Odd coset quantum mechanics

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ABSTRACT

The standard quantum states of $n$ complex Grassmann variables with a free-particle Lagrangian transform as a spinor of $SO(2n)$. However, the same `free-fermion' model has a non-linearly realized $SU(n|1)$ symmetry; it can be viewed as the mechanics of a `particle' on the Grassmann-odd coset space $SU(n|1)/U(n)$. We implement a quantization of this model for which the states with non-zero norm transform as a representation of $SU(n|1)$, the representation depending on the $U(1)$ charge of the wave-function. For $n = 2$ the wave-function can be interpreted as a BRST superfield.

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1 Introduction

Pseudo-classical mechanics models, with anticommuting variables, have found various applications. One class of applications is to the pseudo-classical description of spin. Consider the ‘free-fermion’ Lagrangian

\[ L = i\gamma \bar{\zeta} \cdot \dot{\zeta} \]  

for \( n \) complex anticommuting variables \( \zeta^i \) and their complex conjugates \( \bar{\zeta}_i \) (\( \gamma \) is a real, positive, dimensionless coupling constant). This Lagrangian has an obvious \( U(n) \) invariance but it is also invariant under the larger group \( SO(2n) \). In a (coherent state) basis for which the quantum operators corresponding to the variables \( \bar{\zeta}_i \) are diagonal, with eigenvalues \( \bar{\zeta}_i \), the Hilbert space of the quantum theory is spanned by anti-holomorphic functions \( \Phi(\{\bar{\zeta}\}) \). This space has dimension \( 2^n \) and carries a spinor representation of \( SO(2n) \).

The above Lagrangian is also invariant, although less obviously, under the following non-linear, and non-analytic, transformations

\[ \delta_\epsilon \zeta^i = \left[ 1 + \zeta \cdot \bar{\zeta} \right]^{1/2} \epsilon^i + \frac{\left( \epsilon \cdot \zeta + \epsilon \cdot \bar{\zeta} \right)}{2 \left[ 1 + \zeta \cdot \bar{\zeta} \right]^{1/2}} \zeta^i \]  

where \( \epsilon^i \) are constant anticommuting parameters. These transformations close on those of \( U(n) \) to form the superalgebra \( SU(n|1) \). In other words, the above free-fermion Lagrangian provides a non-linear realization of the supergroup \( SU(n|1) \), with \( \zeta^i \) parametrizing the Grassmann-odd coset \( SU(n|1)/U(n) \).

The full symmetry group is actually much larger than either \( SO(2n) \) or \( SU(n|1) \); it is the supergroup of supersymplectic diffeomorphisms of a superspace of real dimension \( (0|2n) \), which is generated by all \( 2^{2n} \) functions on the Grassmann-odd phase-space. An alternative characterization of it is as the closure of its two subgroups \( SO(2n) \) and \( SU(n|1) \). Both subgroups contain \( U(n) \), which acts in an obvious way on quantum wave-functions, so the Hilbert space decomposes into representations of \( U(n) \). In the standard quantum theory these representations combine to yield the spinor of \( SO(2n) \) and the Hilbert space norm is the standard scalar product of two spinors.

However, one could attempt to combine the \( U(n) \) representations into representations of \( SU(n|1) \) rather than \( SO(2n) \). In this case, the ‘Hilbert’ space would be a vector superspace of dimension \( (2^{n-1}|2^{n-1}) \) rather than a vector space of dimension \( 2^n \), so this quantization of the free-fermion model would be very different from the standard one; a motivation for considering this possibility is that the \( n = 2 \) ‘Hilbert’ space would then carry a representation of the Euclidean BRST group \( SU(2|1) \cong OSp(2|2) \) (see e.g. \([1]\)).

We shall show here that this non-standard quantization can be implemented, but the result depends on the resolution of an operator ordering ambiguity which leads to an ambiguity in the definition of the \( U(1) \) charge or, equivalently, the assignment...
of $U(1)$ charge to the wave-function. There is a ‘natural’ choice, for which the $U(1)$ charge is the direct quantum analog of the $U(1)$ Noether charge of the Lagrangian (1) but, for completeness, we consider other choices too. In many cases the ‘Hilbert’ space has zero norm states so the physical states in ‘Hilbert’ space should be taken to be the equivalence classes of states with non-zero norm modulo the addition of zero norm states. The $SU(n|1)$ representation content of the physical ‘Hilbert’ space depends on the $U(1)$ charge assigned to the wave-function. For the ‘natural’ resolution of the operator ordering, and $\gamma = n - 1$, we find that the physical Hilbert space is an $SU(n|1)$ singlet!

