Memory and Self Organization

Matteo Marsili, Guido Caldarelli

*International School for Advanced Studies (SISSA)*

*V. Beirut 2-4, 34014 Trieste, Italy*

Abstract

The main result of this letter is that SOC naturally arises as a result of memory effects.

We show that memory effects provide the mechanism for self organization. A general procedure to investigate this issue in models that display self organized critical behaviour is proposed and applied to some example. The simplest class of models exhibiting self organized criticality through this mechanism is introduced and discussed in some detail.

PACS: 02.50-r, 05.40+j, 05.40jk

S.I.S.S.A. Ref. 162/94/CM
A large amount of efforts have been recently devoted to uncover the mechanism underlying the tendency of large statistical systems to self organize into a critical state. This issue has a great relevance since self organized criticality (SOC) manifests itself in a large variety of phenomena ranging from earthquakes to magnetic systems, from interface growth to biological evolution.

Much interest has focused on recently proposed models that involve quenched disorder and whose dynamics leads spontaneously to a SOC state. The occurrence of critical aspects in connection with a dynamics in a random environment is not a peculiarity of these models: invasion percolation (IP) is known to reproduce the critical clusters of standard percolation right at the percolation threshold; non trivial space–time correlations also appear in spin glasses dynamics, charge density waves and in zero temperature dynamics of magnetic systems in quenched disorder. Criticality is related to the presence of memory in these systems. By this we mean that the dynamic of local variables is sensible to a long period of the past history of the process.

This letter we inquire on the relation between dynamics in quenched disorder, memory effects and SOC. First we discuss how memory arises in models that evolve in a random environment. Then we show that a SOC behavior does not necessarily imply the presence of memory. We suggest that instead the converse is true: Memory, i.e. the dependence of local dynamics on the whole history of the process, implies a self organized critical behavior. This issue is analyzed with the introduction of a model that contains the effects of memory explicitly. In this model there is no reference to quenched random variables. For a particular value of the exponent that tunes the effects of the past history on the evolution, we reproduce the result of the corresponding dynamics in quenched disorder. The occurrence of SOC behaviour for a whole range of this exponent suggests that criticality is not a peculiarity of dynamics in quenched disorder but rather it arises as a result of memory effects.

The simplest model of dynamics in quenched disorder is perhaps the Bak Sneppen model (BSM) originally devised to model biological evolution: assign a uniformly distributed random variable (RV) \( \eta_i \) on each site \( i \) of a \( d \)-dimensional lattice. At each time step select
the smallest RV and replace it and the RV’s on the neighboring sites with newly extracted uniform RV’s. The system self organizes to a “critical” steady state in which almost all RV’s are above a certain threshold value \( p_c \). This state is characterized by long range correlations, both in space and in time, that have been studied by different techniques.

Imagine to assign to each site of the lattice a counter variable \( k_i \). At time \( t \) the variable \( k_i \) is set to zero if the variable \( \eta_i \) is updated and it is increased by one otherwise. In this way \( k_i(t) \) is the time elapsed since the last update on site \( i \).

In a system that evolves probing a random environment, as the BSM and invasion percolation where the extreme statistics of a random field is selected at each time, it is natural to think that the evolution will take place more often on recently updated regions than in older ones. This is because a site whose RV has been checked a large \( k_i \gg 1 \) number of times in the search for the minimum RV will probably have a large RV. It still has a probability of being the smallest in the future but this probability gets smaller and smaller as time goes on. This implies that the probability that a site with a counter equal to \( k \) is selected decreases with \( k \).

