SINGULAR INTEGRALS ON REGULAR CURVES
IN THE HEISENBERG GROUP

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ABSTRACT. We show that smooth $\frac{d}{d} \log$-homogeneous horizontally odd kernels in the Heisenberg group induce Calderón-Zygmund operators on regular curves. This extends a theorem of G. David from 1984 to the Heisenberg group.

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1. INTRODUCTION

This paper concerns the $L^2$ boundedness of certain singular integral operators (SIOs) on regular curves in the Heisenberg group $(\mathbb{H}, d) = (\mathbb{R}^3, \cdot, d)$. For a brief introduction to the space $(\mathbb{H}, d)$, see Section 3.2. We recall that a closed set $E$ in a metric space $(X, d)$ is $s$-regular, for $s \geq 0$, if there exists a constant $C \geq 1$ such that

$$C^{-r^s} \leq \mathcal{H}^s(E \cap B(x, r)) \leq Cr^s, \quad x \in E, \ 0 < r \leq \text{diam}(E).$$

**Definition 1.1.** A closed set $\gamma$ in a metric space $(X, d)$ is a regular curve if $\gamma$ is a 1-regular set, and also the Lipschitz image of a closed subinterval of $\mathbb{R}$.

The study of SIOs on regular curves in $\mathbb{R}^n$ has a long history. Calderón [3] in 1977 proved that the Cauchy transform $Cf(z) = f * \frac{1}{z}$ defines an operator bounded on $L^2(\Gamma)$, whenever $\Gamma \subset \mathbb{C}$ is the graph of Lipschitz function with small Lipschitz constant. Coifman, McIntosh, and Meyer [13] removed the “small constant” assumption in 1982. Coifman, David, and Meyer [12] then proved the same with the Cauchy kernel “$\frac{1}{z}$” replaced by any smooth $-1$-homogeneous odd function $k: \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$. David [16] extended the results to all regular curves $\gamma \subset \mathbb{R}^n$, see also [17]. The results in [16, 17] imply that if $\gamma \subset \mathbb{R}^n$ is a regular curve, $\mu := \mathcal{H}^1|_{\gamma}$, and $k$ is as above, then the sublinear operator

$$T^s_{k, \mu}f(x) := \sup_{\epsilon > 0} \left| \int_{\{y||x-y| > \epsilon\}} k(x - y)f(y) \, d\mu(y) \right|, \quad f \in C_c(\mathbb{R}^n), \tag{1.2}$$

called the maximal SIO induced by $(k, \mu)$, extends to a bounded operator on $L^p(\mu)$, for any $1 < p < \infty$. In the sequel, we will abbreviate the $L^p(\mu)$ boundedness of $T^s_{k, \mu}$, $1 < p < \infty$, by writing that $k$ is a Calderón-Zygmund (CZ) kernel for $\mu$.

1.1. Overview of proofs in $\mathbb{R}^n$. David’s approach in [16] was to reduce the problem on regular curves to the one on Lipschitz graphs: the main ideas were that regular curves have big pieces of Lipschitz graphs (BPLG), and that CZ kernels for Lipschitz graphs are also CZ kernels for 1-regular sets with BPLG. Another “reduction” proof of this type is due to Semmes [45] from 1990. He introduced the notion of sets which admit corona decompositions by Lipschitz graphs (CDLG), and showed that CZ kernels for Lipschitz graphs are CZ kernels for 1-regular sets admitting CDLG.

An alternative strategy was found by Jones [32, 33]. He introduced the notion of $\beta$-numbers: given a set $K \subset \mathbb{R}^n$, and a ball $B(x, r)$ centred on $K$, the $\beta$-number $\beta_K(B(x, r))$ measures the deviation of $K \cap B(x, r)$ from the best-approximating line. Jones proved in
[33] that the $\beta$-numbers on regular curves in $\gamma \subset \mathbb{C}$ satisfy the following square function estimate:

$$
\int_0^R \int_{B(x_0, R)} \beta_\gamma(B(x, r))^2 \, dH^1_{\gamma}(x) \frac{dr}{r} \lesssim R, \quad B(x_0, R) \subset \mathbb{C}.
$$

(1.3)

The case of Lipschitz graphs was already contained in [32], where Jones deduced the $L^2$-boundedness of $C$ on Lipschitz graphs from the geometric condition (1.3). The square function estimate (1.3) is also valid for regular curves in $\mathbb{R}^n$, as shown by Okikiolu [42].

More recently, Tolsa [48] introduced the notion of $\alpha$-numbers. These are, roughly speaking, measure-theoretic versions of Jones’ $\beta$-numbers. Tolsa showed that odd $m$-dimensional $C^2$-smooth kernels in $\mathbb{R}^n$ are CZ kernels for any $m$-regular measure $\mu$ on $\mathbb{R}^n$ whose $\alpha$-numbers satisfy a square function estimate analogous to (1.3). This improves on the result of David [17], since only $C^2$-regularity of the kernel is required. Moreover, as in Jones’ argument, the proof deduces the $L^2$-boundedness of SIOs directly from bounds on a square function involving the $\alpha$-numbers, without passing via Lipschitz graphs. However, the oddness of the kernels seems to be indispensable.

Investigating the connections between Lipschitz graphs, sets with BPLG, or admitting CDLG, square function estimates involving $\alpha$’s, $\beta$’s, or other geometric quantities, and the $L^2$-boundedness of SIOs, is known as the theory of uniform rectifiability. For more information, see [14, 15, 49].

1.2. Singular integrals on regular curves in $\mathbb{H}$. What are the natural kernels in $\mathbb{H}$? In $\mathbb{R}^n$, the oddness assumption is prevalent, so why not study odd kernels in $\mathbb{H}$? In $\mathbb{R}^n$, oddness is not only a matter of technical convenience: instead, it stems from the existence of “natural” odd kernels in $\mathbb{R}^n$, such as the Cauchy kernel $\frac{1}{x}$, and its higher-dimensional counterpart, the Riesz kernel $\nabla|x|^{2-n}$. SIOs associated to these kernels are of key importance in the theory of partial differential equations, see for example [21, 50]. Following this train of thought in $\mathbb{H}$, one is led to consider the “$\mathbb{H}$-Riesz kernel” $k(p) = \nabla_{\mathbb{H}}\|p\|^{-2}$, and the question of whether $k$ is a CZ kernel for 3-regular surfaces in $\mathbb{H}$. This problem was first raised in [4]. Here, and in the introduction, $\|p\|$ refers to the Korányi norm of $p \in \mathbb{H}$. For currently up-to-date results on the $\mathbb{H}$-Riesz kernel, see [23].

The kernel $k(p) = \nabla_{\mathbb{H}}\|p\|^{-2}$ is not odd. Instead, as noted in [5], it is horizontally odd:$^1$

$$
k(-x, -y, t) = -k(x, y, t), \quad (x, y, t) \in \mathbb{H} \setminus \{0\}.
$$

(1.4)

Horizontal oddness appears to work well with the geometry of $\mathbb{H}$: in [5], it was shown that 3-dimensional horizontally odd kernels are CZ kernels for “intrinsic $C^{1,\alpha}$-graphs”, for $\alpha > 0$. We also study horizontally odd kernels in this paper.

Definition 1.5 (Good kernels). A function $k : \mathbb{H} \setminus \{0\} \to \mathbb{C}$ is a good kernel if

1. $k \in C^\infty(\mathbb{H} \setminus \{0\}) = C^\infty(\mathbb{R}^3 \setminus \{0\})$,
2. $k$ is horizontally odd in the sense (1.4),
3. $k$ is $-1$-homogeneous with respect to dilations in $\mathbb{H}$ (see Section 3.2).

Here is the main result of the paper:

Theorem 1.6. Good kernels are CZ kernels for regular curves in $\mathbb{H}$.

$^1$The terminology horizontally antisymmetric was used in [5].
The property of a good kernel "$k$ being a CZ kernel for a regular curve $\gamma$" means the same as above, namely that the maximal SIO induced by $(k, \mathcal{H}^1|_\gamma)$ defines an operator bounded on $L^p(\mathcal{H}^1|_\gamma)$, for $1 < p < \infty$. See Definition 2.15 for a more formal treatment.

1.3. Previous work. Above, we already discussed previous work concerning SIOs on 3-regular surfaces in $\mathbb{H}$. SIOs on 1-regular subsets of $\mathbb{H}$ were first studied by Chousionis and Li in [7]. The kernels $k: \mathbb{H} \setminus \{0\} \to \mathbb{C}$ considered in [7] are not "good" in the sense of Definition 1.5. Instead, they are non-negative $-1$-homogeneous kernels of the form

$$k_\alpha(x, y, t) = \left(\frac{\sqrt{|p|/\|p\|}}{|p|}\right)^\alpha, \quad p = (x, y, t) \in \mathbb{H} \setminus \{0\}, \quad \alpha \geq 1.$$ 

Chousionis and Li proved that $k_8$ is a CZ kernel for regular curves $\gamma \subset \mathbb{H}$, and with Zimmermann they found a generalisation of this result to arbitrary Carnot groups [8]. Conversely, they also showed in [7] that if $E \subset \mathbb{H}$ is 1-regular, and $k_2$ is a CZ kernel for $E$, then $E$ is contained on a regular curve. It may sound astounding that non-negative kernels could ever be CZ kernels. A partial explanation comes from noting that $k_\alpha$ vanishes identically on the plane $\{(x, y, t) : t = 0\}$. Consequently, if $t \subset \mathbb{H}$ is a horizontal line (see Definition 3.37), then the (maximal) SIO induced by $(k_\alpha, \mathcal{H}^1|_t)$ is the zero operator. In contrast, our good kernels vanish identically on the axis $\{(x, y, t) : x = 0 = y\}$, and the induced SIOs on horizontal lines can behave like the Hilbert transform.

It is natural to long for a result which simultaneously generalises the work in [7], and the present paper. Here is one suggestion (caveat emptor):

**Question 1.** Let $k: \mathbb{H} \setminus \{0\} \to \mathbb{C}$ be a smooth $-1$-homogeneous function which is a CZ kernel for horizontal lines, with uniform constants. Is $k$ then a CZ kernel for regular curves?

For a related result in $\mathbb{H}$, see [5, Theorem 2.10].

1.4. The proof of Theorem 1.6: an outline. In Section 1.1, we mentioned two approaches for studying SIOs on regular curves in $\mathbb{R}^n$: either reduce matters to the special case of Lipschitz graphs via "big piece" or "corona" methods, or take a more direct route via geometric square functions ($\alpha$-numbers or $\beta$-numbers). In this paper, we take the former approach(es), as the latter appears to be difficult to execute for two separate reasons:

- The oddness of kernels in $\mathbb{R}^n$ is critical in "quasiothogonality" arguments, see [48], and horizontal oddness seems to be a poor substitute in this regard.
- Analogues of Jones’ $\beta$-numbers have been extensively studied in $\mathbb{H}$, see [24, 35, 37, 38, 36]. A surprising example of Juillet [35] shows that the $L^2$-integral of the $\beta$-numbers appearing in (1.3) need not be bounded by $\mathcal{H}^1(\gamma)$, for rectifiable curves $\gamma \subset B(x_0, R)$. Instead, Li and Schul [37] proved a version of (1.3) where the exponent "2" is replaced by "4". This fact was used in [7] to show that $k_8$ is a CZ kernel for regular curves, but we were not able to employ it in the proof of Theorem 1.6 (except to deduce the "WGL", see Lemma 6.53).

We then discuss the former approach. Heisenberg analogues of Lipschitz graphs are known as *intrinsic Lipschitz graphs* (iLGs), and they were introduced by Franchi, Serapioni, and Serra Cassano [25] in 2006. Their rectifiability properties, both qualitative and quantitative, have been investigated vigorously in recent years, see [6, 11, 22, 26, 39, 40, 43, 44]. However, many of these papers have focused on 1-co-dimensional iLGs, whereas the objects relevant here are the 1-dimensional iLGs over *horizontal subgroups of*
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The first objective en route to Theorem 1.6 is to establish the result in the special case of 1-dimensional iLGs in \( \mathbb{H} \):

**Theorem 1.7.** Good kernels are CZ kernels for iLGs over horizontal subgroups in \( \mathbb{H} \).

This result is the main news of the paper. Once it has been established, we still need to complete David’s approach in [16], and prove the following statements:

**Theorem 1.8.** Regular curves in \( \mathbb{H} \) have big pieces of intrinsic Lipschitz graphs (BPiLG) over horizontal subgroups.

"Theorem". Let \((X, d)\) be a proper metric space, let \(G\) be a family of \(m\)-regular sets in \((X, d)\), and let \(K\) be an \(m\)-dimensional standard kernel on \(X\) which is a CZ kernel for all \(G \in G\), uniformly. Then \(K\) is a CZ kernel for any \(m\)-regular set \(B \subset X\) which has "big pieces" of sets in \(G\).

For a more precise statement, see Theorem 6.3. The proof is a straightforward adaptation of [18, Proposition 3.2] to proper metric spaces, and we claim very little originality: the main point is to check that the Besicovitch covering theorem is not used in an essential way. Regarding Theorem 1.8, we follow an approach of David and Semmes [20], by showing, first, that regular curves have big horizontal projections (BHP), and satisfy the weak geometric lemma for Jones’ \(\beta\)-numbers. Then, a combination of these properties yields BPiLG. These arguments are quite well-known, and have even been adapted to 1-co-dimensional iLGs in \(\mathbb{H}^n\), see [6, 22]. Only verifying the BHP property for regular curves produces a minor "new" problem. The details are contained in Section 6.2.

So, the heart of the matter is Theorem 1.7, whose proof indeed takes up most of the paper. Adapting some arguments from [12], the proof of Theorem 1.7 may be reduced to a problem concerning certain 1-dimensional SIOs on \(\mathbb{R}\). More precisely, one is led to consider the standard kernel

\[
K_B(x, y) = \frac{1}{x - y} \exp \left( 2\pi i \frac{B_2(x) - B_2(y) - \frac{1}{2} (B_1(x) + B_1(y))(x - y)}{(x - y)^2} \right),
\]

where \(B = (B_1, B_2): \mathbb{R} \to \mathbb{R}^2\) is a tame map. This simply means that \(B_1\) is Lipschitz, and \(\hat{B}_2 = B_1\). Tame maps are quite entertaining, and they are thoroughly investigated in Section 3.1. The kernel \(K_B\) is not antisymmetric, but we nevertheless manage to prove in Theorem 4.10 that \(K_B\) is a CZ kernel on \(\mathbb{R}\). Unfortunately, this is not quantitative enough: to apply the kernels \(K_B\) in the context of Theorem 1.7, we need to know that the CZ constant of \(K_B\), denoted \(\|K_B\|_{C.Z.}\), depends polynomially on the "tameness constant" of \(B\). A similar problem for Lipschitz functions (and graphs) already appears in David’s work [16, 18], but the solution is easier there: it is based on the "big piece theorem" stated below Theorem 1.8, plus the simple – and ingenious – observation that "\(L\)-Lipschitz graphs have big pieces of \( \frac{8}{15}L\)-Lipschitz graphs", see [18, p. 66]. We were not able to prove an analogue of this property for tame maps, see Question 2.

Instead, we found a weaker substitute: tame maps admit "corona decompositions" by tame maps with a smaller constant. More precise statements can be found in Section 3.1.1. We mentioned in Section 1.1 that Semmes [45] used corona decompositions (by Lipschitz graphs) to reduce SIO problems on regular curves to SIO problems on Lipschitz graphs. Applying his mechanism, and the tame-corona decomposition mentioned above, we can finally infer the polynomial dependence of \(\|K_B\|_{C.Z.}\) on the "tameness" of \(B\). We refer to Section 5 for details.
We have now summarised the proof of Theorem 1.6, and explained most of the structure of the paper. Let us add that in Section 2, we merely collect standard preliminaries on Calderón-Zygmund theory. In Section 3, we introduce tame maps, the Heisenberg group, and intrinsic Lipschitz graphs, and prove the corona decomposition for tame maps. In Section 4, we reduce the proof of Theorem 1.7 to the study of the kernel $K_B$ — or, as it really turns out, $K_{A,B}$ — and establish the "qualitative" fact that $K_{A,B}$ is a CZ kernel on $\mathbb{R}$. The quantitative version is the main content of Section 5, and this section concludes the proof of Theorem 1.7. Finally, in Section 6, we prove the "BPiLG" Theorem 1.8 and use it to deduce Theorem 1.6 from Theorem 1.7.

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2. Preliminaries on singular integral operators

2.1. Standard kernels. We define standard kernels and Calderón-Zygmund operators, and recall some of their standard properties.

Definition 2.1. Let $(X, d)$ be a metric space, write $\triangle := \{(x, x) : x \in X\}$, and let $k > 0$. A $k$-dimensional standard kernel ($k$-SK) on $X$ is a Borel function

$$K : X \times X \setminus \triangle \to \mathbb{C}$$

for which there exist constants $C > 0$ and $\alpha \in (0, 1]$ such that the following holds:

1. $|K(x, y)| \leq \frac{C}{d(x, y)^\alpha}$, for all $(x, y) \in X \times X \setminus \triangle$,
2. $\max \{|K(x, y) - K(x', y)|, |K(y, x) - K(y, x')|\} \leq C \frac{d(x, x')^\alpha}{d(x, y)^{\alpha+1}}$,

whenever $x, x', y \in X$ and $d(x, x') \leq d(x, y)/2$. The smallest constant "$C" above will be denoted by $\|K\|_{\alpha, \text{strong}}$.

A standard kernel (SK), without reference to the dimension, will mean a 1-SK.

An important class of SKs are those induced by good kernels $k : \mathbb{H} \setminus \{0\} \to \mathbb{C}$, recall Definition 1.5. Setting $K(p, q) := k(q^{-1} \cdot p)$, one obtains an SK satisfying Definition 2.1(1)-(2) with $\alpha = \frac{1}{2}$, see Proposition 3.35. Further, the kernels $K(p, q) = k(q^{-1} \cdot p)$ “evaluated on” intrinsic Lipschitz graphs yield another class of interesting SKs, this time in $\mathbb{R}$. We record some details right away:

Example 2.2. Let $A : \mathbb{R} \to \mathbb{R}$ be an $M$-Lipschitz function, and let $B = (B_1, B_2) : \mathbb{R} \to \mathbb{R}^2$ be an $N$-tame function (here we just need to know that $B_1$ is $N$-Lipschitz, and $B_2 = B_1$; see Section 3.1), where $M, N \geq 1$. Let $k : \mathbb{R} \times \mathbb{R} \setminus \triangle \to \mathbb{C}$ be an SK. Then, the kernel $K_{k,A,B}(x, y) := k(x, y) e_{A,B}(x, y)$ is an SK, with $\|K_{k,A,B}\|_{\alpha, \text{strong}} \lesssim \|k\|_{\alpha, \text{strong}} \max\{M, N\}$. To see this, fix $x, x', y \in \mathbb{R}$ with $|x - x'| \leq |x - y|/2$, and write

$$|K_{k,A,B}(x, y) - K_{k,A,B}(x', y)| \leq |k(x, y) - k(x', y)| + |k(x', y)||e_{A,B}(x, y) - e_{A,B}(x', y)|,$$
and use the SK estimates for k. The problem then reduces to estimating \(|e_{A,B}(x,y) - e_{A,B}(x',y)|\), which further reduces (using that \(t \mapsto e^{2\pi it}\) is 2\(\pi\)-Lipschitz) to finding upper bounds for

\[
    a(x, x', y) := \left| \frac{A(x') - A(y)}{x' - y} - \frac{A(x) - A(y)}{x - y} \right|
\]

and

\[
    b(x, x', y) := \left| \frac{B_2(x') - B_2(y) - \frac{1}{2} [B_1(x') + B_1(y)] (x' - y)}{(x' - y)^2} - \frac{B_2(x) - B_2(y) - \frac{1}{2} [B_1(x) + B_1(y)] (x - y)}{(x - y)^2} \right|.
\]

We leave it to the reader to check that \(a(x, x', y) \lesssim M|x' - x|/|x - y|\). To see that also \(|b(x, x', y)| \lesssim N|x' - x|/|x - y|\), we first infer from the tameness of \(B\) that \(B_2 \in C^1(\mathbb{R})\), and \(\tilde{B}_2 = B_1\), see Remark 3.2. Therefore, for \(x \neq y\),

\[
    \frac{B_2(x) - B_2(y) - \frac{1}{2} [B_1(x) + B_1(y)] (x - y)}{(x - y)^2} = \int_x^y \frac{B_1(s) - B_1(s) - 2B_1(s)}{2(x - y)^2} \, ds.
\]

The tameness of \(B\) also implies that \(B_1\) is \(N\)-Lipschitz, so a little computation shows that the \(x\) and \(y\) derivatives of the right hand side are \(\lesssim N/|x - y|\) almost everywhere. Now it follows from the fundamental theorem of calculus that \(b(x, x', y) \lesssim N|x - x'|/|x - y|\), as claimed.

The kernel \(K_{k,A,B}\) with \(k(x, y) = (x - y)^{-1}\) will have special significance in the paper, and it will be denoted simply \(K_{A,B}\).

2.2. Generalised standard kernels and CZOs. In Section 5, we will encounter kernels which are not quite SKs in the sense above, but satisfy the following relaxed conditions:

**Definition 2.4.** Let \((X, d)\) be a proper metric space. A Borel function \(K : X \times X \setminus \triangle \to \mathbb{C}\) is a \(k\)-dimensional generalised standard kernel (k-GSK) if the "size" condition in Definition 2.1(1) holds with constant \(C \geq 1\), and moreover \(K\) satisfies the following two inequalities for all Radon measures \(\mu\) on \(X\), for all \(f \in L^1_{loc}(\mu)\), and for all closed balls \(B \subset X\):

\[
    \int_{X \setminus 2B} |K(x,y) - K(x_0, y)||f(y)| \, d\mu(y) \leq CM_{\mu,k} f(x_0), \quad x, x_0 \in B, \quad (2.5)
\]

and

\[
    \int_{X \setminus 2B} |K(y,x) - K(y, x_0)||f(y)| \, d\mu(y) \leq CM_{\mu,k} f(x_0), \quad x, x_0 \in B. \quad (2.6)
\]

Here \(M_{\mu,k}\) is the "radial" maximal function of order \(k\):

\[
    M_{\mu,k} f(x) := \sup_{r > 0} \frac{1}{r^k} \int_{B(x, r)} |f(y)| \, d\mu(y), \quad x \in X.
\]

The best constant "\(C\)" here will be denoted by \(\|K\|\).

On first sight, it may appear odd that the constant "\(C\)" needs to be independent of the choice of the Radon measure \(\mu\) on \(X\). However, Proposition 2.7 below shows that any \(k\)-SK \(K : X \times X \setminus \triangle \to \mathbb{C}\) is a k-GSK, with

\[
    \|K\| \lesssim_\alpha \|K\|_{\alpha, \text{strong}}.
\]

**Proposition 2.7.** Let \((X, d)\) be a proper metric space, let \(k > 0\), and let \(K : X \times X \setminus \triangle \to \mathbb{C}\) a \(k\)-SK. Then (2.5)-(2.6) hold with a constant \(C \lesssim_\alpha \|K\|_{\alpha, \text{strong}}\).
By symmetry, we only need to verify (2.5). Fix $B, \mu, f,$ and $x, x_0 \in B$ as in Definition 2.4. Then,
\[
\int_{X \setminus 2B} |K(x, y) - K(x_0, y)||f(y)| \, d\mu(y) \lesssim \|K\|_{\alpha, \text{strong}} \int_{X \setminus B(x_0, r)} \frac{d(x, x_0)^\alpha}{d(x, y)^{k+\alpha}} |f(y)| \, d\mu(y).
\]
We used $B(x_0, r) \subset 2B$ and the Hölder estimate for $K$. The latter is a priori only valid for $y \in X \setminus 2B$ with $d(x, x_0) \leq d(x_0, y)/2$, but if $d(x_0, y)/2 < d(x, x_0)$, then $d(x, x_0) \sim d(x, y) \sim d(x_0, y) \sim r$, and we can apply the size bounds $|K(x, y)| \lesssim \|K\|_{\alpha, \text{strong}} d(x, y)^{-k}$ and $|K(x_0, y)| \lesssim \|K\|_{\alpha, \text{strong}} d(x_0, y)^{-k}$. Decomposing $X \setminus B(x_0, r)$ into dyadic annuli, we further estimate
\[
\int_{X \setminus B(x_0, r)} \frac{d(x, x_0)^\alpha}{d(x, y)^{k+\alpha}} |f(y)| \, d\mu(y) \lesssim 2^k \sum_{j=0}^\infty \frac{1}{2^j} \frac{1}{(2^j+1)r^k} \int_{B(x_0, 2^j+1r)} |f(y)| \, d\mu(y),
\]
from where (2.5) follows. \qed

The main point about GSKs vs. SKs is that GSKs are stable under "sharp" truncations:

**Lemma 2.8.** Let $K : X \times X \setminus \triangle \to \mathbb{C}$ be a k-GSK, and let $D : X \times X \to [0, \infty)$ be a $\frac{1}{2}$-Lipschitz function in the metric $d_{X \times X}((x, y), (x', y')) = \max\{d(x, x'), d(y, y')\}$. Then, the kernel $K^D$, defined by
\[
K^D(x, y) := K(x, y)1_{\{d(x, y) \geq D(x, y)\}}(x, y),
\]
is a k-GSK with $\|K^D\| \lesssim \|K\|$. \hspace{1cm}

**Proof.** By symmetry, it suffices to verify (2.5). Fix $B \subset X, x, x_0 \in B$, and a Radon measure $\mu$ on $X$. We claim that there are two, roughly dyadic, annuli $A_1, A_2$ centred at $x_0$ such that either
\[
K^D(x, y) = K(x, y) \quad \text{and} \quad K^D(x_0, y) = K(x_0, y),
\]
(2.9)
or
\[
K^D(x, y) = K^D(x_0, y) = 0
\]
(2.10)
for all $y \in (2B)^c$ with $y \notin [A_1 \cup A_2]$. The lemma follows from this, and the computation
\[
\int_{(2B)^c} |K^D(x, y) - K^D(x_0, y)||f(y)| \, d\mu(y)
\]
\[
\lesssim \int_{(2B)^c} |K(x, y) - K(x_0, y)||f(y)| \, d\mu(y)
\]
\[
+ \|K\| \int_{(2B)^c \cap [A_1 \cup A_2]} \frac{|f(y)|}{d(x_0, y)^k} \, d\mu(y) \lesssim \|K\|_{\mu, k} f(x_0).
\]
The points $y \in (2B)^c$ such that both (2.9) and (2.10) fail are contained in the union of
\[
B_1 := \{y \in (2B)^c : D(x, y) \leq d(x_0, y) \quad \text{and} \quad d(x, y) < D(x, y)\}
\]
and
\[
B_2 := \{y \in (2B)^c : D(x, y) \leq d(x, y) \quad \text{and} \quad d(x_0, y) < D(x, y)\}.
\]
We will next show that
\[
B_1 \subset \{y \in (2B)^c : r_1 \leq d(x_0, y) \leq 100r_1\} =: A_1
\]
(2.11)
with \( r_1 := \inf \{ d(x_0, y) : y \in B_1 \} \). To this end, fix \( \varepsilon \in (0, 1) \) and pick \( y_1 \in B_1 \subset (2B)^c \) such that
\[
   r := d(x_0, y_1) \in [r_1, (1 + \varepsilon) r_1].
\]
Consider now any \( y \in (2B)^c \) with
\[
   d(x_0, y) > 100r,
\]
and note that \( d(x, y) \geq d(x_0, y) - d(x_0, x) \geq 100r - 2d(x_0, y_1) = 98r \), because \( d(x_0, x) \leq 2d(x_0, y_1) \). We claim that then \( d(x, y) \geq D(x, y) \), so that \( y \notin B_1 \). Indeed, using that \( D \) is \( \frac{1}{2} \)-Lipschitz, and \( d(x_0, x) \leq 2d(x_0, y_1) \), we have
\[
   D(x, y) \leq D(x_0, y_1) + \frac{d(x_0, x)}{2} + \frac{d(y_1, y)}{2} \leq r + \frac{2r}{2} + \frac{r + d(x_0, y)}{2}
\]
\[
   \leq \frac{d(x, y)}{98} + \frac{d(x, y)}{98} + \frac{d(x, y)}{196} + \frac{d(x_0, y)}{2}
\]
\[
   \leq \frac{5d(x, y)}{196} + \frac{d(x_0, x)}{2} + r
\]
\[
   \leq \left( \frac{105}{196} \right) d(x, y) < d(x, y).
\]
We deduce that the points \( y \in B_1 \) must satisfy \( d(x_0, y) \leq 100r = 100(1 + \varepsilon) r_1 \), and letting \( \varepsilon \to 0 \), we have established (2.11). A symmetric argument yields that
\[
   B_2 \subseteq \{ y \in (2B)^c : r_2 \leq d(x, y) \leq 100r_2 \} =: A_2'
\]
with \( r_2 := \inf \{ d(x, y) : y \in B_2 \} \). Since \( r_2 \geq \text{dist}(x, (2B)^c) \geq d(x_0, x)/2 \), it is easy to see that \( A_2' \subset A_2 \), where \( A_2 \) is a slightly fatter annulus around \( x_0 \) with radius comparable to \( r_2 \). This completes the proof. \( \square \)

**Definition 2.13** (Induced operators and Calderón-Zygmund operators). Let \((X, d)\) be a proper metric space, let \( k > 0 \), and let \( K : X \times X \setminus \Delta \to \mathbb{C} \) be a **bounded** \( k \)-GSK. Let \( \mu \) be a Borel regular measure on \( X \) satisfying
\[
   \mu(B(x, r)) \leq Cr^k, \quad x \in X, \ r > 0,
\]
for some constant \( C \geq 1 \). We associate to \( K \) and \( \mu \) the following operator \( T_\mu \):
\[
   T_\mu f(x) := \int K(x, y) f(y) \, d\mu(y), \quad f \in \bigcup_{1 < p < \infty} L^p(\mu), \ x \in X.
\]
It is easy to see, using Hölder’s inequality, (2.14), and the "size" bound in Definition 2.1(1), that if \( 1 < p < \infty \) and \( f \in L^p(\mu) \), then the integral defining \( T_\mu f(x) \) is absolutely convergent. We say that \( T_\mu \) is the **operator induced by** \((K, \mu)\).