Presumably, these results could be derived by a direct attempt to implement the $SU(n|1)$ symmetry on the ‘Hilbert’ space found by canonical quantization of (1) but the non-analyticity of the non-linear transformations (2) makes it difficult to see how to do this. We can overcome this problem by introducing the new variables

$$\xi^i = \frac{\zeta^i}{\left[1 + \zeta \cdot \bar{\zeta}\right]^\frac{1}{2}},$$

in terms of which the $SU(n|1)$ supersymmetry transformations are analytic:

$$\delta_e \xi^i = e^i + \bar{e} \cdot \xi \xi^i.$$  \hspace{1cm} (4)

The Lagrangian in these new variables is\footnote{There is some similarity to the QM reduction of the Volkov-Akulov model \cite{2} which realizes Poincaré supersymmetry non-linearly in terms of a Goldstino variable.}

$$L = i \gamma \left[1 + \bar{\xi} \cdot \xi\right]^{-1} \bar{\xi} \cdot \dot{\xi}.$$  \hspace{1cm} (5)

This Lagrangian can be shown to be the 1-dimensional pullback of the $U(1)$ connection one-form in the nonlinear realization of $SU(n|1)$ in the Grassmann-odd coset space $SU(n|1)/U(n)$. It is shifted by a total derivative under the $SU(n|1)$ transformations and so is a sort of 1-dimensional Wess-Zumino (WZ) term. It contains ‘interactions’ which complicate the canonical quantization procedure, but this problem is easily solved in a way that will now be described.

## 2 Analytic quantization

An equivalent phase-space form of the Lagrangian (5) is

$$L = \left\{i \pi \cdot \dot{\xi} - \lambda^i \varphi_i\right\} + \text{c.c.}$$  \hspace{1cm} (6)

where $\lambda^i$ are Lagrange multipliers for $n$ complex phase space constraints, with constraint functions

$$\varphi_i = \pi_i - \frac{\gamma}{2} \left[1 + \bar{\xi} \cdot \xi\right]^{-1} \bar{\xi}_i.$$  \hspace{1cm} (7)
Solving these constraints returns us to the original Lagrangian, up to a total derivative. The $SU(n|1)$-supersymmetry transformations of the new Lagrangian are

\[
\begin{align*}
\delta \xi^i &= \bar{\epsilon}^i + \bar{\epsilon} \cdot \xi \xi^i, \\
\delta \pi_i &= \frac{\gamma}{2} \bar{\epsilon}^i + \bar{\epsilon} \cdot \xi \pi - \bar{\epsilon} \cdot \xi \pi_i, \\
\delta \lambda^i &= \bar{\epsilon} \cdot \xi \lambda^i + \bar{\epsilon} \cdot \lambda \xi^i.
\end{align*}
\tag{8}
\]

The $n$ complex constraint functions $\{\varphi\}$ are equivalent to $2n$ real constraint functions that are second class, in Dirac's terminology. However, the $n$ complex constraint functions are in involution; it is only when we consider their complex conjugates that the system of constraints becomes second class. In [3, 4] it was shown that when $2n$ real second class constraints can be separated into two sets of $n$ real constraints in involution then one may quantize without constraints on canonical space variables by imposing one set of $n$ constraints on the Hilbert space states and discarding the other set\(^2\). Here we shall adopt an 'analytic' version of this procedure which, for Grassman variables, actually preceded the formulation of the method for the real case; in this context it has been called 'Gupta-Bleuler quantization' [6, 7].

As the method involves working with an unconstrained phase space, the anticommutation relations follow directly from the canonical Poisson brackets, and these may be realized by setting\(^3\)

\[
\begin{align*}
\pi_i &= \frac{\partial}{\partial \xi^i}, \\
\bar{\pi}^i &= \frac{\partial}{\partial \bar{\xi}^i}.
\end{align*}
\tag{9}
\]

To take the constraints into account we require that physical states be annihilated by the $n$ operators $\varphi_i$; this is equivalent to the 'analyticity' conditions\(^4\)

\[
\frac{\partial \Psi}{\partial \xi^i} = \frac{\gamma}{2} \left[1 + \bar{\xi} \cdot \xi\right]^{-1} \bar{\xi}^i \Psi, \quad i = 1, \ldots, n,
\tag{10}
\]

on wave-functions $\Psi(\{\xi\}, \{\bar{\xi}\})$. These conditions have the solution

\[
\Psi = \left[1 + \bar{\xi} \cdot \xi\right]^{-\frac{\gamma}{2}} \Phi
\tag{11}
\]

for anti-analytic $\Phi$, which has the expansion

\[
\Phi = c_{(0)} + \bar{\xi}^i c_{(1)}^i + \cdots + \bar{\xi}^i_1 \cdots \bar{\xi}^i_{n-1} c_{(n-1)}^{i_1 \cdots i_{n-1}} + \bar{\xi}^i_1 \cdots \bar{\xi}^i_n c_{(n)}^{i_1 \cdots i_n},
\tag{12}
\]

\(^2\)To our knowledge, for systems with Grassmann second-class constraints the possibility of such a quantization scheme was mentioned for the first time in [5]. The idea behind it is that one set of constraints can be interpreted as the $n$ gauge-fixing conditions for $n$ gauge-invariances generated by the other set.

