Linear and Dynamic Programs for Risk-Sensitive Cost Minimization

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Abstract—We derive equivalent linear and dynamic programs for infinite horizon risk-sensitive control for minimization of the asymptotic growth rate of the cumulative cost.

I. INTRODUCTION

Risk-sensitive control problems that seek to minimize over an infinite time horizon the asymptotic growth rate of mean exponentiated cumulative cost of a controlled Markov chain were first studied in [5], [6], which also pioneered the most popular approach to such problems, viz., to use the celebrated ‘log-transformation’ to convert it to a zero sum stochastic game with long run average or ‘ergodic’ payoffs. An equivalent alternative approach that treats the corresponding reward maximization problem as a nonlinear eigenvalue problem was developed in [1]. This leads to an equivalent ergodic reward maximization problem and an associated linear program. For the finite state-action case, the complete details of the latter were worked out in [3]. Unfortunately the techniques therein do not extend to the cost minimization problem, which is equivalent to a zero sum ergodic stochastic game. It may be recalled that unlike the classical criteria such as discounted or ergodic, risk-sensitive reward maximization cannot be converted to a cost minimization and vice versa, by a simple sign flip. Thus the two are not equivalent.

In this work, we make the key observation that the aforementioned zero sum ergodic game belongs to a very special subclass thereof, viz., a single controller game wherein one agent affects only the payoff and not the dynamics. This case is indeed amenable to a linear programming formulation as pointed out in [12]. We exploit this fact to derive the counterparts of the results of [3] for the cost minimization problem. It may be noted that an LP formulation for risk-sensitive cost or reward is not a priori obvious because unlike the classical criteria such as discounted or ergodic, where the uncontrolled problems lead to linear ‘one step analysis’ (or the Poisson equation), risk-sensitive control leads to an eigenvalue problem which is already nonlinear. This leads to a second ‘dynamic programming’ equation coupled to the usual one in what is a counterpart of the corresponding system of equations for ergodic control without irreducibility ([10], Chapter 9). The interesting twist here is the appearance of the so called ‘twisted’ transition kernel.

II. RISK-SENSITIVE COST MINIMIZATION

Consider a controlled Markov chain \( \{X_n\} \) on a finite state space \( S := \{1, 2, \ldots, s\} \), controlled by a control process \( \{Z_n\} \) taking values in a finite action space \( \mathcal{U} \), with running cost \( c(i, u), i \in S, u \in \mathcal{U} \). Let

\[
(i, u, j) \in S \times \mathcal{U} \times S \mapsto p(j \mid i, u) \in [0, 1],
\]

with \( \sum_j p(j \mid i, u) = 1 \) \( \forall (i, u) \in S \times \mathcal{U} \) be its controlled transition kernel, that is, the following ‘controlled Markov property’ holds:

\[
P(X_{n+1} = j \mid X_n, Z_n, m \leq n) = p(j \mid X_n, Z_n), \quad n \geq 0.
\]

Such \( \{Z_n\} \) will be called admissible controls. We call \( \{Z_n\} \) a stationary (randomized) policy if

\[
P(Z_n = u \mid X_m, m \leq n; Z_m, m < n) = \varphi(u \mid X_n)
\]

for some \( \varphi : i \in S \mapsto \varphi(\cdot \mid i) \in \mathcal{P}(\mathcal{U}) \), with \( \mathcal{P}(\mathcal{U}) \) denoting the simplex of probability vectors on \( \mathcal{U} \). A stationary policy is called pure or deterministic if \( Z_n = \varphi(X_n) \) for all \( n \geq 0 \), for some \( \varphi : S \mapsto \mathcal{U} \), equivalently, when \( \varphi(\cdot \mid i) = \delta_{\varphi(i)}(\cdot) \forall i \), i.e., a Dirac measure at \( \nu(i) \forall i \). We let \( \mathcal{U}_\text{ad} \) and \( \mathcal{U}_\text{st} \) denote the class of all stationary and pure policies, respectively. By abuse of terminology, stationary policies, resp. pure policies, are identified with the map \( \varphi \), resp. \( v \), in the preceding definition.

