Memorizing Gaussians with no over-parameterization via gradient decent on neural networks

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March 31, 2020

Abstract

We prove that a single step of gradient decent over depth two network, with q hidden neurons, starting from orthogonal initialization, can memorize \( \Omega \left( \frac{dq}{\log^4(d)} \right) \) independent and randomly labeled Gaussians in \( \mathbb{R}^d \). The result is valid for a large class of activation functions, which includes the absolute value.

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1 Introduction

In recent years, much attention has been given to the ability of neural networks, trained with gradient methods, to memorize datasets (e.g. [21, 9, 7, 5, 16, 10, 1, 2, 6, 22, 18, 11, 17, 4, 13, 6, 15, 14, 8]). The main question is “how large the networks should be in order to memorize a given dataset $S = \{(x_1, y_1), \ldots, (x_m, y_m)\} \subset \mathbb{R}^d \times \{\pm 1\}$?” Here, an example is considered memorized if $y_i h(x_i) > 0$ for the learned function $h$.

In order to memorize even just slightly more that half of the $m$ examples we need a network with at least $m$ parameters (up to poly-log factors). In this paper we will focus on the regime in which the number of parameters is $\tilde{O}(m)$. We will refer to this regime as near optimal memorization. To the best of our knowledge, there are very few results that proves near optimal memorization: Brutzkus et al. [5] implies near optimal memorization of linearly independent points (in particular, $m \leq d$). Ge et al. [11] implies near optimal memorization of $m \leq d^2$ points in general position if the activation is quadratic. Lastly, Daniely [8] shows near optimal memorization of random points in the sphere, for many activation functions, but requires weights initialization that is far from standard, and essentially makes the optimization process equivalent to NTK optimization [12].

In this paper we prove near optimal memorization of $m$ ($d$-dimensional) Gaussians, by depth-two network trained with gradient decent, starting from standard orthogonal initialization, and for a large family of activation functions.

Main Result. The input examples are denoted $(x_1, y_1), \ldots, (x_m, y_m)$. We assume that the $x_i$’s sampled independently from $\mathcal{N}(0, I_d)$, and the $y_i$’s are independent Rademacher random variables. The initial matrix $W \in M_{q,d}$ is assumed to be orthonormal. The activation $\sigma : \mathbb{R} \to \mathbb{R}$ is assumed to be (1) $O(1)$ Lipschitz, (2) piecewise twice differentiable with finitely many pieces and a uniform bounded on the second derivative in any piece, and (3) satisfies $\mathbb{E}_{X \sim \mathcal{N}(0,1)} \sigma'(X) = 0$. An example for such an activation function is the absolute value.

We consider depth two network which calculates the function

$$h_W(x) = \frac{1}{\sqrt{q}} \sum_{i=1}^{q} a_i \sigma(\langle w_i, x \rangle)$$

Where $a_i \in \{\pm 1\}$ satisfy $\sum_{i=1}^{q} a_i = O(\sqrt{q})$ (note that this is valid w.h.p. is the $a_i$’s are random). We consider a single gradient step on $W$, with step size of $\eta = \frac{m \ln(d)}{d}$, w.r.t. the scaled hinge loss $\ell : \mathbb{R} \times \{\pm 1\} \to [0, \infty)$, given by

$$\ell(\hat{y}, y) = (\ln(d) - \hat{y}y)_+$$

We denote by $W^+$ the weights after this single gradient step

**Theorem 1.** Assume that $m \leq \frac{dq}{\log^4(d)}$ and that $q \geq \log^4(d)$. We have that w.p. $1 - o(1)$, for every $i \in [m], y_i h_{W^+}(x_i) = \Omega(\ln(d))$. 

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Open Questions  Several obvious open questions arise from our work: To generalize the result to stochastic gradient decent, to more activation functions (and in particular, to the ReLU activation), to non-Gaussian inputs, and to more initialization schemes.

2  Proof of theorem 1

2.1  Some Tail Inequalities

Proof of all claims made in this section can be found in chapters 2 and 5 of Vershynin [20]. For a real random variable $X$ and $p \geq 1$ we denote

$$\|X\|_{\Psi_p} = \inf\{t : \mathbb{E}\exp(|X|^p/t^p) \leq 2\}$$

We say that $X$ is $\sigma$-Sub-gaussian if $\|X - \mathbb{E} X\|_{\Psi_2} \leq \sigma$. Likewise, we say that $X$ is $\sigma$-Sub-exponential if $\|X - \mathbb{E} X\|_{\Psi_1} \leq \sigma$. We will use the following facts. In the following claims $c$ and $C$ denote positive universal constants.