\(^3\)The classical anticommuting variable $\bar{\pi}$ is the complex conjugate of the variable $\pi$, whereas the complex conjugate of $\partial/\partial \xi$ is, for standard complex conjugation conventions, $-\partial/\partial \bar{\xi}$. The conjugation in the quantum case should be understood with respect to the properly defined $SU(n|1)$ invariant scalar product, as discussed in section 4.

\(^4\)This is analogous to the chirality condition on 4D chiral superfields, which arises in a similar way from analytic quantization of the 4D superparticle [8, 9]. See [10] for other analogous aspects of the superparticle case.
where
\[ c_{i_1 \ldots i_{n-1}}^{(n-1)} = \frac{1}{(n-1)!} \varepsilon^{i_1 \ldots i_{n-1} i_n} c_{c(n-1) i_n}, \quad c_{i_1 \ldots i_n}^{(n)} = \frac{1}{n!} \varepsilon^{i_1 \ldots i_n} c_{c(n)} \quad \ldots . \quad (13) \]

In principle each of the \(2^n\) coefficients could have any Grassmann parity but to implement \(SO(2n)\) invariance we would have to choose all of them to have the same Grassmann parity, which must be even for a positive definite norm. In this way we would recover the standard free-fermion Hilbert space, as a \(2^n\)-dimensional vector space, although the \(SO(2n)\) invariance is not manifest in our approach and has to be imposed. Here however, we wish to explore the alternative possibility that the ‘Hilbert’ space carries some representation of the supergroup \(SU(n|1)\). For this to be possible we must take the anti-analytic function \(\Phi\) to have a definite Grassmann parity\(^5\). In this case the ‘Hilbert’ space is a vector superspace of dimension \((2^n-1|2^n-1)\); for a reason to be made clear later, we assume that \(\Phi\) is Grassmann-even for \(n\) even and Grassmann-odd for \(n\) odd. Our next task is to determine how \(SU(n|1)\) acts in this ‘Hilbert’ space.

3 \(SU(n|1)\) in ‘Hilbert’ space

The linear \(U(n)\) transformations of the canonical variables of the Lagrangian (6) are generated by the Noether charges
\[ J_{i \bar{j}} = \bar{\xi}_j \bar{\pi}^i - \xi^i \pi_j . \quad (14) \]
The corresponding quantum \(U(n)\) generators are the differential operators
\[ \hat{J}_{i \bar{j}} = \bar{\xi}_j \frac{\partial}{\partial \bar{\xi}_i} - \xi^i \frac{\partial}{\partial \xi_j} . \quad (15) \]
For wave-functions of the form (11) we have
\[ \hat{J}_{i \bar{j}} \Psi = \left[ 1 + \bar{\xi} \cdot \xi \right]^{-\frac{\omega}{2}} \xi_j^{\bar{j}} \frac{\partial \Phi}{\partial \bar{\xi}_i} , \quad (16) \]
from which we deduce the \(U(n)\) transformation of \(\Phi\) to be
\[ \delta_\omega \Phi = \omega_i^{\bar{j}} \xi_j^{\bar{j}} \frac{\partial \Phi}{\partial \bar{\xi}_i} \quad (17) \]
where \(\omega_i^{\bar{j}} = -\omega_j^{\bar{i}}\).

