Nonlinear problems of equilibrium for axisymmetric membranes

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Abstract. The problems for equilibrium of circular and annular membranes made of elastic-orthotropic and elastoplastic isotropic materials are considered in this paper in a geometrically and physically nonlinear formulation meeting the Föppl-Karman assumptions. The outer contour is considered to be stationary, while the inner one being free, or reinforced by either an elastic ring or a fixed disk. The load is assumed to be uniformly distributed over the membrane surface.

The results of calculations of circular and annular membranes with large deflections in comparison with thickness, deformations and squares of angular displacement to be comparable with each other but small in comparison with the unit are presented in the first part thereof. The resulting boundary value problems are reduced to the Cauchy problem to be numerically integrated by the shooting method under the iterative Steffensen algorithm. For circular membrane structures an analytical solution as a power series similar to the Prescott series for a circular membrane is assumed either.

The results of numerical solutions of problems are presented as dimensionless characteristic functions. Analysis of this information will enable to solve the problem of the support, being optimal in a certain sense when the membrane is close to be equal in strength.

Problems of elastoplastic equilibrium of isotropic membranes are presented in the second part thereof. The idealized Prandtl diagram is used as the main characteristic of the mechanical properties of the membrane material. The Tresca-Sant-Venant and Huber-Mises-Hencky conditions were used as criteria for the transition of a material from an elastic state to a plastic one. Solutions of physically nonlinear equilibrium problems for axisymmetric membranes are also presented as dimensionless characteristic functions.

1. Relevance

The results of calculating the problems of equilibrium of circular and annular membranes with greater deflections as compared with thickness of membrane, deformations and squares of angular displacement to be comparable with each other but small in comparison with the unit are presented herein. Quite a very large number of papers are devoted to such problems but we do not focus thereon referring the reader to the detailed review contained in [1, 2, 3]. Main attention was paid by the researchers to the solution of equilibrium problems for isotropic membranes in a geometrically nonlinear formulation meeting the well-known Föppl assumptions. The main objective of the first part hereof is to study under the same assumptions the effect of elastic orthotropy of the membrane material on the stress-strain behavior. While the second part of the paper specifies the solutions of several problems of elastoplastic equilibrium of axisymmetric isotropic membranes on the basis of the deformation theory of plasticity. When setting these solutions the calculation methods proposed in [4, 5, 6, 7] were used.
2. Calculation of circular and annular membranes

We consider the problem of elastic equilibrium of an orthotropic membrane under a uniform load \( q \). Let us refer the membrane to the cylindrical coordinate system \( r\Theta z \) (Figure 1). The dependences between membrane deformations and displacements are described by equations:

\[
\varepsilon_\Theta = \frac{u}{r}, \quad \varepsilon_r = \frac{du}{dr} + \frac{1}{2} \left( \frac{dw}{dr} \right)^2.
\]  

(1)

Here, \( r, u, w \) are the radial coordinate, the corresponding displacement and deflection of the membrane related to the \( R_1 \)-radius of the outer contour of the membrane. For the case of polar symmetric orthotropy, the dependences between deformations and dimensionless strains are represented as

\[
\varepsilon_\Theta = \frac{n_\Theta}{\lambda} - v n_r, \quad \varepsilon_r = -v n_\Theta + n_r.
\]  

(2)

Here, \( n_\Theta = N_\Theta / E_r H; \quad n_r = H_r / E_r; \quad \lambda = E_\Theta / E_r; \quad v = v_\Theta = v_r (E_r / E_\Theta) \) where, \( N_\Theta \) and \( N_r \) are strains, \( H \) is a membrane thickness, \( E_\Theta \) and \( E_r \) are elastic modulus of the materials thereof in both annular and radial directions, \( v_\Theta \) and \( v_r \) are Poisson’s ratios. The equilibrium equations for a membrane element may be presented as

\[
n_\Theta = \frac{d}{dr}(n_r r), \quad \frac{d}{dr}(n_r r \sin \varphi) = 2Q_r,
\]  

(3)

where \( \varphi \) is an angle between the \( z \)-axis and normal to the deformed surface (Figure 1); \( Q \) is dimensionless load parameter determined by \( Q = qR_1 / 2 E_r H \). Each equation from Table 1 depending on geometric scheme of the membrane and kind of load case may be easily reduced to one of the equations (4) or (5). The given table is covering all kinds of problems solved herein.

