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François Digne, Jean Michel

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COMMUTATION OF SHINTANI DESCENT AND JORDAN DECOMPOSITION

FRANÇOIS DIGNE AND JEAN MICHEL

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Abstract. Let $G^F$ be a finite group of Lie type, where $G$ is a reductive group defined over $F$, and $F$ is a Frobenius root. Lusztig’s Jordan decomposition parametrizes the irreducible characters in a rational series $\mathcal{E}(G^F, (s)_{G^F})$ where $s \in G_{\text{ss}}^F$ by the series $\mathcal{E}(C_{*}(s)_{G^F}, 1)$. We conjecture that the Shintani twisting preserves the space of class functions generated by the union of the $\mathcal{E}(G^F, (s')_{G^F})$ where $(s')_{G^F}$ runs over the semi-simple classes of $G_{\text{ss}}^F$ geometrically conjugate to $s$; further, extending the Jordan decomposition by linearity to this space, we conjecture that there is a way to fix Jordan decomposition such that it maps the Shintani twisting to the Shintani twisting on disconnected groups defined by Deshpande, which acts on the linear span of $\prod_s \mathcal{E}(C_{*}(s')_{G^F}, 1)$. We show a non-trivial case of this conjecture, the case where $G$ is of type $A_{n-1}$ with $n$ prime.

1. Deshpande’s approach to Shintani descent

We follow [3]. Let $H$ be an algebraic group over an algebraically closed field $k$; we identify $H$ to its points $H(k)$ over $k$. Let $\gamma_1, \gamma_2$ be two commuting bijective isogenies on $H$. We define the following subset of $(H \times \langle \gamma_1 \rangle) \times (H \times \langle \gamma_2 \rangle)$:

$$R_{\gamma_1, \gamma_2} = \{(x \gamma_1, y \gamma_2) \mid x, y \in H, [x \gamma_1, y \gamma_2] = 1\}$$

where $[u, v]$ is the commutator $uvu^{-1}v^{-1}$. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$ defines a map

$$R_{\gamma_1, \gamma_2} \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto R_{\gamma_1^a \gamma_2, \gamma_1^b \gamma_2} : (x \gamma_1, y \gamma_2) \mapsto ((x \gamma_1)^a (y \gamma_2)^b), (x \gamma_1)^b (y \gamma_2)^d)$$

There is an action of $H$ on $R_{\gamma_1, \gamma_2}$ by simultaneous conjugation: $(x \gamma_1, y \gamma_2) \xrightarrow{ad g} (gx \gamma_1 g^{-1}, gy \gamma_2 g^{-1})$, which commutes with the maps $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and we denote by $R_{\gamma_1, \gamma_2}/\sim H$ the space of orbits under this action.

Assume now that $H$ is connected and $\gamma_1$ is a Frobenius root (an isogeny such that some finite power is a Frobenius morphism). Then, by the Lang-Steinberg theorem any $x \gamma_1 \in H \gamma_1$ is $H$-conjugate to $\gamma_1$, and we can take as representatives of the $H$-orbits pairs of the form $(\gamma_1, y \gamma_2)$; on these pairs there is only an action of the fixator of $\gamma_1$, that is $H^{*1}$. Further, the condition $[\gamma_1, y \gamma_2] = 1$ is equivalent to $y \in H^{*1}$ (since $\gamma_1$ and $\gamma_2$ commute). We can thus interpret $R_{\gamma_1, \gamma_2}$ as the $H^{*1}$-conjugacy classes on the coset $H^{*1} \gamma_2$, which we denote by $H^{*1} \gamma_2/\sim H^{*1}$. If $a, c$ and $\gamma_2$ are such that $\gamma_1^a \gamma_2$ is still a Frobenius root, we can thus interpret the map...
given by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) as a map \( H^{\gamma_1} \sim H^{\gamma_1} \to H^{\gamma_2} \gamma_1 \gamma_2 \sim H^{\gamma_2} \), a “generalised Shintani descent”.

We are interested here in the case of \( R_F,Id \) where \( F \) is a Frobenius root. The matrix \( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \) defines a map, that we call the Shintani twisting and will just denote by \( Sh : R_{F,Id} \to R_{F,Id} : (xF, y) \mapsto (xFy, y) \). With the identification above of \( R_{F,Id} \sim H \) with the pairs \( (F, y) \) which identifies it with \( H^F \sim H \), if we write \( y = \lambda F \lambda^{-1} \) using the Lang-Steinberg theorem, \( Sh \) maps \( (F, y) \) to \( (F, y) = (F \lambda F \lambda^{-1}, \lambda F \lambda^{-1}) \) which is conjugate by \( \lambda F \lambda^{-1} \) to \( (F, F \lambda^{-1} \lambda) \), thus we recover the usual definition of the Shintani twisting \( Sh \) as being induced by the correspondence \( \lambda F \lambda^{-1} \mapsto \lambda F \lambda^{-1} \).

The advantage of Deshpane’s approach is that the map \( Sh : R_{F,Id} \to R_{F,Id} \) still makes sense when \( H \) is disconnected; this time the interpretation of \( R_{F,Id} \sim H \) is different: the \( H \)-orbits on \( HF \) are parametrised by \( H^1(F, H/H^0) \). If \( \sigma \in H \) is a representative of such an \( H \)-orbit, the \( H \)-orbits of pairs \( (F, y) \) are in bijection with \( H^F \sim H \), so we see that we must consider all rational forms of \( H \) corresponding to the various representatives \( \sigma F \in H^1(F, H/H^0) \) together, and \( Sh \) will act on the disjoint union of the conjugacy classes of each of these forms. Let \( \sigma F \) be such a representative, let \( H_1 \) be a \( F \)-stable coset of \( H^0 \) in \( H \), let \( \sigma' \in H_1^F \) and let us compute the image by \( Sh \) of a commuting pair \( (\sigma F, \sigma') \) where \( y \in H_0^0 \) (thus \( y \in (H^0)^{\sigma F} \)). Let us write using the Lang-Steinberg theorem \( y = \lambda \sigma \sigma F \lambda^{-1} \).

