Invariants of mixed representations of quivers I

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Introduction

The concept of a representation of a quiver was introduced in [Gab]. If we consider all representations of a quiver of given dimension as an affine variety provided with the action of its automorphism group then the points of the corresponding categorical quotient can be parametrized by semisimple representations. Moreover, this quotient is also an affine variety and its coordinate algebra is generated by all polynomial invariants. In the characteristic zero case invariants of representations of quivers were first described in [PrB1, PrB2]. This result was applied to investigate an etale local structure of categorical quotients of quiver representation spaces [PrB1, PrB2].

The modular case was explored in [Don1, Zub4]. In [Don1] invariants of arbitrary quiver were described over any infinite field. In [Zub4] all defining relations between them are described too. We note that the last result was proved independently in [Dom] for the characteristic zero case. Finally, in [DZ2] the main results from [PrB1, PrB2] concerning an etale local structure of invariants of a quiver were extended to the case of any algebraically closed field.

No doubt the next step should be to generalize these statements for other classical groups, specifically to the orthogonal and symplectic groups. It is clear that one has to start with the action of $O(n)$ or $Sp(n)$ on $m$-tuples of $n \times n$ matrices by simultaneous conjugation. Using the so-called transfer principle [Gr] one can reduce this problem to a representation of some quiver. This representation is a new type of representations of quivers called mixed representations.

Recall some necessary definitions and notations (see [Gab, Don1, PrB1, PrB2]). A quiver is a quadruple $Q = (V, A, i, t)$, where $V$ is a vertex set and $A$ is an arrow set of $Q$. Let the maps $i, t : A \to V$ associate to each arrow $a \in A$ its origin $i(a) \in V$ and its end $t(a) \in V$. We enumerate elements of the vertex set as $V = \{1, \ldots, n\}$.

We consider a collection of vector spaces $E_1, \ldots, E_n$ over an algebraically closed field $K$. Set $\dim E_1 = d_1, \ldots, \dim E_n = d_n$. Denote by $\mathbf{d}$ the vector $(d_1, \ldots, d_n)$. This vector is called a dimension vector. For two dimension vectors $\mathbf{d}(1), \mathbf{d}(2)$ we write $\mathbf{d}(1) \geq \mathbf{d}(2)$ iff $\forall i \in V, d(1)i \geq (2)i$. 

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Denote by $GL(d)$ the group $GL(E_1) \times \ldots \times GL(E_n) = GL(d_1) \times \ldots \times GL(d_n)$. The representation space of a quiver $Q$ of dimension $d$ is $R(Q, d) = \prod_{a \in A} \text{Hom}_K(E_{i(a)}, E_{t(a)})$.

The group $GL(d)$ acts on $R(Q, d)$ by the rule:

$$(y_a)_{a \in A} = (g_{t(a)}y_ag_{h(a)})_{a \in A}, g = (g_1, \ldots, g_n) \in GL(d),$$

$$(y_a)_{a \in A} \in R(Q, d).$$

For example, if our quiver $Q$ has one vertex and $m$ loops which are incident to this vertex then the $d = (d)$-representation space of this quiver is isomorphic to the space of $m \times d$-matrices with respect to the diagonal action of the group $GL(d)$ by conjugation.

The coordinate ring of the affine variety $R(Q, d)$ is isomorphic to $K[y_{ij}(a) \mid 1 \leq j \leq d_{i(a)}, 1 \leq i \leq d_{t(a)}, a \in A]$. For any $a \in A$ denote by $Y_d(a)$ the general matrix $(y_{ij}(a))_{1 \leq j \leq d_{i(a)}, 1 \leq i \leq d_{t(a)}}$. The action of $GL(d)$ on $R(Q, d)$ induces the action on the coordinate ring by the rule $Y_d(a) \mapsto g_{t(a)}^{-1}Y_d(a)g_{h(a)}, a \in A$. We omit the lower index $d$ if it does not lead to confusion. For example, we write $Y(a)$ instead of $Y_d(a)$.

Let us partition the vertex set of the quiver $Q$ into several disjoint subsets. To be precise, let $V = V_{ord} \bigcup (\bigcup_{q \in \Omega} V_q)$. The vertices from $V_{ord}$ are said to be ordinary. We require that all subsets $V_q$ have cardinality two, that is for any $q \in \Omega$ $V_q = \{i_q, j_q\}$.

A dimension vector $d$ is said to be compatible with this partition of $V$ if for any $q \in \Omega, d_{i_q} = d_{j_q} = d_q$. From now on all dimension vectors are compatible with some fixed partition $V = V_{ord} \bigcup (\bigcup_{q \in \Omega} V_q)$ unless otherwise stated.

The next step is to replace all $E_{jq}, q \in \Omega$ by their duals. To indicate that some vertices correspond to the duals of vector spaces we introduce a new dimension vector $t = (t_1, \ldots, t_l)$, where $t_i = d_i$ iff we assign to $i$ the space $E_i$, otherwise $t_i = d_i^*$. We call $t$ the vector underlying $t$. This notation will be used throughout.

By definition, the $t$-dimensional representation space of the quiver $Q$ is equal to the space $R(Q, t) = \prod_{a \in A} \text{Hom}_K(W_{i(a)}, W_{t(a)})$, where $W_i = E_i$ iff $t_i = d_i$, otherwise $W_i = E_i^*$.

The space $R(Q, t)$ is a $G = GL(d)$-module under the same action

$$(y_a)_{a \in A} = (g_{t(a)}y_ag_{h(a)})_{a \in A}, g = (g_1, \ldots, g_l) \in G,$$

$$(y_a)_{a \in A} \in R(Q, t).$$

If $\Omega = \emptyset$ then $t = d$ and $R(Q, t) = R(Q, d)$. Without loss of generality one can identify the coordinate algebras $K[R(Q, t)]$ and $K[R(Q, d)]$.

Finally, replacing all subfactors $GL(E_{i_q}) \times GL(E_{j_q}) = GL(d_q) \times GL(d_q)$ of the group $G = GL(d)$ by their diagonal subgroups we get a new group $H(t)$. The space $R(Q, t)$ with respect to the action of the group $H(t)$ is called the mixed representation space of the quiver $Q$ of dimension $t$ relative to the partition $V = V_{ord} \bigcup (\bigcup_{q \in \Omega} V_q)$. 

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Example 1 Let $V_{ord} = \emptyset, \Omega = \{q\}, i_q = 1, j_q = 2, A = \{a_1, \ldots, a_m b, c\}, i(a_k) = t(a_k) = i(b) = t(c), t(b) = i(c) = 2$. The mixed representation space of this quiver of dimension $t = (d,d^*)$ can be identified with $M(d)^m \times M(d)^2$, where the first $m$ $d \times d$ matrix coordinates correspond to the loops $a_1, \ldots, a_m$ but the two last ones correspond to the arrows $b, c$ respectively. The group $GL(t) = GL(d)$ acts on this mixed representation space by the rule

$$(A_1, \ldots, A_m, B, C)^g = (gA_1g^{-1}, \ldots, gA_mg^{-1}, (g^t)^{-1}Bg^{-1}, gCg^t),$$

$A_i, B, C \in M(d), g \in GL(d), 1 \leq i \leq m.$

This special case of mixed representations of quivers first appeared in [Zub5] to compute the invariants of orthogonal or symplectic groups acting diagonally by conjugations on several matrices.

We formulate the following.

Problem 1 What are the generators and the defining relations between them for the ring $J(Q, t) = K[R(Q, t)]^{H(t)}$?

The principal aim of this article is to answer the first part of this question as well as to prepare some necessary facts to answer the second part in the next article.

To formulate the main result of this article we need some additional definition. Let us define a doubled quiver $Q^{(d)}$. The vertex set $V^{(d)}$ of this quiver is equal to $V \cup V_{ord}^*$, where $V_{ord}^* = \{i^* \mid i \in V_{ord}\}$. Respectively, the arrow set $A^{(d)}$ of $Q^{(d)}$ is equal to $A \cup \overline{A}$, where $\overline{A} = \{\overline{a} \mid a \in A\}$. Further, if $i(a), t(a) \in V_{ord}$ then $i(\overline{a}) = t(a)^*, t(\overline{a}) = i(a)^*$ but if $i(a)$ or $t(a)$ lies in some $V_q, q \in \Omega$, then

$$i(\overline{a}) = \left\{ \begin{array}{ll} j_q, t(a) = i_q, \\
                 i_q, t(a) = j_q \end{array} \right.$$ 

and symmetrically

$$t(\overline{a}) = \left\{ \begin{array}{ll} j_q, i(a) = i_q, \\
                 i_q, i(a) = j_q \end{array} \right..$$

Finally, for any $a \in A^{(d)}$ we suppose $Z(a) = Y(a)$ if $a \in A$ otherwise $a = \overline{b}, b \in A$ and $Z(a) = Y(b)^t$, where $Y(b)^t$ is transpose of $Y(b)$.

A product $Z(a_m) \ldots Z(a_1)$ is said to be admissible if $a_m, \ldots, a_1$ is a closed path in $Q^{(d)}$, that is if $t(a_i) = i(a_{i+1}), i = 1, \ldots, m-1$ and $i(a_1) = t(a_m)$. A pair $Z(a)Z(b)$ is said to be linked if $t(b) = i(a)$. It is clear that $Z(a_m) \ldots Z(a_1)$ is admissible iff all pairs $Z(a_i)Z(a_i), i = 1, \ldots, m-1$, and $Z(a_1)Z(a_m)$ are linked.

Using the theory of modules with good filtration as well as some reductions developed in [Don2, Zub5] we prove
Theorem 1 The algebra $J(Q, t)$ is generated by the elements $\sigma_j(Z(a_r \ldots Z(a_1)))$, where $1 \leq j \leq \max \{d_i\}$, $a_r, \ldots, a_1$ is a closed path in the double quiver $Q^{(d)}$ and $\sigma_j$ is $j$-th coefficient of characteristic polynomial.

One can define more general supermixed representations of quivers involving as special cases mixed representations of quivers and orthogonal (symplectic) representations of symmetric quivers introduced in [DW3]. To be precise, let $R(Q, t)$ be the mixed representation space of a quiver $Q$ of dimension $t = (t_1, \ldots, t_q)$ with respect to some partition of $V$, say $V = V_{ord} \sqcup (\sqcup_{q \in \Omega} V_q)$, as above. By definition $t$ is compatible with this partition.

Replace some factors of the group $H = H(t) = (\prod_{i \in \text{ord}} GL(d_i)) \times (\prod_{q \in \Omega} GL(d_q))$ by orthogonal or symplectic subgroups requiring additionally that the characteristic of the ground field is odd if at least one factor is replaced by an orthogonal group. Denote the subgroup of $H$ obtained in this way as $G = (\prod_{i \in \text{ord}} G_i) \times (\prod_{q \in \Omega} G_q)$, where each factor $G_i(G_q)$ is either group $GL(d_i)$, orthogonal group, or symplectic group of given dimension.

Next, let us extract among all components $\text{Hom}_K(W_{h(a)}, W_{t(a)})$, $a \in A$, those having property $i(a), t(a) \in V_q, q \in \Omega$. Let $i(a) = i, t(a) = j$. We have three cases: $G_q = GL(d_q)$, $G_q = O(d_q)$ or $G_q = Sp(d_q)$.

Let us consider the first case $G_q = GL(d_q)$. Let $i = j_q, j = i_q$ or $i = j_q, j = j_q$, that is $t_i = d_q^*$, $t_j = d_q$ or $t_i = d_q, t_j = d_q^*$. Identifying $\text{Hom}_K(W_i, W_j)$ with $M(d_q)$ one can replace this space by its subspaces of symmetric or skew-symmetric matrices. In notations of [DW3] these subspaces can be identified with $S^2(V)(S^2(V^*))$ or $\Lambda^2(V)(\Lambda^2(V^*))$ respectively in obvious way as a $GL(d_q)$-modules, where $V = E_{i_q} = E_{j_q}$.

In two remaining cases it does not matter if $(t_i, t_j)$ coincide with $(d_q^*, d_q)$ or with $(d_q, d_q^*)$. Indeed, $V \cong V^*$ as a $O(V)$ or $Sp(V)$-module. If $G_q = O(d_q)$ then one can replace the space $\text{Hom}_K(W_i, W_j) = M(d_q)$ by its subspaces of symmetric or skew-symmetric matrices again.

In the case $G_q = Sp(d_q)$ one can replace the space $\text{Hom}_K(W_i, W_j) = M(d_q)$ by its subspaces $\text{Lie}(Sp(d_q)) = \{A \in M(d_q) \mid AJ$ is symmetric matrix$\}$ or $\{A \in M(d_q) \mid AJ$ is a skew-symmetric matrix$\}$, where $J = J_{d_q}$ is a $d_q \times d_q$ skew-symmetric matrix of the bilinear form defining the group $Sp(d_q)$.

Denote a subspace of $R(Q, t)$ obtained with the help of some replacements described above by $S$. A pair $(S, G)$ is said to be a supermixed representation space of the quiver $Q$ with respect to the induced action of the group $G$.

Example 2 The space of $m \times d \times d$ matrices with respect to the diagonal action of $O(d)$ or $Sp(d)$ by conjugation is a supermixed representation space of the quiver $Q$ with one vertex and $m$ loops incident to this vertex.

The invariants of the supermixed representation space from Example 2 can be obtained by specialization of invariants of mixed representation space from Example 1 [Zub5]. This case is typical. In fact, we prove
Theorem 2 Let \((S,G)\) be a supermixed representation space of a quiver \(Q\). There exists a quiver \(Q'\) such that the algebra \(K[S]^G\) is an epimorphic image of \(K[R(Q',t)]^{H(t)}\) for some dimension vector \(t\).

The mixed or supermixed representations of quivers naturally arose from the actions of \(O(n)\) or \(Sp(n)\) on several \(n \times n\) matrices by simultaneous conjugation. If we replace \(O(n)\) by its subgroup \(SO(n)\) then we will have to investigate semi-invariants of mixed representations of quivers. In other words, one can set the problems to find the generators and defining relations between them for semi-invariants of mixed representations of quivers.

The problem to describe semi-invariants of ordinary representations of quivers was very popular during the last 20 years starting with the remarkable Kac’s article [Ka]. Important results were obtained in [S1, S2]. There is also an extensive literature on semi-invariants of Dynkin and Euclidean (or extended Dynkin) quivers, see [As], [Ri], [Ko1], [Ko2], [HH], [SwWl], [SkW]. The complete descriptions of semi-invariants for an arbitrary quiver were obtained in [DW1, DW2] and [DZ]. In the characteristic zero case the similar result was proved in [SV]. I believe that the method of this article will also make possible to describe semi-invariants of mixed or supermixed representations of quivers.

1 Preliminaries

1.1 Induced modules and good filtrations

Let \(G\) be an algebraic group, \(H\) a closed subgroup of \(G\), and \(A\) a rational \(H\)-module. Then \(K[G] \otimes A\) is naturally a rational \(G \times H\)-module with respect to the action \((g,h) \cdot f \otimes a = f^{(g,h)} \otimes ha\), where \(g \in G, h \in H, f \in K[G], a \in A\) and \(f^{(g,h)}(x) = f(g^{-1}xh)\). The set of \(H\)-fixed points is a rational \(G\)-submodule, called the induced module \(\text{ind}_G H A = (K[G] \otimes A)^H\) [Gr].

Proposition 1.1 ([Gr], Theorem 9.1) If \(X\) is an affine \(G\)-variety, that is \(G\) acts rationally on \(X\), then the invariant algebra \(K[X]^G\) is isomorphic to \((K[X] \otimes k[G/H])^G\), where \(G\) acts on \(K[G/H]\) by left translation. The isomorphism is given by \(a \otimes f \mapsto af(eH)\).

Let \(G\) be a reductive group. Fix some maximal torus of the group \(G\), say \(T\), and a Borel subgroup \(B\) containing \(T\). The group \(B\) has a semi-direct decomposition \(B = T \bowtie U\), where \(U\) is a maximal unipotent subgroup of the group \(B\). Denote by \(X(T)\) the character group of the torus \(T\) and by \(X(T)^+\) the dominant weight subset of \(X(T)\) corresponding to \(B\). If \(\mu \in X(T)^+\) then denote by \(\nabla(\mu)\) the induced module \(\text{ind}_{B^-}^G K_\mu\), where \(B^-\) is the opposite Borel subgroup and \(K_\mu\) is the one-dimensional \(B^-\)-module with respect to the action \((tu) \cdot x = \mu(t)x, t \in T, u \in U^-, x \in K_\mu\).

We say that a \(G\)-module \(V\) has a good filtration (briefly GF) if there is some filtration with at most countable number of members.
such that \( \forall i \geq 1, V_i / V_{i-1} \cong \triangledown(\mu_i) \). Respectively, we say that a \( G \)-module \( W \) has a \textit{Weyl filtration} (briefly WF) if there is some filtration with at most countable number of members

\[
0 \subseteq W_1 \subseteq W_2 \subseteq \ldots, \bigcup_{i=1}^{\infty} W_i = W
\]

such that \( \forall i \geq 1, W_i / W_{i-1} \cong \triangleleft(\mu^*_i) \), where \( \triangleleft(\mu^*) \), \( \mu^* = -w_0(\mu) \) and \( w_0 \) is the longest element of the Weyl group \( W(G, T) = N_G(T)/T \).

It is clear that a finite-dimensional \( G \)-module \( V \) has WF iff the dual module \( V^* \) has GF. A finite-dimensional module \( V \) is called a \textit{tilting} one if both \( V \) and \( V^* \) are with GF. In other words, \( V \) has good and Weyl filtrations simultaneously.

We list some standard properties of modules having GF [Jan, Don3, Don5, Mat1].

**Theorem 1.1**

1. If

\[
0 \to V \to W \to S \to 0
\]

is a short exact sequence of \( G \)-modules and \( V \) has GF, then the diagram

\[
0 \to V^G \to W^G \to S^G \to 0
\]

is exact.

2. If \( W \) is a \( G \)-module with GF and \( V \) is a submodule of \( W \) with GF, then the quotient \( W/V \) is also a \( G \)-module with GF.

