A $(1 - e^{-1} - \varepsilon)$-Approximation for the Monotone Submodular Multiple Knapsack Problem

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Abstract

We study the problem of maximizing a monotone submodular function subject to a Multiple Knapsack constraint (SMKP). The input is a set $I$ of items, each associated with a non-negative weight, and a set of bins, each having a capacity. Also, we are given a submodular, monotone and non-negative function $f$ over subsets of the items. The objective is to find a subset of items $A \subseteq I$ and a packing of the items in the bins, such that $f(A)$ is maximized.

SMKP is a natural extension of both Multiple Knapsack and the problem of monotone submodular maximization subject to a knapsack constraint. Our main result is a nearly optimal polynomial time $(1 - e^{-1} - \varepsilon)$-approximation algorithm for the problem, for any $\varepsilon > 0$. Our algorithm relies on a refined analysis of techniques for constrained submodular optimization combined with sophisticated application of tools used in the development of approximation schemes for packing problems.

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1 Introduction

Submodular optimization has recently attracted much attention as it provides a unifying framework capturing many fundamental problems in combinatorial optimization, economics, algorithmic game theory, networking, and other areas. Furthermore, submodularity also captures many real world practical applications where economy of scale is prevalent. Classic examples of submodular functions are coverage functions [12], matroid rank functions [3] and graph cut functions [10]. A recent survey on submodular functions can be found at [1].

Submodular functions are defined over sets. Given a ground set $I$, a function $f : 2^I \rightarrow \mathbb{R}_{\geq 0}$ is called submodular if for every $A \subseteq B \subseteq I$ and $u \in I \setminus B$, $f(A + u) - f(A) \geq f(B + u) - f(B)$.

This reflects the diminishing returns property: the marginal value from adding $x \in I$ to a solution diminishes as the solution set becomes larger. A set function $f : 2^I \rightarrow \mathbb{R}$ is monotone if for any $A \subseteq B \subseteq I$ it holds that $f(A) \leq f(B)$. While in many cases, such as coverage and matroid rank function, the submodular function is monotone, this is not always the case (cut functions are a classic example).

The focus of this work is optimization of monotone submodular functions. In [21] Nemhauser and Wolsey presented a greedy based $(1 - \epsilon^{-1})$-approximation for maximizing a monotone submodular function subject to a cardinality constraint, along with a matching lower bound in the oracle model. A $(1 - \epsilon^{-1})$ hardness of approximation bound is also known for the problem under $P \neq \text{NP}$, due to the hardness of max-$k$-cover [11] which is a special case. The greedy algorithm of [21] was later generalized to monotone submodular optimization with a knapsack constraint [18, 23].

A major breakthrough in the field was the continuous greedy algorithm presented in [24]. Initially used to derive a $(1 - \epsilon^{-1})$-approximation for maximizing a monotone submodular function subject to a matroid constraint, the algorithm has become a primary tool in the development of monotone submodular maximization algorithms subject to various other constraints. These include $d$-dimensional knapsack constraints [19], and combinations of $d$-dimensional knapsack and matroid constraints [8]. A variant of the continuous greedy algorithm for non-monotone functions is given in [13].

In the multiple knapsack problem (MKP) we are given a set of items, where each item has a weight and a profit, and a set of bins of arbitrary capacities. The objective is to find a feasible packing of a subset of items in the bins which yields a maximum profit. The problem is one of the most natural extensions of the classic Knapsack problem.

A polynomial time approximation scheme (PTAS) for MKP was first presented by Chekuri and Khanna [3]. The authors also ruled out the existence of a fully polynomial time approximation scheme for the problem. An efficient PTAS (EPTAS) for the problem was later developed by Jansen [16, 17].

1.1 Our Results

In this paper we consider the submodular multiple knapsack problem (SMKP). The input consists of a set of items $I = \{1, 2, \ldots, n\}$ and $m$ bins $B = \{1, 2, \ldots, m\}$. Each item $i \in I$ is associated with a weight $w_i \geq 0$, and each bin $b \in B$ has a capacity $W_b \geq 0$. We are also given an oracle to a non-negative monotone submodular function $f : 2^I \rightarrow \mathbb{R}_{\geq 0}$. A feasible solution to the problem is a tuple of $m$ subsets $A_1, \ldots, A_m \subseteq I$ such that for every $b \in B$ it holds that $\sum_{i \in A_b} w_i \leq W_b$. The value of a solution $A_1, \ldots, A_m$ is $f(\bigcup_{b \in B} A_b)$. The goal is to find a feasible solution of maximum value.

The problem is a natural generalization of both Multiple Knapsack [6] (where $f$ is modular), and the problem of monotone submodular maximization subject to a knapsack constraint [25] (where $m = 1$).

Our result is summarized in the following theorem.

**Theorem 1.1.** For any $\epsilon > 0$, there is a randomized $(1 - \epsilon^{-1} - \epsilon)$-approximation algorithm for SMKP.

As already mentioned, a $(1 - \epsilon^{-1})$ hardness of approximation bound is known for the problem under $P \neq \text{NP}$, due to the hardness of max-$k$-cover [11] which is a special case of SMKP.

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1 Equivalently, for every $A, B \subseteq I$: $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$. 
1.2 Tools and Techniques

Our algorithm relies on a refined analysis of techniques for submodular optimization subject to $d$-dimensional knapsack constraints \[13\] [5]. [8], combined with sophisticated application of tools used in the development of approximation schemes for packing problems [9].

The problem of \textit{submodular maximization subject to $d$-dimensional knapsack constraint ($d$-SUB)} is defined as follows. The input consists of a set $I$ of items, where each item $i \in I$ is associated with a $d$-dimensional weight vector $\mathbf{w}_i \in \mathbb{R}^d_0$. Also, we are given a $d$-dimensional capacity vector $\mathbf{W} \in \mathbb{R}^d$, and an oracle to a monotone, non-negative submodular function $f : 2^I \to \mathbb{R}_{\geq 0}$. The objective is to find a subset of items $A \subseteq I$, such that $\sum_{i \in A} \mathbf{w}_i \leq \mathbf{B}$ for any dimension $1 \leq j \leq d$, and $f(A)$ is maximized. A randomized $(1 - \epsilon^{-1} - \varepsilon)$-approximation for the problem, for any $\varepsilon > 0$, was given in [14] when $d$ is a fixed constant.

At the heart of our algorithm lies the observation that SMKP for a large number of identical bins (i.e., $\forall b \in B$, $W_b = W$ for some $W \geq 0$) can be easily approximated via a reduction to 2-SUB. Given such an instance and $\varepsilon > 0$, we partition the items to \textit{small} and \textit{large}, where an item $i \in I$ is small if $w_i \leq \varepsilon W$ and large otherwise. We further define a \textit{configuration} to be a subset of large items which fits into a single bin, and let $C$ be the set of all configurations. It follows that for fixed $\varepsilon > 0$, the number of configurations is polynomial.

We use the above observation to define a new submodular optimization problem, to which we refer as the \textit{reduced} problem. We define a new universe $E$ which consists of all configurations $C$ and all small items $E = C \cup \{\{i\} \mid i \text{ is small}\}$. We also define a new submodular function $g : 2^E \to \mathbb{R}_{\geq 0}$ by $g(T) = f(\bigcup_{A \in T} A)$. Now, we seek a subset of elements $T \subseteq E$ such that $T$ has at most $m$ configurations, i.e., $|T \cap C| \leq m$, and the total weight of sets selected is at most $m \cdot \mathbf{W}$; namely, $\sum_{A \in T} w(A) \leq m \cdot \mathbf{W}$, where $w(A) = \sum_{i \in A} w_i$.

It is easy to see that the optimal value of the reduced problem is at least the value of the optimum for the original instance. Essentially, a solution $T$ for the reduced problem can be used to generate a solution for the SMKP instance with only a small loss in value. As there are no more than $m$ configurations, and all other items are small, the items in $T$ can be easily packed into $(1 + \varepsilon)m + 1$ bins of capacity $\mathbf{W}$ using First Fit. Then, it is possible to remove $\varepsilon m + 1$ of the bins while maintaining at least $\frac{m}{\varepsilon m + 1} \geq \frac{1}{1 + 2\varepsilon}$ of the solution value, for $m \geq \frac{1}{\varepsilon}$. Once these $\varepsilon m + 1$ bins are removed, we have a feasible solution for the SMKP instance.

The reduced problem is a $d$-SUB instance (with $d = 2$), and therefore a $(1 - \epsilon^{-1} - \varepsilon)$-approximate solution can be found efficiently.

Our approximation algorithm for SMKP is based on a generalization of the above. We refer to a set of bins of identical capacity as a \textit{block}, and show how to reduce an SMKP instance into a $d$-SUB instance in which $d$ is twice the number of blocks plus a constant. While, generally, $d$-SUB cannot be solved for non-constant $d$, we use a refined analysis of known algorithms [13] [5] to show that the problem can be efficiently solved if the blocks admit a certain structure, to which we refer as \textit{leveled}.

