Microscopic Calculations of the Finite-Size Spectrum in the Kondo Problem

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The finite-size spectrum in the Kondo problem is obtained from the Bethe-ansatz solution of the exactly solved models. We investigate the Anderson model, the highly correlated SU(ν) Anderson model and the s-d exchange model. For all these models we find that the spectra exhibit the properties characteristic of the Fermi liquid fixed point, and hence our microscopic calculations are in accordance with the results obtained by boundary conformal field theory with current algebra symmetry.

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I. INTRODUCTION

The Kondo problem for dilute magnetic impurities in metals has attracted continuous interest for years. Many fascinating aspects of the problem have been revealed by various methods such as the renormalization group approach [1], the local Fermi liquid theory [2,3] and the Bethe-ansatz solution [4,5]. Recently a series of works by Affleck and Ludwig has shed a new light on our understanding of the Kondo effect [9,10,11]. Their new machinery is a technique of two-dimensional boundary conformal field theories (CFT) [12]. Various critical properties of the Kondo problem including the multichannel overscreened as well as the underscreened case have been studied extensively.

The first crucial step made by Affleck in this approach was to recognize that for the ordinary Kondo problem the impurity effect is incorporated as modifying boundary conditions on the otherwise free conduction electrons in the low-energy effective theory [3]. From the CFT viewpoint, therefore, the impurity effect should be explicitly observed in the finite-size spectrum of the Kondo Hamiltonian. Assuming that the low-energy effective Hamiltonian is expressed in the Sugawara form of the current algebra Affleck and Ludwig verified the finite-size spectrum making use of the fusion rule hypothesis [10]. On the other hand, the Bethe-ansatz solutions to various models for the Kondo problem have been obtained [4,5]. Thus the finite-size spectrum should be directly obtained starting with the microscopic models. It is then worth comparing the results with those predicted by boundary CFT.

In this paper we address ourselves to this task and derive the finite-size spectrum in the Kondo problem from the Bethe-ansatz solution of variants of impurity models. We will show that the essential properties of the spectrum predicted by boundary CFT indeed agree with the exact results deduced by the Bethe-ansatz method. In Sec.II we begin with the Anderson model which is more tractable than the conventional s-d exchange model (Kondo model) to calculate the finite-size spectrum. We see that to the first order in $1/L$ with $L$ being the system size the excitation spectrum coincides with that of free electrons modified by the phase shift arising from the impurity scattering. A similar observation was made
earlier for the \( s-d \) exchange model in [9]. Applying the finite-size scaling we then obtain the canonical (integer) exponents characteristic of the local Fermi liquid. We also argue that the anomalous exponents related to the \( X \)-ray absorption singularity are read off from the finite-size spectrum. On the other hand, the correlation effects due to the impurity go into the \( 1/L^2 \) piece in the spectrum. Here the spin and charge susceptibilities for the impurity govern the excitation spectrum. In Sec.III our analysis is extended to the \( SU(\nu) \) Anderson model (degenerate Anderson model). We shall point out that the excitation spectrum is written in terms of the Cartan matrix of the \( \text{OSp}(\nu, 1) \) Lie superalgebra, which turns out to be responsible for the canonical exponents inherent in the local Fermi liquid. In Sec.IV we turn to the \( s-d \) exchange model in which the charge degrees of freedom of the impurity are completely frozen. We will see that a careful treatment of the charge degrees of freedom is necessary in this case. Some technical details are summarized in Appendix A.

II. ANDERSON MODEL

A. Bethe-ansatz solution

In this section our purpose is to calculate the finite-size spectrum of the Anderson model. We first recapitulate the Bethe-ansatz solution of the model while emphasizing the characteristic properties common to the Bethe-ansatz solutions to exactly solved models for the Kondo problem. The Anderson model for the magnetic impurity is defined by the Hamiltonian

\[
H = \sum_{k, \sigma} \epsilon_k c_{k, \sigma}^{\dagger} c_{k, \sigma} + V \sum_{k, \sigma} (c_{k, \sigma}^{\dagger} d_{\sigma} + d_{\sigma}^{\dagger} c_{k, \sigma}) + \epsilon_d \sum_{\sigma} d_{\sigma}^{\dagger} d_{\sigma} + U d_{\uparrow}^{\dagger} d_{\uparrow} d_{\downarrow}^{\dagger} d_{\downarrow},
\]

with standard notations [9]. The model describes free conduction electrons coupled with correlated \( d \)-electrons at the impurity via the resonant hybridization \( V \). After reducing the model to one dimension with the use of partial waves, we linearize the spectrum of conduction electrons as \( \epsilon_k = v k \) with \( v = 1 \) near the Fermi point, which is equivalent to the assumption of the constant density of states. The hopping operator in the kinetic term is
then replaced by $-i\partial/\partial x$ in the coordinate representation. These simplifications enable us to apply the Bethe-ansatz method to diagonalize the Hamiltonian $[4,5]$. Upon diagonalization we are led to introduce two types of rapidity variables, $k_j$ and $\Lambda_\alpha$, for the charge and spin degrees of freedom, respectively. For the Hamiltonian $[4]$ on a circle of circumference $L$ the Bethe-ansatz equations were obtained by Wiegmann $[6]$, and by Kawakami and Okiji $[7]$

$$k_jL + \delta(k_j) = 2\pi N_j - \sum_{\beta=1}^{M} \theta_1(B(k_j) - \Lambda_\beta), \quad (2)$$

$$2\pi J_\alpha + \sum_{\beta=1}^{M} \theta_2(\Lambda_\alpha - \Lambda_\beta) + \sum_{j=1}^{N-2M} \theta_1(\Lambda_\alpha - B(k_j)) = -2\text{Re}\tilde{k}(\Lambda_\alpha)L - 2\text{Re}\delta(\tilde{k}(\Lambda_\alpha)), \quad (3)$$

where $\theta_n = 2\tan^{-1}(2x/n)$ and $\delta(k) = -2\tan^{-1}(V^2/(k - \epsilon_d))$ is the bare phase shift due to the potential scattering by the impurity at $U = 0$, which gives rise to the resonance width $\Gamma = V^2/2$. The total number of (up-spin) electrons is denoted as $N$ ($M$). Here $\tilde{k}(\Lambda_\alpha)$ is the complex solution with $\text{Im}\tilde{k} > 0$ of the charge rapidity which satisfies $iB(\tilde{k}) - i\Lambda - U\Gamma = 0$

$$B(k) = k(k - U - 2\epsilon_d). \quad (4)$$

Comparing Eqs. (2) and (3) with the ordinary Bethe-ansatz equations we observe additional phase shift terms $\delta(k)$ proportional to $1/L$ whose existence is indeed quite relevant to describe the physical properties of the impurity part.

**B. Ground-state energy**

The exact ground-state energy of the Anderson model is given by $[6,7,5]$

$$E_0 = \sum_{j=1}^{N-2M} k_j + 2 \sum_{\alpha=1}^{M} x(\Lambda_\alpha), \quad (5)$$

where

$$x(\Lambda) = \epsilon_d + U/2 - \left[\Lambda + (\epsilon_d + U/2)^2 + [(\Lambda + (\epsilon_d + U/2)^2 + U^2\Gamma^2/4]^{1/2}\right]^{1/2}. \quad (6)$$
Note that this expression is divergent because of the constant density of states with charge rapidities $k_j$ distributed over $[-\infty, \beta]$ in the thermodynamic limit. In order to evaluate the finite-size corrections to the ground-state energy, therefore, we have to regularize the sum (5) upon taking the large-$L$ limit while keeping the $1/L$ scaling term which is expected to be universal. Most naively one may cut off the $k$-integration; $k \in [-D, \beta]$ with $D$ being the bandwidth. This prescription, however, is not suitable for our purpose since introducing the explicit momentum cutoff in this way would break conformal invariance.

