WHEN IS $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ A MORI DREAM SPACE?

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Abstract. We prove that the moduli space of $n$-pointed stable maps $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is a Mori dream space whenever the moduli space $\overline{M}_{0,n+3}$ of $(n+3)$-pointed rational curves is. We also show that $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is a log Fano variety for $n \leq 5$.

1. Introduction

Given a normal projective $\mathbb{Q}$-factorial variety $X$ over an algebraically closed field $\mathbb{K}$ of any characteristic, $X$ is a Mori dream space if $X$ has the following properties (see for instance [3]): (1) The Picard group $\text{Pic}(X)$ of $X$ is finitely generated, and $\text{Pic}(X)_\mathbb{Q} = N_1(X)_\mathbb{Q}$; (2) the cone of nef divisors $\text{Nef}(X)$ is generated by a finite number of semi-ample divisors; and (3) there are finitely many small, $\mathbb{Q}$-factorial modifications $f_i : X \to X_i$ of $X$ such that each $X_i$ has properties (1) and (2) and the moving cone of $X$ is the union of the pullbacks of the nef cones of the $X_i$. In other words, if $X$ is a Mori dream space, then one would be able, at least in principle, to explicitly describe the birational models $X_i$ of $X$, which are isomorphic to $X$ in codimension one, and to use them to describe the nef cone of $X$. It would also follow that the effective cone of $X$ is polyhedral. The nef and effective cones of divisors are crucial in understanding the birational geometry of a variety. In particular, for moduli spaces of curves an understanding of how these cones relate to each other was a crucial ingredient in the proof that the moduli space of stable curves $\overline{M}_g$ is of general type for $g = 22$ and $g \geq 24$ (see for instance [6]). Even for $g = 0$ the moduli space $\overline{M}_{0,n}$, parameterizing stable rational curves with $n$ ordered marked points and not too far from being a toric variety, presents a surprisingly rich birational geometry. The partial results obtained in two decades of intensive investigation range from the positive side (for instance, $\overline{M}_{0,n}$ is a Mori dream space for $n \leq 6$, see [12] and [2]) to the negative one (as the breakthrough in [4] that $\overline{M}_{0,n}$ is not a Mori dream space for $n \geq 134$, later improved to $n \geq 13$ in [9] and then to $n \geq 10$ in [11]).

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Here we address the same question for a different but closely related moduli space. As it is well-known (see for instance [13], Remark 1.4), the Kontsevich moduli space $M_{0,n}(\mathbb{P}^1, 1)$ parameterizing $n$-pointed stable maps to $\mathbb{P}^1$ of genus 0 and degree 1 is isomorphic to the Fulton-MacPherson compactification $\mathbb{P}^1[n]$ of the configuration space of $n$ distinct ordered points in $\mathbb{P}^1$. The natural projection $M_{0,n}(\mathbb{P}^1, 1) \to \overline{M}_{0,n}$ implies by [14] that if $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is a Mori dream space then $\overline{M}_{0,n}$ is a Mori dream space too. In particular, it follows that $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is not a Mori dream space for $n \geq 10$.

In Section 2 we establish a converse statement: if $\overline{M}_{0,n+3}$ is a Mori dream space then $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is a Mori dream space too (see Proposition 1). In order to do so, we introduce a natural birational map $\overline{M}_{0,n+3} \to \overline{M}_{0,n}(\mathbb{P}^1, 1)$ which is surjective in codimension one and we apply [15]. In particular, from the known results for $\overline{M}_{0,n}$ we recover the fact that $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is a Mori dream space for $n \leq 3$, which is already well understood: indeed, $\mathbb{P}^1[1] \cong \mathbb{P}^1$, $\mathbb{P}^1[2] \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^1[3]$ appears in the list of smooth Fano threefolds (for instance [13], p. 108), so it is a Mori dream space by [1], Corollary 1.3.2.

