Improving photon detector efficiency using a high-fidelity optical CNOT gate

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A significant problem for optical quantum computing is inefficient, or inaccurate photo-detectors. It is possible to use CNOT gates to improve a detector by making a large cat state then measuring every qubit in that state. In this paper we develop a code that compares five different schemes for making multiple measurements, some of which are capable of detecting loss and some of which are not. We explore how each of these schemes performs in the presence of different errors, and derive a formula to find at what probability of qubit loss is it worth detecting loss, and at what probability does this just lead to further errors than the loss introduces.

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I. INTRODUCTION

Quantum computers exploit quantum superpositions to process every single input at the same time. However, when a measurement is performed this superposition collapses, and only a single result is obtained. As such, a standard quantum algorithm is run multiple times before meaningful data can be extracted. Despite this, there are several problems which can be solved using a quantum computer significantly faster than using a classical computer. The standard example of this being Shor’s factoring algorithm, which runs exponentially faster than any known classical factoring algorithm [1]. Since each run of the quantum computer requires a significant amount of time, it is preferable to reduce the number of runs required. Depending on context there are two techniques for doing this, the first is clever data extraction algorithms such as phase estimation [2], while the second is quantum state tomography [3]. While both of these techniques aim to minimize the number of runs of the computer required, they are both negatively impacted by inaccurate or missing measurements. As such it is important to make the measurement procedure of our quantum computer as efficient and accurate as possible.

There have been many architectures proposed for building a quantum computer all of which have their own advantages and disadvantages. A significant problem across architectures is the efficiency and accuracy of the detector [4–7]. In this paper we will explore how the efficiency of a detector can be improved using a miniature error correcting code. This miniature code can be layered on top of standard concatenated error corrections. We will concentrate on applying this miniature code to an optical quantum system, where our qubits are formed in the polarization basis of photons. In this scenario there is a unique problem that if a detector fails to make a measurement the qubit is lost and it becomes impossible to detect its state. In this paper we assume a detector efficiency of 90% which is optimistic compared to currently available detectors [4]. Photons also become lost during a calculation, and we need to be able to distinguish between loss caused in the detector, and loss that occurred earlier in the calculation.

Previous work has explored a similar idea from a statistical perspective [7, 8]. While Deuar and Munro [7] concentrated on an ionic system, Schaetz et al. [8] looked at a photonic system in the dual rail basis. Schaetz et al. concentrated on an ionic system. We improve upon both pieces of work in two ways. Firstly we conduct full simulations, rather than simple probabilistic calculations, our results therefore take into account any backwards propagation of error. Secondly we propose several adaptations of the standard scheme to be suitable for architectures where there is loss. Depending on the function of the entangling gate this is essential to distinguish between when the qubit we are trying to measure is in state $|0\rangle$ and when it is lost. In recent work Ghosh et al. [9] discussed how using entangling gates for loss detection can help in topological error correction.

We develop a code that simulates five different forms of improved detection, these five schemes are shown in table 4. The schemes are simulated acting under the following errors; probability that the measurement qubits are present, probability that the measurement qubits are initialized correctly, probability that the $X$ gate has an error, probability that the CNOT gate has an error, probability of loss in the $X$ gate, probability of loss and distribution of said loss in the CNOT gate, probability the detector looses the qubit, probability of a bit flip error in the detector, and the state and probability of loss of the initial photon. This code can be used for any system but we chose error parameters most

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TABLE I: We use five different measurement schemes which are illustrated here for a total of four detectors consistent with a photonic system. We then used this code to derive a formula to work out at what loss probability it becomes worth attempting to detect loss, and at what loss probability detecting loss leads to a greater error than the loss itself.

In section II we discuss what previous work has been done looking at compensating for inaccurate detectors, we also introduce a statistical formula for working out the probability of obtaining a correct reading in the presence of a CNOT gate with uncorrelated errors. In section III we introduce our error model which includes correlated errors on our CNOT gate, something previous work has not taken into account. Then, in section IV we introduce five possible arrangements of CNOT gates, detectors and $X$ gates that can be used for our mini-error correcting code. In section V we discuss the simulation code we developed, and show the results it produces if we use detector number as our variable. Section VI contains the results we obtained for deriving a formula to find the loss probability at which attempting to detect loss introduces less errors than the errors due to the loss itself. Finally, we conclude in section VII.

