Abstract—In this technical note, a recursive set membership filtering algorithm for discrete-time nonlinear dynamical systems subject to unknown but bounded process and measurement noise is proposed. The nonlinear dynamics is represented in a pseudo-linear form using the state dependent coefficient (SDC) parameterization. Matrix Taylor expansions are utilized to expand the unknown state dependent matrices about the corresponding state estimates. Upper bounds on the remainders in the matrix Taylor expansions are calculated on-line using a non-adaptive random search algorithm at each time step. Utilizing these upper bounds and the ellipsoidal set description of the uncertainties, a two-step filter is derived that utilizes the ‘correction-prediction’ structure of the standard Kalman Filter variants. At each time step, correction and prediction ellipsoids are constructed that contain the true state of the system by solving the corresponding semi-definite programs (SDPs). Sufficient conditions for boundedness of those ellipsoidal sets are derived. Finally, a simulation example is included to illustrate the effectiveness of the proposed approach.

Index Terms—Set membership filtering, bounding ellipsoids, unknown but bounded noise, state dependent coefficient parameterization.

I. INTRODUCTION

In this technical note, the ellipsoidal state estimation problem is considered and the terminology set membership filter (SMF) is adopted. Over the years, set membership filtering for linear systems has attracted significant attention and the theory is well-established (see, e.g., [1]–[9] and the references therein). Particularly, the filter design proposed in this note is motivated by [5], [8], [9] where the set estimation problems were converted into recursive algorithms that require solutions to semi-definite programs (SDPs) at each time step. Recently, several extensions of this approach have emerged in the literature (see, e.g., [10]–[12]).

On the other hand, set membership filtering for discrete-time nonlinear systems has received less attention. For discrete-time nonlinear systems, similar to the Extended Kalman Filter (EKF), set membership filtering approaches typically involve linearizing the nonlinear dynamics about the state estimate trajectory [12]–[15]. An extended set membership filter (ESMF) was developed in [13] by linearizing the state dynamics about the state estimates and bounding the linearization errors using interval analysis. An improvement over the algorithm proposed in [13] was provided in [14]. The SDP based approach for discrete-time nonlinear systems was introduced in [15] with a prediction-correction form. Recently, this approach was extended in [12] where the linearization errors were bounded in ellipsoids by solving two optimization problems at each time step.

State dependent coefficient (SDC) parameterization can be utilized to represent a nonlinear system in a pseudo-linear form with state dependent system matrices [16], [17]. The parameterization is non-unique and the non-uniqueness can be utilized to enhance performance of the controller or filter design (see [17] and the references therein). Although SDC parameterization has been utilized for filter design in a stochastic framework for discrete-time nonlinear systems (see [18], [19]), set membership filtering using the SDC parameterization has not been addressed in the existing open literature to the best of the authors’ knowledge.

Motivated by the above discussion, a recursive set membership filter utilizing the SDC parameterization (SMF-SDC) is proposed in this note for discrete-time nonlinear systems subject to unknown but bounded process and measurement noise. A two-step correction-prediction form is developed, similar to the Kalman Filter variants [20]. The proposed filter requires solution to two SDPs at each time step, similar to [5], [8], [9], [12], [15]. The contributions of this technical note are three fold and are summarized as follows.

1) A single SDC parameterization of the nonlinear system is utilized to obtain a pseudo-linear representation which preserves the nonlinearity in the governing equations. To the best of our knowledge, this is the first set membership filter for discrete-time nonlinear systems that utilizes the SDC parameterization.

2) Instead of the conventional EKF approach of linearizing the state trajectory about the state estimates as in [12]–[15], the state dependent matrices are expanded about the state estimates in matrix Taylor expansions using Vetter calculus [21]. The upper bounds on the remainders of the Taylor expansions are calculated on-line at each time step and those bounds are utilized in the filter design at every recursion. This approach is different from the approaches in [13], [14] where interval analysis were utilized to bound the linearization errors and from the recent approach in [12] where the linearization errors were bounded in ellipsoidal sets.

3) Sufficient conditions are derived that a priori guarantee the boundedness of the ellipsoids that contain the true state at the correction and prediction steps. Those sufficient conditions, in turn, are utilized to choose the SDC parameterization for the nonlinear system.

The rest of this technical note is organized as follows. Section II describes the preliminaries and problem formulation for the SMF-SDC. Section III discusses the main results for the proposed SMF-SDC and formulates the SDPs to be solved at each time-step to find the ellipsoidal sets containing the true state of the system. Section IV establishes the boundedness of the ellipsoidal sets using uniform observability properties. Finally, Section V includes a simulation example and Section VI presents the concluding remarks.

Notation: The symbol $\mathbb{Z}$ denotes the set of non-negative integers. For a square matrix $X$, the notation $X > 0$ (respectively, $X \geq 0$) means $X$ is symmetric and positive definite (respectively, positive semi-definite). Similarly, $X < 0$ (respectively, $X \leq 0$) means $X$ is symmetric and negative definite (respectively, negative semi-definite). The notations $\text{diag}(\cdot)$, $I_n$, $O_n$, and $0_n$ denote block-diagonal matrices, the $n \times n$ identity matrix, the $n \times n$ null matrix, and the vector of zeros of dimension $n$, respectively. The symbol $|| \cdot ||$ denotes the spectral norm for matrices and the Euclidean norm for vectors. Ellipsoids are denoted by $E(c, P) = \{x \in \mathbb{R}^n : (x - c)^T P^{-1} (x - c) \leq 1\}$ where $c \in \mathbb{R}^n$ is the center of the ellipsoid and $P > 0$ is the shape matrix that characterizes the orientation and size of the ellipsoid in $\mathbb{R}^n$. Also, notations $\text{tr}(\cdot)$, $\text{rank}(\cdot)$ denote trace and rank of a matrix, respectively, and $\otimes$ denotes the Kronecker product. The superscript $T$ means vector or matrix transpose.
II. PRELIMINARIES AND PROBLEM FORMULATION