To go further we need to implement a different strategy. After rephrasing in Section 3 the characterization of ample divisors on $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ provided by [5], in Section 4 we check that $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is a log Fano variety for $n \leq 5$ but not for $n = 6$. We conclude that $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is a Mori dream space for $n \leq 5$ (see Corollary 1) and we point out that new ideas are required to address the remaining open cases $6 \leq n \leq 9$ (see Remark 1).

We work over the complex field $\mathbb{C}$.

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2.

First we recall the definition and the basic properties of both $\overline{M}_{0,n}$ and $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ following [7].

The moduli space $\overline{M}_{0,n}$ parameterizes isomorphism classes of stable curves of genus 0 with $n$ ordered marked points:

$$(C, p_1, \ldots, p_n).$$

For every subset $S \subset \{1, \ldots, n\}$ with $2 \leq |S| \leq n - 2$ the boundary component $\Delta_S$ is the closure in $\overline{M}_{0,n}$ of the locus of stable curves

$$(C_1 = \mathbb{P}^1, (p_i)_{i \in S}) \cup (C_2 = \mathbb{P}^1, (p_i)_{i \in S^c}).$$

The moduli space $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ parameterizes isomorphism classes of stable maps of degree 1 from curves of genus 0 with $n$ ordered marked points to $\mathbb{P}^1$:

$$(C, p_1, \ldots, p_n, f : C \to \mathbb{P}^1).$$
collapsing the first component to the point $C_1 \cap C_2$ and mapping the second component identically to $\mathbb{P}^1$.

Both $\overline{M}_{0,n}$ and $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ are smooth projective varieties and in both cases the union of the boundary components is a normal crossing (see for instance [10], Theorem 2.3).

**Proposition 1.** If $\overline{M}_{0,n+3}$ is a Mori dream space then $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is a Mori dream space.

**Proof.** By [15], Proposition 1.3 and Remark 2.2, the claim follows if there is a birational map $\overline{M}_{0,n+3} \dashrightarrow \overline{M}_{0,n}(\mathbb{P}^1, 1)$ which is surjective in codimension one.

Let

$$U_0 := \{(C_0 \cup \ldots \cup C_k, p_1, \ldots, p_{n+3}) \in \overline{M}_{0,n+3} : p_{n+1}, p_{n+2}, p_{n+3} \in C_0\}$$

and notice that $U_0$ is dense in $\overline{M}_{0,n+3}$ since it contains the open part $M_{0,n+3} \subset \overline{M}_{0,n+3}$ parameterizing smooth rational curves.

Consider the natural rational map:

$$\Phi : U_0 \dashrightarrow \overline{M}_{0,n}(\mathbb{P}^1, 1)$$

$$(C_0 = \mathbb{P}^1, (p_i)_{i \in S_0}, p_{n+1}, p_{n+2}, p_{n+3}) \cup \bigcup_{j=1}^{k}(C_j = \mathbb{P}^1, (p_i)_{i \in S_j})$$

$$\mapsto (C_0 = \mathbb{P}^1, (\pi(p_i))_{i \in S_0}, \text{id}) \cup \bigcup_{j=1}^{k}(C_j = \mathbb{P}^1, (p_i)_{i \in S_j}, \text{pt})$$

where $k \geq 0$ (for $k = 0$ we adopt the standard convention $\bigcup_{j=1}^{0} = \emptyset$), $S_0 \cup \ldots \cup S_k = \{1, \ldots, n\}$, $\pi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is the automorphism of $\mathbb{P}^1$ such that $\pi(p_{n+1}) = 0$, $\pi(p_{n+2}) = 1$, $\pi(p_{n+3}) = \infty$, $\text{id} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ is the identity on $\mathbb{P}^1$ and $\text{pt} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ collapses $\mathbb{P}^1$ to a point.

