Efficiently Sampling and Estimating from Substructures using Linear Algebraic Queries

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Abstract

Given an unknown $n \times n$ matrix $A$ having non-negative entries, the inner product (IP) oracle takes as inputs a specified row (or a column) of $A$ and a vector $v \in \mathbb{R}^n$, and returns their inner product. A derivative of IP is the induced degree query in an unknown graph $G = (V(G), E(G))$ that takes a vertex $u \in V(G)$ and a subset $S \subseteq V(G)$ as input and reports the number of neighbors of $u$ that are present in $S$. The goal of this paper is to understand the strength of the inner product oracle. Our results in that direction are as follows: (i) IP oracle can solve bilinear form estimation, i.e., estimate the value of $x^T A y$ given two vectors $x, y \in \mathbb{R}^n$ with non-negative entries and can sample almost uniformly entries of a matrix with non-negative entries; (ii) We tackle for the first time weighted edge estimation and weighted sampling of edges that follow as an application to the bilinear form estimation and almost uniform sampling problems, respectively; (iii) induced degree query, a derivative of IP, can solve edge estimation and an almost uniform edge sampling in induced subgraphs. To the best of our knowledge, these are the first set of oracle-based query complexity results for induced subgraphs. We show that IP/induced degree queries over the whole graph can simulate local queries in any induced subgraph; (iv) Apart from the above, we also show that IP can solve several problems related to matrix, like testing if the matrix is diagonal, symmetric, doubly stochastic, etc. The induced degree query has its roots in the queries that deal with the relation between a vertex and a subset of vertices of a graph as in Ben-Eliezer et al. [SODA’08] and Nissan [SODA’21], whereas, the IP oracle is in the class of linear algebraic queries used lately in a series of works by Rashchichian et al. [RANDOM’20], Sun et al. [ICALP’19], and Shi and Woodruff [AAAI’19]. The IP oracle can be used to estimate the Hamming distance between matrices [RANDOM’21].

Key words: Query complexity, inner product oracle, bilinear form estimation, sampling, weighted edge estimation

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1 Introduction

Let $G = (V(G), E(G))$ be an unknown graph on $n$ vertices with oracle access, and consider a query on a substructure of $G$. Suppose we want to know for a vertex $v \in V(G)$, the number of neighbors in a community, as in social networks, represented as a subset $S \subseteq V(G)$. This gives rise to the induced degree query oracle that takes a vertex $u \in V(G)$ and a subset $S \subseteq V$ as input and reports the number of neighbors of $u$ that are present in $S$. On the other hand, let us consider another query, named inner product (IP) query oracle, the main focus of the paper, with a linear algebraic flavor and show its connection to the induced degree query. But let us first deal with the notations used.

**Notations.** In this paper, we denote the set \{1, \ldots, t\} by [t] and \{0, \ldots, t\} by [t]. For a (directed) graph $G$, $V(G)$ and $E(G)$ denote the vertex set and edge sets of $G$, we will use $V$ and $E$ when the graph is clear from the context. For a vertex $u$, let $d_G(u)$ denote the degree of $u$ in $G$ and $N_G(u)$ denote the set of neighbors of $u$ in $G$. For a subset $S$ of $V(G)$, the subgraph of $G$ induced by $S$ is denoted by $G_S = (S, E_S)$ such that $E_S := \{\{u, v\} \in E(G) \mid u \in S \land v \in S\}$. The local queries for a graph $G = (V(G), E(G))$ are: (i) degree query: given $u \in V(G)$, the oracle reports the degree of $u$ in $V(G)$; (ii) neighbor query: given $u \in V(G)$ and an integer $i$, the oracle reports the $i$-th neighbor of $u$, if it exists; otherwise, the oracle reports $\perp$. (iii) adjacency query: given $u, v \in V(G)$, the oracle reports whether $\{u, v\} \in E(G)$.

For a non-empty set $X$ and a given parameter $\epsilon \in (0, 1)$, an almost uniform sample of $X$ means each element of $X$ is sampled with probability values that lie in the interval $[(1 - \epsilon)/|X|, (1 + \epsilon)/|X|]$. For a matrix $A$, $A_{ij}$ denotes the element in the $i$-th row and $j$-th column of $A$. $A_i$ and $A_j$ denote the $i$-th row vector and $j$-th column vector of the matrix $A$, respectively. $A \in [\rho]^{n \times n}$ means $A_{ij} \in [\rho]$ for each $i, j \in [n]$, $\rho \in \mathbb{N}$. Throughout this paper, the number of rows or columns of a square matrix $A$ is $n$, that will be clear from the context. Vectors are matrices of order $n \times 1$ and will be represented using bold face letters. Without loss of generality, we consider $n$ to be a power of 2. The $i$-th coordinate of a vector $x$ is denoted by $x_i$. We denote by $1$ the vector with all coordinates 1. Let $\{0, 1\}^n$ be the set of $n$-dimensional vectors with entries either 0 or 1. For $x \in \mathbb{R}^n$, $1_x$ is a vector in $\{0, 1\}^n$ whose $i$-th coordinate is 1 if $x_i \neq 0$ and 0 otherwise; $\text{nnz}(x) = |\{i \in [n] : x_i \neq 0\}|$ denotes the number of non-zero components of the vector. By $\langle x, y \rangle$, we denote the standard inner product of $x$ and $y$, that is, $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$. $P$ is a $(1 + \epsilon)$-approximation to $Q$ means $|P - Q| \leq \epsilon \cdot Q$. With high probability means that the probability of success is at least $1 - \frac{1}{n^c}$, where $c$ is a positive constant. $\tilde{O}()$ and $\tilde{\Omega}()$ hides a poly$(\log n, \frac{1}{\epsilon})$ term in the upper bound.

1.1 Definition and motivation of inner product oracle

**Inner product (IP) query definition.** Let $A \in [\rho]^{n \times n}$, $\rho \in \mathbb{N}$, be a matrix whose size is known but the entries are unknown. Now given a row index $i \in [n]$ (or, a column index $j \in [n]$) and a vector $v \in \mathbb{R}^n$, with non-negative entries as input, the inner product query to $A$ reports the value of $\langle A_{i\ast}, v \rangle$ (\(\langle A_{\ast j}, v \rangle\)). If the input index is for row (column), we refer the corresponding query as row (column) IP query.

Observe that induced degree query can be implemented for a graph $G$ by IP as a dot product with $1_S$ (indicator vector for the set $S$) and the corresponding row of the matrix $A$ that is the 0-1 adjacency matrix of $G$.

**Motivation of the IP query.** The IP has both graph theoretic and linear algebraic flavors to it and we will highlight them shortly. It may be mentioned here that IP has been already used to estimate the Hamming distance between two matrices \[BGM21]\.

From a practical point of view, Rashtchian et al. \[RWZ20]\ mentions that vector-matrix-vector queries would most likely be useful in the context of specialized hardware or distributed environments. Needless to say, the same carries over to IP. There are many computer architectures that

\*The ordering of neighbors of the vertices are unknown to the algorithm.

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allow us to compute inner products in one cycle of computation with more parallel processors. Inner product computation can be parallelized using single instruction multiple data (SIMD) architecture \[\text{HP12}\]. Modern day GPU processors use instruction level parallelism. Nvidia GPUs precisely do that by providing a single API call to compute inner products \[\text{SK10} \text{ and } \text{cud}\]. There are many such architectures where IP query has been given to users directly. Similarly, there are programming languages built on SIMD framework that can compute inner products \[\text{sim}\].

### 1.2 The problems, results and paper organization

The crux of this paper is in showing that IP can efficiently solve problems in graph theory that \textsc{local} queries and random edge query can not, foremost among them being estimation problems in induced subgraphs, weighted edge estimation, etc. In Section \textbf{3}, we show this separation with the aid of two lower bounds which give a clear separation in power of IP from \textsc{local} queries. The problems we consider for showing this separation are edge estimation and sampling problems in induced subgraphs. We show that IP can solve these problems using \(\tilde{O}(|S|/\sqrt{m_S})\) many IP queries to the adjacency matrix of the graph \(G\), where \(S \subseteq V(G)\) is the set of vertices of interest and \(m_S = |ES|\). Another crucial takeaway from our work is that IP oracle and its derivative, the \textsc{induced degree} oracle act like a local query in an induced graph.

Our work also involves estimating the bilinear form \(x^T A y\) and sampling an element of a matrix almost uniformly using IP. Bilinear form estimation has huge importance in numerical linear algebra (see \textbf{FMR14} and the references therein) because of its use in calculating node centrality measures like resolvent subgraph centrality and resolvent subgraph communicability \textbf{BK13}, \textbf{EH10}, Katz score for adapting it to PageRank computing \textbf{BEG12}, etc. In this paper, we give both upper and lower bounds for several important problems in the context of matrices and graphs when we have IP query access to the corresponding matrix and adjacency matrix of the graph, respectively. The main highlights are as follows.

#### Matrix problems. The main matrix related problems considered in this work and defined below are bilinear form estimation and sampling an element of a matrix uniformly at random.

| Problem         | Query complexity                           | Comments                                 |
|-----------------|--------------------------------------------|------------------------------------------|
| \(\text{BFE}_A(x, y)\) | \(\tilde{\Theta}\left(\sqrt{\frac{\rho}{\gamma} (\text{nnz}(x) + \text{nnz}(y))}\right)\sqrt{x^T A y}\) | Theorem \textbf{4.16} and \textbf{4.17} |
| \(\text{SAU}_A(x, y)\) | \(\tilde{\Theta}\left(\sqrt{\frac{\rho}{\gamma} (\text{nnz}(x) + \text{nnz}(y))}\right)\sqrt{x^T A y}\) | Theorem \textbf{4.10} and \textbf{4.17} |

Table 1: Query complexities of \(\text{BFE}_A(x, y)\) and \(\text{SAU}_A(x, y)\).

For the above problems, we give both upper and (almost) tight lower bounds (see Table \textbf{1}) in Section \textbf{3}. We also discuss weighted edge estimation and weighted sampling of edges as their applications in Appendix \textbf{A}. Apart from these, we also discuss (in Appendix \textbf{B}) several important matrix problems using IP oracle that were studied using stronger queries like \textit{matrix vector} and \textit{vector matrix vector} queries \textbf{SWYZ19a}, \textbf{RWZ20}.

#### Graph problems. Section \textbf{3} discusses our results for graph problems and establishes tight separation between \textsc{induced degree} query and \textsc{local} query oracle. To establish the fact that \textsc{local} query access (to the entire unknown graph) can not solve problems in induced subgraphs...
efficiently, we prove lower bounds for local query access to solve Edge Estimation and Edge Sampling in induced subgraphs in Section 3.1. In Section 3.2, we will discuss that IP/induced degree for the whole graph can simulate local queries in any induced subgraph, and describe its implication in solving problems in induced subgraph.

2 Inner product oracle vis-a-vis other query oracles

Graph parameter estimation, where the graph can be accessed through query oracles only, has been an active area of research in sub-linear algorithms for a while [GR08, ELRS17, ERS18, RSW18]. There are different granularities at which the graph can be accessed – the query oracle can answer properties about graph that are local or global in nature. By now, the local queries have been used for edge [GR08], triangle [ELRS17], clique estimation [ERS18] and has got a wide acceptance among researchers. Apart from the local queries, in the last few years, researchers have also used the random edge query [ABG+18, AKK19], where the oracle returns an edge in the graph G uniformly at random. Notice that the randomness will be over the probability space of all edges, and hence, it is difficult to classify a random edge query as a local query. On the other hand, global queries come in different forms. Starting with the subset queries [Sto83, Sto85, RT16a], there have been other queries like bipartite independent set query, independent set query [BHR+18], gis query [BGK+18, DLM20], cut query [RSW18], etc. Linear measurements or queries [ACK21, AGM12], based on dot product, have been used for different graph problems.

To this collection of query oracles, we introduce a new oracle called inner product (IP) oracle that is a natural oracle to consider for linear algebraic and graph problems. Using this oracle, we solve hitherto unsolved problems (by an unsolved problem, we mean that no non-trivial algorithm was known before) with graph theoretic and linear algebraic flavor, like (a) edge estimation in induced sub-graph; (b) bilinear form estimation; (c) sampling entries of matrices with non-negative entries. We also show weighted edge estimation and edge estimation in induced subgraph as applications of bilinear form estimation. Our lower bound result, for Edge Estimation in induced subgraph with only local query access, implies that there is a separation between the powers of local query and induced degree query. We will show that our newly introduced inner product query oracle can solve problems that can not be solved by the three local queries mentioned even coupled with the random edge query.

Our current survey of the literature (here we do not claim exhaustivity!) shows that a query related to a subgraph was first used in Ben-Eliezer et al. [BKKR08], and named as group query, where one asks if there is at least one edge between a vertex and a set of vertices. We found the latest query in this league to be the demand query (in bipartite graphs where the vertex set are partitioned into two parts left vertices and right vertices) introduced by Nissan [Nis19] – a demand query accepts a left vertex and an order on the right vertices and returns the first vertex in that order that is a neighbor of the left vertex. One can observe that the group and demand queries are polylogarithmically equivalent. Staying on this line of study related to the relation of a vertex with a subset of vertices, we focus on the induced degree query which we feel handles many natural questions.

Query oracle based graph algorithms access the graph at different granularities – this gives rise to a whole gamut of queries with different capacities, ranging from local queries like degree, neighbor, adjacency queries [Fei06, GR08] to global queries like independent set based queries [BHR+18, DLM20], random edge queries [ABG+18], and others like group [BKKR08] and demand queries [Nis19]. This rich landscape of queries has unravelled many interesting algorithmic and complexity theoretic results [Fei06, GR08, ABG+18, BKKR08, Nis19, BHR+18, DLM20, RT16a]. With this in mind, if we turn our focus to the landscape of linear algebraic queries, the most natural query is the matrix entry query where one gives an index of the matrix and asks for the value there. Lately, a series of works [RWZ21, SWYZ19a, SW19, BLWZ19] have
used linear algebraic queries like \textit{vector-matrix-vector query} and \textit{matrix-vector} queries. The IP oracle is also motivated by these new query oracles. Notice the huge difference in power between \textit{matrix entry} query and \textit{vector-matrix-vector query} and \textit{matrix-vector} queries. Note that IP is strictly weaker than these matrix queries but stronger than the \textit{matrix entry} query. We feel there is a need to study linear algebraic queries with intermediate power – the IP query fits in that slot.

