HOLONOMY AND
3-SASAKIAN HOMOGENEOUS MANIFOLDS
VERSUS SYMPLECTIC TRIPLE SYSTEMS

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Dedicated to the memory of Professor Thomas Friedrich

Abstract. Our aim is to support the choice of two remarkable connections with torsion in a 3-Sasakian manifold, proving that, in contrast to the Levi-Civita connection, the holonomy group in the homogeneous cases reduces to a proper subgroup of the special orthogonal group, of dimension considerably smaller. We realize the computations of the holonomies in a unified way, by using as a main algebraic tool a nonassociative structure, that of a symplectic triple system.

1. Introduction

3-Sasakian geometry is a natural generalization of Sasakian geometry introduced independently by Kuo and Udriste in 1970 [23], [25]. 3-Sasakian manifolds are very interesting objects. In fact, any 3-Sasakian manifold has three orthonormal Killing vector fields which span an integrable 3-dimensional distribution. Under some regularity conditions on the corresponding foliation, the space of leaves is a quaternionic Kähler manifold with positive scalar curvature. And conversely, over any quaternionic Kähler manifold with positive scalar curvature, there is a principal SO(3)-bundle that admits a 3-Sasakian structure [10]. The canonical example is the sphere $S^{4n+3}$ realized as a hypersurface in $H^{n+1}$. Besides, any 3-Sasakian manifold is an Einstein space of positive scalar curvature [22].

In spite of that, during the period from 1975 to 1990, approximately, 3-Sasakian manifolds lived in relative obscurity, probably due to the fact that, according to

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the authors of the monograph [10], Boyer and Galicki, the holonomy group of a
3-Sasakian manifold never reduces to a proper subgroup of the special orthogonal
group. There was the idea that manifolds should be divided into different classes
according to their holonomy group, being special geometries: those with holonomy
group not of general type.

The revival of 3-Sasakian manifolds occurred in the nineties. On one hand,
Boyer and Galicki began to study 3-Sasakian geometry because it appeared as a
natural object in their quotient construction for certain hyperkähler manifolds [12].
This construction helped to find new collections of examples, and, since then, the
topology and geometry of these manifolds have been continuously studied. On the
other hand, during the same period, the condition of admitting three Killing spinors
in a compact Riemannian manifold of dimension 7 was proved to be equivalent to
the existence of a 3-Sasakian structure [21]. This result leads to a relation between
the 3-Sasakian manifolds and the spectrum of the Dirac operator. Recall that Dirac
operators have become a powerful tool for the treatment of various problems in
geometry, analysis and theoretical physics [2]. Much of this work was developed
by the Berlin school around Th. Friedrich.

In [5] Friedrich himself, together with Agricola, began to study in a systematic
way the holonomy of connections with skew-symmetric torsion. Our work on 3-
Sasakian manifolds follows this approach. Recall that the holonomy group of
the Levi-Civita connection being of general type indicates that it is not well-
adapted to the 3-Sasakian geometry, because, when a connection is well-adapted
to a particular geometric structure, then there are parallel tensors so that the
holonomy reduces. Thus, in this work, we compute the holonomy groups of several
affine connections with skew-symmetric torsion reportedly better adapted to the
3-Sasakian geometry than the Levi-Civita connection, in order to support their
usage. This paper is a natural continuation of the work [18], which looks for
good affine metric connections on a 3-Sasakian manifold, with nonzero torsion.
The notion of torsion of a connection is due to Elie Cartan [14], who investigates
several examples of connections preserving geodesics (an equivalent condition to
the skew-symmetric torsion), explaining also how the connection should be adapted
to the geometry under consideration. Although there are several types of geometric
structures on Riemannian manifolds admitting a unique metric connection with
skew-symmetric torsion preserving the structure (see references in the complete
survey [2] on geometries with torsion), this is not the situation for a 3-Sasakian
manifold: the 3-Sasakian structure is not parallel for any metric connection with
skew-torsion. Hence, it is natural to wonder whether there is a best affine metric
connection on a 3-Sasakian manifold. This is one of the targets in [18], where some
affine connections have been suggested, like the distinguished connection \( \nabla^S \) in [18,
Thm. 5.6], or supported [18, Thm. 5.7], as the canonical connection \( \nabla^c \) defined in
[4]. Both have interesting geometrical properties in any 3-Sasakian manifold, such
as parallelizing the Reeb vector fields or the torsion, respectively.

Now we revisit these affine connections in the homogeneous cases, in which \( \nabla^S \)
is also the unique invariant metric connection with skew-torsion parallelizing the
Reeb vector fields (and \( \nabla^c \) is one of the few ones parallelizing the torsion). To be
precise, we compute their curvature operators \( R^S \) and \( R^c \) in terms of an interesting
algebraic structure which is hidden behind the 3-Sasakian homogeneous manifolds, namely, that of a symplectic triple system (Section 2). We follow the suggestions in [18, Rem. 4.11]. Surprisingly, the expressions of the curvature operators (after complexification) turn out to be very simple in terms of these symplectic triple systems. The precise computations of the holonomy algebras (in general a very difficult task) is not only feasible but also made in a unified way, independently of the considered homogeneous 3-Sasakian manifold. Our main result is essentially the following one:

For any simply-connected 3-Sasakian homogeneous manifold $M^{4n+3} = G/H$,

- If $\nabla^S$ is the distinguished connection, the holonomy group is isomorphic to $H \times \text{SU}(2)$.
- If $\nabla^c$ is the canonical connection, the holonomy group is isomorphic to $H \times \text{SU}(2)$ too (but embedded in $\text{SO}(4n+3)$ in a different way).

We provide an additional proof that if $\nabla^g$ is the Levi-Civita connection, the holonomy group is the whole group $\text{SO}(4n+3)$. This case is included for completeness, besides providing a nice expression for the curvature operators.

There are no general results about the holonomy group of connections with torsion, in spite of the results on some concrete examples in [5]. For instance, the holonomy algebra on $\mathbb{R}^n$ is semisimple, regardless of the considered metric connection with skew-torsion. A consequence of our results is that the semisimplicity is also a feature of the holonomy algebras attached to either $\nabla^S$ or $\nabla^c$ if $G \neq \text{SU}(m)$. In contrast, if $G = \text{SU}(m)$, both the holonomy algebras attached to either $\nabla^S$ or $\nabla^c$ have a one-dimensional center.

Another precedent on the computation of the holonomy algebras of invariant affine connections with torsion on homogeneous spaces is [9], which deals with the natural connection on the eight reductive homogeneous spaces of $G_2$, in particular on $G_2/\text{Sp}(1)$. It proves that its related holonomy algebra [9, Prop. 5.20] is the whole orthogonal algebra, but takes into consideration that the metric considered in [9] is homothetic to the Killing form, so that it is not the Einstein metric considered in the 3-Sasakian manifold $G_2/\text{Sp}(1)$ (compare with Eq. (3)). Also, the curvatures on some invariant connections with skew-torsion on the 3-Sasakian manifold $S^7 \cong \text{Sp}(2)/\text{Sp}(1)$ have been computed in [17, §5.2], including some Ricci-flat not flat affine connections different from $\nabla^S$ and some flat affine connections on $S^7$ (see also [6]).

Although our objectives in this paper are mainly to use algebraic structures to understand better some aspects of the geometry of the 3-Sasakian manifolds, we can also read our results in terms of the links between algebra and geometry. Symplectic triple systems began to be studied in the construction of 5-graded Lie algebras. So, Lie algebras gave birth to this kind of triples. This is not unusual; it happened also with Freudenthal triple systems and some ternary algebras in the seventies [26]. The common origin was to investigate algebraic characterizations of the metasymplectic geometry due to H. Freudenthal, from a point of view of ternary structures (see the works by Meyberg, Faulkner, Ferrar and Freudenthal himself). But, afterwards, these structures became relevant by their own merits, since they have been used to construct new simple Lie algebras in prime characteristic, new
Lie superalgebras, and so on. The philosophy is that, at the end, geometry has influenced the algebraical development of certain algebraic structures. And vice versa, many algebraic structures (not necessarily binary structures) could help in the study of Differential Geometry.

The paper is organized as follows. The structure of symplectic triple systems is recalled in Section 2, together with a collection of examples exhausting the classification of complex simple symplectic triple systems, and their standard enveloping Lie algebras. Section 3 recalls the notion of a 3-Sasakian manifold and the classification of the homogeneous ones, relating each 3-Sasakian homogeneous manifold with the corresponding symplectic triple system (Proposition 2). The distinguished and canonical affine connections are introduced and translated to an algebraical setting, and their geometrical properties are recalled, such as the invariance. Finally, Section 4 contains the above mentioned results on the holonomy algebras related to such affine connections. Some comparisons have been added on the related Ricci tensors and scalar curvatures, together with a table comparing the dimensions of the holonomy groups.

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2. Background on symplectic triple systems

This subsection is essentially extracted from [19].

Definition 1. Let $\mathbb{F}$ be a field and $T$ an $\mathbb{F}$-vector space endowed with a nonzero alternating bilinear form $(\cdot, \cdot): T \times T \to \mathbb{F}$, and a triple product $[\cdot, \cdot, \cdot]: T \times T \times T \to T$. It is said that $(T, [\cdot, \cdot, \cdot], (\cdot, \cdot))$ is a symplectic triple system if it satisfies

\begin{align}
[x, y, z] &= [y, x, z], \\
[x, y, z] - [x, z, y] &= (x, z)y - (x, y)z + 2(y, z)x, \\
[x, y, [u, v, w]] &= [[[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]], \\
((x, y, u), v) &= -([u, [x, y, v]]),
\end{align}

for any $x, y, z, u, v, w \in T$.