Then \( Sh \) maps \( (\sigma F, \sigma') \) to \( (\sigma \sigma F, \sigma') = (\sigma \sigma F \lambda \sigma \sigma F \lambda^{-1}, \sigma \lambda \sigma \sigma F \lambda^{-1}) \) (using \([\sigma F, \sigma'] = 1\) which is conjugate by \( \sigma \sigma F \lambda^{-1} \) to \( (\sigma' \sigma F, \sigma') \)). Thus

**Proposition 1.** \( Sh \) is induced on \( \sigma' (H^0)^{\sigma F} \), where \( \sigma \) and \( \sigma' \) are two representatives of \( H/H^0 \) such that \( \sigma F \) and \( \sigma' \) commute, by the correspondence between the element \( \sigma' \sigma \sigma F \lambda^{-1} \) of \( \sigma' (H^0)^{\sigma F} \) and the element \( \sigma' \sigma F \lambda^{-1} \) of \( \sigma' (H^0)^{\sigma F} \).

Assume now that \( H/H^0 \) is commutative and that we can lift all elements of \( H/H^0 \) to commuting representatives, such that \( F \)-stable elements lift to \( F \)-stable representatives; then we can lift all pairs \( (\sigma F, \sigma') \) as in Proposition 1 which commute in \( H/H^0 \) to commuting pairs, thus we can see \( Sh \) as a linear map on the space \( C(H, F) := \oplus_{\sigma \in H^1(F, H/H^0)} C(H^F_F) \), the direct sum of the spaces of class functions on the various rational forms \( H^{\sigma F} \).

**Definition 2.** Let \( H \) be a algebraic group with a Frobenius root \( F \), then we say that \( H \) satisfies condition (*) if we can lift elements of \( H/H^0 \) to commuting elements of \( H \), such that \( F \)-stable elements of \( H/H^0 \) lift to \( F \)-stable elements.

Beware that \( Sh \) on class functions is defined by \( Sh(f)(x) = f(Sh(x)) \) thus it maps class functions on \( \sigma' (H^0)^{\sigma F} \) to class functions on \( \sigma (H^0)^{\sigma F} \).

Note finally that when \( H^0 \) is trivial and \( F \) acts trivially on \( H/H^0 \), the computations of this section recover the well-known action of \( GL_2(\mathbb{Z}) \) on the Drinfeld double of the finite group \( H/H^0 = H \). See, for example [2, 8.4.2].

2. Conjectures

**Definition 3.** We extend the definition of Deligne-Lusztig characters to a disconnected reductive group \( H \) with a Frobenius root \( F \) by \( RH_T(s) := Ind_{H^0}^{H} RH_T(s) \) where \( T \) runs over \( F \)-stable maximal tori of \( H \) and \( s \) over \( T^* F^* \). We call unipotent
characters the irreducible components of the $R^H_T(1)$ and we denote by $E(H^F, 1)$ the set of unipotent characters of $H^F$.

From now on, $G$ will be a connected reductive group with a Frobenius root $F$ and $s$ an $F^*$-stable semi-simple element of $G^*$, the group dual to $G$. We denote by $E(G^F, (s)_{G^*, F^*})$ the $G^*$-Lusztig series associated with $s$; this series consists of the irreducible components of the $R^G_T(s)$ where $T^*$ runs over $F$-stable maximal tori of $C_{G^*}(s)$. We denote by $E(G^F, (s))$ the geometric Lusztig series associated with $s$, that is the union of the series $E(G^F, (s'))_{G^*, F^*}$ where $(s')_{G^*, F^*}$ runs over the $G^*$-classes in the geometric class of $s$.

Let $\varepsilon_H := (-1)^{F^*\text{-rank } G}$. Lusztig’s Jordan decomposition of characters in a reductive group with a not necessarily connected centre is given by the following theorem (see for example [5, Theorem 11.5.1]).

**Theorem 4.** Let $G$ be a connected reductive group with Frobenius root $F$ and $(G^*, F^*)$ be dual to $G$; for any semi-simple element $s \in G^*$, there is a bijection from $E(G^F, (s)_{G^*, F^*})$ to $E(C_{G^*}(s)^{F^*}, 1)$. This bijection may be chosen such that, extended by linearity to virtual characters, it sends $\varepsilon_H R^G_T(s)$ to $\varepsilon_H R^{G^*}_T(s)^{F^*}(1)$ for any $F^*$-stable maximal torus $T^*$ of $C_{G^*}(s)$.

**Conjecture 5.** Let $G$ be a connected reductive group with Frobenius root $F$. Then $Sh$, viewed as a linear operator on $C(G^F)$, preserves the subspace spanned by $E(G^F, (s))$ for each geometric class $(s)$.

**Proposition 6.** Assume that $F$ is the Frobenius morphism corresponding to an $F_q$-structure on $G$. Then Conjecture 5 holds when the characteristic is almost good for $G$ in the sense of [7 1.12] and $ZG$ is connected or when $G$ is of type A and either $q$ is large or $F$ acts trivially on $ZG/(ZG)^0$.

**Proof.** The case of type A when $F$ acts trivially on $ZG/(ZG)^0$ results from [11 Théorème 5.5.4]. Let us prove the other cases. In [11 Section 3.2] it is shown that the characteristic functions of character sheaves are eigenvectors of $Sh$; then in [9 Theorem 5.7] and [10 Theorem 3.2 and Theorem 4.1] (resp. [11 Corollaire 24.11]) it is shown that when $ZG$ is connected and the characteristic is almost good (resp. in type A with $q$ large) the characteristic functions of character sheaves coincide up to scalars with “almost characters”. Since for each $s$, the space $C(E(G^F, (s)))$ is spanned by a subset of almost characters, this gives the result. □

It is probable that the above proof applies to more reductive groups (see [11, 12, and also 11 Introduction]), but it is difficult to find appropriate statements in the literature.