\(^5\)The same requirement is made in the standard quantization of the superparticle, in contrast to the ‘spinning particle’. In fact, as the 4-dimensional spinning particle and superparticle Lagrangians can be shown to be classically equivalent [11, 12, 13], the difference between the two can be ascribed to different quantization procedures, in close analogy to the ‘free fermion’ model considered here.
The nonlinear supersymmetry transformations of (8) are generated by the Grassmann-odd Noether charges

\[ S_i = \pi_i + \frac{\gamma}{2} \xi_i - \bar{\xi}_i \sum_j \xi_j \bar{\pi}_j, \quad \bar{S}^i = \bar{\pi}^i + \frac{\gamma}{2} \xi^i + \xi^i \sum_j \bar{\xi}_j \pi_j. \]  

(18)

Note the presence of the terms linear in \( \xi \) and \( \bar{\xi} \); these arise from the fact that the supersymmetry variation of the Lagrangian is not zero but rather a total time derivative. These terms have no effect on the transformations of \( \xi_i \) generated by \( \hat{S} \) and \( \hat{\bar{S}} \), which are those of (4), but they do contribute to the \( U(1) \) charge in the \( SU(n|1) \) superalgebra of Poisson brackets of Noether charges. In fact, one finds that the \( U(1) \) charge is

\[ B = \left( \frac{1}{n} - 1 \right) J_i + \gamma \]  

(19)

where the shift by \( \gamma \) is directly attributable to the \( \gamma \)-dependent linear terms in \( \hat{S} \) and \( \hat{\bar{S}} \). In passing to the quantum theory, the coefficients of these terms become ambiguous because of operator ordering ambiguities. This ambiguity is partially fixed by requiring that physical wave-functions \( \Psi \) of the form (11) transform into physical wave-functions, i.e.

\[ \delta_\epsilon \Psi \equiv - \left( \bar{\epsilon} \cdot \hat{\bar{S}} + \epsilon \cdot S \right) \Psi = \left[ 1 + \bar{\xi} \cdot \xi \right]^{-\frac{1}{2}} \delta_\epsilon \Phi. \]  

(20)

This leaves us with the following quantum supersymmetry generators, parametrized by a real number \( \alpha \):

\[ \hat{S}_i = \frac{\partial}{\partial \xi_i} + \frac{\alpha}{2} \bar{\xi}_i - \bar{\xi}_i \left( \bar{\xi} \cdot \frac{\partial}{\partial \xi} \right), \]

\[ \hat{\bar{S}}^i = \frac{\partial}{\partial \bar{\xi}_i} + \frac{\gamma}{2} \xi^i + \xi^i \left( \xi \cdot \frac{\partial}{\partial \bar{\xi}} \right). \]  

(21)

These have the anticommutation relation

\[ \{ \hat{S}_i, \hat{\bar{S}}^j \} = \left[ j^i_k - \frac{1}{n} \delta^i_j j^k \right] + \delta^i_j \hat{B} \]  

(22)

where \( \hat{B} \) is the quantum \( U(1) \) generator

\[ \hat{B} = \left( \frac{1}{n} - 1 \right) j^k_k + \frac{1}{2} (\gamma + \alpha). \]  

(23)

One sees from this that the choice

\[ \alpha = \gamma \]  

(24)

is ‘natural’ because it leads to a quantum \( U(1) \subset SU(n|1) \) generator that is the direct quantum counterpart of the classical \( U(1) \) charge \( B \) of (19). Nevertheless, we shall consider the case of general \( \alpha \) in what follows.
We now compute the action of the charges $\hat{S}_i$, $\hat{\bar{S}}^i$, on physical wave-functions. One finds that
\[
\hat{S}^i\Psi = \left[1 + \bar{\xi} \cdot \xi\right]^{-\frac{1}{2}} \left[\frac{\partial \Phi}{\partial \bar{\xi}_i} + \frac{1}{2} (\gamma + \alpha) \bar{\xi}_i - \bar{\xi}_i \left(\bar{\xi} \cdot \frac{\partial}{\partial \bar{\xi}}\right)\right] \Phi. \tag{25}
\]

These results yield the following $SU(n|1)$-supersymmetry transformation of $\Phi$:
\[
\delta_{\epsilon} \Phi = -\left[q(\epsilon \cdot \bar{\xi}) + \bar{\epsilon} \cdot \frac{\partial}{\partial \bar{\xi}} - (\epsilon \cdot \bar{\xi}) \bar{\epsilon} \cdot \frac{\partial}{\partial \bar{\xi}}\right] \Phi \tag{26}
\]

where
\[
q = \frac{1}{2} (\gamma + \alpha). \tag{27}
\]

For component fields in the expansion (12) this transformation implies
\[
\delta_{\epsilon} c^{i_1...i_k}_{(k)} = (-1)^k \left\{ [k - 1 - q]c^{i_1...i_k}_{(k-1)} - (1 - \delta_{k,n})(k + 1) \bar{\epsilon}_j c^{j_1...i_k}_{(k+1)} \right\} \quad (k \geq 2),
\delta_{\epsilon} c^{0}_{(1)} = -\bar{\epsilon}_{i} c^{i}_{(1)}, \quad \delta_{\epsilon} c^{j}_{(2)} = q\epsilon^{j} c^{0}_{(0)} + 2\bar{\epsilon}_{j} c^{j}_{(2)}. \tag{28}
\]

The full set of $SU(n|1)$ transformations of $\Phi$ are such that
\[
\Phi'([\xi']) = e^{iqs([\bar{\xi}])} \Phi([\bar{\xi}]) \tag{29}
\]

where $s$ is a local function of $\bar{\xi}$. Thus, $\Phi$ is a scalar anti-analytic superfield when $q=0$; for other values of $q$, including the ‘natural’ value $q = \gamma$, one may consider $\Phi$ as a charged scalar superfield, with charge $q$.

