A NEW BASIS FOR THE SPACE OF MODULAR FORMS

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Abstract. Let $G_{2n}$ be the Eisenstein series of weight $2n$ for the full modular group $\Gamma = SL_2(\mathbb{Z})$. It is well-known that the space $M_{2k}$ of modular forms of weight $2k$ on $\Gamma$ has a basis $\{G_{4\alpha}^\alpha G_{6\beta}^\beta \mid \alpha, \beta \in \mathbb{Z}, \alpha, \beta \geq 0, 4\alpha + 6\beta = 2k\}$. In this paper we will exhibit another (simpler) basis for $M_{2k}$. It is given by $\{G_{2k}\} \cup \{G_{4i}^i G_{2k} - 4i \mid i = 1, 2, \ldots, d_k\}$ if $2k \equiv 0 \pmod{4}$, and $\{G_{2k}\} \cup \{G_{4i+2}^i G_{2k} - 4i - 2 \mid i = 1, 2, \ldots, d_k\}$ if $2k \equiv 2 \pmod{4}$ where $d_k = 1 + \dim_C M_{2k}$.

1. Introduction and statement of results

Modular forms of one variable have been studied for a long time. They appear in many areas of mathematics and in theoretical physics. In this paper we consider the space $M_{2k}$ of modular forms of weight $2k$, and find a simple basis for $M_{2k}$ in terms of Eisenstein series, which is different from the classically known standard basis. A motivation for looking for a new basis will be explained below.

Throughout the paper, we use the following notation:

- $k$ is an integer greater than or equal to 1,
- $\Gamma := SL_2(\mathbb{Z})$ (the full modular group),
- $M_{2k} :=$ the $\mathbb{C}$-vector space of modular forms of weight $2k$ on $\Gamma$,
- $S_{2k} :=$ the $\mathbb{C}$-vector space of cusp forms of weight $2k$ on $\Gamma$,
- $S_{2k}^* := \text{Hom}_C(S_{2k}, \mathbb{C})$ (the dual space of $S_{2k}$),
- $d_k := \begin{cases} \left\lfloor \frac{k}{6} \right\rfloor - 1 & \text{if } 2k \equiv 2 \pmod{12} \\ \left\lfloor \frac{k}{6} \right\rfloor & \text{if } 2k \not\equiv 2 \pmod{12} \end{cases}$ (mod 12)

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding $x \in \mathbb{R}$. We note that

$$\dim_C S_{2k} = d_k \quad \text{and} \quad \dim_C M_{2k} = d_k + 1.$$ 

Let $B_{2n}$ be the $2n$th Bernoulli number and $\sigma_{2n-1}(m)$ is the $(2n - 1)$th divisor function. Namely,

$$\sigma_{2n-1}(m) := \sum_{0 < d | m} d^{2n-1} \quad (n \geq 1).$$

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Then the Eisenstein series of weight $2n$ for $\Gamma$ is defined by

$$G_{2n}(z) := -\frac{B_{2n}}{4n} + \sum_{m=1}^{\infty} \sigma_{2n-1}(m)e^{2\pi imz}$$

where $z \in \mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$.

The classically well-known basis for $M_{2k}$ is the following set (Serre [8, p. 89]):

$$\{G_\alpha^\alpha G_\beta^\beta \mid \alpha, \beta \in \mathbb{Z}, \alpha, \beta \geq 0, 4\alpha + 6\beta = 2k\}.$$

However, the Fourier coefficients of these forms are not so simple when we write down the coefficients as sums of products of divisor functions. This will motivate us to look for a new simpler basis for $M_{2k}$, consisting of modular forms whose Fourier coefficients are convolution sums of two divisor functions. Our result is formulated in the following theorem:

**Theorem 1.1.**

1. If $2k \equiv 0 \pmod{4}$ then
   $$\{G_{2k}\} \cup \{G_{4i}G_{2k-4i} \mid i = 1, 2, \ldots, d_k\}$$
   form a basis for $M_{2k}$.
2. If $2k \equiv 2 \pmod{4}$ then
   $$\{G_{2k}\} \cup \{G_{4i+2}G_{2k-4i-2} \mid i = 1, 2, \ldots, d_k\}$$
   form a basis for $M_{2k}$.

