Time-scales Herglotz type Noether theorem for delta derivatives of Birkhoffian systems

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The time-scales theory provides a powerful theoretical tool for studying differential and difference equations simultaneously. With regard to Herglotz type variational principle, this generalized variational principle can deal with non-conservative or dissipative problems. Combining the two tools, this paper aims to study time-scales Herglotz type Noether theorem for delta derivatives of Birkhoffian systems. We introduce the time-scales Herglotz type variational problem of Birkhoffian systems firstly and give the form of time-scales Pfaff–Herglotz action for delta derivatives. Then, time-scales Herglotz type Birkhoff’s equations for delta derivatives are derived by calculating the variation of the action. Furthermore, time-scales Herglotz type Noether symmetry for delta derivatives of Birkhoffian systems are defined. According to this definition, time-scales Herglotz type Noether identity and Noether theorem for delta derivatives of Birkhoffian systems are proposed and proved, which can become the ones for delta derivatives of Hamiltonian systems or Lagrangian systems in some special cases. Therefore, it is shown that the results of Birkhoffian formalism are more universal than Hamiltonian or Lagrangian formalism. Finally, the time-scales damped oscillator and a non-Hamiltonian Birkhoffian system are given to exemplify the superiority of the results.

1. Introduction

In 1988, Hilger proposed the definition of a time scale \( \mathbb{T} \), which is an arbitrary non-empty closed subset of the real numbers \( \mathbb{R} \), in order to analyse continuous and discrete systems uniformly [1]. For instance, if we choose a continuous time scale, i.e. \( \mathbb{T} = \mathbb{R} \), this time-scale calculus is the same as the calculus of the
classical continuous system; if $T = \mathbb{Z}$, the calculus is changed to that of the discrete system with step size $\mu = 1$; if $T = \mathbb{R}^n$ ($q > 1$), this calculus can solve the problems of quantum systems. Therefore, compared with a single scale, more general results can be obtained based on different time scales. Moreover, the physical essence of those systems can be depicted more accurately by using the time scales theory. The time-scales dynamic equations can provide mathematical models for some processes dependent on continuous-time variables, discrete-time variables and piecewise continuous-time variables, such as the logistic model in biology and the cobweb model in economics [1,2]. Thus it can be seen that the time-scales theory has important theoretical significance and extensive application prospect in various fields [3–6]. Bartosiewicz & Torres [7] found the Noether conserved quantity for delta derivatives based on the time-scales calculus of variation in 2008. It is well known that Noether theorem reveals that conservation quantities of mechanics are directly related to the invariance of actions under infinitesimal transformations. Time-scales Noether theorems are proved not only for delta derivatives, but for nabla derivatives by Martins & Torres [8] in 2010. After that, Malinowska & Martins [9] put forward the second time-scales Noether theorem for delta derivatives in 2013. In the same year, time-scales Noether theorem for delta derivatives of non-conservative non-holonomic systems was studied by Cai et al. [10]. Then, the study of time-scales symmetries and conservation quantities was extended to Birkhoffian systems [11] and Hamiltonian systems [12–14].

In the majority of above articles, their variational principles are the classical extremum principles, for example, the famous Hamilton principle, whose action is defined by an integral. However, in general, the Hamilton principle of non-conservative systems is an instability action principle, because the absence of a functional makes its variation equal to zero [15]. Whereas, Heriglotz type variational principle can deal with this problem to give the variational description for non-conservative systems by the action functional defined by a differential equation [16]. Based on the Heriglotz variational problem, Lagrangians and Hamiltonians with physical meaning can be established for non-conservative systems. Lazo et al. obtained the generalized Einstein’s field equations for a non-conservative gravity by using the Lagrangian of Heriglotz type and applied them to cosmology and gravitational waves [17]. In addition, they constructed Lagrangians of Heriglotz type with physical meaning, such as vibrating string under viscous forces, non-conservative electromagnetic theory, non-conservative Schrödinger equation and Klein–Gordon equation, to describe non-conservative systems and quantum systems [18]. Moreover, when these functions do not depend on the action functional, Heriglotz variational principle can be reduced to the classical integral variational principle, which can deal with conservative problems. Since Heriglotz type variational principle provides a new method for studying non-conservative systems, Heriglotz type Noether theorems of mechanical systems have been investigated in recent decades, including non-conservative Lagrangian systems [19,20], non-conservative Hamiltonian systems [21], Birkhoffian systems [15,22], non-conservative non-holonomic systems [23] and other complex systems [24–31]. But so far, time-scales Heriglotz variational principle is rarely studied, and the results are limited to Lagrangian formalism [32,33] and Hamiltonian formalism [34].

In 1927, Birkhoff, Poincare’s successor, proposed a new type of integral variational principle and a new set of differential equations of motion [35]. The new variational principle was named as Pfaff–Birkhoff principle, and the new equations were called Birkhoff’s equations by Santilli [36]. And a mechanical system that describes motion or a physical system that describes state with Birkhoff’s equations was called a Birkhoffian system, so Birkhoffian mechanics was born. The new mechanics has a number of nice properties. For example, Birkhoff’s equations are not only self-adjoint but also autonomous, and semi-autonomous Birkhoffian systems have Lie algebraic structures and exact symplectic forms [37]. Thus, Birkhoffian mechanics has developed rapidly and has wide applications in many fields, for instance, hadron physics, statistical mechanics, engineering mechanics and biophysics [36]. Another thing to notice is that a Birkhoffian system is a more extensive mechanical system, which can be applicable to the Lagrangian system, Hamiltonian system, holonomic system and non-holonomic system [38].

