Finsleroid-regular space developed. Berwald case

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Abstract

The Finsleroid–Finsler space becomes regular when the norm $||b|| = c$ of the input 1-form $b$ is taken to be an arbitrary positive scalar $c(x) < 1$. By performing required direct evaluations, the respective spray coefficients have been obtained in a simple and transparent form. The adequate continuation into the regular pseudo-Finsleroid domain has been indicated. The Finsleroid-regular Berwald space is found under the assumptions that the Finsleroid charge is a constant and the 1-form $b$ is parallel.

*Keywords:* Finsler metrics, spray coefficients, curvature tensors.
1. Description of new conclusions

In the Finsleroid-Finsler space $\mathcal{F}_g$ (as well in its relativistic counterpart space $\mathcal{F}_g^{SR}$) constructed and studied in the previous papers [1-4] the consideration was referred everywhere to the case $|b| = 1$, that is, the vector field $b_i(x)$ involved in the 1-form $b = b_i(x)y^i$ was assumed to be of unit length. In the present paper, we expand the restrictive case to get the possibility $|b| < 1$.

Namely, we shall deal with the Finsler space notion specified by the condition that the Finslerian metric function $K(x,y)$ be of the functional dependence

$$K(x,y) = \Phi\left(g(x), b_i(x), a_{ij}(x), y\right)$$

subject to the conditions

$$a^{ij}(x)b_i(x)b_j(x) = c^2(x)$$

and

$$0 < c(x) < 1.$$  \hspace{1cm} (1.3)

In (1.1), $a_{ij}(x)$ is a Riemannian metric tensor and $g(x)$ plays the role of the Finsleroid charge. The explicit form of the $\Phi$ is specified by the representations written out explicitly in Appendix A in the positive-definite case, and in Appendix B in the relativistic case, respectively we obtain the spaces $\mathcal{F}R_{g;c}^{PD}$ and $\mathcal{F}R_{g;c}^{SR}$. They fulfill the correspondence

$$\mathcal{F}R_{g;c=1}^{PD} = \mathcal{F}F_g^{PD}$$

and

$$\mathcal{F}R_{g;c=1}^{SR} = \mathcal{F}F_g^{SR}.$$ \hspace{1cm} (1.5)

The Riemannian squared metric function

$$S^2 = b^2 + q^2$$

underlines the positive-definite Finsler space $\mathcal{F}R_{g;c}^{PD}$ under study, and the pseudo-Riemannian $S^2 = b^2 - q^2$ is to relate to the space $\mathcal{F}R_{g;c}^{SR}$.

This scalar $c(x)$ proves to play the role of the regularization factor. Indeed, in the space $\mathcal{F}R_{g;c}^{PD}$ the metric function $K$ is constructed in accordance with the formulas (A.19)-(A.23) which involve the square-root variable $q(x,y) = \sqrt{S^2 - b^2}$ (see (1.6)). Differentiating various tensors of the space $\mathcal{F}R_{g;c}^{PD}$ gives rise, therefore, to appearance of degrees of the fraction $1/q$. Because of the inequality (A.5), the fractions $1/q$ do not produce any singularities as far as $c < 1$. If, however, $c = 1$, we have $q(x,y) = 0$ when the vector $y \in T_xM$ is a factor of the vector $b^i(x)$, so that in the space $\mathcal{F}F_g^{PD}$ the singularities can appear when the vector $y$ belongs to the Finsleroid axis.

In contrast, in the space $\mathcal{F}R_{g;c}^{PD}$ which uses $c \neq 1$ the variable $q$ does not vanish anywhere over all the slit tangent bundle $T_M \setminus 0$, and we obtain the function $\Phi$ which is smooth of the class $C^\infty$.

