A MODULAR CHARACTERIZATION OF SUPERSOLVABLE LATTICES

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Abstract. We characterize supersolvable lattices in terms of a certain modular type relation. McNamara and Thomas earlier characterized this class of lattices as those graded lattices having a maximal chain that consists of left-modular elements. Our characterization replaces the condition of gradedness with a second modularity condition on the maximal chain of left-modular elements.

1. Introduction and background

A supersolvable lattice is one that is well-behaved in a similar way to the subgroup lattice of a supersolvable group. Stanley in a 1972 paper [17] defined a lattice $L$ to be supersolvable if $L$ admits a maximal chain $m$, which we call a chief chain (or $M$-chain), so that the sublattice generated by $m$ and any other chain $c$ is distributive. Since then, this class of lattices has been characterized in a number of different ways. We first give the list of characterizations, and will explain the terminology used shortly after.

Theorem 1.1 (Liu, McNamara and Thomas). For a finite lattice $L$ and maximal chain $m$, the following are equivalent.

1. $m$ is a chief chain (and so $L$ is supersolvable),
2. $L$ admits an $EL$-labeling with ascending chain $m$ so that every maximal chain is labeled with a permutation of a fixed label set $[n]$, and
3. $L$ is graded and $m$ consists of left-modular elements.

The direction (3) $\implies$ (2) was first proved in Liu’s PhD thesis [12]; see also [13]. McNamara showed (1) $\iff$ (2) as part of his own PhD work, and published the results in [14]. McNamara and Thomas noticed the connection with Liu’s work in [15], completing Theorem 1.1.

Stanley’s paper [17] on supersolvable lattices has been cited hundreds of times. Supersolvability is useful for understanding examples such as non-crossing partition lattices [10, 11]. Recent papers that discuss supersolvable lattices include [16].

We now explain the definitions used in Theorem 1.1. A edge labeling over $\mathbb{Z}$ of a lattice assigns an integer label to each edge of the Hasse diagram. Thus, we associate a word in the label set to each maximal chain on an interval. An $EL$-labeling is an edge labeling over $\mathbb{Z}$ so that each interval has a unique lexicographically earliest maximal chain, and so that this chain is the only ascending maximal chain on the interval. We will not discuss $EL$-labelings further here, although closely-related notions were a main motivation for the definition of supersolvable lattices [3, 17, 18].

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Modularity requires more discussion. A lattice is \textit{modular} if for every three elements \(z\) and \(x < y\), we have \((x \lor z) \land y = x \lor (z \land y)\). Equivalent to this condition, though not immediately so, is that for every such \(z\) and \(x < y\), we have either \(z \land x \neq z \land y\) or \(z \lor x \neq z \lor y\). Thus, the sublattice generated by \(x, y\), and \(z\) is not a pentagon, so we see that a modular lattice is exactly one with no pentagon sublattice.

One may think of a modular lattice as being analogous to the subgroup lattice of an abelian group. There are multiple conditions on lattice elements that are analogous to normality of a subgroup in the subgroup lattice of a finite group, most or all of which come down to requiring the modular identity \(x \lor (z \land y) = (x \lor z) \land y\) for certain elements \(z\) and \(x < y\). A pair of lattice elements \((z, y)\) is a modular pair if whenever \(x < y\), we have \(x \lor (z \land y) = (x \lor z) \land y\). Now we say that an element \(m\) of the lattice \(L\) is \textit{left-modular} if \((m, y)\) forms a modular pair for each \(y \in L\), and that \(m\) is \textit{right-modular} if \((z, m)\) is a modular pair for each \(z \in L\). If \(m\) satisfies both, we say it is \textit{two-sided modular}.