The pervasiveness of Birkhoffian system motivates us to study time-scales Heriglotz variational principle of Birkhoffian systems and its Noether theorem. As we know, all conserved quantities in mechanics are directly related to the invariance of action under a series of infinitesimal transformations, such as energy conservation, momentum conservation and conservation of moment of momentum. The time-scales Heriglotz type Noether theorem of Birkhoffian systems has important practical applications for conservative and non-conservative processes in continuous and discrete cases, for example, finding a new solution from a known one, reducing equations, testing computer code and so on. The outline of this paper is as follows. In §2, the time-scales preliminaries of delta derivatives and exponential functions are recalled. Section 3 is our main results: firstly, we introduce the time-scales Heriglotz variational problem for delta derivatives of Birkhoffian systems; secondly, the time-scales Heriglotz type
Birkhoff’s equations for delta derivatives are deduced; then, the time-scales Herglotz type Noether identity and theorem for delta derivatives of Birkhoffian systems are formulated. In §4, the results of Hamiltonian systems and Lagrangian systems are listed to account for the relationship of Hamiltonian, Lagrangian and Birkhoffian systems. Section 5 gives the time-scales damped oscillator of Birkhoffian system and a non-Hamiltonian system as examples. Finally, we offer some conclusions in §6.

2. Time-scales preliminaries

A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the set $\mathbb{R}$ of real numbers. Let $\mathbb{T}$ be a time scale, for $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T} \in \mathbb{T}$; the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by $\rho(t) = \sup \{s \in \mathbb{T} : s < t\}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, if $\inf \mathbb{T} \in \mathbb{T}$. If $\sigma(t) > 0$, $\rho(t) = 0$, $\rho(t) > 0$, or $\rho(t) = 0$, then $t$ is called right-scattered, right-dense, left-scattered and left-dense, respectively. The graininess function $\mu : \mathbb{T} \rightarrow \mathbb{R}$ is defined by $\mu(t) = \sigma(t) - t$, $\mu(t) \geq 0$. For delta derivative, the set $\mathbb{T}^k$ is defined by $\mathbb{T}^k = \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T})$ if $\sup \mathbb{T} < \infty$, and $\mathbb{T}^k = \mathbb{T}$ if $\sup \mathbb{T} = \infty$. If $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function, then $f^\sigma : \mathbb{T} \rightarrow \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e. $f^\sigma = f \circ \sigma$.

**Definition 2.1.** Assume $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function and $t \in \mathbb{T}^k$. $f^\Delta(t)$ is called the delta derivative of $f$ at $t$ if for any given $\varepsilon > 0$, there is a neighbourhood $U$ of $t$ (i.e. $U = (t - \delta, t + \delta) \cap \mathbb{T}$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$ 

Generally, we can denote $f^\Delta(t)$ by $(\Delta f)(t)$. And we call $f$ delta differentiable on $\mathbb{T}^k$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}^k$. Note that if $\mathbb{T} = \mathbb{R}$, for any $t \in \mathbb{R}$, then $\sigma(t) = \rho(t) = t$, $\mu(t) \equiv 0$ and $f^\Delta(t) = f'(t)$. And if $\mathbb{T} = \mathbb{Z}$, for each $t \in \mathbb{Z}$, then $\sigma(t) = t + 1$, $\mu(t) \equiv 1$ and $f^\Delta(t) = f(t + 1) - f(t)$.

**Definition 2.2.** A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous provided it is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by $C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$. The set of functions $f : \mathbb{T} \rightarrow \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R})$.

**Definition 2.3.** A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called regulated provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The indefinite integral of a regulated function $f$ is defined by

$$\int f(t)\Delta t = F(t) + c,$$

where $c$ is an arbitrary constant. And the definite integral of $f$ is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all } a, b \in \mathbb{T}.$$ 

A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided

$$F^\Delta(t) = f(t)$$

holds for all $t \in \mathbb{T}^k$.

**Lemma 2.4.** Assume $f, g : \mathbb{T} \rightarrow \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$. The properties of delta derivatives are:

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t);$$

$$c^\Delta(t) = c f^\Delta(t), \quad c \in \mathbb{R};$$

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t);$$

and

$$f^\Delta_{\frac{h}{t}}(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}, \quad g(t)g^\sigma(t) \neq 0.$$

**Definition 2.5.** For $h > 0$, $C_h = \{x \in \mathbb{C} : x \neq -(1/h)\}$, let $Z_h$ be the strip $Z_h := \{x \in \mathbb{C} : - (\pi/h) < \text{Im}(x) \leq (\pi/h)\}$. The cylinder transformation $\xi_h : \mathbb{C}_h \rightarrow Z_h$ is defined by

$$\xi_h(x) = \frac{1}{h} \text{Log}(1 + xh).$$
Here, \( \log \) is a principal logarithm function. For \( h = 0 \), let \( Z_0 := \mathbb{C} \), then \( \xi_0(x) = x \) is defined for all \( x \in \mathbb{C} \).

**Definition 2.6.** A function \( \gamma : \mathbb{T} \rightarrow \mathbb{R} \) is regressive if \( 1 + \mu(t)\gamma(t) \neq 0 \) for all \( t_0 \in \mathbb{T}^\circ \) holds. If \( \gamma \in \mathcal{R} \), the exponential function is defined by

\[
e_s(t, s) = \exp \left( \int_s^t \xi_{\mu(t)}(\gamma(\theta)) \Delta \theta \right) \quad \text{for } s, t \in \mathbb{T}.
\]

Here, the set of rd-continuous and regressive functions \( f : \mathbb{T} \rightarrow \mathbb{R} \) are denoted by \( \mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}) \).

