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Abstract

It is shown that the inclusive rate of the rare weak radiative decays $B \to X_s \gamma$ contains a series of nonperturbative corrections, whose ‘short distance’ scale is set by $m_c^{-1}$, rather than by $m_b^{-1}$. The first correction in this series is expressed through the chromomagnetic interaction of the $b$ quark inside the $B$ meson and the relative magnitude of the effect is determined by the ratio $(\langle B|\bar{b}\sigma \cdot G b|B\rangle/m_c^2$. Though the magnitude of this first correction is suppressed by a numerical coefficient, the sensitivity of the decay rate to the distance scale $m_c^{-1}$ may significantly limit the accuracy of purely perturbative predictions for the rate.
The rare radiative decays $B \to X_s \gamma$ associated with the underlying FCNC quark process $b \to s \gamma$ are well known to be sensitive to the top quark mass as well as to a possible ‘new physics’ that might exist beyond the Standard Model. The current state of the experimental knowledge about these processes and of their understanding within the Standard Model can be summarized by the measured value of the inclusive branching ratio of these decays

$$B(B \to X_s \gamma) = (2.32 \pm 0.57 \pm 0.35) \times 10^{-4} \quad (1)$$

and by the perturbative Standard Model quark decay rate, whose calculation has been recently taken to the next-to-leading order (NLO) in $\alpha_s \ln(m_W/m_b)$,

$$B(b \to X_s \gamma) = (3.28 \pm 0.33) \times 10^{-4} \quad (2)$$

Additionally, it has been argued that the nonperturbative effects, which distinguish between the heavy quark decay and the decay of a heavy meson, are of relative magnitude $\Lambda_{QCD}/m_c^2$. Thus in the subsequent studies (see Ref. and references therein) it has been assumed that the perturbative total quark decay rate approximates the total inclusive meson decay rate with accuracy better than 10%, though the $O(1/m_b^2)$ nonperturbative effects are naturally expected to be significant in the photon spectrum near the endpoint.

The purpose of this paper is to point out that there in fact exists a series of nonperturbative corrections to the inclusive rate of the decays $B \to X_s \gamma$, whose ‘short distance’ scale is set by $m_c^{-1}$, rather than by $m_b^{-1}$. This series thus generates an expansion for the corrections to the decay rate in powers of $\Lambda_{QCD}/m_c$ and of $\Lambda_{QCD} m_b/m_c^2$. The first term in this expansion is $O(\Lambda_{QCD}^2/m_c^2)$ and is found in terms of the known chromomagnetic energy of the $b$ quark. The subsequent terms, however are expressed trough unknown matrix elements of operators of higher dimension. The relative magnitude of the calculable first correction to the total decay rate is found here in the lowest order in $\alpha_s$ and is given by

$$\frac{\delta \Gamma(B \to X_s \gamma)}{\Gamma(b \to X_s \gamma)} = \frac{1}{27 C_7} \frac{\mu_g^2}{m_c^2} \approx -0.025 \quad (3)$$

where $\mu_g^2$ is the standard parameter for the strength of the chromomagnetic interaction of the $b$ quark inside the hadron: $\mu_g^2 \equiv \langle B|b g \sigma_{\mu\nu} G_{\mu\nu}^a (\lambda^a/2)|B\rangle/2 = 3 (M_B^2 - M_{B^{*}}^2)/4 \approx 0.4 \text{GeV}^2$. (The nonrelativistic normalization for the heavy quark states is assumed throughout this paper, so that $\langle B|b^\dagger b|B\rangle = 1$.) The quantity $C_7$ in the denominator in eq.(3) is the coefficient of the operator

$$P_7 = \frac{e}{16\pi^2} m_b (\bar{s}_L \sigma_{\mu\nu} b_R) F_{\mu\nu} \quad (4)$$
in the perturbative effective Lagrangian for the $b$ quark decay normalized at a scale $\mu \sim m_b$:

$$L_{\text{eff}} = \frac{4 G_F}{\sqrt{2}} V_{ts}^* V_{tb} \sum_{i=1}^{8} C_i(\mu) P_i(\mu) ,$$  \hspace{1cm} (5)

