POSITIVITY IN OPERATOR SYSTEMS GENERATED BY TOEPLITZ-CUNTZ ISOMETRIES

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Abstract. In this paper we characterize positive elements in \( M_p(S) \), for \( p \in \mathbb{N} \), where \( S \) is the operator system generated by \( n \) isometries with Toeplitz-Cuntz relation, by constructing a complete order isomorphism between \( S \) and some quotient operator system. That is, we prove a necessary and sufficient condition for element in \( M_p(S) \) to be positive.

1. Introduction

The Toeplitz-Cuntz algebra with \( n \) generators, denoted by \( \mathcal{TO}_n \), is the universal \( C^* \)-algebra generated by \( n \) isometries \( S_1, \ldots, S_n \) satisfying the Toeplitz-Cuntz relation, i.e., \( \sum_{i=1}^n S_i S_i^* \leq I \), with the following universal property: If \( T_1, \ldots, T_n \) are \( n \) isometries satisfying \( \sum_{i=1}^n T_i T_i^* \leq I \), then there is a surjective \( * \)-homomorphism \( \pi : C^*(\{S_1, \ldots, S_n\}) \to C^*(\{T_1, \ldots, T_n\}) \) such that \( \pi(S_i) = T_i \) for \( 1 \leq i \leq n \).

Now, let \( S_1, \ldots, S_n \) be \( n \) isometries with \( \sum_{i=1}^n S_i S_i^* \leq I \), where \( I \) denotes the identity and we set \( S = \text{span}\{I, S_1, \ldots, S_n, S_1^*, \ldots, S_n^*\} \) so that \( S \) is the operator system generated by \( S_1, \ldots, S_n \). We try to obtain a necessary and sufficient condition for elements in \( M_p(S) \) to be positive for each \( p \in \mathbb{N} \).

To this end, we will build a quotient operator system. The concept of quotient operator system is introduced in [5] and here we present some basic definitions we shall need in our proofs.

Definition 1.1. [5] Given an operator system \( S \), we call \( J \subseteq S \) a kernel, if \( J = \ker \phi \) for an operator system \( T \) and some unitally completely positive map \( \phi : S \to T \).

Proposition 1.2. [5] Let \( S \) be an operator system and \( J \subseteq S \) be kernel, if we define a family of matrix cones on \( S/J \) by setting

\[
C_n = \{(x_{ij} + J) \in M_n(S/J) : \text{for each } \epsilon > 0, \text{ there exists } (k_{ij}) \in M_n(J) \text{ such that } \epsilon \otimes I_n + (x_{ij} + k_{ij}) \in M_n(S)^+\}
\]

then \( (S/J, \{C_n\}_{n=1}^\infty) \) is a matrix ordered \( * \)-vector space with an Archimedean matrix unit \( 1 + J \), and the quotient map \( q : S \to S/J \) is completely positive.

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Definition 1.3. Let $\mathcal{S}$ be an operator system and $J \subseteq \mathcal{S}$ be kernel. We call the operator system $(\mathcal{S}/J, \{C_n\}_{n=1}^{\infty}, 1 + J)$ defined in Proposition 1.2 the quotient operator system.

Definition 1.4. Let $J$ be a kernel and define

$$D_n = \{(x_{ij} + J) \in M_n(\mathcal{S}/J) : \text{there exists } y_{ij} \in J \text{ such that } (x_{ij} + y_{ij}) \in M_n(\mathcal{S})^+\},$$

then $J$ is completely order proximinal if $C_n = D_n$ for all $n \in \mathbb{N}$.

Also, we need the concept of complete order isomorphism.

Definition 1.5. Let $\mathcal{S}$ and $\mathcal{T}$ be operators systems. A map $\phi : \mathcal{S} \to \mathcal{T}$ is called a (unital) complete order isomorphism if $\phi$ is a linear isomorphism and both $\phi$ and $\phi^{-1}$ are (unitally) completely positive, and we say that $\mathcal{S}$ is (unitaly) completely order isomorphic to $\mathcal{T}$ if such $\phi$ exists.

Thus, according to the definitions above, in order to construct a quotient operator system, we need to find a kernel $J$. This $J$, as we shall see in the following proofs, is related to $I - \sum_{i=1}^{n} S_i S_i^*$. So it is reasonable to divide the argument into two cases: $\sum_{i=1}^{n} S_i S_i^* = I$ and $\sum_{i=1}^{n} S_i S_i^* < I$, i.e., $\sum_{i=1}^{n} S_i S_i^* < I$ but $\sum_{i=1}^{n} S_i S_i^* \neq I$.

If $\sum_{i=1}^{n} S_i S_i^* = I$ and $n \geq 2$, then J. Cuntz in [2] showed that the $C^*$-algebra $\mathcal{O}_n$ (also called the Cuntz algebra) generated by $S_1, \ldots, S_n$ is unique. That is, for any two set of $n$ isometries, say $S_1, \ldots, S_n$ and $T_1, \ldots, T_n$, both satisfying $\sum_{i=1}^{n} S_i S_i^* = I$ and $\sum_{i=1}^{n} T_i T_i^* = I$, the map $\pi(S_i) = T_i$ extends to an isomorphism from $C^*(\{S_1, \ldots, S_n\})$ onto $C^*(\{T_1, \ldots, T_n\})$. This means that the operator systems generated by $S_1, \ldots, S_n$ and $T_1, \ldots, T_n$ are unitally completely order isomorphic. So an arbitrary set of $n$ isometries with Cuntz relation will suffice for our discussion.

