SUPERINTEGRABLE HAMILTONIAN SYSTEMS WITH NONCOMPACT INVARIANT SUBMANIFOLDS. KEPLER SYSTEM

G. SARDANASHVILY

Department of Theoretical Physics, Physics Faculty, Moscow State University
117234 Moscow, Russia

The Mishchenko–Fomenko theorem on superintegrable Hamiltonian systems is generalized to superintegrable Hamiltonian systems with noncompact invariant submanifolds. It is formulated in the case of globally superintegrable Hamiltonian systems which admit global generalized action-angle coordinates. The well known Kepler system falls into two different globally superintegrable systems with compact and noncompact invariant submanifolds.

1 Introduction

Let $(Z, \Omega)$ be a $2n$-dimensional connected symplectic manifold. Given a superintegrable system

$$F = (F_1, \ldots, F_k), \quad n \leq k < 2n,$$

(1.1)
on $(Z, \Omega)$ (Definition 2.1), the well known Mishchenko – Fomenko theorem (Theorem 2.6) states the existence of (semi-local) generalized action-angle coordinates around its connected compact invariant submanifold [2, 4, 18]. If $k = n$, this is the case of completely integrable systems (Definition 2.2).

The Mishchenko – Fomenko theorem has been extended to superintegrable systems with noncompact invariant submanifolds (Theorem 2.5) [6]. These submanifolds are diffeomorphic to a toroidal cylinder

$$\mathbb{R}^{m-r} \times T^r, \quad m = 2n - k, \quad 0 \leq r \leq m.$$

(1.2)

Partially and completely integrable systems with noncompact invariant submanifolds were studied in [6, 10, 24]. Our goal here is the following.

We formulate Theorem 2.5 in the case of globally superintegrable Hamiltonian systems, which admit global generalized action-angle coordinates (Definition 4.1, Theorem 4.4). Here-with, Theorem 4.2 establishes the sufficient condition of the existence of global generalized action-angle coordinates [7] (see [3, 4] for the case of compact invariant submanifolds).

Note that the Mishchenko – Fomenko theorem is mainly applied to superintegrable systems whose integrals of motion form a compact Lie algebra. The group generated by flows of their Hamiltonian vector fields is compact. Since a fibration of a compact manifold possesses compact fibers, invariant submanifolds of such a superintegrable system are compact. With Theorems 2.5 and 4.4, one can describe superintegrable Hamiltonian system with an arbitrary Lie algebra of integrals of motion.
It may happen that a Hamiltonian system falls into different superintegrable Hamiltonian systems on different open subsets of a symplectic manifold. This is just the case of the Kepler system considered in Section 7. It contains two different globally superintegrable systems on different open subsets of a phase space $\mathbb{R}^4$. Their integrals of motion form the Lie algebras $so(3)$ and $so(2, 1)$ with compact and non-compact invariant submanifolds, respectively.

2 The Mishchenko – Fomenko theorem in a general setting

Throughout the paper, all functions and maps are smooth, and manifolds are real smooth and paracompact. We are not concerned with the real-analytic case because a paracompact real-analytic manifold admits the partition of unity by smooth functions. As a consequence, sheaves of modules over real-analytic functions need not be acyclic that is essential for our consideration.

Definition 2.1. Let $(Z, \Omega)$ be a $2n$-dimensional connected symplectic manifold, and let $(C^\infty(Z), \{,\})$ be the Poisson algebra of smooth real functions on $Z$. A subset $F$ (1.1) of the Poisson algebra $C^\infty(Z)$ is called a superintegrable system if the following conditions hold.

(i) All the functions $F_i$ (called the generating functions of a superintegrable system) are independent, i.e., the $k$-form $\wedge^k dF_i$ nowhere vanishes on $Z$. It follows that the map $F : Z \to \mathbb{R}^k$ is a submersion, i.e.,

$$F : Z \to N = F(Z)$$

(2.1)

is a fibered manifold over a domain (i.e., contractible open subset) $N \subset \mathbb{R}^k$ endowed with the coordinates $(x_i)$ such that $x_i \circ F = F_i$.

(ii) There exist smooth real functions $s_{ij}$ on $N$ such that

$$\{F_i, F_j\} = s_{ij} \circ F, \quad i, j = 1, \ldots, k.$$  \hfill (2.2)

(iii) The matrix function $s$ with the entries $s_{ij}$ (2.2) is of constant corank $m = 2n - k$ at all points of $N$.

If $k = n$, then $s = 0$, and we are in the case of completely integrable systems as follows.

Definition 2.2. The subset $F$, $k = n$, (1.1) of the Poisson algebra $C^\infty(Z)$ on a symplectic manifold $(Z, \Omega)$ is called a completely integrable system if $F_i$ are independent functions in involution.

If $k > n$, the matrix $s$ is necessarily nonzero. Therefore, superintegrable systems also are called noncommutative completely integrable systems. If $k = 2n - 1$, a superintegrable system is called maximally superintegrable.

The following two assertions clarify the structure of superintegrable systems [4, 6].

Proposition 2.3. Given a symplectic manifold $(Z, \Omega)$, let $F : Z \to N$ be a fibered manifold such that, for any two functions $f, f'$ constant on fibers of $F$, their Poisson bracket $\{f, f'\}$ is so. Then $N$ is provided with an unique coinduced Poisson structure $\{,\}_N$ such that $F$ is a Poisson morphism [23].
Since any function constant on fibers of $F$ is a pull-back of some function on $N$, the superintegrable system (1.1) satisfies the condition of Proposition 2.3 due to item (ii) of Definition 2.1. Thus, the base $N$ of the fibration (2.1) is endowed with a coinduced Poisson structure of corank $m$. With respect to coordinates $x_i$ in item (i) of Definition 2.1 its bivector field reads

$$w = s_{ij}(x_k)\partial^i \wedge \partial^j.$$  

(2.3)

Proposition 2.4. Given a fibered manifold $F: Z \to N$ in Proposition 2.3, the following conditions are equivalent [4, 15]:

(i) the rank of the coinduced Poisson structure $\{.,\}_N$ on $N$ equals $2\dim N - \dim Z$,

(ii) the fibers of $F$ are isotropic,

(iii) the fibers of $F$ are maximal integral manifolds of the involutive distribution spanned by the Hamiltonian vector fields of the pull-back $F^*C$ of Casimir functions $C$ of the coinduced Poisson structure (2.3) on $N$.

It is readily observed that the fibered manifold $F$ (2.1) obeys condition (i) of Proposition 2.4 due to item (iii) of Definition 2.1, namely, $k - m = 2(k - n)$.

Fibers of the fibered manifold $F$ (2.1) are called the invariant submanifolds.

Remark 2.1. In many physical models, condition (i) of Definition 2.1 fails to hold. Often, it is replaced with that a subset $Z_R \subset Z$ of regular points (where $\wedge dF_i \neq 0$) is open and dense. Let $M$ be an invariant submanifold through a regular point $z \in Z_R \subset Z$. Then it is regular, i.e., $M \subset Z_R$. Let $M$ admit a regular open saturated neighborhood $U_M$ (i.e., a fiber of $F$ through a point of $U_M$ belongs to $U_M$). For instance, any compact invariant submanifold $M$ has such a neighborhood $U_M$. The restriction of functions $F_i$ to $U_M$ defines a superintegrable system on $U_M$ which obeys Definition 2.1. In this case, one says that a superintegrable system is considered around its invariant submanifold $M$.

Given a superintegrable system in accordance with Definition 2.1, the above mentioned generalization of the Mishchenko – Fomenko theorem to noncompact invariant submanifolds states the following [6].

Theorem 2.5. Let the Hamiltonian vector fields $\vartheta_i$ of the functions $F_i$ be complete, and let the fibers of the fibered manifold $F$ (2.1) be connected and mutually diffeomorphic. Then the following hold.

(I) The fibers of $F$ (2.1) are diffeomorphic to a toroidal cylinder (1.2).

(II) Given a fiber $M$ of $F$ (2.1), there exists its open saturated neighborhood $U$ which is a trivial principal bundle with the structure group (1.2).