3 A query model for induced subgraph problems

To the best of our knowledge, our work is a first attempt towards solving estimation problems in induced subgraphs. We start by showing a separation between local query and induced degree query using the problems of \textit{Edge Estimation} and \textit{Edge Sampling} in induced subgraph. We now define \textit{Induced Edge Estimation} and \textit{Induced Edge Sampling}.

**INDUCED EDGE ESTIMATION**

| Input: A parameter $\epsilon \in (0, 1)$ and a subset $S$ of the vertex set $V$ of a graph $G$. |
| Output: A $(1 \pm \epsilon)$-approximation to the number of edges $E_S$ in the induced subgraph. |

**INDUCED EDGE SAMPLING**

| Input: A parameter $\epsilon \in (0, 1)$ and a subset $S$ of the vertex set $V$ of a graph $G$. |
| Output: Sample each edge $e \in E_S$ with probability between $\frac{1-\epsilon}{|E_S|}$ and $\frac{1+\epsilon}{|E_S|}$. |

One of the main contributions of this paper is to show, using a lower bound argument, that local queries together with random edge query are inefficient for both \textit{Induced Edge Estimation} and \textit{Induced Edge Sampling}. The lower bound results follow.

**Theorem 3.1** (Lower bound for \textit{Induced Edge Estimation} using local queries). Let us assume that $s, m_s \in \mathbb{N}$ be such that $1 \leq m_s \leq \left(\frac{\epsilon}{2}\right)$ and the query algorithms have access to degree, neighbor, adjacency and random edge queries to an unknown graph $G = (V(G), E(G))$. Any query algorithm that can decide for all $S \subseteq V(G)$, with $|S| = \Theta(s)$, whether $|E_S| = m_s$ or $|E_S| = 2m_s$, with probability at least 2/3, requires $\Omega\left(\frac{\epsilon^2}{m_s}\right)$ queries.

**Theorem 3.2** (Lower bound for \textit{Induced Edge Sampling} using local queries). Let us assume that $s, m_s \in \mathbb{N}$ be such that $1 \leq m_s \leq \left(\frac{\epsilon}{2}\right)$ and the query algorithms have access to degree, neighbor, adjacency and random edge queries to an unknown graph $G = (V(G), E(G))$. Any query algorithm that for any $S \subseteq V(G)$, with $|S| = \Theta(s)$, samples the edges in $E_S$ $\epsilon$-almost uniformly\footnote{Each edge in $E_S$ is sampled with probability between $(1-\epsilon)\frac{1}{|E_S|}$ and $(1+\epsilon)\frac{1}{|E_S|}$.}, with probability at least 99/100, will require $\Omega\left(\frac{\epsilon^2}{m_s}\right)$ queries\footnote{Let $U$ denote the uniform distribution on $E_S$. The lower bound even holds even if the goal is to get a distribution that is $\epsilon$ close to $U$ with respect to $\ell_1$ distance.}. Note that $\epsilon \in (0, 1)$ is given as an input to the algorithm.

**Remark 1.** When $S = V$, \textit{Induced Edge Estimation} and \textit{Induced Edge Sampling} are \textit{Edge Estimation} and \textit{Edge Sampling} problems, respectively. Both \textit{Edge Estimation} and \textit{Edge Sampling} can be solved with high probability by using $\tilde{\Theta}\left(\frac{|V|^2}{|E|}\right)$ adjacency queries\footnote{\cite{Gol17}}. Notice that these bounds match the lower bounds. Contrast this with the fact that \textit{Edge Estimation} and \textit{Edge Sampling} can be solved with high probability by using $\tilde{\Theta}\left(|V|/\sqrt{|E|}\right)$ local queries, where each local query is either a degree or a neighbor or an adjacency query\footnote{\cite{GR08,ER18}}. Thus, we observe that for \textit{Induced Edge Estimation} and \textit{Induced Edge Sampling}, the adjacency query is as good as the entire gamut of local queries and random edge query. On a different note, our results on \textit{Bilinear Form Estimation} and \textit{Almost Uniformly Sampling} using IP query (see Table\footnote{\cite{P}}) generalize the above cases.
mentioned results on Edge Estimation and Edge Sampling using local queries. Note that IP oracle is a natural query oracle for graphs where the unknown matrix is the adjacency matrix of a graph, and we will discuss that in Remark 3 that IP query on the adjacency matrix graphs is stronger than the local queries.

In Section 3.1 we prove Theorems 3.1 and 3.2 by reduction from a problem in communication complexity. In Section 3.2 we discuss the way in which INDUCED DEGREE query simulates local queries in any induced subgraph (see Remark 2). This will imply that the lower bound results in Theorems 3.1 and 3.2 can be overcome if we have an access to INDUCED DEGREE query to the whole graph (see Corollary 3.7). However, the implication is more general and will be discussed in Section 3.2.

3.1 Proofs of Theorems 3.1 and 3.2

The proofs of the lower bounds use communication complexity. We provide a rudimentary introduction to communication complexity in Appendix C and for more details see [KN97]. We will use the following problem in our lower bound proofs.

**Definition 3.3** (k-Intersection). Let k, N ∈ ℕ such that k ≤ N. Let S = {(x, y) : x, y ∈ {0, 1}^N, ∑_i x_i y_i = k or 0}. The k-Intersection function over N bits is a partial function denoted by k-Intersection : S → {0, 1}, and is defined as follows: k-Intersection(x, y) = 1 if ∑_i x_i y_i = k and 0, otherwise.

**Lemma 3.4.** [KN97] Let k, N ∈ ℕ such that k ≤ N. The randomized communication complexity of k-Intersection function on N bits is Ω(N/k).

**Proof of Theorem 3.1.** We give a reduction from m_s-Intersection problem over N = s^2 bits. Let x = (x_{ij}) ∈ {0, 1}^N be such that i, j ∈ [s]. Similarly, let y ∈ {0, 1}^N. It is promised that Alice and Bob will be given x and y such that there are either 0 intersections or exactly m_s intersections, i.e., either ⟨x, y⟩ = 0 or m_s. Now we define a graph G_{(x,y)}(V(G), E(G)) as follows where ⋓ denotes disjoint union.

- |V(G)| = Θ(s). V(G) = S_A ⋓ S_B ⋓ T_A ⋓ T_B ⋓ C such that S_A, S_B, T_A, T_B are independent sets and |S_A| = |S_B| = |T_A| = |T_B| = s and |C| = Θ(s). Note that V(G) is independent of x and y;

- The subgraph (of G_{(x,y)}) induced by C is a fixed graph, independent of x and y, having exactly m_s edges. Also there are no edges in G_{(x,y)} between the vertices of C and V(G)\C.

- The edges in the subgraph (of G_{(x,y)}) induced by V(G) \ C = S_A ⋓ T_A ⋓ S_B ⋓ T_B depend on x and y as follows. Let S_A = {s^A_i : i ∈ [s]}, T_A = {t^A_i : i ∈ [s]}, S_B = {s^B_i : i ∈ [s]} and T_B = {t^B_i : i ∈ [s]}. For i, j ∈ [s], if x_{ij} = y_{ij} = 1, then (s^A_i, t^B_j) ∈ E(G) and (s^B_i, t^A_j) ∈ E(G). For i, j ∈ [s] if either x_{ij} = 0 or y_{ij} = 0, then (s^A_i, t^A_j) ∈ E(G) and (s^B_i, t^B_j) ∈ E(G);

The graph G_{(x,y)} can be uniquely generated from x and y. Moreover, Alice and Bob need to communicate to learn useful information about G_{(x,y)}. Observation 3.3 follows from the construction that shows the relation between the number of edges in the subgraph induced by S_A ⋓ T_B ⋓ C with (x, y), where x, y ∈ {0, 1}^N are such that either ⟨x, y⟩ = 0 or ⟨x, y⟩ = m_s.

**Observation 3.5.** (i) |S_A ⋓ T_B ⋓ C| = Θ(s), (ii) irrespective of x and y: |E_{S_A}| = |E_{S_B}| = |E_{T_A}| = |E_{T_B}| = 0, also the degree of each vertex in S_A ⋓ T_A ⋓ S_B ⋓ T_B is same (i.e., s), (iii) if ⟨x, y⟩ = 0, then |E_{S_A ⋓ T_B ⋓ C}| = m_s, (iv) if ⟨x, y⟩ = m_s, then |E_{S_A ⋓ T_B ⋓ C}| = 2m_s.

The following observation completes the proof of the theorem.
Observation 3.6. Alice and Bob can deterministically determine answer for each local query to graph $G_{(S,X)}$ by communicating $O(1)$ bits.

Proof. DEGREE query: By Observation 3.5 (ii), the degree of every vertex in $V(G) \setminus C$ is $s$. Also, the subgraph induced by $C$ is a fixed graph disconnected from the rest. That is Alice and Bob know the degree of every vertex in $C$. Therefore, any DEGREE query can be simulated without any communication.

NEIGBOR query: Observe that Alice and Bob can get the answer to any neighbor query involving a vertex in $C$ without any communication. Now, consider the set $S_A$. The labels of the $j$ many neighbors of any vertex in $s^A_i \in S_A$ are as follows: for $j \in [s]$, the $j$-th neighbor of $s^A_i$ is either $t^B_j$ or $s^A_j$ depending on whether $x_{ij} = y_{ij} = 1$ or not, respectively. So, any NEIGBOR query involving vertex in $S_A$ can be answered by 2 bits of communication. Similar arguments also hold for the vertices in $S_B \cup T_A \cup T_B$.

ADJACENCY query: Observe that each adjacency query can be answered by at most 2 bits of communication, and it can be argued like the NEIGBOR query.

RANDOM EDGE query: By Observation 3.5 (ii), the degree of every vertex in $V(G) \setminus C$ is $s$ irrespective of the inputs of Alice and Bob. Also, they know the entire subgraph induced by the vertex set $C$. Also, $C$ is disconnected from the rest. Alice and Bob use shared randomness to sample a vertex in $V$ proportional to its degree. Let $r \in V$ be the sampled vertex. They again use shared randomness to sample an integer $j$ in $[d(v)]$ uniformly at random. Then they determine the $j$-th neighbor of $r$ using NEIGBOR query. Observe that this procedure simulates a RANDOM EDGE query by using at most 2 bits of communication.

Proof of Theorem 3.2. For clarity, we prove the theorem for $\epsilon = 1/4$. However, the proof can be extended for any $\epsilon \in (0, 1/2)$. We use the same set up and construction as in Theorem 3.1 with $S = S_A \cup S_B \cup C$. Let $A$ be an algorithm that almost uniformly samples edges from the induced graph $G_S = (S, E_S)$ making $T$ queries, with probability $99/100$. Using $A$ we give another algorithm $A'$ that decides whether $|E_S| = m_s$ or $|E_S| = 2m_s$ by using $O(T)$ queries, with probability at least $2/3$. From the reduction presented in Theorem 3.1 Alice and Bob can use $A'$ to solve $m_s$-INTERSECTION over $N = s^2$ bits, and hence $T = \Omega(m_s^2)$.

$A'$ runs $A$ 10 times independently to obtain edges $e_1, \ldots, e_{10}$. Note that each edge $e_i$ is sampled almost uniformly. If at least one edge $e_i$ satisfies $e_i \in E_{S_A \cup T_B}$, then $A'$ reports that $|E_S| = 2m_s$. Otherwise, $A'$ reports that $|E_S| = m_s$. The query cost of $A'$ is $\Theta(T)$.

If $|E_S| = m_s$, then there is no edge in the subgraph induced by $S_A \cup T_B$. So, in this case, all the edges reported by $A'$ are from the subgraph induced by $C$. Now consider when $|E_S| = 2m_s$. In this case, the subgraph induced by $S_A \cup T_B$ and $C$ have exactly $m_s$ edges each. So, by the assumption of the algorithm $A$, the probability that any particular $e_i$ is present in the subgraph induced by $S_A \cup T_B$ is at least $1/2 - \epsilon = 1/4$ (since, we are analyzing for $\epsilon = 1/4$). So, under the conditional space that all the ten runs of $A$ succeed, the probability that none of the ten edges sampled by $A$ is from the subgraph induced by $S_A \cup T_B$ is at most $(1 - 1/4)^{10} < 1/10$. As each run of algorithm $A$ succeeds with probability at least $99/100$, all the ten runs of the algorithm $A$ succeeds with probability at least 9/10. So, the probability that algorithm $A'$ succeeds is at least $9/10 \cdot (1 - 1/10) > 2/3$.

3.2 A query model for induced subgraphs

We will first show that INDUCED DEGREE query can simulate any LOCAL query in any induced subgraph.
Remark 2. Let us have an induced degree query oracle access to an unknown graph $G(V, E)$. Consider any $X \subseteq V(G)$ and $G_X$, the subgraph of $G$ induced by $X$. Then

(i) Any query to $G_X$, which is either a degree or adjacency, can be answered by one induced degree query to $G$.

(ii) Moreover, any neighbor query to $G_X$ can be answered by $O(\log |X|)$ many induced degree queries to $G$ by binary search.

The above remark together with the edge estimation result of Goldreich and Ron [GR08], and edge sampling result of Eden and Rosenbaum [ER18], gives us the following result as a corollary.

Corollary 3.7 (Upper bound for induced degree estimation and sampling). Let us assume that the query algorithms have access to induced degree query to an unknown graph $G = (V(G), E(G))$. Then, there exists an algorithm that takes a subset $S \subseteq V(G)$ and $\epsilon \in (0, 1)$ as inputs, and outputs a $(1 + \epsilon)$-approximation to $|E_S|$, with high probability, using $\tilde{O}(|S|/\sqrt{|E_S|})$ induced degree queries to $G$. Also, there exists an algorithm that $\epsilon$-almost uniformly samples edges in $E_S$, with high probability, using $\tilde{O}(|S|/\sqrt{|E_S|})$ induced degree queries to $G$.