An ideal of the symplectic triple system $T$ is a subspace $I$ of $T$ such that $[T, T, I] + [T, I, T] \subset I$; and the system is said to be simple if $[T, T, T] \neq 0$ and it contains no proper ideal. If $\dim T \neq 1$ and the field is either $\mathbb{R}$ or $\mathbb{C}$, the simplicity of $T$ is equivalent to the nondegeneracy of the bilinear form [19, Prop. 2.4].

There is a close relationship between symplectic triple systems and certain $\mathbb{Z}_2$-graded Lie algebras [26]. We denote by

$$d_{x,y} := [x, y, \cdot] \in \text{End}_\mathbb{F}(T).$$

Observe that the above set of identities can be read in the following way. By (1d), $d_{x,y}$ belongs to the symplectic Lie algebra

$$\text{sp}(T, (\cdot, \cdot)) = \{ d \in \mathfrak{gl}(T) \mid (d(u), v) + (u, d(v)) = 0 \ \forall u, v \in T \},$$
which is a subalgebra of the general linear algebra $\mathfrak{gl}(T) = (\text{End}_T(T), [\cdot, \cdot])$; and by (1c), $d_{x,y}$ belongs to the Lie algebra of derivations of the triple too, i.e.,

$$\text{der}(T, [\cdot, \cdot, \cdot]) := \{ d \in \mathfrak{gl}(T) \mid d([u, v, w]) = [d(u), v, w] + [u, d(v), w] + [u, v, d(w)] \forall u, v, w \in T \},$$

which is also a Lie subalgebra of $\mathfrak{gl}(T)$. We call the set of inner derivations the linear span

$$\text{inder}(T) := \left\{ \sum_{i=1}^{n} d_{x_i,y_i} \mid x_i, y_i \in T, n \geq 1 \right\},$$

which is a Lie subalgebra of $\text{der}(T, [\cdot, \cdot, \cdot])$, again taking into account (1c). Now, consider $(V, \langle \cdot, \cdot \rangle)$ a two-dimensional vector space endowed with a nonzero alternating bilinear form, and the vector space

$$\mathfrak{g}(T) := \mathfrak{sp}(V, \langle \cdot, \cdot \rangle) \oplus \text{inder}(T) \oplus (V \otimes T).$$

Then $\mathfrak{g}(T)$ is endowed with a $\mathbb{Z}_2$-graded Lie algebra structure ([19, Thm. 2.9]) such that the even part is $\mathfrak{g}(T)_0 := \mathfrak{sp}(V, \langle \cdot, \cdot \rangle) \oplus \text{inder}(T)$, a direct sum of two ideals, and the odd part is $\mathfrak{g}(T)_1 := V \otimes T$. The product is given by the usual bracket on $\mathfrak{g}(T)_0$, the natural action of $\mathfrak{g}(T)_0$ on $\mathfrak{g}(T)_1$, and the product of two odd elements as follows

$$(a \otimes x, b \otimes y) = (x, y) \gamma_{a,b} + \langle a, b \rangle d_{x,y} \in \mathfrak{g}(T)_0,$$

if $a, b \in V$ and $x, y \in T$, where $\gamma_{a,b} \in \mathfrak{sp}(V, \langle \cdot, \cdot \rangle)$ is defined by

$$\gamma_{a,b} := \langle a, \cdot \rangle b + \langle b, \cdot \rangle a.$$

The Lie algebra $\mathfrak{g}(T)$ is called the standard enveloping algebra related to the symplectic triple system $T$. Moreover, $\mathfrak{g}(T)$ is simple if and only if so is $(T, [\cdot, \cdot, \cdot], \langle \cdot, \cdot \rangle)$ ([19, Thm. 2.9]).

Let us recall which $\mathbb{Z}_2$-graded Lie algebras are the standard enveloping algebras related to some symplectic triple system. As above, take $(V, \langle \cdot, \cdot \rangle)$ a two-dimensional vector space endowed with a nonzero alternating bilinear form. If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $\mathbb{Z}_2$-graded Lie algebra such that there is $\mathfrak{s}$ an ideal of $\mathfrak{g}_0$ with $\mathfrak{g}_0 = \mathfrak{sp}(V, \langle \cdot, \cdot \rangle) \oplus \mathfrak{s}$ and there is $T$ an $\mathfrak{s}$-module such that $\mathfrak{g}_1 = V \otimes T$ as $\mathfrak{g}_0$-module, then most of the time $T$ can be endowed with a symplectic triple system structure. To be precise, the invariance of the Lie bracket in $\mathfrak{g}$ under the $\mathfrak{sp}(V, \langle \cdot, \cdot \rangle)$-action provides the existence of

- $(\cdot, \cdot) : T \times T \to \mathbb{F}$, alternating,
- $d : T \times T \to \mathfrak{s}$, symmetric,

such that Eq. (1) holds for any $x, y \in T$, $a, b \in V$. Now, if $(\cdot, \cdot) \neq 0$ and we consider the triple product on $T$ defined by $[x, y, z] := d_{x,y}(z) \equiv d(x, y) \cdot z \in \mathfrak{s} \cdot T \subset T$, then $(T, [\cdot, \cdot, \cdot], \langle \cdot, \cdot \rangle)$ is proved to be a symplectic triple system.
Remark 1. Another algebraic structure involved in our study, better known than the one of a symplectic triple system, is that of a Lie triple system, which can be identified with the tangent space to a symmetric space. Its relevance now is because \((V \otimes T, [\cdot, \cdot, \cdot])\) is a Lie triple system for the triple product given by
\[
[a \otimes x, b \otimes y, c \otimes z] = \gamma_{a,b}(c) \otimes (x,y)z + \langle a, b \rangle c \otimes [x,y,z].
\]
Thus some interesting properties hold, as
\[
\sum_{\text{cyclic}}_{a,b,c} [a \otimes x, b \otimes y, c \otimes z] = 0.
\]

Now we describe some important examples of simple symplectic triple systems, extracted from [20].

Example 1. Let \(W\) be a vector space over \(\mathbb{F}\) endowed with a nondegenerate alternating bilinear form \((\cdot, \cdot)\). Then \(T = W\) is a symplectic triple system with the triple product given by
\[
[x,y,z] := (x,z)y + (y,z)x,
\]
if \(x,y,z \in W\). It is easy to check that the algebra of inner derivations \(\text{ind}_{\text{tr}}(W)\) is isomorphic to \(\mathfrak{sp}(W)\) and the standard enveloping algebra \(\mathfrak{g}(W)\) is isomorphic to the symplectic Lie algebra \(\mathfrak{sp}(V \oplus W)\), where the alternating bilinear form on \(V \oplus W\) is defined by
\[
(v_1 + w_1, v_2 + w_2) = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle.
\]
This symplectic triple system \(W\) is called of symplectic type.

Example 2. Let \(W\) be a vector space over \(\mathbb{F}\) endowed with a nondegenerate symmetric bilinear form \(b: W \times W \to \mathbb{F}\). Then \(T = V \otimes W\) is a symplectic triple system with
\[
(u \otimes x, v \otimes y) := \frac{1}{2} \langle u, v \rangle b(x,y),
\]
\[
[u \otimes x, v \otimes y, w \otimes z] := \frac{1}{2} \gamma_{u,v}(w) \otimes b(x,y)z + \langle u, v \rangle w \otimes (b(x,z)y - b(y,z)x),
\]
for any \(u, v, w \in V\) and \(x, y, z \in W\). Besides, the algebra of inner derivations \(\text{ind}_{\text{tr}}(T)\) is isomorphic to \(\mathfrak{sp}(V) \oplus \mathfrak{so}(W, b)\) and the standard enveloping algebra \(\mathfrak{g}(T)\) is isomorphic to the orthogonal algebra \(\mathfrak{so}((V \otimes V) \oplus W, B)\), where the considered symmetric bilinear form \(B\) is given by
\[
B(v_1 \otimes v_2 + w_1, v'_1 \otimes v'_2 + w_2) = \langle v_1, v'_1 \rangle \langle v_2, v'_2 \rangle + b(w_1, w_2).
\]
The symplectic triple system \(T = V \otimes W\) is called of orthogonal type.
Example 3. Let $W$ be any vector space over $\mathbb{F}$ and $W^*$ its dual vector space. Then $T = W \oplus W^*$ is a symplectic triple system for the alternating map defined by $(x, y) := 0 =: (f, g), (f, x) := f(x)$ if $x, y \in W$ and $f, g \in W^*$; and triple product given by

$$[x, y, T] := 0 =: [f, g, T],$$

$$[x, f, y] := f(x)y + 2f(y)x,$$

$$[x, f, g] := -f(x)g - 2g(x)f.$$

Besides, the algebra of inner derivations $\text{indet}(T)$ is isomorphic to the general linear algebra $\mathfrak{gl}(W)$ and the standard enveloping algebra $\mathfrak{g}(T)$ is isomorphic to the special linear algebra $\mathfrak{sl}(V \oplus W)$. The symplectic triple system $T = W \oplus W^*$ is called of special type.