## II. PREVIOUS WORK

Deuar and Munro \[8\] consider a copying device that is functionally equivalent to a CNOT gate in the vacuum–single-photon basis. Their aim is to determine photon presence in the presence of an inefficient detector and an error prone CNOT gate. If the detector has an efficiency given by $\eta$, and the copier has an error given by $\epsilon$ then as the number of measurements ($N$) tends to infinity, they find that the limiting efficiency is given by

$$\lim_{N \to \infty} \eta = 2 - \frac{1}{\epsilon}.$$  \hspace{1cm} (1)

When $\eta = 0.6$ and $\epsilon = 0.714$ they find that three CNOT gates are needed before the cost of adding more copies no longer becomes worth the gain. Our model differs significantly from the model being considered in that paper, in that our CNOT gate introduces errors onto the initial state being copied. This is more physically realistic and negatively impacts the maximum fidelity that can be achieved, as well as implying that an infinite number of measurements...
would decrease the fidelity with respect to the optimal point. Deuar and Munro [8] are also only considering detection in one basis, here we want to distinguish between three possible states, $|0\rangle$, $|1\rangle$ and loss.

In their paper on experimentally detecting the state of an ionic qubit, Schaeetz et al. [7] derive a formula for getting $m$ correct answers out of $M$ measurements where the fidelity of the detector is given by $F$

$$P_m = \frac{F^m(1-F)^{M+1-m}(M+1)!}{m!(M+1-m)!}.$$  \hspace{1cm} (2)

To find the probability that the state is detected correctly they sum all the probabilities for $m > M/2$. This formula assumes a perfect CNOT gate, and that the error in the detector is a bit-flip rather than a loss error. We can adapt the formula for an error-prone CNOT gate with a lossy detector, the scenario considered for photonic qubits. The probability that the $m$th detector gives the correct answer is given by

$$C_m = SK \sum_{j=0}^{[m/2]} \binom{m}{m-2j} [1-W][1-T]^{2j}(W+T[1-W])^{m-2j}$$ \hspace{1cm} (3)

where $S$ is the probability that the second qubit is present, $K$ is the probability that the detector works, $W$ is the probability that the CNOT gate has no error, and $T$ is the probability that any error in the CNOT gate is in the $Z$-basis, so has no impact on the measurement result in the $|0\rangle$, $|1\rangle$ basis. To find the probability of a correct majority vote we need to sum all scenarios where there are more correct correct readings than incorrect readings while ignoring losses. The formula works on the principle that an even number of errors in the CNOT gates preceding the desired measurement lead to a correct answer. For this to work we need to consider errors on only one qubit, and therefore assume that the CNOT error acts independently on each qubit that it occurs on. In reality this is not the case and we often have a correlated error model. This is one reason that we need to move from a statistical model to a simulation. Another is that this model only works for a simple $|0\rangle$, $|1\rangle$ input and we want to consider the efficiency of a more general input. This general input becomes more important when we consider attempting to detect loss, as the behavior of our CNOT gate means loss will often give a false reading of $|0\rangle$.

### III. DESCRIPTION OF THE ERROR MODEL

In this paper we use a standard Pauli model The error model we use for the $X$ gate is given by

$$\rho_f = \sum_i \sigma_i K_i \rho K_i^\dagger \sigma_i$$ \hspace{1cm} (4)

where $K_0 = \sqrt{1-p_z} I$ and $K_n = \sqrt{p_z/3} \sigma_n$ where $n = x, y, z$.

The error model for the CNOT gate is given by

$$\rho_f = \sum_i CL_i \rho L_i^\dagger C$$ \hspace{1cm} (5)

where $C$ is the CNOT gate. Here we have $L_0 = \sqrt{1-p_z} I \otimes I$ and $L_n = \sqrt{p_z/15} \sigma_n \otimes \sigma_m$ where $n = I, x, y, z$ and $m = x, y, z$ or $n = x, y, z$ and $m = I, x, y, z$. We note that equation (5) results in correlated errors between the qubits that the CNOT gate acts upon. This means the model we derived in equation (3) is not valid, we therefore look at full simulations rather than just a statistical model. If one of the qubits going into the CNOT gate is lost we assume that the ideal CNOT gate performs an identity operation on the other qubit with the same error distribution as the standard CNOT gate.