We resort to the grouping technique of Fernandez de la Vega and Lueker [9] to convert a general SMKP instance to a leveled instance. We sort the bins in decreasing order by capacity and then partition them into levels, where level $t$, $t \geq 0$, has $N^{2+t}$ bins, divided into $N^2$ consecutive blocks, each containing $N^t$ bins. We decrease the capacity of each bin to the smallest capacity of a bin in the same block. While the decrease in capacity generates the leveled structure required for our algorithm to work, it only slightly decreases the optimal solution value. The main idea is that given an optimal solution, each block of decreased capacity can now be used to store the items assigned to the subsequent block on the same level. Also, the items assigned to $N$ blocks from each level can be evicted, while only causing a reduction of $\frac{1}{N}$ to the profit (as only $N$ of the $N^2$ blocks of the level are evicted). These evicted blocks are then used for the items assigned to the first block in the next level.

While the reduced problem provides the foundation of our algorithm, it hinders difficulties for natural extension.

In the case of \textit{non-monotone submodular multiple knapsack problem}, where the function $f$ is non-monotone, the function $g$ used by the reduced problem is not submodular. Thus rendering submodular optimization tools ineffective.

Another natural generalization of the problem is SMKP with a matroid constraint. An instance to
the problem is an SMKP instance and a matroid $\mathcal{M}$; a solution is a solution $A_1, \ldots, A_m$ to the SMKP instance, with the additional constraint that $A_1, \ldots, A_m \in \mathcal{M}$. In terms of the reduced problem, the matroid constraint translates to the constraint $T \in Q$ with $Q = \{S \subseteq E | \cup_{A \in S} A \in \mathcal{M}\}$. However, the set $Q$ is not an independent set of a matroid. As a consequence, matroid optimization tools cannot be directly applied on this reduced problem.

1.3 Related Work

While SMKP is a natural extension of both Multiple Knapsack and submodular optimization subject to a single knapsack constraint, little attention was given to the problem. Feldman presented in [14] a $(1-1/e-\Theta(1))$ approximation for the problem. Additionally, Dobzinski [4] presented a $(1-\frac{1}{e} - o(1)) \approx 0.24$-approximation for the special case of identical bin capacities, along with a $\frac{1}{7}$-approximation for the general case. A $(1-e^{-1} - o(1)) \approx 0.468$-approximation for the problem can be derived for the problem using the techniques of [5]. To the best of our knowledge, this is the best known approximation ratio for the problem.

Simultaneously to our work, Sun et. al. [22] presented a deterministic greedy based $(1-e^{-1} - \varepsilon)$-approximation for the special case of identical bins. Their work also includes a randomized $(1-e^{-1} - \varepsilon)$-approximation for other special cases of SMKP.

2 Preliminaries

Our analysis relies on several basic properties of monotone function.

**Claim 2.1.** Let $f : 2^I \rightarrow \mathbb{R}_{\geq 0}$ be monotone and submodular function, then for any $A \subseteq B \subseteq N$ and $S \subseteq N$ it holds that $f(A \cup S) - f(A) \geq f(B \cup S) - f(B)$.

As the lemma is essential for our algorithm, it is important to emphasize it does not hold for non-monotone submodular function. Lemma 2.1 is used to show the following.

**Claim 2.2.** Let $f : 2^I \rightarrow \mathbb{R}_{\geq 0}$ be a non-negative, monotone and submodular function, and let $E \subseteq 2^I \times X$ for some set $X$ (each element of $E$ is a pair $(S, h)$ with $S \subseteq I$ and $h \in X$). Then function $g : 2^E \rightarrow \mathbb{R}_{\geq 0}$ defined by $g(A) = f(\cup_{(S, h) \in A} S)$ is non-negative, monotone and submodular.

The proof of Claim 2.2 can be found in Appendix A.

Many modern submodular optimization algorithms rely on the submodular Multilinear Extension ([11, 19, 20, 25, 13, 2]). Given function $f : 2^N \rightarrow \mathbb{R}_{\geq 0}$, its multilinear extension is $F : [0, 1]^I \rightarrow \mathbb{R}_{\geq 0}$ defined as:

$$F(\bar{x}) = \sum_{S \subseteq I} f(S) \prod_{i \in S} \bar{x}_i \prod_{i \notin I \setminus S} (1 - \bar{x}_i)$$

The multilinear extension can be interpreted as an expectation of a random variable. Given $\bar{x} \in [0, 1]^I$ we say that a random set $X$ is distributed according to $\bar{x}$, $X \sim \bar{x}$, if $\Pr(i \in X) = \bar{x}_i$ and the events $(i \in X)_{i \in I}$ are independent. It follows that $F(\bar{x}) = \mathbb{E}_{X \sim \bar{x}}[f(X)]$.

The continuous greedy of [3] can be used to find approximate solution for maximization problem of the form $\max F(\bar{x})$ s.t. $\bar{x} \in P$, where $F$ is the multilinear extension of a monotone submodular function $f$, and $P$ is a down-monotone polytope. The algorithm uses two oracles, one to $f$ and an another which given $\lambda \in \mathbb{R}^I$ return a vector $\bar{x} \in P$ such that $\bar{x} \cdot \lambda$ is maximal. The algorithm evaluates $\bar{x} \in P$ that $F(\bar{x}) \geq (1 - e^{-1}) \max_{\hat{y} \in P} F(\hat{y})$, where $A^*$ is an optimal solution for the problem.

In the analysis of the algorithm we utilize two known Chernoff-like bounds.

**Lemma 2.3** (Theorem 3.1 in [14]). Let $X = \sum_{i=1}^n X_i \cdot \lambda_i$ where $(X_i)_{i=1}^n$ is a sequence of independent Bernoulli random variable and $\lambda_i \in [0, 1]$ for $1 \leq i \leq n$. Then for any $\varepsilon \in (0, 1)$ and $\eta \geq \mathbb{E}[X]$ it holds that

$$\Pr(X > (1 + \varepsilon)\eta) < \exp \left(-\frac{\varepsilon^2}{3}\eta\right)$$
Lemma 2.4 (Theorem 1.3 in [7]). Let \( I = \{1, \ldots, n\}, \) \( v > 0 \) and \( f : 2^I \to \mathbb{R}_+ \) be a monotone submodular function such that \( f(\{i\}) - f(\emptyset) \leq v \) for any \( i \in I \). Let \( X_1, \ldots, X_n \) be independent random variables and \( \eta = \mathbb{E}[f(\{i \in I|X_i = 1\})] \). Then for any \( \varepsilon > 0 \) it holds that

\[
\mathbb{E}[f(\{i \in I|X_i = 1\})] \leq (1 - \varepsilon)\eta \leq \exp\left(-\frac{\eta \cdot \varepsilon^2}{2v}\right)
\]

3 The Approximation Algorithm

In this section we present our approximation algorithm for SMKP. First, given an instance of the problem, we observe that there exists a subset of constant size of an optimal solution to the instance, where the remaining solution has the property that the value gained from any item belonging to it is small compared to the optimal value. Thus, the first step of our algorithm is an enumeration over all possible partial assignments of constant size, for each assignment we find a completion to an approximate solution. Among all possible assignments and completions pairs, we pick the best solution. Hence, from now on we restrict our attention to completing a solution to the residual problem, obtained after having fixed the initial partial assignment.

Instances of SMKP become easier to solve when having a small number of different bin capacities, e.g., uniform bin capacities, leading us to define the notion of a block:

**Definition 3.1.** For a given instance of SMKP we say that a subset of bins \( \tilde{B} \subseteq B \) is a block if all the bins in \( \tilde{B} \) have the same capacity. I.e, for bins \( b_1 \) and \( b_2 \) belonging to the same block it holds that \( W_{b_1} = W_{b_2} \).

Following the enumeration, our algorithm reduces the number of blocks by altering the bin capacities. To this end we use a specific structure we call leveled, defined as follows.

**Definition 3.2.** For any \( N \in \mathbb{N} \) we say that capacities \( (W_b)_{b \in B} \) of an SMKP instance are \( N \)-leveled if the set of bins can be partitioned into blocks

\[
\{B_{t,j} \mid 0 \leq t \leq \ell - 1, \, 1 \leq j \leq N^2\} \cup \{B_{t,j} \mid 1 \leq j \leq k\}
\]

for some \( \ell \in \mathbb{N} \) and \( k \leq N^2 \), \( |B_{t,j}| = N^t \) for \( 0 \leq t \leq \ell \) and \( 1 \leq j \leq N^2 \) \( (j < k \text{ for } t = \ell) \) and \( |B_{t,k}| \leq N^\eta \).

In Section 3.1 we define a function \( R_N \) that maps a set of bin capacities into a leveled set. This is done in Algorithm 2 (that runs in polynomial time), and incurs only a mild decrease to the value of an optimal solution. We prove the following.