We use

REGULARITY DEFINITION. The Finsler space under study is regular in the following sense: over all the slit tangent bundle $T_M \setminus 0$, the Finsler metric function $K(x,y)$ of the space is smooth of the class $C^\infty$ and the entailed Finsler metric tensor $g_{ij}(x,y)$ is positive-definite: $\det(g_{ij}) > 0$. 

We use
Attentive direct calculations of the induced spray coefficients $G^i = \gamma^i_{nm} y^n y^m$, where $\gamma^i_{nm}$ denote the associated Finslerian Christoffel symbols, can be used to arrive at the following result.

**THEOREM 1.** In the Finsleroid-regular space $\mathcal{FR}_{g;c}^{PD}$ the spray coefficients $G^i$ can explicitly be written in the form

$$G^i = \frac{g}{v} \left(y^j y^h \nabla_j b_h + gq b^i f_j \right) v^i - gq f^i + E^i + a^i_{nm} y^n y^m. \quad (1.7)$$

We use the notation

$$v^i = y^i - bb^i \quad (1.8)$$

and

$$f_j = f^j_n y^n, \quad f^i = f^i_n y^n, \quad f^i_n = a^{ik} f^k_n, \quad f_{mn} = \nabla_m b_n - \nabla_n b_m \equiv \frac{\partial b_n}{\partial x^m} - \frac{\partial b_m}{\partial x^n}; \quad (1.9)$$

where $\nabla$ means the covariant derivative in terms of the associated Riemannian space $R_N = (M, S)$ (see (A.83)); $a^i_{nm}$ stands for the Riemannian Christoffel symbols (A.84) constructed from the input Riemannian metric tensor $a_{ij}(x)$; the coefficients $E^i$ involving the gradients $g_h = \partial g/\partial x^h$ of the Finsleroid charge can be taken as

$$E^i = M(yg) y^i + K \frac{b^2 w^2}{g B} (yg) X A^i - \frac{1}{2} MK^2 g_h g^i h, \quad (1.10)$$

where $(yg) = g_h y^h$; $X$ is the function given in (A.67); $w$ is the variable (A.38); the function $M$ entered (1.10) is defined by $\partial K/\partial g = (1/2)MK$.

The difference $G^i - E^i - a^i_{nm} y^n y^m$ involves the crucial terms linear in the covariant derivative $\nabla_j b_h$.

Also, the following theorem is valid.

**THEOREM 2.** In the Finsleroid-regular space $\mathcal{FR}_{g;c}^{PD}$ the equality

$$G^i = a^i_{nm} y^n y^m \quad (1.11)$$

holds if and only if

$$g = \text{const} \quad \text{and} \quad \nabla_m b_n = 0. \quad (1.12)$$

When the equality (1.11) holds, one says that the Finsler space is the Berwald space (see [5]). The above theorem yields a simple and attractive example of the regular Berwald space.

The sufficiency of the conditions (1.12) is obvious from the representation (1.7). To verify the necessity, it is worth noting that in the Berwald case the covariant derivative of the Cartan tensor $A_{ijk}$ vanishes identically (see [5,6]), in which case the representations (A.71) and (A.72) just entail $g = \text{const}$, which in turn yields $E^i = 0$. With this observation, it is easy to see that the representation (1.7) reduces to (1.11) if only $\nabla_m b_n = 0$, as far as the Finsleroid charge $g$ is kept differing from zero (the choice $g = 0$ would reduce the Finsleroid-Finsler space to a Riemannian space).
Appendix A: Distinguished $\mathcal{F}R_{g,x}^{PD}$-notions

Let $M$ be an $N$-dimensional $C^\infty$ differentiable manifold, $T_x M$ denote the tangent space to $M$ at a point $x \in M$, and $y \in T_x M \setminus 0$ mean tangent vectors. Suppose we are given on $M$ a Riemannian metric $S = S(x, y)$. Denote by $\mathcal{R}_N = (M, S)$ the obtained $N$-dimensional Riemannian space. Let us also assume that the manifold $M$ admits a non–vanishing 1-form $b = b(x, y)$, denote by

$$c = ||b||_{\text{Riemannian}}$$

(A.1)

the respective Riemannian norm value. Assuming

$$0 < c < 1,$$

(A.2)

we get

$$S^2 - b^2 > 0$$

(A.3)

and may conveniently use the variable

$$q := \sqrt{S^2 - b^2}.$$  

(A.4)

Obviously, the inequality

$$q^2 \geq \frac{1 - c^2}{c^2} b^2$$

(A.5)

is valid.

