SUPERSPECIAL ABELIAN VARIETIES OVER FINITE PRIME FIELDS

CHIA-FU YU

Abstract. In this paper we determine the number of isomorphism classes of superspecial abelian varieties $A$ over the prime field $\mathbb{F}_p$ such that the relative Frobenius morphism $\pi_A$ satisfies $\pi_A^2 = -p$.

1. Introduction

Let $p$ be a rational prime number and $g \geq 1$ be a positive integer. An abelian variety over a field $k$ of characteristic $p$ is said to be superspecial if it is isomorphic to a product of supersingular elliptic curves over an algebraic closure $\overline{k}$ of $k$. Let $E_0$ be a supersingular elliptic curve over $\mathbb{F}_p$ such that $\pi_{E_0}^2 = -p$, where $\pi_{E_0}$ is the relative Frobenius endomorphism of $E_0$. Such an elliptic curve exists (see Deuring [3] or use the Honda-Tate theory [17, Theorem 1, p. 96]) and the condition $\pi_{E_0}^2 = -p$ is automatic if $p > 3$; the latter follows from the Hasse-Weil bound for eigenvalues of the Frobenius. Let $S$ be the set of isomorphism classes of $g$-dimensional superspecial abelian varieties $A$ over $\mathbb{F}_p$ such that there is an isogeny from $E_g^0$ to $A$ over $\mathbb{F}_p$.

Using a theorem of Tate [18, Theorem 1 (c), p. 139], the condition for $A$ isogenous to $E_g^0$ over $\mathbb{F}_p$ is equivalent to that the relative Frobenius morphism $\pi_A$ of $A$ over $\mathbb{F}_p$ satisfies $\pi_A^2 = -p$ (also see Lemma 2.2). In this paper we calculate the number of these superspecial abelian varieties.

Theorem 1.1. Notation as above, we have

$$|S| = \begin{cases} h(\sqrt{-p}), & \text{if } p = 2 \text{ or } p \equiv 1 \pmod{4}, \\ (g + 1)h(\sqrt{-p}), & \text{if } p \equiv 7 \pmod{8} \text{ or } p = 3, \\ (g + 3)h(\sqrt{-p}), & \text{if } p \equiv 3 \pmod{8} \text{ and } p \neq 3, \end{cases}$$

where $h(\sqrt{-p})$ denotes the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$.

It remains to discuss some background on the topic. First of all, it is well-known (due to Deuring [3]) that every supersingular elliptic curve over an algebraically closed field of characteristic $p$ has a model defined over $\mathbb{F}_{p^2}$. Also, if we let $B_{p,\infty}$ denote the quaternion algebra over $\mathbb{Q}$ ramified exactly at $p$ and $\infty$, then the set of isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_p$ is in one-to-one correspondence with the set of ideal classes of a maximal order of the quaternion algebra $B_{p,\infty}$. The same picture can be generalized to higher dimensions: (a) there is only one isomorphism class of superspecial abelian varieties of dimension $> 1$ over $\overline{\mathbb{F}}_p$ (this is due to Deligne, Ogus and Shioda), and (b) the set $\Lambda_g$ of isomorphism

Date: April 14, 2010.

2000 Mathematics Subject Classification. 14K10, 11G10, 14G15, 11E41.

Key words and phrases. superspecial abelian varieties, class numbers, finite fields.
classes of superspecial principally polarized abelian varieties over $\mathbb{F}_p$ has the similar description as a double coset space for an algebraic group $G$ associated to certain quaternion hermitian form (see Ibukiyama-Katsura-Oort [11, Theorem 2.10]). The class number $|\Lambda_g|$ is calculated by Deuring [3, 4] and Eichler [5] for $g = 1$ and by Hashimoto and Ibukiyama [9] for $g = 2$. It is believed after [9] that calculating the class number $|\Lambda_g|$ for higher genus $g$ is an extremely difficult task. However, exploring some structures and relationships among them arising from $\Lambda_g$ is still interesting.

The well-known Deuring-Eichler mass formula suggests that instead of calculating the class number $|\Lambda_g|$ itself, the weighted version, $\text{Mass}(\Lambda_g)$, in which one associates each object $(A, \lambda)$ to the weight $\# \text{Aut}(A, \lambda)^{-1}$ and sums over the objects in $\Lambda_g$, should be more accessible (see [25, Introduction] for a discussion). This is done for $\Lambda_g$, namely the case of Siegel modular varieties, by Ekedahl [6, p. 159] and some others (see Hashimoto-Ibukiyama [9, Proposition 9, p. 568], and Katsura-Oort [12, Section 2, Theorems 5.1 and 5.3]). The analogous formula for the mass associated to the set of superspecial polarized abelian varieties with real multiplication is showed in [21, Theorem 3.7 and Subsection 4.6]. This result is applied for determining the number of supersingular components (those are entirely contained in the supersingular locus) of the reduction of Hilbert modular varieties with Iwahori level structure; see [23, Section 4]. The geometric mass formula is generalized to good reduction of quaternion Shimura varieties; see [22, Theorem 1.2] for the precise formula. The proof of this mass formula uses an arithmetic mass formula obtained by Shimura [16, Introduction, p. 68] and the connection between geometric and arithmetic masses (see [25, Theorem 2.2] for the precise statement).

In the function field analogue where superspecial abelian varieties are replaced by supersingular Drinfeld modules, the mass formula is obtained by J. Yu and the author [20, Theorem 2.1, p. 906] (for any rank and function fields), based on some earlier works of Gekeler [7, 8].

Another viewpoint to depart concerning counting superspecial points (still over $\mathbb{F}_p$) is that the superspecial locus itself is an $\ell$-adic Hecke orbit, where $\ell$ is a prime $\neq p$, in the fine moduli space (see Chai [1] for the definition and results about $\ell$-adic Hecke orbits). It turns out that the similar formalism allows one to calculate the cardinality of each supersingular Hecke orbit, provided one knows the underlying $p$-divisible group structure explicitly. The case of genus $g = 2$ is analyzed in J.-D. Yu and the author [24, Theorem 1.1] using the Moret-Bailly family [14].