If $\sum_{i=1}^{n} S_i S_i^* < I$, then we can view $\{S_i, I : 1 \leq i \leq n\}$ as a Toeplitz-Cuntz-Krieger family for a graph with $n$ loops and a single vertex, and Theorem 4.1 from [4] implies that the $C^*$-algebra generated by $S_1, \ldots, S_n$ is $\mathcal{T}O_n$. Hence, we may choose any set of $n$ isometries with $\sum_{i=1}^{n} S_i S_i^* < I$ to discuss.

In the following, Sections 2, 3 and 4 are proofs for the case: $\sum_{i=1}^{n} S_i S_i^* = I$; Section 5, 6 and 7 are proofs for the case: $\sum_{i=1}^{n} S_i S_i^* < I$.

2. Construction of the Quotient Operator System $\mathcal{S}/J$

Consider the operator system $\mathcal{E} \subseteq M_{n+1} := M_{n+1}(\mathbb{C})$ defined by the following,

$$\mathcal{E} = \text{span}\{E_{00}, E_{0i}, E_{00}, \sum_{i=1}^{n} E_{ii} : 1 \leq i \leq n\},$$

where $E_{ij}$’s are matrix units in $M_{n+1}$. So every element in $\mathcal{S}$ is of the following form,

$$\begin{pmatrix}
a_{00} & a_{01} & \cdots & a_{0,n} \\
a_{10} & b & & \\
\vdots & & \ddots & \\
a_{n0} & & \cdots & b
\end{pmatrix}.$$
We let \( S_1, \ldots, S_n \) be \( n \) (\( n \geq 2 \)) isometries in \( B(\mathcal{H}) \) with \( \sum_{i=1}^n S_i S_i^* = I \), where \( B(\mathcal{H}) \) denotes the space of all bounded linear operators on some Hilbert space \( \mathcal{H} \), and \( S \) be the operator system generated by \( S_1, \ldots, S_n \), i.e., \( S = \text{span}\{I, S_1, \ldots, S_n, S_1^*, \ldots, S_n^*\} \).

Then, we define an operator \( R : \mathcal{H}^{(n+1)} \to \mathcal{H} \) by \( R := (\frac{1}{\sqrt{2}}I, \frac{1}{\sqrt{2}}S_1^*, \ldots, \frac{1}{\sqrt{2}}S_n^*) \). So we know that

\[
R^* R = \begin{pmatrix}
\frac{1}{2} I & \frac{1}{2} S_1^* & \cdots & \frac{1}{2} S_n^* \\
\frac{1}{2} S_1 & \cdots & & \\
& \vdots & \ddots & \\
& & & \frac{1}{2} S_n^* \\
\end{pmatrix},
\]

is positive in \( M_{n+1}(B(\mathcal{H})) \).

Now, we can define a map \( \phi : M_{n+1} \to B(\mathcal{H}) \) by

\[
(\phi(E_{ij}))_{i,j=0}^n = R^* R,
\]

and extend it linearly to \( M_{n+1} \). Since \( E_{ij} \)'s are matrix units, we know \( \phi \) is well-defined and linear. Also \( \phi \) is unital. Moreover, notice that \( (\phi(E_{ij})) \) is the Choi matrix of \( \phi \), and Choi’s theorem tells us that this map is also completely positive. Hence, \( \phi \) is unitaly completely positive.

**Lemma 2.1.** Denote \( \ker \phi \) as \( J \), then we have that \( J = \text{span}\{E_{00} - \sum_{i=1}^n E_{ii}\} \).

**Proof.** It is easy to check that \( J \subseteq \ker \phi \), since \( S_i \)’s obey the Cuntz relation. So what left for us to show is that \( \ker \phi \subseteq J \).

For \( A \in \ker \phi \), write it as \( A = \sum_{i,j=1}^n a_{ij} E_{ij} + \sum_{i=1}^n b_i E_{0i} + \sum_{i=1}^n c_i E_{i0} + d E_{00} \). Then,

\[
\phi(A) = \sum_{i,j=1}^n \frac{a_{ij}}{2} S_i S_j^* + \sum_{i=1}^n \frac{b_i}{2} S_i^* + \sum_{i=1}^n \frac{c_i}{2} S_i + \frac{d}{2} I = 0.
\]

Suppose \( b_k \neq 0 \) for some \( 1 \leq k \leq n \). Notice that \( S_i^* S_i = I \) and \( S_i^* S_j = 0 \) for \( i \neq j \), we can check that

\[
S_k^* \phi(A) S_k = \frac{a_{kk}}{2} I + \frac{b_k}{2} S_k^* + \frac{c_k}{2} S_k + \frac{d}{2} I = 0.
\]

This means

\[
S_k^* = -\frac{a_{kk} + d}{b_k} I - \frac{c_k}{b_k} S_k.
\]

Hence, we have \( S_k S_k^* = S_k^* S_k = I \), which is a contradiction. So \( b_i = 0 \) for every \( 1 \leq i \leq n \). Consider \( \phi(A)^* = 0 \) and use the same technique, we also see that \( c_i = 0 \) for every \( 1 \leq i \leq n \).

Therefore, \( \phi(A) = \sum_{i,j=1}^n \frac{a_{ij}}{2} S_i S_j^* + \frac{d}{2} I = 0 \). For \( i \neq j \), \( S_i^* \phi(A) S_j = 0 \) implies that \( a_{ij} = 0 \). For each \( 1 \leq i \leq n \), \( S_i^* \phi(A) S_i = 0 \) implies \( a_{ii} I + dI = 0 \), and hence \( a_{ii} = -d \).