(III) The neighborhood $U$ is provided with the bundle (generalized action-angle) coordinates $(I_{\lambda}, p_s, q^s, y^\lambda)$, $\lambda = 1, \ldots, m$, $s = 1, \ldots, n - m$, such that (i) the action coordinates $(I_{\lambda})$ are values of Casimir functions of the coinduced Poisson structure $\{.,\}_N$ on $F(U)$, (ii) the generalized angle coordinates $(y^\lambda)$ are coordinates on a toroidal cylinder, and (iii) the symplectic form $\Omega$ on $U$ reads

$$\Omega = dI_\lambda \wedge dy^\lambda + dp_s \wedge dq^s.$$  

(2.4)
Proof. It follows from item (iii) of Proposition 2.4 that every fiber $M$ of the fibered manifold (2.1) is a maximal integral manifold of the involutive distribution spanned by the Hamiltonian vector fields $\upsilon_\lambda$ of the pull-back $F^*C_\lambda$ of $m$ independent Casimir functions $\{C_1, \ldots, C_m\}$ of the Poisson structure $\{,\}_N$ (2.3) on an open neighborhood $N_M$ of a point $F(M) \in N$. Let us put $U_M = F^{-1}(N_M)$. It is an open saturated neighborhood of $M$. Consequently, invariant submanifolds of a superintegrable system (1.1) on $U_M$ are maximal integral manifolds of the partially integrable system

$$C^* = (F^*C_1, \ldots, F^*C_m), \quad 0 < m \leq n, \quad (2.5)$$

on a symplectic manifold $(U_M, \Omega)$. Therefore, statements (I) – (III) of Theorem 2.5 are the corollaries of forthcoming Theorem 2.8. Its condition (i) is satisfied as follows. Let $M'$ be an arbitrary fiber of the fibered manifold $F: U_M \to N_M$ (2.1). Since

$$F^*\upsilon_\lambda(z) = (C_\lambda \circ F)(z) = C_\lambda(F_i(z)), \quad z \in M', \quad (2.6)$$

the Hamiltonian vector fields $\upsilon_\lambda$ on $M'$ are $\mathbb{R}$-linear combinations of Hamiltonian vector fields $\vartheta_i$ of the functions $F_i$. It follows that $\upsilon_\lambda$ are elements of a finite-dimensional real Lie algebra of vector fields on $M'$ generated by the vector fields $\vartheta_i$. Since vector fields $\vartheta_i$ are complete, the vector fields $\upsilon_\lambda$ on $M'$ also are complete (see forthcoming Remark 2.2). Consequently, these vector fields are complete on $U_M$ because they are vertical vector fields on $U_M \to N$. The proof of Theorem 2.8 shows that the action coordinates $(I_\lambda)$ are values of Casimir functions expressed into the original ones $C_\lambda$. □

Remark 2.2. If complete vector fields on a smooth manifold constitute a basis for a finite-dimensional real Lie algebra, any element of this Lie algebra is complete [21].

Remark 2.3. Since an open neighborhood $U$ in item (II) of Theorem 2.5 is not contractible, unless $r = 0$, the generalized action-angle coordinates on $U$ sometimes are called semi-local.

Remark 2.4. The condition of the completeness of Hamiltonian vector fields of the generating functions $F_i$ in Theorem 2.5 is rather restrictive (see the Kepler system in Section 7). One can replace this condition with that the Hamiltonian vector fields of the pull-back onto $Z$ of Casimir functions on $N$ are complete.

If the conditions of Theorem 2.5 are replaced with that the fibers of the fibered manifold $F$ (2.1) are compact and connected, this theorem restarts the Mishchenko – Fomenko one as follows.

Theorem 2.6. Let the fibers of the fibered manifold $F$ (2.1) be connected and compact. Then they are diffeomorphic to a torus $T^m$, and statements (II) – (III) of Theorem 2.5 hold.

Remark 2.5. In Theorem 2.6, the Hamiltonian vector fields $\upsilon_\lambda$ are complete because fibers of the fibered manifold $F$ (2.1) are compact. As well known, any vector field on a compact manifold is complete.
If $F (1.1)$ is a completely integrable system, the coinduced Poisson structure on $N$ equals zero, and the generating functions $F_i$ are the pull-back of $n$ independent functions on $N$. Then Theorems 2.6 and 2.5 come to the Liouville–Arnold theorem [1, 14] and its generalization (Theorem 2.7) to the case of noncompact invariant submanifolds [5, 11], respectively. In this case, the partially integrable system $C^* (2.5)$ is exactly the original completely integrable system $F$.

**Theorem 2.7.** Given a completely integrable system $F$ in accordance with Definition 2.2, let the Hamiltonian vector fields $\vartheta_i$ of the functions $F_i$ be complete, and let the fibers of the fibered manifold $F (2.1)$ be connected and mutually diffeomorphic. Then items (I) and (II) of Theorem 2.5 hold, and its item (III) is replaced with the following one.

(Ill') The neighborhood $U$ is provided with the bundle (generalized action-angle) coordinates $(I_\lambda, y^\lambda), \lambda = 1, \ldots, n$, such that the angle coordinates $(y^\lambda)$ are coordinates on a toroidal cylinder, and the symplectic form $\Omega$ on $U$ reads

$$\Omega = dI_\lambda \wedge dy^\lambda. \quad (2.7)$$

Turn now to above mentioned Theorem 2.8. Recall that a collection $\{S_1, \ldots, S_m\}$ of $m \leq n$ independent smooth real functions in involution on a symplectic manifold $(Z, \Omega)$ is called a partially integrable system. Let us consider the map

$$S : Z \to W \subset \mathbb{R}^m. \quad (2.8)$$

Since functions $S_\lambda$ are everywhere independent, this map is a submersion onto a domain $W \subset \mathbb{R}^m$, i.e., $S (2.8)$ is a fibered manifold of fiber dimension $2n - m$. Hamiltonian vector fields $v_\lambda$ of functions $S_\lambda$ are mutually commutative and independent. Consequently, they span an $m$-dimensional involutive distribution on $Z$ whose maximal integral manifolds constitute an isotropic foliation $F$ of $Z$. Because functions $S_\lambda$ are constant on leaves of this foliation, each fiber of a fibered manifold $Z \to W (2.8)$ is foliated by the leaves of the foliation $F$. If $m = n$, we are in the case of a completely integrable system, and leaves of $F$ are connected components of fibers of the fibered manifold (2.8). The Poincaré–Lyapounov–Nekhoroshev theorem [8, 19] generalizes the Liouville–Arnold one to a partially integrable system if leaves of the foliation $F$ are compact. It imposes a sufficient condition which Hamiltonian vector fields $v_\lambda$ must satisfy in order that the foliation $F$ is a fibered manifold [8, 9]. Extending the Poincaré–Lyapounov–Nekhoroshev theorem to the case of noncompact invariant submanifolds, we in fact assume from the beginning that these submanifolds form a fibered manifold [10, 11].

**Theorem 2.8.** Let a partially integrable system $\{S_1, \ldots, S_m\}$ on a symplectic manifold $(Z, \Omega)$ satisfy the following conditions.

(i) The Hamiltonian vector fields $v_\lambda$ of $S_\lambda$ are complete.

(ii) The foliation $F$ is a fibered manifold

$$\pi : Z \to N \quad (2.9)$$

whose fibers are mutually diffeomorphic.
Then the following hold.

(I) The fibers of $F$ are diffeomorphic to a toroidal cylinder (1.2).

(II) Given a fiber $M$ of $F$, there exists its open saturated neighborhood $U$ which is a trivial principal bundle with the structure group (1.2).

(III) The neighborhood $U$ is provided with the bundle (generalized action-angle) coordinates

$$(I_\lambda, p_s, q^s, y^\lambda) \rightarrow (I_\lambda, p_s, q^s), \quad \lambda = 1, \ldots, m, \quad s = 1, \ldots, n - m,$$

such that: (i) the action coordinates $(I_\lambda)$ (3.13) are expressed into the values of the functions $(S_\lambda)$, (ii) the angle coordinates $(y^\lambda)$ (3.13) are coordinates on a toroidal cylinder, and (iii) the symplectic form $\Omega$ on $U$ reads

$$\Omega = dI_\lambda \wedge dy^\lambda + dp_s \wedge dq^s. \quad (2.10)$$

**Proof.** See Section 3 for the proof. □

If one supposes from the beginning that leaves of the foliation $F$ are compact, the conditions of Theorem 2.8 can be replaced with that $F$ is a fibered manifold due to the following.

**Proposition 2.9.** Any fibered manifold whose fibers are diffeomorphic either to $\mathbb{R}^r$ or a connected compact manifold $K$ is a fiber bundle [17].

### 3 Proof of Theorem 2.8

(I) In accordance with the well-known theorem [20, 21], complete Hamiltonian vector fields $v_\lambda$ define an action of a simply connected Lie group $G$ on $Z$. Because vector fields $v_\lambda$ are mutually commutative, it is the additive group $\mathbb{R}^m$ whose group space is coordinated by parameters $s^\lambda$ of the flows with respect to the basis $\{e_\lambda = v_\lambda\}$ for its Lie algebra. The orbits of the group $\mathbb{R}^m$ in $Z$ coincide with the fibers of a fibered manifold $F$ (2.9). Since vector fields $v_\lambda$ are independent everywhere on $U$, the action of $\mathbb{R}^m$ in $Z$ is locally free, i.e., isotropy groups of points of $Z$ are discrete subgroups of the group $\mathbb{R}^m$. Given a point $x \in \pi(U)$, the action of $\mathbb{R}^m$ in the fiber $M_x = \pi^{-1}(x)$ factorizes as

$$\mathbb{R}^m \times M_x \rightarrow G_x \times M_x \rightarrow M_x \quad (3.1)$$

through the free transitive action in $M_x$ of the factor group $G_x = \mathbb{R}^m/K_x$, where $K_x$ is the isotropy group of an arbitrary point of $M_x$. It is the same group for all points of $M_x$ because $\mathbb{R}^m$ is a commutative group. Clearly, $M_x$ is diffeomorphic to the group space of $G_x$. Since the fibers $M_x$ are mutually diffeomorphic, all isotropy groups $K_x$ are isomorphic to the group $\mathbb{Z}_r$ for some fixed $0 \leq r \leq m$. Accordingly, the groups $G_x$ are isomorphic to the additive group $\mathbb{R}^{m-r} \times \mathbb{T}^r$. This proves statement (I) of Theorem 2.8.

