Balanced condition in networks leads to Weibull statistics

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PACS 02.10.Yn – Matrix Theory
PACS 02.50.-r – Probability Theory, Stochastic Process and Statistics
PACS 89.75.Hc – Networks and Geological Trees

Abstract – The importance of the balance in inhibitory and excitatory couplings in the brain has increasingly been realized. Despite the key role played by inhibitory-excitatory couplings in the functioning of brain networks, the impact of a balanced condition on the stability properties of underlying networks remains largely unknown. We investigate properties of the largest eigenvalues of networks having such couplings, and find that they follow completely different statistics when in the balanced situation. Based on numerical simulations, we demonstrate that the transition from Weibull to Fréchet via the Gumbel distribution can be controlled by the variance of the column sum of the adjacency matrix, which depends monotonically on the denseness of the underlying network. As a balanced condition is imposed, the largest real part of the eigenvalue emulates a transition to the generalized extreme value statistics, independent of the inhibitory connection probability. Furthermore, the transition to the Weibull statistics and the small-world transition occur at the same rewiring probability, reflecting a more stable system.

Introduction. – The largest eigenvalue of network adjacency matrices plays a bridge between dynamical and structural properties of an underlying system. For example, the inverse of the largest eigenvalue of a network characterizes the threshold for phase transition of the virus spread \(^{[1]}\). Recently Goltsev et. al. have demonstrated the importance of the largest eigenvalue in determining disease spread in complex networks \(^{[2]}\). Furthermore, in coupled oscillators the threshold for phase transition to synchronized behaviour is determined by the inverse of the largest eigenvalue \(^{[3]}\). The dynamical properties of neurons have been shown to be highly influenced by a change in the spectra of underlying synaptic matrices constructed from randomly distributed numbers \(^{[4]}\). A remarkable, fundamental direction to analyze the stability of ecological systems was put forward by May \(^{[5]}\), where the largest real part of the eigenvalues (\(R_{\text{max}}\)) establishes a relationship between the stability and complexity of the underlying system. Later, the impact of various types of interactions was demonstrated to deduce stability criteria in terms of \(R_{\text{max}}\) \(^{[6]}\). Mathematically, matrices obeying some constraints satisfy the stability criteria \(^{[7]}\), but real-world systems have an underlying interaction matrix that is too complicated to obey these constraints; hence, the study of fluctuations in \(R_{\text{max}}\) is crucial to understanding stability of a system as well as the stability properties of an individual network in that ensemble. Recent efforts in this direction reveal the similarity of the maximal Lyapunov exponent of synaptic matrices defined for neural networks with their topological complexities \(^{[8]}\). A very recent work investigates the statistical properties of random matrices within the framework of extreme value theory, thereby providing an estimate about the resolution in complex dynamics for a finite system size \(^{[9]}\).

Balanced condition and its role in stability. The balanced condition in the brain refers to a situation in which for each neuron the weight of the inhibitory signal is equal to the excitatory signal \(^{[10][11]}\). Ref. \(^{[12]}\) demonstrates that this condition forces outliers of the spectra to appear inside the bulk, leading to a stable underlying neural system. Further analysis of a dynamical model of cortical networks with the balanced condition for various ratios of inhibitory and excitatory neurons reveals a connection between the spectra of connectivity matrix and the dynamical response \(^{[13]}\). Balance between recurrent excitation and inhibition generates stable periods of activity \(^{[14]}\). There have been several discussions on how synaptic matrices in the brain achieve the balanced condition; for instance, it has been demonstrated that the balanced...
condition in sensory pathways and memory networks is maintained through a plasticity mechanism at inhibitory synapses [15]. In addition to the research emphasizing the importance of the balanced condition, there exist papers discussing the relevance of different ratios of inhibitory and excitatory neurons in brain; for example, cortical neurons consist of only 20 – 30% inhibitory neurons [16].