Instead we are led to introduce a smooth cutoff function $\varphi(k)$ which is unity near the Fermi point and behaves as $|k|^{-(2+\varepsilon)}$ ($\varepsilon > 0$) when $|k| \to \infty$. Eq.(5) is now replaced by

$$E_0 = \sum_{j=1}^{N-2M} \varphi(k_j)k_j + 2\sum_{\alpha=1}^{M} \varphi(\Lambda_\alpha)x(\Lambda_\alpha).$$

(7)

We apply the Euler-Maclaurin summation formula to Eqs.(2), (3) and (5) to evaluate the finite-size corrections. The details of computations are relegated to Appendix A. The final result is given by

$$E_0 = L\varepsilon_0 - \frac{\pi v_c}{12L} - \frac{\pi v_s}{12L},$$

(8)

where $v_c$ and $v_s$ are the velocities of massless charge and spin excitations. It is shown that these velocities take the same value $v$ which is equal to unity in our convention (see Appendix A). To compare the result (8) with the finite-size scaling law in CFT [13] one has to replace $L$ with $2l$ since $L$ has been defined as the periodic length of the system. Then we find the scaling behavior of boundary CFT [13] for the charge and spin sectors, respectively, but with the same Virasoro central charge $c = 1$.

C. Excitation energy

Our next task is to calculate the finite-size corrections to the excitation energies. For this it is sufficient to take the thermodynamic limit of the Bethe-ansatz solution. Introducing density functions $\rho(k)$ and $\sigma(\Lambda)$ for the charge and spin rapidities, respectively, we write the Bethe-ansatz equations in the thermodynamic limit. Standard calculations yield [8]...
\[ \rho(k) = \frac{1}{2\pi} + \frac{1}{L} \Delta(k) + B'(k) \int_{\alpha}^{\infty} d\Lambda a_1(B(k) - \Lambda) \sigma(\Lambda), \quad (9) \]
\[
\sigma(\Lambda) + \int_{\alpha}^{\infty} d\Lambda' a_2(\Lambda - \Lambda') \sigma(\Lambda') + \int_{-\infty}^{\beta} dk a_1(\Lambda - B(k)) \rho(k) = A(\Lambda) + \frac{1}{L} Z(\Lambda), \quad (10)
\]

where
\[
a_n(x) = \frac{1}{\pi} \frac{nU}{x^2 + (nU)^2}, \quad (11)
\]
\[
A(\Lambda) = \int_{-\infty}^{\infty} dk a_1(B(k) - \Lambda), \quad (12)
\]
\[
Z(\Lambda) = \int_{-\infty}^{\infty} dk a_1(B(k) - \Lambda) \Delta(k), \quad (13)
\]

and \[ \Delta(k) = \frac{\Gamma}{\pi}[\frac{1}{2}(k - \epsilon_d)^2 + \Gamma^2]^{-1}. \] It is clear from Eqs.(9), (10) that the density functions can be decomposed as \[ \sigma = \sigma_h + (1/L)\sigma_{\text{imp}} \] and \[ \rho = \rho_h + (1/L)\rho_{\text{imp}} \] corresponding to the host and impurity contributions. For later convenience let us rewrite Eq.(10) in such a way that the \( k \)-integral is taken over the region \[ [\beta, \infty] \] instead of \[ [-\infty, \beta] \]. This can be easily performed by the Fourier transform. Then Eq.(10) turns out to be
\[
\sigma(\Lambda) + \int_{\alpha}^{\infty} d\Lambda' a_2(\Lambda - \Lambda') \sigma(\Lambda') = \int_{\beta}^{\infty} dk a_1(\Lambda - B(k)) \rho(k). \quad (14)
\]

The number of conduction electrons \( N_h \) is expressed in terms of the density functions of the host part
\[
\frac{N_h}{L} = \mathcal{N} - \left( \int_{\beta}^{\infty} dk \rho_h(k) - 2 \int_{\alpha}^{\infty} d\Lambda \sigma_h(\Lambda) \right), \quad (15)
\]
where \[ \mathcal{N} = \int_{\infty}^{-\infty} dk \rho_h(k) \] is an irrelevant infinite constant. Similarly for the number of down-spin electrons \( M_h \) we get
\[
\frac{M_h}{L} = \int_{\alpha}^{\infty} d\Lambda \sigma_h(\Lambda). \quad (16)
\]

The total energy has contributions from both of the host and impurity part
\[
E = L \int_{\beta}^{\infty} dk \left[ \frac{1}{2\pi} + \frac{1}{L} \Delta(k) \right] \varepsilon_s(k), \quad (17)
\]
where we have introduced the dressed energies $\varepsilon_s(k)$ and $\varepsilon_c(\Lambda)$ satisfying [8,9,5]

$$
\varepsilon_s(k) = -\varphi(k)k + \int_{\alpha}^{\infty} d\Lambda a_1(\Lambda - B(k))\varepsilon_c(\Lambda),
$$

(18)

$$
\varepsilon_c(\Lambda) = 2\epsilon_d + U + \int_{\beta}^{\infty} dk B'(k)a_1(B(k) - \Lambda)\varepsilon_s(k) - \int_{\alpha}^{\infty} d\Lambda a_2(\Lambda - \Lambda')\varepsilon_c(\Lambda').
$$

(19)

Let us now evaluate the finite-size corrections in the excitation energy spectrum. We first expand the total energy (17) to the second order in the variations $\Delta\alpha$ and $\Delta\beta$ of the integration limits $\alpha$ and $\beta$. The first order terms give a chemical potential which is usually set to zero. However, since the shift of the chemical potential due to the impurity is of order $1/L$ we must incorporate it as the finite-size corrections to the energy spectrum. In order to estimate this, it is crucial to notice the relation [8,9,5]

$$
\frac{1}{2\pi}(2\epsilon_d + U) = \int_{-\infty}^{\alpha} d\Lambda \sigma_h(\Lambda),
$$

(20)

which holds for the host density function. This relation implies that the impurity level $\epsilon_d$ will change if we vary the number of conduction electrons by tuning $\alpha$ since our convention is to measure the impurity level $\epsilon_d$ from the Fermi level. Using Eqs.(15) and (20) one obtains the energy shift due to the impurity

$$
\Delta E^{(1)} = \frac{dE}{d\alpha} \Delta \alpha = -\frac{\pi}{L} \frac{\partial E}{\partial \epsilon_d} \Delta N_h = -\frac{\pi n_d}{L} \Delta N_h,
$$

(21)

where $n_d (= \sum_{\sigma} \langle d_{\sigma}^\dagger d_{\sigma} \rangle)$ is the average number of localized electrons given by

$$
n_d = 1 - \int_{-\infty}^{\alpha} d\Lambda \sigma_{imp}(\Lambda),
$$

(22)

at the impurity site. Using the Friedel sum rule $n_d = 2\delta_F/\pi$ with $\delta_F$ being the phase shift at the Fermi level we now find

$$
\Delta E^{(1)} = -\frac{2\delta_F}{L} \Delta N_h.
$$

(23)

The second order terms are obtained from Eqs.(17) ~ (19). The result reads

$$
\frac{\Delta E^{(2)}}{L} = -\sigma(\alpha) \frac{\partial \varepsilon_s(\Lambda)}{\partial \alpha} \bigg|_{\Lambda=\alpha} \frac{(\Delta \alpha)^2}{2} - \rho(\beta) \frac{\partial \varepsilon_s(k)}{\partial \beta} \bigg|_{k=\beta} \frac{(\Delta \beta)^2}{2}.
$$

(24)
It is convenient for our purpose to express $\Delta \beta$ and $\Delta \alpha$ in terms of the change of the number of down-spin electrons and the magnetization $S_h (= N_h/2 - M_h)$ of host electrons. Taking the derivatives of Eqs. (15), (16) with respect to $\alpha$ and $\beta$ we have

\[
\frac{2}{L} \frac{\partial S_h}{\partial \beta} = \rho_h(\beta) - \int_\beta^\infty dk \frac{\partial \rho_h(k)}{\partial \beta} = \rho_h(\beta) \xi_{ss}(\beta),
\]

