Minimizing the number of copies of \( K_r \) in an \( F \)-saturated graph

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Abstract

This paper considers two important questions in the well-studied theory of graphs that are \( F \)-saturated. A graph \( G \) is called \( F \)-saturated if \( G \) does not contain a subgraph isomorphic to \( F \), but the addition of any edge creates a copy of \( F \). We first resolve the most fundamental question of minimizing the number of cliques of size \( r \) in a \( K_s \)-saturated graph for all sufficiently large numbers of vertices, confirming a conjecture of Kritschgau, Methuku, Tait, and Timmons. We also go further and prove a corresponding stability result. We then move on to a central and longstanding conjecture in graph saturation made by Tuza, which states that for every graph \( F \), the limit \( \lim_{n \to \infty} \frac{\text{sat}(n, F)}{n} \) exists, where \( \text{sat}(n, F) \) denotes the minimum number of edges in an \( n \)-vertex \( F \)-saturated graph. Pikhurko made progress in the negative direction by considering families of graphs instead of a single graph, and proved that there exists a graph family \( F \) of size 4 for which \( \lim_{n \to \infty} \frac{\text{sat}(n, F)}{n} \) does not exist (for a family of graphs \( F \), a graph \( G \) is called \( F \)-saturated if \( G \) does not contain a copy of any graph in \( F \), but the addition of any edge creates a copy of a graph in \( F \), and \( \text{sat}(n, F) \) is defined similarly). We make the first improvement in 15 years by showing that there exist infinitely many graph families of size 3 where this limit does not exist. Our construction also extends to the generalized saturation problem when we minimize the number of fixed-size cliques.

1 Introduction

Extremal graph theory focuses on finding the extremal values of certain parameters of graphs under certain natural conditions. One of the most well-studied conditions is \( F \)-freeness. For graphs \( G \) and \( F \), we say that \( G \) is \( F \)-free if \( G \) does not contain a subgraph isomorphic to \( F \). This gives rise to the most fundamental question of finding the Turán number \( \text{ex}(n, F) \), which asks for the maximum number of edges in an \( n \)-vertex \( F \)-free graph. The asymptotic answer is known for most graphs \( F \), with the exception of bipartite \( F \) where the most intricate and unsolved cases appear (see, e.g., \( [9] \) and \( [14] \) for nice surveys). Recently, Alon and Shikhelman

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introduced a natural generalization of the Turán number. They systematically studied \( \text{ex}(n, H, F) \), which denotes the maximum number of copies of \( H \) in an \( n \)-vertex \( F \)-free graph. Note that the case \( H = K_2 \) is the standard Turán problem, i.e., \( \text{ex}(n, K_2, F) = \text{ex}(n, F) \).

While the Turán number asks for the maximum number of edges in an \( F \)-free graph, another very classical problem concerns the minimum number of edges in an \( F \)-free graph with a fixed number of vertices. This problem is not interesting as stated because the empty graph is the obvious answer. In most literature, this issue is resolved by imposing the additional condition that adding any edge to \( G \) will create a copy of \( F \). With this additional condition, we say that \( G \) is \( F \)-saturated. A moment’s thought will convince the reader that when maximizing the number of edges, this additional condition does not change the problem at all. On the other hand, this new condition makes the edge minimization problem very interesting, and this area of research is commonly known as graph saturation. Let the saturation function \( \text{sat}(n, F) \) denote the minimum number of edges in an \( n \)-vertex \( F \)-saturated graph. Erdős, Hajnal, and Moon [7] started the investigation of this area with the following beautiful result.

**Theorem 1.1** (Erdős, Hajnal, and Moon 1964). For every \( n \geq s \geq 2 \), the saturation number \( \text{sat}(n, K_s) = (s - 2)(n - s + 2) + \binom{s-2}{2} \). Furthermore, there is a unique \( K_s \)-saturated graph on \( n \) vertices with \( \text{sat}(n, K_s) \) edges: the join of a clique with \( s - 2 \) vertices and an independent set with \( n - s + 2 \) vertices.

The join \( G_1 \ast G_2 \) of two graphs \( G_1 \) and \( G_2 \) is obtained by taking the disjoint union of \( G_1 \) and \( G_2 \) and adding all the edges between them. The original authors of the above theorem completely solved the problem for complete graphs by using a clever induction. A novel approach to prove this theorem is due to Bollobás [3], who developed an interesting tool based on systems of intersecting sets. Graph saturation has been studied extensively since this result appeared half a century ago (see, e.g., [8] for a very informative survey). Alon and Shikhelman’s generalization of the Turán number motivated Kritschgau, Methuku, Tait, and Timmons [10] to start the systematic study of the function \( \text{sat}(n, H, F) \), which denotes the minimum number of copies of \( H \) in an \( n \)-vertex \( F \)-saturated graph. Here again note that \( \text{sat}(n, K_2, F) = \text{sat}(n, F) \). Towards generalizing Theorem 1.1, Kritschgau, Methuku, Tait, and Timmons proved the following lower and upper bounds, which differ by a factor of about \( r - 1 \), and conjectured that the upper bound (achieved by the same construction as Theorem 1.1) is correct.

