COMPACTIFYING MODULI OF HYPERELLIPTIC CURVES

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Abstract. We construct a new compactification of the moduli space $H_g$ of smooth hyperelliptic curves of genus $g$. We compare our compactification with other well-known remarkable compactifications of $H_g$.

1. Introduction

Let $H_g$ be the moduli space of smooth hyperelliptic curves of genus $g \geq 3$. Several compactifications of $H_g$ have been constructed. For example, there exists a moduli space $B_m$ of GIT-semistable binary forms of degree $m$, where a binary form of degree $m$ is a homogeneous polynomial of degree $m$ in two variables over $\mathbb{C}$, up to non-trivial constants. In particular, $B_{2g+2}$ contains $H_g$ as a dense open subset. Recall that a $m$-pointed stable curve of genus zero $(Y,p_1,\ldots,p_m)$ is a curve $Y$ of genus 0 with an ordered set of distinct smooth points $p_i \in Y$ such that $|Y_i \cap Y| + |Y_i \cap \{p_1,\ldots,p_m\}| \geq 3$ for every irreducible component $Y_i$ of $Y$. A $m$-marked stable curve of genus zero $(Y,p_1,\ldots,p_m)$ up to the action of the symmetric group $S_m$ on $p_1,\ldots,p_m$. A natural compactification of $H_g$ is given by its closure within the moduli space of Deligne-Mumford stable curves. This compactification is isomorphic to the moduli space $N_{0,2g+2}$ of $2g+2$-marked stable curves of genus zero. There exists also a fine moduli space $M_{0,m}$ of $m$-pointed stable curves of genus zero and $N_{0,m} = M_{0,m}/S_m$.

As pointed out in [AL], $N_{0,2g+2}$ and $B_{2g+2}$ are different schemes. We construct a compactification $\overline{J}_g$ of $H_g$, given in term of configurations of plane lines and we compare it to $N_{0,2g+2}$ and $B_{2g+2}$. Indeed, consider $(C,p_1,\ldots,p_{2g+2})$, a smooth plane conic and $p_i \in C$ distinct points. Pick the $h_g = \binom{2g+2}{2}$ lines spanned by $p_i,p_j$ for $1 \leq i < j \leq 2g+2$. By taking the closure within $\text{Sym}^h(\mathbb{P}^2)^\vee$, we obtain configurations associated to $(C,p_1,\ldots,p_{2g+2})$, $C$ singular or $p_i = p_j$. The variety $\overline{J}_g$ is the GIT-quotient of the set of GIT-semistable configurations of lines, with respect to the action of $SL(3)$. A boundary point of $\overline{J}_g$ is a configuration containing at least a non-reduced line. For example, if $C$ is smooth and $p_1 = p_2 \neq p_j$ for $j \geq 3$, the associated configuration contains span$(p_1,p_j)_{j \geq 3}$ as double lines. The boundary points of $\overline{J}_g$ have the following geometric meaning. If $C$ is smooth and $p_i \in C$ are distinct, consider the double cover $\varphi : X \to C$ branched at the $2g+2$ points $p_i$. From [ACGH] pag. 288 and [M] Proposition 6.1, we have that $\mathcal{O}_X(\varphi^*(p_1+p_j))$ is a $(g-1)$-th root of $\omega_X$. If $g = 3$, they are the 28 odd theta characteristics of the hyperelliptic curve $X$. Thus $\overline{J}_g$ is a compactification of $H_g$ given in terms of limits of configurations of higher spin curves of order $g - 1$, in the sense of [CCC].

In [H] and [AL], the authors construct a geometrical meaningful morphism $F_g : N_{0,2g+2} \to B_{2g+2}$. In Theorem 3.3 Theorem 4.3 and Theorem 4.7 we construct rational maps $\beta_g : N_{0,2g+2} \dashrightarrow \overline{J}_g \dashrightarrow B_{2g+2}$, giving a factorization of $F_g$. The
construction of $\alpha_g$ follows from Lemma 3.2 proving that it is possible to recover $(C, p_1, \ldots, p_{2g+2})$, $C$ smooth conic and $p_i \in C$, from its configuration of lines. In particular, Lemma 3.2 extends the results of [CS1] and [L], stating that a smooth plane quartic can be recovered from its bitangents, to double conics. In fact, the stable reduction of a general one-parameter deformation of a double conic $C$ is an hyperelliptic curve, the double cover of $C$ branched at 8 points. The limits of the bitangents give rise to the configuration of lines associated to $C$ and the 8 points. We point out that a different generalization of bitangents for any genus is theta hyperplanes, used in [CS2] and [GS].

In short, in Section 2 we show properties of twisters of curves. In Section 3, we construct $J_g$ and the map $\alpha_g$. In Section 4, we construct the map $\beta_g$, showing that $\alpha_g$ and $\beta_g$ provide a factorization of $F_g$.

1.1. Notation. We work over $\mathbb{C}$. A family of curves is a proper and flat morphism $f : C \to B$ whose fibers are curves. If $0$ is a point of a scheme $B$, set $B^* := B - 0$. A smoothing of a curve $C$ is a family $f : C \to B$, where $B$ is a smooth, connected, affine curve of finite type, with a distinguished point $0 \in B$, such that $f^{-1}(0)$ is isomorphic to $C$ and $f^{-1}(b)$ is smooth for $b \in B^*$. A general smoothing is a smoothing with smooth total space. Let $Y$ be a scheme and $X \to Y$ be a $Y$-scheme. Denote by $Sym^m_X := X^m / S_m$ the quotient of $X^m = X \times Y X \times Y \cdots \times Y X$ (the $m$-fiber product) by the symmetric group $S_m$. If a group $G$ acts on a variety $X$, denote by $X^{ss}$ the set of GIT-semistable points. If $p, q \in \mathbb{P}^2$, $p \neq q$, we set $\overline{pq} = \text{span}(p, q)$. For a positive integer $g$, set $m_g = 2g + 2$ and $h_g = \binom{m_g}{2}$.

