A TWISTED FIRST HOMOLOGY GROUP OF THE GOERITZ GROUP OF $S^3$

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Abstract. Given a genus-$g$ Heegaard splitting of a 3-sphere, the genus-$g$ Goeritz group $\mathcal{G}_g$ is defined to be the group of the isotopy classes of orientation preserving homeomorphism of the 3-sphere that preserve the splitting. In this paper, we determine the twisted first (co)homology group of the genus-2 Goeritz group of 3-sphere.

1. Introduction

Mapping class group. Let $H_g$ be a 3-dimensional handlebody of genus $g$, and $\Sigma_g$ be the boundary surface $\partial H_g$. We denote by $\mathcal{M}_g$ the mapping class group of $\Sigma_g$, the group of isotopy classes of orientation preserving homeomorphisms of $\Sigma_g$. Dehn [2] proved that $\mathcal{M}_g$ is generated by finitely many Dehn twists. Furthermore Lickorish [19, 20] proved that $3g - 1$ Dehn twists generate $\mathcal{M}_g$. Humphries [16] found that $2g + 1$ Dehn twists generate $\mathcal{M}_g$.

We denote by $\mathcal{H}_g$ the handlebody mapping class group, the subgroup of mapping class group $\mathcal{M}_g$ of boundary surface $\partial H_g$ defined by isotopy classes of those orientation preserving homeomorphisms of $\partial H_g$ which can be extended to homeomorphisms of $H_g$. It turns out that $\mathcal{H}_g$ can be identified with the group of isotopy classes of orientation preserving homeomorphisms of $H_g$. A finite presentation of the handlebody mapping class group $\mathcal{H}_g$ was obtained by Wajnryb [1].

Goeritz group. Let $H_g$ and $H_g^*$ be 3-dimensional handlebodies, and $M = H_g \cup H_g^*$ be a Heegaard splitting of a closed orientable 3-manifold $M$. Let $\mathcal{M}_g$ be the mapping class group of the boundary surface $\partial H_g = \Sigma_g$. The group of mapping classes $[f] \in \mathcal{M}_g$ such that there is an orientation preserving self-homeomorphism $F$ of $(M, H_g)$ satisfying $[F|_{\partial H_g}] = [f]$ is called the genus-$g$ Goeritz group of $M = H_g \cup H_g^*$. When a manifold $M$ admits a unique Heegaard splitting of genus $g$ up to isotopy, we can define the genus-$g$ Goeritz group of the manifold without mentioning a specific splitting. For example, the 3-sphere, $S^1 \times S^2$ and lens spaces are known to be such manifolds from [6], [4] and [5].

In studying Goeritz groups, finding their generating sets or presentations has been an interesting problem. However the generating sets or the presentation of those groups have been obtained only for a few manifolds with their splittings of small genera. A finite presentation of the genus-2 Goeritz group of 3-sphere was obtained [8]. In an arbitrary genus, first Powell [8] and then Hirose [17] claimed that they have found a finite generating set for the genus-$g$ Goeritz group of 3-sphere, though serious gaps in both arguments were found by Scharlemann. Establishing the existence of such generating sets appears to be an open problem.

In addition, finite presentations of the genus-2 Goeritz groups of each lens spaces $L(p, 1)$ were obtained [12], other lens spaces were obtained [15] and the genus-2 Heegaard splittings of non-prime 3-manifolds were obtained [14]. Recently a finite presentation of the genus-2 Goeritz group of $S^1 \times S^2$ was obtained [13].
Homology of mapping class group. Computing homology of mapping class groups is an interesting topic of studying mapping class groups. Harer [9] determined the second homology group of mapping class group $\mathcal{M}_g$:

$$H_2(\mathcal{M}_g; \mathbb{Z}) \cong \mathbb{Z} \text{ if } g \geq 4.$$  

In fact, Harer proved a more general theorem for surfaces with multiple boundary components and arbitrarily many punctures.