Example 4. To simplify, assume $\mathbb{F} = \mathbb{C}$ and take either $J = \mathbb{C}$ or $J = H_3(C) = \{ x = (x_{ij}) \in \text{Mat}_{3 \times 3}(C^\mathbb{C}) : x_{ji} = \overline{x_{ij}} \}$ with $C \in \{ \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \}$. Then the vector space

$$T_J = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}, a, b \in J \right\}$$

becomes a symplectic triple system with the alternating map and triple product given by

$$(x_1, x_2) : = \alpha_1\beta_2 - \alpha_2\beta_1 - t(a_1, b_2) + t(b_1, a_2),$$

$$[x_1, x_2, x_3] : = \begin{pmatrix} \gamma(x_1, x_2, x_3) & c(x_1, x_2, x_3) \\ -c(x_1^t, x_2^t, x_3^t) & -\gamma(x_1^t, x_2^t, x_3^t) \end{pmatrix}$$

where, if $x_i = \begin{pmatrix} \alpha_i & a_i \\ b_i & \beta_i \end{pmatrix} \in T_J$, we denote $x_i^t := \begin{pmatrix} \beta_i & b_i \\ a_i & \alpha_i \end{pmatrix}$ and

$$\gamma(x_1, x_2, x_3) = \begin{cases} -3(\alpha_1\beta_2 + \beta_1\alpha_2) + t(a_1, b_2) + t(b_1, a_2) & \alpha_3 \\ + 2(\alpha_1t(b_2, a_3) + \alpha_2t(b_1, a_3) - t(a_1 \times a_2, a_3)) \end{cases}$$

$$c(x_1, x_2, x_3) = \begin{cases} -3(\alpha_1\beta_2 + \beta_1\alpha_2) + t(a_1, b_2) + t(b_1, a_2) & a_3 \\ + 2((t(b_2, a_3) - \beta_2\alpha_3)a_1 + (t(b_1, a_3) - \beta_1\alpha_3)a_2) \\ + 2(\alpha_3(a_1 \times a_2) \times x_3 + (a_1 \times a_3) \times x_3 + (a_1 \times a_3) \times x_1). \end{cases}$$

Here, if $J = \mathbb{C}$, then $t(a, b) = 3ab$ and $a \times b = 0$. And, if $J = H_3(C^\mathbb{C})$, then $t(a, b) = \frac{1}{2} \text{tr}(ab + ba)$ and $a \times b = \frac{1}{2} (ab + ba - \text{tr}(a)b - \text{tr}(b)a + (\text{tr}(a) \text{tr}(b) - t(a, b))I_3)$, for $I_3 \in J$ the identity matrix. Now the pair $(\mathfrak{g}(T), \text{indet}(T))$ is described respectively by

$$(\mathfrak{g}_2^C, \mathfrak{sl}(2, \mathbb{C})), \quad (\mathfrak{f}_4^C, \mathfrak{sp}(6, \mathbb{C})), \quad (\mathfrak{e}_6^C, \mathfrak{sl}(6, \mathbb{C})), \quad (\mathfrak{e}_7^C, \mathfrak{so}(12, \mathbb{C})), \quad (\mathfrak{e}_8^C, \mathfrak{e}_7^C).$$

$^1$C$^\mathbb{C}$ is isomorphic, respectively, to $\mathbb{C} \times \mathbb{C}$, $\text{Mat}_{2 \times 2}(\mathbb{C})$ and the Zorn algebra.

$^2$Thus $(a \times a) \cdot a = n(a)I_3$, for $n$ the cubic norm and $\cdot$ the symmetrized product $a \cdot b = \frac{1}{2}(ab + ba)$, so that $\times$ can be seen as a kind of symmetric cross product on the Jordan algebra $(J, \cdot)$.
The symplectic triple system $T_J$ is called of exceptional type.

According to [19, Thm. 2.30], the described symplectic triple systems in Examples 1, 2, 3, and 4, exhaust the classification of simple symplectic triple systems over the field $\mathbb{C}$.

### 3. 3-Sasakian homogeneous manifolds

**Definition 2.** A triple $S = \{\xi, \eta, \varphi\}$ is called a Sasakian structure on a Riemannian manifold $(M, g)$ when

- $\xi \in \mathfrak{X}(M)$ is a unit Killing vector field (called the Reeb vector field),
- $\varphi$ is the endomorphism field given by $\varphi(X) = -\nabla^g_X \xi$ for all $X \in \mathfrak{X}(M)$, denoting by $\nabla^g$ the Levi-Civita connection,
- $\eta$ is the 1-form on $M$ metrically equivalent to $\xi$, i.e., $\eta(X) = g(X, \xi)$,

and the following condition is satisfied

$$(\nabla^g_X \varphi)(Y) = g(X, Y) \xi - \eta(Y) X$$

for any $X, Y \in \mathfrak{X}(M)$. A Sasakian manifold is a Riemannian manifold $(M, g)$ endowed with a fixed Sasakian structure $S$.

**Definition 3.** A 3-Sasakian structure on $(M, g)$ is a family of Sasakian structures $S = \{\xi_\tau, \eta_\tau, \varphi_\tau\}_{\tau \in \mathbb{S}^2}$ on $(M, g)$ parametrized by the unit sphere $\mathbb{S}^2$ and such that, for $\tau, \tau' \in \mathbb{S}^2$, the following compatibility conditions hold

$$g(\xi_\tau, \xi_{\tau'}) = \tau \cdot \tau' \quad \text{and} \quad [\xi_\tau, \xi_{\tau'}] = 2\xi_{\tau \times \tau'}, \quad (2)$$

where "×" and "\times" are the standard inner and cross products in $\mathbb{R}^3$. Again if $S$ is fixed, $(M, g)$ is said to be a 3-Sasakian manifold.

Recall that having a 3-Sasakian structure on a Riemannian manifold $(M, g)$ is equivalent to fix three Sasakian structures $S_k = \{\xi_k, \eta_k, \varphi_k\}$, for $k = 1, 2, 3$, such that $g(\xi_i, \xi_j) = \delta_{ij}$ and $[\xi_i, \xi_j] = 2\epsilon_{ijk} \xi_k$. The compatibility conditions imply

$$\varphi_k = \varphi_i \circ \varphi_j - \eta_j \otimes \xi_i = -\varphi_j \circ \varphi_i + \eta_i \otimes \xi_j, \quad \xi_k = \varphi_i(\xi_j) = -\varphi_j(\xi_i),$$

for any even permutation $(i, j, k)$ of $(1, 2, 3)$ (i.e., $\epsilon_{ijk} = 1$).

3-Sasakian homogeneous manifolds were classified by Boyer, Galicki and Mann in [13]. They showed that every homogeneous 3-Sasakian manifold fibers over a homogeneous quaternionic Kähler manifold of positive scalar curvature. By [7], all the base spaces are symmetric, so that the classification is achieved by using Wolf’s classification of symmetric quaternionic Kähler manifolds ([27]). Surprisingly, there is a one-to-one correspondence between compact simple Lie algebras and simply-connected 3-Sasakian homogeneous manifolds.

**Theorem 1.** Any 3-Sasakian homogeneous manifold is one of the following coset manifolds:

$$\begin{align*}
\text{Sp}(n+1)/\text{Sp}(n), & \quad \text{Sp}(n+1)/\text{Sp}(n) \times \mathbb{Z}_2, & \quad \text{SU}(m)/\left(\text{SU}(m-2) \times \text{U}(1)\right), & \quad \text{SO}(k)/\left(\text{SO}(k-4) \times \text{Sp}(1)\right).
\end{align*}$$
for \( n \geq 0, m \geq 3 \) and \( k \geq 7 \) (\( \text{Sp}(0) \) denoting the trivial group).

Strong algebraical implications are suggested by the above result. For our purposes, [18, Rem. 4.11] can be read as follows:

**Proposition 2.** If \( \mathfrak{g} \) and \( \mathfrak{h} \) are the Lie algebras of \( G \) and \( H \), for \( M = G/H \) a 3-Sasakian homogeneous manifold, then there is a complex symplectic triple system \( T \) such that \((\mathfrak{g}^C, \mathfrak{h}^C) = (\mathfrak{g}(T), \text{in\,der}(T))\). More precisely,

1. If \( M \in \left\{ \frac{\text{Sp}(n+1)}{\text{Sp}(n)}, \frac{\text{Sp}(n+1)}{\text{Sp}(n) \times \mathbb{Z}_2} \right\} \) for some \( n \geq 0 \), then \( T \) is a symplectic triple system of symplectic type as in Example 1.

2. If \( M = \frac{\text{SU}(m)}{\text{S}(\text{U}(m-2) \times \text{U}(1))} \) for some \( m \geq 3 \), then \( T \) is a symplectic triple system of special type as in Example 3.

3. If \( M = \frac{\text{SO}(k)}{\text{SO}(k-4) \times \text{Sp}(1)} \) for some \( k \geq 7 \), then \( T \) is a symplectic triple system of orthogonal type as in Example 2.

4. If \( M \in \left\{ \frac{\text{G}_2}{\text{Sp}(1)}, \frac{\text{F}_4}{\text{Sp}(3)}, \frac{\text{E}_6}{\text{SU}(6)}, \frac{\text{E}_7}{\text{Spin}(12)}, \frac{\text{E}_8}{\text{E}_7} \right\} \), then \( T \) is a symplectic triple system of exceptional type as in Example 4.

We want to apply the algebraic structure of symplectic triple systems to make easy the computations on curvatures and holonomies. First, we recall Nomizu’s Theorem on invariant affine connections on reductive homogeneous spaces [24], since, in this setting, the curvature and torsion tensors are easily written in algebraical terms.