**Lemma 2.** 1. $\|X - \mathbb{E} X\|_{\Psi_1} \leq C \|X\|_{\Psi_2}$ and $\|X - \mathbb{E} X\|_{\Psi_2} \leq C \|X\|_{\Psi_2}$

2. $\|XY\|_{\Psi_1} \leq \|X\|_{\Psi_2} \|Y\|_{\Psi_2}$

3. If $X \sim \mathcal{N}(0, \sigma)$ then $\|X\|_{\Psi_2} \leq C \sigma$

4. $\|X\|_{\Psi_2} \leq C \|X\|_{\infty}$

5. $\Pr(|X| \geq t) \leq 2 \exp\left(-\frac{ct^2}{\|X\|_{\Psi_2}^2}\right)$

6. $\Pr(|X| \geq t) \leq 2 \exp\left(-\frac{ct}{\|X\|_{\Psi_1}}\right)$

**Theorem 3** (Hoeffding). For independent and centered real random variables $X_1, \ldots, X_N$ we have

$$\left\|\sum_{i=1}^{N} X_i\right\|_{\Psi_2}^2 \leq C \sum_{i=1}^{N} \|X_i\|_{\Psi_2}^2$$

In particular,

$$\Pr\left(\left|\sum_{i=1}^{N} X_i\right| \geq t\right) \leq 2 \exp\left(-\frac{ct^2}{\sum_{i=1}^{N} \|X_i\|_{\Psi_2}^2}\right)$$

**Theorem 4** (Bernstein). For independent and centered real random variables $X_1, \ldots, X_N$ we have

$$\Pr\left(\left|\sum_{i=1}^{N} X_i\right| \geq t\right) \leq 2 \exp\left(-c \min\left(\frac{t^2}{\sum_{i=1}^{N} \|X_i\|_{\Psi_1}^2}, \frac{t}{\max_i \|X_i\|_{\Psi_1}}\right)\right)$$

**Theorem 5** (Gaussian Concentration). Suppose that $X \sim \mathcal{N}(0, I_n)$ and that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is $L$-Lipschitz. Then

$$\|f(X) - \mathbb{E} f(X)\|_{\Psi_2} \leq CL$$
2.2 Proof

We first note that

**Lemma 6.** W.p. $1 - o(1)$, for all $i \in [m]$, $|h_{W}(x_i)| < O\left(\sqrt{\ln(d)}\right)$.

**Proof.** If $x \sim \mathcal{N}(0, I_d)$ then, since $W$ is orthogonal, $\langle w_1, x \rangle, \ldots, \langle w_q, x \rangle$ are independent standard Gaussians. Hence, since $\sigma$ is $O(1)$-Lipschitz and by theorem 5, $h_{W}(x) = \frac{1}{\sqrt{q}} \sum_{i=1}^{q} a_i \sigma(\langle w_i, x \rangle)$ is a sum of $q$ independent $O\left(\frac{1}{\sqrt{q}}\right)$-subgaussians. By theorem 3, $h_{W}(x)$ is $O(1)$-subgaussian. Likewise, $\mathbb{E}_x h_{W}(x) = \sum_{i=1}^{m} a_i \mathbb{E}_{X \sim \mathcal{N}} \sigma(x) = O(\sqrt{\eta}) O(1) = O(1)$. By lemma 2, for large enough universal constant $C > 0$, $\Pr\left(|h_{W}(x)| > C\sqrt{\ln(d)}\right) < \frac{1}{m}$. It follows that $\Pr\left(\exists i \in [m], |h_{W}(x_i)| > \frac{1}{m}\right)$.

It follows that w.p. $1 - o(1)$, for all examples, the hinge loss is in the non-zero part, and we have that $W^+ = W + \eta G$ where

$$G = \sum_{i=1}^{m} G^i \text{ for } G^i = \frac{1}{m\sqrt{q}} y_i \text{diag}(a)\sigma'(Wx_i)x_i^T$$

It is therefore enough to prove the following lemma:

**Lemma 7.** Assume that $m \leq \frac{dq}{\log^4(d)}$ and that $q \geq \log^4(d)$. We have that w.p. $1 - o(1)$, for every $i \in [m]$, $y_i h_{W+\eta G}(x_i) = \Omega\left(\ln(d)\right)$.

