Alpha-clustered hypernuclei and chiral SU(3) dynamics

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Abstract

Selected light hypernuclei are studied using an $\alpha$ cluster model approach (the Hyper-THSR wave function) in combination with a density-dependent $\Lambda$ hyperon-nuclear interaction derived from chiral $SU(3)$ effective field theory. This interaction includes important two-pion exchange processes involving $\Sigma N$ intermediate states and associated three-body mechanisms as well as effective mass and surface terms arising in a derivative expansion of the in-medium $\Lambda$ self-energy. Applications and calculated results are presented and discussed for $^{13}_A\Lambda C$ and $^9_A\Lambda Be$. The lightest $\alpha$ clustered hypernucleus, $^5_A\Lambda He$, is also discussed in this context.
§1. Introduction

The physics of $\Lambda$ hypernuclei has a long and well-established history. With the increasing precision of hypernuclear spectroscopy, accurate constraints on the effective interaction of the $\Lambda$ with nucleons in the nuclear medium emerge\cite{1,2,3,4,5}. Empirical single-particle energies of a $\Lambda$ bound in hypernuclei are well described in terms of an attractive mean field that is about half as strong ($U_{\Lambda} \approx -30$ MeV) as the single-particle potential for nucleons in a nucleus. The empirical $\Lambda$-nuclear spin-orbit interaction is extremely weak compared to that of a nucleon in the nucleus. All these features have been described quite successfully, at least for medium-mass and heavy hypernuclei, in various phenomenological mean-field approaches inspired by the shell-model picture. In recent years an equally successful theoretical framework has been developed\cite{6} based on chiral SU(3) dynamics, the effective field theory at the interface of three-flavor, low-energy QCD and nuclear physics with strangeness degrees of freedom. When converting the energy density derived in this framework into a hypernuclear density functional, this theory provides a quantitative description of $\Lambda$ hypernuclei over a wide mass range\cite{7} from $^{16}_A$O to $^{208}_A$Pb.

For lighter hypernuclei such as $^{13}_A$C this approach works still reasonably well but turns out to be less accurate. A primary reason for this is the more complex structure of the corresponding core nuclei. Many previous investigations have shown that clustering correlations play an important role in light nuclei\cite{8,9}. A typical example is $^{12}_C$ which is known not to be a good shell-model nucleus. Its first excited $0^+$ state (the famous Hoyle state) has a pronounced cluster structure displaying a strong component of three alpha clusters in its wave function. Such a structure also emerges in \textit{ab initio} Monte Carlo lattice computations based on chiral effective field theory\cite{10}. Recent works support the picture that the Hoyle state can, to a good approximation, be described as a product state of weakly interacting alpha particles occupying the lowest 0S-orbit of a mean-field potential, with a relatively low average density, only about $\frac{1}{3}$ to $\frac{1}{4}$ of the nuclear saturation density\cite{11,12,13}. The $^{12}_C$ ground state, while displaying a leading shell-model configuration, has nonetheless a pronounced component of three strongly correlated alpha clusters\cite{13}. Likewise, the $^8$Be nucleus features a prominent $\alpha\alpha$ clustering substructure. When a $\Lambda$ hyperon is added to the nuclear core, significant changes of the ground state configuration can be induced as discussed e.g. in refs.\cite{15,16}.

The aim of the present study is to investigate the interaction of a $\Lambda$ with such clustered core nuclei, based on a density-dependent Hamiltonian derived from chiral SU(3) dynamics. Apart from a central potential, this interaction features a characteristic surface (derivative) coupling which is sensitive to the detailed shape of the nuclear density profile. This density
distribution, and in particular its surface shape, is in turn influenced by the microscopic structure of the core wave function. The primary focus in this investigation is on $^{13}_ΛC$, while calculations are also performed for $^9_ΛBe$, starting from realistic dynamical cluster wave functions of their $^{12}_C$ and $^8_Be$ cores. Since the relative proportions of surface and volume change significantly in these two nuclei, they provide a testing ground for a detailed study of the interplay between bulk and surface terms in the chiral SU(3) based $Λ$-nuclear interaction.

For the lightest $Λα$ compound, the $^5_ΛHe$ hypernucleus, the question arises whether the present effective $Λ$-nuclear interaction and its pronounced derivative term apply also to this more compact system. Calculations of the $^5_ΛHe$ binding energy have so far frequently used a schematic Gaussian type density for the $α$ core, with its size parameter fixed to reproduce the observed charge radius of $^4He$. However, in a recent four-body calculation\textsuperscript{17} it was found that the strong $NN$ correlations in $^4He$ imply a significant deviation of the resulting density from a Gaussian form, with consequences also for the detailed density profile and its surface. It is therefore of interest to calculate, in addition, the binding energy of $^5_ΛHe$ using the wave function of $^4He$ generated by the four-body calculation.

