A Korn-type inequality in SBD for functions with small jump sets

Manuel Friedrich

February 15, 2016

Abstract

We present a Korn-type inequality in a planar setting for special functions of bounded deformation. We prove that for each function in $SBD^2$ with sufficiently small jump set the distance of the function and its derivative from an infinitesimal rigid motion can be controlled in terms of the linearized elastic strain outside of a small exceptional set of finite perimeter. Particularly the result shows that each function in $SBD^2$ has bounded variation away from an arbitrarily small part of the domain.

Keywords. Functions of bounded deformation, Korn’s inequality, Korn-Poincaré inequality, brittle materials, variational fracture.

AMS classification. 74R10, 49J45, 70G75

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1Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria. manuel.friedrich@univie.ac.at
1 Introduction

The space $BD(\Omega, \mathbb{R}^d)$ of functions of bounded deformation, which consists of all functions $u \in L^1(\Omega, \mathbb{R}^d)$ whose symmetrized distributional derivative $Eu := \frac{1}{2}((Du)^T + Du)$ is a finite $\mathbb{R}^{d \times d}_{\text{sym}}$-valued Radon measure, has been introduced for the investigation of geometrically linear problems in plasticity theory and fracture mechanics (see [3, 5]). Variational damage or fracture problems are widely formulated in the subspace $SBD^2(\Omega, \mathbb{R}^d)$ (for the definition and properties of this space we refer to Section 2.1 below). In the spirit of the seminal work [24] the modeling essentially concentrates on the competition between elastic bulk contributions $\|e(u)\|_{L^2(\Omega)}$ given in terms of the linear elastic strain $e(u) := \frac{1}{2}(\nabla u^T + \nabla u)$ and surface terms that assign energy contributions on the crack paths comparable to the size of the crack $\mathcal{H}^{d-1}(J_u \cap \Omega)$, where $J_u$ denotes the ‘jump set’ of $u$ (see e.g. [6, 8, 9, 23, 39]).

A major additional difficulty of these problems compared to models in SBV (see [4] for the definition and basic properties of the space of special functions of bounded variation) is the lack of control on the skew symmetric part of the distributional derivative $Du^T - Du$. In fact, it is a natural and important question to analyze in which circumstances the displacement field $u$ or the absolutely continuous part of its derivative $\nabla u$ can be controlled by $\|e(u)\|_{L^2(\Omega)}$ and $\mathcal{H}^{d-1}(J_u)$. Apart from establishing compactness results, such properties may contribute to gain a profound understanding of the relation between SBD and SBV functions which is highly desirable since in contrast to (S)BV fine properties in BD appear not to be well understood by now. (We refer to the recent paper [14] for a thorough discussion and some results in that direction.)

The key estimate providing a relation between the symmetric and the full part of the gradient is known as Korn’s inequality. In its basic version it states that for a bounded connected Lipschitz set $\Omega$ and $p \in (1, \infty)$ there is a constant $C(\Omega, p)$ depending only on $p$ and the domain $\Omega \subset \mathbb{R}^d$ such that for all $u \in W^{1,p}(\Omega, \mathbb{R}^d)$ there is some $A \in \mathbb{R}^{d \times d}_{\text{skew}}$ with

$$\|\nabla u - A\|_{L^p(\Omega)} \leq C(\Omega, p)\|e(u)\|_{L^p(\Omega)}.$$  

(1)

(See e.g. [37] for a proof and [12, 29, 30, 35] for generalizations of this result into various directions.) It turns out that the statement is false in $W^{1,1}$, i.e. one can construct functions with $e(u) \in L^1(\Omega)$, but $\nabla u \notin L^1(\Omega)$ (cf. [13, 32, 38]). On the one hand, these observations particularly show that BD is not contained in BV. On the other hand, it raises the natural question if in the space $SBD^2$ an estimate similar to (1) can be established due to the higher integrability for the elastic strain $e(u)$.

However, simple examples e.g. in [3] or the piecewise rigidity result proved in [11] show that (1) cannot hold for general functions in $SBD^2$ since the behavior of small pieces being almost or completely detached from the bulk part of the specimen might not be controlled. In the recently appeared contributions [10, 25]
it has been proved that for displacement fields having small jump sets with respect to the size of the domain the distance of the function from an infinitesimal rigid motion can be estimated in terms of the linearized elastic energy outside of a small exceptional set $E$. However, these Korn-Poincaré-type estimates being essentially of the form
\[ \|u - (A \cdot + b)\|_{L^2(\Omega \setminus E)} \leq C(\Omega) \|e(u)\|_{L^2(\Omega)} \] (2)
for $u \in SBD^2(\Omega, \mathbb{R}^d)$ and corresponding $A \in \mathbb{R}^{d \times d}_\text{skew}$, $b \in \mathbb{R}^d$, are significantly easier as in contrast to (1) no derivative is involved. The goal of the present article is to provide a generalization of (2) to an inequality of Korn’s-type in a planar setting where one additionally controls $\nabla u$ away from a small exceptional set of finite perimeter. Our main result is the following.

**Theorem 1.1** Let $\Omega \subset \mathbb{R}^2$ be a connected, bounded set with Lipschitz boundary and let $p \in [1,2)$, $q \in [1,\infty)$. Then there is a constant $C = C(\Omega, p, q)$ such that for all $u \in SBD^2(\Omega, \mathbb{R}^2)$ there is a set of finite perimeter $E \subset \Omega$ with
\[ \mathcal{H}^1(\partial^* E) \leq C\mathcal{H}^1(J_u), \quad |E| \leq C(\mathcal{H}^1(J_u))^2 \] (3)
and $A \in \mathbb{R}^{2 \times 2}_\text{skew}, b \in \mathbb{R}^2$ such that
\[ (i) \quad \|u - (A \cdot + b)\|_{L^p(\Omega \setminus E)} \leq C\|e(u)\|_{L^2(\Omega)}, \]
\[ (ii) \quad \|\nabla u - A\|_{L^p(\Omega \setminus E)} \leq C\|e(u)\|_{L^2(\Omega)}, \] (4)
where $e(u)$ denotes the part of the strain $Eu = \frac{1}{2}(Du^T + Du)$ which is absolutely continuous with respect to $\mathcal{L}^2$ and $\partial^* E$ is the essential boundary of $E$.

Let us first mention that we establish the result only in two dimensions as we employ a modification technique for special functions of bounded deformation (see [25]) which was only derived in a planar setting due to technical difficulties concerning the topological structure of crack geometries in higher dimensions.

Note that we can control the length of the boundary $\mathcal{H}^1(\partial^* E)$ of the exceptional set $E$ which is associated to the parts of $\Omega$ being detached from the bulk part of $\Omega$ by $J_u$. Consequently, the result is adapted for the usage of compactness theorems for SBV and SBD functions (see [4, 5, 17]).

Although the main goal of the work at hand is the derivation of the estimate for the derivative in (4)(ii), we also provide a generalization for the integrability exponent $q$. In [25] the exponent was restricted to $q = 2$ due to the application of a BD Korn-Poincaré inequality and in [10] the arguments were based on slicing techniques similar to those used in the proof of Sobolev embeddings and led to an exponent $q = \frac{2d}{d-1}$. In the present context we obtain the estimate for $q < \infty = 2^*$ as in the usual Sobolev setting.

To our knowledge the first inequality of this kind, i.e., also involving an estimate on $\nabla u$, has been presented in [27], which is the preprint version of the
present paper. Subsequently, the result has been extended to the critical exponent \( p = 2 \) in [15]. However, as the proof techniques are quite different, we believe that also the present article may be interesting for the community.

As an application we discuss that Theorem 1.1 together with an approximation result shows that \( SBD^2 \) functions have bounded variation outside an arbitrarily small exceptional set of finite perimeter (see Theorem 5.1 below). Hereby we give another contribution to the relation between SBV and SBD functions which appears to go in a slightly different direction than the results presented in [14]. We note that this statement does not immediately follow from the main theorem since a bound on \( \nabla u \) does not automatically ensure that \( u \) has bounded variation due to the fact that an equivalent of Alberti’s rank one property in BV (see [2]) is not known in BD.

Similarly as the previously mentioned results [10, 25] or the SBV Poincaré inequality (cf. [20]), Theorem 1.1 establishes an estimate only for functions whose jump set is small with respect to the size of the domain. Indeed, for larger jump sets the body may be separated into different parts of comparable size (cf. [11, 26, 29] for related problems). In the general case we expect a ‘piecewise Korn inequality’ to hold, i.e. the body may be broken into different sets and on each connected component the distance of the displacement field from a certain infinitesimal rigid motion can be controlled. We defer the analysis of this problem, for which Theorem 1.1 is a key ingredient, to a subsequent work [28].

The paper is organized as follows. In Section 2 we first recall the definition and basic properties of functions of bounded variation and deformation (Section 2.1). Then in Section 2.2 we introduce the notion of John domains being a class of sets with possibly highly irregular boundary (see e.g. [31, 34]). It is convenient to formulate Korn’s and Poincaré’s inequality for these sets since there are good criteria to obtain uniform control over the involved constants independently of the particular shape of the domain (cf. [1]). Finally, in Section 2.3 we present the modification technique proved in [25] which shows that after a small alteration of the displacement field and the jump set the jump heights of an SBD function can be controlled solely by \( \|e(u)\|_{L^2(\Omega)} \) and \( \mathcal{H}^1(J_u) \).

The rest of the paper then contains the proof of Theorem 1.1. We first establish a local estimate on a square and after a subsequent analysis of the problem near the boundary of the Lipschitz set the main theorem follows by a standard covering argument.

In Section 3 we concern ourselves with the local estimate and first see that by an approximation argument (cf. [2]) it suffices to consider SBD functions with regular jump set. The main strategy is then to modify a function with the techniques presented in Section 2.3. Consequently, using a Korn-Poincaré inequality BD (see [33, 11]) we find good approximations of the displacement field by infinitesimal rigid motions in neighborhoods of the jump set. Then drawing ideas from [29] we can iteratively modify the configuration on various mesoscopic length scales to find a Sobolev function on the square which coincides with the
original displacement field outside of a small exceptional set. Finally, the local estimate follows by application of the standard inequality (1).