Dedekind’s modular identity from group theory says that if \(A\) and \(B \subseteq C\) are subgroups of a group \(G\), then \(B(A \cap C) = BA \cap C\). It follows easily from Dedekind’s modular identity that a normal subgroup in the dual of a subgroup lattice is two-sided modular. (It may be helpful to recall here that if \(A\) normalizes \(B\), then \(BA = AB = A \lor B\).) Stanley showed in \cite{Stanley} that if \(L\) admits a maximal chain of two-sided modular elements, then \(L\) is supersolvable. It is easy to find counterexamples to the converse, as supersolvability is a self-dual property, but right modularity is not. Theorem \cite{Stanley} \cite{2} is a partial converse to Stanley’s condition, but requires the global lattice property of gradedness.

In this paper, we add two conditions to the list of characterizations of supersolvable lattices. First, we give a characterization of supersolvability purely in terms of modular conditions on elements. We say that a chain of elements \(m\) in lattice \(L\) is \textit{right chain-modular} if for every pair \(x < y\) of elements in \(m\) and every \(z \in L\), we have \(x \lor (z \land y) = (x \lor z) \land y\). For convenience, we say that the chain \(m\) is \textit{chain-modular} if it is right chain-modular and if every element in the chain is left-modular.

\begin{theorem}

The maximal chain \(m\) of finite lattice \(L\) is a chief chain if and only if (4) \(m\) is a chain-modular maximal chain.

\end{theorem}

We will give two different proofs of Theorem \cite{12} both showing directly that condition (4) implies condition (1), and also that it implies condition (3). A consequence will be a new proof of the equivalence of these two conditions that avoids the theory of \(EL\)-labelings. Thomas previously gave such a proof in \cite{20} (see also \cite{21}), but this proof required the theory of free lattices. Our proofs will be more elementary.

An additional result of this paper will be useful as an intermediate step in some of our proofs, and extends conditions well-known in the special case of geometric lattices (see \cite{17} Corollary 2.3] and \cite{5} Theorem 3.3], also \cite{2}). We say that an element \(m\) of the
A graded lattice \( L \) with rank function \( \rho \) is rank modular if for every \( x \) in \( L \) the identity 
\[
\rho(m \lor x) + \rho(m \land x) = \rho(m) + \rho(x)
\]
holds. We will show:

**Lemma 1.3.** Let \( L \) be a graded finite lattice with rank function \( \rho \). The element \( m \) of \( L \) is left-modular if and only if \( m \) is rank modular.

We say that a maximal chain is rank modular if it consists of rank modular elements. In view of Theorem 1.1 (3), an immediate consequence of Lemma 1.3 will be:

**Theorem 1.4.** The maximal chain \( m \) of finite lattice \( L \) is a chief chain if and only if 
(5) \( L \) is graded, and \( m \) is a rank modular maximal chain.

**Remark 1.5.** The role of a chief chain in a supersolvable lattice generalizes that of the chief series in the subgroup lattice of a supersolvable group. This explains our terminology. Stanley [17, 19] and other authors call this chain an \( M \)-chain. Theorem 1.2 explains exactly the extent to which this \( M \) stands for “modular.”

A basic example of a supersolvable lattice is that of the partition lattice \( \Pi_n \), consisting of all partitions of \([n]\), ordered by refinement. The elements in \( \Pi_n \) having at most one non-singleton block are easily seen to coincide with those satisfying the rank modular condition. Björner [4] and Haiman [8] extended the family of finite partition lattices to a non-discrete limit lattice. They observed that a similar condition to that of Theorem 1.4 holds in this setting with respect to a generalized rank function. It would be interesting to extend other characterizations of supersolvable lattices to the non-discrete setting.

**Organization.** This paper is organized as follows. In Section 2 we lay out a lemma and other preliminary material for use throughout. In Section 3 we prove that Condition (4) is equivalent to Condition (1), which gives one proof of Theorem 1.2. In Section 4 we first show that Conditions (3) and (5) are equivalent, then give a proof, independent of Section 3, that both are equivalent to (4). This gives a second proof of Theorem 1.2. When combined with Section 3, the results of this section give a new simple proof that Condition (3) implies Condition (1). In Section 5, we give a simple condition for checking chain-modularity of a maximal chain. We close in Section 6 by asking about generalizations to non-discrete lattices.