**Theorem 1.2** (Kritschgau, Methuku, Tait, and Timmons 2018). For every \( s > r \geq 3 \), there exists a constant \( n_{r,s} \) such that for all \( n \geq n_{r,s} \),

\[
\max \left\{ \frac{(s-2)}{r-1} \cdot n - 2 \left( \frac{s-2}{r-1} \right) \cdot \frac{(s-3)}{r-2} \cdot n \right\} \leq \text{sat}(n, K_r, K_s) \leq (n - s + 2) \left( \frac{s-2}{r-1} \right) + \left( \frac{s-2}{r} \right).
\]

Our first main contribution confirms their conjecture for sufficiently large \( n \) by showing that the upper bound is indeed the correct answer. We also show that the natural construction is the unique extremal graph for this generalized saturation problem for large enough \( n \).
Furthermore, we prove a corresponding stability result for sufficiently large $n$ which shows that even if we allow up to some $cn$ more copies of $K_r$ than $\text{sat}(n, K_r, K_s)$ in an $n$-vertex $K_s$-saturated graph, the extremal graph will still be the same and unique. It is worth noting that there are relatively few stability results in the area of graph saturation, essentially only [1] by Amin, Faudree, Gould, and Sidorowicz, and [3] by Bohman, Fonoberova, and Pikhurko. In the notation of joins, the extremal graph in our problem is $K_{s-2} \ast \overline{K}_{n-s+2}$, i.e., the join of a clique with $s-2$ vertices and an independent set with $n-s+2$ vertices.

**Theorem 1.3.** For every $s > r \geq 2$, there exists a constant $n_{r,s}$ such that for all $n \geq n_{r,s}$, we have $\text{sat}(n, K_r, K_s) = (n-s+2)\binom{s-2}{r-1} + \binom{s-2}{r}$. Moreover, there exists a constant $c_r > 0$ such that the only $K_s$-saturated graph with up to $\text{sat}(n, K_r, K_s) + c_r n$ many copies of $K_r$ is $K_{s-2} \ast \overline{K}_{n-s+2}$.

**Remark.** The second part of this theorem is tight in the sense that there exists a constant $C_{r,s}$ for which it is not true. To see that consider the graph $G$ on $n$ vertices which is the join of two graphs $G_1$ and $G_2$, where $G_1$ is $K_{s-1}$ minus an edge, and $G_2$ is an independent set on $n-s+1$ vertices. Clearly, $G$ is $K_s$-saturated, with $(2\binom{s-3}{r-2} + \binom{s-3}{r-1}) (n-s+1) + 2\binom{s-3}{r-1} + \binom{s-3}{r}$ many copies of $K_r$.

In the process of proving Theorem 1.3, we consider a more general setting and prove and use an intermediate result, which may also be of independent interest. The condition that $G$ is $F$-saturated can be weakened by removing the condition that $G$ is $F$-free (as also studied in [4] and [17]). Perhaps counterintuitively, despite the fact that this is a weaker condition, the literature calls $G$ strongly $F$-saturated if adding any edge to $G$ creates a new copy of $F$. Following the notation in the literature, we write $\text{ssat}(n, H, F)$ to denote the minimum number of copies of $H$ in an $n$-vertex strongly $F$-saturated graph. It is obvious that $\text{ssat}(n, H, F) \leq \text{sat}(n, H, F)$. We have the following asymptotic result for the function ssat for cliques.

**Theorem 1.4.** For every $s > r \geq 2$, we have $\text{ssat}(n, K_r, K_s) = n\binom{s-2}{r-1} - o(n)$.

Next we turn our attention to a long-standing yet very fundamental conjecture made by Tuza [15, 16]. In contrast to the Turán number, one of the inherent challenges in studying the saturation number $\text{sat}(n, H)$ for general graphs $H$ is that this function lacks monotonicity properties that one might hope for. For example, Pikhurko [13] showed that there is a pair of connected graphs $F_1 \subset F_2$ on the same vertex set such that $\text{sat}(n, F_1) > \text{sat}(n, F_2)$ for large $n$, violating monotonicity in the second parameter. Regarding non-monotonicity in the first parameter, Kászonyi and Tuza [11] observed that $\text{sat}(2k-1, P_3) = k+1 > k = \text{sat}(2k, P_3)$ where $P_3$ is the path with 3 edges. Moreover, Pikhurko showed a wide variety of examples of irregular behavior of the saturation function in [13]. All of this non-monotonicity makes proving statements about the saturation function difficult, in particular because inductive arguments generally do not work. However, in order to find some smooth behavior of the saturation function Tuza conjectured the following.

**Conjecture 1.5** (Tuza 1986). For every graph $F$, the limit $\lim_{n \to \infty} \frac{\text{sat}(n, F)}{n}$ exists.

Not much progress has been made towards settling the conjecture. The closest positive attempt was made by Truszczyński and Tuza [18], who showed that for every graph $F$, if
\[
\liminf_{n \to \infty} \frac{\text{sat}(n,F)}{n} < 1, \text{ then } \lim_{n \to \infty} \frac{\text{sat}(n,F)}{n} \text{ exists and is equal to } 1 - \frac{1}{p} \text{ for some positive integer } p.
\]

Pikhurko considered the saturation number for graph families to make progress in the negative direction of Conjecture 1.5. For a family of graphs \(F\), the saturation number \(\text{sat}(n,F)\) is defined to be the minimum number of edges in an \(n\)-vertex \(F\)-saturated graph, where a graph \(G\) is called \(F\)-saturated if \(G\) does not contain a copy of any graph in \(F\) and adding any edge in \(G\) will create a copy of a graph in \(F\). Pikhurko first showed in [12] that there exists an infinite family \(F\) of graphs for which \(\lim_{n \to \infty} \frac{\text{sat}(n,F)}{n}\) does not exist, and later in [13] proved the same for a graph family of size only 4. We make the first progress in 15 years, moving one step closer.