We remark that for the nonlinear estimate [29] it was not possible to gain control over the full part of the gradient as the approximating rigid motions had to be adapted after each iteration step leading to a continual increase of the involved constant. In the present context, however, the affine mappings are found a priori and are fixed during the modification procedure whereby a bound for $\nabla u$ can be established using Hölder’s and a scaled Young’s inequality in the case $p < 2$. Moreover, let us mention that our approach to derive the local estimate for the Korn inequality differs from the one proposed in [15], where a suitable triangulation of the domain and a corresponding linear interpolation are constructed.

Section 4 contains the main estimate at the boundary. We consider a Whitney covering of the domain and apply the result obtained in Section 3 on every square where the jump set $J_u$ is small. Hereby we can again construct a Sobolev function outside a small exceptional set $E$. For the application of (1) now an additional difficulty occurs as we have to control the shape of domain. In this context we show that choosing $E$ appropriately we find that the complement is a John domain for a universal John constant and therefore we can derive a uniform estimate of the form (1).

In Section 5 we then give the main proof and discuss an application to the relation of SBV and SBD functions. The standard examples for (S)BD functions not having bounded variation are given by configurations where small balls are cut out from the bulk part with an appropriate choice of the functions on these specific sets (see e.g. [3, 14]). We prove that each $SBD^2$ function has bounded variation away from an arbitrarily small part of the body essentially showing that the mentioned construction provides the only way to obtain functions not lying in SBV.

2 Preliminaries

In this preparatory section we first recall the definition and basic properties of functions of bounded variation and state Korn’s and Poincaré’s inequality for John domains. Afterwards, we recall a result obtained in [25] providing modifications of SBD functions for which the jump heights can be controlled in terms of the linear elastic strain.

2.1 Special functions of bounded variation

In this section we collect the definitions of SBV and SBD functions. Let $\Omega \subset \mathbb{R}^d$ open, bounded with Lipschitz boundary. Recall that the space $SBV(\Omega, \mathbb{R}^d)$, abbreviated as $SBV(\Omega)$ hereafter, of special functions of bounded variation consists
of functions $u \in L^1(\Omega, \mathbb{R}^d)$ whose distributional derivative $Du$ is a finite Radon measure, which splits into an absolutely continuous part with density $\nabla u$ with respect to Lebesgue measure and a singular part $D^j u$. The Cantor part $D^c u$ of $D^j u$ vanishes and thus we have

$$D^j u = [u] \otimes \xi_u \mathcal{H}^{d-1} | J_u,$$

where $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure, $J_u$ (the ‘crack path’) is an $\mathcal{H}^{d-1}$-rectifiable set in $\Omega$, $\xi_u$ is a normal of $J_u$ and $[u] = u^+ - u^-$ (the ‘crack opening’) with $u^\pm$ being the one-sided limits of $u$ at $J_u$. If in addition $\nabla u \in L^p(\Omega)$ for $1 < p < \infty$ and $\mathcal{H}^{d-1}(J_u) < \infty$, we write $u \in SBV^p(\Omega)$. Moreover, $SBV_{loc}(\Omega)$ denotes the space of functions which belong to $SBV(\Omega')$ for every open set $\Omega' \subset \subset \Omega$.

Furthermore, we define the space $GSBV(\Omega)$ of generalized special functions of bounded variation consisting of all $\mathcal{C}^d$-measurable functions $u : \Omega \to \mathbb{R}^d$ such that for every $\phi \in \mathcal{C}^1(\mathbb{R}^d)$ with the support of $\nabla \phi$ compact, the composition $\phi \circ u$ belongs to $SBV_{loc}(\Omega)$ (see [19]). Likewise, we say $u \in GSBV^p(\Omega)$ for $u \in GSVB(\Omega)$ if $\nabla u \in L^p(\Omega)$ and $\mathcal{H}^{d-1}(J_u) < \infty$. See [4] for the basic properties of theses function spaces.

We now state a version of Ambrosio’s compactness theorem in $GSBV$ adapted for our purposes (see e.g. [4, 18]):

**Theorem 2.1** Let $\Omega \subset \mathbb{R}^d$ open, bounded and let $1 < p < \infty$. Let $(u_k)_k$ be a sequence in $GSBV^p(\Omega)$ such that

$$\|\nabla u_k\|_{L^p(\Omega)} + \mathcal{H}^{d-1}(J_{u_k}) + \|u_k\|_{L^1(\Omega)} \leq C$$

for some constant $C$ not depending on $k$. Then there is a subsequence (not relabeled) and a function $u \in GSVB^p(\Omega)$ such that $u_k \to u$ a.e. and $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$. If in addition $\|u_k\|_\infty \leq C$ for all $k \in \mathbb{N}$, we find $u \in SBV^p(\Omega) \cap L^\infty(\Omega)$.

An important subset of SBV is given by the indicator functions $\chi_W$ for measurable $W \subset \Omega$ with $\mathcal{H}^{d-1}(\partial^* W) < \infty$, where $\partial^* W$ denotes the essential boundary of $W$ (see [4, Definition 3.60]). Sets of this form are called sets of finite perimeter. As a consequence of Rellich’s theorem in BV and the lower semicontinuity of the perimeter we get the following result (see [4, Proposition 3.38]).

**Theorem 2.2** Let $\Omega \subset \mathbb{R}^d$ open, bounded. Let $(W_k)_k \subset \Omega$ be a sequence of measurable sets with $\mathcal{H}^{d-1}(\partial^* W_k) \leq C$ for some constant $C$ independent of $k$. Then there is a subsequence (not relabeled) and a measurable set $W$ such that $\chi_{W_k} \to \chi_W$ in measure for $k \to \infty$ and $\mathcal{H}^{d-1}(\partial^* W) \leq \liminf_{k \to \infty} \mathcal{H}^{d-1}(\partial^* W_k)$.

We say that a function $u \in L^1(\Omega, \mathbb{R}^d)$ is in $BD(\Omega, \mathbb{R}^d)$ if the symmetrized distributional derivative $E u := \frac{1}{2}((Du)^T + Du)$ is a finite $\mathbb{R}^{d \times d}_{\text{sym}}$-valued Radon
measure. Likewise, we say \( u \) is a **special function of bounded deformation** if \( E u \) has vanishing Cantor part \( E^c u \). Then \( E u \) can be decomposed as

\[
E u = e(u) \mathcal{L}^d + E^s u = e(u) \mathcal{L}^d + [u] \odot \xi_u \mathcal{H}^{d-1}|_{J_u},
\]

where \( e(u) \) is the absolutely continuous part of \( E u \) with respect to the Lebesgue measure \( \mathcal{L}^d \), \([u] \), \( \xi_u \), \( J_u \) as before and \( a \odot b = \frac{1}{2}(a \otimes b + b \otimes a) \). If in addition \( e(u) \in L^p(\Omega) \) for \( 1 < p < \infty \) and \( \mathcal{H}^{d-1}(J_u) < \infty \), we write \( u \in SBD^p(\Omega, \mathbb{R}^d) \), abbreviated by \( SBD^p(\Omega) \) hereafter. For basic properties of this function space we refer to [3, 5].

We recall a Korn-Poincaré inequality in \( \text{BD} \) (see [33, 41]).

**Theorem 2.3** Let \( \Omega \subset \mathbb{R}^d \) bounded, connected with Lipschitz boundary and let \( P : L^2(\Omega, \mathbb{R}^d) \to L^2(\Omega, \mathbb{R}^d) \) be a linear projection onto the space of infinitesimal rigid motions. Then there is a constant \( C > 0 \), which is invariant under rescaling of the domain, such that for all \( u \in BD(\Omega, \mathbb{R}^d) \)

\[
\|u - Pu\|_{L^{\frac{d}{d-1}}(\Omega)} \leq C|Eu|(\Omega),
\]

where \( |Eu|(\Omega) \) denotes the total variation of \( E u \).

### 2.2 Poincaré’s and Korn’s inequality

A key idea in our analysis will be the replacement of displacement fields in \( \text{SBD} \) by suitable Sobolev functions and then the application of well know Poincaré and Korn inequalities. As the estimates will be employed on different Lipschitz sets, we need to provide uniform bounds for the constants involved in the inequalities. To this end, we introduce the notion of **John domains**.

**Definition 2.4** Let \( \Omega \subset \mathbb{R}^d \) be an open, bounded set and let \( x_0 \in \Omega \). We say \( \Omega \) is a \( c \)-**John domain** with respect to the **John center** \( x_0 \) and with the constant \( c \) if for all \( x \in \Omega \) there exists a rectifiable curve \( \gamma : [0, l_\gamma] \to \Omega \), parametrized by arclength, such that \( \gamma(0) = x \), \( \gamma(l_\gamma) = x_0 \) and \( t \leq c \text{dist}(\gamma(t), \partial \Omega) \) for all \( t \in [0, l_\gamma] \).

Domains of this form were introduced by John in [31] to study problems in elasticity theory and the term was first used by Martio and Sarvas in [34]. Roughly speaking, a domain is a John domain if it is possible to connect two arbitrary points without getting too close to the boundary of the set. This class is much larger than Lipschitz domains and contains sets which may possess fractal boundaries or internal cusps (external cusps are excluded), e.g. the interior of Koch’s snow flake is a John domain.

Although in the present work only Lipschitz sets occur, it is convenient to consider this more general notion as the constants in Poincaré’s and Korn’s inequalities only depend on the John constant. More precisely, we have the following statement (see e.g. [11, 7, 21]).
Theorem 2.5 Let $\Omega \subset \mathbb{R}^d$ be a $c$-John domain. Let $p \in (1, \infty)$ and $q \in (1, d)$. Then there is a constant $C = C(c, p, q) > 0$ such that for all $u \in W^{1,p}(\Omega)$ there is some $A \in \mathbb{R}^{d \times d}$ skew such that

$$\|\nabla u - A\|_{L^p(\Omega)} \leq C\|e(u)\|_{L^p(\Omega)}.$$ 

Moreover, for all $u \in W^{1,q}(\Omega)$ there is some $b \in \mathbb{R}^d$ such that

$$\|u - b\|_{L^{q^*}(\Omega)} \leq C\|\nabla u\|_{L^q(\Omega)},$$

where $q^* = \frac{dq}{d-q}$. The constant is invariant under rescaling of the domain.