**Lemma 2.7.** If \( t, s, r \in \mathbb{T}, \gamma \in \mathcal{R}, \) and \( t_0 \in \mathbb{T} \) is fixed, we list the following properties of exponential functions:

\[
e_\gamma^\Delta(t, t_0) = \gamma(t) e_\gamma(t, t_0); \tag{2.5}
e_\gamma^\Delta(t, s) = e_\gamma(\sigma(t), s) = (1 + \mu(t)\gamma(t)) e_\gamma(t, s); \tag{2.6}
\]

\[
1 e_\gamma^\Delta(t, s) = e_\gamma(s, t); \tag{2.7}
e_\gamma^\Delta(t, s) e_\gamma(s, r) = e_\gamma(t, r); \tag{2.8}
\]

and

\[
[e_\gamma^\Delta(t, s)]^\gamma = \frac{\gamma}{e_\gamma^\Delta(t, s)}. \tag{2.9}
\]

**Lemma 2.8.** Suppose \( e_\gamma^\Delta(t, s) \) is regressive. Let \( t_1 \in \mathbb{T} \) and \( y_1 \in \mathbb{R} \), the unique solution of the initial value problem

\[
y^\Delta = \gamma(t)y + f(t) \quad \text{and } y(t_1) = y_1
\]

is given by

\[
y(t) = e_\gamma(t, t_1)y_1 + \int_{t_1}^t e_\gamma^\Delta(t, \theta) : f(\theta) \Delta \theta
\]

**Lemma 2.9.** Let \( g \in \mathcal{C}_\text{rd}, \ g : [a, b] \rightarrow \mathbb{R}^n \), then

\[
\int_a^b g^\gamma(t) \gamma^\Delta t = 0 \quad \text{for all } \eta \in \mathcal{C}_\text{rd}^1 \text{ with } \eta(a) = \eta(b) = 0
\]

holds if and only if

\[
g(t) = c \quad \text{for } c \in \mathbb{R}^n.
\]

The above definitions, lemmas and the specific proof processes of lemmas can be referred to in the literature [1].

### 3. Main results

First, we indicate that the time-scales Herglotz variational problem for delta derivatives of Birkhoffian systems is a functional extremum problem of determining the function \( a_v(t) \) that extremizes \( z(t_2) \), where the action \( z(t) \) is a solution of

\[
z^\Delta(t) = R_v(t, \nu, \omega; a_v(t), z(t)), \quad (\nu, \omega = 1, 2, \ldots, 2n) \tag{3.1}
\]

with the boundary conditions

\[
a_v(t)|_{t=t_1} = a_{1v}, \quad a_v(t)|_{t=t_2} = a_{2v}, \quad (\nu = 1, 2, \ldots, 2n)
\]

and the initial condition

\[
z(t)|_{t=t_0} = z_1. \tag{3.3}
\]

Here, \( a_v^\mu(t) = (a_{v \nu} \circ \sigma)(t), \ t \in \mathbb{T}, \ R_v : \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R} \rightarrow \mathbb{R} \) are the time-scales Herglotz type Birkhoff’s functions, and \( B : \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R} \rightarrow \mathbb{R} \) is the time-scales Herglotz type Birkhoffian. \( a_{1v}, a_{2v} \) and \( z_1 \) are constants.
**Definition 3.1.** The functional \( z \) determined by equation (3.1) is called the time-scales Pfaff–Herglotz action.

Next, we derive the time-scales Herglotz type Birkhoff’s equations. From the calculation of isochronous variation on both sides of equation (3.1), it follows that

\[
\delta z^\Delta = \left( \frac{\partial R_v}{\partial a_w^\Delta} \delta a_w^\Delta + \frac{\partial R_v}{\partial z} \delta z \right) a_w^\Delta + R_v \delta a_v^\Delta - \frac{\partial B}{\partial a_w^\Delta} \delta a_w^\Delta + \frac{\partial B}{\partial z} \delta z. \tag{3.4}
\]

Considering the exchange relationships [10]

\[
\frac{\Delta}{\Delta t} (\delta q) = \delta (\frac{\Delta}{\Delta t} q) = \delta q^\Delta \quad \text{and} \quad (\delta q)^\nu = \delta q^\nu.
\]

Formula (3.4) can be written as

\[
(\delta z)^\Delta = A(t) + \left( \frac{\partial R_v}{\partial z} a_w^\Delta - \frac{\partial B}{\partial a_w^\Delta} \right) \delta z, \tag{3.5}
\]

where

\[
A(t) = \left( \frac{\partial R_v}{\partial a_w^\Delta} a_w^\Delta + \frac{\partial B}{\partial a_w^\Delta} \right) \delta a_w^\Delta + R_v \delta a_v^\Delta. \tag{3.6}
\]

According to the condition (3.3), equation (3.5) satisfies the initial value condition

\[
\delta z(t_1) = 0. \tag{3.7}
\]

Let \( \gamma(t) = \frac{\partial R_v}{\partial z} a_w^\Delta - \frac{\partial B}{\partial a_w^\Delta} \), by lemma 2.8 and the properties (2.6), (2.7), (2.8), the solution of equations (3.5) and (3.7) is

\[
\delta z(t) = e_\gamma(t, t_1) \int_{t_1}^t e_\gamma^\Delta(t_1, \theta) \cdot A(\theta) \Delta \theta. \tag{3.8}
\]

From the boundary conditions (3.2), we have \( \delta z(t_2) = 0 \). And consider that the action \( z(t) \) yields its extremum at \( t = t_2 \), so that

\[
\int_{t_1}^{t_2} e_\gamma^\Delta(t_1, \theta) \Delta \theta = 0. \tag{3.9}
\]

Substituting formula (3.6) into equation (3.9), it follows that

\[
\int_{t_1}^{t_2} e_\gamma^\Delta(t_1, \theta) \left[ \left( \frac{\partial R_v}{\partial a_w^\Delta} a_w^\Delta - \frac{\partial B}{\partial a_w^\Delta} \right) \delta a_w^\Delta + R_v \delta a_v^\Delta \right] \Delta t = 0. \tag{3.10}
\]

