On the time continuous evolution of the universe if time is discrete and irreversible in nature

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Abstract. The time evolution of the universe is usually mathematically described under a continuous time and thus time reversible. Here, the consequences of studying the evolution of a homogenous isotropic universe by time continuous reversible physics are studied if time is actually discrete and irreversible in nature. The discrete dynamical time concept of Lee and its continuous time limit to the continuous time case is applied to the Newtonian limit of the general relativity theory. By doing so, the cosmic constant as well as the inflation of the universe arise and are predicted quantitatively well by assuming the smallest time step to be the Planck time and by using the current size of the universe.

1. Introduction
Because experimentally measured time is composed of an array of events, which can be exemplified in physics only by an energy-consuming clock measurement, and thus can not be measured continuously because of the uncertainty principle between energy and time ($\Delta E \Delta t > \hbar / 2$ with $\hbar$ being the Planck constant) time may be regarded discrete in nature (see for example [1-10]). It is however not a very popular concept, because the continuity of time and space enables powerful mathematical tools. Nonetheless, several approaches on discrete time physics have been elaborated on ([6]- [12]). Here, the evolution of a homogenous isotropic universe is explored under a discrete dynamical time and its consequences are discussed if continuous reversible physics is applied to an universe that expands in time steps. The discrete dynamical time concept of Lee [6] is thereby used and its limit to the continuous time case is applied to the Newtonian limit of the general relativity theory [13]. This approach is therefore not a quantum mechanical-based or a quantum-loop gravity-based theory [12], while it circumvents principle issues on discrete time physics such as the Lorentz transformation, but yields an alternate explanation on the apparent inflation of the universe and the cosmic constant.

After a short summary on several definitions and laws in cosmology (2.1) and the introduction of a discrete dynamical time (2.2), the request on time reversibility to Newton’s equation is discussed by introducing a scaling of time (2.2). In 2.3 and 2.4, this concept is applied to the gravitational force yielding the cosmic constant as ad hoc introduced by Einstein [14]. In 2.5 the evolution of the universe is described yielding an evolutionary phase of the universe as postulated by Guth [15] and in 2.6 the apparent acceleration of the universe is discussed.
2. Theory

2.1. The standard universe evolution under a continuous time

In the classical Newtonian description of gravity, the gravitational force $F_G(r)$ on a symbolic point particle of mass $m$ at radius $r$ from the center of the mass $M$ of a point-like particle or a spherically symmetric object (with size smaller than $r$) is given by

$$F_G(r) = m \ddot{r} = -\frac{4G}{3\pi} \rho m = -\frac{GMm}{r^2 |r|} \quad (1)$$

with $G$ the gravitational constant and $r$ the vector with length $r$ from the origin of reference at the center of mass $M$, which is considered here the mass of the universe, to the coordinates of the point particle (please note, vectors are written in bold) yielding for $r = \begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} = \begin{pmatrix} r^x \\ r^y \\ r^z \end{pmatrix}$ with $r_x = \frac{r}{\sqrt{3}}$. With the theorem of Gauss and the gravitational field potential $\Phi(r) = -\frac{GM}{|r|}$ the following Poisson equation can be obtained:

$$\nabla^2 \Phi(r) = \Phi(r) = 4 \pi G \rho(r) \quad (2)$$

with $\rho(r)$ being the mass density of the system with mass $M$. Thus, a Newtonian gravitational field can be derived from a scalar potential $\Phi$ that follows the Poisson equation.

In contrast, the Poisson equation derived from the Einstein field equations of the theory of general relativity at the Newtonian limit [16] is

$$\Phi(r) = 4 \pi G \rho(r) - \Lambda \quad (3)$$

with $\Lambda$ defined as the cosmic constant, which is constant by definition. Nowakowski et al. [13] showed using the Schwarzschild solution that the potential for a point-like and/or spherical symmetric object with mass $M$ at the Newtonian limit is

$$\Phi(r) = -\frac{GM}{|r|} - \frac{1}{6} \Lambda r^2 \quad (4)$$

This yields the following gravitational force acting on the point particle of mass $m$ at radius $r$:

$$F_G(r) = m \ddot{r} = -\frac{GMm}{r^2 |r|} + \frac{m}{3} \Lambda r \quad (5)$$

Another equation of interest here is the space time metric in case of a flat space time geometry (restricted to a flat space time geometry for simplicity reasons) given by

