A reductionistic approach to quantum computation

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In the reductionistic approach, mechanisms are divided into simpler parts interconnected in some standard way (e.g. by a mechanical transmission). We explore the possibility of porting reductionism in quantum operations. Conceptually, first parts are made independent of each other by assuming that all “transmissions” are removed. The overall state would thus become a superposition of tensor products of the eigenstates of the independent parts. Transmissions are restored by projecting off all the tensor products which violate them. This would be performed by particle statistics; the plausibility of this scheme is based on the interpretation of particle statistics as projection. The problem of the satisfiability of a Boolean network is approached in this way. This form of quantum reductionism appears to be able of taming the quantum whole without clipping its richness.

I. INTRODUCTION

Reducing the apparently more complex to the less so is the aim of the reductionistic approach. In (classical) applied mechanics, this can be exemplified by the notion of mechanical transmission. At the same time, transmissions divide the whole into simpler parts and reconstruct it – they introduce a “divide and conquer” strategy. In fig. 1, a crank-shaft is the mechanical transmission which imposes a correlation between the positions of parts \( r \) and \( s \) (here discretized as 0 and 1).

Fig. 1

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Things can be more difficult in quantum mechanisms, when the mechanical transmission can become an interaction Hamiltonian which does not commute with the Hamiltonians of the simpler parts. One can make this comparison:

- Classical mechanisms:
  - advantage: the whole is approachable by reductionism, there is a long-standing tradition of taming it through a divide and conquer strategy;
  - disadvantage: the “richness” of the overall functionality grows (roughly) polynomially with the number of parts.

- Quantum mechanisms:
  - disadvantage: the whole is, in general, difficult to tame without clipping its richness;
  - advantage: the “richness” of the overall functionality can grow exponentially with the number of parts; this is the case of the quantum algorithms found so far.\cite{1,2,3,4,5,6,7}

Is there a way of getting the advantages and avoiding the disadvantages of either kind of mechanism? Namely: is the quantum whole tamable through a divide and conquer strategy?

The answer given in this work is affirmative, although speculative. By applying a sort of reverse engineering, we will identify a feature that would be nice to have in quantum mechanisms. Using a top down approach might be justified since the problem at stake, beyond its physical implications, is an engineering one. The nice-to-have thing will be a “quantum mechanical transmission” which, at the same time, divides the whole into simpler parts and reconstructs it; it will be inspired by a special interpretation of identical particles (parts) statistics. The form of reductionism thus introduced turns out to be deeper than the classical one.

Then we ask ourselves whether or not the feature is physical. The answer reached in this work is that it is physical as long as Hermitean matrices can stand in place of Hamiltonians.
II. DEFINITION OF QUANTUM MECHANICAL TRANSMISSION

We consider the mechanism of fig. 1. The “quantum transmission” should establish a constraint between two quantum parts which would otherwise be independent: part $r$ with eigenstates $|0\rangle_r$ and $|1\rangle_r$, and part $s$ with eigenstates $|0\rangle_s$ and $|1\rangle_s$ (fig. 2). We claim that the allowed states of this quantum transmission have the form

$$|\varphi\rangle = \alpha |0\rangle_r |1\rangle_s + \beta |1\rangle_r |0\rangle_s$$

with $|\alpha|^2 + |\beta|^2 = 1$. (1)

We should note that the eigenvalues of each tensor product in (1) satisfy the invertible Boolean NOT function.

Fig. 2

Let us assume that the transmission is temporarily removed. The generic state of each independent part is

$$|\Psi\rangle_r = \alpha_r |0\rangle_r + \beta_r |1\rangle_r, \quad |\Psi\rangle_s = \alpha_s |0\rangle_s + \beta_s |1\rangle_s$$

The whole, unentangled, state in the Hilbert space

$$H = \text{span} \{ |0\rangle_r |0\rangle_s, |0\rangle_r |1\rangle_s, |1\rangle_r |0\rangle_s, |1\rangle_r |1\rangle_s \}$$

is

$$|\Psi\rangle = \alpha_0 |0\rangle_r |0\rangle_s + \alpha_1 |0\rangle_r |1\rangle_s + \alpha_2 |1\rangle_r |0\rangle_s + \alpha_3 |1\rangle_r |1\rangle_s,$$  

with $\alpha_o = \alpha_r \alpha_s$, etc. Some tensor products violate the NOT function (here also called “symmetry”) characterizing this quantum transmission, some do not. The transmission is restored by projecting the compound state $|\Psi\rangle$ on the “symmetric” subspace $H_s = \text{span} \{ |0\rangle_r |1\rangle_s, |1\rangle_r |0\rangle_s \}$. Let us define the “symmetrization projection” $A_{rs}$ by the following equations:

$$A_{rs} |0\rangle_r |1\rangle_s = |0\rangle_r |1\rangle_s, \quad A_{rs} |1\rangle_r |0\rangle_s = |1\rangle_r |0\rangle_s, \quad A_{rs} |0\rangle_r |0\rangle_s = 0, \quad A_{rs} |1\rangle_r |1\rangle_s = 0$$
Given $|\Psi\rangle$ (eq. 2), its projection on $H_s$ is obtained (in a somewhat peculiar way which will be clearly motivated in Section IV) by submitting a free normalized vector $|\varphi\rangle$ of $H$ (whose amplitudes on the basis vectors of $H$ are free and independent variables up to normalization) to the following, mathematically simultaneous, conditions:

i) $A_{rs} |\varphi\rangle = |\varphi\rangle$

ii) the distance between the vector before projection $|\Psi\rangle$ and that after projection $|\varphi\rangle$ should be minimum.