and the values $C_7 = -0.30$ \footnote{The notation and conventions of Ref.\cite{2} are used here as well as the numerical value of $C_7(m_b)$. Furthermore, eq.(3) is written in the approximation where $\Gamma(b \to X_s \gamma) = \Gamma(b \to s \gamma)$ and the latter rate is generated only by the operator $P_7$. In NLO the rate also receives a small contribution from other operators in $L_{\text{eff}}$. However, since the correction is calculated here only in the leading order in $\alpha_s$, taking into account those additional contributions to $\Gamma(b \to X_s \gamma)$ would exceed the accuracy of the present calculation.} and $m_c \approx 1.4$ GeV are used in the numerical estimate in eq.(3)\footnote{\ref{2}}. Clearly, the results of the present paper, in particular eq.(3), are at variance with the generally accepted viewpoint that the relative magnitude of the nonperturbative corrections to the inclusive decay rate of $B \to X_s \gamma$ goes to zero in the limit $m_b \to \infty$. The latter behavior would be true if the relevant contributions to the $B$ meson decay were exhausted by the effective Lagrangian in eq.(5), which is the case for the perturbative $b$ quark decay.

At the level of nonperturbative effects there are additional terms in the effective Lagrangian, among which there are the ones of the type considered in this paper, whose distance scale is set by $m_c^{-1}$, so that their (relative) contribution does not vanish in the limit $m_b \to \infty$. The correction in eq.(3) is small due to a small numerical factor. Apriori there is no reason to expect that similar coefficients are small in subsequent terms. Moreover, the subsequent terms contain average values of operators of the generic form $(\bar{b}(q D)^n Gb)/m_c^{2n+2}$, where $q$ is the momentum of the gluon and the covariant derivative $D$ is acting on the gluon field tensor $G$. After averaging over the phase space in the decay these contributions produce series in $(\Lambda_{QCD}^2/m_c^2) (\Lambda_{QCD} m_b/m_c)^n$. Since in reality the parameter $\Lambda_{QCD} m_b/m_c^2$ is of order one, these terms bring a considerable uncertainty in predictions for the decay rate.

The consideration of the nonperturbative effects in the inclusive decay rate is performed using the standard method\cite{6,7}. Namely, both $m_b$ and $m_c$ are assumed to be large: $m_b, m_c \gg \Lambda_{QCD}$, and one performs the operator product expansion in the inverse powers of the heavy quark masses for the effective operator

$$T = 2 \text{Im} \left[ i \int d^4x e^{iqx} T \left\{ L_{\text{eff}}^\dagger(x), L_{\text{eff}}(0) \right\} \right]. \hspace{1cm} (6)$$

The total decay rate for a heavy hadron $X_b$ is then given by the diagonal matrix element of $T$ over $X_b$:

$$\Gamma(X_b) = \langle X_b | T | X_b \rangle . \hspace{1cm} (7)$$

The notation and conventions of Ref.\cite{2} are used here as well as the numerical value of $C_7(m_b)$. Furthermore, eq.(3) is written in the approximation where $\Gamma(b \to X_s \gamma) = \Gamma(b \to s \gamma)$ and the latter rate is generated only by the operator $P_7$. In NLO the rate also receives a small contribution from other operators in $L_{\text{eff}}$. However, since the correction is calculated here only in the leading order in $\alpha_s$, taking into account those additional contributions to $\Gamma(b \to X_s \gamma)$ would exceed the accuracy of the present calculation.
Different types of decay are separated within this method by picking up the appropriate terms in $L_{\text{eff}}$ to correlate in eq. (1). The leading term in the expansion generates the perturbative ‘parton’ decay rate, while the terms suppressed by inverse powers of heavy quark masses describe the nonperturbative corrections to the decay rate.

