SECTIONS OF SERRE FIBRATIONS WITH 2-MANIFOLD FIBERS

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Abstract. It was proved by H. Whitney in 1933 that a Serre fibration of compact metric spaces admits a global section provided every fiber is homeomorphic to the unit interval \([0,1]\). Results of this paper extend Whitney theorem to the case when all fibers are homeomorphic to a given compact two-dimensional manifold.

1. Introduction

The following problem is one of the central problems in geometric topology [8]. Let \(p: E \to B\) be a Serre fibration of separable metric spaces. Suppose that the space \(B\) is locally \(n\)-connected and all fibers of \(p\) are homeomorphic to a given \(n\)-dimensional manifold \(M^n\). Is \(p\) a locally trivial fibration?

For \(n = 1\) an affirmative answer to this problem follows from results of H. Whitney [21].

Conjecture (Ščepin). A Serre fibration with a locally arcwise connected metric base is locally trivial if every fiber of this fibration is homeomorphic to a given manifold \(M^n\) of dimension \(n \leq 4\).

In dimension \(n = 1\) Ščepin’s Conjecture is valid even for non-compact fibers [17]. Ščepin also proved that the positive solution of the Conjecture in dimension \(n\) implies positive solutions of both the CE-problem and the Homeomorphism Group problem in dimension \(n\) [18, 8]. Since the CE-problem was solved in negative by A.N. Dranishnikov, it follows that there are dimensional restrictions in Ščepin’s Conjecture.

The first step toward proving Ščepin’s Conjecture in dimension \(n = 2\) was made in [8], where existence of local sections of the fibration was proved under the assumption that the base space is an ANR. We improve this result in two directions. First we prove that any section of the fibration over closed subset \(A\) of the base space can be extended to

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a section over some neighborhood of $A$. Secondly, we prove a theorem on global sections of the fibration.

**Theorem 4.4.** Let $p: E \to B$ be a Serre fibration of locally connected compacta with all fibers homeomorphic to given two-dimensional manifold. If $B \in ANR$, then any section of $p$ over a closed subset $A \subset B$ can be extended to a section of $p$ over a neighborhood of $A$.

**Theorem 4.9.** Let $p: E \to B$ be a Serre fibration of locally connected compactum $E$ onto an ANR-compactum $B$ with all fibers homeomorphic to given two-dimensional manifold $M$. If $M$ is not homeomorphic to the sphere or the projective plane, then $p$ admits a global section provided one of the following conditions holds:

(a) $\pi_1(M)$ is abelian and $H^2(B; \pi_1(F_b)) = 0$
(b) $\pi_1(M)$ is non-abelian, $M$ is not homeomorphic to the Klein bottle and $\pi_1(B) = 0$.
(c) $M$ is homeomorphic to the Klein bottle and $\pi_1(B) = \pi_2(B) = 0$.

Our strategy of constructing a section of a Serre fibration is as follows. We consider the inverse (multivalued) mapping and find its compact submapping admitting continuous approximations. Then we take very close continuous approximation and use it to find again a compact submapping with small diameters of fibers admitting continuous approximations. When we continue this process we get a sequence of compact submappings with diameters of fibers tending to zero. This sequence will converge to the desired singlevalued submapping (selection).

Section 3 is devoted to continuous approximations of multivalued maps. We prove filtered finite dimensional approximation theorem (Theorem 3.13) and then apply it in a usual way (compare with [10]) to prove an approximation theorem for maps of ANR-spaces. Since we are going to use singular filtrations of multivalued maps instead of usual filtrations, our Theorem 3.13 generalizes the Filtered approximation theorem proved in [19]. But the proof of our singular version of filtered approximation theorem in full generality requires a lot of technical details. Consequently we decided to present only the version that we need – for compact maps of metric spaces.

Let us recall some definitions and introduce our notation. All spaces will be separable metrizable. By a mapping we understand a continuous single-valued mapping. We equip the product $X \times Y$ with the metric

$$\text{dist}_{X \times Y}(((x, y), (x', y'))) = \text{dist}_X(x, x') + \text{dist}_Y(y, y').$$

By $O(x, \varepsilon)$ we denote the open $\varepsilon$-neighborhood of the point $x$. 
A multivalued mapping \( F: X \rightarrow Y \) is called a submapping (or selection) of a multivalued mapping \( G: X \rightarrow Y \) if \( F(x) \subseteq G(x) \) for every \( x \in X \). The gauge of a multivalued mapping \( F: X \rightarrow Y \) is defined as \( \text{cal}(F) = \sup\{\text{diam}(F(x)) \mid x \in X\} \). The graph of multivalued mapping \( F: X \rightarrow Y \) is the subset \( \Gamma_F = \{(x,y) \in X \times Y \mid y \in F(x)\} \) of the product \( X \times Y \). For arbitrary subset \( U \subset X \times Y \) denote by \( U(x) \) the subset \( \text{pr}_Y(U \cap (\{x\} \times Y)) \) of \( Y \). Then for the graph \( \Gamma_F \) we have \( \Gamma_F(x) = F(x) \).

A multivalued mapping \( G: X \rightarrow Y \) is called complete if all sets \( \{x\} \times G(x) \) are closed with respect to some \( G_\delta \)-set \( S \subset X \times Y \) containing the graph of this mapping. A multivalued mapping \( F: X \rightarrow Y \) is called upper semicontinuous if for any open set \( U \subset Y \) the set \( \{x \in X \mid F(x) \subset U\} \) is open in \( X \). A compact mapping is an upper semicontinuous multivalued mapping with compact images of points.