We have supposed up to now that $\alpha$ and $\gamma$ are arbitrary real variables but one might expect the combination $q$ to be quantized\(^6\). As we shall see, the representation content of the physical ‘Hilbert’ space depends on $q$ and simplifications, associated with the existence of zero norm states, occur for special integer values of $q$.

4 \quad \textbf{$SU(n|1)$-invariant norm}

In order to construct an invariant inner product one must first obtain an $SU(n|1)$ invariant measure under the coordinate transformations (4). With the help of the lemma
\[
\delta_{\epsilon} \left[1 + \bar{\xi} \cdot \xi\right] = \left(\bar{\epsilon} \cdot \xi - \epsilon \cdot \bar{\xi}\right) \left[1 + \bar{\xi} \cdot \xi\right], \tag{30}
\]
\[\text{For example, we have } q = \gamma \text{ for the 'natural' choice } \alpha = \gamma \text{ but, as mentioned earlier, } \gamma \text{ is the coefficient of a WZ term. By analogy with the bosonic WZ terms, this coefficient is expected to be quantized, though the origin of this phenomenon can differ according to the case considered.}\]
it is not difficult to show that the $SU(n|1)$ invariant measure is such that

$$\int d\mu = \int d\mu_0 \left[ 1 + \bar{\xi} \cdot \xi \right]^{n-1},$$

(31)

where

$$\int d\mu_0 = \prod_i \frac{\partial}{\partial \bar{\xi}_i} \frac{\partial}{\partial \xi_i}.$$  

(32)

However, because the transformation (26) involves a $U(1)$ weight term, an additional factor is needed in the measure when $q \neq 0$. Let us replace $\Phi$ by $\Phi(q)$ to remind us that $\Phi$ carries $U(1)$ charge $q$. Then the following bilinear form is $SU(n|1)$ invariant:

$$||\Phi(q)||^2 = \int d\mu \left[ 1 + \bar{\xi} \cdot \xi \right]^{-q} |\Phi(q)|^2.$$  

(33)

Note that the additional factor in the measure is unity precisely when $q = 0$ but is non-trivial otherwise.

In terms of the original wave-functions $\Psi = \left[ 1 + \bar{\xi} \cdot \xi \right]^{-\frac{1}{2}} \Phi(q)$, the $SU(n|1)$ invariant scalar product corresponding to the definition (33) reads

$$< \Omega | \Psi > = \int d\mu_0 \left[ 1 + \bar{\xi} \cdot \xi \right]^\kappa \Omega^\dagger \Psi$$

(34)

where

$$\kappa = \gamma - q + n - 1$$

(35)

and $\Omega(\xi, \bar{\xi})$ is another vector in the same `Hilbert space’. It is straightforward to check that the quantum generators (21) are mutually conjugate with respect to this scalar product

$$(< \Omega | \hat{S}_i | \Psi >) = < \Psi | \hat{S}_i | \Omega >.$$  

(36)

On the other hand, for $\kappa \neq 0$ the fermionic momentum operators $\partial/\partial \xi^i$ and $\partial/\partial \bar{\xi}^i$ are not mutually conjugate with respect to (34). Note, however, that (34) is defined modulo the following similarity transformation (change of basis) in ‘Hilbert space’

$$\Psi = \left[ 1 + \bar{\xi} \cdot \xi \right]^\lambda \Psi(\lambda), \quad \Omega = \left[ 1 + \bar{\xi} \cdot \xi \right]^\lambda \Omega(\lambda), \quad (\lambda^\dagger = \lambda).$$

(37)

This amounts to the substitution $\Omega, \Psi \rightarrow \Omega', \Psi'$ and shift $\kappa \rightarrow \kappa + 2\lambda$ in the definition (34), as well as a corresponding change in the observables. The conjugacy property (36) of the $SU(n|1)$ supersymmetry generators is evidently basis-independent. In contrast, an analogous conjugacy property holds for the fermionic momentum operators only for the special choice of basis corresponding to $\lambda = -\kappa/2$:

$$(< \Omega_{(-\kappa/2)} | \partial/\partial \xi^i | \Psi_{(-\kappa/2)} >)^\dagger = < \Psi_{(-\kappa/2)} | \partial/\partial \bar{\xi}^i | \Omega_{(-\kappa/2)} >.$$  

