Determinations of rational Dedekind-zeta invariants of hyperbolic manifolds and Feynman knots and links

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Abstract We identify 998 closed hyperbolic 3-manifolds whose volumes are rationally related to Dedekind zeta values, with coprime integers \(a\) and \(b\) giving

\[
\frac{a}{b} \text{vol}(\mathcal{M}) = \frac{(-D)^{3/2}}{(2\pi)^{2n-4}} \frac{\zeta_K(2)}{2\zeta(2)}
\]

for a manifold \(\mathcal{M}\) whose invariant trace field \(K\) has a single complex place, discriminant \(D\), degree \(n\), and Dedekind zeta value \(\zeta_K(2)\). The largest numerator of the 998 invariants of Hodgson–Weeks manifolds is, astoundingly, \(a = 2^4 \times 23 \times 37 \times 691 = 9,408,656\); the largest denominator is merely \(b = 9\). We also study the rational invariant \(a/b\) for single-complex-place cusped manifolds, complementary to knots and links, both within and beyond the Hildebrand–Weeks census. Within the censi, we identify 152 distinct Dedekind zetas rationally related to volumes. Moreover, 91 census manifolds have volumes reducible to pairs of these zeta values. Motivated by studies of Feynman diagrams, we find a 10-component 24-crossing link in the case \(n = 2\) and \(D = -20\). It is one of 5 alternating platonic links, the other 4 being quartic. For 8 of 10 quadratic fields distinguished by rational relations between Dedekind zeta values and volumes of Feynman orthoschemes, we find corresponding links. Feynman links with \(D = -39\) and \(D = -84\) are missing; we expect them to be as beautiful as the 8 drawn here. Dedekind-zeta invariants are obtained for knots from Feynman diagrams with up to 11 loops. We identify a sextic 18-crossing positive Feynman knot whose rational invariant, \(a/b = 26\), is 390 times that of the cubic 16-crossing non-alternating knot with maximal \(D_9\) symmetry. Our results are secure, numerically, yet appear very hard to prove by analysis.

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1 Introduction

This paper began with a study of hyperbolic links whose complementary volumes result from evaluations of Feynman diagrams [1]. For each of these volumes we found, by empirical means, a closed–form evaluation in terms of a few Clausen functions, evaluated at rational multiples of \( \pi \). Then, thanks to generous advice from Don Zagier, we came to appreciate that these results form part of a much larger scaffold, whose unifying feature is the existence of rational relations between hyperbolic volumes and Dedekind zeta values. While the existence of such relations is understood [2, 3, 4], their precise forms appear to be unpredictable, thus far, by deductive mathematics. They are therefore ripe for the application of experimental mathematics. The work reported herein has involved a – to the authors – quite intoxicating mix of tools, allowing conjectures to be tested against known data and rapidly confirmed or rejected as the research progressed. We will return to these methodological issues in the conclusion.

For any single-complex-place field \( K \), with discriminant \( D \) and degree \( n \), we define

\[
Z_K := \frac{(-D)^{3/2}}{(2\pi)^{2n-4}} \frac{\zeta_K(2)}{2\zeta(2)}
\]

where the Dedekind zeta value \( \zeta_K(2) \) is the sum of the inverse squares of the norms of the ideals \( \mathfrak{a}, \mathfrak{b} \) of \( K \), and \( \zeta(2) := \sum_{n>0} 1/n^2 = \pi^2/6 \) is the corresponding Riemann zeta value. It follows from [2] that \( Z_K \) is reducible, with unspecified\(^1\) rational coefficients, to Bloch–Wigner dilogarithms \( \{D(z_k) \mid z_k \in K\} \). The volume of a hyperbolic manifold, for which the single-complex-place field \( K \) is the invariant trace field [7], is systematically [8] reducible to such dilogarithms and, moreover, is expected to be some unspecified rational multiple of the very specific construct (1). Accordingly, we seek coprime integers \( a \) and \( b \) such that

\[
\frac{a}{b} \text{vol}(\mathcal{M}) = Z_K
\]

for a manifold \( \mathcal{M} \) with a single-complex-place invariant trace field \( K \). We call \( a/b \) the rational Dedekind-zeta invariant of \( \mathcal{M} \). It evaluates to unity for manifold m003(−3, 1), of conjecturally smallest volume. It will be seen to take some remarkable values.

In Section 2, we report findings of the rational invariant for 998 closed manifolds.

In Section 3, we find a further 12 single-complex-place fields from cusped manifolds, and compute corresponding rational invariants. We also identify 91 census manifolds whose volumes are reducible to pairs of Dedekind zeta values.

Section 4 concerns links, with complementary manifolds of a richer structure than those recorded in the Hildebrand–Weeks [3, 10] census of cusped manifolds triangulated by no more than 7 tetrahedra. In particular we identify a 24-crossing 10-component link, triangulated by 54 ideal tetrahedra with shapes in the quadratic field \( \mathbb{Q}(\sqrt{-5}) \) and determine its rational Dedekind-zeta invariant. We call it a platonic link, since its 10 components mimic the vertices and edges of a tetrahedron. The remaining 4 platonic

\(^1\)Ratios of coefficients may be specified; here we ask for values.
Section 5 concerns connections [1, 11, 12, 13, 14] between Feynman diagrams, knots and links. Recent results [1] from quantum field theory suggested a connection between Dedekind zeta values and volumes of orthoschemes. We find 10 quadratic fields that forge such connections and in 8 cases now know links that have corresponding rational Dedekind-zeta invariants. The remaining 2 cases are quadratic fields with discriminants \( D = -39 \) and \( D = -84 \); these await the discovery of corresponding Feynman links. Hyperbolic knots with up to 18 crossings, from Feynman diagrams with up to 11 loops, are also analyzed. For crossing numbers greater than 9, the physics in [11, 12, 13, 14] is a more fertile source of Dedekind zeta values than an analysis of the maximally symmetric knots found in [15].

Section 6 offers some conclusions and suggestions for further study.

2 Dedekind-zeta invariants of closed manifolds

We have found rational Dedekind-zeta invariants for 998 manifolds in the Hodgson–Weeks census, which lists 11,031 closed orientable manifolds, with 9,218 distinct volumes, all less than 6.5, and with all their geodesics having lengths greater than 0.3. These 998 manifolds have 224 distinct volumes, which are rationally related, via \([12]\), to 140 distinct Dedekind zeta values, each corresponding to an invariant trace field of at least one census manifold. Table 1 enumerates the fields by degree. Tables 2–12 give, for each field \( K \), a generating polynomial, together with its discriminant and the rational relation of \( Z_K \) to the volume of the first manifold in the census with invariant trace field \( K \). The 224 distinct volumes are given in Tables 13a–f, where it can be observed that numerators and denominators of the 998 invariants are bounded by \( a \leq 9408656 \) and \( b \leq 9 \). Our methods were as follows.

2.1 Examination of 1,200 single-complex-place fields

It became clear, from studying \([16, 17, 18, 19]\), that there are unspecified rational relations of the form \([2]\) between Dedekind zeta values and volumes of some, but not all, of the manifolds in the Hodgson–Weeks census. We soon found that 30 of the first 32 census manifolds have rational Dedekind-zeta invariants, with \( a \leq 46 \) and \( b \leq 3 \). The 2 exceptions are manifolds \( m003(-4,1) \) and \( m004(+7,1) \), whose invariant trace fields have 2 complex places.

The next step was to obtain systematic listings of fields with precisely one complex place. Thanks to the numberfields directory\(^2\) at the University of Bordeaux, we found files that order single-complex-place fields, for each degree \( n \leq 7 \), by the magnitudes, \( -D \), of the discriminant. We selected the first 200 fields for each degree \( n \in \{2, 3, 4, 5, 6, 7\} \) and used the \texttt{zetak} command of Pari to compute 19 digits of the Dedekind zeta values, \( \zeta_K(2) \),

\(^2\)\url{ftp://megrez.math.u-bordeaux.fr/pub/numberfields}
of these 1,200 fields. Forming $Z_K$, defined in (1), we made $1200 \times 11031 = 13,237,200$ indiscriminate comparisons between these target fields and the census entries. This took 19 seconds on a 233MHz Pentium. We found 842 manifolds, with 175 distinct volumes, rationally related to 96 distinct Dedekind zeta values, with numerators and denominators of the rational invariants bounded by $a \leq 46$ and $b \leq 5$.

Since the census provided 17-digit volumes, the probability of any of these simple rational results being spurious is comfortably less than $10^{-12}$. Conversely, one needs only 5-digit precision for the volume to discover the rational Dedekind-zeta invariant in these 842 cases, and hence to obtain an accurate volume from (1). Only for degrees $n \geq 8$ do precision, core memory and CPUtime become issues. In the $n = 12$ case of Table 12, with $|D| = 12, 476, 239, 474, 594, 496$, these issues require close attention.

The 96 fields that were caught by this 19-second trawl comprise all those of Tables 2 and 3 and all those above the lines drawn in Tables 4–7. Below the lines, and in Tables 8–12, we took assistance from Melbourne [7].

### 2.2 Examination of 44 fields found by Snap

For the remaining $140 - 96 = 44$ results of Tables 2–12, we made reference to the impressive body of files\(^3\) at the University of Melbourne. These were obtained by Coulson, Goodman, Hodgson and Neumann\([7]\) (hereafter referred to as CGHN) using 50-digit precision, in marked distinction to the 5 digits which suffice for the findings above.

CGHN make no reference to Dedekind zeta values; they do, however, find single-complex-place invariant trace fields for about 9% of the Hodgson–Weeks manifolds. The next step was to examine the overlap between our single-complex-place fields and theirs. We found that the CGHN file `closed_census_algebras` contains 95 of our 96 single-complex-place fields. The exception was the quintic field with $D = -14103$, for which our low-precision method had readily yielded the rational invariant $a/b = 2$ of the manifold s784(+5, 2), which was absent from the file `closed_census_algebras`, obtained by CGHN from 50-digit searches for fields of degree\[^4\] not exceeding 16. We then referred to the file `closed.fields`, where s784(+5, 2) indeed appears, with the expected invariant trace field, $x^5 - 2x^4 + 2x^3 - x^2 - 2x - 1$.

Thus assured that all our 96 finds were genuine, we extracted all of the invariant trace fields from the CGHN files `closed_census_algebras` and `closed.fields`, and asked Pari to determine their signatures and discriminants. This revealed 44 cases beyond the ranges of our search, namely the 32 below the lines in Tables 4–7 and the 12 in Tables 8–12. Within our selected ranges, CGHN had found all and only our 96 single-complex-place fields, indicating the reliability of both their methods and ours.

The next step was obvious, yet computationally demanding: to determine rational Dedekind-zeta invariants for the manifolds associated by Snap to the remaining 44 large-discriminant single-complex-place fields. The `zetak` command of Pari was entirely ade-

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\[^3\]http://www.ms.unimelb.edu.au/~snap

\[^4\]We later ran Snap at 100-digit precision, and found a group field of degree 20.
quate to complete Tables 4–7, at 19-digit precision, yielding numerators and denominators bounded by $a \leq 976$ and $b \leq 5$, for which merely 7-digit precision would have been amply sufficient. The 12 fields of Tables 8–12 presented a much tougher computational challenge.

At degree $n = 8$, Pari-GP 2.0.11 began to falter, with zetak yielding results drastically below the requested precision (as warned in the Pari manual) or running out of memory (when allocated 100MB of core). Accordingly we devised a method based on the dirzetak command, which returns the multiplicities, $m_K(n)$, of the ideals of $K$ with norm $n$, up to some requested maximum, $N$, that may comfortably extend to $N = 10^6$. These multiplicities are the numerators of the Dirichlet series

$$\zeta_K(2) = \sum_{n>0} \frac{m_K(n)}{n^2}$$

We then computed the truncated sums $S_N := \sum_{n=1}^N m_K(n)/n^2$, and accelerated their convergence by forming $T_{N,M} := (S_N N - S_M M)/(N - M)$, with $M \approx \frac{1}{2} N$. This removes a predictable $O(1/N)$ truncation error, leaving an $O(1/N^{3/2})$ “random-walk” error. With $N \leq 10^6$, we were able to achieve 10-digit precision, by averaging out fluctuations in $T_{N,M}$. Even in the most demanding case of Table 12, where the rational invariant is a 7-digit integer, we were left with 3 vanishing decimal places for $a/b = 9408656.000 \pm 0.001$. Hence we completed the task for Hodgson–Weeks manifolds. Tables 2–12 give 140 single-complex-place invariant trace fields of closed census manifolds, of which 96 were detected by us without reference to CGHN. The 224 distinct census volumes rationally related to the Dedekind zeta values of these fields are given in Tables 13a-f. These 224 distinct volumes correspond to 998 of the 11,031 entries of the Hodgson–Weeks census file ClosedManifolds available from the Geometry Center at the University of Minnesota.

2.3 A special case: manifold v3066($-1, 2$)

We note a peculiarity of manifold v3066($-1, 2$), with the same volume as v3066($+1, 2$). The latter has unit Dedekind-zeta invariant and hence a volume

$$Z_{K_+} := \frac{59^{3/2}}{2(2\pi)^2} \zeta_{K_+}(2) = 5.137941201873417769841348339 \ldots$$

where the cubic invariant trace field of v3066($+1, 2$) is

$$K_+ := x^3 + 2x - 1$$

with discriminant $-59$. Snap, working at 100-digit precision, gave the invariant trace field of v3066($-1, 2$) as a join of (5) with $Q(\sqrt{-59})$, generated by a root of the sextic

$$K_- := x^6 - 3x^5 + 10x^4 - 15x^3 + 21x^2 - 14x + 4$$

with discriminant $(-59)^3$ and 3 complex places. The Borel regulator [7, 20] of v3066($-1, 2$) was given as $[Z_{K_+}, -\frac{1}{2}Z_{K_+}, -\frac{1}{2}Z_{K_+}]$. It thus appears that a single-complex-place invariant trace field is sufficient, yet not necessary, to give a rational Dedekind-zeta invariant.

5The 1995 file ftp://ftp.geom.umn.edu/pub/software/snappea/tables/ClosedManifolds contains 11,031 entries; unfortunately some of the Dehn fillings do not correspond to those packaged, internally, with SnapPea. We enlisted Snap, to check all the surgeries given in Tables 2–13.
2.4 How may one derive the rational invariant?

Some of our results contain non-trivial factors, with 4-digit and 3-digit primes in $3 \times 1223 = 3669$ and $2^4 \times 3 \times 5 \times 967 = 232,080$, from Table 10, and $2 \times 3 \times 7 \times 11 \times 149 = 68,838$, from Table 11. Their origins are obscure. The duodecadic example of Table 12 entails $a/b = 2^4 \times 23 \times 37 \times 691 = 9,408,656$. We note, though cannot explain, the circumstance that its largest prime factor also occurs in $\zeta(12)/\pi^{12} = \frac{691}{638512875}$. Reverse engineering gave 50 decimal places of

$\zeta(K_{12}) = 1.0609569959254003575169823863253126692628705990 \ldots$ (7)

$K_{12} := x^{12} - 3x^{11} - 8x^{10} + 17x^9 + 27x^8 - 19x^7 - 50x^6$

$- 24x^5 + 44x^4 + 37x^3 - 5x^2 - 8x - 1$ (8)

from high-precision triangulation of manifold $v2824(+4, 1)$. We know of no easy way to check (7) beyond the first 10 digits, confirmed by accelerated convergence of a million truncations of (3).

Arguments from $K$-theory [16, 19] appear powerless to derive values for the Dedekind $d$-zeta invariant, $a/b$, though they imply [2, 3, 4] its rationality, for every manifold whose invariant trace field has a single complex place. It thus remains a challenge to derive the simple value $a/b = 4$, for the quadratic manifold $m036(-4, 3)$, with invariant trace field $Q(\sqrt{-7})$ and volume observed, at 1800-digit precision, to coincide with

$\frac{1}{4}Z_{Q(\sqrt{-7})} = \frac{7}{4} \left\{ \text{Cl}_2 \left( \frac{2\pi}{7} \right) + \text{Cl}_2 \left( \frac{4\pi}{7} \right) - \text{Cl}_2 \left( \frac{6\pi}{7} \right) \right\}$ (9)

$= 2.66674478344905979079671246261065004409838388855263 \ldots$ (10)

whose reduction to

$\text{Cl}_2(\theta) := \sum_{n>0} \frac{\sin(n\theta)}{n^2}$ (11)

is derived in Section 4. Triangulation gives

$\text{vol}(m036(-4, 3)) = 2D \left( -e^{-i\theta_7} \right) + D \left( -\frac{1}{2}e^{-i\theta_7} \right)$ (12)

$\theta_7 := 2 \arctan \sqrt{7}$ (13)

where the Bloch–Wigner dilogarithm

$D(z) := \text{Li}_2(z) + \log |z| \text{Li}(1 - z) = \text{Li} \sum_{n>0} \left( \frac{1}{n^2} - \frac{\log |z|}{n} \right) z^n$ (14)

gives the volume of an ideal tetrahedron whose essential dihedral angles are the arguments of $\{z, 1/(1-z), 1-1/z\}$. The 1800-digit agreement of (12) with (3) is compelling evidence that, with $\theta_7 := 2 \arctan \sqrt{7}$,

$\frac{3}{2} \text{Cl}_2(\theta_7) - \frac{3}{2} \text{Cl}_2(2\theta_7) + \frac{1}{2} \text{Cl}_2(3\theta_7) = \frac{7}{4} \left\{ \text{Cl}_2 \left( \frac{2\pi}{7} \right) + \text{Cl}_2 \left( \frac{4\pi}{7} \right) - \text{Cl}_2 \left( \frac{6\pi}{7} \right) \right\}$ (15)
An elementary proof of this remarkable relation between Clausen values has escaped both us, and also, it appears, the authors of [16, 17, 18, 19]. To cast it in a less classical form, we multiply (15) by 2, and transform to Bloch–Wigner functions, obtaining

\[ 4D \left( \frac{3 + i\sqrt{7}}{4} \right) + 2D \left( \frac{3 + i\sqrt{7}}{8} \right) = 7D \left( \frac{1 + i\tan\frac{\pi}{7}}{3 - \tan^2\frac{\pi}{7}} \right) \]

where the argument on the r.h.s. is a root of the totally complex sextic

\[ K_6 := x^6 - 4x^5 + 9x^4 - 8x^3 + 4x^2 - 2x + 1 \]

with discriminant \((-7)^5\). The l.h.s. of (16) is the volume of the cusped manifold s776, which is complementary to the 3-component 6-crossing link \(6_1\). Its triangulation immediately yields 6 ideal tetrahedra, with 2 distinct shapes in the quadratic field \(\mathbb{Q}(\sqrt{-7})\). The challenge is to prove that by combining these 6 one forms the same volume as from the 7 congruent sextic shapes on the r.h.s. of (16). This is a modern version of our puzzle (15) in classical analysis: how may one relate Clausen values at multiples of \(\theta_7 := 2\arctan\sqrt{7}\) to Clausen values at multiples of \(2\pi/7\), when there appears to be no non-trivial relation between trigonometric functions of the two sets of angles? A far greater challenge would be to derive, rather than merely measure, the integer Dedekind-zeta invariant \(2^4 \times 23 \times 37 \times 691\) of the closed manifold \(v2824(+4,1)\).