An increasing sequence (finite or countably infinite) of subspaces
\[
Z_0 \subset Z_1 \subset Z_2 \subset \cdots \subset Z
\]
is called a filtration of space \( Z \). A sequence of multivalued mappings \( \{F_k: X \rightarrow Y\} \) is called a filtration of multivalued mapping \( F: X \rightarrow Y \) if for any \( x \in X \), \( \{F_k(x)\} \) is a filtration of \( F(x) \).

We say that a filtration of multivalued mappings \( G_i: X \rightarrow Y \) is complete (resp. compact) if every mapping \( G_i \) is complete (resp. compact).

2. Local properties of multivalued mappings

Let \( \gamma \) be a property of a topological space such that every open subspace inherits this property: if a space \( X \) satisfies \( \gamma \), then any open subspace \( U \subset X \) also satisfies \( \gamma \). We say that a space \( Z \) satisfies \( \gamma \) locally if every point \( z \in Z \) has a neighborhood with this property.

For a multivalued map \( F: X \rightarrow Y \) to satisfy \( \gamma \) locally we not only require that every point-image \( F(x) \) has this property locally, but for any points \( x \in X \) and \( y \in F(x) \) there exist a neighborhood \( W \) of \( y \) in \( Y \) and \( U \) of \( x \) in \( X \) such that \( W \cap F(x') \) satisfies \( \gamma \) for every point \( x' \in U \). And we use the word ”equi” for local properties of multivalued maps.

The first example of such a property is the local compactness.

**Definition 2.1.** A space \( X \) is called locally compact if every point \( x \in X \) has a compact neighborhood. We say that a multivalued map \( F: X \rightarrow Y \) is equi locally compact if for any points \( x \in X \) and \( y \in F(x) \) there exists a neighborhood \( W \) of \( y \) in \( Y \) and \( U \) of \( x \) in \( X \) such that \( W \cap F(x') \) is compact for every point \( x' \in U \).
Another local property we are going to use is the hereditary asphericity. Recall that a compactum $K$ is called \emph{approximately aspherical} if for any (equivalently, for some) embedding of $K$ into an ANR-space $Y$ every neighborhood $U$ of $K$ in $Y$ contains a neighborhood $V$ with the following property: any mapping of the sphere $S^n$ into $V$ is homotopically trivial in $U$ provided $n \geq 2$.

\textbf{Definition 2.2.} We call a space $Z$ \emph{hereditarily aspherical} if any compactum $K \subset Z$ is approximately aspherical.

A space $Z$ is said to be \emph{locally hereditarily aspherical} if any point $z \in Z$ has a hereditarily aspheric neighborhood.

It is easy to prove that the 2-dimensional Euclidean space is hereditarily aspherical. Note that any 2-dimensional manifold is locally hereditarily aspherical.

\textbf{Definition 2.3.} We say that a multivalued map $F: X \to Y$ is \emph{equi locally hereditarily aspherical} if for any points $x \in X$ and $y \in F(x)$ there exists a neighborhood $W$ of $y$ in $Y$ and $U$ of $x$ in $X$ such that $W \cap F(x')$ is hereditarily aspherical for every point $x' \in U$.

Now we consider different properties of pairs of spaces and define the corresponding local properties for spaces and multivalued maps. We follow definitions and notations from [9].

\textbf{Definition 2.4.} An ordering $\alpha$ of subsets of a space $Y$ is \emph{proper} provided:

\begin{itemize}
\item[(a)] If $W \alpha V$, then $W \subset V$;
\item[(b)] If $W \subset V$, and $V \alpha R$, then $W \alpha R$;
\item[(c)] If $W \alpha V$, and $V \subset R$, then $W \alpha R$.
\end{itemize}

\textbf{Definition 2.5.} Let $\alpha$ be a proper ordering.

\begin{itemize}
\item[(a)] A space $Y$ is \emph{locally of type $\alpha$} if, whenever $y \in Y$ and $V$ is a neighbourhood of $y$, then there a neighbourhood $W$ of $y$ such that $W \alpha V$.
\item[(b)] A multivalued mapping $F: X \to Y$ is \emph{lower $\alpha$-continuous} if for any points $x \in X$ and $y \in F(x)$ and for any neighbourhood $V$ of $y$ in $Y$ there exist neighbourhoods $W$ of $y$ in $Y$ and $U$ of $x$ in $X$ such that $(W \cap F(x')) \alpha (V \cap F(x'))$ provided $x' \in U$.
\end{itemize}

For example, if $W \alpha V$ means that $W$ is contractible in $V$, then locally of type $\alpha$ means locally contractible. Another topological property which arises in this manner is $\text{LC}^n$ (where $W \alpha V$ means that every continuous mapping of the $n$-sphere into $W$ is homotopic to a constant mapping in $V$) and the corresponding lower $\alpha$-continuity of multivalued
map is called lower \((n+1)\)-continuity. For the special case \(n = -1\) the property \(W\alpha V\) means that \(V\) is non-empty, and lower \(\alpha\)-continuity is the lower semicontinuity.

The following result is weaker than Lemma 3.5 from [4]. We will use it with different properties \(\alpha\) in Section 4.

**Lemma 2.6.** Let a lower \(\alpha\)-continuous mapping \(\Phi : X \to Y\) of compactum \(X\) to a metric space \(Y\) contains a compact submapping \(F\). Then for any \(\varepsilon > 0\) there exists a positive number \(\delta\) such that for every point \((x, y) \in O(\Gamma F, \delta)\) we have \((O(y, \delta) \cap \Phi(x)) \alpha (O(y, \varepsilon) \cap \Phi(x))\).

