Partition function zeros of the $Q$-state Potts model on the simple-cubic lattice

Seung-Yeon Kim

School of Computational Sciences, Korea Institute for Advanced Study,
207-43 Cheongryangri-dong, Dongdaemun-gu, Seoul 130-012, Korea

Abstract

The $Q$-state Potts model on the simple-cubic lattice is studied using the zeros of the exact partition function on a finite lattice. The critical behavior of the model in the ferromagnetic and antiferromagnetic phases is discussed based on the distribution of the zeros in the complex temperature plane. The characteristic exponents at complex-temperature singularities, which coexist with the physical critical points in the complex temperature plane for no magnetic field ($H_q = 0$), are estimated using the low-temperature series expansion. We also study the partition function zeros of the Potts model for nonzero magnetic field. For $H_q > 0$ the physical critical points disappear and the Fisher edge singularities appear in the complex temperature plane. The characteristic exponents at the Fisher edge singularities are calculated using the high-field, low-temperature series expansion. It seems that the Fisher edge singularity is related to the Yang-Lee edge singularity which appears in the complex magnetic-field plane for $T > T_c$.

PACS numbers: 05.50.+q; 64.60.Cn; 75.10.Hk; 11.15.Ha

Keywords: partition function zeros; ferromagnetic transition; antiferromagnetic phases; complex-temperature singularity; Fisher edge singularity; Yang-Lee edge singularity

* Electronic address: sykim@kias.re.kr
The $Q$-state Potts model\textsuperscript{[1]} in two and three dimensions exhibits a rich variety of critical behavior and is very fertile ground for the analytical and numerical investigation of first- and second-order phase transitions. With the exception of the two-dimensional $Q = 2$ Potts (Ising) model in the absence of an external magnetic field, exact solutions for arbitrary $Q$ are not known. However, some exact results at the critical temperature have been established for the two-dimensional $Q$-state Potts model for no magnetic field. From the duality relation the ferromagnetic critical temperature is known to be $T_c = J/k_B \ln(1 + \sqrt{Q})$ for the isotropic square lattice. The ferromagnetic Potts model has a first-order phase transition for $Q > 4$ in two dimensions\textsuperscript{[2]}, and has been the most important means to understand first-order transitions which are omnipresent in nature, not only in condensed-matter physics but also in subatomic physics, for example, the deconfining or chiral symmetry breaking transition in quantum chromodynamics\textsuperscript{[3, 4]} and phase transitions in the very early Universe\textsuperscript{[5]}. In three dimensions the ferromagnetic $Q$-state Potts model for $Q > Q_c \approx 2.45(1)$\textsuperscript{[6]} has a first-order transition, and the three-state Potts model is related to the order parameter (the expectation value of the Wilson line) for the deconfinement transition in quantum chromodynamics\textsuperscript{[7]}. On the other hand, the antiferromagnetic Potts model is much less well understood than the ferromagnetic model. One of the most interesting properties of the antiferromagnetic model is that for $Q > 2$ the ground-state is highly degenerate and the ground-state entropy is nonzero. The Baxter conjecture\textsuperscript{[8]} for the critical temperature of the Potts antiferromagnet ($J < 0$) on the square lattice, $T_c = J/k_B \ln(\sqrt{4 - Q} - 1)$, gives the known exact value for $Q = 2$, a critical point at zero temperature for $Q = 3$, and no critical point for $Q > 3$. In three dimensions the Potts model on the simple-cubic lattice has the antiferromagnetic ordered phase for $Q \leq 4$\textsuperscript{[9]}. It is now commonly accepted that the order-disorder phase transition of the three-state Potts antiferromagnet on the simple-cubic lattice belongs to the universality class of the three-dimensional $XY$ model\textsuperscript{[9, 10]}. Recently Rosengren \textit{et al.}\textsuperscript{[11, 12]} claimed the existence of two additional phase transitions below the order-disorder phase transition for the simple-cubic three-state Potts antiferromagnet. However, Kolesik and Suzuki\textsuperscript{[13]}, Heilmann \textit{et al.}\textsuperscript{[14]}, and Oshikawa\textsuperscript{[15]} reported different results in disagreement with observations by Rosengren’s group.

By introducing the concept of the zeros of the partition function in the \textit{complex} magnetic-
field plane, Yang and Lee\textsuperscript{[16]} proposed a mechanism for the occurrence of phase transitions in the thermodynamic limit and yielded a new insight into the unsolved problem of the Ising model in an arbitrary nonzero external magnetic field. It has been shown\textsuperscript{[13, 14, 15, 16, 17, 18, 19, 20, 21]} that the distribution of the zeros of a model determines its critical behavior. Lee and Yang\textsuperscript{[17]} also formulated the celebrated circle theorem which states that the partition function zeros of the Ising ferromagnet lie on the unit circle in the complex magnetic-field $(X = e^{\beta H})$ plane. In three dimensions the properties of the zeros in the complex $X$ plane of the Ising model have been studied by exact enumeration\textsuperscript{[22]} and series expansion\textsuperscript{[23]}. However, for the $Q$-state Potts model with $Q > 2$ the partition function zeros in the complex $X$ plane lie close to, but not on, the unit circle with the two exceptions of the critical point $X = 1$ ($H = 0$) itself and the zeros in the limit $T = 0$\textsuperscript{[24, 25, 26]}.

Fisher\textsuperscript{[27]} emphasized that the partition function zeros in the complex temperature $(Y = e^{-\beta J})$ plane are also very useful in understanding phase transitions, and showed that for the square lattice Ising model in the absence of an external magnetic field the zeros in the complex $Y$ plane lie on two circles (the ferromagnetic circle $Y_{FM} = -1 + \sqrt{2}e^{i\theta}$ and the antiferromagnetic circle $Y_{AFM} = 1 + \sqrt{2}e^{i\theta}$) in the thermodynamic limit. In particular, using the Fisher zeros both the ferromagnetic phase and the antiferromagnetic phase can be considered at the same time. Fisher also showed that the logarithmically infinite specific heat singularity of the Ising model results from the properties of the density of zeros. The zeros in the complex $Y$ plane of the simple-cubic Ising model have been studied extensively to understand the analytical properties and critical behavior of the model by exact enumeration\textsuperscript{[28, 29, 30, 31, 32]} and to determine the critical point and exponents by Monte Carlo simulation\textsuperscript{[21, 33, 34, 35, 36]}. Recently it has been shown that for self-dual boundary conditions near the ferromagnetic critical point $Y_c = 1/(1 + \sqrt{Q})$ the zeros of the $Q$-state Potts model on a finite square lattice in the absence of an external magnetic field lie on the circle with center $-1/(Q - 1)$ and radius $\sqrt{Q}/(Q - 1)$ in the complex $Y$ plane, while the antiferromagnetic circle of the Ising model completely disappears for $Q > 2$\textsuperscript{[37, 38, 39, 40, 41]}. Using the distribution of the zeros in the complex temperature plane of the square-lattice Potts model, the critical behavior in both the ferromagnetic and antiferromagnetic phases has been studied\textsuperscript{[39, 41, 42]}, and the unknown properties of the model for nonzero external magnetic field have been investigated\textsuperscript{[39]}. The partition function zeros in the complex $Y$ plane of the three-state Potts model on the simple-cubic lattice have been studied to under-
stand the ferromagnetic critical behavior by exact enumeration for the $3 \times 3 \times 9$ simple-cubic lattice\cite{12} and by Monte Carlo simulation\cite{21, 43}.

