Energy Theorem for 2+1 dimensional gravity

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Abstract

We prove a positive energy theorem in 2+1 dimensional gravity for open universes and any matter energy-momentum tensor satisfying the dominant energy condition. We consider on the space-like initial value surface a family of widening Wilson loops and show that the energy-momentum of the enclosed subsystem is a future directed time-like vector whose mass is an increasing function of the loop, until it reaches the value \(1/4G\) corresponding to a deficit angle of \(2\pi\). At this point the energy-momentum of the system evolves, depending on the nature of a zero norm vector appearing in the evolution equations, either into a time-like vector of a universe which closes kinematically or into a Gott-like universe whose energy momentum vector, as first recognized by Deser, Jackiw and ’t Hooft is space-like. This treatment generalizes results obtained by Carroll, Fahri, Guth and Olum for a system of point-like spinless particle, to the most general form of matter whose energy-momentum tensor satisfies the dominant energy condition. The treatment is also given for the anti de Sitter 2+1 dimensional gravity.

I. INTRODUCTION.

The energy-momentum and angular momentum of a system in general relativity in \(n + 1\) dimensions with \(n \geq 3\) is defined by means of the flux of a pseudotensor through a \(n − 1\) dimensional space-like surface located at space infinity. It is well known that such energy-momentum can be defined when the metric approaches at space infinity the minkowskian metric \([1]\). Such a definition cannot be taken over directly to 2 + 1 dimensional gravity; in fact it is true that the space outside the sources is exactly flat, but the global structure of the space time at space infinity is that of a cone with a non zero deficit angle. In \([2]\) it is proved that for a stationary, static distribution of matter such a deficit angle equals the space integral of \(T^0_0\) i.e. the total amount of mass of the matter present in the system. Such result however is no longer true in the general situation. With regard to the angular momentum the situation in
more critical; one can consider a rather artificial situation in which the deficit angle is zero in which case one can carry over the usual procedure of 3+1 dimension and thus one is able in the stationary case to identify the angular momentum with the time jump which occurs in a synchronous reference system when one performs a closed trip around the source.

A more general definition of energy-momentum and angular momentum in 2+1 dimension can be derived by computing the holonomies of the $ISO(2,1)$ group which were introduced in connection to the formulation of pure 2+1 dimensional gravity as a Chern-Simon theory. Such idea of defining mass and angular momentum in 2+1 dimensional gravity by means of holonomies is already contained in the seminal paper for a system of point-like spinless particles. From the Lorentz holonomy one can extract a three component vector which transforms like a Lorentz vector under local Lorentz transformations. Such a vector does not depend on deformations of the loop keeping the origin fixed provided such loop extends only outside the matter, and its square can be naturally identified with the square of the mass of the system. Change in the origin of the loop is equivalent to a Lorentz transformation on such a vector. The above described definition of the energy-momentum of a system in 2+1 dimensional gravity has been applied in for analyzing general properties of a system of spinless point particles which form an open universe. The important result of their investigation is that if a subsystem of such a collection of particles has space-like momentum the whole system must possess total space-like momentum; in particular if a Gott pair exists in the system, necessarily the universe is either closed, or if open it possesses a space-like momentum.

Here adopting the same definition of the energy-momentum we shall derive general properties for a system whose matter energy-momentum tensor satisfies the dominant energy condition (DEC). We recall that the dominant energy condition states that for any time-like future directed vector $v^\mu$ its contraction with the energy momentum tensor $T^\mu_\nu v^\nu$ is time-like future directed. We shall prove a positive energy theorem in 2+1 dimension which can be stated as follows: Given a space-like initial data two dimensional surface consider on it a family of expanding Wilson loops; if the matter energy-momentum tensor satisfies the dominant energy
condition (DEC), the energy-momentum of the subsystem enclosed by the loop is a time-like future directed vector whose norm increases as the loop embraces more and more matter. This monotonic increase occurs until the mass of the system reaches the critical value $1/4G$ corresponding to a deficit angle $2\pi$. After that point, the evolution equations we derive in Sec.III show that two alternatives can occur: either the universe still increases its mass, going over to a closed universe or it goes over to a Gott-like universe whose energy-momentum is space-like. The choice depends on the nature of a vector in the evolution equation whose norm at the critical point is zero. If such a vector is the zero vector then the universe evolves into a closed one; if it is light-like it goes over to the Gott-like situation. Then we obtain again for a general matter energy-momentum tensor satisfying the DEC, the result of [4] that once a subsystem of the universe has space-like momentum by adding matter one can never recover an open time-like universe.

From the ISO$(2, 1)$ holonomy one can extract [3] another scalar invariant which is given by the scalar product of the above described energy-momentum vector with another three component object which obeys inhomogeneous transformation properties. Such a second invariant can be related to the total angular momentum of the system which in 2+1 dimension has just one component. Evolution equation of similar type are also written for the angular momentum.

As the situation in 2+1 dimensions is different from the one occurring in higher dimensions, one needs to support such definitions of the energy-momentum and angular momentum.

For the energy-momentum one can produce the following arguments 1) For weak sources the given definition goes over the definition in Minkowski space [2]. 2) In the stationary static case the mass of the system coincides with the sum of the masses of the sources (see [2] and Sect.III of the present paper). This is satisfactory because in 2+1 dimensions there are no gravitational forces on thus no gravitational potential. Thus one expects the above result in the static case. 3) The theorem of Sect.III of the present paper proves that if the sources satisfy the DEC and the total mass of the system is less that the critical value $1/4G$ such vector in addition to being conserved is time-like future directed.
With regard to the here adopted definition of angular momentum one can produce the following arguments to support it. 1) It is a conserved quantity. 2) The vector whose scalar product with the energy-momentum vector gives the angular momentum is subject to transformation properties which are formally identical to those that occur in Minkowski space (see Appendix B of the present paper). 3) The solution of the evolution equations for weak sources gives for the angular momentum the same result as in Minkowski space.