In order to use results from [19] we need a local property called polyhedral \(n\)-connectedness. A pair of spaces \(V \subset U\) is called polyhedrally \(n\)-connected if for any finite \(n\)-dimensional polyhedron \(M\) and its closed subpolyhedron \(A\) any mapping of \(A\) into \(V\) can be extended to a map of \(M\) into \(U\). Note that for spaces being locally polyhedrally \(n\)-connected is equivalent to be \(LC^{n-1}\) (it follows from Lemma 2.7). The corresponding local property of multivalued maps is called polyhedral lower \(n\)-continuity.

**Lemma 2.7.** Any lower \(n\)-continuous multivalued mapping is lower polyhedrally \(n\)-continuous.

**Proof.** The proof easily follows from the fact that in a connected filtration \(Z_0 \subset Z_1 \subset \cdots \subset Z_n\) of spaces the pair \(Z_0 \subset Z_n\) is polyhedrally \(n\)-connected. Given a mapping \(f : A \to Z_0\) of subpolyhedron \(A\) of \(n\)-dimensional polyhedron \(P\), we extend it successively over skeleta \(P^{(k)}\) of \(P\) such that the image of \(k\)-dimensional skeleton \(P^{(k)}\) is contained in \(Z_k\). Resulting map gives us an extension \(\tilde{f} : P \to Z_n\) of \(f\) which proves that the pair \(Z_0 \subset Z_n\) is polyhedrally \(n\)-connected.

A filtration of multivalued maps \(\{F_i\}\) is called polyhedrally connected if every pair \(F_{i-1}(x) \subset F_i(x)\) is polyhedrally \(i\)-connected. A filtration \(\{F_i\}\) is called lower continuous if for any \(i\) the mapping \(F_i\) is lower \(i\)-continuous.

**Lemma 2.8.** If \(p : E \to B\) is a Serre fibration of LC\(^0\)-compacta with fibers homeomorphic to a given 2-dimensional compact manifold, then the multivalued mapping \(p^{-1} : B \to E\) is

- equi locally hereditarily aspherical
- polyhedrally lower 2-continuous

**Proof.** Since every open proper subset of a two-dimensional manifold is aspherical, every compact proper subset of 2-manifold is approximately aspherical. Therefore, the mapping \(F\) is equi locally hereditarily aspherical.
It follows from a theorem of McAuley [14] that the mapping \( p^{-1} \) is lower 2-continuous. By Lemma 2.7, the mapping \( p^{-1} \) is polyhedrally lower 2-continuous. □

3. Singlevalued approximations

Definition 3.1. A singular pair of spaces is a triple \((Z, \phi, Z')\) where \( \phi: Z \to Z' \) is a mapping.

We say that a space \( Z \) contains a singular filtration of spaces if a finite sequence of pairs \( \{(Z_i, \phi_i)\}_{i=0}^n \) is given where \( Z_i \) is a space and \( \phi_i: Z_i \to Z_{i+1} \) is a map (we identify \( Z_{n+1} \) with \( Z \)).

For a multivalued map \( F: X \to Y \) it is useful to consider its graph fibers \( \{x\} \times F(x) \subset \Gamma_F \) instead of usual fibers \( F(x) \subset Y \). While the graph fibers are always homeomorphic to the usual fibers, different graph fibers do not intersect (the usual fibers may intersect in \( Y \)). We denote the graph fiber of the map \( F \) over a point \( x \in X \) by \( F_{\Gamma}(x) \).

To define the notion of singular filtration for multivalued maps we introduce a notion of fiberwise transformation of multivalued maps.

Definition 3.2. For multivalued mappings \( F \) and \( G \) of a space \( X \) a fiberwise transformation from \( F \) to \( G \) is a continuous mapping \( T: \Gamma_F \to \Gamma_G \) such that \( T(F_{\Gamma}(x)) \subset G_{\Gamma}(x) \) for every \( x \in X \).

A fiber \( T(x) \) of the fiberwise transformation \( T \) over the point \( x \in X \) is a mapping \( T(x): F(x) \to G(x) \) determined by \( T \).

We say that a multivalued mapping \( F: X \to Y \) contains a singular filtration of multivalued maps if a finite sequence of pairs \( \{(F_i, T_i)\}_{i=0}^n \) is given where \( F_i: X \to Y_i \) is a multivalued mapping and \( T_i \) is a fiberwise transformation from \( F_i \) to \( F_{i+1} \) (we identify \( F_{n+1} \) with \( F \)).

To construct continuous approximations of multivalued maps we need the notion of approximate asphericity.

Definition 3.3. A pair of compacta \( K \subset K' \) is called approximately \( n \)-aspherical if for any embedding of \( K' \) into ANR-space \( Z \) for every neighborhood \( U \) of \( K' \) in \( Z \) there exists a neighborhood \( V \) of \( K \) such that any mapping \( f: S^n \to V \) is homotopically trivial in \( U \).

A compactum \( K \) is approximately \( n \)-aspherical if the pair \( K \subset K \) is approximately \( n \)-aspherical.

The following is a singular version of approximate asphericity.

Definition 3.4. A singular pair of compacta \((K, \phi, K')\) is called approximately \( n \)-aspherical if for any embeddings \( K \subset Z \) and \( K' \subset Z' \) in ANR-spaces and for any extension of \( \phi \) to a map \( \tilde{\phi}: OK \to Z' \) of some neighborhood \( OK \) of \( K \) the following holds: for every neighborhood \( U \)
of $K'$ in $Z'$ there exists a neighborhood $V$ of $K$ in $OK$ such that for any mapping $f: S^n \to V$ the spheroid $\bar{\phi} \circ f: S^n \to U$ is homotopically trivial in $U$.

Following R.C. Lacher \cite{13}, one can prove that this notion does not depend neither on the choices of ANR-spaces $Z$ and $Z'$ nor on the embeddings of $K$ and $K'$ into these spaces.