3. For given \( G \)-modules with GF their tensor product with respect to the diagonal action of the group \( G \) is also a module with GF.

4. If \( V \) is a \( G \)-module with GF and \( H \) is a Levi subgroup or the commutator subgroup of \( G \), then \( V \) has GF as a \( H \)-module.

**1.2 Necessary facts of representation theory of products of general linear groups**

Let \( G = GL(k) \) and \( T(k) = \{ \text{diag}(t_1, \ldots, t_k) \mid t_1, \ldots, t_k \in K^* \} \) is the standard torus of \( G \). We fix the Borel subgroup \( B(k) \) consisting of all upper triangular matrices. It is clear that \( B^-(k) \) consists of all lower triangular matrices.

Any character \( \lambda \in X(T(k)) \) can be regarded as a vector \((\lambda_1, \ldots, \lambda_k)\) with integer coordinates. By definition \( \lambda(t) = t_1^{\lambda_1} \ldots t_k^{\lambda_k}, t \in T(k) \). It is known that
\[ \lambda \in X(T(k))^+ \text{ iff } \lambda_1 \geq \ldots \geq \lambda_k \] [Don2]. If additionally \( \lambda_k \geq 0 \) then \((\lambda_1, \ldots, \lambda_k)\) is called an ordered partition and \( \nabla(\lambda) \) is isomorphic to so-called Schur module \( L_\lambda(K^k) \) (see the next subsection), where \( \lambda \) is the partition conjugated to \( \lambda \). To be precise, if \( \lambda_1 = \ldots = \lambda_{s_1} > \lambda_{s_1+1} = \ldots = \lambda_{s_1+s_2} > \ldots > \lambda_{s_1+\ldots+s_f+1} = \ldots = \lambda_k \) then \( \tilde{\lambda} = (k^{\lambda_k}, (s_1 + \ldots + s_f)^{\lambda_{s_f}}, \ldots, s_1^{\lambda_{s_1}}) \), where \( k^\ell \) means \( \underbrace{l \ldots l}_\ell \).

**Example 1.1** If \( \lambda = (1^t, 0^{k-t}) \) then \( \nabla(\lambda) = L_\lambda(K^k) = \Lambda^t(K^k) \) the \( t \)-th exterior power of the space \( K^k \). Moreover, \( \Lambda^t(K^k)^* \cong \Lambda^{k-t}(K^k) \otimes \det^{-1} = \nabla(0^{k-t}, -1^t) \) has GF. In particular, \( \Lambda^t(K^k) \) is a tilting \( GL(k) \)-module. Using Theorem 1.1(3) we obtain that all tensor products of such modules are also tilting.

The Weyl group \( W(GL(k), T(k)) \) is isomorphic to the group \( S_k \) consisting of all permutations on \( k \) symbols.

More generally, one can describe some fragment of the representation theory of any group \( GL(d) \). A maximal torus of the group \( GL(d) \) is \( T(d) = T(d_1) \times \ldots \times T(d_n) \). Respectively, \( B(d) = B(d_1) \times \ldots \times B(d_n) \) is a Borel subgroup and then \( B^-(d) = B^-(d_1) \times \ldots \times B^-(d_n) \). The characters of the group \( T(d) \) are collections \( \lambda = (\lambda_1, \ldots, \lambda_n) \), where each \( \lambda_i \) is a character of the corresponding torus \( T(d_i) \), \( i = 1, 2, \ldots, n \). It is obvious that the root data of \( GL(d) \) is the direct product of the root data of the groups \( GL(d_i) \). In particular, \( X(T(d))^+ \) coincides with \( X(T(d_1))^+ \times \ldots \times X(T(d_n))^+ \). Moreover, for any weight \( \lambda \in X(T(d))^+ \) we have an isomorphism \( \nabla_d(\lambda) \cong \nabla(\lambda_1) \otimes \ldots \otimes \nabla(\lambda_n) \) and \( \Delta_d(\lambda) \cong \Delta(\lambda_1) \otimes \ldots \otimes \Delta(\lambda_n) \). Therefore, if all \( \lambda_i \) are ordered partitions we see that \( \nabla_d(\lambda) \cong L_{\lambda_1}(E_1) \otimes \ldots L_{\lambda_n}(E_n) \). The Weyl group \( W(GL(d), T(d)) \) is the direct product of the Weyl groups of all factors \( GL(E_i) \). In particular, we have \( \tilde{\lambda}^* = (\lambda_1^*, \ldots, \lambda_n^*) \).

Consider dimensional vectors \( t(1), t(2) \) such that \( d(1) \geq d(2) \). We define the Schur functor \( d_{d(1), d(2)} \) by the following rule. For any \( GL(d(1))-\)module \( V \) we put \( d_{d(1), d(2)}(V) = \sum_{\mu \in L} V_{\tilde{\mu}} \). The set \( L \) consists of all \( \tilde{\mu} = (\mu_1, \ldots, \mu_n) \) such that for any \( i \) all coordinates of \( \tilde{\mu} \) beginning with \( d(2)_i \) + 1-th coordinate are equal to zero and \( \sum_{\mu \in X(T(d(1)))} V_{\tilde{\mu}} \) is the weight decomposition of \( V \). Identifying the group \( GL(d(2)) \) with a subgroup of \( GL(d(1)) \) (see the subsection 1.4) we obtain that \( d_{d(1), d(2)}(V) \) is a \( GL(d(2))-\)module. Besides, one can define a linear endomorphism of \( V \) which takes any \( v = \sum_{\mu \in X(T(d(1)))} v_{\tilde{\mu}} \in V \) to \( \sum_{\mu \in L} v_{\tilde{\mu}} \). Denote this endomorphism by the same symbol \( d_{d(1), d(2)} \). It is not hard to prove that if all coordinates \( \mu_i \) of \( \tilde{\lambda} \) are some ordered partitions then \( d_{d(1), d(2)}(\Delta_d(\lambda)) \neq 0 \) iff each "component" \( \lambda_i \) has all coordinates with numbers \( \geq d(2)_i + 1 \) equal to zero. In the last case we have \( d_{d(1), d(2)}(\Delta_d(\lambda)) = \Delta_d(\lambda) \). The same is valid for the induced modules \( \nabla_d(\lambda) \) as well as for its simple socle. The reader can find the detailed proof in [Green] for the case \( n = 1 \). The general case is a trivial consequence of the case \( n = 1 \).
1.3 ABW-filtrations

For any vector \( \lambda = (\lambda_1, \ldots, \lambda_s) \) with integral coordinates denote by \( | \lambda | \) its degree \( \lambda_1 + \ldots + \lambda_s \). If all coordinates of \( \lambda \) are non-negative integers we denote by \( \Lambda^\lambda(V) \) the tensor product \( \Lambda^\lambda_1(V) \otimes \ldots \otimes \Lambda^\lambda_s(V) \).

Recall the standard embedding of an exterior power \( \Lambda^p(V) \) into \( V^{\otimes p} \). This map is defined by the rule

\[
i_p : v_1 \wedge \ldots \wedge v_p \mapsto \sum_{\sigma \in S_p} (-1)^\sigma v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(p)}, v_1, \ldots, v_p \in V.
\]

Obviously, it is an \( GL(V) \)-equivariant. One can define more general embedding \( i_\lambda : \Lambda^\lambda(V) \to V^{\otimes p} \), where \( \lambda = (\lambda_1, \ldots, \lambda_i) \) is any (non-ordered) partition, \( p = | \lambda | \) and \( i_\lambda = \bigotimes_{1 \leq q \leq i} i_{\lambda_q} \). Denote by \( p_\lambda \) the canonical epimorphism from \( V^{\otimes p} \) onto \( \Lambda^\lambda(V) \).

Let \( S^r(V \otimes W) \) be a homogeneous component of degree \( r \) of the symmetric algebra \( S(V \otimes W) \), where \( V,W \) are any vector spaces. For any ordered partition \( \lambda \) of degree \( r \) we define the map

\[
d_\lambda : \Lambda^\lambda(V) \otimes \Lambda^\lambda(W) \to S^r(V \otimes W)
\]

by \( d_\lambda = d_{\lambda_1} \otimes \ldots \otimes d_{\lambda_s} \), where \( d_{\lambda_i} : \Lambda^{\lambda_i}(V) \otimes \Lambda^{\lambda_i}(W) \to S^{\lambda_i}(V \otimes W), i = 1, \ldots, s \), and the symbol \( \otimes \) means the product map \( S^{\lambda_1}(V \otimes W) \otimes \ldots \otimes S^{\lambda_s}(V \otimes W) \to S^r(V \otimes W) \).

Here, for any non-negative integer \( t \) the map \( d_t : \Lambda^t(V) \otimes \Lambda^t(W) \to S^t(V \otimes W) \) is defined by the rule

\[
d_t((v_1 \wedge \ldots \wedge v_t) \otimes (w_1 \wedge \ldots \wedge w_t)) = \sum_{\sigma \in S_t} (-1)^\sigma v_1 \otimes w_{\sigma(1)} \otimes \ldots \otimes v_t \otimes w_{\sigma(t)},
\]

\( v_i \in V, w_i \in W, 1 \leq i \leq t \).

Let \( M_\gamma = \sum_{\gamma \geq \lambda} \text{Im} d_\gamma \) and \( \hat{M}_\lambda = \sum_{\gamma \geq \lambda} \text{Im} d_\gamma \). The symbol \( \geq \) means the lexicographical order from left to right on the set of partitions. The \( GL(V) \times GL(W) \)-module \( S^r(V \otimes W) \) has the filtration

\[
0 \subseteq M_{(r)} \subseteq M_{(r-1,1)} \subseteq \ldots \subseteq M(1,\ldots,1) = S^r(V \otimes W)
\]

with quotients

\[
M_\lambda/\hat{M}_\lambda \cong L_\lambda(V) \otimes L_\lambda(W),
\]

where \( L_\lambda(V) \) is the Schur module (see [Ak]). We call this filtration Akin-Buchsbaum-Weyman filtration or briefly, an ABW-filtration.

**Remark 1.1** All these statements remain the same if we replace the field \( K \) by any commutative ring \( R \) and require that both \( V,W \) are free \( R \)-modules. The functor
\( V \to L_\lambda(V) \) is universally free, that is, \( L_\lambda(V) \) is a free \( R \)-module and commutes with change of the base ring \( R \) \([Ak]\). In particular, for any homomorphism \( R \to R' \) the functor \( R' \otimes_R - \) takes ABW-filtrations to ABW-filtrations.

We define a superpartition, say \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_n) \), where \( \lambda_i = (\lambda_{i1}, \ldots, \lambda_{is_i}) \), \( \lambda_{i1} \geq \cdots \geq \lambda_{is_i} \geq 0, i = 1, \ldots, n \). By definition, \( |\tilde{\lambda}| = |\lambda_1| + \cdots + |\lambda_n| \). One can endow the space

\[
\Lambda^{\tilde{\lambda}}(f) = \prod_{i=1}^{n} \mathop{\oplus} (\Lambda^{\lambda_i}(V_i)) = \prod_{i=1}^{n} \mathop{\oplus} (\prod_{j=1}^{s_i} \Lambda^{\lambda_{ij}}(V_i))
\]

with a \( GL(f) \)-module structure, where \( f = (f_1, \ldots, f_n) \), \( \dim V_i = f_i, 1 \leq i \leq n \). To be precise, each factor \( GL(V_i) \) of the group \( GL(f) \) acts on the corresponding tensor product \( \prod_{j=1}^{s_i} \Lambda^{\lambda_{ij}}(V_i) \) diagonally.

It is not hard to prove that \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_n) \) is a highest weight of the \( GL(f) \)-module \( \Lambda^{\tilde{\lambda}}(f) \). Moreover, its multiplicity is equal to 1. Since \( \Lambda^{\tilde{\lambda}}(f) \) is a tilting \( GL(f) \)-module there are good and Weyl filtrations of this module such that the last quotient of the first filtration (respectively the first non-zero member of the second one) is isomorphic to \( \nabla_f(\tilde{\lambda}) \) (respectively to \( \triangle_f(\tilde{\lambda}) \)) \([Don4, Don6, Zub3, Zub4]\).

Denote by \( R_f(\tilde{\lambda}) \) the kernel of the corresponding epimorphism

\[
\Lambda^{\tilde{\lambda}}(f) \to \nabla_f(\tilde{\lambda})
\]

and by \( S_f(\tilde{\lambda}) \) the cokernel of the inclusion

\[
\triangle_f(\tilde{\lambda}) \to \Lambda^{\tilde{\lambda}}(f).
\]

The \( GL(f) \)-modules \( R_f(\tilde{\lambda}) \) and \( S_f(\tilde{\lambda}) \) have GF and WF respectively. Moreover, the module \( R_f(\tilde{\lambda}) \) and the inclusion of \( \triangle_f(\tilde{\lambda}) \) are uniquely defined \([Zub4], \text{Proposition 1.1}\). We have the short exact sequence

\[
0 \to S_f(\tilde{\lambda})^* \to \Lambda^{\tilde{\lambda}}(f)^* \to \triangle_f(\tilde{\lambda})^* \to 0.
\]

By definition, \( \triangle_f(\tilde{\lambda})^* \cong \nabla_f(\tilde{\lambda})^* \). The unique highest weight of the module \( \Lambda^{\tilde{\lambda}}(f)^* \cong \Lambda^{\lambda_1}(V_1^*) \otimes \cdots \otimes \Lambda^{\lambda_n}(V_n^*) \) is equal to \( \tilde{\lambda}^* \) and since \( \Lambda^{\tilde{\lambda}}(f) \) is a tilting module we get that \( S_f(\tilde{\lambda})^* \) is uniquely defined by the same Proposition 1.1 from \([Zub4]\).

Let us consider another group \( GL(g), g = (g_1, \ldots, g_m) \) and some superpartition \( \tilde{\mu} = (\mu_1, \ldots, \mu_m), i = 1, \ldots, m \). We have the short exact sequence of \( GL(f) \times GL(g) \)-modules

\[
0 \to D_{f,g}(\tilde{\lambda}, \tilde{\mu}) \to \Lambda^{\tilde{\lambda}}(f) \otimes \Lambda^{\tilde{\mu}}(g)^* \to \nabla_f(\tilde{\lambda}) \otimes \nabla_g(\tilde{\mu})^* \cong \nabla_f(\tilde{\lambda}) \otimes \triangle_g(\tilde{\mu})^* \to 0,
\]

where

\[
D_{f,g}(\tilde{\lambda}, \tilde{\mu}) = R_f(\tilde{\lambda}) \otimes \Lambda^{\tilde{\mu}}(g)^* + \Lambda^{\tilde{\lambda}}(f) \otimes S_g(\tilde{\mu})^*.
\]
Proposition 1.2  The kernel $D_{f,g}(\bar{\lambda}, \bar{\mu})$ is uniquely defined and has GF.

Proof. Use the same Proposition 1.1 from [Zub4] and Proposition 2.3 from [DZ].

Remark 1.2 Notice that $L_{\lambda}(V^*) \cong \triangle(\bar{\lambda})^*$ [Zub1] as a $GL(V)$-module. In particular, the $GL(g)$-module $\triangle(g)(\bar{\mu})^*$ is isomorphic to $L_{\lambda_1}(U_1^*) \otimes \ldots L_{\lambda_m}(U_m^*)$, where $\dim U_j = g_j, 1 \leq j \leq m$.

Remark 1.3 In notations of Remark 1.1 there is a nondegenerate pairing of $GL(V)$-modules $\Lambda^t(V^*) \times \Lambda^t(V) \rightarrow R$ defined by the rule

$$<f_1 \wedge \ldots \wedge f_t, v_1 \wedge \ldots \wedge v_t> = \det(f_i(v_j)), v_i \in V, f_j \in V^*, 1 \leq i, j \leq t.$$  

Here $V^* = Hom_R(V, R)$ and $V$ is a free $R$-module. Tensoring we obtain a nondegenerate pairing $\Lambda^s(V^*) \times \Lambda^t(V) \rightarrow R$. Thus $\Lambda^s(V^*)$ is isomorphic to $\Lambda^s(V)^*$ as a $GL(V)$-module.

1.4 Specializations

For given $d(1) \geq d(2)$ define an epimorphism

$$p_{t(1),t(2)} : K[R(Q,t(1))] \rightarrow K[R(Q,t(2))]$$

by the following rule. Take any arrow $a \in A$. Let $i(a) = i$ and $t(a) = j$. For the sake of simplicity denote $d_i(s)$ and $d_j(s)$ by $m_s$ and $l_s$ respectively, $s = 1, 2$. We know that $m_1 \geq m_2$ and $l_1 \geq l_2$. Then our epimorphism maps $y_{sr}(a)$ to zero if either $s > l_2$ or $r > m_2$. On the remaining variables our epimorphism is the identical map.

On the other hand, one can define the isomorphism $i_{t(2),t(1)}$ of the variety $R(Q,t(2))$ onto a closed subvariety of $R(Q,t(1))$ by the dual rule, that is the epimorphism defined above is the comorphism $i_{t(2),t(1)}$.

By almost the same way as $i_{t(2),t(1)}$ one can define the isomorphism $j_{t(2),t(1)}$ of the group $H(t(2))$ onto a closed subgroup of the group $H(t(1))$ just bordering any invertible $d_i(2) \times d_j(2)$ matrix by the $d_i(1) - d_i(2)$ additional rows and columns which are zero outside of the diagonal tail of length $d_i(1) - d_i(2)$. The entries on this diagonal tail should be 1’s. It is not hard to check that $i_{t(2),t(1)}(\phi^g) = i_{t(2),t(1)}(\phi)^{\lambda(2),\lambda(1)}(g)$ for any $g \in H(t(2))$ and $\phi \in R(Q,t(2))$. The analogous equation is valid for the epimorphism $p_{t(1),t(2)}$.

1.5 Young subgroups

Decompose an interval $[1,k] = \{1, \ldots, k\}$ into some disjoint subsets, say $[1,k] = \bigcup_{1 \leq j \leq m} T_j$. Define the Young subgroup $S_T = S_{T_1} \times \ldots \times S_{T_m}$ of the group $S_k$ as the subgroup which consists of all permutations $\sigma \in S_k$ such that $\sigma(T_j) = T_j, 1 \leq j \leq m$. 