In Sect.V the described treatment is extended to 2+1 dimensional gravity in presence of a (negative) cosmological constant i.e. the anti de Sitter case. Here due to the fact that the anti de Sitter group is the direct product of two $SO(2,1)$ groups, one obtains two independent equations similar to the one which intervene in the Poincaré case and whose discussion to a large extent can be carried through similarly to the previous case.

II. ENERGY IN 2+1 DIMENSIONS

The usual definition of energy which works in dimension equal or higher than 3+1, does not apply to 2+1 dimensional gravity because except for the empty space, the space is never asymptotically minkowskian. A definition of the energy in 2+1 dimensions similar to the one given in 3+1 is obtained in [6] replacing the asymptotic flat minkowski space with the conical space and computing in cartesian coordinates

$$\Theta_c = \frac{1}{2} \varepsilon_{abc} \oint \Gamma_{ab} \, dx^i;$$  \hspace{1cm} (1)

where $\Gamma_{ab}(x)$is the spin-connection and the integral is evaluated on a loop at space infinity. An alternative expression equivalent to (1) which is fully covariant in the sense that it can be computed in any coordinate system is provided by the Lorentz holonomy

$$W = P \exp \left[ -i \oint J_a \Gamma^a_{\mu} \, dx^\mu \right] = e^{-i J_a \Theta^a}$$ \hspace{1cm} (2)

where $\Gamma^a_{\mu} = \frac{1}{2} \varepsilon^a_{\ b} \Gamma^b_{\ c\mu}$. The Lorentz holonomy in (2) is along a closed contour but the trace is not taken. The energy-momentum vector is defined by the $\Theta^a$ appearing on the right hand side.
of (2) divided by $8\pi G$. The reason why in conical coordinates (2) reduce to (1) is that in such coordinate system $\Gamma^a_\mu dx^\mu = 8\pi G M \delta^a_0 \frac{r \wedge dl}{r^2}$. In the most general case the Lorentz holonomy cannot be written as a simple exponential, but for example in the fundamental representation on must allow for a $\pm$ sign in front of it \cite{4}. The direction of $\Theta^a$ is identified by the eigenvector (with eigenvalue 1) belonging to the adjoint representation. Once the direction of $\Theta_a$ is established we can distinguished three cases according to the norm of $\Theta^a$. In the following we always use the fundamental representation of $SU(1,1)$ with commutation relations given by

$$[J_a, J_b] = i \epsilon_{abc} J^c$$

and the traces given by

$$\text{Tr}(J_a J_b) = -\frac{1}{2} \eta^{ab}$$

where $\eta^{ab} \equiv \text{diag}(-1, 1, 1)$.

**Case 1.** $\Theta^a$ is a null vector. In this case the expansion of (2) in the fundamental representation is given by

$$W = I - i\Theta^a J_a$$

all the remaining terms being equal to zero and taking the trace with $J^b$ one obtains

$$\Theta^b = -2i \text{Tr}(W J^b).$$

**Case 2.** $\Theta^a$ is space-like.

$$W = \cosh \frac{\sqrt{\Theta^a \Theta_a}}{2} I - 2i \frac{\sinh \sqrt{\Theta^a \Theta_a}/2}{\sqrt{\Theta^a \Theta_a}} (J_a \Theta^a).$$

Taking the trace

$$\text{Tr}(J^b W) = i \frac{\sinh \sqrt{\Theta^a \Theta_a}/2}{\sqrt{\Theta^a \Theta_a}} \Theta^b$$

This determines completely $\Theta^b$ as sinh is an odd monotonic function.

**Case 3.** $\Theta^a$ is time-like. The expansion in the fundamental representation is
\[ W = \cos \frac{\sqrt{-\Theta^a \Theta_a}}{2} I - 2i \frac{\sin \sqrt{-\Theta^a \Theta_a}/2}{\sqrt{-\Theta^a \Theta_a}} (J_a \Theta^a). \] (9)

Taking the trace

\[ \text{Tr}(J^b W) = i \frac{\sin \sqrt{-\Theta^a \Theta_a}/2}{\sqrt{-\Theta^a \Theta_a}} \Theta^b. \] (10)

Eq. (10) is not sufficient to determine \( \Theta^a \), but combining it with the trace of \( W \)

\[ \text{Tr}(W) = 2 \cos \frac{\sqrt{-\Theta^a \Theta_a}}{2} \] (11)

\( \Theta^a \) is fixed if one imposes that \( \delta \equiv 8\pi GM = \sqrt{-\Theta^a \Theta_a} \) satisfies \( 0 \leq \delta \leq 2\pi \) (open universe). In this way we are also able to establish whether \( \Theta^a \in V^+ \) or \( V^- \). The use of the adjoint representation would have left us with the ambiguity \((\delta, \hat{n}) \leftrightarrow (2\pi - \delta, -\hat{n})\), where \( \hat{n} \) is the versor corresponding to the vector defined by (11). In the following construction, however, this ambiguity can be avoided by continuity reasons. On the other hand for a closed universe the total energy is defined as the Euler number of the space 2-manifold.

**III. A POSITIVE ENERGY THEOREM IN 2+1 DIMENSION**

We shall assume the existence of an edgeless space-like initial data two dimensional surface and we shall investigate here the consequences on the energy of imposing on the matter energy-momentum tensor the dominant energy condition. In our analysis we shall use eq.(2) in defining on it the energy. Given the space-like initial data two dimensional surface \( \Sigma \), we consider on it a family of closed contours \( x^\mu(s, \lambda) \) with \( 0 \leq \lambda \leq 1 \) \( x^\mu(s, 0) = x^\mu(s, 1) \) and such that for increasing \( s \) \( (s_1 < s_2) \) the contour \( x^\mu(s_1, \lambda) \) is completely contained in \( x^\mu(s_2, \lambda) \) and which shrinks to a single point for \( s = 0 \).