**Theorem 1.6.** There exist infinitely many graph families \(F\) of size 3 such that the ratio \(\frac{\text{sat}(n,F)}{n}\) does not converge as \(n\) tends to infinity.

In the spirit of considering the generalized saturation number, it is natural to ask the more general question of whether \(\lim_{n \to \infty} \frac{\text{sat}(n,K_r,F)}{n}\) exists for every graph \(F\). We remark that this problem is interesting since the order of \(\text{sat}(n,K_r,F)\) is linear in \(n\) for every graph family \(F\), which can easily be shown by considering the same construction used by Kászonyi and Tuza in [11], who showed the same for \(r = 2\). We show that our construction of graph families of size 3 can be extended to this scenario.

**Theorem 1.7.** For every \(r \geq 3\), there exist infinitely many graph families \(F\) of size 3 such that the ratio \(\frac{\text{sat}(n,K_r,F)}{n}\) does not tend to a limit as \(n\) tends to infinity.

The remainder of this paper is organized as follows. In the next section we prove an asymptotically tight lower bound on \(\text{ssat}(n,K_r,K_s)\). Then, we use the results and notations of that section to determine \(\text{sat}(n,K_r,K_s)\) exactly for sufficiently large \(n\) in Section 3. In Section 4, we construct infinitely many graph families \(F\) of size 3 for which the ratio \(\frac{\text{sat}(n,F)}{n}\) does not converge. We then extend this construction with the help of Theorem 1.4 in Section 5. We finish with a few open problems and concluding remarks in Section 6.

## 2 Asymptotic result

In this section, we prove Theorem 1.4. Let \(G = (V, E)\) be an \(n\)-vertex strongly \(K_s\)-saturated graph. Our aim is to find a lower bound on the number of \(K_r\)'s in \(G\). Note that if there is an edge \(e \in E\) such that \(e\) is not in a copy of \(K_r\), then \(e\) does not contribute to the number of copies of \(K_r\). It turns out that a careful analysis of the edges which are in a copy of \(K_r\) saves us the required factor of \(r - 1\) when we compare against the previous best result (Theorem 1.2). So, it is natural to split the edge set \(E\) into two parts in the following manner. Let \(E_1\) denote the set of edges which are at least in one copy of \(K_r\). Let \(E_2 = E \setminus E_1\) be the remaining edges in \(G\). Now we will prove a simple but powerful lemma which will be useful throughout the current and next sections.

**Lemma 2.1.** Every edge of \(E_2\) would not be in a copy of \(K_s\) even if any non-edge were added to \(G\).
Proof. Fix an arbitrary edge $uv$ of $E_2$ and an arbitrary non-edge $ab$ of $G$. Note that the sets \{u, v\} and \{a, b\} can overlap, but without loss of generality $b \not\in \{u, v\}$. Assume for the sake of contradiction that after adding the missing edge $ab$ we create a copy of $K_s$ containing both $u$ and $v$. Now if we remove the vertex $b$ from the created copy of $K_s$, we will find a copy of $K_{s-1}$ in $G$ which contains both $u$ and $v$. So $uv$ is in a copy of $K_{s-1}$ in $G$, which contradicts the fact that $uv$ is not in a copy of $K_r$, because $r \leq s - 1$. \hfill \Box

It will be convenient to define a couple of sets which we will use throughout this section and the next section. For $i = 1, 2$, let $G_i$ denote the graph on the same vertex set $V$ with the edge set $E_i$. For a graph $H$, it will be convenient to use the notation $d_H(v)$ to denote the degree of $v$ in $H$. It will be useful to split the vertices according to their degree in $G_1$, so we define

$$A = \{v \in V : d_{G_1}(v) \leq n^{\frac{4}{3}}\}.$$  \hfill (2.1)

We can observe that $A$ consists of almost all vertices of $G$, i.e., $|A| = n - o(n)$, because $|E_1| \leq \binom{|V|}{2}$ sat($n, K_r, K_s$) = $O(n)$ by the construction used in Theorems 1.1 and 1.2, and so $|V \setminus A| = O(n^{\frac{2}{3}})$. Now our aim is to show that almost every vertex of $A$ is in a copy of $K_{s-1}$ which has only one vertex of $A$. Note that the extremal graph $K_{s-2} \ast K_{n-s+2}$ has this property. Formally, we define the following:

$$B = \{v \in A : \exists a_1, \ldots, a_{s-2} \in V \setminus A \text{ such that } v, a_1, \ldots, a_{s-2} \text{ induce a copy of } K_{s-1}\}.$$  \hfill (2.2)

**Lemma 2.2.** Almost all vertices are in $B$, in the sense that $|B| = n - o(n)$.

Proof. Let $m = |A|$. Let $R$ denote the set of vertices in $A$ with degree more than $m - 2n^{\frac{4}{3}}$ in the induced subgraph of $G_2$ on $A$. Now we claim that $R$ has at most $2rn^{\frac{4}{3}}$ vertices. Assume for the sake of contradiction that $|R| > 2rn^{\frac{4}{3}}$; then with a simple greedy process we will find a copy of $K_r$ in $G_2$. Start with any vertex $v_1 \in R$, and let $R_1 \subseteq R$ denote the set of vertices in $R$ which are neighbors of $v_1$. Clearly, $|R_1| > 2(r - 1)n^{\frac{4}{3}}$ because $v_1$ has less than $2n^{\frac{4}{3}}$ non-neighbors in $R$. For $2 \leq i \leq s$, we repeat this process, i.e., at step $i$ we take a vertex $v_i \in R_{i-1}$, and let $R_i \subseteq R_{i-1}$ denote the set of vertices in $R_{i-1}$ which are neighbors of $v_i$. Clearly, $|R_i| > 2(r - i)n^{\frac{4}{3}}$. Now observe that $v_1, v_2, \ldots, v_r$ induce a copy of $K_r$ in $G_2$ which is the desired contradiction. So $|R| \leq 2rn^{\frac{4}{3}}$.

