Conductivity of weakly disordered metals close to a “ferromagnetic” quantum critical point

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We calculate analytically the conductivity of weakly disordered metals close to a “ferromagnetic” quantum critical point in the low temperature regime. Ferromagnetic in the sense that the effective carrier potential $V(q, \omega)$, due to critical fluctuations, is peaked at zero momentum $q = 0$. Vertex corrections, due to both critical fluctuations and impurity scattering, are explicitly considered. We find that only the vertex corrections due to impurity scattering, combined with the self-energy, generate appreciable effects as a function of the temperature $T$ and the control parameter $\tau$, which measures the proximity to the critical point. Our results are consistent with resistivity experiments in several materials displaying typical Fermi liquid behavior, but with a diverging prefactor of the $T^2$ term for small $\tau$.

Keywords: Conductivity calculation, Vertex Corrections, Quantum Critical Point, Fermi Liquid, Weak Disorder

1. Introduction

Itinerant electron systems display non-trivial behaviour close to a quantum critical point (QCP). E.g. some observables may diverge upon approaching the QCP. Our work is motivated by a number of experiments on several materials\textsuperscript{1-10}, which display typical Fermi liquid (FL) behaviour for appropriately low temperature $T$. That is, quadratic in $T$ resistivity and linear in $T$ specific heat. These materials include CeCoIn\textsubscript{5}\textsuperscript{11-13}, Sr\textsubscript{3}Ru\textsubscript{2}O\textsubscript{7}\textsuperscript{2-3}, YbRh\textsubscript{2}Si\textsubscript{2}\textsuperscript{14-16}, La\textsubscript{2-x}Ce\textsubscript{x}CuO\textsubscript{4}\textsuperscript{5}, Tl\textsubscript{2}Ba\textsubscript{2}CuO\textsubscript{6+}%\textsubscript{3+}, CeAuSb\textsubscript{5}, YbAlB\textsubscript{4} and Sr\textsubscript{2}Ru\textsubscript{2}O\textsubscript{7}\textsuperscript{3}. However, the prefactors of these quantities diverge in the vicinity of the respective QCP’s as power laws of the criticality parameter $\tau$, which measures the proximity to the QCP. $\tau$ may be determined by the electron filling factor, the pressure, or the magnetic field $H$ (which is related to filling, through the Zeeman term\textsuperscript{11,12}). The $T^2$ resistivity appears within various material and $H$ dependent ranges. E.g. up to 70 mK for YbRh\textsubscript{2}Si\textsubscript{2}, between 0 - 1.2 K for $H=5-14$ T, respectively, for CeCoIn\textsubscript{5}, for up to 10 K and $H \leq 1$ T for Sr\textsubscript{3}Ru\textsubscript{2}O\textsubscript{7}, up to 15 K at $H = 25$ T for CeAuSb\textsubscript{5}, and up to 100 K at $H = 45$ T for Tl\textsubscript{2}Ba\textsubscript{2}CuO\textsubscript{6+}\textsubscript{3+}. It is possible that in this regime $T$ is less or at most of the order of the impurity scattering rate $\tau_o^{-1}$.

We have shown in\textsuperscript{12}, via analytic diagrammatic calculations, that this critical FL behaviour can be consistently understood as arising from the exchange of relevant ferromagnetic fluctuations with small momentum $q$ among the quasi-particles. Our approach assumes that we deal with weakly disordered metallic systems. Herein we extend our previous calculation of the conductivity in the low $T$ regime, via a more comprehensive inclusion of vertex corrections. The latter are due both to the fluctuation potential $V(q, \omega)$ and to elastic (spinless) disorder scattering. The part of vertex corrections due to $V(q, \omega)$ yields no essential modifications on the results already obtained in\textsuperscript{12}.

2. The model

Henceforth, all momenta are 3-D or 2-D vectors, though we do not use bold letters. We consider the Green’s function

$$G^{R,A}_{\omega,k}(\epsilon) = \frac{1}{\epsilon - \xi_k \pm i/2\tau_o}, \quad \xi_k = \epsilon_k - \epsilon_F,$$  

\hspace{1cm} (1)

\hspace{1cm} with $\epsilon_k$ the quasiparticle dispersion, $\epsilon_F$ the Fermi energy, and $\tau_o$ the momentum relaxation time due to impurities. In the weak disorder regime\textsuperscript{13,14} $\epsilon_F \tau_o \gg 1$, $\tau_o^{-1}$ is important as a regulator in the calculations. In fact, the characteristic FL $T^2, \epsilon^2$ dependence of Im $\Sigma$ in eq. (1) is due to the finite $\tau_o^{-1}$.

