SINGULAR INTEGRALS ASSOCIATED TO HYPERSURFACES: $L^2$ THEORY

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Abstract. We consider singular integrals associated to a classical Calderón-Zygmund kernel $K$ and a hypersurface given by the graph of $\varphi(\psi(t))$ where $\varphi$ is an arbitrary $C^1$ function and $\psi$ is a smooth convex function of finite type. We give a characterization of those Calderón-Zygmund kernels $K$ and convex functions $\psi$ so that the associated singular integral operator is bounded on $L^2$ for all $C^1$ functions $\varphi$.

1. Introduction

The main purpose of this paper is to investigate the $L^2$ boundedness of singular integral operators associated to hypersurfaces in $\mathbb{R}^n$, $n \geq 3$. Let $\Gamma(t)$ be a $C^1$ mapping from a neighborhood of the origin in $\mathbb{R}^{n-1}$ into $\mathbb{R}^n$ with $\Gamma(0) = 0$. For $x$ in $\mathbb{R}^n$ and $f$ a $C^1$ function with compact support in $\mathbb{R}^n$, we set

$$Hf(x) = \lim_{\epsilon \to 0} \int_{\epsilon \leq |t| \leq 1} f(x - \Gamma(t))K(t) \, dt$$

where $K(t)$ is a Calderón-Zygmund kernel in $\mathbb{R}^{n-1}$. That is, $K$ is smooth ($C^\infty$) away from the origin,

$$\int_{a \leq |t| \leq b} K(t) \, dt = 0$$

for every $0 < a < b$, and

$$K(\lambda t) = \lambda^{-n+1}K(t)$$

for every $\lambda > 0$.

It is known that if $\Gamma(t)$ is smooth and the vectors $\left\{\frac{\partial^\alpha \Gamma}{\partial t^{\alpha}}(0)\right\}$, given by the derivatives of $\Gamma$ at the origin, span $\mathbb{R}^n$, then

$$\|Hf\|_{L^p} \leq A_p \|f\|_{L^p}, \quad 1 < p < \infty.$$ 

See [St] for this result. Our main interest is studying what happens when the vectors $\frac{\partial^\alpha \Gamma}{\partial t^{\alpha}}(0)$ do not span $\mathbb{R}^n$. We shall consider surfaces of the form

$$\Gamma(t) = (t, \varphi(\psi(t)))$$

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where $t = (t_1, \ldots, t_{n-1})$ and $\psi(t)$ is a smooth convex function of finite type with $\psi(0) = \nabla \psi(0) = 0$. (We say that $\psi(t)$ is of finite type if the graph $t_n = \psi(t)$ has no lines tangent to infinite order.) If $\psi(t) = |t|^2 = t_1^2 + \cdots + t_{n-1}^2$, then
\[
\|Hf\|_{L^2} \leq A \|f\|_{L^2}
\]
for any $\varphi$. The details of this easy calculation can be found in [KWWZ].

The main purpose of this paper is to decide for what convex functions $\psi$ of finite type and Calderón-Zygmund kernels $K$ do we have
\[
\|Hf\|_{L^2} \leq A \|f\|_{L^2}
\]
for all $C^1$ functions $\varphi$ with $\varphi(0) = 0$. To give the answer to this problem we introduce certain sets which were considered by Schulz, [Sch]. Let
\[
E_\ell = \{v \in \mathbb{R}^{n-1} \mid \psi(sv) = O(s^{\ell+1}) \text{ for small } s > 0\}.
\]
From the convexity of $\psi$, each $E_\ell$ is a linear subspace of $\mathbb{R}^{n-1}$. Clearly $E_1 = \mathbb{R}^{n-1}$, $E_{\ell+1} \subseteq E_\ell$ and $\bigcap E_\ell = \{0\}$ (from the finite type condition). We let $\ell_0$ be the smallest value of $\ell$ such that $E_\ell$ is not all of $\mathbb{R}^{n-1}$. We then have the following theorem.

**Theorem 1.** If the codimension of $E_{\ell_0}$ in $\mathbb{R}^{n-1}$ is at least 2, then
\[
\|Hf\|_{L^2} \leq A \|f\|_{L^2}
\]
for all $C^1$ functions $\varphi$ with $\varphi(0) = 0$.

If the codimension of $E_{\ell_0}$ is 1, then $H$ is bounded on $L^2$ for all $C^1$ functions $\varphi$ if and only if $K$ satisfies an additional cancellation condition.

**Theorem 2.** Suppose the codimension of $E_{\ell_0}$ is 1, and let $v$ be a non-zero vector in $E_{\ell_0}^\perp$. Then
\[
\|Hf\|_{L^2} \leq A \|f\|_{L^2}
\]
for all $C^1$ functions $\varphi$ with $\varphi(0) = 0$ if and only if $K(t)$ satisfies the additional cancellation condition
\[
\int_{v \cdot t \geq 0}^{a \leq |t| \leq b} K(t) \, dt = 0
\]
for all $0 < a < b$.

**Remarks.**

(1) The positive assertions in Theorems 1 and 2 hold for the more general operators
\[
Hf(x) = \int b(\psi(t)) K(t) f(x - \Gamma(t)) \, dt
\]
for any bounded function $b$, with no change in the proof.
(2) Theorem 1 is vacuous and Theorem 2 is trivially true when \( n = 2 \), and so nothing new is being proved for singular integrals along curves in the plane.

**Examples.**

(1) \( \psi(x, y, z) = x^2 + y^2 + z^4 \) is a convex function of finite type where \( \ell_0 = 2 \) and 
\[
E_{\ell_0} = \{(0, 0, z) \mid z \in \mathbb{R}\}
\]
has codimension 2 and so Theorem 1 applies.

(2) \( \psi(x, y, z) = x^2 + y^4 + z^4 \) is also a convex function of finite type where \( \ell_0 = 2 \) but 
\[
E_{\ell_0} = \{(0, y, z) \mid y, z \in \mathbb{R}\}
\]
has codimension 1 and so Theorem 2 applies with \( v = (1, 0, 0) \).

Next we turn to examine what happens when the cancellation condition (1.1) is not satisfied.

**Theorem 3.** Let \( \varphi(s) = \varphi(s^{\ell_0}) \). Assume the codimension of \( E_{\ell_0} \) is 1, and the cancellation condition (1.1) fails. Then if \( \varphi(s) \) is convex, 
\[
\|Hf\|_{L^2} \leq A \|f\|_{L^2}
\]
if and only if 
\[
\varphi'(Cs) \geq 2\varphi'(s)
\]
for some \( C \geq 1 \) and all \( 0 < s \leq 1 \).

**Remarks.**

(1) The significance of the power \( \ell_0 \) is that \( \frac{1}{\ell_0} \) is the smallest power \( \alpha \) such that 
\[
[\psi(t)]^\alpha \text{ is a convex function.}
\]

(2) When \( \phi(s) = |s|^2 \) and so \( n = 1 \), Theorem 3 was proved in [NVWW].

If \( E_{\ell_0} = \{0\} \), which means that \( \psi \) is approximately homogeneous of degree \( \ell_0 \), we obtain \( L^p \) results for \( H \) and the corresponding maximal function 
\[
Mf(x) = \sup_{0 < h \leq 1} \frac{1}{h} \int_{|t| \leq h} |f(x - \Gamma(t))| \, dt.
\]

We again set \( \varphi(s) = \varphi(s^{\ell_0}) \).

**Theorem 4.** Suppose \( n \geq 3 \), \( E_{\ell_0} = \{0\} \) and \( \varphi(s) \) is convex. Then 
\[
\|Hf\|_{L^p} \leq A_p \|f\|_{L^p}, \quad 1 < p < \infty,
\]
and 
\[
\|Mf\|_{L^p} \leq A_p \|f\|_{L^p}, \quad 1 < p \leq \infty.
\]
Remark. It is known that the assertion of Theorem 4 fails in general if the hypothesis that $\bar{\phi}$ is convex is dropped, even if $\psi(t) = |t|^2$. See [SWWZ].

Finally we make one observation in $\mathbb{R}^3$ in the case that $\psi(t)$ is not of finite type. Let $t_0$ be a point on the curve $\psi(t) = 1$ and $\ell(t_0)$ denote the line tangent to $\psi(t) = 1$ at $t_0$. Set

$$E(t_0, \epsilon) = \{ s \in \mathbb{R}^2 \mid \psi(s) = 1 \text{ and } \text{dist}(s, \ell(t_0)) \leq \epsilon \}.$$

**Theorem 5.** Assume $\psi(t)$ is convex and homogeneous of degree 1. Then if

$$\sup_{t_0} \int_0^1 |E(t_0, \epsilon)| \frac{d\epsilon}{\epsilon} < \infty,$$

$$\|Hf\|_{L^2} \leq A \|f\|_{L^2}$$

for every $C^1$ function $\varphi$.

**Example.** Consider a smooth convex function $\psi(x, y)$, homogeneous of degree 1, such that for $|x| << |y|$,

$$\psi(x, y) = \sqrt{x^2 + y^2} \exp \left( - \left( \frac{\sqrt{x^2 + y^2}}{|x|} \right)^\alpha \right).$$

Clearly $\psi$ is not of finite type and the integrability condition in Theorem 5 is satisfied exactly when $\alpha < 1$.

In section 2 we will prove Theorems 1 and 2 in the special cases where $\psi(x, y, z) = x^2 + y^2 + z^4$ (for Theorem 1) and $\psi(x, y, z) = x^2 + y^4 + z^4$ (for Theorem 2), where the main direction of the proof is not clouded by intricate estimates. The proof for Theorem 1 in the general case will be given in section 3. Theorems 2 and 3 will be proved in section 4 and sections 5 and 6 contain the proofs of Theorems 4 and 5 respectively.

Our work is heavily dependent on ideas of Schulz, [Sch]. We would like to thank A. Iosevich for bringing the paper [Sch] to our attention. We would also like to thank Professor A. Carbery for evaluating a determinant for us.