**Lemma 3.3.** For every \( N \in \mathbb{N}, N > 0 \), function \( R_N \) maps every sequence of bin capacities \( W_1, \ldots, W_m \) to \( \tilde{W}_1, \ldots, \tilde{W}_m \) such that

1. \( \tilde{W}_1, \ldots, \tilde{W}_m \) is \( N \)-leveled.
2. For every \( b \in B, \tilde{W}_b \leq W_b \).
3. For any SMKP instance, let \( \tilde{W}_1, \ldots, \tilde{W}_m = R_N(W_1, \ldots, W_m) \). Then there is a solution \( A_1, \ldots, A_m \) to the instance such that \( \sum_{i \in A_b} w_i \leq \tilde{W}_b \) for any \( b \in B \) and \( f(A_1 \cup \ldots \cup A_m) \geq (1 - \frac{1}{N}) OPT \), where \( OPT \) is the value of the optimal solution to the instance. Also, if \( \ell \geq 1 \) then \( A_b = \emptyset \) for any \( b \in B_{\ell,k} \) (the last block is empty).

Afterwards, we use a randomized rounding algorithm presented in Section 3.2. The algorithms efficiently utilizes the leveled structure of the instance. Instead of having a separate constraint for each bin in a block, there are only two constraints for each block. The first constraint is a knapsack constraint for the total capacity of a block, and the second constraint restricts the number of configurations assigned to the block. Thus, the number of constraints significantly decreases if the blocks are large. Since leveled instances also have a constant number of blocks with a single bin, those are handled separately via the notion of \( \delta \)-restricted SMKP.
The $\delta$-restricted SMKP is a variant of SMKP in which there is an additional set of restricted bins $B^r \subseteq B$. A solution to $\delta$-restricted SMKP is an assignment $A_1, \ldots, A_m$ such that $\forall b \in B^r, i \in A_b$, it holds that $w_i \leq \delta W_b$. That is, only items that consume up to a $\delta$ fraction of the capacity can be assigned to restricted bins.

We treat the blocks with a single bin as restricted. While items of weight greater than $\delta W_b$ may be assigned to these blocks in an optimal solution, the overall number of such items can be bounded by a constant. The initial enumeration provides a guarantee that evicting these items from an optimal solution only causes a moderate decrease in the optimal solution values. Therefore, allowing us to consider the instance as $\delta$-restricted.

In Section 3.2 we show the following bound on the performance of Algorithm 3, our randomized rounding algorithm. The algorithm is parameterized by $\mu \in (0, 0.1)$, a value we will set later. Suppose that a $\delta$-restricted SMKP instance is given such that:

- The unrestricted bins are partitioned into blocks, $B \setminus B^r = B_1 \cup \ldots \cup B_k$.
- $v = \max_{i \in I} f\{i\} - f(\emptyset)$ and $\mathcal{O}$ is the value of an optimal solution to the instance.

**Lemma 3.4.** For $\mu \in (0, 0.1)$, Algorithm 3 returns a feasible solution $S_1, \ldots, S_m$ such that $f(S_1 \cup \ldots \cup S_m) \geq (1 - e^{-1})(1 - \mu^2)\mathcal{O}$ with probability at least $1 - \gamma$ where

$$\gamma = \exp\left(-\frac{\mu^3}{16} \cdot \frac{\mathcal{O}}{v}\right) + |B^r| \exp\left(-\frac{\mu^2}{12} \cdot \frac{1}{\delta}\right) + 2 \cdot \sum_{j=1}^{k} \exp\left(-\frac{\mu^2}{12} |B_j|\right)$$

We are now ready to present the approximation algorithm for a general SMKP instance.

**Algorithm 1:** Algorithm for SMKP

**Configuration:** The algorithm uses several configuration parameters: $N, P, \delta, \mu$ which will be set in the proof of the lemma.

**input:** An SMKP instance: $I = \{1, \ldots, n\}$, $(w_i)_{i \in I}$, $B = \{1, \ldots, m\}$, $(W_b)_{b \in B}$, and a monotone submodular function $f$.

1. **forall** feasible assignments $A = (A_1, \ldots, A_m)$ such that $| \cup_{j=1}^{m} A_j | \leq P$ do
   1. Let $Y = \{i \in I \mid f(A \cup \{i\}) - f(A) \geq P^{-1}f(A)\}$
   2. Let $W' = W' = W_b - w(A_j) \ \forall b \in B$. Evaluate $(\tilde{W}_1, ..., \tilde{W}_m) = R_N(W'_1, ..., W'_m)$, and let $\tilde{B}_{t,j}$ be the partition of the bins with capacities $(\tilde{W}_b)_{b \in B}$ to levels and blocks, where $\ell$ is the number of levels and $k$ is the number of blocks in the last level.
   3. Define a residual $\delta$-restricted SMKP instance as follows. The items are $\tilde{I} = I \setminus (\cup_{j=1}^{m} A_j \cup Y)$ with and $\tilde{w}_i = w_i$ for any $i \in \tilde{I}$, the bins are $\tilde{B} = B \setminus B_{\ell,k}$ with capacities $\tilde{W}_b$ for $b \in B$ as in the previous line. The submodular function is $\tilde{f}(S) = f(S \cup A) - f(A)$. The restricted bins are $B^r = \cup_{1 \leq j \leq N2} \tilde{B}_{0,j}$ (the bins of level 0).
   4. Solve the residual instance using Algorithm 3 with $\mu$ (the algorithm’s configuration parameter), and the partition $\tilde{B} = B^r \cup \bigcup_{(t,j),t \geq 1} \tilde{B}_{(t,j)}$. Denote the returned assignment by $S = S_1, ..., S_m$, where $S_b = \emptyset$ for $b \in B \setminus \tilde{B}$.
   5. If $A_1, ..., A_m$ is a feasible solution $f(\cup_{b \in B}(A_b \cup S_b))$ (or $f(\cup_{b \in B}A_b))$ is higher than the current best solution value, set it as the best solution.
   6. Return the maximal value among the solutions found.

**Theorem 3.5.** For any $\varepsilon > 0$, there are parameters $N, P, \delta, \mu$ such that Algorithm 3 is a randomized $(1 - e^{-1} - \varepsilon)$-approximation algorithm for SMKP.

**Proof.** We start by providing a lower bound for the approximation ratio of the algorithm given arbitrary configuration parameters. The values of the parameters will be set afterwards.
Let $A_1^*, \ldots, A_m^*$ be an optimal solution to the input instance and set $A^* = \bigcup_{b \in B} A_b^*$. Order the items in $A^*$ according to their marginal profit. That is, $A^* = \{a_1, \ldots, a_r\}$ with $f_{T_j^{-1}}(a_j) = \max_{a \in A^*} f_{T_j^{-1}}(a)$, where $T_j = \{a_1, \ldots, a_j\}$, and $f_T(S) = f(S \cup T) - f(T)$.

Our analysis focuses on the iteration in which $A = \{a_1, \ldots, a_P\}$ ($P$ is the one of the configuration parameters) and $A_b = A_b^* \cap A$. We will assume this is the case for the rest of the analysis. As $f$ is submodular, for every $a_j \in A^*$, $j > P$, it holds that $\tilde{f}(a_j) \leq \frac{1}{P} f(A)$ (see, e.g., [19]). Therefore, $A^* \setminus A \subseteq I$.

It can be easily verified that $A_1^* \setminus A, \ldots, A_m^* \setminus A$ is a feasible solution to the SMKP instance with the items $\tilde{I}$, weight $w_i$ for $i \in \tilde{I}$, bins $B$ of capacities $W'_b$ for $b \in B$ and the submodular function $\tilde{f}$. Therefore by Lemma 3.3 there is a feasible solution $D_1, \ldots, D_m$ to this instance, such that $\tilde{f}(\bigcup_{b \in B} D_b) \geq (1 - \frac{\delta}{N}) \tilde{f}(\bigcup_{b \in B} A_b) = (1 - \frac{\delta}{N}) \tilde{f}(A^* \setminus A)$, $w(D_b) \leq \tilde{W}_b$ for any $b \in B$ and $D_b = \emptyset$ for every $b \in B \setminus I$. Therefore $\tilde{f}(\bigcup_{b \in B} D_b) \geq (1 - \frac{\delta}{N}) \tilde{f}(A^* \setminus A)$.

It follows from Lemma 3.3 that the residual instance is $N$-leveled. Let $\tilde{B} = \{1, \ldots, m\}$, Clearly $m \leq m$. We note that $D_1, \ldots, D_m$ is not necessarily a feasible solution to the residual instance as there may be $b \in B^r$, $i \in D_b$ such that $w_i > \delta \tilde{W}_b$. We construct a feasible solution $\tilde{D}_1, \ldots, \tilde{D}_m$ to the residual problem by $\tilde{D}_b = D_b$ for $b \in B \setminus B^r$ and $\tilde{D}_b = D_b \setminus \{i|w_i > \delta \tilde{W}_b\}$. It follows that $\tilde{D}_1, \ldots, \tilde{D}_m$ is a feasible solution to the residual problem. Furthermore, as $\tilde{f}$ is submodular it holds that

$$\tilde{f}(\bigcup_{b \in B} \tilde{D}_b) \geq \tilde{f}(\bigcup_{b \in B} D_b) - \sum_{i \in D_b: w_i > \delta \tilde{W}_b, b \in B^r} \tilde{f}(i) \geq \left(1 - \frac{1}{N}\right) \tilde{f}(A^* \setminus A) - \frac{N^2}{\delta} \cdot \frac{f(A)}{P}. \tag{1}$$

The second inequality holds since $|B^r| \leq N^2$ and for every $b \in B^r$ there are at most $\frac{1}{\delta}$ items $i$ in $D_b$ such that $w_i > \delta \tilde{W}_b$. Furthermore, we have $\tilde{f}(\{i\}) \leq \frac{f(A)}{P}$ for any $i \in \tilde{I}$ by definition.

