Quantifying Causal Influence in Quantum Mechanics

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We extend Pearl’s definition of causal influence to the quantum domain, where two quantum systems $A, B$ with finite-dimensional Hilbert space are embedded in a common environment $C$ and propagated with a joint unitary $U$. For finite dimensional Hilbert space of $C$, we find the necessary and sufficient condition on $U$ for a causal influence of $A$ on $B$ and vice versa. We introduce an easily computable measure of the causal influence and use it to study the causal influence of different quantum gates, its mutuality, and quantum superpositions of different causal orders. For two two-level atoms dipole-interacting with a thermal bath of electromagnetic waves, the space-time dependence of causal influence almost perfectly reproduces the one of reservoir-induced entanglement.

Introduction.— To infer causes from effects constitutes a key task of science. Classically, causal influence (CI) is defined between random variables (RVs) that take certain values in a set of possible outcomes of randomized experiments. In a causal model, the RVs sit on vertices of a graph, and the forward-in-time-only CI in the classical world is represented by an arrow in a directed acyclic graph. According to Pearl [1] (see p.276), a RV $x$ has a CI on another RV $y$, if $y$ “listens” to $x$, meaning that there is a functional relationship of the form $y = f(x) + z$, where $f$ is some function and $z$ another RV. Apart from direct CI, correlations between $x$ and $y$ might arise additionally due to common causes. These can be eliminated in practice by “do-interventions”, where an RV is randomly set by the experimenter and one examines if $y$ reacts. The corresponding (do-)probabilities, obtained by randomized controlled experiments or via the do-calculus, are the basis of Pearl’s definition of CI as well as a measure of the average causal effect [2].

Recently, there has been large interest in generalizing causal analysis to the quantum world [3–22]. One of the most exciting perspectives is to superpose different temporal orders, and hereby create “indefinite causal order”. To that end, process matrices were introduced with separate input and output Hilbert spaces of each laboratory, and the possibility to “wire” CI in opposite directions (e.g. from the output of Alice to the input of Bob or vice versa) [4]. The “quantum switch” was invented [33, 54] and experimentally verified [35, 59]. Here a control qubit enables opposite temporal order of two quantum gates, which can improve the communication capacity of quantum channels, and lead to a computational or metrological advantage [35, 80, 11]. Common to most of these developments is, however, that the actual CI remained unexplored, and the “indefinite causal order” refers to indefinite (i.e. superposed, or mixed) temporal orders. In order to study superpositions of different causal orders $x \rightarrow y$ and $y \rightarrow x$, one needs to define what is meant with a CI $x \rightarrow y$ in quantum mechanics (QM). In [12, 19, 30] definitions of CI in QM based on the Choi-Jamiołkowski representation of a unitary channel propagating Alice’s and Bob’s system were given, following earlier work in [42, 44]. Here we give a clear operational definition of CI based simply on density matrices, generalizing the one by Judea Pearl from classical statistical analysis [1, 2]. We prove a theorem that gives the necessary and sufficient condition for CI in QM and introduce an easily computable measure of CI. We use it to analyse the CI of standard quantum gates, examine some of the measure’s statistical properties, as well as a quantum causal switch that superposes two different CIs. At the example of a 2-spin-boson model, we examine propagation of causal influence. We find that substantial CI arrives only far behind the light-cone, and, surprisingly, almost perfectly in sync with reservoir-induced entanglement.

We take a conservative approach based on standard QM (using density matrices rather than process matrices) and the admission of do-interventions. Probability distributions of classical RVs are replaced by quantum states, since observables have, in general, no determined value in QM until they are measured [45–47]. On the other hand, quantum states encode all that can be known about a quantum system, and hence it is natural to base a theory of CI in QM on states: “causal influence” in QM will be understood in the sense that the final quantum state of the causally influenced system “listens to”, i.e. depends on the initial quantum state of the influencer. Below we make this idea mathematically precise and introduce a measure of CI that we then explore. In principle one could base a definition of CI also on correlation functions. If all correlations are included, this is equivalent to using the quantum state, but at the same time appears to be more cumbersome and less fundamental: fundamentally the world is quantum, and quantum computers will one day probably be able to exchange quantum information without doing measurements. This motivates our attempt to base a definition of CI in quantum mechanics directly on quantum states.

No-causal-influence condition.— Consider two quantum systems $A$ (Alice) and $B$ (Bob) described by their respective density matrices $\rho^A$ and $\rho^B$ and a third system $C$, the joint environment, with a fixed initial state $\rho^C$ that can e.g. correspond to the physical system that propagates the CI and creates an effective interaction, but also leads to decoherence. The joint quantum system lies in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, with $\dim(\mathcal{H}_J) = d_J$, $J \in \{A,B,C\}$,
with finite $d_A$, $d_B$, whereas $d_C$ can be infinite. For further purposes, we define $[d_j] := \{0, \ldots, d_j - 1\}$.

**Definition 1.** Let $A$, $B$, $C$ at initial time $t_0$ be in the state $\rho(t_0) = \rho_B \otimes \rho_{AC}$, where $\rho_B$ is a state set by Bob in a do-intervention. We say that $B$ at time $t_0$ does not causally influence $A$ at time $t$ for a given initial $\rho_A$ if and only if the reduced state $\rho_A^B$ of the system $A$ after the propagation from $t_0$ to $t$ is independent of $\rho_B$ for any density matrix $\rho_B$. If this condition is fulfilled for any initial $\rho_A$, we say that system $B$ at time $t_0$ does not causally influence system $A$ at time $t$, shortly denoted by $B(t_0) \not\rightarrow A(t)$. Otherwise we say that $B$ at time $t_0$ causally influences $A$ at time $t$, denoted by $B(t_0) \rightarrow A(t)$. Analogously one defines $A(t_0) \rightarrow B(t)$ and $A(t_0) \rightarrow B(t)$, leading to a total of four possible cases.

Note that if one wants to find out whether there is a CI from $A$ at time $t_0$ to $B$ at time $t$, one needs an initial state $\rho(t_0) = \rho_A^B \otimes \rho_{BC}$. If one wants to find out if there is a CI from $A$ at time $t_0$ to $B$ at time $t$ or from $B$ at time $t_0$ to $A$ at time $t$, the initial state created by a do-intervention on either side factors completely, $\rho(t_0) = \rho_A \otimes \rho_B \otimes \rho_C$, which is the form we assume from now on. The initially factoring states avoids the problem that for an initially entangled state a measurement by Bob collapses the state also on Alice’s side and would signal CI, while it is well known that in this scenario no information can be transmitted.

When we are not concerned with time dependence we may skip the arguments $t$, $t_0$ and simply consider initial and final states of a joint evolution. Def. 1 of no CI is similar to the one of a “semicausal map” in \[\text{[12]},\] but is based on the dependence of a final state directly on an initial state rather than a local channel. The generalization of Def. 1 as well as of Theorem 1 and expression (3) below to $N$ qubit systems is given in the Supplemental Material (SM).