Let \( G \) be a Lie group acting transitively on a manifold \( M \). An affine connection \( \nabla \) on \( M \) is said to be \( G \)-invariant if, for each \( \sigma \in G \) and for all \( X,Y \in \mathfrak{X}(M) \),

\[
\tau_\sigma(\nabla_XY) = \nabla_{\tau_\sigma(X)}\tau_\sigma(Y).
\]

Here \( \tau_\sigma : M \to M \) denotes the diffeomorphism given by the action, \( \tau_\sigma(p) = \sigma \cdot p \) if \( p \in M \); and the vector field \( \tau_\sigma(X) \in \mathfrak{X}(M) \) is defined by \( (\tau_\sigma(X))_p := (\tau_\sigma)_*(X_{\sigma^{-1} \cdot p}) \) at each \( p \in M \). If \( H \) is the isotropy subgroup at a fixed point \( o \in M \), then there exists a diffeomorphism between \( M \) and \( G/H \). For \( H \) connected, the homogeneous space \( M = G/H \) is said to be reductive if the Lie algebra \( \mathfrak{g} \) of \( G \) admits a vector space decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \), for \( \mathfrak{h} \) the Lie algebra of \( H \) and \( \mathfrak{m} \) an \( \mathfrak{h} \)-module. The differential map \( \pi_* \) of the projection \( \pi : G \to M = G/H \) gives a linear isomorphism \( (\pi_*)_e|_{\mathfrak{m}} : \mathfrak{m} \to T_oM \), where \( o = \pi(e) \). Under these conditions, Nomizu’s Theorem asserts:

**Theorem 3.** There is a one-to-one correspondence between the set of \( G \)-invariant affine connections \( \nabla \) on the reductive homogeneous space \( M = G/H \) and the vector space of bilinear maps \( \alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m} \) such that \( \text{ad}(\mathfrak{h}) \subset \text{der}(\mathfrak{m}, \alpha) \).
The torsion and curvature tensors of $\nabla$ can then be computed in terms of the related map $\alpha^\nabla$ as follows:

\[
T^\nabla(X,Y) = \alpha^\nabla(X,Y) - \alpha^\nabla(Y,X) - [X,Y]_m,
R^\nabla(X,Y)Z = \alpha^\nabla(X,\alpha^\nabla(Y,Z)) - \alpha^\nabla(Y,\alpha^\nabla(X,Z)) - \alpha^\nabla([X,Y]_m,Z) - [[X,Y]_h,Z],
\]
for any $X,Y,Z \in m$, where $[\cdot,\cdot]_h$ and $[\cdot,\cdot]_m$ denote the composition of the bracket $([m,m] \subset g)$ with the projections $\pi_h, \pi_m: g \to g$ of $g = h \oplus m$ on each summand, respectively.

Note that the above expressions give $T^\nabla$ and $R^\nabla$ at the point $o = \pi(e)$, but the invariance permits us to recover the whole tensors on $M$. In particular, every homogeneous 3-Sasakian manifold (all of them described in Theorem 1) is a reductive homogeneous space, and its invariant affine connections have been thoroughly studied in [18] and described in terms of the related reductive decomposition $g = h \oplus m$. Note also that

\[
g = g_0 \oplus g_1
\]
is in any case a $\mathbb{Z}_2$-graded Lie algebra such that

\[
g_0 = h \oplus \mathfrak{sp}(1) \quad \text{and} \quad m = \mathfrak{sp}(1) \oplus g_1.
\]

The (invariant) metric $g$ is determined by $g_o: T_oM \times T_oM \to \mathbb{R}$, which, under the identification $(\pi_*)_e|_m$, is given by

\[
g|_{\mathfrak{sp}(1)} = -\frac{1}{4(n+2)} \kappa, \quad g|_{g_1} = -\frac{1}{8(n+2)} \kappa, \quad g|_{\mathfrak{sp}(1) \times g_1} = 0, \tag{3}
\]
for $\kappa$ the Killing form of $g$ ([18, Thm. 4.3ii])

Let us emphasize certain invariant affine connections, combined with their related bilinear maps through Nomizu’s theorem, which have been distinguished in [18] since they satisfy certain geometrical properties.

**Example 5.** The Levi-Civita connection $\nabla^g$ is related to the bilinear map $\alpha^g: m \times m \to m$, defined by, for any $\xi, \xi' \in \mathfrak{sp}(1)$ and $X, Y \in g_1$,

\[
\alpha^g(\xi + X, \xi' + Y) = \frac{1}{2}[\xi,\xi'] + [X,\xi'] + \frac{1}{2}[X,Y]_m,
\]
according to [18, Thm. 4.3]. Note that $\alpha^g$ is not a skew-symmetric map.

If an affine connection is compatible with the metric, that is, the tensor $g$ is parallel ($\nabla g = 0$), then the connection is determined by its torsion, because, as proved in [1, Prop. 2.1], the difference $(1,2)$-tensor $A = \nabla - \nabla^g$ is the only one such that

\[
g(A(X,Y),Z) = \frac{1}{2}(\omega_\nabla(X,Y,Z) - \omega_\nabla(Y,Z,X) + \omega_\nabla(Z,X,Y)),
\]
where we use the notation
\[ \omega_\nabla(X, Y, Z) := g(T^\nabla(X, Y), Z). \] (5)

Among the metric affine connections, we have studied those with totally skew-symmetric torsion, or briefly, \textit{skew-torsion}, that is, those where \( \omega_\nabla \) is a differential 3-form on \( M \). This characterizes the remarkable fact that \( \nabla \) and \( \nabla^g \) share their (parametrized) geodesics. Algebraically, an invariant affine connection is metric if and only if \( \alpha^\nabla(X, \cdot) \in \mathfrak{so}(m, g) \) for all \( X \in m \); and it also has totally skew-symmetric torsion when the trilinear map \( g((\alpha^\nabla - \alpha^g)(\cdot, \cdot, \cdot), m \times m \times m \to \mathbb{R} \) is alternating, so providing an element in \( \text{Hom}_g(A^3(m), \mathbb{R}) \).

Denote by \( \{\xi_i\}_{i=1}^3 \) the \( G \)-invariant vector fields on \( M \) corresponding to the following basis of \( \mathfrak{sp}(1) = \mathfrak{su}(2) \):
\[ \xi_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (6)

Then, according to [18, Thm. 4.3], the endomorphism field \( \varphi_i = -\nabla^g \xi_i \) satisfies
\[ \varphi_i|_{\mathfrak{sp}(1)} = \frac{1}{2} \text{ad} \xi_i, \quad \varphi_i|_{\mathfrak{g}_1} = \text{ad} \xi_i, \]
for each \( i = 1, 2, 3 \), and \( \mathcal{S}_i = \{\xi_i, \eta_i, \varphi_i\} \) is a Sasakian structure for \( \eta_i = g(\xi_i, \cdot) \) and \( g: m \times m \to \mathbb{R} \) given by Eq. (3). Also, \( \Phi_i: m \times m \to m \) given by \( \Phi_i(X, Y) = g(X, \varphi_i(Y)) \) is a 2-form which coincides with \( \frac{1}{2} d\eta_i \), and is usually called the \textit{fundamental form} associated to the Sasakian structure.

Denote by \( \alpha_o, \alpha_{rs}: m \times m \to m \) the alternating bilinear maps determined by the nondegeneracy of the metric \( g \) and
\[ \eta_1 \wedge \eta_2 \wedge \eta_3(X, Y, Z) = g(\alpha_o(X, Y), Z), \quad \eta_r \wedge \Phi_s(X, Y, Z) = g(\alpha_{rs}(X, Y), Z), \] (7)
for any \( X, Y, Z \in m \) and \( r, s \in \{1, 2, 3\} \). A key result [18, Cor. 5.3] states that the set of bilinear maps related by Theorem 3 to the invariant (metric) affine connections with skew-torsion coincides with the set
\[ \left\{ \alpha^g + a\alpha_o + \sum_{r,s=1}^3 b_{rs}\alpha_{rs} \mid a, b_{rs} \in \mathbb{R} \right\}, \] (8)
for any 3-Sasakian homogeneous manifold of dimension at least 7 except for the case \( M = \frac{SU(m)}{S(U(m - 2) \times U(1))} \), in which the set of bilinear maps related to the invariant affine connections with skew-torsion contains strictly the set in (8). For further use, recall from [18, Eq. (40)],
\[ \alpha_o(X, Y) = 0, \quad \alpha_{rs}(X, Y) = \Phi_s(X, Y)\xi_r, \]
\[ \alpha_o(X, \xi_j) = 0, \quad \alpha_{rs}(X, \xi_j) = \delta_{rj}\varphi_sX, \]
\[ \alpha_o(\xi_i, \xi_{i+1}) = \xi_{i+2}, \quad \alpha_{rs}(\xi_i, \xi_{i+1}) = -\delta_{rs}\xi_{i+2}, \] (9)
for any \( X, Y \in \mathfrak{g}_1 \), where the sum of the indices is taken modulo 3.
Example 6. According to [4, Thm. 4.1.1], the canonical connection of a 3-Sasakian manifold is the (metric) affine connection with skew-torsion characterized by
\[ \nabla_X \varphi_i = -2(\eta_{i+2}(X)\varphi_{i+1} - \eta_{i+1}(X)\varphi_{i+2}). \] (10)
Its torsion $T^c$ is determined by the 3-form
\[ \omega^c = \sum_{i=1}^{3} \eta_i \wedge d\eta_i, \]
as in [18, Rem. 5.18]. A key property is that it has parallel torsion ($\nabla^c T^c = 0$).
Take into account that, if the dimension of the homogeneous 3-Sasakian manifold is strictly greater than 7, then the Levi-Civita connection, the characteristic connection of any of the involved Sasakian structures and $T^c$ are the only invariant affine connections with parallel skew-torsion ([18, Thm. 5.7]).
The related bilinear map $\alpha^c : m \times m \to m$ is
\[ \alpha^c = \alpha^g + \sum_{r=1}^{3} \alpha_{rr}, \]
for $\alpha_{rr}$ the skew-symmetric bilinear map defined in Eq. (7).

