Convergence of a Cahn–Hilliard type system to a parabolic-elliptic chemotaxis system with nonlinear diffusion

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Abstract. This paper deals with a parabolic-elliptic chemotaxis system with nonlinear diffusion. It was proved that there exists a solution of a Cahn–Hilliard system as an approximation of a nonlinear diffusion equation by applying an abstract theory by Colli–Visintin [Comm. Partial Differential Equations 15 (1990), 737–756] for a doubly nonlinear evolution inclusion with some bounded monotone operator and subdifferential operator of a proper lower semicontinuous convex function (cf. Colli–Fukao [J. Math. Anal. Appl. 429 (2015), 1190–1213]). Moreover, Colli–Fukao [J. Differential Equations 260 (2016), 6930–6959] established existence of solutions to the nonlinear diffusion equation by passing to the limit in the Cahn–Hilliard equation. However, Cahn–Hilliard approaches to chemotaxis systems with nonlinear diffusions seem not to be studied yet. This paper will try to derive existence of solutions to a parabolic-elliptic chemotaxis system with nonlinear diffusion by passing to the limit in a Cahn–Hilliard type chemotaxis system.

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1. Introduction

A relation between a nonlinear diffusion equation and a Cahn–Hilliard equation has been studied. Colli–Fukao \[5,6\] considered the nonlinear diffusion equation

$$u_t - \Delta u = g, \xi \in \beta(u) \quad \text{in}\ \Omega \times (0,T) \hspace{1cm} \text{(E)}$$

and the Cahn–Hilliard type of approximate equation

$$\begin{aligned}
&\begin{cases}
(u_\varepsilon)_t - \Delta u_\varepsilon = 0 & \text{in } \Omega \times (0,T), \\
\mu_\varepsilon = -\varepsilon \Delta u_\varepsilon + \xi_\varepsilon + \pi_\varepsilon(u_\varepsilon) - f, \xi_\varepsilon \in \beta(u_\varepsilon) & \text{in } \Omega \times (0,T),
\end{cases} \\
&\quad \text{(E)}_\varepsilon
\end{aligned}$$

where \(\Omega \subset \mathbb{R}^d (d = 1, 2, 3)\) is a bounded domain, \(\varepsilon \in (0,1], T > 0, \beta : \mathbb{R} \to \mathbb{R}\) is a multivalued maximal monotone function, \(\pi_\varepsilon : \mathbb{R} \to \mathbb{R}\) is an anti-monotone function which goes to 0 in some sense as \(\varepsilon \to 0\), \(f : \Omega \times (0,T) \to \mathbb{R}\) is a given function. To prove existence of solutions to \((E)_\varepsilon\) they used one more approximation

$$\begin{aligned}
&\begin{cases}
(u_{\varepsilon,\lambda})_t - \Delta u_{\varepsilon,\lambda} = 0 & \text{in } \Omega \times (0,T), \\
\mu_{\varepsilon,\lambda} = \lambda(u_{\varepsilon,\lambda})_t - \varepsilon \Delta u_{\varepsilon,\lambda} + \beta_\lambda(u_{\varepsilon,\lambda}) + \pi_\varepsilon(u_{\varepsilon,\lambda}) - f & \text{in } \Omega \times (0,T),
\end{cases} \\
&\quad \text{(E)_{\varepsilon,\lambda}}
\end{aligned}$$

where \(\lambda > 0\) and \(\beta_\lambda\) is the Yosida approximation of \(\beta\) on \(\mathbb{R}\). In \([5]\) they proved existence of solutions to \((E)_{\varepsilon,\lambda}\) by applying an abstract theory by Colli–Visintin \([9]\) for the doubly nonlinear evolution inclusion:

$$Au'(t) + \partial\psi(u(t)) \ni k(t)$$

with some bounded monotone operator \(A\) and subdifferential operator \(\partial\psi\) of a proper lower semicontinuous convex function \(\psi\). Next, in \([6]\) they derived existence of solutions to \((E)_{\varepsilon}\) and \((E)\) by passing to the limit in \((E)_{\varepsilon,\lambda}\) as \(\lambda \to 0\) and in \((E)_{\varepsilon}\) as \(\varepsilon \to 0\) individually.

On the other hand, relations between chemotaxis systems with nonlinear diffusions and Cahn–Hilliard type chemotaxis systems seem not be studied yet.

In this paper we consider the parabolic-elliptic chemotaxis system with nonlinear diffusin

$$\begin{aligned}
&\begin{cases}
\begin{aligned}
&u_t - \Delta \beta(u) + \eta \nabla \cdot (u \nabla v) = g & \text{in } \Omega \times (0,T), \\
0 = \Delta v - v + u & \text{in } \Omega \times (0,T), \\
\partial_\nu \beta(u) = \partial_\nu v = 0 & \text{on } \partial \Omega \times (0,T), \\
u(0) = u_0 & \text{in } \Omega,
\end{aligned}
\end{cases} \\
&\quad \text{(P)}
\end{aligned}$$

where \(\Omega \subset \mathbb{R}^d (d = 1, 2, 3)\) is a bounded domain with smooth boundary \(\partial \Omega\), \(T > 0, \eta \in \mathbb{R}\), \(\beta : \mathbb{R} \to \mathbb{R}\) is a single-valued maximal monotone function, \(\partial_\nu\) denotes differentiation with respect to the outward normal of \(\partial \Omega\), \(g : \Omega \times (0,T) \to \mathbb{R}\) and \(u_0 : \Omega \to \mathbb{R}\) are given.
functions. Also, in reference to [6], we deal with the Cahn–Hilliard type chemotaxis system

$$
\begin{aligned}
(u_\varepsilon)_t - \Delta u_\varepsilon + \eta \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) &= 0 \quad \text{in } \Omega \times (0, T), \\
\mu_\varepsilon &= -\varepsilon \Delta u_\varepsilon + \beta(u_\varepsilon) + \pi_\varepsilon(u_\varepsilon) - f \quad \text{in } \Omega \times (0, T), \\
0 &= \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon \quad \text{in } \Omega \times (0, T), \\
\partial_\nu \mu_\varepsilon &= \partial_\nu u_\varepsilon = \partial_\nu v_\varepsilon = 0 \quad \text{on } \partial \Omega \times (0, T), \\
u_\varepsilon(0) &= u_{0\varepsilon} \quad \text{in } \Omega,
\end{aligned}
$$

where $\varepsilon \in (0, 1]$, $\pi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is an anti-monotone function, $f : \Omega \times (0, T) \to \mathbb{R}$ and $u_{0\varepsilon} : \Omega \to \mathbb{R}$ are given functions. Moreover, in reference to [6], in this paper we assume that

(A1) $\beta : \mathbb{R} \to \mathbb{R}$ is a single-valued maximal monotone function with effective domain $D(\beta)$ such that there exists a proper lower semicontinuous convex function $\hat{\beta} : \mathbb{R} \to [0, +\infty]$ with effective domain $D(\hat{\beta})$ satisfying that $\hat{\beta}(0) = 0$ and $\beta = \partial\hat{\beta}$, where $\partial\hat{\beta}$ is the subdifferential of $\hat{\beta}$.

(A2) There exist constants $c_1, c_2 > 0$ such that

$$
\hat{\beta}(r) \geq c_1 |r|^4 - c_2 \quad \text{for all } r \in \mathbb{R}.
$$

(A3) $g \in L^2(0, T; L^2(\Omega))$ and $\int_\Omega g(t) \, dx = 0$ for a.a. $t \in (0, T)$. Then we fix a solution $f \in L^2(0, T; H^2(\Omega))$ of

$$
\begin{aligned}
-\Delta f(t) &= g(t) \quad \text{a.e. in } \Omega, \\
\partial_\nu f(t) &= 0 \quad \text{in the sense of traces on } \partial \Omega
\end{aligned}
$$

for a.a. $t \in (0, T)$, that is,

$$
\int_\Omega \nabla f(t) \cdot \nabla z = \int_\Omega g(t)z \quad \text{for all } z \in H^1(\Omega).
$$

(A4) $\pi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and there exists a constant $c_3 > 0$ such that

$$
|\pi_\varepsilon(0)| + \|\pi_\varepsilon'\|_{L^\infty(\mathbb{R})} \leq c_3 \varepsilon \quad \text{for all } \varepsilon \in (0, 1].
$$

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The three functions

\[
\beta_1(r) = r |r|^{m-1}, \quad r \in \mathbb{R}, \quad \beta_2(r) = \ln \frac{1 + r}{1 - r}, \quad r \in (-1, 1), \\
\beta_3(r) = |r| \ln \frac{1 + r}{1 - r}, \quad r \in (-1, 1),
\]

where \( m \geq 3 \) is some constant, are examples of \( \beta \). Indeed, we have that

\[
\ln \frac{1 + r}{1 - r} \geq \frac{8}{3} r^3 \quad \text{for all} \quad r \in (-1, 1).
\]

The function \( \beta_1 \) appears in the porous media equation (see e.g., [1, 13, 19, 20]). Let us define the Hilbert spaces

\[
H := L^2(\Omega), \quad V := H^1(\Omega)
\]

with inner products

\[
(u_1, u_2)_H := \int_{\Omega} u_1 u_2 \, dx \quad (u_1, u_2 \in H), \\
(v_1, v_2)_V := \int_{\Omega} \nabla v_1 \cdot \nabla v_2 \, dx + \int_{\Omega} v_1 v_2 \, dx \quad (v_1, v_2 \in V),
\]

respectively, and with the related Hilbertian norms. Moreover, we use the notation

\[
W := \left\{ z \in H^2(\Omega) \mid \partial_\nu z = 0 \quad \text{a.e. on } \partial \Omega \right\}.
\]

The notation \( V^* \) denotes the dual space of \( V \) with duality pairing \( \langle \cdot, \cdot \rangle_{V^*, V} \). Moreover, in this paper, the bijective mapping \( F : V \rightarrow V^* \) and the inner product in \( V^* \) are defined as

\[
\langle F v_1, v_2 \rangle_{V^*, V} := (v_1, v_2)_V \quad (v_1, v_2 \in V), \\
\langle v_1^*, v_2^* \rangle_{V^*, V^*} := \langle v_1^*, F^{-1} v_2^* \rangle_{V^*, V} \quad (v_1^*, v_2^* \in V^*).
\]

This article employs the Hilbert space

\[
V_0 := \left\{ z \in H^1(\Omega) \mid \int_{\Omega} z = 0 \right\}
\]
with inner product
\[(v_1, v_2)_{V_0} := \int_\Omega \nabla v_1 \cdot \nabla v_2 \, dx \quad (v_1, v_2 \in V_0)\]
and with the related Hilbertian norm. The notation \(V_0^*\) denotes the dual space of \(V_0\) with duality pairing \(\langle \cdot, \cdot \rangle_{V_0^*, V_0}\). Moreover, in this paper, the bijective mapping \(N : V_0^* \rightarrow V_0\) and the inner product in \(V_0^*\) are specified by
\[
\langle v^*, v \rangle_{V_0^*, V_0} = \int_\Omega \nabla N v^* \cdot \nabla v \quad (v^* \in V_0^*, v \in V_0),
\]
\[
(v_1^*, v_2^*)_{V_0^*} := \langle v_1^*, N v_2^* \rangle_{V_0^*, V_0} \quad (v_1^*, v_2^* \in V_0^*).
\]

We define weak solutions of (P) and \((P)_\varepsilon\) as follows.