Now our aim is to show that $A \setminus R \subseteq B$, which will be sufficient to finish the proof of this lemma. To this end, fix an arbitrary vertex $v \in A \setminus R$. We will first show that there is $w \in A$ such that $vw$ is not an edge of $G$ and there is no $z \in A$ such that $vz$ and $zw$ are both in $E_1$. This is because there are at least $2n^{\frac{4}{3}}$ non-neighbors of $v$ in $A$ from the definition of $R$, and in the induced graph of $G_1$ on $A$, there can be at most $n^{\frac{4}{3}} + n^{\frac{2}{3}}$ vertices at distance at most 2 from $v$ due to the fact that $d_{G_1}(v) \leq n^{\frac{4}{3}}$ from (2.1). Fix such a vertex $w$. As $G$ is $K_s$-saturated, if we added the edge $vw$, then we would create a copy of $K_s$. Furthermore, that $K_s$ cannot contain any vertex from $A$ except $v$ and $w$, because if it contained some $z \in A$, then at least one of $vz$ or $zw$ is in $E_2$, contradicting Lemma 2.1. Hence there is a copy of $K_{s-1}$ induced by $v$ together with $s - 2$ vertices from $V \setminus A$, and so $v \in B$. Therefore, $|B| \geq |A| - |R| \geq n - o(n)$. \hfill \Box

Proof of Theorem 1.4. For an arbitrary vertex $v \in B$, the number of $K_s$’s induced by $v$ together with $r - 1$ vertices from $V \setminus A \subseteq V \setminus B$ is at least $\binom{s-2}{r-1}$ from (2.2). So by Lemma
the number of $K_r$’s in $G$ is at least $(s-2)\binom{r}{s-2}|B| = (s-2)n - o(n)$. This matches the upper bound from Theorem 1.2, completing the proof.

Note that by defining the set $A$ in (2.1) optimally, the best lower bound we can achieve with this argument is that $\text{ssat}(n, K_r, K_s) \geq (s-2)\binom{n}{s-2} n - O(\sqrt{n})$. Also note that Theorem 1.4 already proves an asymptotically tight lower bound on $\text{sat}(n, K_r, K_s)$, because:

$$n\left(\frac{s-2}{r-1}\right) - o(n) \leq \text{ssat}(n, K_r, K_s) \leq \text{sat}(n, K_r, K_s) \leq (n - s + 2)\left(\frac{s-2}{r-1}\right) + \left(\frac{s-2}{r}\right).$$

3 Exact result

In this section, we will find the exact value of $\text{sat}(n, K_r, K_s)$ for all sufficiently large $n$, proving Theorem 1.3. The same argument will also show that the graph $K_{s-2} \ast \overline{K}_{n-s+2}$ is the unique extremal graph. Moreover we will prove a stability result, i.e., the same graph is also the unique graph among $K_s$-saturated graphs even if we allow up to some $cn$ more copies of $K_r$ than $\text{sat}(n, K_r, K_s)$. We will start with the structural knowledge we developed in the last section and successively deduce more structure to finally reach the exact structure.

Define $c = \frac{1}{\sqrt{r}}$ and consider an $n$-vertex $K_s$-saturated graph $G$ with at most $\text{sat}(n, K_r, K_s) + cn$ copies of $K_r$. By defining the sets $A$ and $B$ as in (2.1) and (2.2) and applying the same arguments we can make the same structural deductions about $G$ as in the last section. In particular, the number of $K_r$’s with one vertex in $B$ and $r - 1$ vertices in $V \setminus A$ is at least:

$$n\left(\frac{s-2}{r-1}\right) - o(n).$$

Next, define

$$C = \{v \in B : d_{G_1}(v) > s - 2\}.$$  

(3.1)

For $v \in C$, fix $s - 2$ neighbors of $v$ in $V \setminus A$ such that those neighbors along with $v$ induce a copy of $K_{s-1}$ in $G$. For each $v \in C$, pick an edge $vw \in E_1$ such that $w$ is not among the $s - 2$ fixed neighbors. Note that the same edge $vw$ can be picked at most once more. Each of these particular edges is in $E_1$, hence these edges are contained in some $K_r$, which is not counted in (3.1). After counting for multiplicity, these extra edges will constitute at least an extra $\frac{|C|}{2\binom{l}{2}}$ many copies of $K_r$’s. Hence, for sufficiently large $n$, $\frac{|C|}{2\binom{l}{2}} \leq 2cn$, which implies that $|C| \leq \frac{n}{2}$. So, the set $B \setminus C$ is non-empty for large enough $n$. We will now prove two more structural lemmas.

**Lemma 3.1.** Let $v$ be an arbitrary vertex in $B \setminus C$, and suppose $x_1, x_2, \ldots, x_{s-2}$ are vertices in $G$ such that $\{v, x_1, \ldots, x_{s-2}\}$ induce a copy of $K_{s-1}$. Then for all $u \in V \setminus \{v\}$ such that $uv$ is not an edge, $u$ is adjacent to all of $x_1, \ldots, x_{s-2}$.

