Any monotone Boolean circuit computing the $n$-dimensional Boolean convolution requires at least $n^2$ and-gates. This matches the obvious upper bound. The previous best bound for this problem was $\Omega(n^{4/3})$, obtained by Norbert Blum in 1981. More generally, exact bounds are given for all semi-disjoint bilinear forms.

Keywords:

Boolean vector convolution, monotone Boolean circuit complexity, semi-disjoint bilinear forms.

1 Introduction

We consider the monotone circuit complexity of Boolean convolution, i.e., the number of logical gates needed in a Boolean circuit which has only and-gates and or-gates to compute the convolution of two Boolean vectors. The Boolean convolution of vectors $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ is $f_1, \ldots, f_{2n-1}$, where

$$f_k = \bigvee_{i+j-1=k} x_i \land y_j.$$  

This shows explicitly that at most $n^2$ and-operations (and’s) and $(n-1)^2$ or-operations (or’s) are needed to compute the Boolean convolution of two length-$n$ vectors. The wrapped Boolean convolution is just the same but gives $n$ functions defined as above but with $\bigvee_{i+j-1=k \pmod{n}}$. The naive circuit for this uses $n^2$ and’s and $n(n-1)$ or’s.

Substantially fewer operations are needed if the restriction to monotone circuits is removed. With negation added, arithmetic operations can be implemented allowing transform techniques which can compute convolution in $O(n \log^2 n \log \log n)$ Boolean operations. While it is generally believed that $n^2$ and’s are required for convolution with monotone circuits, the best lower bound known to date for the and-complexity is $\Omega(n^{4/3})$,
due to Blum [1] over thirty years ago. Very recently Lingas [4] has given a bound of \( \Omega(n^{2-\epsilon}) \), but conditional on a depth bound for conjunctions. There are larger lower bounds known for the \( \lor \)-complexity, e.g., \( n^{3/2} \) due to Weiss [7], and, \( \Omega(n^2/\log^6 n) \) due to Grinchuk and Sergeev [2, 3].

The problem seems similar to finding the Boolean complexity of matrix multiplication. The naive algorithm for multiplying two \( n \times n \) Boolean matrices requires \( n^3 \) plus's for monotone circuits, whereas, with negations permitted, fast algorithms for matrix multiplication over rings can be implemented yielding circuits with only \( \widetilde{O}(n^\omega) \) operations where \( \omega < 2.373 \). In this case however it was shown over forty years ago ([6, 5]) that exactly \( n^3 \) plus's are necessary in monotone Boolean circuits.

Here we give lower bounds for conjunctive complexity matching exactly the obvious upper bounds for monotone Boolean convolution (both wrapped and unwrapped). More generally, exact results are also shown for all semi-disjoint bilinear forms.

2 Preliminaries

Let \( X \) and \( Y \) be disjoint sets of variables and \( Z = X \cup Y \).

Definition 1. A set \( F \) of functions is a semi-disjoint bilinear form over \( X \) and \( Y \) if

1. each prime implicant of functions in \( F \) has the form \( x \land y \) where \( x \in X \) and \( y \in Y \),
2. for each function \( f \) in \( F \) and each \( z \) in \( Z \) there is at most one prime implicant of \( f \) containing \( z \), and
3. the sets of prime implicants of functions in \( F \) are disjoint.

We will often omit \( \land \) symbols and represent conjunctions by juxtaposition.

Definition 2. The domain of \( F \), \( \text{dom}(F) \) is the set of pairs \( (i, j) \) such that \( x_i y_j \) is a prime implicant of a function in \( F \). We define the domain size of \( F \) to be \( D(F) = |\text{dom}(F)| \).

To consider Boolean circuits, we will use the equivalent formulation of Boolean chains, i.e., sequences of Boolean functions where each function is the conjunction or disjunction of two previous functions in the sequence. More formally, we define a sequence \( s_{-q}, \ldots, s_0, s_1, \ldots, s_T \), where the inputs, \( x_0, \ldots, x_{T-1}, y_0, \ldots, y_{J-1} \), appear as \( s_{-q}, \ldots, s_0 \) and each function \( s_k \), for \( 1 \leq k \leq T \), is either \( s_i \land s_j \) or \( s_i \lor s_j \) for some \( i, j < k \). A Boolean chain computes a set of functions \( F \) if each \( f \in F \) occurs in the chain.

The \( \land \)-complexity of a chain is the number of \( \land \) operations used in the chain, and similarly for the \( \lor \)-complexity. The complexity (\( \land \)- or \( \lor \)-) of a set \( F \) of Boolean functions is the minimal such complexity of a chain which computes \( F \).

3 Motivating examples

It would simplify our investigations into the \( \land \)-complexity of semi-disjoint bilinear forms if we could ignore implicants of degree three or more. Our first example shows that this would lead to the loss of an exact lower bound.
Example 1. Computing $w$, given by

$$w = x_1(y_2 \lor y_3 \lor y_4) \lor y_1(x_2 \lor x_3 \lor x_4)$$

requires two conjunctions. But now

$$w(x_2 \lor y_2) = x_1y_2 \lor x_2y_1 \lor (x_1x_2y_3 \lor x_1x_2y_4 \lor x_3y_1y_2 \lor x_4y_1y_2)$$

and we note that the terms in parenthesis on the right are all of degree three. Similarly we have

$$w(x_3 \lor y_3) = x_1y_3 \lor x_3y_1 \lor (\text{higher order terms})$$

and

$$w(x_4 \lor y_4) = x_1y_4 \lor x_4y_1 \lor (\text{higher order terms}).$$

Hence, were we to ignore higher order terms we could use a total of only 5 conjunctions to evaluate a semi-disjoint bilinear form with domain size 6.

We seek to define an appropriate measure of progress along a Boolean chain towards the final goal. We will define $\text{measure}_{i,j}(G)$ to represent the contribution of a set of functions $G$ towards the computation of the prime implicant $x_i y_j$ in the bilinear form $F$. Initially each $\text{measure}_{i,j}$ is to be zero, and finally each will be 1, i.e.,

$$\text{measure}_{i,j}(\emptyset) = 0 \text{ and } \text{measure}_{i,j}(G) = 1 \text{ for all } (i, j) \in \text{dom}(F) \text{ if } G \supseteq F.$$

A disjunction should not increase any measure and we will want to show that the total progress made by one conjunction is at most 1. For hints towards an appropriate progress measure, the following examples from a convolution computation are instructive.