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## 2. Preliminaries

In this section, we give some basic results on modular elements in a lattice. Additional background can be found in textbooks such as [19, Chapter 3] or [7, Chapter V]. We generally follow the notation of [19, Chapter 3].

First, in any lattice \( L \) and for any \( x < y \) and \( z \), we have the modular inequality
\[
x \lor (z \land y) \leq (x \lor z) \land y.
\]

Thus, the modular conditions on elements that we have discussed give sufficient conditions for the modular inequality to be an equality.
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Figure 2.1. A strict modular inequality on \( x < y \) and \( z \) yields a pentagon sublattice.

The general form of a lattice generated by \( x < y \) and \( z \) is shown in Figure 2.1 (see [7, Figure 6]). It is easy to see from this figure that an element is left-modular if and only if it avoids being the short side of a pentagon sublattice, but that the situation with right-modularity or right chain-modularity is somewhat more complicated. Here, by the short side of the pentagon sublattice, we mean the element \( z \) of Figure 2.1; similarly, by the long side we mean the elements \( x < y \).

Lemma 2.1 (Pentagonal characterization of left-modularity [13, Theorem 1.4]). An element \( z \) of a lattice \( L \) is left-modular if and only if for each \( x < y \) either \( x \lor z \neq y \lor z \) or \( x \land z \neq y \land z \).

If \( z \) is left-modular and in addition \( x \leq y \), then exactly one of the equalities \( x \lor z = y \lor z \) and \( x \land z = y \land z \) holds.

3. Equivalence with supersolvability

In this section, we show the equivalence of Conditions (1) and (4). That is, we show that a maximal chain \( m \) is a chief chain if and only if it is chain-modular.

If \( m \) is a chief chain, then the sublattice generated by \( m \) and any other chain is distributive. Since distributive lattices have no pentagon sublattices, it follows immediately by considering 1-element chains that \( m \) is right chain-modular, and by considering 2-element chains that \( m \) consists of left-modular elements.

Conversely, if \( m \) is a right chain-modular maximal chain then, following Stanley [17, Proposition 2.1] and Birkhoff [2, pp. 65-66], it suffices to prove a pair of dual identities, as follows. For the reader’s convenience, we use notation similar to that in [17].
Lemma 3.1. Let $L$ be a lattice and $m$ be a chain-modular chain in $L$. If $a_1 \geq \cdots \geq a_r$ are elements from $m$ and $b_1 \leq \cdots \leq b_r$ are elements from an arbitrary chain in $L$, then

\begin{align*}
(3.1) & \quad (b_1 \lor a_1) \land \cdots \land (b_r \lor a_r) = b_1 \lor (a_1 \land b_2) \lor \cdots \lor (a_{r-1} \land b_r) \lor a_r, \quad \text{and} \\
(3.2) & \quad (a_1 \land b_1) \lor \cdots \lor (a_r \land b_r) = a_1 \land (b_1 \lor a_2) \land \cdots \land (b_{r-1} \lor a_r) \land b_r.
\end{align*}

Proof. We proceed by simultaneous induction on $r$. We may assume without loss of generality that $\hat{0}, \hat{1} \in m$.

By duality, it suffices to prove the first identity, so we take $x$ to be the left-hand side element of (3.1). We apply several modular identities. First, since $a_1 \leq (b_2 \lor a_2) \land \cdots (b_r \land a_r)$, by left-modularity of $a_1$ we can rewrite

\[ x = b_1 \lor \left( a_1 \land (b_2 \lor (a_2 \land b_3) \lor \cdots (a_{r-1} \land b_r) \lor a_r) \right), \]

where the latter equality is by induction. Now, as $a_r \leq a_1$ are in $m$, we can apply right chain-modularity to rewrite

\[ x = b_1 \lor \left( a_1 \land \left( b_2 \lor (a_2 \land b_3) \lor \cdots (a_{r-1} \land b_r) \lor a_r \right) \right) \lor a_r. \]