Remark 3. More generally, Remarks 2 implies that any problem $\mathcal{P}$ on a graph $G$ that can be solved by using $f(|V(G)|, |E(G)|)$ many local queries, can also be solved on any induced subgraph $G_S$, where $S \subseteq V(G)$, of $G$ by using $f(|V(G_S)|, |E(G_S)|) \cdot O(\log |V(G_S)|)$ many induced degree queries.

4 Bilinear form estimating and sampling entries of a matrix

4.1 Algorithm for Bilinear Form Estimation

To give the main ideas behind the algorithm for BILINEAR FORM ESTIMATION, we will discuss, in this section, the algorithm for estimating $1^T A 1$ using IP access to $A$, with $A$ being symmetric. In Appendix C we show how the algorithm for this special case can be extended for the general problem of estimating $x^T A y$, where $A \in [\rho]^{n \times n}$, $x \in [\gamma_1]^n$ and $y \in [\gamma_2]^n$. We will give an outline of the proof of the following theorem.

Theorem 4.1. There exists a query algorithm for BFE that takes $\epsilon \in (0, 1/2)$ as input and determines a $(1 \pm \epsilon)$-approximation to $1^T A 1$ with high probability by using $\tilde{O} \left(\frac{\sqrt{n}}{\epsilon^2 \sqrt{T_A}}\right)$ many IP queries to a symmetric matrix $A \in [\rho]^{n \times n}$. Moreover, the algorithm only uses IP query of the form $\langle A_k, u \rangle$ for some $k \in [n]$ and $u \in \{0, 1\}^n$.

The algorithms for BILINEAR FORM ESTIMATION and SAMPLE ALMOST UNIFORMLY (Section 4.3) will use a subroutine, which takes as input a given row $i \in [n]$ of $A$ and a non-empty set $S \subseteq [n]$, and outputs $A_{ij}$, where $j \in S$, with probability $A_{ij}/(\sum_{j \in S} A_{ij})$.

Observation 4.2. There exists an algorithm REGR (See Algorithm 2) that takes $i \in [n]$ and $x \in \{0, 1\}^n$ as inputs, outputs $A_{ij}$ with probability $A_{ij}x_j/(\sum_{j \in [n]} A_{ij}x_j)$ by using $O(\log n)$ many IP queries to matrix $A$.

We will now discuss in the following paragraphs the details of the algorithm (Algorithm 2) for estimating $1^T A 1$. The ingredients, to prove the correctness of Algorithm 2 is formally stated in Lemma 4.6. The approximation guarantee of Algorithm 2 which matches the guarantee mentioned in Theorem 4.1 is given in Claim 4.7.
To define the large and small buckets, we require a lower bound \( \ell \) for replacement, and for each sampled row \( S \), let \( \tilde{S} \), a partition of rows of \( A \) induced by the indices present in \( S \), be the indices of the rows present in large and small buckets, respectively. Let \( \tilde{B} \) be the set of indices of the rows present in a particular bucket, and every row in a particular bucket has approximately the same total weight. Consider a standard technique from property testing. For details, see Appendix \( \text{??} \).

**Fact 4.3.** For every \( i \in [t] \), \((1 + \beta)^{i-1} |B_i| \leq \sum_{j \in B_i} |A_{ij}, 1| < (1 + \beta)^i |B_i| \).

Based on the number of rows in a bucket, we classify the buckets to be either large or small. To define the large and small buckets, we require a lower bound \( \ell \) on the value of \( m = 1^T A 1 \). Moreover, let us assume that, \( m/6 \leq \ell \leq m \). However, this restriction can be removed by using standard techniques from property testing. For details, see Appendix \( ?? \).

**Definition 4.4.** We fix a threshold \( \theta = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{\frac{8}{5} \cdot \frac{2}{\rho}} \). For \( i \in [t] \), we define the set \( B_i \) to be a large bucket if \(|B_i| \geq \theta n \). Otherwise, the set \( B_i \) is defined to be a small bucket. Thus, the set of large buckets \( L \) is defined as \( L = \{ i \in [t] : |B_i| \geq \theta n \} \), and \( [t] \setminus L \) is the set of small buckets.

Let \( V, U \subseteq [n] \) be the sets of indices of rows that lie in large and small buckets, respectively. For \( I \subseteq [n] \), let \( x_I \) denote the sub-vector of \( x \) induced by the indices present in \( I \). Similarly, for \( I, J \subseteq [n] \), let \( A_{IJ} \) denote the sub-matrix of \( A \) where the rows and columns are induced by the indices present in \( I \) and \( J \), respectively. Observe that, \( 1^T A 1 = 1^T V A_{VV} 1_V + 1^T V A_{VU} 1_U + 1^T U A_{UV} 1_V + 1^T U A_{UU} 1_U \).

**2-Approximation of** \( 1^T A 1 \) **Note that at this point we know** \( \beta \) **and, upon querying** \( \langle A_j, 1 \rangle \), **we can determine the bucket to which** \( j \) **belongs, for** \( j \in [n] \). **The algorithm begins by sampling a subset** \( S \) **of rows of** \( A \), **such that** \(|S| = K \), **independently and uniformly at random with replacement, and for each sampled row** \( j \), **the algorithm determines** \( \langle A_j, 1 \rangle \) **by using IP oracle. This determines the bucket in which each sampled row belongs. Depending on the number of sampled rows present in different buckets, our algorithm classifies each bucket as either large or small.**
Note that the algorithm does not find $\widetilde{V}$ and $\widetilde{U}$ explicitly – these are used only for analysis purpose.

Observe that, $1^T A 1 = 1^T_U A_{\widetilde{V} \widetilde{U}} 1_{\widetilde{V}} + 1^T_{\widetilde{U}} A_{\widetilde{U} \widetilde{V}} 1_{\widetilde{U}} + 1^T_U A_{\widetilde{U} \widetilde{V}} 1_{\widetilde{U}}$. We can show that $1^T_U A_{\widetilde{U} \widetilde{V}} 1_{\widetilde{U}}$ is at most $\frac{2}{\ell}$, where $\ell$ is a lower bound on $1^T A 1$. Thus, $1^T A 1 \approx 1^T_U A_{\widetilde{V} \widetilde{U}} 1_{\widetilde{V}} + 1^T_{\widetilde{U}} A_{\widetilde{U} \widetilde{V}} 1_{\widetilde{U}}$.

Lemma 4.5 shows that for a sufficiently large $K$, with high probability, the fraction of rows in any large bucket is approximately preserved in the sampled set of rows. Also observe that we know tight (upper and lower) bounds on $\langle A_{xz}, 1 \rangle$ for every row $j$, where $j \in \widetilde{V}$. Thus, the random sample of $S$ rows, such that $|S| = K$, approximately preserves $1^T_U A_{\widetilde{V} \widetilde{U}} 1_{\widetilde{V}} + 1^T_{\widetilde{U}} A_{\widetilde{U} \widetilde{V}} 1_{\widetilde{U}}$. Observe that this is already 2-approximation of $1^T A 1$.

**Using REGR for tight approximation** In order to get a $(1 \pm \epsilon)$-approximation to $1^T A 1$, we need to estimate $1^T_U A_{\widetilde{V} \widetilde{U}} 1_{\widetilde{V}}$, which is same as estimating $1^T_{\widetilde{U}} A_{\widetilde{U} \widetilde{V}} 1_{\widetilde{U}}$ since $A$ is a symmetric matrix. We estimate $1^T_{\widetilde{U}} A_{\widetilde{U} \widetilde{V}} 1_{\widetilde{U}}$, that is, the sum of $A_{ij}$s such that $i \in \widetilde{V}$ and $j \in \widetilde{U}$, as follows. For each bucket $B_i$ that is declared as large by the algorithm, we select enough number of rows randomly with replacement from $S_i = S \cap B_i$, invoke REGR for each selected row in $S_i$ and increase the count by 1 if the element $A_{ij}$ reported by REGR be such that $j \in \widetilde{U}$. A formal description of our algorithm is given in Algorithm 2. Now, we focus on the correctness proof of our algorithm for BFE.

**Algorithm 2: BFE $\ell, \epsilon$**

**Input:** An estimate $\ell$ for $1^T A 1$ and $\epsilon \in (0, 1/2)$.

**Output:** $\hat{m}$, which is a $(1 \pm \epsilon)$-approximation of $1^T A 1$.

1 begin
2 Independently select $K = \Theta \left( \sqrt{\frac{\delta}{\epsilon^6}} \cdot \epsilon^{-4.5} \cdot \log^2(\rho n) \cdot \log(1/\epsilon) \right)$ rows of $A$ uniformly at random and let $S$ denote the multiset of the selected indices (of rows) sampled. For $i \in [t]$, let $S_i = B_i \cap S$.
3 Let $L = \left\{ i : \frac{|S_i|}{|S|} \geq \frac{1}{2} + \frac{\sqrt{2}}{\sqrt{8} \cdot \sqrt{\epsilon}} \right\}$. Note that $\tilde{L}$ is the set of buckets that the algorithm declares to be large. Similarly, $[t] \setminus \tilde{L}$ is the set of buckets declared to be small by the algorithm.
4 For every $i \in L$, select $|S_i|$ samples uniformly at random from $S_i$, with replacement, and let $Z_i$ be the set of samples obtained. For each $z \in Z_i$, make a REGR$(z, 1)$ query and let $A_{zk} = \text{REGR}(z, 1)$. Let $Y_z$ be a random variable that takes value 1 if $k \in \widetilde{U}$ and 0, otherwise.
5 // If we take $Z_i = S_i$, then also the correctness can be proved. But comparatively, the correctness proof is slightly clean as because of the way we are generating $Z_i$.
6 Determine $\tilde{\alpha}_i = \frac{\sum_{z \in Z_i} Y_z}{|S_i|}$.
7 Output $\hat{m} = \frac{K}{n} \sum_{i \in \tilde{L}} (1 + \tilde{\alpha}_i) \cdot |S_i| \cdot (1 + \beta)^i$.
8 end

To prove that $\hat{m}$ is a $(1 \pm \epsilon)$-approximation of $m = 1^T A 1$, we need the following definition and the technical Lemma 4.6

**Definition 4.5.** For $i \in L$, $\alpha_i$ is defined as $\frac{\sum_{z \in B_i}(A_{uz}, V)}{\sum_{z \in B_i}(A_{uz}, V)}$.

**Lemma 4.6.** For a suitable choice of constant in $\Theta(\cdot)$ for selecting $K$ samples in Algorithm 2, the followings hold with high probability:

(i) For each $i \in L$, $(1 - \frac{\epsilon}{4}) \frac{|B_i|}{n} \leq \frac{|S_i|}{K} \leq (1 + \frac{\epsilon}{4}) \frac{|B_i|}{n}$.
(ii) For each $i \in [t] \setminus L$, \( \frac{|S_i|}{n} < \frac{1}{n} \cdot \frac{\varepsilon}{2} \cdot \frac{n}{p} \).

(iii) We have \(|\tilde{U}| < \sqrt{\frac{t}{4}} \cdot \frac{\ell}{p} \), where \( \tilde{U} = \{ j \in B_i : i \in [t] \setminus \tilde{L} \} \).

(iv) For every $i \in \tilde{L}$, (a) if $\alpha_i \geq \frac{\varepsilon}{8}$, then $(1 - \frac{1}{4}) \alpha_i \leq \tilde{\alpha}_i \leq (1 + \frac{\varepsilon}{4}) \alpha_i$, and (b) if $\alpha_i < \epsilon/8$, then $\tilde{\alpha}_i < \epsilon/4$.

Proof. (i) Recall that $K = |S| = \Theta \left( \frac{\sqrt{n}}{\sqrt{q}} \cdot \epsilon^{-4.5} \cdot \log^2 (\rho n) \cdot \log(1/\epsilon) \right)$. Observe that $E[|S_i|] = \frac{|S_i|}{n} |B_i|$. Here $i \in L$. By the definition of $L$ (See Definition 1.3), $|B_i| \geq \frac{1}{t} \cdot \sqrt{\frac{t}{4}} \cdot \frac{\ell}{p}$. So, $E[|S_i|] \geq \frac{|S_i|}{nt} \sqrt{\frac{t}{4}} \cdot \frac{\ell}{p}$. Using the facts that $|S| = \Theta \left( \frac{\sqrt{n}}{\sqrt{q}} \cdot \epsilon^{-4.5} \cdot \log^2 (\rho n) \cdot \log(1/\epsilon) \right)$ and $t = \lceil \log_{1+\epsilon/8}(\rho n) \rceil$, then applying Chernoff bound as mentioned in Lemma 1.6(ii) in Appendix D, we get the desired result.

(ii) In this case, $E[|S_i|] < \frac{|S_i|}{nt} \sqrt{\frac{t}{4}} \cdot \frac{\ell}{p}$. Now applying Chernoff bound as mentioned in part (a) of Lemma 1.6(ii) in Appendix D, we get the the desired result.

(iii) By the definition of $\tilde{U}$, \( |\tilde{U}| = \left| \left\{ j \in B_i : \frac{|S_i|}{|S|} < \frac{1}{t} \cdot \frac{\varepsilon}{4} \cdot \frac{\ell}{p} \right\} \right| \). Applying Lemma 1.6(i) and the definition of $L$, we get \( |\tilde{U}| \leq \left| \left\{ j \in B_i : \frac{|B_i|}{|S_i|} < (1 - \frac{\varepsilon}{4})^{-1} \cdot \frac{1}{t} \cdot \frac{\varepsilon}{4} \cdot \frac{\ell}{p} \right\} \right| \). As there are at most $t$ many buckets,

\[ |\tilde{U}| \leq \left| \left\{ j \in B_i : |B_i| < \frac{1}{t} \cdot \frac{\varepsilon}{4} \cdot \frac{\ell}{p} \right\} \right| \leq \sqrt{\frac{t}{4}} \cdot \frac{\ell}{p}. \]

(iv) From the description of the algorithm, for every $i \in \tilde{L}$, we select $|S_i|$ many samples uniformly at random from $S_i$, with replacement, and let $Z_i$ be the set of samples obtained. For each $z \in Z_i$, we make a REGR$(z, 1)$ query and let $A_{z,k} = \text{REGR}(z, 1)$. Let $Y_z$ be a random variable that takes value 1 if $k_z \in \tilde{U}$ and 0, otherwise. Also, \( \tilde{\alpha}_i = \frac{\sum_{u \in B_i} Y_u}{|B_i|} \).