The dominant electron-electron interaction is assumed to be the “ferromagnetic” fluctuation potential (or fluctuation propagator)\textsuperscript{12,15,16} peaked at $q = 0$

$$V(q, \omega) = \frac{g}{-i\omega/(Dq^2 + r) + \xi^2 q^2 + a},$$  

\hspace{1cm} (2)

[Note: The text continues with further details and equations related to the conductivity calculation and a more comprehensive analysis of the vertex corrections and their impact on the conductivity.]
with $g$ the coupling constant, $\xi$ the correlation length and $a$ measuring the distance from the QCP. The criticality parameter $a$ depends on e.g. $H$, as in the systems of interest mentioned below, like $a = h^s$, $h = |H/H_c - 1|$, $s > 0$, where $H_c$ is the critical field. The factor $D q^2$ indicates disorder induced diffusion of the quasiparticles, with diffusion coefficient $D^{14,17}$.

For the purpose of our calculations, we will treat $\xi$ and $a$ as independent parameters. This procedure, also followed in ref.12, is entirely consistent, as can be seen from the details of the calculations below. Also, after eq. (11), we discuss the role of the Gaussian regime $\xi^2 a = \text{const}^{11,15,16}$.

We have shown in ref.12 that, for the self-energy $\Sigma = \text{Tr} G_o V$, the quasi-particle scattering rate is

$$\text{Im} \Sigma(x, a) = F_d(a, \xi) x^2, \quad x = \text{max}\{T, \epsilon\}.$$  (3)

Here $F_d(a, \xi)$ scales like a negative power of the criticality parameter $a$ in $d = 2, 3$ dimensions. We obtained $F_d \propto a^{-2} \ln (\xi q_{\max}/\sqrt{a})$ for $r = 0$, $D > 0$, and $F_d \propto a^{-1} \xi^{-2}$ for $r > 0$, $D = 0$. This result can be also considered in the frame of the Gaussian regime, though it was derived without assuming any dependence between $\xi$ and $a$. In Appendix A we explicitly derive the result corresponding to eq. (3) for the case $\epsilon > T$.

In the following, we consider the total quasi-particle scattering rate

$$2S = \tau^{-1}(T, a) = \tau_{o,i}^{-1} + 2 \text{Im} \Sigma(\epsilon = 0, T, a),$$  (4)

with $\tau_{o,i}^{-1}$ due to impurity scattering. Then the Green’s function is taken as

$$G^{R,A}(k, \epsilon) = \frac{1}{\epsilon - \xi_k \pm iS},$$  (5)

i.e. it includes the self-energy of eq. (1) due to the fluctuation potential $V(q, \omega)$.

3. Calculation of the vertex corrections

We wish to calculate the conductivity $\sigma$, by including vertex corrections. Our treatment is similar to the one of Mahan\textsuperscript{18} for electron-phonon scattering. However, ours is different in a number of aspects, due to the different $V(q, \omega)$ and $G(k, \epsilon)$ considered here, the scattering by impurities, the specific functions $f(\epsilon), n(\omega)$ defined below etc. Dell’Anna and Metzner\textsuperscript{18} have treated the conductivity with vertex corrections for a scattering potential similar to our $V(q, \omega)$. However, disorder is not included in their Green’s function, our self-energy differs from theirs (while d-wave form factors are included in their potential), and our results differ significantly (this is also due to the different approximations made). $\sigma$ is given by\textsuperscript{18}

$$\sigma = \frac{2 e^2}{3} \lim_{\omega_0 \to 0} \frac{\text{Im} \Pi(\omega_0)}{\omega_0},$$  (6)

where we analytically continue $i\omega_l \to \omega_0$ in

$$\Pi(i\omega_l) = T \sum_{\epsilon_n} \sum_k \nu_k \Gamma(k, i\epsilon_n + i\omega_l) G(k, i\epsilon_n + i\omega_l) G(k, i\epsilon_n).$$  (7)

C.f. fig. 1. Here $\epsilon$ is the charge of the electron and $\nu_k = \nabla_k \epsilon_k$. The vector vertex function $\Gamma(k, i\epsilon_n, i\epsilon_n + i\omega_l)$ (with $\omega_l$ the energy difference between upper and lower lines) depends on the interactions - c.f. below. We consider scattering both via $V(q, \omega)$ and from the impurities. Here the Matsubara energies are $\epsilon_n = (2n + 1)\pi T, \omega_m = 2\pi m T$ and $\omega_l = 2\pi l T$.

