Phase transition of RN-ERN coupled network in failure recovery process

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Abstract. We simulate the failure recovery process and study the phase transition from high network activity (HNA) state to low network activity (LNA) state of the coupled network consisting of a regular network (RN) and an Erdos-Rényi network (ERN). Two recovery mechanisms are considered, the internal recovery and the external recovery, in the failure recovery process. We find that the evolution of the coupled network is crucially dependent on the coupling strength $c$ between the two subnetworks and the external recovery possibility $r$ since RN is sensitive to external recovery. A weak coupling can drive coevolution of the subnetworks when $r \approx 1.0$, while the synchronous HNA-LNA phase transition needs stronger coupling to occur when $r$ is smaller. The internal recovery enhances the robustness of network which makes coevolution more difficult to take place.

1. Background
Networks are ubiquitous in the real world, such as the social networks, the Internet, the traffic, the power systems, the neural networks in our bodies and so on. These systems consist of abundant entities which are connected by some kinds of interactions. To study the properties of such complex systems, one may map the complex system to a simple mathematical model which is called a complex network. In the past few decades, several celebrated complex networks have been proposed, such as the Erdos-Rényi network [1], the Watts-Strogatz “small-world” network [2] and the Barabási-Albert scale-free network [3], which pioneers the comprehensive studies of complex networks. Nowadays, complex network has focused interests from many subjects of research. The study of complex networks plays a very important role in understanding the structures and functions of complex systems.

In recent years, cascading failure in the network has been widely investigated [4-6]. These works concentrate on the irreversible failure induced by abrupt dynamic events. However, there exists a number of complex dynamic systems which can automatically recover after a crash for some time in the real world. For example, a large number of banks collapse in financial crisis, but the world’s economy will return prosperity after a few years [7]; Severe weather may cause road congestion, but the traffic jam will disappear after a period of time [8]; Human’s brain can recover spontaneously after an epileptic seizure. Majdandzic et al. [9] study a failure recovery model where failed nodes will recover after a time period. They find a strong hysteresis behavior and phase switching phenomena induced by internal recovery. Xiao et al. [10] argue that an external recovery mechanism should also be considered. They improve Majdandzic et al.’s model to study the dynamic process of complex networks recovering from the failure.

In Xiao et al.’s work, three kinds of complex networks are considered: the regular network, the Erdos-Rényi network and the scale-free network. However, a single network can hardly exist
independently in the real world, and the couplings between networks are becoming increasingly strong, such as the couplings among the systems of power, water supply, transport and communication [11]. These coupled systems can be described by an interdependent network [12]. The cascading failure in interdependent networks is quite different from that in independent networks. Once a node in a network fails, the impact of the failure will be spread and amplified by the connections between the networks, which may affect the functions of other networks and eventually cause catastrophic consequences to all the networks. For instance, the blackout is the consequence of the cascading failure in the coupled systems of power and communication [13]. Buldyre et al. [14] study the cascading failure process in the coupled network by percolation theory. They observe a first-order phase transition of the coupled network which indicates that the coupled network is less robust than the independent network. They also find that increasing the average degrees of coupled network leads to less robustness against random attack, which is totally different from the case of the independent network. Inspired by Buldyre et al.’s work, cascading failure processes in different coupled networks are widely studied [15-20]. However, the recovery process is not considered in most of these works as well, which is the main purpose of this paper.

2. Model and methods
Motivated by Majdandzic et al.’s work, the failure recovery model studied in this paper contains three basic assumptions:

1) At a time \( t \), any active node in the network can fail with a possibility of failure \( p \) regardless of other nodes.

2) There exist two kinds of recovery mechanisms, one is the internal recovery: if an active node fails at time \( t \), then it will be activated independently at \( t + \tau \), where \( \tau \) denotes the internal recovery period. The other is the external recovery: if a failed node has a good neighborhood, which means the number of the active neighbors is not less than half of the degree of the node, it will recover at the next time with a possibility of \( r \).

3) Within a time interval, all failed nodes can be activated gradually, but each node only has one chance to recover.

In this paper, we consider a coupled network consisting of a regular network (RN) and an Erdos-Rényi network (ERN). Each subnetwork is composed of \( N = 1000 \) nodes with same the average degree \( \langle k \rangle \), the possibility of failure \( p \) and the internal recovery period \( \tau \). The strength of the coupling between the subnetworks is defined by the coupling coefficient \( c = \frac{E_c}{N} \), where \( E_c \) denotes the number of edges connected between the subnetworks. It should be noticed that for \( c \leq 1.0 \), any node in the coupled network cannot be connected more than one extra edge.