In this paper we study the partition function zeros of the $Q$-state Potts model for $2 \leq Q \leq 7$ on the simple-cubic lattice to unveil some of the rich structures of the model. The partition function zeros of the three-dimensional Potts model have never been obtained for $Q > 3$ in the literature. In the next section we describe briefly the microcanonical transfer matrix to evaluate the density of states, from which the exact partition function of the simple-cubic Potts model is obtained. In Section 3 we discuss the ferromagnetic and antiferromagnetic phase transitions and the complex-temperature singularity (nonphysical critical point) of the simple-cubic Potts model using the partition function zeros in the absence of an external magnetic field. We also discuss the properties of the complex-temperature singularity using the low-temperature series expansion. In section 4 we study the unknown properties of the simple-cubic Potts model for nonzero magnetic field using the partition function zeros and the high-field, low-temperature series expansion. We discuss the Yang-Lee edge singularity and the Fisher edge singularity, which are nonphysical critical points for nonzero magnetic field, and relationship between the Yang-Lee and Fisher edge singularities.

II. DENSITY OF STATES

The $Q$-state Potts model in an external magnetic field $H_q$ on a lattice $G$ with $N_s$ sites and $N_b$ bonds is defined by the Hamiltonian

$$\mathcal{H}_Q = -J \sum_{\langle i, j \rangle} \delta(\sigma_i, \sigma_j) - H_q \sum_{k=1}^{N_s} \delta(\sigma_k, q), \quad (1)$$

where $J$ is the coupling constant (ferromagnetic model for $J > 0$ and antiferromagnetic model for $J < 0$), $\langle i, j \rangle$ indicates a sum over nearest-neighbor pairs, $\delta$ is the Kronecker delta, $\sigma_i = 1, 2, ..., Q$, and $q$ is a fixed integer between 1 and $Q$. The partition function of the model is

$$Z_Q = \sum_{\{\sigma_n\}} e^{-\beta \mathcal{H}_Q}, \quad (2)$$

where $\{\sigma_n\}$ denotes a sum over $Q^{N_s}$ possible spin configurations and $\beta = (k_B T)^{-1}$. If we define the restricted density of states with energy $0 \leq E \leq N_b$ and magnetization
which takes on only integer values, then the partition function can be written as
\[
Z_Q(Y, X) = Y^{-N_b} \sum_{E=0}^{N_b} \sum_{M=0}^{N_s} \Omega_Q(E, M) Y^E X^M,
\]
where \( Y = e^{-\beta J} \) and \( X = e^{\beta H_q} \), and states with \( E = 0 \) (\( E = N_b \)) correspond to the ferromagnetic (antiferromagnetic) ground states. From Eq. (4) it is clear that \( Z_Q(Y, X) \) is simply a polynomial in \( Y \) and \( X \). If we sum the restricted density of states over \( M \), we obtain the density of states
\[
\Omega_Q(E) = \sum_{M=0}^{N_s} \Omega_Q(E, M),
\]
whose sum over \( E \) is equal to the \( Q^{N_s} \) spin configurations,
\[
\sum_{E=0}^{N_b} \Omega_Q(E) = Q^{N_s}.
\]
In the absence of an external magnetic field, the partition function of the model is given by
\[
Z_Q(Y) = Y^{-N_b} \sum_{E=0}^{N_b} \Omega_Q(E) Y^E,
\]
which is again a polynomial in \( Y \).

Here we describe briefly the microcanonical transfer matrix \((\mu \text{TM})\) to obtain exact integer values for the density of states \( \Omega_Q(E) \) of the \( Q \)-state Potts model on an \( L \times L \times N \) simple-cubic lattice with periodic boundary conditions in the \( x-y \) plane (horizontal area of \( L \times L \)) and free boundaries in the \( z \) direction (height of \( N \)). First, an array, \( \omega^{(1)} \), which is indexed by the energy \( E \) and variables \( \sigma_i, 1 \leq i \leq L^2 \) for the first \( x-y \) plane of sites is initialized as
\[
\omega^{(1)}(E; \sigma_1, \sigma_2, \ldots, \sigma_{L \times L}) = \delta(E - \sum_{i=1}^{L \times L} [1 - \delta(\sigma_i, \sigma_{i+1})]).
\]
Now each spin in the plane is traced over in turn, introducing a new spin variable from the next \( x-y \) plane,
\[
\omega^{(2)}(E; \sigma_1', \sigma_2, \ldots, \sigma_{L \times L}) = \sum_{\sigma_1} \omega^{(1)}(E - [1 - \delta(\sigma_1', \sigma_1)]; \sigma_1, \sigma_2, \ldots, \sigma_{L \times L}).
\]
This procedure is repeated until all the spins in the first plane have been traced over, leaving a new function of the $L \times L$ spins in the second plane. The bonds connecting the spins in the second $x$-$y$ plane are then taken into account by shifting the energy,

$$\omega^{(2)}(E; \sigma_1', \sigma_2', ..., \sigma_{L\times L}') = \tilde{\omega}^{(2)}(E - \sum_{i=1}^{L\times L} [1 - \delta(\sigma_i', \sigma_{i+1}')] ; \sigma_1', \sigma_2', ..., \sigma_{L\times L}').$$

This procedure is then applied to each $x$-$y$ plane in turn until the final ($N$th) plane is reached. The density of states is then given by

$$\Omega_Q(E) = \sum_{\sigma_1'} \sum_{\sigma_2'} ... \sum_{\sigma_{L\times L}'} \omega^{(N)}(E; \sigma_1', \sigma_2', ..., \sigma_{L\times L}').$$

The permutation symmetry of the $Q$-state Potts model allows us to freeze the last spin $\sigma_{L\times L} = 1$ of each $x$-$y$ plane. Now we need to consider only $Q^{L\times L-1}$ possible spin configurations in each plane instead of $Q^{L\times L}$ configurations, and we save a great amount of memory and CPU time. For example, Table I shows the density of states $\Omega_{Q=7}(E)$ of the seven-state Potts model on the $3 \times 3 \times 3$ simple-cubic lattice.

The above procedure can be modified in a straightforward way to calculate exact integer values for the restricted density of states $\Omega(E, M)$\cite{15, 47}. For the first $x$-$y$ plane one has, instead of Eq. (8),

$$\omega^{(1)}(E, M; \sigma_1, \sigma_2, ..., \sigma_{L\times L}) = \delta(E - \sum_{i=1}^{L\times L} [1 - \delta(\sigma_i, \sigma_{i+1})]) \delta(M - \sum_{k=1}^{L\times L} \delta(\sigma_k, q)).$$

The spins in the first plane are traced over in sequence just as in Eq. (9) leading to a new function of the spins in the second $x$-$y$ plane;

$$\tilde{\omega}^{(2)}(E, M; \sigma_1', \sigma_2', ..., \sigma_{L\times L}') = \sum_{\sigma_1'} \omega^{(1)}(E - [1 - \delta(\sigma_1', \sigma_1)], M - \delta(\sigma_1', q); \sigma_1, \sigma_2, ..., \sigma_{L\times L}).$$