With $f(\epsilon) = (1/2) \tanh(1/2T)$ and $\delta \to 0^+$, it can be shown that

$$\sigma = \frac{2 e^2}{3 \pi} \int_{-\infty}^{\infty} d\epsilon \frac{df(\epsilon)}{d\epsilon} \sum_k \nu_k \left\{ \Gamma(k, \epsilon - i\delta, \epsilon + i\delta) G^{R}(k, \epsilon) G^{A}(k, \epsilon) - \text{Re} \left[ \Gamma(k, \epsilon + i\delta, \epsilon + i\delta) \left( G^{R}(k, \epsilon) \right)^2 \right] \right\}.$$  (8)

This expression contains two different variants of the vertex function, with different energy arguments. Writing

$$\Gamma(k, \epsilon + i\delta, \epsilon + i\delta) = \nu_k \Lambda(k, \epsilon + i\delta, \epsilon + i\delta),$$  (9)

and using the Ward relation $\Lambda(k, \epsilon + i\delta, \epsilon + i\delta) = 1 + (\partial/\partial \xi_k) \Sigma(k, \epsilon)$ (c.f. ref.18, eq. (7.1.27) and after eq. (7.3.4)), we obtain

$$\Lambda(k, \epsilon + i\delta, \epsilon + i\delta) = \Lambda(k, \epsilon - i\delta, \epsilon - i\delta) = 1.$$  (10)
As mentioned in eq. (12), after eq. (14), the dependence of Im $\Sigma$ on $k$ is negligible for $k$ within a thick layer around the Fermi momentum $k_F$. We note that $\Gamma(k, \epsilon - i\delta, \epsilon + i\delta)$ is not given by a Ward identity. To calculate it, we turn to the respective ladder diagram approximation, without crossing interaction lines, in which $\Gamma(k, i\epsilon_n, i\epsilon_n + i\omega_l)$ obeys the equation shown in fig. 2

$$\Gamma(k, i\epsilon_n, i\epsilon_n + i\omega_l) = \Gamma^0(k, i\epsilon_n, i\epsilon_n + i\omega_l) + n_i \sum_q U_i(q)^2 G(k + q, i\epsilon_n + i\omega_l) G(k + q, i\epsilon_n) \Gamma(k + q, i\epsilon_n, i\epsilon_n + i\omega_l)$$

$$+ T \sum_q \sum_{\omega_m} V(q, i\omega_m) G(k + q, i\epsilon_n + i\omega_m + i\omega_l) G(k + q, i\epsilon_n + i\omega_m) \Gamma(k + q, i\epsilon_n + i\omega_m, i\epsilon_n + i\omega_m + i\omega_l) \tag{11}$$

$U_i(q)$ is the impurity scattering potential and $n_i$ the concentration of impurities.

The relevant Aslamazov-Larkin (AL) diagrammatic contribution to the vertex $\Gamma$ has been discussed in refs. 21, 22. However, it was shown that for the charge vertex, and in the $q = 0$ limit, where $q$ is the momentum difference of the two fermion lines at the vertex, the AL contribution vanishes. Hence we do not consider it here.

We make the usual assumption that

$$\Gamma(k, i\epsilon_n, i\epsilon_n + i\omega_l) = v_k \Lambda(k, i\epsilon_n, i\epsilon_n + i\omega_l) \ , \ \Gamma^0(k, i\epsilon_n, i\epsilon_n + i\omega_l) = v_k \tag{12}$$

i.e. the vector dependence is just given by $v_k$.

For the solution of eq. (11), we first look at the term involving $V(q, \omega)$

$$W = T \sum_{\omega_m} V(q, i\omega_m) G(k + q, i\epsilon_n + i\omega_m + i\omega_l) G(k + q, i\epsilon_n + i\omega_m) \Lambda(k + q, i\epsilon_n + i\omega_m, i\epsilon_n + i\omega_m + i\omega_l) \tag{13}$$

In order to evaluate it, we consider $n(\omega) = (1/2) \coth(\omega/2T)$, and the function of the complex variable $z$

$$F(z) = n(z) V(q, z) G(k + q, i\epsilon_n + z + i\omega_l) G(k + q, i\epsilon_n + z) \Lambda(k + q, i\epsilon_n + z, i\epsilon_n + z + i\omega_l) \tag{14}$$

Then we apply Cauchy’s residue theorem for a closed contour $C$ at infinity, thus obtaining

$$\frac{1}{2\pi i} \oint_C dz \ F(z) = W + I_V + I_1 + I_2 + R_V + R_1 + R_2 + L = 0 \tag{15}$$

The integrals $I_V, I_1, I_2$ are along the branch cuts of $V$ and the two $G$’s, and are given below. $R_V, R_1, R_2$ are the residues of $F(z)$ due to the poles of $V$ and the two $G$’s respectively. They are negligible, as discussed in Appendix B. $L$ is the contribution from the poles of $\Lambda$, which will also turn out to be negligible, as shown in Appendix B.
We have

\[ I_V = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega \ \text{Im} \ V^R(q, \omega) \ n(\omega) \ G(k + q, i\varepsilon_n + \omega + i\omega_l) \ G(k + q, i\varepsilon_n + \omega) \ \Lambda(k + q, i\varepsilon_n + \omega, i\varepsilon_n + \omega + i\omega_l). \]  

(16)