In the sequel we denote

$$\tilde{G} = \sum_{i=1}^{m-1} G^i$$

Likewise, we denote by $\tilde{g}_j, g_j$ and $g^i_j$ the $j$'th row of $\tilde{G}, G$ and $G^i$.

**Fact 8.** (e.g. chapter 5 in [19]) There are subsets $S_{d,\epsilon} \subset \mathbb{S}^{d-1}$ of size $(\frac{1}{\epsilon})^{\Theta(d)}$ such that for every matrix $W \in M_{q \times d}$ we have

$$\|W\| \leq (1 + \epsilon) \max_{u \in S_{q,\epsilon}, z \in S_{d,\epsilon}} \langle u, W, z \rangle$$

**Lemma 9.** We have that $\|\eta \tilde{G}\| \leq 2$ w.p. $\exp\left(O(d) - \Omega\left(\frac{d^2 \eta}{m \ln^2(d)}\right)\right)$

**Proof.** Let $S_{q,1}, S_{d,1}$ be the sets from fact 8. We have

$$\|\eta \tilde{G}\| \leq 2 \max_{u \in S_{q,1}, z \in S_{d,1}} \langle u, \eta \tilde{G} z \rangle$$

$$= 2 \max_{u \in S_{q,1}, z \in S_{d,1}} \ln(d) \frac{\max_{i=1}^{m-1} y_i (\text{diag}(a)\sigma'(Wx_i), u) \langle x_i, z \rangle}{\sqrt{q}}$$

$$= 2 \max_{u \in S_{q,1}, z \in S_{d,1}} \ln(d) \frac{\max_{i=1}^{m-1} y_i \langle \sigma'(Wx_i), u \rangle \langle x_i, z \rangle}{\sqrt{q}}$$

Fix $u \in S_{q,1}$ and $z \in S_{d,1}$. We claim that
Claim 10. $\sum_{i=1}^{m} y_i \langle \sigma'(Wx_i), u \rangle \langle xi, z \rangle$ is a sum of $m - 1$ independent and centered $O(1)$-Sub-exponential random variables

Proof. Clearly, $\sum_{i=1}^{m-1} y_i \langle \sigma'(Wx_i), u \rangle \langle xi, z \rangle$ is a sum of $m - 1$ independent and centered random variables. It remains to prove $O(1)$-Sub-exponentiality. By lemma 2 it is enough to show that $\langle \sigma'(Wx_i), u \rangle$ and $\langle xi, z \rangle$ are $O(1)$-Sub-gaussian. Indeed, $\langle xi, z \rangle \sim N(0,1)$ and hence by lemma 2 it is $O(1)$-Sub-gaussian. As for $\langle \sigma'(Wx_i), u \rangle = \sum_{j=1}^{q} u_j \sigma'(\langle w_j, x_i \rangle)$, we have that $\langle w_1, x_i \rangle, \ldots, \langle w_q, x_i \rangle$ are independent since $x_i$ is Gaussian and $W$ is orthogonal. Hence, $\langle \sigma'(Wx_i), u \rangle$ is a sum of independent random variables. Furthermore, for every $j$, $\langle w_j, x_i \rangle \sim N(0,1)$, and since we assume that $\mathbb{E}_{X \sim N(0,1)} \sigma'(X) = 0$, we conclude that $\langle \sigma'(Wx_i), u \rangle$ is a sum of independent and centered random variables. We can now use lemma 2 and theorem 3 to conclude that

$$\|\langle \sigma'(Wx_i), u \rangle\|^2_{\Psi_2} = \left\| \sum_{j=1}^{q} u_j \sigma'(\langle w_j, x_i \rangle) \right\|^2_{\Psi_2}$$

$$\leq C \sum_{j=1}^{q} u_j^2 \|\sigma'(\langle w_j, x_i \rangle)\|^2_{\Psi_2}$$

$$\leq C \|\sigma'\|_{\infty} \sum_{j=1}^{q} u_j^2$$

$$= C \|\sigma'\|_{\infty}$$

We can now use Bernstein inequality to conclude that

$$\Pr \left( \left| \frac{\ln(d)}{d^\sqrt{q}} \sum_{i=1}^{m-1} y_i \langle \sigma'(Wx_i), u \rangle \langle xi, z \rangle \right| \geq t \right) \leq \exp \left( -\Omega \left( \min \left( \frac{t^2 d^2 q}{m \ln^2(d)}, \frac{td\sqrt{q}}{\ln(d)} \right) \right) \right)$$