Note that the $n$th Fourier coefficients of $G_{4i}G_{2k-4i}$ is

$$\sum_{l=0}^{n} \sigma_{4i-1}(l)\sigma_{2k-4i-1}(n-l)$$

where we set $\sigma_{2n-1}(0) := -B_{2n}/(4n)$ by convention.

We will also find a new basis for the space of cusp forms on $\Gamma$ in the following theorem:

**Theorem 1.2.**

1. If $2k \equiv 0 \pmod{4}$ then
   $$\{G_{4i}G_{2k-4i} + \frac{B_{4i}}{4i} \frac{B_{2k-4i}}{2k-4i} \frac{k}{B_{2k}} G_{2k} \mid i = 1, 2, \ldots, d_k\}$$
   form a basis for $S_{2k}$.
2. If $2k \equiv 2 \pmod{4}$ then
   $$\{G_{4i+2}G_{2k-4i-2} + \frac{B_{4i+2}}{4i+2} \frac{B_{2k-4i-2}}{2k-2i-2} \frac{k}{B_{2k}} G_{2k} \mid i = 1, 2, \ldots, d_k\}$$
   form a basis for $S_{2k}$.

We note that, for $\Gamma = \Gamma_0(2)$, similar but slightly different formulas were given in [4, Theorem 1.6].

**Example 1.3.** For $M_{36}$, we have a basis

$$\{G_{36}, G_{4}G_{32}, G_{8}G_{28}, G_{12}G_{24}\},$$

and for $S_{36}$,

$$\{ G_{4}G_{32} - \frac{1479565184900325423}{28631015449722183818240} G_{36}, G_{8}G_{28} - \frac{651138973032093}{12210286006168135010720} G_{36}, G_{12}G_{24} - \frac{114819293577343}{1149451061437375891652640} G_{36} \}$$
is a basis.

2. Preliminaries

Let \( f \) be an element of \( S_{2k} \). We write \( f \) as a Fourier series
\[
f(z) = \sum_{l=1}^{\infty} a_l e^{2\pi i l z}.
\]

Let \( L(f, s) \) be the L-series of \( f \). Namely \( L(f, s) \) is the analytic continuation of
\[
\sum_{l=1}^{\infty} \frac{a_l}{l^s} \quad (\Re(s) \gg 0).
\]

Then \( n \)th period of \( f \), \( r_n(f) \), is defined by
\[
r_n(f) := \int_0^{i\infty} f(z) z^n \, dz = \frac{n!}{(-2\pi i)^{n+1}} L(f, n+1) \quad (n = 0, 1, \ldots, w).
\]

Each period \( r_n \) can be regarded as a linear map from \( S_{2k} \) to \( \mathbb{C} \), that is,
\[
r_n \in S_{2k}^* = \text{Hom}_\mathbb{C}(S_{2k}, \mathbb{C}).
\]

Here we recall the result of Eichler [2], Shimura [9] and Manin [6]:

**Theorem 2.1** (Eichler-Shimura-Manin). The maps
\[
r^+ : S_{2k} \rightarrow \mathbb{C}^k
\]
\[
f \quad \mapsto \quad (r_0(f), r_2(f), \ldots, r_{2k-2}(f))
\]
and
\[
r^- : S_{2k} \rightarrow \mathbb{C}^{k-1}
\]
\[
f \quad \mapsto \quad (r_1(f), r_3(f), \ldots, r_{2k-3}(f))
\]
are both injective.

In other words,

(1) the even periods
\[
r_0, r_2, \ldots, r_{2k-2}
\]
span the vector space \( S_{2k}^* \):

(2) the odd periods
\[
r_1, r_3, \ldots, r_{2k-3}
\]
also span \( S_{2k}^* \).

However, these periods are not linearly independent. A natural question was raised in [3]: which periods form a basis for \( S_{2k}^* \)? A satisfactory answer was obtained in the same paper [3].

To state the result in [3] we need the following notation and convention:

**Definition 2.1.** For an integer \( i \) such that \( 1 \leq i \leq d_k \), let
\[
4i \pm 1 := \begin{cases} 
4i + 1 & \text{if } 2k \equiv 2 \pmod{4} \\
4i - 1 & \text{if } 2k \equiv 0 \pmod{4}.
\end{cases}
\]

Now we can state our result in [3]:
Theorem 2.2 (3). \[ \{r_{4i\pm 1} \mid i = 1, 2, \ldots, d_k\} \]

form a basis for \( S_{2k} \).