Taking into account that

\[ n(\omega - i\varepsilon_n) = f(\omega), \]

we also have

\[ I_1 = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\omega \ V(q, \omega - i\varepsilon_n - i\omega_l) \ f(\omega - i\omega_l) \ G(k + q, \omega - i\omega_l) \ \{G^R(k + q, \omega) \ \Lambda(k + q, \omega - i\omega_l, \omega + i\delta) - G^A(k + q, \omega) \ \Lambda(k + q, \omega - i\omega_l, \omega - i\delta)\}. \]

(18)

\[ I_2 = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d\omega \ V(q, \omega - i\varepsilon_n) \ f(\omega) \ G(k + q, \omega + i\omega_l) \ \{G^R(k + q, \omega) \ \Lambda(k + q, \omega + i\delta, \omega + i\omega_l) - G^A(k + q, \omega) \ \Lambda(k + q, \omega - i\delta, \omega + i\omega_l)\}. \]

(19)

We perform the analytic continuation

\[ i\varepsilon_n \to \varepsilon - i\delta, \ i\omega_l \to \omega_0 + i\delta, \ i\varepsilon_n + i\omega_l \to \varepsilon + \omega_0 + i\delta, \]

which yields

\[ I_V = \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega \ \text{Im} \ V^R(q, \omega) \ n(\omega) \ G^R(k + q, \varepsilon + \omega) \ G^A(k + q, \varepsilon + \omega) \ \Lambda(k + q, \varepsilon + \omega - i\delta, \varepsilon + \omega + i\delta), \]

(21)

\[ 2\pi i \ I_1 = \int_{-\infty}^{+\infty} d\omega \ V(q, \omega - \varepsilon - \omega_0) \ f(\omega - \omega_0) \ G(k + q, \omega - \omega_0 - i\delta) \ \{G^R(k + q, \omega) \ \Lambda(k + q, \omega - \omega_0 - i\delta, \omega + i\delta) - G^A(k + q, \omega) \ \Lambda(k + q, \omega - \omega_0 - i\delta, \omega - i\delta)\}. \]

(22)

\[ 2\pi i \ I_2 = \int_{-\infty}^{+\infty} d\omega \ V(q, \omega - \varepsilon) \ f(\omega) \ G(k + q, \omega + \omega_0 + i\delta) \ \{G^R(k + q, \omega) \ \Lambda(k + q, \omega + i\delta, \omega + \omega_0 + i\delta) - G^A(k + q, \omega) \ \Lambda(k + q, \omega - i\delta, \omega + \omega_0 + i\delta)\}. \]

(23)

Combining \( I_1 \) and \( I_2 \) we have

\[ 2\pi i \ (I_1 + I_2) = \int_{-\infty}^{+\infty} d\omega \ V(q, \omega - \varepsilon) \ f(\omega) \ K_0, \]

(24)

with

\[ K_0 = G(k + q, \omega - i\delta) \ \{G^R(k + q, \omega + \omega_0) \ \Lambda(k + q, \omega - i\delta, \omega + \omega_0 + i\delta) - G^A(k + q, \omega + \omega_0) \ \Lambda(k + q, \omega - i\delta, \omega + \omega_0 - i\delta)\} + G(k + q, \omega + \omega_0 + i\delta) \ \{G^R(k + q, \omega) \ \Lambda(k + q, \omega + i\delta, \omega + \omega_0 + i\delta) - G^A(k + q, \omega) \ \Lambda(k + q, \omega - i\delta, \omega + \omega_0 + i\delta)\}. \]

(25)

We want \( \Lambda(k + q, \omega - i\delta, \omega + i\delta) \), which enters the formula for the conductivity. Taking \( \omega_0 \to 0 \), we see that the term \( G^R G^A \ \Lambda(k + q, \omega - i\delta, \omega + i\delta) \) is multiplied by a total zero prefactor, due to the opposite signs of the contributions from \( I_1 \) and \( I_2 \). The only surviving contribution is

\[ K_0 \to K_1 = G^R(k + q, \omega + \omega_0) \ G^R(k + q, \omega) \ \Lambda(k + q, \omega + i\delta, \omega + \omega_0 + i\delta) - G^A(k + q, \omega + \omega_0) \ G^A(k + q, \omega) \ \Lambda(k + q, \omega - i\delta, \omega + \omega_0 - i\delta). \]

(26)
Now we use eq. (10), and we recall that the derivative $df(\epsilon)/d\epsilon$ in eq. (8) yields $\epsilon \simeq 0$ for low $T$ in $V(q, \omega - \epsilon)$. Then, using $1/(x+iS)^2 - 1/(x-iS)^2 = -4i \times S/(x^2 + S^2)^2$, with $x = \omega - \xi_{k+q}$ and $S$ from eq. (11), for the term $I_1 + I_2$ we make the approximation

$$
\int_{-\infty}^{+\infty} d\omega \, V(q, \omega - \epsilon) \, f(\omega) \left[ (G^R(k + q, \omega))^2 - (G^A(k + q, \omega))^2 \right] 
\simeq -4i \times S \, (G^R(k + q, \omega) = 0) G^A(k + q, \omega = 0))^2 \int_{-C_0}^{+C_0} d\omega \, V(q, \omega) \, f(\omega) \, (\omega - \xi_{k+q}) ,
$$