This yields the usual result $|\varphi\rangle = k (\alpha_1 |0\rangle_r |1\rangle_s + \alpha_2 |1\rangle_r |0\rangle_s)$, where $k$ is the renormalization factor, which is of course an allowed transmission state.

A continuous operation on the state of either part $r$ or $s$, associated with simultaneous, continuous projection of the whole state on $H_s$ [conditions (i) and (ii)] will perform the quantum mechanical transmission. This is inspired by a special interpretation of particle statistics. See also ref. [8, 9].

### III. PARTICLE STATISTICS AS A PROJECTION OPERATOR

Let

$$|\Psi(t)\rangle = \alpha |0\rangle_1 |0\rangle_2 + \beta |1\rangle_1 |1\rangle_2 + \gamma (|0\rangle_1 |1\rangle_2 + |1\rangle_1 |0\rangle_2)$$

be a triplet state at some time $t$, where 1 and 2 are two identical spinless particles; 0/1 stand for, say, horizontal/vertical polarization. There is a well known didascalic way of deriving the symmetrical form of state (3). Particles 1 and 2 are first considered to be independent:

$$|\Psi\rangle_1^\prime = \alpha_1 |0\rangle_1 + \beta_1 |1\rangle_1, \quad |\Psi\rangle_2^\prime = \alpha_2 |0\rangle_2 + \beta_2 |1\rangle_2.$$  

Their compound, unentangled, state is

$$|\Psi\rangle^\prime = \gamma_0 |0\rangle_1 |0\rangle_2 + \gamma_1 |0\rangle_1 |1\rangle_2 + \gamma_2 |1\rangle_1 |0\rangle_2 + \gamma_3 |1\rangle_1 |1\rangle_2,$$
where $\gamma_0 = \alpha_1 \alpha_2$, etc. This state must be submitted to *symmetrization* for the permutation of the two particles, namely to the operator $S_{12} = 1 + P_{12}$ followed by renormalization, and this indeed yields a (entangled) triplet state like that of eq. (3).

The interpretation of particle statistics as the result of projection amounts to taking this didascalic procedure literally. Symmetry is viewed as the result of a *watchdog effect* which continuously *projects* the overall state of the two particles on the three-dimensional Hilbert space

$$H_t = \text{span} \left\{ |0\rangle_1 |0\rangle_2 , |1\rangle_1 |1\rangle_2 , \frac{1}{\sqrt{2}} (|0\rangle_1 |1\rangle_2 + |1\rangle_1 |0\rangle_2) \right\}.$$  

Therefore $|\Psi (t)\rangle$ symmetry is the result of the continuous projection

$$\forall t : S_{12} |\Psi (t)\rangle = |\Psi (t)\rangle . \quad (4)$$

This is different from saying that a particle statistics symmetry is an *initial condition*, that was there from the beginning and has been conserved as a constant of motion. When symmetry is viewed as the result of projection, no independent initial condition is required. Any initial state would itself be the result of a projection satisfying eq. (4), that is satisfying the symmetry. As for the rest, this projection interpretation appears to be consistent with the conventional interpretation. However, the notion of a quantum mechanical transmission is a peculiar case, and will have to rely strictly on the projection interpretation.

By way of exemplification, we shall illustrate a sort of quantum mechanical transmission simply related to particle statistics. Let 1 and 2 be two free, identical and non-interacting spin 1/2 particles. At a given time, the overall spatial wave function $\Psi (x_1, x_2)$ is a linear combination of the spatial wave functions of the two free particles: $\Psi (x_1) = e^{ikAx_1}$, $\Psi (x_2) = e^{ikBx_2}$; $x_1$ and $x_2$ are the spatial coordinates of the two particles:

$$\Psi (x_1, x_2) = e^{ikAx_1} e^{ikBx_2} \pm e^{ikAx_2} e^{ikBx_1} ,$$

the $+$ ($-$) sign goes with the spin singlet (triplet) state (normalization is disregarded). It can be seen that
\[ \| \Psi (x_1, x_2) \|^2 = \cos^2 kx, \text{ for the singlet state,} \]
\[ \| \Psi (x_1, x_2) \|^2 = \sin^2 kx, \text{ for the triplet state,} \]

where \( x = x_1 - x_2, \) \( k = k_A - k_B \) (see fig. 3).

**Fig. 3**

For example, in the surrounding of the origin, close (separated) particles are more likely to be found in a singlet (triplet) state. There is a sort of quantum mechanical transmission inducing a correlation between the mutual distance of the two particles and the character of their spin state. This is of course a consequence of the entanglement between spatial and spin wave functions created by particle statistics.

“Divide and conquer” would work as follows. At the same time the whole is divided into parts (the non-interacting particles) and reconstructed by particle statistics, or projection. Noticeably, this quantum mechanical transmission would fall apart if the two particles were not identical.

**IV. BEHAVIOUR OF THE QUANTUM MECHANICAL TRANSMISSION**

Let us go back to the transmission \( r, s \) defined in Section II. We consider an operation performed on just one qubit, say, \( r \). Let \(|0\rangle_i \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i, \) \(|1\rangle_i \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}_i \) with \( i = r, s, Q_r (\varphi) \equiv \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}_r \). We should think \( Q_r (\varphi) \) to be a continuous rotation, with \( \varphi = \omega t \) and time \( t \) ranging from 0 to \( \frac{\varphi}{\omega} \).