$$dx^2 = (cdt)^2 - R(t)^2 dv^2 \quad (6)$$

with $dx$ the space time interval, which is the distance between two events, $c$ the speed of light, and $R(t)$ the scaling factor. It describes the radial evolution of an isotropic homogenous universe under a continuous time under the theory of general relativity [14, 16]. We further mention the Hubble law [21] with the time-dependent Hubble constant given by

$$H(t) = \frac{\dot{R}(t)}{R(t)} \quad (7)$$
2.2. Under a dynamic discrete time

If time is a discrete dynamical variable $\hat{t}_n$ the continuous space function $r(t)$ of a homogenous isotropic universe is replaced by a sequence of discrete values $r_n = r(\hat{t}_n)$ with [6]:

$$(r_0, \hat{t}_0), (r_1, \hat{t}_1), \ldots, (r_n, \hat{t}_n), \ldots, (r_{N+1}, \hat{t}_{N+1})$$  (8)

with $(r_0, \hat{t}_0)$ the initial and $(r_{N+1}, \hat{t}_{N+1})$ the final position. In this description $r_n$ is still continuous, while $\hat{t}_n$ is discrete and as requested a dynamic variable described by a diagonal tensor. It is noted here, that the dynamic part of the time and its tensor character just introduced is only needed to show the effect of the use of a continuous time in describing the time evolution of the universe under the assumption that time is discrete in nature with a constant step size $\Delta t = \text{const}$ (for example $\Delta t$ could be the Planck time $5.4 \times 10^{-44}$ s). With other words, it is assumed here that the sequence of events happening are in general time irreversible and and can be described by

$$(r_0, t_0), (r_1, t_1), \ldots, (r_n, t_n), \ldots, (r_{N+1}, t_{N+1})$$

with $t_n$ being a scalar and the step size between time points is constant with $\Delta t$. Thus, the exercise that follows with a dynamic tensor description of time and the scaling variable is regarded only a mathematical trick and not real in nature.

The dynamic part of the time can be described by a time scaling tensor variable denoted $s_n$ defined by

$$s_n \Delta t = \hat{t}_n - \hat{t}_{n-1}$$  (9)

with $\Delta t = \text{const}$.

For simplicity reasons the system of interest is the gravitational force of an object of mass $M$ that acts on a test particle at a distance of radius $r$, which enables by proper selection of the coordinate system the reduction of the dynamic part of the time to only a single variable $s_n$ within the tensor notation as elaborated on more detailed in the Appendix (please note, the Appendix also includes the more general case).

In discrete mechanics there are many possible definitions of the velocity $\dot{r}_n$. The following definition at time point $n$ follows ref [11]:

$$\dot{r}_n = \frac{r_n - r_{n-1}}{\Delta t}$$  (10)

It seems to be a plausible one because it is in line with the causality argument and our daily experience that the presence is determined by the past and presence (i.e. to describe a present state of a system information can experimentally-derived only from the past and presence). With eq. (10) the velocity is defined backward in time and thus time asymmetric, permitting a forward progressing description of the system from past and present information. Please note, that this definition contrasts other theories as for example quantum loop gravity theory [12]. In presence of a discrete dynamic time variable described by the additional time scaling variable $s_n$ a Lagrangian must be derived that is able to describe adequately the system. Such a Lagrangian has been introduced by Nosé and Hoover [17]-[20] and its discrete analog has been introduced in ref. [11]:

$$L_n^N = L_n^N(r_n, r_{n-1}; s_n, s_n) = s_n \left( \frac{1}{2} m \dot{r}_n^2 - V(r_n) + \frac{1}{2} Q \frac{s_n^2}{s_n^3} - N_{df} k_B T \ln s_n \right)$$  (11)

where $V(r_n)$ is the acting potential (in the case of a Newtonian gravitational potential $V(r_n) = \Phi_m$ ), $k_B$ is the Boltzmann constant, $Q$ is a constant with units $J s^2$ (energy*seconds*seconds), which has been described as a “mass”-like term for the motion of $s_n$ with $Q \hat{c} = 0$, $N_{df}$ is the degree of freedom of the system, $\dot{s}_n = \frac{s_{n+1} - s_n}{\Delta t}$ is the velocity of the
scaling factor with the unit $s^{-1}$, and $T$ is the temperature of the system defined through the kinetic energy of the system. A corresponding Hamiltonian without the last two terms of eq. 11, which are not relevant here, has been described by Elze et al. [10].

