Homogeneous Plane Waves

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Abstract

Motivated by the search for potentially exactly solvable time-dependent string backgrounds, we determine all homogeneous plane wave (HPW) metrics in any dimension and find one family of HPWs with geodesically complete metrics and another with metrics containing null singularities. The former generalises both the Cahen-Wallach (constant $A_{ij}$) metrics to time-dependent HPWs, $A_{ij}(x^+)$, and the Ozsvath-Schücker anti-Mach metric to arbitrary dimensions. The latter is a generalisation of the known homogeneous metrics with $A_{ij} \sim 1/(x^+)^2$ to a more complicated time-dependence. We display these metrics in various coordinate systems, show how to embed them into string theory, and determine the isometry algebra of a general HPW and the associated conserved charges. We review the Lewis-Riesenfeld theory of invariants of time-dependent harmonic oscillators and show how it can be deduced from the geometry of plane waves. We advocate the use of the invariant associated with the extra (timelike) isometry of HPWs for lightcone quantisation, and illustrate the procedure in some examples.

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1 Introduction

It has long been recognised [1, 2] that gravitational wave metrics provide potentially exact and exactly solvable string theory backgrounds [1]. More recently the discovery of the maximally supersymmetric BFHP [10] plane wave solution of IIB string theory, and the recognition that string theory in this RR background is also exactly solvable [11], has led to renewed interest in this subject, in particular with the realisation that the BFHP solution arises [12] as the Penrose-Gueven limit [13, 14, 15] of $AdS_5 \times S^5$, and that this gives rise to a novel explicit form of the AdS/CFT correspondence [16], the BMN plane wave / CFT correspondence.

The metric of a plane wave in $d$ dimensions is

$$ds^2 = 2dx^+ dx^- + A_{ij}(x^+) z^i z^j(dx^+)^2 + dz^2,$$

where $A_{ij}(x^+)$ is a symmetric matrix and $z^i$ label the flat transverse coordinates. In the lightcone gauge, the particle or string action is quadratic in the $z^i$ [2], and hence the theory is, at least in principle, exactly solvable. In particular, the dynamics of relativistic particles is that of an harmonic oscillator with (possibly time-dependent) frequencies given by $A_{ij}(x^+)$. In practice, however, string theory on generic time-dependent plane wave backgrounds is difficult to understand, even in the lightcone gauge, and the emphasis has been on studying metrics with a constant $A_{ij}$ (see e.g. [17, 18], but also e.g. [19, 20, 21] for some notable early exceptions).

Nevertheless, time-dependent plane waves are of considerable interest, as potentially exactly solvable time-dependent string backgrounds, and because they arise as Penrose limits of various relevant supergravity configurations [13, 22, 23, 24, 25, 26]. It is therefore natural to look for plane wave backgrounds leading to a dynamics with complexity intermediate between that of constant $A_{ij}$ and that of generic time-dependent plane waves. To see what might characterise such backgrounds, recall that generically plane waves have a $(2d - 3)$-dimensional Heisenberg algebra

$$[X^{(k)}, X^{* (l)}] = -\delta_{kl} Z$$

of isometries (generated by $Z = \partial_-$ and less manifest transverse translations and null rotations). This isometry algebra acts transitively on the null hyperplanes $x^+ = \text{const.}$ Plane Waves with constant $A_{ij}$, on the other hand, are Lorentzian symmetric (Cahen-Wallach) spaces [27, 28], and as such have many more isometries. In particular, the additional Killing vector $X = \partial_+$ extends the Heisenberg algebra to the oscillator algebra, with $X$ playing the role of the Hamiltonian. In the lightcone gauge, $X$ indeed

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1 See e.g. [3, 4, 5, 6, 7, 8] and [9] for a review of exact solutions of string theory.
becomes the non-relativistic oscillator Hamiltonian, and it is this intimate relation between spacetime symmetries and worldsheet dynamics that makes string theory in these backgrounds work so beautifully \[11, 16, 17\] and that lies at the heart of the BMN correspondence \[16, 29\].

The existence of this additional Killing vector $X$, generating translations in the $x^+$-directions, renders these plane waves homogeneous. This motivates us to look for other Lorentzian homogeneous (but not Lorentzian symmetric) plane waves, i.e. plane waves with at least one additional Killing vector $X$ with a non-zero $x^+$-component. One might hope that the existence of the associated conserved charge, related to the lightcone Hamiltonian, and the corresponding extended isometry algebra, simplify and enrich the quantisation of string theory also in such backgrounds.

A special class of homogeneous plane wave with $A_{ij} \neq \text{const.}$ is given by (1.1) with

$$A_{ij}(x^+) = \frac{B_{ij}}{(x^+)^2}$$

which obviously has an additional scaling symmetry generated by $X = x^+ \partial_+ - x^- \partial_-$. This kind of plane wave metric arises e.g. as the Penrose limit of the fundamental string soliton \[13\], the near horizon limit of dilatonic $p$-branes \[26\] and the spatially flat FRW metric \[15\]. String theory in this background (with $B_{ij} \sim \delta_{ij}$) has very recently been analysed in detail in \[30\]. It has been shown to be exactly solvable and displays a wealth of interesting phenomena, related to the presence of null singularities and the additional isometry.

Our first aim in this paper is to obtain a complete classification of all homogeneous plane wave (HPW) metrics. From a direct analysis of the Killing equations, we find that there are two families of solutions, one generalising the Cahen-Wallach metrics with constant $A_{ij}$, the other generalising the metrics of type (1.3). The metrics in both families are parametrised by a constant symmetric matrix $(A_0)_{ij}$ and a constant antisymmetric matrix $f_{ij}$. Metrics in the first family are of the form

$$ds^2 = 2dx^+dx^- + (e^{x^+f}A_0e^{-x^+f})_{ij}z^iz^j(dx^+)^2 + d\vec{z}^2.$$  \hspace{1cm} (1.4)

They are completely smooth and geodesically complete. They are simultaneously generalisations of the geodesically complete Cahen-Wallach metrics to time-dependent HPWs and generalisations of the Ozsvath-Schücking anti-Mach metric \[31, 32\] to arbitrary dimensions.

Metrics in the second family are of the form

$$ds^2 = 2dx^+dx^- + (e^{f \log x^+}A_0e^{-f \log x^+})_{ij}z^iz^j(dx^+)^2 + d\vec{z}^2.$$  \hspace{1cm} (1.5)
They have null singularities at \( x^+ = 0 \) and are not geodesically complete. They generalise the metrics of the type (1.3) to which they reduce when \( f_{ij} = 0 \). The behaviour of the metric near the singularity differs from that in (1.3). Another novel feature of both types of metrics is that they are essentially non-diagonal, i.e. \( A_{ij}(x^+) \) cannot be diagonalised by a coordinate transformation preserving the general form (1.1) of the metric.

We then study various elementary properties of these metrics. We describe in some detail the emergence of the Heisenberg isometry algebra from the harmonic oscillator equation, as the explicit construction and parametrisation of the generators will be helpful at various other points. We then determine the isometry algebra of a general HPW and the corresponding conserved charges for a particle moving in this background, show how these metrics can be embedded into supergravity (this is straightforward for any plane wave metric), and display the metrics in some other coordinate systems.

Our second main aim is to study the quantisation of particles (and ultimately strings, but here we restrict ourselves to the point-particle case) in these backgrounds, to study how this interacts with the geometry of plane waves, and in particular to understand and subsequently exploit the simplifying features of HPWs. We already mentioned that lightcone quantisation of particles or strings in plane wave backgrounds gives rise to time-dependent harmonic oscillators. In general one can quantise these systems using the theory of invariants developed by Lewis and Riesenfeld [33, 34]. This theory has already been employed in the present context in [24, 30].

Since this harmonic oscillator equation describes both the geodesics and the isometries of a plane wave background, embedding the problem of a time-dependent harmonic oscillator into the plane wave setting equips it with a rich geometric structure, and links the dynamics of the harmonic oscillator to the conserved charges associated with the Killing vectors. We will show that this provides a natural geometric explanation of the entire Lewis-Riesenfeld procedure (partially anticipated already in [24, 30]).

Moreover, in the case of HPWs there is a preferred invariant, namely the conserved charge associated to the Killing vector \( X \) and related to the lightcone Hamiltonian. One of our motivations for looking (for and) at homogeneous plane wave space-times was precisely the belief that the invariant \( I_X \) associated with the extra Killing vector \( X \) would lead us to a natural quantisation of particles (and subsequently string theory) on these backgrounds. To illustrate this, we consider two examples. One is the particle moving in the background (1.3). This complements rather than duplicates the results of [30] as we naturally end up in the range of frequencies not covered there (\( k > 1/4 \) in the notation of [30]). The other example is the anti-Mach metric (2.54,3.68) [31]. In both cases we will see that using the invariant \( I_X \) leads to a simple and natural quantisation
of the system.

In an Appendix, we discuss the relation between plane waves in Brinkmann coordinates (as in (1.1)) and Rosen coordinates, as this also turns out to involve the harmonic oscillator equation in a nice way.

Invariably, in our discussion of the simplest examples of HPWs, namely Cahen-Wallach spaces and the metrics (1.3), there is some overlap with the discussion in [30]. Since our main interest is in the generalisation of these metrics, we have tried to keep the overlap to the absolute minimum necessary for our purposes. Some recent papers dealing with other geometrical and general relativity aspects of plane waves are [35, 36, 37, 38, 39].

2 Classification of Homogeneous Plane Waves

The classification of pp-wave spacetimes according to their isometries goes back (at least) to the classic work of Jordan, Ehlers and Kundt [40] (see [41] for a detailed exposition and [42, 43] for a summary of the results), who classified all vacuum pp-waves in $d = 4$ and exhibited them in ‘normal form’. However, the methods used in [40, 41] are tailored to four dimensions and neither they nor the results immediately lend themselves to a higher-dimensional generalisation.

As we are not aware of any such classification in higher dimensions, in this section we analyse the Killing equations of the metric (1.1). Our aim is to find all plane wave metrics which are homogeneous in the sense that they admit one additional Killing vector $X$ with a non-zero $x^+$-component. We will refer to these metrics simply as homogeneous plane waves (or HPWs for short).

2.1 Preliminary Considerations

The metric of a plane wave in Brinkmann coordinates is

$$ds^2 = 2dx^+dx^- + A_{ij}(x^+)z^iz^j(dx^+)^2 + d^2z^2,$$

where $z^i, i = 1, \ldots, d - 2$ label the transverse coordinates, the ‘polarisation tensor’ $A_{ij}(x^+)$ is a (generically $x^+$-dependent) symmetric matrix and $d^2z^2$ is the flat metric on the transverse space.

This metric is characterised by the existence of a covariantly constant, hence Killing, null vector $Z$, $Z = \partial_{x^-}$ and a planar symmetry in the transverse directions. This transverse extension to impulsive gravitational waves [43] is non-trivial and may well be of interest in the string theory context.
planar symmetry is somewhat hidden in Brinkmann coordinates but manifest in Rosen coordinates in which the metric takes the form

\[ ds^2 = 2dudv + C_{ij}(u)dy^idy^j, \]  

where \( y^i \) label the transverse coordinates, and the symmetric matrix \( C_{ij}(u) \) is positive definite and non-degenerate in the range of validity of the Rosen coordinates. The relation between these two coordinate systems is discussed in detail in the Appendix.

As already mentioned in the Introduction, generically the plane wave metric has \( 2d - 3 \) linearly independent Killing vectors \( X^{(k)}, X^{*{(k)}}, Z \). In a suitable basis these generate the Heisenberg algebra

\[ [X^{(k)}, X^{*(l)}] = -\delta_{kl}Z \]  

with central element \( Z \). In Rosen coordinates, half (plus one) of these symmetries are manifest (and independent of \( C_{ij} \)), and the remaining ones can be expressed in simple closed form in terms of \( T C^{ij}(u) \) (see e.g. [15]). In Brinkmann coordinates, \( Z = \partial_x^+ \equiv \partial_+ \), but none of the other symmetries are manifest.\(^3\) In that case the \( X^{(k)}, X^{*{(k)}} \) are constructed from the \( 2(d-2) \) linearly independent solutions of the differential equation

\[ \ddot{b}_i^{(f)}(x^+) = A_{ij}(x^+)b_j^{(f)}(x^+) \]  

As this construction and this oscillator equation are central to our discussion, we will recall and rederive this result in section 3.2.

This isometry algebra acts transitively on the null hyperplanes \( u = \text{const.} \) or \( x^+ = \text{const.} \), with a simply transitive Abelian translation subalgebra generated e.g. by \( \{X^{(k)}, Z\} \) or \( \{X^{*{(k)}}, Z\} \). For special choices of \( C_{ij}(u) \) or \( A_{ij}(x^+) \), the plane wave metric will have additional Killing vectors. These could obviously arise from ‘internal’ symmetries of \( C_{ij} \) or \( A_{ij} \), giving rise to more Killing vectors in the transverse directions. For example, if \( A_{ij}(x^+) = A(x^+)\delta_{ij} \), as in the BFHP solution, the plane wave metric will have an additional \( SO(d-2) \) isometry.

Of more interest to us is that for particular non-trivial profiles \( (x^+\text{-dependence}) \) of the plane wave, there can be isometries with a \( \partial_u \) or \( \partial_+ \) component. In these cases the plane wave metric will be homogeneous (away from the fixed points of this additional Killing vector), and we are interested in determining the most general form of a plane wave metric homogeneous in this sense. Such additional symmetries are occasionally somewhat more manifest in Brinkmann than in Rosen coordinates.