The new variables are used in the table 1: \( y = B n_r r^2, \quad x = 1/2 r^2 \). Here, \( B \) is an arbitrary constant not yet defined, introduced to give both the resolving equation and the subsequent solution of greater generality. In addition, the following notations have been introduced: \( \alpha = 1/4(\lambda-1), \quad \beta = (E_r H / E_c A_c)R_1 \) where \( A_c \) is the sectional area of the ring supporting the membrane, and \( E_c \) is the elastic modulus of the material thereof. The values corresponding to the inner and outer contours are marked by indices 0 and 1, respectively.

Equations (4) and (5) under the corresponding boundary conditions are integrated numerically. Therewith, the boundary value problem is reduced to the Cauchy problem with the use of the shooting method based on the iterative Steffensen algorithm [8, 9, 10].

Figure 1. Original calculating scheme.

The solution to the problem of elastic equilibrium of an isotropic (\( \alpha = 0 \)) of the uniformly loaded circular membrane under large deflections (4) may also be expressed as infinite power series proposed by D. Prescott [8]:
\[ y = \sum_{n=1}^{\infty} a_n x^n. \]

Table 1. Kinds of problems solved.

| Calculated schemes | Resolving equations | Boundary conditions |
|--------------------|---------------------|---------------------|
| a                  | \[ y'' - \alpha \frac{y}{x^2} + \left(1 + \nu \lambda \right) \frac{y}{y^2} = 0 \] | \( x = 0 : y = 0 \) |
| b                  | \( x = x_0 : 2y_0' - \left(1 + \nu \lambda \right) \frac{y_0}{x_0} = 0 \) | \( x = x_1 : 2y_1' - \left(1 + \nu \lambda \right) \frac{y_1}{x_1} = 0 \) |
| c                  | \( y'' - \alpha \frac{y}{x^2} - \left(1 + \nu \lambda \right) \frac{y}{y^2} = 0 \) | \( x = x_0 : 2y_0' - \left(1 + \nu \lambda \right) \frac{y_0}{x_0} = \beta \frac{y_0}{(2x_0)^{1/2}} \) |
| d                  | \( x = x_1 : 2y_1' - \left(1 + \nu \lambda \right) \frac{y_1}{x_1} = 0 \) | \( x = x_0 : y = y_0 \) |
| e                  | \( x = x_1 : 2y_1' - \left(1 + \nu \lambda \right) \frac{y_1}{x_1} = 0 \) | \( x = x_0 : y = y_0 \) |

In problems of annular membranes equilibrium (5) a similar method may be applied, with the same series therewith being formally obtained, but relating to a new argument:

\[ y = \sum_{n=1}^{\infty} a_n (x - x_0)^n. \]

An expression for the general term of a similar series is obtained:

\[ a_n x^n = -\frac{x^n}{n(n-1)a_1^2} \sum_{k=1}^{k=2} \sum_{i=k-1,j=k-1}^{i=n-1,j=n} k(k-1)a_k a_i a_j. \] (6)

This expression enables to get values \( a_n \) starting with \( n = 3 \). The first coefficient of the series is found from the boundary conditions of the problem, and \( a_2 = -1/2a_1^2 \). Reasonable good convergence (within 3%) of analytical and numerical solutions for the problems of elastic equilibrium of annular membranes is achieved when 10 terms of the series are taken into account.

Below are the results of the numerical solution of specific problems (for membranes with the inner contour radius of \( r_0 = 0.5 \)) as characteristic functions:
\[ f_1 = \frac{n_0}{Q^3} = \left( \frac{\lambda}{4} \right)^{\frac{1}{2}} \left[ 2y' - \frac{y}{x} \right], \quad f_2 = \frac{n_2}{Q^3} = \left( \frac{\lambda}{4} \right)^{\frac{1}{2}} \frac{y}{x}; \]
\[ f_4 = \frac{w}{Q^3} = \left( \frac{4}{\lambda} \right) \int \frac{y}{x} \left( x - x_0 \right) dx, \quad (7) \]
\[ f_5 = \frac{u}{Q^3} = \left( \frac{1}{2\lambda} \right)^{\frac{1}{2}} \left[ 2y' - \frac{(1 + \nu \lambda) y}{x} \right] \left( 2x \right)^{\frac{1}{2}}. \]