(II) Because $F$ is a fibered manifold, one can always choose an open fibered neighborhood $U$ of its fiber $M$ such that $\pi(U)$ is a domain and a fibered manifold

$$\pi : U \rightarrow \pi(U) \subset N \quad (3.2)$$
admits a section $\sigma$. Let us bring the fibered manifold (3.2) into a principal bundle with the structure group $G_0$, where we denote $\{0\} = \pi(M)$. For this purpose, let us determine isomorphisms $\rho_x : G_0 \to G_x$ of the group $G_0$ to the groups $G_x, x \in \pi(U)$. Then a desired fiberwise action of $G_0$ in $U$ is defined by the law

$$G_0 \times M_x \to \rho_x(G_0) \times M_x \to M_x.$$ (3.3)

Generators of each isotropy subgroup $K_x$ of $\mathbb{R}^m$ are given by $r$ linearly independent vectors of the group space $\mathbb{R}^m$. One can show that there exist ordered collections of generators $(v_1(x), \ldots, v_r(x))$ of the groups $K_x$ such that $x \mapsto v_i(x)$ are smooth $\mathbb{R}^m$-valued fields on $\pi(U)$. Indeed, given a vector $v_i(0)$ and a section $\sigma$ of the fibered manifold (3.2), each field $v_i(x) = (s^a(x))$ is the unique smooth solution of the equation

$$g(s^a)\sigma(x) = \sigma(x), \quad (s^a(0)) = v_i(0),$$
on on an open neighborhood of $\{0\}$. Let us consider the decomposition

$$v_i(0) = B^a_i(0)e_a + C^j_i(0)e_j, \quad a = 1, \ldots, m-r, \quad j = 1, \ldots, r,$$

where $C^j_i(0)$ is a non-degenerate matrix. Since the fields $v_i(x)$ are smooth, there exists an open neighborhood of $\{0\}$, say $\pi(U)$ again, where the matrices $C^j_i(x)$ are non-degenerate. Then

$$A_x = \begin{pmatrix} \text{Id} & (B(x) - B(0))C^{-1}(0) \\ 0 & C(x)C^{-1}(0) \end{pmatrix}$$ (3.4)

is a unique linear morphism of the vector space $\mathbb{R}^m$ which transforms the frame $v_\lambda(0) = \{e_a, v_i(0)\}$ into the frame $v_\lambda(x) = \{e_a, v_i(x)\}$. Since it also is an automorphism of the group $\mathbb{R}^m$ sending $K_0$ onto $K_x$, we obtain a desired isomorphism $\rho_x$ of the group $G_0$ to the group $G_x$. Let an element $g$ of the group $G_0$ be the coset of an element $g(s^a)$ of the group $\mathbb{R}^m$. Then it acts in $M_x$ by the rule (3.3) just as the element $g((A_x^{-1})^{\lambda}_{\beta}s^\beta)$ of the group $\mathbb{R}^m$ does. Since entries of the matrix $A_x$ (3.4) are smooth functions on $\pi(U)$, this action of the group $G_0$ in $U$ is smooth. It is free, and $U/G_0 = \pi(U)$. Then the fibered manifold $U \to \pi(U)$ is a trivial principal bundle with the structure group $G_0$.

(III) Given a section $\sigma$ of the principal bundle $U \to \pi(U)$, its trivialization $U = \pi(U) \times G_0$ is defined by assigning the points $\rho^{-1}(g_x)$ of the group space $G_0$ to the points $g_x\sigma(x), g_x \in G_x$, of a fiber $M_x$. Let us endow $G_0$ with the standard coordinate atlas $(r^\lambda) = (t^a, \varphi^i)$ of the group $\mathbb{R}^{m-r} \times T^r$. Then we provide $U$ with the trivialization

$$U = \pi(U) \times (\mathbb{R}^{m-r} \times T^r) \to \pi(U)$$ (3.5)

with respect to the fiber coordinates $(t^a, \varphi^i)$. The vector fields $v_\lambda$ on $U$ relative to these coordinates read

$$v_a = \partial_a, \quad v_i = -(BC^{-1})^a_i(x)\partial_a + (C^{-1})^k_i(x)\partial_k.$$ (3.6)

In order to specify coordinates on the base $\pi(U)$ of the trivial bundle (3.5), let us consider the fibered manifold $S$ (2.8). It factorizes as

$$S : U \xrightarrow{\pi} \pi(U) \xrightarrow{\pi'} S(U), \quad \pi' = S \circ \sigma,$$
through the fiber bundle $\pi$. The map $\pi'$ also is a fibered manifold. One can always restrict the domain $\pi(U)$ to a chart of the fibered manifold $\pi'$, say $\pi(U)$ again. Then $\pi(U) \to S(U)$ is a trivial bundle $\pi(U) = S(U) \times V$, and so is $U \to S(U)$. Thus, we have the composite bundle

$$U = S(U) \times V \times (\mathbb{R}^{m-r} \times T^r) \to S(U) \times V \to S(U).$$

Let us provide its base $S(U)$ with the coordinates $(J_\lambda)$ such that

$$J_\lambda \circ S = S_\lambda.$$  

(3.8)

Then $\pi(U)$ can be equipped with the bundle coordinates $(J_\lambda, x^A)$, $A = 1, \ldots, 2(n - m)$, and $(J_\lambda, x^A, t^a, \varphi^i)$ are coordinates on $U$ (3.7). Since fibers of $U \to \pi(U)$ are isotropic, a symplectic form $\Omega$ on $U$ relative to the coordinates $(J_\lambda, x^A, r^\lambda)$ reads

$$\Omega = \Omega^{\alpha\beta} dJ_{\alpha} \wedge dJ_{\beta} + \Omega_{\beta} dJ_{\alpha} \wedge dr^\beta + \Omega_{AB} dx^A \wedge dx^B + \Omega_{\lambda} dJ_{\lambda} \wedge dx^A + \Omega_{\lambda\beta} dx^A \wedge dr^\beta. \quad (3.9)$$

The Hamiltonian vector fields $v_\lambda = v_\lambda^\mu \partial_\mu$ (3.6) obey the relations $v_\lambda|\Omega = -dJ_\lambda$ which result in the coordinate conditions

$$\Omega^{\alpha\beta}_\lambda v_\lambda^\beta = \delta^\alpha_\lambda, \quad \Omega_{AB} v_\lambda^\beta = 0. \quad (3.10)$$

The first of them shows that $\Omega^{\alpha\beta}_\lambda$ is a non-degenerate matrix independent of coordinates $y^\lambda$. Then the second one implies that $\Omega_{AB} = 0$.

By virtue of the well-known Künneth formula for the de Rham cohomology of manifold products, the closed form $\Omega$ (3.9) is exact, i.e., $\Omega = d\Xi$ where the Liouville form $\Xi$ is

$$\Xi = \Xi^\alpha(J_\lambda, x^B, r^\lambda) dJ_{\alpha} + \Xi_i(J_\lambda, x^B) d\varphi^i + \Xi_A(J_\lambda, x^B, r^\lambda) dx^A.$$ 

Since $\Xi_\alpha = 0$ and $\Xi_i$ are independent of $\varphi^i$, it follows from the relations

$$\Omega_{AB} = \partial_A \Xi_B - \partial_B \Xi_A = 0$$

that $\Xi_A$ are independent of coordinates $t^a$ and are at most affine in $\varphi^i$. Since $\varphi^i$ are cyclic coordinates, $\Xi_A$ are independent of $\varphi^i$. Hence, $\Xi_i$ are independent of coordinates $x^A$, and the Liouville form reads

$$\Xi = \Xi^\alpha(J_\lambda, x^B, r^\lambda) dJ_{\alpha} + \Xi_i(J_\lambda) d\varphi^i + \Xi_A(J_\lambda, x^B) dx^A. \quad (3.11)$$

Because entries $\Omega^{\alpha\beta}_\lambda$ of $d\Xi = \Omega$ are independent of $r^\lambda$, we obtain the following.

(i) $\Omega^\lambda_i = \partial^\lambda \Xi_i - \partial_i \Xi^\lambda$. Consequently, $\partial_\alpha \Xi^\lambda$ are independent of $\varphi^i$, and so are $\Xi^\lambda$ since $\varphi^i$ are cyclic coordinates. Hence, $\Omega^\lambda_i = \partial^\lambda \Xi_i$ and $\partial_i|\Omega = -d\Xi_i$. A glance at the last equality shows that $\partial_i$ are Hamiltonian vector fields. It follows that, from the beginning, one can separate $r$ generating functions on $U$, say $S_i$ again, whose Hamiltonian vector fields are tangent to invariant tori. In this case, the matrix $B$ in the expressions (3.4) and (3.6) vanishes, and the Hamiltonian vector fields $v_\lambda$ (3.6) read

$$v_a = \partial_a, \quad v_i = (C^{-1})^k_i \partial_k. \quad (3.12)$$
Moreover, the coordinates \( t^a \) are exactly the flow parameters \( s^a \). Substituting the expressions (3.12) into the first condition (3.10), we obtain
\[
\Omega = \Omega_\beta \alpha dJ_\beta \wedge dJ_\alpha + C_i^a dJ_i \wedge d\varphi^k + \Omega_{AB} dx^A \wedge dx^B + \Omega_\lambda^A dJ_\lambda \wedge dx^A.
\]
It follows that \( \Xi_i \) are independent of \( J_a \), and so are \( C_i^k = \partial_i \Xi_k \).

(ii) \( \Omega^\lambda_a = -\partial_a \Xi^\lambda = \delta^\lambda_a \). Hence, \( \Xi^a = -s^a + E^a(J_\lambda) \) and \( \Xi^i = E^i(J_\lambda, x^B) \) are independent of \( s^a \).

