Classification of real low dimensional Jacobi-Lie bialgebras

A. Rezaei-Aghdam∗ and M. Sephid †

Department of Physics, Faculty of Science, Azarbaijan Shahid Madani University , 53714-161, Tabriz, Iran

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Abstract

We have reformulated the definition of the Jacobi (generalized)-Lie bialgebras ((g, φ0), (g∗, X0)) in terms of structure constants of the Lie algebras g and g∗ and components of their 1-cocycles X0 ∈ g and φ0 ∈ g∗ in the basis of the Lie algebras. Then, using adjoint representations and automorphism Lie groups of Lie algebras, we give a method for classification of real low dimensional Jacobi-Lie bialgebras. In this way, we have obtained and classified all real two and three dimensional Jacobi-Lie bialgebras.
1 Introduction

Jacobi structure on a manifold $M$ (Jacobi manifold) has been introduced by Lichnerowicz [1] and also as a local Lie algebra structure on $C^\infty(M,R)$ by Kirillov [2]. Jacobi structures have all properties of the Poisson structures, except that they are not necessarily derivation. The generalization of the Poisson-Lie group (a Lie group such that the Poisson structure on it, is compatible with the group structure [3,4]) to Jacobi-Lie group have been done in [5]. Of course, firstly, this work has been performed for the Lie bialgebroids i.e. the relation between Jacobi structure and Lie bialgebroids has been studied as the generalized Lie bialgebroids [6] (Jacobi bialgebroid [7]) have been defined; then the generalized Lie bialgebras and their group structure (i.e. Jacobi-Lie groups) have been defined in [5]. Generalized Lie bialgebras (where we call these in this paper Jacobi-Lie bialgebras) defined in [5] are the algebraic structure of the Jacobi-Lie groups, like Lie bialgebras as the algebraic structures of the Poisson-Lie groups [3], [4]. In [5], a generalizing Yang-Baxter equation method has been proposed to obtain Jacobi-Lie bialgebras and some examples of Jacobi-Lie bialgebras have been given. Here, we have reformulated the definition of the Jacobi-Lie bialgebra $(g, \phi_0), (g^*, X_0)$ in terms of structure constants of the Lie algebras $g$ and $g^*$ and components of their 1-cocycles $X_0 \in g$ and $\phi_0 \in g^*$ in the basis of the Lie algebras. In this way, using the adjoint representations and the automorphism Lie groups of these Lie algebras, we have obtained a method for classifying Jacobi-Lie bialgebras; then we classify all real two and three dimensional Jacobi-Lie bialgebras. The outline of the paper is as follow.

In section two, we reformulate the formal definition of the Jacobi-Lie bialgebra $(g, \phi_0), (g^*, X_0)$ in terms of the structure constants of the Lie algebras $g$ and $g^*$ and the components of 1-cocycles $X_0 \in g$ and $\phi_0 \in g^*$ in the basis of the Lie algebras; and at the end of this section we give a proposition about equivalence of Jacobi-Lie bialgebras using automorphism of Lie algebras. Then, in section three we give the matrix representation of the obtained relations (in section two) using the adjoint representations. Also, we give three steps for obtaining and classifying real low dimensional Jacobi-Lie bialgebras. In section four, in order to clarifying our method, we give a detailed example, then we obtain and classify all real two and three dimensional Jacobi-Lie bialgebras by this method. Some remarks are addressed in conclusion.

2 Jacobi-Lie bialgebra

In this section, we review the basic definitions of Jacobi (generalized)-Lie bialgebras $(g, \phi_0), (g^*, X_0))$. Then we reformulate these definitions in terms of structure constants of Lie algebras $g$ and $g^*$ and the components of 1-cocycles $X_0 \in g$ and $\phi_0 \in g^*$ in the basis of the Lie algebras.

Definition 1 [3]: A Jacobi-Lie bialgebra (generalized Lie bialgebra defined in [5]) is a pair $(g, \phi_0), (g^*, X_0))$, where $(g, [\cdot, \cdot]^g)$ is a real Lie algebra of finite dimension with Lie bracket $[\cdot, \cdot]^g$ so that the dual space $g^*$ is also a Lie algebra with bracket $[\cdot, \cdot]^{g*}$ and $X_0 \in g$ and $\phi_0 \in g^*$ are 1-cocycles on $g^*$ and $g$ respectively, and $\forall X, Y \in g$ we have

\[ d_{\phi_0}X_0[X, Y]^g = [X, d_{\phi_0}X]X_0^g - [Y, d_{\phi_0}X]X_0^g, \]

(1)

\[ \phi_0(X_0) = 0, \]

(2)

\[ i_{\phi_0}(d_{\phi_0}X) + [X_0, X] = 0. \]

(3)

Where $i_{\phi_0}P$ is contraction of a $P \in \wedge^k g$ to a tensor $\wedge^{k-1} g$; furthermore $d_{\phi_0}$ being the Chevalley-Eilenberg differential of $g^*$ acting on $g$ and $d_{\phi_0}X_0$ is its generalization [6] such that we have

\[ d_{\phi_0}X_0Y = d_{\phi_0}Y + X_0 \wedge Y, \]

(4)

meanwhile $[\cdot]_{\phi_0}^g$ is $\phi_0$-Schouten-Nijenhuis bracket with the following properties [5]

\[ \forall P \in \wedge^k g, P' \in \wedge^{k'} g, P'' \in \wedge^{k''} g, \]

\[ [P, P']_{\phi_0} = [P, P'] + (-1)^{k+1}(k-1)P \wedge i_{\phi_0}P' - (k'-1)i_{\phi_0}P \wedge P', \]

(5)

\[ [P, P']_{\phi_0} = (-1)^{kk'}[P', P]_{\phi_0}, \]

(6)
\[ [P, P' \wedge P'']_{\phi_0} = [P, P']_{\phi_0} \wedge P'' + (-1)^{k(k+1)}P' \wedge [P, P'']_{\phi_0} - (i_{\phi_0} P) \wedge P' \wedge P'', \quad (7) \]

\[ (-1)^{kk''} \left([P, P']_{\phi_0}, P''\right)_{\phi_0} + (-1)^{k'k''} \left([P', P']_{\phi_0}, P''\right)_{\phi_0} + (-1)^{k'k}\left([P', P''], P\right)_{\phi_0} = 0. \quad (8) \]

Furthermore, in the above definition the \( \phi_0(X_0) \) means the natural inner product of the dual spaces \( g \) and \( g^* \).

Note that, the above definition is symmetric with respect to \((g, \phi_0) \) and \( (g^*, X_0) \) i.e. if \(((g, \phi_0), (g^*, X_0)) \) is a Jacobi-Lie bialgebra then \(((g^*, X_0), (g, \phi_0)) \) is also a Jacobi-Lie bialgebra \([5]\). For the last case we have \( d \) as the Chevalley-Eilenberg differential of \( g \) acting on \( g^* \) and have the following \( \phi_0 \in g^* \) generalization \([5]\)

\[ \forall w \in \wedge^k g^* \quad d_{\phi_0} w = dw + \phi_0 \wedge w, \quad (9) \]

with the Schouten-Nijenhuis bracket replaced by

\[ [Q, Q']^g_{X_0} = [Q, Q']^{g^*} + (-1)^{k+1}(k - 1)Q \wedge i_{X_0} Q' - (k' - 1)i_{X_0} Q \wedge Q', \quad (10) \]

\[ \forall Q \in \wedge^k g^*, Q' \in \wedge^k g^{**} \text{, with the same properties \([6, 8]\) as for } [\_\_]_{\phi_0}. \quad (11) \]