Let us examine the effect of applying \( Q_r (\varphi) \) to qubit \( r \),

\[ \rho_r (t) = Q_r (\omega t) \rho_r (0) Q_r^\dagger (\omega t), \quad (5) \]

under continuous \( A_{rs} \) projection of the overall state; \( \rho_r (t) \) is qubit \( r \) density matrix at time \( t \), we will be concerned with the interval \( 0 \leq \varphi \leq \frac{\pi}{2} \).
Let the initial state of the transmission be the “symmetrical” state (whose tensor products satisfy the Boolean NOT, or symmetry $A_{rs}$):

$$|\Psi (0)\rangle = \cos \vartheta |0\rangle_r |1\rangle_s + \sin \vartheta |1\rangle_r |0\rangle_s .$$  

(6)

For all times $t$, the state of the transmission is obtained by submitting a free normalized vector $|\Psi (t)\rangle$ of the Hilbert space $H$ (Section II) to the following mathematically simultaneous conditions (by a rotation of $\rho_r$ under continuous $A_{rs}$ projection, we assume that the two operations are mathematically simultaneous — see ref. 8,9); therefore, for all $t$ or $\varphi$:

i) $A_{rs} |\Psi (t)\rangle = |\Psi (t)\rangle ,$

ii) $\rho_r (t) = Tr_s (|\Psi (t)\rangle \langle \Psi (t)|) = \cos^2 (\vartheta + \varphi) |0\rangle_r \langle 0|_r + \sin^2 (\vartheta + \varphi) |1\rangle_r \langle 1|_r ,$ where $Tr_s$ means partial trace over $s,$

iii) the distance between the vectors before and after projection must be minimum. This is a condition moving in time together with the continuous projection; the notion of the minimization of such a distance is clearer by thinking in terms of finite increments.

Conditions (i) and (ii) yield

$$|\Psi (t)\rangle = \cos (\vartheta + \varphi) |0\rangle_r |1\rangle_s + e^{i\delta} \sin (\vartheta + \varphi) |1\rangle_r |0\rangle_s ,$$

where $\delta$ is a still unconstrained phase, as can be readily checked; condition (iii), given the transmission initial state (6), sets $\delta = 0,$ thus establishing rotation additivity:

$$|\Psi (t)\rangle = \cos (\vartheta + \varphi) |0\rangle_r |1\rangle_s + \sin (\vartheta + \varphi) |1\rangle_r |0\rangle_s .$$  

(7)

This is the behaviour required from a “good” transmission. The rotation of qubit $r$ is identically transmitted to the other qubit $s$. In fact

$$Tr_r (|\Psi (t)\rangle \langle \Psi (t)|) = \rho_s (t) = \sin^2 (\vartheta + \varphi) |0\rangle_s \langle 0|_s + \cos^2 (\vartheta + \varphi) |1\rangle_s \langle 1|_s ,$$  

(8)

where $|\Psi (t)\rangle$ is given by eq. (7). Eigenvalues 0 and 1 are interchanged: of course we are using a transmission where one qubit is the NOT of the other. See also ref. [9,10,11].
Interestingly, by simultaneously rotating also the other extremity of the transmission by the same amount \( Q_s (\omega t) \rho_s (0) Q^\dagger_s (\omega t) \), the same result is obtained (eq. 7). Actually, this means adding eq. (8) as a condition, but this is of course redundant with respect to condition (ii) — it is derived from conditions (i) and (ii). Whereas, two different rotations of the two transmission extremities give an impossible mathematical system; this resembles a rigid classical transmission.

It should be noted that a rotation \( \varphi \) of qubit (part) \( r \) under \( A_{rs} \) projection, is equivalent to applying the unitary operator \( Q (\varphi) \) to the overall state \( |\Psi (t)\rangle \):

\[
Q (\varphi) \equiv \begin{pmatrix}
\cos \varphi & \sin \varphi & 0 & 0 \\
-\sin \varphi & \cos \varphi & 0 & 0 \\
0 & 0 & \cos \varphi & -\sin \varphi \\
0 & 0 & \sin \varphi & \cos \varphi
\end{pmatrix}, \quad \text{with}
\]

\[
|0\rangle_r |1\rangle_s \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle_r |0\rangle_s \equiv \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.
\]

\( Q (\varphi) \) brings the overall state from \( |\Psi (0)\rangle \) (eq. 6) to \( |\Psi (t)\rangle \) (eq. 7) without ever violating (for all \( \varphi \)) \( A_{rs} \). It cannot be represented as the product of two transformations operating separately on the two qubits. As a matter of fact, it can be seen that the result of operating on either qubit is conditioned by the state of the other. There is a form of conditional logic implicit in this unitary transformation, which operates on the whole state in an irreducible way. Finding \( Q (\varphi) \) amounts to solving a problem.

We have thus ascertained a peculiar fact. Our operation on a part, blind to its effect on the whole, performed together with continuous \( A_{rs} \) projection, generates a unitary transformation which is, so to speak, wise to the whole state, to how it should be transformed without violating \( A_{rs} \). Of course \( A_{rs} \) ends up commuting with the resulting overall unitary propagator (shaped by it). This is clearly an application of the divide and conquer strategy,
whose advantages will become clear in the next section.

V. QUANTUM COMPUTATION NETWORKS

Let us consider the reversible Boolean network of fig. 4, fully deployed in space. This is different from sequential computation, where the Boolean network appears in the space-time diagram of the computation process. Time is now orthogonal to the network lay-out (for a computation of such networks by means of projectors, see ref. 8).

Fig. 4

The network nodes \( t, u, v \) and \( r \) make the input and the output of a controlled NOT; \( r \) and \( s \) belong to a quantum mechanical transmission as defined above.

