\textbf{\textit{\varepsilon\text{-APPROXIMABILITY OF HARMONIC FUNCTIONS IN $L^p$ IMPLIES UNIFORM RECTIFIABILITY}}}

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\textbf{Abstract.} Suppose that $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, is an open set satisfying the corkscrew condition with an $n$-dimensional ADR boundary, $\partial \Omega$. In this note, we show that if harmonic functions are $\varepsilon$-approximable in $L^p$ for any $p > n/(n-1)$, then $\partial \Omega$ is uniformly rectifiable. Combining our results with those in [HT] gives us a new characterization of uniform rectifiability which complements the recent results in [HMM], [GMT] and [AGMT].

1. Introduction

The purpose of this note is to answer a question posed in [HT]: If $E \subset \mathbb{R}^{n+1}$ is an $n$-ADR set, does $\varepsilon$-approximability of harmonic functions in $L^p$ for some fixed $p$ in $\mathbb{R}^{n+1} \setminus E$ imply uniform rectifiability of $E$? We answer this question affirmatively with the following theorem.

\textbf{Theorem 1.1.} Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with $n$-dimensional ADR boundary such that $\Omega$ satisfies the corkscrew condition. Let $p > n/(n-1)$ and suppose that every harmonic function is $\varepsilon$-approximable in $L^p$ for every $\varepsilon \in (0, 1)$: there exist constants $C_\varepsilon$ and $C_p$ and a function $\varphi = \varphi^\varepsilon \in BV_{\text{loc}}(\Omega)$ such that
\[
\left\{ \begin{array}{l}
\|N_\ast(u - \varphi)\|_{L^p(\partial \Omega, \sigma)} \leq \varepsilon C_p \|N_\ast u\|_{L^p(\partial \Omega, \sigma)}, \\
\|C(\nabla \varphi)\|_{L^p(\partial \Omega, \sigma)} \leq C_\varepsilon C_p \|N_\ast u\|_{L^p(\partial \Omega, \sigma)}.
\end{array} \right.
\]
Then $\partial \Omega$ is uniformly rectifiable. Here $N_\ast u$ is the non-tangential maximal function (see Definition 2.7) and
\[
C(\nabla u)(x) := \sup_{r > 0} \frac{1}{r^d} \int_{B(x, r) \cap \Omega} |\nabla \varphi| \, dY.
\]

As usual, if $\varphi$ is not differentiable, $\int_{B(x, r) \cap \Omega} |\nabla \varphi| \, dY$ means the total variation of $\varphi$ in $B(x, r) \cap \Omega$.

\[
\int_{B(x, r) \cap \Omega} |\nabla \varphi| \, dY := \sup_{\overline{\Psi} \in C_0^1(B(x, r) \cap \Omega)} \int_{B(x, r) \cap \Omega} \varphi \text{div} \overline{\Psi} \, dY,
\]
and we denote $\varphi \in BV(\Omega)$ if the total variation over $\Omega$ is finite and $\varphi \in BV_{\text{loc}}(\Omega)$ if the total variation over any open relatively compact $\Omega' \subset \Omega$ is finite.

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Our proof draws strongly on the ideas of [GMT] and [HR]. In particular, we will follow a central idea in [GMT] and prove the following result:

**Theorem 1.2.** Suppose that the hypotheses of Theorem 1.1 hold. Then the harmonic measure $\omega$ admits a Corona decomposition (see Definition 4.2).

By [GMT, Proposition 5.1], Theorem 1.2 is enough to imply Theorem 1.1.

We provide some context to the results herein. There has been significant and continued interest in characterizations of geometric properties by properties of solutions to elliptic PDEs and/or elliptic measure. In light of this, a fundamental question at the interface of harmonic analysis and geometric measure theory is the following: what PDE properties serve to characterize uniform rectifiability of the boundary of an open set with ADR boundary? Recently, powerful tools from harmonic analysis and geometric measure theory have provided several characterizations [HMM, GMT]. Among these is the following theorem.

**Theorem 1.3 ([GMT, HMM]).** Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain satisfying the interior corkscrew condition with $n$-dimensional ADR boundary, $\partial\Omega$. Then $\partial\Omega$ is uniformly rectifiable if and only if bounded harmonic functions in $\Omega$ are $\varepsilon$-approximable (see Definition 1.4).

The direction uniform rectifiability implies $\varepsilon$-approximability was proved in [HMM], and the converse in [GMT].

A remarkable thing about this theorem, is that it does not have any assumptions on the connectivity of $\Omega$ or $\partial\Omega$. For a long time, giving up connectivity assumptions was a serious obstacle in the field since uniform rectifiability is not enough to imply absolute continuity of the elliptic measure with respect to the surface measure [BJ]. Without absolute continuity, analyzing the properties of the solutions to elliptic PDE becomes considerably more difficult.

Let us recall the definition of the usual $L^\infty$ type $\varepsilon$-approximability:

**Definition 1.4.** Suppose that $\Omega \subset \mathbb{R}^{n+1}$ is a domain satisfying the interior corkscrew condition with $n$-dimensional ADR boundary $\partial\Omega$ (see Definition 2.1) and let $\varepsilon \in (0, 1)$. We say that a function $u$ is $\varepsilon$-approximable if there exists a constant $C_\varepsilon$ and a function $\varphi = \varphi^\varepsilon \in BV_{loc}(\Omega)$ satisfying

$$\|u - \varphi\|_{L^\infty(\Omega)} < \varepsilon \quad \text{and} \quad \sup_{x \in E, r > 0} \frac{1}{r^n} \int_{B(x,r) \cap \Omega} |\nabla \varphi(Y)| dY \leq C_\varepsilon.$$

The notion of $\varepsilon$-approximability was first introduced by Varopoulos who proved that every harmonic function in $\mathbb{R}^{n+1}_+$ is $\varepsilon_0$-approximable for some $\varepsilon_0 \in (0, 1)$ [Var]. He used the property to prove the so called Varopoulos extension theorem which gives an alternative characterization of $BMO$ functions. In his work, it was not necessary to have the approximability property for all $\varepsilon \in (0, 1)$ but in later developments a sharper version of the property has been crucial. Garnett [Gar] extended Varopoulos’s result for all $\varepsilon \in (0, 1)$ and his result in turn was generalized to bounded Lipschitz domains by Dahlberg [Dah]. The property has been used to e.g. explore the absolute continuity properties of elliptic measures\(^1\).

