Critical evaluation of the neoclassical model for the equilibrium electrostatic field in a tokamak

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Abstract

The neoclassical prescription to use an equation of motion to determine the electrostatic field within a tokamak plasma is fraught with difficulties. Herein, we examine two popular expressions for the equilibrium electrostatic field so determined and show that one fails to withstand a formal scrutiny thereof and the other makes a prediction for the electron temperature profile which does not compare well to that commonly seen in a tokamak discharge. Reconsideration of the justification for the presence of the equilibrium electrostatic field indicates that no field is needed for a neutral plasma when considering the net bound current defined as the curl of the magnetization. With any shift in the toroidal magnetic flux distribution, a dynamic electric field is generated with both radial and poloidal components.

1 Introduction

The neoclassical prescription to use an equation of motion to determine the equilibrium electrostatic field within a tokamak plasma, rather than Gauss’s law or Poisson’s equation [1 2 3], is fraught with difficulties alleviated by insisting on the neutral fluid limit rather than the quasineutral approximation [4 5]. Herein, we examine two popular expressions for the electrostatic
field so determined, the first established from electron momentum conservation and the second from the Pfirsch-Schlüter current, and show that one fails to withstand a formal scrutiny thereof and the other makes a prediction for the electron temperature profile which does not compare well to that commonly seen in a tokamak discharge. Reconsideration of the justification for the presence of the equilibrium electrostatic field indicates that no field is needed when all bound currents are considered and that the field must either vanish or result from sources for a model based upon the physics of a neutral, conducting fluid. Shifting of the toroidal magnetic flux density results in a circulating electric field with both radial and poloidal components.

At issue is the validity of the quasineutral approximation, which allows for a divergenceful electric field in the absence of a non-vanishing space charge density \( \rho_e \equiv \sum_s n_s e_s \), formally expressed as \( \nabla \cdot E \neq 0 \) for \( \rho_e = 0 \), and requires the determination of the electrostatic potential from an equation of motion. However, the quasineutral approximation does not respect the mathematics of electrodynamic field theory,

\[
\nabla \cdot E = \rho_e / \varepsilon_0 .
\]

From a particle physicist’s field-theoretic perspective \[6\,7\,9\,10\], the gauge invariant Maxwell field tensor \( F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \) is known to carry only 3 scalar degrees of freedom in media, not 3 for each of the electric and magnetic fields, embodied by the four-potential \( A^\mu \equiv (\Phi/c, A) \) subject to the gauge condition and coupled to sources given by the conserved four-current \( J^\mu \equiv (c\rho_e, J) \) through the inhomogeneous Maxwell equations \( \partial_{\nu} F^{\mu\nu} = \mu_0 J^\mu \), and the homogeneous equations are recognized as the Bianchi identity for electromagnetism given by the field equation for the dual tensor \( \partial_{\nu} \tilde{F}^{\mu\nu} = 0 \) and are satisfied identically when written in terms of the electromagnetic potential hence do not determine any degrees of freedom, thus the electrostatic field is determined by the space charge density \( \rho_e \) and not by an equation of motion. The equations by Maxwell may be expressed succinctly using intrinsic, geometric notation as \( \mathbf{d}^* \mathbf{d} A = J \) in terms of the exterior derivative \( \mathbf{d} \), the Hodge dual \( ^* \), the connection 1-form \( A \), and the current 3-form \( J \), as given in many standard quantum field theory texts, such as Ryder \[6\], or more esoteric monographs, such as Davis \[7\]. What this shows is that Gauss’s law may not be isolated from the remainder of the source bearing Maxwell field equations, \( \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \partial \mathbf{E} / \partial t = \mu_0 \mathbf{J} \), as they are but one unit of truth.
Note that we are not criticising the kinetic approach to plasma calculations, which in its original inception as the Vlasov-Maxwell system [11] fully respects the microscopic electrodynamic field theory, but rather the neoclassical (non-classical) fluid model based on the quasineutral approximation, which does not respect the macroscopic electrodynamic field theory. While various approximations are made in the drift kinetic equations [12, 13], most practical numerical evaluations address Poisson’s equation directly [14]. With this manuscript, we examine in detail the mathematical difficulties one encounters when following the neoclassical prescription. Similar discussions on the existence of the whistler oscilliton in geophysical plasmas [15, 16, 17, 18] and of the helicon wave in propulsion devices [19, 20] are noted. Criticism of the quasineutral approach in a cosmological context [21, 22] has also recently appeared, and it is time for the fusion community to address these difficulties head on.

2 Electrostatic field from the species equations of motion

Some geometric nomenclature and vector identities are given in the Appendix. The first equilibrium $\partial / \partial t \to 0$ electrostatic field under consideration is one commonly used in the analysis of tokamak experiments [23, 24, 25, 26], determined by integration of the electron poloidal and ion radial equations of motion in the large aspect ratio, concentric circular flux surface approximation. The use of concentric circular flux surfaces with a toroidal integrating measure is pursued herein to remain consistent with the model as presented in the literature, as are the expansions of the electron density $n_e = n^0_e(r) \left[1 + n^c_e(r) \cos \theta + n^s_e(r) \sin \theta \right]$ and electrostatic potential. Reference [23] states that “the electron momentum balance can be solved for $\tilde{\Phi}^{c,s} \equiv \Phi^{c,s} / \varepsilon = n^{c,s}_e / \varepsilon (e \Phi^0 / T_e)$, which represents the poloidal asymmetry in the electrostatic potential.” Let us examine that statement in detail.