With respect to natural local coordinates in the space $\mathcal{R}_N$ we have the local representations

$$a^{ij} b_i b_j = c^2$$

(A.6)

and

$$b = b_i(x) y^i, \quad S = \sqrt{a_{ij}(x) y^i y^j}.$$  

(A.7)

The reciprocity $a^{in} a_{nj} = \delta^i_j$ is assumed, where $\delta^i_j$ stands for the Kronecker symbol. The covariant index of the vector $b_i$ will be raised by means of the Riemannian rule $b^i = a^{ij} b_j$, which inverse reads $b_i = a_{ij} b^j$. We also introduce the tensor

$$r_{ij}(x) := a_{ij}(x) - b_i(x) b_j(x)$$

(A.8)

to have the representation

$$q = \sqrt{r_{ij}(x) y^i y^j}.$$  

(A.9)

The equalities

$$r_{ij} b^i = (1 - c^2) b_i, \quad r_{in} r^{nj} = r^j_i - (1 - c^2) b^j b_i$$

(A.10)

hold.

We introduce on the background manifold $M$ a scalar field $g = g(x)$ subject to ranging

$$-2 < g(x) < 2,$$

(A.11)

and apply the convenient notation

$$h(x) = \sqrt{1 - \frac{1}{4}(g(x))^2}, \quad G(x) = \frac{g(x)}{h(x)}$$

(A.12)
(compare with (2.10) in [3]). The Finsleroid-regular space is underlined by the characteristic quadratic form

$$B(x, y) := b^2 + gqb + q^2 \equiv \frac{1}{2} \left[ (b + g_q)^2 + (b + g_q)^2 \right]$$

(A.13)

(cf. (2.11) in [3]), where $g_+ = \frac{1}{2}g + h$ and $g_- = \frac{1}{2}g - h$, and the discriminant $D_{\{B\}}$ of the quadratic form $B$ is negative:

$$D_{\{B\}} = -4h^2 < 0.$$  

(A.14)

Therefore, the quadratic form $B$ is positively definite. In the limit $g \to 0$, the definition (A.13) degenerates to the quadratic form of the input Riemannian metric tensor:

$$B|_{g=0} = b^2 + q^2 \equiv S^2.$$  

(A.15)

Also,

$$\eta g|_{g^2=b} = c^2,$$  

(A.16)

where

$$\eta = \frac{1}{1 + gc\sqrt{1 - c^2}}.$$  

(A.17)

On the definition range (A.11) of the $g$, we have

$$\eta > 0.$$  

(A.18)

Under these conditions, we introduce the following definition.

**DEFINITION.** The scalar function $K(x, y)$ given by the formulas

$$K(x, y) = \sqrt{B(x, y)} J(x, y)$$

(A.19)

and

$$J(x, y) = e^{-\frac{1}{2}G(x) f(x, y)},$$

(A.20)

where

$$f = -\arctan \frac{G}{2} + \arctan \frac{L}{hb}, \quad \text{if} \quad b \geq 0,$$

(A.21)

and

$$f = \pi - \arctan \frac{G}{2} + \arctan \frac{L}{hb}, \quad \text{if} \quad b \leq 0,$$

(A.22)

with

$$L = q + \frac{g}{2}b.$$  

(A.23)

is called the Finsleroid-regular metric function.