Theorem 1.1 is not a weighted version. It sits between the trivial case (a) and the extremely difficult case (b) above. It is worth noting that the set $S$ is not always a single $\ell$-adic Hecke orbit over $\mathbb{F}_p$ (here two abelian varieties over $\mathbb{F}_p$ lie in the same $\ell$-adic Hecke orbit over $\mathbb{F}_p$ if there is an $\ell$-quasi-isogeny from one to another over $\mathbb{F}_p$), rather it consists of $g + 1$ Hecke orbits in some cases (see Proposition 5.3). A crucial ingredient which makes our situation simpler is that the strong approximation holds for the algebraic group $R_{E/Q} \text{SL}_{n,E}$ (see Lemma 5.1), where $E$ is a number field and $n > 1$. Note that we impose a condition on $S$ which also makes the computation simpler. As any superspecial abelian variety $A$ is isogenous (even isomorphic) to $E_0^g$ over $\mathbb{F}_p$, this condition is unseen the geometric setting. Suppose that one does not impose this isogenous condition, and let $S'$ be the set of isomorphism classes of $g$-dimensional superspecial abelian varieties over $\mathbb{F}_p$. In the case $g = 1$ Theorem 1.1
gives the size of $S'$ for $p > 3$, as there is only one isogeny class in $S'$. It is not hard to treat the cases $p = 2$ and $3$ separately, and we get

$$|S'| = \begin{cases} 2h(\sqrt{-1}) + h(\sqrt{-2}), & \text{if } p = 2, \\ 4h(\sqrt{-3}), & \text{if } p = 3. \end{cases} \tag{1.2}$$

For the genus $g > 1$, the set $S'$ is classified by the cohomology

$$H^1(\text{Gal}(\overline{F}_p/F_p), \text{Aut}_{F_p}(E^0_g)), \tag{1.3}$$

due to the fact (a). This is a single example of $k$-forms of quasi-projective algebraic varieties over a perfect field $k$, and this problem may not be very interesting, unless one finds some structure on (1.3) which allows us to compute it more effectively.

The proof of Theorem 1.1 uses the classification of modules over certain non-maximal order $R$ of a number field $E$. The classification is of interest on its own right; on the other hand, that is also useful to determine isomorphism classes in other isogeny classes over $F_p$ (see Theorem 4.1).

Another closely related problem studied is about the field of definition of isomorphism classes over $\overline{F}_p$. Deuring showed that the number $h'$ of isomorphism classes of supersingular elliptic curves over $\overline{F}_p$ which have a model over $F_p$ is as follows (see Deuring [4], also c.f. Ibukiyama-Katsura [10, Remark 3, p. 42]) For simplicity, one has for $p > 3$,

$$h' = \begin{cases} \frac{1}{2}h(\sqrt{-p}), & \text{if } p \equiv 1 \pmod{4}, \\ h(\sqrt{-p}), & \text{if } p \equiv 7 \pmod{8}, \\ 2h(\sqrt{-p}), & \text{if } p \equiv 3 \pmod{8}. \end{cases} \tag{1.4}$$

The relationship between (1.1) and (1.4) for $g = 1$ is that each isomorphism class over $\overline{F}_p$ which has a model over $F_p$ has exactly two isomorphism classes over $F_p$. Deuring [4] also showed the following equality

$$h' = 2t - h, \tag{1.5}$$

where $t$ is the type number of the quaternion algebra $B_{p,\infty}$ and $h$ is the class number of $B_{p,\infty}$. Ibukiyama and Katsura further generalized the result of Deuring to higher genus $g$. They showed that the same relation (1.5) holds for the number $H'$ of the objects in $\Lambda_g$ which have a model over $\overline{F}_p$, the type number $T$ of the quaternion unitary algebraic group $G$ in question, and the class number $H$ of the group $G$; see [10, Theorems 1 and 2]. However, as mentioned before, this is a relationship we could have, and no explicit formula for each term is given when $g > 2$.

Acknowledgments. The author thanks Frans Oort for helpful comments on an earlier manuscript. The manuscript is prepared during the author’s stay at l’Institut des Hautes Études Scientifiques. He acknowledges the institution for kind hospitality and excellent working conditions. The research was partially supported by grants NSC 97-2115-M-001-015-MY3 and AS-99-CDA-M01.

2. Preliminaries

Let the set $S$ and the elliptic curve $E_0$ be as in §1. Let $G$ be the Galois group $\text{Gal}(\overline{F}_p/F_p)$. We define a set $\Phi_v$ for each prime $v$ of $\mathbb{Q}$. If $\ell$ is a prime $\neq p$, let $\Phi_\ell$ be the set of isomorphism classes of Tate modules $T_\ell(A)$ as $\mathbb{Z}[G]$-modules for all $A \in S$. Let $\Phi_p$ be the set of isomorphism classes of Dieudonné modules $M(A)$ for
all $A \in S$. In this paper we use covariant Dieudonné modules. We refer the reader to Demazure [2] and Manin [13] for a basic account of Dieudonné theory.

Let $M$ be a Dieudonné module over a perfect field $k$ of characteristic $p$. We recall that the $a$-number of $M$, denoted by $a(M)$, is the dimension of the $k$-vector space $M/(F,V)M$. The $a$-number of an abelian variety over $k$, denoted by $a(A)$, is defined to be the $a$-number $a(M(A))$ of the Dieudonné module $M(A)$ associated to $A$.

**Theorem 2.1.** Let $A$ be an abelian variety of dimension $g$ over an algebraically closed field of characteristic $p$. If $a(A) = g$, then $A$ is superspecial.

**Proof.** This is Theorem 2 of Oort [15].

**Lemma 2.2.** Let $A$ be a $g$-dimensional abelian variety over $\mathbb{F}_p$. Then $A$ is isogenous to $E_0^g$ over $\mathbb{F}_p$ if and only if the relative Frobenius endomorphism $\pi_A$ of $A$ over $\mathbb{F}_p$ satisfies $\pi_A^2 + p = 0$. Furthermore, in this case the abelian variety $A$ is superspecial.