This model [23, 24, 25] writes the equilibrium poloidal equation of motion for arbitrary species $s$ as

$$[n_s m_s (V_s \cdot \nabla) \nabla + \nabla \cdot \Pi_s ] \cdot \hat{\theta} + \partial p_s / r \partial \theta - F_s \theta + n_s e_s (V_s r B_\phi - E_\theta) = 0 ,$$

where $p_s = n_s T_s$ for $T_s \leftarrow k_B T_s$ and $F_s$ is the friction term, and takes the poloidal component of the electrostatic field on a flux surface at $r$ in Coulomb
gauge as

\[
E_\theta \equiv -\frac{\partial \Phi(r, \theta)}{r \partial \theta} = -\frac{\partial}{r \partial \theta} \Phi^0(r) \left[ 1 + \Phi^c(r) \cos \theta + \Phi^s(r) \sin \theta \right]
\]

\[
\equiv -\frac{\Phi^0(r)}{r} \left[ \Phi^s(r) \cos \theta - \Phi^c(r) \sin \theta \right],
\]

where \( \Phi \) is the electrostatic potential, indicating an expansion around \( \Phi^0(r) \equiv -\int_a^r dr E^0_r \neq 0 \) for a last closed flux surface at \( r = a \), where the radial electrostatic field is calculated from an ion equation of motion \[26\]. The resulting evaluation of the flux surface unity, cosine, and sine moments of the electron poloidal equation of motion with \( \partial T_e/\partial \theta = 0 \) (where other terms are assumed negligible at equilibrium),

\[
T_e \partial n_e/\partial \theta = -e n_e r E_\theta,
\]

defined by the expressions \( \langle A \rangle_{\{U,C,S\}} \equiv \oint d\theta \{1, \cos \theta, \sin \theta \} (1 + \varepsilon \cos \theta) A/2\pi \), yields three equations which have only trivial solution. Specifically, we have the system of equations

\[
U : \quad \varepsilon n^s_e T_e = e\Phi^0(\varepsilon \Phi^s + n^c_e \Phi^s - n^s_e \Phi^c),
\]

\[
C : \quad n^s_e T_e = e\Phi^0(4\Phi^s + 3\varepsilon n^c_e \Phi^s - \varepsilon n^s_e \Phi^c)/4,
\]

\[
S : \quad n^c_e T_e = e\Phi^0(4\Phi^c + \varepsilon n^c_e \Phi^c - \varepsilon n^s_e \Phi^s)/4,
\]

valid \( \forall \varepsilon, n^c_e, n^s_e, \) and \( T_e \). The terms with factors of \( \varepsilon \equiv r/R_0 \) above are strictly due to the toroidal geometry and would disappear for a cylindrical plasma column \( R_0 \to \infty \); one cannot address the extension to toroidal geometry of other aspects of the model \[27\] without addressing the extension here. Solution in pairs given finite (fixed) \( \Phi^0 \) yields inconsistent values of \( \Phi^c,s \) and an overdetetermined system, which therefor has no solution, thus the poloidal electrostatic field in this neoclassical model, which fails to consider the \( O(\varepsilon) \) terms within the cosine and sine moment equations, is unphysical. Failing to include the \( O(\varepsilon) \) terms indicates expressions applicable only on the magnetic axis \( r = 0 \), where \( \varepsilon = 0 \) for \( R_0 \neq \infty \), yet this neoclassical model is commonly used to address the physics near the edge of the confinement region \[23\]. This system may be put into linear, homogeneous form \( Ax = 0 \) by dividing through by \( \Phi^0 \),

\[
\begin{bmatrix}
-\varepsilon n^s_e T_e/e & -n^s_e & \varepsilon + n^c_e \\
-4n^s_e T_e/e & -\varepsilon n^s_e & 4 + 3\varepsilon n^c_e \\
-4n^c_e T_e/e & 4 + \varepsilon n^c_e & -\varepsilon n^s_e
\end{bmatrix}
\begin{bmatrix}
1/\Phi^0 \\
\Phi^c \\
\Phi^s
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]

(9)
thus its only exact solution for $\varepsilon \neq 0$ is trivial $(1/\Phi^0, \Phi^c, \Phi^s) \equiv (0, 0, 0)$, which we interpret to mean exactly what it says, that it is solved when $\Phi^0(r) = \pm \infty$, displaying its unphysical definition when determined from the electron poloidal equation of motion. The matrix has rank 3, thus the solution has 0 free parameters [28]. For a cylindrical column $\varepsilon \to 0$, one recovers a matrix of rank 2 and the solution $\Phi^{c,s} = n_{e}^{c,s}(T_e/e\Phi^0)$ with $\Phi^0$ a free parameter. If one assumes that the smallness of the coefficients may be represented by $\varepsilon$, e.g. $n_{e}^{c,s} \equiv \varepsilon n_{e}^{c,s}$, then the offending terms near the edge where $\varepsilon$ approaches 1/2 represent up to a 19% correction to the leading order equations. As non-vanishing $\Phi^{c,s}$ are an integral part of the development of this neoclassical model and appear in its remaining equations without a prefactor of $\Phi^0$ through substitution, the validity of its conclusions is in jeopardy.

Note that a putative non-vanishing radial electrostatic field without poloidal variation demands the existence of no poloidal electrostatic field, else the poloidal variation to the potential ruins the poloidal symmetry of the radial field; if that radial field is determined from a radial equation of motion then the associated poloidal field is determined by the poloidal dependence of that equation.