**Remark 2** \([\text{[2]}] \): In the above definition the \( X_0 \) and \( \phi_0 \) are 1-cocycles on \( g^* \) and \( g \) respectively i.e. we must have

\[ d_{\phi_0} X_0 = 0, \quad (11) \]

\[ d_{\phi_0} 0 = 0. \quad (12) \]

**Remark 3** : In the case of \( \phi_0 = 0 \), \( X_0 = 0 \) the definition 1 recovers the concept of a Lie bialgebra \([2]\), that is, a pair of dual Lie algebras \((g, g^*)\) such that relation (11) reduces to the following one

\[ d_* X, Y]_g = [X, d_* Y]_g - [Y, d_* X]_g. \quad (13) \]

For the above case there is a correspondence between Lie bialgebra \((g, g^*)\) and the Manin triple \((g \oplus g^*, g, g^*)\) such that the direct sum \( g \oplus g^* \) is a Lie algebra where \( g \) and \( g^* \) are isotropic subspaces of \( g \oplus g^* \) with respect to \( ad\)-invariant symmetric pairing \([4]\). But for the Jacobi-Lie bialgebra \(((g, \phi_0), (g^*, X_0)) \) where in the sense of Tan and Liu \([8]\), we have the following bilinear skew-symmetric bracket on the space \( g \oplus g^* \)

\[ [X \oplus \zeta, Y \oplus \eta]^{g \oplus g^*} = \left( [X, Y]_g^* + \left(L_{\phi_0} \right) \zeta Y - \left(L_{\phi_0} \right) \eta X - \frac{1}{2} \left( \zeta (Y) - \eta (X) \right) X_0 \right) \]

\[ \oplus \left( [\zeta, \eta]^{g^*} + \left(L_{\phi_0} \right) \eta Y - \left(L_{\phi_0} \right) \zeta X + \frac{1}{2} \left( \zeta (Y) - \eta (X) \right) \phi_0 \right), \quad (14) \]

\[ \forall X, Y \in g \text{ and } \zeta, \eta \in g^* \text{; such that Lie derivative } L_{x_0} (L_{\phi_0}) \text{ of } g^* (g) \text{ on } g (g^*) \text{ are defined as follow}\]

\[ \forall w \in \wedge^k g^*, X \in g \quad (L_{\phi_0})_X w = (d_{\phi_0} \circ i_X + i_X \circ d_{\phi_0}) w, \quad (15) \]

and

\[ \forall \xi \in g^* \quad (L_{\phi_0})_{\xi} P = (d_{\phi_0} \circ i_{\xi} + i_{\xi} \circ d_{\phi_0}) P, \quad (16) \]

where, for \( P \in \wedge^k g \) (or \( w \in \wedge^k g^* \)), \( \phi_0 \) (or \( X_0 \))-Lie derivative are defined by \( \phi_0 \) (or \( X_0 \))-Schouten Nijenhuis brackets in the following forms

\[ (L_{\phi_0})_X P = [X, P]_g^*_{\phi_0}, \quad (17) \]

and

\[ (L_{\phi_0})_X w = [\xi, w]^{g^*}_{X_0}, \quad (18) \]

in general the \((g \oplus g^*, [\_\_]^{g \oplus g^*})\) is not a Lie algebra, i.e. the Jacobi identities don’t satisfy the algebra \( g \oplus g^* \). Now, choosing the basis of the Lie algebras \( g \) and \( g^* \) as \( \{X_i\} \) and \( \{X^i\} \) respectively, we try to reformulate the above definitions in terms of structure constants. We have

\[ \footnote{For general definition of the differential and the Lie derivative associated with a 1-cocycle, one can see \([6, 8]\).} \]


\[ [X_i, X_j] = f_{ij}^k X_k, \quad [\tilde{X}^i, \tilde{X}^j] = \tilde{f}_{ij}^k \tilde{X}^k, \]

where \( f_{ij}^k \) and \( \tilde{f}_{ij}^k \) are the structure constants of the Lie algebras \( g \) and \( g^* \) respectively, such that they satisfy the following Jacobi identities

\[ f_{ij}^k f_{km}^n + f_{ik}^n f_{mj}^k + f_{jk}^m f_{im}^k = 0, \quad (20) \]

\[ \tilde{f}_{ij}^k \tilde{f}_{km}^n + \tilde{f}_{ik}^n \tilde{f}_{mj}^k + \tilde{f}_{jk}^m \tilde{f}_{im}^k = 0. \quad (21) \]

Furthermore, according to duality between \( g \) and \( g^* \) we have

\[ < X_i, \tilde{X}^j > = \delta^j_i. \quad (22) \]

On the other hand, we know that for the Lie bialgebras by choosing \( [10] \)

the relation \( (13) \) can be rewritten in terms of \( f_{ij}^k \) and \( \tilde{f}_{ij}^k \) as the following mixed-Jacobi identities

\[ f_{ij}^k \tilde{f}_{mni}^k = f_{ik}^n \tilde{f}_{jm}^k + f_{jk}^m \tilde{f}_{im}^k + f_{mj}^n \tilde{f}_{ik}^k + f_{im}^n \tilde{f}_{jk}^k, \quad (24) \]

and using the \( ad \)-invariant symmetric bilinear form on the Manin triple of Lie bialgebras \( g \oplus g^* \) one can find the following commutation relation \( [3] \)

\[ [X_i, \tilde{X}^j] = \tilde{f}_{ik}^l X_l + f_{kl}^i \tilde{X}^l, \quad (25) \]

where relation \( (21) \) together with \( (20) \) and \( (24) \) are the Jacobi identities on the Lie algebra \( g \oplus g^* \). Now, for Jacobi-Lie bialgebra \( [11]-[13] \) and \( [11]-[12] \) one can also apply relation \( (23) \) as Chevalley-Eilenberg differential. In this way, expanding \( X_0 \in g, \phi_0 \in g^* \) in terms of the basis of the Lie algebras \( g \) and \( g^* \)

\[ X_0 = \alpha^i X_i, \quad \phi_0 = \beta_j \tilde{X}^j, \quad (26) \]

and using \( [4], [19] \) and \( [23] \), after some calculations, relations \( [11]-[3] \) and \( [11]-[12] \) can be rewritten as follow, respectively,

\[ f_{ij}^k \tilde{f}_{mni}^k - f_{ik}^n \tilde{f}_{jm}^k + f_{jk}^m \tilde{f}_{im}^k + f_{mj}^n \tilde{f}_{ik}^k + f_{im}^n \tilde{f}_{jk}^k + \alpha^m f_{ij}^n - \alpha^n f_{ij}^m + 2(\alpha^k f_{ik}^m - \alpha^m \beta_j \delta_j^m - 2(\alpha^k f_{jk}^m - \alpha^m \beta_j) \delta_i^m = 0, \quad (27) \]

\[ \alpha^i \beta_i = 0, \quad (28) \]

\[ \alpha^n f_{mni}^m - \beta_n \tilde{f}_{mni}^m = 0, \quad (29) \]

\[ \alpha^i \tilde{f}_{mni}^m = 0, \quad (30) \]

\[ \beta_i f_{mni}^i = 0. \quad (31) \]

Furthermore, from \( [14]-[16] \) and \( [19],[22] \) after some calculation, one can find the commutation relations between \( \{X_i \} \) and \( \{ \tilde{X}^j \} \) as follows

\[ [X_i, \tilde{X}^j] = (\tilde{f}_{jk}^i + \frac{1}{2} \alpha^k \delta_i^j - \alpha^j \delta_i^k) X_k + (f_{ki}^j - \frac{1}{2} \beta_k \delta_i^j + \beta_i \delta_k^j) \tilde{X}^k. \quad (32) \]

Note that relations \( [20]-[21] \) and \( [27]-[31] \) and \( [32] \) are the algebraic definitions for the Jacobi-Lie bialgebras in the terms of basis \( \{X_i \} \) and \( \{ \tilde{X}^j \} \) and in this sense, these are a generalization of the ordinary Lie bialgebras \( [19]-[21] \) and \( [24]-[25] \).[2]

\[ ^2 \text{Note that relations} \ [27]-[32] \text{for} \ \alpha^i = \beta_i = 0, \text{reduce to} \ [21] \text{and} \ [25]. \]
In this way, one can apply the definition 4 and proposition 5 for obtaining and classifying the Jacobi-Lie bialgebras in low dimensions as for the Lie bialgebras and Lie super bialgebras in low dimensions [11][12]. The following proposition will help us for this aim.