**Proof.** Using a theorem of Tate [18, Theorem 1 (c), p. 139], $A$ is isogenous to $E_0^g$ over $\mathbb{F}_p$ if and only if the characteristic polynomial of the endomorphism $\pi_A$ is equal to that of $\pi_{E_0^g}$, which is $(X^2 + p)^g$. On the other hand, since the Tate space $V_\ell(A) := T_\ell(A) \otimes \mathbb{Q}_\ell$ is semi-simple as $\mathbb{Q}_\ell[G]$-modules, that the characteristic polynomial of $\pi_A$ equal to $(X^2 + p)^g$ implies that the minimal polynomial of $\pi_A$ is $X^2 + p$. This shows the first statement.

For the second statement, we use Theorem 2.1. Since $F^2 = -p$ on $M(A)$, we get $VM(A) = FM(A)$ and hence $a(A) = g$. This proves the lemma.

Put $R := \mathbb{Z}[X]/(X^2 + p) = \mathbb{Z}[\pi]$ and $E := R \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{-p})$. Let $O_E$ be the ring of integers of $E$. For each finite place $v$ of $\mathbb{Q}$, write $R_v$, $E_v$, and $O_{E_v}$ for $R \otimes_{\mathbb{Z}} \mathbb{Z}_v$, $E \otimes_{\mathbb{Q}} \mathbb{Q}_v$, and $O_E \otimes_{\mathbb{Z}} \mathbb{Z}_v$, respectively.

Let $A$ be an object of $S$. Let $\sigma_p \in G$ be the arithmetic Frobenius automorphism $x \mapsto x^p$. We have $\sigma_p x = \pi_A x$ for all $x \in T_\ell(A)$, where $\ell$ is any prime $\neq p$.

Since $\pi_A^2 + p = 0$, the action of $G$ on $T_\ell(A)$ factors through the epi-morphism $\mathbb{Z}_\ell[G] \to \mathbb{Z}_\ell[X]/(X^2 + p)$. Therefore, we may classify Tate modules $T_\ell(A)$ as $R_\ell$-modules. On the other hand the Tate space $V_\ell(A)$ is a free $E_\ell$-module of rank $g$; this follows from the fact that $tr(a; V_\ell(A)) = g \cdot tr(a; E_\ell)$ for all $a \in R$. From this, if $R_\ell$ is the maximal order of $E_\ell$, then $T_\ell(A)$ is a free $R_\ell$-module of rank $g$. In this case the set $\Phi_\ell$ consists of single element.

In the case $v = p$, the Dieudonné module $M(A)$ is a free $\mathbb{Z}_p$-module of rank $2g$, together with a $\mathbb{Z}_p$-linear operator $F$ satisfying $F^2 + p$, and hence $M(A)$ is simply a $\mathbb{Z}_p$-free finite $R_p$-module. Since $R_p \otimes \mathbb{Q}_p = \mathbb{Q}_p(\sqrt{-p}) = E_p$ is a field and $R_p$ is always the maximal order of $E_p$, the Dieudonné module $M(A)$ is $R_p$-free of rank $g$. This shows that the set $\Phi_p$ also consists of single element.

As an elementary result, the ring $R_v$ is not the maximal order if and only if $v = 2$ and $p \equiv 3 \pmod{4}$. We conclude

**Lemma 2.3.** The set $\Phi_v$ consists of single element except when $v = 2$ and $p \equiv 3 \pmod{4}$. 

3. Classification of $\Phi_2$

In this section we assume that $p \equiv 3 \pmod{4}$. To simplify the notation, we write $R := \mathbb{Z}_2[X]/(X^2 + p) = \mathbb{Z}_2[\pi]$, $E := R \otimes_{\mathbb{Z}_2} \mathbb{Q}_2$ and $O_E$ for the ring of integers of $E$. We have

$$O_E = \mathbb{Z}_2[\alpha] = \mathbb{Z}_2[X]/(X^2 + X + (p + 1)/4).$$

Put $\omega := \pi - 1$, and one has

$$R = \mathbb{Z}_2[\omega] = \mathbb{Z}_2[X]/(X^2 + 2X + (p + 1)) \quad \text{and} \quad 2\alpha = \omega.$$

We shall classify $R$-modules $M$ which is finite and free as $\mathbb{Z}_2$-modules. We divide the classification into two cases:

**Case (a):** $p \equiv 3 \pmod{8}$. In this case, $E$ is a unramified quadratic extension of $\mathbb{Q}_2$. We have (at least) two indecomposable $\mathbb{Z}_2$-free finite $R$-modules: $R$ and $O_E$ as $R$-modules. The $R$-module structure of $O_E$ is given as follows: write $O_E = \langle 1, \alpha \rangle$, then

$$\omega 1 = 2\alpha \quad \text{and} \quad \omega \alpha = -2\alpha - (p + 1)/2.$$  

If $M = R^r \oplus O_E^s$, then the non-negative integers $r$ and $s$ are uniquely determined by $M$. Indeed, we have $r + s = \dim_E M \otimes_{\mathbb{Z}_2} \mathbb{Q}_2$, and $M/(2, \omega)M = (\mathbb{F}_2)^r \otimes (\mathbb{F}_2^r \oplus \mathbb{F}_2^s)$.

**Case (b):** $p \equiv 7 \pmod{8}$. In this case, $E = \mathbb{Q}_2 \times \mathbb{Q}_2$. Write

$$X^2 + X + (p + 1)/4 = (X - \alpha_1)(X - \alpha_2),$$

where $\alpha_1, \alpha_2 \in \mathbb{Z}_2$. By switching the order, we may assume that $\alpha_1$ is a unit and $\alpha_2 \in 2\mathbb{Z}_2$. We have the isomorphisms

$$O_E = \mathbb{Z}_2[\alpha] \cong O_E/(\alpha - \alpha_1) \times O_E/(\alpha - \alpha_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$ 

Therefore,

$$X^2 + 2X + (p + 1) = (X - 2\alpha_1)(X - 2\alpha_2).$$

We have (at least) three indecomposable $\mathbb{Z}_2$-free finite $R$-modules:

$$R, \quad R/(\omega - 2\alpha_1), \quad \text{and} \quad R/(\omega - 2\alpha_2).$$