Example 2. Given $w_1 = (x_1 \lor x_2)y_2$ and $w_2 = x_1(y_1 \lor y_2)$, what should $\text{measure}_{i,j}\{w_1, w_2\}$ be? We note that

$$w_3 = (w_1 \lor x_2)(w_2 \lor y_1) = (x_1y_2 \lor x_2)(x_1y_2 \lor y_1) = x_1y_2 \lor x_2y_1.$$ 

This suggests that the conjunction producing $w_3$ increases $\text{measure}_{2,1}$ from zero to one, and so $w_1$ and $w_2$ should already jointly provide measure of one for $x_1 y_2$, i.e., $\text{measure}_{1,2}\{w_1, w_2\} = 1$. Hence by symmetry we should regard the computation of $w_1$ as providing measure 1/2 to each of $x_1y_2$ and $x_2y_2$.

Example 3. What should $\text{measure}_{1,k}(x_1y_k \lor x_{k-1} \lor x_k)$ be?

Suppose we take this measure to be $\alpha$, and similarly $\text{measure}_{i,k+1-1}(x_i y_{k+1-i} \lor x_{k-1} \lor x_k) = \alpha$ for $1 \leq i \leq k - 2$. The “or” of all these terms is $v_1 = v \lor x_{k-1} \lor x_k$, where $v = x_1y_k \lor \cdots \lor x_{k-2}y_3$. We expect the total measure, $\text{measure}(v_1) = \sum_{i,j} \text{measure}_{i,j}(v_1)$ to be $(k-2)\alpha$.

We define $v_2, v_3,$ and $v_4$ similarly, so that

$$v_1 = v \lor x_{k-1} \lor x_k,$$

$$v_2 = v \lor y_2 \lor x_k,$$

$$v_3 = v \lor x_{k-1} \lor y_1,$$

and

$$v_4 = v \lor y_2 \lor y_1.$$
Then measure(\{v_1, v_2, v_3, v_4\}) is at most 4(k-2)\alpha. However with three more conjunctions the term \(v_1v_2v_3v_4 = x_1y_k \lor \cdots \lor x_ky_1\) is generated which shows that \(4(k-2)\alpha + 3 \geq k\), i.e., \(\alpha \geq \frac{k-3}{4(k-2)}\). This suggests that an appropriate choice of \(\alpha\) would be 1/4.

More generally, we expect that

\[
\text{measure}_{i,j}(x_iy_j \lor z_1 \lor \cdots \lor z_k) \geq 2^{-k},
\]

where \(x_i, y_j \notin \{z_1, \ldots, z_k\} \subset \mathbb{Z}\). We choose to take this measure to be \(2^{-k}\).

**Notation.** We will identify a Boolean function \(f\) with the set \(f^{-1}(1)\), i.e., the set of arguments which \(f\) maps to 1. Then \(f \land g = f \cap g\) and \(f \lor g = f \cup g\). Set inclusion corresponds with implication: \(f \subseteq g\) is equivalent to \(f \land g = f\) or \(f \lor g = g\) or \(f \to g\).

### 4 Definitions

For a semi-disjoint bilinear set of functions \(F = f_1, \ldots, f_K\) and \((i, j) \in \text{dom}(F)\), we define \(h(i, j) = k\) where \(f_k\) is the unique function in \(F\) with prime implicant \(x_iy_j\). For example, in the notation we are using for Boolean convolution, we have \(h(i, j) = i + j - 1\).

**Definition 3.** For any Boolean function \(g\), any semi-disjoint bilinear set of functions \(F\) and any \((i, j) \in \text{dom}(F)\), we can partition the prime implicants, \(PI(g)\), of \(g\) into the “harmless” ones \(H_{i,j}(g)\) and the “bad” ones \(B_{i,j}(g)\), where \(H_{i,j}(g) = \{p \in PI(g) : p \subseteq f_{h(i,j)}\}\) and \(B_{i,j}(g) = \{p \in PI(g) : p \not\subseteq f_{h(i,j)}\}\).

We can further partition \(B_{i,j}(g)\) into those prime implicants which are dependent on \(x_i\), those that are dependent on \(y_j\), and those that are independent of \(x_i\) and \(y_j\). Any prime implicants involving both \(x_i\) and \(y_j\) are clearly in \(H_{i,j}(g)\). The \((i, j)\)-decomposition of \(g\) is the 4-tuple \((h, a, b, c)\) of functions such that \(g = h \lor x_ia \lor y_jb \lor c\), corresponding to the partition described, and where \(a, b, c\) are independent of \(x_i\) and \(y_j\).

Let \(G\) be a set of Boolean functions and \(F\) a semi-disjoint bilinear set of functions. We need to measure the progress of \(G\) towards the computation of each prime implicant \(x_iy_j\) of \(F\). It is natural that such a progress measure should be dependent only on functions in \(G\) which have \(x_iy_j\) as a prime implicant. What is less natural but technically convenient is that our definitions for measuring the progress of \(G\) depend only on the conjunction of these functions.

We will write \(\land J\) and \(\lor J\) for the conjunction and disjunction, respectively, of a set \(J\) of functions.

**Definition 4.** The \((i, j)\)-support of a set of functions \(G\), denoted \(S_{i,j}(G)\), is the conjunction of the subset of functions that have \(x_iy_j\) as a prime implicant,

\[
S_{i,j}(G) = \land\{g \in G : x_iy_j \in PI(g)\}.
\]

Note that this support either has \(x_iy_j\) as a prime implicant or is trivial, i.e., 1. Suppose the \((i, j)\)-support of \(G\) is nontrivial and \((h, a, b, c)\) is its \((i, j)\)-decomposition, then \(h\) has \(x_iy_j\) as a prime implicant and contains only implicants of \(f_{h(i,j)}\). The other terms \(a, b, c\) contain
non-implicants of $f_{h(i,j)}$, which we regard as “pollutants”. At the end of the computation we must have eliminated these pollutants, i.e., $a \lor b \lor c = 0$. Our $(i,j)$-measure is designed to quantify progress in pollutant elimination.

In this paper, a projection maps some subset of the variables to 0 and leaves the rest unchanged.

**Definition 5.** Attenuation for a monotone Boolean function $g$ is the random projection which, independently for each variable $z$, maps $z$ to 0 with probability $1/2$ and leaves $z$ unchanged otherwise.

**Definition 6.** The vacuity, $\text{vac}(g)$, of a monotone Boolean function $g$ is the probability that the attenuation of $g$ is 0, i.e., the zero function corresponding to the empty set $\emptyset$.