For finite $d_C$, we consider the evolution of $A$, $B$, $C$ via a joint unitary transformation $U ∈ U(d_A d_B d_C)$. The uncorrelated initial state is mapped to $U(\rho_A \otimes \rho_B \otimes \rho_C)^U$. Using from now on Einstein’s sum convention, we write the initial states as $\rho_A = \rho_A^i |i⟩⟨i|$, and $\rho_B = \rho_B^k |k⟩⟨l|$, for $i, j ∈ [d_A], k, l ∈ [d_B]$, and take $\rho_C = |0⟩C$. Then, Alice’s final reduced state is $\rho_A^U = \rho_A^i U_{i′k}^A U_{k′l}^B U_{l′j}^C |0⟩C$,\label{eq:rhoA}

where the prime indices run over the same range as the respective non-prime and $m′ ∈ [d_C]$ and thus we see that $B \rightarrow A$ if (1) does not depend on $\rho_B$.

**Theorem 1.** Let $A$ and $B$ be quantum systems in a common environment with initial state $|0\rangle_{E}$, and let $F^{U}_{k,l}(|i^{'}, j^{'}) := U_{i′k}^A U_{k′l}^B U_{l′j}^C |0⟩C$. Then, $B \rightarrow A$ if and only if, for all $i, j, k, l, i’, j’$, (no sum over $k$)

$$F^{U}_{k,l}(|i^{'}, j^{'}) = δ_{kl} F^{U}_{k,k}(|i^{'}, j^{'}) \label{eq:correlation}$$

The statement for $A \rightarrow B$ is analogous with the function $\hat{F}^{U}_{i,j}(|k^{'}, l^{'}) = U_{i′k}^A U_{k′l}^B U_{l′j}^C |0⟩C$, replacing $F^{U}_{k,l}(|i^{'}, j^{'})$. On the other hand, if the evolution of the initial uncorrelated state is given by Kraus operators $K^{\mu}, \rho_B^A K = K^{\mu} \rho_B^A K^{\nu}$, Theorem 1 holds with $F^{U}_{k,l}(|i^{'}, j^{'}) := K^{\mu}_{k,m} K^{\nu}_{l,n} |0⟩C$. The proof of Theorem 1 is based on straightforward linear algebra and is given in the SM.

**Mutuality.**—As an example, consider $U = U_A \otimes U_{BC}$, for $U_A ∈ U(d_A), U_{BC} ∈ U(d_B d_C)$, the unitary group. Then $F^{U}_{k,l}(|i^{'}, j^{'})$ becomes $δ_{kl} U_{i′j}^A U_{j′i}^A$, such that condition (2) is fulfilled, and therefore, by Theorem $1, B \not\rightarrow A$ as expected. Similarly one finds $A \rightarrow B$. However, tensor products of unitary matrices are a set of Haar measure zero in $U(d_A d_B d_C)$. Unitary matrices that allow one-way CI exist as well. An example where $A \rightarrow B$, is given by the unitary transformation $U(\text{132})$ corresponding to the permutation $|ikm⟩ \rightarrow |kim⟩$. See SM for another example which does not correspond to a permutation. Among $10^5$ Haar distributed random unitary $2^3 \times 2^3$ matrices via QuTip \[\text{[45]},\] all of them permitted CI in both directions, which is the generic situation.

**No transitivity.**—Let $A$, $B$, and $C$ be quantum systems and $t_0 < t_1 \leq t_2$. $A(t_0) → B(t_1)$ and $B(t_1) → C(t_2)$ does not imply $A(t_0) → C(t_2)$. As a counterexample, consider that at $t_1$ one applies the CNOT gate with $B$ the control and $A$ the target to the initial state $\rho_A^B \otimes \rho_B \otimes \rho_C$. The reduced density matrix of $B$, $\rho_B(t_1)$, depends on $\rho_A^B$ only in the off-diagonal elements (see SM). Similarly, starting at time $t_1$ with a product state $\rho_A^B \otimes \rho_B \otimes \rho_C$, if one applies at $t_2$ the CNOT gate with $C$ the target system and $B$ the control, one easily shows $B(t_1) → C(t_2)$, but the reduced density matrix of system $C$ only depends on the diagonal components of $\rho_B$, which do not depend on $\rho_A^B$ after the first CNOT, and therefore $A(t_0) \rightarrow C(t_2)$. However, one can easily construct corresponding examples in the classical case. Hence, CI must be distinguished from implications as if these are transitive.

**Measure of the causal influence.**—Consider the two qubit systems $A$ (control) and $B$ (target) and apply the CNOT gate on a pure state. One might suspect that CNOT permits only influence from $A$ to $B$. However, applying Theorem 1, we see that both CI directions are allowed. This is not merely a mathematical consequence of the definition but corresponds to a well-known and real physical effect called phase kickback in quantum circuits \[\text{[13]},\] where the phase $β$ in the initial state $(|0⟩ + |1⟩) \otimes (|0⟩ + e^{i\beta} |1⟩)/2$, ends up in the state of Alice. More generally, in the SM we give the reduced state after the CNOT gate for both parties for an arbitrary initial product state and one sees that in both cases these depend on components of their partner’s initial state. Nevertheless, while Alice’s final state only carries Bob’s dependence in the off-diagonal components (see (C.1) in the SM), all his components after the CNOT depend on Alice’s initial state. This is in sync with the intuition
that the CI from control to target is ‘stronger’ than the other way round, and motivates the introduction of a measure of CI. Although one could argue that a CNOT with reversed roles of control and target is a CNOT conjugated with local Hadamard transformations and hence expect equal influence in both directions, we prove below that it is reasonable not to request invariance of a measure of CI under local pre-propagation on the influencing system, as varying its initial state is part of the process of examining the influence. A natural way to quantify the CI is via

$$\frac{\partial \rho_{A'}}{\partial \rho_{B'}}$$

for $h, f \in [d]$. We base our definition on all pure initial states $\rho^A$ of Alice. Note that due to linear propagation the $\rho_{A'}^B$ are holomorphic functions of the $\rho_B^B$ such that these complex derivatives are always well-defined.

For finite $d$, this measure average $E[I_{A \rightarrow B}(U)] = \int d\rho(U') \sum_{h,f} |\partial \rho_{A'}^B/\partial \rho_{B'}^B|^2$.