From the property (2.3) of delta derivatives, we obtain

\[
\int_{t_1}^{t_2} e_\gamma^\Delta(t_1, \theta) \left( \frac{\partial R_v}{\partial a_w^\Delta} a_w^\Delta - \frac{\partial B}{\partial a_w^\Delta} \right) \delta a_w^\Delta \Delta t
\]

\[
= \int_{t_1}^{t_2} \left[ \left( \int_{t_1}^{t_2} e_\gamma^\Delta(t_1, \theta) \left( \frac{\partial R_v}{\partial a_w^\Delta} a_w^\Delta - \frac{\partial B}{\partial a_w^\Delta} \right) \Delta \theta \right) \delta a_w^\Delta \right] \Delta t
\]

\[
= \left( \int_{t_1}^{t_2} e_\gamma^\Delta(t_1, \theta) \left( \frac{\partial R_v}{\partial a_w^\Delta} a_w^\Delta - \frac{\partial B}{\partial a_w^\Delta} \right) \Delta \theta \right) \left( \delta a_w^\Delta \right)^2 \Delta t
\]

\[
= - \left( \int_{t_1}^{t_2} e_\gamma^\Delta(t_1, \theta) \left( \frac{\partial R_v}{\partial a_w^\Delta} a_w^\Delta - \frac{\partial B}{\partial a_w^\Delta} \right) \Delta \theta \right) \left( \delta a_w^\Delta \right)^2 \Delta t. \tag{3.11}
\]

Substituting formula (3.11) into equation (3.10), we have

\[
\int_{t_1}^{t_2} e_\gamma^\Delta(t_1, \theta) R_w - \int_{t_1}^{t_2} e_\gamma^\Delta(t_1, \theta) \left( \frac{\partial R_v}{\partial a_w^\Delta} a_w^\Delta - \frac{\partial B}{\partial a_w^\Delta} \right) \Delta \theta \left( \delta a_w^\Delta \right)^2 \Delta t = 0. \tag{3.12}
\]
By lemma 2.9, we obtain

$$e^\omega(t_1, t)R_w - \int_{t_1}^t e^\omega(t_1, \theta) \left( \frac{\partial R_w^a}{\partial a^w} a^\omega \Delta \theta \right) \Delta \theta = \text{const}, \quad (\omega = 1, 2, \ldots, 2n).$$  

(3.13)

By delta differentiation of both sides of equation (3.13), we derive the time-scales Herglotz type Birkhoff’s equations

$$\frac{\Delta}{\Delta t} \left[ e^\omega(t_1, t)R_w - e^\omega(t_1, t) \left( \frac{\partial R_w^a}{\partial a^w} a^\omega \right) \right] = 0, \quad (\omega = 1, 2, \ldots, 2n).$$  

(3.14)

**Remark 3.2.** If $T = R^n$, then $\sigma(t) = t$, $\mu(t) = 0$, $e^\omega(t_1, t) = \exp \left(-\int_{t_1}^t ((\partial R_w^a/\partial z) \bar{a}^w - (\partial B/\partial z)) d\theta \right)$. Thus, equations (3.14) become the time-scales Herglotz type Birkhoff’s equations in the continuous case [22]

$$\exp \left[-\int_{t_1}^t \left( \frac{\partial R_w^a}{\partial z} \bar{a}^w - \frac{\partial B}{\partial z} \right) d\theta \right] \left( \left( \frac{\partial R_w^a}{\partial a^w} \right) \bar{a}^w + \frac{\partial B}{\partial a^w} \frac{\partial R_w}{\partial t} \right) \bar{a}^w - \frac{\partial R_w^a}{\partial z} B + \frac{\partial B}{\partial z} \frac{\partial R_w}{\partial t} = 0,$$

$$ (\omega = 1, 2, \ldots, 2n).$$  

(3.15)

**Remark 3.3.** If the time-scales Herglotz type Birkhoffian and Birkhoff’s functions do not contain $z$, i.e. $z^\lambda(t) = R(t, a^w_\lambda(t), a^a_\lambda(t)) - B(t, a^a_\lambda(t))$, then $\gamma(t) = 0$, $e^\omega(t_1, t) = 1$. Thus, equations (3.14) change to the time-scales Birkhoff’s equations based on the traditional variational problem [11]

$$R^a_w - \frac{\partial R_w^a}{\partial a^w} a^\omega + \frac{\partial B}{\partial a^w} = 0, \quad (\omega = 1, 2, \ldots, 2n).$$  

(3.16)

Then, we study the time-scales Herglotz type Noether theorem for delta derivatives of Birkhoffian systems. Let $U$ be a set of $C^1_{\text{rd}}$ functions $a^\omega : [t_1, t_2] \rightarrow R^n$. We introduce the infinitesimal transformations of the one-parameter group with respect to time $t$ on $U$

$$\tilde{t} = t + \varepsilon \tau(t, a^\omega, z) \quad \text{and} \quad \tilde{a}^\omega(t) = a^\omega(t) + \varepsilon \tilde{\xi}(t, a^\omega, z),$$  

(3.17)

where $\tau$, $\xi$ are infinitesimal generators, and $\varepsilon$ is an infinitesimal parameter. Then, under the transformations (3.17), we can write the time-scales Pfaff–Herglotz action $z(t)$ as $\tilde{z}(t) = z(t) + \Delta z(t)$, where $\Delta$ denotes total variation. According to the literature [10], we know $\Delta \tilde{q} = \Delta q + \varepsilon^3 \Delta \tilde{t}$.

**Definition 3.4.** If the time-scales Pfaff–Herglotz action $z$ is acted on by the infinitesimal transformations (3.17), and $\Delta z(t_b) = 0$ holds for any subinterval $[t_a, t_b] \subseteq [t_1, t_2]$ with $t_a, t_b \in T$, then the invariance is called the time-scales Herglotz type Noether symmetry of Birkhoffian systems under the infinitesimal transformations.

**Theorem 3.5.** If the time-scales Pfaff–Herglotz action $z$ is invariant on $U$ under the infinitesimal transformations (3.17), then

$$\left( \frac{\partial R_w^a}{\partial t} a^\omega + \frac{\partial B}{\partial t} \right) \tau + \left( \frac{\partial R_w^a}{\partial a^w} a^\omega + \frac{\partial B}{\partial a^w} \right) \tilde{\xi}^a + R_w \tilde{\xi}^a + (\mu R_w^a a^\omega - B^a) \tau = 0$$  

holds for all $t \in [t_1, t_2]$. Formula (3.18) is called the time-scales Herglotz type Noether identity for delta derivatives of Birkhoffian systems.