(25)

\[
\frac{2}{L} \frac{\partial S_h}{\partial \alpha} = - \int_\beta^\infty dk \frac{\partial \rho_h(k)}{\partial \alpha} = \sigma_h(\alpha) \xi_{sc}(\alpha),
\]

(26)

\[
\frac{1}{L} \frac{\partial M_h}{\partial \beta} = \int_\alpha^\infty d\Lambda \frac{\partial \sigma_h(\Lambda)}{\partial \beta} = - \rho_h(\beta) \xi_{cs}(\beta),
\]

(27)

\[
\frac{1}{L} \frac{\partial M_h}{\partial \alpha} = - \sigma_h(\alpha) + \int_\alpha^\infty d\Lambda \frac{\partial \sigma_h(\Lambda)}{\partial \alpha} = - \sigma_h(\alpha) \xi_{cc}(\alpha),
\]

(28)

where the $2 \times 2$ dressed charge matrix $(\xi_{ij})$ is given by the solutions to the integral equations

\[
\xi_{ss}(k) = 1 + \int_\alpha^\infty d\Lambda a_1(\Lambda - B(k)) \xi_{cc}(\Lambda),
\]

(29)

\[
\xi_{sc}(\Lambda) = \int_\beta^\infty dk B'(k) a_1(B(k) - \Lambda) \xi_{ss}(k) - \int_\alpha^\infty d\Lambda a_2(\Lambda - \Lambda') \xi_{sc}(\Lambda'),
\]

(30)

\[
\xi_{cs}(k) = \int_\alpha^\infty d\Lambda a_1(B(k) - \Lambda) \xi_{cc}(\Lambda),
\]

(31)

\[
\xi_{cc}(\Lambda) = 1 + \int_\beta^\infty dk B'(k) a_1(B(k) - \Lambda) \xi_{cs}(k) - \int_\alpha^\infty d\Lambda a_2(\Lambda - \Lambda') \xi_{cc}(\Lambda').
\]

(32)

Consequently $\Delta \alpha$ and $\Delta \beta$ are written in terms of $\Delta S_h$ and $\Delta M_h$ which represent the deviations from the ground-state values. From Eqs. (24)~(28) we thus obtain

\[
\Delta E^{(2)} = \frac{2 \pi v_c}{L} \frac{1}{2} \left( \frac{2 \xi_{cs} \Delta S_h + \xi_{ss} \Delta M_h}{\det \{ \xi_{ij} \}} \right)^2 + \frac{2 \pi v_s}{L} \frac{1}{2} \left( \frac{2 \xi_{cc} \Delta S_h + \xi_{sc} \Delta M_h}{\det \{ \xi_{ij} \}} \right)^2,
\]

(33)

where the "velocities" of spin and charge excitations are defined as

\[
2 \pi v_c = - \left[ \frac{1}{\sigma_h(\alpha)} + \frac{1}{L} \frac{\sigma_{imp}(\alpha)}{\sigma_h(\alpha)^2} \right] \left| \frac{\partial \varepsilon_c(\Lambda)}{\partial \alpha} \right|_{\Lambda=\alpha},
\]

(34)
\[ 2\pi v_s = -\left[ \frac{1}{\rho_h(\beta)} + \frac{1}{L} \frac{\rho_{imp}(\beta)}{\rho_h(\beta)^2} \right] \frac{\partial \varepsilon_s(k)}{\partial \beta} \bigg|_{k=\beta}. \]  

(35)

For vanishing magnetic fields, we can solve Eqs.(29) \sim (32) using the standard Wiener-Hopf method. Note first that the dressed charges in these equations become independent of \( \alpha \) in the limit of zero magnetic field for which \( \beta \to -\infty \). Hence we can simply set \( \alpha = -\infty \), which corresponds to the so-called symmetric model with \( 2\epsilon_d + U = 0 \). The dressed charge matrix is thus cast in a simple formula

\[
\begin{pmatrix}
\xi_{ss} & \xi_{cs} \\
\xi_{sc} & \xi_{cc}
\end{pmatrix}
= \begin{pmatrix}
\xi(\beta) & \frac{\xi(\beta)}{2} \\
0 & \frac{1}{\sqrt{2}}
\end{pmatrix},
\]

(36)

with \( \beta \to -\infty \). Here \( \xi(\beta) \) is obtained from the solution to the equation

\[
\xi(k) = 1 + \frac{1}{2\pi} \int_{\beta}^{\infty} dk B'(k) K(B(k) - B(k')) \xi(k'),
\]

(37)

with the kernel given by

\[
K(x) = \frac{1}{2UT} \int_0^\infty dp \frac{e^{-p/2}}{\cosh p/2} \cos \frac{px}{2UT}.
\]

(38)

One can readily solve this equation by the Wiener-Hopf technique in the limit of \( \beta \to -\infty \), obtaining

\[
\begin{pmatrix}
\xi_{ss} & \xi_{cs} \\
\xi_{sc} & \xi_{cc}
\end{pmatrix}
= \begin{pmatrix}
\sqrt{2} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}}
\end{pmatrix},
\]

(39)

It is remarkable that the dressed charges obtained here are independent of parameters such as the Coulomb repulsion and the electron filling. As we shall see momentarily this fact ensures the canonical exponents for conduction electrons inherent in the (local) Fermi liquid.

The particle-hole type excitations are readily taken into account. Let us define non-negative integers \( n^+_c \) and \( n^+_s \) to specify their contributions in the charge and spin sectors. Substituting now the explicit values of the dressed charge into Eq.(33) we write down the finite-size spectrum in the following form

\[
\Delta E^{(2)} = \frac{2\pi v_c}{L} \left[ \frac{(\Delta N_h)^2}{4} + n^+_c \right] + \frac{2\pi v_s}{L} \left[ (\Delta S_h)^2 + n^+_s \right].
\]

(40)
Furthermore we notice that the “velocities” $v_c$, $v_s$ can be written in a transparent form with
the use of susceptibilities. Recall that the host and impurity parts of the charge and spin
susceptibilities are given by \[34,35\]
\[\chi^h_c = -\frac{\partial N^h_c}{\partial \epsilon_d} = -4\sigma^h_c(\alpha)\left.\left(\frac{\partial \varepsilon^h_c(\Lambda)}{\partial \alpha}\right)\right|_{\Lambda=\alpha}^{-1}, \tag{41}\]
\[\chi^{imp}_c = -\frac{\partial n^d}{\partial \epsilon_d} = -4\sigma^{imp}_c(\alpha)\left.\left(\frac{\partial \varepsilon^c(\Lambda)}{\partial \alpha}\right)\right|_{\Lambda=\alpha}^{-1}, \tag{42}\]
\[\chi^h_s = \frac{\partial S^h_s}{\partial H} = -\rho^h_s(\beta)\left.\left(\frac{\partial \varepsilon^s(k)}{\partial \beta}\right)\right|_{k=\beta}^{-1}, \tag{43}\]
\[\chi^{imp}_s = \frac{\partial s^z_s}{\partial H} = -\rho^{imp}_s(\beta)\left.\left(\frac{\partial \varepsilon^s(k)}{\partial \beta}\right)\right|_{k=\beta}^{-1}, \tag{44}\]
where the magnetization $s_z$ of the impurity part reads
\[s_z = \frac{1}{2} \int_{-\infty}^\beta dk \rho^{imp}_s(k). \tag{45}\]
We find
\[v_c = v \left[ 1 + \frac{\chi^{imp}_c}{\chi^h_c} \right], \tag{46}\]
\[v_s = v \left[ 1 + \frac{\chi^{imp}_s}{\chi^h_s} \right], \tag{47}\]
where the velocity of free electrons is $v = 1$ in the present formulation.