Consider discrete-time, nonlinear dynamical systems of the form

\[ x_{k+1} = f(x_k) + w_k \]

\[ y_k = h(x_k) + v_k \]

(1)

where \( k \in \mathbb{Z}_+ \), \( x_k \in \mathbb{R}^n \) is the state of the system, \( w_k \in \mathbb{R}^n \) is the process noise or (matched) input disturbance, \( y_k \in \mathbb{R}^p \) is the measured output, and \( v_k \in \mathbb{R}^p \) is the measurement noise. The first task is to cast the nonlinear dynamics (1) into a pseudo-linear form using the state dependent coefficient (SDC) parameterization [16], [17] as

\[ x_{k+1} = A(x_k)x_k + w_k \]

\[ y_k = H(x_k)x_k + v_k \]

(2)

where \( f(x_k) = A(x_k)x_k \) and \( h(x_k) = H(x_k)x_k \), \( \forall k \in \mathbb{Z}_+ \). Note that the parameterization is non-unique for \( n > 1 \) and a convex combination of multiple parameterizations can be utilized to improve filter performance or avoid loss of observability (see, e.g., [22]). However, a single parameterization is utilized here and requirements for that choice will be discussed in the sequel (see Section IV).

Consider the nominal (‘noise-free’) system associated with the system (2)

\[ \dot{x}_k = A(x_k)x_k \]

\[ y_k = H(x_k)x_k \]

(3)

with the same initial state as for system (2), i.e., \( x_0 = x_0 \). Next, the following assumption is introduced for the state dynamics of the nominal system (3).

**Assumption 1:** [23] There exist compact sets \( D, \mathcal{D} \subset \mathbb{R}^n \) and \( \epsilon > 0 \) such that \( x_0 \in \mathcal{D} \) implies \( \bar{x}_k + \epsilon B(\bar{x}_k) \subset \mathcal{D}, \forall k \in \mathbb{Z}_+ \) where \( B(\bar{x}_k) \) is the closed unit ball in \( \mathbb{R}^n \) centered at \( \bar{x}_k \).

Assumption 1 implies that the nominal state \( \bar{x}_k \) evolves within a compact set \( \mathcal{D} \) which is not necessarily small [23]. The state of the system (2) satisfies \( x_k \in \mathcal{D}, \forall k \in \mathbb{Z}_+ \), provided the process noise is sufficiently small. This statement is made more precise in the following assumptions for the system (2).

**Assumption 2:**

1. \( x_0 \) is unknown but belongs to a known ellipsoid, i.e., \( x_0 \in E(\bar{x}_0, P_0) \) where \( \bar{x}_0 \) is a given initial estimate and \( P_0 \) is known.
2. \( w_k \) and \( v_k \) are unknown but belongs to known ellipsoids, i.e., \( w_k \in E(0, Q_k) \) and \( v_k \in E(0, R_k) \), \( \forall k \in \mathbb{Z}_+ \), where \( Q_k, R_k \) are known.
3. \( Q_k \leq q I_n \) and \( R_k \leq r I_p, \forall k \in \mathbb{Z}_+ \) hold with some \( q, r > 0 \).

Assumption 2.2 and 2.3 mean that the process and measurement noise acting on the system (2) are uniformly bounded. With that, a sufficiently small \( q \) ensures that \( x_k \in \mathcal{D}, \forall k \in \mathbb{Z}_+ \) holds. This is utilized in Section IV.

A. SMF-SDC Objectives

The objective is to develop an SMF-SDC for the system (2) having a correction-prediction form, similar to the Kalman Filter variants [20]. The filtering objectives are as follows.

1) **Correction Step:** At each time step \( k \in \mathbb{Z}_+ \), upon receiving the measurement \( y_k \) with \( v_k \in E(0, R_k) \) and given \( x_k \in E(\bar{x}_k, P_k) \), the objective is to find a correction ellipsoid such that \( x_k \in E(\hat{x}_k, P_k) \). The corrected state estimate is given by

\[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(y_k - H(\hat{x}_{k|k-1})\hat{x}_{k|k-1}) \]

(4)

where \( L_k \) is the filter gain.

2) **Prediction Step:** At each time step \( k \in \mathbb{Z}_+ \), given \( x_k \in E(\bar{x}_k, P_k) \) and \( w_k \in E(0, Q_k) \), the objective is to find a prediction ellipsoid such that \( x_{k+1} \in E(\hat{x}_{k+1|k}, P_{k+1|k}) \) where the predicted state estimate is given by

\[ x_{k+1|k} = A(\hat{x}_{k|k})\hat{x}_{k|k} \]

(5)

Initialization is provided by \( \hat{x}_{0|0} = \bar{x}_0 \) and \( P_{0|0} = P_0 \) [20] which form the initial prediction ellipsoid due to Assumption 2.1. Then, \( x_k \in E(\hat{x}_{k|k-1}, P_{k|k-1}) \) follows directly from the recursive nature of the filtering problem.

B. Matrix Taylor Expansions of the SDC Matrices

Assume the state of the system (2) at time step \( k \) belongs to the prediction ellipsoid of time step \( k - 1 \), i.e., \( x_k \in E(\hat{x}_{k|k-1}, P_{k|k-1}) \) where \( \hat{x}_{k|k-1} \) and \( P_{k|k-1} \) are known. Then, there exists a \( z_{k|k-1} \in \mathbb{R}^n \) with \( ||z_{k|k-1}|| \leq 1 \) such that

\[ x_k = \hat{x}_{k|k-1} + E_{k|k-1}(z_{k|k-1}) \]

(6)

where \( E_{k|k-1} \) is the Cholesky factorization of \( P_{k|k-1} \), i.e., \( P_{k|k-1} = E_{k|k-1}E_{k|k-1}^T \) [5], [8]. Utilizing the matrix Taylor expansion in [21], \( H(x_k) = H(\hat{x}_{k|k-1} + E_{k|k-1}(z_{k|k-1})) \) can be expanded about the state estimate \( \hat{x}_{k|k-1} \) as

\[ H(x_k) = H(\hat{x}_{k|k-1}) + \left( D_{x_k}H(\hat{x}_{k|k-1}) \right)(z_{k|k-1} \otimes I_n) \]

\[ + R_{H}(z_{k|k-1}, x_k) \]