**2. Special Cases**

In this section we will prove Theorems 1 and 2 in the special cases $\psi(x, y, z) = x^2 + y^2 + z^4$ and $\psi(x, y, z) = x^2 + y^4 + z^4$. We begin with $\psi(x, y, z) = x^2 + y^2 + z^4$. Here no further cancellation condition is required for the Calderón-Zygmund kernel $K$. We need to show

$$\left| \int_{x^2 + y^2 + z^2 \leq 1} e^{ix\varphi(x^2+y^2+z^4)} e^{i\eta z} e^{i(\xi_1 x + \xi_2 y)} K(x, y, z) \, dx \, dy \, dz \right| \leq B$$

(2.1)
uniformly in $\xi = (\xi_1, \xi_2), \eta, \gamma$ and $\epsilon > 0$. Introducing polar coordinates in the $(x, y)$ integral, the integral in (2.1) becomes

\begin{equation}
(2.2) \int_{\epsilon \leq r^2 + z^2 \leq 1} e^{i \gamma \varphi (r^2 + z^4)} e^{i n z} \int_0^{2\pi} e^{i n (\xi_1 \cos \theta + \xi_2 \sin \theta)} K(r \cos \theta, r \sin \theta, z) \, d\theta \, dr \, dz.
\end{equation}

We split the integral in (2.2) as a sum of two integrals $I_1 + I_2$ where the $r$ integration in $I_1$ is restricted to $r|\xi| \geq 1$ and where the integration in $I_2$ is over the complementary range. Using the fact that the $\theta$ integral in (2.2) is the Fourier transform of a smooth density on the unit circle, we see that

$$|I_1| \leq C \int_{r|\xi| \geq 1} \frac{r^2}{(r|\xi|)^{1/2}} \int_{-\infty}^{\infty} \frac{1}{(r^2 + z^2)^{3/2}} + \frac{1}{r(r^3 + |z|^3)} \, dz \, dr = C \int_{r|\xi| \geq 1} \frac{1}{(r|\xi|)^{1/2}} \, dr \leq C.$$

In $I_2$ we replace $e^{i n (\xi_1 \cos \theta + \xi_2 \sin \theta)}$ with 1, creating an error at most a multiple of

$$\int_{r|\xi| \leq 1} r^2 |\xi| \int_{-\infty}^{\infty} \frac{1}{(r^2 + z^2)^{3/2}} \, dz \, dr = C \int_{r|\xi| \leq 1} |\xi| \, dr = C.$$

Therefore the integral in (2.2) is

$$\int_0^{2\pi} \int_{\epsilon \leq r^2 + z^2 \leq 1} e^{i \gamma \varphi (r^2 + z^4)} e^{i n z} K(r \cos \theta, r \sin \theta, z) \, dr \, dz \, d\theta + O(1).$$

Furthermore the $(r, z)$ integration may be further restricted to the region where $|z| \leq \delta r^{1/2}$ since integrating $K$ over the complementary region is at most

$$\int_{\delta r^{1/2} \leq |z| \leq 1} \int_{r|\xi| \leq 1} \frac{1}{(r^2 + z^2)^{3/2}} r \, dr \, dz \leq \int_{|z| \leq 1} \frac{1}{|z|^3} \int_{r \leq \left(\frac{1}{\delta} |z|\right)^2} r \, dr \leq C.$$

With the restriction $|z| \leq \delta r^{1/2}$ for small $\delta > 0$, we may make the change of variables $\lambda = \sqrt{r^2 + z^4}$ (so that $\lambda \sim r$) in the $r$ integral to reduce matters to showing that the integral

$$I = \int_0^{2\pi} \int_{0 \leq |\xi| \leq 1} e^{i \gamma \varphi (\lambda^2)} e^{i n z} K(\sqrt{\lambda^2 - z^4} \cos \theta, \sqrt{\lambda^2 - z^4} \sin \theta, z) \, dz \, d\lambda \, d\theta$$

$$= \int_{0 \leq \lambda \leq 1} e^{i \gamma \varphi (\lambda^2)} e^{i n z} K(\lambda, \lambda \sin \theta, \lambda \cos \theta) \, d\lambda \, d\theta$$
is uniformly bounded in $\gamma, \eta, \xi$ and $\epsilon > 0$. Replacing $\sqrt{\lambda^2 - z^4}$ by $\lambda$ in $I$ creates an error at most

$$C \int_0^1 \int_{|z| \leq \lambda^{1/2}} \frac{z^4}{(z^2 + \lambda^2)^{4/2}} \, dz \, d\lambda \leq C \int_0^1 \lambda^{1/2} \, d\lambda \leq C$$

and so

$$I = \int_0^{2\pi} \int_0^\frac{\lambda |\xi| \leq 1}{\lambda |\xi| \leq 1} e^{i\nu \varphi(\lambda^2)} \lambda \int_{|z| \leq \delta \lambda^{1/2}} e^{inz} K(\lambda \cos \theta, \lambda \sin \theta, z) \, dz \, d\lambda \, d\theta + O(1).$$

Next we will see that we can replace the oscillatory factor $e^{inz}$ with 1 in the above integral if we further restrict the $\lambda$ integration to $\lambda \leq \frac{1}{|\eta|}$. In fact we can integrate by parts in the $z$ integral to see that the part of the integral where $\lambda |\eta| \geq 1$ is at most

$$C \frac{1}{|\eta|} \int_{\lambda |\eta| \geq 1} \lambda \int_{-\infty}^{\infty} \frac{1}{(z^2 + \lambda^2)^{4/2}} \, dz \leq C \frac{1}{|\eta|} \int_{\lambda |\eta| \geq 1} \frac{1}{\lambda^2} \leq C.$$  

For $\lambda |\eta| \leq 1$, replacing $e^{inz}$ by 1 creates an error at most

$$C |\eta| \int_{\lambda |\eta| \leq 1} \lambda \int_{-\infty}^{\infty} \frac{|z|}{(\lambda^2 + z^2)^{3/2}} \, dz \, d\lambda \leq C |\eta| \int_{\lambda |\eta| \leq 1} \, d\lambda \leq C.$$
Therefore

\[
I = \int_{0}^{2\pi} \int_{\lambda|\xi| \leq 1}^{2\pi} e^{i\gamma \varphi(\lambda^2)} \lambda \int_{\lambda|\eta| \leq 1}^{\infty} K(\lambda \cos \theta, \lambda \sin \theta, z) d\lambda d\theta + O(1)
\]

\[
= \int_{0}^{2\pi} \int_{\lambda|\xi| \leq 1}^{2\pi} e^{i\gamma \varphi(\lambda^2)} \frac{1}{\lambda} \int_{\lambda|\eta| \leq 1}^{\infty} K(\cos \theta, \sin \theta, s) ds d\lambda d\theta + O(1)
\]

\[
= \int_{0}^{2\pi} \int_{\lambda|\xi| \leq 1}^{2\pi} e^{i\gamma \varphi(\lambda^2)} \frac{1}{\lambda} \int_{\lambda|\eta| \leq 1}^{\infty} K(\cos \theta, \sin \theta, s) ds d\lambda d\theta + O(1)
\]

\[
= - \int_{\lambda|\xi| \leq 1}^{2\pi} e^{i\gamma \varphi(\lambda^2)} \frac{1}{\lambda} \int_{0}^{\pi} \int_{0}^{\frac{\pi}{2}} \frac{K(\sin \psi \cos \theta, \sin \psi \sin \theta, \cos \psi) \sin \psi d\psi d\theta d\lambda + O(1).}
\]

Here we made the change of variables \( z = s\lambda \) followed by \( s = \cot \psi \) in the \( z \) integral. Using the fact that

\[
0 = \int_{0}^{2\pi} \int_{0}^{\pi} K(\sin \psi \cos \theta, \sin \psi \sin \theta, \cos \psi) \sin \psi d\psi d\theta
\]

we easily see (by splitting the \( \lambda \) integration at \( \lambda = \sqrt{\epsilon} \)) that \( I \) is uniformly bounded in \( \gamma, \xi, \eta \) and \( \epsilon > 0 \). This finishes the proof of Theorem 1 in the case where \( \psi(x, y, z) = x^2 + y^2 + z^4 \).

For the example \( \psi(x, y, z) = x^2 + y^4 + z^4 \) we will show that the integral

\[
(2.3) \quad \int_{\epsilon \leq x^2 + y^4 + z^4 \leq 1} e^{i\gamma \varphi(x^2 + y^4 + z^4)} e^{i\eta x} e^{i(\xi_1 y + \xi_2 z)} K(x, y, z) dx dy dz
\]

is uniformly bounded in \( \gamma, \eta, \xi = (\xi_1, \xi_2) \) and \( \epsilon > 0 \) under the additional hypothesis that for all \( 0 < a < b \)

\[
(2.4) \quad \int_{a \leq x^2 + y^4 + z^4 \leq b} K(x, y, z) dx dy dz = 0.
\]

We would like to make the change of variables \( \lambda^2 = x^2 + y^4 + z^4 \) in the \( x \) integral. In order to do this first observe that the integral in (2.3) over the region \( \delta|x|^{1/2} \leq
\( \sqrt{y^2 + z^2} \) is uniformly bounded. In fact
\[
\int \int \int_{\delta|x|^{1/2} \leq \sqrt{y^2 + z^2} \leq 1} |K(x, y, z)| \, dx \, dy \, dz
\]
\[
\leq C \int \int \int_{\delta|x|^{1/2} \leq \sqrt{y^2 + z^2} \leq 1} \frac{1}{(y^2 + z^2)^{3/2}} \, dx \, dy \, dz
\]
\[
\leq C \int \int \int_{\sqrt{y^2 + z^2} \leq 1} \frac{1}{\sqrt{y^2 + z^2}} \, dy \, dz \leq C.
\]
Hence it suffices to show the uniform boundedness of
\[
II = \int_{\epsilon \leq x^2 + |y|^2 \leq 1} e^{i\gamma \varphi(x^2 + y^4 + z^4)} e^{i\eta x} e^{i\xi \cdot \bar{y}} K(x, \bar{y}) \, dx \, d\bar{y}
\]
where \( \bar{y} = (y, z) \). We write \( II = II_+ + II_- \) where the integration in \( II_+ \) is over positive values of \( x \). We first concentrate on \( II_+ \), making the change of variables
\[
\lambda = \sqrt{x^2 + y^4 + z^4}, \quad x = x(\lambda, \bar{y}) = \sqrt{\lambda^2 - y^4 - z^4}
\]
in the \( x \) integral so that \( x \sim \lambda \). Then
\[
II_+ = \int_0^1 e^{i\gamma \varphi(\lambda^2)} \int_{\epsilon \leq \lambda^2 + |\bar{y}|^2 \leq 1} e^{i\eta x(\lambda, \bar{y})} e^{i\xi \cdot \bar{y}} K(x(\lambda, \bar{y}), \bar{y}) \frac{\partial x}{\partial \lambda} \, d\bar{y} \, d\lambda + O(1).
\]
In order to analyze this integral we make the following simple observations regarding \( x(\lambda, \bar{y}) \) in the region \( |\bar{y}| \leq \delta \lambda^{1/2} \):

(a) \( x(\lambda, 0) = \lambda, \quad \frac{\partial x}{\partial \lambda}(\lambda, 0) = 1 \),

(b) \[ \left| \frac{\partial x}{\partial \bar{y}} \right| \sim \frac{|\bar{y}|^3}{\lambda}, \quad \left| \frac{\partial x}{\partial \lambda \partial \bar{y}} \right| \sim \frac{|\bar{y}|^3}{\lambda^2}, \]

and

(c) \[ \frac{\partial^4 x}{\partial \bar{y}^4} \sim \frac{1}{\lambda}, \quad \frac{\partial^4 x}{\partial z^4} \sim \frac{1}{\lambda}. \]
Using (a) and (b) we may replace $\frac{\partial x}{\partial \lambda}$ by 1 in $II_+$ with an error at most

$$C \int_0^1 \frac{1}{\lambda^2} \int_{|\bar{y}| \leq \lambda^{1/2}} \frac{|\bar{y}|^4}{\lambda^3 + |\bar{y}|^3} d\bar{y} d\lambda \leq C \int_0^1 \lambda \int_{|s| \leq \lambda^{-1/2}} \frac{|\bar{s}|^4}{1 + |\bar{s}|^4} d\bar{s} d\lambda$$

$$\leq C \int_0^1 \frac{1}{\sqrt{\lambda}} d\lambda \leq C.$$

Also replacing $x(\lambda, \bar{y})$ with $\lambda$ in the kernel $K$ creates an error at most

$$C \int_0^1 \frac{1}{\lambda} \int_{|\bar{y}| \leq \lambda^{1/2}} \frac{|\bar{y}|^4}{\lambda^4 + |\bar{y}|^4} d\bar{y} d\lambda \leq C \int_0^1 \lambda \int_{|s| \leq \lambda^{-1/2}} \frac{|\bar{s}|^4}{1 + |\bar{s}|^4} d\bar{s} d\lambda$$

$$\leq C \int_0^1 d\lambda = C.$$

Therefore

$$II_+ = \int_0^1 e^{i\gamma \varphi(\lambda^2)} \int_{\lambda \leq \lambda^{1/3} \geq 1} e^{i\eta x(\lambda, \bar{y})} e^{i\xi \cdot \bar{y}} K(\lambda, \bar{y}) d\bar{y} d\lambda + O(1).$$

