Position-space approach to hadronic light-by-light scattering in the muon $g - 2$ on the lattice

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Hadronic Light-by-Light Contribution

\[ \text{gyromagnetic moment: } \mu = g \frac{e}{2m} S \]

\[ \text{anomalous magnetic moment } a_\mu = \frac{g_{\mu-2}}{2} \]

| contribution         | \[a_\mu [10^{-10}]\] | reference                  |
|----------------------|------------------------|-----------------------------|
| QED (leptons)        | 11 658 471.8853 ± 0.0036 | Aoyama et al '12           |
| HVP LO               | 690.75 ± 4.72          | Jegerlehner and Szafron et al '11 |
| HVP NLO              | −10.03 ± 0.22          | Jegerlehner and Szafron et al '11 |
| HVP NNLO             | 1.24 ± 0.01            | Kurz et al '14              |
| HLBL LO              | 11.6 ± 4.0             | Jegerlehner and Nyffeler '09 |
| HLBL NLO             | 0.3 ± 0.2              | Colangelo et al '14         |
| EW                   | 15.36 ± 0.10           | Gnendiger et al '13         |
| total                | 11 659 181.1 ± 6.2     |                             |
| experimental         | 11 659 208.9 ± 6.3     | Bennett et al '06          |

\[ \approx 3 \text{ standard deviations discrepancy } \rightarrow \text{ new physics?} \]
State of the Art for HLbL

- not fully related to any cross section
- until now only model dependent estimates
- → large uncertainties

**phenomenology:** reduce model uncertainties for dominant contribution \((\pi^0, \eta, \eta', \pi\pi)\)

using experimental input → dispersion relations
Colangelo *et al* '14 '14 '15;
Pauk and Vanderhaeghen '14

**lattice QCD**

can provide a model independent first-principle estimate
*only publications from one group so far:* Blum *et al* '15
(talk by Luchang Jin earlier this session)
two independent developments by

- Blum et al '15
- our group (NA talk at DPG meeting March '15 and Green et al Lattice 2015 [arXiv:1510.08384], Asmussen et al in preparation)

similarities

- get directly $F_2(q^2 = 0)$
- no cancellation of an $O(\alpha^2)$ term
- position space
- perturbative treatment of the QED part
Our Approach

how
- QCD blob: lattice regularization
- everything else: position space perturbation theory in Euclidean formulation (most natural choice!)

strengths
- QED part computed in infinite volume in continuum
- no power law effects in the volume

challenges
- need to calculate a four-point function

I will focus on the perturbative part.
In Euclidean space:

\[
\langle \mu^-(p', s') | j_\rho(0) | \mu^-(p, s) \rangle = -\bar{u}^{s'}(p') \left[ \gamma_\rho F_1(k^2) + \frac{\sigma_{\rho\tau} k_\tau}{2m} F_2(k^2) \right] u^s(p)
\]

**first steps**

- start by the Feynman rules for QED
- project out \( F_2(0) \) (cf. Kinoshita et al ’70)

\[
\Pi_{\mu\nu\lambda\rho}(q_1, q_2, k-q_1-q_2) = -k_\sigma \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k-q_1-q_2)
\]

\[
F_2(0) = \frac{-i}{48m} \text{Tr}\{[\gamma_\rho, \gamma_\tau](-i\slashed{p} + m)\Gamma_{\rho\tau}(p, p)(-i\slashed{p} + m)\}
\]

- on-shell muon momentum \( p = im\hat{e} \ (p^2 = -m^2) \)
- Fourier transform
Vertex function in momentum space

**vertex function**

\[
\Gamma_{\rho\sigma}(p', p) = -e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2 - k)^2} \frac{1}{(p' - q_1)^2 + m^2} \frac{1}{(p' - q_1 - q_2)^2 + m^2} \\
\gamma_\mu (i p' - i q_1 - m) \gamma_\nu (i p - i q_1 - i q_2 - m) \gamma_\lambda \frac{\partial}{\partial k_\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, k - q_1 - q_2) \]

\[
\Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \int_{x_1, x_2, x_3} e^{-i(q_1 x_1 + q_2 x_2 + q_3 x_3)} \langle j_\mu(x_1) j_\nu(x_2) j_\lambda(x_3) j_\sigma(0) \rangle
\]
Vertex function in position space

\[ \Gamma_{\rho \sigma}(p, p) = -e^6 \int_{x, y} K_{\mu \nu \lambda}(x, y, p) \Pi_{\rho; \mu \nu \lambda \sigma}(x, y) \]