At this point, we notice that $b_2 = \hat{1} \land b_2$, so we can apply (3.2) inductively to rewrite

\[ x = b_1 \lor \left( a_1 \land \left( (b_2 \lor a_2) \land \cdots (b_{r-1} \lor a_{r-1}) \land b_r \right) \right) \lor a_r \]

\[ = b_1 \lor \left( a_1 \land (b_2 \lor a_2) \land \cdots (b_{r-1} \lor a_{r-1}) \land b_r \right) \lor a_r. \]

Now a second inductive application of (3.2) yields the desired. \qed

Remark 3.2. We replace the two separate arguments of Stanley [17] Proposition 2.1 with a simultaneous and self-dual induction, where the $r$th case of (3.1) depends on the $(r-1)$st case of both (3.1) and (3.2). This argument seems basically simpler to us than that of Stanley, and we were surprised not to find it in the literature. We do comment that Thomas gave a self-dual argument from generally similar hypotheses in [20, Lemma 9].

Since it does not seem to be as well-known as it should be, we briefly overview the rest of the proof from [2, pp. 65-66]; this proof is also discussed in [7, Theorem 363]. Let $m = \{m_i\}$ be a chain satisfying the conclusion of Lemma 3.1 and $c = \{c_j\}$ be any other chain. Assume without loss of generality that $m$ and $c$ both contain $\hat{0}$ and $\hat{1}$. Birkhoff considers the set $S$ of elements that can be written as a join of elements of the form $m_i \land c_j$, and the set $S^*$ of elements that can be written as a meet of elements of the form $m_i \lor c_j$. He uses (a restricted version of) Lemma 3.1 to show that $S = S^*$, and so that both coincide with the sublattice generated by $m$ and $c$.

It is easy to see (with no modularity assumption) that if $x = (m_{i_1} \land c_{j_1}) \lor \cdots \lor (m_{i_t} \land c_{j_t})$, then the $i$ indices may be chosen to be increasing and the $j$ indices to be decreasing. It follows after a bit of work that the sublattice generated by $m$ and $c$ is a homomorphic image of the lattice of order ideals in the direct product $m \times c$. Here,
the homomorphism sends the principal order ideal below the pair \((m_i, c_j)\) to \(m_i \land c_j\). Thus, the sublattice generated by \(m\) and \(c\) is the homomorphic image of a lattice of sets under the operations \(\cup\) and \(\cap\), hence is distributive.

4. Equivalence with graded left-modularity

In this section, we show directly the equivalence of Condition (4) with Conditions (3) and (5).

We first prove Lemma 1.3, which immediately gives equivalence of Conditions (3) and (5). We begin by observing:

Lemma 4.1. Let \(L\) be a graded lattice with rank function \(\rho\). If \(z\) together with \(x < y\) form a pentagon sublattice of \(L\) (so that \(z \land x = z \land y\) and \(z \lor x = z \lor y\)), then we have

\[
\rho(z \lor x) + \rho(z \land x) \neq \rho(z) + \rho(x) \quad \text{or} \quad \rho(z \lor y) + \rho(z \land y) \neq \rho(z) + \rho(y).
\]

Proof. The left-hand sides agree, but \(\rho(x) < \rho(y)\).

Lemma 1.3 follows easily from Lemma 4.1 together with Lemma 2.1.

Proof (of Lemma 1.3). If \(m\) is not left-modular, then \(L\) has a pentagon sublattice containing \(m\), which then fails the rank modular identity by Lemma 4.1.

Conversely, if \(m\) is a left-modular element and \(y\) is some other element of \(L\), then consider a maximal chain \(x_0 \ll \cdots \ll x_k\) on \([m \land y, y]\). As each \(x_i\) satisfies \(m \land x_i = m \land y\), repeated application of Lemma 2.1 shows that \(x_k \lor m < x_1 \lor m < \cdots < x_k \lor m\) are distinct elements on \([m, m \lor y]\). Thus, \(k = \rho(y) - \rho(m \land y) \leq \rho(m \lor y) - \rho(m)\). Now the same argument on the dual lattice yields that \(\rho(y) - \rho(m \land y) \geq \rho(m \lor y) - \rho(m)\), as desired.