Once all the spins in an $x$-$y$ plane have been traced over, the function for the next plane is completed as in Eq. (10)

$$\omega^{(2)}(E, M; \sigma_1', \sigma_2', ..., \sigma_{L\times L}') = \tilde{\omega}^{(2)}(E - \sum_{i=1}^{L\times L} [1 - \delta(\sigma_i', \sigma_{i+1})], M; \sigma_1', \sigma_2', ..., \sigma_{L\times L}'),$$

and the restricted density of states is given by

$$\Omega_Q(E, M) = \sum_{\sigma_1'} \sum_{\sigma_2'} ... \sum_{\sigma_{L\times L}'} \omega^{(N)}(E, M; \sigma_1', \sigma_2', ..., \sigma_{L\times L}').$$
III. PARTITION FUNCTION ZEROS IN THE COMPLEX TEMPERATURE PLANE

In this section we discuss the properties of the $Q$-state Potts models on the simple-cubic lattice using the partition function zeros \( \{ Y_0 \} \) or \( \{ A_0 \} \) of the models in the complex temperature \( (Y = e^{-\beta J} \text{ or } A = e^{\beta J}) \) plane which are obtained by solving the equation

\[
Z_Q(Y_0) = Y_0^{-N_b} \sum_{E=0}^{N_b} \Omega_Q(E)Y_0^E = 0
\]

for the exact partition functions on finite lattices. Figure 1 shows the partition function zeros in the complex $Y = e^{-\beta J}$ plane of the $Q$-state Potts models for $Q=2$, $4$, and $7$ on the $3 \times 3 \times 3$ simple-cubic lattice with periodic boundary conditions in the $x$-$y$ plane and free boundary conditions in the $z$ direction. In Figure 1, near the positive real axis, the partition function zeros show one-dimensional distributions for all the Potts models which become denser as $Q$ increases. Since the value of latent heat of a system is proportional to the density of zeros in the complex temperature plane\[19\], for $Q \geq 3$ the latent heat increases at the transition temperature as $Q$ increases, and the first-order phase transition becomes stronger. Moreover, the first zeros of these linear distributions will cut the real axis and be the ferromagnetic (FM) critical points in the thermodynamic limit. In Figure 1, as $Q$ increases, the first zeros become closer to the real axis and to the values of the FM critical temperatures estimated in the literature (Table II). This fact suggests that the first zero obtained for a smaller size of simple-cubic lattice approach very closely the real critical point in the limit $Q \to \infty$ where the mean-field theory is known to be exact\[48, 49\].

Even for the three-dimensional Ising model the critical temperature has never been known exactly except for some conjectures, for example, the Rosengren conjecture\[50\]. However, for the simple-cubic lattice Hajduković conjectured\[51\] the FM critical temperature

\[
Y_{c}^{FM}(Q) = \frac{1}{(Q + \sqrt{2Q-1})^{1/3}}
\]

of the $Q$-state Potts model based on two assumptions. The first assumption is that $Y_{c}^{FM}(Q)$ reduces to the mean-field result $Y_{c}^{FM} = Q^{-1/3}$ in the limit $Q \to \infty$\[48, 49\]. And the second assumption is that in $d$ dimensions $Y_{c}^{FM}(Q)$ is the solution of an equation

\[
Y^{-d} - c_1(d)Y^{-d/2} - c_2(d) = Q - 2,
\]
where \( c_1(d) \) and \( c_2(d) \) have to be determined, for example, \( c_1 = 2 \) and \( c_2 = 1 \) for \( d = 2 \). As shown in Table II the Hajduković conjecture is close to numerical estimates in the literature for \( 2 \leq Q \leq 7 \). However, for \( Q = 1 \) the Hajduković conjecture, \( Y_{c}^{FM} = 0.7937... \), completely disagrees with the numerical estimate \( 0.753 \). We can guess a simpler form than the Hajduković conjecture for the critical temperature,

\[
Y_{c}^{FM}(Q) = \frac{1}{1/\pi + Q^{1/3}},
\]

satisfying the first assumption. Eq. (19) reasonably agrees with numerical estimates for \( 1 \leq Q \leq 7 \) as shown in Table II. Even though the Hajduković conjecture and Eq. (19) are incorrect expressions for the FM critical temperature of the \( Q \)-state Potts model on the simple-cubic lattice, we can infer that the true expression for the critical temperature may be something between the Hajduković conjecture and Eq. (19).

Recently using cluster-variation calculations Rosengren and Lapinskas claimed the existence of two additional phase transitions at \( T_{RL1} = 0.787(-J/k_B) \) and \( T_{RL2} = 0.7715(-J/k_B) \) below the well-known antiferromagnetic (AFM) critical temperature \( T_{c}^{AFM} = 1.2259(7)(-J/k_B) \) for the three-state AFM Potts model \( (J < 0) \) on the simple-cubic lattice. Using the Monte Carlo method Kundrotas, Rosengren, and Lapinskas obtained \( T_{KRL} = 0.68(-J/k_B) \) and supported their earlier claim. However, Kolesik and Suzuki and Heilmann et al. observed different results by Monte Carlo simulations which are in disagreement with observations by Rosengren’s group, and obtained \( T_{KH} = 1.0(-J/k_B) \) as the intermediate transition temperature in the region below \( T_{c}^{AFM} \). Furthermore, based on a renormalization-group picture Oshikawa argued that there exists only one transition at \( T_{c}^{AFM} \) for the three-state AFM Potts model on the simple-cubic lattice. Figure 2 shows the partition function zeros in the complex \( A = e^{\beta J} \) plane of the three-state Potts model on the \( 3 \times 3 \times 30 \) simple-cubic lattice with periodic boundary conditions in the \( x \)-\( y \) plane and free boundary conditions in the \( z \) direction. The interval \( 0 \leq A \leq 1 \) \( (0 \leq T \leq \infty) \) is antiferromagnetic \( (J < 0) \) while the interval \( 1 \leq A \leq \infty \) \( (\infty \geq T \geq 0) \) ferromagnetic \( (J > 0) \). There exists a clear one-dimensional distribution of zeros near \( A_{KRL} = 0.2298 \), in Figure 2, which favors the existence of additional phase transitions below \( A_{c}^{AFM} = 0.4424 \). However, we do not know if the first zero of the distribution near \( A_{KRL} \) cuts the real axis in the thermodynamic limit, and if there exist the true phase transitions below \( A_{c}^{AFM} \).
We have also obtained two pairs of the complex-temperature (CT) singularities (Figure 2) using the low-temperature series\[58, 62, 63\] for the three-state Potts model on the simple-cubic lattice. Figure 2 exhibits accumulations of partition function zeros near the CT singularities. These accumulations imply that some thermodynamic quantities may diverge at the CT singularities which are not physical. The critical exponents associated with the CT singularities, which we call the CT critical exponents, are defined in the usual way,

\[
C_0^{CT} \sim \left(1 - \frac{Y}{Y_{CT}}\right)^{-\alpha_{CT}},
\]

\[
m_0^{CT} \sim \left(1 - \frac{Y}{Y_{CT}}\right)^{\beta_{CT}},
\]

and

\[
\chi_0^{CT} \sim \left(1 - \frac{Y}{Y_{CT}}\right)^{-\gamma_{CT}},
\]

where \(Y_{CT}\) is the location of a CT singularity, and \(C_0^{CT}\), \(m_0^{CT}\) and \(\chi_0^{CT}\) are the singular parts of the specific heat, spontaneous magnetization, and susceptibility, respectively. Here, the superscript 0 represents no external magnetic field. Because the density of zeros near a CT singularity is given by\[27\]

\[
g(Y) \sim \left(1 - \frac{Y}{Y_{CT}}\right)^{1-\alpha_{CT}},
\]

the density of zeros diverges at the CT singularity if \(\alpha_{CT} > 1\).