Assuming conjecture 5, a choice of a Jordan decomposition as specified by Theorem 4 will map the operator $Sh$ on $C(G^F, (s))$ to a linear operator on the space

$$C(C_{G^*}(s)^{F^*}, F^*, 1) := \oplus_{s' \in H^* \cap C_{G^*}(s)^{F^*}, C_{G^*}(s)} C(E(C_{G^*}(s)^{F^*}, F^*, 1)).$$

**Conjecture 7.** Let $s \in G^*$ be a $F^*$-stable semi-simple element. Then $C_{G^*}(s)$ satisfies (\*) and $Sh$ on $C(C_{G^*}(s), F^*)$ preserves the subspace $C(C_{G^*}(s)^{F^*}, F^*, 1)$, and the choice of a Jordan decomposition in Theorem 4 may be refined so that it maps $Sh$ on $C(E(G^F, (s)))$ to $Sh$ on $C(C_{G^*}(s)^{F^*}, F^*, 1)$.
3. The case of a group of type $A_{n-1}$ when $n$ is prime.

Proposition 8. Conjectures 5 and 7 hold for any reductive group of type $A_{n-1}$, with $n$ prime, if they hold for $\text{SL}_n$.

Proof. We start with two lemmas.

Lemma 9. Let $\mathbf{G}$ be a connected reductive group with a Frobenius root $F$. Let $\mathbf{G}$ be a closed connected $F$-stable subgroup of $\mathbf{G}$ with same derived group. Let $F^*$ be a Frobenius root dual to $F$ and let $s \in \mathbf{G}^{*F^*}$ be such that $C_{\mathbf{G}^*}(s)^{F^*} = C_{\mathbf{G}}^*(s)^{F^*}$. Then the characters in $\mathbf{G}^F$ be a closed connected subgroup of $\mathbf{G}$. Let $\mathbf{F}$ be a connected reductive group of type $A$.

Proof. By definition the characters in $\mathbf{G}^F$ are the restrictions from $\mathbf{G}$ to $\mathbf{G}^F$ of the characters in $\mathbf{E}(\mathbf{G}^F, (\tilde{s})_{\mathbf{G}^F})$ where $\tilde{s} \in \mathbf{G}^{*F^*}$ lifts $s \in \mathbf{G}^{*F^*}$.

Lemma 10. Consider a quotient $1 \rightarrow Z \rightarrow \mathbf{G}_1 \xrightarrow{\pi} \mathbf{G} \rightarrow 1$, where $Z$ is a connected subgroup of $\mathbf{G}_1$. Assume that $\mathbf{G}$ and $\mathbf{G}_1$ have Frobenius roots both denoted by $F$ and that $\pi$ commutes with $F$; then $\pi$ is surjective from the conjugacy classes of $\mathbf{G}_1^F$ to that of $\mathbf{G}^F$, and commutes with $\text{Sh}$.

Proof. The group $Z$ is $F$-stable since $\pi$ commutes with $F$. Let $x \in \mathbf{G}^F$. Since $Z$ is connected, by the Lang-Steinberg theorem $\pi^{-1}(x)^F$ is non-empty whence the surjectivity of $\pi$ on elements, hence on conjugacy classes. If $y \in \pi^{-1}(x)^F$ the image by $\pi$ of $\text{Sh}(y)$ is $\text{Sh}(x)$, whence the commutation of $\text{Sh}$ with $\pi$. 

Let now $\mathbf{G}$ be an arbitrary reductive group of type $A_{n-1}$. Since $n$ is prime, the only semi-simple connected reductive groups of type $A_{n-1}$ are $\text{SL}_n$ and $\text{PGL}_n$; thus any connected reductive group $\mathbf{G}$ of type $A_{n-1}$ is the almost direct product of the derived group $\mathbf{G}'$ of $\mathbf{G}$, equal to $\text{SL}_n$ or $\text{PGL}_n$, by a torus $\mathbf{S}$.

In the case $\mathbf{G}' = \text{PGL}_n$, the almost direct product is direct since $\text{PGL}_n$ has a trivial centre. If $\mathbf{T}$ is an $F$-stable maximal torus of $\mathbf{G}$, then it has a decomposition $\mathbf{T}_1 \times \mathbf{S}$ where $\mathbf{T}_1 = \mathbf{T} \cap \mathbf{G}'$ is $F$-stable since $\mathbf{G}'$ is $F$-stable. It is possible to find an $F$-stable complement $\mathbf{S}'$ of $\mathbf{T}_1$ (see for example the proof of [6 2.2]) and then $\mathbf{G}$ has an $F$-stable product decomposition $\mathbf{G}' \times \mathbf{S}'$. Since $\text{Sh}$ is trivial on a torus the conjectures are reduced to the case of $\mathbf{G}' = \text{PGL}_n$.

The same argument reduces the case of a direct product $\mathbf{G} = \mathbf{G}' \times \mathbf{S}$ where $\mathbf{G}' = \text{SL}_n$ to the case of $\text{SL}_n$. The other possibility when $\mathbf{G}' = \text{SL}_n$ is an almost direct product $\text{SL}_{n_1} \times \mathbf{S}$ amalgamated by $Z\text{SL}_{n_2}$; this is isomorphic to a product of the form $\text{GL}_{n_1} \times \mathbf{S}'$; in such a group all centralisers are connected, as well as in
the dual group (isomorphic to $\mathbf{G}$) thus both conjectures are trivial since $\text{Sh}$ is the identity (by for example [4 IV, 1.1]).

Finally, in $\operatorname{PGL}_n$, the action of $\text{Sh}$ is trivial by Lemma 10 applied to the quotient $\operatorname{GL}_n \twoheadrightarrow \operatorname{PGL}_n$, and in the dual $\operatorname{SL}_n$ semisimple elements have connected centralisers which are Levi subgroups of $\operatorname{SL}_n$; considering the embedding of these centralisers in the corresponding Levi of $\operatorname{GL}_n$, on which $\text{Sh}$ is trivial since it is isomorphic to a product of $\operatorname{GL}_n$, we get that $\text{Sh}$ is trivial on the unipotent characters of these centralisers by Lemma 9. 

