Hard Lefschetz theorem for valuations and related questions of integral geometry.

Semyon Alesker
Department of Mathematics, Tel Aviv University, Ramat Aviv
69978 Tel Aviv, Israel
e-mail: semyon@post.tau.ac.il

Abstract
We continue studying the properties of the multiplicative structure on valuations. We prove a new version of the hard Lefschetz theorem for even translation invariant continuous valuations, and discuss problems of integral geometry staying behind these properties. Then we formulate a conjectural analogue of this result for odd valuations.

0 Introduction.
In this article we continue studying properties of the multiplicative structure on valuations introduced in [3]. In [4] we have proven a certain version of the hard Lefschetz theorem for translation invariant even continuous valuations. Its statement is recalled in Section 4 of this article (Theorem 4.1). The main result of this article is Theorem 2.1 where we prove yet another version of it for even valuations which is more closely related to the multiplicative structure. As a consequence, we obtain a version of it for valuations invariant under a compact group acting transitively on the unit sphere and containing the operator $-Id$ (Corollary 2.5). Then in Section 3 we state a conjecture which is an analogue of the hard Lefschetz theorem for odd valuations.

It turns out that behind the hard Lefschetz theorem for even valuations stay results about the Radon and the cosine transforms on the Grassmannians (see the proof of Theorem 2.1 in this article, and also the proof of Theorem 1.1.1 in [4] which is also a version of the hard Lefschetz theorem). One
passes from even valuations to functions on Grassmannians using the Klain-Schneider imbedding (see Section 1). The case of odd valuations turns out to be related to integral geometry on partial flags which seems to be not well understood. On the other hand, various integral geometric transformations on such spaces can be interpreted sometimes as intertwining operators (or their compositions) for certain representations of \( GL_n(\mathbb{R}) \), and thus can be reduced to a question of representation theory. This point of view was partly used in [6] in the study of the cosine transform on the Grassmannians, and more explicitly in [5] in the study of the generalized cosine transform. It would be of interest to understand these problems for odd valuations.

Note also that the connection between the hard Lefschetz theorem for valuations and integral geometry turns out to be useful in both directions. Thus in [4] it was applied to obtain an explicit classification of unitarily invariant translation invariant continuous valuations.

For the general background on convexity we refer to the book by Schneider [22]. For the classical theory of valuations we refer to the surveys by McMullen and Schneider [21] and McMullen [20].

1 Background.

In this section we recall some definitions and known results. Let \( V \) be a real vector space of finite dimension \( n \). Let \( \mathcal{K}(V) \) denote the family of convex compact subsets of \( V \).

1.1 Definition. 1) A function \( \phi : \mathcal{K}(V) \to \mathbb{C} \) is called a valuation if for any \( K_1, K_2 \in \mathcal{K}(V) \) such that their union is also convex one has

\[
\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).
\]

2) A valuation \( \phi \) is called continuous if it is continuous with respect the Hausdorff metric on \( \mathcal{K}(V) \).

3) A valuation \( \phi \) is called translation invariant if \( \phi(K + x) = \phi(K) \) for every \( x \in V \) and every \( K \in \mathcal{K}(V) \).

4) A valuation \( \phi \) is called even if \( \phi(-K) = \phi(K) \) for every \( K \in \mathcal{K}(V) \).

5) A valuation \( \phi \) is called homogeneous of degree \( k \) (or \( k \)-homogeneous) if for every \( K \in \mathcal{K}(V) \) and every scalar \( \lambda \geq 0 \), we have \( \phi(\lambda \cdot K) = \lambda^k \phi(K) \).

We will denote by \( Val(V) \) the space of translation invariant continuous valuations on \( V \). Equipped with the topology of uniform convergence on
compact subsets of $\mathcal{K}(V)$ it becomes a Fréchet space. We will also denote by $Val_k(V)$ the subspace of $k$-homogeneous valuations from $Val(V)$. We will need the following result due to P. McMullen [18].

1.2 Theorem ([18]). The space $Val(V)$ decomposes as follows

$$Val(V) = \bigoplus_{k=0}^{n} Val_k(V)$$

where $n = \dim V$.

In particular note that the degree of homogeneity is an integer between 0 and $n = \dim V$. It is known that $Val_0(V)$ is one-dimensional and it is spanned by the Euler characteristic $\chi$, and $Val_n(V)$ is also one-dimensional and is spanned by a Lebesgue measure [13]. One has a further decomposition with respect to parity:

$$Val_k(V) = Val_k^{ev}(V) \oplus Val_k^{odd}(V),$$

where $Val_k^{ev}(V)$ is the subspace of even $k$-homogeneous valuations, and $Val_k^{odd}(V)$ is the subspace of odd $k$-homogeneous valuations.