Combining Eqs.(23) and (40) we finally arrive at the simple expression for the finite-size
spectrum
\[E = E_0 + \Delta E^{(1)} + \Delta E^{(2)} = E_0 + \frac{1}{L} E_1 + \frac{1}{L^2} E_2, \tag{48}\]
where the terms of order $1/L$ and $1/L^2$ are given respectively by
\[\frac{1}{L} E_1 = \frac{2\pi v}{L} \left[ \frac{1}{4} \left( \Delta N^h - 2\frac{\delta_F}{\pi} \right)^2 - \frac{(\delta_F^c)^2}{\pi} + n^+_c \right] + \frac{2\pi v}{L} [(\Delta S^h)^2 + n^+_s], \tag{49}\]
and
\[\frac{1}{L^2} E_2 = \frac{2\pi v}{L^2} \frac{\chi^{imp}_c}{\chi^h_c} \left[ \frac{(\Delta N^h)^2}{4} + n^+_c \right] + \frac{2\pi v}{L^2} \frac{\chi^{imp}_s}{\chi^h_s} [(\Delta S^h)^2 + n^+_s]. \tag{50}\]
We remark that the bulk electrons do not contribute to the $1/L^2$ piece because of the
linearized dispersion. Hence the terms of order $1/L$ and order $1/L^2$ give the finite-size
spectrum of bulk electrons and impurity electrons, respectively.
D. Critical exponents and universal relations

We now wish to discuss our results (48)∼(50) in view of the finite-size scaling in CFT. It has already been found that the low-energy critical behavior in the Kondo effect is described by boundary CFT in which we have only left (or right) moving sector of CFT [9,10,11]. The \(1/L\) formula (49) in fact exhibits the scaling behavior predicted by boundary CFT. Notice again that when comparing with CFT formula one has to replace \(L\) with \(2l\) as pointed out before. We observe clearly from Eqs.(8) and (49) that the charge sector is described by \(c = 1\) CFT with U(1) symmetry and the spin sector by \(c = 1\) SU(2) Kac-Moody CFT at low energies.

In Eq.(49) it is seen that the finite-size spectrum of the bulk electron part depends on the non-universal phase shift \(\delta_F\). We should remember that this constant phase shift \(\delta_F\) is equivalent to the chemical potential due to the impurity, see Eq.(23). This implies that the effect of the phase shift amounts to merely imposing twisted boundary conditions on conduction electrons [9,10,11]. When reading off from Eq.(49) the dimension of the scaling operator associated with conduction electrons, we should thus ignore the \(\delta_F\) dependence. Then the scaling dimension \(x\) of the conduction electron field is determined by choosing the quantum numbers

\[
\Delta N_h = 1, \quad \Delta S_h = 1/2, \quad (51)
\]

which yields \(x = 1/2\) upon ignoring \(\delta_F\). This is consistent with the fact that the system is described by the strong-coupling fixed point of the local Fermi liquid [2,4].

According to boundary CFT [12], however, there exists a one-to-one correspondence between energy eigenstates and boundary scaling operators. Such operators have scaling dimensions explicitly dependent of the phase shift. Our intention now is to identify a boundary operator which is specified in particular by a “single-electron” quantum number (51). This type of excitation could be relevant to describe the long-time asymptotic behavior of the electron correlator in which the hybridization \(V\) in Eq.(11) is switched on suddenly at time
\( t = 0 \) and \( V(t) = V \) for \( t > 0 \). The correlation exponent will be obtained by setting \( \Delta N_h = 1 \) and \( \Delta S_h = 1/2 \) in Eq.(49), but without ignoring \( \delta_F \). We get

\[
\eta_s = 1 - 2\delta_F/\pi. \tag{52}
\]

Actually such phenomena have been well known to occur as the *orthogonal catastrophe* which is induced by the time-dependent local perturbation, e.g. a power-law singularity in the X-ray absorption profile \([14]\). It is quite remarkable that the critical exponent for the excitonic correlation part is precisely given by Eq.(52) \([13]\). Therefore we observe interesting evidence that boundary CFT will also play a role to clarify the critical behavior related to the orthogonal catastrophe.

We point out another important property characteristic of the local Fermi liquid, i.e. the universal relations among the static quantities. To see this we first recall that in Tomonaga-Luttinger liquids \((c = 1 \text{ CFT})\) there exist universal relations among the velocities and the susceptibilities such as \([16,17,18,19]\)

\[
v_c \chi_c = \frac{2K_\rho}{\pi}, \quad v_s \chi_s = \frac{1}{2\pi}, \tag{53}
\]

for zero magnetic fields, where \( K_\rho \) is a Gaussian coupling constant which features the \( U(1) \) conformal critical line in the charge sector. An important point is that the above relations hold in the present case not only for the conduction electron part but also for the impurity part with the fixed value \( K_\rho = 1 \) independent of the interaction. We can see this from the fact that the term proportional to \( 1/L^2 \) of the energy spectrum \([51]\) with \( n_c^+ = n_s^+ = 0 \) can be cast into the expression analogous to the \( 1/L \) term

\[
\Delta E^{(2)}_{\text{imp}} = 2\pi v_c^{\text{imp}} \frac{(\Delta n_d)^2}{4} + 2\pi v_s^{\text{imp}} (\Delta s_z)^2. \tag{54}
\]

Here we have introduced the velocities of the impurity part through

\[
v_c^{\text{imp}} = \frac{\pi}{3\gamma_c^{\text{imp}}}, \quad v_s^{\text{imp}} = \frac{\pi}{3\gamma_s^{\text{imp}}}, \tag{55}
\]

with \( \gamma_c^{\text{imp}}, \gamma_s^{\text{imp}} \) being the specific heat coefficients of the charge and spin degrees of freedom of the impurity electrons. This expression is essentially the same as in the bulk. From \([12]\),
and (54) it is shown that the universal relations for static quantities (53) with $K_{\rho} = 1$ hold for the impurity part. This is nothing but the properties of the local Fermi liquid [2].

III. SU($\nu$) ANDERSON MODEL

An integrable generalization of the infinite-$U$ Anderson model with SU($\nu$) spin degrees of freedom is given by the Hamiltonian [8]

$$
H = \sum_{m=1}^{\nu} \int dx c_{m}^\dagger(x)(-i{\partial \over \partial x})c_{m}(x) + \epsilon_{f} \sum_{m=1}^{\nu} |m\rangle\langle m|
+ V \sum_{m=1}^{\nu} \int dx \delta(x) \left( |m\rangle\langle 0|c_{m}(x) + c_{m}^\dagger(x)|0\rangle\langle m| \right),
$$

(56)

where impurity electrons have $\nu$-fold degeneracy. $|0\rangle$ and $|m\rangle$ ($m = 1, 2, \cdots, \nu$) represent the unoccupied and single occupied states of the impurity site, respectively, and the double occupancy is forbidden by infinite repulsive Coulomb interactions among impurity electrons. The Hamiltonian (56) is presented in the coordinate space and the spectrum for conduction electrons has already been linearized. This model has often been used to investigate the Kondo effect for rare-earth impurities in a metal.

The diagonalization of the SU($\nu$) Anderson model can be performed by introducing $\nu - 1$ spin rapidities $\Lambda_{\alpha}^{(l)}$ ($l = 0, 1, \cdots, \nu - 2$) besides the charge rapidity $\Lambda_{\alpha}^{(\nu - 1)}$. According to Schlottmann [8] the Bethe-ansatz equations take the form

$$
(l + 1)\Lambda_{\alpha}^{(l)}L = 2\pi I_{\alpha}^{(l)} - [\pi + 2\tan^{-1}(\epsilon_{f})/(l + 1)\Gamma])
+ 2\sum_{q=0}^{\nu-1} \sum_{p=0}^{K_{q}} \sum_{\beta=1}^{K_{l}} \tan^{-1}((\Lambda_{\alpha}^{(l)} - \Lambda_{\beta}^{(q)})/(l + q - 2p)\Gamma),
\alpha = 1, 2, \cdots, K_{l}, \quad K_{l} = N_{l} - N_{l+1}, \quad N_{\nu} = 0, \quad l = 0, 1, \cdots, \nu - 1,
$$

(57)

where integers $I_{\alpha}^{(l)}$ are quantum numbers associated with the spin and charge degrees of freedom, and $N_{l}$ is the number of electrons with each spin.