2. On some properties of conic twisters

A twister of a curve $Y$ is a $T \in \text{Pic}(Y)$ such that there exists a smoothing $Y'$ of $Y'$ such that $T \simeq O_Y(D) \otimes O_Y$, where $D$ is a Cartier divisor of $Y'$ supported on irreducible components of $Y$. If $Y$ is of compact type, it is well-known that a twister depends only on its multidegree. A conic twister is a twister whose degrees on the irreducible components are positive and sum up to 2.

**Proposition 2.1.** Let $Y$ be a genus zero curve and $T$ a conic twister of $Y$. Then:

(i) if $d_1, \ldots, d_N$ are positive integers summing up to 2, then there exists a conic twister $T$ of $Y$ such that $\deg (\omega_Y^N \otimes T) = (d_1, \ldots, d_N)$;

(ii) the linear system $[\omega_Y^N \otimes T]$ is base point free, two-dimensional and induces a morphism $Y \to \mathbb{P}^2$ realizing $Y$ as plane conic.

**Proof.** (i) Given two components $Y_1, Y_2 \subset Y$ such that $Y_1 \cap Y_2 = \{p_1 \sim p_2\}$, the class $[p_1] - [p_2]$ is a twister $T$ such that $T|_{Y_1} = O_{Y_1}(1)$, $T|_{Y_2} = O_{Y_2}(-1)$ and $T$ is trivial on the other components of $Y$; the claim follows from the connectivity of $Y$.

(ii) If $\deg Y_i T = 2$ for some $Y_i \subset Y$, then $|O_{Y_i}(T|_{Y_i})| = |O_{\mathbb{P}^2}(2)| \simeq \mathbb{P}^2$, and the map $Y_1 \to \mathbb{P}^2$ is degree 2. This map extends to $Y$ because the dual graph of the components of $Y$ is a tree. If $\deg Y_1 T = \deg Y_2 T = 1$ for some $Y_1, Y_2 \subset Y$, let $\Gamma$ be the unique path in the dual graph of the components of $Y$ connecting $Y_1, Y_2$, and let $p_1 \in Y_1, p_2 \in Y_2$ be the unique points in $Y_1, Y_2$ which sit on the dual of $\Gamma$ in $Y$. Note that $T$ induces isomorphisms $\pi_i : Y_i \simeq \mathbb{P}^1 \cup_{\pi_i(p_1) \sim \pi_i(p_2)} \mathbb{P}^1$ for $i = 1, 2$. Since the dual graph of the components of $Y$ is a tree, there is a unique map $\pi : Y \to \mathbb{P}^1 \cup_{\pi_1(p_1) \sim \pi_2(p_2)} \mathbb{P}^1$ extending $\pi_1, \pi_2$. Since up to projective morphisms there is a unique embedding $\mathbb{P}^1 \cup_{\pi_1(p_1) \sim \pi_2(p_2)} \mathbb{P}^1 \to \mathbb{P}^2$, we are done. □
3. The first map

Let \( C \to \mathbb{P}^5 \cong |O_{\mathbb{P}^2}(2)| \) be the universal plane conic. For any integer \( m \geq 2 \), consider the variety \( \text{Sym}^m C \) and the morphism \( \rho : \text{Sym}^m C \to \mathbb{P}^5 \). If \( k \in \text{Sym}^m C \), let \( \text{supp}(k) \) be the conic parametrized by \( \rho(k) \). The points of \( k \) are called markings and \( (k, \text{supp}(k)) \), a conic with markings. A marking has a weight, i.e. the number of times it appears in \( k \). We call the markings \( p_{\min} \) and \( p_{\max} \) with minimal and maximal weight, the minimal and maximal markings of \( k \). Recall that \( m_g = 2g + 2 \) and \( h_g = \binom{m_g}{2} \), where \( g \geq 3 \). Set \( \mathbb{P}_{h_g} = \text{Sym}^{h_g}(\mathbb{P}^2)^\vee \). Consider the rational map:

\[
\psi : \text{Sym}^m C \dashrightarrow \mathbb{P}_{h_g}
\]

where, if \( k \) has markings \( \{p_i\}_{1 \leq i \leq s} \) of weight \( m_i = 1 \) and \( \{p_i\}_{s < i \leq r} \) of weights \( m_i > 1 \), then:

\[
\psi(k) = (\cdots \overbrace{p_1, \cdots, p_1}^{m_1, m_1 \text{ times, } \frac{s_i}{m_i}} \cdots \overbrace{p_{s_i}, \cdots, p_{s_i}}^{m_{s_i}, m_{s_i} \text{ times, } \frac{s_i}{m_i}} \cdots )_{1 \leq s_i < \frac{s_i}{m_i} \leq r}.
\]

