NOTE ABOUT STIEFEL-WHITNEY CLASSES ON REAL BOTT MANIFOLDS

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Abstract. Real Bott manifolds is a class of flat manifolds with holonomy group \( \mathbb{Z}_k^2 \) of diagonal type. In this paper we want to show how we can compute even Stiefel-Whitney classes on real Bott manifolds. This paper is an answer to the question of professor Masuda if is it possible to extend \([4]\) and compute any Stiefel-Whitney classes for real Bott manifolds. It also extends results of \([6]\).

1. Introduction

Let \( M_n \) be a flat manifold of dimension \( n \), i.e. a compact connected Riemannian manifold without boundary with zero sectional curvature. From the theorem of Bieberbach ([1], [9]) the fundamental group \( \pi_1(M_n) = \Gamma \) determines a short exact sequence:

\[
0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0,
\]

where \( \mathbb{Z}^n \) is a maximal torsion free abelian subgroup of rank \( n \) and \( G \) is a finite group which is isomorphic to the holonomy group of \( M_n \). The universal covering of \( M_n \) is the Euclidean space \( \mathbb{R}^n \) and hence \( \Gamma \) is isomorphic to a discrete cocompact subgroup of the isometry group \( \text{Isom}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n = E(n) \). In that case \( p : \Gamma \to G \) is a projection on the first component of the semidirect product \( O(n) \ltimes \mathbb{R}^n \) and \( \pi_1(M_n) = \Gamma \) is a subgroup of \( O(n) \ltimes \mathbb{R}^n \). Conversely, given a short exact sequence of the form \([1]\), it is known that the group \( \Gamma \) is (isomorphic to) the fundamental group of a flat manifold. In this case \( \Gamma \) is

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called a Bieberbach group. We can define a holonomy representation \( \phi: G \to \text{GL}(n, \mathbb{Z}) \) by the formula:

\[
\phi(g)(e) = \tilde{g}e(\tilde{g})^{-1},
\]

for all \( e \in \mathbb{Z}^n, g \in G \) and where \( p(\tilde{g}) = g \). In this article we shall consider Bieberbach groups of rank \( n \) with holonomy group \( \mathbb{Z}_k^2 \), \( 1 \leq k \leq n-1 \), and \( \phi(\mathbb{Z}_k^2) \subset D \subset \text{GL}(n, \mathbb{Z}) \). Here \( D \) is the group of matrices with \( \pm 1 \) on the diagonal.

The main result is the formula for even Stiefel-Whitney classes for real Bott manifolds. This formula is generalization of the one from our previous paper ([4], Lemma 2.1). It was suggested to us by M. Masuda. The author thanks Andrzej Szczepański for discussion.

2. STIEFEL-WHITNEY CLASSES FOR REAL BOTT MANIFOLDS

Let \( \gamma_i \) be the canonical line bundle over \( M_i \) and we set \( x_i = w_1(\gamma_i) \) (\( w_1 \) is the first Stiefel-Whitney class). Since \( H^1(M_{i-1}, \mathbb{Z}_2) \) is additively generated by \( x_1, x_2, \ldots, x_{i-1} \) and \( L_{i-1} \) is a line bundle over \( M_{i-1} \), we can uniquely write

\[
w_1(L_{i-1}) = \sum_{l=1}^{i-1} a_{il}x_l
\]

where \( a_{il} \in \mathbb{Z}_2 \) and \( i = 2, 3, \ldots, n \).

From the above we obtain the matrix \( A = [a_{il}] \) which is an \( n \times n \) strictly upper triangular matrix whose diagonal entries are 0 and remaining entries are either 0 or 1. One can observe (see [7]) that the tower (3) is completely determined by the matrix \( A \) and therefore we may denote the real Bott manifold \( M_n \) by \( M_n(A) \). From [7, Lemma 3.1]
we can consider $M_n(A)$ as the orbit space $M_n(A) = \mathbb{R}^n / \Gamma(A)$, where $\Gamma(A) \subset E(n)$ is generated by elements

$$s_i = \left( \text{diag}[1, \ldots, (-1)^{a_{i,i+1}}, \ldots, (-1)^{a_{i,n}}], \left(0, \ldots, 0, \frac{1}{2}, 0, \ldots, 0 \right)^T \right),$$

where $(-1)^{a_{i,i+1}}$ is in the $(i+1, i+1)$ position and $\frac{1}{2}$ is the $i$th coordinate of the last column, $i = 1, 2, \ldots, n - 1$. $s_n = (I, (0, 0, \ldots, 0, \frac{1}{2})) \in E(n)$. From [7, Lemma 3.2, 3.3] $s_1^2, s_2^2, \ldots, s_n^2$ commute with each other and generate a free abelian subgroup $\mathbb{Z}^n$. In other words $M_n(A)$ is a flat manifold with holonomy group $\mathbb{Z}^k_2$ of diagonal type. Here $k$ is a number of non zero rows of a matrix $A$.

We have the following two lemmas.

**Lemma 2.1** ([7], Lemma 2.1). The cohomology ring $H^*(M_n(A), \mathbb{Z}_2)$ is generated by degree one elements $x_1, \ldots, x_n$ as a graded ring with $n$ relations

$$x_j^2 = x_j \sum_{i=1}^n a_{ij} x_i,$$

for $j = 1, \ldots, n$.