(38)

Thus, the fermionic momentum operators (9) are mutually conjugate in the sense that there is a basis in ‘Hilbert space’ for which they satisfy (38).
Let us now turn to the analysis of the field content of $\Phi$ implied by the invariant norm (33). In general, there are contributions to (33) from all coefficients in the expansion (12), but zero norm states occur for special values of $q$. For example, when $q = n - 1$ we have

$$||\Phi_{(n-1)}||^2 = |c_{(n)}|^2.$$  

(39)

As $\delta_{\xi} c_{(n)} = 0$ for this choice of $q$ we see that the physical Hilbert space is an $SU(n|1)$ singlet. All functions $\Phi_{(n-1)}$ with $c_{(n)} = 0$ have zero norm. If instead we set $q = n - 2$ then we find

$$||\Phi_{(n-2)}||^2 = |c_{n}|^2 - \bar{c}_{(n-1)}^i c_{(n-1)}^i.$$  

(40)

Again there are zero norm states because the $SU(n)$ representation content appearing in the norm is restricted to $n \oplus 1$; these $SU(n)$ representations combine to form the fundamental $n + 1$ representation of $SU(n|1)$. As a final example, consider $q = 0$. In this case we have

$$||\Phi_{(0)}||^2 = (-1)^n (n - 1)! \sum_{k=1}^{n} (-1)^k k \bar{c}_{(k)}^i_{i_1 \ldots i_k} c_{(k)}^{i_1 \ldots i_k}.$$  

(41)

The $SU(n)$ representation content is $n \oplus n(n - 1)/2 \oplus \ldots \oplus n \oplus 1$.

An inspection of the transformations (28) leads to the following general conclusions about the structure of the ‘Hilbert spaces’ corresponding to different choices of $q$. For the choice

$$q = (\hat{k} - 1),$$  

(42)

for integer $\hat{k}$ in the range $0 \leq \hat{k} \leq n$ (the examples above correspond to $\hat{k} = n, n-1, 1$, respectively) there is an invariant irreducible subspace spanned by

$$c_{(k)}^{i_1 \ldots i_k}, \ldots, c_{(n)}^{i_1 \ldots i_n}.$$  

(43)

For $\hat{k} = 0$ this subspace is the full space of coefficients of $\Phi$ but otherwise it is not, and the remaining coefficients are transformed into the above set; this shows that the representation of $SU(n|1)$ carried by $\Phi_{(\hat{k}-1)}$ is reducible but not fully reducible. The norm $||\Phi_{(\hat{k}-1)}||$ includes only the components (43), so there exist zero norm states unless $\hat{k} = 0$.

As the set (43) is irreducible under the action of $SU(n|1)$, we can consistently set them to zero:

$$c_{(\hat{k})}^{i_1 \ldots i_k} = c_{(\hat{k}+1)}^{i_1 \ldots i_{k+1}} = \ldots = c_{(n)}^{i_1 \ldots i_n} = 0.$$  

(44)

The complementary set of coefficients then forms an irreducible set on its own, and one would expect there to exist a corresponding $SU(n|1)$-invariant norm. However, the ‘obvious’ norm, defined by (33), is identically zero when (44) is satisfied; this is easily seen by rewriting (44) in the superfield form

$$\frac{\partial^{\hat{k}} \Phi_{(q)}}{\partial \xi_{i_1} \ldots \partial \xi_{i_\hat{k}}} = 0 \quad \text{(and c.c.)}.$$  

(45)
One can check that these constraints are covariant under (26) provided that the condition (42) holds. Of course, they do not correspond to constraints in the Lagrangian (6) so what we are now doing cannot be considered as a quantization of \textit{that} Lagrangian but one could add to it the classical constraints corresponding to (45), for which the constraint functions are polynomials in \( \bar{\pi} \).

It is remarkable that for \( \Phi_{(k-1)} \) constrained by (45) there exists the following \textit{alternative} norm
\[
|||\Phi_{(k-1)}|||^2 = \int d\mu_0 (1 + \bar{\xi} \cdot \xi)^{n-k} \ln(1 + \bar{\xi} \cdot \xi) \bar{\Phi}_{(k-1)} \Phi_{(k-1)} ,
\]  
(46)
Taking into account that
\[
\delta_\epsilon \ln(1 + \bar{\xi} \cdot \xi) = (\bar{\epsilon} \cdot \xi - \epsilon \cdot \bar{\xi}) ,
\]  
(47)
it is straightforward to prove invariance of (46) given the constraints (45), which are crucial to the result. It is interesting that the ‘Lagrangian density’ in (46) is not a tensor one as in (33), but has an additional variation into a total derivative, as is typical for WZ or Chern-Simons lagrangians.