Analogously one defines $I_{A \rightarrow B}$. Notice that for either $A$ or $B$ corresponding to the trivial (1-dimensional) Hilbert space, there is a single pure state so the derivatives vanish and the influence is always 0. Therefore we restrict ourselves to $d_A, d_B > 1$. Based on an average rather than a maximization procedure, definition 2 allows for straightforward evaluation. While not all do-interventions are physically realizable [21], both definitions [22] can be applied experimentally, as long as Alice and Bob can generate arbitrary local states, e.g. through measurement and experimentally, as long as Alice and Bob can generate arbitrary local states, e.g. through measurement and

$$I_{B \rightarrow A} = \int d\rho(V) \sum_{h,f} |\partial \rho_{A'}^B/\partial \rho_{B'}^B|^2.$$  (3)

2. $I_{B \rightarrow A}((U_A^A \otimes U_B^B \otimes I_C^C)U(U_A^A \otimes I_B^B)) = I_{B \rightarrow A}(U)$, for all $U_A^A, U_A^A \in U(d_A), U_B^B \in U(d_B) U \in U(d_A d_B d_C)$, and analogously for $I_{A \rightarrow B}$. These properties are natural: after the propagation, any local action should not change the CI. And since we consider all initial states of Alice, the measure should be invariant under her local pre-propagation unitaries. On the other hand, since the changes of $\rho^B$ reflect Bob’s do-interventions, $I_{B \rightarrow A}$ need not be invariant under local unitaries of Bob before the propagation, and indeed, in general $I_{B \rightarrow A}(U(U_A^A \otimes U_B^B)) \neq I_{B \rightarrow A}(U)$. 3. The natural scale of $I_{B \rightarrow A}$ is given by the Haar-measure average $E[I_{B \rightarrow A}(U)] = \int d\rho(U) I_{B \rightarrow A}(U)$. $E[I_{B \rightarrow A}(U)] = \frac{d_B - 1}{d_A d_B d_C - 1}[2d_A^2 d_B^2 d_C - 2d_A^2 d_C] + d_A(d_B - 2)^2(d_B^2 d_C - 1).$ (5)

$E[I_{A \rightarrow B}(U)]$ is obtained by permuting $A \leftrightarrow B$. Fig. 1(b) illustrates the behaviour of $E[I_{A \rightarrow B}(U)]$. Notice that it is strictly increasing with $d_B$, which is reasonable as one expects that the larger the Hilbert space dimension of the influencing system, the more it can influence. A particular case of interest is such that either one of the systems $A$ or $B$, or the environment has a dimension much greater than the others. $E[I_{B \rightarrow A}(U)]$ tends to $(d_B - 1)/d_C$, $\infty$ and $(d_B - 1)^2/d_A$ in these limits, respectively. Fig. 2 shows a histogram of $I_{A \rightarrow B}$ for random Haar generated unitary matrices. The relatively narrow distribution confirms that $E[I_{B \rightarrow A}(U)]$ represents a good scale for $I_{B \rightarrow A}(U)$ for given Hilbert space dimension. For three qubits, there are gates such as CNOT or Csgn with almost twice the value of $E[I_{B \rightarrow A}(U)]$.

Quantum causal switch.—In analogy to the quantum switch [34] one can create a unitary evolution $U_{sup}$ that superposes the CI $A \supset B$ and $A \supset B$, controlled by an ancilla qubit $|\phi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi} \sin(\theta/2)|1\rangle$, for $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$. In the SM we investigate the behavior of such a “quantum causal switch” and show how one can interpolate continuously between the two directions of the CI as func-

| $I_{B \rightarrow A}$ | $I_{A \rightarrow B}$ |
|-----------------|-----------------|
| CNOT | 0.66 | 1.33 |
| Csgn | 1.33 | 1.33 |
| $\sqrt{SWAP}$ | 1.66 | 1.66 |
| Fredkin | 0.33 | 1.00 |
| Toffoli | 0.33 | 0.33 |

FIG. 1. (Color online) (a) CI for some quantum gates. The environment for the three-qubit gates is initially in state $|0\rangle_c$, and, when required, $A$ is the control qubit and $B$, the target. (b) $E[I_{A \rightarrow B}]$ as function of $d_A d_B$ and $d_C$ (eq. (3)), where $d_A$ is coded in the sequence of points for each $d_B, d_C$ with $2 \leq d_A \leq 6$ from top to bottom (3 $\leq d_B \leq 8$) or bottom to top ($d_B = 2$).
tion of the state of the control qubit. For almost all states of the control qubit, the CI is in both directions.

Propagator of CI — In order to illustrate the propagation of CI within an exactly solvable physical model, compare it to the well-known creation of reservoir-induced entanglement, and to verify that indeed entanglement does not arrive before the CI, of reservoir-induced entanglement, and to verify that ical model, compare it to the well-known creation propagation of CI within an exactly solvable phys-

periodically oscillates as function of \( \phi \) and is not \( \phi \) for all CI generates entanglement. For values of \( \phi \) and \( \varphi \) such that there is no causal influence, no entanglement between the qubits is generated. However, the converse is not true, i.e. not all CI generates entanglement. \( I^\text{path}_{B \rightarrow A} \) takes a maximum value of 4/3, periodically oscillates as function of \( \varphi \), and decays ex-

potentially for large \( f \) but does not vanish exactly, whereas the generated entanglement does vanish exactly as the state approaches the fully mixed state [52, 53].

For the physical example of two double quantum dots (DQD) at distance \( x \) from each other coupled with dipole interaction to black-body radiation, with a UV frequency cut-off \( y_m = \omega_{\text{max} \tau} \), where \( \tau = h/k_B T \) and \( T \) is the temperature, the functions \( f(t) \) and \( \varphi(t) \) can be obtained analytically (see SM) if one approximates \( \text{coth} \simeq 1 \) for \( y_m \gg 1 \) [53]. Physically, it is clear that CI can only arise inside the light-cone, \( x = ct \), and indeed, the CI plotted for this system in Fig. 3 shows that there is no CI for space-like separated points, \( t < x/c \). Surprisingly, however, significant CI is generated only far behind the light-cone, namely for \( t \gtrsim 10^{12}(x/c)^3 \). This is reminiscent of reservoir-induced entanglement [53, 57] that also arises only far behind the light-cone [55] (i.e. “Entanglement harvesting” [58–68], i.e. entanglement creation outside the light-cone, is typically not possible here without “reservoir engineering”). In fact, the space-time dependence of CI and entanglement of formation (EOF) \( E \), created as long as both initial states of \( \rho^{AB} \) and \( \rho^{CD} \) contain components of both \( |0\rangle \) and \( |1\rangle \), are almost perfectly in sync, see Fig. 3b). Minimization of \( \sum_{i,j} \rho_{ii,jj}^2 \) over a regular grid of 51 x 51 points in the space-time regions shown with \( \delta = \lambda_{B \rightarrow A} - E \) gives \( \lambda \approx 0.795481 \), and the remaining differences are less than about 0.05 in absolute value over the 12 orders of magnitude of \( x/c \) considered. Under time reversal, \( t \rightarrow -t \), \( f(t) \) remains invariant, whereas \( \varphi(-t) = -\varphi(t) \), which leads to complex conjugate matrix elements of \( \rho^{AB} \) as usual. Formally there is hence exactly as much CI in positive time direction as in negative time direction. This can be seen already from 2 with \( \rho^{AB} \rightarrow \rho^{AB^*} \) under time-reversal. Hence, reasons for the apparent purely forward CI in Nature must be sought outside quantum mechanics [69]. Indeed, causality is, even in our most fundamental established theories, implemented by hand by choosing advanced Green’s functions only.