A **Calderón-Zygmund operator** (CZO) is an operator \( T_\mu \) induced by \((K, \mu)\), as above, which also happens to be bounded on \( L^2(\mu) \). For a CZO \( T_\mu \), we write
\[
   \|T_\mu\|_{C,Z} := \|T_\mu\|_{L^2(\mu) \to L^2(\mu)} + \|K\|.
\]
Definition 2.15 (ε-SIOs and CZ kernels). Let $K : X \times X \setminus \triangle \to \mathbb{C}$ be a $k$-GSK, not necessarily bounded, and let $\mu$ be a Borel measure on $X$ satisfying (2.14). For $\epsilon > 0$, we define $T_{\mu,\epsilon}$ to be the operator induced by $(K_{\epsilon}, \mu)$, where
\[
K_{\epsilon}(x,y) := K(x,y)1_{\{d(x,y) > \epsilon\}}(x,y), \quad (x,y) \in X \times X \setminus \triangle .
\]
The operator $T_{\mu,\epsilon}$ is called the ε-SIO induced by $(K, \mu)$. We also define the maximal SIO
\[
T_{\mu}^* f(x) := \sup_{\epsilon > 0} |T_{\mu,\epsilon} f(x)|, \quad f \in \bigcup_{1 < p < \infty} L^p(\mu), \ x \in X .
\]

If the ε-SIOs are uniformly bounded on $L^2(\mu)$,
\[
\sup_{\epsilon > 0} \|T_{\mu,\epsilon}\|_{L^2(\mu) \to L^2(\mu)} < \infty , \tag{2.16}
\]
we say that $K$ is a Calderón-Zygmund kernel (CZ kernel) for $\mu$, and we write
\[
\|K\|_{C.Z.} := \sup_{\epsilon > 0} \|T_{\mu,\epsilon}\|_{L^2(\mu) \to L^2(\mu)} + \|K\|.
\]

Remark 2.17. In the introduction – notably the statements of the main theorems – we used the terminological convention that $K$ is a CZ kernel for $\mu$ if $\|T_{\mu}^*\|_{L^p(\mu) \to L^p(\mu)} < \infty$ for all $1 < p < \infty$. There is no serious conflict: if $\mu$ is a measure on a proper metric space $(X, d)$ satisfying the growth condition (2.14), and $K : X \times X \setminus \triangle \to \mathbb{C}$ is a $k$-SK, then the condition (2.16) implies that $\|T_{\mu,\epsilon}\|_{L^p(\mu) \to L^p(\mu)} < \infty$ for all $1 < p < \infty$, see [41, Theorem 1.1]. In particular, all of this is true for kernels of the form $(p, q) \mapsto k(q^{-1} \cdot p)$, where $k : \mathbb{R} \setminus \{0\} \to \mathbb{C}$ is a good kernel, and for $\mathcal{H}^1$ measures restricted to regular curves in $\mathbb{R}$.

The reason why we chose to define "CZ kernels" as in Definition 2.15 is that we, sometimes, want to apply the definition to GSKs: the maximal SIO characterisation above may well remain valid in this generality, but at least we have not seen it written down.

For a big part of this paper, we will only be concerned with CZOs, ε-SIOs, and maximal SIOs induced by GSKs on $\mathbb{R}$, and the measure $\mu = \mathcal{L}^1$. We will drop the sub-index "$\mathcal{L}^1$" in this situation, and write $T, T_\epsilon, T_\epsilon^*$ in place of $T_{\mathcal{L}^1}, T_\epsilon, T_\epsilon^*$, also, on $\mathbb{R}$, we will only consider CZ kernels for $\mathcal{L}^1$, and write $\|K\|_{C.Z.} := \|K\|_{C.Z.(\mathcal{L}^1)}$.

We will now gather some basic facts about the case $X = \mathbb{R}$ (although many of these statements have generalisations to metric spaces, see for example [41]).

Proposition 2.18. Let $T$ be a CZO on $\mathbb{R}$. Then $T$ is bounded $L^1(\mathbb{R}) \to L^{1,\infty}(\mathbb{R})$ with norm
\[
\|T\|_{L^1 \to L^{1,\infty}} \lesssim \|T\|_{C.Z.}.
\]

Proof. Applying (2.6) with $f \equiv 1$ yields Hörmander’s condition
\[
\int_{(2B)^c} |K(x,y) - K(x,y_0)| \leq \|T\|_{C.Z.}, \quad y, y_0 \in I .
\]
It follows that $\|T\|_{L^1 \to L^{1,\infty}} \lesssim \|T\|_{C.Z.}$, see for example [28, Exercise 8.2.4]. \hfill \Box

Lemma 2.19 (Cotlar’s inequality). Let $K : \mathbb{R} \times \mathbb{R} \setminus \triangle \to \mathbb{C}$ be a bounded GSK, and let $T$ be the CZO induced by $K$. Then, there exists an absolute constant $C \geq 1$ such that
\[
T^* f(x) \leq C [M(|T f|)(x) + \|T\|_{C.Z.} M f(x)], \quad f \in L^2(\mathbb{R}), \ x \in \mathbb{R} . \tag{2.20}
\]
Here $M$ is the (non-centred) Hardy-Littlewood maximal function on $\mathbb{R}$.

For a proof, see for instance [34, p. 56].
**Theorem 2.21 (T1 theorem).** Let $T$ be an operator induced by a bounded SK $K: \mathbb{R} \times \mathbb{R} \setminus \triangle \to \mathbb{C}$. Then, $T$ is a CZO if and only if $T^1, T^4 \in \text{BMO}$, and $T$ satisfies the weak boundedness property (WBP). In this case,

$$
\|T\|_{L^2 \to L^2} \lesssim_{\alpha} \|T^1\|_{\text{BMO}} + \|T^4\|_{\text{BMO}} + \|T\|_{\text{WBP}} + \|K\|_{\alpha, \text{strong}}.
$$

(2.22)

For a proof, see [28, Theorem 8.3.3], or the original reference [19].

**Definition 2.23 (Definitions of $T^1$, $T^4$, and WBP).** Under the assumptions of the T1 theorem, the condition $T^1 \in \text{BMO}$ means that there exists a constant $C \geq 1$ with the following property. If $\varphi \in C^\infty(\mathbb{R})$ is a "smooth $H^1$-atom" supported on a ball $B_0$, i.e. satisfies

$$
spt \varphi \subset B_0, \quad \int_{B_0} \varphi = 0, \quad \text{and} \quad \|\varphi\|_{L^2} \leq |B_0|^{-1},
$$

(2.24)

and $b \in C^\infty(\mathbb{R})$ satisfies $1_{2B_0} \leq b \leq 1_{3B_0}$, then

$$
|\langle T(b), \varphi \rangle| \leq C.
$$

(2.25)

The best constant "$C"$, as above, is the definition of the quantity \"$\|T^1\|_{\text{BMO}}" in (2.22). The condition $T^4 \in \text{BMO}$ means, by definition, that \( \langle T^4(\varphi), b \rangle \) on the left hand side. Finally, the WBP means that if $\varphi, \psi$ are smooth non-negative functions supported on $B(0,1) \subset \mathbb{R}$, with $\max\{\|\varphi\|_{C^5}, \|\psi\|_{C^5}\} \leq 1$, then

$$
\|\langle T(\varphi, \psi)\rangle\|_{L^p} \leq C_r^{-1}, \quad x \in \mathbb{R}, \ r > 0.
$$

(2.26)

Here $f_x(r) := r^{-1} f((y-x)/r)$. The best constant "$C" in (2.26) is the definition of the quantity \"$\|T\|_{\text{WBP}}" in (2.22).

**2.2.1. Verifying the T1 testing conditions in practice.** Let $K: \mathbb{R} \times \mathbb{R} \setminus \triangle \to \mathbb{C}$ be an SK, not necessarily bounded, let $\varepsilon > 0$, and let $\varphi \in C^\infty(\mathbb{R})$ be a fixed, even, bump function satisfying $1_{B(0,1/2)} \leq \varphi \leq 1_{B(0,1)}$. Writing $\psi_\varepsilon := 1 - \varphi_\varepsilon$, we define the smooth $\varepsilon$-SIO $\tilde{T}_\varepsilon$ to be the operator induced by the bounded SK $K_\varepsilon(x,y) := \psi_\varepsilon(x-y)K(x,y)$. We also define the formal adjoint $\tilde{T}_\varepsilon^*$ by replacing $K(x,y)$ by $\overline{K(y,x)}$ in the definition above. We record the standard fact that $\|K_\varepsilon\|_{\alpha, \text{strong}} \lesssim K_\alpha, \text{strong}$, where the constants do not depend on $\varepsilon > 0$. Now, assume that we can prove the following for some constant $C \geq 1$: if $B_0$ is a ball, and $b \in C^\infty(\mathbb{R})$ satisfies $1_{2B_0} \leq b \leq 1_{3B_0}$, then

$$
\int_{B_0} |\tilde{T}_\varepsilon(b)| \leq C \quad \text{and} \quad \int_{B_0} |\tilde{T}_\varepsilon^*(b)| \leq C.
$$

(2.27)

We claim that

$$
\max\{\|\tilde{T}_\varepsilon 1\|_{\text{BMO}}, \|\tilde{T}_\varepsilon^* 1\|_{\text{BMO}}\} \leq C \quad \text{and} \quad \|\tilde{T}_\varepsilon\|_{\text{WBP}} \lesssim C + \|K\|.
$$

The first inequality is immediate from the definitions. To infer the second, fix $x_0 \in \mathbb{R}$, $r > 0$, write $B_0 := B(x_0, r)$, and find $b$ as above (2.27). Then, since $spt \varphi_{x,r} \subset B_0$ (as in (2.26)), we may write

$$
|\tilde{T}_\varepsilon(\varphi_{x,r})(x)| = |\tilde{T}_\varepsilon[b\varphi_{x,r}](x)|
\leq |\varphi_{x,r}(x)| \cdot \tilde{T}_\varepsilon(b)(x) + \left| \int_{B(x_0,r)} b(y) \varphi_{x,r}(y) - \varphi_{x,r}(x) \right| K_\varepsilon(x,y) dy \right|. \quad (2.28)
$$
Here,
\[
\langle \varphi_{x_0,t} \cdot \tilde{T}_t(b), \psi_{x_0,t} \rangle \leq \frac{1}{r^2} \int_{B_0} |\tilde{T}_t(b)| \lesssim C r^{-1}
\]
by (2.25). But since \([\varphi_{x_0,t} - \varphi_{x_0,s}(x)] K_t(x,y)] \lesssim r^{-2} \|K\|, \) and \(b|_{B_0} = 1\), the second term on line (2.28) is bounded, for every \(x \in B_0\), by \(|B_0| r^{-2} \|K\| \sim r^{-1} \|K\|\). It follows that the WBP (2.26) holds with constant at most \(\lesssim C + \|K\|\), as claimed.

We have established the following corollary of the T1 theorem:

**Corollary 2.29.** Let \(K : \mathbb{R} \times \mathbb{R} \to \mathbb{C}\) be an SK, and assume that the testing conditions (2.27) hold for some \(C \geq 1\), uniformly for \(\varepsilon > 0\). Then \(\|K\|_{C.Z.} \lesssim_\alpha C + \|K\|_{\alpha,\text{strong}}\).

**Proof.** Theorem 2.21 gives the uniform bound \(\|\tilde{T}_\varepsilon\| \lesssim_\alpha C + \|K\|_{\alpha,\text{strong}}\). This implies (2.16) (for \(\mu = L^1\)) with roughly the same constants, since \(\|T_\varepsilon - T_0\| \lesssim \|K\| Mf\). \(\square\)

### 3. Intrinsic Lipschitz Graphs and Tame Maps

#### 3.1. Tame maps

We say that a map \((\phi_1, \phi_2) : E \to \mathbb{R}^2\), defined on \(E \subset \mathbb{R}\), is \(L\)-tame if
\[
\left| \frac{\phi_2(x) - \phi_2(y)}{x - y} - \phi_1(x) \right| + \left| \frac{\phi_2(x) - \phi_2(y)}{x - y} - \phi_1(y) \right| \leq L|x - y|, \quad x, y \in E, \ x \neq y. \tag{3.1}
\]

**Remark 3.2.** We make a few hopefully clarifying remarks about the definition of tameness. First, condition (3.1) is implied (with twice the constant) by a "1-sided" version of itself:
\[
\left| \frac{\phi_2(x) - \phi_2(y)}{x - y} - \phi_1(x) \right| \leq L|x - y|, \quad x, y \in E, \ x \neq y. \tag{3.3}
\]

Indeed, just apply the inequality above to both \((x, y)\) and \((y, x)\) to arrive at (3.1). Second, (3.1) implies that \(\phi_1\) is \(L\)-Lipschitz (by the triangle inequality). Third, assume that \(E\) contains an open interval \(I\). Then (3.1) clearly implies that \(\phi_2\) exists on \(I\), and \(\phi_2 = \phi_1\). Conversely, assume that \(\phi = (\phi_1, \phi_2) : I \to \mathbb{R}^2\), where \(I \subset \mathbb{R}\) is an open interval, \(\phi_1\) is \(L\)-Lipschitz, and \(\phi_2 = \phi_1\). Then (3.3) is satisfied, because, for \(x < y\),
\[
[\phi_2(x) - \phi_2(y)] - \phi_1(x)(x - y) \leq \int_x^y |\phi_1(s) - \phi_1(x)| \, ds \leq L|x - y|^2. \tag{3.4}
\]

So, (3.1) and (3.3) are essentially short ways of writing that \(\phi_2 = \phi_1\) for a \(L\)-Lipschitz function \(\phi_1\) without actually mentioning the derivative of \(\phi_2\). We also note for future reference that the class of \(L\)-tame maps is preserved under the following operations:

1. Pre-composing with a translation in \(\mathbb{R}\).
2. Adding a map of the form \(L_{a,b}(x) := (a, ax + b)\), with \(a, b \in \mathbb{R}\).

In fact, the second point is just a special case of the fact that adding an \(L_1\)-tame map to an \(L_2\)-tame map produces an \((L_1 + L_2)\)-tame map: note that \(L_{a,b}\) is 0-tame for any \(a, b \in \mathbb{R}\).

The next lemma observes that tameness is preserved under parabolic rescaling:

**Lemma 3.5.** Let \(B = (B_1, B_2) : E \to \mathbb{R}^2\) be \(L\)-tame, where \(E \subset \mathbb{R}\), and let \(r > 0\). Then, the map \(B^r : r^{-1} \cdot E \to \mathbb{R}^2\), defined by
\[
B^r(x) := (B_1^r(x), B_2^r(x)) := \left( \frac{1}{r} B_1(rx), \frac{1}{r^2} B_2(rx) \right)
\]
is also \(L\)-tame.
Proposition 3.6. An L-tame map defined on $E \subset \mathbb{R}$ extends to an $18L$-tame map defined on $\mathbb{R}$.

Proof. Let $\phi = (\phi_1, \phi_2) : E \to \mathbb{R}^2$ be $L$-tame. By assumption, $\phi_1$ is Lipschitz, and also $\phi_2$ is locally Lipschitz by (3.1). So, extending $\phi_1, \phi_2$ to continuous maps on $\overline{E}$ is no problem, and then (3.1) remains valid on $\overline{E}$. So, we may assume that $E$ is closed to begin with, and we write

$$\mathbb{R} \setminus E = \bigcup_{I \in \mathcal{I}} I,$$

where $\mathcal{I}$ are the components of $\mathbb{R} \setminus E$. We will extend $\phi$ to each interval in $\mathcal{I}$ individually. There are at most two unbounded intervals $I \in \mathcal{I}$. Both of them have an endpoint in $E$, and we define $\phi_1$ on $I$ to be the constant attained at the endpoint, say $x$. Then, we define

$$\phi_2(y) := \int_x^y \phi_1(s) \, ds, \quad y \in I.$$

Evidently $\phi_1$ remains $L$-Lipschitz, and we will worry about condition (3.1) later. Next, fix $I = [x, y] \in \mathcal{I}$ with $x, y \in E$ and $x < y$. Assume for minor notational convenience that

$$\phi_1(x) = \phi_2(x) = x = 0. \quad (3.7)$$

This can be achieved by applying the operations (1)-(2) described above. To understand the problem we are now facing, consider any extension of $\phi = (\phi_1, \phi_2)$ to $I$, denoted by $\phi^I = (\phi_1^I, \phi_2^I)$. Then, if $\phi^I$ is supposed to be tame, we should have $\phi_2^I = \phi_1^I$, and this forces

$$\phi_2(y) = \phi_1^I(y) = \int_0^y \phi_1^I(s) \, ds. \quad (3.8)$$

So, $\phi_1^I$ needs to be chosen so that (3.8) holds – and on the other hand $\phi_1^I$ needs to be a $\sim L$-Lipschitz extension of $\phi_1$. In fact, we claim that $\phi_1^I$ can be taken $7L$-Lipschitz. Let us first attempt the linear extension

$$\tilde{\phi}_1^I(s) := \frac{\phi_1(y)s}{y}, \quad s \in I.$$

This is an $L$-Lipschitz extension of $\phi_1$, but

$$\int_0^y \tilde{\phi}_1^I(s) \, ds = \frac{\phi_1(y)y}{2}, \quad (3.9)$$

which may not agree with $\phi_2(y)$, i.e. the left hand side of (3.8). However, we are not too far off the mark. Recalling (3.7), and then using the tameness assumption (3.1), we have

$$\left| \phi_2(y) - \frac{\phi_1(y)y}{2} \right| \leq |y| \left| \frac{\phi_2(y) - \phi_2(0)}{y - 0} - \phi_1(0) \right| + \frac{|y|\phi_1(y) - \phi_1(0)|}{2} \leq \frac{3L|y|^2}{2} \quad (3.10)$$

We then record an extension result:

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Proof. For $x, y \in \mathbb{R}, x \neq y$, fixed, we note that

$$\left| \frac{B_2^I(x) - B_2^I(y)}{x - y} - B_1^I(x) \right| = \frac{1}{r} \left| \frac{B_2(rx) - B_2(ry)}{rx - ry} - B_1(rx) \right| \leq \frac{L}{r} |rx - ry| = L|x - y|,$$

as desired. \qed

We then record an extension result:

Proposition 3.6. An $L$-tame map defined on $E \subset \mathbb{R}$ extends to an $18L$-tame map defined on $\mathbb{R}$. 

Proof. Let $\phi = (\phi_1, \phi_2) : E \to \mathbb{R}^2$ be $L$-tame. By assumption, $\phi_1$ is Lipschitz, and also $\phi_2$ is locally Lipschitz by (3.1). So, extending $\phi_1, \phi_2$ to continuous maps on $\overline{E}$ is no problem, and then (3.1) remains valid on $\overline{E}$. So, we may assume that $E$ is closed to begin with, and we write

$$\mathbb{R} \setminus E = \bigcup_{I \in \mathcal{I}} I,$$

where $\mathcal{I}$ are the components of $\mathbb{R} \setminus E$. We will extend $\phi$ to each interval in $\mathcal{I}$ individually. There are at most two unbounded intervals $I \in \mathcal{I}$. Both of them have an endpoint in $E$, and we define $\phi_1$ on $I$ to be the constant attained at the endpoint, say $x$. Then, we define

$$\phi_2(y) := \int_x^y \phi_1(s) \, ds, \quad y \in I.$$

Evidently $\phi_1$ remains $L$-Lipschitz, and we will worry about condition (3.1) later. Next, fix $I = [x, y] \in \mathcal{I}$ with $x, y \in E$ and $x < y$. Assume for minor notational convenience that

$$\phi_1(x) = \phi_2(x) = x = 0. \quad (3.7)$$

This can be achieved by applying the operations (1)-(2) described above. To understand the problem we are now facing, consider any extension of $\phi = (\phi_1, \phi_2)$ to $I$, denoted by $\phi^I = (\phi_1^I, \phi_2^I)$. Then, if $\phi^I$ is supposed to be tame, we should have $\phi_2^I = \phi_1^I$, and this forces

$$\phi_2(y) = \phi_1^I(y) = \int_0^y \phi_1^I(s) \, ds. \quad (3.8)$$

So, $\phi_1^I$ needs to be chosen so that (3.8) holds – and on the other hand $\phi_1^I$ needs to be a $\sim L$-Lipschitz extension of $\phi_1$. In fact, we claim that $\phi_1^I$ can be taken $7L$-Lipschitz. Let us first attempt the linear extension

$$\tilde{\phi}_1^I(s) := \frac{\phi_1(y)s}{y}, \quad s \in I.$$

This is an $L$-Lipschitz extension of $\phi_1$, but

$$\int_0^y \tilde{\phi}_1^I(s) \, ds = \frac{\phi_1(y)y}{2}, \quad (3.9)$$

which may not agree with $\phi_2(y)$, i.e. the left hand side of (3.8). However, we are not too far off the mark. Recalling (3.7), and then using the tameness assumption (3.1), we have

$$\left| \phi_2(y) - \frac{\phi_1(y)y}{2} \right| \leq |y| \left| \frac{\phi_2(y) - \phi_2(0)}{y - 0} - \phi_1(0) \right| + \frac{|y|\phi_1(y) - \phi_1(0)|}{2} \leq \frac{3L|y|^2}{2} \quad (3.10)$$

We then record an extension result:
Now, to fix the discrepancy between (3.9) and (3.8), we choose a $6L$-Lipschitz function $\eta_I: [0, y] \to \mathbb{R}$ satisfying
\[
\eta_I(0) = 0 = \eta_I(y) \quad \text{and} \quad \int_0^y \eta_I(s) \, ds = \phi_2(y) - \frac{\phi_1(y)y}{2}.
\]
(3.11)
For example, one can take $\eta_I = c\eta_0$, where $|c| \leq 1$, and
\[
\eta_0(s) = \begin{cases} 
6Ls, & s \in [0, \frac{y}{2}], \\
6L(y - s), & s \in [\frac{y}{2}, y], 
\end{cases}
\]
(3.12)
because
\[
\int_0^y \eta_0(s) \, ds = \frac{3L|y|^2}{2},
\]
which coincides with the upper bound in (3.10). Finally, we set
\[
\phi_1^I := \hat{\phi}_1 + \eta_I,
\]
which is a $7L$-Lipschitz extension of $\phi_1$ (by the first point in (3.11)), and we define $\phi_2^I$ in the only possible way:
\[
\phi_2^I(s) := \int_0^s \phi_2^I(r) \, dr, \quad s \in I.
\]
This function extends $\phi_2$ by a combination of (3.9) and the second point in (3.11).

It remains to check that the tameness condition (3.1) is satisfied on $\mathbb{R}$, with constant $18L$; in fact, we check the 1-sided condition (3.3) with constant $9L$. Pick distinct $x, y \in \mathbb{R}$. If $x, y \in E$, there is nothing to prove. The same is true if $x, y$ are contained on (the closure of) a common interval in $I$, because $\phi_2 = \phi_1$ on these intervals, and recalling the estimate (3.4). So, assume that $x \in E$ and $y \in I \in \mathcal{I}$ with $x < y$, say. Let $x_1 \in E \cap [x, y)$ be the left endpoint of $I$. Then, use the triangle inequality multiple times:
\[
|\phi_2(x) - \phi_2(y)| - \phi_1(x)(x - y) | \leq |\phi_2(x) - \phi_2(x_1)| - \phi_1(x)(x - x_1) | \\
+ |\phi_2(x_1) - \phi_2(y)| - \phi_1(x_1)(x_1 - y) | \\
+ |\phi_1(x_1) - \phi_1(x)|(x_1 - y) | \\
\leq L|x - x_1|^2 + 7L|x_1 - y|^2 + L|x - x_1||x_1 - y| \\
\leq 9L|x - y|^2.
\]
This completes the proof. 

\[ \square \]

3.1.1. Corona decomposition for tame maps. In this section, we prove the first main result of this paper, a corona decomposition for maps that are tame in the sense of (3.1). We start with the following rather obvious definition:

Definition 3.13 (Tame-linear and tame-affine maps). A map $\phi = (\phi_1, \phi_2): \mathbb{R} \to \mathbb{R}^2$ is called tame-linear (or affine) if $\phi_1: \mathbb{R} \to \mathbb{R}$ is linear (or affine) and $\hat{\phi}_2 = \hat{\phi}_1$. A tame-linear map is $L$-tame-linear if $\phi_1$ is $L$-Lipschitz.

It would be nice to know the answer to the following question:

Question 2. Does there exist a constant $\delta > 0$ with the following property? Let $\phi: [0, 1] \to \mathbb{R}^2$ be $1$-tame. Then there exist a tame-linear map $L: \mathbb{R} \to \mathbb{R}^2$ and a $(1 - \delta)$-tame map $\phi_3: [0, 1] \to \mathbb{R}^2$ such that
\[
|\{x \in [0, 1] : \phi(x) = [\phi_3 + L](x)\}| \geq \delta.
\]
In other words: do 1-tame maps have big pieces of \((1-\delta)\)-tame maps (up to subtracting a tame-linear map)? Since we were not able to answer this question, we show something slightly weaker, namely that 1-tame maps admit a "corona decomposition" with \(\eta\)-tame maps, for any \(\eta > 0\). To formulate the statement, we recall some terminology.

**Definition 3.14** (Dyadic intervals and trees). We write "\(D\)" for the standard dyadic intervals of \(\mathbb{R}\). For \(j \in \mathbb{Z}\), we further write \(D_j \subset D\) for the dyadic intervals \(Q\) of length \(|Q| = 2^{-j}\). A collection \(T \subset D\) is called a tree if

1. \(T\) contains a "top interval" \(Q(T)\), that is, a unique maximal element.
2. \(T\) is "coherent": if \(Q \in T\), then \(Q' \in T\) for all \(Q \subset Q' \subset Q(T)\).
3. If \(Q \in T\), then either both, or neither, of the children of \(Q\) lie in \(T\).

Now we are prepared to formulate the statement of the corona decomposition:

**Theorem 3.15.** For every \(\eta \in (0,1)\), there exists a constant \(C \geq 1\) such that the following holds. Let \(\phi: \mathbb{R} \to \mathbb{R}^2\) be 1-tame. Then, there exists a decomposition \(D = B \cup G\) with the following properties. First, the intervals in \(B\) satisfy a Carleson packing condition:

\[
\sum_{Q \in B} |Q| \leq C|Q_0|, \quad Q_0 \in D. \tag{3.16}
\]

Second, the intervals in \(G\) can be decomposed into a "forest" \(F\) of disjoint trees \(T\),

\[
G = \bigcup_{T \in F} T, \tag{3.17}
\]

whose top intervals satisfy a Carleson packing condition:

\[
\sum_{Q \in F, Q(T) \subset Q_0} |Q(T)| \leq C|Q_0|, \quad Q_0 \in D. \tag{3.18}
\]

For every \(T \in F\) there exists a 2-tame-linear map \(L_T: \mathbb{R} \to \mathbb{R}^2\) and an \(\eta\)-tame map \(\psi_T: \mathbb{R} \to \mathbb{R}^2\) such that \(\psi_T + L_T\) approximates \(\phi\) well at the resolution of the intervals in \(T\):

\[
d_\pi(\phi(s), [\psi_T + L_T](s)) \leq \eta|Q|, \quad s \in 2Q, \ Q \in T. \tag{3.19}
\]

In (3.19), \(d_\pi\) refers to the parabolic metric on \(\mathbb{R}^2\):

\[
d_\pi((x, s), (y, t)) := \max\{|x - y|, \sqrt{|s - t|}\}, \quad (x, s), (y, t) \in \mathbb{R}^2,
\]

and \(2Q\) is the interval with the same midpoint but twice the length of \(Q\). The proof of Theorem 3.15 uses, as a black box, the corona decomposition for \(\mathbb{R}\)-valued Lipschitz functions on \(\mathbb{R}\). This statement looks very similar to the one of Theorem 3.15:

**Theorem 3.20.** For every \(\eta \in (0,1)\), there exists a constant \(C \geq 1\) such that the following holds. Let \(\phi: \mathbb{R} \to \mathbb{R}\) be 1-Lipschitz. Then, there exists a decomposition \(D = B \cup G\) with the properties (3.16), (3.17), (3.18), and such that the following holds. For every \(T \in F\) there exists a 2-Lipschitz linear function \(L_T: \mathbb{R} \to \mathbb{R}\) and an \(\eta\)-Lipschitz function \(\psi_T: \mathbb{R} \to \mathbb{R}\) such that

\[
|\phi(s) - (\psi_T + L_T)(s)| \leq \eta|Q|, \quad s \in 2Q, \ Q \in T. \tag{3.21}
\]

This statement follows, after a moment’s thought, from the corona decomposition in [15, p.61, (3.33)]. We give the details in Appendix A. Before proving Theorem 3.15, we record version of Theorem 3.15 for \(N\)-tame maps with \(N \geq 1\). The main point here is that
the Carleson packing constants do not depend on “N”, which only makes an appearance in the “quality of approximation” in (3.23).