Example 7. Let $\mathcal{S} = \{\xi_\tau, \eta_\tau, \varphi_\tau\}_{\tau \in \mathbb{S}^2}$ be a 3-Sasakian structure on a 3-Sasakian homogeneous manifold $(M, g)$ of dimension at least 7. The unique $G$-invariant affine connection with skew-torsion on $M$ such that $\xi_\tau$ is parallel for any $\tau \in \mathbb{S}^2$, is denoted by $\nabla^\mathcal{S}$. The related bilinear map is given by
\[ \alpha^\mathcal{S} = \alpha^g + 2\alpha_o + \sum_{r=1}^{3} \alpha_{rr}. \]
The above connection $\nabla^\mathcal{S}$ admits trivially a generalization for not necessarily homogeneous 3-Sasakian manifolds, and then $\nabla^\mathcal{S}$ becomes
- Einstein with skew-torsion, with symmetric Ricci tensor, if $\dim M = 7$;
- $\mathcal{S}$-Einstein, for arbitrary dimension.

The concept of Einstein with skew-torsion is introduced in [3], where it is proved that the metric connections such that $(M, g, \nabla)$ is Einstein with skew-torsion are the critical points of certain variational problems. An affine connection with totally skew-symmetric torsion is Einstein with skew-torsion if the symmetric part of the Ricci tensor is a multiple of the metric, while it is $\mathcal{S}$-Einstein [18, Def. 5.2] when the Ricci tensor is multiple of the metric both in the horizontal and vertical distributions. There are $\mathcal{S}$-Einstein invariant affine connections in any homogeneous 3-Sasakian manifold but there are Einstein with skew-torsion invariant affine connections only if the homogeneous 3-Sasakian manifold has dimension 7 [18, Thm. 5.4i]).

Our purpose is to study the holonomy groups of $\nabla^g$, $\nabla^\mathcal{S}$, and $\nabla^c$, in order to figure out which of these connections is in some way better adapted to the 3-Sasakian geometry. A good sign of an affine connection adapted to a geometry is that its holonomy group is small.
4. Curvatures and holonomies

We begin by recalling some well-known facts in order to unify the notation. Given a piecewise smooth loop $\gamma : [0,1] \to M$ based at a point $p \in M$, a connection $\nabla$ defines a parallel transport map $P_\gamma : T_p M \to T_p M$, which is both linear and invertible. The holonomy group of $\nabla$ based at $p$ is defined as $\text{Hol}_p(\nabla) = \{ P_\gamma \in \text{GL}(T_p M) | \gamma$ is a loop based at $p \}$. In our case the holonomy group does not depend on the basepoint (up to conjugation) since $M$ is connected, and $\text{Hol}(\nabla) = \text{Hol}_p(\nabla)$ turns out to be a Lie group which can be identified with a subgroup of $\text{GL}(m)$. The Ambrose–Singer Theorem [8] gives a way of computing the holonomy group in terms of the curvature tensor of the connection. The Lie algebra of the holonomy group $\text{Hol}(\nabla)$, denoted by $\mathfrak{hol}(\nabla)$ and called the holonomy algebra, turns out to be the smallest Lie subalgebra of $\mathfrak{gl}(m)$ containing the curvature tensors $R^{\nabla}(X,Y)$ for any $X,Y \in m$ and closed under commutators with the left multiplication operators $\alpha^\nabla(X,\cdot)$, for $X \in m$. If $\nabla$ is compatible with the (invariant) metric $g$, we already mentioned that $\alpha^\nabla(X,\cdot) \in \mathfrak{so}(m,g)$ for all $X \in m$, so that $R^{\nabla}(X,Y) \in \mathfrak{so}(m,g)$ too, and hence the holonomy algebra is a subalgebra of the orthogonal Lie algebra.

Our main tool in this section will be complexification, since the complex Lie algebra $\mathfrak{hol}(\nabla)^C$ is the smallest Lie subalgebra of $\mathfrak{gl}(m^C)$ containing the curvature maps for any $X,Y \in m^C$,

$$R^\nabla(X,Y) = [\alpha^\nabla_X,\alpha^\nabla_Y] - \alpha^\nabla_{[X,Y],m} - \text{ad}[X,Y]_{\mathfrak{h}}, \quad (11)$$

which is closed under commutators with the left multiplication operators $\alpha^\nabla_X := \alpha^\nabla(X,\cdot)$ for $X \in m^C$ (we use the same notation for $R^\nabla$, $\alpha^\nabla$, $[\cdot,\cdot]_m$ and $[\cdot,\cdot]_{\mathfrak{h}}$, to avoid complicating the notation). Our setting is $m^C = \mathfrak{sp}(V,\langle \cdot,\cdot \rangle) \oplus (V \otimes T)$ and $\mathfrak{h}^C = \text{iderv}(T)$, for $T$ a complex symplectic triple system and $(V,\langle \cdot,\cdot \rangle)$ a two-dimensional complex vector space endowed with a nonzero alternating bilinear form. This makes it very easy to compute the curvature maps. From Eq. (1), the projections of the bracket on $\mathfrak{h}^C$ and $m^C$ are

$$[\xi + a \otimes x,\xi' + b \otimes y]_{\mathfrak{h}} = \langle a,b \rangle d_{x,y},$$

$$[\xi + a \otimes x,\xi' + b \otimes y]_m = [\xi,\xi'] + (x,y)\gamma_{a,b} + \xi(b) \otimes y - \xi'(a) \otimes x,$$

for any $\xi,\xi' \in \mathfrak{sp}(V,\langle \cdot,\cdot \rangle)$, $a,b \in V$, $x,y \in T$. (Recall the definitions of $\gamma_{a,b} \in \mathfrak{sp}(V,\langle \cdot,\cdot \rangle)$ and $d_{x,y} \in \text{iderv}(T)$ in Section 2.)

4.1. The Levi-Civita connection

We enclose this case for completeness, and for providing a unified treatment, including the algebraical expressions of the curvature operators. These expressions are of independent interest, and were used in [16] to prove that $\text{Ric}^g = (\dim M - 1)g$ in a slightly different setting. The fact that the holonomy group related to the Levi-Civita connection is general is well-known (recall the words of Boyer and Galicki in [10] mentioned above in the Introduction, or see, for instance, [10, Cor. 14.1.9]); but we were not able to find a direct and explicit proof in the literature.
Proposition 4. After complexifying, the curvature operators become

\[
R^g(\xi, \xi')(\xi'') = -\frac{1}{4}[[\xi, \xi'], \xi''],
\]
\[
R^g(\xi, \xi')(a \otimes x) = 0,
\]
\[
R^g(a \otimes x, \xi)(\xi') = g(\xi, \xi')a \otimes x,
\]
\[
R^g(a \otimes x, b \otimes y)(\xi) = 0,
\]
\[
R^g(a \otimes x, b \otimes y)(c \otimes z) = \frac{\gamma_{a,c}(b) \otimes (x, z)y - \gamma_{b,c}(a) \otimes (y, z)x}{2} - \langle a, b \rangle c \otimes [x, y, z],
\]
for any \(\xi, \xi', \xi'' \in \mathfrak{sp}(V, \langle \cdot, \cdot \rangle)\), \(a, b, c \in V\), \(x, y, z \in T\).

Proof. These computations are straightforward, by taking in mind the precise expression of \(\alpha^g\) in Eq. (4). First, we compute

\[
R^g(a \otimes x, b \otimes y)(c \otimes z)
= \alpha^g_{a \otimes x}(\frac{1}{2}(y, z)g_{b,c}) - \alpha^g_{b \otimes y}(\frac{1}{2}(x, z)g_{a,c}) - \alpha^g_{(x, y)\gamma_{a,b}}(c \otimes z) - [\langle a, b \rangle dx, y, c \otimes z]
\]

and also

\[
R^g(a \otimes x, b \otimes y)(\xi)
= \alpha^g_{a \otimes x}(\xi(b) \otimes y) - \alpha^g_{b \otimes y}(\xi(a) \otimes x) - \alpha^g_{(x, y)\gamma_{a,b}}(\xi) - 0
\]
\[
= -\frac{1}{2}(x, y)\gamma_{a,\xi}(b) + \frac{1}{2}(y, x)\gamma_{b,\xi}(a) - (x, y)\frac{1}{2}[\gamma_{a,b}, \xi]
\]
\[
= \frac{1}{2}(x, y)(\gamma_{a,\xi}(b) - \gamma_{b,\xi}(a) + [\xi, \xi, \gamma_{a,b}]) = 0.
\]

Another curvature operator is

\[
R^g(\xi, \xi')(\xi'' + a \otimes x) = \alpha^g_{\xi}(\frac{1}{2}[\xi', \xi'']) - \alpha^g_{\xi'}(\frac{1}{2}[\xi, \xi'']) - \alpha^g([\xi, \xi'], \xi'' + a \otimes x)
\]
\[
= \frac{1}{4}([\xi, [\xi', \xi'']] - [\xi', [\xi, \xi'']] - \frac{1}{2}[[\xi, \xi'], \xi'']
\]
\[
= (\frac{1}{4} - \frac{1}{2})[[\xi, \xi'], \xi''].
\]

Finally, recall \(\alpha^g(\xi) = 0\) (again Eq. (4)) to get

\[
R^g(a \otimes x, \xi)(\xi' + b \otimes y)
= \alpha^g_{a \otimes x}(\frac{1}{2}[\xi, \xi']) - \alpha^g_{\xi}(\frac{1}{2}(x, y)\gamma_{a,b}) - \alpha^g_{-\xi(a) \otimes x}(\xi' + b \otimes y)
\]
\[
= -\frac{1}{2}([\xi, \xi'](a) \otimes x - \frac{1}{2}(x, y)[\xi, \gamma_{a,b}] - \xi'(a) \otimes x + \frac{1}{2}(x, y)\gamma_{a,b}, b)
\]
\[
= -\frac{1}{2}(\xi + \xi')(a) \otimes x + \frac{1}{2}(x, y)\gamma_{a,\xi}(b) + \gamma_{a}(b)
\]
\[
= g(\xi, \xi')a \otimes x + \frac{1}{2}(x, y)(\gamma_{a,\xi}(b) - \gamma_{a,\xi(b)}).
\]