Among them, we have

$$O_E \cong R/(\omega - 2\alpha_1) \oplus R/(\omega - 2\alpha_2)$$

as $R$-modules. If $M = R^r \oplus [R/(\omega - 2\alpha_1)]^s \oplus [R/(\omega - 2\alpha_2)]^t$, then the non-negative integers $r$, $s$ and $t$ are uniquely determined by $M$. Indeed, we have

$$\text{rank}_{\mathbb{Z}_2} M = 2r + s + t, \quad M/(2, \omega)M = \mathbb{F}_2^r \oplus \mathbb{F}_2^s \oplus \mathbb{F}_2^t,$$

and

$$M/(\omega - 2\alpha_1)M = [R/(\omega - 2\alpha_1)]^{r+s} \oplus (\mathbb{F}_2)^t.$$

Conversely, we show that the indecomposable finite $R$-modules described in Cases (a) and (b) exhaust all possibilities.

**Theorem 3.1.** Let $M$ be a $\mathbb{Z}_2$-free finite $R$-module. Then

1. **Case (a).** The $R$-module $M$ is isomorphic to $R^r \otimes O_E^s$ for some non-negative integers $r$ and $s$. Moreover, the integers $r$ and $s$ are uniquely determined by $M$. 

(2) Case (b). The $R$-module $M$ is isomorphic to
\[ R^r \oplus [R/(\omega - 2\alpha_1)]^s \oplus [R/(\omega - 2\alpha_2)]^t \]
for some non-negative integers $r$, $s$ and $t$. Moreover, the integers $r$, $s$ and $t$ are uniquely determined by $M$.

**Proof.** The unique determination of integers $r$, $s$ and $t$ has been showed. We prove the first part of each statement.

(1) Let
\[
\overline{M} := M/2M = (\mathbb{F}_2[\omega]/\omega^2)^r \oplus (\mathbb{F}_2)^s\]
be the decomposition as $R/2R = \mathbb{F}_2[\omega]/\omega^2$-modules. We first show that if $s = 0$, then $M \cong R^r$. Since $r = \dim M \otimes_R \mathbb{F}_2 = \dim_E M \otimes_R E$ and $R$ is a local Noetherian domain, the module $M$ is free.

Now suppose $s > 0$. Choose an element $\bar{a} \neq 0 \in (\mathbb{F}_2)^s$ and let $x \in M$ be an element such that $\bar{x} = \bar{a}$. As $\omega x/2 = 0$, the element $\omega x/2 \in M$. Put $M_1 := \langle x, \omega x/2 \rangle_{\mathbb{Z}_2}$; it is an $R$-module and is isomorphic to $O_E$. Let $\omega'$ be the conjugate of $\omega$, one has $\omega' = -2 - \omega$ and $\omega' = (1 + p)$. Note that $(1 + p)/4$ is a unit. Since $\bar{x} \notin \omega \overline{M}$, one has $x \notin \omega M$. We show that $\omega x/2 \neq 0$. Suppose not, then $\omega x = 4y$ for some $y \in M$. Applying $\omega'$, we get $x = \omega' y'$ for some $y' \in M$, contradiction. Since $x$ and $\omega x/2$ are $\mathbb{Z}_2$-linearly independent, the $\mathbb{F}_2$-vector space $M_2 = \langle x, \omega x/2 \rangle_{\mathbb{Z}_2}$ has dimension 2, and hence the quotient $\overline{M}/M_1$ has dimension decreased by 2. On the other hand, the $\mathbb{Z}_2$-rank of $M/M_1$ also decreases by 2. This shows that $M/M_1$ is free as $\mathbb{Z}_2$-modules. If the integer $s$ in (3.2) for $M/M_1$ is positive, then we can find an $R$-submodule $M_2 = \langle x, \omega x/2 \rangle_{\mathbb{Z}_2}$, which is not contained in the vector space $EM_1$ such that $M/(M_1 + M_2)$ is free as $\mathbb{Z}_2$-modules. Continuing this process, we get $R$-submodules $M_1, \ldots, M_{s'}$, which are isomorphic to $O_E$, such that $M_1 + \cdots + M_{s'} = M_1 + \cdots + M_{s'}$ and $M/(M_1 + \cdots + M_{s'})$ is a free $R$-module. It follows that $M \cong O_E^r \oplus R^{s'}$. Since $s'$ and $t'$ are determined by $M$ as before, the integers $s'$ and $t'$ are actually equal to $s$ and $r$ in (3.2), respectively. This proves (1).

(2) Let
\[
M_1 := \{ x \in M \mid (\omega - 2\alpha_1)x = 0 \},
\]
and
\[
M_1 := \{ x \in M \mid (\omega - 2\alpha_2)x = 0 \}.
\]
Using the relation
\[
2 = (\omega - 2\alpha_1)(2\alpha_2 + 1)^{-1} - (\omega - 2\alpha_2)(2\alpha_1 + 1)^{-1},
\]
one shows that $2M \subset M_1 + M_2$, and hence the quotient $M/(M_1 + M_2)$ is an $\mathbb{F}_2$-vector space, say of dimension $r$. Let $x_1, \ldots, x_r$ be elements of $M$ such that the images $\bar{x}_1, \ldots, \bar{x}_r$ form an $\mathbb{F}_2$-basis for $M/(M_1 + M_2)$. Put $F_0 := \langle x_1, \ldots, x_r \rangle_R$, which is isomorphic to $R^r$, as $\bar{x}_i's$ form a basis for $F_0/(M_1 + M_2) = F_0/(2, \omega)F_0 \cong \mathbb{F}_2$. Now $(\omega - 2\alpha_2)F_0 \subset M_1$, we choose elements $y_1, \ldots, y_s$ in $M_1$ so that the images $\bar{y}_1, \ldots, \bar{y}_s$ form an $R/(\omega - 2\alpha_1)$-basis for $M_1/(\omega - 2\alpha_2)F_0$, and put $F_1 := \langle y_1, \ldots, y_s \rangle_R$. We have
\[
M_1 = (\omega - 2\alpha_2)F_0 \oplus F_1, \quad \text{and} \quad F_0 \cap F_1 = 0.
\]
Similarly, we have a free $R/(\omega - 2\alpha_2)$-submodule $F_2$ of $M_2$, of rank $t$, such that
\[
M_2 = (\omega - 2\alpha_1)F_0 \oplus F_2, \quad \text{and} \quad F_0 \cap F_2 = 0.
\]
We have \((F_0 + F_1) \cap F_2 = F_0 \cap F_2 = 0\) and \(M = F_0 + F_1 + F_2\), and hence \(M = F_0 \oplus F_1 \oplus F_2\). This proves (2). ■

We retain the notation as in § 1 and 2.