It is easy to prove (see Appendix A) that the Wilson loop under the deformation induced by the parameter \( s \) changes according to the following equation

\[
\frac{dW}{ds}(s, 1) \equiv \frac{dW}{ds}(s, 1) + i [J_a \Gamma^a \mu(s, 1) \frac{dx^\mu}{ds}, W(s, 1)] = \]

\[
iW(s, 1) \int_0^1 d\lambda W(s, \lambda)^{-1} R_{\mu\nu}(s, \lambda) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{ds} W(s, \lambda) \] (12)
where $R_{\mu\nu}$ is the curvature form in the fundamental representation. In 2+1 dimensions the curvature 2-form is given directly by the energy-momentum tensor

$$R_{\mu\nu} = 8\pi G \eta_{\mu\nu\rho} T^{\alpha\rho} J_a$$

where $\eta_{\mu\nu\rho} \equiv \sqrt{-g} \epsilon_{\mu\nu\rho}$ and $T^{\alpha\rho}$ is the energy momentum tensor contracted with the dreibein, i.e. $T^{\alpha\rho} = \epsilon_{\mu} J_a (13)$. Substituting (13) into (12) we have

$$\frac{DW}{ds}(s, 1) = 8\pi i G W(s, 1) \int_0^1 d\lambda W(s, \lambda)^{-1} \eta_{\mu\nu\rho} T^{\alpha\rho} J_a \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{ds} W(s, \lambda)$$

where $\eta_{\mu\nu\rho}$ is the area vector $N_\rho$ corresponding to the 2-dimensional surface $\Sigma$, and thus it is time-like. But then

$$q^a(s, \lambda) \equiv T^{\alpha\rho} N_\rho$$

according to the DEC is time-like and future directed. By substituting eq. (15) into eq. (14) we have

$$\frac{DW}{ds}(s, 1) = -8\pi i G W(s, 1) \int_0^1 d\lambda W(s, \lambda)^{-1} J_a q^a(s, \lambda) W(s, \lambda) = -8\pi i G W(s, 1) J_a Q^a(s)$$

where due to the similitude transformation $W^{-1} J_a q^a W$, also the integrated $Q^a$ is time-like and future directed.

If we parametrize $W$ in the fundamental representation as $W(s, 1) = w(s) I - 2i J_a \theta^a(s)$ the evolution equation gives

$$\frac{dw(s)}{ds} = 4\pi G Q_a(s) \theta^a(s)$$

and

$$\frac{D\theta^b(s)}{ds} = \frac{d\theta^b(s)}{ds} + \Gamma^{b}_{c\mu}(s) \frac{dx^\mu}{ds} \theta^c(s) = 4\pi G \left( w(s) Q^b(s) + \epsilon^b_{\imath m} \theta^\imath(s) Q^m(s) \right)$$

The initial condition $W(0, 1) = I$ imposes that $w(0) = 1$ and $\theta^b(0) = 0$. During the evolution in $s$ the quantity $w(s)^2 - \theta_b(s) \theta^b(s)$ is conserved and the initial conditions fix its value to 1. This simply means that $\det(W)$ is equal to 1 for every $s$. 

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We define $\bar{s}$ as the lower bound of the support of $Q^a(s)$, (if the point $(\lambda = 0, s = 0)$ is inside the matter $\bar{s}$ coincides with $s = 0$, otherwise not). For $s \geq \bar{s}$, using the initial condition $w(\bar{s}) = 1$ and $\theta^a(\bar{s}) = 0$, from (17) and (18) we obtain
\[
\begin{align*}
\frac{dw(s)}{ds} &= 1 + 4\pi G \int_{\bar{s}}^{s} \theta^a(s') Q_a(s') ds' \\
\theta^a(s) &= 4\pi G \int_{\bar{s}}^{s} Q^a(s') ds' + \text{higher order in } (s - \bar{s})^2.
\end{align*}
\]
Using this inequalities eq. (17) becomes
\[
\begin{align*}
\frac{dw(s)}{ds} &\leq -4\pi G m(s) \sqrt{1 - w(s)^2} \leq 0
\end{align*}
\] from which we see that for $-1 \leq w(s) \leq 1$, $w(s)$ is a decreasing function of $s$.

Physically the monotonic decrease of $w(s)$ from 1 to $-1$ means that by adding matter satisfying the DEC the mass of the system increases up to its final value which for an open universe is less than $\frac{1}{4G}$, i.e. $\delta \leq 2\pi$ and thus the final value of $w$ is larger than $-1$. From
this we have also for an open universe that $\theta^a(s)$ and thus $\Theta^a(s)$ is a future directed time-like vector. In conclusion for an open universe with matter sources satisfying the DEC the total energy-momentum vector defined by (2) is time-like.

To prove eq. (21) we write using $-1 \leq w(s) \leq 1$, $w(s) = \cos(f(s))$ with $0 \leq f(s) \leq \pi$. Substituting in the inequality (23) we have

$$-\sin(f(s)) f'(s) \leq -4\pi G m(s) \sin(f(s))$$

(24)

But then

$$f'(s) \geq 4\pi G m(s) \rightarrow f(s) \geq 4\pi G \int_0^s m(s') ds'$$

(25)

from which (21) follows. Physically such a relation means that for an open universe with matter satisfying the DEC the total mass of the universe is higher or equal to $\int_0^\infty m(s') ds'$, a result which is common to Minkowski space, despite the non linear composition law of momenta in 2+1 gravity. Such inequality is optimal in absence of the other hypothesis, as it is saturated by the static universe [2].