Therefore, by Lemma 3.4 with probability at least $1 - \gamma$ it holds that $S_1, \ldots, S_m$ is a feasible solution and $\tilde{f}(S_1 \cup \ldots \cup S_m) \geq (1 - \epsilon)^2 \left(1 - \frac{\delta}{1 + \mu}\right) \tilde{f}(\bigcup_{b \in B} D_b)$ with

$$\gamma \leq \exp\left(\frac{\mu^3 \cdot O \cdot P}{f(A)}\right) + N^2 \exp\left(-\frac{\mu^2}{12} \cdot \frac{1}{\delta}\right) + 2N^2 \cdot \sum_{t=1}^{\infty} \exp\left(-\frac{\mu^2}{12}N^t\right). \tag{2}$$

where $O$ is the optimal value for the residual problem.

Let $\epsilon \in (0, 0.1)$, therefore there is $\mu \in (0, 0.1)$ such that $\frac{(1 - \epsilon)^2}{1 + \mu} > (1 - \epsilon^2).$ By the Monotone Convergence Theorem $\lim_{N \to \infty} 2N^2 \cdot \sum_{t=1}^{\infty} \exp\left(-\frac{\mu^2}{12}N^t\right) = \sum_{t=1}^{\infty} \lim_{N \to \infty} 2N^2 \exp\left(-\frac{\mu^2}{12}N^t\right) = 0.$ This limit is a key for the parameters’ selection. It follows that there are $N > \frac{1}{\epsilon^2}$ and $\delta > 0$ such that

$$N^2 \exp\left(-\frac{\mu^2}{12} \cdot \frac{1}{\delta}\right) + 2N^2 \sum_{t=1}^{\infty} \exp\left(-\frac{\mu^2}{12}N^t\right) < \epsilon^2. \tag{3}$$

Last, we select $P$ such that $P \geq \frac{N^2}{\epsilon^2}$ and $\exp\left(-\frac{\mu^3}{16} \cdot P\right) \leq \epsilon^2$.

If $O \leq \epsilon f(A)$ then as $\tilde{D}_1, \ldots, \tilde{D}_m$ is a feasible solution to the residual problem and following (1) we have

$$\epsilon f(A) \geq \tilde{f}(\bigcup_{b \in B} \tilde{D}_b) \geq (1 - \epsilon^2) \tilde{f}(A^* \setminus A) - \epsilon^2 f(A) = (1 - \epsilon^2)(f(A^*) - f(A)) - \epsilon^2 f(A).$$

And by rearranging the terms we have $f(A) \geq \frac{1 - \epsilon^2}{1 + \epsilon - \epsilon^2} \geq (1 - e^{-1}) f(A^*)$. As $A_1, \ldots, A_m$ is considered as a solution (Line 6), the algorithm surely returns an $(1 - e^{-1})$ approximation in this case.

Otherwise, we have $O \geq \epsilon f(A)$ and thus by (2) and (3), $\gamma \leq \exp\left(-\frac{\mu^3}{16} \cdot P\right) + \epsilon^2 \leq 2\epsilon^2$. Thus the
expected value of the solution return by the algorithm is at least,

\[
\Pr \left( S_1, \ldots, S_m \text{ is feasible and } \tilde{f}(S_1 \cup \ldots \cup S_m) \geq (1 - e^{-1}) \frac{(1 - \mu)^2}{1 + \mu} \left( (1 - \frac{1}{\mu}) \tilde{f}(A^* \setminus A) - \frac{N^2}{\delta^2} f(A) \right) \right).
\]

\[
\mathbb{E} \left[ f(A) + \tilde{f}(\cup_{b \in B} S_b) \mid S_1, \ldots, S_m \text{ is feasible and } \tilde{f}(S_1 \cup \ldots \cup S_m) \geq (1 - e^{-1}) \frac{(1 - \mu)^2}{1 + \mu} \left( (1 - \frac{1}{\mu}) \tilde{f}(A^* \setminus A) - \frac{N^2}{\delta^2} f(A) \right) \right] \\
\geq (1 - \gamma) \left( f(A) + (1 - e^{-1})(1 - \epsilon^2) \left( (1 - \epsilon^2) (f(A^*) - f(A)) - \epsilon^2 f(A) \right) \right) \\
\geq (1 - e^{-1})(1 - 2\epsilon^2)(1 - \epsilon^2)^2 f(A^*) + (1 - 2\epsilon^2) f(A)(1 - (1 - e^{-1})(1 - \epsilon^2)^2 - \epsilon^2) \\
\geq (1 - e^{-1} - \epsilon) f(A^*).
\]

That is, in both cases \( \mathcal{O} \geq \epsilon f(A) \) and \( \mathcal{O} \leq \epsilon f(A) \) the algorithm returns a feasible solution with expected value at least \((1 - e^{-1} - \epsilon) f(A^*)\). It also holds that the algorithm is polynomial tie for fixed parameters. Thus completing the proof of the lemma.

\[ \square \]

### 3.1 Structuring the Instance

In this section we prove Lemma 3.3. That is, we define the function \( R_N \) and show it fulfills the properties in the lemma. Our technique for constructing \( R_N \) is inspired by the grouping technique of Fernandez de la Vega and Lueker 9.

For a fixed value \( N \), we group the bins into \emph{leveled blocks}. We assume the bins are ordered by deceasing capacity and then partition them into levels, where level \( t, t \geq 0 \), has \( N^2 + t \) bins, divided into \( N^2 \) consecutive blocks, each containing \( N \) bins, with the exception of the last level and last block which may be incomplete. That is, level 0 is a set of \( N^2 \) blocks, where each block is a single bin, level 1 is a set of \( N^2 \) blocks where each block has \( N \) bins, and so on. We associate the bins to levels and blocks consecutively. For example, the first bin belongs to the first block of level 0, the second bin belongs to the second block of level 0 and bins \( N^2 + 1 \) to \( N^2 + N \) form block 1 of level 1.

Formally, for \( 1 \leq j \leq N^2 \) and \( t \geq 0 \) define

\[
B_{t,j} = \left\{ b \mid N^2 \sum_{t'=0}^{t-1} N^{t'} + (j - 1) N^t + 1 \leq b \leq N^2 \sum_{t'=0}^{t-1} N^{t'} + N^t \right\}.
\]

Though the sets \( B_{t,j} \) are defined independently of an instance, we refer to their elements as bins. It can be easily observed that the sets \( B_{t,j} \) for \( t \geq 0 \) and \( 1 \leq j \leq N \) form a partition of the positive integers. Furthermore, for any \( t \geq 0 \) and \( j < N^2 \) it holds that the bins in \( B_{t,j+1} \) are consecutive to the bins in \( B_{t,j} \), the bins in \( B_{t+1,j} \) are consecutive to the bins in \( B_{t,N^2} \) and \( |B_{t,j}| = N^t \).

Given a tuple \( W_1, \ldots, W_m \) of capacities, we assume w.l.o.g that the capacities are sorted in decreasing order \( W_1 \geq W_2 \geq \ldots \geq W_m \), and decrease the capacity of bin \( b \) to the minimal capacity of a bin in the same block. That is, for \( b \in B_{t,j} \) for some \( t \geq 0 \) and \( 1 \leq j \leq N^2 \) \( \bar{W}_b = \min_{b' \in B_{t,j}} W_{b'} \). If the last block, \( B_{t,k} \), is not full we set the capacity of its bin to zero, i.e., \( \bar{W}_b = 0 \) for \( b \in B_{t,k} \). We define \( R_N(W_1, \ldots, W_m) = (W_1, \ldots, W_m) \).

The above procedure is encapsulated in the following algorithm.

**Algorithm 2**: for \( N \in \mathbb{N} \)

1. Order \( B \) in non-increasing order (by bin capacity).
2. Set \( B_{t,j} = \left\{ b \mid N^2 \sum_{t'=0}^{t-1} N^{t'} + (j - 1) N^t + 1 \leq b \leq N^2 \sum_{t'=0}^{t-1} N^{t'} + N^t \right\} \).
3. \( \bar{W}_b = \min_{b' \in B_{t,j}} W_{b'} \) for \( b \in B_{t,j} \), \( 1 \leq t \leq t - 1, 1 \leq j \leq N^2 \) and \( t = \ell, 1 \leq j \leq k - 1 \).
4. \( \bar{W}_b = 0 \) for \( b \in B_{t,k} \).
5. Return \((\bar{W}_1, \ldots, \bar{W}_m)\).