By definition, $\Phi$ is injective, it is surjective onto the open part $M_{0,n}(\mathbb{P}^1, 1) \subset \overline{M}_{0,n}(\mathbb{P}^1, 1)$ parameterizing stable maps with smooth domain $\mathbb{P}^1$ and for every subset $S \subset \{1, \ldots, n\}$ with $2 \leq |S| \leq n$ the image $\Phi(\Delta_S \cap U_0)$ is dense in $B_S$, so that every boundary component of $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is dominated by $\Phi$. It follows that $\Phi$ induces a birational map $\overline{M}_{0,n+3} \dashrightarrow \overline{M}_{0,n}(\mathbb{P}^1, 1)$ which is surjective in codimension one. 

\[\square\]
According to [5], the ample cone of $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ can be described in terms of natural maps:

$$\alpha : \overline{M}_{0,n+1} \rightarrow \overline{M}_{0,n}(\mathbb{P}^1, 1)$$

$$\beta_i : \mathbb{P}^1 \rightarrow \overline{M}_{0,n}(\mathbb{P}^1, 1), \ i = 1, \ldots, n$$

defined in [5], 2.1 and 2.2. Indeed, by [5], Theorem 2.3, a divisor $H$ on $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is ample if and only if $\alpha^*H$ is ample on $\overline{M}_{0,n+1}$ and $\beta_i^*H$ is ample on $\mathbb{P}^1$ for $i = 1, \ldots, n$.

In addition to the divisors $\Delta_S \subset \overline{M}_{0,n}$ and $B_S \subset \overline{M}_{0,n}(\mathbb{P}^1, 1)$, for $i = 1, \ldots, n$ we introduce also the classes

$$\psi_i := c_1(T_i^*),$$

where $T_i^*$ is the line bundle on $\overline{M}_{0,n}$ whose fiber over $(C, p_1, \ldots, p_n)$ is the cotangent space $(T_pC)^*$, and

$$L_i := \{(C, p_1, \ldots, p_n, f : C \rightarrow \mathbb{P}^1) \in \overline{M}_{0,n}(\mathbb{P}^1, 1) : f(p_i) = 0\}.$$  

The pullback of the classes $B_S$ and $L_i$ under the maps $\alpha$ and $\beta_i$ is computed in [5], Proposition 2.5 (see also [5], Table 1), in terms of the classes $\Delta_S$ and $\psi_i$, namely:

$$\alpha^*B_S = \Delta_S \text{ if } |S| \leq n - 1$$

$$\alpha^*B_S = -\psi_{n+1} \text{ if } |S| = n$$

$$\alpha^*L_i = 0 \text{ for every } i = 1, \ldots, n$$

$$\beta_i^*B_S = O_{\mathbb{P}^1}(-1) \text{ for } S = \{1, \ldots, n\} \text{ and } S = \{i\}^c$$

$$\beta_i^*B_S = 0 \text{ otherwise}$$

$$\beta_i^*L_i = O_{\mathbb{P}^1}(1)$$

$$\beta_j^*L_i = 0 \text{ for every } j \neq i.$$

The canonical class of $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is

$$K_n = -2L + \sum_{s=3}^{n} (s - 2)B[s]$$

where

$$L := \sum_{i=1}^{s} L_i$$

$$B[s] := \sum_{|S|=s} B_S, \ 2 \leq s \leq n$$
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Hence we have

$$\alpha^*K_n = -(n - 2)\psi_{n+1} + \sum_{s=3}^{n-1} (s - 2) \sum_{|S|=s} \Delta_S$$

$$\beta^*K_n = \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(-(n - 2)) \otimes \mathcal{O}_{\mathbb{P}^1}(-(n - 3)) = \mathcal{O}_{\mathbb{P}^1}(-(2n - 3)).$$

According to Fulton’s conjecture (see [8], Conjecture 0.2), a divisor on $\overline{M}_{0,n}$ is ample if and only if it has positive intersection with all one-dimensional strata, parameterizing $n$-pointed rational curves with at least $n - 4$ singular points. More explicitly, let

$$H = \sum_{|S| \geq 1} c_S \Delta_S,$$

where we adopt the convention $\Delta_{\{i\}} := -\psi_i$ for every $i = 1, \ldots, n$. By [8], Theorem 2.1, the divisor $H$ has positive intersection with all one-dimensional strata if and only if

$$c_{I \cup J} + c_{I \cup K} + c_{I \cup L} - c_I - c_J - c_K - c_L > 0$$

for every partition $I \cup J \cup K \cup L$ of $\{1, \ldots, n\}$.