This paper is organized as follows. Section 2 briefly summarizes the $Λ$-nuclear interaction derived from chiral SU(3) dynamics and employed in this work. In Sections 3 and 4, the model wave functions and density distributions for $^{13}_ΛC$ and $^9_ΛBe$ are introduced. Results and discussions are presented in Section 5 followed by a summary in Section 6.

\textbf{§2. The $Λ$-nuclear interaction from chiral SU(3) dynamics}

In previous work\textsuperscript{6,7} the interaction of a $Λ$ hyperon with a nuclear medium has been derived using the chiral SU(3) meson-baryon effective Lagrangian at next-to-leading order (NLO) as a starting point. An important element of this approach is the systematic treatment of kaon and two-pion exchange processes governing the in-medium $ΛN$ interactions. While direct single-pion exchange in the $ΛN$ system is isospin-forbidden, iterated pion exchange driven by the second-order tensor force and involving an intermediate $Σ$ hyperon provides the dominant mid-range attraction. Short-distance dynamics, not resolved at the relevant nuclear Fermi momentum scales, are encoded in a few contact terms with coefficients adjusted to reproduce bulk properties of hypernuclei. The remaining parameters of the theory are the known structure constants of the pseudoscalar meson octet (the pion and kaon decay constants in vacuum) and the axial vector coupling constants of the baryon octet (determined by nucleon and hyperon beta decays). A calculation of all NLO contributions has been performed at two-loop order for the $Λ$-nuclear (central and spin-orbit) mean fields, with full account of important Pauli-blocking effects in the nuclear medium\textsuperscript{9}. 
Within this framework, the self-energy of the $\Lambda$ interacting with the nuclear many-body system has been constructed. Its dependence on the nuclear density, $\rho_N = 2k_F^3/3\pi^2$, can be represented in the form of an expansion in powers of the nucleon Fermi momentum, $k_F$. In a subsequent step this self-energy has been translated into a $\Lambda$-nucleus potential $U_\Lambda(r)$ for applications to hypernuclei, using a derivative expansion in terms of the local density, $\rho_N(r)$, of the nuclear core. The result is as follows:

$$U_\Lambda(r) = U_c(r) - \frac{1}{2M_\Lambda} \nabla \cdot R(\rho_N) \nabla - D(\rho_N)(\nabla^2 \rho_N(r)),$$  \hspace{1cm} (2.1)

where the central part is written as an expansion in fractional powers of $\rho_N$:

$$U_c(r) = U_0 \frac{\rho_N(r)}{\rho_0} \times$$

$$\left[ 1 + 0.351 \left( \frac{\rho_N(r)}{\rho_0} \right)^{1/3} - 0.359 \left( \frac{\rho_N(r)}{\rho_0} \right)^{2/3} - 0.033 \left( \frac{\rho_N(r)}{\rho_0} \right) \right],$$  \hspace{1cm} (2.2)

with $U_0 = -30.56$ MeV, about half the strength of the single-particle potential for nucleons in nuclei. A slightly more attractive potential, $U_0 \simeq -35$ MeV, was found in the systematic analysis of hypernuclear binding energies using a similar approach. In the expansion Eq. (2.2) the core density $\rho_N$ is expressed in units of normal nuclear matter density, $\rho_0 = 0.16$ fm$^{-3}$. The leading term linear in $\rho_N$ is characteristic of the Hartree mean field approximation. Non-trivial terms beyond this linear density dependence arise from two-pion exchange dynamics in the medium with inclusion of Pauli-blocking effects.

The derivative terms in Eq. (2.1) reflect the momentum dependence of the $\Lambda$ self-energy in the nuclear medium at order $p^2$. In r-space these derivative terms represent non-local effects beyond the simplest local density approximation. Such terms are expected to become increasingly important as the proportion of surface to bulk increases in light nuclei. The piece proportional to $R(\rho_N)$ contributes to the (density dependent) effective mass of the $\Lambda$ hyperon. When combined with the kinetic energy piece of the (free) $\Lambda$ Hamiltonian, one has

$$H_{\Lambda,\text{kin}} = -\frac{1}{2M_\Lambda} \nabla \cdot [1 + R(\rho_N)] \nabla.$$

The explicit analytical expression for $R(\rho_N)$ can be found in the appendix. For the present purpose it is well approximated by the series:

$$R(\rho_N(r)) = -0.073 \left( \frac{\rho_N(r)}{\rho_0} \right) - 0.098 \left( \frac{\rho_N(r)}{\rho_0} \right)^{4/3} - 0.101 \left( \frac{\rho_N(r)}{\rho_0} \right)^{5/3} + 0.056 \left( \frac{\rho_N(r)}{\rho_0} \right)^2.$$

(2.4)
At typical densities, $\rho_N \sim \rho_0/2$, this gives a small but significant correction to the effective $A$ mass, $M_A^*(\rho_N)/M_A = [1 + R(\rho_N)]^{-1}$, of about 10%.