(7)

where \( D_{x_k}H(\hat{x}_{k|k-1}) \) is the derivative matrix evaluated at \( \hat{x}_{k|k-1} \), \( z_{k|k-1} \otimes I_n \) is the Kronecker product of \( z_{k|k-1} \) and \( I_n \) (see Section 6 in [21]). Similarly, the matrix \( A(x_k) = A(\hat{x}_{k|k}) + E_{k|k}(z_{k|k}) \) is expanded as

\[ A(x_k) = A(\hat{x}_{k|k}) + \left( D_{x_k}A(\hat{x}_{k|k}) \right)(z_{k|k} \otimes I_n) + R_{A}(z_{k|k}, x_k) \]

where \( z_{k|k} \) is given by \( E_{k|k}(z_{k|k}) \) with \( P_{k|k} = E_{k|k}E_{k|k}^T \) and \( ||z_{k|k}|| \leq 1 \). For notational simplicity, following definitions are introduced.

\[ K_1(\hat{x}_{k|k-1}) = D_{x_k}H(\hat{x}_{k|k-1}) \]

\[ \Delta_1(z_{k|k-1}) = (z_{k|k-1} \otimes I_n) \]

Then, the state dependent matrices can be expressed in matrix Taylor expansions as

\[ H(x_k) = H(\hat{x}_{k|k-1}) + K_1(\hat{x}_{k|k-1})\Delta_1(z_{k|k-1}) \]

\[ + R_{H}(z_{k|k-1}, x_k) \]

\[ A(x_k) = A(\hat{x}_{k|k}) + K_2(\hat{x}_{k|k})\Delta_2(z_{k|k}) + R_{A}(\hat{x}_{k|k}, x_k) \]

Assumption 3: There exist \( p_1, p_2, p_3, p_4 > 0 \) such that \( P_{k|k-1} \) and \( F_{k|k} \) satisfy the following bounds \( \forall k \in \mathbb{Z}_+ \):

\[ p_1 I_n \leq P_{k|k-1} \leq p_2 I_n \]

\[ p_3 I_n \leq P_{k|k-1} \leq p_4 I_n \]

with \( p_1 \leq p_2 \) and \( p_3 \leq p_4 \).

This assumption states that the correction and prediction ellipsoids remain uniformly bounded. This is closely related to uniform observability of the nonlinear system [13], [24], [25]. The sufficient conditions for satisfying this assumption *a priori* are given in the sequel (see Section IV).

**Assumption 4:** Consider a compact subset \( \mathcal{D} \subset \mathbb{R}^n \). There exist \( a, h, k_1, k_2 > 0 \) such that following holds:

\[ h = \sup_{a \in \mathcal{D}} ||H(a)||, \]

\[ a = \sup_{a \in \mathcal{D}} ||A(a)||, \]

\[ k_1 = \sup_{a \in \mathcal{D}} ||K_1(a)||, \]

\[ k_2 = \sup_{a \in \mathcal{D}} ||K_2(a)||, \forall x \in \mathcal{D}. \]

Assumption 4 provides uniform upper bounds for the matrices in (8) on a compact subset \( \mathcal{D} \). The arguments of \( K_1(\cdot) \) and \( \Delta_1(\cdot) \) (\( i = 1, 2 \)) have been dropped in the subsequent analysis to avoid clumsy notations.
C. Upper Bounds on the Remainders of Matrix Taylor Expansions

At each time step, the upper bounds on the remainders in (8) are calculated and utilized in the SMF-SDC design. Before elaborating on that, let us state the following Proposition that establishes the uniform upper boundedness of the remainders in (8).

Proposition 1: Consider a compact subset $\mathbb{D} \subset \mathbb{R}^n$. Assume that the state of the system (2) and corresponding state estimates in (4), (5) satisfy $x_k, \hat{x}_{k|k}, \hat{x}_{k|k-1} \in \mathbb{D}, \forall k \in \mathbb{Z}_c$. Further, let the Assumptions 3 and 4 hold. Then, the remainders in (8) are uniformly upper bounded.

Proof: Consider the remainder $R_{A_2}(\hat{x}_{k|k}, x_k)$ in (8), expressed as

$$R_{A_2}(\hat{x}_{k|k}, x_k) = A(x_k) - A(\hat{x}_{k|k}) - K_2\Delta_2$$

where $x_k = \hat{x}_{k|k} + E_k|k\hat{z}_{k|k}$. Taking the norm leads to

$$\|R_{A_2}(\hat{x}_{k|k}, x_k)\| \leq 2\alpha + k_2\|\Delta_2\|$$

Utilizing the definition of $\Delta_2$, the following holds:

$$\|\Delta_2\| = \|\langle \xi_k \otimes I_k \rangle\| = \|\xi_k\| \|I_k\| \leq \|E_k|k\| \|z_{k|k}\|$$

(9)

where the identity $\|A \otimes B\| = \|A\| \|B\|$ has been utilized which holds for the spectral norm [26]. Denoting $\|E_k|k\| = \gamma_k|k - 1$, (9) becomes $\|\Delta_2\| \leq \gamma_k|k - 1|\|z_{k|k}\|$ where $0 < \gamma_k|k - 1| < \infty$, $\forall k \in \mathbb{Z}_c$ due to Assumption 3. Then, the norm of the remaining satisfies

$$\|R_{A_2}(\hat{x}_{k|k}, x_k)\| \leq 2\alpha + k_2\gamma_k|k - 1|\|z_{k|k}\|$$

(10)

Carrying out the same analysis for $R_{H_2}(\hat{x}_{k|k-1}, x_k)$ yields

$$\|R_{H_2}(\hat{x}_{k|k-1}, x_k)\| \leq 2h + k_1\gamma_{k-1}$$

where $\|E_k|k-1\| = \gamma_{k-1}$ with $0 < \gamma_{k-1} < \infty$, $\forall k \in \mathbb{Z}_c$ due to Assumption 3. This completes the proof.