**Extreme value theory and its relevance.** The extremal eigenvalue statistics are widely used in various disciplines of science. The generalized extreme value distribution (GEV) is applied to model extrema of independent, identically distributed random variables. GEV statistics have been realized in many real-world and model systems. For example, the radius of the bulk of complex eigenvalues of non-Hermitian random matrices has been shown to follow the Gumbel distribution [17]. Recent research revealing a rich network architecture has given way to the spectral studies of matrices deviating from a random structure. One of these studies demonstrates that statistics of largest eigenvalue of matrices with entries following the power-law distribution displays a transition from the Fréchet to the Tracy-Widom distribution at a threshold governed by the power-law exponent [18]. The statistics of the inverse of the largest eigenvalue for an ensemble of scalefree networks follows the Weibull distribution [19]. Some of the studies pertaining to sparse random graphs, and gain matrices in the context of brain networks, are shown to deviate from GEV statistics and follow normal distribution instead [20]. The statistical properties of $R_{max}$ of synaptic matrices capturing inhibitory and excitatory couplings reveal a transition to the extreme value distribution [21]. However, the extreme value distribution is not observed for a larger parameter regime, thereby restricting the applicability of the results for a more realistic underlying network construction.

Extreme value statistics for independent, identically distributed random variables can be formulated entirely in terms of three universal types of probability functions: the Fréchet, Gumbel and Weibull distributions, also known as GEV statistics depending upon whether the tail of the distribution is power-law, faster than any power-law, and bounded or unbounded respectively [22]. GEV statistics with a location parameter $\mu$, scale parameter $\sigma$ and shape parameter $\xi$ have often been used to model unnormalized data from a given system. The probability density function for extreme value statistics with these parameters is given by [22]

$$\rho(x) = \begin{cases} \frac{1}{\xi} \left[1 + \left(\frac{x - \mu}{\xi \sigma} \right)^{-1 - \frac{1}{\xi}} \right] \exp \left[-\left(1 + \left(\frac{x - \mu}{\xi \sigma} \right)^{-1} \right) \right] & \text{if } \xi \neq 0 \\ \frac{1}{\sigma} \exp \left(-\frac{x - \mu}{\sigma} \right) \exp \left[-\exp \left(-\frac{x - \mu}{\sigma} \right) \right] & \text{if } \xi = 0. \end{cases}$$

(1)

Distributions associated with $\xi > 0$, $\xi = 0$ and $\xi < 0$ are characterized by the Fréchet, Gumbel, and Weibull distributions respectively.

In this Letter, we investigate the statistical properties of $R_{max}$ for networks in the balanced condition. The factors affecting the balanced condition are monitored. We witness the Weibull distribution for the strictly balanced condition. Observed behaviour is not much affected by the change in underlying architecture; rather it depends more on the denseness of connections. We present results for Erdős-Rényi random networks, small-world networks and scalefree networks.

**Model.** – The balanced condition is attained by assigning a fixed weight to inhibitory and excitatory connections in the following manner [13]. When a node is defined as inhibitory with probability $p_n$, the corresponding entry in that row of the matrix $A$ is replaced by $1 - 1/p_n$. In the matrices constructed as above, most of the column sum would be fluctuating closely about the zero value for $p_n$, lying in the vicinity of 0.50. However, for lesser values of $p_n$, there may be some columns that have only excitatory connections, yielding only zero or +1 entries, which hinder the achievement of the balanced condition. Furthermore, we achieve a strictly balanced condition by subtracting a constant term from each non-zero element of a column, which restricts the sum of the column entries to a zero value. The strictly balanced condition is defined to resolve situations where the arrangement of inhibitory and excitatory couplings leads to a fluctuation around the zero value for the column sum, even after imposing the balanced condition.