Corollary 3.22 (Corona for N-tame maps). For every $\eta \in (0, 1)$, there exists a constant $C \geq 1$ such that the following holds. Let $\phi : \mathbb{R} \to \mathbb{R}^2$ be N-tame, $N \geq 1$. Then, there exists a decomposition $D = B \cup G$ with the properties (3.16), (3.17), (3.18), and such that the following holds. For every $T \in \mathcal{F}$, there exists a 2N-tame-linear map $L : \mathbb{R} \to \mathbb{R}^2$ and an $(\eta N)$-tame map $\psi_T : \mathbb{R} \to \mathbb{R}^2$ such that

$$d_\pi(\phi(s), [\psi_T + L_T](s)) \leq (\eta N)|Q|, \quad s \in 2Q, \; Q \in \mathcal{T}. \tag{3.23}$$

Proof. The map $\tilde{\phi} := N^{-1} \phi : \mathbb{R} \to \mathbb{R}^2$ is 1-tame, so Theorem 3.15 applies to it verbatim. This yields the desired decomposition $D = B \cup G$ and, for each $T \in \mathcal{F}$, a 2-tame-linear map $\tilde{L}_T : \mathbb{R} \to \mathbb{R}^2$, and an $\eta$-tame map $\tilde{\psi}_T : \mathbb{R} \to \mathbb{R}^2$, such that (3.19) holds for $\tilde{\phi}, \tilde{\psi}_T, \tilde{L}_T$.

Now, we define the $(\eta N)$-tame map $\psi_T := \eta N \tilde{\psi}_T$, and the 2N-tame-linear map $L_T := \eta N \tilde{L}_T$. Then,

$$d_\pi(\phi(s), [\psi_T + L_T](s)) \leq \eta d_\pi(\phi(s), [\tilde{\psi}_T + \tilde{L}_T](s)) \leq (\eta N)|Q|$$

for $s \in 2Q$ with $Q \in \mathcal{T}$. In the first inequality, we used $N \geq 1$ to infer that $\sqrt{N} \leq N$. \qed

There is also a similar version of Theorem 3.20 for M-Lipschitz functions, $M \geq 1$, but we omit stating this explicitly. We then turn to the proof of Theorem 3.15.

Proof of Theorem 3.15. Write $\phi = (\phi_1, \phi_2)$, where now $\phi_1 : \mathbb{R} \to \mathbb{R}$ is 1-Lipschitz. We apply the Lipschitz corona decomposition, Theorem 3.20, to $\phi_1$ with the parameter $\delta := \min\{\eta^2/5, \eta/17\} > 0$. The result is a decomposition $D = B \cup G$ of the type desired in the statement Theorem 3.15, accompanied with the trees $T \in \mathcal{F}$, and corresponding $\delta$-Lipschitz functions $\phi_T : \mathbb{R} \to \mathbb{R}$ and linear 2-Lipschitz maps $L_T : \mathbb{R} \to \mathbb{R}$ with the property that

$$|\phi_1(s) - [\phi_T + L_T](s)| \leq \delta |Q|, \quad s \in 2Q, \; Q \in \mathcal{T}. \tag{3.24}$$

Fix a tree $T \in \mathcal{F}$, and consider the top interval $Q(T) = [x, y]$. Based on the existence of the function $\phi_T$, we would now like to produce an $\eta$-tame function $\psi_T : [x, y] \to \mathbb{R}^2$ satisfying (3.19). The tame-linear part will be defined in the obvious way: $L_T = (L_T, P_T) : \mathbb{R} \to \mathbb{R}^2$, where

$$P_T(s) := \int_x^s L_T(r) \, dr, \quad s \in \mathbb{R}.$$

To define $\psi_T$, probably the first idea to try is to set $\psi_1 := \phi_T$, and define

$$\psi_2(s) := \phi_2(x) + \int_x^s \psi_1(r) \, dr = \phi_2(x) + \int_x^s \phi_T(r) \, dr, \quad s \in \mathbb{R}. \tag{3.25}$$

The good news are that $\psi_2 = \psi_1$, and $\psi_2(x) = \phi_2(x)$, so at least (3.19) is satisfied for $s = x$ (recalling that (3.24) holds, and noting that $\phi_2(x) = \psi_2(x) + P_T(x)$). The bad news is that there is no a priori reason why $[\psi_2 + P_T](s) - \phi_2(s)$ would be small for any $s \in (x, y]$. To fix this, we in fact need to modify $\phi_T$ slightly before defining $\psi_1$ and $\psi_2$ exactly as above.

Let $S(T)$ be the collection of minimal intervals in $T$ (possibly an empty collection). Also, write

$$E := Q(T) \setminus \bigcup_{S \in S(T)} S.$$
for the set of points in \( Q(T) \) in "infinite branches" of \( T \). Observe that, by (3.24), we have
\[
\phi_1(s) = [\phi_T + L_T](s), \quad s \in E.
\]
Now, for \( S \in S(T) \) fixed, we will slightly modify the restriction of \( \phi_T \) to \( \frac{1}{2}S \), which is the interval with the same centre but half the length as \( S \). The geometric feature of \( \frac{1}{2}S \) needed in the future is that if \( Q \in T \) with \(|Q| < |S|\), then
\[
2Q \cap \frac{1}{2}S = \emptyset.
\]
This is clear, because \(|Q| < |S|\) forces \( Q \cap S = \emptyset \) by the minimality of \( S \in S(T) \).

While modifying \( \phi_T \), we want to maintain the property that \( \phi_T \) is 17\( \delta \)-Lipschitz, and that (3.24) holds with "5\( \delta \)" replaced by "5\( \delta \)". However, in addition, we want to arrange that
\[
\int_S \phi_1(s) \, ds = \int_S [\phi_T + L_T](s) \, ds.
\]
This is easily done, using the "triangle" function familiar from (3.12), and observing that
\[
\left| \int_S \phi_1(s) - [\phi_T + L_T](s) \, ds \right| \leq \delta |S|^2
\]
by (3.24). Now, if we replace \( \phi_T \) by \( \phi_T + \eta_S \) on \( S \), we find that the "new" \( \phi_T \) is 17\( \delta \)-Lipschitz, and (3.27) holds. Moreover, since \( \|\eta_S\|_{L^\infty(S)} \leq 4\delta|S| \), there is some hope that (3.24) remains valid with "5\( \delta \)" replaced by "5\( \delta \)". To prove this carefully, fix \( Q \in T \) and \( s \in 2Q \). During the procedure above, we only modified \( \phi_T \) on sets of the form \( \frac{1}{2}S \), with \( S \in S(T) \). So, if \( s \notin \frac{1}{2}S \) for any \( S \in S(T) \), then (3.24) is certainly valid, with original constant. So, assume that \( s \in \frac{1}{2}S \) for some \( S \in S(T) \). Then \( s \in 2Q \cap \frac{1}{2}S \), so (3.26) forces \(|S| \leq |Q|\). Consequently,
\[
\|\eta_S\|_{L^\infty} \leq 4\delta|S| \leq 4\delta|Q|.
\]
Since the "original" \( \phi_T \) only differs from the "new" \( \phi_T \) on \( \frac{1}{2}S \) by the function \( \eta_S \), we see that
\[
|\phi_1(s) - [\phi_T + L_T](s)| \leq \delta|Q| + 4\delta|Q| = 5\delta|Q|,
\]
as desired.

Now, assume that similar modifications to \( \phi_T \) have been performed inside all intervals \( S \in S(T) \), and in particular (3.27) holds for all \( S \in S(T) \). We infer the following corollary: if \( s \in Q(T) \), and either
\[
s \in E \quad \text{or} \quad s \in \partial S \text{ with } S \in S(T),
\]
then
\[
\int_x^s \phi_1(s) \, ds = \int_x^s [\phi_T + L_T](s) \, ds.
\]
Recall that \( x \) is the left endpoint of \( Q(T) \). Now, with the fine-tuned definition of \( \phi_T \), we proceed as planned, setting \( \psi_1 := \phi_T \) and defining \( \psi_2 \) as in (3.25). Since the map
ψ = (ψ₁, ψ₂): Q(T) → ℝ is now 17δ-tame, and 17δ ≤ η by definition, it remains to check that (3.19) holds for all x ∈ Q ∈ T. This amounts to checking that

\[ |φ₂(s) - [ψ₂ + P_T](s)| \leq η^2|Q|^2, \quad s \in 2Q ∈ T. \]  (3.29)

First, consider s ∈ E. Then, since φ₂ = φ₁, we have

\[ φ₂(s) = φ₂(x) + \int_x^s φ₁(r) dr = φ₂(x) + \int_x^s [φ_T + L_T](r) dr = ψ₂(s) + P_T(s). \]  (3.30)

So, the difference in (3.29) is zero, as it should be. Next, fix some Q ∈ T, and consider s ∈ 2Q. Then, there exists a point

\[ s₁ ∈ Q \cap \left[ E \cup \bigcup_{S ∈ S(T)} \partial S \right] \]

satisfying |s - s₁| ≤ |Q|. Then φ₂(s₁) = ψ₂(s₁) + P_T(s₁), repeating the computation on line (3.30). Consequently,

\[ |φ₂(s) - [ψ₂ + P_T](s)| = |φ₂(s) - φ₂(s₁) - ([ψ₂ + P_T](s) - [ψ₂ + P_T](s₁))| \]

\[ = \int_{s₁}^s |φ₁(r)| dr - \int_{s₁}^s [φ_T + L_T](r) dr \]

\[ \leq \int_{s₁}^s |φ₁(r) - [φ_T + L_T](r)| dr \leq 5δ|Q|^2, \]

noting in the last inequality that [s₁, s] ⊂ 2Q, so (3.24) (with “5δ” in place of “δ”) holds for all points in [s₁, s]. We conclude from this estimate and (3.24) that

\[ d_π(φ(s), [ψ + L_T](s)) \leq \max\{5δ|Q|, \sqrt{5δ}|Q|\} \leq η|Q|, \quad s \in 2Q, \quad Q ∈ T, \]

recalling that \( \sqrt{5δ} \leq η \). The proof is complete. \qed

Tame maps will now go away for a moment, but they will return in Section 3.3, where we relate them to intrinsic Lipschitz functions on the Heisenberg group.

3.2. The Heisenberg group.

Definition 3.31 (Heisenberg group, dilations, and distance). The Heisenberg group \( \mathbb{H} \) is the group \( (\mathbb{R}^3, \cdot) \) with

\[ (x, y, t) \cdot (x', y', t') := (x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y)), \quad (x, y, t), (x', y', t') ∈ \mathbb{R}^3. \]

The Heisenberg dilations \( (δ_λ)_{λ > 0} \) are the group automorphisms

\[ δ_λ: \mathbb{H} → \mathbb{H}, \quad δ_λ(x, y, t) = (λx, λy, λ^2t). \]

We define the Heisenberg metric \( d: \mathbb{H} → [0, +∞) \) by setting \( d(p, q) := \|q^{-1} \cdot p\| \), where

\[ \|(x, y, t)\| := \max\{\sqrt{x^2 + y^2}, \sqrt{|t|}\}. \]  (3.32)

Remark 3.33. In the introduction, we used the notation "\| \cdot \|" for the Korányi norm \( \|(x, y, t)\| = ((x^2 + y^2)^2 + 16t^2)^{1/4} \), which is a quantity comparable to the "max-norm" in (3.32). From now on, \( \| \cdot \| \) always refers to the quantity in (3.32).
Definition 3.34 (Horizontal gradient). Let $\Omega \subset \mathbb{H}$ be an open set. The horizontal gradient of a $C^1$ function $u : \Omega \to \mathbb{R}$ is defined by

$$\nabla_{\mathbb{H}} u = (X u, Y u),$$

where

$$X := \partial_x - \frac{p}{q} \partial_t \quad \text{and} \quad Y := \partial_y + \frac{p}{q} \partial_t.$$

We record that good kernels (Definition 1.5) give rise to SKs.

Proposition 3.35. If $k$ is a good kernel on $\mathbb{H}$, then $K : \mathbb{H} \times \mathbb{H} \setminus \triangle \to [0, +\infty)$, defined by $K(p, q) := k(q^{-1} \cdot p)$, is an SK on $(\mathbb{H}, d)$ with $\alpha = 1/2$.

Proof. The first claim follows immediately from the $-1$-homogeneity of $k$, as

$$|k(p)| = |k(\delta_{|p|}(\delta_{|p|-1}(p)))| = \|p\|^{-1} |k(\delta_{|p|-1}(p))| \leq \|p\|^{-1} \sup_{|q|=1} |k(q)|, \quad p \in \mathbb{H}\setminus\{0\},$$

and hence, $|k(q^{-1} \cdot p)| \lesssim d(p, q)^{-1}$. The second claim can be proven by arguments analogous to [4, Proposition 3.11] and [5, Lemma 2.1], observing that the horizontal gradient $\nabla_{\mathbb{H}}$ from Definition 3.34 satisfies

$$\nabla_{\mathbb{H}}(k \circ \delta_\lambda)(p) = \lambda \nabla_{\mathbb{H}} k(\delta_\lambda(p)) \quad \text{and} \quad \nabla_{\mathbb{H}} \lambda^{-1} k(p) = \lambda^{-1} \nabla_{\mathbb{H}} k(p), \quad p \in \mathbb{H}\setminus\{0\}, \lambda > 0,$$

and hence, since $k \circ \delta_\lambda = \lambda^{-1} k$,

$$|\nabla_{\mathbb{H}} k(p)| \lesssim \|p\|^{-2}, \quad p \in \mathbb{H}\setminus\{0\}.$$

The exponent $\alpha = \frac{1}{2}$ arises when verifying the Hölder continuity of $q \to K(q^{-1} \cdot p)$. □

Definition 3.36 (Homogeneous subgroups). A subgroup of $\mathbb{H}$ is homogeneous if it is closed under dilations. Homogeneous subgroups of $\mathbb{H}$ are either contained in the $xy$-plane, in which case they are called horizontal, or they contain the $t$-axis, in which case they are said to be vertical.

Definition 3.37 (Horizontal lines). A left translate of a non-trivial horizontal subgroup $\mathbb{V} \subset \mathbb{H}$ is called a horizontal line in $\mathbb{H}$.

Definition 3.38 (Projections and components). Let $\mathbb{W} \subset \mathbb{H}$ be a vertical subgroup of topological dimension 2. We associate to $\mathbb{W}$ the unique horizontal subgroup $\mathbb{L} \subset \mathbb{W}$, and the complementary horizontal subgroup $\mathbb{V}$. The choice of $\mathbb{V}$ is somewhat arbitrary, but we declare here $\mathbb{V}$ to be the Euclidean orthogonal complement of $\mathbb{L}$ in the $xy$-plane. We write $\mathbb{T}$ for the $t$-axis. Then, every point $p \in \mathbb{H}$ has a unique “coordinate” decomposition

$$p = v \cdot w = v \cdot l \cdot t,$$

where $w = l \cdot t = t \cdot l \in \mathbb{W}$ with $l \in \mathbb{L}$ and $t \in \mathbb{T}$, and $v \in \mathbb{V}$. This decomposition gives rise to the vertical projections $\pi_{\mathbb{W}} : \mathbb{H} \to \mathbb{W}$ and $\pi_{\mathbb{T}} : \mathbb{H} \to \mathbb{T}$, given by $p \mapsto w$ and $p \mapsto t$, and the horizontal projections $\pi_{\mathbb{V}} : \mathbb{H} \to \mathbb{V}$ and $\pi_{\mathbb{L}} : \mathbb{H} \to \mathbb{L}$, given by $p \mapsto v$ and $p \mapsto l$, respectively. The horizontal projections are 1-Lipschitz group homomorphisms, while $\pi_{\mathbb{W}}$ and $\pi_{\mathbb{T}}$ are neither Lipschitz maps nor group homomorphisms. Nevertheless, $\pi_{\mathbb{T}}$ and $\pi_{\mathbb{W}}$ satisfy

$$\|\pi_{\mathbb{T}}(p)\| \leq \|\pi_{\mathbb{W}}(p)\| \leq C\|p\|, \quad p \in \mathbb{H} \quad (3.39)$$

for some absolute constant $C \geq 1$. If $\phi : X \to \mathbb{W}$ is a map, where $X$ is any set, we define the first and second components of $\phi$ to be the functions $\phi_1 = \pi_{\mathbb{L}} \circ \phi : X \to \mathbb{L}$ and $\phi_2 = \pi_{\mathbb{T}} \circ \phi : X \to \mathbb{T}$.
Remark 3.40. If \( W = L \times T \) is a vertical subgroup with complementary subgroup \( V \), we will write in coordinates \( W = \{ y \cdot t : y \in L \text{ and } t \in T \} \cong \{ (y, t) : y, t \in \mathbb{R} \} = \mathbb{R}^2 \). Similarly, \( V \) will be identified with \( \mathbb{R} \). Under these identifications, the components \( \phi_1 : V \to L \) and \( \phi_2 : V \to T \) are parametrised by \( (y, t) \). The horizontal projection \( p \) is called the horizontal projection \( p \) over horizontal subgroups have nicer properties than those over vertical subgroups, essentially because \( \pi_V \) is a group homomorphism.

3.3. Intrinsic Lipschitz graphs. We define intrinsic Lipschitz functions and graphs over horizontal subgroups in \( \mathbb{H} \). On the one hand, this is just a special case of a definition of Franchi, Serapioni, and Serra Cassano [25]. On the other hand, intrinsic Lipschitz functions over horizontal subgroups have nicer properties than those over vertical subgroups, essentially because \( \pi_V \) is a group homomorphism.

Definition 3.41 (Intrinsic L-Lipschitz graphs and functions). For \( W, V \) as in Definition 3.38, and \( \alpha > 0 \), we define the cone

\[
C_V(\alpha) := \{ p \in \mathbb{H} : \| \pi_V(p) \| \leq \alpha \| \pi_W(p) \| \}.
\]

A set \( \Gamma \subset \mathbb{H} \) is called an intrinsic L-Lipschitz graph over \( V \), or simply an intrinsic Lipschitz graph, if there exists \( L > 0 \) such that

\[
(p \cdot C_V(\alpha)) \cap \Gamma = \{ p \}, \quad \text{for all } p \in \Gamma \text{ and all } \alpha < \frac{1}{L}.
\]

(3.42)

Let \( \phi : E \to W \) be a map, where \( E \subset V \). The function \( \phi \) is called intrinsic L-Lipschitz if \( \Gamma(\phi) := \{ v \cdot \phi(v) : v \in E \} \) is an intrinsic L-Lipschitz graph. The map \( v \mapsto \Phi(v) := v \cdot \phi(v) \) is called the graph map of \( \phi \).

Proposition 3.43. A set \( \Gamma \subset \mathbb{H} \) is an intrinsic Lipschitz graph over a horizontal subgroup \( V \) if and only if the horizontal projection \( \pi_V \) restricted to \( \Gamma \) is injective with metric Lipschitz inverse \( \Phi : \pi_V(\Gamma) \to \Gamma \).

Proof. Let \( \Gamma \subset \mathbb{H} \) be an intrinsic L-Lipschitz graph over \( V \). If \( p, q \in \Gamma \) then

\[
\| \pi_V(q) - \pi_V(p) \| = \| \pi_V(q \cdot p) \| \leq \| \pi_V(q \cdot p) \| \leq \frac{1}{L} \| \pi_W(q \cdot p) \|.
\]

(3.44)

which implies by the triangle inequality that \( \| q^{-1} \cdot p \| \leq (1 + L)\| \pi_V(q)^{-1} \cdot \pi_V(p) \| \). Consequently, the projection \( \pi_V \) restricted to \( \Gamma \) is bilipschitz, so the map \( \Phi : \pi_V(\Gamma) \to \Gamma \), given by the relation \( \pi_V(\Phi(v)) = v \), is well-defined and \( (1 + L) \)-Lipschitz.

Conversely, assume that \( \Gamma \subset \mathbb{H} \) is a set such that the horizontal projection \( \pi_V \) restricted to \( \Gamma \) is injective with L-Lipschitz inverse \( \Phi \). Then, if \( p = \Phi(v), q = \Phi(v') \in \Gamma \), we have

\[
\| \pi_W(\Phi(v')^{-1} \cdot \Phi(v)) \| \leq C \| \Phi(v')^{-1} \cdot \Phi(v) \| \leq C L \| (v')^{-1} \cdot v \| = C L \| \pi_V(q^{-1} \cdot p) \|.
\]

which shows that \( \Gamma \) is an intrinsic CL-Lipschitz graph over \( V \). \( \square \)

Remark 3.45. We record that every intrinsic L-Lipschitz graph \( \Gamma \subset \mathbb{H} \) can be parametrised by an intrinsic L-Lipschitz function defined on \( E := \pi_V(\Gamma) \subset V \). Simply, let \( \Phi : E \to \Gamma \) be the map defined in Proposition 3.43, and let

\[
\phi_T(v) := \pi_W(\Phi_T(v)).
\]

(3.46)

Then \( \Phi_T(v) = \pi_V(\Phi_T(v)) \cdot \pi_W(\Phi_T(v)) = v \cdot \phi_T(v) \) for \( v \in E \), so indeed \( \Gamma = \Phi(\phi) \). Thus, \( \Gamma \) is parametrised by \( \phi \), and \( \phi \) is intrinsic L-Lipschitz by definition.
Lemma 3.47. Let $\phi: E \to \mathbb{W}$ be an intrinsic $L$-Lipschitz function, with $E \subset \mathbb{V}$. Then the first component $\phi_1$, recall Definition 3.38, is $L$-Lipschitz. Consequently, under the identification from Remark 3.40, the function $\phi_1 : \mathbb{R} \to \mathbb{R}$ is Euclidean Lipschitz.

Proof. Indeed, recall from (3.46) that $\phi(v) = \pi_\mathbb{W}(\Phi(v))$, where $\Phi : E \to \Gamma$ is the graph map of $\Gamma(\phi)$. Consequently $\phi_1 = \pi_\mathbb{L} \circ \Phi$. Then, using the fact that $\pi_\mathbb{L}$ is a group homomorphism, we infer that

$$
\|\phi_1(v')^{-1} \cdot \phi_1(v)\| = \|\pi_\mathbb{L}(\Phi(v'))^{-1} \cdot \pi_\mathbb{L}(\Phi(v))\|
\leq \|\pi_\mathbb{W}(\Phi(v')^{-1} \cdot \Phi(v))\|
\tag{3.44}
\leq L\|\pi_\mathbb{V}(\Phi(v'))^{-1} \cdot \pi_\mathbb{V}(\Phi(v))\| = L\|v'\|^{-1} \cdot v
$$

for all $v, v' \in E$. □

We conclude this section with an area formula for intrinsic Lipschitz graphs over horizontal subgroups.

Proposition 3.48. Let $\phi = (\phi_1, \phi_2) : I \subset \mathbb{V} \to \mathbb{W}$ be an intrinsic Lipschitz map defined on an interval $I \subset \mathbb{V}$, and let $\Phi$ be its graph map. Then, $\Phi(I)$ is a 1-regular subset of $(\mathbb{H}, d)$ and

$$
\mathcal{H}^1(\Phi(A)) = \int_A \left(1 + \phi_1(v)^2\right)^{1/2} dv, \quad A \subset I \text{ Borel.}
$$

(3.49)

Proof. By Proposition 3.43, the map $\Phi : I \to (\mathbb{H}, d)$ is a Lipschitz curve. Since $\Phi$ is injective, the length with respect to the metric $d$ of a subcurve $\Phi([a, b])$, $[a, b] \subset I$, agrees with $\mathcal{H}^1(\Phi([a, b]))$, see for instance [2, Theorem 2.6.2.]. Moreover,

$$
\text{length}_{I|\iota}(\pi(\Phi([a, b]))) \leq \text{length}_{d}(\Phi([a, b])) \leq \text{length}_{cc}(\Phi([a, b])),
$$

(3.50)

where the left-hand side denotes the Euclidean length of the image of $\Phi([a, b])$ under the projection $\pi : \mathbb{H} \to \mathbb{R}^2$, $(x, y, t) \mapsto (x, y)$, and $d_{cc}$ is the standard sub-Riemannian distance on $\mathbb{H}$, see [1]. Since $\pi \circ \Phi$ is (Euclidean) Lipschitz, the left-hand side of (3.50) equals

$$
\int_a^b |(\pi \circ \Phi)'(v)| dv,
$$

and the same is true for the right-hand side, cf. e.g. [29]. Using

$$
|(\pi \circ \Phi)'(v)| = \left(\|\pi_\mathbb{V}(\Phi(v))'\|^2 + \|\pi_\mathbb{L}(\Phi(v))'\|^2\right)^{1/2} = \left(1 + \phi_1(v)^2\right)^{1/2},
$$

we have thus established (3.49) for $A = [a, b]$. The case of Borel sets $A \subset I$ follows by approximation. □

3.3.1. Connection between tame maps and intrinsic Lipschitz graphs. In this section, let $\mathbb{W} = \{(0, y, t) : y, t \in \mathbb{R}\}$, $\mathbb{L} = \{(0, y, 0) : y \in \mathbb{R}\}$, and $\mathbb{V} = \{(x, 0, 0) : x \in \mathbb{R}\}$. As we discussed in Remark 3.40, we will identify $\mathbb{W} \cong \mathbb{R}^2$ and $\mathbb{V} \cong \mathbb{R} \cong \mathbb{L}$. With these identifications, we have the following relationship between intrinsic Lipschitz functions and tame maps.

Proposition 3.51. Let $E \subset \mathbb{V}$. If $\phi = (\phi_1, \phi_2) : E \to \mathbb{W}$ is intrinsic $L$-Lipschitz, then $\phi_1, \phi_2) : E \to \mathbb{R}^2$ is $2L^2$-tame.
Proof. A formula for the vertical projection $\pi_\mathcal{W}$ is

$$
\pi_\mathcal{W}(x, y, t) = (y, t - \frac{\phi_1(v)}{2}), \quad (x, y, t) \in \mathbb{H},
$$

while $\pi_\mathcal{V}(x, y, t) = x$. The graph map of $\phi$ is given by

$$
\Phi(v) = v \cdot \phi(v) = (v, \phi_1(v), \phi_2(v) + \frac{\phi_1(v)}{2}), \quad v \geq (v, 0, 0) \in E,
$$

and consequently $\Phi(v_1)^{-1} \cdot \Phi(v_2) =

$$
\left( v_2 - v_1, \phi_1(v_2) - \phi_1(v_1), \phi_2(v_2) - \phi_2(v_1) + \frac{\phi_1(v_1) + \phi_1(v_2)}{2}(v_2 - v_1) \right).
$$

(3.52)

Since $\phi : E \to \mathcal{W}$ is intrinsic $L$-Lipschitz, $\Phi(E)$ is an intrinsic $L$-Lipschitz graph, which means that

$$
\|\pi_\mathcal{W}(\Phi(v_1)^{-1} \cdot \Phi(v_2))\| \leq L\|\pi_\mathcal{V}(\Phi(v_1)^{-1} \cdot \Phi(v_2))\|, \quad v_1, v_2 \in E.
$$

Spelling out the last condition, one finds that

$$
|\phi_1(v_2) - \phi_1(v_1)| \leq L|v_2 - v_1|, \quad v_1, v_2 \in E,
$$

(3.53)

and

$$
\left| \frac{\phi_2(v_2) - \phi_2(v_1)}{v_2 - v_1} + \phi_1(v_1) \right| \leq L^2|v_2 - v_1|, \quad v_1, v_2 \in E, \quad v_1 \neq v_2.
$$

(3.54)

But (3.54) is exactly the 1-sided tameness condition (3.3) for the map $(\phi_1, -\phi_2)$. \qed

Remark 3.55. Recall from Remark 3.2 that the first component of an $L$-tame functions is automatically $L$-Lipschitz. Thus, if conditions (3.53)-(3.54) hold for some $L < 1/2$, then actually (3.53) holds with the better constant "$2L^2$"! On the other hand, assume that (3.53)-(3.54) hold for some $L \geq 1$, and $E$ contains an open interval $I$. Then $\hat{\phi}_2(v) = -\phi_1(v)$ for $v \in I$ which implies, by the calculation in (3.4), that (3.54) actually holds with constant "$L$" for $v_1, v_2 \in I$.

In conclusion, if $E$ is an interval, the best constants in the inequalities (3.53) and (3.54) are actually within a multiple of "2" from each other.

Thanks to the connection between tame maps and intrinsic Lipschitz functions, Proposition 3.6 (extension of tame maps) implies an extension result for intrinsic Lipschitz graphs over horizontal subgroups.

Proposition 3.56. Let $\phi : E \to \mathcal{W}$ be an intrinsic $L$-Lipschitz function. Then there exists an intrinsic $L'$-Lipschitz function $\tilde{\phi} : \mathcal{V} \to \mathcal{W}$ for $L' \leq \max\{L, L^2\}$ such that $\tilde{\phi}|E = \phi$.