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\(^1\)See also [Gar], where similar ideas were introduced in the formulation and proof of a quantitative Fatou theorem.
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[KKoPT, HKMP15] and, as stated above, give a new characterization of uniform rectifiability [HMM, GMT]. The $L^p$ version of ε-approximability was recently introduced by Hytönen and Rosén [HR, Theorem 1.3] who showed that any weak solution to certain elliptic partial differential equations in $\mathbb{R}^{n+1}$ are ε-approximable in $L^p$ for every $\varepsilon \in (0, 1)$ and every $p \in (1, \infty)$. More recently, Steve Hofmann and the second author proved an analog of [HR, Theorem 1.3] for harmonic functions in the rougher setting of a domain with uniformly rectifiable boundary which satisfies the interior corkscrew condition. We note that the result in [HT] is stated for the complement of an ADR set and in that setting the interior corkscrew condition is automatically satisfied. However, it is straightforward to show that the result holds in the current context, too.

Combining our main result with results in [HMM, HMM2, GMT] and [HT] gives us the following characterization result:

**Theorem 1.5.** Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set with $n$-dimensional ADR boundary such that $\Omega$ satisfies the corkscrew condition. Then the following conditions are equivalent.

(a) $\partial \Omega$ is uniformly rectifiable.

(b) Every bounded harmonic function $u$ satisfies the following Carleson measure estimate:

$$
\sup_{x \in \partial \Omega, r > 0} \frac{1}{r^n} \int_{B(x,r) \setminus \partial \Omega} |\nabla u(Y)|^2 \text{dist}(Y, \partial \Omega) \, dY \lesssim \|u\|_{L^2(\Omega)}^2.
$$

(c) For any fixed $p \in [2, \infty)$, every function $u \in C_0(\overline{\Omega})$ that is harmonic in $\Omega$ satisfies the following $S$-estimate:

$$
\|Su\|_{L^p(\partial \Omega)} \leq C_p \|N_* u\|_{L^p(\partial \Omega)}.
$$

(d) Every bounded harmonic function on $\Omega$ is ε-approximable for all $\varepsilon > 0$.

(e) For any fixed $p > n/(n-1)$, every harmonic function on $\Omega$ is ε-approximable in $L^p$ for all $\varepsilon > 0$.

We remark that at least the conditions (a), (b) and (d) remain equivalent if we assume the conditions both for certain types of elliptic operators and their adjoints as was proven in [AGMT]. The methods we use to prove Theorem 1.1 should extend to the class elliptic operators considered in [AGMT], but we choose to treat the case of the Laplacian for the sake of brevity. We note that in the proof of Theorem 1.1 we need the approximability property only for some sufficiently small $\varepsilon_1 \in (0, 1)$ depending on $p$ and the structural constants. The restriction $p > n/(n-1)$ is due to the proof of Lemma 3.3; a suitable version of the Serrin-Weinberger theory [SW] would allow one to reach $p = n/(n-1)$ (see [HR, Proposition 5.1]).

2. Preliminaries and Definitions

**Definition 2.1 (ADR (Ahlfors-David regular) sets).** We say that a set $E \subset \mathbb{R}^{n+1}$, of Hausdorff dimension $n$, is ADR if it is closed, and if there is some uniform constant $C$ such that

$$
\frac{1}{C} r^n \leq \sigma(\Delta(x, r)) \leq C r^n, \quad \forall r \in (0, \text{diam}(E)), \ x \in E,
$$

where $\sigma(\cdot)$ denotes the $n$-dimensional Hausdorff measure.
where \( \text{diam}(E) \) may be infinite. Here, \( \Delta(x, r) := E \cap B(x, r) \) is the “surface ball” of radius \( r \), and \( \sigma := H^n|_E \) is the “surface measure” on \( E \), where \( H^n \) denotes \( n \)-dimensional Hausdorff measure.

**Definition 2.3** (UR). Following [DS1, DS2], we say that an ADR set \( E \subset \mathbb{R}^{n+1} \) is UR (uniformly rectifiable) if it contains “big pieces of Lipschitz images” (BPLI) of \( \mathbb{R}^n \): there exist constants \( \theta, M > 0 \) such that for every \( x \in E \) and \( r \in (0, \text{diam}(E)) \) there is a Lipschitz mapping \( \rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1} \), with Lipschitz norm no larger that \( M \), such that

\[
H^n(E \cap B(x, r) \cap \rho([-r, r]^n)) \geq \theta r^n.
\]

**Definition 2.4** (Corkscrew Condition). We say that an open set \( \Omega \subset \mathbb{R}^{n+1} \) satisfies the corkscrew condition if there exists a constant \( M \) such that for every \( x \in \partial \Omega \) and \( r \in (0, \text{diam}(\partial \Omega)) \) there exists a point \( Y \in \Omega \) such that \( B(Y, \frac{r}{M}) \subset B(x, r) \cap \Omega \). If \( Y \) is as above we say \( Y \) is a corkscrew point relative to \( x \) at scale \( r \).