**Definition 3.5.** A singular filtration of compacta $\{(K_i, \phi_i)\}_{i=0}^n$ is called *approximately connected* if for every $i < n$ the singular pair $(K_i, \phi_i, K_{i+1})$ is approximately $i$-aspherical.

Clearly, a singular pair of compacta $(K, \phi, K')$ is approximately $n$-aspherical in either of the following three situations: compactum $K$, compactum $K'$, or the pair $\phi(K) \subset K'$ is approximately $n$-aspherical.

**Definition 3.6.** A singular filtration $\mathcal{F} = \{(F_i, T_i)\}_{i=0}^n$ of compact mappings $F_i: X \to Y_i$ is said to be *approximately connected* if for every point $x \in X$ the singular filtration of compacta $\{(F_i(x), T_i(x))\}_{i=0}^n$ is approximately connected.

An approximately connected singular filtration $\mathcal{F} = \{(F_i: X \to Y_i, T_i)\}_{i=0}^n$ is said to be *approximately $\infty$-connected* if the mapping $F_n$ has approximately $k$-aspherical point-images $F_n(x)$ for all $k \geq n$ and all $x \in X$.

Note that if a singular filtration $\mathcal{F} = \{(F_i, T_i)\}_{i=0}^n$ is approximately $\infty$-connected, then the mapping $F_n$ contains an approximately connected singular filtration of any given finite length.

We will reduce our study of singular filtrations to the study of usual filtrations using the following cylinder construction.

**Definition 3.7.** For a continuous singlevalued mapping $f: X \to Y$ we define a *cylinder* of $f$ denoted by $\text{cyl}(f)$ as a space obtained from the disjoint union of $X \times [0, 1]$ and $Y$ by identifying each $\{x\} \times \{1\}$ with $f(x)$.

Note that the cylinder $\text{cyl}(f)$ contains a homeomorphic copy of $Y$ called the *bottom* of the cylinder, and a homeomorphic copy of $X$ as $X \times \{0\}$ called the *top* of the cylinder.

**Remark 3.8.** There is a natural deformation retraction $r: \text{cyl}(f) \to Y$ onto the bottom $Y$. Clearly, the fiber of the mapping $r$ over a point $y \in Y$ is either one point $\{y\}$ or a cone over the set $f^{-1}(y)$. Therefore, if the map $f$ is proper, then $r$ is $UV^\infty$-mapping.

**Remark 3.9.** Suppose that $X$ is embedded into Banach space $B_1$ and $Y$ is embedded into Banach space $B_2$. Then we can naturally embed
the cylinder \( \text{cyl}(f) \) into the product \( B_1 \times \mathbb{R} \times B_2 \). The embedding is clearly defined on the top as embedding into \( B_1 \times \{0\} \times \{0\} \) and on the bottom as embedding into \( \{0\} \times \{1\} \times B_2 \). We extend these embeddings to the whole cylinder by sending its point \( \{x\} \times \{t\} \) to the point \( \{(1-t) \cdot x\} \times t \times \{t \cdot f(x)\} \).

**Lemma 3.10.** If a singular pair of compacta \((K, \phi, K')\) is approximately \(n\)-aspherical, then the pair \( K \subset \text{cyl}(\phi) \) is approximately \(n\)-aspherical.

**Proof.** Let us fix embeddings of \( K \) into Banach space \( B_1 \), of \( K' \) into Banach space \( B_2 \), and of the cylinder \( \text{cyl}(\phi) \) into the product \( B = B_1 \times \mathbb{R} \times B_2 \) as described in Remark 3.9. Fix a neighborhood \( U \) of \( \text{cyl}(\phi) \) in \( B \). Extend the mapping \( \phi \) to a map \( \phi_1 : B_1 \rightarrow B_2 \). Take a neighborhood \( V_1 \) of the top of our cylinder in \( B_1 \) such that the cylinder \( \text{cyl}(\phi_1|_{V_1}) \) is contained in \( U \). Using approximate \( n\)-asphericity of the pair \((K, \phi, K')\) we find for a neighborhood \( U \cap \{0\} \times \{1\} \times B_2 \) of \( K' \) in \( \{0\} \times \{1\} \times B_2 \) a neighborhood \( V' \) of \( K \) in \( B_1 \times \{0\} \times \{0\} \). Let \( \varepsilon \) be a positive number such that the product \( V = V' \times (-\varepsilon, \varepsilon) \times O(0, \varepsilon) \) is contained in \( U \).

Given a spheroid \( f : S^n \rightarrow V \) we retract it into \( V' \times \{0\} \times \{0\} \), then retract it to the bottom of the cylinder \( \text{cyl}(\phi_1|_{V_1}) \) using Remark 3.8 and finally contract it to a point inside \( U \cap \{0\} \times \{1\} \times B_2 \). Clearly, the whole retraction sits inside \( U \), as required. \( \square \)

**Definition 3.11.** Let \( \mathcal{F} = \{(F_i : X \rightarrow Y_i, T_i)\}_{i=0}^n \) be a singular filtration of a multivalued mapping \( F : X \rightarrow Y = Y_{n+1} \). If all the spaces \( Y_i \) are Banach, then for a multivalued mapping \( \mathbb{F} \) from \( X \) to \( Y = Y \times \prod_{i=0}^n (Y_i \times \mathbb{R}) \) defined as \( \mathbb{F}(x) = \bigcup_{k=0}^n \text{cyl}(T_k(x)) \) we can define a cylinder \( \text{cyl}(\mathcal{F}) \) as a filtration of multivalued maps \( \{\mathbb{F}_i\}_{i=0}^n \) defined as follows:

\[
\mathbb{F}_0 = F_0 \quad \text{and} \quad \mathbb{F}_i(x) = \bigcup_{k=0}^{i-1} \text{cyl}(T_k(x)).
\]

It is easy to see that for a singular filtration \( \mathcal{F} = \{(F_i, T_i)\}_{i=0}^n \) of compact mappings \( F_i \) the filtration \( \text{cyl}(\mathcal{F}) \) consists of compact mappings \( \mathcal{F}_i \).