Now, we conclude that \( A = -d \sum_{i=1}^n E_{ii} + d E_{00} \in J \). So \( \ker \phi \subseteq J \), meaning that \( \ker \phi = J \). \( \square \)

Let \( \psi = \phi|_E \), then \( \psi : E \to B(\mathcal{H}) \) is unitaly completely positive.

It is easy to see that \( J \subseteq E \), so by the lemma above we can form the quotient operator system \( E/J \) and the map \( \psi : E/J \to B(\mathcal{H}) \) defined by \( \psi(A + J) = \psi(A) \) is one-to-one.
Also, by the definition of \( \psi \), \( \tilde{\psi} \) maps \( \mathcal{E}/J \) onto the operator system \( \mathcal{S} \). Since, \( \psi \) is unital completely positive, we also know that \( \tilde{\psi} \) is unitally completely positive. So, in sum, we have constructed a map \( \tilde{\psi} : \mathcal{E}/J \to \mathcal{S} \) which is one-to-one, onto and unitally completely positive.

3. Complete Order Isomorphism Between \( \mathcal{E}/J \) and \( \mathcal{S} \)

In this section, we will show that \( \tilde{\psi}^{-1} \) is also completely positive so that \( \mathcal{E}/J \) is unitally completely order isomorphic to \( \mathcal{S} \).

First, we let \( I_{n+1} \) denote the identity matrix of \( M_{n+1} \) and represent \( T_i := 2E_{i0} + J, I_C := I_{n+1} + J \in C_C^*(\mathcal{E}/J) \) via some unital complete order injection, where \( C_C^*(\mathcal{E}/J) \) is the \( C^* \)-envelope of \( \mathcal{E}/J \) defined in Chapter 15 in [7]. Here, we only need to know that \( C_C^*(\mathcal{E}/J) \) is a \( C^* \)-algebra and there is a unital complete order isomorphism \( \varphi \) from \( \mathcal{E}/J \) onto the range of \( \varphi \) in \( C_C^*(\mathcal{E}/J) \).

The proof of the following lemma is quite similar to that of Lemma 3.1 in [7], so we omit the proof.

**Lemma 3.1.** Let \( \mathcal{H}, \mathcal{K} \) be Hilbert spaces and \( T \in B(\mathcal{H}, \mathcal{K}) \). Also, denote \( I_{\mathcal{H}} \) and \( I_{\mathcal{K}} \) as the Identity operators on \( \mathcal{H} \) and \( \mathcal{K} \) respectively. Then we have \( \|T\| \leq 1 \) if and only if

\[
\begin{pmatrix}
I_{\mathcal{H}} & T^* \\
T & I_{\mathcal{K}}
\end{pmatrix}
\]

is positive in \( B(\mathcal{H} \oplus \mathcal{K}) \).

**Proposition 3.2.** We have that \( (T_1, \ldots, T_n) \) is a row contraction, i.e., \( \sum_{i=1}^{n} T_i T_i^* \leq I_C \).

**Proof.** We represent \( T_i \)'s on a Hilbert space \( \mathcal{H} \) via some unital one-to-one \( * \)-homomorphism, so \( (T_1, \ldots, T_n) \in B(\mathcal{H}^{(n)}, \mathcal{H}) \). By the above lemma, equivalently, we show the following,

\[
\begin{pmatrix}
I_C & T_1 & \cdots & T_n \\
T_1^* & I_C & & \\
\vdots & & \ddots & \\
T_n^* & & & I_C
\end{pmatrix}
\geq 0.
\]

Notice that \( E_{00} + \sum_{i=1}^{n} E_{ii} = I_{n+1} \) and \( E_{00} + J = \sum_{i=1}^{n} E_{ii} + J \), so

\[
E_{00} + J = \sum_{i=1}^{n} E_{ii} + J = \frac{1}{2} I_{n+1} + J
\]

Since the quotient map \( q : M_{n+1} \to M_{n+1}/J \) is completely positive, we just need to show

\[
\begin{pmatrix}
2 \sum_{i=1}^{n} E_{ii} & 2E_{10} & \cdots & 2E_{n0} \\
2E_{01} & 2E_{00} & & \\
\vdots & & \ddots & \\
2E_{0n} & & & 2E_{00}
\end{pmatrix}
\geq 0.
\]
To this end, we write this matrix as a sum of $n$ matrices

\[
\left( \sum_{i=1}^{n} E_{ii} \right. \begin{array}{cccc}
E_{11} & E_{10} & \cdots & E_{10} \\
E_{01} & E_{00} & & \\
& \ddots & & \\
E_{0n} & & & E_{00}
\end{array} \left. \right) = \begin{pmatrix}
E_{nn} & 0 & \cdots & 0 & E_{n0} \\
0 & \ddots & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \ddots & 0 \\
E_{0n} & 0 & \cdots & 0 & E_{00}
\end{pmatrix}
\]

\[
+ \cdots + \begin{pmatrix}
E_{11} & E_{10} & 0 & \cdots & 0 \\
E_{01} & E_{00} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

From the equation above, we can see that each summand on the right is positive, so the block matrix on the left is positive and the conclusion follows. \qed

Lemma 3.3. [8] For any sequence $\{T_\lambda\}_{\lambda \in \Lambda}$ of operators on a Hilbert space $\mathcal{H}$ such that $\sum_{\lambda \in \Lambda} T_\lambda T_\lambda^* \leq I_\mathcal{H}$, there exists an isometric dilation $\{V_\lambda\}_{\lambda \in \Lambda}$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ such that $\sum_{\lambda \in \Lambda} V_\lambda V_\lambda^* = I_\mathcal{K}$

Proposition 3.4. The map $\tilde{\psi} : \mathcal{E}/J \to \mathcal{S}$ is a unital complete order isomorphism.

