator

\[
D\lambda_o = -\frac{i}{\hbar} \theta \otimes \lambda_o \tag{39}
\]

where \( \hbar \) is the Planck constant divided by \( 2\pi \) and \( \theta \) is the Liouville form.
Since \( \theta \) is real, we can normalize the section by the condition

\[
H(\lambda_o, \lambda_o) = 1, \tag{40}
\]

which uniquely defines the hermitian form \( H \) on the line bundle and fixes
the scalar product of the sections (pre-quantum wave-functions \[4, 8\]).
As is well-known, the pre-quantum wave-functions depend on all phase space variables and consequently they do not yet give an adequate quantum description of the physical system: it is necessary to find the space of wave-functions depending on, roughly speaking, the spectrum of a maximal algebra of commuting observables. Technically this is achieved by choosing a polarization, i.e. an involutive lagrangian distribution $F \subset TP$, and imposing the condition on the sections to be covariantly constant along $F$.

In our case we want to represent the algebra of functions on the group manifold $G$. Then it is natural to take $F$ to be spanned by the two vector fields corresponding to the Poisson commuting functions $a$ and $b$:

$$F := \text{span}\{X_a, X_b\}.$$  \hfill (41)

This way, the variable $w$ turns out not to be independent, but the corresponding operator will be a function of $a, b, a^*, b^*$. The non-commutative algebra formed by the operators $a, b$ and their conjugates (which as we shall see will generate the spectrum of the system) will correspond to the "algebra of functions" on a 'quantum' $S^3$ manifold, but not on a quantum $SU(2)$ group in the sense of Hopf algebras.

With this polarization, the wave functions

$$\Psi = \varphi \lambda_o$$  \hfill (42)

have to satisfy the conditions

$$\nabla_{X_a, b} \Psi = 0$$  \hfill (43)

which are equivalent to

$$\frac{\partial \varphi}{\partial w} = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial z} - \frac{i}{2} (1 + z \bar{z})^{-1} \bar{z} \frac{\partial}{\partial q}\right) \varphi = 0.$$ \hfill (44)

They are satisfied by any superposition of the following fundamental solutions

$$\varphi_k = (1 + z \bar{z})^{-k/2} e^{ikq} h(\bar{z}) \quad \text{(no sum over} \ k),$$  \hfill (45)

where $h$ is a holomorphic function of $\bar{z}$. The sections are then given by

$$s_k = \varphi_k \otimes \lambda_o.$$  \hfill (46)

\footnotetext{The other choice is to take $a^*$ and $b^*$, which for a single sector amounts to a change in sign of $w$. The choice in the other sector (see further) then should be adapted to this.}
To normalize these wave functions, one can not use the symplectic scalar product defined by the density $|\omega|^2$, since it includes integration of $w$-independent functions along the non-compact Weyl chamber. Within the framework of geometric quantization, there are two methods of dealing with this difficulty. The first multiplies the wave-functions by half-densities [8, 20], the second one uses half-forms [19, 8]. In our case, after application of the half-form method, one obtains the following representation for the scalar product:

$$(s, s') = \int_{S^3} (1 + z\bar{z})^{-2} \varphi \varphi' | d\bar{z} \wedge dz \wedge dq |.$$  

(47)

Following the standard procedure of GQu, the operators corresponding to the classical functions $a$ and $b$ are (by our choice of polarisation) simply multiplication operators:

$$a \rightarrow \hat{a} = \text{mult } a$$

$$b \rightarrow \hat{b} = \text{mult } b ,$$  

(48)

whereas

$$w \rightarrow \hat{w} = h(-i \frac{\partial}{\partial q} + 1).$$  

(49)