The function (A.23) obeys the identity

$$L^2 + h^2b^2 = B.$$  

(A.24)

**DEFINITION.** The arisen space

$$\mathcal{FR}_{g;c}^{FD} := \{ \mathcal{R}_N; b_i(x); g(x); K(x, y) \}$$

(A.25)
is called the Finsleroid-regular space.

**DEFINITION.** The space $\mathcal{R}_N$ entering the above definition is called the associated Riemannian space.

The associated Riemannian metric tensor $a_{ij}$ has the meaning

$$a_{ij} = g_{ij} \big|_{g=0}. \quad (A.26)$$

**DEFINITION.** Within any tangent space $T_x M$, the Finsleroid-regular metric function $K(x, y)$ produces the regular Finsleroid

$$\mathcal{F}R^{PD}_{g,c\{x\}} := \{ y \in \mathcal{F}R^{PD}_{g,c\{x\}} : y \in T_x M, K(x, y) \leq 1 \}. \quad (A.27)$$

**DEFINITION.** The regular Finsleroid Indicatrix $IR^{PD}_{g,c\{x\}} \subset T_x M$ is the boundary of the regular Finsleroid, that is,

$$IR^{PD}_{g,c\{x\}} := \{ y \in IR^{PD}_{g,c\{x\}} : y \in T_x M, K(x, y) = 1 \}. \quad (A.28)$$

**Definition.** The scalar $g(x)$ is called the Finsleroid charge. The 1-form $b = b_i(x)y^i$ is called the Finsleroid-axis 1-form.

We shall meet the function

$$\nu := q + (1 - c^2)gb \quad (A.29)$$

for which

$$\nu > 0 \quad \text{when} \quad |g| < 2. \quad (A.30)$$

Indeed, if $gb > 0$, then the right-hand part of (A.29) is positive. When $gb < 0$, we may note that at any fixed $c$ and $b$ the minimal value of $q$ equals $\sqrt{1 - c^2}\frac{|b|}{c}$ (see (A.5)), arriving again at (A.30).

The identities

$$\frac{c^2S^2 - b^2}{q\nu} = 1 - (1 - c^2)\frac{B}{q\nu}, \quad gb(c^2S^2 - b^2) = qB - \nu S^2 \quad (A.31)$$

are valid.

In evaluations it is convenient to use the variables

$$u_i := a_{ij}y^j, \quad v^i := y^i - bb^i, \quad v_m := u_m - bb^m = r_{mn}y^n \equiv a_{mn}v^n. \quad (A.32)$$

We have

$$r_{ij} = \frac{\partial v_i}{\partial y^j}, \quad (A.33)$$

$$u_i v^i = v_i y^i = q^2, \quad v_i b^i = v^i b_i = (1 - c^2)b, \quad (A.34)$$

$$r_{in} v^n = v_i - (1 - c^2)bb_i, \quad v_k v^k = q^2 - (1 - c^2)b^2, \quad (A.35)$$
\[
\frac{\partial b}{\partial y^i} = b_i, \quad \frac{\partial q}{\partial y^i} = \frac{v_i}{q}.
\] (A.36)

Under these conditions, we are to explicitly extract from the function \( K \) the distinguished Finslerian tensors, and first of all the covariant tangent vector \( \hat{y} = \{ y_i \} \), the Finslerian metric tensor \( \{ g_{ij} \} \) together with the contravariant tensor \( \{ g^{ij} \} \) defined by the reciprocity conditions \( g_{ij} g^{jk} = \delta_i^k \), and the angular metric tensor \( \{ h_{ij} \} \), by making use of the following conventional Finslerian rules in succession:

\[
y_i := \frac{1}{2} \frac{\partial K^2}{\partial y^i}, \quad g_{ij} := \frac{1}{2} \frac{\partial^2 K^2}{\partial y^i \partial y^j}, \quad h_{ij} := g_{ij} - y_i y_j \frac{1}{K^2}.
\] (A.37)