For examples, if $z_1, z_2, \ldots$ are distinct variables,

- $\text{vac}(0) = 1$,
- $\text{vac}(1) = 0$,
- $\text{vac}(z_1) = 1/2$,
- $\text{vac}(z_1 \lor \ldots \lor z_k) = 2^{-k}$,
- $\text{vac}(z_1 \ldots z_k) = 1 - 2^{-k}$,
- $\text{vac}(z_1(z_2 \lor z_3)) = 5/8$, and
- $\text{vac}(z_1(z_2 \lor \ldots \lor z_k)) = 1/2 + 2^{-k}$.

Vacuity is used to give a measure of progress for a prime implicant.

**Lemma 1.** The function $\text{vac}$ is (i) monotone decreasing and (ii) modular, i.e., for all Boolean functions $U$ and $V$,

(i) if $U \subseteq V$ then $\text{vac}(U) \geq \text{vac}(V)$; and

(ii) $\text{vac}(U) + \text{vac}(V) = \text{vac}(U \lor V) + \text{vac}(U \land V)$.

**Proof.** For (i), since any projection $\pi$ is monotone increasing, if $U \subseteq V$ and $\pi(V) = \emptyset$ then $\pi(U) = \emptyset$. Therefore $\text{vac}(U) \geq \text{vac}(V)$.

For (ii), we see that for any projection $\pi$:

(a) if $\pi(U) = 0$ and $\pi(V) = 0$ then $\pi(U \lor V) = 0$ and $\pi(U \land V) = 0$,
(b) if $\pi(U) = 0$ and $\pi(V) \neq 0$ (or vice versa) then $\pi(U \lor V) \neq 0$ and $\pi(U \land V) = 0$, and
(c) if $\pi(U) \neq 0$ and $\pi(V) \neq 0$ then $\pi(U \lor V) \neq 0$ and $\pi(U \land V) \neq 0$.

In each case $\pi$ makes an equal contribution to the probability on each side of the equation. \qed
Corollary 1. For all Boolean functions $U$, $V$ and $W$, 

(i) $\text{vac}(UV) \leq \text{vac}(U) + \text{vac}(V)$;

(ii) $\text{vac}(U \lor VW) = \text{vac}((U \lor V)(U \lor W)) \leq \text{vac}(U \lor V) + \text{vac}(U \lor W)$.

Definition 7. Let $p_{i,j}(g) = a \lor b \lor c$ where $(h, a, b, c)$ is the $(i, j)$-decomposition of the function $g$. It is convenient to abbreviate $\text{vac}(p_{i,j}(g))$ as $\text{vac}_{i,j}(g)$. Recall that $a \lor b \lor c$ does not depend on $x_i$ or $y_j$.

Definitions 8. For a set of Boolean functions $G$, $\text{measure}_{i,j}(G) = \text{vac}_{i,j}(S_{i,j}(G))$, i.e., $\text{measure}_{i,j}(G) = \text{vac}(p_{i,j}(S_{i,j}(G)))$.

For a set $G$ of Boolean functions, the total measure of $G$ is

$$\text{measure}(G) = \sum_{(i,j) \in \text{dom}(F)} \text{measure}_{i,j}(G).$$

Revisiting Example 1, we see that

$$\text{measure}_{1,2}(w(x_2 \lor y_2)) = \text{vac}(x_2(y_3 \lor y_4) \lor y_1(x_3 \lor x_4)) = \frac{5}{8} \cdot \frac{5}{8} = 25/64.$$ 

So the conjunction with $x_2 \lor y_2$ increases $\text{measure}_{1,2}$, and similarly $\text{measure}_{2,1}$, by $1/4$ from

$$\text{vac}(y_3 \lor y_4 \lor y_1(x_2 \lor x_3 \lor x_4)) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{9}{16} = 9/64$$

to $25/64$. This conjunction therefore increases the total measure by $1/2$.

5 First results

Suppose $G$ is a set of Boolean functions and $g^l = g' \circ g''$ is a new function where $\{g', g''\} \subseteq G$ and $\circ \in \{\land, \lor\}$. Let $G^l = G \cup \{g^l\}$. Suppose $(i, j) \in \text{dom}(F)$.

We begin by showing some elementary properties of our progress measures.

Lemma 2. For all $i, j$, $p_{i,j}$ is a monotonic increasing function, i.e., if $g_1 \subseteq g_2$ then $p_{i,j}(g_1) \leq p_{i,j}(g_2)$.

Proof. Monotonicity follows from the observations that if $(h, a, b, c)$ is the $(i, j)$-decomposition of $g$ then $ax_i \lor by_j \lor c = g \setminus f_{h(i,j)}$ and $p_{i,j}(g) = a \lor b \lor c$ is obtained from $ax_i \lor by_j \lor c$ by setting $x_i = y_j = 1$. 

Lemma 3. The following inequalities hold.

(i) If $\circ = \lor$, then $\text{measure}_{i,j}(G^l) = \text{measure}_{i,j}(G)$.

(ii) $\text{measure}_{i,j}(G^l) \geq \text{measure}_{i,j}(G)$.
Proof.
For (i), since \( S_{i,j}(G^1) = S_{i,j}(G) \), the result is clear.

For (ii),

\[
G^1 \supseteq G \implies S_{i,j}(G^1) \subseteq S_{i,j}(G) \quad \text{(immediate from Definition 4)}
\]
\[
\implies p_{i,j}(S_{i,j}(G^1)) \subseteq p_{i,j}(S_{i,j}(G)) \quad \text{(by Lemma 2)}
\]
\[
\implies \text{vac}(p_{i,j}(S_{i,j}(G^1))) \geq \text{vac}(p_{i,j}(S_{i,j}(G))) \quad \text{(by Lemma 4(i))}
\]
\[
\iff \text{measure}_{i,j}(G^1) \geq \text{measure}_{i,j}(G)
\]

The next lemmas establish limits on the amount of progress which can be made with a single conjunction.

**Lemma 4.** For all \((i, j) \in \text{dom}(F)\), \(\text{measure}_{i,j}\) is supermodular, i.e., for all sets of functions \(G_1, G_2\),

\[
\text{measure}_{i,j}(G_1 \cup G_2) + \text{measure}_{i,j}(G_1 \cap G_2) \leq \text{measure}_{i,j}(G_1) + \text{measure}_{i,j}(G_2).
\]

**Proof.** We observe that \(S_{i,j}(G_1 \cup G_2) = S_{i,j}(G_1) \cap S_{i,j}(G_2)\) and \(S_{i,j}(G_1 \cap G_2) \supseteq S_{i,j}(G_1) \lor S_{i,j}(G_2)\).