3 Cusped manifolds and join fields

Now we turn to the study of cusped manifolds, complementary to knots and links. Packaged with SnapPea [21], there are the 4,815 orientable cusped manifolds of the Hildebrand–Weeks census [3], triangulated by no more than 7 ideal tetrahedra, and 114 non-orientable manifolds, triangulated by no more than 5. Making \(4929 \times 1200\) comparisons with the Dedekind zeta values of Section 2.1, we found 7 new single-complex-place fields, beyond those from the Hodgson–Weeks census. Then the CGHN file \(\text{cusped.fields}\) confirmed these finds and yielded the remaining 5 fields of Table 14, with larger discriminants. In Table 14, we give the rational Dedekind-zeta invariant of a selected cusped manifold for each of the 12 new fields.

Our tally of single-complex-place fields is now \(96 + 44 + 7 + 5 = 152\). Among the 4,929 cusped manifolds, we found 312 whose volumes are rationally related to one of the 152 Dedekind zeta values, with numerators and denominators in (2) bounded by \(a \leq 2853\) and \(b \leq 7\).

We also sought integer relations of the form

\[ a \text{ vol}(\mathcal{M}) = b_1Z_{K_1} + b_2Z_{K_2} \]

corresponding to a cusped or closed manifold \(\mathcal{M}\) whose invariant trace field is the join of single-complex-place fields \(K_1\) and \(K_2\), or is a subfield of this join. Restricting \(K_1\) and \(K_2\) to the 152 fields of Tables 2–12 and Table 14, we found 91 census manifolds, with 26
distinct volumes, rationally related, via \((18)\), to pairs of Dedekind zeta values. Details of the 29 cusped and 62 closed manifolds are provided by Tables 15–18. Two of the 26 integer relations entail join fields noted in \([7]\), namely the first in Table 16 and the second in Table 17. We used Pari’s \texttt{nfisincl} command to confirm that all 6 of the quartic invariant trace fields in Table 18 are subfields of the octadic joins. In 4 of these 6 cases, distinct values of \(b_1/b_2\) occur, for the same invariant trace field.

We believe that we have exhausted the 3-term relations between census volumes and pairs of the 152 target Dedekind zeta values. David Bailey’s impressively efficient, arbitrary precision \([22]\), implementation of PSLQ \([23]\) found the 91 relations in 40 minutes and then took 17 hours to exhaust the \((4815 + 114 + 11031) \times \binom{152}{2} = 183,156,960\) indiscriminate possibilities. The PSLQ search rate was thus a healthy 3 kHz. No reduction to 3 distinct Dedekind zeta values was detected, presumably because the census volumes are kept deliberately small.

\section{Links whose volumes link Clausen values}

Our order of presentation is the reverse of the order in which we obtained results. Our primary motivation was to elucidate connections between Clausen values, revealed by studies of Feynman diagrams \([1]\), on which we comment in Section 5.

This section concerns links whose rational Dedekind-zeta invariants link Clausen values. An example is provided by the attractive alternating\(^6\) link of Fig. 1, discovered as a result of work on light-by-light scattering, reported in Section 5. A second is provided by the non-alternating\(^7\) daisy-chain link of Fig. 2. These form part of our study of Dedekind-zeta invariants of quadratic links.

Next in chronological sequence, came the higher-degree Dedekind-zeta invariants of cusped manifolds, in Section 3. Finally, we undertook the systematic study of closed manifolds, culminating with the 12th-degree Dedekind-zeta invariant of Section 2. Like most stories, it benefits from signposting.

\subsection{Signpost: integer sequence A003657}

Mathematically speaking, the results of Section 2, while numerically striking, are of a rational character \([2, 3, 4]\) that was expected by specialists of \(K\)-theory (which we are not). The challenge is not to understand why a rational Dedekind-zeta invariant exists, but to learn how to derive (as opposed to measure) it. The cusped results of Section 3 were of the same character as for closed manifolds: hyperbolic knots and links have rational Dedekind-zeta invariants if their complementary cusped manifolds have single-complex-place invariant trace fields. Here, in Section 4, we address the question raised by the sparsity of quadratic entries in Tables 2 and 14: where are the quadratic links with

\(^6\)The viewer is asked to alternate all the crossings in Fig. 1 and Figs. 3–8.

\(^7\)The viewer is asked to supply the uniquely non-alternating non-trivial crossings in Fig. 2.
(negated) discriminants beyond the first 4 entries in the integer sequence A003657 of Neil Sloane’s on-line encyclopedia.

\[ 3, 4, 7, 8, 11, 15, 19, 20, 23, 24, 31, 35, 39, 40, 43, 47, 51, 52, 55, 56, 59, 67, 68, 71, 79, 83, 84 \ldots \]

(19)

Only the first 4 cases figure in the Hildebrand–Weeks census. Here we identify 4 more underlined cases, with \(-D \in \{11, 15, 20, 24\}\), where the platonic link of Fig. 1 furnishes an example with \(D = -20\), and the daisy-chain link of Fig. 2 furnishes an example with \(D = -24\). Then, in Section 5, we shall explain how we were led to Figs. 1 and 2, by consideration of the physical process of light-by-light scattering, and why its Feynman diagrams suggest that the remaining 2 underlined cases, with \(D = -39\) and \(D = -84\), will be of similar symmetrical appeal, and similar analytical mystery.

### 4.2 Dirichlet character

The simplest example of a number field with a single complex place is an imaginary-quadratic field \(\mathbb{Q}(\sqrt{-d})\), where \(d\) is a square-free positive integer. When \(d \equiv 3 \pmod{4}\), the discriminant is \(D = -d\); otherwise it is \(D = -4d\).

It is proven in [24] that

\[
\frac{\zeta_K(2)}{\zeta(2)} = \sum_{n>0} \frac{\chi_K(n)}{n^2} \tag{20}
\]

where \(\chi_K\) is the real Dirichlet character of the group of units of the field \(\mathbb{Z}/|D|\) and \(D\) is the discriminant of the imaginary-quadratic field \(K\). The Dirichlet character is related to the Jacobi (or Kronecker) symbol by

\[
\chi_K(n) = \left( \frac{D}{n} \right) \tag{21}
\]

which vanishes if \(\gcd(D, n) > 1\). When \(D\) is odd, one may use the alternative form \(\left( \frac{n}{|D|} \right)\).

Using (20) in (1), at \(n = 2\), one happily disposes of powers of \(\pi\). We dispose of \(\sqrt{-D}\), by using the imaginary-quadratic result [24]

\[
\sum_{n>0} \frac{\chi_K(n)}{n^2} = \frac{1}{\sqrt{-D}} \sum_{-D > k > 0} \chi_K(k) \text{Cl}_2 \left( \frac{2\pi k}{|D|} \right) \tag{22}
\]

which yields a finite sum over the elements of the group of units. Finally, we dispose of a factor 2, via the reflection relations \(\chi_K(k) = -\chi_K(|D| - k)\) and \(\text{Cl}_2(\theta) = -\text{Cl}_2(2\pi - \theta)\), to obtain the readily computable result

\[
Z_K = Z_{|D|} := D \sum_{-D > 2k > 0} \left( \frac{D}{k} \right) \text{Cl}_2 \left( \frac{2\pi k}{D} \right) \tag{23}
\]

with the magnitude of the discriminant identifying the imaginary-quadratic field.

\[\text{http://www.research.att.com/~njas/sequences}\]
Thanks to David Bailey [22], it is now a routine matter to evaluate (23) to high precision\[^9\] using \[^{25}\]
\[
\frac{\text{Cl}_2(\theta)}{\theta} = 1 - \frac{1}{\pi} \log(\theta^2) + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n^2 + n} \left(\frac{\theta}{2\pi}\right)^{2n}
\]
which converges faster than \(1/9^n\), for \(\theta \in [0, 2\pi/3]\). An angle in \([2\pi/3, \pi]\) may be transformed to a pair in \([0, 2\pi/3]\), by rearrangement of the duplication formula
\[
\frac{1}{2} \text{Cl}_2(2\theta) = \text{Cl}_2(\theta) - \text{Cl}_2(\pi - \theta)
\]
Rather than compute \(\zeta(2n)/(2\pi)^{2n} = \frac{1}{2}|B_{2n}|/(2n)!\) recursively, from the Bernoulli numbers, we used a FFT program written by David Bailey to evaluate \(\{\zeta(2n) \mid 0 \leq n < 2^m\}\), in one fell swoop, by a multi-dimensional generalization of Newton’s method. With \(m = 11\), the initial outlay, to tabulate 2048 coefficients to 1800 digits, was 8 minutes; then 1800 good digits of any Clausen value in \(\{\text{Cl}_2(\theta) \mid 0 < \theta < 2\pi/3\}\) are obtainable in less than 2 seconds on a 533MHz machine. Low-precision results for \(Z_D\) are given in Table 19.

It follows from (23) that every discovery of a rational Dedekind-zeta invariant for a quadratic link gives a relation between a set of Clausen values at angles whose tangents involve \(\sqrt{d}\), and a set of Clausen values at angles which are multiples of \(2\pi/|D|\). We have already seen this, rather strikingly, in the case \(-D = d = 7\) of (15), where the hyperbolic 6-crossing 3-component link \(6^3_1\) mediates between Clausen values, and hence the Bloch–Wigner dilogarithms of (16), with very different types of argument on the left and right.

We now address the question: which links have volumes that link Clausen values?

### 4.3 Figure-8 knot at \(D = -3\)

The first case, at \(D = -3\), is elementary. The unique\[^{11}\] arithmetic knot is the figure-8 knot of Fig. 3. Its volume is the first entry of Table 19. It links Clausen values at \(2\arctan \sqrt{3}\) and \(2\pi/3\). This relation clearly poses no analytical puzzle, since the angles are identical.

### 4.4 Whitehead link at \(D = -4\)

Just as the simplest hyperbolic knot, \(4_1\), gave the answer at \(D = -3\), so the simplest hyperbolic 2-component link, namely the Whitehead link \(5^2_1\) of Fig. 4, gives the answer at \(D = -4\). Its volume is the second entry of Table 19. It links Clausen values at \(2\arctan \sqrt{1}\) and \(2\pi/4\), which are again identical.

\[^9\]http://science.nasa.gov/Groups/AAA/db.webpage/mpdist/mpdist.html
\[^{10}\]We use the notation of [26], whose appendix give drawings of links up to 9 crossings.
\[^{11}\]See Helaman Ferguson’s sculpture at http://www.mtholyoke.edu/acad/math/ma/sculpture.htm
4.5 Link $6_3^1$ at $D = -7$

The Dedekind-zeta invariant of the link $6_3^1$, illustrated in Fig. 5, is measured to be 2, corresponding to

$$2 \left\{ 3 \text{Cl}_2(\theta_7) - 3 \text{Cl}_2(2\theta_7) + \text{Cl}_2(3\theta_7) \right\} = Z_7 = 7 \left\{ \text{Cl}_2 \left( \frac{2\pi}{7} \right) + \text{Cl}_2 \left( \frac{4\pi}{7} \right) - \text{Cl}_2 \left( \frac{6\pi}{7} \right) \right\}$$

(26)

with $\theta_7 := 2 \arctan \sqrt{7}$. At 1800-digit precision, PSLQ finds this to be the sole integer relation between the 6 Clausen values. The combination on the left is selected by hyperbolic geometry; that on the right by the Dirichlet character. Thus the rational Dedekind-zeta invariant of link $6_3^1$ encodes a highly non-trivial analytical relation between Clausen values. A derivation of $a/b = 2$ would prove (26). Conversely, a proof of (26) would derive $a/b = 2$. Where may one find any such proof?

4.6 Link $9_{40}^2$ at $D = -8$

As exemplar of the next non-trivial relation between Clausen values, we select the 9-crossing 2-component link $9_{40}^2 := (\sigma_1^2 \sigma_2^{-1})^3$ of Fig. 6. Its Dedekind-zeta invariant is observed to be unity, which implies that its volume is equal to each side of

$$\frac{1}{2} \left\{ 27 \text{Cl}_2(\theta_2) - 9 \text{Cl}_2(2\theta_2) + \text{Cl}_2(3\theta_2) \right\} = Z_8 = 8 \left\{ \text{Cl}_2 \left( \frac{2\pi}{8} \right) + \text{Cl}_2 \left( \frac{6\pi}{8} \right) \right\}$$

(27)

with $\theta_2 := 2 \arctan \sqrt{2}$. As in the case of (26), a proof is lacking.

4.7 A 12-crossing 4-component link at $D = -11$

In [7] it was noted that neither the closed nor the cusped census entails the field $\mathbb{Q}(\sqrt{-11})$. In [1] it was observed that the 12-crossing 4-component link of Fig. 7, with braid word $(\sigma_1 \sigma_2^{-2} \sigma_3 \sigma_2^{-2})^2$, entails this field. Its Dedekind-zeta invariant is observed to be unity, corresponding to the relation

$$15 \text{Cl}_2(\theta_{11}) - 10 \text{Cl}_2(2\theta_{11}) + \text{Cl}_2(5\theta_{11}) = Z_{11} = 11 \sum_{k=1}^{5} \left( \frac{k}{11} \right) \text{Cl}_2 \left( \frac{2\pi k}{11} \right)$$

(28)

where $\theta_{11} := 2 \arctan \sqrt{11}$ and $\left( \frac{k}{11} \right)$ is the Jacobi (or Legendre) symbol for the Dirichlet character. Again, we lack a proof.

4.8 A 12-crossing 3-component link at $D = -15$

The next case likewise comes from [1]. The 12-crossing 3-component link of Fig. 8, with braid word $(\sigma_1^2 \sigma_2^{-2})^3$, has Dedekind-zeta invariant empirically equal to 2, at 1800-digit precision. This corresponds to the relation

$$24 \text{Cl}_2(\theta_{5,3}) - 12 \text{Cl}_2(2\theta_{5,3}) - 8 \text{Cl}_2(3\theta_{5,3}) + 6 \text{Cl}_2(4\theta_{5,3}) = Z_{15} = 15 \sum_{k=1}^{7} \left( \frac{k}{15} \right) \text{Cl}_2 \left( \frac{2\pi k}{15} \right)$$

(29)
with \( \theta_{5,3} := 2 \arctan \sqrt{5/3} \). Again, where is the proof?

### 4.9 A self-dual platonic link at \( D = -20 \)

The reader might now expect us to consider the case \( D = -19 \). For reasons that will be given in Section 5, we skip to \( D = -20 \), where we were rewarded by the splendidly symmetric alternating platonic link drawn in Fig. 1. Its Dedekind-zeta invariant was measured to be unity. This innocent-sounding statement translates to our 5th unproven relation

\[
36 \text{Cl}_2(\theta_5) - 30 \text{Cl}_2(2\theta_5) + 4 \text{Cl}_2(3\theta_5) + 3 \text{Cl}_2(4\theta_5) = Z_{20} = 20 \sum_{k \in \{1, 3, 7, 9\}} \text{Cl}_2 \left( \frac{2\pi k}{20} \right)
\]

with \( \theta_5 := 2 \arctan \sqrt{5} \). Between these 8 Clausen values, PSLQ found no other relation.

We found that, far from complicating the result, the 4 Clausen values on the left simplify the 54 Bloch–Wigner values of the triangulation in \( \mathbb{Q}(\sqrt{-5}) \). There is, of course, an infinity of rewritings of (30), obtained by adding, on the left, combinations of Bloch–Wigner values that algebraically sum to zero, by virtue of special cases of the classical \( \mathbb{Z}_2 \)-variable 5-dilogarithm relation of Abel, which is easily proved by differentiation. Here, as elsewhere, we expose the classical analysis that remains to be done.

### 4.10 A 48-crossing daisy-chain link at \( D = -24 \)

The attentive reader will now expect us to skip the case \( D = -23 \) and jump to the 8th underlining in (19), at \( D = -24 \). This is precisely what we did, though the character of the result was not what we first supposed. It seemed to us that the hallmark of past success, in finding links that link Clausen values, was to achieve the largest possible symmetry group. What, we asked ourselves, could pack a better symmetry-to-crossing ratio than the remaining 4 platonic alternating links?

Let us denote the tetrahedral link of Fig. 1 by \( T := 24^{10}_{\text{tet}} \). The 4 non-self-dual alternating platonic links have components that mimic the vertices and edges of the cube \( C := 48^{20}_{\text{cub}} \), octahedron \( O := 48^{18}_{\text{oct}} \), dodecahedron \( D := 120^{50}_{\text{dod}} \), and icosahedron \( I := 120^{42}_{\text{ico}} \), where, for example, the last has 120 alternating crossings of its 42-components, which mimic the 12 vertices and 30 edges of the perfect solid with 20 faces. These links are highly symmetrical, yet none of their 4 volumes yielded the sought-for rational relation to a quadratic Dedekind zeta value. Later, we show that they are quartic links.

Instead, we found an answer to our 8th question by a more child-like construction: a non-alternating daisy chain. By a daisy chain we mean a link each of whose components has 2 crossings with a neighbour on one side, and 2 crossings with a neighbour on the other, with the whole forming a circle, as in Fig. 2. By a non-alternating daisy chain, we mean one in which the 4 crossings of each component occur in an order over-over-under-under, while still linking with neighbours. A little doodling should convince the reader that non-alternating daisy-chain links must have an even number of components.
A delightful feature of Jeff Weeks’ program SnapPea [21] is that it enables one to draw such chains quickly, and then ask whether the resultant non-alternating link has quadratic shapes in its triangulation. Using A.C. Manoharan’s port of SnapPea to Windows95 we inferred, from instances with up to 96 crossings, that the non-alternating daisy-chain link with $2n \geq 6$ components, and hence $4n \geq 12$ crossings, has hyperbolic volume

$$V_{2n} = 8nD \left( i \tan \frac{(n-2)\pi}{4n} \right)$$  \hspace{1cm} (31)

The first 4 hyperbolic non-alternating daisy chains yield results with $-D \in \{3, 4, 8, 20\}$:

$$V_6 = 4Z_4$$  \hspace{1cm} (32)
$$V_8 = 2Z_8$$  \hspace{1cm} (33)
$$V_{10} = \frac{1}{2}Z_{20} + V_6$$  \hspace{1cm} (34)
$$V_{12} = 20Z_3$$  \hspace{1cm} (35)

Then nothing interesting happens until we reach 24 components, with 48 crossings, where

$$V_{24} = Z_{24} + V_8$$  \hspace{1cm} (36)

hits the target, at $D = -24$.

SnapPea illustrates this nicely. Drawing daisy chains with 6, 8, 10, 12, 24 components, and inspecting the triangulations, one detects the square roots of 1, 2, 5, 3, 6. The last, illustrated in Fig. 2, delivers a result for $Z_{24} = V_{24} - V_8$, namely

$$60Cl_2(\theta_{3,2}) - 18Cl_2(2\theta_{3,2}) - 4Cl_2(3\theta_{3,2}) + 3Cl_2(4\theta_{3,2}) = Z_{24} = 24 \sum_{k \in \{1,5,7,11\}} Cl_2 \left( \frac{2\pi k}{24} \right)$$  \hspace{1cm} (37)

with $\theta_{3,2} := 2 \arctan \sqrt{3/2}$. So, for the 6th time, we have a relation that is as easy to check numerically as it appears hard to derive.