**Proposition 5** Two Jacobi-Lie bialgebras \(((g, φ_0), (g^*, X_0))\) and \(((g, φ_0'), (g^{∗′}, X_0'))\) are equivalent, if there exist \(A ∈ \text{Aut}(g)\) (automorphism group of the Lie algebra \(g\)) such that

\[
\tilde{f}^{ij}_{n(g^{∗′})} = (A^{-t})^i_k f_{kl}^m(g^*) (A^{-t})^j_l (A^t)^m_n,
\]

\[
α'^n = (A^{-t})^j_l mα^m_l,
\]

\[
β'_i = A_i^m β_m^l,
\]

where \(A_m^m\) are the elements of the automorphism matrix \(A\) for the Lie algebra \(g\) and \(X_0' = α'^n X_i, φ_0' = β'_i \tilde{X}^j\) so that \{\tilde{X}^j\} is the basis of \(g^{∗′}\).

**Proof:** From the definition of automorphism of the Lie algebra; \(A : g → g\) in terms of the basis \(X_j\) we have

\[
AX_i = A_i^j X_j,
\]

where \(A_i^j\) satisfy the following relation

\[
A_i^m f_{mn}^k A_k^n = f_{ij}^l A_i^k.
\]

Now, applying (37) in (27), one can obtain relations (33)-(35) where \(α'^n, β'_i\) and \(f_{ij}^{n(g^{∗′})}\) are satisfied relations (27)-31 and these show that \(((g, φ_0), (g^*, X_0))\) is also a Jacobi-Lie bialgebra and is equivalent to the Jacobi-Lie bialgebra \(((g, φ_0), (g^*, X_0))\).

The above proposition in the case of \(φ_0 = 0, X_0 = 0\) recovers the equivalency between Lie bialgebras \((g, δ)\) and \((g, δ')\) [10] i.e. the relation

\[
δ' = (A ⊗ A) ◦ δ ◦ A^{-1}.
\]

In this way, one can apply the definition 4 and proposition 5 for obtaining and classifying the Jacobi-Lie bialgebras directly.
3 Calculation of Jacobi-Lie bialgebras using adjoint representation

In this section, using the adjoint representation of the Lie algebras \( \mathfrak{g} \) and \( \mathfrak{g}^* \) in [12] and [27]-[31], we organize a way as in [12] for calculating and classifying low dimensional Jacobi-Lie bialgebras. Because of tensorial form of mentioned relations, working with them is very difficult and we suggest writing those equations in matrix forms using the following adjoint representations for the Lie algebras \( \mathfrak{g} \) and \( \mathfrak{g}^* \)

\[(X_i)_j^k = -f_{ij}^k, \quad (Y_j^i)_j^k = -f_{ij}^k, \quad (\tilde{X}_i)_j^k = -\tilde{f}_{ij}^k, \quad (\tilde{Y}_j^i)_j^k = -\tilde{f}_{ij}^k,\]

(39)

Then, the matrix forms of the relations (21) and (27)-(31) become as follow, respectively,

\[(\tilde{X}_i)_j^k \tilde{X}^k + \tilde{X}^i \tilde{X}^j - \tilde{X}^j \tilde{X}^i = 0,\]

(41)

\[(D^{mn})_{ij} + 2 C_i^m \delta_j^n - 2 C_j^m \delta_i^n = 0,\]

(42)

\[\text{Tr}(AB^t) = 0,\]

(43)

\[\alpha^i (X_i)^t - \beta_i \tilde{X}^i = 0,\]

(44)

\[\alpha^i \tilde{Y}_i = 0,\]

(45)

\[\beta_i \tilde{J}^n = 0,\]

(46)

where the matrices \( C, D^{mn} \) have the following forms

\[C = \alpha^k X_k^t - \mathcal{B} A^t,\]

\[D^{mn} = (\tilde{X}^m)^n_k \tilde{Y}^k + \tilde{Y}^m \tilde{X}^n - \gamma^m \tilde{X}^n + (\tilde{X}^m)^n \gamma^m - (\tilde{X}^{mn}) \gamma^m + \mathcal{B}(\tilde{F}^{mn})^t - \tilde{F}^{mn} B^t + \alpha^n \gamma^m - \alpha^m \gamma^n,\]

and \( A, B, \tilde{F} \) are used to represent the following column matrices (where \( d \) is dimension of Lie algebras \( \mathfrak{g} \) and \( \mathfrak{g}^* \)).

\[
A = \begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_d
\end{pmatrix}, \quad
B = \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\vdots \\
\beta_d
\end{pmatrix}, \quad
\tilde{F}^{mn} = \begin{pmatrix}
\tilde{f}_{mn}^1 \\
\tilde{f}_{mn}^2 \\
\vdots \\
\tilde{f}_{mn}^d
\end{pmatrix}.
\]

(48)

Now, by substituting the structure constants of Lie algebra \( \mathfrak{g} \) in the matrix equations (42)-(46) and solving these equations simultaneously with (41); we obtain the structure constants of dual Lie algebras \( \mathfrak{g}^* \) and the matrices \( A, B \) so that \( (\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0) \) is Jacobi-Lie bialgebra. By this method, we will classify two and three dimensional Jacobi-Lie bialgebras. We perform this work in the following three steps similar to [12].

**Step 1:** Solving the equations (41)-(46) and determining the Lie algebras \( \mathfrak{g}^* \) which are isomorphic with dual solutions \( \mathfrak{g}^* \)

With the solution of matrix equations (41)-(46) for obtaining matrices \( \tilde{X}^i, A \) and \( B \), some structure constants of \( \mathfrak{g}^* \) and also some coefficients of \( \alpha^i \) and \( \beta_i \) are obtained to be zero, some are unknown and some are obtained in terms of each other. In order to know whether \( \mathfrak{g}^* \) is one of the known Lie algebras of the table or isomorphic

\[3\] Note that in [5] in order to obtain the Jacobi-Lie bialgebras, one must at first find the algebraic \( r \)-matrix; but here one can find directly the Jacobi-Lie bialgebras by using only the structure constants \( f_{ij}^k \) (similar to the ordinary Lie bialgebras [11] and [12]).
we have of the elements of matrices $A$.

Now, for obtaining Jacobi-Lie bialgebras $(\mathfrak{g}, \phi_0, \phi_1, X_0)$, we have

$$\tilde{X}^i = C^i \tilde{X}^j,$$

then we obtain the following matrix equations for isomorphism

$$C \left( C^i_k \tilde{X}^k \right) = X^i_{\mathfrak{g}'^*} C.$$

(50)

Where $X^i_{\mathfrak{g}'}$ are adjoint matrices of the known Lie algebra $\mathfrak{g}'$ of the classification table. Solving equation (50), we obtain some extra conditions on $f^k_{ij}$ that were obtained from (32) and (33).