\(^3\)There are of course also intermediate possibilities - see e.g. [14, 15] for applications to supergravity and the BMN correspondence.
2.2 Basic Examples

1. The most obvious examples of HPW metrics in Brinkmann coordinates are

\[ ds^2 = 2dx^+ dx^- + A_{ij} z^i z^j (dx^+)^2 + d\vec{z}^2 , \quad (2.5) \]

where \( A_{ij} \) is constant (independent of \( x^+ \)). These evidently have the additional Killing vector \( X = \partial_+ \). In fact these metrics are not only homogeneous but are actually Lorentzian symmetric (Cahen-Wallach [27]) metrics (for a nice exposition see [28]). Since \( A_{ij} \) is \( x^+ \)-independent, it can be diagonalised by an \( x^+ \)-independent orthogonal transformation acting on the \( z^i \). Moreover, the overall scale of \( A_{ij} \) can be changed, \( A_{ij} \rightarrow \mu^2 A_{ij} \), by the coordinate transformation

\[ (x^+, x^-, z^i) \rightarrow (\mu x^+, \mu^{-1} x^-, z^i) . \quad (2.6) \]

Thus these metrics are classified by the eigenvalues of \( A_{ij} \) up to an overall scale and permutations of the eigenvalues.

In Rosen coordinates, metrics which lead to a constant (and hence without loss of generality also diagonal) \( A_{ij} \),

\[ A_{ij} = A_i \delta_{ij} . \quad (2.7) \]

have

\[ C_{ij}(u) = a_i(u)^2 \delta_{ij} \quad (2.8) \]

with

\[ A_i = +\alpha_i^2 : \quad a_i(u) = b_i \cosh \alpha_i u + c_i \sinh \alpha_i u \]

\[ = b_i' e^{\alpha_i u} + c_i' e^{-\alpha_i u} \]

\[ A_i = -\alpha_i^2 : \quad a_i(u) = b_i \cos \alpha_i u + c_i \sin \alpha_i u \quad (2.9) \]

Even in this simple case, in Rosen coordinates the additional Killing vector \( \partial_+ \) is typically much less manifest. E.g. for the special case \( A_{ij} = -\delta_{ij} \),

\[ ds^2 = 2dudv + \sin^2 u d\vec{y}^2 \quad (2.10) \]

(the BFHP solution [10] for \( d = 10 \)) one has [12]

\[ X = \partial_u - \frac{y^2}{2} \partial_v - y^i \cot u \partial_{y^i} . \quad (2.11) \]

A similar result holds when \( \sin^2 u \) is replaced by \( \sinh^2 u \) or \( \cosh^2 u \). However, when \( C_{ij}(u) \) is an exponential function of \( u \), say

\[ C_{ij}(u) = e^{2\alpha u} \delta_{ij} \quad (2.12) \]
(isometric to the sinh$^2 \alpha u$ metric) then the metric is obviously invariant under a shift in $u$ combined with a scaling of the $y^i$-coordinates,

$$(u, v, y^i) \rightarrow (u + \lambda, v, e^{-\alpha \lambda} y^i) ,$$

(2.13)
corresponding to

$$X = \partial_u - \alpha y^i \partial_{y^i} .$$

(2.14)

2. Another case where there is an obvious symmetry in $x^+$ in Brinkmann coordinates is when the metric is of the form

$$ds^2 = 2 dx^+ dx^- + B_{ij} z^i z^j \frac{(dx^+)^2}{(x^+)^2} + d\vec{z}^2 ,$$

(2.15)

where $B_{ij}$ is constant. The geometry of these backgrounds has recently been discussed in detail in [30], to which we refer for additional details. In this case the metric is invariant under the boost (2.6) generated by the Killing vector

$$X = x^+ \partial_+ - x^- \partial_-. $$

(2.16)

Hence the overall scale of $B_{ij}$ cannot be changed by a coordinate transformation, but again, without loss of generality, we can assume that $B_{ij}$ is diagonal,

$$B_{ij} = B_i \delta_{ij} .$$

(2.17)

In Rosen coordinates, the corresponding $C_{ij}(u)$ is of the form (2.8) with

$$B_i = -\frac{1}{4} + \beta_i^2 \quad a_i(u) = u^{1/2}(b_i \cosh \beta_i \log u + c_i \sinh \beta_i \log u)$$

$$= b'_i u^{\alpha_i} + c'_i u^{1-\alpha_i} \quad (\alpha_i = \frac{1}{2} + \beta_i)$$

(2.18)

As in the first example, generically in Rosen coordinates the symmetry in $x^+$ is not particularly manifest. However, when $C_{ij}(u)$ is a homogeneous function of $u$,

$$C_{ij}(u) = u^{2\alpha} \delta_{ij} ,$$

(2.19)

(this correspond to a special choice of Rosen coordinates for $B_i \geq -1/4$) the metric

$$ds^2 = 2 du dv + u^{2\alpha} d\vec{y}^2 .$$

(2.20)

is clearly invariant under the scaling

$$(u, v, y^i) \rightarrow (\lambda u, \lambda^{-1} v, \lambda^{-\alpha} y^i) ,$$

(2.21)

generated by the Killing vector

$$X = u \partial_u - v \partial v - \alpha y^i \partial_{y^i} .$$

(2.22)
Many of the features of the above examples are prototypical of the general situation. For example, we will see from an analysis of the Killing equations for a plane wave metric that the $x^+$-component $X^+$ of a Killing vector $X$ in Brinkmann coordinates can be at most linear in $x^+$,

$$X^+ = a_0 x^+ + b_0,$$  

(2.23)

with $a_0, b_0$ constants, corresponding to a constant scaling or shift of $x^+$ (and likewise for $u$). Similarly, an $x^-$-component, if it occurs, will only appear in the combination $x^+ \partial_+ - x^- \partial_-$, corresponding to the boost (2.6).

We noted that there may be different representations of the same metric in Rosen coordinates, and that what may be a manifest $u$-symmetry for one (leading to a simple expression for the Killing vector) may be much more obscure for another. It is this non-uniqueness of the Rosen coordinate system that makes it more convenient to work in Brinkmann coordinates, even though there the Heisenberg algebra generators are initially somewhat more hidden.

In one respect our above examples are non-generic, and this is the fact that they all lead to a (essentially) diagonal $A_{ij}$ in Brinkmann coordinates. For such $A_{ij}(x^+)$, the $x^+$-dependence we have seen above, namely $A_{ij}$ either constant or proportional to $(x^+)^{-2}$, is the only one compatible with homogeneity. However, it is known that in $d = 4$ a genuinely different $x^+$-dependence is possible for HPWs with non-diagonal $A_{ij}$ [30, 41, 32], and one of our aims in the following will be to find these more general homogeneous plane waves in arbitrary dimensions.

2.3 Analysis of the Killing Equations

In Brinkmann coordinates,

$$ds^2 = 2dx^+ dx^- + H(x^+, z)(dx^+)^2 + d\mathbf{z}^2$$

$$H(x^+, z) = A_{ij}(x^+) z^i z^j,$$  

(2.24)

the Killing equations

$$(AB)L_X g_{AB} = X^C \partial_C G_{AB} + \partial_A X^C g_{CB} + \partial_B X^C g_{AC} = 0$$  

(2.25)

for a Killing vector

$$X = X^+ \partial_+ + X^- \partial_- + X^i \partial_i$$  

(2.26)

(when working in Brinkmann coordinates, $\partial_i \equiv \partial_{z^i}$) are

$$(++) (X^+ \partial_+ + X^i \partial_i) H + 2\partial_+ X^- + 2H \partial_+ X^+ = 0$$
\begin{align*}
\partial_- X^+ &= 0 \\
\partial_+ X^+ + \partial_- X^- &= 0 \\
\partial_+ X^i + \partial_i X^- + H \partial_i X^+ &= 0 \\
\partial_- X^i + \partial_i X^+ &= 0 \\
\partial_i X^j + \partial_j X^i &= 0 .
\end{align*}

From the \( \partial_- \) derivative of \((+-)\) we learn that \( X^- \) is at most linear in \( x^- \), and from the \( \partial_- \) derivative of \((ij)\) that \( X^+ \) is at most linear in the \( z^i \). Using \((+-)\) itself, one finds that

\begin{align*}
X^+ &= a_i(x^+) z^i + a_0(x^+) \\
X^- &= -(\dot{a}_i(x^+) z^i + \dot{a}_0(x^+)) x^- + g(x^+, z) .
\end{align*}

The equation \((-i)\) implies that

\begin{equation}
X^i = -a_i(x^+) x^- + e^i(x^+, z) .
\end{equation}

As a consequence, \((+i)\) has one part linear in \( x^- \) which has to vanish separately. This imposes \( \dot{a}_i = 0 \) or \( a_i = c_i = \text{const.} \). The other part of the equation leads to

\begin{equation}
e^i = -\int (\partial_i g(x^+, z) + c_i H(x^+, z)) + f^i(z) ,
\end{equation}

so that, with \( b = -\int g \),

\begin{align*}
X^+ &= c_i z^i + a_0(x^+) \\
X^- &= -\dot{a}_0(x^+) x^- - \dot{b}(x^+, z) \\
X^i &= -c_i x^- + \partial_i b(x^+, z) - c_i \int H(x^+, z) + f^i(z) .
\end{align*}

Here \( \int f(x^+) \) is short for \( \int x^+ dx f(x) \). Now all of the above equations apart from the first \((++)\) and the last \((ij)\) are satisfied. Since everything else in sight is polynomial in the \( z^i \), we can expand \( X^i(z) \) as

\begin{equation}
X^i(z) = -c_i x^- + x_i(x_+) + x_{ik}(x^+) z^k + \frac{1}{2} x_{ikl}(x^+) z^k z^l + \frac{1}{3} x_{iklm}(x^+) z^k z^l z^m + \ldots
\end{equation}

The condition \((ij)\) then implies that \( x_{ij} \) is antisymmetric, and antisymmetry in the first two indices and symmetry in all indices but the first dictates that the higher \( x_{ijkl\ldots} \) are zero,

\begin{equation}
x_{ij} = -x_{ji} , \quad x_{ijkl\ldots} = 0 .
\end{equation}

Thus \( X^i \) can be at most linear in the \( z^i \), with antisymmetric coefficient for the linear part. In particular, therefore, since \( H \) is quadratic, either \( c_i = 0 \) or \( c_i \neq 0 \) and the
term \( c_i \int H \) is cancelled by a cubic term in \( b(x^+,z) \). The latter possibility gives rise to a Killing vector which is a null rotation in the \((+,i)\) directions and can only occur if \( A_{ij} \) is degenerate, so that the plane wave decomposes into the product of a lower-dimensional plane wave metric and a Euclidean space. Thus without loss of generality we can assume \( c_i = 0 \), and we obtain

\[
\begin{align*}
X^+ &= a_0(x^+) \\
X^- &= -\dot{a}_0(x^+)x^- - \dot{b}(x^+,z) \\
X^i &= \partial_i b(x^+,z) + f^i(z) .
\end{align*}
\] (2.34)

Expanding

\[
\partial_i b(x^+,z) = b_i(x^+) + b_{ik}(x^+)z^k ,
\] (2.35)

we see that \( b_{ik} = 0 \) because it has to be both antisymmetric (from \((ij)\)) and symmetric, because \( b_{ij} = \partial_i \partial_j b \). \( f^i(z) \) is not restricted in this way. Absorbing its constant part in \( b_i(x^+) \) and calling \( \int c(x^+) \) the integration ‘constant’ arising from integrating \( \partial_i b(x^+,z) \) to \( b(x^+,z) \) we obtain

\[
\begin{align*}
X^+ &= a_0(x^+) \\
X^- &= -\dot{a}_0(x^+)x^- - \dot{b}_i(x^+)z^i + c(x^+) \\
X^i &= b_i(x^+) + f_{ik}z^k .
\end{align*}
\] (2.36)

Now all the equations apart from \((++)\) are satisfied. That equation will contain one, and only one, term linear in \( x^- \), arising from \( \partial_+ X^- \), namely \( \ddot{a}_0(x^+) \). Since this term has to vanish separately, we learn that \( a_0(x^+) \) is at most linear in \( x^+ \),

\[
a_0(x^+) = a_0 x^+ + b_0 .
\] (2.37)

Now the remainder of the \((++)\) equations splits into three parts, those quadratic in the \( z^i \), linear in the \( z^i \), and independent of the \( z^i \). The latter two produce the equations

\[
\begin{align*}
\ddot{b}_i(x^+) &= A_{ij}(x^+)b_j(x^+) \\
\dot{c}(x^+) &= 0 ,
\end{align*}
\] (2.38)

giving rise, as we will discuss in section 3.2, to the Heisenberg algebra Killing vectors.