The risk-sensitive cost minimization problem we are interested in seeks to determine

\[
\bar{\lambda}^* := \max_{i \in S} \lambda^*_i, \quad (1)
\]

\[
\lambda^*_i := \inf_{(Z_m)} \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E}_i \left[ e^{\sum_{m=1}^{n-1} c(X_m, Z_m)} \right], \quad (2)
\]

where the infimum is over all admissible controls, and \( \mathbb{E}_i[\cdot] \) denotes the expectation with \( X_0 = i \). We restrict ourselves to stationary policies. For a stationary policy \( v \in \mathcal{U}_\text{st} \), we use the notation

\[
c_v(i) := \sum_{u \in \mathcal{U}} c(i, u)v(u \mid i),
\]

\[
p_v(j \mid i) := \sum_{u \in \mathcal{U}} p(j \mid i, u)v(u \mid i). \quad (3)
\]
Let
\[
\lambda_i^v := \lim_{n \to \infty} \frac{1}{n} \log E^v_i \left[ e^{\sum_{m=0}^{n-1} c_n(X_m)} \right],
\]
where \(E^v_i[\cdot]\) indicates the expectation under the policy \(v \in \mathcal{U}_{nm}\) with \(X_0 = i\). Thus
\[
\lambda_i^* = \min_{v \in \mathcal{U}_{nm}} \max_{i \in S} \lambda_i^v. \tag{4}
\]

**Definition 2.1:** Let \(Q\) denote the class of stochastic matrices \(q = [q_{ij}]_{i,j \in S}\) such that
\[
q_{ij} = 0 \quad \text{if} \quad \max_{u \in U} p(j \mid i, u) = 0.
\]
Also, \(M_q\) denotes the set of invariant probability vectors of \(q \in Q\).

Using the equivalent notation \(q(j \mid i) = q_{ij}\), we define
\[
\tilde{c}(i, q, u) = c(i, u) - D(q(\cdot \mid i) \parallel p(\cdot \mid i, u)), \tag{5}
\]
if \(q(\cdot \mid i) \ll p(\cdot \mid i, u)\), and \(\tilde{c}(i, q, u) = -\infty\), otherwise. Here,
\[
D(q(\cdot \mid i) \parallel p(\cdot \mid i, u)) = \sum_{j \in S} q(j \mid i) \frac{\log q(j \mid i)}{\log p(j \mid i, u)}
\]
denotes the Kullback-Leibler divergence. For \(v \in \mathcal{U}_{nm}\), we let \(\tilde{c}_v(i, q)\) be defined analogously to (3), that is,
\[
\tilde{c}_v(i, q) := \sum_{u \in U} \tilde{c}(i, q, u) v(u \mid i).
\]
Specializing [1, Theorem 3.3] to the above, we have
\[
\max_{i \in S} \lambda_i^v = \max_{q \in Q} \max_{\pi \in M_q} \sum_{i \in S} \pi(i) \tilde{c}_v(i, q). \tag{6}
\]
The reason that we can restrict the maximization to the set \(Q\) is the following. Suppose \((\hat{q}, \hat{\pi})\) is a pair where the maximum in (6) is attained. Without loss of generality we may assume that \(\hat{\pi}\) is an ergodic measure. It is clear then that we must have \(\hat{q}(\cdot \mid i) \ll p_v(\cdot \mid i)\) on the support of \(\hat{\pi}\), otherwise \(\max_{i \in S} \lambda_i^v = -\infty\), which is not possible.

Equations (4)–(6) suggest an ergodic game for a controlled Markov chain which we describe next.

**Definition 2.2:** The model for the controlled Markov chain \(\{X_n\}\) is as follows:

- The state space is \(S\).
- The action space is \(Q(i) \times U\), for \(i \in S\), where
  \[
  Q(i) := \{q_{ij} : j \in S, q \in Q\}. \tag{7}
  \]
- The controlled transition probabilities are dictated by \(q \in Q\). Note then that \(Q\) may be viewed as the set of stationary policies with action spaces \(\{Q(i), i \in S\}\). It is clear that in this space there is no difference between randomized and pure policies.
- The running reward is \(\tilde{c}(i, q, u)\) defined in (5).