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By definition, \( S_T = \{ \sigma \in S_k \mid \sigma(T) = T, \forall j \not\in T \; \sigma(j) = j \} \) for arbitrary subset \( T \). The subsets \( T_1, \ldots, T_m \) are said to be the layers of the group \( S_T \) [Zub1, Zub4].

The group \( S_T \) can be introduced in other way. In fact, let \( f \) be a map from \([1, k]\) onto \([1, m]\) defined by the rule \( f(T_j) = j, j = 1, \ldots, m \). Then \( S_T = \{ \sigma \in S_k \mid f \circ \sigma = f \} \). Sometimes we will denote \( S_T \) by \( S_f \).

For a given superpartition \( \lambda = (\lambda_1, \ldots, \lambda_n) \), where \( \lambda_i = (\lambda_{i1}, \ldots, \lambda_{is_i}), \lambda_{i1} \geq \ldots \geq \lambda_{is_i} \geq 0, i = 1, \ldots, n \), denote by \( S_\lambda \) the Young subgroup of \( S_{|\lambda|} \) corresponding to the decomposition of \([1, |\lambda|]\) into sequential subintervals of lengths \( \lambda_{i1}, \ldots, \lambda_{1s_1}, \lambda_{21}, \ldots, \lambda_{2s_2}, \ldots \).

For any group \( G \) and its subgroup \( H \) we denote by \( G/H \) some fixed representative set of the left cosets of \( H \) if it does not lead to confusion. For any \( g \in G \) denote by \( \bar{g} \in G/H \) the representative of the left coset \( gH \).

### 1.6 Schur duality and a lemma

Let \( V \) be a projective module over a commutative ring \( R \) such that, if \( f(x) \in R[x] \) is a polynomial which vanishes on \( R \) then \( f(x) \) is identically zero. We have a ring homomorphism \( \psi : R[S_d] \to \text{End}_{GL(V)}(V^{\otimes d}) \) defined by the rule: \( \sigma \mapsto \bar{\sigma}, \sigma \in S_d \), where \( \bar{\sigma} = \text{Im}\phi_{\psi} = \psi_{\bar{\sigma}^{-1}(1)} \otimes \cdots \otimes \psi_{\bar{\sigma}^{-1}(d)} \). For the sake of simplicity we will omit the upper tilde.

**Theorem 1.2** ([Pr]) The homomorphism \( \psi \) is surjective. If \( V \) is a free module of rank \( p \) then the kernel \( I_{p+1} \) of this epimorphism is not equal to zero iff \( d > p \) and in this case it is generated (as a two-sided ideal) by the element \( \sum_{\tau \in S_{p+1}} (-1)^\tau \tau, \) where \( S_{p+1} = S_{[1, p+1]} \).

Let \( R \) be a principal ideal domain of odd or zero characteristic and \( V, W \) are free \( R \)-modules of finite ranks. For given partitions \( \lambda, \mu \) of degree \( r \) one can define an inclusion \( \Phi_{\lambda, \mu} \) of \( \text{Hom}_R(\Lambda^\lambda(V), \Lambda^\mu(W)) \) into \( \text{Hom}_R(V^{\otimes r}, W^{\otimes r}) \) by the rule \( \phi \mapsto i_\mu \phi i_\lambda, \phi \in \text{Hom}_R(\Lambda^\lambda(V), \Lambda^\mu(W)) \).

**Lemma 1.1** The image of \( \text{Hom}_R(\Lambda^\lambda(V), \Lambda^\mu(W)) \) in \( \text{Hom}_R(V^{\otimes r}, W^{\otimes r}) \) coincides with \( \{ \phi \in \text{Hom}_R(V^{\otimes r}, W^{\otimes r}) \mid \forall \tau_1 \in S_\lambda, \forall \tau_2 \in S_\mu, \tau_2 \phi \tau_1 = (-1)^{\tau_1}(-1)^{\tau_2} \phi \} \).

Proof. Denote the submodule \( \{ \phi \in \text{Hom}_R(V^{\otimes r}, W^{\otimes r}) \mid \forall \tau_1 \in S_\lambda, \forall \tau_2 \in S_\mu, \tau_2 \phi \tau_1 = (-1)^{\tau_1}(-1)^{\tau_2} \phi \} \) by \( M \). By Remark 1.3 we have the standard isomorphisms

\[
\text{Hom}_R(\Lambda^\lambda(V), \Lambda^\mu(W)) \cong \Lambda^\lambda(V^*) \otimes \Lambda^\mu(W), \text{Hom}_R(V^{\otimes r}, W^{\otimes r}) \cong (V^*)^{\otimes r} \otimes W^{\otimes r}.
\]

Then \( \Phi_{\lambda, \mu} \) can be identified with \( i_\lambda \otimes i_\mu \). The groups \( S_\mu \) and \( S_\lambda \) act on \( (V^*)^{\otimes r} \otimes W^{\otimes r} \) in obvious way. It remains to notice that for any free \( R \)-module \( U \) and partition \( \chi \) (of \( r \)) we have \( i_\chi(\Lambda^\lambda(U)) = \{ x \in U^{\otimes r} \mid \forall \tau \in S_\lambda, \tau(x) = (-1)^r x \} \). In fact, \( \text{Im}\Phi_{\lambda, \mu} = \text{Im}i_\lambda \otimes i_\mu = M \).
2 Auxiliary computations

2.1 Generators, free invariant algebras and relations

We start with some simplification of the space \( R(Q, t) \). To be precise, let \( a \in A \) and \( i(a) = i, t(a) = j \). We have the following possibilities:

1. If \( W_i = E_i, W_j = E_j \) then \( H = H(t) \) acts on the component \( K[\operatorname{Hom}_K(E_i, E_j)] = K[Y(a)] = K[y_{lt}(a) \mid 1 \leq l \leq d_j, 1 \leq t \leq d_i] \) by the rule \( Y(a) \mapsto g^{-1}Y(a)h \), \( g \in GL(d_j), h \in GL(d_i) \). It can easily be checked that \( K[Y(a)] \cong S(E^*_j \otimes E^*_i) \) and this isomorphism of \( GL(d_j) \times GL(d_i) \)-modules is defined by the rule \( y_{lt}(a) \mapsto e^*_l \otimes f^*_t \), where \( e_1, \ldots, e_{d_j} \) and \( f_1, \ldots, f_{d_i} \) are some fixed bases of the spaces \( E_j \) and \( E_i \) respectively. The basis \( e^*_1, \ldots, e^*_{d_j} \) is the dual relative to \( e_1, \ldots, e_{d_j} \).

2. If \( W_i = E_i, W_j = E^*_j \) then \( K[Y(a)] \cong S(E_j \otimes E_i) \) with respect to the identification \( y_{lt}(a) \mapsto e_l \otimes f_t \). In other words, \( H \) acts on \( Y(a) \) by \( Y(a) \mapsto g^t Y(a)h \).

Other cases are listed without any comments.

3. \( W_i = E^*_i, W_j = E_j \), \( K[Y(a)] \cong S(E^*_j \otimes E^*_i) \), \( y_{lt}(a) \mapsto e_l^* \otimes f_t^* \), \( Y(a) \mapsto g^{-1}Y(a)(h^t)^{-1} \).

4. \( W_i = E^*_i, W_j = E^*_j \), \( K[Y(a)] \cong S(E_j \otimes E^*_i) \), \( y_{lt}(a) \mapsto e_l \otimes f_t^* \), \( Y(a) \mapsto g^t Y(a)(h^t)^{-1} \).

**Lemma 2.1** Up to some changing of \( Q \) one can eliminate the fourth case.

Proof. By the definition there should be some \( q, q' \in \Omega \) such that \( i = j_q, j = j_{q'} \). Redefine the maps \( i, t \) on any arrow \( a \) which goes from \( i \) to \( j \) by \( : i'(a) = i^q_q, t'(a) = i_q \) and \( i', t' \) coincide with \( i, t \) on the remaining arrows. We get a new quiver \( Q' \).

Let us consider the representation space of this new quiver of the same dimension \( t \). It is clear that this space can be produced from \( R(Q, t) \) by replacing all summands \( \operatorname{Hom}_K(E^*_i, E^*_j) \) by \( \operatorname{Hom}_K(E_j, E_i) \).

The group \( H \) remains the same. Moreover, the algebra \( K[R(Q, t)] \) is isomorphic to \( K[R(Q', t)] \). To be precise, we map each \( y_{lt}(a) \) to \( z_{lt}(a) \), where \( Z(a) = Z_i(a), i'(a) = i^q, t'(a) = i_q \). The remaining generators of \( K[R(Q, t)] \) and \( K[R(Q', t)] \) coincide with one another. It can easily be checked that this isomorphism is \( H \)-equivariant. After repeating this procedure as many times as we need one can see that the fourth case does not happen at all. The lemma is proved.

Decompose the arrow set \( A \) into three subsets \( A_i, i = 1, 2, 3 \), where \( A_1 = \{a \in A \mid W_{i(a)} = E_{i(a)}, W_{t(a)} = E_{t(a)} \} \), \( A_2 = \{a \in A \mid W_{i(a)} = E_{i(a)}, W_{t(a)} = E^*_{t(a)} \} \) and \( A_3 = \{a \in A \mid W_{i(a)} = E^*_{i(a)}, W_{t(a)} = E_{t(a)} \} \).

The algebra \( K[R(Q, t)] \) is isomorphic to the tensor product

\[
\prod_{1 \leq k \leq 3} \otimes_{a \in A_k} K[Y(a)]
\]
or to
\[
\prod_{1 \leq k \leq 3} \otimes (\otimes_{a \in A_k} (\otimes_{ra} K[Y(a)](r_a))) \cong (\prod_{a \in A_1} \otimes (\otimes_{ra} S^{ra}(E_{t(a)} \otimes E_{i(a)})) \otimes \\
(\prod_{a \in A_2} \otimes (\otimes_{ra} S^{ra}(E_{t(a)} \otimes E_{i(a)})) \otimes (\prod_{a \in A_3} \otimes (\otimes_{ra} S^{ra}(E_{t(a)} \otimes E_{i(a)}))),
\]
as a \(H\)-module.

Fix a multidegree \(\bar{r} = (r_a)_{a \in A}\). Sometimes we will rewrite it as \((\bar{r}_1, \bar{r}_2, \bar{r}_3)\), where \(\bar{r}_i = (r_a)_{a \in A_i}, i = 1, 2, 3\). Denote \(\sum_{a \in A} r_a\) by \(r\) and \(\sum_{a \in A_i} r_a\) by \(r_i, i = 1, 2, 3\). The \(\bar{r}\)-homogeneous component of the algebra \(K[R(Q, t)]\) is isomorphic to
\[
(\prod_{a \in A_1} \otimes S^{ra}(E_{t(a)}^* \otimes E_{i(a)})) \otimes (\prod_{a \in A_2} \otimes S^{ra}(E_{t(a)} \otimes E_{i(a)})) \otimes (\prod_{a \in A_3} \otimes S^{ra}(E_{t(a)}^* \otimes E_{i(a)}^*)).
\]

Tensoring ABW-filtrations of all factors in this tensor product we see that the \(\bar{r}\)-homogeneous component of the algebra \(K[R(Q, t)]\) has a filtration with quotients
\[
\prod_{a \in A_1} \otimes (L_{\lambda_a}(E_{t(a)}^*) \otimes L_{\lambda_a}(E_{i(a)})) \otimes \prod_{b \in A_2} \otimes (L_{\mu_b}(E_{t(a)}) \otimes L_{\mu_b}(E_{i(a)}) \otimes \\
\otimes \prod_{c \in A_3} \otimes (L_{\gamma_c}(E_{t(c)}^*) \otimes L_{\gamma_c}(E_{i(c)}))
\]
as a \(\prod_{1 \leq t \leq 3}(\prod_{a \in A_t}(GL(d_{t(a)}) \times GL(d_{i(a)})))\)-module, where by definition \(\forall a \in A_1, \forall b \in A_2, \forall c \in A_3 | \lambda_a | = r_a, | \mu_b | = r_b, | \gamma_c | = r_c\) (see [DZ], Proposition 2.3). We enumerate the members of this filtration by the triples (superpartitions) \(\Theta = (\lambda_{A_1}, \mu_{A_2}, \gamma_{A_3})\), where \(\lambda_{A_1} = (\lambda_a)_{a \in A_1}, \mu_{A_2} = (\mu_a)_{a \in A_2}, \gamma_{A_3} = (\gamma_a)_{a \in A_3}\), say
\[
\ldots \subseteq M_{\Theta}(t) = M_{\Theta} \subseteq \ldots \tag{1}
\]

Denote by \(\Lambda_1(\Theta, t)\) and \(\Lambda_2(\Theta, t)\) the spaces
\[
\prod_{a \in A_1} \otimes (\Lambda^\lambda_a(E_{i(a)}) \otimes \prod_{a \in A_2} \otimes (\Lambda^\mu_a(E_{t(a)}) \otimes \Lambda^\mu_a(E_{i(a)})
\]
and
\[
\prod_{a \in A_1} \otimes (\Lambda^\lambda_a(E_{t(a)}) \otimes \prod_{a \in A_3} \otimes (\Lambda^\gamma_a(E_{t(a)}) \otimes \Lambda^\gamma_a(E_{i(a)}))
\]
respectively. Sometimes we will omit the indices \(t, d\) or \(\Theta\) if it does not lead to confusion.

By Remark 1.3 one can identify the dual space \(\Lambda_2(\Theta, t)^*\) with the space
\[
\prod_{a \in A_1} \otimes (\Lambda^\lambda_a(E_{t(a)}^*) \otimes \prod_{a \in A_3} \otimes (\Lambda^\gamma_a(E_{t(a)}^*) \otimes \Lambda^\gamma_a(E_{i(a)}^*)).
\]

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Arrange the tensor factors of the quotient \( M_{\Theta}/\hat{M}_{\Theta} \) into groups by the following way:

\[
\prod_{\alpha \in A_1} \otimes (L_{\lambda_{\alpha}}(E_{i(\alpha)})) \otimes \prod_{\alpha \in A_2} \otimes (L_{\mu_{\alpha}}(E_{i(\alpha)}) \otimes L_{\mu_{\alpha}}(E_{i(\alpha)})) \otimes \prod_{\alpha \in A_1} \otimes (L_{\lambda_{\alpha}}(E_{i(\alpha)})^*) \otimes \prod_{\alpha \in A_3} \otimes (L_{\gamma_{\alpha}}(E_{i(\alpha)})^*) \otimes L_{\gamma_{\alpha}}(E_{i(\alpha)})^*),
\]

The first factor

\[
\prod_{\alpha \in A_1} \otimes (L_{\lambda_{\alpha}}(E_{i(\alpha)})) \otimes \prod_{\alpha \in A_2} \otimes (L_{\mu_{\alpha}}(E_{i(\alpha)}) \otimes L_{\mu_{\alpha}}(E_{i(\alpha)}))
\]

is a \((\prod_{\alpha \in A_1} GL(d_{i(\alpha)})) \times (\prod_{\alpha \in A_2} GL(d_{i(\alpha)}) \times GL(d_{i(\alpha)}))\)-module. Denote this group by \( G_1 = G_1(d) \).

Analogously, the second factor

\[
\prod_{\alpha \in A_1} \otimes (L_{\lambda_{\alpha}}(E_{i(\alpha)})^*) \otimes \prod_{\alpha \in A_3} \otimes (L_{\gamma_{\alpha}}(E_{i(\alpha)})^*) \otimes L_{\gamma_{\alpha}}(E_{i(\alpha)})^*),
\]

is a \((\prod_{\alpha \in A_1} GL(d_{i(\alpha)})) \times (\prod_{\alpha \in A_3} GL(d_{i(\alpha)}) \times GL(d_{i(\alpha)}))\)-module. Denote it by \( G_2 = G_2(d) \).

For any \( \Theta \) we have a homomorphism of \( G_1 \times G_2 \)-modules

\[
d_{\Theta} : \Lambda_1(\Theta) \otimes \Lambda_2(\Theta)^* \to K[R(Q, t)](\bar{r}). \tag{2}
\]

The homomorphism \( d_{\Theta} \) induces an epimorphism

\[
\Lambda_1(\Theta) \otimes \Lambda_2(\Theta)^* \to M_{\Theta}/\hat{M}_{\Theta} \to 0.
\]

Denote by \( \Theta_1, \Theta_2 \) the superpartitions \((\lambda_{A_1}, \mu_{A_2}, \mu_{A_2})\) and \((\lambda_{A_1}, \gamma_{A_3}, \gamma_{A_3})\) respectively.

The \( G_1(d) \)-module

\[
(\prod_{\alpha \in A_1} \otimes L_{\lambda_{\alpha}}(E_{i(\alpha)})) \otimes \prod_{\alpha \in A_2} \otimes (L_{\mu_{\alpha}}(E_{i(\alpha)}) \otimes L_{\mu_{\alpha}}(E_{i(\alpha)}))
\]

coincides with

\[
\nabla((d_{i(\alpha)})_{\alpha \in A_1}, (d_{i(\alpha)})_{\alpha \in A_2}, (d_{i(\alpha)})_{\alpha \in A_2}) (\tilde{\Theta}_1).
\]

Similarly, the \( G_2(d) \)-module

\[
(\prod_{\alpha \in A_1} \otimes L_{\lambda_{\alpha}}(E_{i(\alpha)})) \otimes \prod_{\alpha \in A_3} \otimes (L_{\gamma_{\alpha}}(E_{i(\alpha)})^*) \otimes L_{\gamma_{\alpha}}(E_{i(\alpha)})^*),
\]

coincides with

\[
\Delta((d_{i(\alpha)})_{\alpha \in A_1}, (d_{i(\alpha)})_{\alpha \in A_3}, (d_{i(\alpha)})_{\alpha \in A_3}) (\tilde{\Theta}_2)^*.
\]
Slightly abusing our notations we denote these modules by $\nabla_d(\Theta)$ and $\Delta_d(\Theta)$ correspondingly.