**Proof.** Since $\{v, x_1, \ldots, x_{s-2}\}$ induces $K_{s-1}$ and $s - 1 \geq r$, every edge $vx_i$ is in $E_1$. As $v \in B \setminus C$, $v$ has no more $E_1$-edges. If we add the non-edge $vw$, we must create a copy of $K_s$. If some vertex $w \notin \{u, v, x_1, \ldots, x_{s-2}\}$ participates in the created copy of $K_s$ we know
that $vw$ must be in $E_2$ since $v$ has no more $E_1$-edges, contradicting Lemma 2.1. So, the only choice for the remaining $s - 2$ vertices of the created copy of $K_s$ would be $x_1, \ldots, x_{s-2}$. Thus $u$ must be adjacent to all of $x_1, \ldots, x_{s-2}$.

Lemma 3.2. All vertices of $B \setminus C$ have no incident edges from $E_2$.

Proof. Assume for the sake of contradiction that $uv \in E_2$, where $v \in B \setminus C$. Since $G$ is $K_s$-saturated, $u$ is in a copy $S$ of $K_{s-1} \supseteq K_s$. Since $uv \in E_2$, $v \not\in S$ by Lemma 2.1. Furthermore, $v$ cannot be adjacent to all the vertices in $S$, or else there would be a copy of $K_s$. Similarly, $v$ is in a copy of $K_{s-1}$, and $u$ is not adjacent to the full set of those vertices. Let $a_1, \ldots, a_k$, $b_1, \ldots, b_k$ and $c_{k+1}, \ldots, c_{s-2}$ be distinct vertices such that $\{u, a_1, \ldots, a_k, c_{k+1}, \ldots, c_{s-2}\}$ and $\{v, b_1, \ldots, b_k, c_{k+1}, \ldots, c_{s-2}\}$ both induce $K_{s-1}$. The above argument shows that $k \geq 1$. Now we claim that there must be at least two non-edges between $v$ and the set $\{a_1, \ldots, a_k\}$, otherwise the neighbors of $v$ in $\{a_1, \ldots, a_k\}$ along with $u, v, c_{k+1}, \ldots, c_{s-2}$ will induce a clique of order at least $s - 1 \geq r$, which contradicts the fact that $uv \in E_2$ due to Lemma 2.1. Without loss of generality, $v$ is not adjacent to both $a_1$ and $a_2$. Now by applying Lemma 3.1 with $v \in B \setminus C$, and $b_1, \ldots, b_k, c_{k+1}, \ldots, c_{s-2}$ as $x_1, \ldots, x_{s-2}$, and $a_1$ as $u$, we see that $a_1$ is adjacent to all of $b_1, \ldots, b_k, c_{k+1}, \ldots, c_{s-2}$. The same is true of $a_2$. So, $a_1, a_2, b_1, \ldots, b_k, c_{k+1}, \ldots, c_{s-2}$ induce a copy of $K_s$ in $G$ which is impossible.

Proof of Theorem 3.3. Fix any vertex $v \in B \setminus C$. There exists a set $S$ of $s - 2$ vertices such that $S \cup \{v\}$ induces a copy of $K_{s-1}$. Since $v \in B \setminus C$, there are no more $E_1$ edges incident to $v$ other than those to $S$. By Lemma 3.2, there are no $E_2$ edges either. By Lemma 3.1, every vertex $u \not\in S \cup \{v\}$ must be adjacent to all vertices in $S$. This is already the graph $K_{s-2} \ast \overline{K}_{n-s+2}$, which is $K_s$-saturated, so $G$ is precisely $K_{s-2} \ast \overline{K}_{n-s+2}$.

4 Family of size 3 with non-converging saturation ratio

In this section, we prove Theorem 1.6. We begin by stating the families of graphs that we will use for the construction.

Definition 4.1. For every positive integer $m \geq 4$, let $\mathcal{F}_m$ be the family of the following three graphs.

- Let $B_{m,m}$ be the disjoint union of two copies of $K_m$ plus one edge joining them (often called a “dumb-bell”).
- Let $V_m$ be a copy of $K_m$ plus two more vertices each with one edge incident to a single fixed vertex of the $K_m$.
- Let $\Lambda_m$ be a copy of $K_m$ plus a single vertex with exactly two edges incident to the $K_m$.

The proof of Theorem 1.6 boils down to the fact that the behavior of $\text{sat}(n, \mathcal{F}_m)$ depends on whether or not $n$ is divisible by $m$. The following two lemmas constitute the proof.

Lemma 4.2. For every $n$ divisible by $m$, we have $\text{sat}(n, \mathcal{F}_m) \leq \frac{n(n-1)}{2m(m-1)}$. 


Proof. Since \( n \) is divisible by \( m \), the graph \( G \) consisting of the disjoint union of \( \frac{n}{m} \) many copies of \( K_m \) is clearly \( F_m \)-saturated, and the number of edges in \( G \) is \( \frac{n}{m} \binom{m}{2} \), which proves the result. \qed

Lemma 4.3. For every \( n \geq m \geq 4 \) where \( n \) is not divisible by \( m \), we have \( \text{sat}(n, F_m) \geq \frac{n-m}{m} \left( \binom{m}{2} + 1 \right) \).

The proof of Lemma 4.3 will easily follow from the next three lemmas about the structure of \( F_m \)-saturated graphs. Let \( G \) be an \( F_m \)-saturated graph on \( n \) vertices. Let \( B \) be the set of all vertices of \( G \) which are contained in any copy of \( K_m \).

Lemma 4.4. The subgraph induced by \( B \) is only a disjoint union of \( K_m \)'s.