With \(\{\tilde{X}_n\}_{n \in \mathbb{N}}\) denoting the chain defined above, and \(E^v_i[\cdot]\) the expectation operator under the policy \(v \in \mathcal{U}_{nm}\) with \(\tilde{X}_0 = i \in S\), define
\[
\hat{\Phi}(q, v) := \max_{i \in S} \lim_{N \to \infty} \frac{1}{N} E^v_i \left[ \sum_{k=0}^{N-1} \tilde{c}_v(\tilde{X}_n, q) \right]. \tag{8}
\]
with \(q \in Q\). The preceding analysis shows that we seek to maximize \(\hat{\Phi}(q, v)\) with respect to \(q \in Q\) and minimize it with respect to \(v \in \mathcal{U}_{nm}\). This forms a single controller zero-sum ergodic game between the agent who chooses \(q\) to maximize the long-term average value of the reward \(\tilde{c}(i, q, u)\) and the agent who chooses \(u\) to minimize it. The reason that it is a single controller game is because the decisions of the second player affect only the payoff and not the transition probability. This facilitates the application of [12] to derive equivalent linear programs, which we do in Section III. It is clear that
\[
\hat{\Phi}(q, v) = \max_{i \in S} \sum_{q \in Q} \pi(i) \tilde{c}_v(i, q). \tag{9}
\]
Suppose we can show that
\[
\min_{v \in \mathcal{U}_{nm}} \max_{q \in Q} \hat{\Phi}(q, v)
\]
is attained at some \(v^* \in \mathcal{U}_p\). Then, in view of (4) and (6), and the fact that
\[
\tilde{c}_v(i, q) = c_v(i) - D(q(\cdot \mid i) \parallel p_v(\cdot \mid i)) \quad \forall v \in \mathcal{U}_p,
\]
we obtain
\[
\hat{\lambda}^* = \min_{v \in \mathcal{U}_{nm}} \max_{q \in Q} \hat{\Phi}(q, v). \tag{10}
\]
In fact, in Section III we show that the game has a value \(\hat{\lambda}^*\), that is,
\[
\hat{\lambda}^* = \inf_{v \in \mathcal{U}_{nm}} \sup_{q \in Q} \hat{\Phi}(q, v) = \sup_{q \in Q} \inf_{v \in \mathcal{U}_{nm}} \hat{\Phi}(q, v), \tag{11}
\]
and there exists \(v^* \in \mathcal{U}_p\) and \(q^* \in Q\) such that
\[
\hat{\lambda}^* = \inf_{v \in \mathcal{U}_{nm}} \sup_{q \in Q} \hat{\Phi}(q^*, v) = \sup_{q \in Q} \hat{\Phi}(q^*, v^*). \tag{12}
\]
In other words, the pair \((q^*, v^*)\) is optimal.

### III. Equivalent Linear Programs

We now adapt the key results of [12] relevant for us. Since [12] works with finite state and action spaces and \(Q\) is not finite, we first replace \(Q\) by a finite approximation \(Q_n\) for \(n > 1\), of transition probability kernels \(q(\cdot \mid \cdot)\) such that for all \(i, j \in S, q(j \mid i)\) takes values in the set of dyadic rationals of the form \(\frac{k}{2^n}\) for some \(0 \leq k \leq 2^n\). Let \(A_n(i)\) be the corresponding action spaces defined as in (7), but with \(Q\) replaced by \(Q_n\). As noted in Definition 2.2, \(Q_n\) may be viewed as the set of stationary policies with action spaces \(\{A_n(i), i \in S\}\). For \((q, v) \in Q_n \times \mathcal{U}_{nm}\), we let
\[
\Phi_n(q, v) := \lim_{N \to \infty} \frac{1}{N} E^{v}_{i} \left[ \sum_{k=0}^{N-1} \tilde{c}_v(\tilde{X}_n, q) \right], \quad i \in S. \tag{13}
\]
We consider the corresponding single controller zero-sum game analogous to the one described in Section II. As we show later, the single controller zero-sum game with the objective in (13) over \((q, v) \in Q_n \times \mathcal{U}_{nm}\) has the following equivalent linear programming formulation.

**Primal program (LP\(_p\))**: The primal variables are
\[
V = (V_1, \ldots, V_s) \in \mathbb{R}^s, \quad \beta = (\beta_1, \ldots, \beta_s) \in \mathbb{R}^s,
\]
and
\[ y = (y_1, \ldots, y_s) : \mathcal{U} \to \mathcal{P}(S), \]
and the linear program is the following:

Minimize \( \sum_{i \in S} \beta_i \) subject to:
\[
\begin{align*}
\beta_i & \geq \sum_{j \in S} q_{ij} \beta_j \quad \forall \ i \in S, \\
V_i & \geq \sum_{u \in \mathcal{U}} \tilde{c}(i, q, u) y_i(u) - \beta_i + \sum_{j \in S} q_{ij} V_j \quad \forall \ i \in S.
\end{align*}
\]

