Positive solutions for nonlinear problems involving the one-dimensional $\phi$-Laplacian∗†‡

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Abstract

Let $\Omega := (a, b) \subset \mathbb{R}$, $m \in L^1(\Omega)$ and $\lambda > 0$ be a real parameter. Let $L$ be the differential operator given by $Lu := -\phi (u')' + r(x) \phi (u)$, where $\phi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism and $0 \leq r \in L^1(\Omega)$. We study the existence of positive solutions for problems of the form

$$\begin{cases}
Lu = \lambda m(x) f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $f : [0, \infty) \to [0, \infty)$ is a continuous function which is, roughly speaking, sublinear with respect to $\phi$. Our approach combines the sub and supersolution method with some estimates on related nonlinear problems.

We point out that our results are new even in the cases $r \equiv 0$ and/or $m \geq 0$.

1 Introduction

Let $\Omega := (a, b) \subset \mathbb{R}$, $m \in L^1(\Omega)$ and $\lambda > 0$ be a real parameter. Let us consider problems of the form

$$\begin{cases}
-\phi (u')' = \lambda m(x) f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

(1.1)

where $\phi : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism and $f : [0, \infty) \to [0, \infty)$ is a continuous function. The existence of positive solutions for problems as (1.1) involving the so-called $\phi$-Laplacian have been widely studied in the literature (see e.g. [1, 2, 4, 5, 11, 14, 15, 16, 23] and the references therein) and appear in diverse applications such as reaction-diffusion systems, nonlinear elasticity, glaciology, population biology, combustion theory, and non-Newtonian fluids.

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see for instance [3] [10] [12] [17]. We mention also that these kind of problems arise naturally in the study of radial solutions for nonlinear equations in annular domains (see e.g. [21] and its references).

When \( \phi(x) = |x|^{p-2}x \) and \( f(x) = x^q \) with \( 1 < p < \infty \) and \( 0 < q < p - 1 \), the existence of positive solutions for (1.1) was considered in [13], even for sign-changing weights (see also [9] [6] for the analogous \( N \)-dimensional problem). We note, however, that for the computations in [13] it was crucial the homogeneity of both \( \phi \) and \( f \), which of course is no longer true here.

Let us now introduce the following assumptions on \( m \) and \( \phi \):

\begin{align}
(M) & \ m \in C(\overline{\Omega}) \text{ with } m \ge 0 \text{ in } \Omega \text{ and } m \not= 0 \text{ on any subinterval of } \Omega, \\
(M') & \ m \in C(\overline{\Omega}) \text{ with } \min_{\Omega} m > 0, \\
(\Phi) & \ \text{There exist increasing homeomorphisms } \psi_1, \psi_2 : [0, \infty) \to [0, \infty) \text{ such that } \\
& \psi_1(t) \phi(x) \le \phi(tx) \le \psi_2(t) \phi(x) \text{ for all } t, x > 0, \\
(\Phi') & \ \text{There exist } p, q \in (0, \infty) \text{ such that } t^p \phi(x) \le \phi(tx) \le t^q \phi(x) \text{ for } t \in [0,1] \\
& \text{and all } x > 0.
\end{align}

Under some standard growth conditions on \( f \) (which allow both sublinear and superlinear nonlinearities) and assuming (M) and (\Phi), it was proved that (1.1) possesses a positive solution for all \( \lambda > 0 \) (see [20] Theorem 1.1), and recently in [22] Theorem 2 the authors extended this result to certain \( m \in L^1_{loc}(\Omega) \) and not requiring that \( \psi_2(0) = 0 \). We point out that these hypothesis impose, in particular, rather strong restrictions on

\[ l(t) := \lim_{\tau \to \infty} \phi(tx)/\phi(x) \quad \text{and} \quad L(t) := \lim_{\tau \to \infty} \phi(tx)/\phi(x). \]

Indeed, the existence of \( \psi_1 \) as above implies that \( l(t) > 0 \) for all \( t \in (0,1) \) and \( \lim_{t \to \infty} l(t) = \infty \), while the existence of \( \psi_2 \) entails that \( L(t) < \infty \) for all \( t > 1 \). Let us note that the first and third of these conditions are not satisfied for instance by exponential-like nonlinearities, and the remaining one does not hold for example for logarithmic-like functions.

On the other side, a similar result was established in [3] Corollary 3.4 assuming (M') and (\Phi'). We observe that the first inequality in (\Phi') also implies that \( l(t) > 0 \) for all \( t \in (0,1) \), while the second one requires that \( \lim_{\tau \to 0^+} L(t) = 0 \) (and this does not occur, for instance, with logarithmic-like nonlinearities). Let us add that in all these works the main tool utilized was some kind of Krasnoselskii-type fixed point theorem in cones.

Following a different approach, in Theorem 3.2 below we shall improve substantially the aforementioned results in the sublinear case, under much weaker conditions on both \( \phi \) and \( m \). In fact, regarding the assumptions on \( m \in L^1(\Omega) \), we shall only require that \( 0 \le m \not= 0 \) in \( \Omega \). Furthermore, we shall see that the solutions \( u_\lambda \to 0 \) in \( C^1(\overline{\Omega}) \) as \( \lambda \to 0^+ \). In order to derive our theorems, we shall rely on the well-known sub and supersolution method, combined with upper and lower estimates on some related nonlinear problems.

Also, under some additional hypothesis on \( \phi \) and \( m \), we shall prove in Theorem 3.3 similar results for the differential operator

\[ Lu := -\phi(u')' + r(x) \phi(u), \quad (1.2) \]
where \(0 \leq r \in L^1(\Omega)\). Moreover, as a consequence of Theorems 3.2 and 3.4, we shall deduce the existence of (nontrivial) nonnegative solutions for sign-changing weights \(m\), see Corollary 3.5.