In twisted case, Morita [18] determined the first homology group with coefficients in the first integral homology group of the surface:

$$H_1(\mathcal{M}_g; H_1(\Sigma_g)) \cong \mathbb{Z} / (2g - 2)\mathbb{Z} \text{ if } g \geq 2.$$  

Recently Ishida and Sato [7] computed the twisted first homology groups of the handlebody mapping class group $\mathcal{H}_g$ with coefficients in the first integral homology group of the boundary surface $\Sigma_g$:

$$H_1(\mathcal{H}_g; H_1(\Sigma_g)) \cong \begin{cases} \mathbb{Z} / (2g - 2)\mathbb{Z} & \text{if } g \geq 4, \\ \mathbb{Z} / 2\mathbb{Z} \oplus \mathbb{Z} / 4\mathbb{Z} & \text{if } g = 3, \\ (\mathbb{Z} / 2\mathbb{Z})^2 & \text{if } g = 2. \end{cases}$$  

Goeritz group of $S^3$. Let $H_g$ and $H_g^*$ be 3-dimensional handlebodies, and $S^3 = H_g \cup H_g^*$ be the Heegaard splitting of the 3-sphere $S^3$. Waldhausen [6] proved that a genus-$g$ Heegaard splitting of $S^3$ is unique up to isotopy. Let $\mathcal{M}_g$ be the mapping class group of the boundary surface $\partial H_g = \Sigma_g$. The group of mapping classes $[f] \in \mathcal{M}_g$ such that there is an orientation preserving self-homeomorphism $F$ of $(S^3, H_g)$ satisfying $[F|_{\partial H_g}] = [f]$ is denoted by $\mathcal{E}_g$. It is called the genus-$g$ Goeritz group of the 3-sphere.

Twisted homology group of $\mathcal{E}_2$. In this paper, we compute the twisted first homology group of $\mathcal{E}_2$ with coefficients in the first integral homology group of the Heegaard surface $\Sigma_2$. The following is the main theorem in this paper.

Theorem 1.1.

$$H_1(\mathcal{E}_2; H_1(\Sigma_2)) \cong (\mathbb{Z} / 2\mathbb{Z})^2.$$  

A finite presentation of the genus-2 Goeritz group of the 3-sphere was obtained from the works of [3]. For the higher genus Goeritz groups of the 3-sphere, it is conjectured that all of them are finitely presented however it is still known to be an open problem.
Let $\Sigma_g$ be a compact connected orientable surface of genus $g \geq 1$ and $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be oriented simple closed curves as in Figure 1. We denote their homology classes in $H_1(\Sigma_g)$ by $x_1 = [\alpha_1], x_2 = [\alpha_2], \ldots, x_g = [\alpha_g], y_1 = [\beta_1], y_2 = [\beta_2], \ldots, y_g = [\beta_g]$. The basis \{ $x_1, ..., x_g, y_1, ..., y_g$ \} of $H_A$ induces an isomorphism $H_A \cong A^{2g}$. For $v \in A^{2g}$, we denote its projection to the $i$-th coordinate of $A^{2g}$ by $v_i$ for $i = 1, 2, \ldots, 2g$.

Akbas gave following presentation for $E_2$ in [3].

**Theorem 1.2** ([3]). The group $E_2$ has four generators $[\alpha], [\beta], [\gamma]$ and $[\delta]$, and the following relations:

- (P1) $[\alpha]^2 = [\beta]^2 = [\delta]^3 = [\alpha\gamma]^2 = 0$.
- (P2) $[\alpha\delta\alpha] = [\delta]$ and $[\alpha\beta\alpha] = [\beta]$.
- (P3) $[\gamma\beta\gamma] = [\alpha\beta]$ and $[\delta] = [\gamma\delta^2\gamma]$.

![Figure 1](image-url)  
**Figure 1**: Surface $\Sigma_g$ and simple closed curves $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$.

![Figure 2](image-url)  
**Figure 2**: Generators of $E_2$. 
We define $\delta$ as follows. Consider the genus-two handlebody as a regular neighborhood of a sphere, centered at the origin, with three holes. The homeomorphism $\delta$ is a $2\pi/3$ rotation of the handlebody about the vertical $z$-axis. See Figure 2. Scharlemann [11] showed that the group $\mathcal{E}_2$ is generated by isotopy classes $[\alpha], [\beta], [\gamma]$ and $[\delta]$. Correspondence of homology classes of (iv) and the others are as follows:

$$\begin{align*}
x_1 &< -x_2 < x_1 - x_2 < x_2 < y_1 + y_2 < y_1 < y_2.
\end{align*}$$