Let \( \Gamma_\psi \) be the closure in \( \text{Sym}^m C \times \mathbb{P}_{h_g} \) of the graph of \( \psi \) and \( \psi : \Gamma_\psi \to \mathbb{P}_{h_g} \) be the second projection. Consider the GIT quotient:

\[
q : p(\Gamma_\psi) \cap \mathbb{P}_{h_g}^{ss} \longrightarrow \mathcal{G} = (p(\Gamma_\psi) \cap \mathbb{P}_{h_g}^{ss})/\text{SL}(3).
\]

We say that \( k \) is degenerate if \( \text{supp}(k) \) is not integral. Consider the open subset \( V \) of \( p(\Gamma_\psi) \cap \mathbb{P}_{h_g}^{ss} \) defined as:

\[
V = \{ r \in p(\Gamma_\psi) \cap \mathbb{P}_{h_g}^{ss} : r \neq p(k, r) \ \forall \ k \ \text{degenerate} \} \subset p(\Gamma_\psi) \cap \mathbb{P}_{h_g}^{ss}
\]

Recall the Hilbert-Mumford criterion [MFK Proposition 4.3] for configurations of plane lines. Let \( r \) be in \( \mathbb{P}_{h_g} \). For a point \( p \in \mathbb{P}^2 \), let \( \mu_p(r) \) be the number of lines of \( r \), with multiplicities, containing \( p \). Let \( \mu_i(r) \) be the multiplicity of a line \( l \) of \( r \). Then \( r \) is GIT-semistable if \( \max_{p \in \mathbb{P}^2} \mu_p(r) \leq 2h_g/3 \) and \( \max_{l \in \mathbb{P}^2} \mu_l(r) \leq h_g/3 \).

Lemma 3.1. Let \( (k, r) \in \Gamma_\psi \). Then:

(i) if \( k \) has a marking \( q \) of weight at least \( g + 1 \), then \( r \) is not GIT-semistable;

(ii) if \( \text{supp}(k) \) is reducible and the set of markings on smooth points of a component is one marking of weight 1, then \( r \) is not GIT-semistable;

(iii) if \( \text{supp}(k) \) is integral and the markings have weight 1, then \( r \in V \).

Proof. (i) We have \( \max_{p \in \mathbb{P}^2} \mu_p(r) \geq \mu_q(r) \geq \binom{g+1}{2} + (g+1)^2 > 2h_g/3 \).

(ii) From (i), we can assume that the node \( n \) of \( \text{supp}(k) \) is a marking of weight at most \( g \). The number of lines of \( r \) not containing \( n \) is at most \( 2g + 1 \) and hence \( \max_{p \in \mathbb{P}^2} \mu_p(r) \geq \mu_n(r) \geq h_g - 2g - 1 > 2h_g/3 \).

(iii) We have that \( r \) is GIT-semistable, because \( \max_{p \in \mathbb{P}^2} \mu_p(r) = 2g+1 < 2h_g/3 \) and \( \max_{l \in \mathbb{P}^2} \mu_l(r) = 1 < h_g/3 \). The property \( \max_{l \in \mathbb{P}^2} \mu_l(r) = 1 \) characterizes the configurations of integral cones with markings of weight 1, hence \( r \in V \).

Lemma 3.2. Consider the rational map \( \psi : \text{Sym}^m C \dashrightarrow \mathbb{P}_{h_g} \). Then the restricted morphism \( \psi^{-1}(V) \to V \) is injective for every \( g \geq 3 \).

Proof. Pick \( k \in \text{Sym}^m C \), where \( m \geq 2 \). Let \( C = \text{supp} k \) be integral. Set \( r = \psi(k) \), as in (3.2). Let \( \{m_1, \ldots, m_r\} \) be the set of the weights of \( k \), where \( m_i \leq m_{i+1} \).

Step 1. Assume that \( \{m_1, \ldots, m_r\} \neq \{1, 1\} \). The goal of the first step is to recover the maximal markings of \( k \) and their weights. We claim that the maximal markings of \( k \) are the points \( p \in \mathbb{P}^2 \) with maximum multiplicity \( \mu_p(r) \). It is easy if \( m_i = 1 \),
for $1 \leq i \leq r$, thus assume that $m_{\text{max}} = \max \{m_i\}_{1 \leq i \leq r} \geq 2$. If $p \in C$, then $\mu_p(r) \leq \mu_{p_{\text{max}}}(r)$, with the equality iff $p$ is a maximal marking of $k$ and we are done. If $p \notin C$, take two markings $p_i, p_j \in C$ of $k$ of weights $m_i$ and $m_j$ such that $p \in \overline{p_i p_j}$ if $p_i \neq p_j$ and $p \in \mathcal{T}_{p,C}$ if $p_i = p_j$. Thus:

i) if $p_i \neq p_j$ and $p_i, p_j \neq p_{\text{max}}$, then $\mu_{p_i p_j}(r) = m_i m_j < m_{\text{max}}(m_i + m_j) = \mu_{p_{\text{max}}}(r) + \mu_{p_{\text{max}}}(r);$

ii) if $p_j = p_{\text{max}}$ and $p_i \neq p_{\text{max}}$, then $\mu_{p_i p_{\text{max}}}(r) < \mu_{p_{\text{max}}}(r) + \mu_{p_{\text{max}}}(r);$

iii) if $p_i = p_j \neq p_{\text{max}}$, then $\mu_{p_i p_{\text{max}}}(r) = (m_i m_j) < m_i m_{\text{max}} = \mu_{p_{\text{max}}}(r);$

iv) two lines $\overline{p_i p_{\text{max}}}$ in two different cases among i) ii) iii) cannot be the same;

v) if $p_i = p_j = p_{\text{max}}$, then $\mathcal{T}_{p_{\text{max}} C}$ contains both $p_{\text{max}}$ and $p$. The case ii) does not hold. If $\overline{p_{\text{max}} C}$ is the only line of $r$ containing $p$, then $\mu_p(r) = \mu_{p_{\text{max}} C}(r) < \mu_{p_{\text{max}} C}(r) + \mu_{p_{\text{max}}}(r) \leq \mu_{p_{\text{max}}}(r)$ for some $C \ni p_k \neq p_{\text{max}}$. If $p$ is in at least two lines of $r$, then at least one case i) or iii) holds.