### 4.11 Quartic platonic links

Our first guess, that the 4 non-self-dual platonic links might yield quadratic number fields, failed. Nonetheless, it is interesting to determine their number fields, and hence obtain accurate volumes. This is clearly a taxing job. To prepare for it, we first tackled a problem of similar complexity, where an accurate answer could be inferred.

We found that by adding $n$ concentric components at each of the 4 “vertices” of Fig. 1, one obtains a volume $Z_{20} + 2nZ_{15}$ for the link with $24 + 24n$ crossings and $10 + 4n$ components, when $n \leq 5$. At $n = 6$, SnapPea was given the 34-component link with 168 crossings, which was triangulated to give 632 ideal tetrahedra, each having a volume

12 [http://home.att.net/Manoharan/SnapPea/snappea.html](http://home.att.net/Manoharan/SnapPea/snappea.html)
$D(z_k)$ with $z_k \in \mathbb{Q}(\sqrt{-5})$ or $z_k \in \mathbb{Q}(\sqrt{-15})$. We output the 632 shapes, and computed, at 1800-digit precision, the $3 \times 632 = 1,896$ Clausen values entailed by

$$D(z) = V(\arg(z), -\arg(1-z))$$

$$V(\theta_1, \theta_2) := \frac{1}{2} \{ \text{Cl}_2(2\theta_1) + \text{Cl}_2(2\theta_2) - \text{Cl}_2(2\theta_1 + 2\theta_2) \}$$

obtaining 1800-digit agreement with the expectation $Z_{20} + 12Z_{15} = 502.408032 \ldots$ for the volume. This gave us confidence that we could process the 950 ideal tetrahedra entailed by the remaining 4 platonic links.

We obtained from SnapPea the low-precision volumes

$$\text{vol}(C) \approx \text{vol}(O) \approx 114.537611$$

$$\text{vol}(D) \approx \text{vol}(I) \approx 310.913145$$

from $128 + 130 + 346 + 346 = 950$ ideal tetrahedra.

The duality between cube and octahedron, and between dodecahedron and icosahedron, is gratifying. Such duality is not restricted to platonic links. More generally, suppose that we have a link, $L$, with an alternating projection in which the components may be separated into two classes: vertex (V) components and edge (E) components, with pairs of crossings only between V and E components, and every E connecting a pair of V's. Thus the crossing number is 4 times the number of E components. Now shrink the V components to true vertices (some of which may be divalent) and the E components to true edges that connect these points, to obtain a planar graph, $P$. Then construct the dual graph, $P^*$, whose vertices lie in the regions of the plane partitioned by the edges of $P$ and whose edges thus cross the edges of $P$. Now thicken $P^*$ to obtain the alternating link $L^*$ dual to $L$. The crossing numbers of $L$ and $L^*$ are equal, but the numbers of components need not be. For example $C := 48^{20}_{\text{cub}}$ has more components than $C^* := O := 48^{18}_{\text{oct}}$, and $D := 120^{10}_{\text{dod}}$ has more than $D^* := I := 120^{12}_{\text{ico}}$. On the basis of these and further tests, we conjecture that $\text{vol}(L) = \text{vol}(L^*)$, for every shrinkable link $L$.

To identify the number fields of the \{C, O\} and \{D, I\} pairs, we first examined the cusp shapes, and found that

$$K_C := \mathbb{Q}\left(\sqrt{8\sqrt{2} - 15}\right)$$

(42)

gave a simple fit to the cusp shapes of \{C, O\}, at the 12-digit precision provided by SnapPea, while

$$K_D := \mathbb{Q}\left(\sqrt{-12\sqrt{5} - 31}\right)$$

(43)

similarly fitted the cusp shapes of \{D, I\}.

We then output $z$-values of the triangulations. Fitting these was not easy, since the data now consisted of numerical values, to 12 decimal places, of $1,900$ real or imaginary parts, each of which was supposed to be fittable by two powers of $\sqrt{15 - 8\sqrt{2}}$, in the cubical and octahedral cases, or $\sqrt{31 + 12\sqrt{5}}$, in the dodecahedral and icosahedral cases.
In each of the 1,900 cases we required a significant integer relation between 3 numbers: the 12-digit SnapPea datum and the two powers, namely 0 and 2 for a real part, or 1 and 3 for an imaginary part. Whether this can be done by a reliable and uniformly automated method depends upon the largest integer. If it has more than 4 digits, the matter is moot, since fitting 12-digit data with 3 integers with up to 4 digits is already a parlous business. In fact our first attempt yielded garbage in a significant fraction of the 1,900 cases.

Fortunately, 3 features of the mathematics enable us to crack this tough nut. First, the 6-fold symmetry of the Bloch–Wigner function

\[ D(z) = D(1/(1 - z)) = D(1 - 1/z) = D(1/z) = D(1 - \tau) = D(\tau/(\tau - 1)) \] (44)

gave us 6 bites at each cherry. Secondly, number theory suggested that we might do well to make the Ansatz

\[ z = \frac{1}{I_3} \sum_{p=0}^{3} I_p u^p \] (45)

with 6 × 950 × 5 = 28,500 integers relating 6 equivalent \( z \) values of 950 ideal volumes to 4 powers of \( u = \frac{1}{2}(1 + i\sqrt{15} - 8\sqrt{2}) \) or \( u = \frac{1}{2}(1 + i\sqrt{31} + 12\sqrt{5}) \). Finally, we were prepared for even the simplest of the 6 fits to any shape to contain the prime factor \( 97 = 15^2 - 2 \times 8^2 \) of the cubical/octahedral discriminant, or \( 241 = 31^2 - 5 \times 12^2 \) in the dodecahedral/icosahedral case.

These 3 features allowed us to devise a diversity of algorithms that yielded, eventually, total fits which we consider to be corporately indubitable. Our confidence came from a fourth mathematical feature, which at first sight appeared to make life difficult, namely that there was virtually no overlap between problems that were supposed to be related by duality. In fact we found only two distinct ideal volumes that contributed to both the dodecahedron and icosahedron, and each of these was unambiguous. The absence of further overlap doubled the computational load, yet made the resulting numerical agreement of volumes, to 1800 digits, a potent signal of success. It seems unlikely that a misidentification of one of the ideal volumes, for the dodecahedron, could produce the same effect as a misidentification of a quite different ideal volume, for the icosahedron.

Thus we believe that we have found exact (and also very lengthy) expressions of the form \( \sum_{k=1}^{n} D(z_k) \), with \( n = 128 \) and \( z_k \in \mathbb{Q}(\sqrt{8\sqrt{2} - 15}) \), for the cubical/octahedral volume of Table 20, and with \( n = 346 \) and \( z_k \in \mathbb{Q}(\sqrt{-12\sqrt{5} - 31}) \), for the dodecahedral/icosahedral volume of Table 21. Thanks to David Bailey’s evaluations of \( \zeta(2n) \), described in Section 4.2, we were able to obtain 1,800 good digits of 2,850 Clausen values in less than 2 hours.

## 5 Knots and links from Feynman diagrams

Finally, we arrive at the motivating idea for this work: values of Feynman diagrams.
5.1 Hyperbolic Feynman links

In [27], Andrei Davydychev and Bob Delbourgo made a fine discovery: the dilogarithms of the box diagram for particle scattering are those which give the volume of a tetrahedron in hyperbolic 3-space (or its analytic continuation).

In the 3-dimensional studies of statistical physics, one-loop Feynman diagrams yield logarithms; in the 4 space-time dimensions of relativistic quantum field theory (QFT), they yield dilogarithms [28]. A connection between the dilogs of QFT and those of hyperbolic geometry was considered in [30]. The achievement of [27] is to derive a fairly simple relation between the value of any scalar box diagram, in 4 space-time dimensions, and the volume of an explicit tetrahedron in a 3-space of constant curvature. There is nothing ideal about this tetrahedron: in general it has 6 essential dihedral angles, determined by the 10 physical quantities in the problem: the 4 external masses, the 4 internal masses, and the Mandelstam variables $s$ and $t$, related to the energy at which the process occurs and the angle of scatter. Already one sees a nice simplification, with 9 dimensionless ratios of physical quantities collapsing to 6 essential dihedral angles. So far, however, the question of number theory does not arise, since in the generic physical situation the kinematic quantities are real variables, and hence no algebraic number field is implied for the arguments of the dilogarithms.

Now consider the process of light-by-light scattering, where the external (photon) masses vanish and there is a common internal (electron) mass, normalized to unity. There remain the Mandelstam variables, $s$ and $t$. Following work that involved comparable Clausen values [31, 32], it was observed in [1] that light-by-light scattering yields remarkable results at $s = t = n \in \{4, 5, 6, 7\}$, where the dilogs give rational multiples of the volumes of links. These links all figure in Section 4, above, where their volumes were rationally related to $Z_{|D|}$, in (23). The $n = 4$ case gives a rational multiple of $Z_4$, corresponding to the Whitehead link; $n = 5$ relates to $Z_{15}$ and the link $(\sigma_1^2 \sigma_2^{-2})^3$; $n = 6$ to $Z_3$ and the figure-8 knot; $n = 7$ to $Z_7$ and $6_3^1$. Moreover, a rational multiple of the volume, $Z_8$, of the link $9_{40}^2 := (\sigma_1^2 \sigma_2^{-1})^3$ is obtained for Mandelstam variables $s = \frac{1}{2}t = 4$.

In the course of the present work we discovered that the hyperbolic volume, $Z_{20}$, of the self-dual alternating platonic link of Fig. 1 corresponds to the values $s = \frac{1}{2}t = 5$ for the Mandelstam variables. Perhaps not even Delbrück and Meitner could have imagined that light-by-light scattering would spawn a tetrahedral hyperbolic link.

5.2 Dedekind-zeta invariants of Feynman orthoschemes

Let us try to disentangle this remarkable circumstance from its physical origin. What is the common characteristic of the Davydychev–Delbourgo (DD) hyperbolic tetrahedron at

For a scattering above the electron-positron threshold, with $s > 4$ and $t < 0$, unitarity makes the amplitude complex. A relation to a real hyperbolic volume is obtained by analytic continuation to $t > 4$.

In 1933, Max Delbrück (1906–81) and Lise Meitner (1878–1968) foresaw non-linear effects in quantum electrodynamics. See [http://www.nobel.se/laureates/medicine-1969-1-bio.html] for Delbrück’s subsequent work on molecular genetics and sensory physiology.
those values of the physical quantities which gave the volumes of links?

In terms of (39), we define the 3-argument dilogarithm

$$S(\psi_1, \psi_2, \psi_3) := V(\delta + \psi_1, \delta - \psi_1) + V(\delta + \psi_3, \delta - \psi_3) + V(\frac{1}{2} \pi + \psi_2 - \delta, \frac{1}{2} \pi - \psi_2 - \delta) + V(\frac{1}{2} \pi - \delta, \frac{1}{2} \pi - \delta)$$

(46)

with an auxiliary angle $\delta \in [0, \pi/2]$ satisfying

$$\tan^2 \delta = \frac{\cos^2 \psi_2}{\cos^2 \psi_1 \cos^2 \psi_3} - \tan^2 \psi_1 \tan^2 \psi_3$$

(47)

Then the Schlafli/Lobachevsky [33, 34, 35, 36] function (16) is 4 times the volume of a bi-rectangular hyperbolic tetrahedron, with essential dihedral angles $\psi_1, \psi_2, \psi_3$. The edges with angles $\psi_1$ and $\psi_3$ are opposite, while that with $\psi_2$ is adjacent to each. The remaining 3 dihedral angles are right angles. With 3 essential angles, a bi-rectangular tetrahedron is an example of an orthoscheme [18]. With a common internal mass, and a common external mass, the DD tetrahedron comprises 4 identical orthoschemes, with (46) giving its volume.

The next step was to calculate the essential angles $\psi_k$ that had given a rational relation to the volume of a link. In all cases we found that $\psi_k \in \{0, \pi/6, \pi/4, \pi/3\}$. The final step was clear: to compute all such instances of (46), in the hope of finding more Feynman orthoschemes that are rationally related to Dedekind zeta values. Taking account of the positivity of (47), and the symmetry $S(\psi_1, \psi_2, \psi_3) = S(\psi_3, \psi_2, \psi_1)$, there are 36 possibilities to consider, of which the physics in [33] had already shown 5 to be fruitful. We had a lively expectation of further rational relations to $Z_{|D|}$. We were thus totally delighted by the following, totally rational, finding.

**Discovery:** All real instances of (46) with $\psi_k \in \{0, \pi/6, \pi/4, \pi/3\}$ are rational multiples of $Z_{|D|}$ with $|D| \in \{3, 4, 7, 8, 11, 15, 20, 24, 39, 84\}$. Table 22 gives the 36 empirical relations. All the relations are realized by Feynman diagrams; in the cases with $\psi_2 \neq 0$, one has merely to give the external particles a suitable common mass.

It was trivial to decide which value of $Z_{|D|}$ to try in each of the 36 cases: one has only to examine the square root of (47) to determine $D$. Had we been less result-oriented we might have taken time out to recast each of the 36 searches in terms of 4 complex arguments of Bloch–Wigner dilogarithms for the 4 ideal parts of (16), and then used algebra [17, 19] to determine, in advance, whether a rational (but unspecified!) number would result from dividing the volume by $Z_{|D|}$. We were content with the faster process of division, which gives the concrete result. Throughout this work, the issue is not the existence of rationals, but their values. The rational numbers of Table 22 in the cases

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\(^{15}\)There is nothing complex, or ideal, in the physical problem that led to our discoveries: integration over real Feynman parameters yields the volume of a tetrahedron, none of whose vertices are at infinity.
with $-D \geq 7$ are at least as hard to derive as those in \cite{26,30,37}. For $D = -39$, we found that
\begin{equation}
24\text{Cl}_2(\theta_{13,3}) - 15\text{Cl}_2(2\theta_{13,3}) + \text{Cl}_2(6\theta_{13,3}) = \frac{1}{6}Z_{39}
\end{equation}
with $\theta_{13,3} := 2 \arctan \sqrt{13/3}$. For $D = -84$, we found that
\begin{equation}
60\text{Cl}_2(\theta_{7,3}) - 36\text{Cl}_2(2\theta_{7,3}) - 4\text{Cl}_2(3\theta_{7,3}) + 3\text{Cl}_2(4\theta_{7,3}) + 2\text{Cl}_2(6\theta_{7,3}) = \frac{1}{6}Z_{84}
\end{equation}
with $\theta_{7,3} := 2 \arctan \sqrt{7/3}$. These are our 7th and 8th unprovens, courtesy of Feynman.

### 5.3 Two of Feynman’s links are missing

The rationale for the underlinings in (19) should now be clear: it was these 10 imaginary-quadratic fields that had been distinguished by the Feynman orthoschemes of Table 22, which initiated our studies. For 4 of these 10 fields, the census gave links; for 2 of the remaining 6, the work in \cite{1} gave links; for 2 of the remaining 4, the alternating platonic link of Fig. 1 and the non-alternating daisy-chain link of Fig. 2 supplied answers; that left us with the $D = -39$ and $D = -84$ links still missing.

It may be imagined that we spent much time looking for Feynman’s two missing links. We find it significant that, in all our exploration, SnapPea never detected a quadratic field beyond those we have reported, with $-D \in \{3, 4, 7, 8, 11, 15, 20, 24\}$. One could, if one was minded, synthesize cusped manifolds with gluing conditions that are satisfied in other quadratic fields, and then assert the existence of links whose complements in $S^3$ are isometric to these manifolds. Our aim was more concrete, and perhaps more old-fashioned. Just as the basic ingredients of Table 22 would have been immediately intelligible to their originators Nikolai Ivanovich Lobachevsky and Thomas Clausen, so would those of Figs. 1 and 2 have been to James Clerk Maxwell and Peter Guthrie Tait.

It was, therefore, very satisfying to progress to $D = -24$, at 48 crossings, and not to encounter any quadratic SnapPea triangulation beyond those we had learnt to expect from Feynman, Davydychev and Delbourgo. By the same token, it was frustrating not to discover two more Feynman links, with volumes rationally related to $Z_{39}$ and $Z_{84}$.

We hope that others will be motivated to search. Table 19 indicates the challenge. The self-dual platonic link of Fig. 1 entails 24 crossings, 10 components, and a volume $Z_{20} = \text{vol}(24_{\text{tet}}) \approx 50.447$; the daisy-chain link of Fig. 2 entails 48 crossings and 24 components, to reach $Z_{24} = V_{24} - V_{8} \approx 62.186$. The reader is left to imagine what might be entailed by $Z_{39} \approx 165.575$ and $Z_{84} \approx 404.736$.

In any case, one now knows that Feynman orthoschemes at Mandelstam variables $s = 3t = 13$ and $s = \frac{3}{2}t = 7$ are rationally related to \cite{23} at $D = -39$ and $D = -84$.

---

\footnote{The geometry of Lobachevsky (1792–1856) gives a model for the Universe that accords with data. Clausen (1801–85) was an astronomer, described by Gauss (1777–1855) as a man of outstanding talents.}

\footnote{Maxwell (1831–79) and Tait (1831–1901) were schoolfriends in Edinburgh. Maxwell later formulated electromagnetic field theory and encouraged Tait’s search for a connection between knots and physics.}

\footnote{The Feynman diagram evaluates to the volume of orthoscheme \cite{13} divided by a square root.}
with coefficients in \([18,19]\) that are as easy to measure, and as hard to prove, as those which are reified in Figs. 1 and 2. It is hard to believe that the 2 missing links will be less beautiful than the 8 which we have already related to Feynman orthoschemes.

### 5.4 Dedekind zeta values from 10-crossing Feynman knots

We found that 7 of the 84 knots with less than 10 crossings have rational Dedekind-zeta invariants. The 6 distinct values of \([\mathbb{H}]\) are

\[
Z_3 = 1 \times \text{vol}(41) \tag{50}
\]

\[
Z_{23,3} = \frac{1}{3} \times \text{vol}(52) = \frac{1}{10} \times \text{vol}(949) \tag{51}
\]

\[
Z_{44,3} = \frac{1}{3} \times \text{vol}(948) \tag{52}
\]

\[
Z_{59,3} = 1 \times \text{vol}(74) \tag{53}
\]

\[
Z_{76,3} = 1 \times \text{vol}(935) \tag{54}
\]

\[
Z_{448,4} = \frac{1}{6} \times \text{vol}(818) \tag{55}
\]

where the subscripts of \(Z_{|D|,n}\) identify the (negated) discriminant and degree of the number field, and we omit the latter in the quadratic case. Two further knots, \(8_{21}\) and \(9_{28}\), have invariant trace fields in Table 18. From these subfields of joins, one may extract

\[
Z_7 = 4 \times \text{vol}(821) - \frac{4}{3} \times \text{vol}(818) \tag{56}
\]

\[
Z_{507,4} = \frac{2}{5} \times \text{vol}(928) - 1 \times \text{vol}(41) \tag{57}
\]

We now report on two very special knots at 10 crossings. Work begun by Dirk Kreimer \([37, 38, 39]\), and extended in collaborations with Broadhurst, Delbourgo and Gracey \([11, 12, 13, 14]\), has established a rich connection between multiple zeta values \([40, 41, 42, 43]\) and positive knots, forged by multi-loop Feynman diagrams in quantum field theory. A positive knot is one with a minimal braid word that entails exclusively positive powers of the generators of the braid group \([14]\). There is an important feature to note: no positive knot with less than 10 crossings is hyperbolic. The 5 positive knots with less than 10 crossings are the \((3, 2), (5, 2), (7, 2), (4, 3)\) and \((9, 2)\) torus\(^{19}\) knots, with 3, 5, 7, 8 and 9 crossings, corresponding to \(\zeta(3), \zeta(5), \zeta(7), \zeta(5, 3)\) and \(\zeta(9)\), where \(\zeta(r, s) := \sum_{j>k>0} j^{-r}k^{-s}\) is a multiple zeta value (MZV) of depth 2 and weight \(r+s\), with \(\zeta(5, 3)\) being the sole irreducible MZV below weight 10.