5 \textit{n = 2 and BRST}

We shall now illustrate the above results with the \( n = 2 \) case; we also choose \( \gamma = 1 \), which means that the ‘natural’ choice of operator ordering corresponds to \( q = 1 \). For \( n = 2 \) we can interpret the odd coset space \( SU(2|1)/U(2) \) as a BRST superspace because \( SU(2|1) \cong OSp(2|2) \) is the Euclidean BRST supergroup. For \( n = 2 \) we have
\[
\Phi_{(q)} = a + \bar{\xi}_i b^i + \bar{\xi}_1 \bar{\xi}_2 c .
\]  
(48)
The coefficients \((b^1, b^2)\), which form an \( SU(2) \) doublet, can be interpreted as (euclidean) Faddeev-Popov ghost and antighost for the \( SU(2) \)-singlet gauge-fixing term \( a \); the other \( SU(2) \) singlet \( c \) is then the ‘Nakanishi-Lautrup’ auxiliary field.

\( \delta \)From (28) we deduce that the \( SU(2|1) \)-supersymmetry transformations for \( q = 0 \) are
\[
\delta_\epsilon a = -\bar{\epsilon}_i b^i , \\
\delta_\epsilon b^i = -\epsilon^{ij} \bar{\epsilon}_j c , \\
\delta_\epsilon c = \epsilon_{ij} \bar{\epsilon}^i b^j ,
\]  
(49)
where \( \epsilon_{12} = \epsilon^{12} = 1 , \, \epsilon^{ik} \bar{\epsilon}_{il} = \delta_k^l \). This is not a reducible representation because \( a \) transforms non-trivially while \((b_1, b_2, c)\) span a 3-dimensional invariant subspace. Observe that
\[
|||\Phi_{(0)}|||^2 = |c|^2 + b^i \bar{b}_i
\]  
(50)

is invariant. Of course, this is not really a norm as the variables \( b^i \) are anticommuting.\(^7\)

In other words, the physical states are vectors in a vector superspace of dimension \((1|2)\) transforming as the fundamental representation of \(SU(2|1)\).

For \( q = 1 \) the transformations (49) become

\[
\begin{align*}
\delta_e a &= -\varepsilon_i b^i , \\
\delta_e b^i &= -\varepsilon^{ij} \tilde{e}_j c + \epsilon^i a , \\
\delta_e c &= 0 .
\end{align*}
\]

(51)

This is the ‘natural’ case for which the physical Hilbert space is a singlet. Indeed, the norm (33) in this case is simply

\[
||\Phi^{(1)}||^2 = |c|^2 .
\]

(52)

Still with \( q = 1 \), we may impose the covariant condition

\[
c = 0 \quad \Leftrightarrow \quad \frac{\partial^2 \tilde{\Phi}^{(1)}}{\partial \xi_i \partial \xi_k} = 0 .
\]

(53)

This leaves us with the irreducible multiplet \((a, b^i)\):

\[
\begin{align*}
\delta_e a &= -\varepsilon_i b^i , \\
\delta_e b^i &= \epsilon^i a .
\end{align*}
\]

(54)

The alternative norm is

\[
|||\tilde{\Phi}^{(1)}|||^2 = - \int d\mu_0 \ln(1 + \tilde{\xi} \cdot \xi) \tilde{\Phi}^{(1)} \tilde{\Phi}^{(1)} = |a|^2 + \tilde{b}_i b^i
\]

(55)

so physical states once again transform as a fundamental \((1 + 2)\) representation of \(SU(2|1)\) (the precise correspondence with the realization (49), (50) is achieved via substitutions \( a \rightarrow \tilde{c}, \ b^i \rightarrow \epsilon^{ik} \tilde{b}_k \), where \( \tilde{c} \) and \( \tilde{b}^i \) are transformed just as \( c \) and \( b^i \)).