Guttmann et al.\[64, 65, 66\] studied the locations of the CT singularities and the CT critical exponents for the Ising model in detail, introducing a new method of low-temperature series analysis. For the Ising model on the simple cubic, body-centered cubic, and face-centered cubic lattices there exist the CT singularities, closer to the origin than the physical critical points, which control the low-temperature series and spoil the analysis of the real critical behavior. Guttmann et al.\[64\] reported that for the Ising model on the simple-cubic lattice the CT singularity is located at \((Y_{CT})^2 = -0.2860(2) \ (|Y_{CT}| < (Y_{FM})^2 = 0.4120)\) with \(\alpha_{CT} \sim 1, \ -\frac{1}{8} \leq \beta_{CT} \leq -\frac{1}{16}, \) and \(\gamma_{CT} \sim \frac{9}{8}\). Itzykson et al.\[30\] studied the zeros of the exact partition function in the complex temperature plane of the Ising model on the \(4 \times 4 \times 4\) simple-cubic lattice, and found that the distribution of zeros supports the real existence of the CT singularities. By the analysis of the longer series they obtained \(\alpha_{CT} = 1.25(15), \ \beta_{CT} = -0.05, \) and \(\gamma_{CT} = 1.1(1)\) for the Ising model on the simple-cubic lattice, and argued that the values of the CT critical exponents of the Ising model are same for the simple cubic, body-centered cubic, and face-centered cubic lattices and satisfy
\( \alpha_{CT} + 2\beta_{CT} + \gamma_{CT} \approx 2 \). Finally, Itzykson et al. conjectured that the behavior at the CT singularities is universal. Recently Guttmann and Enting obtained \((Y_{CT})^2 = -0.2853(3)\) with \(\alpha_{CT} \approx 1.03, \beta_{CT} \approx -0.01, \) and \(\gamma_{CT} \approx 1.01\) using the low-temperature series generated by the finite-lattice method of series expansion for the simple-cubic Ising model. They proposed \(\alpha_{CT} = 1, \beta_{CT} = 0, \) and \(\gamma_{CT} = 1\) with logarithmic corrections, satisfying the Rushbrooke scaling law \(\alpha_{CT} + 2\beta_{CT} + \gamma_{CT} = 2\).

Using Dlog Padé approximants to the low-temperature series of the specific heat, spontaneous magnetization, and susceptibility for the three-state Potts model on the simple-cubic lattice, we found the locations of two pairs of the CT singularities

\[
Y_{CT}^{(1)} = 0.0254(11) \pm 0.4948(9)i
\]

and

\[
Y_{CT}^{(2)} = -0.4981(45) \pm 0.2453(35)i.
\]

Since \(|Y_{CT}^{(1)}| = 0.4955(9)\) and \(|Y_{CT}^{(2)}| = 0.5552(42)\), these CT singularities lie closer to the origin than the estimated values of the FM critical point shown in Table II. The CT critical exponents for the simple-cubic three-state Potts model are same at \(Y_{CT}^{(1)}\) and \(Y_{CT}^{(2)}\) as shown in Table III, and satisfy the scaling law \(\alpha_{CT} + 2\beta_{CT} + \gamma_{CT} \approx 2\). Because in three dimensions the values of the CT critical exponents for the three-state Potts model are very close to those for the Ising model, it is difficult not to suggest that the values of the CT critical exponents are independent of \(Q\) in three dimensions. That is, our results support the universality of the CT singularities in three dimensions. For the two-dimensional \(Q\)-state Potts model on the square lattice it has been shown that two pairs of the CT singularities of the model are mapped each other through the dual transformation

\[
Y \rightarrow \frac{1 - Y}{1 + (Q - 1)Y},
\]

which determines the FM critical temperature

\[
Y_{c}^{FM}(Q) = \frac{1}{1 + \sqrt{Q}}.
\]

Therefore, one expects the true expression for the FM critical points of the simple-cubic Potts model to provide the mapping between \(Y_{CT}^{(1)}\) and \(Y_{CT}^{(2)}\).
IV. PARTITION FUNCTION ZEROS FOR AN EXTERNAL MAGNETIC FIELD

In this section we discuss the properties of the simple-cubic $Q$-state Potts models for nonzero magnetic field using the high-field, low-temperature series expansions and the zeros of the exact partition function $Z_Q(Y, X)$ on a finite lattice in the complex temperature ($Y = e^{-\beta J}$) or magnetic-field ($X = e^{\beta H_q}$) plane. Figure 3 shows the partition function zeros in the complex $X$ plane of the three-state Potts model on the $3\times3\times3$ simple-cubic lattice with periodic boundary conditions in the $x$-$y$ plane and free boundary conditions in the $z$ direction. The zeros lie close to the unit circle at lower temperatures, but they move away from the positive real axis and the unit circle as the temperature is increased. The zeros at $T = 0$ are uniformly distributed on the circle with radius $(Q - 1)^{1/N_s}$ which approaches unity in the thermodynamic limit, independent of $Q$, whereas the zeros at $T = \infty$ are $N_s$-degenerate at $X = 1 - Q$, independent of lattice size.

The edge zero, which we call the first zero, and its complex conjugate of a circular distribution of zeros in the complex $X$ plane cut the positive real axis at the physical critical point $X_c = 1$ for $T \leq T_c$ in the thermodynamic limit. However, for $T > T_c$ the first zero does not cut the positive real axis in the thermodynamic limit, that is, there is a gap in the distribution of zeros around the positive real axis. Within this gap, the free energy is analytic and there is no phase transition. Kortman and Griffiths carried out the first systematic investigation of the magnetization at the first zero, based on the high-field, high-temperature series expansion for the Ising model on the square lattice and the diamond lattice. They found that above $T_c$ the magnetization at the first zero diverges for the square lattice and is singular for the diamond lattice. For $T > T_c$ we rename the first zero as the Yang-Lee edge singularity. The divergence of the magnetization at the Yang-Lee edge singularity means the divergence of the density of zeros, which does not occur at a physical critical point. Fisher proposed the idea that the Yang-Lee edge singularity can be thought of as a new second-order phase transition with associated critical exponents and the Yang-Lee edge singularity can be considered as a conventional critical point. The critical point of the Yang-Lee edge singularity is associated with a $\phi^3$ theory, different from the usual critical point associated with the $\phi^4$ theory. The crossover dimension of the Yang-Lee edge singularity is $d_c = 6$. The edge critical exponent $\sigma_e = 1/\delta_e = (d - 2 + \eta_e)/(d + 2 - \eta_e)$ at a Yang-Lee edge
singularity $X_e$ for $Y > Y_c$ is defined by

$$m_e^X \sim \left(1 - \frac{X}{X_e} \right)^{\sigma_e},$$

where $m_e$ is the singular part of the magnetization in the complex $X$ plane. The study of the Yang-Lee edge singularity has been extended to the classical $n$-vector model, the quantum Heisenberg model, the spherical model, the quantum one-dimensional transverse Ising model, the hierarchical model, the one-dimensional Potts model, branched polymers, fluid models with repulsive-core interactions, etc. Dhar calculated the edge critical exponent $\sigma_e$ in two dimensions by solving a particular model of three-dimensional directed animals and mapping the solution to the hard hexagon model. Using Fisher’s idea and conformal field theory, Cardy studied the Yang-Lee edge singularity for a two-dimensional $\phi^3$ theory. It is generally accepted that the value of the edge critical exponent $\sigma_e$ depends only on dimension. The values of $\sigma_e$ are known to be $\sigma_e = -\frac{1}{2}$ in one dimension and $\sigma_e = -\frac{1}{4}$ in two dimensions, whereas $\sigma_e \approx 0.08$ in three dimensions.