FIG. 2. (Color online) Histogram of \( I_{B \rightarrow A} \), for \( 10^4 \) random Haar distributed unitary matrices of dimension \( D = d_Ad_Bd_C \), for \( D = 2 \cdot 2 \cdot 2 \) (light blue) and \( D = 3 \cdot 2 \cdot 2 \) (dark blue), their mean values are 0.76 and 0.896, the standard deviations 0.19 and 0.095, and their expected values (eq. (5)) 16/21 \( \simeq 0.76 \) and 128/143 \( \simeq 0.895 \), respectively.

FIG. 3. (Color online) (a) \( I_{B \rightarrow A} \) for two initially non-interacting DQD with \( d = 10 \text{nm} \) coupled to the black body radiation at \( T = 2.73 K \) with \( \omega_m = 4250 \ (\text{nm} = 1 \text{ ev}) \), parametrized by \( x/c \) and \( t \). The corresponding plot for the entanglement of formation looks almost identical (see Fig. 2 in [50]). Dotted line: \( t = 10^{11}(x/c)^3 \). Full line: light-cone \( ct = x \). (b) Difference \( \delta = 0.795481 I_{B \rightarrow A} - E \), see text, \( E \) for the initial state \( |0\rangle \kron [0\rangle + |1\rangle \kron [0\rangle + [1]\rangle \rangle/2 \). Same parameters as in (a).
In summary, we gave a definition of causal influence (CI) in the quantum world based on reduced density matrices. We derived a necessary and sufficient condition on the joint evolution operator of Alice, Bob, and an environment that a given quantum system can causally influence another one. Moreover, we introduced a measure of the CI, analysed it in detail, showed the possibility of superposing opposite directions of CI, and applied it to particular cases of both finite and infinite dimensional environments. For the example of two degenerate double-quantum dots at distance $x$ dipole-interacting with thermal black body radiation we found that the space-time dependence of the CI is almost perfectly in sync with the reservoir induced entanglement. Both arrive long after ($t \propto (x/c)^3$) the light cone $ct = x$. Just as entanglement measures, classicality measures and measures based on other resource theories have had a large impact in theoretical physics for the last three decades. Similarly, we hope that having a causality measure in quantum mechanics opens the path to study many new things, starting from its properties, over Bell-like inequalities, to the relationship to entanglement hardentment measures, classicality measures and measures of CI in theoretical physics for the last three decades.

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SUPPLEMENTARY MATERIAL

A. PROOF OF THEOREM 1

Let \( \alpha_{ij}(i',j') := \rho_{B}^{\mathcal{U}}_{k'k'ij} \). Then from (1), \( \rho_{ij}^{A} = \rho_{ij}^{A}(i',j') \). Imposing \( B \rightarrow A \) for all \( \rho^{A} \), \( B \rightarrow A \) must hold in particular for \( \rho_{ij}^{A} = \delta_{ij} \delta_{il} \) and thus, \( \rho_{ij}^{A} = \alpha_{ij}(i',j') \) must be independent of \( (\rho^{B}) \) \( \forall i_{0} \). Since \( i_{0} \) is arbitrary \( \in \{d_{A}\} \), we need that \( \alpha_{ij}(i',j')(\rho^{B}) \) must hold for all \( \rho^{B} \). Moreover, consider the previous density matrix but with two non-vanishing off-diagonal components \( \rho_{ij}^{A} \) and, due to Hermiticity, \( \rho_{ji}^{A} \), for \( i \neq j \). Then, imposing the independence condition, it is necessary that

\[
(\rho_{ij}^{A}(i',j') + \rho_{ji}^{A}(i',j'))(\rho^{B})
\]

(7)

(without implicit sum). Thus, writing \( \rho_{ij}^{A} \) as the sum of its real part and \( \sum_{l \neq k} \rho_{kl}^{B} F_{kl,ij}(i',j') \), as their difference, one sees that it is necessary that \( \alpha_{ij}(i',j')(\rho^{B}) \) and \( \alpha_{ij}(i',j')(\rho^{B}) \), leading to the conclusion that \( \alpha_{ij}(i',j')(\rho^{B}) \) must hold for all \( i, j, i', j' \). Applying the definition of \( F_{kl,ij}(i',j') \),

\[
\alpha_{ij}(i',j') = \rho_{kk}^{B} F_{kk,ij}(i',j') + \sum_{l \neq k} \rho_{kl}^{B} F_{kl,ij}(i',j'),
\]

(8)

its independence of \( \rho^{B} \) is fulfilled if and only if

\[
F_{kk,ij}(i',j') = F_{kk,ij}(i',j') \quad \text{and} \quad F_{kl,ij}(i',j') = 0,
\]

(9)

for all \( k \neq l, k, i, j, i', j' \). The first condition comes from the trace one of the density matrices, thus, any \( \rho_{kk}^{B} \)

\[
\sum_{k \neq k_{0}} \rho_{kk_{0}}^{B} F_{kk_{0},ij}(i',j') + (1 - \sum_{k \neq k_{0}} \rho_{kk_{0}}^{B}) F_{kk_{0},ij}(i',j').
\]

(10)

Since \( \rho_{kk}^{B} \) for \( k \neq k_{0} \) can vary independently, it is necessary that \( F_{kk,ij}(i',j') = F_{kk_{0},ij}(i',j') \) for all \( k \), then

\[
\sum_{k} \rho_{kk}^{B} F_{kk,ij}(i',j') = F_{kk_{0},ij}(i',j') \sum_{k} \rho_{kk}^{B} = F_{kk_{0},ij}(i',j') \text{Tr} [\rho^{B}] = F_{kk_{0},ij}(i',j').
\]

(11)

Finally, the second condition comes from a similar argument derived for the independence of \( \alpha_{ij}(i',j') \) with \( i \neq j \).

B. EXAMPLE OF A SINGLE DIRECTION OF INFLUENCE

The unitary matrix \( U(2 \cdot 2 \cdot 2) \) is such that \( A \rightarrow B \) but \( B \rightarrow A \), i.e., it only allows one influencing direction. Moreover, \( I_{A \rightarrow B}(U) = 1.5 \) and \( I_{B \rightarrow A}(U) = 0.0 \).

\[
U = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

(12)

C. QUBIT DENSITY MATRICES AFTER THE APPLICATION OF THE CNOT GATE

Writing Alice's and Bob's initial states in the computational basis so that \( \rho^{A} = \rho_{ij}^{A} |i\rangle \langle j| \) and \( \rho^{B} = \rho_{kl}^{B} |l\rangle \langle k| \), for \( i, j \in [2], k, l \in [2] \), the joint state after the application of the CNOT gate to the state \( \rho^{A} \otimes \rho^{B} \) is given by

\[
CNOT(\rho^{A} \otimes \rho^{B})(CNOT)^{\dagger} \quad \text{and, tracing out systems } B \text{ and } A, \text{ respectively, one has}
\]

\[
\rho^{A} = \begin{pmatrix}
\rho_{00}^{A} & 0 & \rho_{01}^{A} & 0 & 0 & 0 \\
0 & \rho_{10}^{A} & 0 & \rho_{11}^{A} & 0 & 0 \\
\rho_{00}^{A} & \rho_{01}^{A} & 0 & 0 & 0 & 0 \\
0 & \rho_{10}^{A} & \rho_{11}^{A} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\rho^{B} = \begin{pmatrix}
\rho_{00}^{B} & \rho_{01}^{B} & 0 & 0 & 0 \\
\rho_{10}^{B} & \rho_{11}^{B} & 0 & 0 & 0 \\
0 & 0 & \rho_{00}^{B} & \rho_{01}^{B} & 0 \\
0 & 0 & \rho_{10}^{B} & \rho_{11}^{B} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

