Variational Study of the Phase Transition at Finite T in the $\lambda\phi^4$-Theory

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1 Introduction

In recent years the question of the nature of the phase transitions in Quantum Field Theories has been a long standing subject of study, specially in gauge theories at finite temperature, because of its cosmological implications on the Baryonic asymmetry of the universe [1].

In spite of a large amount of work done in this subject [2], we do not have a conclusive answer about the nature of the phase transition even in toy models like the scalar electrodynamics at finite temperature. The numerical approach has been extremely elusive, and we are still waiting for trustable predictions which are not sensitive to the method used. At \( T = 0 \) the conventional loop expansion fails to predict spontaneous symmetry breaking in the scalar \( \lambda \Phi^4 \) theory: the non-trivial minimum lies outside the validity region of the one-loop expansion [3]. In addition the re-summation of certain classes of diagrams, daisy or super-daisy diagrams does not improve the situation: a first order phase transition is found in contradiction with the evidence supported by a large amount of numerical as well as analytical renormalization group studies [4].

We propose an alternative method based on the triviality of the 4-dimensional \( \lambda \Phi^4 \) theory and the use of the so called Feynman-Bogoliubov method [5], which consists in the solution of a gap equation to compute the effective potential non-perturbatively [6]. To implement this, we consider the \( \lambda \Phi^4 \) theory as a function of a finite cut off \( \Lambda \) lying within the scaling region, or area in the phase diagram where the low energy physical amplitudes depends only weakly on the cutoff, and apply a variational method to compute the effective potential.

One obtains an expression for the effective potential which depends on the plasma mass, which is given self-consistently by a gap-equation. This approach should be specially accurate very close to the phase transition as it is shown in field theory at zero dimensions [7]. Its perturbative solutions come from summations of certain classes of superdaisy diagrams [8].

We analyze numerically the shape of the effective potential to obtain information of the phase transition. The cutoff is determined by an stability condition on the gap equation for the plasma mass and it is expressed by a simple function of the bare parameters.
2 The Method

We start with the Euclidean action of the one-component $\lambda \phi^4$ theory expressed as a function of the fluctuation field $\zeta$, which represents the high frequency component of the fundamental Higgs field $\phi$ and has no zero-momentum Fourier component:

$$S[\zeta] = \int dz \left[ \frac{1}{2} (\partial_\mu \zeta(z))^2 + \frac{1}{2} m^2(\Phi)\zeta^2(z) \right. $$
$$+ \frac{\lambda}{4!} \left( 4\Phi \zeta^3(z) + \zeta^4(z) \right) + U_{cl}(\Phi) \left. \right]$$

(1)

where $U_{cl}$ is the classical potential

$$U_{cl}(\Phi) = \frac{1}{2} m^2(\Phi) \Phi^2 + \frac{\lambda}{4!} \Phi^4 \quad , \quad m^2(\Phi) = U_{cl}'(\Phi)$$

The constraint effective potential $U_{eff}(\Phi)$ for finite volume $V$ gives the probability distribution of the magnetization $\Phi = \frac{1}{V} \int \phi(z) \, dz$ and is defined as the delta constrained path integral [9]:

$$e^{-V U_{eff}(\Phi)} = \int D\zeta \, \delta(C\zeta) e^{-S[\zeta]}$$

(2)

$$= e^{-V U_{cl}(\Phi)} \int d\mu_\Gamma(\zeta) \exp \left[ - \int dz \frac{\lambda}{4!} \left( 4\Phi \zeta^3(z) + \zeta^4(z) \right) \right]$$

where $d\mu_\Gamma(\zeta)$ is the gaussian measure with covariance $\Gamma$, the propagator of the fluctuation field $\zeta$. If $C$ denotes the averaging operator an $C^\dagger$ its adjoint, we write explicitly:

$$d\mu_\Gamma(\zeta) = \exp \left( - \frac{1}{2} (\zeta, \Gamma^{-1} \zeta) \right) D\zeta \quad ; \quad \Gamma = \lim_{\kappa \to \infty} (-\Delta + \kappa C^\dagger C)^{-1}$$

The limit $\kappa \to \infty$ can be taken if desired or kept $\kappa$ finite for lattice computations [9].
The expression (2) has the adequate form to apply the Feynman-Bogoliubov method to find the best quadratic approximation $S_o [\zeta] = \frac{1}{2} (\zeta, J\zeta)$ around which to expand. The optimal quadratic approximation $J$ is determined by the extreme condition on the Peierls inequality [11]:

$$\left\langle \frac{\delta^2 S[\zeta]}{\delta \zeta(z_1) \delta \zeta(z_2)} \right\rangle_0 = J(z_1, z_2)$$

Herein, $< .. >_o$ is the expectation value in the theory with optimal action $S_o [\zeta]$. This is equivalent to the self consistent equation:

$$J(z_1, z_2) = \Gamma^{-1}(z_1, z_2) + \delta(z_1 - z_2) \left\{ m^2(\Phi) + \frac{\lambda}{2} J^{-1}(z_1, z_1) \right\}$$