2.3 A modification with controllable jump heights

A main strategy of our proof will be the application of Theorem 2.3 in certain regions of the domain. It first appears that this inequality is not adapted for the estimates in (4) as in $|Eu|_{(\Omega)}$ not only the elastic but also the surface energy depending on the jump height is involved. However, in [25] we have shown that one can indeed find bounds on the jump heights in terms of the elastic energy after a suitable modification of the displacement field.

In the following $d(W)$ denotes the diameter of a set $W \subset \mathbb{R}^2$ and for $\mu > 0$ we define $Q_\mu := (-\mu, \mu)^2$. We recall the following result (see [25, Theorem 2.2]).

Theorem 2.6 Let $\lambda > 0$. Then there is a constant $C = C(\lambda)$ and a universal constant $c > 0$ both independent of $\mu$ such that for all $\varepsilon > 0$, $\delta > 0$ and all $u \in SBD^2(Q_\mu)$ the following holds: There are paraxial rectangles $R_1, \ldots, R_n$ with

$$\sum_{j=1}^n d(R_j) \leq (1 + c\lambda)(\mathcal{H}^1(J_u) + \varepsilon^{-1}\|e(u)\|_{L^2(Q_\mu)}^2)$$

and a modification $\tilde{u} \in SBV^2(\hat{Q})$ with $J_{\tilde{u}} \subset \bigcup_{j=1}^n \partial R_j$ and

$$\|\tilde{u} - u\|_{L^2(\hat{Q}\setminus F)} \leq \delta, \quad \|e(\tilde{u})\|_{L^2(\hat{Q})} \leq \|e(u)\|_{L^2(Q_\mu)} + \delta,$$

where $F := \bigcup_{j=1}^n R_j$ and $\hat{Q} := (-\bar{\mu}, \bar{\mu})^2$ with $\bar{\mu} = \max\{\mu - 3\sum_j d(R_j), 0\}$, such that for all measurable sets $D \subset \hat{Q}$ we have

$$\left(|E\tilde{u}|(D)\right)^2 \leq c|D|\|e(\tilde{u})\|_{L^2(D)}^2 + C\varepsilon \mathcal{H}^1(D \cap J_{\tilde{u}}) \sum_{R_j \in \mathcal{R}(D)} (d(R_j))^2,$$

where $\mathcal{R}(D) := \{R_j : D \cap \overline{R_j} \neq \emptyset\}$. Moreover, if $u \in L^\infty(Q_\mu)$, we can choose the modification such that $\|\tilde{u}\|_{L^\infty(\hat{Q}\setminus F)} \leq \|u\|_{L^\infty(Q_\mu)}$. 

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3 The local estimate on a square

This section is devoted to the derivation of a local estimate on a square. Recall $Q_\mu = (-\mu, \mu)^2$ for $\mu > 0$.

**Theorem 3.1** Let $p \in [1, 2), q \in [1, \infty)$ and $\mu > 0$. Then there is a constant $C = C(p,q)$ independent of $\mu$ such that for all $u \in SBD^2(Q_\mu)$ there is a set of finite perimeter $E \subset Q_\mu$ with

$$\mathcal{H}^1(\partial^* E) \leq C\mathcal{H}^1(J_u), \quad |E| \leq C(\mathcal{H}^1(J_u))^2$$

and $A \in \mathbb{R}^{2 \times 2}$, $b \in \mathbb{R}^2$ such that

\begin{equation}
(i) \quad \|u - (A \cdot + b)\|_{L^q(Q_\mu \setminus E)} \leq C\mu^{\frac{2}{p}}\|e(u)\|_{L^2(Q_\mu)},
(ii) \quad \|\nabla u - A\|_{L^p(Q_\mu \setminus E)} \leq C\mu^{\frac{2}{p} - 1}\|e(u)\|_{L^2(Q_\mu)},
\end{equation}

where $Q_\mu = (-\mu, \mu)^2$ with $\bar{\mu} = \max\{\mu - C\mathcal{H}^1(J_u), 0\}$. Moreover, we obtain $u\chi_{Q_\mu \setminus E} \in GSBV^p(Q_\mu)$. If in addition $u \in L^\infty(Q_\mu)$, we find $u\chi_{Q_\mu \setminus E} \in SBV^p(Q_\mu)$.

Observe that the additional statement that $u\chi_{Q_\mu \setminus E}$ lies in $(G)SBV^p$ does not directly follow from (10)(ii) as a Korn-type inequality for $\nabla u$ does not automatically guarantee that $u$ has bounded variation. In fact, an equivalent of Alberti’s rank one property in BV (see [2]) stating $|D^e u|(Q_\mu) \leq \sqrt{2}|E^c u|(Q_\mu)$ for BV functions is not known in BD. In the present context we circumvent this problem by an approximation of SBD functions.

We first show the result for modifications given by Theorem 2.6 and afterwards we prove the general version of Theorem 3.1 by considering sequences of modifications.

3.1 Local estimate for modifications

We first introduce some further notation. For $s > 0$ we partition $\mathbb{R}^2$ up to a set of measure zero into squares $Q^s(p) = p + s(-1, 1)^2$ for $p \in I^s := s(1, 1) + 2s\mathbb{Z}^2$. Let $\theta \in 2^{-\mathbb{N}}$ small, fixed and define $s_i = \mu \theta^i$ for $i \geq 0$. We let

$$Q_i = \{Q = Q^s(p) : p \in I^{s_i}\}. \quad (11)$$

For each $Q \in Q_i$ we introduce enlarged squares $Q \subset Q'' \subset Q'$ defined by

$$Q'' = \frac{5}{4}Q, \quad Q' = \frac{3}{2}Q, \quad (12)$$

where $\lambda Q$ denotes the square with the same center and $\lambda$-times the sidelength of $Q$. Moreover, by $\text{dist}(A, B)$ we denote the euclidian distance between $A, B \subset \mathbb{R}^2$. In the sequel infinitesimal rigid motions $A x + b, A \in \mathbb{R}^{2 \times 2}, b \in \mathbb{R}^2$, will appear frequently and we will often write $a(x) = a_{A,b}(x) = A x + b$ for the sake of brevity. We now prove the local result for modifications.
Theorem 3.2. There is a constant $C = C(p, q) > 0$ such that for each $\delta > 0$ and all $u \in SBD^2(Q_\mu)$ there is a finite union of rectangles $F \subset Q_\mu$ with $\mathcal{H}^1(\partial F) \leq C\mathcal{H}^1(J_u)$ and a modification $\bar{u} \in SBV^2(\bar{Q})$ for $\bar{\mu} = \max\{\mu - C\mathcal{H}^1(J_u), 0\}$ with

\[
\begin{align*}
(i) & \quad \|\bar{u} - u\|_{L^2(Q_\mu \setminus F)} \leq \delta, \quad \|e(\bar{u})\|_{L^2(Q_\mu)} \leq \|e(u)\|_{L^2(Q_\mu)} + \delta, \\
(ii) & \quad \mathcal{H}^1(J_{\bar{u}}) \leq C\mathcal{H}^1(J_u)
\end{align*}
\]

such that there is a set $E \subset Q_\mu$ satisfying \(\ref{eq:9}\) and $A \in \mathbb{R}^{2 \times 2}$, $b \in \mathbb{R}^2$ with

\[
\mu^{-\frac{q}{2}} \|\bar{u} - (A \cdot + b)\|_{L^q(Q_\mu \setminus E)} + \mu^{1 - \frac{q}{2}} \|\nabla \bar{u} - A\|_{L^q(Q_\mu \setminus E)} \leq C(\|e(u)\|_{L^2(Q_\mu)} + \delta).
\]

Proof. Let $u \in SBD^2(Q_\mu)$ and $\delta > 0$ be given. Without restriction we can assume $\mathcal{H}^1(J_u) > 0$ as otherwise the statement follows from Theorem 2.5. We first apply Theorem 2.6 with $\epsilon = (\mathcal{H}^1(J_u))^{-1}\|e(u)\|^2_{L^2(Q_\mu)}$ and $\lambda = 1$ to obtain a modification $\bar{u} \in SBV^2(\bar{Q})$ such that $\ref{eq:6}$-$\ref{eq:8}$ hold with $\bar{Q} \subset Q_\mu$ as in Theorem 2.6. Letting $\bar{J} := \bigcup_{l=1}^n \partial R_l$ and $F := \bigcup_{l=1}^n R_l$ we particularly have by $\ref{eq:6}$

\[
\mathcal{H}^1(\partial F) \leq \mathcal{H}^1(\bar{J}) \leq 2\sqrt{2} \sum_{l=1}^n d(R_l) \leq 4\sqrt{2}(1 + c)\mathcal{H}^1(J_u).
\]

Choosing the constant $C > 0$ in the assertion large enough (depending on $\theta$) we get by $\ref{eq:15}$

\[
Q_{\bar{\mu}} \subset \bar{Q}, \quad \sum_{l=1}^n d(R_l) \leq \frac{\theta}{24} \text{dist}(\partial \bar{Q}, \partial Q_{\bar{\mu}}).
\]

Then in view of $\ref{eq:7}$, $\ref{eq:15}$ and $J_u \subset \bar{J}$ we get $\ref{eq:13}$.

We now derive $\ref{eq:14}$ for the modification $\bar{u}$, where we will regard $\bar{u}$ as a function defined on $\bar{Q}$. We may assume that $Q_{\bar{\mu}} \neq \emptyset$ as otherwise the assertion of the lemma is trivial. Then $\ref{eq:16}$ implies $\sum_i d(R_i) \leq \frac{\theta}{24} \mu = \frac{1}{24} s_1$. We start with the identification of regions where $\bar{J}$ is too large (Step I). Afterwards, we will use $\ref{eq:8}$ to apply Theorem 2.3 on these specific sets (Step II). This Korn-Poincaré estimate will then enable us to define a suitable modification (Step III) for which Korn’s inequality for Sobolev functions can be used (Step IV).