Next we will display a basis for \( S_{2k} \). For \( f, g \in S_{2k} \), let \( (f, g) \) denote the Petersson scalar product. Then there is a cusp form \( R_n \), which is characterized by the formula:

\[ r_n(f) = (R_n, f) \quad \text{for any} \quad f \in S_{2k}. \]

Passing to the dual space, we obtain a basis for \( S_{2k} \).

Theorem 2.3 (3). \[ \{R_{4i\pm 1} \mid i = 1, 2, \ldots, d_k\} \]

form a basis for \( S_{2k} \).

This theorem will be needed to prove Theorem 1.1. Finally some remark on the Petersson scalar product might be in order.

Remark 2.1. Let \( f \) and \( g \) be modular forms in \( M_{2k} \) with at least one of them a cusp form. Then the Petersson scalar product \( (f, g) \) is defined by

\[ (f, g) = \int_{\Gamma \backslash \mathbb{H}} f(z) \overline{g(z)} y^{2k-2} \, dx \, dy \]

where \( z = x + iy \). We note that the Petersson scalar product of an Eisenstein series and a cusp form is always zero (refer to [1, p. 183]).

However, there is a natural extension of the Petersson scalar product from the space of cusp forms to the space of all modular forms (Zagier [10, pp. 434–435]). This extended scalar product is always non-degenerate, and furthermore, it is positive definite if and only if \( 2k \equiv 2 \pmod{4} \).

Petersson scalar products considered in this article are those of extended one in the above sense which are always non-degenerate.

3. Proof of Theorems 1.1 and 1.2

In this section, we will give proofs of Theorems 1.1 and 1.2. We need the following lemma:

Lemma 3.1. Let \( V \) be a \( \mathbb{C} \)-vector space of dimension \( n \) and

\[ B : V \times V \to \mathbb{C} \]

be a non-degenerate bilinear form. Let \[ \{u_i \in V \mid i = 1, \ldots, n\} \quad \text{and} \quad \{v_i \in V \mid i = 1, \ldots, n\} \]

be two sets of vectors in \( V \). Then the determinant

\[ |B(u_i, v_j)|_{i,j=1,2,\ldots,n} \neq 0 \]

if and only if both \( \{u_i \in V \mid i = 1, \ldots, n\} \) and \( \{v_i \in V \mid i = 1, \ldots, n\} \) are sets of linearly independent vectors.

The proof of this lemma is quite standard and we omit it.
Proof of Theorem 2.3. First we assume that $2k \equiv 0 \pmod{4}$. We consider two sets of modular forms:

$$\{G_{2k}\} \cup \{G_{4i}G_{2k-4i} \mid i = 1, 2, \ldots, d_k\} \quad \text{and} \quad \{G_{2k}\} \cup \{R_{4i-1} \mid i = 1, 2, \ldots, d_k\}.$$ 

We would like to verify that $G_{2k}$, $G_{4i}G_{2k-4i}$ ($i = 1, 2, \ldots, d_k$) are linearly independent. By virtue of Lemma 3.1, it is sufficient to show that the determinant

$$\begin{vmatrix}
(G_{2k}, G_{2k}) & (R_{4-1}, G_{2k}) & \cdots & (R_{4d_k-1}, G_{2k}) \\
(G_{2k}, G_4G_{2k-4}) & (R_{4-1}, G_4G_{2k-4}) & \cdots & (R_{4d_k-1}, G_4G_{2k-4}) \\
\cdots & \cdots & \cdots & \cdots \\
(G_{2k}, G_{4d_k}G_{2k-4d_k}) & (R_{4-1}, G_{4d_k}G_{2k-4d_k}) & \cdots & (R_{4d_k-1}, G_{4d_k}G_{2k-4d_k})
\end{vmatrix} \neq 0.$$

Since $(G_{2k}, G_{2k}) \neq 0$ and $(R_{4i-1}, G_{2k}) = 0$ as mentioned in Remark 2.1, (3.1) is equivalent to

$$\begin{vmatrix}
(R_{4-1}, G_4G_{2k-4}) & (R_8-1, G_4G_{2k-4}) & \cdots & (R_{4d_k-1}, G_4G_{2k-4}) \\
(R_{4-1}, G_8G_{2k-8}) & (R_8-1, G_8G_{2k-8}) & \cdots & (R_{4d_k-1}, G_8G_{2k-8}) \\
\cdots & \cdots & \cdots & \cdots \\
(R_{4-1}, G_{4d_k}G_{2k-4d_k}) & (R_8-1, G_{4d_k}G_{2k-4d_k}) & \cdots & (R_{4d_k-1}, G_{4d_k}G_{2k-4d_k})
\end{vmatrix} \neq 0.$$