For $t = \frac{1}{2}$ and $m \geq \frac{d\sqrt{q}}{\ln(d)}$ we get

$$\Pr \left( \left| \frac{\ln(d)}{d^\sqrt{q}} \sum_{i=1}^{m-1} y_i \langle \sigma'(Wx_i), u \rangle \langle xi, z \rangle \right| \geq \frac{1}{2} \right) \leq \exp \left( -\Omega \left( \frac{d^2 q}{m \ln^2(d)} \right) \right)$$

Via a union bound on $S_{d,1} \times S_{d,1}$ we get that

$$2 \max_{u \in S_{d,1}, x \in S_{d,1}} \left| \frac{\ln(d)}{d^\sqrt{q}} \sum_{i=1}^{m-1} y_i \langle \sigma'(Wx_i), u \rangle \langle xi, z \rangle \right| \leq 1$$

w.p. $\exp \left( O(d + q) - \Omega \left( \frac{d^2 q}{m \ln^2(d)} \right) \right) = \exp \left( O(d) - \Omega \left( \frac{d^2 q}{m \ln^2(d)} \right) \right)$. Finally, the case $m < d\sqrt{q}$ can be reduced to the case $m \geq d\sqrt{q}$ by adding $(d\sqrt{q} - m)$ random variables which are identically 0, and noting that we are still left with a sum of independent and centered $O(1)$-subexponential random variables.
Lemma 11. Assume that \( m \leq dq \). For every \( i \) we have that

1. \( 1 \leq \mathbb{E} \|w_i + \eta g_i\| \leq 1 + \frac{m \ln^2(d)}{d q} \|\sigma'\|_\infty^2 \). Furthermore, the probability that \( \|w_i + \eta g_i\| \geq \frac{1}{\sqrt{d}} \) deviates from its expectation is at most \( \exp \left( -\Omega \left( \frac{d q^2}{m^2 \ln^2(d)} \right) \right) \)

2. \( \mathbb{E} \|\eta g_i\|^2 \leq \frac{m \ln^2(d)}{d q} \|\sigma'\|_\infty^2 \). Furthermore, the probability that \( \|\eta g_i\|^2 \geq \frac{1}{\sqrt{d}} \) deviates from its expectation is at most \( \exp \left( -\Omega \left( \frac{d q^2}{m^2 \ln^2(d)} \right) \right) \)

**Proof.** We will prove the first part of the lemma. The proof of second part is very similar. Denote \( g = g_i \) and \( w = w_i \). Since the input distribution is invariant to orthogonal transformations, we can assume w.l.o.g. we assume that \( w = e_1 \). We also assume that \( a_i = 1 \). The case \( a_i = -1 \) is similar. We have

\[
\|w + \eta g\|^2 = \left( w(1) + \frac{\ln(d)}{d q} \sum_{i=1}^{m-1} y_i \sigma'(x_i(1))x_i(1) \right)^2 + \sum_{j=2}^d \left( w(j) + \frac{\ln(d)}{d q} \sum_{i=1}^{m-1} y_i \sigma'(x_i(1))x_i(j) \right)^2
\]

\[
= \left( 1 + \frac{\ln(d)}{d q} \sum_{i=1}^{m-1} y_i \sigma'(x_i(1))x_i(1) \right)^2 + \sum_{j=2}^d \left[ \left( \frac{\ln(d)}{d q} \sum_{i=1}^{m-1} y_i \sigma'(x_i(1))x_i(j) \right)^2 - \frac{\ln^2(d)}{d q} \sum_{i=1}^{m-1} (\sigma'(x_i(1)))^2 \right]
\]

\[
+ \frac{\ln^2(d)(d-1)}{d^2 q} \sum_{i=1}^{m-1} (\sigma'(x_i(1)))^2
\]