Step I (Identification of ‘bad’ sets): We first identify squares of various length scales where $\bar{J}$ is too large. Recalling $\ref{eq:11}$ and $\ref{eq:12}$ we introduce the sets

\[
\mathcal{A}_i = \{Q \in \mathcal{Q} : Q'' \subset \bar{Q}, \quad \sum_{l=1}^n \mathcal{H}^1(Q'' \cap \partial R_l) \geq \frac{1}{8} \theta s_1\}
\]

for $i \in \mathbb{N}$. Let $A_i = \bigcup_{Q \in \mathcal{A}_i} \overline{Q}$. Then we define

\[
\mathcal{B}_i = \{Q \in \mathcal{A}_i : Q \cap Q_{\bar{\mu}} \neq \emptyset, \quad Q \cap \bigcup_{j=1}^{i-1} A_j = \emptyset\}
\]
and accordingly let \( B_i = \bigcup_{Q \in B_i} \overline{Q} \) and \( B'_i = \bigcup_{Q \in B_i} \overline{Q} \) for \( i \in \mathbb{N} \). For later we note that \( \bigcup_{i \geq 1} A_i \cap Q_\mu = \bigcup_{i \geq 1} B_i \cap Q_\mu \). We now show that for some \( I \in \mathbb{N} \) sufficiently large we have

\[
\begin{align*}
(i) \quad \sum_{R_i \in \mathcal{F}} d(R_i) &\leq s_i \quad \text{for all } Q \in B_i, \: i \geq 1, \\
(ii) \quad B_i = \emptyset \quad \text{for } i > I,
\end{align*}
\]

\[
(iii) \quad J_\mu \cap Q_\mu \subset J \cap Q_\mu \subset \bigcup_{i \geq 1} B_i,
\]

\[
(iv) \quad \mathcal{H}^1(\partial (\bigcup_{i \geq 1} B_i')) \leq C\theta^{-1} \sum_{i=1}^n d(R_i) \leq C\mu,
\]

\[
(v) \quad |B'_i| \leq Cs_i\theta^{-1} \sum_{i=1}^n d(R_i) \leq Cs_i\mu \quad \text{for all } i \geq 1
\]

for a constant \( C > 0 \) independent of \( \theta \). We first confirm (i). We assume \( \sum_{R_i \in \mathcal{F}} d(R_i) > s_i \) for some \( Q \in B_i \), where \( \mathcal{F} = \{ R_i : R_i \cap Q'' \neq \emptyset \} \) and derive contradictions treating two different cases:

a) Suppose there is some \( R_k \in \mathcal{F} \) with \( d(R_k) > \frac{1}{3} s_{i-1} \). Then choose \( j \leq i - 1 \) such that \( \frac{1}{3}s_{j} < d(R_k) \leq \frac{1}{3}s_{j-1} \) and observe that by the remark below (16) we find \( j \geq 2 \). Moreover, we select \( Q_+ \in Q_{j-1} \) such that \( Q \subset Q_+ \). As \( s_{j-1} < \theta d(R_k) \leq \frac{1}{3} \text{dist}(\partial \overline{Q}, \partial Q_\mu) \) by (16) and \( Q_+ \cap Q_\mu \neq \emptyset \), we find \( Q'' \subset \overline{Q} \) recalling (12). Since \( R_k \cap Q'' \neq \emptyset \) and \( Q \subset Q_+ \), we get \( \overline{R_k} \subset Q''_+ \) and thus

\[
\sum_{i} \mathcal{H}^1(\partial R_i \cap Q''_+) \geq \mathcal{H}^1(\partial R_k) \geq d(R_k) > \frac{1}{3} \theta s_{j-1}.
\]

This yields \( Q_+ \in A_{j-1} \). Consequently, as \( Q \subset Q_+ \), this implies \( Q \notin B_i \) giving a contradiction.

b) Now assume \( d(R_i) \leq \frac{1}{3} s_{i-1} \) for all \( R_i \in \mathcal{F} \). Then similarly as before we see \( \overline{R_i} \subset Q''_+ \), where \( Q_+ \in Q_{i-1} \) such that \( Q \subset Q_+ \). Arguing as in a) we derive \( \sum_{i} \mathcal{H}^1(\partial R_i \cap Q''_+) \geq \sum_{R_i \in \mathcal{F}} d(R_i) > s_i = \theta s_{i-1} \). Moreover, we get \( s_{i-1} < \theta^{-1} \sum_{i} d(R_i) \leq \frac{1}{21} \text{dist}(\partial \overline{Q}, \partial Q_\mu) \) and thus \( Q'' \subset \overline{Q} \). This gives \( Q_+ \in A_{i-1} \) leading again to a contradiction.

Likewise, we obtain (ii): Assume there was some \( Q \in B_i \) for \( i \) so large that \( s_i \leq \theta \min_{i=1}^n d(R_i) \). Then we find some \( R_k \) with \( R_k \cap Q'' \neq \emptyset \) with \( d(R_k) \geq s_i \) which as in a) above yields a contradiction.

Moreover, the definition in (17) implies \( J \cap Q_\mu \subset \bigcup_{i \geq 1} A_i \cap Q_\mu \) and thus (iii) follows from the property \( \bigcup_{i \geq 1} A_i \cap Q_\mu = \bigcup_{i \geq 1} B_i \cap Q_\mu \) and the fact that \( J_\mu \subset J \).

To show (iv) we define \( \hat{B}_i = \{ Q \in B_i : \overline{Q} \notin \bigcup_{j=1}^{i-1} B'_j \} \) as well as \( \hat{B}'_i = \bigcup_{Q \in \hat{B}_i} \overline{Q} \) and observe \( \bigcup_{i \geq 1} B'_i = \bigcup_{i \geq 1} \hat{B}'_i \). It is elementary to see that for \( \theta \) small enough each \( x \in \overline{Q} \) is contained in at most one set \( \bigcup_{Q \in \hat{B}_i} \overline{Q} \), \( i \in \mathbb{N} \), and thus contained in at most four different squares \( Q'' \) with \( Q \in \bigcup_{i \geq 1} \hat{B}_i \). Consequently, we derive
using (17) for a universal constant $C > 0$ large enough

$$
\mathcal{H}^1(\partial (\bigcup_{i \geq 1} B_i')) \leq \sum_{Q \in \mathcal{U}_{i \geq 1}} \mathcal{H}^1(\partial Q') \leq C\theta^{-1} \sum_{Q \in \mathcal{U}_{i \geq 1}} \left( \sum_{l=1}^n \mathcal{H}^1(Q'' \cap \partial R_l) \right)
$$

$$
\leq C\theta^{-1} \sum_{l=1}^n \mathcal{H}^1(\bar{Q} \cap \partial R_l) \leq C\theta^{-1} \sum_{l=1}^n d(R_l) \leq C\mu,
$$

where in the last step we employed (16). Finally, (v) follows from a similar argumentation, again using (17).

**Step II (Korn-Poincaré inequality):** Recall that by the definition in (17) we find $Q'' \subset Q$ for every $Q \in \mathcal{B}_i$. Thus, using Theorem 2.3 on $Q''$ for $Q \in \mathcal{B}_i$, we obtain by (8) and (18)(i) infinitesimal rigid motions $a_Q = a_{A_Q, B_Q}$ such that

$$
\| \bar{u} - a_Q \|_{L^2(Q''')}^2 \leq C(\| E\bar{u} \|_{L^2(Q''')})^2 \leq C_s^2 \varepsilon \| e(\bar{u}) \|_{L^2(Q''')}^2 + C\varepsilon \mathcal{H}^1(J_{\bar{u}} \cap Q''') \sum_{R_i \cap Q''' \neq \emptyset} d(R_i))^2
$$

$$
\leq C_s^2 \varepsilon \| e(\bar{u}) \|_{L^2(Q''')}^2 + \varepsilon \mathcal{H}^1(\bar{J} \cap Q''')
$$

for all $Q \in \mathcal{B}_i$, where in the last step we employed $J_{\bar{u}} \subset \bar{J}$. For shorthand we let $E := \| e(u) \|_{L^2(Q_{\mu})} + \varepsilon$. Summing over all squares in $\mathcal{B}_i$, using (7) and recalling that each $x \in \bar{Q}$ is contained in at most four different $Q''', Q \in \mathcal{B}_i$, we get for all $i \geq 1$

$$
\sum_{Q \in \mathcal{B}_i} \| \bar{u} - a_Q \|_{L^2(Q''')}^2 \leq C_s^2 \varepsilon \| e(\bar{u}) \|_{L^2(Q_{\mu})}^2 + \| e(u) \|_{L^2(Q_{\mu})}^2 \leq C_s^2 E^2,
$$

(19)

where we used $\varepsilon = (\mathcal{H}^1(J_{\bar{u}}))^{-1} \| e(u) \|_{L^2(Q_{\mu})}^2 \leq C(\mathcal{H}^1(J))^{-1} \| e(u) \|_{L^2(Q_{\mu})}^2$ (cf. (15)).

**Step III (Modification):** We now show that we can ‘heal’ the discontinuities of $\bar{u}$ in $Q_{\mu}$. The strategy is to modify the displacement field inductively. Let $1 < p < 2$ be given. Let $I \in \mathbb{N}$ be the largest index such that $B_I \neq \emptyset$ (see (18)(ii)). Define $\bar{u}_0 = \bar{u}$ and assume $\bar{u}_j \in SBD^2(\bar{Q})$ has already been constructed satisfying

$$
(i) \quad J_{\bar{u}_j} \cap Q_{\mu} \subset \bigcup_{k=1}^{I-j} B_k,
$$

(20)

$$
(ii) \quad \sum_{Q \in \mathcal{B}_i} \| \bar{u}_j - a_Q \|_{L^2(Q''')}^2 \leq \bar{C} s^2 \varepsilon \prod_{k=0}^{j} (1 + \eta^{I-j-k}) E^2 \forall i \leq I - j,
$$

$$
(iii) \quad \| e(\bar{u}_j) \|_{L^p(Q_{\mu})}^p \leq \bar{C} \mu^{2-p} \prod_{k=0}^{j} (1 + \eta^{I-k}) E^p
$$

for some $\bar{C}$ large enough, where for shorthand $\eta = \theta_{\frac{1}{2} - \frac{x}{4}} < 1$. Clearly, by (13), (18)(ii),(iii), (19) and Hölder’s inequality together with $|Q_{\mu}| \leq 4\mu^2$ we find that (20) holds for $j = 0$.

