AN IDENTITY CONCERNING THE RIEMANN-ZETA FUNCTION

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ABSTRACT. For a certain function \( J(s) \) we prove that the identity
\[
\frac{\zeta(2s)}{\zeta(s)} - \left( s - \frac{1}{2} \right) J(s) = \frac{\zeta(2s+1)}{\zeta(s+1/2)},
\]
holds in the half-plane \( \text{Re}(s) > 1/2 \) and both sides of the equality are analytic in this half-plane.

1. Introduction

Let \( s = \sigma + it \), \( \zeta(s) \), as usual, the Riemann zeta-function and \( \lambda(n) \) the Liouville function, that is, a completely multiplicative function with \( \lambda(1) = 1 \) and \( \lambda(p) = -1 \) at all primes \( p \).

It is well know that \( \lambda(n) \) is deeply related to the Riemann hypothesis, or simply, RH and also to the Prime Number Theorem (PNT). For instance, RH is equivalent to
\[
\sum_{n \leq x} \lambda(n) = O(x^{1/2}) + O(x^{1/2} \log x),
\]
for all \( \epsilon > 0 \). Whereas the PNT is equivalent to (11)
\[
\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} = 0.
\]
We also refer to [11, p.179] for more details on the PNT.

Any improvements in the zero-free region for \( \zeta(s) \) will immediately imply improvements in the error term of the prime number theorem. For example, if the Riemann Hypothesis is true then we obtain
\[
\pi(x) = \int_{2}^{x} \frac{du}{\log(u)} + O(\sqrt{x} \log x),
\]
and this last is not only implied by the RH but actually implies the RH itself.

The identity
\[
(1.1) \quad \frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \quad \text{Re}(s) > 1,
\]

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was investigated by Pólya [5] and Turán [6] aiming information about the RH. Pólya studied the change of sign of sum

\[ P(x) = \sum_{n \leq x} \lambda(n) \]

and its relation to the RH. He remarked that the RH would follow if one could establish that \( P(x) \) eventually has constant sign. The assertion that \( P(x) \leq 0 \) for \( x \geq 2 \) is often called ‘Pólya’s conjecture’ in the literature, although it appears that Pólya never in fact stated this as a conjecture ([3]). Similarly, in connection with some studies of partial sums of the Riemann zeta function, Turán investigated

\[ T(x) = \sum_{n \leq x} \frac{\lambda(n)}{n}. \]

He showed that if there exists a positive constant \( c \) such that \( T(n) > -c/\sqrt{n} \) for all sufficiently large \( n \), then the RH would follow. He also reported that several assistants had verified that \( T(n) > 0 \) for \( n \leq 1000 \). ([3])

However, the main issue in both approaches is the determination of the sign of \( P(x) \) and \( T(x) \). In 1958, Haselgrove [2] proved that both \( P(x) \) and \( T(x) \) change sign infinitely often. Consequently one can not rely on this to infer about the RH.

These approaches to the RH depend upon an application of a Landau’s result about the domain of convergence of Dirichlet integrals (Lemma 2.1 below). Let us summarize the idea behind the approach which consists of two main features: the first, is to represent the ratio \( \frac{\zeta(2s)}{\zeta(s)} \) by means of a Dirichlet integral, that is,

\[
\int_{1}^{\infty} A(u) u^{-s} \, du
\]

for all \( \text{Re}(s) > 1 \). The second, is that \( A(u) \) does not change sign for all \( u \) large. By partial summation it easy to see that

\[
\frac{\zeta(2s)}{\zeta(s)} = s \int_{1}^{\infty} \frac{P(u)}{u} u^{-s} \, du \quad \text{and} \quad \frac{\zeta(2s)}{\zeta(s)} = (s - 1) \int_{1}^{\infty} T(u) u^{-s} \, du,
\]

\( \text{Re}(s) > 1 \). Therefore, if \( P(x) \) or \( T(x) \) do not change sign for all \( x \) large (which is not the case), by Landau’s result, since \( \frac{\zeta(2s)}{\zeta(s)} \) has real singularity at \( s = \frac{3}{2} \), then \( \frac{\zeta(2s)}{\zeta(s)} \) is analytic in the half-plane \( \text{Re}(s) > \frac{3}{2} \). Hence, \( \zeta(s) \neq 0 \) in this half-plane. In other words, we would have the validity of the RH.