(3)

We seek for a translational invariant solution: $J(z_1, z_2) = J(z_1 - z_2)$. Inserting eqn. (3) into the r.h.s. Peierls inequality, we obtain the constraint effective potential:

$$U_{eff}^{FB}(\Phi) = U_{cl}(\Phi) + \frac{1}{V} Tr \ln J + \frac{1}{2} m^2(\Phi) J^{-1}(0)$$

$$+ \frac{1}{2} \int dz \ \Gamma^{-1}(z) J^{-1}(z) + \frac{\lambda}{8} (J^{-1}(0))^2$$

(4)

In the infinite volume limit this expression coincides with the gaussian effective potential of ref [12]. His analysis amount to discuss the triviality of the $\lambda \phi^4$- theory and find suitable renormalization conditions in continuum in different dimensions.

The quadratic operator $J^{-1}(z_1, z_2)$ is given by:

$$J^{-1}(z_1, z_2) = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip(z_1-z_2)}}{p^2 + m_p^2(\Phi)}$$

Thus the value of $m_p^2(\Phi)$ is given by the self consistent gap equation:

$$m_p^2(\Phi) = m_o^2(\Phi) + \frac{\lambda}{2} \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m_p^2(\Phi)}$$

(5)

This gap equation has perturbative solutions which comes from summation of superdaisy diagrams.
The introduction of a finite cutoff allows to compute explicitly the integrals appearing in eqns. (4) and (5). We obtain the final expression for $U_{\text{eff}}^F(\Phi)$:

$$U_{\text{eff}}^F(\Phi) = U_{cl}(\Phi) + \frac{1}{64\pi^2} \left\{ \Lambda^4 \ln \left( m_p^2(\Phi) + \Lambda^2 \right) - \frac{\Lambda^4}{2} + m_p^2(\Phi) \Lambda^2 - m_p^2(\Phi) \ln \left( 1 + \frac{\Lambda^2}{m_p^2(\Phi)} \right) \right\}$$

$$+ \frac{\lambda}{2048\pi^2} \left\{ \Lambda^4 - 2\Lambda^2 m_p^2(\Phi) \ln \left( 1 + \frac{\Lambda^2}{m_p^2(\Phi)} \right) \right\}$$

$$+ m_p^4(\Phi) \ln \left( 1 + \frac{\Lambda^2}{m_p^2(\Phi)} \right) \right\}$$

(6)

Here the plasma mass given by

$$m_p^2(\Phi) = m_o^2(\Phi) + \frac{\lambda}{32\pi^2} \left\{ \Lambda^2 - m_p^2(\Phi) \ln \left( 1 + \frac{\Lambda^2}{m_p^2(\Phi)} \right) \right\}$$

(7)

### 3 Renormalization

In order to compute the curves of constant physics or renormalization group trajectories one has to find suitable physical quantities to keep constant \[13\]. For instance we use the renormalized coupling constants which parametrize the action. We define the renormalized mass $m_R$ and self coupling $\lambda_R$ constant through the second and fourth derivative respectively at the non trivial minimum in the broken phase to avoid infrared divergencies, and at the origin in the symmetric phase.

The minimum of $U_{\text{eff}}^F(\Phi)$ is given by the condition

$$\frac{dU_{\text{eff}}^F(\Phi)}{d\Phi} = \Phi \left( m_p^2(\Phi_{\text{min}}) - \frac{\lambda}{3} \Phi^2_{\text{min}} \right) = 0$$

(8)

The explicit expression for $m_R^2$ and $\lambda_R$ can be given in a closed form
\[ m_R^2 = m_p^2 - \lambda \Phi_{\text{min}}^2 + \frac{\lambda \Phi_{\text{min}}^2}{1 + \frac{1}{2} I_2(m_p^2)} \]  

\[ \lambda_R = -2\lambda + \frac{3\lambda}{(1 + \lambda I_2(m_p^2)/2)} + \frac{6\lambda^3 \Phi_{\text{min}}^2 I_3(m_p^2)}{(1 + \lambda I_2(m_p^2)/2)^3} - \frac{3\lambda^4 \Phi_{\text{min}}^4 I_4(m_p^2)}{(1 + \lambda I_2(m_p^2)/2)^4} + \frac{3\lambda^5 \Phi_{\text{min}}^5 I_5(m_p^2)}{(1 + \lambda I_2(m_p^2)/2)^5} \]  

where \( m_p^2 = m_p^2(\Phi_{\text{min}}) \) and \( I_n(m_p^2) = \int \frac{d^4p}{(2\pi)^4} (p^2 + m_p^2)^{-n} \). In particular:

\[ I_2(m_p^2) = \frac{1}{8\pi^2} \left\{ \frac{-m_p^2}{2 (m_p^2 + \Lambda^2)} - \frac{1}{2} \ln \left( \frac{1 + \Lambda^2}{m_p^2} \right) \right\} \]

and

\[ I_3(m_p^2) = \frac{1}{32\pi^2 m_p^2} \quad , \quad I_4(m_p^2) = \frac{1}{96\pi^2 m_p^4} \]