Remark 2: Consider the system (2). Utilizing the matrix Taylor expansions in (8), the governing equations utilized for the SMF-SDC design can be expressed as

$$x_{k+1} = A(\hat{x}_{k|k})x_k + \tilde{w}_k$$
$$y_k = H(\hat{x}_{k|k-1})x_k + \tilde{v}_k$$

(14)

where

$$\tilde{w}_k = w_k + K_2\Delta_2x_k + R_{A_2} (\hat{x}_{k|k}, x_k)x_k$$
$$\tilde{v}_k = v_k + K_1\Delta_1x_k + R_{H_2}(\hat{x}_{k|k-1}, x_k)x_k$$

To compare (14) with the governing equations utilized for the EKF and ESMEF designs, see Section 8.2 in [20] and Section 3 in [13], respectively. The bounds on the terms in $\tilde{w}_k$, $\tilde{v}_k$ and the ellipsoidal set description of the true state $x_k$ are utilized in the next section to derive the SMF-SDC.

III. Main Results

This section formulates the SDPs to be solved at each time step for the correction and prediction steps. The arguments of $R_{A_2} (\cdot)$ and $R_{H_2} (\cdot)$ are omitted in the subsequent analysis for notational simplicity. With that, let us state Theorem 1 that summarizes the filtering problem at the correction step.

Theorem 1: Consider the system (2) under the Assumptions 2.1 and 2.2. At each time step $k \in \mathbb{Z}_c$, upon receiving the measurement $y_k$ with $y_k \in \mathcal{E}(0, \mathcal{R}_k)$ and given $x_k \in \mathcal{E}(\hat{x}_{k|k-1}, \mathcal{P}_{k|k-1})$, the state $x_k$ is bounded in the correction ellipsoid given by $\mathcal{E}(\hat{x}_{k|k}, \mathcal{P}_{k|k})$, if there exist $\mathcal{P}_{k|k} > 0$, $L_k$, $\tau_i \geq 0$, $i = 1, 2, 3, 4, 5, 6$ as solutions to the following SDP:

$$\min_{P_{k|k}, L_k, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6} \text{trace}(\mathcal{P}_{k|k})$$
subject to

$$P_{k|k} > 0$$
$$\tau_i \geq 0, \ i = 1, 2, 3, 4, 5, 6$$
$$-P_{k|k} \Pi_{k|k-1} \Theta(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) \Pi_{k|k-1}^T \leq 0$$

(15)

where $\Pi_{k|k-1}$ and $\Theta(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6)$ are given by

$$\Pi_{k|k-1} = \left[ \begin{array}{c}
0_n \ E_{k-1|k-1} - L_k H(\hat{x}_{k|k-1}) E_{k-1|k-1} - L_k \\
- L_k K_1 - L_k - L_k K_1 - L_k
\end{array} \right]$$

$$\Theta(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = \text{diag}\left(1 - \tau_1 - \tau_2 - \tau_5 \gamma_{k-1}^2 - \tau_2 \gamma_{k-1}^2 \hat{x}_{k|k-1}^T \hat{x}_{k|k-1} - \tau_2 \gamma_{k-1}^2 E_{k-1|k-1}^T E_{k-1|k-1} - \tau_4 \gamma_{k-1}^2 E_{k-1|k-1}^T E_{k-1|k-1} - \tau_3 I_n^2, \tau_4 I_{n_2}, \tau_5 I_{n_2}, \tau_6 I_{n_2} \right)$$

(16)

Furthermore, the center of the correction ellipsoid is given by the corrected state estimate

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k (y_k - H(\hat{x}_{k|k-1})\hat{x}_{k|k-1})$$

(17)
The above inequalities are expressed in terms of $\zeta$ as follows

\[
\begin{align*}
\mathcal{C}^T \text{diag} & \left(-1, I_n, O_p, O_p, O_p, O_p, O_p, O_p, O_p\right) \zeta \leq 0, \\
\mathcal{C}^T \text{diag} & \left(-1, O_n, R^{-1}, O_n, O_n, O_p, O_p, O_p\right) \zeta \leq 0, \\
\mathcal{C}^T \text{diag} & \left(-\gamma_0^k, -\gamma_1^k, -\gamma_1^k, -\gamma_1^k, -\gamma_1^k, -\gamma_1^k, -\gamma_1^k, -\gamma_1^k, -\gamma_1^k\right) \zeta \leq 0, \\
\mathcal{C}^T \text{diag} & \left(-r_1^k, -r_1^k, -r_1^k, -r_1^k, -r_1^k, -r_1^k, -r_1^k, -r_1^k, -r_1^k\right) \zeta \leq 0, \\
\mathcal{C}^T \text{diag} & \left(-r_2^k, -r_2^k, -r_2^k, -r_2^k, -r_2^k, -r_2^k, -r_2^k, -r_2^k, -r_2^k\right) \zeta \leq 0.
\end{align*}
\]

Next, the $\Pi$-procedure (see, e.g., [30]) is applied to the inequalities in (22) and (24). The inequality in (22) holds if there exist $\tau_1 \geq 0, \tau_2 \geq 0, \tau_3 \geq 0, \tau_4 \geq 0, \tau_5 \geq 0$ such that the following is true:

\[
\begin{align*}
\Pi_{k+1}^T P_{k+1}^{-1} \Pi_{k+1} & = \Theta (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) \leq 0, \\
\Pi_{k+1}^T & \leq \Theta (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5).
\end{align*}
\]

The above inequality can be expressed in a compact form as

\[
\begin{align*}
\Pi_{k+1}^T P_{k+1}^{-1} \Pi_{k+1} & = \Theta (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) \leq 0,
\end{align*}
\]

where $\Theta (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$ is given in (16). Utilizing the Schur complement (see, e.g., [30]), the inequality in (25) can be equivalently expressed as

\[
\begin{align*}
\begin{bmatrix}
-P_{k+1} & \Pi_{k+1}^T \\
\Pi_{k+1} & -\Theta (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)
\end{bmatrix}
\leq 0.
\end{align*}
\]

Solving the inequality in (26) yields a correction ellipsoid that contains the true state of the system. To obtain the minimal set containing the true state, the sum of the squared lengths of semiaxes of the correction ellipsoid is minimized by minimizing the trace of $P_{k+1}$. This completes the proof.