**Definition 1.1.** A pair \((u, \mu)\) with
\[u \in H^1(0, T; V^*) \cap L^\infty(0, T; L^4(\Omega)), \quad \mu \in L^2(0, T; V)\]
is called a weak solution of (P) if \((u, \mu)\) satisfies
\[
\langle u'(t), z \rangle_{V^*, V} + \int_\Omega \nabla \mu(t) \cdot \nabla z - \eta \int_\Omega u(t) \nabla (1 - \Delta)^{-1} u(t) \cdot \nabla z = 0
\]
for all \(z \in V\) and a.a. \(t \in (0, T)\), \((1.1)\)
\[
\mu = \beta(u) - f \quad \text{a.e. on } \Omega \times (0, T), \quad (1.2)
\]
\[
u(0) = u_0 \quad \text{a.e. on } \Omega. \quad (1.3)
\]

**Definition 1.2.** A pair \((u_\varepsilon, \mu_\varepsilon)\) with
\[u_\varepsilon \in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad \mu_\varepsilon \in L^2(0, T; V)\]
is called a weak solution of \((P)_\varepsilon\) if \((u_\varepsilon, \mu_\varepsilon)\) satisfies
\[
\langle u'_\varepsilon(t), z \rangle_{V^*, V} + \int_\Omega \nabla \mu_\varepsilon(t) \cdot \nabla z - \eta \int_\Omega u_\varepsilon(t) \nabla (1 - \Delta)^{-1} u_\varepsilon(t) \cdot \nabla z = 0
\]
for all \(z \in V\) and a.a. \(t \in (0, T)\), \((1.4)\)
\[
\mu_\varepsilon = -\varepsilon \Delta u_\varepsilon + \beta(u_\varepsilon) + \pi_\varepsilon(u_\varepsilon) - f \quad \text{a.e. on } \Omega \times (0, T), \quad (1.5)
\]
\[
u_\varepsilon(0) = u_{0\varepsilon} \quad \text{a.e. on } \Omega. \quad (1.6)
\]
Now the main results read as follows.

**Theorem 1.1.** Assume (A1)-(A5). Then for all $\varepsilon \in (0, 1]$ there exists a weak solution $(u_\varepsilon, \mu_\varepsilon)$ of $(P)_\varepsilon$. In addition, there exists a constant $M > 0$, depending only on the data, such that

$$\varepsilon \|u_\varepsilon(t)\|_{V}^2 + \|u_\varepsilon(t)\|_{L^4(\Omega)}^4 \leq M,$$

(1.7)

$$\int_0^t \|u'_\varepsilon(s)\|_{V^*, \Omega}^2 \, ds \leq M,$$

(1.8)

$$\varepsilon^2 \int_0^t \|u_\varepsilon(s)\|_{W}^2 \, ds \leq M,$$

(1.9)

$$\int_0^t \|\mu_\varepsilon(s)\|_{V}^2 \, ds + \int_0^t \|\beta(u_\varepsilon(s))\|_{H}^2 \, ds \leq M$$

(1.10)

for all $t \in [0, T]$ and all $\varepsilon \in (0, 1]$.

**Theorem 1.2.** Assume (A1)-(A5). Then there exists a weak solution $(u, \mu)$ of $(P)$.

**Remark 1.1.** In the case that $\beta(r) = r$ and $d = 1$ Osaki–Yagi [16] established existence of a finite-dimensional attractor and proved that global existence and boundedness hold for all smooth initial data, which implies that blow-up solutions do not exist in the 1-dimensional setting. In the case that $\beta$ is nonlinear Marinoschi [14] proved local existence of solutions to $(P)$ for unbounded initial data by applying the nonlinear semigroup theory. Moreover, Yokota–Yoshino [21] established not only local but also global existence of solutions to $(P)$ for unbounded initial data by improving the method used in [14], while this paper derives global existence of solutions to $(P)$ for unbounded initial data by a Cahn–Hilliard approach.

This paper is organized as follows. Section 2 considers a suitable approximation of $(P)_\varepsilon$ in terms of a parameter $\lambda > 0$ and introduces a time discretization scheme in reference to [8]. In Section 3 we establish existence for the discrete problem. In Section 4 we derive uniform estimates for the time discrete solutions and show existence for the approximation of $(P)_\varepsilon$ by passing to the limit as the time step tends to zero. Finally, in Section 5 we prove existence for $(P)_\varepsilon$ and $(P)$ by taking the limit in the approximation of $(P)_\varepsilon$ as $\lambda \searrow 0$ and in $(P)_\varepsilon$ as $\varepsilon \searrow 0$ individually.
2. Approximate problems and preliminaries

We consider the approximation

\[
\begin{aligned}
(u_{\varepsilon,\lambda})_t - \Delta u_{\varepsilon,\lambda} + \eta \nabla \cdot (u_{\varepsilon,\lambda} \nabla v_{\varepsilon,\lambda}) &= 0 & & \text{in } \Omega \times (0, T), \\
\mu_{\varepsilon,\lambda} &= \lambda (u_{\varepsilon,\lambda})_t - \varepsilon \Delta u_{\varepsilon,\lambda} + \beta (u_{\varepsilon,\lambda}) + \pi_\varepsilon (u_{\varepsilon,\lambda}) - f & & \text{in } \Omega \times (0, T), \\
0 &= \Delta v_{\varepsilon,\lambda} - v_{\varepsilon,\lambda} + u_{\varepsilon,\lambda} & & \text{in } \Omega \times (0, T), \\
\partial_\nu \mu_{\varepsilon,\lambda} &= \partial_\nu u_{\varepsilon,\lambda} = \partial_\nu v_{\varepsilon,\lambda} = 0 & & \text{on } \partial \Omega \times (0, T), \\
u_{\varepsilon,\lambda}(0) &= u_{0,\varepsilon} & & \text{in } \Omega,
\end{aligned}
\]

where \( \lambda \in (0, \varepsilon) \). The definition of weak solutions to \((P)_{\varepsilon,\lambda}\) is as follows.

**Definition 2.1.** A pair \( (u_{\varepsilon,\lambda}, \mu_{\varepsilon,\lambda}) \) with

\[
u_{\varepsilon,\lambda} \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W),
\]

\[
\mu_{\varepsilon,\lambda} \in L^2(0, T; V)
\]
is called a **weak solution** of \((P)_{\varepsilon,\lambda}\) if \( (u_{\varepsilon,\lambda}, \mu_{\varepsilon,\lambda}) \) satisfies

\[
(u'_{\varepsilon,\lambda}(t), z)_H + \int_\Omega \nabla \mu_{\varepsilon,\lambda}(t) \cdot \nabla z - \eta \int_\Omega u_{\varepsilon,\lambda}(t) \nabla (1 - \Delta)^{-1} u_{\varepsilon,\lambda}(t) \cdot \nabla z = 0
\]

for all \( z \in V \) and a.a. \( t \in (0, T) \), (2.1)

\[
\mu_{\varepsilon,\lambda} = \lambda u'_{\varepsilon,\lambda} - \varepsilon \Delta u_{\varepsilon,\lambda} + \beta (u_{\varepsilon,\lambda}) + \pi_\varepsilon (u_{\varepsilon,\lambda}) - f \quad \text{a.e. on } \Omega \times (0, T), \\
u_{\varepsilon,\lambda}(0) = u_{0,\varepsilon} \quad \text{a.e. on } \Omega.
\]

Moreover, to prove existence of weak solutions to \((P)_{\varepsilon,\lambda}\) we employ a time discretization scheme. More precisely, in reference to [8], we will deal with the following problem: find

\[(u_{\lambda,n+1}, \mu_{\lambda,n+1}) \in W \times W \text{ such that }
\]

\[
\begin{aligned}
\delta_h u_{\lambda,n} + h \delta_h \mu_{\lambda,n} - \Delta \mu_{\lambda,n+1} + \eta \nabla \cdot (u_{\lambda,n} \nabla (1 - \Delta)^{-1} u_{\lambda,n}) &= 0 & & \text{in } \Omega, \\
\mu_{\lambda,n+1} &= \lambda \delta_h u_{\lambda,n} - \varepsilon \Delta u_{\lambda,n+1} + \beta (u_{\lambda,n+1}) + \pi_\varepsilon (u_{\lambda,n+1}) - f_{n+1} & & \text{in } \Omega, \\
\partial_\nu \mu_{\lambda,n+1} &= \partial_\nu u_{\lambda,n+1} = 0 & & \text{on } \partial \Omega
\end{aligned}
\]

for \( n = 0, \ldots, N - 1 \), where \( h = \frac{T}{N}, N \in \mathbb{N}, \)

\[
u_{\lambda,0} := u_{0,\varepsilon}, \quad \mu_{\lambda,0} := 0, \quad \delta_h u_{\lambda,n} := \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h}, \quad \delta_h \mu_{\lambda,n} := \frac{\mu_{\lambda,n+1} - \mu_{\lambda,n}}{h},
\]