The rest of the article is organized as follows. In the next section we collect some auxiliary results, while in Section 3 we shall state and prove our main theorems. Finally, at the end of the paper we present several examples illustrating our conditions and their relations with the ones already mentioned (see also Remarks 3.4 and 3.5).

2 Preliminaries

Let \(\phi : \mathbb{R} \to \mathbb{R}\) be an odd increasing homeomorphism and \(h \in L^1(\Omega)\). We start compiling some necessary facts about the problem

\[
\begin{cases}
-\phi (v')' = h(x) & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(2.1)

**Remark 2.1.** For every \(h \in L^1(\Omega), \) (2.1) admits a unique solution \(v \in C^1(\overline{\Omega})\) such that \(\phi (v')\) is absolutely continuous and that the equation holds pointwise a.e. \(x \in \Omega\). In fact, one can see that

\[
v(x) = \int_a^x \phi^{-1}\left( c_h - \int_a^y h(t) \, dt \right) \, dy,
\]

(2.2)

where \(c_h\) is the unique constant such that \(v(b) = 0\). Furthermore, the solution operator \(S_\phi : L^1(\Omega) \to C^1(\overline{\Omega})\) is continuous (see e.g. [7, Lemma 2.1]).

The following lemma shows that \(S_\phi\) is a nondecreasing operator. Although this result should probably be well-known, we have not been able to find a proof in the literature.

**Lemma 2.2.** Let \(h_1, h_2 \in L^1(\Omega)\) with \(h_1 \leq h_2\) a.e. \(x \in \Omega\). Then \(S_\phi(h_1) \leq S_\phi(h_2)\) in \(\overline{\Omega}\).

**Proof.** Let \(v_i := S_\phi(h_i), \ i = 1, 2, \) and suppose by contradiction that \(O := \{x \in \Omega : v_1 > v_2\} \neq \emptyset\). Let \(O_c\) be a connected component of \(O\). Note that, either by the continuity of \(v_1\) and \(v_2\) or by the boundary condition in (2.1), \(v_1 = v_2\) on \(\partial O_c\). Taking into account this, multiplying (2.1) (with \(h_1\) in place of \(h\)) by \(v_1 - v_2\) and integrating by parts we get

\[
\int_{O_c} \phi (v') (v_1 - v_2) \, dy = \int_{O_c} h_1 (x) (v_1 - v_2) \, dy.
\]

Since we can argue in the same way with the equation involving \(h_2\) and \(h_1 \leq h_2\) in \(\Omega\), recalling that \(\phi\) is increasing we infer that

\[
0 \leq \int_{O_c} (\phi (v'_1) - \phi (v'_2)) (v_1 - v_2) = \int_{O_c} (h_1 (x) - h_2 (x)) (v_1 - v_2) \leq 0
\]

and thus \(v'_1 = v'_2\) in \(O_c\). Furthermore, \(v_1 = v_2\) in \(O_c\) because \(v_1 = v_2\) on \(\partial O_c\).

Contradiction. ■
For \( h \in L^1(\Omega) \) with \( 0 \leq h \neq 0 \) we define

\[
\mathcal{A}_h := \{ x \in \Omega : h(y) = 0 \text{ a.e. } y \in (a,x) \},
\]

\[
\mathcal{B}_h := \{ x \in \Omega : h(y) = 0 \text{ a.e. } y \in (x,b) \},
\]

and

\[
\alpha_h := \begin{cases} \sup \mathcal{A}_h & \text{if } \mathcal{A}_h \neq \emptyset, \\ a & \text{if } \mathcal{A}_h = \emptyset, \end{cases}
\]

\[
\beta_h := \begin{cases} \inf \mathcal{B}_h & \text{if } \mathcal{B}_h \neq \emptyset, \\ b & \text{if } \mathcal{B}_h = \emptyset, \end{cases}
\]

\[
\vartheta_h := \min \left\{ \frac{1}{\beta_h - a}, \frac{1}{b - \alpha_h} \right\}, \quad \overline{\vartheta}_h := \frac{\alpha_h + \beta_h}{2}.
\]

(2.3)

Observe that, since \( h \neq 0 \), \( \vartheta_h \) is well defined and \( \alpha_h < \beta_h \) (and so, \( \theta_h \in (\alpha_h, \beta_h) \)). Let us also set

\[
\delta_{\Omega}(x) := \text{dist}(x, \partial \Omega) = \min (x - a, b - x).
\]

The next lemma provides some useful upper and lower bounds for \( S_{\phi}(h) \) when \( h \) is nonnegative.

**Lemma 2.3.** Let \( 0 \leq h \in L^1(\Omega) \) with \( h \neq 0 \). Then in \( \Omega \) it holds that

\[
\vartheta_h \min \left\{ \int_a^\vartheta h \phi^{-1}(\int_y^\vartheta h) \, dy, \int_b^\vartheta \phi^{-1}(\int_y^\vartheta h) \, dy \right\} \delta_{\Omega} \leq S_{\phi}(h) \leq \phi^{-1}(\int_b^\vartheta h) \delta_{\Omega}.
\]

(2.4)

**Proof.** Let \( v := S_{\phi}(h) \). Since \( \phi^{-1} \) is increasing and \( h \geq 0 \) in \( \Omega \), using (2.2) we see that \( v'(x) = \phi^{-1}(c_h - \int_a^x h(t) \, dt) \) is nonincreasing and so \( v \) is concave in \( \Omega \). Hence, since \( v = 0 \) on \( \partial \Omega \) and \( v \neq 0 \) we deduce that \( v'(b) < 0 < v'(a) \) and therefore

\[
0 < c_h < \int_a^b h(t) \, dt.
\]

(2.5)

Employing again the fact that \( \phi \) is increasing and (2.2) we find that

\[
v'(a), |v'(b)| \leq \phi^{-1}(\int_a^b h)\]

and thus from the concavity of \( v \) we obtain the second inequality in (2.4).