\[ K_{\mu \nu \lambda}(x, y, p) = \gamma_\mu (i p + \partial (x) - m) \gamma_\nu (i p + \partial (x) + \partial (y) - m) \gamma_{\lambda} \mathcal{I}(\hat{\epsilon}, x, y) \]

\[ \mathcal{I}(\hat{\epsilon}, x, y) = \int_{q, k} \frac{1}{q^2 k^2 (q+k)^2} \frac{1}{(p-q)^2+m^2} \frac{1}{(p-q-k)^2+m^2} e^{-i(qx+ky)} \]

\[ \Pi_{\rho; \mu \nu \lambda \sigma}(x, y) = \int_{z} i z_{\rho} \langle j_\mu (x) j_\nu (y) j_{\sigma} (z) j_{\lambda} (0) \rangle \]

- \( \mathcal{I} \) is logarithmic infrared divergent for \( p^2 = -m^2 \) (introduce regulator)
- \( K_{\mu \nu \lambda} \) is infrared finite
Details of the calculation

**evaluating \( I(\hat{\epsilon}, x, y) \)**

\[
I(\hat{\epsilon}, x, y) = \int_{u, \text{IR-reg}} G_0(u - y) J(\hat{\epsilon}, u) J(\hat{\epsilon}, x - u)
\]

\[
J(\hat{\epsilon}, y) = \int_x G_0(x + y) e^{-m\hat{\epsilon} \cdot x} G_m(x)
\]

Chebyshev expansion of \( J \): \( J(\hat{\epsilon}, y) = \sum_{n \geq 0} z_n(y^2) U_n(\hat{\epsilon} \cdot \hat{y}) \)

\( U_n \) = Chebyshev polynomials of the second kind
(special case of the Gegenbauer polynomials)

\[
G_m(x) = \frac{m}{4\pi^2|x|} K_1(m|x|) \quad (K_1 \text{ is a modified Bessel function})
\]

\( z_n \) = linear combination of products of two modified Bessel functions.
master formula

\[ a_{\mu}^{\text{HLbl}} = F_2(0) = \frac{me^6}{3} \int_y \int_x \tilde{L}_{[\rho,\sigma];\mu\nu\lambda}(x, y) i\Pi_{\rho;\mu\nu\lambda\sigma}(x, y) \]

- after contracting the Lorentz indices the integration reduces to a 3-dimensional integration over \( x^2, y^2, x \cdot y \)

QCD four-point function

\[ i\Pi_{\rho;\mu\nu\lambda\sigma}(x, y) = -\int_z z_\rho \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0) \rangle \]

QED kernel function

\[ \tilde{L}_{[\rho,\sigma];\mu\nu\lambda}(x, y) \]

- weights the position-space vertex
- averaged over the direction of the muon momentum
- we have computed it once and for all
or how to handle the Lorentz structure of $\tilde{L}$

tensor decomposition

\[
\tilde{L}_{[\rho,\sigma];\mu\nu\lambda}(x, y) = \langle \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(\hat{\epsilon}, x, y) \rangle_{\hat{\epsilon}} = \sum_{A=I,II,III} G^A_{\delta\rho\sigma\mu\alpha\nu\beta\lambda} T^A_{\alpha\beta\delta}(x, y)
\]

\[
\langle (\cdots) \rangle_{\hat{\epsilon}} = \frac{1}{2\pi^2} \int d\Omega_{\hat{\epsilon}} (\cdots) \text{ average over the direction of the muon momentum}
\]

\[
G^{I,II,III}_{\delta\rho\sigma\mu\alpha\nu\beta\lambda} = \text{sums of products of Kronecker deltas}
\]

\[
T^I_{\alpha\beta\delta}(x, y) = \partial^{(x)}_{\alpha}(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta}) V_\delta(x, y)
\]

\[
T^{II}_{\alpha\beta\delta}(x, y) = m\partial^{(x)}_{\alpha}( T_{\beta\delta}(x, y) + \frac{1}{4} \delta_{\beta\delta} S(x, y))
\]

\[
T^{III}_{\alpha\beta\delta}(x, y) = m(\partial^{(x)}_{\beta} + \partial^{(y)}_{\beta})( T_{\alpha\delta}(x, y) + \frac{1}{4} \delta_{\alpha\delta} S(x, y))
\]
Form Factors

scalar \[ S(x, y) = \langle I \rangle \hat{\epsilon} \quad \text{(IR regulated)} \]

vector \[ V_\delta(x, y) = \langle \hat{\epsilon}_\delta I \rangle \hat{\epsilon} \]

tensor \[ T_{\beta\delta}(x, y) = \langle (\hat{\epsilon}_\delta \hat{\epsilon}_\beta - \frac{1}{4} \delta_{\delta\beta}) I \rangle \hat{\epsilon} \]