In this paper we follow a similar idea to investigate the RH. Precisely, we prove that there exist certain functions \( F(x) \) and \( J(s) \), with \( \lim_{x \to \infty} F(x) = -1 \) as a consequence of the PNT, for which the identity

\[
\left(s - \frac{1}{2}\right)^{-1/2} \left(\frac{\zeta(2s)}{\zeta(s)} - 1\right) + J(s) = \int_{1}^{\infty} F(u) u^{-s - 1/2} \, du,
\]

holds in the half plane \( \text{Re}(s) > 1 \). Hence, since, \( F(x) < 0 \) for all \( x \) large and the integral on the right hand side converges absolutely for \( s > 1/2 \) but diverges at \( s = 1/2 \), as a consequence of Landau’s theorem for Dirichlet’s integrals (Lemma 2.1) we can conclude that both sides of the identity are analytic in the half-plane \( \text{Re}(s) > 1/2 \).

The same ideas also hold if we use the Möbius function \( \mu(n) \) instead \( \lambda(n) \), with some adaptations.
2. Auxiliary results

Let us present some results that will be needed throughout this paper.

For the next result, which is an analogue of Landau’s theorem concerning Dirichlet series with non-negative coefficients, we refer [4, Lemma 15.1]. This result is the main tool used to obtain the central results of this paper.

Lemma 2.1. Suppose that \( G(u) \) is bounded Riemann-integrable function on every compact interval \([1, a]\), and that \( G(u) \geq 0 \) for all \( u > M \) or \( G(u) \leq 0 \) for all \( u > M \). Let \( \sigma_c \) denote the infimum of those \( \sigma \) for which \( \int_M^\infty G(u)u^{-\sigma} \, du \) converges. Then the function

\[
\varphi(s) = \int_1^\infty G(u)u^{-s} \, du
\]

is analytic in the half-plane \( \text{Re}(s) > \sigma_c \) but not at \( s = \sigma_c \).

Our first result is just an observation derived from the identity (1.1) and an application of the PNT.

Lemma 2.2. Let \( \lambda(n) \) be the Liouville function and \( a(n) \) be the arithmetic function defined as

\[
a(n) = \begin{cases} 
0, & n = 1 \\
\lambda(n), & n \geq 2.
\end{cases}
\]

We have that,

\[
\frac{\zeta(2s)}{\zeta(s)} - 1 = \sum_{n=1}^\infty \frac{a(n)}{n^s}, \quad \text{Re}(s) > 1.
\]

In particular,

\[
\sum_{n=1}^\infty \frac{a(n)}{n} = -1.
\]

Proof. Since \( \lambda(1) = 1 \), it is immediate that

\[
\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^\infty \frac{\lambda(n)}{n^s} = 1 + \sum_{n=2}^\infty \frac{\lambda(n)}{n^s} = 1 + \sum_{n=1}^\infty \frac{a(n)}{n^s}, \quad \text{Re}(s) > 1.
\]

The conclusion of the proof follows the definition of \( a(n) \) and from the fact that the PNT is equivalent to

\[
\sum_{n=1}^\infty \frac{\lambda(n)}{n} = 0.
\]

\[\square\]

Let \( x \geq 1 \) and \( b(n) \) an arithmetical function. Consider the Dirichlet polynomials

\[
F_x(\alpha) = \sum_{n \leq x} b(n)n^{-\alpha}, \quad \alpha \in \mathbb{R}.
\]

A first consequence that can be extracted from Lemma 2.2 is an alternative proof for the well-known fact that \( \zeta(s) \neq 0 \) for \( \text{Re}(s) > 1 \). Indeed, by partial summation

\[
(1 - \frac{1}{\zeta(s)}) (s - 1)^{-1} = \int_1^\infty F_u(1)u^{-s} \, du, \quad \text{Re}(s) > 1.
\]
Since \( \lim_{x \to \infty} F_u(1) = -1 \), it is clear that \( F_u(1) < 0 \) for all \( x \) large. Also note that, since the inequalities

\[
\frac{1}{\sigma - 1} < \zeta(\sigma) < \frac{\sigma}{\sigma - 1}
\]

hold for all \( \sigma > 0 \) [4, p.25], the left hand side of (2.2) has a pole at \( \sigma = 1 \) but is analytic on the real line for \( \sigma > 1 \). Hence, by an application of Lemma 2.1 (2.2) holds for \( \text{Re}(s) > 1 \) and both sides of this identity are analytic in this half-plane. Therefore, \( \zeta(s) \neq 0 \) for \( \text{Re}(s) > 1 \).