The first 2 hyperbolic Feynman knots are \(10_{139} := \sigma_1 \sigma_2^3 \sigma_1^3 \sigma_2^3\) and \(10_{152} := \sigma_1^2 \sigma_2 \sigma_1^3 \sigma_2^3\), with volumes

\[
\text{vol}(10_{139}) = 4.8511707573327375670583270521153124788452830277699\ldots \tag{58}
\]

\[
\text{vol}(10_{152}) = 8.5360653472056086031441819205493259949649913969140\ldots \tag{59}
\]

that are rather modest, compared with most of the other 162 hyperbolic 10-crossing knots.

\(^{19}\)Non-hyperbolic knots are torus or satellite knots, with the latter beginning at 13 crossings.
We do not know how many of the 164 hyperbolic 10-crossing knots enjoy single-complex-place invariant trace fields, though we may estimate the fraction from 3 previous results. We found rational relations for $\frac{998}{11031} \approx 9\%$ of the closed census manifolds, for $\frac{312}{4929} \approx 6\%$ of the cusped census manifolds, and for $\frac{7}{79} \approx 9\%$ of the hyperbolic knots with less than 10 crossings. It thus seems unlikely that more than 10% of 10-crossing knots have rational Dedekind invariants. Had we selected a pair at random, the odds on both having single-complex-place invariant trace fields would be of order 100 : 1 against. Yet the volumes $\{58,59\}$ were not chosen at random; they come from the unique pair of positive hyperbolic 10-crossing knots.

It is, therefore, both notable and gratifying that the Feynman knots $10_{139}$ and $10_{152}$ both yield simple rational Dedekind-zeta invariants, namely 1 and $\frac{1}{2}$, corresponding to the Dedekind zeta values

$$
\zeta_{K_{139}}(2) = 1 \times \frac{(2\pi)^6 \text{vol}(10_{139})}{12 \times 688^{3/2}}
$$

$$
\zeta_{K_{152}}(2) = \frac{1}{2} \times \frac{(2\pi)^8 \text{vol}(10_{152})}{12 \times 8647^{3/2}}
$$

for the single-complex-place quartic and quintic fields

$$
K_{139} := x^4 - 2x - 1
$$

$$
K_{152} := x^5 - 3x^3 - 2x^2 + 2x + 1
$$

where $K_{139}$ is also the invariant trace field of the link $8_2^2$, with the same volume as $10_{139}$.

There is an important distinction between the 2 hyperbolic Feynman knots at 10 crossings and the sole hyperbolic Feynman knot at 11 crossings, associated with the irreducible $[10, 41]$ triple sum $\zeta(3, 5, 3) = \sum_{j>k>l>0} j^{-3}k^{-5}l^{-3}$. At 10 crossings it has not yet been possible to identify the generalized polylogarithms of weight 10 in the 7-loop Feynman diagrams $[11]$ that skein to give these knots. We know that each involves more than MZVs, since $\zeta(7, 3)$ is accounted for by the $(5, 3)$ torus knot, $10_{124}$.

Plans are afoot to compute, to high precision, the numbers associated by QFT to $10_{139}$ and $10_{152}$. Discussion with Andrei Davydychev and Dirk Kreimer suggests that it may be possible to extract values from high-order $\varepsilon$-expansions of multi-loop dressings of 2-loop skeletons in $4 - 2\varepsilon$ spacetime dimensions, with dressings deliberately chosen to frustrate reduction to MZVs. If that project bears fruit, there will be scope for PSLQ searches, beyond what is possible with current 10-digit data from 7-loop diagrams.

It is believed that promising targets for such integer-relation searches might be provided by volumes of polytopes in hyperbolic spaces of odd dimensions substantially greater than 3, and perhaps as large as 13, which may present a computational challenge to geometers as great as that confronting quantum field theorists at 7 loops. However, it will do no harm to include in PSLQ searches the easily computed weight-10 quintic Dedekind zeta value $\zeta(7, 3)$, from 3 dimensions, which would correspond to the less likely hypothesis that the associated geometry is simpler than appears from the 7-loop physics.

---

$^{20}$The loop number, $L$, is the number of 4-dimensional integrations over internal momenta; the crossing number does not exceed $2L - 3$. Numerical analysis of 28-dimensional integrals is rather taxing.
We noted that at 9 crossings the 3 knots with rational Dedekind-zeta invariants comprise the 3, 3, 3 pretzel knot 9_{35}, in (54), and a pair of non-alternating knots, namely 9_{48} and 9_{49}, in (51,52). Accordingly, we sought for further single-complex-place fields among the 10-crossing pretzels, namely 10_{46} := 5, 3, 2 and 10_{61} := 4, 3, 3, and the remaining non-alternating hyperbolic 10-crossing knots, namely \{10_n|165 \geq n \geq 125; n \neq 139; n \neq 152\}. From these 41 knots we obtained only 2 results:

\[ Z_{31,3} = \frac{1}{3} \times \text{vol}(10_{157}) = \frac{1}{3} \times \text{vol}(7_1^3) \]  
\[ Z_{29963,5} = 4 \times \text{vol}(10_{153}) \]  
providing a measure of how privileged is the Feynman [11, 14] pair, 10_{139} and 10_{152}. There was one join, already found in (57) at 9 crossings, with

\[ Z_3 + Z_{507,4} = \frac{1}{2} \times \text{vol}(10_{155}) = \frac{2}{5} \times \text{vol}(9_{28}) \]  
We note that the first example of degenerate volumes, provided by

\[ \text{vol}(10_{132}) = \text{vol}(9_{42}) \]  
does not yield a rational invariant, since the quintic invariant trace fields are generated by \(x^5 - x^4 - 2x^2 + 2x - 1\), with 2 complex places.

We also remark on the alternating knot 10_{123}. Like 9_{41}, it has a two-complex-place invariant trace field, generated by a root of \(x^4 - x^3 + x^2 - x + 1\). Thus its volume is provably related to Clausen values at multiples of \(\pi/5\). Interestingly, both knots have volumes that are rationally related to instances of the orthoscheme (46), with

\[ S(\frac{3}{4} \pi, \frac{1}{10} \pi, \frac{1}{10} \pi) = \frac{1}{10} \times \text{vol}(10_{123}) = \frac{2}{3} \text{Cl}_2(\frac{3}{5} \pi) + \frac{1}{3} \text{Cl}_2(\frac{\pi}{5}) \]  
\[ S(\frac{3}{10} \pi, \frac{1}{5} \pi, \frac{1}{10} \pi) = \frac{1}{10} \times \text{vol}(9_{41}) = \frac{1}{3} \text{Cl}_2(\frac{2}{5} \pi) + \frac{3}{5} \text{Cl}_2(\frac{4}{5} \pi) \]  
The corresponding relations between Clausen values are straightforward to derive, and hence quite unlike those with \(-D \geq 7\) in Table 22, which involve the 8 dramatic switches between number fields recorded in [26,30,37,48,49].

5.5 Dedekind zeta values from 12-crossing Feynman knots

At 12 crossings, corresponding to 8 loops in QFT, Broadhurst and Kreimer [14] found that 7 of the 2,176 hyperbolic knots are positive. We found that 3 of these have rational Dedekind-zeta invariants, with

\[ Z_{23,3} = \frac{1}{3} \times \text{vol}\left(\sigma_1 \sigma_2 \sigma_1 \sigma_2^7\right) \]  
\[ Z_{848,4} = 1 \times \text{vol}\left(\sigma_1 \sigma_2 \sigma_1 \sigma_2^5\right) \]  
\[ Z_{2068,4} = 2 \times \text{vol}\left(\sigma_1^3 \sigma_2 \sigma_1^3 \sigma_2^3\right) \]  
The first two are those identified as Feynman knots in [13,14]. Again, we find it uncanny that, with odds of at least 100 : 1 against, both Feynman knots proved to have single-complex-place invariant trace fields. It is clear that the Dedekind single-complex-place
criterion and the Feynman positivity criterion are strongly related. The origin of this association is, however, quite unclear to us.

Most notable is the result (70) for the 12-crossing positive knot with braid word \( \sigma_1 \sigma_2 \sigma_1 \sigma_2 \), which is associated \([13]\) with the irreducible MZV \( \zeta(9, 3) \) in QFT. It has a rather small, and very special, volume:

\[
\text{vol}(\sigma_1 \sigma_2 \sigma_1) = \frac{3}{10} \times \text{vol}(9_{49}) = \frac{3}{7} \times \text{vol}(7_2^2) = 3Z_{23,3} \approx 2.828122 \tag{73}
\]

which is precisely 3 times the volume of the closed Weeks manifold, m003(−3, 1), conjectured to be the smallest of all hyperbolic manifolds. The knot \( \sigma_1 \sigma_2 \sigma_1 \sigma_2 \) is the first in the sequence of hyperbolic Feynman knots \([14]\) \( F_{2n} := \sigma_1 \sigma_2 \sigma_1 \sigma_2^{2n-5} \), with \( 2n \geq 12 \) crossings, associated with \( \zeta(2n - 3, 3) \) in Feynman diagrams with \( n + 2 \geq 8 \) loops.

We found that the equality of the volumes of \( F_{12} \) and \( 5_2 \) generalizes to

\[
\text{vol}(\sigma_1 \sigma_2 \sigma_1 \sigma_2^{2n-5}) = \text{vol}(\sigma_1 \sigma_2 \sigma_1 \sigma_2^{11-2n}) \tag{74}
\]

where the knot on the r.h.s. is formally obtained by \( n \to 8 - n \) and has no more than \( 2n - 6 \) crossings. For \( 2n \geq 12 \) we also found that

\[
\text{vol}(F_{12}) \leq \text{vol}(F_{2n}) \leq \text{vol}(9_{60}^2) = \frac{1}{2}Z_7 \approx 5.333489 \tag{75}
\]

where \( Z_7 \) is the pivot of (24), which is the first of the non-trivial Clausen relations from 1-loop box diagrams. At large \( n \), the behaviour was measured to be

\[
\text{vol}(F_{2n}) = \frac{1}{2}Z_7 - \frac{C}{(\frac{1}{4}n - 1)^2} + O(1/n^4) \tag{76}
\]

with \( C \approx 0.8160 \), found from 12-digit SnapPea results, with up to 500 crossings.

The asymptote suggested that the manifolds complementary to the series \( F_{2n} \) of Feynman knots might be isometric to a series of Dehn fillings of a manifold with volume \( \frac{1}{2}Z_7 \). The drill command of Snap suggested the manifold s785, which we found to be isometric to the complement of the link \( 9_{60}^2 \). Performing the surgeries \((-2, 1)\ldots(-21, 1)\) on its second cusp, we obtained 20 manifolds and asked SnapPea to compare them with the manifolds complementary to the Feynman knots \( F_{2n} \), with crossing numbers from 12 to 50. The result was isometry, in all 20 cases. It was then possible to compute 64 good digits of

\[
C = 0.8160162119959694990691941006445603982758744790599736680521553757 \ldots \tag{77}
\]

from 20 high-precision Snap triangulations of s785(\(4-n, 1\)), with \( n = O(10^3) \). It would be interesting to obtain analytical results for asymptotic changes \([45]\) in volume, such as that given by (77).

Now one sees the origin of the invariance (74) of volumes, under the transformation \( n \to 8 - n \). This merely flips a sign of the Dehn surgery on the torus \([10]\) curve \((4-n, 1)\). The fixed point, at \( n = 4 \), is the non-hyperbolic longitudinal surgery \((0, 1)\), corresponding to the \((4, 3)\) torus knot, \( F_8 := (\sigma_1 \sigma_2^3)^2 \sim (\sigma_1 \sigma_2)^4 = 8_{19} \), which is the first 3-braid
Feynman knot, found at 6 loops in QFT, where it signals the appearance of the first irreducible MZV, $\zeta(5,3)$, in the counterterms of $\phi^4$-theory. At 7 loops, with $n = 5$, the surgery $(-1,1)$ is likewise non-hyperbolic, corresponding to the $(5,3)$ torus knot $F_{10} := \sigma_1 \sigma_2^3 \sigma_1 \sigma_2^5 \sim (\sigma_1 \sigma_2)^5 = 10_{124}$, associated with $\zeta(7,3)$. Only at 12 crossings, and hence 8 loops, does this series of Feynman knots start to be hyperbolic. One thus expects to find a rather special volume at 12 crossings, as is indeed seen in (73).

5.6 Dedekind zeta values from maximally symmetric knots

Jim Hoste, Morwen Thistlethwaite and Jeff Weeks (hereafter HTW) have recently completed an impressive symmetry analysis of all 1,701,936 prime knots with up to 16 crossings. From the University of Tennessee at Knoxville, we obtained files that identify highly symmetric alternating and non-alternating knots at crossing numbers from 11 to 16. Before analyzing the most symmetric of these, we comment on the situation up to 10 crossings, in territory charted by Dale Rolfsen and predecessors.

Table 23 gives the hyperbolic alternating and non-alternating knots of maximal symmetry at crossing numbers from 4 to 10, together with the symmetry groups, invariant trace fields, signatures and discriminants of their complementary manifolds. Where the field has a single complex place (i.e. signature $[n-2,1]$ at degree $n$) we give the rational Dedekind-zeta invariant, $a/b$. Table 24 shows our remaining finds of Dedekind-zeta invariants, for knots up to 10 crossings. The latter are likely to be complete up to 9 crossings; at 10 crossings we analyzed all non-alternating knots, but only about 20% of the alternating knots.

Some comments are in order.

1. Up to 9 crossings, maximal symmetry is a good – yet far from infallible – diagnostic of a single-complex-place field.

2. The maximally symmetric knots designated by Rolfsen as $6_3$, $7_7$, $8_{21}$, $9_{40}$ and $10_{123}$ have invariant trace fields with more than one complex place.

3. In (56) it is shown that the volume of $8_{21}$ reduces to a pair of Dedekind zeta values.

4. In (59) it is shown that the volume of $10_{123}$ reduces to that of a simple orthoscheme.

5. Only one Dedekind-zeta knot, namely $9_{49}$, escapes the sieve of maximal symmetry up to 9 crossings. Moreover, (51) shows that it yields a Dedekind zeta value already encountered at fewer crossings.

6. At 10 crossings, we found 3 non-alternating Dedekind-zeta knots with less than maximal symmetry: the Feynman pair $11_4$, $14$ pair, $10_{139}$ and $10_{152}$, with the modest symmetries $D_2$ and $Z_2$, and $10_{153}$, which SnapPea declared to be devoid of symmetry.

21http://www.math.utk.edu/~morwen/knotscape.html
22Below 8 crossings, all hyperbolic knots are alternating.
In the light of the above, we were prepared for a dwindling yield from maximal symmetry, above 10 crossings. The harvest proved to be even more meagre than we anticipated. Table 25 shows that none of the 11 maximally – and visibly \[15\] – symmetric alternating knots from 11 to 16 crossings has a single-complex-place invariant trace field. From the 10 maximally – and covertly \[15\] – symmetric non-alternating knots, we obtained 4 single-complex-place fields, yet only one of these entailed a new Dedekind zeta value, namely

$$Z_{643,4} = \frac{1}{5} \times \text{vol}(n14.13191)$$

while those in

$$Z_{44,3} = \frac{1}{4} \times \text{vol}(n12.642) \quad (79)$$

$$Z_{448,4} = \frac{1}{10} \times \text{vol}(n15.112310) \quad (80)$$

$$Z_{31,3} = \frac{1}{15} \times \text{vol}(n16.1007813) \quad (81)$$

had already been obtained in \([52,53,54]\). The last of these duplicates merits further comment.

There are more \([15]\) than a million non-alternating hyperbolic 16-crossing knots. Amid this plethora, HTW identified the 1,007,813th (in their lexicographic ordering of Dowker codes) as uniquely maximally symmetric. It enjoys the 18-fold dihedral symmetry group \(D_9\) of the nonagon, exquisitely disguised in any 16-crossing projection. If the reader has access to SnapPea, s/he should certainly not miss the opportunity to click in a depiction of the non-alternating knot in Fig. 7 of the highly readable article\(^{23}\) by HTW. Then the power of Jeff Weeks’ topological engine \([21]\) becomes apparent, when its symmetry analyzer announces \(D_9\). Morwen Thistlethwaite’s website\(^{24}\) renders this more visible, by resort to a 5-braid presentation.

Intuition told us to expect a Dedekind zeta value from the HTW knot with symmetry group \(D_9\). In this respect, we were not disappointed: the volume of the knot is 15 \(Z_{31,3}\), giving a Dedekind-zeta invariant \(a/b = 1/15\) that is smaller than any we had previously encountered. We allayed the disappointment, at having already encountered \(Z_{31,3}\), by the following ex post facto considerations. The \(D_9\) knot is so special that it merits a simple invariant trace field. The quadratic field \(\mathbb{Q}(\sqrt{-3})\) was taken by the figure-8 knot, in \([20]\). The cubic field with \(D = -23\) had already been engaged by Feynman, at 12 crossings, in \([73]\). The next cubic discriminant, \(D = -31\), provides an eminently suitable resting place for the Hoste–Thistlethwaite–Weeks \(D_9\) knot. Colleagues engaged on commensurability \([7]\) analyses may now investigate the wondrously long chain

$$Z_{31,3} = \frac{1}{15} \times \text{vol}(n16.1007813)$$

$$= \frac{1}{3} \times \text{vol}(10_{157}) \quad (82)$$

$$= \frac{1}{4} \times \text{vol}(v3183) \quad (83)$$

$$= \frac{1}{3} \times \text{vol}(v7^2) \quad (84)$$

$$= \frac{1}{2} \times \text{vol}(m034) \quad (85)$$

\(^{23}\) http://www.pitzer.edu/\texttt{\~jhoste}\ also gives access to \([15]\).