Now, by theorem 3 we have that \( \frac{\ln(d)}{d q} \sum_{i=1}^{m-1} y_i \sigma'(x_i(1))x_i(1) \) is \( O \left( \frac{\sqrt{m \ln(d)}}{d q} \right) \)-sub-gaussian. This implies that the probability that

\[
\left( 1 + \frac{\ln(d)}{d q} \sum_{i=1}^{m-1} y_i \sigma'(x_i(1))x_i(1) \right)^2 \epsilon \text{-deviates from its expectation is bounded by } \exp \left( -\Omega \left( \frac{d q^2}{m^2 \ln^2(d)} \right) \right) + \exp \left( -\Omega \left( \frac{d q^2}{m^2 \ln^2(d)} \right) \right) = \exp \left( -\Omega \left( \frac{d q^2}{m^2 \ln^2(d)} \right) \right).
\]

Likewise,

\[
\mathbb{E} \left( 1 + \frac{\ln(d)}{d q} \sum_{i=1}^{m-1} y_i \sigma'(x_i(1))x_i(1) \right)^2 = 1 + \frac{\ln^2(d)}{d^2 q} \sum_{i=1}^{m-1} \mathbb{E} (\sigma'(x_i(1))x_i(1))^2 \leq 1 + \frac{\ln^2(d) \|\sigma'\|_\infty^2}{d^2 q} \sum_{i=1}^{m-1} \mathbb{E} (x_i(1))^2 = 1 + \frac{\ln^2(d)(m-1)\|\sigma'\|_\infty^2}{d^2 q}
\]

Theorem 3 also implies that the last line is \( O \left( \frac{\ln^2(d)\sqrt{m}}{dq} \right) \)-sub-gaussian. Thus, the probability that it \( \epsilon \)-deviates from its expectation is bounded by \( \exp \left( -\Omega \left( \frac{d q^2\sqrt{m}}{m^2 \ln^2(d)} \right) \right) \). Likewise, its expectation is bounded by \( \frac{\ln^2(d)(d-1)m\|\sigma'\|_\infty^2}{d^2 q} \) from above and by 0 from below.
Finally, given $x_1(1), \ldots, x_{m-1}(1)$ and $y_1, \ldots, y_{m-1}$, the middle line is a sum of $d-1$ independent random variables. Each of which has zero mean and is $O\left(\frac{m \ln^2(d)}{d^2 q}\right)$-sub-exponential. By Berstein inequality, the probability that it $\epsilon$-deviates from its expectation is bounded by $\exp\left(-\Omega\left(\frac{d^2 \epsilon^2}{m^2 \ln^2(d)}\right)\right) + \exp\left(-\Omega\left(\frac{d^2 \epsilon^2}{m \ln^2(d)}\right)\right)$. Choosing $\epsilon = \frac{1}{\sqrt{d}}$, we conclude that the probability that $\|w + \eta u\|_2 \frac{1}{\sqrt{d}}$-deviates from its expectation is at most $\exp\left(-\Omega\left(\frac{d^2 \epsilon^2}{m^2 \ln^2(d)}\right)\right)$. As for the expectation, since the expectation of the middle line is 0, the total expectation is bounded by

$$1 + \frac{\ln^2(d) m\|\sigma'\|_\infty^2}{d^2 q} + 0 + \frac{\ln^2(d)(d-1)m\|\sigma'\|_\infty^2}{d^2 q} = 1 + \frac{m \ln^2(d)}{d q} \|\sigma'\|_\infty^2$$

from above and by $1 + 0 + 0 = 1$ from below.

We are now ready to prove lemma 7, and therefore also theorem 1.

**Proof.** (of lemma 7) We will prove the theorem under the assumption that $\sigma$ is twice differentiable everywhere. We will later explain how to amend the proof in the case that it is only piece-wise twice differentiable. It is enough to show that w.p. $1 - o\left(\frac{1}{m}\right)$, $y_m h_{W + \eta G}(x_m) = \Omega\left(\ln(d)\right)$. Throughout the proof, w.h.p., means “w.p. $1 - o\left(\frac{1}{m}\right)$”. Note that if $O(1)$ events holds w.h.p., then so is their union. We have

$$h_{W + \eta G}(x_m) = h_{W + \eta G + \eta G_m}(x_m) - h_{W + \eta G_m}(x_m) + h_{W + \eta G}(x_m)$$

The proof of the lemma follows from the following two claims.