It is possible, keeping track of the evolution of the statistics of the RV on each site, to pursue this argument further and to evaluate the probability of each selection event, that is the probability that a random variable \( \eta_i \), that has a counter \( k_i(t) \), is the smallest one. This in principle provides a stochastic formulation of the process, that is deterministic for each realization of the randomness. We will not enter the details of this (the interested reader is referred to) but just note that the probability that site \( i \) is the smallest can be labeled by \( k_i \) and evaluated for invasion percolation, under approximations of mean field type, with the result

\[
\text{Prob}\{\eta_i = \min[\eta_j; \forall j]\} = \mu_{k_i,t} \sim k_i^{-\alpha}
\]

for \( t \gg k_i \gg 1 \) with \( \alpha = 2 \). It is worth to stress here that \( \mu_{k,t} \) is not a function of \( k \) alone in models like invasion percolation. In a single realization it actually depends on finer details of the past history. However, on average, it displays a fairly stable power law dependence.
The distribution $\mu_{k,t}$ is a directly accessible quantity in a computer simulations. Indeed, the fraction of times selection occur on a site with $k_i = k$ will be $n_{k,t} \mu_{k,t}$, where $n_{k,t}$ is the number of sites with counter $k_i(t) = k$ at time $t$.

We will be concerned mainly with the stationary state of a system of linear size $L$ with periodic boundary condition. In the steady state the above distributions attain a constant value $\mu_k(L)$ and $n_k(L)$. These satisfy the normalization conditions $\sum_k \mu_k(L)n_k(L) = 1$ and $\sum_k n_k(L) = L^d$ for a system of linear size $L$.

A measure of the effect of memory is given by the first moment of the distribution $n_k(L)$

$$T_1(L) = \frac{1}{L^d} \sum_{k=0}^{\infty} kn_k(L) \sim L^{d(1+\zeta)}.$$  

(2)

The exponent $\zeta$ is a measure of the presence of memory effects. On the average the local dynamic of the variable on site $i$ is sensible to a period of the past history of the process. If the length of this period, measured in units of $L^d$ individual events, increases with $L$, i.e. if $\zeta > 0$, the state of the infinite system will depend on the whole history. If $\zeta = 0$ we can say that no memory effect is present.

Counter variables can be introduced in any system. Consider e.g. the Metropolis dynamics\cite{17} of the Ising model. A variable is selected on average once every $L^d$ attempts. With a probability that does not depend on $L$ the move is accepted and the spin flipped. Thus we expect $\zeta = 0$ for this model and in general for equilibrium dynamics. Consider next the prototype model of SOC, the sandpile model\cite{1}: sand is added on randomly chosen sites. A site cannot store more than $2d - 1$ grains and it “topples” when it receives the $2d^{th}$ one, i.e. it distributes one grain to each of its neighbor sites causing eventually “toppling” on these sites as a result. After a toppling a site is empty. Before it will topple again it needs to store enough sand. Thus the probability that it will topple after $k$ toppling events grows with $k$ and on the average it will topple once every $L^d$ toppling events. Then $\zeta = 0$ also in this case. This result is consistent with the abelian nature of this model\cite{15}. The BSM has instead a non–abelian evolution and figure\cite{11} indeed shows that the situation is different in
this case. $\mu_k(L)$ actually decays as a power law with $k$ with an exponent $\alpha_{BSM} = 1.30 \pm 0.02$ and $n_k(L)$ satisfies the scaling behavior

$$n_k(L) = k^{-\beta} f\left(k/L^{1+\zeta}\right) \text{ for } k > 0$$

with $\beta_{BSM} = 0.58 \pm 0.01$.

The scaling function $f(x)$ drops quickly to zero for large arguments and tends to a constant $f(0) \cong n_1(L)$ for $x \to 0$. The assumption of a single time scale $T_\infty(L)$ for a given size is implicit in (3). If $\beta < 1$, the normalization condition on $n_k(L)$ easily yields the exponent relation

$$\zeta = \frac{\beta}{1 - \beta}. \quad (4)$$

This yields $\zeta_{BSM} \approx 1.44$ in fair agreement with the direct measure $\zeta_{BSM} = 1.46 \pm 0.03$ using eq.(2).

If $\zeta$ is the indicator of the relevance of memory, the self organized nature is usually related to the occurrence of avalanche events. An avalanche event is made of a spatially and causally connected series of events. In the BSM the selection and update of one site at time $t$ may generate RV’s that are smaller than the one that has just been selected. The evolution will naturally select these RV at time $t + 1$. The same may happen for a certain period and as a result selection events will be localized in a small region. At each time one avalanche starts so a number of nested avalanches are active at each time. In the SOC state the duration $s$ of an avalanche follows a power law distribution $N(s) \sim s^{-\tau}$ that defines the exponent $\tau$. An avalanche that lasts a time $s$ typically extends on a region $\xi \sim s^h$.