The next Theorem summarizes the filtering problem at the prediction step.

**Theorem 2:** Consider the system (2) with the state $x_k$ in the correction ellipsoid $\mathcal{E}(\hat{x}_k|k, P_k|k)$ and $\omega_k \in \mathcal{E}(O_n, Q_k)$. Then the successor state $x_{k+1}$ belongs to a prediction ellipsoid $\mathcal{E}(\hat{x}_{k+1}|k, P_{k+1}|k)$, if there exist $P_{k+1}|k > 0, \tau_i \geq 0, i = 7, 8, 9, 10, 11, 12$ as solutions to the following SDP:

\[
\begin{align*}
\min_{P_{k+1}|k} \text{trace}(P_{k+1}|k) \\
\text{subject to} \\
P_{k+1}|k > 0, \tau_i \geq 0, i = 7, 8, 9, 10, 11, 12 \\
\begin{bmatrix}
-P_{k+1}|k & \Pi_{k+1}^T \\
\Pi_{k+1} & -\Psi (\tau_7, \tau_8, \tau_9, \tau_{10}, \tau_{11}, \tau_{12})
\end{bmatrix}
\leq 0.
\end{align*}
\]
where $\Pi_{k|k}$ and $\Psi(\tau_7, \tau_8, \cdots, \tau_{1t}, \cdots, \tau_{12})$ are given by

$$
\Pi_{k|k} = \begin{bmatrix} O_n & A(\bar{x}_{k|k})E_{k|k} & I_n & K_2 & I_n & K_2 & I_n \end{bmatrix},
$$

$$
\Psi(\tau_7, \tau_8, \cdots, \tau_{1t}, \cdots, \tau_{12}) = \text{diag} \left( 1 - \tau_7 - \tau_8 - \tau_9 \gamma_k \| \bar{x}_{k|k} \| - \tau_{10} \gamma_k \| \bar{x}_{k|k} \| - \tau_{11} \beta_k \| \bar{x}_{k|k} \|, \right)

$$

Furthermore, the center of the prediction ellipsoid is given by the predicted state estimate

$$
\hat{x}_{k+1} = A(\bar{x}_{k|k})\hat{x}_{k|k}.
$$

Proof: The proof is similar to Theorem 1 and is omitted.

These SDPs in (15) and (27) can be solved efficiently using interior point methods [31]. The recursive SMF-SDC algorithm is summarized as follows.

**Algorithm 1 SMF-SDC Algorithm**

1. (Initialization) Given the initial values $(\bar{x}_0, P_0)$, set $k = 0$, $\hat{x}_{k|k-1} = \bar{x}_0$, $E_{k|k-1} = E_0$ where $P_0 = E_0 E_0^T$, and $\gamma_k |_{k-1} = \|E_0\|$.
2. Calculate $r_{nk}$ by solving (13). Find $P_{k|k}$ and $L_k$ by solving the SDP in (15).
3. Calculate $\hat{x}_{k|k}$ using (17). Also, calculate $E_{k|k}$ using $P_{k|k} = E_{k|k} E_{k|k}^T$ and set $\gamma_k |_{k} = \| E_{k|k} \|$.\n4. Calculate $r_{nk}$ by solving (12). With that, given $\hat{x}_{k|k}$, $E_{k|k}$, $\gamma_k$, solve the SDP in (27) to obtain $P_{k+1|k}$.
5. Calculate $\hat{x}_{k+1|k}$ using (28). Set $E_{k+1|k}$ using $P_{k+1|k} = E_{k+1|k} E_{k+1|k}^T$ and $\gamma_{k+1|k} = \| E_{k+1|k} \|$.\n6. Set $k = k + 1$ and go to Step 2.

IV. BOUNDEDNESS OF THE ELLIPTOIDS

In this section sufficient conditions for boundedness of the correction and prediction ellipsoids are provided. The approach is similar to the ones in [24],[25] for the EKF and in [13] for the ESMF. First, let us consider the uniform observability condition for discrete-time linear time varying (LTV) systems [24],[32].

**Definition 1:** Consider a discrete-time LTV system with the time-varying matrices $A_k, H_k \in \mathbb{R}^{n \times n}$, $\forall k \in Z_s$. Define the observability gramian as

$$
\mathcal{M}_{k+s,k} = \sum_{i=k}^{k+s} \Phi_{i,k}H_i^TP_{i,k}(\Phi_{i,k})^T
$$

where $s \in Z_s \setminus \{0\}$, $\Phi_{i,k}$ is the state transition matrix with $\Phi_{k,k} = I_n$ and

$$
\Phi_{i,k} = A_{i-1} \cdots A_k
$$

for $i > k$. Then, the pair $(A_k, H_k)$ is said to satisfy the uniform observability condition if there exist some $s \in Z_s \setminus \{0\}$ and $\beta_1, \beta_2 > 0$ such that

$$
\beta_1 I_n \leq \mathcal{M}_{k+s,k} \leq \beta_2 I_n
$$

holds. Furthermore, if $A_k^T A_k > 0$, $\forall k \in Z_s$, holds then the gramian $\mathcal{M}_{k+s,k}$ can be expressed as

$$
\mathcal{M}_{k+s,k} = \mathcal{O}_{k+s,k}^T \mathcal{O}_{k+s,k}
$$

where $\mathcal{O}_{k+s,k}$ is given by

$$
\mathcal{O}_{k+s,k} =
\begin{bmatrix}
H_k \\
H_{k+1}A_k \\
\vdots \\
H_{k+s}A_{k+s-1} \cdots A_k
\end{bmatrix}
$$

Next, the following assumption is made on the SDC parameterization utilized for the SMF-SDC design.