We denote by $A$ the ring $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for an integer $n \geq 2$, and set $H_A = H_1(\Sigma_g; A)$. For a group $G$ and a left $G$-module $N$, a map $d : G \to N$ is called a crossed homomorphism if it satisfies $d(\phi\psi) = d(\phi) + \phi d(\psi)$ for $\phi, \psi \in G$. Now let $Z^1(\mathcal{E}_g; H_A)$ be the set of all crossed homomorphisms $d : \mathcal{E}_g \to H_A$. Namely

$$Z^1(\mathcal{E}_g; H_A) = \{d : \mathcal{E}_g \to H_A; d(\phi\psi) = d(\phi) + \phi d(\psi), \phi, \psi \in \mathcal{E}_g\}.$$ 

Let $\pi : H_A \to Z^1(\mathcal{E}_g; H_A)$ be the homomorphism defined by

$$\pi(u)(\phi) = \phi u - u$$

for $u \in H_A$. Then as is well known we have

$$H^1(\mathcal{E}_g; H_A) = Z^1(\mathcal{E}_g; H_A)/\text{Im } \pi$$

(cf. K.S. Brown [10]).

We consider the case $g = 2$. Then we have the homomorphism $\mathcal{E}_2 \to \text{Aut}(H_1(\Sigma_2; \mathbb{Z}))$ induced by the action of the group $\mathcal{E}_2$ on $H_1(\Sigma_2; \mathbb{Z})$. The action of $\alpha, \beta, \gamma$, and $\delta$ is as follows:

- $\alpha_* : \alpha_*(x_i) = -x_i$ and $\alpha_*(y_i) = -y_i$ ($i = 1, 2$).
- $\beta_* : \beta_*(x_1) = x_1, \beta_*(x_2) = -x_2, \beta_*(y_1) = y_1, \beta_*(y_2) = -y_2$.
- $\gamma_* : \gamma_*(x_1) = -x_2, \gamma_*(x_2) = -x_1, \gamma_*(y_1) = -y_2, \gamma_*(y_2) = -y_1$.
- $\delta_* : \delta_*(x_1) = -x_1 + x_2, \delta_*(x_2) = -x_1, \delta_*(y_1) = y_2, \delta_*(y_2) = -y_1 - y_2$.

For a group $G$ and a left $G$-module $N$, the coinvariant $N_G$ is quotient module of $N$ by the subgroup $\{gn - n|g \in G, n \in N\}$.
Lemma 2.1.

\[ H_1(\Sigma_2)_{E_2} = 0. \]

Proof. Since we have \( \alpha_s \delta_s^2(-x_2) = x_1 - x_2 \) and \( \gamma_s(x_1) = -x_2 \), we obtain \( x_1 = x_2 = 0 \in H_1(\Sigma_2)_{E_2} \). And we have \( \gamma_s \delta_s^2(y_1) = y_1 + y_2 \) and \( \gamma_s(y_1) = -y_2 \). Hence we also obtain \( y_1 = y_2 = 0 \in H_1(\Sigma_2)_{E_2} \).

\[ \square \]

Lemma 2.2. Let \( G_i, H_i \) and \( K \) be \( G \)-modules \((i = 1, 2, 3)\), and let

\[ \cdots \to G_3 \to G_2 \to K \to G_1 \to 0 \quad \text{and} \quad 0 \to H_1 \to K \to H_2 \to H_3 \to \cdots \]

be exact sequences. If \( G_2 \to K \to H_2 \) is an isomorphism, then we have \( H_1 \to K \to G_1 \) is isomorphism.

Proof. Now we have the following diagram. Set \( \Phi = g_1 \circ f_1 \) and \( \Psi = f_2 \circ g_2 \).

\[ \begin{array}{c}
\cdots \to G_3 \xrightarrow{g_3} G_2 \xrightarrow{g_2} K \xrightarrow{g_1} G_1 \to 0 \\
\downarrow f_3 \downarrow f_2 \downarrow f_1 \\
H_2 \xrightarrow{g_1} H_1 \xrightarrow{g_1} \]

Note that \( g_2 \) is injective and \( f_2 \) is surjective if \( G_2 \to K \to H_2 \) is an isomorphism. Thus we have \( g_3 = 0 \) and \( f_3 = 0 \). Since \( \Psi \) is isomorphism, we have \((\Psi^{-1} \circ f_2) \circ g_2 = id_{G_2} \) and \( f_2 \circ (g_2 \circ \Psi^{-1}) = id_{H_2} \). Hence those exact sequences are split and we have

\[ H_1 \oplus H_2 \xrightarrow{\cong} K : \quad (h_1, h_2) \mapsto f_1(h_1) + g_2 \circ \Psi^{-1}(h_2), \]

\[ K \xrightarrow{\cong} G_1 \oplus G_2 : \quad k \mapsto (g_1(k), \Psi^{-1} \circ f_2(k)). \]