The remaining (quadratic) equation is independent of the \( b_i \) and \( c \). For Killing vectors with \( X^+ = 0 \), i.e. \( a_0 = b_0 = 0 \), this equation becomes

\[
A_{ik}(x^+)f_{kj} - f_{ik}A_{kj}(x^+) = 0 ,
\] (2.39)

\footnote{Indeed, for this cancellation to be possible, \( c_i A_{kl} \) has to be totally symmetric, \( c_i A_{kl} = c_k A_{li} \). Assuming that \( A_{ij} \) is non-degenerate and contracting this with the inverse of \( A_{kl} \) one obtains \((d-2)c_i = c_l \) and thus \( c_i = 0 \) for \( d > 3 \). For \( d = 3 \) (a single transverse dimension), the vanishing of \( c_i \) follows from the cubic part (in \( z \)) of the equation \((++)\).}
and has non-trivial solutions only when $A_{ij}$ is invariant under some subgroup of $SO(d-2)$. The $f_{ik}$ are the corresponding Lie algebra elements and give rise to the rotational Killing vectors
\[ X_f = f_{ik}z^k \partial_i. \] (2.40)

For $X^+ \neq 0$, the remaining equations to solve is
\[ (a_0x^+ + b_0)\partial_+ A_{ij}(x^+) + 2a_0A_{ij}(x^+) + A_{ik}(x^+)f_{kj} - f_{ik}A_{kj}(x^+) = 0, \] (2.41)

and the corresponding Killing vector is
\[ X = (a_0x^+ + b_0)\partial_+ - a_0x^- \partial_- + f_{ik}z^k \partial_i. \] (2.42)

Without loss of generality we can assume that either $a_0 = 0, b_0 = 1$ or $a_0 = 1, b_0 = 0$. First of all, let us note that for $f_{ij} = 0$ these equations are in perfect agreement with what we already know about Killing vectors of HPWs from the examples in section 2.2. In particular, for $a_0 = f_{ik} = 0$, the matrix $A_{ij}$ is constant (Cahen-Wallach metrics), and there is the translational Killing vector $X = \partial_+$. And for $b_0 = f_{ik} = 0$, $A_{ij}(x^+)$ has to be homogeneous of degree $-2$, and there is the Killing vector $X = x^+ \partial_+ - x^- \partial_-$ generating the invariance (2.4) of the metric (2.15). These are the only possible solutions for $d = 3$ and, more generally, for a diagonal (isable) $A_{ij}(x^+)$. However, there are also new solutions with $f_{ik} \neq 0$ and $A_{ij}$ genuinely not diagonal. For instance, there will be a Killing vector with $b_0$ and $f_{ik}$ non-zero if an $x^+$-derivative of $A_{ij}$ generates an infinitesimal rotation of $A_{ij}$. Likewise there can be (and there are) solutions with $a_0$ and $f_{ik}$ non-zero, when acting with the Euler operator $x^+ \partial_+$ on $A_{ij}$ generates a rotation plus scaling of $A_{ij}$. We will now construct the general solution to (2.41) and thus obtain all HPW metrics.

For $a_0 = 0, b_0 = 1$, the matrix equation to solve is
\[ \partial_+ A(x^+) + [A(x^+), f] = 0, \] (2.43)

with $A$ symmetric, $A^T = A$ and $f$ antisymmetric, $f^T = -f$. The solution is
\[ A(x^+) = e^{x^+f}A_0e^{-x^+f}, \] (2.44)

with $A_0$ a constant symmetric matrix, and thus the metric is
\[ ds^2 = 2dx^+dx^- + (e^{x^+f}A_0e^{-x^+f})_{ij}z^i z^j(dx^+)^2 + d\tilde{z}^2. \] (2.45)

We now consider the case $a_0 = 1, b_0 = 0$. In that case it is convenient to change variables from $x^+$ to $r = \log x^+$. With the ansatz
\[ A(r) = e^{-2r}B(r), \] (2.46)
\[ \text{(2.41)} \] reduces to
\[ \partial_r B(r) + [B(r), f] = 0 \quad , \] (2.47)
i.e., the same equation as in the other case. Hence the solution is
\[ A(r) = e^{-2r} e^{rf} A_0 e^{-rf} \quad . \] (2.48)
and the metric is
\[ ds^2 = 2dx^+dx^- + (e^{f \log x^+} A_0 e^{-f \log x^+})_{ij} z^i z^j (dx^+)^2 + d\vec{z}^2 \quad . \] (2.49)

2.4 All Homogeneous Plane Waves

From the above analysis we deduce the existence of two families of HPWs. The metrics in both families are parametrised by a constant symmetric matrix \( A_{ij} \) to which they reduce for \( f_{ij} = 0 \) and as for the Cahen-Wallach metrics we can diagonalise \( A_0 \) by orthogonal \( x^+ \)-independent transformations of the \( z^i \). We can also scale the eigenvalues by the boost \((2.6)\) which is an invariance of the metric if accompanied by a scaling of the matrix \( f \). If some of the eigenvalues of \( A_0 \) are equal, the metric will have additional rotational isometries and not all the \( f_{ij} \) will lead to distinct metrics. Nevertheless, all in all we find a
\[ n - 1 + n(n - 1)/2 = (n + 2)(n - 1)/2 \quad . \] (2.52)
dimensional family of HPW metrics, parametrised by the eigenvalues of \( A_0 \) (up to an overall scale) and the elements \( f_{ik} \) of the Lie algebra of \( SO(n) \). For example, for \( d = 10 \) \((n = 8)\), there is a 35-parameter family of such metrics.

Clearly all of these HPW metrics are completely non-singular and geodesically complete, and they will be solutions to the vacuum Einstein equations iff \( A_0 \) is traceless. An
example of a vacuum solution is the anti–Mach \textsuperscript{5} metric of Ozsvath and Schücking \cite{31},
with
\[ A(x^+) = \begin{pmatrix} \cos 2x^+ & -\sin 2x^+ \\ -\sin 2x^+ & \cos 2x^+ \end{pmatrix}, \tag{2.53} \]
or, explicitly,
\[ ds^2 = 2dx^+dx^- + \left[ ((z_1^2 - (z_2^2)) \cos 2x^+ - 2z_1 z_2^2 \sin 2x^+) \right] \left( \frac{dx^+}{(x^+)^2} \right)^2 + (dz_1^2 + dz_2^2)^2. \tag{2.54} \]
This is of the above form, with
\[ f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.55} \]
The additional Killing vector in this case is
\[ X = \partial_+ + z^2 \partial_1 - z_1 \partial_2. \tag{2.56} \]
We now turn to the properties of the second family of HPW metrics (2.13). These generalise the metrics (2.13) with \( A_{ij} \sim (x^+)^{-2} \), to which they reduce for \( f_{ij} = 0 \). \( A_0 \) can again be diagonalised by a constant orthogonal transformation. However, as for the metrics (2.13), the overall scale cannot be changed by a coordinate transformation. Thus the second family of HPW metrics has
\[ n + n(n-1)/2 = n(n+1)/2 \] (2.57)
parameters.
From the usual arguments \cite{2} (see also \cite{37, 38} ) one sees that these metrics are singular at \( x^+ = 0 \) and geodesically incomplete because geodesics starting off at some finite \( x^+ \) will reach the singularity at \( x^+ = 0 \) in finite proper time.
An example is the vacuum solution with
\[ A(x^+) = (x^+)^{-2} \begin{pmatrix} \cos 2 \log x^+ & -\sin 2 \log x^+ \\ -\sin 2 \log x^+ & \cos 2 \log x^+ \end{pmatrix}, \tag{2.58} \]
i.e.
\[ ds^2 = 2dx^+dx^- + \left[ ((z_1^2 - (z_2^2)) \cos 2 \log x^+ - 2z_1 z_2^2 \sin 2 \log x^+) \right] \left( \frac{dx^+}{(x^+)^2} \right)^2 + (dz_1^2 + dz_2^2)^2, \tag{2.59} \]
which has the additional Killing vector
\[ X = x^+ \partial_+ - x^- \partial_- + z^2 \partial_1 - z_1 \partial_2. \tag{2.60} \]
We see from this example the general feature that the \( f_{ij} \neq 0 \) metrics of the second family are slightly less singular than their \( f_{ij} = 0 \) counterparts (2.13) with \( A(x^+) \sim (x^+)^{-2} \) because of the oscillatory part.
\textsuperscript{5}It is anti-Machian in the sense that there is inertia without (distant) matter.
3 Various Properties of Homogeneous Plane Waves

3.1 Homogeneous Plane Waves as String Backgrounds

Since the only non-vanishing component of the Ricci tensor of a plane wave metric in Brinkmann coordinates is (see e.g. the Appendix)

\[ R_{++}(x^+) = - \text{Tr} A(x^+) \quad , \quad (3.1) \]

it is trivial to realise any plane wave metric (with a non-positive trace) as a solution of supergravity in a variety of ways. For any \( p \)-form field \( A^p \) one makes the ansatz that its field strength is of the form

\[ F^{p+1} = dx^+ \wedge \varphi(x^+) \quad , \quad (3.2) \]

where \( \varphi(x^+) \) has only transverse components. Then the Einstein equations reduce to

\[ - \text{Tr} A(x^+) = c_p ||\varphi(x^+)||^2 \quad (3.3) \]

for some constant \( c_p \), and all the other equations of motion and Bianchi identities are identically satisfied (for the RR five-form field strength one has to impose the self-duality condition).

This is completely general and true for any plane wave metric, but there are two things worth noting about this in the context of homogeneous plane waves. First of all, due to the special form of the metric (2.50) or (2.51), the Ricci tensor of the general HPW is actually independent of \( f_{ik} \) and coincides with that of the simple metric (2.5) or (2.15),

\[ R_{++}(x^+) = - \text{Tr} e^{x^+ f} A_0 e^{-x^+ f} = - \text{Tr} A_0 \]

\[ R_{++}(x^+) = - e^{-2 \log x^+} \text{Tr} e^{\log x^+ f} A_0 e^{-f \log x^+} \]

\[ = -(x^+)^2 \text{Tr} A_0 \quad . \quad (3.4) \]

One thus finds the rather remarkable fact that any string background for either of the special metrics (2.5) or (2.15) will also automatically provide a background for the most general HPW metric (2.50) or (2.51). Of course, to actually obtain a non-trivial metric, \( A_0 \) should not be proportional to the identity matrix, but any generic choice of (diagonal) matrix will do.

The simplest possibility is to add only an \( x^+ \)-dependent dilaton field. These solutions were described in [30] (for \( A_0 \) proportional to the identity matrix, but the generalisation is obvious since only the trace of the matrix enters the Einstein equations).
are also generalisations of the BFHP plane wave [10], a solution of IIB supergravity supported by the metric (2.50) with $f_{ij}$ arbitrary,

$$(A_0)_{ij} = A_i \delta_{ij}$$ \hspace{1cm} (3.5)$$

with $\text{Tr} A_0 < 0$, and a self-dual five-form

$$F_5 = \lambda dx^+ \wedge (1 + *_8) dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4$$ \hspace{1cm} (3.6)$$

with $\sum A_i$ and $\lambda$ related by (3.3). Likewise, there is a corresponding solution for (2.51) based on a solution for (2.15) (also noted in [30]), namely $A_0$ as above, $f_{ij}$ arbitrary, and

$$F_5 = \lambda \frac{dx^+}{x^+} (1 + *_8) dz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4$$ \hspace{1cm} (3.7)$$

Similarly, one can construct supergravity solutions with other RR fields, or with the NS B-field. All these solutions are, like a generic plane wave background, half-supersymmetric.

Finally, we remark that the field strengths of the RR or NS fields will be invariant under the additional isometry generated by $X$ if they are invariant under the rotation generated by $f_{ik}$, so only then will the string background as a whole be homogeneous. A non-constant dilaton, on the other hand, will of course always break the homogeneity of the supergravity configuration.

### 3.2 The Heisenberg Algebra

We will now show that the Killing vectors which arise as solutions to (2.38) describe the Heisenberg isometry algebra of a generic plane wave. While this is well known, for later applications, e.g. the determination of the full isometry algebra of a homogeneous plane wave and the lightcone quantisation in these backgrounds, we find it useful and necessary to be quite explicit about this construction.

The second of the equations (2.38) gives rise to the obvious Killing vector $Z = \partial_-$ characterising a pp-wave. The first is a second order harmonic oscillator matrix differential equation for the $(d - 2)$-vector $b_i$,

$$\ddot{b}_i(x^+) = A_{ij}(x^+)b_j(x^+)$$ \hspace{1cm} (3.8)$$

Let us denote the $2(d - 2) \equiv 2n$ solutions to (3.8) by $b^{(J)}_i$, $J = 1, \ldots, 2n$. Then to each solution $b^{(J)}$ we can associate the Killing vector

$$X^{(J)} \equiv X(b^{(J)}) = \dot{b}^{(J)}_i \partial_i - \dot{b}^{(J)}_i z^i \partial_-$$ \hspace{1cm} (3.9)$$
These Killing vectors and \( Z = \partial_{-} \) satisfy the algebra

\[
[X^{(J)}, X^{(K)}] = W(b^{(J)}, b^{(K)})Z \tag{3.10}
\]
\[
[X^{(J)}, Z] = 0 . \tag{3.11}
\]

Here \( W(b^{(J)}, b^{(K)}) \), the Wronskian of the two solutions, is defined by

\[
W(b^{(J)}, b^{(K)}) = \sum_{i} (\dot{b}_{i}^{(J)} \dot{b}_{i}^{(K)} - \ddot{b}_{i}^{(K)} \dot{b}_{i}^{(J)}) . \tag{3.12}
\]

It is constant (independent of \( x^{+} \)) courtesy of the differential equation (3.8). Thus \( W(b^{(J)}, b^{(K)}) \) is a constant, non-degenerate, even-dimensional antisymmetric matrix. Hence it can be put into standard (Darboux) form. Explicitly, a canonical choice of basis for the solutions \( b^{(J)} \) is obtained by splitting the \( b^{(J)} \) into two sets of solutions \( \{b^{(k)}\} \rightarrow \{b^{(k)}, b^{*,(k)}\} \) (3.13)

characterised by the initial conditions

\[
\begin{align*}
b_{i}^{(k)}(x_{0}^{+}) &= \delta_{ik} \quad \dot{b}_{i}^{(k)}(x_{0}^{+}) = 0 \\
\dot{b}_{i}^{*,(k)}(x_{0}^{+}) &= 0 \quad \ddot{b}_{i}^{*,(k)}(x_{0}^{+}) = \delta_{ik} . \tag{3.14}
\end{align*}
\]