This c-NOT is not the usual time sequential gate, as its input and output coexist. It is a quantum object of four qubits, and four eigenstates which map the Boolean relation imposed by the gate. The Hilbert space of the gate states is thus

\[
H_g = \text{span} \{ |0\rangle_t |0\rangle_u |0\rangle_v |0\rangle_r, |0\rangle_t |1\rangle_u |0\rangle_v |1\rangle_r, |1\rangle_t |0\rangle_u |1\rangle_v |1\rangle_r, |1\rangle_t |1\rangle_u |1\rangle_v |0\rangle_r \}.
\]

Model Hamiltonians of similar elementary gates have been developed in ref [12]. Of course these gates differ from the gates used in the sequential approach, where inputs and outputs are successive states of the same register.\(^{[13,14,15,16]}\)

The usual way of stating a problem with such a space-deployed network, is by constraining part of the input and part of the output (actually, there is no preferred input-output direction) and asking whether this constrained network admits a solution. Let \( u = 1 \) and \( s = 1 \) be such partial constraints. \( u = 1 \ (s = 1) \) propagates a conditional logical implication from left to right (right to left). The logical implication is conditioned by the possible values of the unconstrained part of the input (output). In order to have a solution, these two propagations must be matched, namely must generate a univocal set of values on all the
nodes of the network — of course comprising the unconstrained input/output nodes. Finding whether the network admits at least one match (one solution) is an NP-complete problem in general. Naturally, it is the well known SAT (satisfiability) problem on a reversible Boolean network.

Possible collisions (mismatch) between the two propagations will be both overcome and reconciled by the transmission, as it will become clear.

Let us assume that the network has just one solution (which is the case here: \( t = 1, u = 1, r = 0, v = 1, s = 1 \)). The procedure to find it is as follows. The output constraint is removed while an arbitrary value, say \( t = 0 \), is assigned to the unconstrained part of the input. The logical propagation of this input toward the output yields \( t = 0, u = 1, r = 1, s = 0 \) (\( v = t \) will be disregarded). This computation is performed off line in polynomial time. It serves to specify the initial state in which the network must be prepared:

\[
|\Psi (0)\rangle = |0\rangle_t |1\rangle_u |1\rangle_r |0\rangle_s ;
\]

of course this state satisfies \( A_{rs} \). However, qubit \( s \) is in the eigenstate \( |0\rangle_s \) rather than \( |1\rangle_s \) (the output constraint). It will be turned into \( |1\rangle_s \) under \( A_{rs} \) projection, by applying the continuous transformation \( 1\rho_u (0) 1Q_s (\omega t) \rho_s (0) Q_s \dagger (\omega t), \) with \( t \) going from 0 to \( \frac{\pi}{2} \). While \( \rho_s \) is rotated, \( \rho_u = |1\rangle_u \langle 1|_u \) is kept fixed. This transformation operates on the network Hilbert space \( H_n \) defined as \( H_n = H_g \otimes H_s, \) where \( H_s = span \{ |0\rangle_s , |1\rangle_s \}. \) Note that all states of \( H_n \) already satisfy the gate, not necessarily the transmission.

At any time \( t, \) the state of the network is obtained by submitting a free normalized state \( |\Psi (t)\rangle \) of \( H_n \) to the conditions:

(i) \( A_{rs} |\Psi (t)\rangle = |\Psi (t)\rangle \)

(ii) \( Tr_{t,u,r} (|\Psi (t)\rangle \langle \Psi (t)|) = \rho_s (t) = Q_s (\omega t) \rho_s (0) Q_s \dagger (\omega t) \)

(iii) \( Tr_{t,r,s} (|\Psi (t)\rangle \langle \Psi (t)|) = \rho_u (0) = |1\rangle_u \langle 1|_u \)

(iv) the distance between the vectors before and after projection should be minimum.
This yields at time $t$:

$$|\Psi(t)\rangle = \cos \varphi |0\rangle_t |1\rangle_u |1\rangle_r |0\rangle_s + e^{i\delta} \sin \varphi |1\rangle_t |1\rangle_u |0\rangle_r |1\rangle_s,$$

with $\varphi = \omega t$, which is readily checked. Condition (iv) and the network initial state set $\delta = 0$. Thus

$$|\Psi(t)\rangle = \cos \varphi |0\rangle_t |1\rangle_u |1\rangle_r |0\rangle_s + \sin \varphi |1\rangle_t |1\rangle_u |0\rangle_r |1\rangle_s$$

(9)

For $\varphi = \frac{\pi}{2}$, one obtains

$$|\Psi\left(\frac{\pi}{2\omega}\right)\rangle = |1\rangle_t |1\rangle_u |0\rangle_r |1\rangle_s,$$

namely the solution.

The “wise” unitary transformation (9) (to how to behave on the whole) brings the state of the network from satisfying only the input to satisfying both the input and the output constraints. It is obtained by “blindly” operating on divided parts of the network, but under $A_{rs}$ projection. The latter is the conquering agent.

It is worth analysing the character of this computation process. We will consider a generic Boolean network where each wire comprises (without loss of generality) a NOT function, and is implemented by a transmission.

First, let us assume that the network has exactly one solution. The network eigenstates — tensor products of eigenstates of the network parts — inherently satisfy all gates (by definition, a gate is a part whose eigenstates map the gate logical input-output function). Network preparation is described by just one tensor product (as in the example of fig. 4). Beyond the gates, this satisfies input constraints and all transmissions, not the output constraint (but for a lucky chance).