Proof. We first apply the above lemma to $T_1, \ldots, T_n$, we know that there exist $n$ isometries $\hat{T}_1, \ldots, \hat{T}_n \in B(\mathcal{K})$ with Cuntz relation which dilate them. Let $\mathcal{E}$ be the operator system generated by $\hat{T}_1, \ldots, \hat{T}_n$. Also, we denote $\hat{I}$ as the identity operator in $B(\mathcal{K})$.

The uniqueness of the Cuntz algebra mentioned in Introduction tells us that we can instead construct the map $\tilde{\psi} : \mathcal{E}/J \to \mathcal{E}$ as the way explained in Section 1 and show that $\psi : \mathcal{E}/J \to \mathcal{E}$ is unital complete order isomorphism. Hence, what left for us to show is that $\tilde{\psi}^{-1}$ is completely positive.

To this end, we first notice the following from the definition $\tilde{\psi}$:

\[
\tilde{\psi}(E_{i0} + J) = \psi(E_{i0}) = \frac{1}{2} \hat{T}_i,
\]

\[
\tilde{\psi}(E_{0i} + J) = \psi(E_{0i}) = \frac{1}{2} \hat{T}_i^*,
\]

\[
\tilde{\psi}(E_{00} + J) = \psi(E_{00}) = \frac{1}{2} \hat{I},
\]

\[
\tilde{\psi}(\sum_{i=1}^{n} E_{ii} + J) = \psi(\sum_{i=1}^{n} E_{ii}) = \frac{1}{2} \sum_{i=1}^{n} \hat{T}_i \hat{T}_i^* = \frac{1}{2} \hat{I}.
\]
Then, for $\tilde{\psi}^{-1}$, we have that
\[
\tilde{\psi}^{-1}(\hat{T}_i) = 2E_{0i} + J = T_i,
\]
\[
\tilde{\psi}^{-1}(\hat{T}_i^*) = 2E_{0i} + J = T_i^*,
\]
\[
\tilde{\psi}^{-1}(\hat{I}) = 2E_{00} + J = 2\sum_{i=1}^n E_{ii} + J = I_{n+1} + J.
\]

Since $T_i$’s are compressions of $\hat{T}_i$’s, it follows that $\tilde{\psi}^{-1}$ is indeed a compression map, which is completely positive. Thus, $\tilde{\psi}$ is a unital complete order isomorphism.

Finally, we conclude that $\mathcal{E}/J$ is unitally completely order isomorphic to $S$. Moreover, the isomorphism sends $2E_{0i} + J$ to $S_i$. □

**Proposition 3.5.** Let $J$ be a finite dimensional $*$-subspace in an operator system $S$ which contains no positive elements other than $0$, then it is a completely order proximinal kernel.

**Lemma 3.6.** For $J$ defined in lemma 2.1, it is completely order proximinal.

**Proof.** According to the above lemma, we just need to show that $J$ contains no positive elements other than $0$. This is clear since by definition for any nonzero element in $J$, its first diagonal entry is opposite to the others. Hence, $J$ is completely order proximinal. □

### 4. Characterization of Positive Elements in $M_p(S)$

We first point out that all isomorphisms used in this section are unital complete order isomorphism.

It has been shown in Section 3 that $\mathcal{E}/J$ is unitally completely order isomorphic to $S$. So we can instead characterize the positive elements in $M_p(\mathcal{E}/J)$ for $p \in \mathbb{N}$. Before doing this, we note that $M_p(\mathcal{E}/J) \cong M_p \otimes (\mathcal{E}/J)$ and every positive element in $M_p \otimes (\mathcal{E}/J)$ is of the following form:

\[
A_0 \otimes I + \sum_{i=1}^n A_i \otimes T_i + \sum_{i=1}^n A_i^* \otimes T_i^*,
\]

where $A_i$’s are matrices in $M_p$, $T_i := 2E_{0i} + J$ and $I := I_{n+1} + J$.

**Proposition 4.1.** We have that $A_0 \otimes I + \sum_{i=1}^n A_i \otimes T_i + \sum_{i=1}^n A_i^* \otimes T_i^* \geq 0$ if and only if there exists $B \in M_p$ such that

\[
\begin{pmatrix}
A_0 & 2A_1^* & \cdots & 2A_n^*
\end{pmatrix}
\begin{pmatrix}
B & -B \\
-B & \ddots
\end{pmatrix}
\begin{pmatrix}
2A_0 & \cdots & 2A_n
\end{pmatrix} \geq 0
\]
Proof. Let $A_k = (a_{ij}^k)$ for $0 \leq k \leq n$, then using the isomorphism $M_p(\mathcal{E}/J) \cong M_p \otimes (\mathcal{E}/J)$, we know $A_0 \otimes I + \sum_{i=1}^{n} A_i \otimes T_i + \sum_{i=1}^{n} A_i^* \otimes T_i^*$ corresponds to

$$(\sum_{k=0}^{n} a_{ij}^k T_k + \sum_{k=1}^{n} a_{ji}^k T_k^*),$$

which is positive.