One might think to alleviate the difficulty by invoking the logarithmic derivative, writing the electron poloidal equation of motion as $\partial (\ln n_e - e\Phi/T_e)/\partial \theta = 0$, with solution $C(r) = n_e(r, \theta) \exp[-e\Phi(r, \theta)/T_e(r)]$. Expanding the exponential gives, for $-\infty < e\Phi/T_e < \infty$,

$$ C(r) = n_e \sum_{k=0}^{\infty} \frac{(e\Phi/T_e)^k}{k!} \approx n_e \left[ 1 - \frac{e\Phi}{T_e} + \frac{1}{2} \left( \frac{e\Phi}{T_e} \right)^2 - \frac{1}{6} \left( \frac{e\Phi}{T_e} \right)^3 + \ldots \right], $$

(10)

and to each order in $e\Phi/T_e$, taking the flux surface moments yields three equations to solve. Using the expansions for $n_e$ and $\Phi$ above, the zeroth order equations give $C(r) = n_e^0$ and $n_{e}^{c,s} = 0$, and the first order equations yield $\varepsilon$-dependent solutions which, upon restriction to the magnetic axis $\varepsilon = 0$, are $C(r) \to n_e^0[1 - (n_e^0)^2/2 - (n_e^s)^2/2](1 - e\Phi^0/T_e)$ and $\Phi^{c,s} \to n_e^{c,s}(T_e/e\Phi^0 - 1)$. Agreement is found with the previous $\Phi^{c,s} = n_{e}^{c,s}(T_e/e\Phi^0)$ provided $e\Phi^0/T_e \ll 1$. Considering now a ratio $e\Phi^0/T_e \sim 10 > 1$, such as may be found in the plasma core [24, 29, 30], the expansion in Equation (10) requires 10 orders before the absolute magnitude of the terms starts decreasing, indicating that a low-order approximation is not an exact solution. Considering the second order equations leads to solutions for $C(r)$ and $\Phi^{c,s}$ only when $\varepsilon = 0$ which diverge when $n_{e}^{c,s} = 0$, and no solution has yet been found to the nonlin-
ear equations for general $\varepsilon$. The difficulty with $\Phi^0$ encountered above has simply been shifted to the “undetermined” function $C(r)$, which is perfectly determinable in principle from the system of equations. We remark that, assuming the model’s system to be well-determined (ie equal numbers of independent equations and degrees of freedom) before invoking the logarithmic derivative, the introduction of $C(r)$ without an additional equation for its determination leaves the system under-determined—if $C(r)$ is identified as an independent degree of freedom, then it needs its own independent equation, else if not, then one of those degrees of freedom must be determined by the unity moment equation here. Pursuing the argument, as the electron density $n_0^e$ divides out of Equation (5), its poloidal variations $n_{c,s}^e$ are determined by continuity $\langle \partial n_e/\partial t + \nabla \cdot n_e V_e - \dot{n}\rangle_{C,S} = 0$ for particle source rate $\dot{n}$, and the thermal energy $T_e$ is determined by the heat equation and given as input from experimental measurement for the analysis, the remaining degree of freedom $\Phi^0$ must be determined by $C(r)$ as given by the poloidal force balance system, Equation (9).

Notwithstanding the above, let us now consider an alternate expansion of the exponential which extracts the $\theta$ dependence using $\xi \equiv \Phi^c \cos \theta + \Phi^s \sin \theta$, so that

$$C(r) = n_e \exp(-\varepsilon \Phi^0 / T_e) \left( \exp \xi \right)^{-\varepsilon \Phi^0 / T_e}, \quad (11)$$

$$\approx n_e \exp(-\varepsilon \Phi^0 / T_e) \left( 1 + \xi + \frac{\xi^2}{2} + \ldots \right)^{-\varepsilon \Phi^0 / T_e}, \quad (12)$$

$$\approx n_0^e \exp(-\varepsilon \Phi^0 / T_e) \left( 1 + n_{c}^e \cos \theta + n_{s}^e \sin \theta \right) \times$$

$$\left[ 1 - \frac{\varepsilon \Phi^0}{T_e} \xi + \frac{1}{2} \left( \frac{\varepsilon \Phi^0}{T_e} \right)^2 \xi^2 + \ldots \right], \quad (13)$$

noting the use of low-order truncations to the two infinite series (which can never be as exact as the algebraic method above) and that technically the expansion found in Equation (10) is more democratic in its treatment of the powers of $\varepsilon \Phi^0 / T_e$. Taking the flux surface moments as before, we find nearly the same first order solutions, whereupon reduction to the magnetic axis (or cylindrical geometry) yields $C(r) \rightarrow n_0^e \exp(-\varepsilon \Phi^0 / T_e) \left[ 1 - (n_c^e)^2 / 2 - (n_s^e)^2 / 2 \right]$ and $\Phi^c,s \rightarrow n_{c,s}^e (T_e / \varepsilon \Phi^0)$, and for the second order equations the solution on the magnetic axis diverges and for general $\varepsilon$ cannot be found. Inserting factors of $\varepsilon$ at the cost of the loss of the cylindrical interpretation of $\varepsilon \rightarrow 0$, which still identifies the magnetic axis in toroidal geometry, yields
Equation (11) in terms of the tilded coefficients (which need not be small),

\[
C(r) \approx n_0 e^{\exp\left(-e\Phi_0^/Te\right)} \left[1 + \varepsilon(\tilde{\Phi}_c \cos \theta + \tilde{\Phi}_s \sin \theta)\right] \times \\
\left[1 - \frac{e\Phi_0}{Te} \varepsilon(\tilde{\Phi}_c \cos \theta + \tilde{\Phi}_s \sin \theta) + \frac{1}{2}\left(\frac{e\Phi_0}{Te}\right)^2 \varepsilon^2(\tilde{\Phi}_c \cos \theta + \tilde{\Phi}_s \sin \theta)^2\right], 
\]

