A NOTE ON THE REALIZATION OF RELATIVE $h_\infty$-DIAGRAMS

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Abstract. We prove a relative version of the realization theorem for $h_\infty$-diagrams in case that the underlying diagram subcategory is factorization-closed.

The rigidification of diagrams which only commute up to homotopy has been investigated by various authors. The most striking result goes back to Dwyer-Kan-Smith-Stover [DKS89] and Devinatz-Hopkins [DH04, Theorem 3.2]. They proved realizability in the case that all mapping spaces have contractible path components. In this brief note we prove a relative version of this theorem. The source subcategory should have the same objects and its morphisms should be factorization-closed. This allows us to neglect morphisms for which the path components are not contractible but for which the diagram already strictly commutes. We will show that the explicit proof given in [DH04, Theorem 3.2] carries over from the absolute to the relative situation.

Definition 1.1. A subcategory $\mathcal{C}$ of a category $\mathcal{D}$ is called factorization-closed if all composable morphisms $\alpha, \beta \in \mathcal{D}$ satisfy

$$\alpha \beta \in \mathcal{C} \implies \alpha \in \mathcal{C} \text{ and } \beta \in \mathcal{C}.$$  

Example 1.2. An important example looks like

where the dashed arrows do not belong to $\mathcal{C}$. That is, $\mathcal{C}$ consists of two copies of the simplicial category $\Delta$ and $\mathcal{D}$ is the category which classifies morphisms between cosimplicial objects.

Suppose $\mathcal{C}$ is a subcategory of a small category $\mathcal{D}$ with the same objects. For the target category of our diagrams, let $C$ be a good choice of an $A_\infty$-operad. More precisely, assume that the augmented simplicial spectrum $C^{\bullet+1}E$ is Reedy cofibrant for all cofibrant $C$-algebras $E$. Write $C$-alg for the category of $A_\infty$-ring spectra. We will study the problem of finding a realization of a diagram $(X, Y)$ of the form

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{X} & C\text{-alg} \\
\downarrow & & \downarrow \pi \\
\mathcal{D} & \xrightarrow{\xi} & Ho(C\text{-alg})
\end{array}$$

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This means we are looking for functors $Y : \mathcal{D} \to \text{C-alg}$ and $X_i : \mathcal{C} \to \text{C-alg}$, $1 \leq i \leq n$ and a zigzag of weak equivalences

$$Y \upharpoonright_{\mathcal{C}} \xrightarrow{\varphi_1} X_1 \leftarrow \xrightarrow{\varphi_2} X_2 \leftarrow \cdots \leftarrow \xrightarrow{\varphi_n} X_n \xrightarrow{\varphi_{n+1}} X$$

with the property that $\varphi_{n+1} \cdots \varphi_2^{-1} \varphi_1 : \pi Y \to \tilde{Y}$ is a natural equivalence. Here, the last arrow will point to the left if $n$ is odd.

**Definition 1.3.** Let $Z = (X, \tilde{Y})$ be a diagram as above. We say that $Z$ is an $h_\infty$-diagram if for each morphism $\alpha : j_1 \to j_2$ in $\mathcal{D}$ which is not contained in $\mathcal{C}$ the space $\text{C-alg}(X_{j_1}, X_{j_2})_{\tilde{Y}_\alpha}$ is contractible. Here, $\text{C-alg}(X_{j_1}, X_{j_2})_{\tilde{Y}_\alpha}$ is the path component of $\text{C-alg}(X_{j_1}, X_{j_2})_{\tilde{Y}_\alpha}$ which contains $\tilde{Y}_\alpha$.

**Theorem 1.4.** Suppose $\mathcal{C}$ is a factorization-closed subcategory of $\mathcal{D}$. Then every $h_\infty$-diagram is realizable.

