On the \((n, d)^{th}\) \(f\)-Ideals *

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Abstract. A square-free monomial ideal \(I\) is called an \(f\)-ideal, if both 
\(\delta_F(I)\) and \(\delta_N(I)\) have the same \(f\)-vector, where \(\delta_F(I)\) (\(\delta_N(I)\), respectively) is the facet (Stanley-Reisner, respectively) complex related to \(I\).

In this paper, we introduce the concepts of perfect set containing \(k\) and 
perfect set without \(k\). We study the \((n, d)^{th}\) perfect sets and show that 
\(V(n, d) \neq \emptyset\) for \(d \geq 2\) and \(n \geq d + 2\). Then we give some algorithms 
to construct \((n, d)^{th}\) \(f\)-ideals and show an upper bound of the \((n, d)^{th}\) 
perfect number.

Key Words and phrases: perfect set containing \(k\); perfect set 
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1 Introduction

Throughout the paper, for a set \(A\), we use \(A_d\) to denote the set of the subsets of \(A\) with 
cardinality \(d\). For a monomial ideal \(I\) of \(S\), let \(sm(I)\) be the set of square-free monomials 
in \(I\). As we know, there is a natural bijection between \(sm(S)\) and \(2^n\), denoted by
\[
\sigma : x_{i_1}x_{i_2} \cdots x_{i_k} \mapsto \{i_1, i_2, \ldots, i_k\},
\]
where \([n] = \{1, 2, \ldots, n\}\) for a positive integer \(n\). A square-free monomial \(u\) is called 
covered by a square-free monomial \(v\), if \(u | v\) holds. For other concepts and notations, see 
references [2, 5, 7, 8, 10, 11].

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Constructing free resolutions of a monomial ideal is one of the core problems in commutative algebra. A main approach to the problem is by taking advantage of the properties of a simplicial complex, so it is important to have a research on the properties of the complex corresponding to the related ideals, see for example, references [4] [6] [9] [12]. There is an important class of ideals called \( f \)-ideals, whose facet complex \( \delta_F(I) \) and Stanley-Reisner complex \( \delta_N(I) \) have the same \( f \)-vector, where \( \delta_F(I) \) is generated by the set \( \sigma(G(I)) \), and \( \delta_N(I) = \{ \sigma(g) | g \in \text{sm}(S) \setminus \text{sm}(I) \} \). Note that the \( f \)-vector of a complex \( \delta_N(I) \), which is not easy to calculate in general, is essential in the computation of the Hilbert series of \( S/I \). Since the correspondence of the complex \( \delta_F(I) \) and the ideal \( I \) is direct and clear, it is more easier to calculate the \( f \)-vector of \( \delta_F(I) \). So, it is convenient to calculate the Hilbert series and study other corresponding properties of \( S/I \) while \( I \) is an \( f \)-ideal.

The formal definition of an \( f \)-ideal first appeared in [1], and it was then studied in [3]. In [7], the authors characterized the \( f \)-ideals of degree \( d \), as well as the \( f \)-ideals in general case. They introduced a bijection between square-free monomial ideals of degree \( 2 \) and simple graphs, and showed that \( V(n, 2) \neq \emptyset \) for any \( n \geq 4 \), where \( V(n, d) \) is the set of \( f \)-ideals of degree \( d \) in \( S = K[x_1, \ldots, x_n] \). The structure of \( V(n, 2) \) was determined, and the characterization of the unmixed \( f \)-ideals is also studied in [7].

In this paper, we give another characterization of unmixed \( f \)-ideals in part two. In part three, we generalize the aforementioned result of [7] by showing that \( V(n, d) \neq \emptyset \) for general \( d \geq 3 \) and \( n \geq d + 2 \). In part four, we introduce some algorithms to construct \((n, d)^{th}\) \( f \)-ideals, and we show an upper bound of the \((n, d)^{th}\) perfect number in part five. In part six, we show some examples of nonhomogeneous \( f \)-ideals, which is still open in [7].

The following propositions are needed in this paper.

**Proposition 1.1.** ([7] Theorem 2.4) Let \( S = K[x_1, \ldots, x_n] \), and let \( I \) be a square-free monomial ideal of \( S \) of degree \( d \) with the minimal generating set \( G(I) \). Then \( I \) is an \( f \)-ideal if and only if \( G(I) \) is \((n, d)^{th}\) perfect and \( |G(I)| = \frac{1}{2}C_n^d \) holds true.

**Proposition 1.2.** ([7] Proposition 3.3) \( V(n, 2) \neq \emptyset \) if and only if \( n = 4k \) or \( n = 4k + 1 \) for some positive integer \( k \).

**Proposition 1.3.** ([7] Proposition 5.3) Let \( S = K[x_1, \ldots, x_n] \). If \( I \) is an \( f \)-ideal of \( S \) of degree \( d \), then \( I \) is unmixed if and only if \( \text{sm}(S)_d \setminus G(I) \) is lower perfect.