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and \( f_k := \frac{1}{h} \int_{(k-1)h}^{kh} f(s) \, ds \) for \( k = 1, ..., N \). Also, putting
\[
\hat{u}_h(0) := u_{\lambda,0} = u_{0\varepsilon}, \quad (\hat{u}_h)_t(t) := \delta_h u_{\lambda,n},
\]
\[
\hat{\mu}_h(0) := \mu_{\lambda,0} = 0, \quad (\hat{\mu}_h)_t(t) := \delta_h \mu_{\lambda,n},
\]
\[
\overline{u}_h(t) := u_{\lambda,n+1}, \quad \underline{u}_h(t) := u_{\lambda,n}, \quad \overline{\mu}_h(t) := \mu_{\lambda,n+1}, \quad \underline{\mu}_h(t) := f_{n+1}
\]
for a.a. \( t \in (nh, (n + 1)h), \) \( n = 0, ..., N - 1 \), we can rewrite \((P)_{\lambda,n}\) as
\[
\begin{cases}
(\hat{u}_h)_t + h(\hat{\mu}_h)_t - \Delta \overline{\mu}_h + \eta \nabla \cdot (\underline{u}_h \nabla (1 - \Delta)^{-1} \hat{u}_h) = 0 & \text{in } \Omega \times (0, T), \\
\overline{\mu}_h = \lambda(\hat{u}_h)_t - \varepsilon \Delta \overline{\mu}_h + \beta(\overline{u}_h) + \pi(\overline{u}_h) - \overline{f}_h & \text{in } \Omega \times (0, T), \\
\partial_n \overline{\mu}_h = \partial_n \overline{u}_h = 0 & \text{on } \partial \Omega \times (0, T), \\
\hat{u}_h(0) = u_{0\varepsilon}, \quad \hat{\mu}_h(0) = 0 & \text{in } \Omega.
\end{cases}
\]

**Remark 2.1.** On account of (2.4)-(2.6), the reader can check directly the following properties:
\[
\|\hat{u}_h\|_{L^\infty(0,T;L^4(\Omega))} = \max\{\|u_{0\varepsilon}\|_{L^4(\Omega)}, \|\overline{u}_h\|_{L^\infty(0,T;L^4(\Omega))}\},
\]
\[
\|\hat{u}_h\|_{L^\infty(0,T;V)} = \max\{\|u_{0\varepsilon}\|_V, \|\overline{u}_h\|_{L^\infty(0,T;V)}\},
\]
\[
\|\hat{\mu}_h\|_{L^\infty(0,T;H)} = \|\overline{\mu}_h\|_{L^\infty(0,T;H)},
\]
\[
\|\overline{u}_h - \hat{u}_h\|_{L^2(0,T;H)}^2 = \frac{h^2}{3} \|\hat{u}_h\|_{L^2(0,T;H)}^2,
\]
\[
h(\hat{u}_h)_t = \overline{\mu}_h - \underline{\mu}_h.
\]

**Lemma 2.1.** For all \( \varepsilon \in (0, 1], \) \( \lambda \in (0, \varepsilon), \) \( h \in (0, \frac{\lambda}{4\varepsilon}] \) there exists a unique solution \((u_{\lambda,n+1}, \mu_{\lambda,n+1})\) of \((P)_{\lambda,n}\) satisfying
\[
u_{\lambda,n+1}, \mu_{\lambda,n+1} \in W \quad \text{for } n = 0, ..., N - 1.
\]

**Lemma 2.2.** For all \( \varepsilon \in (0, 1] \) and all \( \lambda \in (0, \varepsilon) \) there exists a weak solution of \((P)_{\varepsilon,\lambda}\).

We provide some basic results which will be applied in this paper.

**Lemma 2.3.** Let \( \beta: \mathbb{R} \to \mathbb{R} \) be a multi-valued maximal monotone function. Then
\[
(-\Delta u, \beta_r(u))_H \geq 0 \quad \text{for all } u \in W \text{ and all } \tau > 0,
\]
where \( \beta_r \) is the Yosida approximation of \( \beta \) on \( \mathbb{R} \). In particular, if \( \beta: \mathbb{R} \to \mathbb{R} \) is a single-valued maximal monotone function, then
\[
(-\Delta u, \beta(u))_H \geq 0 \quad \text{for all } u \in W \text{ with } \beta(u) \in H.
\]
Proof. It follows from Okazawa [15, Proof of Theorem 3 with $a = b = 0$] that

$$(-\Delta u, \beta_r(u))_H \geq 0 \quad \text{for all } u \in W \text{ and all } \tau > 0.$$ 

In the case that $\beta : \mathbb{R} \to \mathbb{R}$ is a single-valued maximal monotone function, noting that $\beta_r(u) \to \beta(u)$ in $H$ as $\tau \searrow 0$ if $\beta(u) \in H$ (see e.g., [4, Proposition 2.6] or [17, Theorem IV.1.1]), we can obtain the second inequality. $\square$

**Lemma 2.4** ([18, Section 8, Corollary 4]). Assume that

$X \hookrightarrow Z \hookrightarrow Y$ with compact embedding $X \hookrightarrow Z$ ($X$, $Z$ and $Y$ are Banach spaces).

(i) Let $K$ be bounded in $L^p(0,T;X)$ and let $\{\frac{du}{dt} \mid v \in K\}$ be bounded in $L^1(0,T;Y)$ with some constant $1 \leq p < \infty$. Then $K$ is relatively compact in $L^p(0,T;Z)$.

(ii) Let $K$ be bounded in $L^\infty(0,T;X)$ and let $\{\frac{du}{dt} \mid v \in K\}$ be bounded in $L'(0,T;Y)$ with some constant $r > 1$. Then $K$ is relatively compact in $C([0,T];Z)$.

3. Existence for the discrete problem

In this section we will prove Lemma 2.1.

**Lemma 3.1.** For all $g \in H$, $\varepsilon \in (0,1]$, $\lambda \in (0,\varepsilon)$, $h \in (0,\frac{\lambda}{2\varepsilon^3})$ there exists a unique solution $u \in W$ of the equation

$$(\lambda + (1 - \Delta)^{-1})u - \varepsilon h \Delta u + h\beta(u) + h\pi_\varepsilon(u) = g.$$

**Proof.** We define the operator $\Psi : V \to V^*$ as

$$\langle \Psi u, z \rangle_{V^*,V} := ((\lambda + (1 - \Delta)^{-1})u, z)_H + \varepsilon h \int_\Omega \nabla u \cdot \nabla z + h(\beta_r(u), z)_H + h(\pi_\varepsilon(u), z)_H$$

for $u, z \in V$, where $\tau > 0$ and $\beta_r$ is the Yosida approximation of $\beta$ on $\mathbb{R}$. Then this operator is monotone, continuous and coercive for all $\varepsilon \in (0,1]$, $\lambda \in (0,\varepsilon)$, $h \in (0,\frac{\lambda}{2\varepsilon^3})$, $\tau > 0$. Indeed, we have from (A4), the monotonicity of $(1 - \Delta)^{-1}$ and $\beta_r$, the Lipschitz continuity of $(1 - \Delta)^{-1}$ and $\beta_r$ that

$$\langle \Psi u - \Psi w, u - w \rangle_{V^*,V}$$

$$= \lambda\|u - w\|_H^2 + ((1 - \Delta)^{-1}(u - w), u - w)_H + \varepsilon h\|\nabla(u - w)\|_H^2$$

$$+ h(\beta_r(u) - \beta_r(w), u - w)_H + h(\pi_\varepsilon(u) - \pi_\varepsilon(w), u - w)_H$$

$$\geq \min\{\lambda - c_3\varepsilon h, \varepsilon h\} \|u - w\|_V^2,$$

$$\langle \Psi u - \Psi w, z \rangle_{V^*,V}$$

$$= \lambda(u - w, z)_H + ((1 - \Delta)^{-1}(u - w), z)_H + \varepsilon h \int_\Omega \nabla(u - w) \cdot \nabla z$$

$$+ h(\beta_r(u) - \beta_r(w), z)_H + h(\pi_\varepsilon(u) - \pi_\varepsilon(w), z)_H$$

$$\leq \max\{\lambda, 1, \varepsilon h, \tau^{-1} h, c_3\varepsilon h\} \|u - w\|_V \|z\|_V$$
and
\[
\langle \Psi u - \Psi 0, u \rangle_{V^*, V} \geq \min \{\lambda - c_3 \varepsilon h, \varepsilon h\} \|u\|^2_V
\]

for all \( u, w, z \in V, \varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, \frac{1}{c_3 \varepsilon}), \tau > 0 \). Thus the operator \( \Psi : V \to V^* \) is surjective for all \( \varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, \frac{1}{c_3 \varepsilon}), \tau > 0 \) (see e.g., [3, p. 37]) and then the elliptic regularity theory yields that for all \( g \in H, \varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, \frac{1}{c_3 \varepsilon}), \tau > 0 \) there exists a unique solution \( u_\tau \in W \) of the equation
\[
(\lambda + (1 - \Delta)^{-1})u_\tau - \varepsilon h \Delta u_\tau + h \beta_\varepsilon(u_\tau) + h \pi_\varepsilon(u_\tau) = g.
\]  
(3.1)

Here, multiplying (3.1) by \( u_\tau \) and integrating over \( \Omega \), we see from (A4) and the Young inequality that
\[
\lambda \|u_\tau\|^2_H + ((1 - \Delta)^{-1}u_\tau, u_\tau)_H + \varepsilon h \|\nabla u_\tau\|^2_H + h(\beta_\varepsilon(u_\tau), u_\tau)_H
\]
\[
= (g, u_\tau)_H - h(\pi_\varepsilon(u_\tau) - \pi_\varepsilon(0), u_\tau)_H - h(\pi_\varepsilon(0), u_\tau)_H
\]
\[
\leq \|g\|_H \|u_\tau\|_H + h(\|\pi_\varepsilon\|_{L^\infty(\Omega)} + |\pi_\varepsilon(0)|) \|u_\tau\|^2_H + \frac{|\pi_\varepsilon(0)| \|\Omega\|}{4} h
\]
\[
\leq \|g\|_H \|u_\tau\|_H + c_3 \varepsilon h \|u_\tau\|^2_H + \frac{c_3 |\Omega|}{4} h
\]
\[
\leq \frac{1}{4c_3 h} \|g\|^2_H + 2c_3 \varepsilon h \|u_\tau\|^2_H + \frac{c_3 |\Omega|}{4} h,
\]

and hence for all \( \varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, \frac{1}{c_3 \varepsilon}) \) there exists a constant \( C_1 = C_1(\varepsilon, \lambda, h) \) such that
\[
\|u_\tau\|^2_V \leq C_1
\]  
(3.2)

for all \( \tau > 0 \). It follows from (3.1), Lemma 2.3, the monotonicity of \( \beta_\varepsilon \) and the Lipschitz continuity of the operator \( (1 - \Delta)^{-1} \) that
\[
\|\beta_\varepsilon(u_\tau)\|^2_H = (\beta_\varepsilon(u_\tau), \beta_\varepsilon(u_\tau))_H
\]
\[
= \frac{1}{h} (g, \beta_\varepsilon(u_\tau))_H - (\pi_\varepsilon(u_\tau), \beta_\varepsilon(u_\tau))_H - \varepsilon(-\Delta u_\tau, \beta_\varepsilon(u_\tau))_H
\]
\[
- \frac{1}{h} (u_\tau, \beta_\varepsilon(u_\tau))_H - \frac{1}{h} (1 - \Delta)^{-1} u_\tau, \beta_\varepsilon(u_\tau))_H
\]
\[
\leq \frac{1}{h} \|g\|_H \|\beta_\varepsilon(u_\tau)\|_H + \|\pi_\varepsilon(u_\tau)\|_H \|\beta_\varepsilon(u_\tau)\|_H + \frac{1}{h} \|u_\tau\|_H \|\beta_\varepsilon(u_\tau)\|_H.
\]  
(3.3)