A composition map \( H_1 \oplus H_2 \to K \to G_1 \oplus G_2 \) is

\[ (h_1, h_2) \mapsto (g_1(f_1(h_1) + g_2 \circ \Psi^{-1}(h_2)), \Psi^{-1} \circ f_2(f_1(h_1) + g_2 \circ \Psi^{-1}(h_2))) = (g_1 \circ f_1(h_1) + g_1 \circ g_2 \circ \Psi^{-1}(h_2)) + \Psi^{-1} \circ f_2 \circ f_1(h_1) + \Psi^{-1} \circ f_2 \circ g_2 \circ \Psi^{-1}(h_2) = (\Phi(h_1), \Psi^{-1}(h_2)). \]

Hence \( \Psi \) is an isomorphism. \( \square \)
Lemma 2.3. In the case $g = 2$, we have

$$H^1(\mathcal{E}_2; H_A) \cong \{ d \in Z^1(\mathcal{E}_2; H_A) ; \quad d([\delta])_1 - d([\alpha])_1 = d([\gamma])_2 - d([\beta])_2 = d([\gamma])_3 - d([\alpha])_3 = d([\beta])_4 - d([\delta])_4 = 0 \}.$$ 

Proof. Let $f : Z^1(\mathcal{E}_2; H_A) \to A^4$ be a homomorphism defined by

$$f(d) = (d([\delta])_1 - d([\alpha])_1, d([\gamma])_2 - d([\beta])_2, d([\gamma])_3 - d([\alpha])_3, d([\beta])_4 - d([\delta])_4).$$

Since we have

$$\begin{align*}
\alpha v - v &= (-2v_1, -2v_2, -2v_3, -2v_4), \\
\beta v - v &= (0, -2v_2, 0, -2v_4), \\
\gamma v - v &= (-v_1 - v_2, -v_1 - v_2, -v_3 - v_4, -v_3 - v_4), \\
\delta v - v &= (-2v_1 + v_2, -v_1 - v_2, -v_3 + v_4, -v_3 - 2v_4),
\end{align*}$$

the composition map $f \circ \pi : H_A \to A^4$ is written as

$$f \circ \pi(v) = (v_2, -v_1 + v_2, v_3 - v_4, -v_3)$$

for $v \in H_A$. This map is an isomorphism. We have the following diagram.

$$\begin{array}{c}
\begin{array}{ccc}
0 & \longrightarrow & H_A \\
\downarrow & & \downarrow \pi \\
H_A & \longrightarrow & Z^1(\mathcal{E}_2; H_A) \\
\downarrow \cong & & \downarrow f \\
0 & \longrightarrow & \text{Coker } \pi \\
\end{array}
\end{array}$$

By Lemma 2.2, we have

$$H^1(\mathcal{E}_2; H_A) = Z^1(\mathcal{E}_2; H_A) / \text{Im } \pi \cong \text{Ker } f.$$

The group $\mathcal{E}_2$ is generated by $[\alpha], [\beta], [\gamma], \text{ and } [\delta]$. Therefore, all crossed homomorphisms $d : \mathcal{E}_2 \to H_A$ are determined by the values $d([\alpha]), d([\beta]), d([\gamma]) \text{ and } d([\delta])$. If $d \in Z^1(\mathcal{E}_2, H_A)$, we can set

$$\begin{align*}
d([\alpha]) &= \omega_{11}x_1 + \omega_{12}x_2 + \omega_{13}y_1 + \omega_{14}y_2, \\
d([\beta]) &= \omega_{21}x_1 + \omega_{22}x_2 + \omega_{23}y_1 + \omega_{24}y_2, \\
d([\gamma]) &= \omega_{31}x_1 + \omega_{32}x_2 + \omega_{33}y_1 + \omega_{34}y_2, \\
d([\delta]) &= \omega_{41}x_1 + \omega_{42}x_2 + \omega_{43}y_1 + \omega_{44}y_2.
\end{align*}$$

Then we have

$$H^1(\mathcal{E}_2; H_A) \cong \{ d \in Z^1(\mathcal{E}_2; H_A) ; \quad \omega_{41} - \omega_{11} = \omega_{32} - \omega_{22} = \omega_{33} - \omega_{13} = \omega_{24} - \omega_{44} = 0 \}. $$
3. Relations of $\omega_{ij}$

In this section we shall consider the case $g = 2$. We denote $d([\phi])$ for $\phi \in \mathcal{E}_2$ simply by $d(\phi)$.