\(^{24}\) http://www.math.utk.edu/\texttt{\~morwen/d9.html}\
\[ \begin{align*}
&= 1 \times \text{vol}(m007(+4,1)) \\
&= \frac{2}{3} \times \text{vol}(m149(+1,2)) \\
&= \frac{2}{7} \times \text{vol}(v1963)
\end{align*} \]

of distinct rational Dedekind-zeta invariants from a common invariant trace field. The last 6 invariants cover 43 census manifolds. The volumes of the \( D_9 \) knot and \( 10_{157} \) are 30 \( D(z) \) and 16 \( D(z) \), where \( z^3 = 1 - z \), with \( 3z > 0 \), and \( D(z) = \frac{1}{2}Z_{31,3} \) is the volume of any one of the 4 ideal tetrahedra that triangulate \( m034 \). Here, as in the Feynman case (73), the Dedekind zeta value collapses to a rational multiple of a single Bloch–Wigner dilogarithm.

### 5.7 Dedekind zeta value from an 18-crossing Feynman knot

It was observed that the positivity criterion of [11, 13, 14] was more fertile than maximal symmetry, at 10 and 12 crossings. We expect further sequences of positive knots whose first hyperbolic instances yield Dedekind zeta values at high crossing numbers. From QFT we inferred a source of such a sequence, namely the 3-parameter family of even-crossing positive 3-braids [14]

\[ R_{k,m,n} := \sigma_1 \sigma_2^{2k} \sigma_1^m \sigma_2^{2n+1} \]

We knew from [14] that \( R_{2,2,1} \) is the 14-crossing torus knot \((7,3)\). Dirk Kreimer helped us show that \( R_{3,2,1} \) is the 16-crossing torus knot \((8,3)\). But \( R_{4,2,1} \) cannot be a torus knot, since 9 and 3 are not coprime. The 18-crossing hyperbolic volume is intriguingly small:

\[ \text{vol}(R_{4,2,1}) = 3.47424776131274229602900855361193191879781770805621\ldots \]

Table 13b immediately identified the invariant trace field as a single-complex-place sextic, with \( D = -753079 \). Table 6 then gave \( x^6 - x^5 - 3x^4 - x^3 + 2x^2 + 2x - 1 \) as the generating polynomial. SnapPea confirmed isometry of the complement of \( R_{4,2,1} \) with manifold \( m082 \). The corresponding rational Dedekind-zeta invariant, \( a/b = 26 \), in

\[ Z_{753079,6} = 26 \times \text{vol}(R_{4,2,1}) \]

is larger than we had found for any graphically generated knot, and is 390 times that for the \( D_9 \) knot, in (82). For the 5th time of asking, a positive Feynman knot gives a Dedekind zeta value. The odds on this being accidental are at least \( 10^5 : 1 \) against.

Lest such a connection between knots, numbers and Feynman diagrams be thought exceptional, we recall that at 8 crossings QFT demanded [11] a positive 3-braid knot and an irreducible depth-2 MZV, which were duly forthcoming, in the shape of \( 8_{19} \) and \( \zeta(5,3) \). At 11 crossings, the demands were similarly imperious: a positive 4-braid knot and an irreducible depth-3 MZV, satisfied by the uniqueness [11] of \( \sigma_1 \sigma_2^3 \sigma_3^2 \sigma_1^2 \sigma_2 \sigma_3 \) and the irreducibility [10] of \( \zeta(3,5,3) \). At 12 crossings, QFT seemed, at first, to require something perverse: an arbitrary choice between an irreducible depth-4 MZV and an irreducible depth-2 alternating [10] Euler sum. Mathematics accommodated, via the remarkable discovery [13, 14, 11] that one is reducible to the other. Compared with these past findings, a new Dedekind zeta value, at 18 crossings, is small fry to the maw of natural philosophy.

24
It took 50 years to discover that the renormalization of QFT is governed by a Hopf algebra \[^{[47]}\], richer than that of noncommutative geometry \[^{[48]}\], and readily automated \[^{[49]}\]. This offers the prospect of elucidating the existence of analytically non-trivial 4-term relations \[^{[50]}\] in realistic (i.e. 4-dimensional) QFT. Hopefully, our latest addition of a Dedekind-zeta connection, to the melting-pot of knot/number/field theory \[^{[52]}\], may also be illuminated by the joint efforts of physicists and mathematicians who are responsive to empirical data.

6 Conclusions and prospects

Perhaps more than in any other piece of research which either author has undertaken, this work has been vitally enabled by the internet. It provided us with the opportunity to blend number theory, topology, geometry, analysis, physics and computer science, in a global empirical mixing bowl, thanks to:

- ready access to significant data at Bordeaux, Claremont CA, Florham Park NJ, Knoxville TN, Melbourne, Minneapolis MN, as detailed in footnotes;
- the ability to run high-level packages, such as Maple, Pari and Reduce, on whatever machine best served our purposes, irrespective of the contingencies of our personal geographic co-ordinates;
- wonderful specialized resources, downloadable as per our footnotes, namely: David Bailey’s superb PSLQ and FFT routines, Oliver Goodman’s port of high-precision Snap to DigitalUnix, and Al Manoharan’s attractive adaptation to Windows95 of Jeff Weeks’ amazing SnapPea program, all supported by generous email advice;
- access to powerful computers in England, Newfoundland\(^{25}\) and Vancouver, yielding high-precision results such as those in \[^{[7]}\] and \[^{[77]}\], and the 1800-digit hyperbolic volumes of Tables 20 and 21, achieved by dedicated multiple-precision \[^{[22]}\] code, after exploratory work enabled by the above.

The facility with which we were able to plug into all these valuable resources advertises how rich the opportunities for empirical mathematics are becoming. The tools came together to offer more patterns than we had dared to hope for. Enterprises such as we have limned promise to be more and more a part of mathematical and physical research in the next few decades.

That said, many of the results which we have exhibited remain tantalizingly far from proof, let alone understanding. Here we repeat 4 of many remaining puzzles:

- How might one begin to derive relations such as \[^{[10]}\], between dilogarithms with arguments in radically different number fields?

\[^{25}\)http://www.cecm.sfu.ca/~jborwein/PUP_report_March15/report.html\]
• Why do 1-loop Feynman diagrams, at very specific values of the Mandelstam variables, generate even more relations than we were able to reify by quadratic links?

• Why do Feynman diagrams, with 7, 8 and 11 loops, lead to 5 Dedekind-zeta knots, at 10, 12 and 18 crossings, with odds of a chance association being at least $10^5 : 1$ against?

• Can the mere dilogarithms of 3-dimensional hyperbolic geometric tell us anything about the unidentified 10th-order polylogarithms of 7-loop quantum field theory?

When faced by such visible expansion of one’s lack of wisdom, it is probably best to concentrate on that which is easiest to state. Hence we conclude with a rewriting of the simplest unproven relation (13) in terms of the Dirichlet series (20). From light-by-light scattering at $s = t = 7$, and – just as mysteriously – from the hyperbolic volume of the link $6_3^1$, we infer that

$$
\sum_{n>0} \left\{ 3 \left( \frac{-3}{4} + \frac{\sqrt{-7}}{4} \right)^n - 3 \left( \frac{-3}{4} + \frac{\sqrt{-7}}{4} \right)^{2n} + \left( \frac{-3}{4} + \frac{\sqrt{-7}}{4} \right)^{3n} \right\} \frac{1}{n^2} = 13 \zeta(2) - 6 \pi \arctan \sqrt{7} + \frac{7\sqrt{-7}}{4} \sum_{n>0} \left( \frac{n}{7} \right) \frac{1}{n^2}
$$

(93)

The real part of this empirical and indubitable equality is easily proven; in its imaginary part, with the Jacobi symbol $\left( \frac{n}{7} \right)$, resides a flinty kernel which is – to us at least – intractable. A proof of (93) and the other quadratic identities would be most welcome.

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Table 1: Dedekind zeta values hereby related to volumes of closed manifolds

| degree | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | ≤12 |
|--------|---|---|---|---|---|---|---|---|----|----|----|-----|
| number | 3 | 11| 32| 38| 25| 19| 5 | 3 | 2  | 1  | 1  | 140 |

Table 2: Quadratic fields

| field            | $-D$ | a | b | manifold          |
|------------------|------|---|---|-------------------|
| $x^2 - x + 1$    | 3    | 2 | 1 | m007(+3, 1)       |
| $x^2 + 1$        | 4    | 2 | 1 | m009(+5, 1)       |
| $x^2 - x + 2$    | 7    | 4 | 1 | m036(−4, 3)       |

Table 3: Cubic fields

| field            | $-D$ | a | b | manifold          |
|------------------|------|---|---|-------------------|
| $x^3 - x^2 + 1$  | 23   | 1 | 1 | m003(−3, 1)       |
| $x^3 + x - 1$    | 31   | 1 | 1 | m007(+4, 1)       |
| $x^3 - x^2 + x + 1$ | 44 | 2 | 1 | m006(+3, 1)       |
| $x^3 + 2x - 1$   | 59   | 4 | 1 | m004(+6, 1)       |
| $x^3 - 2x - 2$   | 76   | 2 | 1 | s784(+1, 2)       |
| $x^3 - x^2 + x - 2$ | 83 | 4 | 1 | m034(−3, 2)       |
| $x^3 - x^2 + 2x + 1$ | 87 | 2 | 1 | s784(−1, 2)       |
| $x^3 - x - 2$    | 104  | 4 | 1 | s297(+1, 3)       |
| $x^3 - x^2 + 3x - 2$ | 107| 4 | 1 | m168(−3, 2)       |
| $x^3 - x^2 - 2$  | 116  | 4 | 1 | s881(−1, 3)       |
| $x^3 - x^2 + x + 2$ | 139| 4 | 1 | v3106(+3, 1)      |
| field                     | $-D$ | $a$ | $b$ | manifold |
|---------------------------|------|-----|-----|----------|
| $x^4 - x^3 + 2x - 1$      | 275  | 2   | 5   | m016(−4, 3) |
| $x^4 - x - 1$             | 283  | 1   | 1   | m003(−2, 3) |
| $x^4 - x^3 + x^2 + x - 1$ | 331  | 1   | 1   | m003(−4, 3) |
| $x^4 - x^2 - 1$           | 400  | 2   | 5   | m400(+4, 1) |
| $x^4 - 2x^3 + x^2 - 2x + 1$ | 448 | 1   | 1   | m010(+3, 2) |
| $x^4 - x^3 - x^2 + 3x - 1$ | 491  | 1   | 1   | m029(−3, 2) |
| $x^4 - x^3 - x^2 - x + 1$ | 507  | 1   | 1   | m160(−3, 2) |
| $x^4 - x^3 + x^2 - x - 1$ | 563  | 1   | 1   | m130(+1, 4) |
| $x^4 - x^3 - 2x + 1$      | 643  | 1   | 1   | m247(−1, 4) |
| $x^4 - 2x - 1$            | 688  | 2   | 1   | m019(+3, 4) |
| $x^4 - x^3 + 2x^2 - 1$    | 731  | 1   | 1   | s649(−3, 4) |
| $x^4 - 2x^3 + x^2 - x - 1$ | 751 | 2   | 1   | m081(−4, 1) |
| $x^4 - x^2 - 2x + 1$      | 848  | 2   | 1   | m207(−1, 3) |
| $x^4 - 2x^3 + 3x^2 - 1$   | 976  | 2   | 1   | m286(−5, 1) |
| $x^4 - x^3 + x^2 - 3x + 1$ | 1099 | 2   | 1   | s663(+1, 2) |
| $x^4 - x^3 - 2x - 1$      | 1107 | 2   | 1   | s928(+4, 1) |
| $x^4 - x^3 - 2x^2 - x + 1$ | 1156 | 4   | 1   | m082(+1, 3) |
| $x^4 - x^3 + 2x^2 + x - 1$ | 1192 | 4   | 1   | m148(−3, 2) |
| $x^4 - x^2 - 3x - 1$      | 1255 | 2   | 1   | v3492(+4, 3) |
| $x^4 + 2x^2 - x - 1$      | 1371 | 2   | 1   | v3489(+2, 3) |
| $x^4 - x^3 + x - 2$       | 1399 | 4   | 1   | m293(+2, 3) |
| $x^4 - x^3 - 3x^2 + 2$    | 1588 | 8   | 1   | m038(+4, 1) |
| $x^4 - x^3 + 3x - 1$      | 1732 | 4   | 1   | v3187(−4, 1) |
| $x^4 - x^3 - x^2 - 2x + 1$ | 1791 | 4   | 1   | s961(+2, 3) |
| $x^4 - x^3 - 2x^2 - 3x + 1$ | 1879 | 4   | 1   | v2914(+2, 3) |
| $x^4 - 2x^3 + x^2 - 3x + 1$ | 1927 | 4   | 1   | v3452(−5, 1) |
| $x^4 - 4x^2 - 2x + 2$     | 1968 | 6   | 1   | s594(+1, 3) |
| $x^4 - x^3 - 2x^2 + 3x + 1$ | 2068 | 8   | 1   | m389(+3, 1) |
| $x^4 - x^2 - 3x - 2$      | 2151 | 12  | 1   | m015(+8, 1) |
| $x^4 - x^3 - 2x^2 + 3x + 2$ | 2319 | 6   | 1   | v2944(−5, 2) |
| $x^4 - 2x^3 - x^2 + 2x - 2$ | 3312 | 12  | 1   | v3209(+2, 3) |
| $x^4 - 5x^2 - 4$          | 6724 | 64  | 1   | s479(−5, 1) |
**Table 5:** Quintic fields

| field                                      | $-D$  | $a$ | $b$ | manifold       |
|--------------------------------------------|-------|-----|-----|----------------|
| $x^5 - x^4 - 2x^2 + 1$                     | 4511  | 1   | 1   | m003(−5, 3)    |
| $x^5 - x^4 - x^3 + 2x^2 - x - 1$          | 4903  | 1   | 1   | m015(+5, 1)    |
| $x^5 - x^4 - x^3 + 3x^2 - 1$              | 5519  | 1   | 1   | m016(+3, 2)    |
| $x^5 - 2x^4 + x^3 + 2x^2 - 2x - 1$        | 5783  | 1   | 1   | m016(+2, 3)    |
| $x^5 - x^3 - x^2 - x + 1$                 | 7031  | 1   | 1   | m160(−4, 1)    |
| $x^5 - 2x^4 + x^3 - 2x + 1$               | 7463  | 1   | 1   | m178(+4, 3)    |
| $x^5 - 3x^3 - 2x^2 + 2x + 1$              | 8647  | 2   | 1   | m016(+4, 1)    |
| $x^5 - x^4 - x^3 + x^2 - 2x + 1$          | 9439  | 1   | 1   | v2759(−3, 1)   |
| $x^5 - 2x^4 + x^2 - 2x - 1$               | 9759  | 3   | 1   | m007(−3, 2)    |
| $x^5 - x^4 - 3x^3 + 3x - 1$               | 10407 | 3   | 1   | m023(−4, 1)    |
| $x^5 - x^4 - 2x + 1$                      | 11243 | 2   | 1   | s090(+5, 1)    |
| $x^5 - x^4 - 2x^3 + 3x^2 - x - 1$         | 11551 | 2   | 1   | m223(−1, 2)    |
| $x^5 + x^3 - x^2 - 3x - 1$                | 12447 | 2   | 1   | s657(−1, 2)    |
| $x^5 - 2x^2 - x + 1$                      | 13219 | 2   | 1   | s645(+1, 3)    |
| $x^5 - x^4 - x^3 - x^2 - 3x + 1$          | 13523 | 2   | 1   | v2530(+1, 3)   |
| $x^5 - x^4 - 2x^3 + x + 2$                | 13883 | 2   | 1   | v3310(+5, 1)   |
| $x^5 - 2x^4 + 2x^3 - x^2 - 2x^2 - 1$      | 14103 | 2   | 1   | s784(+5, 2)    |
| $x^5 - x^4 - 3x^3 + 3x^2 - 1$             | 14631 | 2   | 1   | s958(+3, 2)    |
| $x^5 - x^4 - 2x^2 - x^2 - 3x + 1$         | 14911 | 2   | 1   | v2704(−5, 1)   |
| $x^5 - x^4 - 2x^3 + x^2 + 2x + 2$         | 17348 | 8   | 1   | m006(−5, 1)    |
| $x^5 - 2x^4 + 4x^2 - x - 1$               | 22331 | 4   | 1   | s884(+2, 3)    |
| $x^5 - 2x^3 - x^2 - x + 1$                | 22424 | 4   | 1   | v3246(−2, 3)   |
| $x^5 - 2x^4 + x^2 - 3x + 1$               | 23103 | 6   | 1   | m223(+1, 1)    |
| $x^5 - x^4 + 2x^2 - 2x - 1$               | 23339 | 4   | 1   | v3199(+3, 1)   |
| $x^5 - 2x^3 - 3x^2 + x + 1$               | 29444 | 8   | 1   | s478(−1, 3)    |
| $x^5 - 2x^4 - 2x^3 + 4x^2 - x - 1$        | 29963 | 16  | 1   | m007(−5, 1)    |
| $x^5 - x^4 + 3x^2 - 6x + 2$               | 31684 | 8   | 1   | v2381(+3, 1)   |
| $x^5 - 2x^4 + 2x^3 + x^2 - 3x - 1$        | 34436 | 8   | 1   | v2794(−2, 3)   |
| $x^5 - 2x^4 + 2x^3 - 3x^2 - x + 4$        | 34779 | 9   | 1   | v3214(+3, 1)   |
| $x^5 - 2x^4 - x^3 + 2x^2 - x + 3$         | 38083 | 14  | 1   | s373(+1, 3)    |
| $x^5 - 2x^4 - x^3 + 4x^2 - 2x - 2$        | 58064 | 40  | 1   | m141(+2, 3)    |
| $x^5 - 4x^4 + 3x^2 - 8x + 4$              | 60803 | 28  | 1   | m148(+5, 1)    |
| $x^5 - 2x^4 - 2x^2 + 4$                   | 70736 | 32  | 1   | v2787(−1, 3)   |
| $x^5 - 5x^3 - 2x^2 + 3x + 2$              | 79952 | 36  | 1   | s594(+3, 2)    |
| $x^5 - 2x^4 - 2x^3 + 4x^2 - x - 2$        | 107264| 52  | 1   | v3454(−5, 1)   |
| $x^5 - 2x^4 - 3x^3 + x^2 + 5x + 2$        | 112919| 88  | 1   | m304(+1, 3)    |
| $x^5 - 4x^4 - 6x^2 - 7x - 2$              | 141791| 104 | 1   | s707(+5, 1)    |
| $x^5 - 6x^3 - 5x^2 - 2x - 4$              | 239639| 184 | 1   | s918(+3, 2)    |
Table 6: Sextic fields