Next we will show that we can replace the oscillation $e^{i\eta x(\lambda, \bar{y})}$ with $e^{i\eta \lambda}$ provided that the $\lambda$ integration is restricted to where $\lambda \eta \geq 1$. In fact using the fact that $\frac{\partial x}{\partial \bar{y}} \sim \frac{1}{\bar{y}}$ and Van der Corput’s lemma (see e.g., [St]) in the $y$ integral we see that the part of the integral where $\lambda \eta \geq 1$ is at most

$$C \frac{1}{|\eta|^{1/4}} \int_{\lambda \eta \geq 1/3} \lambda^{1/4} \left[ \int \frac{1}{\lambda^4 + |\bar{y}|^4} d\bar{y} + \int \frac{1}{\lambda^3 + |z|^3} dz \right] d\lambda \leq C \frac{1}{|\eta|^{1/4}} \int_{\lambda \eta \geq 1/3} \frac{\lambda^{1/4}}{\lambda^2} d\lambda \leq C.$$

For the part where $\lambda \eta \geq 1$ we expect only to replace $e^{i\eta x(\lambda, \bar{y})}$ with $e^{i\eta \lambda}$ in the region where $|\bar{y}| \leq \left( \frac{\lambda}{|\eta|} \right)^{1/4}$ since using (b)

$$|e^{i\eta x(\lambda, \bar{y})} - e^{i\eta \lambda}| \leq |\eta||x(\lambda, \bar{y}) - x(\lambda, 0)| \leq |\eta| \frac{|\bar{y}|^4}{\lambda}.$$
In the complementary region, $\lambda|\eta|^{1/3} \leq 1$ and $|\vec{y}| \geq \left(\frac{\lambda}{|\eta|}\right)^{1/4}$ we see that $K$ is uniformly integrable. In fact
\[
\int_{\lambda|\eta|^{1/3} \leq 1} \int_{|\vec{y}| \geq \left(\frac{\lambda}{|\eta|}\right)^{1/4}} |K(\lambda, \vec{y})| \, d\vec{y} \, d\lambda \leq \int_{\lambda|\eta|^{1/3} \leq 1} \int_{|\vec{y}| \geq \left(\frac{\lambda}{|\eta|}\right)^{1/4}} \frac{1}{\lambda^3 + |\vec{y}|^3} \, d\vec{y} \, d\lambda
\]
\[
\leq C \int_{\lambda|\eta|^{1/3} \leq 1} \int_{|\vec{s}| \geq \left(\frac{\lambda}{|\eta|}\right)^{1/4}} \frac{1}{1 + |\vec{s}|^3} \, d\vec{s} \, d\lambda
\]
\[
\leq C |\eta|^{1/4} \int_{\lambda|\eta|^{1/3} \leq 1} \frac{1}{\lambda^{1/4}} \, d\lambda \leq C.
\]
Replacing $e^{i\eta \chi(\lambda, \vec{y})}$ with $e^{i\eta \lambda}$ in the region $\lambda|\eta|^{1/3} \leq 1$ and $|\vec{y}| \leq \left(\frac{\lambda}{|\eta|}\right)^{1/4}$ creates an error at most
\[
C |\eta| \int_{\lambda|\eta|^{1/3} \leq 1} \frac{1}{\lambda} \int_{|\vec{y}| \leq \left(\frac{\lambda}{|\eta|}\right)^{1/4}} \frac{|\vec{y}|^4}{\lambda^3 + |\vec{y}|^3} \, d\vec{y} \, d\lambda
\]
\[
\leq C |\eta| \int_{\lambda|\eta|^{1/3} \leq 1} \lambda^2 \int_{|\vec{s}| \geq \left(\frac{\lambda}{|\eta|}\right)^{1/4}} \frac{|\vec{s}|^4}{1 + |\vec{s}|^3} \, d\vec{s} \, d\lambda
\]
\[
\leq C \frac{|\eta|}{|\eta|^{3/4}} \int_{\lambda|\eta|^{1/3} \leq 1} \frac{1}{\lambda^{1/4}} \, d\lambda \leq C.
\]
Therefore
\[
II_+ = \int_{\lambda|\eta|^{1/3} \leq 1} e^{i\chi \lambda^2} e^{i\eta \lambda} \int_{\epsilon \leq \lambda^2 + |\vec{y}|^2 \leq 1} e^{i\chi \cdot \vec{y}} K(\lambda, \vec{y}) \, d\vec{y} \, d\lambda + \mathcal{O}(1).
\]
A similar but easier argument allows us to replace $e^{i\chi \cdot \vec{y}}$ with 1 if we further restrict the $\lambda$ integration where $\lambda|\xi| \leq 1$. Hence making the change of variables $\vec{y} = \lambda \vec{s}$,
\[
II_+ = \int_{0 \leq \lambda \leq 1} \int_{\lambda \leq \min(|\xi|^{-1}, |\eta|^{-1/3})} e^{i\gamma \phi(\lambda^2)} e^{i\eta \lambda} \int_{\epsilon \leq \lambda^2 + |\vec{y}|^2 \leq 1} \int_{|\vec{s}| \leq \delta \lambda^{1/2}} K(\lambda, \vec{y}) \, d\vec{y} \, d\lambda + \mathcal{O}(1)
\]
\[
= \int_{0 \leq \lambda \leq 1} \int_{\lambda \leq \min(|\xi|^{-1}, |\eta|^{-1/3})} e^{i\gamma \phi(\lambda^2)} e^{i\eta \lambda} \frac{1}{\lambda} \int_{\epsilon \leq \lambda^2 (1 + |\vec{s}|^2) \leq 1} K(1, \vec{s}) \, d\vec{s} \, d\lambda + \mathcal{O}(1).
\]
Here we used the fact that
\[ \int_{|\bar{s}| \geq \lambda^{-1/2}} |K(1, \bar{s})| \, d\bar{s} = \Theta(\lambda^{1/2}). \]

Making a polar change of coordinates \( \bar{s} = (r \cos \theta, r \sin \theta) \) followed by \( r = \tan \psi \), \( 0 \leq \psi < \pi/2 \), in the \( \bar{s} \) integral allows us to write
\[ \mathcal{II} + \mathcal{II} = \int_{0 \leq \lambda \leq 1} \int_{\lambda \leq \min(|\xi|^{-1}, |\eta|^{-1/3})} \int_{0 \leq \psi \leq \pi/2} K(\cos \psi, \sin \psi \cos \theta, \sin \psi \sin \theta) \sin \psi \, d\psi \, d\theta \, d\lambda + \Theta(1) \]
\[ = \int_{\sqrt{\tau} \leq \lambda \leq 1} \int_{\lambda \leq \min(|\xi|^{-1}, |\eta|^{-1/3})} \int_{0 \leq \psi \leq \pi/2} K(\cos \psi, \sin \psi \cos \theta, \sin \psi \sin \theta) \sin \psi \, d\psi \, d\theta \, d\lambda + \Theta(1). \]

A similar analysis for \( \mathcal{II} - \mathcal{I} \) shows
\[ \mathcal{II} - \mathcal{I} = \int_{\sqrt{\tau} \leq \lambda \leq 1} \int_{\lambda \leq \min(|\xi|^{-1}, |\eta|^{-1/3})} \int_{0 \leq \psi \leq \pi/2} K(\cos \psi, \sin \psi \cos \theta, \sin \psi \sin \theta) \sin \psi \, d\psi \, d\theta \, d\lambda + \Theta(1). \]

Therefore
\[ \mathcal{II} = \mathcal{II} + \mathcal{II} - \mathcal{I} = \int_{\sqrt{\tau} \leq \lambda \leq 1} \int_{\lambda \leq \min(|\xi|^{-1}, |\eta|^{-1/3})} \int_{0 \leq \psi \leq \pi/2} K(\cos \psi, \sin \psi \cos \theta, \sin \psi \sin \theta) \sin \psi \, d\psi \, d\theta \, d\lambda + \Theta(1) \]
\[ = 0 + \Theta(1) \]
by (2.4). Note that when the additional cancellation condition for \( K \) is not satisfied, we are left with a truncated Hilbert transform along the curve \( (\lambda, \varphi(\lambda^2)) \) and so we might expect to be able to use the analysis in [NVWW] when \( \varphi(\lambda^2) \) is convex.

### 3. Proof of Theorem 1

According to Schulz [Sch], after a rotation of coordinates, we may write
\[ \varphi(t) = P(t) + R(t) \]
where
\[ P(t) = \sum_{j=1}^{r} a_j t_0^j + \sum_{j=r+1}^{n-1} a_j t_j^m + P_1(t). \]

\( P(t) \) is a convex polynomial, \( P(t) > 0 \) for \( t \neq 0 \), \( a_j > 0 \) for \( 1 \leq j \leq n-1 \), \( \ell_0 < m_j \) for \( r+1 \leq j \leq n-1 \), \( P_1(t) \) has no pure powers of \( t \), and if \( A t_1^{\alpha_1} \ldots t_{n-1}^{\alpha_{n-1}} \) is a monomial of \( P_1(t) \),
\[ \frac{1}{\ell_0} \sum_{j=1}^{r} a_j + \sum_{j=r+1}^{n-1} \frac{a_j}{m_j} = 1. \]

\( R(t) \) is smooth and if \( A t_1^{\alpha_1} \ldots t_{n-1}^{\alpha_{n-1}} \) is a term in the Taylor expansion of \( R(t) \)
\[ \frac{1}{\ell_0} \sum_{j=1}^{r} a_j + \sum_{j=r+1}^{n-1} \frac{a_j}{m_j} > 1. \]

To prove our theorems, we may assume \( \psi(t) \) has the form (3.1). The hypothesis of Theorem 1 asserts that \( r \geq 2 \). Let \( H(t) \) be the part of \( P(t) \) which is homogeneous of degree \( \ell_0 \). Then \( H(t) \) is a function of only \( t_1, \ldots, t_r \). In fact if
\[ A t_1^{\alpha_1} \ldots t_r^{\alpha_r} t_{r+1}^{\alpha_{r+1}} \ldots t_{n-2}^{\alpha_{n-2}} t_{n-1}^{\ell_0 - (\alpha_1 + \cdots + \alpha_{n-2})} \]
were a monomial of \( H(t) \), then
\[ \frac{1}{\ell_0} \sum_{j=1}^{r} \alpha_j + \sum_{j=r+1}^{n-2} \frac{\alpha_j}{m_j} + \frac{\ell_0 - (\alpha_1 + \cdots + \alpha_{n-2})}{m_{n-1}} = 1. \]

This identity would clearly hold if \( m_{r+1} = \ldots m_{n-1} = \ell_0 \), so it could not hold if one of the \( m \)'s were bigger than \( \ell_0 \). Similarly every monomial of \( P(t) \) which depends only on \( t_1, \ldots, t_r \) belongs to \( H \). So
\[ H(t) = H(t_1, \ldots, t_r) = P(t_1, \ldots, t_r, 0, \ldots, 0) \]
is convex and positive if some \( t_j \) is nonzero.