Using the fact that we choose $S$ independently and uniformly at random, set $S_1 = S \cap B_i$ and sample the elements in $Z_i$ from $S_1$, we get

\[ E[Y_z] = \frac{|S_i|}{|B_i|} \cdot \frac{1}{|S_i|} \sum_{u \in B_i} \langle A_{u,1} \rangle = \frac{1}{|B_i|} \cdot \sum_{u \in B_i} \langle A_{u,1} \rangle. \]

So, $E[\tilde{\alpha}_i] = \sum_{u \in B_i} \frac{1}{|B_i|} \cdot \frac{\langle A_{u,1} \rangle}{\langle A_{u,1} \rangle}$. Since $u \in B_i$, $(1 + \beta)^{i-1} \leq \langle A_{u,1} \rangle < (1 + \beta)^i$. Also, by Fact 1,

\[ |B_i| (1 + \beta)^{i-1} \leq \sum_{u \in B_i} \langle A_{u,1} \rangle \leq |B_i| (1 + \beta)^i. \]

Thus,

\[ \sum_{u \in B_i} \langle A_{u,1} \rangle \leq E[\tilde{\alpha}_i] \leq \frac{\sum_{u \in B_i} \langle A_{u,1} \rangle}{{|B_i|}(1 + \beta)^{i-1}}. \]

or

\[ \frac{1}{1 + \beta} \cdot \sum_{u \in B_i} \langle A_{u,1} \rangle \leq E[\tilde{\alpha}_i] \leq \frac{\sum_{u \in B_i} \langle A_{u,1} \rangle}{{|A_{u,1}|}(1 + \beta)}. \]

Using $\alpha_i = \frac{\sum_{u \in B_i} \langle A_{u,1} \rangle}{{|A_{u,1}|}}$ (Definition 1.6) and the fact that $\beta \leq \epsilon/8$, we get

\[ (1 - \frac{\epsilon}{4}) \alpha_i \leq E[\tilde{\alpha}_i] \leq (1 + \frac{\epsilon}{4}) \alpha_i. \]
Proof. (i) Recall that 
\[ m \leq \hat{m} \leq (1 + \frac{\epsilon}{4}) m, \] 
where \( m = 1^T A 1 \).

### Claim 4.7

With high probability, we have,

(i) \( \hat{m} \geq (1 - \frac{\epsilon}{4}) (m - \frac{\epsilon}{4} \ell), \) and

(ii) \( \hat{m} \leq (1 + \frac{3\epsilon}{4}) m \).

Proof. (i) Recall that \( \hat{m} = \frac{1}{K} \sum_{i \in L} (1 + \tilde{\alpha}_i) \cdot |S_i| \cdot (1 + \beta)^i \) is the estimate returned by Algorithm 2.

Using Lemma 4.6 (i), we have

\[ \hat{m} \geq \sum_{i \in L} (1 + \tilde{\alpha}_i) \left( 1 - \frac{\epsilon}{4} \right) (1 + \beta)^i |B_i| \]

\[ = \left( 1 - \frac{\epsilon}{4} \right) \sum_{i \in L} (1 + \tilde{\alpha}_i) (1 + \beta)^i |B_i| \]

\[ = \left( 1 - \frac{\epsilon}{4} \right) \left( \sum_{i \in L, \alpha_i \geq \epsilon/8} (1 + \tilde{\alpha}_i) (1 + \beta)^i |B_i| + \sum_{i \in L, \alpha_i < \epsilon/8} (1 + \tilde{\alpha}_i) (1 + \beta)^i |B_i| \right) \]

\[ \geq \left( 1 - \frac{\epsilon}{4} \right) \left( \sum_{i \in L, \alpha_i \geq \epsilon/8} (1 + (1 - \frac{\epsilon}{4}) \alpha_i) (1 + \beta)^i |B_i| + \sum_{i \in L, \alpha_i < \epsilon/8} (1 - \frac{\epsilon}{4})(1 + \frac{\epsilon}{4}) (1 + \beta)^i |B_i| \right) \]

\[ > \left( 1 - \frac{\epsilon}{4} \right) \left( \sum_{i \in L, \alpha_i \geq \epsilon/8} ((1 - \frac{\epsilon}{4})(1 + \alpha_i)) (1 + \beta)^i |B_i| + \sum_{i \in L, \alpha_i < \epsilon/8} (1 - \frac{\epsilon}{4})(1 + \alpha_i) (1 + \beta)^i |B_i| \right) \]

\[ = \left( 1 - \frac{\epsilon}{4} \right)^2 \sum_{i \in L} (1 + \alpha_i) (1 + \beta)^i |B_i| \]

\[ \geq \left( 1 - \frac{\epsilon}{4} \right)^2 \sum_{i \in L} \left( 1 + \alpha_i \right) \sum_{k \in B_i} \langle A_{k*}, 1 \rangle \]

Using \( \sum_{k \in B_i} \langle A_{k*}, 1 \rangle = \alpha_i \sum_{k \in B_i} \langle A_{k*}, 1 \rangle \) we have:

\[ \hat{m} \geq \left( 1 - \frac{\epsilon}{4} \right)^2 \sum_{i \in L} \left( 1 + \alpha_i \right) \sum_{k \in B_i} \langle A_{k*}, 1 \rangle \]

\[ = \left( 1 - \frac{\epsilon}{4} \right)^2 \sum_{i \in L} \left( \sum_{k \in B_i} \langle A_{k*}, 1 \rangle + \alpha_i \sum_{k \in B_i} \langle A_{k*}, 1 \rangle \right) \]

\[ = \left( 1 - \frac{\epsilon}{4} \right)^2 \sum_{i \in L} \left( \sum_{k \in B_i} \langle A_{k*}, 1 \rangle + \sum_{k \in B_i} \langle A_{k*}, 1 \rangle \right) + \alpha_i \sum_{k \in B_i} \langle A_{k*}, 1 \rangle \]

\[ = \left( 1 - \frac{\epsilon}{4} \right)^2 \left( \sum_{i \in L} \sum_{k \in B_i} \langle A_{k*}, 1 \rangle + 2 \sum_{k \in B_i} \langle A_{k*}, 1 \rangle \right) \]

\[ = \left( 1 - \frac{\epsilon}{4} \right)^2 \left( \sum_{i \in L} \sum_{k \in B_i} \langle A_{k*}, 1 \rangle + 2 \sum_{i \in L} \sum_{k \in B_i} \langle A_{k*}, 1 \rangle \right) \]

\[ = \left( 1 - \frac{\epsilon}{4} \right)^2 \left( 1^T A \bar{V} \bar{V}^T 1 - 2 \bar{V}^T A \bar{V} \bar{U} \bar{U} \right) \]

\[ = \left( 1 - \frac{\epsilon}{4} \right)^2 \left( 1^T A 1 - 1^T A \bar{U} \bar{U} \right) \]
Since |\(\hat{U}\)| $< \sqrt{\frac{m}{\ell}}$ (From Lemma 4.6 (iii)) and $\forall i, j \in [n]$, $|A_{ij}| \leq \rho$ we have $1^T_{\hat{U}}A_{\hat{U},\hat{U}}1_{\hat{U}} \leq \frac{\rho}{10} \cdot \ell$. So, we have $N \geq (1 - \frac{\rho}{10})^2 (m - \frac{\rho}{10} \ell)$.

(ii) Using Lemma 4.6 (i), we have

$$N = \sum_{i \in L} (1 + \alpha_i) + \frac{\rho}{10} \sum_{i \in L} \langle A_{k^*}, 1 \rangle$$

Since $\beta \leq \frac{\rho}{8}$ and $\sum_{i \in B_1} \langle A_{k^*}, 1 \rangle = \alpha_1 \sum_{i \in B_1} \langle A_{k^*}, 1 \rangle$ we have:

$$N \leq (1 + \frac{\rho}{8})^2 (1 + \frac{\rho}{10}) \sum_{i \in L} \langle A_{k^*}, 1 \rangle$$

Recall that we have assumed $m/6 \leq \ell \leq m$. Under this assumption, the above claim says that $N$ is in fact a $(1 \pm \epsilon)$-approximation to $m$. From the description of Algorithm 2 the number of IP queries made by the algorithm is $\tilde{O} \left( \frac{m^2}{\sqrt{\ell} \cdot A_1} \right) = \tilde{O} \left( \frac{m}{\sqrt{\ell} \cdot A_1} \right)$ as $m/10 \leq \ell \leq m$. We will discuss how to remove the assumption, that $m/6 \leq \ell \leq m$, by using a standard technique in property testing in the following.
4.2 Proof of Theorem 4.1. How to remove the assumption that \( \ell \) is a correct lower bound on \( m = 1^T A1 \)?

Algorithm 3: \( \text{BFE} (\epsilon) \)

Input: \( \epsilon \in (0, 1) \).
Output: \( \hat{m} \), which is an \((1 \pm \epsilon)\)-approximation of \( 1^T A1 \).

1. \begin{align*}
\text{Initialize } \ell & = \rho n^2/2. \\
\text{while } (\ell \geq 1) & \text{ do} \\
\text{Call BFE } (\ell, \epsilon) \text{ (Algorithm 2) and let } \hat{m}_{\ell} & \text{ be the output.} \\
\text{if } (\ell \leq \frac{\hat{m}_{\ell}}{1 + \frac{3\epsilon}{4}}) & \text{ then} \\
\quad & \text{Report } \hat{m}_{\ell} \text{ as the output and QUIT.} \\
\text{else} & \text{ Set } \ell \text{ as } \ell/2 \text{ and CONTINUE.} \\
\end{align*}

2. \begin{align*}
\text{end} \\
\text{Compute } m = 1^T A1 \text{ exactly, by making } n \text{ many IP queries, and return it as } \hat{m}. \\
\end{align*}

Algorithm 3 (\( \text{BFE} (\epsilon) \)) is our algorithm to determine an \((1 \pm \epsilon)\) approximation to \( m = 1^T A1 \) with high probability. Note that \( \text{BFE} (\epsilon) \) calls Algorithm 2 (\( \text{BFE} (\ell, \epsilon) \)) recursively at most \( \mathcal{O}(\log(\rho n^2)) \) times for different values of \( \ell \) until we have \( \ell \leq \frac{\hat{m}_{\ell}}{1 + \frac{3\epsilon}{4}} \).

Observation 4.8. \( \text{BFE} (\epsilon) \) does not QUIT as long as \( \ell > m \), with high probability.

Proof. Consider a fix \( \ell \) with \( \ell > m \). By Claim 2 (ii), \( \hat{m}_{\ell} \leq (1 + \frac{3\epsilon}{4}) m \) with high probability. As \( \ell > m \), \( \hat{m}_{\ell} < (1 + \frac{3\epsilon}{4}) \ell \) with high probability. This implies \( \ell > \frac{\hat{m}_{\ell}}{1 + \frac{3\epsilon}{4}} \), that is, \( \text{BFE} (\epsilon) \) does not QUIT for this fixed \( \ell \) with high probability. As there can be at most \( \mathcal{O}(\log(\rho n^2)) \) many \( \ell \)'s with \( \ell > m \) such that \( \text{BFE} (\epsilon) \) calls \( \text{BFE} (\ell, \epsilon) \), we are done with the proof. \( \square \)

Observation 4.9. Let \( \ell \leq m/3 \). \( \text{BFE} (\epsilon) \) quits and reports \( \hat{m}_{\ell} \) as the output with high probability, where \( \hat{m}_{\ell} \) denotes the output of \( \text{BFE} (\ell, \epsilon) \).

Proof. By Claim 2, with high probability, \((1 - \frac{\epsilon}{2}) (m - \frac{\epsilon}{4} \ell) \leq \hat{m} \leq (1 + \frac{3\epsilon}{4}) m \). As \( \ell \leq m/3 \), with high probability, we have \((1 - \frac{\epsilon}{2}) m \leq \hat{m} \leq (1 + \frac{3\epsilon}{4}) m \). So, \( \ell \leq \frac{\hat{m}}{1 + \frac{3\epsilon}{4}} \) with high probability. As \( \epsilon \in (0, \frac{1}{2}) \) (See the statement of Theorem 2), we have \( \ell \leq \frac{\hat{m}}{1 + \frac{3\epsilon}{4}} \) with high probability. Hence, \( \text{BFE} (\epsilon) \) quits and reports \( \hat{m}_{\ell} \) as the output with high probability. \( \square \)

From Observation 4.8 and 4.9 \( \text{BFE} (\epsilon) \) does not quit when \( \text{BFE} (\ell, \epsilon) \) is called for any \( \ell > m \) and quits when \( \text{BFE} (\ell, \epsilon) \) is called for some \( m/6 \leq \ell \leq m \), with high probability. As \( \text{BFE} (\epsilon) \) reduces \( \ell \) by a factor of 2 each time it does not quit, \( \text{BFE} (\ell, \epsilon) \) is called \( \mathcal{O}(\log(\rho n^2)) \) times. The number of queries made by \( \text{BFE} (\epsilon) \) for a call to \( \text{BFE} (\ell, \epsilon) \) is \( \tilde{O} \left( \frac{\sqrt{\tau n}}{\sqrt{m}} \right) \). Observe that the query complexity of \( \text{BFE} (\epsilon) \) is dominated by the query complexity of last call to \( \text{BFE} (\ell, \epsilon) \), that is, when \( m/6 \leq \ell \leq m \). Hence, the total number of queries made by \( \text{BFE} (\epsilon) \) is \( \tilde{O} \left( \frac{\sqrt{\tau n}}{\sqrt{m}} \right) \).