The partition function zeros in the complex $X$ plane for real $Y$ have been extensively studied and well understood. However, the partition function zeros in the complex $X$ plane for real $X$ are much less well understood than the zeros in the complex $X$ plane. Figure 4 shows the partition function zeros in the complex $Y$ plane of the three-state Potts model for an external magnetic field $X \geq 1 (H_q \geq 0)$ on the $3 \times 3 \times 3$ simple-cubic lattice with periodic boundary conditions in the $x$-$y$ plane and free boundary conditions in the $z$ direction. For $X = 1$ some zeros distribute along the one-dimensional locus near $Y_c$, and the first zero of the distribution cuts the positive real axis at $Y_c$ in the thermodynamic limit as explained in the previous section. As $H_q$ increases, all the zeros move away from the origin. In the limit $H_q \to \infty (X \to \infty)$ the field $H_q$ favors the state $q$ for every site and the $Q$-state Potts model is transformed into the one-state model whose partition function zeros are the roots of the equation $Y^{-N_b} = 0$. That is, $|Y|$ for all the zeros increases without bound as $X$ increases. Note that for $X > 1$ there is accumulation of the zeros as we approach the first zero. Matveev and Shrock studied the properties of the thermodynamic functions at the first zero in the complex temperature plane for the square-lattice Ising model with $X > 1$. They observed that the thermodynamic functions and the density of zeros diverge at the first zero when $X > 1$, and that the values of the critical exponents associated with
the first zero for $X > 1$ are nearly independent of $X$ and satisfy the Rushbrooke scaling law approximately. We call the first zero in the complex temperature plane for $X > 1$ as the Fisher edge singularity. The edge critical exponents $\alpha_e$, $\beta_e$, and $\gamma_e$ at the Fisher edge singularities are given by

$$C_e \sim \left(1 - \frac{Y}{Y_e}\right)^{-\alpha_e},$$  \hspace{1cm} (29)$$

$$m_e^Y \sim \left(1 - \frac{Y}{Y_e}\right)^{\beta_e},$$  \hspace{1cm} (30)$$

and

$$\chi_e \sim \left(1 - \frac{Y}{Y_e}\right)^{-\gamma_e},$$  \hspace{1cm} (31)$$

where $Y_e$ is the location of a Fisher edge singularity, and $C_e$, $m_e^Y$ and $\chi_e$ are the singular parts of the specific heat, magnetization, and susceptibility, respectively, for $X > 1$. Kim and Creswick\[39] extended the study of the Fisher edge singularity to the $Q$-state Potts model ($Q \geq 3$) on the square lattice. They found that the values of the edge critical exponents at the Fisher edge singularities are nearly independent of $Q$ ($Q \geq 2$).

In three dimensions to study the critical behavior at the Fisher edge singularity we have used the high-field, low-temperature series expansions for the Ising model\[70, 71, 72] and the three-state Potts model\[73, 74] on the simple-cubic lattice. Table IV shows the locations ($Y_e$) and the edge critical exponents ($\alpha_e$, $\beta_e$, and $\gamma_e$) of the Fisher edge singularities estimated by Dlog Padé approximants to the specific heat, magnetization, and susceptibility series for $x = 100, 200, \text{and} 500$. The edge critical exponents, which are independent of $X$, have the values of $\alpha_e = 1.0 \sim 1.1$, $\beta_e \approx -0.4$, and $\gamma_e \approx 1.0$ for $Q = 2$ and for $Q = 3$ similar values of $\alpha_e$ and $\gamma_e$ and $\beta_e \approx -0.6$. So in three dimensions we seem to have a strong violation of the scaling law $\alpha_e + 2\beta_e + \gamma_e = 2$. Another interesting observation is that there exist apparently three pairs of the Fisher edge singularities in three dimensions (Figure 4 and Table IV). The estimates of the edge critical exponents at each of the three edge singularities indicate that they are equal within error bars.

On the other hand, everything can be evaluated exactly in one dimension, and there exist only one pair of the Fisher edge singularities characterized by the edge critical exponents $\alpha_e = \frac{3}{2}$, $\beta_e = -\frac{1}{2}$, and $\gamma_e = \frac{3}{2}$, all independent of $Q$. These values satisfy the scaling law $\alpha_e + 2\beta_e + \gamma_e = 2$. Table V shows the locations and the edge critical exponents of the Fisher edge singularities for the $Q$-state Potts models on the square lattice. In two dimensions we have also found two pairs of the Fisher edge singularities which have the same values of the
edge critical exponents within error bars. The edge critical exponents have the values of \( \alpha_e = 1.13 \sim 1.24 \), \( \beta_e = -0.11 \sim -0.20 \), and \( \gamma_e = 1.11 \sim 1.26 \). In two dimensions the edge critical exponents also satisfy the scaling law \( \alpha_e + 2\beta_e + \gamma_e = 2 \). Now we can conclude that the number of pairs of the Fisher edge singularities is equal to the dimensionality.

Until now the Yang-Lee and Fisher edge singularities have been studied and understood independently. However, the Yang-Lee edge singularity seems to have a direct relationship to the Fisher edge singularity. From the edge critical exponents of the Fisher edge singularity in one dimension we get \( y^e_t = 2 \) and \( y^e_h = 2 \), which also follows from \( \sigma_e = -\frac{1}{2} \) at the Yang-Lee edge singularity. The equivalence of the Yang-Lee and Fisher edge singularities is not surprising in light of the transformation which maps the temperature and field variables into each other in one dimension\([81]\). In two dimensions it is known that \( \sigma_e = -\frac{1}{6} \) from which it follows that \( y^e_h = \frac{12}{5} \). If we assume \( y^e_t = y^e_h = \frac{12}{5} \) for the Fisher edge singularity, we have \( \alpha_e = \frac{7}{6} \), \( \beta_e = -\frac{1}{6} \), and \( \gamma_e = \frac{7}{6} \), which are not far from the values of the exponents estimated by series expansions. In three dimensions if we take \( \alpha_e = 1.0 \) then \( y^e_t = 3 \), and \( \sigma_e \sim 0 \) (logarithmic singularity), which seems to be the historical trend, then \( y^e_h = 3 \) as well. So in \( d = 1, 2 \) and 3 we seem to find \( y^e_t \sim y^e_h \). If \( y^e_t = y^e_h \), the Fisher and Yang-Lee edge singularities are not independent behaviors, and they seem to have only one relevant variable.