**Corollary 3.2.** Assume \(p \equiv 3 \pmod{4}\), then the Tate module \(T_2(A)\) of an object \(A\) in \(S\) is isomorphic to \(O_{\mathbb{R}} \oplus O_{\mathbb{C}}\) for some non-negative integers \(r\) and \(s\) such that \(r + s = g\). Moreover, the integers \(r\) and \(s\) are uniquely determined by \(T_2(A)\).

**Proof.** Since the Tate space \(V_2(A)\) is a free \(\mathbb{Q}_2\)[\(\alpha]\]-module, the numbers \(s\) and \(t\) in Theorem 3.1 (2) above are the same. Therefore, the corollary follows. ■

**Lemma 3.3.** Assume \(p \equiv 3 \pmod{4}\). For any non-negative integers \(r\) and \(s\) with \(r + s = g\), there exists an abelian variety \(A_r\) in \(S\) such that the Tate module \(T_2(A_r)\) of \(A_r\) is isomorphic to \(O_{\mathbb{R}} \oplus O_{\mathbb{C}}\).

**Proof.** Choose a supersingular elliptic curve \(E_0\) over \(\mathbb{F}_p\) such that the endomorphism ring \(\text{End}_{\mathbb{F}_p}(E_0)\) is equal to \(O_E\), and a supersingular elliptic curve \(E_1\) over \(\mathbb{F}_p\) such that the endomorphism ring \(\text{End}_{\mathbb{F}_p}(E_1)\) is equal to \(R\) (see Waterhouse [19, Theorem 4.2 (3), p. 539]). Put \(A_r = E_r^1 \times E_0^s\), then the superspecial abelian variety \(A_r\) has the desired property. ■

4. **Abelian Varieties over \(\mathbb{F}_p\)**

For our purpose we need to describe abelian varieties up to isomorphism over \(\mathbb{F}_p\). The Honda-Tate theory [18, 17] has described isogeny classes of abelian varieties over finite fields. Therefore, we may focus on isogeny classes in one single isogeny class over \(\mathbb{F}_p\). We describe this in terms of modules so that we can count them explicitly. In this section the ground field considered is \(\mathbb{F}_p\).

Let \(A_0\) be a fixed abelian variety, and denote by \(\text{Isog}(A_0)\) the set of isomorphism classes in the isogeny class of \(A_0\). Recall that an **quasi-isogeny** \(\varphi : A \to A_0\) is an element \(\varphi \in \text{Hom}(A, A_0) \otimes \mathbb{Q}\) such that \(n\varphi\) is an isogeny for some integer \(n\). We identify two quasi-isogenies \(\varphi_i : A_i \to A_0\), \(i = 1, 2\) as the same element if there is an (necessarily unique) isomorphism \(\rho : A_1 \to A_2\) such that \(\varphi_2 \circ \rho = \varphi_1\). Therefore, it makes sense to talk about the **set** of quasi-isogenies to \(A_0\), which we denote \(\text{Q-isog}(A_0)\). The set \(\text{Q-isog}(A_0)\) can be parametrized by pairs \((H, n)\) where \(H\) is a subgroup scheme of the dual abelian variety \(A_0^\vee\) and \(n\) is a positive integer. The quasi-isogeny represented by the pair \((H, n)\) is \((1/n)\pi^n\), where \(\pi : A_0^\vee \to A_0^\vee/H\) is the canonical homomorphism and \(\pi^t : (A_0^\vee/H)^t \to A_0\) is the dual of \(\pi\).

Let \(P(X) \in \mathbb{Z}[X]\) be the minimal polynomial of the relative Frobenius endomorphism \(\pi_0 \in \text{End}(A_0)\). Put \(S = \mathbb{Z}[\pi_0] = \mathbb{Z}[X]/(P(X))\) and \(F := S \otimes \mathbb{Q}\); it is a finite-dimensional commutative semi-simple algebra over \(\mathbb{Q}\).

**Theorem 4.1.** There is a finite \(F\)-module \(V\), unique up to isomorphism, such that \(V \otimes \mathbb{Q}_l \simeq V_l(A_0)\) as \(F_l\)-modules for all primes \(\ell \neq p\), and \(V \otimes \mathbb{Q}_p \simeq M(A_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\) as \(F_p\)-modules, where \(M(A_0)\) is the Dieudonné module of \(A_0\). Moreover, there is a one-to-one correspondence between the set of quasi-isogenies \(\varphi : A \to A_0\) and the set of \(S\)-lattices in \(V\). In this correspondence, isomorphism classes in the isogeny class of \(A_0\) are in bijection with isomorphism classes in the \(S\)-lattices in \(V\).
Proof. Let the abelian variety $A_0$ be isogenous to $\prod_{i=1}^n A_0^{\ast}$, where each abelian variety $A_i$ is simple and $A_i$ is not isogenous to $A_j$ for $i \neq j$. Then the endomorphism algebra $\text{End}^0(A) := \text{End}(A) \otimes \mathbb{Q}$ is isomorphic to $\prod_{i=1}^n M_{n_i}(D_i)$, where $D_i := \text{End}^0(A_i)$ is a finite-dimensional division algebra over $\mathbb{Q}$. The center of the endomorphism algebra $\text{End}^0(A)$ is equal to $F$. If we let $F_i$ be the center of $D_i$, then $F = \prod_{i=1}^n F_i$. The field $F_i$ is the subfield of $D_i$ generated by the Frobenius endomorphism $\pi_i$ of $A_i$ over $\mathbb{Q}$. Let $e_i := [D_i : F_i]^{1/2}$ and $m_i := n_i e_i$. Then, by a theorem of Tate [18, Main Theorem, p. 134]

$$\text{End}^0(A_0^n) \otimes \mathbb{Q}_\ell = M_{m_i}(F_i \otimes \mathbb{Q}_\ell) \simeq \text{End}_{\mathbb{Q}[\pi_i] \otimes \mathbb{Q}_\ell}(V_f(A_0^n)).$$