(15)

and affords additional control over the terms retained in the expansion. The resulting system is of the form 0 = \(U_0 + U_2 \varepsilon^2 + U_4 \varepsilon^4 = C_1 \varepsilon + C_3 \varepsilon^3 = S_1 \varepsilon + S_3 \varepsilon^3\) and has not solution until reduced to order \(O(\varepsilon^2)\) which, as a factor of \(\varepsilon\) cancels out of the \(\langle \ldots \rangle_{C,S}\) moments, is the first appearance of an explicit \(\varepsilon\) dependence in the equations. Its expression for \(\tilde{\Phi}_c\) contains terms with \(1/\varepsilon^2\), thus the physical \(\Phi_c \rightarrow \infty\) when \(\varepsilon = 0\). Only when each equation is reduced to its leading term does the system return the solutions \(C(r) = n_0 e^{\exp(-e\Phi_0^/Te)}\) and \(\Phi_{c,s} = n_0^{c,s}(Te/e\Phi_0)\). Comparing the leading coefficients in Equation (16) gives the ratios 1 : 10 \(\varepsilon : 55\varepsilon^2\) for \(e\Phi_0^/Te \sim 10\), and allowing for \(e\Phi_0^/Te \sim 1\) yields 1 : \(\varepsilon : \varepsilon^2\) which still represents up to a 25% correction. The lesson here is that knowledge of a series’ convergence need not imply that a low-order approximation is an adequate representation of the function. Concern over the expansion in no way detracts from the observation that the physics of the situation is embodied by the algebraic Equations (6) which contain no exponential factor thus represent a more exact method of solution.

Finally, we consider the consistency of using two different species’ equations of motion to determine the species independent electrostatic potential, taking the radial electrostatic field from the equation of motion for arbitrary ion species \(j\) as

\[
E_r = \frac{1}{n_j e_j} \frac{\partial p_j}{\partial r} + V_{\theta j} B_\theta - V_{\phi j} B_\phi ,
\]

(17)

and dropping the convective term as is standard practice in the field \[24, 25, 26\]. The usual evaluation of that expression from experimental measurements neglects any poloidal dependence; however, when the intrinsic variation of quantities induced by the geometry is considered \[31\], given in the large aspect ratio \(\varepsilon \ll 1\), concentric circular \(R_r \equiv R_0\), flux surface approximation \(B = (0, B_\theta, B_\phi)\) by

\[
B = B^0/(1+\varepsilon \cos \theta) , \hspace{1cm} V_{\theta j} = V_{\theta j}^0/(1+\varepsilon \cos \theta) , \hspace{1cm} V_{\phi j} = V_{\phi j}^0(1+\varepsilon \cos \theta) , \hspace{1cm} n_j = n_j^0 , \hspace{1cm}
\]

(18)

where \(A^0\) is the average of the values of \(A\) on the vertical midplane, a geometric dependence is introduced. Using these values (\(r\) dependence implied),
we find
\[ E_r(r, \theta) = \frac{p_j^0}{n_j^0 e_j} + V_{\phi j}^0 B_{\phi}^0 - V_{\theta j}^0 B_{\phi}^0/(1 + \varepsilon \cos \theta)^2 , \]  
(19)

for species pressure gradient \( p_j^0 \equiv \partial n_j / \partial r \), thus \( E_r(\theta) \neq E_r^0 \). Expanding the denominator reveals a power series in \( \varepsilon \cos \theta \),
\[ E_r(r, \theta) \approx \frac{p_j^0}{n_j^0 e_j} + V_{\phi j}^0 B_{\phi}^0 - V_{\theta j}^0 B_{\phi}^0 (1 - 2\varepsilon \cos \theta + 3\varepsilon^2 \cos^2 \theta - 4\varepsilon^3 \cos^3 \theta) , \]  
(20)

and as integration with respect to \( r \) does not affect the \( \theta \) dependency, the potential associated with the radial electrostatic field relative to its central value, \( \Phi_{E_r}(r, \theta) \equiv - \int_0^r dr E_r(r, \theta) \), may be written as a cosine series,
\[ \Phi_{E_r}(r, \theta) \approx \Phi_{E_r}^0 + \Phi_{E_r}^1 \cos \theta = \Phi_{E_r}^0 (1 + \Phi_{E_r}^c \cos \theta) , \]  
(23)

where \( \Phi_{E_r}^c = \int_0^r dr E_r^1(r)/\int_0^r dr E_r^0 \neq 0 \) in general, noting that the potential on the last closed flux surface at \( r = a \) is not single valued. Returning now to the electrostatic potential appearing in the poloidal equation of motion, expressed to leading order as
\[ \Phi_{E_\theta}(r, \theta) = \Phi_{E_\theta}^0(r)[1 + \Phi_{E_\theta}^c(r) \cos \theta + \Phi_{E_\theta}^s(r) \sin \theta] , \]  
(24)

where \( \Phi_{E_\theta}^c(r) \equiv - \int_a^r dr E_\theta^0 \neq 0 \) is the potential relative to that of the last closed flux surface, with solution \( \Phi_{E_\theta}^c = n_e^{c,s}(T_e/e\Phi) \). No loss of generality ensues if one redefines the potential relative to its central value. With vanishing extrinsic poloidal dependence to the electron density, \( n_e^{c,s} \rightarrow 0 \) as above, the potential retains no explicit poloidal dependence, \( \Phi_{E_\theta}(r, \theta) \rightarrow \Phi_{E_\theta}^0(r) \). Thus, we conclude that the electrostatic potentials associated with the radial and poloidal electrostatic fields evaluated from the ion and electron equations of motion by this neoclassical model are inconsistent, \( \Phi_{E_r} \neq \Phi_{E_\theta} \), as Equation (23) does not equal Equation (24) in the case of vanishing density asymmetries \( n_{e,j}^{c,s} = 0 \).
3 Equilibrium field from the Ohm’s law equation