**Step 2:** Obtaining general form of the transformation matrices $B : \mathfrak{g}^* \rightarrow \mathfrak{g}'^*; i$ such that $(\mathfrak{g}, \phi_0, \mathfrak{g}'^* - \mathfrak{g}^*, X_0)$ is a Jacobi-Lie bialgebra.

As the second step, we transform Jacobi-Lie bialgebra $((\mathfrak{g}, \phi_0, \mathfrak{g}'^*, X_0))$ where in the Lie algebra $\mathfrak{g}$ we impose extra conditions obtained in the step one to Jacobi-Lie bialgebra $((\mathfrak{g}, \phi_0, \mathfrak{g}'^*, X_0'))$ (where $\mathfrak{g}'^* - \mathfrak{g}^*$ is isomorphic as Lie algebra with $\mathfrak{g}'$) using an automorphism $A$ of the Lie algebra $\mathfrak{g}$. As the inner product $\langle \cdot, \cdot \rangle$ is invariant, we have $A^{-t} : \mathfrak{g}^* \rightarrow \mathfrak{g}'^*; i$,

$$X'_i = A^{-t} X_k,$$

$$\tilde{X}'^j = \langle A^{-t} \rangle^i_j \tilde{X}'^i, \quad < X'_i, \tilde{X}'^j > = \delta_i^j,$$

(51)

where $A^{-t}$ is inverse transpose of every matrix $A \in \text{Aut}(\mathfrak{g})$. Thus we have the following relation

$$(A^{-t})^i_j f^k_{ij}(\mathfrak{g}'^*) m(A^{-t})^n_i = f^{ij}(\mathfrak{g}'^*) n(A^{-t})^n_i m.$$

(52)

Now, for obtaining Jacobi-Lie bialgebras $((\mathfrak{g}, \phi_0, \mathfrak{g}'^*, X_0'))$, we must obtain Lie algebras $\mathfrak{g}'^* - \mathfrak{g}^*$ or transformations $B : \mathfrak{g}^* \rightarrow \mathfrak{g}'^*; i$ such that

$$B^i_k f^k_{ij}(\mathfrak{g}') \! m B^j_l = f^{ij}(\mathfrak{g}'^*) n B^m_i,$$

(53)

for this purpose, it is enough to omit $f^{ij}(\mathfrak{g}'^*) n$ between (32) and (33). Then we will have the following matrix equation for $B$

$$(A^{-t})^i_j m X^j_{\mathfrak{g}'} \! A^{-1} = (B^i_A)^{-1} (B^i_k X^k_{\mathfrak{g}'}^i) B^j.$$

(54)

Solving (54), we obtain the general form of matrix $B$ with the condition $det B \neq 0$. In solving equation (54), one can obtain conditions on the elements of the matrix $A$; we must only consider those conditions under which we have $det A \neq 0$.

**Step 3:** Obtaining the non-equivalent Jacobi-Lie bialgebras.

Having solved (54), we obtain the general form of the matrix $B$ so that its elements are written in terms of the elements of matrices $A$, $C$ and structure constants $f^k_{ij}$ and some $\alpha^i$ and $\beta_i$. Now with substituting $B$ in (33), we obtain structure constants $f^{ij}(\mathfrak{g}'^*) n$ of the Lie algebra $\mathfrak{g}'^*$ in terms of elements of matrices $A$ and $C$ and some $f^k_{ij}$ $k$s and also using (34) and (35), we obtain the $\alpha^i$ and $\beta_i$ or column matrices $\mathfrak{A}'$ and $\mathfrak{B}'$. Then we check whether it is possible to equalize some structure constants $f^{ij}(\mathfrak{g}'^*) n$ with each other and with $\pm 1$ such that $det A \neq 0$, $det B \neq 0$ and $det C \neq 0$. In this way, we obtain matrices $B_t, B_{t2},...$ and also $\mathfrak{A}''$ and $\mathfrak{B}''$,... . Note that in obtaining $B_t$s, we impose the condition $BB_t^{-1} \in \text{Aut}(\mathfrak{g})$ (where $\text{Aut}(\mathfrak{g})$ is the transpose of $\text{Aut}(\mathfrak{g})$); if this condition is not satisfied, we can not impose it on the structure constants because $B$ and $B_t$ are not equivalent.

Now using isomorphism matrices $B_1, B_2, ...$ and also $\mathfrak{A}', \mathfrak{A}''$ and $\mathfrak{B}', \mathfrak{B}''$, ... we can obtain Jacobi-Lie bialgebras $((\mathfrak{g}, \phi_0'), (\mathfrak{g}'^*, X_0'))$, $((\mathfrak{g}, \phi_0''), (\mathfrak{g}'^*, X_0''),... ...$. But, there is a question: which of these Jacobi-Lie bialgebras are equivalent? For answering this question, we use the matrix form of the relation (33). Consider the two Jacobi-Lie bialgebras $((\mathfrak{g}, \phi_0'), (\mathfrak{g}'^*, X_0'))$, $((\mathfrak{g}, \phi_0''), (\mathfrak{g}'^*, X_0''))$; using

$$A(X_i) = A^j_i X_j,$$

(55)

the relation (33) will have the following matrix form

$$A^t((A^t)^i_{\mathfrak{g}'^*}^k X^k_{\mathfrak{g}'^*}^i) = X^i_{\mathfrak{g}'^*}^j A^t.$$

(56)
On the other part, the transformation matrix between \( g', \dot i \) and \( g', \ddot ii \) is \( B_2 B_1^{-1} \) if \( B_1 : g' \rightarrow g', \dot i \) and \( B_2 : g' \rightarrow g', \ddot ii \); then we have
\[
(B_2 B_1^{-1})( (B_2 B_1^{-1})^t, \mathcal{A}(g', i)^k ) = \mathcal{A}(g', ii)^l (B_2 B_1^{-1}).
\] (57)

A comparison of (57) with (56) reveals that if \( B_2 B_1^{-1} \in A' \) holds such that we have also \( A'' = A'^{-1}A' \) and \( B'' = AB' \), then the Jacobi-Lie bialgebras \( ((g, \phi'_0), (g', i, X'_0)) \) and \( ((g, \phi'_0), (g', ii, X'_0)) \) are equivalent. In this way, we obtain nonequivalent class of \( B'_g \)s and \( \mathcal{A}_s \) such that we consider only one element of this class. Note that, for obtaining and fixing \( \alpha^i, \beta_i \) we must impose conditions which were obtained for the elements \( f^{ij} (g', i)^8 \) and elements of automorphism group in relations (34) and (35) and try to fix elements of \( \alpha'^0 \) and \( \beta'_i \) with constant values \( (0,1,\ldots) \), using those relations. In this manner, we obtain \( \phi'_0, X'_0 \) and then Jacobi-Lie bialgebras \( ((g, \phi'_0), (g', i, X'_0)) \); such that one can classify all Jacobi-Lie bialgebras. In the next section, we apply this formulation to classify real two and three dimensional Jacobi-Lie bialgebras.