In [7], a method for finding an \((n, 2)^{th}\) perfect set with the smallest cardinality is provided, namely, first, decompose the set \([n]\) into a disjoint union of two subsets \( B \) and \( C \) uniformly, i.e., \( ||B| - |C|| \leq 1 \) holds; then set \( A = \{ x_i x_j | i, j \in B, \text{or} \ i, j \in C \} \). Finally, \( A \) is an \((n, 2)^{th}\) perfect set whose cardinality is equal to the \((n, 2)^{th}\) perfect number \( N_{(n, 2)} \), where

\[
N_{(n, 2)} = \begin{cases} 
  k^2 - k, & \text{if } n = 2k; \\
  k^2, & \text{if } n = 2k + 1.
\end{cases}
\]
Note that any set $D$ with $A \subseteq D \subseteq sm(S)_2$ is also an $(n, 2)^{th}$ perfect set.

2 $(n, d)^{th}$ unmixed $f$-ideals

For a positive integer $d$ greater than 2, an $(n, d)^{th}$ $f$-ideal may be not unmixed, see Example 5.1 of [7] for a counterexample. So, it is interesting to characterize the unmixed $f$-ideals. In this section, we show a characterization of unmixed $f$-ideals by the corresponding simplicial complex, by taking advantage of the bijection $\sigma$ between square-free monomial ideals and simplicial complexes.

Recall that a simplicial complex is a $d$-flag complex if all of its minimal non-faces contain $d$ elements. Recall that $\Delta^\vee$ denotes the Alexander dual of a simplicial complex $\Delta$, see [8] for details.

**Proposition 2.1.** Let $S = K[x_1, \ldots, x_n]$, and let $I$ be a square-free monomial ideal of $S$ of degree $d$. $I$ is an $(n, d)^{th}$ unmixed $f$-ideal if and only if the followings hold:

1. $|G(I)| = C_n^d/2$;
2. $\dim \delta_F(I)^\vee = n - d - 1$;
3. $\langle \sigma(u) \mid u \in sm(S)_d \setminus G(I) \rangle$ is a $d$-flag complex.

**Proof.** We claim that the following two results hold true: First, the condition (2) holds if and only if $G(I)$ is lower perfect. Second, the condition (3) holds if and only if $G(I)$ is upper perfect and $sm(S)_d \setminus G(I)$ is lower perfect. If the above two results hold true, then it is easy to see that the conclusion holds by Proposition 1.2 and Proposition 1.3.

For the first claim, if $G(I)$ is lower perfect, then for each minimal non-face $F$ of $\delta_F(I)$, $|F| \geq d$ holds. By the definition of the Alexander dual, $G$ is a face of $\delta_F(I)^\vee$ if and only if $[n] \setminus G$ is a non-face of $\delta_F(I)$. So, for each facet $L$ of $\delta_F(I)^\vee$, $|L| \leq n - d$. Since $|G(I)| \neq C_n^d$, there exists some non-face of $\delta_F(I)$ with cardinality $d$, so there exists some facet of $\delta_F(I)^\vee$ with cardinality $n - d$. Thus $\dim(\delta_F(I)^\vee) = n - d - 1$.

Conversely, assume $\dim(\delta_F(I)^\vee) = n - d - 1$. By a similar argument, one can see that the smallest cardinality of non-faces of $\delta_F(I)$ is $d$, hence $G(I)$ is lower perfect.

For the second claim, if $sm(S)_d \setminus G(I)$ is lower perfect, then for the complex $\Delta = \langle \sigma(u) \mid u \in sm(S)_d \setminus G(I) \rangle$, the cardinality of a non-face is not less than $d$. Since $G(I)$ is upper perfect, for each non-face $F$ of $\Delta$, there exists $v \in G(I)$ such that $\sigma(v) \subseteq F$. Note that $\sigma(v)$ is a non-face of $\Delta$, so all the minimal non-faces of $\Delta$ have cardinality $d$. Hence $\Delta$ is a $d$-flag complex.

Conversely, assume that $\Delta = \langle \sigma(u) \mid u \in sm(S)_d \setminus G(I) \rangle$ is a $d$-flag complex. In a similar way, one can see that $G(I)$ is upper perfect and $sm(S)_d \setminus G(I)$ is lower perfect. □
3 Existence of $(n, d)^{th}$ f-ideals

For a subset $M$ of $sm(S)_d$, denote $M' = \{ \sigma^{-1}(A) \mid A = [n] \setminus \sigma(u) \text{ for some } u \in M \}$. The following lemma is essential in the proof of our main result in this section.