Let us recall the imbedding of the space of valuations into the space of functions on partial flags essentially due to D. Klain [14], [15] and R. Schneider [23]. It will be used in Section 2 to reduce the hard Lefschetz theorem for even valuations to integral geometry of Grassmannians.

Let us denote by $Gr_i(V)$ the Grassmannian of real linear $i$-dimensional subspaces in $V$. For a manifold $X$ we will denote by $C(X)$ (resp. $C^\infty(X)$) the space of continuous (resp. infinitely smooth) functions on $X$. Assume now that $V$ is a Euclidean space. Let us describe the imbedding of $Val_k^{ev}(V)$ into the space of continuous functions $C(Gr_k(V))$ which we call Klain’s imbedding. For any valuation $\phi \in Val_k^{ev}(V)$ let us consider the function on $Gr_k(V)$ given by $L \mapsto \phi(D_L)$ where $D_L$ denotes the unit Euclidean ball inside $L$. Thus we get a map $Val_k^{ev}(V) \to C(Gr_k(V))$. The nontrivial fact due to D. Klain [15] (and heavily based on [14]) is that this map is injective.

Now we would like to recall the Schneider imbedding of $Val_k^{odd}(V)$ into the space of functions on a partial flag manifold. Let us denote by $\tilde{F}(V)$ the manifold of pairs $(\omega, M)$ where $M \in Gr_{k+1}(V)$, and $\omega \in M$ is a vector of unit length. Let us denote by $C^-(\tilde{F}(V))$ the space of continuous functions on $\tilde{F}(V)$ which change their sign when one replaces $\omega$ by $-\omega$. Let us
describe the imbedding of $Val_k^{odd}(V)$ into $C^-(\tilde{F}(V))$ (following [2]) which we call Schneider’s imbedding since its injectivity is an easy consequence of a non-trivial result due to R. Schneider [23] about characterization of odd translation invariant continuous valuations. Fix a valuation $\phi \in Val_k^{odd}(V)$. Fix any subspace $M \in Gr_k(V)$. Consider the restriction of $\phi$ to $M$. By a result of P. McMullen [19] any $k$-homogeneous translation invariant continuous valuation $\psi$ on $(k + 1)$-dimensional space $M$ has the following form. There exists a function $f \in C(S(M))$ (here $S(M)$ denotes the unit sphere in $M$) such that for any subset $K \in \mathcal{K}(M)$

$$\psi(K) = \int_{S(M)} f(\omega) dS_k(K, \omega).$$

Moreover the function $f$ can be chosen to be orthogonal to any linear functional (with respect to the Haar measure on the sphere $S(M)$), and after this choice it is defined uniquely. We will always make such a choice of $f$. If the valuation $\psi$ is odd then the function $f$ is also odd. Thus applying this construction to $\phi|_M$ for any $M \in Gr_k(V)$ we get a map $Val_k^{odd}(V) \to C^-(\tilde{F}(V))$ defined by $\phi \mapsto f$. This map turns out to be continuous and injective (see [2] Proposition 2.6, where the injectivity is heavily based on [23]).

Let us recall the definition of the Radon transform on the Grassmannians. The orthogonal group acts transitively on $Gr_i(V)$, and there exists a unique $O(n)$-invariant probability measure (the Haar measure).

The Radon transform $R_{j,i} : Gr_i(V) \to Gr_j(V)$ for $j < i$ is defined by $(R_{j,i}f)(H) = \int_{F \supseteq H} f(F) \cdot dF$. Similarly for $j > i$ it is defined by $(R_{j,i}f)(H) = \int_{F \subseteq H} f(F) \cdot dF$. In both cases the integration is with respect to the invariant probability measure on all subspaces containing (or contained in) the given one. The Radon transform on real Grassmannians was studied in [7], [12].