The third term in Eq. (2.1), the one proportional to the Laplacian acting on the density, is sensitive to the detailed surface profile of $\rho_N(r)$. The analytical expression for $D(\rho_N)$ can again be found in the appendix. From previous analyses of intermediate-mass and heavy hypernuclei, the surface coupling strength $D(\rho_N)$ turns out to be approximately constant, i.e. independent of density. Values of $D$ are in the range $D \sim (0.2 - 0.4) \text{ fm}^4 \sim (40 - 80) \text{ MeV} \cdot \text{fm}^5$ or smaller depending on the attractive strength of the central (local) $A$-nuclear mean field. While it is not possible to determine the surface coupling strength $D$ more accurately from heavier hypernuclei for which the fraction of surface-to-volume is small, it is of interest to analyse in more detail the interplay between the strengths $U_0$ and $D$ of the central and surface potentials, respectively, for lighter hypernuclei. This is the primary task of the present study which combines the input $A$-nuclear interaction with a nuclear core wave function constructed from a microscopic cluster model.

§3. Hypernuclear alpha cluster structure

Consider as a starting point a Hamiltonian for $N = Z$ nuclei with $4n$ nucleons ($n = 2$ for Be and $n = 3$ for C) plus a $A$ hyperon, composed of kinetic energies $-\frac{1}{2M} \nabla_i^2$ (with nucleon mass $M$) and $-\frac{1}{2M_A} \nabla_A^2$ (with $A$ hyperon mass $M_A$), the Coulomb potential $V_{ij}^C$, the effective nucleon-nucleon interaction $V_{ij}^{NN}$, and the $A$-nucleon ($AN$) interaction $V_{ij}^{AN}$:

$$H = -\sum_{i=1}^{4n} \frac{1}{2M} \nabla_i^2 - \frac{1}{2M_A} \nabla_A^2 - T_G + \sum_{i<j}^{4n} V_{ij}^C + \sum_{i<j}^{4n} V_{ij}^{NN} + \sum_{i=1}^{4n} V_{i}^{AN}.$$  \hspace{1cm} (3.1)

The center-of-mass kinetic energy $T_G$ is properly subtracted. We neglect the small $AN$ spin-orbit interaction. In the actual calculation the Volkov No.2 NN-force for $^8\text{Be}$ is used, and a slightly modified version of this force for $^{12}\text{C}$. The $AN$ interaction is provided by the phenomenological Nijmegen potential (model D) for the purpose of computing wave functions and density distributions for $^9\Lambda\text{Be}$ and $^{13}\Lambda\text{C}$. With this phenomenological input the calculation of the $A$ binding energy yields 6.69 MeV (exp.: 6.71 MeV) for $^9\Lambda\text{Be}$, and 11.68 MeV (exp.: 11.71 MeV) for $^{13}\Lambda\text{C}$, respectively. Ultimately these microscopic cluster calculations should be updated replacing phenomenological forces by new baryon-baryon interactions derived consistently from chiral dynamics.

In the present work the focus is on the interaction of the $A$ with the nuclear core based
on chiral SU(3) effective field theory, replacing

\[ H_A = -\frac{1}{2M_A} \vec{\nabla}^2 A + \sum_{i=1}^{4n} V_{i}^{AN} \rightarrow -\frac{1}{2M_A} \vec{\nabla} \cdot [1 + R(\rho_N)] \vec{\nabla} + U_c(\vec{r}) - D (\vec{\nabla}^2 \rho_N(\vec{r})) \],

where the gradients in the first term are understood to act on the \( \Lambda \) hyperon coordinate and the remaining expressions are as specified in the previous section. Taking expectation values of this new interaction with calculated wave functions, the aim is then to study in particular the role of the genuine surface term of Eq.(3.2). The effect of the derivative term in light hypernuclei is examined here for the first time. The importance of this term has been established in previous calculations for a \( \Lambda \) in slightly inhomogeneous nuclear matter and for hypernuclei ranging from \( {^{16}\Lambda}O \) to \( {^{208}\Lambda}Pb \).