Proof. Since $\phi = (\phi_1, \phi_2)$ is intrinsic $L$-Lipschitz by assumption, the map $(\phi_1, -\phi_2)$ is $2L^2$-tame according to Proposition 3.51. The extension result from Proposition 3.6 then allows us to find a $36L^2$-tame map $(\tilde{\phi}_1, \tilde{\phi}_2) : \mathbb{R} \to \mathbb{R}^2$ with $(\tilde{\phi}_1, -\tilde{\phi}_2)|E = (\phi_1, -\phi_2)$. Thus, $\tilde{\phi} = (\tilde{\phi}_1, \tilde{\phi}_2)$ satisfies the conditions (3.53) and (3.54) for all $v_1, v_2 \in \mathbb{R}$, $v_1 \neq v_2$, with "$L$" replaced by $L' = \max\{6L, 36L^2\}$. \qed
4. The exponential kernel appears

4.1. Good kernels and intrinsic Lipschitz graphs. We fix a good kernel $k: \mathbb{H} \setminus \{0\} \to \mathbb{C}$, and gradually start proving that it is a CZ kernel for $(\mathcal{H}^1 \text{ restricted to})$ any intrinsic Lipschitz graph over a horizontal subgroup in $\mathbb{H}$. We fix a horizontal subgroup $\mathcal{V}$ with complementary vertical subgroup $\mathcal{W}$, and an intrinsic $L$-Lipschitz function $\phi = (\phi_1, \phi_2): \mathcal{V} \to \mathcal{W}$, for $L \geq 1$. We assume with no loss of generality that $\mathcal{V} \cong \{(x,0,0) : x \in \mathbb{R}\} \cong \mathbb{R}$ and $\mathcal{W} \cong \{(0,y,t) : y,t \in \mathbb{R}\} \cong \mathbb{R}^2$. The main point of this section is to show how Theorem 1.7 can be reduced to a statement involving only Lipschitz functions and tame maps defined on $\mathbb{R}$, see Theorem 4.8 below.

Let $\Phi$ be the graph map of $\phi$, and let $\Gamma = \Phi(\mathcal{V}) \subset \mathbb{H}$ be the intrinsic graph of $\phi$. Write $\mu := \mathcal{H}^1|_{\gamma}$, and let $\hat{K}: \mathbb{H} \times \mathbb{H} \setminus \Delta \to \mathbb{C}$ be the SK $\hat{K}(p,q) := k(q^{-1}p)$. We start by inferring from the area formula, Proposition 3.48, that

$$T_{\mu, \varepsilon}g(\Phi(w)) = \int_{\{v \in \mathbb{R} : d(\Phi(v), \Phi(w)) > \varepsilon\}} K(\Phi(w), \Phi(v))g(\Phi(v)) \left(1 + |\phi_1(v)|^2\right)^{1/2} dv$$

for all $w \in \mathbb{R}$ and $g \in \bigcup_{1 < p < \infty} L^p(\mathbb{R})$. Since

$$1 \leq \sqrt{1 + |\phi_1(v)|^2} \leq \sqrt{1 + L^2} \text{ for } \mathcal{H}^1 \text{ a.e. } v,$$

we are reduced to considering the $\varepsilon$-SIO $T_\varepsilon := T_{\varepsilon, \mathcal{L}^1}$ induced by the kernel $(v,w) \mapsto K(\Phi(w), \Phi(v))$, namely

$$T_\varepsilon f(w) = \int_{|v-w| > \varepsilon} K(\Phi(w), \Phi(v))f(v)dv, \quad f \in \bigcup_{1 < p < \infty} L^p(\mathbb{R}), \quad w \in \mathbb{R}.$$ (4.2)

The truncations appearing in (4.1) and (4.2) are different, but the proof of Proposition 3.43 shows that

$$|v-w| \leq d(\Phi(v), \Phi(w)) \leq (1 + L)|v-w|, \quad v, w \in \mathbb{R}.$$  

A standard maximal function argument then implies that there is a constant $C \geq 1$, depending only on $K$ and $L$, such that

$$\sup_{\varepsilon > 0} \|T_{\mu, \varepsilon}\|_{L^2(\mu) \to L^2(\mu)} \leq \sup_{\varepsilon > 0} C\|T_\varepsilon\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} + C.$$ (4.3)

So, to prove that $K$ is a CZ kernel for $\mu$, it suffices to show that $(v,w) \mapsto K(\Phi(w), \Phi(v))$ is a CZ kernel (for $L^1$). Recalling (3.52), and using the $-1$-homogeneity and horizontal oddness of $k$, we obtain the following explicit expression for the kernel of interest:

$$K(\Phi(w), \Phi(v)) = k(\Phi(v)^{-1} \cdot \Phi(w)) = k\left(\frac{w-v, \phi_1(w) - \phi_1(v), \phi_2(w) - \phi_2(v) + \frac{\phi_1(v) + \phi_2(v)}{2}(w-v)}{2L(w-v)}\right),$$

$$= \frac{2L}{w-v}k\left(\frac{1}{2L} \frac{\phi_1(w) - \phi_1(v), \phi_2(w) - \phi_2(v) + \frac{1}{2}[\phi_1(v) + \phi_1(w)](w-v)}{(2L)(w-v)}\right)$$

(4.4)

Here $\phi_1$ is $L$-Lipschitz by (3.53), and $(\phi_1, -\phi_2): \mathbb{R} \to \mathbb{R}^2$ is a $2L^2$-tame function by Proposition 3.51, so the terms

$$\frac{\phi_1(w) - \phi_1(v)}{(2L)(w-v)} \text{ and } \frac{\phi_2(w) - \phi_2(v) + \frac{1}{2}[\phi_1(v) + \phi_1(w)](w-v)}{4L^2(w-v)^2}$$

(4.5)
are bounded by 1 in absolute value. So, the values of \( k((2L)^{-1}, \theta_1, \theta_2) \) for \((\theta_1, \theta_2) \in \mathbb{R}^2 \) outside \([-1, 1]^2 \) never appear in the final expression on line (4.4), and having already arrived on this line, we may assume that \((\theta_1, \theta_2) \mapsto k((2L)^{-1}, \theta_1, \theta_2)\) is \(2\pi\)-periodic in both variables in \(\theta_1, \theta_2\) (and evidently smooth as a function on \( \mathbb{R}^2 \)). We learned this trick from \([18, \text{p. } 54]\). Under this assumption, we may expand \((\theta_1, \theta_2) \mapsto k((2L)^{-1}, \theta_1, \theta_2)\) as a Fourier series

\[
k((2L)^{-1}, \theta_1, \theta_2) = \sum_{n \in \mathbb{Z}^2} c_n e^{2\pi i (\theta_1, \theta_2) \cdot n}.
\] (4.6)

Here

\[
c_n = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} k((2L)^{-1}, \theta_1, \theta_2) e^{-2\pi i (\theta_1, \theta_2) \cdot n} d\theta_1 d\theta_2.
\]

Since \((\theta_1, \theta_2) \mapsto k((2L)^{-1}, \theta_1, \theta_2)\) is smooth, the constants \(c_n\) decay rapidly as \(|n| \to \infty\). Now, going back to the original kernel \(K(\Phi(w), \Phi(v))\), we note by combining (4.4) and (4.6) that

\[
\frac{1}{2L} \cdot K(\Phi(w), \Phi(v))
\]

\[
= \sum_{n \in \mathbb{Z}^2} c_n \exp \left( 2\pi i \left[ \frac{\phi_1(w) - \phi_1(v)}{(2L)(w-v)}, \frac{\phi_2(w) - \phi_2(v) + \frac{1}{4}[\phi_1(v) + \phi_1(w)](w-v)}{4L^2(w-v)^2} \right] \cdot n \right)
\]

\[
= \sum_{n \in \mathbb{Z}^2} c_n \cdot K_n(w, v).
\] (4.7)

Due to the rapid decay of the coefficients \(c_n\) as \(|n| \to \infty\), it remains to show that \(\|K_n\|_{\text{C.Z.}} \lesssim \text{poly}(|n|)\). This can be deduced from the subsequent proposition, whose proof follows by combining techniques developed by Christ \([10]\), David \([17]\), Hofmann \([30]\), and Semmes \([46]\):

**Theorem 4.8.** There exists a constant \(C \geq 1\) such that the following holds. Let \(M, N \geq 1\). Let \(A : \mathbb{R} \to \mathbb{R}\) be \(M\)-Lipschitz, and let \(B : \mathbb{R} \to \mathbb{R}^2\) be \(N\)-tame. Then the kernel

\[
K_{A,B}(v, w) := \frac{1}{w-v} \exp \left( 2\pi i \left[ \frac{A(w)-A(v)}{w-v}, \frac{B_2(w)-B_2(v)}{(w-v)^2} \right] \right)
\] (4.9)

is a CZ kernel for \(L^1\) with

\[
\|K_{A,B}\|_{\text{C.Z.}} \leq C \max\{M, N\}^C.
\]

Theorem 4.8 will be proven in Section 5, in more general form, see Theorem 5.4.

**Proof of Theorem 1.7 assuming Theorem 4.8.** From (4.3) and (4.7), we infer that

\[
\sup_{\epsilon > 0} \|T_{\mu, \epsilon}\|_{L^2(\mu) \to L^2(\mu)} \lesssim \sup_{\epsilon > 0} \|T_\epsilon\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} \lesssim \sum_{n \in \mathbb{Z}^2} c_n \cdot \|K_n\|_{\text{C.Z.}}.
\]

To see that the right hand side is finite, it suffices by the discussion above to show that there exists a constant \(C \geq 1\) such that, for every \(n = (n_1, n_2) \in \mathbb{Z}^2\), the kernel

\[
K_n(w, v) = \frac{c_n}{w-v} \exp \left( 2\pi i \left[ \frac{\phi_1(w) - \phi_1(v)}{(2L)(w-v)}, \frac{\phi_2(w) - \phi_2(v) + \frac{1}{4}[\phi_1(v) + \phi_1(w)](w-v)}{4L^2(w-v)^2} \right] \cdot n \right)
\]

is a Calderón-Zygmund kernel for \(L^1\) with

\[
\|K_n\|_{\text{C.Z.}} \leq C(1 + |n|)^C.
\]
This follows from Theorem 4.8 applied to
\[ A := \frac{n_1}{2L} \phi_1 \quad \text{and} \quad B := \frac{n_2}{4L^2} (\phi_1, -\phi_2), \]
observing that \( A \) is \( n_1/2 \)-Lipschitz, and \( B \) is \( n_2/2 \)-tame by the comment below (4.4). \( \square \)

4.2. Calderón commutators appear. Let \( A : \mathbb{R} \to \mathbb{R} \) be Lipschitz, let \( B : \mathbb{R} \to \mathbb{R}^2 \) be tame, and consider the SK
\[
K_{A,B}(x, y) := \frac{1}{x-y} \exp \left( 2\pi i \left( \frac{A(x) - A(y)}{x-y} + \frac{B_2(x) - B_2(y)}{x-y} - \frac{1}{2} [B_1(x) + B_1(y)](x-y) \right) \right),
\]
familiar from Example 2.2 with \( k(x) = \frac{1}{x} \).

**Theorem 4.10.** Let \( A : \mathbb{R} \to \mathbb{R} \) be a 1-Lipschitz function, and let \( B : \mathbb{R} \to \mathbb{R}^2 \) be a 1-tame map. Then \( \|K_{A,B}\|_{C,Z} \leq C \) for some absolute constant \( C \geq 1 \).

We mention that Theorem 4.8 does not immediately, or even easily, follow from Theorem 4.10, because we are interested in the polynomial dependence on \( M \) and \( N \). The sharper result will be derived "by induction" in Section 5, and the main result of this section will be the "base case" of that induction.

We will show the CZ property of \( K_{A,B} \) by decomposing the kernel into a sum of simpler ones, resembling Calderón commutators, then proving separately that they are CZ kernels, and finally summing up the results. In fact, using that \( e^{2\pi ix} = \sum_{n \geq 0} (2\pi i)^n/n! \), we first write
\[
K_{A,B}(x, y) = \sum_{n \geq 0} \frac{(2\pi i)^n}{n!} S_n(x, y),
\]
where
\[
S_n(x, y) := \frac{1}{x-y} \left[ \frac{A(x) - A(y)}{x-y} + \frac{B_2(x) - B_2(y)}{x-y} - \frac{1}{2} [B_1(x) + B_1(y)](x-y) \right]^n.
\]
(4.11)

Then, the terms \( S_n \) are further decomposed as follows:
\[
S_n(x, y) = \sum_{m=0}^{n} \binom{n}{m} \frac{1}{x-y} \left[ \frac{A(x) - A(y)}{x-y} \right]^m \left[ \frac{B_2(x) - B_2(y)}{x-y} - \frac{1}{2} [B_1(x) + B_1(y)](x-y) \right]^{n-m}.
\]
Motivated by this decomposition, we define the standard kernels
\[
C_{m,n}(x, y) := \frac{1}{x-y} \left[ \frac{A(x) - A(y)}{x-y} \right]^m \left[ \frac{B_2(x) - B_2(y)}{x-y} - \frac{1}{2} [B_1(x) + B_1(y)](x-y) \right]^{n}.
\]
(4.12)

**Example 4.13.** It is easy to check that if \( K : \mathbb{R} \times \mathbb{R} \setminus \triangle \to \mathbb{C} \) is an SK, \( A : \mathbb{R} \to \mathbb{R} \) is \( M \)-Lipschitz, and \( B = (B_1, B_2) : \mathbb{R} \to \mathbb{R}^2 \) is \( N \)-tame, then both
\[
K_A(x, y) = K(x, y) \left[ \frac{A(x) - A(y)}{x-y} \right]
\]
and
\[
K_B(x, y) = K(x, y) \left[ \frac{B_2(x) - B_2(y)}{x-y} - \frac{1}{2} [B_1(x) + B_1(y)](x-y) \right]
\]
are standard kernels with
\[
\|K_A\|_{\alpha,\text{strong}} \lesssim (1 + M)\|K\|_{\alpha,\text{strong}} \quad \text{and} \quad \|K_B\|_{\alpha,\text{strong}} \lesssim (1 + N)\|K\|_{\alpha,\text{strong}}.
\]
For the second inequality, use expansion (2.3), which reduces matters to the Lipschitz constant of $B_1$ (i.e., $N$). It follows, by iteration, that if $A$ is 1-Lipschitz and $B$ is 1-tame, the kernel $C_{m,n}$ satisfies

$$\|C_{m,n}\|_{\text{strong}} \leq C^{m+n+1}$$

for some absolute constant $C \geq 1$.

The proof of the following theorem will occupy most of this section.

**Theorem 4.14.** Let $A : \mathbb{R} \to \mathbb{R}$ be 1-Lipschitz, let $B = (B_1, B_2) : \mathbb{R} \to \mathbb{R}^2$ be 1-tame, and let $m, n \geq 0$. Then $\|C_{m,n}\|_{C.Z} \leq C^{m+n+1}$, where $C \geq 1$ is an absolute constant.

It follows immediately from Theorem 4.14 that $S_n$ is a also a CZ-kernel with

$$\|S_n\|_{C.Z} \leq C^{n+1} \sum_{m=0}^{n} \binom{n}{m} \leq (2C)^{n+1},$$

and finally that $K_{A,B}$ is a CZ kernel with

$$\|K_{A,B}\|_{C.Z} \lesssim \sum_{n=0}^{\infty} \frac{2\pi}{m} \|S_n\|_{C.Z} < \infty.$$

So, Theorem 4.10 follows from Theorem 4.14. We start with a few preparations to prove the latter.

**4.3. Reminder on $\beta$-numbers.** Let $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. For $x \in \mathbb{R}, s > 0$, we define

$$\beta_A(B(x, s)) := \inf_{a,b \in \mathbb{R}} \sup \left\{ \left| A(y) - \left[ ay + b \right] \right| : y \in B(x, s) \right\}. \tag{4.15}$$

The $\beta$-numbers satisfy the following Carleson packing condition:

$$\int_0^r \int_{B(x,r)} \beta_A(B(y, s))^2 dy \frac{ds}{s} \lesssim \text{Lip}(A)^2, \quad x \in \mathbb{R}, \quad r > 0. \tag{4.16}$$

This is a special case of Jones’ traveling salesman theorem [33], but the case for Lipschitz graphs in $\mathbb{R}^2$ is much simpler; see the book of Garnett-Marshall, [27, Chapter X, Lemma 2.4]. The quadratic dependence on $\text{Lip}(A)$ follows from the $\text{Lip}(A) = 1$ case by scaling (noting that $\beta_{cA}(B(x, s)) = c\beta_A(B(x, s)))$. The following lemma shows that the $\beta$-number in (4.15) also controls deviations from affine maps defined via averaging the gradient:

**Lemma 4.17.** Let $\psi \in C^\infty(\mathbb{R})$ be a standard bump function:

$$\int \psi = 1, \quad \psi \geq 0 \quad \text{spt} \psi \subset B(0, 1), \quad \text{and} \quad \psi(-z) = \psi(z). \tag{4.18}$$

For $s > 0$, let $\psi_s(x) := s^{-1} \cdot \psi(x/s)$. For a Lipschitz function $A : \mathbb{R} \to \mathbb{R}, x \in \mathbb{R},$ and $s > 0$, define the linear map

$$y \mapsto L_{x,s}(y) := P_s(A')(x)y,$$

where $P_s(A')(x) := (A' \ast \psi_s)(x).$\(^2\) Then,

$$\frac{|A(x) - A(y) - L_{x,s}(x - y)|}{s} \lesssim \psi \beta_A(B(x, s)), \quad y \in B(x, s).$$

\(^2\)This seems to be standard notation in our sources [10, 30], so we chose to follow it.
Proof. To simplify notation, assume, without loss of generality, that $x = 0 = A(x)$. Let $y \mapsto ay + b$ be the best approximating affine map associated to the number $β_A(B(0, s))$, that is,

$$|A(y) - (ay + b)| \leq s \cdot β_A(B(0, s)), \quad y \in B(0, s).$$

Applying this with $y = 0$ gives $|b| \leq s \cdot β_A(B(0, s))$. Further,

$$|A(y) - L_{0,s}(y)| \leq |A(y) - (ay + b)| + |b| + |ay - L_{0,s}(y)|$$

$$\leq 2s \cdot β_A(B(0, s)) + s \cdot \left| \int \hat{ψ}_s(z)[A'(z) - a] \, dz \right|.$$ (4.19)

To treat the last term, integrate by parts:

$$\left| \int \hat{ψ}_s(z)[A'(z) - a] \, dz \right| \leq \int |\hat{ψ}_s(z)||A(z) - (az + b)| \, dz$$

$$\lesssim \frac{1}{s^2} \int_{B(0, s)} s \cdot β_A(B(0, s)) \, dz = 2β_A(B(0, s)).$$

Plugging this last estimate to (4.19) completes the proof. □

4.4. Boundedness of the Calderón commutators. In this section, we prove Theorem 4.14. To a large extent, we can use arguments in [10] and [30], but the details look a little different, so we record them fairly completely. Fix a 1-Lipschitz function $A: \mathbb{R} \to \mathbb{R}$, a 1-tame map $B = (B_1, B_2): \mathbb{R} \to \mathbb{R}^2$, and $m, n \geq 0$. We abbreviate $C_{m,n}(x, y) := \left( \frac{A(x) - A(y)}{x - y} \right)^m \left[ \frac{B_2(x) - B_2(y)}{x - y} - \frac{1}{2} \frac{B_1(x) + B_1(y)}{(x - y)^2} \right]^{n}$. (4.20)

Remark 4.20. The kernel $K$ looks a little like the kernel of the standard Calderón commutator, see e.g. [10, p. 56], but there is a qualitative difference worth pointing out. Consider the case $m = 0$. Then,

$$K(x, y) = \frac{1}{y - x} \left[ \frac{B_2(x) - B_2(x) - \frac{1}{2} \frac{B_1(x) + B_1(x)}{(y - x)^2}}{y - x} \right]^{n} = (-1)^{n+1} K(x, y),$$ (4.21)

so $K$ is antisymmetric only when $n$ is even. On the other hand, the kernels of standard Calderón commutators (i.e. the kernels $K$ above with $n = 0$) are always antisymmetric.

Proof. We plan to verify the testing conditions (2.27), so let $K_ε(x, y) := \psi_ε(x - y)K(x, y)$, as above (2.27), where $ψ$ is even, and

$$1_{\mathbb{R} \setminus B(0, ε)} \leq ψ_ε \leq 1_{\mathbb{R} \setminus B(0, ε/2)}. \quad (4.22)$$

For simplicity of notation, the smooth ε-SIO is denoted $T$:

$$Tf(x) := \int K_ε(x, y) f(y) \, dy, \quad f \in S.$$
In fact, the value of the kernel satisfies
\[ B_1(0) = 0 \quad \text{and} \quad \operatorname{spt} B_1 \subset B(0, 10). \] (4.24)

In fact, the value of the kernel \( K_\varepsilon(x, y) \) remains unchanged if replace \( B \) by \( B - \xi \), where \( \mathcal{L}(x) = (B_1(0), B_1(0)x) \) is a 0-tame-affine map. Next, already using that \( B_1(0) = 0 \), it is easy to show that there exists a 1-Lipschitz function \( \tilde{B}_1 \) with \( \operatorname{spt} \tilde{B}_1 \subset B(0, 10) \) which agrees with \( B_1 \) on \( B(0, 3) \). Since only the values of \( B_1 \) on \( B(0, 3) \) appear in (4.23), we may replace \( B_1 \) by \( \tilde{B}_1 \) without changing the value of (4.23). We will only use the tameness condition \( \tilde{B}_2(z) = B_1(z) \) for \( z \in B(0, 3) \) (see (4.28)), and this now remains valid with \( \tilde{B}_1 \) instead. Alternatively, we could redefine \( B_2 \) on \( \mathbb{R} \) so that \( B_2 = \tilde{B}_1 \) on \( \mathbb{R} \), and hence acquire a new 1-tame function \( \tilde{B} : \mathbb{R} \to \mathbb{R}^2 \) satisfying (4.24), but this is a little overkill.

To prove (4.23), we start as in the proof of [10, Theorem 10, p. 58], and fix an auxiliary function \( \eta \in C^\infty(\mathbb{R}) \) satisfying
\[ \operatorname{spt} \eta \subset \left[ \frac{1}{4}, 1 \right], \quad \int_0^\infty \eta(s) \frac{ds}{s} = 1, \quad \text{and} \quad \int_0^\infty \eta(s) \frac{ds}{s} = 0. \] (4.25)

Then, for \( x, y \in \mathbb{R} \) with \( x \neq y \) fixed, we note that
\[ \int_0^\infty \eta \left( \frac{|x - y|}{s} \right) \frac{ds}{s} = \int_0^\infty \frac{\eta(r)}{r} \frac{dr}{r} = 1. \]

In particular, for \( x \in B(0, 1) \) (as in (4.23)) fixed, we may write
\[ T \eta(x) = \int_0^\infty \eta \left( \frac{|x - y|}{s} \right) \int_\mathbb{R} K_\varepsilon(x, y) b(y) \, dy \frac{ds}{s}, \]
\[ = \int_0^\infty \int_\mathbb{R} \eta \left( \frac{|x - y|}{s} \right) K_\varepsilon(x, y) b(y) \, dy \frac{ds}{s}. \]

Let us point out that the integrals above are absolutely convergent, because, first, a necessary condition for \( \eta(|x - y|/s)K_\varepsilon(x, y) \neq 0 \) is \( \varepsilon/2 < |x - y| < s \), so the integral over \( s \leq \varepsilon/2 \) contributes zero. Second, observe that if \( s > 16 \), then \( s^{-1}|x - y| < \frac{3}{4} \) for all pairs \( x \in B(0, 1) \) and \( y \in \operatorname{spt} b \subset B(0, 3) \), so the integral over \( s > 16 \) also contributes zero. Also, the integration over \( s \in (\varepsilon/2, 4\varepsilon) \cup (1, 16) \) only yields an absolute constant, so we have reduced (4.23) to showing that
\[ \int_{B(0, 1)} \int_{4\varepsilon}^1 \int_\mathbb{R} \eta \left( \frac{|x - y|}{s} \right) K_\varepsilon(x, y) b(y) \, dy \frac{ds}{s} \, dx \leq C(m + 1). \] (4.26)

The lower bound "4\varepsilon" is convenient, because whenever \( |x - y|/s \in \operatorname{spt} \eta \) and \( s \geq 4\varepsilon \), we have \( |x - y| \geq s/4 \geq \varepsilon \), and hence \( \psi_\varepsilon(x - y) = 1 \) by (4.22). Consequently, the value of (4.26) does not change if and when we replace \( K_\varepsilon \) by \( K \). Finally, since \( \operatorname{spt} [1 - b] \subset \mathbb{R} \setminus B(0, 2) \),
we have $|x - y| \geq 1$ for all $x \in B(0, 1)$ and $y \in \text{spt}[1 - b]$. Consequently $\eta(|x - y|/s) = 0$ whenever $s \in [0, 1]$, $x \in B(0, 1)$, and $y \in \text{spt}[1 - b]$, and it follows that
\[
\int_{B(0,1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \eta \left( \frac{|x-y|}{s} \right) K(x,y)[1-b](y) \, dy \, ds \, dx = 0.
\]
Therefore, (4.26) reduces to proving that
\[
\int_{B(0,1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \eta \left( \frac{|x-y|}{s} \right) K(x,y) \, dy \, ds \, dx \leq C(m+1). \tag{4.27}
\]
To prove (4.27), fix $x \in B(0, 1)$. Recall the exponents $m, n \geq 0$ from the definition of the kernel $K$. The case $n \geq 2$ turns out to be easy, see the Section 4.5, and the case $n = 0$ is the case of "standard" Calderón commutators. So, the case $n = 1$ contains the main news.

4.5. The case $n \geq 2$. In this case, we make the following rather crude estimate for (4.27):
\[
(4.27) \lesssim \int_{B(0,1)} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \frac{B_2(x) - B_2(y) - \frac{1}{2}[B_1(x) + B_1(y)](x-y)}{(x-y)^2} \right|^2 \, dy \, ds \, dx.
\]
To proceed with this expression, we first use the tameness condition $B_2 = B_1$ to write
\[
\frac{B_2(x) - B_2(y) - \frac{1}{2}[B_1(x) + B_1(y)](x-y)}{(x-y)^2} = -\int_x^y B_1(x) + B_1(y) - 2B_1(r) \, dr. \tag{4.28}
\]
It is easy to check that the right hand side on (4.28) vanishes if $B_1$ is affine. In particular,
\[
\left| \int_x^y B_1(x) + B_1(y) - 2B_1(r) \, dr \right| \leq \int_x^y \left| B_1(x) - B_{x,s}(x) \right| + \left| B_1(y) - B_{x,s}(y) \right| + 2\left| B_1(r) - B_{x,s}(r) \right| \, ds, \tag{4.29}
\]
where $B_{x,s}(y) = ay + b$ is an affine map minimising the $\beta$-number (introduced in (4.15)) of $B_1$ in $B(x, s)$. Therefore, we have
\[
\left| \frac{B_2(x) - B_2(y) - \frac{1}{2}[B_1(x) + B_1(y)](x-y)}{(x-y)^2} \right| \lesssim \beta_{B_1}(B(x, s)) \tag{4.30}
\]
for $x \in B(0, 1)$ and $\frac{1}{t} \leq |x-y| \leq s$, and consequently
\[
(4.27) \lesssim B^2 := \int_{B(0,1)} \int_0^1 \beta_{B_1}(B(x, s))^2 \, ds \, dx \lesssim 1. \tag{4.31}
\]
by Jones’ estimate (4.16).

4.6. The case $n \in \{0, 1\}$. We then consider the case $n \in \{0, 1\}$ and $m \geq 0$. We view $n \in \{0, 1\}$ as "fixed", and write
\[
K_m(x, y) := \frac{1}{x-y} \left[ A(x) - A(y) \right] \left[ \frac{B_2(x) - B_2(y) - \frac{1}{2}[B_1(x) + B_1(y)](x-y)}{(x-y)^2} \right]^m.
\]
Let $\psi \in C^\infty(\mathbb{R})$ be a "standard bump function" as in (4.18). Then, as in Lemma 4.17, we consider the linear maps
\[
L_{x,s}(y) := (A' * \psi_s)(x)y =: P_s(A')(x)y, \quad s \in (0, 1).
\]
The plan is to reduce the treatment of the kernel (4.32) to the case \( m = 0 \). To accomplish this, assume that initially \( m \geq 1 \). Then, for \( x \in B(0,1) \) and \( s \in (0,1) \) fixed, we write
\[
\left[ \frac{A(x) - A(y)}{x - y} \right]^m = \left[ \frac{A(x) - A(y)}{x - y} \right]^{m-1} \left[ \frac{A(x) - A(y) - L_{x,s}(x - y)}{x - y} \right] + \left[ \frac{A(x) - A(y)}{x - y} \right]^{m-1} P_s(A')(x).
\]

(4.33)

Here, for \( y \in B(x,s) \),
\[
\left[ \frac{A(x) - A(y)}{x - y} \right]^{m-1} \left[ \frac{A(x) - A(y) - L_{x,s}(x - y)}{x - y} \right] \leq \beta_A(B(x,s))
\]
by Lemma 4.17. We plug this this information into (4.27), and use the triangle inequality, to obtain two terms (4.27)1 and (4.27)2. For (4.27)1, we combine (4.30) and (4.34) to infer that
\[
(4.27)_1 \lesssim \int_{B(0,1)} \int_0^1 \beta_A(B(x,s)) \beta_{B_1}(B(x,s)) \frac{ds}{s} \, dx \lesssim 1,
\]
by Cauchy-Schwarz, and Jones’ estimate (4.16). Let us then consider
\[
(4.27)_2 = \int_{B(0,1)} \int_0^1 \left[ \int_{4e} P_s(A') \int_{4e} \eta \left( \frac{|x - y|}{s} \right) K_{m-1}(x,y) \frac{dy}{s} \right] \, dx.
\]

(4.36)

If still \( m - 1 \geq 0 \), we repeat the same procedure as in (4.33), separating one power of \( (A(x) - A(y))/(x - y) \) from \( K_{m-1} \), adding and subtracting \( L_{x,s}(x - y) \), and then repeating the estimates (4.34)-(4.35). This operation yields two terms, one "error" term dominated, as before, by \( \lesssim 1 \) (also using that \( \|P_s(A')\|_{L^2} \leq 1 \)), and then the "main" term
\[
\int_{B(0,1)} \int_0^1 \left[ \int_{4e} P_s(A') \int_{4e} \eta \left( \frac{|x - y|}{s} \right) K_{m-2}(x,y) \frac{dy}{s} \right] \, dx.
\]

(4.37)

Comparing (4.36) and (4.37), we note that if \( j \geq 1 \), we can reduce the study of \( K_j \) to the study of \( K_{j-1} \) at the cost of

1. committing an additive error of magnitude \( \lesssim 1 \), and
2. replacing \( P_s(A')^2 \) by \( P_s(A')^{j+1} \) in (4.37).