Dual program (LP'): The dual variables are
\[
(\mu(i, q), \nu(i, q) : (i, q) \in S \times A_n(i)),
\]
and \( w = (w_1, \ldots, w_s) \in \mathbb{R}^s \), and the dual linear program is:

Maximize \( \sum_{i \in S} w_i \) subject to:
\[
\begin{align*}
\sum_{(i, q) \in S \times A_n(i)} (\delta_{ij} - \tilde{p}(j \mid i, q)) \mu(i, q) & = 0 \quad \forall j \in S, \\
\sum_{(i, q) \in S \times A_n(i)} (\delta_{ij} - \tilde{p}(j \mid i, q)) \nu(i, q) & = 1 \quad \forall j \in S, \\
\sum_{q \in A_n(i)} \tilde{c}(i, q, u) \mu(i, q) & \geq w_i \quad \forall (i, u) \in S \times \mathcal{U}, \\
\mu(i, q), \nu(i, q) & \geq 0 \quad \forall (i, q) \in S \times A_n(i).
\end{align*}
\]

In the above constraints, \( \delta_{ij} = 1 \) if \( i = j \), and equals 0 otherwise.

The programs in (LP) and (LP') are exactly as given in [12, Section 2], with the notation adapted to the current setting. Arguing as in [12, Lemma 2.1], we deduce that both linear programs are feasible and have bounded solutions. We note that \( \tilde{c} \) is extended-valued here, whereas it is and real-valued and bounded in [12]. Nevertheless, note that \( q' \in A_n(i) \) can always be selected so that \( \tilde{c}(i, q', u) > -\infty \), and this shows that the solution \( \beta \) is bounded.

Definition 3.1: Let \((V^n, \beta^n, y^n), (\mu^n, \nu^n, w^n)\) denote solutions for (LP) and (LP'), resp., for each \( n \in \mathbb{N} \). Define
\[
\alpha^n_i(q) := \begin{cases} 
\sum_{q \in A_n(i)} \mu^n(i, q), & \text{if } \tilde{c}_i^n \neq 0, \\
\sum_{q \in A_n(i)} \nu^n(i, q), & \text{otherwise.}
\end{cases}
\]
and
\[
\Phi_i^n(q) = \inf_{v \in \mathcal{V}} \sup_{\nu \in \mathcal{Q}_n} \Phi_i(q, v).
\]

The following lemma follows from the results in [12], some of them drawn from [2], [9], [11].

Lemma 3.1: The single controller zero-sum game with the objective in (13) over \((q, v) \in \mathcal{Q}_n \times \mathcal{V}_n\) has a value
\[
\Phi_i^n = (\Phi_1^n, \ldots, \Phi_s^n) \in \mathbb{R}^s,
\]
that is,
\[
\Phi_i^n = \inf_{v \in \mathcal{V}} \sup_{\nu \in \mathcal{Q}_n} \Phi_i(q, v)
= \sup_{q \in \mathcal{Q}_n} \inf_{v \in \mathcal{V}} \Phi_i(q, v),
\]
and the following hold:
(a) We have \( \beta^n = \Phi_i^n \), where \( \beta^n \) is the solution of (LP).
(b) A pair of optimal stationary policies \((q^n_1, v^n) \in \mathcal{Q}_n \times \mathcal{V}_n\) exists.
(c) The inner supremum (resp., infimum) in the left (resp., right) hand side of (15) is attained at a stationary (nonrandomized) policy.
(d) For any solution \((V^n, \beta^n, y^n)\) of (LP), \( y^n \) is an optimal policy for player 2. In other words, \( v^n_q(\cdot | i) = \Phi^n_q(\cdot) \) for all \( i \in S \). Moreover \( y^n \) can be selected so as to induce a pure Markov policy.
(e) For any solution \((\mu^n, \nu^n, w^n)\) of (LP'), \( q^n_0 \) can be selected as
\[
q^n_0(\cdot | i) = \sum_{q \in A_n(i)} q(\cdot | i) \alpha^n_i(q),
\]
with \( \alpha^n_i \) as defined in (14).

Proof: The proof is based on the results in [12]. However, the roles of the players should be interchanged, since it is player 1 that does not influence the transition probabilities in [12]. But if we define the expected average payoff \( V \) as
\[
V(v, q) = -\Phi(q, v),
\]
then with \( v \in \mathcal{V}_n \) the stationary strategies of player 1, and \( q \in \mathcal{Q}_n \) those of player 2, the model matches exactly that of [12].