Our standing assumption will be that \( \Omega \subset \mathbb{R}^{n+1} \) is an open set with \( n \)-dimensional ADR boundary \( \partial \Omega \) and that \( \Omega \) satisfies the corkscrew condition with constant \( M \). Let us fix some notation:

- We will use lowercase letters \( x, y, z \ldots \) to denote points on \( \partial \Omega \) and capital letters \( X, Y, Z, \ldots \) to denote generic points in \( \mathbb{R}^{n+1} \).
- We use the standard notation \( B(x, r) \) for usual Euclidean balls in \( \mathbb{R}^{n+1} \) and \( \Delta(x, r) := B(x, r) \cap \partial \Omega \) for surface balls. As usual, we use the notation \( \kappa B(X, r) := B(X, \kappa r) \) and \( \kappa \Delta(x, r) := \Delta(x, \kappa r) \) for the dilations of the balls.
- We set \( \delta(X) := \text{dist}(X, \partial \Omega) \) for every \( X \in \Omega \).
- We let \( \omega = \omega^X \) to denote the harmonic measure for \( \Omega \).

We will use actively well-known dyadic techniques on \( \partial \Omega \). We note that in our context there is no canonical choice for the dyadic system but it is not necessary to know the exact structure of the “cubes”.

**Lemma 2.5** (Dyadic systems for ADR sets). Suppose that \( E \subset \mathbb{R}^{n+1} \) is closed \( n \)-dimensional ADR set. Then there exist constants \( a_0 > 0 \) and \( a_1 < \infty \), depending only on dimension and the ADR constant, such that for each \( k \in \mathbb{Z} \) there is a collection of Borel sets (“cubes”)

\[
\mathcal{D}_k := \{ Q^k_j : j \in \mathcal{S}_k \},
\]

where \( \mathcal{S}_k \) denotes some (possibly finite) index set depending on \( k \), satisfying

1. \( E = \bigcup \mathcal{Q}_k^k \) for each \( k \in \mathbb{Z} \) and the union is disjoint.
2. If \( m \geq k \) then either \( \mathcal{Q}_m^m \subset \mathcal{Q}_k^k \) or \( \mathcal{Q}_m^m \cap \mathcal{Q}_k^k = \emptyset \).
3. For each \( (j, k) \) and each \( m < k \), there is a unique \( i \) such that \( Q^k_j \subset Q^m_i \).
4. For each \( (j, k) \) there exists a point \( x_k^{j} \in Q^k_j \) such that
\[
\Delta(x_k^{j}, a_0 2^{-k}) \subset Q^k_j \subset \Delta(x_k^{j}, a_1 2^{-k}) := B(x_k^{j}, a_1 2^{-k}) \cap E.
\]

In the literature, there exist numerous proofs for this result with many additional properties (see e.g. [DS1, DS2, Chr, HK, HMMP]) but we listed only the ones that we actually need in this paper. Let us fix some notation related to the dyadic system.
If the set $E$ is bounded, then we simply have $Q^k_j = E$ for all sufficiently small $k \in \mathbb{Z}$. Because of this, we denote $\mathbb{D} := \bigcup_k \mathbb{D}_k$, where the union runs over the $k$ such that $2^{-k} \leq \text{diam}(E)$.

- For each $Q := Q^k_j \in \mathbb{D}$ we denote

$$x_Q := x^k_j, \quad B_Q := B(x_Q, a_1 2^{-k}), \quad \Delta_Q := B_Q \cap E.$$ 

We shall refer to the point $x_Q$ as the “center” of $Q$.

- For each $Q := Q^k_j \in \mathbb{D}$, we set $\ell(Q) := 2^{-k}$. We refer to this quantity as the “side length” of $Q$. Since we assume that $\ell(Q) \leq \text{diam}(E)$ for every $Q \in \mathbb{D}$, we have $\ell(Q) = \text{diam}(Q)$ and $\ell(Q)^n \approx \sigma(Q)$.

- For each $R \in \mathbb{D}$, we let $\mathbb{D}_R \subset \mathbb{D}$ be the collection of the subcubes of $R$.

Next, we define “dyadic” Whitney regions similar to those in [HM]: for each $Q \in \mathbb{D}$ we choose a bounded number of usual Whitney cubes $I^k_Q \subset \Omega$ such that $\bigcup_I I^k_Q$ acts as a substitute for a region of the type $P \times (\ell(P)/2, \ell(P))$, the standard Whitney region in the upper half-space. Our Whitney regions are not disjoint but they have a bounded overlap property which is enough for us. In [HM, HMM, HT], it was crucial to fatten the cubes $I^k_Q$ to ensure that most of the regions break into components with strong geometric properties (see particularly [HMM, Lemma 3.24]). In our situation, we do not have to fatten the cubes but we note that we have to make sure that the Whitney regions are large enough for the non-tangential maximal function we use (see Definition 2.7) to be equivalent in $L^p$ sense with the non-tangential maximal function used in [HT]. See [HT, Lemma 1.23] and its proof for more details about this and corresponding Fefferman-Stein [FS] type arguments.