**Proof Sketch of Lemma 3.3** It follows immediately that \( \bar{W}_1, \ldots, \bar{W}_m \) is \( N \)-leveled and \( \bar{W}_b \leq W_b \) for every \( b \in B \).
To prove the third property, we consider an optimal solution $A^*_1, \ldots, A^*_m$ for the SMKP instance and alter it through a series of steps, following the last alternation the solution fulfills the requirements in the third property of the Lemma. Let $\ell, k$ be the maximal values (lexicographically) such that $B \cap B_{\ell,k} \neq \emptyset$. We refer to $\ell$ as the last level. We use the term block to refer to the bins in $B_{t,j}$ for some $t$ and $j$, even though it is not formally a block with respect to $W_1, \ldots, W_m$. Similarly, we use the term level to refer to a the blocks $(B_{t,j})_{1 \leq j \leq N^2}$ for a some $t \geq 0$. The alternation steps are as follows (we ignore some handling of corner cases in this sketch).

1. **Eviction:** From each level except the last we evict all the items from $N$ consecutive blocks. That is, for every $0 \leq t < \ell$ we select a value $1 \leq r^*_t \leq N$. We then generate a new solution $T_1, \ldots, T_m$ in which blocks $N(r^*_t - 1) + 1$ to $N \cdot r^*_t$ are empty. For every $0 \leq t < \ell$, $N(r^*_t - 1) + 1 \leq j \leq N \cdot r^*_t$ and $b \in B_{t,j}$ set $T_b = \emptyset$. For any other bin $b \in B$ set $T_b = A_b$. Since $f$ is submodular and in each level only $N$ out of $N^2$ bins have been evicted, we can select the values $r^*_t$ such that $f(T_1 \cup \ldots \cup T_m) \geq (1 - \frac{1}{N}) f(A_1 \cup \ldots \cup A_m)$.

2. **Shuffling:** We generate a new assignment $\tilde{T}_1, \ldots, \tilde{T}_m$ such $\cup_{b \in B} \tilde{T}_b = \cup_{b \in B} T_b$ and the last $N$ blocks in each level (except the last) are empty. This is attained by moving the assignments of bins in blocks $N^2 - N + 1, \ldots, N^2$ to the bins in blocks $N(r^*_t - 1) + 1, \ldots, N \cdot r^*_t$. As the latter are empty due to the previous step, this will not change the set of items in the solution. Also, since the bins are ordered by decreasing capacities, the capacity constraints are preserved.

3. **Shifting:** In this step we generate the assignment $A_1, \ldots, A_m$ which fulfills the requirements in the lemma. As the last $N$ block in each level (except the last level) are vacant in $\tilde{T}_1, \ldots, \tilde{T}_m$, we use them for the assignment of the first block of the following level. This can be done since $N$ blocks of level $t$ contain the same number of bins as a single block of level $t + 1$. We also use blocks in levels greater than 0 which are not the last $N$ blocks to store the assignment of the subsequent block in the level. Formally, set $A_{b-N^2} = \tilde{T}_b$ for any $0 < t \leq \ell$, $1 \leq j \leq N^2$ and $b \in B_{t,j}$. Also, $A_b = \tilde{T}_b$ for $b \in B_{0,j}$ and $1 \leq j \leq N^2 - N$. As the decrease capacities of each block are greater than the original capacities of the subsequent block, we obtain an assignment such that $w(A_b) \leq \tilde{W}_b$ for any $b \in B$. Also, if there was more than one complete level $\ell \geq 1$, blocks of the last bin are not use for the assignment.

A complete and formal proof of Lemma 3.4 is provided in Appendix A.

### 3.2 Solving a Continuous Relaxation and Rounding

The aim of this section is to provide an algorithm fulfilling the requirements in Lemma 3.4. As stated in the lemma, the input for the algorithm is a restricted SMKP instance along with a partition $B = B^* \cup B_1 \cup \ldots \cup B_k$ of the bins, so that $B_j$ is a block for every $1 \leq j \leq k$.

We use the following definitions. Denote the capacity of the bins in $B_j$ by $W_j^*$ for $1 \leq j \leq k$. That is for any $b \in B_j$ it holds that $W_j^* = W_b$. Let $1 \leq j \leq k$, we say an item $w \in I$ is $j$-small if $w_i \leq \mu \cdot W_j^*$, otherwise we say it is $j$-large. Define $I_j = \{i \mid w \subseteq I, i \text{ is } j\text{-small}\}$.

A $j$-configuration is a subset of $j$-large elements which can be packed into a single bin in $B_j$. That is, $C \subseteq I$ is a $j$-configuration if any item $i \in C$ is $j$-large and $w(C) \leq W_j^*$, where $w(C) = \sum_{i \in C} w_i$. Let $C_j$ be the set of all $j$-configurations. As any $j$-configuration has at most $\mu^{-1}$ items it follows that $|C_j| \leq n^{\mu^{-1}}$, i.e., the number of configurations is polynomial in the input size. Furthermore, for $A \subseteq I$ such that $w(A) \leq W_j^*$ it holds that $A = C \cup S$ where $C$ is a $j$-configuration and all the items in $S$ are $j$-small. Our algorithm exploits this property.

Last, for $b \in B^*$ define $I_b^* = \{i \mid w \subseteq I, w_i \leq \delta W_b\}$.

We use the above definitions to define a new submodular function over a different universe of elements. Let

$$E = \{(S,j) \mid S \in C_j \cup I_j, 1 \leq j \leq k\} \cup \{(S,-b) \mid b \in B^*, S \in I_b^*\}$$ (4)
and define $g : 2^E \rightarrow \mathbb{R}_{\geq 0}$ by $g(T) = f \left( \bigcup_{(S,h) \in T} S \right)$. It follows that $g$ is a submodular, monotone and non-negative function from Claim 2.2. Informally, the element $(S, j) \in E$ represents an assignment of all the items in $S$ to a single bin $b \in B_j$ and the element $\{i\}, -b \in E$ represent the assignment of $i$ to the bin $b \in B'$. Let $G : [0,1]^E \rightarrow \mathbb{R}_{\geq 0}$ be the multilinear extention of $g$.

We define a polytope $P$ for the instance as follows.

$$P = \left\{ \bar{x} \in [0,1]^E \left| \begin{array}{ll} \sum_{S \in B} w(S) \cdot \bar{x}_{(S-b)} \leq W_b & \forall b \in B' \\
\sum_{C \in C_j} \bar{x}_{(C,j)} \leq |B_j| & \forall 1 \leq j \leq k \\
\sum_{S \in C_j \cup I} w(S) \cdot \bar{x}_{(S,j)} \leq |B_j| \cdot W_j & \forall 1 \leq j \leq k \end{array} \right. \right\}$$

The polytope represents a relaxed version of the bins capacity constraints. For each set of bins $B_j$ it only requires that the number of configuration is not greater than the number of bins in $B_j$, and that the total wights of elements assigned to bins in $B_j$ does not exceed the bins total capacity. We note that the polytope is down-monotone, a necessary property since we solve the relaxation using the continuous greedy.

We are now ready to present the rounding algorithm.

**Algorithm 3:** Rounding

**Configuration:** A parameter $\mu > 0$

1. Find a solution $\bar{x}^*$ for $x \in \arg\max_{x \in [0,1]^E} G(\bar{x})$ using the continuous greedy of [3].
2. Sample a set $T$ according to $\bar{x}^*$. That is, $(S, h) \in T$ with probability $\bar{x}^*_i (S, h)$ and $(S, h) \in T$ and $(S', h') \in T$ are independent for $(S, h) \neq (S', h')$.
3. Set $S_1, \ldots, S_m = \emptyset$. Iterate over the elements $(S, h) \in T$ in decreasing order of $w(S)$. Set $S_b \leftarrow S_b \cup S$ where $b = \arg \min_{b \in B_h} w(S_b)$ if $1 \leq h \leq j$ and $b = -h$ otherwise.
4. Return $S_1, \ldots, S_m$.

We first note that Algorithm 3 is a polynomial time algorithm. Thus, we are left to show it returns a solution with high value with the probability stated in Lemma 3.4. The approach used to prove the inequality is similar to the approach taken in [7]. We note it is possible to prove a variant claim using the approach of [19]. This will eliminate the dependency in $\nu$, but will result in a more involved proof.

Let $A_1^* \ldots A_m^* \subseteq I$ be an optimal solution to the restricted SMKP instance given as an input to the algorithm, and $A^* = A_1^* \cup \ldots \cup A_m^*$. Also, let $S_1, \ldots, S_m$ be the solution returned by the algorithm in Line 4. Our analysis is based on the following Lemmas.

**Lemma 3.6.** If the following conditions hold:

1. For any $b \in B'$, $\sum_{(S-b) \in T} w(S) \leq (1 - \mu)W_b$.
2. For any $1 \leq j \leq k$, $\sum_{(S,j) \in T} w(S) \leq (1 - \mu)W_j \cdot |B_j|$.
3. For any $1 \leq j \leq k$, $\sum_{(S_j) \in T} w(S) \leq (1 - \mu)W_j \cdot |B_j|$.

Then $S_1 \cup \ldots \cup S_m$ is a feasible solution to the restricted SMKP instance.

**Proof.** Assume the conditions in the lemma hold. For any $b \in B'$ it holds that $S_b = \{i|\{i\}, -b \in T\}$, and it follows from condition 1 of the lemma that $w(S_b) \leq (1 - \mu) \cdot W_b \leq W_b$.