To this end it is convenient to use the variable

\[
w = \frac{q}{b},
\] (A.38)

obtaining

\[
\frac{\partial w}{\partial y^i} = \frac{z_i}{b^2 q}, \quad z_i = b v_i - q^2 b_i \equiv b u_i - S^2 b_i,
\] (A.39)

and

\[
y^i z_i = 0, \quad b^i z_i = b^2 - c^2 S^2,
\]

\[
a^{ij} z_i z_j = S^2 (c^2 S^2 - b^2).
\]

We also introduce the \( \eta \)-tensors given by

\[
\eta_{ij} := r_{ij} - \frac{1}{q^2} v_i v_j, \quad \eta^i_j := r^i_j - \frac{1}{q^2} v^i v^j, \quad \eta^{ij} := r^{ij} - \frac{1}{q^2} v^i v^j,
\] (A.40)

which obey the identities

\[
\eta^m_j = a^{mn} \eta_{nj}, \quad \eta^{ij} = a^{in} \eta^j_n,
\] (A.41)

\[
\eta_{mi} y^i = 0,
\] (A.42)

\[
\eta_{ij} b^i = -(1 - c^2) \frac{1}{q^2} z_i, \quad \eta_{ij} z^j = (1 - c^2) \frac{S^2}{q^2} z_i,
\] (A.43)

and

\[
\frac{\partial \left( \frac{1}{q} v_k \right)}{\partial y^j} = \frac{1}{q} \eta_{kj}, \quad \frac{\partial z_i}{\partial y^k} = b \eta_{ik} + \frac{1}{q^2} v_k z_i + \frac{1}{b} (b_k z_i - z_k b_i).
\] (A.44)

Using the generating metric function \( V(x, w) \) defined from the representation

\[
K = b V(x, w),
\] (A.45)

we obtain

\[
\frac{\partial K}{\partial y^i} = b_i V + \frac{1}{b q} z_i V', \quad \frac{\partial^2 K}{\partial y^i \partial y^j} = \frac{1}{q} \eta_{ij} V'' + \frac{1}{b^2 q^2} z_i z_j V''',
\] (A.46)
where
\[ V' = \frac{\partial V}{\partial w}, \quad V'' = \frac{\partial^2 V}{\partial w^2}. \] (A.47)

Taking into account the explicit derivatives of the function \( V \):
\[ VV' = w \frac{K^2}{B}, \quad VV'' = \frac{b^2}{B} \frac{K^2}{B} \] (A.48)

(use (A.19)-(A.23)), we find the representations
\[ y_i = \left( B b_i + z_i \right) \frac{K^2}{b B}, \] (A.49)
\[ g_{ij} = \frac{K^2}{B} \eta_{ij} + \frac{K^2}{b^2} b_i b_j + \frac{K^2}{b^2 B} (b_i z_j + b_j z_i) + \frac{B - gbq K^2}{b^2 q^2} \frac{K^2}{B^2} z_i z_j, \] (A.50)
\[ h_{ij} = \frac{K^2}{B} \left( \eta_{ij} + \frac{1}{B q^2} z_i z_j \right), \] (A.51)

which entail
\[ y_i = \left( v_i + (b + gbq b_i) \right) \frac{K^2}{B}, \] (A.52)
\[ g_{ij} = \left[ a_{ij} + \frac{gb}{B} \left( q(b + gbq b_i) b_j + q(b_i v_j + b_j v_i) - b v_i v_j \right) \right] \frac{K^2}{B}, \] (A.53)

and
\[ g^{ij} = \left[ a^{ij} + \frac{gb}{B} \left( -b q b b^j - q(b^i v^j + b^j v^i) + (b + gc^2 q) \frac{v^i v^j}{\nu} \right) \right] \frac{B}{K^2}. \] (A.54)

The determinant of the metric tensor is everywhere positive:
\[ \det(g_{ij}) = \nu \left( \frac{K^2}{B} \right)^N \det(a_{ij}) > 0 \] (A.55)

with the function \( \nu \) given by (A.29).