We now construct $\bar{u}_{j+1}$. In the following $C > 0$ denotes a generic constant which is always independent of $\theta$. We consider a partition of unity $\{ \varphi_0 \} \cup$
(\varphi_Q)_{Q \in B_{I-j}} \subset C^\infty(\mathbb{R}^2) \) with the properties

\[(i) \quad \varphi_0(x) + \sum_{Q \in B_{I-j}} \varphi_Q(x) = 1 \quad \text{for all } x \in \tilde{Q},
(ii) \quad Q \subset \text{supp}(\varphi_Q) \subset Q'' \quad \text{for all } Q \in B_{I-j},
(iii) \quad \text{supp}(\varphi_0) \subset \mathbb{R}^2 \setminus B_{I-j},
(iv) \quad \|\nabla \varphi_0\|_{\infty}, \|\nabla \varphi_Q\|_{\infty} \leq Cs_{I-j}^{-1} \quad \text{for all } Q \in B_{I-j}.
\]

Then we define

\[
\bar{u}_{j+1}(x) = \bar{u}_j(x) + \sum_{Q \in B_{I-j}} \varphi_Q(x)(A_Q x + b_Q - \bar{u}_j(x)) = \bar{u}_j + \sum_{Q \in B_{I-j}} \varphi_Q(a_Q - \bar{u}_j)
\]

for all \( x \in \tilde{Q} \). As \( \bar{u}_{j+1} \) is smooth in \( B_{I-j} \) by (21)(i),(iii), (20)(i) holds. Employing (20)(ii) for \( j \) and \( i = I - j \) we obtain

\[
\|\bar{u}_{j+1} - \bar{u}_j\|_{L^2(\tilde{Q})}^2 \leq C \sum_{Q \in B_{I-j}} \|\bar{u}_j - a_Q\|_{L^2(Q'')}^2 \leq C\bar{C}s_{I-j}^2 \prod_{k=0}^j (1 + \eta^{j-k})E^2 \leq C\bar{C}s_{I-j}E^2,
\]

where in the first step we used that each \( x \in \tilde{Q} \) is contained in at most four different enlarged squares. Using \( \eta = \theta^{1 - \frac{p}{q}} \geq 2 \), we get for \( \theta \) small enough (recall that \( C \) is independent of \( \theta \))

\[
Cs_{I-j}^2 \leq C\bar{s}_i^2 \eta^{2(I-j-i)} \leq \frac{1}{2}s_i^2 \eta^{2(I-j-i-1)}
\]

for all \( 1 \leq i \leq I - j - 1 \). The last estimate together with (20)(ii) for \( j \), (22) and a scaled version of Young's inequality of the form \((a + b)^2 \leq (1 + \delta)a^2 + (1 + \frac{1}{\delta})b^2 \) \((a, b \in \mathbb{R}, \delta > 0)\) yields for \( \delta = \frac{1}{2}\eta^{I-i-j-1} \)

\[
\sum_{Q \in B_i} \|\bar{u}_{j+1} - a_Q\|_{L^2(Q'')}^2 \\
\leq (1 + \frac{1}{2}\eta^{I-i-j-1}) \sum_{Q \in B_i} \|\bar{u}_j - a_Q\|_{L^2(Q'')}^2 + C\eta^{-I-i-j-1}\|\bar{u}_{j+1} - \bar{u}_j\|_{L^2(\tilde{Q})}^2 \\
\leq \bar{C}s_i^2 \prod_{k=0}^j (1 + \eta^{j-i-k})(1 + \frac{1}{2}\eta^{j-i-i-j-1})E^2 + \bar{C}\eta^{j-i-j-1}s_i^2E^2 \\
\leq \bar{C}s_i^2 \prod_{k=0}^{j+1} (1 + \eta^{j-i-k})E^2
\]

for all \( 1 \leq i \leq I - j - 1 \). This shows (20)(ii). To confirm (20)(iii) we first note that by Hölder's inequality, (20)(ii) and the fact that \( |B'_{I-j}| \leq Cs_{I-j} \) (see (18)(v)) we obtain

\[
\sum_{Q \in B_{I-j}} \|\bar{u}_j - a_Q\|_{L^p(Q'')}^p \leq C|B'_{I-j}|^{\frac{p}{2}} \left( \sum_{Q \in B_{I-j}} \|\bar{u}_j - a_Q\|_{L^2(Q'')}^2 \right)^{\frac{p}{2}} \\
\leq C\bar{C}s_i^2 \mu^{1-\frac{p}{2}}s_{I-j}E^p,
\]

13
where we again used that each $x \in \tilde{Q}$ is contained in at most four different enlarged squares. We calculate the derivative

$$\nabla \tilde{u}_{j+1} = \nabla \tilde{u}_j \varphi_0 + \sum_{Q \in \mathcal{B}_{l-j}} \varphi_Q A_Q + (a_Q - \tilde{u}_j) \otimes \nabla \varphi_Q.$$ 

Now we again apply a scaled version of Young’s inequality of the form $|a + b|^p \leq (1 + \delta)|a|^p + (1 + \delta^2)|b|^p$ for $a, b \in \mathbb{R}, \delta > 0$. Consequently, similarly as before using (20)(ii), (21)(iv) and (23) we find with $\delta^2 = \frac{1}{2} \eta^{l-j-1}$

$$\|e(\tilde{u}_{j+1})\|_{L^p(\tilde{Q}_\mu)} \leq (1 + \delta^2 \|e(\tilde{u}_j)\|_{L^p(\tilde{Q}_\mu)} + C \delta^{-\frac{p}{2}} s_{l-j}^{-p} \sum_{Q \in \mathcal{B}_{l-j}} \|\tilde{u}_j - a_Q\|_{L^p(\mu)}^p \leq (1 + \delta^2 \|e(\tilde{u}_j)\|_{L^p(\tilde{Q}_\mu)} + C \delta^{-\frac{p}{2}} \tilde{C} \frac{1}{2} \mu^{1-\frac{p}{2}} \bar{E}^p \| \leq \tilde{C} \mu^{2-p} \prod_{k=0}^{j} (1 + \eta^{l-k})(1 + \frac{1}{2} \eta^{l-j-1}) \bar{E}^p + \tilde{C} \frac{1}{2} \eta^{l-j-1} \mu^{2-p} \bar{E}^p \leq \tilde{C} \mu^{2-p} \prod_{k=0}^{j+1} (1 + \eta^{l-k}) \bar{E}^p,$$

where we used $\eta^2 = \theta^{1-\frac{p}{2}}$ and thus $C s_{l-j}^2 = C \mu^{1-\frac{p}{2}} \eta^{2(l-j)} \leq \frac{1}{2} \mu^{1-\frac{p}{2}} \eta^{2(l-j-1)}$ for $\theta$ sufficiently small since $C$ is independent of $\theta$.

**Step IV (Korn’s inequality):** Assume $\theta > 0$ has been fixed according to Step III. We define $\hat{u} = \tilde{u}_I$ and observe that by (20)(i),(iii) we have $\hat{u}|_{\tilde{Q}_\mu} \in W^{1,p}(\tilde{Q}_\mu)$ with

$$\|e(\hat{u})\|_{L^p(\tilde{Q}_\mu)} \leq C \mu^{-\frac{p}{2}} - 1 E = C \mu^{-\frac{p}{2}} - 1 (\|e(u)\|_{L^2(\tilde{Q}_\mu)} + \delta)$$

for some $C = C(p)$. Moreover, we define $E = \bigcup_{i=1}^j B'_i$ and get that $\hat{u} = \tilde{u}_{I} \in \tilde{Q}_\mu \setminus E$ due to the construction of the functions $(\tilde{u}_j)_j$. By (15) and (18)(iv) we obtain $H^1(\partial E) \leq C H^1(J_u)$. In view of the definition in (17) we find some $i_0 \in \mathbb{N}$ with $s_{i_0} \leq c \sum_{k=1}^{j} \mathcal{H}^1(\partial R_k \cap \tilde{Q})$ for a sufficiently large $c$ such that $B'_{i_0} = \emptyset$ for all $i \leq i_0$. Thus, using (15), (18)(v) we find

$$|\bigcup_{i=1}^j B'_i| \leq C \sum_{i} d(R_i) \sum_{k=i+1}^j s_k \leq C \mathcal{H}^1(J_u) s_{i_0} \leq C (\mathcal{H}^1(J_u))^2.$$ 

This yields $|E| \leq C (\mathcal{H}^1(J_u))^2$ and shows (9). We now apply Poincaré’s and Korn’s inequality (see Theorem 2.5) and find $A \in \mathbb{R}^{2 \times 2}, b \in \mathbb{R}^2$ such that by a standard scaling argument

$$\|
abla \tilde{u} - A\|_{L^p(\tilde{Q}_\mu \setminus E)} \leq \|
abla \tilde{u} - A\|_{L^p(\tilde{Q}_\mu)} \leq C \|e(\hat{u})\|_{L^p(\tilde{Q}_\mu)},$$

$$\|
\tilde{u} - (A \cdot + b)\|_{L^q(\tilde{Q}_\mu \setminus E)} \leq \|
\tilde{u} - (A \cdot + b)\|_{L^q(\tilde{Q}_\mu)} \leq C \mu^{-\frac{p}{2} - 1} \|e(\hat{u})\|_{L^p(\tilde{Q}_\mu)}$$

for $q \leq \frac{2p - 1}{2p}$. Then the second part of (14) holds for $p \in (1, 2)$ by (24) and the case $p = 1$ directly follows. Likewise, the first part of (14) also holds for $q \in [1, \infty)$ since $p \in [1, 2)$.\[\square\]

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3.2 General case

To prove the local estimate for a general function we consider a sequence of modifications and show that the properties in Theorem 3.1 can be recovered in the limit.