**Lemma 3.1.** We have relations:

\[
\begin{align*}
\omega_{11} + \omega_{12} &= \omega_{31} + \omega_{32}, \\
\omega_{13} + \omega_{14} &= \omega_{33} + \omega_{34}.
\end{align*}
\]

*Proof.* By the relations $(\alpha \gamma)^2 = 0$ in (P1), we have $d((\alpha \gamma)^2) = 0$. The equation

\[
d((\alpha \gamma)^2) = d(\alpha \gamma) + \alpha \gamma d(\alpha \gamma) = d(\alpha) + \alpha d(\alpha) + \alpha \gamma d(\alpha) + \alpha \gamma \delta d(\gamma)
\]

\[
= \omega_{11} x_1 + \omega_{12} x_2 + \omega_{13} y_1 + \omega_{14} y_2 + (-\omega_{31} x_1 - \omega_{32} x_2 - \omega_{33} y_1 - \omega_{34} y_2)
\]

\[
+ \omega_{12} x_1 + \omega_{11} x_2 + \omega_{14} y_1 + \omega_{13} y_2 + (-\omega_{32} x_1 - \omega_{31} x_2 - \omega_{34} y_1 - \omega_{33} y_2)
\]

\[
= (\omega_{11} + \omega_{12} - \omega_{31} - \omega_{32}) x_1 + (\omega_{11} + \omega_{12} - \omega_{31} - \omega_{32}) x_2
\]

\[
+ (\omega_{13} + \omega_{14} - \omega_{33} - \omega_{34}) y_1 + (\omega_{13} + \omega_{14} - \omega_{33} - \omega_{34}) y_2
\]

\[
= 0
\]

holds. Hence, we obtain (1a) and (1b). \hfill \Box

**Lemma 3.2.** We have relations:

\[
\begin{align*}
2\omega_{11} + \omega_{12} &= 2\omega_{41}, \\
-\omega_{11} + \omega_{12} &= 2\omega_{42}, \\
\omega_{13} + \omega_{14} &= 2\omega_{43}, \\
-\omega_{13} + 2\omega_{14} &= 2\omega_{44}.
\end{align*}
\]

*Proof.* By the relations $\alpha \delta \alpha = \delta$ in (P2), we have $d(\alpha \delta \alpha) = d(\delta)$ and

\[
d(\alpha \delta \alpha) = d(\alpha) + \alpha d(\delta) + \alpha \delta d(\alpha)
\]

\[
= \omega_{11} x_1 + \omega_{12} x_2 + \omega_{13} y_1 + \omega_{14} y_2 + (-\omega_{41} x_1 - \omega_{42} x_2 - \omega_{43} y_1 - \omega_{44} y_2)
\]

\[
+ \omega_{11} (x_1 - x_2) + \omega_{12} x_1 - \omega_{13} y_2 + \omega_{14} (y_1 + y_2)
\]

\[
= (2\omega_{11} - \omega_{41} + \omega_{12}) x_1 + (\omega_{12} - \omega_{42} - \omega_{11}) x_2
\]

\[
+ (\omega_{13} - \omega_{43} + \omega_{14}) y_2 + (\omega_{44} - \omega_{44} - \omega_{13}) y_2,
\]

\[
d(\delta) = \omega_{41} x_1 + \omega_{42} x_2 + \omega_{43} y_1 + \omega_{44} y_2.
\]

Comparing $d(\alpha \delta \alpha)$ and $d(\delta)$, we obtain (2a) – (2d). \hfill \Box

**Lemma 3.3.** We have relations:

\[
\begin{align*}
2\omega_{21} &= 2\omega_{23} = 0, \\
2\omega_{12} &= 2\omega_{22}, \\
2\omega_{14} &= 2\omega_{24}.
\end{align*}
\]
Proof. By the relation $\alpha\beta\alpha = \beta$ in (P2), we have $d(\alpha\beta\alpha) = d(\beta)$ and

$$d(\alpha\beta\alpha) = d(\alpha) + \alpha d(\beta\alpha)$$
$$= d(\alpha) + \alpha d(\beta) + \alpha\beta d(\alpha)$$
$$= \omega_{11}x_1 + \omega_{12}x_2 + \omega_{13}y_1 + \omega_{14}y_2$$
$$-\omega_{21}x_1 - \omega_{22}x_2 - \omega_{23}y_1 - \omega_{24}y_2$$
$$-\omega_{11}x_1 + \omega_{12}x_2 - \omega_{13}y_1 + \omega_{14}y_2$$
$$= -\omega_{21}x_1 + (2\omega_{12} - \omega_{22})x_2 - \omega_{23}y_1 + (2\omega_{14} - \omega_{24})y_2,$$
$$d(\beta) = \omega_{21}x_1 + \omega_{22}x_2 + \omega_{23}y_2 + \omega_{24}y_2.$$