**Lemma 3.12.** If a singular filtration \( \mathcal{F} = \{(F_i, T_i)\}_{i=0}^n \) of compact maps is approximately connected, then the filtration \( \text{cyl}(\mathcal{F}) = \{\mathbb{F}_i\}_{i=0}^n \) is approximately connected.

**Proof.** Using Remark 3.8 it is easy to define a deformation retraction \( r : \mathbb{F}_i(x) \rightarrow F_i(x) \) which is \( UV^\infty \)-mapping. This retraction defines \( UV^\infty \)-mapping of pairs \( (\mathbb{F}_i(x), \mathbb{F}_{i+1}(x)) \rightarrow (F_i(x), \text{cyl}(T_i(x))) \). By
Lemma 3.10 the pair $F_i(x) \subset \text{cyl}(T_i(x))$ is approximately $i$-aspherical, and by Pairs Mapping Lemma from [19] the pair $F_i(x) \subset F_{i+1}(x)$ is also approximately $i$-aspherical. □

Theorem 3.13. Let $H: X \to Y$ be a multivalued mapping of metric space $X$ to a Banach space $Y$. If $\dim X \leq n$ and $H$ contains approximately connected singular filtration $\mathcal{H} = \{(H_i: X \to Y_i, T_i)\}_{i=0}^n$ of compact mappings, then any neighborhood $U$ of the graph $\Gamma_H$ contains the graph of a single-valued and continuous mapping $h: X \to Y$.

Proof. Without loss of generality we may assume that all spaces $Y_i$ are Banach spaces. We consider $Y$ as a subspace of the product $Y = Y \times \prod_{i=0}^n (Y_i \times \mathbb{R})$. Clearly, $H$ is a submapping of a multivalued mapping $\mathbb{H}: X \to Y$ defined as $\mathbb{H}(x) = \cup_{k=0}^n \text{cyl}(T_k(x))$ and $\Gamma_H$ admits a deformation retraction $R$ onto $\Gamma_H$. Fix a neighborhood $U$ of the graph $\Gamma_H$ in $X \times Y$. Since all maps $H_i$ are compact, $\mathbb{H}$ is also compact and the graph $\Gamma_H$ is closed in $X \times Y$. Extend the mapping $\text{pr}_Y \circ R: \Gamma_H \to Y$ to some neighborhood $W$ of $\Gamma_H$ in $X \times Y$ and denote by $R'$ the map of $W$ to $X \times Y$ such that $\text{pr}_Y \circ R'$ is our extension. Clearly, we may assume that $R'(W)$ is contained in $U$.

By Lemma 3.12 the multivalued map $\mathbb{H}$ admits approximately connected filtration $\text{cyl}H$ of compact multivalued maps. By Single-Valued Approximation Theorem from [19] there exists a singlevalued continuous mapping $h: X \to Y$ with $\Gamma_h \subset W$. Define a singlevalued continuous map $h$ by the equality $\Gamma_h = R'(\Gamma_h)$. Clearly, $\Gamma_h$ is contained in $R'(W) \subset U$. □

Theorem 3.14. Suppose that a compact mapping of separable metric ANRs $F: X \to Y$ admits a compact singular approximately infinite-connected filtration. Then for any compact space $K \subset X$ every neighborhood of the graph $\Gamma_F(K)$ contains the graph of a single-valued and continuous mapping $f: K \to Y$.

Proof. Let $U$ be an open neighborhood of the graph $\Gamma_F(K)$ in the product $X \times Y$. Since $F$ is upper semicontinuous, there is a neighborhood $OK$ of compactum $K$ such that $\Gamma_F(OK)$ is contained in $U$. Since any open subset of separable ANR-space is separable ANR-space [12], we can denote $OK$ by $X$ and consider $U$ as an open neighborhood of the graph $\Gamma_F$.

For every point $x \in X$ take open neighborhoods $O_x \subset X$ of the point $x$ and $V_x \subset X$ of the compactum $F(x)$ such that the product $O_x \times V_x$ is contained in $U$. Using upper semicontinuity of $F$ we can choose $O_x$ so small that the following inclusion holds: $F(O_x) \subset V_x$. Fix an open covering $\omega_1$ of the space $X$ which is starlike refinement of $\{O_x\}_{x \in X}$.
Let $\omega_2$ be a locally finite open covering of the space $X$ which is starlike refined into $\omega_1$.

There exist a locally finite simplicial complex $L$ and mappings $r : X \to L$ and $j : L \to X$ such that the map $j \circ r$ is $\omega_2$-close to $\text{id}_X$ [12]. Fix a finite subcomplex $N \subset L$ containing the compact set $r(K)$. Define a compact mapping $\Psi : N \to Y$ by the formula $\Psi = F \circ j$. Clearly, the mapping $\Psi$ admits a compact approximately connected singular filtration of any length (particularly, of the length $\dim N$). Let us define a neighborhood $W$ of the graph $\Gamma_\Psi$. For every point $q \in N$ we put $W(q) = \bigcap \{ U(y) \mid y \in \text{st}_{\omega_1}(\text{St}_{\omega_2}(j(q))) \}$.