Since the Wronskian of these functions is independent of \( x^{+} \), it can be determined by evaluating it at \( x_{0}^{+} \). Hence one can immediately read off that

\[
W(b^{(k)}, b^{(l)}) = W(b^{*(k)}, b^{*(l)}) = 0 \\
W(b^{(k)}, b^{*(l)}) = -\delta_{kl} . \tag{3.15}
\]

Thus the corresponding Killing vectors

\[
X^{(k)} \equiv X(b^{(k)}) , \quad X^{*(k)} \equiv X(b^{*(k)}) \tag{3.16}
\]

and \( Z \) satisfy the canonically normalised Heisenberg algebra

\[
\begin{align*}
[X^{(k)}, X^{(l)}] &= [X^{*(k)}, X^{*(l)}] = 0 \\
[X^{(k)}, X^{*(l)}] &= -\delta_{kl} Z . \tag{3.17}
\end{align*}
\]

### 3.3 The Isometry Algebra of a Homogeneous Plane Wave

As a first step towards determining the isometry algebra of a homogeneous plane wave, we study the closure of the algebra generated by the Heisenberg algebra vectorfields

\[
\begin{align*}
X(b^{(l)}) &= b_{i}^{(l)} \partial_{i} - \dot{b}_{i}^{(l)} z^{i} \partial_{-} \\
Z &= \partial_{-} , \tag{3.18}
\end{align*}
\]

Non-degeneracy is implied by the linear independence of the solutions \( b^{(J)} \).
and
\[ X = (a_0 x^+ + b_0) \partial_+ - a_0 x^- \partial_- + f_{ik} z^k \partial_i, \]
where we recall that the coefficients are subject to the conditions
\[ \ddot{b}_i^{(l)}(x^+) = A_{ij}(x^+) b_j^{(l)}(x^+) \]
\[ (a_0 x^+ + b_0) \partial_+ A_{ij}(x^+) + 2a_0 A_{ij}(x^+) + A_{ik}(x^+) f_{kj} - f_{ik} A_{kj}(x^+) = 0. \]

Clearly
\[ [X, Z] = a_0 Z, \]
so that we need only look at \([X, X^{(l)}].\) One finds
\[ [X, X(b^{(l)})] = X(c^{(l)}) \]
where
\[ c_i^{(l)} = (a_0 x^+ + b_0) \dot{b}_i^{(l)} - f_{ik} b_k^{(l)}. \]
As a consequence of the two conditions \[ (3.20), \]
\[ c_i^{(l)} \] also solves the oscillator equation
and is thus a linear combination of the \(b^{(l)}\) with constant coefficients \(m^{(l)}_{(j)},\) as it should
be, so that
\[ [X, X(b^{(l)})] = m^{(l)}_{(j)} X(b^{(j)}) \]

Using the basis for the \(b^{(l)}\) introduced in the previous section, it is possible to be completely explicit about this algebra. Indeed, we have
\[ [X, X(b^{(k)})] = X(c^{(k)}) \]
\[ [X, X(b^{*^{(k)}})] = X(c^{*^{(k)}}), \]
with
\[ c_i^{(k)} = (a_0 x^+ + b_0) \dot{b}_i^{(k)} - f_{ij} b_j^{(k)} \]
\[ c_i^{*^{(k)}} = (a_0 x^+ + b_0) \dot{b}_i^{*^{(k)}} - f_{ij} b_j^{*^{(k)}}. \]
Evidently \(c^{(k)}\) and \(c^{*^{(k)}}\) satisfy the initial conditions
\[ c_i^{(k)}(x_0^+) = -f_{ik} \]
\[ c_i^{*^{(k)}}(x_0^+) = (a_0 x_0^+ + b_0)(A_0)_{ik} \]
\[ c_i^{*^{(k)}}(x_0^+) = (a_0 x_0^+ + b_0) \delta_{ik} \]
and therefore
\[ c_i^{(k)} = f_{kl} b_i^{(l)} + (a_0 x_0^+ + b_0)(A_0)_{kl} b_i^{*^{(l)}} \]
\[ c_i^{*^{(k)}} = (a_0 x_0^+ + b_0) b_i^{(k)} + (a_0 \delta_{kl} + f_{kl}) b_i^{*^{(l)}}. \]
Thus the complete isometry algebra of a homogeneous plane wave is

\[
\begin{align*}
[X^{(k)}, X^{(l)}] &= [X^{*(k)}, X^{*(l)}] = 0 \\
[X^{(k)}, Z] &= [X^{*(k)}, Z] = 0 \\
[X^{(k)}, X^{*(l)}] &= -\delta_{kl}Z \\
[X, X^{(k)}] &= f_{kl}X^{(l)} + (a_0x_0^+ + b_0)(A_0)_{kl}X^{*(l)} \\
[X, X^{*(k)}] &= (a_0x_0^+ + b_0)X^{(k)} + (a_0\delta_{kl} + f_{kl})X^{*(l)} \\
[X, Z] &= a_0Z .
\end{align*}
\]

(3.29)

Let us consider some examples.

1. The first example is the Cahen-Wallach metric (2.5)

\[
ds^2 = 2dx^+ dx^- + A_{ij}z^iz^j(dx^+)^2 + d\vec{z}^2,
\]

(3.30)

with \(A_{ij}\) constant. We have \(a_0 = f_{ik} = 0\), choose \(b_0 = 1\) and find that the non-zero commutators are

\[
\begin{align*}
[X^{(k)}, X^{*(l)}] &= -\delta_{kl}Z \\
[X, X^{(k)}] &= A_{kl}X^{*(l)} \\
[X, X^{*(k)}] &= X^{(k)}
\end{align*}
\]

(3.31)

This is the ‘standard’ twisted Heisenberg algebra, the extension of the Heisenberg algebra by the outer automorphism \(X\) which rotates the generators \(X^{(k)}\) and \(X^{*(k)}\). We will also refer to it as the harmonic oscillator algebra, with \(X\) playing the role of the harmonic oscillator Hamiltonian or number operator.

Let us note that, due to our choice of basis, the stabiliser of the action of the isometry algebra at a point of the spacetime on the line \((x^+ = x_0^+, x^-, z^k = 0)\), i.e. the subalgebra of the isometry algebra whose Killing vectors vanish at \(x_0^+\), is precisely the Abelian subalgebra spanned by the \(X^{*(k)}\). This algebra is evidently symmetric, confirming that the Cahen-Wallach spaces are Lorentzian symmetric spaces.

As for the metric, the isometry algebra depends only on the eigenvalues of \(A_{ij}\), up to an overall scale. In particular, for \(d = 3\), the algebra depends only on the sign of \(A_{11}\). For \(A < 0\), this algebra can also be considered as the central extension \(E_2^c\) of the Euclidean algebra of two-dimensional translations and rotations, the Nappi-Witten algebra, or \(A_{4,10}\) in the classification of \([16]\). Likewise, for \(A > 0\) the algebra is the central extension \(P_2^c = A_{4,8}\) of the two-dimensional Poincaré algebra. Among the twelve four-dimensional Lie algebras \([14]\), the two algebras
$A_{4,8}$ and $A_{4,10}$ occurring as isometry algebras of Cahen-Wallach spaces are the only ones which admit a non-degenerate invariant scalar product.

2. Our next example is the metric (2.13),

$$ds^2 = 2dx^+dx^- + B_{ij}z^iz^j \left(\frac{dx^+}{x^+}\right)^2 + d\vec{z}^2 ,$$

(3.32)

where $B_{ij}$ is constant. This corresponds to $b_0 = f_{ik} = 0$ and we choose $a_0 = x_0^+ = 1$ so that $(A_0)_{kl} = B_{kl}$. In this case the non-zero commutators are

$$[X^{(k)}, X^{*(l)}] = -\delta_{kl}Z$$

$$[X, X^{(k)}] = B_{kl}X^{*(l)}$$

$$[X, X^{*(k)}] = X^{(k)} + X^{*(k)}$$

$$[X, Z] = Z .$$

(3.33)

We note the interesting feature that $Z$, which usually plays the role of Planck's constant in the oscillator algebra, is now no longer central. In particular, this implies that the stabiliser subalgebra (once again generated by the $X^{*(k)}$), is not symmetric. As expected, these space-times are Lorentzian homogeneous but not Lorentzian symmetric.

If all the eigenvalues of $B_{ij}$ are equal, $B_{ij} = B\delta_{ij}$, (this is the case considered in [30]), this isometry algebra contains the simply transitive subalgebra spanned by \{ $X, Z, X^{(k)} + (1-B)X^{*(k)}$ \}. This is related to the observation made in [30] that in this case the spacetime can be identified with the corresponding group manifold, equipped with a left-invariant metric.

For $d = 3$, the algebra depends explicitly on $B_{11} = B$, which we parametrise, as in section 2.2, as

$$B \geq -\frac{1}{4} : \quad B = -\frac{1}{4} + \beta^2 = \alpha(\alpha - 1)$$

$$B \leq -\frac{1}{4} : \quad B = -\frac{1}{4} - \beta^2 .$$

(3.34)

For $B > -1/4$, one finds the algebra $A_{4,9}$ with $a = (1 - \alpha)/\alpha$, and for $B < -1/4$ the algebra $A_{4,11}$ with $a = 1/2\beta$. In the remaining case $B = -1/4$, the isometry algebra is $A_{4,7}$.

Interestingly these three (families of) algebras have no invariants at all, in particular no quadratic Casimir. Finally let us note that the five (families

\footnote{It is not particularly evident from the above presentation of the algebra that $B = -1/4$ is a special point where the structure of the algebra changes. However, to put the algebra into canonical form, as given in [46], requires (among other things) a rescaling of the generators which becomes singular for precisely this value.}
of) four-dimensional Lie algebras we have found as the isometry algebras of three-dimensional HPWs are precisely the five algebras whose derived algebra is the Heisenberg algebra [46].

3. Now let us consider the isometry algebra of the family of metrics (2.50), i.e. the algebra (3.29) in the case $a_0 = 0$ (and we choose $b_0 = 1$, $x_0^+ = 0$). This algebra is

\[
\begin{align*}
[X^{(k)}, X^{*(l)}] &= -\delta_{kl} Z \\
[X, X^{(k)}] &= f_{kl} X^{(l)} + (A_0)_{kl} X^{*(l)} \\
[X, X^{*(k)}] &= X^{(k)} + f_{kl} X^{*(l)} \\
[X, Z] &= 0 .
\end{align*}
\]

(3.35)

In particular, for the anti-Mach metric (2.54) a convenient basis is \{\(X^{(k)}, Y^{(k)}\)} with

\[
\begin{align*}
Y^{(1)} &= X^{*(1)} + X^{(2)} \\
Y^{(2)} &= -X^{*(2)} - X^{(1)} ,
\end{align*}
\]

in terms of which the algebra reads

\[
\begin{align*}
[X^{(k)}, Y^{(l)}] &= -(A_0)_{kl} Z \\
[X, X^{(k)}] &= Y^{(k)} \\
[X, Y^{(1)}] &= 0 \\
[X, Y^{(2)}] &= -2X^{(2)} \\
[X, Z] &= [X^{(k)}, Z] = [Y^{(k)}, Z] = 0 .
\end{align*}
\]

(3.37)

Note that this algebra has the simply transitive ‘Siberian’ \(A_{4,1}\) subalgebra spanned by \{\(X, Z, X^{(1)}, Y^{(1)}\)} , allowing an identification of this spacetime with the corresponding group manifold, equipped with a left-invariant metric. Note also that there are three commuting elements, namely \(X, Z, Y^{(1)}\). As we will see in section 4.5, quantisation of a particle moving in this background is particularly simple in a basis where the three corresponding operators are diagonal.

3.4 Homogeneous Plane Waves in Rosen Coordinates and Null Cosmology

In general, finding Rosen coordinates is not straightforward. However, as explained in the Appendix, it is not necessary to perform explicitly the coordinate transformation of the metric from Brinkmann to Rosen coordinates. Rather, given the solutions \(b^{(J)}\) to the oscillator (or Killing) equation

\[
\ddot{b}_i(u) = A_{ij}(u)b_j(u) ,
\]

(3.38)
one can algebraically construct the metric in Rosen coordinates. It takes the form

$$C_{ik} = b_j^{(J_i)} b_j^{(J_k)}$$

(3.39)

where the $b^{(J)}$ are any $n = d - 2$ of the $2n$ solutions $b^{(J)}$ having zero Wronskian. Clearly the metric in Rosen coordinates is not unique. In terms of the basis of solutions introduced in section 3.2, a natural choice is $b^{(J_k)} = b^{(k)}$ or $b^{(J_k)} = b^{*(k)}$.