Let us prove the first inequality in (2.4). We first claim that

\[
v \geq \vartheta_h \|v\|_\infty \delta_{\Omega} \quad \text{in } \Omega.
\]

(2.6)

In order to verify this, let \( \xi \in \Omega \) be some point where \( v \) reaches its maximum (and so \( v'(\xi) = 0 \)). We note that \( \xi > \alpha_h \). Indeed, when \( \mathcal{A}_h = \emptyset \) this is obvious. If \( \mathcal{A}_h \neq \emptyset \), then by (2.2) we have \( v(x) = \phi^{-1}(c_h)(x - a) \) for all \( x \in (a, \alpha_h) \), with \( \phi^{-1}(c_h) > 0 \) by (2.5). In particular, \( v \) is increasing for such \( x \) and thus \( \xi > \alpha_h \) as asserted. Hence, recalling the concavity of \( v \) we get that for all \( x \in [\xi, b] \),

\[
v(x) \geq \frac{v(\xi)(b - x)}{b - \xi} \geq \frac{\|v\|_\infty}{b - \alpha_h} \delta_{\Omega}(x).
\]
Analogously, for \( x \in [a, \xi] \),

\[
v(x) \geq \frac{v(\xi)(x-a)}{\xi-a} \geq \|v\|_{\infty} \delta_{\Omega}(x)
\]

and the claim is proved.

Suppose now that \( \xi \geq \overline{\theta} \). Taking into account that \( \phi \) is an homeomorphism with \( \phi(0) = 0 \), that \( v'(x) = \phi^{-1}(c_h - \int_a^x h) \) and \( v'(\xi) = 0 \), we derive that \( c_h = \int_a^\xi h \). Then, recalling (2.2), that \( \phi \) is increasing and \( h \geq 0 \),

\[
v(\overline{\theta}) = \int_a^{\overline{\theta}} \phi^{-1} \left( \int_a^x h - \int_y^b h \right) dy \geq \int_a^{\overline{\theta}} \phi^{-1} \left( \int_y^b h \right) dy.
\]

(2.7)

Assume now that \( \xi \leq \overline{\theta} \). In this case we rewrite \( v \) as

\[
v(x) = \int_x^b \phi^{-1} \left( \phi_h - \int_y^b h(t) dt \right) dy,
\]

where \( \phi_h \) is the unique constant such that \( v(a) = 0 \). Moreover, reasoning as in the previous paragraph we see that \( \phi_h = \int_x^\xi h \). Therefore,

\[
v(\overline{\theta}) = \int_{\phi_h}^{\overline{\theta}} \phi^{-1} \left( \int_{\xi}^\xi h - \int_y^b h \right) dy \geq \int_{\phi_h}^{\overline{\theta}} \phi^{-1} \left( \int_y^b h \right) dy.
\]

(2.8)

Taking into account (2.6), (2.7) and (2.8) we may infer the first inequality in (2.4) and this concludes the proof. ■

**Remark 2.4.** Let \( 0 \leq h \in L^1(\Omega) \) with \( h \not\equiv 0 \).

(i) Observe that, since \( \overline{\theta}_h \in (\alpha_h, \beta_h) \), the constant that appears in the first term of the inequalities in (2.4) is strictly positive.

(ii) For any \( g \in C(\overline{\Omega}) \) with \( g > 0 \) in \( \Omega \), note that \( \alpha_h = \alpha_{hg} \) and \( \beta_h = \beta_{hg} \).

Therefore, by the above lemma we have that

\[
S_{\phi}(hg) \geq \overline{\theta}_h \min \left\{ \int_{\phi_h}^{\overline{\theta}_h} \phi^{-1}(\int_y^{\phi_h} h g) dy, \int_{\phi_h}^{b} \phi^{-1}(\int_y^{\phi_h} h g) dy \right\} \delta_{\Omega} \quad \text{in } \Omega.
\]

Let \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function (that is, \( f(\cdot, \xi) \) is measurable for all \( \xi \in \mathbb{R} \) and \( f(x, \cdot) \) is continuous for a.e. \( x \in \Omega \)). Let \( \mathcal{L} \) be as in (1.2), and let us now consider problems of the form

\[
\begin{cases}
\mathcal{L}u = f(x,u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\]

(2.9)

We say that \( v \in C(\overline{\Omega}) \) is a *subsolution* of (2.9) if there exists a finite set \( \Sigma \subset \Omega \) such that \( \phi(v') \in AC_{loc}(\overline{\Omega}\setminus \Sigma) \), \( v'(\tau^+) := \lim_{x \to \tau^+} v'(x) \in \mathbb{R} \), \( v'(\tau^-) := \lim_{x \to \tau^-} v'(x) \in \mathbb{R} \) for each \( \tau \in \Sigma \), and

\[
\begin{cases}
\mathcal{L}v \leq f(x,v(x)) & \text{a.e. } x \in \Omega, \\
v \leq 0 & \text{on } \partial\Omega, \\
v'(\tau^-) < v'(\tau^+) & \text{for each } \tau \in \Sigma.
\end{cases}
\]

(2.10)
If the inequalities in (2.10) are inverted, we say that \( v \) is a supersolution of (2.9).

For the reader’s convenience we state the following existence theorem in the presence of well-ordered sub and supersolutions (for a proof, see for instance [18, Theorem 7.16]).

**Theorem 2.5.** Let \( v \) and \( w \) be sub and supersolutions respectively of (2.9) such that \( v(x) \leq w(x) \) for all \( x \in \Omega \). Suppose there exists \( g \in L^1(\Omega) \) such that

\[
|f(x, \xi)| \leq g(x) \quad \text{for a.e. } x \in \Omega \text{ and all } \xi \in [v(x), w(x)].
\]

Then there exists \( u \in C^1(\Omega) \) solution of (2.9) with \( v \leq u \leq w \) in \( \Omega \).