That the game has a value and parts (a) and (b) then follow from [12, Theorem 2.15]. Part (c) is the statement of [12, Lemma 1.2]. Part (d) then follows by considering the second constraint in (LP) together with [12, Lemma 2.14]. Part (e) follows from the definitions (2.4)-(2.10) following the proof of [12, Lemma 2.2] together with [12, Lemma 2.9]. This completes the proof.

\[ \square \]

A. The semi-infinite linear programs

Letting \( n \nearrow \infty \), we obtain a pair of semi-infinite linear programs with \( \mathcal{Q}_n \) replaced by \( \mathcal{Q} \) in (LP), that is, linear programs with finitely many variables, but infinitely many constraints. These are as follows:

Primal program (LP): The primal variables are as in (LP), and the program is the following:

Minimize \( \sum_{i \in S} \beta_i \) subject to:
\[
\begin{align*}
\beta_i & \geq \sum_{j \in S} \tilde{p}(j \mid i, q') \beta_j, \\
V_i & \geq \sum_{u \in \mathcal{U}} \tilde{c}(i, q', u) y_i(u) - \beta_i + \sum_{j \in S} \tilde{p}(j \mid i, q') V_j, \\
\forall q' \in \mathcal{Q}(i), \forall i \in S.
\end{align*}
\]
Dual program (LP'): The dual variables are
\[(\mu(i, q), \nu(i, q) : (i, q) \in S \times Q(i))\],
and \(w = (w_1, \ldots, w_s) \in \mathbb{R}^s\), and the dual linear program is:

Maximize \(\sum_{i \in S} w_i\) subject to:
\[
\sum_{(i, q) \in S \times Q(i)} (\delta_{ij} - \tilde{p}(j | i, q)) \mu(i, q) = 0 \quad \forall j \in S, \\
\sum_{(i, q) \in S \times Q(i)} (\delta_{ij} - \tilde{p}(j | i, q)) \nu(i, q) + \sum_{q \in Q(i)} \mu(j, q) = 1 \quad \forall j \in S, \\
\sum_{q \in Q(i)} c(i, u, q) \mu(i, q) \geq w_i \quad \forall (i, u) \in S \times U, \\
\mu(i, q), \nu(i, q) \geq 0 \quad \forall (i, q) \in S \times Q(i).
\]

(LP')

With an eye on the passage from the approximate linear programs (LP\(_n\)) and (LP\(_n'\)) on \(Q_n\) to the analogous semi-infinite linear programs (LP) and (LP') over \(Q\), we need the following two lemmas.

**Lemma 3.2:** The sequence \(\{\beta^n\}_{n \in \mathbb{N}}\) converges monotonically to some \(\beta \in \mathbb{R}^s\) in each component. Moreover, \(\hat{\beta}\) is the infimum of all feasible values of (LP).

**Proof:** Since any solution \((V^n, \beta^n, y^n)\) of (LP\(_n\)) is feasible for the program LP\(_{n+1}\), it is clear that \(\beta^n\) is nonincreasing in \(n\) in each component. It also follows by the definition in (5) that there exists a constant \(M\) such that
\[
\min_{(i, u) \in S \times U} \max_{q \in A(i)} c(i, u, q) \geq M.
\]

Thus, since \(\beta^n_i\) is clearly bounded below by \(M\) for each \(i \in S\), and \(n \in \mathbb{N}\), there exists a limit
\[
\hat{\beta} := \lim_{n \to \infty} \beta^n.
\]

Now, it is straightforward to show that any feasible solution \(\beta\) of (LP) satisfies \(\beta \geq \hat{\beta}\). Indeed, if some \(\beta\) with \(\beta_i < \hat{\beta}_i\) is feasible for (LP), one can find a \(\tilde{\beta}\) arbitrarily close to \(\hat{\beta}\) which is feasible for (LP\(_n\)) for large enough \(n\) by continuity. This of course contradicts the fact that \(\beta^n \geq \hat{\beta}\) for all \(n \in \mathbb{N}\), and completes the proof.