Using Proposition 1.2 we obtain the uniquely defined short exact sequence of $G_1 \times G_2$-modules with GF

$$0 \to D_{d,d}(\Theta) = D(\Theta) \to \Lambda_1(\Theta) \otimes \Lambda_2(\Theta)^* \to M_\Theta / \dot{M}_\Theta \to 0.$$  

Here,

$$D_{d,d}(\Theta) = D((d(t_a))_{a \in A_1},(d(t_a))_{a \in A_2},((d(t_a))_{a \in A_1},(d(t_a))_{a \in A_2},(d(t_a))_{a \in A_3},(d(t_a))_{a \in A_3})(\Theta_1, \Theta_2)).$$

To turn to the group $H = H(d)$ we have to replace the group $G_1$ ($G_2$) by some subgroup. Indeed, represent, say $G_1$, as $\times_{i \in V} GL(d_i)^{w_i}$, where $w_i$ is the number of factors of $G_1$ coinciding with $GL(d_i), i \in V$. The next step is to replace any subproduct $GL(d_i)^{w_i}, i \in V_{ord}$, or $GL(d_q)^{v_q}, q \in \Omega$, where $v_q = w_{i_q} + w_{j_q}$, by the corresponding diagonal subgroup.

Using Theorem 1.1(3) we obtain that any $G_i$-module has GF (respectively – any $G_i$-module has WF) retains this property under the restriction to the group $H$, $i = 1, 2$. Referring to Theorem 1.1(1) we get

**Proposition 2.1** The short sequence

$$0 \to D(\Theta)^H \to (\Lambda_1(\Theta) \otimes \Lambda_2(\Theta)^*)^H \to Z(\Theta) \to 0$$

is exact. Here, $Z(\Theta) = (M_\Theta / \dot{M}_\Theta)^H = M_\Theta^H / \dot{M}_\Theta^H$.

The same arguments show that $\Lambda_1$ and $\Lambda_2$ are tilting $H$-modules and all quotients of the filtration (1) are $H$-modules with GF.

One can rewrite the exact sequence from Proposition 2.1 as

$$0 \to D(\Theta)^H \to \text{Hom}_H(\Lambda_2(\Theta), \Lambda_1(\Theta)) \to Z(\Theta) \to 0. \quad (3)$$

Sometimes, if it is necessary to indicate that the original representation space has dimension $t$, we write $Z_t(\Theta)$.

**Theorem 2.1** ([Don2]) The epimorphism $p_{t(1),t(2)} : K[R(Q,t(1))]) \to K[R(Q,t(2))]$ induces the epimorphism $\phi_{t(1),t(2)} : J(Q,t(1)) \to J(Q,t(2))$.

Proof. For given pair of compatible dimension vectors $d(1) \geq d(2)$ one can define at least three Schur functors $d, d_1, d_2$ for the groups $H, G_1, G_2$ correspondingly. Nevertheless, it is not hard to see that the action of the Schur functor $d$ coincides with the actions of both functors $d_i, i = 1, 2$ on the short exact sequences

$$0 \to R_{d(1)}(\Theta) \to \Lambda_1(t(1)) \to \nabla_{d(1)}(\Theta) \to 0.$$
and
\[ 0 \to \triangle_{d(1)}(\Theta) \to \Lambda_2(t(1)) \to S_{d(1)}(\Theta) \to 0. \]
It is obvious for modules \( \Lambda_1(\Theta, t(1)), \Lambda_2(\Theta, t(1)), \nabla_{d(1)}(\Theta), \triangle_{d(1)}(\Theta) \) and it follows for \( R_{d(1)}(\Theta), S_{d(1)}(\Theta) \) since all Schur functors are exact [Green]. Moreover, one can identify the exact sequences
\[ 0 \to d(R_{d(1)}(\Theta)) \to d(\Lambda_1(t(1))) \to d(\nabla_{d(1)}(\Theta)) \to 0 \]
and
\[ 0 \to d(\triangle_{d(1)}(\Theta)) \to d(\Lambda_2(t(1))) \to d(S_{d(1)}(\Theta)) \to 0 \]
with
\[ 0 \to R_{d(2)}(\Theta) \to \Lambda_1(t(2)) \to \nabla_{d(2)}(\Theta) \to 0 \]
and
\[ 0 \to \triangle_{d(2)}(\Theta) \to \Lambda_2(t(2)) \to S_{d(2)}(\Theta) \to 0 \]
respectively since \( R \) and \( S \) are uniquely defined in all these sequences.

Let \( \psi : \Lambda_1 \otimes \Lambda_2^* \to d(\Lambda_1) \otimes d(\Lambda_2)^* \) be a map given by \( \psi(v \otimes \alpha) = d(v) \otimes \alpha \mid_{d(\Lambda_2)} \), \( v \in \Lambda_1, \alpha \in \Lambda_2^* \). In other words, it is the map \( \Hom_K(\Lambda_2, \Lambda_1) \to \Hom_K(d(\Lambda_1), d(\Lambda_2)) \) defined by \( \phi \mapsto d \circ \phi \mid_{d(\Lambda_2)} \).

If \( \phi \in \Hom_H(\Lambda_2, \Lambda_1) \) then \( \phi(d(\Lambda_2)) \subseteq d(\Lambda_1) \) since \( \phi \) commutes with the torus action. In particular, \( \psi \) is the restriction map on \( \Hom_H(\Lambda_2, \Lambda_1) \). Moreover, \( \psi(D_{d(1)}) \subseteq D_{d(2)} \). Indeed, it is clear for the summand \( R \otimes \Lambda_2^* \). Let \( v \otimes \alpha \in \Lambda_1 \otimes S^* \). The space \( S^* \) is identified with a subspace of \( \Lambda_2^* \) by the rule \( \alpha \mapsto \alpha \circ p \), where \( p \) is the epimorphism of the \( G_2 \)-modules \( \Lambda_2 \to S \to 0 \). In particular, \( p(d(\Lambda_2)) = d(S) \) and \( (\alpha \circ p) \mid_{d(\Lambda_2)} = \alpha \mid_{d(S)} \circ p \mid_{d(\Lambda_2)} \).

Consider the filtration \( 0 \subseteq R \otimes S^* \subseteq D \) of \( H \)-module \( D \) with quotients \( R \otimes S^* \) and \( (R \otimes \triangle^*) \oplus (\nabla \otimes S^*) \). These quotients can be identified with \( \Hom_K(S, R) \) and \( \Hom_K(\triangle, R) \oplus \Hom_K(S, \nabla) \) respectively and the map \( \psi \) induces the maps

\[ \Hom_K(S, R) \to \Hom_K(d(S), d(R)), \Hom_K(\triangle, R) \to \Hom_K(d(\triangle), d(R)), \]

\[ \Hom_K(S, \nabla) \to \Hom_K(d(S), d(\nabla)). \]

All these arguments show that we have the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & D \\
\downarrow & & \downarrow \\
0 & \to & \Hom_K(\Lambda_2, \Lambda_1) \\
\downarrow & & \downarrow \\
0 & \to & \Hom_K(d(\Lambda_2), d(\Lambda_1)) \\
\end{array}
\]

\[
\begin{array}{cccc}
& & \nabla \otimes \triangle^* & \to 0 \\
& & \downarrow & \\
\to & d(D) & \to & \Hom_K(d(\Lambda_2), d(\Lambda_1)) \\
& & \downarrow & \\
& & \to & d(\nabla) \otimes d(\triangle)^* \to 0.
\end{array}
\]
If we identify the last right members of the horizontal sequences with the corresponding quotients of the filtrations of \( K[R(Q, t(1))] \) and \( K[R(Q, t(2))] \) respectively then the last right vertical arrow is induced by the epimorphism \( p_{t(1), t(2)} \).

Indeed, the map \( d = d_{d(1), d(2)} \) takes a basis vector of \( \Lambda_1 = \Lambda_1(t(1)) \) or \( \Lambda_2 = \Lambda_2(t(1)) \) to zero if its record contains at least one vector \( e^{(i)}_j \) or \( (e^{(i)}_j)^* \), where \( j \geq d(2)_i + 1 \) and \( e^{(i)}_1, \ldots, e^{(i)}_{d(1)_i} \) is a fixed basis of \( E_i \), \( 1 \leq i \leq n \). It remains to apply the rule of the identification of the algebra \( K[R(Q, t)] \) with the corresponding symmetric algebra.

Finally, we have the following commutative diagram

\[
\begin{array}{ccccccc}
0 & \to & D^H_{d(1), d(1)}(t(1)) & \to & \text{Hom}_{H(t(1))}(\Lambda_2(t(1)), \Lambda_1(t(1))) & \to & Z_{t(1)} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & D^H_{d(2), d(2)}(t(2)) & \to & \text{Hom}_{H(t(2))}(\Lambda_2(t(2)), \Lambda_1(t(2))) & \to & Z_{t(2)} & \to & 0.
\end{array}
\] (4)

Repeating word by word the proof of Proposition 1 from [Zub1] (and using Lemma 1.1 from [Zub4] as well) we see that all vertical arrows in the last diagram are epimorphisms. This concludes the proof.

We have an inverse spectrum of algebras:

\[ \{J(Q, t), \phi_{t(1), t(2)} \mid d(1) \geq d(2) \}. \]

Moreover, because of epimorphisms \( \phi_{t(1), t(2)} \) are homogeneous we have the countable set of spectrums:

\[ \{J(Q, t)(r), \phi_{t(1), t(2)} \mid d(1) \geq d(2) \}, r = 0, 1, 2, \ldots \}. \]

The inverse limit of \( r \)-th spectrum denote by \( J(Q)(r) \). It is clear that \( J(Q) = \bigoplus_{r \geq 0} J(Q)(r) \) can be endowed with an algebra structure in obvious way. The algebra \( J(Q) \) is called a free invariant algebra of mixed representations of the quiver \( Q \).

**Remark 2.1** From the geometrical point of view we have a commutative diagram

\[
\begin{array}{ccc}
R(Q, t(2)) & \to & R(Q, t(2))/H(t(2)) \\
\downarrow & & \downarrow \\
R(Q, t(1)) & \to & R(Q, t(1))/H(t(1))
\end{array}
\]

which is dual to

\[
\begin{array}{ccc}
J(Q, t(2)) & \to & K[R(Q, t(2))] \\
\uparrow & & \uparrow \\
J(Q, t(1)) & \to & K[R(Q, t(1))].
\end{array}
\]

In the first diagram horizontal sequences are categorical quotients with respect to the corresponding reductive group actions and vertical arrows are isomorphisms onto closed subvarieties. The algebra \( J(Q) \) can be regarded as a coordinate algebra of an infinitely dimensional variety which is the direct limit of varieties Spec \( (J(Q, t)) = \)
$R(Q,t)/H(t)$ or as an invariant algebra $K[R(Q)]^{H(Q)}$, where $K[R(Q)]$ is the homogeneous inverse limit of the algebras $K[R(Q,t)]$ defined by the same way as above and $H(Q)$ is the direct limit of the groups $H(t)$.

It is clear that any $J(Q,t)$ is an epimorphic image of $J(Q)$. Denote the kernel of this epimorphism by $T(Q,t)$.

**Remark 2.2** It is not necessary to consider the algebra $J = J(Q)$ as the inverse limit over all compatible dimensional vectors $t$. One can replace the set of all dimensional vectors by any cofinal subset. For example, we can take $\{N = (T_1, \ldots, T_n) \mid N \geq 2\}$, where $T_i = N$ iff $t_i = d_i$ otherwise $T_i = N^s$.

From now on we suppose that $t(1) = N$ and $t(2) = t$, where the number $N$ is sufficiently large, say $N \geq r$. Denote by $d$ the underlying vector of $t$.

Finally, denote the image $d_\Theta(\phi)$ of any $\phi \in \text{Hom}_{H(t)}(A_2(t), A_1(t))$ in the homogeneous component $K[R(Q,t)][\bar{r}]$ by $c(\phi)$.

**Lemma 2.2** If $\text{Hom}_{H(t)}(A_2(\Theta), A_1(\Theta)) \neq 0$ then $r_2 = r_3 = | \mu_{A_2} | = | \gamma_{A_3} |$. Moreover, the following conditions are satisfied:

1. $\forall i \in V_{\text{ord}}, \sum_{a \in A,i(a)=i} r_a = \sum_{a \in A,i(a)=i} r_a = p_a$.
2. $\forall q \in \Omega, \sum_{a \in A,t(a)=i_q} r_a + \sum_{a \in A,i(a)=j_q} r_a = \sum_{a \in A,i(a)=i_q} r_a + \sum_{a \in A,t(a)=j_q} r_a = p_q$.

Proof. It is clear that the group $H(N)$ contains the diagonal subgroup which is isomorphic to $GL(N)$. The requirement $\text{Hom}_{H(t)}(A_2(\Theta), A_1(\Theta)) \neq 0$ implies $\text{Hom}_{H(N)}(A_2(N), A_1(N)) \neq 0$ for sufficiently large $N$. Therefore, we obtain that $\text{Hom}_{GL(N)}(A_2(N), A_1(N)) \neq 0$ and the degrees of the polynomial $GL(N)$-modules (see [Green] for definitions) $A_1(\Theta)$ and $A_2(\Theta)$ should be the same. In other words, $r_2 = r_3 = | \mu_{A_2} | = | \gamma_{A_3} |$.

The space $\text{Hom}_{H(t)}(A_2(\Theta), A_1(\Theta))$ can be represented as

$$\otimes_{i \in V_{\text{ord}}} \text{Hom}_{GL(d_i)}(\otimes_{a \in A,i(a)=i} \Lambda^{x_a}(E_i), \otimes_{a \in A,i(a)=i} \Lambda^{x_a}(E_i))$$

$$\otimes_{q \in \Omega} \text{Hom}_{GL(d_q)}((\otimes_{a \in A,t(a)=i_q} \Lambda^{x_a}(E_{i_q}))) \otimes ((\otimes_{a \in A,i(a)=j_q} \Lambda^{x_a}(E_{j_q}))),$$

where $\chi_a$ is equal to $\lambda_{a}, \mu_{a}$ or $\gamma_{a}$ if $a \in A_1, a \in A_2$ or $a \in A_3$ respectively. All tensor multipliers are not equal to zero. It remains to compare the degrees of all modules in the corresponding groups of homomorphisms.

Denote $| \lambda_{A_1} |$ by $t$. Then $r = t + 2s$.

As in [Zub4] we extend the set of matrix variables $\{Y(a) \mid a \in A\}$ in the following way. Replace each $Y(a)$ by some new set of matrices having the same size as $Y(a)$. 

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The cardinality of this set is equal to $r_a$. Simultaneously, we replace each arrow $a$ by $r_a$ new arrows with the same origin and end as $a$ and set them in one-to-one correspondence with these new matrices. So we get a new quiver $\hat{Q}$. The vertex set of $\hat{Q}$ coincides with $V$ but the arrow set $\hat{A}$ can be different from $A$.

Take any linear order on $A$. Denote this order by usual symbol $<$. We enumerate arrows of the quiver $\hat{Q}$ by numbers $1, \ldots, r$. One can assume that for any $a \in A$ the corresponding set of new arrows is enumerated by the numbers from the segment $[\hat{a}, a] = [\sum_{b < a} r_b + 1, \sum_{b < a} r_b]$. We obtain some specialization $f : [1, r] = \hat{A} \rightarrow A$ defined by $f(j) = a$ iff $j \in [\hat{a}, a], a \in A$.

In the same way one can define the specialization $Y(j) \mapsto Y(a)$ iff $j \in [\hat{a}, a], a \in A$. Denote the last specialization by the same symbol $f$.

Without loss of generality it can be assumed that $\forall a \in \hat{A}_1, b \in \hat{A}_2, c \in \hat{A}_3, a < b < c$. Thus $\hat{A}_1 = [1, t], \hat{A}_2 = [t+1, t+s], \hat{A}_3 = [t+s+1, r]$. Moreover, $f([1, t]) = A_1$, $f([t+1, s+t]) = A_2$ and $f([s+t+1, r]) = A_3$. It is clear that $i(j) = i$ or $t(j) = i$ iff $i(f(j)) = i$ or $t(f(j)) = i$ respectively, $j \in \hat{A} = [1, \ldots, r], i \in V$. Set

$$T(i) = \{ j \in \hat{A} \mid t(j) = i \}, I(i) = \{ j \in \hat{A} \mid i(j) = i \}, i \in V_{ord}. $$

Analogously, write

$$T(q) = \{ j \in \hat{A} \mid t(j) = i_q \text{ or } i(j) = j_q \}, I(q) = \{ j \in \hat{A} \mid i(j) = i_q \text{ or } t(j) = j_q \}, q \in \Omega. $$

It is obvious that $p_i = |T(i)| = |I(i)|$ for each $i \in V_{ord}$ and $p_q = |T(q)| = |I(q)|$ for all $q \in \Omega$.

We have the inclusion $\Phi_\Theta = \Phi_{\Theta_2, \Theta_1}$ of the space $Hom_{H(t)}(\Lambda_2(\Theta), \Lambda_1(\Theta))$ into

$$Hom_{H(t)}((\otimes_{a \in A_1} E_{t(a)}^{\otimes r_a}) \otimes (\otimes_{a \in A_2} E_{t(a)}^{\otimes r_a}) \otimes (\otimes_{a \in A_3} E_{i(a)}^{\otimes r_a}),$$

$$(\otimes_{a \in A_1} E_{t(a)}^{\otimes r_a}) \otimes (\otimes_{a \in A_2} E_{t(a)}^{\otimes r_a}) \otimes (\otimes_{a \in A_3} E_{i(a)}^{\otimes r_a})).$$

Denote the last space by $Hom(t)$.

The multilinear component of degree $r$ of the algebra $J(\hat{Q}, t)$ is isomorphic to

$$Hom_{H(t)}((\otimes_{a \in \hat{A}_1} E_{t(a)}) \otimes (\otimes_{a \in \hat{A}_2} E_{t(a)}) \otimes (\otimes_{a \in \hat{A}_3} E_{i(a)}),$$

$$(\otimes_{a \in \hat{A}_1} E_{t(a)}) \otimes (\otimes_{a \in \hat{A}_2} E_{t(a)}) \otimes (\otimes_{a \in \hat{A}_3} E_{i(a)})).$$

It is clear that this space coincides with $Hom(t)$.