### 3 Main results

Before proving our main results, let us introduce the following conditions on \( \phi \) and \( f \).

**H1.** There exist \( t_1 > 0 \) and an increasing homeomorphism \( \psi \) defined in \([0, t_1]\) such that \( \psi(0) = 0 \) and

\[
\phi(tx) \leq \psi(t) \phi(x) \quad \text{for all } t \in [0, t_1], x \geq 0. \tag{3.1}
\]

**H1’.** There exists \( p > 0 \) such that

\[
\lim_{t \to 0^+} \frac{t^p}{\phi(t)} > 0, \quad \text{and} \quad \lim_{t \to \infty} \frac{\phi(ct)}{t} < \infty, \quad \text{where } c_\Omega := \frac{b - a}{2}. \tag{3.2}
\]

**H2.** There exist \( t_2, M > 0 \) such that

\[
\phi(tx) \leq M\phi(t) \phi(x) \quad \text{for all } t \in [0, t_2], x \in [0, c_\Omega]. \tag{3.4}
\]

**F1.** There exist \( T, k_1, k_2, q > 0 \) such that

\[
k_1 t^{q} \leq f(t) \quad \text{for } t \in [0, T] \quad \text{and} \quad f(t) \leq k_2 t^{q} \quad \text{for all } t \geq 0. \tag{3.5}
\]

**F1’.** There exist \( T, k_1, k_2, q_1, q_2 > 0 \) such that

\[
k_1 t^{q_1} \leq f(t) \quad \text{for } t \in [0, T] \quad \text{and} \quad f(t) \leq k_2 \phi(t)^{q_2} \quad \text{for all } t \geq 0. \tag{3.6}
\]

We notice that \( c_\Omega = \max_{\Omega} \delta_\Omega \). Let us also mention that the inequality in (3.4) appears (but for large values of \( t \) and \( x \)) in the so-called \( \Delta' \) condition referred to Young functions (see e.g. [19]).

**Remark 3.1.**

(i) Note that if \( |\Omega| \leq 2 \) the condition (3.3) holds automatically since \( \phi \) is increasing and thus in that case \( H1' \) reduces to (3.2). On the other hand, if \( H1 \) is true with \( \psi(t) = ct^p \) for some \( c, p > 0 \), fixing \( x = 1 \) in (3.1) we see that \( H1 \) implies (3.2). In other words, in this particular case, in “small” domains \( H1 \) is stronger than \( H1' \). However, in general, these hypothesis are independent (see examples (a2) and (d) at the end of the paper).
(ii) Suppose that $\phi$ fulfills $H_1'$ or $H_1$ with $\psi(t) = ct^p$ for some $c, p > 0$. Then the condition
\[
\lim_{x \to 0^+} \frac{\phi(x)}{x^p} > 0
\] (3.7)
is sufficient in order for $H_2$ to hold. Indeed, in any case we may assume (3.2) (see (i)). Hence, given any $t_0 > 0$, there exists $M_{t_0} > 0$ such that $\phi(t) \leq M_{t_0} t^p$ for all $t \in [0, t_0]$. Also, (3.7) implies that for every $x_0 > 0$ there exists $N_{x_0} > 0$ such that $x^p \leq N_{x_0} \phi(x)$ for all $x \in [0, x_0]$. It follows that for all $t \in [0, 1]$ and $x \in [0, c_0]$,
\[
\phi(tx) \leq M_{c_0} (tx)^p \leq M_{c_0} N_{c_0} \phi(t) \phi(x),
\]
and thus $H_2$ is valid. We observe however that (3.7) is not necessary for $H_2$ to be true (see examples (a4), (b) and (c) below).

(iii) Let us point out that if $\phi$ is differentiable in $(0, c_0)$ and
\[
\sup_{t \in (0, 1), x \in (0, c_0)} \frac{t \phi'(tx)}{\phi'(t) \phi'(x)} := M < \infty,
\]
then one can readily verify that $H_2$ holds with $t_2 = 1$.

(iv) It is not difficult to check that the hypothesis $H_1$ and $H_2$ are independent, and that the same is true for $H_1'$ and $H_2$, see examples (a), (a2) and (d).

Our results shall provide us with solutions that lie in the interior of the positive cone of $C_c^1(\Omega) := \{ u \in C^1(\Omega) : u = 0 \text{ on } \partial \Omega \}$, which is denoted by
\[
\mathcal{P}^o := \{ u \in C_c^1(\Omega) : u > 0 \text{ in } \Omega \text{ and } u'(b) < 0 < u'(a) \}.
\]

**Theorem 3.2.** Let $0 \leq m \in L^1(\Omega)$ with $m \neq 0$.
(i) Assume $H_1$ and $F_1$ with
\[
\lim_{|t| \to 0^+} \frac{t^q}{\psi(t)} = \infty.
\] (3.8)
Then for all $\lambda > 0$ there exists $u = u_\lambda \in \mathcal{P}^o$ solution of (1.1).