Comparing $d(\alpha\beta\alpha)$ and $d(\beta)$, we obtain (3a) – (3c). \[\square\]

Lemma 3.4. We have relations:

$$\begin{align*}
\omega_{31} - \omega_{32} - \omega_{22} & = \omega_{11} + \omega_{21}, \quad (4a) \\
\omega_{32} + \omega_{31} - \omega_{21} & = \omega_{12} - \omega_{22}, \quad (4b) \\
\omega_{33} - \omega_{34} - \omega_{24} & = \omega_{13} + \omega_{23}, \quad (4c) \\
\omega_{34} + \omega_{33} - \omega_{23} & = \omega_{14} - \omega_{24}. \quad (4d)
\end{align*}$$

Proof. By the relation $\gamma\beta\gamma = \alpha\beta$ in (P3), we have $d(\gamma\beta\gamma) = d(\alpha\gamma)$ and

$$d(\gamma\beta\gamma) = d(\gamma) + \gamma d(\beta\gamma)$$
$$= d(\gamma) + \gamma d(\beta) + \gamma\beta d(\gamma)$$
$$= \omega_{31}x_1 + \omega_{32}x_2 + \omega_{33}y_1 + \omega_{34}y_2$$
$$-\omega_{31}x_2 - \omega_{32}x_1 - \omega_{33}y_2 - \omega_{34}y_1$$
$$= (\omega_{31} - \omega_{32} - \omega_{22})x_1 + (\omega_{31} + \omega_{32} - \omega_{21})x_2$$
$$+(\omega_{33} - \omega_{34} - \omega_{24})y_1 + (\omega_{33} + \omega_{34} - \omega_{23})y_2,$$
$$d(\alpha\beta) = d(\alpha) + \alpha d(\beta)$$
$$= (\omega_{11} + \omega_{21})x_1 + (\omega_{12} - \omega_{22})x_2 + (\omega_{13} - \omega_{23})y_1 + (\omega_{14} + \omega_{24})y_2.$$n

Comparing $d(\gamma\beta\gamma)$ and $d(\alpha\beta)$, we obtain (4a) – (4d). \[\square\]

Lemma 3.5. We have relations:

$$\begin{align*}
2\omega_{31} + \omega_{32} & = 2\omega_{11} + \omega_{42}, \quad (5a) \\
\omega_{33} & = \omega_{43}. \quad (5b)
\end{align*}$$
Proof. By the relation $\gamma \delta^2 \gamma = \delta$ in (P3), we have $d(\gamma \delta^2 \gamma) = d(\delta)$ and
\[
d(\gamma \delta^2 \gamma) = d(\gamma) + \gamma d(\delta^2 \gamma)
\]
\[
= d(\gamma) + \gamma d(\delta) + \gamma \delta d(\delta) + \gamma \delta^2 d(\gamma)
\]
\[
= \omega_{31} x_1 + \omega_{32} x_2 + \omega_{33} y_1 + \omega_{34} y_2
\]
\[
- \omega_{42} x_1 - \omega_{41} x_2 - \omega_{44} y_1 - \omega_{43} y_2
\]
\[
- \omega_{41} x_1 + (\omega_{41} + \omega_{42}) x_2 + (-\omega_{43} + \omega_{44}) y_1 + \omega_{44} y_2
\]
\[
+ (\omega_{31} + \omega_{32}) x_1 - \omega_{32} x_2 + \omega_{33} y_1 + (\omega_{33} - \omega_{34}) y_2
\]
\[
= (2\omega_{31} - \omega_{41} - \omega_{42} + \omega_{32}) x_1 + \omega_{42} x_2
\]
\[
+ (2\omega_{33} - \omega_{43}) y_1 + (\omega_{33} + \omega_{44} - \omega_{43}) y_2,
\]
\[
d(\delta) = \omega_{41} x_1 + \omega_{42} x_2 + \omega_{43} y_2 + \omega_{44} y_2.
\]
Comparing $d(\gamma \delta^2 \gamma)$ and $d(\delta)$, we obtain (5a) and (5b).