We henceforth concentrate on the excitation spectrum. As mentioned in the previous SU(2) case, we can work with the rapidity density functions $\sigma^{(l)}(\Lambda)$ defined in the thermody-
namic limit when calculating the finite-size corrections to excitation energies. These density functions satisfy \( \nu \)-coupled linear equations \[8\]

\[
\sigma^{(l)}(\Lambda) = \frac{l + 1}{2\pi} + \frac{1}{\pi L} \frac{(l + 1)\Gamma}{(\Lambda - \epsilon_f)^2 + [(l + 1)\Gamma]^2} \\
- \sum_{p=0}^{\nu-1} \sum_{q=0}^{p_l} \int_{-\infty}^{B_l} d\Lambda' \sigma^{(q)}(\Lambda') K_{pq}^{(l)}(\Lambda - \Lambda'),
\]

\( l = 0, 1, \cdots, \nu - 1, \)

where \( p_{lq} = \min(l, q) - \delta_{lq} \) and \( \Gamma = V^2/2 \). The integral kernel is defined by

\[
K_{pq}^{(l)}(x) = \frac{1}{\pi} \frac{(l + q - 2p)\Gamma}{x^2 + [(l + q - 2p)\Gamma]^2}.
\]

We note here again that the density functions are decomposed into the sum of the host part and the impurity part: \( \sigma^{(l)} = \sigma_h^{(l)} + (1/L)\sigma_{imp}^{(l)} \). The total number of host electrons is then given by

\[
\frac{N_h}{L} = \sum_{l=0}^{\nu-1} (l + 1) \int_{-\infty}^{B_l} d\Lambda \sigma_h^{(l)}(\Lambda).
\]

On the basis of these equations we compute the finite-size spectrum following the method explained in the preceding sections. To proceed with clarity we will treat the charge and spin sectors separately.

A. Charge excitation

Let us first evaluate the finite-size corrections in the charge sector. For this purpose we can put \( B_l \to -\infty \) for \( l = 0, 1, \cdots, \nu - 2 \) while \( B_{\nu-1} = Q \), which corresponds to the case of zero magnetic fields. The value of \( Q \) is determined by the number of host electrons through Eq.\((60)\). The total energy including the impurity contribution is simply expressed as \[8\]

\[
\frac{E}{L} = \int_{-\infty}^{Q} d\Lambda \left[ \frac{\nu}{2\pi} + \frac{1}{\pi L} \frac{\nu\Gamma}{(\Lambda - \epsilon_f)^2 + (\nu\Gamma)^2} \right] \epsilon^{(\nu-1)}(\Lambda),
\]

in terms of the dressed energies \( \epsilon^{(l)}(\Lambda) \) (\( l = 0, 1, \cdots, \nu - 1 \)) which are defined by the following equations
The finite-size corrections are computed by expanding Eq. (61) in \( \Delta Q \) to the second order. Similarly to the SU(2) model, the first order term in \( \Delta Q \) arises from the impurity part. Expanding the energy to the linear term in \( \Delta Q \) we obtain

\[
\Delta E^{(1)} = -\frac{\partial E}{\partial Q} \Delta Q = -\frac{\nu}{\pi} \Delta Q,
\]

where we have used the Friedel sum rule \( \texttt{[8]} \)

\[
2\nu \delta_F = 2\pi \frac{\partial E}{\partial Q} = 2\pi n_f,
\]

with the average number \( n_f \) of localized electrons given by

\[
n_f = \nu \int_{-\infty}^{Q} \sigma_h^{(\nu-1)}(\Lambda) \, d\Lambda.
\]

We next rewrite \( \Delta Q \) in terms of the change of the host electron number \( \Delta N_h \). From Eq. (60) one finds

\[
\frac{1}{L} \frac{\partial N_h}{\partial Q} = \nu \sigma_h^{(\nu-1)}(Q) + \nu \int_{-\infty}^{Q} d\Lambda \frac{\partial \sigma_h^{(\nu-1)}(\Lambda)}{\partial Q} = \sigma_h^{(\nu-1)}(Q) \xi_{\nu-1,\nu-1}(Q),
\]

where the dressed charge \( \xi_{\nu-1,\nu-1} \) is determined by

\[
\xi_{\nu-1,\nu-1}(\Lambda) = \nu - \int_{-\infty}^{Q} d\Lambda' \sum_{l=0}^{\nu-2} K_{l,\nu-1}^{(\nu-1)}(\Lambda - \Lambda') \xi_{\nu-1,\nu-1}(\Lambda').
\]

The Wiener-Hopf solution to this equation yields \( \xi_{\nu-1,\nu-1}(Q) = \sqrt{\nu} \texttt{[8]} \). Hence we get

\[
\Delta Q = \frac{\Delta N_h}{\sqrt{\nu} \sigma_h^{(\nu-1)}(Q)L}.
\]

Furthermore the host electron density function \( \sigma_h^{(\nu-1)} \) at \( \Lambda = Q \) is similarly evaluated as \( \sigma_h^{(\nu-1)}(Q) = \sqrt{\nu}/2\pi \). We now obtain the corrections linear in \( 1/L \)

\[
\Delta E^{(1)} = - \frac{2\delta_F}{L} \Delta N_h.
\]

Turning to the second order terms we expand the energy as
\[
\frac{\Delta E^{(2)}}{L} = -\sigma^{(\nu-1)}(Q) \frac{\partial \varepsilon^{(\nu-1)}(\Lambda)}{\partial Q} \bigg|_{\Lambda=Q} \frac{(\Delta Q)^2}{2}.
\] (70)

Using (68) we obtain

\[
\Delta E^{(2)} = \frac{2\pi v_c (\Delta N_h)^2}{\nu},
\] (71)

where the velocity is defined by

\[
2\pi v_c = -\left[ \frac{1}{\sigma_h^{(\nu-1)}(Q)} + \frac{1}{L \sigma_h^{(\nu-1)}(Q)^2} \right] \frac{\partial \varepsilon^{(\nu-1)}(\Lambda)}{\partial Q} \bigg|_{\Lambda=Q}.
\] (72)

Combining Eqs.(71) and (69), and repeating similar manipulations as in the SU(2) case we obtain the finite-size spectrum for the charge sector

\[
E = E_0 + \frac{1}{L} E_1^c + \frac{1}{L^2} E_2^c,
\] (73)

where the terms of order 1/L and 1/L^2 are given respectively by

\[
\frac{1}{L} E_1^c = \frac{2\pi v_c}{L} \left[ \frac{(\Delta N_h - \nu \delta_F/\pi)^2}{2\nu} + n_c^+ \right] - \frac{\pi v_c}{L} \nu \left( \frac{\delta_F}{\pi} \right)^2,
\] (74)

and

\[
\frac{1}{L^2} E_2^c = \frac{2\pi v_c^{imp}}{L^2} \chi^{imp}_c \left[ \frac{(\Delta N_h)^2}{2\nu} + n_c^+ \right].
\] (75)

Here the velocity of free electrons is \(v = 1\) and a non-negative integer \(n_c^+\) represents the particle-hole excitation in the charge sector. Putting \(n_c^+ = 0\) we can express the \(1/L^2\)-order term in an analogous fashion to the \(1/L\)-order term

\[
\frac{1}{L^2} E_2^c = 2\pi v_c^{imp} (\Delta n_f)^2,\]

(76)

where the velocity of the impurity part is \(v_c^{imp} = v \chi_c^h / \chi_c^{imp}\).

