Iterative Solution of the Supereigenvalue Model

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ABSTRACT

An integral form of the discrete superloop equations for the supereigenvalue model of Alvarez–Gaumé, Itoyama, Mañes and Zadra is given. By a change of variables from coupling constants to moments we find a compact form of the planar solution for general potentials. In this framework an iterative scheme for the calculation of higher genera contributions to the free energy and the multi–loop correlators is developed. We present explicit results for genus one.

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1 Introduction

After the successful application of matrix models to 2d gravity and bosonic string theory \cite{1}, it was natural to ask what may be done for the supersymmetric case. Unfortunately the description of discretized two–dimensional random surfaces by the hermitian matrix model to date has no analogue in terms of a supersymmetric matrix model describing a discretization of super–Riemann surfaces.

Nevertheless the supereigenvalue model proposed by Alvarez–Gaum´ e, Itoyama, Mañes and Zadra \cite{2} appears to have all the virtues of a discrete approach to 2d supergravity. Guided by the prominent role the Virasoro constraints \cite{3} played for the hermitian matrix model, the authors constructed a partition function built out of $N$ Grassmann even and odd variables (the “supereigenvalues”) as well as even and odd coupling constants obeying a set of super–Virasoro constraints. As it is unknown whether there exists a matrix–based formulation of this model, we do not have a geometric interpretation of it at hand. However, the study of the supereigenvalue model revealed that in its continuum limit it describes 2d supergravity coupled to minimal superconformal field theories \cite{2, 4}. Moreover, a precise dictionary between continuum $N = 1$ super–Liouville amplitudes and supereigenvalue correlators has been developed \cite{5}.

Just as the hermitian matrix model, the supereigenvalue model admits an expansion in $1/N^2$, termed the genus expansion. The super–Virasoro constraints of the supereigenvalue model are equivalent to a set of superloop equations for the superloop correlators, generalizing the well–known loop equations of the hermitian matrix model \cite{3, 6, 7}. The superloop equations were solved in the planar $N \to \infty$ limit for general potentials away from the double scaling limit in \cite{4}. Higher genera results were obtained in the double scaling limit. An alternative approach was pursued by the authors of \cite{2, 10}, who managed to directly integrate out the Grassmann–odd variables on the level of the partition function. Interestingly enough this analysis revealed that the free energy of the supereigenvalue model depends at most quadratically on the fermionic coupling constants. Moreover, Becker and Becker \cite{9} showed that this free energy is related in a simple way to the free energy of the hermitian matrix model.

In this paper we shall be interested in the supereigenvalue model away from the double scaling limit. We develop an iterative procedure to solve the superloop equations for general potentials genus by genus. It represents a generalization of the very effective iterative scheme of Ambjørn, Chekhov, Kristjansen and Makeenko \cite{11} to calculate higher genus contributions to the multi–loop correlators and the free energy of the hermitian one matrix model. Our approach is based on a new integral form of the superloop equations and the introduction of superloop insertion operators. Similarly to \cite{11} the key point in this scheme is the change of variables from the coupling constants to the so-called moments of the bosonic and fermionic potentials. The advantage of these new variables is that a dependence on an infinite number of coupling constants arranges itself nicely into a finite
number of moments at each genus. With these methods at hand, we derive a new quite compact form of the solution of the superloop equations in the planar limit. In addition we present explicit results for the free energy and the one–superloop correlators at genus one. Moreover, it is shown that the genus $g$ contribution to the free energy depends on $2(3g)$ bosonic and $2(3g+1)$ fermionic moments.

The paper is organized as follows: In section 2 we introduce the basic ingredients of our iterative solution, the superloop insertion operators and the superloop equations. Section 3 then contains the planar solution. A set of basis functions and our new variables – the moments – are introduced. The iterative procedure is developed in Section 4. In parallel we present explicit results for genus one. Moreover we state some details on the general structure of the free energy and the superloop correlators at genus $g$. Finally in section 6 we conclude.

2 Superloop Equations

Our solution of the supereigenvalue model proposed by Alvarez–Gaumé, Itoyama, Mañes and Zadra [2] is based on an integral form of the superloop equations. Generalizing the approach of Ambjørn, Chekhov, Kristjansen and Makeenko [11] for the hermitian one–matrix model we develop an iterative procedure which allows us to calculate the genus $g$ contribution to the $(n|m)$-superloop correlators for (in principle) any $g$ and any $(n|m)$ and (in practice) for any potential. The possibility of going to arbitrarily high genus is provided by the superloop equations, whereas the possibility of obtaining arbitrary $(n|m)$-superloop results is due to the superloop insertion operators introduced below. A change of variables from the coupling constants to moments allows us to explicitly present results for arbitrary potentials.

2.1 Superloop Insertion Operators

The supereigenvalue model [2] is built out of a set of $N$ bosonic and fermionic variables, denoted by $\lambda_i$ and $\theta_i$ respectively. $N$ is even. The partition function is given by

$$Z = e^{N^2 F} = \int (\prod_{i=1}^{N} d\lambda_i d\theta_i) \prod_{i<j} (\lambda_i - \lambda_j - \theta_i\theta_j) \exp\left(-N \sum_{i=1}^{N} [V(\lambda_i) - \theta_i\Psi(\lambda_i)]\right)$$

where

$$V(\lambda_i) = \sum_{k=0}^{\infty} g_k \lambda_i^k \quad \text{and} \quad \Psi(\lambda_i) = \sum_{k=0}^{\infty} \xi_{k+1/2} \lambda_i^k,$$

the $g_k$ and $\xi_{k+1/2}$ being Grassmann even and odd coupling constants, respectively. This model obeys a set of super–Virasoro constraints $L_k Z = 0$ and $G_{k-1/2} Z = 0$. 


for \( k \geq 0 \) by construction. Here the super–Virasoro operators are represented as differential operators in the coupling constants \( g_k \) and \( \xi_{k+1/2} \).

Expectation values are defined in the usual way by

\[
\langle O(\lambda_j, \theta_j) \rangle = \frac{1}{Z} \int \left( \prod_{i=1}^{N} d\lambda_i d\theta_i \right) \Delta(\lambda_i, \theta_i) \, O(\lambda_j, \theta_j) \, \exp\left(-N \sum_{i=1}^{N} [V(\lambda_i)-\theta_i \Psi(\lambda_i)]\right),
\]

(2.3)

where we write \( \Delta(\lambda_i, \theta_i) = \prod_{i<j} (\lambda_i - \lambda_j - \theta_i \theta_j) \) for the measure. We introduce the one–superloop correlators

\[
\hat{W}(p |) = N \left\langle \sum_i \frac{\theta_i}{p - \lambda_i} \right\rangle \quad \text{and} \quad \hat{W}(|p) = N \left\langle \sum_i \frac{1}{p - \lambda_i} \right\rangle,
\]

(2.4)

which act as generating functionals for the one–point correlators \( \langle \sum_i \lambda_i^k \rangle \) and \( \langle \sum_i \theta_i \lambda_i^k \rangle \) upon expansion in \( p \). This easily generalizes to higher–point correlators with the \((n|m)\)–superloop correlator

\[
\hat{W}(p_1, \ldots, p_n | q_1, \ldots, q_m) = \frac{1}{Z} \frac{\delta}{\delta \Psi(p_1)} \cdots \frac{\delta}{\delta \Psi(p_n)} \frac{\delta}{\delta V(q_1)} \cdots \frac{\delta}{\delta V(q_m)} Z,
\]

(2.5)

Quite analogously to the bosonic case \[\text{[11]}\] these correlators may be obtained from the partition function \( Z \) by application of the superloop insertion operators \( \delta/\delta V(p) \) and \( \delta/\delta \Psi(p) \):

\[
\hat{W}(p_1, \ldots, p_n | q_1, \ldots, q_m) = \frac{1}{Z} \frac{\delta}{\delta \Psi(p_1)} \cdots \frac{\delta}{\delta \Psi(p_n)} \frac{\delta}{\delta V(q_1)} \cdots \frac{\delta}{\delta V(q_m)} Z,
\]

(2.6)

where

\[
\frac{\delta}{\delta V(p)} = -\sum_{k=0}^{\infty} \frac{1}{p^{k+1}} \frac{\partial}{\partial g_k} \quad \text{and} \quad \frac{\delta}{\delta \Psi(p)} = -\sum_{k=0}^{\infty} \frac{1}{p^{k+1}} \frac{\partial}{\partial \xi_{k+1/2}}.
\]

(2.7)

In particular equation (2.4) can now be written as \( \hat{W}(|p) = \delta \ln Z/\delta V(p) \) and \( \hat{W}(p |) = \delta \ln Z/\delta \Psi(p) \).