In view of items (i)–(ii), the Liouville form \( \Xi \) (3.11) reads
\[
\Xi = (-s^a + E^a(J_\lambda, x^B)) dJ_a + E^i(J_\lambda, x^B) dJ_i + \Xi^i(J_j) d\varphi^j + \Xi_A(J_\lambda, x^B) dx^A.
\]

Since the matrix \( \partial^k \Xi_i \) is non-degenerate, we can perform the coordinate transformations
\[
I_a = J_a, \quad I_i = \Xi_i(J_j), \quad r^a = -s^a + E^a(J_\lambda, x^B), \quad r^i = \varphi^i - E^j(J_\lambda, x^B) \frac{\partial J_j}{\partial I_i}.
\]

These transformations bring \( \Omega \) into the form
\[
\Omega = dI_\lambda \wedge dr^\lambda + \Omega_{AB}(I_\mu, x^C) dx^A \wedge dx^B + \Omega_\lambda^A(I_\mu, x^C) dI_\lambda \wedge dx^A.
\]

Since functions \( I_\lambda \) are in involution and their Hamiltonian vector fields \( \partial_\lambda \) mutually commute, a point \( z \in M \) has an open neighbourhood \( U_z = \pi(U_z) \times O_z, O_z \subset \mathbb{R}^{m-r} \times T^r \), endowed with local Darboux coordinates \((I_\lambda, p_s, q_s, y^\lambda), \) \( s = 1, \ldots, n-m \), such that the symplectic form \( \Omega \) (3.14) is given by the expression
\[
\Omega = dI_\lambda \wedge dy^\lambda + dp_s \wedge dq_s.
\]

Here, \( y^\lambda(I_\lambda, x^A, r^{\alpha}) \) are local functions
\[
y^\lambda = r^{\alpha} + f^\lambda(I_\lambda, x^A)
\]
on \( U_z \). With the group \( G \), one can extend these functions to the open neighborhood
\[
\pi(U_z) \times \mathbb{R}^{k-m} \times T^m
\]of \( M \), say \( U \) again, by the law
\[
y^\lambda(I_\lambda, x^A, G(z)^\alpha) = G(z)^\lambda + f^\lambda(I_\lambda, x^A).
\]

Substituting the functions (3.16) on \( U \) into the expression (3.14), one brings the symplectic form \( \Omega \) into the canonical form (3.15) on \( U \).

4 **Globally superintegrable systems**

To study a superintegrable system, one conventionally considers it with respect to generalized action-angle coordinates. A problem is that, restricted to an action-angle coordinate chart on an open subbundle \( U \) of the fibered manifold \( Z \rightarrow N \) (2.1), a superintegrable system becomes different from the original one since there is no morphism of the Poisson algebra.
\( C^\infty(U) \) on \((U, \Omega)\) to that \( C^\infty(Z) \) on \((Z, \Omega)\). Moreover, a superintegrable system on \( U \) need not satisfy the conditions of Theorem 2.5 because it may happen that the Hamiltonian vector fields of the generating functions on \( U \) are not complete. To describe superintegrable systems in terms of generalized action-angle coordinates, we therefore follow the notion of a globally superintegrable system.

**Definition 4.1.** A superintegrable systems \( F \) (1.1) on a symplectic manifold \((Z, \Omega)\) by Definition 2.1 is called globally superintegrable if there exist global generalized action-angle coordinates

\[
(I_\lambda, x^A, y^\lambda), \quad \lambda = 1, \ldots, m, \quad A = 1, \ldots, 2(n - m),
\]

such that: (i) the action coordinates \( I_\lambda \) are expressed into the values of some Casimir functions \( C_\lambda \) on the Poisson manifold \((N, \{, \}_N)\), (ii) the angle coordinates \( y^\lambda \) are coordinates on the toroidal cylinder (1.2), and (iii) the symplectic form \( \Omega \) on \( Z \) reads

\[
\Omega = dI_\lambda \wedge dy^\lambda + \Omega_{AB}(I_\mu, x^C)dx^A \wedge dx^B.
\]

It is readily observed that the semi-local generalized action-angle coordinates on \( U \) in Theorem 2.5 are global in accordance with Definition 4.1.

Forthcoming Theorem 4.2 provides the sufficient conditions of the existence of global generalized action-angle coordinates of a superintegrable system on a symplectic manifold \((Z, \Omega)\) [7].

**Theorem 4.2.** A superintegrable system \( F \) on a symplectic manifold \((Z, \Omega)\) is globally superintegrable if the following conditions hold.

(i) Hamiltonian vector fields \( \vartheta_i \) of the generating functions \( F_i \) are complete.

(ii) The fibered manifold \( F \) (2.1) is a fiber bundle with connected fibers.

(iii) Its base \( N \) is simply connected and the cohomology \( H^2(N, \mathbb{Z}) \) is trivial

(iv) The coinduced Poisson structure \( \{, \}_N \) on a base \( N \) admits \( m \) independent Casimir functions \( C_\lambda \).

**Proof.** Theorem 4.2 is a corollary of Theorem 4.3 below which is a global generalization of Theorem 2.8. In accordance with Theorem 4.3, we have a composite fibered manifold

\[
Z \xrightarrow{F} N \xrightarrow{C} W,
\]

where \( C : N \to W \) is a fibered manifold of level surfaces of the Casimir functions \( C_\lambda \) (which coincides with the symplectic foliation of a Poisson manifold \( N \)). The composite fibered manifold (4.3) is provided with the adapted fibered coordinates \((J_\lambda, x^A, r^\lambda)\) (5.4), where \( J_\lambda \) are values of independent Casimir functions and \((r^\lambda) = (t^a, \varphi^i)\) are coordinates on a toroidal cylinder. Since \( C_\lambda = J_\lambda \) are Casimir functions on \( N \), the symplectic form \( \Omega \) (5.6) on \( Z \) reads

\[
\Omega = \Omega_\alpha^\beta dJ_\alpha \wedge dr^\beta + \Omega_{\alpha A} dx^\alpha \wedge dx^A + \Omega_{AB} dx^A \wedge dx^B.
\]

In particular, it follows that transition functions of coordinates \( x^A \) on \( N \) are independent of coordinates \( J_\lambda \), i.e., \( C : V \to W \) is a trivial bundle. By virtue of Lemma 5.1 below, the symplectic form (4.4) is exact, i.e., \( \Omega = d\Xi \), where the Liouville form \( \Xi \) (5.7) is

\[
\Xi = \Xi^\lambda J_\alpha dJ_\lambda + \Xi_i J_\alpha d\varphi^i + \Xi_A (x^B) dx^A.
\]
Then the coordinate transformations (5.8):

\[ I_a = J_a, \quad I_i = \Xi_i(J_j), \quad y^a = -\Xi^a = t^a - E^a(J_\lambda), \quad y^i = \varphi^i - \Xi^j(J_\lambda) \frac{\partial J_j}{\partial I_i}, \quad (4.5) \]

bring \( \Omega \) (4.4) into the form (4.2). In comparison with the general case (5.8), the coordinate transformations (4.5) are independent of coordinates \( x^A \). Therefore, the angle coordinates \( y^i \) possess identity transition functions on \( N \). □

Theorem 4.2 restarts Theorem 2.5 if one considers an open subset \( V \) of \( N \) admitting the Darboux coordinates \( x^A \) on the symplectic leaves of \( U \).

Note that, if invariant submanifolds of a superintegrable system are assumed to be connected and compact, condition (i) of Theorem 4.2 is unnecessary since vector fields \( v_\lambda \) on compact fibers of \( F \) are complete. Condition (ii) also holds by virtue of Proposition 2.9. In this case, Theorem 4.2 reproduces the well known result in [3].

If \( F \) in Theorem 4.2 is a completely integrable system, the coinduced Poisson structure on \( N \) equals zero, the generating functions \( F_i \) are the pull-back of \( n \) independent functions on \( N \), and Theorem 4.2 coincides with Theorem 4 in [7].

Turn now to the above mentioned Theorem 4.3.

**Theorem 4.3.** Let a partially integrable system \( \{S_1, \ldots, S_m\} \) on a symplectic manifold \((Z, \Omega)\) satisfy the following conditions.

(i) The Hamiltonian vector fields \( v_\lambda \) of \( S_\lambda \) are complete.

(ii) The foliation \( \mathcal{F} \) is a fiber bundle

\[ \pi : Z \to N. \quad (4.6) \]

(iii) Its base \( N \) is simply connected and the cohomology \( H^2(N, \mathbb{Z}) \) is trivial.

Then the following hold.

(I) The fiber bundle \( \mathcal{F} \) is a trivial principal bundle with the structure group (1.2), and we have a composite fibered manifold

\[ S = \zeta \circ \pi : Z \to N \twoheadrightarrow W, \quad (4.7) \]

where \( N \to W \) however need not be a fiber bundle.

(II) The fibered manifold (4.7) is provided with the fibered generalized action-angle coordinates

\[ (I_\lambda, x^A, y^\lambda) \to (I_\lambda, x^A) \to (I_\lambda), \quad \lambda = 1, \ldots, m, \quad A = 1, \ldots, 2(n-m), \]

such that: (i) the action coordinates \( (I_\lambda) \) (5.8) are expressed into the values of the functions \( (S_\lambda) \) and they possess identity transition functions, (ii) the angle coordinates \( (y^\lambda) \) (5.8) are coordinates on a toroidal cylinder, (iii) the symplectic form \( \Omega \) on \( U \) reads

\[ \Omega = dI_\lambda \wedge dy^\lambda + \Omega^\lambda_A dI_\lambda \wedge dx^A + \Omega_{AB} dx^A \wedge dx^B. \quad (4.8) \]

**Proof.** See Section 5 for the proof. □
It follows from the proof of Theorem 4.3 that its condition (iii) and, accordingly, condition (iii) of Theorem 4.2 guarantee that fiber bundles $F$ in conditions (ii) of these theorems are trivial. Therefore, Theorem 4.2 can be reformulated as follows.