Let us be more precise. Let $\eta \ll 1 \ll K$ be parameters depending on $n$, $M$ and the ADR constant whose values we will determine later in the proofs. Suppose that $\mathcal{W} := \{I\}$ is a Whitney decomposition of $\Omega$, that is, $\{I\}$ is a collection of closed $(n + 1)$-dimensional Euclidean cubes whose interiors are disjoint such that $\bigcup_I I = \mathbb{R}^{n+1} \setminus \partial \Omega$ and

$$4 \text{diam}(I) \leq \text{dist}(4I, \partial \Omega) \leq \text{dist}(I, \partial \Omega) \leq 40 \text{diam}(I), \quad \forall I \in \mathcal{W}$$

and

$$1/4 \text{diam}(I_1) \leq \text{diam}(I_2) \leq 4 \text{diam}(I_1)$$

whenever $I_1 \cap I_2 \neq \emptyset$. For every $Q \in \mathbb{D}(\partial \Omega)$ we set

$$W_Q(\eta, K) = \{I \in \mathcal{W} : \eta^{1/4} \ell(Q) \leq \ell(I) \leq K^{1/2} \ell(Q), \text{dist}(I, Q) \leq K^{1/2} \ell(Q)\},$$

and

$$U_Q(\eta, K) = \bigcup_{I \in W_Q(\eta, K)} I.$$ 

We call $U_Q(\eta, K)$ the Whitney region relative to $Q \in \mathbb{D}$.

**Remark 2.6.** We put some initial restrictions on the parameters $\eta$ and $K$ to ensure that $U_Q(\eta, K)$ is non-empty for every $Q \in \mathbb{D}$. We notice that by the corkscrew condition for every $Q \in \mathbb{D}$ there exists a corkscrew point $X_Q$ such that $|X_Q - x_Q| < \ell(Q)$ and $\text{dist}(X_Q, \partial \Omega) > M^{-1} \ell(Q)$. It follows that $X_Q \in I$ for some $I \in \mathcal{W}$ with $\text{dist}(I, Q) \approx \ell(Q)$. 

Choosing \( \eta \ll 1 \ll K \) depending on the corkscrew condition and \( n \) we obtain that \( U_Q(\eta, K) \neq \emptyset \). We later impose further assumptions on \( \eta, K \), but these will continue to only depend on \( n \), the ADR constant and the corkscrew condition. We will drop the \( \eta, K \) from \( U_Q(\eta, K) \) for notational convenience.

Finally, let us define cones and the non-tangential maximal operator. We first denote the usual cone of aperture \( \alpha > 1 \) at \( x \in \partial \Omega \) by \( \tilde{\Gamma}_\alpha(x) := \{ Y \in \Omega : |x - Y| < \alpha \cdot \delta(Y) \} \). We notice that if \( \partial \Omega \) is bounded, then we have \( \mathbb{R}^{n+1} \setminus B(x, R) \subset \tilde{\Gamma}_\alpha(x) \) for large enough \( R \) and every \( x \in \partial \Omega \). Thus, since we only construct the regions \( U_Q \) for such \( Q \) that \( \ell(Q) \ll \text{diam}(\partial \Omega) \), we set \( \Gamma(x) := \bigcup_{Q \in \mathcal{D}, \Omega \ni x} U_Q, \) if \( \text{diam}(\partial \Omega) = \infty \), \( \bigcup_{Q \in \mathcal{D}, \Omega \ni x} U_Q \cup (\mathbb{R}^{n+1} \setminus B(x, R_0)), \) if \( \text{diam}(\partial \Omega) < \infty \),

for a fixed large enough \( R_0 \). It is straightforward to verify that by choosing the constants \( \eta \) and \( K \) in a suitable way, there exist \( \alpha_1 > \alpha_0 > 1 \) such that \( \tilde{\Gamma}_{\alpha_0}(x) \subset \Gamma(x) \subset \tilde{\Gamma}_{\alpha_1}(x) \) for every \( x \in \partial \Omega \).

**Definition 2.7 (Non-tangential maximal function).** For any function \( g : \Omega \to \mathbb{R} \) we define the non-tangential maximal function \( N^*_g : \partial \Omega \to \mathbb{R} \) by \( N^*_g(y) := \sup_{X \in \Gamma(y)} |g(X)|. \)

### 3. Two lemmas

In order to present the proof of Theorem 1.2 we prove two simple lemmas.

**Lemma 3.1.** Let \( Q \in \mathcal{D} \). Suppose that \( f \) is a Borel function such that \( |f| \leq 1_Q \) and set \( u(X) = \int_{\partial \Omega} f(y) \, d\omega^X(y) \) for every \( X \in \Omega \). Then

\[
|u(X)| \leq 1_{4B_Q}(X) + 1_{(4B_Q)^c}(X) \left( \frac{\ell(Q)}{|X - x_Q|} \right)^{n-1},
\]

where the implicit constants depend on \( n \) and the ADR constant.

**Proof.** Let us set

\[
H(X) = \frac{1}{\ell(Q)} \int_{B_Q} E(X, y) \, d\sigma(y),
\]

where

\[
E(X, Y) := \frac{c_n}{|X - Y|^{n-1}}
\]

is the fundamental solution to the Laplacian in \( \mathbb{R}^{n+1} \). By the ADR condition and the local \( \sigma \)-integrability of \( E \), we know that \( H \) is bounded in \( \mathbb{R}^{n+1} \) and we have

\[
H(y) \gtrsim 1, \quad \forall y \in Q.
\]

We also notice that

\[
0 \leq H(X) \lesssim \left( \frac{\ell(Q)}{|X - x_Q|} \right)^{n-1}, \quad \forall X \in (4B_Q)^c,
\]
where the implicit constants above depend on $n$ and the ADR constant. It is straightforward to show that $H$ is harmonic in $\Omega$ and continuous in $\overline{\Omega}$. Thus, we have

$$H(X) = \int_{\partial \Omega} H(y) \, d\omega^X(y).$$

In particular, since $|f| \leq 1_Q$, we get

$$|u(X)| \leq \int_Q |f(y)| \, d\omega^X(y) \overset{(3.2)}{\leq} H(X) \leq 1_{4B_0}(X) + 1_{4B_0}(X) \left( \frac{\ell(Q)}{|X - x_Q|} \right)^{n-1}$$

by the boundedness of $H$.