We have thus shown that we only need to consider $\operatorname{SL}_n$. Up to isomorphism there are two possible $\mathbb{F}_p$-structures on $\operatorname{SL}_n$ (only one if $n = 2$) thus $F$ will be one of the Frobenius endomorphisms $F_+$ or $F_-$ where $\operatorname{SL}_n^{F_+} = \operatorname{SL}_n(\mathbb{F}_q)$ and $\operatorname{SL}_n^{F_-} = \operatorname{SU}_n(\mathbb{F}_q)$. When we want to consider both cases simultaneously but keep track whether $F = F_+$ or $F = F_-$, we will denote the Frobenius by $F_\varepsilon$ with $\varepsilon \in \{-1, 1\}$, where we always take $\varepsilon = 1$ if $n = 2$. We will use the dual group $\operatorname{PGL}_n$, the inclusion $\operatorname{SL}_n \subset \operatorname{GL}_n$, and the quotient $\operatorname{GL}_n \rightarrow \operatorname{PGL}_n$. Since $(\operatorname{GL}_n, F)$ is its own dual, we will write $F$ (instead of $F^*$) for the Frobenius map on the dual of $\operatorname{GL}_n$ and on $\operatorname{PGL}_n$. We choose for $F_+$ the standard Frobenius which raises all matrix entries to the $q$-th power, and choose for $F_-$ the map given by $x \mapsto F_+(x^{-1})$. This choice is such that on the torus of diagonal matrices $T^*$ of $\mathbf{G}^*$, $F_\varepsilon$ acts by raising all the eigenvalues to the power $-q$, and acts trivially on $W_{\mathbf{G}^*}(T^*)$. The torus $T^*$ is split for $F_+$ and of type $w_0$ (the longest element of the Weyl group) with respect to a quasi-split torus for $F_-$. 

**Proposition 11.** For the group $\operatorname{SL}_n$, $n$ prime, with Frobenius $F = F_\varepsilon$ conjecture $\mathbf{5}$ holds. Further, conjecture $\mathbf{7}$ holds if it holds when $q \equiv \varepsilon \pmod{n}$ for the series $E(\operatorname{SL}_n^{F_\varepsilon}, (s))$ with $s \in \operatorname{PGL}_n^{F_\varepsilon}$ geometrically conjugate to $\operatorname{diag}(1, \zeta, \zeta^2, \ldots, \zeta^{n-1})$ where $\zeta \in \overline{\mathbb{F}_q}$ is a non-trivial $n$-th root of $1$.

**Proof.** Lemma 9 applied with $\mathbf{G} = \operatorname{SL}_n$ and $\tilde{\mathbf{G}} = \operatorname{GL}_n$ shows that conjectures 5 and 7 hold when $C_{\operatorname{PGL}_n}(s)^F = C_{\operatorname{PGL}_n}^0(s)^F$. Indeed, in this case, since $\text{Sh}$ is trivial in $\operatorname{GL}_n$ and commutes with the restriction to $\operatorname{SL}_n$ by Lemma 10, Lemma 9 shows that $\text{Sh}$ is trivial on $E(\operatorname{SL}_n^{F_\varepsilon}, (s))$: on the other hand, since $C_{\operatorname{PGL}_n}^0(s)$ is the quotient with central kernel of a product of $\operatorname{GL}_n$, the unipotent characters of $C_{\operatorname{PGL}_n}(s)^F = C_{\operatorname{PGL}_n}^0(s)^F$ can be lifted to this product (see [5 Proposition 11.3.8]). Since $\text{Sh}$ is trivial in $\operatorname{GL}_n$ and commutes with such a quotient with connected central kernel by Lemma 10, it is trivial on the unipotent characters of $C_{\operatorname{PGL}_n}(s)^F$.

We have thus reduced the study of conjectures 5 and 7 to the case of semi-simple elements $s \in \operatorname{PGL}_n$ with $C_{\operatorname{PGL}_n}(s)^F \neq C_{\operatorname{PGL}_n}^0(s)^F$.

By [5 Lemma 11.2.1(iii)], since $n$ is prime a semi-simple element $s \in \operatorname{PGL}_n$ has a non-connected centraliser if and only if $|C_{\operatorname{PGL}_n}(s)/C_{\operatorname{PGL}_n}^0(s)| = n$. Since this group of components is a subquotient of the Weyl group $\mathfrak{S}_n$, it has order $n$ if and only if it is a subgroup of $\mathfrak{S}_n$ generated by an $n$-cycle. An easy computation shows that such an $s$ is as in the statement of Proposition 11. Now the centraliser of $s$ is the semidirect product of $T^*$ with the cyclic group generated by the $n$-cycle $c = (1, \ldots, n)$. Assume that the geometric class of $s$ has an $F$-stable representative $s'$ in a torus of type $w \in \mathfrak{S}_n$ with respect to $T^*$ (or equivalently that $wF^k s = s$). Then $C_{\operatorname{PGL}_n}(s')^F \simeq C_{\operatorname{PGL}_n}(s)^{wF}$ hence is equal to $C_{\operatorname{PGL}_n}^0(s')^F$ unless $w$ commutes with $c$, that is $w = c^i$ for some $i$. Since $c$ centralises $s$ we get that $s$ is $F$-stable,
which means for $F = F_2$ that $n$ divides $q - \varepsilon$, in particular Conjecture 7 holds when $q \not\equiv \varepsilon \pmod{n}$; we recall that we always take $\varepsilon = 1$ when $n = 2$.

Finally, conjecture 5 holds in any case, since $\mathrm{Sh}$ preserves all geometric series except possibly that of $s$ and preserves orthogonality of characters, thus preserves also the series $E(SL_n^F, (s)).$ $\square$

Henceforth we assume that $n$ is prime and divides $q - \varepsilon$, and study conjecture 7 for the particular $s$ of Proposition 11.