3.1 Derivation

Examining the expression of another leading contender for the equilibrium electrostatic field \[3\] evaluated from the Ohm’s law equation and the Pfirsch-Schlüter current, one may put its poloidal component

\[ E_\theta = \left( \frac{E_\phi B_\phi}{B_\theta} \right) B_\theta \rho_{\parallel} \eta_{\parallel} \left( \frac{1}{B_\theta} B_\phi - \frac{1}{B_\phi} \right), \quad (25) \]

into the form

\[ E_\theta = E_\theta^c \left[ 2\varepsilon \cos \theta - \left( \varepsilon^2 / 2 \right) \cos 2\theta \right], \quad (26) \]

when the Shafranov shift is neglected, as in the concentric circular flux surface approximation of above, upon application of Stokes’ theorem to Faraday’s law \[32\], i.e. by requiring \( \oint d\theta E_\theta = 0 \). Note that this neoclassical model for the poloidal electrostatic field differs distinctly from that of the previous section in detailed functional form. Inserting Equation (26) into Equation (5) and taking the flux surface Fourier moments yields three equations which have a nontrivial solution only when expanded to order \( O(\varepsilon^3) \), given by

\[
\begin{bmatrix}
 n_c^e \\
n_s^e \\
 E_\theta^c
\end{bmatrix}
= \begin{bmatrix}
 \frac{\varepsilon^3}{(6\varepsilon^2 - 8)} \\
 \pm \varepsilon \sqrt{3\varepsilon^4 - 168\varepsilon^2 + 192/(18\varepsilon^2 - 24)} \\
 \pm 4(T_e/eR_0)/\varepsilon \sqrt{3\varepsilon^4 - 168\varepsilon^2 + 192}
\end{bmatrix},
\quad (27)
\]

thus the presence of a poloidal electrostatic field of that form should be accompanied by a potentially measurable shift in the electron density profile.

Note that the derivation immediately preceding is slightly inconsistent, as the associated electrostatic potential \( \Phi_{axi} = \Phi_0(2\varepsilon \sin \theta - (\varepsilon^2 / 4) \sin 2\theta) \) is of the correct harmonic form \[33, 34, 35, 36\] for axial geometry, as is easily verified in \( (Z = -r \sin \theta, R = R_0 + r \cos \theta, z) \) coordinates via application of the axial Laplacian \( \nabla_{axi}^2 \equiv \partial^2 / \partial Z^2 + \partial^2 / \partial R^2 \) to \( \Phi_{axi} = \Phi_0 Z[(R - R_0)/2R_0^2 - 2/R_0] \), yet the flux surface average is done in toroidal geometry. Note that this \( \Phi_0 \) is not the \( \Phi^0 \) of the preceding section but is a unit bearing constant which sets the scale.

In order to achieve the correct harmonic form for tokamak geometry, the potential must satisfy the toroidal Laplacian, which in cylindrical coordinates \[32, 37, 38\] is given by \( \nabla_{tor}^2 \equiv \nabla_{axi}^2 + \partial / \partial R \partial R \) (note that the
expression for $\Delta^* \star$ given by Hopcraft [2] is not the toroidal Laplacian and differs by the sign of the additional term), and the form of the geometric term hints at the solution. Direct integration yields the toroidal potential $\Phi_{\text{tor}} \equiv \Phi_{\text{axi}}(R \to \ln R)$, from which $E_Z = -\Phi_0[(\ln R - R_0)/2R_0^2 - 2/R_0]$ and $E_R = -\Phi_0 Z/2R_0^2$, noting that the introduction of the logarithm breaks the usually obvious relation between the symbol for the magnitude of a quantity and the units associated with that quantity—carefully pulling the units beside the leading coefficients of expressions ensures that they are respected. From these, we determine the poloidal field to be

$$E_\theta \equiv -\frac{1}{r} \frac{\partial \Phi}{\partial \theta} = \left( -\frac{\Phi_0}{r} \right) \left( \frac{\partial Z}{\partial \theta} \frac{\partial}{\partial Z} + \frac{\partial R}{\partial \theta} \frac{\partial}{\partial R} \right) \frac{\Phi}{\Phi_0} = E_c^\theta \left[ (R - R_0) \frac{E_Z}{\Phi_0} - \frac{Z E_R}{\Phi_0} \right],$$

(28)

where we identify $E_c^r,\theta \equiv -\Phi_0/r$, and the corresponding radial field is

$$E_r \equiv -\frac{\partial \Phi}{\partial r} = \left( \frac{-\Phi_0}{r} \right) r \left( \frac{\partial Z}{\partial r} \frac{\partial}{\partial Z} + \frac{\partial R}{\partial r} \frac{\partial}{\partial R} \right) \frac{\Phi}{\Phi_0} = E_c^r \left[ -Z E_Z \frac{\Phi}{\Phi_0} - (R - R_0) \frac{E_R}{\Phi_0} \right].$$

(29)

In $(r, \theta, \phi)$ coordinates, we have $\Phi_{\text{tor}} = \Phi_0 r \sin \theta [\ln (R_0 + r \cos \theta) + 5R_0]/2R_0^2$, and

$$E_\theta(r, \theta) = \frac{\Phi_0}{2rR_0^2} \left\{ r \cos \theta \left[ \ln (R_0 + r \cos \theta) - 5R_0 \right] - \frac{r^2 \sin^2 \theta}{R_0 + r \cos \theta} \right\}, \quad (30)$$

$$E_r(r, \theta) = \frac{\Phi_0}{2rR_0^2} \left\{ r \sin \theta \left[ \ln (R_0 + r \cos \theta) - 5R_0 \right] + \frac{r^2 \cos \theta \sin \theta}{R_0 + r \cos \theta} \right\}. \quad (31)$$

As this is an electrostatic field within a neutral medium (ie one for which the net charge on a differential volume element vanishes) at equilibrium, Maxwell’s equations $\nabla \times \mathbf{E} = 0$ and $\nabla \cdot \mathbf{E} = 0$ are satisfied.