This shows that $\mu_p(r) < \mu_{p_{\text{max}}}(r)$. Thus, we recover the maximal markings of $k$ as the points of $\mathbb{P}^2$ with maximum multiplicity. In particular, we find also the number $N$ of maximal markings of $k$. To recover their weight $m_{\text{max}}$, consider $m = \sum_{1 \leq i \leq r} m_i$, and the subconfiguration $r'$ of $r$ of the lines containing no maximal markings of $k$. If $r' = \emptyset$, then $r'$ is the configuration of lines associated to $(k', C)$, where $k'$ are the non-maximal markings of $k$. We know the sum of the multiplicities of the lines of $r'$, thus we know also the sum $m'$ of the weights of $k'$ and $m_{\text{max}} = (m - m')/N$. If $r' = \emptyset$, then either $m_i = m_j$ for $1 \leq i, j \leq r$ and $m_{\text{max}} = m/N$, or $m_1 = 1 < m_i = m_j$ for $1 \leq i, j \leq r$ and $m_{\text{max}} = (m - 1)/N$.

**Step 2.** Pick $k \in \psi^{-1}(V)$ for $m = 2g + 2$. We recover the markings of $k$, with the exception of the marking of multiplicity $m_1$, if $m_1 = 1 < m_2$, and of multiplicity $m_1$ and $m_2$, if $m_1 = m_2 = 1 < m_3$. In fact, using Step 1 we find the maximal markings of $k$ and their weights. Now, consider $(k', C)$, where $k'$ are the non-maximal markings of $k$. We find the maximal markings of $k'$ and their weights, using Step 1. By iterating, we find the configuration $r_0$ associated to the markings of $k$ with minimal weights. If either $m_1 = m_2 = 1$ or $m_1 = m_2 = m_3 = 1$, we find the markings of $k$ and their weights. Otherwise, let $\{p_1, \ldots, p_s\}$ be the set of the recovered markings, where $s \geq 2$, by Lemma 3.1 (i). Consider the subconfiguration $r''$ of $r$ obtained by getting rid of the lines containing $p_3, \ldots, p_s$ and $\overline{p_1 p_2}$. We have three cases. In the first case, $r_0 = \emptyset$ and $m_1 = 1 < m_2$. In the second case, $r_0$ is a line $l$ of multiplicity $\mu_l(r_0) > 1$ and $3 \leq m_1 < m_2$. The marking with weight $m_1$ is the point contained in 3 lines of $r''$. We recover also its weight $m_1$, because $\mu_{l}(r_0) = \binom{m_1}{2}$. In the third case, $r_0$ is a line $l$ of multiplicity $\mu_l(r_0) = 1$ and either $m_1 = 2 < m_2$ or $m_1 = m_2 = 1 < m_3$. We have $m_1 = 2 < m_2$ iff $r''$ has 5 lines. The marking with weight 2 is the unique point contained in 3 lines of $r''$.

**Step 3.** Pick $k \in \psi^{-1}(V)$ for $m = 2g + 2$. If, using Step 2, we find at least 5 markings, we recover $(k, C)$, because we find also the markings with multiplicity 1 as the points on $C$ with multiplicity $2g + 1$. Assume that, using Step 2, we find at most 4 markings and their weights sum up to $2g + 2$, i.e. they are all the markings of $k$. They are at least 3 markings of weights at least 2, by Lemma 3.1 (i) and Step 2. Their tangents are the lines which do not contain a pair of markings. We find at least 3 markings and 3 tangents to the markings, then we recover $(k, C)$. If, using Step 2, we find at most 4 markings of $k$ and their weights do not sum up to $2g + 2$, do
then either \( m_1 = 1 < m_2 \) or \( m_1 = m_2 = 1 < m_3 \). There are 5 cases. Notice that \( k \) has at least 4 markings, by Lemma 3.1 (i).

a) 2 recovered markings and the weights are \( \{1, 1, m_3, m_4\} \), \( 2 \leq m_3 \leq m_4 \).

Let \( p_3, p_4 \) be the markings with multiplicities \( m_3, m_4 \). If \( m_3 \neq 3 \), then \( \mathcal{T}_{p_3} C \) is the line through \( p_3 \) not containing \( p_4 \) and whose multiplicity is not \( m_3 \). Similarly, we determine also \( \mathcal{T}_{p_4} C \). Consider the 4 lines of the configuration \( r \) different from \( \mathcal{T}_{p_3} C, \mathcal{T}_{p_4} C, \mathcal{P}_{p_2} p_3 \) and containing either \( p_3 \) or \( p_4 \). The pairwise intersections of these lines are 6 points: two points of multiplicity \( 2g + 1 \), two of multiplicity \( 2g \) and \( p_3, p_4 \). The points of multiplicity \( 2g + 1 \) are the markings with weight 1. Thus we recover 4 markings and 2 tangents to the markings, hence also \( (k, \text{supp} \ k) \). If \( m_3 = 3 \), then also \( m_4 = 3 \) from Lemma 3.1 (i) and \( g = 3 \). It is easy to see that there is only one conic with markings having \( r \) as associated configuration.

b) 3 recovered markings and the weights are \( \{1, m_2, m_3, m_4\} \), \( 2 \leq m_2 \leq m_3 \leq m_4 \).