However, it is convenient to work with the connected part of the \((n|m)\)–superloop correlators, denoted by \( W \). They may be obtained from the free energy \( F = N^{-2} \ln Z \) through

\[
W(p_1, \ldots, p_n | q_1, \ldots, q_m) = \frac{\delta}{\delta \Psi(p_1)} \cdots \frac{\delta}{\delta \Psi(p_n)} \frac{\delta}{\delta V(q_1)} \cdots \frac{\delta}{\delta V(q_m)} F
\]

(2.8)

\[
= N^{n+m-2} \left\langle \sum_i \frac{\theta_i}{p_i - \lambda_i} \cdots \sum_n \frac{\theta_n}{p_n - \lambda_n} \sum_j \frac{1}{q_j - \lambda_j} \cdots \sum_m \frac{1}{q_m - \lambda_m} \right\rangle_C.
\]
Note that \( n \leq 2 \) due to the structure of \( F \) mentioned below.

With the normalizations chosen above, one assumes that these correlators enjoy the genus expansion

\[
W(p_1, \ldots, p_n \mid q_1, \ldots, q_m) = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} W_g(p_1, \ldots, p_n \mid q_1, \ldots, q_m). \tag{2.9}
\]

Similarly one has the genus expansion

\[
F = \sum_{g=0}^{\infty} \frac{1}{N^{2g}} F_g \tag{2.10}
\]

for the free energy.

### 2.2 Superloop Equations

The superloop equations of our model are two Schwinger–Dyson equations, which we derive in appendix A. They were first stated in [4, 4], and we present them in an integral form for the loop correlators \( W(p \mid ) \) and \( W(\mid p) \).

The Grassmann–odd superloop equation reads

\[
\oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p - \omega} W(\omega \mid ) + \oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{p - \omega} W(\mid \omega) = W(p \mid ) W(\mid p) + \frac{1}{N^2} W(p \mid p) \tag{2.11}
\]

and its counterpart, the Grassmann–even superloop equation, takes the form

\[
\oint_C \frac{d\omega}{2\pi i} \frac{V(\omega)}{p - \omega} W(\mid \omega) + \Psi'(\omega) W(\mid p) - \frac{1}{2} \frac{d}{dp} \oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega) W(\omega \mid )}{p - \omega} =
\]

\[
\frac{1}{2} \left[ W(\mid p)^2 - W(p \mid ) W'(p \mid ) + \frac{1}{N^2} \left( W(\mid p, p) - \frac{d}{dq} W(p, q \mid ) \right|_{p=q} \right] . \tag{2.12}
\]

In the derivation we have assumed that the loop correlators have one–cut structure, i.e. in the limit \( N \to \infty \) we assume that the eigenvalues are contained in a finite interval \([x, y]\). Moreover \( C \) is a curve around the cut.

Note the similarity to the loop equations for the hermitian matrix model [8]. The equations (2.11) and (2.12) are equivalent to the superloop equations discussed in [3, 4], which may be seen by performing the contour integrals and taking the residues at \( \omega = p \) and \( \omega = \infty \). Moreover eqs. (2.11) and (2.12) simply encode the super–Virasoro constraints \( G_{k-1/2} Z = 0 \) and \( L_k Z = 0 \) for \( k \geq 0 \), which the model obeys by construction.

The key to the solution of these complicated equations order by order in \( N^{-2} \) is the observation made in [3, 10] that the free energy \( F \) depends at most quadratically on fermionic coupling constants. Via eq. (2.8) this directly translates to the one-loop correlators, which we from now on write as
\[ W(p) = v(p) \quad (2.13) \]
\[ W(p) = u(p) + \tilde{u}(p). \quad (2.14) \]

Here \( v(p) \) is of order one in fermionic couplings, whereas \( u(p) \) is taken to be of order zero and \( \tilde{u}(p) \) of order two in the fermionic coupling constants \( \xi_{k+1/2} \). This observation allows us to split up the two superloop equations (2.11) and (2.12) into a set of four equations, sorted by their order in the \( \xi_{k+1/2} \)'s. Doing this we obtain

**Order 0:**
\[
\oint_{C} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} u(\omega) = \frac{1}{2} u(p)^2 + \frac{1}{2} \frac{1}{N^2} \frac{\delta}{\delta V(p)} u(p) - \frac{1}{2} \frac{1}{N^2} \frac{d}{dq} \frac{\delta}{\delta \Psi(p)} v(q) \bigg|_{p=q}
\]
\[ (2.15) \]

**Order 1:**
\[
\oint_{C} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} v(\omega) + \oint_{C} \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{p-\omega} u(\omega) = v(p) u(p) + \frac{1}{N^2} \frac{\delta}{\delta V(p)} v(p)
\]
\[ (2.16) \]

**Order 2:**
\[
\oint_{C} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} \tilde{u}(\omega) + \oint_{C} \frac{d\omega}{2\pi i} \frac{\Psi'(\omega)}{p-\omega} v(\omega) = \frac{1}{2} \frac{d}{dp} \oint_{C} \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{p-\omega} v(\omega) =
\]
\[ u(p) \tilde{u}(p) - \frac{1}{2} v(p) \frac{d}{dp} v(p) + \frac{1}{2} \frac{1}{N^2} \frac{\delta}{\delta V(p)} \tilde{u}(p) \]
\[ (2.17) \]

**Order 3:**
\[
\oint_{C} \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{p-\omega} \tilde{u}(\omega) = v(p) \tilde{u}(p).
\]
\[ (2.18) \]

It is the remarkable form of these four equations which allows us to develop an iterative procedure to determine \( u_g(p), v_g(p), \tilde{u}_g(p) \) and \( F_g \) genus by genus. Plugging the genus expansions into these equations lets them decouple partially, in the sense that the equation of order 0 at genus \( g \) only involves \( u_g \) and lower genera contributions. The order 1 equation then only contains \( v_g, u_g \) and lower genera results and so on. The first thing to do, however, is to find the solution for \( g = 0 \).
3 The Planar Solution

In the following the planar solution for the superloop correlators is given for a general potential. It was first obtained in [4]. We present it in a very compact integral form augmented by the use of new variables to characterize the potentials, the moments.