Let $1 \leq j \leq k$ and $b \in B_j$. Assume by negation that $w(S_b) > W_b = W_j$. Let $(S, h) \in T$ be the last element in $T$ such that $S \neq \emptyset$ and $S$ was added to $S_b$ in Line 3. We can conclude that $w(S_b \setminus S) > 0$, as otherwise $w(S_b) = w(S) \leq W_b$, by the definition of $E$ [4]. Therefore there are at least $|B_j|$ elements $(S', j) \in T$ such that $w(S') \geq w(S)$ (else, on the iteration of $(S, h)$ there must be $b \in B_j$ with $S_b = \emptyset$). If $S \in C_j$ then $w(S) > \mu \cdot W_j$ and thus

$$\left| \{S' \neq \emptyset | (S', j) \in T, S' \in C_j \right| \geq \left| \{S' | (S', j) \in T, w(S') \geq w(S) \right| > |B_j|,$$

contradicting condition 3 of the lemma.
Therefore $S \notin C_j$, and we an conclude that $S = \{i\}$ with $w_i \leq \mu \cdot W_j^*$. Thus, $w(S_b \setminus S) > (1 - \mu) \cdot W_j^*$. As $S$ was allocated to a set $S_b$, $b' \in B_j$ such that $w(S_{b'})$ is minimal, it holds that for any $b' \in B_j$ we have $w(S_{b'}) \geq w(S_b) > (1 - \mu) \cdot W_j^*$. Therefore,

$$\sum_{(S',j) \in T} w(S') \geq \sum_{b' \in B_j} w(S_{b'}) > |B_j|(1 - \mu) \cdot W_j^*$$

contradicting condition 2 of the lemma. Thus we can conclude that $w(S_b) \leq W_b$.

Also, by definition, we have that for any $b \in B^r$ and $i \in S_b$ it holds that $w_i \leq \delta W_b$. Hence, $S_1, \ldots, S_m$ is a solution to the restricted SMKP instance.

\[ \square \]

**Lemma 3.7.** $E[g(T)] \geq (1 - e^{-1}) \frac{\mu}{1 + \mu} f(A^*)$

The proof of the lemma is standard, and provided in Appendix A.

**Proof of Lemma 3.3.** For any $e \in E$ define $X_e$ to be a random variable such that $X_e = 1$ if $e \in T$ and $X_e = 0$ otherwise. It follows that $(X_e)_{e \in E}$ are independent Bernoulli random variables, $E[X_e] = \bar{x}_e^*$ and $T = \{e \in E|X_e = 1\}$.

For any $b \in B^r$, since $\bar{x}^* \in \frac{1 - e^{-1}}{1 + \mu} P$, it follows that,

$$E \left[ \sum_{(S, b) \in E} \frac{w(S)}{\delta \cdot W_b} \cdot X_{(S, b)} \right] = \sum_{(S, b) \in E} \frac{w(S)}{\delta \cdot W_b} \cdot \bar{x}_{(S, b)}^* \leq \frac{1 - \mu}{1 + \mu} \cdot \frac{1}{\delta},$$

as well as $\frac{w(S, b)}{\delta \cdot W_b} \leq 1$ for every $(S, b) \in E$. Therefore by Chernoff bound (Lemma 2.3) we have,

$$\Pr \left( \sum_{(S, b) \in E} w(S) > (1 - \mu) W_b \right) = \Pr \left( \sum_{(S, b) \in E} \frac{w(S)}{\delta \cdot W_b} \cdot X_{(S, b)} > (1 + \mu) \frac{(1 - \mu)}{(1 + \mu)} \cdot \frac{1}{\delta} \right) \leq \exp \left( -\frac{\mu^2}{3} \cdot \frac{1 - \mu}{1 + \mu} \cdot \frac{1}{\delta} \right) \leq \exp \left( -\frac{\mu^2}{12} \cdot \frac{1}{\delta} \right) \tag{6}$$

Note we use $\mu \in (0, 0.1)$ in the last inequality. Similarly, for any $1 \leq j \leq R$ and every $(S, j) \in E$ it holds that $\frac{w(S_j)}{W_j^*} \leq 1$. Also, since $\bar{x}^* \in \frac{1 - e^{-1}}{1 + \mu}$,

$$E \left[ \sum_{(S, j) \in E} \frac{w(S)}{W_j^*} \cdot X_{(S, j)} \right] = \sum_{(S, j) \in E} \frac{w(S)}{W_j^*} \cdot \bar{x}_{(S, j)}^* \leq \frac{1 - \mu}{1 + \mu} \cdot |B_j|,$$

and

$$E \left[ \sum_{(S, j) \in E: S \in C_j} 1 \cdot X_{(S, j)} \right] = \sum_{(S, j) \in E: S \in C_j} \bar{x}_{(S, j)}^* \leq \frac{1 - \mu}{1 + \mu} \cdot |B_j|.$$ 

Therefore by Chernoff bound (Lemma 2.3) we have,

$$\Pr \left( \sum_{(S, j) \in T} w(S) > (1 - \mu) |B_j| \cdot W_j^* \right) = \Pr \left( \sum_{(S, j) \in E} \frac{w(S)}{W_j^*} X_{(S, j)} > (1 + \mu) \frac{(1 - \mu)}{(1 + \mu)} |B_j| \right) \leq \exp \left( -\frac{\mu^2}{3} \cdot \frac{1 - \mu}{1 + \mu} \cdot |B_j| \right) \leq \exp \left( -\frac{\mu^2}{12} \cdot |B_j| \right) \tag{7}$$

and
\[ \Pr \left( \sum_{(S,j) \in T: S \in C_j} 1 > (1 - \mu)|B_j| \right) = \Pr \left( \sum_{(S,j) \in T: S \in C_j} 1 > (1 + \mu) \frac{1 - \mu}{1 + \mu}|B_j| \right) \]
\[ \leq \exp \left( -\frac{\mu^2}{3} \cdot \frac{1 - \mu}{1 + \mu} |B_j| \right) \leq \exp \left( -\frac{\mu^2}{12} |B_j| \right) \] (8)

For any \((S,h) \in E\) we have \(|S| \leq \mu^{-1}\), and following the submodularity of \(f\) we have \(g(\{(S,h)\}) - g(\emptyset) \leq \mu^{-1}v\) (recall that \(v\) was defined in Lemma 3.4). Therefore, by the concentration bound of Lemma 3.7 we have,
\[ \Pr \left( g(T) \leq (1 - \mu) \frac{1 - \mu}{1 + \mu} f(A^*) \right) = \Pr \left( g(\{e \in E|x_e = 1\}) \leq (1 - \mu) \frac{1 - \mu}{1 + \mu} f(A^*) \right) \]
\[ \leq \Pr \left( g(\{e \in E|x_e = 1\}) \leq (1 - \mu)G(\bar{x}^*) \right) \leq \exp \left( -\frac{\mu^3 \cdot G(\bar{x}^*)}{2v} \right) \]
\[ \leq \exp \left( -\frac{\mu^3(1 - e^{-1})}{2v} \frac{1 - \mu}{1 + \mu} f(A^*) \right) \leq \exp \left( -\frac{\mu^3}{16 \cdot v} f(A^*) \right) \] (9)

The first and third inequality are due to Lemma 3.7.

Therefore, by applying the union bound over (6), (7), (8) and (9), we have that with probability at least
\[ 1 - \left| B^c \right| \exp \left( \frac{\mu^2}{12} \cdot \frac{1}{\delta} \right) - 2 \cdot \sum_{j=1}^{k} \exp \left( -\frac{\mu^2}{12} |B_j| \right) - \exp \left( -\frac{\mu^3}{16} \cdot \frac{1}{v} f(A^*) \right) = 1 - \gamma \]

The conditions of Lemma 3.6 hold and \(g(T) \geq \frac{(1 - \mu)^2}{1 + \mu}(1 - e^{-1}) f(A^*)\). Therefore, by applying Lemma 3.6 in probability of at least \(1 - \gamma\) algorithm returns a solution \(S_1, \ldots, S_m\). And such \(g(T) \geq \frac{(1 - \mu)^2}{1 + \mu}(1 - e^{-1}) f(A^*)\). By the construction of \(S_1, \ldots, S_m\) in Line 3 we have \(S_1 \cup \ldots \cup S_m = \bigcup_{(S,h) \in T} S\), therefore \(f(S_1, \ldots, S_m) = g(T) \geq \frac{(1 - \mu)^2}{1 + \mu}(1 - e^{-1}) f(A^*)\) and \(S_1, \ldots, S_m\) is a solution with probability \(1 - \gamma\). As required.

4 Discussion

In this paper we presented a randomized \((1 - e^{-1} - \varepsilon)\)-approximation for the monotone submodular multiple knapsack problem. Our algorithm relies on three main building blocks. The structuring technique (Section 3.1) which converts a general instance to a leveled instance, the reduction to the block-constraint problem (Section 3.2) and a refined analysis of known algorithms for submodular optimization with a d-dimensional knapsack constraint (Section 3.3). While the structuring technique and the refined analysis seem to be fairly robust, the reduction to the block-constraint problem proved to be limiting when generalizations of the problem have been considered.