| field                                                                 | $-D$   | $a$ | $b$ | manifold       |
|----------------------------------------------------------------------|--------|-----|-----|----------------|
| $x^6 - x^5 - 2x^4 + 3x^3 - x^2 - 2x + 1$                            | 92779  | 1   | 1   | $m222(-6,1)$   |
| $x^6 - 2x^4 - 2x^3 + 3x + 1$                                        | 94363  | 1   | 1   | $m345(+1,2)$   |
| $x^6 - x^5 - x^4 + 2x^3 - 2x^2 - x + 1$                              | 104483 | 1   | 1   | $s648(+1,2)$   |
| $x^6 - 2x^5 + x^4 - 2x^3 - x^2 + 3x - 1$                             | 161939 | 2   | 1   | $s682(+3,1)$   |
| $x^6 - x^5 - 2x^4 - 2x^3 + x^2 + 3x + 1$                             | 215811 | 6   | 1   | $m015(-3,2)$   |
| $x^6 - x^5 - 4x^4 + 4x^3 + 4x^2 - 2x - 1$                           | 238507 | 7   | 1   | $m034(+4,1)$   |
| $x^6 - x^5 - 2x^4 - 3x^3 + 3x^2 + 3x - 3$                            | 365263 | 26  | 1   | $m004(+3,2)$   |
| $x^6 - 2x^5 - 2x^4 + 6x^3 - 2x^2 - 5x + 3$                          | 417467 | 13  | 2   | $v3375(-5,2)$  |
| $x^6 - 2x^5 - x^4 + 5x^3 - 3x^2 - 3x + 2$                            | 463471 | 16  | 1   | $m249(+4,1)$   |
| $x^6 - 2x^5 - 2x^4 + 5x^3 - 3x^2 + 3x - 1$                          | 561863 | 10  | 1   | $v3239(+3,2)$  |
| $x^6 - 2x^5 + 6x^3 - 3x^2 - 2x + 1$                                 | 629952 | 18  | 1   | $s386(+5,2)$   |
| $x^6 - x^5 - 3x^3 - 3x^2 + 5x - 1$                                  | 661831 | 16  | 1   | $v3361(+1,3)$  |
| $x^6 - x^5 - x^4 + 5x^3 + x^2 - 3x - 1$                              | 688927 | 14  | 1   | $v3243(-3,1)$  |
| $x^6 - 4x^4 - 2x^3 + 3x^2 + 5x + 1$                                 | 709783 | 20  | 1   | $s952(-4,1)$   |
| $x^6 - x^5 - 3x^4 - x^3 + 2x^2 + 2x - 1$                             | 753079 | 26  | 1   | $m337(-3,1)$   |
| $x^6 - 2x^5 + 2x^3 + 5x^3 - 2x^2 - 3x + 2$                          | 899447 | 44  | 1   | $m189(+3,2)$   |
| $x^6 - 2x^5 - 2x^4 + 7x^3 - x^2 - 5x + 1$                            | 1014119| 38  | 1   | $m286(-6,1)$   |
| $x^6 - x^5 + 5x^3 - 4x^2 - 4x + 2$                                  | 1107052| 50  | 1   | $v1315(-4,1)$  |
| $x^6 - 2x^5 - 2x^4 + 4x^3 - 2x^2 - 2x + 1$                           | 1290496| 80  | 1   | $m285(-4,1)$   |
| $x^6 - 2x^5 + x^4 - 3x^3 - x^2 + 5x - 2$                             | 1494223| 56  | 1   | $v3036(+3,2)$  |
| $x^6 - x^5 - 3x^4 + 5x^3 + x^2 - 4x - 1$                             | 1825672| 100 | 1   | $m358(-5,3)$   |
| $x^6 - 9x^3 - 10x^2 - x + 1$                                        | 2803244| 236 | 1   | $m192(-5,2)$   |
| $x^6 - x^5 - 4x^4 + 8x^3 + 9x^2 - 10x - 7$                           | 4241707| 296 | 1   | $v3428(-4,1)$  |
| $x^6 + x^4 - 7x^3 - 2x^2 + 7x + 2$                                  | 5873596| 688 | 1   | $v1858(+6,1)$  |
| $x^6 - x^5 - 3x^4 + 8x^3 - 3x^2 - 7x + 1$                            | 7792864| 976 | 1   | $v2789(-2,3)$  |
### Table 7: Septic fields

| field                                                                 | $-D$   | $a$ | $b$ | manifold     |
|----------------------------------------------------------------------|--------|-----|-----|--------------|
| $x^7 - x^6 - 3x^5 + 4x^4 + 2x^3 - 6x^2 + x + 1$                      | 3685907| 14  | 1   | m006(-3,2)  |
| $x^7 - 2x^5 + 5x^4 - 6x^2 + x + 1$                                   | 3998639| 17  | 1   | m004(+5,2)  |
| $x^7 - 3x^5 - 3x^4 + 4x^3 + 5x^2 - 2x - 1$                          | 4297259| 10  | 1   | m221(-1,2)  |
| $x^7 - 2x^5 - 3x^4 - 3x^3 + 3x^2 + 4x + 1$                          | 4795631| 13  | 1   | m032(+5,2)  |
| $x^7 - 3x^5 - 4x^4 - 3x^3 + 3x^2 + 4x + 1$                          | 4795631| 13  | 1   | m032(+5,2)  |
| $x^7 - x^6 - 5x^5 - 2x^4 + 6x^3 + 5x^2 - 2x + 1$                    | 6515927| 11  | 1   | s900(+3,2)  |
| $x^7 - 2x^6 - 3x^5 + 3x^4 + 5x^3 - x^2 - 3x + 1$                    | 7215127| 46  | 1   | m004(+1,2)  |
| $x^7 - 2x^6 + 4x^4 - 5x^3 - 2x^2 + 4x + 1$                          | 7557047| 14  | 1   | v2221(-1,3) |
| $x^7 - x^6 - 5x^5 + 6x^4 + 6x^3 - 2x^2 + 4x + 1$                    | 7729991| 32  | 1   | m038(+1,2)  |
| $x^7 - 3x^6 - 4x^5 + 6x^4 + 6x^3 - 4x^2 - 3x + 1$                   | 9429911| 20  | 1   | s838(-2,3)  |

**Notes:**

- Table entries are rounded to the nearest integer.
- The manifold column indicates the specific manifold associated with each field.
- The $-D$ column lists the negative discriminant values for each field.
Table 8: Octadic fields

| Field                                      | $-D$     | $a$ | $b$ | Manifold     |
|--------------------------------------------|----------|-----|-----|--------------|
| $x^8 - 6x^6 - 5x^5 + 7x^4$                 | 202734487| 100 | 1   | m038(+3, 2) |
| $+ 10x^3 - x^2 - 4x - 1$                   |          |     |     |              |
| $x^7 - 3x^6 + 7x^5 - 2x^4$                 | 948381887| 496 | 1   | s900(-2, 3) |
| $- 8x^3 + 4x^2 + 2x - 1$                   |          |     |     |              |
| $x^6 - 2x^5 - 3x^4 + 11x^3 - 4x^2$        | 209521867| 2240| 1   | s901(+4, 1) |
| $- 13x^3 + 8x^2 + 5x - 2$                  |          |     |     |              |
| $x^5 - 3x^4 - 16x^3 - 11x^2 - x^1$        | 240125983| 1543| 1   | s649(-5, 3) |
| $- 21x^3 + 16x^2 + 12x - 1$                |          |     |     |              |
| $x^4 - 4x^3 - 4x^2 + 14x^3 + 23x^4$       | 810519653| 571 | 1   | m115(-5, 2) |
| $- 13x^3 - 32x^2 - 6x + 4$                 |          |     |     |              |

Table 9: Nonadic fields

| Field                                      | $-D$     | $a$ | $b$ | Manifold     |
|--------------------------------------------|----------|-----|-----|--------------|
| $x^9 - 3x^8 - 2x^7 + 11x^6 - 5x^5$         | 884365279| 571 | 1   | m115(-5, 2) |
| $- 10x^4 + 11x^3 - x^2 - 4x + 1$          |          |     |     |              |
| $x^8 - 3x^7 - 4x^6 + 16x^5 - 11x^4$       | 485028103| 5230| 1   | v3157(+5, 1) |
| $- 22x^4 + 6x^3 - 4x^2 - 4x + 4$          |          |     |     |              |
| $x^7 - 3x^6 - 3x^5 + 13x^4$               | 999619203| 1543| 1   | s649(-5, 3) |
| $- 13x^4 + 2x^3 - 2x^2 + 3x + 1$         |          |     |     |              |

Table 10: Decadic fields

| Field                                      | $-D$     | $a$ | $b$ | Manifold     |
|--------------------------------------------|----------|-----|-----|--------------|
| $x^{10} - 4x^9 - 5x^8 + 5x^7 + 19x^6$     | 271488204| 3669| 1   | m006(-5, 2) |
| $- 2x^4 - 21x^3 + x^2 + 6x - 1$           |          |     |     |              |
| $x^{10} - 4x^9 - x^8 + 20x^7 - 15x^6 - 23x^5$ | 774868765| 2320| 1   | s900(+2, 3) |
| $+ 29x^4 - 4x^3 - 7x^2 + 6x - 1$         |          |     |     |              |

Table 11: Endecadic field

| Field                                      | $-D$     | $a$ | $b$ | Manifold     |
|--------------------------------------------|----------|-----|-----|--------------|
| $x^{11} - 3x^{10} - 5x^9 + 20x^8 + 3x^7 - 42x^6$ | 219904978| 6883| 1   | m007(-5, 2) |
| $+ 14x^5 + 28x^4 - 17x^3 - x^2 + 4x - 1$   |          |     |     |              |

Table 12: Duodecadic field

| Field                                      | $-D$     | $a$ | $b$ | Manifold     |
|--------------------------------------------|----------|-----|-----|--------------|
| $x^{12} - 3x^{11} - 8x^{10} + 17x^9$      | 124762394| 9408| 1   | v2824(+4, 1) |
| $+ 27x^8 - 19x^7 - 50x^6 - 24x^5$         |          |     |     |              |
| $+ 44x^4 + 37x^3 - 5x^2 - 8x - 1$         |          |     |     |              |
Table 13a: Rational relations of Dedekind zeta values to volumes

| −D  | n   | a   | b   | \( M \)          | vol(\( M \)) |
|------|-----|-----|-----|------------------|--------------|
| 23   | 3   | 1   | 1   | m003(−3, 1)      | 0.942707362776927720921299603 |
| 283  | 4   | 1   | 1   | m003(−2, 3)      | 0.98136882889223208091452189 |
| 3    | 2   | 2   | 1   | m007(+3, 1)      | 1.014941606409653625021202554 |
| 331  | 4   | 1   | 1   | m003(−4, 3)      | 1.263709238658034655884716346 |
| 59   | 3   | 4   | 1   | m004(+6, 1)      | 1.284485300468354442460337084 |
| 7215127 | 7 | 46  | 1   | m004(+1, 2)      | 1.39850888415086640509594326  |
| 23   | 3   | 2   | 3   | m009(+4, 1)      | 1.41406104416539158138149404  |
| 365263 | 6  | 26  | 1   | m004(+3, 2)      | 1.440699006727364875282370223 |
| 3998639 | 7  | 17  | 1   | m004(+5, 2)      | 1.52947732943002626282492629  |
| 4511  | 5   | 1   | 1   | m003(−5, 3)      | 1.54356891174518504328472943  |
| 31   | 3   | 1   | 1   | m007(+4, 1)      | 1.583166660624812836166028851 |
| 44   | 3   | 2   | 1   | m006(+3, 1)      | 1.58864639300162988176913812  |
| 3685907 | 7  | 14  | 1   | m006(−3, 2)      | 1.64960971580664120798395881  |
| 4903  | 5   | 1   | 1   | m015(+5, 1)      | 1.75712602918845136287474593  |
| 9759  | 5   | 3   | 1   | m007(−3, 2)      | 1.8243443220291961274957217   |
| 42   | 2   | 2   | 1   | m009(−5, 1)      | 1.83193118835438030109207029  |
| 29963 | 5   | 16  | 1   | m007(−5, 1)      | 1.843585972326677938720454754 |
| 23   | 3   | 1   | 2   | m016(−3, 2)      | 1.885414725553855441842599206 |
| 17348 | 5   | 8   | 1   | m006(−5, 1)      | 1.941503084027467793730320127 |
| 283  | 4   | 1   | 2   | m006(+2, 3)      | 1.96273765778446176182904379  |
| 10407 | 5   | 3   | 1   | m023(+4, 1)      | 2.014336583776842504278826477 |
| 271488204251 | 10 | 3669 | 1 | m006(−5, 2)      | 2.028853091474922845790767756 |
| 3    | 2   | 1   | 1   | m006(−3, 2)      | 2.029883212819307250042405108 |
| 448  | 4   | 1   | 1   | m010(−3, 2)      | 2.05848436819303362456050739  |
| 21990497831723 | 11 | 68838 | 1 | m007(−5, 2)      | 2.065670838488380741576307932 |
| 5519  | 5   | 1   | 1   | m016(+3, 2)      | 2.114567693110222328090213031 |
| 8647  | 5   | 2   | 1   | m016(+4, 1)      | 2.134016336081402150786045480 |
| 238507 | 5   | 7   | 1   | m034(+4, 1)      | 2.184755575062588397026324600 |
| 83   | 3   | 4   | 1   | m034(−3, 2)      | 2.20766623872639212474919817  |
| 215811 | 6  | 6   | 1   | m015(−3, 2)      | 2.226717903919389683617840551 |
| 7729991 | 7 | 32  | 1   | m038(+1, 2)      | 2.25976713259575572056728360  |
| 5783  | 5   | 1   | 1   | m016(−2, 3)      | 2.272631863586558174554150421 |
| 1588  | 4   | 8   | 1   | m038(+4, 1)      | 2.27795944493692655279329057  |
| 275   | 4   | 2   | 5   | m016(−4, 3)      | 2.343017136901306265356245324 |
| 23   | 3   | 2   | 5   | m019(+1, 4)      | 2.35676840694231930230324907  |
| 2151  | 4   | 12  | 1   | m015(+8, 1)      | 2.362700792554500476496595823 |
| 31   | 3   | 2   | 3   | m149(+1, 2)      | 2.374749990937219254249043277 |
| 688   | 4   | 2   | 1   | m19(+3, 4)       | 2.425585378666368783529163526 |
| 491   | 4   | 1   | 1   | m029(−3, 2)      | 2.468232196680908678928523005  |
| 202734487 | 8 | 100  | 1 | m038(+3, 2)      | 2.502659305372821115596395708  |
Table 13b: Rational relations of Dedekind zeta values to volumes

| $-D$ | $n$ | $a$ | $b$ | $\mathcal{M}$ | $\mathrm{vol}(\mathcal{M})$ |
|------|-----|-----|-----|----------------|-----------------|
| 331  | 4   | 1   | 2   | $m070(-3,1)$   | 2.527418477316087311769442693 |
| 59   | 3   | 2   | 1   | $m039(+6,1)$   | 2.5689706093608884920674169 |
| 507  | 4   | 1   | 1   | $m160(-3,2)$   | 2.59538593667421393993834 |
| 32775179 | 7 | 316 | 1   | $m026(-5,2)$   | 2.60918123951303362390193601 |
| 4795631 | 7 | 13  | 1   | $m032(+5,2)$   | 2.62940539528839872278830854 |
| 4903 | 5   | 2   | 3   | $m034(-5,2)$   | 2.6356890438267704312119897 |
| 7    | 2   | 4   | 1   | $m036(-4,3)$   | 2.66674478344905979076712462 |
| 58360112 | 7 | 788 | 1   | $m034(+5,2)$   | 2.679475805755312597797042568 |
| 751  | 4   | 2   | 1   | $m081(-4,1)$   | 2.78183931236079791875337802 |
| 1156 | 4   | 4   | 1   | $m082(+1,3)$   | 2.7868054561556852185509429 |
| 12558899 | 7 | 68  | 1   | $m070(-3,2)$   | 2.812516496543210373175617462 |
| 23   | 3   | 1   | 3   | $m221(+3,1)$   | 2.828122088330783162763889809 |
| 4297259 | 7 | 10  | 1   | $m221(-1,2)$   | 2.9133321143066936871943484 |
| 1192 | 4   | 4   | 1   | $m148(-3,2)$   | 2.92151142892383082004060113 |
| 283  | 4   | 1   | 3   | $m130(-2,3)$   | 2.94410648667669626427456569 |
| 848  | 4   | 2   | 1   | $m207(-1,3)$   | 2.95837286759139434763630075 |
| 60803 | 5   | 28  | 1   | $m148(+5,1)$   | 2.97032110188936428725353762 |
| 3    | 2   | 2   | 3   | $m149(-4,1)$   | 3.044824819228960560676662 |
| 3998639 | 7 | 17  | 2   | $m286(-4,1)$   | 3.058954658860505252549875259 |
| 563  | 4   | 1   | 1   | $m130(-4,1)$   | 3.05933805777895567338809625 |
| 4511 | 5   | 1   | 2   | $m119(+3,2)$   | 3.08713782294371048656945886 |
| 7031 | 5   | 1   | 1   | $m160(-4,1)$   | 3.104808522680010091051472945 |
| 8843652791 | 9 | 571 | 1   | $m115(-5,2)$   | 3.123273828139318565185117904 |
| 58064 | 5   | 40  | 1   | $m141(+2,3)$   | 3.1333496486809635137196912 |
| 331  | 4   | 2   | 5   | $m146(+5,1)$   | 3.15927309664510913971790867 |
| 31   | 3   | 1   | 2   | $s119(+4,1)$   | 3.16633332124962567232057703 |
| 44   | 3   | 1   | 1   | $m141(+4,1)$   | 3.177293278600325976353826724 |
| 11243 | 5   | 2   | 1   | $s090(+5,1)$   | 3.25290804847164592803755603 |
| 107  | 3   | 4   | 1   | $m168(-3,2)$   | 3.27587164393934323965063070 |
| 3685907 | 7 | 7   | 1   | $m222(+3,2)$   | 3.29921431673282415971763 |
| 23   | 3   | 2   | 7   | $m178(-2,3)$   | 3.29947576971924702324548610 |
| 1290496 | 6 | 80  | 1   | $m285(-4,1)$   | 3.341002200879537591226767603 |
| 463471 | 6   | 16  | 1   | $m249(+4,1)$   | 3.36209320442704804370589278 |
| 899447 | 6   | 44  | 1   | $m189(+3,2)$   | 3.383197893650556120356333505 |
| 7463 | 5   | 1   | 1   | $m178(+4,3)$   | 3.402991251166455752574894719 |
| 5783 | 5   | 2   | 3   | $m175(-1,3)$   | 3.408947795379837261831225632 |
| 2068 | 4   | 8   | 1   | $m389(+3,1)$   | 3.410187936572092377001210125 |
| 283  | 4   | 2   | 7   | $m189(-5,2)$   | 3.43479090112281230832082664 |
| 976  | 4   | 2   | 1   | $m286(-5,1)$   | 3.454319317492031906374420374 |
| 753079 | 6 | 26  | 1   | $m337(-3,1)$   | 3.474247761312742296029008553 |
Table 13c: Rational relations of Dedekind zeta values to volumes

