DIOPHANTINE PROPERTIES OF IETS AND GENERAL SYSTEMS: QUANTITATIVE PROXIMALITY AND CONNECTIVITY

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ABSTRACT. We present shrinking targets results for general systems with the emphasis on applications for IETs (interval exchange transformations) \((I, T), I = [0, 1]\). In particular, we prove that if an IET \((I, T)\) is ergodic (relative to the Lebesgue measure \(\lambda\)), then the equality

\[
\liminf_{n \to \infty} n |T^n(x) - y| = 0
\]

holds for \(\lambda \times \lambda\)-a.a. \((x, y) \in I^2\). The ergodicity assumption is essential: the result does not extend to all minimal IETs. The factor \(n\) in (A1) is optimal (e.g., it cannot be replaced by \(n \ln(\ln(\ln(n))\).

On the other hand, for Lebesgue almost all 3-IETs \((I, T)\) we prove that for all \(\epsilon > 0\)

\[
\liminf_{n \to \infty} n^\epsilon |T^n(x) - T^n(y)| = \infty,
\]

for Lebesgue a.a. \((x, y) \in I^2\).

This should be contrasted with the equality \(\liminf_{n \to \infty} |T^n(x) - T^n(y)| = 0\), for a.a. \((x, y) \in I^2\), which holds since \((I^2, T \times T)\) is ergodic (because generic 3-IETs \((I, T)\) are weakly mixing).

We also prove that no 3-IET is strongly topologically mixing.

CONTENTS

1. Introduction
2. Notation and Definitions
2.1. Systems and Invariant Measures
2.2. Scale Sequences and Gauges.
3. The extremality of the connectivity and the proximality gauges
3.1. Properties of Scale Sequences
3.2. The Extremality Gauge Theorem (EGT)
3.3. Proximality Constant of a Weakly Mixing m.m.p.-system
4. Results on IETs
4.1. A brief introduction
4.2. The Connectivity Gauges
4.3. Results on Proximity Gauges
4.4. Absence of Topological Mixing for 3-IETs
5. Contact Gauges and Decisive Measures
6. From Quantitative to Connectivity Gauges in General Systems
6.1. Quantitative Recurrence Results, Review
6.2. Quantitative Connectivity Results
7. Two Results on the Proximality Constant
8. Proof of Theorem 3
9. Proof of Theorem 5
10. Proof of Theorem 4
11. The \(\tau\)-entropy of an IET
12. No 3-IET is Topologically Mixing
13. Questions
References
1. Introduction

Let \((X, d)\) be a metric space with a Borel probability measure \(\mu\) on it. Let \(\alpha \geq 0\) be a constant. Let \(T: X \to X\) be a Borel measurable map preserving measure \(\mu\). We study the distribution of the following functions (called the connectivity, proximality and recurrence gauges, respectively):

- \(\phi_\alpha(x, y) = \lim_{n \to \infty} n^\alpha d(T^n(x), y), \quad x, y \in X;\)
- \(\psi_\alpha(x, y) = \lim_{n \to \infty} n^\alpha d(T^n(x), T^n(y)), \quad x, y \in X;\)
- \(\rho_\alpha(x) = \phi(x, x) = \lim_{n \to \infty} n^\alpha d(T^n(x), x), \quad x \in X.\)

We show that for \(\alpha > 0\) the values taken by the first two gauges \(\mu \times \mu\)-almost surely lie in the set \(\{0, \infty\}\) (Theorem 4, Corollary 11); under some mild conditions all three gauges are almost surely constant (Section 6).

If \(X\) is a smooth manifold with a probability Borel measure \(\mu\) on it and if \(T: X \to X\) is Borel measurable, \(\mu\)-ergodic map which is differentiable (not necessarily continuously) \(\mu\)-almost everywhere then for any \(\alpha > 0\) either \(\phi_\alpha(x, y) = 0\) a.s., or \(\phi_\alpha(x, y) = \infty\) a.s.. (See Corollary 6 in Section 4 for more general result; here “a.s.” stands for “for \(\mu \times \mu\)-a.a. pairs \((x, y)\) \in X^2\)).

The above general results are applied to IETs (interval exchange transformations) and more accurate results are obtained. In particular, if \(T\) is an IET and \(\mu\) is a \(T\)-ergodic measure then the result quoted in the previous paragraph implies only that each of the gauges \(\phi_\alpha(x, y)\) is a.s. a constant, either 0 or \(\infty\). It turns out (Theorem 3) that for \(\alpha = 1\) (in the setting of \(\mu\)-ergodic IETs) the constant is always 0:

\[
\phi_1(x, y) = \lim_{n \to \infty} n d(T^n(x), y) = \lim_{n \to \infty} n|T^n(x) - y| = 0 \quad \text{(mod } \mu \times \mu).\]

The conditions of the above statement are proper. The measure \(\mu\) cannot be replaced by Lebesgue measure, even for minimal IETs (Theorem 4). Moreover, the “scale sequence” \(n\) in the statement of this result is optimal (Theorem 5); in particular, it cannot be replaced by \(n \log(\log n)\). Related issues have been considered in [30], [32], [2] and [17]. This is discussed in Section 11.

We also prove that for any \(\alpha > 0\) the relation \(\psi_\alpha(x, y) = \infty\) holds for Lebesgue almost all 3-IETs and for Lebesgue almost all pairs \((x, y)\) (Theorem 6). One should mention that almost every 3-IET is weakly mixing [24] and therefore \(\psi_0(x, y) = 0\) holds for almost all pairs \((x, y)\). This is also discussed in Section 4.

Lastly, we show that no 3-IET is topologically mixing (Theorem 13). (Note that the second author J. Chaika has recently constructed a topologically mixing 4-IET [18].)

Our motivation in studying the gauges \(\phi\) and \(\psi\) partially came from [8] where the connection between the Hausdorff dimension of the invariant measure \(\mu\) and the distribution of the recurrence gauge \(\rho\) was established (see Proposition 11 for a related result).

In this paper we establish a variety of general results for the connectivity and proximality gauges (Sections 4 and especially 4.6) in addition to strong results for IETs that are of independent interest and showcase the applications of these methods (Sections 4.7 and 11).

2. Notation and Definitions

Denote by \(\mathbb{R}, \mathbb{Z}\) the sets of real and integer numbers, respectively. Denote by \(I\) the unit interval (circle) \(I = [0, 1) = \mathbb{R} \mod \mathbb{Z}\) and by \(\lambda\) the Lebesgue measure on it.

2.1. Systems and Invariant Measures. In what follows, \(X, Y\) are separable metrizable spaces.

- \(B(X)\) stands for the family \((\sigma\text{-algebra})\) of Borel subsets of \(X\).
- \(\mathcal{P}(X)\) stands for the set of Borel probability \(\sigma\text{-additive}\) measures on \(X\) (defined on \(B(X)\)).

For a measure \(\mu \in \mathcal{P}(X)\), we write \(\text{supp}(\mu) \subset X\) for the (minimal closed) support of \(\mu\). If \(\text{supp}(\mu)\) is a singleton \(\{x\} \subset X\), the measure \(\mu\) is called atomic and is denoted by \(\delta_x\).

A map \(f: X \to Y\) is called measurable if it is Borel measurable, i.e. if \(A \in B(Y) \Rightarrow f^{-1}(A) \in B(X)\). Given such a map, denote by \(f^*: \mathcal{P}(X) \to \mathcal{P}(Y)\) the (push forward) map defined by the formula

\[
(f^*(\mu))(B) = \mu(f^{-1}(B)) \quad \text{(for all } \mu \in \mathcal{P}(X), B \in \mathcal{B}(Y)),
\]
and denote by $\hat{f}(\mu)$ the support of the measure $f^*(\mu)$:

$$(2.1b) \quad \hat{f}(\mu) = \text{supp}(f^*(\mu)) \subset Y \quad (\text{for } \mu \in \mathcal{P}(X)).$$

If $f^*(\mu) = \delta_y$ for some $y \in Y$ (i.e. if $f^*(\mu)$ is an atomic measure), $f$ is called $\mu$-constant (or just a. s. constant if there is no doubt as to the measure implied).

For two measurable maps $f,g : X \to Y$ we write $f = g \pmod{\mu}$ to signify the relation $f(x) = g(x)$ for $\mu$-a. a. $x \in X$.

By a metric system $(X,T) = (X,d,T)$ we mean a metric space $X = (X,d)$ together with a measurable map $T : X \to X$. Denote by

$$\mathcal{P}(X,T) = \mathcal{P}(T) = \{ \mu \in \mathcal{P}(X) \mid T^*(\mu) = \mu \}$$

the set of $T$-invariant Borel probability measures on $X$. (Note that $\mathcal{P}(X,T) = \emptyset$ is possible.)

The following abbreviations are used:

- **m. m.-space** (metric measure space $(X,\mu)$) – a metric space $X$ (together with a measure $\mu \in \mathcal{P}(X)$, $\mu$-ergodic if $\mu(S) = \{ 0,1 \}$ for every $T$-invariant measurable set $S \in \mathcal{B}(X)$). An equivalent condition is that there is no presentation $\mu = \frac{\mu_1 + \mu_2}{2}$ with $\mu_1, \mu_2 \in \mathcal{P}(X)$, $\mu_1 \neq \mu_2$.

- **m. m. p.-system** (metric measure preserving system $(X,T,\mu)$) – a metric system $(X,T)$ with an invariant measure $\mu \in \mathcal{P}(X,T)$.

Given a metric space $(X,T)$, a measure $\mu \in \mathcal{P}(T)$ is called $T$-ergodic if $\mu(S) \in \{ 0,1 \}$ for every $T$-invariant measurable set $S \in \mathcal{B}(X)$. An equivalent condition is that there is no presentation $\mu = \frac{\mu_1 + \mu_2}{2}$ with $\mu_1, \mu_2 \in \mathcal{P}(X)$, $\mu_1 \neq \mu_2$.

### 2.2. Scale Sequences and Gauges.

**Definition 1.** By a scale sequence we mean a sequence $s = \{ s_n \}_1^\infty$ of positive real numbers such that $\lim_{n \to \infty} s_n = \infty$. For $\alpha > 0$, the sequence $s_\alpha = \{ n^\alpha \}_1^\infty$ will be referred to as the $\alpha$-power sequence.

**Definition 2.** Let $(X,d)$ be a metric space. By a gauge (on $X$) we mean a measurable map $f : X \to [0,\infty]$. Let $\mu \in \mathcal{P}(X)$ be a measure. A gauge $f$ is called

- **$\mu$-extreme:** if $\hat{f}(\mu) = \text{supp}(f^*(\mu)) \subset \{ 0,\infty \}$ (i.e. if $\mu(f^{-1}(\{ 0,\infty \}) = 1$);
- **$\mu$-constant:** if $f^*(\mu)$ is an atomic measure, i.e. if $f^*(\mu) = \delta_c$ for some $c \in [0,\infty]$;
- **$\mu$-trivial:** if $f^*(\mu) \in \{ \delta_0, \delta_\infty \}$.

When using the above terminology we suppress referring to the measure $\mu$ if there are no doubts as to the measure implied. Thus, given a measure $\mu \in \mathcal{P}(X)$, a gauge on $X$ is trivial if and only if it is both extreme and constant.

Given a metric system $(X,T)$ and a scale sequence $s = \{ s_n \}_1^\infty$, two gauges $\phi, \psi$ on $X^2 = X \times X$ and one gauge $\rho$ on $X$ are introduced as follows:

$$(2.2a) \quad \phi(x,y) = \Phi(x,y,s,T) = \liminf_{n \to \infty} s_n d(T^n(x),y) \quad (\text{the connectivity gauge}),$$

$$(2.2b) \quad \psi(x,y) = \Psi(x,y,s,T) = \liminf_{n \to \infty} s_n d(T^n(x),T^n(y)) \quad (\text{the proximality gauge}),$$

$$(2.2c) \quad \rho(x) = \Phi(x,x,s,T) = \liminf_{n \to \infty} s_n d(T^n(x),x) \quad (\text{the recurrence gauge}).$$

The above gauges are measurable (because $T$ is), so that, given measures $\mu, \nu \in \mathcal{P}(X)$, the following measures in $\mathcal{P}([0,\infty])$

$$(2.2d) \quad \phi^*(\mu,\nu) \overset{\text{def}}{=} \phi^*(\mu \times \nu), \quad \psi^*(\mu,\nu) \overset{\text{def}}{=} \psi^*(\mu \times \nu), \quad \rho^*(\mu)$$

and the subsets $\hat{\phi}(\mu,\nu), \hat{\psi}(\mu,\nu), \hat{\rho}(\mu) \subset [0,\infty]$ are well defined (see 2.1; here $\mu \times \nu \in \mathcal{P}(X^2)$ stands for the product measure).

We often add subscript $\alpha > 0$ to functional symbols ($\phi, \psi$ or $\rho$) of the above three gauges to specify the $\alpha$-power sequence $s = s_\alpha = \{ n^\alpha \}_1^\infty$ to be used as a scale sequence:

$$(2.3a) \quad \phi_\alpha(x,y) = \Phi(x,y,s_\alpha,T) = \liminf_{n \to \infty} n^\alpha d(T^n(x),y) \quad (\text{the } \alpha\text{-connectivity gauge}),$$

$$(2.3b) \quad \psi_\alpha(x,y) = \Psi(x,y,s_\alpha,T) = \liminf_{n \to \infty} n^\alpha d(T^n(x),T^n(y)) \quad (\text{the } \alpha\text{-proximality gauge}),$$

$$(2.3c) \quad \rho_\alpha(x) = \Phi(x,x,s_\alpha,T) = \liminf_{n \to \infty} n^\alpha d(T^n(x),x) \quad (\text{the } \alpha\text{-recurrence gauge}).$$
Note that the equations (2.3) make sense and define gauges for \( \alpha = 0 \) (even though the constant sequence \( s_n = 1 \) is not a scale sequence).