**Lemma 3.1.** $M$ is a perfect subset of $sm(S)_d$ if and only if $M'$ is a perfect subset of $sm(S)_{n-d}$.

**Proof.** For the necessary part, if $M$ is a subset of $sm(S)_d$, then it follows from definition that $M'$ is a subset of $sm(S)_{n-d}$. In order to check that $M'$ is upper perfect, we will show for each monomial $u \in sm(S)_{n-d+1}$ that $u \in \cup(M')$ holds. This is equivalent to showing that there exists some $v \in M'$, such that $\sigma(v) \subseteq \sigma(u)$ holds. In fact, since $M$ is lower perfect, for the monomial $u' = \sigma^{-1}([n] \setminus \sigma(u)) \in sm(S)_{d-1}$, there exists some $w \in M$ such that $u' | w$ holds. Hence $\sigma(u') \subseteq \sigma(w)$ holds. Now let $v = \sigma^{-1}([n] \setminus \sigma(w))$ and then it is easy to see that $\sigma(v) = [n] \setminus \sigma(w) \subseteq [n] \setminus \sigma(u') = \sigma(u)$ hold. This shows that $M'$ is upper perfect. In a similar way, one can prove that $M'$ is lower perfect.

The sufficient part is similar to prove, and we omit the details. □

By the proof of the above lemma, one can see that $M$ is an upper (lower, respectively) perfect subset of $sm(S)_d$ if and only if $M'$ is a lower (upper, respectively) perfect subset of $sm(S)_{n-d}$.

**Corollary 3.2.** If $I$ is a square-free monomial ideal of $S$ of degree $d$, then $I$ is an $(n, d)^{th}$ f-ideal if and only if $|G(I)| = C_n^d/2$ and $G(I)'$ is a perfect subset of $sm(S)_{n-d}$.

Denote $sm(S\{\hat{k}\})_d = \{ u \in sm(S)_d \mid x_k \upharpoonright u \}$, and $sm(S\{k\})_d = \{ u \in sm(S)_d \mid x_k | u \}$. For a subset $X = \{i_1, \ldots, i_j\}$ of $[n]$, denote

$$sm(S\{X\})_d = \{ u \in sm(S)_d \mid x_k \upharpoonright u \text{ for every } k \in X \},$$

and let $sm(S\{X\})_d = \{ u \in sm(S)_d \mid x_k \uparrow u \text{ for every } k \in X \}$.

**Definition 3.3.** For a subset $M$ of $sm(S\{\hat{k}\})_d$, if $sm(S\{\hat{k}\})_{d+1} \subseteq \uplus(M)$ holds, then $M$ is called *upper perfect without $k$*. Dually, a subset $M$ of $sm(S\{\hat{k}\})_d$ is called *lower perfect without $k$*, if $sm(S\{\hat{k}\})_{d-1} \subseteq \cap(M)$ holds. A subset $M$ of $sm(S\{k\})_d$ is called *upper perfect containing $k$*, if $sm(S\{k\})_{d+1} \subseteq \uplus(M)$ holds; a subset $M$ of $sm(S\{k\})_d$ is called *lower perfect containing $k$*, if $sm(S\{k\})_{d-1} \subseteq \cap(M)$ holds. If $M$ is not only upper but also lower perfect without $k$, then $M$ is called *perfect without $k*$. Similarly, if $M$ is both upper and lower perfect containing $k$, then $M$ is called *perfect containing $k*.

For a subset $X$ of $[n]$, we can define the upper perfect (lower perfect, perfect, respectively) set without $X$ (containing $X$) similarly. For a subset $A$ of $sm(S)_d$, let $A\{\hat{X}\} = A \cap sm(S\{\hat{X}\})_d$, and let $A\{X\} = A \cap sm(S\{X\})_d$. 


Proposition 3.4. Let $A$ be a subset of $sm(S)_d$, and let $X = \{i_1, \ldots, i_j\}$ be a subset of $[n]$. Then the following statements hold:

1. $A\{\bar{X}\} = A\{\bar{i}_1\}\{\bar{i}_2\} \ldots \{\bar{i}_j\},$ and $A\{X\} = A\{i_1\}\{i_2\} \ldots \{i_j\},$
2. If $A$ is upper perfect, then $A\{\bar{X}\}$ is upper perfect without $X$;
3. If $A$ is lower perfect, then $A\{X\}$ is lower perfect containing $X$;
4. If $A$ is upper (lower, respectively) perfect without $X$, then $A'$ is lower (upper, respectively) perfect containing $X$. Furthermore, the converse also holds true.

Proof. (1) and (2) are easy to see by the corresponding definitions.

In order to prove (3), it is sufficient to show that $A\{k\}$ is a lower perfect set containing $k$ for each $k \in [n]$. In fact, since $A$ is lower perfect, for each monomial $u \in sm(S\{k\})_{d-1}$, there exists a monomial $v$ in $A$ such that $u \mid v$. Note that $x_k \mid u$ holds, so $x_k \mid v$ also holds, which implies that $v \in sm(S\{k\})_d$ holds. Hence $A\{k\}$ is a lower perfect set containing $k$.

