Vortex Strings and Nonabelian sine-Gordon Theories

Q-Han Park\textsuperscript{1} and H.J. Shin\textsuperscript{2}

Department of Physics and Research Institute of Basic Science
Kyung Hee University, Seoul 130-701, Korea

ABSTRACT

We generalize the Lund-Regge model for vortex string dynamics in 4-dimensional Minkowski space to the arbitrary n-dimensional case. The n-dimensional vortex equation is identified with a nonabelian sine-Gordon equation and its integrability is proven by finding the associated linear equations of the inverse scattering. An explicit expression of vortex coordinates in terms of the variables of the nonabelian sine-Gordon system is derived. In particular, we obtain the n-dimensional vortex soliton solution of the Hasimoto-type from the one soliton solution of the nonabelian sine-Gordon equation.

\textsuperscript{1}e-mail: qpark@nms.kyunghee.ac.kr
\textsuperscript{2}e-mail: hjshin@nms.kyunghee.ac.kr
The relativistic motion of vortex strings in a superfluid was first modeled by Lund and Regge in 1976 [1]. Among many vortex models, their model is distinguished in that it is an exactly integrable model and it becomes the Nambu string model in the no-coupling limit. In the context of string theory, the nonvanishing coupling term also received an interpretation as describing the interaction of a string with background antisymmetric tensor fields [2]. Lund and Regge has proven the integrability of the model by recognizing the vortex equation as the integrability equation of Gauss and Codazzi [1, 3]. They have identified the vortex equation with the complex sine-Gordon equation [4] and found the associated linear equations of the inverse scattering. Since then, the complex sine-Gordon theory has been studied intensively [3]-[10], with applications to nonlinear optics [11]. Extensions to more general cases of the nonabelian sine-Gordon theories have been also made by associating them with symmetric spaces [12] and their properties were investigated in detail [13, 14].

However, the vortex model by Lund and Regge is defined only in the 3+1-dimensional Minkowski space and the higher-dimensional model which generalizes the n+1-dimensional Nambu string model is not known. Even in the 3+1-dimensional case, the identification of the vortex equation with the complex sine-Gordon equation is not complete. The variables of the complex sine-Gordon equation has been given in terms of the vortex string coordinates, but the expression of the vortex string coordinates in terms of the complex sine-Gordon variables is not known. Since exact solutions, e.g. solitons and breathers, have been constructed only in the context of the complex sine-Gordon equation, the explicit correspondence to the vortex string coordinates is critical in obtaining exact solutions of the vortex equation systematically using the inverse scattering method.

In this letter, we resolve these two problems. We first present an n+1-dimensional generalization of the vortex equation which reduces to the Nambu string in the no-coupling limit. We identify the n+1-dimensional vortex equation with the nonabelian sine-Gordon equation and prove the integrability by finding the associated linear equations of the inverse scattering. In doing so, we obtain an expression of vortex coordinates in terms of the variables of the nonabelian sine-Gordon system. Using this relation, we obtain explicitly an n-dimensional Hasimoto-type vortex soliton from the one soliton solution of

\footnote{Many integrable equations arise from the study of the surface embedding problem in differential geometry which provides a clear geometrical meaning to integrable equations. For the modern formulation of surface embedding problem, see for example in [4].}
the nonabelian sine-Gordon equation.

We begin with a review of the vortex model by Lund and Regge. The relativistic motion of vortices in a uniform static field is governed by the equation of motion (in a Lorentz frame in which $X^0 = \tau$):

$$\left(\partial^2_\tau - \partial^2_\sigma\right)X^i + c\epsilon_{ijk}\partial_\tau X^j\partial_\sigma X^k = 0; \ i = 1, 2, 3$$

and also by the quadratic constraints:

$$\left(\partial_\tau X^i\right)^2 + \left(\partial_\sigma X^i\right)^2 = 1, \ \left(\partial_\sigma X^i\right)\left(\partial_\tau X^i\right) = 0.$$  \hfill (2)