Now let us analyse the detailed form of the wave-functions eq.(46). Let $s_k = (1 + z\bar{z})^{-k/2}e^{ikq}h_\lambda o$, where $k \in \mathbb{Z}$ has some fixed value and $\hat{h} = h(\bar{z})$ is holomorphic. Let us transform $s_k$ from the neighbourhood $U_-$ to $U_+$ (see eq.(24)):

$$s_k(q_-, z_-, \bar{z}_-) = (1 + z_-\bar{z}_-)^{-k/2} \exp(ikq_-)h(z_-f)\lambda o =$$

$$= |z_+|^{k} (1 + z_+\bar{z}_+)^{-k/2} \exp(ikq_+)(\frac{\bar{z}_+}{z_+})^k h(\bar{z}_+^{-1})\lambda o$$

$$= (1 + z_+\bar{z}_+)^{-k/2} \exp(ikq_+)z_+^k h(\bar{z}_+^{-1})\lambda o.$$  

(50)

Clearly in order to keep the holomorphic property of the section $s_k$ one should restrict $h$ to be a polynomial of degree not higher than $k$ and $k$ has to be non-negative. This argument is of a purely geometrical nature. One can arrive at the same conclusion for the degree of $h$ using a normalisation condition: the sections eq.(46) have finite norm with respect to the scalar product eq.(47) if and only if the above condition on $h$ is satisfied. Consequently the vectors

$$s_{k,j} := e^{ikq}(\bar{z})^j(1 + z\bar{z})^{-k/2}\lambda o ; \quad j \leq k$$  

(51)
form a basis in the Hilbert space of quantum states. For the scalar product of the basis elements we get:

\[
(s_{k,n}, s_{m,l}) = \int e^{(m-k)q(z)(\bar{z})^l}(1 + z\bar{z})^{-(m+k+4)/2} |d\bar{z}dzdq| = (2\pi)^2 \delta_{m,k} \delta_{j,l} \left[(\frac{k}{j})(k + 1)\right]^{-1}.
\]

(52)

We normalize the states and relabel them according to the rule:

\[
s_{k,n} \Rightarrow \psi_{j,j_3}
\]

(53)

where

\[
j = \frac{k}{2}; \quad j_3 = \frac{k}{2} - n.
\]

(54)

The values of \(j\) are then non-negative half-integers and for fixed \(j\) the value of \(j_3\) ranges from \(-j\) to \(j\) step 1.

The action on these states by the quantum operators corresponding to the generators of the algebra of functions on \(S^3\) is then given as (here and in the sequel we identify \(a\) with \(\hat{a}\) etc):

\[
a\psi_{j,j_3} = \left(\frac{j + j_3 + 1}{2j + 2}\right)^{1/2} \psi_{j + \frac{1}{2}, j_3 + \frac{1}{2}}
\]

\[
a^*\psi_{j,j_3} = \left(\frac{j + j_3}{2j + 1}\right)^{1/2} \psi_{j - \frac{1}{2}, j_3 - \frac{1}{2}}
\]

\[
b\psi_{j,j_3} = \left(\frac{j - j_3 + 1}{2j + 2}\right)^{1/2} \psi_{j + \frac{1}{2}, j_3 - \frac{1}{2}}
\]

\[
b^*\psi_{j,j_3} = \left(\frac{j - j_3}{2j + 1}\right)^{1/2} \psi_{j - \frac{1}{2}, j_3 + \frac{1}{2}}.
\]

(55)

The \(a^*\) operators can most easily be obtained from the fact that the quantization method, including the scalar product eq. \([52]\), guarantees that it is the hermitian conjugate of \(a\). The other method, using a varying polarization, is much more involved.

Notice that \(\Psi = \psi_{0,0}\) is the unique state annihilated by both \(a^*\) and \(b^*\). The whole Hilbert space of states can be generated by the successive action of \(a, b, a^*, b^*\) operators, hence we will call their algebra the spectrum generating algebra (SGA). We can find explicitly the relations satisfied by the generators of this algebra. First of all, the condition that shows the classical \(SU(2)\) group to be a sphere, reappears in the form

\[
a^*a + b^*b = 1,
\]

(56)
with the ordering of the operators fixed. With the opposite ordering, the left hand side differs from 1 by a term of order $\hbar$, see eq.\((58)\). Hence the sphere has become a 'quantum sphere'.