4.3 Algorithm for Sample Almost Uniformly

In this section, we will be proving the following theorem on almost uniformly sampling the entries of a symmetric matrix \( A \in [\rho]_{n \times n} \). In Appendix C, we show how this algorithm can be extended to solve the more general \( \text{SAU}_A(X, Y) \) problem.
Theorem 4.10. Let $A \in [\rho]^{n \times n}$ be an unknown symmetric matrix with IP query access. There exists an algorithm that takes $\epsilon \in (0, 1)$ as input and with high probability outputs a sample from a distribution on $[n] \times [n]$, such that each $(i, j) \in [n] \times [n]$ is sampled with probability $p_{ij}$ satisfying:

$$(1 - \epsilon) \sum_{1 \leq j \leq n} A_{ij} \leq p_{ij} \leq (1 + \epsilon) \sum_{1 \leq j \leq n} A_{ij}.$$ 

Moreover, the algorithm makes $\tilde{O}\left(\frac{\rho n}{\sqrt{1 + T} A_1}\right)$ IP queries to the matrix $A$ of the form $\langle A_k, u \rangle$ for some $k \in [n]$ and $u \in \{0, 1\}^n$.

Our algorithm for Sample Almost Uniformly is a generalization of Eden and Rosenbaum’s algorithm for sampling edges of an unweighted graph [ER18]. First, consider the following strategy by which we sample each ordered pair $(i, j) \in [n] \times [n]$ proportional to $A_{ij}$ when the matrix $A$ is such that $\langle A_i, 1 \rangle$ is the same for each $i \in [n]$.

**Strategy-1:** Sample $r \in [n]$ uniformly at random and then sample an ordered pair of the form $(r, j)$ from the $r$-th row using REGR query. Observe that this strategy fails when $\langle A_i, 1 \rangle$’s are not the same for every $i \in [n]$. So, the modified strategy is as follows.

**Strategy-2:** Sample $r \in [n]$ with probability $\frac{\langle A_r, 1 \rangle}{1^T A_1}$ and then sample an ordered pair of the form $(r, j)$ from the $r$-th row by using REGR query.

Note that Strategy-2 samples each ordered pair $(i, j)$ proportional to $A_{ij}$. However, there are two challenges in executing Strategy-2:

(i) We do not know the value of $1^T A_1$.

(ii) We need $\Omega(n)$ queries to determine $\langle A_r, 1 \rangle$ for each $r \in [n]$ for all $r \in [n]$.

The first challenge can be taken care of by finding an estimate $\hat{m}$ for $1^T A_1$, with high probability, by using Theorem 4.1 such that $\hat{m} = \Theta(1^T A_1)$. To cope up with the second challenge, we partition the elements as well as rows into two classes as defined in Definition 4.11. In what follows, we consider a parameter $\tau$ in terms of which we base our discussion as well as algorithm. $\tau$ is a function of $\hat{m}$ that will evolve over the calculation and will be $\tau = \sqrt{\frac{2m}{\tau}}$.

**Definition 4.11.** The $i$-th row of the matrix is light if $\langle A_i, 1 \rangle$ is at most $\tau$. Otherwise, the $i$-th row is heavy. Any order pair $(i, j)$, for a fixed $i$, is light (heavy) if the $i$-th row is light (heavy).

We denote the set of all light (heavy) ordered pairs by $L$ ($H$). Also, let $I(L)$ ($I(H)$) denote the set of light (heavy) rows of the matrix $A$. Let $w(L) = \sum_{A_{ij} \in L} A_{ij}$ and $w(H) = \sum_{A_{ij} \in H} A_{ij}$.

Our algorithm consists of repeated invocation of two subroutines, that is, Sample-Light and Sample-Heavy. Both Sample-Light and Sample-Heavy succeed with good probability and sample elements from $L$ and $H$ almost uniformly, respectively. The threshold $\tau$ is set in such a way that there are large$^8$ number of light rows and small number of heavy rows. In Sample-Light, we select a row uniformly at random, and if the selected row is light, then we sample an ordered pair from the selected row randomly using REGR. This gives us an element from $L$ uniformly. However, the same technique will not work for Sample-Heavy as we have few heavy rows. To cope up with this problem, we take a row uniformly at random and if the selected row is light, we sample an ordered pair from the selected row randomly using REGR. Let $(i, j)$ be the output of the REGR query. Then we go to the $j$-th row, if it is heavy, and then select an ordered pair from the $j$-th row randomly using REGR query.

The formal algorithms for Sample-Light and Sample-Heavy are given in Algorithm 4 and Algorithm 5, respectively. The formal correctness proofs of Sample-Light and Sample-Heavy are given in Lemmas 4.12 and 4.13, respectively. We give the final algorithm along with its proof of correctness in Theorem 4.10.

$^8$Large is parameterized by $\tau$. 

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Algorithm 4: SAMPLE-LIGHT

Input: An estimate $\hat{m}$ for $1^T A 1$ and a threshold $\tau$.
Output: $(i, j) \in L$ with probability $\frac{A_{ij}}{n\tau}$.

1 begin
2 Select a row $r \in [n]$ uniformly at random.
3 if $(r \in I(L), \text{ that is, } \langle A_r, 1 \rangle \text{ is at most } \tau)$ then
4 Return FAIL with probability $p = \frac{\tau - \langle A_r, 1 \rangle}{\tau}$, and Return REGR$(r, 1)$ with probability $1 - p$ as the output.
5 end
6 Return FAIL
7 end

Lemma 4.12. SAMPLE-LIGHT succeeds with probability $\frac{w(L)}{n\tau}$. Let $Z_L$ be the output in case it succeeds. Then $\mathbb{P}(Z_L = (i, j)) = \frac{A_{ij}}{n\tau}$ if $(i, j) \in L$, and $\mathbb{P}(Z_L = (i, j)) = 0$, otherwise. Moreover, SAMPLE-LIGHT makes $O(\log n)$ many queries.

Proof. Consider an ordered pair $(i, j) \in L$. The probability of $(i, j)$ returned by SAMPLE-LIGHT is

$$\mathbb{P}(Z_L = (i, j)) = \mathbb{P}(r = i) \cdot \mathbb{P}($$ SAMPLE-LIGHT returns REGR$(r, 1)$ $) \cdot \mathbb{P}($ REGR$(r, 1)$ returns $(i, j)$ $) = \frac{1}{n} \cdot \frac{\langle A_{rs}, 1 \rangle}{\tau} \cdot A_{ij} = \frac{A_{ij}}{n\tau}$$

Hence, the probability that SAMPLE-LIGHT does not return FAIL is $\sum_{A_{ij} \in L} \frac{A_{ij}}{n\tau} = \frac{w(L)}{n\tau}$. Now the query complexity of SAMPLE-LIGHT follows from the query complexity of REGR given in Lemma 4.2.

Lemma 4.13. SAMPLE-HEAVY succeeds with probability at most $\frac{w(H)}{n\tau}$ and at least $\left(1 - \frac{\rho \hat{m}}{\tau^2}\right) \frac{w(H)}{n\tau}$. Let $Z_L$ be the output in case it succeeds. Then $\left(1 - \frac{\rho \hat{m}}{\tau^2}\right) \frac{A_{ij}}{n\tau} \leq \mathbb{P}(Z_L = (i, j)) \leq \frac{A_{ij}}{n\tau}$ for each $(i, j) \in H$, and $\mathbb{P}(Z_L = (i, j)) = 0$, otherwise. Moreover, SAMPLE-HEAVY makes $O(\log n)$ many queries.

Proof. For each $k \in I(H)$, note that, $\langle A_{ks}, 1 \rangle$ is more than $\tau$. So,

$$|I(H)| \leq \frac{1^T A 1}{\tau} \leq \frac{\hat{m}}{\tau}.$$

Note that

$$\langle A_{ks}, 1 \rangle = \sum_{u \in I(L)} A_{ku} + \sum_{v \in I(H)} A_{kv}.$$

Observe that

$$\sum_{v \in I(H)} A_{kv} \leq \rho |I(H)| \leq \frac{\rho \hat{m}}{\tau} \leq \frac{\rho \hat{m} \langle A_{ks}, 1 \rangle}{\tau^2}.$$

So, we have the following Observation.

Observation 4.14. $\sum_{u \in I(L)} A_{ku} \geq \left(1 - \frac{\rho \hat{m}}{\tau^2}\right) \langle A_{ks}, 1 \rangle$, where $k \in I(H)$. 

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Let us consider some ordered pair \((i, j) \in \mathcal{H}\). The probability that \((i, j)\) is returned by the algorithm is

\[
P(Z_h = (i, j)) = \mathbb{P}(s = i) \cdot \mathbb{P}(\text{REGR}(s, 1) \text{ returns } (i, j))
\]

\[
= \left( \sum_{u \in I(\mathcal{L})} \mathbb{P}(r = u) \cdot \frac{\langle A_{rs}, 1 \rangle}{\tau} \cdot \mathbb{P}(\text{REGR}(r, 1) \text{ returns } (r, i)) \right) \cdot \frac{A_{ij}}{\langle A_{is}, 1 \rangle}
\]

\[
= \frac{1}{n} \cdot \frac{A_{ij}}{\langle A_{is}, 1 \rangle} \cdot \sum_{u \in I(\mathcal{L})} \frac{\langle A_{us}, 1 \rangle}{\tau} \cdot \frac{A_{ui}}{\langle A_{us}, 1 \rangle} = \frac{A_{ij}}{n \tau} \sum_{u \in I(\mathcal{L})} A_{iu} \frac{\langle A_{us}, 1 \rangle}{\langle A_{is}, 1 \rangle} \tau \cdot \frac{\langle A_{is}, 1 \rangle}{A_{ij}}
\]

The last equality follows from the fact that \(A\) is a symmetric matrix.

Using the fact that \(\sum_{u \in I(\mathcal{L})} A_{iu} \leq \langle A_{i*}, 1 \rangle\) and Observation 4.14, we have

\[
1 - \rho \frac{\hat{m}}{\tau^2} \leq \frac{1}{\langle A_{i*}, 1 \rangle} \sum_{u \in I(\mathcal{L})} A_{iu} \leq 1.
\]

Putting everything together, we get

\[
\left(1 - \rho \frac{\hat{m}}{\tau^2}\right) \cdot \frac{A_{ij}}{n \tau} \leq \mathbb{P}(Z_h = (i, j)) \leq \frac{A_{ij}}{n \tau}.
\]

So, the probability that \textsc{Sample-Heavy} succeeds is \(\sum_{A_{ij} \in \mathcal{H}} \mathbb{P}(Z = (i, j))\), which lies between \(\left(1 - \rho \frac{\hat{m}}{\tau^2}\right) \frac{w(\mathcal{H})}{n \tau}\) and \(\frac{w(\mathcal{H})}{n \tau}\). The query complexity of \textsc{Sample-Heavy} follows from the query complexity of \textsc{REGR} (Lemma 4.2).

\[\square\]

**Algorithm 5: \textsc{Sample-Heavy} (\hat{m})**

**Input:** An estimate \(\hat{m}\) for \(1^T A 1\) and a threshold \(\tau\).

**Output:** \(A_{ij} \in \mathcal{H}\) with probability at most \(\frac{A_{ij}}{n \tau}\) and at least \(\left(1 - \rho \frac{\hat{m}}{\tau^2}\right) \frac{A_{ij}}{n \tau}\).

1 begin
2 Select a row \(r \in [n]\) uniformly at random;
3 if \((r \in I(\mathcal{L}), \text{ that is, } \langle A_{rs}, 1 \rangle \text{ is at most } \tau)\) then
4 Return \textsc{FAIL} with probability \(p = \tau \frac{\langle A_{rs}, 1 \rangle}{\langle A_{is}, 1 \rangle}\), and with probability \(1 - p\) do the following;
5 \(A_{rs} = \text{REGR}(r, 1)\)
6 If \(s \in I(\mathcal{H}), \text{ that is, } \langle A_{ss}, 1 \rangle > \tau\), then Return \textsc{REGR}(s, 1) as the output.
7 Otherwise, Return \textsc{FAIL};
8 end
9 end

Now we will prove Theorem 4.10.

**Proof of Theorem 4.10.** Our algorithm first finds a rough estimate \(\hat{m}\) for \(1^T A 1\), with high probability, by using Theorem 4.1 such that \(\hat{m} = \Theta(1^T A 1)\). For the rest of the proof, we work on the conditional probability space that \(\hat{m} = \Theta(1^T A 1)\). We set \(\tau = \sqrt{\rho \hat{m}}\) and do the following for \(\Gamma\) times, where \(\Gamma\) is a parameter to be set later. With probability \(1/2\), we invoke \textsc{Sample-Light} and with probability \(1/2\), we invoke \textsc{Sample-Heavy}. If the ordered pair \((i, j)\) is reported as
the output by either Sample-Light or Sample-Heavy, we report that. If we get Fail in all the trials, we report Fail.

Now, let us consider a particular trial and compute the probability of success \( \mathbb{P}(S) \), which is \( \mathbb{P}(S) = \frac{1}{4}(\mathbb{P}('\text{Sample-Light} succeeds') + \mathbb{P}('\text{Sample-Heavy succeeds}')). \) Observe that from Lemmas 4.12 and 4.13, we have, \( \frac{1}{7} \left\{ \frac{u(A)}{n^2} + \left( 1 - \frac{\alpha_n}{n^2} \right) \frac{u(H)}{n^2} \right\} \leq \mathbb{P}(S) \leq \frac{1}{7} \left\{ \frac{u(A)}{n^2} + \frac{u(H)}{n^2} \right\} \). This implies \( (1 - \epsilon)^{-\frac{1}{2n^2}} \mathbb{P}(S) \leq \mathbb{P}(S) \leq (1 - \epsilon)^{-\frac{1}{2n^2}} \mathbb{P}(S) \) as \( \epsilon = \sqrt{\frac{\alpha_n}{n^2}} \) and using \( u(L) + u(H) = 1^T A 1 \).

Now, let us compute the probability of the event \( \mathcal{E}_{ij} \), that is, the algorithm succeeds and it returns \( A_{ij} \). If \( A_{ij} \in L \), by Lemma 4.12 we have \( \mathbb{P}(Z = (i, j)) = \frac{1}{2} \cdot \frac{A_{ij}}{n^2} \). Also, if \( A_{ij} \in H \), by Lemma 4.13 we have, \( (1 - \frac{\alpha_n}{n^2}) \frac{A_{ij}}{n^2} \leq \mathbb{P}(Z = (i, j)) \leq \frac{A_{ij}}{n^2} \). So, for any \( (i, j) \), we get \( (1 - \epsilon) \frac{A_{ij}}{2n^2} \mathbb{P}(\mathcal{E}_{ij}) \leq \mathbb{P}(\mathcal{E}_{ij}) \leq \frac{A_{ij}}{2n^2} \). Let us compute the probability of \( \mathcal{E}_{ij} \) on the conditional probability space that the algorithm succeeds, that is, \( \mathbb{P}(Z = (i, j) \mid S) = \frac{\mathbb{P}(\mathcal{E}_{ij})}{\mathbb{P}(S)} \), which lies in the interval \( [\frac{1}{2}, (1 - \epsilon)^{\frac{1}{2n^2}} \cdot (1 + \epsilon) \frac{A_{ij}}{1^T A 1}] \).