By Theorem 3.13 there exists a single-valued continuous mapping $\psi : N \to Y$ such that the graph $\Gamma_\psi$ is contained in $W$. Put $f = \psi \circ r : X \to Y$. For any point $x \in K$ we have $\psi(r(x)) \in \bigcap \{ U(x') \mid x' \in \text{St}_{\omega_2}(j \circ r(x)) \}$. Since $x \in \text{St}_{\omega_2}(j \circ r(x))$, then $\psi(r(x)) \in U(x)$. That is, the graph of $f$ is contained in $U$.

4. Fibrations with 2-manifold fibers

The following Lemma is a weak form of Compact Filtration Lemma from [19].

**Lemma 4.1.** Any polyhedrally connected lower continuous finite filtration of complete mappings of a compact space contains a compact approximately connected subfiltration of the same length.

**Lemma 4.2.** Let $F : X \to Y$ be equi locally hereditarily aspherical, lower 2-continuous complete multivalued mapping of ANR-space $X$ to Banach space $Y$. Suppose that a compact submapping $\Psi : A \to Y$ of $F|_A$ is defined on a compactum $A \subset X$ and admits continuous approximations. Then for any $\varepsilon > 0$ there exists a neighborhood $OA$ of $A$ and a compact submapping $\Psi' : OA \to Y$ of $F|_{OA}$ such that $\Gamma_\Psi \subset O(\Gamma_\Psi, \varepsilon)$, $\Psi'$ admits a compact approximately $\infty$-connected filtration, and $\text{cal} \Psi' < \varepsilon$.

**Proof.** Fix a positive number $\varepsilon$. Apply Lemma 2.6 with $\alpha$ being equi local hereditary asphericity to get a positive number $\varepsilon_2 < \varepsilon/4$. By Lemma 2.7 the mapping $F$ is lower polyhedrally 2-continuous. Subsequently applying Lemma 2.6 with $\alpha$ being polyhedral $n$-continuity for $n = 2, 1, 0$, we find positive numbers $\varepsilon_1, \varepsilon_0$, and $\delta$ such that $\delta < \varepsilon_0 < \varepsilon_1 < \varepsilon_2$ and for every point $(x, y) \in O(\Gamma_\Psi, \delta)$ the pair $(O(y, \varepsilon_1) \cap F(x), O(y, \varepsilon_2) \cap F(x))$ is polyhedrally 2-connected, the pair...
(O(y, ε_0) ∩ F(x), O(y, ε_1) ∩ F(x)) is polyhedrally 1-connected, and the intersection O(y, ε_0) ∩ F(x) is not empty.

Let f: K → Y be a continuous single-valued mapping whose graph is contained in O(Γ_y, δ). Let f': O K → Y be a continuous extension of the mapping f over some neighborhood O K such that the graph of f' is contained in O(Γ_y, δ). Now we can define a polyhedrally connected filtration G_0 ⊂ G_1 ⊂ G_2: O K → Y of the mapping F|O K by the equality

\[ G_i(x) = O(f'(x), \varepsilon_i) \cap F(x). \]

Since the set \( \bigcup_{x \in O K} \{ x \} \times O(f'(x), \varepsilon_i) \) is open in the product O K × Y and the mapping F is complete, then G_i is also complete. Clearly, calG_2 < 2\varepsilon_2 < \varepsilon and for any point x ∈ K the set G_i(x) is contained in O(Γ_y, ε). Now, applying Lemma 4.1 to the filtration G_0 ⊂ G_1 ⊂ G_2, we obtain a compact approximately connected subfiltration F_0 ⊂ F_1 ⊂ F_2: O K → Y. By the choice of ε_2 the mapping F_2 has approximately aspherical point-images. Therefore, the filtration F_0 ⊂ F_1 ⊂ F_2 is approximately \( \infty \)-connected. Finally, we put \( \Psi' = F_2 \).

**Theorem 4.3.** Let F: X → Y be equi locally hereditarily aspherical, lower 2-continuous complete multivalued mapping of locally compact ANR-space X to Banach space Y. Suppose that a compact submapping \( \Psi: A → Y \) of F|A is defined on compactum A ⊂ X and admits continuous approximations. Then for any \( \varepsilon > 0 \) there exists a neighborhood OA of A and a single-valued continuous selection s: OA → Y of F|OA such that OA \( \subset O(Γ_y, ε) \).

**Proof.** Consider a \( G_δ \)-subset G ⊂ X × Y such that all fibers of F are closed in G and fix open sets \( G_i \subset X \times Y \) such that \( G = \bigcap_{i=1}^{∞} G_i \). Fix \( \varepsilon > 0 \) such that O(Γ_y, ε) \( \subset G_1 \). By Lemma 4.2 there is a neighborhood \( U_1 \) of A in X and a compact submapping \( \Psi_1: U_1 → Y \) of F|U_1 such that Γ_y,1 \( \subset O(Γ_y, ε) \), \( \Psi_1 \) admits a compact approximately \( \infty \)-connected filtration, and calΨ_1 < ε. Since X is locally compact and A is compact, there exists a compact neighborhood OA of A such that OA \( \subset U_1 \). By Theorem 3.14 the mapping \( \Psi_1|O A \) admits continuous approximations. Take \( \varepsilon_1 < \varepsilon \) such that the neighborhood \( U_i = O(Γ_y, 1(OA), \varepsilon_1) \) lies in O(Γ_y, ε). Clearly, \( U_i \subset G_1 \).