As electrostatic fields necessarily require a supporting charge density [6, 32, 39, 89, 90], one may very well ask where these charges are. As the solution to Laplace’s equation is uniquely determined by the boundary condition, we will find them on the boundary of the region under consideration, which in this case is the $R/R_0$ weighted circle representing our outermost flux surface at normalized minor radius $r/a = 1$ upon collapse of the toroidal dimension, giving us an inverse Dirichlet problem, which is usually defined as solving for the potential given the boundary condition. The source of these charges is intended to be the accumulation resulting from the divergence of the pressure gradient driven diamagnetic current in toroidal geometry, and
with $-\partial \rho_e / \partial t = \nabla \cdot \mathbf{J}_{PS} \propto \nabla \cdot \mathbf{E}$, we find the rate of surface charge accumulation $\dot{\sigma}_b \propto -\hat{\mathbf{r}} \cdot \mathbf{E}(r = a)$, noting that quantity represents the charge density on the weighted circle, so to get the physical charge density on a poloidal circle of the toroidal flux surface we take $\dot{\sigma}_b \rightarrow \dot{\sigma}_b R_0 / R$. The Pfirsch-Schlüter current is supposed to flow along the electrostatic field at equilibrium to cancel this charge accumulation.

From the expression for the poloidal field, one may determine a prediction for the associated electron temperature profile by solving the flux surface Fourier moments of the poloidal equation of motion as in the previous section. With electron density $n_e = n_e^0 (1 + n_e^c \cos \theta + n_e^s \sin \theta)$ and temperature $T_e$, we solve the unity, cosine, and sine moments $\langle T_e \partial n_e / r \partial \theta + n_e e E_\theta \rangle_{U,C,S} = 0$ for (the pure numbers) $n_e^c, n_e^s,$ and $E_\theta^c$, finding

$$\begin{bmatrix} n_e^c \\ n_e^s \\ E_\theta^c \end{bmatrix} = \begin{bmatrix} 3\varepsilon^3 / G_2 \\ \pm \sqrt{6}\varepsilon G_1 / 2(F + 2)G_2 \\ \pm 8\sqrt{6}T_e / e\varepsilon G_1 \end{bmatrix}, \quad (32)$$

where $F \equiv 5R_0 - \ln R_0$, in terms of the functions $G_2(r) \equiv [(6F - 1)\varepsilon^2 - 8F]$ and

$$G_1(r) \equiv \sqrt{(F + 2)[\{16F[4F - (1 + 3F)\varepsilon^2]\} + 9\varepsilon^4]}, \quad (33)$$

from which we obtain $\Phi_0 = \mp 8\sqrt{6}R_0 T_e / eG_1$. Note that the quantities appearing in, between, and following Equations (32) and (33) carry no units, as the units cancel out of the system of equations solved. The factor of $R_0$ appearing as a scale factor in $\Phi_0$ has a physical interpretation—for an equivalent supporting charge density and minor radius, as one scales the system by $R_0$, the magnitude of the charge distribution on the weighted circle increases as the major circumference of the torus, thus increasing the magnitude of the potential by a numerical factor of $R_0$. As $\Phi_0$ is constant $\forall r$, the explicit $r \equiv \varepsilon R_0$ dependence of $G_1$ must cancel the implicit $r$ dependence of $T_e(r)$, allowing us to write $T_e(r) / G_1(r) \equiv T_e(0) / G_1(0)$, thus $G_1$ determines the $r$ dependence of the electron temperature as a prediction of the hypothesis that the electrostatic field in a tokamak is of a form similar to that of Equation (26). Note that a similar compatibility relationship obtains from the bottommost of Equations (27) given by the radical factor.

### 3.2 Evaluation

The following presents the results of a numerical evaluation of the expressions in the preceding subsection. The scale is set by $R_0 = 2$ and $T_e(0)/e = 1$, and
Figure 1: The electrostatic potential $\Phi_{\text{tor}}$ for toroidal geometry in arbitrary units seen from multiple views (a)-(d). The magnitude of the associated electrostatic field in arbitrary units seen from multiple views (e)-(h).

$r/a$ is the normalized minor radius. Some useful geometry is presented in the Appendix. The toroidal electrostatic potential $\Phi_{\text{tor}}$ is shown in Figure 1(a)-(d)—note the correct harmonic form of a stretched circular membrane (eg a drum-head put on a hoop given by the boundary condition), consistent with the physics of potential theory [33, 34], and the obvious lack of bumps or poles, consistent with a neutral medium (the term “pole” actually means something physical [32, 34, 35], namely the effect on the potential of an isolated flux-source which ultimately is quantized in units of the charge). We stress that these are multiple views of the same single object and that the boundary is determined by the potential and not vice versa. In Figure 1(e)-(h) we display the magnitude of the associated electrostatic field, and in Figure 2(a)-(d) we display the electrostatic field and the associated supporting charge density.