Then, we start operating on the density matrices of those input/output qubits which must satisfy the external constraints, under the $A_{rs}$ projections (one for each transmission), in order to keep the constrained inputs unaltered, while bringing the output in match with its constraint. This computation process works on the basis vectors of the network Hilbert
space, which satisfy all gates, and cannot originate a tensor product which violates an $A_{rs}$ symmetry. In counterfactual reasoning, if there had been such a “bad” tensor product, its amplitude would have been canceled by continuous $A_{rs}$ projection and, through renormalization, would have gone to feed the amplitudes of the “good” tensor products. Computation becomes thus a unitary evolution of the overall network state, driven by the operations on some parts (on the input/output qubits which must satisfy the external constraints) and shaped by $A_{rs}$ projections.

As a result of this process, $A_{rs}$ symmetries become constants of motion which commute with the propagation at all times. They are also pairwise commuting, being applied to disjoint Hilbert spaces. However, the cause should not be confused with the effect. $A_{rs}$ projections shape or forge the unitary evolution with which they commute. This, according to the projection interpretation of particle statistics (Section III).

It is clear from this picture that the two propagations of conditional logical implication, from input to output and vice-versa, never originate a collision (a tensor product where a transmission is in an inconsistent state). The amplitudes of collisions must be zero because of continuous projection. Computation can never go into a deadlock. It is of course a unitary evolution toward the solution, driven, as we have seen, by operating on the parts and shaped by projecting the whole.

If the network admits no solution, conditions (i) through (iv) make up an impossible system. Measuring the network final state — when the operations on the parts are completed ($t = \frac{\pi}{2\omega}$) — gives some random result which is not a solution. This is checkable in polynomial time and tells that the network is not satisfiable.

If the network admits many solutions, the final state can be a linear combination thereof. Which linear combination, depends on the network initial state through condition (iv). This can be seen through numerical calculation. However this is irrelevant since measurement anyhow gives one solution (that it is a solution is checkable in polynomial time).

We should better clarify how parts should be connected by transmissions. Fig. 5 illustrates the appropriate arrangement (the network of fig. 4 was simplified). All elementary
gates and the constrained input/output qubits are interconnected through transmissions. Namely any network wire connecting nodes say $r$ and $s$, becomes a transmission, or projector $A_{rs}$.

It should be made clear that this is not necessary, but useful. Two or more elementary gates could be interconnected without the interposition of transmissions. However, this would require “building” an exponentially more complex gate and the number of its eigenstates would be the product of the number of eigenstates of the individual gates. Transmissions are exactly meant to reduce network complexity (exponentially with the number of transmissions). Clearly, they introduce a divide and conquer strategy. This quantum form of reductionism would be more rewarding than (the known) classical reductionism. At this abstract level, it would make NP-complete $\equiv$ P. Of course we are applying reverse engineering and we have just highlighted a nice-to-have feature, namely the $A_{rs}$ projection. This raises the problem whether this feature is physical. Until now, we have seen that these projections share the nature of particle statistics symmetries, under a special interpretation thereof.

VI. INDUCED SYMMETRY

We will show that the “artificial” symmetry $A_{rs}$ of a quantum mechanical transmission is an epiphenomenon of fermionic antisymmetry in a special physical situation. This is generated by submitting a couple of identical fermions 1 and 2 to a suitable Hamiltonian $[9]$. We assume that each fermion has two compatible, binary degrees of freedom $\chi$ and $\lambda$. Just for the sake of visualization, we can assume that each fermion is a spin 1/2 particle and can occupy one of either two sites of a spatial lattice. In this case $\chi$ becomes the particle spin component $\sigma_z = 0$ (down), 1 (up) and $\lambda = r, s$ the label of the site occupied by the particle.

Therefore the Hilbert space of the spatial states of the two fermions is:

$$H_\lambda = \text{span} \left\{ |r\rangle_1 |r\rangle_2, |s\rangle_1 |s\rangle_2, \frac{1}{\sqrt{2}} (|r\rangle_1 |s\rangle_2 - |s\rangle_1 |r\rangle_2), \frac{1}{\sqrt{2}} (|r\rangle_1 |s\rangle_2 + |s\rangle_1 |r\rangle_2) \right\},$$

where $|r\rangle_1 |r\rangle_2$ means that particle 1 is in site $r$, particle 2 is also in site $r$, etc. The Hilbert space of the spin states of the two fermions is:
$H_X = \text{span} \left\{ |0\rangle_1 |0\rangle_2, |1\rangle_1 |1\rangle_2, \frac{1}{\sqrt{2}} (|0\rangle_1 |1\rangle_2 + |1\rangle_1 |0\rangle_2), \frac{1}{\sqrt{2}} (|0\rangle_1 |1\rangle_2 - |1\rangle_1 |0\rangle_2) \right\}$,

$|0\rangle_1 |0\rangle_2$ means that the $\sigma_z$ component of particle 1 is 0 (down), etc. The overall Hilbert space $H_{\lambda X} = H_{\lambda} \otimes H_X$ has 16 basis vectors. The following is the list of the states which do not violate statistics. States are represented in first and second quantization and, when applicable, in qubit notation:

- Both particles in the same site, spin (necessarily) in the singlet state:

  $|a\rangle = \frac{1}{\sqrt{2}} (|0\rangle_1 |1\rangle_2 - |1\rangle_1 |0\rangle_2) |r\rangle_1 |r\rangle_2 = a^\dagger_{0r} a^\dagger_{1r} |0\rangle$,

  $|b\rangle = \frac{1}{\sqrt{2}} (|0\rangle_1 |1\rangle_2 - |1\rangle_1 |0\rangle_2) |s\rangle_1 |s\rangle_2 = a^\dagger_{0s} a^\dagger_{1s} |0\rangle$.