A remarkable fact is that for an arbitrary m.m.p.-system \((X, T, \mu)\) and all \( \alpha > 0 \), the \( \alpha \)-connectivity and \( \alpha \)-proximity gauges \((\phi_\alpha, \psi_\alpha)\) are always extreme (Theorem 1) while the same assertion does not need to hold for the \( \alpha \)-recurrence gauges \( \rho_\alpha \) (Example 1, page 5).

Given a m.m.p.-system \((X, T, \mu)\), we define three constants (taking values in the set \([0, \infty]\)) in the following way:

\[
\begin{align*}
(2.4a) & \quad C_\phi = C_\phi((X, T, \mu)) = \sup \{0 \cup \{\alpha > 0 | \phi_\alpha^*(\mu, \mu) = \delta_0\}\} \quad \text{(the connectivity constant)} \\
(2.4b) & \quad C_\psi = C_\psi((X, T, \mu)) = \sup \{0 \cup \{\alpha > 0 | \psi_\alpha^*(\mu, \mu) = \delta_0\}\} \quad \text{(the proximity constant)} \\
(2.4c) & \quad C_\rho = C_\rho((X, T, \mu)) = \sup \{0 \cup \{\alpha > 0 | \rho_\alpha^*(\mu) = \delta_0\}\} \quad \text{(the recurrence constant)}
\end{align*}
\]

According to notation in (2.1), (2.2d) and (2.3), \( \phi_\alpha^*(\mu, \mu) = \delta_0 \) is a shortcut for \( \phi_\alpha(x, y) = 0 \) (mod \( \mu \times \mu \)). Both expressions mean that \( \phi_\alpha(x, y) = \lim_{n \to \infty} n^\alpha d(T^n(x), y) = 0 \), for \( \mu \times \mu \)-a. a. \( x, y \in X \). The same interpretation applies for the expressions \( \psi_\alpha^*(\mu, \mu) = \delta_0 \) and \( \rho_\alpha^*(\mu) = \delta_0 \).

The recurrence constant has been previously connected to the Hausdorff dimension of a system \([8]\) and to entropy for symbolic dynamical systems \([36]\). This paper links the connectivity and proximity constants to the Hausdorff dimension (Proposition 11 and Remark 8).

It also links the proximity constant to complexity properties (Propositions 12 and 13).

The terminology chosen in the paper (connectivity, proximity and recurrence) is suggestive: larger gauge constants indicate stronger manifestation of the corresponding property. For instance the proximity constant of any distal system is 0.

3. The extremality of the connectivity and the proximity gauges

In this section we prove that the “power” gauges \( \phi_\alpha \) and \( \psi_\alpha \) are always extreme (Theorem 1). The main tool is the EGT (the Extremality Gauge Theorem, Theorem 2), a result of independent interest. The proof of the EGT is based on the method of “a single orbit analysis” in the spirit of the book \([48]\).

3.1. Properties of Scale Sequences. Many results regarding the gauges \( \phi, \psi \) and \( \rho \) will be proved in a larger generality than scales sequences \( s = s_\alpha = \{n^\alpha\} \). We need the following definitions detailing some properties a scale sequence may satisfy.

**Definition 3.** A scale sequence \( s = \{s_n\}^\infty_1 \) is called:

- monotone: if \( s_{n+1} \geq s_n \), for \( n \in \mathbb{N} \) large enough,
- steady: if \( \lim_{n \to \infty} \frac{s_{n+1}}{s_n} = 1 \),
- two-jumpy: if \( s \) is monotone and if \( \lim_{n \to \infty} \frac{s_{n+1}}{s_n} > 1 \),
- bounded-ratio: if \( \sup_{n \geq 1} \frac{s_{n+1}}{s_n} < \infty \),
- nice: if \( s \) is two-jumpy and bounded-ratio.

Power scale sequences \( s_\alpha \) are “all of the above”: monotone, steady, two-jumpy, bounded-ratio and nice. The sequence \( \{2^n\} \) is nice but not steady. The sequence of primes \( \{2, 3, 5, \ldots\} \) is “all of the above”.

**Theorem 1.** Let \((X, T)\) be a metric system and let \( s = \{s_n\} \) be a two-jumpy scale sequence. Let both \( \mu, \nu \in P(X, T) \) be \( T \)-invariant measures. Then the connectivity and the proximity gauges \((\phi, \psi)\) are both \( \mu \times \nu \)-extreme. In particular, for all \( \alpha > 0 \), both \( \phi_\alpha \) and \( \psi_\alpha \) are \( \mu \times \nu \)-extreme.

**Corollary 1.** Let \((X, T, \mu)\) be a m.m.p-system and let \( s = \{s_n\} \) be a two-jumpy scale sequence. Then both gauges \( \phi, \psi \) are extreme (i.e., \( \mu \times \nu \)-extreme). In particular, for all \( \alpha > 0 \), both \( \phi_\alpha, \psi_\alpha \) are extreme.

As the following example shows, the analogue of the Theorem 1 fails for the recurrence gauge. In what follows, we write \( ||x|| = \min_{n \in \mathbb{Z}} |x - n| \), for \( x \in \mathbb{R} \).
Example 1. Let \((I, T)\) be the golden mean rotation \(T(x) = x + \alpha \pmod{1}\), \(\alpha = \frac{\sqrt{5} - 1}{2}\), of the unit interval (circle) \(I = [0, 1)\). Then \(\rho_1(x) = \liminf_{n \to \infty} n \| n \alpha \| = \frac{1}{\sqrt{5}}\), for all \(x \in I\); see e.g. \cite{EGT} Chap.1, §6. Thus the 1-recurrence gauge for the m.m.p.-system \((I, T, \lambda)\) is constant but not extreme. (Note that the Lebesgue measure \(\lambda \in \mathcal{P}(I, T)\) is \(T\)-invariant).

Theorem 1 follows easily from the EGT (Theorem 2 below).

3.2. The Extremality Gauge Theorem (EGT). One needs the following definition (cf. Definition 2).

Definition 4. Let \((X, \mathcal{B}, \mu)\) be a probability measure space. A map \(f: X \to [0, \infty]\) is called a gauge (on \(X\)) if \(f\) is measurable (i.e., for every open subset \(U \subset \mathbb{R}\), \(f^{-1}(U) \in \mathcal{B}\)). Such an \(f\) is said to be an extreme gauge if \(\mu(f^{-1}(0, \infty)) = 1\).

The following theorem (the Extremality Gauge Theorem) claims that certain procedure to derive new gauges always leads to extreme gauges; this theorem plays an important role in this paper.

Theorem 2 (EGT). Let \((X, \mathcal{B}, \mu, T)\) be a measure preserving system and let \(f: X \to [0, \infty]\) be a gauge. Let \(s = \{s_n\}_{n=1}^{\infty}\) be a two-jumpy scale sequence. Then the gauge \(F(x) = \liminf_{n \to \infty} s_n f(T^n x)\) is extreme.

Remark. The assumption for \(s\) to be two-jumpy is important. The claim of Theorem 2 fails for the scale sequence \(s = (\log(n + 1))_n^{\infty}\) and the golden mean rotation \((I, T, \lambda)\) (see Example 1 above) with \(f(x) = -1/\log \|x\|\). (It is not hard to show that in this case \(F(x) \equiv 1 \pmod{\lambda}\)). In fact there are similar examples for almost every IET \cite{2}. These ‘logarithm laws’ are widely studied and one can view some of this paper’s results as exploring finer analogues.

On the other hand, the EGT (Theorem 2) holds if \(s = \{s_n\}_{n=0}^{\infty}\) is a power scale sequence: \(s_n = n^\alpha\), with some \(\alpha > 0\). More generally, if a sequence \(s = \{s_n\}\) is regularly growing (in the sense e.g. that it is defined as the restriction \(s_n = g(n)\) of a function \(g(x)\) in a Hardy field, see e.g. \cite{3} for introduction in the subject), then a necessary and sufficient condition for a scale sequence \(s\) to be two-jumpy is that \(s_n > n^\alpha\), for some \(\alpha > 0\) and all large \(n\).

Proof of Theorem 1. Set \(Y = X^2\) and consider the maps \(T': Y \to Y\) and \(T'': Y \to Y\) defined by the formula \(T'(x_1, x_2) = (Tx_1, x_2)\) and \(T''(x_1, x_2) = (Tx_1, Tx_2)\). Clearly, \(\mu \times \mu \in \mathcal{P}(Y, T')\) and also \(\mu \times \mu \in \mathcal{P}(Y, T'')\). In order to prove the inclusion \(\hat{\phi}(\mu, \mu) \subset \{0, \infty\}\), one applies Theorem 2 with \(T = T'\), \(X = Y\) and \(f(x) = d(x_1, x_2)\) where \(x = (x_1, x_2) \in Y\). To prove the inclusion \(\psi(\mu, \mu) \subset \{0, \infty\}\), one applies Theorem 2 with \(T = T''\) and with the same \(X\) and \(f (X = Y\) and \(f(x) = d(x_1, x_2))\).

Proof of Theorem 2. Under the standard convention that \(\frac{1}{0} = \infty\) and \(\frac{1}{\infty} = 0\), set the gauges \(g(x) = 1/f(x)\) and \(G(x) = \limsup_{n \to \infty} \frac{g(T^n x)}{s_n}\). Since \(G(x) = 1/F(x)\), it would suffice to prove that the gauge \(G(x)\) is extreme.

Without loss of generality we may assume that the system \((X, T)\) is ergodic (by passing to the ergodic decomposition of \((X, T)\)).

Consider the sets \(S = g^{-1}(\infty) = \{x \in X \mid g(x) = \infty\}\), \(S_1 = \bigcup_{n=0}^\infty T^{-n}(S)\) and \(S_2 = \bigcap_{n=0}^\infty T^{-n}(S_1)\).

If \(\mu(S) > 0\), then \(\mu(S_1) = \mu(S_2) = 1\) in view of the ergodicity of \(T\). Since \(G(x) = \infty\) for all \(x \in S_2\), it follows that \(G^*(\mu) = 0\), i.e. that \(G\) is extreme. It remains to consider the case \(\mu(S_1) = 0\).

By replacing \(X\) with \(X \setminus S_1\), we may assume without loss of generality that \(S = \emptyset\). For \(k \geq 1\), denote \(G_k(x) = \sup_{n \geq 1} \frac{g_k(T^n x)}{s_n}\), where \(g_k(x) = \begin{cases} g(x), & \text{if } g(x) \geq k \\ 0, & \text{otherwise}. \end{cases}\)

Observe that \(\{G_k(x)\}_{k=1}^\infty\) is a non-increasing sequence of gauges on \(X\) pointwise converging to \(G(x)\). On the other hand, \(G(T(x)) \geq G(x)\) (because \(s\) is increasing). By the ergodicity of \(T\), \(G(x)\) must be constant: \(G(x) = c \pmod{\mu}\), for some \(c \in [0, \infty)\). We have to show that \(c \in \{0, \infty\}\).
Assume to the contrary that \( 0 < c < \infty \). Since \( s \) is two-jumpy, we have \( \liminf_{n \to \infty} \frac{s_{2n}}{s_n} > 1 \). Observe that the definition of \( G \) is insensitive to modifications of a finite number of the terms of the sequence \( s \). Therefore we may assume that there exists a constant \( M > 1 \) such that \( \frac{s_{2n}}{s_n} > M \) for all \( n \geq 1 \). Set

\[ \frac{1}{2} > \epsilon = \frac{M-1}{2M} > 0, \quad c' = M(1-\epsilon) c = \frac{(M+1)}{2} c > c, \]

\[ A_k = \{ x \in X \mid G_k(x) \geq c \}, \quad A = \bigcap_{k \geq 1} A_k, \]

\[ B_k = \{ x \in X \mid G_k(x) \geq c' \}, \quad B = \bigcap_{k \geq 1} B_k. \]

Since \( \{G_k(x)\}_{k=1}^{\infty} \) is a non-increasing sequence of functions converging to the constant \( c \) and since \( c' > c \), it follows that the sequences of sets \( (A_k)_{k=1}^{\infty} \) and \( (B_k)_{k=1}^{\infty} \) are non-increasing: \( A_k \supset A_{k+1} \supset A, ~ B_k \supset B_{k+1} \supset B \) \((k \geq 1)\), and that \( \mu(A) = 1 \) and \( \mu(B) = 0 \).

Now let \( x \in A \). For every \( k \geq 1 \), we have \( x \in A_k \) and hence one can select \( n_k \geq 1 \) such that

\[ \frac{g_k(T^{n_k}x)}{s_{n_k}} \geq c(1-\epsilon). \]

Since \( c(1-\epsilon) > 0 \), it follows that \( g(T^{n_k}x) \geq k \) and therefore \( \lim_{k \to \infty} n_k = \infty \). (Here we use the assumption that \( S = g^{-1}(\infty) = \emptyset \).)