V. CONCLUSION

We have investigated the interesting properties of the \( Q \)-state Potts model on the simple-cubic lattice using the zeros of the exact partition function on a finite lattice and series expansions. The critical behavior of the Potts model in the ferromagnetic and antiferromagnetic phases has been studied at the same time based on the distribution of the partition function zeros in the complex temperature plane. The distribution of the zeros in the complex temperature plane reveals the possibility of the intermediate phase transitions below the order-disorder phase transition of the Potts antiferromagnet. The behavior at the CT singularities, where some thermodynamic quantities diverge with the characteristic CT exponents \( \alpha_{CT}, \beta_{CT}, \) and \( \gamma_{CT}, \) has been discussed using the partition function zeros and the low-temperature series expansion of the Potts model in the absence of an external magnetic field. In the complex temperature plane the CT singularities coexist with the physical
critical points for $H_q = 0$.

We have also studied the veiled properties of the Potts model for nonzero magnetic field using the partition function zeros and the high-field, low-temperature series expansion. As $H_q$ increases, all the zeros in the complex $Y = e^{\beta J}$ plane move away from the origin. For $H_q > 0$ the physical critical points $Y_{c}^{FM}$ and $Y_{c}^{AFM}$ disappear and the Fisher edge singularities $Y_e$ appear in the complex temperature plane. For different values of $X = e^{\beta H_q}$ we have calculated the locations and the edge critical exponents of the Fisher edge singularities of the Ising and three-state Potts models on the simple-cubic lattice. The values of the edge critical exponents $\alpha_e$, $\beta_e$, and $\gamma_e$ at a Fisher edge singularity are universal, that is, independent of $X$ and $Q$ but dependent on dimension $d$. We found that the number of pairs of the Fisher edge singularities is equal to $d$. We have considered the Yang-Lee edge singularity $X_e$, which appears in the complex $X$ plane for $Y > Y_c$ instead of the physical critical point $X_c = 1$, and its edge exponent $\sigma_e = 1/\delta_e$. In one, two, and three dimensions the Yang-Lee edge singularity is related to the Fisher edge singularity. They seem to have only one relevant variable $y_e^c \sim y_e^h$.

ACKNOWLEDGMENTS

The author is indebted to Prof. R. J. Creswick for very helpful discussions. The author is grateful to Prof. A. J. Guttmann for kindly providing him with the missed coefficients in the paper [58]. The author also wishes to thank Prof. M. E. Fisher for valuable comments and for kindly drawing his attention to the paper [84].
[1] F. Y. Wu, Rev. Mod. Phys. 54 (1982) 235.
[2] R. J. Baxter, J. Phys. C 6 (1973) L445.
[3] F. R. Brown, N. H. Christ, Y. Deng, M. Gao, and T. J. Woch, Phys. Rev. Lett. 61 (1988) 2058.
[4] N. A. Alves, B. A. Berg, and S. Sanielevici, Phys. Rev. Lett. 64 (1990) 3107.
[5] E. W. Kolb and M. S. Turner, The Early Universe (Addison-Wesley, Reading, 1994).
[6] J. Lee and J. M. Kosterlitz, Phys. Rev. B 43 (1991) 1286.
[7] B. Svetitsky and L. G. Yaffe, Nucl. Phys. B 210 (1982) 423.
[8] R. J. Baxter, Proc. R. Soc. London A 383 (1982) 43.
[9] J. R. Banavar, G. S. Grest, and D. Jasnow, Phys. Rev. B 25 (1982) 4639.
[10] J.-S. Wang, R. H. Swendsen, and R. Kotecký, Phys. Rev. B 42 (1990) 2465.
[11] A. Rosengren and S. Lapinskas, Phys. Rev. Lett. 71 (1993) 165.
[12] P. J. Kundrotas, A. Rosengren and S. Lapinskas, Phys. Rev. B 52 (1995) 9166.
[13] M. Kolesik and M. Suzuki, J. Phys. A 28 (1995) 6543.
[14] R. K. Heilmann, J.-S. Wang, and R. H. Swendsen, Phys. Rev. B 53 (1996) 2210.
[15] M. Oshikawa, Phys. Rev. B 61 (2000) 3430.
[16] C. N. Yang and T. D. Lee, Phys. Rev. 87 (1952) 404.
[17] T. D. Lee and C. N. Yang, Phys. Rev. 87 (1952) 410.
[18] R. J. Creswick and S.-Y. Kim, Phys. Rev. E 56 (1997) 2418.
[19] R. J. Creswick and S.-Y. Kim, in: D. P. Landau, K. K. Mon, and H.-B. Schüttler (Ed.), Computer Simulation Studies in Condensed-Matter Physics, Vol. 10, Springer, Berlin, 1998, p. 224.
[20] R. J. Creswick and S.-Y. Kim, Comput. Phys. Commun. 121 (1999) 26.
[21] W. Janke and R. Kenna, J. Stat. Phys. 102 (2001) 1211.
[22] M. Suzuki, C. Kawabata, S. Ono, Y. Karaki, and M. Ikeda, J. Phys. Soc. Japan 29 (1970) 837.
[23] P. J. Kortman and R. B. Griffiths, Phys. Rev. Lett. 27 (1971) 1439.
[24] S.-Y. Kim and R. J. Creswick, Phys. Rev. Lett. 81 (1998) 2000.
[25] S.-Y. Kim and R. J. Creswick, Phys. Rev. Lett. 82 (1999) 3924.
[26] S.-Y. Kim and R. J. Creswick, Physica A 281 (2000) 252.
[27] M. E. Fisher, in: W. E. Brittin (Ed.), Lectures in Theoretical Physics, Vol. 7c, University of Colorado Press, Boulder, 1965, p. 1.
[28] S. Ono, Y. Karaki, M. Suzuki, and C. Kawabata, J. Phys. Soc. Japan 25 (1968) 54.
[29] R. B. Pearson, Phys. Rev. B 26 (1982) 6285.
[30] C. Itzykson, R. B. Pearson, and J. B. Zuber, Nucl. Phys. B 220 (1983) 415.
[31] P. P. Martin, Nucl. Phys. B 220 (1983) 366.
[32] G. Bhanot and S. Sastry, J. Stat. Phys. 60 (1990) 333.
[33] E. Marinari, Nucl. Phys. B 235 (1984) 123.
[34] G. Bhanot, R. Salvador, S. Black, P. Carter, and R. Toral, Phys. Rev. Lett. 59 (1987) 803.
[35] N. A. Alves, B. A. Berg, and R. Villanova, Phys. Rev. B 41 (1990) 383.
[36] N. A. Alves, J. R. Drugowich de Felicio, and U. H. E. Hansmann, J. Phys. A 33 (2000) 7489.
[37] P. P. Martin, J. Phys. A 19 (1986) 3267.
[38] C.-N. Chen, C.-K. Hu, and F. Y. Wu, Phys. Rev. Lett. 76 (1996) 169.
[39] S.-Y. Kim and R. J. Creswick, Phys. Rev. E 58 (1998) 7006.
[40] S.-Y. Kim and R. J. Creswick, Phys. Rev. E 63 (2001) 066107.
[41] P. P. Martin, Nucl. Phys. B 225 (1983) 497.
[42] N. A. Alves, B. A. Berg, and R. Villanova, Phys. Rev. B 43 (1991) 5846.
[43] G. Bhanot, J. Stat. Phys. 60 (1990) 55.
[44] B. Strošić, S. Milošević, and H. E. Stanley, Phys. Rev. B 41 (1990) 11466.
[45] L. Stodolsky and J. Wosiek, Nucl. Phys. B 413 (1994) 813.
[46] R. J. Creswick, Phys. Rev. E 52 (1995) 5735.
[47] L. Mittag and M. J. Stephen, J. Phys. A 7 (1974) L109.
[48] P. A. Pearce and R. B. Griffiths, J. Phys. A 13 (1980) 2143.
[49] A. Rosengren, J. Phys. A 19 (1986) 1709.
[50] D. Hajduković, J. Phys. A 16 (1983) L193.
[51] D. S. Gaunt and H. Ruskin, J. Phys. A 11 (1978) 1369.
[52] D. Stauffer, Phys. Rep. 54 (1979) 1.
[53] D. P. Landau, Physica A 205 (1994) 41.
[54] H. G. Ballesteros, L. A. Fernandez, V. Martin-Mayor, A. Munoz Sudupe, G. Parisi, and J. J.
Ruiz-Lorenzo, J. Phys. A 32 (1999) 1.