A notable example is the non-monotone submodular multiple knapsack problem, in which the set function \(f\) is non-monotone. Unfortunately, when \(f\) is non-monotone the function \(g\) used for solving the block-constraint problem is not submodular. A variant of the block-constraint problem which does not alter the set function may be used to overcome this hurdle. However, this variant limits the knapsacks utilization and degrades the approximation ratio. Our preliminary results for the non-monotone case guarantee an approximation ratio of \(\frac{1}{2} \cdot e^{-\frac{1}{2}} - \varepsilon \approx 0.303 - \varepsilon\) using this approach.

Another natural generalization of SMKFP is monotone submodular optimization subject to a multiple knapsack and a matroid constraints, in which the solution \((A_b)_{b \in B}\) must also satisfy \(\bigcup_{b \in B} A_b \in M\) for a matroid \(M\). However, the matroid properties are not preserved throughout the reduction to the block-constraint problem, rendering existing techniques for submodular optimization with matroid and d-dimensional knapsack constraints [7] ineffective.

On the positive side, we believe the techniques in this paper can be extended to handle the problem for maximizing a monotone submodular function subject to a multiple knapsack constraint and an additional d-dimensional knapsack constraint, for a fixed d. We defer the details to the full version of the paper.
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A Omitted Proofs

Proof of Claim 2.1. By the submodularity of \( f \) we have

\[
f(A \cup S) + f(B) \geq f(A \cup S \cup B) + f((A \cup S) \cap B) \geq f(B \cup S) + f(A)
\]

where the second inequality follows from \( A \subseteq (A \cup S) \cap B \) and the monotonicity of \( f \). By rearranging the terms in the above we get

\[
f(A \cup S) - f(A) \geq f(B \cup S) - f(B)
\]

as required.

Proof of Claim 2.2. It is easy to see that \( g \) is non-negative, as \( f \) is non negative. In addition, for any two subsets \( A \subseteq B \subseteq E \) we have \( \cup_{(S,h) \in A S} \subseteq \cup_{(S,h) \in B S} \). Thus, since \( f \) is monotone, \( g \) is monotone as well.

All that is left to prove is that \( g \) is submodular. Consider subsets \( A \subseteq B \subseteq E \) and \( (S,h) \in E \setminus B \).

\[
g(A \cup \{(S,h)\}) - g(A) = f(\cup_{(S',h') \in A S'} \cup S) - f(\cup_{(S',h') \in B S'} \cup S) \\
\leq f(\cup_{(S',h') \in B S'} \cup S) - f(\cup_{(S',h') \in B S'}) \\
= g(B \cup \{(S,h)\}) - g(B)
\]

where the inequality follows Claim 2.1 and \( \cup_{(S',h') \in A S'} \subseteq \cup_{(S',h') \in B S'} \).

Proof of Lemma 3.7. First we define a solution \( \bar{z} \in P \) such that \( G(\bar{z}) = f(A^*) \). Denote \( L_j = \{ i \in I \mid i \text{ is } j \text{-large} \} \) and define \( \bar{z} \in [0,1]^E \) by

\[
\bar{z}(S,h) = \begin{cases} 1 & -h \in B^*, S = \{ i \} \text{ and } i \in A_b^* \\ 1 & 1 \leq h \leq k, S \in C_h, \exists b \in B_h: S = A_b^* \cap L_h \\ 1 & 1 \leq h \leq k, S \notin C_h, \exists b \in B_h: S \subseteq A_b^* \\ 0 & \text{otherwise} \end{cases}
\]

(10)

It follows that \( G(\bar{z}) = g(T_{\bar{z}}) \) with

\[
T_{\bar{z}} = \{(i,-b) | b \in B^*, i \in A_b^* \} \cup \bigcup_{j=1}^k \bigcup B_j \{(A_b^* \cap L_j,j)\} \cup \{(i,j) | \exists b \in B_j : (i) \in A_b^* \setminus L_j \}.
\]

Therefore \( G(\bar{z}) = g(T_{\bar{z}}) = f(A^*) \). Since \( A_1^*, \ldots, A_n^* \) is a solution to the instance, it can be easily verified that \( \bar{z} \) is in \( P \), the polytope of the instance (3). As the second derivatives of the multilinear extension are non-positive (see [5]) it follows that \( \varphi(\lambda) = G(\lambda\bar{z}) \) is concave on \([0,1] \). Therefore, holds that

\[
\max_{\bar{z} \in [0,1]^E} G(\bar{z}) \geq G \left( \frac{1 - \mu}{1 + \mu} \bar{z} \right) \geq \frac{1 - \mu}{1 + \mu} G(\bar{z}) = \frac{1 - \mu}{1 + \mu} f(A^*).
\]

Since the continuous greedy in Line 14 of the algorithm produces \((1 - e^{-1})\)-approximation it holds that \( G(\bar{x}^*) \geq (1 - e^{-1}) \frac{1 - \mu}{1 + \mu} f(A^*) \). Thus, \( g(T) = G(\bar{x}^*) \geq (1 - e^{-1}) \frac{1 - \mu}{1 + \mu} f(A^*) \).

\[\square\]

Lemma A.1. Let \( h : 2^\Omega \rightarrow \mathbb{R}_{\geq 0} \) be a monotone submodular function and let \( S_{i,1}, \ldots, S_{i,N} \subseteq \Omega \) for \( 1 \leq i \leq M \). Then for every \( 1 \leq i \leq M \) there is \( 1 \leq j^*_i \leq N \) such that

\[
h \left( \bigcup_{i=1}^M \bigcup_{1 \leq j \leq N} S_{i,j} \right) \geq \left( 1 - \frac{1}{N} \right) h \left( \bigcup_{i=1}^M \bigcup_{j=1}^N S_{i,j} \right)
\]

To prove Lemma A.1 we first prove a special case of the lemma.
Lemma A.2. Let \( h : 2^\Omega \to \mathbb{R}_{\geq 0} \) be a submodular monotone and non-negative function, and let \( S_1, \ldots, S_N \subseteq \Omega \). Then there is \( 1 \leq j^* \leq N \) such that

\[
f \left( \bigcup_{1 \leq j \leq N, j \neq j^*} S_j \right) \geq \left( 1 - \frac{1}{N} \right) f(S_1 \cup \ldots \cup S_N).
\]

Proof. As \( h \) is submodular and monotone we have,

\[
h(S_1 \cup \ldots \cup S_N) - f(\emptyset) = \sum_{j=1}^{N} (h(S_1 \cup \ldots \cup S_j) - h(S_1 \cup \ldots \cup S_{j-1})) \geq \sum_{j=1}^{N} \left( h \left( \bigcup_{j' \leq N, j' \neq j} S_{j'} \right) - h \left( \bigcup_{1 \leq j \leq N, j \neq j^*} S_j \right) \right)
\]

Therefore there is \( 1 \leq j^* \leq N \) such that

\[
h \left( \bigcup_{j=1}^{N} S_j \right) - h \left( \bigcup_{1 \leq j \leq N, j \neq j^*} S_j \right) \leq \frac{1}{N} (h(S_1 \cup \ldots \cup S_N) - f(\emptyset)).
\]

By rearranging the terms we have,

\[
h \left( \bigcup_{1 \leq j \leq N, j \neq j^*} S_j \right) \geq \left( 1 - \frac{1}{N} \right) h(S_1 \cup \ldots \cup S_N),
\]

as required. \( \square \)

In the proof of Lemma A.3 we use the following.

Proof of Lemma A.3. Let \( h : 2^\Omega \to \mathbb{R}_{+} \) be a submodular, non-negative and monotone function, and \( S_i, \ldots, S_i, N \subseteq \Omega \) for every \( 1 \leq i \leq M \).

For any \( T \subseteq \Omega \) we use \( h_T \) to denote the function \( h_T(S) = h(S \cup T) - h(T) \). It follows that for any \( T \subseteq \Omega \) the function \( h_T \) is also submodular, monotone and non-negative. It also holds that if \( T_1 \subseteq T_2 \subseteq \Omega \) then for any \( S \subseteq \Omega \) it holds that \( h_{T_1}(S) \geq h_{T_2}(S) \).