Replace $T_i$ by $2E_{i0} + J$ and $I$ by $I_{n+1} + J$, we have

$$(a_{ij}^0 I + \sum_{k=1}^{n} a_{ij}^k T_k + \sum_{k=1}^{n} a_{ji}^k T_k^*) = (a_{ij}^0 I_{n+1} + \sum_{k=1}^{n} 2a_{ij}^k E_{k0} + \sum_{k=1}^{n} 2a_{ji}^k E_{0k} + J).$$

Proposition 3.6 together with definition of positivity in quotient operator system imply that there exists $(J_{ij}) \in M_p(J)$ such that

$$(a_{ij}^0 I_{n+1} + \sum_{k=1}^{n} 2a_{ij}^k E_{k0} + \sum_{k=1}^{n} 2a_{ji}^k E_{0k}) + (J_{ij}) \in M_p(\mathcal{E})^+.$$

Now we let $J_{ij} = b_{ij}(E_{00} - \sum_{k=1}^{n} E_{kk})$ and the isomorphism $M_p(\mathcal{E}) \cong M_p \otimes \mathcal{E}$ implies that the above block matrix corresponds to

$$(a_{ij}^0) \otimes I_{n+1} + \sum_{k=1}^{n} 2(a_{ij}^k) \otimes E_{k0} + \sum_{k=1}^{n} 2(a_{ji}^k) \otimes E_{0k} + (b_{ij}) \otimes (E_{00} - \sum_{k=1}^{n} E_{kk}) \in (M_p \otimes \mathcal{E})^+.$$

Finally, using the isomorphism $M_p \otimes \mathcal{E} \cong \mathcal{E} \otimes M_p$ and noticing that $(a_{ji}^k)^* = A_i^*$, we know that the above tensor product corresponds to the following block matrix in $M_{n+1}(M_p)$ which is positive,

$$\begin{pmatrix}
A_0 & 2A_1^* & \cdots & 2A_n^* \\
2A_1 & A_0 & & \\
\vdots & & \ddots & \\
2A_n & & & A_0
\end{pmatrix}
+ \begin{pmatrix}
B & -B \\
- & \ddots & -B
\end{pmatrix}.$$

Here, $B = (b_{ij})$. Since isomorphism is used in each step, we know that our conclusion is in fact an “if and only if” statement. \qed

We remember that the unital complete order isomorphism between $\mathcal{E}/J$ and $\mathcal{S}$ sends $T_i$ to $S_i$. So if we write every positive element in $M_p \otimes \mathcal{S}$ as the following form:

$$A_0 \otimes I + \sum_{i=1}^{n} A_i \otimes S_i + \sum_{i=1}^{n} A_i^* \otimes S_i^*,$$

where $A_i$’s are matrices in $M_p$, and keep in mind that $M_p(\mathcal{S}) \cong M_p \otimes \mathcal{S}$, then we immediately have the following theorem which characterizes positive elements in $M_p(\mathcal{S})$ for every $p \in \mathbb{N}$.
Theorem 4.2. The matrix of operators $A_0 \otimes I + \sum_{i=1}^{n} A_i \otimes S_i + \sum_{i=1}^{n} A_i^* \otimes S_i^* \in M_p(S)$ is positive if and only if there exists $B \in M_p$ such that
\[
\begin{pmatrix}
A_0 & 2A_1^* & \cdots & 2A_n^* \\
2A_1 & A_0 & & \\
\vdots & & \ddots & \\
2A_n & & & A_0
\end{pmatrix} + 
\begin{pmatrix}
B & & \\
& -B & \\
& & \ddots & \\
& & & -B
\end{pmatrix}
\]
is positive in $M_n(M_p)$.

Next, we use similar methods to derive the characterization of positivity for the case: $\sum_{i=1}^{n} S_i S_i^* < I$.

5. Construction of the Quotient Operator System $\mathcal{E}'/J'$

Now consider the following operator system $\mathcal{E}' \subset M_{n+2} := M_{n+2}(\mathbb{C})$,
\[
\mathcal{E}' := \text{span}\{E_{11}, E_{1i}, E_{i1}, E_{00} + \sum_{k=2}^{n+1} E_{kk} : 2 \leq i \leq n + 1\}.
\]
So every element in $\mathcal{S}$ is of the following form,
\[
\begin{pmatrix}
b & 0 & \cdots & \cdots & 0 \\
0 & a_{11} & a_{12} & \cdots & a_{1,n+1} \\
\vdots & a_{21} & b & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & a_{n+1,1} & & \cdots & b
\end{pmatrix}.
\]
Let $S_1, \ldots, S_n$ be $n$ isometries in some $B(\mathcal{H})$ with $\sum_{i=1}^{n} S_i S_i^* < I$ and $\mathcal{S}'$ be the operator system generated by $S_1, \ldots, S_n$, that is $\mathcal{S}' = \text{span}\{I, S_1, \ldots, S_n, S_1^*, \ldots, S_n^*\}$.

We define the following map $\phi : M_{n+2} \rightarrow B(\mathcal{H})$ by
\[
\phi((E_{ij}))_{i,j=0}^{n+1} = \frac{1}{2} \begin{pmatrix}
I - \sum_{i=1}^{n} S_i S_i^* & 0 & \cdots & \cdots & 0 \\
0 & I & S_1^* & \cdots & S_n^* \\
\vdots & S_1 & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & (S_i S_j^*) & \vdots \\
0 & S_n & \cdots & \cdots & I
\end{pmatrix}
\]
and extend it linearly to $M_{n+2}$. The map $\phi$ is easily seen to be unital and Choi’s theorem guarantees that $\phi$ is completely positive, so $\phi$ is unitally completely positive.

We first compute the kernel of $\phi$ in the following lemma.

Lemma 5.1. We have that $\ker \phi = \text{span}\{E_{00} + \sum_{i=2}^{n+1} E_{ii} - E_{11}, E_{0i}, E_{i0} : 1 \leq i \leq n + 1\}$.

Proof. It is easy to check that $\text{span}\{E_{00} + \sum_{i=2}^{n+1} E_{ii} - E_{11}, E_{0i}, E_{i0} : 1 \leq i \leq n + 1\} \subseteq \ker \phi$. 