**B. Spin excitation**

In order to deal with the spin sector we take the integer-valence limit, \(n_f \to 1\) \((Q \to \infty)\), to facilitate our calculations. In this limit the Bethe-ansatz equations for the spin part reduce to [8]
\[ \sigma^{(l)}(\Lambda) = \frac{1}{2\pi} \frac{\delta_{l,0}}{(\Lambda + 1/J)^2 + 1/4} - \sum_{q=0}^{\nu-2} \int_{B_q} d\Lambda A_{lq}(\Lambda - \Lambda') \sigma^{(q)}(\Lambda'), \]

\[ l = 0, 1, \cdots, \nu - 2, \quad (77) \]

where \( J = -\Gamma/\epsilon_f \) and \( A_{lq}(\Lambda) \) is the Fourier transform of

\[ \tilde{A}_{lq}(x) = \delta_{l,q}(1 + e^{-|x|}) - (\delta_{l+1,q} + \delta_{l-1,q})e^{-|x|/2}. \quad (78) \]

The values of \( B_l \)'s are determined by the total number of electrons \( N_l \) [8]. We have

\[ \frac{N_l}{L} - \frac{N_{l+1}}{L} = \int_{B_{l-1}} d\Lambda \sigma^{(l-1)}(\Lambda) - 2 \int_{B_l} d\Lambda \sigma^{(l)}(\Lambda) + \int_{B_{l+1}} d\Lambda \sigma^{(l+1)}(\Lambda), \quad (79) \]

for \( l = 1, 2, \cdots, \nu - 3 \) and

\[ \frac{N_0}{L} - \frac{N_1}{L} = 1 - 2 \int_{B_0} d\Lambda \sigma^{(0)}(\Lambda) + \int_{B_1} d\Lambda \sigma^{(1)}(\Lambda), \quad (80) \]

\[ \frac{N_{\nu-2}}{L} - \frac{N_{\nu-1}}{L} = -2 \int_{B_{\nu-2}} d\Lambda \sigma^{(\nu-2)}(\Lambda) + \int_{B_{\nu-3}} d\Lambda \sigma^{(\nu-3)}(\Lambda). \quad (81) \]

We define the dressed energy function

\[ \varepsilon^{(l)} = -\nu \Lambda \delta_{l,\nu-2} - \sum_{q=0}^{\nu-2} \int_{B_q} d\Lambda' A_{lq}(\Lambda - \Lambda') \varepsilon^{(q)}(\Lambda'), \quad (82) \]

so as to express the total energy in the form

\[ \frac{E}{L} = \int_{B_0} d\Lambda \frac{1}{2\pi} \frac{1}{(\Lambda + 1/J)^2 + 1/4} \varepsilon^{(0)}(\Lambda). \quad (83) \]

We observe here that Eqs. (77) \sim (83) take the form common to integrable SU(\( \nu \)) symmetric models such as the SU(\( \nu \)) Heisenberg model [20,21] and the degenerate Hubbard model [22]. Based on the analysis of these SU(\( \nu \)) models let us introduce the following quantities

\[ \frac{M^{(l)}_h}{L} = \int_{B_l} d\Lambda \sigma^{(l)}_h(\Lambda), \quad l = 0, 1, \cdots, \nu - 2, \quad (84) \]

as quantum numbers characterizing the spin degrees of freedom. Making use of the SU(\( \nu \)) results [21,22] we find the finite-size spectrum of the spin sector

\[ E - E_0 = \frac{2\pi}{L} \left[ \frac{1}{2} \Delta M_h^T (Z_{\nu}^{-1})^T V Z_{\nu}^{-1} \Delta M_h + \sum_{l=0}^{\nu-2} v_l n_l^+ \right], \quad (85) \]
where $\Delta M_h^T = (\Delta M_h^{(0)}, \cdots, \Delta M_h^{(\nu-2)})$, $V = \text{diag}(v_0, v_1, \cdots, v_{\nu-2})$ and $n_l^+ \geq 0$ are non-negative integers representing particle-hole type excitations in the spin sector. The dressed charge matrix $Z_\nu$ is defined as $(Z_\nu)_{lm} = \xi_{lm}(B_l)$ where

$$
\xi_{lm}(\Lambda) = \delta_{lm} - \sum_{q=0}^{\nu-2} \int_{B_q}^{\infty} d\Lambda A_{qm}(\Lambda - \Lambda') \xi_{lq}(\Lambda').
$$

We note that the matrix $Z_\nu$ enjoys a nice property of $(Z_\nu^{-1})^T Z_\nu^{-1}$ being equal to the Cartan matrix for the $\text{SU}(\nu)$ Lie algebra $[21]$. In Eq.(85) the velocities of spin excitations have been defined by

$$
v_l = -\left[ \frac{1}{\sigma_h^{(l)}(B_l)} + \frac{\sigma^{(l)}_{\text{imp}}(B_l)}{L \sigma_h^{(l)}(B_l)^2} \right] \frac{\partial \varepsilon^{(l)}(\Lambda)}{\partial B_l} \bigg|_{\Lambda = B_l}, \quad l = 0, 1, \cdots, \nu - 2,
$$

which are independent of the spin index $l$ for zero magnetic fields. Consequently the finite-size spectrum for the spin sector reads

$$
E - E_0 = \frac{2\pi v}{L} \frac{1}{2} \Delta M_h^T C_\nu \Delta M_h + \frac{2\pi v_{\text{imp}}}{L} \frac{1}{2} (L \Delta m^T) C_\nu (L \Delta m),
$$

where we have set $n_l^+ = 0$ for simplicity, $C_\nu$ is the $(\nu - 1) \times (\nu - 1)$ $\text{SU}(\nu)$ Cartan matrix

$$
C_\nu = \begin{pmatrix}
2 & -1 & 0 \\
-1 & \ddots & \ddots \\
& \ddots & -1 \\
0 & -1 & 2
\end{pmatrix},
$$

and $\Delta m$ denote the variations of the $(\nu - 1)$-component vector $m$ defined by

$$
m^{(l)} = \int_{B_l}^{\infty} d\Lambda \sigma^{(l)}_{\text{imp}}(\Lambda),
$$

for the impurity part. This completes the calculation of the spin excitation spectrum.

**C. Finite-size spectrum**

We are now ready to present the final result for the finite-size spectrum. The $1/L$ piece in the spectrum is obtained by combining the charge and spin excitations given by Eqs.(74) and (88). Setting $n_c^+ = n_l^+ = 0$ for simplicity we write the result in the $\nu \times \nu$ matrix form.
\[
\frac{1}{L} E_1 = \frac{2 \pi v}{L} \Delta M^T C_f \Delta M - \frac{\pi v}{L} \nu \left( \frac{\delta_F}{\pi} \right)^2,
\]
(91)

where \( \Delta M^T \equiv (\Delta M^T_h, \Delta M_h^{(\nu-1)}) \) with \( \Delta M_h^{(\nu-1)} = \Delta N_h - \nu \delta_F / \pi \). Here the \( \nu \times \nu \) matrix \( C_f \) is given by

\[
C_f = \begin{pmatrix}
2 & -1 & 0 \\
-1 & \ddots & \ddots \\
\ddots & 2 & -1 \\
0 & -1 & 1
\end{pmatrix}.
\]
(92)

It is amusing that this matrix coincides with the Cartan matrix for the OSp(\(\nu, 1\)) Lie superalgebra. This matrix characterizes the energy spectrum of free electron systems as well as the electron models based on the OSp(\(\nu, 1\)) superalgebra. That is, the OSp(\(\nu, 1\)) Cartan matrix in the finite-size spectrum gives rise to canonical exponents for the correlation functions. In fact the critical exponents for electron correlators \( \langle v_j^\dagger(t) c_j(0) \rangle \simeq t^{-\eta_n} \) are obtained by specifying the quantum numbers \( \Delta M_h^{(l)} = 1 \) (or 0) for \( l \geq n \) (or \( l < n \)). We then find the critical exponents \( \eta_n = 1 \) irrespective of \( n (= 0, 1, \ldots, \nu - 1) \). We see that the finite-size spectrum for the host part (1/L order) has the form in accordance with boundary CFT with U(1) symmetry in the charge sector and level-1 SU(\(\nu\)) Kac-Moody symmetry in the spin sector.

On the other hand, the 1/L^2 part does not take such a simple form since the spin and charge velocities are different from each other due to the electron correlation effects. It is worth, however, considering the hypothetical situation where these two velocities are equal, say \( v_c^{imp} = v_s^{imp} = 1 \). Then the excitation energy is again cast into the form of the OSp(\(\nu, 1\)) Cartan matrix

\[
\Delta E^{(2)} = \frac{2 \pi v}{L^2} \frac{1}{2} (L \Delta m^T) C_f (L \Delta m).
\]
(93)

Since the critical exponents of correlation functions do not depend on the velocities this result indicates that the spectrum of scaling dimensions are solely characterized by the Cartan matrix. Furthermore note that the spectrum for the impurity part (1/L^2 order) is
essentially given by the same formula as that for free electrons except for the modification of velocities. This result nicely fits with the local Fermi-liquid picture of Nozières [2].