Once again, the first non-trivial example is the anti-Mach metric (2.54). The solutions to the oscillator equation satisfying the initial conditions (3.14) are

$$b_1^{(1)} = \cos u + u \sin u$$
$$b_2^{(1)} = - \sin u + u \cos u$$
$$b_1^{(2)} = \sin u \cos \sqrt{2} u - (1/\sqrt{2}) \cos u \sin \sqrt{2} u$$
$$b_2^{(2)} = \cos u \cos \sqrt{2} u + (1/\sqrt{2}) \sin u \sin \sqrt{2} u$$

$$b_1^{*(1)} = \sin u - b_1^{(2)}$$
$$b_2^{*(1)} = \cos u - b_2^{(2)}$$

$$b_1^{*(2)} = \cos u \cos \sqrt{2} u + \sqrt{2} \sin u \sin \sqrt{2} u - b_1^{(1)}$$
$$b_2^{*(2)} = - \sin u \cos \sqrt{2} u + \sqrt{2} \cos u \sin \sqrt{2} u - b_2^{(1)}$$

(3.40)

The most natural choice $b^{(J_k)} = b^{(k)}$ gives a rather complicated expression. The choice leading to the simplest form of the metric appears to be to take $b^{*(1)} + b^{(2)}$ and $b^{*(2)} + b^{(1)}$ (which also have zero Wronskian), leading to

$$C = \begin{pmatrix}
1 & \sqrt{2} \sin \sqrt{2} u \\
\sqrt{2} \sin \sqrt{2} u & 1 + \sin^2 \sqrt{2} u
\end{pmatrix},$$

(3.41)

or

$$ds^2 = 2dudv + (dy^1)^2 + 2\sqrt{2} \sin \sqrt{2} u dy^1 dy^2 + (1 + \sin^2 \sqrt{2} u)(dy^2)^2.$$  

(3.42)

A similar, but slightly more involved, calculation shows that the metric (2.59) in Rosen coordinates can e.g. take the form

$$ds^2 = 2dudv + u[(2+\cos \sqrt{3} \log u)(dy^1)^2 - 2 \sin \sqrt{3} \log u dy^1 dy^2 + (2-\cos \sqrt{3} \log u)(dy^2)^2].$$

(3.43)

The determinant of the metric is $3u^2$, showing that, along the lines proposed in [30], one can consider this as a model of a null cosmology, albeit with a rather bizarre oscillatory
behaviour near the initial singularity at $u = 0$. For the hyperbolic counterparts of these metrics, the behaviour is different as the individual components of the metric appear to blow up as $u \to 0$.

It is quite plausible that both the quantitative and qualitative behaviour of string propagation in the singular HPW backgrounds (2.51) differs from that in the spacetime (2.13) studied in [30]. For instance, it was argued in [20] that string propagation in potentials behaving near the singularity like $1/(x^+)^{(2+a)}$ is quite different for $a \geq 0$ and $a < 0$. It would be interesting to gain a better understanding of this for these examples.

3.5 Geodesics and Conserved Charges in a Homogeneous Plane Wave

Since any plane wave has a Heisenberg algebra of isometries, there will be corresponding conserved charges also satisfying the Heisenberg algebra. These will provide a useful basis for quantisation in plane wave backgrounds. Given the extra Killing vector $X$, there will be one more conserved charge, related to the energy, for particles moving in a homogeneous plane wave. To exhibit these, we need to quickly review the standard (and elementary) discussion of geodesics in general plane wave backgrounds.

We begin with the Lagrangian

$$L = \dot{x}^+ \dot{x}^- + \frac{1}{2} A_{ij}(x^+) z^i z^j (\dot{x}^+)^2 + \frac{1}{2} (\dot{z})^2,$$

(3.44)

where an overdot denotes a derivative with respect to the affine parameter $t$. Evidently the lightcone momentum

$$P_- = \frac{\partial L}{\partial \dot{x}^-} = \dot{x}^+,$$

(3.45)

is conserved. For $P_- = 0$, the particle obviously does not feel the curvature and the geodesic equations reduce to $\ddot{x}^- = \ddot{z}^i = 0$. When $P_- \neq 0$, for present purposes nothing is gained by carrying around $P_-$, and we choose $x^+ = t$.

The constraint $L = 0$ for a massless particle implies the $x^-$ equation of motion, and one has

$$P_- = 1 \quad \dot{x}^- + \frac{1}{2} A_{ij} z^i z^j + \frac{1}{2} (\dot{z})^2 = 0$$

$$\ddot{z}^i = A_{ij} \dot{z}^j.$$  

(3.46)

Multiplying the second equation by $z^i$ and inserting this into the first equation, one gets

$$\dot{x}^- + \frac{1}{2} z^i \dot{z}^i + \frac{1}{2} z^2 = 0,$$

(3.47)

which can be integrated to

$$x^- = -\frac{1}{2} (z^i \dot{z}^i) + c.$$  

(3.48)
for some constant $c$.

Associated with any Killing vector $Y$ one has a conserved quantity

$$Q(Y) = Y^M \dot{x}_M$$  \hfill (3.49)

where $x^M = (x^+, x^-, z^i)$. The conserved quantity $P_-$ is that associated with the null Killing vector $Z = \partial_-$ because

$$Q(Z) = Z^M \dot{x}_M = \dot{x}^+ = P_- .$$  \hfill (3.50)

The conserved quantities associated with the Heisenberg algebra Killing vectors are

$$Q(X^{(J)}) = p^i b_i^{(J)} - \dot{b}_i^{(J)} z^i ,$$  \hfill (3.51)

where $p^i = \dot{z}^i$. These are indeed conserved quantities because the $b_i^{(J)}$ satisfy the same oscillator equation with respect to $x^+ = t$ as the $z^i$ with respect to $t$, and $Q(X^{(J)})$ is just the Wronskian $W(z, b^{(J)})$.

Lastly, associated with the extra Killing vector $X$, if it exists, there is yet another conserved quantity $Q(X)$. For Cahen-Wallach spaces (constant $A_{ij}$), one has $X = \partial_+$, thus $X^M = g_{M+}$ and therefore

$$Q(\partial_+) = \dot{x}^- + A_{ij} z^i \dot{z}^j = \frac{\partial L}{\partial \dot{x}^+} = P_+ .$$  \hfill (3.52)

Substituting for $\dot{x}^-$, one finds, none too surprisingly, that for a time-independent harmonic oscillator potential the associated conserved quantity $P_+$ is just the non-relativistic harmonic oscillator Hamiltonian. With our sign conventions

$$P_+ = -H_{osc}(p, z)$$

$$H_{osc}(p, z) = \frac{1}{2} (\delta_{ij} p_i p_j - A_{ij} z^i \dot{z}^j) .$$  \hfill (3.53)

A special feature of the time-dependent harmonic oscillators appearing for HPWs is that, in spite of their time-dependence, there is a conserved quantity which is associated with $x^+$ and hence, while not equal to the Hamiltonian, at least closely related to it.

In particular, when $X$ takes the form $X = x^+ \partial_+ - x^- \partial_-$ instead (i.e. for the metrics (2.13)), then $X^M = x^+ g_{M+} - x^- g_{M-}$, so that, evidently,

$$Q(x^+ \partial_+ - x^- \partial_-) = x^+ P_+ - x^- P_- .$$  \hfill (3.54)

Substituting for $P_+$ and $x^-$ and dropping the irrelevant constant $c$, one sees that this is

$$Q(x^+ \partial_+ - x^- \partial_-) = -t H_{osc}(p, z) + \frac{1}{4} (p^i z^i + z^i p^i) .$$  \hfill (3.55)
Anticipating the appearance of this quantity as an operator in the quantum theory, where it also defines an invariant, we have already symmetrized the second term.

For the metrics (2.50), the Killing vector is $X = \partial_+ + f_{ik}z^k\partial_i$, and the associated conserved charge is

$$Q(\partial_+ + f_{ik}z^k\partial_i) = -(H_{osc}(p, z) - f_{ik}z^kp^i) ,$$

and likewise for the remaining class of metrics (2.51),

$$Q(x^+\partial_+ - x^-\partial_+ + f_{ik}z^k\partial_i) = -tH_{osc}(p, z) + \frac{1}{4}(p^iz^i + z^ip^i) + f_{ik}z^kp^i .$$

### 3.6 Homogeneous Plane Waves in Rotated (Stationary) Coordinates

Additional insight into the structure of homogeneous plane waves is gained by exhibiting them in other coordinate systems. We already briefly described the form of the metric in Rosen coordinates in section 3.4. Here we consider a coordinate system which is useful for describing the quantisation of the point-particle (and strings) in the lightcone gauge.

Let us first consider the solution (2.50). It is natural to go to a rotating coordinate system,

$$z^i \rightarrow w^i = (e^{-x^+f})_{ik}z^k .$$

In these coordinates, the metric takes the stationary form

$$ds^2 = 2dx^+dx^- + ((A_0)_{ij} - f_{ik}f_{kj})w^iw^j(dx^+)^2 + dw^2 - 2w^if_{ik}dw^kdx^+ ,$$

and the additional isometry in $x^+$ is manifest, with

$$X = \partial_+ + f_{ik}z^k\partial_i \rightarrow \partial_+ .$$

Thus in these coordinates, the metric is of the Cahen-Wallach type, with an additional rotation term. Note that in this case of time-dependent harmonic oscillators we cannot, as in the Landau problem, trade a magnetic field for an oscillator term or vice-versa.

In the lightcone gauge this leads to time-independent mass terms (frequencies) for the scalars, and additionally there is an interaction with the constant magnetic field $f_{ik}$. In particular, in the point-particle case the conserved quantity associated with $X$ is now simply the time-independent Hamiltonian of this system,

$$Q(X = \partial_+) = -H(\pi, w) .$$

Here $\pi^i$ are the canonical momenta,

$$\pi^i = \dot{w}^i + f_{ik}w^k .$$
and the Hamiltonian is
\[ H(\pi, w) = \frac{1}{2}(\pi^2 - (A_0)_{ij} w^i w^j) + f_{ik} w^i \pi^k \] (3.63)

Expressing this in terms of the original variables \( z^i \) and
\[ p^i = \dot{z}^i = (e_x + f)_{ik} \pi^k \] (3.64)
one finds that this is precisely the conserved quantity (3.56) derived above,
\[ H(\pi(p), w(z)) = H_{osc}(p, z) + f_{ik} z^i p^k . \] (3.65)

We will argue in section 4 that using this invariant simplifies significantly the quantisation in this class of backgrounds.

In particular, for the anti-Mach metric we find
\[ ds^2 = 2dx^+ dx^- + 2(w^1)^2(dx^+)^2 + dw^2 - 2(w^1 dw^2 - w^2 dw^1)dx^+ . \] (3.66)

By shifting \( x^- \),
\[ w^- = x^- + w^1 w^2 \] (3.67)
(this amounts to adding a total derivative to the lightcone Lagrangian, or to changing the gauge for the constant magnetic field) we can eliminate the explicit dependence of the metric on \( w^2 \),
\[ ds^2 = 2dx^+ dw^- + 2(w^1)^2(dx^+)^2 + dw^2 - 4w^1 dw^2 dx^+ . \] (3.68)
This is precisely the form of the anti-Mach metric found originally in [31]. In these coordinates, translations in \( x^+, w^- \) and \( w^2 \) are manifest symmetries of the metric. These are the three commuting isometries we had already deduced from the isometry algebra (3.37).

An analogous rotation puts the metrics with \( A(x^+) \) of the type (2.51) into the form
\[ ds^2 = 2dx^+ dx^- + ((A_0)_{ij} - f_{ik} f_{kj}) w^i w^j \frac{(dx^+)^2}{(x^+)^2} + dw^2 - 2w^i f_{ik} dw^k \frac{dx^+}{x^+} , \] (3.69)
the additional isometry
\[ X = x^+ \partial_+ - x^- \partial_- + f_{ik} z^k \partial_i \rightarrow x^+ \partial_+ - x^- \partial_- \] (3.70)
taking the same form as for the metric (2.15). It is possible to go to adapted coordinates for \( X \), but in these coordinates the null isometry generated by \( Z = \partial_- \) will no longer be manifest - since \( X \) and \( Z \) do not commute there is no coordinate system adapted to both simultaneously. In the lightcone gauge this will lead to a combination of the time-dependent model analysed in detail in [30] and the magnetic field models studied e.g. in [7]. A possible alternative approach to quantisation of this model is to use the invariant associated with the Killing vector \( X \) in the Lewis-Riesenfeld procedure (see below).
4 Lightcone Quantisation and the Plane Wave Geometry underlying the Lewis-Riesenfeld Procedure

4.1 Preliminary Remarks

It is known that lightcone quantisation of particles or strings in plane wave backgrounds gives rise to, in general time-dependent, harmonic oscillators [2]. We saw this in our analysis in the previous section of the relativistic particle in these backgrounds. In the general case one can quantise these systems by using the theory of invariants for time-dependent oscillators developed by Lewis and Riesenfeld [33, 34], already employed in the present context in [24, 30]. This construction is based on the simple but remarkable observation that for any oscillator Hamiltonian $H(t)$ with a time-dependent frequency,

$$H(t) = \frac{1}{2}(p^2 + \omega(t)^2 z^2),$$

there exist invariants, i.e. explicitly time-dependent quantum operators $I(q(t), p(t), t)$ satisfying

$$i\frac{dI(t)}{dt} = i\frac{\partial I(t)}{\partial t} + [I(t), H(t)] = 0.$$  \hspace{1cm} (4.2)

Lewis and Riesenfeld (LR) give an algorithm which provides a quadratic invariant for any time-dependent harmonic oscillator (and more general systems), and which moreover has the feature that $I(t)$ itself has the form of a time-independent harmonic oscillator. Then it is straightforward to determine the spectra and eigenstates of $I(t)$. The second ingredient in the LR procedure is the construction of all the solutions to the time-dependent Schrödinger equation for $H(t)$ from the eigenstates of $I(t)$.

Embedding the problem of a time-dependent harmonic oscillator into the plane wave setting equips it with a rich geometric structure. Indeed, as we have seen, the oscillator equation describes both the geodesics and the isometries of a plane wave background. This links the dynamics of the harmonic oscillator to the conserved charges associated with these symmetries and, as we will see, provides a natural geometric explanation of the entire LR procedure (to a certain extent this has already been recognised in [24, 30]).