3.1 Solution for \( u_0(p) \) and \( v_0(p) \)

In the limit \( N \to \infty \) the order 0 equation (2.13) becomes

\[
\oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p - \omega} u_0(\omega) = \frac{1}{2} u_0(\omega)^2. \tag{3.1}
\]

This equation is well known, as up to a factor of \( 1/2 \) it is nothing but the planar loop equation of the hermitian matrix model. With the above assumptions on the one–cut structure and by demanding that \( u(p) \) behaves as \( 1/p \) for \( p \to \infty \) one finds [6]

\[
u_0(p) = \oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p - \omega} \left[ \frac{(p - x)(p - y)}{(\omega - x)(\omega - y)} \right]^{1/2}, \tag{3.2}\]

where the endpoints \( x \) and \( y \) of the cut on the real axis are determined by the following requirements:

\[
0 = \oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{\sqrt{(\omega - x)(\omega - y)}}, \quad 1 = \oint_C \frac{d\omega}{2\pi i} \frac{\omega V'(\omega)}{\sqrt{(\omega - x)(\omega - y)}}, \tag{3.3}\]

deduced from our knowledge that \( W(\mid p) = 1/p + \mathcal{O}(p^{-2}) \).

The order 1 equation (2.16) in the \( N \to \infty \) limit determining the odd loop correlator \( v_0(p) \) reads

\[
\oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p - \omega} v_0(\omega) + \oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{p - \omega} u_0(\omega) = v_0(p) u_0(p). \tag{3.4}\]

It is solved by

\[
v_0(p) = \oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{p - \omega} \left[ \frac{(\omega - x)(\omega - y)}{(p - x)(p - y)} \right]^{1/2} + \frac{\chi}{\sqrt{(p - x)(p - y)}}. \tag{3.5}\]

Here \( \chi \) is a constant not determined by eq. (3.4), in fact \( \chi = N^{-1} \langle \sum_i \theta_i \rangle \) in the planar limit. It will be determined in the analysis of the two remaining equations (2.17) and (2.18). We verify the above solution in appendix B.
3.2 Moments and Basis Functions

Let us now define new variables characterizing the potentials $V(p)$ and $Ψ(p)$. Instead of the couplings $g_k$ we introduce the bosonic moments $M_k$ and $J_k$ defined by

$$M_k = \oint_C \frac{dω}{2πi} \frac{V'(ω)}{(ω-x)^k \left[(ω-x)(ω-y)\right]^{1/2}}, \quad k \geq 1 \quad (3.6)$$

$$J_k = \oint_C \frac{dω}{2πi} \frac{V'(ω)}{(ω-y)^k \left[(ω-x)(ω-y)\right]^{1/2}}, \quad k \geq 1, \quad (3.7)$$

and the couplings $ξ_{k+1/2}$ are replaced by the fermionic moments

$$Ξ_k = \oint_C \frac{dω}{2πi} \frac{Ψ(ω)}{(ω-x)^k \left[(ω-x)(ω-y)\right]^{1/2}}, \quad k \geq 1 \quad (3.8)$$

$$Λ_k = \oint_C \frac{dω}{2πi} \frac{Ψ(ω)}{(ω-y)^k \left[(ω-x)(ω-y)\right]^{1/2}}, \quad k \geq 1. \quad (3.9)$$

The main motivation for introducing these new variables is that, for each term in the genus expansion of the free energy and the correlators, the dependence on an infinite number of coupling constants arranges itself nicely into a function of a finite number of moments.

We further introduce the basis functions $χ^{(n)}(p)$ and $Ψ^{(n)}(p)$ recursively

$$χ^{(n)}(p) = \frac{1}{M_1} \left( \phi_x^{(n)}(p) - \sum_{k=1}^{n-1} \chi^{(k)}(p) M_{n-k+1} \right), \quad (3.10)$$

$$Ψ^{(n)}(p) = \frac{1}{J_1} \left( \phi_y^{(n)}(p) - \sum_{k=1}^{n-1} Ψ^{(k)}(p) J_{n-k+1} \right), \quad (3.11)$$

where

$$\phi_x^{(n)}(p) = (p-x)^{-n} \left[(p-x)(p-y)\right]^{-1/2}, \quad (3.12)$$

$$\phi_y^{(n)}(p) = (p-y)^{-n} \left[(p-x)(p-y)\right]^{-1/2}, \quad (3.13)$$

following [11].

It is easy to show that for the linear operator $\hat{V}'$ defined by

$$\hat{V}' f(p) = \oint_C \frac{dω}{2πi} \frac{V'(ω)}{p-ω} f(ω) \quad (3.14)$$

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and appearing in the superloop equations we have

\begin{align}
\left( \widetilde{V}' - u_0(p) \right) \chi^{(n)}(p) &= \frac{1}{(p-x)^n}, \quad n \geq 1, \\
\left( \widetilde{V}' - u_0(p) \right) \Psi^{(n)}(p) &= \frac{1}{(p-y)^n}, \quad n \geq 1.
\end{align}

(3.15) (3.16)

Moreover, \( \phi'_2(0) = \phi'(0) \equiv \phi(0) \) lies in the kernel of \( (\widetilde{V}' - u_0(p)) \).

### 3.3 Solution for \( \hat{u}_0 \) and \( \chi \)

Next consider the order 2 equation (2.17) at genus 0

\begin{align}
\left( \widetilde{V}' - u_0(p) \right) \hat{u}_0 &= \frac{1}{2} \frac{d}{dp} \oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{p-\omega} v_0(\omega) - \oint_C \frac{d\omega}{2\pi i} \frac{\Psi'(\omega)}{p-\omega} v_0(\omega) \\
&\quad - \frac{1}{2} v_0(p) \frac{d}{dp} v_0(p). 
\end{align}

(3.17)

Plugging eq. (3.5) into the right hand side of this equation yields after a somewhat lengthy calculation

\begin{align}
\left( \widetilde{V}' - u_0(p) \right) \hat{u}_0 &= \frac{1}{2} \Xi_2 \left( \Xi_1 - \chi \right) \frac{1}{(x-y)} \frac{1}{p-x} - \frac{1}{2} \Lambda_2 \left( \Lambda_1 - \chi \right) \frac{1}{(x-y)} \frac{1}{p-y}.
\end{align}

(3.18)

With eqs. (3.15) and (3.16) this immediately tells us that

\begin{align}
\hat{u}_0(p) &= \frac{1}{2} \Xi_2 \left( \Xi_1 - \chi \right) \chi^{(1)}(p) - \frac{1}{2} \Lambda_2 \left( \Lambda_1 - \chi \right) \psi^{(1)}(p).
\end{align}

(3.19)

There can be no contributions proportional to the zero mode \( \phi(0)(p) \), as we know that \( \hat{u}(p) \) behaves as \( \mathcal{O}(p^{-2}) \) for \( p \to \infty \).

Finally we determine the odd constant \( \chi \). This is done by employing the order 3 equation for \( g = 0 \), i.e.

\begin{align}
\oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{p-\omega} \hat{u}_0(\omega) - v_0(p) \hat{u}_0(p) = 0.
\end{align}

(3.20)

After insertion of eqs. (3.3) and (3.19) one can show that

\begin{align}
0 &= \left( \hat{\Psi} - v_0(p) \right) \hat{u}_0(p) = \frac{1}{2} \Xi_2 \left( \Xi_1 - \chi \right) \left( \Lambda_1 - \chi \right) \frac{M_1}{(x-y)^3 (p-y)} - \frac{1}{2} \Lambda_2 \left( \Lambda_1 - \chi \right) \left( \Xi_1 - \chi \right) \\
&\quad \frac{1}{2} \Lambda_2 \left( \Lambda_1 - \chi \right) \left( \Xi_1 - \chi \right) J_1 (x-y)^3 (p-x).
\end{align}

(3.21)
\[ \hat{\Psi} f(p) = \oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{p-\omega} f(\omega), \]  

(3.22)

in accordance to \( \hat{V}' \). The result (3.21) lets us finally read off the coefficient \( \chi \) as

\[ \chi = \frac{1}{2} (\Xi_1 + \Lambda_1). \]  

(3.23)

Putting it all together, we may now write down the complete genus 0 solution for the one–superloop correlators \( W(\mid p) \) and \( W(p \mid) \):

\[ W_0(\mid p) = \oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} \left[ \frac{(p-x)(p-y)}{(\omega-x)(\omega-y)} \right]^{1/2} + \frac{1}{4} \Xi_2 (\Xi_1 - \Lambda_1) \phi_x^{(1)}(p) + \frac{1}{4} \Lambda_2 (\Xi_1 - \Lambda_1) \phi_y^{(1)}(p) \]  

(3.24)

\[ W_0(p \mid) = \oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{p-\omega} \left[ \frac{(\omega-x)(\omega-y)}{(p-x)(p-y)} \right]^{1/2} + \frac{1}{2} \frac{\Xi_1 + \Lambda_1}{(p-x)(p-y)} \right]^{1/2}. \]  

(3.25)

One can show that this solution is equivalent to the one obtained by Alvarez–Gaumé, Becker, Becker, Emparan and Mañes in [4].