The normalized electron temperature profile in a tokamak often compares favorably (from unpublished results of the analysis behind Reference [24]) with the formula $T_e(r)/T_e(0) \approx (1 - r^4)^2$, which misses the effect of the pedestal $T_e^{\text{ped}}/T_e(0) \approx 0.2$ but works surprisingly well for $r/a < .8$, and so that
Figure 2: The electrostatic field and associated supporting charge density in arbitrary units seen from multiple views (a)-(d). Comparison (e) of the neoclassical prediction for the normalized electron temperature profile, $G_1$, versus an approximation to that found in tokamak experiments, $(1 - r^4)^2$.

is the assumed profile to which we will make our comparison in Figure 2(e). The normalized predicted profile for the temperature is $G_1(r)/G_1(0)$ and fails to fall sufficiently quickly to match the shape of the expected temperature profile. Thus, while comparison with our approximate formula hardly qualifies as a true experimental test of the hypothesis, the existence of an electrostatic field of form similar to that of Equation (26) is not supported by this analysis, even allowing for a pedestal at 20%. The observed electron temperature at the edge of the confinement region would have to exceed 90% of its central value to be compatible with the model of this section.

Regardless of the preceding argument, relying as it does on an abbreviated form of the electron poloidal equation of motion, the supporting charge density above is reminiscent of the surface charge on a current carrying conductor, a topic of recent interest [40, 41, 42, 43, 44]. Note that these authors rely extensively on Poisson’s equation, ie Gauss’s law. Despite the poloidal appearance of the initial expression, Equation (25), the resultant electric field is primarily vertical and should lead to an $E \times B$ drift force helping to bal-
ance the magnetic pressure force in the major radial direction. Investigations are proceeding along these lines to determine whether the return flux of the fringing field from the finite central solenoid [45] can induce such effect.

The motivation for the existence of the Pfirsch-Schlüter current leading to the poloidal electric field of this section is the cancellation of charge accumulation arising from the pressure gradient driven diamagnetic current \( J_{\nabla p} = -\nabla p \times B / B^2 \) in toroidal geometry [3], \( \nabla_{\text{tor}} \cdot J_{\nabla p} = -\partial \rho / \partial t \neq 0 \) for

\[
\nabla \cdot J_{\nabla p} = \nabla \cdot \left( \frac{B \times \nabla p}{B^2} \right) = \nabla p \cdot (\nabla \times B) - B \cdot (\nabla \times \nabla p) \frac{1}{B^2} + (B \times \nabla p) \cdot \nabla \frac{1}{B^2},
\]

where the final term is nonzero due to the poloidal dependence of \( B \). However, \( J_{\nabla p} \) is but one component of the total diamagnetic current, properly defined as the curl of the magnetization \( J_{\text{dia}} = \nabla \times M \) where \( M = -(p / B^2)B \), which includes the effects of the pressure gradient driven current as well as the curvature and \( \nabla B \) drift currents [46] and remains divergence-free regardless of the geometry \( \nabla \cdot J_{\text{dia}} = 0 \), thus there is no space charge accumulation and no motivation for a cancelling current. (The common procedure of adding the particle drift currents to the fluid diamagnetic current we feel represents an over-counting of the underlying phenomenon, as all currents are the curl of either an \( H \) or an \( M \).) The error here lies in not fully distinguishing the free and bound charges and currents as they appear in the Heaviside notation of the Maxwell equations and in trying to model a fully ionized medium as both a conductor and dielectric at zero frequency.

4 Dynamic electric field

Relaxing our requirement of equilibrium slightly, we consider now the effect of a shift in the magnetic flux density when the total magnetic flux remains constant. As Gauss’s law is inviolate, any electric fields within the neutral, conducting medium of a tokamak plasma necessarily are driven by changes in the magnetic flux density at some location in space, giving us

\[
\nabla \cdot E = 0, \ \nabla \times E = -\partial B / \partial t.
\]

These changes may result for fixed \( H_\phi \) by a shift in the pressure distribution leading to \( \partial M_\phi / \partial t \neq 0 \), as \( B / \mu_0 = (H - M) \hat{\mathbf{h}} \). Noting the similarity to
the laws of Thomson and Ampere allows one to define an electric vector potential $\mathbf{E} = \nabla \times \mathbf{F}$ in the analogue of Coulomb gauge $\nabla \cdot \mathbf{F} = 0$. Then $\nabla^2 \mathbf{F} = \partial \mathbf{B} / \partial t$ has the solution \[45, 47, 48, 49\] for a circular loop source at $(Z_0, R_0)$ given by

$$F_{\phi}/F_{\phi}^0 = \frac{4\sqrt{a + b}}{b} \left[ \left( \frac{a}{a + b} \right) K(k) - E(k) \right], \quad (37)$$

expressed in terms of the complete elliptic integrals \[50\] with parameter $k^2 = 2b/(a + b)$, where $a = (Z - Z_0)^2 + R^2 + R_0^2$ and $b = 2RR_0$ and its magnitude $F_{\phi}^0 = -(\partial B_{\phi} / \partial t)\Delta^2 R_0/4\pi$ is supposed constant over a differential area $\Delta^2$.

On a uniform grid $(Z, R)$ in meters with spacing $\Delta = .01$ m, we wish to evaluate $F_{\phi}$ for sources with opposite polarity around $(\pm Z_0, R_0)$ for $Z_0 = 0.25$ aligned to the vertical midplane at $R_0 = 2$. Expressing the plasma magnetization \[51\] as $M/H = (1 - \sqrt{1 - 4p/\mu_0 H^2})/2$ for $\mathbf{h} = \hat{\phi} \cos \zeta + \hat{\theta} \sin \zeta$ and supposing the pitch angle $\zeta = \varepsilon \pi/2$ for $\varepsilon = Z_0/R_0$ and $\partial \mathbf{H} / \partial t = 0$ allows
\[ \frac{\partial B_\phi}{\partial t} = - (\cos \zeta)(H^2 - 4p/\mu_0)^{-1/2}\frac{\partial p}{\partial t}. \]  