**Claim 12.** W.h.p. $h_{W + \eta G}(x_m) = O\left(\sqrt{\log(d)}\right)$.

**Proof.** By lemma 9, and since $m \leq \frac{d q}{\log(d)}$, we have that w.h.p. $\|W + \eta G\| \leq 3$. Likewise, lemma 11 implies that w.h.p., for all $i$, $\|w_i + \eta \tilde{g}_i\| = \sqrt{\mathbb{E} \|w_i + \eta \tilde{g}_i\|^2} + \delta_i$, where $\delta_i = O\left(\frac{1}{\sqrt{q}}\right)$. We will show that the claim holds w.h.p. given these two events.

First, since $\|W + \eta \tilde{G}\| \leq 3$, $x \mapsto h_{W + \eta \tilde{G}}(x)$ is $O(1)$-Lipschitz, as a composition of the $O(1)$-Lipschitz functions $x \mapsto (W + \eta \tilde{G}) x$, $x \mapsto \sigma(x)$, and $x \mapsto \frac{1}{\sqrt{q}} \sum_{i=1}^q a_i x_i$.

It follows that, w.h.p., by Lipschitz Gaussian concentration (theorem 5) we have that $h_{W + \eta G}(x_m)$, is $O(1)$ Sub-Gaussian. Hence, w.h.p., its distance from its expectation is $O\left(\sqrt{\log(d)}\right)$. It therefore enough to show that $\mathbb{E}_{x_m} h_{W + \eta \tilde{G}}(x_m) = O(1)$. Since $\|w_i + \eta \tilde{g}_i\| = \sqrt{\mathbb{E} \|w_i + \eta \tilde{g}_i\|^2} + \delta_i$, where $\delta_i = O\left(\frac{1}{\sqrt{q}}\right)$, we can write $\langle w_i + \eta \tilde{g}_i, x_m \rangle = X + Y_i$, where $X$ is a centered Gaussian of variance $\mathbb{E} \|w_i + \eta \tilde{g}_i\|^2$, and $Y_i$ is a centered Gaussian of variance
\( O \left( \frac{1}{d} \right) \). We have that
\[
\mathbb{E}_{x_m} h_{W + \eta G}(x_m) = \frac{1}{\sqrt{q}} \sum_{j=1}^{q} a_i \mathbb{E}_{x_m} \sigma(\langle w_i + \eta \tilde{g}_i, x_m \rangle)
\]
\[
= \frac{1}{\sqrt{q}} \sum_{j=1}^{q} a_i \mathbb{E}_{X,Y} \sigma(X + Y_i)
\]
\[
= \frac{1}{\sqrt{q}} \sum_{j=1}^{q} a_i \mathbb{E}_{X,Y} \sigma(X) + \sigma(X + Y_i) - \sigma(X)
\]
\[
\sum_{j=1}^{q} a_i = O(\sqrt{q})
\]
\[
\Rightarrow O(1) + \frac{1}{\sqrt{q}} \sum_{j=1}^{q} a_i \mathbb{E}_{X,Y} \sigma(X + Y_i) - \sigma(X)
\]

Now, for every fixed \( x \) we have, since \( \sigma \) is \( O(1) \)-Lipschitz,
\[
\left| \mathbb{E}_{Y_i} \sigma(x + Y_i) - \sigma(x) \right| \leq O(1) \mathbb{E}_{Y_i} |Y_i| = O \left( \frac{1}{\sqrt{d}} \right)
\]

It therefore follows that \( \mathbb{E}_{x_m} h_{W + \eta G}(x_m) = O(1) + O \left( \frac{\sqrt{q}}{d} \right) = O(1) \)

**Claim 13.** W.h.p. \( y_m \left[ h_{W + \eta G + \eta G_m}(x_m) - h_{W + \eta G}(x_m) \right] = \Omega \left( \log \left( \frac{d}{q} \right) \right) \)