(27)

where the integration cutoff $C_0$ is of the order of $\epsilon_F$. Here we assumed that the main $\omega$ dependence comes from the integrand shown. The product $(G^R \: G^A)^2$ acts as an additional cut-off for $|\omega| > C_0$, hence this energy range is omitted.

To simplify the notation, we write

$$
\Lambda(k, \epsilon) \equiv \Lambda(k, \epsilon + i\delta, \epsilon - i\delta) .
$$

(28)

For the term $I_V$ we also make an approximation similar to the one in eq. (27)

$$
\int_{-\infty}^{+\infty} d\omega \, \text{Im} \, V^R(q, \omega) \, n(\omega) \, G^R(k + q, \epsilon + \omega) \, G^A(k + q, \epsilon + \omega) \, \Lambda(k + q, \epsilon + \omega)
\simeq G^R(k + q, \epsilon) \, G^A(k + q, \epsilon) \, \Lambda(k + q, \epsilon) \int_{-C_0}^{+C_0} d\omega \, \text{Im} \, V^R(q, \omega) \, n(\omega) .
$$

(29)

Now we introduce approximate forms for the functions $f(x)$ and $n(x)$. Namely we consider

$$
f(x) \rightarrow f_A(x) = x/(4T) , \text{ for } |x| < 2T , \quad f_A(x) = \text{sgn}(x)/2 \text{ for } |x| \geq 2T ,
$$

$$
n(x) \rightarrow n_A(x) = T/x , \text{ for } |x| < 2T , \quad n_A(x) = \text{sgn}(x)/2 \text{ for } |x| \geq 2T .
$$

(30)

The functions $f_A(x)$ and $n_A(x)$ are continuous and asymptotically exact for $|x| \ll 2T$ and $|x| \gg 2T$. They differ from the original $f(x)$ and $n(x)$ mostly at $x = 2T$. Namely $f_A(x = 2T) = 1/2 = (1/c_F)f(x = 2T)$ and $n_A(x = 2T) = 1/2 = (1/c_B)n(x = 2T)$, where $f(x = 2T) = 0.3808$ and $n(x = 2T) = 0.6565$. The “correction” constants are

$$
c_F = 0.762 , \quad c_B = 1.31 .
$$

(31)

Using these $f_A(x)$ and $n_A(x)$ we obtain the analytical expressions for $P(q)$ and $P_{12}(q)$ below. If we wish to consider the substitution $f(x) \rightarrow f_A(x)$ and $n(x) \rightarrow n_A(x)$ at face value, we should take $c_F = c_B = 1$ hereafter. Else, we consider the values given in eq. (31), and we note that $c_F$ and $c_B$ are introduced by hand in the following expressions, in order to compensate for the discrepancy, due to the approximation in eqs. (31), around $x = 2T$. Overall the difference between these two cases has an upper limit of $c_B - 1 = 0.31$ for the appropriate terms in $P(q)$, $P_{12}(q)$ and $R_{1k}$ below.

Thus we obtain

$$
I_V = G^R(k + q, \epsilon) \, G^A(k + q, \epsilon) \, \Lambda(k + q, \epsilon) \, P(q) ,
$$

(32)

$$
P(q) = \frac{g}{\pi} \left\{ c_B \frac{2 \, T \, \hbar_q}{\hbar_q} \tan^{-1} \left( \frac{2 \, T}{a_q \, \hbar_q} \right) + \frac{h_q^2}{2} \ln \left( \frac{a_q \, \hbar_q}{a_q \, \hbar_q}^2 + 4T^2 \right) \right\} ,
$$

(33)

with $\hbar_q = r + Dq^2$ and $a_q = a + \xi^2 q^2$.