**Theorem 4.4.** A superintegrable system $F$ on a symplectic manifold $(Z, \Omega)$ is globally superintegrable iff the following conditions hold.

(i) The fibered manifold $F$ (2.1) is a trivial fiber bundle.

(ii) The coinduced Poisson structure $\{\cdot, \cdot\}_N$ on a base $N$ admits $m$ independent Casimir functions $C_\lambda$ such that Hamiltonian vector fields of their pull-back $F^*C_\lambda$ are complete.

**Remark 4.1.** It follows from Remark 2.2 and condition (ii) of Theorem 4.4 that a Hamiltonian vector field of the the pull-back $F^*C$ of any Casimir function $C$ on a Poisson manifold $N$ is complete.

### 5 Proof of Theorem 4.3

Following part (I) of the proof of Theorem 2.8, one can show that a typical fiber of the fiber bundle (4.6) is the toroidal cylinder (1.2).

Let us bring the fiber bundle (4.6) into a principal bundle with the structure group (1.2). Generators of each isotropy subgroup $K_x$ of $\mathbb{R}^m$ are given by $r$ linearly independent vectors $u_i(x)$ of the group space $\mathbb{R}^m$. These vectors are assembled into an $r$-fold covering $K \to N$. This is a subbundle of the trivial bundle

$$N \times \mathbb{R}^m \to N \tag{5.1}$$

whose local sections are local smooth sections of the fiber bundle (5.1). Such a section over an open neighborhood of a point $x \in N$ is given by a unique local solution $s^\lambda(x')e_\lambda$ of the equation

$$g(s^\lambda)\sigma(x') = \exp(s^\lambda u_\lambda)\sigma(x') = \sigma(x'), \quad s^\lambda(x)e_\lambda = u_i(x),$$

where $\sigma$ is an arbitrary local section of the fiber bundle $Z \to N$ over an open neighborhood of $x$. Since $N$ is simply connected, the covering $K \to N$ admits $r$ everywhere different global sections $u_i$ which are global smooth sections $u_i(x) = u_i^\lambda(x)e_\lambda$ of the fiber bundle (5.1). Let us fix a point of $N$ further denoted by $\{0\}$. One can determine linear combinations of the functions $S_\lambda$, say again $S_\lambda$, such that $u_i(0) = e_i$, $i = m - r, \ldots, m$, and the group $G_0$ is identified to the group $\mathbb{R}^{m-r} \times T^r$. Let $E_x$ denote the $r$-dimensional subspace of $\mathbb{R}^m$ passing through the points $u_1(x), \ldots, u_r(x)$. The spaces $E_x, x \in N$, constitute an $r$-dimensional subbundle $E \to N$ of the trivial bundle (5.1). Moreover, the latter is split into the Whitney sum of vector bundles $E \oplus E', \text{ where } E'_x = \mathbb{R}^m/E_x$ [13]. Then there is a global smooth section $\gamma$ of the trivial principal bundle $N \times GL(m, \mathbb{R}) \to N$ such that $\gamma(x)$ is a morphism of $E_0$ onto $E_x$, where $u_i(x) = \gamma(x)(e_i) = \gamma_i^\lambda e_\lambda$. This morphism also is an automorphism of the group $\mathbb{R}^m$ sending $K_0$ onto $K_x$. Therefore, it provides a group isomorphism $\rho_x : G_0 \to G_x$. With these isomorphisms, one can define the fiberwise action of the group $G_0$ on $Z$ given by the law

$$G_0 \times M_x \to \rho_x(G_0) \times M_x \to M_x. \tag{5.2}$$
Namely, let an element of the group $G_0$ be the coset $g(s^\lambda)/K_0$ of an element $g(s^\lambda)$ of the group $\mathbb{R}^m$. Then it acts on $M_x$ by the rule (5.2) just as the coset $g((\gamma(x)^{-1})^\lambda\beta s^\beta)/K_x$ of an element $g((\gamma(x)^{-1})^\lambda\beta s^\beta)$ of $\mathbb{R}^m$ does. Since entries of the matrix $\gamma$ are smooth functions on $N$, the action (5.2) of the group $G_0$ on $Z$ is smooth. It is free, and $Z/G_0 = N$. Thus, $Z \rightarrow N$ (4.6) is a principal bundle with the structure group $G_0 = \mathbb{R}^{m-r} \times T^r$.

Furthermore, this principal bundle over a paracompact smooth manifold $N$ is trivial as follows. In accordance with the well-known theorem [13], its structure group $N$ element $g$ (4.6) is a principal bundle with the structure group $G$ are defined as

$$\xi = \Omega = \Omega(\gamma) = \Omega^\lambda s^\lambda$$

reducible to the maximal compact subgroup $T^r$, which also is the maximal compact subgroup of the group product $\times GL(1, \mathbb{C})$. Therefore, the equivalence classes of $T^r$-principal bundles $\xi$ are defined as

$$c(\xi) = c(\xi_1 \oplus \cdots \oplus \xi_r) = (1 + c_1(\xi_1)) \cdots (1 + c_1(\xi_r))$$

by the Chern classes $c_1(\xi_i) \in H^2(N, \mathbb{Z})$ of $U(1)$-principal bundles $\xi_i$ over $N$ [13]. Since the cohomology group $H^2(N, \mathbb{Z})$ of $N$ is trivial, all Chern classes $c_1$ are trivial, and the principal bundle $Z \rightarrow N$ over a contractible base also is trivial. This principal bundle can be provided with the following coordinate atlas.

Let us consider the fibered manifold $S : Z \rightarrow W$ (2.8). Because functions $S_\lambda$ are constant on fibers of the fiber bundle $Z \rightarrow N$ (4.6), the fibered manifold (2.8) factorizes through the fiber bundle (4.6), and we have the composite fibered manifold (4.7). Let us provide the principal bundle $Z \rightarrow N$ with a trivialization

$$Z = N \times \mathbb{R}^{m-r} \times T^r \rightarrow N,$$  

(5.3)

whose fibers are endowed with the standard coordinates $(r^\lambda) = (t^a, \varphi^i)$ on the toroidal cylinder (1.2). Then the composite fibered manifold (4.7) is provided with the fibered coordinates

$$(J_\lambda, x^A, t^a, \varphi^i),$$

(5.4)

\[\lambda = 1, \ldots, m, \quad A = 1, \ldots, 2(n-m), \quad a = 1, \ldots, m-r, \quad i = 1, \ldots, r,\]

where $J_\lambda$ (3.8) are coordinates on the base $W$ induced by Cartesian coordinates on $\mathbb{R}^m$, and $(J_\lambda, x^A)$ are fibered coordinates on the fibered manifold $\zeta : N \rightarrow W$. The coordinates $J_\lambda$ on $W \subset \mathbb{R}^m$ and the coordinates $(t^a, \varphi^i)$ on the trivial bundle (5.3) possess the identity transition functions, while the transition function of coordinates $(x^A)$ depends on the coordinates $(J_\lambda)$ in general.

The Hamiltonian vector fields $v_\lambda$ on $Z$ relative to the coordinates (5.4) take the form

$$v_\lambda = v_\lambda^a(x) \partial_a + v_\lambda^i(x) \partial_i,$$

(5.5)

Since these vector fields commute (i.e., fibers of $Z \rightarrow N$ are isotropic), the symplectic form $\Omega$ on $Z$ reads

$$\Omega = \Omega_\beta^\alpha dJ_\alpha \wedge dr^\beta + \Omega_A d^A \wedge dx^A + \Omega_\alpha^\beta dJ_\alpha \wedge dJ_\beta + \Omega^A_\alpha dJ_\alpha \wedge dx^A + \Omega_{AB} dx^A \wedge dx^B.$$  

(5.6)

**Lemma 5.1.** The symplectic form $\Omega$ (5.6) is exact.
Proof. In accordance with the well-known Künneth formula, the de Rham cohomology group of the product (5.3) reads

\[ H^2(Z) = H^2(N) \oplus H^1(N) \otimes H^1(T^r) \oplus H^2(T^r). \]

By the de Rham theorem [13], the de Rham cohomology \( H^2(N) \) is isomorphic to the cohomology \( H^2(N, \mathbb{R}) \) of \( N \) with coefficients in the constant sheaf \( \mathbb{R} \). It is trivial since \( H^2(N, \mathbb{R}) = H^2(N, \mathbb{Z}) \otimes \mathbb{R} \) where \( H^2(N, \mathbb{Z}) \) is trivial. The first cohomology group \( H^1(N) \) of \( N \) is trivial because \( N \) is simply connected. Consequently, \( H^2(Z) = H^2(T^r) \). Then the closed form \( \Omega \) (5.6) is exact since it does not contain the term \( \Omega_{ij} d\varphi^i \wedge d\varphi^j \).