Since \(p_{i,j}\) is monotonic increasing (Lemma 2),

\[
\begin{align*}
\text{measure}_{i,j}(G_1 \cup G_2) + \text{measure}_{i,j}(G_1 \cap G_2) &= \text{vac}(p_{i,j}(S_{i,j}(G_1 \cup G_2))) + \text{vac}(p_{i,j}(S_{i,j}(G_1 \cap G_2))) \\
&\leq \text{vac}(p_{i,j}(S_{i,j}(G_1) \cap S_{i,j}(G_2))) + \text{vac}(p_{i,j}(S_{i,j}(G_1) \lor S_{i,j}(G_2))) \\
&= \text{vac}(p_{i,j}(S_{i,j}(G_1))) + \text{vac}(p_{i,j}(S_{i,j}(G_2))) \quad \text{(by Lemma 4(ii))} \\
&= \text{measure}_{i,j}(G_1) + \text{measure}_{i,j}(G_2).
\end{align*}
\]

**Lemma 5.** The progress for \(\text{measure}_{i,j}\) from \(G\) to \(G^1\) is at most the progress seen by considering the last operation in isolation, i.e.,

\[
\text{measure}_{i,j}(G^1) - \text{measure}_{i,j}(G) \leq \text{measure}_{i,j}(\{g', g'', g^1\}) - \text{measure}_{i,j}(\{g', g''\}).
\]

**Proof.** Since \(\text{measure}_{i,j}\) is supermodular (Lemma 4),

\[
\begin{align*}
\text{measure}_{i,j}(G^1) + \text{measure}_{i,j}(\{g', g''\}) &= \text{measure}_{i,j}(G \cup \{g', g'', g^1\}) + \text{measure}_{i,j}(G \cap \{g', g'', g^1\}) \\
&\leq \text{measure}_{i,j}(G) + \text{measure}_{i,j}(\{g', g'', g^1\}).
\end{align*}
\]

When \(g^1 = g' \lor g''\), no progress is made since \(S_{i,j}(G^1) = S_{i,j}(G)\). So now we need only analyse the single operation \(g^1 = g' \land g''\).

**Lemma 6.** In the following cases, no progress is made with respect to \(\text{measure}_{i,j}\), i.e., \(\text{measure}_{i,j}(\{g', g''\}) = \text{measure}_{i,j}(\{g', g'', g^1\})\), where \(g^1 = g' \land g''\):

(i) \(x_iy_j \not\in PI(g^1)\);
(ii) \( x_i y_j \in PI(g') \) and \( x_i y_j \in PI(g'') \);
(iii) \( x_i y_j \in PI(g') \) and \( x_i \in PI(g'') \) and \( y_j \in PI(g'') \);
(iv) \( x_i y_j \in PI(g'') \) and \( x_i \in PI(g') \) and \( y_j \in PI(g') \).

Proof. Cases (i) and (ii) are obvious since \( S_{i,j}(\{g', g''\}) = S_{i,j}(\{g', g'', g'\}) \).

For Case (iii), let \( g' = h' \lor x_i y_j \lor x_i a' \lor y_j b' \lor c' \) and \( g'' = h'' \lor x_i \lor y_j \lor c'' \), corresponding to their \((i, j)\)-decompositions \((h' \lor x_i y_j, a', b', c')\) and \((h'', 1, 1, c'')\). Then \( g^1 = g' \land g'' \) has \((i, j)\)-decomposition \((h^* \lor x_i y_j, a' \lor c', b' \lor c', c^*) \) where \( c^* \subseteq c' \subseteq c'' \) and \( h^* \subseteq f_{h(i,j)} \). Hence

\[
p_{i,j}(g' \land g'') = a' \lor c' \lor b' \lor c' \lor c^* = a' \lor b' \lor c = p_{i,j}(g'),
\]
so \( measure_{i,j}(\{g', g''\}) = measure_{i,j}(\{g', g'', g'\}) \). Case (iv) is similar. \( \square \)

There are six remaining cases where \( measure_{i,j} \) may be improved. These are characterised by the significant occurrences of \( x_i \) and \( y_j \) given below. A precise description is given by the decompositions shown in the subsequent case analysis.

The six cases

XY: \( x_i \in PI(g') \) and \( y_j \in PI(g'') \);
QY: \( x_i y_j \in PI(g') \) and \( y_j \in PI(g'') \);
XQ: \( x_i \in PI(g') \) and \( x_i y_j \in PI(g'') \);
YX: \( y_j \in PI(g') \) and \( x_i \in PI(g'') \);
YQ: \( y_j \in PI(g') \) and \( x_i y_j \in PI(g'') \);
QX: \( x_i y_j \in PI(g') \) and \( x_i \in PI(g'') \).

Let

\[
g' = \bigvee X' \lor \bigvee Y' \lor \bigvee m'_{u,v} x_u y_v \lor c' \quad \text{and} \quad g'' = \bigvee X'' \lor \bigvee Y'' \lor \bigvee m''_{u,v} x_u y_v \lor c'',
\]

where \( X' \subseteq X, X'' \subseteq X, Y' \subseteq Y, Y'' \subseteq Y, m' \) and \( m'' \) are \((0, 1)\)-valued matrices, and \( c' \) and \( c'' \) contain no terms linear both in \( x \)'s and \( y \)'s. We define the integers

\[
n'_X = |X'|, \quad n'_Y = |Y'|, \quad n''_X = |X''|, \quad n''_Y = |Y''|,
\]

and also, for each \( i \) and \( j \), the linear functions

\[
R'_i = \bigvee m'_{u,i} x_u, \quad S'_i = \bigvee m'_{i,v} y_v, \quad R''_j = \bigvee m''_{u,j} x_u, \quad S''_j = \bigvee m''_{i,v} y_v,
\]

and the corresponding integers

\[
r'_i = \sum m'_{u,i}, \quad s'_i = \sum m'_{i,v}, \quad r''_j = \sum m''_{u,j} \quad \text{and} \quad s''_j = \sum m''_{i,v}.
\]

It will be convenient in the following to abbreviate, e.g., \( \bigvee X' \) by \( X' \) for a subset \( X' \) of variables, where this presents no ambiguity.
Lemma 7. In Case XY, \( x_i \in X' \) and \( y_j \in Y'' \), and
\[
\text{measure}_{i,j}(g^\dagger) \leq 2^{-(n_X' - 1 + n_Y' - 1 + \max(n_X'', r_j')} + \max(n_Y', s_i')
\leq 2^{-(n_X' - 1 + n_Y' - 1 + n_X' + n_Y')}.
\]

Proof. We have
\[
g' \supseteq x_i \lor (X' \setminus x_i) \lor Y' \lor y_j R_j', \quad g'' \supseteq y_j \lor (Y'' \setminus y_j) \lor x_i S_i'',
\]
and
\[
g^\dagger = g' \land g'' \supseteq x_i y_j \lor x_i (X'' \setminus y_j) \lor S_i'' \lor y_j ((X' \setminus x_i) \lor Y' \lor R_j').
\]
Hence
\[
\text{measure}_{i,j}(g^\dagger) \leq \text{vac}((X' \setminus x_i) \lor (Y'' \setminus y_j) \lor X'' \lor R_j' \lor Y' \lor S_i'')
= 2^{-(n_X' - 1 + n_Y' - 1 + |X'' \cup R_j'| + |Y' \cup S_i''|)},
\]
since the six sets of variables involved are disjoint except possibly for the two unions indicated. The inequalities of the lemma follow immediately.