Set

\[ I_k = \left[ \frac{n_k}{2}, n_k \right) \cap \mathbb{N} = \{ m \in \mathbb{N} \mid \frac{n_k}{2} \leq m < n_k \}, \quad J_N = \bigcup_{k \geq N} I_k, \quad N \geq 1. \]

We claim that \( T^m(x) \in B_k \) for \( m \in I_k \). Indeed,

\[ \frac{g_k(T^{n_k-m}(T^m x))}{s_{n_k-m}} \geq \frac{g_k(T^{n_k})}{s_{n_k}} \cdot \frac{s_{n_k}}{s_{n_k-m}} \geq c(1-\epsilon) \cdot \frac{s_{2(n_k-m)}}{s_{n_k-m}} \geq c(1-\epsilon)M = c'. \]

Fix \( N \geq 1 \). It follows that \( T^m(x) \in B_N \) for all \( m \in J_N = \bigcup_{k \geq N} I_k \) (since \( \{B_k\} \) is a non-increasing sequence of sets). Note that \( J_N \subset \mathbb{N} \) is a subset of upper density at least \( 1/2 \). Since \( x \in A \) is arbitrary and \( \mu(A) = 1 \), the ergodicity of \( T \) implies \( \mu(B_N) \geq 1/2 \) which is in contradiction with \( \lim_{N \to \infty} \mu(B_N) = \mu(B) = 0. \)

### 3.3. Proximality Constant of a Weakly Mixing m. m. p.-system.

**Proposition 1** (The invariance of the the proximality gauge). Let \((X,T,\mu)\) be a m. m. p.-system and let \( s = \{s_n\}_{k=1}^{\infty} \) be either a monotone or steady scale sequence. Then the proximality gauge

\[ (3.1) \quad \psi(x,y) = \liminf_{n \to \infty} s_n d(T^n x,T^n y), \quad x,y \in X, \]

is \( T \times T \)-invariant: \( \psi(Tx,Ty) \equiv \psi(x,y) \) \((\text{mod } \mu \times \mu)\).

**Proposition 2** (The constancy of the proximality gauge). Let \((X,T,\mu)\) be a weakly mixing m. m. p.-system and let \( s = \{s_n\}_{k=1}^{\infty} \) be either a monotone or steady scale sequence. Then the measure \( \psi^*(\mu,\mu) \) is atomic, i.e., the proximality gauge \( \psi \) (see (2.1)) is \( \mu \times \mu \)-constant. In particular, \( \psi^*(\mu,\mu) = \delta_0 \) if \( 0 \leq \alpha < C_\psi \) and \( \psi^*(\mu,\mu) = \delta_\infty \) if \( C_\psi < \alpha \). \( \psi = C_\psi(X,T,\mu) \) is the proximality constant (see (2.4b)).

**Corollary 2.** Under the conditions of the above theorem, assume that the scale sequence \( s \) is two-jumpy. Then the proximality gauge is trivial, i.e., \( \psi^*(\mu,\mu) \in \{\delta_0,\delta_\infty\} \).

**Proof of Proposition 1.** Denote \( Y = X^2 \) and \( T'' = T \times T, \nu = \mu \times \mu \). The assumption on \( s \) implies the inequality \( \psi(T'' y) \leq \psi(y) \), for all \( y \in Y \). Since \( \nu \in P(Y,T'') \), the claim of Proposition 1 follows. \( \square \)

**Proof of Proposition 2.** Follows from Proposition 1 because \( T \times T \) is ergodic. \( \square \)

**Proof of Corollary 2.** It is enough to show that \( \psi \) is both \( \mu \times \mu \)-extreme and \( \mu \times \mu \)-constant. The first assertion follows from Theorem 1; the second assertion is a consequence of Proposition 2. \( \square \)
4. Results on IETs

4.1. A brief introduction. Several results in the paper concern the systems on the unit interval \( X = I = [0, 1) \), and in particular the systems called interval exchange transformations (IETs). Let \( \{r, L, \pi\} \) be a triple such that

- \( r \geq 2 \) is an integer,
- \( \pi \in S^r \) is a permutation on the set \( J_r = \{1, 2, \ldots, r\} \),
- \( L \) is a positive probability vector, \( L = (\ell_1, \ell_2, \ldots, \ell_r) \in \mathbb{R}^r \) with \( \sum_{i=1}^r \ell_i = 1 \) and all \( \ell_i > 0 \).

An IET is a map \( T: I \to I \) completely determined by the parameters \( \{r, L, \pi\} \) in the following way. Set \( s_0 = 0 \) and \( s_k = \sum_{i=1}^k \ell_i \), for \( 1 \leq k \leq r \). This way the interval \( I = [0, 1) \) is partitioned into \( r \) subintervals \( I_k = [s_{k-1}, s_k), 1 \leq k \leq r \). Define \( T = T(L, \pi) \) by the formula

\[
T(x) = x - \sum_{i<k} \ell_i + \sum_{\pi(i')<\pi(k)} \ell_{i'}, \quad \text{for } x \in I_k.
\]

A map \( T \) constructed this way is called the \( (L, \pi) \)-IET (sometimes, less informatively, an \( r \)-IET or just an IET). It exchanges the intervals \( I_k \) in accordance with permutation \( \pi \). It follows from the fact that IETs are invertible piecewise isometries that \( \lambda \in \mathcal{P}(T) \) for every IET \( T \). (Every IET preserves the Lebesgue measure). If \( \mathcal{P}(T) = \{\lambda\} \), a singleton, \( T \) is called uniquely ergodic. (The equivalent condition is that all orbits of \( T \) are uniformly distributed). An IET \( T \) is called ergodic if its is \( \lambda \)-ergodic.

A permutation \( \pi \in S^r \) is called irreducible if for every \( k \in J_{r-1} = \{1, 2, \ldots, r-1\} \) there exists \( i \in J_{r-1} \) such that \( i \leq k < \pi(i) \). An IET \( T \) is called minimal if every orbit is dense. For \( n \geq 2 \), the irreducibility of \( \pi \) is a necessary condition for the minimality of \( T = (L, \pi) \). On the other hand, for irreducible \( \pi \in S_r \), linear independence of the \( r \) entries of \( L \) (over \( \mathbb{Q} \)) is a sufficient condition for minimality of \( (L, \pi) \), see [25]. Thus \((L, \pi)\) is minimal for irreducible \( \pi \) and Lebesgue almost all \( L \).

Every \( \lambda \)-ergodic IET must be minimal but the converse does not hold. A minimal IET does not need to be uniquely ergodic or even \( \lambda \)-ergodic [33] and [25]. Nevertheless, “most” minimal IETs are uniquely ergodic. More precisely, for an irreducible permutation \( \pi \) the IET \((L, \pi)\) is uniquely ergodic for Lebesgue almost all \( L \). This result (often referred to as Keane’s conjecture) has been proved independently by Mazur [34] and Veech [35]; see also [6] for an elementary proof.

For \( n \geq 3 \) and irreducible \( \pi \in S_r \), which are not rotations (the technical condition is \( \pi(k) + 1 \neq \pi(k+1) \)) for \( 1 \leq k < r \) the \((L, \pi)\)-IETs have been recently shown to be weakly mixing for Lebesgue a.a. \( L \) (see [1]). In particular, Lebesgue a.a. 3-IETs with \( \pi(k) = 4-k \) are weakly mixing [24].

4.2. The Connectivity Gauges. Given an IET \((I, T)\), recall that \( \phi_1(x, y) = \liminf_{n \to \infty} n |T^n(x) - y| \), for \( x, y \in I \). The following is the central result on the connectivity gauges for IETs.

**Theorem 3.** Let \( T \) be an IET and assume that \( \mu \in \mathcal{P}(T) \) is a \( T \)-ergodic measure. Then \( \phi_1^*(\mu, \mu) = \delta_0 \).

Recall that \( \phi_1^*(\mu, \mu) = \delta_0 \) in our notation means that \( \phi_1(T^n x, y) = 0 \pmod{\mu \times \mu} \) (Section 2).

**Corollary 3** (The case \( \mu = \lambda \), the Lebesgue measure). Let \((I, T)\) be a \( \lambda \)-ergodic IET. Then \( \phi_1^*(\lambda, \lambda) = \delta_0 \).

**Remark 2.** A special case of Corollary 3 (for the irrational rotations \( T \) of the unit circle \( I \)) is already known [30]. (Irrational rotations are naturally identified with 2-IETs).

Note that the ergodicity assumption in Theorem 3 (and Corollary 3) is crucial (see Theorem 4), and the choice of the factor \( s_n = n \) in the formula for \( \phi(x, y) \) is optimal (see Theorem 5).

The next theorem shows that even the special case of Corollary 3 does not extend to all minimal IETs. Nevertheless, for minimal IETs a weaker assertion is possible (see Proposition 3).

**Theorem 4.** There exists a minimal 4-IET \((I, T)\) such that \( \phi_1^*(\lambda, \lambda) = b \delta_0 + (1-b) \delta_\infty \), with some \( b \in \left[ \frac{1}{4}, \frac{3}{4} \right] \).
In what follows, $[r]$ stands for the integer part of $r \in \mathbb{R}$: $[r] = \min\{k \in \mathbb{Z} \mid k \geq r\}$.

**Proposition 3.** Let $(I, T)$ be a minimal $r$-IET. Then $(\lambda \times 1)(\phi^{-1}_1(0)) = \phi^*(\lambda, \lambda)(\{0\}) \geq \frac{1}{[r/2]} \geq 2/r > 0$.

Denote by $R_\beta : I \rightarrow I$ the $\beta$-rotation map $R_\beta(x) = x + \beta \;(\mod 1)$. (It can be viewed as a 2-IET).

**Theorem 5.** Let $s = \{s_n\}_1^\infty$ be a scale sequence such that $\lim_{n \rightarrow \infty} \frac{s_n}{n} = \infty$. Then there exists an irrational $\beta \in \mathbb{R} \setminus \mathbb{Q}$ such that the $\beta$-rotation $T = R_\beta$ satisfies $\phi^*(\lambda, \lambda) = \delta_\infty$.

Recall that $\phi^*(\lambda, \lambda) = \delta_\infty$ means that $\phi(x, y) = \liminf_{n \rightarrow \infty} s_n d(T^n x, y) = \infty$ (mod $\lambda(2)$).

Note that, under the conditions of Theorem 5 the relation $\phi_1^*(\lambda, \lambda) = \delta_0$ holds (by Corollary 3 even though $\phi^*(\lambda, \lambda) = \delta_\infty$. It follows that in Proposition 4 and Corollary 3 the factor $n$ (underlined in the formula $\liminf_{n \rightarrow \infty} |T^n x - y|$ for $\phi_1(x, y)$) cannot be replaced by one approaching infinity faster.

**Remark 4.** The conclusion of Theorem 5 can be presented in the following equivalent form: There exists an irrational $\alpha \in I = [0, 1)$ such that $\lim_{n \rightarrow \infty} s_n \|n \alpha - y\| = \infty$ for $\lambda$-a. a. $y \in I$.

To put Proposition 3 and Theorems 4 and 5 in perspective, we state the following three general propositions.

**Proposition 4.** Let $s = \{s_n\}_1^\infty$ be a scale sequence, let $(I, T)$ be a metric system (on the unit interval). Let $\mu \in \mathcal{P}(I).$ Then the gauge $\phi(x, y) = \liminf_{n \rightarrow \infty} s_n d(T^n x, y)$ is extreme relative to the measure $\mu \times \lambda \in \mathcal{P}(I^2)$ (i. e., $\hat{\phi}(\mu, \lambda) \subset \{0, \infty\}.$)

Note that in Proposition 4 no assumptions are imposed on the scale sequence $s$, and the measures $\mu, \lambda$ are not assumed to be $T$-invariant (cf. EGT, Theorem 2). Proposition 4 is a special case of Theorem 7 (presented and proved in Section 6).

**Proposition 5.** Let $s = \{s_n\}_1^\infty$ be a steady scale sequence, let $(I, T)$ be an $r$-IET and let $\mu, \nu \in \mathcal{P}(T)$ (be invariant measures). Then the set $\hat{\phi}(\mu, \nu) \subset [0, \infty]$ must be finite.

This follows from Proposition 4 and the fact that there are at most $\frac{2}{r}$ non-atomic ergodic measures.

Note that if $s$ is two-jumpy then the cardinality of $\hat{\phi}(\mu, \nu)$ does not exceed 2 because of the inclusion $\hat{\phi}(\mu, \nu) \subset \{0, \infty\}$ holds by Theorem 4.

**Proposition 6.** Let $s = \{s_n\}_1^\infty$ be a steady scale sequence, let $(I, T)$ be an IET and let $\mu, \nu \in \mathcal{P}(T)$ be two ergodic measures. Then the measure $\phi^*(\mu, \nu)$ is atomic, i. e. the set $\hat{\phi}(\mu, \nu)$ must be a singleton in $[0, \infty]$.

Note that the claim of the above theorem holds in a more general setting of m. m. p.-system which are local contractions (see Theorem 3 in Section 6). The next proposition shows that under certain assumption on $s$ the conclusion of Proposition 6 can be strengthened.

**Proposition 7.** Let $s = \{s_n\}_1^\infty$ be a nice scale sequence, let $(I, T)$ be an IET and let $\mu, \nu \in \mathcal{P}(T)$ be two ergodic measures. Then $\phi^*(\mu, \nu) \in \{\delta_0, \delta_\infty\}$.

Recall that $s$ is called nice if it is bounded-ratio and two-jumpy (Definition 1). Under the stronger condition that $s$ is steady and two-jumpy the claim of Proposition 7 follows immediately from Theorem 4 and Proposition 3. In Section 6 a more general version of Proposition 7 is stated and proved (Theorem 10).

By Theorem 7, $\phi_1^*(\mu, \mu) = \delta_0$, for any ergodic $\mu \in \mathcal{P}(T)$. It is possible to have $\phi_1^*(\mu, \nu) = \delta_\infty$ for some minimal IET $T$ and $T$-ergodic measures $\mu, \nu \in \mathcal{P}(T)$ (Proposition 14 see also Theorem 4).

The following questions are open:

**Question 1.** Under the conditions of Proposition 6 the measure $\phi^*(\mu, \nu)$ must be atomic: $\phi^*(\mu, \nu) = \delta_c$ with some $c \in [0, \infty]$. May $c$ be a finite positive number, $0 < c < \infty$?

Note that the metric on $I = [0, 1)$ in the above question is assumed to be the standard one, otherwise the answer is “yes” (a counterexample is possible on the basis of Remark 1 page 5). Note that $\phi^*(\mu, \nu) \in \{0, \infty\}$ holds if the measure $\nu$ is decisive, e. g. if $\nu$ is absolutely continuous relative to $\lambda$ (see Corollary 5 page 12).
Question 2. Does the equality \( \phi^* (\mu, \nu) = \phi^* (\nu, \mu) \) need to hold (perhaps, under some conditions on the scale sequence \( \mathbf{s} \) and the measures \( \mu, \nu \in \mathcal{P}(T) \))? 