Thus, we can write

\[ \Omega = d\Xi, \quad \Xi = \Xi^\lambda(J_\alpha, x^B, r^\alpha) dJ_\lambda + \Xi_\lambda(J_\alpha, x^B) dr^\lambda + \Xi_A(J_\alpha, x^B, r^\alpha) dx^A. \]  

(5.7)

Up to an exact summand, the Liouville form \( \Xi \) (5.7) is brought into the form

\[ \Xi = \Xi^\lambda(J_\alpha, x^B, r^\alpha) dJ_\lambda + \Xi_i(J_\alpha, x^B) d\varphi^i + \Xi_A(J_\alpha, x^B, r^\alpha) dx^A, \]

i.e., it does not contain the term \( \Xi_a dt^a \).

The Hamiltonian vector fields \( v_\lambda \) (5.5) obey the relations \( v_\lambda | \Omega = -dJ_\lambda \), which result in the coordinate conditions (3.10). Then following the proof of Theorem 2.8, we can show that a symplectic form \( \Omega \) on \( Z \) is given by the expression (4.8) with respect to the coordinates

\[ I_a = J_a, \quad I_i = \Xi_i(J_j), \quad y^a = -\Xi^a = t^a - E^a(J_\lambda, x^B), \quad y^i = \varphi^i - \Xi^j(J_\lambda, x^B) \frac{\partial J_j}{\partial I_i}. \]

(5.8)

### 6 Superintegrable Hamiltonian systems

In autonomous (symplectic) Hamiltonian mechanics, one considers superintegrable systems whose generating functions are integrals of motion, i.e., they are in involution with a Hamiltonian \( H \), and the functions \((H, F_1, \ldots, F_k)\) are nowhere independent, i.e.,

\[ \{H, F_i\} = 0, \]

(6.1)

\[ dH \wedge (\wedge dF_i) = 0. \]

(6.2)

In order that an evolution of Hamiltonian system can be defined at any instant \( t \in \mathbb{R} \), one supposes that the Hamiltonian vector field of its Hamiltonian is complete. By virtue of Remark 4.1 and forthcoming Proposition 6.1, a Hamiltonian of a superintegrable system always satisfies this condition.

**Proposition 6.1.** It follows from the equality (6.2) that a Hamiltonian \( H \) is constant on the invariant submanifolds. Therefore, it is the pull-back of a function on \( N \) which is a Casimir function of the Poisson structure (2.3) because of the conditions (6.1).

Proposition 6.1 leads to the following.
Proposition 6.2. Let $\mathcal{H}$ be a Hamiltonian of a globally superintegrable system provided with the generalized action-angle coordinates $(I_\lambda, x^A, y^\lambda)$ (4.1). Then a Hamiltonian $\mathcal{H}$ depends only on the action coordinates $I_\lambda$. Consequently, the equations of motion of a globally superintegrable system take the form

$$\dot{y}^\lambda = \frac{\partial \mathcal{H}}{\partial I_\lambda}, \quad I_\lambda = \text{const.,} \quad x^A = \text{const.}$$

Following the original Mishchenko–Fomenko theorem, let us mention superintegrable systems whose generating functions $\{F_1, \ldots, F_k\}$ form a $k$-dimensional real Lie algebra $\mathcal{G}$ of corank $m$ with the commutation relations

$$\{F_i, F_j\} = c^h_{ij} F_h, \quad c^h_{ij} = \text{const.} \quad (6.3)$$

Then $F$ (2.1) is a momentum mapping of $Z$ to the Lie coalgebra $\mathcal{G}^*$ provided with the coordinates $x_i$ in item (i) of Definition 2.1 [11, 12]. In this case, the coinduced Poisson structure $\{,\}_N$ coincides with the canonical Lie–Poisson structure on $\mathcal{G}^*$ given by the Poisson bivector field

$$w = \frac{1}{2} c^h_{ij} x_h \partial^i \wedge \partial^j.$$ 

Let $V$ be an open subset of $\mathcal{G}^*$ such that conditions (i) and (ii) of Theorem 4.4 are satisfied. Then an open subset $F^{-1}(V) \subset Z$ is provided with the generalized action-angle coordinates.

Remark 6.1. Let Hamiltonian vector fields $\vartheta_i$ of the generating functions $F_i$ which form a Lie algebra $\mathcal{G}$ be complete. Then they define a locally free Hamiltonian action on $Z$ of some simply connected Lie group $G$ whose Lie algebra is isomorphic to $\mathcal{G}$ [20, 21]. Orbits of $G$ coincide with $k$-dimensional maximal integral manifolds of the regular distribution $\mathcal{V}$ on $Z$ spanned by Hamiltonian vector fields $\vartheta_i$ [22]. Furthermore, Casimir functions of the Lie–Poisson structure on $\mathcal{G}^*$ are exactly the coadjoint invariant functions on $\mathcal{G}^*$. They are constant on orbits of the coadjoint action of $G$ on $\mathcal{G}^*$ which coincide with leaves of the symplectic foliation of $\mathcal{G}^*$.

Theorem 6.3. Let a globally superintegrable Hamiltonian system on a symplectic manifold $Z$ obey the following conditions.

(i) It is maximally superintegrable.

(ii) Its Hamiltonian $\mathcal{H}$ is regular, i.e., $d\mathcal{H}$ nowhere vanishes.

(iii) Its generating functions $F_i$ constitute a finite dimensional real Lie algebra and their Hamiltonian vector fields are complete.

Then any integral of motion of this Hamiltonian system is the pull-back of a function on a base $N$ of the fibration $F$ (2.1). In other words, it is expressed into the integrals of motion $F_i$.

Proof. The proof is based on the following. A Hamiltonian vector field of a function $f$ on $Z$ lives in the one-codimensional regular distribution $\mathcal{V}$ on $Z$ spanned by Hamiltonian vector fields $\vartheta_i$ iff $f$ is the pull-back of a function on a base $N$ of the fibration $F$ (2.1). A
Hamiltonian $H$ brings $Z$ into a fibered manifold of its level surfaces whose vertical tangent bundle coincide with $\mathcal{V}$. Therefore, a Hamiltonian vector field of any integral of motion of $H$ lives in $\mathcal{V}$. □

It may happen that, given a Hamiltonian $\mathcal{H}$ of a Hamiltonian system on a symplectic manifold $Z$, we have different superintegrable Hamiltonian systems on different open subsets of $Z$. For instance, this is the case of the Kepler system.

## 7 Kepler system

We consider the Kepler system on a plane $\mathbb{R}^2$. Its phase space is $T^*\mathbb{R}^2 = \mathbb{R}^4$ provided with the Cartesian coordinates $(q_i, p_i)$, $i = 1, 2$, and the canonical symplectic form

$$\Omega = \sum_i dp_i \wedge dq_i. \quad (7.1)$$

Let us denote

$$p = (\sum_i (p_i)^2)^{1/2}, \quad r = (\sum_i (q_i^2)^{1/2}, \quad (p, q) = \sum_i p_i q_i.$$

A Hamiltonian of the Kepler system reads

$$H = \frac{1}{2} p^2 - \frac{1}{r}. \quad (7.2)$$

The Kepler system is a Hamiltonian system on a symplectic manifold

$$Z = \mathbb{R}^4 \setminus \{0\} \quad (7.3)$$

endowed with the symplectic form $\Omega$ (7.1).

Let us consider the functions

$$M_{12} = -M_{(21)} = q_1 p_2 - q_2 p_1, \quad (7.4)$$

$$A_i = \sum_j M_{ij} p_j - \frac{q_i}{r} = q_i p^2 - p_j (p, q) - \frac{q_i}{r}, \quad i = 1, 2, \quad (7.5)$$

on the symplectic manifold $Z$ (7.3). It is readily observed that they are integrals of motion of the Hamiltonian $H$ (7.2). One calls $M_{12}$ the angular momentum and $(A_i)$ the Rung–Lenz vector. Let us denote

$$M^2 = (M_{12})^2, \quad A^2 = (A_1)^2 + (A_2)^2 = 2M^2 H + 1. \quad (7.6)$$

Let $Z_0 \subset Z$ be a closed subset of points where $M_{12} = 0$. A direct computation shows that the functions $(M_{12}, A_i)$ (7.4) – (7.5) are independent on an open submanifold

$$U = Z \setminus Z_0. \quad (7.7)$$

of $Z$. At the same time, the functions $(H, M_{12}, A_i)$ are nowhere independent on $U$ because it follows from the expression (7.6) that

$$H = \frac{A^2 - 1}{2M^2} \quad (7.8)$$
on $U$ (7.7). The well known dynamics of the Kepler system shows that the Hamiltonian vector field of its Hamiltonian is complete on $U$ (but not on $Z$).

The Poisson bracket of integrals of motion $M_{12}$ (7.4) and $A_i$ (7.5) obeys the relations

\[
\{ M_{12}, A_i \} = \eta_{2i} A_1 - \eta_{1i} A_2, \tag{7.9}
\]
\[
\{ A_1, A_2 \} = 2HM_{12} = \frac{A^2 - 1}{M_{12}}, \tag{7.10}
\]

where $\eta_{ij}$ is an Euclidean metric on $\mathbb{R}^2$. It is readily observed that these relations take the form (2.2). However, the matrix function $s$ of the relations (7.9) – (7.10) fails to be of constant rank at points where $H = 0$. Therefore, let us consider the open submanifolds $U_- \subset U$ where $H < 0$ and $U_+ \subset U$ where $H > 0$. Then we observe that the Kepler system with the Hamiltonian $H$ (7.2) and the integrals of motion $(M_{ij}, A_i)$ (7.4) – (7.5) on $U_-$ and the Kepler system with the Hamiltonian $H$ (7.2) and the integrals of motion $(M_{ij}, A_i)$ (7.4) – (7.5) on $U_+$ are superintegrable Hamiltonian systems. Moreover, these superintegrable systems can be brought into the form (6.3) as follows.