When $\Lambda(k, \epsilon)$ is inserted in eq. (8) for $\sigma$, the dominant momenta are $k \sim k_F$, with $k_F$ the Fermi momentum. In this way $v_F^2$ can be inserted in the integrand below, and we obtain the following equation for $\Lambda(k, \epsilon)$

$$
\Lambda(k, \epsilon) = 1 + \sum_q \{ n_i \, U_i^2(q) - P(q) \} \, \Lambda(k + q, \epsilon) \, G^R(k + q, \epsilon) \, G^A(k + q, \epsilon) \left( \frac{v_{k+q} - v_k}{v_k^2} \right) + \sum_q (G^R(k + q, 0) \, G^A(k + q, 0))^2 \left( \frac{v_{k+q} - v_k}{v_k^2} \right) \, P_{12}(q) ,
$$

(34)
where
\[
P_{12}(q) = \frac{2g S}{\pi} \left\{ a_\lambda h_\lambda \ln \left( \frac{(h_\lambda a_\lambda)^2 + C_0^2}{(h_\lambda a_\lambda)^2 + 4T^2} \right) + c_P \left[ a_\lambda h_\lambda - \frac{a_\lambda^2 h_\lambda^2}{2T} \tan^{-1} \left( \frac{2T}{a_\lambda h_\lambda} \right) \right] \right\} .
\]

Further, we assume that, for \( k \sim k_F \), \( \Lambda \) is very weakly dependent on \(|q| \ll |k|\), i.e., \( \Lambda(k + q, \epsilon) \approx \Lambda(k, \epsilon) \). This assumption means that \( \Lambda(k, \epsilon) \) is a smooth function of \( k \sim k_F \), which is consistent with what follows, and is common in related derivations\(^{10}\). Also we note that, as far as the integration over \( q \) is concerned, the contribution from \( G^R G^A \) is subleading compared to the other terms. As a consequence
\[
\Lambda(k, \epsilon) = \frac{1 + Q_k}{1 - R_k G^R(k, \epsilon) G^A(k, \epsilon)} ,
\]
where
\[
R_k = \sum_q \{ n_i U_i^2(q) - P(q) \} \left( \frac{v_{k+q} v_k}{v_k^2} \right) , \quad Q_k = \{ G^R(k, 0) G^A(k, 0) \}^2 \sum_q \left( \frac{v_{k+q} v_k}{v_k^2} \right) P_{12}(q) .
\]

4. Calculation of the conductivity

Taking into account eqs. \(^{3},^{30}\), \( \sigma \) is given by
\[
\sigma = \frac{\epsilon^2}{3 \pi} \int_{-\infty}^{+\infty} \frac{de}{dc} \sum_k v_k^2 \left\{ \frac{(1 + Q_k) G^R(k, \epsilon) G^A(k, \epsilon)}{1 - R_k G^R(k, \epsilon) G^A(k, \epsilon)} - \text{Re} \left( G^R(k, \epsilon) \right)^2 \right\} .
\]

This is the central result of this work. Considering the limit of low \( T \) we have
\[
\sigma = \frac{\epsilon^2}{3 \pi} \sum_k v_k^2 \left\{ \frac{(1 + Q_k) G^R(k, 0) G^A(k, 0)}{1 - R_k G^R(k, 0) G^A(k, 0)} - \text{Re} \left( G^R(k, 0) \right)^2 \right\} .
\]

Overall, this is a decent approximate formula, valid for intermediate \( T \) as well. In the relevant terms \( P(q) \) and \( P_{12}(q) \) explicit \( T^2 \) terms were kept. The derivative of the Fermi distribution was taken as a delta function, which is also a reasonable approximation for intermediate \( T \).

We write
\[
R_k = R_{1k} + R_{2k} ,
\]
where
\[
R_{1k} = - \sum_q P(q) ,
\]
\[
R_{2k} = \sum_q n_i U_i^2(q) \left( \frac{v_{k+q} v_k}{v_k^2} \right) + \sum_q P(q) \left( 1 - \frac{v_{k+q} v_k}{v_k^2} \right) .
\]

Incidentally, we note that the transport scattering rate, due to the impurities, \( \tau_{tr}^{-1} = \sum_q n_i U_i^2(q) \left( 1 - \frac{v_{k+q} v_k}{v_k^2} \right) \) comes from the term \( R_{2k} \).

Considering \(|q| \ll |k|\), we have \( v_{k+q} v_k = v_k^2 + B_{1k} q + B_{2k} q^2 + ... \) (where \( B_{1k}, B_{2k} \) are coefficients of a Taylor expansion) and the dominant contribution for the criticality parameter \( a \rightarrow 0 \) comes from the term \( R_{1k} \). This is the case because higher powers of \( q \) in the numerator of the integrand in eq. \(^{37}\) yield terms less singular in the parameter \( a \).

We evaluate \( R_{1k} \). The interesting contribution, including negative powers of \( a \), arises from the low \( T \) limit, with \( 2T < a_\lambda h_\lambda \). Hence we consider a minimum \( q_T \) given by \( 2T = a_{\ell} h_{\ell} \). As in \(^{12}\) we consider a maximum \( q_{\max} = 1/2\tau_a v_F \), where \( v_F \) is the Fermi velocity. Also we approximate the logarithm in \( P_q \) as \( l_0 \simeq \ln(C_0/a_0 h_0) \), where \( a_0 = a_\lambda, h_0 = h_\lambda \) with \( q = q_{\max} \).