If the discrete analog of the Lagrangian equation for each coordinate component $i$ is defined as in ref. [11]

$$\frac{1}{\Delta t} \left( \frac{\partial L^N}{\partial \dot{r}_{i+1}^n} (r_{n+1}, \dot{r}_{n+1}) - \frac{\partial L^N}{\partial \dot{r}_i^n} (r_n, \dot{r}_n) \right) = \frac{\partial L^N}{\partial r_i^n} \tag{12}$$

the discrete Newton’s law can be obtained using the discrete Lagrangian from eq. 10

$$\frac{1}{\Delta t} \left( \dot{r}_{n+1} \frac{s_{n+1}}{s_n} - \dot{r}_{n+1} + \dot{r}_n \right) = \frac{1}{m} F(r_n) \tag{13}$$

$$\ddot{r}_n = \frac{1}{\Delta t} (\dot{r}_{n+1} - \dot{r}_n) = \frac{1}{m} F(r_n) - \dot{r}_{n+1} \left( \frac{s_{n+1}}{s_n} - 1 \right) \frac{1}{\Delta t} \tag{14}$$

$$\ddot{r}_n = \frac{1}{m} F(r_n) - \gamma_n \dot{r}_{n+1} \tag{15}$$

with $\gamma_n = \frac{s_{n+1}}{s_n}$ and $s_n = \frac{s_{n+1} - s_n}{\Delta t}$.

If there is no scaling of time (i.e. $s_i = 1$ for all $i = 1...N+1$) the Newton’s law under a discrete time with constant time steps is of the form $m \ddot{r}_n = F(r_n)$ which resembles its continuous analog. In presence of a scaling of time (unequal to 1) however, the Newton’s law has an additional term, which can be regarded an acceleration or a friction term in dependence on the sign of $\gamma_n$. It is this friction term, which enables discrete time physics to be time reversible with the following request (see ref. [11]) that holds for the symmetric case selected.

$$\frac{S_n}{s_{n+1}} = \frac{F^x(r_{n+1})}{F^x(r_n)} i = x, y, z \tag{16}$$

with $F(r_n) = \begin{bmatrix} F^x(r_n) \\ F^y(r_n) \\ F^z(r_n) \end{bmatrix} = \begin{bmatrix} F^x(r_n) \\ F^y(r_n) \\ F^z(r_n) \end{bmatrix}$. This equation (eq. 16) and its more general analog in the Supplementary Material (eq. 54) are called the reversibility axiom. If the reversibility axiom is fulfilled, the introduced discrete time physics is time reversible [11].

By incorporating the reversibility axiom into the discrete Newtonian equation (eq. 15) the following expression is obtained:

$$\ddot{r}_n = \frac{1}{\Delta t} (\ddot{r}_{n+1} - \ddot{r}_n) = \frac{1}{m} F(r_n) + \dot{r}_{n+1} \frac{\ddot{F}^x(r_{n+1})}{\ddot{F}^x(r_n)} \tag{17}$$

with $\ddot{F}^x(r_{n+1}) = \frac{F^x(r_{n+1}) - F^x(r_n)}{\Delta t}$. Eq. 17 reflects thereby the time reversible evolution of a system under a discrete time. It is highlighted that due to the dynamic nature of the discrete time with the variable $s_n$ time reversibility is guaranteed.

As in the case of Newton’s law (eq. 17) the introduction of a dynamic discrete time has an impact also on other time-dependent formulas such as the well known differential equation $\frac{2}{s} = b$ with $b = $ const having as solution the exponential function $x(t) = x_0 e^{bt}$ with $x_0$ the value at time 0. The corresponding differential equation under a discrete dynamic time (dependent on eq. 16) is given by

$$\frac{\dot{x}_n}{x_n} = bs_n \tag{18}$$
This can be derived from $x_n(t) = x_0 \prod_{i=1}^{n} (1 + b s_i \Delta t) \approx x_0 e^{\sum_{i=1}^{n} s_i \Delta t}$ with the discrete definition of $\dot{x}_n$ given above for $\dot{r}_n$ (eq. 10). The latter equation is true at the limit $\lim \Delta t \to 0$.

2.3. Under a dynamic continuous time

In order to elaborate on the consequences of using a time continuous description of the evolution of the universe if time is actually discrete and non dynamic in nature, the above description with a dynamic discrete time is set to the limit with $\lim \Delta t \to 0$.