Lemma 8. The contribution to the total measure under Case XY, is at most
\[
\sum_{\{i,j\} | x_i \in X' \land y_j \in Y''} 2^{-(n_X' - 1 + n_Y' - 1 + \max(n_X'', r_j')} + \max(n_Y', s_i')}
\leq \frac{n_X'}{2^{n_X' - 1}} \frac{n_Y''}{2^{n_Y' - 1}} 2^{-(n_X' + n_Y')} \leq 2^{-(n_X' + n_Y')}.
\]

Proof. The first expression is immediate from Lemma 7. The next inequality replaces the max’s by their first argument, removing dependency on \( i \) and \( j \). The final inequality follows since \( n/2^{n-1} \leq 1 \) for all \( n \geq 0 \).

The analysis of Case QY is more complicated since we need to bound the difference, \( \text{measure}_{i,j}(g^\dagger) - \text{measure}_{i,j}(g'') \), whereas in Case XY \( \text{measure}_{i,j}(g') = \text{measure}_{i,j}(g'') = 0 \).

Lemma 9. In Case QY, suppose \( g' = h' \lor x_i a' \lor y_j b' \lor c' \) and \( g'' = h'' \lor x_i a'' \lor y_j \lor c'' \) corresponding to their \( (i,j) \)-decompositions, where \( x_i y_j \in h' \), then the \( (i,j) \)-progress made in this step is
\[
\text{measure}_{i,j}(g^\dagger) - \text{measure}_{i,j}(g'') \leq \text{vac}_{i,j}(a'(a'' \lor c'') \lor b' \lor c') - \text{vac}_{i,j}(a' \lor b' \lor c').
\]

Proof. We have \( g'' \supseteq x_i a'' \lor y_j \lor c'' \), so
\[
g^\dagger = g' \land g'' \supseteq x_i y_j \lor x_i a'(a'' \lor c'') \lor y_j (b' \lor c').
\]
Hence
\[
\text{measure}_{i,j}(g^\dagger) \leq \text{vac}_{i,j}(a'(a'' \lor c'') \lor b' \lor c') \quad \text{and} \quad \text{measure}_{i,j}(g') = \text{vac}_{i,j}(a' \lor b' \lor c').
\]
Example 4. We denote by \( \pi \) the unique index such that \( h(e, \pi) = h(i, j) \), and assume \( i \) and \( j \) are disjoint from \( 1, 2, 3, \overline{1}, \overline{2}, \overline{3} \). Suppose

\[
\begin{align*}
g' &= x_iy_j \lor x_ja' \quad \text{where} \quad a' = x_1x_2 \lor x_2x_3 \lor x_3x_1, \\
g'' &= x_ia'' \lor y_j \quad \text{where} \quad a'' = y_{\overline{1}} \lor y_{\overline{2}} \lor y_{\overline{3}}, \quad \text{and so} \\
g' \uparrow &= g'g'' = x_iy_j \lor x_ia''.
\end{align*}
\]

For this example, \( vac(a'' \lor c'' \lor b' \lor d') = vac(a'') = 1/8 \). Note that \( a' \) is the majority function of \( x_1, x_2, x_3 \), and so \( vac_{i,j}(g') = vac(a') = 1/2 \), and hence \( vac(g'g'') = 1/2 + 1/2 \cdot 1/8 = 1/2 + 1/16 \). However,

\[
a'a'' \subseteq f_{h(i,j)} \lor x_1x_2y_{\overline{2}} \lor x_2x_3y_{\overline{1}} \lor x_3y_{\overline{3}}
\]

so

\[
\begin{align*}
vac_{i,j}(g'g'') &= vac(x_1x_2y_{\overline{1}} \lor x_2x_3y_{\overline{1}} \lor x_3x_1y_{\overline{3}}) = 45/64 = 1/2 + 13/64,
\end{align*}
\]

giving an increase in \( measure_{i,j} \) of \( 13/64 \), which exceeds \( vac(a'') \). (The value 45/64 can be shown by considering the three cases: (i) at most one \( x \) is 1, (ii) exactly two \( x \)’s are 1 and (iii) all three \( x \)’s are 1. The corresponding contributions to \( vac \) are 1/2, 3/8 \( \cdot \) 1/2 and 1/8 \( \cdot \) 1/8.)

We can however obtain a weaker inequality provided that \( V \), say, is linear, i.e., a disjunction of variables. We first need the following monotonicity property.

**Lemma 10.** For any \((i, j) \in dom(F)\), \( vac_{i,j}(UV \lor W) - vac_{i,j}(U \lor W) \) is monotone decreasing in \( V \) and \( W \).

(Not that Example 4 shows that this expression is neither monotone decreasing nor increasing in \( U \). With \( V = a'' \) and \( W = 0 \), setting \( U \) to 0, \( a' \), 1 gives values of 0, 13/64, 1/8 respectively.)

**Proof.** If \( V \supseteq V_0 \) and \( W \supseteq W_0 \) then for any projection \( \pi \), if \( \pi(UV) \equiv 0 \) and \( \pi(W) \equiv 0 \) and \( \pi(U) \equiv 0 \) then \( \pi(UV_0) \equiv 0 \) and \( \pi(W_0) \equiv 0 \) and \( \pi(U) \equiv 0 \). Hence \( vac_{i,j}(UV_0 \lor W_0) - vac_{i,j}(U \lor W_0) \geq vac_{i,j}(UV \lor W) - vac_{i,j}(U \lor W) \). \( \square \)

6 Technical crux

For expressing our upper bound in Case QY, we need the function

\[
q(k) = (3^k + 1)/2^{2k+1}.
\]
Lemma 11. For any \((i, j) \in \text{dom}(F)\), if \(V\) is the disjunction of \(k\) variables then

\[
vac_{i,j}(U \land V) - vac_{i,j}(U) \leq q(k).
\]

Proof. Within this proof it is convenient to define the mate \(\hat{z}\) of a variable \(z \in Z\) as the unique variable such that \(z\hat{z} \subseteq f_{h(i,j)}\), i.e., \(z\hat{z} \equiv 0\). For example, \(\hat{y}_2\) is \(x_{\bar{\tau}}\) (in the notation of Example 4), and for all \(z, \hat{z} = z\).