The analysis of solving all the problems indicated in Table 1 also enabled to solve the problem of the support, in a certain sense the optimal support, when the membrane is close to being equal in strength. Maximum radial \((\beta = 0)\) and \((\beta = \infty)\) tangential stresses correspond to extremes \(c\) and \(e\), in both cases resulting in large values of stress intensity. It should also be noted that for these cases we have the largest rangeability in the stress intensity during the transition from the inner contour of the membrane to the outer one. All the above, as well as the fact that the stress functions in both cases are increasing from the outer contour to the inner one, made it possible to take the equality of radial tangential stresses on the inner contour of the membrane as the criterion of "optimality". As a result, an analytical dependence was obtained to determine the optimal supporting parameter that for the case of an isotropic annular membrane coincides with the expression obtained in \([4, 7, 9]\) with a similar criterion of "optimality":
\[ \beta = \frac{1 - \nu \lambda}{\lambda r_0}. \quad (8) \]

For membranes made of materials with strong orthotropy of mechanical properties \((\nu \lambda > 1)\), the solution of problems using this criterion is possible only with additional radial load case of the ring.

Under the calculations it was assumed \(\lambda = 3.152; \nu \lambda = 1.18 \quad [4]\). Characteristic functions for an isotropic membrane (dotted line) made of material with Poisson's ratio \(\nu = 0.37\) are also shown in Figure 2-4 to demonstrate the effect of anisotropy \([6, 11]\).

Let us consider elastoplastic problems of equilibrium of isotropic membranes under a uniform load. As the main characteristic of the mechanical properties of the membrane material, the idealized Prandtl diagram is used, thereby enabling to apply solutions for membranes made of materials with a very small strain hardening modulus (steel, aluminum and alloys thereof) \([7]\).

The biggest modulo constant shear stress condition (the Tresca-Sant-Venant condition) may be used as a criterion for the transition of a material from an elastic state to a plastic one, to be written as:
\[ \max \left( |n_1|, |n_0|, |n_n - n_0| \right) = n_s = \frac{\sigma_s}{E}. \quad (9) \]
or constant shear stress intensity condition (the Huber-Mises-Hencky condition), which may be written as
\[ n^2 + n_0^2 - n_0 n_n = n^2_s. \quad (10) \]

Here, \(\sigma_s\) is the yield point of the membrane material. When solving the elastoplastic problem for a uniformly loaded solid membrane, both conditions coincide and are simplified to
\[ n_s = n_0 = n_n, \quad (11) \]
which is proved using the first of the equilibrium equations (3) and the condition of equality of stresses in the center of the membrane \([5]\). This plasticity condition (11) is simultaneously a solution
for stresses. The annular area of plastic deformation, first appearing at the center of the membrane with increasing load, is extending to the periphery [8].

Geometric equations (1) using the basic relations of the theory of small elastoplastic deformations in the plastic area are transformed into the resolving equation with respect to $\varepsilon_\Theta$

$$\frac{d\varepsilon_\Theta}{dr} + \frac{1}{r} \varepsilon_\Theta \left[ \varepsilon_\Theta - \frac{\varepsilon_\Theta (2n_\Theta - n_\Theta) - (1 - 2\nu)(n_\Theta^2 - n_\Theta^2)}{(2n_\Theta - n_\Theta)} \right] = - \frac{1}{2r} \left( \frac{dv}{dr} \right)^2. \quad (12)$$

Comparing this equation with the solution obtained in [2], where the Tresca-Sant-Venant plasticity condition was used we come to the conclusion that assuming the smallness of deformations and thereby neglecting the effect of changes in the membrane thickness on the stress state and form thereof, the compressibility will affect the very law of thickness variation only [9, 11].

The solutions for the plastic area for deflection and radial displacement are obtained as a result of integration, taking into account expression (11) of the second of the equilibrium equations and ratio (12). These solutions are represented (although the constants of integration and parameters $x_s$ and $y_s$ depend on the external load) as functions similar to the above characteristic ones for convenience of conjugation with an elastic solution:

$$f_3 = \left( \frac{1}{2} \right)^{\frac{3}{2}} \left[ e_3 - 2 \frac{x^2}{y_s} x \right] (2x)^{\frac{1}{2}}, \quad f_4 = \left( \frac{1}{2} \right)^{\frac{3}{2}} \left[ e_4 - 2 \frac{x}{y_s} x \right]. \quad (13)$$