For this, the derived discrete Newtonian equation (eq. 17) is transformed into its corresponding continuous analog:

$$\ddot{\mathbf{r}} = \frac{1}{m} \mathbf{F}(\mathbf{r}) - \gamma \dot{\mathbf{r}}$$

(19)

with $g = \frac{\dot{x}}{x} = -\frac{\dot{r}}{r^2}$ with $s(t)$.

Correspondingly, in this time continuous frame the differential equation of the exponential function is then given by $\frac{\dot{x}}{x} = b s(t)$ and thus the Hubble law of eq. 5 is given by

$$\frac{\dot{R}}{R} = H(t) s(t)$$

(20)

Please note, within the system of interest, which is homogenous isotropic expanding universe with the argumentation put forward in eq. 22 below, the sin eq. 19 and eq. 20 are the same. The introduction of the dynamic continuous time resolves also the problem of the immediate request for a discrete space that comes along with a discrete time through the Lorentz transformation as has been done for example by the quantum loop gravity theory [12]. However, by the present translation from the discrete time to a dynamic continuous time with $\lim \Delta t \to 0$ the space can be described as usual because no need for a dynamic continuous space is required as space has not the problem of irreversibility.

2.4. From the Poisson equation to the Evolution of the Universe

Applying the continuous formulation of the modified Newton’s equation of eq. 17 to the gravitational force with $\mathbf{F}(\mathbf{r}) = \mathbf{F}_G(\mathbf{r})$ for the system of interest here (using $\dot{F}_G(\mathbf{r}) = -2 \frac{GMm}{r^3} \dot{r}$) the following equation is obtained:

$$\ddot{\mathbf{r}} = \frac{1}{m} \mathbf{F}(\mathbf{r}) - 2 \frac{\dot{r}_x}{r_x} \dot{r}_x = -\frac{GM}{r^2} \frac{\mathbf{r}}{r} + 2 \left(\frac{\dot{r}_x}{r_x}\right)^2 \mathbf{r}$$

(21)

In an isotropic homogenous and spherically expanding universe at the Newtonian limit (eq. 5: $\ddot{\mathbf{r}} = -\frac{GM}{r^2} \frac{\mathbf{r}}{r} + \frac{1}{3} \Lambda \mathbf{r}$) the scaling factor $R$ is proportional to $r$ and $\dot{R}$ proportional to $\dot{r}$ and thus we obtain for the second term of eq. 19 (using also eq. 20)

$$2 \left(\frac{\dot{r}_x}{r_x}\right)^2 = 2 \left(\frac{\dot{R}}{R}\right)^2 = 2 H(t)^2 s(t)^2$$

(22)

which in combination with the Newtonian limit of the poisson equation derived for the general relativity theory (eq. 3) yields

$$\frac{1}{3} \Lambda = 2 H(t)^2 s(t)^2$$

(23)

This results under a slow changing force limit (i.e. $s \approx 1$) to
\[ \Lambda = 6 \ H(t)^2 \]  
(24)

or under a constant changing force with \( s(t) = \sqrt{1/2} \)

\[ H = \sqrt{\frac{\Lambda}{3}} \]  
(25)

usually obtained under standard descriptions [16] (obviously \( s(t) = \sqrt{1/2} \) was chosen by purpose albeit meaningless and not required for the further derivations; alternatively, the geometry of the expanding universe can be altered such that \( 2 (\dot{\rho}_R)^2 = (\frac{\dot{R}}{R})^2 \)).

Concluding the first part of the presented theory, when time continuous physics at the Newtonian limit is applied to a flat isotropic homogenous universe that evolves in discrete time steps under slow changing force, it appears that the universe is expanding with a positive cosmic constant of \( \Lambda = 6 \ H^2 \). With other words, if time is discrete, the cosmic constant is a consequence of applying time reversible continuous physics to the expansion of the universe. It is further interesting to note, that under the given condition with the cosmic constant being constant by definition, the Hubble constant is independent of time if the gravitational force does not and never did change fast.