4 The Iterative Procedure

Our iterative solution of the superloop equations results in a certain representation of the free energy and the loop correlators in terms of the moments and basis functions defined in section 3.2. We will show that it suffices to know \( u_g(p) \) and \( v_g(p) \) only up to a zero mode in order to calculate \( F_g \). We give explicit results for genus one.

4.1 The Iteration for \( u_g \) and \( v_g \)

The correlators \( u_g(p) \) and \( v_g(p) \) are determined by the order 0 and order 1 equations (2.15) and (2.16) after insertion of the genus expansions (2.9) of these operators. We find

\[ \left( \hat{V}' - u_0(p) \right) u_g(p) = \sum_{g'=1}^{g-1} u_{g'}(p) u_{g-g'}(p) + \frac{1}{2} \frac{\delta}{\delta V(p)} u_{g-1}(p) \]  

\[ - \frac{1}{2} \frac{d}{dq} \frac{\delta}{\delta \Psi(p)} v_{g-1}(q) \bigg|_{p=q} \]  

(4.1)
and

\[
\left( \hat{V}' - u_0(p) \right) v_g(p) = - \left( \hat{\Psi} - v_0(p) \right) u_g(p) + \sum_{g'=1}^{g-1} v_{g'}(p) u_{g-g'}(p)
\]

\[
+ \frac{\delta}{\delta V(p)} v_{g-1}(p)
\]

(4.2)

at genus \( g \geq 1 \). From the structure of these equations we directly deduce that \( u_g(p) \) and \( v_g(p) \) will be linear combinations of the basis functions \( \chi^{(n)}(p) \) and \( \Psi^{(n)}(p) \). By eqs. (3.13) and (3.14) the coefficients of this linear combination may be read off the poles \((p - x)^{-k}\) and \((p - y)^{-k}\) of the right hand sides of eqs. (4.1) and (4.2) after a partial fraction decomposition.

Let us demonstrate how this works for \( g = 1 \). According to eq. (4.1) for \( u_1(p) \) we first calculate \( \delta u_0/\delta V(p) \). We then need to know the derivatives \( \delta x/\delta V(p) \) and \( \delta y/\delta V(p) \). They can be obtained from eq. (3.3) and read

\[
\frac{\delta x}{\delta V(p)} = \frac{1}{8} \frac{1}{(p - x)^2} + \frac{1}{8} \frac{1}{(p - y)^2} - \frac{1}{4 d} \frac{1}{(p - x)} + \frac{1}{4 d} \frac{1}{(p - y)},
\]

(4.3)

Using the relation

\[
\frac{\delta}{\delta V(p)} V'(\omega) = \frac{d}{dp} \frac{1}{p - \omega}
\]

(4.4)

one finds

\[
\frac{\delta}{\delta V(p)} u_0(p) = \frac{1}{8} \frac{1}{(p - x)^2} + \frac{1}{8} \frac{1}{(p - y)^2} - \frac{1}{4 d} \frac{1}{(p - x)} + \frac{1}{4 d} \frac{1}{(p - y)},
\]

(4.5)

where \( d = x - y \).

Next we determine \( \delta v_0(q)/\delta \Psi(p) \). Using the relation

\[
\frac{\delta}{\delta \Psi(p)} \Psi(q) = - \frac{1}{p - q}
\]

(4.6)

and the result

\[
\frac{\delta \Xi_k}{\delta \Psi(p)} = \delta_k - \frac{[(p - x)(p - y)]^{1/2}}{(p - x)^k}
\]

(4.7)

\[
\frac{\delta \Lambda_k}{\delta \Psi(p)} = \delta_k - \frac{[(p - x)(p - y)]^{1/2}}{(p - y)^k}
\]

(4.8)

for \( k \geq 1 \), one finds
\[
\frac{d}{dq} \frac{\delta}{\delta \Psi(p)} v_0(q) \bigg|_{p=q} = -\frac{\delta}{\delta V(p)} u_0(p). \tag{4.9}
\]

This enables us to write down \( u_1(p) \),

\[
 u_1(p) = \frac{1}{8} \chi^{(2)}(p) + \frac{1}{8} \Psi^{(2)}(p) - \frac{1}{4d} \chi^{(1)} + \frac{1}{4d} \Psi^{(1)}(p). \tag{4.10}
\]

Note that up to the overall factor of two this is identical to the one–loop correlator of the hermitian matrix model \([11]\), as it has to be \([9]\).

Now we solve eq. (4.2) at \( g = 1 \) for \( v_1(p) \). It is important to realize that generally eq. (4.2) fixes \( v_g(p) \) only up to a zero mode contribution \( \kappa_g \phi^{(0)}(p) \). This comes from the fact that, unlike for the bosonic \( u(p) \), we do not know the coefficient of the \( p^{-1} \) term for \( v(p) \). The zero mode coefficient \( \kappa_g \) will be fixed by requiring \( v_g(p) \) to be a total derivative of the free energy \( F_g \).

In order to calculate \( \delta v_0 / \delta V(p) \) we make use of the relation

\[
\frac{\delta \Xi_k}{\delta V(p)} = (k - \frac{1}{2}) \Xi_{k+1} \frac{1}{M_1} \phi_x^{(1)}(p) + \frac{1}{2} \left[ \sum_{r=2}^{k} \frac{\Xi_r}{(-d)^{1+k-r}} + \frac{\Xi_1 - \Lambda_1}{(-d)^k} \right] \frac{1}{J_1} \phi_y^{(1)}(p), \tag{4.11}
\]

as well as \( \delta \Lambda_k / \delta V(p) \) obtained from the above by the replacements \( x \leftrightarrow y \), \( M_k \leftrightarrow J_k \), \( \Xi_k \leftrightarrow \Lambda_k \) and \( d \rightarrow -d \). The derivatives \( \delta M_k / \delta V(p) \) and \( \delta J_k / \delta V(p) \) were calculated in \([11]\)

\[
\frac{\delta M_k}{\delta V(p)} = -\frac{1}{2} \left( p - x \right)^{-k-1/2} \left( p - y \right)^{-3/2} - \left( k + 1/2 \right) \phi_x^{(k+1)}(p)
+ \frac{1}{2} \left[ \frac{1}{(-d)^k} - \sum_{i=1}^{k} \frac{1}{(-d)^{k-i+1}} \frac{M_i}{J_1} \right] \phi_y^{(1)}(p)
+ \left( k + 1/2 \right) \frac{M_{k+1}}{M_1} \phi_x^{(1)}(p), \tag{4.12}
\]