**Assumption 5:** The SDC parameterization is such that $A(x)$ and $H(x)$ are continuous on $S \subseteq \mathbb{R}^n$. Thus, $\forall \chi \in S$, $\forall p_1, p_2 > 0$ there exists $\lambda > 0$ such that $\|x - \zeta\| < \lambda$ imply $\|A(x) - A(\zeta)\| < p_1$ and $\|H(x) - H(\zeta)\| < p_2$, $\forall \chi \in S$.

Note that Assumption 5 requires the nonlinear functions $f_d$ and $h_d$ to be sufficiently smooth. The following Proposition provides conditions for uniform observability of the nominal system (3).

**Proposition 2:** Consider the nominal system (3) under the Assumption 1 with $x_0 \in \mathbb{D}_0$. Let $A(\bar{x}_k), H(\bar{x}_k)$ satisfy Assumption 5 with $S = \mathbb{D}$ and let $A(\bar{x}_k)$ be full rank $\forall k \in Z_s$. Denote $A_k = A(\bar{x}_k)$, $H_k = H(\bar{x}_k)$ and assume that the pair $(A_k, H_k)$ satisfies the condition

$$
\text{rank}(\mathcal{O}_{k+n-1,k}) = n, \forall k \in Z_s
$$

where $\mathcal{O}_{k+n-1,k}$ is as given in (33) with $s = n - 1$. Then, the pair $(A_k, H_k)$ satisfies the uniform observability condition with $s = n - 1$.

Proof: With $A_k$ full rank $\forall k \in Z_s$, the condition $A_k^T A_k > 0, \forall k \in Z_s$ holds. Therefore, the gramian $\mathcal{M}_{k+s,k}$ can be expressed as in (32). With that, from the rank condition in (34),

$$
\text{rank}(\mathcal{O}_{k+n-1,k}) = n
$$

holds $\forall k \in Z_s$. Moreover, Assumption 1 with $x_0 \in \mathbb{D}_0$ imply $\bar{x}_k \in \mathbb{D}, \forall k \in Z_s$. Therefore, the following holds due to the compactness of $\mathbb{D}$ and continuity of $A_k, H_k$ (see Section 4 in [25]):

$$
\mu_1 I_n \leq \mathcal{O}_{k+n-1,k}^T \mathcal{O}_{k+n-1,k} \leq \mu_2 I_n, \forall k \in Z_s
$$

where $\mu_1, \mu_2 > 0$ with $\mu_2 \geq \mu_1$. Hence, the pair $(A_k, H_k)$ satisfies the uniform observability condition with $s = n - 1$.

**Remark 3:** Note that $A_k = A(\bar{x}_k)$ is required to be full rank $\forall k \in Z_s$ such that the gramian can be expressed as in (32). That was crucial for the result in Proposition 2. Therefore, the SDC parameterization should be chosen such that $A_k$ is full rank $\forall k \in Z_s$.

**Remark 4:** Note the similarity of the proposed rank condition in (34) with the one recently introduced for discrete-time LTV systems in [33]. Also, the rank condition in (34) is different from the nonlinear observability definitions utilized in [13],[23]–[25]. Checking the rank condition in (34) for systems with large dimensions would not be a trivial task. However, for systems with small dimensions, this can be achieved by the choice of the SDC parameterization (see Section V).

Next, consider the system (2) with only the measurement noise as

$$
\hat{x}_{k+1} = A(\bar{x}_k)\hat{x}_k + y_k
$$

$$
y_k = H(\bar{x}_k)\bar{x}_k + v_k
$$

where $\bar{x}_0 = x_0$. The next Lemma relates the rank condition in (34) with the uniform observability of the pair $(A(\bar{x}_{k|k}), H(\bar{x}_{k|k}^{-1}))$.

**Lemma 1:** Consider the system (37) under the Assumption 1 with $x_0 \in \mathbb{D}_0$ and the corresponding state estimates in (17), (28). Let $A(\bar{x}_k), H(\bar{x}_k)$ satisfy Assumption 5 with $S = \mathbb{D}$ and let $A(\bar{x}_k)$ be full rank $\forall k \in Z_s$. Denote $A_k = A(\bar{x}_k), H_k = H(\bar{x}_k)$ and assume that the pair $(A_k, H_k)$ satisfies the rank condition in (34). Then, there exists a $0 < \delta < \epsilon$ such that the pair $(A(\bar{x}_{k|k}), H(\bar{x}_{k|k}^{-1}))$
satisfies the uniform observability condition with \( s = n - 1 \), provided \( \| \hat{x}_k - \hat{x}_{k|k-1} \| \leq \delta \) and \( \| \hat{x}_k - \hat{x}_{k|k-1} \| \leq \delta, \forall k \in \mathbb{Z}_+ \).

Proof: Assumption 1 with \( x_0 \in D_0 \) implies \( \hat{x}_k \in D_0, \forall k \in \mathbb{Z}_+ \).

Then, the condition \( \| \hat{x}_k - \hat{x}_{k|k} \| \leq \delta \) and \( \| \hat{x}_k - \hat{x}_{k|k-1} \| \leq \delta, \forall k \in \mathbb{Z}_+ \) with \( 0 < \delta \leq \epsilon \) leads to \( \| \hat{x}_{k|k} - \hat{x}_{k|k-1} \| \leq \epsilon, \forall k \in \mathbb{Z}_+ \). The rest of the proof follows from that of Proposition 4.1 in [25].

Remark 5: Lemma 1 requires the initial estimation error to be sufficiently small such that \( \| x_0 - \hat{x}_0 \| \leq \delta \) where \( \hat{x}_0 = \hat{x}_{0|0} \) (see Section II). Also, the measurement noise is required to be sufficiently small (Assumption 2.3) and the filter gain \( L_k \) is required to be bounded such that \( \| \hat{x}_k - \hat{x}_{k|k} \| \leq \delta \) and \( \| \hat{x}_k - \hat{x}_{k|k-1} \| \leq \delta, \forall k \in \mathbb{Z}_+ \).