We identify the space $Hom_{H(N)}(\Lambda_2(N), \Lambda_1(N))$ with its image in $Hom(N)$.

**Lemma 2.3** If both modules $\Lambda_2(t), \Lambda_1(t)$ are not equal to zero then the kernel of the map

$$Hom_{H(N)}(\Lambda_2(N), \Lambda_1(N)) \rightarrow Hom_{H(t)}(\Lambda_2(t), \Lambda_1(t))$$

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is the intersection of \( \text{Hom}_{H(N)}(\Lambda_2(N), \Lambda_1(N)) \) with the kernel of the epimorphism \( \text{Hom}(N) \to \text{Hom}(t) \).

Proof. It is sufficient to look at the following commutative diagram

\[
\begin{array}{ccc}
0 & \to & \text{Hom}_{H(N)}(\Lambda_2(N), \Lambda_1(N)) \\
\downarrow & & \downarrow \\
0 & \to & \text{Hom}_{H(t)}(\Lambda_2(t), \Lambda_1(t))
\end{array}
\]

Here, the horizontal arrows are the inclusions defined above and the vertical arrows are the surjective restriction maps.

Notice that the space \( \text{Hom}(t) \) can be represented as

\[
\otimes_{i \in \mathcal{V}_{ord}} \text{End}_{GL(d_i)}(E_i^{\otimes p_i}) \otimes \otimes_{q \in \Omega} \text{End}_{GL(d_q)}(E_q^{\otimes p_q}).
\]

Here \( E_q \cong E_{iq} \cong E_{jq} \). By Theorem 1.2 \( \text{Hom}(N) \) is isomorphic to \( \otimes_{i \in \mathcal{V}_{ord}} K[S_{p_i}] \otimes \otimes_{q \in \Omega} K[S_{p_q}] \) since we assumed that \( N \geq r \). The kernel of the epimorphism \( \text{Hom}(N) \to \text{Hom}(t) \) is isomorphic to

\[
I_{t+1} = \sum_{i \in \mathcal{V}_{ord}, p_i > d_i} \cdots \otimes I_{d+1} \otimes \cdots + \sum_{q \in \Omega, p_q > d_q} \cdots \otimes I_{d+1} \otimes \cdots .
\]

the place of \( k[S_{p_i}] \)

the place of \( k[S_{p_q}] \)

Using Lemma 2.3 we get

**Proposition 2.2** If \( N' \geq N \geq r \) then the epimorphism \( \text{Hom}(N') \to \text{Hom}(N) \) is an isomorphism. The same is valid for all epimorphisms

\[
\text{Hom}_{H(N')}(\Lambda_2(N'), \Lambda_1(N')) \to \text{Hom}_{H(N)}(\Lambda_2(N), \Lambda_1(N)).
\]

In particular, the \( \bar{r} \)-homogeneous component of the algebra \( J(Q, N) \) does not depend on the number \( N \) and can be identified with the \( \bar{r} \)-homogeneous component of the free invariant algebra \( J(Q) \).

Using the commutative diagram (4) from Theorem 2.1 and repeating again the proof of Proposition 1 from [Zub1] we get

**Proposition 2.3** The \( \bar{r} \)-homogeneous component of the ideal \( T(Q, t) \) is generated as a vector space by the elements \( c(\phi) \), where \( \phi \in \text{Hom}_{H(N)}(\Lambda_2(\Theta, N), \Lambda_1(\Theta, N)) \neq 0 \) and \( \Theta \) runs over all superpartitions of multidegree \( \bar{r} \). In addition, one has to require that either at least one of the modules \( d(\Lambda_2(\Theta, N)) = \Lambda_2(\Theta, t) \), \( d(\Lambda_1(\Theta, N)) = \Lambda_1(\Theta, t) \) is equal to zero or \( \phi |_{\Lambda_2(\Theta, t)} = 0 \).

**Corollary 2.1** The algebra \( J(Q, t) \) is generated by all \( c(\phi) \) without any restrictions on \( \phi \in \text{Hom}_{H(t)}(\Lambda_2(t), \Lambda_1(t)) \) or by all \( p_{N,t}(c(\phi')) \), \( \phi' \in \text{Hom}_{H(N)}(\Lambda_2(N), \Lambda_1(N)) \).
2.2 Multilinear invariants

Denote the image of a \( \phi \in \text{Hom}(t) \) in \( J(\hat{Q}, t) \) by \( tr^*(\phi) \) and its specialization under \( f \) by \( tr^*(\phi, f) \).

Given operator \( \sigma \) from \( \text{Hom}(N) \subseteq \text{End}_{GL(N)}(E^\otimes r) = K[S_r] \), where \( E \) is a \( N \)-dimensional space, it can be written as

\[
\sum_{1 \leq j_1, \ldots, j_r \leq N} \left( \prod_{1 \leq k \leq t} y(k)_{j_k; j_{k-1}(k)} \right) \left( \prod_{t+1 \leq k \leq t+s} y(k)_{j_{k-1}(k); j_{k-1}(k+s)} \right) \left( \prod_{t+s+1 \leq k \leq r} y(k)_{j_{k-s}; j_k} \right).
\]

In order to contract this sum into a product of ordinary traces one can use the following rule (see \([Zub5]\)). We consider the formal product of pairs:

\[
\prod_{1 \leq k \leq t} (k, \sigma^{-1}(k)) \prod_{t+1 \leq k \leq t+s} (\sigma^{-1}(k), \sigma^{-1}(k+s)) \prod_{t+s+1 \leq k \leq r} (k-s, k).
\]

The next step is to partition this product into cyclic subproducts. By definition, these subproducts are \( \prod_{1 \leq f \leq l} (a_f, b_f) \) such that \( b_f = a_{f+1}, 1 \leq f \leq l-1 \) and \( b_l = a_1 \). If it is necessary, one can change the initial order of coordinates of any pair. This partition is possible because of the following fact: each symbol \( k \) appears two times in the original product. Finally, each subproduct \( \prod_{1 \leq f \leq l} (a_f, b_f) \) corresponds to a trace \( tr(Z(j_1) \ldots Z(j_l)) \), where \( j_f \) is the number of the pair \( (a_f, b_f) \) in the original product and \( Z(j_f) \) coincides with \( Y(j_f) \) iff the initial order of the coordinates of this pair was not changed, otherwise \( Z(j_f) = Y(j_f)^t, 1 \leq f \leq l \).

It is more convenient for further computations to denote \( Y^t \) by \( \overline{Y} \).

**Example 2.1** Let \( t = 3, s = 2, r = 7, \sigma = (1726)(354) \in S_7 \). The formal product of pairs corresponding to \( \sigma \) is (16)(27)(34)(52)(31)(46)(57). Decomposing it into cyclic subproducts we get (16)(64)(43)(31) \( \cdot (27)(75)(52) \). Therefore,

\[
tr^*(\sigma) = tr(Y(1)\overline{Y}(6) \overline{Y}(3)Y(5))tr(Y(2)\overline{Y}(7)Y(4))
\]

or

\[
tr^*(\sigma) = tr(Z(1)Z(6)Z(3)Z(5))tr(Z(2)Z(7)Z(4))
\]

with respect to the notations from the introduction.
Notice that if \( s = 0 \) then \( tr^*(\sigma) = tr(\sigma) \), where \( tr(\sigma) = tr(Y(a) \ldots Y(b)) \ldots tr(Y(c) \ldots Y(d)) \) and \((a \ldots b) \ldots (c \ldots d)\) is a cyclic decomposition of \( \sigma^{-1} \).

We set

\[
T(i_q) = \{ j \in \hat{A} \mid t(j) = i_q \}, I(i_q) = \{ j \in \hat{A} \mid i(j) = i_q \},
\]

\[
T(j_q) = \{ j \in \hat{A} \mid i(j) = j_q \}, I(j_q) = \{ j \in \hat{A} \mid t(j) = j_q \}.
\]

It is clear that \( \forall q \in \Omega, T(q) = T(i_q) \cup T(j_q), I(q) = I(i_q) \cup I(j_q) \).

**Lemma 2.4** Any \( \sigma \in S \) lies in \( \text{Hom}(\mathbf{N}) \) iff the following equations are satisfied:

1. \( \forall i \in V_{ord}, \sigma((T(i) \cap \hat{A}_1) \cup (T(i) \cap \hat{A}_3 - s)) = (I(i) \cap \hat{A}_1) \cup (I(i) \cap \hat{A}_2 + s) \).
2. \( \forall q \in \Omega, \sigma((T(i_q) \cap \hat{A}_1) \cup (T(i_q) \cap \hat{A}_3 - s) \cup T(j_q)) = (I(i_q) \cap \hat{A}_1) \cup (I(i_q) \cap \hat{A}_2 + s) \cup I(j_q) \).

Proof. For given \( \sigma \in \text{Hom}(\mathbf{N}) \) its record (5) can be rewritten in a more refined way

\[
\sum_{1 \leq j_1, \ldots, j_r \leq N} \otimes_{i \in V_{ord}} ((\otimes_{1 \leq k \leq t, t(k) = i} (e^i_{j_k})^*) \otimes (\otimes_{k+1 \leq t \leq s, t(k+s) = i} (e^i_{j_k})^*) \otimes (\otimes_{1 \leq k \leq t, i(k) = t} e^i_{j_{k-1}}))
\]

\[
\otimes_{(\otimes_{k+s+1 \leq t \leq r, i(k) = j_q} (e^i_{j_k})^*)} \otimes (\otimes_{1 \leq k \leq t, i(k) = i_q} e^i_{j_{k-1}}) \otimes (\otimes_{k+1 \leq t \leq s, i(k) = i_q} e^i_{j_{k-1}})
\]

\[
\otimes (\otimes_{k+s+1 \leq t \leq r, t(k) = j_q} (e^i_{j_k})^*) \otimes (\otimes_{1 \leq k \leq t, t(k) = t} e^i_{j_{k-1}}) \otimes (\otimes_{k+1 \leq t \leq s, t(k) = i_q} e^i_{j_{k-1}})
\]

It remains to notice that each factor \( e^z_{yz} \) in every summand of this sum must appear on the dual side of the same summand, that is like \( (e^z_{yz})^* \). This completes the proof.

For the sake of convenience denote the right hand side sets of these equations by \( \mathcal{I}(i), \mathcal{I}(q) \) and the left hand side sets, that is the arguments of the substitution \( \sigma \), by \( \mathcal{T}(i), \mathcal{T}(q) \) respectively. Then they can be rewritten as \( \sigma(\mathcal{T}(u)) = \mathcal{I}(u) \), where \( u \in V_{ord} \cup \Omega \). In other words, for any arrow \( j \) the end of \( \sigma^{-1}(j) \) coincides with the origin of \( j \) (see [Zub5]).

**Lemma 2.5** Any \( tr(Z(a_m) \ldots Z(a_1)) \) occurs as a factor of some multilinear trace products from \( J(Q, t_0) \) iff \( Z(a_m) \ldots Z(a_1) \) is admissible.
Proof. Fix some subproduct $UVW$ of any cyclic permutation of the product $Z(a_m)\ldots Z(a_1)$ consisting of three factors. Without loss of generality one can assume that $V = Y(j)$. Otherwise one can transpose the product $Z(a_m)\ldots Z(a_1)$. The following list contains all admissible cases to occupy both places around $(a_j)$.

1. If $j \in \hat{A}_1 = \{1, \ldots, t\}$ then $U$ can be occupied by $Y(\sigma(j))$. It happens iff $\sigma(j) \in \hat{A}_1$. Let $t(j) = i$. Then we have either $j \in T(i) \cap \hat{A}_1$ or $i = i_q, j \in T(i_q) \cap \hat{A}_1$. In both cases $\sigma(j) \in I(i)$, that is the product $Y(\sigma(j))Y(j)$ is linked. The matrix $U$ can be equal to $Y(j')$ or $Y(j')$, where $j' \in \hat{A}_2 = [t + 1, \ldots, t + s]$. The case $U = Y(j')$ takes place iff $\sigma(j) = j' + s \in \hat{A}_3$. It means that either $j' \in I(i) \cap \hat{A}_2$ or $i = i_q, j' \in I(i_q) \cap \hat{A}_2$ and in both cases the product $Y(j')Y(j)$ is also linked.

Finally, let $U = \overline{Y(j')}$. It means that $\sigma(j) = j'$. In this case $i = i_q$ only and $j' \in I(i_q) \cap \hat{A}_2$, that is the product $\overline{Y(j')}Y(j)$ is linked again. As for $W$ the possibilities are the following: $Y(\sigma^{-1}(j)), Y(j')$ or $Y(j')$, where $j' \in \hat{A}_3$. As above it can easily be checked that $VW$ is linked. Briefly one can describe all these ways of occupying as $\{\hat{A}_2, \hat{A}_3, \hat{A}_1\}$.

Other cases are listed without any comments. The interested reader can check them very easily.

2. If $j \in \hat{A}_2$ then either $U = \overline{Y(j')}, j' \in \hat{A}_1, t(j') = i_q, t(j) = j_q$ or $U = Y(j'), \overline{Y(j')}, j' \in \hat{A}_3$. In the last case either $t(j) = i(j')$ or $t(j) = j_q, t(j') = i_q$. For $W$ we have the following possibilities: $W = Y(j')$, $j' \in \hat{A}_1$, or $W = Y(j'), \overline{Y(j')}, j' \in \hat{A}_3$. The first possibility is described in the previous item, the second one implies either $t(j') = i(j)$ or $i(j') = j_q, i(j) = i_q$. Briefly, $\{\hat{A}_2, \hat{A}_3, \hat{A}_1\}$.

3. If $j \in \hat{A}_3$ then either $U = Y(j'), j' \in \hat{A}_1, t(j) = i(j)$ or $U = Y(j'), \overline{Y(j')}, j' \in \hat{A}_2$. The last case is considered in the second item up to some transposition. For $W$ we have the following possibilities: $W = \overline{Y(j')}, j' \in \hat{A}_1$, or $W = Y(j'), \overline{Y(j')}, j' \in \hat{A}_2$. The first possibility is described in the first item up to some transposition, the second possibility is described in the second item. Briefly, $\{\hat{A}_2, \hat{A}_3, \hat{A}_1\}$.

It is clear that in all cases listed above the products $UV, VW$ are linked. The lemma is proved.

In other words, the conditions 1, 2 in Lemma 2.4 are equivalent to the conditions of admissibility in Lemma 2.5.

A trace product $u = tr(Z(a_r)\ldots Z(a_k)) \ldots tr(Z(a_m)\ldots Z(a_1))$ from $J(\hat{Q}, \hat{N})(1^r)$ can be written in many ways. Fix some standard record of each product as follows.
Any matrix $Z(a)$ is equal either to $Z(j) = Y(j)$ or to $Z(\bar{j}) = Y(\bar{j})$. Let us ascribe to $Z(a)$ this number $j$. The record of $tr(Z(a) \ldots Z(b))$ is called right if the matrix with maximal number, say $j$, occupies the first place. Moreover, $Z(a) = Y(j)$ otherwise one has to transpose the product $Z(a) \ldots Z(b)$. Let us call $j$ by the number associated to $tr(Z(a) \ldots Z(b))$. The record of $u$ is called right iff all its factors are right and their associated numbers increase on passing by this product from left to right.

**Proposition 2.4** The right trace products form a basis of the vector space $J(\hat{Q}, N)(1^r)$. In particular, they span $J(\hat{Q}, t)(1^r)$ for any $t$.

Proof. The first assertion has been proved in [Zub5]. The second one is a trivial consequence of Corollary 2.1.

For the sake of convenience we will omit the symbol $tr$ in the record of any multilinear invariant from $J(\hat{Q}, N)(1^r)$ if it does not lead to confusion. We replace any matrix $Y(j)$ or its transposed $Y(\bar{j})$ by the number $j$ or its transposed $\bar{j}$, $1 \leq j \leq r$. For example, the invariant $tr(Y(1)Y(6)Y(3)Y(5))tr(Y(2)Y(7)Y(4))$ given above can be rewritten as (1635)(274).

We suppose by definition that $i = i, i = 1, \ldots, r$ and $[\bar{1}, \bar{r}] = \{\bar{1}, \ldots, \bar{r}\}$.

We reformulate the contracting rules mentioned above as follows.

**Proposition 2.5** Let $\sigma \in \text{Hom}(N)$ and $tr^*(\sigma) = (a \ldots b) \ldots (c \ldots d)$, where $a, \ldots, b, c, \ldots, d \in [1, r][[\bar{1}, \bar{r}]$. All we need is to define exactly what is a right hand side neighbor of any symbol $j$ in a cyclic record of $tr^*(\sigma)$? If $j$ is an ordinary symbol, that is if $j \in [1, r]$, then we have

1. If $j \in \hat{A}_1$ then $(\ldots jk \ldots)$, where

$$k = \begin{cases} 
\sigma^{-1}(j), \sigma^{-1}(j) \in \hat{A}_1, \\
\sigma^{-1}(j) + s, \sigma^{-1}(j) \in \hat{A}_2, \\
\frac{\sigma^{-1}(j)}{\sigma^{-1}(j)}, \sigma(j) \in \hat{A}_3.
\end{cases}$$

2. If $j \in \hat{A}_2$ then $(\ldots jk \ldots)$, where

$$k = \begin{cases} 
\sigma^{-1}(j + s), \sigma^{-1}(j + s) \in \hat{A}_1, \\
\sigma^{-1}(j + s) + s, \sigma^{-1}(j + s) \in \hat{A}_2, \\
\frac{\sigma^{-1}(j + s)}{\sigma^{-1}(j + s)}, \sigma^{-1}(j + s) \in \hat{A}_3.
\end{cases}$$

3. If $j \in \hat{A}_3$ then $(\ldots jk \ldots)$, where

$$k = \begin{cases} 
\bar{\sigma(j)}, \sigma(j) \in \hat{A}_1, \\
\frac{\sigma(j)}{\sigma(j)}, \sigma(j) \in \hat{A}_2, \\
\sigma(j) - s, \sigma(j) \in \hat{A}_3.
\end{cases}$$

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If \( j = \bar{l} \) then the corresponding rules are:

1. If \( l \in \hat{A}_1 \) then \((...jk...)\), where

\[
   k = \begin{cases} 
   \sigma(l), & \sigma(l) \in \hat{A}_1, \\
   \sigma(l), & \sigma(l) \in \hat{A}_2, \\
   \sigma(l) - s, & \sigma(l) \in \hat{A}_3. 
\end{cases}
\]

2. If \( l \in \hat{A}_2 \) then \((...jk...)\), where

\[
   k = \begin{cases} 
   \sigma^{-1}(l), & \sigma^{-1}(l) \in \hat{A}_1, \\
   \sigma^{-1}(l) + s, & \sigma^{-1}(l) \in \hat{A}_2, \\
   \sigma^{-1}(l), & \sigma^{-1}(l) \in \hat{A}_3. 
\end{cases}
\]

3. If \( l \in \hat{A}_3 \) then \((...jk...)\), where

\[
   k = \begin{cases} 
   \sigma(l - s), & \sigma(l - s) \in \hat{A}_1, \\
   \sigma(l - s), & \sigma(l - s) \in \hat{A}_2, \\
   \sigma(l - s) - s, & \sigma(l - s) \in \hat{A}_3. 
\end{cases}
\]

Proof. The check of these rules is obvious. For example, let \( j = \bar{l} \) and \( \sigma^{-1}(l) \in \hat{A}_3 \). Then the corresponding pair is \((\sigma^{-1}(l+s), \sigma^{-1}(l))\). If its right hand side neighbor (up to the order) is \((k, \sigma^{-1}(k))\), \( k \in \hat{A}_1 \), then either \( \sigma^{-1}(l) = k \in \hat{A}_3 \) or \( l = k \in \hat{A}_2 \). Both cases drive to a contradiction so this neighbor should be \((k, k - s), k \in \hat{A}_3 \). The other cases can be considered in the same way.