For (4), we only show that $A'$ is lower perfect containing $k$ when $A$ is upper perfect without $k$, and the remaining implications are similar to prove. In fact, for each monomial $u \in sm(S\{k\})_{n-d-1} \subseteq sm(S)_{n-d-1}$, $u' \in sm(S)_{d+1}$, note that $x_k \mid u$ implies $x_k \mid u'$ holds true, hence $u' \in sm(S\{\bar{k}\})_{d+1}$ also hold. Since $A$ is upper perfect without $k$, there exists a monomial $v \in A$ such that $v \mid u'$ holds, hence $u \mid v'$ holds, where $v' \in A'$. This completes the proof. \square

Remark 3.5. For a perfect subset $A$ of $sm(S)_d$, $A\{\bar{X}\}$ needs not to be a lower perfect set without $X$, and $A\{X\}$ needs not to be an upper perfect set containing $X$, see the following for counter-examples:

Example 3.6. Let $S = K[x_1, \ldots, x_6]$, let

$A = \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_3x_4x_5, x_1x_2x_6, x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\},$

and let $B = A \setminus \{x_1x_2x_6\}$. It is easy to see

$A\{6\} = B\{\bar{6}\} = \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_3x_4x_5\},$

$A\{6\} = \{x_1x_2x_6, x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\},$ and $B\{\bar{6}\} = \{x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\}$. Also, it is direct to check that both $A$ and $B$ are perfect sets, and that both $A\{6\}$ and $B\{\bar{6}\}$ are perfect sets without 6. Note that $A\{6\}$ is a perfect set containing 6, but $B\{\bar{6}\}$ is not upper perfect.

By Proposition 3.4, we have the following example by mapping $A$, $B$ to $A'$, $B'$ respectively.

Example 3.7. Let $S = K[x_1, \ldots, x_6]$, and let

$A' = \{x_1x_2x_3, x_1x_4x_5, x_2x_4x_5, x_3x_4x_5, x_1x_2x_6, x_3x_4x_6, x_3x_5x_6, x_4x_5x_6\},$
and \( B' = A' \setminus \{x_3x_4x_5\} \). It is easy to see that

\[
A'\{6\} = \{x_1x_2x_3, x_1x_4x_5, x_2x_4x_5, x_3x_4x_5\}, \quad B'\{6\} = \{x_1x_2x_3, x_1x_4x_5, x_2x_4x_5\},
\]

and \( A'\{6\} = B'\{6\} = \{x_1x_2x_6, x_3x_4x_6, x_3x_5x_6, x_4x_5x_6\} \). It is direct to check that both \( A' \) and \( B' \) are perfect sets, and that both \( A'\{6\} \) and \( A'\{6\} \) are perfect sets containing 6. Note that \( A'\{6\} \) is a perfect set without 6, but \( B'\{6\} \) is not lower perfect.

In order to obtain the main result of this section, we need a further fact and we omit the verification.

**Lemma 3.8.** Let \( S = K[x_1, \ldots, x_n] \), and let \( A \) be a subset of \( \text{sm}(S)_d \). If \( A\{k\} \) is a perfect subset of \( \text{sm}(S\{k\})_d \) without \( k \), and \( A\{k\} \) is a perfect subset of \( \text{sm}(S\{k\})_d \) containing \( k \) for some \( k \in [n] \), then \( A \) is a perfect subset of \( \text{sm}(S)_d \).

**Theorem 3.9.** For any integer \( d \geq 2 \) and any integer \( n \geq d + 2 \), there exists an \((n, d)\)th perfect set with cardinality less than or equal to \( C^d_n / 2 \).

**Proof.** We prove the result by induction on \( d \).

If \( d = 2 \), the conclusion holds true for any integer \( n \geq 4 \) by Proposition 1.2. In the following, assume \( d > 2 \).

Assume that the conclusion holds true for any integer less than \( d \). For \( d \), we claim that the conclusion holds true for any integer \( n \geq d + 2 \). We will show the result by induction on \( n \).

If \( n = d + 2 \), then \( C^d_n = C^2_n \). Note that for any integer \( n \geq 4 \), there exists an \((n, 2)\)th perfect set \( M \), such that \( |M| \leq C^2_n / 2 \). By Lemma 3.1, \( M' \) is an \((n, d)\)th perfect set. Note that \( |M'| = |M| \leq C^2_n / 2 = C^d_n / 2 \) holds.

Now assume that the conclusion holds true for any integer less than \( n \). Then by Lemma 3.8, it will suffice to show that there is a perfect subset \( A \) of \( \text{sm}(S\{\hat{n}\})_d \) without \( n \) and a perfect subset \( B \) of \( \text{sm}(S\{n\})_d \) containing \( n \), such that \( |A| \leq |\text{sm}(S\{n\})_d| / 2 = C^d_{n-1} / 2 \) and \( |B| \leq |\text{sm}(S\{n\})_d| / 2 = C^d_{n-1} / 2 \) hold.