For the reversed inclusion, we pick $A \in \ker \phi$ and write it as $A = \sum_{i,j=1}^{n} a_{ij} E_{i+1,j+1} + \sum_{i=1}^{n+1} b_{i} E_{i0} + \sum_{i=1}^{n+1} c_{i} E_{i0} + d E_{00} + m E_{11} + \sum_{i=1}^{n} p_{i} E_{1,i+1} + \sum_{i=1}^{n} q_{i} E_{i+1,1}$. Then,

$$\phi(A) = \sum_{i,j=1}^{n} \frac{a_{ij}}{2} S_{i} S_{j}^{*} + \sum_{i=1}^{n} \frac{p_{i}}{2} S_{i}^{*} + \sum_{i=1}^{n} \frac{q_{i}}{2} S_{i} + \frac{d}{2} (I - \sum_{i=1}^{n} S_{i} S_{i}^{*}) + m I = 0.$$ 

Suppose $p_{k} \neq 0$ for some $1 \leq k \leq n$. Notice that $S_{i}^{*} S_{i} = I$ and $S_{i}^{*} S_{j} = 0$ if $i \neq j$, we can check that

$$S_{k}^{*} \phi(A) S_{k} = \frac{a_{kk}}{2} I + \frac{p_{k}}{2} S_{k}^{*} + \frac{q_{k}}{2} S_{k} + \frac{m}{2} I = 0.$$ 

This means

$$S_{k}^{*} = -\frac{a_{kk} + m}{p_{k}} I - \frac{q_{k}}{p_{k}} S_{k}.$$ 

Hence, we have $S_{k} S_{k}^{*} = S_{k}^{*} S_{k} = I$, which is a contradiction. So $b_{i} = 0$ for every $1 \leq i \leq n$. Consider $\phi(A)^{*} = 0$ and use the same technique, we also see that $c_{i} = 0$ for every $1 \leq i \leq n$.

Therefore, $\phi(A) = \sum_{i,j=1}^{n} \frac{a_{ij}}{2} S_{i} S_{j}^{*} + \frac{d}{2} (I - \sum_{i=1}^{n} S_{i} S_{i}^{*}) + \frac{m}{2} I = 0$. For $i \neq j$, $S_{i}^{*} \phi(A) S_{j} = 0$ implies that $a_{ij} = 0$. Then, for each $1 \leq i \leq n$, $S_{i}^{*} \phi(A) S_{i} = 0$ implies $a_{ii} I + m I = 0$, and hence $a_{ii} = -m$.

Now, we have seen that $\phi(A) = -m \sum_{i=1}^{n} S_{i} S_{i}^{*} + \frac{d}{2} (I - \sum_{i=1}^{n} S_{i} S_{i}^{*}) + \frac{m}{2} I = 0$. In order for this equality to hold, we must have $d = -m$. Thus, we conclude that $A = -m \sum_{i=1}^{n+1} E_{i+1,i+1} + \sum_{i=1}^{n+1} b_{i} E_{i0} + \sum_{i=1}^{n+1} c_{i} E_{i0} - m E_{00} + m E_{11}$. This means $A \in \text{span}\{E_{00} + \sum_{i=1}^{n+1} E_{ii} - E_{11}, E_{0i}, E_{i0}; 1 \leq i \leq n + 1\}$ and the lemma is proved.

Let $\psi = \phi|_{\mathcal{E}'}$, then $\psi : \mathcal{E}' \to B(\mathcal{H})$ is unitally completely positive. Also, by the lemma we just proved, we can see that $\ker \psi = \text{span}\{E_{00} + \sum_{i=2}^{n+1} E_{ii} - E_{11}\}$. Let $J' = \ker \psi$. Then we can form the quotient operator system $\mathcal{E}'/J'$ and obtain the unitally completely positive map $\tilde{\psi} : \mathcal{E}'/J' \to S'$, which is one-to-one and onto.

In the next section, we will prove that $\tilde{\psi}$ is a complete order isomorphism. To this end, we need to show that $\tilde{\psi}^{-1}$ is completely positive.

6. Complete Order Isomorphism Between $\mathcal{E}'/J'$ and $S'$

We let $I_{n+2}$ denote the identity matrix in $M_{n+2}$ and represent $T_{i} := 2 E_{i+1,i+1} + J'$, $I_{C} := I_{n+2} + J' \in \mathcal{C}(\mathcal{E}'/J')$, where $\mathcal{C}(\mathcal{E}'/J')$ is the $\mathcal{C}^{*}$-envelope of $\mathcal{E}'/J'$, then we have the following proposition.

**Proposition 6.1.** $(T_{1}, \ldots, T_{n})$ is a row contraction, i.e. $\sum_{i=1}^{n} T_{i}^{*} T_{i} \leq I_{C}$.

**Proof.** Equivalently, we show the following,

$$ \begin{pmatrix} I_{C} & T_{1} & \cdots & T_{n} \\ T_{1}^{*} & I_{C} \\ \vdots & \ddots & \ddots \\ T_{n}^{*} & \cdots & I_{C} \end{pmatrix} \geq 0.$$
Notice that $E_{00} + E_{11} + \sum_{i=2}^{n+1} E_{ii} = I_{n+2}$ and $E_{00} + \sum_{i=2}^{n+1} E_{ii} + J' = E_{11} + J'$, so

$$E_{00} + \sum_{i=2}^{n+1} E_{ii} + J' = E_{11} + J' = \frac{1}{2} I_{n+2} + J'.$$