Here, $X^\mu(\sigma, \tau); \mu = 0, 1, 2, 3$ are the vortex coordinates and $\sigma, \tau$ are local coordinates on the string world-sheet. In the no-coupling limit ($c = 0$), this equation describes the transverse modes of the 4-dimensional Nambu-Goto string in the orthonormal gauge. The critical step leading to the integration of the vortex equation (1) and (2) was to interpret the equation as the Gauss-Codazzi integrability condition for the embedding of a surface, i.e. the embedding of the string world-sheet projected down to the $X^0 = \tau$ hypersurface into the 3-dimensional Euclidean space, $X^0 = \tau$. The induced metric on the projected world-sheet is given by

$$ds^2 = (\partial_\sigma \vec{X})^2 d\sigma^2 + 2(\partial_\sigma \vec{X} \cdot \partial_\tau \vec{X}) d\sigma d\tau + (\partial_\tau \vec{X})^2 d\tau^2,$$

or

$$ds^2 = \cos^2 \phi d\sigma^2 + \sin^2 \phi d\tau^2$$

by parameterizing $(\partial_\sigma \vec{X})^2 = \cos^2 \phi, \ (\partial_\tau \vec{X})^2 = \sin^2 \phi$ according to Eq. (2). The unit tangent vectors, $\vec{N}_1$ and $\vec{N}_2$, spanning the plane tangent to the surface, and the unit normal vector $\vec{N}_3$ consisting a moving frame are given by

$$\vec{N}_1 = \frac{1}{|\partial_\sigma \vec{X}|} \partial_\sigma \vec{X}, \ \vec{N}_2 = \frac{1}{|\partial_\tau \vec{X}|} \partial_\tau \vec{X}, \ \vec{N}_3 = \frac{1}{|\partial_\sigma \vec{X} \times \partial_\tau \vec{X}|} \partial_\sigma \vec{X} \times \partial_\tau \vec{X}.$$  \hfill (5)

The vectors ($\vec{N}_i; \ i = 1, 2, 3$), given coordinates $u_1, u_2$ on the surface, satisfy the equation of Gauss and Weingarten:

$$\frac{\partial \vec{N}_i}{\partial u_k} = \Gamma^i_{jk} \vec{N}_j + L_{ik} \vec{N}_3, \ \frac{\partial \vec{N}_3}{\partial u_k} = -g^{ij} L_{kj} \vec{N}_i.$$  \hfill (6)
where $\Gamma^l_{ik}$ are the Christoffel symbols and the $L_{ij}$ are the components of the extrinsic curvature tensor. They are a set of overdetermined linear equations and the consistency of which requires the Gauss-Codazzi equation:

$$R_{ijkl} = L_{ik}L_{jl} - L_{il}L_{jk}, \quad L_{ij:k} = L_{ik:j},$$

(7)

where the semicolon denotes covariant differentiation on the surface and $R_{ijkl}$ are the components of its Riemann tensor. From Eq. (7), it follows that there exists a field $\eta$ such that

$$L_{12} = \cot \phi \frac{\partial \eta}{\partial u_2}, \quad \frac{1}{2}(L_{11} + L_{22}) = \cot \phi \frac{\partial \eta}{\partial u_1},$$

(8)

We introduce the light-cone coordinates $z = (\sigma + \tau)/2$, $\bar{z} = (\sigma - \tau)/2$ and make the coordinate transformation: $z \to z/\lambda$, $\bar{z} \to \lambda \bar{z}$ under which the Gauss-Codazzi equation is invariant due to its Lorentz invariance. In this case, the Gauss-Weingarten equation in the spin-1/2 representation changes into the linear equation of the inverse scattering [3]:

$$\partial \Phi = -(U_0 + \lambda U_1)\Phi, \quad \bar{\partial} \Phi = -(V_0 + \lambda^{-1} V_{-1})\Phi,$$