Let \( L = K[x_1, \ldots, x_{n-1}] \). Then clearly, \( \text{sm}(S\{\hat{n}\})_d = \text{sm}(L)_d \) holds. By induction on \( n \), there exists an \((n-1, d-1)\)th perfect subset \( A \) of \( \text{sm}(L)_d \), such that \( |A| \leq C^d_{n-1} / 2 \). It is easy to see that \( A \) is a perfect subset of \( \text{sm}(S\{\hat{n}\})_d \) without \( n \). By induction on \( d \), there exists an \((n-1, d-1)\)th perfect subset \( B_1 \) of \( \text{sm}(L)_{d-1} \), such that \( |B_1| \leq C^d_{n-1} / 2 \) holds. Let \( B = \{\sigma^{-1}(D) \mid D \in \sigma(u) \cup \{n\} \text{ for some } u \in B_1 \} \). It is easy to see that \( B \) is a perfect subset of \( \text{sm}(S\{n\})_d \) containing \( n \), and \( |B| = |B_1| \leq C^d_{n-1} / 2 \).

Finally, by Lemma 3.8, \( A \cup B \) is a perfect subset of \( \text{sm}(S)_d \), and \( |A \cup B| = |A| + |B| \leq C^d_{n-1} / 2 + C^d_{n-1} / 2 = C^d_n / 2 \). This completes the proof. \( \square \)

By Proposition 1.1 and Theorem 3.9, the following corollary is clear.

**Corollary 3.10.** For any integer \( d \geq 2 \) and any integer \( n \geq d + 2 \), \( V(n, d) \neq \emptyset \) if and only if \( 2 \mid C^d_n \).
4 Algorithms for constructing examples of \((n, d)^{th}\) \(f\)-ideals

In this section, we will show some algorithms to construct \((n, d)^{th}\) \(f\)-ideals. We discuss the following cases:

**Case 1**: \(d = 2\). An \((n, 2)^{th}\) \(f\)-ideal is easy to construct by [7]. For readers convenience, we repeat it as the following: Decompose the set \([n]\) into a disjoint union of two subsets \(B\) and \(C\) uniformly, namely, \(|B| - |C| \leq 1\). Then set \(A = \{x_i x_j \mid i, j \in B, \text{ or } i, j \in C\}\) to obtain an \((n, 2)^{th}\) perfect set. Note that \(|A| = N_{(n, 2)} \leq C_n^2 / 2\), choose a subset \(D\) of \(sm(S)_2 \setminus A\) randomly, such that \(|D| = C_n^2 / 2 - N_{(n, 2)}\) holds. It is easy to see that \(A \cup D\) is still a perfect set, and \(|A \cup D| = C_n^2 / 2\). By Proposition [11] the ideal generated by \(A \cup D\) is an \((n, 2)^{th}\) \(f\)-ideal. Note that each \((n, 2)^{th}\) \(f\)-ideal can be obtained in this way except \(C_5\) by [7].

**Case 2**: \(d > 2\) and \(n = d + 2\).

**Algorithm 4.1.** In order to build an \(f\)-ideal \(I \in V(d+2, d)\), we obey the following steps:

Step 1: Calculate \(C_{d+2}^d / 2\). Note that \(C_{d+2}^d / 2 = C_{d+2}^d / 2\).

Step 2: As in the case 1, find a perfect subset \(B\) of \(sm(S)_2\) such that \(|B| \leq C_{d+2}^d / 2\), where \(S = K[x_1, \ldots, x_{d+2}]\).

Step 3: Let \(A = B'\). Then \(A\) is a perfect subset of \(sm(S)_d\) by Lemma [3.1] and \(|A| = |B| \leq C_{d+2}^d / 2 = C_{d+2}^d / 2\).

Step 4: Choose a subset \(D\) of \(sm(S)_d \setminus A\) randomly, such that \(|D| = C_{d+2}^d / 2 - |A|\) holds. It is easy to see that \(M = A \cup D\) is still a perfect set, and \(|A \cup D| = C_{d+2}^d / 2\).

Step 5: Let \(I\) be the ideal generated by \(A \cup D\). By Proposition [12] again, \(I\) is an \((d + 2, d)^{th}\) \(f\)-ideal.

Note that in this way, we constructed almost all \((d + 2, d)^{th}\) \(f\)-ideals.