Recall now the definition of the cosine and sine of the angle between two subspaces. Let $E \in Gr_i(V)$, $F \in Gr_j(V)$. Assume that $i \leq j$. Let us call cosine of the angle between $E$ and $F$ the following number:

$$|\cos(E, F)| := \frac{vol_i(Pr_F(A))}{vol_i(A)},$$

where $A$ is any subset of $E$ of non-zero volume, $Pr_F$ denotes the orthogonal projection onto $F$, and $vol_i$ is the $i$-dimensional measure induced by the Euclidean metric. (Note that this definition does not depend on the choice
of a subset $A \subset E$). In the case $i \geq j$ we define the cosine of the angle between them as cosine of the angle between their orthogonal complements:

$$|\cos(E, F)| := |\cos(E^\perp, F^\perp)|.$$

(It is easy to see that if $i = j$ both definitions are equivalent.)

Let us call sine of the angle between $E$ and $F$ the cosine between $E$ and the orthogonal complement of $F$:

$$|\sin(E, F)| := |\cos(E, F^\perp)|.$$

The following properties are well known (and rather trivial):

$$|\cos(E, F)| = |\cos(F, E)| = |\cos(E^\perp, F^\perp)|,$$

$$|\sin(E, F)| = |\sin(F, E)| = |\sin(E^\perp, F^\perp)|,$$

$$0 \leq |\cos(E, F)|, |\sin(E, F)| \leq 1.$$

For any $1 \leq i, j \leq n - 1$ one defines the cosine transform

$$T_{j,i} : C(Gr_i(V)) \rightarrow C(Gr_j(V))$$

as follows:

$$(T_{j,i}f)(E) := \int_{Gr_j(V)} |\cos(E, F)| f(F) dF,$$

where the integration is with respect to the Haar measure on the Grassmannian. The cosine transform was studied in \[16\], \[17\], \[8\], \[9\], \[10\], \[6\].

Let us state some facts on the multiplicative structure on valuations. Let us briefly recall a construction of multiplication from \[3\], Section 1. A measure $\mu$ on a linear space $V$ is called polynomial if it is absolutely continuous with respect to a Lebesgue measure, and the density is a polynomial. Let $\mu$ and $\nu$ be two polynomial measures on $V$ (the case of Lebesgue measures would be sufficient for the purposes of this article). Let $A, B \in K(V)$. Consider the valuations $\phi(K) = \mu(K + A)$, $\psi(K) = \nu(K + B)$ where $+$ denotes the Minkowski sum of convex sets. Let $\Delta : V \hookrightarrow V \times V$ denote the diagonal imbedding. Then the product of valuations is computed as follows:

$$(\phi \cdot \psi)(K) = (\mu \boxtimes \nu)(\Delta(K) + (A \times B))$$

where $\mu \boxtimes \nu$ denotes the usual product measure. Product of linear combinations of measures of the above form is defined by distributivity. The product
turns out to be well defined and it is determined uniquely by the above expressions since the valuations of the above form are dense in the space of polynomial valuations with respect to some natural topology.

We will discuss now only the translation invariant case though in [3] the product was defined on a wider class of polynomial valuations. Actually in this article we will need only the following result.

1.3 Proposition. Let \( p_1 : V \rightarrow W_1 \) and \( p_2 : V \rightarrow W_2 \) be surjective linear maps. Let \( \mu_1 \) and \( \mu_2 \) be Lebesgue measures on \( W_1 \) and \( W_2 \) respectively. Consider on \( V \) the valuations \( \phi_i(K) := \mu_i(p_i(K)) \), \( i = 1, 2 \). Then

\[
(\phi_1 \cdot \phi_2)(K) = (\mu_1 \otimes \mu_2)((p_1 \oplus p_2)(K))
\]

where \( p_1 \oplus p_2 : V \rightarrow W_1 \oplus W_2 \) is given by \( (p_1 \oplus p_2)(v) = (p_1(v), p_2(v)) \) and \( \mu_1 \otimes \mu_2 \) is the usual product measure on \( W_1 \oplus W_2 \).

Proof. First recall the following well known formula (which can be easily checked). Let \( I \) be a segment of unit length in the Euclidean space \( V \) orthogonal to a hyperplane \( H \). Then one has

\[
\frac{\partial}{\partial \lambda} \bigg|_0 \text{vol}(K + \lambda I) = \text{vol}_{n-1}(Pr_H K)
\]

where \( Pr_H \) denotes the orthogonal projection onto \( H \). Let now \( I_1, \ldots, I_k \) be pairwise orthogonal unit segments in \( V \). Let \( L \) be the orthogonal complement to their span. Thus \( \dim L = n - k \). By the inductive application of the above formula one obtains

\[
\frac{\partial^k}{\partial \lambda_1 \ldots \partial \lambda_k} \bigg|_0 \text{vol}_n(K + \sum_{j=1}^k \lambda_j I_j) = \text{vol}_{n-k}(Pr_L K).
\]