### 2.3 \( Z \)-forms

In the definition of the representation space of a quiver \( Q \) of dimension \( d \) one can replace all spaces by free \( Z \)-modules of the same ranks \( d_1, \ldots, d_n \). We denote these modules by the same symbols \( E_1, \ldots, E_n \). Then the free \( Z \)-module \( R_Z(Q, d) = \prod_{a \in A} \text{Hom}_Z(E_{i(a)}, E_{t(a)}) \) can be regarded as a \( Z \)-form of \( R(Q, d) \), that is \( R(Q, d) = K \otimes Z R_Z(Q, d) \) and the dimension of the space \( R(Q, d) \) coincides with the rank of the free \( Z \)-module \( R_Z(Q, d) \). The same is true for \( R_Z(Q, t) = \prod_{a \in A} \text{Hom}_Z(W_{i(a)}, W_{t(a)}) \), where as above \( W_i = E_i \) iff \( t_i = d_i \), otherwise \( W_i = E^*_i \).

It is clear that the ring \( Z[R(Q, t)] = Z[y_{ij}(a) \mid 1 \leq j \leq d_{i(a)}, 1 \leq i \leq d_{t(a)}, a \in A] \) is a \( Z \)-form of \( K[R(Q, t)] \). Moreover, \( Z[R(Q, t)] \) can be identified with

\[
   (\prod_{a \in A_1} \otimes S(E^*_{i(a)} \otimes E_{i(a)})) \otimes (\prod_{a \in A_2} \otimes S(E_{i(a)} \otimes E_{i(a)})) \otimes (\prod_{a \in A_3} \otimes S(E^*_{i(a)} \otimes E^*_{i(a)}))
\]

by the same rule as in Section 2. By Remark 1.1 any homogeneous component \( Z[R(Q, t)](\bar{\bar{r}}) \) has an ABW-filtration.
such that (1) can be obtained from (6) by base change. Analogously, for any super-partition Θ of degree  ¯r we have a homomorphism

\[ d_{Θ,Z} : \text{Hom}_Z(Λ_2(Θ, Z), Λ_1(Θ, Z)) → Z[R(Q, t)]( ¯r) \]  

such that \( d_{Θ} = K ⊗ Z d_{Θ,Z} \). Here \( Λ_1(Θ, Z), Λ_2(Θ, Z) \) or, more precisely, \( Λ_1(Θ, t, Z), Λ_2(Θ, t, Z) \) are obvious \( Z \)-forms of \( Λ_1(Θ, t), Λ_2(Θ, t) \). If \( φ ∈ \text{Hom}_Z(Λ_2(Θ, Z), Λ_1(Θ, Z)) \) we denote the element \( d_{Θ,Z}(φ) \) by \( c_Z(φ) \). Now it is easy to guess what \( tr^*_Z \) means.

Finally, as above one can define an inclusion \( Φ_{Θ,Z} : \text{Hom}_Z(Λ_2(Θ, Z), Λ_1(Θ, Z)) → B_Z(t) \), where

\[ B_Z(t) = \text{Hom}_Z((\otimes_{a∈A_1} E_{t(a)}^{⊗ r_a}) \otimes (\otimes_{a∈A_2} E_{t(a)}^{⊗ r_a}) \otimes (\otimes_{a∈A_3} E_{t(a)}^{⊗ r_a}), ) \]

such that the restriction of \( K ⊗ Z Φ_{Θ,Z} \) on \( \text{Hom}_{H(t)}(Λ_2(Θ), Λ_1(Θ)) \) coincides with \( Φ_Θ \).

### 3 Proof of Theorem 1

The proof of Theorem 1 is organized as follows. Using the above identification of the space \( \text{Hom}(N) \) with a group algebra of a product of several symmetric groups one can find some filtration of this algebra such that the ideal \( I_{t+1} \) corresponds to a segment of this filtration. We consider the above group algebra as a weight subspace of some tensor product of symmetric algebras with respect to a torus action. The last tensor product has an ABW-filtration. The intersection of its terms with our group algebra gives the required filtration of \( \text{Hom}(N) \). We notice (see Lemma 3.3 below) that the members of the last filtration are invariant subspaces of terms of the initial ABW-filtration with respect to an action of a reductive group which is a product of general linear groups. By Theorem 1.1(1) a basis of \( \text{Hom}(N) \) or \( I_{t+1} \) can be produced as a union of bases of invariant subspaces of some sequential quotients of this ABW-filtration. Following this way we get the generators of \( J(Q) \) as a vector space such that some subset of them generate the ideal \( T(Q, t) \). It remains to simplify these generators. We reduce this problem to the computation of invariants of ordinary representations of quivers and refer to [Don2, Zub4].
3.1 Suitable generators

Fix some \( \sigma_0 \in \text{Hom}(N) \). By Lemma 2.4 we have \( \text{Hom}(N) = \sigma_0 \cdot (\otimes_{i \in V_{ord}} K[S_\tau]) \otimes (\otimes_{q \in \Omega} K[S_{q\tau}]) \). Moreover, the ideal \( I_{t+1} \) is equal to

\[
\sigma_0 \cdot \left( \sum_{i \in V_{ord}, \mu_i > d_i} \otimes \cdots \otimes I_{d_i+1} \otimes \cdots + \sum_{q \in \Omega, \mu_q > d_q} \otimes \cdots \otimes I_{d_q+1} \otimes \cdots \right).
\]

Denote by \( S_\tau \) the group \( (\prod_{i \in V_{ord}} S_{\tau_i}) \times (\prod_{q \in \Omega} S_{q\tau}) \) and by \( B(t) \) the space \( K \otimes Z B_Z(t) \).

**Remark 3.3** Notice that the layers of the group \( S_{\Theta_1} \) form a subdecomposition of the space \( \text{Hom}_{H(N)}(\Lambda_2(N), \Lambda_1(N)) \) in \( B(N) \) equals \{ \( | \forall \tau_1 \in S_{\Theta_1}, \forall \tau_2 \in S_{\Theta_2}, \tau_1 \phi \tau_2 = (-1)^n (-1)^n \phi \} \). The image of \( \text{Hom}_{H(N)}(\Lambda_2(N), \Lambda_1(N)) \) in the space \( \text{Hom}(N) \) is equal to \( N_\Theta \). On the other hand, both \( \text{Hom}_K(\Lambda_2(N), \Lambda_1(N)) \) and \( B(N) \) are \( H(N) \)-modules with GF and their formal characters do not depend on the characteristic of the ground field. In particular, the dimensions of the spaces \( \text{Hom}_{H(N)}(\Lambda_2(N), \Lambda_1(N)) \) and \( \text{Hom}(N) \) are equal to multiplicities of the trivial character and also do not depend on the characteristic. It remains to notice that in the characteristic zero case our statement is obviously true.

Up to the beginning of Proposition 3.2 we denote by \( E_r \) a vector space of dimension \( r \) with a fixed basis \( e_1, \ldots, e_r \). For any subset \( T \subseteq \{1, \ldots, r\} \) denote by \( E_T \) the subspace of \( E_r \) generated by all vectors \( e_j, j \in T \). Let us identify the group algebra \( K[S_r] \) with a subspace of the homogeneous component \( S^r(E_r \otimes E_r) \) by the rule \( \sigma \mapsto \prod_{i=1}^{r \times r} e_{\sigma(i)} \otimes e_i \). Denote by \( GL(T) \) the group \( \prod_{i \in V_{ord}} GL(E_{T(i)}) \times \prod_{q \in \Omega} (E_{T(q)}) \).

We consider the space \( S^r(E_r \otimes E_r) \) as a \( GL(r) \times GL(r) \)-module. The group \( S_r \) acts on the space \( E_r \) by the rule \( \sigma(e_i) = e_{\sigma(i)}, \sigma \in S_r, 1 \leq i \leq r \). In other words, we identify the group \( S_r \) with a subgroup of the group of permutation matrices by the rule \( \sigma \mapsto \sum_{1 \leq i \leq r} e_{\sigma(i),i} \), where \( e_{kl} \) is a matrix unit which has zero entries outside of \( k \)-th row or \( l \)-th column but the remaining entry is 1. Denote the matrix \( \sum_{1 \leq i \leq r} e_{\sigma(i),i} \) by the same symbol \( \sigma \).

The inclusion \( K[S_r] \rightarrow S^r(E_r \otimes E_r) \) is a morphism of \( S_r \times S_r \)-modules. It can easily be checked that \( K[S_r] \) coincides with the weight subspace \( S^r(E_r \otimes E_r)^{(1)} \times (1) \) under the induced action of the standard torus \( T(r) \times T(r) \). In the same way the space \( K[S_T] \) coincides with the subspace
Let $GL(\Theta_1)$ (respectively, $GL(\Theta_2)$) be a subgroup of the group $GL(r)$ consisting of all block diagonal matrices which satisfy the following requirement: if we decompose the interval $[1, r]$ into sequential subintervals whose lengths equal to the sizes of their blocks considered from top to bottom then we get the layers of the superpartition $\Theta_1$ (respectively, the layers of the superpartition $\Theta_2$).

Lemma 3.2 The space $\sigma_0^{-1} \cdot N_\Theta$ can be identified with

$$\{ g \in (\otimes_{i \in V_{ord}} S^{p_i}(E_{T_i} \otimes E_{T_i})) \otimes (\otimes_{q \in \Omega} S^{p_q}(E_{T_q} \otimes E_{T_q})) \mid \forall x \in GL(\Theta_1), \forall y \in GL(\Theta_2), g^{(\sigma_0^{-1} \sigma_0, y)} = \det(x) \det(y) g \}. $$

Proof. One has to check it on elements from $T(r) \times T(r)$ and transvections from $GL(\Theta_1)$ and $GL(\Theta_2)$.

Let us construct some filtration in $K[S_T]$. We divide each $T(z)$, $z \in V_{ord} \sqcup \Omega$, into some sublayers in a monotonic way. In other words, let $T(z) = \sqcup_{1 \leq j \leq l} \beta_{jz}$, where $\max \beta_{jz} = \min \beta_{jz}$ as soon as $j_1 < j_2$, and $\max(\min) \beta_{jz}$ means the maximal (minimal) number from this sublayer. Joining over all indices $z$ we obtain a decomposition of the segment $[1, r]$.

Denote by $S_{\bar{\beta}}$ the Young subgroup $\prod_{i \in V_{ord}} (\prod_{1 \leq j \leq l_i} S_{\beta_{ij}}) \times \prod_{q \in \Omega} (\prod_{1 \leq j \leq l_q} S_{\beta_{qj}})$. As in [Zub1] we call this subgroup by base subgroup.

Denote by $\Lambda_{\bar{\beta}}$ the space $\otimes_{i \in V_{ord}, q \in \Omega} (\otimes_{1 \leq j \leq l_i} \Lambda^{p_{ij}}(E_{T(i)})) \otimes (\otimes_{1 \leq j \leq l_q} \Lambda^{p_{qj}}(E_{T(q)}))$, where $p_{ij} = | \beta_{ij} |$. The restriction of the pairing map $\delta_{\bar{\beta}}$ on the space $\Lambda_{\bar{\beta}} \times \Lambda_{\bar{\beta}}$ is denoted by the same symbol.

Repeating all arguments concerning ABW-filtrations from Section 2 we define a filtration $\{ M_{\bar{\beta}} \}$ of the space $\otimes_{i \in V_{ord}} S^{p_i}(E_{T_i} \otimes E_{T_i}) \otimes (\otimes_{q \in \Omega} S^{p_q}(E_{T_q} \otimes E_{T_q}))$. For any $\bar{\beta}$ we have an exact sequence of $GL(T) \times GL(T)$-modules

$$0 \to \ker \delta_{\bar{\beta}} \to \Lambda_{\bar{\beta}} \otimes \Lambda_{\bar{\beta}} \to M_{\bar{\beta}}/\hat{M}_{\bar{\beta}} \to 0. \quad (8)$$

All these $GL(T) \times GL(T)$-modules have GF. Denote by $G$ the group $\sigma_0^{-1}GL(\Theta_1)\sigma_0 \times GL(\Theta_2)$.

We have the filtration $\{ M_{\bar{\beta}}^{(1')} \}$ of the space $K[S_T]$ and $\sigma_0^{-1} \cdot I_{t+1}$ is a union of members of this filtration whose indices $\bar{\beta}$ satisfy the following condition: there is some $i \in V_{ord}$ or $q \in \Omega$ such that at least one subset $\beta_{ij}$ or $\beta_{qj}$ has the cardinality $p_{ij} \geq d_i + 1$ or $p_{qj} \geq d_q + 1$ respectively. Combining with Lemma 3.2 we get

Lemma 3.3 The space $\sigma_0^{-1}(N_\Theta \cap I_{t+1})$ has the filtration $\{(M_{\bar{\beta}} \otimes D)^G\}$, where $D = \det^{-1} \otimes \det^{-1}$ and $\bar{\beta}$ satisfies the conditions formulated above.
By Remark 3.1 both groups $\sigma_0^{-1}GL(\Theta_1)\sigma_0$ and $GL(\Theta_2)$ are Levi subgroups of the $GL(T)$. By Theorem 1.1(4) all modules of the exact sequence (8) are $G$-modules with GF. Therefore, we obtain the following short exact sequence

$$0 \to (\ker \delta_\beta \otimes D)^G \to (\Lambda^\beta \otimes \Lambda^\beta \otimes D)^G \to (M_\beta / \bar{M}_\beta \otimes D)^G \to 0.$$ 

In particular, all we need is to find a $(1') \times (1')$-weight subspace of the space $(\Lambda^\beta \otimes \Lambda^\beta \otimes D)^G$ which is equal to $(\Lambda^\beta \otimes (\det)^{-1})\sigma_0^{-1}GL(\Theta_1)\sigma_0 \otimes (\Lambda^\beta \otimes (\det)^{-1})GL(\Theta_2)$ and then we should compute the image of this subspace under the pairing map $\delta_\beta$. It can easily be checked that this subspace consists of all vectors $x$ from $(\Lambda^\beta \otimes \Lambda^\beta)^{(1') \times (1')}$ such that $x(\sigma_0^{-1} \tau_1 \sigma_0, \tau_2) = (-1)^{\tau_1}(-1)^{\tau_2}x$, for all $\tau_1 \in S_{\Theta_1}, \tau_2 \in S_{\Theta_2}$ [Zub1, Zub4]. Denote this subspace by $V_\beta$.

Let $\pi \in S_{[1,t+1]}$ and $(a \ldots d)$ be its cyclic decomposition. Denote by $\pi + s$ the element $(a \ldots b + s) \ldots (c + s \ldots d + c) \in S_{[t+1,t+s]}$. Analogously, any $\pi = (a \ldots b) \ldots (c \ldots d) \in S_{[t+s+1,r]}$ has a shifted double $\pi - s = (a - s \ldots b - s) \ldots (c - s \ldots d - s) \in S_{[t+1,t+s]}$.

For any Young subgroup $S_\lambda \leq S_{[t+1,t+s]}$ denote by $S_\lambda^{s+s}$ the Young subgroup of $S_{[t+s+1,r]}$ consisting of all elements $\pi + s, \pi \in S_\lambda$. In the same way, $S_{\lambda-s} = \{ \pi - s \mid \pi \in S_\lambda \}$ if $S_\lambda \leq S_{[t+s+1,r]}$.

It is clear that the groups $S_{\Theta_2}$ and $S_{\Theta_1}$ coincide with $S_{\lambda A_1} \times S_{\gamma A_3-s} \times S_{\gamma A_4}$ and $S_{\lambda A_1} \times S_{\mu A_2} \times S_{\mu A_2}^{s+s}$ respectively. Thus any element $\pi \in S_{\Theta_2}$ can be written as the product $\pi_1 \pi_2 \pi_3$, where $\pi_1 \in S_{\lambda A_1}, \pi_2 \in S_{\gamma A_3-s}, \pi_3 \in S_{\gamma A_4}$. Analogously, any element $\pi \in S_{\Theta_1}$ can be written as the product $\pi_1 \pi_2 \pi_3$, where $\pi_1 \in S_{\lambda A_1}, \pi_2 \in S_{\mu A_2}, \pi_3 \in S_{\mu A_2}^{s+s}$.

Denote the groups $S_{\Theta}$ and $S_{[1,t]} \times S_{[t+1,t+s]} \times S_{[t+s+1,r]}$ by $S$ and $S_0$ respectively and define two homomorphisms $\rho_1, \rho_2$ from $S_0$ into the group $S_r$. The first homomorphism is given by $\pi \mapsto \pi_1 \pi_2 (\pi_2 + s)$. The second one takes any $\pi$ to $\pi_1 (\pi_3 - s) \pi_3$.