Remark 6: Lemma 1 holds for system (2) if the process noise is sufficiently small so that \( \hat{x}_k \) remains close to \( x_k \), i.e., there exists an \( \epsilon_1 > 0 \) such that \( \| x_k - \hat{x}_k \| \leq \epsilon_1, \forall k \in \mathbb{Z}_+ \). With this, Lemma 1 can be applied with the following modifications [13], [25] \( \| \hat{x}_k - \hat{x}_{k|k} \| \leq \| \hat{x}_k - x_k \| + \| x_k - \hat{x}_{k|k} \| \) \( \| \hat{x}_k - \hat{x}_{k|k-1} \| \leq \| \hat{x}_k - x_k \| + \| x_k - \hat{x}_{k|k-1} \| \).

The next Lemma states the sufficient conditions for the boundedness of the correction and prediction ellipsoids obtained as solutions to the SDPs in (15) and (27).

Lemma 2: Consider the system (2) under the Assumption 2 and consider the corresponding state estimates in (17), (28). Also, let the nominal state dynamics associated with system (2) satisfy Assumption 1 with \( x_0 \in D_0 \). Denote \( A_k = A(\hat{x}_k) \) and \( H_k = H(\hat{x}_k) \). Let the following conditions hold:

1. \( \| x_0 - \hat{x}_0 \| \leq \delta_0 \) with a \( \delta_0 > 0 \) independent of \( \epsilon \) be such that \( \hat{x}_0 \in D_0 \).
2. \( \| L_k \| \leq l \) with a \( l > 0 \) and \( q, r \) in Assumption 2.3 be such that \( \hat{x}_k, \hat{x}_{k|k}, \hat{x}_{k|k-1} \in D_0, \forall k \in \mathbb{Z}_+ \).
3. \( A_k, H_k \) satisfy Assumption 5 with \( S = D_0, \forall k \in \mathbb{Z}_+ \).
4. \( A_k \) is full rank \( \forall k \in \mathbb{Z}_+ \), and the pair \( (A_k, H_k) \) satisfies the rank condition in (34).
5. \( K_1 \) and \( K_2 \) remain bounded \( \forall k \in \mathbb{Z}_+ \).
6. \( P_{k|k} \) remains bounded for \( 0 \leq k \leq n - 1 \) and \( P_{k|k-1} \) remains bounded for \( 1 \leq k \leq n \).

Then, there exist \( p_1, p_2, p_3, p_4 > 0 \) such that \( P_{k|k-1} \) and \( P_{k|k} \) satisfy the following bounds:

\[
\begin{align*}
& p_1 I_n \leq P_{k|k} \leq p_2 I_n, \quad \forall k \geq n \\
& p_3 I_n \leq P_{k|k-1} \leq p_4 I_n, \quad \forall k \geq n + 1
\end{align*}
\]

with \( p_1 \leq p_2 \) and \( p_3 \leq p_4 \).

Proof: The proof is inspired from that of Lemma 5.1 in [32] and a brief sketch is given here. Lower bounds on \( P_{k|k}, P_{k|k-1} \) directly follow from positive definiteness and let us focus on the proof of upper boundedness. Consider the alternate representation of the system (2) given in (14) and compare that with the system (5.1) in [32]. Conditions 1, 2, 3, and 4 together with Lemma 1 imply that the pair \( (A(\hat{x}_k), H(\hat{x}_{k|k-1})) \) satisfies the uniform observability condition with \( s = n - 1 \). Conditions 1, 3, 5, and 6 lead to \( \hat{\omega}_k \) being bounded for \( 0 \leq k \leq n - 1 \) and \( \hat{\omega}_k \) being bounded for \( 0 \leq k \leq n \).

Now, let us analyze the correction step at \( k = n \). Consider the smoothed estimate in [32] with \( k = 1, s = n - 1 \) and some \( t \in \mathbb{Z}_+ \) with \( t \leq s \) where \( k \) is replaced by the dummy variable \( \kappa \in \mathbb{Z}_+ \). Since the noise terms for the system (14) are bounded (as discussed above) and the matrices \( A(\hat{x}_k), H(\hat{x}_{k|k-1}) \) are bounded due to Conditions 2, 3, the term \( \kappa \) is bounded. Then, the error in the smoothed estimate \( \hat{x}_{1:t} \) is bounded due to the uniform observability of the pair \( (A(\hat{x}_k), H(\hat{x}_{k|k-1})) \), as shown in [32]. Thus, \( \| \hat{x}_{k+1} - \hat{x}_{k|k} \| < \alpha \) holds with \( \alpha > 0 \). This leads to \( \| x_n - \hat{x}_{n|n} \| < \beta \) with \( \beta > 0 \) which implies \( (x_n - \hat{x}_{n|n})(x_n - \hat{x}_{n|n})^T < \beta^2 I_n \). However, due to the ellipsoidal set description of the true state \( (x_n - \hat{x}_{n|n})^T P_{n|n}^{-1} (x_n - \hat{x}_{n|n}) \leq 1 \) is true. This can be equivalently expressed using Schur complements [10] as \( (x_n - \hat{x}_{n|n})^T (x_n - \hat{x}_{n|n}) \leq P_{n|n}. \) Denote \( e_{n|n} = (x_n - \hat{x}_{n|n}) \). Then, \( e_{n|n} P_{n|n}^{-1} e_{n|n} < \beta^2 I_n \) leads to \( e_{n|n} e_{n|n}^T - P_{n|n} + P_{n|n}^{-1} \beta^2 I_n < 0. \) Since \( e_{n|n} e_{n|n}^T - P_{n|n} \leq 0, (P_{n|n}^{-1} \beta^2 I_n < 0 \) must hold. Hence, there exists a \( p_2 > 0 \) such that \( P_{n|n} \leq p_2 I_n \).