Here we use $|0\rangle$ to represent the horizontal polarization of a photon, and $|1\rangle$ to represent the vertical polarization of a photon. Our detection scheme assumes measurement in the $|0\rangle$–$|1\rangle$ basis. Since we are considering measurement at the end of a calculation this is not a problem as a necessary transform can be applied to the qubit we wish to detect and then a measurement made in the standard $Z$-basis. We have not included this error in the detection calculation, as we assume it is part of the computational error that is accounted for using standard error-correction or multiple runs of the computer. In quantum optical systems the two qubit gates requires a nonlinearity that is typically introduced through measurement. However there are a few alternative ways of introducing this nonlinearity, such techniques include using a photonic module [10], using the Zeno effect [11] (although in this case measurement is still often required to improve efficiency) and using a cross-Kerr nonlinearity, as done in qubus systems [12].
FIG. 1: Five possible schemes for detecting the state of a qubit in the $|0\rangle$, $|1\rangle$ basis. Circuits (a) and (d) have no loss detection, while circuits (b), (c) and (e) can detect loss.

IV. ALTERNATIVE DETECTION SCHEMES

Schaetz et al. [7] discussed the fact that it should be possible to detect loss by using a simple bit flip operation (a Pauli $X$ gate) between successive CNOT gates. In the case where the gates are error free then attempting to detect loss is clearly the best strategy. However, in reality both the CNOT and the $X$ gate will be subject to errors, and there is no reason to assume that these errors are identical. To detect loss we need to use at least one additional $X$ gate and this results in a worse performance than a scheme which does not detect loss. As such we simulate five different techniques for improving our detector to see which one functions best under different error conditions. These techniques are illustrated in figure 1 and summarised in table I in this section we discuss each technique and its advantages and disadvantages.

Technique one, shown in figure 1a is the standard technique of using CNOT gates and measuring. We make the assumption that the CNOT gates are performed on qubit one sequentially with no hold operations between them. The second qubit in the CNOT gate is prepared on demand and measured straight after the CNOT gate. This technique has the advantage of using the minimal number of gates possible for a majority vote but has the disadvantage of being unable to detect when the initial qubit is lost.

Techniques two and three use a combination of CNOT and $X$ gates. In technique two shown in figure 1b an $X$ gate is performed on qubit one after every CNOT gate. Technique three shown in figure 1c splits the total number of CNOT gates into two performing half of them, then an $X$ gate on qubit one, then the rest of the CNOT gates. Both technique two and three function best with an even number of CNOTs and detectors. One again we perform the operations sequentially with no hold operations between them. The second qubit in each CNOT gate is prepared on demand and measured straight after the CNOT gate. Both technique two and three should be suitable for detecting loss but if we assume a tight majority vote even a single error will mean that we will get a false reading of either $|0\rangle$ or $|1\rangle$.

Finally we consider techniques four and five. Both of these techniques use the minimum number of CNOT gates and detectors possible. In technique four shown in figure 1d we perform CNOT gates and detections until one detector gives a reading. This reading alone is then used to determine our results. This technique uses considerably less operations on average than required for a majority vote thus is more effective for CNOT gates with a high error rate. Similar to technique one, technique four cannot detect loss.

Technique five, shown in figure 1e is an adaptation of technique four that is also able to detect when qubit one is lost. We consider performing CNOT gates and detectors until a single measurement is made. This reading alone is then used to determine our results. This technique uses considerably less operations on average than required for a majority vote thus is more effective for CNOT gates with a high error rate. Similar to technique one, technique four cannot detect loss.

V. NUMBER OF DETECTORS

To compare our five possible improved detector schemes, which are summarised in table II we created a code that simulates the five schemes and compares the fidelity to the original input state. The fidelity calculation is computed
Fidelity

$0.992\quad 0.994\quad 0.996\quad 0.998\quad 1.000$

Number of detectors

For up to six detectors then has only gradual increases. This is because the probability of obtaining at least two losses in the detector. We chose a starting state of $|\psi\rangle = (k_1|\psi\rangle + (1 - k_1)|L\rangle(L) \otimes (k_2p_0)0\rangle0\rangle + k_2(1 - p_0)1\rangle1\rangle + (1 - k_2)|L\rangle(L)$

where $k_1$ is the probability that the qubit we are trying to detect is present, $k_2$ is the probability that the qubits used for detection are present, and $p_0$ is the probability that the photons used for detection are correctly initialized in $|0\rangle$. The wave-function $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is the state of the qubit we are trying to detect, and $|L\rangle$ is loss. Each measurement is modelled as a projective measurement, and is followed by a partial trace so that there is never any more than two qubits in the system.