We want now to inquire what happens if $w(s)$ instead of stopping in its evolution before reaching $-1$, reaches this value at $s = s_0$. In $w(s_0) = -1$, $\theta^a(s_0)\theta_a(s_0) = 0$ which means that $\theta^a(s_0)$ is either the zero vector or a light-like vector. First we consider the zero vector alternative. From (18) we see that for $s = s_0 + \epsilon$, $\theta^a(s)$ is a past-directed time-like vector, thus giving $w(s_0 + \epsilon) > -1$. With identical reasoning as before one can prove that $w(s)$ is monotonically non-decreasing function until it reaches the value 1 which correspond to the closure of the universe because in this case the total mass of the universe is $1/2G$ (i.e. $\delta = 4\pi$). Even if $\theta_a(s)$ for $s > s_0$ is time-like past-directed, the energy-momentum vector of our subsystem (contained in the contour s)

$$\Theta^a(s) = \frac{M(s)}{\sin M(s)/2} \theta^a(s)$$

(26)

is future-directed because now $\sin \frac{M(s)}{2} < 0$. We come now to the case in which $\theta_a(s_0) \neq 0$ and light-like, necessarily future directed because the limit of a time-like future directed vector. From eq. (17) we have that $w(s_0+\epsilon) < -1$ and thus $\theta_a(s_0+\epsilon)$ is space-like. The simplest example
of such a situation is the Gott-pair in which two particles each having time-like momentum give rise to a universe with total space-like momentum \cite{[8]}. We recall that for such system there exists space-like surfaces \cite{[8],[9]} which satisfy the condition of our theorem.

Adding matter satisfying the DEC we cannot return to an open universe (with $\delta < 2\pi$). In fact in such a circumstance there is a value of $s$, $s_1 > s_0$, where $w(s_1) = -1$ and it is increasing. At $s = s_1$ we also have either $\theta^a(s_1) \in L^+$ or $\theta^a(s_1) = 0$. For $\theta^a(s_1) \in L^+$, $Q_a\theta^a(s_1) < 0$ giving $\frac{dw}{ds} < 0$ which contradicts the fact that $w(s)$ increases at $s = s_1$. For $\theta_a(s_1) = 0$ we have $w(s) = -1 - 8\pi^2G^2Q_a(s_1)Q^a(s_1)(s-s_1)^2 + o((s-s_1)^2)$ which contradicts again the assumption that $w(s)$ increases at $s = s_1$. On the other hand by properly adding matter one can land to universe with $M(s) > 1/4G$. This is achieved when $\theta^a(s_1) \in L^-$. In fact now from \cite{[17]} $\frac{dw}{ds} > 0$, thus $w(s)$ increases above $-1$ and $\theta^a(s) \in V^-$ which means that the mass has increased above $2\pi$. This theorem provides a generalization of the theorem \cite{[4]} stating that for an open universe composed by a collection of spinless point particles, with total time-like momentum, no subsystem can possess space-like momentum. Here the proof is given for any distribution of matter satisfying the DEC. As final check of our definition of energy we show how in this formalism one can recover the usual ansatz for the static problem. We notice that in the static case the dreibeins can be chosen independent of time and of the form $e^a_0 = N\delta^a_0$, $e^0_i = 0$ and we have \cite{[2]} for the energy-momentum tensor $T^{00} = NT^{00}$ where $T^{00}$ is the 00 component of the energy-momentum tensor in the base given by the coordinates. Substituting into eq. \cite{[12]} we have

$$\frac{DW}{ds}(s, 1) = 8\pi iG W(s, 1) \int_0^1 d\lambda W(s, \lambda)^{-1} \sqrt{-g}N T^{00} J_0 \left(\frac{dx^1}{d\lambda} \frac{dx^2}{ds} - \frac{dx^2}{d\lambda} \frac{dx^1}{ds}\right) W(s, \lambda) \tag{27}$$

which is solved by

$$W = V^{-1}(s) \exp(-8\pi iGJ_0 \int T_0^0 \sqrt{-\gamma}dx^1dx^2) V(s) \tag{28}$$

where $\gamma$ is the determinant of the space metric and $V(s)$ is the holonomy of $\Gamma^a(s, 1)$ computed along $x^\mu(s, 1)$. Then we have

$$\delta = 8\pi GM = 8\pi G \int T_0^0 \sqrt{-\gamma}dx^1dx^2 = 4\pi G \int R^{(2)} \sqrt{-\gamma}dx^1dx^2 \tag{29}$$
in agreement with the result of [2].

IV. POINCARÉ GROUP AND A DEFINITION OF ANGULAR MOMENTUM

This section is devoted to an invariant definition of the angular momentum of the system by means of the Wilson loop. To this purpose we introduce the complete holonomy associated to the ISO(2, 1) group i.e.

\[ W = \text{Pexp} \left( -i \oint J_a \Gamma^a_{\mu} dx^\mu + P_a e^a_{\mu} dx^\mu \right) \]  

(30)

computed on a loop that encloses all matter, e.g. a loop at space infinity. Such a loop (30) can always be written in the form

\[ W = \pm \exp \left( -iJ_a \Theta^a - iP_a \Xi^a \right) \]  

(31)

As shown in Appendix B, \( \Theta^a \) is the total energy-momentum defined in the previous sections. Now we shall see that it is natural to identify \( \Xi^a \) with the total angular momentum of the system; however this identification requires some caution. Contrary to what happens for \( \Theta^a \), which, up to a Lorentz transformation, is independent of the loop chosen to calculate it, \( \Xi^a \) and its modulus depend on the point \( O \) where the loop closes. (See Appendix B for a complete treatment of the transformation law of \( \Xi^a \)). This fact is neither contradictory nor unexpected if we recall what happens to the total angular momentum of a mechanical system in special relativity [10]. Also in that case it depends both on the origin of the reference frame and on the frame itself. The only quantity, which is invariant and meaningful, is the component of the angular momentum along the total momentum. In our formalism this quantity is

\[ \mathcal{J} = -\frac{\Theta_a \Xi^a}{8\pi G \sqrt{-\Theta_a \Theta^a}} \]  