Let us return to the situation of our proposition. We may assume that \( W_i, i = 1, 2 \), are subspaces of \( V \) and \( p_i \) are orthogonal projections onto them. We may also assume that the measures \( \mu_i \) coincide with the volume forms induced by the Euclidean metric. Let us fix \( I_1^{(i)}, \ldots, I_{m_i}^{(i)}, i = 1, 2 \), pairwise orthogonal unit segments in the orthogonal complement to \( W_i \). Then

\[
\phi_i(K) = \frac{\partial^{m_i}}{\partial \lambda_1 \ldots \partial \lambda_{m_i}} \bigg|_0 \text{vol}_n(K + \sum_{j=1}^{m_i} \lambda_j I_j^{(i)}).
\]
Hence using the construction of the product described above we get

\[(\phi_1 \cdot \phi_2)(K) = \]

\[\frac{\partial^{m_1}}{\partial \lambda_1 \ldots \partial \lambda_{m_1}} \frac{\partial^{m_2}}{\partial \mu_1 \ldots \partial \mu_{m_2}} \left|_0 \right. \text{vol}_{2n}(\Delta(K) + \sum_{j=1}^{m_1} \lambda_j (I_j^{(1)} \times 0) + \sum_{l=1}^{m_2} \mu_l (0 \times I_l^{(2)})) =
\]

\[\text{vol}_{2n-m_1-m_2}(Pr_{W_1 \oplus W_2}(\Delta(K))) = \text{vol}_{2n-m_1-m_2}((p_1 \oplus p_2)(K)).\]

Q.E.D.

2 Hard Lefschetz theorem for even valuations.

To formulate our main result let us recall a definition from the representation theory. Let \(G\) be a Lie group. Let \(\rho\) be a continuous representation of \(G\) in a Fréchet space \(F\). A vector \(v \in F\) is called \(G\)-smooth if the map \(G \rightarrow F\) defined by \(g \mapsto g(v)\) is infinitely differentiable. It is well known (and easy to prove) that smooth vectors form a linear \(G\)-invariant subspace which is dense in \(F\). We will denote it by \(F^\text{sm}\). It is well known (see e.g. [25]) that \(F^\text{sm}\) has a natural structure of a Fréchet space, and the representation of \(G\) in \(F^\text{sm}\) is continuous with respect to this topology. Moreover \((F^\text{sm})^\text{sm} = F^\text{sm}\).

In our situation the Fréchet space is \(F = \text{Val}(V)\) with the topology of uniform convergence on compact subsets of \(K(V)\), and \(G = GL(V)\). The action of \(GL(V)\) on \(\text{Val}(V)\) is the natural one, namely for any \(g \in GL(V)\), \(\phi \in \text{Val}(V)\) one has \((g(\phi))(K) = \phi(g^{-1}K)\).

2.1 Theorem. Let \(0 \leq i < n/2\). Then the multiplication by \((V_1)^{n-2i}\) induces an isomorphism \(\text{Val}^e(V)^{sm} \rightarrow \text{Val}^e_{n-i}(V)^{sm}\). In particular for \(p \leq n-2i\) the multiplication by \((V_1)^p\) is an injection from \(\text{Val}^e_{i+p}(V) \hookrightarrow \text{Val}^e_{i+p}(V)\).

2.2 Remark. We would like to explain the use of the name ”hard Lefschetz theorem”. The classical hard Lefschetz theorem is as follows (see e.g. [11]). Let \(M\) be a compact Kähler manifold of complex dimension \(n\) with Kähler form \(\omega\). Let \([\omega] \in H^2(M, \mathbb{R})\) be the corresponding cohomology class. Then for \(0 \leq i < n\) the multiplication by \([\omega]^{2(n-i)}\) induces an isomorphism \(H^i(M, \mathbb{R}) \rightarrow H^{2n-i}(M, \mathbb{R})\).

First recall that in [3], Theorem 2.6, we have shown that \((V_1)^i\) is proportional to \(V_j\) with a non-zero constant of proportionality. Recall also the
Cauchy-Kubota formula (see e.g. [21]):

$$V_k(K) = c \int_{E \in \text{Gr}_k(V)} \text{vol}_k(\text{Pr}_E K) dE.$$ 

2.3 Lemma. Let $F \in \text{Gr}_i(V)$. Let $\phi(K) = \text{vol}_i(\text{Pr}_F(K))$. Then the image of $V_k \cdot \phi$ in $C(\text{Gr}_{k+i})$ under the Klain imbedding is given by the function

$$g(L) = c \int_{E \in \text{Gr}_k(V)} |\cos(L, R)| \cdot |\sin(E, F)| dE$$

where $dR$ is the (unique) $O(i) \times O(n-i)$-invariant probability measure on the Grassmannian of $(k+i)$-subspaces in $V$ containing $F$, and $c$ is a non-zero normalizing constant.