IV. S-D EXCHANGE MODEL

This section is devoted to the analysis of the finite-size spectrum of the traditional s-d exchange model. The reader may be somewhat curious about why the analysis of the simplest model comes at the end. As we shall proceed it is seen that the simplest nature of the model gives rise to peculiarity when trying to extract the finite-size spectrum based on the Bethe-ansatz solution.

The Hamiltonian of the s-d exchange model is given by

\[ H = \sum_\sigma \int dx c_\sigma^\dagger(x) \left( -i \frac{\partial}{\partial x} \right) c_{\sigma}(x) + J \sum_{\sigma\sigma'} S \cdot c_{\sigma'}^\dagger(0) \sigma_{\sigma'} c_{\sigma}(0), \]  

where conduction electrons are scattered by the impurity spin located at the origin and the exchange coupling is assumed to be antiferromagnetic. Applying the Bethe-ansatz technique Andrei [23] and Wiegmann [24] obtained the Bethe-ansatz equations

\[ k_j L = 2\pi I_j + \sum_{\beta=1}^{M} \theta(2\Lambda_\beta - 2), \quad j = 1, 2, \cdots, N_h, \]  

\[ N_h \theta(2\Lambda_\alpha - 2) + \theta(2\Lambda_\alpha) = -2\pi J_\alpha + \sum_{\beta=1}^{M} \theta(\Lambda_\alpha - \Lambda_\beta), \quad \alpha = 1, 2, \cdots, M, \]

where \( \theta(x) = -2\tan^{-1}(x/c) \) with \( c = 2J/(1 - J^2) \), and \( N_h, M \) are the number of host electrons and the number of down spins, respectively. In contrast to the Anderson model the charge degrees of freedom for the impurity are completely frozen, so we need a careful treatment of the charge excitation in the Bethe-ansatz equations. A peculiar aspect in the Bethe equations is that the charge and spin degrees of freedom are completely decoupled. Hence a standard technique of the dressed charge cannot be applied in a straightforward manner to the charge sector of the model. For example, a naive calculation of the dressed charge yields \( \xi = 1 \) for the charge sector, which is the value for spinless fermions. This seems apparently in contradiction to \( \xi = \sqrt{2} \) for free electrons.
Thus we deal with the charge degrees of freedom directly without relying on the dressed-charge approach. The total energy $E$ is derived from the Bethe-ansatz solution \[23,24\]. When there are no magnetic fields we have

$$E = \frac{2\pi}{L} \sum_{j=1}^{N_h} n_j + N_h \int_{-\infty}^{\infty} d\Lambda [\theta(2\Lambda - 2) - \pi] \sigma(\Lambda),$$

(97)

where the density function $\sigma(\Lambda)$ obeys

$$\sigma(\Lambda) = \frac{1}{2\pi} \left( \frac{N_h}{L} \frac{4c}{(2\Lambda - 2)^2 + c^2} + \frac{1}{L} \frac{4c}{(2\Lambda)^2 + c^2} - \int_{-\infty}^B d\Lambda' \frac{2c}{(\Lambda - \Lambda')^2 + c^2} \sigma(\Lambda') \right),$$

(98)

with $B = -\infty$ for zero magnetic fields.

Let us first examine the charge excitation. In the limit $B = -\infty$ one can solve Eq.(98) using the Fourier transform. Having obtained the explicit form of $\sigma(\Lambda)$ it is not difficult to evaluate the $1/L$ contribution in the charge sector. One finds

$$\frac{1}{L} \Delta E^c_1 = \frac{2\pi v}{L} \frac{1}{4} \left( \Delta N_h - 1 \right)^2 - \frac{\pi v}{2L}.$$  

(99)

Notice that the shift $\delta N_h \to \delta N_h - 1$ originates from the $\pi/2$ phase shift.

Consider next the spin excitation spectrum with $N_h$ being kept fixed. The spin part of the total energy in the presence of a magnetic field is \[23,24\]

$$E^s = N_h \int_{-\infty}^{B} [\theta(2\Lambda - 2) - \pi] \sigma(\Lambda),$$

(100)

where $B$ is determined through the number of down spins $M_h$ of host electrons which is given by

$$\frac{M_h}{L} = \int_B^{\infty} d\Lambda \sigma_h(\Lambda).$$

(101)

Here the density function is separated into a host and an impurity part: $\sigma(\Lambda) = \sigma_h(\Lambda) + (1/L)\sigma_{imp}(\Lambda)$. Using the dressed energy defined by

$$\epsilon(\Lambda) = \frac{N_h}{L} [\theta(2\Lambda - 2) - \pi] - \frac{1}{2\pi} \int_{B}^{\infty} d\Lambda' \frac{2c}{(\Lambda - \Lambda')^2 + c^2} \epsilon(\Lambda'),$$

(102)

we can express the total energy as \[23,24\]
\[ E_s = L \int_B^\infty d\Lambda \left\{ \frac{N_h}{\pi L (2\Lambda - 2)^2 + c^2} + \frac{1}{\pi L (2\Lambda)^2 + c^2} \right\} \varepsilon(\Lambda). \] (103)

The dressed charge is determined by the following equation

\[ \xi_s(\Lambda) = 1 - \frac{1}{2\pi} \int_B^\infty d\Lambda' \frac{2c}{(\Lambda - \Lambda')^2 + c^2} \xi_s(\Lambda'), \] (104)

which gives \( \xi_s(B) = 1/\sqrt{2} \) reflecting SU(2) symmetry. The variation \( \Delta B \) is expressed in terms of \( \Delta S_h \), i.e. \( \Delta S_h = -\sigma_h(B)\xi_s(B)\Delta B \). As in the preceding sections, we then obtain the finite-size spectrum of the spin sector

\[ \frac{1}{L}E_1^e + \frac{1}{L^2}E_2^e = \frac{2\pi v}{L} (\Delta S_h)^2 + \frac{2\pi v_{\text{imp}}}{L^2} (L\Delta s_z)^2, \] (105)

where we have suppressed the particle-hole excitation just for convenience and the velocity of free fermion \( v = 1 \). The velocity \( v_{\text{imp}} \) of spin excitation for the impurity part is written in terms of the dressed energy

\[ 2\pi v_{\text{imp}} = \left. \frac{-1}{\sigma_{\text{imp}}(B)} \frac{\partial \varepsilon(\Lambda)}{\partial B} \right|_{\Lambda = B}. \] (106)

One can see from Eqs.(99) and (105) that the finite-size spectrum of the \( s-d \) exchange model is essentially identical to that of the Anderson model under \( \delta_F = \pi/2 \). Our results are in agreement with those in [9] for the \( s-d \) exchange model.

V. SUMMARY

We have studied the finite-size corrections to the energy spectrum for the Kondo problem based on the Anderson model, the degenerate Anderson model and the \( s-d \) exchange model. All these models are known to exhibit the local Fermi-liquid properties at low energies. The basic picture of local Fermi liquid was already established and confirmed by the exact evaluation of various static quantities. A recent CFT approach to the Kondo problem by Affleck and Ludwig has clarified how the Fermi-liquid picture emerges and is described in terms of boundary CFT. Our calculations demonstrate that the finite-size spectrum consistent with boundary CFT is derived directly from the microscopic models. We have also pointed out
that the critical behavior related to the X-ray absorption singularities will also be described by boundary CFT. A natural extension of the present work is to analyze the multichannel Kondo problem including the overscreened case. It is in principle possible to work out the finite-size spectrum using the Bethe-ansatz solution. We hope to turn to this issue in the near future.