(32)

and, as expected, it does not depend on the loop used to compute it. The proof of the invariance of \( \mathcal{J} \) defined by two equivalent loops can be obtained using the results of Appendix B. In the following when we speak about angular momentum we refer to the definition (32).
check for our definition is to compute it for the geometry of a localized source (cosmic string). Using for convenience as closed contour a circle of radius $R$ on the surface $t = \text{const}$ and keeping in mind that the metric is independent of the angular variable $\theta$ we obtain

$$W = \exp (-i2\pi J_a \Gamma_a^\theta - 2\pi i P_a e_a^\theta)$$  \hspace{1cm} (33)

Using the explicit expression of $e_a^\theta$ and $\Gamma_a^\theta$ for a cosmic string we obtain

$$\mathcal{J} = -\frac{\Gamma_a^\theta e_a^\theta}{8\pi G \sqrt{\Gamma_a^\theta \Gamma_a^\theta}} = J$$  \hspace{1cm} (34)

where $J$ is the usual angular momentum defined by the time-shift which appears in a synchronous coordinate system when one encircles the source. Coming back to the general problem, also in this case we are able to write an evolution equation for the Wilson loop under a deformation of the contour. Generalizing the argument of the previous section we obtain

$$\frac{dW}{ds}(s,1) = \frac{dW}{ds} + i[\Gamma_s, W] + i[e_s, W] =$$

$$= iW(s,1) \int_0^1 d\lambda W(s,\lambda)^{-1}\{J_a R_{\lambda s}^a(s,\lambda) + P_a s_{\lambda s}^a(s,\lambda)\}W(s,\lambda)$$  \hspace{1cm} (35)

where we have projected the group curvature on the usual geometric curvature. Taking into account that our theory is torsionless and that

$$R_{\mu\nu} = 8\pi G \eta_{\mu\nu\rho} T^{\rho\nu} J_a$$  \hspace{1cm} (36)

we obtain

$$\frac{dW}{ds}(s,1) = -8\pi i G W(s,1) \int_0^1 d\lambda W(s,\lambda)^{-1} J_a q^a(s,\lambda) W(s,\lambda)$$  \hspace{1cm} (37)

where $q^a(s,\lambda)$ is the same quantity defined in sec. 3. The substantial difference between this equation and eq. (16) is that now $W$ are Poincaré transformations. This prevents us from

\footnote{We notice that eq. (34) can be considered a simple formula which gives the angular momentum for all axially symmetric geometries, once expressed in a reference in which the metric is independent of $\theta$.}
rewriting the integral in eq. (37) simply as a future directed time-like vector $Q^a(s)$. In fact the translation part of $W(s, \lambda)$ generates a new term which has no particular properties even when we impose that our energy-momentum tensor satisfies the DEC. In detail, recalling the structure of the adjoint representation of Poincaré group, we get

$$\frac{DW}{ds} = -8\pi i G W \int_0^1 d\lambda (J_a A^a_l(s, \lambda) q^l(s, \lambda) + P_a l^a(s, \lambda)) = -8\pi i GW (J_a Q^a + P_a L^a) \quad (38)$$

where $Q^a(s)$ is the same vector as the one defined in the previous section and $l^a$ is given by

$$l^a(s) = \epsilon_{bc} A^b_l(s, \lambda) q^l(s, \lambda) \Xi^c(s, \lambda) \quad (39)$$

where $A^b_l(s, \lambda)$ is the matrix action for the operator $\exp(i J_a \Theta^a(s, \lambda))$ in the adjoint representation. To write explicitly eq. (38) we introduce the following $4 \times 4$ representation

$$J_a = \begin{pmatrix} I_a & 0 \\ 0 & I_a \end{pmatrix}, \quad P_a = \begin{pmatrix} 0 & I_a \\ 0 & 0 \end{pmatrix} \quad (40)$$

where $I_a$ are the generators of $SO(2,1)$ in the fundamental representation. It is easy to verify that the matrix defined in eq. (38) satisfy the algebra of $ISO(2,1)$ and in addition $P^a$ is nilpotent. Now we parametrize the generic transformation in this representation as follows

$$W = w(s) - 2i J_a \theta^a(s) + u(s) K - 2i P_a v^a(s) \quad (41)$$

where

$$K = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (42)$$

Substituting the previous parametrization and the explicit form of generators in eq. (38) we recover the same equations of sec. 3 for the sector regarding the Lorentz transformations, and two new equations for the sector regarding the translations

$$\frac{du(s)}{ds} = 4\pi G(v_a(s) Q^a(s) + \theta^a(s) L_a(s))$$

$$\frac{Dv^a(s)}{ds} = 4\pi G(u(s) Q^a(s) + \epsilon^a_{bc} v^b(s) Q^c(s) + w(s) L^a(s) + \epsilon^a_{bc} \theta^b(s) L^c(s)). \quad (43)$$
Eq. (43) relate the angular momentum to the spacial distribution of energy and momentum. Here we shall use eq. (43) to prove that for weak sources the given definition of angular momentum identifies with usual definition of special relativity [10]. In fact to first order writing explicitly the covariant derivative we have

$$\frac{dv^a(s)}{ds} = 4\pi GL^a(s) + \epsilon_{\mu b}^a \frac{dx^\mu(s,1)}{ds} \theta^b(s)$$

(44)

and form eq. (18) with the same degree of approximation

$$\theta^b(s) = 4\pi G \int_0^s ds' \int_0^1 d\lambda q^b(s',\lambda).$$

(45)