Secondly, corresponding to the classical eq.\((31)\),

\begin{align*}
[a, b] &= 0, \\
[a, a^*] &= (aa^* + bb^* - 1)b^*b, \\
[b, b^*] &= (aa^* + bb^* - 1)a^*a, \\
[a, b^*] &= -(aa^* + bb^* - 1)b^*a. \quad (57)
\end{align*}

These quartic relations (and similar ones with the bracket on the right hand side at the end) could be written in a pseudo-quadratic form, like the classical ones, eq.\((31)\), using $w$, which parametrizes the Weyl chamber, and which is an independent variable classically. Here however, $w$ is not an independent operator,

\[ w = -\hbar(aa^* + bb^* - 1)^{-1}. \quad (58) \]

It has a discrete spectrum given by

\[ \{\hbar(2j + 1), \quad j \in \left\lfloor \frac{Z}{2} \right\rfloor \}. \quad (59) \]

It commutes with $aa^*$ and $bb^*$. Although the quantum system is still living on a 'quantum group-manifold', eq.\((53)\), the relations eq.\((57)\) do not define a quantum group structure. This was already visible at the classical level, as the Poisson structure was not a Lie-Poisson structure \([13]\), and moreover the right group action does not preserve the symplectic structure, so one is not able to define the group multiplication in a compatible way.

The quantum hamiltonian of a free particle can be represented in terms of the generators of the SGA:

\[ H = -\frac{\hbar^2}{2} \frac{(aa^* + bb^* - 2)(aa^* + bb^*)}{(aa^* + bb^* - 1)^2} \quad (60) \]

and its eigenstates, with eigenvalues equal to $\frac{1}{2}j(j + 1)\hbar^2$, are $\psi_{j,js}$. Consequently the time evolution is diagonalised. The operators corresponding to the classical left momenta eq.\((73)\) are the following:

\[ J_3 = \frac{\hbar}{2} \frac{aa^* - bb^*}{2aa^* + bb^* - 1} \]

\[ J_+ = \frac{\hbar}{aa^* + bb^* - 1} \]

\[ J_- = J^*_+ \quad (61) \]
and their action on the states is standard:

\[ J_3 \psi_{j,j_3} = h_3 \psi_{j,j_3}, \]
\[ J_{\pm} \psi_{j,j_3} = h((j \mp j_3)(j \pm j_3 + 1))^{1/2} \psi_{j,j_3 \pm 1}. \]  

(62)

Comparing eq.(62) with eq.(55) one can see that the operators \( a \) and \( b \) together with their conjugated form a kind of 'square roots' of the Lie algebra formed by \( J_3, J_{\pm} \).

In conclusion, the Hilbert space generated by the SGA out of the vacuum \( \Psi \) is a direct sum of all irreducible unitary representations (with multiplicity 1) of the group \( SU(2) \). Symbolically

\[ SGA \, \Psi = \mathcal{H} = \bigoplus_{j \in \mathbb{Z}/2} \mathcal{H}_j \]  

(63)

This ends the application of the geometric quantisation method to the left sector of our extended phase space. The procedure can be repeated for the right sector. The quantum description is obtained by exchanging the roles of \( a \) with \( a^* \) and \( b \) with \( b^* \). This is not a matter of choice, but is imposed by the difference in sign of the symplectic forms of the right and left sector, see eq.(94) and page 4.