Let \(P^n\) denote the transition matrix induced by (LP\(_n'\)) via the optimal policy defined in Lemma 3.1 (d). Note that this satisfies \(\beta^n = P^n \beta^n\) for all \(n \in \mathbb{N}\) by (LP\(_n\)) (see [12, Lemma 2.9]). Recall also that \(y^n\) is pure Markov, and can be identified with \(y^n_\ast\) as asserted in Lemma 3.1 (d). We continue with the following lemma.

**Lemma 3.3:** Any limit point \((\hat{\beta}, \tilde{P}, \hat{\gamma})\) of \((\beta^n, P_n, y^n)\) along a subsequence, as \(n \to \infty\) is feasible for (LP).

**Proof:** Let \(Q_n\) be defined by
\[
Q_n := \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} P^n_k,
\]
and similarly define \(\hat{Q}\) relative to the stochastic matrix \(\tilde{P}\). Also let \(\hat{c}_n\) denote the running cost under \(P^n\) and \(y^n\). It is clear that \(\hat{c}_n\) converges to some \(\hat{c}\) as \(n \to \infty\) along the same subsequence. Since \((\beta^n, P_n, y^n)\) is optimal for (LP\(_n\)), we have (in vector notation) \(\beta^n = Q_n \hat{c}_n\), and \(\beta^n = P^n \beta^n\) for all \(n \in \mathbb{N}\) by (LP\(_n\)) (see [12, Lemma 2.9]). Thus taking limits as \(n \to \infty\), we obtain
\[
\hat{\beta} = \hat{Q} \hat{c}, \quad \text{and} \quad \{\hat{\beta}\} = \{\hat{P}\} \hat{\gamma}.
\]

Note that \(V^n\) can be selected as
\[
V^n = (I - P_n + Q_n)^{-1} (I - Q_n) \hat{c}_n.
\]

Taking limits as \(n \to \infty\), it follows that \(V^n \to \hat{V}\) which satisfies
\[
\hat{V} := (I - \hat{P} + \hat{Q})^{-1} (I - \hat{Q}) \hat{c}.
\]

Inserting the dependence of \(\hat{c}_n\) and \(P_n\) explicitly in the notation, the second constraint in (LP\(_n\)) can be written as
\[
V^n + \beta^n \geq \hat{c}_n(q) + P(q)V^n
\]
for all \(q \in Q_n\). Now, fix some \(m \in \mathbb{N}\) and \(q \in Q_m\). Taking limits in (16) as \(n \to \infty\), we obtain
\[
\hat{V} + \hat{\beta} \geq \hat{c}(q) + P(q)\hat{V}.
\]

Since \(q \in Q_m\) is arbitrary and \(\cup_{m \in \mathbb{N}} Q_m\) is dense in \(Q\), it follows that (17) holds for all \(q \in Q\). Hence the second constraint in (LP) is satisfied. Similarly, starting from
\[
\beta^n \geq P(q) \beta^n \quad \forall q \in Q_n,
\]
and repeating the same argument, we see that the first constraint in (LP) is also satisfied. This completes the proof of the lemma.

**Remark 3.1:** It is also possible to start from a solution \((\mu^n, \nu^n, w^n)\) of the dual program (LP\(_n'\)), and then take limits as \(n \to \infty\). Note that \(\mu^n\) is a Dirac mass, so convergence to (say) \(\tilde{\mu}\) is interpreted in the weak sense. Same for \(\nu^n \to \tilde{\nu}\).

It is easy to see then that any subsequential limit \((\tilde{\mu}, \tilde{\nu}, \tilde{w})\) satisfies (LP') by continuity.

By Lemmas 3.2 and 3.3, the linear programs in (LP) and (LP') are feasible and have bounded solutions. This allows us to extend Lemma 3.1 as follows.

**Theorem 3.1:** The single controller zero-sum game with the objective in (13) over \((q, v) \in Q \times \mathcal{U}_m\) has a value \(\Phi^* = (\Phi^*_1, \ldots, \Phi^*_s) \in \mathbb{R}^s\), that is,
\[
\Phi^*_i = \inf_{v \in \mathcal{U}_m} \sup_{q \in \mathcal{Q}} \Phi_i(q, v) = \sup_{q \in \mathcal{Q}} \inf_{v \in \mathcal{U}_m} \Phi_i(q, v),
\]
and the following hold:
(i) \(\Phi^* = \beta\), the solution to (LP).
(ii) A pair of optimal stationary policies \((q^*, v^*) \in Q \times \mathcal{U}_g\) exists.
(iii) The analogous statements of parts (c)–(e) in Lemma 3.1 hold.