**Proof.** We first note that by lemma 11 we have that, w.h.p., for every \( i \in [q] \), \( \| \eta g_i \| = O \left( \frac{1}{\log(d)} \right) \). Hence, w.h.p., for every \( i \in [q] \), \( |\langle \eta \tilde{g}_i, x_m \rangle| = O \left( \frac{1}{\sqrt{\log(d)}} \right) \). Fix \( i \in [q] \). Recall that
\[
\eta g_i^m = \frac{a_i y_m \log(d)}{\sqrt{qd}} \sigma' \left( \langle w_i, x_m \rangle \right) x_m.
\]
Likewise, w.h.p., \( \frac{\|x_m\|^2}{d} = 1 + o(1) \). We have that, w.h.p.,
\[
\sigma(\langle w_i + \eta \tilde{g}_i + \eta g_i^m, x_m \rangle) - \sigma(\langle w_i + \eta \tilde{g}_i, x_m \rangle)
\]
\[
= \sigma' \left( \langle w_i + \eta \tilde{g}_i, x_m \rangle \right) (\eta g_i^m, x_m) + O(1) \left( |\langle \eta \tilde{g}_i, x_m \rangle| \right)^2
\]
\[
= \frac{\ln(d)}{d \sqrt{q}} \sigma' \left( \langle w_i + \eta \tilde{g}_i, x_m \rangle \right) |\langle \eta \tilde{g}_i, x_m \rangle| \frac{\|x_m\|^2}{d} + O(1) \left( \frac{\ln(d)}{d \sqrt{q}} \right)^2
\]
\[
= \frac{y_m a_i \ln(d)(1 + o(1))}{\sqrt{q}} \sigma' \left( \langle w_i + \eta \tilde{g}_i, x_m \rangle \right) \sigma' \left( \langle w_i, x_m \rangle \right)
\]
\[
+ O \left( \frac{\ln^2(d)}{q} \right)
\]
\[
\text{\( \sigma' \) is \( O(1) \)-Lip.}
\]
\[
= \frac{y_m a_i \ln(d)(1 + o(1))}{\sqrt{q}} \sigma' \left( \langle w_i, x_m \rangle \right) + O \left( \frac{\ln^2(d)}{d} \right)
\]
\[
= \frac{y_m a_i \ln(d)(1 + o(1))}{\sqrt{q}} \sigma' \left( \langle w_i, x_m \rangle \right) + o(1) \sigma' \left( \langle w_i, x_m \rangle \right)
\]
\[
+ O \left( \frac{\ln^2(d)}{q} \right)
\]
\[
= \frac{y_m a_i \ln(d)(1 + o(1))}{\sqrt{q}} \sigma' \left( \langle w_i, x_m \rangle \right)^2 + \frac{o(\log(d))}{\sqrt{q}} + O \left( \frac{\ln^2(d)}{q} \right)
\]
\[
q \geq \log^4(d)
\]
\[
= \frac{y_m a_i \ln(d)(1 + o(1))}{\sqrt{q}} \sigma' \left( \langle w_i, x_m \rangle \right)^2 + \frac{o(\log(d))}{\sqrt{q}}
\]

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It follows that

\[
h_{W+\eta\hat{G}+\eta G_m}(x_m) - h_{W+\eta\hat{G}}(x_m) = \frac{1}{\sqrt{q}} \sum_{i=1}^{q} a_i \left( \sigma \left( \langle w_i + \eta g_i \rangle_{\mu} + \eta g_{m,i} \right), x_m \right) - \sigma \left( \langle w_i + \eta g_i \rangle_{\mu}, x_m \right)
\]

\[
= o(\log(d)) + \frac{y_m \ln(d)(1+o(1))}{q} \sum_{i=1}^{q} \left( \sigma' \left( \langle w_i, x_m \rangle \right) \right)^2
\]

\[
= o(\log(d)) + y_m \ln(d)(1+o(1)) \mathbb{E}_{X \sim N(0,1)} \left( \sigma'(X) \right)^2
\]

\[
= y_m \ln(d)(1+o(1)) \mathbb{E}_{X \sim N(0,1)} \left( \sigma'(X) \right)^2
\]

To handle the case that \( \sigma \) is only piece-wise twice differentiable (with finitely many pieces), one should observe that \( 1 - o(1) \) of the neurons we have that \( \langle w_i, x_m \rangle \) is well inside one of the pieces, so that the estimation of \( \sigma \left( \langle w_i + \eta g_i \rangle_{\mu} + \eta g_{m,i} \right), x_m \rangle - \sigma \left( \langle w_i + \eta g_i \rangle_{\mu}, x_m \rangle \right) \) is still valid. Likewise, the remaining neurons effect \( h_{W+\eta\hat{G}+\eta G_m}(x_m) - h_{W+\eta\hat{G}}(x_m) \) by \( o(\log(d)) \), and hence the estimation of \( h_{W+\eta\hat{G}+\eta G_m}(x_m) - h_{W+\eta\hat{G}}(x_m) \) remains valid.