A map \( T: \mathcal{I} \to \mathcal{I} \) is called \( \alpha \)-collapsing if \( \phi^\alpha (x, y) = 0 \) for all \( x, y \in \mathcal{I} \). We can prove that there are no 1-collapsing measurable maps with \( \mathcal{P}(T) \neq \emptyset \). Using the axiom of choice we can construct a 1-collapsing (non-measurable) map.

Question 3. Does there exist a 1-collapsing measurable map \( T: \mathcal{I} \to \mathcal{I} \)? (Such a map \( T \) cannot have invariant measures: \( \mathcal{P}(T) = \emptyset \)).

In fact, we know of no example of a 1-collapsing m. m. p.-system \( (\mathcal{I}, T, \lambda) \) (on the unit interval).

All minimal 2-IETs (equivalently, irrational rotations) are \( \alpha \)-collapsing for all \( 0 < \alpha < 1 \) (because the sets \( S_q = \{ x + i \alpha \ (\text{mod} \ 1) \mid 0 \leq k < q \} \subset \mathcal{I} \) are \( 2/q \)-dense in \( \mathcal{I} \) for infinitely many \( q \in \mathbb{N} \); in particular, for all denominators \( q = q_\alpha \) of the convergents for \( \alpha \). No 2 or 3-IET is 1-collapsing [13], and it is conjectured by the authors that the answer to Question 3 is negative.

4.4. Results on Proximality Gauges. Recall that a 3-IET is determined by a pair \( (L, \pi) \) where \( L \in \mathbb{R}^3 \) is a probability vector and \( \pi \in S_3 \) is a permutation on the set \( \{1, 2, 3\} \). In what follows, the implied permutation for 3-IETs is assumed to be \( \pi = (3 \ 2 \ 1) \) (reversing the order of the exchanged intervals).

We show that that for “generic” 3-IETs \( (\mathcal{I}, T) \) and all \( \alpha > 0 \), the relation

\[
\psi^\alpha (x, y) = \liminf_{n \to \infty} n^\alpha d(T^n(x), T^n(y)) \equiv \infty \quad (\text{mod } \lambda \times \lambda)
\]

holds (see (2.36)).

Theorem 6. For Lebesgue almost all 3-IETs (with \( \pi = (3 \ 2 \ 1) \)), \( \psi^\alpha (\lambda, \lambda) = \delta_\infty \), for all \( \alpha > 0 \).

The result is quite surprising, especially when compared with Corollary 3 (on page 7). Note that “most” 3-IETs \( (\mathcal{I}, T) \) are weakly mixing [24], so that \( \liminf_{n \to \infty} d(T^n(x), T^n(y)) \equiv 0 (\text{mod } \lambda (2)) \) (since \( T \times T \) is ergodic).

Theorem 6 can be restated in the following way.

Corollary 4. For Lebesgue almost all 3-IETs \( T \) (with \( \pi = (3 \ 2 \ 1) \)), the proximality constant \( C_\psi(T) \) vanishes.

Proposition 12 shows that the proximality constant of any IET is less than or equal to \( \frac{1}{4} \). In Section 11 we introduce the notion of the \( \tau \)-entropy of an IET and show that its value provides an upper bound for its proximality constant. \( \tau \)-entropy is connected to the convergence of ergodic averages for the characteristic function of subintervals (Theorem 15).

4.4. Absence of Topological Mixing for 3-IETs. We prove that no 3-IET can be strongly topologically mixing (Theorem 14 page 21). Note that the second author (J. Chaika) has recently established the existence of topologically mixing 4-IETs [18]. It is known that no IET is (measure-theoretically) strongly mixing [24] and that “many” 3-IETs are weakly mixing [24].

5. Contact Gauges and Decisive Measures

Given a sequence \( \mathbf{x} = \{x_n\}_1^\infty \) in a metric space \( (X, d) \) and a scale sequence \( \mathbf{s} = \{s_n\}_1^\infty \), by the contact gauge (of a sequence \( \mathbf{x} \) relative to a scale sequence \( \mathbf{s} \)) we mean the map \( \omega: X \to [0, \infty] \) defined as follows:

\[
(5.1) \quad \omega(x) = \Omega(x, \mathbf{s}, \mathbf{x}) = \liminf_{n \to \infty} s_n d(x_n, x) \quad (\text{the contact gauge}).
\]

Definition 5. Let a metric space \( (X, d) \) be given. A measure \( \mu \in \mathcal{P}(X) \) is called decisive if for every sequence \( \mathbf{x} = \{x_k\}_1^\infty \) in \( X \) and for every scale sequence \( \mathbf{s} = \{s_n\}_1^\infty \) the contact gauge \( \omega(x) = \Omega(x, \mathbf{s}, \mathbf{x}) \) is \( \mu \)-extreme (i.e., the equality \( \mu(\omega^{-1}([0, \infty})) = 1 \) holds).

Thus a measure \( \mu \in \mathcal{P}(X) \) on a metric space \( X \) is decisive if all contact gauges on it are extreme (whatever the choices for \( \mathbf{x} \) and \( \mathbf{s} \) are made).
Proposition 8. The Lebesgue measure on the unit interval \( \lambda \in \mathcal{P}(I) \) is decisive.

There are many examples of decisive measures (absolutely continuous measures on open subsets of \( \mathbb{R}^n \) and smooth manifolds, natural measures on self similar fractals etc., see [11]). Since the emphasis in the present paper is on the unit interval, Proposition 8 is confined to this case.

Denote by \( B_d(a) = \{ x \in I \mid ||x-a|| < \delta \} \) the open ball of radius \( \delta > 0 \) around \( a \in I \). (Recall that \( ||s|| = \min_{n \in \mathbb{Z}} |s-n| \) stands for the distance of \( s \in \mathbb{R} \) to the closest integer).

Proof. We have to prove that every contact gauge \( \omega(x) = \lim_{n \to \infty} s_n|x_n-x| \) on the unit interval is \( \lambda \)-extreme, i.e. that \( \lambda(E) = 0 \) where \( E = \omega^{-1}((0,\infty)) \).

For \( a > 0 \) and \( N \in \mathbb{N} \) denote \( E(a) = \omega^{-1}((a,2a)) \) and \( E(a,N) = \{ x \in E(a) \mid \inf_{n>N} s_n|x_n-x| > a \} \).

It would suffice to prove that \( \lambda(E(a,N)) = 0 \) for all \( a > 0 \) and \( N \in \mathbb{N} \) because the set \( E \) is the union of a countable family of sets \( E(a,N) \). Indeed, one verifies that \( E = \bigcup_{k \in \mathbb{Z}} E((3/2)^k) \) and that \( E(a) = \bigcup_{N \in \mathbb{N}} E(a,N) \) for all \( a > 0 \).

Let \( a > 0 \) and \( N \in \mathbb{N} \) be fixed. Select a point \( x \in E(a,N) \). Observe that for all \( n \) in the infinite set \( S = \{ n \in \mathbb{N} : s_n|x_n-x| < 2a \} \) the inclusion \( B_{\frac{3}{2}a}(\frac{2a}{s_n}) \subset B_2(\frac{3a}{s_n}) \) holds. (The set \( S \) is infinite because \( x \in E \)).

It follows that the set \( \{ n \in \mathbb{N} \mid B_{\frac{3}{2}a}(\frac{2a}{s_n}) \subset B_{2a}(\frac{3a}{s_n}) \text{ and } E(a,N) \cap B_{\frac{3}{2}a}(\frac{2a}{s_n}) = \emptyset \} \) is also infinite because it contains the set \( \{ n \in S \mid n > N \} \).

Taking in account that \( \lambda(B_{\frac{3}{2}a}(\frac{2a}{s_n})) = \frac{\lambda(B_{2a}(\frac{3a}{s_n}))}{3} \) and that \( \lim_{n \to \infty} \frac{3a}{s_n} = 0 \), we conclude that \( x \) cannot be a (Lebesgue) density point for the set \( E(a,N) \). Since \( x \in E(a,N) \) is arbitrary, \( \lambda(E(a,N)) = 0 \) (by the Lebesgue density theorem), and the proof of the proposition is complete.

Now Proposition 1 becomes (in view of Proposition 8) a special case of the following theorem.

Theorem 7. Let \( s = (s_n)_{n=1}^{\infty} \) be a scale sequence, let \( (X,T) \) be a metric system and let \( \mu,\nu \in \mathcal{P}(X) \) (not necessarily \( T \)-invariant). Assume that \( \nu \) is decisive (in \( X \)). Then \( \phi(\mu,\nu) \subset (0,\infty) \), i.e. \( \phi^\ast(\mu,\nu) = c\delta_0 + (1-c)\delta_\infty \), for some \( 0 \leq c \leq 1 \).

Proof. For every fixed \( x \in X \), the values \( \phi(x,y) = \lim_{n \to \infty} s_n d(T^n x,y) = \Omega(y,s,\{T^n x\}_n^\infty) \) lie in \( (0,\infty) \) for \( \nu \)-a.a. \( y \in X \) since \( \nu \) is decisive (Definition 1). (Equivalently, \( \hat{\phi}(\delta x,\nu) \subset (0,\infty) \)). Since \( \phi(x,y) \in \mathcal{B}(X) \), Fubini’s theorem applies to conclude that \( \phi(x,y) \in (0,\infty) \), for \( (\mu \times \nu) \)-a.a. pairs \( (x,y) \in X^2 \), completing the proof of the theorem.

Note that in the special case when \( s \) is a two-jumpy scale sequence (e.g., if \( s \) is a power scale sequence: \( s_n = n^\alpha \), for some \( \alpha > 0 \), no assumption on “decisiveness” of \( \nu \) is needed because of Theorem 1).

In the next section we collect basic facts on the gauges \( \phi,\psi,\rho \) which hold for general metric systems. This sets a framework for the proof and comparison with more specific results on IETs, and also motivates these results.

6. From Quantitative to Connectivity Gauges in General Systems

In this section we present general results, mostly on the connectivity gauges. The motivation comes partially from [8] and [48] where Quantitative Recurrence study has been initiated.

6.1. Quantitative Recurrence Results, Review. In particular, a very general lower bound for the recurrence constant is found:

Theorem 8. Let \( (T,X,\mu,d) \) be an m.m.p system and \( \alpha \) be the Hausdorff dimension of \( \mu \) (with respect to the metric \( d \)). Then \( C_\rho \geq \frac{1}{\alpha} \).

An analogous version of this for the connectivity gauge is Proposition 11 (see also Remarks 6 and 8 in the end of the section).

It is known that for the m.m.p.-systems on the unit interval the 1-recurrence gauge \( \rho_1 \) must be finite almost everywhere; moreover, it is bounded by 1 if \( \mu = \lambda \).
**Proposition 9** ([3]). For every m. m. p. system \((I, T, \mu)\), the inequality \(\rho_1(x) < \infty\) holds for \(\mu\)-a. a. \(x \in I\). Moreover, in the special case \(\mu = \lambda\) the gauge \(\rho_1\) is essentially bounded: \(\rho_1(x) \leq 1 \pmod{\lambda}\).

Note that the inequality \(\rho_1(x) \leq 1\) in the above proposition can be replaced by \(\rho_1(x) \leq 1/2\) (unpublished), and the authors conjecture that the optimal constant is \(\sqrt{\lambda}\).

**Theorem 8** (and its versions) have been used in the study of recurrent properties of specific dynamical systems ([8], [12], [5], [16], [37] and [39]); there are several refinements of it, often under the additional assumption (see [4], [3], [20], [21], [38], [55]).

Shkredov in [10], [41], [12] obtained quantitative pointwise multiple recurrence results (of Szemeredi-Furstenberg type) using Gower’s quantitative estimates ([22], [10]) and the approach in [3]. Kim [31] has recently extended Theorem 1 to arbitrary group actions.

### 6.2. Quantitative Connectivity Results.

Given a metric system \((X, d, T)\), the dilation gauge \(D_T: X \to [0, \infty)\) is defined by the formula

\[
D_T(x) = \begin{cases} 
\limsup_{y \to x \atop y \in X} \frac{d(T(y), T(x))}{d(y, x)}, & \text{if } x \text{ is a limit point of } X, \\
0, & \text{if } x \text{ is an isolated point of } X.
\end{cases}
\]

Let \(\mu \in \mathcal{P}(X)\) be a measure. The map \(T\) is said to be a local contraction (mod \(\mu\)) if \(D_T(x) \leq 1\) for \(\mu\)-a. a. \(x \in X\). For example, every IET is a local contraction (mod \(\mu\)) for every continuous measure \(\mu \in \mathcal{P}(I)\).

The map \(T\) is said to be locally \(\lambda\)-lipschitz (mod \(\mu\)) if \(D_T(x) < \infty\), for \(\mu\)-a. a. \(x \in X\). For example, a piecewise monotone map \(T: I \to I\) must be locally \(\lambda\)-lipschitz (mod \(\lambda\)).

For \(x \in X\) and \(\mu \in \mathcal{P}(X)\), we abbreviate \(\phi^*(\mu, \delta_x) = \phi^*(\mu \times \delta_x)\) to just \(\phi^*(\mu, x)\) (for notation see (2.1) and (2.2)). In the similar way, \(\phi^*(\delta_x, \mu)\) is abbreviated to \(\phi^*(x, \mu)\). (Recall that \(\delta_x \in \mathcal{P}(X)\) stands for the atomic measure supported by the point \(x \in X\)).

Recall that a scale sequence is called nice if it is both two-jumpy and bounded-ratio (Definition [3] page 4).