Proof. Using the Cauchy-Kubota formula and Proposition 1.3 we have

$$(V_k \cdot \phi)(K) = c \int_{E \in \text{Gr}_k(V)} \text{vol}_{k+i}((\text{Pr}_F \oplus \text{Pr}_E)(K)) dE.$$ 

Let $K = D_L$ be the unit ball in a subspace $L \in \text{Gr}_{k+i}(V)$. Then the image of $V_k \cdot \phi$ in $\text{Gr}_{k+i}(V)$ under the Klain imbedding is

$$g(L) = (V_k \cdot \phi)(D_L) = c \int_{E \in \text{Gr}_k(V)} \text{vol}_{k+i}((\text{Pr}_E \oplus \text{Pr}_F)(D_L)) dE.$$ 

2.4 Claim.

$$\text{vol}_{k+i}((\text{Pr}_E \oplus \text{Pr}_F)(D_L)) = \kappa |\cos(L, (E + F))| \cdot |\sin(E, F)|$$

where $\kappa$ is a non-zero constant.

Let us postpone the proof of this claim and let us finish the proof of Lemma 2.2. We get

$$g(L) = c' \int_{E \in \text{Gr}_k(V)} |\cos(L, (E + F))| \cdot |\sin(E, F)| dE.$$ 

For a fixed subspace $F \in \text{Gr}_i(V)$ let us denote by $U_F$ the open dense subset of $\text{Gr}_k(V)$ consisting of subspaces intersecting $F$ trivially. Clearly the complement to $U_F$ has smaller dimension. We have the map $T : U_F \to \text{Gr}_k(V/F)$ given by $T(E) := (E + F)/F$. Clearly $T$ commutes with the action of
the stabilizer of $F$ in the orthogonal group $O(n)$ (which is isomorphic to $O(i) \times O(n-i)$). Let $m : Gr_k(V/F) \rightarrow Gr_{k+i}(V)$ be the map sending a subspace in $V/F$ to its preimage in $V$ under the canonical projection $V \rightarrow V/F$. Then in this notation we have

$$g(L) = c' \int_{E \in U_F} |\cos(L, (m \circ T)(E))| \cdot |\sin(E, F)|dE.$$ 

Let us consider the following submanifold $M \subset Gr_k(V/F) \times Gr_k(V)$ given by

$$M = \{(R, E) | E \subset m(R)\}.$$ 

Then $U_F$ is isomorphic to $M$ via the map $T_1 : U_F \rightarrow M$ defined by $T_1(E) = (T(E), E)$. Clearly $T_1$ commutes with the natural action of the group $O(i) \times O(n-i)$ which is the stabilizer of $F$ in $O(n)$. We have also the projection $t : M \rightarrow Gr_k(V/F)$ given by $t(R, E) = R$. Then

$$g(L) = c' \int_{R \in Gr_k(V/F)} dR |\cos(L, m(R))| \left[ \int_{E \in Gr_k(t^{-1}(R))} |\sin(E, F)|d\mu_R(E) \right]$$

where $\mu_R$ is a measure on $Gr_k(t^{-1}(R))$. Note that the integral in square brackets is positive and does not depend on $R$ since the map $T = t \circ T_1$ commutes with the action of $O(i) \times O(n-i)$. Hence $g(L) = c'' \int_{R \in Gr_k(V/F)} |\cos(L, m(R))|dR$. Lemma 2.2 is proved. Q.E.D.