By [12], Theorem 1.2(3), Fulton’s conjecture holds for $n \leq 7$.

Finally we are going to check that $\overline{M}_{0,n}(\mathbb{P}^1, 1)$ is a log Fano variety (hence a Mori dream space) for $n \leq 5$ but not for $n = 6$.

**Lemma 1.** On $\overline{M}_{0,4}(\mathbb{P}^1, 1)$ the divisor $K_4 + B[4]$ is anti-ample, hence $\overline{M}_{0,4}(\mathbb{P}^1, 1)$ is log Fano.

**Proof.** By [5], Proposition 2.5, we have

$$\alpha^*(K_4 + B[4]) = -3\psi_5 + \sum_{|S|=3} \Delta_S$$

$$\beta^*(K_4 + B[4]) = \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(-3) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) = \mathcal{O}_{\mathbb{P}^1}(-6).$$

It is clear that $\beta^*(K_4 + B[4])$ is anti-ample on $\mathbb{P}^1$; on the other hand, in order to check that $\alpha^*(K_4 + B[4])$ is anti-ample on $\overline{M}_{0,5}$, by [8], Theorem 2.1, we have to consider the following partitions $I \cup J \cup K \cup L$ of $\{1, \ldots, 5\} = \{a, b, c, d, 5\}$:

- $\{a\} \cup \{b\} \cup \{c\} \cup \{d, 5\}$
- $\{a\} \cup \{b\} \cup \{5\} \cup \{c, d\}$.

If

$$\alpha^*(K_4 + B[4]) = \sum_{|S| \geq 1} c_S \Delta_S$$
then
\[ c_{I \cup J} + c_{I \cup K} + c_{I \cup L} - c_I - c_J - c_K - c_L = -1 \]
in both cases listed above, hence \( \alpha^*(K_4 + B[4]) \) is anti-ample by [12], Theorem 1.2(3), and \( K_4 + B[4] \) is anti-ample by [3], Theorem 2.3.

\[ \square \]

**Lemma 2.** Let \( D = \frac{1}{4}B[2] + \frac{1}{4}B[4] + B[5] \) on \( \overline{M}_{0,5}(\mathbb{P}^1,1) \). The divisor \( K_5 + D \) is anti-ample, hence \( \overline{M}_{0,5}(\mathbb{P}^1,1) \) is log Fano.

**Proof.** By [5], Proposition 2.5, we have

\[
\alpha^*(K_5 + D) = -4\psi_6 + \frac{1}{4} \sum_{|S|=2} \Delta_S + \sum_{|S|=3} \Delta_S + \left(2 + \frac{1}{4}\right) \sum_{|S|=4} \Delta_S
\]

\[
\beta_1^*(K_5 + D) = \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(-4) \otimes \mathcal{O}_{\mathbb{P}^1}\left(-2 - \frac{1}{4}\right)
\]

\[ = \mathcal{O}_{\mathbb{P}^1}\left(-8 - \frac{1}{4}\right) \).

It is clear that \( \beta_1^*(K_5 + D) \) is anti-ample on \( \mathbb{P}^1 \); on the other hand, in order to check that \( \alpha^*(K_5 + D) \) is anti-ample on \( \overline{M}_{0,6} \), by [8], Theorem 2.1, we have to consider the following partitions \( I \cup J \cup K \cup L \) of \( \{1, \ldots, 6\} = \{a, b, c, d, e, 6\} \):

- \( \{a\} \cup \{b\} \cup \{c\} \cup \{d, e, 6\} \)
- \( \{a\} \cup \{b\} \cup \{6\} \cup \{c, d, e\} \)
- \( \{a\} \cup \{b\} \cup \{c, d\} \cup \{e, 6\} \)
- \( \{a\} \cup \{6\} \cup \{b, c\} \cup \{d, e\} \).