In terms of the set \( \{ b_i, u_i = a_{ij} y^j \} \), we obtain the alternative representations
\[ y_i = \left( u_i + gb b_i \right) \frac{K^2}{B}, \] (A.56)
\[ g_{ij} = \left[ a_{ij} + \frac{g}{B} \left( (gb^2 - \frac{b S^2}{q}) b_i b_j - \frac{b}{q} u_i u_j + \frac{S^2}{q} (b_i u_j + b_j u_i) \right) \right] \frac{K^2}{B}, \] (A.57)

and
\[ g^{ij} = \left[ a^{ij} + \frac{gb}{\nu} \left( b^i b^j - b^j b^i - b^i y^j + b^j y^i \right) + \frac{g}{B \nu} (b + gc^2 q) y^i y^j \right] \frac{B}{K^2}, \] (A.58)

together with
\[ h_{ij} = \left[ a_{ij} + \frac{1}{qB} \left( -gb S^2 b_i b_j - (q + gb) u_i u_j + gb^2 (b_i u_j + b_j u_i) \right) \right] \frac{K^2}{B}, \] (A.59)
which entails

\[ h_{ij}b^j = -\frac{\nu z_i K^2}{Bq B}, \quad g^{ij}z_iz_j = \frac{q}{\nu} \left( c^2 S^2 - b^2 \right) \frac{B^2}{K^2} \]  

(A.60)

and

\[ g^{ij}b_j = \frac{1}{K^2} \left[ S^2 b^i - \frac{g}{\nu} \left( c^2 S^2 - b^2 \right) v^i \right]. \]  

(A.61)

Given any vector \( t_j \), we have

\[ g^{ij}t_j = \left[ Ba^{ij}t_j + \frac{g}{\nu} \left( B(b^i b^j - b^i y^j - b^j y^i) + (b + gc^2q)y^iy^j \right) t_j \right] \frac{1}{K^2}, \]  

or

\[ g^{ij}t_j = \left[ Ba^{ij}t_j - gq(yt)b^i + \frac{g}{\nu} \left( -B(bt) + (b + gc^2q)(yt) \right) v^i \right] \frac{1}{K^2}. \]  

(A.62)

Also,

\[ b + gc^2q = \frac{1}{b}(B - q\nu), \quad \frac{\partial B}{\partial y^k} = \frac{2B}{K^2}y_k + \frac{g}{q}z_k, \]  

(A.63)

\[ v_k = \frac{q^2}{K^2}y_k + \frac{b}{B} \frac{g}{q}z_k, \]  

(A.64)

\[ g^{ij}u_j = \frac{1}{K^2} \left( By^i - gq \left[ S^2 b^i - \frac{g}{\nu} \left( c^2 S^2 - b^2 \right) v^i \right] \right), \]  

(A.65)

and

\[ g^{ij}v_j = \frac{1}{K^2} \left( By^i - (b + gq) \left[ S^2 b^i - \frac{g}{\nu} \left( c^2 S^2 - b^2 \right) v^i \right] \right). \]  

(A.66)

Using the function \( X \) given by

\[ \frac{1}{X} = N + \frac{(1 - c^2)B}{q\nu}, \]  

(A.67)

we can evaluate the Cartan tensor

\[ A_{ijk} := \frac{K}{2} \frac{\partial g^{ij}}{\partial y^k} \]  

(A.68)

and the contraction

\[ A_i := g^{ik}A_{ijk} = K \frac{\partial \ln \left( \sqrt{\det(g_{mn})} \right)}{\partial y^i}. \]  

(A.69)

From (A.55) it follows that

\[ A_i = \frac{Kg}{2qBX} \frac{1}{X} (q^2 b_i - bv_i), \]  