In particular, as we have seen, every plane wave metric has a Heisenberg isometry algebra. Promoting the corresponding conserved charges (which we determined in section 3.5) to quantum operators, these are already themselves invariants (and correspond to the invariant oscillators used in [24, 30]). Therefore, any quadratic operator built from these operators will be a quadratic invariant. Note that to ‘see’ these invariants geometrically, one has to extend the harmonic oscillator configuration space (spanned by the $z^k$) by $x^-$.  

28
In any case, this makes it evident that there are many invariants, and any one of
them can be used as a basis for quantisation. Ultimately, and in principle, the result
of calculating a physically observable quantity does not depend on which invariant one
chooses. In practice, however, one choice may be more convenient than another, perhaps
because one invariant is more simply related to the Hamiltonian $H(t)$ than another,
or perhaps because one invariant has an oscillator representation while another has
hypergeometric eigenfunctions. For example, in the case of time-independent harmonic
oscillators (Cahen-Wallach spaces), one would be foolish to base quantisation on some
quadratic invariant (possibly explicitly time-dependent and related in a complicated
way to the Hamiltonian of the system) other than the Hamiltonian itself.

In the case of HPWs, and hence time-dependent harmonic oscillators arising from
HPWs, there is a natural and preferred invariant $I_X$ associated with the extra Killing
vector $X$. As we have seen above, this extra invariant is in all cases closely related
to the light-cone Hamiltonian, and may lead to a natural quantisation of particles and
subsequently string theory on these backgrounds.

What is not guaranteed, however, as already mentioned above, is that $I_X$ has a standard
oscillator realisation. If it has, the better, and the construction of the eigenstates is
routine. If it does not, but is nevertheless sufficiently simple, then one can just construct
the eigenstates directly. In these cases, our construction is a simple generalisation of the
construction that one uses for a time-independent harmonic oscillator. If $I_X$ is neither
of oscillator type nor particularly simple in some other sense, then one can of course
always choose some other invariant. We will see examples of all of these possibilities in
the following.

4.2 Review of the Lewis–Riesenfeld Procedure

Even though we are advocating the point of view that for the purposes of lightcone
quantisation in plane wave backgrounds one can bypass much of the LR procedure alto-
tgether and just make use of the symmetries of plane waves, in order to make this point,
and to provide a more detailed comparison with the geometric plane wave approach, we
need to first review the salient aspects of this construction. We will then show how to
recover all of these results from the plane wave geometry and its Heisenberg algebra of
isometries.

Let us first assume that an invariant $I(t)$ satisfying (4.2) exists and that it is hermitian.
We choose a complete set of eigenstates labelled by the real eigenvalues $\lambda$ of $I$,

$$I(t)|\lambda\rangle = \lambda|\lambda\rangle$$

(4.3)
It follows from (4.2) that the eigenvalues $\lambda$ are time-independent, and that

$$i<\lambda'|\frac{\partial}{\partial t}|\lambda> = <\lambda'|H|\lambda>, \quad (4.4)$$

for $\lambda \neq \lambda'$. We would like this equation to be true also for the diagonal elements, in which case the corresponding eigenvectors are solutions of the time-dependent Schrödinger equation for $H(t)$. We need to slightly modify the eigenfunctions to satisfy this condition and so introduce a time-dependent phase,

$$|\lambda>_{\alpha} = e^{i\alpha(t)}|\lambda>. \quad (4.5)$$

It can be seen immediately that this phase factor does not change the off-diagonal matrix elements of $i\frac{\partial}{\partial t} - H$ and leads to an equation for $\alpha_{\lambda}(t)$,

$$\frac{d\alpha_{\lambda}}{dt} = <\lambda|i\frac{\partial}{\partial t} - H|\lambda>. \quad (4.6)$$

Solving this equation, the general solution to the Schrödinger equation is

$$|t> = \sum_{\lambda} c_{\lambda}e^{i\alpha_{\lambda}(t)}|\lambda>, \quad (4.7)$$

where the $c_{\lambda}$ are constants.

Coming to the second ingredient in the LR procedure, consider the one-dimensional time-dependent oscillator

$$H_{osc}(t) = \frac{1}{2}(p^2 + \omega(t)^2z^2) \quad (4.8)$$

with canonical commutation relations $[z,p] = i$, and let $\sigma(t)$ be any solution to the differential equation

$$\ddot{\sigma}(t) + \omega(t)^2\sigma(t) = c\sigma(t)^{-3}, \quad (4.9)$$

where $c$ is a constant. This constant can be scaled by a positive number by scaling $\sigma$, so only the sign of $c$ is relevant.\footnote{And usually $c$ is absorbed into $\sigma$ by $\sigma(t) = e^{ct/4}\rho(t)$, upon which (4.9) becomes independent of $c$. Here we do not yet do this as we don’t want to prejudice the sign of $c$.}

Then it can be checked by a straightforward calculation, using the Heisenberg equations of motion, that

$$I(t) = \frac{1}{2}(cz^2\sigma^{-2} + (\sigma p - \dot{\sigma} z)^2) \quad (4.10)$$

is an invariant in the sense of (4.2). Up to a scale, this is the most general invariant of a time-dependent harmonic oscillator that is a homogeneous quadratic form in $z$ and $p$.\footnote{We now introduce the raising and lowering operators $a^\dagger, a,$}

$$a = \frac{1}{\sqrt{2}}(\alpha(t)p + \beta(t)z), \quad a^\dagger = \frac{1}{\sqrt{2}}(\alpha^*(t)p + \beta^*(t)z). \quad (4.11)$$
where \( \alpha(t) \) and \( \beta(t) \) are complex functions, and try to write \( I \) in oscillator form,

\[
I = a^\dagger a + \frac{1}{2}
\]

\[
= \frac{1}{2}(|\alpha|^2 p^2 + |\beta|^2 z^2) + \frac{1}{4}(\alpha^* \beta + \alpha \beta^*)(zp + pz) \quad (4.12)
\]

The condition \([a, a^\dagger] = 1\) imposes the requirement

\[
\alpha^* \beta - \alpha \beta^* = 2i \quad (4.13)
\]

and comparison of (4.10) and (4.12) yields the conditions

\[
|\alpha|^2 = \sigma^2 \\
|\beta|^2 = \dot{\sigma}^2 + c\sigma^{-2} \\
\alpha^* \beta + \alpha \beta^* = -2\sigma \dot{\sigma} \quad (4.14)
\]

By calculating \(|\alpha|^2 |\beta|^2\) in two different ways from these equations, one finds the condition \(c = 1\). Thus while any solution to (4.9) gives an invariant, it is only the solutions with \(c = 1\) (or positive \(c\)) that have an oscillator realisation in terms of oscillators for which \(a^\dagger\) is the hermitian conjugate of \(a\). The oscillator representation also imposes constraints on the coefficients of the different terms in (4.12) which we will deduce in the next section.

With \(c = 1\), the general solution to (4.9) can be written in terms of any two (real or complex) linearly independent solutions to the harmonic oscillator equation for \(H_{osc}(t)\) (this is (4.9) with \(c = 0\)). Denoting these two solutions by \(f_1\) and \(f_2\), and normalising their Wronskian to \(\pm 1\), the general solution is

\[
\sigma = \pm \left[c_1^2 f_1^2 + c_2^2 f_2^2 + 2(c_1^2 c_2^2 - 1)^{1/2} f_1 f_2\right]^{1/2} \quad (4.15)
\]

where \(c_i\) are constants (subject to the condition that the solution is real) and the signs can be chosen independently. For any such solution, a possible expression for the oscillators is

\[
a = \frac{1}{\sqrt{2}}(z\sigma^{-1} + i(\sigma p - \dot{\sigma} z)), \quad a^\dagger = \frac{1}{\sqrt{2}}(z\sigma^{-1} - i(\sigma p - \dot{\sigma} z)) \quad (4.16)
\]

These can still be multiplied by (possibly time-dependent) phases, and we will exploit this freedom below. From this one finds that the relation between \(H(t)\) and \(I(t)\) is

\[
H(t) = c(t)(a)^2 + c(t)^*(a^\dagger)^2 + d(t)(a^\dagger a + \frac{1}{2})
\]

\[
c(t) = c_1(t) + ic_2(t)
\]

\[
c_1(t) = \frac{1}{4}(\omega(t)^2 \sigma(t)^2 + \dot{\sigma}(t)^2 - \sigma(t)^{-2})
\]

\[
= \frac{1}{4}(\dot{\sigma}(t)^2 - \sigma(t)\ddot{\sigma}(t))
\]

\[
c_2(t) = -\frac{1}{2}\sigma(t)^{-1}\dot{\sigma}(t)
\]

\[
d(t) = \frac{1}{2}(\omega(t)^2 \sigma(t)^2 + \dot{\sigma}(t)^2 + \sigma(t)^{-2}) \quad (4.17)
\]
The eigenfunctions $|s\rangle$ of $I$ can be constructed in the standard way from the ground state $|0\rangle$ with $a|0\rangle = 0$,

$$I|s\rangle = (s + \frac{1}{2})|s\rangle, \quad a|s\rangle = s^{\frac{1}{2}}|s - 1\rangle, \quad a^\dagger|s\rangle = (s + 1)^{\frac{1}{2}}|s + 1\rangle, \quad s = 0, 1, \ldots \quad (4.18)$$

Then the non-vanishing matrix elements of the Hamiltonian are

$$\langle s|H(t)|s\rangle = (s + \frac{1}{2})d(t)$$
$$\langle s|H(t)|s + 2\rangle = (s + 2)^{1/2}(s + 1)^{1/2}c(t)$$
$$\langle s|H(t)|s - 2\rangle = s^{1/2}(s - 1)^{1/2}c(t)^* \quad (4.19)$$

To determine the phases which relate these eigenfunctions to solutions of the Schrödinger equation we need to solve the differential equation $[4.10]$ for the phase factor. For this we need to know the diagonal matrix elements of $H(t)$, which we have already determined, and $\partial_t$. The latter can be expressed recursively in terms of $\langle 0|\partial_t|0\rangle$,

$$\langle s|\partial_t|s\rangle = \langle 0|\partial_t|0\rangle - 2isc_1(t) \quad (4.20)$$

The state $|0,t\rangle$ is only fixed up to a time-dependent phase. For example, in the z-representation, the ground state at time $t$ has the form

$$\langle z|0,t\rangle = (\pi \sigma^2)^{-1/4}e^{i\phi(t)}e^{-z^2(1 - i\sigma\dot{\sigma})/2\sigma^2} \quad (4.21)$$

where $\phi(t)$ is an arbitrary time dependent phase. One can for instance choose $\phi(t) = 0$ and then calculate $\langle 0|\partial_t|0\rangle$. Alternatively, in $[13]$ a particular choice for $\phi(t)$ is made which has the property that $\langle s|\partial_t|s\rangle$ vanishes for constant $\sigma(t)$ (hence $\omega(t)$ constant) and makes a 'zero-point' contribution to $[4.20]$, resulting in

$$\langle s|\partial_t|s\rangle = -2i(s + \frac{1}{2})c_1(t) \quad (4.22)$$

Another natural choice, which we will adopt here, is to set this zero-point contribution to zero. Then $[4.10]$ becomes

$$\frac{d\alpha_s}{dt} = s(2c_1(t) - d(t)) = -s\sigma(t)^{-2} \quad , (4.23)$$

or, up to an irrelevant constant,

$$\alpha_s(t) = -s \int\limits_0^t dt'\sigma(t')^{-2} \quad , (4.24)$$

and the solutions of the time-dependent Schrödinger equation for $H(t)$ are linear combinations of the states

$$|t, s\rangle = e^{i\alpha_s(t)}|s\rangle \quad (4.25)$$
One way of summarising this entire construction is to note that the oscillators defined in (4.11) are not invariant. Rather, one has
\[
\frac{d}{dt} a = -i\sigma^{-2} a \quad \frac{d}{dt} a^\dagger = i\sigma^{-2} a^\dagger ,
\]
so that
\[
\tilde{a} = e^{-i\alpha_1(t)} a \quad \tilde{a}^\dagger = e^{i\alpha_1(t)} a^\dagger
\]
are invariant. \(I(t)\) has the same expression in terms of these oscillators as in terms of the \(a, a^\dagger\),
\[
I(t) = a^\dagger a + \frac{1}{2} = \tilde{a}^\dagger \tilde{a} + \frac{1}{2} ,
\]
which makes it manifest that \(I(t)\) is an invariant. Moreover the oscillator states \(|\tilde{s}\rangle\) constructed using this oscillator basis are precisely the states \(|s\rangle\),
\[
|\tilde{s}\rangle = e^{i\alpha_\ell(t)} |s\rangle = |t, s\rangle
\]
which solve the Schrödinger equation.

Thus, if one could somehow construct these invariant oscillators directly, then one could bypass the bulk of the LR procedure. And indeed plane waves provide a way of doing just this.