Finally, we shall consider \( q = -1 \), for which the transformation law (26) becomes

\[
\begin{align*}
\delta_e a &= -\varepsilon_i b^i , \\
\delta_e b^i &= -\varepsilon^{ij} \tilde{e}_j c - \epsilon^i a , \\
\delta_e c &= 2 \varepsilon_{ij} \epsilon^i b^j
\end{align*}
\]

(56)

and the invariant norm calculated by the formula (33) is

\[
||\Phi^{(-1)}||^2 = |c|^2 + 2|a|^2 - 2\tilde{b}_i b^i .
\]

(57)

In this case one cannot single out any invariant subspace and so ends up with a 4-dimensional irreducible multiplet \((b_1, b_2, c, a)\) of \(SU(2|1)\).

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\(^7\)For odd \( n \) this feature presents a difficulty because in this case the \( c_{(n)} \) are Grassmann odd and the \( c'_{(n-1)} \) are Grassmann even, but this difficulty is overcome by changing the Grassmann parity of \( \Phi \); this is why we earlier required \( \Psi \) and \( \Phi \) to be even for \( n = 2k \) and odd for \( n = 2k + 1 \). With this definition, the norm for bosonic variables is always positive semi-definite.
We have shown that the mechanics of \( n \) free complex Grassmann-odd variables provides a non-linear realization of the supergroup \( SU(n|1) \). It can be viewed as the mechanics of a ‘particle’ with the Grassmann-odd coset space \( SU(n|1)/U(n) \) as its phase space. This model is trivial in the sense that its Hamiltonian vanishes but it is the simplest of a class of models that realize \( SU(n|1) \) non-linearly and for which the Hamiltonian is generically non-zero. The particle on \( SU(2|1)/[U(1) \times U(1)] \) is an example, and one that will be considered in a future publication. Part of the motivation of this paper was to exhibit some of the properties of these models in the simplest possible setting. Another motivation is that coset-spaces of the \( n = 2 \) supergroup \( SU(2|1) \) can be interpreted as BRST superspaces.

We have shown that there exists an alternative quantization of the ‘free fermion’ model that implements the classical \( SU(n|1) \) symmetry. In contrast to the standard quantization, for which the states transform as a spinor of \( SO(2n) \), the states of the alternative quantum theory are vectors in a vector superspace transforming under \( SU(n|1) \). The specific \( SU(n|1) \) representation content depends on the resolution of an operator ordering ambiguity, which amounts to a choice of \( U(1) \) charge for the wave-function. There is a natural choice, given the initial classical Lagrangian, because this Lagrangian can be viewed as a WZ term for \( U(1) \subset SU(n|1) \) and this leads to specific shift in the \( U(1) \) generator that is naturally identified with the \( U(1) \) charge of the quantum wave-function. For this choice, and a particular choice of the ‘coupling constant’, the ‘Hilbert’ space contains zero norm states and the physical ‘Hilbert’ space is an \( SU(n|1) \) singlet.

For other choices of \( U(1) \) charge assignment (and other choices of coupling constant) one gets other representations of \( SU(n|1) \), picked out by an \( SU(n|1) \) invariant norm. We showed that there exists a class of integer \( U(1) \) charge assignments for which the representation is irreducible. Remarkably, in this case the complementary representation contained in the wave-function, again irreducible, could be picked out by a different invariant, but not manifestly-invariant, norm provided that the original representation was constrained to be absent; this case corresponds to the quantization of the original free-fermion Lagrangian with additional phase space constraints.

Any quantization of Grassmann-odd variables has to take into account (explicitly or implicitly) second-class phase-space constraints. In our case these were non-trivial because of a redefinition of variables needed for analyticity of the \( SU(n|1) \) transformations. We dealt with these constraints by a variant of the ‘gauge unfixing’ method involving a separation of the constraints into analytic and anti-analytic subsets in involution. It may be helpful if we sketch here how this method can be used to covariantly quantize the massless 4D superparticle, as done in [8, 9]. The fermionic constraint operators are the supercovariant derivatives \( D_\alpha \) and their complex conjugates \( \bar{D}_{\dot{\alpha}} \). These are not all second class (given \( p^2 = 0 \)) because the combinations \( p^{\alpha\dot{\alpha}} \bar{D}_{\dot{\alpha}} \) and \( p^{\alpha\dot{\alpha}} D_\alpha \) are first class. Although we should require that both of these first class operators annihilate physical states \( \Phi \) we need only impose \( p^{\alpha\dot{\alpha}} D_\alpha \Phi = 0 \) explicitly if we also impose \( \bar{D}_{\dot{\alpha}} \Phi = 0 \), as required for ‘analytic quantization’, because the...
other one is then implied. The independent constraints are therefore $\dot{D}_a \Phi = 0$ and $p^{\alpha \dot{\alpha}} D_\alpha \Phi = 0$ (because these imply $p^2 = 0$), but these are just the free field equations for a massless chiral superfield.

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