A VASE OF CATENOIDS

PETER CONNOR

Abstract. In this note we construct a vase of catenoids - a symmetric immersed minimal surface with planar and catenoid ends.

2000 Mathematics Subject Classification. Primary 53A10; Secondary 49Q05, 53C42.

Key words and phrases. Minimal surface, catenoid.

1. Introduction

The building blocks for minimal surfaces with finite total curvature and embedded ends are planes and catenoids. One can consider the possible arrangements of planar and catenoid ends that yields a minimal surface. This note proves the existence of two beautiful families of minimal surfaces on punctured spheres. The first, which we call a vase of catenoids, has a horizontal planar end, a downward pointing catenoid end with vertical normal, and symmetrically placed upward pointing catenoid ends with non-vertical normals. See figure 1.1.

Figure 1.1. Vase of catenoids
The second is a variation of the vase of catenoids. Imagine taking two copies of a vase of catenoids. Cut off the bottom catenoid end on each copy, and glue the copies together along the resulting closed curves. See figure [1.2] for the resulting surface. It has two horizontal planar ends, symmetrically placed downward pointing catenoid ends with non-vertical normals, and symmetrically placed upward pointing catenoid ends with non-vertical normals.

The construction of these surfaces was inspired by the Finite Riemann minimal surface constructed in [4] with a horizontal planar end together with two catenoid ends with non-vertical normals, and also by the $k$-noid surface constructed by Jorge and Meeks [1] with catenoid ends at each root of unity with horizontal normals in the direction of that root of unity. None of these surfaces are embedded. By the Lopez-Ros theorem [2], the plane and catenoid are the only complete embedded minimal surfaces on punctured spheres.

2. Weierstrass Representation

We use the Weierstrass Representation of minimal surfaces on a punctured sphere, which may be written as

$$X(z) = \text{Re} \int_{z_0}^{z} \left( \frac{1}{2} \left( \frac{1}{G} - G \right) dh, \frac{i}{2} \left( \frac{1}{G} + G \right) dh, dh \right)$$
where $z, z_0 \in \Sigma = \mathbb{C} - \{p_1, p_2, \ldots, p_n\}$. The points $p_1, p_2, \ldots, p_n$ are the ends of the surface, $G$ is the composition of stereographic projection with the Gauss map, and $dh$ is a meromorphic one-form called the height differential. A good reference for the Weierstrass representation is [3]. One issue is that $X$ depends on the path of integration. The map $X : \Sigma \to \mathbb{R}^3$ is well defined provided that

$$\text{Re} \int_{\gamma} \left( \frac{1}{2} \left( \frac{1}{G} - G \right) dh, \frac{i}{2} \left( \frac{1}{G} + G \right) dh, dh \right) = (0, 0, 0)$$

for all closed curves $\gamma$ in $\Sigma$. This is called the period problem, and it can be expressed as

$$\text{Res}_{p_j} \left( \left( \frac{1}{G} - G \right) dh \right) \in \mathbb{R}, \text{Res}_{p_j} \left( i \left( \frac{1}{G} + G \right) dh \right) \in \mathbb{R}, \text{Res}_{p_j} (dh) \in \mathbb{R}$$

for $j = 1, 2, \ldots, n$.

A second issue is that we want $X$ to be regular. This is ensured by requiring that $G$ has either a zero or pole at $p \in \Sigma$ if and only if $dh$ has a zero at $p$ with the same multiplicity.

3. Constructions

One can use the desired geometry of a minimal surface to create potential Weierstrass data for a minimal surface. A vertical normal at $p_j$ corresponds to $G(p_j) = 0$ (downward pointing normal) or $G(p_j) = \infty$ (upward pointing normal). If $X$ has a horizontal planar end at $p_j$ then $G$ has a zero or pole of order $k + 1$ and $dh$ has a zero of order $k - 1$ at $p_j$. If $X$ has a catenoid end at $p_j$ with vertical normal then $G$ has a simple pole or zero and $dh$ has a simple pole at $p_j$. If $X$ has a catenoid end at $p_j$ with non-vertical normal then $G$ has neither a pole or zero and $dh$ has a double order pole at $p_j$.

When the domain is a punctured sphere, the sum of zeros of $G$ equals the sum of poles of $G$ and the sum of zeros of $dh$ is two less then the sum of poles of $dh$.

We can use the images of our desired surfaces to construct the Weierstrass data $G$ and $dh$. For the vase of catenoids, place the horizontal planar end at $z = \infty$ and the downward pointing catenoid end with vertical normal at $z = 0$. Place the upward pointing catenoid ends with non-vertical normals at each root of unity. Fix $G(0) = 0$. Then $G(\infty) = \infty$. There will also be a point on each catenoid at the roots of unity with vertical downward pointing normal. In keeping with the symmetry of the surface, fix

$$G \left( ae^{i2\pi j/k} \right) = 0$$

for $j = 0, 1, \ldots, k - 1$. Let $G(z)$ have simple zeros at $z = 0$ and the roots of $z^k = a^k$, with $a \in \mathbb{R}$. Then, $G$ has a pole of order $k + 1$ at $z = \infty$, and

$$G(z) = \rho z (z^k - a^k).$$

The height differential has simple zeros at the roots of $z^k = a^k$, double order poles at the $k$th-roots of unity, and a simple pole at $z = 0$. This forces $dh$ to have a zero of order $k - 1$ at $z = \infty$, and

$$dh = \frac{a^k - z^k}{z(z^k - 1)^2} dz.$$
Theorem 3.1. For each positive integer \( k > 1 \) and real number \( a \in (0,1) \) there exists a \( \rho > 0 \) such that

\[
G(z) = \rho z (z^k - a^k)
\]

\[
dh = \frac{a^k - z^k}{z(z^k - 1)^2} dz
\]

is the Weierstrass data for a minimal surface with a horizontal planar end at \( z = \infty \), a downward pointing vertical catenoid end at \( z = 0 \), and upward pointing non-vertical catenoid ends at the roots of unity.