To boost the probability of success, we set \( \mathcal{G} = \mathcal{O}\left(\frac{\sqrt{m}}{\sqrt{1 - \epsilon} \sqrt{\epsilon n}} \log n\right) \) for a suitable large constant in \( \mathcal{O}(\cdot) \) notation. The query complexity of each call to Sample-Light and Sample-Heavy is \( \mathcal{O}(\log n) \). Also note that our algorithm for Sample Almost Uniformly makes at most \( \mathcal{O}\left(\frac{\sqrt{m}}{(1 - \epsilon) \sqrt{\epsilon n}} \log n\right) \) many invocation to Sample-Light and Sample-Heavy. Hence, the total query complexity of our algorithm is \( \tilde{\mathcal{O}}\left(\frac{\sqrt{m}}{\sqrt{1 - \epsilon} \sqrt{\epsilon n}} \right) \).

We will first discuss the extensions of our results, from the previous sections, to solve \( \text{Bfe}_A(1, 1) \) and \( \text{Sau}_A(1, 1) \) when \( A \in [\mu]^{n \times n} \) is not necessarily a symmetric matrix. Consider the symmetric matrix \( B = \frac{A^T + A}{2} \). Using Theorem 4.1, we can solve \( \text{Bfe}_B(1, 1) \) and \( \text{Sau}_B(1, 1) \) only using row IP queries on the matrix \( B \). Observe that one can simulate the row IP access for \( B \) by using two IP queries to \( A \) as \( \langle B_{ix}, v \rangle = \langle A_{ix}, v \rangle + \langle A_{ix}, v \rangle \). So, we can solve both \( \text{Bfe}_B(1, 1) \) and \( \text{Sau}_B(1, 1) \) using \( \tilde{\mathcal{O}}\left(\frac{\sqrt{m}}{\sqrt{x^T By}} \right) \) many row and column IP queries to the matrix \( A \). Note that the query algorithms for both \( \text{Bfe}_B(1, 1) \) and \( \text{Sau}_B(1, 1) \) problems uses queries of the form \( \langle A_{ix}, v \rangle \) or \( \langle A_{ix}, v \rangle \) where \( i \in [n] \) and \( v \in \{0, 1\}^n \).

Now, let us return to the problems \( \text{Bfe}_A(1, 1) \) and \( \text{Sau}_A(1, 1) \) that we wanted to solve. We will now convert the query algorithms for matrix \( B \) to algorithms for the matrix \( A \).

- As \( 1^T A 1 = 1^T B 1 \), we can directly use the query algorithm for \( \text{Bfe}_B(1, 1) \) for the problem \( \text{Bfe}_A(1, 1) \).

- Now consider the problem \( \text{Sau}_A(1, 1) \). We will first run the query algorithm for \( \text{Sau}_B(1, 1) \). Suppose the algorithm outputs \( (i, j) \in [n] \times [n] \), which happens with probability \( p_{ij} \) where \( p_{ij} \in \left[ (1 - \epsilon) \frac{B_{ij}}{1^T B 1}, (1 + \epsilon) \frac{B_{ij}}{1^T B 1} \right] \). Now we want to use the fact that \( B_{ij} = \frac{A_{ij} + A_{ji}}{2} \). We find \( A_{ij} \) and \( A_{ji} \) by two row IP queries to the matrix \( A \) as \( A_{ij} = \langle A_{ix}, a_i \rangle \) and \( A_{ji} = \langle A_{jx}, a_j \rangle \), where \( a_i \) (\( a_j \)) is the vector in \( \{0, 1\}^n \) having 1 only in the \( i \)-th (\( j \)-th) entry. We report \( (i, j) \) and \( (j, i) \) with probability \( \frac{A_{ij}}{A_{ij} + A_{ji}} \) and \( \frac{A_{ji}}{A_{ij} + A_{ji}} \), respectively. Observe that \( A_{ij} \) can be the output only when \( B_{ij} \) or \( B_{ji} \) is the output of the query algorithm for the problem \( \text{Sau}_B(1, 1) \). Using the facts that \( B_{ij} = B_{ij} = \frac{A_{ij} + A_{ji}}{2} \) and \( 1^T A 1 = 1^T B 1 \), we report \( A_{ij} \) with probability in the range \( \left[ (1 - \epsilon) \frac{A_{ij}}{1^T A 1}, (1 + \epsilon) \frac{A_{ij}}{1^T A 1} \right] \).

- Therefore the query algorithms for \( \text{Bfe}_A(1, 1) \) and \( \text{Sau}_A(1, 1) \) uses \( \tilde{\mathcal{O}}\left(\frac{\sqrt{m}}{\sqrt{x^T By}} \right) = \tilde{\mathcal{O}}\left(\frac{\sqrt{m}}{\sqrt{x^T A y}} \right) \) (as \( 1^T A 1 = 1^T B 1 \)) many row and column IP queries to the matrix \( A \).
Finally, we now consider the general problems $\text{Bfe}_A(x, y)$ and $\text{Sau}_A(x, y)$ where $A \in [\rho]^{n \times n}$ (not necessarily symmetric), $x \in [\gamma_1]$, and $y \in [\gamma_2]$. Observe that $1^T C_1 = x^T A y$, where $C_{ij} = A_{ij}$. So, if we have IP oracle access to the matrix $C$, then we can design query algorithms for $\text{Bfe}_A(x, y)$ and $\text{Sau}_A(x, y)$ problems by using the query algorithms for $\text{Bfe}_C(1, 1)$ and $\text{Bfe}_C(1, 1)$ respectively. Note that, as we have already discussed, that even if $C$ is not symmetric we can still design efficient query algorithms for $\text{Bfe}_C(1, 1)$ and $\text{Bfe}_C(1, 1)$.

But, we do not directly have IP query access to $C$. However, we can simulate any IP query to $C$ by using an appropriate IP query to $A$ by the following observation.

Observation 4.15. $\langle C_{k*}, a \rangle = \langle A_{k*}, a' \rangle$, where $a'_i = x_k a_i y_i$ for each $i \in [n]$. And $\langle C_{s*}, a \rangle = \langle A_{s*}, a' \rangle$, where $a'_i = x_s a_i y_i$ for each $i \in [n]$.

Using Observation 4.15 (i.e., IP query to $C$ can be simulated by IP query to $A$), Theorems 4.11 and 4.10 and the fact that $C_{ij} \in [\rho \gamma_1 \gamma_2]$, we get the following result.

Theorem 4.16. Let $A \in [\rho]^{n \times n}$ be an unknown matrix (not necessarily symmetric) with an IP query access. There exist query algorithms for $\text{Bfe}_A(x, y)$ and $\text{Sau}_A(x, y)$ that takes $x \in [\gamma_1]^n$, $y \in [\gamma_2]^n$, and $\epsilon \in (0, 1/2)$ as inputs, and gives the correct output with probability at least $2/3$, using $\tilde{O}(\sqrt{n^2 \gamma_2^{n+1} + n n z(y)})$ many IP queries to $A$. Additionally, if $x, y \in \{0, 1\}^n$, the query algorithms only uses queries of the form $\langle A_{k*}, u \rangle$ or $\langle A_{s*}, u \rangle$ for some $k \in [n]$ and $u \in \{0, 1\}^n$.

We also show the following lower bound that matches the upper bound of Theorem 4.16.

Theorem 4.17. Let $A \in [\rho]^{n \times n}$ be an unknown matrix with an IP query access. Any algorithm for $\text{Bfe}_A(x, y)$ and $\text{Sau}_A(x, y)$ that takes $x \in [\gamma_1]^n$, $y \in [\gamma_2]^n$, and $\epsilon \in (0, 1/2)$ as inputs, and gives the correct output with probability at least $2/3$, will require $\Omega(\sqrt{n^2 \gamma_2^{n+1} + n n z(y)})$ many IP queries to $A$.

Proof. We show that the stated lower bound for $\text{Bfe}_A(x, y)$ holds even if (i) $A$ is a symmetric matrix, (ii) $x_i = \gamma_1$ for each $i \in [n]$, that is, $n n z(x) = n$, and (iii) $y_i = \gamma_2$ for each $i \in [n]$, that is, $n n z(y) = n$. Without loss of generality assume that $m = o(\gamma_1 \gamma_2 n^2)$.

We prove by giving a reduction from Disjointness in communication complexity: Alice is given $a \in \{0, 1\}^t$ and Bob is given $b \in \{0, 1\}^t$, where $t = \frac{\sqrt{m^2 n^2}}{\sqrt{m}} - 1$. The players want to output 1 if there is an $i \in [t]$ such that $a_i = b_i = 1$. Disjointness admits a randomized communication complexity of $\Omega(t)$ (See Definition 3.3 and Lemma 3.4). Consider a block-diagonal matrix $A \in [\rho]^{n \times n}$ with $A^1, \ldots, A^t, A^{t+1} \in [\rho]^{K \times K}$ diagonal blocks. For $i \in [t]$, if $a_i = b_i = 1$, then $(A^i)_{r,s} = \rho$ for all $r, s \in [K]$; and if $a_i = b_i \neq 1$, then $A^i = 0$. Also, $(A^{t+1})_{r,s} = \rho$ for all $r, s \in [K]$. Observe that, for $x = \gamma_1^n$ and $y = \gamma_2^n$, $x^T A y \geq 2m$ if $a, b$ intersect. Otherwise, $x^T A y = m$.

We will be done with the stated lower bound proof for $\text{Bfe}_A(x, y)$ by showing that Alice and Bob can determine the answer to any IP query, to matrix $A$, with 2 bits of communication. Let $\langle A_{i*}, v \rangle$ be an IP query. From the construction of matrix $A$, there exists a block diagonal matrix $A^j$, $j \in [t + 1]$, such that row $A^j$ can completely determined if we know $A^j$. If $j = t + 1$, then Alice and Bob need not communicate as it is known that $(A^{t+1})_{r,s} = \rho$ for all $r, s \in [K]$.

If $1 \leq j \leq t$, $A^j$ depends on $x_j$ and $y_j$. So, Alice and Bob can determine matrix $A^j$, and hence, $\langle A_{i*}, v \rangle$ with 2 bits of communication. Similar argument holds for any IP query of the form $\langle A_{i*}, v \rangle$.

Recall the argument that we used to prove the lower bound for Induced Edge Sampling (Theorem 3.2) by using the construction of the lower bound proof of Induced Edge Estimation (Theorem 3.1). One can use similar argument to show the stated lower bound for $\text{Sau}_A(x, y)$ by using the construction of our lower bound proof for $\text{Bfe}_A(x, y)$. \square
5 Conclusion and discussions

5.1 Other Matrix Problems

Recently, vector-matrix query \cite{SWYZ19b} and vector-matrix-vector query \cite{RWZ20} were introduced to study a bunch of matrix, graph and statistics problems. As noted earlier, IP query oracle is in the same linear algebraic framework of vector-matrix query and vector-matrix-vector query, but these queries are stronger than IP query. Study of the various matrix, graph and statistics problems, introduced in \cite{SWYZ19b, RWZ20}, using IP query will be of independent interest. As a first step in that direction, in Appendix B we study the query complexity of the following problems using IP queries.

- **Symmetric Matrix**: Is $A \in \{0,1\}^{n \times n}$ a symmetric matrix?
- **Diagonal Matrix**: Is $A$ a diagonal matrix?
- **Trace**: Compute the trace of the matrix $A$.
- **Permutation Matrix**: Is $A \in \{0,1\}^{n \times n}$ a permutation matrix?
- **Doubly stochastic matrix**: Is $A \in \{0,1\}^{n \times n}$ a doubly stochastic matrix?
- **Identical columns**: Does there exist two columns in $A \in \{0,1\}^{n \times n}$ that are identical?
- **All ones column**: Does there exist a column in $A$ all of whose entries are 1?

Table 2 (in Appendix B) gives the query complexity of solving the above matrix problems using IP oracle. We also do a comparative study of IP with respect to these stronger queries.

5.2 Data structure complexity and open problems

**Data structure complexity**: Besides property testing, there have been extensive work concerning vector-matrix-vector product in data structure complexity and other models of computation like the cell probe model \cite{CKL18, CKLM18, DLM20, LW17, RR20}. For the purposes of this paper, it is an interesting question to find a pre-processing scheme for the matrix such the IP queries on the matrix can be answered efficiently.

**Open questions**: One of the open problem (that is left from our work) is to design algorithm and/or prove lower bound for BILINEAR FORM ESTIMATION and SAMPLING when the entries of the matrices are not necessarily positive. Here we would like to that our technique does not work for matrix with both positive and negative entries.

Some other natural open question are

- Are there some special kind of matrices where we can solve BILINEAR FORM ESTIMATION and SAMPLING using fewer queries?
- Can we solve some other linear algebraic problems using IP queries?
- Are there other graph problems, where INDUCED DEGREE outperforms LOCAL queries?
References

[ABG+18] M. Aliakbarpour, A. S. Biswas, T. Gouleakis, J. Peebles, R. Rubinfeld, and A. Yodpinyanee. Sublinear-Time Algorithms for Counting Star Subgraphs via Edge Sampling. *Algorithmica*, 80(2):668–697, 2018.

[ACK21] Sepehr Assadi, Deeparnab Chakrabarty, and Sanjeev Khanna. Graph connectivity and single element recovery via linear and OR queries. In Petra Mutzel, Rasmus Pagh, and Grzegorz Herman, editors, *29th Annual European Symposium on Algorithms, ESA 2021, September 6-8, 2021, Lisbon, Portugal (Virtual Conference)*, volume 204 of LIPIcs, pages 7:1–7:19. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[AGM12] Kook Jin Ahn, Sudipto Guha, and Andrew McGregor. Analyzing Graph Structure via Linear Measurements. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms, SODA*, pages 459–467, 2012.