Thus we derive from (3.3), (A4), the Young inequality and (3.2) that for all \( \varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, \frac{1}{c_3 \varepsilon}) \) there exists a constant \( C_2 = C_2(\varepsilon, \lambda, h) \) such that
\[
\|\beta_\varepsilon(u_\tau)\|^2_H \leq C_2
\]  
(3.4)
Thus proving Lemma 2.1 is equivalent to derive existence and uniqueness of solutions to (Q).\( u \) there exists a unique solution \((3.4)\), and the elliptic regularity theory imply that for all \(\varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, \frac{\lambda}{2c_{\varepsilon}})\) there exists a constant \(C_3 = C_3(\varepsilon, \lambda, h)\) such that
\[
\|u_\tau\|^2_W \leq C_3
\] (3.5)
for all \(\tau > 0\). Hence by (3.4) and (3.5) there exist \(u \in W\) and \(\xi \in H\) such that
\[
u \rightarrow u \quad \text{weakly in } W,
\]
(3.6)
\[
\beta_\tau(u_\tau) \rightarrow \xi \quad \text{weakly in } H
\] (3.7)
as \(\tau = \tau_j \searrow 0\). Here, owing to (3.2) and the compact embedding \(W \hookrightarrow H\), it holds that
\[
u \rightarrow u \quad \text{strongly in } H
\] (3.8)
as \(\tau = \tau_j \searrow 0\). Moreover, the convergences (3.7) and (3.8) mean that \((\beta_\tau(u_\tau), u_\tau)_H \rightarrow (\xi, u)_H\) as \(\tau = \tau_j \searrow 0\), whence we have that
\[
\xi = \beta(u) \quad \text{a.e. on } \Omega
\] (3.9)
(see e.g., [2, Lemma 1.3, p. 42]).

Therefore we infer from (3.1), (3.6), (3.7), (3.8), the Lipschitz continuity of \(\pi_\varepsilon\), and (3.9) that there exists a solution \(u \in W\) of the equation
\[
(\lambda + (1 - \Delta)^{-1})u - \varepsilon h\Delta u + h\beta(u) + h\pi_\varepsilon(u) = g.
\]
Moreover, we can check that the solution \(u\) of this problem is unique. \(\square\)

**Proof of Lemma 2.1.** The problem \((P)_{\lambda, n}\) can be written as
\[
\begin{cases}
(\lambda + (1 - \Delta)^{-1})u_{\lambda,n+1} - \varepsilon h\Delta u_{\lambda,n+1} + h\beta(u_{\lambda,n+1}) + h\pi_\varepsilon(u_{\lambda,n+1}) \\
= hf_{n+1} + \lambda u_{\lambda,n} + (1 - \Delta)^{-1}u_{\lambda,n} + h(1 - \Delta)^{-1}\mu_{\lambda,n} \\
- \eta h(1 - \Delta)^{-1}\nabla \cdot (u_{\lambda,n}\nabla(1 - \Delta)^{-1}u_{\lambda,n}),
\end{cases}
\]
\((Q)_{\lambda,n}\)
Thus proving Lemma 2.1 is equivalent to derive existence and uniqueness of solutions to \((Q)_{\lambda,n}\) for \(n = 0, ..., N - 1\). It suffices to consider the case that \(n = 0\). By Lemma 3.1 there exists a unique solution \(u_{\lambda,1} \in W\) of the equation
\[
(\lambda + (1 - \Delta)^{-1})u_{\lambda,1} - \varepsilon h\Delta u_{\lambda,1} + h\beta(u_{\lambda,1}) + h\pi_\varepsilon(u_{\lambda,1}) \\
= hf_1 + \lambda u_{\lambda,0} + (1 - \Delta)^{-1}u_{\lambda,0} + h(1 - \Delta)^{-1}\mu_{\lambda,0} \\
- \eta h(1 - \Delta)^{-1}\nabla \cdot (u_{\lambda,0}\nabla(1 - \Delta)^{-1}u_{\lambda,0}).
\]
Therefore, putting \(\mu_1 := (1 - \Delta)^{-1}(\mu_{\lambda,0} - \frac{u_{\lambda,1} - u_{\lambda,0}}{h} - \eta\nabla \cdot (u_{\lambda,0}\nabla(1 - \Delta)^{-1}u_{\lambda,0}))\), we conclude that there exists a unique solution \((u_{\lambda,1}, \mu_{\lambda,1})\) of \((Q)_{\lambda,n}\) in the case that \(n = 0\). \(\square\)
4. Uniform estimates and passage to the limit

In this section we will show Lemma 2.2. We will establish a priori estimates for \((P)_h\) to prove existence for \((P)_{\varepsilon,\lambda}\) by passing to the limit in \((P)_h\).

**Lemma 4.1.** There exists a constant \(C > 0\) depending on the data such that for all \(\varepsilon \in (0, 1]\) and all \(\lambda \in (0, \varepsilon)\) there exists \(h_1 \in (0, \min\{1, \frac{1}{2c_\lambda}\})\) such that

\[
\left\| (\hat{u}_h)_t + h(\hat{\mu}_h)_t \right\|^2_{L^2(0,T;V)} + \lambda \left\| (\hat{u}_h)_t \right\|^2_{L^2(0,T;H)} + \varepsilon \left\| \hat{u}_h \right\|^2_{L^\infty(0,T;V)} + \varepsilon h \left\| (\hat{u}_h)_t \right\|^2_{L^2(0,T;V)} + \left\| \hat{u}_h \right\|^4_{L^4(0,T;L^4(\Omega))}
\]

\[
+ h \left\| \nabla \hat{\mu}_h \right\|^2_{L^\infty(0,T:H)} + h^2 \left\| (\hat{\mu}_h)_t \right\|^2_{L^2(0,T;H)} + \left\| \nabla \nabla \hat{\mu}_h \right\|^2_{L^2(0,T;H)} \leq C
\]

for all \(h \in (0, h_1)\).

**Proof.** We multiply the first equation in \((P)_{\lambda,n}\) by \(h\mu_{\lambda,n+1}\), integrate over \(\Omega\) and use the Young inequality to obtain that

\[
(u_{\lambda,n+1} - u_{\lambda,n}, \mu_{\lambda,n+1})_H + \frac{h}{2}\left( \left\| \mu_{\lambda,n+1} \right\|_H^2 - \left\| \mu_{\lambda,n} \right\|_H^2 + \left\| \mu_{\lambda,n+1} - \mu_{\lambda,n} \right\|_H^2 \right)
\]

\[
+ h\left\| \nabla \mu_{\lambda,n+1} \right\|^2_H
\]

\[
= \eta h \int_\Omega u_{\lambda,n} \nabla (1 - \Delta)^{-1} u_{\lambda,n} \cdot \nabla \mu_{\lambda,n+1}
\]

\[
\leq \frac{h}{4} \left\| \nabla \mu_{\lambda,n+1} \right\|^2_H + \eta^2 h \int_\Omega |u_{\lambda,n} \nabla (1 - \Delta)^{-1} u_{\lambda,n}|^2.
\]

(4.1)

Here multiplying the second equation in \((P)_{\lambda,n}\) by \(u_{\lambda,n+1} - u_{\lambda,n}\) and integrating over \(\Omega\) lead to the identity

\[
(u_{\lambda,n+1} - u_{\lambda,n}, \mu_{\lambda,n+1})_H
\]

\[
= \lambda h \left\| \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} \right\|^2_H + \frac{\varepsilon}{2}\left( \left\| u_{\lambda,n+1} \right\|^2_V - \left\| u_{\lambda,n} \right\|^2_V + \left\| u_{\lambda,n+1} - u_{\lambda,n} \right\|^2_V \right)
\]

\[
+ (\beta(u_{\lambda,n+1}), u_{\lambda,n+1} - u_{\lambda,n})_H
\]

\[
+ (\pi_\varepsilon(u_{\lambda,n+1}) - f_{n+1} - \varepsilon u_{\lambda,n+1}, u_{\lambda,n+1} - u_{\lambda,n})_H.
\]

(4.2)

It follows from (A1) and the definition of the subdifferential that

\[
(\beta(u_{\lambda,n+1}), u_{\lambda,n+1} - u_{\lambda,n})_H \geq \int_\Omega \beta(u_{\lambda,n+1}) - \int_\Omega \beta(u_{\lambda,n}).
\]

(4.3)

Since the first equation in \((P)_{\lambda,n}\) means that

\[
\int_\Omega \left( \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} + \mu_{\lambda,n+1} - \mu_{\lambda,n} \right) = 0,
\]