(ii) Assume $H_1'$ and $F_1'$ with
\[
q_1 \in (0, p) \quad \text{and} \quad q_2 \in (0, 1).
\] (3.9)
Then for all $\lambda > 0$ there exists $u = u_\lambda \in \mathcal{P}^o$ solution of (1.1).
Moreover, in both (i) and (ii) it holds that
\[
\lim_{\lambda \to 0^+} \| u_\lambda \|_{C^1(\Omega)} = 0.
\] (3.10)

**Remark 3.3.** When $\phi$ is the $p$-Laplacian, i.e. $\phi(t) = |t|^{p-2} t$ with $p > 1$, clearly $H_1$ (with $\psi(t) = t^{p-1}$) and $H_1'$ (with $p-1$ in place of $p$ in (3.2)) hold. Furthermore, (3.8) is valid if and only if $q < p - 1$, so in this case we have the usual growth condition that characterizes the sublinear problems. Observe also that, since for the $p$-Laplacian in (ii) we can take any $q_1 \in (0, p-1)$ and $1 > q_2 > 1$, Theorem 3.2 (i) and (ii) provide here the same result.
Proof. Let $\lambda > 0$. We start proving (i). Let $\psi, t_1, T, k_1, k_2, q > 0$ be given by H1 and F1 accordingly. By the the continuity of $\phi^{-1}$ and the fact that $\phi^{-1}(0) = 0$, there exists $\varepsilon > 0$ such that

$$
\phi^{-1}(\varepsilon \int_a^b m \delta_\Omega) \leq \frac{T}{c_\Omega}
$$

for all $\varepsilon \in (0, \varepsilon_0]$, where $c_\Omega$ is given by (5.3). Also, let $\theta_m$ and $\bar{\theta}_m$ be as in (2.3) and set

$$
M_\Omega := \min \left\{ \int_a^b \phi^{-1}(\int_y^b m \delta_\Omega) dy, \int_a^b \phi^{-1}(\int_y^a m \delta_\Omega) dy \right\}.
$$

It follows from the definition of $\bar{\theta}_m$ that $M_\Omega > 0$. Let us also write

$$
M := \max \left\{ \frac{1}{\lambda k_1(\theta_m M_\Omega)^q}, \lambda k_2(\phi^{-1}(\int_a^b m \delta_\Omega))^q \right\}.
$$

We now observe that by (3.8) there exists $\varepsilon_0 > 0$ such that

$$
M \psi(\varepsilon) \leq \varepsilon^q
$$

for all $\varepsilon \in [0, \varepsilon_0]$. We notice next that H1 says that $t \phi^{-1}(x) \leq \phi^{-1}(\psi(t)x)$ for all $t \in [0, t_1]$ and $x \geq 0$, and therefore

$$
\psi^{-1}(r) \phi^{-1}(x) \leq \phi^{-1}(rx)
$$

for all $r \in [0, \psi'(t_1)]$ and $x \geq 0$.

Let us choose $0 < \varepsilon \leq \min \left\{ 1, \pi, \psi(\varepsilon_0), \psi(t_1) \right\}$, and for such $\varepsilon$ define $v := S_\theta(\varepsilon m \delta_\Omega)$. Since $\varepsilon \leq \pi$ and $\delta_\Omega \leq c_\Omega$ in $\Omega$, the second inequality in (2.4) and (3.11) tell us that $\|v\|_{\infty} \leq T$. Consequently, taking into account (3.12), (3.13) and (3.14), employing F1 and Remark 2.4 (ii) we deduce that

$$
\lambda m(x) f(v) \geq \lambda k_1 m(x) v^q \geq \lambda k_1 m(x) \left[ \theta_m \min \left\{ \int_a^b \phi^{-1}(\varepsilon \int_y^b m \delta_\Omega) dy, \int_a^b \phi^{-1}(\varepsilon \int_y^a m \delta_\Omega) dy \right\} \right]^q \geq \lambda k_1 m(x) (\theta_m \psi^{-1}(\varepsilon) M_\Omega \delta_\Omega)^q \geq \varepsilon m(x) \delta_\Omega^q = -\phi(v') \quad \text{in } \Omega.
$$

In other words, $v$ is a subsolution of (1.1).

On the other side, we see that H1 yields that $\phi(x)/\psi(t) \leq \phi(x/t)$ for all $t \in (0, t_1]$ and $x \geq 0$. Thus, $\phi^{-1}(x/\psi(t)) \leq \phi^{-1}(x)/t$ for such $t$ and $x$ and so,

$$
\phi^{-1}(\frac{x}{r}) \leq \frac{\phi^{-1}(x)}{\psi^{-1}(r)}
$$

(3.16)
for all $r \in (0, \psi(t_1)]$ and $x \geq 0$. Let now $w := S_\phi \left( \varepsilon^{-1} m \delta^q_{\Omega} \right)$. Recalling (3.12), (3.14) and (3.16) and utilizing again F1 and Lemma 2.3, we get that

$$
\lambda m(x) f(w) \leq \lambda k_2 m(x) w^q \leq \lambda k_2 m(x) \left( \phi^{-1} \left( \frac{1}{\varepsilon} \int_a^b m \delta^q_{\Omega} \right) \right)^q \leq \lambda k_2 m(x) \left( \frac{1}{\varepsilon} \int_a^b m \delta^q_{\Omega} \right)^q \leq \frac{1}{\varepsilon} m(x) \phi^{-1} \left( \int_a^b m \delta^q_{\Omega} \right) \leq \frac{1}{\varepsilon} m(x) \phi^{-1} \left( \int_a^b m \delta^q_{\Omega} \right) \leq \frac{1}{\varepsilon} m(x) \phi^{-1} \left( \int_a^b m \delta^q_{\Omega} \right) $$

and hence $w$ is a supersolution of (1.1). Moreover, since $\varepsilon \leq 1$ and $S_\phi$ is nondecreasing (see Lemma 2.2) we infer that $v$ where

$$v$$

and since

$$w$$

as in (3.15) we derive that

$$v$$

and similarly to (i) we define $v := S_\phi \left( \varepsilon m \delta^q_{\Omega} \right)$, picking

$$0 < \varepsilon \leq \min \left\{ \bar{\varepsilon}, \left( \lambda k_1 \left( \frac{\theta_m N_{\Omega}}{K^{1/p}} \right)^{q_1} \right)^{p/(p-q_1)} \right\}, \quad (3.18)$$

where $\bar{\varepsilon} > 0$ is such that $\phi^{-1}(\bar{\varepsilon}) \int_a^b m \delta^q_{\Omega} \leq \bar{\varepsilon}/\varepsilon$. As in the proof of (i) we have that $\|v\|_{\infty} \leq \bar{\varepsilon}$. Thus, taking into account (3.19), (3.17) and (3.18) and arguing as in (3.11) we derive that

$$\lambda m(x) f(v) \geq \lambda k_1 m(x) v^q \geq \lambda k_1 m(x) \left( \frac{\theta_m (\varepsilon / K)^{1/p} N_{\Omega} \delta^q_{\Omega}}{v^q} \right)^{q_1} \geq \varepsilon m(x) \delta^q_{\Omega} \left( \phi^{-1}(\varepsilon) \right)^q \geq \varepsilon m(x) \phi^q (x) = -\phi (v')$$

in $\Omega$.