**Random Networks.** – Erdős-Rényi random networks of size $N$ are constructed where pairs of nodes are connected with a probability $p$. Figure 1 plots the statistical properties of networks with 0, 1 and $1-1/p_n$ entries. The data is fitted with the Gaussian and GEV distributions (Eq. 1). Figure 1 is plotted for various values of $p_n$ while keeping other network parameters the same. The nature of the distribution is normal for $p_n = 0$. As inhibitory connections are introduced, thereby inducing directional-
The behaviour of the column sum statistics provides an understanding of the impact of network structure on the shape parameter of a GEV distribution. For the balanced condition, the mean and variance of the column sum are zero and \( Np(1/p_{in} - 1) \) respectively. The maximum and minimum values of the column sum for a particular matrix in the ensemble are denoted by \( C_{\text{max}} \) and \( C_{\text{min}} \). The Weibull, Gumbel, and Fréchet distributions are observed in Fig. 2(a), (b) and (c), respectively for an ensemble consisting of realizations of matrices generated by imposing three different restrictions on \( C_{\text{max}} \) by keeping \( p_{in} \), \( p \) and \( N \) the same. An additional limitation on \( C_{\text{min}} \) to a particular value shifts the shape parameter \( \xi \) towards a positive value yielding Gumbel and Fréchet statistics as illustrated by Fig. 2(d), (e) and (f). The implications of restricting \( C_{\text{min}} \) and \( C_{\text{max}} \) are that they characterize the deviation from the strictly balanced condition, and interestingly, decide the shape parameter of the statistics regardless of the denseness of underlying networks. Note that for the lower \( p_{in} \) values, the network has a significant number of nodes connected to only excitatory nodes, thus failing to satisfy the strictly balanced condition. In order to avoid this situation, only those realizations are chosen that lead to columns with at least one negative entry. Fig. 3 depicts the statistics under the strictly balanced condition for a network of size \( N = 50 \) and \( p = 0.20 \). The Weibull distribution is observed in the regime \( 0.20 \lesssim p_{in} \lesssim 0.50 \). The statistics witness a sharp transition from the Gaussian to the Weibull at \( p_{in} = 0.2 \). The estimated parameters of both the types of statistics and the detailed information of fitting are addressed in [23].

Tail behaviour. The nature of extreme value distribution can further be explained by the tail behaviour of the parent distribution [22]. In the case of the Weibull distribution the tail of the parent distribution follows a power-law with bounded maxima. Fig. 4 plots tail behaviour extracted from the real part of the eigenvalues for the matrices associated with the Erdős-Rényi networks at different values of \( p_{in} \), which confirms the Weibull statistics as expected from the extremal eigenvalues of this ensemble.

Random matrices. Fig. 5 plots \( \rho(R_{\text{max}}) \) for random matrices generated using Gaussian distributed random numbers under the strictly balanced condition. This matrix represents the case of when coupling weights of inhibitory and excitatory connections are taken from Gaussian distributed random numbers with mean \( 1 - 1/p_{in} \) and 1, and standard deviation 0.05, as can be seen in the Ref. [12]. The nature of the \( R_{\text{max}} \) distribution remains normal
Fig. 5: (Colour online) Statistics of $R_{\text{max}}$ at different values of $p_{in}$ for random matrices. The histograms are numerical results; blue and red lines correspond to normal and GEV fit, respectively. For each cases the statistics display the Weibull distribution, except $p_{in} = 0$ for which normal distribution is observed. All plots are obtained for 5000 realizations of matrices with $N = 400$.

for $p_{in} = 0$, whereas the Weibull distribution is observed for $0.10 \lesssim p_{in} \lesssim 0.50$. The robustness of the Weibull statistics in this parameter regime is indicated by a fixed value for the $\xi$ parameter. The mean and variance of the data remains constant for the strictly balanced condition in the range $0.10 \leq p_{in} \leq 0.50$.

Fig. 6: (Colour online) Statistics of $R_{\text{max}}$ at different values $p_{in}$, for small-world networks with the strictly balanced condition. The histograms are numerical results; blue and red lines correspond to normal and GEV fit, respectively. For each case, the statistics exhibit the Weibull distribution. The rewiring probability is fixed at $p_r = 0.0256$ characterizing small-world transition. All plots are obtained for 5000 realizations of matrices with $N = 500$ and $\langle k \rangle = 20$.

This result confirms the robustness of the Weibull distribution for the strictly balanced condition, as it leads to a distribution independent of whether the matrix is modelled using a random network, i.e. entries being 0 and 1, or a random matrix, where entries are the Gaussian distributed random numbers. The left panel of the phase diagram in Fig. 7 illustrates this behaviour for various values of $p$ and $p_{in}$ for Erdős-Rényi networks under the strictly balanced condition. Very small values of $p_{in}$ may yield a situation in which some columns have only positive entries, and which do not allow the strictly balanced condition to be imposed, thereby making these values of $p_{in}$ out of the scope of the present study.