**Proposition 10.** Let a metric system \((X, d, T)\) and a scale sequence \(s = \{s_n\}_1^\infty\) be given. Then:

1. Assume that the scale sequence \(s\) is either monotone or steady. Then for every \(T\)-ergodic measure \(\mu \in \mathcal{P}(T)\) and a point \(y \in X\), the measure \(\phi^*(\mu, y) = \phi^*(\mu \times \delta_y) \in \mathcal{P}([0, \infty])\) is atomic. Moreover, if \(s\) is two-jumpy then \(\phi^*(\mu, y) \in \{\delta_0, \delta_\infty\}\).

2. Assume that \(s\) is a steady scale sequence and that \(T\) is a local contraction (mod \(\nu\)), for some \(T\)-ergodic measure \(\nu \in \mathcal{P}(T)\). Then for every \(x \in X\), the measure \(\phi^*(x, \nu) \in \mathcal{P}([0, \infty])\) is atomic.

3. Assume that:

   - (a) \(s = \{s_n\}_1^\infty\) is a nice scale sequence,
   - (b) The measure \(\nu \in \mathcal{P}(T)\) is decisive (Definition [5] page 2) and \(T\)-ergodic,
   - (c) \(T\) is locally \(\lambda\)-lipschitz (mod \(\nu\)).

Then \(\phi^*(x, \nu) \in \{\delta_0, \delta_\infty\}\) for every \(x \in X\).

**Proof of Proposition 10**

Proof of (1). Observe that for all \(x, y \in X\) the inequality

\[
\phi(Tx, y) = \liminf_{n \to \infty} s_n d(T^n(Tx), y) = \liminf_{n \to \infty} \frac{s_n}{s_{n+1}} (s_{n+1} d(T^{n+1}x, y)) \leq \phi(x, y)
\]

holds. For a fixed \(y\), the ergodicity of \(T\) implies that \(\phi^*(\mu, y) \in \mathcal{P}([0, \infty])\) is an atomic measure. Finally, if \(s = \{s_n\}_1^\infty\) is two-jumpy, the EGT (Theorem 2) implies that \(\phi^*(\mu, y) \in \{\delta_0, \delta_\infty\}\).

Proof of (2). Fix \(x \in X\). Then for \(\nu\)-a. a. \(y \in X\) we have

\[
\phi(x, Ty) = \liminf_{n \to \infty} s_n d(T^n x, Ty) = \liminf_{n \to \infty} \frac{s_n}{s_{n-1}} \frac{d(T^n x, Ty)}{d(T^{n-1}x, y)} (s_{n-1} d(T^{n-1}x, y)) \leq D_T(y) \phi(x, y) \leq \phi(x, y).
\]

Since \(\nu\) is \(T\)-ergodic, \(\phi^*(x, \nu) \in \mathcal{P}([0, \infty])\) is atomic.
Corollary 5. Some additional assumptions are needed.

Let \( \phi \) be two \( \mu, \nu \)-trivial. Assume that (at least) one of the following conditions holds:

- for \( \mu \)-a. a. \( x \in X \) the measures \( \phi^*(x, \nu) \) are atomic;
- for \( \nu \)-a. a. \( y \in Y \) the measures \( \phi^*(\mu, y) \) are atomic.

Then the measure \( \phi^*(\mu, \nu) \) is atomic too.

Proof. Set \( f(x) = \int_X \phi(x, y) \, d\nu(y) \), for \( x \in X \). Set \( g(y) = \int_X \phi(x, y) \, d\mu(x) \), for \( y \in Y \).

Then \( f(x) = \phi(x, y) = g(y) \) (mod \( \mu \times \nu \)), and the claim of the lemma becomes obvious by Fubini’s Theorem.

Theorem 9. Let a metric system \( (X, d, T) \) and a steady scale sequence \( s = \{s_n\}_1^\infty \) be given. Let \( \mu, \nu \in \mathcal{P}(T) \) be two \( T \)-ergodic measures. Assume that \( T \) is a local contraction (mod \( \nu \)). Then the measure \( \phi^*(\mu, \nu) \) is atomic.

Proof. Since \( s \) is steady, it follows from Proposition \( 10(1) \) that the measures \( \phi^*(\mu, y) \) are atomic, for every \( y \in Y \). On the other hand, by Proposition \( 10(2) \) the measures \( \phi^*(x, \nu) \) are atomic too, for every \( x \in X \).

It follows from Lemma \( 1 \) that the measure \( \phi^*(\mu, \nu) \) is atomic.

Note that under the conditions of Theorem \( 9 \) one cannot claim that \( \phi^*(\mu, \nu) \in \{\delta_0, \delta_\infty\} \) (see Example \( 1 \)). Some additional assumptions are needed.

Corollary 5. Under the conditions of Theorem \( 10 \), assume that (at least) one of the following conditions holds:

1. The measure \( \nu \) is decisive (see Definition \( 5 \), page \( 4 \)).
2. The scale sequence \( s \) is two-jumpy.

Then \( \phi^*(\mu, \nu) \in \{\delta_0, \delta_\infty\} \), i.e. the gauge \( \phi \) is \( \mu \times \nu \)-trivial.

Proof. By Theorem \( 10 \), \( \phi^*(\mu, \nu) = \delta_c \), for some \( c \in [0, \infty] \). Assuming (1), we have \( \phi^*(x, \nu) \subset \{\delta_0, \delta_\infty\} \), for all \( x \in X \). It follows that \( \phi(\mu, \nu) \subset \{\delta_0, \delta_\infty\} \). Thus \( c \in [0, \infty] \). Assuming (2), the inclusion \( \hat{\phi}(\mu, \nu) \subset \{0, \infty\} \) holds in view of Theorem \( 1 \). Thus \( c \in \{0, \infty\} \). The proof is complete.

Theorem 10. Let \( (X, d, T) \) be a metric system and let \( \mu, \nu \in \mathcal{P}(T) \) be two \( T \)-ergodic measures. Assume that \( T \) is locally Lipschitz (mod \( \nu \)) and that \( s = \{s_n\}_1^\infty \) is a nice scale sequence. Then \( \phi^*(\mu, \nu) \in \{\delta_0, \delta_\infty\} \) (i.e., the gauge \( \phi \) is \( \mu \times \nu \)-trivial).

Proof. The scale sequence \( s \) is two-jumpy, hence monotone. Proposition \( 10(1) \) applies to conclude that the set \( \phi^*(\mu, y) \) is atomic for all \( y \in X \). Moreover, the inclusion

\[
\phi^*(\mu, y) \in \{\delta_0, \delta_\infty\}\quad (\text{for all } y \in X),
\]

holds in view of the EGT (Theorem \( 2 \)).

To conclude the proof we only need to show that \( \phi^*(x, \nu) \) is atomic for \( \nu \)-a. a. \( x \in X \) (see Lemma \( 1 \)). This task is accomplished by exhibiting the set

\[
X' = \{x \in X \mid \phi(x, Ty) \leq \phi(x, y) \quad \text{for} \ \nu \text{-a. a. } y \in X\},
\]

with the following two properties:

1. \( \mu(X') = 1 \).
2. For every \( x \in X' \), \( \phi^*(x, \nu) \) is atomic.
Remark 5 to the metric $d$

Proposition 11. Theorem 10 because power scale sequences \( \alpha > 0 \) follows from Theorem 10 because power scale sequences \( \alpha \phi \) form \( \alpha \)

Let Corollary 6.

Just as in the proof of Proposition 10(3), the inequality

\[
\phi(x, Ty) \leq \kappa D_T(y) \phi(x, y) \quad (\text{mod } \nu(y))
\]

holds for every \( x \in X \). It follows that

\[
(6.5) \quad \phi(x, Ty) \leq \kappa D_T(y) \phi(x, y) \quad (\text{mod } \mu \times \nu).
\]

In view of (6.4) and the fact that \( D_T(y) < \infty \) for \( \nu \text{-a.} \) \( y \in X \), the relation (6.5) implies

\[
\phi(x, Ty) \leq \phi(x, y) \quad (\text{mod } \mu \times \nu),
\]

validating the property (p1). The \( T \)-ergodicity of \( \nu \) implies (p2). The proof is complete. \( \square \)

An important special case of Theorem 10 when \( s = s_\alpha = \{n^{\alpha}\}_1^{\infty} \) is a power scale sequence is outlined in the following theorem. Recall that the \( \alpha \)-connectivity gauge (the connectivity gauge with \( s = s_\alpha \)) takes the form \( \phi_\alpha(x, y) = \liminf_{n \to \infty} n^\alpha d(T^n x, y) \) (see (2.3)).

Corollary 6. Let \((X, d, T)\) be a metric system and let \( \mu, \nu \in \mathcal{P}(T) \) be two \( T \)-ergodic measures. Assume that \( T \) is locally Lipschitz (mod \( \nu \)). Then all power connectivity gauges \( \phi_\alpha \) are \( \mu \times \nu \)-trivial (i.e., for every \( \alpha > 0 \) the inclusion \( \phi_\alpha^\ast(\mu, \nu) \in \{\delta_0, \delta_\infty\} \) holds).

Proof. Follows from Theorem 10 because power scale sequences \( s = \{s_n\} \) are nice. (Alternative argument: It also follows from Corollary 5(2) because power scale sequences \( s = \{s_n\} \) are both two-jumpy and steady). \( \square \)

Remark 5. A. Qnas has constructed a \( \lambda \)-ergodic transformation of \( I \) such that \( \phi_\lambda^\ast(\lambda) = \delta_\infty \) (personal communication).

Proposition 11. Let \((T, X, \mu, d)\) be an m.m.p system and \( \alpha \) be the Hausdorff dimension of \( \mu \) (with respect to the metric \( d \)). Then \( C_\phi \leq \frac{1}{\alpha} \).

Proof. Let \( s < C_\phi(\mu) \). So \( \mu(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B(T^i x, \frac{1}{n^\alpha})) = 1 \) for \( \mu \) almost every \( x \). Because \( \sum_{i=1}^{\infty} \left( \frac{1}{n^\alpha} \right)^{1+\epsilon} \) converges for any \( \epsilon > 0 \), and \( \bigcup_{i=1}^{\infty} B(T^i x, \frac{1}{n^\alpha}) \) covers \( \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B(T^i x, \frac{1}{n^\alpha}) \) for all \( t \), the Hausdorff dimension of \( \bigcup_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B(x, \frac{1}{n^\alpha}) \leq \frac{1}{\alpha}. \) Thus \( \mu \) assigns full measure to a set of \( H_{dim} \leq \frac{1}{\alpha} \). Because \( s \) can be arbitrarily close to \( C_\phi \) the proposition follows. \( \square \)

Remark 6. One can define the upper connectivity constant by \( \inf \{ \alpha : \phi_\alpha^\ast(\mu, \nu)(0) > 0 \} \). One can define the lower Hausdorff dimension of a measure to be the infimum of the Hausdorff dimensions of all Borel sets of positive \( \mu \) measure. By a similar argument one obtains that the upper connectivity constant is less than the reciprocal of the lower Hausdorff dimension of \( \mu \). It is easy to construct Hölder examples where the upper and lower connectivity constants are different (these examples can show that the Lipschitz assumption in Theorem 10 and Corollary 6 is important).

Remark 7. The inequality in this proposition is sharp. For every \( n \geq 2 \) there are (Lebesgue) ergodic rotations of the \( n \)-torus that have connectivity constant 0. (For \( n = 1 \), the connectivity constant is always 1). Lebesgue measure on the \( n \)-torus has Hausdorff dimension \( n \).

Remark 8. An analogous proof shows that \( C_\phi \leq \frac{1}{H_{dim}(\mu)} \). (Compare with Proposition 11 and Theorem 8).

7. Two Results on the Proximality Constant

Proposition 12. Let \( T \) be an IET then \( C_\psi(T) \leq \frac{1}{\delta_t} \).

Proof. Consider \( U_n = \{(x, y) : d(T^n x, T^n y) < \min(d(T^{n-1} x, T^{n-1} y), \frac{1}{n^\alpha})\} \). If \( x \) appears in the first coordinate then \( d(x, \delta_t) < \frac{1}{n^\alpha} \) for some discontinuity \( \delta_t \). The measure of such points is \( \frac{2(r-1)}{n^{\alpha-1}} \) (if \( T \) is an \( r \)-IET). For a fixed \( x \), the inequality \( \lambda(x \times [0,1) \cap S_n) \leq \frac{2}{n^\alpha} \) holds. Therefore \( \lambda(\psi(U_n)) \leq \frac{4(r-1)}{n^{\alpha-1}}, \) and \( \sum \lambda(\psi(U_n)) \) converges if \( c > \frac{1}{2} \). By the Borel-Cantelli Theorem, \( C_\psi(T) \leq \frac{1}{\delta_t} \). \( \square \)
Proposition 13. Let $S : X \to X$ be a linear recurrent dynamical system and $\bar{d}$ be the standard metric, $\bar{d}(\bar{x}, \bar{y}) = \min\{1, \bar{x}, \bar{y}\}$ and $\mu$ its unique ergodic measure. Then $C_{\phi}(X, \mu, \bar{d}) \leq \frac{1}{2}$.

We review some facts about linear recurrent transformations. Linear recurrent transformations are uniquely ergodic. If $\mu$ is the unique invariant measure then $\frac{1}{C_n} < \mu(x_1, x_2, x_3, ..., x_n, *) < \frac{C}{n}$ where $C$ is a constant depending only on the transformation $(x_1, x_2, ..., x_n, *)$ denotes all words beginning $(x_1, x, ..., x_n)$.

They have linear block growth. Systems of linear block growth have bounded difference between the number of allowed $k + 1$ blocks and allowed $k + 1$ blocks [14].

Proof. Consider $U_n = \{(x, y) \in d(S^n \bar{x}, S^n \bar{y}) < \min(d(T^{-1}n \bar{x}, T^{-n} \bar{y}), \frac{1}{n})\}$. If $(\bar{x}, \bar{y}) \in U_n$ then $x_{n-1} \neq y_n$ and $x_{n+k} = y_{n+k}$ for $0 \leq k < n^\epsilon$. Let $K$ be the largest difference between the number of allowed $k + 1$ blocks and $k$ blocks. Then $\mu(U_n) \leq \frac{k^2 C}{n^\epsilon}$.