Moreover we notice that in eq. (38)

$$L^a(s) = \int \epsilon_{\mu b}^a (x(s,\lambda) - x(s,0)) q^b(s,\lambda) d\lambda$$

(46)

because to first order in eq. (39) $\Xi^a(s,\lambda) = x^a(s,\lambda) - x^a(s,0)$ and the rotation matrix can be taken equal to the identity. We now substitute eqs. (45) and (46) into eq. (44) and integrate in $s$. Integrating by parts the $\theta$ term we obtain

$$\Xi^a(s) \approx 2v^a(s) = -8\pi G \int_0^s ds' \int_0^1 d\lambda \epsilon_{\mu b}^a x^\mu(s',\lambda) q^b(s',\lambda) + 2\epsilon_{\mu b}^a x^\mu(s,0) \theta^b(s)$$

(47)

If now we take into account the relation between $q^a(\lambda,s)$ and the energy-momentum tensor and the eq. (20) to the lowest order in $G$ the previous formula can be rewritten as

$$\Xi^a(s) \approx 2v^a(s) = -8\pi G \int dx^1 dx^2 \epsilon_{\mu \nu}^a x^\mu T^{\nu 0} + 8\pi G \epsilon_{\mu \nu}^a x^\mu(s,0) \int dx^1 dx^2 T^{\nu 0}$$

(48)

which is $-8\pi G$ times the expression of angular momentum in special relativity [10]. The last term means that we are not computing the angular momentum with respect to the origin but with respect to the point where the loop closes. This can be easily understood if we keep in mind that only this point is strictly related to the holonomy.

V. ANTI DE SITTER 2+1 DIMENSIONAL GRAVITY

The problem of defining the energy in de Sitter and anti de Sitter 3+1 dimensional gravity has been treated by Abbot and Deser [11], with the result that it is possible to define an energy
associated to the generator \( J_{04} \), which is de Sitter rather than Poincaré covariant. The results of sect. 3 can be easily extended to anti de Sitter 2+1 dimensional gravity due to the fact that the anti de Sitter algebra is the algebra of \( SU(1, 1) \times SU(1, 1) \). In this case it is meaningful to consider only the Wilson loop the full connection \( J_a \Gamma^a + \frac{1}{1} P_a e^a \), because the loop of the \( J_a \Gamma^a \) does not generate invariants under the full anti de Sitter group. We adopt the following Lie algebra

\[
[J_a, J_b] = i\varepsilon_{abc} J^c \tag{49}
\]

\[
[J_a, P_b] = i\varepsilon_{abc} P^c \tag{50}
\]

\[
[P_a, P_b] = il^2 \varepsilon_{abc} J^c. \tag{51}
\]

Following the procedure of the previous section we derive the evolution equation for the Wilson loop

\[
\frac{dW}{ds} \equiv \frac{dW}{ds} + i[\Gamma_s, W] + i[e_s, W] =
\]

\[
iW(s, 1) \int_0^1 d\lambda W(s, \lambda)^{-1} \{J_a R^a_{\lambda s}(s, \lambda) + P_a S^a_{\lambda s}(s, \lambda) + i[e_\lambda(s, \lambda)), e_s(s, \lambda)]\} W(s, \lambda) \tag{52}
\]

where \( \Gamma = J_a \Gamma^a \) and \( e = P_a e^a \) and \( R^a_{\lambda s} \) is the usual curvature and \( S^a_{\lambda s} \) is the torsion which in the following will be set to 0.

Using Einstein’s equations

\[
R_{\mu\nu} + i[e_\mu, e_\nu] = 8\pi G \eta_{\mu\nu} T^{\alpha\beta} J_a \tag{53}
\]

we can rewrite eq. (52) as

\[
\frac{dW}{ds} = -8\pi i GW(s, 1) \int_0^1 d\lambda W(s, \lambda)^{-1} q^a(s, \lambda) J_a W(s, \lambda) \tag{54}
\]

Defining in the anti de Sitter case

\[
J^+_a = \frac{1}{2} \left( J_a + \frac{1}{l} P_a \right) \tag{55}
\]
we have that

\[
[J^\pm_a, J^\pm_b] = i\epsilon_{abc} J^\pm c
\]  \hspace{1cm} (56)

while

\[
[J^\pm_a, J^\mp_b] = 0.
\]  \hspace{1cm} (57)

Using these generators the previous equation separates into two independent equations

\[
\frac{D W^\pm}{ds} = -8\pi i G W^\pm(s, 1) \int_0^1 d\lambda W^\pm(s, \lambda)^{-1} q^a(s, \lambda) J^\pm_a W^\pm(s, \lambda) =
\]

\[
-8\pi i W^\pm(s, 1) J^\pm_a Q^\pm a(s)
\]  \hspace{1cm} (58)

where

\[
W^\pm = \text{Pexp} \left( -i \int J^\pm_a (\Gamma^a_{\mu} \pm le^a_{\mu}) dx^\mu \right).
\]  \hspace{1cm} (59)

We remark that \(Q^\pm a(s)\) are future directed time like vectors because \(W^\pm\) induce a similitude transformation which is a real Lorentz transformation on \(Q^a\). Parametrizing \(W^\pm\) in the fundamental representation as follows

\[
W^\pm = e^{-iJ^\pm_a \theta^\pm a} = w^\pm - 2i J^\pm_a \theta^\pm a
\]  \hspace{1cm} (60)

we obtain the equations

\[
\frac{dw^\pm(s)}{ds} = 4\pi G Q^\pm a(s) \theta^\pm a(s)
\]  \hspace{1cm} (61)

and

\[
\frac{D \theta^\pm a(s)}{ds} = \frac{d\theta^\pm a(s)}{ds} + (\Gamma^a_{b\mu} \pm le^a_{bc} e^c_{\mu}) \theta^\pm b(s) \frac{dx^\mu(s, 1)}{ds} =
\]

\[
4\pi G (w^\pm(s) Q^\pm a(s) + \epsilon^a_{\imath m} \theta^\pm \imath (s) Q^\pm m(s)).
\]  \hspace{1cm} (62)