Let us replace the integrals of motions $A_i$ with the integrals of motion

\[
L_i = \frac{A_i}{\sqrt{-2H}} \tag{7.11}
\]
on $U_-$, and with the integrals of motion

\[
K_i = \frac{A_i}{\sqrt{2H}} \tag{7.12}
\]
on $U_+$.

The superintegrable system $(M_{12}, L_i)$ on $U_-$ obeys the relations

\[
\{ M_{12}, L_i \} = \eta_{2i} L_1 - \eta_{1i} L_2, \tag{7.13}
\]
\[
\{ L_1, L_2 \} = -M_{12}. \tag{7.14}
\]

Let us denote $M_{i3} = -L_i$ and put the indexes $\mu, \nu, \alpha, \beta = 1, 2, 3$. Then the relations (7.13) – (7.14) are brought into the form

\[
\{ M_{\mu\nu}, M_{\alpha\beta} \} = \eta_{\mu\beta} M_{\nu\alpha} + \eta_{\nu\alpha} M_{\mu\beta} - \eta_{\nu\beta} M_{\mu\alpha} - \eta_{\mu\alpha} M_{\nu\beta} \tag{7.15}
\]

where $\eta_{\mu\nu}$ is an Euclidean metric on $\mathbb{R}^3$. A glance at the expression (7.15) shows that the integrals of motion $M_{12}$ (7.4) and $L_i$ (7.11) constitute the Lie algebra $\mathcal{G} = so(3)$. Its corank equals 1. Therefore the superintegrable system $(M_{12}, L_i)$ on $U_-$ is maximally superintegrable. The equality (7.8) takes the form

\[
M^2 + L^2 = -\frac{1}{2H}. \tag{7.16}
\]

The superintegrable system $(M_{12}, K_i)$ on $U_+$ obeys the relations

\[
\{ M_{12}, K_i \} = \eta_{2i} K_1 - \eta_{1i} K_2, \tag{7.17}
\]
\[
\{ K_1, K_2 \} = M_{12}. \tag{7.18}
\]
Let us denote $M_{i3} = -K_i$ and put the indexes $\mu, \nu, \alpha, \beta = 1, 2, 3$. Then the relations (7.17) – (7.18) are brought into the form

$$\{M_{i\mu}, M_{i\nu}\} = \rho_{\mu\nu}M_{i\alpha} + \rho_{\nu\alpha}M_{i\beta} - \rho_{\mu\beta}M_{i\alpha} - \rho_{\nu\alpha}M_{i\beta}$$

(7.19)

where $\rho_{\mu\nu}$ is a pseudo-Euclidean metric of signature $(+, +, -)$ on $\mathbb{R}^3$. A glance at the expression (7.19) shows that the integrals of motion $M_{i12}$ (7.4) and $K_i$ (7.12) constitute the Lie algebra $so(2, 1)$. Its corank equals 1. Therefore the superintegrable system $(M_{i12}, K_i)$ on $U_+$ is maximally superintegrable. The equality (7.8) takes the form

$$K^2 - M^2 = \frac{1}{2H}.$$  

(7.20)

Thus, the Kepler system on a phase space $\mathbb{R}^4$ falls into two different maximally superintegrable systems on open submanifolds $U_-$ and $U_+$ of $\mathbb{R}^4$. We agree to call them the Kepler superintegrable systems on $U_-$ and $U_+$, respectively.

Let us study the first one. Put

$$F_1 = -L_1, \quad F_2 = -L_2, \quad F_3 = -M_{i12},$$

(7.21)

$$\{F_1, F_2\} = F_3, \quad \{F_2, F_3\} = F_1, \quad \{F_3, F_1\} = F_2.$$  

(7.22)

We have the fibered manifold

$$F: U_- \to N \subset G^*,$$  

(7.23)

which is the momentum mapping to the Lie coalgebra $G^* = so(3)^*$, endowed with the coordinates $(x_i)$ such that integrals of motion $F_i$ on $G^*$ read $F_i = x_i$. A base $N$ of the fibered manifold (7.23) is an open submanifold of $G^*$ given by the coordinate condition $x_3 \neq 0$. It is a union of two contractible components defined by the conditions $x_3 > 0$ and $x_3 < 0$. The coinduced Lie–Poisson structure on $N$ takes the form

$$w = x_2 \partial^3 \wedge \partial^1 + x_3 \partial^1 \wedge \partial^2 + x_1 \partial^2 \wedge \partial^3.$$  

(7.24)

The coadjoint action of $so(3)$ on $N$ reads

$$\varepsilon_1 = x_3 \partial^2 - x_2 \partial^3, \quad \varepsilon_2 = x_1 \partial^3 - x_3 \partial^1, \quad \varepsilon_3 = x_2 \partial^1 - x_1 \partial^2.$$  

The orbits of this coadjoint action are given by the equation

$$x_1^2 + x_2^2 + x_3^2 = \text{const}.$$  

They are the level surfaces of the Casimir function

$$C = x_1^2 + x_2^2 + x_3^2$$

and, consequently, the Casimir function

$$h = -\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)^{-1}.$$  

(7.25)
A glance at the expression (7.16) shows that the pull-back $F^*h$ of this Casimir function (7.25) onto $U_-$ is the Hamiltonian $H$ (7.2) of the Kepler system on $U_-$.

As was mentioned above, the Hamiltonian vector field of $F^*h$ is complete. Furthermore, it is known that invariant submanifolds of the superintegrable Kepler system on $U_-$ are compact. Therefore, the fibered manifold $F$ (7.23) is a fiber bundle in accordance with Proposition 2.9. Moreover, this fiber bundle is trivial because $N$ is a disjoint union of two contractible manifolds. Consequently, it follows from Theorem 4.4 that the Kepler superintegrable system on $U_-$ is globally superintegrable, i.e., it admits global generalized action-angle coordinates as follows.

The Poisson manifold $N$ (7.23) can be endowed with the coordinates

$$(I, x_1, \gamma), \quad I < 0, \quad \gamma \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2},$$

defined by the equalities

$$I = -\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)^{-1}, \quad x_2 = (-\frac{1}{2I} - x_1^2)^{1/2} \sin \gamma, \quad x_3 = (-\frac{1}{2I} - x_1^2)^{1/2} \cos \gamma. \quad (7.26)$$

It is readily observed that the coordinates (7.26) are the Darboux coordinates of the Lie–Poisson structure (7.24) on $U_-$, namely,

$$w = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial \gamma}. \quad (7.27)$$

Let $\vartheta_I$ be the Hamiltonian vector field of the Casimir function $I$ (7.26). By virtue of Proposition 2.4, its flows are invariant submanifolds of the Kepler superintegrable system on $U_-$. Let $\alpha$ be a parameter along the flows of this vector field, i.e.,

$$\vartheta_I = \frac{\partial}{\partial \alpha}. \quad (7.28)$$

Then $N$ is provided with the generalized action-angle coordinates $(I, x_1, \gamma, \alpha)$ such that the Poisson bivector associated to the symplectic form $\Omega$ on $N$ reads

$$W = \frac{\partial}{\partial I} \wedge \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial \gamma}. \quad (7.29)$$

Accordingly, Hamiltonian vector fields of integrals of motion $F_i$ (7.21) take the form

$$\vartheta_1 = \frac{\partial}{\partial \gamma},$$

$$\vartheta_2 = \frac{1}{4I^2}(-\frac{1}{2I} - x_1^2)^{-1/2} \sin \gamma \frac{\partial}{\partial \alpha} - x_1(-\frac{1}{2I} - x_1^2)^{-1/2} \sin \gamma \frac{\partial}{\partial \gamma} - \frac{1}{2I} \frac{\partial}{\partial x_1},$$

$$\vartheta_3 = \frac{1}{4I^2}(-\frac{1}{2I} - x_1^2)^{-1/2} \cos \gamma \frac{\partial}{\partial \alpha} - x_1(-\frac{1}{2I} - x_1^2)^{-1/2} \cos \gamma \frac{\partial}{\partial \gamma} + \frac{1}{2I} \frac{\partial}{\partial x_1}.$$
A glance at these expressions shows that the vector fields $\vartheta_1$ and $\vartheta_2$ fail to be complete on $U_-$ (see Remark 2.4).

One can say something more about the angle coordinate $\alpha$. The vector field $\vartheta_I$ (7.28 reads
\[
\frac{\partial}{\partial \alpha} = \sum_i \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right).
\]
This equality leads to the relations
\[
\frac{\partial q_i}{\partial \alpha} = \frac{\partial H}{\partial p_i}, \quad \frac{\partial p_i}{\partial \alpha} = -\frac{\partial H}{\partial q_i},
\]
which take the form of the Hamilton equations. Therefore, the coordinate $\alpha$ is a cyclic time $\alpha = t \mod 2\pi$ given by the well-known expression
\[
\alpha = \phi - \frac{3}{2} e \sin(\phi - \frac{3}{2} \phi), \quad r = a(1 - e \cos(\phi - \frac{3}{2} \phi)) \quad a = -\frac{1}{2l}, \quad e = (1 + 2IM^2)^{1/2}.
\]

Now let us turn to the Kepler superintegrable system on $U_+$. It is a globally superintegrable system with noncompact invariant submanifolds as follows.