Now by induction with the use of Lemma 4.2 we construct a sequence of neighborhoods \( U_1 \supset U_2 \supset U_3 \supset \ldots \) of the compactum OA, a sequence of compact submappings \( \{ Ψ_k: U_k → Y \}_{k=1}^{∞} \) of the mapping F, and a sequence of neighborhoods \( U_k = O(Γ_y, 1(OA), \varepsilon_k) \) such that for every \( k ≥ 2 \) we have calΨ_k < \( \varepsilon_{k-1}/2 < ε/2^k \), and \( U_k(OA) \) is contained in \( U_{k-1}(OA) \cap G_k \). It is not difficult to choose the neighborhood \( U_k \) of
the graph $\Gamma_{\Psi_k}$ in such a way that for every point $x \in U_k$ the set $U_k(x)$ has diameter less than $\frac{3}{2^k}$.

Then for any $m \geq k \geq 1$ and for any point $x \in OA$ we have $\Psi_m(x) \subset O(\Psi_k(x), \frac{3}{2^k})$; this implies the fact that the sequence $\{\Psi_k|_{OA}\}_{k=1}^{\infty}$ is a Cauchy sequence. Since $Y$ is complete, there exists the limit $s: OA \to Y$ of this sequence. The mapping $s$ is single-valued by the condition $cal\Psi_k < \frac{1}{2^k}$ and is upper semicontinuous (and, therefore, is continuous) by the upper semicontinuity of all the mappings $\Psi_k$. Clearly, for any $x \in OA$ the point $s(x)$ lies in $G(x)$ and is a limit point of the set $F(x)$. Since $F(x)$ is closed in $G(x)$, then $s(x) \in F(x)$, i.e. $s$ is a selection of the mapping $F$.

\[ \square \]

**Theorem 4.4.** Let $p: E \to B$ be a Serre fibration of locally connected compacta with all fibers homeomorphic to some fixed two-dimensional manifold. If $B \in ANR$, then any section of $p$ over closed subset $A \subset B$ can be extended to a section of $p$ over some neighborhood of $A$.

**Proof.** Let $s: A \to E$ be a section of $p$ over $A$. Embed $E$ into Hilbert space $l_2$ and consider a multivalued mapping $F: B \to l_2$ defined as follows:

$$F(b) = \begin{cases} s(b), & \text{if } b \in A \\ p^{-1}(b), & \text{if } x \in B \setminus A. \end{cases}$$

Since every fiber $p^{-1}(b)$ is compact, the mapping $F$ is complete. By Lemma 2.8 the mapping $F$ is equi locally hereditarily aspherical and lower 2-continuous. We can apply Theorem 4.3 to the mapping $F$ and its submapping $s$ to find a single-valued continuous selection $\tilde{s}: OA \to l_2$ of $F|_{OA}$. By definition of $F$, we have $\tilde{s}|_{A} = F|_{A} = s$. Clearly, $\tilde{s}$ defines a section of the fibration $p$ over $OA$ extending $s$. \[ \square \]

**Definition 4.5.** For a mapping $p: E \to B$ we say that $s: B \to E$ is $\varepsilon$-section if the map $p \circ s$ is $\varepsilon$-close to the identity $id_B$.

The following proposition easily follows from Theorem 4.1 of the paper [15].

**Proposition 4.6.** If $p: E \to B$ is a locally trivial fibration of finite-dimensional compacta with locally contractible fiber, then there is $\varepsilon > 0$ such that an existence of $\varepsilon$-section for $p$ implies an existence of a section for $p$.

We will use the following two propositions in the proof of existence of global sections in Serre fibrations. For the definition and basic properties of Menger manifolds we refer the reader to [2]. Proofs of these two propositions follow from Bestvina’s construction of Menger manifold [2] and Dranishnikov’s triangulation theorem for Menger manifolds [7].
Proposition 4.7. Let $X$ be a compact 2-dimensional Menger manifold. For any $\varepsilon > 0$ there exist a finite polyhedron $P$ and maps $g: X \to P$ and $h: P \to X$ such that $h \circ g$ is $\varepsilon$-close to the identity. If $\pi_1(X) = 0$, then we may choose $P$ with $\pi_1(P) = 0$.

Proposition 4.8. Let $X$ be a compact 3-dimensional Menger manifold with $\pi_1(X) = \pi_2(X) = 0$. For any $\varepsilon > 0$ there exist a finite polyhedron $P$ with $\pi_1(P) = \pi_2(P) = 0$ and maps $g: X \to P$ and $h: P \to X$ such that $h \circ g$ is $\varepsilon$-close to the identity.

Theorem 4.9. Let $p: E \to B$ be a Serre fibration of locally connected compactum $E$ onto an ANR-compactum $B$ with all fibers homeomorphic to a given two-dimensional manifold $M$. If $M$ is not homeomorphic to the sphere or the projective plane, then $p$ admits a global section provided one of the following conditions holds:

(a) $\pi_1(M)$ is abelian and $H^2(B; \pi_1(F_b)) = 0$
(b) $\pi_1(M)$ is non-abelian, $M$ is not homeomorphic to the Klein bottle and $\pi_1(B) = 0$.
(c) $M$ is homeomorphic to the Klein bottle and $\pi_1(B) = \pi_2(B) = 0$.

Proof. Embed $E$ into the Hilbert space $l_2$ and consider a multivalued mapping $F: B \to l_2$ defined as $F = p^{-1}$. Since every fiber $p^{-1}(b)$ is compact, the mapping $F$ is complete. It follows from Lemma 2.8 that the mapping $F$ is equi locally hereditarily aspherical and lower 2-continuous.