We notice that \(w^\pm\) and \(\theta^\pm\) obey the same initial conditions as in the Poincaré case and except for the term \(\pm le^a_{bc} e^c_{\mu} \theta^\pm b(s) \frac{dx^\mu(s, 1)}{ds}\) the same equations. As the additional term contains the \(\epsilon^a_{bc}\) we still have as in the Poincaré case \(w^\pm 2 - \theta^\pm 2 = 1\).
We can now carry over exactly the same discussion given in Sec.III to the vectors $\theta^\pm$. Again the conclusion is that $w^\pm$ which are initially equal to 1, as matter is included in the loop, decrease below 1 while $\theta^\pm$ become time-like future directed vectors. The monotonic decrease of $w^\pm$ goes on until they reach the value -1. Thus from eq.(62) we see that for $s < s_0^+$, $s < s_0^-$ i.e. before any of the two $w^\pm$ reaches -1, the mass of the system is monotonic non decreasing. If above $s_0^\pm$ both $\Theta^a_+$ and $\Theta^a_-$ remain time-like we have still a monotonic increase of mass. Instead if one of the two vectors at the critical point goes over to a space-like vector, the situation is no longer clear cut because we cannot decide a priori whether the square of the invariant mass remains positive.

The usual mass $M$ and angular momentum $J$ can be recovered by the following two combinations of the invariants associated to $\Theta^a_\pm$

\[ M = \frac{1}{8\pi G} \sqrt{-\Theta^{a_+} \Theta^{a_+} + \Theta^{-a} \Theta^{-a}} \quad (63) \]

and

\[ J = -\frac{1}{256\pi^2 G^2 lM} (\Theta^{a_+} \Theta^{a_+} - \Theta^{-a} \Theta^{-a}) \quad (64) \]

If the cosmological constant goes to zero, these equations becomes the definitions given in the previous section for the Poincaré group. In the region where the deficit angles, associated to $\Theta^a_+$ and $\Theta^a_-$, are less then $2\pi$ the following inequality between mass and angular momentum holds

\[ |J| \leq M/2l \quad (65) \]

where $l^2$ is the cosmological constant. This fact allows the existence of black-holes in the anti de Sitter gravity [12].

**VI. CONCLUSIONS**

Due to the angular deficit at infinity which appears in 2+1 dimensional gravity, it is not possible to carry over from gravity in 3 + 1 dimensions the usual procedure for defining energy
and momentum. The holonomies provide an invariant, consistent procedure for defining the total energy and angular momentum for a system in 2 + 1 dimensions. They give rise to conserved quantities which for small masses go over to the minkowski results. Given an initial value space-like surface, we considered a family of widening loops and proved that for any matter energy-momentum tensor that satisfies the dominant energy condition, the energy-momentum of the subsystem enclosed by the loop is a time-like future directed vector, whose norm increases as the loop widens.

By enclosing more and more matter satisfying the DEC, one reaches the angular deficit of $2\pi$ and then one can either go over to a time-like universe which, having an angular deficit greater that $2\pi$ closes kinematically, or to a space-like universe of the Gott type.

By adding more matter one can never come back to a space-like open universe; this provides an extension of the result by Carroll, Fahri, Guth and Olum to the most general matter energy-momentum tensor satisfying the DEC, i.e. that in 2+1 dimensions if a subsystem has space-like momentum the universe is either closed or space-like.

Without much changes the above mentioned results are carried over to the anti de Sitter case.

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**APPENDIX A:**

In this appendix we derive the equation for the change of a Wilson loop under a continuous deformation. Consider a family of curves parametrized by a parameter $\lambda$ which describes the curve and a parameter $s$ which labels the member of the family of curves. Given a connection $A(x)$ on the surface described by the two parameters $\lambda$ and $s$ we can consider the two
components $A_\lambda$ and $A_s$ of the connection according to the formulae

$$A_\lambda(s, \lambda) = A_\mu(x) \frac{dx^\mu}{d\lambda} \quad \text{and} \quad A_s(s, \lambda) = A_\mu(x) \frac{dx^\mu}{ds} \quad (A1)$$

The path ordered integral along the generic arc is given by

$$W(s, \lambda) = \text{Pexp}\left(-i \int_0^\lambda A_\lambda(s, \lambda')d\lambda'\right) \quad (A2)$$

One now considers the derivative of $W(s, \lambda)$ with respect to $s$; actually it is more significant to consider the so called covariant derivative of $W(s, \lambda)$ given by

$$\frac{DW(s, \lambda)}{ds} = \frac{\partial W(s, \lambda)}{\partial s} + i(A_s(s, \lambda)W(s, \lambda) - W(s, \lambda)A_s(s, 0)) =$$

$$= \frac{\partial W(s, \lambda)}{\partial s} + iW(s, \lambda)(W(s, \lambda)^{-1}A_s(s, \lambda)W(s, \lambda) - A_s(s, 0)) \quad (A3)$$

The first term on the r.h.s. of (A3) is given by (A3)

$$-iW(s, \lambda) \int_0^\lambda d\lambda'W(s, \lambda)^{-1}\frac{\partial A_\lambda(s, \lambda')}{\partial s}W(s, \lambda') \quad (A4)$$