| $-D$ | $n$ | $a$ | $b$ | $\mathcal{M}$ | $\text{vol}(\mathcal{M})$ |
|------|-----|-----|-----|-------------|------------------|
| 23103 | 5   | 6   | 1   | m223(+5,1)  | 3.476375673391812878562982302 |
| 4903  | 5   | 1   | 2   | m220(+5,2)  | 3.51425205837690272574993187 |
| 11551 | 5   | 2   | 1   | m223(-1,3)  | 3.5440817346457986884505442  |
| 2803244 | 6  | 236  | 1   | m192(-5,2)  | 3.5504301411866437562339763  |
| 81589747 | 7  | 828  | 1   | m213(-5,2)  | 3.573600148051412384835850940 |
| 643   | 4   | 1    | 1   | m247(-1,4)  | 3.5817073255636530514272152  |
| 92779 | 6   | 1    | 1   | m222(-6,1)  | 3.608689606770784943522497291 |
| 4     | 2   | 1    | 1   | s942(-2,1)  | 3.663862376708876060218414059 |
| 1399  | 4   | 4    | 1   | m293(+2,3)  | 3.675645605949870731162818740 |
| 29963 | 5   | 8    | 1   | m310(+1,2)  | 3.687119446535587440905096   |
| 94363 | 6   | 1    | 1   | m345(+1,2)  | 3.70289732185694061261639443   |
| 1014119 | 6  | 38   | 1   | m286(-6,1)  | 3.71997765434257771107282353   |
| 104  | 3   | 4    | 1   | s297(+1,3)  | 3.758844948237284271433258100 |
| 23   | 3   | 1    | 4   | s645(-2,1)  | 3.770829451107710883685198412 |
| 331  | 4   | 1    | 3   | s254(+5,1)  | 3.7911271597430967654149040   |
| 59   | 3   | 4    | 3   | s296(-5,1)  | 3.853455901405063327381011254 |
| 17348 | 5   | 4    | 1   | s403(-1,2)  | 3.883006168054935587460640255 |
| 283  | 4   | 1    | 4   | m339(-2,3)  | 3.92547531556892835236580759   |
| 112919 | 5  | 88   | 1   | m304(+1,3)  | 3.933950637784033249426550295 |
| 76   | 3   | 2    | 1   | s784(+1,2)  | 3.970289623890655394010469558 |
| 12447 | 5   | 2    | 1   | s657(-1,2)  | 3.97812785235913152636783361 |
| 38083 | 5   | 14   | 1   | s437(+1,3)  | 4.003979154528882088138782858 |
| 1099 | 4   | 2    | 1   | s663(+1,2)  | 4.018817238361670351502527938 |
| 10407 | 5   | 3    | 2   | m322(+3,2)  | 4.02867316755365008557652954 |
| 6724 | 4   | 64   | 1   | s479(-5,1)  | 4.04398689431318652280561166 |
| 271488204251 | 10 | 3669 | 2   | s394(+5,2)  | 4.057706182949845691594135513 |
| 3   | 2   | 1    | 2   | s912(+0,1)  | 4.059766425638614500084810217 |
| 629952 | 6  | 18   | 1   | s386(+5,2)  | 4.061101845242785944343938727 |
| 448 | 4   | 1    | 2   | s566(+2,3)  | 4.1196873638066624912101479   |
| 1107052 | 6  | 50   | 1   | v1315(-4,1) | 4.15667542681794279137038689   |
| 13219 | 5   | 2    | 1   | s645(+1,3)  | 4.17132040132250074447402835 |
| 751 | 4   | 4    | 3   | s850(-3,1)  | 4.172750868594119687813006703 |
| 664674571 | 7  | 480  | 1   | s523(-6,1)  | 4.201030415031789207437923442 |
| 5519 | 5   | 1    | 2   | s648(+5,1)  | 4.229135386220444476180426062 |
| 161939 | 6  | 2    | 1   | s682(+3,1)  | 4.23021683478397300285746784 |
| 23   | 3   | 2    | 9   | s702(-3,1)  | 4.242183132496174744145848213 |
| 87   | 3   | 2    | 1   | s784(-1,2)  | 4.252582946954347612792724802 |
| 1825672 | 6  | 100  | 1   | m358(-5,3)  | 4.259629131248291580063933525 |
| 8647 | 5   | 1    | 1   | v0940(-5,2) | 4.268032673602804301572090960 |
| 400 | 4   | 2    | 5   | m400(+4,1)  | 4.306207600730808652919837159 |
Table 13d: Rational relations of Dedekind zeta values to volumes

| $-D$ | $n$ | $a$ | $b$ | $\mathcal{M}$ | $\text{vol}(\mathcal{M})$ |
|------|-----|-----|-----|--------------|-----------------|
| 365263 | 6 | 26 | 3 | s961(+1, 2) | 4.322097020182094625847110669 |
| 104483 | 6 | 1 | 1 | s648(+1, 2) | 4.33009508377546093234851467 |
| 238507 | 6 | 7 | 2 | s649(-1, 4) | 4.36951150125176794502649201 |
| 4297259 | 7 | 20 | 3 | s730(-1, 2) | 4.36999817145910403530790226 |
| 29444 | 5 | 8 | 1 | s478(-1, 3) | 4.37494651198360519146415444 |
| 1968 | 4 | 6 | 1 | s594(+1, 3) | 4.40315501669485810999843642 |
| 83 | 3 | 2 | 1 | s869(-1, 2) | 4.41533247745385824949839635 |
| 283 | 4 | 2 | 9 | s650(-1, 3) | 4.41615973015044396411534854 |
| 331 | 4 | 2 | 7 | m400(+4, 3) | 4.4229823350315279596507213 |
| 50052727 | 7 | 296 | 1 | v2200(-3, 2) | 4.45445308448592499547644665 |
| 116 | 3 | 4 | 1 | s881(-1, 3) | 4.46465891154868077205392692 |
| 7729991 | 7 | 16 | 1 | s645(+4, 3) | 4.519534265195114413456721 |
| 5783 | 5 | 1 | 2 | v2203(+3, 1) | 4.54526372717311639108300843 |
| 1588 | 4 | 4 | 1 | v2641(-4, 1) | 4.55591888987385310455865811 |
| 79952 | 5 | 36 | 1 | s594(+3, 2) | 4.60646915244949889747572026 |
| 22331 | 5 | 4 | 1 | s884(+2, 3) | 4.61102426189563805070980503 |
| 731 | 4 | 1 | 1 | s649(-3, 4) | 4.62565090127753961466468128 |
| 4511 | 5 | 1 | 3 | s646(+5, 2) | 4.63070673441556522985418829 |
| 75117248 | 7 | 616 | 1 | v1788(+3, 2) | 4.64632995114468981239774385 |
| 1107 | 4 | 2 | 1 | s928(+4, 1) | 4.66228929094737107682574690 |
| 99961920379 | 9 | 15436 | 1 | s649(-5, 3) | 4.67874307221514332201617143 |
| 141791 | 5 | 104 | 1 | s707(+5, 1) | 4.68221862938611969193724560 |
| 275 | 4 | 1 | 5 | v1251(+4, 3) | 4.68603427380261253071249064 |
| 23 | 3 | 1 | 5 | s944(-1, 2) | 4.713536813884638604606498015 |
| 2151 | 4 | 6 | 1 | s882(+4, 1) | 4.72540158510900095299319164 |
| 31 | 3 | 1 | 3 | s874(+4, 1) | 4.74949998187443850849808655 |
| 44 | 3 | 2 | 3 | v1368(+2, 3) | 4.765939917900488964530714437 |
| 2095218667 | 8 | 2240 | 1 | s901(-3, 2) | 4.80936703360246965210995820 |
| 14103 | 5 | 2 | 1 | s784(+5, 2) | 4.81476802388802892872629731 |
| 688 | 4 | 1 | 1 | s944(-3, 2) | 4.85117075733273756705832705 |
| 31684 | 5 | 8 | 1 | v2381(+3, 1) | 4.86885685109806320082695666 |
| 1879 | 4 | 3 | 1 | v2914(+2, 3) | 4.87575815910615723935629509 |
| 11243 | 5 | 4 | 3 | v2447(+1, 3) | 4.87936207270746888571103259 |
| 9429911 | 7 | 20 | 1 | s838(-2, 3) | 4.88338697153947650563987375 |
| 283 | 4 | 1 | 5 | v3215(+3, 1) | 4.906844144661604405726094 |
| 239639 | 5 | 184 | 1 | s918(+3, 2) | 4.924074751099801277306001435 |
| 491 | 4 | 1 | 2 | s900(-1, 3) | 4.936464393618735785704601 |
| 3685907 | 7 | 14 | 3 | v2335(+3, 2) | 4.94882914742599236239518764 |
| 6515927 | 7 | 11 | 3 | s900(+3, 2) | 4.95301036813667974247291854 |
| 23339 | 5 | 4 | 1 | v3199(+3, 1) | 4.967241778215442305200322817 |
| $-D$  | $n$  | $a$  | $b$  | $\mathcal{M}$ | vol($\mathcal{M}$) |
|-------|------|------|------|--------------|-------------------|
| 5873596 | 6    | 688  | 1    | v1858 (+6,1) | 4.988809551743365540758322561 |
| 948381887 | 8    | 496  | 1    | v2486 (-3,2) | 4.989804907785885826671653092 |
| 7557047 | 7    | 14   | 1    | v2221 (-1,3) | 5.002053292789971186662200159 |
| 202734487 | 8    | 50   | 1    | s900 (+1,3)  | 5.00531861074564231192791416  |
| 331 | 4    | 1    | 4    | v2402 (+3,2) | 5.05483695463217462358865387  |
| 2401259831 | 8    | 1978 | 1    | v3184 (+4,1) | 5.056718039142583479359939301 |
| 3    | 2    | 2    | 5    | v2422 (+1,3) | 5.0747083204826812510612771  |
| 7463 | 5    | 2    | 3    | v2344 (+5,2) | 5.104486876749683628862342079 |
| 7748687650003 | 10  | 232080 | 1  | s900 (+2,3) | 5.10922754154802289584464043 |
| 1463 | 5    | 2    | 1    | s958 (+3,2)  | 5.112724558199085851711904721 |
| 43210364 | 7    | 242  | 1    | v3305 (-1,2) | 5.17242878697906186835081520 |
| 709783 | 6    | 20   | 1    | s952 (-4,1)  | 5.1752305825967761096843350  |
| 507 | 4    | 1    | 2    | v3347 (+3,1) | 5.190775187373484276603987668 |
| 12476239474594496 | 12  | 9408656 | 1  | v2824 (+4,1) | 5.1942145752011244541895549  |
| 139 | 3    | 4    | 1    | v3106 (+3,1) | 5.1984336640442561125403410328 |
| 9439 | 5    | 1    | 1    | v2759 (-3,1) | 5.200723713644593903398311904 |
| 1732 | 4    | 4    | 1    | v3187 (-4,1) | 5.2024968424808233112092464  |
| 32775179 | 7    | 158  | 1    | v2725 (-4,1) | 5.218362479026066724780387202 |
| 46692071 | 7    | 250  | 1    | v2825 (-4,1) | 5.24017030245763905979008831  |
| 14911 | 5    | 2    | 1    | v2704 (-5,1) | 5.262436101311908292112209718 |
| 4903 | 5    | 1    | 3    | s944 (-5,2)  | 5.27137808756535488624239780 |
| 13523 | 5    | 2    | 1    | v2530 (+1,3) | 5.2879362705261273562285367  |
| 70736 | 5    | 32   | 1    | v2787 (-1,3) | 5.28893750721863751409544257  |
| 7 | 2    | 2    | 1    | v3390 (+3,1) | 5.333489566898119581593424925 |
| 58360112 | 7    | 394  | 1    | v3462 (-1,2) | 5.35895161151062519594085316  |
| 1791 | 4    | 4    | 1    | s961 (+2,3)  | 5.36369322179581744185777767 |
| 7792864 | 6    | 976  | 1    | v2789 (-2,3) | 5.387253764656890006255173367 |
| 34779 | 5    | 9    | 1    | v3214 (+3,1) | 5.426764227123089981537817177 |
| 9759 | 5    | 1    | 1    | v3031 (+3,1) | 5.47303296608735883824871653 |
| 4 | 2    | 2    | 3    | v3412 (+5,1) | 5.49579356063314090327621089 |
| 13883 | 5    | 2    | 1    | v3310 (+5,1) | 5.504748837818013496725737495 |
| 34436 | 5    | 8    | 1    | v2794 (-2,3) | 5.50967698952557537121576098 |
| 4241707 | 6    | 296  | 1    | v3428 (-4,1) | 5.517978522012509993351570954 |
| 31 | 3    | 2    | 7    | v3091 (-2,3) | 5.5410833121864492658110981 |
| 751 | 4    | 1    | 1    | v3277 (-2,3) | 5.563667824792159583750675604 |
| 1156 | 4    | 2    | 1    | v3183 (-3,2) | 5.5736091123137043711018858 |
| 688927 | 6    | 14   | 1    | v3243 (-3,1) | 5.57625962636003943109360827 |
| 12558899 | 7    | 34   | 1    | v3520 (+4,1) | 5.625032993086420746351234925 |
| 48502810352 | 9   | 5230 | 1    | v3157 (+5,1) | 5.646678958454479680102882510 |
Table 13f: Rational relations of Dedekind zeta values to volumes

| $-D$ | $n$ | $a$ | $b$ | $\mathcal{M}$ | $\text{vol}(\mathcal{M})$ |
|------|-----|-----|-----|-----------------|--------------------------|
| 7729991 | 7   | 64  | 5   | v3264(+4,1)     | 5.649417831489938930141820901 |
| 23   | 3   | 1   | 6   | v3375(-3,2)     | 5.65624417661566325527797618 |
| 2319 | 4   | 6   | 1   | v2944(-5,2)     | 5.67188354356654169608496809 |
| 81051965432 | 8   | 636704 | 1   | v3107(+2,3)     | 5.683882556138987145289674657 |
| 331 | 4   | 2   | 9   | v3438(-3,1)     | 5.686691573961196451481223560 |
| 1494223 | 6   | 56  | 1   | v3036(+3,2)     | 5.68728205652596214890785149 |
| 3312 | 4   | 12  | 1   | v3209(+2,3)     | 5.724388054820026594606305363 |
| 97569124 | 7   | 736 | 1   | v3214(+2,3)     | 5.73679504714267027771627006 |
| 22424 | 5   | 4   | 1   | v3246(-2,3)     | 5.74500010444903663416596800 |
| 365263 | 6   | 13  | 2   | v3184(-3,2)     | 5.762796026909459501129480892 |
| 561863 | 6   | 10  | 1   | v3239(+3,2)     | 5.804174400001677001746529114 |
| 1192 | 4   | 2   | 1   | v3214(-4,3)     | 5.8430228578586616408120227 |
| 283 | 4   | 1   | 6   | v3431(-2,3)     | 5.88821297353392528548713138 |
| 848 | 4   | 1   | 1   | v3477(+4,1)     | 5.916745735182788695272260151 |
| 661831 | 6   | 16  | 1   | v3361(+1,3)     | 5.920105898675782923200528147 |
| 1927 | 4   | 1   | 1   | v3452(-5,2)     | 5.93446388389947249746225266 |
| 417467 | 6   | 13  | 2   | v3375(-5,2)     | 5.999880841314256220228307651 |
| 10407 | 5   | 1   | 1   | v3418(+6,1)     | 6.043009751330527512836479431 |
| 1371 | 4   | 2   | 1   | v3489(+2,3)     | 6.087435457969104353387839 |
| 3 | 2   | 1   | 3   | v3492(-4,1)     | 6.089649638457921750127215325 |
| 107264 | 5   | 52  | 1   | v3454(-5,1)     | 6.120178528987339574492892096 |
| 1255 | 4   | 2   | 1   | v3492(+4,3)     | 6.223431719518907660423581730 |
| 13219 | 5   | 4   | 3   | v3543(+1,3)     | 6.256980601983750111671104253 |
| 58064 | 5   | 20  | 1   | v3526(+2,3)     | 6.266699297321792702743593825 |

Table 14: Further single-complex-place fields, from cusped manifolds

| field | $-D$ | $a$ | $b$ | manifold |
|-------|------|-----|-----|----------|
| $x^2 + 2$ | 8    | 2   | 1   | v2787    |
| $x^4 - x^2 - 2x - 2$ | 152   | 4   | 1   | v3526    |
| $x^4 - 3x^2 - 2x + 1$ | 1328  | 2   | 1   | v2631    |
| $x^4 - x^3 + x^2 - 6x - 4$ | 2375  | 4   | 1   | v3545    |
| $x^4 + 5x^2 - 3$ | 4107  | 15  | 1   | v1143    |
| $x^6 - x^2 + 3x^2 - 2x - 1$ | 7367  | 1   | 1   | m052     |
| $x^6 - x^3 + 3x^2 + 2x - 1$ | 303619 | 5   | 1   | s281     |
| $x^6 - x^5 - 2x^4 + 4x^3 - 10x^2 + 6x + 3$ | 13266363 | 1362 | 1 | v1461 |
| $x^5 - 2x^4 - 6x^3 + 9x^2 + 12x^2 - 9x^2 - 11x + 2$ | 161329612 | 1832 | 1 | v3418 |
| $x^5 - 4x^4 + 5x^3 - 6x^2 + 9x^2 - 4x + 1$ | 74671875 | 12 | 1 | m283 |
| $x^8 - 3x^7 + x^6 + 4x^5 + 8x^4 - 4x^3 - 8x^2 + 1$ | 397538359 | 142 | 1 | v2824 |
| $x^8 - x^7 - 6x^6 + 8x^5 - 12x^3 + 23x^2 - 9x - 3$ | 2597840403 | 2853 | 1 | s311 |
### Table 15: Joins of quadratics

| quartic join       | $n_1$ | $-D_1$ | $n_2$ | $-D_2$ | $a$ | $b_1$ | $b_2$ | manifolds                     |
|--------------------|-------|--------|-------|--------|-----|-------|-------|-------------------------------|
| $x^4 - x^2 + 1$    | 2     | 3      | 2     | 4      | 2   | 2     | 1     | m350($-1,3$)                  |
|                    |       |        |       |        |     |       |       | m360($-2,3$)                  |
|                    |       |        |       |        |     |       |       | s913, v2274                  |
| $x^4 - x^3 - x^2 - 2x + 4$ | 2    | 3      | 2     | 7      | 4   | 4     | 1     | v2274($\pm 4,1$)             |
| $x^4 - 3x^2 + 4$   | 2     | 4      | 2     | 7      | 4   | 2     | 1     | v1859($\pm 3,1$)             |
|                    |       |        |       |        |     |       |       | v1859($\pm 1,3$)             |
|                    |       |        |       |        |     |       |       | m314–5                        |