After repeating these steps \( m \) times, we see that (4.27) is bounded by \( \lesssim m \) plus either
\[
\int_{B(0,1)} \int_0^1 \left[ \int_{4e} P_s(A') \int_{4e} \eta \left( \frac{|x - y|}{s} \right) \frac{1}{x - y} \frac{dy}{s} \right] \, dx
\]
in the case \( n = 0 \), or
\[
\int_{B(0,1)} \int_0^1 \left[ \int_{4e} P_s(A') \int_{4e} \eta \left( \frac{|x - y|}{s} \right) \frac{1}{x - y} \frac{dy}{s} \right] \left( \int_x^y B_1(x) + B_1(y) - 2B_1(r) \frac{dr}{2(x - y)} \right) \, dx
\]
in the case \( n = 1 \). In the latter case we already plugged in (4.28). The case \( n = 1 \) is, of course, the "main case"; in fact, after a few changes of variables, we will reduce the treatment of (4.39) to the expression on line (4.45) below, which is more general than (4.38) (taking \( B_1(x) = x \)). So, we can, and will, ignore the case \( n = 0 \). To proceed...
estimating the expression in (4.39), we concentrate for the moment on the three innermost integrals. We make the change-of-variables \( r \to uy + (1 - u)x \) in the \( r \)-integration, and then use Fubini’s theorem, to find the expression

\[
\left| \int_0^1 \int_{4e}^1 \int_{\mathbb{R}} \eta \left( \frac{|x - y|}{s} \right) \frac{B_1(x) + B_1(y) - 2B_1(uy + (1 - u)x)}{2(x - y)^2} \, dy \, ds \, du \right|. \tag{4.40}
\]

Some cancellation, coming from the choice of \( \eta \), allows us to simplify the expression. Recalling the third point in (4.25), we compute as follows:

\[
\int_{\mathbb{R}} \eta \left( \frac{|x - y|}{s} \right) \frac{B(x)}{(x - y)^2} \, dy \left. \right|_{y = z + x} \int_{\mathbb{R}} B(x) \eta(|z|) \frac{dz}{z^2} = 0. \tag{4.41}
\]

Thus,

\[
(4.40) \leq \frac{1}{2} \int_{4e}^1 \int_{\mathbb{R}} \eta \left( \frac{|x - y|}{s} \right) \frac{B_1(y)}{(x - y)^2} \, dy \, ds \mid_{y = z + x} + \int_0^1 \int_{4e}^1 \int_{\mathbb{R}} \eta \left( \frac{|x - y|}{s} \right) \frac{B_1(uy + (1 - u)x)}{(x - y)^2} \, dy \, ds \, du. \tag{4.42}
\]

We want to bring (4.43) in the same form as (4.42), so we perform the change-of-variables \( y \mapsto [z - (1 - u)x]/u \):

\[
(4.43) = \int_0^1 \int_{4e}^1 \int_{\mathbb{R}} \eta \left( \frac{|x - z|}{su} \right) \frac{B_1(z)u^2}{(x - z)^2} \, dz \, ds \mid_{s = u} + \int_0^u \int_{4e}^1 \int_{\mathbb{R}} \eta \left( \frac{|x - z|}{t} \right) \frac{B_1(z)}{(x - z)^2} \, dz \, dt \, du. \tag{4.43}
\]

After these computations, we find that

\[
(4.40) \leq \frac{1}{2} \int_{4e}^1 \int_{\mathbb{R}} \eta \left( \frac{|x - y|}{s} \right) \frac{B_1(y)}{(x - y)^2} \, dy \, ds \mid_{y = z + x} + \int_0^1 u \int_{4e}^1 \int_{\mathbb{R}} \eta \left( \frac{|x - y|}{s} \right) \frac{B_1(y)}{(x - y)^2} \, dy \, ds \, du. \tag{4.44}
\]

The two terms are essentially of the same form; the \( u \)-averaging in the second term is completely harmless. For the time being, we concentrate on handling the first term. We plug it back into (4.39), and also use the observation (4.41), to find the expression

\[
\int_{B(0,1)} \int_{4e}^1 (P_\kappa(A')(x))^m \int_{\mathbb{R}} \eta \left( \frac{|x - y|}{s} \right) \frac{B_1(x) - B_1(y)}{(x - y)^2} \, dy \, ds \, dx. \tag{4.45}
\]

This expression is of the same form as treated in Christ’s lectures [10], see the bottom of [10, p. 59], so we may follow his argument verbatim from now on. First, the \( y \)-integral is written as

\[
\int_{\mathbb{R}} \eta \left( \frac{|x - y|}{s} \right) \frac{B_1(x) - B_1(y)}{(x - y)^2} \, dy = (B_1' * \Psi_\kappa)(x) =: Q_\kappa(B_1')(x),
\]

where

\[
\Psi(x) = \text{sgn}(x) \cdot \int_{|x|}^\infty \frac{\eta(t)}{t} \frac{dt}{t}.
\]
and $\Psi_s(x) = s^{-1} \cdot \Psi(x/s)$. Thus,

$$(4.45) = \int_{B(0,1)} \left| \int_{4\epsilon}^{1} (P_s(A')(x))^{m} \cdot Q_s(B'_1)(x) \frac{ds}{s} \right| \, dx$$

$$\lesssim m \left( \int_{B(0,1)} \left[ \int_{4\epsilon}^{1} \frac{1}{m}(P_s(A')(x))^{m} \cdot Q_s(B'_1)(x) \frac{ds}{s} \right]^2 \, dx \right)^{1/2}. \tag{4.46}$$

At this point, still following Christ’s argument, we use [10, Proposition 9]. It states that if $\{F_s : s > 0\}$ is a family of $C^1(\mathbb{R})$ functions, depending continuously on $s$, with $\|F_s\|_{L^\infty} + \|F'_s\|_{L^\infty} \leq C$, and $a \in L^\infty(\mathbb{R})$, then the operator

$$f \mapsto \int_0^\infty F_s(P_s(a)) \cdot Q_s(f) \frac{ds}{s} \tag{4.47}$$

is bounded on $L^2$ with operator norm $\lesssim C \|a\|_{L^\infty}$. We essentially see such an operator on line (4.46), with

$$F_s(t) := \begin{cases} \frac{1}{m} t^m, & \text{for } |t| \leq 1, \\ 0, & \text{for } |t| > 2, \end{cases} \quad \text{for } s \in [4\epsilon, 1] \quad \text{and} \quad F_s(t) \equiv 0 \text{ for } s \in (0, \epsilon) \cup [2, \infty).$$

and $a = A'$ (noting that $|P_s(a)| \leq 1$). The values of $F_s(t)$ for the remaining parameters $(s, t)$ are irrelevant, as long as the dependence on $s$ is continuous, and $F_s \in C^1(\mathbb{R})$ with $\|F_s\|_{L^\infty} + \|F'_s\|_{L^\infty} \lesssim 1$. Under these assumptions, the operator

$$f \mapsto \int_{|t| \leq 2} F_s(P_s(A'))^{m} \cdot Q_s(f) \frac{ds}{s} \tag{4.48}$$

is clearly bounded on $L^2$, with absolute constants, and the difference between the operators appearing in (4.46)-(4.47) is exactly (4.48). So, from this discussion, and [10, Proposition 9], it follows that

$$(4.46) \lesssim m. \tag{4.49}$$

Finally, we briefly remark that the term on line (4.44) can be handled in the same way, first re-introducing the term $B_1(x)$ inside the $y$-integration. Then, in place of (4.46), one ends up with the expression

$$m \int_0^1 u \left( \int_{B(0,1)} \left[ \int_{4\epsilon u}^{u} (P_s(A')(x))^{m} \cdot Q_s(B'_1)(x) \frac{ds}{s} \right]^2 \, dx \right)^{1/2} \, du,$$

which is easily bounded by $\lesssim m$, as above. This completes the proof of the first estimate in (4.23), and consequently the proof of the theorem. \qed

5. THE EXPONENTIAL KERNEL RETURNS

In Theorem 4.10, we showed that if $A : \mathbb{R} \to \mathbb{R}$ is 1-Lipschitz, and $B : \mathbb{R} \to \mathbb{R}^2$ is 1-tame, then $K_{A,B}$ is a CZ-kernel. In this section, we prove Theorem 4.8, which stated that $\|K_{A,B}\|_{C,Z} \leq \text{poly}(M, N)$ whenever $A : \mathbb{R} \to \mathbb{R}$ is $M$-Lipschitz, and $B : \mathbb{R} \to \mathbb{R}^2$ is $N$-tame. The result will be reduced to the case $M = 1 = N$ via the corona decompositions
for Lipschitz functions and tame maps from Section 3.1.1. In fact, this manner of reasoning works, without extra effort, in slightly higher generality. Let us fix, for the entire section, an SK $k: \mathbb{R} \times \mathbb{R} \setminus \Delta \to \mathbb{R}$ such that $\|k\|_{\alpha,\text{strong}} \leq 1$, $\alpha \in (0, 1]$. We also assume that
\[
k(x, y) = 0, \quad |x - y| \leq \epsilon, \tag{5.1}\]
for some fixed $\epsilon > 0$. Then, let us (re-)define
\[
K_{A,B}(x, y) := k(x, y) \exp \left( 2\pi i \left[ \frac{A(x) - A(y)}{x - y} + \frac{B_2(x) - B_2(y) - \frac{1}{2}(B_1(x) + B_1(y))(x-y)}{(x-y)^2} \right] \right), \tag{5.2}
\]
where $A: \mathbb{R} \to \mathbb{R}$ is Lipschitz, and $B: \mathbb{R} \to \mathbb{R}^2$ is tame. The main point here is that the "homogeneity" of the specific kernel $k(x, y) = (x - y)^{-1}$ is not needed in this section. For $M, N \geq 1$, and the fixed kernel $k$, we define
\[
\varphi_k(M, N) := \varphi(M, N) := \sup\{\|K_{A,B}\|_{C, Z} : A \text{ is } M\text{-Lipschitz and } B \text{ is } N\text{-tame}.\}
\]
Thus, Theorem 4.10 implies that $\varphi_{1/(x-y)}(1, 1) < \infty$. Without additional requirements on $k$, this is certainly not true, so we assume it a priori in this section:
\[
C_0(k) := \varphi(1, 1) < \infty. \tag{5.3}
\]
Now, we arrive at the main result of the section:

**Theorem 5.4.** There exists a constant $C_1 := C_1(k) \geq 1$, depending only on $C_0(k)$ in (5.3) and $\alpha \in (0, 1]$, such that the following holds. Let $M, N \geq 1$. Let $A: \mathbb{R} \to \mathbb{R}$ be $M$-Lipschitz, and let $B: \mathbb{R} \to \mathbb{R}^2$ be $N$-tame. Then
\[
\|K_{A,B}\|_{C, Z} \leq C_1 \max\{M, N\}^{C_1}. \tag{5.5}
\]

Theorem 5.4 will be inferred from the following recursive statement:

**Theorem 5.6.** Let $M, N \in 2\mathbb{N}$. Then, there exists a constant $C = C_\alpha \geq 1$ such that
\[
\varphi(M, N) \leq \min\{C_{M/2,N}, C_{M,N/2}\}, \tag{5.7}
\]
where
\[
C_{M,N} := C \max\{M, N^2, \varphi(M, N)\}.
\]

Let us quickly deduce Theorem 5.4 from Theorem 5.6.

**Proof of Theorem 5.4 assuming Theorem 5.6.** Let $C_1 := \max\{C_0(k), 2\log_2 C, 2\}$. Assume that we already have (5.5) with constant "$C_1" for some $M = N \in 2\mathbb{N}$, that is, $\varphi(N, N) \leq C_1 N^{C_1}$. This is true for $M = 1 = N$ by (5.3). From two applications of (5.7), the inductive hypothesis, and noting that $2^{C_1} \geq C^2$, we find that
\[
\varphi(2N, 2N) \overset{(5.7)}{\leq} C \max\{2N, N^2, \varphi(2N, N)\} \overset{(5.7)}{\leq} C \max\{2N, N^2, C \max\{N^2, \varphi(N, N)\}\} \leq C \max\{2N, N^2, C \max\{N^2, C_1 N^{C_1}\}\} = C^2 C_1 N^{C_1} \leq C_1 (2N)^{C_1}.
\]
This completes the proof. \qed

For the remainder of the section, we will view the Hölder continuity parameter $\alpha \in (0, 1]$ as "fixed", so any "absolute constants" are actually allowed to depend on $\alpha$. 


5.1. **Proof of Theorem 5.6: getting started.** We begin the proof of Theorem 5.6. The argument is based on ideas from Semmes’ paper [45], although our setting allows for some simplifications. We fix an $M$-Lipschitz function $A: \mathbb{R} \to \mathbb{R}$, and an $N$-tame map $B = (B_1, B_2): \mathbb{R} \to \mathbb{R}^2$, with $M, N \in 2^\mathbb{N}$. Write

$$Tf(x) := \int K_{A, B}(x, y) f(y) \, dy,$$

which is well-defined for e.g. $f \in L^2(\mathbb{R})$ due to (5.1). In the sequel, we abbreviate $K(x, y) := K_{A, B}(x, y)$. The plan will be to show that for any dyadic interval $Q_0 \in D$, the $T1$ testing condition

$$\int_{Q_0} |T(b)| \, dx \leq \min\{C_{M/2, N}, C_{M, N/2}\}, \quad (5.8)$$

familiar from (2.27), holds for all functions $b \in C^\infty(\mathbb{R})$ with $1_{2Q_0} \leq b \leq 1_{3Q_0}$. The estimate (5.8) (and a similar, completely symmetric, estimate for $T^f$) imply by Corollary 2.29 that $\|T\|_{L^2 \to L^2} \lesssim \min\{C_{M/2, N}, C_{M, N/2}\} + \|K_{A, B}\|_{\text{strong}}$. To conclude from here, recall from Example 2.2 that $\|K_{A, B}\|_{\text{strong}} \lesssim \max\{M, N\} \leq \min\{C_{M/2, N}, C_{M, N/2}\}$. So, (5.7) follows.

Fix $b \in C^\infty(\mathbb{R})$, as in (5.8). Now, (5.8) is actually composed of two distinct inequalities: we will mostly concentrate on proving the inequality

$$\int_{Q_0} |T(b)| \, dx \leq C_{M,N/2}. \quad (5.9)$$

that is, the one where the “tameness constant” is reduced by a factor of 2. The argument for the other inequality in (5.8) is virtually the same, and we will indicate the small differences in Section 5.4.6. To show (5.9), we start by applying the tame corona decomposition, Theorem 3.15 – or more precisely its Corollary 3.22 – to the $N$-tame function $B$, with parameter $\eta = \frac{1}{2}$. The result is a decomposition $D = \mathcal{B} \cup \mathcal{G}$, as explained in the statement of Theorem 3.15, a collection $\mathcal{F}$ of trees $\mathcal{T} \subset D$, and for each tree a function $\Psi_\mathcal{T} = \psi_\mathcal{T} + \mathcal{L}_\mathcal{T}$, where $\psi_\mathcal{T}$ is $(N/2)$-tame, $\mathcal{L}_\mathcal{T}$ is tame-linear, and the good approximation property (3.23) holds. To recap:

$$d_\varepsilon(B(s), \Psi_\mathcal{T}(s)) \leq \frac{1}{2} N|Q|, \quad s \in 2Q, \; Q \in \mathcal{T} \subset \mathcal{F}. \quad (5.10)$$

We were proving the second inequality in (5.8), we would, instead, start with the corona decomposition in Theorem 3.20 of the $M$-Lipschitz function $A$, at level $M/2$.

To benefit from the decomposition $D = \mathcal{B} \cup \mathcal{G}$, we will now decompose the operator $T$ in an analogous manner. For $j \in \mathbb{Z}$, we first define the operator $T_j$ by

$$T_j f(x) := \int_{\{y: 2^{-j} \leq |x-y| \leq 2^{-j+1}\}} K(x, y) f(y) \, dy.$$

Then, we set

$$T_Q f := \chi_Q T_j f, \quad Q \in \mathcal{D}_j, \; j \in \mathbb{Z},$$

and write

$$Tf = \sum_{Q \in \mathcal{D}} T_Q f = \sum_{Q \in \mathcal{B}} T_Q f + \sum_{\mathcal{T} \in \mathcal{F}} \sum_{Q \in \mathcal{T}} T_Q f =: \sum_{Q \in \mathcal{B}} T_Q f + \sum_{\mathcal{T} \in \mathcal{F}} T_\mathcal{T} f. \quad (5.11)$$
We begin by disposing of the first sum. Note that for $Q \in D_{j}$, we have
\[ |T_{Q}(b)(x)| \lesssim 1_{Q}(x) \int_{B(x,2^{j}+1)} |b(y)| \, dy \lesssim 1_{Q}(x), \]
using that $|K(x,y)| \leq |x-y|^{-1}$ and $|b|_{L^{\infty}} \leq 1$. Therefore, for $g \in L^{\infty}(Q_{0})$ with $\|g\|_{L^{\infty}(Q_{0})} = 1$, we have
\[ \left| \int_{Q_{0}} \left[ \sum_{Q \in B} T_{Q}(b) \right] g \right| \lesssim \sum_{Q \in B} \langle |g| \rangle_{Q} |Q| + \sum_{Q \in D} \langle |g| \rangle_{Q} |Q| \lesssim |Q|. \]

The implicit constants only depend on the Carleson packing constant of the family $B$. This is better than what we need for (5.9).

We then concentrate on the second sum in (5.11). We claim that for individual trees $T \in F$, we have the estimate
\[ \|T_{T}\|_{L^{2} \rightarrow L^{2}} \leq C_{M,N/2}. \]  
(5.12)

This will imply that
\[ \int_{Q_{0}} \sum_{T \in F} |T_{T}(b)| \, dx \lesssim C_{M,N/2} |Q_{0}|, \]  
(5.13)
as we will next check, and hence complete the proof of (5.9). Assume then for the moment that (5.12) holds, and write
\[ \int_{Q_{0}} \sum_{T \in F} |T_{T}(b)| \, dx = \int_{Q_{0}} \sum_{T \in F_{0}} |T_{T}(b)| \, dx + \int_{Q_{0}} \sum_{T \in F \setminus F_{0}} |T_{T}(b)| \, dx, \]  
(5.14)
where $F_{0} = \{ T \in F : Q(T) \subset Q_{0} \}$. The second term in (5.14) is straightforward to estimate, so we start from there. If $T \in F \setminus F_{0}$ is tree satisfying
\[ \int_{Q_{0}} |T_{T}(b)| \, dx \neq 0, \]  
(5.15)
then $Q_{0} \subset Q(T)$, since $T_{T}(b)$ is supported on $Q(T)$. In addition, there exists $Q \in T$ and $x \in Q_{0}$ such that
\[ T_{Q}(b)(x) = 1_{Q}(x) \int_{\{ y : |y| |y-x| \leq 2|Q| \}} K(x,y) b(y) \, dy \neq 0. \]  
(5.16)

Hence $x \in Q \cap Q_{0}$, so either $Q \subset Q_{0}$, or $Q_{0} \subset Q$. In the second case, (5.16) forces $|Q| \lesssim |Q_{0}|$, because $\text{spt} \ b \subset 3Q_{0}$. In the first case, since $Q_{0} \subset Q(T)$, there anyway exists a parent $Q' \in T$ of $Q$ such that $Q_{0} \subset Q'$ and $|Q'| \sim |Q_{0}|$. We conclude that whenever (5.15) holds for some $T \in F \setminus F_{0}$, there exists $Q \in T$ with $Q_{0} \subset Q$ and $|Q| \lesssim |Q_{0}|$. But since the trees $T \in F$ are disjoint, this implies that (5.15) can only occur for boundedly many $T \in F \setminus F_{0}$. Hence, the second sum in (5.14) is bounded by a constant times
\[ \int_{Q_{0}} |T_{T}b| \, dx \lesssim |Q_{0}|^{1/2} \|T_{T}(b)\|_{L^{2}(R)} \leq C_{M,N} |Q_{0}|^{1/2} \|b\|_{L^{\infty}(Q_{0})} \lesssim C_{M,N} |Q_{0}|, \]
as desired. To estimate the first sum in (5.14), we use the Carleson packing condition for the top intervals $Q(T)$ with $T \in F_{0}$. Recalling that $|b|_{L^{\infty}} \leq 1$ and $\text{spt} \ b \subset 3Q_{0}$, and also observing that
\[ T_{T}(b) = 1_{Q(T)} T_{T}(1_{5Q(T)} b), \quad T \in F, \]
we estimate as follows:
\[
\int_{Q_0} \sum_{T \in F_0} |T_T(b)| \, dx \leq \sum_{T \in F_0} \left( \int_{Q(T)} |T_T(b)|^2 \, dx \right)^{1/2} |Q(T)|
\]
\[
\lesssim C_{M,N} \sum_{T \in F_0} \left( \int_{5Q(T)} |b|^2 \, dx \right)^{1/2} |Q(T)|
\]
\[
\leq C_{M,N} \sum_{T \in F_0} |Q(T)| \lesssim C_{M,N}|Q_0|.
\]

The implicit constants only depend on the Carleson packing constant of the top intervals \( Q(T), \, T \in F \). We have now reduced (5.13) to proving (5.12).

To prove (5.12), fix \( T \in F \) and \( f \in L^2(\mathbb{R}) \), and write \( j_0 \) for the generation of \( Q(T) \), that is, \( Q(T) \in D_{j_0} \). Note that
\[
T_T f(x) = \sum_{Q \in \mathcal{F}} T_Q f(x) = \sum_{Q \in \mathcal{F}} \int_{\{y : |Q| \leq |x-y| \leq 2|Q|\}} K(x, y) f(y) \, dy
\]
\[
= 1_{Q(T)}(x) \int_{\{y : h(x) \leq |x-y| \leq \rho\}} K(x, y) f(y) \, dy, \quad x \in \mathbb{R},
\]
where \( \rho = 2^{-j_0 + 1} = 2|Q(T)| \), and
\[
h(x) := \inf \{|Q| : x \in Q \in \mathcal{F}\}, \quad \text{for } x \in Q(T). \tag{5.17}
\]

Now, following an idea in [45], we want to “replace” \( T_T \) by the somewhat regularised operator
\[
\hat{T}_T f(x) := 1_{Q(T)}(x) \int_{\{y : D(x,y) \leq |x-y| \leq \rho\}} K(x, y) f(y) \, dy, \tag{5.18}
\]
where
\[
D(x,y) := \frac{d(x) + d(y)}{4}, \tag{5.19}
\]
and \( d : \mathbb{R} \to \mathbb{R} \) is the 1-Lipschitz function
\[
d(x) = \inf \{|Q| + \text{dist}(x,Q) : Q \in \mathcal{F}\}, \quad x \in \mathbb{R}. \tag{5.20}
\]

By “replacement”, we mean that \( |T_T|_{L^2 \to L^2} \lesssim |\hat{T}_T|_{L^2 \to L^2} + \max\{M, N\} \), so it will suffice to prove (5.12) for \( \hat{T}_T \) in place of \( T_T \). Let us now see carefully how to dominate \( T_T \) by \( \hat{T}_T \).

**Lemma 5.21.** If \( x, y \in \mathbb{R} \) with \( x \in Q(T) \) and \( |x-y| \geq h(x) \), then \( |x-y| \geq D(x,y) \).

**Proof.** We use the facts that \( d \) is 1-Lipschitz, and \( d(x) \leq h(x) \) to estimate as follows:
\[
D(x,y) \leq \frac{d(x) + d(y) + |x-y|}{4} \leq \frac{h(x)}{2} + \frac{|x-y|}{4} \leq \frac{3|x-y|}{4} \leq |x-y|.
\]
\[\square\]

**Corollary 5.22.** Consider the kernel \( K^{D,\rho}(x,y) := K(x,y) 1_{D(x,y) \leq |x-y| \leq \rho}(x,y) \). Then,
\[
|T_T f(x)| \leq \sup_{\delta > 0} \int_{\{y : |x-y| \geq \delta\}} K^{D,\rho}(x,y) f(y) \, dy =: \hat{T}_T^\rho f(x), \quad x \in \mathbb{R}. \tag{5.23}
\]
The reader should protest that the right hand side of (5.23) is, in fact, the kernel of $K_{A,\Psi}$ instead of $K_{A,\Psi}$. Have we forgotten about the tame-linear part $L = (L, P)$ altogether? No: recalling that $L$ is linear, and $P = L$, one easily checks that

$$P(x) - P(y) - \frac{1}{2}[L(x) + L(y)](x - y) \equiv 0.$$ 

In other words,

$$K_{A,\Psi} = K_{A,\Psi}.$$  

Proof. The estimate (5.23) is clear if $x \notin Q(T)$, since then $T_T f(x) = 0$, so we assume in the following that $x \in Q(T)$. Choose $\delta := \max\{h(x), \epsilon\} > 0$, where $\epsilon > 0$ was the a priori truncation from (5.1) (in other words, $K(x, y) = 0$ whenever $|x - y| < \epsilon$). Then, if $h(x) \leq |x - y| \leq \rho$, and $K(x, y) \neq 0$, we also have $|x - y| \geq \delta$, and $D(x, y) \leq |x - y| \leq \rho$ by the previous lemma. This shows that

$$\int_{\{y: h(x) \leq |x - y| \leq \rho\}} K(x, y)f(y)\,dy = \int_{\{y: |x - y| \geq \delta\}} K_{D,\rho}(x, y)f(y)\,dy,$$

and the claim follows. \qed

So, at least $T_T$ is dominated by $\hat{T}_T^\rho$. But since $D, \rho$ are $\frac{1}{2}$-Lipschitz functions ($\rho$ being a 0-Lipschitz function), we find from Lemma 2.8 that $K_{D,\rho}$ is a GSK with

$$\|K_{D,\rho}\| \lesssim \|K\| \lesssim \max\{M, N\},$$

and hence Cotlar’s inequality (2.20) applies:

$$\hat{T}_T f(x) \lesssim M(\|T_T f\|)(x) + \|T_T\|_{C, Z} M(x), \quad f \in L^2(\mathbb{R}), \ x \in \mathbb{R}.$$ 

Here $\|T_T\|_{C, Z} = \|K_{D,\rho}\| + \|\hat{T}_T\|_{L^2 \to L^2}$ by definition. Combining this inequality with (5.23) and (5.24), we infer that

$$\|T_T\|_{L^2 \to L^2} \lesssim \|\hat{T}_T\|_{L^2 \to L^2} + \max\{M, N\},$$

as desired. Consequently, (5.12) will follow (with a slightly worse constant) once we manage to establish that

$$\|\hat{T}_T\|_{L^2 \to L^2} \leq C_{M,N/2}.$$  

(5.25)

To simplify notation a little bit, we will, from now on, write “$T_T$” in place of “$\hat{T}_T$” for the operator associated to the $D(x, y)$-truncation. This should cause no confusion, because there will be no further reference to the original operator $T_T$.

5.2. Applying the corona decomposition. To prove (5.25), we recall the functions

$$\Psi_T := \Psi = \psi_T + L_T =: \psi + L$$

associated to the fixed tree $T$, where $\psi = (\psi_1, \psi_2): \mathbb{R} \to \mathbb{R}^2$ is $(N/2)$-tame, and $L = (L, P) := \mathbb{R} \to \mathbb{R}^2$ is $2N$-tame-linear. We recall from (3.23) that

$$d_\pi(B(s), \Psi(s)) \leq (N/2)|Q|, \quad s \in 11Q, \ Q \in T.$$  

(5.26)

To be accurate, (3.23) only gives (5.26) for $s \in 2Q$, but enlarging the constant from $"^2$ to "^1" is a standard trick, see e.g. the argument on [14, p. 20]. Alternatively, one could just prove (3.23) directly with constant "^1". To establish the good $L^2$-bound for $T_T$, we want to compare it to a suitable operator $T_\Psi$ associated to the kernel

$$K_{A,\Psi}(x, y) = k(x, y) \exp \left(2\pi i \left[\frac{A(x) - A(y)}{x - y} + \frac{\psi_1(x) - \psi_2(y) - \frac{1}{2}(\psi_1(x) + \psi_1(y))(|x - y|)}{(x - y)^2}\right]\right).$$  

(5.27)

The reader should protest that the right hand side of (5.27) is, in fact, the kernel of $K_{A,\Psi}$ instead of $K_{A,\Psi}$. Have we forgotten about the tame-linear part $L = (L, P)$ altogether? No: recalling that $L$ is linear, and $P = L$, one easily checks that

$$P(x) - P(y) - \frac{1}{2}[L(x) + L(y)](x - y) \equiv 0.$$ 

In other words,

$$K_{A,\Psi} = K_{A,\Psi}.$$  

(5.28)
This is crucial: the kernel $K_{A,\psi}$ approximates $K_{A,B}$ well (using information from the corona decomposition, as we will soon see), while $\tilde{K}_{A,\psi}$ is a kernel associated to an $(N/2)$-tame function $\psi$. On the other hand, $\Psi$ can be, at worst, $2N$-tame, so without knowing (5.28), the kernel $K_{A,\psi}$ would be no better than $K_{A,B}$!