It is now easy to connect the original game in (8) to the game with the objective in (13). Since the maximum of \(\Phi_i(q, v)\) over \(i \in S\), is attained in some ergodic class
Theorem 3.1

Theorem 3.1 for (8) implies that (20) holds. Theorem 4.1 satisfies (1). Theorem 4.1 that the 22 has the value (8) and (20). The proof of (ii) is optimal for simply match the constraints and in general.

In addition, the fact that \( \ast \in U_p \) as asserted in Theorem 3.1 (ii) implies that (10) holds. So, in summary, the risk-sensitive value \( \bar{\lambda}^* \) defined in (1) satisfies

\[
\bar{\lambda}^* = \inf_{i \in S} \sup_{q \in Q} \Phi(q, v) = \sup_{q \in Q} \inf_{i \in S} \Phi(q, v)
\]

\[
= \inf_{i \in S} \sup_{q \in Q} (\Phi(q^*, v)) = \sup_{q \in Q} (\Phi(q^*, v^*)).
\]

IV. DYNAMIC PROGRAMMING

It can be seen from the linear program (LP) that the values \( \{\Phi_i^*: i \in S\} \) can be calculated by nested dynamic programming equations (see [10], pp. 442–443). We simplify the notation and write the stochastic matrix \( q \) as \([q_{ij}]\).

We have the following theorem.

Theorem 4.1: It holds that

\[
\bar{\lambda}^* = \max_{i \in S} \lambda_i^* = \max_{i \in S} \Phi_i^*,
\]

where \( \{\Phi_i^*: i \in S\} \) solves, for all \( i \in S \),

\[
\Phi_i^* = \max_{q \in Q} \sum_{j \in S} q_{ij} \Phi_j^*,
\]

\[
\Phi_i^* + V_i = \min_{u \in U} \max_{q \in Q} \left[ c(i, q, u) + \sum_{j \in S} q_{ij} V_j \right],
\]

with

\[
B_i := \left\{ q \in Q : \sum_{j \in S} q_{ij} \Phi_j^* = \Phi_i^* \right\}.
\]

Note that (19) and (20) simply match the constraints in (LP), so that existence of a solution to these equations follows from Theorem 3.1. The proof of Theorem 4.1 again goes through a sequence of finite approximations of \( \mathcal{A} \) so that the aforementioned results from [10] apply.

Care should be taken when performing the maximization over \( q \in Q \) in (20) explicitly using the Gibbs variational principle (see Proposition 2.3, [4]), since the variables \( q \) in (20) are not free but depend on the maximization in (19). Reorder the solution \( \{\Phi_i^*\} \) so that over a partition \( \{I_1, \ldots, I_m\} \) of \( S \), we have

\[
\Phi_i^* = \beta_i^* \quad \forall i \in I_k
\]

and

\[
\beta_1^* < \beta_2^* < \cdots < \beta_m^*.
\]

It is clear then that

\[
B_i = \{q_{ij} \in Q : j \in I_1\} \quad \forall i \in I_1,
\]

and in general

\[
B_i = \{q_{ij} \in Q : j \in I_k\} \quad \forall i \in I_k.
\]

Let

\[
\hat{\rho}(j | i, u) := \begin{cases} \hat{\rho}(j | i, u) & \text{if } i, j \in I_k \text{ for } k \in \{1, \ldots, m\} \\ 0 & \text{otherwise.} \end{cases}
\]

Note that the matrix \( [\hat{\rho}] \) is block-diagonal. Thus we can write the maximum in (20) as

\[
q^*(j | i, u) := \frac{\hat{\rho}(j | i, u) e^{c(i, u)V_j}}{\sum_k \hat{\rho}(j | i, u) e^{c(i, u)V_k}}.
\]

Substituting this back into (19) and (20) along with the change of variables \( \Psi_i = e^{V_i}, \Lambda_i = e^{\beta_i^*}, \) and \( \Lambda^* = e^{\bar{\lambda}^*} \), we get

\[
\Lambda^* = \max_{i \in S} \Lambda_i,
\]

\[
\Lambda_i \Psi_i = \min_{u \in U} \left( \sum_{j \in S} \hat{\rho}(j | i, u) e^{c(i, u)\Psi_j} \right),
\]

\[
\Lambda_i = \min_{u \in B_i} \left( \sum_{j \in S} \hat{\rho}(j | i, u) e^{c(i, u)\Psi_j} \right) \Lambda_j
\]

for \( i \in S \), where \( B_i^* \) is the set of minimizers in (22). As in [3], the important observation here is the appearance of a ‘twisted kernel’ for averaging in (22).