(4.4)
we have from (A4) and the Young inequality that there exists a constant $C_1 > 0$ such that
\[
- (\pi_\varepsilon(u_{\lambda,n+1}) - f_{n+1} - \varepsilon u_{\lambda,n+1}, u_{\lambda,n+1} - u_{\lambda,n})_H
\]
\[
= - h (\pi_\varepsilon(u_{\lambda,n+1}) - f_{n+1} - \varepsilon u_{\lambda,n+1}, \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} + \mu_{\lambda,n+1} - \mu_{\lambda,n})_H
\]
\[
+ h (\pi_\varepsilon(u_{\lambda,n+1}) - f_{n+1} - \varepsilon u_{\lambda,n+1}, \mu_{\lambda,n+1} - \mu_{\lambda,n})_H
\]
\[
= - h \left( \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} + \mu_{\lambda,n+1} - \mu_{\lambda,n}, \pi_\varepsilon(u_{\lambda,n+1}) - f_{n+1} - \varepsilon u_{\lambda,n+1} - \frac{1}{|\Omega|} \int_{\Omega} (\pi_\varepsilon(u_{\lambda,n+1}) - f_{n+1} - \varepsilon u_{\lambda,n+1}) \right)_{V_0^\ast, V_0}
\]
\[
+ h (\pi_\varepsilon(u_{\lambda,n+1}) - f_{n+1} - \varepsilon u_{\lambda,n+1}, \mu_{\lambda,n+1} - \mu_{\lambda,n})_H
\]
\[
\leq \frac{h}{8} \left\| \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} + \mu_{\lambda,n+1} - \mu_{\lambda,n} \right\|_{V_0^\ast}^2 + \frac{h}{4} \left\| \mu_{\lambda,n+1} - \mu_{\lambda,n} \right\|_H^2 + C_1 \varepsilon^2 h \| u_{\lambda,n+1} \|_V^2
\]
\[
+ C_1 h \| f_{n+1} \|_V^2 + C_1 \varepsilon^2 h
\]
for all $\varepsilon \in (0, 1], \lambda \in (0, \varepsilon)$, $h \in (0, \frac{1}{2 \varepsilon \lambda})$. We see from the first equation in (P)_{\lambda,n} and the Young inequality that
\[
h \left\| \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} + \mu_{\lambda,n+1} - \mu_{\lambda,n} \right\|_{V_0^\ast}^2
\]
\[
= h \left( \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} + \mu_{\lambda,n+1} - \mu_{\lambda,n}, \nabla \mathcal{N} \left( \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} + \mu_{\lambda,n+1} - \mu_{\lambda,n} \right) \right)_{V_0^\ast, V_0}
\]
\[
= - h \int_{\Omega} \nabla \mu_{\lambda,n+1} \cdot \nabla \mathcal{N} \left( \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} + \mu_{\lambda,n+1} - \mu_{\lambda,n} \right)
\]
\[
+ \eta h \int_{\Omega} u_{\lambda,n} \nabla (1 - \Delta)^{-1} u_{\lambda,n} \cdot \nabla \mathcal{N} \left( \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} + \mu_{\lambda,n+1} - \mu_{\lambda,n} \right)
\]
\[
\leq \frac{h}{2} \left\| \nabla \mu_{\lambda,n+1} \right\|_H^2 + \eta^2 h \int_{\Omega} |u_{\lambda,n} \nabla (1 - \Delta)^{-1} u_{\lambda,n}|^2
\]
\[
+ \frac{3 h}{4} \left\| \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} + \mu_{\lambda,n+1} - \mu_{\lambda,n} \right\|_{V_0^\ast}^2
\]
(4.6)
We infer from the continuity of the embedding $W^{2,4}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ and standard elliptic regularity theory ([10]) that there exist constants $C_2, C_3 > 0$ such that
\[
\eta^2 h \int_{\Omega} |u_{\lambda,n} \nabla (1 - \Delta)^{-1} u_{\lambda,n}|^2 \leq \eta^2 h \| u_{\lambda,n} \|_{L^4(\Omega)}^2 \| \nabla (1 - \Delta)^{-1} u_{\lambda,n} \|_{L^4(\Omega)}^2
\]
\[
\leq C_2 \eta^2 h \| u_{\lambda,n} \|_{L^4(\Omega)}^2 \| (1 - \Delta)^{-1} u_{\lambda,n} \|_{W^{2,4}(\Omega)}^2
\]
\[
\leq C_3 \eta^2 h \| u_{\lambda,n} \|_{L^4(\Omega)}^2 \| (1 - \Delta) (1 - \Delta)^{-1} u_{\lambda,n} \|_{L^4(\Omega)}^2
\]
\[
= C_3 \eta^2 h \| u_{\lambda,n} \|_{L^4(\Omega)}^4
\]
(4.7)
for all $\varepsilon \in (0, 1]$, $\lambda \in (0, \varepsilon)$, $h \in (0, \frac{\lambda}{2 \varepsilon^3})$. Thus we derive from (4.1)-(4.3), (4.5)-(4.7) that

$$
\frac{h}{8} \left[ u_{\lambda,n+1} - u_{\lambda,n} \right] + \mu_{\lambda,n+1} - \mu_{\lambda,n} \right]_{V_0^*}^2 + \lambda h \left[ u_{\lambda,n+1} - u_{\lambda,n} \right]_H^2
+ \varepsilon \left( \left[ u_{\lambda,n+1} \right]_{V_0^*}^2 - \left[ u_{\lambda,n} \right]_{V_0^*}^2 + \left[ u_{\lambda,n+1} - u_{\lambda,n} \right]_{V_0^*}^2 \right) + \int_\Omega \tilde{\beta}(u_{\lambda,n+1}) - \int_\Omega \tilde{\beta}(u_{\lambda,n})
+ \frac{h}{2} \left( \left[ \mu_{\lambda,n+1} \right]_{H}^2 - \left[ \mu_{\lambda,n} \right]_{H}^2 \right) + \frac{h}{4} \left[ \mu_{\lambda,n+1} - \mu_{\lambda,n} \right]_{H}^2 + \frac{h}{4} \left[ \nabla \mu_{\lambda,n+1} \right]_H^2
\leq C_1 \varepsilon h \left[ u_{\lambda,n+1} \right]_{V_0}^2 + 2C_3 \varepsilon^2 h \left[ u_{\lambda,n} \right]_{L^4(\Omega)}^4 + C_1 h \left[ f_{n+1} \right]_{V_0}^2 + C_1 T + 2c_4
$$

for all $\varepsilon \in (0, 1]$, $\lambda \in (0, \varepsilon)$, $h \in (0, \frac{\lambda}{2 \varepsilon^3})$ and $m = 1, ..., N$, whence by (A2) there exists a constant $C_4 > 0$ such that for all $\varepsilon \in (0, 1]$ and all $\lambda \in (0, \varepsilon)$ there exists $h_1 \in (0, \min\{1, \frac{\lambda}{2 \varepsilon^3}\})$ such that

$$
\frac{h}{8} \left[ u_{\lambda,n+1} - u_{\lambda,n} \right] + \mu_{\lambda,n+1} - \mu_{\lambda,n} \right]_{V_0^*}^2 + \lambda h \left[ u_{\lambda,n+1} - u_{\lambda,n} \right]_H^2
+ \varepsilon \left[ u_{\lambda,n} \right]_{V_0^*}^2 + \varepsilon h^2 \left( \left[ u_{\lambda,n+1} \right]_{V_0^*}^2 - \left[ u_{\lambda,n} \right]_{V_0^*}^2 + \left[ u_{\lambda,n+1} - u_{\lambda,n} \right]_{V_0^*}^2 \right) + \int_\Omega \tilde{\beta}(u_{\lambda,n+1}) - \int_\Omega \tilde{\beta}(u_{\lambda,n})
+ \frac{h}{2} \left( \left[ \mu_{\lambda,n+1} \right]_{H}^2 - \left[ \mu_{\lambda,n} \right]_{H}^2 \right) + \frac{h}{4} \left[ \mu_{\lambda,n+1} - \mu_{\lambda,n} \right]_{H}^2 + \frac{h}{4} \left[ \nabla \mu_{\lambda,n+1} \right]_H^2
\leq C_4 \varepsilon h \left[ u_{\lambda,n} \right]_{V_0}^2 + C_4 h \left[ u_{\lambda,n} \right]_{L^4(\Omega)}^4 + C_4
$$

for all $h \in (0, h_1)$ and $m = 1, ..., N$. Therefore, owing to the discrete Gronwall lemma (see
e.g., [12, Prop. 2.2.1]), it holds that there exists a constant $C_5 > 0$ such that

$$h \sum_{n=0}^{m-1} \left\| \frac{u_{\lambda,n+1}}{h} - \frac{u_{\lambda,n}}{h} + \mu_{\lambda,n+1} - \mu_{\lambda,n} \right\|_{V_0^*}^2 + \lambda h \sum_{n=0}^{m-1} \left\| \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} \right\|_H^2$$

$$+ \varepsilon \left\| u_{\lambda,m} \right\|_V^2 + \varepsilon h^2 \sum_{n=0}^{m-1} \left\| \frac{u_{\lambda,n+1} - u_{\lambda,n}}{h} \right\|_V^2 + \left\| u_{\lambda,m} \right\|_{L^4(\Omega)}^4$$

$$+ h \left\| \mu_{\lambda,m} \right\|_H^2 + h^3 \sum_{n=0}^{m-1} \left\| \frac{\mu_{\lambda,n+1} - \mu_{\lambda,n}}{h} \right\|_H^2 + h \sum_{n=0}^{m-1} \left\| \nabla \mu_{\lambda,n+1} \right\|_H^2 \leq C_5$$

for all $\varepsilon \in (0, 1]$, $\lambda \in (0, \varepsilon)$, $h \in (0, h_1)$ and $m = 1, ..., N$.

**Lemma 4.2.** Let $h_1$ be as in Lemma 4.1. Then there exists a constant $C > 0$ depending on the data such that

$$\left\| u_h \right\|_{L^\infty(0,T;L^4(\Omega))} \leq C$$

for all $\varepsilon \in (0, 1]$, $\lambda \in (0, \varepsilon)$, $h \in (0, h_1)$.

**Proof.** We can obtain this lemma by Lemma 4.1 and (A5).

**Lemma 4.3.** Let $h_1$ be as in Lemma 4.1. Then there exists a constant $C > 0$ depending on the data such that

$$\left\| \tilde{u}_h(t) \right\|_{L^2(0,T;V^*)}^2 \leq C$$

for all $\varepsilon \in (0, 1]$, $\lambda \in (0, \varepsilon)$, $h \in (0, h_1)$.