By Lemma 3.1, 3.2, 3.3, 3.4 and 3.5, we obtain the following equations.

\[
\begin{align*}
\omega_{11} + \omega_{12} &= \omega_{31} + \omega_{32}, \quad (1a) \\
\omega_{13} + \omega_{14} &= \omega_{33} + \omega_{34}, \quad (1b)
\end{align*}
\]
\[
\begin{align*}
2\omega_{11} + \omega_{12} &= 2\omega_{41}, \quad (2a) \\
-\omega_{11} + \omega_{12} &= 2\omega_{42}, \quad (2b) \\
\omega_{13} + \omega_{14} &= 2\omega_{43}, \quad (2c) \\
-\omega_{13} + 2\omega_{14} &= 2\omega_{44}, \quad (2d)
\end{align*}
\]
\[
\begin{align*}
2\omega_{21} &= 2\omega_{23} = 0, \quad (3a) \\
2\omega_{12} &= 2\omega_{22}, \quad (3b) \\
2\omega_{14} &= 2\omega_{24}, \quad (3c)
\end{align*}
\]
\[
\begin{align*}
\omega_{31} - \omega_{32} - \omega_{22} &= \omega_{11} + \omega_{21}, \quad (4a) \\
\omega_{31} + \omega_{32} - \omega_{21} &= \omega_{12} - \omega_{22}, \quad (4b) \\
\omega_{33} - \omega_{34} - \omega_{24} &= \omega_{13} + \omega_{23}, \quad (4b) \\
\omega_{34} + \omega_{33} - \omega_{23} &= \omega_{14} - \omega_{24}, \quad (4b)
\end{align*}
\]
\[
\begin{align*}
2\omega_{31} + \omega_{32} &= 2\omega_{11} + \omega_{42}, \quad (5a) \\
\omega_{31} &= \omega_{43}, \quad (5b)
\end{align*}
\]

4. Calculation of cohomology

In this section we prove that $H^1(\mathcal{E}_2; H_A) \cong \text{Hom}(\mathbb{Z}/2\mathbb{Z})^2, A)$. The universal coefficient theorem implies $H_1(\mathcal{E}_2; H_1(\Sigma_2)) \cong (\mathbb{Z}/2\mathbb{Z})^2$. To determine the twisted first cohomology group of $\mathcal{E}_2$, we solve equations (1a) – (5b) and the condition $d \in \text{Ker} f$, i.e.

\[
\omega_{41} - \omega_{11} = \omega_{32} - \omega_{22} = \omega_{33} - \omega_{13} = \omega_{24} - \omega_{41} = 0.
\]

Lemma 4.1. We have a relation of $\text{Ker} f$

\[
\omega_{12} = 0.
\]

Proof. Using (2a) and $\omega_{11} = \omega_{41}$ by (*), we obtain

\[
\omega_{12} = 0.
\]

\[\square\]
Lemma 4.2. The elements $\omega_{21}$, $\omega_{22}$, $\omega_{31}$, $\omega_{32}$ and $\omega_{42}$ have order 2 and
$$\omega_{22} = \omega_{31} = \omega_{32} = -\omega_{21} = -\omega_{42}.$$  

Proof. By (2a) and (2b), we have $\omega_{11} + 2\omega_{12} = 2\omega_{41} + 2\omega_{42}$. By (5a) we have $2\omega_{41} + 2\omega_{42} = 2\omega_{31} + \omega_{32} + \omega_{42}$. Using these two equations and (1a), we have $\omega_{12} = \omega_{31} + \omega_{42}$. Since $\omega_{12} = 0$, the equation
$$\omega_{31} + \omega_{42} = 0 \quad (4.2.1)$$  
holds. Using (2b) and Lemma 4.1, we obtain
$$\omega_{11} + 2\omega_{42} = 0. \quad (4.2.2)$$  
The equation
$$\omega_{31} + \omega_{32} \overset{(1a)}{=} \omega_{11} + \omega_{12} \overset{(Lem4.1)}{=} \omega_{11} \overset{(4.2.2)}{=} -2\omega_{42} \overset{(4.2.1)}{=} 2\omega_{31}$$  
holds. So we obtain
$$\omega_{31} - \omega_{32} = 0. \quad (4.2.3)$$  
We have
$$\omega_{21} \overset{(4a)}{=} \omega_{31} - \omega_{32} - \omega_{11} - \omega_{22} \overset{(4.2.3)}{=} -\omega_{11} - \omega_{22} \overset{(**)}{=} -2\omega_{22} - \omega_{22} = -3\omega_{22} \overset{(3b)}{=} -\omega_{22} - 2\omega_{12} \overset{(Lem4.1)}{=} -\omega_{22}. \quad \text{(4.2.3)}$$  
Here, (**) is the equation $\omega_{11} = 2\omega_{22}$. This equation is obtained as follows:
$$\omega_{11} \overset{(4.2.2)}{=} -\omega_{42} \overset{(4.2.1)}{=} 2\omega_{31} \overset{(4.2.3)}{=} 2\omega_{32} \overset{(*)}{=} 2\omega_{22}.$$  
By (3b) and Lemma4.1, we have $-2\omega_{22} = 0$. Hence we obtain
$$2\omega_{21} = -2\omega_{22} = 0.$$  
□