Proof of Theorem 3.1. Let $p \in (1, 2)$, $q \in [1, \infty)$ and let $u \in SBD^2(Q_n)$ be given. By Theorem 3.2 for $\delta = \frac{1}{n}$ we obtain modifications $\bar{u}_n$ and exceptional sets $F_n$ such that (13) holds. Moreover, we find $A_n \in \mathbb{R}^{2 \times 2}_{skew}$, $b_n \in \mathbb{R}^2$ as well as exceptional sets $E_n \subset Q_n$ with $H^1(\partial E_n) \leq C H^1(J_u)$, $|E_n| \leq C(H^1(J_u))^2$ such that

\begin{align}
& (i) \| \bar{u}_n - (A_n \cdot + b_n) \|^2_{L^p(Q_n \setminus E_n)} \leq C \mu^2 \left( \| e(u) \|_{L^2(Q_n)} + \frac{1}{n} \right), \\
& (ii) \| \nabla \bar{u}_n - A_n \|^2_{L^p(Q_n \setminus E_n)} \leq C \mu^{\frac{2}{p}-1} \left( \| e(u) \|_{L^2(Q_n)} + \frac{1}{n} \right),
\end{align}

where $Q_n = (-\bar{\mu}, \bar{\mu})^2$ with $\bar{\mu} = \max \{ \mu - C H^1(J_u), 0 \}$ independently of $n$. Define $G_n = E_n \cup F_n$ and observe $|G_n| \leq C(H^1(J_u))^2$, $H^1(G_n) \leq C H^1(J_u)$ for a sufficiently large constant. Then by Theorem 2.2 we find a set of finite perimeter $E \subset Q_n$ with $H^1(\partial^* E) \leq C H^1(J_u)$, $|E| \leq C(H^1(J_u))^2$ such that $\chi_{G_n} \to \chi_E$ in measure for $n \to \infty$ for a not relabeled subsequence.

Moreover, letting $v_n := (\bar{u}_n - (A_n \cdot + b_n)) \chi_{Q_n \setminus E_n} \in GSBV^p(Q_n)$ and using (13), (25) we apply Ambrosio’s compactness result in GSBV (see Theorem 2.1) to find a function $v \in GSBV^p(Q_n)$ such that passing to a further (not relabeled) subsequence we obtain $v_n \to v$ a.e. and $\nabla v_n \rightharpoonup \nabla v$ weakly in $L^p$. In particular, we derive by Fatou’s lemma

\begin{align}
& (i) \| v \|^2_{L^p(Q_n \setminus E)} \leq \liminf_{n \to \infty} \| v_n \|^2_{L^p(Q_n \setminus E)} \leq C \mu^2 \| e(u) \|^2_{L^2(Q_n)}, \\
& (ii) \| \nabla v \|_{L^p(Q_n \setminus E)} \leq \liminf_{n \to \infty} \| \nabla v_n \|_{L^p(Q_n \setminus E)} \leq C \mu^{\frac{2}{p}-1} \| e(u) \|_{L^2(Q_n)}.
\end{align}

Consequently, to finish the proof it suffices to show $v = (u - a) \chi_{Q_n \setminus E}$ for some infinitesimal rigid motion $a = a_{A,b}$. (Observe that as before the assertion then holds also for $p = 1$.)

Possibly passing to a further subsequence we can assume $\chi_{G_n} \to \chi_E$ pointwise a.e. and thus we find a measurable set $H$ with $|H| > 0$ such that $H \subset Q_{\bar{\mu}} \setminus E$, $H \subset Q_{\bar{\mu}} \setminus G_n$ (up to a set of negligible measure) for $n$ large enough. By (13), (25) (i) and Hölder’s inequality this implies

$$\| A_n \cdot + b_n \|_{L^1(H)} \leq C(\| e(u) \|^2_{L^2(Q_n)} + \frac{1}{n}) + C \| \bar{u}_n \|_{L^1(H)} \leq C$$

for $C = C(\mu) > 0$ large enough. Consequently, we obtain $A_n \to A$, $b_n \to b$ for some $A \in \mathbb{R}^{2 \times 2}_{skew}$ and $b \in \mathbb{R}^2$. As $\bar{u}_n \chi_{Q_n \setminus E_n} \to u \chi_{Q_n \setminus E}$ in $L^1$ by (13) and $v_n \to v = v \chi_{Q_n \setminus E}$ a.e., we conclude $v = (u - (A \cdot + b)) \chi_{Q_n \setminus E}$.

As $GSBV^p(Q_n)$ is a vector space (see Proposition 2.3), we then get $u \chi_{Q_n \setminus E} \in GSBV^p(Q_n)$. To see the additional statement that $u \chi_{Q_n \setminus E} \in SBV^p(Q_n)$ if $u \in L^\infty(Q_n)$, we observe that $|A_n|, |b_n| \leq C$ and $\| \bar{u}_n \|_{L^\infty(Q_n \setminus F_n)} \leq \| u \|_{L^\infty}$ (see Theorem 2.6) imply $\| v_n \| \leq C$ independently of $n \in \mathbb{N}$ and the claim follows from Theorem 2.7. \qed
4 Estimate at the boundary

In this section we give a refined estimate which holds up to the boundary of Lipschitz sets. This together with a standard covering argument will then lead to the proof of the main theorem. We first give an elementary estimate about the difference of infinitesimal rigid motions which we state in arbitrary space dimensions.

Lemma 4.1 Let $p \in [1, \infty)$ and $\bar{c} > 0$. Then there is a constant $C = C(p, \bar{c})$ such that for all $x \in \mathbb{R}^d$, $R > 0$ and measurable $\Omega \subset Q^x_R := x + (-R, R)^d$ with $|\Omega| \geq \bar{c}R^d$ and all affine mappings $a : \mathbb{R}^d \to \mathbb{R}^d$ one has

$$\|a\|_{L^p(Q^x_R)} \leq C\|a\|_{L^p(\Omega)}.$$  

Although similar estimates have already been used (see e.g. [10, 25]) we include the elementary proof here for the sake of completeness.

Proof. We first note that by an elementary translation argument it suffices to consider cubes $Q^x_R := (-R, R)^d$ centered at the origin. Assume the statement was false. Then there would be sequences $(R_k)$, $(\Omega_k)$ with $\Omega_k \subset Q^{R_k}_x$, $|\Omega_k| \geq \bar{c}R_k^d$ and a sequence of affine mappings $(a_k)$ with

$$\|a_k\|_{L^p(Q^x_{R_k})} > k\|a_k\|_{L^p(\Omega_k)}.$$  

We define $c_k(x) = a_k(R_kx)$ as well as $\Omega'_k = \frac{1}{R_k}\Omega_k$ and obtain by transformation

$$\|c_k\|_{L^p(Q_1)} > k\|c_k\|_{L^p(\Omega'_k)}.$$  

Then we define the affine mappings $d_k = \frac{c_k}{\|c_k\|_{L^p(Q_1)}}$ and derive

$$1 = \|d_k\|_{L^p(Q_1)} > k\|d_k\|_{L^p(\Omega'_k)}.$$  

As $(d_k)$ are affine, we find that $\|d_k\|_{W^{1,p}(Q_1)}$ is uniformly bounded and thus a compactness result yields (after passage to a not relabeled subsequence) $d_k \to d$ in $L^p(Q_1)$ for some affine mapping $d$. Moreover, there is a measurable function $f$ with $f \geq 0$, $\|f\|_{L^1(Q_1)} \geq \bar{c}$ such that $\chi_{\Omega'_k} \rightharpoonup f$ weakly in $L^\infty(Q_1)$. Consequently, we find $1 = \|d\|_{L^p(Q_1)}$ and $0 = \|d \cdot f\|_{L^1(Q_1)}$ which gives a contradiction. □

We are now in a position to give the boundary estimate.

Theorem 4.2 Let $p \in [1, 2)$, $q \in [1, \infty)$. Let $\mu > 0$ and $\psi : (-2\mu, 2\mu) \to [\mu, \infty)$ Lipschitz with $\|\psi'\|_{\infty} \leq \bar{c}$ and $\inf \psi = \mu$. Let

$$U = \{(x_1, x_2) : -2\mu < x_1 < 2\mu, -2\mu \leq x_2 \leq \psi(x_1)\},$$

$$U' = \{(x_1, x_2) : -\mu < x_1 < \mu, -\mu \leq x_2 \leq \psi(x_1)\}.$$  

(26)
Then there is a constant $C = C(p, q, \hat{c})$ independent of $\mu$ such that for all $u \in SBD^2(U)$ there is a set of finite perimeter $F \subset U$ with $\mathcal{H}^1(\partial^* F) \leq C \mathcal{H}^1(J_u)$, $|F| \leq C(\mathcal{H}^1(J_u))^2$ and for suitable $A \in \mathbb{R}^{2\times 2}$, $b \in \mathbb{R}^2$

\begin{align}
(i) \quad & \|u - (A \cdot + b)\|_{L^q(U \setminus F)} \leq C \mu_{1+}^2 \|e(u)\|_{L^2(U)}, \\
(ii) \quad & \|\nabla u - A\|_{L^p(U \setminus F)} \leq C \mu_{1+}^{2-1} \|e(u)\|_{L^2(U)}.
\end{align}

Proof. Recall the definition of the sets $Q_i$, $i \in \mathbb{N}$, and the enlarged squares $Q \subset Q'' \subset Q'$ in (11), (12). Moreover, by $d(B)$ we again denote the diameter of a set $B \subset \mathbb{R}^2$. Let $Q_W \subset \bigcup_{i>1} Q_i$ be a Whitney-type covering of $U$, i.e. $\bigcup_{Q \in Q_W} Q' = U$ such that (cf. e.g. [11, 22, 40])

\begin{align}
(i) \quad & d(Q) \leq \text{dist}(Q, \partial U) \leq C d(Q) \quad \text{for all } Q \in Q_W, \\
(ii) \quad & \# \{Q \in Q_W : x \in Q'\} \leq N \quad \text{for all } x \in U, \\
(iii) \quad & Q_1 \cap Q_2 \neq \emptyset \quad \text{for } Q_1, Q_2 \in Q_W \Rightarrow \frac{1}{c}d(Q_1) \leq d(Q_2) \leq C d(Q_1).
\end{align}