Define \( T_i = \bigcup_{j=1}^{N} S_{i,j} \). Now,

\[
h \left( \bigcup_{i=1}^{M} \bigcup_{j=1}^{N} S_{i,j} \right) - h(\emptyset) = \sum_{i=1}^{M} h \left( \bigcup_{j=1}^{N} S_{i,j} \right) = \sum_{i=1}^{M} h \left( \bigcup_{j=1}^{N} S_{i,j} \right)
\]

By Lemma A.2 for every \( 1 \leq i \leq M \) there is \( 1 \leq j_i^* \leq N \) such that

\[
h \left( \bigcup_{j=1}^{N} S_{i,j} \right) \geq \left( 1 - \frac{1}{N} \right) h \left( \bigcup_{j=1}^{N} S_{i,j} \right).
\]

Therefore,

\[
h \left( \bigcup_{i=1}^{M} \bigcup_{j=1}^{N} S_{i,j} \right) - h(\emptyset) = \sum_{i=1}^{M} h \left( \bigcup_{j=1}^{N} S_{i,j} \right) \geq \sum_{i=1}^{M} \left( 1 - \frac{1}{N} \right) h \left( \bigcup_{j=1}^{N} S_{i,j} \right) \geq \left( 1 - \frac{1}{N} \right) \sum_{i=1}^{M} h \left( \bigcup_{j=1}^{N} S_{i,j} \right).
\]

\[\square\]
As $h$ is non-negative we can conclude,

$$h \left( \bigcup_{i=1}^{M} \bigcup_{1 \leq j \leq M, j \neq j_i^*} S_{i,j} \right) \geq \left( 1 - \frac{1}{N} \right) \cdot h \left( \bigcup_{i=1}^{M} \bigcup_{j=1}^{N} S_{i,j} \right).$$

\[ \square \]

**Proof of Lemma 3.3.** It follows immediately that $\tilde{W}_1, \ldots, \tilde{W}_m$ is $N$-leveled. The partition for the definition is $B_{t,j} = B_{t,j} \cap B$, for $0 \leq t \leq \ell$ and $1 \leq j \leq N^2$ $(1 \leq j \leq k$ for $t = \ell$), with $(t, k)$ is set to the maximal (lexicographic) value such that $B_{t,k} \cap B \neq \emptyset$. Clearly, $B_{t,j}$ is always a block. It also follows that properties 3.3, 1, and 2 of Lemma 3.3 hold (we note that $R_N$ can be trivially computed in polynomial time).

Let $(f, I = \{1, \ldots, n\}, B = \{1, \ldots, m\}, (w_i)_{i \in I}, (W_b)_{b \in B})$ be an SMKP instance, assume w.l.o.g. that $W_1 \geq W_2 \geq \ldots \geq W_m$ and let $A_1^*, \ldots, A_m^*$ be an optimal solution. To show property 3 we use $A_1^*, \ldots, A_m^*$ to construct a new solution $A_1, \ldots, A_m$ to the SMKP instance such that $w(A_b) \leq \tilde{W}_b$ for any $b \in B$, while only incurring a small loss in value.

For simplicity we say that $(t, j)$ is a valid block if $1 \leq t \leq \ell - 1$ and $1 \leq j \leq N^2$ or $1 \leq j \leq k$ if $t = \ell$. We say that a valid block $(t_1, j_1)$ precedes a valid block $(t_2, j_2)$ if either $t_1 < t_2$ or both $t_1 = t_2$ and $j_1 < j_2$. Since the capacities $W_1, \ldots, W_m$ are sorted, for any valid blocks $(t_1, j_1)$ which precedes a valid block $(t_2, j_2)$ and $b_1 \in B_{t_1,j_1}, b_2 \in B_{t_2,j_2}$ it holds that $W_{b_1} \geq W_{b_2}$ and thus $w(A_{b_2}) \leq w(A_{b_1})$.

This suggests the following approach, we can “shift” the assignment of a block back. That is we can construct a new solution $D_1, \ldots, D_m \subseteq I$ by $D_{b-N} = A_b$ for any $b \in B_{t+1,j+1}$ (recall that $N^2$ is the number of bins in a block of level $t$). It follows that $w(D_b) \leq W_{b+N^2} \leq \tilde{W}_b$ for any $b \in B$. However, by doing so all the items in the first block of each level have been removed from the solution, possibly incurring a large decrease to the solution’s value.

To overcome the last issue we first evict $N$ blocks from each level except the last. The evicted blocks are conceptually used for the first block of the subsequent level. Since there are $N^2$ in each level and $f$ is submodular, we can do so while only reducing the solution’s value by a factor of $\frac{1}{N}$, as shown in Lemma \ref{lem:eviction}.

Denote $L = \bigcup_{j=1}^{N} \bigcup_{b \in B_{t,j}} A_b^*$, the items assigned to the last level in the optimal solution and set $f_L(S) = f(S \cup L) - f(S)$. It follows that $f_L$ is also submodular, non-negative and monotone. In order to use Lemma \ref{lem:eviction} we gather every $N$ consecutive blocks together; let $K_{t,r} = \bigcup_{j \leq N \left( r-\ell \right)+1} B_{t,j}$ and $D_{t,r} = \bigcup_{b \in K_{t,r}} A_b^*$ for $0 \leq t < \ell$ and $1 \leq r \leq N$. By Lemma \ref{lem:eviction} for every $0 \leq t < \ell$ there is $r_t^*$ such that $f_L \left( \bigcup_{j=0}^{N} \bigcup_{r \leq r_t^*} D_{t,r} \right) \geq \left( 1 - \frac{1}{N} \right) f_L \left( \bigcup_{j=0}^{N} \bigcup_{r \neq r_t^*} D_{t,r} \right)$.

Consider the solution $T_1, \ldots, T_m$ defined by $T_b = \emptyset$ for $b \in K_{t,r_t^*}$ for some $0 \leq t < \ell$ and $T_b = A_b^*$ otherwise. Clearly, $L \subseteq T_1 \cup \ldots \cup T_m$, hence,

$$f(T_1 \cup \ldots \cup T_m) = f_L(T_1 \cup \ldots \cup T_m) + f(L)$$

$$= f_L \left( \bigcup_{t=0}^{\ell-1} \bigcup_{1 \leq r \leq N, r \neq r_t^*} D_{t,r} \right) + f(L) \geq \left( 1 - \frac{1}{N} \right) \left( f_L \left( \bigcup_{t=0}^{\ell-1} \bigcup_{1 \leq r \leq N} D_{t,r} \right) + f(L) \right)$$

$$= \left( 1 - \frac{1}{N} \right) f(A_1^* \cup \ldots \cup A_m^*).$$

Clearly, for every $b \in B$ we have $w(T_b) \leq w(A_b^*) \leq W_b$. We first use the empty bins in layer $t$ to generate a new solution $\tilde{T}_1, \ldots, \tilde{T}_m$ in which that last $N$ blocks in each level (except the last) are empty. This new solution is generated keeping all the bins preceding $K_{t,r_t^*}$ in place, and shifting all the bins succeeding $K_{t,r_t^*}$ by $|K_{t,r_t^*}| = N^{t+1}$ bins. Formally, for every valid block $(t, j)$ and $b \in B_{t,j}$ if $t = \ell$ then set $\tilde{T}_b = T_b$. For $t < \ell$ set $\tilde{T}_b = T_b$ if $j \leq (r_t^* - 1)N$, $\tilde{T}_b = T_{b+N}A_{b}^*$ if $(r_t^* - 1)N < j \leq N^2 - N$, and $\tilde{T}_b = \emptyset$ for $N^2 - N < j \leq N^2$. It follows that for every, $b \in B$, $\tilde{T}_b = T_{b'}$ for $b' \geq b$ or $\tilde{T}_b = \emptyset$, therefore $w(\tilde{T}_b) \leq w(T_{b'}) \leq W_{b'} \leq W_b$. Also, it easy to verify that $\bigcup_{b \in B} \tilde{T}_b = \bigcup_{b \in B} T_b$.

We generate $A_1, \ldots, A_m$ from $\tilde{T}_1, \ldots, \tilde{T}_m$ by shifting the content of each block from level 1 and above to the block preceding it. We place the first block of each level (except the first block of the
level 0) in place of the last $N$ block of the previous level. Formally, for every valid block $(t, j)$ and $b \in \tilde{B}_{t,j}$, if $j \leq N^2 - N$ and $t \geq 1$ we set $A_b = \tilde{T}_{b+N^t}$ (empty set if the second term does not exist), if $j > N^2 - N$ then $A_b = \tilde{T}_{b+N^t+1}$, and otherwise, the case where $t = 0$ and $j \leq N^2 - N$, we set $A_b = \tilde{T}_b$.

It follows that if $\ell \geq 1$, then for any $b \in B_{\ell,k}$ we have $A_b = \emptyset$.

For any $b \in \tilde{B}_{0,j}$ valid block $(0, j)$, $j \leq N^2 - N$ we have $w(A_b) = w(\tilde{T}_b) \leq W_b = \tilde{W}_b$. It follows that Also, for any valid block $(t, j)$ with $t \geq 1$ or $j > N^2 - N$, and $b \in \tilde{B}_{t,j}$ it holds that $A_b = \tilde{T}_b'$ with $b' \in \tilde{B}_{t',j'}$ and $t, j$ precedes $t', j'$. Therefore $w(A_b) = w(\tilde{T}_b') \leq W_{b'} \leq \min_{c \in B_{t,j}} W_c = \tilde{W}_b$. Also, for every $b \in B$, either $\tilde{T}_b = \emptyset$ or there is $b' \in B$ such that $A_{b'} = \tilde{T}_b$. Therefore,

$$f(A_1 \cup \ldots \cup A_m) \geq f(\tilde{T}_1 \cup \ldots \cup \tilde{T}_m) \geq \left(1 - \frac{1}{N}\right) f(A_1^* \cup \ldots \cup A_m^*).$$