This lemma readily yields a bound also for the non-tangential maximal operator acting on functions of the same type:

**Lemma 3.3.** Let $Q \in \mathbb{D}$. Suppose that $f$ is a Borel function satisfying $|f| \leq 1_Q$ and set $u(X) = \int_{\partial \Omega} f(y) \, d\omega^X(y)$ for every $X \in \Omega$. Then for all $p > n/(n - 1)$ we have

$$\|N_* u\|_{L^p(\partial \Omega, \sigma)} \leq C_1 \sigma(Q)^{1/p},$$

where $C_1$ depends on $n$, the ADR constant, $\eta$, $K$ and $p$.

**Proof.** Let $y \in \partial \Omega$ and suppose that $X \in \Gamma(y)$. Then by the definition of $\Gamma(y)$ we have that

$$|X - y| \approx_{\eta, K} \delta(X) \leq |X - x_Q|.$$  

It follows that

$$|y - x_Q| \leq |X - y| + |X - x_Q| \leq |X - x_Q|.$$ 

Thus, Lemma 3.1 yields

$$|u(X)| \leq 1_{4B_0}(X) + 1_{4B_0}(X) \left( \frac{\ell(Q)}{|X - x_Q|} \right)^{n-1} \leq 1_{4B_0}(X) + 1_{4B_0}(X) \left( \frac{\ell(Q)}{|y - x_Q|} \right)^{n-1}, \quad \forall X \in \Gamma(y).$$

Let us set $A_k := 2^{k+1} \Delta Q \setminus 2^k \Delta Q$ for every $k \geq 2$. By the ADR property, we get

$$\int_{\partial \Omega} N_* u^p \, d\sigma \leq \int_{4\Delta Q} N_* u^p \, d\sigma + \sum_{k=2}^{\infty} \int_{A_k} N_* u^p \, d\sigma \leq \sigma(4\Delta Q) + \sum_{k=2}^{\infty} \int_{A_k} \left( \frac{\ell(Q)}{|y - x_Q|} \right)^{p(n-1)} \, d\sigma$$

$$\leq \sigma(Q) + \sum_{k=2}^{\infty} \int_{A_k} \frac{1}{2^{k(n-1)p}} \, d\sigma$$

$$\leq \sigma(Q) + \sigma(Q) \sum_{k=2}^{\infty} 2^{kn - k(n-1)p} \leq \sigma(Q),$$

since $p > n/(n - 1)$.

□
4. Corona decomposition for harmonic measure and the proof of Theorem 1.2

In this section, we prove Theorem 1.2 which, as we pointed out earlier, is enough to imply Theorem 1.1. Before the proof, we recall some definitions and results from [GMT].

Definition 4.1. [DS2]. Let $S \subset \mathbb{D}$. We say that $S$ is coherent if the following conditions hold:

(a) $S$ contains a unique maximal element $Q(S)$ which contains all other elements of $S$ as subsets.
(b) If $Q$ belongs to $S$ and $Q \subset Q' \subset Q(S)$ for any $Q' \in \mathbb{D}$, then $Q' \in S$.
(c) Given a cube $Q^k_j \in S$, either all of its children (i.e. the cubes $P \in D_{k+1}$ such that $P \subset Q^k_j$) belong to $S$, or none of them do.

Definition 4.2 (Corona decomposition for harmonic measure [GMT]). Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set satisfying the corkscrew condition with $n$-dimensional ADR boundary, and let $\omega$ be the harmonic measure for $\Omega$. We say that $\omega$ admits a corona decomposition if $\mathbb{D}$ can be decomposed into disjoint coherent subcollections $S$ such that the following two conditions holds.

1. The maximal cubes, $Q(S)$, satisfy a Carleson packing condition
   \[
   \sum_{Q(S) \subset R} \sigma(Q(S)) \leq C\sigma(R), \quad \forall R \in \mathbb{D}(\partial \Omega).
   \]

2. For each $Q(S)$ there exists $P_{Q(S)} \in \Omega$ such that
   \[
   c^{-1}\ell(Q(S)) \operatorname{dist}(P_{Q(S)}, Q(S)) \leq \operatorname{dist}(P_{Q(S)}, \partial \Omega) \leq c\ell(Q(S)),
   \]
   \[
   \omega^{P_{Q(S)}}(3R) \approx \frac{\sigma(R)}{\sigma(Q(S))} \quad \forall R \in S,
   \]
   where the implicit constants above and $c$ are uniform in $S$ and $R$.

Let us fix the value of $\epsilon \ll 1$ later. For each cube in $\mathbb{D}$, let $P_Q \in B_Q$ be a corkscrew point at scale $\epsilon\ell(Q)$ relative to $x_Q$. Then we have
\[
\frac{\epsilon}{M} \ell(Q) \leq \delta(P_Q) \leq \epsilon\ell(Q),
\]
Let $y_Q \in \partial \Omega$ be the touching point for the point $P_Q$, that is, $|y_Q - P_Q| = \delta(P_Q)$. By choosing $\epsilon \ll 1$ to be small enough, we may assume that
\[
B(y_Q, |y_Q - P_Q|) \subset B(x_Q, \frac{\epsilon}{\delta(P_Q)^2} \ell(Q)).
\]
Next, for some parameter $\tau$ to be chosen later depending on $M$, dimension and the ADR constant, we set
\[
S_Q := y_Q + 2\tau(P_Q - y_Q),
V_Q := B(P_Q, (1 - \tau)\delta(P_Q)),
V^1_Q := B(P_Q, (1/2)\delta(P_Q)),
V^2_Q := B(S_Q, \tau\delta(P_Q)).
\]
Notice that $V^1_Q, V^2_Q \subset V_Q$. We then have:

$^2$Note that $B(x_Q, \delta(P_Q))$ is exactly “$B_0^c$” from [GMT]. See Lemma 2.5 (4).
Lemma 4.3 ([GMT, Lemma 3.3, proof of Lemma 3.7]). Suppose that $\tau \ll 1$ and $\varepsilon \ll \tau$ are chosen appropriately depending on $M$, dimension and the ADR constant. Then if $E_Q \subset Q \in \mathbb{D}$ is such that

$$\omega^{p_Q}(E_Q) \geq (1 - \varepsilon)\omega^{p_Q}(Q),$$

there exists a non-negative harmonic function $u_Q$ on $\Omega$ and a Borel function $f_Q$ with

$$u_Q(x) = \int_{\partial \Omega} f_Q d\omega^x, \quad 0 \leq f_Q \leq 1_{E_Q}$$

satisfying

$$|m_{V_Q} u_Q - m_{V_Q} u_Q| \geq c_1,$$

where $c_1$ depends only on $M$, $n$ and the ADR constant and $m_{B_i} u = \int_{B_i} u dX$.

Definition 4.5 (Low density cubes [GMT]). Let $0 < \delta \ll 1$ be a fixed constant. For a cube $R \in \mathbb{D}$ we say that a subcube $Q \in \mathbb{D}_R$ is a low density cube and write $Q \in LD(R)$ if $Q$ is a maximal cube (with respect to containment) satisfying

$$\frac{\omega^{p_Q}(Q)}{\sigma(Q)} \leq \delta \frac{\omega^{p_R}(R)}{\sigma(R)}.$$

For any cube $R \in \mathbb{D}$, we denote $LD^0(R) = [R]$ and define $LD^k(R), k \geq 1$, inductively by

$$LD^k(R) = \bigcup_{Q \in LD^{k-1}(R)} LD(Q).$$

In the proof of the corona decomposition for harmonic measure in [GMT], $\varepsilon$-approximability is used only to prove a packing condition for the low density cubes [GMT, Lemma 3.7]. Thus, we actually have:

Lemma 4.6. Suppose that for any $m \geq 1$ and $R \in \mathbb{D}$ we have

$$\sum_{k=0}^m \sum_{Q \in LD^k(R)} \sigma(Q) \leq C \sigma(R),$$

where $C$ is independent of $m$ and $R$. Then $\omega$ admits a corona decomposition.

We now prove Theorem 1.2.

Proof of Theorem 1.2. Let us start by fixing $\eta$ and $K$ depending on $\tau$ and $\varepsilon$ so that

$$V_Q \subset U_Q(\eta, K).$$

This can be done since every point $Y \in V_Q$ has the property that

$$\delta(Y) \approx_{\varepsilon, \tau} \ell(Q).$$

As $\varepsilon$ and $\tau$ depend only on $M$, $n$ and the ADR constant so do $\eta$ and $K$. We also note that by the construction of the regions $U_Q(\eta, K)$ we have

$$\sum_{Q \in \mathbb{D}} 1_{V_Q}(x) \leq 1, \quad \forall x \in \Omega,$$

where the implicit constant depends on $\eta, K, M, n$ and the ADR constant.
By Lemma 4.6, to prove the theorem it is enough to show (4.7). We do this now. For every cube \( Q \in \mathbb{D} \) and every \( m \geq 1 \) we set
\[
\mathcal{A}(Q, m) := \frac{1}{\sigma(Q)} \sum_{Q' \in \bigcup_{k=1}^{m} \text{LD}(Q)} \sigma(Q').
\]
Then (4.7) is equivalent to the statement
\[
\mathcal{A}(m) := \sup_{Q \in \mathbb{D}} \mathcal{A}(Q, m) \leq C,
\]
where \( C \) is independent of \( m \).

Let us fix \( R \in \mathbb{D} \) and set \( \mathcal{F}_{1,m} := \bigcup_{k=1}^{m} \text{LD}(R) \). For \( Q \in \mathcal{F}_{1,m} \) we set \( L_Q := \bigcup_{Q' \in \text{LD}(Q)} Q' \) and
\[
E_Q := Q \setminus L_Q.
\]
The sets \( \{E_Q\}_{Q \in \mathcal{F}_{1,m}} \) are pairwise disjoint by definition. Moreover, by the definition of \( \text{LD}(Q) \), we have
\[
\omega^0(L_Q) \leq \sum_{Q' \in \text{LD}(Q)} \omega^0(Q') \leq \delta \sum_{Q' \in \text{LD}(Q)} \frac{\sigma(Q')}{\sigma(Q)} \omega^0(Q) \leq \delta \omega^0(Q)
\]
and hence
\[
(4.10) \quad \omega^0(E_Q) \geq (1 - \delta) \omega^0(Q) \geq (1 - \varepsilon) \omega^0(Q)
\]
as long as we choose \( \delta \leq \varepsilon \). Then we may apply Lemma 4.3 to obtain a collection of functions \( \{u_Q\}_{Q \in \mathcal{F}_{1,m}} \) such that
\[
u f_Q \, d\omega^X, \quad 0 \leq f_Q \leq 1_{E_Q}
\]
for some Borel function \( f_Q \) and
\[
|\nu f_Q - \nu f_Q| \geq c_1.
\]