and \( \delta J_k / \delta V(p) \) is obtained by the usual replacements. Using these and the earlier results one has

\[
\frac{\delta v_0}{\delta V(p)} = W_0(p \mid p) = \left[ -\frac{(\Xi_1 - \Lambda_1)}{4d M_1} \right] \frac{1}{(p - x)^3} + \left[ -\frac{(\Xi_1 - \Lambda_1)}{4d J_1} \right] \frac{1}{(p - y)^3}
+ \left[ \frac{\Xi_2}{4d M_1} \right] \frac{1}{(p - x)^2} + \left[ -\frac{\Lambda_2}{4d J_1} \right] \frac{1}{(p - y)^2}
+ \left[ \frac{\Lambda_2}{4d^2 J_1} - \frac{\Xi_2}{4d^2 M_1} \right] \frac{1}{(p - x)}
+ \left[ \frac{\Xi_2}{4d^2 M_1} - \frac{\Lambda_2}{4d^2 J_1} \right] \frac{1}{(p - y)}. \tag{4.13}
\]
For the evaluation of the right hand side of eq. (4.2) at genus $g$ we also need to know how the operator $(\hat{\Psi} - v_0(p))$ acts on the functions $\phi^{(n)}_x(p)$ and $\phi^{(n)}_y(p)$, in terms of which $u_g(p)$ is given. A straightforward calculation yields

$$
(\hat{\Psi} - v_0(p)) \phi^{(n)}_x(p) = \sum_{k=1}^{n+1} \frac{1}{(p-x)^k} \left[ -\frac{(\Xi_1 - \Lambda_1)}{2(d)^{n+2-k}} - \sum_{l=2}^{n+2-k} \frac{\Xi_l}{(d)^{n+3-k-l}} \right]
+ \frac{1}{(p-y)} \left[ -\frac{(\Xi_1 - \Lambda_1)}{2(d)^{n+1}} \right] \quad (4.14)
$$

as well as the analogous expression for $(\hat{\Psi} - v_0(p)) \phi^{(n)}_y(p)$ obtained from eq. (4.14) by the replacements $x \leftrightarrow y$, $M_k \leftrightarrow J_k$, $\Xi_k \leftrightarrow \Lambda_k$ and $d \rightarrow -d$.

We now have collected all the ingredients needed to evaluate the right hand side of eq. (4.2). After a partial fraction decomposition we may read off the poles at $x$ and $y$, and therefore obtain the coefficients of the linear combination in the basis functions. We arrived at the result for $g = 1$ with the aid of Maple, namely

$$
v_1(p) = \sum_{i=1}^{3} \left( B_1^{(i)} \chi^{(i)}(p) + E_1^{(i)} \Psi^{(i)}(p) \right) + \kappa_1 \phi^{(0)}(p), \quad (4.15)
$$

where the coefficients $B_1^{(i)}$ and $E_1^{(i)}$ are given by

$$
B_1^{(1)} = \frac{1}{8} \frac{\Xi_3}{d M_1} + \frac{1}{8} \frac{\Xi_2}{d^2 M_1} + \frac{1}{4} \frac{\Lambda_2}{d^2 J_1}
+ \frac{1}{8} \frac{M_2 \Xi_2}{d M_1^2} - \frac{1}{16} \frac{M_2 (\Xi_1 - \Lambda_1)}{d^2 M_1^2} + \frac{1}{16} \frac{J_2 (\Xi_1 - \Lambda_1)}{d^2 J_1^2}
- \frac{3}{16} \frac{(\Xi_1 - \Lambda_1)}{d^3 M_1} - \frac{3}{16} \frac{(\Xi_1 - \Lambda_1)}{d^3 J_1},
$$

$$
B_1^{(2)} = \frac{1}{8} \frac{\Xi_2}{d M_1} + \frac{1}{16} \frac{M_2 (\Xi_1 - \Lambda_1)}{d M_1^2} + \frac{3}{16} \frac{(\Xi_1 - \Lambda_1)}{d^2 M_1},
$$

$$
B_1^{(3)} = -\frac{5}{16} \frac{(\Xi_1 - \Lambda_1)}{d M_1}, \quad (4.16)
$$

and $E_1^{(i)} = B_1^{(i)}(M \leftrightarrow J, \Xi \leftrightarrow \Lambda, d \rightarrow -d)$.

Yet $\kappa_1$ is still undetermined. To compute it and the remaining doubly fermionic part $\tilde{u}_g(p)$ of the loop correlator $W_\gamma(\mid p)$ one can employ the order 2 and order 3 eqs. (2.17) and (2.18) at genus $g$. It is, however, much easier to construct the free energy $F_g$ at this stage from our knowledge of $u_g(p)$ and $v_g(p)$.

### 4.2 The Computation of $F_g$ and $\kappa_g$

As mentioned earlier, the free energy of the supereigenvalue model depends at most quadratically on the fermionic coupling constants. In this subsection we
present an algorithm which allows us to determine \( F_g \) and \( \kappa_g \) as soon as the results for \( u_g(p) \) and \( v_g(p) \) (up to the zero mode coefficient \( \kappa_g \)) are known.

One can show that the purely bosonic part of the free energy \( F_g \) is just twice the free energy of the hermitian matrix model. By using the results of Ambjorn et al. [11] one may then compute the bosonic part of \( F_g \) from \( u_g(p) \).

The strategy for the part of \( F_g \) quadratic in fermionic couplings consists in rewriting \( v_g(p) \) as a total derivative in the fermionic potential \( \Psi(p) \). We know that eqs. (4.7) and (4.8) imply

\[
\frac{\delta (\Xi_1 - \Lambda_1)}{\delta \Psi(p)} = -d \phi^{(0)}(p)
\]

and

\[
\frac{\delta \Xi_k}{\delta \Psi(p)} = -\varphi^{(k)}_x(p), \quad \frac{\delta \Lambda_k}{\delta \Psi(p)} = -\varphi^{(k)}_y(p), \quad k \geq 2
\]

where

\[
\varphi^{(k)}_x(p) = (p - x)^{-k} [(p - x)(p - y)]^{1/2} \quad k \geq 1,
\]

\[
\varphi^{(k)}_y(p) = (p - y)^{-k} [(p - x)(p - y)]^{1/2} \quad k \geq 1.
\]

Let us again specialize to \( g = 1 \). Using the above we can reexpress eq. (4.13) as

\[
\frac{\delta}{\delta \Psi(p)} F_1 + \kappa_1 \frac{1}{d} \frac{\delta}{\delta \Psi(p)} (\Xi_1 - \Lambda_1) = \sum_{i=1}^{3} \left( B^{(i)}_1 \chi^{(i)}(p) + E^{(i)}_1 \Psi^{(i)}(p) \right)
\]

\[
= \sum_{r=2}^{4} \left( \beta^{(r)}_1 \varphi^{(r)}_x(p) + \epsilon^{(r)}_1 \varphi^{(r)}_y(p) \right) + \gamma_1 [v^{(1)}_x(p) - \varphi^{(1)}_y(p)],
\]

with the new coefficients \( \beta^{(r)}_1, \epsilon^{(r)}_1 \) and \( \gamma_1 \) completely determined by the known coefficients \( B^{(i)}_1 \) and \( E^{(i)}_1 \). As the new functions \( \varphi^{(r)}_x(p) \) and \( \varphi^{(r)}_y(p) \) are total derivatives in \( \Psi(p) \), this equation allows us to calculate \( \kappa_1 \) and \( F_1 \).