**Example 4.2.** Show an \(f\)-ideal \(I \in V(8, 6)\).

Note that \(8 = 6 + 2\), we obey the Algorithm [4.1]

Note that \(C_8^6 / 2 = 14\). Find a perfect subset \(B\) of \(sm(S)_2\) such that \(|B| \leq C_8^6 / 2 = 14\), where \(S = K[x_1, \ldots, x_8]\). It is easy to see that

\[
B = \{x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, x_3 x_4, x_5 x_6, x_5 x_7, x_5 x_8, x_6 x_7, x_6 x_8, x_7 x_8\}
\]

is a perfect subset of \(sm(S)_2\), with \(|B| = 12\). Let

\[
A = B' = \{x_3 x_4 x_5 x_6 x_7 x_8, x_2 x_4 x_5 x_6 x_7 x_8, x_2 x_3 x_5 x_6 x_7 x_8, x_1 x_4 x_5 x_6 x_7 x_8, \\
x_1 x_3 x_5 x_6 x_7 x_8, x_1 x_2 x_5 x_6 x_7 x_8, x_1 x_2 x_3 x_4 x_7 x_8, x_1 x_2 x_3 x_4 x_6 x_8, \\
x_1 x_2 x_3 x_4 x_6 x_7, x_1 x_2 x_3 x_4 x_5 x_8, x_1 x_2 x_3 x_4 x_5 x_7, x_1 x_2 x_3 x_4 x_5 x_6\}
\]

is a perfect subset of \(sm(S)_6\). Choose \(D = \{x_1 x_2 x_3 x_5 x_6 x_7, x_1 x_2 x_4 x_5 x_6 x_8\}\), then the ideal \(I\) generated by \(A \cup D\) is an \((8, 6)^{th}\) \(f\)-ideal.
Case 3: \( d > 2 \) and \( n > d + 2 \). Let \( S^{[k]} = K[x_1, \ldots, x_k] \), and let \( S = S^{[n]} = K[x_1, \ldots, x_n] \).

**Algorithm 4.3.** For an integer \( n > d + 2 \), we construct an \((n, d)\)\(th\) \(f\)\(-ideal\) by using the following steps:

Step 1: Let \( t = n, l = d \) and \( E = \emptyset \). Set \( \mathcal{B} = \{B(t, l, E)\} \).

Step 2: Assign \( \mathcal{C} = \mathcal{B} \), and denote \( i = |\mathcal{C}| \).

Step 3: Choose each \( B(t, l, E) \in \mathcal{C} \) one by one, deal with each one obeying the following rules:

If \( l = 2 \) or \( t = l + 2 \), don’t change anything.

If \( l \neq 2 \) and \( t > l + 2 \), then cancel \( B(t, l, E) \) from \( \mathcal{B} \), and add \( B(t - 1, l, E \cup \{t\}) \) into \( \mathcal{B} \).

After \( i \) times, i.e., when \( B(t, l, E) \) goes through all the element of \( \mathcal{C} \), make a judgement:

If \( l = 2 \) or \( t = l + 2 \) for each \( B(t, l, E) \in \mathcal{B} \), then go to step 4, else return to step 2.

Step 4: Choose \( B(t, l, E) \in \mathcal{B} \) one by one, deal with each one obeying the following rules:

If \( l = 2 \), assign \( B(t, l, E) \) a perfect subset of \( sm(S^{[l]}_d) \) as case 1.

If \( l \neq 2 \) and \( t = l + 2 \), assign \( B(t, l, E) \) a perfect subset of \( sm(S^{[l]}_d) \) as case 2.

Step 5: For each \( B(t, l, E) \in \mathcal{B} \), denote \( \mathcal{B}^{*}(t, l, E) = \{ux \in B(t, l, E)\} \), where \( x_E = \prod_{j \in E} x_j \). Denote \( \mathcal{B}^{*} = \cup_{B(t, l, E) \in \mathcal{B}} \mathcal{B}^{*}(t, l, E) \). It is direct to check that \( \mathcal{B}^{*} \) is a perfect subset of \( sm(S)_d \), and \( |\mathcal{B}^{*}| \leq C^d_n / 2 \). Choose a subset \( D \) of \( sm(S)^{c}_d \ \mathcal{B}^{*} \) randomly, such that \( |D| = C^d_n / 2 - |\mathcal{B}^{*}| \) holds.

Step 6: Let \( I \) be the ideal generated by \( \mathcal{B}^{*} \cup D \). By Proposition 1.1 again, \( I \) is an \((n, d)\)\(th\) \(f\)\(-ideal\).