4.3 Deducing the Lewis–Riesenfeld Procedure from the Plane Wave Geometry

We begin by recalling that the conserved charges (3.51) associated with the Heisenberg algebra Killing vectors are
\[
Q(X^{(J)}) = p^i b_i^{(J)} - \dot{b}_i^{(J)} z^i .
\]
Promoting these to quantum operators and using the basis of solutions introduced in section 3.2, we thus have the operators
\[
\hat{Q}(X^{(k)})(t) = p^i \hat{b}_i^{(k)}(t) - \hat{\dot{b}}_i^{(k)}(t) z^i
\]
\[
\hat{Q}(X^{* (k)})(t) = p^i \hat{b}_i^{*(k)}(t) - \hat{\dot{b}}_i^{*(k)}(t) z^i .
\]
These operators are invariants in the sense of (4.2),
\[
\frac{d}{dt} \hat{Q}(X^{(k)})(t) = \frac{d}{dt} \hat{Q}(X^{* (k)})(t) = 0 ,
\]
and as a consequence of \([z^k, p^j] = i\delta_{kl}\) and the initial conditions (3.14) they satisfy the algebra
\[
[\hat{Q}(X^{(k)})(t), \hat{Q}(X^{* (l)})(t)] = i\delta_{kl} .
\]
Noting that
\[
\hat{Q}(X^{(k)})(t_0) = p^k \quad \hat{Q}(X^{(k)*})(t_0) = -z^k ,
\]
we are led to define the quantum operators
\[
Z^k(t) \equiv -\hat{Q}(X^{(k)*})(t) \quad P^k(t) = \hat{Q}(X^{(k)})(t) ,
\]
with
\[
[Z^k(t), P^l(t)] = i\delta_{kl} .
\]
Now any quadratic operator in these variables (with constant coefficients) is a quadratic invariant. In the one-dimensional case \((d = 3)\), we can suggestively write this invariant as
\[
\hat{I}(Z(t), P(t)) = \frac{1}{2M} P(t)^2 + \frac{\Omega^2}{2} Z(t)^2 + \frac{J}{4} (P(t) Z(t) + Z(t) P(t)) ,
\]
where \(M, \Omega^2\) and \(J\) are (not necessarily positive) constants. Thus given any quadratic invariant \(I(z(t), p(t), t)\) of the original quantum system, say
\[
I(z(t), p(t), t) = \frac{1}{2m(t)} p(t)^2 + \frac{\omega(t)^2}{2} z(t)^2 + \frac{j(t)}{4} (p(t) z(t) + z(t) p(t)) ,
\]
it must be possible to express it in the form \((4.37)\). Comparing \(\hat{I}\) with \(I\) at \(t = t_0\), one finds that in terms of \(Z\) and \(P\) the invariant \(I\) is \(\hat{I}\) with \(M = m(t_0)\), \(\Omega^2 = \omega(t_0)^2\), \(J = j(t_0)\), or
\[
I(z(t), p(t), t) = I(Z(t), P(t), t_0) .
\]
Expanding \((4.37)\) in terms of the linearly independent solutions \(b\) and \(b^*\) and the original variables \((z, p)\), one finds an expression for \(\hat{I}\) analogous to \((4.10)\). We will now show that when \(\hat{I}\) has an oscillator representation, i.e.
\[
\hat{I} = A^\dagger A + \frac{1}{2} ,
\]
we reproduce precisely the LR invariant \((4.10)\) with \(\sigma\) given by \((4.15)\).

First let us note that not every invariant can be written in this way. Indeed, writing
\[
A = \frac{1}{\sqrt{2}}(\alpha P + \beta Z), \quad A^\dagger = \frac{1}{\sqrt{2}}(\alpha^* P + \beta^* Z) ,
\]
(as in \((1.11)\), but now with constant coefficients), \((1.12)\) implies constraints on the coefficients of a quadratic invariant that has a standard oscillator realisation. The obvious constraints are that the coefficients of \(P^2\) and \(Z^2\) both be positive. But there is also a less obvious constraint on the relative magnitude of the \(P^2\) and \(Z^2\) terms versus the \((PZ + ZP)\) term, as a consequence of
\[
(\alpha \beta^* + \alpha^* \beta)^2 \leq 4|\alpha|^2 |\beta|^2 .
\]
\(^9\)In \(d > 3\) we would also need to include angular momentum terms.
Given that $\alpha\beta^*$ cannot be real, this is actually a strict inequality. Here this means that this ansatz implies the constraints $M > 0$ and $\Omega^2 > 0$, as well as

$$\Omega^2 > \frac{MJ^2}{4}. \quad (4.43)$$

If this inequality is satisfied, then $\hat{I}$ takes the form (4.10) of the general Lewis-Riesenfeld invariant. We can identify what $\sigma^2$ is by identifying it with the coefficient of $p^2$ in the expansion of $\hat{I}$ in terms of $p$ and $z$. The upshot is that $\sigma$ has precisely the form given in (4.15) with $f_1 = b^*, f_2 = b, c_1^2 = \Omega^2, c_2^2 = 1/M$, provided that

$$\Omega^2 = \frac{MJ^2}{4} + M. \quad (4.44)$$

This can always be achieved by an overall rescaling of $\hat{I}$.

We have thus come full circle. Starting with the conserved charges associated with the Heisenberg algebra Killing vectors, we have constructed the most general quadratic invariant and have reproduced the general Lewis-Riesenfeld invariant in those cases where the invariant has an oscillator realisation. Constructing the Fock space in the usual way, one then obtains all the solutions to the time-dependent Schrödinger equation by choosing the phase factor of the vacuum appropriately.

The extra feature in the case of HPWs is that there is a preferred invariant, namely the operator $I_X$ associated to the conserved charge $Q(X)$ on which one might like to base the quantisation. For the first family (2.50) of HPWs, this invariant is

$$I_X = H_{osc}(p, z) + f_{ik} z^i p^k, \quad (4.45)$$

and for the second family (2.51)

$$I_X = t H_{osc}(p, z) - \frac{1}{4}(z^i p^i + p^i z^i) + f_{ik} z^i p^k. \quad (4.46)$$

We see that these invariants that we have found by geometric reasoning have an advantage over the general LR invariants in that they are very simply related to the Hamiltonian of the associated non-relativistic quantum mechanical system in which we are interested. Basing the quantisation of this time-dependent system on $I_X$ is thus as close as one might hope to get to the standard quantisation of a time-independent harmonic oscillator.

4.4 COMMENTS ON LIGHTCONE QUANTISATION FOR THE $1/(x^+)^2$ POTENTIAL

In this section, we will illustrate some of the aspects of the procedure outlined above in the case of the metric (2.15). More specifically, we will choose

$$B_{ij} = -\omega^2 \delta_{ij}. \quad (4.47)$$
This example has already been studied in considerable detail in [30], also employing the LR procedure, so we will limit ourselves to some comments which, we believe, complement the discussion in [30].

Without loss of generality, we consider only the three-dimensional case, i.e. the one-dimensional harmonic oscillator with Hamiltonian

$$H_{osc}(p,z) = \frac{1}{2}(p^2 + \omega^2 z^2) .$$

(4.48)

As discussed in section 3.5, the spacetime invariant associated to the Killing vector $X = x^+ \partial_+ - x^- \partial_-$ is

$$I_X = t H_{osc}(p,z) - \frac{1}{4}(pz + zp) .$$

(4.49)

Comparing with the general expression (4.10) for the invariant, we see that the above expression looks like it is associated with a solution $\sigma(t)$ to (4.9) which is of the form

$$\sigma(t) = bt^{1/2}$$

(4.50)

for some constant $b$. This is indeed a remarkably simple solution to this equation, here

$$\ddot{\sigma}(t) + \omega^2 t^{-2}\sigma(t) = c\sigma(t)^{-3} ,$$

(4.51)

provided that

$$\omega^2 - \frac{1}{4} = \frac{c}{b^2} .$$

(4.52)

We see that this has a solution for positive $c$ (or $c = 1$) if and only if $\omega^2 > 1/4$, and thus only in this case does our invariant have a standard oscillator realisation, the precise relation being

$$I(\sigma(t) = bt^{1/2}) = b^2 I_X = (\omega^2 - \frac{1}{4})^{-1/2} I_X .$$

(4.53)

In the range $\omega^2 > 1/4$, the general solution to the oscillator equation has been given in (2.18). Comparison of (1.52) and (2.18) shows that $b^2 = 1/\beta$, and our $\sigma(t)$ arises from the general solution (1.15) for the particular choice

$$f_1 = bt^{1/2} \sin b^{-2} \log t$$

$$f_2 = bt^{1/2} \cos b^{-2} \log t ,$$

(4.54)

and $c_1 = c_2 = 1$.

We already know that the value $\omega^2 = 1/4$ is special in many respects. In the notation of section 2.2 it corresponds to the limiting logarithmic case $B_i = -1/4$ between a trigonometric and a power-law behaviour in Rosen coordinates - see (2.18). We also know from the discussion of the isometry algebras in section 3.3, that the isometry algebra of the metric for $B_i = 1/4$ is a singular limit of the isometry algebras for
B_i \neq 1/4. The special role of this frequency was also noted in \[30\] where the emphasis was on the range 0 < \omega^2 < 1/4.

In any case, provided that \omega^2 > 1/4, we can quantise the system in a straightforward way using the simple invariant above. For example, the phase factor is (with some choice for the zero-point contribution)

\[
\alpha_s(t) = -(s + \frac{1}{2})b^{-2} \log t ,
\]

and thus the general solution to the Schrödinger equation is a linear combination of the states

\[
|t, s\rangle = t^{-i(s+\frac{1}{2})/b^2} |s\rangle .
\]

The expectation value of the Hamiltonian \( H(t) \) in any such state is

\[
\langle t, s| H(t)|t, s\rangle = (s + \frac{1}{2})b^2 \omega^2 t^{-1} ,
\]

which diverges as \( t \to 0 \). It is interesting to observe that in the limit \( t \to 0 \) the leading order \( t \)-dependence of the invariant used in \[30\] in the complementary frequency range 0 < \omega^2 < 1/4 is that of \( I_X \) for all \( t \). However, the behaviour as \( t \to \infty \) is different in that here we find no divergence.

4.5 Lightcone Quantisation of a Particle on the anti-Mach Spacetime

We now consider the lightcone quantisation of a massless particle in the anti-Mach metric \[3.68\]. As we discussed in section 3.6, the invariant \[3.56\] for the anti-Mach metric,

\[
I_X = H_{osc}(p, z) + f_{ik} z^i p^k ,
\]

is equal to the Hamiltonian \[3.63\]

\[
H(\pi, w) = \frac{1}{2}(\pi^2 - (A_0)_{ij} w^i w^j) + f_{ik} w^i \pi^k
\]

in the rotated (stationary) coordinates. Thus, in this case our strategy of adopting this invariant as a basis for the quantisation of the system is in a sense really very much a standard quantum mechanical treatment of the particle in the anti-Mach metric in stationary coordinates. This is precisely what we wanted, a quantisation procedure for HPWs modelled on the general Lewis-Riesenfeld procedure, but nevertheless as close as possible to what one does in the case of time-independent harmonic oscillators. And translating the wave functions back to the original Brinkmann coordinates, we will obtain quite non-trivial solutions to the corresponding time-dependent Schrödinger equation.
As we saw (3.37), there are three commuting isometries in this case. Hence, even though
the anti-Mach metric looks quite complicated, in an appropriate basis the Schrödinger
equation will be an ordinary differential equation for a single variable. We will work at
fixed lightcone momentum $P_− = 1$, corresponding to the Killing vector $Z$. The other
isometry commuting with $I_X$ is generated by (3.36)

$$Y = X^{(1)} + X^{(2)},$$

(4.60)

This is the Heisenberg algebra Killing vector $X(b)$ associated to the solution $b = b^{<(1)} +
b^{(2)} = (\sin t, \cos t)$ (see (3.40)) so that after the coordinate transformation (3.58) the
corresponding conserved quantity is,

$$Q(Y) = \pi^2 - w_1.$$  

(4.61)

As $[I_X,Q(Y)] = 0$ we can simultaneously diagonalize these two conserved quantities,
now thought of as quantum operators, and we will do so in the position representation.

It is simple to deal with $Q(Y)$. Noting that there is no explicit time-dependence in $I_X$
or $Q(Y)$ we make the ansatz

$$\Psi(w,t) = e^{-iEt}\psi(w)$$

(4.62)

for the eigenfunctions. The equation for $Q(Y)$ eigenfunctions is,

$$Q(Y)\psi = y\psi = \left(\frac{1}{i} \frac{\partial}{\partial w^2} - w_1\right)\psi,$$

(4.63)

so that

$$\psi(w) = e^{i\nu^2(w_1+y)}\rho(w_1).$$

(4.64)

Diagonalizing also $I_X$,

$$I_X\psi(w) = E\psi(w)$$

(4.65)

we obtain the equation for $\rho$,

$$\frac{1}{2}((\pi^1)^2 + (w_1 + y)^2 - 2w_2\pi_1 + 2w_1(w_1 + y) - (w_1)^2 + (w_2)^2)\rho = E\rho,$$

(4.66)

or

$$\left(-\frac{1}{2} \frac{\partial}{\partial w_1}\right)^2 + (w_1 + y)^2)\rho = (E + \frac{y^2}{2})\rho. $$

(4.67)

This is clearly an harmonic oscillator equation, with a shifted center, as in the case of
a particle in a magnetic field. We can therefore immediately read off the eigenvalues,
$\sqrt{2}(s + \frac{1}{2})$, and corresponding solutions,

$$\rho_{s,y}(w_1) = 2^{-s/2}(s!)^{-1/2}2^{1/8}e^{-\frac{(w_1+y)^2}{\sqrt{2}}}H_{s}(2^{1/4}(w_1 + y)),$$

(4.68)
where $H_s$ are Hermite polynomials. The eigenvalues of $I$ are then
\[ E = \sqrt{2}(s + \frac{1}{2}) - \frac{y^2}{2}. \]  
(4.69)

It is interesting to note that, as one may have anticipated from the metric in the coordinates of (3.68), the wave-functions are a combination of an harmonic oscillator in the $w^1$ direction, and a free particle in the $w^2$ direction, the two directions however tied together by the eigenvalue of $Q(Y)$ which shifts the centre of the oscillator from $w^1$ to $w^1 + y$.