**Proof.** On the basis of definition 3.4, we know $\Delta z(t_b) = 0$. Then, from equation (3.1), we obtain

$$\Delta z^\lambda = \tilde{\Delta} (R(a^\omega, a^a, B) = \tilde{\Delta} R(a^\omega, a^a) + \tilde{R}_w \tilde{\Delta} a^\omega - \tilde{B}$$

$$= \left( \frac{\partial R_w^a}{\partial t} \tilde{\Delta} t + \frac{\partial R_w^a}{\partial a^w} \tilde{\Delta} a^\omega + \frac{\partial R_w}{\partial z} \tilde{\Delta} z \right) a^\omega + R_w \tilde{\Delta} a^\omega$$

$$- \frac{\partial B}{\partial t} \tilde{\Delta} t - \frac{\partial B}{\partial a^w} \tilde{\Delta} a^\omega - \frac{\partial B}{\partial z} \tilde{\Delta} z.$$  

(3.19)
Because of the property (2.3), we can calculate

\[
\Delta z^\Delta = \frac{\Delta}{\Delta t} (\Delta z) - z^\Delta \frac{\Delta}{\Delta t} (\Delta t), \tag{3.20}
\]

Taking into account equations (3.20) and (3.1), formula (3.19) can change to

\[
\frac{\Delta}{\Delta t} (\Delta z) = \left( \frac{\partial R_v}{\partial t} \Delta t + \frac{\partial R_v}{\partial a^\sigma_w} \Delta a^\sigma_w \right) a^\nu_v + R_v \Delta a^\nu_v - \partial B \Delta t - \frac{\partial B}{\partial a^\sigma_w} \Delta a^\sigma_w \\
+ \frac{\partial R_v}{\partial z_v} \frac{\partial B}{\partial z_v} \Delta z + (R_v a^\nu_v - B^\sigma) \frac{\Delta}{\Delta t} (\Delta t), \tag{3.21}
\]

where \( \Delta t = e(t, a_v, z) \), \( \Delta a_v = e \xi_v(t, a_v, z) \). Note the initial condition \( \Delta z(t_0) = 0 \) that the solution of equation (3.21) is

\[
\Delta z(t) = e_v(t, t_0) \int_{t_0}^{t} e_{\gamma}^v(t, t, \theta) \left[ \left( \frac{\partial R_v}{\partial t} \Delta t + \frac{\partial R_v}{\partial a^\sigma_w} \Delta a^\sigma_w \right) a^\nu_v + R_v \Delta a^\nu_v \\
- \frac{\partial B}{\partial t} \Delta t - \frac{\partial B}{\partial a^\sigma_w} \Delta a^\sigma_w + (R_v a^\nu_v - B^\sigma) \frac{\Delta}{\Delta t} (\Delta t) \right] \Delta \theta. \tag{3.22}
\]

Then, when \( t = t_0 \), we have

\[
\int_{t_0}^{t_0} e_{\gamma}^v(t, t_0) \left[ \left( \frac{\partial R_v}{\partial t} \Delta t - \frac{\partial B}{\partial a^\sigma_w} \Delta a^\sigma_w \right) a^\nu_v + R_v \Delta a^\nu_v + (\mu R_v^2 a^\sigma_v - B^\sigma) \Delta \theta \right] \Delta \theta = 0. \tag{3.23}
\]

According to the arbitrariness of integral interval, we obtain

\[
e_{\gamma}^v(t, t, \theta) \left[ \left( \frac{\partial R_v}{\partial t} \Delta t - \frac{\partial B}{\partial a^\sigma_w} \Delta a^\sigma_w \right) a^\nu_v + R_v \Delta a^\nu_v + (\mu R_v^2 a^\sigma_v - B^\sigma) \Delta \theta \right] \Delta \theta = 0. \tag{3.24}
\]

Since \( e_{\gamma}^v(t, t, \theta) > 0 \), theorem 3.5 is proved.

**Theorem 3.6.** If the transformations (3.17) correspond to the time-scales Herglotz type Noether symmetry for delta derivatives of Birkhoffian systems, then there exists a conserved quantity in the form of

\[
I_N = e_{\gamma}^v(t, t, \theta) R_w e_v + \int_{t_0}^{t} \left[ \left( e_{\gamma}^v(t_0, t, \theta) \left( \frac{\partial R_v}{\partial \theta} a^\nu_v - \frac{\partial B}{\partial \theta} \right) \right) a^\nu_v + \right. \\
+ e_{\gamma}^v(t_0, t, \theta) \left( \frac{\partial R_v}{\partial a^\sigma_w} a^\nu_v - \frac{\partial B}{\partial a^\sigma_w} \right) a^\sigma_w - \frac{\Delta}{\Delta t} (e_{\gamma}^v(t_0, t, \theta) R_w a^\nu_v) \Delta \theta \\
+ e_{\gamma}^v(t_0, t, \theta) (\mu R_v^2 a^\sigma_v - B^\sigma) a^\nu_v \Delta \theta \right] \Delta \theta = \text{const.} \tag{3.25}
\]