**Example 4.4.** Show a \((6, 3)\)\(th\) \(f\)\(-ideal\).

Let \( S = K[x_1, \ldots, x_6] \). By the above algorithm, we will choose a perfect subset \( B(5, 3, \emptyset) \) of \( sm(S^3)_3 \) and a perfect subset \( B(5, 2, \{6\}) \) of \( sm(S^5)_2 \). Set \( B(5, 3, \emptyset) = \{x_3x_4x_5, x_2x_4x_5, x_1x_4x_5, x_1x_2x_3\} \), and set \( B(5, 2, \{6\}) = \{x_1x_2, x_1x_3, x_2x_3, x_4x_5\} \). Correspondingly, \( \mathcal{B}^{*}(5, 3, \emptyset) = B(5, 3, \emptyset) \) and

\[
\mathcal{B}^{*}(5, 2, \{6\}) = \{x_1x_2x_6, x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\}.
\]

Hence

\[
\mathcal{B}^{*} = \{x_3x_4x_5, x_2x_4x_5, x_1x_4x_5, x_1x_2x_3, x_1x_2x_6, x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\}
\]

is a perfect subset of \( sm(S)_3 \). Note that \( C^3_6/2 = 10 \), and \( |\mathcal{B}^{*}| = 8 \). Set \( D = \{x_1x_2x_4, x_1x_2x_5\} \). The ideal \( I \) generated by \( \mathcal{B}^{*} \cup D \) is a \((6, 3)\)\(th\) \(f\)\(-ideal\).

Note that the \((6, 3)\)\(th\) \(f\)\(-ideal\) given in the above example is not unmixed. In fact, consider the simplicial complex \( \sigma(sm(S)_3 \setminus G(I)) \), and note that \( \{1, 2\} \) is a non-face of \( \sigma(sm(S)_3 \setminus G(I)) \), which implies that \( \sigma(sm(S)_3 \setminus G(I)) \) is not a 3-flag complex. So, \( I \) is not unmixed by Proposition 2.1.
5 An upper bound of the perfect number $N_{(n,d)}$

For a positive integer $k$ and a pair of positive integers $i \leq j$, denote by $Q_{[i,j]}^k$ the set of square-free monomials of degree $k$ in the polynomial ring $K[x_i, x_{i+1}, \ldots, x_j]$. Note that $Q_{[i,j]}^k = \emptyset$ holds for $i > j$. For a pair of monomial subsets $A$ and $B$, denote by $A \bullet B = \{uv | u \in A, v \in B\}$. If $B = \emptyset$, then assume $A \bullet B = A$. The following theorem gives an upper bound of the $(n,d)^{th}$ perfect number for $n > d + 2$.

**Theorem 5.1.** Given a integer $d > 2$, and a integer $n \geq d + 2$. The following statements about the perfect number $N_{(n,d)}$ hold:

1. If $n = d + 2$, then
   \[ N_{(n,d)} = N_{(n,2)} = \begin{cases} k^2 - k, & \text{if } n = 2k; \\ k^2, & \text{if } n = 2k + 1. \end{cases} \tag{2} \]

2. If $n > d + 2$, then
   \[ N_{(n,d)} \leq \sum_{i=5}^{n-d+2} N_{(i,2)} C_{n-i-1}^{d-3} + \sum_{j=3}^{d} N_{(j+2,2)} C_{n-j-3}^{d-j}, \tag{3} \]

where $C_0^0 = 1$.

**Proof.** By Lemma 3.1 and the equation 1 in the first section, (1) is clear.

In order to prove (2), it will suffice to show that there exists a perfect set with cardinality $t = \sum_{i=5}^{n-d+2} N_{(i,2)} C_{n-i-1}^{d-3} + \sum_{j=3}^{d} N_{(j+2,2)} C_{n-j-3}^{d-j}$.

Let $P_{(i,2)}$ be an $(i,2)^{th}$ perfect set with cardinality $N_{(i,2)}$ for $5 \leq i \leq n - d + 2$, and let $P_{(j+2,j)}$ be a $(j+2, j)^{th}$ perfect set with cardinality $N_{(j+2,2)}$ for $3 \leq j \leq d$. We claim that the set
\[ M = (\cup_{i=5}^{n-d+2} P_{(i,2)} \bullet x_{i+1} \bullet Q_{[i+2,n]}^{d-3}) \cup (\cup_{j=3}^{d} P_{(j+2,2)} \bullet Q_{[j+4,n]}^{d-j}) \]
is an $(n,d)^{th}$ perfect set, with cardinality $t$. It is easy to check that the cardinality of $M$ is $t$. It is only necessary to prove that $M$ is perfect.

For each $w \in sm(S)_{d+1}$, denote by $n_k(w)$ the cardinality of the set $\{x_i | i \leq k \text{ and } x_i \mid w\}$. If $n_5(w) \geq 4$, then choose the smallest $k$ such that $n_{k+3}(w) = n_{k+2}(w) = k + 1$. Clearly, $3 \leq k \leq d$. It is direct to check that $w$ is divided by some monomial in $P_{(k+2,k)} \bullet Q_{[k+4,n]}^{d-k}$. If $n_5(w) \leq 3$, then choose the smallest $k$ such that $n_k(w) = 3$ and $n_{k+1}(w) = 4$. Clearly, $5 \leq k \leq n - d + 2$. It is not hard to check that $w$ is divided by some monomial in $P_{(k,2)} \bullet x_{k+1} \bullet Q_{[k+2,n]}^{d-3}$. Hence $M$ is upper perfect.