**Proof.** Since it follows from the first equation in $(P_h)$ that

$$\left\| \tilde{u}_h(t) + h(\tilde{\mu}_h)(t) \right\|_{V^*}^2$$

$$= ((\tilde{u}_h)(t) + h(\tilde{\mu}_h)(t), F^{-1}((\tilde{u}_h)(t) + h(\tilde{\mu}_h)(t)))_{V^*,V}$$

$$= -\int_\Omega \nabla \tilde{\eta}_h \cdot \nabla F^{-1}((\tilde{u}_h)(t) + h(\tilde{\mu}_h)(t))$$

$$+ \eta \int_\Omega \tilde{u}_h(t) \nabla(1 - \Delta)^{-1} \tilde{u}_h(t) \cdot \nabla F^{-1}((\tilde{u}_h)(t) + h(\tilde{\mu}_h)(t)),$$

we see from the Young inequality, Lemmas 4.1 and 4.2 that there exists a constant $C_1 > 0$ such that

$$\left\| \tilde{u}_h(t) + h(\tilde{\mu}_h)(t) \right\|_{L^2(0,T;V^*)}^2 \leq C_1$$

for all $\varepsilon \in (0, 1]$, $\lambda \in (0, \varepsilon)$, $h \in (0, h_1)$. Then Lemma 4.1 leads to Lemma 4.3.
Lemma 4.4. Let $h_1$ be as in Lemma 4.1. Then there exists a constant $C > 0$ depending on the data such that

$$
\varepsilon^2 \| \overline{u}_h \|^2_{L^2(0,T;W)} \leq C
$$

for all $\varepsilon \in (0, 1]$, $\lambda \in (0, \varepsilon)$, $h \in (0, h_1)$.

Proof. We have from the second equation in (P)$_{\lambda,n}$ that

$$
\varepsilon^2 h \| \Delta u_{\lambda,n+1} \|^2_H = \varepsilon^2 h (-\Delta u_{\lambda,n+1}, -\Delta u_{\lambda,n+1})_H \\
= \varepsilon h (\mu_{\lambda,n+1}, -\Delta u_{\lambda,n+1})_H - \frac{\varepsilon \lambda}{2} (\| \nabla u_{\lambda,n+1} \|^2_H - \| \nabla u_{\lambda,n} \|^2_H + \| \nabla u_{\lambda,n+1} - \nabla u_{\lambda,n} \|^2_H) \\
- \varepsilon h (\beta(u_{\lambda,n+1}), -\Delta u_{\lambda,n+1})_H - \varepsilon h (\pi_\varepsilon(u_{\lambda,n+1})), -\Delta u_{\lambda,n+1})_H \\
+ \varepsilon h (f_{n+1}, -\Delta u_{\lambda,n+1})_H. 
$$

(4.9)

Here Lemma 2.3 implies that

$$
-\varepsilon h (\beta(u_{\lambda,n+1}), -\Delta u_{\lambda,n+1})_H \leq 0. 
$$

(4.10)

We deduce from the first equation in (P)$_{\lambda,n}$ that

$$
\varepsilon h (\mu_{\lambda,n+1}, -\Delta u_{\lambda,n+1})_H = \varepsilon h (-\Delta \mu_{\lambda,n+1}, u_{\lambda,n+1})_H \\
= -\frac{\varepsilon}{2} (\| \mu_{\lambda,n+1} \|^2_H - \| \mu_{\lambda,n} \|^2_H + \| \mu_{\lambda,n+1} - \mu_{\lambda,n} \|^2_H) - \varepsilon h^2 \left( \frac{\mu_{\lambda,n+1} - \mu_{\lambda,n}}{h}, u_{\lambda,n+1} \right)_H \\
+ \eta \varepsilon h \int_\Omega u_{\lambda,n} \nabla (1 - \Delta)^{-1} u_{\lambda,n} \cdot \nabla u_{\lambda,n+1}. 
$$

(4.11)

We infer from the continuity of the embedding $W^{2,4}(\Omega) \hookrightarrow W^{1,4}(\Omega)$, standard elliptic regularity theory ([10]), Lemmas 4.1 and 4.2 that there exist constants $C_1, C_2, C_3 > 0$ such that

$$
\eta \varepsilon h \int_\Omega u_{\lambda,n} \nabla (1 - \Delta)^{-1} u_{\lambda,n} \cdot \nabla u_{\lambda,n+1} \\
\leq \eta \varepsilon h \| u_{\lambda,n} \|_{L^4(\Omega)} \| \nabla (1 - \Delta)^{-1} u_{\lambda,n} \|_{L^4(\Omega)} \| \nabla u_{\lambda,n+1} \|_H \\
\leq C_1 \eta \varepsilon h \| u_{\lambda,n} \|_{L^4(\Omega)} \| (1 - \Delta)^{-1} u_{\lambda,n} \|_{W^{2,4}(\Omega)} \| \nabla u_{\lambda,n+1} \|_H \\
\leq C_2 \eta \varepsilon h \| u_{\lambda,n} \|_{L^4(\Omega)}^2 \| \nabla u_{\lambda,n+1} \|_H \\
\leq C_3 \eta h
$$

(4.12)

for all $\varepsilon \in (0, 1]$, $\lambda \in (0, \varepsilon)$, $h \in (0, h_1)$. The condition (A4) and Lemma 4.1 yield that there exists a constant $C_4 > 0$ such that

$$
-\varepsilon h (\pi_\varepsilon(u_{\lambda,n+1})), -\Delta u_{\lambda,n+1})_H \leq c_3 \varepsilon h \| \nabla u_{\lambda,n+1} \|^2_H \leq C_4 h
$$

(4.13)
for all $\varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, h_1)$. Therefore by (4.9)-(4.13), the Young inequality, summing over $n = 0, \ldots, m - 1$ with $1 \leq m \leq N$, (A5) and Lemma 4.1 there exists a constant $C_5 > 0$ such that

$$
\varepsilon^2 \|\Delta \pi_h\|_{L^2(0,T,H)}^2 \leq C_5
$$

for all $\varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, h_1)$ and then we can obtain Lemma 4.4 by Lemma 4.1 and the elliptic regularity theory.

**Lemma 4.5.** Let $h_1$ be as in Lemma 4.1. Then there exists a constant $C > 0$ depending on the data such that

$$
\|\beta(\overline{u}_h)\|_{L^2(0,T;L^1(\Omega))^2} \leq C
$$

for all $\varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, h_1)$.

**Proof.** Let $\tau > 0$ and let $\beta_\tau$ be the Yosida approximation of $\beta$ on $\mathbb{R}$. Then there exist constants $C_1, C_2 > 0$ such that

$$
\beta_\tau(r) (r - m_0) \geq C_1|\beta_\tau(r)| - C_2
$$

for all $r \in \mathbb{R}$ and all $\tau > 0$ (see e.g., [11, Section 5, p. 908]), where $m_0$ is as in (A5). Thus, since $\beta : \mathbb{R} \to \mathbb{R}$ is single-valued maximal monotone, it holds that $\beta_\tau(r) \to \beta(r)$ in $\mathbb{R}$ as $\tau \searrow 0$ for all $r \in D(\beta)$ and then the inequality

$$
\beta(r) (r - m_0) \geq C_1|\beta(r)| - C_2
$$  \hspace{1cm} (4.14)

holds for all $r \in D(\beta)$. The second equation in (P)$_{\lambda,n}$ leads to the identity

$$
h^{1/2}(\beta(u_{\lambda,n+1}), u_{\lambda,n+1} - m_0)_H
= h^{1/2}(\mu_{\lambda,n+1}, u_{\lambda,n+1} + h\mu_{\lambda,n+1} - m_0)_H - h^{3/2}\|\mu_{\lambda,n+1}\|_H^2
- \lambda h^{1/2}\left(\frac{u_{\lambda,n+1}}{h}, u_{\lambda,n+1} - m_0\right)_H - h^{1/2}\varepsilon\|\nabla u_{\lambda,n+1}\|_H^2
- h^{1/2}(\pi_\varepsilon(u_{\lambda,n+1}), u_{\lambda,n+1} - m_0)_H + h^{1/2}(f_{n+1}, u_{\lambda,n+1} - m_0)_H. \hspace{1cm} (4.15)
$$

Here, since it follows from (4.4) and (A5) that

$$
\frac{1}{|\Omega|} \int_\Omega (u_{\lambda,j} + h\mu_{\lambda,j}) = \frac{1}{|\Omega|} \int_\Omega u_{0\varepsilon} = m_0
$$

for $j = 0, \ldots, N$, we derive from the continuity of the embedding $H \hookrightarrow V_0^*$ that there exists a constant $C_3 > 0$ such that

$$
h^{1/2}(\mu_{\lambda,n+1}, u_{\lambda,n+1} + h\mu_{\lambda,n+1} - m_0)_H
= h^{1/2}\left(u_{\lambda,n+1} + h\mu_{\lambda,n+1} - m_0, \mu_{\lambda,n+1} - \frac{1}{|\Omega|} \int_\Omega \mu_{\lambda,n+1}\right)_V_{0,0}
\leq h^{1/2}\|u_{\lambda,n+1} + h\mu_{\lambda,n+1} - m_0\|_{V_0^*}\|\nabla \mu_{\lambda,n+1}\|_H
\leq C_3 h^{1/2}\|u_{\lambda,n+1} + h\mu_{\lambda,n+1} - m_0\|_H\|\nabla \mu_{\lambda,n+1}\|_H. \hspace{1cm} (4.16)
$$
for all $\varepsilon \in (0, 1], \lambda \in (0, \varepsilon)$, $h \in (0, h_1)$. Therefore we can obtain Lemma 4.5 by (4.14)-(4.16) and Lemma 4.1.

**Lemma 4.6.** Let $h_1$ be as in Lemma 4.1. Then there exists a constant $C > 0$ depending on the data such that

$$\|\overline{\mu}_h\|_{L^2(0,T;V)}^2 + \|\beta(\overline{u}_h)\|_{L^2(0,T;H)}^2 \leq C$$

for all $\varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, h_1)$.

**Proof.** We see from the second equation in $(P)_h$ that

$$\int_{\Omega} \overline{\mu}_h = \lambda \int_{\Omega} (\hat{u}_h)_t + \int_{\Omega} \beta(\overline{u}_h) + \int_{\Omega} \pi_\varepsilon(\overline{u}_h) - \int_{\Omega} \mathcal{J}_h.$$

Thus Lemmas 4.1 and 4.5 mean that there exists a constant $C_1 > 0$ such that

$$\int_0^T \left| \int_{\Omega} \overline{\mu}_h \right|^2 \leq C_1$$

for all $\varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, h_1)$, and hence we have from Lemma 4.1 and the Poincaré–Wirtinger inequality that there exists a constant $C_2 > 0$ such that

$$\|\overline{\mu}_h\|_{L^2(0,T;V)} \leq C_2$$

(4.17)

for all $\varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, h_1)$. Therefore the second equation in $(P)_h$, Lemmas 4.1 and 4.4, and (4.17) that there exists a constant $C_3 > 0$ such that

$$\|\beta(\overline{u}_h)\|_{L^2(0,T;H)}^2 \leq C_3$$

for all $\varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, h_1)$.