We write \( y = (t_1, \ldots, t_r) \) in \( \mathbb{R}^r \) and \( x = (t_{r+1}, \ldots, t_{n-1}) \) in \( \mathbb{R}^{n-1-r} \). We shall suppose \( n-r-1 \geq 1 \), otherwise the proof is similar but simpler. We then write
\[ P(x, y) = H(y) + P_2(x, y). \]

To prove Theorem 3, we must show for \( \xi \) in \( \mathbb{R}^r \) and \( \eta \) in \( \mathbb{R}^{n-1-r} \),
\[ \left| \int_{\mathbb{R}^r \times \mathbb{R}^{n-1-r}} e^{i \gamma \phi(x, y)} e^{i \eta \xi} e^{i \epsilon \xi} K(y, x) \, dy \, dx \right| \leq B \]
uniformly in \( \xi, \eta, \gamma \) and \( \epsilon > 0 \).
We begin by introducing polar coordinates in the $y$ variables. That is, we write $y = s\omega$ where $s$ goes from 0 to 1 and $\omega$ runs over the surface $H(w) = 1$. The integral in (3.2) becomes

$$\int_{H(\omega) = 1} \int_{\epsilon \leq |x|^2 + s^2|\omega|^2 \leq 1} e^{i\gamma \varphi(\psi(x, s\omega))} e^{i(s\xi \omega + \eta x)} K(s\omega, x) s^{r-1} h(\omega) \, ds \, dx \, d\omega \tag{3.3}$$

where $h$ is a smooth function. We let $m$ be the smallest of the values among the $m_j$’s, $r + 1 \leq j \leq n - 1$, and choose $\sigma > 0$ so that $\frac{\ell_0}{m} + \sigma < 1$. We may restrict the integration in (3.3) to $|x| \leq s^{\ell_0/m + \sigma}$ since integrating $K$ over the complementary region is at most

$$C \int_{\frac{\ell_0}{s^m + \sigma} \leq |x| \leq 1} \frac{1}{|x|^{n-1}} s^{r-1} \, ds \, dx \leq C \int_{|x| \leq 1} \frac{1}{|x|^{n-1}} \int_{s \leq |x|^\lambda} s^{r-1} \, ds \, dx \leq C \int_{|x| \leq 1} \frac{1}{|x|^{n-r\lambda-1}} \, dx \leq C$$

since

$$\lambda = \frac{1}{\frac{\ell_0}{m} + \sigma} > 1 \quad \text{and} \quad x \in \mathbb{R}^{n-r-1}.$$

Furthermore in the region $|x| \leq s^{\ell_0/m + \sigma}$,

$$\psi(x, s\omega) = s^{\ell_0} + O(s^{\ell_0 + \sigma}) \tag{3.4}$$

and

$$\frac{\partial \psi}{\partial s}(x, s\omega) = \ell_0 s^{\ell_0 - 1} + O(s^{\ell_0 - 1 + \sigma}). \tag{3.5}$$

In fact

$$\psi(x, s\omega) = s^{\ell_0} + P_2(x, s\omega) + R(x, s\omega)$$

and every monomial in $P_2$ or any monomial in $R$ of the form $x^\alpha (s\omega)^\beta$ with $|\alpha| > 0$ has the bound

$$|x^\alpha (s\omega)^\beta| \leq s^{\frac{\ell_0}{m} + \sigma |\alpha| + |\beta|} \leq s^{\ell_0 + \sigma}.$$

Also any monomial in $R$ of the form $(s\omega)^\beta$ is $O(s^{\ell_0 + 1})$. Therefore we may make the change of variables

$$\lambda^{\ell_0} = \psi(x, s\omega) \tag{3.6}$$
in $s$ for fixed $x$ and $\omega$ and write (3.3) as

$$\int_{H(\omega)=1} h(\omega) \int_0^1 e^{i\gamma \varphi(\lambda^0)} \int_{|x|^{2+\epsilon} + |\omega|^2 \leq 1} e^{i\eta \cdot x} e^{is \xi \omega} K(s\omega, x) s^{-1} \frac{\partial s}{\partial \lambda} \, dx \, d\lambda \, d\omega + O(1)$$

where $s = s(\lambda, \omega, x)$. From (3.4) and (3.5) we have, for some $\epsilon > 0$,

$$s = \lambda + O(\lambda^{1+\epsilon}) \quad \text{and} \quad \frac{\partial s}{\partial \lambda} = 1 + O(\lambda^\epsilon).$$

Also we have

$$\frac{|\partial s|}{\partial x} \leq C \lambda^{1-\frac{\ell_0}{m}}$$

which follows by differentiating (3.6) with respect to $x$, giving

$$0 = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial s} \frac{\partial s}{\partial x}.$$

(3.3) implies $\frac{\partial \psi}{\partial s} \sim \lambda^{\ell_0 - 1}$ and a similar argument which established (3.4) and (3.5) shows $\frac{\partial \psi}{\partial x} = O(\lambda^{\ell_0 - \frac{\eta}{m}})$ and thus (3.8).

Arguing as in section 2, using the above estimates on the derivatives of $s(\lambda, \omega, x)$ and $s$ itself, shows that we may replace $s(\lambda, \omega, x)$ by $\lambda$ (except in the oscillation $e^{is \xi \omega}$) and $\frac{\partial s}{\partial \lambda}$ by 1 with a uniformly bounded error. Thus the integral in (3.7) is

$$\int_0^1 e^{i\gamma \varphi(\lambda^0)} \lambda^{r-1} \int_{|x| \leq \lambda^{\frac{\eta}{m} + \sigma}} \int_{|\omega|^2 \leq 1} e^{is(\lambda, \omega, x) \xi \omega} K(\lambda \omega, x) h(\omega) \, d\omega \, dx \, d\lambda + O(1).$$

Consider first the contribution to (3.9) from those values of $\lambda$ where $\lambda |\xi| \geq 1$. Since $H(\omega) = 1$ is of finite type we may for each $\omega_0$ on $H(\omega) = 1$ parametrize $H(\omega) = 1$ in a neighborhood of $\omega_0$ as

$$\omega_0 + (\tau_1, \ldots, \tau_{r-1}, g(\tau_1, \ldots, \tau_{r-1}))$$

where $g(0) = 0$, $\nabla g(0) = 0$, and for some $j_0 \geq 2$,

$$\frac{\partial^j g}{\partial \tau_{j_0}^j}(0) = 0$$

for $1 \leq j \leq j_0 - 1$ and

$$\frac{\partial^j g}{\partial \tau_{j_0}^j}(0) \neq 0.$$
It follows that we may assume
\[
\frac{\partial^{\alpha} g}{\partial \tau^{\alpha}} \neq 0
\]
for all \( \tau \) in a neighborhood of 0. Therefore since \( s(\lambda, \omega, x) \sim \lambda \),
\[
\left| \int_{|\omega-\omega_0| \leq \delta} e^{i\lambda \omega \cdot \xi} h(\omega) \, d\omega \right| \leq C \frac{1}{(|\lambda| |\xi|)\delta}
\]
for some positive \( \delta \) by Van der Corput’s lemma. Integrating by parts now shows that the contribution to the integral in (3.9) from those \( \lambda \) where \( |\lambda| |\eta| \geq 1 \) is at most
\[
C \int_{|\lambda| |\xi| \geq 1} \frac{1}{\lambda^r} \frac{1}{\lambda} \int_{|\omega| \leq \lambda \sqrt{\frac{\lambda}{m} + \sigma}} e^{i\lambda \eta \cdot z} K(\omega, z) h(\omega) \, dz \, d\omega \, d\lambda \leq C.
\]
Thus the proof of Theorem 1 reduces to showing that the integral
\[
I = \int_{|\lambda| |\xi| \leq 1} \frac{1}{\lambda^r} \frac{1}{\lambda} \int_{H(\omega) = 1} e^{i\lambda \eta \cdot z} K(\omega, x) h(\omega) \, dx \, d\omega \, d\lambda
\]
is uniformly bounded in \( \gamma, \xi, \eta \) and \( \epsilon > 0 \). Putting \( x = \lambda z \) makes
\[
I = \int_{|\lambda| |\xi| \leq 1} \frac{1}{\lambda^r} \frac{1}{\lambda} \int_{H(\omega) = 1} e^{i\lambda \eta \cdot z} K(\omega, x) h(\omega) \, dx \, d\omega \, d\lambda
\]
and using the fact
\[
\int_{|z| \leq 1} |K(\omega, z)| \, dz = O\left( \frac{\lambda}{C} \right)^{r\delta}
\]
three times, first with \( \delta = 1 \) and \( C = \sqrt{\epsilon} \), then with \( \delta = 1 \) and \( C = 1 \), and finally with \( \delta = 1 - \left( \left( \frac{\lambda}{m} \right) + \sigma \right) \) and \( C = 1 \), we see that
\[
I = \int_{A, \sqrt{\frac{\lambda}{m} \leq \lambda \leq \frac{1}{\eta}}} \frac{1}{\lambda} \int_{H(\omega) = 1} e^{i\lambda \eta \cdot z} K(\omega, z) h(\omega) \, dz \, d\omega \, d\lambda + O(1)
\]
if \( A \) is chosen large enough. An integration by parts in the \( z \) integral shows that the part of the integral where \( |\lambda| |\eta| \geq 1 \) is at most (up to boundary terms)
\[
C \int_{|\lambda| |\eta| \geq 1} \frac{1}{\lambda^2} \int_{\mathbb{R}^{n-1}} \sup_{\omega} |\nabla K(\omega, z)| \, dz \, d\lambda \leq C \int_{|\lambda| |\eta| \geq 1} \frac{1}{\lambda^2} \int_{|z| \geq 1} \frac{1}{|z|^n} \, dz \, d\lambda \leq C,
\]
and so
\[ I = \int_{A^{\sqrt{r} \leq \lambda \leq 1} \lambda \leq \min \left( \frac{1}{|\xi|}, \frac{1}{|\eta|} \right)} e^{i\gamma(\lambda^0)} \frac{1}{\lambda} \int H(\omega) = 1 \int e^{i\lambda \eta \cdot z} K(\omega, z) h(\omega) d\omega d\lambda + \mathcal{O}(1). \]

The boundary terms are handled similarly. Replacing $e^{i\lambda \eta \cdot z}$ by 1 creates an error at most
\[ C|\eta| \int_{|\lambda| \eta| \leq 1} H(\omega) = 1 \int \frac{|z|}{|\omega|^{n-1} + |z|^{n-1}} d\omega d\lambda \leq C \]
since $r \geq 2$. Therefore
\[ I = \int_{A^{\sqrt{r} \leq \lambda \leq 1} \lambda \leq \min \left( \frac{1}{|\xi|}, \frac{1}{|\eta|} \right)} e^{i\gamma(\lambda^0)} \frac{1}{\lambda} \int H(\omega) = 1 \int K(\omega, z) h(\omega) d\omega d\lambda + \mathcal{O}(1), \]
and so it suffices to show
\[ (3.10) \quad \int_{H(\omega) = 1} K(\omega, z) h(\omega) d\omega = 0. \]

Now for any $\delta > 0$
\[ 0 = \int_{1-\delta \leq |y|^2 + |z|^2 \leq 1} K(y, x) \, dy \, dx \]
\[ = \int_{H(\omega) = 1} \int_{1-\delta \leq \lambda^2 |\omega|^2 + |z|^2 \leq 1} \lambda^{r-1} K(\lambda \omega, x) h(\omega) \, d\lambda \, d\omega \, dx \]
\[ = \int_{H(\omega) = 1} \int_{1-\delta \leq \lambda^2 (|\omega|^2 + |z|^2) \leq 1} \frac{K(\omega, z) h(\omega)}{\lambda} \, d\omega \, dz \]
\[ = \int_{H(\omega) = 1} \int_{\mathbb{R}^{n-r-1}} h(\omega) \int_{|\omega|^2 + |z|^2 \leq \lambda^2 \leq |\omega|^2 + |z|^2} \frac{d\lambda}{\lambda} \, dz \, d\omega. \]