The definition of an avalanche is particularly simple in terms of the variables $k_i$. Consider the avalanche started at time $t_0$. This will be active at time $t_0 + s$ if all sites $i(t)$ selected at times $t_0 < t \leq t_0 + s$ had a counter $k_{i(t)}(t) \leq t - t_0$, i.e. all these sites where generated after the avalanche began. On the other hand the selection of a site with a counter $k$ at time $t$ terminates all avalanches that started after time $t - k$. The size of an avalanche that lasts for $s$ time steps is simply evaluated as the size of the region with counters smaller than $k_i \leq s$. 

5
The key points we have reached up to now with the introduction of counter variables are:
i) dynamics in quenched disorder leads to memory effects and is characterized by power law behavior in both $\mu_k(L)$ and $n_k(L)$. ii) the mechanism of self organization is not the same in sandpile models\(^1\), that display no memory effect, and in the BSM. iii) We can describe both memory effects and avalanche events in terms of counter variables alone. These observations motivates the introduction of a new model defined in terms of counter variables alone to study the interplay between the effects of memory and self organization.

The model we are going to discuss is defined as follows: assign a counter variable $k_i$ to each site $i = 1, \ldots, L$ of a 1 dimensional lattice. At each time step one site is selected, with a probability $\mu_k$, that depends on the value of the counter

$$\mu_k = \mu_0(k + 1)^{-\alpha}. \quad (5)$$

When a site is selected its counter variables and that of its neighbor sites are set to zero. All other variables are increased by one:

$$k_{i+\delta}(t+1) = 0 \quad \text{for } \delta = 0, \pm 1 \quad (6)$$

$$k_j(t+1) = k_j(t) + 1 \quad \text{else}. \quad (7)$$

The dependence of the selection probability on $k$ is devised to generalize the situation occurring when the dynamics is driven by the extreme statistics of a random field. The larger the time a region has been tested for selection the smallest the probability it will be selected. This is the simplest way to account for a dependence of the local dynamics on the history of the process and the rules of the model may apply also to situations where no disorder is present.

For a finite $L$ the system gets to a steady state that is characterized by a distribution of counters $n_k(L)$ for which we shall assume the scaling form eq.(3). Of course $n_0(L) = 3$ since three counters are updated at each time step. The normalization of the selection probability

$$\sum_{k=0}^\infty n_k(L)\mu_k(L) = L$$

fixes the value of $\mu_0(L)$:

$$\mu_0(L) = \frac{1}{3 + \sum_{k=1}^\infty n_k(L)(k + 1)^{-\alpha}}. \quad (8)$$
Let us start the discussion of the model from the $\alpha = 0$ case. Clearly $\mu_0(L) = 1/L$ at all times. The $n_k(L)$ instead decays exponentially. A simple explanation of this comes from the relation between the number of sites with $k_i(t + 1) = k + 1$ and $k_i(t) = k$. In the steady state this reads $n_{k+1}(L) = n_k(L)(1 - 3/L)$. This immediately yields $n_k(L) \approx 3 \exp(3k/L)$ and $\zeta(\alpha = 0) = \beta(\alpha = 0) = 0$. Of course the probability of a connected event of $s$ steps also goes to zero exponentially with $s$, i.e. $\tau(\alpha = 0) = 0$. In conclusion neither memory nor avalanches are present in the model for $\alpha = 0$. The same behavior is expected to persist for small values of $\alpha$. Let us focus on a site with $k_i = k$ and consider the probability $P_k(s)$ that it will not be selected in the next $s$ steps, under the condition that in this period it will not be updated because of its neighbors. Clearly $P_k(s) = \prod_{j=k+1}^{k+s}(1 - \mu_0 j^{-\alpha})$. It is not difficult to check that, if $\alpha < 1$, $P_k(s) \to 0$ as $s \to \infty$ while it tends to a constant if $\alpha > 1$. So this site, if it is not updated by its neighbors, will surely be selected sooner or later. This suggests that for $\alpha < 1$ the average time $T_\infty(L)$ between two updates of the same site stays finite and $\zeta(\alpha < 1) = \beta(\alpha < 1) = 0$ (see eq.(4)). The occurrence of SOC can be excluded as well for $\alpha < 1$. The probability of local, causally connected events (the avalanches) depends on $\mu_0(L)$. The existence of such events on all length and time scale requires this probability to stay finite independent of $L$. Supposing the scaling form (3) for $n_k(L)$ in eq.(8), it is easy to see that if $\alpha + \beta \geq 1$, as $L \to \infty$, $\mu_0(L) \to \mu_0(\infty) > 0$, while in the opposite case $\mu_0(\infty) = 0$.