Put
\[
S_1 = -K_1, \quad S_2 = -K_2, \quad S_3 = -M_{12}, \quad \{S_1, S_2\} = -S_3, \quad \{S_2, S_3\} = S_1, \quad \{S_3, S_1\} = S_2.
\]

We have the fibered manifold
\[
S : U_+ \to N \subset G^*, \quad (7.31)
\]
which is the momentum mapping to the Lie coalgebra $G^* = so(2,1)^*$, endowed with the coordinates $(x_i)$ such that integrals of motion $S_i$ on $G^*$ read $S_i = x_i$. A base $N$ of the fibered manifold (7.32) is an open submanifold of $G^*$ given by the coordinate condition $x_3 \neq 0$. It is a union of two contractible components defined by the conditions $x_3 > 0$ and $x_3 < 0$. The coinduced Lie–Poisson structure on $N$ takes the form
\[
w = x_2 \partial^3 \wedge \partial^1 - x_3 \partial^1 \wedge \partial^2 + x_1 \partial^2 \wedge \partial^3.
\]

The coadjoint action of $so(2,1)$ on $N$ reads
\[
\varepsilon_1 = -x_3 \partial^2 - x_2 \partial^3, \quad \varepsilon_2 = x_1 \partial^3 + x_3 \partial^1, \quad \varepsilon_3 = x_2 \partial^1 - x_1 \partial^2.
\]
The orbits of this coadjoint action are given by the equation
\[
x_1^2 + x_2^2 - x_3^2 = \text{const}.
\]
They are the level surfaces of the Casimir function
\[
C = x_1^2 + x_2^2 - x_3^2.
\]
and, consequently, the Casimir function

\[ h = \frac{1}{2} (x_1^2 + x_2^2 - x_3^2)^{-1}. \]  

(7.34)

A glance at the expression (7.20) shows that the pull-back \( S^* h \) of this Casimir function (7.34) onto \( U_+ \) is the Hamiltonian \( H \) (7.2) of the Kepler system on \( U_+ \).

As was mentioned above, the Hamiltonian vector field of \( S^* h \) is complete. Furthermore, it is known that invariant submanifolds of the superintegrable Kepler system on \( U_+ \) are diffeomorphic to \( \mathbb{R} \). Therefore, the fibered manifold \( S (7.32) \) is a fiber bundle in accordance with Proposition 2.9. Moreover, this fiber bundle is trivial because \( N \) is a disjoint union of two contractible manifolds. Consequently, it follows from Theorem 4.4 that the Kepler superintegrable system on \( U_+ \) is globally superintegrable, i.e., it admits global generalized action-angle coordinates as follows.

The Poisson manifold \( N (7.32) \) can be endowed with the coordinates

\[ (I, x_1, \lambda), \quad I > 0, \quad \lambda \neq 0, \]

defined by the equalities

\[ I = \frac{1}{2} (x_1^2 + x_2^2 - x_3^2)^{-1}, \quad x_2 = \left( \frac{1}{2I} - x_1^2 \right)^{1/2} \cosh \lambda, \quad x_3 = \left( \frac{1}{2I} - x_1^2 \right)^{1/2} \sinh \lambda. \]

These coordinates are the Darboux coordinates of the Lie–Poisson structure (7.33) on \( N \), namely,

\[ w = \frac{\partial}{\partial \lambda} \wedge \frac{\partial}{\partial x_1}. \]

(7.35)

Let \( \vartheta_I \) be the Hamiltonian vector field of the Casimir function \( I \) (7.26). By virtue of Proposition 2.4, its flows are invariant submanifolds of the Kepler superintegrable system on \( U_+ \). Let \( \tau \) be a parameter along the flows of this vector field, i.e.,

\[ \vartheta_I = \frac{\partial}{\partial \tau}. \]

(7.36)

Then \( N (7.32) \) is provided with the generalized action-angle coordinates \( (I, x_1, \lambda, \tau) \) such that the Poisson bivector associated to the symplectic form \( \Omega \) on \( U_+ \) reads

\[ W = \frac{\partial}{\partial I} \wedge \frac{\partial}{\partial \tau} + \frac{\partial}{\partial \lambda} \wedge \frac{\partial}{\partial x_1}. \]

(7.37)

Accordingly, Hamiltonian vector fields of integrals of motion \( S_i \) (7.30) take the form

\[ \vartheta_1 = -\frac{\partial}{\partial \lambda}, \]

\[ \vartheta_2 = \frac{1}{4I^2} \left( \frac{1}{2I} - x_1^2 \right)^{-1/2} \cosh \lambda \frac{\partial}{\partial \tau} + x_1 \left( \frac{1}{2I} - x_1^2 \right)^{-1/2} \cosh \lambda \frac{\partial}{\partial \lambda} + \]

\[ \left( \frac{1}{2I} - x_1^2 \right)^{1/2} \sinh \lambda \frac{\partial}{\partial x_1}, \]

\[ \vartheta_3 = \frac{1}{4I^2} \left( \frac{1}{2I} - x_1^2 \right)^{-1/2} \sinh \lambda \frac{\partial}{\partial \tau} + x_1 \left( \frac{1}{2I} - x_1^2 \right)^{-1/2} \sinh \lambda \frac{\partial}{\partial \lambda} + \]

\[ \left( \frac{1}{2I} - x_1^2 \right)^{1/2} \cosh \lambda \frac{\partial}{\partial x_1}. \]
Similarly to the angle coordinate $\alpha (7.28)$, the generalized angle coordinate $\tau (7.36)$ obeys the Hamilton equations

$$\frac{\partial q_i}{\partial \tau} = \frac{\partial H}{\partial p_i}, \quad \frac{\partial p_i}{\partial \tau} = -\frac{\partial H}{\partial q_i}.$$ 

Therefore, it is the time $\tau = t$ given by the well-known expression

$$\tau = s - a^{3/2}e \sinh(a^{-3/2}s), \quad r = a(e \cosh(a^{-3/2}s) - 1) \quad a = \frac{1}{2I}, \quad e = (1 + 2IM^2)^{1/2}.$$ 

**References**

[1] V. Arnold (Ed.), *Dynamical Systems III, IV* (Springer-Verlag, Berlin, 1990).

[2] A. Bolsinov and B. Jovanović, Noncommutative integrability, moment map and geodesic flows, *Ann. Global Anal. Geom.* 23 (2003) 305.

[3] P. Dazord and T. Delzant, Le probleme general des variables actions-angles, *J. Diff. Geom.* 26 (1987) 223.

[4] F. Fassó, Superintegrable Hamiltonian systems: geometry and applications, *Acta Appl. Math.* 87 (2005) 93.

[5] E. Fiorani, G. Giachetta and G. Sardanashvily, The Liouville – Arnold – Nekhoroshev theorem for noncompact invariant manifolds, *J. Phys. A* 36 (2003) L101.

[6] E. Fiorani and G. Sardanashvily, Noncommutative integrability on noncompact invariant manifold, *J. Phys. A* 39 (2006) 14035.

[7] E. Fiorani and G. Sardanashvily, Global action-angle coordinates for completely integrable systems with noncompact invariant manifolds, *J. Math. Phys.* 48 (2007) 032001.

[8] G. Gaeta, The Poincaré – Lyapounov – Nekhoroshev theorem, *Ann. Phys.* 297 (2002) 157.

[9] G. Gaeta, The Poincaré–Nekhoroshev map, *J. Nonlin. Math. Phys.* 10 (2003) 51.

[10] G. Giachetta, L. Mangiarotti and G. Sardanashvily, Bi-Hamiltonian partially integrable systems, *J. Math. Phys.* 44 (2003) 1984.

[11] G. Giachetta, L. Mangiarotti and G. Sardanashvily, *Geometric and Algebraic Topologi- cal Methods in Quantum Mechanics* (World Scientific, Singapore, 2005).

[12] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics* (Cambr. Univ. Press, Cambridge, 1984).

[13] F. Hirzebruch, *Topological Methods in Algebraic Geometry* (Springer-Verlag, Berlin, 1966).
[14] V. Lazutkin, *KAM Theory and Semiclassical Approximations to Eigenfunctions* (Springer-Verlag, Berlin, 1993).

[15] P. Libermann and C.-M. Marle, *Symplectic Geometry and Analytical Mechanics* (D.Reidel Publishing Company, Dordrecht, 1987).

[16] L. Mangiarotti and G. Sardanashvily, *Gauge Mechanics* (World Scientific, Singapore, 1998).

[17] G. Meigniez, Submersions, fibrations and bundles, *Trans. Amer. Math. Soc.* 354 (2002) 3771.

[18] A. Mishchenko and A. Fomenko, Generalized Liouville method of integration of Hamiltonian systems, *Funct. Anal. Appl.* 12 (1978) 113.

[19] N. Nekhoroshev, The Poincaré – Lyapounov – Liouville – Arnold theorem, *Funct. Anal. Appl.* 28 (1994) 128.

[20] A. Onishchik (Ed.), *Lie Groups and Lie Algebras I. Foundations of Lie Theory, Lie Transformation Groups* (Springer-Verlag, Berlin, 1993).

[21] R. Palais, A global formulation of Lie theory of transformation groups, *Mem. Am. Math. Soc.* 22 (1957) 1.

[22] H. Sussmann, Orbits of families of vector fields and integrability of distributions, *Trans. Amer. Math. Soc.* 180 (1973) 171.

[23] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds* (Birkhäuser, Basel, 1994).

[24] A. Vinogradov and B. Kupershmidt, The structure of Hamiltonian mechanics, *Russian Math. Surveys* 32 (4) (1977) 177.