Since the quotient map $q : \mathcal{E}' \to \mathcal{E}'/J'$ is completely positive, we just need to show

$$\begin{pmatrix}
2E_{00} + 2 \sum_{i=2}^{n+1} E_{ii} & 2E_{21} & \cdots & 2E_{n+1,1} \\
2E_{12} & 2E_{11} & & \\
& \ddots & \ddots & \\
2E_{1, n+1} & & 2E_{11}
\end{pmatrix} \geq 0.$$

We can drop the coefficient 2 and write this block matrix as a sum of $n$ block matrices

$$\begin{pmatrix}
E_{00} + \sum_{i=2}^{n+1} E_{ii} & E_{21} & \cdots & E_{n+1,1} \\
E_{12} & E_{11} & & \\
& \ddots & \ddots & \\
E_{1, n+1} & & E_{11}
\end{pmatrix} = \begin{pmatrix}
E_{00} & 0 & \cdots & 0 \\
0 & 0 & & \\
& \ddots & \ddots & \\
0 & & 0 & 0
\end{pmatrix} + \begin{pmatrix}
E_{n+1, n+1} & 0 & \cdots & 0 \\
0 & \ddots & & \\
& \ddots & \ddots & \\
0 & & 0 & E_{11}
\end{pmatrix}$$

$$+ \begin{pmatrix}
E_{n, n} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
& \ddots & \ddots & \\
0 & & 0 & E_{11}
\end{pmatrix} + \cdots$$

It is easy to see that each summand on the right is positive, so the block matrix on the right is positive and the conclusion follows. \hfill \Box

Lemma 3.3 again implies the existence of $n$ isometries $\tilde{T}_1, \ldots, \tilde{T}_n$ with $\sum_{i=1}^{n} \tilde{T}_i \tilde{T}_i^* = \tilde{I}$ which dilate $T_1, \ldots, T_n$.

Let $\mathcal{T}$ be the operator system generated by $\tilde{T}_1, \ldots, \tilde{T}_n$, then the map $\theta : \mathcal{T} \to \mathcal{E}'/J'$ defined by

$$\theta(\tilde{T}_i) = 2E_{i+1,1} + J', \quad \theta(I) = I_{n+2} + J'$$

is unitally completely positive, since $T_i := 2E_{i+1,1} + J'$ is the compression of $\tilde{T}_i$ for each $1 \leq i \leq n$.  

The universal property of $\mathcal{TO}_n$ guarantees that there exists a surjective $*$-homomorphism $\pi: \mathcal{TO}_n \to C^*((\tilde{T}_1, \ldots, \tilde{T}_n))$ such that $\pi(S_i) = \tilde{T}_i$ for $1 \leq i \leq n$. Define $\tilde{\pi} := \pi|_{S'}$, we know that $\tilde{\pi}$ is unitally completely positive from $S'$ to $\mathcal{T}$ and $\tilde{\pi}(S_i) = \tilde{T}_i$.

Now, we consider the map $\tilde{\pi} \circ \theta: S' \to \mathcal{E}'/J'$. From the last two paragraphs, it is easy to check that $\tilde{\pi} \circ \theta$ is unitally completely positive and moreover

$$\tilde{\pi} \circ \theta(S_i) = 2E_{i+1,1} + J'.$$

Hence, $\tilde{\pi} \circ \theta = \tilde{\psi}^{-1}$. This means $\tilde{\psi}: \mathcal{E}'/J' \to S'$ is a unital complete order isomorphism.

Using Proposition 3.5, it is easy to prove the following lemma.

Lemma 6.2. The kernel $J'$ is completely order proximinal.

Proof. Notice that by definition the second diagonal entry of a nonzero element in $J'$ has to be opposite to the others, so $J'$ has no nonzero positive elements. \qed

7. Characterization of Positive Elements in $M_p(S')$

Since we have proved in the last section that $S'$ and $\mathcal{E}'/J'$ are completely order isomorphic, it suffices to characterize positive elements in $M_p(\mathcal{E}'/J')$ for $p \in \mathbb{N}$.

Note that $M_p(\mathcal{E}'/J') \cong M_p \otimes (\mathcal{E}'/J')$ and every positive element in $M_p \otimes (\mathcal{E}/J)$ is of the following form:

$$A_0 \otimes I + \sum_{i=1}^{n} A_i \otimes T_i + \sum_{i=1}^{n} A_i^* \otimes T_i^*,$$

where $A_i$'s are matrices in $M_n$.

Use the same argument as in the proof of proposition 4.1, it is easy to derive the following proposition, so we omit the proof.

Proposition 7.1. We have that $A_0 \otimes I + \sum_{i=1}^{n} A_i \otimes T_i + \sum_{i=1}^{n} A_i^* \otimes T_i^* \in M_p^+(\mathcal{E}'/J')$ if and only if there exists $B \in M_p$ such that

$$
\begin{pmatrix}
A_0 & 0 & \cdots & \cdots & 0 \\
0 & A_0 & 2A_1^* & \cdots & 2A_n^* \\
\vdots & 2A_1 & A_0 & \ddots & \\
\vdots & \vdots & \ddots & \ddots & \\
0 & 2A_n & \cdots & A_0 & \\
\end{pmatrix}
+ \begin{pmatrix}
-B & 0 & \cdots & \cdots & 0 \\
0 & B & 0 & \cdots & 0 \\
\vdots & 0 & -B & \ddots & \\
\vdots & \vdots & \ddots & \ddots & \\
0 & 0 & \cdots & -B & \\
\end{pmatrix} \succeq 0.
$$