**Assumption 12.** We choose $\zeta$, a primitive $n$-th root of unity in $\mathbb{F}_q$. We choose an isomorphism $\mathbb{F}_q^\times \simeq (\mathbb{Q}/\mathbb{Z})_{p'}$ which maps $\zeta$ to $1/n$ and a group embedding $\mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$ which maps $1/n$ to $\zeta^q := \exp(2i\pi/n)$.

We denote by $T$ the diagonal torus of $SL_n$ and by $\hat{T}$ the diagonal torus of $GL_n$; we choose the dual torus $T^*$ to be the diagonal torus of $PGL_n$. We let $s \in T^*$ be the image of $\mathrm{diag}(1, \zeta, \zeta^2, \ldots, \zeta^{n-1}) \in \hat{T}$.

**Sh on $C_{PGL_n}(s)$.**

We first study $\mathrm{Sh}$ on $\mathcal{C}(C_{PGL_n}(s)^F, F, 1)$. We have $C_{PGL_n}(s) = T^* \times_c (c)$ where $c$ is the permutation matrix representing the cycle $(1,2,\ldots,n) \in S_n$, which acts on $T^*$ by sending $\mathrm{diag}(t_1, \ldots, t_n)$ to $\mathrm{diag}(t_n, t_1, \ldots, t_{n-1})$. On $T^*$ the Frobenius $F_\mathfrak{c}$ acts by $t \mapsto t^{q-1}$, and acts trivially on $W_{G_\mathfrak{c}}(T^*)$. Note that $C_{G_\mathfrak{c}}(s)$ satisfies condition (a) since $c$ is $F$-stable. Since $C_{PGL_n}(s)/C_{PGL_n}^0(s) = (c)$, and $F$ acts trivially on $c$, we have $H^1(F, C_{PGL_n}(s)/C_{PGL_n}^0(s)) = (c)$ and the geometric class of $s$ splits into $n$ rational classes parametrised by the powers of $c$. A representative of the class parametrised by $c^j$ is $\bar{s}_j$ where $x$ is such that $x^{-1}F_x = c^j$. This representative lies in a maximal torus $T^*_c = x^T$ of type $c^j$ with respect to $T^*$. Choosing the $PGL_n$-conjugacy by $x^{-1}$ to identify $(T^*_c, F)$ with $(T^*, c^jF)$, we identify back the representative of the class parametrised by $c^j$ with the element $s$ in $T^*c^jF$. We have

$$\mathcal{C}(C_{PGL_n}(s)^F, F, 1) = \bigoplus_{j=0}^{n-1} \mathcal{C}E(T^*c^jF \times_c (c), 1).$$

Since $T^*$ is a torus, the unipotent characters of $T^*c^jF \times_c (c)$ are the $n$ extensions of the trivial character of $T^*c^jF$. We parametrise these by $Z SL_n \times \mathbb{Z}/n\mathbb{Z}$ in the following way: if $z_0 = \mathrm{diag}(\zeta, \ldots, \zeta)$, we call $\hat{\theta}_{z_0, k}$ the character of $T^*c^jF \times_c (c)$ which is trivial on $T^*c^kF$ and equal to $\zeta^k$ on $c$. This allows us to define another basis of $\mathcal{C}(C_{PGL_n}(s)^F, F, 1)$, the “Mellin transforms”, defined for $j, k \in \mathbb{Z}/n\mathbb{Z}$ by $\hat{\theta}_{j, k} := \sum_{z \in Z SL_n} \hat{s}_j(z)\hat{\theta}_{z, k}$, where $\hat{s}_j$ is the character of $T^*c^jF$ corresponding to the element $s \in T^*c^jF$ through duality. The point is that it is more convenient to compute the action of $\mathrm{Sh}$ on the Mellin transforms:

**Proposition 13.** We have $\mathrm{Sh} \hat{\theta}_{j, i} = \hat{\theta}_{j, i+j}$ unless $n = 2$ (thus $\varepsilon = 1$) and $q \equiv -1 \pmod{4}$. In this last case we have $\mathrm{Sh} \hat{\theta}_{j, i} = \hat{\theta}_{j, i+j-1}$.

**Proof.**

**Lemma 14.** With the conventions of Assumption 12, when $n$ is odd, we have $\hat{s}_j(z_0^j) = \zeta^j$; when $n = 2$ (thus $\varepsilon = 1$) we have $\hat{s}_j(z_0) = \begin{cases} (-1)^{(q-1)/2} & \text{if } j = 0 \\ (-1)^{(q+1)/2} & \text{if } j = 1 \end{cases}$. 


Lemma 15. If $\kappa$ is an integer, then $\kappa$ divides $\varepsilon q - 1$, we have $\frac{(\varepsilon q)^n - 1}{n(\varepsilon q - 1)} \equiv 1 \pmod{n^k}$.

Proof. Let us write $\varepsilon q = 1 + an^k$. We have $\frac{(\varepsilon q)^n - 1}{\varepsilon q - 1} = 1 + \varepsilon q + \cdots + (\varepsilon q)^{n-1} \equiv n^k + n(n-1)n^k \pmod{n^k}$. Since $n$ is odd it divides $n(n-1)/2$, so that $\frac{(\varepsilon q)^n - 1}{\varepsilon q - 1} \equiv \varepsilon q^{k+1} \pmod{n^k(k+1)}$ which, dividing by $n$, gives the result.

This lemma applied with $k = 1$ shows that $\hat{s}_j$ maps $\tilde{z}_0$ to $\hat{\zeta}_C^{kJ}$ when $n$ is odd. If $n = 2$ (thus $\varepsilon = 1$) we get $\hat{s}_1(\text{diag}(-1, -1))$ when $n$ is odd.
We now compute the Mellin transforms $\theta_{j,i}$. If $n$ is odd, we have by Lemma \[14\]

$$
\theta_{j,i}(c^k) = \sum_{t \in \mathbb{Z}/n\mathbb{Z}} \hat{s}_j(z_0^t) \theta_{j,i}(e^k) = \sum_{t \in \mathbb{Z}/n\mathbb{Z}} \hat{s}_j(z_0^t) = \begin{cases} 0 & \text{if } j + k \neq 0, \\ n & \text{if } j = -k. 
\end{cases}
$$

We see that $\theta_{j,i}$ is a function supported by the coset $T^{cF}.c^{-j} \subset T^{cF} \ltimes \langle c \rangle$ and is constant equal to $n$ on this coset. By proposition \[1\] $\text{Sh}$ maps a constant function on this coset to the constant function on $T^{c^{i+j}F}.c^{-j}$ with same value, hence $\text{Sh} \theta_{j,i} = \theta_{j,i+j}$.