### 4 Results and Discussion

The starting point is to allocate in the space of the bare parameters, which of course depend on the cutoff, the critical line where the renormalized mass vanishes. This is known to be a non-perturbative problem. The Feynman-Bogoliubov method is expected to describe very accurately the theory very close to the transition line. The enlighten point to observe is that the renormalized mass equals the plasma mass given by the gap equation when the transition line is approached from the symmetric phase. Therefore we seek the values of the bare parameters where the stability condition for the gap equation is reached:

\[ \lambda_o = -32\pi^2 \frac{m_o^2}{\Lambda^2} \]  

We use the lattice parametrization \( m_{\text{lat}}^2 = m_o^2 a^2 \) with \( a = \pi/\Lambda \), and plot in figure 1 the equation (11). A remarkable correspondence is found when comparing with the more accurate determination of the critical line by Lüscher
et al. [4], which is a meaningful test of the self-consistency of our approach that transcend perturbation theory.

For an arbitrary cutoff which only ensures the validity of the stability condition, figure 2 shows the shape of the effective potential computed from equation (2) for values of the bare parameter very close to the critical value. A second order phase transition is clearly found as the non trivial minimum evolves continuously to the origin of the desired field Φ, as the critical line is approached. This result is independent of the choice of the region in the bare parameter space provided they are close enough to the critical line. This is in agreement with the large amount of numerical evidence on the transition of the $\lambda \phi^4$-theory at zero temperature [4].

In figure 3 the renormalized coupling constant $\lambda_R$ is plotted as a function of the cutoff $\Lambda$ for fixed values of the renormalized mass $m_R$ and the lattice bare mass $m_{\text{latt}}$. We conclude that it is not possible to increase indefinitely the UV cutoff without entering an instability region. This clearly suggests the existence of a triviality bound for $\Lambda$ or equivalently the triviality of the lattice $\lambda \phi^4$-theory.

By choosing periodic boundary conditions in time direction with period $\beta = 1/T$, which leads to the well known sums on Matsubara frequencies $\omega_n = 2\pi n/\beta$ one includes finite temperature effects [11]. In this case we obtain a modified equation for the critical line at finite temperature:

$$\lambda_o = -\frac{m_{\text{latt}}^2}{(1/32\pi^2 + T^2_B/24\Lambda^2)} \quad (12)$$

Moreover from figure 4 we see that the symmetry restoration at high temperature is driven by a weak first order phase transition. In this situation the strength of the transition is sensitive to the value of the cutoff.

Our concluding remarks about this method is to emphasize its simplicity and beauty, and the remarkable agreement with numerical simulations much more computer time consuming. Stimulated by this work, we are extending the method to gauge theories.

Acknowledgments

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References

[1] A. Linde, Elementarteilchen und inflationärer Kosmos, Spektrum Akademischer Verlag (1990), and references therein

[2] C. Wetterich, Nuc. Phys. B352 (1991) 529; N. Tetradis and C. Wetterich, Nuc. Phys. B398 (1993) 659; K. Takahashi, Z. Phys. C26 (1985) 601; M.E. Carrington, Phys Rev. D45 (1992) 2933; P. Arnold and O. Espinosa, Phys. Rev D45 (1993) 3546; M. Dine, Phys. Lett. B303 (1993) 308-314

[3] S. Coleman and E. Weinberg Phys. Rev. D7 (1973) 1888; L. Dolan and R. Jackiw: Phys. Rev. D9 (1974) 3320

[4] M. Lüscher and P. Weisz: Nuc. Phys. B290 (1987) 25; M. Lüscher and P. Weisz: Nuc. Phys. B295 (1988) 65

[5] S.V. Tyablikov, Methods in Quantum Theory of magnetism, Plenum, NY 1967; W. Rühl, Z. Phys. C32 (1986) 265

[6] U. Kerres, G. Mack and G. Palma, Nucl. Phys. B42, Proc. Supplements (1995)584-586

[7] V. Cardenas and G. Palma, in: Proceedings of the IX Chilen Physical Simposium, Temuco 1994, p. 89

[8] W. Buchmüller, T. Helbig and D. Walliser, Nucl. Phys. B407 (1993)387; W. Buchmuller, T. Helbig, Int. J. Mod. Phys. C3 (1992) 799

[9] R. Fukuda, E. Kyriapolis, Nuc. Phys. B85 (1975) 354; Y. Fujimoto, L. O’Raifeartaigh and G. Paravicini, Nucl. Phys. B212 (1983) 268; G. Palma, Z. Phys. C54 (1992) 679

[10] P. Hasenfratz and F. Niedermayer, Nucl. Phys. B293 (1994) 785

[11] U. Kerres, G. Mack and G. Palma, DESY preprint DESY 94-226 (1994)

[12] P.M. Stevenson, Phys. Rev. D32 (1985) 1389

[13] I. Montvay, Nucl. Phys. B293 (1987) 479
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