### 4 Classification of real two and three dimensional Jacobi-Lie bialgebras

In this section, we will use the classification of real two and three dimensional Lie algebras and their automorphism groups. It should be noted that for real two dimensional Lie algebras, we will use the classification of the ref [13] as in table 1 and for real three dimensional case we will use the Bianchi classification [14] as in table 2.

| Lie Algebra | Commutation relations | Comments |
|-------------|-----------------------|----------|
| \( A_1 \)   | \([X_1, X_2] = 0\)    |          |
| \( A_2 \)   | \([X_1, X_2] = X_1 \) |          |

| Lie Algebra | Commutation relations | Comments |
|-------------|-----------------------|----------|
| \( I \)     | \([X_1, X_2] = 0\)    |          |
| \( II \)    | \([X_2, X_3] = X_1 \) |          |
| \( III \)   | \([X_1, X_2] = -(X_2 + X_3), [X_1, X_3] = -(X_2 + X_3)\) |          |
| \( IV \)    | \([X_1, X_2] = -(X_2 - X_3), [X_1, X_3] = -X_3\) |          |
| \( V \)     | \([X_1, X_2] = -X_2, [X_1, X_3] = -X_3\) |          |
| \( VI_0 \)  | \([X_1, X_3] = X_2, [X_2, X_3] = X_1\) |          |
| \( VI_1 \)  | \([X_1, X_2] = -(aX_2 + X_3), [X_1, X_3] = -(X_2 + aX_3)\) | \( a \in \mathbb{R} - \{1\}, a > 0\) |
| \( VII_0 \) | \([X_1, X_3] = -X_2, [X_2, X_3] = X_1\) |          |
| \( VII_1 \) | \([X_1, X_2] = -(aX_2 - X_3), [X_1, X_3] = -(X_2 + aX_3)\) | \( a \in \mathbb{R}, a > 0\) |
| \( VIII \)  | \([X_1, X_2] = -X_3, [X_1, X_3] = -X_2, [X_2, X_3] = X_1\) |          |
| \( IX \)    | \([X_1, X_2] = X_3, [X_1, X_3] = -X_2, [X_2, X_3] = X_1\) |          |

As mentioned in the previous section, for obtaining Jacobi-Lie bialgebras, automorphism groups are necessary. These automorphism groups have been calculated using the transformation (57), or in the matrix form (with condition \( det A \neq 0 \)) (10) (see also [15]) using the following relation
\[
A_k^j A^l = \mathcal{Y}^j A_k^l.
\] (58)
The results are given in table 3.
Table 3: Automorphism groups of the real two and three dimensional Lie algebras

| Lie Algebra | Automorphism groups | Comments |
|-------------|--------------------|----------|
| $A_1$       | $GL(2, R)$         |          |
| $A_2$       | $GL(2, R)$         | $a \in \mathbb{R} - \{0\}$ |
| $I$         | $GL(3, R)$         |          |
| $II$        | $\begin{pmatrix} \beta & 0 & 0 \\ \alpha & c & d \\ d & e & f \end{pmatrix}$ | $a, b, c, d, e, f \in \mathbb{R}, \beta \neq ce$ |
| $III$       | $\begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & d & c \end{pmatrix}$ | $a, b, c, d \in \mathbb{R}, c \neq \pm d$ |
| $IV$        | $\begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & c & 0 \end{pmatrix}$ | $a, b, d \in \mathbb{R}, c \in \mathbb{R} - \{0\}$ |
| $V$         | $\begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & e & f \end{pmatrix}$ | $a, b, c, d, f \in \mathbb{R}, cf \neq ed$ |
| $VI_0$      | $\begin{pmatrix} a & b & 0 \\ b & a & 0 \\ c & d & 1 \end{pmatrix}$ | $-a$ |
| $VI_a$      | $\begin{pmatrix} 1 & b & c \\ 0 & d & e \\ 0 & e & d \end{pmatrix}$ | $b, c, d, e \in \mathbb{R}, d \neq \pm e$ |
| $VII_0$     | $\begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ c & d & 1 \end{pmatrix}$ | $-a$ |
| $VII_a$     | $\begin{pmatrix} 1 & b & c \\ 0 & d & -e \\ 0 & e & d \end{pmatrix}$ | $b, c, d, e \in \mathbb{R}, d^2 + e^2 \neq 0$ |
| $VIII$      | $SL(2, R)$         |          |
| $IX$        | $SO(3)$            |          |

Now, using the method considered in section 3 and applying maple program for solving our equations (41)-(46), we could classify real two and three dimensional Jacobi-Lie bialgebras. Let us for explaining the method and steps mentioned in the previous section, investigate an example.

4.1 An example

In the following, we explain our method for this classification by describing the details of the calculations for obtaining the Jacobi-Lie bialgebra $((III, \phi_0), (V.i, X_0))$. By substituting the structure constants of Lie algebra $III$ in the matrix equations (41)-(46), we obtain the following form for the structure constants of $g^*$ and matrices $A, B$

$$f^{12}_1 = f^{13}_1 = \alpha, f^{23}_1 = \beta, f^{23}_2 = -f^{23}_3 = \gamma, \ A = \begin{pmatrix} 0 \\ -\alpha \\ -\alpha \end{pmatrix}, \ B = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix},$$

such that using (50) the obtained Lie algebra $g^*$ is isomorphic with the Lie algebra $V$ with the following isomorphism matrix

$$C = \begin{pmatrix} c_{11} & \frac{c_{21} - c_{31}}{\gamma} & c_{13} \\ c_{21} & -c_{23} & c_{23} \\ c_{31} & -c_{33} & c_{33} \end{pmatrix}$$

(59)
with the conditions \( \alpha = \gamma \) and \( \beta = 0 \). Now, by substituting the above results and the automorphism group of the Lie algebra \( III \) in (54) one can obtain the following form for the matrix \( B \)

\[
B = \begin{pmatrix}
0 & b_{12} & b_{13} \\
b_{22} & b_{23} & b_{33} \\
b_{32} & b_{33} & b_{33}
\end{pmatrix}
\]  

(61)

where condition \( \text{det}B \neq 0 \) requires \( \gamma \neq 0 \). Then, using (59) and (54) we have the following commutation relations for the algebra \( g \).

\[ [\tilde{X}_1, \tilde{X}_2] = \alpha' \tilde{X}_1, [\tilde{X}_1, \tilde{X}_3] = \alpha' \tilde{X}_1, [\tilde{X}_2, \tilde{X}_3] = \alpha'(\tilde{X}_2 - \tilde{X}_3), \quad \mathcal{A}' = \begin{pmatrix}
0 \\
-\alpha' \\
-\alpha'
\end{pmatrix}, \quad B' = \begin{pmatrix}
2 \\
0 \\
0
\end{pmatrix}.
\]  

(62)

where \( \alpha' = \frac{\gamma}{c+d} \) such that \( \alpha' \neq 0 \). Now, by choosing \( \alpha' = 1 \) i.e. \( \gamma = c + d \) we have

\[ B' = \begin{pmatrix}
0 & b_{12}' & b_{13}' \\
1 & b_{22}' & b_{23}' \\
1 & b_{32}' & b_{33}'
\end{pmatrix}, \quad \mathcal{A}'' = \begin{pmatrix}
0 \\
-1 \\
-1
\end{pmatrix}, \quad B'' = \begin{pmatrix}
-2 \\
0 \\
0
\end{pmatrix}.
\]  

(63)

Since \( B'B^{-1} \in \mathcal{A}' \) then \( B' \) is equivalent to \( B \) and according to relations (54) and (55), \( \mathcal{A}' \) and \( B' \) is equivalent to \( \mathcal{A}'' \) and \( B'' \), respectively, where \( A \) is automorphism group of Lie algebra \( III \), then we can choose \( \alpha' = 1 \). In this way we obtain the Jacobi-Lie bialgebra \(((III, \phi_0), (V.i, X_0))\) where \( X_0 = -(X_2 + X_3) \) and \( \phi_0 = -2\tilde{X}_1 \) and commutation relations for \( V.i \) as

\[ [\tilde{X}_1, \tilde{X}_2] = \tilde{X}_1, [\tilde{X}_1, \tilde{X}_3] = \tilde{X}_1, [\tilde{X}_2, \tilde{X}_3] = \tilde{X}_2 - \tilde{X}_3. \]  