(13)
D. INTEGRATION OVER THE UNITARY GROUP

Using the definition of $D_{kl,hf}$, the CI (3) can be written as

$$I_{B\to A}(U) = \int d\mu(V) \sum_{h,j'} |V_{i0}V_{j0}^* F_{kl,j}(i',j') D_{kl,hf}|^2. \tag{14}$$

Expanding the square, one has order 2 terms of the functions $F$ and $D$ and integrals of the type $\int d\mu(V)V_{i0}^*V_{j0} V_{i0}V_{j0}$. In [50] is shown that the integrals $\int d\mu(U)U_{i1,j1}^*...U_{p+1,j1}^* U_{k1,l1}...U_{kq,lq}$ denoted by $\langle I,J|K,L \rangle$ vanish unless $q = p$, which will be assumed to be the case. Even more, such integrals are non-vanishing if, in addition, $K = I_{\sigma_1}$ and $L = J_{\sigma_2}$, for $\sigma_1, \sigma_2 \in S_p$, and since $U_{i1,j1}...U_{p+1,j1} = U_{(i1)\tau(1)}...U_{(p)\tau(p)}$ for any $\tau \in S_p$ due to the fact that it is a multiplication of complex numbers, we may assume $\sigma_1 = id$, then the non zero integrals are of the form $\langle I,J|I,J_{\sigma_2} \rangle$. Since the Haar measure is invariant under transposition, a permutation between rows and columns does not change the integral, i.e. $\langle I,J|K,L \rangle = \langle J,I|L,K \rangle$. Furthermore, the integral is affected by whether the indices take on the same or different values, but independent of what these values are, for this reason it is convenient to use the graphical representation introduced in [50]:

1. The distinct values in the index set $I$ are represented as dots in a column and, on its right, the distinct values of the index set $J$ as dots in a column (since $J_{\sigma_2}$ is a permutation of $J$, it has the same distinct values as $J$).

2. Factors $U_{i',j'}$, and $U_{i',\sigma_2(j')}$, for $r = 1,...,p$, are represented by thin (solid) and dotted lines, respectively. The power of the matrix element, if greater than 1, will be represented above the solid line or below the dotted line, correspondingly. When a pair $U_{i',j'}, U_{i'',j''}$ occurs together, the thin and dotted lines will be replaced by a thick solid line, whose multiplicity will be understood as the power of this pair.

Using this graphical representation, the non-vanishing integrals $I$ of $p = 2$ are those shown in Figure[4] together with their values. Then, the non-vanishing integrals $\int d\mu(V)V_{i0}^*V_{j0}^* V_{i0}V_{j0} = \langle ji_1,00|ji_1,00 \rangle$, for $V \in U(d_A)$, are such that $(i,j_1) = \sigma(j_1,i_1)$, for $\sigma \in S_2$, i.e. either

1. $(i,j_1) = (j,i_1)$, so that the integral is $\langle ji_1,00|ii_1,00 \rangle$, and it takes the values

$$\langle ji_1,00|ji_1,00 \rangle = \begin{cases} \frac{I(c)}{d_A d_A + 1} & \text{if } j = i_1, \\ \frac{I(d)}{d_A d_A + 1} & \text{if } j \neq i_1, \end{cases} \tag{15}$$

whose value can be compactly written as $\frac{1+\delta_{i_1,j_1}}{d_A d_A + 1}$.

2. or $(i,j_1) = (i_1,j)$, and, since the integral is independent of what the values of the indices are, it takes the same value as in case 1.

Thus, considering both options and not overcounting the intersection,

$$\int d\mu(V)V_{i0}^*V_{j0}^* V_{i0}V_{j0} = \delta_{i_1,j_1} \delta_{j_1,j} I(c) + (1-\delta_{j_1,j}) I(d) + \delta_{i_1,j_1} \delta_{j_1,j} I(c) + (1 - \delta_{j_1,j}) I(d) - \delta_{i_1,j} \delta_{i_1,j_1} I(c) = \frac{1}{d_A d_A + 1} (\delta_{i_1,j_1} + \delta_{i_1,j_1} \delta_{j_1,j}). \tag{16}$$

E. GENERALIZATION TO $n$ QUANTUM SYSTEMS

Consider $n$ quantum systems described by their respective density matrices $\rho^1,...,\rho^n$ and a system $e$, the joint environment, with a fixed initial state $\rho^e$ corresponding to the physical system that propagates the causal influence and creates an effective interaction, but also leads to decoherence. The joint quantum system lies in the Hilbert space $H_1 \otimes ... \otimes H_n \otimes H_e$, with dim($H_r$) = $d_r$, $r \in \{1,...,n,e\}$. Definition 1 naturally extends as: $l \to k$ if and only if, after the propagation of the initial state $\rho^1 \otimes ... \otimes \rho^n \otimes \rho^e$, the reduced state of system $k$ fulfills that $\rho^k(\hat{\rho})$ for any density matrix $\rho^l$ describing system $l$.

Definition 2 is generalized as: Let $\rho^k = V^k |00\rangle \langle 00| V^{k*}$ for, where $V^k \in U(d_k)$ so that $\rho^k_{i_0,j_0} = V^k_{i_0,j_0} (V^{k*})^\dagger$. Then, the causal influence from $l$ to $k$ is

$$I_{l\to k} = \int d\mu(V^k) \sum_{i',j',i,j} \left| \frac{\partial \rho^k_{i',j'}}{\partial \rho^l_{i,j}} \right|^2. \tag{17}$$
Using Einstein’s sum convention, we write the initial state of system \( i \):

\[ \rho_{i_1 \ldots i_n}^{\rho} = \rho_{i_1 \ldots i_n}^{\rho} U_{i_1 \ldots i_n} \]

Then, the elements of the reduced density matrix of the system \( k \) become

\[ \rho_{i_i' j_j}^k = \rho_{i_1 \ldots i_n}^{\rho} U_{i_1 \ldots i_n} \rho_{i_1 \ldots i_n}^{\rho} U_{i_1 \ldots i_n}^* \]

and, from \([19]\) and \([20]\),

\[ \frac{\partial \rho_{i_i' j_j}^k}{\partial \rho_{i_i'' j_j''}^l} \]

thus,

\[ I_{l \rightarrow k} = \int d\rho^k \rho_{i_i' j_j}^k F_{i_1 \ldots i_n}^U(i_i', j_j) \prod_{r \neq k} \rho_{i_i' j_j}^r F_{i_1 \ldots i_n}^U(i_i', j_j) \sum_{r, r \neq k, l} \rho_{i_i' j_j}^r \rho_{i_i'' j_j''}^l \int d\mu(V^k) \rho_{i_i' j_j}^k \rho_{i_i'' j_j''}^l \]

where the values of the integral is given by \([16]\).