Then in 3-D
\[
R_{1k} = - \frac{g}{2\pi^2} \left( c_B \frac{2T^2}{\xi^3 \sqrt{a}} \left\{ \tan^{-1} \left( \frac{\xi q_{\max}}{\sqrt{a}} \right) - \frac{1}{\xi q_{\max}} \right\} + q_{\max}^3 l_0 \left( \frac{v^2}{3} + \frac{2\pi D q_{\max}^2}{5} + \frac{D^2 q_{\max}^4}{7} \right) \right) ,
\]

\(^{10}\) We shall therefore refer to the constant \( \frac{1}{2} \) in eq. (41) as \( \eta \).

\(^{12}\) We refer to the constant \( \frac{1}{2} \) in eq. (43) as \( \eta \).

\(^{3}\) We refer to the constant \( \frac{1}{2} \) in eq. (45) as \( \eta \).
while in 2-D
\[
R_{ik} = -\frac{g}{2\pi} \left( e_B \frac{2T^2}{\xi^2} \left\{ \frac{1}{a} - \frac{1}{\xi q_{\text{max}}} \right\} + \frac{q_{\text{max}}^2}{2} \left( \frac{\tau^2}{2} + \frac{\tau D q_{\text{max}}^2}{2} + \frac{D^2 q_{\text{max}}^6}{6} \right) \right). \tag{44}
\]

We note that, upon assuming the Gaussian regime $\xi^2 a = \text{const.}$ \cite{11,15,16}, there is no diverging factor in $R_{ik}$ for $a \to 0$. This possibility only arises if $\xi$ and $a$ are independent parameters - c.f. also \cite{12}. We do not explicitly evaluate the integral in $Q_k$ of eq. (37) because it does not yield any diverging factor for $a \to 0$. As discussed below, overall vertex corrections due to $V(q, \omega)$ do not modify appreciably the conductivity in the vicinity of the critical point.

To further evaluate the conductivity, we assume a parabolic dispersion relation $\epsilon_k$ so that $v_L = k/m$, with $m$ the mass of the electrons, as in eq. (18) in \cite{12}. Then, with $x = \epsilon_k - \epsilon_F$, $N_F$ the density of states at the Fermi level and now taking both $R_k \to R_F$ and $Q_{12} \to Q_F$ independent of $k$ and evaluated at $k = k_F$, we obtain
\[
\sigma = \frac{2e^2 N_F}{3 \pi m} \int_{-\epsilon_F}^{\infty} dx (x + \epsilon_F) \left\{ \frac{1 + Q_F}{x^2 + S^2 - R_F} + \frac{S^2 - x^2}{(x^2 + S^2)^2} \right\}. \tag{45}
\]
This yields
\[
\sigma = \frac{2e^2 N_F}{3 \pi m} \left\{ \left( \frac{1 + Q_F}{\epsilon_F} \right) \frac{\pi}{\rho_0} + \tan^{-1} \left( \frac{\epsilon_F}{\rho_0} \right) \right\} + \frac{1 + Q_F}{2} \frac{\rho_0}{\epsilon_F^2 + S_0^2} \ln \left( \frac{E_0^2 + S_0^2}{\epsilon_F^2 + S_0^2} \right) + 1 - \frac{3}{2} \frac{\rho_0}{\epsilon_F^2 + S_0^2} \ln \left( \frac{E_0^2 + S_0^2}{\epsilon_F^2 + S_0^2} \right), \tag{46}
\]
with $E_0 \sim O(\epsilon_F)$ (the upper limit of integration was taken as $E_0$ for the $\ln(...) \text{ terms, which are ultraviolet divergent}$) and $S_0 = \sqrt{S^2 - R_F}$. Of course, eq. (46) is not exact, due to the use of the parabolic dispersion instead of the actual crystalline one. However, it is advantageous in that it allows to discern more clearly the essential dependence on $a$ and $T$. Eq. (46) can be simplified, for reasons explained in the paragraph after next, with the result
\[
\sigma \approx \frac{2e^2 N_F \epsilon_F}{3mS_0}. \tag{47}
\]

These two expressions are very similar to eq. (18) in \cite{12} (modulo a shear numerical prefactor), which includes a part of the vertex corrections due to impurity scattering, as we explain in the following. For reference, the final simplified expression for the conductivity in \cite{12}, given after eq. (18) therein, is $\sigma = 4\pi e^2 N_F \epsilon_F/(m \sqrt{S^2 - u_0})$ (where $u_0 = n_i V_i^2$, with $V_i$ the typical value of the impurity scattering potential $U_i(q)$).