Proof. All that remains is to solve the period problem. As the residues of \( dh \) are all real, \( dh \) has no periods. The residues of \( G dh \) and \( 1/G dh \) are zero at \( z = 0 \) and \( z = \infty \). Assuming \( \rho \in \mathbb{R} \), the symmetries of the surface we are constructing reduce the period problem to the equation

\[
0 = \text{Res}_1 \left( \left( \frac{1}{G} + G \right) dh \right) = \frac{\rho (a^k - 1)(ka^k + k - a^k + 1)}{k^2} + \frac{k + 1}{\rho k^2}
\]

which is solved when

\[
\rho = \sqrt{\frac{k + 1}{(1 - a^k)(ka^k + k - a^k + 1)}}.
\]

Examining figure 1.2 the second family has horizontal planar ends at \( z = 0 \) and \( z = \infty \), downward pointing catenoid ends with non-vertical normal at the roots of \( z^k = b^k \), and upward pointing catenoid ends with non-vertical normal at the roots of \( z^k = 1/b^k \). The Gauss map is 0 at the roots of \( z^k = a^k \) and \( \infty \) at the roots of \( z^k = 1/a^k \). The height differential \( dh \) has double order poles at the roots of \( z^k = b^k \) and \( z^k = 1/b^k \) and simple zeros at the roots of \( z^k = a^k \) and \( z^k = 1/a^k \). In order for the surface to have horizontal planar ends at 0 and \( \infty \), we need \( dh \) to have zeros at 0 and \( \infty \). Thus, set \( dh \) with zeros of order \( k - 1 \) at 0 and \( \infty \). This forces \( G \) to have a zero of order \( k + 1 \) at \( z = 0 \) and a pole of order \( k + 1 \) at \( z = \infty \). Hence,

\[
G(z) = \frac{\rho z^{k+1}(z^k - a^k)}{a^k z^k - 1}
\]

and

\[
dh = \frac{b^{2k} z^{k-1}(z^k - a^k)(a^k z^k - 1)}{a^k(z^k - b^k)^2(b^k z^k - 1)^2} dz.
\]

If \( \rho = 1 \) then, similar to the first example, the period problem reduces to a single equation.
Theorem 3.2. For each positive integer $k > 1$ and real number $b \in (0, 1)$ there exists an $a > 0$ such that

$$G(z) = \frac{z^{k+1}(z^k - a^k)}{a^k z^k - 1},$$

$$dh = \frac{b^{2k} z^{k-1}(z^k - a^k)(a^k z^k - 1)}{a^k(z^k - b^k)^2(b^k z^k - 1)^2} \, dz$$

is the Weierstrass data for a minimal surface with horizontal planar ends at $z = 0$ and $z = \infty$, upward pointing non-vertical catenoid ends at solutions to $z^k = b^k$, and downward pointing non-vertical catenoid ends at solutions to $z^k = 1/b^k$.

Proof. As with the vase of catenoids, the symmetries of the surface reduce the period problem to the equation

$$0 = \text{Res}_b \left( \frac{1}{G} + G \right) \, dh$$

$$= \frac{b^{2k} (k - 1 + b^{2+2k}(k - 1) + (b^2 + b^{2k})(k + 1)) a^{2k}}{a^k b(b^k - 1)^3(b^k + 1)^3 k^2}$$

$$+ \frac{2b^k (1 + b^{2+4k} - (b^{2k} + b^{2+2k})(2k + 1)) a^k}{a^k b(b^k - 1)^3(b^k + 1)^3 k^2}$$

$$+ \frac{-k - 1 + b^{2k} + b^{2+4k} - b^{2+6k} + 3kb^{2k} + 3kb^{2+4k} - kb^{2+6k}}{a^k b(b^k - 1)^3(b^k + 1)^3 k^2}$$

which is solved when

$$a = \sqrt{-1 - b^{2+4k} + (b^{2k} + b^{2+2k})(2k + 1) + (1 - b^{2k}) \sqrt{k^2 + b^2(1 - b^{2k})^2(2k + 1) + k^2 b^2(1 + b^{1k} + b^{2+4k})}}$$

$$b^c(k - 1 + b^{2+2k}(k - 1) + (b^2 + b^{2k})(k + 1))$$

and $0 < b < 1$. We do need $a > 0$. Assume that $k > 1$ and $0 < b < 1$. Then the denominator in $a$ is clearly positive, and the numerator in $a$ is positive because it is greater than

$$-1 - b^{2+4k} + (b^{2k} + b^{2+2k})(2k + 1) + (1 - b^{2k}) \sqrt{k^2} = k - 1 + b^{2k}(k + 1 - b^{2+2k} + b^2(2k + 1)) > 0.$$
[4] M. Weber. Classical minimal surfaces in euclidean space by examples. geometric and computational aspects of the weierstrass representation. In Global theory of minimal surfaces, pages 19–64. American Mathematical Society, 2005.

PETER CONNOR, DEPARTMENT OF MATHEMATICAL SCIENCES, INDIANA UNIVERSITY SOUTH BEND, 1700 MISHAWAKA AVE, SOUTH BEND, IN 46634, USA, PCONNOR@IUSB.EDU