8. Proof of Theorem 3

The following lemma (implicit in [23]) will be used in the proof.

Lemma 2. Let $(I, T)$ be a minimal $r$-IET and let $\mu \in \mathcal{P}(T)$ be an invariant measure. Then, for every $\epsilon > 0$, there are a subinterval $J \subset \mathcal{I}$ and an integer $N \in \mathbb{N}$ such that:

1. $\mu(J) < \epsilon$;
2. $\mu\left(\bigcup_{n=0}^{N-1} T^n(J)\right) \geq \frac{1}{r};$
3. the sets $J, T(J), ..., T^{N-1}(J)$ are pairwise disjoint subintervals of $\mathcal{I}$.

Proof. Pick a subinterval $I \subset \mathcal{I}$, $0 < \mu(I) < \epsilon$, such that the induced map $T'$ on $I$ is an $s$-IET, for some $2 \leq s \leq r$. Let $I_k \subset I$, $1 \leq k \leq s$, be the subintervals of $I$ exchanged by $T'$, $T'(I_k) = T^{N_k}I_k$.

By minimality, the images $T^n(I_k)$ cover $\mathcal{I} = [0, 1)$ before returning to $I$. More precisely, the family of subintervals $\{T^n(I_k) \mid 1 \leq k \leq s, 0 \leq n \leq N_k\}$ partition the interval $\mathcal{I} = [0, 1)$.

Thus $\mu\left(\bigcup_{n=0}^{N_k} T^n(I_k)\right) \geq \frac{1}{s} \geq \frac{1}{r}$, for some $k$, $1 \leq k \leq s$. To complete the proof of the lemma, one takes $J = I_k$ and $N = N_k$.

Proof of Theorem 3. Given an IET $T$ with a $T$-ergodic measure $\mu \in \mathcal{P}(T)$, we have to show that $\phi_1^*(\mu, \mu) = \delta_0$.

Since $\mu$ is ergodic, it is supported by a minimal component $K$ which is either a finite set (permuted cyclically by $T$), or a finite collection of subintervals of $\mathcal{I}$. (It is well known that the domain of an IET admits partition into a finitely many periodic and continuous components; each such component is a finite union of subintervals. For an algorithmic way of finding this decomposition in certain situations see e.g. [10]).

In the first case $\phi_1^* = \delta_0$ is immediate (because then $\phi(x, y) = 0$ for all $x, y \in K$).

In the second case, $T|_K$ is itself a minimal IET (after making the subintervals of $K$ adjacent to each other and rescaling). Thus without loss of generality we may assume that the original $T$ is a minimal IET (and then $\mu$ is continuous).

We first claim that the gauge $\phi_1(x, y) = \lim_{n} n|T^n(x) - y|$, is $\mu \times \mu$-trivial, i.e. that $\phi_1^*(\mu, \mu) \in \{\delta_0, \delta_\infty\}$.

This follows from Corollary 3.2 (alternatively, from Corollary 6).

It remains to show that $\phi_1^*(\mu, \mu) = \delta_\infty$ leads to a contradiction.

For every integer $k \geq 2$, one takes an interval $J_k$ and an integer $N_k$ as in Lemma 2 with $\epsilon = \epsilon_k = \frac{1}{rN_k}$.

Then the inequality $N_k |J_k| \geq \frac{1}{r}$ implies $\epsilon_k > |J_k| \geq \frac{1}{rN_k}$ whence $N_k > 3k$. Set $m_k = \lceil N_k/3 \rceil \geq k$ and define the sets

$A_k = \bigcup_{i=0}^{m_k-1} T^i(J_k), \quad B_k = \bigcup_{i=0}^{N_k-1} T^i(J_k), \quad C_k = A_k \times B_k \subset \mathbb{T}^2$.

We estimate $\mu(A_k) = \mu(B_k) = m_k \mu(J_k) \geq \frac{m_k}{rN_k} \geq \frac{m_k}{r(3m_k + 2)} \geq \frac{1}{5r}$ whence $(\mu \times \mu)(C_k) \geq \frac{1}{25r^2}$, for all $k$.

It follows that $(\mu \times \mu)(D) \geq \frac{1}{25r^2}$ where $D = \lim_{k \to \infty} C_k = \bigcap_{n \geq 2} \left(\bigcup_{k=n}^\infty C_k\right) \subset \mathbb{T}^2$. 

Observe that if \((x, y) \in C_k\) then there is an \(m, \frac{N_k}{3} < m < N_k\), such that the points \(T^m(x)\) and \(y\) lie in the same subinterval \(T^j(J_k)\) (for some \(j \in [N_k - m, N_k]\)) of length \(|J_k| = \frac{1}{N_k}\).

It follows that for all \(k \geq 2\)

\[
\min_{\frac{N_k}{3} \leq m < N_k} m |T^m(x) - y| \leq \min_{\frac{N_k}{3} \leq m < N_k} N_K |T^m(x) - y| < 1, \quad \text{for } (x, y) \in C_k.
\]

Since \(N_k > k\), the definition of \(D\) and the last inequality imply

\[
\phi_1(x, y) = \liminf_{n \to \infty} n |T^n(x) - y| \leq 1, \quad \text{for } (x, y) \in D.
\]

Now the assumption \(\phi'_{\mu, \mu} = \delta_\infty\) would imply that \((\mu \times \mu)(D) = 0\), a contradiction with the earlier conclusion that \((\mu \times \mu)(D) \geq \frac{2}{\pi^2}\).

\[\square\]

Remark 9. The above proof and Theorem 3 extend to many finite rank transformations.

9. Proof of Theorem 5

Given a sequence \(s = (s_k)_{k=1}^\infty\) of positive real numbers such that

\[(9.1) \quad \lim_{n \to \infty} s_n/n = \infty,
\]

we have to construct an irrational \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\) such that \(\lim_{n \to \infty} s_n/|n\alpha - y| = \infty\), for Lebesgue a.a. \(y \in \mathcal{I}\).

For \(a, r \in \mathbb{R}\), denote \(B(a, r) = \{x \in \mathcal{I} \mid \|x - a\| < r\}\). It is easy to verify that

\[(9.2) \quad \lambda(B(a_1, r) \setminus B(a_2, r)) \leq \|a_2 - a_1\|, \quad \text{for all } a_1, a_2, r \in \mathbb{R},
\]

and that \(\lambda(B(a, r)) = 2r\) if \(0 \leq r \leq 1/2\).

We may assume that the sequence \(s = (s_k)\) is non-decreasing. (Otherwise we replace it by the sequence \(s' = (s'_k)\) where \(s'_k = k \cdot (\min s_n)/n\). The new sequence \(s'\) is strictly increasing, is dominated by \(s\) and \((9.1)\) implies that \(\lim_{n \to \infty} s'_n/n = \infty\) holds).

The required \(\alpha \in \mathcal{I} \cap (\mathbb{R} \setminus \mathbb{Q})\) will be determined by its continued fraction

\[(9.3) \quad \alpha = [0, a_1, a_2, \ldots] \defeq \frac{1}{a_1 + \frac{1}{a_2 + \ldots}} \quad a_k \in \mathbb{N}.
\]

Recall some basic facts from the theory of continued fractions [29]. The numbers \(\alpha \in \mathcal{I} \cap (\mathbb{R} \setminus \mathbb{Q})\) are in the one-to-one correspondence with the sequences \((a_k)_{k=1}^\infty\), their continued fraction expansions. The convergents of \(\alpha\) are defined by formula \(a_k = p_k/q_k = \left[0, a_1, \ldots, a_k\right]_i\), where the sequences \((p_k), (q_k)\) are defined by the initial conditions \(p_0 = 0, q_0 = 1, p_1 = 1, q_1 = a_1\) and the recurrent relation \(x_k = a_k x_{k-1} + x_{k-2}\), satisfied by both for \(k \geq 2\).

The following inequality (standard in the theory of continued fractions) will be used later:

\[(9.4) \quad \|\alpha q_n\| < 1/q_{n+1}, \quad n \geq 1.
\]

For an irrational number \(\alpha \in \mathcal{I}\), define

\[A_{k,c}(\alpha) = \{x \in \mathcal{I} \mid s_n/\|n\alpha - x\| < c, \text{ for some } n \in [q_k, q_{k+1}) \cap \mathbb{N}\},
\]

where \(q_k\) (the denominators of convergents) are determined by \(\alpha\). By Borel Cantelli lemma, in order to prove Theorem 5 it would suffice to find an irrational number \(\alpha \in \mathcal{I}\) such that

\[(9.5) \quad \sum_{k \geq 1} \lambda(A_{k,c}(\alpha)) < \infty, \quad \text{for all } c > 0.
\]

We exhibit such an \(\alpha\) explicitly by its continued fraction \((9.3)\) by setting

\[(9.6) \quad a_k = \max(N_k, 3k^2), \quad k \geq 1,
\]

where \(N_k = \min S_k\) and \(S_k = \{m \in \mathbb{N} \mid s_n/n \geq k^4, \text{ for all } n \geq m\} \subset \mathbb{N}\). Note that the sets \(S_k\) are not empty in view of the assumption \((9.4)\). (One can verify that, in fact, the inequalities \(a_k \geq \max(N_k, 3k^2)\), rather than \((9.6)\), would suffice for our task).
Set $m_k = \lceil q_{k+1}/k^2 \rceil$ (where $q_k$ are the denominators of the convergents for $\alpha$ determined by the expansion (9.3)). Then
\[(9.7)\]
\[q_{k+1} \geq m_k \geq 3q_k > q_k, \quad \text{for } k \geq 1.\]
(Indeed, $q_{k+1} \geq \lceil q_{k+1}/k^2 \rceil = m_k \geq \lceil a_{k+1}q_k/k^2 \rceil \geq \lceil a_{k+1}/k^2 \rceil \cdot q_k \geq 3q_k.$)

Denote $B_n = B(n\alpha, \frac{\alpha}{s_n})$ and set
\[S_1 = \lambda\left( \bigcup_{n=q_k}^{m_k} B_n \right), \quad S_2 = \lambda\left( \bigcup_{n=m_k}^{q_{k+1}} B_n \right), \quad S_3 = \lambda\left( \bigcup_{n=q_k}^{2q_k} B_n \right) \quad \text{and} \quad S_4 = \lambda\left( \bigcup_{n=q_k}^{m_k} B_n \setminus B_{n-q_k} \right).\]

Then $\lambda(A_{k,c}(\alpha)) \leq S_1 + S_2$ and $S_1 \leq S_3 + S_4$. It follows that $\lambda(A_{k,c}(\alpha)) \leq S_2 + S_3 + S_4$.

Since $m_k = \lceil q_{k+1}/k^2 \rceil$, the inequalities $m_k > q_k > a_k \geq N_k$ imply
\[S_2 \leq \frac{2c q_{k+1}}{s_{m_k}} < \frac{2c q_{k+1}}{m_k + 1}. \quad \frac{2m_k}{s_{m_k}} \leq 4ck^2, \quad \frac{m_k}{s_{m_k}} \leq \frac{4ck^2}{k^4} = 4c^{-2}.\]

An estimate for $S_3$ follows from the inequalities $q_k > a_k \geq N_k$:
\[S_3 \leq \frac{c q_k + 1}{s_{q_k}} < (c+1)q_k/s_{q_k} \leq (c+1)/k^4.\]

In order to estimate $S_4$, we first observe that $s_{n-q_k} \leq s_n$ implies
\[\lambda(B_n \setminus B_{n-q_k}) = \lambda(B(n\alpha, c/s_n) \setminus B((n-q_k)\alpha, c/s_{n-q_k})) \leq \lambda(B(n\alpha, c/s_n) \setminus B((n-q_k)\alpha, c/s_{n}),\]
whence $\lambda(B_n \setminus B_{n-q_k}) \leq \alpha q_n \leq 1/q_{k+1}$ (see (9.2) and (9.4)). Since $m_k = \lceil q_{k+1}/k^2 \rceil$, we get
\[S_4 \leq \frac{m_k - 2q_{k+1}}{q_{k+1}} < \frac{m_k}{q_{k+1}} \leq 1/k^2.\]

It follows that $\lambda(A_{k,c}(\alpha)) \leq S_2 + S_3 + S_4 < (4c+1)/k^2 + (c+1)/k^4$, so that (9.3) holds.

The proof of Theorem 4 is complete.

10. Proof of Theorem 4

The following proposition shows that particular choices of $(x, y)$ need not satisfy Theorem 8. It relies on a class of examples constructed in [26] and its proof relies on results about a subclass of these examples in [16]. To prove Theorem 4 we establish the following proposition and Remark 11 finishes the argument.

Proposition 14. There exists $T$, a minimal 4-IET with 2 ergodic measures, $\mu_{\text{sing}}, \lambda$ such that for $\mu_{\text{sing}} \times \lambda$, a.e. pair $(x, y)$, $\liminf_{n \to \infty} n|T^n(x) - y| = \infty$.

Compare it with Chebyshev’s Theorem (29 Theorem 24): Theorem 11. For an arbitrary irrational number $\alpha$ and real number $\beta$ the inequality $|n\alpha - m - \beta| < \frac{2}{n}$ has an infinite number of integer solutions $(n, m)$.

In the language of this paper Chebyshev’s Theorem states that if $T_\alpha$ is an irrational rotation, $\liminf_{n \to \infty} n|T^n_\alpha(x) - y| \leq 3$ for any $y$.