On the other hand, let $N := \sup_{t \geq 1} \phi (c_t t) / \phi (t) < \infty$ (by (3.3)). For all $t \geq 1$ we have $\phi (c_t t) \leq N \phi (t)$ and so

$$\phi (c_t \phi^{-1} (t)) \leq N t \quad (3.20)$$

for all $t \geq \phi (1)$. Let $w := S_\phi (\gamma m)$ with

$$\gamma \geq \max \left\{ \frac{\phi (1)}{m}, \left( \lambda k_2 (N \int_a^b m)^{1/(1-q_3)} \right) \right\}. \quad (3.21)$$

Recalling F1’, the upper bound given by Lemma 2.3 and that $q_2 < 0$ and
δΩ ≤ cΩ in Ω, employing \((3.20)\) and \((3.21)\) we infer that

\[
\lambda m(x) f(w) \leq \lambda k_2 m(x) \phi(w)^q \leq \\
k_2 m(x) \left( \phi(\phi^{-1}(\gamma \int_a^b m)\delta_\Omega) \right)^q \leq \\
k_2 m(x) \left( \phi(c_\Omega \phi^{-1}(\gamma \int_a^b m)) \right)^q \leq \\
k_2 m(x) \left( N \gamma \int_a^b m \right)^q \leq \gamma m(x) = -\phi(w)' \text{ in } \Omega.
\]

Furthermore, enlarging γ if necessary so that \(\gamma \geq c_\Omega^{\alpha}\) and utilizing Lemma 2.2 we can achieve that \(w \geq v\) in \(\overline{\Omega}\) and thus we obtain a solution \(u_\lambda \in \mathcal{P}^\circ\) of \((1.1)\).

Finally, let us prove (3.10). Let \(\lambda_0 > 0\) be fixed, and consider \(\lambda \in (0, \lambda_0)\). We first observe that the solutions \(u_\lambda\) obtained in either (i) or (ii) can be chosen such that \(\|u_\lambda\|_\infty \leq C\) with \(C\) independent of \(\lambda\). Indeed, since \(u_{\lambda_n} \in \mathcal{P}^\circ\) is a supersolution of \((1.1)\) for any \(\lambda \in (0, \lambda_0)\), and since the above part of the proof provides arbitrary small subsolutions of \((1.1)\) (that converge to 0 in \(\mathcal{C}(\overline{\Omega})\) as \(\varepsilon \to 0\), by the second inequality in \((2.4)\), it follows from Theorem 2.5 that there exist \(u_{\lambda} \in \mathcal{P}^\circ\) solutions of \((1.1)\) such that \(0 \leq u_{\lambda} \leq u_{\lambda_0}\) for all \(\lambda \in (0, \lambda_0)\). So, \(\|u_\lambda\|_\infty \leq C\) as claimed. Taking into account this, the upper estimate in Lemma 2.3 yields that

\[
0 \leq u_{\lambda}(x) = S_{\phi}(\lambda m f(u_\lambda))(x) \leq \phi^{-1}\left( \lambda \int_a^b m f(u_\lambda) \right) \delta_{\Omega}(x) \to 0
\]

uniformly in \(\overline{\Omega}\) as \(\lambda \to 0^+\) and so \(\lim_{\lambda \to 0^+} \|u_{\lambda}\|_\infty = 0\).

We choose next \(\xi = \xi_\lambda \in \Omega\) such that \(u_{\lambda}'(\xi) = 0\). Integrating \((1.1)\) over \((a, \xi)\) we get that \(u_{\lambda}'(a) = \phi^{-1}\left( \lambda \int_a^\xi m f(u_\lambda) \right)\) and hence by the above paragraph we see that \(u_{\lambda}'(a) \to 0\) as \(\lambda \to 0^+\). Now, for any \(x \in \overline{\Omega}\), we integrate \((1.1)\) over \((a, x)\) to find that

\[
u_{\lambda}'(x) = \phi^{-1}\left( \phi(u_{\lambda}'(a)) + \lambda \int_a^x m f(u_\lambda) \right) \to 0
\]

uniformly when \(\lambda \to 0^+\). Thus, the proof of \((3.10)\) is complete. \(\blacksquare\)

We next consider the case \(r \in L^1(\Omega)\) with \(r \geq 0\), that is, the problem

\[
\begin{cases}
-\phi(u')' + r(x) \phi(u) = \lambda m(x) f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(3.22)

**Theorem 3.4.** Let \(0 \leq m \in L^1(\Omega)\) with \(m \not\equiv 0\). Assume that \(\phi\) fulfills H2, and suppose \(\phi\) and \(f\) satisfy the hypothesis of Theorem 3.2 (i) or (ii), with \(\psi(t) = ct^p\) for some \(c, p > 0\) in case (i). If either \(r \leq m + \Omega\) or \(m, r \in L^\infty(\Omega)\) and \(\inf_{\Omega} m > 0\), then for all \(\lambda > 0\) there exists \(u = u_{\lambda} \in \mathcal{P}^\circ\) solution of \((3.22)\). Moreover, these \(u_{\lambda}\) satisfy \((3.10)\).