[AKK19] S. Assadi, M. Kapralov, and S. Khanna. A Simple Sublinear-Time Algorithm for Counting Arbitrary Subgraphs via Edge Sampling. In *ITCS*, pages 6:1–6:20, 2019.

[BEG+12] F. Bonchi, P. Esfandiar, D. F. Gleich, C. Greif, and L. V.S. Lakshmanan. Fast Matrix Computations for Pairwise and Columnwise Commute Times and Katz Scores. *Internet Mathematics*, 1-2(8):73–112, 2012.

[BGK+18] A. Bishnu, A. Ghosh, S. Kolay, G. Mishra, and S. Saurabh. Parameterized Query Complexity of Hitting Set Using Stability of Sunflowers. In *ISAAC*, pages 25:1–25:12, 2018.

[BGM21] Arijit Bishnu, Arijit Ghosh, and Gopinath Mishra. Distance Estimation Between Unknown Matrices Using Sublinear Projections on Hamming Cube. In *Accepted in International Conference on Randomization and Computation, RANDOM*, 2021.

[BHR+18] P. Beame, S. Har-Peled, S. N. Ramamoorthy, C. Rashtchian, and M. Sinha. Edge Estimation with Independent Set Oracles. In *ITCS*, pages 38:1–38:21, 2018.

[BK13] M. Benzi and C. Klymko. Total Communicability as a Centrality Measure. *J. Complex Networks*, 1:124—-149, 2013.

[BKKR08] Ido Ben-Eliezer, Tali Kaufman, Michael Krivelevich, and Dana Ron. Comparing the strength of query types in property testing: the case of testing k-colorability. In Shang-Hua Teng, editor, *Proceedings of the Nineteenth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2008, San Francisco, California, USA, January 20-22, 2008*, pages 1213–1222. SIAM, 2008.

[BLWZ19] M.-F. Balcan, Y. Li, D. P. Woodruff, and H. Zhang. Testing Matrix Rank, Optimaly. In *SODA*, pages 727–746, 2019.

[CKL18] D. Chakraborthy, L. Kamma, and K. G. Larsen. Tight Cell Probe Bounds for Succinct Boolean Matrix-Vector Multiplication. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC*, pages 1297–1306, 2018.

[CKLM18] A. Chattopadhyay, M. Koucký, B. Loff, and S. Mukhopadhyay. Simulation Beats Richness: New Data-Structure Lower Bounds. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing, STOC*, pages 1013–1020, 2018.
[cud] NVIDIA Developer: Cg 3.1 Toolkit Documentation. [https://developer.download.nvidia.com/cg/dot.html]

[DLM20] H. Dell, J. Lapinskas, and K. Meeks. Approximately Counting and Sampling Small Witnesses using a Colourful Decision Oracle. In SODA, pages 2201–2211, 2020.

[DP09] D. Dubhashi and A. Panconesi. Concentration of Measure for the Analysis of Randomized Algorithms. Cambridge University Press, 1st edition, 2009.

[EH10] E. Estrada and D. J. Higham. Network Properties Revealed through Matrix Functions. SIAM Rev., 52:696–714, 2010.

[ELRS17] T. Eden, A. Levi, D. Ron, and C. Seshadhri. Approximately Counting Triangles in Sublinear Time. SICOMP, 46(5):1603–1646, 2017.

[ER18] T. Eden and W. Rosenbaum. On Sampling Edges Almost Uniformly. In SOSA, pages 7:1–7:9, 2018.

[ERS18] T. Eden, D. Ron, and C. Seshadhri. On Approximating the Number of k-Cliques in Sublinear Time. In STOC, pages 722–734, 2018.

[Fei06] U. Feige. On Sums of Independent Random Variables with Unbounded Variance and Estimating the Average Degree in a Graph. SICOMP, 35(4):964–984, 2006.

[FMR14] P. Fika, M. Mitrouli, and P. Roupa. Estimates for the Bilinear Form $x^T A^{-1} y$ with Applications to Linear Algebra Problems. Electronic Transactions on Numerical Analysis, 43:70–89, 2014.

[Gol17] O. Goldreich. Introduction to Property Testing. Cambridge University Press, 2017.

[GR08] O. Goldreich and D. Ron. Approximating Average Parameters of Graphs. Random Structures & Algorithms, 32(4):473–493, 2008.

[HP12] John L. Hennessy and David A. Patterson. Computer Architecture - A Quantitative Approach (5. ed.). Morgan Kaufmann, 2012.

[KN97] E. Kushilevitz and N. Nisan. Communication Complexity. Cambridge University Press, 1997.

[LW17] K. G. Larsen and R. R. Williams. Faster Online Matrix-Vector Multiplication. In Philip N. Klein, editor, Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA, pages 2182–2189, 2017.

[Nis19] Noam Nisan. The demand query model for bipartite matching. CoRR, abs/1906.04213, 2019.

[RR20] S. N. Ramamoorthy and C. Rashtchian. Equivalence of Systematic Linear Data Structures and Matrix Rigidity. In Proceedings of the 11th Innovations in Theoretical Computer Science Conference, ITCS, volume 151, pages 35:1–35:20, 2020.

[RSW18] A. Rubinstein, T. Schramm, and S. M. Weinberg. Computing Exact Minimum Cuts Without Knowing the Graph. In ITCS, pages 39:1–39:16, 2018.

[RT16a] D. Ron and G. Tsur. The Power of an Example: Hidden Set Size Approximation Using Group Queries and Conditional Sampling. TOCT, 8(4):15:1–15:19, 2016.

[RT16b] Dana Ron and Gilad Tsur. The power of an example: Hidden set size approximation using group queries and conditional sampling. ACM Trans. Comput. Theory, 8(4):15:1–15:19, 2016.
[RWZ20] C. Rashtchian, D. P. Woodruff, and H. Zhu. Vector-Matrix-Vector Queries for Solving Linear Algebra, Statistics, and Graph Problems. In *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques*, APPROX/RANDOM, volume 176, pages 26:1–26:20, 2020.

[sim] Intel C++ Compiler Classic Developer Guide and Reference: Floating Point Dot Product Intrinsics. [https://software.intel.com/content/www/us/en/develop/documentation/cpp-compiler-developer-guide-and-reference/floatin g-point-dot-product-intrinsics.html](https://software.intel.com/content/www/us/en/develop/documentation/cpp-compiler-developer-guide-and-reference/floatin g-point-dot-product-intrinsics.html).

[SK10] Jason Sanders and Edward Kandrot. *CUDA by Example: An Introduction to General-Purpose GPU Programming*. Addison-Wesley, Upper Saddle River, NJ, 2010.

[Sto83] L. J. Stockmeyer. The Complexity of Approximate Counting (Preliminary Version). In *STOC*, pages 118–126, 1983.

[Sto85] L. J. Stockmeyer. On Approximation Algorithms for \#P. *SICOMP*, 14(4):849–861, 1985.

[SW19] X. Shi and D. P. Woodruff. Sublinear Time Numerical Linear Algebra for Structured Matrices. In *AAAI*, pages 727–746, 2019.

[SWYZ19a] X. Sun, D. P. Woodruff, G. Yang, and J. Zhang. Querying a Matrix Through Matrix-Vector Products. In *ICALP*, pages 94:1–94:16, 2019.

[SWYZ19b] X. Sun, D. P. Woodruff, G. Yang, and J. Zhang. Querying a Matrix Through Matrix-Vector Products. In *Proceedings of the 46th International Colloquium on Automata, Languages, and Programming*, ICALP, volume 132, pages 94:1–94:16, 2019.

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A  Weighted edge estimation and weighted edge sampling

As an application, consider the Weighted Edge Estimation problem on a graph $G$, with non-negative weights, formally defined as follows: Given IP oracle access to the adjacency matrix $A$ of a graph $G$, the objective is to estimate the quantity $Q = \sum_{\{i,j\} \in E(G)} A_{ij}$. Observe that as $2Q = \sum_{1 \leq i, j \leq n} A_{ij}$. From our results on Bilinear Form Estimation as mentioned in Table 1, $Q$ can be estimated by making $\tilde{O} \left( \sqrt{\frac{n}{\rho V(G)}} \right)$ many IP queries, where the weights on the edges of $G$ are in $[\rho]$. This is a generalization of the edge estimation results using local queries by Feige [Fei06], and Goldreich and Ron [GR08]. Also, according to our results on Sample Almost Uniformly, we can design an almost uniform sampler of $E(G)$, i.e., an edge $\{i, j\} \in E(G)$ is sampled with probability $p_{ij}$ satisfying the following inequality:

$$(1 - \epsilon) \frac{A_{ij}}{\sum_{\{k,l\} \in E(G)} A_{kl}} \leq p_{ij} \leq (1 + \epsilon) \frac{A_{ij}}{\sum_{\{l,k\} \in E(G)} A_{kl}}.$$  

The sampler also makes $\tilde{O} \left( \sqrt{\frac{n V(G)}{\sqrt{Q}}} \right)$ many queries to the IP oracle. This is a generalization of a result in the unweighted graph setting by Eden and Rosenbaum [ER18].

B  Other Matrix Problems

Recently, vector-matrix query [SWYZ19b] and vector-matrix-vector query [RWZ20] were introduced to study a bunch of matrix, graph and statistics problems. As noted earlier, IP query oracle is in the same linear algebraic framework of vector-matrix query and vector-matrix-vector query, but these queries are stronger than IP query. Study of the various matrix, graph and statistics problems, introduced in [SWYZ19b, RWZ20], using IP query will be of independent interest. As a first step in that direction, we study the query complexity of the following problems using IP queries.

- **Symmetric Matrix**: Is $A \in \{0, 1\}^{n \times n}$ a symmetric matrix?
- **Diagonal Matrix**: Is $A$ a diagonal matrix?
- **Trace**: Compute the trace of the matrix $A$.
- **Permutation Matrix**: Is $A \in \{0, 1\}^{n \times n}$ a permutation matrix?
- **Doubly stochastic matrix**: Is $A \in \{0, 1\}^{n \times n}$ a doubly stochastic matrix?
- **Identical columns**: Does there exist two columns in $A \in \{0, 1\}^{n \times n}$ that are identical?
- **All ones column**: Does there exist a column in $A$ all of whose entries are 1?

Table 2 gives the query complexity of solving the above matrix problems using IP oracle. We also do a comparative study of IP with respect to these stronger queries. The details are as follows:

**Theorem B.1 (Symmetric Matrix).** There exists an algorithm, that given an IP access to an unknown matrix $A \in \{0, 1\}^{n \times n}$, decides whether $A$ is symmetric or not with high probability by using $\tilde{O}(n)$ many IP queries. Also, any algorithm that has IP query access to an unknown matrix $A$ and decides whether $A$ is symmetric or not with probability $2/3$, requires $\Omega(n)$ many IP queries to $A$. 

\footnote{Assume that $G$ is a complete graph such that the weights on $\{i, j\} \notin E(G)$ is 0. Also, in the adjacency matrix of $A$, $A_{ij}$ is the weight on the edge $\{i, j\}$.}
Table 2: Other matrix problems.

| Problem                  | Query | Comments |
|--------------------------|-------|----------|
| Symmetric Matrix         | $\Theta(n)$ | Theorem B.1 |
| Diagonal Matrix          | $\Theta(n)$ | Theorem B.2 |
| Trace                    | $\Theta(n)$ | Theorem B.3 |
| Permutation Matrix       | $\Theta(n)$ | Theorem B.4 |
| Doubly Stochastic matrix | $\Theta(n)$ | Theorem B.5 |
| Identical Columns        | $\Theta(n)$ | Theorem B.6 |
| All Ones Columns         | $\Theta(n)$ | Theorem B.7 |

Proof. First we prove the upper bound result. Let $t = \Theta(\log n)$. Pick $v_1, \ldots, v_t \in \{0,1\}^n$ uniformly at random. For each $j \in [n]$ and $k \in [t]$, find $D_{vk}(j) = \langle A_{sj}, v_k \rangle - \langle A_{sj}, v_k \rangle \mod 2$. If there exists one $(j, k) \in [n] \times [t]$ such that $D_{vk}(j) \neq 0$, then we report $A$ is not a symmetric matrix. Otherwise, we report $A$ is symmetric. Observe that the number of IP queries made by the algorithm is $O(n)$.

Now we prove the correctness. If $A$ is symmetric, then $D_{vk}(j) = 0$ for each $(j, k) \in [n] \times [t]$ with probability 1. Now consider when $A$ is not symmetric. Then there exists $j_1 \in [n]$ such that the $j_1$-th row of $A$ is not same as that of the $j_1$-th column, that is, $A_{j_1} \neq A_{j_1j_1}$. It can be shown that, for any fixed $k \in [t]$, $(A_{sj}, v_k) \mod 2 = \langle A_{sj}, v_k \rangle \mod 2$, that is, $D_{vk}(j) = 0$ with probability $1/2$. As we are taking $t = \Theta(\log n)$ many random vectors, $v_1, \ldots, v_t \in \{0,1\}^n$, the probability that there exists $k \in [t]$ such that $D_{vk}(j) \neq 0$ is $1 - (1/2)^{\Theta(\log n)}$, that is, our algorithm reports that $A$ is symmetric with high probability.

For the lower bound, consider a variation of classical DISJOINTNESS \footnote{DISJOINTNESS is same as $k$-INTERSECTION (See Definition \ref{def:k-intersection} in Section \ref{sec:k-intersection}) when $k = 1$.} in communication complexity, where Alice is given $x \in \{0,1\}^n$ and Bob is given $y \in \{0,1\}^n$. The players want to output 1 if there is an $i \in [n]$ such that $x_i = y_{i+1} = 1$, where the indices of $x$ and $y$ are modulo $n$. Furthermore, it is promised that either there is a unique $i \in [n]$ such that $x_i = y_{i+1} = 1$ or there is no such $i$. We name the above problem as SHIFTED DISJOINTNESS. It admits a randomized communication complexity of $\Omega(n)$. This follows from the randomized communication complexity of standard DISJOINTNESS.