V. COMMENTS ON A COUNTEREXAMPLE OF [7]

We discuss the counterexample in [7, Example 2.1], which is for an uncontrolled model.

Example 5.1: Let

\[
p_{21} = 1 - \rho, \quad p_{22} = \rho, \quad p_{11} = 1, \quad c(2) = 1 \quad c(1) = 0,
\]

with \( \rho \in (0, 1) \). Solving (19) and (20) for \( i = 1 \), we obtain

\[
\Phi_1^* = 0, \quad q_{11} = 1, \quad V_1 = \text{any constant}.
\]

The equations for \( i = 2 \) are

\[
\Phi_2^* = \max_{q \in Q} \left( q_{22} \Phi_2^* \right),
\]

\[
\Phi_2^* + V_2 = \max_{q \in B_2} \left[ 1 - q_{22} \log \frac{q_{22}}{\rho} - q_{21} \log \frac{q_{21}}{1 - \rho} \right.
\]

\[
\left. + q_{22} V_2 + q_{21} V_1 \right].
\]

Thus we must have \( q_{22} = 1 \) if \( \Phi_2^* \neq 0 \). In this case, from (19) and (20), we get

\[
\Phi_2^* = 1 + \log \rho, \quad q_{22} = 1, \quad V_2 = \text{any constant}.
\]

If \( \log \rho > -1 \), then the first hitting time to state 1 (from state 2) does not have an exponential moment, and \( \lambda_i^* = 1 + \log \rho \), while of course \( \lambda_i^* = 0 \).

On the other hand if \( \log \rho < -1 \), then \( q_{22} \neq 1 \), and we get \( \Phi_2^* = 0, \) and \( q \equiv q_{22} \in (0, 1) \) solves

\[
\log \frac{q}{1 - q} - \log \frac{\rho}{1 - \rho} + \frac{1}{1 - q} (1 - 2q) V_1 - B(q) = 0.
\]

1 This also serves as a ‘correction note’ to the derivation of (11)-(12) in [3]. The treatment of dynamic programs in ibid. is flawed and should be replaced by the exact counterpart of the above.
with

$$B(q) := 1 - q \log \frac{q}{\rho} - (1 - q) \frac{1 - q}{1 - \rho}.$$ 

Also $V_2 = \frac{q V_1 + B(q)}{1 - q}$.

Thus, in either case, $\bar{\lambda}^* = \max \{ \Phi_1^*, \Phi_2^* \}$.

However, as noted in [7, Example 2.1], the multiplicative Poisson equation does not have a solution when $\log \rho > -1$, because there is no pair of numbers $(h_1, h_2)$ that even solves the inequality

$$e^{\rho h_2} = e^{\lambda_2^* e^{h_2}} \geq e^{e^{(2)} \left[ p_{22} e^{h_2} + p_{21} e^{h_1} \right]} = e^{\rho e^{h_2} + (1 - \rho) e^{h_1}}.$$ 

We compare Theorem 4.1 with the results in [8]. As shown in [8, Theorem 3.5], under a Doeblin hypothesis, it holds that

$$\lambda_2^* = \inf_{g \in \mathcal{G}} g(i),$$

where $\mathcal{G}$ is the class of functions satisfying

$$g(i) = \min_{u \in U} \left( \max \{ g(j) : p(j \mid i, u) > 0 \} \right),$$

and

$$e^{g(i) + h_i} \geq \min_{u \in B_g(i)} \left[ e^{e(i, u)} \sum_{j \in S} p(j \mid i, u) e^{h_j} \right],$$

where $h = (h_1, \ldots, h_s) \in \mathbb{R}^s$ is a vector possibly depending on $g$, and

$$B_g(i) := \{ u \in U : g(i) = \max \{ g(j) : p(j \mid i, u) > 0 \} \}.$$

It is important to note that the infimum in (24) might not be realized in $\mathcal{G}$. This is what Example 5.1 shows in the case $\log \rho > -1$.

VI. FUTURE DIRECTIONS

One interesting problem that still remains is to show optimality of stationary or pure policies under very general conditions that do not require irreducibility. Yet another interesting direction is an extension of this paradigm to general state spaces and to continuous time risk-sensitive control.

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