(64)

Note that, we have two classes of Jacobi-Lie bialgebras; first class is Jacobi-Lie bialgebras with \( X_0, \phi_0 \neq 0 \) and second class is Jacobi-Lie bialgebras with \( X_0 = 0 \) or \( \phi_0 = 0 \). We give all real two and three dimensional Jacobi-Lie bialgebras in tables 4, 5 and 6, 7 respectively, as follow.

| Table 4: Real two dimensional Jacobi-Lie bialgebras with \( X_0, \phi_0 \neq 0 \) |
|---|
| \( g \) | \( g^* \) | Commutation relations of \( g^* \) | \( X_0 \) | \( \phi_0 \) | Comments |
|---|
| \( A_1 \) | \( A_1 \) | \([X^1, X^1] = 0\) | \( X_2 \) | \( X^1 \) | |
| \( A_2 \) | \( A_2, i \) | \([\tilde{X}_1, \tilde{X}_2] = \tilde{X}_2\) | \(-\alpha X_1 \) | \( \alpha \tilde{X}_2 \) | \( \alpha \in \mathbb{R} - \{0\} \) |

| Table 5: Real two dimensional Jacobi-Lie bialgebras with \( \phi_0 = 0 \) |
|---|
| \( g \) | \( g^* \) | Commutation relations of \( g^* \) | \( X_0 \) | Comments |
|---|
| \( A_1 \) | \( A_1 \) | \([X^1, X^1] = 0\) | \( X_1 + X_2 \) | |
| \( A_1 \) | \( A_2 \) | \([\tilde{X}_1, \tilde{X}_2] = \tilde{X}_1\) | \( \alpha X_2 \) | \( \alpha \in \mathbb{R} - \{0\} \) |

\(^4\)Here in the above relations, \( a, b, c, d \) are elements of automorphism group of Lie algebra \( III \) (see table 3).

\(^5\)Since, if \(((g, \phi_0), (g^*, X_0))\) is Jacobi-Lie bialgebra then \(((g^*, X_0), (g, \phi_0))\) is also Jacobi-Lie bialgebra, therefore, in this paper (for state with \( X_0 \) or \( \phi_0 = 0 \)) we have only presented Jacobi-Lie bialgebras with \( \phi_0 = 0 \).
| $g$    | $g^*$ | Commutation relations of $g^*$ | $X_0$        | $\phi_0$ | Comments |
|--------|-------|--------------------------------|--------------|-----------|----------|
| I      | $III$ | $[X^1, X^2] = -(X^2 + X^3), [X^1, X^3] = -(X^2 + X^3)$ | $-2X_1$     | $-(X^2 - X^3)$ |          |
| II     | $III$ | $[X^1, X^2] = -(X^2 + X^3), [X^1, X^3] = -(X^2 + X^3)$ | $-2X_1$     | $-(X^2 - X^3)$ |          |
| III.i  |       | $[\tilde{X}^2, \tilde{X}^3] = -\frac{a_2}{a}(\tilde{X}^2 + \tilde{X}^3)$ | $-(X_2 - X_3)$ | $a\tilde{X}^1$ | $\alpha \in \mathbb{R} - \{0\}$ |
| III.ii |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1$ | $-2X_2$     | $-2\tilde{X}^1$ |          |
| III.iii|       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1$ | $-(X_2 + X_3)$ | $-2\tilde{X}^1$ |          |
| III.iv |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = -2\tilde{X}^1$ | $X_2 + X_3$ | $-(X^2 - X^3)$ |          |
| IV.i   |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2 - \tilde{X}^3$ | $-(X_2 + X_3)$ | $-2\tilde{X}^1$ |          |
| IV.v   |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1$ | $-X_3$      | $-\tilde{X}^1$ |          |
| IV.vi  |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1$ | $-(X_2 + X_3)$ | $-\tilde{X}^1$ |          |
| IV.vii |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2$ | $-X_3$      | $-\tilde{X}^1$ |          |
| IV.a.i |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \frac{a+1}{a+1}\tilde{X}^2$ | $-X_3$      | $-\tilde{X}^1$ | $\epsilon = 1, 2, \alpha > 0$ |
| IV.a.ii|       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \frac{a+1}{a+1}\tilde{X}^2$ | $-X_3$      | $-\tilde{X}^1$ | $\epsilon = 1, 2, \alpha > 0$ |
| V.i    |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1$ | $-X_3$      | $-\tilde{X}^1$ | $\epsilon = 1, 2$ |
| V.ii   |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^2$ | $-X_3$      | $-\tilde{X}^1$ | $\epsilon = 1, 2$ |
| V.a.i  |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \frac{a+1}{a+1}\tilde{X}^2$ | $-X_3$      | $-\tilde{X}^1$ | $\epsilon = 1, 2$ |
| V.a.ii |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \frac{a+1}{a+1}\tilde{X}^2$ | $-X_3$      | $-\tilde{X}^1$ | $\epsilon = 1, 2$ |
| V.a.iii|       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \frac{a+1}{a+1}\tilde{X}^2$ | $-X_3$      | $-\tilde{X}^1$ | $\epsilon = 1, 2$ |
| V.a.iv |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \frac{a+1}{a+1}\tilde{X}^2$ | $-X_3$      | $-\tilde{X}^1$ | $\epsilon = 1, 2$ |
| V.a.v  |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \frac{a+1}{a+1}\tilde{X}^2$ | $-X_3$      | $-\tilde{X}^1$ | $\epsilon = 1, 2$ |
| V.a.vi |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \frac{a+1}{a+1}\tilde{X}^2$ | $-X_3$      | $-\tilde{X}^1$ | $\epsilon = 1, 2$ |
| V.a.vii|       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \frac{a+1}{a+1}\tilde{X}^2$ | $-X_3$      | $-\tilde{X}^1$ | $\epsilon = 1, 2$ |
| V.a.viii|     | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3$ | $-(X_1 + X_2)$ | $\tilde{X}^3$ |          |
| V.a.ix |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3$ | $-(X_1 - X_2)$ | $\tilde{X}^3$ |          |
| V.a.x  |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3$ | $-X_1$      | $\tilde{X}^3$ |          |
| V.a.xi |       | $[\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3$ | $-\epsilon(X_1 - X_2)$ | $\epsilon\tilde{X}^3$ | $\epsilon = 1, -2$ |
| \( g \) | \( g^* \) | Commutation relations of \( g^* \) | \( X_0 \) | \( \delta_0 \) | Comments |
|-----|-----|-------------------|-------|-------|---------|
| \( VI_0 \) | \( VI_{0, iii} \) | \( [\tilde{X}^1, \tilde{X}^2] = -\frac{a}{a+1}(\tilde{X}^3 + \tilde{X}^2), [\tilde{X}^1, \tilde{X}^3] = -\tilde{X}^3, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3 \) | \(-X_1 - X_2\) | \( X^3 \) | \( a > 0, a \neq 1 \) |
| \( VI_0 \) | \( VI_{0, iv} \) | \( [\tilde{X}^1, \tilde{X}^2] = -\frac{a}{a+1}(\tilde{X}^3 + \tilde{X}^2), [\tilde{X}^1, \tilde{X}^3] = -\tilde{X}^3, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3 \) | \(-X_1 - X_2\) | \( X^3 \) | \( a > 0, a \neq 1 \) |
| \( VI_0 \) | \( VI_{0, iii} \) | \( [\tilde{X}^1, \tilde{X}^2] = -\frac{a}{a+1}(\tilde{X}^3 + \tilde{X}^2), [\tilde{X}^1, \tilde{X}^3] = -\tilde{X}^3, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3 \) | \(-\frac{2}{a+1}(X_1 - X_2)\) | \( \frac{2}{a+1}X^3 \) | \( a > 0, a \neq 1, 3 \) |
| \( VI_0 \) | \( VI_{0, iv} \) | \( [\tilde{X}^1, \tilde{X}^2] = -\frac{a}{a+1}(\tilde{X}^3 + \tilde{X}^2), [\tilde{X}^1, \tilde{X}^3] = -\tilde{X}^3, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3 \) | \(-\frac{2}{a+1}(X_1 - X_2)\) | \( \frac{2}{a+1}X^3 \) | \( a > 0, a \neq 1 \) |
| \( VI_a \) | \( III_{0, i} \) | \( [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1 \) | \(-X_2 + X_3\) | \(- (a+1)\tilde{X}^1 \) | \( a > 0, a \neq 1 \) |
| \( VI_a \) | \( III_{0, ii} \) | \( [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1 \) | \(-\frac{1}{a+1}(X_2 - aX_3)\) | \(- (a-1)\tilde{X}^1 \) | \( a > 0, a \neq 1 \) |
| \( VI_a \) | \( III_{0, iii} \) | \( [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1 \) | \(-\frac{1}{a+1}(X_2 - aX_3)\) | \(- (a-1)\tilde{X}^1 \) | \( a > 0, a \neq 1 \) |
| \( VI_a \) | \( III_{0, iv} \) | \( [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^1 \) | \(-\frac{1}{a+1}(X_2 - aX_3)\) | \(- (a-1)\tilde{X}^1 \) | \( a > 0, a \neq 1 \) |
| \( VI_a \) | \( III_{0, v} \) | \( [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1 \) | \(-\frac{a+1}{a+1}(X_2 - X_3)\) | \(- (a+1)\tilde{X}^1 \) | \( a > 0, a \neq 1 \) |
| \( VI_a \) | \( III_{0, vi} \) | \( [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1 \) | \(-\frac{a+1}{a+1}(X_2 - X_3)\) | \(- (a+1)\tilde{X}^1 \) | \( a > 0, a \neq 1 \) |
| \( VI_a \) | \( III_{0, vii} \) | \( [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1 \) | \(-\frac{2(ab+1)}{b+1}(X_2 + X_3)\) | \( \frac{2(ab+1)}{b+1}\tilde{X}_1 \) | \( a > 0, a \neq 1 \) |
| \( VI_a \) | \( III_{0, viii} \) | \( [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1 \) | \(-\frac{2(ab+1)}{b+1}(X_2 + X_3)\) | \( \frac{2(ab+1)}{b+1}\tilde{X}_1 \) | \( a > 0, a \neq 1 \) |
| \( VI_a \) | \( III_{0, i} \) | \( [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1 \) | \(-\frac{2(ab+1)}{b+1}(X_2 + X_3)\) | \( \frac{2(ab+1)}{b+1}\tilde{X}_1 \) | \( a > 0, a \neq 1 \) |
| \( VI_a \) | \( III_{0, ii} \) | \( [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1 \) | \(-\frac{2(ab+1)}{b+1}(X_2 + X_3)\) | \( \frac{2(ab+1)}{b+1}\tilde{X}_1 \) | \( a > 0, a \neq 1 \) |
| \( VI_a \) | \( III_{0, iii} \) | \( [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1 \) | \(-\frac{2(ab+1)}{b+1}(X_2 + X_3)\) | \( \frac{2(ab+1)}{b+1}\tilde{X}_1 \) | \( a > 0, a \neq 1 \) |
| \( VI_a \) | \( III_{0, iv} \) | \( [\tilde{X}^1, \tilde{X}^2] = \tilde{X}^1, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^1 \) | \(-\frac{2(ab+1)}{b+1}(X_2 + X_3)\) | \( \frac{2(ab+1)}{b+1}\tilde{X}_1 \) | \( a > 0, a \neq 1 \) |
Table 7: Real three dimensional Jacobi-Lie bialgebras with $\phi_0 = 0$