We prove the the desired lower bound by a reduction from SHIFTED DISJOINTNESS. Consider a matrix $A \in \{0,1\}^{n \times n}$ as follows. For each $i, j \in [n]$ with $j \neq i + 1$, $A_{ij} = 0$. If $x_i = y_{i+1} = 1$, then $A_{i(i+1)} = 1$, and 0, otherwise. Note that the indices of $x$ and $y$ are modulo $n$. Observe that $A$ is a symmetric matrix if and only if $x$ and $y$ do not intersect.

We will be done by showing that Alice and Bob can determine the answer to any IP query, to matrix $A$, with 2 bits of communication. Let $\langle A_{is}, v \rangle$ be an IP query. From the construction of matrix $A$, answer to this query depends on $x_i$ and $y_{i+1}$. So, Alice and Bob can determine $\langle A_{is}, v \rangle$ with 2 bits of communication. \hfill $\Box$

Theorem B.2 (Diagonal Matrix). There exists an algorithm, that given an IP access to an unknown matrix $A$, decides whether $M$ is diagonal or not deterministically by using $O(n)$ many IP queries. Also, any algorithm, that has IP query access to an unknown matrix $A$ and decides whether $A$ is a diagonal matrix or not with probability $2/3$, requires $\Omega(n)$ many IP queries to $A$.

Proof. First, let us discuss the upper bound. For each $i \in [n]$, we determine $A_{ii} = \langle A_{is}, e_i \rangle$ and $\sum_{j=1}^n A_{ij} = \langle A_{is}, 1 \rangle$. If there exists an $i \in [n]$ with $A_{ii} \neq \sum_{j=1}^n A_{ij}$, then the algorithm reports $A$ is not a diagonal matrix. Otherwise, the algorithm reports $A$ is a diagonal matrix. Note that the algorithm makes exactly $2n$ many IP queries. The correctness of the algorithm follows from its description.

**$e_i$** is the indicator vector for the $i$-th coordinate.
The proof of the lower bound is similar to the proof of the lower bound part of Theorem B.3.

**Theorem B.3 (Trace).** There exists an algorithm, that given an IP access to an unknown matrix $A$, determines the value of the trace of $A$ deterministically by using $O(n)$ many IP queries. Also, any algorithm, that has IP query access to an unknown matrix $A$ and finds the value of the trace of $A$ with probability $2/3$, requires $\Omega(n)$ many IP queries to $A$.

**Proof.** The upper bound, in the above theorem, follows from the fact that the trace of $A$ is

$$\sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n} \langle A_{is}, e_i \rangle.$$

We prove the lower bound by giving a reduction from **DISJOINTNESS**. Consider a matrix $A \in \{0,1\}^{n \times n}$ as follows. For each $i,j \in [n]$ with $j \neq i$, $A_{ij} = 0$ [1]. If $x_i = y_i = 1$, then $A_{ii} = 1$, and 0, otherwise. Observe that the trace of $A$ is 1 if $x$ and $y$ intersect. Otherwise, the trace of $A$ is 0. We will be done with the proof by showing that Alice and Bob can determine the answer to each IP query with 2 bits of communication as follows. To determine $\langle A_{is}, v \rangle$, Alice and Bob needs to determine $A_{ii}$ that depends on $x_i$ and $y_i$. It is because both Alice and Bob know $A_{ij} = 1$ with $j \neq i$.

**Theorem B.4 (Permutation matrix).** There exists an algorithm, that given an IP access to an unknown matrix $A$, decides whether $A$ is a permutation matrix or not deterministically by using $O(n)$ many IP queries. Also, any algorithm that has IP query access to an unknown matrix $A$ and decides whether $A$ is a permutation matrix with probability $2/3$, requires $\Omega(n)$ many IP queries to $A$.

**Proof.** A matrix $A$ is a permutation matrix if and only if each row and column of $A$ has exactly one 1. The upper bound part of the above theorem follows from the following algorithm: for each $i \in [n]$, we check whether both $\langle A_{is}, 1 \rangle = 1$ and $\langle A_{si}, 1 \rangle = 1$. If for all $i \in [n]$, $\langle A_{is}, 1 \rangle = \langle A_{si}, 1 \rangle = 1$, then $A$ is a permutation matrix. Otherwise, $A$ is not a permutation matrix. The number of queries made by the algorithm is $2n$. The correctness of the algorithm follows from the description of the algorithm.

For the lower bound, we prove by giving a reduction from **DISJOINTNESS**. Consider a matrix $A \in \{0,1\}^{n \times n}$ as follows. For each $i,j \in [n]$ with $j \neq i$, $A_{ij} = 0$ [2]. If $x_i = y_i = 1$, then $A_{ii} = 0$, and 1, otherwise. Observe that $A$ is a permutation matrix if and only if $x$ and $y$ do not intersect. Observe that Alice and Bob can determine the answer to each IP query with 2 bits of communication. The argument is same as that of in the proof of Theorem B.3.

**Theorem B.5 (Doubly stochastic).** There exists an algorithm, that given an IP access to an unknown matrix $A \in \{0,1\}^{n \times n}$, decides whether $A$ is doubly stochastic or not deterministically by using $O(n)$ many IP queries. Also, any algorithm that has IP query access to an unknown matrix $A$ and decides whether $A$ is a doubly stochastic matrix with probability $2/3$, requires $\Omega(n)$ many IP queries to $A$.

**Proof.** A matrix $A$ is a doubly stochastic matrix if and only if the sum of the entries in all rows and columns are same. The upper bound part of the above theorem follows from the following algorithm: for each $i \in [n]$, determine $\sum_{j=1}^{n} A_{ij} = \langle A_{is}, 1 \rangle$ and $\sum_{j=1}^{n} A_{ji} = \langle A_{si}, 1 \rangle$. If for each $i \in [n]$, $\langle A_{is}, 1 \rangle = \langle A_{si}, 1 \rangle = 1$, then the algorithm reports that $A$ is doubly stochastic. Otherwise, the algorithm reports $A$ is not doubly stochastic. The number of queries made by the algorithm is $2n$. The correctness of the algorithm follows from the description of the algorithm.

The proof of the lower bound is similar to that of the proof of the lower bound part of Theorem B.3.

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[1] Here 0 is arbitrary. Any other real number is also ok for our purpose.

[2] Here 0 is arbitrary. We can take any real number here.
Theorem B.6 (Identical Columns). There exists an algorithm, that given an IP access to an unknown matrix $A \in \{0,1\}^{n \times n}$, decides whether $A$ has two columns that are identical or not with high probability by using $\tilde{O}(n)$ many IP queries. Also, any algorithm that has IP query access to an unknown matrix $A$ and decides whether $A$ has a pair of identical columns or not with probability 2/3, requires $\Omega(n)$ many IP queries to $A$.

Proof. First, we discuss the upper bound. Let $t = \Theta(\log n)$. Pick $v_1, \ldots, v_t \in \{0,1\}^n$ uniformly at random. For each $(r,s) \in [n] \times [n]$, such that $r \neq s$ and $k \in [t]$, find $D_{v_k}(r,s) = \langle A_{sr}, v_k \rangle - \langle A_{ss}, v_k \rangle \mod 2$. If there exists $(r,s) \in [n] \times [n]$ such that $D_{v_k}(r,s) = 2$ for all $k \in [t]$, then we report $A$ has two identical columns. Otherwise, we report no two columns of $A$ are identical. Observe that the number of IP queries made by the algorithm is $\tilde{O}(n)$. Now we prove the correctness of the above algorithm. If two columns of $A$ are identical, say $r, s \in [n]$, $r \neq s$, then $D_{v_k}(r,s) = 0$ for all $k \in [t]$ with probability 1. Now consider when no two columns of $A$ are identical. Consider a fixed $(r,s) \in [n] \times [n]$ with $r \neq s$ and $k \in [t]$. $\langle A_{sr}, v_k \rangle \mod 2 = 2 = \langle A_{ss}, v_k \rangle \mod 2$, that is $D_{v_k}(r,s) = 0$ holds probability 1/2. As we are taking $t = \Theta(\log n)$ many random vectors, $v_1, \ldots, v_t \in \{0,1\}^n$, the probability that there exists $k \in [t]$, such that $D_{v_k}(r,s) \neq 0$ is $1 - (1/2)^\Theta(\log n)$, that is, the algorithm decides that the $r$-th column is different from $s$-th column with probability $1 - (1/2)^\Theta(\log n)$. Using union bound over all $(r,s) \in [n] \times [n]$, the algorithm reports no two columns are identical with probability at least $1 - n^2(1/2)^\Theta(\log n)$. The algorithm, described above, not only decides whether there exists two identical columns, but also can compute the number of pairs of identical columns with high probability.

We prove the lower bound by reducing from DISJOINTNESS problem. Consider a matrix $A \in \{0,1\}^{n \times n}$ constructed from $I \in \{0,1\}^{n \times n}$ as follows, where $I$ is the $(n \times n)$ identity matrix. If there exists $i \in [n]$ such that $x_i = y_i = 1$, then $A$ has the $i$-th and $(i+1)$-th column of $I$ changed to 1, where the indices $i$ and $(i+1)$ are mod $n$. Otherwise $A = 1$. Note that the indices of $x$ and $y$ are mod $n$. Observe that $A$ has two identical columns if and only if $x$ and $y$ intersect.

We will be done with the proof by showing that Alice and Bob can determine the answer to each IP query with 4 bits of communication as follows. To determine $\langle A_{ii}, v \rangle$, Alice and Bob need to determine $A_{ii}$ that depends only on either $x_i$ and $y_i$, or $x_{i-1}$ and $y_{i-1}$.

Theorem B.7 (All One Columns). There exists an algorithm, that given an IP access to an unknown matrix $A$, decides whether $A$ has at least one all 1’s columns or not deterministically by using $O(n)$ many IP queries. Note that here the entries of matrix $A$ are 0 or 1. Also, any algorithm that has IP query access to an unknown matrix $A$ and decides whether $A$ has an all 1’s column with probability 2/3, requires $\Omega(n)$ many IP queries to $A$.

Proof. For each $i \in [n]$, determine $\sum_{j=1}^n A_{ji} = \langle A_{ii}, 1 \rangle$. If for each $i \in [n]$ with $\langle A_{ii}, 1 \rangle = \langle A_{ii}, 1 \rangle = 1$, then the algorithm reports that $A$ has at least one all 1’s column. Otherwise, the algorithm reports $A$ does not have an all 1’s column. The number of queries made by the algorithm is $n$. The correctness of the algorithm follows from the description of the algorithm.

The lower bound proof is similar to the proof of the lower bound part of Theorem B.6.

Comparison with other matrix queries

We investigated the problems mentioned in Table 3 using IP. It will be interesting to study other matrix and graph problems mentioned in SWYZ19, RWZ20 using IP.

\footnote{Upper and lower bounds results for IDENTICAL COLUMNS problem in RWZ20, SWYZ19 uses vectors from $\{0,1\}^n$ in their respective queries.}
Table 3: Comparing IP with matrix-vector and vector-matrix-vector queries.

| Problem                      | IP Query | $x^TAy$ Query $[RWZ20]$ | $Ay$ Query $[SWYZ19b]$ |
|------------------------------|----------|-------------------------|------------------------|
| Symmetric Matrix             | $\Theta(n)$ (Theorem B.1) | $O(1)$                  | $O(1)$                 |
| Diagonal Matrix              | $\Theta(n)$ (Theorem B.2) | $O(1)$                  | $O(1)$                 |
| Trace                        | $\Theta(n)$ (Theorem B.3) | $\Omega\left(\frac{n}{\log n}\right)$ | $\Omega\left(\frac{n}{\log n}\right)$ |
| Permutation Matrix           | $\Theta(n)$ (Theorem B.4) | $O(1)$                  | $O(1)$                 |
| Doubly Stochastic matrix     | $\Theta(n)$ (Theorem B.5) | $O(1)$                  | $O(1)$                 |
| Identical columns            | $\Theta(n)$ (Theorem B.6) | $\Theta(n)$             | $\Theta(n)$            |
| All Ones Columns             | $\Theta(n)$ (Theorem B.7) | $\Omega\left(\frac{n}{\log n}\right)$ | $\Omega\left(\frac{n}{\log n}\right)$ |

C Communication Complexity

In two-party communication complexity there are two parties, Alice and Bob, that wish to compute a function $\Pi : \{0,1\}^N \times \{0,1\}^N \rightarrow \{0,1\}$. Alice is given $x \in \{0,1\}^N$ and Bob is given $y \in \{0,1\}^N$. Let $x_i (y_i)$ denotes the $i$-th bit of $x (y)$. While the parties know the function $\Pi$, Alice does not know $y$, and similarly Bob does not know $x$. Thus they communicate bits following a pre-decided protocol $P$ in order to compute $\Pi(x, y)$. We say a randomized protocol $P$ computes $\Pi$ if for all $(x, y) \in \{0,1\}^N \times \{0,1\}^N$ we have $\mathbb{P}[P(x, y) = \Pi(x, y)] \geq 2/3$. The model provides the parties access to common random string of arbitrary length. The cost of the protocol $P$ is the maximum number of bits communicated, where maximum is over all inputs $(x, y) \in \{0,1\}^N \times \{0,1\}^N$. The communication complexity of the function is the cost of the most efficient protocol computing $\Pi$. For more details on communication complexity see [KN97].

D Probability Results

Lemma D.1 (See [DP09]). Let $X = \sum_{i \in [n]} X_i$ where $X_i, i \in [n],$ are independent random variables, $X_i \in [0,1]$ and $\mathbb{E}[X]$ is the expected value of $X$. Then

(i) For $\epsilon > 0$, $\mathbb{P}[|X - \mathbb{E}[X]| > \epsilon \mathbb{E}[X]] \leq \exp\left(-\frac{\epsilon^2}{3} \mathbb{E}[X]\right)$.

(ii) Suppose $\mu_L \leq \mathbb{E}[X] \leq \mu_H$, then for $0 < \epsilon < 1$

(a) $\mathbb{P}[X > (1 + \epsilon)\mu_H] \leq \exp\left(-\frac{\epsilon^2}{2} \mu_H\right)$.

(b) $\mathbb{P}[X < (1 - \epsilon)\mu_L] \leq \exp\left(-\frac{\epsilon^2}{2} \mu_L\right)$. 

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