4 Fusion

The classical fusion eq.(10) was straightforward, doing a symplectic reduction. Now we investigate the quantum version of this reduction. The first step is to form the quantum analog of the symplectic product of the classical sectors. Since at the classical level these sectors are completely separated, i.e. the observables of different sectors Poisson-commute, it is natural to take the tensor product of Hilbert spaces as the Hilbert space for the quantum product:

\[ \mathcal{H} = \mathcal{H}_l \bigotimes \mathcal{H}_r. \]  

(64)

The reduction should now identify the Weyl chamber elements of the left and right sectors, which means that we should extract from this space the kernel of the quantum constraint \( w_l - w_r = 0 \). Since \( w \) had a discrete spectrum, eq.(59), this is quite trivial:

\[ \mathcal{H}_o = \bigoplus_{j \in \mathbb{Z}/2} \mathcal{H}_j \otimes \mathcal{H}_j. \]  

(65)
This space is isomorphic with $\mathcal{L}^2(SU(2), d\nu)$, where $\nu$ is an invariant measure. The SGA of this space is simply the commutative algebra of the matrix elements of all the unitary representations of $SU(2)$ \cite{4}. Therefore one should be able to recover the generators of this algebra out of the generators of SGAs of the left and right sector. At the classical level the prescription is given by the formula eq.(97) of symplectic reduction. To extend it to the quantum level we need the antipode - the analog of the group inverse for the algebra eq.(57). Let us introduce matrices with operator entries:

\[
(g_{i,j}^r) = \begin{bmatrix} a_r & -b_r^* \\ b_r & a_r^* \end{bmatrix}
\]

(66)

and let

\[
(S(g^r))_{i,j} = \begin{bmatrix} a_r^*(1 + \frac{\hbar}{w_r}) & b_r^*(1 + \frac{\hbar}{w_r}) \\ -b_r & a_r^* \end{bmatrix}
\]

(67)

where $w_r = -\hbar(a_r^*a_r + b_r^*b_r - 1)^{-1}$. It can be verified that

\[
(S(g^r))_{i,j}g_{j,k}^r = g_{i,j}^rS(g^r)_{j,k} = \delta_{i,k}
\]

(68)

which means that $S$ is the desired antipode (although not in the sense of the Hopf algebras). Of course there is a similar operation in the left sector. The generators of the SGA for the fused model are the entries of the following 'coproduct' matrix (compare eq.(10)):

\[
G_{i,j} := g_{i,k}^l \otimes S(g^r)_{k,j} \equiv \begin{bmatrix} A & C \\ B & D \end{bmatrix}
\]

(69)

Explicitly, they are given by

\[
A = a_l \otimes a_r^*(1 + \frac{\hbar}{w_r}) + b_l^* \otimes b_r
\]

\[
B = b_l \otimes a_r^*(1 + \frac{\hbar}{w_r}) - a_l^* \otimes b_r
\]

\[
C = -b_l^* \otimes a_r + a_l \otimes b_r^*(1 + \frac{\hbar}{w_r})
\]

\[
D = a_l^* \otimes a_r + b_l \otimes b_r^*(1 + \frac{\hbar}{w_r}).
\]

(70)

They form a commutative algebra on the kernel of the constraint operator, as can be checked by straightforward but rather tedious calculations. The
determinant of this matrix is equal to 1 and there are no other quadratic relations between the matrix elements. Note that this coproduct matrix is not the operator that one obtains by quantising the classical matrix elements, obtained by putting $\hbar = 0$ in eq. (70). Instead, it was directly constructed out of the quantum operators, with the intention to obtain a commuting set.

We now make a comparison between the results of the present split quantization (SQ) and the perhaps more straightforward one, which consists in direct quantization (DQ) in the cotangent bundle, without decomposing it in left and right sectors first. The classical theories are of course completely the same, see the reduction theorem in the appendix. Both quantizations give rise to a $2 \times 2$ matrix of commuting functions. Also, both matrices have determinant equal to 1. What is different however, is that DQ also automatically keeps the unitarity of this matrix, whereas SQ violates it, as can be seen easily from the formulas for $G_{i,j}$. This is the price paid for the fact that the antipode, eq. (68), is not related to $g$ by matrix conjugation. We recall that its form was instead designed to obtain the matrix of commuting quantities $G$, eq. (69). If one wishes to restore the unitarity one has to modify the scalar product in $H_o$. This can be done by redefining the scalar product in one of the sectors, or both, for example:

$$\langle \psi, \psi' \rangle := \langle \psi, \frac{1}{w_r} \psi' \rangle_{\text{old}}$$