2.5. The expansion of the universe with its apparent inflation

The condition that the gravitational force is not changing fast is of course not correct in the early time of the expansion of the universe, which is studied next. For this purpose we use the following relationship \( \gamma = \frac{\dot{\rho}}{\rho} = -\frac{\dot{F}}{F} = -\frac{\dot{R}}{R} \) yielding

\[ \frac{s(t)}{s(t)} = \frac{\dot{R}(t)}{R(t)} = H(t) \]  
(26)

and correspondingly

\[ s(t) = s_0 e^{\sqrt{\frac{\Lambda}{6}} t} \]  
(27)

with \( s_0 \) the scaling at time 0 (please note, that this \( s_0 \) is from the continuous description of time, while in the discrete description \( s_n \) i used with \( n \) an integer starting from 1)

Since

\[ H(t) = \frac{\sqrt{\frac{\Lambda}{6}}}{s(t)} = \sqrt{\frac{\Lambda}{6} \frac{1}{s_0} e^{-\sqrt{\frac{\Lambda}{6}} t}} \]  
(28)

If one now takes the Hubble constant of eq. 36 and applies it for the determination of the scaling factor by using the standard Hubble law of eq. 5 we obtain the following scaling factor of the metric of the universe

\[ R(t) = R_0 e^{\int_0^t \sqrt{\frac{\Lambda}{6}} dt} = R_0 e^{\frac{\sqrt{\Lambda}}{s_0} \int_0^t e^{-\sqrt{\frac{\Lambda}{6}} t} dt} = R_0 e^{\frac{1}{s_0} [1-e^{-\sqrt{\frac{\Lambda}{6}} t}]} \]  
(29)

Hence, when time continuous physics is applied to a flat isotropic homogenous universe that evolves in discrete time steps there is an apparent expansion of the universe following eq. 29. This artifact is constant for large time \( t_{lim} \rightarrow \infty \) \( R(t) \approx R_0 e^{\frac{1}{s_0} t} \) as postulated by Guth [15].

There are three so far three undetermined constants left in eq. 29 (i.e. \( R_0, \ s_0, \) and \( \Lambda \ )) that need to be discussed for a quantitative time continuous description of the evolution of
the universe taking into account that time is actually discrete and irreversible. First, \( R_0 \) is considered. Because the scaling factor \( R(t) \) is in frame of the reference at time point 0, \( R_0 = 1 \). Next, the unknown constant \( s_0 \) is determined by assuming that the first time step of the universe is the Planck time \( t_p = h/\sqrt{\hbar c} = 5.3 \times 10^{-44} \) s (with \( h \) Planck’s Wirkungsquantum and \( c \) the velocity of light in vacuum) and that the universe was expanding with maximal velocity, i.e. light velocity \( c \). Furthermore, it is assumed that the universe at time point \( t = 0 \) had the minimal size of a system possible, which is regarded the Planck length \( l_p \). Since by definition \( l_p = c t_p \) at time point \( t_1 \) the size of the universe was thus \( l_p + c \Delta t_p = 2 l_p = R(t_1) l_p \) yielding \( R(t_1) = 2 \).

We further describe \( R(t_1) \) by a Taylor expansion first order and using \( \dot{R}(t) = R(t) e^{\sqrt{\frac{A}{6}} t} \sqrt{\frac{A}{6} \frac{1}{s_0}} \) starting with time \( t = 0 \) yielding

\[
R(t_1) \approx R_0 + \dot{R}(0) \Delta t_p = 1 + \sqrt{\frac{A}{6} \frac{1}{s_0}} \Delta t_p
\]

which results in

\[
s_0 = \sqrt{\frac{A}{6}} \Delta t_p
\]

since \( R(t_1) = 2 \). Following these assumptions, the quantitative expression of the expansion of the universe is given by

\[
R(t) = e^{\sqrt{\frac{A}{6}} t} [1 - e^{-\sqrt{\frac{A}{6}} t}]
\]

The scaling factor for large time \( t \) is thus given by \( \lim_{t \to \infty} R(t) = R_{inf} = e^{\sqrt{\frac{A}{6}} t} \) and constant, while for small times using a Taylor expansion 1. order of the exponential function the universe appears to expand exponentially with \( R(t) \approx e^{\frac{A}{6} t} \) as requested by Guth [15]. The last unknown is the cosmic constant \( \Lambda \), which can be gathered from measurements and the Hubble law with an \( \Lambda \) in the order of \( 10^{-35} \) s\(^{-2} \) assumed in the following to be \( 3 \times 10^{-35} \) s\(^{-2} \) [16]. This allows to calculate the evolution of the universe using eq. 32 as demonstrated in Figure 1. Figure 1 shows that when time continuous physics is applied to a flat isotropic homogenous universe that evolves in discrete time steps, it appears that the universe is expanding through an inflationary phase during the first \( 3 \times 10^{-35} \) s after which it approximates its maximal expansion at an \( R(t) \) of \( 10^{60} = R_{inf} \). Please note, that the presented approach does neither take special relativity nor quantum effects into account, which in particular at early time points such as during the grand unification theory (GUT) time, may have to be considered. Nevertheless, assuming a universe size of \( 10^l_p \) at time point 0 the expansion of the universe follows closely accepted inflationary universe models [15],[16]. Furthermore, the model predicts well the current size of the universe having \( R(t_{present}) \approx R_{inf} = 4 \times 10^{60} \) yielding a size of the universe of \( R(t) l_p = R_{inf} c \Delta t_p \approx 1 \times 10^{27} m \). Of course, the return process starting on the measured current size of the universe in combination with eq. 32 is also possible and results in the cosmic constant with expected size.
1. The scaling factor $R(t)$ versus time $t$ using eq. 32. In addition, the maximal expansion factor $\lim_{t \to \infty} R(t) = e^{\sqrt{\frac{k}{M}}}$ is indicated.