Now, we abbreviate
\[
\tilde{K}(x,y) := K_{A,\psi}(x,y) = K_{A,\psi}(x,y),
\]
and define the operator $T_{\Psi}$ with the same $D(x,y)$-truncation as in the definition of $T_T$:
\[
T_{\Psi} f(x) = 1_{Q(T)}(x) \int_{\{y: D(x,y) \leq |x-y| \leq \rho\}} \tilde{K}(x,y)f(y) \, dy.
\] (5.29)

To prove (5.25), we will establish that
\[
\|T_T\|_{L^2 \to L^2} \lesssim \|T_{\Psi}\|_{L^2 \to L^2} + \max\{M, N^2\} \lesssim \varphi(M, N/2) + \max\{M, N^2\}.
\] (5.30)

The second inequality in (5.30) is virtually a consequence of the definition of the number $\varphi(M, N/2)$, and (5.28), since $A$ is $M$-Lipschitz, and $\psi$ is $(N/2)$-tame. A little technicality is the presence of the $D(x,y)$-truncation, but we can dispose of it by easy maximal function tricks, as follows. Recalling that $D(x,y) = (d(x) + d(y))/4$, we claim that
\[
\left| T_{\Psi} f(x) - \int_{\{y: d(x)/4 \leq |x-y| \leq \rho\}} \tilde{K}(x,y)f(y) \, dy \right| \lesssim M f(x).
\] (5.31)

Indeed, since $D(x,y) \geq d(x)/4$, the left hand side of (5.31) is bounded by
\[
\int_{\{y: d(x)/4 \leq |x-y| < D(x,y)\}} |\tilde{K}(x,y)||f(y)| \, dy \leq \frac{4}{d(x)} \int_{B(x,d(x))} |f(y)| \, dy \lesssim M f(x).
\]

We used that $D(x,y) \leq d(x)/2 + |x-y|/4$, so $|x-y| \leq D(x,y)$ implies that $|x-y| \leq d(x)$. Now, it follows from (5.31) and Cotlar’s inequality that
\[
\|T_{\Psi}f\|_{L^2} \lesssim \|T_{A,\psi}f\|_{L^2} + \|f\|_{L^2} \lesssim \|T_{A,\psi}f\|_{L^2} + (1 + \|\tilde{K}\|)\|f\|_{L^2}.
\]

Here $\|T_{A,\psi}f\|_{L^2} = \|T_{A,\psi}f\|_{L^2} \lesssim \varphi(M, N/2)\|f\|_{L^2}$ by (5.28) and the definition of $\varphi(M, N/2)$, while $\|\tilde{K}\| \lesssim \max\{M, N\}$. This completes the proof of the second part of (5.30), and the rest of the section is devoted to establishing the first part.

5.3. A Whitney decomposition. Recall that $d(x) = \inf\{\dist(x,Q) + |Q| : Q \in \mathcal{T}\}$, so $d$ is $1$-Lipschitz, and well-defined on $\mathbb{R}$. However, the set
\[
E := \{x \in \mathbb{R} : d(x) = 0\}
\]
is a compact subset of $\overline{Q(\mathcal{T})}$. It follows easily from (5.26) that
\[
\Psi(s) = B(s), \quad s \in E.
\] (5.32)

In this short section, we perform a Whitney type decomposition of $\mathbb{R} \setminus E$. Fix $x \in \mathbb{R} \setminus E$. Since $0 < d(x) \leq \dist(x, Q(\mathcal{T})) + |Q(\mathcal{T})| < \infty$, and $d$ is continuous (hence $d$ stays positive in a neighbourhood of $x$), there exists a maximal dyadic interval $I \ni x$ with
\[
\inf_{y \in I} d(y) = \inf_{y \in I} \inf_{Q \in \mathcal{T}} \{d(y, Q) + |Q|\} \geq |I|.
\] (5.33)

These intervals are disjoint and cover $\mathbb{R} \setminus E$, and we will denote them $S$. We first observe that
\[
|S| \leq d(y) \leq 4|S|, \quad y \in S \in S.
\] (5.34)
Indeed, the lower bound is immediate from the definition (5.33). To see the upper bound, note that by the maximality of $S \in S$ there exists $y'$ in the parent $\hat{S}$ of $S$ with $d(y') < |\hat{S}| = 2|S|$, whence $d(y) \leq d(y') + |\hat{S}| \leq 4|S|$, as claimed. We next observe that

$$S \in S \text{ and } S < 11Q(T) \implies d_x(B(s), \Psi(s)) \lesssim N|S|, \quad s \in S. \quad (5.35)$$

Indeed, fix $x \in S$ and, based on (5.34), find $Q \in T$ with $d(x, Q) + |Q| \leq 5|S|$. Then, let $Q' \in T$ be the minimal ancestor of $Q$ in $T$ with $S < 11Q'$ (this exists because $S \subset 11Q(T)$). It is easy to check that $|Q'| \sim |S|$, and now (5.35) follows from (5.26) applied to $s \in 11Q'$.

5.4. **Comparing $T_T$ and $T_\Psi$.** Recall that $T_T$ and $T_\Psi$ are the operators defined in (5.18) and (5.29), respectively. To prove the first inequality in (5.30), that is,

$$\|T_T\|_{L^2 \to L^2} \lesssim \|T_\Psi\|_{L^2 \to L^2} + \max\{M, N^2\},$$

we fix $f, g \in L^2(\mathbb{R})$. It suffices to show that

$$\left| \int (T_T f)g - \int (T_\Psi f)g \right| \lesssim \max\{M, N^2\}\|f\|_{L^2}\|g\|_{L^2}. \quad (5.36)$$

Since $T_T(f) = 1_{Q(T)} T_T(f 1_{5Q(T)})$ and $T_\Psi(f) = 1_{Q(T)} T_\Psi(f 1_{5Q(T)})$, which follows from the upper $\rho$-truncation in (5.18) and (5.29) (recall: $\rho = 2|Q(T)|$), it moreover suffices to prove (5.36) for $f, g$ satisfying

$$\text{spt } f \cup \text{spt } g \subset 5Q(T).$$

To estimate the difference in (5.36), we introduce the following auxiliary notation. If $x \in E$, we define $S(x) = \{x\}$, and otherwise $S(x)$ is the unique element in $S$ containing $x$. If $h : \mathbb{R} \to \mathbb{R}$ is a function, and $x \in \mathbb{R}$, we then define

$$h_{\geq x}(y) := h(y) 1_{\{|S(y)| \geq |S(x)|\}}(y) \quad \text{and} \quad h_{> x}(y) := h(y) 1_{\{|S(y)| > |S(x)|\}}(y).$$

The functions $h_{\leq x}$ and $h_{< x}$ are defined similarly, swapping the inequalities. Note that $h_{> x} \equiv 0$ for any $x \in \mathbb{R}$, and $h_{< x} \equiv 0$ whenever $x \in E$. With this notation, we have

$$\int (T_T f)(x)g(x) \, dx = \int (T_T f_{\geq x})(x)g(x) \, dx + \int (T_T f_{< x})(x)g(x) \, dx,$$

where further

$$\int (T_T f_{< x})(x)g(x) \, dx = \int g(x) \left[ \int_{\{D(x,y) < |x-y| < \rho\}} K(x,y) f(y) 1_{\{|S(y)| < |S(x)|\}}(y) \, dy \right] \, dx$$

$$= \int f(y) \left[ \int_{\{D(x,y) < |x-y| < \rho\}} K(x,y) g(x) 1_{\{|S(x)| > |S(y)|\}}(x) \, dx \right] \, dy$$

$$= \int (T_T^* g_{> y})(y) f(y) \, dy.$$

The same calculation works if "$T$" is replaced with "$\Psi$". Consequently,

$$\int (T_T f) - \int (T_\Psi f) = \int [T_T f_{\geq x} - T_\Psi f_{\geq x}](x)g(x) \, dx \quad (5.37)$$

$$+ \int [T_T^* g_{> y} - T_\Psi^* g_{> y}](y) f(y) \, dy. \quad (5.38)$$
We will only estimate the term on line (5.37), since the argument for the second term is virtually the same. This is actually a reason why we introduced the ‘symmetric’ $D(x, y)$-truncation: to make the term on line (5.38) look as similar to (5.37) as possible.

5.4.1. Estimate for (5.37). The plan is to fix $x \in \text{spt } g \subset 5Q(T)$, and obtain pointwise bounds for the expression $[T_T f_{\geq x} - T_{\Psi} f_{\geq x}](x)$, which we spell out as follows:

\[
[T_T f_{\geq x} - T_{\Psi} f_{\geq x}](x) = \sum_{S \in S} \int_{\{y \in S : d(x, y) \leq |x - y| \leq \rho \}} K(x, y) f(y) - \tilde{K}(x, y) f(y) \, dy. \tag{5.39}
\]

But is this also accurate when $x \in E$, that is, when $|S(x)| = 0$? Then, the a priori correct expression for $[T_T f_{\geq x} - T_{\Psi} f_{\geq x}](x)$ is actually

\[
\sum_{S \in \mathcal{S}} \int_{\{y \in S : d(x, y) \leq |x - y| \leq \rho \}} K(x, y) f(y) - \tilde{K}(x, y) f(y) \, dy + \int_E f(y)[K(x, y) - \tilde{K}(x, y)] \, dy.
\]

However, when $x, y \in E$, as in the second integration, then $B(x) = \Psi(x)$ and $B(y) = \Psi(y)$ by (5.32), so $K(x, y) = \tilde{K}(x, y)$. Consequently, the second integral contributes nothing, and (5.39) is indeed true even when $x \in E$.

We will now write “$I_x(S)$” for the individual terms in (5.39), with $|S| \geq |S(x)|$. Note that intervals $S \in \mathcal{S}$ with $S \cap 5Q(T) = \emptyset$ contribute nothing to (5.39), so they can be discarded. But if $S \cap 5Q(T) \neq \emptyset$, then $d(y) \leq \text{dist}(y, Q(T)) + |Q(T)| \leq 3|Q(T)|$ for some $y \in S$. This implies by (5.34) that $|S| \leq 3|Q(T)|$, and consequently,

\[
S \subset 11Q(T). \tag{5.40}
\]

In fact this inclusion explains our choice of the constant “11” in (5.26). We proceed to estimate the pieces $I_x(S)$ in a manner adapted from [45], eventually proving the following claim: the intervals $S \in \mathcal{S}$ with $|S| \geq |S(x)|$ and $S \subset 5Q(T)$ can be split into two groups $\mathcal{G}_1(x)$ and $\mathcal{G}_2(x)$, where

\[
|I_x(S)| \lesssim \max\{|M, N^2\}|S| \|f\|_{L^2} \int_S |f(y)| \, dy, \quad S \in \mathcal{G}_1(x), \tag{5.41}
\]

and

\[
\sum_{S \in \mathcal{G}_2(x)} |I_x(S)| \lesssim M f(x). \tag{5.42}
\]

The estimate (for (5.37)) concerning group $\mathcal{G}_2(x)$ is straightforward:

\[
\int |g(x)| \sum_{S \in \mathcal{G}_2(x)} |I_x(S)| \, dx \lesssim \int |g(x)| M f(x) \, dx \lesssim \|g\|_{L^2} \|f\|_{L^2}.
\]

Before proceeding with the proofs of (5.41)-(5.42), let us briefly see that the estimate (5.41) leads to essentially the same conclusion (up to multiplication by $\max\{|M, N^2\}$):

**Lemma 5.43.** Let $1 < p < \infty$, and $1/p + 1/q = 1$. Then, for $g \in L^p$ and $f \in L^q$, we have

\[
\int |g(x)| \left[ \sum_{S \in \mathcal{S}} \frac{|S|}{\text{dist}(x, S)^2 + |S|^2} \int_S |f(y)| \, dy \right] \, dx \lesssim_p \|g\|_{L^p} \|f\|_{L^q}. \tag{5.44}
\]
Proof. We start by rewriting and estimating the left hand side as follows:

$$
\text{L.H.S. of (5.44)} = \sum_{S \in S} \left( \int_S \frac{|S| |g(x)| \, dx}{\text{dist}(x, S)^2 + |S|^2} \left( \int_S |f(y)| \, dy \right) |S| \right)
\leq \left( \sum_{S \in S} \left( \int_S \frac{|S| |g(x)| \, dx}{\text{dist}(x, S)^2 + |S|^2} \right)^p |S| \right)^{1/p} \left( \sum_{S \in S} \left( \int_S |f(y)| \, dy \right)^q |S| \right)^{1/q}.
$$

Since the intervals in $S$ are disjoint, the second factor is evidently controlled by $\|Mf\|_{L^q} \lesssim_p \|f\|_{L^q}$. The first factor is also dominated by the maximal function, since for $S \in S$ fixed,

$$
\int \frac{|S| |g(x)| \, dx}{\text{dist}(x, S)^2 + |S|^2} \lesssim \int \frac{|g(x)| \, dx}{\text{dist}(x, S)^2 + |S|^2} + \sum_{j \geq 0} \frac{1}{2^{2j}|S|} \int_{\{x \colon 2^j|S| \leq \text{dist}(x, S) \leq 2^{j+1}|S|\}} |g(x)| \, dx
\lesssim \sum_{j \geq 0} 2^{-j} \left( \inf_{y \in S} Mg(y) \right) \lesssim \inf_{y \in S} Mg(y),
$$

and consequently

$$
\left( \sum_{S \in S} \left( \int_S \frac{|S| g(x) \, dx}{\text{dist}(x, S)^2 + |S|^2} \right)^p |S| \right)^{1/p} \lesssim \left( \sum_{S \in S} \int_S [Mg(y)]^p \, dy \right)^{1/p} \lesssim_p \|g\|_{L^p},
$$

as desired. \(\square\)

This allows us to conclude the estimate for (5.37) (but see Section 5.4.5 for a final "wrap-up" of the whole argument). We then begin to verify the estimates (5.41)-(5.42). We fix $x \in 5Q(T)$ and $S \in \mathcal{S}$ with $|S| \geq |S(x)|$ and $S \subset 11Q(T)$. Since $S(x) \cap 5Q(T) \neq \emptyset$, the argument above (5.40) also yields

$$
S(x) \subset 11Q(T). \tag{5.45}
$$

5.4.2. Case where $\text{dist}(S(x), S) \geq 2|S|$ and $\{y \in S : D(x, y) \leq |x - y| \leq 2\rho\} = S$. This is the "main case", and we write

$$
I_x(S) = \int_{\{y \in S : D(x, y) \leq |x - y| \leq \rho\}} K(x, y) f(y) - \tilde{K}(x, y) f(y) \, dy
= \int_S [K(x, y) - K(x, y_0)] f(y) + [K(x, y_0) - \tilde{K}(x, y)] f(y) \, dy, \tag{5.46}
$$

where $y_0$ is the midpoint of $S$. In particular, $|y - y_0| \leq |S| \leq |x - y_0|/2$. We give pointwise estimates for the two differences of the kernels in (5.46). The first difference is easier, as the same kernel "$K$" appears twice, and

$$
|K(x, y) - K(x, y_0)| \lesssim \frac{\max\{M, N\}|S|}{\text{dist}(x, S)^2} \tag{5.47}
$$

follows from standard estimates for $K$. We claim a similar estimate also for the second difference in (5.46), and we start by writing

$$
|K(x, y) - \tilde{K}(x, y_0)| \leq |K(x, y) - K(x, y_0)| + |K(x, y_0) - \tilde{K}(x, y_0)|. \tag{5.48}
$$
The first term is the same as \((5.47)\), so let us concentrate on the second one. Recalling the definitions, and writing \(\Psi = \psi + \mathcal{L} =: (\Psi_1, \Psi_2)\), this term equals
\[
|K(x, y_0) - \tilde{K}(x, y_0)|
\]
\[
\leq \frac{1}{|x - y_0|} \left| \exp \left( 2\pi i \left[ \frac{A(x) - A(y_0)}{x - y_0} + \frac{B_2(x) - B_2(y_0) - \frac{1}{2}[B_1(x) + B_1(y_0)](x - y_0)}{(x - y_0)^2} \right] \right) 
- \exp \left( 2\pi i \left[ \frac{A(x) - A(y_0)}{x - y_0} + \frac{\Psi_2(x) - \Psi_2(y_0) - \frac{1}{2}[\Psi_1(x) + \Psi_1(y_0)](x - y_0)}{(x - y_0)^2} \right] \right) \right|.
\]

To estimate the difference, we just use that \(t \mapsto \exp(2\pi it)\) is \(2\pi\)-Lipschitz. The ensuing upper bound for \((5.49)\) is
\[
\frac{2\pi}{|x - y_0|} \left( \frac{|B_2(x) - \Psi_2(x)| + |B_2(y_0) - \Psi_2(y_0)|}{|x - y_0|^2} + \frac{|B_1(x) - \Psi_1(x)| + |B_1(y_0) - \Psi_1(y_0)|}{2|x - y_0|} \right).
\]

To estimate these terms, we plug in the information from the corona decomposition on the quality of approximation of \(B\) by \(\Psi\). Since \(x \in S(x) \subset 11Q(T)\) (by \((5.45)\)) and \(y_0 \in S \subset 11Q(T)\), and \(|S(x)| \leq |S|\), we deduce from \((5.35)\) that
\[
|B_2(x) - \Psi_2(x)| \lesssim N^2|S(x)|^2 \lesssim N^2|S|^2 \quad \text{and} \quad |B_2(y_0) - \Psi_2(y_0)| \lesssim N^2|S|^2.
\]

For the same reasons,
\[
|B_1(x) - \Psi_1(x)| \lesssim N|S| \quad \text{and} \quad |B_1(y_0) - \Psi_1(y_0)| \lesssim N|S|.
\]

Combining these estimates, and recalling that \(|x - y_0| \geq \text{dist}(S(x), S) \geq 2|S|\), we infer that
\[
|K(x, y_0) - \tilde{K}(x, y_0)| \lesssim \frac{N^2|S|^2}{\text{dist}(x, S)^3} + \frac{N|S|}{\text{dist}(x, S)^2} \lesssim \frac{N^2|S|}{\text{dist}(x, S)^2}.
\]

Combining \((5.47)\) and the estimate above, we conclude that
\[
I_x(S) \lesssim \frac{\max\{M, N^2\}|S|}{\text{dist}(x, S)^2 + |S|^2} \int_S |f(y)| \, dy.
\]

This matches the estimate in \((5.41)\), so in this case \(S \in \mathcal{G}_1(x)\).

5.4.3. Case where \(\text{dist}(S(x), S) \leq 2|S|\). Recall from \((5.34)\) that \(d(y) \sim |S|\) for all \(y \in S\). Therefore, if \(|x - y| \geq D(x, y) = (d(x) + d(y))/4\), we certainly have \(|x - y| \gtrsim |S|\). Since \(\max\{|K(x, y)|, |\tilde{K}(x, y)|\} \lesssim |x - y|^{-1}\), and \(d(x, S) \leq |S(x)| + \text{dist}(S(x), S) \leq 3|S|\), we conclude that
\[
I_x(S) \lesssim \frac{1}{|S|} \int_S |f(y)| \, dy \lesssim \frac{|S|}{d(x, S)^2 + |S|^2} \int_S |f(y)| \, dy.
\]

This matches the estimate in \((5.41)\), so again \(S \in \mathcal{G}_1(x)\).
5.4.4. Case where \( \text{dist}(S(x), S) \geq 2|S| \) and \( \{ y \in S : D(x, y) \leq |x - y| \leq \rho \} \neq S \). This case \textit{a priori} divides into two further sub-cases: either

\[
|x - y_0| < D(x, y_0) \quad \text{or} \quad |x - y_0| > \rho
\]

for some \( y_0 \in S \). We assume that the former option holds, and pick \( y_0 \in S \) with \( |x - y_0| < D(x, y_0) = (d(x) + d(y_0))/4 \). Then, using the 1-Lipschitz property of \( d \), we first deduce that

\[
|x - y_0| < \frac{d(x) + d(y_0)}{4} \leq \frac{d(x)}{2} + \frac{|x - y_0|}{4},
\]

and consequently

\[
d(x) \geq \frac{3}{2}|x - y_0| \geq \frac{3}{2} \text{dist}(x, S).
\]

Since \( |S| \leq \text{dist}(x, S) \), this implies that \( S \subset B(x, 3d(x)) \). Consequently, also noting that the integration in \( I_\varepsilon(S) \) only takes into account such \( y \in \mathbb{R} \) with \( |x - y| \geq D(x, y) \geq d(x) \), we find from the estimates that

\[
\sum_{S \subset 11Q(T) \atop \inf_{y \in S}|x - y_0| - D(x, y_0)| < 0} |I_x(S)| \lesssim \rho^{-1} \int_{5Q(T)} |f(y)| \, dy \lesssim M f(x).
\]

This is the estimate desired in (5.42), so we can include all \( S \in S \) with \( \inf_{y \in S}|x - y| - D(x, y)| < 0 \) to the collection \( \mathcal{G}_2(x) \).

Finally, assume that the second option in (5.51) is realised, and pick \( y_0 \in S \) accordingly. If \( |S| \leq \rho/2 \), then \( \inf_{y \in S}|x - y| \geq \rho/2 \) by the triangle inequality. But even if \( |S| \geq \rho/2 \), we have \( \inf_{y \in S}|x - y| = \text{dist}(x, S) \geq 2|S| \geq \rho \) by the case assumption. So, we have

\[
\sum_{S \subset 11Q(T) \atop \sup_{y \in S}|x - y_0| > \rho} |I_x(S)| \lesssim \rho^{-1} \int_{5Q(T)} |f(y)| \, dy \lesssim M f(x),
\]

which is the same estimate as in (5.52). The proof of this – final – case is complete.

5.4.5. \textit{Summary.} We have now proven that all the intervals \( S \in S \) with \( |S| \geq |S(x)| \) and \( S \subset 11Q(T) \), for \( x \in 5Q(T) \), can be split into the groups \( \mathcal{G}_1(x) \) and \( \mathcal{G}_2(x) \) so that (5.41)-(5.42) hold. As we saw directly under (5.41)-(5.42), we can then conclude the estimate

\[
\int |T_T f \mathbb{1}_x - T_T f \mathbb{1}_y(x)||g(x)| \, dx \lesssim \int |g(x)| \sum_{|S| \geq |S(x)|} I_x(S) \, dx \lesssim \max\{M, N^2\} \|f\|_{L^2} \|g\|_{L^2}.
\]

Repeating rather verbatim the same argument, we could also show that

\[
\int |(T_T f \mathbb{1}_y - T_T g \mathbb{1}_y)(y)||f(y)| \, dy \lesssim \max\{M, N^2\} \|f\|_{L^2} \|g\|_{L^2},
\]

and consequently the splitting in (5.37) shows that

\[
\left| \int (T_T f) g - \int (T_T f) g \right| \lesssim \max\{M, N^2\} \|f\|_{L^2} \|g\|_{L^2}.
\]

Since \( f, g \in L^2(\mathbb{R}) \) were arbitrary functions, this allows us to conclude the first inequality in (5.30), namely that \( \|T_T f\|_{L^2 \rightarrow L^2} \lesssim \|T_T f\|_{L^2 \rightarrow L^2} + \max\{M, N^2\} \). Since we already established the second inequality in (5.30), we may then infer (5.25), which then implies (5.12), and finally (5.9) (one of the two inequalities in (5.8)).
5.4.6. The second inequality in (5.8). As we explained above, we have now established one of the two inequalities claimed in (5.8). We still need to establish the second:

$$\int_{Q_0} |T(b)| \, dx \leq C_{M/2,N}. \quad (5.53)$$

As we noted below (5.9), the first step is to apply Theorem 3.20 to the $M$-Lipschitz function $A$ at level $M/2$, and then decompose the operator $T$ with respect to the ensuing families of intervals $B$ and $\{T\}_{\mathcal{T} \in \mathcal{F}}$, as in (5.11). For each tree $\mathcal{T} \in \mathcal{F}$, the corona decomposition yields an $(M/2)$-Lipschitz function $\psi_T : \mathbb{R} \to \mathbb{R}$, and a linear map $L_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}$. However, the proof presented above makes no explicit reference to these "approximating" functions before the introduction of the kernel $K_{A,\psi}$ in (5.27). So, the argument is literally the same until that point. In proving (5.53), the relevant "approximating" kernel is

$$\tilde{K}(x,y) = k(x,y) \exp \left(2\pi i \left[ \frac{(\psi+L)(x)-(\psi+L)(y)}{x-y} + \frac{B_2(x)-B_2(y)-\frac{1}{2}(B_1(x)+B_2(y))(x-y)}{(x-y)^2} \right] \right),$$

because $|A(x)-(\psi+L)(x)|$ is the quantity controlled by the corona information for $x \in 2Q$ and $Q \in \mathcal{T}$, recall the estimates in Section 5.4.2. As before, the crux of the proof is to prove the analogue of (5.30), namely

$$\|T_{\mathcal{T}}\|_{L^2 \to L^2} \lesssim \|T_{\mathcal{F}}\|_{L^2 \to L^2} + \max\{M,N\} \lesssim \varphi(M/2,N) + \max\{M,N\}. \quad (5.54)$$

Here $T_{\mathcal{T}}$ is precisely the same object as in the previous sections, and

$$T_{\mathcal{T}} f(x) = \int_{\{y : D(x,y) \leq |x-y| \leq \rho\}} \tilde{K}(x,y) f(y) \, dy.$$

The proof of the first inequality in (5.54) is virtually the same as above: the formula of the kernel $K$ only plays a role in Section 5.4.2, and the upper bound for $|A(x)-(\psi+L)(x)|$, coming from the corona decomposition, is exactly of the form applicable in (5.49). So, one can conclude (5.50), in fact with constant "$\max\{M,N\}" in place of "\max\{M,N^2\}".

The proof of the second inequality in (5.54) contains the only essential, albeit easy, difference in the proofs. Namely, recall from the discussion around (5.28) that the equation $K_{A,\psi} = K_{A,\psi}$ was crucially important. Now, the same is not true, but we have something comparable, and good enough. Namely, if $L(x) = cx$, we have

$$K_{\psi+L,B}(x,y) = e^{2\pi ic} K_{\psi,B}(x,y), \quad x,y \in \mathbb{R}, \ x \neq y.$$ 

Thus, even though $\psi + L$ is not $(M/2)$-Lipschitz, the $L^2 \to L^2$ operator norm of

$$T_{\psi+L,B} f(x) = \int K_{\psi+L,B}(x,y) f(y) \, dy = e^{2\pi i c} \int K_{\psi,B}(x,y) f(y) \, dy$$

is bounded from above by $\varphi(M/2,N)$. This fact (in combination with Cotlar’s inequality, as discussed after (5.30)) allows us to conclude the second inequality in (5.54). This completes the proof of (5.53), and hence the proof of (5.8) and of Theorem 5.6.

6. Regular curves and big pieces of intrinsic Lipschitz graphs

In this section, we prove Theorem 1.6, which states that certain SKs in $\mathbb{H}$ are CZ kernels for (Hausdorff measures on) regular curves. The plan is to reduce the assertion to its special case concerning intrinsic Lipschitz graphs, Theorem 1.7, through the observation that regular curves have big pieces of intrinsic Lipschitz graphs (Theorem 6.42). Further,
the transition from “intrinsic Lipschitz graphs” to sets with “big pieces of intrinsic Lipschitz graphs” is based on an abstract argument, originally due to David [16, 17] in $\mathbb{R}^n$. We will record a version of this argument in all proper metric spaces $(X, d)$, see Theorem 6.3 below, although the case $X = \mathbb{H}$ suffices for our application.

6.1. David’s big piece theorem in metric spaces.

**Definition 6.1** (Regular measures). Let $(X, d)$ be a metric space, and let $k > 0$. We write $\Sigma_k$ for the class of $k$-regular measures on $X$, that is, Borel regular measures $\mu$ on $X$ with the property that there exists a finite constant $C \geq 1$ such that

$$C^{-1}r^k \leq \mu(B(x, r)) \leq C r^k, \quad x \in \text{spt } \mu, \ r > 0. \quad (6.2)$$

The smallest constant $C \geq 1$ such that (6.2) holds will be denoted $\text{reg}_k(\mu)$, or just $\text{reg}(\mu)$.

If $\mu \in \Sigma_k$, then spt $\mu$ is a $k$-regular set and, since the lower bound is required to hold for arbitrarily large $r > 0$, it follows that $\text{diam}(X, d) \geq \text{diam}(\text{spt } \mu) = \infty$. This is a matter of technical convenience. Anyway, our focus will be on 1-regular curves in the metric space $X = \mathbb{H}$, and every such curve is contained in an unbounded 1-regular curve.

**Theorem 6.3.** Let $(X, d)$ be a proper metric space, and let $k > 0$. Let $K : X \times X \setminus \triangle \to \mathbb{C}$ be a $k$-GSK, and assume that $\mu \in \Sigma_k$ has the following properties. There exist constants $0 < \theta < 1$, $C \geq 1$ and, for each $1 < p < \infty$, a finite constant $A_p \geq 0$ such that the following is true. For every closed ball $B$ centred on spt $\mu$, there exists a Borel regular measure $\sigma$ on $X$, and a compact set $E \subset B \cap \text{spt } \mu$, such that

1. $\sigma \in \Sigma_k$ with $\text{reg}(\sigma) \leq C$,
2. $\mu(E) \geq \theta \mu(B)$,
3. $\mu(A \cap E) \leq \sigma(A)$ for all $A \subset X$,
4. $\|T^*_\sigma f\|_{L^p(\sigma)} \leq A_p \|f\|_{L^p(\sigma)}$ for $f \in C_c(X)$.