Moreover, we consider a corresponding partition of unity $(\varphi_Q)_{Q \in Q_W} \subset C^\infty(U)$ with $\sum_{Q \in Q_W} \varphi_Q(x) = 1$ for $x \in U$ and

\begin{align}
(i) \quad & Q \subset \text{supp}(\varphi_Q) \subset Q'' \quad \text{for all } Q \in Q_W, \\
(ii) \quad & \|\nabla \varphi_Q\|_{\infty} \leq C d(Q)^{-1} \quad \text{for all } Q \in Q_W.
\end{align}

for a universal constant $C > 0$. Let

$$B = \{Q \in Q_W : \mathcal{H}^1(Q' \cap J_u) \geq \hat{c}d(Q)\}$$

be the ‘bad’ squares for some $\hat{c}$ sufficiently small to be specified below. For each enlarged square $Q' = p + (r, r)^2$, $Q \in B$, we define $P_Q = (p + (r, r^2) \times (-r, \infty)) \cap U$. Employing (28)(i) and using $\|\psi\|_{\infty} \leq \hat{c}$ we then observe that $\mathcal{H}^1(\partial P_Q) \leq C d(Q)$ for $C = C(\hat{c})$ and thus $\mathcal{H}^1(\partial P_Q) \leq C \mathcal{H}^1(Q' \cap J_u)$ for some $C = C(\hat{c}, \hat{c})$. Letting $P = \bigcup_{Q \in B} P_Q$ we obtain by (28)(ii)

$$\mathcal{H}^1(\partial P) \leq C \mathcal{H}^1(J_u)$$

and using the isoperimetric inequality we also find $|P| \leq C(\mathcal{H}^1(J_u))^2$. We let $V = U' \setminus P$.

Observe that we can assume $Q \notin B$ for all $Q \in Q_W$ with $Q \cap (-\mu, \mu) \times \{0\} \neq \emptyset$. In fact, these squares satisfy $d(Q) \geq c \mu$. Consequently, if $Q \in B$, we find $\mathcal{H}^1(J_u) \geq \hat{c}c^2 \mu$ and in this case the claim of the theorem holds with the choice $F = U$ if in the assertion $C$ is chosen large enough.

We now see that $V$ is a John domain with center 0 and a constant only depending on $\hat{c}$. In fact, fix some $x = (x_1, x_2) \in V$ and $Q \in Q_W$ such that $x \in \overline{Q}$. We consider a vertical chain $C_1 = \{Q_1^1 = Q_1, \ldots, Q_n^1\}$ of squares in $Q_W$ intersecting $\{x_1\} \times [0, x_2]$ together with a horizontal chain $C_2 = \{Q_1^2 = Q_1^2, \ldots, Q_n^2\}$.
\[Q_1, \ldots, Q_2\} \text{ of squares intersecting } [x_1, 0] \times \{0\} \text{ such that } \overline{Q^j_k} \cap \overline{Q^j_{k+1}} \neq \emptyset \text{ for } 1 \leq k \leq n_j - 1, j = 1, 2.

Now in view of (28) we see that 
\[d(Q^j_k) \leq d(Q^j_{k+1}) \text{ for all } 1 \leq k_1 \leq k_2 \leq n_j, \quad j = 1, 2, \quad \text{and } d(Q^j_{k_2}) \leq \theta d(Q^j_{k_1}) \text{ for all } k_2 \geq k_1 + l \text{ for some } l = l(\hat{c}) \in \mathbb{N}.
\]

Consequently, it is elementary to construct a curve \(\gamma\) starting in \(x\), ending in 0 and intersecting the midpoints of the squares in \(C_1 \cup C_2\) such that the condition given in Definition 2.4 holds (cf. Figure 1).

Let \(\mathcal{G} = Q_W \setminus \mathcal{B}\). For each \(Q \in \mathcal{G}\) we apply Theorem 3.1 on \(Q'\) to find infinitesimal rigid motions \(a_Q = a_{A_Q, b_Q}\) and exceptional sets \(E_Q\) so that by (10)

\[
(i) \quad \|u - (A_Q \cdot + b_Q)\|_{L^2(Q'' \setminus E_Q)} \leq Cd(Q)^{\frac{2}{p}}\|e(u)\|_{L^2(Q')},
(ii) \quad \|\nabla u - A_Q\|_{L^p(Q'' \setminus E_Q)} \leq Cd(Q)^{\frac{2}{p} - 1}\|e(u)\|_{L^2(Q')}.
\]

(For \(\hat{c}\) sufficiently small in (30) we can in fact assume that \(Q''\) is contained in the shrunken square given by Theorem 3.1.) Moreover, by (9) and (28)(ii) we get that \(\hat{E} := \bigcup_{Q \in \mathcal{G}} E_Q\) fulfills \(\mathcal{H}^1(\partial^* E) \leq C\mathcal{H}^1(J_n)\) and \(|E| \leq C(\mathcal{H}^1(J_n))^2\).

We now estimate the difference of the infinitesimal rigid motions. Consider some \(Q \in \mathcal{G}\) and let \(\mathcal{N}(Q) = \{Q \in \mathcal{G} \setminus \{Q\} : Q'' \cap \hat{Q}'' \neq \emptyset\}\). Recall that \(\frac{1}{c}d(\hat{Q}) \leq d(\hat{Q}) \leq Cd(\hat{Q})\) for all \(\hat{Q} \in \mathcal{N}(Q)\) by (28)(iii) which also implies \#\(\mathcal{N}(Q)\) \leq C for some \(C > 0\) large enough. Since the covering \(Q_W\) consists of dyadic squares, \(Q'' \cap \hat{Q}''\) contains a ball \(B\) with radius larger than \(cd(\hat{Q})\) for some small \(c > 0\). In view of (9) we find choosing \(\hat{c}\) in (30) sufficiently small that \(|\hat{E} \cap B| \leq \frac{1}{2}|B|\), where \(\hat{E} = E_{\hat{Q}} \cup E_Q\). Therefore, by (32)(i) for \(p = q\), (28)(iii) and the triangle

\[\text{Figure 1: Illustration of a part of } U \text{ and } Q_W \cap \bigcup_{i=1}^3 Q_i. \text{ The squares in } B \text{ are depicted in dark gray and the corresponding set } P \text{ in light gray. Moreover, a John curve } \gamma \text{ connecting } x \text{ with } 0 \text{ is sketched.}
\]
inequality we derive
\[ \|a_Q - a_{\bar{Q}}\|_{L^p(B \setminus \bar{E})} = \|(A_Q \cdot b_Q) - (A_{\bar{Q}} \cdot b_{\bar{Q}})\|_{L^p(B \setminus \bar{E})} \leq C d(Q)^{\frac{2}{p}} \|e(u)\|_{L^2(Q \cup \bar{Q}')} \]
and thus by Lemma \ref{lem:4.1}
\[ \|a_Q - a_{\bar{Q}}\|_{L^p(Q')} \leq C d(Q)^2 \|e(u)\|_{L^2(Q \cup \bar{Q}')} \] (33)
for some \( C = C(p) \). Let \( N_Q = \bigcup_{Q \in \mathcal{N}(Q)} \bar{Q}' \cup Q' \) and observe that by (28) each \( x \in U \) is contained in a bounded number of different sets \( N_Q \). Moreover, we observe that \( \sum_{Q \in \mathcal{G}} d(Q)^2 \leq C |V| \leq C \mu^2 \). Summing over all squares, recalling \( \# \mathcal{N}(Q) \leq C \) and using Hölder’s inequality we then find
\[ \sum_{Q \in \mathcal{G}} \sum_{Q \in \mathcal{N}(Q)} d(Q)^{-p} \|a_Q - a_{\bar{Q}}\|_{L^p(Q')} \leq C \sum_{Q \in \mathcal{G}} d(Q)^{-p} \|e(u)\|_{L^2(N_Q)}^{\frac{p}{2}} \leq C \mu^2 \|e(u)\|_{L^2(U)}^{\frac{p}{2}} \] (34)
We observe that \( \sum_{Q \in \mathcal{G}} \varphi_Q(x) = 1 \) for all \( x \in V \). In fact, we recall that \( \varphi_Q \in \mathcal{Q}_w \) is a partition of unity and \( \text{supp}(\varphi_Q) \subset Q'' \subset U \setminus V \) for all \( Q \in \mathcal{B} = \mathcal{Q}_w \setminus \mathcal{G} \) by construction. Similarly as in the proof of Theorem \ref{thm:3.2} we define
\[ \bar{u}(x) = \sum_{Q \in \mathcal{G}} \varphi_Q(x) (A_Q x + b_Q) = \sum_{Q \in \mathcal{G}} \varphi_Q(x) a_Q(x) \]
for all \( x \in V \). Clearly, \( \bar{u} \) is smooth in \( V \). Using \( \sum_{Q \in \mathcal{G}} \nabla \varphi_Q = 0 \) we find that
\[ \nabla \bar{u} = \sum_{Q \in \mathcal{G}} \left((a_Q - f) \otimes \nabla \varphi_Q + \varphi_Q A_Q\right) \] (35)
for all functions \( f \). Consequently, letting \( f(x) = a_Q(x) \) for \( x \in Q \cap V \), \( Q \in \mathcal{G} \), we derive using once more (28)(ii),(iii) and applying (29), (34)
\[ \|e(\bar{u})\|_{L^p(V)}^{\frac{p}{2}} \leq C \sum_{Q \in \mathcal{G}} d(Q)^{-p} \sum_{Q \in \mathcal{N}(Q)} \|a_Q - a_{\bar{Q}}\|_{L^p(Q')}^{\frac{p}{2}} \leq C \mu^2 \|e(u)\|_{L^2(U)}^{\frac{p}{2}}. \]
We compute using (28)(ii), (29)(i) and (32)(i)
\[ \|\bar{u} - u\|_{L^p(V \setminus E)} \leq C \sum_{Q \in \mathcal{G}} \|a_Q - u\|_{L^p(Q'' \setminus E)} \leq C \sum_{Q \in \mathcal{G}} d(Q)^{\frac{2}{p}} \|e(u)\|_{L^2(Q')} \]
\[ \leq C \mu^{\frac{2}{p}} \|e(u)\|_{L^2(U)}. \] (36)
Likewise, using (35) for \( f = u \) and (32) for \( q = p \) we find repeating the Hölder-type estimate in (34)
\[ \|\nabla \bar{u} - \nabla u\|_{L^p(V \setminus E)} \leq C \sum_{Q \in \mathcal{G}} d(Q)^{-p} \|a_Q - u\|_{L^p(Q'' \setminus E)}^{\frac{p}{2}} \]
\[ + C \sum_{Q \in \mathcal{G}} \|\nabla u - A_Q\|_{L^p(Q'' \setminus E)}^{\frac{p}{2}} \leq C \sum_{Q \in \mathcal{G}} d(Q)^{\frac{2}{p}} \|e(u)\|_{L^2(Q')}^{\frac{p}{2}} \leq C \mu^2 \|e(u)\|_{L^2(U)}^{\frac{p}{2}}. \] (37)
As $\bar{u}$ is smooth in $V$ and $V$ is a John domain with constant only depending on $c$, we can apply Theorem 2.5 and we find $A \in \mathbb{R}^{2 \times 2}$, $b \in \mathbb{R}^2$ such that by a scaling argument
\[
\mu^{-\frac{2}{q} - 1 + \frac{2}{p}} \|\bar{u} - (A \cdot + b)\|_{L^q(V)} + \|\nabla \bar{u} - A\|_{L^p(V)} \leq C\|e(\bar{u})\|_{L^p(V)} \leq C\mu^{\frac{2}{p} - 1} e(u)\|_{L^2(U)}
\]
for $C = C(p, q, \bar{c})$. We now define $F = E \cup P$ and by (31) and the remark below (32) we obtain $|F| \leq C(H^1(J_u))^2$ as well as $H^1(\partial^* F) \leq C H^1(J_u)$. Finally, (27) follows from (36) and (37).