| $g$ | $g^*$ | Commutation relations of $g^*$ | $X_0$ | Comments |
|-----|-------|--------------------------------|-------|----------|
| $I$ | $I$   | $[X^1, X^2] = 0$               | $X_1$ |          |
| $I$ | $II$  | $[X^2, X^3] = X^1$             | $X_3$ |          |
| $I$ | $III$ | $[X^1, X^2] = -(\tilde{X}^2 + \tilde{X}^3), [X^1, X^3] = -(\tilde{X}^2 + \tilde{X}^3)$ | $bX_1$ | $b \in \mathbb{R} - \{0\}$ |
| $I$ | $III$ | $[X^2, X^3] = -(\tilde{X}^2 + \tilde{X}^3), [X^1, X^3] = -(\tilde{X}^2 + \tilde{X}^3)$ | $-X_2 - X_3$ |          |
| $I$ | $IV$  | $[X^1, X^2] = -(X^2 - \tilde{X}^3), [X^1, \tilde{X}^3] = -\tilde{X}^3$ | $bX_1$ | $b \in \mathbb{R} - \{0\}$ |
| $I$ | $V$   | $[X^1, X^2] = -X^2, [X^1, \tilde{X}^2] = -\tilde{X}^3$ | $bX_1$ | $b \in \mathbb{R} - \{0\}$ |
| $I$ | $VI_0$| $[X^1, X^3] = \tilde{X}^2, [X^2, \tilde{X}^3] = \tilde{X}^1$ | $bX_3$ | $b > 0$ |
| $I$ | $VI_a$| $[X^1, X^2] = -(a\tilde{X}^2 + \tilde{X}^3), [X^1, \tilde{X}^3] = -(\tilde{X}^2 + a\tilde{X}^3)$ | $bX_1$ | $a > 0, a \neq 1$ |
| $I$ | $VII_0$| $[X^1, X^3] = -\tilde{X}^2, [X^2, \tilde{X}^3] = \tilde{X}^1$ | $bX_3$ | $b > 0$ |
| $I$ | $VII_a$| $[X^1, X^2] = -(a\tilde{X}^2 - \tilde{X}^3), [X^1, \tilde{X}^3] = -(\tilde{X}^2 + a\tilde{X}^3)$ | $bX_1$ | $a > 0$ |
| $I$ | $I$   | $[X^1, \tilde{X}^2] = 0$       | $X_1$ |          |
| $I$ | $II.$ | $[X^2, \tilde{X}^3] = \tilde{X}^2$ | $X_1$ |          |
| $I$ | $II.$ | $[X^1, \tilde{X}^3] = -\tilde{X}^2$ | $X_1$ |          |
| $I$ | $III.$| $[X^1, X^2] = -(\tilde{X}^2 + \tilde{X}^3), [X^1, \tilde{X}^3] = -(\tilde{X}^2 + \tilde{X}^3)$ | $bX_1$ | $b \in \mathbb{R} - \{0\}$ |
| $I$ | $IV.$ | $[X^1, X^2] = -(\tilde{X}^2 - \tilde{X}^3), [X^1, \tilde{X}^3] = -\tilde{X}^3$ | $bX_1$ | $b \in \mathbb{R} - \{0\}$ |
| $I$ | $IV.$ | $[X^1, X^2] = \tilde{X}^2 - \tilde{X}^3, [X^2, \tilde{X}^3] = \tilde{X}^3$ | $bX_1$ | $b \in \mathbb{R} - \{0\}$ |
| $I$ | $V.$  | $[X^1, X^2] = X^2 - \tilde{X}^3, [X^1, \tilde{X}^3] = -\tilde{X}^3$ | $bX_1$ | $b \in \mathbb{R} - \{0\}$ |
| $I$ | $VI_{0.iii}$ | $[X^2, \tilde{X}^3] = \tilde{X}^3, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^2$ | $bX_1$ | $b > 0$ |
| $I$ | $VI_a$ | $[X^1, X^2] = -(a\tilde{X}^2 + \tilde{X}^3), [X^1, \tilde{X}^3] = -(\tilde{X}^2 + a\tilde{X}^3)$ | $bX_1$ | $a > 0, a \neq 1$ |
| $I$ | $VII_{0.i}$ | $[X^1, X^2] = -\tilde{X}^3, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^2$ | $bX_1$ | $b \in \mathbb{R} - \{0\}$ |
| $I$ | $VII_{0.ii}$ | $[X^1, X^2] = \tilde{X}^3, [\tilde{X}^1, \tilde{X}^3] = -\tilde{X}^2$ | $bX_1$ | $b > 0$ |
| $I$ | $VII_a$ | $[X^1, X^2] = -(a\tilde{X}^2 - \tilde{X}^3), [X^1, \tilde{X}^3] = -(\tilde{X}^2 + a\tilde{X}^3)$ | $bX_1$ | $a > 0$ |
| $I$ | $VII_{a.ii}$ | $[X^1, X^2] = a\tilde{X}^2 - \tilde{X}^3, [\tilde{X}^1, \tilde{X}^3] = \tilde{X}^2 + a\tilde{X}^3$ | $bX_1$ | $a > 0$ |
| III | $III.$ | $[\tilde{X}^1, X^3] = \tilde{X}^1, [\tilde{X}^2, \tilde{X}^3] = \tilde{X}^3$ | $\frac{1}{2}(X_2 - X_3)$ |          |
| III | $III.x$ | $[\tilde{X}^1, X^2] = \tilde{X}^1, [\tilde{X}^3, \tilde{X}^3] = -\tilde{X}^1$ | $-(X_2 - X_3)$ |          |
| III | $IV.$ | $[\tilde{X}^1, X^2] = -\tilde{X}^3, [\tilde{X}^1, X^3] = \tilde{X}^1, [\tilde{X}^2, X^3] = \tilde{X}^1 + \tilde{X}^2 + \tilde{X}^3$ | $X_2 - X_3$ |          |
| III | $V.$ | $[\tilde{X}^1, X^2] = \tilde{X}^3, [\tilde{X}^1, X^3] = \tilde{X}^1, [\tilde{X}^2, X^3] = \tilde{X}^2 + \tilde{X}^3$ | $X_2 - X_3$ |          |
| III | $VI_{0.iii}$ | $[\tilde{X}^1, X^2] = \tilde{X}^3, [\tilde{X}^1, X^3] = -\tilde{X}^1, [\tilde{X}^2, X^3] = -\frac{a+1}{a+3}(\tilde{X}^2 + X^3)$ | $-(X_2 - X_3)$ | $a > 0, a \neq 1$ |
| III | $VI_{a.ii}$ | $[\tilde{X}^1, X^2] = \tilde{X}^3, [\tilde{X}^1, X^3] = \tilde{X}^1, [\tilde{X}^2, X^3] = -\frac{a+1}{a+3}(\tilde{X}^2 + X^3)$ | $-(X_2 - X_3)$ | $a > 0, a \neq 1$ |