**Proof** of Claim 2.3. We may assume that $E \cap F = 0$. Set for brevity $p := Pr_E \oplus Pr_F$. Then $p$ factorizes as $p = q \circ Pr_{E+F}$ where $q : E + F \rightarrow E \oplus F$ is the restriction of $Pr_E \oplus Pr_F$ to the subspace $E + F$. Then we have

$$vol_{k+i}((Pr_E \oplus Pr_F)(D_L)) = vol_{k+i}(Pr_{E+F}(D_L)) \cdot \frac{vol_{k+i}(q(D_{E+F}))}{vol_{k+i}(D_{E+F})} =$$

$$vol_{k+i}(D_L) : |\cos(L, (E + F))| \cdot \frac{vol_{k+i}(q(D_{E+F}))}{vol_{k+i}(D_{E+F})}.$$ 

Let us compute the last term in the above expression. Note that the dual map $q^* : E \oplus F \rightarrow E + F$ is given by $q^*((x, y)) = x + y$. Clearly

$$\frac{vol(q^*(D_{E\oplus F}))}{vol(D_{E\oplus F})} = |\sin(E, F)|.$$ 

But the left hand side in the last expression is equal to $\frac{vol(q(D_{E+F}))}{vol(D_{E+F})}$. Thus Claim 2.3 is proved. Q.E.D.

We will need one more lemma.
2.5 Lemma. Let $\phi \in Val_{i}^{ev}(V)$ be a valuation given by

$$\phi(K) = \int_{F \in Gr_{i}(V)} f(F) \text{vol}_{i}(Pr_{F}K) dF$$

with $f \in C(Gr_{i}(V))$. Then the image of $V_{k} \cdot \phi$ in $C(Gr_{i+k}(V))$ under the Klain imbedding is given by

$$g(L) = cT_{k+i,k+i} \circ R_{k+i,i}(f)$$

where $c$ is a non-zero normalizing constant.

Proof. By Lemma 2.2 one has

$$g(L) = c \int_{F \in Gr_{i}(V)} dF f(F) \int_{Gr_{k+i} \ni R \supset F} |\cos(L,R)| dR =$$

$$c \int_{R \in Gr_{k+i}(V)} dR |\cos(L,R)| \int_{F \in Gr_{i}(R)} f(F) dF = cT_{k+i,k+i} \circ R_{k+i,i}(f).$$

Q.E.D.

Proof of Theorem 2.1. Let us consider the Klain imbedding $Val_{i}^{ev}(V)^{sm} \hookrightarrow C^{\infty}(Gr_{i}(V))$. In [6] it was shown that the image of $Val_{i}^{ev}(V)^{sm}$ is a closed subspace which coincides with the image of the cosine transform $T_{i,i}: C^{\infty}(Gr_{i}(V)) \rightarrow C^{\infty}(Gr_{n-i}(V))$. By Lemma 2.4, $V_{n-2i} \cdot T_{i,i}(f) = T_{n-i,n-i} \circ R_{n-i,i}(f)$. It was shown in [7] that $R_{n-i,i}: C^{\infty}(Gr_{i}(V)) \rightarrow C^{\infty}(Gr_{n-i}(V))$ is an isomorphism. Hence the image under the Klain imbedding of $V_{n-2i} \cdot Val_{i}^{ev}(V)^{sm}$ coincides with the image of $T_{n-i,n-i} : C^{\infty}(Gr_{n-i}) \rightarrow C^{\infty}(Gr_{n-i})$ which is equal to $Val_{n-i}^{ev}(V)^{sm}$. Hence the multiplication by $V_{n-i}^{n-2i} : Val_{i}^{ev}(V)^{sm} \rightarrow Val_{n-i}^{ev}(V)^{sm}$ is onto. Let us check that the kernel is trivial. Indeed this operator commutes with the action of the orthogonal group $O(n)$. The spaces $Val_{i}^{ev}(V)^{sm}$ and $Val_{n-i}^{ev}(V)^{sm}$ are isomorphic as representations of $O(n)$ (see [4], Theorem 1.2.2). Since every irreducible representation of $O(n)$ enters with finite multiplicity (in fact, at most 1) then any surjective map must be injective. Q.E.D.

Now let us discuss an application to valuations invariant under a group. Let $G$ be a compact subgroup of the orthogonal group $O(n)$. Let us denote by $Val^{G}(V)$ the space of $G$-invariant translation invariant continuous valuations. Assume that $G$ acts transitively on the unit sphere. Then it was shown in [1] that $Val^{G}(V)$ is finite dimensional. Also it was shown in [4] (see the proof
of Corollary 1.1.3) that \( \text{Val}^G(V) \subset \text{Val}^{sm}(V) \). We have also McMullen’s decomposition with respect to the degree of homogeneity:

\[
\text{Val}^G(V) = \oplus_{i=0}^n \text{Val}^G_i(V)
\]

where \( \text{Val}^G_i(V) \) denotes the subspace of \( i \)-homogeneous \( G \)-invariant valuations. Then it was shown in [3], Theorem 0.9 that \( \text{Val}^G(V) \) is a finite dimensional graded algebra (with grading given by the degree of homogeneity) satisfying the Poincaré duality (i.e. it is a so called Frobenius algebra). A version of the hard Lefschetz theorem was given in [4]. Now we would like to state another version of it.