For each $w \in sm(S)_{d-1}$, if $n_5(w) \geq 2$, then choose the smallest $k$ such that $n_{k+3}(w) = n_{k+2}(w) = k - 1$. Clearly, $3 \leq k \leq d$. It is direct to check that $w$ is covered by some monomial in $P_{(k+2,k)} \bullet Q_{[k+4,n]}^{d-k}$. If $n_5(w) \leq 1$, then choose the smallest $k$ such that $n_k(w) = 1$ and $n_{k+1}(w) = 2$. Clearly, $5 \leq k \leq n - d + 2$ holds. It is not hard to check.
that \( w \) is covered by some monomial in \( P_{(k,2)} \cdot x_{k+1} \cdot Q_{[k+2,n]}^{d-3} \). Hence \( M \) is lower perfect.

\[ \square \]

The following figure may help to interpret the above theorem intuitively.

Figure 1. Upper Bound

In this figure, there is a boundary consisting of the line \( l = 2 \) and the line \( t = l + 2 \). From the point \( (d, n) \) to a point of the boundary, every directed chain \( C \) denotes a set of monomials \( M(C) \) by the following rules:

1. Every arrow of \( C \) is from \( (l, t) \) to either \( (l, t - 1) \) or \( (l - 1, t - 1) \).
2. If the arrow is from \( (l, t) \) to \( (l, t - 1) \), then each monomial in \( M(C) \) is not divided by \( x_t \). Correspondingly, if it is from \( (l, t) \) to \( (l - 1, t - 1) \), then each monomial in \( M(C) \) is divided by \( x_t \).
3. Each point \( (l, t) \) of the boundary is a \( (t, l) \)-th perfect set.

Actually, the figure shows us a class of \( (n, d) \)-th perfect sets. For each point \( (l, t) \) of the boundary, if we choose the corresponding perfect set to be a \( (t, l) \)-th perfect set with cardinality \( N_{(t,l)} \), then the cardinality of the \( (n, d) \)-th perfect set is exactly

\[
N_{t+2} N_{j+2} \left( C_{n-j-3} \right) + \sum_{j=3}^{d} N_{j+2} \left( C_{n-j-3} \right) \]

**Example 5.2. Calculation of the \( (6, 3) \)-th perfect number.**

Let \( A \) be a \( (6, 3) \)-th perfect set. By Proposition 3.3(2), \( A\{6\} \) is an upper perfect set without 6. Hence \(|A\{6\}| \geq N_{(5,3)} = 4\). By Proposition 3.3(3), \( A\{6\} \) is a lower perfect set containing 6. Note that for the monomials of \( \{x_1, x_2, x_3, x_4, x_5\} \), each monomial in \( A\{6\} \)
covers at most two of them. So, $|A\{6\}| \geq 3$. Hence $|A| \geq |A\{6\}| + |A\{6\}| \geq 7$. Actually, as showed in Example 3.6 there exists a $(6,3)^{th}$ perfect set

$$B = \{x_1x_2x_3, x_1x_2x_4, x_1x_2x_5, x_3x_4x_5, x_1x_3x_6, x_2x_3x_6, x_4x_5x_6\}$$

with cardinality 7. Thus $N_{(6,3)} = 7$. Note that the upper bound given by Proposition 5.1(2) is 8, and is not bad for the perfect number in the case.

6 Nonhomogeneous $f$-ideal

In [7], a characterization of $f$-ideals in general case is shown, but it is still not easy to show an example of nonhomogeneous $f$-ideal. In fact, the interference from monomials of different degree makes the computation complicated. Anyway, we worked out the following examples:

**Example 6.1.** Let $S = K[x_1, x_2, x_3, x_4, x_5]$, and let

$$I = \langle x_1x_2, x_3x_4, x_1x_3x_5, x_2x_4x_5 \rangle.$$

It is direct to check that

$$\delta_F(I) = \langle \{1, 2\}, \{3, 4\}, \{1, 3, 5\}, \{2, 4, 5\} \rangle$$

and

$$\delta_N(I) = \langle \{1, 3\}, \{2, 4\}, \{1, 4, 5\}, \{2, 3, 5\} \rangle.$$

It is easy to see they have the same $f$-vector, and hence $I$ is an $f$-ideal, which is clearly nonhomogeneous.

In fact, there are a lot of nonhomogeneous $f$-ideals. We will show another example to end this section.

**Example 6.2.** Let $S = K[x_1, x_2, x_3, x_4, x_5, x_6]$, and let

$$I = \langle x_1x_2, x_2x_3, x_1x_3, x_4x_5, x_1x_4x_6, x_1x_5x_6, x_2x_4x_6 \rangle.$$

Note that

$$\delta_N(I) = \langle \{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5, 6\}, \{3, 4, 6\}, \{3, 5, 6\} \rangle.$$

It is direct to check that $I$ is also a nonhomogeneous $f$-ideal.
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