**Proof.** We adopt the notation of [DH04] and write $\Gamma T$ for the realization of the singular simplicial set of an unbased topological space $T$. For a morphism $\alpha : j \to j'$ in $\mathcal{D}$ set

$$M(X_j, X_{j'})_\alpha = \begin{cases} \{X\alpha\} & \text{if } \alpha \in \mathcal{C} \\ \Gamma \text{-alg}(X_j, X_{j'})_{\tilde{Y}_\alpha} & \text{else} \end{cases}$$

Let $\alpha : j' \to j''$ be another morphism. Then the composition of maps induces a pairing

$$\circ : M(X_{j'}, X_{j''})_\alpha \times M(X_j, X_{j'})_\alpha \to M(X_j, X_{j''})_{\alpha'\alpha}$$

which is associative since $\mathcal{C}$ is factorization-closed. Define a cosimplicial $\text{C-alg}$ $\Pi^\bullet_h Z$ by

$$\Pi^0_h Z = \Pi_{j \in \mathcal{C}} X_j; \quad \Pi^i_h Z = \Pi_{\mathcal{D}_n} F(M\alpha, X_{j_0})$$

where $\mathcal{D}_n$ is the set of diagrams

$$\alpha : j_0 \xrightarrow{\alpha_1} j_1 \xleftarrow{\alpha_2} j_2 \cdots \xleftarrow{\alpha_n} j_n$$

in $\mathcal{D}$, $M\alpha = M(X_{j_0}, X_{j_1})_{\alpha_1} \times \cdots \times M(X_{j_n}, X_{j_{n-1}})_{\alpha_n}$ and $F(T, X)$ is the function spectrum, that is, the cotensor product of the space $T$ with the spectrum $X$. The cofaces $d^i$ are induced by the composition $\circ$ and are defined in exactly the same way as in [DH04] Construction 3.3: For $0 < i < n+1$ the coface $d^i : \Pi^i_h Z \to \Pi^{i+1}_h Z$ is defined in the factor indexed by $\alpha : j_0 \leftarrow \cdots \leftarrow j_{n+1}$ by the composite

$$\Pi^i_h Z \xrightarrow{\pi^i} F(M\alpha', X_{j_0}) \xrightarrow{d^i} F(M\alpha, X_{j_0})$$

where

$$\alpha' : j_0 \xrightarrow{\alpha_1'} j_1 \leftarrow \cdots \leftarrow j_{i-1} \xleftarrow{\alpha_{i+1}'} j_{i+1} \xleftarrow{\alpha_{i+2}} \cdots \xleftarrow{\alpha_{n+1}'} j_{n+1}$$

and

$$(d^i\alpha)(f_1, \ldots, f_{n+1}) = g(f_1, \ldots, f_i, f_{i+1}, \ldots, f_{n+1}).$$

For $i = 0$ it is defined in the same way with

$$\alpha' : j_1 \xrightarrow{\alpha_2'} j_2 \leftarrow \cdots \xleftarrow{\alpha_{n+1}'} j_{n+1}$$

and

$$(d^0\alpha)(f_1, \ldots, f_{n+1}) = f_1(g(f_2, \ldots, f_{n+1})).$$
Finally, for $i = n + 1$ set
\[ \alpha' : j_0 \xleftarrow{\alpha_i} j_1 \leftarrow \cdots \xleftarrow{\alpha_n} j_n \]
and
\[ (d_{\alpha}^{n+1} g)(f_1, \ldots, f_{n+1}) = g(f_1, \ldots, f_n). \]
Associativity ensures that the cosimplicial identities hold. The codegeneracies are defined via the evaluations on identity maps. Then \cite[Lemma 3.6]{DH04} shows that $\Pi^*_n Z$ is fibrant.