[56] D. Kim and R. I. Joseph, J. Phys. A 8 (1975) 891.

[57] W. Janke and R. Villanova, Nucl. Phys. B 489 (1997) 679.

[58] A. J. Guttmann and I. G. Enting, J. Phys. A 27 (1994) 5801. This paper has the longest low-temperature series for the specific heat and susceptibility of the simple-cubic three-state Potts model. The coefficients of $(Y^2)^{36}$ were unintentionally omitted by publisher. The missed coefficients are $5858732$, $-58983558$, and $281636092$ for $\lambda_{36}$, $m_{36}$, and $c_{36}$, respectively. And the coefficient $c_{21} = 36922$ is a misprint. It should be $-36922$.

[59] R. V. Ditzian and L. P. Kadanoff, J. Phys. A 12 (1979) L229.

[60] H. Park and D. Kim, J. Korean Phys. Soc. 15 (1982) 55.

[61] H. H. Chen, F. Lee, and Y. M. Kao, Phys. Rev. B 52 (1995) 39.

[62] C. Vohwinkel, Phys. Lett. B 301 (1993) 208. This paper has the low-temperature series only for the spontaneous magnetization of the simple-cubic three-state Potts model. But it is the longest series for the spontaneous magnetization.

[63] G. Bhanot, M. Creutz, U. Glässner, I. Horvath, J. Lacki, K. Schilling, and J. Weckel, Phys. Rev. B 48 (1993) 6183. In this paper, the coefficient $c_{29} = 12285816$ for the susceptibility series is a misprint, instead, $-12285816$ is correct.

[64] C. J. Thompson, A. J. Guttmann, and B. W. Ninham, J. Phys. C 2 (1969) 1889.

[65] A. J. Guttmann, J. Phys. C 2 (1969) 1900.

[66] C. Domb and A. J. Guttmann, J. Phys. C 3 (1970) 1652.

[67] A. J. Guttmann and I. G. Enting, J. Phys. A 26 (1993) 807.

[68] A. J. Guttmann, in: C. Domb and J. Lebowitz (Ed.), Phase Transitions and Critical Phenomena, Vol. 13, Academic, New York, 1989, p. 1.

[69] V. Matveev and R. Shrock, Phys. Rev. E 54 (1996) 6174.

[70] M. F. Sykes, J. W. Essam, and D. S. Gaunt, J. Math. Phys. 6 (1965) 283.

[71] M. F. Sykes, D. S. Gaunt, J. W. Essam, and C. J. Elliott, J. Phys. A 6 (1973) 1507.

[72] G. Bhanot, M. Creutz, and J. Lacki, Phys. Rev. Lett. 69 (1992) 1841.

[73] I. G. Enting, J. Phys. A 7 (1974) 1617.

[74] J. P. Straley, J. Phys. A 7 (1974) 2173.

[75] M. E. Fisher, Phys. Rev. Lett. 40 (1978) 1610.

[76] D. A. Kurtze and M. E. Fisher, Phys. Rev. B 20 (1979) 2785.
[77] D. A. Kurtze and M. E. Fisher, J. Stat. Phys. 19 (1978) 205.
[78] K. Uzelac, P. Pfeuty, and R. Jullien, Phys. Rev. Lett. 43 (1979) 805.
[79] G. A. Baker, Jr., M. E. Fisher, and P. Moussa, Phys. Rev. Lett. 42 (1979) 615.
[80] L. Mittag and M. J. Stephen, J. Stat. Phys. 35 (1984) 303.
[81] Z. Glumac and K. Uzelac, J. Phys. A 27 (1994) 7709.
[82] G. Parisi and N. Sourlas, Phys. Rev. Lett. 46 (1981) 871.
[83] S.-N. Lai and M. E. Fisher, J. Chem. Phys. 103 (1995) 8144.
[84] Y. Park and M. E. Fisher, Phys. Rev. E 60 (1999) 6323.
[85] D. Dhar, Phys. Rev. Lett. 51 (1983) 853.
[86] J. L. Cardy, Phys. Rev. Lett. 54 (1985) 1354.
[87] V. Matveev and R. Shrock, Phys. Rev. E 53 (1996) 254.
| $E$ | $\Omega_{Q=7}(E)$ |
|-----|------------------|
| 0   | 7                |
| 1   | 0                |
| 2   | 0                |
| 3   | 0                |
| 4   | 0                |
| 5   | 756              |
| 6   | 378              |
| ... | ...              |
| 61  | 8297099724190402024200 |
| 62  | 8931912475351510844640 |
| 63  | 8562381313020489303720 |
| ... | ...              |
| 70  | 60500919374783049960 |
| 71  | 9822183432973269120 |
| 72  | 794516261799074640 |
TABLE II: The ferromagnetic critical temperatures $Y^\text{FM}_c$ of the $Q$-state Potts models on the simple-cubic lattice. $Y^\text{FM}_c(\text{numer.})$ are the values of the critical temperatures estimated by various numerical methods in the literature.

| $Q$ | $Y^\text{FM}_c = 1/(Q + \sqrt{2Q - 1})^{1/3}$ | $Y^\text{FM}_c = 1/(\pi^{-1} + Q^{1/3})$ | $Y^\text{FM}_c(\text{numer.})$ |
|-----|-----------------------------------------------|-----------------------------------------------|--------------------------------|
| 1   | 0.793700...                                  | 0.758546...                                  | 0.753 [52, 53]                |
| 2   | 0.644689...                                  | 0.633620...                                  | 0.6419 [54, 55]              |
| 3   | 0.575879...                                  | 0.568001...                                  | 0.571 [56]                   |
|     |                                               |                                               | 0.5766 [57]                   |
|     |                                               |                                               | 0.5767 [58]                   |
| 4   | 0.531886...                                  | 0.524738...                                  | 0.523 [59]                   |
|     |                                               |                                               | 0.524 [60]                   |
|     |                                               |                                               | 0.532 [60]                   |
| 5   | $\frac{1}{2}$                                | 0.493027...                                  | 0.501 [61]                   |
| 6   | 0.475240...                                  | 0.468289...                                  | 0.472 [61]                   |
|     |                                               |                                               | 0.477 [61]                   |
| 7   | 0.455151...                                  | 0.448181...                                  | 0.456 [61]                   |