Dividing by $\delta$ and letting $\delta \to 0$ gives (3.10) and this finishes the proof of Theorem 1. \[ \square \]
4. The proof of Theorems 2 and 3

We may again assume \( \psi(t) \) is of the general form (3.1), where now \( r = 1 \). The cancellation condition (1.1) now becomes

\[
\int_{\Sigma^+} K(t) \, d\sigma(t) = 0
\]

where \( \Sigma^+ = \{ t \in \mathbb{R}^{n-1} \mid |t| = 1, t_1 > 0 \} \) is the “upper” hemisphere of \( S^{n-2} \). It will be convenient to let \( t_1 \) be denoted by \( y \) and \( (t_2, \ldots, t_{n-1}) = x \in \mathbb{R}^{n-2} \). We are then concerned with the uniform boundedness of

\[
\int \int e^{i\gamma \varphi(y,x)} e^{i\xi \cdot x} e^{in_\eta y} K(y,x) \, dy \, dx = \int \int_{y>0} + \int \int_{y<0} = I + II.
\]

Theorems 2 and 3 will then follow if we can prove for some \( b, 0 < b < 1 \),

\[
(4.1) \quad I = \int \int_{0 \leq \lambda \leq 1} e^{i\gamma \varphi(\lambda^0)} e^{in_\eta(\lambda)} \frac{d\lambda}{\lambda} \int_{\Sigma^+} K(\omega) d\sigma(\omega) + \mathcal{O}(1),
\]

\[
(4.2) \quad II = -\int \int_{0 \leq \lambda \leq 1} e^{i\gamma \varphi(\lambda^0)} e^{-in_\eta(\lambda)} \frac{d\lambda}{\lambda} \int_{\Sigma^+} K(\omega) d\sigma(\omega) + \mathcal{O}(1),
\]

and for \( \bar{\varphi}(\lambda) = \varphi(\lambda^0) \) convex,

\[
(4.3) \quad \int \int_{0 \leq \lambda \leq 1} e^{i\gamma \bar{\varphi}(\lambda)} \sin(n\eta(\lambda)) \frac{d\lambda}{\lambda}
\]

is uniformly bounded in \( \gamma, \eta \) and \( \xi \) if and only if

\[
\bar{\varphi}'(C\lambda) \geq 2\bar{\varphi}'(\lambda)
\]

for some \( C \geq 1 \) and \( 0 < \lambda \leq 1 \). Here \( q(\lambda) = \lambda + \mathcal{O}(\lambda^{1+\epsilon}) \) and \( q'(\lambda) = 1 + \mathcal{O}(\lambda^\epsilon) \).

We begin with the proof of (1.1). It will convenient to write (3.1) in the form

\[
(4.4) \quad \psi(y,x) = A y^{\ell_0} + \sum_{j=1}^{n-2} a_j x_j^{m_j} + \sum_{j=1}^{n-2} b_j x_j^{\alpha_j} y^{\beta_j} + P_2(y,x) + R(y,x)
\]

where \( \ell_0 < m_j, 1 \leq j \leq n - 2 \), \( A > 0 \), \( a_j > 0 \), \( b_j \neq 0 \), \( 1 \leq j \leq n - 2 \) and each monomial of \( P_2(y,x) \) has the form \( x_j^\alpha y^\beta \) with \( \alpha > \alpha_j \), or contains powers of at least
two different $x_j$'s. Let $m = \min_{1 \leq j \leq n-2}(m_j)$ and choose $\sigma > 0$ such that $\frac{\ell_0}{m} + \sigma < 1$.

Again

$$\int_{|x| \geq \frac{\ell_0}{m}} |K(x, y)| \, dx \, dy = O(1)$$

and so it suffices to study $I$ in the region $|x| \leq \frac{\ell_0}{m} + \sigma$. In this region we wish to make a change of variables

$$\lambda^\ell_0 = \psi(y, x)$$

in the $y$ integral. As in section 3, $|x| \leq \frac{\ell_0}{m} + \sigma$ implies that $y = y(x, \lambda)$ defined implicitly by (4.5) satisfies

$$y(x, \lambda) = A^{-1}\lambda + O(\lambda\sigma),$$

and

$$\partial y / \partial \lambda = A^{-1}\lambda + O(\lambda\sigma).$$

Therefore, as before, making the change of variables (4.5), shows

$$I = \int_{0 \leq \lambda \leq 1} e^{i\gamma\phi(\lambda^\ell_0)} \int_{|x| \leq \frac{\ell_0}{m} + \sigma} e^{ix \cdot \xi} e^{iy(x, \lambda) K(x, \lambda)} \, dx \, d\lambda + O(1).$$

To study (4.9), it is necessary to have information on the derivatives of $y$ with respect to the $x$ variables.

**Lemma 6.** Suppose $|x| \leq \lambda^{\frac{\ell_0}{m} + \sigma}$.

1. For $\delta > 0$ small,

$$\left| \frac{\partial^k y}{\partial x_j^k}(x, \lambda) \right| \leq \delta \lambda^{1-k\frac{\ell_0}{m_j}}, \quad 1 \leq k \leq \alpha_j - 1, \quad 1 \leq j \leq n - 2.$$

2. $\frac{\partial y}{\partial x_j}(x, \lambda) \sim \lambda^{1-\alpha_j\frac{\ell_0}{m_j}}, \quad 1 \leq j \leq n - 2$.

3. For every $\beta = (\beta_1, \ldots, \beta_{n-2})$ with $0 \leq \beta_j \leq \alpha_j, \quad 1 \leq j \leq n - 2$,

$$\left| \frac{\partial^\beta y}{\partial x^\beta}(x, \lambda) \right| \leq C\lambda^{1-\ell_0\sum_{j=1}^{n-2} \frac{\beta_j}{m_j}}.$$
(4) For every \( \beta = (\beta_1, \ldots, \beta_{n-2}) \) with \( 0 \leq \beta_j \leq \alpha_j - 1, 1 \leq j \leq n-2 \), either
\[
\frac{\partial^\beta y}{\partial x^\beta}(0, \lambda) \sim \lambda^{p_\beta}
\]
for some \( p_\beta > -|\beta| \), or
\[
\frac{\partial^\beta y}{\partial x^\beta}(0, \lambda) = O(\lambda^N)
\]
for every \( N \).

Proof of lemma. For \( M \) large, write
\[
\psi(y, x) = Ay^\ell_0 + \sum_{j=1}^{n-2} b_j x_j^{\alpha_j} y^{\beta_j} + \sum_{u, \beta} c_{u, \beta} x^u y^\beta + O(|x|^M) + O(y^M)
\]
where \( A > 0, b_1, \ldots, b_{n-2} \neq 0, \frac{\alpha_j}{m_j} + \frac{\beta_j}{\ell_0} = 1 \) for \( 1 \leq j \leq n-2 \), each \( x^u y^\beta \) satisfies \( \sum_{j=1}^{n-2} \frac{u_j}{m_j} + \frac{\beta}{\ell_0} \geq 1 \) and if \( u = (0, \ldots, u_j, \ldots, 0) \), then \( u_j > \alpha_j \). To prove (1), we will show inductively that in the larger region, \( |x_j| \leq \epsilon \lambda^{\ell_{m_j}} \), \( 1 \leq j \leq n-2 \),
\[
(4.10) \quad \left| \frac{\partial^k y}{\partial x_j^k} \right| \leq \delta \lambda^{1-k} \lambda^{\ell_{m_j}}, \quad 1 \leq k \leq \alpha_j - 1,
\]
provided \( \epsilon = \epsilon(\delta) > 0 \) is small enough. We first prove (4.10) for \( k = 1 \) (so \( \alpha_j \geq 2 \) or there is nothing to prove). If we differentiate (4.5) with respect to \( x_j \), noting \( y \sim \lambda \) from (4.6), we obtain
\[
0 = C_1 \frac{\partial y}{\partial x_j} + C_2
\]
where
\[
C_1 = A\ell_0 y^{\ell_0 - 1} + O(\lambda^{\ell_0 + 1}) + E
\]
and \( E \) is a finite sum of terms of the form \( x^u y^{\beta - 1} \) where \( u = (u_1, \ldots, u_{n-2}) \neq 0 \) and \( \sum_{j=1}^{n-2} \frac{u_j}{m_j} + \frac{\beta}{\ell_0} \geq 1 \). Hence for \( \epsilon > 0 \) small enough, \( |x^u y^{\beta - 1}| \leq \delta \lambda^{\ell_0 - 1} \) and therefore
\[
C_1 \sim \lambda^{\ell_0 - 1}.
\]
\( C_2 \) is \( O(\lambda^{\ell_0}) \) plus a finite sum of terms of the form \( x_j^{-1} x^u y^\beta \) with \( u_j \geq 2 \) and \( \frac{|u|}{m_j} + \frac{\beta}{\ell_0} \geq 1 \). Thus for \( \epsilon > 0 \) small enough,
\[
|x_j^{-1} x^u y^\beta| \leq \delta \lambda^{\frac{|u|}{m_j} + \beta - \ell_0} \lambda^{-\frac{\ell_0}{m_j}} \leq \delta \lambda^{\ell_0 - \frac{\ell_0}{m_j}}
\]
and so \( C_2 = O(\delta \lambda^{\ell_0 - \frac{\ell_0}{m_j}}) \) which proves (4.10) with \( k = 1 \).
Next we assume (4.10) for \( k \leq k_0 - 1 \) where \( k_0 \leq \alpha_j - 1 \), and prove (4.10) for \( k = k_0 \). Differentiating (4.5) \( k_0 \) times with respect to \( x_j \), we again obtain

\[
0 = D_1 \frac{\partial^{k_0} y}{\partial x_j^{k_0}} + D_2
\]

where as before \( D_1 \sim \lambda^{\ell_0 - 1} \). \( D_2 \) consists of a finite sum of products of terms involving either a positive power of \( x \) or a derivative of order at most \( k_0 - 1 \) of \( y \) with respect to \( x_j \). In the first case we pick up an \( \epsilon \) from the powers of \( x \) and in the second case we pick up a \( \delta \) from the induction hypothesis. So we only need to determine the magnitude of each term in \( D_2 \). Since each term in the expression for \( \psi(y, x) \) is \( O(\lambda^{\ell_0}) \), we only need to understand how the powers of \( \lambda \) decrease when we differentiate a product involving \( x^u \) and \( (\frac{\partial^{k_j} y}{\partial x_j^{k_j}})^p \), \( 1 \leq k \leq k_0 - 2 \), with respect to \( x_j \). Differentiating \( x^u \) gives \( x_j^{-1} x^u \), losing \( \lambda^{-\frac{\ell_0}{m_j}} \) and differentiating

\[
\left( \frac{\partial^k y}{\partial x_j^k} \right)^p \text{ gives } \left( \frac{\partial^k y}{\partial x_j^k} \right)^{p-1} \frac{\partial^{k+1} y}{\partial x_j^{k+1}}, \text{ losing } \lambda^{-(1-k\frac{\ell_0}{m_j})} \lambda^{1-(k+1)\frac{\ell_0}{m_j}} = \lambda^{-\frac{\ell_0}{m_j}}
\]

by induction. Therefore each term in \( D_2 \) is \( O(\delta \lambda^{\ell_0 - k_0\frac{\ell_0}{m_j}}) \) and this finishes the proof of (4.10) and thus (1) of the lemma.

The proof of (2) follows in the same way as the proof of (1). The only difference is that differentiating the term \( b_j x_j^{\alpha_j} y^{\beta_j} \), \( b_j \neq 0 \), contributes a term \( b_j \alpha_j^1 y^{\beta_j} \sim \lambda^{\beta_j} \) and so

\[
D_2 \sim \lambda^{\beta_j} + O(\delta \lambda^{\ell_0 - k_0\frac{\ell_0}{m_j}}) \sim \lambda^{\ell_0 - \alpha_j \frac{\ell_0}{m_j}}
\]

since \( \frac{\alpha_j}{m_j} + \frac{\beta_j}{\ell_0} = 1 \). This shows (2).