**Proof.** The proof follows the lines of the proof of Theorem 3.2 and hence we only indicate the minor changes that are needed.
Let $\lambda > 0$ and suppose the hypothesis of Theorem 3.2 (i) hold. Let $t_2, M > 0$ be given by H2 and pick $\varepsilon > 0$ such that
\[ \phi^{-1}(\varepsilon \int_{a}^{b} m\delta_{\Omega}^{q}) \leq t_2. \]
For such $\varepsilon$ define $v := \mathcal{S}_{\phi}(\varepsilon m\delta_{\Omega}^{q})$. Taking $x = 1$ in (3.1) (and recalling that here $\psi(t) = ct^{p}$ for some $c, p > 0$) we get that there exists $K > 0$ such that
\[ \phi(t) \leq Kt^{p} \quad \text{for all } t \in [0, c_{\Omega}], \quad (3.23) \]
where $c_{\Omega}$ is given by (3.3). Taking into account that $\delta_{\Omega} \leq c_{\Omega}$ in $\Omega$, using Lemma 2.3 and H2 we derive that
\[ \phi(v) \leq \varepsilon MK\delta_{\Omega}^{p} \int_{a}^{b} m\delta_{\Omega}^{q}. \quad (3.24) \]
Now, assume first that $r \leq m$ in $\Omega$. By (3.8) we have that $q < p$. Thus, making $\varepsilon$ smaller if necessary, since $\psi^{-1}(t) = (t/c)^{1/p}$, from (3.15) and (3.24) we get that
\[ \lambda m(x) f(v) - r(x) \phi(v) \geq \lambda k_{1}(\theta_{m}(\varepsilon c^{1/p})M_{\Omega}\delta_{\Omega})^{q} - \varepsilon MK\delta_{\Omega}^{p} \int_{a}^{b} m\delta_{\Omega}^{q} \geq \varepsilon m(x) \delta_{\Omega}^{q}(x) = -\phi(v'). \]
On the other hand, if $r, m \in L^{\infty}(\Omega)$ and $m := \inf_{\Omega} m > 0$, for all $\varepsilon$ sufficiently small, also from (3.15) and (3.24) we deduce that
\[ \lambda m(x) f(v) - r(x) \phi(v) \geq \lambda_{1} k_{1}(\theta_{m}(\varepsilon c^{1/p})M_{\Omega}\delta_{\Omega})^{q} - \|r\|_{\infty} \varepsilon MK\delta_{\Omega}^{p} \int_{a}^{b} m\delta_{\Omega}^{q} \geq \varepsilon \|m\|_{\infty} \delta_{\Omega}^{q}(x) \geq -\phi(v'). \]
Hence, in any case we obtain a subsolution of (3.22) which belongs to $\mathcal{P}^{0}$. Furthermore, these subsolutions tend uniformly to zero (by Lemma 2.3) as $\varepsilon \to 0$. Therefore, since the solutions given by Theorem 3.2 (which also lie in $\mathcal{P}^{0}$) are supersolutions of (3.22). Theorem 2.5 yields the desired solution $u_{\lambda}$. Moreover, it also follows that $\lim_{\lambda \to 0} \|u_{\lambda}\|_{L^{\infty}(\Omega)} = 0$, and similar computations to those in the last part of the proof of Theorem 3.2 show that $u_{\lambda}$ satisfy (3.10).

Suppose now the assumptions of Theorem 3.2 (ii) hold. Then we set $v := \mathcal{S}_{\phi}(\varepsilon m\delta_{\Omega}^{q_{1}})$, where $q_{1}$ is given by H1'. Since (3.24) is true by (3.2), proceeding as in (3.24) we have that
\[ \phi(v) \leq \varepsilon MK\delta_{\Omega}^{p} \int_{a}^{b} m\delta_{\Omega}^{q_{1}}. \]
Therefore, employing (3.15) in place of (3.15) and arguing as in the above two paragraphs we can construct arbitrarily small subsolutions and thus the proof can be completed as before. ■
As a direct consequence of Theorems 3.2 and 3.4 we are able to provide an existence result also for
\[ \begin{align*}
L u &= \lambda m(x)f(u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*} \tag{3.25}\]
where \( L u = -\phi(u') \) or \( L u = -\phi(u') + r(x)\phi(u) \) accordingly, in the case where \( m \) changes sign in \( \Omega \). As usual, we write \( m = m^+ - m^- \) with \( m^\pm := \max(\pm m, 0) \).

**Corollary 3.5.** Let \( m \in L^1(\Omega) \) such that there exists an open interval \( \Omega_0 \subset \Omega \) with \( 0 \leq m \neq 0 \) in \( \Omega_0 \). Suppose the hypothesis of one of the above theorems are satisfied, with \( m^+ \) in place of \( m \). Then for all \( \lambda > 0 \) there exists \( u = u_\lambda \in C^1(\Omega) \) nonnegative (and nontrivial) solution of (3.25). Moreover, these \( u_\lambda \) satisfy (3.10).

**Proof.** Let \( \lambda > 0 \), and let \( \varpi = \varpi_\lambda \in P^\circ \) be the solution of (3.25) with \( m^+ \) in place of \( m \), provided by some of the above theorems. It is clear that \( \varpi \) is a supersolution of (3.25).

On the other side, since \( 0 \leq m \neq 0 \) in \( \Omega_0 \), an inspection of the proofs of the aforementioned theorems show that we can find some \( z = z_\lambda \in C^1(\Omega_0) \) with \( z_\lambda \leq \varpi_\lambda \) in \( \Omega_0 \) and such that
\[ \begin{align*}
L z &\leq \lambda m(x)f(z) \quad \text{in } \Omega_0, \\
z &= 0 \quad \text{on } \partial \Omega_0.
\end{align*} \]
Define now \( \vartheta_\lambda \in C(\Omega) \) by \( \vartheta_\lambda := z_\lambda \in \Omega_0 \) and \( \vartheta_\lambda := 0 \) in \( \Omega \setminus \Omega_0 \). Then \( \vartheta_\lambda \) is a subsolution of (3.25) and this yields the existence assertion.