The conjugation with respect to the new product (denoted by $\dagger$) is related to the old one (which was denoted by $^*$):

$$a^\dagger_r = a^*_r (1 + \hbar \frac{1}{w_r})$$

$$b^\dagger_r = b^*_r (1 + \hbar \frac{1}{w_r})$$

The tensor product (64) and also the Hilbert space (65) are then equipped with a pairing, with respect to which $D$ is conjugated to $A$ and $-C$ is conjugated to $B$. The unimodularity property becomes then equivalent to unitarity. The new scalar product is weaker than the old one so the new Hilbert space contains more vectors in the completion of the set of states with fixed spin. Since $w$ commutes with the hamiltonian, and also with the spin operators, transition probabilities are not influenced by this change. This might
conceivably be different if this model is used as an ingredient for an interacting theory.

5 Conclusion and outlook

In this paper we have shown how the classical quadratic Poisson algebra of the chiral sectors of a particle model can be quantized by applying a systematic procedure, geometric quantization. It gives not only 'quantum deformed' algebraic relations, but also the representation of the corresponding operators in a Hilbert space. We carried out this procedure very explicitly for $SU(2) \sim S^3$, obtaining a quantum version of the sphere relation, and a quantum version of the inverse. The exchange relations, written in terms of independent variables, are quartic. We were not able to push through the quantization procedure in the general case of an arbitrary compact Lie group. For $SU(2)$ we could easily specify the polarization by the Poisson–commuting functions $a$ and $b$, but the generalization to arbitrary groups is not obvious. It is of course reasonable to expect that in the general case, the Hilbert space of states of the chiral sector will be a direct sum of all irreducible representation modules of corresponding Lie algebra. For the chiral sectors, it is not clear that the exchange algebra of the quantized matrix elements will be quartic in general. It might well be that the degree of these relations is correlated with the rank of $G$.

We may add that the geometrical mechanism of the chiral splitting of the canonical WZW theory is exactly the same as in the case of free point particle treated in detail in this paper. The (affine) orbits of the loop group describing the conjugacy classes of chiral momenta are also parametrized by (single) Weyl chamber. In fact the chiral sector of WZW is composed of a point particle and fluctuation modes, described by the loops based at unity [22]. Both are coupled, and this coupling can be seen to underlie a deformation of the classical exchange algebra [23] of the zero modes of WZW. The point particle exchange algebra can be recovered after shrinking the loop to a point [24].

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18
Appendix

In this appendix, we present a point of view on the classical model that is slightly different from the one in the text. At the same time, we provide a more detailed description of splitting and fusion, with proofs of the main statements.

To describe the system we trivialize the bundle $TG$ by means of the left (right) action of $G$ on itself:

$$L_{go}g := g_o g \in G \quad ; \quad R_{go}g := gg_o^{-1} \in G \quad ; \forall g_o \in G.$$  \hspace{1cm} (73)

Both are lifted canonically to actions on $TG$. We shall use the one corresponding to the left action to identify globally $TG$ with $G \times G$.

In these trivializing coordinates each element of $TG$ is represented by a pair $(g, p)$ and the Lagrangian is given by:

$$L(g, p) = \frac{1}{2}K(p, p)$$  \hspace{1cm} (74)

where this $K$ is the $Ad$-invariant form on $G$.

Let us now pass to the Hamiltonian analysis of the system. By means of the $Ad$-invariant form $K$ we identify $G$ with $G^*$. The canonical Liouville one-form is then given by

$$\alpha = K(p, g^{-1}dg).$$  \hspace{1cm} (75)

The symplectic form $\Omega$ is the exterior differential of $\alpha$:

$$\Omega = K(dp, g^{-1}dg) - K(p, g^{-1}dg \wedge g^{-1}dg).$$  \hspace{1cm} (76)

Together with the hamiltonian

$$H = \frac{1}{2}K(p, p),$$  \hspace{1cm} (77)

this structure describes the dynamics (kinematics) of a free particle on the group $G$.