This shows that the Tate space $V_f(A_0^n)$ is a free $F_i \otimes \mathbb{Q}_\ell$ of rank $m_i$. It follows from another theorem of Tate [19, Theorem, p. 525] that

$$\text{End}^0(A_0^n) \otimes \mathbb{Q}_p = M_{n_i}(D_i \otimes \mathbb{Q}_p) \simeq \text{End}_{\mathbb{Q}[\pi_i] \otimes \mathbb{Q}_p}(M(A_0^n) \otimes \mathbb{Q}_p).$$

This shows that the Dieudonné space $M(A_0^n) \otimes \mathbb{Q}_p$ is a free $F_i \otimes \mathbb{Q}_p$ of rank $m_i$. The integer $m_i$ is independent of $\ell$ and $p$. Put $V := \oplus F_i^{m_i}$ as a finite $F$-module, and then $V$ has the desired property.

For any abelian variety $A$, we write $T(A) := M(A) \times \prod_{\ell \neq p} T_\ell(A)$. We fix an isomorphism

$$(*) \quad T(A_0) \otimes \mathbb{A}_f \simeq V \otimes \mathbb{A}_f$$

as $F \otimes \mathbb{A}_f$-modules. Then $T(A_0)$ is an $S \otimes \mathbb{Z}$-lattice and there is an $S$-lattice $M$ in the vector space $V$ such that $M \otimes \mathbb{Z} \otimes \mathbb{Z}$ is equal to $T(A_0)$ under the fixed rational isomorphism.

Let $\varphi : A \to A_0$ be a quasi-isogeny. Then the image $L := \varphi_*(T(A)) \subset T(A_0) \otimes \mathbb{A}_f \simeq V \otimes \mathbb{A}_f$ is an $S \otimes \mathbb{Z}$-lattice. Two quasi-isogenies $\varphi_1, \varphi_2$ induce the same lattice $L$ if and only if they are the same. Conversely, given an $S \otimes \mathbb{Z}$-lattice $L$ in $V \otimes \mathbb{A}_f$, then by a theorem of Tate, there is an abelian variety $A$ together with a quasi-isogeny $\varphi : A \to A_0$ such that the image $\varphi_*(T(A))$ is equal to $L$ under the isomorphism $(*)$. Since there is a national one-to-one correspondence between the set of $S$-lattices in $V$ and the set of $S \otimes \mathbb{Z}$-lattices in $V \otimes \mathbb{A}_f$. We proved the second statement. The last statement follows from the basic fact: if a quasi-isogeny $\varphi : A_1 \to A_2$ induces an isomorphism $T(A_1) \simeq T(A_2)$, then $\varphi : A_1 \to A_2$ is an isomorphism. This proves the theorem.

It should be clear that the proof of Theorem 1.1 is straightforward. However, it seems that this simple result has not yet been used to classify abelian varieties up to isomorphism (over $\mathbb{F}_p$).

5. Proof of Theorem 1.1

5.1. Proof of Theorem 1.1. We keep the notation as in § 1 and 2. Since every abelian variety over $\mathbb{F}_p$ isogenous to $E_0^9$ is superspecial (Lemma 2.2), the set $\mathcal{S}$ classifies isomorphism classes of abelian varieties $A$ over $\mathbb{F}_p$ in the isogeny class of the abelian varieties $E_0^9$. By Theorem 4.1 the set $\mathcal{S}$ is in bijection with the set of isomorphism classes of $R$-lattices in the vector space $V = E^9$.

Recall that a genus of $R$-lattices in $V$ is a maximal set of $R$-lattices in which any two $R$-lattices are mutually isomorphic locally everywhere.
Lemma 5.1. Let $n \geq 1$ be an integer, and $K$ be an open compact subgroup of $\text{GL}_n(\mathbb{A}_{E,f})$, where $\mathbb{A}_{E,f}$ is the finite adele ring of the field $E = \mathbb{Q}(\sqrt{-p})$. Then the determinant map $\det : \text{GL}_n(E) \setminus \text{GL}_n(\mathbb{A}_{E,f})/K \to E^\times \setminus \mathbb{A}_{E,f}^\times /\det(K)$.

Proof. We may assume that $n \geq 2$. Clearly that the induced map is surjective. We show the injectivity. Let $[a]$ be an element in the target space. Fix a section $s : \mathbb{A}^\times_{E,f} \to \text{GL}_n(\mathbb{A}_{E,f})$ of the determinant map. Then the inverse image $T[a]$ of the class $[a]$ consists of elements $\text{GL}_n(E)[gs(a)]_K$ for all $g \in \text{SL}_n(\mathbb{A}_{E,f})$. The surjective map $g \mapsto \text{GL}_n(E)[gs(a)]_K$ induces a surjective map

$$\alpha : \text{SL}_n(E) \setminus \text{SL}_n(\mathbb{A}_{E,f})/K' \to T[a],$$

where $K' := s(a)Ks(a)^{-1} \cap \text{SL}_n(\mathbb{A}_{E,f})$. Since the group $\text{SL}_n$ is simply connected and $\text{SL}_n(E \otimes \mathbb{R})$ is not compact, the strong approximation holds for the algebraic group $R_E/Q \text{SL}_{n,E}$. Therefore, $T[a]$ consists of single elements and one proves the lemma.

Case (i): $p = 2$ or $p = 1 \pmod{4}$. In this case, the ring $R$ is the maximal order in $E$. Since any $R_v$-lattice in $V_v$ for a finite place $v$ of $Q$ is free, there is only one genus of $R$-lattices in $V$. The set of equivalence classes in this genus is expressed as

$$\text{GL}_2(E) \setminus \text{GL}_2(\mathbb{A}_{E,f})/\text{GL}_2(\hat{O}_E).$$

By Lemma 5.1 this double coset space is isomorphic to $E^\times \setminus \mathbb{A}_{E,f}^\times /\hat{O}_E^\times$ and hence has the cardinality $h(\sqrt{-p})$.