Let us now consider the opposite case: $\alpha = \infty$. In this case $\mu_k = 0 \forall k > 0$ and $\mu_0 = 1/3$. The model describes a random walk on a $d = 1$ lattice. It is not difficult to find $\zeta(\infty) = 1$ and a distribution $n_k(L)$ that follows eq.(3) with $\beta(\infty) = 1/2$. With respect to SOC, the evolution is a single connected event: every avalanche lasts for an infinite time. For a finite $\alpha > 1$ it is convenient to generalize the avalanche distribution to account for infinite avalanches:

$$N(s) = (1 - N_\infty)N_f(s) + N_\infty \delta_{s,\infty}. \quad (9)$$

$N_\infty$ is the fraction of avalanches that never stop. The lack of characteristic lengths in the model suggests that the distribution of finite avalanches $N_f(s)$ will in general follow a power
law distribution. An avalanche of duration $s$ is terminated when a site with $k_i > s$ is selected. If $k_{\text{max}}(t) = \max[k_i(t), i = 1, \ldots, L]$, all the avalanches that began before time $t - k_{\text{max}}(t)$ will never be terminated. The probability that an avalanche is still active after $s$ events is

$$P_{\text{act}}(s) = \prod_{t=1}^{s} \left[ 1 - \sum_{k>t} n_k(L)\mu_k(L) \right].$$

Supposing again eq. (3) for $n_k(L)$, the sum in eq. (10) goes as $k^{1-\alpha-\beta}$ so that $P_{\text{act}}(s) \to 0$ if $\alpha + \beta(\alpha) < 2$. For large $\alpha$ a finite fraction of avalanche events will never stop while we expect a finite interval $\alpha \in [1, \alpha_c]$ where $N_\infty(\alpha) = 0$ and the usual scenario of SOC applies.

This picture was checked by computer simulation of the model. For $\alpha < 1$, as expected, $\mu_0(L) \sim L^{-\omega}$ vanishes with $\omega(\alpha) \cong 1 - \alpha$. Table lists the values obtained for the exponents $\beta$ and $\zeta$ by numerical simulations of the model for sizes up to $L = 256$ and $\alpha > 1$. The statistical uncertainty gets large as $\alpha = 1$ is approached from above. The relation (4) is satisfied fairly well. $\alpha + \beta(\alpha)$ gets bigger than 1 for $\alpha \cong 1.4$. Correspondingly, as expected, $N_\infty$ becomes positive. This is shown in fig. 2 where we report also the estimate of the exponent $\tau$ obtained for $L = 128$ and 256. The distribution $N_f(s)$ was found to decay as a power law for more than two decades but the value of the exponent actually decreases as $L$ increases. For small $\alpha$ the statistics of avalanches gets scarcer and scarcer and a reliable estimate was not possible. Even though of a qualitative nature, the behavior of $\tau(\alpha)$ is quite evident from figure 2. $\tau$ reaches a minimum approximately in the same region where $\alpha + \beta \cong 1$ and $N_\infty$ starts to increase. The last occurrence would naturally arises as a result of the divergence of the normalization integral of $N(s)$ that occurs for $\tau = 1$. This would imply a systematic error in our $\tau$ data of approximately $+0.3$. With this proviso we find that, for $\alpha = \alpha_{BSM} = 1.30$, the exponent should be $\tau_{BSM} \approx 1.1$ in good agreement with numerical results for the BSM. With respect to the BSM, it is to note that the values of $\beta$ and $\zeta$ agree with those obtained for $\alpha = 1.30$. We confirmed numerically that $N_\infty(L) \sim L^{-0.98}$ goes to zero for the BSM.
FIGURE 1. The distributions $n_k(L)$ and $\mu_k(L)$ for the BSM.
FIG. 2. Fraction of infinite avalanches $N_{\infty}(\alpha)$ in a system with $L = 128$ and $\tau(\alpha)$ exponent.
| $\alpha$ | $\beta$   | $\zeta$   |
|---------|-----------|-----------|
| 1.10    | 0.3 ± 0.1 | 0.5 ± 0.2 |
| 1.20    | 0.48 ± 0.01 | 0.90 ± 0.04 |
| 1.30    | 0.58 ± 0.01 | 1.40 ± 0.06 |
| 1.40    | 0.619 ± 0.005 | 1.53 ± 0.03 |
| 1.50    | 0.613 ± 0.005 | 1.47 ± 0.03 |
| 1.75    | 0.571 ± 0.005 | 1.31 ± 0.03 |
| 2.00    | 0.545 ± 0.005 | 1.17 ± 0.02 |
| 2.50    | 0.510 ± 0.005 | 1.06 ± 0.02 |
REFERENCES