Similarly, upper boundedness of \( P_{n+1|n} \) can be proved with the predicted state estimate defined as \( \hat{x}_{n+1|n} = \phi_{n+1|n} \hat{x}_{n|n} = \hat{x}_{n|n} + \hat{x}_{n|n} \) with \( \kappa = 1, s = n - 1 \). Also, the SMF-SDC is required to be non-divergent during the initialization period, which is similar to the requirements for the EKF in [25]. If the system dimension is not large, this requirement has to be satisfied for only a few recursions of the SMF-SDC. Note that satisfaction of Conditions 1, 2, 6 can only be verified on-line during the state estimation process.

![Fig. 1. Simulation results for the Van der Pol equation (Case-1).](image)

V. SIMULATION EXAMPLE

A simulation example is provided in this section to illustrate the effectiveness of the proposed approach. All the simulations are carried out on a desktop computer with a 16.00 GB RAM and a 3.40 GHz Intel(R) Xeon(R) E-2124 G processor running MATLAB R2019a. The SDPs in (15) and (27) are solved utilizing ‘YALMIP’ [34] with the ‘SDPT3’ solver in the MATLAB framework.

Consider the discretized Van der Pol equation in [23] with \( \mu = 2 \) and discretization time step \( \Delta t = 0.1 \) seconds. The Van der Pol equation admits a stable limit cycle, thus satisfying Assumption 1. The SDC parameterization for the nonlinear system is chosen as

\[
\begin{align*}
\dot{x}_{k+1} &= \begin{bmatrix} 1 & \Delta t \\ -9 \Delta t & 1 + \mu \Delta t(1 - x_{2k}^2) \end{bmatrix} \begin{bmatrix} x_{2k} \\ x_{2k+1} \end{bmatrix} + \begin{bmatrix} 0 \\ u_k \end{bmatrix} \\
&= A(x_k) x_{k+1} + u_k \\
y_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k = H(x_k) x_k + v_k
\end{align*}
\]

where \( A(\cdot) \) is full rank \( \forall \Delta t \neq 1 \). Also, it is easy to verify that the SDC parameterization satisfies Assumption 5 and the rank condition
in (34). With the above SDC parameterization, the matrices $K_1$ and $K_2$ are given by

$$K_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 & -2\mu\delta t\hat{x}_{1_{k|k}} & 0 & 0 \end{bmatrix}.$$ 

Thus, $K_1$ and $K_2$ remain bounded if $\hat{x}_{1_{k|k}}$ does not diverge. With these, the Conditions 3, 4, 5 in Lemma 2 are satisfied. Rest of the conditions in Lemma 2 are verified in the simulations. For comparison, two cases are considered with different levels of noises and initial estimation error. Points on the unit circle are parameterized as $x_{k|k} = [\cos(\theta_i) \sin(\theta_i)]^T$ for the random research algorithm in (12). Consider the initial condition $P_0 = 2I_2$ and $\hat{x}_0 = [1 \ 1]^T$. Utilizing this initial condition, the random search algorithm in (12) is solved at $k = 0$ by randomly sampling different numbers of $\theta_i$, $i = 1, 2, ..., N$. The results are summarized as follows: (i) $r_{A_k} = 0.3996298150741$ for $N = 100$; (ii) $r_{A_k} = 0.39999997007$ for $N = 10,000$; (iii) $r_{A_k} = 0.4$ for $N = 1,000,000$. Based on these results, $N = 10,000$ is utilized at each time step for all the simulations shown here.

1) Case-1: In this case, the initial condition is $P_0 = 2I_2$, $x_0 = [2 \ 0]^T$, and $\hat{x}_0 = [1 \ 1]^T$. The process and measurement noise related quantities are chosen as $\omega_k = [0 \ 0.01]^T$, $v_k = 0.01$, $Q_k = 0.01I_2$, $R_k = 0.01$. The true state components along with the corresponding corrected state estimates and bounds are shown in Fig. 1 as a function of time. Clearly, $x_{1_{k}}$, $x_{2_{k}}$ remain within the bounds at all times and the bounds for $x_2$ are large. Fig. 2 depicts the true state trajectory and the corrected state estimate trajectory in the phase plane. Note that, at $k = 0$, the correction step brings the corrected state estimate close to the initial true state. Also, it is obvious that the corrected state estimate trajectory converges close to the true state trajectory after a few recursions of the filter.

2) Case-2: In this case, the process and measurement noise are 10 times higher compared to Case-1, i.e., $\omega_k = [0 \ 0.1]^T$, $v_k = 0.1$, $Q_k$ and $R_k$ are kept unchanged. Also, the initial condition is $P_0 = 8I_2$, $x_0 = [2.5 \ 1]^T$, and $\hat{x}_0 = [0 \ 0]^T$. Thus, the initial estimation error is higher for this case. Time histories of the true state components along with the corresponding corrected state estimates and bounds are shown in Fig. 3. Again, $x_{1_{k}}$ and $x_{2_{k}}$ remain within the bounds at all times and the bounds are comparable with those in Fig. 1. Fig. 4 depicts the true state trajectory and the corrected state estimate trajectory in the phase plane. The corrected state estimate trajectory does not converge as close to the true state trajectory as in Fig. 2.

A comparison between the estimation errors at the correction step for the two cases is shown in Fig. 5. Clearly, Case-2 results in higher estimation errors due to the larger process and measurement noise. Also, it is interesting to observe that $||e_{1_{k|k}}||$ converges to a neighborhood of $||r_k||$ in both the cases.

VI. CONCLUSION

A recursive set membership filtering algorithm for discrete-time nonlinear dynamical systems subject to unknown but bounded process and measurement noise has been derived utilizing the state dependent coefficient (SDC) parameterization. At each time step, the filtering problem has been transformed into two semi-definite programs (SDPs) using the S-procedure and Schur complement. Optimal (minimum trace) ellipsoids have been constructed that contain the true state of the system at the correction and prediction steps. Sufficient conditions for boundedness of those ellipsoidal sets have been derived. Finally, an illustrative simulation example is provided which show that the proposed filter performs adequately...
under different noise levels and initial estimation errors. Our future research will involve investigation of the steady state behavior of ellipsoids as well as the state estimates. Also, sufficient conditions for the boundedness of the ellipsoids with a known control input, acting through a possibly non-square state dependent control matrix, would be investigated.

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