**Lemma 4.7.** Let $h_1$ be as in Lemma 4.1. Then there exists a constant $C > 0$ depending on the data such that

$$\lambda \|\hat{u}_h\|_{H^1(0,T;H)}^2 + \|\overline{\mu}_h\|_{L^\infty(0,T;V)}^2 + \|\overline{\mu}_h\|_{L^\infty(0,T;L^4(\Omega))}^4 + \|\overline{\mu}_h\|_{H^1(0,T;V^*)}^2$$

$$+ h \|\hat{\mu}_h\|_{L^\infty(0,T;H)}^2 + h^2 \|\hat{\mu}_h\|_{H^1(0,T;H)}^2 \leq C$$

for all $\varepsilon \in (0, 1], \lambda \in (0, \varepsilon), h \in (0, h_1)$.

**Proof.** This lemma holds by (2.7)-(2.9), Lemmas 4.1 and 4.3.

Now we set the operator $\Phi : L^4(\Omega) \rightarrow V^*$ as

$$\langle \Phi u, z \rangle_{V^*, V} := -\eta \int_{\Omega} u \nabla (1 - \Delta)^{-1} u \cdot \nabla z \quad \text{for } u \in L^4(\Omega), \ z \in V.$$
Proof of Lemma 2.2. Let $\varepsilon \in (0, 1]$ and let $\lambda \in (0, \varepsilon)$. Then by Lemmas 4.1, 4.2, 4.4, 4.6, 4.7, recalling (2.10), (2.11) and the compactness of the embedding $V \hookrightarrow H$ there exist some functions $u_{\varepsilon, \lambda}$, $\mu_{\varepsilon, \lambda}$, $\xi_{\varepsilon, \lambda}$ such that

\[ u_{\varepsilon, \lambda} \in H^1(0, T; H) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad \mu_{\varepsilon, \lambda} \in L^2(0, T; V), \]

\[ \xi_{\varepsilon, \lambda} \in L^2(0, T; H) \]

and

\begin{align*}
\hat{u}_h &\to u_{\varepsilon, \lambda} \quad \text{weakly* in } H^1(0, T; H) \cap L^\infty(0, T; V), \\
\hat{u}_h &\to u_{\varepsilon, \lambda} \quad \text{strongly in } C([0, T]; H), \\
h\hat{\mu}_h &\to 0 \quad \text{strongly in } L^\infty(0, T; H), \\
h\hat{\mu}_h &\to 0 \quad \text{weakly in } H^1(0, T; H), \\
\beta(\hat{\pi}_h) &\to \xi_{\varepsilon, \lambda} \quad \text{weakly in } L^2(0, T; H), \\
\hat{\pi}_h &\to u_{\varepsilon, \lambda} \quad \text{weakly* in } L^\infty(0, T; L^4(\Omega)), \\
\hat{\pi}_h &\to u_{\varepsilon, \lambda} \quad \text{weakly in } L^2(0, T; W), \\
u_h &\to u_{\varepsilon, \lambda} \quad \text{weakly* in } L^\infty(0, T; L^4(\Omega))
\end{align*}

as $h = h_j \searrow 0$. We can verify (2.3) by (4.20). Now we check (2.2). It follows from (2.10), Lemma 4.1 and (4.20) that

\[
\|\hat{\pi}_h - u_{\varepsilon, \lambda}\|_{L^2(0,T;H)} \leq \|\hat{\pi}_h - \hat{u}_h\|_{L^2(0,T;H)} + \|\hat{u}_h - u_{\varepsilon, \lambda}\|_{L^2(0,T;H)}
\]

\[
= \frac{h}{\sqrt{3}} \|\hat{u}_h\|_{L^2(0,T;H)} + \|\hat{u}_h - u_{\varepsilon, \lambda}\|_{L^2(0,T;H)}
\]

\[
\to 0 \quad (4.26)
\]

as $h = h_j \searrow 0$. Therefore we see from (4.23) and (4.26) that

\[
\int_0^T (\beta(\hat{\pi}_h(t)), \hat{\pi}_h(t))_H \, dt \to \int_0^T (\xi_{\varepsilon, \lambda}(t), u_{\varepsilon, \lambda}(t))_H \, dt
\]

as $h = h_j \searrow 0$, whence it holds that

\[ \xi_{\varepsilon, \lambda} = \beta(u_{\varepsilon, \lambda}) \quad \text{a.e. on } \Omega \times (0, T) \quad (4.27) \]

(see e.g., [2, Lemma 1.3, p. 42]). Thus we can obtain (2.2) by the second equation in (P)$h$, (4.19), (4.22)-(4.24), (4.27), the Lipschitz continuity of $\pi_\varepsilon$, (4.26), and by observing that $\hat{\pi}_h \to \hat{f}$ strongly in $L^2(0, T; V)$ as $h \searrow 0$ (cf. [7, Section 5]).

Next we prove (2.1). Owing to standard elliptic regularity theory ([10]) and Lemma 4.7, there exists a constant $C_1 > 0$ such that

\[
\|F^{-1}(\hat{u}_h)_t\|_{L^2(0,T;V)} = \|(\hat{u}_h)_t\|_{L^2(0,T;V^*)} \leq C_1,
\]

\[
\|F^{-1}\hat{u}_h\|_{L^\infty(0,T;W^{2,4}(\Omega))} = \|(-\Delta + 1)^{-1}\hat{u}_h\|_{L^\infty(0,T;W^{2,4}(\Omega))} \leq C_1
\]

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for all $h \in (0, h_1)$ and then we have from $W^{2,4}(\Omega) \hookrightarrow W^{1,4}(\Omega) \hookrightarrow V$ with compactness embedding $W^{2,4}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ and Lemma 2.4 that

$$F^{-1}\hat{u}_h \to F^{-1}u_{\varepsilon,\lambda} \quad \text{strongly in } C([0,T];W^{1,4}(\Omega))$$

as $h = h_j \searrow 0$, which yields that

$$(1 - \Delta)^{-1}\hat{u}_h \to (1 - \Delta)^{-1}u_{\varepsilon,\lambda} \quad \text{strongly in } L^2(0,T;W^{1,4}(\Omega))$$

as $h = h_j \searrow 0$. Thus we derive from (2.11), the continuity of the embedding $W \hookrightarrow W^{1,4}(\Omega)$, the elliptic regularity theory, (2.10), Lemma 4.1 and (4.28) that

$$\|(1 - \Delta)^{-1}u_h - (1 - \Delta)^{-1}u_{\varepsilon,\lambda}\|_{L^2(0,T;W^{1,4}(\Omega))}$$

$$\leq h\|(1 - \Delta)^{-1}(u_h)\|_{L^2(0,T;W^{1,4}(\Omega))} + \|(1 - \Delta)^{-1}(\hat{u}_h - \bar{u}_h)\|_{L^2(0,T;W^{1,4}(\Omega))}$$

$$+ \|(1 - \Delta)^{-1}\bar{u}_h - (1 - \Delta)^{-1}u_{\varepsilon,\lambda}\|_{L^2(0,T;W^{1,4}(\Omega))}$$

$$\leq C_2h\|(1 - \Delta)^{-1}(u_h)\|_{L^2(0,T;W)} + C_2\|(1 - \Delta)^{-1}(\bar{u}_h - \hat{u}_h)\|_{L^2(0,T;W)}$$

$$+ \|(1 - \Delta)^{-1}\bar{u}_h - (1 - \Delta)^{-1}u_{\varepsilon,\lambda}\|_{L^2(0,T;W^{1,4}(\Omega))}$$

$$\leq C_3h\|\bar{u}_h\|_{L^2(0,T;H)} + \|(1 - \Delta)^{-1}\bar{u}_h - (1 - \Delta)^{-1}u_{\varepsilon,\lambda}\|_{L^2(0,T;W^{1,4}(\Omega))}$$

$$\to 0$$

as $h = h_j \searrow 0$, where $C_2, C_3 > 0$ are some constants. Hence we infer that

$$\nabla(1 - \Delta)^{-1}u_h \to \nabla(1 - \Delta)^{-1}u_{\varepsilon,\lambda} \quad \text{strongly in } L^2(0,T;L^4(\Omega))$$

as $h = h_j \searrow 0$. Moreover, Lemma 4.2, the convergences (4.25) and (4.29) imply that

$$\left|\int_0^T \int_\Omega (u_h \nabla(1 - \Delta)^{-1}u_h - u_{\varepsilon,\lambda} \nabla(1 - \Delta)^{-1}u_{\varepsilon,\lambda}) \cdot \nabla\psi \right|$$

$$\leq \left|\int_0^T \int_\Omega (u_h - u_{\varepsilon,\lambda}) \nabla(1 - \Delta)^{-1}u_{\varepsilon,\lambda} \cdot \nabla\psi \right|$$

$$+ \left|\int_0^T \int_\Omega u_h \nabla(1 - \Delta)^{-1}(u_h - u_{\varepsilon,\lambda}) \cdot \nabla\psi \right|$$

$$\leq \left|\int_0^T (u_h(t) - u_{\varepsilon,\lambda}(t)) \nabla(1 - \Delta)^{-1}u_{\varepsilon,\lambda}(t) \cdot \nabla\psi(t) \right|_{L^4(\Omega),L^{4/3}(\Omega)} dt$$

$$+ C_4\|\nabla(1 - \Delta)^{-1}(u_h - u_{\varepsilon,\lambda})\|_{L^2(0,T;L^4(\Omega))} \|\nabla\psi\|_{L^2(0,T;H)}$$

$$\to 0$$

as $h = h_j \searrow 0$, where $C_4 > 0$ is some constant. Therefore it follows from the first equation in (P)$_h$, (4.19), (4.21), (4.22), (4.30) and (4.18) that

$$(u_{\varepsilon,\lambda})_t + (F - I)u_{\varepsilon,\lambda} + \Phi(u_{\varepsilon,\lambda}) = 0 \quad \text{in } L^2(0,T;V^*)$$

which leads to (2.1).
5. Proof of main theorems

In this section we will prove Theorems 1.1 and 1.2.