Proof. First, note that no subgraph of \( G \) is isomorphic to \( F_{m,j} \) for any \( j \in \{1, 2, \ldots, m-1\} \), where \( F_{m,j} \) denotes the union of 2 copies of \( K_m \) overlapping in exactly \( j \) common vertices. This is because each \( F_{m,j} \) contains a copy of \( V_m \) or \( \Lambda_m \). As \( B \) does not have any copies of \( F_{m,j} \) for all \( j \), all copies of \( K_m \) induced by \( B \) are pairwise disjoint. Furthermore, the subgraph of \( G \) that \( B \) induces is just a disjoint union of \( K_m \)'s, because any other edge would create a copy of \( B_{m,m} \) in \( G \). \qed

Now let \( A \) be the set of all vertices not in \( B \). Since the structure in \( B \) is so simple, our lower bound will follow by independently lower-bounding the number of edges induced by \( A \), and the number of edges between \( A \) and \( B \). We start with \( A \).

Lemma 4.5. Let \( k \) be the number of disjoint copies of \( K_m \) in \( B \). Then \( A \) has at most \( m \) vertices, or \( A \) is \( K_m \)-saturated.

Proof. If \( A \) is complete, then the number of vertices in \( A \) is at most \( m \), or else \( G \) contains a copy of \( K_{m+1} \), and hence \( G \) contains a copy of \( \Lambda_m \), which is a contradiction.

So, suppose \( A \) is not complete. We claim that adding any edge to the induced graph on \( A \) must create a copy of \( K_m \) in \( G \). Suppose for the sake of contradiction that there is a non-edge \( uv \) with \( u, v \in A \) such that adding \( uv \) does not create a copy of \( K_m \) in \( G \). However, it must create a copy of one of the graphs \( B_{m,m}, V_m \), or \( \Lambda_m \), hence one of \( u \) or \( v \) must be in a copy of a \( K_m \) in \( G \) (because these three graphs have the property that for all edges \( ab \), either \( a \) or \( b \) is in a copy of \( K_m \)), which contradicts the definition of \( A \).

Finally, we show that adding any edge to the induced graph on \( A \) creates a copy of \( K_m \) which entirely lies in \( A \). Suppose for the sake of contradiction that there is a non-edge \( uv \) in the induced graph on \( A \) which, if added, would create a copy of \( K_m \) which intersects \( B \). Let \( w \in B \) be a vertex which lies in a created copy of \( K_m \) after adding the edge \( uv \). That means that \( G \) has the edges \( uw \) and \( vw \), and the copy of \( K_m \) in \( B \) containing \( w \), together with the edges \( uw \) and \( vw \), creates a copy of \( V_m \). So, this is not possible, and we conclude that the induced subgraph on \( A \) is indeed \( K_m \)-saturated. \qed

It only remains to bound the number of edges between \( A \) and \( B \). We have the following structural lemma.

Lemma 4.6. If \( A \) is non-empty, then each copy of \( K_m \) in \( B \) has at least one edge to \( A \).
Proof. Assume for the sake of contradiction that there is a copy $U$ of $K_m$ in $B$ which does not have an edge to $A$. Consider arbitrary vertices $u \in U$ and $v \in A$. Then one of the following situations must happen.

Case 1: Adding $uv$ creates a copy of $K_m$. Then it is easy to check that $U$ has an edge to $A$, which is a contradiction.

Case 2: Adding $uv$ creates a copy of $B_{m,m}$ with $uv$ being the middle edge connecting the copies of $K_m$. This would imply that $v$ is in a copy of $K_m$, which is a contradiction.

Case 3: Adding $uv$ creates a copy of $V_m$ or $\Lambda_m$ with $uv$ being one of the two extra edges outside of the copy of $K_m$. Then if $uv$ becomes one of the extra edges, the other extra edge should already be there and will connect $U$ and $A$, giving a contradiction.

Since all the cases give contradictions, we are done.

We now combine the previous three lemmas to prove Lemma 4.3, which then finishes the proof of Theorem 1.6.

Proof of Lemma 4.3. Let $n$ and $m$ satisfy the conditions of Lemma 4.3. Clearly $A$ must be non-empty because the number of vertices in $B$ is a multiple of $m$ by Lemma 4.4 and so Lemma 4.6 implies that there are at least $k$ edges between $B$ and $A$, where $k$ is the number of disjoint copies of $K_m$ in $B$. Now from Lemma 4.5 we have two situations. When $A$ has at most $m$ vertices, using Lemmas 4.4 and 4.6, the number of edges in $G$ is at least $\left\lfloor \frac{n}{m} \right\rfloor \left( \left\lfloor \frac{m}{2} \right\rfloor + 1 \right) \geq \frac{n-m}{m} \left( \left( \frac{m}{2} \right) + 1 \right)$. Otherwise, $A$ is $K_m$-saturated, so Theorem 1.1 implies that for all $m \geq 4$ the number of edges in $G$ is at least:

\[
\begin{align*}
&k \left( \frac{m}{2} \right) + k + (n - (k+1)m + 2)(m-2) \\
&\quad > (km) \frac{m-1}{2} + k + (n - (k+1)m) \left( \frac{m-1}{2} + \frac{1}{m} \right) \\
&\quad = (km) \frac{m-1}{2} + (n - (k+1)m) \frac{m-1}{2} + k + (n - (k+1)m) \frac{1}{m} \\
&\quad = \frac{n-m}{m} \left( \left( \frac{m}{2} \right) + 1 \right)
\end{align*}
\]

This completes the proof.