§4. Derivation of the nuclear core density

The quantity of key importance is now the nuclear core density distribution \( \rho_N(\vec{r}) \) in the hypernucleus. The model wave function used here to calculate this density is the so-called Hyper-THSR (Tohsaki-Horiuchi-Schuck-Röpke) wave function. It is based on the deformed THSR wave function\(^ {11},24)\) describing nuclei with \( 4n \) nucleons as follows:

\[ \Phi^{\text{THSR}}_{\alpha n}(\vec{B}) \propto A \left\{ \prod_{i=1}^{n} \exp \left[ - \sum_{k=x,y,z} \frac{2}{B_k^2} (X_{ik} - X_{Gk})^2 \right] \phi(\alpha_i) \right\} , \]  

(4.1)

with the antisymmetrizer \( A \) operating on all nucleons and \( \phi(\alpha_i) \) the intrinsic wave function of the \( i \)-th \( \alpha \) cluster:

\[ \phi(\alpha_i) \propto \exp \left[ - \sum_{1 \leq k < \ell \leq 4} (\vec{r}_{i,k} - \vec{r}_{i,\ell})^2/(8b^2) \right] . \]  

(4.2)

In Eq. (4.1), \( \vec{X}_i \) denotes the center-of-mass coordinates of the \( i \)-th \( \alpha \) particle. The spurious total center-of-mass coordinate \( \vec{X}_C \) is properly eliminated. The center-of-mass motions of the \( n \) alpha clusters occupy the same deformed orbit, \( \exp[- \sum_{k=x,y,z} \frac{2}{B_k^2} (X_k - X_{Gk})^2] \), displaying a product arrangement of the \( n \alpha \) particles when \( \vec{B} \) is so large that the effect of the antisymmetrizer becomes negligible.\(^ {13} \) In the limiting case \( B_k \to \infty \) \((k = x, y, z)\), this wave function corresponds to a free \( n\alpha \) state in which the \( \alpha \) particles are uncorrelated. In the symmetric limit \( B_x = B_y = B_z \to b \) the normalized THSR wave function coincides with the shell model Slater determinant.

The wave function, Eqs.(4.1,4.2), succeeded in describing light \( N = Z \) nuclei such as \( {}^8\text{Be}, {}^{12}\text{C}, {}^{16}\text{O} \) and \( {}^{20}\text{Ne} \). In particular the first excited \( 0^+ \) state in \( {}^{12}\text{C} \) is described correctly by this wave function, with a loosely coupled structure of the three \( \alpha \) particles reminiscent of
a gas occupying the lowest $0S$ orbit of a mean-field potential for the $\alpha$ particle.[12] Apart from this example, the THSR wave function is known to give a valid description of the ground states of those nuclei since their shell model configurations are properly taken into account due to the antisymmetrization of nucleons, together with the $\alpha$-like ground state correlations. In particular, for the ground state rotational bands of $^{12}\text{C}$ and $^{20}\text{Ne}$, the THSR wave functions[13] give large squared overlap (close to 100 %) with the corresponding microscopic cluster model wave functions such as RGM (Resonating Group Method) and GCM (Generator Coordinate Method).[26]

The Hyper-THSR wave function describing the $4\pi$ hypernuclei is then introduced as follows:

\[
\Phi_{H^{-\text{THSR}}}^{n\alpha-L}(B, \kappa) = \Phi_{n\alpha}^{\text{THSR}}(B) \varphi_L(\kappa), \quad (4.3)
\]

where the $\Lambda$ particle is assumed to couple to the $n\alpha$ core nucleus in an $S$ wave. This approximation is supported by previous calculations[27] for $^{13}\Lambda\text{C}$ and $^{9}\Lambda\text{Be}$ where it was found that the ground states of these hypernuclei are dominated (to 94.4 % for $^{9}\Lambda\text{Be}$ and 98.8 % for $^{13}\Lambda\text{C}$) by configurations with the $\Lambda$ in an s-orbit coupled to a nuclear $0^+$ core. This also underlines the justification for an approximate treatment of such systems as effective two-body problems.

The radial part of the wave function Eq. (4.3) is expanded in Gaussian basis functions, $\varphi_L(\kappa) = (\pi/2\kappa)^{-3/4} \exp(-\kappa r^2)$. In the practical calculations we use an axially-symmetric function in Eq. (4.1) with $B_z = B_y \equiv B_\perp \neq B_z$. In this way the intrinsic deformation of the core wave function is taken into account. The parameter $b$ in Eq. (4.2) is fixed to almost the same size as the one of the $\alpha$ particle in free space. The total wave function for quantum states of the $n\alpha + \Lambda$ nucleus can then be expressed as the superposition of the angular-momentum projected wave function of Eq. (4.3) with different values of the parameters, $B_\perp, B_z$ and $\kappa$, as follows:

\[
\Psi_{n\alpha-L}^{H^{-\text{THSR}}}(J^+_\pi) = \sum_{B_\perp, B_z, \kappa} f_\lambda(B_\perp, B_z, \kappa) \hat{P}^J \Phi_{n\alpha-L}^{H^{-\text{THSR}}}(B_\perp, B_z, \kappa), \quad (4.4)
\]