2.6. The apparent acceleration of the universe at late time points

Using the standard model of general relativity in combination with experimental measures it is often suggested that the universe is currently in an accelerating phase [16]. The presented description of the universe assuming a continuous time while time is postulated discrete in nature yields however a maximal scale factor of $\lim_{t \to \infty} R(t) = e^{\sqrt{\frac{k}{M}}}$ (see also Figure 1). While it depends on the cosmic constant $\Lambda$, within the description presented the size of the cosmic constant does not determine whether the universe is accelerating in size, expanding forever, or may collapse. Within this model, there is thus no support and no requirement for the proclaimed current accelerated universe.

3. Conclusion

The present work studied the artifacts that appear when time continuous reversible physics is applied to a universe that evolves irreversibly under a discrete time. The artifacts are (i) a cosmic constant and (ii) an inflationary evolution of the universe at early time - two prominent postulates in cosmology of which physical origins and requests are puzzling. Several recent attempts to resolve these issues have been suggested as for example [22-24]. The presented work suggests a simple solution by demanding a discrete time. The consequences are manifold such as that the present universe is not expanding under acceleration as believed, nor that an inflationary expansion of the universe is necessary to describe the evolution of the universe.

We invite the reader to apply the concept of a discrete time to further areas in physics and cosmology.

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Appendix

The following time reversible discrete time physics has been published in ref [11] and is repeated here for the example of interest.

If time is a discrete dynamical variable $\hat{t}_n$, the continuous space function $r(t)$ of a homogenous isotropic universe is replaced by a sequence of discrete values $r_n = r(\hat{t}_n)$ with [6]:

\[ R(t) = \exp(\sqrt{\frac{k}{M}} t) \]
Lagrangian (in real space) is given by the following expression [11]:

\[
L^s(t) = \frac{1}{2} \dot{r}^s(t) \dot{r}_s(t) - V(r(t))
\]

This definition contrasts other theories as for example quantum loop gravity theory [12]. In progressing description of the system from past and present information. Please note, that the velocity is defined backward in time and thus time asymmetric, permitting a forward information can experimentally-derived only from the past and presence. With eq. (37) presence is determined by the past and presence (i.e. to describe a present state of a system). The more general case is treated below.

In discrete mechanics there are many possible definitions of the velocity \( \dot{r}_n \). The following definition at time point \( n \) seems to be a plausible definition:

\[
\dot{r}_n = \frac{r_n - r_{n-1}}{\Delta t}
\]

because it is in line with the causality argument and our daily experience that the presence is determined by the past and presence (i.e. to describe a present state of a system information can experimentally-derived only from the past and presence). With eq. (37) the velocity is defined backward in time and thus time asymmetric, permitting a forward progressing description of the system from past and present information. Please note, that this definition contrasts other theories as for example quantum loop gravity theory [12]. In presence of a discrete dynamic time variable described by the additional time scaling variable \( s_n \) a Lagrangian must be derived that is able to describe adequately the system. Such a Lagrangian under an artificial continuous scaled time \( s \) has been introduced by Nosé and Hoover (\( L^N = s \dot{r}^N - V(r) + \frac{1}{2} \dot{r}^N V - N \dot{s} \dot{k_b T} \ln s \)) [17]-[20]. The discrete analog of the Nosé-Hoover Lagrangian (in real space) is given by the following expression [11]:
where $V(r_n)$ is the acting potential (in the case of a Newtonian gravitational potential $V(r_n) = Fm$ ), $k_B$ is the Boltzmann constant, $Q$ is a constant with units $Js^2$ (energy*seconds*seconds), which has been described as a “mass”-like term for the motion of $s_n$ with $Q \ i \ 0$, $N_{df}$ is the degree of freedom of the system, $\dot{s}_n = \frac{s_{n+1} - s_n}{\Delta t}$ is the velocity of freedom of the system, and $T$ is the temperature of the system defined through the kinetic energy of the system. A corresponding Hamiltonian without the last two terms of eq. 38 has been described by Elze et al. [10].