Fig. 7: (Left panel) Statistical behaviour of $R_{\text{max}}$ for Erdős-Rényi networks with $N = 50$ at different values of $p_{in}$ and $p$. Region 1 denotes the parameter region for which the strictly balanced condition cannot be defined, because some columns have only non-negative entries. Region 2 corresponds to the distribution that is Weibull or close to Weibull. Region 3 stands for the undefined distribution. (Right panel) For small-world networks with $N = 500$ and $\langle k \rangle = 20$ at different values of $p_{in}$ and rewiring probability $p_r$. Regions 4, 5 and 6 represent Gumbel (or close to it), Weibull and normal distributions, respectively. All plots are obtained for 5000 realizations of the networks.

For some cases, the KS test accepts the normal as well as the Weibull distribution. This happens because a particular shape parameter range, the Weibull distribution complies closely with the normal distribution [24]. For very high values of $p_{in}$, the conformation space of a network’s structure is reduced, which results in a lack of variation in network topology for an ensemble. This might be a reason for the undefined shape of the statistics for higher $p$ values. Model systems having a lower average degree could be modelled by the Weibull distribution, which would be due to many configuration possibilities in the ensemble leading to more fluctuations in the $R_{\text{max}}$, leading to a smooth shape of the statistics. The fact that most of the real-world networks are sparse [25], implies that they too can be modelled by this ensemble, which exhibits the Weibull distribution. We further analyze the effects of different network configurations on the statistics of extremal eigenvalues under the strictly balanced condition.

Small-World Networks. – First we consider small-world networks generated using the Watts-Strogatz algorithm. Properties of many real-world networks, including the brain, are prescribed by this small-world model [26]. This type of network maintains the clustering coefficient close to that of the regular lattices, whereas the average diameter is close to that of the random networks. Small-world networks have been found in C. elegans, cat cortex, and macaque cortex and it has been shown that the efficiency of the brain to rapidly integrate information from both locally and distantly specialized brain areas increases with the organization of small-world topology [25].
We generate small-world networks using the Watts-Strogatz model \cite{20}. For $N = 500$ and $\langle k \rangle = 20$, the rewiring probability is chosen as $p_r = 0.0256$, which corresponds to the small-world transition. Fig. 4 confirms that for all the $p_{in}$ values, the statistics remains Weibull. All the plots of Fig. 4 except that which corresponds to $p_{in} = 0.0$, satisfy the strictly balanced condition. The mean and variance of the $R_{max}$ decrease monotonically with $p_{in}$ for the strictly balanced condition. However, the shape parameter ($\xi$) is most negative for $p_{in} = 0.20$, which corresponds closely to a real brain situation \cite{10}, reflecting a less right-skewed Weibull statistics. Information pertaining to the parameters estimated in the statistics are referred in \cite{23}.

**Phase diagram for small-world networks.** The phase diagram in Fig. 7 (right panel) demonstrates the behaviour of the statistics for various values of $p_r$ and $p_{in}$. For $p_r = 0$, only inhibitory couplings contribute to the statistics, and as a result the statistics of $R_{max}$ is found close to the Gumbel. The tail behaviour displays an exponential decay, thereby supporting the observed Gumbel distribution (Fig. 8). As $p_r$ increases, the contribution of structural variation also increases yielding more variation in the $R_{max}$ statistics. Because the nature of the statistics is determined by the shape parameter $\xi$, for a fixed value of disorder ($p_i$), the occurrence of all the three statistics are possible. Increased disorder in network structure leads to an enhancement of the value of $\xi$, for a fixed total number of degrees occurring in all the realizations for networks generated using the small-world model. Note that for Erdős-Rényi networks, a particular $p$ value there exists a fluctuation in the total degree

Fig. 8: (Colour online) The tail behaviour of the real part of the eigenvalues at various values of $p_r$ for small-world networks under the strictly balanced condition. For each case, $N = 100$ and $p_{in} = 0.50$.