On the other hand, applying an analogous reasoning as in the proof of Theorem 1, one finds the extension of Theorem 1: \( I \rightarrow k \) if and only if

\[ \prod_{r \neq k, l} \rho_{i_i' j_j}^r F_{i_1 \ldots i_n}^U(i_i', j_j) = \delta_{i_i j_j} \prod_{r \neq k, l} \rho_{i_i' j_j}^r F_{i_1 \ldots i_n}^U(i_i', j_j) \]

for all \( i_i, j_j, i'_{i_i}, j'_j \).

**F. PROOF OF PROPERTY 1. OF THE MEASURE: \( I = 0 \) IF AND ONLY IF THERE IS NO CAUSAL INFLUENCE**

By Def 1, if there is no causal influence, all the derivatives in (3) vanish and therefore \( I \) is the integral over 0, which is 0. Conversely, suppose that \( \partial \rho_{i_i' j_j}^A / \partial \rho_{i_i'' j_j''}^B \neq 0 \) for a particular combination of indices \( h, f \) and a specific value of \( \rho_{i_i'' j_j''}^B \). Since \( \rho_{i_i' j_j}^A \) is a linear function of all \( \rho_{i_i'' j_j''}^B \), \( \partial \rho_{i_i' j_j}^A / \partial \rho_{i_i'' j_j''}^B \) is independent of \( \rho_{i_i'' j_j''}^B \). Hence, \( \partial \rho_{i_i' j_j}^A / \partial \rho_{i_i'' j_j''}^B > 0 \) for a set of finite measure, and hence \( I_{B \rightarrow A} \neq 0 \).
G. PROOF OF PROPERTY 2. OF THE MEASURE: \( I_{B \rightarrow A}((U_A^A \otimes U_B^B \otimes 1^C)U(U_A^A \otimes 1^B)) = I_{B \rightarrow A}(U) \).

The invariance follows from the following three invariances:

1. \( I_{B \rightarrow A}((U_A^A \otimes 1^B)U) = I_{B \rightarrow A}(U) \).

   According to (4), the dependence of the measure of the causal influence on the unitary relies on the sum of the functions \( F \). Let \( W := (U_A^A \otimes 1^B) \cdot U \), and consider the sum
   \[
   \sum_{i,j,i',j'} W_{i'k'm',ik0} W^*_{i'k'm',j0} \left( \sum_{k',\tilde{n}'} W_{j'k',\tilde{n}'i0} W^*_{j'k',\tilde{n}'j10} \right). \tag{24}
   \]
   The components of \( W \) are given by \( W_{i'k'm',ik0} = \sum_{\tau_1=0}^{d_A-1} U^A_{\tau_1} U_{\tau_1} k'm',ik0, \) in such a way that (24) becomes
   \[
   \sum_{i,j,k',m'} \left( \sum_{\tau_1=0}^{d_A-1} U^A_{\tau_1} \sum_{i'\tau_1} U^B_{\tau_1} \right) \left( \sum_{\tau_2=0}^{d_A-1} U^A_{\tau_2} U^B_{\tau_2} \right), \tag{25}
   \]
   using that \( U^A \) is a unitary matrix, we have that the sums over \( i' \) and \( j' \) are \( \delta_{r_1,r_3} \) and \( \delta_{r_2,r_4} \), respectively. Therefore, since \( r_1 \) and \( r_2 \) are mute indices, we can replace them by \( i' \) and \( j' \), respectively and, with implicit sum over repeated indices, the sum (24) can be written as
   \[
   U_{i'k'm',ik0} U^*_{i'k'm',j0} U_{j'k'm',j10} = \sum_{i,j,i',j'} \sum_{k',\tilde{n}'} W_{i'k'm',ik0} \rightarrow U_{i'k'm',ik0} U_{j'k'm',j10}, \tag{26}
   \]
   where the dependence on \( U^A \) has vanished, and the expression depends only on the components of \( U \). Therefore, using (26) in equation (4), we have that \( I_{B \rightarrow A}((U_A^A \otimes 1^B) : U) = I_{B \rightarrow A}(U) \). Analogously, one finds the local invariance for \( I_{A \rightarrow B} \).

2. \( I_{B \rightarrow A}(1^B) = I_{B \rightarrow A}(U) \).

   According to Def 2, \( I_{B \rightarrow A} = \int d\mu(V) \sum_{h1'i'j'} |\partial \rho^A_{i'j'}/\rho^B_{hj}|^2 \). Alice’s reduced final state is obtained tracing out systems \( B \) and \( C \) after the evolution, nevertheless both \( (1^B \otimes 1^C)U \) and \( U \) lead to the same reduced state of Alice and therefore \( \partial \rho^A_{i'j'}/\rho^B_{hj} \) remains unaltered, meaning that \( I_{B \rightarrow A}(U) \) is the same in both cases.

3. \( I_{B \rightarrow A}(U(1^A)) = I_{B \rightarrow A}(U) \).

   It follows from the (Haar) integration over all possible initial Alice’s states.

H. PROOF OF PROPERTY 3. \( E[I_{A \rightarrow B}(U)] \)

From equation (4),
\[
I_{B \rightarrow A}(U) = \frac{1}{d_A(d_A+1)} D_{kl,hf} D_{i1,hj}(\delta_{1i1} + \delta_{3i1}) F_{kl,ij}(i',j'), \tag{27}
\]
and, applying the definition of the first moment,
\[
E[I_{B \rightarrow A}(U)] = \int d\mu(U) I_{B \rightarrow A}(U) = \frac{1}{d_A(d_A+1)} \sum_{i,j,i,j} D_{kl,hf} D_{i1,hj}(\delta_{1i1} \delta_{3i1}) \int d\mu(U) F_{kl,ij}(i',j') F_{kl,ij}(i',j'). \tag{28}
\]
Using the definition of the \( F \) functions, the integral over \( U \) can be written as
\[
\sum_{k',\tilde{n}',m',\tilde{n}'} \int d\mu(U) U^*_{i'k'm',j0} U_{j'k'm',ik0} U_{i'k'm',ik0} U_{j'k'm',j10} = \sum_{k',\tilde{n}',m',\tilde{n}'} \langle p_1 p_2, q_1 q_2 | r_1 r_2, s_1 s_2 \rangle, \tag{29}
\]
where the integrals of the form \( \int d\mu(V) V_{i1,l1}^* \cdots V_{i1,l1}^* V_{k1,l1} \cdots V_{k1,l1} \) have been denoted by \( \langle i_1 \ldots i_p, j_1 \ldots j_p | k_1 \ldots k_p \rangle, (l_1 \ldots l_p) := (I, J| K, L), \) and where, for short, it has been introduced the notation...
\[ p_1 = j'k'm', \; p_2 = i'k'm', \; q_1 = jk0, \; q_2 = ijk0, \; r_1 = i'k'm', \; r_2 = j'k'm', \; s_1 = ik0 \text{ and } s_2 = jk10. \] The non-vanishing integrals are those such that \((r_1, r_2) = \sigma_1(p_1, p_2)\) and \((s_1, s_2) = \sigma_2(q_1, q_2)\), for \(\sigma_1, \sigma_2 \in S_2 = \{id, \sigma\}\).