5 Family of size 3 for generalized saturation ratio

Inspired by the construction for Theorem 1.6, we extend the construction to prove Theorem 1.7. One of the key challenges is to find an appropriate extension of Theorem 1.1. Fortunately, our Theorem 1.4 rescues us. We start by stating the the families of graphs that we will use for the construction, which are not quite the straightforward generalizations of the families used in Theorem 1.6. For notational brevity, let $r \geq 2$ be a fixed integer for the remainder of this section.

Definition 5.1. For every positive integer $m \geq 2r^2 + 2r$, let $\mathcal{F}_m$ be the family of the following three graphs.
Let $B_{m,m}$ be the same “dumb-bell” graph from Definition 4.1.

Let $V_{m,r}$ be the union of a copy of $K_m$ and a copy of $K_{m-r+1}$ overlapping in exactly one common vertex.

Let $\Lambda_{m,r}$ be a copy of $K_m$ plus a single vertex with exactly $r$ edges incident to the $K_m$.

Note that for $r = 2$, we have $\Lambda_{m,2} = \Lambda_m$. However, $V_{m,r}$ is not quite a generalization of $V_m$, and in fact $V_m$ is a subgraph of $V_{m,2}$. We considered $V_m$ instead of $V_{m,2}$ in the case of $r = 2$ to make the analysis simpler and more elegant. So, the above construction actually gives different families of three graphs with non-converging saturation ratio for $r = 2$.

We proceed to the proof of Theorem 1.7. It turns out that the behavior of $\text{sat}(n, K_r, F_m)$ is similar to before, i.e., it depends on whether or not $n$ is divisible by $m$. The following two lemmas constitute the proof.

**Lemma 5.2.** For every $n$ divisible by $m$, we have $\text{sat}(n, K_r, F_m) \leq \frac{n}{m} \binom{m}{r}$.

**Proof.** The same graph used in the proof of Lemma 4.2, i.e., the disjoint union of $\frac{n}{m}$ many copies of $K_m$, gives us the desired upper bound. \qed

**Lemma 5.3.** For every $n \geq m \geq 2r^2 + 2r$ where $n$ is not divisible by $m$, we have that $\text{sat}(n, K_r, F_m) \geq \frac{n}{m} \left( \binom{m}{r} + 1 \right) - o(n)$.

Similarly to Lemma 4.3, the proof of Lemma 5.3 will follow from the next couple of structural lemmas about $F_m$-saturated graphs. Let $G$ be an $F_m$-saturated graph on $n$ vertices. Let $B$ be the set of all vertices of $G$ which are contained in any copy of $K_m$. The subgraph induced by $B$ is only a disjoint union of $K_m$'s, by essentially the same proof as Lemma 4.4. Now let $A$ be the set of all vertices not in $B$. Motivated by Lemma 5.5 we have the following lemma.

**Lemma 5.4.** Let $k$ be the number of disjoint copies of $K_m$ in $B$. Then $A$ has at most $m$ vertices, or $A$ is strongly $K_{m-r}$-saturated.

**Remark.** This is in contrast to Lemma 4.5 which got that the induced graph on $A$ was $K_m$-saturated. Here we only get strongly $K_{m-r}$-saturated (recall that despite its counterintuitive name, strong saturation is a weaker condition), but we can later use our Theorem 1.4 to lower-bound the number of copies of $K_r$ in $A$.

**Proof.** If $A$ is complete, then the number of vertices in $A$ is at most $m$, or else $G$ contains a copy of $K_{m+1}$, and hence $G$ contains a copy of $\Lambda_{m,r}$, which is a contradiction. So, suppose $A$ is not complete. Fix a non-edge $uv$ in the induced graph on $A$. We consider two cases.

**Case 1:** Adding $uv$ would create a copy of $K_m$ in $G$. We will show that the copy of $K_m$ would lie entirely in $A$, giving the required $K_{m-r}$ in $A$. Indeed, assume for the sake of contradiction that there is a non-edge $uv$ in the induced graph on $A$ which, if added, would create a copy of $K_m$ which intersects $B$. That implies that there is a copy $T$ of $K_{m-1}$ which contains the vertex $u$ and intersects $B$. Clearly $T$ can intersect only a single copy $U$ of $K_m$ in $B$, because the induced graph on $B$ is just a disjoint union of $K_m$'s. Now, if $|T \cap U| \geq r$, then $T \cup U$
contains a copy of $\Lambda_{m,r}$, which is a contradiction. Otherwise, $|T \cap U| < r$, and so $T \cup U$ contains a copy of $V_{m,r}$, which is also a contradiction.

**Case 2:** Adding $uv$ would not create a copy of $K_{m}$ in $G$. If adding $uv$ creates a copy of $B_{m,m}$ or $\Lambda_{m,r}$ in $G$, then one of $u$ or $v$ must be in a copy of a $K_{m}$ in $G$, which contradicts the definition of $A$. Alternatively, if adding $uv$ creates a copy of $V_{m,r}$ in $G$, then that copy of $V_{m,r}$ would contain a copy of $K_{m}$ in $B$, together with $m - r$ vertices in $A$. These $m - r$ vertices would clearly induce a copy of $K_{m-r}$ after adding $uv$. Hence we are done.

Next, following the proof of Lemma 4.3 we bound the number of $K_{r}$’s that intersect both $A$ and $B$.

**Lemma 5.5.** Suppose $m \geq 2r + 1$. If $A$ is non-empty, then for each copy $U$ of $K_{m}$ in $B$, there is at least one copy of $K_{r}$ intersecting both $U$ and $A$.