We now show that Condition (5) implies Condition (4) by a straightforward calculation. If \(m_i < m_j\) are rank modular elements in the graded lattice \(L\) and \(z\) is some other element, then three applications of the rank modular identity yields that

\[
\rho(m_i \lor (z \land m_j)) = \rho(m_i) + \rho(z \land m_j) - \rho(z \land m_i)
\]

\[
= \rho(z \land m_j) - \rho(z) + \rho(z \lor m_i)
\]

\[
= \rho(m_j) - \rho(z \lor m_j) + \rho(z \lor m_i).
\]

A single application of the same identity then yields that \(\rho((m_i \lor z) \land m_j) = \rho(m_i \lor (z \land m_j))\). Since the ranks agree, the modular inequality is an equality, as desired.

It remains only to show that Condition (4) implies Condition (5). Let \(m\) consisting of \(0 = m_0 < m_1 < \cdots < m_n = \hat{1}\) be a chain-modular maximal chain. Define a function \(\rho : L \to \mathbb{N}\) by

\[
\rho(y) = \# \{ i : m_{i+1} \land y > m_i \land y \}.
\]

Since when \(x < y\), it holds that \(m_{i+1} \land x > m_i \land x\) only if \(m_{i+1} \land y > m_i \land y\), the function \(\rho\) is weakly order preserving. We’ll show that indeed \(\rho\) is the rank function, and that \(m\) is rank modular.

We first verify the rank modularity condition. It is enough to show that \(\rho(x)\) and \(\rho(x \lor m_j) + \rho(x \land m_j) - \rho(m_j)\) count the same subset of the modular chain. Indeed, for any \(x \in L\), we notice that:
• $\rho(x \lor m_j) = \# \{i : (m_j \lor x) \land m_{i+1} > (m_j \lor x) \land m_i\}$. Those $i$ with $i < j$ clearly always satisfy the condition. When $i \geq j$, the condition becomes $m_j \lor (x \land m_{i+1}) > m_j \lor (x \land m_i)$ by right chain-modularity. Clearly if $x \land m_{i+1} = x \land m_i$ then $i$ does not satisfy the condition (and is not counted). Conversely, if $x \land m_{i+1} > x \land m_i$, then since $m_j \land (x \land m_{i+1}) = m_j \land (x \land m_i)$, Lemma 2.1 yields that $m_j \lor (x \land m_{i+1}) > m_j \lor (x \land m_i)$.

In short, $\rho(x \lor m_j)$ counts the number of $i \geq j$ for which $m_{i+1} \land x > m_i \land x$, together with all $i < j$.

• $\rho(x \land m_j) = \# \{i : (m_j \land x) \land m_{i+1} > (m_j \land x) \land m_i\}$ clearly counts the number of $i < j$ for which $x \land m_{i+1} > x \land m_i$.

• $\rho(m_j) = j$ counts the $m_i$'s with $i < j$.

Thus, among the indices counted by $\rho(x)$, we see that $(\rho(x \lor m_j) - \rho(m_j))$ counts exactly those indices $i$ with $i \geq j$, while $\rho(x \land m_j)$ counts the indices $i$ with $i < j$.

Now we finish the proof by using the rank modular condition to show that if $y > x$, then also $\rho(y) = (\rho(x) + 1)$. For in this case, let $i$ be the least index for which $x \land m_{i+1} < y \land m_{i+1}$. Then $x \lor m_{i+1} = y \lor m_{i+1}$ by Lemma 2.1. As $m_i \land x \land m_{i+1} = m_i \land y \land m_{i+1}$ by minimality of $i$, a second application of Lemma 2.1 gives that $m_i \lor x \land m_{i+1} < m_i \lor y \land m_{i+1}$. Thus, $m_i \land x \land m_{i+1} = m_i$ and $m_i \lor y \land m_{i+1} = m_{i+1}$. We apply rank modularity twice to calculate

$$
\rho(y) - \rho(x) = \rho(y \land m_{i+1}) - \rho(x \land m_{i+1})
\quad = \rho(m_i \lor y \land m_{i+1}) - \rho(m_i \lor x \land m_{i+1})
\quad = \rho(m_{i+1}) - \rho(m_i) = (i + 1) - i = 1,
$$

as desired.