In the leading logarithm approximation (LLA) the perturbative quark decay rate of $B \to X_s \gamma$ is obtained by retaining in $L_{\text{eff}}$ only the term with the operator $P_7$ in eq.(5) and substituting it in the correlator $T$ in eq.(6) with the result

$$T_{s\gamma}^{(0)} = \frac{\alpha}{32\pi^4} G_F^2 m_b^5 |V_{ts}^* V_{tb}|^2 |C_7|^2 (\bar{b} b) .$$

(8)

This expression gives the standard perturbation theory formula for the inclusive rate of $B \to X_s \gamma$ upon using noticing that up to terms $O(m_b^{-2})$ one has $\langle B | (\bar{b} b) | B \rangle = 1$.

The reasoning here for terms suppressed by only $m_c^{-2}$ rather than by $m_b^{-2}$ is based on considering the ‘gluon - photon penguin’ type of mechanism, which provides additional to eq.(5) terms in the effective Lagrangian. This mechanism arises through graphs of the type shown in Fig. 1. The gluon momentum $k$ in this graph is assumed to be small, and the expansion in this momentum gives rise to a series of operators in $L_{\text{eff}}$ with the gluon field strength tensor $G$, and its derivatives. One can readily see, that already starting from first such term the loop integration for the coefficients in this expansion is convergent in the local four-fermion limit of the weak interaction. In particular, the corresponding graph with the top quark becomes irrelevant. In the local limit of the weak interaction the loop with the $c$ quark is kinematically equivalent (after the Fierz transform of the four-fermion vertex) to the $c$ quark loop for coupling of axial current to two vector currents. Since the photon momentum $q$ is on-shell: $q^2 = 0$, and the gluon momentum is infinitesimal, the external kinematical invariants for the loop, $q^2$, $k^2$, and $(q + k)^2$ are either zero or infinitesimal. Thus the coefficients of expansion of the loop in powers of $k$ are determined by the mass of the quark in the loop, i.e. by $m_c$. Furthermore, an expansion of the loop in the invariant $(q + k)^2$ gives rise to terms with powers of the ratio $(q \cdot k)/m_c^2$, resulting in the operator terms of the form $(\bar{b} (q D)^n G b)/m_c^{2n+2}$, which give rise to corrections to the rate of the relative magnitude $(\Lambda_{\text{QCD}}^2/r_{\text{QCD}} m_b/m_c^2)^n$.

By calculating the graph of Fig. 1 and the one with the photon and the gluon permuted, and expanding to the first order in the gluon momentum $k$, one explicitly finds the first term in this series of corrections as an additional contribution to $L_{\text{eff}}$ of the form

$$L_{\text{eff}}^{(s\gamma)} = \frac{\epsilon}{16 \pi^2} \sqrt{2} G_F V_{cs}^* V_{cb} (s_L \gamma_\mu \frac{\lambda^a}{2} b_L) \frac{1}{3m_c^2} g G_\mu^a \epsilon_{\mu\rho\sigma} \partial_\lambda F_{\rho\sigma} .$$

(9)
where \( Q_c = \frac{2}{3} \) is the electric charge of the charmed quark and the expression is written under the convention about the sign of \( G_F \) as in Ref.\[3\], i.e. where the bare four-fermion Lagrangian is written as

\[
L(b \rightarrow c \bar{c} s) = -(4G_F/\sqrt{2})V^*_{cs} V_{cb} \bar{s}_L \gamma_\mu c_L \bar{c}_L \gamma_\mu b_L.
\]

When correlated according to eq.(3) through the on-shell (massless) \( s \) quark and the photon with the term with the operator \( P_7 \) in \( L_{\text{eff}} \) of eq.(5) the newly found term \( L^{(s g \gamma)} \) gives an interference contribution to the effective operator \( T \). This contribution is found here as

\[
T^{(1)}_{\gamma} = -\frac{\alpha}{32 \pi^4} G_F^2 m_b^5 \frac{1}{27} \Re \left[V^*_{cs} V_{cb} V^*_{ts} V_{tb} C_7\right] \frac{\bar{b} g \sigma_{\mu\nu} G_{\mu\nu} \frac{\lambda_c}{2} b}{2 m_c^2}.
\]

Ignoring the terms in the unitarity relation for quark mixing associated with the \( u \bar{u} \) pair contribution, which are of the relative order \( |V_{ub}/V_{cb}| \sin \theta_c \), one can write

\[
V^*_{cs} V_{cb} = -V^*_{ts} V_{tb}.
\]

Then a comparison of the equations (10) and (8) gives the expression in eq.(3) for the relative nonperturbative correction.