(9)

where

$$U_0 + \lambda U_1 = -\begin{pmatrix} ic\lambda/4 + i\partial \eta \cos 2\phi/2\sin^2 \phi & -\partial \phi + i\partial \eta \cot \phi \\ \partial \phi + i\partial \eta \cot \phi & -ic\lambda/4 - i\partial \eta \cos 2\phi/2\sin^2 \phi \end{pmatrix},$$

$$V_0 + \lambda^{-1} V_{-1} = -\frac{i}{4}\begin{pmatrix} -c\cos 2\phi/\lambda - 2\partial \eta/\sin^2 \phi & -c\sin 2\phi/\lambda \\ -c\sin 2\phi/\lambda & c\cos 2\phi/\lambda + 2\partial \eta/\sin^2 \phi \end{pmatrix}. \quad (10)$$

The integrability equation:

$$[\partial + U_0 + \lambda U_1, \ \bar{\partial} + V_0 + \lambda^{-1} V_{-1}] = 0,$$

(11)

then becomes the complex sine-Gordon equation:

$$\partial \bar{\partial} \phi - \frac{c^2}{2} \sin 2\phi + \frac{\cos \phi}{\sin^2 \phi} \partial \eta \bar{\partial} \eta = 0$$

$$\bar{\partial} (\cot^2 \phi \partial \eta) + \partial (\cot^2 \phi \bar{\partial} \eta) = 0.$$

(12)

This reduces to the well-known sine-Gordon equation when $\eta = 0$.

Even though the vortex equation as in Eqs. (1) and (2) has been identified with the complex sine-Gordon equation as in Eq. (12), the explicit correspondence between variables of each equations is not well understood. In particular, it is not known how to
write the vortex coordinates $X^i$ from the variables of the complex sine-Gordon system.

In order to resolve this problem, and also to extend the vortex equation to the higher-dimensional case, we first consider the linear equation in Eq. (9) and assume that matrices $U_0, U_1$ and $V_0, V_{-1}$ are valued in a certain Lie algebra $\mathfrak{g}$ but otherwise arbitrary. Define

$$F \equiv \Phi^{-1}(z, \bar{z}, \lambda) \lambda \frac{\partial}{\partial \lambda} \Phi(z, \bar{z}, \lambda).$$  \hspace{1cm} (13)

Then, using Eq. (9), we have

$$\partial F = -\lambda \Phi^{-1} U_1 \Phi, \quad \bar{\partial} F = \frac{1}{\lambda} \Phi^{-1} V_{-1} \Phi.$$  \hspace{1cm} (14)

Also, using Eqs. (9) and (11), we obtain

$$\partial \bar{\partial} F = \Phi^{-1} [U_1, V_{-1}] \Phi = [\bar{\partial} F, \partial F].$$  \hspace{1cm} (15)

Let $F = \sum \alpha X^i T^i$ ($i = 1, ..., n = dim \mathfrak{g}$) where $T^i$ are generators of the Lie algebra $\mathfrak{g}$ normalized by $\text{Tr} T^i T^j = 2 \delta_{ij}$ and $\alpha$ is some constant. Then, Eq. (15) becomes

$$\partial \bar{\partial} X^i = \alpha f^{ijk} \partial X^j \partial X^k,$$  \hspace{1cm} (16)

where $f^{ijk}$ are structure constants of the Lie algebra $\mathfrak{g}$. Note that for $\mathfrak{g} \approx so(3)$ this becomes precisely the vortex equation in Eq. (1). The constraints as in Eq. (2), after the coordinate transformation $z \rightarrow z/\lambda, \quad \bar{z} \rightarrow \lambda \bar{z}$, are equivalent to the condition:

$$\text{Tr} (\partial F)^2 = \lambda^2 \text{Tr} (U_1)^2 = 2 \lambda^2 \alpha^2, \quad \text{Tr} (\bar{\partial} F)^2 = \frac{1}{\lambda^2} \text{Tr} (V_{-1})^2 = \frac{1}{\lambda^2} 2 \alpha^2.$$  \hspace{1cm} (17)