Now we proceed by presenting an application of the Mean Value Theorem which plays an important role in the justification of the main results that will be presented in this paper.

**Lemma 2.3.** Let \( \beta > \alpha \) and \( b(n) \) an arithmetic function. There exists a sequence \( \xi = (\xi(n)) \subset [\alpha, \beta] \) such that

\[
F_x(\beta) - F_x(\alpha) = - (\beta - \alpha) \sum_{n \leq x} \frac{b(n) \log(n)}{n^{\xi(n)}},
\]

for all \( x \geq 1 \). Moreover, \( \xi \) decreases and

\[
\lim_{n \to \infty} \xi(n) = \alpha.
\]

**Proof.** Let \( n > 1 \) and \( \beta > \alpha \). By the Mean Value Theorem

\[
n^{-\beta} - n^{-\alpha} = -(\beta - \alpha) \log(n) n^{-\xi(n)},
\]

for some \( \xi(n) \in [\alpha, \beta] \). That is, for each \( n \geq 1 \), there exists \( \xi(n) \in [\alpha, \beta] \) for which

\[
F_x(\beta) - F_x(\alpha) = - (\beta - \alpha) \sum_{n \leq x} \frac{b(n) \log(n)}{n^{\xi(n)}},
\]

for all \( x \geq 1 \).

The limit and the decreasigness of \( \xi \) follows from the equality

\[
\xi(n) = \frac{\log \left( (\beta - \alpha) \log(n) \frac{n^{\alpha + \beta}}{n^{\alpha - \beta}} \right)}{\log(n)} = \frac{\log(\log(n))}{\log(n)} + \frac{\log \left( \frac{1}{1 - n^{\beta - \alpha}} \right)}{\log(n)} + \alpha,
\]

for \( n > 1 \).

\[ \square \]

3. Main results

Let \( a(n) \) be as in Lemma 2.2 and we now fix the notation

\[
F_x(\alpha) = \sum_{n \leq x} \frac{a(n)}{n^\alpha},
\]

with \( \alpha > 0 \) and \( x \geq 1 \).

**Lemma 3.1.** Let \( \text{Re}(s) > 1 \). The following identity holds true

\[
\left( s - \frac{1}{2} \right)^{-1} \left( \frac{\zeta(2s)}{\zeta(s)} - 1 \right) = \int_1^\infty F_u(1/2) u^{s-1/2} du.
\]
Proof. Let $x \geq 1$. By partial summation we obtain that
\[
\sum_{n \leq x} \frac{a(n)}{n^s} = F_x(1/2) x^{-s+1/2} + \left( s - \frac{1}{2} \right) \int_1^x F_u(1/2) u^{-s-1/2} du,
\]
for $\text{Re}(s) > 1$. Since
\[
\lim_{x \to \infty} F_x(1/2) x^{-s+1/2} = 0,
\]
whenever $\text{Re}(s) > 1$, we obtain that
\[
\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \left( s - \frac{1}{2} \right) \int_1^\infty F_u(1/2) u^{-s-1/2} du,
\]
holds in the half-plane $\text{Re}(s) > 1$. By Lemma 2.2 we have that
\[
\frac{\zeta(2s)}{\zeta(s)} - 1 = \left( s - \frac{1}{2} \right) \int_1^\infty F_u(1/2) u^{-s+1/2} du,
\]
holds in the half-plane $\text{Re}(s) > 1$.

In the following result we prove an identity that concerning the analyticity of $\frac{\zeta(2s)}{\zeta(s)}$ in the half-plane $\text{Re}(s) > 1/2$.

**Theorem 3.2.** There exists a decreasing sequence $\xi = (\xi(n)) \subset ]1/2, 1[$ such that
\[
\lim_{n \to \infty} \xi(n) = 1/2
\]
for which
\[
J_\xi(s) = \int_1^\infty L_u(\xi) u^{-s-1/2} du, \quad \text{Re}(s) > 1
\]
holds for $\text{Re}(s) > 1/2$ and both sides of the equality are analytic in this half-plane.