where $\hat{P}^J$ is the angular-momentum projection operator onto the $J^\pi$ subspace with positive parity, $\pi = +$ (note that the Hyper-THSR wave function, Eq.(4.3), has positive intrinsic parity). The coefficients $f_\lambda(B_\perp, B_z, \kappa)$ are then determined by solving the following Griffin-Hill-Wheeler equation:[28]

\[
\sum_{B'_\perp, B'_z, \kappa'} \langle \Phi_{n\alpha-L}^{H^{-\text{THSR}}}(B_\perp, B_z, \kappa) | \hat{H} - E^\lambda | \hat{P}^J \Phi_{n\alpha-L}^{H^{-\text{THSR}}}(B'_\perp, B'_z, \kappa') \rangle f_\lambda(B'_\perp, B'_z, \kappa') = 0. \quad (4.5)
\]

Using the wave function (4.4) the (radial) nucleon density distribution of the core, averaged
over angular dependence, is introduced as:

\[
\rho_N(r) = \langle \psi_{n\alpha - A}^{H\text{THSR}}(J^+_N) | \frac{1}{4\pi r^2} \sum_{i=1}^{4n} \delta(r - |r_i - X_C|) | \psi_{n\alpha - A}^{H\text{THSR}}(J^+_N) \rangle,
\]

(4.6)

where \( X_C = (r_1 + \cdots + r_{4n})/(4n) \) is the center-of-mass coordinate of the \( n\alpha \) core nucleus. Note again that the wave functions entering Eq. (4.6) are angular-momentum projected. The nuclear core density \( \rho_N \) is normalized as usual to the total number of nucleons, \( \int d^3r \rho_N(r) = 4n \).

Fig. 1. Nucleon density distribution (multiplied by \( r^2 \)) of \(^{13}_A\Lambda\text{C}\) defined by Eq. (4.6) (solid curve). For comparison, the density of the \(^{12}_A\text{C}\) nucleus is also shown by the dotted curve.

In Figs. [1] and [2] we show \( r^2 \) times the density distributions \( \rho_N \) of the nuclear core for the ground states of \(^{13}_A\text{C}\) and \(^9_4\text{Be}\), i.e. for \( n = 3, J = 0 \) and \( n = 2, J = 0 \) in Eq. (4.6), respectively. In both figures, the nucleon density distributions of the ground states of \(^8\text{Be}\) and \(^{12}_A\text{C}\) are shown for comparison. They are also obtained by solving the Griffin-Hill-Wheeler equation based on the 3\( \alpha \) or 2\( \alpha \) THSR wave functions. The calculated \(^{12}_A\text{C}\) charge density actually compares quite well with the empirical density deduced from electron scattering data\(^{29}\). For \(^9_4\text{Be}\) one observes that a significant spatial shrinkage is induced by injecting the \( \Lambda \) particle into \(^8\text{Be}\). The calculated root mean square (rms) radius for the \(^8\text{Be}\) core in \(^9_4\text{Be}\) is 2.35 fm, almost 20% smaller than that for the \(^8\text{Be}\) nucleus (2.87 fm) which has a pronounced 2\( \alpha \) cluster structure. This shrinkage effect is far less significant for \(^{13}_A\text{C}\): the density of the \(^{12}_A\text{C}\) core in the hypernucleus is close to the normal density of the \(^{12}_A\text{C}\) nucleus. The calculated rms radius for the \(^{12}_A\text{C}\) core in \(^{13}_A\text{C}\) is 2.32 fm, not much different from the rms radius (2.40
Fig. 2. Nucleon density distribution (multiplied by $r^2$) of $^9\Lambda$Be defined by Eq. (4.6) (solid curve). For comparison, the density of $^8\text{Be}$ is also shown by the dotted curve.

(fm) of the $^{12}\text{C}$ nucleus, but this difference is nonetheless of some significance concerning the effect of the $\nabla^2 \rho_N$ term.

§5. Results and Discussion

Given the density distributions introduced in the previous section, we can now focus on the detailed investigation of the $\Lambda$-nuclear derivative coupling terms. Our primary example is $^{13}\Lambda\text{C}$ for which a direct comparison with the mean-field calculation of Ref. 7) is at hand and the derivative expansion with a local density $\rho_N$ is expected to work. For even lighter nuclei one expects larger uncertainties as the limits of applicability of this expansion may be encountered.