Thus, we define the n-dimensional generalization of the vortex equation in terms of Eqs. (13)-(17). This equation is integrable in the sense that it arises from the linear equation (4) of the inverse scattering. In order to better understand the model defined by Eq. (9) and the constraint in Eq. (17), we first solve the constraint by fixing $U_1$ and $V_{-1}$. By a gauge transformation, we can always set $U_1 = T$ for some constant element $T \in \mathfrak{g}$ satisfying $\text{Tr} T^2 = 2 \alpha^2$. The remaining constraint, $\text{Tr} (V_{-1})^2 = 2 \alpha^2$, may be solved for $V_{-1} = g^{-1} \bar{T} g$ for some constant element $\bar{T}$ satisfying $\text{Tr} \bar{T}^2 = 2 \alpha^2$ and an arbitrary group variable $g(z, \bar{z})$. The zero curvature condition in Eq. (11) should hold for any $\lambda$, that is, each coefficients of the polynomial in $\lambda$ should vanish. Thus, the coefficients of the $\lambda$ and the $\lambda^{-1}$ terms give rise to respectively

$$[\partial + U_0, \ g^{-1} \bar{T} g] = 0 \quad \text{and} \quad [\bar{\partial} + V_0, \ T] = 0,$$  \hspace{1cm} (18)
which we solve for \( U_0 = g^{-1} \partial g + g^{-1} Ag \) and \( V_0 = \tilde{A} \) for some fields \( A \) and \( \tilde{A} \) satisfying the relation \([A, \tilde{T}] = 0 \) and \([\tilde{A}, T] = 0 \). Finally, the zeroth-order term results in

\[
[\partial + g^{-1} \partial g + g^{-1} Ag, \tilde{\partial} + \tilde{A}] + [T, g^{-1} \tilde{T} g] = 0.
\]

(19)

This is precisely the nonabelian sine-Gordon equation introduced in the Ref. [9]. We emphasize that, at this stage, \( A \) and \( \tilde{A} \) are regarded only as background fields which commute with arbitrary constant elements \( \tilde{T} \) and \( T \) respectively. Further specifications of these variables and their physical meanings are given below. For the field theory formulation, one can readily check that the nonabelian sine-Gordon equation (19) arises from the gauged Wess-Zumino-Novikov-Witten action plus a potential term:

\[
S = S_{WZNW}(g) + S_{gauge} - S_{pot}
\]

(20)

\[
S_{WZNW}(g) = -\frac{1}{4\pi} \int_{\Sigma} dzd\bar{z} \text{Tr} (g^{-1} \partial g g^{-1} \partial g) - \frac{1}{12\pi} \int_{B} \text{Tr} (\tilde{g}^{-1} \bar{g} \wedge \tilde{g}^{-1} \bar{g} \wedge \tilde{g}^{-1} \bar{g}),
\]

\[
S_{gauge} = \frac{1}{2\pi} \int \text{Tr} (-A \bar{\partial} g g^{-1} + \bar{A} g^{-1} \partial g + Ag \bar{A} g^{-1} - A\bar{A}),
\]

\[
S_{pot} = \frac{1}{2\pi} \int dzd\bar{z} \text{Tr}(g \tilde{T} g^{-1} \tilde{T}),
\]

where \( S_{WZNW}(g) \) is the usual Wess-Zumino-Novikov-Witten action.