If the discrete analog of the Lagrangian equation for each coordinate component $i$ is defined as in ref. [11]

$$\Delta t \left( \frac{\partial L_n^N(r_{n+1}, \dot{r}_{n+1})}{\partial \dot{r}_{n+1}^i} - \frac{\partial L_n^N(r_n, \dot{r}_n)}{\partial \dot{r}_n^i} \right) = \frac{\partial L_n^N}{\partial r_n^i}$$

the discrete Newton’s law can be obtained using the discrete Lagrangian from eq. 10

$$\Delta t (\ddot{r}_{n+1} - \ddot{r}_n) = \frac{1}{m} F(r_n)$$

$$\ddot{r}_n = \frac{1}{m} F(r_n) - \frac{1}{m} F(r_n) - \frac{s_{n+1}}{s_n} - 1 \frac{1}{\Delta t}$$

with $g_n = \frac{s_{n+1}}{s_n}$ and $s_n = \frac{s_{n+1} - s_n}{\Delta t}$.

An alternative derivation results in

$$\ddot{r}_n = \frac{s_n}{m s_{n+1}} F(r_n) - \frac{1}{m} \left( 1 - \frac{s_n}{s_{n+1}} \right) \frac{1}{\Delta t}$$

If there is no scaling of time (i.e. $s_i = 1$ for all $i = 1...N + 1$) the Newton’s law under a discrete time with constant time steps is of the form $m \ddot{r}_n = F(r_n)$ which resembles its continuous analog. In presence of a scaling of time unequal to 1 however, the Newton’s law has an additional term, which can be regarded an acceleration or a friction term in dependence on the sign of $g_n$. It is this friction term, which enables discrete time physics to be time reversible as demonstrated in the following.

Time reversibility can be described by a two step process having one step forward followed by a step backward. Let us consider the evolution of the discrete Newton’s law with two steps forwards. Following eq. 14 by solving $\ddot{r}_n = \frac{1}{\Delta t} (\dddot{r}_{n+1} - \dddot{r}_n)$ for

(i)

$$\dddot{r}_{n+1} = \dddot{r}_n + \frac{s_n}{m s_{n+1}} \Delta t F(r_n) - (1 - \frac{s_n}{s_{n+1}}) \dddot{r}_n$$

For the second time step we take the second expression of the discrete Newton’s law (eq. 13)

(ii)

$$\dddot{r}_{n+2} = \dddot{r}_{n+1} + \frac{1}{m} \Delta t F(r_{n+1}) - \dddot{r}_{n+2} \left( \frac{s_{n+2}}{s_{n+1}} - 1 \right)$$
If the second step is now backward in time

\[
\dot{r}_{n+2} = \dot{r}_{n+1} - \frac{1}{m} \Delta t F(r_{n+1}) - \dot{r}_{n+2}(\frac{s_{n+2}}{s_{n+1}} - 1) \quad (A.14)
\]

and if time reversibility is requested (i.e. \(\dot{r}_n = \dot{r}_{n+2}\) and \(s_{n+2} = s_n\))

\[
\dot{r}_n = \dot{r}_{n+2} = \dot{r}_{n+1} - \frac{1}{m} \Delta t F(r_{n+1}) - \dot{r}_n(\frac{s_n}{s_{n+1}} - 1) \quad (A.15)
\]

\[
\dot{r}_n = \dot{r}_n + \frac{s_n}{m s_{n+1}} \Delta t F(r_n) - (1 - \frac{s_n}{s_{n+1}})\dot{r}_n - \frac{1}{m} \Delta t F(r_{n+1}) - \dot{r}_n(\frac{s_n}{s_{n+1}} - 1) \quad (A.16)
\]

\[
\frac{s_n}{s_{n+1}} F(r_n) = F(r_{n+1}) \quad (A.17)
\]

and thus

\[
\frac{s_n}{s_{n+1}} = \frac{F(r_{n+1})}{F(r_n)} \quad i = x, y, z \quad (A.18)
\]

with \(F(r_n) = \begin{bmatrix} F^x(r_n) \\ F^y(r_n) \\ F^z(r_n) \end{bmatrix} = \begin{bmatrix} F^x(r_n) \\ F^x(r_n) \end{bmatrix} \).