Consider the expectation values of the $\Lambda$ Hamiltonian in Eq. (3.2) taken with the correlated THSR wave functions for $^{13}\Lambda\text{C}$ and $^9\Lambda\text{Be}$, using the $\Lambda$-nuclear interaction in Eq. (2.1) derived from chiral SU(3) effective field theory. We employ the nuclear core densities, $\rho_N(r)$, determined relative to the center of mass of the $^{12}\text{C}$ and $^8\text{Be}$ cores in $^{13}\Lambda\text{C}$ and $^9\Lambda\text{Be}$, respectively. As an exploratory sideline, the $^5\Lambda\text{He}$ prototype hypernucleus will also be discussed.

We first study the case of $^{13}\Lambda\text{C}$. Its ground state is considered to be a compact shell-model-like configuration for which the previously introduced $\Lambda$-core interaction is supposed to work well. It has been pointed out\(^ {30, 20}\) that the density distribution of a compact state such as $^{13}\Lambda\text{C}$ does not experience a dynamical contraction when adding the $\Lambda$ particle to the nuclear
core. This feature is in fact realized in the present calculation, as seen in Fig. 1.

Let us now examine the $\Lambda$ interaction with the $^{12}$C core in more detail. For example, with only the central piece $U_c$ of the interaction and choosing $U_0 = -35$ MeV, the resulting calculated $\Lambda$ binding energy in $^{13}\Lambda$C is 15.3 MeV. In the next step, including the second term of Eq. (2.1) that contributes to the effective mass $M^*_\Lambda$, the kinetic energy is reduced by slightly less than 10% and so the binding increases to $B_\Lambda = 16.7$ MeV. This overbinding is then compensated by the (repulsive) surface term proportional to $\nabla^2 \rho_N$. Choosing a surface coupling constant $D = 54$ MeV·fm$^5$ brings $B_\Lambda$ back to its empirical value, 11.7 MeV.

Fig. 3 demonstrates a subtle balance between the depth of the central potential, $U_0$, and the surface coupling strength $D$. Pairs of values $(U_0, D)$ that reproduce the empirical $B_\Lambda = 11.71$ MeV are correlated linearly as shown in this figure. For example, the combination of $U_0 = -32$ MeV with a surface term $D = 25$ MeV·fm$^5$ fits the empirical $^{13}\Lambda$C binding energy of the $\Lambda$ equally well. From Ref. [7], we recall that, in the chiral $SU(3)$ effective field theory approach, the surface coupling strength $D(\rho_N)$ deduced from best-fit mean-field results for a variety of heavier hypernuclei ranges between about 40 and 80 MeV fm$^5$. The present analysis is consistent with such values if the strength of the central piece $U_c$ lies in the range $U_0 = -33$ to $-35$ MeV. This result is based on the nuclear core density distribution Eq. (4.6) as it emerges from the full calculation of the hypernuclear THSR wave function. For comparison, Fig. 3 also shows a study in which the nuclear $A = 12$ core density in $^{13}\Lambda$C hypernucleus is simply replaced by the density distribution $\rho_N$ of the free, isolated $^{12}$C nucleus, either calculated using the THSR cluster model or using a parameterization of the empirical $^{12}$C charge density. In this case the suggested values of the surface coupling $D$ would be reduced. One observes that the relatively small difference seen in Fig. 1 between the density distribution of $^{12}$C and the nuclear core in $^{13}\Lambda$C has nonetheless a pronounced effect on the derivative term proportional to $\nabla^2 \rho_N$.

The $r$-dependence of the $D\nabla^2 \rho_N(r)$ term resulting from the $^{13}\Lambda$C core density is displayed in Fig. 4 together with the one from the calculated $^{12}$C density and in comparison with the one derived from the empirical $^{12}$C density distribution. The sensitivity of this derivative term (multiplied by $r^2$ as it appears in the relevant integral) with respect to the detailed behaviour of $\rho_N(r)$ is evident. Note that when combined with the square of the $\Lambda$ wave function in $^{13}\Lambda$C, the weight in the relevant matrix element is dominantly in the range $r \sim 1 - 3$ fm, resulting in a net repulsive correction to the $\Lambda$ binding energy. It is also instructive to examine in more detail the effect of this derivative term on the $\Lambda$ binding energy with varying strength parameter $D$ as displayed in Table I.

Next, consider $^{9}\Lambda$Be. The linear relationship between $U_0$ and $D$ is also observed in this case as shown in Fig. 5, but with a displacement towards smaller values of $D$ as compared
Fig. 3. Correlation between the $\Lambda$-nuclear potential depth $U_0$ and the strength $D$ of the surface term for $^{13}_{\Lambda}$C. The straight lines connect points that reproduce the empirical $\Lambda$ binding energy $B_\Lambda = 11.71$ MeV. The following cases using different $A = 12$ nuclear core densities $\rho_N$ are displayed in comparison: full hypernucleus calculation using Eq. (4.6) (THSR $^{13}_{\Lambda}$C); calculated $^{12}$C core density using THSR cluster wave function (THSR $^{12}$C); empirical $^{12}$C core density deduced from electron scattering data (dashed curve).