Let \( \varepsilon_1 > 0 \) to be chosen. Let \( \Xi \) denote the collection of sequences \( \{a = (a_Q) : Q \in \mathcal{F}_{1,m}, a_Q \in \{-1, +1\}\} \) and let \( \lambda \) be a probability measure on \( \Xi \) which assigns equal probability to \( -1 \) and \( +1 \). For every \( a \in \Xi \) we set
\[
u u_Q(X) = \sum_{Q \in \mathcal{F}_{1,m}} a_Q u_Q(X).
\]
Note that since \( f_Q \) are Borel functions with disjoint supports, \( \sum_{Q \in \mathcal{F}_{1,m}} a_Q f_Q \) is a Borel function and clearly
\[
|u_a(X)| \leq \int_{\partial \Omega} \sum_{Q \in \mathcal{F}_{1,m}} |a_Q| f_Q \, d\omega^X \leq \sum_{Q \in \mathcal{F}_{1,m}} \omega^X(E_Q) \leq 1, \quad \forall X \in \Omega.
\]
We now apply the \( \varepsilon \)-approximability in \( L^p \) property (see Theorem 1.1) with “\( \varepsilon \)” = \( \varepsilon_1 \): for each \( a \in \Xi \) there exists \( \varphi_a \in \text{BV}_{\text{loc}} \) such that
\[
\begin{align*}
\|N_a(u_a - \varphi_a)\|_{L^p(\partial \Omega, \sigma)} & \leq \varepsilon_1 C_p \|N_a u_a\|_{L^p(\partial \Omega, \sigma)} \quad \text{and} \\
\|C(\nabla \varphi_a)\|_{L^p(\partial \Omega, \sigma)} & \leq C_p \|N_a u_a\|_{L^p(\partial \Omega, \sigma)}.
\end{align*}
\]
By Lemma 3.3 we also have
\[
(4.11) \quad \|N_a(u_a - \varphi_a)\|_{L^p(\partial \Omega, \sigma)} \leq \varepsilon_1 \|N_a u_a\|_{L^p(\partial \Omega, \sigma)} \leq C_1 \varepsilon_1 \sigma(R)^{1/p}
\]
and

\[ \|C(\nabla \varphi_a)\|_{L^p(\partial Q, \sigma)} \leq C_{C_1}C_1 \sigma(R)^{1/p}, \]

where \( C_1 \) depends on \( n \), the ADR constant, \( \eta, K \) and \( p \). By Chebyshev’s inequality and (4.11), for each \( a \in \Xi \) we have

\[ \sigma(\{x \in R : N_a(u_a - \varphi_a)(x) > C_2 \varepsilon_1\}) \leq \frac{C_p^p}{C_2^p} \sigma(R). \]

We will fix the exact value of \( \gamma \in (0, 1) \) later but regardless of the exact value, we may choose \( C_2 \) so that \( C_p^p/C_2^p < \gamma/2 \). Let us set \( \varepsilon_0 := C_2 \varepsilon_1 \). It follows that for each \( a \in \Xi \) there exists a set \( F(R, a) \subset R \) such that \( \sigma(F(R, a)) > (1 - \gamma)\sigma(R) \) and for all \( y \in F(R, a) \)

\[ |u_a(X) - \varphi_a(X)| \leq \varepsilon_0, \quad \forall X \in \Gamma(y). \]

Now for each \( a \in \Xi \) we let \( \mathcal{F}_{2,m}(a) \) be the collection of cubes \( Q \in \mathcal{F}_{1,m} \) such that \( Q \cap F(R, a) = \emptyset \). We then let \( \mathcal{F}_{2,m}(a) \) be the collection of maximal cubes in \( \mathcal{F}_{2,m}(a) \) with respect to inclusion and \( \mathcal{F}_{3,m}(a) = \mathcal{F}_{1,m} \setminus \mathcal{F}_{2,m}(a) \). By maximality,

\[ \sum_{Q \in \mathcal{F}_{2,m}(a)} \sigma(Q') \leq \gamma \sigma(R). \]

Suppose that \( Q \in \mathcal{F}_{3,m}(a) \). Then there exists \( y \in F(R, a) \cap Q \). It follows that \( U_Q \subset \Gamma(y) \) and hence

\[ |u_a(X) - \varphi_a(X)| \leq \varepsilon_0, \quad \forall X \in U_Q. \]

In particular, by (4.8) we have for all \( Q \in \mathcal{F}_{3,m}(a) \)

\[ |u_a(X) - \varphi_a(X)| \leq \varepsilon_0, \quad \forall X \in V_Q. \]

Now, using (4.4) and Khintchine’s inequality we obtain for every \( Q \in \mathcal{F}_{1,m} \)

\[ c_1 \leq |m_{V_Q} u_Q - m_{V'_Q} u_Q| \]

\[ \leq \left( \sum_{Q' \in \mathcal{F}_{1,m}} \left| m_{V_Q} u_{Q'} - m_{V'_Q} u_{Q'} \right|^2 \right)^{1/2} \]

\[ \leq \frac{1}{c_3} \int_{\Xi} \left| \sum_{Q' \in \mathcal{F}_{1,m}} a_{Q'} \left| m_{V_Q} u_{Q'} - m_{V'_Q} u_{Q'} \right| \right| d\lambda(a) \]

\[ = \frac{1}{c_3} \int_{\Xi} |m_{V_Q} u_a - m_{V'_Q} u_a| d\lambda(a), \]
where $c_3$ is a universal constant. Then it follows that integrating over $V_Q$ (note that $\text{diam}(V_Q) \approx \ell(Q)$) and summing in $Q$ we have

$$
\sum_{Q \in F_{1,m}} \sigma(Q) \leq \sum_{Q \in F_{1,m}} \ell(Q)^n
$$

$$
\leq \int_{\mathbb{E}} \sum_{Q \in \mathcal{F}_{2,m}(a)} \int_{V_Q} \frac{1}{\ell(Q)} |m_{V_Q} u_{a} - m_{V_Q} u_{a}| dX d\lambda(a)
$$

$$
\leq \int_{\mathbb{E}} \sum_{Q \in \mathcal{F}_{2,m}(a)} \int_{V_Q} \frac{1}{\ell(Q)} |m_{V_Q} u_{a} - m_{V_Q} u_{a}| dX d\lambda(a)
$$