**Proof.**

\[
\frac{\Delta}{\Delta t} I_N = \frac{\Delta}{\Delta t} (e_{\gamma}^v(t_0, t, R_w) e_v + (e_{\gamma}^v(t_0, t) R_w) e_v) + \left[ e_{\gamma}^v(t_0, t) \left( \frac{\partial R_v}{\partial t} \Delta t - \frac{\partial B}{\partial a^\sigma_w} \Delta a^\sigma_w \right) a^\nu_v + \right. \\
+ e_{\gamma}^v(t_0, t) \left( \frac{\partial R_v}{\partial a^\sigma_w} a^\nu_v - \frac{\partial B}{\partial a^\sigma_w} \right) a^\sigma_w - \frac{\Delta}{\Delta t} (e_{\gamma}^v(t_0, t) R_w a^\nu_v) \Delta \theta + e_{\gamma}^v(t_0, t) (\mu R_v^2 a^\sigma_v - B^\sigma) a^\nu_v \Delta \theta \\
= e_{\gamma}^v(t_0, t) \left( \frac{\partial R_v}{\partial t} a^\nu_v - \frac{\partial B}{\partial a^\sigma_w} \right) a^\nu_v \Delta t + R_w e_v + (\mu R_v^2 a^\sigma_v - B^\sigma) \Delta \theta \\
+ \frac{\Delta}{\Delta t} (e_{\gamma}^v(t_0, t) R_w) (e_v - a^\nu_v \Delta t) + e_{\gamma}^v(t_0, t) \left( \frac{\partial R_v}{\partial a^\sigma_w} a^\nu_v - \frac{\partial B}{\partial a^\sigma_w} \right) a^\sigma_w \Delta t \\
= e_{\gamma}^v(t_0, t) \left( \frac{\partial R_v}{\partial t} a^\nu_v - \frac{\partial B}{\partial a^\sigma_w} \right) a^\nu_v \Delta t + \frac{\partial R_v}{\partial a^\sigma_w} a^\nu_v - \frac{\partial B}{\partial a^\sigma_w} \right) a^\sigma_w \Delta t \\
+ R_w e_v + (\mu R_v^2 a^\sigma_v - B^\sigma) \Delta \theta \\
\Rightarrow \frac{\Delta}{\Delta t} (e_{\gamma}^v(t_0, t) R_w) - e_{\gamma}^v(t_0, t) \left( \frac{\partial R_v}{\partial a^\sigma_w} a^\nu_v - \frac{\partial B}{\partial a^\sigma_w} \right) (e_v - a^\nu_v \Delta t).
Considering formulae (3.14) and (3.18), we can easily get

$$\frac{\Delta}{\Delta t} I_N = 0.$$  

Integrating the above formula, therefore the theorem is proved.

**Remark 3.7.** If $T = \mathbb{R}$, i.e. $\sigma(t) = t$, $\mu(t) = 0$, then formula (3.18) becomes the Herglotz type Noether identity of Birkhoffian systems in the continuous case [22]

$$\left(\frac{\partial R_v}{\partial t} \dot{a}_v - \frac{\partial B}{\partial t}\right) \tau + \left(\frac{\partial R_v}{\partial a_v} \dot{a}_v - \frac{\partial B}{\partial a_v}\right) \xi_v + R_v \dot{\xi}_v - B \tau = 0. \quad (3.26)$$

And the conserved quantity (3.25) changes to the Herglotz type Noether conserved quantity of classical Birkhoffian systems [22]

$$I_N = \exp \left( - \int_t^\tau \left( \frac{\partial R_v}{\partial z} \dot{a}_v - \frac{\partial B}{\partial z}\right) d\theta \right) (R_v \dot{\xi}_v - B \tau) = \text{const.} \quad (3.27)$$

**Remark 3.8.** If $T = h\mathbb{Z}$, $h > 0$, i.e. $\sigma(t) = t+h$, $\mu(t) = h$, then formula (3.18) can be written as

$$\left(\frac{\partial R_v}{\partial t} \dot{a}_v - \frac{\partial B}{\partial t}\right) \tau(t) + \left(\frac{\partial R_v}{\partial a_v} \dot{a}_v - \frac{\partial B}{\partial a_v}\right) \xi_v(t+h) + R_v \dot{\xi}_v(t) + [\mu R_v^\Delta \dot{a}_v^\Delta (t+h) - B(t+h)] \tau^\Delta = 0. \quad (3.28)$$

And the conserved quantity (3.25) changes to

$$I_N = e_y(t, t, t+h) R_u \dot{\xi}_u + \int_t^\tau \left\{ e_y(t, \theta+h) \left( \frac{\partial R_v}{\partial \theta} \dot{a}_v^\Delta - \frac{\partial B}{\partial \theta}\right) + e_y(t, \theta+h) \right\} \left(\frac{\partial R_v}{\partial a_v(\theta+h)} \right) a_v^\Delta - \frac{\Delta}{\Delta \theta} e_y(t, \theta+h) R_u a_v^\Delta \right\} \tau + e_y(t, \theta+h) [h R_v^\Delta \dot{a}_v^\Delta (\theta+h) - B(\theta+h)] \tau \Delta = \text{const.} \quad (3.29)$$

Formulæ (3.28) and (3.29) are the Herglotz type Noether identity and Noether conserved quantity of Birkhoffian systems in the discrete case.

### 4. Some special cases

The above results of Birkhoffian systems can be applied to Hamiltonian systems and Lagrangian systems under certain cases.

**Case 1:** Let

$$a_v^\gamma = \begin{cases} q_v^\gamma, & (v = 1, 2, \ldots, n) \\ p_{v-n}, & (v = n+1, n+2, \ldots, 2n) \\ \end{cases} \quad (4.1)$$

$$R_v = \begin{cases} p_v, & (v = 1, 2, \ldots, n) \\ 0, & (v = n+1, n+2, \ldots, 2n) \end{cases} \quad (4.2)$$

and

$$B(t, a_v^\gamma(t), z(t)) = H(t, q_v^\gamma(t), p_v(t), z(t)), \quad (s = 1, 2, \ldots, n). \quad (4.3)$$

Here, $H$ is the time-scales Hamiltonian for delta derivatives, $q_v(s = 1, 2, \ldots, n)$ are generalized coordinates, and $p_v(s = 1, 2, \ldots, n)$ are generalized momenta. From equations (3.14), we can obtain

$$\frac{\partial H}{\partial p_s} - q_v^\gamma = 0, \quad \frac{\Delta}{\Delta t} (e_v^\gamma(t_1, t) p_v) + e_v^\gamma(t_1, t) \frac{\partial H}{\partial q_v^\gamma} = 0, \quad (s = 1, 2, \ldots, n), \quad (4.4)$$

where $\gamma(t) = -\langle \partial H / \partial z \rangle$. Equations (4.4) are the time-scales Herglotz type Hamilton canonical equations for delta derivatives [34].