In the last step we have used \(\xi + \xi' = -2g(\xi, \xi')id_V\) by Eq. (6), because two different elements in the orthonormal basis \(\{\xi_i\}_{i=1}^3\) anticommute. For simplifying the expression, note that, for any \(a, b, c \in V\),

\[
\langle a, b \rangle c + \langle b, c \rangle a + \langle c, a \rangle b = 0,
\]

(12)
because \( \dim V = 2 \) (we can assume \( c \in \{a, b\} \) by trilinearity) and \( \langle \cdot, \cdot \rangle \) is alternating. Thus, using \( \langle \xi(a), c \rangle + \langle a, \xi(c) \rangle = 0 \) when \( \xi \) belongs to the symplectic Lie algebra, and Eq. (12),

\[
(\gamma_{\xi(a), b} - \gamma_{a, \xi(b)}) (c) = \langle \xi(a), c \rangle b + \langle b, c \rangle \xi(a) - \langle a, c \rangle \xi(b) - \langle \xi(b), c \rangle a
\]

\[
= -\langle a, \xi(c) \rangle b + \xi(\langle b, c \rangle a) - \langle a, b \rangle \xi(c) + \langle b, \xi(c) \rangle a
\]

\[
= \langle \xi(c), a \rangle b + \xi(-\langle a, b \rangle c) + \langle b, \xi(c) \rangle a
\]

\[
= -\langle a, b \rangle \xi(c) - \langle a, b \rangle \xi(c).
\]

So \( \gamma_{\xi(a), b} - \gamma_{a, \xi(b)} = -2\langle a, b \rangle \xi \).

Let us introduce some convenient notation. Take \( \varphi_{a, b} := g(a, \cdot)b - g(b, \cdot)a \). As \( g: \mathfrak{m} \times \mathfrak{m} \to \mathbb{R} \) is a bilinear symmetric map, then \( \varphi_{a, b} \) belongs to \( \mathfrak{so}(m, g) \). Moreover, these maps span the whole orthogonal Lie algebra, i.e., \( \mathfrak{so}(m, g) = \{ \sum_i \varphi_{a_i, b_i} \mid a_i, b_i \in \mathfrak{m} \} \equiv \mathfrak{so}_{m, m} \). If \( \mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \) is a decomposition as a sum of vector subspaces, the general Lie algebra \( \mathfrak{gl}(\mathfrak{m}) \) admits a \( \mathbb{Z}_2 \)-grading \( \mathfrak{gl}(\mathfrak{m}) = \mathfrak{gl}(\mathfrak{m})_0 \oplus \mathfrak{gl}(\mathfrak{m})_1 \) where \( \mathfrak{gl}(\mathfrak{m})_i = \{ f \in \mathfrak{gl}(\mathfrak{m}) \mid f(\mathfrak{m}_j) \subset \mathfrak{m}_{i+j} \forall j = 0, 1 \} \) for any \( i \in \mathbb{Z}_2 \). If such vector space decomposition is orthogonal for \( g \), then the orthogonal Lie algebra \( \mathfrak{so}(m, g) \) inherits the \( \mathbb{Z}_2 \)-grading, \( \mathfrak{so}(m, g) = \mathfrak{so}(m, g)_0 \oplus \mathfrak{so}(m, g)_1 \), being \( \mathfrak{so}(m, g)_i = \mathfrak{gl}(\mathfrak{m})_i \cap \mathfrak{so}(m, g) \).

Then it is clear that

\[
\mathfrak{so}(m, g)_0 = \varphi_{m_0, m_0} + \varphi_{m_1, m_1}, \quad \mathfrak{so}(m, g)_1 = \varphi_{m_0, m_1}. \tag{13}
\]

We are going to apply the above to

\[
m_0 = \mathfrak{sp}(1) = \mathfrak{g}_0 \cap \mathfrak{m} \quad \text{and} \quad m_1 = \mathfrak{g}_1,
\]

since they are orthogonal relative to the Killing form, and then, orthogonal for \( g \). An interesting remark is that \( \alpha_{m_0}^g \subset \mathfrak{gl}(m)_0 \) and \( \alpha_{m_1}^g \subset \mathfrak{gl}(m)_1 \) (see (4)). Note also that \( [m_i, m_j] \subset \mathfrak{g}_{i+j} \), and that \( \pi_{h_i}, \pi_\mathfrak{m}: \mathfrak{g} \to \mathfrak{g} \) are grade preserving maps, so that \( R^g(m_i, m_j, m_k) \subset m_{i+j+k} \), or equivalently, \( R^g(m_i, m_j) \subset \mathfrak{gl}(m_{i+j}) \). Thus, as the curvature operators are orthogonal maps, then \( R^g(m_i, m_j) \subset \mathfrak{so}(m, g)_{i+j} \).

With the introduced notation, the condition, for a Riemannian manifold \( (M, g) \), of being a Sasakian manifold is equivalent to the existence of a Killing vector field \( \xi \) of unit length such that \( \nabla^g_X \xi = -\varphi(X) \) and \( \nabla^g_X \varphi = -\varphi_X, \xi \), and is also equivalent to the existence of a Killing vector field \( \xi \) of unit length such that \( R^g(X, \xi) = -\varphi_X, \xi \), according to [11, Prop. 1.1.2]. This is the main clue in the next theorem. Taking the advantage, we have computed the concrete expressions of the curvature operators in the above proposition, so we can check directly that, for any \( \xi, \xi' \in \mathfrak{sp}(V, \langle \cdot, \cdot \rangle) \), \( a \in V \), \( x \in T \),

\[
R^g(\xi, \xi') = -\varphi_{\xi, \xi'}, \tag{14a}
\]

\[
R^g(a \otimes x, \xi) = -\varphi_{a \otimes x, \xi}. \tag{14b}
\]

Indeed, \( R^g(\xi, \xi')|_{\mathfrak{g}_1} = 0 = -\varphi_{\xi, \xi'}|_{\mathfrak{g}_1} \), and both \( R^g(\xi_i, \xi_{i+1})|_{\mathfrak{sp}(1)} = -\frac{1}{2} \text{ad} \xi_{i+2} \) and \( -\varphi_{\xi_i, \xi_{i+1}} = g(\xi_{i+1}, \cdot)\xi_i - g(\xi_i, \cdot)\xi_{i+1} \) send

\[
\xi_i \rightarrow -\xi_{i+1}, \quad \xi_{i+1} \rightarrow \xi_i, \quad \xi_{i+2} \rightarrow 0.
\]
In order to prove (14b), we would need to know if the complexification of the Killing form restricted to the odd part of the grading is \( \kappa(a \otimes x, b \otimes y) = -4(n + 2) \langle a, b \rangle (x, y) \), because

\[
-\varphi_{a \otimes x, \xi}(\xi' + b \otimes y) = g(\xi, \xi')a \otimes x + \frac{1}{8(n + 2)} \kappa(a \otimes x, b \otimes y)\xi.
\]

Without doing explicit computations on traces, classical arguments of representation theory provide the existence of \( s \in \mathbb{C}^\times \) such that

\[
\kappa(a \otimes x, b \otimes y) = s \langle a, b \rangle (x, y) \tag{14}
\]

for all \( a, b \in V, \; x, y \in T \). By the associativity of \( \kappa \), the restriction \( \kappa: \mathfrak{g}_1^C \times \mathfrak{g}_1^C \to \mathbb{C} \) is a \( \mathfrak{g}_0^C \)-invariant bilinear symmetric form, which permits us to identify \( \mathfrak{g}_1^C \) with \( (\mathfrak{g}_1^C)^* \). Also the map given by \( (a \otimes x, b \otimes y) \to (x, y) \langle a, b \rangle \) is a nonzero \( \mathfrak{g}_0^C \)-invariant bilinear symmetric form, which provides another identification between \( \mathfrak{g}_1^C \) with \( (\mathfrak{g}_1^C)^* \), and hence, when composing, an element in \( \text{Hom}_{\mathfrak{g}_0^C}(\mathfrak{g}_1^C, \mathfrak{g}_1^C) \), which coincides with \( \text{Cid} \) by Schur’s Lemma ([18, Lem. 3.4]), since \( \mathfrak{g}_1^C \) is an irreducible \( \mathfrak{g}_0^C \)-module. This provides the required \( s \). Consider now the map \( \rho = R^g(a \otimes x, \xi) + \varphi_{a \otimes x, \xi} \in \mathfrak{so}(\mathfrak{m}^C, g) \), which satisfies \( \rho|_{\text{sp}(V, \langle \cdot, \cdot \rangle)} = 0 \) and \( \rho|_{V \otimes T} = (4(n + 2)/s + 1) \varphi_{a \otimes x, \xi} \). Hence we have

\[
0 = g(\rho(\xi), b \otimes y) + g(\xi, \rho(b \otimes y)) = \left( \frac{4(n + 2)}{s} + 1 \right) g(a \otimes x, b \otimes y)g(\xi, \xi),
\]

which implies \( s = -4(n + 2) \), \( \rho = 0 \), and Eq. (14b). Now, it is not difficult to find the holonomy algebra.