The proof of (3) follows similarly. We use induction on the partial ordering \( u = (u_1, \ldots, u_{n-2}) \leq \beta = (\beta_1, \ldots, \beta_{n-2}) \) if and only if \( u_j \leq \beta_j, 1 \leq j \leq n-2 \). (1) and (2) show (3) is true for all pure derivatives, \( \beta = (0, \ldots, \beta_j, \ldots, 0) \). The arguments used in proving (1) and (2) show that if (3) is true for all \( u \leq \beta \), then differentiating (4.3) shows

\[
0 = D_1 \frac{\partial^{\beta} y}{\partial x^\beta} + D_2
\]

where \( D_1 \sim \lambda^{\ell_0 - 1} \) and

\[
D_2 = O \left( \lambda^{\ell_0 - \ell_0 \sum_{j=1}^{n-2} \frac{\beta_j}{m_j}} \right),
\]

proving (3).
Finally to prove (4), we first note that (3) implies that it is enough to show (4) for any power $p_{β}$. Again we use induction on the partial ordering $\leq$, supposing (4) is true for all $u \leq β$. Rewriting (4.4) expresses (4.5) as

$$
\lambda^{ℓ_{0}} = A y^{ℓ_{0}} + \sum_{u \leq β} a_{u}(y)x^{u} + O(|x|^{β} + 1)
$$

where $a_{0}(y) = O(y^{ℓ_{0} + 1})$, $a'_{0}(y) = O(y^{ℓ_{0}})$ and the $a_{u}$'s are smooth. Taking the $β$-th derivative of (4.11) gives

$$
0 = [Aℓ_{0}y^{ℓ_{0} - 1}(0, λ) + a'_{0}(y(0, λ))] \frac{∂^β y}{∂x^β}(0, λ) + C(λ)
$$

where $C(λ)$ is a finite sum of terms of the form

$$
a^{(s)}(y(0, λ)) \prod_{u \leq β} \left( \frac{∂^{u} y}{∂x^{u}}(0, λ) \right)^{q_{u}}
$$

for some non-negative integers $q_{u}$. Here $a(y)$ is either a power of $y$ or one of the $a_{u}$'s. Using the fact that $y(0, λ) \sim λ^{1}$ and the inductive hypothesis, we see that $C(λ) \sim λ^{p}$ for some $p$ or $C(λ) = O(λ^{N})$ for every $N$. Since

$$
\frac{∂^β y}{∂x^β}(0, λ) = - \frac{C(λ)}{[Aℓ_{0}y^{ℓ_{0} - 1}(0, λ) + a'_{0}(y(0, λ))]}\n$$

and $a'_{0}(y) = O(λ^{ℓ_{0}})$, we have shown (4) and this finishes the proof of the lemma. □

We now turn back to the proof of (4.1) where we are examining the integral in (4.3). Let us write

$$
y(x, λ) = M_{1}(x, λ) + M_{2}(x, λ)
$$

where $M_{1}(x, λ)$ is a polynomial in $x_{1}$ of degree $α_{1} - 1$ and $M_{2}$ is that part of the Taylor expansion of $y(x, λ)$ in the variable $x_{1}$ that is $O(|x|^{α_{1}})$. We wish to replace the integral in (4.3) by a similar integral where $y(x, λ)$ is replaced by $M_{1}(x, λ)$ and the $λ$ integral is restricted to

$$
λ \leq \left( \frac{1}{|η|} \right)^{-1 + α_{1}(α_{1})\kappa(α_{1})^{-1}}
$$

where $κ(α_{1}) = 1 - α_{1} \frac{ℓ_{0}}{m_{1}}$. Note that $α_{1} + κ(α_{1}) > 1$. Since

$$
\frac{∂^{α_{1}} y}{∂x_{1}^{α_{1}}} \sim λ^{1 - α_{1} \frac{ℓ_{0}}{m_{1}}}
$$
by part (2) of Lemma 1 and since \( \alpha_1 \geq 2 \), an application of Van der Corput’s lemma together with integration by parts shows

\[
\left| \int_{\lambda \geq \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1 + \kappa(\alpha_1)}}} e^{i\gamma_\varphi(\lambda^\alpha)} \int_{|x| \leq y^{\frac{\alpha_0}{\alpha_1}}} e^{i\xi x} e^{i\eta y(x,\lambda)} K(x_1 \lambda) \, dx \, d\lambda \right|
\]

\[
\leq C \left( \frac{1}{|\eta|} \right)^{1/\alpha_1} \int_{\lambda \geq \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1 + \kappa(\alpha_1)}}} \frac{1}{\lambda^{\frac{\alpha_0}{\alpha_1}}} \left[ 1 - \alpha_1 \eta \frac{\alpha_0}{\alpha_1} \right] \int_{\mathbb{R}^{n-2}} \frac{1}{|x|^n + \lambda^n} \, dx \, d\lambda
\]

\[
\leq C \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1}} \int_{\lambda \geq \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1 + \kappa(\alpha_1)}}} \frac{\lambda^{\frac{\alpha_0}{\alpha_1}}}{\lambda^{2 + \frac{1}{\alpha_1}}} \, d\lambda \leq C \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1}} |\eta|^{\frac{1}{\alpha_1} \left[ \frac{\alpha_1 + \kappa(\alpha_1)}{\alpha_1 + \kappa(\alpha_1)} \right]} = C.
\]

In the region

\[
\lambda \leq \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1 + \kappa(\alpha_1)}}
\]

we would like to replace \( e^{i\eta y(x,\lambda)} \) by \( e^{i\eta M_1(x,\lambda)} \). We expect to be able to replace \( e^{i\eta y(x,\lambda)} \) by \( e^{i\eta M_1(x,\lambda)} \) with a bounded error when

\[
|x| \leq \left( \frac{1}{|\eta| \lambda^{\kappa(\alpha_1)}} \right)^{\frac{1}{\alpha_1}}
\]

since

\[
|e^{i\eta y(x,\lambda)} - e^{i\eta M_1(x,\lambda)}| \leq C|\eta|\lambda^{1-\alpha_1} \frac{\alpha_0}{\alpha_1} |x|^{\alpha_1} \leq C
\]

when \( |x| \leq \left( \frac{1}{|\eta| \lambda^{\kappa(\alpha_1)}} \right)^{\frac{1}{\alpha_1}} \). However in the complementary region, when

\[
\lambda \leq \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1 + \kappa(\alpha_2)}} \quad \text{and} \quad |x| \geq \left( \frac{1}{|\eta| \lambda^{\kappa(\alpha_1)}} \right)^{\frac{1}{\alpha_1}}
\]
$K$ is uniformly integrable. In fact

$$\int_{\lambda \leq \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1 + \kappa(\alpha_1)}}} \int_{|x| \geq \left( \frac{1}{|\eta\lambda^{\kappa(\alpha_1)}} \right)^{\frac{1}{\alpha_1}}} |K(x, \lambda)| \, dx \, d\lambda$$

$$\leq C \int_{\lambda \leq \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1 + \kappa(\alpha_1)}}} \int_{|x| \geq \left( \frac{1}{|\eta\lambda^{\kappa(\alpha_1)}} \right)^{\frac{1}{\alpha_1}}} \frac{1}{|x|^{\alpha_1 + 1}} \, dx \, d\lambda$$

$$\leq \int_{\lambda \leq \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1 + \kappa(\alpha_1)}}} (|\eta| \lambda^{\kappa(\alpha_1)})^{\frac{\kappa(\alpha_1)}{\alpha_1} + 1} \frac{d\lambda}{\lambda} \leq C$$

since

$$\frac{\kappa(\alpha_1)}{\alpha_1} = 1 - \frac{\alpha_1 \ell_0}{m_1} > -1.$$ 

Replacing $e^{i\eta y(x, \lambda)}$ by $e^{i\eta M_1(x, \lambda)}$ when

$$\lambda \leq \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1 + \kappa(\alpha_1)}} \quad \text{and} \quad |x| \leq \left( \frac{1}{|\eta| \lambda^{\kappa(\alpha_1)}} \right)^{\frac{1}{\alpha_1}}$$

creates an error at most

$$C|\eta| \int_{\lambda \leq \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1 + \kappa(\alpha_1)}}} \lambda^{\kappa(\alpha_1)} \int_{|x| \leq \left( \frac{1}{|\eta| \lambda^{\kappa(\alpha_1)}} \right)^{\frac{1}{\alpha_1}}} \frac{|x|^{\alpha_1}}{|x|^{\alpha_1 + 1}} \, dx \, d\lambda$$

$$\leq C|\eta| \int_{\lambda \leq \left( \frac{1}{|\eta|} \right)^{\frac{1}{\alpha_1 + \kappa(\alpha_1)}}} \lambda^{\kappa(\alpha_1)} \frac{d\lambda}{\lambda^{\alpha_1 + 1} \lambda^{\alpha_1 - 1}} \leq C$$

since

$$\frac{\alpha_1 + \kappa(\alpha_1)}{\alpha_1} = 1 - \frac{\ell_0}{m_1} + \frac{1}{\alpha_1} > 0.$$
Therefore
\[
I = \int_{\lambda \leq \frac{1}{1+\lambda^{1/\lambda}}} \int_{0 \leq \lambda \leq 1} e^{i\gamma \varphi(\lambda^0)} e^{i\xi x} e^{i\eta M_1(x, \lambda)} K(x, \lambda) \, dx \, d\lambda + O(1).
\]

Since
\[
M_1(x, \lambda) = \sum_{k=0}^{\alpha_1-1} \frac{1}{k!} \frac{\partial^k y}{\partial x_1^k}(0, x_2, \ldots, x_{n-2}) x_1^k,
\]
we see that for \(2 \leq j \leq n-2\),
\[
\frac{\partial^{\alpha_j} M_1}{\partial x_j^{\alpha_j}} (x, \lambda) = \frac{\partial^{\alpha_j} y}{\partial x_j^{\alpha_j}}(0, x_2, \ldots, x_{n-2}) + \sum_{k=1}^{\alpha_1-1} \frac{1}{k!} \frac{\partial^{k+\alpha_j} y}{\partial x_1^k \partial x_j^{\alpha_j}}(0, x_2, \ldots, x_{n-2}) x_1^k.
\]

Also since \(|x_1| \leq \lambda^{\alpha_0/\ell_0}\), we have by part (3) of Lemma 1,
\[
\left| \sum_{k=1}^{\alpha_1-1} \frac{1}{k!} \frac{\partial^{k+\alpha_j} y}{\partial x_1^k \partial x_j^{\alpha_j}}(0, x_2, \ldots, x_{n-2}) x_1^k \right| \leq \epsilon \lambda^{\alpha_0/\ell_0} \lambda^{1-\ell_0} \left( \frac{\ell_0 - \frac{\alpha_j}{\ell_0}}{m_j} \right) \leq \epsilon \lambda^{1-\ell_0} \frac{\alpha_j}{m_j},
\]
and since
\[
\frac{\partial^{\alpha_j} y}{\partial x_j^{\alpha_j}} \sim \lambda^{1-\ell_0} \frac{\alpha_j}{m_j}
\]
by part (2) of Lemma 1, we conclude that for \(2 \leq j \leq n-2\),
\[
\frac{\partial^{\alpha_j} M_1}{\partial x_j^{\alpha_j}} (x, \lambda) \sim \lambda^{1-\ell_0} \frac{\alpha_j}{m_j}.
\]