- One particle per site, antisymmetrical spatial state vector, spin (necessarily) in a triplet state:

  $|c\rangle = \frac{1}{\sqrt{2}} |0\rangle_1 |0\rangle_2 (|r\rangle_1 |s\rangle_2 - |s\rangle_1 |r\rangle_2) = a^\dagger_{0r} a^\dagger_{1s} |0\rangle = |0\rangle_r |0\rangle_s$,

  $|d\rangle = \frac{1}{\sqrt{2}} |1\rangle_1 |1\rangle_2 (|r\rangle_1 |s\rangle_2 - |s\rangle_1 |r\rangle_2) = a^\dagger_{1r} a^\dagger_{1s} |0\rangle = |1\rangle_r |1\rangle_s$,

  $|e\rangle = \frac{1}{2} (|0\rangle_1 |1\rangle_2 + |1\rangle_1 |0\rangle_2) (|r\rangle_1 |s\rangle_2 - |s\rangle_1 |r\rangle_2) = \frac{1}{\sqrt{2}} (a^\dagger_{0r} a^\dagger_{1s} + a^\dagger_{1r} a^\dagger_{0s}) |0\rangle = \frac{1}{\sqrt{2}} (|0\rangle_r |1\rangle_s + |1\rangle_r |0\rangle_s)$.

- One particle per site, symmetrical spatial state vector, spin (necessarily) in the singlet state:

  $|f\rangle = \frac{1}{2} (|0\rangle_1 |1\rangle_2 - |1\rangle_1 |0\rangle_2) (|r\rangle_1 |s\rangle_2 + |s\rangle_1 |r\rangle_2) = \frac{1}{\sqrt{2}} (a^\dagger_{0r} a^\dagger_{1s} - a^\dagger_{1r} a^\dagger_{0s}) |0\rangle = \frac{1}{\sqrt{2}} (|0\rangle_r |1\rangle_s - |1\rangle_r |0\rangle_s)$.

Creation/annihilation operators form the algebra:

$$\{a^\dagger_i, a^\dagger_j\} = \{a_i, a_j\} = 0, \quad \{a^\dagger_i, a_j\} = \delta_{i,j}. \quad (10)$$

Now we introduce the Hamiltonian

$$H_{rs} = E_a |a\rangle \langle a| + E_b |b\rangle \langle b| + E_c |c\rangle \langle c| + E_d |d\rangle \langle d|.$$

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or, in second quantization,

\[ H_{rs} = -\left( E_a a_0^r a_0^s a_1^r + E_b a_0^s a_1^s a_0^r + E_c a_0^r a_0^s a_1^s + E_d a_1^r a_1^s a_1^r \right), \]
with \( E_a, E_b > E_c, E_d \) discretely above 0.

This leaves us with two degenerate ground eigenstates:

\[ \left| e \right\rangle = \frac{1}{\sqrt{2}} \left( \left| 0 \right\rangle_r \left| 1 \right\rangle_s + \left| 1 \right\rangle_r \left| 0 \right\rangle_s \right) \text{ and } \left| f \right\rangle = \frac{1}{\sqrt{2}} \left( \left| 0 \right\rangle_r \left| 1 \right\rangle_s - \left| 1 \right\rangle_r \left| 0 \right\rangle_s \right). \]

Alternatively, their linear combinations \( \left| 0 \right\rangle_r \left| 1 \right\rangle_s \) and \( \left| 1 \right\rangle_r \left| 0 \right\rangle_s \) can be used as the two orthogonal ground eigenstates.

The generic ground state is thus:

\[ \left| \Psi \right\rangle = \alpha \left| 0 \right\rangle_r \left| 1 \right\rangle_s + \beta \left| 1 \right\rangle_r \left| 0 \right\rangle_s, \text{ with } |\alpha|^2 + |\beta|^2 = 1. \]

It can be seen that \( \left| \Psi \right\rangle \) satisfies \( A_{rs} \) symmetry.

Let \( A_{12} = 1 - P_{12} \) and renormalization be the antisymmetrization projector, where \( P_{12} \) permutes particles 1, 2. Then \( A_{12} \left| \Psi \right\rangle = \left| \Psi \right\rangle \) means that \( \left| \Psi \right\rangle \) is antisymmetric.

Due to the anticommutation relations (10), \( A_{12} \left| 0 \right\rangle_r \left| 1 \right\rangle_s = \left| 0 \right\rangle_r \left| 1 \right\rangle_s \) and \( A_{12} \left| 1 \right\rangle_r \left| 0 \right\rangle_s = \left| 1 \right\rangle_r \left| 0 \right\rangle_s \). Also, \( A_{12} \left| 0 \right\rangle_r \left| 0 \right\rangle_s = \left| 0 \right\rangle_r \left| 0 \right\rangle_s \) and \( A_{12} \left| 1 \right\rangle_r \left| 1 \right\rangle_s = \left| 1 \right\rangle_r \left| 1 \right\rangle_s \), without forgetting that these are excited states.