For each detection scheme we find the probability of each possible measurement combination. A simple summing procedure is then used to find the conclusion that an experimenter would draw from that set of results. In the case of scheme one this scheme simple involves using a dummy variable, $St$, that starts at zero. Every time a $|0\rangle$ is measured one is added to $St$, every time a $|1\rangle$ is measured one is subtracted from $St$, and when there is no detection $St$ is left constant. In the schemes with $X$ gates the procedure is flipped after every $X$ gate with a reading of $|0\rangle$ requiring the alternating addition or subtraction of one from $St$. We then sum the probability for all combinations where $St$ is greater than zero, exactly zero, or less than zero and use this to form the probability of concluding $|0\rangle$, mixed, or $|1\rangle$ respectively. In the case of schemes one and four, a mixed result is automatically an error, while in the cases of schemes two, three and five, a mixed result represents a conclusion of loss.

If, for some reason we want a system that has a higher chance of detecting loss at the expensive of a false positive for loss, it is possible to change the summation points. For example we could use $|S| \geq 1$ for the loss interval. We have not considered that here as our primary aim is to detect the state of our qubit in the $|0\rangle, |1\rangle$ basis, and therefore the false positive readings on loss would mean throwing away too much information. At the same time our code can easily consider these scenarios with minimal adaptations. We note that in all cases, if our CNOT has a higher error in the $Z$-basis than the detector it is impossible to get any improvements.

Given that our CNOT error model means equation $\mathbf{3}$ is inaccurate, we want to look at how all five schemes perform as the number of detectors increases. An example case is shown in figure $\mathbf{2}$ where we start with our first qubit in the initial state $|1\rangle$ and have a Pauli error of 0.001 on both the CNOT and $X$ gate, and a 0.1 probability of loss in the detector. We chose a starting state of $|1\rangle$ because in schemes one and four, where loss is not detected, loss always leads to a false reading of $|0\rangle$; we therefore consider the worst possible case. The photons used for detection are present with a 0.999 probability. We see that scheme four, where the minimum number of detectors is used always outperforms scheme one, which is a majority vote based scheme. This is because scheme one typically uses more CNOT gates. For a detector efficiency of 0.9 we find that schemes one through four reach a maximum fidelity at four detectors. Scheme five, which uses the two first readings with an $X$-gate between them, has significant improvements for up to six detectors then has only gradual increases. This is because the probability of obtaining at least two readings without loss increasing significantly until six detectors are used. Scheme three, which only uses one $X$ gate
FIG. 3: Here we see how the five techniques perform in the presence of different values of loss. Here we have a detector error of 0.1 and the probability of a Pauli error or loss error in the $X$ and CNOT gate is 0.001. The red undashed line represents scheme one, the green closely-dashed line scheme two, the purple medium-dashed line scheme three, the blue wide-dashed line scheme four and the orange widest dashed line scheme five.

always outperforms scheme two which uses multiple $X$ gates; this is because the extra $X$ gates in scheme two provide no significant benefits but contribute to the net error.

A particular thing to note is the poor performance of scheme five that uses the first two non-loss detections, relative to scheme three, and even scheme two, for a low number of detectors. Both scheme two and three use a fixed number of detectors and $X$ gates. This poor performance of scheme five is due to the fact that we have to have at least two detections before we get a reading, and a single CNOT or $X$-error will lead to a false reading of loss. Using this we can see that to first order, with four detectors we only expect a reading (including an incorrect one) with 0.999 probability compared to the 0.9999 probability of obtaining a reading for the other schemes. Similarly a single error on any of the gates in scheme five will lead to an inaccurate reading, while the impact of an error on schemes two and three will depend on whether it occurred on the initial qubit (at which point it will effect all other measurements) or the qubit being detected. Therefore, despite the fact that taking the first reading (scheme four) gives significant improvements over a majority vote (scheme one), when there is no loss a majority vote (scheme two) out-performs using the first two correct detections (scheme five) for less than seven detectors. Unsurprisingly, when we increase the detector efficiency then scheme five begins to perform relatively well compared to schemes two and three, particularly in the cases of high loss.

As we increase the detector efficiency we reduced the optimal number of detectors needed, this is because each gate used introduces an error, so we want to use the minimum number of gates possible. Scheme four gets around this problem to some degree by using the minimum possible number of gates independent of the detector efficiency. Logically if we had no detector error then it would be better to not perform this enhanced measurement scheme and instead just measure the original qubit. Since we want to consider a range of detector efficiencies between 0.9 and 1, we will consider using four detectors for future calculations. This gives an accurate reading in the case of a 10% detector without introducing too many errors in the case of low detector error. For higher detector errors more detectors would be required.