Therefore we may proceed in the same manner to find that up to a bounded error
\[
I = \int_{\lambda \leq \frac{1}{|\eta|^0}} \int_{|x| \leq \lambda^{\alpha_0/|\eta|^0}} e^{i\gamma \varphi(\lambda^0)} e^{i\xi x} e^{i\eta Q(x, \lambda)} K(x, \lambda) \, dx \, d\lambda
\]
for some \(0 < \delta < 1\) where
\[
Q(x, \lambda) = \sum_{\beta \leq \alpha_j - 1} \frac{1}{\beta!} \frac{\partial^\beta y}{\partial x^\beta}(0, \lambda) x^\beta.
\]

By part (4) of Lemma 1, we have for each \(\beta\) with \(\beta_j \leq \alpha_j - 1\), \(1 \leq j \leq n - 2\), either
\[
\frac{\partial^\beta y}{\partial x^\beta}(0, \lambda) \sim \lambda^{p_\beta}
\]
for some \(p_\beta > -|\beta|\) or
\[
\frac{\partial^\beta y}{\partial x^\beta}(0, \lambda) = O(\lambda^N)
\]
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for every $N$. If the latter occurs, then up to a bounded error, we may clearly replace $e^{i\eta \frac{\partial}{\partial x}(0,\lambda)x}$ by 1. When the former occurs, that is, when $\frac{\partial^\alpha y}{\partial x^\alpha}(0,\lambda)$ behaves like a power of $\lambda$, we can repeat the above argument to see that for some (other) $\delta$, $0 < \delta < 1$,

$$I = \int_{y \leq \frac{1}{|y|^\beta}} e^{i\gamma y(0,\lambda)} \int_{|x| \leq \frac{t_0}{\lambda^{n+\sigma}}} e^{i\sum_{j=1}^{n-2} (x_j + \lambda^\delta \eta_j) x_j} K(x,\lambda) dx d\lambda + \mathcal{O}(1).$$

For each $x_j$ integral, by splitting the $\lambda$ integral where $\lambda$ is smaller or larger than $|x_j|^{1/\eta_j}$, we can once again repeat the same argument to conclude that

$$I = \int_{\lambda \leq \min\left(\frac{1}{|x_j|^{1/\eta_j}}, \frac{1}{|x_j|^{1/\eta_j}}\right)} e^{i\gamma y(0,\lambda)} \int_{|x| \leq \frac{t_0}{\lambda^{n+\sigma}}} e^{i\sum_{j=1}^{n-2} (x_j + \lambda^\delta \eta_j) x_j} K(x,\lambda) dx d\lambda + \mathcal{O}(1)$$

for some $0 < b < 1$. Thus the proof of (4.1) will be finished once we establish the identity

(4.12) $\int_{\mathbb{R}^{n-2}} K(x,1) dx = \int_{\Sigma^+} K(\omega) d\sigma(\omega).$

This is done by making the change of variables

$$x_j = \frac{s_j}{1 - |s|^2}, \quad 1 \leq j \leq n - 2.$$

In evaluating the Jacobian of this change of variables, we need to observe that if an $r \times r$ matrix $(\alpha_{j,k})$ is defined by $\alpha_{j,k} = s_j s_k$ for $j \neq k$ and $\alpha_{j,j} = 1 - |s|^2 - s_j^2$, then

$$\det(\alpha_{j,k}) = (1 - |s|^2)^{r-1}.$$

This calculation was shown to us by A. Carbery and is carried out in the appendix. This establishes (4.12) and finishes the proof of (4.1). The proof of (4.2) is similar. It remains to prove (4.3).

Suppose first that there is no constant $C_0$ so that $\varphi'(C_0 \lambda) \geq 2\varphi'(\lambda)$ for $0 < \lambda \leq 1$. Then there exists a sequence of points $\lambda_j \searrow 0$ such that

$$\frac{\lambda_j \varphi'(\lambda)}{\lambda_j \varphi'(\lambda_j) - \varphi(\lambda_j)} \to \infty.$$

See, e.g., [NVWW]. Let

$$\gamma_j = \pi \frac{1}{\lambda_j \varphi'1(\lambda_j) - \varphi(\lambda_j)}, \quad \eta_j = \gamma_j \varphi'(\lambda_j),$$

for every $N$. If the latter occurs, then up to a bounded error, we may clearly replace $e^{i\eta \frac{\partial}{\partial x}(0,\lambda)x}$ by 1. When the former occurs, that is, when $\frac{\partial^\alpha y}{\partial x^\alpha}(0,\lambda)$ behaves like a power of $\lambda$, we can repeat the above argument to see that for some (other) $\delta$, $0 < \delta < 1$,
and choose $\xi_j$ so that
\[
\frac{1}{\xi_j} = \min \left( \frac{\lambda_j}{\eta_j}, \frac{1}{\eta_j^{1+\epsilon}} \right)
\]
where $\epsilon > 0$ is chosen so that $q(\lambda) = \lambda + \mathcal{O}(\lambda^{1+\epsilon})$. Then
\[
\left| \int_0^1 e^{i\gamma_j \varphi(\lambda)} \sin(\eta_j q(\lambda)) \frac{d\lambda}{\lambda} \right| = \left| \int_0^1 e^{i(\gamma_j \varphi(\lambda)-\eta_j \lambda)} \frac{d\lambda}{\lambda} \right| + \mathcal{O}(1)
\]
\[
\geq A \log \left( \frac{\eta_j}{\xi_j} \right) \geq A \log(\lambda_j \eta_j) \to +\infty
\]
for some $A > 0$ since $\lambda_j \eta_j \to \infty$ and
\[
0 \leq \eta_j \lambda - \gamma_j \varphi(\lambda) \leq \frac{\pi}{4}
\]
for all $0 \leq \lambda \leq \lambda_j$.

Finally let us turn to the proof of the sufficiency of (4.3) and assume $\bar{\varphi}(0) = \bar{\varphi}'(0) = 0$, $\varphi'(C_0 t) \geq 2 \varphi'(t)$ for some $C_0 \geq 1$. It suffices to show that the integral
\[
(4.13) \quad II = \int_{\frac{1}{\eta} \leq t \leq 1} e^{i\gamma \varphi(t)} e^{-i\eta q(t)} \frac{dt}{t}
\]
is uniformly bounded in $\gamma, \eta > 0$. First assume $10 \gamma > \eta$. Choosing $t_0$ such that $\bar{\varphi}'(t_0) = \frac{\eta}{\gamma}$ we write
\[
II = \int_{\frac{1}{\eta} \leq t \leq t_0} + \int_{t_0 \leq t \leq C_0 t_0} + \int_{C_0 t_0 \leq t \leq 1} = A + B + D.
\]
For $\frac{1}{\eta} \leq t \leq \frac{\eta}{C_0}$, $\frac{d}{dt}(\eta t - \gamma \varphi(t)) = \eta - \gamma \varphi'(t) \geq \eta - \gamma \varphi'(\frac{\eta}{C_0}) \geq \frac{\eta}{2}$, and so integrating by parts shows
\[
|A| \leq \frac{1}{\eta} \int_{\frac{1}{\eta} \leq t \leq 1} \eta|q'(t)| - 1 \left| \frac{dt}{t} + \frac{1}{\eta} \int_{\frac{1}{\eta} \leq t} \frac{1}{t^2} dt \right| + C
\]
\[
\leq C \int_{0}^{\frac{1}{\eta}} t^\epsilon \frac{dt}{t} + C \leq C.
\]
Also
\[
|B| \leq \int_{\frac{\eta}{C_0} \leq t \leq C_0 t_0} \frac{1}{t} dt \leq 2 \log(C_0).
\]
For $C_0 t_0 \leq t$, $\frac{d}{dt}(\gamma \varphi(t) - \eta t) = \gamma \varphi'(t) - \eta \geq \frac{\eta}{\gamma} \varphi'(t)$, and so integrating by parts show

$$|D| \leq \frac{1}{\gamma} \int_{\frac{\eta}{\gamma} \leq t \leq t_0} \left( \frac{\varphi''(t)}{t[\varphi'(t)]^2} + \frac{1}{\varphi'(t)} + \frac{\eta|q'(t)| - 1|}{\varphi'(t)} \right) dt + \frac{\eta}{\gamma} \frac{1}{\varphi'(t_0)}$$

$$\leq \frac{\eta}{\gamma} \int_{t_0 \leq t} \frac{\varphi''(t)}{[\varphi'(t)]^2} dt + \frac{1}{\varphi'(t_0)} \left[ \frac{\eta}{\gamma} + C \frac{\eta}{\gamma} \int_{0}^{t_0} \frac{1}{t} dt \right] + \frac{\eta}{\gamma} \frac{1}{\varphi'(t_0)}$$

$$\leq C \frac{\eta}{\gamma} \frac{1}{\varphi'(t_0)} \leq C$$

since $\frac{\eta}{\gamma} = \varphi'(t_0)$. Next suppose $10 \gamma \leq \eta$. Then in a neighborhood of the origin, $\frac{d}{dt}[\eta t - \gamma \varphi(t)] \geq \frac{\eta}{2}$, and so integrating by parts shows

$$|II| \leq \frac{1}{\eta} \left[ \int_{\frac{\eta}{2} \leq t} |q'(t)| - 1 dt - \int_{\frac{\eta}{2} \leq t} \frac{1}{t} dt \right] + C$$

$$\leq C \int_{0}^{1} t^2 \frac{dt}{t} + C \leq C.$$

This completes the proof of Theorems 2 and 3. □

5. Proof of Theorem 4

We will prove the $L^p$ boundedness of the maximal function

$$Mf(x', x_n) = \sup_{0 < h \leq 1} \frac{1}{h^{n-1}} \int_{|t| \leq h} f(x' - t, x_n - \varphi(\psi(t))) dt.$$

The proof for the singular integral is similar.

When $E_{\ell_0} = \{0\}$, the main term $P(t)$, in the decomposition (3.1) for $\psi(t)$, $\psi(t) = P(t) + R(t)$, is a positive homogeneous polynomial of degree $\ell_0$. $R(t)$ consists of all the terms in the Taylor expansion of $\psi$ with degree greater than $\ell_0$. The proof of $L^p$ boundedness for $M$ in the case $P(t) = |t|^2$ and $R(t) \equiv 0$ is carried out in [KWWZ]. We will see that slight modifications of the arguments given in [KWWZ] work for the general case.