Here, the vertex correction term $Q_F \propto S \left| G'(k_F,0) \right|^2 = 1/S^3$, where $S$ in eq. (41) contains a negative power law of $a \to 0$ (times $T^2$). Hence $Q_F$ is negligible. The two remaining logarithmic terms in eq. (46) practically cancel each other (the remainder is just $x_1 - x_2 - O(x_1^2) + O(x_2^2)$, where $x_1 = R_F/|\epsilon_F^2 + S_0^2|$ and $x_2 = R_F/|E_0^2 + S_0^2|$). Further, the factor $R_F$, which also emanates from the vertex corrections, enters in the combination $S^2 - R_F$ in the final expression for the conductivity. It does not modify in an essential manner the dependence on either $T$ or $a \to 0$. Notably the square of $S$, yielding the main $a$ and $T$ dependence, is combined with the linear in $R_F$ term. Manifestly $R_F$ is less singular than $S^2$ for $a \to 0$, and overall of smaller magnitude. In other words, as in \cite{12}, the main dependence of the conductivity $\sigma$ on $T$ and $a$ is due to the combination of the self-energy of eq. (41) and of the vertex corrections from impurity scattering. The contribution of the vertex corrections from the fluctuation potential $V(q, \omega)$ is not essential.

Here the resistivity is taken as $\rho = \rho_0 + A T^2$ and the specific heat is $C = \gamma T$. We note that our theory yields a Kadowaki-Woods ratio $A/\gamma^2$ which is constant for $a \to 0$ (possibly times a $\ln(a)$ term) in 3-D only in \cite{12}, and this is consistent with experiments \cite{2,4,5,9}.

5. Overview

We calculate the conductivity, including vertex corrections due to both critical ferromagnetic fluctuations and disorder, in a weakly disordered metal close to a quantum critical point. We explicitly show that no appreciable effect results due to the fluctuation part of the vertex corrections. Our results are in very good agreement with relevant experiments in several materials \cite{4,10}, and complement our previous calculation which did not explicitly consider vertex corrections \cite{12} due to $V(q, \omega)$. The characteristic Fermi liquid $A T^2$ dependence for the resistivity, with a prefactor $A$ diverging as $a \to 0$, found therein thus remains valid.

Appendix A: On the calculation of the scattering rate

The derivation below follows that of \cite{12}, i.e. (I), and equation numbers refer to (I) as well. In the limit $T \to 0$ the thermal function $X = \coth(\omega/2T) + \tan(\epsilon/2T)$ in eq. (I-4) becomes $X = 2$ for $2T < \omega < \epsilon$, and $X = 0$ for
\( \omega < -2T \) and \( \omega > \epsilon \). Then the integration over \( \omega \) - compare with eq. (I-7) - amounts to

\[
2 \int_{-2T}^{2T} d\omega \, \text{Im} \, V(q, \omega) \, R(q, \omega) \simeq g \, R_0 \, \ln \left( \frac{(h_q \, a_q)^2 + \epsilon^2}{(h_q \, a_q)^2 + 4T^2} \right) \simeq g \, R_0 \, \frac{\epsilon^2}{(h_q \, a_q)^2} ,
\]

(48)

for \( h_q \, a_q > \epsilon \). The rest of the algebra proceeds as in eq. (I-8) and onwards. Thus the scattering rate scales like \( \epsilon^2 \) as well, as expected for the FL regime.

Appendix B : The terms \( R_V, R_1, R_2, L \) in eq. (I.5)

The terms \( R_1, R_2 \) each contain a single propagator \( G \). Hence, upon the final integration over momentum \( k \) they both yield a small contribution. This is the case because this integration is similar to an integration over \( \epsilon_k \) from \(-\infty \) to \(+\infty \), which can be taken as part of a contour integral closing at infinity. That contour can be taken such that the pole of the \( G \) in the integrand lies outside of it, and hence yields a zero contribution. C.f. also ref. 18.

The term \( R_V \) is due to the residue from the pole \( z = z_0 = -i \, a_q \, h_q \) of \( V(q, z) \). Here both \( G \)'s enter the formula for the residue. However, their poles are on the same semi-plane (i.e. in a combination \( G^A G^A \)), and the argument for \( R_1, R_2 \) applies as well.

The term \( L \) is the residue from the 2 poles \( z = z_k, z_k^* \) of \( \Lambda(k, z) \) - c.f. eqs. (50), (57) - with

\[
z_k = \xi_k + i \, W_k \ , \quad W_k^2 = S^2 - R_k .
\]

(49)

Considering the function

\[
H(z) = n(z) \, V(q, z) \, G(k + q, i\epsilon_n + z + i\omega) \, G(k + q, i\epsilon_n + z) \, R_{k+q}
\]

we have

\[
L = \frac{H(z_{k+q})}{z_{k+q} - z_{k+q}^*} + \frac{H(z_{k+q}^*)}{z_{k+q}^* - z_{k+q}} .
\]

(51)

This term is much smaller than \( I_V \) because \( |\text{Im} \, H(z)| \ll |\text{Re} \, H(z)| \).

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