Now \(vac_{i,j}(U \land V) - vac_{i,j}(U)\) is the proportion of projections \(\pi\) such that \(\pi(UV) \equiv 0\) and \(\pi(U) \not\equiv 0\). Suppose that \(V = z_1 \lor \cdots \lor z_k\). We begin by observing that we need only consider functions \(U\) which depend only on \(\hat{z}_1, \ldots, \hat{z}_k\). Suppose that \(U\) depends on the variable \(z\), and \(U = U_0 \lor U_1z\) where \(U_0\) and \(U_1\) are independent of \(z\). Then \(UV = U_0V \lor U_1zV\).

Case (i): \(z \in \{z_1, \ldots, z_k\}\).

Then \(zV = z\) and so \(UV = U_0V \lor U_1z\). Hence

\[
vac_{i,j}(UV) - vac_{i,j}(U) = vac_{i,j}(U_0V \lor U_1z) - vac_{i,j}(U_0 \lor U_1z) \leq vac_{i,j}(U_0V) - vac_{i,j}(U_0)
\]

by Lemma 10.

Case (ii): \(z \not\in \{z_1, \ldots, z_k, \hat{z}_1, \ldots, \hat{z}_k\}\).

We can assume that \(U_1\) is independent of \(\hat{z}\) since any \(t\hat{z} \in PI(U_1)\) yields the term \(t\hat{z}z \equiv 0\) in \(U_1z\), and so, for any projection \(\pi\) such that \(\pi(z) = z\), \(\pi(U_1zV) \equiv 0\) if and only if \(\pi(U_1V) \equiv 0\). Then

\[
vac_{i,j}((U_0 \lor U_1z)V) - vac_{i,j}(U_0 \lor U_1z)
= \frac{1}{2}(vac_{i,j}(U_0V) - vac_{i,j}(U_0) + vac_{i,j}((U_0 \lor U_1)V) - vac_{i,j}(U_0 \lor U_1)),
\]
i.e., the average of the terms when the projection of \(z\) is zero or nonzero.

In each case we see that \(vac_{i,j}(UV) - vac_{i,j}(U) \leq vac_{i,j}(U'V) - vac_{i,j}(U')\) for some \(U'\) derived from \(U\) by eliminating the variable \(z\). Hence it is sufficient for the proof to assume that \(U\) depends only on \(\hat{z}_1, \ldots, \hat{z}_k\). We denote \(\{1, \ldots, k\}\) by \(I_k\) and define Boolean variables \(b_S\) for \(S \subseteq I_k\) by the disjunctive normal form

\[
U = \bigvee_{S \subseteq I_k} \left( \bigwedge_{i \in S} \hat{z}_i \right) b_S.
\]

By definition, if \(b_S = 1\) then \(b_{S'} = 0\) for any \(S' \subset S\).

For any projection \(\pi\), we define

\[
S_\pi = \{i \mid \pi(z_i) = z_i\} \text{ and } \hat{S}_\pi = \{i \mid \pi(\hat{z}_i) = \hat{z}_i\}.
\]

Consider a projection \(\pi\) such that \(\pi(U) \not\equiv 0\) but \(\pi(UV) \equiv 0\). Then for some \(S \subseteq \hat{S}_\pi\), \(\pi(\bigwedge_{i \in S} \hat{z}_i b_S) \not\equiv 0\). Hence \(\{\hat{z}_i \mid i \in S\}\) cannot contain a pair of mates and \(b_S = 1\).

Since \(\pi(UV) \equiv 0\), \(\bigwedge_{i \in S} \hat{z}_i \bigvee_{i \in S_\pi} z_i \equiv 0\), which implies \(S_\pi \subseteq S\). So \(S_\pi \subseteq S \subseteq \hat{S}_\pi\), and hence \(S_\pi \subseteq \hat{S}_\pi\).
The probability that $|S_\pi| = r$ is $2^{-k}(k)$, and with the extra condition that $\{\hat{z}_i| i \in S_\pi\}$ does not contain a pair of mates (since $S_\pi \subseteq S$), this probability is at most $2^{-k}(k)$. The probability when $|S_\pi| = r$ that $\hat{S}_\pi \supseteq S_\pi$ is $2^{-r}$. Hence

$$vac_{i,j}(U \wedge V) - vac_{i,j}(U) \leq 2^{-k} \sum_{0 \leq r \leq k} \binom{k}{r} 2^{-r} = 2^{-k}(1 + 1/2)^k = (3/4)^k.$$ 

We improve this bound using the following observation. Consider a pair of dual projections $\pi_1$ and $\pi_2$, i.e., such that $S_{\pi_1} = I_k \setminus \hat{S}_{\pi_2}$ and $S_{\pi_2} = I_k \setminus \hat{S}_{\pi_1}$. If both $\pi_1$ and $\pi_2$ contribute to $vac_{i,j}(U \wedge V) - vac_{i,j}(U)$ then

(a) $\bigvee_{S \subseteq \hat{S}_{\pi_1}} b_S = 1$, since $\pi_1(U) = \bigvee_{S \subseteq \hat{S}_{\pi_1}} \bigwedge_{i \in S} \hat{z}_i b_S \neq 0$, and

(b) $\bigvee_{S \supseteq S_{\pi_2}} b_S = 0$, since $\pi_2(UV) = \bigvee_{S \supseteq \hat{S}_{\pi_2}} \bigwedge_{i \in S} \hat{z}_i b_S \bigwedge_{i \in S_{\pi_2}} z_i = 0$.

From (a), there exists some $S_1$ such that $S_1 \subseteq \hat{S}_{\pi_1} = (I_k \setminus S_{\pi_2})$ and $b_{S_1} = 1$. From (b), $b_{S_1} = 1$ implies that $S_1 \supseteq S_{\pi_2}$. Hence $S_{\pi_2} \subseteq S_1 \subseteq (I_k \setminus S_{\pi_2})$, which implies that $S_{\pi_2} = S_1 = \emptyset$, and $S_{\pi_1} = \emptyset$ by a symmetric argument.