**Lemma 2.2** ([7], Lemma 2.2). The real Bott manifold $M_n(A)$ is orientable if and only if the sum of entries is $0(\text{mod} 2)$ for each row of the matrix $A$.

The $k$th Stiefel-Whitney class [8, page 3, (2.1)] is given by the formula

(5) \[ w_k(M(A)) = (B(p))^* \sigma_k(y_1, y_2, \ldots, y_n) \in H^k(M(A); \mathbb{Z}_2), \]

where $\sigma_k$ is the $k$–th elementary symmetric function, $B(p)$ is a map induced by $p$ on the classification space and

(6) \[ y_i := w_1(L_{i-1}) \]

for $i = 2, 3, \ldots, n$.

Follow [2], if we consider $H^*(M_j(A), \mathbb{Z})$ as a subring of $H^*(M_n(A), \mathbb{Z})$ through the projection in [3], we see that

(7) \[ H^*(M_n(A), \mathbb{Z}) \]

$$= \mathbb{Z}[x_1, \ldots, x_n]/ \left( x_j^2 - x_j \sum_{i=1}^n a_{ij} x_i : j = 1, 2, \ldots, n \right).$$
From the above we get

Lemma 2.3. \[2\] Let $k$ be positive integer less or equal to $\frac{n}{2}$. The the set
\[\{x_{i_1}x_{i_2}\ldots x_{i_{2k}} : 1 \leq i_1 < i_2 < \ldots < i_{2k} \leq n\}\]
is an additive basis of $H^{2k}(M_n(A), \mathbb{Z}_2)$.

Let $A_{i_1i_2\ldots i_{2k}}$ denotes the $(n \times n)$ matrix consisting of $i_1, i_2, \ldots, i_{2k}$ rows of matrix $A$. Then non zero entries are only in $i_1, i_2, \ldots, i_{2k}$ rows of the matrix $A_{i_1i_2\ldots i_{2k}}$ and we have the following main result.

Theorem 2.1. Let $A$ be an $(n \times n)$ the Bott matrix. Then,
\[w_{2k}(M_n(A)) = \sum_{1 \leq i_1 < i_2 < \ldots < i_{2k} \leq n} w_{2k}(M_n(A_{i_1i_2\ldots i_{2k}})).\]

Proof.

From (\[2\] Lemma 2.1) we have that the $2k$ cohomology group of $H^{2k}(M_n(A), \mathbb{Z}_2)$ has a basis
\[\mathcal{B} = \{x_{i_1}x_{i_2}\ldots x_{i_{2k}} : 1 \leq i_1 < i_2 < \ldots < i_{2k} \leq n\}.
\]
Moreover, also from Lemma 2.1, $x_j^2$ can be expressed by a linear combination of $x_kx_j$ for $k < j$. Note that this combination always contains an $x_j$-term. Hence, we get that $w_{2k}(M_n(A))$ is a sum of linear elements
\[w_{2k}(M_n(A)) = \sum_{1 \leq i_1 < i_2 < \ldots < i_{2k} \leq n} x_{i_1}x_{i_2}\ldots x_{i_{2k}},\]
Each term $x_{i_1}x_{i_2}\ldots x_{i_{2k}}$ of this sum is an element from basis $\mathcal{B}$ and it is equal to the $2k$ Stiefel-Whitney class of the real Bott manifold $M_n(A_{i_1i_2\ldots i_{2k}})$, so we get
\[w_{2k}(M_n(A)) = \sum_{1 \leq i_1 < i_2 < \ldots < i_{2k} \leq n} w_{2k}(M_n(A_{i_1i_2\ldots i_{2k}})).\]
Thus, the $2k$th Stiefel-Whitney class of the real Bott manifold $M_n(A)$ is equal to the sum of $2k$th Stiefel-Whitney classes of elementary components $M_n(A_{i_1i_2\ldots i_{2k}})$, $1 \leq i_1 < i_2 < \ldots < i_{2k} \leq n$.

At the end of the paper we give an example.
Example 2.1. For

\[
A = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

we get \( w_4(M(A)) = x_2x_3x_4x_5 + x_1x_3x_4x_5 + x_1x_2x_3x_5 + x_1x_2x_3x_4. \) For the matrix \( A \) we have the following

\[
A_{1234} = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad w_4(M(A_{1234})) = x_1x_2x_3x_4,
\]

\[
A_{1235} = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad w_4(M(A_{1235})) = x_1x_2x_3x_5,
\]

\[
A_{1245} = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad w_4(M(A_{1245})) = 0,
\]
\[ A_{1345} = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad w_4(M(A_{1345})) = x_1x_3x_4x_5, \]

\[ A_{2345} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad w_4(M(A_{2345})) = x_2x_3x_4x_5. \]

So we have

\[ w_4(M(A_{1234})) + w_4(M(A_{1235})) + w_4(M(A_{1245})) + w_4(M(A_{1345})) + w_4(M(A_{2345})) = x_1x_2x_3x_4 + x_1x_2x_3x_5 + x_1x_3x_4x_5 + x_2x_3x_4x_5 = w_4(M(A)). \]

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