From the results of Lemma 4.2 and (4.2.2), we have
$$\omega_{11} \overset{(*)}{=} \omega_{41} = 0.$$  

Lemma 4.3. We have a relation:
$$\omega_{13} = \omega_{14} = \omega_{33} = \omega_{34} = \omega_{43} = 0.$$
Proof. By (5b) and (*), we obtain

\[ \omega_{13} = \omega_{33} = \omega_{43} = \omega_{14}. \]

Using this equation and (1b), we have \( \omega_{14} = \omega_{34} \). So the equation

\[ \omega_{13} = \omega_{33} = \omega_{43} = \omega_{14} = \omega_{34} \quad (4.3.1) \]

holds. The equation

\[ 2\omega_{14} \overset{(2d)}{=} 2\omega_{14} - \omega_{13} \overset{(4.3.1)}{=} \omega_{14} \]

holds. Hence we have

\[ 4\omega_{14} = 2\omega_{14} \overset{(3c)}{=} 2\omega_{24} = 2\omega_{44}. \]

So we obtain

\[ 2\omega_{44} = 0. \]

Since the equation \( 2\omega_{44} = \omega_{14} \) holds, we complete the proof of Lemma 4.3. □

Lemma 4.4. The elements \( \omega_{23}, \omega_{24} \) and \( \omega_{14} \) have order 2 and are equal to each other:

\[ \omega_{23} = \omega_{24} = \omega_{14}, \quad 2\omega_{23} = 0. \]

Proof. By (3a), the element \( \omega_{23} \) have order 2. Since the equation

\[ \omega_{24} \overset{(4b)}{=} \omega_{14} - \omega_{33} - \omega_{34} + \omega_{23} \overset{(Lem4.3)}{=} \omega_{23}, \]

holds, we obtain

\[ \omega_{23} = \omega_{24} \overset{(*)}{=} \omega_{44}. \]

□

Proof of Theorem 1.1. From Lemmas 3, 7, 4.1, 4.2, 4.3 and 4.4, we have

\[
\begin{align*}
d(\alpha) & = 0, \\
d(\beta) & = \omega_{22}(-x_1 + x_2) + \omega_{23}(y_1 + y_2), \\
d(\gamma) & = \omega_{22}(x_1 + x_2), \\
d(\delta) & = \omega_{42}x_2 + \omega_{23}y_2,
\end{align*}
\]

where \( -\omega_{22} = \omega_{42} \) and \( 2\omega_{22} = 2\omega_{23} = 0 \). Hence it follows

\[ H^1(\mathcal{E}_2; H_A) \cong \text{Ker } f \cong \{(\omega_{22}, \omega_{23}) \in A^2; 2\omega_{22} = 2\omega_{23} = 0\}. \]

So we obtain

\[ H_1(\mathcal{E}_2; H_1(\Sigma_2)) \cong (\mathbb{Z}/2\mathbb{Z})^2. \]

This isomorphism follows from the short exact sequence

\[ 0 \rightarrow \text{Ext}(H_0(\mathcal{E}_2; H_1(\Sigma_2)), A) \rightarrow H^1(\mathcal{E}_2; H_A) \rightarrow \text{Hom}(H_1(\mathcal{E}_2; H_1(\Sigma_2)), A) \rightarrow 0 \]

since we have \( H_0(\mathcal{E}_2; H_1(\Sigma_2)) = H_1(\Sigma_2)e_2 = 0 \) by Lemma 2.1. we complete the proof of Theorem 1.1.
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