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APPENDIX A: FINITE-SIZE CORRECTIONS TO THE GROUND-STATE ENERGY OF THE ANDERSON MODEL

We present the details of computations of the finite-size corrections to the ground-state energy of the Anderson model in Sec.I. Applying the Euler-Maclaurin formula to Eqs.(2) and (3) one obtains the finite-size form of the Bethe-ansatz integral equations. The result reads

\[
\rho_L(k) = \frac{1}{2\pi} + \frac{1}{L} \Delta(k) + B'(k) \int_{\Lambda}^{\infty} d\Lambda a_1(B(k) - \Lambda) \sigma_L(\Lambda) \\
+ \frac{1}{24L^2} \left\{ \frac{B'(k)a_1'(B(k) - \Lambda^+)}{\sigma_h(\Lambda^+)} - \frac{B'(k)a_1'(B(k) - \Lambda^-)}{\sigma_h(\Lambda^-)} \right\},
\]

(A1)

\[
\sigma_L(\Lambda) + \int_{\Lambda}^{\infty} d\Lambda' a_2(\Lambda - \Lambda') \sigma_L(\Lambda') + \frac{1}{24L^2} \left\{ \frac{a_2'(\Lambda - \Lambda^+)}{\sigma_h(\Lambda^+)} - \frac{a_2'(\Lambda - \Lambda^-)}{\sigma_h(\Lambda^-)} \right\} \\
+ \int_{-\infty}^{k^+} dk a_1(\Lambda - B(k)) \rho_L(k) + \frac{1}{24L^2} \left\{ \frac{B'(k^+)a_1'(\Lambda - B(k^+))}{\rho_h(k^+)} - \frac{B'(k^-)a_1'(\Lambda - B(k^-))}{\rho_h(k^-)} \right\} \\
= A(\Lambda) + \frac{1}{L} Z(\Lambda),
\]

(A2)

with \( k^+ = \beta \) and \( \Lambda^- = \alpha \). We let \( k^- \to -\infty \) as well as \( \Lambda^+ \to +\infty \) at the end of the calculation. The solutions of Eqs.(A1) and (A2) are written as

\[
\rho_L(k) = \rho_h(k) + \frac{1}{L} \rho_{\text{imp}}(k) + \frac{1}{24L^2} \left\{ \rho_1^+(k) \frac{\rho_1^-(k)}{\rho_h(k^-)} + \rho_2^+(k) \frac{\rho_2^-(k)}{\rho_h(k^+)} + \rho_1^-(k) \frac{\rho_2^-(k)}{\sigma_h(\Lambda^-)} + \rho_2^+(k) \frac{\rho_2^+(k)}{\sigma_h(\Lambda^+)} \right\},
\]

(A3)

\[
\sigma_L(\Lambda) = \sigma_h(\Lambda) + \frac{1}{L} \sigma_{\text{imp}}(\Lambda) + \frac{1}{24L^2} \left\{ \frac{\sigma_1^+(\Lambda)}{\rho_h(k^+)} + \frac{\sigma_1^-(\Lambda)}{\rho_h(k^-)} + \frac{\sigma_2^-(\Lambda)}{\sigma_h(\Lambda^-)} + \frac{\sigma_2^+(\Lambda)}{\sigma_h(\Lambda^+)} \right\}.
\]

(A4)

Here \( \rho_i^+(k) \) and \( \sigma_i^+(\Lambda) \) \((i = 1, 2)\) satisfy

\[
\rho_i^+(k) = \rho_i^{0+} + B'(k) \int_{\alpha}^{\infty} d\Lambda a_1(B(k) - \Lambda) \sigma_i^{0+}(\Lambda),
\]

(A5)

\[
\sigma_i^{0+}(\Lambda) + \int_{\alpha}^{\infty} d\Lambda' a_2(\Lambda - \Lambda') \sigma_i^{0+}(\Lambda') + \int_{-\infty}^{\beta} dk a_1(\Lambda - B(k)) \rho_i^0(k) = \sigma_i^{0+},
\]

(A6)

where \( \rho_i^{0+} = 0, \rho_2^{0+} = \pm B'(k)a_1'(B(k) - \Lambda^\pm), \sigma_1^{0+} = \pm B'(k^+)a_1'(B(k^+) - \Lambda) \) and \( \sigma_2^{0+} = \pm a_2'(\Lambda^\pm - \Lambda) \).
Let us next apply the Euler-Maclaurin formula to Eq.(7). We then obtain the finite-size form of the ground-state energy as follows:

\[ E_0 = L \left\{ \int_{-\infty}^{\beta} dk \varphi(k) k \rho_L(k) + 2 \int_{\alpha}^{\infty} d\Lambda x(\Lambda) \varphi(x(\Lambda)) \sigma_L(\Lambda) \right\} \]  

\[ - \frac{1}{24L \rho_h(k^+)} + \frac{2x'(\Lambda^-)}{24L \sigma_h(\Lambda^-)}. \]

To derive this we have used the assumed property \( \varphi(k^-), \varphi'(k^-) \to 0 \) as \( k^- \to -\infty \) and \( \varphi(x(\Lambda^-)), \varphi'(x(\Lambda^-)) \to 0 \) as \( \Lambda^- \to +\infty \). Using Eqs.(A3) and (A4) we express the ground-state energy in the form

\[ E_0 = L \varepsilon_0 + \frac{1}{24L \sigma_h(\Lambda^-)} \left\{ 2x'(\Lambda^-) + \int_{-\infty}^{\beta} dk \varphi(k) k \rho'_2(k) + 2 \int_{\alpha}^{\infty} d\Lambda x(\Lambda) x(\Lambda) \sigma_2^-(\Lambda) \right\} \]

\[ + \frac{1}{24L \sigma_h(\Lambda^+)} \left\{ \int_{-\infty}^{\beta} dk \varphi(k) k \rho'_2^+(k) + 2 \int_{\alpha}^{\infty} d\Lambda x(\Lambda) x(\Lambda) \sigma_2^+(\Lambda) \right\} \]

\[ + \frac{1}{24L \rho_h(k^+)} \left\{ -1 + \int_{-\infty}^{\beta} dk \varphi(k) k \rho'_1^+(k) + 2 \int_{\alpha}^{\infty} d\Lambda x(\Lambda) x(\Lambda) \sigma_1^+(\Lambda) \right\} \]

\[ + \frac{1}{24L \rho_h(k^-)} \left\{ \int_{-\infty}^{\beta} dk \varphi(k) k \rho'_1^-(k) + 2 \int_{\alpha}^{\infty} d\Lambda x(\Lambda) x(\Lambda) \sigma_1^-(\Lambda) \right\}, \quad (A8) \]

where

\[ \varepsilon_0 = \int_{-\infty}^{\beta} dk \varphi(k) k \rho(k) + 2 \int_{\alpha}^{\infty} d\Lambda x(\Lambda) \varphi(x(\Lambda)) \sigma(\Lambda). \]  

(A9)

It is easily seen from Eqs.(A5) and (A6) that the second and fourth lines of (A8) vanish in the limit \( k^- \to -\infty, \Lambda^+ \to +\infty \). Furthermore the first and third lines can be expressed in terms of the dressed energy functions (18) and (19)

\[ E_0 = L \varepsilon_0 + \frac{1}{24L \sigma_h(\alpha)} \frac{\partial \varepsilon_c(\Lambda)}{\partial \Lambda} \bigg|_{\Lambda=\alpha} + \frac{1}{24L \rho_h(\beta)} \frac{\partial \varepsilon_s(k)}{\partial k} \bigg|_{k=\beta}. \]  

(A10)

Here we note the relations

\[ \frac{1}{\sigma_h(\alpha)} \frac{\partial \varepsilon_c(\Lambda)}{\partial \Lambda} \bigg|_{\Lambda=\alpha} = -2\pi, \quad \frac{1}{\rho_h(\beta)} \frac{\partial \varepsilon_s(k)}{\partial k} \bigg|_{k=\beta} = -2\pi, \]  

(A11)

which can be checked from Eqs.(9), (10), (18) and (19). Thus we arrive at Eq.(8) in the text.
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