Acknowledgments

This research is partially supported by ISF grant 2258/19

References

[1] Zeyuan Allen-Zhu, Yuanzhi Li, and Yingyu Liang. Learning and generalization in overparameterized neural networks, going beyond two layers. arXiv preprint arXiv:1811.04918, 2018.

[2] Zeyuan Allen-Zhu, Yuanzhi Li, and Zhao Song. A convergence theory for deep learning via over-parameterization. arXiv preprint arXiv:1811.03962, 2018.

[3] A. Andoni, R. Panigrahy, G. Valiant, and L. Zhang. Learning polynomials with neural networks. In Proceedings of the 31st International Conference on Machine Learning, pages 1908–1916, 2014.

[4] Sanjeev Arora, Simon S Du, Wei Hu, Zhiyuan Li, and Ruosong Wang. Fine-grained analysis of optimization and generalization for overparameterized two-layer neural networks. arXiv preprint arXiv:1901.08584, 2019.

[5] Alon Brutzkus, Amir Globerson, Eran Malach, and Shai Shalev-Shwartz. Sgd learns over-parameterized networks that provably generalize on linearly separable data. arXiv preprint arXiv:1710.10174, 2017.
[6] Yuan Cao and Quanquan Gu. Generalization bounds of stochastic gradient descent for wide and deep neural networks. *arXiv preprint arXiv:1905.13210*, 2019.

[7] Amit Daniely. Sgd learns the conjugate kernel class of the network. In *Advances in Neural Information Processing Systems*, pages 2422–2430, 2017.

[8] Amit Daniely. Neural networks learning and memorization with (almost) no over-parameterization. *arXiv preprint arXiv:1911.09873*, 2019.

[9] Amit Daniely, Roy Frostig, and Yoram Singer. Toward deeper understanding of neural networks: The power of initialization and a dual view on expressivity. In *NIPS*, 2016.

[10] Simon S Du, Xiyu Zhai, Barnabas Poczos, and Aarti Singh. Gradient descent provably optimizes over-parameterized neural networks. *arXiv preprint arXiv:1810.02054*, 2018.

[11] Rong Ge, Runzhe Wang, and Haoyu Zhao. Mildly overparametrized neural nets can memorize training data efficiently. *arXiv preprint arXiv:1909.11837*, 2019.

[12] Arthur Jacot, Franck Gabriel, and Clément Hongler. Neural tangent kernel: Convergence and generalization in neural networks. In *Advances in neural information processing systems*, pages 8571–8580, 2018.

[13] Ziwei Ji and Matus Telgarsky. Polylogarithmic width suffices for gradient descent to achieve arbitrarily small test error with shallow relu networks. *arXiv preprint arXiv:1909.12292*, 2019.

[14] Jaehoon Lee, Lechao Xiao, Samuel S Schoenholz, Yasaman Bahri, Jascha Sohl-Dickstein, and Jeffrey Pennington. Wide neural networks of any depth evolve as linear models under gradient descent. *arXiv preprint arXiv:1902.06720*, 2019.

[15] Chao Ma, Lei Wu, et al. A comparative analysis of the optimization and generalization property of two-layer neural network and random feature models under gradient descent dynamics. *arXiv preprint arXiv:1904.04326*, 2019.

[16] Samet Oymak and Mahdi Soltanolkotabi. Overparameterized nonlinear learning: Gradient descent takes the shortest path? *arXiv preprint arXiv:1812.10004*, 2018.

[17] Samet Oymak and Mahdi Soltanolkotabi. Towards moderate overparameterization: global convergence guarantees for training shallow neural networks. *arXiv:1902.04674 [cs, math, stat]*, February 2019. URL http://arxiv.org/abs/1902.04674. arXiv: 1902.04674.

[18] Zhao Song and Xin Yang. Quadratic suffices for over-parametrization via matrix chernoff bound. *arXiv preprint arXiv:1906.03593*, 2019.

[19] Ramon van Handel. Probability in high dimension. Technical report, PRINCETON UNIV NJ, 2014.
[20] Roman Vershynin. High-dimensional probability, 2019.

[21] Bo Xie, Yingyu Liang, and Le Song. Diverse neural network learns true target functions. *arXiv preprint arXiv:1611.03131*, 2016.

[22] Difan Zou and Quanquan Gu. An improved analysis of training over-parameterized deep neural networks. *arXiv preprint arXiv:1906.04688*, 2019.