Using some typical parameters \( n_0 = 10^{19} \text{m}^{-3}, T_e = 3 \text{keV}, T_i = 9 \text{keV}, B_\phi = 2 \text{T}, B_\theta = 0.2 \text{T}, \) and supposing the rate of change to the pressure is 1% per millisecond, lets one give a numerical estimate to \( F^0_\phi \) hence the magnitude of the electric field. From Gauss’s law one defines the flux function \( \chi \) such that

\[ \mathcal{K}_\chi \equiv \hat{\phi} \times \nabla \chi = R E = R \nabla \times F, \]  

whence \( \chi = -RF_\phi \) upto an unphysical constant, where \( \mathcal{K} \) is the contour operator \[45\]. The results of this evaluation are shown in Figure 3, where one sees that an electric field with magnitude approaching 1 mV/m for these parameters may arise on the vertical midplane. Pursuing the analogy between \( E \sim B \) and \( J \sim -\partial M/\partial t \), a force of repulsion

\[ F_{++} = -\mu_0\varepsilon_0(\partial M_+ / \partial t) \times E_+ \]  

should appear between line sources of opposite polarity by action of the macroscopic Lorentz force \[51, 52, 53\], where the subscript indicates the source location, thus a perturbation to the pressure does not necessarily coalesce.

5 Conclusions and outlook

From the preceding analysis, we find that the use of an equation of motion to determine the equilibrium electrostatic field leads to inconsistencies, either internally or with expected experimental measurement of the electron temperature profile. Part of the problem lies in treating the poloidal and radial components of the field separately, when there is only one electrostatic potential from which both components may be determined. The remainder lies in the neglect of Gauss’s law popularly established within the plasma physics community \[11, 2, 3, 11, 52, 53\], which relegates the defining relation for the electrostatic field to a position of subsidiary moment. Any model which treats the plasma as a neutral, conducting fluid, where neutrality is understood to hold down to some scale smaller than the differential volume element used to define the continuum quantities, needs to respect all of Maxwell’s laws, which are manifestly Lorentz covariant. Taking
a field theoretic perspective implies that the electrostatic field within a neutral medium must either vanish or result from sources and be supported by dielectric polarization. In conclusion, we determine that we are unsatisfied by either neoclassical model for the equilibrium electrostatic field within a tokamak plasma on the grounds that they do not respect the mathematics of field theory. An alternative description in terms of shifting magnetic flux may possibly account for any direct experimental observations of an electric field within the device [60, 61]. The thrust of our argument is simply that, if there is an electric field within a tokamak, then it must play by the same rules as every other electric field found in nature.

A Appendix

There are (at least) three useful sets of coordinate axes to describe a toroidal magnetic confinement device with concentric circular flux surfaces, namely \((\hat{Z}, \hat{R}, \hat{\phi})\), \((\hat{r}, \hat{\theta}, \hat{\phi})\), and \((\hat{r}, \hat{\perp}, \hat{\parallel})\), and in the infinite aspect ratio limit \((R_0 \to \infty)\) we have the axial coordinate axes \((\hat{Z}, \hat{R}, \hat{z})\) and \((\hat{r}, \hat{\theta}, \hat{z})\). Note that the term “toroidal coordinates” means something very different to a mathematician than those commonly applied to a tokamak, which we call “tokamak coordinates.” For a plasma with coaxial applied electric and magnetic fields and free current driving a circulating field, we note that \((\hat{r}, \hat{\theta}, \hat{z}) = (-E \times B / EB, -E \times (E \times B) / E^2 B, E / E)\) and \((\hat{r}, \hat{\perp}, \hat{\parallel}) = (-E \times B / EB, -B \times (E \times B) / EB^2, B / B)\). Cylindrical coordinate labels \((Z, R, \phi)\) relate to tokamak coordinates \((r, \theta, \phi)\) via \(Z = -r \sin \theta\) and \(R = R_0 + r \cos \theta\) in the concentric circular approximation, where \(\hat{r}, \hat{\theta},\) and \(\hat{\phi}\) give the radial, poloidal, and toroidal directions, respectively. The outermost minor radius of the confined plasma, given in meters by \(a\), defines the normalized minor radius \(r/a\), and \(R_0\) is its centroid. The magnetic field \(B\) and current density \(J\) lie in isobaric surfaces given by \(\nabla p = J \times B\) for a stationary equilibrium, defining the “flux surface” at radius \(r\). In general, the nested flux surfaces are neither circular nor concentric. The relationship between vector components in the tokamak coordinates \((\hat{Z}, \hat{R}, \hat{\phi}) \leftrightarrow (\hat{r}, \hat{\theta}, \hat{\phi}) \leftrightarrow (\hat{r}, \hat{\perp}, \hat{\parallel})\) may be succinctly expressed by

\[
\begin{bmatrix}
F_Z \\
F_R \\
F_\phi
\end{bmatrix} = \begin{bmatrix}
-\sin \theta & -\cos \theta & 0 \\
\cos \theta & -\sin \theta & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & b_{\phi} & b_{\theta} \\
0 & -b_{\theta} & b_{\phi}
\end{bmatrix} \begin{bmatrix}
F_r \\
F_\perp \\
F_\parallel
\end{bmatrix},
\]

(41)
where $\hat{\|} \equiv \mathbf{B}/B \equiv (0, b_\theta, b_\phi)$. Various useful relationships are

$$
Z = -r \sin \theta , \quad \partial Z/\partial r = Z/r , \quad \partial Z/\partial \theta = -(R - R_0) , \\
R - R_0 = r \cos \theta , \quad \partial R/\partial r = (R - R_0)/r , \quad \partial R/\partial \theta = Z , \\
$$

(42)

and for the logarithm, we have $\partial \ln R/\partial R = 1/R$ where the units on the left are carried by the differential operator and the units on the right are carried by the result, which shows that the $\ln R$ is a pure number which carries no units.

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