On the other hand the second term in (A3) can be written as

$$iW(s, \lambda) \int_0^\lambda \frac{\partial}{\partial \lambda'}(W(s, \lambda')^{-1}A_s(s, \lambda')W(s, \lambda'))d\lambda' \quad (A5)$$

because $W(s, 0) = I$. Performing the derivative in the previous equation and adding to eq. (A4) we have

$$\frac{DW(s, \lambda)}{ds} = -iW(s, \lambda) \int_0^\lambda d\lambda'W(s, \lambda')^{-1}\left[\frac{\partial A_\lambda(s, \lambda')}{\partial s} - \frac{\partial A_s(s, \lambda')}{\partial \lambda}\right] +$$

$$iA_\lambda(s, \lambda')A_s(s, \lambda') - iA_s(s, \lambda')A_\lambda(s, \lambda')\right]W(s, \lambda') =$$

$$-iW(s, \lambda) \int_0^\lambda d\lambda'W(s, \lambda')^{-1}F_{s\lambda}(s, \lambda')W(s, \lambda') \quad (A6)$$

Obviously these equations are valid in $n$ dimensions. The simplifying feature in 2+1 dimensions, that is exploited in the text, is that one can express the curvature directly in terms of the matter energy-momentum tensor.
APPENDIX B:

In this appendix we discuss the transformation properties of the energy momentum vector and the angular momentum, defined in sec. 2, under the change of origin of the loop. In the following we give the complete treatment for the Poincaré case, however all the results can be easily extended to anti de Sitter case $SO(2,2)$.

Given a closed contour $\Lambda$ we consider two points $O$ and $O'$ on it. The Wilson loop $W_O$ with origin $O$ is related to that with origin $O'$ by the following rule

$$W_O = V_{O'O}^{-1} W_{O'} V_{O'O}$$

(B1)

where $V_{O'O}$ is the gauge transformation associated to the path from $O$ to $O'$ along the loop $\Lambda$. Due to eq. (B1) $W_O$ and $W_{O'}$ have the same secular polynomial and so possess the same invariants. In other words the two loops define the same total mass and component of angular momentum along the total momentum. The explicit form of $V_{O'O}$ is given by the following path ordered integral

$$V_{O'O} = \text{Pexp} \left( -i \int_{O'}^O J_a \Gamma^a_{\mu} dx^\mu + P_a e^a_{\mu} dx^\mu \right).$$

(B2)

We can explicitly calculate the sector regarding the translation generators because they commute among themselves and we obtain the formula

$$V_{O'O} = \text{Pexp} \left( -i \int_{O'}^O J_a \Gamma^a_{\mu} dx^\mu \right) \times \exp \left( -i \int_{O}^{O'} P_a T^a_l (\lambda) e^l_{\mu} dx^\mu \right)$$

(B3)

where $T^a_l = [\text{Pexp} \left( -i \int_{O}^{P} J_a \Gamma^a_{\mu} dx^\mu \right)]^{-1}$ in the adjoint representation and $P$ is a generic point between $O$ and $O'$. This result can be easily found using the formula of [13] which relates the Pexp-integral of the sum $A + B$ of two quantities $A$ and $B$ to the Pexp-integral of $A$ and that of $B$ and recalling that $[P_a, P_b] = 0$. For brevity we define

$$S^a = \int_{O}^{O'} T^a_l (\lambda) e^l_{\mu} dx^\mu \quad \text{and} \quad A = \text{Pexp} \left( -i \int_{O}^{O'} J_a \Gamma^a_{\mu} dx^\mu \right)$$

(B4)

and we rewrite
\[ V_{O'O} = A \times \exp \left( -iP_a S^a \right) \] (B5)

This formula has a simple geometrical interpretation: \( S^a \) is the shift between the two origins and \( A \) is the Lorentz transformation which relates the dreibein of the observer in \( O \) with that of the observer in \( O' \).

In the following we consider the case in which our Wilson loop exponentiates, i.e. it can be parametrized as

\[ W_O = \exp(-iJ_a \Theta^a_O - iP_a \Xi^a_O), \] (B6)

and the same parametrization holds for \( W_{O'} \). Substituting eq. (B6), its analogous for \( W_{O'} \) and eq. (B5) into eq. (B1) we get the following transformation rules for \( \Theta^a \) and \( \Xi^a \)

\[ \Theta^a_{O'} = A_i^a \Theta^i_O \] (B7)

\[ \Xi^a_{O'} = A_i^a (\Xi^i_O + \epsilon^l_{bc} S^b \Theta^c_O). \] (B8)

where \( A_i^a \) is the second operator in eq. (B4) in the adjoint representation. If we recall the classical transformation law for the energy-momentum vector and angular momentum under a change of the origin and a rotation of reference frame, we see that they have the same form of eq.(B7) and eq.(B8). This fact is a further hint for identifying \( \Theta^a \) and \( \Xi^a \) with energy-momentum vector and angular momentum of the system.

We consider also the relation between the \( \Theta^a \) and \( \Xi^a \) for two equivalent loops, i.e, two loop which can be deformed one into the other without crossing the matter. The loop are similar in this case too, i.e.

\[ W = V^{-1} W' V \] (B9)

This is a well known fact, but one can show it also using eq. (A6) of appendix A. It gives

\[ \frac{D\hat{W}}{ds} \equiv \frac{\partial \hat{W}}{\partial s} - i[\hat{W}, e_s] - i[\hat{W}, \Gamma_s] = 0 \] (B10)
where $\hat{W}(0) = W$ and $\hat{W}(1) = W'$. This equation gives exactly the relation (B9) between the two loops with

$$V = \text{Pexp}\left(-i \int_{0}^{1} ds (Ja \Gamma^a_s + Pa e^a_s)\right).$$

(B11)

where in (B11) the integral runs along the trajectory of the origin of the loop as $s$ varies. The structure of $V$ is completely analogous to that of $V_{O'O}$, which implies that the transformation laws between the total momentum and angular momentum, defined by the two loops, are identical to those found in the previous case.
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