Therefore, unless $S_{\pi_1} = \emptyset$ and $S_{\pi_2} = \emptyset$, we find $\pi_1$ and $\pi_2$ cannot both contribute to $vac_{i,j}(U \wedge V) - vac_{i,j}(U)$, and so the two projections together contribute at most $2^{-2k}$ to $vac_{i,j}(U \wedge V) - vac_{i,j}(U)$ instead of each contributing $2^{-2k}$.

So we can successfully match dual pairs $(\pi_1, \pi_2)$ of projections except when $S_{\pi_1} = \emptyset$ and $S_{\pi_2} = \emptyset$. The number of such pairs is $(3^k - 1)/2$ which yields $q(k) = (3^k + 1)/2^{2k+1}.$

**Lemma 12.** The function $q$ has the following monotonicity properties:

(i) for all $k \geq 1$, $q(k - 1) \geq q(k)$,

(ii) for all $k \geq 1$, $k q(k - 1) \geq (k + 1) q(k)$,

(iii) for all $k \geq 1$ and all $t \geq 0$, $k q(k - 1 + t) \leq 1$.

**Proof.** For (i),

$$(3^{k-1} + 1)/2^{2k-1} \geq (3^k + 1)/2^{2k+1} \iff 4(3^{k-1} + 1) \geq 3^k + 1 \iff 3^{k-1} + 3 \geq 0.$$ 

For (ii),

$$k(3^{k-1} + 1)/2^{2k-1} \geq (k + 1)(3^k + 1)/2^{2k+1} \iff 4k(3^{k-1} + 1) \geq (k + 1)(3^k + 1) \iff 4k - (k + 1) \geq (3(k + 1) - 4k)3^{k-1} \iff 3k - 1 \geq (3 - k)3^{k-1}.$$ 

The last inequality holds for $k \geq 1$.

Inequality (iii) follows from inequalities (i) and (ii), and checking the value for $k = 1$ and $t = 0$, i.e., $1q(0) = 1$.  


Lemma 13. For any \((i, j) \in \text{dom}(F)\), if \(V\) and \(W\) are sets of variables, where \(|W| = m\) and \(|V \setminus W| = k\) then

\[
vac_{i,j}(UV \lor W) - vac_{i,j}(U \lor W) \leq (1/2)^m q(k).
\]

Proof. We need to estimate the probability of a projection \(\pi\) such that \(\pi((UV \lor W) \equiv 0\) but \(\pi(U \lor W) \equiv 0\), i.e., \(\pi(W) \equiv 0\) and \(\pi(UV) \equiv 0\) but \(\pi(U) \neq 0\) and \(\pi(V) \neq 0\). We can regard any such \(\pi\) as the composition of a projection \(\pi_1\) over the variables of \(W\), a projection \(\pi_2\) over the variables of \(|V \setminus W|\) and a projection \(\pi_3\) over the remaining variables. The required result is reached with probability \((1/2)^m\) for \(\pi_1\), probability at most \(q(k)\) for \(\pi_2\) and probability at most 1 for \(\pi_3\). \(\square\)

We use the monus (limited subtraction) notation, i.e., \(r - s = \max(r - s, 0)\).

Lemma 14. In Case QY, \(x_i y_j \in PI(g')\) and \(y_j \in Y''\), and

\[
\text{measure}_{i,j}(g') - \text{measure}_{i,j}(g') \leq (1/2)^m q(k),
\]

where \(m = r_j' - 1 + n'_X + n'_Y\) and

\[
k = (n''_Y - 1 + (n''_X - (r_j' - 1))) + (s''_i - n'_Y).
\]

Proof. If \(g' = h' \lor x_i a' \lor y_j b' \lor c'\) and \(g'' = h'' \lor x_i a'' \lor y_j c''\) then since \(b' \supseteq (R'_j \setminus x_i)\), \(c' \supseteq X' \lor Y'\), \(a'' \supseteq S''_i\) and \(c'' \supseteq X'' \lor (Y'' \setminus y_j)\), from Lemmas 13 and 11 we have

\[
\text{measure}_{i,j}(g') - \text{measure}_{i,j}(g') = vac_{i,j}(a'(a'' \lor c'') \lor b' \lor c') - vac_{i,j}(a' \lor b' \lor c')
\leq vac_{i,j}(a'V \lor W) - vac_{i,j}(a' \lor W),
\]

where \(V = (Y'' \setminus y_j) \cup X'' \cup S''_i\) and \(W = (R'_j \setminus x_i) \cup X' \cup Y'.\) Now \(|W| = r_j' - 1 + n'_X + n'_Y\) and

\[
|V \setminus W| = |(Y'' \setminus y_j) \cup (X'' \setminus (R'_j \setminus x_i)) \cup (S''_i \setminus Y')|
\geq n''_Y - 1 + (n''_X - (r_j' - 1)) + (s''_i - n'_Y).
\]

By Lemmas 13 and 11(i),

\[
vac_{i,j}(a'V \lor W) - vac_{i,j}(a' \lor W) \leq (1/2)^{|W|} q(|V \setminus W|),
\]

which establishes the result. \(\square\)

Lemma 15. The contribution to the total measure under Case QY, is at most

\[
\sum_{\{i,j\}|x_i \in R'_j \land y_j \in Y''} 2^{-(r_j' - 1 + n'_X + n'_Y)} q(n''_Y - 1 + (n''_X - (r_j' - 1)) + (s''_i - n'_Y))
\]

\[
\leq \sum_{\{j|y_j \in Y''\}} r_j' 2^{-(r_j' - 1 + n'_X + n'_Y)} q(n''_Y - 1 + (n''_X - (r_j' - 1)))
\]

\[
\leq 2^{-(n'_X + n'_Y)},
\]

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Lemma 16. The total contribution made in one step of the Boolean chain is at most 2.

Proof. Lemmas 8 and 15 together with symmetry show that

1. if \( n'_X > 0 \) and \( n''_Y > 0 \) then \( C_{XY} \leq 2^{-(n''_X + n'_Y)} \) else \( C_{XY} = 0 \),

2. if \( n''_Y > 0 \) then \( C_{QY} \leq 2^{-(n'_X + n''_Y)} \) else \( C_{QY} = 0 \),

3. if \( n'_X > 0 \) then \( C_{XQ} \leq 2^{-(n''_X + n'_Y)} \) else \( C_{XQ} = 0 \),

4. if \( n'_Y > 0 \) and \( n''_X > 0 \) then \( C_{YX} \leq 2^{-(n''_X + n'_Y)} \) else \( C_{YX} = 0 \),

5. if \( n''_Y > 0 \) then \( C_{YQ} \leq 2^{-(n''_X + n'_Y)} \) else \( C_{YQ} = 0 \), and

6. if \( n''_X > 0 \) then \( C_{QX} \leq 2^{-(n'_X + n''_Y)} \) else \( C_{QX} = 0 \).