### Table 16: Joins of a quadratic and cubic

| sextic join        | $n_1$ | $-D_1$ | $n_2$ | $-D_2$ | $a$ | $b_1$ | $b_2$ | manifolds                     |
|--------------------|-------|--------|-------|--------|-----|-------|-------|-------------------------------|
| $x^6 - x^5 + x^4 - 2x^3 + x^2 + 1$ | 2    | 3      | 3     | 23     | 1   | 1     | 3     | v2274($\pm 3,2$)             |
|                    |       |        |       |        |     |       |       | v2274($\pm 6,1$)             |
|                    |       |        |       |        |     |       |       | v2274($\pm 2,3$)             |
| $x^6 - 2x^4 - 2x^3 + 4x^2 + 2x + 1$ | 2    | 3      | 3     | 59     | 2   | 2     | 1     | v2274($\pm 1,2$)             |
|                    |       |        |       |        |     |       |       | s636($-1,4$)                 |
|                    |       |        |       |        |     |       |       | s618($+1,4$)                 |
| $x^6 - 4x^4 + 4x^2 + 1$ | 2    | 4      | 3     | 59     | 2   | 1     | 1     | s518($-1,4$)                 |
|                    |       |        |       |        |     |       |       | s530($-1,4$)                 |
|                    |       |        |       |        |     |       |       | v3066                         |
| $x^6 - 6x^4 - 5x^3 + 16x^2 + 8x + 8$ | 2    | 7      | 3     | 59     | 4   | 1     | 2     | v3066($\pm 4,1$)             |

### Table 17: Joins of cubics

| nonadic join       | $n_1$ | $-D_1$ | $n_2$ | $-D_2$ | $a$ | $b_1$ | $b_2$ | manifolds                     |
|--------------------|-------|--------|-------|--------|-----|-------|-------|-------------------------------|
| $x^9 + 2x^7 - 2x^6 + 8x^5 + 4x^4 + 11x^3 + 4x^2 - 1$ | 3    | 23     | 3     | 59     | 2   | 6     | 1     | v3066($\pm 3,2$)             |
|                    |       |        |       |        |     |       |       | v3066($\pm 2,3$)             |
|                    |       |        |       |        |     |       |       | v3066($\pm 6,1$)             |
| $x^9 - 2x^7 - 5x^6 + 12x^5 + 8x^4 + 15x^3 + 4x^2 + 2x - 1$ | 3    | 44     | 3     | 59     | 2   | 2     | 1     | v3066($\pm 3,2$)             |
|                    |       |        |       |        |     |       |       | v3066($\pm 2,3$)             |
|                    |       |        |       |        |     |       |       | v3066($\pm 6,1$)             |
| quartic ⊂ octadic join | \( n_1 - D_1 \) | \( n_2 - D_2 \) | \( a \) | \( b_1 \) | \( b_2 \) | manifolds |
|-------------------------|----------------|----------------|-----|------|------|--------|
| \( x^4 - 2x^3 + 2 \) \( \subset x^8 - 2x^7 + 2x^6 + 2x^5 - 2x^4 + 2x^3 + 2x^2 - 2x + 1 \) | 2 | 4 | 4 | 400 | 1 | 5 | m135(±1,4) v1859(±4,1) v1859(±1,4) v2942–7 |
| \( x^4 - x^3 + x + 1 \) \( \subset x^8 + 4x^6 + x^4 - 6x^2 + 4 \) | 2 | 7 | 4 | 448 | 8 | 1 | 8 | m235(−4,1) m234(−1,3) m305(−4,1) s719(+7,1) v1373(−2,3) v3505–7 |
| \( x^4 - x^3 - x^2 + x + 1 \) \( \subset x^8 - x^7 + 2x^6 + 3x^5 - x^4 + 3x^3 + 2x^2 - x + 1 \) | 2 | 3 | 4 | 507 | 2 | 1 | 1 | m023(−5,1) m022(+2,3) m038(−5,1) s645(−2,3) s646(−2,3) s648(−7,1) s649(+2,3) v1809(−4,1) m345 v3461–2 |
| \( x^4 + x^2 - 2x + 1 \) \( \subset x^8 + 5x^6 + 4x^4 + 5x^2 + 1 \) | 2 | 4 | 4 | 1156 | 4 | 4 | 1 | v3318–9 |
| \( x^4 - x^3 - 2x^2 + 3 \) \( \subset x^8 - 5x^6 + 28x^4 + 15x^2 + 9 \) | 2 | 3 | 4 | 4107 | 30 | 45 | 1 | s869 |
| \( x^4 - 2x^3 - x^2 + 2x + 2 \) \( \subset x^8 + 13x^6 + 40x^4 + 52x^2 + 16 \) | 2 | 4 | 4 | 6724 | 128 | 32 | 1 | m135(±2,3) v1859(±2,3) v1859(±3,2) v3431(+3,2) v3217(−5,1) v3213(−5,1) v3212(+4,3) v3209(+4,3) v3210(+5,1) v3387(−2,3) v3207(+5,1) v3208(+4,3) s937–41 v2573–6 |
Table 19: Numerical values of $Z_{|D|} := Z_K$ for imaginary-quadratic fields

| $-D$ | $Z_{|D|}$ |
|------|-----------|
| 3    | 2.02988321281930725004240510854904057188337861506059 |
| 4    | 3.66386237670887606021841405972953644309659749712668 |
| 7    | 10.66697913379623916318684985044260017639353555421055 |
| 8    | 12.0460920400943776472683786292359423099605804944499 |
| 11   | 16.59129969483175048405984013396780188163367504042159 |
| 15   | 37.663366735357521501108052592233790231511162680252581 |
| 20   | 50.44763111371256002113427103608556540514680566830894 |
| 24   | 62.18607477383502595106662726058243112965063718233095 |
| 39   | 165.57570369926833581678917631121833077904271386139253 |
| 84   | 404.73628202464445448608555478460494570828840108112927 |
Table 20: Volume of the cubical and octahedral links

| Volume                      |
|-----------------------------|
| 114.537611451236294322424724789936294357498808242 |
| 2049231250851942121127501027441161281234687892520 |
| 99421046573180631580080289831770300702374092992825 |
| 0084586402839836513300827269407599687597029206065 |
| 315315807593241507574143878837313712948751658431691 |
| 7174309377284113318628123855416344693593219229652 |
| 66073261071127716611194216966602756637015575724347 |
| 304955834611690928188894213894359522005281897761327 |
| 87481780722702810697927326811113193211506085618819 |
| 839515005238805037656220588050105992474582567995845 |
| 24852839534204871794340415879015640464651359451611 |
| 96357246969286008273298748180867981112146197992045 |
| 62818897243423342214341286694709315899967336066828 |
| 402880934941073033594494398978521250331659884338246 |
| 44517641804581874967125904277881159661925353870806 |
| 55344565593252496994635391954711081416596646456178 |
| 99682627682970247100159721685353632474114663301702 |
| 6722701001876954880260061616569740593057768744203 |
| 1978242453550110127124720720911167848941140775 |
| 8040952575817031150502822254452974913073005473499436 |
| 49644923893596219534503892584099468539202882445560 |
| 67043214403394086453561263528712168065777974863536 |
| 2042521380099213497676263816718536484989562580836367 |
| 68477163430037441223365797732060301796216060089171 |
| 2728475261553738525106609455831808384146439018576726 |
| 913897525757344557924510007575910156390208650947188 |
| 8339672676577814651597087067941025010073059637530 |
| 81422959270328728282984281618973146607449652158522 |
| 78007793547439090335284739905795279447511546662295 |
| 82830239755327066934515195333231267833826837558054 |
| 31913571195670329744639917173868105940089705910151 |
| 2489698280122687947344758903589206619826480243308 |
| 26876906144012908721342540125943215276251702819029 |
| 18946884384227781081283744525707741068411779905750 |
| 35583054992839794034527044721173256021542771843790 |
| 61496448437835045757339558623479750202375305175456299 |
Table 21: Volume of the dodecahedral and icosahedral links

| Volume                                      |
|---------------------------------------------|
| 310.91314542589224185606657163408037891071979627546 |
| 95386653927493346989744424412131110973939975971507 |
| 98643269709265823832914978469249739351095948216115 |
| 51135502802407570241546036301498765323898439299669 |
| 4578995636598157798923976701962012526567526838465 |
| 4854880027177130344283892380456577259183652869728 |
| 54084324901367747475606352594813054449587164285970 |
| 715636114110644568634341052373960417941929693593205 |
| 40772124188342369610921968892286063808089013825878 |
| 1523046741668499343158745655959847908069713326 |
| 5085876565063065478811216589047871297150603197258 |
| 23113347663540660063185505122390690532209472702746 |
| 84653584725667817894996050207455301963078167128747 |
| 884976936056715275992273025680729781457595362466734 |
| 5120482003278540179777583942295000755158363428262 |
| 345666072119904539897586783267155795634292042596 |
| 40210765494283340093378373891402748722649628433618 |
| 99776582050245619951448951311088720526789005381822 |
| 562867040261825821549522339738817785994375329737 |
| 22530736120788811969510398747558459417006190535250 |
| 16187420583314157401670875056552682715430880984829 |
| 281870610436634787078038597334383346993815502724798 |
| 25927982584761501331976838167147091050425539561024 |
| 58379417142495796685978885556943637012844570671457 |
| 05212348658694246666829668517274002954751117140581 |
| 88231035672842687990189221246859638768219728313394 |
| 012158162662629356688361101821397595969743744268728 |
| 61897426110723765254305981655479282036799500110828 |
| 577925545228354860924831805226203355633583617840 |
| 7455296702184547644390948427990004868424690548734 |
| 32941297657691799943811633733468108748967009155145 |
| 80688308119967508389482982167982511015711538776266 |
| 89089501076905467146741212155703510844506629664 |
| 88328174086883755702372138686832584538642639144251 |
| 47198193809500775033230947560721000905153519571355423 |
### Table 22: Empirical reductions of Feynman orthoschemes to the values of Table 19

| $\psi_1$ | $\psi_2$ | $\psi_3$ | $S(\psi_1,\psi_2,\psi_3)$ | $\psi_1$ | $\psi_2$ | $\psi_3$ | $S(\psi_1,\psi_2,\psi_3)$ | $\psi_1$ | $\psi_2$ | $\psi_3$ | $S(\psi_1,\psi_2,\psi_3)$ |
|--------|--------|--------|----------------|--------|--------|--------|----------------|--------|--------|--------|----------------|
| 0      | 0      | 0      | $Z_4$         | 0      | 0      | $\frac{\pi}{6}$ | $\frac{5}{3}Z_3$ | 0      | 0      | $\frac{\pi}{4}$ | $\frac{1}{4}Z_8$ |
| 0      | 0      | $\frac{2}{3}$ | $\frac{2}{3}Z_4$ | 0      | $\frac{4}{6}$ | 0 | $\frac{2}{3}Z_3$ | 0      | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{5}{6}Z_4$ |
| 0      | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{1}{24}Z_{21}$ | 0      | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{2}{3}Z_3$ | 0      | $\frac{4}{4}$ | 0 | $\frac{1}{4}Z_8$ |
| 0      | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{1}{24}Z_{21}$ | 0      | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{1}{2}Z_4$ | 0      | $\frac{4}{4}$ | $\frac{3}{4}$ | $\frac{1}{12}Z_8$ |
| 0      | $\frac{4}{3}$ | 0      | $\frac{2}{3}Z_4$ | 0      | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{5}{6}Z_3$ | 0      | $\frac{3}{3}$ | $\frac{3}{3}$ | $\frac{1}{12}Z_8$ |
| 0      | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{1}{6}Z_4$ | $\frac{4}{6}$ | 0 | $\frac{6}{6}$ | $\frac{1}{12}Z_{15}$ | $\frac{4}{6}$ | 0 | $\frac{6}{6}$ | $\frac{1}{144}Z_{24}$ |
| $\frac{4}{6}$ | 0      | $\frac{4}{6}$ | $\frac{1}{12}Z_{21}$ | $\frac{6}{6}$ | $\frac{6}{6}$ | $\frac{6}{6}$ | $\frac{1}{6}Z_{11}$ | $\frac{6}{6}$ | $\frac{6}{6}$ | $\frac{6}{6}$ | $\frac{1}{16}Z_{15}$ |
| $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{1}{6}Z_{11}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{5}{24}Z_7$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{5}{12}Z_4$ |
| $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{1}{24}Z_{15}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{1}{2}Z_3$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{5}{24}Z_3$ |
| $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{1}{12}Z_{3}$ | $\frac{4}{6}$ | 0 | $\frac{6}{6}$ | $\frac{5}{4}Z_3$ | $\frac{4}{6}$ | 0 | $\frac{6}{6}$ | $\frac{1}{24}Z_{20}$ |
| $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{1}{12}Z_{3}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{5}{8}Z_3$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{1}{12}Z_4$ |
| $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{1}{12}Z_{4}$ | $\frac{4}{6}$ | 0 | $\frac{6}{6}$ | $\frac{1}{6}Z_7$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{4}{6}$ | $\frac{1}{3}Z_3$ |

### Table 23: Maximally symmetric knots to 10 crossings

| Rolfsen | HTW | sym | invariant trace field | sig | $D$ | $a$ | $b$ |
|--------|-----|-----|----------------------|-----|-----|-----|-----|
| $4_1$  | a4.1| $D_4$ | $x^2 - x + 1$        | [0, 1] | $-3$ | $1$ | $1$ |
| $5_2$  | a5.1| $D_2$ | $x^3 - x^2 + 1$      | [1, 1] | $-23$ | $1$ | $3$ |
| $6_3$  | a6.1| $D_4$ | $x^6 - x^5 - x^4 + 2x^3 - x + 1$ | [0, 3] | $-10571$ | $\times$ | $\times$ |
| $7_4$  | a7.6| $D_4$ | $x^7 + 2x - 1$       | [1, 1] | $-59$ | $1$ | $1$ |
| $7_7$  | a7.1| $D_4$ | $x^4 + x^2 - x + 1$  | [0, 2] | $257$ | $\times$ | $\times$ |
| $8_{21}$ | a8.12 | $D_8$ | $x^8 - 2x^7 + x^2 - 2x + 1$ | [2, 1] | $-448$ | $1$ | $6$ |
| $9_{35}$ | a9.40 | $D_6$ | $x^9 - 2x - 2$       | [1, 1] | $-76$ | $1$ | $1$ |
| $9_{40}$ | a9.37 | $D_6$ | $x^9 + 2x^2 - 2x + 1$ | [0, 2] | $592$ | $\times$ | $\times$ |
| $9_{48}$ | a9.6 | $D_6$ | $x^9 - x^2 + x + 1$  | [1, 1] | $-44$ | $1$ | $3$ |
| $10_{123}$ | a10.121 | $D_{10}$ | $x^2 - x^3 + x^2 - x + 1$ | [0, 2] | $125$ | $\times$ | $\times$ |
| $10_{157}$ | a10.42 | $D_4$ | $x^3 + x - 1$       | [1, 1] | $-31$ | $1$ | $8$ |

### Table 24: Less symmetric non-alternating Dedekind-zeta knots

| Rolfsen | HTW | sym | invariant trace field | sig | $D$ | $a$ | $b$ |
|--------|-----|-----|----------------------|-----|-----|-----|-----|
| $9_{49}$ | n9.8 | $D_3$ | $x^3 - x^2 + 1$     | [1, 1] | $-23$ | $1$ | $10$ |
| $10_{139}$ | n10.27 | $D_3$ | $x^4 - 2x - 1$     | [2, 1] | $-688$ | $1$ | $1$ |
| $10_{152}$ | n10.36 | $Z_2$ | $x^5 + 3x^3 - 2x^2 + x$ | [3, 1] | $-8647$ | $1$ | $2$ |
| $10_{153}$ | n10.10 | triv. | $x^5 - 2x^4 - 2x^3 + 4x^2 - x + 1$ | [3, 1] | $-29963$ | $4$ | $1$ |
Table 25: Maximally symmetric knots from 11 to 16 crossings

| HTW knot | sym | invariant trace field | sig |
|----------|-----|-----------------------|-----|
| a11.366  | $D_6$ | $x^6 - x^5 + 4x^4 - 3x^3 + 4x^2 - 2x - 1$ | $[2, 2]$ |
| n11.126  | $D_3$ | $x^5 - x^4 + 3x^2 - 2x + 1$ | $[1, 2]$ |
| n11.133  | $D_3$ | $x^5 - x^4 + x^2 - x + 1$ | $[1, 2]$ |
| a12.503  | $D_6$ | $x^6 + 2x^5 + 2x^3 - x^2 - x - 1$ | $[1, 3]$ |
| a12.561  | $D_6$ | $x^7 - 2x^5 - x^4 + 2x^3 + x^2 + x - 1$ | $[1, 3]$ |
| a12.1019 | $D_6$ | $x^{10} - 2x^9 + 2x^8 - 2x^7 - 3x^6 + 7x^5 - 3x^4 - 2x^3 + 2x^2 - 2x + 1$ | $[2, 4]$ |
| a12.1202 | $D_6$ | $x^8 - x^7 + x^6 + 3x^4 + x^2 - x + 1$ | $[0, 4]$ |
| n12.555  | $D_8$ | $x^5 + x^3 - x^2 + 2x + 1$ | $[1, 2]$ |
| n12.642  | $D_8$ | $x^3 - x^2 + x + 1$ | $[1, 1]$ |
| a13.1786 | $D_8$ | $x^8 + 2x^6 - x^5 + 3x^4 - x^3 + x^2 - 2x + 1$ | $[0, 4]$ |
| a13.4877 | $D_8$ | $x^6 + 4x^4 - x^3 + 4x^2 - 3x - 1$ | $[2, 2]$ |
| n13.4051 | $D_6$ | $x^5 - x^4 - x^3 + x^2 + 3x + 1$ | $[1, 2]$ |
| a14.19470| $D_{14}$ | $x^6 - 2x^5 + 2x^4 - 3x^3 + 2x^2 - 2x + 1$ | $[2, 2]$ |
| n14.13191| $D_5$  | $x^4 - x^3 - 2x + 1$ | $[2, 1]$ |
| n14.17159| $D_5$  | $x^6 - 2x^5 + x^4 + x^3 + x^2 - 2x - 1$ | $[2, 2]$ |
| a15.84903| $D_{10}$| $x^8 - 2x^7 + x^6 - 2x^5 + x^4 - 4x^3 + 6x^2 - 4x + 4$ | $[2, 3]$ |
| a15.85262| $D_{10}$| $x^6 - 3x^4 - 4x^3 + x^2 + 6x + 4$ | $[2, 2]$ |
| n15.99226| $D_{10}$| $x^7 - x^5 + 2x^4 - x^3 + 3x^2 - 2x + 1$ | $[1, 3]$ |
| n15.112310| $D_{10}$ | $x^4 - 2x^3 + x^2 - 2x + 1$ | $[2, 1]$ |
| a16.379778| $D_{16}$| $x^8 - 4x^7 + 6x^6 - 11x^4 + 16x^3 - 10x^2 + 4x - 1$ | $[2, 3]$ |
| n16.1007813| $D_9$ | $x^3 + x - 1$ | $[1, 1]$ |
Fig. 1: Alternating platonic link from light-by-light scattering

Fig. 2: Non-alternating daisy-chain link
Fig. 3: Alternating figure-8 knot at $D = -3$

Fig. 4: Alternating Whitehead link at $D = -4$
Fig. 5: Alternating link $6_1^3$ at $D = -7$

Fig. 6: Alternating link $9_4^2$ at $D = -8$
Fig. 7: Alternating link \((\sigma_1 \sigma_2^{-2} \sigma_3 \sigma_2^{-2})^2\) at \(D = -11\)

Fig. 8: Alternating link \((\sigma_1^2 \sigma_2^{-2})^3\) at \(D = -15\)