TABLE III: The locations ($A_{CT} = Y_{CT}^{-1}$) and the corresponding critical exponents of the complex-temperature (CT) singularities for the three-state Potts models on the simple-cubic lattice.

| $A_{CT}$                                 | $\alpha_{CT}$ | $\beta_{CT}$ | $\gamma_{CT}$ | $\alpha + 2\beta + \gamma$ |
|------------------------------------------|----------------|---------------|----------------|-------------------------------|
| $0.1036(49) \pm 2.0157(35)i$             | 1.107(8)       | $-0.063(1)$   | 1.161(38)      | 2.142(39)                     |
| $-1.6158(119) \pm 0.7958(142)i$          | 1.208(212)     | $-0.063(2)$   | 1.237(166)     | 2.319(269)                    |
TABLE IV: The locations and the edge critical exponents of the Fisher edge singularities for the Ising ($Q = 2$) and three-state Potts ($Q = 3$) models on the simple-cubic lattice for an external magnetic field $X = e^{\beta H_q}$.

| $X$    | $Y_e$             | $\alpha_e$ | $\beta_e$ | $\gamma_e$ | $\alpha_e+2\beta_e+\gamma_e$ |
|--------|-------------------|-------------|------------|-------------|--------------------------------|
| 100($Q = 2$) | 1.249(9) ± 0.642(4)$i$ | 1.079(54)   | −0.418(27) | 1.024(10)  | 1.267(67)                          |
|         | ±1.323(12)$i$      | 1.047(64)   | −0.410(86) | 1.075(78)  | 1.303(157)                          |
|         | −1.249(9) ± 0.642(4)$i$ | 1.079(54)   | −0.418(27) | 1.024(10)  | 1.267(67)                          |
| 200($Q = 2$) | 1.433(10) ± 0.737(4)$i$ | 1.046(62)   | −0.422(24) | 1.031(9)   | 1.233(71)                           |
|         | ±1.500(12)$i$      | 1.052(2)    | −0.415(107) | 1.058(125) | 1.280(197)                          |
|         | −1.433(10) ± 0.737(4)$i$ | 1.046(62)   | −0.422(24) | 1.031(9)   | 1.233(71)                           |
| 500($Q = 2$) | 1.649(11) ± 0.876(5)$i$ | 1.057(24)   | −0.425(24) | 1.038(9)   | 1.246(42)                           |
|         | ±1.771(21)$i$      | 1.050(5)    | −0.418(76) | 1.058(114) | 1.271(157)                          |
|         | −1.649(11) ± 0.876(5)$i$ | 1.057(24)   | −0.425(24) | 1.038(9)   | 1.246(42)                           |
| 100($Q = 3$) | 1.217(35) ± 0.573(21)$i$ | 1.065(141)  | −0.611(50) | 1.003(33)  | 0.846(161)                          |
|         | 0.052(13) ± 1.201(32)$i$ | 0.999(155)  | −0.608(14) | 1.076(134) | 0.858(206)                          |
|         | −1.070(25) ± 0.603(17)$i$ | 1.070(158)  | −0.603(17) | 1.064(60)  | 0.927(171)                          |
| 200($Q = 3$) | 1.342(36) ± 0.660(22)$i$ | 1.075(35)   | −0.605(16) | 0.998(82)  | 0.863(92)                           |
|         | 0.054(11) ± 1.359(38)$i$ | 0.996(245)  | −0.610(26) | 1.064(137) | 0.839(283)                          |
|         | −1.199(29) ± 0.682(21)$i$ | 1.039(227)  | −0.610(20) | 1.066(66)  | 0.885(238)                          |
| 500($Q = 3$) | 1.535(39) ± 0.787(24)$i$ | 1.071(17)   | −0.605(48) | 1.012(22)  | 0.872(73)                           |
|         | 0.056(8) ± 1.601(41)$i$ | 1.058(134)  | −0.615(32) | 1.076(106) | 0.905(177)                          |
|         | −1.397(33) ± 0.802(22)$i$ | 1.096(66)   | −0.613(37) | 1.063(155) | 0.934(177)                          |
TABLE V: The locations and the edge critical exponents of the Fisher edge singularities for the $Q$-state Potts models on the square lattice for $X = 100$.

| $Q$ | $Y_e$ | $\alpha_e$ | $\beta_e$ | $\gamma_e$ | $\alpha_e+2\beta_e+\gamma_e$ |
|-----|-------|-------------|-----------|-----------|---------------------------|
| 3   | $1.232(2) \pm 1.048(3)i$ | $1.217(19)$ | $-0.197(6)$ | $1.205(33)$ | $2.029(39)$ |
|     | $-1.063(13) \pm 1.068(6)i$ | $1.238(26)$ | $-0.172(25)$ | $1.257(69)$ | $2.151(82)$ |
| 4   | $1.159(5) \pm 0.929(10)i$ | $1.181(76)$ | $-0.180(4)$ | $1.121(77)$ | $1.941(76)$ |
|     | $-0.908(13) \pm 0.959(10)i$ | $1.136(17)$ | $-0.127(11)$ | $1.206(16)$ | $2.088(28)$ |
| 5   | $1.103(4) \pm 0.859(14)i$ | $1.181(109)$ | $-0.173(4)$ | $1.110(98)$ | $1.946(109)$ |
|     | $-0.834(18) \pm 0.889(27)i$ | $1.131(176)$ | $-0.109(28)$ | $1.207(166)$ | $2.120(245)$ |

FIG. 1: Partition function zeros in the complex $Y = e^{\beta J}$ plane of the $Q$-state Potts models on the $3 \times 3 \times 3$ simple-cubic lattice. The solid squares ($Y_c$) show the locations of the ferromagnetic critical points estimated in the literature for $Q = 2$[54, 55], 4[56], and 7[57].
FIG. 2: Partition function zeros in the complex $A = e^{\beta J}$ plane of the three-state Potts model on the $3 \times 3 \times 30$ simple-cubic lattice. The open triangles show the locations of the ferromagnetic ($A_{c}^{FM} = 1.7342$) and antiferromagnetic ($A_{c}^{AFM} = 0.4424$) critical points estimated in the literature. The X marks are the suggested locations of additional transition temperatures at $A_{KH} = 0.3679$, $A_{RL1} = 0.2806$ and $A_{RL2} = 0.2736$ (Because $A_{RL1}$ and $A_{RL2}$ are too close, they are not clearly distinguished in the Figure.), and $A_{KRL} = 0.2298$ below $A_{c}^{AFM}$ for the antiferromagnetic model. The open circles are the locations of two pairs of the complex-temperature (CT) singularities estimated from the series analysis.
FIG. 3: Partition function zeros in the complex $X = e^{\beta H_q}$ plane of the three-state Potts model on the $3 \times 3 \times 3$ simple-cubic lattice for $Y = e^{-\beta J} = 0.1$, $Y = Y_c \approx 0.5766^{57}$, $Y = 0.7$, and $Y = 0.9$. 

---

25
FIG. 4: Partition function zeros in the complex $Y = e^{-\beta J}$ plane of the three-state Potts model on the $3 \times 3 \times 3$ simple-cubic lattice for an external magnetic field $X = e^{\beta H_q}$. The closed circles show the locations of three pairs of the Fisher edge singularities estimated from the series analysis for $X = 100$. 