The nonabelian sine-Gordon model in Eq. (19) associated with a Lie algebra \( g \) is rather general for the practical purpose of obtaining exact solutions. Thus, we make further restrictions by specifying \( T \) and \( \tilde{T} \) as follows; we assume that \( T \) and \( \tilde{T} \) belong to the Cartan subalgebra of \( g \) and the subalgebra \( h \subset g \) is the common centralizer of \( T \) and \( \tilde{T} \), i.e. \( h = \{ h \subset g : [T, h] = 0, [\tilde{T}, h] = 0 \} \). We also assume \( A \) and \( \tilde{A} \) to be valued in \( h \) so that the gauged Wess-Zumino-Novikov-Witten action, \( S_{WZNW}(g) + S_{gauge} \), becomes the \( G/H \)-WZNW action for the coset conformal field theory [15]. Note that the potential term \( S_{pot} \) is invariant under the \( H \)-group action so that the whole action \( S \) possesses the group \( H \)-vector gauge invariance if we treat \( A \) and \( \tilde{A} \) as gauge connections. Moreover, we could further restrict the model by treating \( A \) and \( \tilde{A} \) as Lagrangian multipliers which give rise to the constraint equations,

\[
(-\bar{\partial} g g^{-1} + g \bar{A} g^{-1} - \tilde{A})_h = 0, \quad (g^{-1} \partial g + g^{-1} Ag - A)_h = 0.
\]

(21)

Here, the subscript \( h \) denotes the projection to the subalgebra \( h \). This restricted nonabelian sine-Gordon model corresponding to the coset \( G/H \) has been named as the symmetric space sine-Gordon(SSSG) model for the type-II symmetric spaces [12]. It has been
also shown that the field strength of \( A, \bar{A} \) vanishes, i.e. \( F_{zz} = [\partial + A, \bar{\partial} + \bar{A}] \). Thus, we may fix the vector gauge invariance by taking \( A = \bar{A} = 0 \). Note that the vortex string coordinates in Eq. (13) are also invariant under the vector gauge transformation: \( \Phi \rightarrow h \Phi \) for an element \( h(z, \bar{z}) \in H \). This means that the vortex solution \( X^i \) can be obtained from the solution of the gauge fixed \( (A = \bar{A} = 0) \) SSSG equation:

\[
- \bar{\partial}(g^{-1}\partial g) + [T, g^{-1}\bar{T}g] = 0,
\]

and the constraints,

\[
(g^{-1}\partial g)_h = 0, \quad (\bar{\partial}gg^{-1})_h = 0.
\]

Other types of symmetries of the vortex and the SSSG models are also interconnected. For example, the symmetry of the SSSG equation under the parity transformation: \( g \rightarrow Pg, \quad \bar{z} \rightarrow -\bar{z}, \) for \( P \) a constant element which anti-commutes with \( T \) and \( \bar{T} \), induces a symmetry of the vortex equation under the exchange (\( \tau \leftrightarrow \sigma \)) of string world-sheet coordinates. On the other hand, the transformation: \( \Phi(\lambda, z, \bar{z}) \rightarrow \Phi(\lambda, z, \bar{z})\tilde{\Phi}(\lambda) \) for a unitary element \( \tilde{\Phi}(\lambda) \), which leaves the linear equation (9) invariant, induces the rotational and the translational transformation a vortex such that

\[
F \rightarrow \tilde{\Phi}(\lambda)^{-1}F\tilde{\Phi}(\lambda) + \lambda\tilde{\Phi}(\lambda)^{-1}\frac{d\tilde{\Phi}(\lambda)}{d\lambda}.
\]

If \( \tilde{\Phi}(\lambda) \) is not unitary, e.g. \( \tilde{\Phi}(\lambda) = f(\lambda) \) for some function \( f(\lambda) \), the trace of \( F \) changes. Thus we can always set the trace to zero by choosing an appropriate \( f(\lambda) \).