Denote \(P = (p_1, p_2), \ Q = (q_1, q_2), \ R = (r_1, r_2)\) and \(S = (s_1, s_2)\), so that the order 2 integrals are of the form \(\langle P, Q|R, S \rangle\) and we split them into the four (three non-equal) following (non-disjoint) sets of integrals: \(\langle P, Q|P, S \rangle, \langle P, Q|R, Q \rangle\) and \(\langle P, Q|P, Q \rangle = \langle PQ|P_\sigma Q_\sigma \rangle\). From set theory, given three sets \(I_1, I_2\) and \(I_3\) the number of elements of its union is
\[
|I_1 \cup I_2 \cup I_3| = |I_1| + |I_2| + |I_3| - |I_1 \cap I_2| - |I_1 \cap I_3| - |I_2 \cap I_3| + |I_1 \cap I_2 \cap I_3|.
\] (30)

Thus, we can write the order 2 integrals in terms of the four (three non-equal) sets of integrals stated above:
\[
\langle P, Q|R, S \rangle = \delta_{PR} \langle P, Q|P, S \rangle + \delta_{QS} \langle P, Q|R, Q \rangle + \delta_{PR} \delta_{QS} \langle P, Q|P_\sigma Q_\sigma \rangle - \delta_{PR} \delta_{QS} \langle P, Q|P, P_\sigma \rangle - \delta_{QS} \delta_{PR} \delta_{QS} \langle P, Q|P, Q_\sigma \rangle - \delta_{QS} \delta_{PR} \delta_{QS} \langle P, Q|P_\sigma Q_\sigma \rangle - \delta_{PR} \delta_{QS} \delta_{QS} \langle P, Q|P_\sigma Q_\sigma \rangle.
\] (31)

The integrals in (31), in terms of integrals depending on \(p_1, p_2, q_1\) and \(q_2\), are

1. \(\langle P, Q|P, S \rangle\), which can be written as
\[
\langle p_1p_2, q_1q_2|p_1p_2, s_1s_2 \rangle = \delta_{q_1s_1} \delta_{q_2s_2} \langle p_1p_2, q_1q_2|p_1p_2, q_1q_2 \rangle + \delta_{q_1s_2} \delta_{q_2s_1} \langle p_1p_2, q_1q_2|p_1p_2, q_1q_2 \rangle
- \delta_{q_1s_2} \delta_{q_1s_1} \langle p_1p_2, q_1q_1|p_1p_2, q_1q_1 \rangle,
\] (32)

2. \(\langle P, Q|R, Q \rangle\), which takes the same value as \(\langle Q, P|Q, R \rangle\), and therefore it is computed as in 1,

3. \(\langle P, Q|P_\sigma Q_\sigma \rangle = \langle P, Q|P, Q \rangle = \langle p_1p_2, q_1q_2|p_1p_2, q_1q_2 \rangle\),

4. \(\langle P, Q|P_\sigma Q_\sigma \rangle = \langle p_1p_2, q_1q_2|p_1p_2, q_2q_1 \rangle\), and

5. \(\langle P, Q|P_\sigma Q_\sigma \rangle\), which takes the same value as \(\langle Q, P|P_\sigma Q_\sigma \rangle\), and therefore it is computed as in 4.

Therefore, we are left with the values of \(\langle p_1p_2, q_1q_2|p_1p_2, q_1q_2 \rangle\) and \(\langle p_1p_2, q_1q_2|p_1p_2, q_2q_1 \rangle\). From Fig. 4 we have that:
\[
\langle p_1p_2, q_1q_2|p_1p_2, q_1q_2 \rangle = \begin{cases} I(c) & \text{if } p_1 = p_2 \text{ and } q_1 = q_2, \\ I(c) & \text{if } p_1 = p_2 \text{ and } q_1 \neq q_2, \\ I(d) & \text{if } p_1 \neq p_2 \text{ and } q_1 = q_2, \\ I(a) & \text{if } p_1 \neq p_2 \text{ and } q_1 \neq q_2, \end{cases}
\] (33)

and
\[
\langle p_1p_2, q_1q_2|p_1p_2, q_2q_1 \rangle = \begin{cases} I(c) & \text{if } p_1 = p_2 \text{ and } q_1 = q_2, \\ I(c) & \text{if } p_1 = p_2 \text{ and } q_1 \neq q_2, \\ I(d) & \text{if } p_1 \neq p_2 \text{ and } q_1 = q_2, \\ I(b) & \text{if } p_1 \neq p_2 \text{ and } q_1 \neq q_2. \end{cases}
\] (34)
Combining (31), (32), (33) and (34), a generic integral of order 2 can be written as

\[
\langle p_1 p_2, q_1 q_2 | r_1 r_2, s_1 s_2 \rangle = \delta_{p_1 r_1} \delta_{p_2 r_2} \left( \delta_{q_1 s_1} \delta_{q_2 s_2} \left\{ \frac{1 + \delta_{p_1 p_2}}{D(D + 1)} + (1 - \delta_{q_1 q_2}) \left[ \frac{\delta_{p_1 p_2}}{D(D + 1)} + \frac{1 - \delta_{p_1 p_2}}{D^2 - 1} \right] \right\} \right)
\]

\[
+ \delta_{q_1 s_2} \delta_{q_2 s_1} \left\{ \frac{1 + \delta_{q_1 q_2}}{D(D + 1)} + (1 - \delta_{q_1 q_2}) \left[ \frac{\delta_{p_1 p_2}}{D(D + 1)} - \frac{1 - \delta_{p_1 p_2}}{D^2 - 1} \right] \right\}
\]

\[
+ \delta_{q_1 s_2} \frac{1 + \delta_{q_1 q_2}}{D(D + 1)} + (1 - \delta_{q_1 q_2}) \left[ \frac{1 - \delta_{p_1 p_2}}{D^2 - 1} \right]
\]

\[
+ \delta_{p_1 r_2} \delta_{p_2 r_1} \left\{ \frac{1 + \delta_{q_1 q_2}}{D(D + 1)} - \frac{1 - \delta_{p_1 p_2}}{D^2 - 1} \right\}
\]

and plugging it on equation (29) and subsequently in (28), we obtain the expected value of the measure of causal influence.