Let \( p_1 \) have weight \( m_1 \). Assume that two weights are not equal to 3, e.g. \( m_2, m_3 \neq 3 \). Consider the subconfiguration \( r' \) of \( r \) of the lines not containing \( p_1 \) and different from \( \mathcal{P}_{p_2} p_3 \). The marking with weight 1 is the point \( p \neq p_2, p_3 \) with multiplicity \( \mu_p(r') = m_2 + m_3 \). If two weights are equal to 3, then \( m_2 = m_3 = 3 \), \( m_4 = 2g - 5 \). If \( m_4 \neq 3 \), then \( \mathcal{T}_{p_3} C \) is the unique line containing \( p_4 \) with multiplicity \( \binom{m_1}{2} \) and different from \( \mathcal{P}_{p_2} p_3, \mathcal{P}_{p_3} p_4, \mathcal{P}_{p_4} p_3 \). Consider the subconfiguration \( r' \) of \( r \) obtained by getting rid of \( \mathcal{T}_{p_3} C \). Then \( r' \) is the configuration of \((k', C)\), where \( k' \) are the markings of \( k \) with set of weights \( \{1, 1, 3, 3\} \), where \( p_4 \) has weight 1 in \( k' \). Using (a), we recover \((k', C)\), hence also the original \((k, C)\). If \( m_2 = m_3 = m_4 = 3 \), it is easy to see that there is only one conic with markings having \( r \) as associated configuration.

c) 3 recovered markings and the weights are \( \{1, m_1, m_3, m_5\} \), \( 2 \leq m_3 \leq m_4 \leq m_5 \).

There is a marking of weight different from 3, otherwise \( 2g + 2 = 11 \), for example \( m_3 \neq 3 \). Let \( r' \) be the subconfiguration of \( r \) obtained by getting rid of lines containing the points with weights \( m_4, m_5 \). Thus the markings with weight 1 are the points \( p, q \) of multiplicity \( \mu_p(r') = \mu_q(r') = m_3 + 1 \). We recover \((k, C)\).

d) 4 recovered markings and the weights are \( \{1, m_2, m_3, m_4, m_5\} \), \( 2 \leq m_2 \cdots \leq m_5 \).

Consider the subconfiguration \( r' \) of \( r \) obtained by getting rid of the lines connecting any two points with multiplicity \( > 1 \). Then the point with multiplicity 1 is the only point contained in 4 lines of \( r' \). We recover \((k, C)\).

e) 4 recovered markings and the weights are \( \{1, 1, m_3, \ldots, m_6\} \), \( 2 \leq m_3 \cdots \leq m_6 \).

If a marking has a weight different form 3, we argue as in c). Otherwise, consider the subconfiguration \( r' \) of \( r \) obtained by getting rid of the lines containing the markings with weight \( 5 \), \( 6 \). Thus \( r' \) is the configuration of \((k', C)\), where \( k' \) has weights \( \{1, 1, 3, 3\} \) and we argue as in a).

Denote by \( J_g \) the subset of \( \overline{\mathcal{J}_g} \) corresponding to the classes of configurations of lines of integral conics with markings of weight 1.

**Theorem 3.3.** There exists a map \( \alpha_g : J_g \rightarrow \overline{\mathcal{B}_{m_g}} \) defined at least over \( J_g \).

**Proof.** Up to restrict \( V \), we have that \( \psi : \psi^{-1}(V) \rightarrow V \) is an isomorphism by Theorem 3.2. Pick \( k \in \psi^{-1}(V) \). Since \( \text{supp}(k) \) is irreducible, Lemma 3.1 (i) implies that \((k, \text{supp}(k))\) is a GIT-stable binary form. Thus \( \mathcal{U} \rightarrow \psi^{-1}(V) \simeq V \) is a family of stable binary forms, where \( \mathcal{U} = \{(k, p) : p \in \text{supp} \ k\} \subset \psi^{-1}(V) \times \mathbb{P}^2 \), hence we get a \( \text{SL}(3) \)-invariant morphism \( V \rightarrow \overline{\mathcal{B}_{m_g}} \), inducing the rational map \( \alpha_g : J_g \rightarrow \overline{\mathcal{B}_{m_g}} \).
To show that $\alpha_g$ is defined over $J_g$, it is enough to show that the differential of $\psi$ is injective over an irreducible conics with markings of weight 1. We show that, if $U$ is the open subset of $\text{Sym}^{m_\sigma}(\mathbb{P}^2)$ of $m_\sigma$ distinct points such that any three of them are not contained in a line, then the differential of $\psi: U \to \mathbb{P}_{h_g}$ is injective, where $\psi(p_1, \ldots, p_{m_\sigma}) = (\ldots, p_i p_j \ldots)_{1 \leq i < j \leq m_\sigma}$. If $k \in U$, set $X = \psi(k)$, the union of $h_g$ distinct lines. If $N_{X/P^2}$ is the normal sheaf of $X$ in $\mathbb{P}^2$, then:

$$T_k U = T_{p_1} \mathbb{P}^2 \oplus \cdots \oplus T_{p_{m_\sigma}} \mathbb{P}^2 \xrightarrow{d\psi} H^0(X, N_{X/P^2}).$$