If $n = 2$ we have

$$
\theta_{j,i}(c^k) = (-1)^k \hat{s}_j(z_0) + \hat{s}_j(1) = 1 + \left\{ \begin{array}{ll}
(-1)^{k+(q-1)/2} & \text{if } j = 0, \\
(-1)^{k+(q+1)/2} & \text{if } j = 1.
\end{array} \right.
$$

Thus $\theta_{j,i}$ is supported by $T^{cF}.c^j$ if $q \equiv 1 \pmod{4}$ and by $T^{cF}.c^{1-j}$ if $q \equiv -1 \pmod{4}$ and is constant equal to $2$ on its support. We get the same result as in the odd case for the action of $\text{Sh}$ when $q \equiv 1 \pmod{4}$. If $q \equiv -1 \pmod{4}$, since $\text{Sh}$ maps functions on $T^{c^i}c^{-j}$ to functions on $T^{c^{i+j}}c^{-j}$ we get $\text{Sh} \theta_{j,i} = \theta_{j,i+j-1}$. □

Proposition \[13\] shows that $\text{Sh}$ preserves the space $C(C_{\mathbb{G}L_n}(s)^F,F,1)$. Note that the computation made in the proof of Lemma \[14\] and Definition \[3\] show that $R_{T_{X,i}}^{T_{r,s}} \text{Id} = \theta_{0,i}$ unless $n = 2$ and $q \equiv -1 \pmod{4}$, in which case $R_{T_{X,i}}^{T_{r,s}} \text{Id} = \theta_{1,i}$.

**Sh on $SL_n^F$.**

For computing the other side of conjecture \[7\] that $\text{Sh}$ is on $SL_n^F$, we first parametrise the characters in $\bigcup_j \mathcal{E}(SL_n^F,(s_j)_{\mathbb{G}L_n^F})$, where $s_j \in (T_{c^j})^F$ is an $F$-stable representative of the rational class that we parametrised above by $s \in T^{cF}c^F$. We use the following notation: for $z \in Z SL_n$ we denote by $\Gamma_z$ the Gelfand-Graev character indexed by $z$ as in \[5\] Definition 12.3.3; that is $\Gamma_z = \text{Ind}_{U^F}^F \psi_z$ where $U$ is the unipotent radical of the Borel subgroup consisting of upper triangular matrices and $\psi_z$ is a regular character of $U^F$ indexed by $z$. This labelling depends on the choice of a regular character $\psi_1$: we have $\psi_z = t^{\psi_1}$ where $t \in T$ satisfies $t^{-1} F t = z$.

Let $\hat{s}_j \in (T_{c^j})^F$ be a lifting of $s_j$. By \[5\] Proposition 11.3.10 $R_{T_{X,i} \cap SL_n}^{SL_n}(s_j)$ is the restriction of $R_{T_{X,i}}^{\mathbb{G}L_n}(\hat{s}_j)$, hence the series $\mathcal{E}(SL_n^F,(s_j)_{\mathbb{G}L_n^F})$ is the set of irreducible components of the restrictions of the elements of $\mathcal{E}(\mathbb{G}L_n^F,(\hat{s}_j))$. Moreover since $C_{\mathbb{G}L_n}(\hat{s}_j)$ is a torus, the character $R_{T_{X,i}}^{\mathbb{G}L_n}(\hat{s}_j)$ is irreducible, hence is the only character in $\mathcal{E}(\mathbb{G}L_n^F,(\hat{s}_j))$; thus this character must be a component of the (unique) Gelfand-Graev character of $\mathbb{G}L_n^F$ by \[5\] Theorem 12.4.12. The restriction of this character to $SL_n^F$ is equal to $\sum_{z \in Z SL_n} \chi_{z,j}$ where $\chi_{z,j}$ is the unique irreducible component of the Gelfand-Graev character $\Gamma_z$ in the series $\mathcal{E}(SL_n^F,(s_j))$ (see \[5\] Corollary 12.4.11). Thus we have $\mathcal{E}(SL_n^F,(s_j)) = \{ \chi_{z,j} \mid z \in Z SL_n \}$. It is again more convenient to compute $\text{Sh}$ on the basis formed by the Mellin transforms $\chi_{j,k} := \sum_{z \in Z SL_n} \hat{s}_j(z) \chi_{z,k}$. 
Proposition 16. We have \( \text{Sh} \chi_{j,i} = \chi_{j,i+j} \) unless \( n = 2 \) (thus \( \varepsilon = 1 \)) and \( q \equiv -1 \) (mod 4). In this last case we have \( \text{Sh} \chi_{j,i} = \chi_{j,i+j-1} \).

Proof. We note first that \( \chi_{z,j}(g) \) is independent of \( z \) if \( C_{\text{SL}_n}(g) \) is connected. Indeed, \( \chi_{z,j} \) and \( \chi_{z',j} \) are conjugate by an element \( x \in \text{GL}_n^F \) since they are two components of the restriction of an irreducible character of \( \text{GL}_n^F \). We have thus \( \chi_{z',j}(g) = \chi_{z,j}(xg) \). We can multiply \( x \) by a central element to obtain an element \( y \in \text{SL}_n \) and, since \( y = xg \in \text{SL}_n^F \), we have \( y^{-1}Fy \in C_{\text{SL}_n}(g) \). If this centraliser is connected, using the Lang-Steinberg theorem we can multiply \( y \) by an element of \( C_{\text{SL}_n}(g) \) to get a rational element \( y' \), whence \( \chi_{z',j}(g) = \chi_{z,j}(y'g) = \chi_{z,z}(g) \). For such an element \( g \) we thus have \( \chi_{j,i}(g) = 0 \) if \( j \neq 0 \).