Now, we are ready to state the following theorem, which gives a necessary and sufficient condition for an element in $M_p(S')$ to be positive, for every $p \in \mathbb{N}$. Notice that the complete order isomorphism $\tilde{\psi}: \mathcal{E}'/J' \to S'$ sends $T_i$ to $S_i$ for each $i$. We write every positive element in $M_p \otimes S'$ as the following form:

$$A_0 \otimes I + \sum_{i=1}^{n} A_i \otimes S_i + \sum_{i=1}^{n} A_i^* \otimes S_i^*,$$

where $A_i$'s are matrices in $M_p$. 
Theorem 7.2. The matrix of operators $A_0 \otimes I + \sum_{i=1}^{n} A_i \otimes S_i + \sum_{i=1}^{n} A_i^* \otimes S_i^* \in M_p^+(S')$ if and only if there exists $B \in M_p$ such that

$$
\begin{pmatrix}
A_0 & 0 & \cdots & \cdots & 0 \\
0 & A_0 & 2A_1^* & \cdots & 2A_n^* \\
\vdots & 2A_1 & A_0 & \ddots & \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 2A_n & \cdots & \cdots & A_0
\end{pmatrix}
+ 
\begin{pmatrix}
-B & 0 & \cdots & \cdots & 0 \\
0 & B & 0 & \cdots & 0 \\
\vdots & 0 & -B & \ddots & \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & -B
\end{pmatrix} 
geq 0.
$$

Remark 7.3. In the discussions in section 2, 3, and 4, we assume that $n \geq 2$, because for $n = 1$, the $C^*$-algebra generated by a unitary may not be isomorphic to another $C^*$-algebra generated by another unitary. However, in this case, we can consider the Toeplitz algebra $T$, which is the universal $C^*$-algebra generated by a unitary $U$ such that if $\mathcal{R}$ is a $C^*$-algebra generated by a unitary $V$, then there exists a surjective $\ast$-homomorphism $\pi : T \to \mathcal{R}$ with $\pi(U) = V$.

If we let $S$ be the operator system generated by this universal unitary $U$ and mimic the proofs in section 2, 3, and 4, then it is not hard to derive the same necessary and sufficient condition as the one we obtained at the end of section 4.

On the other hand, we notice that

$$
\begin{pmatrix}
A_0 & 0 & \cdots & \cdots & 0 \\
0 & A_0 & 2A_1^* & \cdots & 2A_n^* \\
\vdots & 2A_1 & A_0 & \ddots & \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 2A_n & \cdots & \cdots & A_0
\end{pmatrix}
+ 
\begin{pmatrix}
-B & 0 & \cdots & \cdots & 0 \\
0 & B & 0 & \cdots & 0 \\
\vdots & 0 & -B & \ddots & \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & -B
\end{pmatrix} 
geq 0
$$

if and only if

$$
\begin{pmatrix}
A_0 & 2A_1^* & \cdots & 2A_n^* \\
2A_1 & A_0 & \ddots & \\
\vdots & \ddots & \ddots & \vdots \\
2A_n & \cdots & \cdots & A_0
\end{pmatrix}
+ 
\begin{pmatrix}
B \\
-B \\
\ddots \\
-B
\end{pmatrix} 
geq 0
$$

Thus, we can combine the Theorem 4.2 and 7.2 into the following theorem.

Theorem 7.4. Let $S$ be the operator system generated by $n$ isometries $S_1, \ldots, S_n$ with Toeplitz-Cuntz relation. If $n = 1$, we assume that $S$ is the operator system generated by a universal unitary $S_1$ or an isometry $S_1$ with $S_1S_1^* < I$. We write every positive element in $M_p \otimes S$ as the following form:

$$
A_0 \otimes I + \sum_{i=1}^{n} A_i \otimes S_i + \sum_{i=1}^{n} A_i^* \otimes S_i^*.
$$
where $A_i$’s are matrices in $M_p$. Then $A_0 \otimes I + \sum_{i=1}^n A_i \otimes S_i + \sum_{i=1}^n A_i^* \otimes S_i^*$ is positive if and only if
\[
\begin{pmatrix}
A_0 & 2A_1^* & \cdots & 2A_n^* \\
2A_1 & A_0 & & \\
\vdots & & \ddots & \\
2A_n & & & A_0
\end{pmatrix} + \begin{pmatrix}
B \\
-B \end{pmatrix}
\]
is positive in $M_{n+1}(M_p)$.

8. Some Further Questions

**Question 8.1.** It can be shown that if $S$ and $T$ are operator systems generated by $n$ ($n \geq 2$) isometries with Cuntz relation, then $S$ is canonically completely order isomorphic to $T$ if and only if $C^*(S)$ is $*$-isomorphic to $C^*(T)$. In fact, $C^*(S)$ is precisely the $C^*$-algebra generated by the $n$ isometries, and so is $C^*(T)$. Hence, a question arises: by using Theorem 7.4 can we prove that the operator system generated by $n$ isometries with Cuntz relation is unique so that we obtain a second proof of the uniqueness of the Cuntz algebra?

**Question 8.2.** The Cuntz $O_A$-algebra is the $C^*$-algebra generated by $n$ partial isometries $S_i$ satisfying
\[
\sum_{i=1}^n S_i S_i^* = I, \quad S_i^* S_i = \sum_{i=1}^n A(i,j) S_j S_j^*, \quad A(i,j) = 0, 1.
\]
A. Huef and I. Raeburn proved in [1] that for every choice of such matrix $A$, there always exists a universal $C^*$-algebra $AO_A$. So can we characterize positive elements in the operator system generated by the $n$ universal partial isometries which generate $AO_A$?

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