It will be convenient for us to use polar coordinates with respect to the surface $P(\omega) = 1$. That is, every $t \neq 0 \in \mathbb{R}^{n-1}$ can be written uniquely as $t = r\omega$ where $r > 0$ and $P(\omega) = 1$. We also introduce a norm $|| \cdot ||$ so that $||t|| = ||r\omega|| = r$. Since the Euclidean norm of $\omega$, $||\omega||$, is bounded above and below as $\omega$ runs over the surface
\[ P(\omega) = 1, \] it is clear that the maximal function \( Mf(x) \) is pointwise comparable to the maximal function defined in terms of averages with respect to the norm \( \| \cdot \| \). Therefore it suffices to consider
\[
Mf(x) = \sup_{k>0} 2^{k(n-1)} \left| \int \chi(2^k \|\|) f(x - \Gamma(t)) \, dt \right| = \sup_{k>0} |f \ast d\mu_k(x)|
\]
where \( \chi \) is a smooth cut-off function supported in \([1, 2] \) and chosen so that
\[
2^{k(n-1)} \int_{\mathbb{R}^{n-1}} \chi(2^k \|\|) \, dt \equiv 1.
\]

To prove \( L^p \) bounds for \( M \) we introduce dilations \( \{\delta(t)\}_{t>0} \), defined by \( \delta(t)(\xi, \gamma) = (t\xi, \varphi(t)\gamma) \). Although the “balls” generated with respect to these dilations do not in general form a space of homogeneous type with respect to Lebesgue measure, an appropriate singular integral and Littlewood–Paley theory for the dilations \( \{\delta(t)\}_{t>0} \) has been worked out in [CCVWW]. Using this theory and well-known techniques, following the arguments detailed in [KWWZ], we reduce ourselves to proving two basic estimates for the Fourier transform of the measures \( \{d\mu_k\} \) defined above:
\[
(5.1) \quad |\widehat{d\mu_k}(\xi, \gamma) - 1| \leq C|\delta(2^{-k+3})(\xi, \gamma)|,
\]
and
\[
(5.2) \quad |\widehat{d\mu_k}(\xi, \gamma)| \leq C|\delta(2^{-k-1})(\xi, \gamma)|^{-\epsilon}
\]
for some \( \epsilon > 0 \). Using polar coordinates \( t = r\omega \),
\[
(5.3) \quad \widehat{d\mu_k}(\xi, \gamma) = 2^{k(n-1)} \int_{\mathbb{R}} \int_{P(\omega)=1} \chi(2^k r)e^{i\xi r \omega}e^{ir\varphi(\psi(r\omega))}r^{n-2}h(\omega) \, d\omega \, dr
\]
where \( h(\omega) \) is some smooth function. Since \( \psi(r\omega) = r^{\ell_0} + O(r^{\ell_0+1}) \), we have for \( k > 0 \) large, \( \varphi(\psi(r\omega)) \leq \varphi(2^{-k+3}) \) when \( 2^{-k} \leq r \leq 2^{-k+1} \). Therefore
\[
|\widehat{d\mu_k}(\xi, \gamma) - 1| \leq C[2^{-k}|\xi| + \varphi(2^{-k+3})|\gamma|] \leq C|\delta(2^{-k+3})(\xi, \gamma)|,
\]
establishing (5.1). To prove (5.2) we make the change of variables
\[
(5.4) \quad \lambda^{\ell_0} = \psi(r\omega) = r^{\ell_0} + R(r\omega)
\]
in the \( r \) integral in (5.3) for fixed \( \omega \). For \( k > 0 \) large this is a good change of variables and so
\[
(5.5) \quad \widehat{d\mu_k}(\xi, \gamma) = 2^{k(n-1)} \int_{\mathbb{R}} e^{ir\varphi(\lambda)} \int_{P(\omega)=1} e^{ir(\lambda, \omega)\xi} \chi(2^k r(\lambda, \omega)) \, \frac{\partial r}{\partial \lambda} r^{n-2}(\lambda, \omega) h(\omega) \, d\omega \, d\lambda.
\]
From (5.4) one easily deduces the following estimates on the derivatives of $r(\lambda, \omega)$:

\[(5.6) \quad r(\lambda, \omega) = \lambda + O(\lambda^2), \quad \frac{\partial r}{\partial \lambda} = 1 + O(\lambda),\]

\[(5.7) \quad \nabla_{\omega} r = O(\lambda), \quad \frac{\partial r}{\partial \lambda \partial \omega} = O(1),\]

\[(5.8) \quad \frac{\partial^2 r}{\partial \lambda^2} = O\left(\frac{1}{\lambda}\right).\]

Since $P(\omega) = 1$ is of finite type, we can argue as in section 3 to find an $\epsilon > 0$ such that

\[
\left| \int_{P(\omega) = 1} e^{ir(\lambda, \omega)\xi \cdot \omega} h(\omega) \, d\omega \right| \leq C \frac{1}{\lambda |\xi|^{2\epsilon}}.
\]

Now integrating by parts, using (5.6) and (5.7), shows

\[
|\widehat{d\mu_k}(\xi, \gamma)| \leq C \left( \frac{1}{2^{-k-1}|\xi|} \right)^{2\epsilon},
\]

establishing (5.2) if

\[
\sqrt{|\gamma|}\bar{\varphi}(2^{-k-1}) \leq C2^{-k-1}|\xi|.
\]

On the other hand, if

\[
C2^{-k-1}|\xi| \leq \sqrt{|\gamma|}\bar{\varphi}(2^{-k-1}),
\]

we perform the $\lambda$ integration first, writing (5.5) as

\[
\widehat{d\mu_k}(\xi, \gamma) = 2^{k(n-1)} \int_{P(\omega) = 1} h(\omega) \int_{\mathbb{R}} e^{i[\gamma\bar{\varphi}(\lambda) + \lambda \xi \cdot \omega]} e^{i[r(\lambda, \omega) - \lambda |\xi| \omega]} \chi(2^k r) \frac{\partial r}{\partial \lambda} r^{n-2} \, d\lambda \, d\omega.
\]

For $2^{-k} \leq r(\lambda, \omega) \leq 2^{-k+1}$, we have

\[
\left| \frac{\partial}{\partial \lambda} [\gamma\bar{\varphi}(\lambda) + \lambda \xi \cdot \omega] \right| \geq \frac{|\gamma|}{2} \bar{\varphi}'(2^{-k-1}) \geq |\gamma| \frac{\bar{\varphi}(2^{-k-1})}{2^{-k}}
\]

since $2^{-k}|\xi| \ll |\gamma|\bar{\varphi}(2^{-k-1})$. Thus integrating by parts, using (5.6) and (5.8), shows

\[
|\widehat{d\mu_k}(\xi, \gamma)| \leq C \left[ \frac{1}{|\gamma|\bar{\varphi}(2^{-k-1})} + \frac{|\xi|2^{-k}}{|\gamma|\bar{\varphi}(2^{-k-1})} \right]
\]

\[
\leq C \frac{1}{\sqrt{|\gamma|\bar{\varphi}(2^{-k-1})}}
\]

\[
\leq C \frac{1}{\sqrt{|\delta(2^{-k-1})(\xi, \gamma)|}}
\]

since $C2^{-k}|\xi| \leq \sqrt{|\gamma|\bar{\varphi}(2^{-k-1})}$. 
This completes the proof of (5.1) and (5.2) from which the $L^p$ boundedness of the maximal function follows as in [KWWZ].

6. Proof of Theorem 5

We need to show that the multiplier for $H$,

\[(6.1)\quad m(\xi, \gamma) = \int \int \frac{1}{|t|} e^{i\gamma \varphi(t)} e^{i\xi \cdot t} K(t) \, dt\]

is uniformly bounded for $\xi \in \mathbb{R}^2$ and $\gamma \in \mathbb{R}$. Introducing polar coordinates with respect to the convex curve $\psi(t) = 1$, we may write (6.1) as

\[(6.2)\quad \int_0^1 e^{i\gamma \varphi(r)} \frac{1}{r} \int_{\psi(\omega) = 1} e^{i\omega \cdot \xi} K(\omega) h(\omega) \, d\omega \, dr\]

for some smooth function $h(\omega)$. The argument used in the proof of Theorem 1 to establish (3.10) shows

\[\int_{\psi(\omega) = 1} K(\omega) h(\omega) \, d\omega = 0\]

and so the part of the integral in (6.2) where $r \leq \frac{1}{|\xi|}$ is at most

\[C \sup_{\psi(\omega) = 1} \int_0^1 \frac{1}{r} \int_{\psi(\omega) = 1} (e^{i\omega \cdot \xi} - 1) K(\omega) h(\omega) \, d\omega \, dr \leq C |\xi| \int_{r \leq \frac{1}{|\xi|}} dr \leq C.\]

For the region where $r \geq \frac{1}{|\xi|}$, we observe that the inner integral in (6.2) is the Fourier transform of a smooth density on the convex curve $\psi(\omega) = 1$ evaluated at $r\xi$. This Fourier transform can be estimated in terms of the "balls," $E(t, \epsilon)$, introduced in section 1. In fact

\[\int_{\psi(\omega) = 1} e^{i\omega \cdot \xi} K(\omega) h(\omega) \, d\omega \leq C \left[ |E(t_1(\xi), \frac{1}{r|\xi|})| + |E(t_2(\xi), \frac{1}{r|\xi|})| \right]\]

where $t_1(\xi)$ and $t_2(\xi)$ are the two points on the curve $\psi(t) = 1$ whose tangent lines are normal to $\xi$. See [BNW]. Therefore the part of the integral in (6.2) where $r \geq \frac{1}{|\xi|}$ can be estimated by

\[C \sup_{\psi(t) = 1} \int_{|t| \leq r} |E(t, \frac{1}{r|\xi|})| \, d\delta \leq C \sup_{r \geq \frac{1}{|\xi|}} \int_0^1 |E(t, \delta)| \, d\delta.\]
Hence the multiplier $m(\xi, \gamma)$ is uniformly bounded in $\xi$ and $\gamma$ if the quantity

$$
\sup_{t} \int_0^1 |E(t, \delta)| \frac{d\delta}{\delta}
$$

is finite. This completes the proof of Theorem 5. \qed 

7. Appendix

In this appendix we will compute the determinant of an $r \times r$ matrix $A = \{\alpha_{j,k}\}$ of the form $A = cI + B$ where $B = \{b_{j,k}\}$ and $b_{j,k} = b s_j t_k$. We will show that

$$
det(A) = c^r + c^{r-1} b \sum_{j=1}^{r} s_j t_j.
$$

(7.1)

For the example we need in this paper, $\alpha_{j,k} = s_j s_k$ for $j \neq k$ and $\alpha_{j,j} = 1 - |s|^2 + s_j^2$. Therefore taking $t_j = s_j$, $c = 1 - |s|^2$ and $b = 1$ in the above formula (7.1) gives us the desired result $det(A) = (1 - |s|^2)^{r-1}$ in this case. To prove (7.1) first note that as a function of $s = (s_1, \ldots, s_r)$, $det(A)$ is an affine function in each of the variables $s_j$ separately. Also computing any pure mixed derivative, e.g., $\frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3}$, of $det(A)$ gives rise to two or more rows being identical and therefore zero. Hence expanding $det(A)$ in its Taylor series in $s$ about the origin, we see that (7.1) follows from the fact that for each $1 \leq j \leq r$, the partial derivative of $det(A)$ with respect to $s_j$ at the origin is $c^{r-1} b t_j$. This is a straightforward computation. \qed

References

[BNW] J. Bruna, A. Nagel, and S. Wainger. Convex hypersurfaces and Fourier transforms. Ann. of Math., 127:333–365, 1988.

[CCVWW] Anthony Carbery, Michael Christ, James Vance, Stephen Wainger, and David K. Watson. Operators associated to flat plane curves: $L^p$ estimates via dilation methods. Duke Math. J., 59(3):675–700, 1989.

[KWWZ] Weon-Ju Kim, Stephen Wainger, James Wright, and Sarah Ziesler. Singular integrals and maximal functions associated to surfaces of revolution. Bull. London Math. Soc., 28(3):291–296, 1996.

[NVWW] A. Nagel, J. Vance, S. Wainger, and D. Weinberg. Hilbert transforms for convex curves. Duke Math. J., 50:735–744, 1983.

[Sch] Helmut Schulz. Convex hypersurfaces of finite type and the asymptotics of their Fourier transforms. Indiana Univ. Math. J., 40(4):1267–1275, 1991.

[SWWZ] A. Seeger, S. Wainger, J. Wright, and S. Ziesler. Classes of singular integrals along curves and surfaces. Trans. Amer. Math. Soc. To appear.

[St] E. Stein. Harmonic Analysis Princeton University Press, Princeton New Jersey, 1993.
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