Let \( \langle \ldots \rangle \) denote the integral \( \langle p_1 p_2, q_1 q_2 | r_1 r_2, s_1 s_2 \rangle \) assuming that the indices are written in terms of the original ones, i.e. \( p_1 = j'k'm' \), etc. and denote the same integral but with a certain index taking the same value as another index writing the equality of indices inside \( \langle \ldots \rangle \) instead of the three dots, i.e. the integral \( \langle p_1 p_2, q_1 q_2 | r_1 r_2, s_1 s_2 \rangle \) for \( i_1 = i \) will be written as \( \langle i_1 = i \rangle \). Then, from equation (28), and denoting by \( \lambda \) the set of indices \( i'j'k' \), we have

\[
d_A (d_A + 1)^E [I_{B \rightarrow A} (U)] = \sum_{\lambda_{ij} j'k' \cdot h f} D_{h f} (\delta_{i_1, j_1} + \delta_{i_1, j_1}) \langle \ldots \rangle
\]

\[
= \sum_{\lambda_{ij} j'k' \cdot h f} D_{h f} (\delta_{i, j} = j) + \sum_{\lambda_{ij} j'k' \cdot h f} D_{h f} (\delta_{j, i} = i_1).
\]

Since, for our purpose, the two terms in (36) can be treated analogously, we will show the computations for the first term. We separate the cases where \( h \neq f \) and \( h = f \). In the former case \( D_{h f} \) takes the value \( \delta_{h f} d_{j f} \) and in the latter, \( (-1)^{\delta_{h 0}} (\delta_{h 0} - 1) \delta_{h f} d_{j f} \). Thus, the first term summed in the above expression can be written as

\[
\sum_{\lambda_{ij} j'k' \cdot h f} \sum_{f : f \neq h} \delta_{h f} d_{j f} \delta_{k h} \delta_{l f} \langle i_1 = i, j_1 = j \rangle + \sum_{\lambda_{ij} j'k' \cdot h f} \sum_{f : f = h} (-1)^{\delta_{k h} + \delta_{k 0}} (1 - \delta_{h 0}) \delta_{h f} d_{j f} \delta_{k h} \langle i_1 = i, j_1 = j \rangle.
\]

The first term of the sum (37) can be written as

\[
\sum_{\lambda_{ij} j'k' \cdot h f} \langle i_1 = i, j_1 = j, k = k_1 = h, l = l_1 = f \rangle - \sum_{\lambda_{ij} j'k' \cdot h f} \langle i_1 = i, j_1 = j, k = k_1 = l = l_1 = f = h \rangle.
\]

and the second term is simplified as

\[
(d_B - 1) \sum_{\lambda_{ij} j'k' \cdot h f} (-1)^{\delta_{h 0} + \delta_{k 0}} \langle i_1 = i, j_1 = j, l = k, l_1 = k_1 \rangle.
\]

Using Mathematica, computing the integrals via (37), one obtains the sums in (38) and (39), e.g. the first term in (38) is \( \frac{d_A^2 d_B}{h f} (d_0^2 + d_{h 0} - 1)(d_0^2 + d_{h 0} - 1) - 1 \), and (5) is recovered. \( E [I_{B \rightarrow A} (U)] \) is obtained by permuting \( A \leftrightarrow B \), i.e. \( E [I_{A \rightarrow B} (U)] (d_A, d_B, d_C) = \tau (E [I_{B \rightarrow A} (U)] (d_A, d_B, d_C)) \), where \( \tau \) is the permutation \( \tau (A) = B, \tau (B) = A \).
I. CAUSAL QUANTUM SWITCH

Consider unitary evolutions $U^{A \rightarrow B}$ and $U^{A \leftarrow B}$ that permit a single influence direction indicated in their superscript. In analogy to the quantum switch one can create a unitary evolution $U^{\text{sup}}$ that superposes the CI $A \rightarrow B$ and $A \leftarrow B$, controlled by an ancilla qubit $|\chi_c\rangle = \cos(\theta/2) |0\rangle + e^{i\phi} \sin(\theta/2) |1\rangle$, for $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi)$.

$$U^{\text{sup}} = |0\rangle\langle 0| \otimes U^{A \rightarrow B} + |1\rangle\langle 1| \otimes U^{A \leftarrow B}. \quad (40)$$

Fig. 5 shows the CI from $A$ to $B$ and vice-versa, with $d_A = d_B = 2$, for $U_1^{\text{sup}} = |0\rangle\langle 0| \otimes U^{(123)} + |1\rangle\langle 1| \otimes U^{(132)}$ depending on the control qubit, where one sees that if $|\chi_c\rangle = |0\rangle$, $I(U^{\text{sup}}, |\chi_c\rangle) = I(U^{(123)})$ and if $|\chi_c\rangle = |1\rangle$, $I(U^{\text{sup}}, |\chi_c\rangle) = I(U^{(132)})$, for $I \in \{I_{A \rightarrow B}, I_{B \rightarrow A}\}$. Classically, CI is usually considered one-way only (represented by an arrow in a directed acyclic graph), as time-ordering of events and forward-in-time-only causal influence is assumed. In [46], both CI are forward in time. When time stamps of events can be exchanged, mixtures of causal influences are also possible classically (lung cancer can be caused by smoking, but also incite people to enjoy some final cigarettes...), but the superpositions introduced here go beyond this. More generally, the four possible CI options could be superposed, and, for open quantum systems, Kraus-operators with different causal influence.

J. PROPAGATION OF CAUSAL INFLUENCE

For the physical example of two double quantum dots (DQD) at distance $l$ from each other coupled with dipole interaction to black-body radiation, with a UV frequency cut-off $y_m = \omega_{\text{max}} \tau$, where $\tau = h/k_B T$ and $T$ is the temperature, the functions $f(t)$ and $\varphi(t)$ can be obtained analytically if one approximates $\coth \approx 1$ for $y_m \gg 1$ [55]. The argument of the sin in (6) becomes

$$\frac{At}{t_0} \{ -2 \sin(y_m t_0) + \text{Si}[y_m(t - t_0)] + 2 \text{Si}(y_m t_0) \} - \text{Si}[y_m(t + t_0)] + \frac{2A}{y_m t_0^2} \sin(y_m t) \sin(y_m t_0), \quad (41)$$

where $t_0 = l/c$ denotes the time of travel of a light signal between the two DQD and $A = \alpha_0 d^2/\pi e^2 \tau^2$, $d$ is the dipole moment of the quantum system divided by the electron charge, and $\alpha_0 \approx 1/137$ and $c$ are the fine-structure constant and the speed of light in vacuum, respectively. In [41] and onwards, both $t$ and $t_0$ are in units of $\tau$. The exponent in (6) becomes $-\alpha_0 d^2 \omega_{\text{max}}^2 / 3c^2$. 

![Fig. 5. (Color online) $I_{A \rightarrow B}$ (dashed line) and $I_{B \rightarrow A}$ (continuous line) of $U^{\text{sup}}$ depending on the control qubit $|\chi_c\rangle = \cos(\theta/2) |0\rangle + e^{i\phi} \sin(\theta/2) |1\rangle$, parametrized by $\theta$.](image_url)