6 form factors

\[ \begin{align*}
S(x, y) &= g^{(0)} \\
V_\delta(x, y) &= x_\delta g^{(1)} + y_\delta g^{(2)} \\
T_{\alpha\beta}(x, y) &= (x_\alpha x_\beta - \frac{x^2}{4} \delta_{\alpha\beta}) j^{(1)} + (y_\alpha y_\beta - \frac{y^2}{4} \delta_{\alpha\beta}) j^{(2)} + (x_\alpha y_\beta + y_\alpha x_\beta - \frac{x \cdot y}{2} \delta_{\alpha\beta}) j^{(3)}
\end{align*} \]

all form factors depend on \( x^2, y^2, x \cdot y \)
Example: Form Factor $g^{(2)}$

$$g^{(2)}(x^2, x \cdot y, y^2) = \frac{1}{8\pi y^2|x| \sin^3 \beta} \int_0^\infty du \int_0^\pi d\phi_1$$

$$\left\{ 2 \sin \beta + \left( \frac{y^2 + u^2}{2|u||y|} - \cos \beta \cos \phi_1 \right) \frac{\log \chi}{\sin \phi_1} \right\} \sum_{n=0}^\infty$$

$$\{ z_n(|u|)z_{n+1}(|x-u|) \left[ |x-u| \cos \phi_1 \frac{U_n}{n+1} + (|u| \cos \phi_1 - |x|) \frac{U_{n+1}}{n+2} \right]$$

$$+ z_{n+1}(|u|)z_n(|x-u|) \left[ (|u| \cos \phi_1 - |x|) \frac{U_n}{n+1} + |x-u| \cos \phi_1 \frac{U_{n+1}}{n+2} \right]\}$$

where

$$x \cdot y = |x||y| \cos \beta, \quad |x-u| = \sqrt{|x|^2 + |u|^2 - 2|x||u| \cos \phi_1}$$

$$\chi = \frac{y^2 + u^2 - 2|u||y| \cos(\beta - \phi)}{y^2 + u^2 - 2|u||y| \cos(\beta + \phi)}, \quad U_n = U_n\left( \frac{|x| \cos \phi_1 - |u|}{|u-x|} \right)$$

$$z_n = \text{linear combination of products of two modified Bessel functions.}$$
Example $g^{(2)}(\{|x|, |y|, \cos \beta\})$

- $\cos \beta = -0.15625$
- $\cos \beta = 0.31250$
- $\cos \beta = 0.87500$

All 6 form factors computed to about 5 digits precision stored once and for all
Numerical test: pion-pole contribution to $a_{\mu}^{Hlbl}$

3d integration
\[ \int_y \rightarrow 2\pi^2 \int_0^\infty d|y||y|^3 \]
\[ \int_x \rightarrow 4\pi \int_0^\infty d|x||x|^3 \int_0^\pi d\beta \sin^2 \beta \]

cutoff for $x$ integration
\[ |x|^{\text{max}} = 4.05 \text{ fm} \]

result for VMD model from momentum-space representation
Lattice aspects

large volume needed

\[ |y|^\text{max} \gtrsim 2 - 3 \text{ fm needed even for } m_\pi = 600 - 900 \text{ MeV} \]

remember we do not have power law effects(!)

computational cost (fully connected contribution)

- With the help of sequential propagators, the computation is arranged so that the \(d^4x\) integral can be evaluated at the sink
- If the 1-dim. integral over \(|y|\) is done with N evaluations of the integrand:
  - (1+N) forward propagators
  - 6(1+N) sequential propagators
Conclusions and Outlook

so far:

- explicit formula for $a_{\mu}^{\text{Hlbl}}$
- kernel function multiplying the position-space correlation function → stored on disc (form factors), ready to be used
- verified the kernel function

what next?

- calculate the four-point correlation function on the lattice
- work out the Hlbl contribution to $a_{\mu}^{\text{Hlbl}}$
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Thank you for your attention!