**Lemma 5.1.** There exists a constant $M > 0$, depending only on the data, such that

\[
\begin{align*}
\lambda \int_0^t \| u'_{\varepsilon,\lambda}(s) \|^2_H \, ds + \varepsilon \| u_{\varepsilon,\lambda}(t) \|^2_V + \| u_{\varepsilon,\lambda}(t) \|^2_{L^4(\Omega)} & \leq M, \\
\int_0^t \| u'_{\varepsilon,\lambda}(s) \|^2_V \, ds & \leq M, \\
\varepsilon^2 \int_0^t \| u_{\varepsilon,\lambda}(s) \|^2_W \, ds & \leq M, \\
\int_0^t \| \mu_{\varepsilon,\lambda}(s) \|^2_V \, ds + \int_0^t \| \beta(u_{\varepsilon,\lambda}(s)) \|^2_H \, ds & \leq M
\end{align*}
\]

(5.1) for all $t \in [0, T]$, $\varepsilon \in (0, 1]$, $\lambda \in (0, \varepsilon)$.

**Proof.** This lemma holds by Lemmas 4.1, 4.3, 4.4 and 4.6. □

**Proof of Theorem 1.1.** Let $\varepsilon \in (0, 1]$. Then we have from Lemma 5.1, the compactness of the embedding $V \hookrightarrow H$ that there exist some functions $u_{\varepsilon}, \mu_{\varepsilon}, \xi_{\varepsilon}$ such that

\[
\begin{align*}
u_{\varepsilon} & \in H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \quad \mu_{\varepsilon} \in L^2(0, T; V), \\
\xi_{\varepsilon} & \in L^2(0, T; H)
\end{align*}
\]

and

\[
\begin{align*}
\lambda u_{\varepsilon,\lambda} & \rightarrow 0 \quad \text{strongly in } H^1(0, T; H), \\
u_{\varepsilon,\lambda} & \rightarrow u_{\varepsilon} \quad \text{weakly* in } H^1(0, T; V^*) \cap L^\infty(0, T; V) \cap L^2(0, T; W), \\
u_{\varepsilon,\lambda} & \rightarrow u_{\varepsilon} \quad \text{strongly in } C([0, T]; H), \\
\mu_{\varepsilon,\lambda} & \rightarrow \mu_{\varepsilon} \quad \text{weakly in } L^2(0, T; V), \\
\beta(u_{\varepsilon,\lambda}) & \rightarrow \xi_{\varepsilon} \quad \text{weakly in } L^2(0, T; H)
\end{align*}
\]

(5.5)\(\text{as } \lambda = \lambda_j \searrow 0\). We can check (1.6) by (5.7). Now we show (1.5). The convergences (5.9) and (5.7) yield that

\[
\int_0^T (\beta(u_{\varepsilon,\lambda}(t)), u_{\varepsilon,\lambda}(t))_H \, dt \rightarrow \int_0^T (\xi_{\varepsilon}(t), u_{\varepsilon}(t))_H \, dt
\]

as $\lambda = \lambda_j \searrow 0$, which means that

\[
\xi_{\varepsilon} = \beta(u_{\varepsilon}) \quad \text{a.e. on } \Omega \times (0, T)
\]

(5.10)
(see e.g., [2, Lemma 1.3, p. 42]). Thus we can confirm that (1.5) holds by (2.2), (5.8), (5.5), (5.6), (5.9), (5.10), the Lipschitz continuity of $\pi_\varepsilon$, and (5.7).

Next we verify (1.4). It follows from standard elliptic regularity theory ([10]) and Lemma 5.1 that there exists a constant $C_1 > 0$ such that

$$\|F^{-1}(u_{\varepsilon,\lambda})_t\|_{L^2(0,T;V^*)} = \|u_{\varepsilon,\lambda}\|_{L^2(0,T;V^*)} \leq C_1,$$

$$\|F^{-1}u_{\varepsilon,\lambda}\|_{L^\infty(0,T;W^{2,4}(\Omega))} = \|(-\Delta + 1)^{-1}u_{\varepsilon,\lambda}\|_{L^\infty(0,T;W^{2,4}(\Omega))} \leq C_1,$$

for all $\lambda \in (0,\varepsilon)$. Thus we deduce from $W^{2,4}(\Omega) \hookrightarrow W^{1,4}(\Omega) \hookrightarrow V$ with compactness embedding $W^{2,4}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ and Lemma 2.4 that

$$F^{-1}u_{\varepsilon,\lambda} \rightarrow F^{-1}u_\varepsilon \quad \text{strongly in } C([0,T];W^{1,4}(\Omega))$$

as $\lambda = \lambda_j \downarrow 0$, which leads to the convergence

$$(1 - \Delta)^{-1}u_{\varepsilon,\lambda} \rightarrow (1 - \Delta)^{-1}u_\varepsilon \quad \text{strongly in } L^4(0,T;W^{1,4}(\Omega))$$

as $\lambda = \lambda_j \downarrow 0$. Then it holds that

$$\nabla(1 - \Delta)^{-1}u_{\varepsilon,\lambda} \cdot \nabla\psi \rightarrow \nabla(1 - \Delta)^{-1}u_\varepsilon \cdot \nabla\psi \quad \text{strongly in } L^{4/3}(0,T;L^{4/3}(\Omega)) \quad (5.11)$$

as $\lambda = \lambda_j \downarrow 0$ for all $\psi \in L^2(0,T;V)$. Therefore we see from (2.1), (5.6), (5.8) and (5.11) that

$$(u_\varepsilon)_t + (F - I)\mu_\varepsilon + \Phi(u_\varepsilon) = 0 \quad \text{in } L^2(0,T;V^*),$$

where $\Phi$ is as in (4.18), and then we can obtain (1.4).

\[ \square \]

**Proof of Theorem 1.2.** By (1.7)-(1.10) and the compactness of the embedding $H \hookrightarrow V^*$ there exist some functions $u$, $\mu$, $\xi$ such that

$$u \in H^1(0,T;V^*) \cap L^\infty(0,T;L^4(\Omega)), \quad \mu \in L^2(0,T;V),$$

$$\xi \in L^2(0,T;H)$$

and

$$u_\varepsilon \rightharpoonup u \quad \text{weakly}^* \text{ in } H^1(0,T;V^*) \cap L^\infty(0,T;L^4(\Omega)), \quad (5.12)$$

$$u_\varepsilon \rightarrow u \quad \text{strongly in } C([0,T];V^*), \quad (5.13)$$

$$\varepsilon u_\varepsilon \rightarrow 0 \quad \text{strongly in } L^\infty(0,T;V), \quad (5.14)$$

$$\varepsilon u_\varepsilon \rightarrow 0 \quad \text{weakly in } L^2(0,T;W), \quad (5.15)$$

$$\mu_\varepsilon \rightharpoonup \mu \quad \text{weakly in } L^2(0,T;V), \quad (5.16)$$

$$\beta(u_\varepsilon) \rightarrow \xi \quad \text{weakly in } L^2(0,T;H)$$

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as $\varepsilon = \varepsilon_j \downarrow 0$. We can prove (1.3) by (5.13) and (A5). Also we can show (1.1) by (1.4), (5.12), (5.15), (1.7), (1.8), $W^{2,4}(\Omega) \hookrightarrow W^{1,4}(\Omega) \hookrightarrow V$ with compactness embedding $W^{2,4}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ and Lemma 2.4 in the same way as in the proof of Theorem 1.1. Now we check (1.2). We have from (A4) and (1.7) that

$$
\left\| \pi_{\varepsilon}(u_{\varepsilon}) \right\|_{L^\infty(0,T;L^4(\Omega))} \leq \frac{1}{4} |\Omega| \left| \pi_{\varepsilon}(0) \right| + \left\| \pi_{\varepsilon}(u_{\varepsilon}) - \pi_{\varepsilon}(0) \right\|_{L^\infty(0,T;L^4(\Omega))}
$$

$$
\leq \frac{1}{4} |\Omega| \left| \pi_{\varepsilon}(0) \right| + \left\| \pi'_{\varepsilon} \right\|_{L^\infty(\mathbb{R})} \left\| u_{\varepsilon} \right\|_{L^\infty(0,T;L^4(\Omega))}
$$

$$
\leq C_1 \varepsilon \rightarrow 0 \quad (5.17)
$$
as $\varepsilon \downarrow 0$, where $C_1 > 0$ is some constant. Hence we derive from (1.5), (5.15), (5.14), (5.16) and (5.17) that

$$
\mu = \xi - f \quad \text{a.e. on } \Omega \times (0,T). \quad (5.18)
$$

Moreover, we infer from (1.5), (5.13), (5.15), (5.17), (5.12) and (5.18) that

$$
\int_0^T (\beta(u_{\varepsilon}(t)), u_{\varepsilon}(t))_H dt = \int_0^T (\mu_{\varepsilon}(t) + \varepsilon \Delta u_{\varepsilon}(t) - \pi_{\varepsilon}(u_{\varepsilon}(t)) + f(t), u_{\varepsilon}(t))_H dt
$$

$$
= \int_0^T \left\langle u_{\varepsilon}(t), \mu_{\varepsilon}(t) \right\rangle_{V^*,V} dt - \varepsilon \int_0^T \left\| \nabla u_{\varepsilon}(t) \right\|_H^2 dt - \int_0^T (\pi_{\varepsilon}(u_{\varepsilon}(t)), u_{\varepsilon}(t))_H dt
$$

$$
+ \int_0^T (f(t), u_{\varepsilon}(t))_H dt
$$

$$
\leq \int_0^T \left\langle u_{\varepsilon}(t), \mu_{\varepsilon}(t) \right\rangle_{V^*,V} dt - \int_0^T (\pi_{\varepsilon}(u_{\varepsilon}(t)), u_{\varepsilon}(t))_H dt + \int_0^T (f(t), u_{\varepsilon}(t))_H dt
$$

$$
\rightarrow \int_0^T \left\langle u(t), \mu(t) \right\rangle_{V^*,V} dt + \int_0^T (f(t), u(t))_H dt = \int_0^T (\xi(t), u(t))_H dt
$$
as $\varepsilon = \varepsilon_j \downarrow 0$. Thus it holds that

$$
\xi = \beta(u) \quad \text{a.e. on } \Omega \times (0,T). \quad (5.19)
$$

(see e.g., [2, Lemma 1.3, p. 42]). Therefore we can verify (1.2) by (5.18) and (5.19). \qed

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