In the equality above we have that
\[
J_\xi(s) = \int_1^\infty L_u(\xi) u^{-s-1/2} du, \quad \text{Re}(s) > 1
\]
and
\[
L_x(\xi) = \frac{1}{2} \sum_{n \leq x} \frac{a(n) \log(n)}{n \xi(n)}, \quad x \geq 1.
\]

Proof. From the previous lemma we have that
\[
\left( s - \frac{1}{2} \right)^{-1} \left( \frac{\zeta(2s)}{\zeta(s)} - 1 \right) = \int_1^\infty F_u(1/2) u^{-s+1/2} du,
\]
in the half-plane $\text{Re}(s) > 1$. For $\beta = 1$ and $\alpha = 1/2$ in Lemma 2.3, there exists a decreasing sequence $\xi = (\xi(n)) \subset ]1/2, 1[$ such that
\[
\lim_{n \to \infty} \xi(n) = 1/2
\]
for which
\[
F_u(1/2) = F_u(1) + L_u(\xi),
\]
for all $u \geq 1$, where
\[
L_u(\xi) = \frac{1}{2} \sum_{n \leq u} \frac{a(n) \log(n)}{n \xi(n)}, \quad u \geq 1.
\]
Hence, we can rewrite (3.2) as

\[(3.3) \quad \left(s - \frac{1}{2}\right)^{-1} \left(\frac{\zeta(2s)}{\zeta(s)} - 1\right) - J_\xi(s) = \int_1^\infty F_u(1)u^{-s-1/2} du,\]

for Re\(s\) > 1, for some sequence \(\xi = (\xi(n)) \subset [1/2, 1]\), with

\[J_\xi(s) = \int_1^\infty L_u(\xi)u^{-s-1/2} du.\]

In order to show that the equality (3.3) extends to the half-plane Re\(s\) > 1/2 and that both sides are analytic there, first note that from Lemma 2.2, \(\lim_{u \to \infty} F_u(1) = -1\). Thus there exists \(M \geq 1\) for which \(F_u(1) < 0\) for all \(u > M\). Moreover, clearly the integral

\[\int_1^\infty F_u(1)u^{-\sigma-1/2} du\]

converges (absolutely) at every \(\sigma > 1/2\) but diverges at \(\sigma = 1/2\). By Lemma 2.1, this implies that the function

\[\int_1^\infty F_u(1)u^{-s-1/2} du\]

is analytic for Re\(s\) > 1/2. Therefore, Lemma 2.1 implies that (3.3) holds for Re\(s\) > 1/2 and both sides of (3.1) are analytic in this half-plane. \(\square\)

Note that by Lemma 2.1 if \(F_\xi(1/2) \leq 0\) for all \(x\) large, then by Lemma 3.1 \(\frac{\zeta(2s)}{\zeta(s)}\) is analytic in the half-plane Re\(s\) > 1/2, which implies the truth of RH. In the following result it is provided another condition to obtain the RH.

**Theorem 3.3.** If there exists \(r > 0\) such that \(L_x(\xi) \leq 1 - r\) for all \(x\) large, then \(\frac{\zeta(2s)}{\zeta(s)}\) is analytic in Re\(s\) > 1/2.

**Proof.** From Lemma 2.2 we have that \(F_\xi(1/2) = F_x(1) + L_x(\xi)\), for all \(x \geq 1\). Lemma 2.2 implies that for any \(\epsilon > 0\), \(F_x(1) < -1 + \epsilon\), for all \(x\) large. Hence, if \(L_x(\xi) \leq 1 - r\), for \(0 < \epsilon \leq r\) sufficiently small \(F_x(1/2) = F_x(1) + L_x(\xi) < -\epsilon - r \leq 0\), for all \(x \geq 1\). By Lemma 2.1 and Lemma 3.1 this implies the analyticity of \(\frac{\zeta(2s)}{\zeta(s)}\) in the half-plane Re\(s\) > 1/2. \(\square\)

Now note that by partial summation, it follows that

\[\sum_{n=1}^\infty \frac{a(n)}{n^{s+1/2}} = (s - 1/2) \int_1^\infty F_u(1)u^{-s-1/2} du\]

for Re\(s\) > 1/2. Hence, as a consequence of Lemma 3.2 and Lemma 2.2, equation (3.1) can be written as

\[(3.4) \quad \frac{\zeta(2s)}{\zeta(s)} - \left(s - \frac{1}{2}\right) J_\xi(s) = \sum_{n=1}^\infty \frac{\lambda(n)}{n^{s+1/2}},\]

Re\(s\) > 1/2, and both sides are analytic in this half-plane. Moreover, from (3.1), this previous conclusions imply that

**Theorem 3.4.**

\[(3.5) \quad \frac{\zeta(2s)}{\zeta(s)} - \left(s - \frac{1}{2}\right) J_\xi(s) = \frac{\zeta(2s + 1)}{\zeta(s + 1/2)},\]

Re\(s\) > 1/2 and both sides are analytic in this half-plane.
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