We will show that a quantum mechanical transmission can be implemented by suitably operating on the ground state (11). Without significant restriction, we assume that the initial (“symmetrical”) state of the transmission is given by eq. (6), repeated here for convenience:

\[ \left| \Psi (0) \right\rangle = \cos \vartheta \left| 0 \right\rangle_r \left| 1 \right\rangle_s + \sin \vartheta \left| 1 \right\rangle_r \left| 0 \right\rangle_s \]

Then we apply transformation (5) to qubit \( r \):

\[ \rho_r (t) = Q_r (\omega t) \rho_r (0) Q_r^\dagger (\omega t). \]

Let \( \left| \Psi (t) \right\rangle \) be a free normalized vector of \( H_{\lambda \chi} \). The transmission state at time \( t \) is obtained by submitting \( \left| \Psi (t) \right\rangle \) to the following mathematically simultaneous conditions:
i) \( A_{12} |\Psi (t)\rangle = |\Psi (t)\rangle \),

ii) \( \rho_r (t) = Tr_s (|\Psi (t)\rangle \langle \Psi (t)|) = \cos^2 (\vartheta + \varphi) |0\rangle_r \langle 0|_r + \sin^2 (\vartheta + \varphi) |1\rangle_r \langle 1|_r \),

iii) the distance between the vectors before and after reduction must be minimum

iv) the expected energy of the transmission in \(|\Psi (t)\rangle : \langle \xi (t)\rangle = \langle \Psi (t)| H_{rs} |\Psi (t)\rangle \), must be minimum

It is readily seen that the solution of this system is still \(|\Psi (t)\rangle\) of eq. (7):

\[
|\Psi (t)\rangle = \cos (\vartheta + \varphi) |0\rangle_r |1\rangle_s + \sin (\vartheta + \varphi) |1\rangle_r |0\rangle_s .
\]

Simultaneous satisfaction of (i), i.e. fermionic antisymmetry viewed as a projection, and (iv) (which is satisfied by \( \langle \xi (t)\rangle = 0 \)) originates the projection \( A_{rs} |\Psi (t)\rangle = |\Psi (t)\rangle \), as is readily seen. Therefore, if \( \langle \xi (t)\rangle \) remains zero, namely the operation on qubit \( r \) is performed \textit{adiabatically}, we obtain a quantum mechanical transmission as described in section IV. This transmission undergoes a unitary transformation driven by \( Q_r (\omega t) \) and shaped by \( A_{rs} \) (i.e. \( A_{12} \) and \( \langle \xi (t)\rangle = 0 \)).

The operation on qubit \( r \) should be gentle enough to be adiabatic. It should be noted that this is a local problem, whereas network size is irrelevant. The consequences of the operation on \( r \) are propagated throughout the network by means of the transmissions, therefore by way of interference (destructive due to projection and constructive due to renormalization). We should note that the tensor products \(|0\rangle_r |0\rangle_s \) and \(|1\rangle_r |1\rangle_s \) that would be projected off since they violate particle statistics (Section V), are not the excited states \(|c\rangle \) and \(|d\rangle \), which satisfy \( A_{12} \). They would be instead the symmetrical states

\[
|0\rangle_r |0\rangle_s = \frac{1}{\sqrt{2}} |0\rangle_1 |0\rangle_2 (|s\rangle_1 |r\rangle_2 + |s\rangle_2 |r\rangle_1 ) \quad \text{and} \quad |1\rangle_r |1\rangle_s = \frac{1}{\sqrt{2}} |1\rangle_1 |1\rangle_2 (|r\rangle_1 |s\rangle_2 + |s\rangle_1 |r\rangle_2 ),
\]

which do not satisfy \( A_{12} \). As a matter of fact, such states would be, so to speak, immediately projected off by statistics. The two kinds of states (antisymmetrical and symmetrical) have the same qubit notations and density matrices.

Given that the symmetric states cannot exist, this is counterfactual reasoning, which might help understanding; the important thing remains that conditions (i) through (iv)
yield the solution (7). To ensure that the operation on qubit \( r \) is performed adiabatically, the transmission density matrix should be kept on ground.

However, let us examine the case that the operation in question is not fully adiabatic. An antisymmetric, excited state of the form \( \gamma |0\rangle_r |0\rangle_s + \delta |1\rangle_r |1\rangle_s \) would appear in superposition with the transmission ground state:

\[
\left| \Psi \left( \frac{\pi}{2\omega} \right) \rightangle = \alpha |0\rangle_r |1\rangle_s + \beta |1\rangle_r |0\rangle_s + \gamma |0\rangle_r |0\rangle_s + \delta |1\rangle_r |1\rangle_s ,
\]

(the antisymmetrical, excited states where both particles are in the same site can be dealt with in a similar way). Let us assume this to be the state at the end of the operations. The probability that the result of measurement gives a transmission “malfunction” (\( |0\rangle_r |0\rangle_s \) or \( |1\rangle_r |1\rangle_s \)) is \( q = |\gamma|^2 + |\delta|^2 \). The expected transmission energy is \( \langle \xi (t) \rangle = |\gamma|^2 E_c + |\delta|^2 E_d \).

In order to keep \( q \) small, it should be \( \langle \xi (t) \rangle \ll E_c, E_d \). Since this computation is reversible, namely it does not dissipate free energy (the result of driving and shaping is a unitary transformation), it seems that \( \langle \xi (t) \rangle \) can be kept as close to zero as desired.

However, in order to keep network complexity down, the number of transmissions should grow linearly with network size. If the probability of a transmission “malfunction” remains constant, the probability that there are no malfunctions at network level (in the measurement outcome) would exponentially decrease with network size — in this case the SAT problem would remain NP-complete.

But transmissions do relax toward ground state, moreover independently of each other, since they also decouple the parts of the network (then re-coupled by projections). The pace of computation \( (\omega) \) could be slowed down so that computation is continuously caught up by the relaxation process of the transmissions. At any time, there should be a fixed desired probability \( p_N \) that a measurement would find all transmissions in the ground state.

Let us assume that \( p = 1 - e^{-\sigma t} \), with \( \sigma > 0 \), is the probability of finding an individual transmission in the ground state at time \( t \) (this exponential law holds when relaxation has reached a constant rate). Under this assumption, it can be shown analytically that, for a given \( p_N \), computation time grows polynomially with network size. This would mean
NP-complete $\equiv P$.