With the help of Maple the zero mode coefficient \( \kappa_1 \) of \( v_1(p) \) becomes

\[
\kappa_1 = \frac{11 \Xi_2}{16 d M_1^2} - \frac{11 \Lambda_2}{16 d J_1^2} + \frac{5 \Lambda_2 J_2}{8 d^2 J_1^3} + \frac{5 \Xi_2 M_2}{8 d^2 M_1^3} - \frac{5 \Xi_2 M_3}{16 d M_1^3}
\]

\[
+ \frac{5 \Lambda_2 J_3}{16 d J_1^3} - \frac{3 \Xi_2 M_2^2}{8 d M_1^4} - \frac{3 \Lambda_2 J_2^2}{8 d J_1^4} + \frac{3 \Lambda_3 J_2}{8 d J_1^3} - \frac{3 \Xi_3 M_2}{8 d M_1^3} + \frac{5 \Xi_4}{16 d M_1^2}
\]
by applying the loop insertion operator

\[ W = \text{loop insertion operator} \]

The remaining part of the loop correlator

\[ \hat{A} = 1 \]

and the doubly fermionic part of \( A \) is constructed as well.

The result for the free energy at genus 1 then reads

\[
F_1 = \frac{1}{12} \ln M_1 - \frac{1}{12} \ln J_1 - \frac{1}{3} \ln d - (\Xi_1 - \Lambda_1) \left\{ \frac{11 \Xi_2}{16 d^4 M_1^2} - \frac{11 A_2}{16 d^4 J_1^2} + \frac{5 A_2 J_2}{8 d^3 J_1^3} + \frac{5 \Xi_2 M_2}{8 d^3 M_1^3} \right. \\
- \frac{5 \Xi_2 M_3}{16 d^2 M_1^3} + \frac{5 A_2 J_3}{16 d^2 M_1^3} \left. \right\} \]

The result for the free energy at genus 1 then reads

\[
F_1 = \frac{1}{12} \ln M_1 - \frac{1}{12} \ln J_1 - \frac{1}{3} \ln d - (\Xi_1 - \Lambda_1) \left\{ \frac{11 \Xi_2}{16 d^4 M_1^2} - \frac{11 A_2}{16 d^4 J_1^2} + \frac{5 A_2 J_2}{8 d^3 J_1^3} + \frac{5 \Xi_2 M_2}{8 d^3 M_1^3} \right. \\
- \frac{5 \Xi_2 M_3}{16 d^2 M_1^3} + \frac{5 A_2 J_3}{16 d^2 M_1^3} \left. \right\} \]

4.3 The Iteration for \( \tilde{u}_g(p) \)

The remaining part of the loop correlator \( W(\mid p) \) is now easily derived from \( F_g \) by applying the loop insertion operator \( \delta/\delta V(p) \) to its doubly fermionic part.

For genus 1 the result is

\[
\tilde{u}_1(p) = \sum_{i=1}^{4} \left( \tilde{A}_1^{(i)} \chi^{(i)}(p) + \tilde{D}_1^{(i)} \Psi^{(i)}(p) \right),
\]

where

\[
\tilde{A}_1^{(4)} = -\frac{35 (\Xi_1 - \Lambda_1) \Xi_2}{32 d^2 M_1^2} \\
\tilde{A}_1^{(3)} = (\Xi_1 - \Lambda_1) \left\{ -\frac{15 \Xi_3}{16 d^2 M_1^2} + \frac{45 \Xi_2}{32 d^3 M_1^2} + \frac{25 \Xi_2 M_2}{32 d^2 M_1^3} - \frac{5 A_2}{32 d^3 J_1 M_1} \right\} \]

\[
\tilde{A}_1^{(2)} = \frac{21 (\Xi_1 - \Lambda_1) \Xi_3}{16 d^2 M_1^2} - \frac{3 (\Xi_1 - \Lambda_1) \Xi_2 M_2^2}{8 d^2 M_1^4} + \frac{3 (\Xi_1 - \Lambda_1) \Xi_2 J_2}{32 d^3 J_1^2 M_1} + \frac{\Xi_2 M_3}{8 d^2 M_1^2} \\
+ \frac{3 A_2 \Xi_3}{8 J_1 M_1 d^3} + \frac{3 (\Xi_1 - \Lambda_1) \Xi_3 M_2}{4 d^2 M_1^3} - \frac{15 (\Xi_1 - \Lambda_1) \Xi_4}{16 d^2 M_1^2} - \frac{\Xi_3 \Xi_2}{4 d^2 M_1^2}
\]
\[ \tilde{A}_1^{(1)} = \frac{-51 (\Xi_1 - \Lambda_1) \Xi_2}{32 d^4 M_1^2} - \frac{\Lambda_2 (\Xi_1 - \Lambda_1) M_2}{32 J_1 d^3 M_1^2} - \frac{\Lambda_2 (\Xi_1 - \Lambda_1)}{4 J_1 d^4 M_1} \\
- \frac{33 (\Xi_1 - \Lambda_1) \Xi_2 M_2}{32 d^3 M_1^3} + \frac{5 (\Xi_1 - \Lambda_1) \Xi_2 M_3}{16 d^2 M_1^3} - \frac{9 (\Xi_1 - \Lambda_1) \Xi_2}{32 J_1 d^4 M_1} \\
- \frac{3 (\Xi_1 - \Lambda_1) \Xi_2 J_2}{16 d^3 J_1^4} + \frac{15 (\Xi_1 - \Lambda_1) \Xi_4 M_2}{32 d^2 M_1^2} + \frac{45 (\Xi_1 - \Lambda_1) \Xi_4}{32 d^3 M_1^2} \\
+ \frac{5 (\Xi_1 - \Lambda_1) \Xi_2}{16 d^5 J_1 M_1} - \frac{7 (\Xi_1 - \Lambda_1) \Lambda_3}{32 d^4 J_1^2} - \frac{\Xi_2 \Lambda_3}{16 d^3 J_1^2} + \frac{\Xi_3 \Xi_2}{16 d^3 M_1^2} \\
+ \frac{51 (\Xi_1 - \Lambda_1) \Xi_2}{32 d^4 M_1^2} - \frac{51 (\Xi_1 - \Lambda_1) \Xi_3}{8 d^3 M_1^2} + \frac{5 \Xi_2 \Xi_4}{16 d^2 M_1^2} \\
- \frac{\Lambda_2 \Xi_2}{2 J_1 M_1 d^4} - \frac{7 \Lambda_2 (\Xi_1 - \Lambda_1) J_2}{32 J_1^3 d^4} + \frac{3 \Lambda_2 \Xi_3}{8 d^3 J_1 M_1} + \frac{7 \Lambda_2 (\Xi_1 - \Lambda_1)}{16 J_1^2 d^5} \\
+ \frac{5 (\Xi_1 - \Lambda_1) \Xi_2 J_2}{16 d^3 J_1^3} + \frac{11 (\Xi_1 - \Lambda_1) \Xi_2}{32 d^3 J_1^2 M_1} + \frac{5 (\Xi_1 - \Lambda_1) \Xi_2 M_3}{32 d^2 M_1^3} \\
- \frac{9 (\Xi_1 - \Lambda_1) \Xi_3 M_2}{16 d^2 M_1^4} + \frac{3 (\Xi_1 - \Lambda_1) \Xi_3 J_2}{32 d^3 J_1^2 M_1} + \frac{9 (\Xi_1 - \Lambda_1) \Xi_3}{32 J_1 M_1 d^4} \\
- \frac{15 (\Xi_1 - \Lambda_1) \Xi_3 M_3}{32 d^2 M_1^3} - \frac{9 (\Xi_1 - \Lambda_1) \Xi_3 M_2}{8 d^3 M_1^3} + \frac{5 (\Xi_1 - \Lambda_1) \Xi_2 J_3}{32 d^3 J_1^3} \\
- \frac{\Xi_2}{2 d} \left( \frac{11 \Lambda_2}{16 d^3 J_1^2} + \frac{5 \Lambda_2 J_2}{8 d^2 J_1^3} + \frac{5 \Lambda_2 J_3}{16 d J_1^4} + \frac{\Lambda_2}{16 d^3 J_1 M_1} \right) \\
- \frac{\Lambda_2 M_2}{16 d^2 M_1^2 J_1} - \frac{3 \Lambda_2 J_2}{8 d J_1^4} + \frac{3 \Lambda_3 J_2}{8 d J_1^3} - \frac{3 \Xi_3 M_2}{8 d M_1^3} + \frac{5 \Xi_4}{16 d M_1^2} \\
- \frac{5 \Lambda_4}{16 d J_1^2} - \frac{5 \Xi_3}{8 d^2 M_1^2} + \frac{5 \Lambda_3}{8 d^2 J_1^2} - \frac{3 (\Xi_1 - \Lambda_1) \Xi_2 M_2}{8 d^3 M_1^4} \\
- \frac{(\Xi_1 - \Lambda_1) \Xi_2 J_2}{8 d^4 J_1^2 M_1} + \frac{\Lambda_2 (\Xi_1 - \Lambda_1) M_2}{16 d^4 J_1 M_1^2} + \frac{11 \Lambda_2 (\Xi_1 - \Lambda_1)}{32 J_1^6 M_1^2} \\
+ (\Xi_1 - \Lambda_1) \left( \frac{3 J_2^2}{8 d^2 J_1^4} - \frac{3 M_2^2}{8 d^2 M_1^4} - \frac{5 M_2}{8 d^3 M_1^3} - \frac{5 J_2}{8 d^3 J_1^3} \\
+ \frac{5 M_3}{16 d^2 M_1^3} - \frac{5 J_3}{16 d^2 J_1^3} + \frac{11}{16 d^4 J_1^2} - \frac{11}{16 d^4 M_1^2} \right) \right) , \\
(4.24) \\
and the analogue expressions for the \( \tilde{D}_1^{(i)} \) obtained from the above by replacing \( M \leftrightarrow J, \Xi \leftrightarrow \Lambda \) and \( d \rightarrow -d \).
4.4 General Structure of $u_g$, $v_g$, $\hat{u}_g$ and $F_g$