Remark 4.3 Similarly as in the local estimate considered in Section 3 one can show that the displacement field restricted to $U' \setminus F$ is an element of $GSBV^p$ or $SBV^p$, respectively. As this property will not be needed in the following, we have omitted the proof.

5 Proof of the main result and application

5.1 Proof of Theorem 1.1

We now combine the local estimate in Theorem 3.1, the boundary estimate (Theorem 4.2) and a standard covering argument to prove the main result. A similar argument may be found, e.g. in [10], where an inequality of Korn-Poincaré type is derived.

Proof of Theorem 1.1 We first choose finitely many $U_1, \ldots, U_n$ being of the form given in (26) (possibly after application of an affine isometry) such that $\partial \Omega$ is covered by $U_1', \ldots, U_n'$. Moreover, we cover $\Omega \setminus \bigcup_{i=1}^n U_i'$ with squares $U_{n+1}', \ldots, U_m'$ such that the squares $U_{n+1} := 2U_{n+1}', \ldots, U_m := 2U_m'$ of double size are still contained in $\Omega$.

By a similar reasoning as in the proof of Theorem 4.2 we may suppose that $H^1(J_u) \leq \hat{c}$ for some $\hat{c} = \hat{c}(p, q, \Omega)$ to be specified below as otherwise we can choose $E = \Omega$ in Theorem 1.1. We now apply Theorem 3.1 and Theorem 4.2, respectively, on the sets $(U_i)_{i=1}^m$ and obtain infinitesimal rigid motions $a_i = a_{A_i, b_i}$ as well as exceptional sets $E_i \subset U_i$ such that
\[
\sum_{i=1}^m (\|u - a_i\|_{L^q(U_i' \setminus E_i)} + \|\nabla u - A_i\|_{L^p(U_i')}) \leq C \sum_{i=1}^m \|e(u)\|_{L^q(U_i')} \leq C\|e(u)\|_{L^2(\Omega)}
\]
for some $C = C(p, q, \Omega)$. In fact, selecting $\hat{c}$ sufficiently small we get that the shrinked squares given in Theorem 3.1 contain $U_i'$ for $i = n + 1, \ldots, m$ (cf. (32) for a similar argument).
Define \( E = \bigcup_{i=1}^{m} E_i \) and observe that \(|E| \leq C(\mathcal{H}^1(J_u))^2\) as well as \(\mathcal{H}^1(\partial^* E) \leq C\mathcal{H}^1(J_u)\) follow from [9] and the similar estimate for the sets at the boundary (see before (27)). Moreover, we can choose \(\tilde{c}\) so small such \(|E| \leq \frac{1}{2}\eta\), where

\[
\eta := \min\{|U_i' \cap U_j'| : U_i', U_j', i \neq j, \text{ with } U_i' \cap U_j' \neq \emptyset\}.
\]

Obviously \(\eta\) only depends on \(\Omega\). Consequently, we obtain \(|(U_i' \cap U_j') \setminus E| \geq \frac{1}{2}\eta\) for all \(U_i', U_j', i \neq j\), with \(U_i' \cap U_j' \neq \emptyset\). As \(\Omega\) is connected, we then find by Lemma 4.1 and (38)

\[
\max_{1 \leq i, j \leq m} \|a_i - a_j\|_{L^q(\Omega)} + \|A_i - A_j\|_{L^p(\Omega)} \leq C\|c(u)\|_{L^2(\Omega)}
\]

for a constant depending only on \(p, q, \eta\) and \(m\). Recalling (38) and the fact that \(\eta, m\) only depend on \(\Omega\) we finally obtain (4) for, e.g., \(A = A_1\) and \(b = b_1\).

\[
\square
\]

5.2 Relation between SBV and SBD functions

Finally, we present a consequence of our main result concerning the relation between SBV and SBD functions. We briefly recall that the typical examples for functions lying in BD but not in BV or likewise lying in SBV but not in BV or SBD, \(p > 1\), are based on the idea to cut out small balls and to choose the displacement field appropriately on these sets (see e.g. [3, 14]). The following result shows that this construction essentially describes the only way to obtain functions of bounded deformation which do not have bounded variation. In particular, we see that for each function in \(SBD^2 \cap L^\infty\) there is a modification \(\tilde{u}\) in SBV such that \(\{u \neq \tilde{u}\}\) is an arbitrarily small set.

**Theorem 5.1** Let \(\varepsilon > 0\) and let \(\Omega \subset \mathbb{R}^2\) open, bounded with Lipschitz boundary. Then for every \(u \in SBD^2(\Omega)\) we find an exceptional set \(E\) with \(|E| \leq \varepsilon\) and \(\mathcal{H}^1(\partial^* E) < +\infty\) such that \(u\chi_{\Omega \setminus E} \in GSBV^p(\Omega)\) for all \(p < 2\). If in addition \(u \in L^\infty(\Omega)\), we obtain \(u\chi_{\Omega \setminus E} \in SBV^p(\Omega)\).

**Proof.** The statement follows from Theorem 3.1 by an additional covering argument. Assume first \(u \in SBD^2(\Omega) \cap L^\infty(\Omega)\). Recalling (11) we cover \(\Omega\) with squares in \(Q_i\) for \(s_i \ll \varepsilon\) to be specified below. We define the bad squares \(B = \{Q \in Q_i : Q' \not\subset \Omega \text{ or } \mathcal{H}^1(J_u \cap Q') \geq \tilde{c}s_i\}\) for a constant \(\tilde{c} > 0\). We let \(F = \Omega \cap \bigcup_{Q \in B} Q\) and observe \(\mathcal{H}^1(\partial F) \leq C\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial \Omega) < +\infty\) as well as \(|F| \leq Cs_i(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial \Omega))\).

Choosing \(\tilde{c}\) sufficiently small we can apply Theorem 3.1 on each enlarged square \(Q', Q \in Q_i \setminus B\), and obtain exceptional sets \(E_Q\) such that \(\{u\chi_{Q' \setminus E_Q}\}\) \(Q' \in SBV^p(Q')\) for all \(Q \in Q_i \setminus B\). (In fact, for \(\tilde{c}\) small we can assume that the shrinked square given in Theorem 3.1 contains \(Q''\).)

Letting \(E = \bigcup_{Q} E_Q \cup F\) we find by (9) that \(\mathcal{H}^1(\partial^* E) < +\infty\) and \(|E| \leq Cs_i\) for a constant depending only on \(\Omega\) and \(u\). Thus, \(|E| \leq \varepsilon\) for \(i \in \mathbb{N}\) sufficiently
large. Then defining \( \bar{u} = u\chi_{\Omega \setminus E} \) in \( \Omega \setminus F \) we derive \( \bar{u} \in SBV^p(\Omega \setminus F) \) for all \( p \in [1, 2) \). Observe that

\[
D(u\chi_{\Omega \setminus E}) = D\bar{u} + (\bar{u} \otimes \xi_F)\mathcal{H}^1|_{\partial F \cap \Omega}
\]  

in \( \Omega \), where \( \xi_F \) denotes the inner normal of \( F \) (see e.g. [4, Theorem 3.87]). As \( \|u\|_\infty < +\infty \), this implies \( u\chi_{\Omega \setminus E} \in SBV^p(\Omega) \). Likewise, in the general case we consider \( \phi \in C^1(\mathbb{R}^2) \) with the support of \( \nabla \phi \) compact and find \( (\phi(u\chi_{\Omega \setminus E}))|_{\Omega \setminus F} \in SBV^p(\Omega \setminus F) \). Then we repeat the argument in (39) to conclude \( \phi(u\chi_{\Omega \setminus E}) \in SBV^p(\Omega) \).

The above result can also be interpreted as an approximation result for SBD functions. On the one hand, it is weaker than standard density results, see e.g. [9], as it does not lead to a fine estimate for the surface energy. On the other hand, whereas in results based on interpolation arguments the approximating sequences typically only converge in \( L^p \), in the present context we see that the functions already coincide up to a set of arbitrarily small measure.

Acknowledgements This work has been funded by the Vienna Science and Technology Fund (WWTF) through Project MA14-009.

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