5 Conclusion

In this paper, we have reformulated the definition of the Jacobi (generalized)-Lie bialgebras $((g, \phi_0), (g^*, X_0))$ in terms of structure constants of the Lie algebras $g$ and $g^*$ and components of their 1-cocycles $X_0 \in g$ and
\( \phi_0 \in g^* \). In this way, we have obtained a method to classify the real low dimensional Jacobi-Lie bialgebras; then by this method we classify all real two and three dimensional Jacobi-Lie bialgebras. Also, there are some open problems in this direction; such as obtaining the classical \( r \)-matrices and Jacobi-Lie groups related to these Jacobi-Lie bialgebras and constructing integrable models, quantizing of these Jacobi-Lie bialgebras, generalizing of Poisson-Lie symmetry \cite{16} to Jacobi-Lie symmetry and so on. Some of these problems are under investigation \cite{17}, \cite{18}.

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References

[1] A. Lichnerowicz, "Les variétés de Jacobi et leurs algébres de Lie associées", J. Math. Pures Appl. 57 (1978), 453-488.
[2] A. Kirillov, "Local Lie algebras", Russ. Math. Surv. 31 (1976), 55-75.
[3] V. G. Drinfel’d, "Hamiltonian Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation", Sov. Math. Dokl. 27 (1983), 68-71.
[4] V. G. Drinfeld, "Quantum groups", Proceedings of the International Congress of Mathematicians, Berkeley, Vol. 1 (1986), 789-820.
[5] D. Iglesias and J. C. Marrero, "Generalized Lie bialgebras and Jacobi structures on Lie groups", Isr. J. Math. 133 (2003), 285-320; arXiv:math/0102171.
[6] D. Iglesias and J. C. Marrero, "Generalized Lie bialgebroids and Jacobi structures", J. Geo. Phys. 40 (2001), 176-199; arXiv:math/0008105.
[7] J. Grabowski and G. Marmo, "Jacobi structures revisited", J. Phys. A:Math.Gen. 34 (2001), 10975-90.
[8] Y. Tan and Z.J. Liu, "Generalized Lie bialgebras", Commu. Alg. 26 (1998), 2293-2319.
[9] M. de León, B. López, J.C. Marrero, E. Padrón, "Lichnerowicz Jacobi cohomology and homology of Jacobi manifolds: modular class and duality"; arXiv:math/9910079.
[10] A. Rezaei-Aghdam, M. Hemmati, A. R. Rastkar, "Classification of real three-dimensional Lie bialgebras and their Poisson-Lie groups", J. Phys. A:Math.Gen. 38 (2005), 3981-3994; arXiv:math-ph/0412092.
[11] M.A. Jafarizadeh and A. Rezaei-Aghdam, "Poisson-Lie T-duality and Bianchi type algebras", Phys. Lett. B. 458 (1999), 477-490; arXiv:hep-th/9903152.
[12] A. Eghbali, A. Rezaei-Aghdam and F. Heidarpour, "Classification of two and three dimensional Lie superbialgebras", J. Math. Phys. 51 (2010), 073503; arXiv:math-ph/0901.4471.
[13] J. Patera and P. Winternitz, "Subalgebras of real three and four dimensional Lie algebras", J. Math. Phys. 18(7) (1977), 1449-1456.
[14] L.D. Landau and E.M. Lifshitz, "The Classical Theory of Fields", (Oxford: Pergamon) (1987).
[15] A. Harvey, "Automorphisms of the Bianchi model Lie groups", J. Math. Phys. 20 (1979), 251-253.
[16] C. Klimčík and P. Ševera,, "Dual non-Abelian duality and the Drinfeld double", Phys. Lett. B. 351 (1995), 455-462.
[17] A. Rezaei-Aghdam and M. Sephid, "Jacobi-Lie symmetry and Jacobi-Lie T dual sigma models", (in preparation).
[18] A. Rezaei-Aghdam and M. Sephid, "Classical r-matrices of real two and three dimensional Jacobi-Lie bialgebras and their Jacobi-Lie groups", (in preparation).