In the following subsection we find the number of moments and basis functions the free energy and the superloop correlators at genus $g$ depend on. For this write $F_g^\text{bos}$ for the purely bosonic and $F_g^\text{ferm}$ for the doubly fermionic parts of the free energy.

Ambjørn et al. [11, 12] have shown that the free energy of the hermitian matrix model depends on $2(3g - 2)$ moments. This directly translates to $F_g^\text{bos}$. Similarly as $u_g = \delta F_g^\text{bos} / \delta V(p)$ and with eq. (4.12) we see that $u_g$ contains $2(3g - 1)$ bosonic moments and basis functions up to order $(3g - 1)$, i.e.

$$u_g(p) = \sum_{k=1}^{3g-1} A_g^{(k)} \chi_g^{(k)}(p) + D_g^{(k)} \Psi_g^{(k)}(p).$$

(4.25)

For the structure of $v_g$ consider the leading–order poles on the RHS of eq. (4.2). Label this order by $n_g$, then with eqs. (4.11), (4.12) and (4.14) the three terms on the RHS of eq. (4.2) give rise to the following poles of leading order

$$\left( \hat{\Psi} - v_0(p) \right) u_g(p) : (3g - 1) + 1$$
$$v_{g'} u_{g-g'} : n_{g'} + (3(g - g') - 1) + 1$$
$$\frac{\delta}{\delta V(p)} v_{g-1} : n_{g-1} + 3.$$

(4.26)

From the above we deduce that $n_g = 3g$, and therefore

$$v_g(p) = \sum_{k=1}^{3g} B_g^{(k)} \chi_g^{(k)}(p) + E_g^{(k)} \Psi_g^{(k)}(p).$$

(4.27)

As the highest bosonic moments in $v_g$ come from the highest–order basis functions, we see that $v_g$ depends on $2(3g)$ bosonic moments. To find the dependence on the number of fermionic moments recall eqs. (4.7) and (4.8). In order for $v_g$ to have a leading contribution of $\chi_g^{(3g)}(p)$ the fermionic part of the free energy $F_g^\text{ferm}$ must contain $\Xi_{3g+1}$. We are thus led to the conclusion that $F_g^\text{ferm}$ and $v_g$ both depend on $2(3g + 1)$ fermionic moments. As the application of the loop insertion operator $\delta / \delta \Psi(p)$ does not change the number of bosonic moments, $F_g^\text{ferm}$ must contain $2(3g)$ bosonic moments.

Knowing the structure of $F_g^\text{ferm}$ then tells us with eqs. (4.11) and (4.12) that $\hat{u}_g$ depends on $2(3g + 1)$ bosonic and $2(3g + 2)$ fermionic moments. For genus $g$ it reads

$$\hat{u}_g(p) = \sum_{k=1}^{3g+1} \hat{A}_g^{(k)} \chi_g^{(k)}(p) + \hat{D}_g^{(k)} \Psi_g^{(k)}(p).$$

(4.28)

The genus $g$ contribution to the $(|s|)$–superloop correlator $W_g(|p, \ldots, p)$ will then depend on $2(3g + s)$ bosonic and $2(3g + s + 1)$ fermionic moments. Similarly
the genus $g$ $(1|s)$–superloop correlator $W_g(p|p,\ldots,p)$ is a function of $2(3g+s)$ bosonic and $(2(3g+s+1)$ fermionic moments.

This concludes our analysis of the iterative process.

5 Conclusions

We have studied the supereigenvalue model away from the double scaling limit. The superloop correlators of this model obey a set of integral equations, the superloop equations. These two equations could be split up into a set of four equations, sorted by their order in fermionic coupling constants. By a change of variables from coupling constants to moments we were able to present the planar solution of the superloop equations for general potentials in a very compact form. The remarkable structure of the superloop equations enabled us to develop an iterative procedure for the calculation of higher–genera contributions to the free energy and the superloop correlators. Here it proved sufficient to solve the two lowest–order equations at genus $g$ for the purely bosonic $u_g(p)$ and the fermionic $v_g(p)$ (up to a zero mode contribution). The zero mode as well as the doubly fermionic part of the free energy could then be found by rewriting $v_g(p)$ as a total derivative in the fermionic potential. The purely bosonic part of the free energy can be calculated with the methods of [11]. In principle the application of loop insertion operators to the free energy then yields arbitrary multi–superloop correlators. As we demonstrated for genus one, in practice these expressions become quite lengthy. We ended with a survey on the general structure of the free energy and the multi–superloop correlators for genus $g$.

We believe that this paper describes a suitable approach to the solution of the supereigenvalue model. The structures of the iterative solutions to the superloop equations and to the loop equations of the hermitian matrix model show interesting relations. This may give new hope for finding a generalized matrix–based formulation of the supereigenvalue model and thus allowing a geometrical interpretation of it. Moreover we think that our results underline the effectiveness of the iterative scheme of Ambjørn et al., which should be applicable to further models of similar type as well.

We expect that the described iterative solution for the supereigenvalue model may also be set up in the double scaling limit. This, however, is left for future work.