For $v_i \in T_{p_i} \mathbb{P}^2$, let $d\psi(v_1, \ldots, v_{m_\sigma}) = 0$, i.e. it is the trivial embedded deformation of $X$, fixing all the components of $X$. This means that $v_i$ is contained in the lines of $X$ containing $p_i$. It is impossible, being $p_i$ contained in at least two lines of $X$. □

4. THE SECOND MAP AND THE FACTORIZATION

A family of $m$-pointed stable curves of genus zero is a family $f: \mathcal{Y} \to B$ of curves of genus zero with sections $\sigma_1, \ldots, \sigma_m$ of $f$ such that $(Y_b, \sigma_1(b), \ldots, \sigma_m(b))$ is a $m$-pointed stable curve of genus zero for $b \in B$. If $T$ is a conic twister of $Y$, let $\varphi_T: \mathcal{Y} \to \mathbb{P}^2$ be the morphism induced by $|\omega_\mathcal{Y} \otimes T|$ as in Lemma 2.1(ii).

**Definition 4.1.** Let $(Y, p_1, \ldots, p_m)$ be a $m$-pointed stable curve of genus zero. A connected subcurve $P \subset Y$ is a principal part if there exists a conic twister $T$ of $Y$ such that the point $k = (\varphi_T(p_1), \ldots, \varphi_T(p_m)) \in \text{Sym}^m_{\mathbb{P}^2} \mathcal{C}$ satisfies $\text{supp } k = \varphi_T(P) = \varphi_T(Y)$ and $\psi(k) \in \mathbb{P}_{h_g}^m$, where $\psi: \text{Sym}^m_{\mathbb{P}^2} \mathcal{C} \to \mathbb{P}_{h_g}$ is the map (3.1).

A principal part $P$ has at most two components. If $P$ is a principal part of $(Y, p_1, \ldots, p_m)$, the associated conic twister $T$ is uniquely determined by the condition $\varphi_T(P) = \varphi_T(Y)$. Since $P$ is connected, $\varphi_T(p_i)$ is not in the singular locus of $\varphi_T(P)$ and hence the map $\psi$ is defined over $k = (\varphi_T(p_1), \ldots, \varphi_T(p_m)).$

**Example 4.2.** Let $(Y, p_1, \ldots, p_{10})$ be a pointed stable curve where $Y = Y_1 \cup Y_2$. Assume that $p_1, \ldots, p_9 \in Y_1$ and $p_{10} \in Y_2$. Both $Y_1$ and $Y_2$ are principal parts. In fact, if we consider $k_1 = (\varphi_T(p_1), \ldots, \varphi_T(p_{10}))$, where $T_1 = \mathcal{O}_{\mathcal{Y}}$ and $T_2$ is the twister given by $Y_2$, then $\psi(k_1)$ and $\psi(k_2)$ are GIT-semistable.

We refer to [K] for a proof of the following Lemma.

**Lemma 4.3.** Let $[f: \mathcal{Y} \to B, \sigma_1, \ldots, \sigma_m]$ be a family of $m$-pointed stable curves of genus zero. Then there exists a unique family $[f': \mathcal{Y}' \to B, \sigma_1', \ldots, \sigma_{m-1}']$ and a $B$-morphism $h: \mathcal{Y} \to \mathcal{Y}'$ such that $h \circ \sigma_i = \sigma_i'$ for $i = 1, \ldots, m - 1$. If $E_b \subset f^{-1}(b)$ is the component with $\sigma_m(b) \in E_b$ and $h_b = h|_{f^{-1}(b)}: f^{-1}(b) \to (f')^{-1}(b)$, then:

(i) $h_b$ contracts $E_b$ iff $|E_b \cap (f^{-1}(b) - E_b)| + |E_b \cap \{\sigma_1(b), \ldots, \sigma_{m-1}(b)\}| \leq 2$;

(ii) if $h_b$ does not contract $E_b$, then $h_b$ is an isomorphism;

(iii) if $h_b$ contracts $E_b$, then $h_b|_{(f')^{-1}(b) - E_b}$ is an isomorphism.

**Lemma 4.4.** The subset of $M_{0, m_\sigma}$ of the curves with a principal part is an open subset containing the locus of the curves with at most two components.

**Proof.** Let $P$ be a principal part of $(Y, p_1, \ldots, p_{m_\sigma})$. Let $[f: \mathcal{Y} \to B, \sigma_1, \ldots, \sigma_{m_\sigma}]$ be a family of $m_\sigma$-pointed stable curves of genus zero with $Y = f^{-1}(0)$ and $p_i = \sigma_i(0)$ for $0 \in B$. Applying Lemma 4.3, we get a family $f': \mathcal{Y}' \to B$ with $P = (f')^{-1}(0)$ and a morphism $h: \mathcal{Y} \to \mathcal{Y}'$. Now, $P$ has at most two components, then up to shrinking $B$ to an open subset containing 0, all the fibers of $\mathcal{Y}'$ have at most two components.
The image of the map $\varphi : Y \to \mathbb{P}(H^0(f^*\omega_Y^\vee)^\vee)$ induced by $|\omega_Y^\vee|$ is a family of conics over $B$. The conic over $b \in B$ has markings $\gamma(b) = (\gamma_1(b), \ldots, \gamma_m(b))$, where $\gamma_i = \varphi \circ h \circ \sigma_i$. By constructing a marking is not a node of a fiber, hence $\psi : \text{Sym}_m^g C \to \mathbb{P}_h^g$ is defined over $\gamma(b)$. Now, $\psi \circ \gamma(0) \in \mathbb{P}_h^g$ because $P$ is a principal part. Thus, up to shrinking again $B$, we have $\psi \circ \gamma(b) \in \mathbb{P}_h^g$ for $b \in B$ and the fiber of $Y' \to B$ over $b$ is a principal part of $f^{-1}(b)$ and we are done.