**2.6 Corollary.** Let \( G \) be a compact subgroup of the orthogonal group \( O(n) \) acting transitively on the unit sphere. Assume that \(-\text{Id} \in G\). Let \( 0 \leq i < n/2 \) where \( n = \dim V \). Then the multiplication by \((V_1)^{n-2i}\) induces an isomorphism \( \text{Val}^G_i(V) \cong \text{Val}^G_{n-i}(V) \). In particular for \( p \leq n - 2i \) the multiplication by \((V_1)^p\) induces an injection \( \text{Val}^G_i(V) \hookrightarrow \text{Val}^G_{i+p}(V) \).

This result follows immediately from Theorem 2.1.

### 3 The case of odd valuations.

In this section we state the conjecture about an analogue of the hard Lefschetz theorem for odd valuations and discuss it briefly.

**Conjecture.** Let \( 0 \leq i < n/2 \). Then the multiplication by \((V_1)^{n-2i}\) is a map \( \text{Val}_i^{odd}(V)^{sm} \to \text{Val}_{n-i}^{odd}(V)^{sm} \) with trivial kernel and dense image. In particular for \( p \leq n - 2i \) the multiplication by \((V_1)^p\) is an injection from \( \text{Val}_i^{odd}(V)^{sm} \hookrightarrow \text{Val}_{i+p}^{odd}(V)^{sm} \).

Let us show that it is enough to prove only either injectivity or density of the image of the multiplication by \((V_1)^{n-2i}\). This is a particular case of the following slightly more general statement.

**3.1 Proposition.** Let \( L : \text{Val}_i^{odd}(V)^{sm} \to \text{Val}_{n-i}^{odd}(V)^{sm} \) be a continuous linear operator commuting with the action of the orthogonal group \( O(n) \). Then it has a dense image if and only if its kernel is trivial.

**Proof.** Note that the spaces \( \text{Val}_i^{odd}(V)^{sm} \) and \((\text{Val}_{n-i}^{odd}(V)^*)^{sm} \otimes \text{Val}_n(V)\) are isomorphic as representation of the full linear group \( GL(V) \) by the Poincaré duality proved in [3]. Let us replace all spaces by their Harish-Chandra...
modules. All irreducible representations of $O(n)$ are selfdual. Hence the subspaces of $O(n)$-finite vectors in $Val_{i}^{\text{odd}}(V)^{\text{sm}}$ and $Val_{n-i}^{\text{odd}}(V)^{\text{sm}}$ are isomorphic as $O(n)$-modules. Moreover each irreducible representation of $O(n)$ enters into these subspaces with finite multiplicity (since by [2] both spaces are realized as subquotients in so called degenerate principal series representations of $GL(V)$, namely representations induced from a character of a parabolic subgroup). Then it is clear that the operator $L$ between the spaces of $O(n)$-finite vectors commuting with the action of $O(n)$ is surjective if and only if it is injective. Q.E.D.

4 Hard Lefschetz theorem for even valuations from [4].

In this short section we remind another form of the hard Lefschetz theorem for even valuations as it was proven in [4]. Note that it also was heavily based on the properties of the Radon and cosine transforms on Grassmannians.

Let us fix on $V$ a Euclidean metric, and let $D$ denote the unit Euclidean ball with respect to this metric. Let us define on the space of translation invariant continuous valuations an operation $\Lambda$ of mixing with the Euclidean ball $D$, namely

$$(\Lambda \phi)(K) := \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} \phi(K + \varepsilon D)$$

for any convex compact set $K$. Note that $\phi(K + \varepsilon D)$ is a polynomial in $\varepsilon \geq 0$ by McMullen’s theorem [18]. It is easy to see that the operator $\Lambda$ preserves parity and decreases the degree of homogeneity by one. In particular we have

$$\Lambda : Val_{k}^{\text{ev}} \rightarrow Val_{k-1}^{\text{ev}}(V).$$

The following result is Theorem 1.1.1 in [4].