If

\[
\alpha^*(K_5 + D) = \sum_{|S|\geq 1} c_S \Delta_S
\]

then

\[ c_{I \cup J} + c_{I \cup K} + c_{I \cup L} - c_I - c_J - c_K - c_L = -\frac{1}{4} \]
in all cases listed above, hence \( \alpha^*(K_5 + D) \) is anti-ample by [12], Theorem 1.2(3), and \( K_5 + D \) is anti-ample by [3], Theorem 2.3.

\[ \square \]

**Lemma 3.** Let \( D = a_2B[2] + a_3B[3] + a_4B[4] + a_5B[5] + a_6B[6] \) on \( \overline{M}_{0,6}(\mathbb{P}^1,1) \) with \( a_i \in \mathbb{Q} \). If \( a_4 \geq 0 \) and \( a_6 \leq 1 \) then \( K_6 + D \) is not anti-ample.

**Proof.** By [3], Proposition 2.5, we have

\[
\alpha^*(K_6 + D) = \sum_{|S|\geq 1} c_S \Delta_S
\]
with
\[
c_S = \begin{cases} 
4 + a_6 & \text{if } S = \{7\} \\
0 & \text{if } |S| = 1, S \neq \{7\} \\
3 + a_5 & \text{if } |S| = 2, 7 \in S \\
a_2 & \text{if } |S| = 2, 7 \notin S \\
2 + a_4 & \text{if } |S| = 3, 7 \in S \\
1 + a_3 & \text{if } |S| = 3, 7 \notin S.
\end{cases}
\]

Consider the following partitions of \(\{1, \ldots, 7\} = \{a, b, c, d, e, f, 7\}\):

(i) \(\{a\} \cup \{b\} \cup \{c\} \cup \{d, e, f, 7\}\)

(ii) \(\{a\} \cup \{b\} \cup \{c, d\} \cup \{e, f, 7\}\)

(iii) \(\{7\} \cup \{a, b\} \cup \{c, d\} \cup \{e, f\}\).

According to [8], Theorem 2.1, the corresponding necessary conditions for \(\alpha^*(K_6 + D)\) to be anti-ample are:

(i) \(3a_2 - a_3 - 1 < 0\)

(ii) \(2a_3 - a_4 < 0\)

(iii) \(3a_4 - 3a_2 - a_6 + 2 < 0\).

Hence we deduce:

(ii) \(a_3 < \frac{a_4}{3}\)

(i) \(a_2 < \frac{1}{3} + \frac{a_3}{3} < \frac{1}{3} + \frac{a_6}{6}\)

(iii) \(a_2 > \frac{2 - a_6}{3} + a_4\)

which is impossible if \(a_4 \geq 0\) and \(a_6 \leq 1\).

\[\square\]

**Corollary 1.** If \(n \leq 5\) then \(\overline{M}_{0,n}(\mathbb{P}^1, 1)\) is a Mori dream space.

**Proof.** If \(n \leq 3\) we exploit the isomorphism \(\overline{M}_{0,n}(\mathbb{P}^1, 1) \cong \mathbb{P}^1[n]\), where \(\mathbb{P}^1[n]\) denotes the Fulton-MacPherson compactification (see [13], Remark 1.4) and the fact that \(\mathbb{P}^1[n]\) is Fano for \(n \leq 3\) (see [13], p. 108). If \(n = 4, 5\) then \(\overline{M}_{0,n}(\mathbb{P}^1, 1)\) is log Fano by Lemma [1] and Lemma [2]. Hence \(\overline{M}_{0,n}(\mathbb{P}^1, 1)\) is a Mori dream space for \(n \leq 5\) by [1], Corollary 1.3.2.

\[\square\]

**Remark 1.** By Lemma [3] there is no hope to deduce from [1], Corollary 1.3.2, that \(\overline{M}_{0,6}(\mathbb{P}^1, 1)\) is a Mori dream space.

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