Next, the transformations (3.17) in phase space can be expressed as

$$\begin{cases} \tilde{t} = t + \epsilon \tau(t, q_j, p_j, z), \\ \tilde{q}_j(t) = q_j(t) + \epsilon \xi_j(t, q_j, p_j, z) \\ \end{cases} \quad (5.4)$$

and

$$\begin{cases} \tilde{p}_j(t) = p_j(t) + \epsilon \eta_j(t, q_j, p_j, z), \\ \end{cases} \quad (5.5)$$

where $\epsilon$, $\xi_j$, $\eta_j$ are infinitesimal generators. By theorems 3.5 and 3.6, we can also obtain the time-scales Herglotz type Noether identity and Noether conserved quantity for delta derivatives of Hamiltonian systems.
Hamiltonian systems

Thus, equations (3.14) become

Theorem 4.2. If the transformations (4.5) correspond to the Herglotz type Noether symmetry for delta derivatives of Hamiltonian systems, then there exists a conserved quantity in the form of

Case 2: Let

H(t, q^\sigma(t), p(t), z(t)) = p_s q^\Delta_s - L(t, q^\sigma(t), q^\Delta(t), z(t))

and

The partial derivative of equation (4.10) with respect to \( q^\sigma_s, p_s, z \), respectively, we have

Thus, equations (3.14) become

where \( \gamma(t) = \frac{\partial}{\partial t} \).

The infinitesimal transformations (3.17) in configuration space can be expressed as

where \( \tau \) and \( \xi \) are infinitesimal generators. Similarly it is possible to obtain the time-scales Herglotz type Noether identity and Noether conserved quantity for delta derivatives of Lagrangian systems according to theorems 3.5 and 3.6.

Theorem 4.4. If the time-scales Hamilton-Herglotz action \( z \) of Lagrangian systems is invariant on \( U \) under the infinitesimal transformations (4.14), then

holds for all \( t \in [t_1, t_2] \). Formula (4.15) is called the time-scales Herglotz type Noether identity for delta derivatives of Lagrangian systems [32].
Theorem 4.5. If the transformations (4.14) correspond to the time-scales Herglotz type Noether symmetry for delta derivatives of Lagrangian systems, then there exists a conserved quantity in the form of

\[ I_N = \int L \exp \left( -\int \frac{\partial L}{\partial \dot{q}_1} dt \right) \left[ \frac{\partial L}{\partial \dot{q}_1} \dot{q}_1 + \frac{\partial L}{\partial q_1} q_1 + \frac{\partial L}{\partial \dot{q}_2} \dot{q}_2 \right] - \frac{\partial L}{\partial q_1} q_1 + \frac{\partial L}{\partial \dot{q}_2} \dot{q}_2 \right] dt = \text{const.} \]  

(4.16)

Remark 4.6. If \( T = \mathbb{R} \), i.e. \( \sigma(t) = t, \mu(t) = 0, \ e_{\tau}(t, \dot{t}) = \exp(-\int_0^t (\dot{L}/\dot{q}_2) dt) \), then formula (4.15) becomes the Herglotz type Noether identity on this time scale as follows:

\[ \frac{\partial L}{\partial t} \dot{q}_1 + \frac{\partial L}{\partial q_1} q_1 + \frac{\partial L}{\partial \dot{q}_2} \dot{q}_2 = 0. \]  

(4.17)

And the conserved quantity (4.16) changes to the Herglotz type Noether conserved quantity of Lagrangian systems

\[ I_N = \exp \left( -\int_{t_0}^t \frac{\partial L}{\partial \dot{q}_1} d\theta \right) \left[ \frac{\partial L}{\partial \dot{q}_1} (\dot{q}_1 - \dot{q}_2 \tau) + L \right] = \text{const.} \]  

(4.18)

5. Examples

Example 5.1. Suppose the time-scales Herglotz type Birkhoffian and Birkhoff’s functions are, respectively,

\[ B = \frac{1}{2} \left( (a_1)^2 + (a_2)^2 \right) + \alpha z, \quad R_1 = a_2, \quad R_2 = 0, \]  

(5.1)

where \( z^2 = a_2^2 a_1^2 - (1/2) [(a_1)^2 + (a_2)^2] - \alpha z \), and \( \alpha \) is a constant.

When \( T = \mathbb{R} \), then \( \sigma(t) = t, \mu(t) = 0, \ e_{\tau}(t, \dot{t}) = \exp[\alpha(t - t_1)]. \) From the Herglotz type Birkhoff’s equations (3.15), we have

\[ \exp[\alpha(t - t_1)] \cdot (a_2 \dot{a} + \dot{a}_2 + a_1) = 0 \quad \text{and} \quad -\exp[\alpha(t - t_1)] \cdot (\dot{a}_1 - a_2) = 0, \]  

(5.2)

i.e.

\[ a_2 \dot{a}_2 + \dot{a}_2 + a_1 = 0 \quad \text{and} \quad \dot{a}_1 - a_2 = 0. \]  

(5.3)

Let \( x = a_1, \dot{x} = a_2; \) the Birkhoff’s equations (5.3) can become the damped oscillator

\[ \ddot{x} + \alpha \dot{x} + x = 0. \]  

(5.4)

And its Noether conserved quantity has been given in the literature [22].