The information about the global constants of motion is very well encoded in terms of momentum mappings [21] corresponding to the group actions eq.(73). In the coordinates on $G \times G$ the lifts of the actions eq.(73) are

$$l_{go}(g, p) = (g_o g, p) \quad ; \quad r_{go}(g, p) = (gg_o^{-1}, Ad_g p).$$  \hspace{1cm} (78)

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They are symplectic with respect to $\Omega$ and their Hamiltonian realizations are given by the following momentum mappings:

$$J_l(g,p) := -\text{Ad}_g p \quad J_r(g,p) := p.$$  \hspace{1cm} (79)

Since the Hamiltonian is $\text{Ad}$-invariant it is obvious that it Poisson-commutes with the momenta eq. (79) (i.e. they are conserved). The mapping $I$:

$$TG \ni (g,p) \mapsto I(g,p) := (-J_l(g,p), J_r(g,p)) = (\text{Ad}_g p, p) \in \mathcal{G}^2$$  \hspace{1cm} (80)

projects the phase space onto the set of constants of motion.

We parametrize the image $I(TG)$ in a special way in two steps. The first step is quite standard, whereas the second step is specific to our purposes.

- Let the map $\beta : \mathcal{G} \ni p \mapsto \beta(p) \in \mathcal{W}$ be the projection onto the space of adjoint orbits. This mapping can be defined by all independent Casimir polynomials. The number $n$ of independent Casimirs (the rank of $\mathcal{G}$) is equal to the dimension of the target for $\beta$. The mapping $\beta$ restricted to the open subset $\mathcal{G}^o$ of regular points defines a smooth fibration of $\mathcal{G}$ over $\mathcal{W}$ with the adjoint orbit $\mathcal{O}$ of maximal dimension as a fiber. The space $\mathcal{W}^o$ of the orbits of maximal dimension is an open, convex subset of $\mathbb{R}^n$ and consequently the fibration $\beta$ is trivial. Therefore we have:

$$\mathcal{O} \hookrightarrow \mathcal{G}^o \approx \mathcal{W}^o \times \mathcal{O} \xrightarrow{\beta} \mathcal{W}.$$  \hspace{1cm} (81)

- Define the fibered product:

$$\Delta_{\beta} := \{ (\tilde{p}, p) \in \mathcal{G}^2 ; \beta(\tilde{p}) = \beta(p) \} = \mathcal{G} \times_\beta \mathcal{G}$$  \hspace{1cm} (82)

with the projection being given by

$$\Delta_{\beta} \ni (\tilde{p}, p) \mapsto \beta(\tilde{p}, p) \equiv \beta(p) \in \mathcal{W}.$$  \hspace{1cm} (83)

For $\Delta_{\beta}^o := \beta^{-1}(\mathcal{W}^o)$ we can write

$$\Delta_{\beta}^o \approx \mathcal{W}^o \times \mathcal{O} \times \mathcal{O}.$$  \hspace{1cm} (84)

It is easily seen that the image of $I$, eq.(80), is $I(TG) = \Delta_{\beta}$. Defining $TG^0 := I^{-1}(\Delta_{\beta}^o)$ we now show:
is a principal fiber bundle with the maximal torus $T$ as typical fiber.

To show this, we begin with some observations about the structure of $\Delta_\beta^o$. We identify $\mathcal{W}^o$ with some Weyl chamber $W \subset \mathcal{G}$. The interior of $W$ (denoted by $W^o$) can be identified with the space of regular orbits. Let $T$ be the maximal torus stabilizing $W^o$. Now we define local sections

$$\tilde{\sigma} \times \sigma : \tilde{U} \times U \rightarrow G$$

where $\tilde{U} \times U \subset G/T \times G/T$ is a local neighbourhood, and let

$$p = \text{Ad}_\sigma w; \tilde{p} = \text{Ad}_{\tilde{\sigma}} w.$$  \hspace{1cm} (87)