Case (ii): $p = 3 \pmod{4}$. In this case, the ring $R$ has index 2 in the maximal order $O_E$. At the place where $v \neq 2$, the ring $R_v$ is the maximal order, and hence any two $R_v$-lattice in $V_v$ are isomorphic. At the place where $v = 2$, by Corollary 5.2, there are $g + 1$ isomorphism classes of $R_2$-lattices in $V_2$, namely $R_2^r \otimes \hat{O}_E^{r^\times}$ for $r = 0, \ldots, g$. Therefore, there are $g + 1$ genera of $R$-lattices in $V$; those are represented by $L_r := R \otimes \hat{O}_E^{r^\times}$ for $r = 0, \ldots, g$. Let $K_r$ be the open compact subgroup of $\text{GL}_2(\mathbb{A}_{E,f})$ which stabilizes the $R \otimes \hat{Z}$-lattice $L_r \otimes \hat{Z}$. Then by Lemma 5.1, we have

$$|S| = \sum_{r=0}^{g} h_r, \quad h_r = \#E^\times \setminus \mathbb{A}_{E,f}^\times /\det(K_r).$$

It is easy to see that

$$\det(K_r) = \begin{cases} \hat{O}_E^\times, & r \neq g; \\ \hat{R}_E^\times, & r = g. \end{cases}$$

Therefore, $h_r = h(\sqrt{-p})$ for $r = 0, \ldots, g - 1$.

In the case $r = g$, we have an exact sequence of finite abelian groups:

$$1 \to \hat{O}_E^\times / (\hat{O}_E^\times \cap E^\times \hat{R}_E^\times) \to \mathbb{A}_{E,f}^\times /E^\times \hat{R}_E^\times \to \mathbb{A}_{E,f}^\times /E^\times \hat{O}_E^\times \to 1.$$
Definition 5.2. Siegel modular varieties are defined and explored.

We conclude that the important reference is Chai [1], where both \( \ell \)-adic and prime-to-\( p \) Hecke orbits in Siegel modular varieties are defined and explored.

5.2. As a final remark, we discuss a bit about Hecke orbits in our case. An important reference is Chai [1], where both \( \ell \)-adic and prime-to-\( p \) Hecke orbits in Siegel modular varieties are defined and explored.

**Definition 5.2.** Let \( k_0 \) be a field, and let \( S \) be a set of abelian varieties over \( k_0 \). Let \( A_0 \) be an abelian variety in the set \( S \), and \( \ell \) be a rational prime, not necessarily different from the characteristic of \( k_0 \). For a field extension \( k \) of \( k_0 \), we define the \( \ell \)-adic Hecke orbit of \( A_0 \) over \( k \) in \( S \) as the subset of \( S \) consisting of all abelian varieties \( A \) in \( S \) such that there is an \( \ell \)-quasi-isogeny from \( A \) to \( A_0 \) over \( k \). An \( \ell \)-quasi-isogeny \( \varphi : A_1 \to A_2 \) of two abelian varieties is a quasi-isogeny such that there is an integer \( m \in \mathbb{N} \) such that \( \ell^m \varphi \) is an isogeny of \( \ell \)-power degree.

We retain the notation as in § 1 and 2. We have \( S = \text{Isog}(E_0^g) \) and a natural map \( \varphi : \text{Isog}(E_0^g) \to S \) which sends any quasi-isogeny \( (\varphi : A \to E_0^g) \) to \([A]\), where the sets \( \text{Isog}(E_0^g) \) and \( \text{Q-isog}(E_0^g) \) are defined in § 4, and \([A]\) denotes the isomorphism class of \( A \) over \( \mathbb{F}_p \). By Theorem 4.1, there is a one-to-one correspondence between \( \text{Isog}(E_0^g) \) and \( \text{R-lattices in } V = E^g \). Under this correspondence, the set \( S \) is in bijection with the set \( \mathcal{L}/ \simeq \text{of isomorphism classes of } \mathcal{R}\text{-lattices of } V \). Let \([L]\) denote the isomorphism classes of an \( \mathcal{R}\text{-lattice } L \) in \( V \). Suppose that \( B \) is an abelian variety in \( S \). We choose a quasi-isogeny \( \varphi_0 : B \to E_0^g \) and let \( L_{\varphi_0} \) be the \( \mathcal{R}\text{-lattice corresponding to } \varphi_0 \). If \( A \) is an abelian variety in \( S \) such that there is an \( \ell \)-quasi-isogeny \( \varphi \) from \( A \) to \( B \) and let \( L \) be the corresponding \( \mathcal{R}\text{-lattice of the quasi-isogeny } \varphi_0 \circ \varphi : A \to E_0^g \), then we have the relative index \([L_{\varphi_0} : L] = \ell^m \) for some \( m \in \mathbb{Z} \). Recall that \([L_{\varphi_0} : L] = [L_{\varphi_0} : L'][L : L']^{-1} \) for any \( \mathcal{R}\text{-lattice } L' \) contained in \( L_{\varphi_0} \cap L \). From this, the \( \ell \)-adic Hecke orbit of \( B \) over \( \mathbb{F}_p \) in \( S \) corresponds to the following set

\[
\mathcal{H}_\ell([L_{\varphi_0}]) := \{ [L] \in \mathcal{L} / \simeq \mid [L_{\varphi_0} : L] = \ell^m \text{ for some } m \in \mathbb{Z} \}.
\]