For an object $j$ of $\mathcal{D}$ let $\mathcal{D}\backslash j$ be the under category of $j$. Its objects are morphisms from $j$ to some object $j'$ and its morphisms are commutative triangles
\[ \begin{array}{ccc}
  j & \xleftarrow{\alpha} & j' \\
  \downarrow & & \downarrow \\
  j'' & \xrightarrow{\beta} & j''
\end{array} \]

Let $(\mathcal{D}\backslash j)_C$ be the subcategory of $(\mathcal{D}\backslash j)$ with the same objects and with morphisms $\alpha \in C$. Note that $(\mathcal{D}\backslash j)_C$ is factorization-closed. Let $\mu_j : \mathcal{D}\backslash j \rightarrow \mathcal{D}$ be the evident forgetful functor which maps the subcategory $(\mathcal{D}\backslash j)_C$ to $C$. The $h_\infty$-diagram $Z = (X, Y)$ provides us with an $h_\infty$-diagram $\mu_j^* Z = (\mu_j^* X, \mu_j^* Y)$ over the pair $(\mathcal{D}\backslash j, (\mathcal{D}\backslash j)_C)$. Set
\[ Y(j) = \text{Tot}(\Pi^*_h \mu_j^* Z). \]
Here, the totalization $\text{Tot}(W)$ of a cosimplicial spectrum $W$ is the spectrum $F(\Delta[s], W)$ of cosimplicial maps from the standard cosimplicial space $\Delta[s]$ to $W$. $Y$ is a functor from $\mathcal{D}$ to $C$-alg in the obvious way: a morphism $f : j \rightarrow j'$ gives a functor $f^* : \mathcal{D}\backslash j' \rightarrow \mathcal{D}\backslash j$ compatible with the subcategories. This functor induces a map on the cosimplicial spaces and hence on their totalizations. We claim that $Y$ is the desired realization.

As in the absolute case there is a map $Y(j) \rightarrow \tilde{Y}(j)$ given by the projection
\[ p_j : \text{Tot}(\Pi^*_h \mu_j^* Z) \rightarrow F(\Delta[0], \Pi^*_0 \mu_j^* Z) = \Pi^*_0 \mu_j^* Z \xrightarrow{pr_0} X_j \]
onto the factor indexed by the identity of $j$. The maps are natural in the homotopy category for the same reason as in the absolute case: the cosimplicial maps of degree one provide the homotopies which make the naturality diagram commute. We claim that the maps $p_j$ are weak equivalences. The Bousfield-Kan spectral sequence \cite[X.6,7]{BK72} takes the form
\[ \pi_{t-s}(\text{Tot}(\Pi^*_h \mu_j^* Z)) \cong E_2^{s,t} \cong \pi^s(\pi_t(\Pi^*_h \mu_j^* Z)) \]
Since $Z$ is an $h_\infty$-diagram each $M_\alpha$ is contractible and hence
\[ \pi^s(\pi_t(\Pi^*_h \mu_j^* Z)) \cong \lim_{\mathcal{D}\backslash j} \pi^s(\mu_j^* Z). \]
Finally, since the identity map of $j$ is initial in $\mathcal{D}\backslash j$ we have that the higher limits vanish and for $s = 0$ it coincides with $\pi_0 X_j$.

It remains to analyze the restriction of $Y$ to $C$. Let $\Pi^*$ be the standard cosimplicial replacement (cf. \cite[XI5.1]{BK72}) of a diagram $F$ on a small category $\mathcal{I}$ given in codimension $n$ by
\[ \Pi^n F = \prod_{\mathcal{I}_n} F_{i_0}. \]
There is an obvious cosimplicial map

$$\Pi^* \mu^*_j Z \longrightarrow \Pi^* \mu^*_j X$$

induced by projections onto the given factors since all $M_\alpha$ are points for morphisms in $\mathcal{C}$. This yields an equivalence on totalizations

$$Y_j \longrightarrow \text{Tot}(\Pi^* \mu^*_j X) = \text{holim}_{C \setminus j} \mu^*_j X$$

which is natural in $C$. The natural map

$$X_j \cong \lim_{C \setminus j} \mu^*_j X \longrightarrow \text{holim}_{C \setminus j} \mu^*_j X$$

is again an equivalence by the Bousfield-Kan spectral sequence. This completes the desired zig-zag. 

\[ \square \]

**References**

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