+ \int_{\mathbb{E}} \sum_{Q \in \mathcal{F}_{2,m}(a)} \int_{V_Q} \frac{1}{\ell(Q)} |m_{V_Q} u_{a} - m_{V_Q} u_{a}| dX d\lambda(a)
$$

$$
\leq \int_{\mathbb{E}} \sum_{Q \in \mathcal{F}_{2,m}(a)} \int_{V_Q} \frac{1}{\ell(Q)} |m_{V_Q} u_{a} - m_{V_Q} u_{a}| dX d\lambda(a)
$$

+ \int_{\mathbb{E}} \sum_{Q \in \mathcal{F}_{2,m}(a)} \sigma(Q) d\lambda(a),
$$

where the implicit constants depend on $M$, $n$ and the ADR constant and we used that $\|u\|_{\infty} \leq 1$ in the last inequality. We have by (4.13) and the definitions of $\tilde{\mathcal{F}}_{2,m}(a)$ and $\mathcal{A}(m)$ that

$$
\frac{1}{\sigma(R)} \sum_{Q \in \mathcal{F}_{2,m}(a)} \sigma(Q) \leq \frac{1}{\sigma(R)} \sum_{Q' \in \tilde{\mathcal{F}}_{2,m}(a)} \sum_{Q \in \bigcup_{Q' \in \mathcal{L}(Q')} \mathcal{L}(Q')} \sigma(Q)
$$

$$
\leq \frac{1}{\sigma(R)} \sum_{Q' \in \tilde{\mathcal{F}}_{2,m}(a)} \sigma(Q') \mathcal{A}(m) \leq \gamma \mathcal{A}(m).
$$

Set $B_R^{**} = B(x_R, 5a_1 \ell(R))$ and $\Delta_R^{**} = B_R^{**} \cap \Omega$ and note that $V_Q \subset B_R^{**}$ for all $Q \in \mathbb{D}(R)$. We immediately have that

$$
\int_{B_R^{**} \cap \Omega} |\nabla \varphi(X)| \, dX \leq \ell(R)^n \int_{S_{R}^{**}} C(\nabla \varphi)(y) \, d\sigma(y)
$$

$$
\approx \sigma(R) \int_{S_{R}^{**}} C(\nabla \varphi)(y) \, d\sigma(y).
$$
Using (4.14), (4.9), the Poincaré inequality\(^3\) and (4.12) we obtain (4.18)

\[
\int \sum_{Q \in F_{3,m}(a)} \int_{V_Q} \frac{1}{\ell(Q)} |m_{V^0} u_a - m_{V^2} u_a| dX d\lambda(a)
\]

\[
\leq \int \sum_{Q \in F_{3,m}(a)} \int_{V_Q} \frac{1}{\ell(Q)} |m_{V^0} \varphi_a - m_{V^2} \varphi_a| dX d\lambda(a) + \varepsilon_0 \int \sum_{Q \in F_{3,m}(a)} \sigma(Q)
\]

\[
\leq \int \sum_{Q \in F_{3,m} \cap \Omega} |\nabla \varphi_a| dX d\lambda + \varepsilon_0 \sum_{Q \in F_{3,m}} \sigma(Q)
\]

\[
\leq \int \sum_{Q \in F_{3,m} \cap \Omega} |\nabla \varphi_a| dX d\lambda + \varepsilon_0 \sum_{Q \in F_{3,m}} \sigma(Q)
\]

\[
\leq \int \sum_{Q \in F_{3,m} \cap \Omega} \left( \int_{\Lambda^*} (C(\nabla \varphi_a)(y))^p d\sigma(y) \right)^{1/p} d\lambda + \varepsilon_0 \sum_{Q \in F_{3,m}(a)} \sigma(Q)
\]

\[
\leq C_{\varepsilon_1} C_1 \sigma(R) + \varepsilon_0 \sum_{Q \in F_{3,m}} \sigma(Q),
\]

where the implicit constant depends only on \(M, n\) and the ADR constant. Dividing (4.16) by \(\sigma(R)\) and using (4.17) and (4.18) we have shown

\[
\mathcal{A}(R, m) \lesssim C_{\varepsilon_1} C_1 + \varepsilon_0 \mathcal{A}(R, m) + \gamma \mathcal{A}(m),
\]

where the implicit constant depends only on \(M, n\) and the ADR constant. Taking the supremum over \(R \in \mathcal{D}\) and recalling that \(\varepsilon_0 = C_2 \varepsilon_1\) we have

\[
\mathcal{A}(m) \lesssim C_{\varepsilon_1} C_1 + C_2 \varepsilon_1 \mathcal{A}(m) + \gamma \mathcal{A}(m).
\]

We recall the order we have chosen the constants and first choose \(\gamma \ll 1\); this choice dictates \(C_2\) (\(C_2\) depends on \(M, p, \gamma, n\), the ADR constant and \(C_1\)). Finally, we choose \(\varepsilon_1 \ll 1\) depending on \(C_2, M, n\) and the ADR constant. Thus,

\[
\mathcal{A}(m) \lesssim C_{M,n,ADR} C_{\varepsilon_1} C_1,
\]

which allows us to conclude the claim. Here we have used the fact that the quantity \(\mathcal{A}(m)\) is finite (to be more precise, we have \(\mathcal{A}(m) \leq m + 1 < \infty\)), which allows us to choose \(\gamma\) and \(\varepsilon_1\) to be so small that \(\mathcal{A}(m) - C_2 \varepsilon_1 + \gamma \mathcal{A}(m) > 0\) for a structural constant \(c\) which was implicit in the estimates. This concludes the proof. \(\square\)

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\(^3\)See e.g. the proof of [Z, Theorem 5.11.1], for the case of BV functions.
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