In the case \( p \equiv 3 \pmod{4} \), there are \( g + 1 \) genera \( L_0, \ldots, L_g \) in \( L \) represented by the \( \mathcal{R}\text{-lattices } L_r = R^r \oplus O_{E}^{2-r} \) for \( r = 0, \ldots, g \). We further assume that \( \ell \neq 2 \). Then any two \( R\text{-lattices in } V_r \) are isomorphic, and we have

\[
\mathcal{H}_\ell([L_r]) \simeq \Gamma_{1/\ell} \setminus \text{GL}_g(E \otimes \mathbb{Q}_\ell)/K_{r, \ell},
\]

where \( \Gamma_{1/\ell} := \text{GL}_g(E) \cap \prod_{r \neq \ell} K_{r, \ell} \) and \( K_{r, \ell} \) is the \( \ell\text{-component of the open compact subgroup } K_r \). The inclusion \( \mathcal{H}_\ell([L_r]) \subset \mathcal{L}_r / \simeq \) is given by

\[
\Gamma_{1/\ell} \setminus \text{GL}_g(E \otimes \mathbb{Q}_\ell)/K_{r, \ell} \subset \text{GL}_g(E) \setminus \text{GL}_g(A_{E, f})/K_r,
\]

and we have \( \mathcal{H}_\ell([L_r]) = \mathcal{L}_r / \simeq \) if and only if

\[
\text{GL}_g(E) \setminus \text{GL}_g(A_{E, f})/K_r = 1.
\]
By Lemma 5.1, this is equivalent to $E^\times \times \mathbb{A}^{\infty}_{E,f} / \det(K_r) = 1$, or Pic($O_E[1/\ell]$) = 1 if $r \neq g$ and Pic($R[1/\ell]$) = 1 if $r = g$. Recall that the Picard group Pic($R'$) of a commutative ring $R'$ is the group of isomorphism classes of locally free $R'$-modules of rank one. Note that the condition Pic($R[1/\ell]$) = 1 implies Pic($O_E[1/\ell]$) = 1.

**Proposition 5.3.** Suppose that $p \equiv 3 \pmod{4}$ and $\ell$ is an odd prime. If Pic($R'[1/\ell]$) = 1, then there are $g + 1$ $\ell$-adic Hecke orbits over $F_p$ in the set $S$.

**References**

[1] C.-L. Chai, Every ordinary symplectic isogeny class in positive characteristic is dense in the moduli. *Invent. Math.* **121** (1995), 439–479.

[2] M. Demazure, *Lectures on $p$-divisible groups*. Lecture Notes in Math., vol. 302, Springer-Verlag, 1972.

[3] M. Deuring, Die Typen der Multiplikatorenringe elliptischer Funktionenkörper. *Abh. Math. Sem. Hamburg* **14** (1941), 197–272.

[4] M. Deuring, Die Anzahl der Typen von Maximalordnungen einer definierten Quaternionenalgebra mit primer Grundzahl. *Jber. Deutsch. Math.* **54** (1950), 24–41.

[5] M. Eichler, Über die Idealklassenzahl total definer Quaternionenalgebren. *Math. Z.* **43** (1938), 102–109.

[6] T. Ekedahl, On supersingular curves and supersingular abelian varieties. *Math. Scand.* **60** (1987), 151–178.

[7] E.-U. Gekeler, On finite Drinfeld modules. *J. Algebra* **141** (1991), 187–203.

[8] E.-U. Gekeler, On the arithmetic of some division algebras. *Comment. Math. Helv.* **67** (1992), 316–333.

[9] K. Hashimoto and T. Ibukiyama, On class numbers of positive definite binary quaternion hermitian forms, *J. Fac. Sci. Univ. Tokyo* **27** (1980), 549–601.

[10] T. Ibukiyama and T. Katsura, On the field of definition of superspecial polarized abelian varieties and type numbers. *Compositio Math.* **91** (1994), 37–46.

[11] T. Ibukiyama, T. Katsura and F. Oort, Supersingular curves of genus two and class numbers. *Compositio Math.* **57** (1986), 127–152.

[12] T. Katsura and F. Oort, Families of supersingular abelian surfaces, *Compositio Math.* **62** (1987), 107–167.

[13] Yu. Manin, Theory of commutative formal groups over fields of finite characteristic. *Russian Math. Surveys* **16** (1963), 1–80.

[14] L. Moret-Bailly, Familles de courbes et de variétés abéliennes sur $\mathbf{P}^1$. Sém. sur les pinceaux de courbes de genre au moins deux (ed. L. Szpiro). *Astérisques* **86** (1981), 109–140.

[15] F. Oort, Which abelian surfaces are products of elliptic curves. *Math. Ann.* **214** (1975), 35–47.

[16] G. Shimura, Some exact formulas for quaternion unitary groups. *J. Reine Angew. Math.* **509** (1999), 67–102.

[17] J. Tate, Classes d’isogenie de variétés abéliennes sur un corps fini (d’après T. Honda). *Sém. Bourbaki Exp.* 352 (1968/69). Lecture Notes in Math., vol. 179, Springer-Verlag, 1971.

[18] J. Tate, Endomorphisms of abelian varieties over finite fields. *Invent. Math.* **2** (1966), 134–144.

[19] W. C. Waterhouse, Abelian varieties over finite fields. *Ann. Sci. École Norm. Sup.* (4) **1969**, 521–560.

[20] C.-F. Yu and J. Yu, Mass formula for supersingular Drinfeld modules. *C. R. Acad. Sci. Paris Sér. I Math.* **338** (2004), 905–908.

[21] C.-F. Yu, On the mass formula of supersingular abelian varieties with real multiplications. *J. Australian Math. Soc.* **78** (2005), 373–392.

[22] C.-F. Yu, An exact geometric mass formula. *Int. Math. Res. Not.* **2008**, Article ID rnn113, 11 pages.

[23] C.-F. Yu, Irreducibility of Hilbert-Blumenthal moduli spaces with parahoric level structure. *J. Reine Angew. Math.* **635** (2009), 187–211.

[24] C.-F. Yu and J.-D. Yu, Mass formula for supersingular abelian surfaces. *J. Algebra* **322** (2009), 3733–3743.
[25] C.-F. Yu, Simple mass formulas on Shimura varieties of PEL-type. To appear in *Forum Math.*

**Institute of Mathematics, Academia Sinica and NCTS (Taipei Office), 6th Floor, Astronomy Mathematics Building, No. 1, Roosevelt Rd. Sec. 4, Taipei, Taiwan, 10617**

_E-mail address:_ chiafu@math.sinica.edu.tw