We consider the space $W_\beta = \{ x \in (\Lambda^\beta \otimes \Lambda^\beta)^{(1') \times (1')} \mid \forall \tau \in S_r, x(\sigma_0^{-1} \rho_1(\tau) \sigma_0, \rho_2(\tau)) = x \}$. It is clear that this space contains the space $V_\beta$.

Denote by $p$ the canonical projection $\otimes_{i \in V_{ord,q} \in \Omega} (E_{T(i)}^{\otimes \rho_1}) \otimes (E_{T(q)}^{\otimes \rho_2}) \to \Lambda^\beta$.

The vectors $e_\sigma = p(e_\sigma)$ form a basis of the space $(\Lambda^\beta)^{(1')}$, where

$$e_\sigma = \otimes_{i \in V_{ord,q} \in \Omega} (\otimes_{j \in T(i)} e_{\sigma(j)}) \otimes (\otimes_{j \in T(q)} e_{\sigma(j)})$$

and $\sigma$ runs over $S_T/S_\beta$.

**Proposition 3.1** The space $W_\beta$ has a basis consisting of all vectors

$$\sum_{\tau \in S_r/(\rho_1^{-1}(S_\beta^{s+s})) \cap \rho_2^{-1}(S_\beta^{s+s}) \cap S} c_{\sigma_0^{-1} \rho_1(\tau) \sigma_0, \rho_2(\tau)} e_{\sigma_0^{-1} \rho_1(\tau) \sigma_0, \rho_2(\tau)}$$

where the pairs $(\sigma_1, \sigma_2)$ range over some subset of $S_T/S_\beta \times S_T/S_\beta$. 

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This basis is decomposed into orbits under the action of the group \( \rho \). Remark 3.2 refer the interested reader to [Zub1, Zub4].

We call the vectors from this proposition suitable generators. By the same arguments one can obtain a basis of the space \( V_\beta \). We omit these computations and refer the interested reader to [Zub1, Zub4].

**Remark 3.2** One can suppose that \( E_r \) is a free \( \mathbb{Z} \)-module with the same basis \( e_1, \ldots, e_r \). In this case \( \Lambda^3 \otimes \Lambda^3 \) is a free \( \mathbb{Z} \)-module and we obtain the same free generators of the free \( \mathbb{Z} \)-modules \( V_\beta \) and \( W_\beta \) as above. In particular, any free generator of \( V_\beta \) is a sum of suitable generators with integral coefficients [Zub1, Zub4].

**Lemma 3.4** The space \( N_\theta \cap I_{t+1} \) is generated by the elements

\[
{h}_{\sigma_1, \sigma_2} = \sum_{\tau \in \mathfrak{S}_2} \sum_{\pi \in S / (\rho_1^{-1}(S_\beta^{\sigma_0 \sigma_1}) \cap \rho_2^{-1}(S_\beta^{\sigma_2}) \cap S)} (-1)^{\tau} \rho_1(\pi) \sigma_0 \sigma_1 \tau \sigma_2^{-1} \rho_2(\pi)^{-1}.
\]

Proof. Applying the map \( \delta_\beta \) to the generators from Proposition 3.1 we obtain the elements

\[
g_{\sigma_1, \sigma_2} = \sum_{\tau \in \mathfrak{S}_2} \sum_{\pi \in S / (\rho_1^{-1}(S_\beta^{\sigma_0 \sigma_1}) \cap \rho_2^{-1}(S_\beta^{\sigma_2}) \cap S)} (-1)^{\tau} \rho_0^{-1}(\pi) \sigma_0 \sigma_1 \tau \sigma_2^{-1} \rho_2(\pi)^{-1}.
\]

It remains to multiply by \( \sigma_0 \).
Proposition 3.2 Let $\phi \in \text{Hom}_Z(A_2(t), A_1(t))$. We have $\text{tr}_Z^*(\Phi_{\Theta,Z}(\phi), f) = |S_\Theta| c_Z(\phi)$.

Proof. Let $e_1^i, \ldots, e_{d_i}^i$ be a free basis of the module $E_i$, $i \in V$. The dual basis of $E_i^*$ is $(e_1^i)^*, \ldots, (e_{d_i}^i)^*$. Let us decompose the interval $[1, r]$ into subintervals by the following rule:

$$[1, r] = (\bigsqcup_{a \in A_1} [\hat{a}, a]) \bigsqcup (\bigsqcup_{a \in A_3} [\hat{a} - s, a - s]) \bigsqcup (\bigsqcup_{a \in A_3} [\hat{a} + s, a + s]),$$

where $[\hat{a} - s, a - s]$ is equal to $[\sum_{b < a} r_b - s + 1, \sum_{b \leq a} r_b - s]$.

In the same way one can decompose the interval $[1, r]$ into other subintervals:

$$[1, r] = (\bigsqcup_{a \in A_1} [\hat{a}, a]) \bigsqcup (\bigsqcup_{a \in A_2} [\hat{a}, a]) \bigsqcup (\bigsqcup_{a \in A_2} [\hat{a} + s, a + s]),$$

where $[\hat{a} + s, a + s]$ is equal to $[\sum_{b < a} r_b + s + 1, \sum_{b \leq a} r_b + s]$.

Let $I, J : [1, r] \to [1, \max_{i \in V} d_i]$ be two maps such that the following conditions are satisfied:

1. $\forall a \in A_1, I([\hat{a}, a]) \subseteq [1, d_i(a)]$ and $J([\hat{a}, a]) \subseteq [1, d_i(a)]$;
2. $\forall a \in A_2, J([\hat{a}, a]) \subseteq [1, d_i(a)]$ and $J([\hat{a} + s, a + s]) \subseteq [1, d_i(a)]$;
3. $\forall a \in A_3, I([\hat{a} - s, a - s]) \subseteq [1, d_i(a)]$ and $I([\hat{a}, a]) \subseteq [1, d_i(a)]$.

Suppose that the restrictions of the maps $I$ and $J$ on all layers of the Young subgroups $S_{\Theta_2}$ and $S_{\Theta_1}$ respectively are injective. Then a typical basis vector of $\text{Hom}_Z(A_2(t), A_1(t))$ is $p_{\Theta_2}(e_1^*) \otimes p_{\Theta_1}(e_J)$, where

$$e_1^* = (\otimes_{a \in A_1} (\otimes_{l \in [\hat{a}, a]} (e_{I(l)}^{(a)}))^*) \otimes (\otimes_{a \in A_3} (\otimes_{l \in [\hat{a} - s, a - s]} (e_{I(l)}^{(a)}))^*) \otimes (\otimes_{a \in A_3} (\otimes_{l \in [\hat{a}, a]} (e_{I(l)}^{(a)}))^*))$$

and

$$e_J = (\otimes_{a \in A_1} (\otimes_{l \in [\hat{a}, a]} e_{J(l)}^{(a)})) \otimes (\otimes_{a \in A_2} (\otimes_{l \in [\hat{a}, a]} e_{J(l)}^{(a)})) \otimes (\otimes_{a \in A_2} (\otimes_{l \in [\hat{a} + s, a + s]} e_{J(l)}^{(a)})).$$

The element $\text{tr}_Z^*(\Phi_{\Theta,Z}(\phi), f)$ equals

$$\sum_{\sigma_1, \sigma_2 \in S_{\lambda_1}} (-1)^{\sigma_1} (-1)^{\sigma_2} \prod_{1 \leq j \leq t} y(f(j)) I_{i(\sigma_1(j)), J(\sigma_2(j))} \times$$

$$\sum_{\sigma_1, \sigma_2 \in S_{\lambda_2}} (-1)^{\sigma_1} (-1)^{\sigma_2} \prod_{t \leq j \leq t+s} y(f(j)) I_{i(\sigma_1(j)), J(\sigma_2(j+s))} \times$$

$$\sum_{\sigma_1, \sigma_2 \in S_{\lambda_3}} (-1)^{\sigma_1} (-1)^{\sigma_2} \prod_{t+s \leq j \leq r} y(f(j)) I_{i(\sigma_1(j-s)), I(\sigma_2(j))},$$

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Ordering the factors of these products with respect to their first subindices we get

\[
|S_\Theta| \sum_{\sigma \in S_{\lambda,1}} (-1)^{\sigma} \prod_{1 \leq j \leq t} y(f(j))_I(j), J(j) \times \\
\sum_{\sigma \in S_{\lambda,2}} (-1)^{\sigma} \prod_{t \leq j \leq t+s} y(f(j))_J(j), J(j+s) \times \\
\sum_{\sigma \in S_{\lambda,3}} (-1)^{\sigma} \prod_{t+s \leq j \leq r} y(f(j))_I(j-s), J(j) = \\
|S_\Theta| \cdot c_Z(\phi).
\]

This concludes the proof.

**Corollary 3.1** Each \(\frac{1}{|S|} tr^*(h_{\sigma_1, \sigma_2}, f)\) belongs to \(Z[R(Q, N)]\). In particular, it can be reduced modulo any \(p\).

Proof. By Lemma 3.1 we see that there is \(\phi \in \text{Hom}_Z(A_2(N, Z), \Lambda_1(N, Z))\) such that \(\Phi_{\Theta,Z}(\phi) = h_{\sigma_1, \sigma_2}\). By Proposition 3.2 we get \(\frac{1}{|S|} tr^*(h_{\sigma_1, \sigma_2}) = \frac{1}{|S|} tr_Z^*(\Phi_{\Theta,Z}(\phi), f) = c_Z(\phi) \in Z[R(Q, N)]\).

Using Remark 3.2, Lemma 3.4 and Corollary 3.1 as well as Proposition 2.2 and Corollary 2.1 we get

**Proposition 3.3** The \(\bar{r}\)-component of \(T(Q, t)\) is generated as a vector space by all elements \(\frac{1}{|S|} tr^*(h_{\sigma_1, \sigma_2}, f)\) which are also called suitable, where \(S_{\bar{3}}\) runs over all Young subgroups of \(S_{\bar{r}}\) satisfying the condition on its layers formulated above. If we ignore this condition we get the generators of \(J(Q)(\bar{r})\) and mapping them into \(J(Q, t)(\bar{r})\) the generators of this last homogeneous component are obtained.

### 3.2 Reductions to ordinary representations of quivers

Let us denote by \(R\) the permutation \(\prod_{i \in \bar{A}_2} (i \ i+s \ \bar{i} \ \bar{s} \ \bar{\bar{i}} \ \bar{\bar{s}})\) from \(S_{[1, r]} \sqcup [\bar{1}, \bar{r}]\). For any \(\pi \in S_{[1, r]} \sqcup [\bar{1}, \bar{r}]\) having a cyclic decomposition \((a \ldots b) \ldots (c \ldots d)\) denote \((\bar{a} \ldots \bar{b}) \ldots (\bar{c} \ldots \bar{d})\) by \(\bar{\pi}\). We have a bijection \(\iota : \pi \mapsto \bar{\pi}^{-1}\) on \(S_{[1, r]} \sqcup [\bar{1}, \bar{r}]\). It is clear that this bijection induces an involution on the group \(S_{[1, r]} \sqcup [\bar{1}, \bar{r}]\). In fact, let us denote by \(a\) the permutation \(\prod_{i \in [1, r]} (i \bar{i})\). Then \(a \pi a^{-1} = \bar{\pi}\) and \(\iota(\pi) = a \pi a^{-1}\).

**Lemma 3.5** Let \(\sigma \in S_r\) and \(tr^*(\sigma) = u = (a \ldots b) \ldots (c \ldots d)\), where \(\{a, b, \ldots, c, d\}\) is a subset of \([1, r] \sqcup [\bar{1}, \bar{r}]\) having cardinality \(r\). Then \(R \sigma^{-1} \sigma R = w \bar{u}^{-1} = u \iota(u)\).
Proof. It can be easily checked that for any \( j \in [1, r] \cup [\bar{1}, \bar{r}] \) its right hand side neighbors in cyclic decompositions of both \( R\sigma^{-1}\bar{\sigma}R \) and \( u \) are the same. For example, let \( j = \bar{l}, l \in \hat{A}_3 \). Then we have the following equations:

\[
R(\bar{l}) = \bar{l} - \bar{s}, \bar{\sigma}(\bar{l} - \bar{s}) = \bar{\sigma}(l - s)
\]

and finally

\[
R(\bar{\sigma}(l - s)) = \left\{ \begin{array}{l}
\bar{\sigma}(l - s), \bar{\sigma}(l - s) \in \hat{A}_1, \\
\bar{\sigma}(l - s), \bar{\sigma}(l - s) \in \hat{A}_2, \\
\bar{\sigma}(l - s) - s, \bar{\sigma}(l - s) \in \hat{A}_3,
\end{array} \right.
\]

that is the result is the same as in the contracting rules defining \( u \) (see Proposition 2.5). Other cases can be checked similarly. Thus follows that any cycle of \( R\sigma^{-1}\bar{\sigma}R \) is a cycle of \( u \) or its transposed \( \bar{u}^{-1} \). This completes the proof.

**Lemma 3.6** Let \( \sigma \in S_r \) and \( R\sigma^{-1}\bar{\sigma}R = u(u) \). Suppose that the cyclic record of \( u \), including trivial cycles, contains two symbols \( i, j \) belonging to the same set \( \hat{A}_l \) or \( \bar{A}_l, l = 1, 2, 3 \). Then \((ij)u((ij)u) = R\sigma^{-1}\bar{\sigma}R \), where either \( \sigma' = (i', j')\sigma \) or \( \sigma' = \sigma(i', j') \) and \( i', j' \) belong to the same \( \hat{A}_f \) or \( \bar{A}_f, f = 1, 2, 3 \). More precisely,

\[
i', j' = \left\{ \begin{array}{l}
i, j, i, j \in \hat{A}_1, \\
\bar{i}, \bar{j}, i, j \in \hat{A}_1,
\end{array} \right.
\]

\[
i', j' = \left\{ \begin{array}{l}
i, j, i, j \in \hat{A}_2, \\
i + s, j + s, i, j \in \bar{A}_2,
\end{array} \right.
\]

\[
i', j' = \left\{ \begin{array}{l}
i - s, j - s, i, j \in \bar{A}_3, \\
\bar{i}, \bar{j}, i, j \in \bar{A}_3.
\end{array} \right.
\]

In particular, \( \sigma \) and \( \sigma' \) have different parities.

Proof. Notice that a decomposition \( u(u) \) of \( R\sigma^{-1}\bar{\sigma}R \) is not uniquely defined. For example, interchanging any cycle \((a \ldots b)\) from a cyclic record of \( u \) with its transposed \((\bar{b} \ldots \bar{a})\) from a record of \( i(u) \) we get some other decomposition \( u'(u') \). Therefore, a left factor \( u \) can be defined as a part of a cyclic decomposition of \( R\sigma^{-1}\bar{\sigma}R \) depending of \( r \) symbols from \([1, r] \cup [\bar{1}, \bar{r}]\) which does not contain any \( i \)-invariant cycles. Let \( i, j \in \hat{A}_3 \), say \( i = \bar{m}, j = \bar{n}, m, n \in \bar{A}_3 \). We have

\[ (ij)u(ij)u = (\bar{m}\bar{n})R\sigma^{-1}\bar{\sigma}R(mn) = RR^{-1}(\bar{m}\bar{n})R\sigma^{-1} \times \bar{\sigma}R(mn)R^{-1}R. \]

Further, \( R^{-1}(\bar{m}\bar{n})R = (\bar{m}R^{-1}\bar{n}R^{-1}) = (mn) \). Thus \((ij)u(ij)u = R\sigma'^{-1}\bar{\sigma}'R \), where \( \sigma' = \sigma(mn) \). All other cases can be checked in the same way.

It remains to prove that \((ij)u \) is correctly defined. Using the identity \((ij)(iC)(jD) = (iCjD) \), where \( C, D \) are some completing fragments of these cycles, we see that the
sets of symbols involved in the records of \( u \) and \((ij)u\) correspondingly are the same.

So it is enough to prove that \((ij)u\) does not contain \( i\)-invariant cycles.

Suppose that \( u = (iCjD)\ldots \). We have \((ij)u = (iC)(jD)\ldots \). If \((iC) = \iota((iC))\) then in the cycle \((iC)\) there are two sequential symbols like \( \bar{z}, z \). Then it is true for \((iCjD)\) excepting the case \( C = C_1\bar{i}. \) In the last case we have \((iCjD) = (iC_1\bar{i}jD)\). But both cases are forbidden because of \( i, j \) or \( \bar{i}, \bar{j} \) belongs to the same set \( A_l, l = 1, 2, 3, \) (see Lemma 2.5). The case \( u = (iC)(jD)\) is symmetrical to the previous one. The lemma is proved.

**Lemma 3.7** ([Zub5]) Let \( \pi \in S_0, \sigma \in S_i, \) and \( \text{tr}(\sigma) = u. \) Then we have \( u_{\pi \times \pi} = \text{tr}^\ast(\rho_1(\pi)\sigma\rho_2(\pi)^{-1}). \)

Proof. It is enough to prove this equation for \( \pi = (ij)\), where \( i, j \) lie in \( \hat{A}_1, \hat{A}_2 \) or \( \hat{A}_1 \) simultaneously. Let \( i, j \in \hat{A}_2. \) Then \( \rho_1(\pi) = (ij)(i + s, j + s) \) and \( \rho_2(\pi) = \text{id}. \)

We have

\[
R((ij)(i + s, j + s))\sigma^{-1}\bar{\sigma}(ij)(i + s, j + s)R = ((ij)(i + s, j + s))^\tau R\sigma^{-1}\bar{\sigma}R((ij)(i + s, j + s))^\tau^{-1},
\]

where \( \tau = (i, i + s, i + s, \bar{i})(j, j + s, j + s, \bar{j}) \). It remains to notice that

\[
((ij)(i + s, j + s))^\tau = (ij)(ij), (ij)(i + s, j + s))^\tau = (ij)(ij).
\]

The other cases can be checked in the same way. The lemma is proved.