Since \( C_{\text{SL}_n}(g) \) is connected we have \( \text{Sh}(g) = g \) thus \( \text{Sh} \chi_{j,i}(g) = \chi_{j,i}(g) \), in particular \( \text{Sh} \chi_{0,i} = \chi_{0,i} \) and if \( j \neq 0 \) we have \( \text{Sh} \chi_{j,i}(g) = \chi_{j,i+j}(g) = 0 \), whence \( \text{Sh} \chi_{j,i}(g) = \chi_{j,i+j}(g) \) for all \( j \).

It remains to consider the conjugacy classes of \( \text{SL}_n \) which have a non-connected centraliser.

Lemma 17. When \( n \) is prime the only elements of \( \text{SL}_n \) which have a non-connected centraliser are the \( zu \) with \( z \in \text{ZSL}_n \) and \( u \) regular unipotent.

Proof. Let \( su \) be the Jordan decomposition of an element of \( \text{SL}_n \) with \( s \) semi-simple and \( u \) unipotent. We have \( C_{\text{SL}_n}(su) = C_{\text{GL}_n}(s)u \). The group \( C_{\text{SL}_n}(s) \) is a Levi subgroup of \( \text{SL}_n \), that is the subgroup of elements of determinant 1 in a product \( \prod_{i=1}^k \text{GL}_{n_i} \). Thus \( C_{\text{SL}_n}(su) \) is the subgroup of elements with determinant 1 in a product \( \prod_{i=1}^k H_i \) where \( H_i \) is the centraliser of \( u_i \) in \( \text{GL}_{n_i} \), which is connected. We claim that if \( k > 1 \) the group \( C_{\text{SL}_n}(su) \) is connected. Indeed, since \( n \) is prime, if \( k > 1 \) the \( n_i \) are coprime so that there exist integers \( a_i \) satisfying \( \sum_{i=1}^k a_i n_i = -1 \). Then the map \( (h_1, \ldots, h_k) \mapsto (h_1 \lambda^{a_1}, \ldots, h_k \lambda^{a_k}) \) where \( \lambda = \det(h_1 h_2 \ldots h_k) \) is an isomorphism from \( H_1 \times \cdots \times H_k \) to \( C_{\text{SL}_n}(su) \times \bar{F}_q \). Hence this last group is connected, thus its projection \( C_{\text{SL}_n}(su) \) is also connected.

It remains to look at the centralisers of elements \( zu \) with \( u \) unipotent and \( z \in \text{ZSL}_n \), that is the centralisers of unipotent elements. By [4, IV, Proposition 4.1], since \( n \) is prime, \( C_{\text{SL}_n}(u) \) is connected unless \( u \) is regular and when \( u \) is regular we have \( C_{\text{SL}_n}(u) = R_u(C_{\text{SL}_n}(u)) \cup Z\text{SL}_n \) which is not connected.

Thus to prove the proposition we have only to consider the classes of the elements \( zu \) with \( u \) regular unipotent and \( z \in \text{ZSL}_n \). Fix a rational regular unipotent element \( u_1 \). The conjugacy classes of rational regular unipotent elements are parametrised by \( H^1(F, C_{\text{SL}_n}(u_1)/C_{\text{SL}_n}(u_1)) = H^1(F, \text{ZSL}_n) = Z\text{SL}_n \) (the last equality since \( q \equiv \varepsilon \) (mod 4)): a representative \( u_z \) of the class parametrised by the \( F \)-class of \( z \in \text{ZSL}_n \) is \( t \gamma_1 \) where \( t \in T \) satisfies \( t^{-1}Ft = z \). By [4, IV, Proposition 1.2] Sh maps the \( SL_n \)-class of \( zu_z \) to that of \( zu_z' \). Now \( \chi_{z,k} \) being a component of the restriction of the irreducible character \( R^\text{GL}_n \) has central character equal to \( \delta_k \), independently of \( z \), whence \( \chi_{z,k}(z'u_z') = \delta_k(z') \chi_{z,k}(u_z') \). By [5 Corollary 12.3.13], there is a family of Gauss sums \( \sigma_z \) indexed by \( Z\text{SL}_n \) such that

\[
\chi_{z,k}(u_z') = \sum_{z' \in H^1(F, \text{ZSL}_n)} \sigma_{z',z-1}((-1)^{F-\text{semi-simple rank}(\text{SL}_n)} D(\chi_{z,k}, \Gamma_{z'})_{\text{SL}_n^F},
\]

where \( D \) is the Curtis-Alvis duality. If \( k \neq 0 \) and \( F = F_+ \) or if \( F = F_- \) we claim that \((-1)^{F-\text{semi-simple rank}(\text{SL}_n)} D(\chi_{z,k}) = \chi_{z,k} \): indeed, in both cases the character
$R_{T_k}^{GL_n}(\hat{s}_k)$ is cuspidal since the torus $\hat{T}_{k}$ is not contained in a proper rational Levi subgroup. Indeed when $F = F_{+}$ and $k \neq 0$ (resp. $F = F_{-}$), the type $e^k$ (resp. the type $u_0 v^k$) of $T_{k}$ with respect to a quasi-split torus is not contained in a standard parabolic subgroup of $\mathfrak{S}_n$. Thus $\chi_{z,k}$ is cuspidal as a component of the restriction to $\mathcal{S}_n^F$ of $R_{T_k}^{GL_n}(\hat{s}_k)$, whence our claim since the duality on cuspidal characters is the multiplication by $(-1)^{F_{\text{semi-simple}} \text{rank}(\mathcal{S}_n)}$ (take $L = \mathcal{S}_n$ in [5, 7.2.9]). If $k = 0$ and $F = F_{+}$, the characters $\chi_{z,0}$ are the irreducible components of $R_{T_k}^{\mathcal{S}_n}(\hat{s}_0) = \text{Ind}^{B_F}_{B_k} \hat{s}_0$, where $B$ is the Borel subgroup of upper triangular matrices and $\hat{s}_0$ has been lifted to $B^F$. The endomorphism algebra of this induced character is isomorphic to the group algebra of $Z / nZ$, hence the components of $\text{Ind}^{\mathcal{S}_n}_{B_k} \hat{s}_0$ are parametrised by the characters of $Z / nZ$ and by [8, Theorem B] the effect of the duality is to multiply the parameters by the sign character of the endomorphism algebra which is trivial if $n$ is odd and is $-1$ if $n = 2$.