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Appendix

A. Derivation of the Superloop Equations

In order to derive the first of the two superloop equations for the model of eq. (2.1) consider the shift in integration variables

\[ \lambda_i \rightarrow \lambda_i + \theta_i \frac{\epsilon}{p - \lambda_i} \quad \text{and} \quad \theta_i \rightarrow \theta_i + \frac{\epsilon}{p - \lambda_i} \]  

(A.1)

where \( \epsilon \) is an odd constant. Under these we find that

\[ \prod_i d\lambda_i d\theta_i \rightarrow (1 - \epsilon \sum_i \frac{\theta_i}{(p - \lambda_i)^2}) \prod_i d\lambda_i d\theta_i \]  

(A.2)

and the measure transforms as

\[ \Delta(\lambda_i, \theta_i) \rightarrow (1 - \epsilon \sum_{i \neq j} \frac{\theta_i}{(p - \lambda_i)(p - \lambda_j)}) \Delta(\lambda_i, \theta_i). \]  

(A.3)

The vanishing of the terms proportional to \( \epsilon \) then gives us the Schwinger–Dyson equation

\[ \langle N \left\{ \sum_i \frac{1}{p - \lambda_i} \left( V'(\lambda_i) \theta_i + \Psi(\lambda_i) \right) \right\} - \sum_i \frac{\theta_i}{p - \lambda_i} \sum_j \frac{1}{p - \lambda_j} \rangle = 0. \]  

(A.4)

Note that \( \langle \sum_i \theta_i (p - \lambda_i)^{-1} \sum_j (p - \lambda_j)^{-1} \rangle = N^{-2} \hat{W}(p \mid p) \) with the above definitions. In order to transform eq. (A.4) into an integral equation we define the bosonic and fermionic density operators

\[ \rho(\lambda) = \frac{1}{N} \sum_i \langle \delta(\lambda - \lambda_i) \rangle \quad \text{and} \quad r(\lambda) = \frac{1}{N} \sum_i \langle \theta_i \delta(\lambda - \lambda_i) \rangle. \]  

(A.5)

With these the first sum in eq. (A.4) may be written as

\[ N^2 \int d\lambda \left( \frac{\rho'(\lambda)}{p - \lambda} + \rho(\lambda) \frac{\Psi'(\lambda)}{p - \lambda} \right) = \]

\[ N^2 \int d\lambda r(\lambda) \left[ \oint_C \frac{d\omega}{2\pi i} \frac{1}{\omega - \lambda} \frac{V'(\omega)}{p - \omega} \right] + N^2 \int d\lambda \rho(\lambda) \left[ \oint_C \frac{d\omega}{2\pi i} \frac{1}{\omega - \lambda} \frac{\Psi'(\omega)}{p - \omega} \right] \]  

(A.6)

where we assume that the real eigenvalues \( \lambda_i \) are contained within a finite interval \( x < \lambda_i < y, \forall i \). Moreover \( C \) is a curve around 0 with radius \( R \geq \max(|x|, |y|) \) and we choose \( |p| > R \). Performing the \( \lambda \) integrals in eq. (A.6) gives us the one–superloop correlators. The full Schwinger–Dyson equation (A.4) may then be expressed in the integral form
\[ \oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} \tilde{W}(\omega |) + \oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{p-\omega} \tilde{W}(| \omega) = N^{-2} \tilde{W}(p | p). \] (A.7)

Rewriting this in terms of the connected superloop correlators \( W \) yields eq. (2.11).

The derivation of the second superloop equation goes along the same lines by performing the shift

\[ \lambda_i \to \lambda_i + \frac{\epsilon}{p - \lambda_i} \quad \text{and} \quad \theta_i \to \theta_i + \frac{1}{2} \frac{\epsilon \theta_i}{(p - \lambda_i)^2} \] (A.8)

with \( \epsilon \) even and infinitesimal. Similar steps as the ones discussed above then lead us to the second superloop equation

\[ \oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} \tilde{W}(| \omega) + \frac{1}{2} \frac{d}{dp} \left( \oint_C \frac{d\omega}{2\pi i} \frac{\Psi(\omega)}{p-\omega} \tilde{W}(| \omega) \right) = \frac{1}{2} N^2 \tilde{W}(| p, p) - \frac{1}{2} N^2 \frac{d}{dq} \tilde{W}(p, q |) \bigg|_{p=q} \] (A.9)

which, after rephrasing in connected quantities, gives eq. (2.12).

**B. Proof of the Solution \( v_0(p) \) for Eq. (3.4)**

Using eqs. (3.2) and (3.5) the right hand side of eq. (3.4) reads

\[ u_0(p) v_0(p) = \oint_{C_1} \frac{d\omega}{2\pi i} \oint_{C_2} \frac{dz}{2\pi i} \left[ \frac{V'(\omega) \Psi(z)}{(p-\omega)(p-z)} \right] \left[ \frac{(z-x)(z-y)}{(\omega-x)(\omega-y)} \right]^{1/2} + \oint_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} \frac{\chi}{[(\omega-x)(\omega-y)]^{1/2}}, \] (B.1)

and the left hand becomes

\[ \oint_{C_1} \frac{d\omega}{2\pi i} \oint_{C_2} \frac{dz}{2\pi i} \frac{V'(\omega) \Psi(z)}{(p-\omega)(p-z)} \left[ \frac{(z-x)(z-y)}{(\omega-x)(\omega-y)} \right]^{1/2} + \oint_{C_1} \frac{d\omega}{2\pi i} \frac{V'(\omega)}{p-\omega} \frac{\chi}{[(\omega-x)(\omega-y)]^{1/2}} + \oint_{C_1} \frac{d\omega}{2\pi i} \oint_{C_2} \frac{dz}{2\pi i} \frac{\Psi(\omega) V'(z)}{(p-\omega)(p-z)} \left[ \frac{(\omega-x)(\omega-y)}{(z-x)(z-y)} \right]^{1/2}. \] (B.2)

Now in the last term pull the contour integral \( C_2 \) over the curve \( C_1 \). One can show that the contribution from the extra pole vanishes. After renaming \( \omega \leftrightarrow z \) and combining the first and third terms one gets eq. (B.1). Thus eq. (3.4) is verified.
References

[1] P. Di Francesco, P. Ginsparg and J. Zinn–Justin, 2D Gravity and Random Matrices, LANL preprint LA–UR–93–1722, hep–th/9306153.

[2] L. Alvarez–Gaumé, H. Itoyama, J. L. Mañes and A. Zadra, Int. J. Mod. Phys. A7 (1992) 5337.

[3] R. Dijkgraaf, H. Verlinde, E. Verlinde, Nucl. Phys. B348 (1991) 435; Yu. Makeenko, A. Marshakov, A. Mironov, A. Morozov, Nucl. Phys. B356 (1991) 574.

[4] L. Alvarez–Gaumé, K. Becker and M. Becker, R. Emparan, J. Mañes, Int. J. Mod. Phys. A8 (1993) 2297.

[5] A. Zadra and E. Abdalla, Nucl. Phys. B432 (1994) 163.

[6] A. A. Migdal, Phys. Rep. 102 (1983) 199.

[7] S. R. Wadia, Phys. Rev. D24 (1981) 970; V. Kazakov, Mod. Phys. Lett. A4 (1989) 2125; F. David, Mod. Phys. Lett. A5 (1990) 1019.

[8] Yu. Makeenko, Mod. Phys. Lett. (Brief Reviews) A6 (1991) 1901.

[9] K. Becker and M. Becker, Mod. Phys. Lett. A8 (1993) 1205.

[10] I. N. McArthur, Mod. Phys. Lett. A8 (1993) 3355.

[11] J. Ambjørn, L. Chekhov, C. F. Kristjansen and Yu. Makeenko, Nucl. Phys. B404 (1993) 127.

[12] J. Ambjørn, L. Chekhov and Yu. Makeenko, Phys. Lett. B282 (1992) 341.