(A.70)

which entails

\[ A_iA_i = \frac{g^2}{4} \frac{1}{X^2} \left( N + 1 - \frac{1}{X} \right). \]  

(A.71)
Also, we find

\[ A_{ijk} = X \left[ A_i h_{jk} + A_j h_{ik} + A_k h_{ij} - \left( N + 1 - \frac{1}{X} \right) \frac{1}{A_h A^h} A_i A_j A_k \right]. \] \tag{A.72}

The equalities

\[ A^i = \frac{g}{2 X K \nu} \left[ B b^i - (b + ggq^2) y^i \right], \] \tag{A.73}

\[ b_i A^i = \frac{g}{2 X K \nu} (c^2 S^2 - b^2), \] \tag{A.74}

and

\[ v_i = \frac{q^2}{K^2 y_i} - q(b + ggq) \frac{2X}{K g} A_i \] \tag{A.75}

are valid.

If we use the vector

\[ e_i := -b_i + \frac{b v_i}{q^2}, \] \tag{A.76}

so that

\[ y^i e_i = 0, \]

we can readily convert the representation (A.72) to the form

\[ \frac{B^2}{K^2} A_{ijk} = -\frac{1}{2} g q (e_k \eta_{ij} + e_i \eta_{kj} + e_j \eta_{ik}) - \frac{gq^3}{B} e_i e_j e_k. \] \tag{A.77}

In various processes of evaluations, it is useful to apply the formulas

\[ Kb_n = b l_n + \frac{2q}{g} X A_n, \quad K^1 v_n = q l_n - \frac{B - q^2}{b} \frac{2}{g} X A_n, \] \tag{A.78}

and

\[ \frac{\partial K}{\partial y^n} = \frac{2}{g b q^2} (B - q^2) X A_n, \] \tag{A.79}

together with

\[ \frac{\partial B}{\partial y^k} = \frac{2B}{K^2 y_k} - \frac{2B}{K} X A_k, \quad \frac{\partial \left( \frac{q^2}{B} \right)}{\partial y^k} = -\frac{2q(2b + ggq)}{g K B} X A_k. \] \tag{A.80}

The formula (A.67) can also be represented in the form

\[ \frac{1}{X} = N + \frac{1 - c^2}{w} \frac{1 + g w + w^2}{w + (1 - c^2) g}. \] \tag{A.81}

Simple straightforward calculation yields

\[ \frac{\partial (X A_k)}{\partial y^n} = -\frac{1}{K} l_k X A_n - \frac{g}{2 K w} h_{kn} + \frac{2}{g K q} (b + ggq) X^2 A_k A_n. \] \tag{A.82}
We use the Riemannian covariant derivative
\[ \nabla_i b_j := \partial_i b_j - b_k a^k_{ij}, \quad (A.83) \]
where
\[ a^k_{ij} := \frac{1}{2} a^{kn}(\partial_j a_{ni} + \partial_i a_{nj} - \partial_n a_{ji}) \quad (A.84) \]
are the Christoffel symbols given rise to by the associated Riemannian metric \( \mathcal{S} \).

**Appendix B: Indefinite \( \mathcal{FR}^{SR}_{g; c} \)-space**

The positive–definite \( \mathcal{FR}^{PD}_{g; c} \)-space described possesses the indefinite (relativistic) version, to be denoted as the \( \mathcal{FR}^{SR}_{g; c} \)-space (with the upperscripts “SR” meaning “special–relativistic”). The underlined space \( \mathcal{R}_N = \{ M, a_{mn} \} \) is now taken to be pseudo–Riemannian, such that the input metric tensor \( \{a_{mn}(x)\} \) is to be pseudo–Riemannian with the time–space signature:
\[ \text{sign}(a_{mn}) = (+-\ldots). \quad (B.1) \]
The definition range \(-2 < g(x) < 2\) and the representation \( h = \sqrt{1 - (1/4)g^2} \) applicable in the positive-definite case (see (A.11) and (A.12)) transform now according to
\[ -\infty < g(x) < \infty, \quad h(x) = \sqrt{1 + \frac{1}{4}(g(x))^2}, \quad G(x) = \frac{g(x)}{h(x)} \]
(such a phenomenon was explained in [3]). The pseudo–Finsleroid-regular characteristic quadratic form
\[ B(x, y) := b^2 - gqb - q^2 \equiv (b + g_+ q)(b + g_- q) \quad (B.2) \]
is now of the positive discriminant
\[ D_{(B)} = 4h^2 > 0 \quad (B.3) \]
(compare these formulas with (A.13) and (A.14)).