Then, there are constants $C_p > 0$, for $1 < p < \infty$, depending only on $(k, p, A_p, C, \text{reg}(\mu), \|K\|, \theta)$ such that

$$\|T^*_\mu f\|_{L^p(\mu)} \leq C_p \|f\|_{L^p(\mu)}, \quad f \in C_c(X). \quad (6.4)$$

Theorem 6.3 in $\mathbb{R}^n$ is due to David [17, Proposition 4 bis.], see also [18, III.3, Proposition 3.2] and [15, Proposition 1.18], and it is based on “good $\lambda$ inequalities”. The proof of the $(X, d)$ version follows David’s proof very closely, and there are no real difficulties. The main differences are:

- David only considers $k$-SKs $K : \mathbb{R}^n \times \mathbb{R}^n \setminus \triangle \to \mathbb{C}$ satisfying

$$|\nabla_x K(x, y)| + |\nabla_y K(x, y)| \lesssim |x - y|^{-1-k}, \quad x \neq y.$$

In contrast, we consider $k$-GSKs, and associated operators $T^*$. In this generality, we do not know if $T^* f$ is lower semicontinuous, which causes minor technical trouble in the proof of Lemma 6.25.

- At one point of the original proof, David seems to refer to the Besicovitch covering theorem, which is not available in metric spaces. However, it turns out that the $5r$-covering theorem suffices, see Lemma 6.8.

Often, when arguments follow [17, Proposition 4 bis.] verbatim, we will omit details.
6.1.1. Proof of Theorem 6.3. The version of the “good $\lambda$ inequalities” which we use in the proof of Theorem 6.3 is borrowed from [18, III, Lemma 3.1]:

**Proposition 6.5.** Let $(X, \mu)$ be a measure space, and let $1 < p < \infty$. Let $u : X \to [0, +\infty]$ be a $\mu$ measurable function that agrees with an $L^p(\mu)$ function outside a set of finite $\mu$ measure, and let $v : X \to [0, +\infty]$ be an $L^p(\mu)$ function. Assume that there exists a constant $0 < \nu < 1$ such that, for all $\varepsilon > 0$, there is a constant $\gamma > 0$ so that, for all $\lambda > 0$,

$$\mu(\{x \in X : u(x) > \lambda + \varepsilon\lambda \text{ and } v(x) \leq \gamma \lambda\}) \leq (1 - \nu)\mu(\{x \in X : u(x) > \lambda\})$$

(6.6)

Then $u \in L^p(\mu)$ with $\|u\|_{L^p(\mu)} \leq C(p, \varepsilon, \nu, \gamma)\|v\|_{L^p(\mu)}$.

A proof for the case $X = \mathbb{R}$ and $\mu = \mathcal{L}^1$ is included below [16, Lemme 12] (we do not need an explicit expression of $C(p, \varepsilon, \nu, \gamma)$ for our purposes). The version for an arbitrary measure space $(X, \mu)$ is proven in the same way (David leaves this as an exercise in [18]). The proof of Theorem 6.3 follows by applying Proposition 6.5 for given $f \in \mathcal{C}_c(X)$ and $1 < p < \infty$ to the functions

$$u := T_{\mu}^* f \quad \text{and} \quad v := M_{\mu,k} f + \left((M_{\mu,k}|f|\mathcal{V})\right)^{\frac{1}{p}}$$

(6.7)

where $M_{\mu,k}$ is the radial maximal function of order $k$ (see Section 2.2). For $\mu \in \Sigma_k$, we will abbreviate $M_{\mu} := M_{\mu,k}$. In order to employ Proposition 6.5, we want to show that $u$ agrees with an $L^p(\mu)$ function outside a compact set, namely outside a closed ball $B(x_\mu, 2R)$, where $x_\mu \in X$, and $R > 0$ is so large that $spt f \subseteq B(x_\mu, 2R)$. Moreover, we have to verify that $u$ and $v$ are $L^p(\mu)$ satisfying (6.6). This will yield Theorem 6.3 since $\|v\|_{L^p(\mu)} \leq C(p, \text{reg}(\mu))\|f\|_{L^p(\mu)}$. We start with some preliminaries.

Whenever $\mu \in \Sigma_k$, the triple $(\text{spt} \mu, \mu, d)$ a doubling metric measure space, and $M_\mu$ is bounded on $L^p(\mu)$ for $1 < p < \infty$. We need a more general version of this result that involves two distinct measures in $\Sigma_k$ with potentially distinct, even disjoint, supports. David states this in [18, Lemma 2.2, p. 58], and writes the proof is easy, and based on the Besicovitch covering theorem. This tool is not available in our generality, but, in fact, the $5r$-covering theorem is good enough.

**Lemma 6.8.** Assume that $(X, d)$ is a proper metric space and $k > 0$. Let $\mu, \sigma \in \Sigma_k$, and $1 < p < \infty$. Then, there exists a constant $0 < C < \infty$, depending only on $p$ and $\text{reg}(\mu), \text{reg}(\sigma)$, such that

$$\|M_\mu f\|_{L^p(\sigma)} \leq C\|f\|_{L^p(\mu)}, \quad f \in L^p(\mu).$$

**Proof.** Lemma 6.8 is proved in the same way as [16, Proposition 4], using Marcinkiewicz interpolation. One has to show that $M_\mu$ maps $L^p(\mu)$ into $L^p(\sigma)$, which is clear (only using $\mu \in \Sigma_k$), and that it also maps $L^1(\mu)$ into $L^{1,\infty}(\sigma)$:

$$\sigma(\{x \in X : M_\mu f(x) > \lambda\}) \leq \frac{C}{\lambda}\|f\|_{L^1(\mu)}, \quad f \in L^1(\mu).$$

(6.9)

This follows from the "standard" proof, and only uses that $\sigma \in \Sigma_k$, but to convince the reader that no Besicovitch covering theorem is needed, let us record the details. Fix $f \in L^1(\mu)$, and consider the ball family

$$\mathcal{B} := \left\{ B(x, r) \subset X : x \in \text{spt} \sigma \text{ and } \frac{1}{r^k} \int_{B(x, r)} |f| \, d\mu > \lambda \right\}.$$
Since \( f \in L^1(\mu) \), the radii of the balls in \( B \) are uniformly bounded. Second, \( B \) is a cover for the set \( E = \{ x \in \text{spt } \sigma : \mathcal{M}_\sigma f(x) > \lambda \} \), which has the same \( \sigma \)-measure as the left hand side of (6.9). Using the \( 5r \)-covering theorem, we extract a countable disjoint subfamily \( \mathcal{B}_0 := \{ B(x_i, r_i) \}_{i \in \mathbb{N}} \subset B \) with \( x_i \in \text{spt } \sigma \), and
\[
E \subset \bigcup_{i \in \mathbb{N}} B(x_i, 5r_i).
\]
Finally,
\[
\sigma(E) \leq \sum_{i \in \mathbb{N}} \sigma(B(x_i, 5r_i)) \leq C \sum_{i \in \mathbb{N}} r_i^k \leq C \frac{1}{X} \sum_{i \in \mathbb{N}} \int_{B(x_i, r_i)} |f| \, d\mu \leq \frac{C}{X} \|f\|_{L^1(\mu)},
\]
as claimed. \( \square \)

Lemma 6.8 yields a “two-measure statement” for SIOs, Proposition 6.16 below. We follow closely David’s proof of [17, Proposition 2] and deduce Proposition 6.16 from two auxiliary lemmas.

**Lemma 6.10.** Let \( (X, d) \) be a proper metric space, \( k > 0 \), and let \( K : X \times X \setminus \triangle \rightarrow C \) a \( k \)-GSK. Assume that \( \sigma \in \Sigma_k \). Then there exists a constant \( C > 0 \), depending only on \( k, \|K\|, \) and \( \text{reg}(\sigma) \), such that
\[
T^\sigma_{\epsilon} f(x_0) \leq C \left( M_{\sigma}(T^\sigma_{\epsilon} f) \right)(x_0) + C M_{\sigma} f(x_0), \quad f \in \mathcal{C}_c(X), \quad x_0 \in X.
\]
(6.11)

The main point is that we can take \( x_0 \in X \setminus \text{spt } \sigma \).

**Proof.** One first shows that there exists a constant \( C_0 > 0 \), depending only on \( k \) and \( \|K\| \), such that for all \( \epsilon > 0 \) and \( x_0 \in X \), one has
\[
|T_{\sigma, \epsilon} f(x_0)| \leq T^\sigma_{\epsilon} f(x) + C_0 M_{\sigma} f(x_0), \quad x \in B(x_0, \epsilon/2).
\]
(6.12)
This can be done as in the proof of [17, Lemma 4].

To show (6.11), we fix \( x_0 \in X \) and write \( d := \text{dist}(x_0, \text{spt } \sigma) \). The proof is divided in three cases, exactly as the proof of [17, Lemme 3]. First, if \( \epsilon \geq 4d \), then \( \sigma(B(x_0, \epsilon/2)) > 0 \). Integrating (6.12) with respect to \( \frac{1}{\sigma(B(x_0, \epsilon/2))} \, d\sigma \) over \( B(x_0, \epsilon/2) \) and using the assumption \( \sigma \in \Sigma_k \), we find a constant \( C > 0 \), depending only on \( C_0 \) in (6.12), \( k \), and \( \text{reg}(\sigma) \), such that
\[
|T_{\sigma, \epsilon} f(x_0)| \leq C \left( M_{\sigma}(T^\sigma_{\epsilon} f) \right)(x_0) + C M_{\sigma} f(x_0).
\]
(6.13)
Second, if \( d/2 \leq \epsilon < 4d \), then by (6.13) for \( \epsilon = 4d \) and the size estimate \( |K(x_0, y)| \lesssim d(x_0, y)^{-k} \) on the annulus \( B(x_0, 4d) \setminus B(x_0, \epsilon) \) yield again a bound of the form (6.13). Third, if \( \epsilon < d/2 \), then \( T_{\sigma, \epsilon} f(x_0) = T_{\sigma, d/2} f(x_0) \), and we are reduced to the second case. \( \square \)

The next lemma is a Cotlar-type inequality. Such inequalities are available in very general settings, cf. [47, I.7.3, Proposition 2], [34, p.56], [9, p.606], and [41], but we are not aware of one that would be precisely in the desired form for our purposes. In particular, we have to deal simultaneously with two measures \( \mu \) and \( \sigma \) in a metric space \( (X, d) \).

**Lemma 6.14.** Let \( (X, d) \) be a proper metric space, \( k > 0 \), and \( \mu \in \Sigma_k \). Let \( \overline{K} : X \times X \setminus \triangle \rightarrow \mathbb{H} \) be a bounded \( k \)-GSK, and let \( \overline{T} \) be the operator induced by \( (K, \mu) \). Let \( \sigma \in \Sigma_k \) with regularity constant \( C_0 \geq 1 \), and assume, for some \( 1 < s < \infty \), that
\[
A := \|\overline{T}\|_{L^s(\mu) \rightarrow L^s(\sigma)} < \infty.
\]
Then, there exists a constant $C = C(A, C_0, k, \|K\|, s)^3$ such that
\[
T_\mu^sf(x) \leq C \left[ M_\sigma(T_\mu f)(x) + M_\mu f(x) + (M_\mu |f|^s)^{\frac{1}{s}}(x) \right], \quad f \in C_c(X), \ x \in \text{spt} \sigma. \tag{6.15}
\]

Proof. The proof is verbatim the same as for [17, Lemma 5]. \hfill \Box

Proposition 6.16. Let $(X, d)$ be a proper metric space, $k > 0$, let $K : X \times X \setminus \triangle \to \mathbb{C}$ be a $k$-GSK, and let $\sigma \in \Sigma_k$. Assume that, for all $1 < p < \infty$, there is a constant $C_p \geq 1$ such that
\[
\|T_\sigma^s f\|_{L^p(\sigma)} \leq C_p \|f\|_{L^p(\sigma)}, \quad f \in C_c(X). \tag{6.17}
\]
Then for all $1 < p < \infty$ and $\mu \in \Sigma_k$, there is a constant $C'_p \geq 1$ such that for all $f \in C_c(X)$,
\[
\begin{align*}
(1) & \|T_\mu^s f\|_{L^p(\mu)} \leq C'_p \|f\|_{L^p(\sigma)}, \\
(2) & \|T_\mu^s f\|_{L^p(\sigma)} \leq C'_p \|f\|_{L^p(\mu)}.
\end{align*}
\]
The constants $C'_p$ depend only on $p, C_p, k, \|K\|, \text{and reg}(\mu), \text{reg}(\sigma)$.

Proof. Part (1) is a straightforward consequence Lemmas 6.10 and 6.8.

Part (2) is proved by duality. Fix $\mu \in \Sigma_k$, $1 < p < \infty$, and let $q = p/(p - 1)$. From the first part of the lemma, we know that the operators $T_{\sigma, \varepsilon}$ are uniformly bounded $L^q(\sigma) \to L^q(\mu)$. Now we define $K^t(x, y) := K(y, x)$, and let $T_{\mu, \varepsilon}^t$ be the (adjoint) $\varepsilon$-SIO induced by $(K^t, \mu)$. Then,
\[
\sup_{\varepsilon > 0} \|T_{\mu, \varepsilon}^t\|_{L^p(\mu) \to L^p(\sigma)} \leq C_p,
\]
As an intermediate step towards (2), we wish to deduce from Lemma 6.16 the corresponding bound for the maximal SIO $T_{\mu, \varepsilon}^t$. A small technical issue is that $K$ is not necessarily a bounded GSK, as required in the hypothesis (to even make sense of $T$). To remedy this, fix $\varepsilon > 0$, and note that $K^t$ is a bounded GSK, with GSK constants independent of $\varepsilon$, by Lemma 2.8. Consequently, Lemma 6.16, applied with $K^t$ and $s := \sqrt{p}$, implies that
\[
\|T_{\mu, \varepsilon}^t f\|_{L^p(\sigma)} \lesssim \|T_{\mu, \varepsilon}^t f\|_{L^p(\mu)} + |M_\mu f|_{L^p(\sigma)} + \|M_\mu (|f|^s)^{1/s}\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\mu)} \tag{6.18}
\]
for $f \in C_c(X)$. Here $T_{\mu, \varepsilon}^t f$ is the maximal SIO associated to $K^t$, and we also used the $L^p(\mu) \to L^p(\sigma)$ and $L^s(\mu) \to L^s(\sigma)$ boundedness of $M_\mu$ from Lemma 6.8, and the $L^p(\sigma) \to L^p(\sigma)$ boundedness of $M_\sigma$. To proceed, we note that
\[
T_{\mu, \varepsilon}^t f(x) = \sup_{\delta > 0} |T_{\mu, \delta}^t f(x)|, \quad f \in L^p(\mu), \ x \in X, \ \varepsilon > 0,
\]
so $T_{\mu, \varepsilon}^t f(x) \not\to T_{\mu}^t f(x)$ as $\varepsilon \downarrow 0$. Now, (6.18) and monotone convergence yield
\[
\|T_{\mu, \varepsilon}^t f\|_{L^p(\sigma)} \lesssim \|f\|_{L^p(\mu)}, \quad f \in C_c(X). \tag{6.19}
\]
This almost looks like (2), except that it concerns $T^t$ in place of $T$. However, applying (6.19) to $\mu := \sigma$, we conclude that also $K^t$ satisfies (6.17). Hence, we can re-run the whole argument with $K^t$! But since $(K^t)^t = K$, this time we end up with (2). \hfill \Box

---

3This constant does not depend on the regularity constant of $\mu$, so the assumption $\mu \in \Sigma_k$ is only made to ensure that $T$ is well-defined. It would suffice to assume that $\mu(B(x, r)) \lesssim r^k$ instead.
Let us continue with the proof of Theorem 6.3. Fix \( \mu \in \Sigma_k \) as in the statement, fix \( 1 < p < \infty \), and let \( f \in C_c(X) \). Our task is to show that

\[
\|T^*_\mu f\|_{L^p(\mu)} \leq C_p \|f\|_{L^p(\mu)}, \quad f \in C_c(X). \tag{6.20}
\]

This will follow from Proposition 6.5 (“good \( \lambda \) inequality”) applied to

\[
u := T^*_\mu f \quad \text{and} \quad v := \mathcal{M}_\mu f + \left( \mathcal{M}_\mu |f|^{\varphi} \right)^{\frac{1}{\varphi}}. \tag{6.21}
\]

The rest of the proof consists of explaining how Proposition 6.16 can be used to verify that the assumptions of Proposition 6.5 are fulfilled.

**Lemma 6.22.** Let \((X, d)\) be a proper metric space, \( k > 0 \), and let \( K: X \times X \setminus \triangle \to \mathbb{C} \) be a \( k\)-GSK. Let \( \mu \in \Sigma_k, f \in C_c(X) \) and \( 1 < p < \infty \). Then \( u := T^*_\mu f \) is a Borel function on \((X, d)\) and it agrees with an \( L^p(\mu) \) function outside a ball, hence outside a set of finite \( \mu \) measure.

**Proof.** First we note that

\[
T^*_\mu f(x) = \sup_{\varepsilon \in \mathbb{Q} \cap (0, +\infty)} |T_{\mu, \varepsilon} f(x)|. \tag{6.23}
\]

Indeed, for every \( \varepsilon \in (0, +\infty) \), there exists a sequence \((\varepsilon_j)_{j \in \mathbb{N}} \subset \mathbb{Q} \) with \( \varepsilon_j \searrow \varepsilon \) as \( j \to \infty \), and it follows that

\[
|T_{\mu, \varepsilon} f(x) - T_{\mu, \varepsilon_j} f(x)| \leq \int_{\varepsilon < d(x, y) \leq \varepsilon_j} |K(x, y)| f(y) \, d\mu(y) \to 0 \quad \text{as } j \to \infty.
\]

Since \( T_{\mu, \varepsilon} f \) is a Borel function for every \( \varepsilon > 0 \), we deduce from (6.23) that \( u \) is a Borel function.

Regarding the second claim, if \( \text{spt } f \subset B(x_0, R) \), the “size” condition for \( K \) alone implies that \( T^*_\mu f(x) \lesssim \mathcal{M}_{\mu,k} f(x) \) for \( x \in X \setminus B(x_0, 2R) \). Now, the claim follows from the \( L^p(\mu) \)-boundedness of \( \mathcal{M}_{\mu,k} \).

**Lemma 6.24.** Let \((X, d)\) be a proper metric space, \( k > 0, \mu \in \Sigma_k, f \in C_c(X) \), and \( 1 < p < \infty \). Then

\[
v := \mathcal{M}_{\mu,k} f + \left( \mathcal{M}_{\mu,k} |f|^{\varphi} \right)^{\frac{1}{\varphi}} \in L^p(\mu)
\]

with \( \|v\|_{L^p(\mu)} \leq C \|f\|_{L^p(\mu)} \), where \( C \) depends only on \( p \) and \( \text{reg}(\mu) \).

**Proof.** This follows from the boundedness of \( \mathcal{M}_{\mu,k} \) on \( L^p(\mu) \) and \( L^{\varphi}(\mu) \).

**Lemma 6.25.** Assume that \((X, d), k > 0, K: X \times X \setminus \triangle \to \mathbb{C} \) and \( \mu \in \Sigma_k \) are as in Theorem 6.3. Then there exists \( \nu \in (0, 1) \), depending only on \( \text{reg}(\mu) \) and the parameter \( \theta > 0 \), such that the following holds. Let \( 1 < p < \infty \), \( f \in C_c(X) \), and define the functions \( u \) and \( v \) as in Lemmas 6.22 and 6.24. Then, for all \( \varepsilon > 0 \), there is \( \gamma = \gamma(\varepsilon) > 0 \) such that

\[
\mu(\{x \in X : u(x) > \lambda + \varepsilon \lambda \text{ and } v(x) \leq \gamma \lambda\}) \leq (1 - \nu) \mu(\{x \in X : u(x) > \lambda\}) \tag{6.26}
\]

for \( \lambda > 0 \). The choice of \( \gamma \) is also allowed to depend on \( p \), and the “data” of Theorem 6.3.

**Proof.** The proof follows [17, p.234ff] closely. The main difference is that \( T^*_\mu f \) may not be lower semicontinuous when \( K \) is only a generalised standard kernel; this causes minor technical issues. Fix \( \varepsilon, \lambda > 0 \) and abbreviate

\[
\Omega := \Omega_{\lambda, \varepsilon} := \{ x \in \text{spt } \mu : u(x) > \lambda \},
\]

and

\[
A := A_{\lambda, \varepsilon, \gamma} = \{ x \in \text{spt } \mu : u(x) > \lambda + \varepsilon \lambda \text{ and } v(x) \leq \gamma \lambda\} \subseteq \Omega.
\]
Our task is to ensure that \( \mu(A) \leq (1 - \nu)\mu(\Omega) \) for some \( \nu = \nu(\operatorname{reg}(\mu), \theta) > 0 \). We may evidently assume that \( \mu(\Omega) > 0 \).

We start by constructing a cover for \( \Omega \). Since \( f \in C_c(X) \), it follows from the "size" estimate \( \lvert K(x, y) \rvert \lesssim d(x, y)^{-k} \), and from \( \mu \in \Sigma_k \), that \( T_{\mu}^* f(x) \to 0 \) as \( \operatorname{dist}(x, \operatorname{spt} f) \to \infty \). Hence \( \Omega \) is a bounded set. On the other hand, for \( \mu \) almost every \( x \in \Omega \),

\[
\lim_{j \to \infty} \frac{\mu(B(x, 2^{-j}) \cap \Omega)}{\mu(B(x, 2^{-j}))} = 1, \tag{6.27}
\]

by Lebesgue differentiation in the doubling metric measure space \((\operatorname{spt} \mu, \mu, d)\). Combining (6.27) and the fact that \( \Omega \) is bounded, it follows that for \( \mu \) almost every \( x \in \Omega \), there exists a maximal dyadic radius \( r_x = 2^{-j_x} \lesssim \theta \mu \), with \( j_x \in \mathbb{Z} \) such that

\[
\frac{\mu(B(x, r_x) \cap \Omega)}{\mu(B(x, r_x))} \geq 1 - \frac{\theta}{2}. \tag{6.28}
\]

In particular, since the reverse inequality already holds for \( 2r_x \), we can find

\[
a_x \in B(x, 2r_x) \cap \Omega'. \tag{6.29}
\]

We then apply the 5r-covering theorem to find a disjoint family \( \{B(x, r_i) : x \in \Omega\} \) with the property that \( \mu \) almost all of \( \Omega \) is contained in

\[
\bigcup_{i \in \mathbb{N}} B(x_i, 5r_i).
\]

We write \( B_i := B(x_i, r_i) \), \( 5B_i := B(x_i, 5r_i) \), and \( a_i := a_{x_i} \). In order to prove (6.26), it suffices to show that

\[
\frac{\mu([B_i \cap \Omega] \setminus A)}{\mu(B_i)} > \frac{\theta}{4}, \quad i \in \mathbb{N}. \tag{6.30}
\]

This will establish (6.26), because

\[
\mu(\Omega \setminus A) \geq \sum_{i} \mu([B_i \cap \Omega] \setminus A) > \frac{\theta}{4} \sum_{i \in \mathbb{N}} \mu(B_i) \geq \mu(\Omega),
\]

and consequently \( \mu(A) \leq (1 - \nu)\mu(\Omega) \) for some \( \nu = \nu(\operatorname{reg}(\mu), \theta) > 0 \), as desired.

We then prove that (6.30) holds if \( \gamma = \gamma(\epsilon) > 0 \) is chosen small enough (recall that \( A = A_{\lambda, \epsilon, \gamma} \)). For now, let \( \gamma > 0 \) be arbitrary, and fix \( B_i \). Note that (6.30) is clear if \( \nu(x) > \gamma \lambda \) for all \( x \in B_i \) (then \( [B_i \cap \Omega] \setminus A = B_i \cap \Omega \), which has density \( \geq 1 - \theta/2 \geq \theta/2 \)), so we may assume that there exists a point \( \xi_i \in B_i \) with

\[
\mathcal{M}_\mu f(\xi_i) + \left( \mathcal{M}_\mu |f|^{\sqrt{q}} \right)^{\frac{1}{q}}(\xi_i) = v(\xi_i) \leq \gamma \lambda. \tag{6.31}
\]

Now, we decompose \( f = f_1 + f_2 \), where \( f_1 = f \phi \), and \( \phi \in C_c(X) \) satisfies

\[
1_{B(\xi_i, 10r_i)} \leq \phi \leq 1_{B(\xi_i, 20r_i)}.
\]

Then

\[
u(x) \leq T_{\mu}^* f_1(x) + T_{\mu}^* f_2(x), \quad x \in B_i, \tag{6.32}
\]

and we will check in a moment that

\[
T_{\mu}^* f_2(x) \leq \lambda + \epsilon/2, \quad x \in B_i, \tag{6.33}
\]

if \( \gamma = \gamma(\epsilon) > 0 \) is small enough. Thus, (6.32)-(6.33) imply that

\[
\{x \in B_i : T_{\mu}^* f_1(x) \leq \epsilon \} \subseteq B_i \setminus A.
\]
and the proof of (6.30) has been reduced to showing that
\[ \mu(\{x \in B_i \cap \Omega : T^*_\mu f_1(x) \leq \epsilon \frac{1}{2}\}) \geq \frac{3}{4} \mu(B_i). \] (6.34)

Before tackling (6.34), we verify (6.33). In fact, (6.33) follows from the chain
\[ T^*_\mu f_2(x) \leq T^*_\mu f(a_i) + C M_\mu f(\xi_i) \leq \lambda + C \gamma \lambda, \quad x \in B_i, \] (6.35)
by choosing \( \gamma \) small enough so that \( C \gamma \leq \epsilon/2 \). The second inequality in (6.35) follows from the choices of \( a_i \in \Omega^c \) and \( \xi_i \) in (6.31). The first inequality can be obtained by writing \( R_i := 10r_i \), and decomposing
\[ |T_\mu f_2(x)| = \left| \int K(x,y)[1 - \phi(y)]f(y) \, d\mu(y) \right| \]
\[ \leq \int_{B(a_i, R_i)^c} K(a_i, y)f(y) \, d\mu(y) + \int \|\phi - 1_{B(a_i, R_i)}\|(y)\|K(a_i, y)||f(y)||d\mu(y) \]
\[ + \int_{B(\xi_i, R_i)^c} |K(a_i, y) - K(\xi_i, y)||f(y)||d\mu(y) \]
\[ + \int_{B(\xi_i, R_i)^c} |K(x, y) - K(\xi_i, y)||f(y)||d\mu(y). \]

The first term is bounded by \( T^*_\mu f(a_i) \), as desired. The three latter ones are bounded by \( \lesssim M_\mu f(\xi_i) \), using the GSK bounds of \( K \), and recalling that \( x, a_i, \xi_i \in 2B_i \subset B(\xi_i, R_i/2) \), and that \( \hat{\phi}|_{B(\xi_i, R_i)} = 1 \). Similar, but slightly messier, estimates also work for \( T^*_{\mu, \delta} \), \( \delta > 0 \), in place of \( T_\mu \), so (6.35) has been confirmed.

Finally, we turn to (6.34), which is based on the “big piece” assumption: there exists a measure \( \sigma = \sigma_{B_i} \in \Sigma_{B_i} \), and a compact set \( E \subseteq B_i \cap \text{spt}\, \mu \) with the property that \( \mu(E) \geq \theta \mu(B_i) \) and such that
\[ \mu \left( \{x \in E : T^*_\mu f_1(x) \leq \epsilon \frac{1}{2}\} \right) \leq \sigma \left( \{x \in E : T^*_\mu f_1(x) > \epsilon \frac{1}{2}\} \right). \] (6.36)

Since \( \mu(\Omega \cap B_i) \geq (1 - \theta/2) \mu(B_i) \), we moreover find that \( \mu(E \cap \Omega) \geq (\theta/2) \mu(B_i) \). We will show that \( \gamma = \gamma(\epsilon) > 0 \) can be chosen small enough so that the assumption (6.31) implies that
\[ \mu \left( \{x \in E : T^*_\mu f_1(x) > \epsilon \frac{1}{2}\} \right) < \frac{3}{4} \mu(B_i). \] (6.37)
This of course yields (6.34):
\[ \mu \left( \{x \in B_i \cap \Omega : T^*_\mu f_1(x) \leq \epsilon \frac{1}{2}\} \right) \geq \mu \left( \{x \in E \cap \Omega : T^*_\mu f_1(x) \leq \epsilon \frac{1}{2}\} \right) \geq \frac{3}{4} \mu(B_i). \]

To prove (6.37), start by combining (6.36) with Chebyshev’s inequality with \( s := \sqrt{\frac{\lambda}{\sigma}} \):
\[ \mu \left( \{x \in E : T^*_\mu f_1(x) > \epsilon \frac{1}{2}\} \right) \leq 2^s \epsilon^{-s} \lambda^{-s} \|T^*_\mu f_1\|_{L^s(\sigma)}. \] (6.38)

To proceed, we plan to apply (6.31). By the hypothesis (4) of Theorem 6.3, \( \|T^*_g \|_{L^p(\sigma)} \leq A_p \|g\|_{L^p(\sigma)} \) for all \( g \in C_c(X) \). This is the assumption (6.17) in Proposition 6.16, so part (2) of that proposition yields
\[ \|T^*_\mu f_1\|_{L^s(\sigma)} \leq C_s \|f_1\|_{L^s(\mu)} \lesssim r^k \gamma \lambda^s \] (6.39)
Combining (6.38)-(6.39), we find that
\[ \mu \left( \{x \in E : T^*_\mu f_1(x) > \epsilon \frac{1}{2}\} \right) \lesssim_p r^k \epsilon^{-s} \lambda^s \lesssim \mu(B_i). \]
Choosing \( \gamma > 0 \) small enough, depending on \( \theta, \varepsilon, p, \) and \( \operatorname{reg}(\mu) \), we conclude the proof of (6.37), and therefore the lemma.