At the beginning of our simulations, it is assumed that every node composing the coupled network is set to be active. As time goes on, the coupled network evolves and finally achieves dynamical equilibrium. We define a parameter \( \langle z \rangle \), which is the fraction of active nodes in the network, to measure the activity of the network when the network achieves dynamical equilibrium.

It should be noticed that even if the network configuration is the same, each evolution should exhibit some differences due to the characteristics of random number generation. To reduce the effects of random, we simulate 100 times of network evolution and take the average of the results for each set of parameters.

3. Results and discussions
In our failure recovery model, if only the internal recovery mechanism is considered, the evolution of the network is only influenced by the possibility of failure \( p \) and the internal recovery period \( \tau \), regardless of the topological configuration of the network. When the network achieves dynamical equilibrium, the activity \( \langle z \rangle \) satisfies \( \langle z \rangle - \langle z \rangle \cdot p + (1 - \langle z \rangle) / \tau = \langle z \rangle \), which solution is \( \langle z \rangle = 1 / (1 + \tau p) \). We would not discuss this case anymore.

Now we consider the situation where only the external recovery mechanism is introduced. Figure 1 demonstrates the relationship curves of the network activities \( \langle z \rangle \) versus the failure possibilities \( p \) for
the subnetworks with different coupling coefficients. The average degree $\langle k \rangle$ is set to be 20 and the external recovery possibility $r$ equals 1.0. As a guide, the red (blue) dashed line represents the independent RN (ERN). From Figure 1 it can be distinguished that the networks could exist in the high network activity (HNA) state ($\langle z \rangle > 0$) or the low network activity (LNA) state ($\langle z \rangle = 0$). The steep decrease of $\langle z \rangle$ indicates a first-order HNA-LNA phase transition occurs. We can find that the curve of RN can even be influenced by a very weak coupling ($c = 0.01$), while the curve of ERN almost keeps the same, which indicates that RN is sensitive to the external recovery and ERN is robust. With the enhancement of $c$, the curve of RN gradually approaches that of ERN. When $c = 0.06$, the subnetworks achieve coevolution.

![Figure 1](image)

Figure 1. The network activities $\langle z \rangle$ as a function of failure possibilities $p$ ($r = 1.0, \langle k \rangle = 20$). Inset shows the HNA-LNA phase transition points $p_c^{RN}$ ($p_c^{ERN}$) of RN (ERN) as a function of coupling coefficients $c$.

It can be explained that when the degrees of nodes are small, the recovery induced by the local fluctuation can be spread throughout the whole RN easily by the external recovery mechanism. Obviously, this process is crucially dependent on the external recovery probability $r$, since once a node fails to recover by the external recovery mechanism, it may not satisfy the external recovery condition due to the failure of its neighbors at the next time. When the degrees of nodes are large, it is difficult to spread the local fluctuation throughout the whole RN. However, since the degrees of nodes are large, a failed node is more likely to satisfy the external recovery condition and has a chance to recover at the next time. As a result, the network is not sensitive to the external recovery probability $r$. In ERN, nodes have different degrees, so the scenarios of both small and large degrees could exist at the same time. Since the nodes of large degrees are much more important, the ERN is dominated by the latter scenario.

The inset in Figure 1 demonstrates the relationship curves of the phase transition points $p_c^{RN}$ ($p_c^{ERN}$) of RN (ERN) versus the coupling coefficients $c$. When the coupling is weak, since ERN is robust and RN is sensitive to the external recovery, RN is easily affected by ERN and the evolution of the coupled network is dominated by ERN. As the coupling between the subnetworks is equivalent to increasing the average degree of ERN, which lessens the effects of the external recovery and reduces the critical point of ERN, the critical point of the coupled network is smaller as well. When the coupling is stronger, the local fluctuation in RN is depressed and the effects of RN should not be ignored any more. Since $p_c^{RN} > p_c^{ERN}$ (if we ignore the coupling), the nodes in RN could be regarded as a good environment of the nodes in ERN and increase the activity of ERN, which increases the transition point of the coupled network.

From above we find that it is easy to achieve coevolution of RN and ERN by tuning the strength of coupling with the external recovery possibility $r = 1.0$. However, it could be very different when $r$ is smaller. Figure 2 demonstrates the similar information as Figure 1 but $r = 0.88$. Since RN is sensitive
to the external recovery while ERN is robust, the critical point $p_c^{RN}$ decreases rapidly when $r$ is smaller while $p_c^{ERN}$ almost doesn’t change. One can observe $p_c^{ERN} > p_c^{RN}$ in Figure 2 and should expect a strong coupling to achieve the coevolution of the subnetworks. Compared with Figure 1, the curve of RN no longer approaches to that of ERN monotonously as the coupling coefficient $c$ becomes bigger. An intuitive insight of this process is presented in the inset of Figure 2. With enhancement of the coupling between the subnetworks, $p_c^{RN}$ firstly decreases and reaches its minimum at $c \approx 1.0$ due to the increment of the average degree $\langle k \rangle$ raised by the coupling, then increases due to the effects of ERN. Since ERN is robust, the behavior of $p_c^{ERN}$ is similar to that in the inset of Figure 1. At $c \approx 1.4$, the subnetworks achieve coevolution.