4.1 Theorem. Let $n \geq k > n/2$. Then $\Lambda^{2k-n} : (Val_{k}^{\text{ev}})^{\text{sm}} \rightarrow (Val_{n-k}^{\text{ev}}(V))^{\text{sm}}$ is an isomorphism. In particular $\Lambda^{i} : Val_{k}^{\text{ev}} \rightarrow Val_{k-i}^{\text{ev}}(V)$ is injective for $1 \leq i \leq 2n - k$.

References

[1] Alesker, Semyon; On P. McMullen’s conjecture on translation invariant valuations. Adv. Math. 155 (2000), no. 2, 239–263.
[2] Alesker, Semyon; Description of translation invariant valuations on convex sets with solution of P. McMullen’s conjecture. Geom. Funct. Anal. 11 (2001), no. 2, 244–272.

[3] Alesker, Semyon; The multiplicative structure on polynomial continuous valuations. Geom. Funct. Anal., to appear. also: math.MG/0301148

[4] Alesker, Semyon; Hard Lefschetz theorem for valuations, complex integral geometry, and unitarily invariant valuations. math.MG/0209263

[5] Alesker, Semyon; The $\alpha$-cosine transform and intertwining integrals on real Grassmannians. Preprint.

[6] Alesker, Semyon; Bernstein, Joseph; Range characterization of the cosine transform on higher Grassmannians. Adv. Math., to appear. also: math.MG/0111031

[7] Gelfand, I. M.; Graev, M. I.; Roşu, R.; The problem of integral geometry and intertwining operators for a pair of real Grassmannian manifolds. J. Operator Theory 12 (1984), no. 2, 359–383.

[8] Goodey, Paul; Howard, Ralph; Processes of flats induced by higher-dimensional processes. Adv. Math. 80 (1990), no. 1, 92–109.

[9] Goodey, Paul R.; Howard, Ralph; Processes of flats induced by higher-dimensional processes. II. Integral geometry and tomography (Arcata, CA, 1989), 111–119, Contemp. Math., 113, Amer. Math. Soc., Providence, RI, 1990.

[10] Goodey, Paul; Howard, Ralph; Reeder, Mark; Processes of flats induced by higher-dimensional processes. III. Geom. Dedicata 61 (1996), no. 3, 257–269.

[11] Griffiths, Phillip; Harris, Joseph; Principles of algebraic geometry. Pure and Applied Mathematics. Wiley-Interscience [John Wiley and Sons], New York, 1978.

[12] Grinberg, Eric L.; Radon transforms on higher Grassmannians. J. Differential Geom. 24 (1986), no. 1, 53–68.

[13] Hadwiger, H.; Vorlesungen über Inhalt, Oberfläche und Isoperimetrie. (German) Springer-Verlag, Berlin-Göttingen-Heidelberg 1957.
[14] Klain, Daniel; A short proof of Hadwiger’s characterization theorem. Mathematika 42 (1995), no. 2, 329–339.

[15] Klain, Daniel; Even valuations on convex bodies. Trans. Amer. Math. Soc. 352 (2000), no. 1, 71–93.

[16] Matheron, G.; Un théorème d’unicité pour les hyperplans poissoniens. (French) J. Appl. Probability 11 (1974), 184–189.

[17] Matheron, G.; Random sets and integral geometry. Wiley Series in Probability and Mathematical Statistics. Wiley, New York-London-Sydney, 1975.

[18] McMullen, Peter; Valuations and Euler-type relations on certain classes of convex polytopes. Proc. London Math. Soc. (3) 35 (1977), no. 1, 113–135.

[19] McMullen, Peter; Continuous translation-invariant valuations on the space of compact convex sets, Arch. Math. (Basel) 34:4 (1980), 377-384.

[20] McMullen, Peter; Valuations and dissections. Handbook of convex geometry, Vol. A, B, 933–988, North-Holland, Amsterdam, 1993.

[21] McMullen, Peter; Schneider, Rolf; Valuations on convex bodies. Convexity and its applications, 170–247, Birkhäuser, Basel, 1983.

[22] Schneider, Rolf; Convex bodies: the Brunn-Minkowski theory. Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1993.

[23] Schneider, Rolf; Simple valuations on convex bodies. Mathematika 43 (1996), no. 1, 32–39.

[24] Schneider, Rolf; Weil, Wolfgang; Integralgeometrie. (German) [Integral geometry] Teubner Skripten zur Mathematischen Stochastik. [Teubner Texts on Mathematical Stochastics] B. G. Teubner, Stuttgart, 1992.

[25] Wallach, Nolan R.; Real reductive groups. I. Pure and Applied Mathematics, 132. Academic Press, Inc., Boston, MA, 1988.