**Figure 2.** Characteristics of the stress-strain behavior of an annular membrane with a rigid inner contour.

**Figure 3.** Characteristics of the stress-strain behavior of an annular membrane with a free inner contour.

**Figure 4.** Characteristics of the stress-strain behavior of an annular membrane with a supported inner contour.
The values related to the boundaries of the area are denoted here and below by the index. The constants of integration are found under the conditions for continuity of deflections and displacements at the boundary of the elastic and plastic areas [1, 12]. For this it is initially required to set up a solution in the field of elastic strains. Equation (4) is numerically integrated under the boundary conditions:

$$x = x_0 : \frac{y_0}{y_1} = \left(\frac{1 + n_{th}}{n}\right), \quad x = x_1 : u_1 = 0,$$

(14)

Given the radius of the plastic area, with a certain load corresponding thereto, we solve the elastic problem \((Q = (n(n / f_s^3)^{3/2})\). Furthermore, using the conditions for continuity of components of strains and displacements, we set up an elastoplastic solution. Figure 5 (curve 1) shows the dependence of the deflection at the center of the membrane \((w_0 = w_{th}^{1/2})\) on the load parameter \((Q = Q^{3/2})\) for a solid membrane made of material with \(v = 0.3\).

It should also be noted that under \(r_s \to 1\), i.e. in the transition to the limit equilibrium, we have a difference in the value characterizing the strains, which is explained by simultaneous fulfillment of \(n_r = n_s = n_r\) and \(n_{th} = 0\) under \(r_s = 1\). When solving the equilibrium problem for an annular solid uniformly loaded membrane under the A. Föppl assumptions, this difference is nonremovable [8].

### 3. Problems of elastoplastic equilibrium of isotropic membranes

Let us further turn to the solution of the problems of elastoplastic equilibrium of annular membranes supported by elastic rings along the inner contour. A detailed series of problems under the condition of Huber-Mises-Hencky plasticity for a noncompressible material was considered earlier [5, 8]. No restrictions were imposed on the compressibility of the membrane material when solving the problem using the Tresca-Sant-Venant plasticity condition. The fairly good convergence of the results again confirms the thesis formulated under the analysis of the equation (12).

When considering the general geometric scheme, we neglect the transverse dimensions of the ring in comparison with the membrane dimensions. The problems of a supported membrane in an elastic approach were considered in article [6, 12]. Proceeding from the analysis of these solutions, depending on the relative stiffness of the support, these problems may be said to have two solutions: the first covers situations of relatively more rigid support (from absolutely rigid, when \(\beta = 0\), to "optimal"), the second one – from "optimal" support to the free boundary situation when \(\beta = \infty\). The problem enabling occurrence of deformations in the support is also reduced to the problem of supporting the relatively light stiffness (both solutions are given here).

The first plastic deformations occur in the annular area near the support and propagate under the load increasing [12, 13]. The above plasticity condition for the first situation takes the form of \(n_r = n_s\).
and for the second one \( n_0 = n_s \). Both conditions give solutions in the field of plastic deformations for one of the strains only, the second one may be obtained from the first equation of the system (3). In the first case we get \( n_0 = n_s \), and in the second one \( n_s = (n_r + c_3) / r \). The constant of integration is determined from the condition of conjugation of strains at the boundary of areas: \( c_3 = -\gamma n_r r \), where \( \gamma = n_s / n_0 \). The solutions for deflections and displacements in the area of plastic deformations are also represented as functions similar to the characteristic ones:

\[
\begin{align*}
f_4 &= \left( \frac{1}{2} \right)^\frac{1}{3} \left\{ c_4 - 2 \frac{x}{\gamma r} \left[ x - 2x_r ln(2x)^\frac{1}{2} \right] \right\}, \\
\gamma &= \frac{\text{radial}}{\text{annular}} \text{forces at the boundary of areas is a value that characterizes the relative stiffness of the supporting ring.}
\end{align*}
\]

Given by \( r_s \) (in both problems) and \( \gamma \) (in the second case only), we solve the problem of elastic equilibrium in area \( r_0 \leq r \leq 1 \). Further, the constants of integration are determined from the conjugation conditions. Then the second of the expressions (15) will enable to determine the radial displacement of the inner contour of membrane \((f_3)_0\), based upon which it is possible to find the value of the relative stiffness of the ring reinforcing the membrane [6] corresponding to a given \( \gamma \).

4. Results
The results of solving the first (curve 3 is corresponding to \( \beta = 0.5 \)) and the second (curve 2 - \( \beta = 1.4 \)) problems are given in Figure 4 for annular membrane with \( r_0 = 0.5 \) and \( \nu = 0.3 \).

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