**Proof.** Assume for the sake of contradiction that there is a copy $U$ of $K_{m}$ in $B$ for which there is no copy of $K_{r}$ intersecting both $U$ and $A$. Consider arbitrary vertices $u \in U$ and $v \in A$. One of the following situations must happen.

**Case 1:** Adding $uv$ creates a copy $T$ of $K_{m-r}$. Then it is easy to check that there is a copy of $K_{m-r-1}$ (and hence a copy of $K_{r}$ if $m \geq 2r + 1$) intersecting both $U$ and $A$, which is a contradiction.

**Case 2:** Adding $uv$ creates a copy of $B_{m,m}$ with $uv$ being the middle edge connecting the copies of $K_{m}$. This case is exactly the same as before, i.e., $v$ is in a copy of $K_{m}$, which is a contradiction.

**Case 3:** Adding $uv$ creates a copy of $\Lambda_{m,r}$ with $uv$ being one of the $r$ extra edges outside of the copy of $K_{m}$. Then if $uv$ becomes one of the extra $r$ edges, the $r-1$ endpoints in $U$ of the remaining $r-1$ extra edges, together with the vertex $v$, induce a copy of $K_{r}$, giving a contradiction.

Since all the cases give contradictions, we are done.

**Proof of Lemma 5.3.** Let $n$ and $m$ satisfy the conditions of Lemma 5.3. Clearly $A$ must be non-empty because the number of vertices in $B$ is a multiple of $m$, and so Lemma 5.5 implies that there are at least $k$ copies of $K_{r}$ intersecting both $B$ and $A$, where $k$ is the number of disjoint copies of $K_{m}$ in $B$. Now from Lemma 5.4, we have two situations. When $A$ has at most $m$ vertices, by Lemma 5.5, the number of copies of $K_{r}$ in $G$ is at least $\left[ \frac{m}{r} \right] \binom{m}{r} + \left[ \frac{m}{m} \right] \geq \frac{2-m}{m} \left( \binom{m}{r} + 1 \right)$. Otherwise, $A$ is strongly $K_{m-r}$-saturated, so Theorem 1.4 implies that $A$ induces at least $\binom{m-r-2}{r-1}(n - km) - o(n)$ many copies of $K_{r}$, and so for all $m \geq 2r^{2} + 2r$ the number of copies of $K_{r}$ in $G$ is at least:

$$k \binom{m}{r} + k + \binom{m-r-2}{r-1}(n - km) - o(n). \quad (5.1)$$

To get the required lower bound, we next prove the simple claim that $\binom{m-r-2}{r-1} \geq \frac{1}{m} \left( \binom{m}{r} + 1 \right)$ for all $m \geq 2r^{2} + 2r$ and $r \geq 2$. The most convenient way to do this is to show that
\[ m^{(m-r-2)} > \binom{m}{r}, \] since both sides of this last inequality are integers. Indeed, let \( m \) and \( r \) satisfy the conditions we just mentioned. Then,

\[
\frac{m-1}{m-r-2} \leq \frac{m-2}{m-r-3} \leq \cdots \leq \frac{m-r+1}{m-2r} \leq \frac{2r^2 + r + 1}{2r^2}.
\]

Hence,

\[
\binom{m}{r} \leq \frac{1}{r} \left( \frac{1}{r} + \frac{1}{2r^2} \right)^{r-1} \leq \frac{1}{r} \cdot e^{\frac{(r+1)(r-1)}{2r}} \leq \frac{1}{r} \cdot \sqrt{e} < 1,
\]

which establishes the claim that \( \binom{m-r-2}{r-1} \geq \frac{1}{m} \left( \binom{m}{r} + 1 \right) \). Using this, we get that (5.1) is at least \( k \binom{m}{r} + k + \frac{1}{m} \left( \binom{m}{r} + 1 \right) (n - km) - o(n) \geq \frac{n}{m} \left( \binom{m}{r} + 1 \right) - o(n) \), completing the proof.

6 Concluding remarks

We end with some open problems. We determined the exact value of \( \text{sat}(n, K_r, K_s) \) for all sufficiently large \( n \), but our arguments do not extend to find the value for small \( n \). So the following question still remains open.

Problem 6.1. For \( s > r \geq 3 \), determine the exact value of \( \text{sat}(n, K_r, K_s) \) for all \( n \).

We have already made a remark on the maximum constant \( c_r \). We can write in the stability result in Theorem 1.3. It would be interesting to determine that maximum constant.

Problem 6.2. For \( s > r \geq 3 \), what is the second smallest number of copies of \( K_r \) in an \( n \)-vertex \( K_s \)-saturated graph?

It might be interesting to consider a more general problem of finding the spectrum (set of possible values) of the number of copies of \( K_r \) in a \( K_s \)-saturated graph. The \( r = 2 \) case, i.e., the edge spectrum of \( K_s \)-saturated graphs, was completely solved in [1] and [5].

Problem 6.3. For \( s > r \geq 3 \), what are the possible numbers of copies of \( K_r \) in an \( n \)-vertex \( K_s \)-saturated graph?

As we mentioned earlier, Conjecture 1.5 is still wide open and likely needs new ideas to settle it. It would be interesting to figure out if the size of the family in Theorem 1.6 can be further reduced to 2. Finally, as we briefly discussed before stating Theorem 1.7, it would be interesting to consider Conjecture 1.5 for the generalized saturation problem.

Problem 6.4. For \( r \geq 2 \), does the limit \( \lim_{n \to \infty} \frac{\text{sat}(n, K_r, F)}{n} \) exist for every graph \( F \)?
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