**Theorem 5.** The complexification of the holonomy Lie algebra of the Levi-Civita connection is \( \mathfrak{ho}(\nabla^g)^C = \mathfrak{so}(\mathfrak{m}^C, g) \); so that

\[
\mathfrak{ho}(\nabla^g) = \mathfrak{so}(\mathfrak{m}^C, g).
\]

**Proof.** Recall that \( \mathfrak{ho}(\nabla^g)^C \) is a Lie subalgebra of \( \mathfrak{so}(\mathfrak{m}^C, g) \). Eq. (14b) implies that \( R^g(m_1^C, m_0^C) = \mathfrak{so}(\mathfrak{m}^C, g)_1 \) by taking into account Eq. (13), so that \( \mathfrak{so}(\mathfrak{m}^C, g)_1 \) is contained in \( \mathfrak{ho}(\nabla^g)^C \). In particular, also \( [\mathfrak{so}(\mathfrak{m}^C, g)_1, \mathfrak{so}(\mathfrak{m}^C, g)_1] \subset \mathfrak{ho}(\nabla^g)^C \). We compute

\[
[\varphi_{\xi, a \otimes x}, \varphi_{\xi', b \otimes y}] = \varphi_{\varphi_{\xi, a \otimes x}(\xi'), b \otimes y} + \varphi_{\xi'}, \varphi_{\xi, a \otimes x}(b \otimes y)
\]

\[
= g(\xi, \xi') \varphi_{a \otimes x, b \otimes y} - g(a \otimes x, b \otimes y) \varphi_{\xi', \xi} \in \mathfrak{ho}(\nabla^g)^C. \tag{15}
\]

But Eq. (14a) gives \( \varphi_{\xi', \xi} \subset \mathfrak{ho}(\nabla^g)^C \). By making the sum with the map in Eq. (15), we also have \( \varphi_{a \otimes x, b \otimes y} \in \mathfrak{ho}(\nabla^g)^C \). Hence, \( \mathfrak{so}(\mathfrak{m}^C, g)_0 = \mathfrak{m}_0, \mathfrak{m}_0 + \varphi_{m_1, m_1} \) is completely contained in \( \mathfrak{ho}(\nabla^g)^C \), which ends the proof. \( \square \)

**Remark 2.** Recall that any 3-Sasakian manifold is an \( n \)-Sasakian manifold for \( n = 3 \) in the sense proposed by [15], so that, \( R^g(X, Y)|_{\text{sp}(1)} = -\varphi_{X, Y} \) for any \( X, Y \in \mathfrak{m} \). But this is only true for the restriction to the vertical part. Put attention on the fact that, for a homogeneous 3-Sasakian manifold, \( R^g(X, Y)|_{\mathfrak{b}_1} \) coincides with \( -\varphi_{X, Y} \)
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if and only if the corresponding symplectic triple system is of symplectic type (corresponding to the sphere, well known for being of constant curvature, or to the projective space, locally undistinguishable). Indeed, for \( x, y, z \in T \) and \( a, b \in V \), then

\[
R^g(a \otimes x, a \otimes y)(b \otimes z) = -\frac{1}{2}(a, b)a \otimes ((x, z)y - (y, z)x) = -\varphi_{a \otimes x, a \otimes y}(b \otimes z),
\]

taking into account Eq. (14). But, for \( e_1, e_2 \in V \) with \( \langle e_1, e_2 \rangle = 1 \), then \( \gamma_{e_1, e_1}(e_2) = 2e_1 \), \( \gamma_{e_2, e_1}(e_1) = -e_1 \) and

\[
R^g(e_1 \otimes x, e_2 \otimes y)(e_1 \otimes z) = e_1 \otimes ((x, z)y + \frac{1}{2}(y, z)x - [x, y, z]),
\]

which will coincide with

\[
-\varphi_{e_1 \otimes x, e_2 \otimes y}(e_1 \otimes z) = g(e_2 \otimes y, e_1 \otimes z)(e_1 \otimes x) = -\frac{1}{2}e_1 \otimes (y, z)x,
\]

if and only if the identity \( [x, y, z] = (x, z)y + (y, z)x \) holds in \( T \). Note that such identity is false for a symplectic triple system not of symplectic type. It is interesting to remark that just this difference, \( [x, y, z] - ((x, z)y + (y, z)x) \), measures how far is a 3-Sasakian manifold from being of constant curvature.

4.2. The distinguished connection

Lemma 6. If \( \xi, \xi' \in \mathfrak{sp}(1) \), \( X, Y \in \mathfrak{g}_1 \), then

\[
\alpha^S(\xi, \xi') = 0, \quad \alpha^S(X, \xi) = 0, \\
\alpha^S(\xi, X) = -\varphi_i(X), \quad \alpha^S(X, Y) = 0.
\]

Proof. Note that the fact \( \alpha^S(\cdot, \xi) = 0 \) is the condition required for the choice of the affine connection in [18, Thm. 5.6]. Anyway, it is easy to check it directly. Indeed, as \( \alpha^S - \alpha^g = 2\alpha_0 + \sum_{r=1}^3 \alpha_{rr} \) is alternating, so Eq. (9) gives

\[
(\alpha^S - \alpha^g)(\xi, \xi') = [\xi, \xi'] - \frac{3}{2}[\xi, \xi'], \\
(\alpha^S - \alpha^g)(X, \xi) = 0 + \varphi_i(X) = [\xi, X], \\
(\alpha^S - \alpha^g)(\xi, X) = 0 - \varphi_i(X), \\
(\alpha^S - \alpha^g)(X, Y) = 0 + \sum_r \Phi_r(X, Y)\xi_r = -\frac{1}{2}[X, Y],
\]

and we finish by taking into account Eq. (4).

Proposition 7. After complexifying, the curvature operators become

\[
R^S(\xi, \xi')(\xi'' + c \otimes z) = 2[\xi, \xi'](c) \otimes z, \\
R^S(a \otimes x, \xi) = 0, \\
R^S(a \otimes x, b \otimes y)(\xi + c \otimes z) = \gamma_{a, b}(c) \otimes (x, y)z - (a, b)c \otimes [x, y, z],
\]

for any \( \xi, \xi', \xi'' \in \mathfrak{sp}(V, \langle \cdot, \cdot \rangle) \), \( a, b, c \in V, \ x, y, z \in T \).
Proof. By recalling $R^S(X,Y) = [\alpha^S_X, \alpha^S_Y] - \alpha^S_{[X,Y]} + \text{ad}[X,Y]_b$, and $\alpha^S_{a \otimes x} = 0$, then

$$R^S(a \otimes x, \xi) = [0, \alpha^S_\xi] - \alpha^S_{-\xi(a) \otimes x} = 0,$$

and

$$R^S(a \otimes x, b \otimes y) = 0 - \alpha^S_{(x,y) \gamma_{a,b}} - \text{ad} (a, b) d_{x,y} = -(x, y) \alpha^S_{\gamma_{a,b}} - \langle a, b \rangle \text{ad} d_{x,y}.$$

Finally we get $[\alpha^S_\xi, \alpha^S_\xi] = -\alpha^S_{[\xi, \xi]}$, because both maps are zero in the vertical part and $[\alpha^S_\xi, \alpha^S_\xi]|_{V \otimes T} = [\text{ad} \xi, \text{ad} \xi'] = [\text{ad} \xi, \xi']$. Thus $R^S(\xi, \xi') = -2\alpha^S_{[\xi, \xi']}$. □

Theorem 8. The complexification of the holonomy Lie algebra of the distinguished affine connection is

$$\mathfrak{hol}(\nabla^S)^C = \{ f \in \mathfrak{gl}(m^C) \mid \exists \xi \in \mathfrak{sp}(V, \langle \cdot, \cdot \rangle) \text{ with } f|_{\mathfrak{sp}(V, \langle \cdot, \cdot \rangle)} = 0, f|_{V \otimes T} = \xi \otimes \text{id}_T \} \oplus \{ f \in \mathfrak{gl}(m^C) \mid \exists d \in \text{ind} \text{er}(T) \text{ with } f|_{\mathfrak{sp}(V, \langle \cdot, \cdot \rangle)} = 0, f|_{V \otimes T} = \text{id}_V \otimes d \} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \text{ind} \text{er}(T),$$

so that

$$\mathfrak{hol}(\nabla^S) \cong \mathfrak{su}(2) \oplus \mathfrak{h}.$$

Proof. The linear span of the curvature operators $R^S(\xi, \xi') = -2\alpha^S_{[\xi, \xi']}$, and $R^S(a \otimes x, b \otimes y) = -(x, y) \alpha^S_{\gamma_{a,b}} - \langle a, b \rangle \text{ad} d_{x,y} \in \mathfrak{gl}(m^C)$ is the vector space (which turns out to be also a Lie algebra)

$$\{ \alpha^S_\xi \mid \xi \in \mathfrak{sp}(V, \langle \cdot, \cdot \rangle) \} \oplus \{ \text{ad} d|_{m^C} \mid d \in \text{ind} \text{er}(T) \}. \quad (16)$$

The complexification of the holonomy algebra, $\mathfrak{hol}(\nabla^S)^C$, becomes the smallest Lie algebra containing such set and closed with brackets with $\alpha^S_\xi = -\text{ad} \xi \circ \pi_{g_1}$ for all $\xi \in \mathfrak{sp}(V, \langle \cdot, \cdot \rangle)$, since $\alpha^S_{a \otimes x} = 0$. As $\text{ad} d_{x,y}$ acts on $T$, and $\text{ad} \xi$ acts on $V$, they commute, so that

$$[R^S(a \otimes x, b \otimes y), \alpha^S_\xi] = -(x, y)[\alpha^S_{\gamma_{a,b}}, \alpha^S_\xi] = (x, y)\alpha^S_{[\gamma_{a,b}, \xi]}.$$

Hence, the set in Eq. (16) is closed for the required brackets so that it coincides with the whole Lie algebra $\mathfrak{hol}(\nabla^S)^C$. □

If we compare $\mathfrak{hol}(\nabla^S)$ with $\mathfrak{hol}(\nabla^g)$, we see that this holonomy algebra is considerably smaller, which indicates that the distinguished connection is better adapted than $\nabla^g$ to the geometry of the 3-Sasakian manifolds, as expected.