**Figure 2.** The network activities $\langle z \rangle$ as a function of failure possibilities $p$ ($r = 0.88, \langle k \rangle = 20$). Inset shows the HNA-LNA phase transition points $p_c^{RN}$ ($p_c^{ERN}$) of RN (ERN) as a function of coupling coefficients $c$.

**Figure 3.** The critical couplings $c^*$ as a function of external recovery possibilities $r$. Inset shows the corresponding phase transition points $p_c$.

Now we discuss the influences of the external recovery on the cascading failure. Figure 3 demonstrates the relationship curves of the critical coupling coefficients $c^*$ when the subnetworks achieve coevolution versus the external recovery possibilities $r$. We can find that the coevolution can be induced by a very weak coupling ($c^* \leq 0.05$) when $r \geq 0.92$ and $c^*$ even gets slightly smaller as $r$
gets bigger. Although the external recovery is weaker, the transition points of the subnetworks get closer to each other, which makes the synchronous phase transition take place more easily. At \( r \approx 0.92 \), there exists a crossover from \( p_{c \text{ERN}}^{\text{RN}} < p_{c \text{RN}}^{\text{ERN}} \) to \( p_{c \text{ERN}}^{\text{RN}} > p_{c \text{RN}}^{\text{ERN}} \) (not illustrated in this paper). Since \( p_{c \text{RN}}^{\text{ERN}} \) can be reduced by weak couplings but increased by strong couplings, the critical coupling experiences a jump to \( c^* \approx 1 \) and increases approximately linearly as \( r \) gets smaller, which is in good agreement with the result shown in the inset of Figure 2. Since RN is less robust and more likely to be affected by ERN when the average degree \( \langle k \rangle \) is smaller, a smaller \( c^* \) is required to achieve coevolution. The inset of Figure 3 demonstrates the corresponding critical points \( p_c \) of \( c^* \). A jump at \( r \approx 0.92 \) can be distinguished as well. As RN is more sensitive to \( r \) when \( \langle k \rangle \) is smaller, the change of \( p_c \) is more dramatic.

![Figure 4](image-url)

**Figure 4.** The critical couplings \( c^* \) as a function of the internal recovery periods \( \tau \). Inset shows the corresponding phase transition points \( p_c \).

At last, we apply both the internal recovery and the external recovery to the coupled network. Figure 4 demonstrates the relationship curves of the critical coupling coefficients \( c^* \) when the subnetworks achieve coevolution versus the internal recovery periods \( \tau \). Here we only consider the case of \( r = 1.0 \). As mentioned above, the synchronous HNA-LNA phase transition is more difficult to take place when the external recovery possibility \( r \) is smaller. Since the internal recovery is obviously in favor of the robustness of networks, a stronger coupling is needed to achieve coevolution. \( c^* \) decreases as \( \tau \) gets bigger. We also find that although the difference between the critical points of the subnetworks for \( \langle k \rangle = 10 \) is larger than that for \( \langle k \rangle = 20 \) (not illustrated in this paper), the synchronous phase transition requires a stronger coupling to take place for \( \langle k \rangle = 20 \) due to the stronger robustness of the networks with larger \( \langle k \rangle \). The corresponding critical points \( p_c \) of \( c^* \) is demonstrated in the inset of Figure 4. When \( \tau \) gets bigger, the effects of the internal recovery are weaker and the coupled network is more easily to exist in the LNA state. As a result, \( p_c \) gets smaller.

**4. Conclusion**

In this paper, we study the HNA-LNA phase transition of the RN-ERN coupled network in the failure-recovery process. Both internal and external recovery mechanisms are considered. We find that the evolution of the coupled network is crucially dependent on the coupling strength between the subnetworks and the external recovery possibility. When the external recovery possibility equals approximately to 1, a weak coupling can drive the coevolution of the subnetworks, while a stronger coupling is required to increase the activity of RN by tuning ERN as the external recovery possibility gets smaller. Since the internal recovery enhances the robustness of networks, the synchronous HNA-LNA phase transition is more difficult to occur when the internal recovery is introduced.

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