Next, we derive a vortex solution from the one soliton solution of the SSSG equation. Instead of applying the method of inverse scattering, we adopt the following B"acklund transformation to derive the one soliton solution; let \( (f, \Phi_f) \) is a solution of the linear equation (9). Then, \( (g, \Phi_g) \) is another solution provided that

\[
\Phi_g = \frac{\lambda}{\lambda - ib}\left(1 + \frac{ib}{\lambda}g^{-1}f\right)\Phi_f
\]

and

\[
\begin{align*}
g^{-1}\partial g - f^{-1}\partial f - ib[g^{-1}f, T] &= 0 \\
ib\bar{\partial}(g^{-1}f) + g^{-1}\bar{T}g - f^{-1}\bar{T}f &= 0,
\end{align*}
\]

where \( b \) is a parameter of the B"acklund transformation. For simplicity, we take a trivial solution for \( (f, \Phi_f) \) such that

\[
f = 1, \quad \Phi_f = \text{exp}(-\lambda Tz - \lambda^{-1}\bar{T}\bar{z}).
\]
One can easily see that this corresponds to the straight vortex line $F_f = -\lambda T z + \lambda^{-1} \bar{T} \bar{z}$.

In order to solve Eq. (26) with the trivial solution in Eq. (27), we use the fact that $g^{-1} \partial g$ is anti-Hermitian so that $[g - g^{-1}, T] = 0$ due to Eq. (26). This may be solved in terms of a Hermitian projection matrix $P$ satisfying $P^2 = P$, $P^\dagger = P$ by

$$g = 2 \cos \theta P - e^{i \theta}.$$  \hfill (28)

where $\theta$ is some constant parameter. The linear equation now changes into

$$(1 - P)(\partial - ibe^{i \theta} T)P = 0, \quad (1 - P)(ibe^{i \theta} \bar{T})P = 0.$$  \hfill (29)

If we consider only the 1-dimensional projection, we may write

$$P_{ij} = s_i s_j^*/\sum_{k=1}^n s_k s_k^*$$  \hfill (30)

so that Eq. (29) becomes

$$(\partial - ibe^{i \theta} T)s = 0, \quad (ibe^{i \theta} \bar{T})s = 0.$$  \hfill (31)

This can be integrated immediately to yield

$$s_i = \sum_{k=1}^n [exp(ib e^{i \theta} T z - ie^{-i \theta} \bar{T} \bar{z}/b)]_{ik} u_k,$$  \hfill (32)

where $u_i$ are constants of integration. Finally, using

$$\Phi_g = f(\lambda) \left( 1 + \frac{ib}{\lambda} (2 \cos \theta P - e^{-i \theta}) \right) exp(-\lambda T z - \lambda^{-1} \bar{T} \bar{z})$$  \hfill (33)

where $f(\lambda)$ is chosen to make $F$ traceless, we obtain the n-dimensional vortex soliton solution of the Hasimoto-type [16]:

$$F = \frac{2ib \lambda \cos \theta}{2b \lambda \sin \theta - \lambda^2 - b^2} \left[ exp(\lambda T z + \lambda^{-1} \bar{T} \bar{z}) P exp(-\lambda T z - \lambda^{-1} \bar{T} \bar{z}) - 1/2 \right] - \lambda T z + \lambda^{-1} \bar{T} \bar{z},$$  \hfill (34)

where $P$ is defined by Eqs. (30) and (32).

Now, we restrict to the case of Lund and Regge by taking $g \approx so(3)$ and $T^i = \sigma_i$ where $\sigma_i$ are Pauli matrices. $c$ and $\alpha$ as in Eqs. (1) and (16) are related by $c = -4i \alpha$. Choosing $T$ and $\bar{T}$ by $T = -\bar{T} = -ic \sigma_3/4$ and also with an appropriate parametrization of an $SU(2)$ element $g_i$, one can readily see that the SSSG equation becomes the complex
sine-Gordon equation in Eq. (12). The vortex coordinates $X^i$ in Eq. (1) are given by the components of $F$ with the following scaling;

$$X_i = -\frac{4i}{c} F_i \left( z = \frac{1}{2\lambda}(\sigma + \tau), \bar{z} = \frac{\lambda}{2}(\sigma - \tau) \right).$$  