VI. WHEN IS IT WORTH DETECTING LOSS?

In the previous section we decided that it was worth limiting the number of detectors we used to four. We now want to work out at what point is it worth detecting loss, and at what point does the increase in the number of gates required to detect the loss cause a greater error than the loss itself. From figure we can see that using the first detector (scheme four) always outperforms a majority vote (scheme one) when we are using four detectors. We therefore consider this as our standard scheme for detecting the state of a qubit without loss. Unsurprisingly we see that scheme three which only uses one $X$ gate always outperforms scheme two which uses mutliple $X$ gates (the only
The probability of loss in the $X$ is given in equation (8). Errors of 0.0005 that the qubit we are trying to detect is lost with a probability of greater than 0.0001. A CNOT and detecting scheme dependent upon the probability that the initial qubit is lost. We show that in a typical model with $P_{\text{loss}} = 0.0001$ the performance starts to drop at $k_1 = 0.99949$, for an error of 0.0005 the performance starts to drop at $k_1 = 0.9957$. Therefore the formula above is very approximate, still it explains intuitively why we would expect the fidelity of the scheme to fall at roughly the point it does.

We find a formula to show at what probability of loss for the initial qubit is it worth trying to detect loss. To do this we find the loss probability where scheme four is greater than one is given by $k_1 = 0.99991$. More generally we would expect the performance of schemes two and three to start decreasing when the probability of a gate error is higher than the probability of the photon being lost.

Assuming that the gates have no loss error then the point where it becomes worth attempting to detect loss is due to the fact that as the detector error increases the number of CNOT gates required by scheme four increases. We find a formula to show at what probability of loss for the initial qubit is it worth trying to detect loss. To do this we find the loss probability where scheme four is greater than one is given by $k_1 = 0.99991$. More generally we would expect the performance of schemes two and three to start decreasing when the probability of a gate error is higher than the probability of the photon being lost.

The model was then checked with random values to ensure it was accurate to $\pm 0.0001$. The linearity in the $X$-error and CNOT error is not surprising since the two schemes differ by a constant number of CNOT and $X$ gates. The quadratic behaviour in the detector errors is due to the fact that as the detector error increases the number of CNOT gates required by scheme four increases. Assuming that the gates have no loss error then the point where it becomes worth attempting to detect loss is given by

$$P_L = 1 - \left[ (0.0128122 - 0.575681 P_c + P_c [-0.656495 + 3.75622 P_c]) P_D^2 + (-0.00117864 + 0.198512 P_c) P_D ight. \\
+ P_c P_D (0.115926 - 0.496572 P_c) + 0.99986 - 0.203882 P_c + P_c (-0.13992 + 0.41898 P_c) \right]$$

In a logical check we find that when all the errors are zero this gives $P_L = 1 - 0.999986 \pm 0.0001$ which is within error bounds of $P_L = 0$. The formula is only valid for $P_D \leq 0.1$, $P_c \leq 0.1$ and $P_c \leq 0.05$. This is not a strong restriction since with errors above these levels it is unlikely that quantum error correction will work, and therefore quantum computing would be impossible. For a more complicated error model our code can be used to find the best performing of the five error correcting schemes at any error probability and loss probability.

VII. CONCLUSIONS

In conclusion given a very good CNOT gate, detector and $X$-error we have derived formulas to find the optimal detecting scheme dependent upon the probability that the initial qubit is lost. We show that in a typical model with a CNOT and $X$-error of 0.001, and a detector error of 0.1 it is worth attempting to detect loss if the probability that the qubit we are trying to detect is lost with a probability of greater than 0.0003. If our CNOT and $X$ gate had errors of 0.05 and the detector had an error of 0.1 the probability of loss is increased to 0.0152. A generalized formula is given in equation (8).

To generate this model we have developed a code that compares all five schemes shown in table I given the following input errors; probability that the measurement qubits are present, probability that the measurement qubits are initialized correctly, probability that the $X$ gate has an error, probability that the CNOT gate has an error, probability of loss in the $X$ gate, probability of loss and distribution of said loss in the CNOT gate, probability the detector looses the qubit, probability of a bit flip error in the detector, and the state and probability of loss of the initial photon. This problem is difficult to solve analytically due to the correlated errors assumed in the CNOT gates, and the fact that loss during the process has different impacts depending on the initial state of our qubit. The code takes into account backward propagation of errors, and can be quickly be adapted to run over any variable and for any number of detectors. It is also possible to change how the majority vote is constructed, so that the loss detection can be made more or less cautious.
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