However, for the time being these are conjectural discussions: an implementation model would be needed in order to move to a less speculative analysis.

Let us now address the problem of creating many transmissions, namely an $H_{rs}$ Hamiltonian per network wire $r, s$ (fig. 5). These Hamiltonians operate on disjoint pairs of qubits. Viewed as projectors (when $\langle \xi (t) \rangle = 0$), they are pairwise commuting in spite of the fact that qubits belonging to different transmissions are mutually bounded by the network gates. Actually, all $A_{rs}$ projectors commute with the propagator of the network state (shaped by the $A_{rs}$).

VII. DISCUSSION

The notion of applying a particle statistics symmetry (seen as a projector) to divide the quantum whole into parts without clipping its richness, introduces an engineering (reductionistic) perspective in the design of quantum mechanisms. For the time being, the development of this idea remains at an abstract level. Finding model Hamiltonians which implement the Hermitean matrix of Section VI could possibly be the next step.

The form of computation propounded seems to imply a somewhat deeper interpretation of the notion of time-reversibility. Indeed the very notion of quantum computation was born as an evolution of the notion of reversible computation$^{[17,18,19,20,21]}$. Now we further conjecture that any quantum computation speed-up is related to the notion of the coexistence of forward and backward in time computation (or propagation).

This can be shown both in the traditional approach of sequential quantum computation (where the Boolean network appears in the time-diagram of the computation process) and in the approach propounder here.

Let us consider sequential computation first. Here computation speed-up can be related to the notion of performing inverse computation (of a hard to invert function) by using direct computation in a time reversed way (which of course requires computation reversibility). In
this way, the time interval used by inverse computation is the same of direct computation.

For example, with Simon’s algorithm, given a (hard to invert) function \( f : B^n \rightarrow B^n \), 2-to-1 with periodicity \( r \), \( r \) can always be found in polynomial time. An essential point of this algorithm is measuring \( f(x) \) in the entangled state.

\[
\sum_x |x\rangle |f(x)\rangle.
\]

Say this yields \( f(x) = k \). This gives, in the left register

\[
\frac{1}{\sqrt{2}} \left( |f^{-1}(k)\rangle + |f^{-1}(k) + r\rangle \right). \tag{12}
\]

Then, after applying the Hadamard transform, measurement of the left register and some repetitions of the entire process bring \( r \) out in the episystem. However, it can be argued that the quantum speed-up is already in the intermediate result (12). Roughly speaking, if this were the printout of a classical computer (with \( f^{-1}(k) \) and \( r \) substituted by the appropriate numerical assignments), computation time would be exponential.

Let us consider fig. 6. Measurement of the right register (right-top of figure 6), after computation, gives no information since the result is random. This result, \( f(x) = k \), goes backward in time along the right branch \( r \) to the interaction region (the cloud), where the direct function is computed. Since the result of measurement now goes back in time, the inverse function is computed (which would require exponential time in classical computation). Naturally, the time interval required is that for computing the direct function. This effectively implies the coexistence of forward and backward in time computation. The result of inverse computation is state (12) in the left register, which goes forward in time along the left branch. Further processing brings the symbol \( r \) out in the episystem. This algorithm clearly computes the inverse function through a time inversion of direct computation.

By the way, this is no time-travel, no information can go to the past. The result of measurement is random and provides no information. Thus, there is no information in the final state of computation. Furthermore, reversible computation\textsuperscript{[17,18,19]} neither destroys nor creates information. To sum up, there is no information whatsoever that can go back in time.
This is much similar to EPR correlations. EPR entanglement can be seen as an elementary computation. One can still use fig. 6 with $\sum_x |x\rangle \otimes |f(x)\rangle = |0\rangle |1\rangle - |1\rangle |0\rangle$ (normalization is disregarded). Measurement on, say, the right particle originates a random result, this is first propagated backward in time to the region of the interaction, then forward in time to the left particle, which carries that result to the left measurement. This EPR interpretation has been formalized by using a two-way propagation model in [22].

This interpretation of the quantum computation speed-up can be extended to all quantum algorithms found so far\cite{23} — readily by using the unified model developed by A. Ekert\cite{24}.

Two-way computation/propagation is also an interpretation of the computation model propounded in this work. The state of the transmission $r,s$ at some time $t$ must change due to the operations performed on a part of it. The new overall state is forged by $A_{rs}$ projection. This projection gives two possible outcomes: one in $H_s$ (satisfying $A_{rs}$), the other in the orthogonal subspace $H_{s\perp}$ (violating $A_{rs}$). Because of continuous projection, the probability of violating $A_{rs}$ is rounded off to sharp zero.

This rounding off is implied in the mathematically simultaneous application of (i) operation on a part and (ii) projection of the whole on a predetermined outcome (conditions (i) through (iv) of Section VI). In other words, the choice between ending up in $H_s$ or $H_{s\perp}$ is predetermined by the requirement that the projection outcome will satisfy a certain final condition ($A_{rs}$ symmetry). Projection on a predetermined outcome ($H_s$) would be “magic” if the state to be projected had cumulated a discrete distance from $H_s$. A condition set in the future would determine a choice in the past. It is less “magic” because projection is continuous. Still, computation speed-up is essentially related to it. So to speak, this is equivalent to always guessing the right choice in a decision process — in pruning a decision tree. This overcomes the “blindness” of the operations on the parts and makes the evolution wise to how to reach (unitarily) the problem solution. This also implies a propagation which is elusively (without time-travel) driven by both the initial and the final conditions\cite{22,25}. See also ref. \cite{9}. Other literature on two-way propagation models can be found in refs. \cite{26,27,28}.

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