be a pair of elements of the orbits corresponding to $w \in W$. Then for the bundle neighbourhood

$$V := I^{-1}(W^o \times \tilde{U} \times U)$$

we have

$$V = \{(g, p); \tilde{p} = \text{Ad}_p w, (\tilde{p}, p) \in W^o \times \tilde{U} \times U\}$$

and the trivializing map:

$$V \ni (g, p) \rightarrow \psi(g, p) := ((\tilde{p}, p), \tilde{\sigma}^{-1}(\tilde{p}) g \sigma(p)) \in (W^o \times \tilde{U} \times U) \times T,$$  \hspace{1cm} (90)

where the identification of $(W^o \times \tilde{U} \times U)$ with a corresponding neighbourhood in $\Delta_\beta^o$ should be clear.

These local trivializations fit together defining a principal bundle structure on $T\mathcal{G}^o$, the transition maps being inherited from the bundle structure of $T \hookrightarrow G \rightarrow G/T$.

The construction above involves the choice of a Weyl chamber. Locally we have

$$T\mathcal{G}^o \approx (W^o \times G/T \times G/T) \times T.$$  \hspace{1cm} (91)

By adding another copy of $W^o$ and $T$, one is able to identify $T\mathcal{G}^o$ with a quotient of $(W^o \times G)^2$ by some suitable relation. We now describe this 'splitting' and 'fusion'.
Let us consider the manifold \( P := W^o \times G \ni (w, g) \), and the 1-form
\[
\theta := K(w, g^{-1}dg). \tag{92}
\]
Its differential \( d\theta \equiv \omega \) defines a symplectic structure, as one can check that it is non-degenerate on \( P \). To write it out more explicitly, let us locally represent an element of \( G \) as a product of the section \( \sigma \) of \( G \to G/T \) and an element \( t \in T \). Then
\[
\omega = K(dw, \sigma^{-1}d\sigma) + K(dw, t^{-1}dt) - K(w, \sigma^{-1}d\sigma \wedge \sigma^{-1}d\sigma). \tag{93}
\]
The last term of the sum is nothing but a local expression for a non-degenerate symplectic form on the adjoint orbit \( G/T \) corresponding to the point \( w \). The middle one canonically couples the Weyl chamber to the torus, and the nondegeneracy can be read off from eq.(12).

We now take two copies of \((P, \omega)\), a ‘left’ and a ‘right’ copy (distinguished by the indices \( r \) and \( l \)), and introduce the symplectic product
\[
(P, \omega_P) := (P_l \times P_r, \omega_l - \omega_r). \tag{94}
\]
The left and right sectors can be fused into the particle phase space:

\[ \text{\textit{TG}^o \text{ is the symplectic reduction of } (P, \omega_P) \text{ by the constraints } w_l = w_r.} \]

The constraints are solved by embedding of \( N := W^o \times G \times G \) in \( P \):
\[
N \ni (w, g_l, g_r) \mapsto i(w, g_l, g_r) := (w, g_l, w, g_r) \in P. \tag{95}
\]
The pull-back of the potential of the symplectic form is given by:
\[
i^*(\theta_P = \theta_l - \theta_r) = K(w, g_l^{-1}dg_l) - K(w, g_r^{-1}dg_r). \tag{96}
\]
It is easy to check that the projection :
\[
N \ni (w, g_l, g_r) \mapsto \pi(w, g_l, g_r) := (g_l g_r^{-1}, Ad_g w) \in (G \times G)^o \tag{97}
\]
satisfies
\[
\pi^*(\alpha) = i^*(\theta_P), \tag{98}
\]
where \( \alpha \) is the Liouville form on \( \text{TG}^o \).

Splitting the phase space into two sectors we are free to split the Hamiltonian \((77)\) to generate dynamics in each of them. This procedure is by no means unique. One can take the ‘left’ (‘right’) Hamiltonian simply as \( h_{l,r} := \frac{1}{2i} K(w_{l,r}, w_{l,r}) \) i.e. they are both proportional to the quadratic Casimir.
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