We bound the total contribution, \( C = C_{XY} + C_{QY} + C_{XQ} + C_{YX} + C_{YQ} + C_{QX} \), by considering the following seven cases. Other cases all follow by symmetry.

(a) \( n'_X, n'_Y, n''_X, n''_Y > 0 \);

\[
C_{XY} + C_{QY} + C_{XQ} + C_{YX} + C_{YQ} + C_{QX} \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{2}.
\]

(b) \( n'_X, n'_Y, n''_X > 0 \) and \( n''_Y = 0 \);

\[
C_{XY} + C_{QY} + C_{XQ} + C_{YX} + C_{YQ} + C_{QX} \leq 0 + 0 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}.
\]

(c) \( n'_X, n'_Y > 0 \) and \( n''_X = n''_Y = 0 \);

\[
C_{XY} + C_{QY} + C_{XQ} + C_{YX} + C_{YQ} + C_{QX} \leq 0 + 0 + 1 + 0 + 1 + 0 = 2.
\]
(d) $n'_X, n''_X > 0$ and $n'_Y = n''_Y = 0$;

$$C_{XY} + C_{QY} + C_{XQ} + C_{YX} + C_{YQ} + C_{QX} \leq 0 + 0 + \frac{1}{2} + 0 + 0 + \frac{1}{2} = 1.$$  

(e) $n'_X, n''_Y > 0$ and $n'_Y = n''_X = 0$;

$$C_{XY} + C_{QY} + C_{XQ} + C_{YX} + C_{YQ} + C_{QX} \leq 1 + \frac{1}{2} + \frac{1}{2} + 0 + 0 + 0 = 2.$$  

(f) $n'_X > 0$ and $n'_Y = n''_X = n''_Y = 0$;

$$C_{XY} + C_{QY} + C_{XQ} + C_{YX} + C_{YQ} + C_{QX} \leq 0 + 0 + 1 + 0 + 0 + 0 = 1.$$  

(g) $n'_X = n'_Y = n''_X = n''_Y = 0$;

$$C_{XY} + C_{QY} + C_{XQ} + C_{YX} + C_{YQ} + C_{QX} \leq 0 + 0 + 0 + 0 + 0 = 0.$$  

7 Main results

The results so far already give a new lower bound for semi-disjoint bilinear forms.

**Theorem 1.** The ∧-complexity for a semi-disjoint bilinear form $F$ is at least $D(f)/2$, i.e., half the number of prime implicants in $F$.

**Proof.** For some Boolean chain computing $F$, suppose that conjunctions are used only at steps $t_1, \ldots, t_k$. Let $M_s = \text{measure}(\{g_t \mid t \leq t_s\})$. Now $M_0 = 0$ and, by Lemma 16, $M_s - M_{s-1} \leq 2$ for all $s > 0$. Hence $M_s \leq 2s$ for all $s$. So $D(F) = \text{measure}(F) = M_k \leq 2k$ implies that $k \geq D(F)/2$.  

**Corollary 2.** Boolean convolution (wrapped or unwrapped) for $n$-vectors requires at least $n^2/2$ conjunctions.

Now we refine Theorem 1 to give the exact bound, i.e., increasing $D(F)/2$ to $D(F)$. We use the stronger bounds given in Lemmas 8 and 15 to show that the contribution to the total measure is at most 1 per step.

The case analysis is long and complicated. (We have used Mathematica for this, but plan eventually to present a readable analysis.) As a preliminary step we use simpler, weaker bounds to eliminate most cases. Define

$$C^*_X = \sum_{\{i,j\mid x_i \in X' \land y_j \in Y''\}} 2^{-(n'_X - 1 + n''_Y - 1 + n'_X + n'_Y)},$$  

$$C^*_Y = \sum_{\{i,j\mid x_i \in R'_X \land y_j \in Y''\}} 2^{-(r'_j - 1 + n'_X + n'_Y)q(n''_Y - 1 + (n'_X - (r'_j - 1))},$$  

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and similarly for $XQ, YX, YQ$, and $QX$. By monotonicity, 
\[ C_W \leq C_W^* \text{ for } W \in \{XY, QY, XQ, YX, YQ, QX\}. \]

We first use $C^* = C_{XY}^* + C_{QY}^* + C_{XQ}^* + C_{YX}^* + C_{YQ}^* + C_{QX}^*$ as an upper bound on the progress per step. Symmetry permits us to assume without loss of generality that $n'_X = \max\{n'_X, n'_Y, n''_X, n''_Y\}$. An analysis based on Mathematica shows that $C^* < 1$ unless $(n'_X, n'_Y, n''_X, n''_Y) =$

(i) $(4, 0, 0, *)$,
(ii) $(4, *, 0, 0)$,
(iii) $(3, *, 0, *)$,
(iv) $(2, *, *, *)$, or
(v) $(1, *, *, *)$, where *’s denote arbitrary integers no larger than $n'_X$.

The better bound $C$ resolves these cases, showing that $C \leq 1$ and that the only values for which $C = 1$ are $(n'_X, n'_Y, n''_X, n''_Y) = (1, 0, 0, 1), (1, 1, 1, 1), (2, 0, 0, 1)$ or $(2, 0, 0, 2)$.

**Lemma 17.** The total contribution made in one step of the Boolean chain is at most 1.

**Proof.** This is shown by the case analysis outlined above. \(\square\)

**Main Theorem.** The $\wedge$-complexity for a semi-disjoint bilinear form $F$ is at least $D(f)$, i.e., the number of prime implicants in $F$.

**Proof.** The proof is as in Theorem 1 but uses the tighter bound in Lemma 17 instead of that in Lemma 16. \(\square\)

**Main Corollary.** Boolean convolution (wrapped or unwrapped) for $n$-vectors requires $n^2$ conjunctions.

### 8 Concluding remarks

We have used a classic approach, giving an explicit “measure of progress” for each successive Boolean operation. We believe that our technique of **attenuation** defining the **vacuity** probability measure is novel in this context. It will be interesting to see if the technique has wider uses.

We have dealt only with bounds for the **conjunctive** complexity for bilinear Boolean forms. The problem of giving an exact bound for the corresponding disjunctive complexity is open.

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