Now let $\{f_i \mid i = 1, 2, \ldots, d_k\}$ be a basis for $S_{2k}$ such that each $f_i$ is a normalized Hecke eigenform. Then, since $\{R_{4i-1} \mid i = 1, 2, \ldots, d_k\}$ is also a basis for $S_{2k}$ by Theorem 2.3, we know that (3.2) is equivalent to

$$\begin{vmatrix}
(f_1, G_4G_{2k-4}) & (f_2, G_4G_{2k-4}) & \cdots & (f_{d_k}, G_4G_{2k-4}) \\
(f_1, G_8G_{2k-8}) & (f_2, G_8G_{2k-8}) & \cdots & (f_{d_k}, G_8G_{2k-8}) \\
\cdots & \cdots & \cdots & \cdots \\
(f_1, G_{4d_k}G_{2k-4d_k}) & (f_2, G_{4d_k}G_{2k-4d_k}) & \cdots & (f_{d_k}, G_{4d_k}G_{2k-4d_k})
\end{vmatrix} \neq 0.$$

To show (3.3), we use the following Rankin’s identity [7, also refer to Kohnen-Zagier 5] noting that their notation of $r_n(f)$ differs from ours by a factor $i^{n+1}$:

$$\langle f, G_{2n}G_{2k-2n} \rangle = \frac{1}{(2i)^{2k-1}}r_{2k-2}(f)r_{2n-1}(f) \quad \text{where } n = 2, 3, \ldots, k - 2.$$ 

From this identity we know that (3.3) is equivalent to

$$\begin{vmatrix}
\frac{r_{2k-2}(f_1)r_{2k-2}(f_2)\cdots r_{2k-2}(f_{d_k})}{(2i)^{2k-1}d_k} & r_{4-1}(f_1) & r_{4-1}(f_2) & \cdots & r_{4-1}(f_{d_k}) \\
r_{8-1}(f_1) & r_{8-1}(f_2) & \cdots & r_{8-1}(f_{d_k}) \\
\cdots & \cdots & \cdots & \cdots \\
r_{4d_k-1}(f_1) & r_{4d_k-1}(f_2) & \cdots & r_{4d_k-1}(f_{d_k})
\end{vmatrix} \neq 0.$$

Finally, (3.5) is equivalent to
Now (3.6) holds, since both \( \{ f_i \mid i = 1, 2, \ldots, d_k \} \) and \( \{ R_{d_k-1} \} \) are bases for \( S_{2k} \). This implies the assertion (1) of Theorem 1.1.

Next we assume that \( 2k \equiv 2 \mod 4 \). The argument similar to the above proves the assertion (2) of Theorem 1.1. This completes the proof. \( \square \)

\textbf{Proof of Theorem 1.2.} In Theorem 1.1 we proved that 
\[
\{ G_{2k} \} \cup \{ G_{4i}G_{2k-4i} \mid i = 1, \ldots, d_k \}
\]
is a basis for \( M_{2k} \) and, in particular, the members are linearly independent. Hence \( \{ G_{2k} \} \cup \{ G_{4i}G_{2k-4i} + \frac{B_{4i}}{4i} \cdot \frac{B_{2k-4i}}{2k-4i} \cdot \frac{B_{2k}}{2k} \cdot G_{2k} \mid i = 1, \ldots, d_k \} \) are linearly independent. This implies \( \{ G_{4i}G_{2k-4i} + \frac{B_{4i}}{4i} \cdot \frac{B_{2k-4i}}{2k-4i} \cdot \frac{B_{2k}}{2k} \cdot G_{2k} \mid i = 1, \ldots, d_k \} \) are again linearly independent. Moreover, since \( G_{4i}G_{2k-4i} + \frac{B_{4i}}{4i} \cdot \frac{B_{2k-4i}}{2k-4i} \cdot \frac{B_{2k}}{2k} \cdot G_{2k} \in S_{2k} \) \( (i = 1, \ldots, d_k) \), these form a basis for \( S_{2k} \). This completes the proof. \( \square \)

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