Fig. 4. Derivative term $D\nabla^2 \rho_N(r)$ multiplied by $r^2$ (with $D = 50$ MeV·fm$^5$). Solid curve: full calculation of $^{13}_{\Lambda}$C using Eq. (4.6) (THSR $^{13}_{\Lambda}$C); dashed curve: calculated $^{12}$C core density using THSR cluster wave function (THSR $^{12}$C); dash-dot curve: using empirical $^{12}$C core density deduced from electron scattering data.

to $^{13}_{\Lambda}$C. The nuclear core density of $^9_{\Lambda}$Be with its compressed distribution has a radius close to that of the core in $^{13}_{\Lambda}$C. Now a central potential depth $U_0 = -33.3$ MeV alone would give $B_\Lambda = 6.24$ MeV, and the empirical $B_\Lambda = 6.71$ MeV is reached already by just adding the
Table I. Binding energies (in MeV) of $^{13}_A\Lambda$C with varying derivative coupling strength $D$ (in MeV·fm$^5$) and central potential $U_0$ ranging between -32 and -35 MeV.

| $^{13}_A\Lambda$C | $U_0$ | $D = 0$ | $D = 30$ | $D = 50$ |
|-------------------|-------|---------|-----------|-----------|
| -32               | 14.15 | 11.29   | 9.47      |           |
| -33               | 15.00 | 12.10   | 10.44     |           |
| -34               | 15.85 | 12.92   | 11.23     |           |
| -35               | 16.70 | 13.75   | 12.04     |           |

Fig. 5. Correlation between the $\Lambda$-nuclear potential depth $U_0$ and the strength $D$ of the surface term for $^9_\Lambda$Be. The straight line connects points that reproduce the empirical $\Lambda$ binding energy, $B_\Lambda = 6.71$ MeV, using the THSR cluster model wave function as described in the text.

As a final point a brief discussion of the $^5_\Lambda$He hypernucleus is also instructive. This is...
a prototype system featuring the interaction of the $\Lambda$ hyperon with the $\alpha$ particle core. In principle, an *ab-initio* calculation of the $^5\Lambda$He binding energy requires solving a five-body problem. Given the chiral SU(3) effective interaction between the $\Lambda$ and the core, this reduces to a two-body problem with the $^4$He core density distribution as input, assuming that this compact distribution does not change much in the presence of the hyperon. Here we use the density resulting from the microscopic four-body calculation\(^{17}\) that reproduces the $^4$He matter radius, $\langle r^2 \rangle^{1/2}_m = [\langle r^2 \rangle_{ch} - \langle r^2 \rangle_p]^{1/2} = 1.45$ fm, derived from the empirical $^4$He charge radius $\langle r^2 \rangle^{1/2}_{ch} = 1.68$ fm together with the proton charge radius $\langle r^2 \rangle^{1/2}_p = 0.85$ fm.

As mentioned, this density profile differs from a simple Gaussian form that reproduces the same radius (see Fig.6). Calculating the $\Lambda$ binding energy and choosing again the potential parameters, $U_0$ and $D$, such as to reproduce the empirical $B_{\Lambda} = 3.12$ MeV for $^5\Lambda$He, one finds once more a linear relationship between $U_0$ and $D$ (see Fig.7). Using $U_0 = -35$ MeV, a surface coupling $D \simeq 23 \text{MeV} \cdot \text{fm}^5$ would be suggested. The stronger correlation between $U_0$ and $D$ in the $^5\Lambda$He case reflects the more compact $\alpha$ particle core in this light hypernucleus with its more pronounced surface gradient. The sensitivity with respect to the $\nabla^2 \rho_N$ term becomes apparent by comparison with a standard Gaussian density. Of course, for such light and compact hypernuclei, one reaches the limit of applicability for the gradient expansion Eq. (2-1) with a local density $\rho_N(r)$, and higher powers of gradients are expected to become non-negligible.
Fig. 7. Correlation between the $\Lambda$-nuclear potential depth $U_0$ and the strength $D$ of the surface term for $^{5}_A\text{He}$. The $^4\text{He}$ core density (see Fig. 6) is either of Gaussian form or taken from a microscopic four-body calculation. The straight lines connect points that reproduce the empirical $\Lambda$ binding energy, $B_\Lambda = 3.12$ MeV.