(35)

Then, the vortex soliton in Eq. (34) becomes

$$X_1 = R \operatorname{sech} \Sigma \cos \Theta,$$
$$X_2 = R \operatorname{sech} \Sigma \sin \Theta,$$
$$X_3 = R \tanh \Sigma + \sigma,$$  

(36)

where

$$R = \frac{4b\lambda \cos \theta}{c(2b\lambda \sin \theta - \lambda^2 - b^2)},$$ $$\Sigma = \frac{c}{4} \cos \theta \left( \frac{b}{\lambda}(\sigma + \tau) + \frac{\lambda}{b}(\sigma - \tau) \right),$$ $$\Theta = \frac{c}{2} \tau + \frac{c}{4} \sin \theta \left( \frac{\lambda}{b}(\sigma - \tau) - \frac{b}{\lambda}(\sigma + \tau) \right).$$  

(37)

In this paper, we have extended the vortex equation by Lund and Regge to the n-dimensional case and proved its integrability by mapping the vortex equation into the nonabelian sine-Gordon equation defined in association with a Lie algebra $\mathfrak{g}$ of dimension $n$. Through the identification, we have obtained explicit correspondence between vortex coordinates and the variables of the nonabelian sine-Gordon system, and also the Hasimoto-type one soliton solution for the vortex equation. Other explicit solutions can be also found through this correspondence with interesting physical implications. This will appear elsewhere [17].

**ACKNOWLEDGEMENT**

We are grateful to K. Lee for many helpful discussions. This work was supported in part by the program of Basic Science Research, Ministry of Education 1998-015-D00073, and by Korea Science and Engineering Foundation, 97-07-02-02-01-3.

**References**

[1] F. Lund and T. Regge, Phys. Rev. D14 (1976) 1524.
[2] A. Zee, Nucl. Phys. B421 (1994) 111.

[3] F. Lund, Phys. Rev. Lett. 38 (1977) 1175; Ann. of Phys. 115 (1978) 251.

[4] A.I. Bobenko, in Harmonic maps and integrable systems, edited by A.P. Fordy and J.C. Wood (Vieweg, 1993).

[5] K. Pohlmeyer, Commun. Math. Phys. 46 (1976) 207.

[6] B.S. Getmanov, JETP Lett. 25 (1977) 119.

[7] H.J. de Vega and J.M. Maillet, Phys. Lett. B101 (1981) 302; Phys. Rev. D28 (1983) 1441.

[8] I. Bakas, Int. J. Mod. Phys. A9 (1994) 3443.

[9] Q-H. Park, Phys. Lett. B328 (1994) 329.

[10] Q-H. Park and H. J. Shin, Phys. Lett. B359 (1995) 125.

[11] Q-H. Park and H.J. Shin, J. Korean. Phys. Soc. 30 (1997) 336; Phys. Rev. A57 (1998) 4621; ibid A57 (1998) 4643.

[12] I. Bakas, Q-Han Park and H. J. Shin, Phys. Lett. B372 (1996) 45.

[13] Q-H. Park and H. J. Shin, Phys. Lett. B347 (1995) 73; Nucl. Phys. B458 (1996) 327; T. J. Hollowood, J. L. Miramontes and Q-H. Park, Nucl. Phys. B445 (1995) 451.

[14] C.R. Fernandez-Pousa, M.V. Gallas, T.J. Hollowood, J. L. Miramontes, Nucl.Phys. B484 (1997) 609; Nucl.Phys. B499 (1997) 673.

[15] D.Karabali, Q-H.Park, H.J.Schnitzer and Z.Yang, Phys. Lett. B216 (1989) 307; K.Gawedski and A.Kupiainen, Nucl. Phys. B320 (1989) 625.

[16] H. Hasimoto, J. Fluid Mech. 51 (1972) 477.

[17] K. Lee, Q-H. Park and H. J. Shin, in preparation.