Now we show that $F$ admits a compact singular approximately $\infty$-connected filtration. In cases (a) and (b) there exists $UV^1$-mapping $\mu$ of Menger 2-dimensional manifold $L$ onto $B$ [4]. Note that $\pi_1(L) = 0$ if $\pi_1(B) = 0$. In case (c) we consider $UV^2$-mapping $\mu$ of Menger 3-dimensional manifold $L$ onto $B$ [6]; note that $\pi_1(L) = \pi_2(L) = 0$ if $\pi_1(B) = \pi_2(B) = 0$. Since $\dim L < \infty$, the induced fibration $p_L = \mu^*(p): E_L \to L$ is locally trivial [11]. By Proposition 4.6 there is $\varepsilon > 0$ such that an existence of $\varepsilon$-section for $p_L$ implies an existence of a section for $p_L$. In cases (a) and (b), by Proposition 4.7 there exist a 2-dimensional finite polyhedron $P$ and continuous maps $g: L \to P$ and $h: P \to L$ such that $h \circ g$ is $\varepsilon$-close to the identity (we assume $\pi_1(P) = 0$ in case $\pi_1(B) = 0$). In case (c) by Proposition 4.8 there exist a 3-dimensional finite polyhedron $P$ with $\pi_1(P) = \pi_2(P) = 0$ and continuous maps $g: L \to P$ and $h: P \to L$ such that $h \circ g$ is $\varepsilon$-close to the identity.

Consider a locally trivial fibration $p_P = h^*(p_L): E_P \to P$.

Claim. The fibration $p_P$ has a section $s_P$. 

Proof. (a) If \( \pi_1(M) \) is abelian and \( H^2(B; \pi_1(F_b)) = 0 = H^2(P; \pi_1(F_b)) \), the fibration \( p_P \) has a section \( s_P \) [20].

(b) Since \( \pi_1(P) = 0 \) and \( \dim P = 2 \), then \( P \) is homotopy equivalent to a bouquet of 2-spheres \( \Omega = \bigvee_{i=1}^m S^2_i \). Let \( \psi: P \to \Omega \) and \( \phi: \Omega \to P \) be maps such that \( \phi \circ \psi \) is homotopic to the identity \( \text{id}_P \). The locally trivial fibration over a bouquet \( p_\Omega = \phi^*(p_P): E_\Omega \to \Omega \) has a global section if and only if it has a section over every sphere of the bouquet. If the fiber \( M \) has non-abelian fundamental group and is not homeomorphic to Klein bottle, then the space of autohomeomorphisms \( \text{Homeo}(M) \) has simply connected identity component \( \Pi \) and therefore any locally trivial fibration over 2-sphere with fiber homeomorphic to \( M \) has a section (in fact, this fibration is trivial). Hence, the fibration \( p_\Omega \) has a section \( s_\Omega \). This section defines a lifting of the map \( \phi \circ \psi: P \to P \) with respect to \( p_P \). Since \( p_P \) is a Serre fibration and \( \phi \circ \psi \) is homotopic to the identity, the identity mapping \( \text{id}_P \) has a lifting \( s_P: P \to E_P \) with respect to \( p_P \) which is simply a section of \( p_P \).

(c) Since \( \pi_1(P) = \pi_2(P) = 0 \) and \( \dim P = 3 \), then \( P \) is homotopy equivalent to a bouquet of 3-spheres \( \Omega = \bigvee_{i=1}^m S^3_i \). Let \( \psi: P \to \Omega \) and \( \phi: \Omega \to P \) be maps such that \( \phi \circ \psi \) is homotopic to the identity \( \text{id}_P \). The locally trivial fibration over the bouquet \( p_\Omega = \phi^*(p_P): E_\Omega \to \Omega \) has a global section if and only if it has a section over every sphere of the bouquet. Since the space of autohomeomorphisms of the Klein bottle \( \text{Homeo}(K^2) \) has \( \pi_2(\text{Homeo}(K^2)) = 0 \) [11], any locally trivial fibration over 3-sphere with fiber homeomorphic to \( K^2 \) has a section (in fact, this fibration is trivial). Hence, the fibration \( p_\Omega \) has a section \( s_\Omega \). This section defines a lifting of the map \( \phi \circ \psi: P \to P \) with respect to \( p_P \). Since \( p_P \) is a Serre fibration and \( \phi \circ \psi \) is homotopic to the identity, the identity mapping \( \text{id}_P \) has a lifting \( s_P: P \to E_P \) with respect to \( p_P \) which is simply a section of \( p_P \).

By the construction of \( P \) the section \( s_P \) defines an \( \varepsilon \)-section for \( p_L \). Therefore, \( p_L \) has a section \( s_L \). Clearly, \( s_L \) defines a lifting \( T: L \to E \) of \( \mu \) with respect to \( p \). Finally, we define compact singular filtration \( \mathcal{F} = \{(F_i, T_i)\}_{i=0}^2 \) of \( F \) as follows:

\[
F_0 = F_1 = \mu^{-1}: B \to L, \quad F_2 = F, \quad T_i = \text{id} \quad \text{for} \ i = 0
\]

and \( T_i \) is defined fiberwise by \( T_i(x) = T_{\mu^{-1}(x)}: \mu^{-1}(x) \to F(x) \). The filtration \( \mathcal{F} \) is approximately connected since for \( i = 0, 1 \) any compactum \( F_i(x) \) is \( UV^1 \). And \( \mathcal{F} \) is approximately \( \infty \)-connected since every compactum \( F(x) \) is an aspherical 2-manifold (and therefore is approximately \( n \)-aspherical for all \( n \geq 2 \)).
Now we can apply Theorem 4.3 to the mapping $F$ to find a single-valued continuous selection $s: B \to I^2$ of $F$. Clearly, $s$ defines a section of the fibration $p$. \hfill □

The following Remark explains the appearance of the condition (c) in Theorem 4.9.

**Remark 4.10.** There exists a locally trivial fibration over 2-sphere with fibers homeomorphic to Klein bottle having no global section.

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