1 P. Bak, C. Tang and K. Weisenfeld, Phys. Rev. Lett. 59, 381, (1987); Phys. Rev. A 38, 364 (1988).

2 T. Hwa, M. Kardar: Phys. Rev. Lett. 62, 1813 (1989)

3 L. Pietronero, A. Vespignani and S. Zapperi, Phys. Rev. Lett. 72, 1690 (1994).

4 M. Marsili: Renormalization Group approach to the self organization of a simple model of biological evolution submitted to Europhys. Lett. 1994.

5 M. Paczuski, S. Maslov and P. Bak Field Theory of Self Organized Criticality preprint.

6 J. M. Carlson and J. S. Langer: Phys. Rev. Lett. 62, 2632 (1989)

7 M. Lederman, R. Orbach, J. M. Hamman, M. Ocio and E. Vincent: Phys. Rev. B 44, 7403 (1991) and references therein.

8 J. P. Sethna, K. Dahmen, S. Kartha, J. A. Krumhansl, B. W. Roberts and J. D. Shore Phys. Rev. Lett. 70, 3347 (1993).

9 K. Sneppen: Phys. Rev. Lett. 69, 3539 (1992), 71, 101 (1993).

10 L.-H. Tang and H. Leshhorn: Phys. Rev. A 45, R8309 (1992), S. V. Buldyrev, A.-L. Barabási, F. Caserta, S. Havlin, H. E. Stanley and T.Vicsek: Phys. Rev. A 45, R8313 (1992).

11 P. Bak and K. Sneppen: Phys. Rev. Lett. 71, 4083 (1993), H. Flyvbjerg, K. Sneppen and P. Bak: Phys. Rev. Lett. 71, 4087 (1993).

12 R. Lenormand and S. Bories: C. R. Acad. Sci. 291, 279 (1980), R. Chandler, J. Koplick, K. Lerman, J. F. Willemsen: J.Fluid Mech. 119, 249 (1982), D. Wilkinson and J. F. Willemsen: J. Phys. A 16, 3365 (1983).

13 G. Kriza and G. Miháli: Phys. Rev. Lett. 56, 2529 (1986); R. M. Fleming and L. F.
Schneemeyer: Phys. Rev. B 33, 2930 (1986); G. Gruner: Rev. Mod. Phys. 60, 1129 (1988).

14 R. Bruinsma and G. Aeppli: Phys. Rev. Lett. 52, 1547 (1984); J. Koplick and H. Levine: Phys. Rev. B 32, 280 (1985); B. Koiller, H. Ji and M. O. Robbins: Phys. Rev. B 46, 5258 (1992).

15 D. Dhar: Phys Rev. Lett. 64, 1613 (1990)

16 M. Marsili: Run Time Statistics in Models of Growth in Disordered Media. To appear in Nov. 1994 issue of J.Stat.Phys.

17 N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. M. Teller, and E. Teller: J. Chem. Phys. 21, 1087 (1953).