We show that $Y$ has a principal part if it has at most two components. If $Y$ is irreducible and $T = \mathcal{O}_Y$, then $\psi(\varphi_T(p_1), \ldots, \varphi_T(p_{m_2}))$ is GIT-stable by Lemma 3.1 (iii), thus $Y$ is a principal part. If $Y = Y_1 \cup Y_2$, let $[f : Y \to B, \sigma_1, \ldots, \sigma_{m_2}]$ be a general smoothing of $(Y, p_1, \ldots, p_{m_2})$ and $C^* \to B^*$ be the family of conics given by $|T f^{-1}_1(B^*)|$. We have a map $\varphi : Y \to C^*$, which is an isomorphism away from $Y$. Now, $\psi(\varphi \circ \sigma_1(b), \ldots, \varphi \circ \sigma_{m_2}(b)) \in \mathbb{P}_h^g$ for $b \in B^*$ by Lemma 3.1 (iii). By the GIT-semistable replacement property, up to a finite base change totally ramified over $0 \in B$, we find a completion $f' : C \to B$ and $k \in \text{Sym}_m^g(C)$, where $C = (f')^{-1}(0)$ such that $\psi(k) \in \mathbb{P}_h^g$. If $\varphi$ induces a morphism $\varphi : Y \to C$, then $\varphi|_Y = \varphi_T$, where either $T = \mathcal{O}_Y$ or $T$ is given by $Y_1$ or $Y_2$ and $Y$ is a principal part. Otherwise, call $\tilde{\varphi} : \tilde{Y} \to C$ the regularization of $\varphi$. Let $\varphi$ be not defined at $p \in Y$ and $E$ be an exceptional components over $p$. Let $p$ be a smooth point of $Y$. Then $\tilde{\varphi}(E) \subset C$ is a component containing at one marking with weight 1 and $\psi(k) \notin \mathbb{P}_h^g$ by Lemma 3.1 (ii), a contradiction. Assume that $p_1, \ldots, p_t \in Y_1$ for $t \leq m_2/2$. Let $p$ be the node of $Y$. If $\tilde{\varphi}$ contracts $Y_2$, then $k$ has a marking with at least $m_2/2$ and $\psi(k) \notin \mathbb{P}_h^g$ by Lemma 3.1 (ii), a contradiction. If $Y_2$ is not contracted, then $C = \tilde{\varphi}(Y_2) \cup E$. If $q = \tilde{\varphi}(p_1) = \cdots = \tilde{\varphi}(p_t)$, then $k = (t, \tilde{\varphi}(p_{t+1}), \ldots, \tilde{\varphi}(p_{m_2}))$ and $\psi(k) \in \mathbb{P}_h^g$. Consider $T = \mathcal{O}_T$ and $k' = (\varphi_T(p_1), \ldots, \varphi_T(p_{m_2}))$. We have $\psi(k') \in \mathbb{P}_h^g$ because $\max_{p \in E} |\mu_p(\psi(k'))| \leq \max_{p \in \Delta} |\mu_p(\psi(k))| \leq |\mu_p(\psi(k))|$, hence $Y$ is a principal part. $\square$

Let $P_g \subset \mathcal{M}_{0,m_g}$ be the open subset of the curves with a principal part and $\overline{M}_{0,m_g} = \mathcal{M}_{0,m_g}/S_{m_g}$ the moduli space of $m_g$-marked stable curves, where $S_{m_g}$ is the symmetric group.

**Theorem 4.5.** There exists a map $\beta_g : \overline{N}_{0,m_g} \to \mathcal{T}_g$ defined at least over $P_g/S_m$.

**Proof.** First of all, we show that if $(Y, p_1, \ldots, p_{m_2}) \in \mathcal{M}_{0,m_g}$ has two principal parts $P_1$ and $P_2$, whose associated configurations of lines are $r_1$ and $r_2$, then:

$$O_{\text{SL}(3)}(r_1) \cap O_{\text{SL}(3)}(r_2) \cap \mathbb{P}_h^g \neq \emptyset,$$

where $O_{\text{SL}(3)}(\cdot)$ denotes the orbit under the action of $\text{SL}(3)$. In fact, consider a general smoothing $[f : Y \to B, \sigma_1, \ldots, \sigma_{m_2}]$ of $(Y, p_1, \ldots, p_{m_2})$. Fix $j = 1, 2$, let $T_j$ be the twist of $P_j$. The morphisms $\varphi_T$, induced by $|\omega_Y^\vee \otimes T_j|$ give rise to two families of conics, which are isomorphic away from the special fiber. Furthermore, $\{\varphi_{T_1} \circ \sigma_1\}$ and $\{\varphi_{T_2} \circ \sigma_1\}$ induce markings on the two families. By construction, the associated families of configurations of lines are $\text{SL}(3)$-conjugate over $B^*$ and $r_1$ and $r_2$ are their special fibers. Thus, $r_1$ and $r_2$ are GIT-semistable limits of conjugate families of GIT-stable configurations, hence (3.3) follows.