We are now in possession of all ingredients necessary for the proof of Theorem 6.3.

**Proof of Theorem 6.3.** Lemmas 6.22, 6.24, and 6.25 show that Proposition 6.5 can be applied to the functions \( u \) and \( v \) as defined in (6.21). This establishes (6.20). \( \square \)

### 6.2. Regular curves and BPiLG

Recall that a closed set \( E \in \mathbb{H} \) is 1-regular if there exists a finite constant \( C \geq 1 \) such that

\[
C^{-1} r \leq \mathcal{H}^1(B(p, r) \cap E) \leq C r, \quad \text{for all } p \in E, \ 0 < r \leq \text{diam}(E).
\]  

(6.40)

The smallest constant \( C \geq 1 \) such that (6.40) holds will be denoted \( \operatorname{reg}(E) \).

Recall further that a regular curve in \( \mathbb{H} \) is a closed 1-regular subset of \( \mathbb{H} \) which has a Lipschitz parametrisation by an interval \( I \subset \mathbb{R} \). In this section, we will use the letter "\( \gamma \)" for both the set, and the Lipschitz map \( I \rightarrow \gamma \). A compact regular curve is a regular curve parametrised by a compact interval \( I \subset \mathbb{R} \).

**Definition 6.41** (Big pieces of intrinsic Lipschitz graphs). A closed 1-regular set \( E \subset \mathbb{H} \) has big pieces of intrinsic Lipschitz graphs (over horizontal subgroups) (BPiLG) if there exist constants \( c, L > 0 \) such that for all \( p \in E \) and all \( 0 < r \leq \text{diam}(E) \) there is an intrinsic \( L \)-Lipschitz graph \( \Gamma \subset \mathbb{H} \) over some horizontal subgroup such that

\[
\mathcal{H}^1(E \cap \Gamma \cap B(p, r)) \geq cr.
\]

In this section, we prove the following:

**Theorem 6.42.** Every regular curve in \( \mathbb{H} \) has BPiLG.

A short proof for the fact that regular curves in \( \mathbb{R}^n \) have big pieces of 1-dimensional Lipschitz graphs can be found in [18, III.4]. It is based on the rising sun lemma, and we did not find a way to adapt it to intrinsic Lipschitz graphs. Instead, we follow [20].

The proof of Theorem 6.42 employs a system \( \mathcal{D} \) of dyadic cubes on a closed 1-regular set \( E \subset \mathbb{H} \), see [6, Section 3.0.1] for a more thorough introduction. These are Borel subsets of \( E \) with the following properties:

- \( \mathcal{D} = \bigcup_j D_j, \ j \in \mathbb{Z}, \) where each \( D_j \) is a partition of \( E \).
- There exist \( 0 < c_0 < C_0 < \infty \), depending on \( \operatorname{reg}(E) \), such that \( \text{diam}(Q) \leq C_0 \ell(Q) \) for \( Q \in D_j \), where \( \ell(Q) := 2^{-j} \). For every \( Q \in D_j \), there exists a "midpoint" \( z_Q \in Q \) such that \( E \cap B(z_Q, C_0 \ell(Q)) \subset Q \).

With this notation, we write \( B_Q := B(z_Q, 2C_0 \ell(Q)) \), so that \( Q \subset B_Q \) (with room to spare). For \( Q \in \mathcal{D} \), we define the horizontal \( \beta \)-number

\[
\beta(Q) := \beta_E(Q) := \inf_{\ell \in \mathcal{L}} \sup_{Q \in B_{z_Q} \cap E} \frac{\text{dist}(q, \ell)}{\ell(Q)},
\]

where the infimum is taken over the horizontal lines familiar from Definition 3.37,

\[
\mathcal{L} := \{ p \cdot \mathbb{V} : \ p \in \mathbb{H}, \ \mathbb{V} \text{ is a horizontal subgroup} \}.
\]

These numbers, notably their summability on horizontal curves, has been investigated extensively, see for example [35, 37] and the discussion in the introduction. Given a system \( \mathcal{D} \) of dyadic cubes on a closed 1-regular set \( E \), we introduce the following subclass of good cubes in \( \mathcal{D} \):
Definition 6.43. Let $E \subset \mathbb{H}$ be a closed 1-regular set with a system $\mathcal{D}$ of dyadic cubes. Given $0 < c, \varepsilon < 1$ and a horizontal subgroup $\mathcal{V}$, we say that $Q \in \mathcal{D}$ is $(c, \varepsilon, \mathcal{V})$-good if

1. $\mathcal{H}^1(\pi_{\mathcal{V}}(Q)) \geq c \mathcal{H}^1(Q)$,
2. $\beta(Q) \leq \varepsilon$.

Here $\pi_{\mathcal{V}}$ is the horizontal projection introduced in Definition 3.38. Recall also the cones $C_{\mathcal{V}}(\alpha)$ from Section 3.3. The next lemma shows that $(c, \varepsilon, \mathcal{V})$ good cubes $Q \in \mathcal{D}$ look like intrinsic Lipschitz graphs over $\mathcal{V}$ at scale $\ell(Q)$.

Lemma 6.44. Let $E \subset \mathbb{H}$ be a closed 1-regular set with a system $\mathcal{D}$ of dyadic cubes. Then for all $c > 0$ and $M \geq 2C_0 \geq 1$, there exists $\alpha, \varepsilon > 0$, depending only on $c$ and $M$, such that the following holds. If $Q \in \mathcal{D}$ is a $(c, \varepsilon, \mathcal{V})$-good cube, then

$$p \in Q, \ q \in B_Q \cap E \text{ and } d(p, q) \geq \ell(Q)/M \implies p^{-1} \cdot q \notin C_{\mathcal{V}}(\alpha).$$

(6.45)

Proof. Using rotations around the $t$-axis, we may, without loss of generality, suppose that $\mathcal{V} = \{(x, 0, 0) : x \in \mathbb{R}\}$. Now, fix $c > 0$ and $M \geq 2C_0$. We also fix arbitrary $\varepsilon, \alpha > 0$ at this point, and we fix a cube $Q \in \mathcal{D}$ such that Definition 6.43(2) is satisfied, that is, $\beta(Q) \leq \varepsilon$. The plan is to show that if (6.45) fails for some $p \in Q$ and $q \in B_Q$ with $d(p, q) \geq \ell(Q)/M$, and if $\alpha, \varepsilon > 0$ are small enough, then $Q$ cannot be a $(c, \varepsilon, \mathcal{V})$-good cube, that is, $\mathcal{H}^1(\pi_{\mathcal{V}}(Q)) < c \mathcal{H}^1(Q)$. Since the constants in Definition 6.43 are invariant under left translations and dilations, we may arrange that

$$p = 0 \in Q \subset E \text{ and } M^{-1} \leq d(0, q) \leq M.$$  

(6.46)

We write in coordinates $q = (x, y, t)$, so that

$$q \in C_{\mathcal{V}}(\alpha) \iff \|(x, 0, 0)\| \leq \alpha \left\|(0, y, t - \frac{xy}{2})\right\|.$$  

(6.47)

If $\alpha = \alpha_M > 0$ is sufficiently small, this implies, together with (6.46), that $\|(0, y, t)\| \sim_M 1$. Next we will use $\beta(Q) \leq \varepsilon$ to infer that $t$ is small, and hence $q$ lies close to $\{(0, y, 0) : y \in \mathbb{R}\}$. But since $Q$ lies close to the segment $[p, q] = [0, q]$, again by $\beta(Q) \leq \varepsilon$, and $\pi_{\mathcal{V}}(\{(0, y, 0) : y \in \mathbb{R}\}) = \{0\}$, this will eventually show that $\mathcal{H}^1(\pi_{\mathcal{V}}(Q)) < c \mathcal{H}^1(Q)$.

We turn to the details. Condition (6.47) implies that

$$|x| \leq \alpha \left(|y| + \sqrt{|t|} + \frac{\sqrt{|x||y|}}{\sqrt{2}}\right).$$  

(6.48)

Now we consider two cases. If $|x| \leq |y|$, then (6.48) implies

$$|x| \leq 2\alpha(|y| + \sqrt{|t|}).$$  

(6.49)

On the other hand, if $|y| \leq |x|$, then (6.48) implies that

$$|x| \left(1 - \alpha(1 + 1/\sqrt{2})\right) \leq \alpha \sqrt{|t|}$$

and hence (6.49) holds true also in this case assuming, as we may, that $\alpha \leq 1/2$. Combined with the assumption that $d(q, 0) \geq M^{-1}$, this shows that

$$|y| + \sqrt{|t|} \geq \frac{M^{-1}}{(1 + 2\alpha)}.$$
To deduce more precise information about the coordinates of the point $q$, we use the assumption $\beta(Q) \leq \varepsilon$, which ensures the existence of a horizontal line $\ell = p_0 \cdot \mathbb{V}$ with the property that
\[
\text{dist}(q', \ell) \leq 2\varepsilon, \quad q' \in B_Q \cap E.
\]
Thus there exist $(a, b) \in \mathbb{R}^2$, $a^2 + b^2 = 1$, $p_0 = (x_0, y_0, t_0) \in \mathbb{H}$, and $s \in \mathbb{R}$, such that
\[
\max \{d(q, p_0 \cdot (as, bs, 0)), d(0, p_0)\} \leq 2\varepsilon.
\]
\begin{equation}
\text{(6.50)}
\end{equation}

Triangle inequality, (6.46), (6.50), and left-invariance of the metric $d$ yield
\[
M^{-1} - 4\varepsilon \leq d(p_0 \cdot (as, bs, 0), p_0) = |s| \leq M + 4\varepsilon.
\]
Take $4\varepsilon < M^{-1}$. The estimates (6.50) then also imply that
\[
|as + x| \leq |x_0| + |x_0 + as - x| \leq 4\varepsilon \quad \text{and} \quad |bs + y| \leq 4\varepsilon.
\]
By what we said before, this yields a non-trivial upper bound for $|a|$ (and lower bound for $|b|$):
\[
|a| (M^{-1} - 4\varepsilon) \leq |a||s| \leq 4\varepsilon + |x| \overset{(6.49),(6.46)}{\leq} 4\varepsilon + 2\alpha M.
\]
\begin{equation}
\text{(6.51)}
\end{equation}
Returning to (6.50), we have established that
\[
d(q, p_0 \cdot (as, bs, 0)) \leq 2\varepsilon,
\]
with $|p_0| \leq 2\varepsilon$, $M^{-1} - 4\varepsilon \leq |s| \leq M + 4\varepsilon$, and $(a, b)$ can be picked as close to $(0, 1)$ as we like by choosing $\alpha, \varepsilon > 0$ small enough. Recall that
\[
\{0, q\} \subseteq Q \subseteq B_Q \cap E \subseteq N(\ell, 2\varepsilon) \cap B_Q \cap E,
\]
\begin{equation}
\text{(6.52)}
\end{equation}
where $N(\ell, 2\varepsilon)$ denotes the $2\varepsilon$-neighborhood of $\ell$ in the metric $d$. It follows from (6.50), (6.51), and (6.52) that
\[
(x', y', t') \in Q \implies |x'| \leq 2\varepsilon + \frac{4\varepsilon + 2\alpha M}{M^{-1} - 4\varepsilon} + 2\varepsilon.
\]
The right hand side gives an upper bound for $\mathcal{H}^1(\pi_{\mathbb{V}}(Q))$ which tends to zero if $M$ is fixed, and $\alpha, \varepsilon \to 0$. For sufficiently small $\alpha, \varepsilon > 0$, we arrive at $\mathcal{H}^1(\pi_{\mathbb{V}}(Q)) < c$, and hence $Q$ is not a $(c, \varepsilon, \mathbb{V})$-good cube. The proof is complete. \hfill \Box

The geometry of horizontal lines in $\mathbb{H}$ enters the proof of Theorem 6.42 only through Lemma 6.44. With this result in hand, intrinsic Lipschitz graphs over horizontal subgroups can be constructed inside regular curves by an abstract coding argument, due to Jones [31]. The construction requires to control the “bad” cubes of $\gamma$ that violate the second condition in Definition 6.43. For that purpose we first recall the following lemma, which follows from [37, Theorem 1], and the observation in [7, Proposition 3.1].

**Lemma 6.53 (Weak geometric lemma (WGL)).** Let $\gamma \subset \mathbb{H}$ be a compact regular curve, and let $\mathcal{D}$ be a system of dyadic cubes on $\gamma$. Then for every $\varepsilon > 0$, we have
\[
\sum_{\beta(Q) > \varepsilon, Q \subseteq Q_0} \ell(Q) \lesssim_{\text{reg}(\gamma), \varepsilon} \ell(Q_0), \quad Q_0 \in \mathcal{D}. \quad \text{(6.54)}
\]

In general, a closed 1-regular set $E \subset \mathbb{H}$ satisfying (6.54) is said to satisfy the WGL. This lemma is the only spot where we need compact regular curves; quite likely the WGL is true for all regular curves, but it has only been stated for compact ones in the literature.
Theorem 6.55. Let $E \subseteq \mathbb{H}$ be a closed 1-regular set satisfying the WGL, let $b > 0$, and let $\mathcal{V} \subset \mathbb{H}$ be a horizontal subgroup. Then there exist $L \geq 1$ and $N \in \mathbb{N}$, depending only on $b, \text{reg}(E)$, and the WGL constants of $E$, such that the following holds: for every $Q_0 \in \mathcal{D}$, there exist intrinsic $L$-Lipschitz graphs $\Gamma_1, \ldots, \Gamma_N \subset \mathbb{H}$ over $\mathcal{V}$ such that

$$\mathcal{H}^1 \left( \pi_{\mathcal{V}} \left( Q_0 \setminus \bigcup_{j=1}^N \Gamma_j \right) \right) \leq b \mathcal{H}^1(Q_0).$$

With the geometric result from Lemma 6.44 in hand, the proof of 6.55 only uses the 1-Lipschitz property of $\pi_{\mathcal{V}}$, and an abstract "coding argument", due to Jones [31], which has been applied to prove variants of Theorem 6.55 for k-regular sets in $\mathbb{R}^d$ ([20, Theorem 2.11]) and for $(2n + 1)$-regular sets in $\mathbb{H}^n$ ([6, Theorem 3.9] or [22]) satisfying natural analogues of the WGL property. The argument, and the notation, is nearly verbatim the same as in the proof of [6, Theorem 3.9], so we refer there for details.

The conclusion of Theorem 6.55 is only meaningful if $\mathcal{H}^1(\pi_{\mathcal{V}}(Q_0))$ is relatively large. If $\gamma \subset \mathbb{H}$ is a regular curve, then Lemma 6.57 below ensures that for every $Q_0 \in \mathcal{D}$, there exists a horizontal subgroup $\mathcal{V} \subset \mathbb{H}$ such that

$$\mathcal{H}^1(\pi_{\mathcal{V}}(Q_0)) \gtrsim_{\text{reg}(\gamma)} \ell(Q_0).$$

The enemy is the possibility $Q_0 \subset \gamma$ "wraps tightly around a vertical line", so that it projects to a set of small $\mathcal{H}^1$ measure on the $xy$-plane, and in particular on every horizontal subgroup $\mathcal{V}$. Yet, heuristically, the regular curve $\gamma$ simply cannot resemble a vertical line that much. This eventually gives the existence of $\mathcal{V}$ such that (6.56) holds.

Lemma 6.57. Let $\gamma \subset \mathbb{H}$ be a regular curve. Then $\gamma$ has big horizontal projections, which means the following. There exists a constant $c \gtrsim_{\text{reg}(\gamma)} 1$ such that for all $p_0 \in \gamma$ and all $0 < r \leq \text{diam}(\gamma)$, there is a horizontal subgroup $\mathcal{V} \subset \mathbb{H}$ such that

$$\mathcal{H}^1(\pi_{\mathcal{V}}(\gamma \cap B(p_0, r))) \geq cr.$$  

Proof of Lemma 6.57. Let $\gamma \subset \mathbb{H}$ be a regular curve parametrised by an interval $I \subset \mathbb{R}$. Write $\pi : \mathbb{H} \to \mathbb{R}^2$ for the projection map $\pi(x, y, t) = (x, y)$. Fix a point $p_0 \in \gamma$, and a radius $0 < r < \kappa \text{diam}(\gamma)$ for a suitable small absolute constant $\kappa > 0$ (if $\text{diam}(\gamma) = \infty$, there is no restriction for $r > 0$). Consider then the projection $\gamma_{\pi} := \pi(\gamma) \subset \mathbb{R}^2$, and write $\gamma_{\pi}(s) := \pi(\gamma(s))$ for $s \in I$.

Assume without loss of generality that $p_0 = \gamma(0) = 0$. Since $r < \kappa \text{diam}(\gamma)$, there exists another point $p_1 = \gamma(s_1) \in \gamma$ with $\|p_1\| \geq r/\kappa$. We choose the smallest parameter $s_1 > 0$ with this property, and we restrict attention to considering $\gamma_{\pi}|_{[0, s_1]}$ and $\gamma_{\pi}|_{[0, s_1]}$. We claim that if $\kappa > 0$ was chosen small enough, depending on $\text{reg}(\gamma)$, then there exists a point $s \in [0, s_1]$ with the property that

$$|\gamma_{\pi}(s)| = r.$$  

We only have to exclude the possibility that the projection $\gamma_{\pi}|_{[0, s_1]}$ stays inside the open disc $U(0, r)$. To see this, assume that (6.59) fails for all $0 < s \leq s_1$. We assume, for example, that the third component $t_1$ of $p_1 = \gamma(s_1)$ is strictly positive. Now comparing the conditions

$$|\pi(p_1)| = |\gamma_{\pi}(s_1)| < r \quad \text{and} \quad \|p_1\| \geq r/\kappa$$

in fact shows that $\sqrt{t_1} \gtrsim r/\kappa$, hence

$$t_1 \gtrsim \frac{r^2}{\kappa^2}.$$
To proceed, cover the box $U(0, r) \times [0, t_1] \subset \mathbb{H}$ with boundedly overlapping balls of radius $2r$ centred on the $t$-axis or, equivalently, with vertical translates of the box $U(0, 2r) \times [-4r^2, 4r^2]$. According to (6.60), the required number of such boxes is $\sim t_1/r^2$. Moreover, since $\gamma|_{[0, s_1]}$ is a continuum satisfying $|\gamma_\pi(s)| < r$ for all $s \in [0, s_1]$, and $\gamma(s_1) = p_1$, it must in fact meet $\geq t_1/r^2$ of the slightly smaller boxes of the type $U(0, r) \times [-r^2, r^2]$. Finally, by the 1-regularity of $\gamma$, we have

$$\gamma \cap [U(0, r) \times [-r^2, r^2]] \neq \emptyset \quad \implies \quad \mathcal{H}^1(\gamma \cap [U(0, 2r) \times [-4r^2, 4r^2]]) \sim r.$$ 

Since also $\sqrt{t_1}$ is much larger than $r$, we on the other hand observe that $U(0, 2r) \times [0, t_1]$ is covered by the single "$\sqrt{t_1}$-ball" $B_{\sqrt{t_1}} := U(0, \sqrt{t_1}) \times [0, t_1]$. This gives us the two-sided estimate

$$\frac{t_1}{r} = \frac{t_1}{r^2} \lesssim \mathcal{H}^1(\gamma \cap [U(0, 2r) \times [0, t_1]]) \leq \mathcal{H}^1(\gamma \cap B_{\sqrt{t_1}}) \lesssim \sqrt{t_1},$$

hence $t_1 \lesssim r^2$. This violates (6.60) for $\kappa > 0$ small enough, and the proof of (6.59) is complete.

Now, we let $s_0 \in [0, s_1]$ be the first parameter such that (6.59) holds, and we again recall that $\gamma(s) \in B(0, r/\kappa)$ for all $s \in [0, s_1]$. Then

$$\{0, \gamma_\pi(s_0)\} \subseteq \gamma_\pi([0, s_0]) \subseteq \pi(B(0, r/\kappa)).$$

Let $\mathbb{V}$ be the horizontal subgroup containing $\gamma_\pi(s_0)$. Then, since $\gamma|[0, s_0] \subset B(0, r/\kappa)$ is a connected set containing $p_0 = 0$ and $\gamma(s_0)$, we have

$$\mathcal{H}^1(\mathbb{V}(\gamma \cap B(0, r/\kappa))) \supseteq \mathcal{H}^1([0, \gamma_\pi(s_0)]).$$

This shows that (6.58) holds with $c = \kappa$, and the proof is complete.

We then put the pieces together to prove Theorem 6.42.

**Proof of Theorem 6.42.** Let $\gamma \subset \mathbb{H}$ be a regular curve. Fix $p \in \gamma$ and $0 < r \leq \text{diam}(\gamma)$. Start by choosing a compact regular curve $\gamma_0 \subset \gamma$ with $\text{reg}(\gamma_0) \lesssim \text{reg}(\gamma)$, which contains $p$, and satisfies $\text{diam}(\gamma_0) \geq r$. Then $\gamma_0$ satisfies the WGL by Lemma 6.53, and, on the other hand, Lemma 6.57 gives a horizontal subgroup $\mathbb{V} \subset \mathbb{H}$ such that $\mathcal{H}^1(\mathbb{V}(\gamma_0 \cap \mathbb{V})) \geq cr$, where $c \gtrsim \text{reg}(\gamma)$ (to be precise, use the version (6.56) for a dyadic cube $Q_0 \subset B(p, r) \cap \gamma_0$ with $t(Q_0) \sim r$). Finally, apply Theorem 6.55 to $\gamma_0$, with parameter $b = 2/2$, and use the 1-Lipschitz property of $\pi\mathbb{V}$ to deduce that $\mathcal{H}^1(\gamma_0 \cap \Gamma_i) \gtrsim c/N$ for some $1 \leq i \leq N$. Since $N$ only depends on the WGL and 1-regularity constants of $\gamma_0$ (both of which are uniform), the proof is complete.

**6.3. Singular integrals on regular curves.** It is now easy to put the pieces together to arrive at the main result, Theorem 1.6, which stated that good kernels are CZ kernels for regular curves in $\mathbb{H}$.

**Proof of Theorem 1.6.** Let $\gamma \subset \mathbb{H}$ be a regular curve. Then $\gamma$ is contained in an unbounded regular curve (attach horizontal half-lines if necessary). Since it suffices to prove the boundedness of any SIO on the extension, we may assume that $\text{diam}(\gamma) = \infty$ to begin with. Therefore, $\mu := \mathcal{H}^1|_\gamma \in \Sigma_1$ in the sense of Definition 6.1. By Theorem 6.42, moreover, $\gamma$ has BPiLG. This means that, for every ball $B$ centred on $\gamma$, there exists an intrinsic Lipschitz graph $\Gamma_B$ with $\mu(\Gamma_B) \geq \theta B$ (with $\text{reg}(\Gamma_B)$ uniformly bounded). By Proposition 3.56 (extension of intrinsic Lipschitz graphs), we may moreover arrange that $\Gamma$ is unbounded, and $\sigma_B := \mathcal{H}^1|_{\Gamma_B} \in \Sigma_1$ (with $\text{reg}(\sigma_B)$ uniformly bounded from above).
Now, let $k: \mathbb{H} \setminus \{0\} \to \mathbb{C}$ be a good kernel, and write $K(p, q) := k(q^{-1} \cdot p)$. We already know, by Theorem 1.7 and Remark 2.17, that the maximal SIO $\mathcal{T}_\mu^w$ induced by $(K, \sigma_B)$ is bounded on $L^p(\sigma_B)$, $1 < p < \infty$, with constants independent of the choice of $B$. Therefore, the hypotheses of Theorem 6.3 are met for $K$ and $\mu$, and (6.4) implies that $K$ is a CZ kernel for $\mu$, as claimed in Theorem 1.6.

\begin{appendix}
\section{On the corona decomposition for Lipschitz functions}

Recall that we needed the following statement regarding 1-Lipschitz functions.

\begin{theorem}[Corona decomposition of David-Semmes] For every $\eta > 0$, there exists a constant $C \geq 1$ such that the following holds. Let $\phi: \mathbb{R} \to \mathbb{R}$ be 1-Lipschitz. Then, there exists a decomposition $\mathcal{D} = B \cup Q$ with the properties (3.16), (3.17), and (3.18). For every $T \in \mathcal{F}$ there exists a $2$-Lipschitz linear map $L_T: \mathbb{R} \to \mathbb{R}^2$ and an $\eta$-Lipschitz map $\psi_T: \mathbb{R} \to \mathbb{R}^2$ such that $\psi_T + L_T$ approximates $\phi$ well at the resolution of the intervals in $T$:

$$|\phi(s) - (\psi_T + L_T)(s)| \leq \eta|Q|, \quad s \in 2Q, \ Q \in T. \quad (A.2)$$

This version looks slightly different to the corona decomposition for Lipschitz graphs in David and Semmes’ monograph, so we explain here briefly, how to bridge the gap. We start by stating the exact corona decomposition in [15, Definition 3.13, p. 55].

\begin{theorem}[Corona decomposition of David-Semmes] For every $\eta > 0$, there exists a constant $C \geq 1$ such that the following holds. Let $\phi: \mathbb{R} \to \mathbb{R}$ be 1-Lipschitz, and write

$$\Phi(x) := (x, \phi(x)), \quad x \in \mathbb{R}.$$ 

There exists a decomposition $\mathcal{D} = B \cup Q$ with the properties (3.16), (3.17), and (3.18). For every $T \in \mathcal{F}$, there exists a possibly rotated $\eta$-Lipschitz graph $\Gamma_T \subset \mathbb{R}^2$ such that

$$\text{dist}(\Phi(s), \Gamma_T) \leq \eta|Q|, \quad s \in 2Q, \ Q \in T. \quad (A.4)$$

To deduce Theorem A.1 from this statement, all we need to do is establish (A.2), that is, find an $\eta$-Lipschitz map $\psi_T: \mathbb{R} \to \mathbb{R}$, and a 2-Lipschitz linear map $L_T: \mathbb{R} \to \mathbb{R}^2$, such that (A.2) holds. We start by applying Theorem A.3 with a sufficiently small parameter $\eta' > 0$, at least so small that $0 < \eta' < \eta/12$. Then, fix $T \in \mathcal{F}$, and $Q \in T$. Let

$$\Gamma_T = R_\phi\{(x, \phi_T(x)) : x \in \mathbb{R}\}$$

be a rotated $\eta'$-Lipschitz graph appearing in (A.4), that is,

$$R_\phi(x, y) = (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta),$$

and $\phi_T: \mathbb{R} \to \mathbb{R}$ is $\eta'$-Lipschitz. We first observe that, if $\eta' > 0$ is small enough, then $|\tan \theta| \leq 2$. Namely, the case $\tan \theta = 2$ and $\eta' = 0$ would imply, by (A.4), that $\phi_T|Q$ is affine with slope in $[-2, 2]$, contradicting the 1-Lipschitz assumption. The case of “small $\eta''$” requires a small additional argument, which we leave to the reader.

Now, we claim that $\Gamma_T$ can be written as the graph of a function of the form $\psi_T + L_T$, where $\psi_T$ is $\eta'$-Lipschitz, and $L_T(x) = x \tan \theta$. To this end, we note that

$$\Gamma_T = \{(z(x), x \sin \theta + \phi_T(x) \cos \theta) : x \in \mathbb{R}\},$$

where $z(x) = x \cos \theta - \phi_T(x) \sin \theta$. Here,

$$|z(x) - z(x')| \geq |\cos \theta| \cdot |\eta'| \sin \theta| |x - x'| \geq \frac{1}{2} |x - x'|, \quad (A.5)$$

\end{appendix}
taking $\eta' > 0$ small enough, since $|\cos \theta| \geq 1/\sqrt{5}$. In particular, the change-of-variables $x \mapsto z(x)$ is bijective, and it now suffices to find a $\eta$-Lipschitz $\psi_T: \mathbb{R} \to \mathbb{R}$ such that
\[ x \sin \theta + \phi_T(x) \cos \theta = \psi_T(z(x)) + z(x) \tan \theta. \]

Plugging in the definition of $z(x) = x \cos \theta - \phi_T(x) \sin \theta$, this requirement is equivalent to
\[ \psi_T(z(x)) = \left[ \cos \theta + \frac{\sin^2 \theta}{\cos \theta} \right] \phi_T(x) = \frac{\phi_T(x)}{\cos \theta}. \]

Finally, $\psi_T$ is indeed $\eta$-Lipschitz:
\[ |\psi_T(z(x)) - \psi_T(z(x'))| = \frac{1}{\cos \theta} |\phi_T(x) - \phi_T(x')| \leq \frac{\eta'}{\cos \theta} |x - x'| \leq \eta |z(x) - z(x')|, \]
using (A.5) in the last estimate, and recalling that $\cos \theta \geq 1/\sqrt{5} \geq 1/3$, and $\eta' < \eta/12$.

Now we have re-parametrised $I_T$ as the graph of the function $\psi_T + L_T$, as desired, but we still need to check that (A.2) holds. This follows easily from (A.4): if $s \in 2Q$, then (A.4) gives us a point $s' \in \mathbb{R}$ with
\[ \max\{|s - s'|, |\phi(s) - (\psi_T + L_T)(s')|\} \leq \eta|Q|. \]

Consequently, using that $\psi_T + L_T$ is $3$-Lipschitz, and $\eta' < \eta/4$,\[ |\phi(s) - [\psi_T + L_T](s)| \leq |\phi(s) - [\psi_T + L_T](s')| + |[\psi_T + L_T](s) - [\psi_T + L_T](s')| \leq \eta|Q|. \]

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