4.3. The canonical connection

Lemma 9. If $\xi, \xi' \in \mathfrak{sp}(1)$, $X, Y \in \mathfrak{g}_1$, then

$$\alpha^c(\xi, \xi') = -[\xi, \xi'], \quad \alpha^c(X, \xi) = 0,$$

$$\alpha^c(\xi, X) = -\varphi_i(X), \quad \alpha^c(X, Y) = 0.$$
Proof. Simply observe that $\alpha^c - \alpha^S = -2\alpha$, so that by Eq. (9),

$$\alpha^c(\xi, \xi') = \alpha^S(\xi, \xi') - [\xi, \xi']$$,

$$\alpha^c(\xi, X) = \alpha^S(\xi, X),$$

and by Lemma 6 the result follows. 

Proposition 10. After complexifying, the curvature operators become

$$R^c(\xi, \xi') = 2\text{ad}[\xi, \xi']|_{\mathfrak{m}^c},$$

$$R^c(a \otimes x, \xi) = 0,$$

$$R^c(a \otimes x, b \otimes y) = \text{ad} ((x, y)\gamma_{a,b} - \langle a, b\rangle d_{x,y})|_{\mathfrak{m}^c},$$

for any $\xi, \xi' \in \mathfrak{sp}(V, \langle \cdot, \cdot \rangle)$, $a, b \in V$, $x, y \in T$.

Proof. First, as $[\xi, \xi']|_{\mathfrak{m}} = [\xi, \xi']|_{\mathfrak{h}}$ and $[\xi, \xi']|_{\mathfrak{h}} = 0$, then by Eq. (11), $R^c(\xi, \xi') = [\alpha^c, \alpha^c'] - \alpha^c[\xi, \xi'] = -\text{ad} \xi - \text{ad} \xi' + \text{ad}[\xi, \xi'] = 2\text{ad}[\xi, \xi']$. Second, $\alpha^c|_{\mathfrak{m}^c} = 0 = \alpha^c(a \otimes x)$, which implies $R^c(a \otimes x, \xi) = 0$. And third, $R^c(a \otimes x, b \otimes y) = 0 - (x, y)\gamma_{a,b} - \langle a, b\rangle \text{ad} d_{x,y}$. 

Remark 3. Observe some analogies between the curvature operators related to the distinguished and canonical connections: $R^S(X, Y)|_{\mathfrak{g}^0} = R^c(X, Y)|_{\mathfrak{g}^0}$ for all $X, Y \in \mathfrak{m}$, although they do not coincide in the vertical part, since $R^S(X, Y)|_{\mathfrak{sp}(1)} = 0$ (the same fact that happens after the complexification).

Recall also that the first Bianchi identity for connections with torsion is given in [1, Thm. 2.6], relating the cyclic sum of the curvature with the torsion, and hence not necessarily zero if the torsion is not zero. For instance, in our case, by using Remark 1,

$$\sum_{\text{cyclic}} R^c(a \otimes x, b \otimes y, c \otimes z) = 2(\gamma_{a,b}(c) \otimes (x, y) z + \gamma_{b,c}(a) \otimes (y, z)x + \gamma_{c,a}(b) \otimes (z, x)y).$$

Theorem 11. The complexification of the holonomy Lie algebra of the canonical affine connection is

$$\mathfrak{hol}(\nabla^c)^C = \text{ad}(\mathfrak{g}(T)_{\bar{0}})|_{\mathfrak{m}^c} \cong \mathfrak{g}(T)_{\bar{0}} = \mathfrak{sp}(V, \langle \cdot, \cdot \rangle) \oplus \text{intert}(T);$$

and hence

$$\mathfrak{hol}(\nabla^c) \cong \mathfrak{su}(2) \oplus \mathfrak{h}.$$

Proof. In this case, the maps $\alpha^c = -\text{ad} \xi|_{\mathfrak{m}^c} \in R^c(\mathfrak{sp}(V, \langle \cdot, \cdot \rangle), \mathfrak{sp}(V, \langle \cdot, \cdot \rangle))$ are obviously included in $\mathfrak{hol}(\nabla^c)^C$, by the above proposition. Hence $R^c(a \otimes x, b \otimes y) - (x, y)\text{ad} \gamma_{a,b} = -\langle a, b\rangle \text{ad} d_{x,y}|_{\mathfrak{m}^c}$ belongs to the holonomy algebra too and the whole $\text{ad}(\mathfrak{g}(T)_{\bar{0}})$ is a subalgebra of $\mathfrak{hol}(\nabla^c)^C$. As this subalgebra is already closed for the bracket with the maps $\alpha^c = -\text{ad} \xi$, then it has to coincide with $\mathfrak{hol}(\nabla^c)^C$. 

\[\square\]
Observe that $\mathfrak{hol}(\nabla^S) \cong \mathfrak{hol}(\nabla^c)$, so that both holonomy algebras are isomorphic Lie subalgebras of $\mathfrak{so}(m, g) \cong \mathfrak{so}(4n + 3)$ but different. The point is that $\nabla^c$ parallelizes the torsion while $\nabla^S$ parallelizes the Reeb vector fields.

The holonomy algebras (and groups) are semisimple in the two considered cases, distinguished and canonical, whenever $T$ is not of special type. But if $M = SU(m) / S(U(m - 2) \times U(1))$, then $\mathfrak{h} \cong \mathfrak{u}(m - 2)$ and hence the holonomy algebras have a one-dimensional center.

**Remark 4.** Note that the invariant affine connection related to the bilinear map $\alpha = 0$ gives a holonomy algebra even smaller, since it can be proved to be isomorphic to $\mathfrak{h}$. But the geometric properties are not very good since, for instance, it does not have skew-symmetric torsion.

**Remark 5.** Another invariant metric affine connection with skew-torsion emphasized in [18] is the characteristic connection $\nabla^ch$ related to any of the Sasakian structures $\{ \xi_\tau, \eta_\tau, \varphi_\tau \}$ (for some $\tau \in S^2$). It is the only one satisfying $\nabla^ch \xi_\tau = 0, \nabla^ch \eta_\tau = 0$ and $\nabla^ch \varphi_\tau = 0$, and moreover, it parallelizes the (skew-symmetric) torsion $T^ch$. In spite of that, we have not included here the study of its holonomy algebra because the related curvature operators turn out to be very intricate so that its holonomy algebra is not small enough.

Here we enclose a table for comparing the dimensions of the obtained holonomy algebras for any (simply-connected) 3-Sasakian homogeneous manifold:

| $n$ | $\frac{\text{Sp}(n+1)}{\text{Sp}(n)}$ | $\frac{\text{SU}(m)}{\text{SU}(m-2) \times \text{U}(1)}$ | $\frac{\text{SO}(k)}{\text{SO}(k-4) \times \text{Sp}(1)}$ |
|-----|----------------------------------|----------------------------------|----------------------------------|
| dim $\mathfrak{hol}(\nabla^g)$ | $8n^2 + 10n + 3$ | $8n^2 + 10n + 3$ | $8n^2 + 10n + 3$ |
| dim $\mathfrak{hol}(\nabla^S)$ | $2n^2 + n + 3$ | $n^2 + 3$ | $\frac{n^2-n}{2} + 6$ |
| dim $\mathfrak{hol}(\nabla^c)$ | $2n^2 + n + 3$ | $n^2 + 3$ | $\frac{n^2-n}{2} + 6$ |

| $n$ | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|-----|-------|-------|-------|-------|-------|
| dim $\mathfrak{hol}(\nabla^g)$ | 55 | 465 | 903 | 2211 | 6555 |
| dim $\mathfrak{hol}(\nabla^S)$ | 6 | 24 | 38 | 69 | 136 |
| dim $\mathfrak{hol}(\nabla^c)$ | 6 | 24 | 38 | 69 | 136 |

Other possible comparisons, for instance the study of the Ricci tensor, are directly extracted from [18, Prop. 5.2 and Cor. 5.4]. The three Ricci tensors vanish in $\mathfrak{sp}(1) \times \mathfrak{g}_1$, as the metric does, and

$$\text{Ric}^g|_{\mathfrak{sp}(1) \times \mathfrak{sp}(1)} = (4n + 2)g, \quad \text{Ric}^g|_{\mathfrak{g}_1 \times \mathfrak{g}_1} = (4n + 2)g,$$

$$\text{Ric}^S|_{\mathfrak{sp}(1) \times \mathfrak{sp}(1)} = 0, \quad \text{Ric}^S|_{\mathfrak{g}_1 \times \mathfrak{g}_1} = (4n - 4)g,$$

$$\text{Ric}^c|_{\mathfrak{sp}(1) \times \mathfrak{sp}(1)} = -16g, \quad \text{Ric}^c|_{\mathfrak{g}_1 \times \mathfrak{g}_1} = (4n - 4)g.$$

In particular, $\nabla^S$ and $\nabla^c$ are $\mathcal{S}$-Einstein invariant affine connections in the sense of [18]. Moreover, $\nabla^S$ is Ricci-flat (but not flat) if $\dim M = 7$, that is, if $M$ is
either the sphere $S^7$, or the projective space $\mathbb{RP}^7$, or the Aloff–Wallach space $\mathcal{W}_{1,1} = \text{SU}(3)/\text{U}(1)$.

If we recall too that the scalar curvature is given by $(4n + 2)(4n + 3) - \frac{3}{2}(a - \text{tr}(B))^2 - 3n\|B\|^2$ for $B = (b_{rs})$ the matrix of the coefficients in Eq. (8), then, for $\dim M = 7$,

$$s^g = 42; \quad s^S = 0; \quad s^c = -48;$$

and, for arbitrary dimension,

$$s^g = (4n + 2)(4n + 3); \quad s^S = 16n(n - 1); \quad s^c = 16(n^2 - n - 3).$$

Again it is interesting that these results only depend on the dimension of $M$, but not on the concrete homogeneous 3-Sasakian manifold we are working with.

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