Now, let $\mathcal{P}_g$ be the functor of families of $m_g$-pointed stable curves with a principal part. We construct a functor trnasformation $\mathcal{P}_g \to \mathcal{M}_\text{or}(-, \mathcal{T}_g)$. For a scheme $B$, pick $[f : Y \to B, \sigma_1, \ldots, \sigma_{m_g}] \in \mathcal{P}_g(B)$. As in the proof of Lemma 4.4 (ii), we get an open covering $B = \cup B_h$ of $B$ and morphisms $t_h : B_h \to \mathbb{P}_h^g$ such that $t_h(b)$ is
Theorem 4.7. The variety $\mathcal{J}_g$ corresponding to a central vertex of $\Gamma$ is the set of connected subgraphs such that $\Gamma$ has a central vertex if and only if there are no edges $e$ of $\Gamma$ such that $\omega(e) = -2$ or of $\Gamma$ such that $\Gamma_1 \cup \Gamma_2 = \Gamma - e$. There exists at most one central vertex.

Let us recall how the morphism $F_g: \overline{N_{0,m_g}} \to \overline{B_{m_g}}$ is defined in [AL]. Let $(Y,p_1,\ldots,p_{m_g}) \in \overline{N_{0,m_g}}$. The weighted dual tree of $Y$ is the dual graph $\Gamma_Y$ of $Y$ and, for each vertex, the number of marked points contained in the corresponding component. If $\Gamma$ is a subset of $\Gamma_Y$, let $\omega(\Gamma)$ be the sum of the weights of the vertices contained in $\Gamma$. We say that $Y$ has a central vertex $v \in \Gamma_Y$ if $\omega(\Gamma) < m_g/2$ for every connected subsets $\Gamma$ of $Y$. The following is [AL] Lemma 3.2.

Lemma 4.6. Let $(Y,p_1,\ldots,p_{m_g}) \in \overline{N_{0,m_g}}$. Then $Y$ has a central vertex if and only if there are no edges $e$ of $\Gamma$ such that $\omega(\Gamma_1) = \omega(\Gamma_2) = m_g/2$, where $\Gamma_1,\Gamma_2$ are the connected subgraphs such that $\Gamma_1 \cup \Gamma_2 = \Gamma - e$. There exists at most one central vertex.

A marked stable curve has a central vertex if it is not contained in the divisor $\Delta$ of $\overline{N_{0,m_g}}$ whose general point has two components containing $g+1$ marked points. If the central vertex exists, then $F_g(Y,p_1,\ldots,p_{2g+2})$ is obtained by contracting all the components of $Y$, which do not correspond to the central vertex. If $Y$ has no central vertex, it is easy to see that the edge disconnecting $\Gamma_Y$ in two subgraphs with weights $g+1$ is unique. We call it the central edge of $\Gamma$.

Theorem 4.7. The variety $\mathcal{J}_g$ is a compactification of $H_g$ and the chain of maps $\overline{N_{0,m_g}} \to \mathcal{J}_g \to \overline{B_{m_g}}$ gives a rational factorization of $F_g: \overline{N_{0,m_g}} \to \overline{B_{m_g}}$.

Proof. The morphism $\beta_g$ restricts to an injection $H_g \to \mathcal{J}_g$ whose image is the subset $J_g$ of Theorem 3.3. The inverse is the morphism of Theorem 3.3.

To prove the factorization, it is enough to show that, if $P$ is an irreducible principal part of a pointed stable curve $(Y,p_1,\ldots,p_{m_g})$, then $P$ is the component corresponding to a central vertex of $\Gamma_Y$. First of all, assume that $Y$ has a (unique) central vertex $v$ and let $C \subset Y$ be the corresponding component. Assume that $P \neq C$ and let $v_P$ be the vertex of $P$. There is an edge $e$ of $\Gamma_Y$ such that, if $\Gamma_1,\Gamma_2$ are the connected subgraphs with $\Gamma_1 \cup \Gamma_2 = \Gamma_Y - e$, then $v_P \in \Gamma_1$ and $v \in \Gamma_2$. Thus $\omega(\Gamma_1) < m_g/2$ by definition of central vertex. Let $T$ be the conic twister such that $\omega_T \otimes T$ has degree 2 on $P$. Now, $\psi(\varphi_T(p_1),\ldots,\varphi_T(p_{m_g})) \in \mathbb{P}^g_{h^+_g}$, by definition of principal part. Since $\varphi_T$ contracts the connected subcurve of $Y$ corresponding to $\Gamma_2$ to a unique marking, we have $\omega(\Gamma_2) < m_g/2$ by Lemma 3.1 (i). Then $m_g = \omega(\Gamma) = \omega(\Gamma_1) + \omega(\Gamma_2) < m_g$, a contradiction. If $Y$ has no central edge, let $e$ be the central edge of $Y$. Set $\Gamma_1 \cup \Gamma_2 = \Gamma_Y - e$ for connected graphs $\Gamma_1,\Gamma_2$. If $T$ is the twister of $P$, then $\varphi_T$ contracts either the component of $\Gamma_1$ or of $\Gamma_2$. Now, $\psi(\varphi_T(p_1),\ldots,\varphi_T(p_{m_g})) \in \mathbb{P}^g_{h^+_g}$ and $(\varphi_T(p_1),\ldots,\varphi_T(p_{m_g}))$ has a marking of weight at least $g+1$, because $\omega(\Gamma_1) = \omega(\Gamma_2) = g + 1$, contradicting Lemma 3.1 (i).

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