We also note that there is a negative contribution to the energy. This should not come as a surprise as we are dealing with a vacuum plane wave metric and thus unavoidably with real and imaginary oscillator frequencies. For any value of $y$ there are a finite number of oscillator states with negative energy. For further discussion of such ‘tachyonic’ modes in plane waves see [37, 38].

As noted in the beginning of this section the Hamiltonian for the stationary (rotated) coordinates is identical to the invariant arising from the extra isometry of the anti-Mach metric and thus our eigenfunctions $\Psi(w, t)$ are solutions to the Schrödinger equation
\[ (i\frac{\partial}{\partial t} - H(\pi, w))\Psi = 0. \]  
(4.70)

The change of coordinates (3.58) and (3.64) that takes one from the original time-dependent form of the anti-Mach metric to the time-independent rotated form is a canonical transformation on the phase space of the non-relativistic system. This simple fact allows us, from the above solution to the Schrödinger equation in $\pi, w$ coordinates, to write down immediately the corresponding solution to the apparently much less trivial equation
\[ (i\frac{\partial}{\partial t} - H(p, z, t))\Phi = 0. \]  
(4.71)
in the $p, z$ coordinates. The solution is
\[ \Phi(z, t) = \Psi(e^{-tf} z, t). \]  
(4.72)

Thus the solution to the Schrödinger equation has a significantly more complicated time-dependence than the simple exponential dependence that appears in $\Psi(w, t)$.

Once again we see that the invariant derived from the additional isometry of an homogeneous plane wave plays an important role in the simplification of the physics.

5 Conclusion Comments

In this paper, motivated by the search for potentially exactly solvable time-dependent plane wave string backgrounds, we have obtained a classification of all homogeneous
plane waves. We found two families of solutions, (2.50) and (2.51), which generalise respectively the Cahen-Wallach metrics (2.5) and the $1/(x^+)^2$-type plane waves (2.15), and we discussed some of the more elementary properties of these new HPW metrics.

We also explained how the Lewis-Riesenfeld approach to the quantisation of time-dependent harmonic oscillators, which govern the lightcone gauge dynamics of any plane wave, can be understood in terms of the Heisenberg isometry algebra of a plane wave geometry. For HPWs we advocated the use of the special invariant $I_X$, associated with the extra Killing vector $X$ and closely related to the lightcone Hamiltonian, as a basis for quantisation, and we illustrated this proposal in two examples.

Clearly there are many other things that can or should be done. Foremost among them is perhaps an analysis of string theory on these backgrounds. In particular, one would like to know if string theory on these HPWs is exactly solvable (in the sense of [30]), as is the case for the Cahen-Wallach metrics [17, 18] and the metrics (2.15) [30]. If the answer to this is affirmative (and we believe that this is quite likely), there are several avenues to explore.

For instance, in the case of the generalised Cahen-Wallach metrics (2.50) one might wonder whether there is an analogue of the BMN correspondence [16], i.e. whether there is a dual gauge theory description of string theory in these backgrounds. In particular, one would like to know if these HPW backgrounds arise from Penrose limits of brane configurations with a known gauge theory description. Given the essentially non-diagonal nature of the HPW metrics, one should look at (perhaps rotating) supergravity solutions which are themselves sufficiently non-diagonal.

It would also be very interesting if string theory in the other family (2.51) of backgrounds turned out to be exactly solvable, as one could then address issues related to the nature of their singularities (stability, mode creation, backreaction, ...) in a string theory setting, in the spirit of recent studies of time-dependent orbifolds and related models [48, 49, 50, 51, 52, 53]. In particular, one could, as in [30], explore the possibility of continuing string theory through the singularity.

Also various geometric and global aspects of HPWs remain to be clarified, such as their causal structure and the nature of their singularities. It would also be nice to exhibit these HPWs explicitly as Lorentzian homogeneous spaces, as was done for the Cahen-Wallach metrics in [28] and for the metrics (2.15) in [30].

Finally, it might be of interest to generalise the analysis of the Killing equation to pp-wave spacetimes. A generic pp-wave has a single Killing vector, $Z$, and while there are no homogeneous pp-waves that are not plane waves, there are pp-waves that have an extra Killing vector akin to the $X$ that we have been considering which would be related
to the lightcone Hamiltonian. This might be of particular interest in the context of the
pp-waves leading to integrable worldsheet theories [44, 55, 56]. One might even wonder
if a geometrisation of the quantisation of integrable models exists by embedding them
into pp-wave backgrounds, along the lines we described for the time-dependent harmonic
oscillator in section 4.3.

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A Rosen and Brinkmann Coordinates

In this Appendix, we briefly outline the relation between the plane wave metric in
Brinkmann coordinates (2.1) and Rosen coordinates (2.2). How to pass from the latter
to the former has already been described in detail e.g. in [14, 15]. We want to emphasise
the role of the oscillator (or Killing vector) equation (3.8) in passing from Brinkmann
to Rosen coordinates.

A.1 From Rosen to Brinkmann coordinates

To pass from Rosen to Brinkmann coordinates, one first chooses a matrix $Q^i_j(u)$ such
that

$$C_{ij}(u)Q^i_k(u)Q^j_l(u) = \delta_{kl}$$  \hspace{1cm} (A.1)

(so that $Q^i_j(u)$ is an inverse vielbein for the metric $C_{ij}(u)$), and subject to the symmetry
condition

$$C_{ij}(u)\dot{Q}^i_k(u)Q^j_l(u) = C_{ij}(u)Q^i_k(u)\dot{Q}^j_l(u)\ ,$$  \hspace{1cm} (A.2)

where an overdot denotes a $u$-derivative. Such a $Q$ can always be found and is unique
up to $u$-independent orthogonal transformations [15].

The coordinate transformation mapping (2.2) to (2.1) is

$$u = x^+$$
$$v = x^- - \frac{1}{2}C_{ij}\dot{Q}^i_kQ^j_lz^kz^l$$
$$y^i = Q^i_jz^j\ ,$$  \hspace{1cm} (A.3)
and the matrices $C_{ij}$ and $A_{ij}$ are related by

$$A_{kl} = -(C_{ij} \dot{Q}^i_k) \dot{Q}^j_l .$$  \hspace{1cm} (A.4)

The matrix $A$ is essentially the Riemann curvature tensor in Rosen coordinates. Indeed, the non-trivial Christoffel symbols are $\Gamma_{iju} = -\Gamma_{uij} = \dot{C}_{ij}$, and the only non-vanishing components of the curvature tensor are

$$R_{uiuj} = -\frac{1}{2} \dot{C}_{ij} + \frac{1}{4} \dot{C}_{ik} C^{kl} \dot{C}_{lj} ,$$  \hspace{1cm} (A.5)

so that

$$A_{kl} = -Q^i_k Q^j_l R_{uiuj}$$  \hspace{1cm} (A.6)

and

$$A_{kl} z^k z^l = -R_{uiuj} y^i y^j .$$  \hspace{1cm} (A.7)

This shows that the Brinkmann coordinates are Riemann normal coordinates (centered at $z^k = 0$), at least as far as the transverse coordinates are concerned, and that for plane waves this kind of Riemann normal coordinate expansion is exact at quadratic order. This truncation to the first non-trivial term in the Riemann normal coordinate expansion for coordinates transverse to the null geodesic is another way of looking at the Penrose limit.

In particular, the only non-vanishing component of the Ricci tensor is

$$R_{uu} = C^{ij} R_{uiuj}$$  \hspace{1cm} (A.8)

in Rosen coordinates, or simply

$$R_{++} = -\text{Tr} A$$  \hspace{1cm} (A.9)

in Brinkmann coordinates, and the metric is flat iff $A_{ij} = 0$. In these coordinates the vacuum Einstein equation thus reduces to a simple algebraic condition on $A_{ij}$, namely that it be traceless.

The above considerations explain why, while the procedure of determining the metric in Brinkmann coordinates from that in Rosen coordinates is in principle straightforward, the converse procedure is more involved as it is essentially equivalent to the problem of finding a metric given its curvature tensor (given $A_{ij}$, find $C_{ij}$).

The equations simplify significantly when the metric $C_{ij}(u)$ is diagonal,

$$C_{ij}(u) = e_i(u)^2 \delta_{ij} .$$  \hspace{1cm} (A.10)

so that one can choose

$$Q^i_j = e^{-1}_i \delta^i_j .$$  \hspace{1cm} (A.11)
In that case
\[ A_{ij} = (\ddot{e}_i / e_i) \delta_{ij} \]  \hspace{1cm} (A.12)

In particular, the metric \( C_{ij}(u) \) solves the vacuum Einstein equations iff
\[ \sum_i (\ddot{e}_i / e_i) = 0 \]  \hspace{1cm} (A.13)

and is it is flat iff \( e_i(u) = a_i u + b_i \) for some constants \( a_i, b_i \).

### A.2 From Brinkmann to Rosen Coordinates

It follows from the above that, given a plane wave metric in Brinkmann coordinates with a diagonal \( A_{ij} \),
\[ A_{ij}(x^+) = a_i(x^+) \delta_{ij} \]  \hspace{1cm} (A.14)

the solution in Rosen coordinates is obtained by solving the differential equations
\[ \ddot{e}_i(u) = a_i(u) e_i(u) \]  \hspace{1cm} (A.15)

We will seek an analogue of this equation for a general \( A_{ij}(x^+) \). It will be useful to employ a shorthand matrix notation in which the relations (A.1,A.2,A.4) take the form
\[ Q^T C Q = I \]  \hspace{1cm} (A.16)
\[ \dot{Q}^T C Q = Q^T C \dot{Q} \]  \hspace{1cm} (A.17)
\[ A = - (\dot{Q}^T C) Q \]  \hspace{1cm} (A.18)

The symmetry condition (A.17) is equivalent to
\[ Q^{-1} \dot{Q} = (Q^{-1} \dot{Q})^T \]  \hspace{1cm} (A.19)

Using this property, one can find an expression for \( A \) in terms of \( Q \) only,
\[ A = 2(Q^{-1} \dot{Q})^2 - (Q^{-1} \ddot{Q}) \]  \hspace{1cm} (A.20)

This equation, regarded as a differential equation for \( Q \), given \( A \), can be linearised by multiplying it on the left by the matrix
\[ E = (Q^T)^{-1} \]  \hspace{1cm} (A.21)
contragredient to \( Q \) and using again the symmetry (A.19). Just as \( Q \) had an interpretation as an inverse vielbein for \( C \), \( E \) is a vielbein for \( C \) (hence the notation),
\[ C = EE^T \]  \hspace{1cm} (A.22)
In terms of $E$, the relation between $C$ and $A$ is simply

$$
\dot{E} = EA \ .
$$

(A.23)

This is the matrix counterpart of the relation (A.15) valid for a diagonal metric. It is interesting to observe the similarity between the $n = (d - 2)$ oscillator or Killing vector equations (3.8),

$$
\ddot{b}_i = b_j A_{ji},
$$

(A.24)

and the $(n \times n)$ matrix equation (A.23), in components

$$
\dot{E}_{ki} = E_{kj} A_{ji} \ .
$$

(A.25)

This shows that a matrix formed from any $n$ of the $2n$ solutions $b^{(J)}$,

$$
E = \begin{pmatrix}
    b^{(J_1)} \\
    \vdots \\
    b^{(J_n)}
\end{pmatrix},
$$

(A.26)

or, in components,

$$
E_{ki} = b^{(J_k)}_i \ ,
$$

(A.27)

solves the differential equation (A.25). However, we cannot yet claim that any such $E$ will give rise to the plane wave metric in Rosen coordinates. Indeed, in the derivation of (A.23,A.25) (as well as in the explicit coordinate transformation relating the two systems of coordinates) a crucial role was played by the symmetry of $Q^{-1} \dot{Q}$. We will see that this (up to now somewhat mysterious) condition has a natural geometric interpretation in the present context.

In terms of $E$, the symmetry condition is

$$
W = \dot{E}E^T - EE^T = 0 \ .
$$

(A.28)

Clearly, $W$ is just the Wronskian,

$$
W_{ki} = W(b^{(J_k)}, b^{(J_l)}) \ .
$$

(A.29)

Hence vanishing of $W$ means that in the construction of the matrix $E$ one is to use the solutions $b^{(J_i)}$ corresponding to any maximal set of commuting Killing vectors, e.g. the $X^{(i)}$ or the $X^{* (j)}$.

This is very natural. Indeed, passing from Brinkmann to Rosen coordinates can be interpreted as passing to coordinates in which half of the translational Heisenberg symmetries are manifest. This is achieved by choosing the (transverse) coordinate lines to be the integral curves of these Killing vectors. This is of course only possible if these
Killing vectors commute and results in a metric which is independent of the transverse coordinates.

In any case, having chosen such a set of Killing vectors, the metric in Rosen coordinates can then be immediately constructed (without having to use an explicit coordinate transformation) as

\[ C_{ik} = E_{ij} E_{kj} \]  \hspace{1cm} (A.30)

where \( E_{ij} \) is constructed from the corresponding functions \( b^{(J_k)} \) according to the above recipe. In terms of the basis of solutions introduced in section 3.2, a natural choice is \( b^{(J_k)} = b^{(k)} \) or \( b^{(J_k)} = b^{*(k)} \), so that e.g.

\[ C_{ik} = b^{(i)} b^{(k)} \equiv b^{(i)} b^{(k)} \]  \hspace{1cm} (A.31)

This expression can be useful in applications even if the \( b^{(i)} \) are not known explicitly.

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