The terminology that will be used is in [16]. Recalling a few key facts from [26] observe that by considering $I^{(k+1)}_j$ as the first induced map of $I^{(k)}$, one can see that the images of $I^{(k+1)}_j$ in $O(I^{(k+1)}_j)$ land $b_{k,4}$ times in $O(I^{(k)}_4)$, then $m_{k+1}b_{k,2}$ times in $O(I^{(k)}_2)$, then $n_{k+1}b_{k,3}$ times in $O(I^{(k)}_3)$ before returning to $I^{(k+1)}$. (Recall that $O(I^{(k)}_j)$ is the Kakutani-Rokhlin tower over the $j$th interval of the $k$th induced map. $b_{k,j}$ represents the number of images of $I^{(k)}_j$ in this tower.)

We place the following conditions on $m_k$ and $n_k$.

1. $(n_k)^3 < m_k$
2. $(b_{k-1,2})^2 < m_k < (b_{k-1,2})^5$
(3) \((b_{k,2})^2 2^{2k} m_k < n_{k+1}\).

These conditions provide the following immediate consequences:

1. \(b_{k,2} \geq b_{k,j}\) for any \(j\) (Lemma 3 of [16]).
2. \((b_{k-1,2})^3 < b_{k+1,2} < 4(b_{k-1,2})^6\) (direct computation with condition 2).
3. \(\lambda(O(I_2^{(k)})) < \frac{2}{(b_{2,5})^2}\) (by Lemma 1 of [16] which is in the proof of Lemma 3 of [26]).
4. \(\sum_{j=1}^{\infty} \frac{n_{k+1} b_{k,2}}{b_{k+1,2}}\) converges \((b_{k+1,2} > m_{k+1} b_{k,2} \text{ and } b_{k,2} > b_{k,3})\).
5. \(\sum_{j=1}^{\infty} \frac{n_k}{m_k}\) converges (condition 1).
6. \(\mu_{\text{sing}}\text{ a.e. point is in } \bigcup_{j=1}^{\infty} \bigcap_{k=1}^{\infty} O(I_2^{(k)})\) (condition 1 and Lemma 6 of [16]).

**Definition 6.** Let \(\mathcal{A}_{x,r,M,N} = \{y : |T^n(x) - y| < \frac{1}{n} \text{ for some } N < n \leq M\}\)

This example relies on showing that \(\lambda(\mathcal{A}_{x,r,M,N})\) is small (see Lemma 3) for a \(\mu_{\text{sing}}\) large set of \(x\). The following definition provides us with a class of \(x\) such that we can control \(\lambda(\mathcal{A}_{x,r,M,N})\) as seen by Lemma 4. This class is also \(\mu_{\text{sing}}\) large as seen by Lemma 5.

**Definition 7.** \(x\) is called \(k\)-good if:

1. \(T^n(x) \in O(I_2^{(k)})\) for all \(0 \leq n \leq b_{k,2}\).
2. \(T^n(x) \in O(I_2^{(k-1)})\) for all \(0 \leq n \leq (n_k b_{k-1,3})^2\)

Consequence (6) shows that \(\mu_{\text{sing}}\) almost every point satisfies condition (1) for \(k\)-good for all \(k > N\). The following lemma shows that condition (2) is also satisfied eventually.

**Lemma 3.** For \(\mu_{\text{sing}}\) a.e. \(x\) there exists \(N\) such that \(x\) is \(k\)-good for all \(k > N\).

**Proof.** The basic reason \(\mu_{\text{sing}}\) a.e. \(x\) is eventually \(k\)-good for all large enough \(k\) is that the images of \(O(I_2^{(k+1)})\) not in \(O(I_2^{(k)})\) are consecutive (by the construction in [16]; see the discussion immediately following the statement of Theorem 14 in this paper). This means we need to avoid \(n_k b_{k,3} + b_{k,4} + (n_k b_{k,3})^2\) image of \(O(I_2^{(k+1)})\). Because

\[
\frac{n_k b_{k,3} + b_{k,4} + (n_k b_{k,3})^2}{b_{k+1,2}} < \frac{(n_k + 1)b_{k,2}}{m_k + b_{k,2}}
\]

is a convergent sum, the Borel Cantelli Theorem implies \(\mu_{\text{sing}}\) almost every \(x\) is \(k\)-good for all big enough \(k\). (The left hand side is the proportion of the images of \(I_2^{(k+1)}\) which are not good. □

The next lemma shows that if \(x\) is \(k + 1\) good then \(A_{x,r,b_{k,2},b_{k+1,2}}\) is small in terms of Lebesgue measure.

**Lemma 4.** If \(x\) is \(k + 1\)-good for all \(k > N\) then \(\lambda(A_{x,r,b_{k,2},b_{k+1,2}})\) forms a convergent sum.

**Proof.** This proof will be carried out by estimating the measure \(A_{x,r,b_{k,2},b_{k+1,2}}\) gains when \(x\) lands in \(O(I_2^{(k)})\) and when it doesn’t. Since \(x\) is \(k + 1\)-good, the Lebesgue measure \(A_{x,r,b_{k,2},b_{k+1,2}}\) gains by not landing in \(O(I_2^{(k)})\) is less than

\[
\frac{2}{(n_{k+1} b_{k,3})^2(n_k b_{k,3} + b_{k,4})} \leq \frac{2(n_k b_{k,3} + 4b_{k-1,2})}{(n_{k+1} b_{k,3})^2} \leq \frac{4}{n_{k+1} b_{k,3}}.
\]

When \(x\) lands in \(O(I_2^{(k)})\) it either lands in one of the \(b_{k-1,2}\) components of which are images of \(I_2^{(k-1)}\) (this is \(O(I_2^{(k-1)})\)) or it doesn’t. On each pass through of the orbit, it lands \(m_k b_{k-1,2}\) in one of the first \(b_{k-1,2}\) images of \(I_2^{(k-1)}\), and \(n_k b_{k-1,3} + b_{k-1,4}\) times it doesn’t. We will estimate \(A_{x,r,b_{k,2},b_{k+1,2}}\) by dividing up the orbit into these pieces.

When \(x\) lands in \(O(I_2^{(k-1)})\) the measure of the points its landings place in \(A_{x,r,b_{k,2},b_{k+1,2}}\) is at most \(\lambda(O(I_2^{(k-1)}))\)
\[ + \frac{2r}{b_{k,2}} b_{k-1,2}. \] (There are \(b_{k-1,2}\) connected components of \(O(I_2^{(k-1)})\).)

Otherwise we approximate the measure by:

\[ \frac{2}{b_{k,2}} \sum_{i=1}^{b_{k+1,2}/b_{k,2}} \frac{n_k b_{k-1,3} + b_{k-1,4}}{i} \leq \frac{n_k b_{k-1,3} + b_{k-1,4}}{b_{k,2}} \ln(b_{k+1,2}) \leq \frac{n_k b_{k-1,3} + b_{k-1,4}}{b_{k,2}} \ln(b_{k,2}). \]

The left hand side is given by estimating the measure gained by hits in \(O(I_2^{(k)})\) \(O(I_2^{(k-1)})\) \((n_k b_{k-1,3} + b_{k-1,4})\) hits each of which contributes at most \(\mu\) on renormalizing. The first inequality is given by the fact that \(\lambda(A_{x,r,b_k})\) is Lebesgue measure and \(\lambda(A_{x,r,b_k})\) is a.e.

\[ \lambda(A_{x,r,b_k}) \leq \frac{4}{n_k+1} + \frac{2r}{b_{k,2}} \ln(b_{k+1,2}) + \frac{n_k b_{k-1,3} + b_{k-1,4}}{b_{k,2}} \ln(b_{k,2}) \]

which forms a convergent series due to the at least exponential growth of \(b_{k,i}\).

**Proof of Proposition 14** \(\mu_{\text{sing}}\) a.e. \(x\) is eventually \(k+1\)-good. By Borel-Cantelli for each of these \(x\), Lebesgue a.e. \(y\) has \(\lim_{n \to \infty} n|T^n(x) - y| = \infty\). The set of all \((x,y)\) such that \(\lim_{n \to \infty} n|T^n(x) - y| = \infty\) is measurable, and so has \(\mu_{\text{sing}} \times \lambda\) measure 1 (by Fubini’s Theorem).

**Remark 10.** One can modify conditions 1-3 to achieve \(\liminf_{n \to \infty} n^\alpha|T^n x - y| = \infty\) for \(0 < \alpha < 1\).

**Remark 11.** Following [44] Section 1, one can renormalize the IET by choosing the IET \(S_p\), with length vector

\[ (p\lambda(I_1) + (1-p)\mu_{\text{sing}}(I_1), p\lambda(I_2) + (1-p)\mu_{\text{sing}}(I_2), p\lambda(I_3) + (1-p)\mu_{\text{sing}}(I_3), p\lambda(I_4) + (1-p)\mu_{\text{sing}}(I_4)) \]

and permutation 4213. \(S_p\) has the same symbolic dynamics and obeys the same type induction procedure as \(T\) (with the same matrices). As a result \(S\) has two ergodic measures \(\mu_{S_p}, \lambda_{S_p}\) such that \(\mu_{S_p}(I_j^{(k)})\) for \(S_p\) is the same as \(\mu_{\text{sing}}(I_j^{(k)})\) for \(T\) and \(\lambda_{S_p}(I_j^{(k)})\) for \(S_p\) is the same as \(\lambda(I_j^{(k)})\). Moreover, if \(0 < p < 1\) then \(\mu_{S_p}\) and \(\lambda_{S_p}\) are both absolutely continuous and supported on disjoint sets of Lebesgue measure 1 - \(p\) and \(p\) respectively. If \(p = 1\) the IET is \(T\), if \(p = 0\) then \(\mu_{S_0}\) is Lebesgue measure and \(\lambda_{S_0}\) is singular. As a result the Lebesgue measure of \(I_j^{(k)}\) for \(S_p\) is at least \(0.5\max\{\lambda(I_j^{(k)}), \mu_{\text{sing}}(I_j^{(k)})\}\) for \(T\). From this it follows that \(\liminf_{n \to \infty} n|S_p^k(x) - y| = \infty\) on a set of \((x,y)\) with measure \(0.25\) (corresponding to \(x\) being chosen from a set of \(\mu_{S_p}\) full measure and \(y\) being chosen from a set of \(\lambda_{S_p}\)). This proves Theorem 4 (See [44] Section 1 for more on renormalizing).

**11. The \(\tau\)-entropy of an IET**

Note that Lebesgue almost all 3-IET’s (corresponding to \(\pi = (321)\) in \(S_3\)) are weakly mixing. This result has been first proved by Katok and Stepin [24], with an explicit sufficient diophantine generic condition on the lengths \(\{\ell_1, \ell_2, \ell_3\}\) (of exchanged intervals) given. A weaker sufficient generic condition (for weak mixing of a 3-IET) has been given in [13].

Next we introduce the notion of a \(\tau\)-entropy, \(\tau(T)\), of an IET \(T\) which provides an upper bound on the proximality constant \(C_{\rho}(T)\) (Theorem 12 below).

We consider the following finite subsets of \(\mathcal{I}\) associated with a given IET \((\mathcal{I}, T)\):

(11.1a) \[ \Delta(T) \overset{\text{def}}{=} \{T(x) \ominus x \mid x \in \mathcal{I}\} \subset \mathcal{I}; \]

(11.1b) \[ \Delta'(T) \overset{\text{def}}{=} \{x \ominus y \mid x, y \in \Delta(T)\} \subset \mathcal{I}; \]

(11.1c) \[ \Delta'_n(T) \overset{\text{def}}{=} \bigcup_{k=1}^n \Delta'(T^k), \quad \text{for } n \geq 1; \]

where \(\oplus, \ominus\) stand for the binary operations on \(\mathcal{I}\) defined as addition and subtraction modulo 1. Clearly, for an \(r\)-IET \(T\), the following inequalities hold: \(\text{card}(\Delta(T)) \leq r\), \(\text{card}(\Delta(T^k)) \leq rk\) (because \(T^k\) is an IET...
exchanging at most \((r - 1)k + 1 \leq rk\) subintervals, and hence \(\text{card}(\Delta_n')(T) < r^2n^3\). The \(\tau\)-entropy of an IET \(T\) is defined by the formula

\[
\tau(T) = \limsup_{n \to \infty} \frac{\log(\text{card}(\Delta_n')(T))}{\log n}
\]

It is clear that the inequalities \(0 \leq \tau(T) \leq 3\) hold for any IET \(T\).

**Definition 1.** An IET \(T\) is called \(\tau\)-deterministic if \(\tau(T) = 0\).

It is clear that all 2-IETs (circle rotations) are \(\tau\)-deterministic (because then \(\Delta_n'(T) = \{0\}\)). It turns out that Lebesgue a. a. 3-IETs also are \(\tau\)-deterministic (see Theorem \ref{thm:3IETs_deterministic}).

**Theorem 12.** If \(T\) is a \(\tau\)-deterministic IET then its proximality constant \(C_{\psi}(T)\) vanishes. More generally, for an arbitrary IET \(T\), \(C_{\psi}(T) \leq \min(1, \tau(T))\).

We will see that certain assumption on the rate the orbits of an IET \(T\) become uniformly distributed implies that \(T\) is \(\tau\)-deterministic (see Theorem \ref{thm:uniform_distribution} below).

Theorem \ref{thm:uniform_distribution} is a special case of the following proposition.

**Proposition 15.** Let \(T\) be an IET. Let \(s = \{s_n\}_{n=1}^{\infty}\) be a scale sequence and let \(\psi: \mathcal{I} \times \mathcal{I} \to [0, \infty)\) be an associated proximality gauge: \(\psi(x, y) = \liminf_{n \to \infty} s_n |T^n(x) - T^n(y)|\). Set \(v_k = |\Delta_{2k}'(T)|/s_k\) and assume that \(\sum_{k \geq 1} v_{2k} < \infty\). Then \(\psi(\lambda, \lambda) = \delta_{\infty}\), i.e. \(\psi(x, y) = \infty \pmod{\lambda \times \lambda}\).