We define some intermediate collection of matrices \( U(l), 1 \leq l \leq m \), where \( m \) is equal to the number of all layers of the group \( G = \rho_1^{-1}(S^i_{\sigma_0\sigma_1}) \cap \rho_2^{-1}(S^j_{\sigma_2}) \cap S \). One can define the new specialization \( g \) which takes any matrix \( Y(j) \) to \( U(l) \) iff \( j \) belongs to the \( l \)-th layer of the group \( G \). It is clear that there is some specialization \( h \) such that \( f = h \circ g \).

**Lemma 3.8** Every \( \frac{1}{|S|}\text{tr}^\ast(h_{\sigma_1, \sigma_2}, f) \) is obtained from \( \frac{1}{|G|}\text{tr}^\ast(\sum_{\tau \in S_\bar{\beta}}(-1)^\tau \sigma_0\sigma_1\tau\sigma_2^{-1}, g) \) by applying \( h \).

Proof. Using Lemma 3.7 we see that

\[
\text{tr}^\ast(\rho_1(\pi)\sigma_0\sigma_1\tau\sigma_2^{-1}\rho_2(\pi)^{-1}, f) = \text{tr}^\ast(\sigma_0\sigma_1\tau\sigma_2^{-1}, f)
\]

because of \( f \circ \pi = f \). The final computations are trivial.

Lemma 3.8 shows that without loss of generality one can assume that \( S = G \leq \rho_1^{-1}(S^i_{\sigma_0\sigma_1}) \cap \rho_2^{-1}(S^j_{\sigma_2}) \) up to some gluing of matrix variables (see [Zub1, Zub4]). Replacing \( \sigma_0 \) by \( \sigma_0\sigma_1 \) one can suppose that \( \sigma_1 = 1 \). Similarly, replacing the group \( S_{\bar{\beta}} \) by the group \( S^j_{\sigma_2} \leq S_T \) and the element \( \sigma_0 \) by the element \( \sigma_0\sigma_2^{-1} \) one can suppose that \( \sigma_2 = 1 \) too.
Lemma 3.9 The invariant $\frac{1}{|S_g|}tr^*(\sum_{\tau \in S_\beta}(-1)^\tau \sigma, g)$ is some partial linearization (briefly PL) of the invariant $\frac{1}{|\rho_1^{-1}(S_\beta^2)\cap \rho_2^{-1}(S_\beta)|}tr^*(\sum_{\tau \in S_\beta}(-1)^\tau \sigma, f')$, where the specialization $f'$ corresponds to the group $\rho_1^{-1}(S_\beta^2)\cap \rho_2^{-1}(S_\beta)$.

Proof. By definition, $S_f' = \rho_1^{-1}(S_\beta^2)\cap \rho_2^{-1}(S_\beta) \leq S_0$. Consider two layers $\alpha, \beta$ of the group $G$ which are contained in some layer of the group $S_f' \cap S_{[t+1,t+s]}$. For the sake of simplicity assume that these layers have numbers $m-1, m$ corresponding. We define the new specialization $g'$ such $g'(j) = m-1$ if $j \in \alpha \cup \beta$ otherwise $g'(j) = g(j)$. Let $x \in S_r$ and $tr^*(x, g') = (g'(a) \ldots g'(b)) \ldots (g'(c) \ldots g'(d))$, where $\{a, \ldots, b, c, \ldots, d\}$ is a subset of $[1, r] \cup [1, r]$ having cardinality $r$. By definition, $g'((j) = g(j), j \in [1, r]$.

Extracting the homogeneous summands of degrees $|\alpha|$ and $|\beta|$ in $U(m-1)$ and $U(m)$ respectively from $tr^*(x, g') |_{U(m-1)+U(m)}$ we get the sum

$$\sum_{\pi \in S_{\alpha\beta}/\alpha \times \beta} (g(\pi(a)) \ldots g(\pi(b))) \ldots (g(\pi(c)) \ldots g(\pi(d))).$$

Using Lemma 3.7 we see that

$$(g(\pi(a)) \ldots g(\pi(b))) \ldots (g(\pi(c)) \ldots g(\pi(d))) = tr^*(\rho_1(\pi)x, g).$$

Thus our PL of the element $\frac{1}{|S_g|}tr^*(\sum_{\tau \in S_\beta}(-1)^\tau \sigma, g')$ is equal to

$$\frac{1}{|S_f'|} \sum_{\tau \in S_\beta} \sum_{\pi \in S_{\alpha\beta}/\alpha \times \beta} (-1)^\tau tr^*(\rho_1(\pi)\sigma, g).$$

Further, $\rho_1(\pi) \in S_\beta^2$, i.e. $\rho_1(\pi) = \sigma y \sigma^{-1}, y \in S_\beta$. In particular, we get

$$\sum_{\tau \in S_\beta} (-1)^\tau tr^*(\rho_1(\pi)\sigma, g) = \sum_{\tau \in S_\beta} (-1)^\tau tr^*(\sigma y \sigma, g) = \sum_{\tau \in S_\beta} (-1)^\tau tr^*(\sigma \tau, g),$$

since $\rho_1(\pi)$ is an even element. Therefore, our PL is equal to the initial invariant $\frac{1}{|G|}tr^*(\sum_{\tau \in S_\beta}(-1)^\tau \sigma, g)$. Repeating these arguments as many times as we need we pass from the group $G$ to the group $S_f'$. This completes the proof.

Summarizing we see that up to some rearrangements, gluings of matrix variables and PL-s the generators of $J(Q)$ ($J(Q, t)$) as well as the generators of $T(Q, t)$ are

$$c(\phi) = \frac{1}{|S_f|} tr^*(\sum_{\tau \in S_\beta} (-1)^\tau \sigma, f),$$

where $S_f = \rho_1^{-1}(S_\beta^2) \cap \rho_2^{-1}(S_\beta)$ and $\phi = \sum_{\tau \in S_\beta} (-1)^\tau \sigma \tau$. As for the generators of $T(Q, t)$ one has to impose the condition on cardinality of layers of $S_\beta$ mentioned above. Notice that if $s = 0$ then these elements are the same as the suitable generators from [Zub1, Zub4].

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We recall some definitions from [Don2]. Let $S_g \leq S_r$ be a Young subgroup corresponding to a map $g : [1, r] \to [1, m]$. Any sequence $p = j_1 \ldots j_s$ of symbols from $[1, m]$ is said to be a primitive cycle if there is no proper subsequence $q$ of $p$ such that $p$ graphically coincides with $q^k = q \ldots q, kd = s, s > k > 1$. For any Young superclass $\tau = (a \ldots b) \ldots (c \ldots d) \in S_r$ we have

$$g(\tau) = (g(a) \ldots g(b)) \ldots (g(c) \ldots g(d)) = \prod_{1 \leq i \leq s_1} (p_{1j}^{k_1}) \ldots \prod_{1 \leq j \leq s_v} (p_{vj}^{k_v}),$$

where each $p_i$ is a primitive cycle uniquely defined up to cyclic permutations of its symbols, $i = 1, \ldots, v$. Two substitutions $\mu, \pi \in S_r$ are called $S_g$-equivalent if there is a sequence $\mu = \tau_1, \ldots, \tau_k = \pi$ such that for any pair $\tau_i, \tau_{i+1}, 1 \leq i < k - 1$, either there is $x \in S_g$ such that $\tau_{i+1} = \tau_i^x$ or for two cycles of $\tau_i (\tau_{i+1})$, say $(a \ldots b), (c \ldots d)$, we have $g(a) \ldots g(b) = (p^f), (g(c) \ldots g(d)) = (p^d)$, where $p$ is a primitive cycle and $\tau_{i+1} = (ac)\tau_i$ (respectively $-\tau_i = (ac)\tau_{i+1}$). It can easily be checked that this relation between elements of $S_r$ is really an equivalence. Donkin calls such equivalence class by Young superclass.

It is clear that all permutations from the same Young superclass $D$ have the same sets of primitive cycles. We denote each of these sets by $P_D$.

For any Young superclass $D$ a formal invariant $\frac{1}{|S_g|} \sum_{x \in D} (-1)^x tr(x, g)$ can be regarded as an invariant of $m N \times N$ matrices of degree $r$ or as an element of the corresponding free invariant algebra due our assumption $N \geq r$.

**Lemma 3.10 ([Don2])** The element $\frac{1}{|S_g|} \sum_{x \in D} (-1)^x tr(x, g)$ can be written as a sum with integer coefficients of products of the elements $\sigma_j(p)$, where $p \in P_D$.

According to our conventions the element $\frac{1}{|S_g|} \sum_{x \in D} (-1)^x tr(x, g)$ can be represented as a sum $\frac{1}{|S_g|} \sum_{x \in D^{-1}} (-1)^x g(x)$, where $D^{-1} = \{x^{-1} \mid x \in D\}$. Notice that $D^{-1}$ is also an Young superclass.

Now, everything is prepared to prove Theorem 1. Without loss of generality one can work in $J(Q)$ or in $J(Q)$ if it is necessary. Let us consider any suitable generator

$$z = c(\phi) = \frac{1}{|S_f|} tr^*(\sum_{\tau \in S_{\beta}} (-1)^{\tau} \sigma_{\tau}, f).$$

Fix a summand $tr^*(\sigma_{\tau}) = u$. One can interpret the element $u$ as an ordinary invariant depending on $r$ matrix variables $Z(j_1), \ldots, Z(j_r), j_1, \ldots, j_r \in [1, r] \cup [1, r]$ or as a permutation from $S_{\{j_1, \ldots, j_r\}}$. It is clear that $\{j_1, \ldots, j_r\} = T_1 \cup T_2$, where $T_1, T_2$ are two subsets of $[1, r]$ such that $T_1 \cup T_2 = [1, r]$. Denote by $S_f'$ the group $S_{f'}$, where $\pi = \Pi_{i \in T_2} (i)$. It can be easily checked that $S_f' = S_{f'}$, where $f' = (f \times f \circ a)_{|j_1, \ldots, j_r|}$.

I claim that Young superclass corresponding to $S_f'$, say $D$, which contains $u$, is a subset of $tr^*(\sigma S_g)$. Indeed, for any $\nu \in S_f$ we have $
u' = \nu \pi, \nu \in S_f$ and $\nu'' = u^{\nu \times \nu} =$
The centralizer \( R \) by example, their definition does not include any action of some orthogonal (symplectic) groups. Moreover, in the last case the groups can be replaced by \( \text{Sp} \). But any layer of \( S \) prove the coincidence of signs. But for any element \( s, j \), \( \rho \) they associate with any generalized quiver of \( \text{tr} \) \( \rho \) of \( \text{tr} \). As we noticed in the introduction one can define more general supermixed representations of quivers. A similar definition was introduced in [DW3]. Briefly speaking, \( \rho \).

4 Proof of Theorem 2

As we noticed in the introduction one can define more general supermixed representations of quivers. A similar definition was introduced in [DW3]. Briefly speaking, they associate with any generalized quiver of \( O(n) \) (\( \text{Sp}(n) \)) orthogonal (symplectic) representations of so-called symmetric quiver. For example, typical components of orthogonal representations of a symmetric quiver are

\[
\text{Hom}_K(V_1, V_2), \text{Hom}_K(V_1, V_2^*), \text{Hom}_K(V_1^*, V_2), \Lambda^2(V) \subseteq \text{Hom}_K(V^*, V),
\]

\[
\Lambda^2(V^*) \subseteq \text{Hom}_K(V, V^*),
\]

\[
\text{Hom}_K(V, W), \text{Hom}_K(V^*, W), \text{Hom}_K(W_1, W_2), \Lambda^2(W) \subseteq \text{Hom}_K(W, W).
\]

The spaces \( V, V_i, W, W_j \) are regarded as standard \( GL(V), GL(V_i), O(W), O(W_j) \)-modules respectively, \( i = 1, 2, j = 1, 2 \). These spaces are isotypical components of the space \( K^n \) with respect to the action of an abelian reductive subgroup \( D \) of \( O(n) \). The centralizer \( R = Z_{O(n)}(D) \) is a product of the same \( GL(V), GL(V_i), O(W), O(W_j) \).

In the symplectic case one has to replace the components \( \Lambda^2(V), \Lambda^2(V^*), \Lambda^2(W) \) by \( S^2(V), S^2(V^*), S^2(W) \) up to some identifications like \( A \leftrightarrow AJ \) mentioned in the introduction. Moreover, in the last case the groups \( O(W), O(W_1), O(W_2) \) must be replaced by \( \text{Sp}(W), \text{Sp}(W_1), \text{Sp}(W_2) \) correspondingly.

It is clear that our definition is more general than Derksen-Weyman’s one. For example, their definition does not include any action of some orthogonal (symplectic) group on symmetric (skew-symmetric) matrix component.
Proposition 4.1 Let $H$ be an orthogonal or symplectic subgroup of the group $GL(n)$. The affine variety $GL(n)/H$ is isomorphic to the affine variety $L$ consisting of all non-degenerate symmetric matrices or skew-symmetric matrices with zero diagonal entries according to which case is considered: $H = O(n)$ or $H = Sp(n)$. This isomorphism is induced by the map $g \mapsto gg^t$ or $g \mapsto gJ_0g^t$ respectively.

Proof. We refer to [Zub5] for this statement.

Notice that the left action of $GL(n)$ on $GL(n)/H$ induces the action of $GL(n)$ on $L$ by the rule $x^g = gxg^t, x \in L, g \in GL(n)$.

Now everything is prepared to prove Theorem 2. We describe the construction of $Q'$ step by step with respect to all the replacements which were used to get $S$ and $G$.

For example, let us consider the case when $G_q = Sp(d_q)$ acts on some component $S_a \subseteq \text{Hom}_K(V_i, V_j), a \in A, i(a) = i, t(a) = j, V_i = V_j = K^{d_q}$ and $S_a$ can be identified with the subspace of symmetric matrices by the rule $A \mapsto AJ, A \in S_a$. With respect to this identification the group $G_q = Sp(d_q)$ acts on $S_a$ by $A^g = gAg^t, g \in G_q$.

Repeating word by word the proof of Lemma 1.3 [Zub5] we have an epimorphism $R^G \rightarrow K[S]^G$, where $R = K[S' \times M(d_q)], S'$ is a product of all components of $S$ except $S_a$, and $G_q$ acts on $M(d_q)$ by the same rule $A^g = gAg^t$.

In fact, $S$ is a closed $G$-subvariety of $S' \times M(d_q)$. Moreover, it is a complete intersection defined by the relations $x_{ij} - x_{ji} = 0, 1 \leq i < j \leq d_q$, where $X = (x_{ij})$ is the general matrix corresponding to the factor $M(d_q)$.

The ideal $I$ of $S$ is generated by $G$-invariant subspace $E = \bigoplus_{1 \leq i < j \leq d_q} K \cdot z_{ij}$, where $z_{ij} = x_{ij} - x_{ji}, 1 \leq i, j \leq d_q$. The algebra $S(E)$ is a $G_q$-module with GF with respect to the induced action $Z \mapsto g^{-1}Z(g^t)^{-1}, Z = (x_{ij} - x_{ji})$. It follows immediately from [Kur1, Kur2]. Using Proposition 1.3b from [Don7] we obtain that $I$ is a $G$-module with GF. In particular, we have the exact sequence

$$0 \rightarrow I^G \rightarrow R^G \rightarrow K[S]^G \rightarrow 0.$$  

Using Proposition 4.1 we replace the group $G_q = Sp(d_q)$ by $GL(d_q)$. In other words, we have to add to the variety $S' \times M(d_q)$ the new factor $GL(d_q)/Sp(d_q)$. It can be identified with a closed subvariety of $M(d_q)^2$ consisting of all pairs of matrices $(x, y)$ such that $xy = I_{d_q}$ and both $x$ and $y$ are skew-symmetric. This subvariety is a complete intersection again so one can use the same Proposition 1.3b from [Don7]. This step was explained in [Zub5] and we omit all details but briefly describe what we get in this case.

The algebra $R^G$ is an epimorphic image of the algebra $R'G'$, where $R' = K[S' \times M(d_q) \times M(d_q)^2], G' = \times_{f, f \neq q} G_f \times GL(d_q)$ and $GL(d_q)$ acts on the additional factor $M(d_q)^2$ by the rule $(x, y)^g = (gxg^t, (g^t)^{-1}yg^{-1}), x, y \in M(d_q), g \in GL(d_q)$.

It means that we add to our quiver $Q$ one vertex, say with the number $n+1$, and two arrows $b, c$ such that $i(b) = t(c) = i, t(b) = i(c) = n+1$. Moreover, the vertex $n+1$ is occupied by the space $E_{n+1} = K^{d_q} = V$ as well as the vertex $i$ is occupied by $V^*$. Our epimorphism is just the specialization $X(b) \mapsto J, X(c) \mapsto -J = J^{-1}$. 

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As above we have the following exact sequence

$$0 \to I^G \to R'^G \to R^G \to 0.$$ 

The ideal $I'$ is generated by the $G'$-invariant subspace

$$E' = (\oplus_{1 \leq i < j \leq d_q} K \cdot z_{ij}) \oplus (\oplus_{1 \leq i \leq d_q} K \cdot z_i) \oplus (\oplus_{1 \leq i, j \leq d_q} K \cdot t_{ij}),$$

where $z_{ij} = x_{ij}(b) + x_{ji}(b), z_i = x_{ii}(b), 1 \leq i \neq j \leq d_q, t_{ij} = \sum_{1 \leq k \leq d_q} x_{ik}(b)x_{kj}(c) - \delta_{ij}, 1 \leq i, j \leq d_q.$ All other cases can be considered in the same way as above. This completes the proof.

**Remark 4.1** There is some integer $M > 0$ depending only on $d$ such that whenever $\text{char} K > M$ the kernel of the epimorphism from Theorem 2 can be described exactly. For example, let us consider the same case $\text{char} K > 0$ and suppose in $[Zub6]$ that

$$\Delta$$

by Lemma 1.2e (ii) from $[Don7].$ It remains to use the long exact sequence of cohomology groups. It was erroneously supposed in $[Zub6]$ that $\Delta$ has GF and therefore $(E \otimes R)^G \to I^G$ is an epimorphism in all characteristics. This question is still open.
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