Fig. 9: (Colour online) Statistics of $R_{max}$ for different values of $p_{in}$ for scalefree networks with the strictly balanced condition. The histograms are numerical results; blue and red lines correspond to normal and GEV fit, respectively. For each case, the statistics show the Weibull distribution, except $p_{in} = 0$. All plots are obtained for 5000 realizations of networks with $N = 100$ and $\langle k \rangle = 8$.

for the different network realizations.

**Scalefree Networks.** In this section we present results for the scalefree network architecture, generated using the preferential growth algorithm \cite{28}. After introducing inhibitory connections with the probability $p_{in}$, the strictly balanced condition is imposed. Fig. 4 demonstrates that for all $p_{in}$ values, the statistics remain Weibull. However, the KS test accepts the normal distribution as well for $p_{in} = 0$. The mean and variance of the data remains constant for the strictly balanced condition and for the different $p_{in}$ values (i.e. 0.30, 0.35, 0.40, 0.45 and 0.50). The reason for discarding cases with $p_{in} \leq 0.20$ is that these values do not yield enough realizations that satisfy the strictly balanced condition. It happens due to the presence of a large number of nodes having a lesser degree, as compared to the Erdős-Rényi networks. The observed statistics can further be explained from the tail behaviour of the parent distributions. Fig. 10 displays a consistent power-law behaviour with the increase in $\langle k \rangle$, whereas the shape parameter of GEV monotonically decreases with an increase in $p_{in}$. The estimated parameters information is referred to \cite{23}.

**Discussions and Conclusion.** The nature of extreme values distribution for many real-world systems is associated with various shape parameters. The right skewness reflects the chances of occurrence of higher values. The effect of the strictly balanced condition dominates the behaviour of extreme value statistics, in particular to a fixed Weibull statistics. This condition is so strong that even changes in the interaction patterns do not affect the distribution behaviour.

Origin of the Weibull distribution for the strictly balanced condition could be explained by the fact that the strictly balanced condition shifts the outliers into the bulk of spectra, i.e. $R_{max}$ becomes bounded \cite{12}. The observed
Weibull statistics is supported further by the tail behaviour of the parent distribution which follows a power-law with bounded maxima.

The strictly balanced condition yields the Weibull distribution for networks with structural variations such as Erdős-Rényi random networks and scalefree networks. The 1-d lattice structure lacks any structural variation in the ensemble leading to a deviation from the Weibull distribution even for the strictly balanced condition. We demonstrated that at the small-world transition, a network has sufficient structural variations or randomness leading to a less right-skewed statistics governed by the Weibull distribution, consequently making the system more stable.

The Weibull distribution does not display any significant change with the change in the average degree of the network in the balanced condition, whereas previous work [21] demonstrates a deviation from the Weibull to the Fréchet distribution via Gumbel as connectivity of the network increases by keeping $p_{in}$ fixed at 0.5. The reasons for the networks with a lower $p$ value (corresponding to a lower average degree) following the Weibull distribution is that such matrices have fewer fluctuations around the strictly balanced condition and exhibit similar statistics to that observed for the strictly balanced condition. However, higher values of $p$ yield matrices with more deviations than those satisfying the strictly balanced condition, and as a result lead to an increased number of outliers from the bulk part of the eigenvalues, consequently resulting in a transition from the Weibull statistics.

The extreme value theory might enhance our understanding of stability properties of real brain systems. For instance, model networks capturing real brain network properties, such as small-world architecture and a 20-80% inhibitory to excitatory ratio, tend to witness fewer right-skewed $R_{max}$ statistics compared to other possible values of $p_{in}$. The higher values of $R_{max}$ are more likely to generate right-skewed statistics with higher variances. The variances can be managed by weight scaling the connections. The combined framework of the network architecture and the strictly balanced situation thus emulates the existence of stable statistics upon capturing realistic brain scenario. Future studies will incorporate other network architectures, particularly those having community structures [29]. Recently the stability of eco-systems has been analyzed using the spectral properties of underlying matrices [6]. The framework presented in this Letter can be extended to have a proper understanding of other such complex systems [30].

Acknowledgment. SJ thanks the DST and CSIR for funding. SKD acknowledges the University grants commission, India for financial support.

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