When \( T = h\mathbb{Z}, h > 0, \) then \( \sigma(t) = t + h, \mu(t) = h, \ e_{\tau}(t, \dot{t}) = \exp[\alpha(t + h - t_1)]. \) From equations (3.14), we obtain

\[ \exp[\alpha(t + h - t_1)] \cdot (a_2 \dot{a} + (a_1)^2 + a_1) = 0 \quad \text{and} \quad -\exp[\alpha(t + h - t_1)] \cdot (a_1^2 - a_2) = 0. \]  

(5.5)

Let \( x = a_1, \dot{x} = a_2^2; \) then

\[ x^2 + \alpha x + x = 0. \]  

(5.6)

The equation (5.6) can be called the time-scales damped oscillator. Now, we study its Noether conserved quantity on time scale \( T = h\mathbb{Z}. \)

From equation (3.18), the Herglotz type Noether identity on this time scale is

\[ \frac{\partial L}{\partial q_1} \frac{\partial L}{\partial \dot{q}_1} + (a_1^2 - a_1) \dot{q}_1 + a_2 \dot{q}_1 + [h(a_2^2) a_1^2 - B] x^2 = 0. \]  

(5.7)

The above equation has a solution

\[ \tau = 1, \quad \xi_1 = \exp\left( \int \frac{a_1}{a_1^2 - h a_1^2} \Delta \theta \right). \]  

(5.8)
There is no limit to $\xi_2$. Thus, by theorem 3.6, the Herglotz type conserved quantity of the Birkhoffian system on this time scale is

$$I_N = \exp\left(\int_{t_1}^t \frac{\partial \mathcal{L}}{\partial a^\gamma} \Delta \theta\right) a^\gamma_2 = \text{const.} \quad (5.9)$$

**Example 5.2.** Let us study a non-Hamiltonian Birkhoffian system, whose time-scales Herglotz type Birkhoffian and Birkhoff’s functions for delta derivatives are, respectively,

$$B = \frac{1}{2}(a^\sigma_1)^2 + \frac{1}{2}(a^\gamma_1)^2 - z, \quad R_1 = a^\sigma_1 + a^\gamma_1, \quad R_2 = a^\sigma_2, \quad R_3 = R_4 = 0, \quad (5.10)$$

where $z$ satisfies the differential equation

$$z^\Delta = (a^\sigma_1 + a^\gamma_1)a^\Delta_1 + a^\sigma_2 a_2^\Delta - \frac{1}{2}(a^\gamma_1)^2 - \frac{1}{2}(a^\sigma_1)^2 + z. \quad (5.11)$$

Now, we study the Herglotz type Noether conserved quantity of the non-Hamiltonian system (5.10) on a time scale of

$$\Sigma = \{2^m : m \in \mathbb{Z}\} \cup \{0\}. \quad (5.12)$$

From the time scale (5.12), it is obvious that $\sigma(t) = 2t$ and $\mu(t) = t$. According to definition 2.6, we have $e_\gamma(t_1, t) = \exp(-\int_{t_1}^t (1/\theta) \Delta \theta)$, where $\gamma(t) = (\partial R_\gamma/\partial z)a^\gamma_1 - (\partial B/\partial z) = 1$. Then, $e_\gamma(t_1, t) = (1 + t)e_\gamma(t_1, t)$. From equations (3.14), the time-scales Herglotz type Birkhoff’s equations of the system can be obtained, as follows:

$$\frac{\Delta}{\Delta t}[e_\gamma^\sigma(t_1, t) \cdot (a^\sigma_1 + a^\gamma_1)] = 0, \quad \frac{\Delta}{\Delta t}[e_\gamma^\sigma(t_1, t) \cdot a^\sigma_1] - e_\gamma^\sigma(t_1, t) a^\Delta_1 = 0,$n

$$- e_\gamma^\sigma(t_1, t) \cdot (a^\sigma_1 - a^\gamma_1) = 0 \quad \text{and} \quad - e_\gamma^\sigma(t_1, t) \cdot (a^\gamma_1 - a^\sigma_1) = 0. \quad (5.13)$$

From equation (3.18), the Herglotz type Noether identity on this time scale is

$$\{\mu[(a^\sigma_2 + a^\gamma_2)\Delta + (a^\gamma_2)\Delta] - B^\Delta\}^\Delta + a^\Delta_2 a^\gamma_2 + a^\Delta_2 a^\gamma_2 + a^\Delta_2 a^\gamma_2 = 0. \quad (5.14)$$

Hence, one solution to the above equation is

$$\tau = 1, \quad \xi_1 = 1, \quad \xi_2 = 0. \quad (5.15)$$

There is no limit to $\xi_3$ and $\xi_4$. Therefore, the Herglotz type conserved quantity of the system on the time scale can be obtained by theorem 3.6, as follows

$$I_N = e_\gamma^\sigma(t_1, t) \cdot (a^\sigma_2 + a^\gamma_2) = \text{const.} \quad (5.16)$$

**6. Conclusion**

The time-scales Herglotz variational principle for delta derivatives of Birkhoffian systems is introduced and its time-scales Pfaff–Herglotz action is put forward. The time-scales Herglotz type Birkhoff’s equations (3.14) are obtained, which can reduce to the Herglotz type Birkhoff’s equations (3.15) of continuous systems or the time-scales Birkhoff’s equations (3.16) based on the traditional variational problem. The time-scales Herglotz type Noether identity and Noether conserved quantity for delta derivatives of Birkhoffian systems, i.e. theorems 3.5 and 3.6, are new main results. Herglotz type Noether identities and Noether conserved quantities of Birkhoffian systems are given in continuous and discrete cases, respectively. On account of the universality of Birkhoffian systems, theorems 3.5, 3.6 of Birkhoffian systems can become theorems 4.1, 4.2 of Hamiltonian systems or theorems 4.4, 4.5 of Lagrangian systems in special cases. Therefore, the correctness and the generality of the results are verified. Because of the advantage of time-scales Herglotz variational principle, the results of this paper not only are applicable to discrete and continuous Birkhoffian systems but also can be used to solve conservative and non-conservative problems. Moreover, it should not be neglected that the results of this paper provide a theoretical basis for computer programming. Similarly, it is possible to expand to study time-scales Herglotz type Noether theorem for nabla derivatives or mixed derivatives by the method of this paper.
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