§6. Summary

The present analysis of light hypernuclei using microscopic cluster model wave functions points to a sensitive interplay between the Hartree-type central $\Lambda$-nuclear potential and terms involving derivatives of the nuclear core density, $\rho_N$, in the hypernucleus. These derivative terms have their well-founded origin in the in-medium $\Lambda$ self-energy derived from chiral SU(3) meson-baryon effective field theory. A part proportional to $\vec{\nabla} \cdot R(\rho_N) \vec{\nabla}$ effectively increases the mass and reduces the kinetic energy of the $\Lambda$ in the hypernucleus, thereby increasing the $\Lambda$ binding energy. A repulsive surface term proportional to $\vec{\nabla}^2 \rho_N$ counteracts this tendency. A systematic linear correlation is found between the strengths of the central attraction and the surface repulsion. The results for $^{13}_A\text{C}$ turn out to be consistent with earlier mean-field calculations using similar input. For the lightest hypernuclei studied in this work ($^{9}_A\text{Be}; ^{5}_A\text{He}$) the linear correlation just mentioned is also found but with a weaker surface term. For $^{9}_A\text{Be}$ the loosely bound $2\alpha$ structure of the $^8\text{Be}$ core nucleus, although compressed by the presence of the $\Lambda$ hyperon, makes this system special. The $^{5}_A\text{He}$ case is presumably at the borderline of applicability of the present approach, as explained.

In summary, a significant result of the present study is that an independent calculation of $^{13}_A\text{C}$ using a microscopic wave function confirms the importance of the $\Lambda$-nuclear derivative coupling terms predicted by in-medium chiral SU(3) effective field theory.
Appendix

In this appendix we provide the analytical expressions for the strength functions $R(\rho_N)$ and $D(\rho_N)$ as obtained by evaluating the momentum-dependent in-medium $\Lambda$ self-energy in SU(3) chiral effective field theory up to two-loop order.

The one-kaon exchange Fock diagram gives:

\[ M_A^{-1} R(\rho_N) = \frac{(D + 3F)^2}{(6\pi f_\pi)^2} \frac{2m_K^2 k_F^2}{(m_K^2 + k_F^2)^2}, \]

(6.1)

with the SU(3) axial vector couplings $D = 0.84$, $F = 0.46$ and the pion decay constant $f_\pi = 92.4$ MeV. The Fermi momentum $k_F$ is related to the nuclear density by $\rho_N = \frac{2k_F^3}{3\pi^2}$.

The iterated pion-exchange diagram with a $\Sigma$ hyperon in the intermediate state gives:

\[ M_A^{-1} R(\rho_N) = \frac{D g_A^2 M_B m_\pi^2}{24\pi^3 f_\pi^4} \left\{ -4 \arctan \frac{\sqrt{u}}{2 + \sqrt{4\delta - u}} + \frac{\sqrt{u}}{[u + (\delta - 1)^2]^2} \left[ 2(1 - \delta)^3 \right. \\
+ u(3 - \delta) - \frac{2\delta^3(1 + \delta)^2}{\sqrt{4\delta - u}} + \sqrt{4\delta - u} \left( \frac{1}{2} (u^2 - 3u + \delta^4 - 5\delta^2) \right. \\
+ u\delta^2 + 2\delta^3 + 3\delta - 1) \right\}, \]

(6.2)

with the dimensionless variables $u = \left( k_F/m_\pi \right)^2$ and $\delta = \left( \Delta/m_\pi \right)^2$, where the small scale $\Delta = 285$ MeV is related to the $\Sigma\Lambda$ mass splitting by $M_\Sigma - M_\Lambda = \Delta^2/M_B$. Furthermore, $g_A = D + F = 1.3$ is the nucleon axial vector coupling constant and $M_B = 1047$ MeV denotes an average baryon mass.

The same (long-range) two-pion exchange mechanism gives for surface coupling strength:

\[ \rho_N D(\rho_N) = \frac{D^2 g_A^2 M_B m_\pi^2}{96\pi^3 f_\pi^4} \left\{ 4(3 - 4\delta + 2u) \arctan \frac{\sqrt{u}}{2 + \sqrt{4\delta - u}} + \frac{\sqrt{u}}{[u + (\delta - 1)^2]^2} \right. \\
\times \left. \left[ - \frac{15u^2}{2} + \frac{u}{6} (\delta^3 - 90\delta^2 + 180\delta - 77) + 2(3 - 4\delta)(\delta - 1)^3 - \frac{\delta^3(1 + \delta)^2}{\sqrt{4\delta - u}} \right. \\
+ \sqrt{4\delta - u} \left( \frac{25u^2}{6} + \frac{u}{12} (99\delta^2 - 188\delta + 78) + \frac{\delta^2}{4} (17\delta^2 - 58\delta + 85) - 13\delta + 3) \right\}, \]

(6.3)

In addition there are (small) Pauli-blocking corrections which reduce the contributions $R(\rho_N)$ and $D(\rho_N)$ with increasing density.

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