5. How to check chain-modularity

In this section, we reduce checking right chain-modularity of a maximal chain $m$ consisting of left-modular elements to checking the condition on the cover relations of $m$.

The main lemma that allows us to do this is as follows:

**Lemma 5.1.** Let $m_1 < m_2 < m_3$ be elements in a lattice $L$, where $m_2$ is left-modular. If $m_1 < m_2$ and $m_2 < m_3$ are both right chain-modular, then also $m_1 < m_2 < m_3$ is right chain-modular.

**Proof.** It remains to check that $m_1 < m_3$ satisfies the required right modularity property. Suppose for contradiction that there is some $y$ so that $m_1 \lor (y \land m_3) < (m_1 \lor y) \land m_3$. Then since

$$
m_1 \lor y \land m_2 = m_1 \lor (y \land m_3) \land m_2 < m_1 \lor (y \land m_3),$$

while

$$m_2 \lor y \land m_3 = m_2 \lor (m_1 \lor y) \land m_3 > m_2 \lor (m_1 \lor y),$$

we have $m_2$ incomparable to $m_1 \lor (y \land m_3) < (m_1 \lor y) \land m_3$. But now these three elements generate a pentagon sublattice, contradicting left-modularity of $m_2$. $\square$

A straightforward inductive argument now yields:
Theorem 5.2. Let $L$ be a finite lattice, and $m$ a maximal chain consisting of $0 = m_0 \leq m_1 \leq \cdots \leq m_n = 1$. If each $m_i$ is left-modular, and each $m_i \leq m_{i+1}$ is right chain-modular, then $m$ is chain-modular.

Checking whether a cover relation is right chain-modular is easier than checking an arbitrary chain of two elements. Indeed, in contrast to right chain-modularity for an arbitrary pair of elements (see Figure 2.1), we can reduce right chain-modularity of a cover relation to the following “no pentagon” condition.

Lemma 5.3. If $m_i \leq m_{i+1}$ is a cover relation in lattice $L$, then the pair is right chain-modular if and only if $m_i \leq m_{i+1}$ does not form the long side of any pentagon sublattice of $L$.

Proof. For $m < m'$ with $m \lor (x \land m') < (m \lor x) \land m'$, the latter two elements together with $x$ form a pentagon sublattice. If $m_i \leq m_{i+1}$ is a cover relation, then $m_i \lor (x \land m_{i+1})$ is either $m_i$ or $m_{i+1}$; similarly for $(m_i \lor x) \land m_{i+1}$. It follows that if $m_i \leq m_{i+1}$ fails to be modular, then $m_i \leq m_{i+1}$ forms the long side of a pentagon sublattice. \hfill $\square$

Thus, Theorem 1.2 says that a chain $m$ of a lattice $L$ is a chief chain if and only if no pentagon sublattice of $L$ has an element of $m$ as its short side, nor any cover relation of $m$ as its long side.

6. Questions

We came to look at finite supersolvable lattices in this light because of an interest in non-discrete lattices, such as the continuous partition lattice of [4, 8]. This lattice is $[0, 1]$-graded, meaning that there is an order-preserving function from $L$ to the real interval $[0, 1]$ that restricts to a bijection on maximal chains. Moreover, the lattice has a maximal chain whose elements satisfy the rank modular condition. Which of our characterizations extend from the finite or discrete case to the $[0, 1]$-graded and other non-discrete cases?

**Question 6.1.** Is there a description of $[0, 1]$-graded lattices having a rank modular maximal chain in terms of element- and/or chain-wise modularity?

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