In terms of these concepts, we propose

**DEFINITION.** The scalar function \( F(x, y) \) given by the formula
\[ F(x, y) := \sqrt{|B(x, y)|} J(x, y) \equiv |b + g_- q|^G_{+2} |b + g_+ q|^{-G_-/2}, \quad (B.4) \]
where
\[ J(x, y) = \left| \frac{b + g_- q}{b + g_+ q} \right|^{-G/4} \quad (B.5) \]
is called the pseudo-Finsleroid-regular metric function. It is convenient to use the quantities
\[ g_+ = -\frac{1}{2} g + h, \quad g_- = -\frac{1}{2} g - h, \quad (B.6) \]
\[ G_+ = \frac{g_+}{h} \equiv -\frac{1}{2} G + 1, \quad G_- = \frac{g_-}{h} \equiv -\frac{1}{2} G - 1. \quad (B.7) \]

Again, the zero–vector \( y = 0 \) is excluded from consideration: \( y \neq 0 \). The positive (not absolute) homogeneity holds:
\[ F(x, \lambda y) = \lambda F(x, y), \quad \lambda > 0, \; \forall x, \; \forall y. \quad (B.8) \]
The function
\[ L(x, y) = q - \frac{g}{2} b \]  
(B.9)
is now to be used instead of (A.23), so that (A.24) changes to read
\[ L^2 - h^2 b^2 = B. \]  
(B.10)

Similarly to (A.25), we introduce

**DEFINITION.** The arisen space
\[ \mathcal{FR}^{SR}_{g;c} := \{ \mathcal{R}_N; b_i(x); g(x); F(x, y) \} \]  
(B.11)
is called the **pseudo–Finsleroid-regular space.**

**DEFINITION.** The space \( \mathcal{R}_N = (M, S) \) entering the above definition (B.11) is called the **associated pseudo–Riemannian space.**

**DEFINITION.** The scalar \( g(x) \) is called the **pseudo–Finsleroid charge**. The 1-form \( b \) is called the **pseudo–Finsleroid–axis 1-form.**

The equality
\[ a_{ij}(x) = g_{ij}(x, y) \bigg|_{g=0} \]  
(B.12)
(cf. (A.26)) is applicable to the pseudo–Finsleroid case.

One can observe the phenomenon that the representations of the components \( y_i, g_{ij}, g^{ij}, A_i, A_{ijk} \) in the \( \mathcal{FR}^{SR}_{g;c} \)-space are directly obtainable from the positive–definite case representations (written in the preceding Appendix A) through the formal change:
\[ g \xrightarrow{PD} \xrightarrow{SR} ig \]  
(B.13)
and
\[ q \xrightarrow{PD} \xrightarrow{SR} iq, \]  
(B.14)
where \( i \) stands for the imaginary unity. Therefore, we may apply the rules
\[ \frac{g}{q} \xrightarrow{PD} \xrightarrow{SR} \frac{g}{q}, \quad gq \xrightarrow{PD} \xrightarrow{SR} -gq. \]  
(B.15)

It is the useful exercise to verify that if we apply these rules to the expression (B.4) of the relativistic function \( F \), we obtain the positive–definite case function \( K \) defined by (A.19).

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