To conclude the proof we note that the last assertion follows similarly to the previous theorems, having in mind that \( u_\lambda \leq \vartheta_\lambda \) in \( \Omega \) and that \( \vartheta_\lambda \rightarrow 0 \) uniformly as \( \lambda \rightarrow 0^+ \). \( \blacksquare \)

**Examples.** We assume that \( x \geq 0 \) since we may extend \( \varphi \) oddly.

(a) Let \( \varphi : [0, \infty) \rightarrow [0, \infty) \) be continuous and nondecreasing, with \( \varphi \) increasing in \((0, x_0)\) for some \( x_0 > 0 \) if \( \varphi(0) = 0 \). Define
\[ \phi(x) := x^p \varphi(x), \quad p > 0. \tag{3.26} \]
Then \( \phi \) fulfills H1 with \( t_1 := 1 \) and \( \psi(t) := t^p \) because
\[ \phi(tx) = (tx)^p \varphi(tx) \leq \psi(t)x^p \varphi(x) = \psi(t)\phi(x) \]
for all \( t \in [0, 1] \) and \( x \geq 0 \). Furthermore, in this case the condition (3.8) of Theorem 3.2 is true if and only if \( p > q \).

Let us note that here \( \phi \) satisfies H2 if and only if \( \varphi \) does. In particular, taking some \( \varphi \) which does not fulfill H2 we obtain a function \( \phi \) that satisfies H1 but not H2 (one such \( \varphi \) is for instance \( \varphi(x) = e^{-1/x} \) for \( x > 0 \) and \( \varphi(0) = 0 \)).

We finally point out that if \( |\Omega| \leq 2 \), the above paragraph together with Remark 3.1 (i) imply the existence of some \( \phi \) which satisfies H1’ but not H2.

Let us exhibit next some interesting particular cases:
(a1) Let
\[ \phi(x) := x^{p_1} + x^{p_2}, \quad p_1, p_2 > 0. \]
Since \( \varphi(x) := \phi(x)/x^{p_2} \) is increasing, by the first paragraph in (a) we get that H1 holds, and it is also clear that H2 is true with  \( M = 1 \) and any  \( t_2 > 0 \).

(a2) Let 
\[ \phi(x) := e^{x^p} - 1, \quad p > 0. \]
A brief computation shows that  \( \phi(x)/x^p \) is increasing and so  \( \phi \) fulfills H1. Moreover, taking into account Remark 3.1 (ii) we see that  \( \phi \) satisfies H2. Note also that if \( |\Omega| > 2 \), then (3.3) is not valid. In particular,  \( \phi \) does not fulfill H1’ in this case (and for any  \( \Omega \),  \( \phi \) neither satisfies the conditions (\( \Phi \)) and (\( \Phi' \)) at the introduction nor the one in [22]).

(b) Let 
\[ \phi(x) := x (|\ln x| + 1). \]

It can be proved that  \( \phi \) cannot be written as in (3.26) with  \( p > 0 \) and  \( \varphi \) nondecreasing. Let us demonstrate, however, that  \( \phi \) satisfies H1 and H2. Let  \( p \in (0, 1) \). We choose  \( t_1 > 0 \) such that  \( |\ln t_1| \leq 1/t_1^{1-p} - 1 \) for all  \( t \in [0, t_1] \). Then, for such  \( t \) and all  \( x \geq 0 \) we have that
\begin{align*}
\phi(tx) &\leq tx (|\ln t| + |\ln x| + 1) \\
tx \left[ (1/t_1^{1-p} - 1) (|\ln x| + 1) + |\ln x| + 1 \right] &\leq t^p \phi(x)
\end{align*}
and H1 holds. Also, employing the first inequality in (3.27) it is easy to see that H2 is true with  \( M = 1 \) and any  \( t_2 > 0 \). Let us finally note that (3.3) is not valid with any  \( p \in (0, 1) \).

(c) Let 
\[ \phi(x) := x - \ln (x + 1). \]

Then  \( \phi \) fulfills (3.2) with  \( p = 1 \) and also (3.3). In other words, H1’ holds (let us remark that, despite it is less direct, one can prove that  \( \phi \) also satisfies H1 with  \( t_1 = 1 \) and  \( \psi(t) = ct \) for some  \( c > 0 \) large enough). Although (3.7) does not hold with  \( p = 1 \), from Remark 3.1 (iii) we deduce that H2 is valid since
\[ \frac{t\phi'(tx)}{\phi(t)\phi'(x)} = \frac{t^2(x + 1)}{(t - \ln (t + 1))(tx + 1)} \leq \frac{t^2(c_\Omega + 1)}{(t - \ln (t + 1))} \leq \frac{c_\Omega + 1}{1 - \ln 2} \]
for all  \( t \in (0, 1) \) and all  \( x \in (0, c_\Omega) \).

(d) Let 
\[ \phi(x) := (\ln (x + 1))^p, \quad p > 0. \]

One can readily check (3.2) and (3.3), and thus H1’ is true. Also, utilizing again Remark 3.1 (ii) we get that H2 holds. Let us observe that  \( \phi \) does not satisfy
H1 (and hence neither fulfills the conditions (Φ) or (Φ′) at the introduction) because
\[ \lim_{x \to \infty} \frac{\phi(tx)}{\phi(x)} = 1 \text{ for all } t > 0 \]
and so there is not a continuous \( \psi \) such that (3.1) is valid and \( \psi(0) = 0 \). Furthermore, this tell us that \( \phi \) neither meets the assumptions in [22].

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