If $n$ is odd or if $n = 2$ and $k \neq 0$, we thus have $\chi_{z,k}(u_{z} u^{v}) = \sigma_{z} u_{z-1}$. If $n = 2$ and $k = 0$ we have $\chi_{z,0}(u_{z} u^{v}) = -\sigma_{z} u_{z}$ where $\{z, z'\} = \{1, \text{diag}(-1, -1)\}$.

We consider first the case $n$ odd. By Lemma 14 we have $\hat{s}_k(z_0) = \zeta^k$. Thus the values of the Mellin transforms are $\chi_{j,k}(z' u_{z} u^{v}) = \sum_i \hat{s}_k(z_0) \chi_{j,k}(z' u_{z} u^{v}) = \sum_i \hat{s}_k(z_0) \chi_{j,k}(z' u_{z} u^{v}) = \sum_i \hat{s}_k(z_0) \chi_{j,k}(z' u_{z} u^{v}) = \zeta^k \sum_i \hat{s}_k(z_0) \chi_{j,k}(z' u_{z} u^{v})$. Let us put $z' = z_0$; we get $\chi_{j,k}(z_0 u_{z} u^{v}) = \sum_i \hat{s}_k(z_0) \chi_{j,k}(z' u_{z} u^{v}) = \sum_i \hat{s}_k(z_0) \chi_{j,k}(z' u_{z} u^{v}) = \sum_i \hat{s}_k(z_0) \chi_{j,k}(z' u_{z} u^{v}) = \chi_{j,k}(z_0 u_{z} u^{v})$, which gives the proposition for $n$ odd.

If $n = 2$ we have $z_0 = \text{diag}(-1, -1)$. We have $\chi_{j,k}(z u_{z} u^{v}) = \hat{s}_k(z)(\hat{s}_j(1) \chi_{j,k}(u_{z}) + \hat{s}_j(z_0) \chi_{j,k}(u_{z}))$. We get $\chi_{j,0}(z u_{z} u^{v}) = \hat{s}_0(z)(-\sigma_{z} u_{z} - \hat{s}_j(z_0) \sigma_{z})$ and $\chi_{j,1}(z u_{z} u^{v}) = \hat{s}_1(z)(\sigma_{z} + \hat{s}_j(z_0) \sigma_{z})$. If $\hat{s}_j$ is the identity character then $\chi_{j,0}$ and $\chi_{j,1}$ are invariant under $\text{Sh}$ since $\text{Sh}(u_{z} u^{v}) = u_{z} u^{v}$ and $\text{Sh}(z_0 u_{z} u^{v}) = z_0 u_{z} u^{v}$. If $\hat{s}_j$ is not trivial then, $\chi_{j,0}(z u_{z} u^{v})$ and $\chi_{j,1}(z u_{z} u^{v})$ are equal if $z = 1$ and opposite if $z \neq 1$, thus, using again $\text{Sh}(u_{z} u^{v}) = u_{z} u^{v}$ and $\text{Sh}(z_0 u_{z} u^{v}) = z_0 u_{z} u^{v}$ we see that $\chi_{j,0}$ and $\chi_{j,1}$ are exchanged by $\text{Sh}$. By Lemma 14 if $q \equiv 1 \pmod{4}$ the character $\hat{s}_0$ is trivial and we get the same result as in the odd case. If $q \equiv -1 \pmod{4}$, the character $\hat{s}_1$ is trivial and $\text{Sh}$ exchanges $\chi_{0,0}$ and $\chi_{0,1}$ and fixes $\chi_{1,0}$ and $\chi_{1,1}$, which is the announced result.

We can now state:

**Proposition 18.** For $s = \text{diag}(1, \zeta, \ldots, \zeta^{n-1})$, the bijection $J : \chi_{j,i} \mapsto \theta_{j,i}$ from $C \mathcal{E}(\mathcal{S}_n^F(s))$ to $C \mathcal{E}(C_{\text{PGL}_n}(s)^F, F, 1)$ restricts on characters to a refinement of the Jordan decomposition which satisfies Conjecture 4.

**Proof.** Propositions 13 and 16 give the commutation of $J$ with $\text{Sh}$. It remains to show that $J(R_{T_k}^{\mathcal{S}_n}(\hat{s}_j)) = R_{T_k}^{\mathcal{E}}(\chi_{j,i} \chi_{j,0})$. As we have noticed after the proof of Proposition 13 we have $R_{T_k}^{\mathcal{E}}(\chi_{j,i} \chi_{j,0}) \text{Id} = \begin{cases} \theta_{0,j} & \text{for } n \text{ odd or } q \equiv 1 \pmod{4}, \\ \theta_{l,j} & \text{for } n = 2, q \equiv -1 \pmod{4}. \end{cases}$ On the other hand for $n$ odd or $q \equiv 1 \pmod{4}$, we have $\chi_{0,j} = \sum \hat{s}_0(z) \chi_{z,j} = \sum \chi_{z,j}$ since in that case $\hat{s}_0$ is the trivial character. For $n = 2, q \equiv -1 \pmod{4}$, we have $\chi_{1,j} = \sum \hat{s}_1(z) \chi_{z,j} = \sum \chi_{z,j}$ since in that case $\hat{s}_1$ is the trivial character. By definition of $\chi_{z,j}$ we have $\sum \chi_{z,j} = R_{T_k}^{\mathcal{S}_n}(\hat{s}_j)$, whence the proposition.
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