**Proof of Proposition \ref{prop:uniform_distribution}** Fix arbitrary \(x_0 \in \mathcal{I}\) and \(t \in \mathbb{R}, t > 0\). For \(k, n \geq 1\), denote

\[
Y_k = \{y \in \mathcal{I} \mid \|T^k y - T^k x_0\| \leq t/s_k\}; \quad Z_n = \bigcup_{k=n}^{2n} Y_k.
\]

Since \(T^k y \ominus T^k x_0 = ((T^k y \ominus y) \ominus (T^k x_0 \ominus x_0)) \ominus (y \ominus x_0) \ominus \Delta_k'(T)\), it follows that

\[
Y_k \subset x_0 \ominus B(\Delta_k'(T), t/s_k)
\]

where \(B(\Delta, \epsilon) = \{x \in \mathcal{I} \mid \text{dist}(x, \Delta) < \epsilon\}\) denotes the \(\epsilon\)-neighborhood of a subset \(\Delta \subset \mathcal{I}\), with the standard convention: \(\text{dist}(x, \Delta) = \inf_{z \in \Delta} ||x-z||\).

Since \(\{s_n\}\) is non-decreasing, the following inclusions hold

\[
Z_n = \bigcup_{k=n}^{2n} Y_k \subset x_0 + \bigcup_{k=n}^{2n} B(\Delta_k'(T), t/s_k) \subset x_0 + B(\bigcup_{k=n}^{2n} \Delta_k'(T), t/s_n) \subset x_0 + B(\Delta_{2n}'(T), t/s_n).
\]

Therefore \(\lambda(Z_n) \leq \text{card}(\Delta_{2n}'(T)) 2t/s_n \leq 2t v_n\).

By the Borel-Cantelli lemma, \(\sum_{k \geq 1} v_{2k} < \infty \implies \lambda(\limsup_{k \to \infty} Z_{2k}) = 0\).

It follows that \(\lambda(\limsup_{n \to \infty} Y_n) = 0\) (because \(\limsup_{n \to \infty} Y_n = \limsup_{n \to \infty} Z_n\), thus \(\lambda(\{y \in \mathcal{I} \mid \psi(x_0, y) < t\}) = 0\). Since \(s > 0\) is arbitrary, we get \(\lambda(\{y \in \mathcal{I} \mid \psi(x_0, y) = \infty\}) = 1\). The claim of the proposition follows from Fubini’s Theorem since \(x_0 \in \mathcal{I}\) is arbitrary.

**Proof of Theorem \ref{thm:uniform_distribution}** Assuming that \(\alpha > \tau\), set \(\beta = (\alpha + \tau)/2 > \tau\), \(\gamma = (\alpha - \tau)/2 > 0\). Then, with notation as in Proposition \ref{prop:uniform_distribution} with \(s_n = n^\alpha\), we obtain \(v_n = |\Delta_{2n}'(T)|/s_n \leq |\Delta_{2n}'(T)|/n^\alpha \leq n^{-\beta}/n^\alpha = n^{-\gamma}\), for all large \(n\). It follows that \(\sum_{k \geq 1} v_{2k} < \infty\) because \(v_{2k} \leq 2^{-\gamma k^\beta}\) for large \(k\).

It remains to consider the case \(\alpha > 1\). Then, with \(s_n = n^\alpha\) and notation as in the proof of Proposition \ref{prop:uniform_distribution} we have \(\lambda(Y_n) \leq tn^{-\alpha}\). By Borel-Cantelli lemma, \(\lambda(\limsup_{n \to \infty} Y_n) = 0\) since \(\sum_{n} tn^{-\alpha} < \infty\).

The completion of the proof is similar to the one of Proposition \ref{prop:uniform_distribution}.

\(\square\)
Theorem 12 provides a motivation for finding estimates for $\tau$-entropies $\tau(T)$, for various IETs $T$. We will show that such estimates are possible in terms of the rate the orbits of $T$ become uniformly distributed (see Theorem 13).

Let $T : \mathcal{I} \to \mathcal{I}$ be a map and $n \in \mathbb{N}$. Set

$$D_n(T) = \sup_{x \in \mathcal{I}} \left| \frac{1}{n} \sum_{k=0}^{n-1} \theta_{a,b}(T^k(x)) - (b - a) \right|$$

where $\theta_{a,b} : \mathcal{I} \to \{0,1\}$ stands for the characteristic function of the interval $(a,b) \in \mathcal{I}$.

By an $\omega$-discrepancy of $T$ we mean the following constant

$$\omega(T) = \limsup_{n \to \infty} \frac{\log (n D_n(T))}{\log n} = 1 + \limsup_{n \to \infty} \frac{\log D_n(T)}{\log n} \in [0,1].$$

Thus, for any $\epsilon > 0$, the inequality $D_n(T) < n^{\omega(T)-1+\epsilon}$ holds for all $n \in \mathbb{N}$ large enough.

Note that an IET $T$ with $\omega(T) < 1$ must be minimal and uniquely ergodic (because every orbit is uniformly distributed in $\mathcal{I}$).

The following lemma is an immediate consequence of the definition $D_n$.

**Lemma 5.** Let $T : \mathcal{I} \to \mathcal{I}$ be a map. Then the following inequality holds for all $x,y,a,b \in \mathcal{I}$ such that $a < b$ and all $n \in \mathbb{N}$:

$$\left| \sum_{k=0}^{n-1} \theta_{a,b}(T^k(x)) - \sum_{k=0}^{n-1} \theta_{a,b}(T^k(y)) \right| \leq 2n D_n(T).$$

In particular, for $\epsilon > 0$, $\left| \sum_{k=0}^{n-1} \theta_{a,b}(T^k(x)) - \sum_{k=0}^{n-1} \theta_{a,b}(T^k(y)) \right| \leq n^{\omega(T)+\epsilon}$, for all $n \in \mathbb{N}$ large enough.

The notions of $\omega$-discrepancy $\omega(T)$ and $\tau$-entropy $\tau(T)$ are motivated by the following result establishing an upper limit on the proximality constant of an IET.

**Theorem 13.** An IET $T$ with $\omega(T) = 0$ must be $\tau$-deterministic, i.e. $\tau(T) = C_\psi(T) = 0$. More generally, if $T$ is an $r$-IET with $\omega(T) = \omega_0$, then $C_\psi(T) \leq \tau(T) \leq (r - 1)\omega_0$.

**Proof.** In view of Theorem 12 all we need is to validate the inequality $\tau(T) \leq (r - 1)\omega_0$.

Let $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_r$ be the subintervals exchanged by $T$. Then $T(x) - x$ is constant on each $\mathcal{I}_k$. Set $h_k = T(x) - x$, for any $x \in \mathcal{I}_k$, $1 \leq k \leq r$. It follows that

$$T^n(x) - x = \sum_{i=0}^{n-1} (T^{i+1}x - T^ix) = \sum_{k=1}^{r} h_k H_{k,n}(x),$$

where $H_{k,n}(x) = \sum_{j=0}^{n-1} \theta_{\mathcal{I}_k}(T^jx) \in \mathbb{Z}$ and $\theta_{\mathcal{I}_k}$ stand for the characteristic functions of intervals $\mathcal{I}_k$.

It is clear that $\sum_{k=1}^{r} H_{k,n}(x) = n$, for all $x \in \mathcal{I}$ and $n \in \mathbb{N}$.

Fix $\epsilon > 0$. Then, for large $n$, $|H_{k,n}(x) - n\lambda(\mathcal{I}_k)| \leq D_n(T) < n^{\omega_0+\epsilon}$, whence $\Delta'(T^n) \subset S_n$ where

$$S_n = \left\{ \sum_{k=1}^{r} h_k H_k \mid H_k \in \mathbb{Z}, |H_k| < 2n^{-\omega_0+\epsilon}, \sum_{k=1}^{r} H_k = 0 \right\}.$$

It follows that $\Delta'_n(T) \subset S_n$ (see equations (11.1)), and hence $\text{card}(\Delta'_n(T)) \leq \text{card}(S_n) < (5n^{\omega_0+\epsilon})^{r-1}$ for large $n$ whence $\tau(T) \leq (r - 1)(\omega_0 + \epsilon)$ (see 11.2). The proof is complete since $\epsilon > 0$ is arbitrary.

**Definition 2.** The type $\nu = \nu(\alpha)$ of an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is defined as the limit

$$\nu = \liminf_{n \to \infty} \left( -\frac{\log \|n\alpha\|}{\log n} \right).$$
It is well known that \( \nu(\alpha) [1, +\infty) \). A number \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) is called Liouville if \( \nu(\alpha) = \infty \); otherwise it is called diophantine. Denote \( R_s = \{ \alpha \in \mathbb{R} \setminus \mathbb{Q} \mid \nu(\alpha) = s \} \). The numbers of type 1 (i.e., the numbers in \( R_1 \)) are also said to be of Roth type. It is easy to show that Lebesgue almost all numbers are of Roth type.

For references on the above definitions and related basic results see [33] Chapter 2, §3, pp.121–125] (in particular, see Lemma 3.1, Theorem 3.2 and Example 3.1).

**Proposition 16.** Let \( \alpha \in \mathbb{R} \), for some \( 1 \leq s < \infty \). Let \( T_\alpha \) denote the 2-IET corresponding to the \( \alpha \)-rotation on \( T = [0, 1) = S^1 \) (i.e., \( T(x) = x + \alpha \) (mod 1), \( x \in T \)). Then \( \omega(T) \leq \frac{\pi}{\alpha} \). In particular, if \( \alpha \) is of Roth type, then \( \omega(T) = 0 \).

**Proof.** By [33] Chapter 2, Theorem 3.2, for any \( \epsilon > 0 \), the inequality \( D_n(T) < n^{s-1/\epsilon} \) holds for large \( n \). The proposition follows from the definition of \( \omega(T) \). \( \square \)

**Proof of Theorem 12** This follows by direct modification of Theorem [13] and the previous proposition. Almost every 3-IET is the induced map of a rotation by a Roth type \( \alpha \) (i.e. \( \alpha \in R_1 \)). Fix \( T \) to be a 3-IET given by the induced map of \( R_\alpha \) on \( [0,b] \) for \( \alpha \in R_1 \). Thus \( T(x) = (x \pm \alpha)/b \) if \( x \in [0,b) \) and \((x \pm 2\alpha)/b \) otherwise.

Therefore the growth of \( \Delta_n'(T) \) is controlled by \( \max_{x,y \in J} \theta_{b,1}R_\alpha(x) - \theta_{b,1}R_\alpha(y) \). Theorem 12 follows. \( \square \)

12. No 3-IET is Topologically Mixing

It is worth mentioning that at certain times the discrepancy of rotations is at most 4 valued [27]. This provides the following theorem:

**Theorem 14.** No 3 IET is Topologically mixing.

**Definition 3.** \( T : [0, 1) \to [0, 1) \) is called topologically mixing if for any (nonempty) open sets \( U \) and \( V \) there exists \( N \) such that \( T^n(U) \cap V \neq \emptyset \) for all \( n > N \).

It is sufficient to consider 3-IETs formed by inducing on irrational (non-periodic) rotations of the circle.

**Lemma 6.** Let \( R_\alpha \) be rotation of the circle \([0,1)\) by \( \alpha \). Then for any interval \( J \subset [0,1) \) the cardinality of the set \( \{ x, x + \alpha, ..., (q_n - 1)\alpha + x \} \cap J \) takes at most 4 consecutive values as \( x \) varies.

Call the largest of these \( b \). This Lemma is a consequence of Theorem 1 of [27] which states:

**Theorem 15.** [27] Each interval \( (\frac{r}{q_n}, \frac{r+1}{q_n}) ; r = 0, 1, ... , q_n - 1 \) contains exactly one point \( k\alpha \) with \( 1 \leq k \leq q_n \).

To prove Lemma 6 observe that \( |\{ x, x + \alpha, ..., (q_n - 1)\alpha + x \} \cap J| \in [\lfloor \frac{q}{q_n} \rfloor - 1, \lfloor \frac{q}{q_n} \rfloor + 2] \cap \mathbb{N} \)

**Lemma 7.** Let \( T \) be a 3-IET obtained by inducing rotation by \( \alpha \) on \([1-t,1) \). Then \( T^{q_n - 1 - b_n}(x) - x \) takes at most 7 consecutive values for each \( n \).

**Proof of Theorem 13** Divide \([0,1) \) into intervals of length \( \frac{1}{20} \). For each such interval, \( J \) and each \( n T^{q_n - 1 - b_n}(J) \) intersects at most fourteen of them. So it misses at least six. Therefore it misses at least one (in fact at least six) infinitely often, violating topological mixing. \( \square \)

13. Questions

We conclude by listing some open problems.

**Problem 1.** For \( r \geq 4 \) and permutations \( \pi \in S_r \), what are possible and what are “typical” values for \( C_\psi(T) \) for \( r \)-IETs \( T \) corresponding to a given permutation \( \pi \)?

In fact, it is quite difficult (but possible) to construct an IET \( T \) with \( C_\psi(T) > 0 \), even though we believe that for “most” \( 4 \)-IETs \( C_\psi(T) > 0 \).

For “most” 3-IETs \( C_\psi(T) = 0 \) (Theorem 13), and we conjecture that \( C_\psi(T) = 0 \) for all 3-IETs. Note that the same question regarding the constants \( C_\phi(T) \) and \( C_\rho(T) \) is well understood.

**Problem 2.** Is it possible to have a uniquely ergodic IET \( T \) with two points \( x, y \) such that \( \psi_1(x, y) = \infty \)? If \( T \) is \( \lambda \)-ergodic but not uniquely ergodic, then \( \psi_1(x, y) = \infty \) may be possible (Proposition 13).

**Problem 3.** Does there exist an 1-collapsing IET \( T \)?

\( T \) is 1-collapsing if \( \phi_1(x, y) = \liminf_{n \to \infty} n|T^nx - y| = 0 \) for all \( x, y \in T \).
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