This equation holds for the symmetric case selected.

This equation (eq. 50) and its more general analog below (eq. 55) are called the reversibility axiom. If the reversibility axiom is fulfilled, the introduced discrete time physics is time reversible.

By incorporating the reversibility axiom on the symmetric case selected (eq. 50) into the discrete Newtonian equation (eq. 42) the following expression is obtained:

\[
\ddot{r}_n = \frac{1}{\Delta t}(\ddot{r}_{n+1} - \ddot{r}_n) = \frac{1}{m} F(r_n) + \dot{r}_{n+1} \frac{F^x(r_{n+1})}{F^x(r_n)} \quad (A.19)
\]

with \(\frac{F^x(r_{n+1})}{F^x(r_n)} = \frac{F^x(r_{n+1}) - F^x(r_n)}{\Delta t}\). Eq. 51 reflects thereby the time reversible evolution of a system under a discrete time. It is thereby highlighted that due to the dynamic nature of the discrete time the variable \(s_n\) time reversibility was obtained.

The approach taken above has been focusing on a spherical potential that acts along all the coordinates the same for simplicity. Here, the more general case is highlighted of an acting force on a single particle starting with time to be a discrete dynamical tensor second order \(\hat{t}_n\) describing a series of events as follows

\[
(r_0, \hat{t}_0), (r_1, \hat{t}_1), \ldots, (r_n, \hat{t}_n), \ldots, (r_{N+1}, \hat{t}_{N+1}) \quad (A.20)
\]

with \((r_0, \hat{t}_0)\) the initial and \((r_{N+1}, \hat{t}_{N+1})\) the final position. the dynamical part of time can then be described by the scaling factor

\[
\hat{s}_n \Delta t = s_n 1 \Delta t = \begin{bmatrix} s_n^x & 0 & 0 \\ 0 & s_n^y & 0 \\ 0 & 0 & s_n^z \end{bmatrix} \Delta t = \begin{bmatrix} t_{n+1}^x & 0 & 0 \\ 0 & t_{n+1}^y & 0 \\ 0 & 0 & t_{n+1}^z \end{bmatrix} - \begin{bmatrix} t_n^x & 0 & 0 \\ 0 & t_n^y & 0 \\ 0 & 0 & t_n^z \end{bmatrix} = \hat{t}_n - \hat{t}_{n-1} \quad (A.21)
\]

This yields the following reversibility axiom
\[
\frac{s^x_n}{s^x_{n+1}} = \frac{F^x(r_{n+1})}{F^x(r_n)}, \quad \frac{s^y_n}{s^y_{n+1}} = \frac{F^y(r_{n+1})}{F^y(r_n)}, \quad \frac{s^z_n}{s^z_{n+1}} = \frac{F^z(r_{n+1})}{F^z(r_n)} \quad (A.22)
\]

By incorporating the reversibility axiom into the discrete Newtonian equation (eq. 42) the following expression is obtained:

\[
\ddot{r}_n = \frac{1}{\Delta t}(\dot{r}_{n+1} - \dot{r}_n) = \frac{1}{m} F(r_n) - \dot{r}_{n+1} \left[ \begin{array}{ccc}
\frac{F^x(r_n)}{F^x(r_{n+1})} - 1 & \frac{1}{\Delta t} & 0 \\
0 & \frac{F^y(r_n)}{F^y(r_{n+1})} - 1 & \frac{1}{\Delta t} \\
0 & 0 & \frac{F^z(r_n)}{F^z(r_{n+1})} - 1 \end{array} \right]
\]

Next, the derived discrete Newtonian equation is transformed into its corresponding continuous analog by \(\lim_{\Delta t \to 0}\):

\[
\ddot{\mathbf{r}} = \frac{1}{m} \mathbf{F}(\mathbf{r}) - \dot{\gamma} \mathbf{r} \quad (A.24)
\]

with

\[
\dot{\gamma} = \left[ \begin{array}{ccc}
-\frac{F^x}{F^x} & 0 & 0 \\
0 & -\frac{F^y}{F^y} & 0 \\
0 & 0 & -\frac{F^z}{F^z} \end{array} \right]
\]

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