Proceeding to a discussion, we first remark that the ignored contribution of the \( u \bar{u} \) pair in fact leads to a hopefully small, but presently uncalculable effect in the rate of \( B \rightarrow X_s \gamma \).

The smallness is obviously related to the suppression in the mixing, while the uncertainty of the contribution arises, when one considers the same loop as in Fig. 1, but with the \( u \) quark instead of the charmed one. In no approximation can the mass of the \( u \) quark be considered as large. Thus this mechanism gives rise to an essentially long distance contribution to the inclusive rate\[3\].

Returning to the discussion of the corrections arising from distances \( O(m_c^{-1}) \), it can be noticed that in general the effect of such distances can also significantly affect the spectrum of inclusive photons through enlarged distance scale for the amplitude of the photon emission. However any detailed consideration of this effect is beyond the scope of the present paper.

The particular numerical value in eq.(3) for the relative correction to the inclusive rate is most certain to be modified by the subsequent terms with derivatives of the gluonic tensor. Apriori those terms have no parametric smallness in comparison with the expression in eq.(3), since the additional factors in those terms are given by powers of the parameter \( \Lambda_{QCD} m_b/m_c^2 \sim O(1) \). The principal difficulty of evaluating those subsequent terms is in that we have no knowledge of the matrix elements of operators with derivatives of the gluonic tensor over the \( B \) meson. Therefore this series of the corrections puts an essential limit on

\[2\text{Possible long distance effects were in fact discussed within a different technique in the literature}[8]. I believe that the used here systematic method of OPE provides a more reliable approach to calculable terms and leaves undetermined only a small contribution due to the light \( u \bar{u} \) pair, which is strongly suppressed by the relative factor \( |V_{ub}/V_{cb}| \sin \theta_c \approx 0.02 \).}
the present calculability of the decay rate.

It can be noted that the presence of a contribution in the rate due to the chromomagnetic interaction of the $b$ quark can in principle be verified experimentally by measuring also analogous inclusive rate for the $\Lambda_b$ baryon: $\Gamma(\Lambda_b \rightarrow X_s \gamma)$. In $\Lambda_b$ the spin of the light component of the baryon is zero, thus the average of the chromomagnetic operator vanishes. Therefore for $\Lambda_b$ the discussed correction is absent, and the difference between the inclusive decay rates $\Gamma(B \rightarrow X_s \gamma) - \Gamma(\Lambda_b \rightarrow X_s \gamma)$ is dominated by the spin-dependent terms in the discussed corrections with further small corrections being $O(m_b^{-2})$ and analogous to those found\cite{5} for the dominant decays. Thus this difference in rates can serve as a measure of importance of the nonperturbative corrections coming from distances $O(m_c^{-1})$.

In summary, it has been shown that there exists a set of nonperturbative corrections to the inclusive rate of the decays $B \rightarrow X_s \gamma$, which are determined by the distance scale $O(m_c^{-1})$. The relative magnitude of these corrections is not vanishing in the limit $m_b \rightarrow \infty$, provided that $m_c$ is kept fixed. The first and calculable, although not necessarily dominant, among these corrections is of the order $O(m_c^{-2})$ and is given in the lowest order in $\alpha_s$ by eq.(3). The presence of the distance scale $m_c^{-1}$ in the ‘penguin’ type decays of the $b$ hadrons may have further implication for the photon radiative decays as well as for the nonleptonic decays of these hadrons.

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**Figure caption.**

**Figure 1.** A graph for the ‘gluon-photon penguin’. The heavy dot represents the four-fermion weak interaction. At small momentum $k$ of the gluon the scale for the expansion in powers of $k$ is set by $m_c$. 
Figure 1