Generic measures for hyperbolic flows on non compact spaces

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Abstract

We consider the geodesic flow on a complete connected negatively curved manifold. We show that the set of invariant Borel probability measures contains a dense $G_δ$-subset consisting of ergodic measures fully supported on the non-wandering set. We also treat the case of non-positively curved manifolds and provide general tools to deal with hyperbolic systems defined on non-compact spaces.

1 Introduction

In the context of hyperbolic dynamics, Axiom A flows admit many ergodic invariant probability measures supported on each connected component $X$ of the non-wandering set: the measure of maximal entropy and the SRB measure are two examples which provide many informations on the asymptotic behaviour of trajectories. K. Sigmund [Si72] showed even more: the ergodic probability measures with full support on $X$ form a $G_δ$ dense subset of the set $\mathcal{M}^1(X)$ of all probability invariant measures.

Our goal is to extend these results to hyperbolic systems defined on non-compact spaces. We build our study on two remarkable properties of hyperbolic systems: the existence of a product structure and the closing lemma. Neither of these two properties relies on the compactness of the ambient space. Yet they are sufficient to obtain a general density theorem, which shows that ergodic probability measures of full support are indeed abundant.

The most famous example of hyperbolic system with noncompact phase space is given by the geodesic flow on a complete connected negatively curved manifold. Our general density theorem implies the following result:

Theorem 1.1 Let $M$ be a negatively curved, connected, complete riemannian manifold. We assume that the non wandering set $Ω$ of the geodesic flow is non empty. Then the $G_δ$ subset of ergodic measures fully supported on $Ω$ is dense in the set of all probability borel measures on $T^1M$ invariant by the geodesic flow.

As the construction ultimately relies on the Baire category theorem, the measures obtained are not explicit. This result was previously known for geodesic flows on geometrically finite negatively curved manifolds. In that setting, there is an explicit construction of an ergodic invariant probability Gibbs measure [C03].

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On the other hand, the theorem does not hold in the context of non-positive curvature. We give a counterexample, and proceed to show that ergodic probability measures of full support are dense, if we assume that the non-wandering set is everything and there is a hyperbolic closed geodesic.

Other examples of hyperbolic systems on non-compact manifolds are obtained by lifting an Anosov or Axiom A flow to some non-compact cover of the phase space. Our density result applies verbatim and gives the density of ergodic measures of full support, if we restrict the system to a connected component of the non-wandering set on the cover.

The set of all borel probability measures invariant by a dynamical system will be denoted by $\mathcal{M}^1(X)$ in the sequel. It is endowed with the weak topology: a sequence $\mu_n$ of probability measures converges to some measure $\mu$ if $\int f d\mu_n \to \int f d\mu$ for all bounded continuous $f$. We refer to the classical treatise of Billingsley [Bi99] for a detailed study of that topology. In particular, we will use the fact that we may restrict ourselves to bounded Lipschitz functions in the definition of weak convergence.

In the first two sections, using first the closing lemma, and then the local product structure of a hyperbolic flow, we prove the density of the Dirac measures supported on periodic orbits in the set of all invariant probability measures. In section 4 we prove theorem 1.1. We then extend this result to rank one manifolds, and to hyperbolic systems on covers. We end by an appendix containing a proof of the closing lemma for rank one vectors on non-positively curved manifolds.

## 2 The closing lemma and ergodic measures

The geodesic flow on the unit tangent bundle of a negatively curved manifold satisfies a property known as a closing lemma. Roughly speaking, a piece of orbit that comes back close to its initial point is in fact close to a periodic orbit. Here is a more formal statement:

**Definition 2.1** A flow $\phi_t$ on a metric space $X$ satisfies the closing lemma if for all points $v \in X$, there exists a neighborhood $V$ of $v$ which satisfies:

- for all $\varepsilon > 0$, there exists a $\delta > 0$ and a $t_0 > 0$ such that for all $x \in V$ and all $t > t_0$ with $d(x, \phi_t x) < \delta$ and $\phi_t x \in V$, there exists $x_0$ and $l > 0$, with $|l - t| < \varepsilon$, $\phi_l x_0 = x_0$, and $d(\phi_s x_0, \phi_s x) < \varepsilon$ for $0 < s < \min(t, l)$.

Note that for such flow, the non-wandering set and the closure of the recurrent points coincide with the closure of the set of periodic points.

The first closing lemma was proven by Hadamard in 1898, for negatively curved surfaces embedded in ambient space. D. V. Anosov generalized the closing lemma to the systems that now bear his name [An67]; for this reason, it is often called the Anosov closing lemma. A proof of the closing lemma valid for rank one manifolds and higher rank locally symmetric spaces can be found in [Eb96]. We recall its argument at the end of the paper.

**Lemma 2.2** Let $X$ be a metric space, $\phi_t$ a flow on $X$ which satisfies the closing lemma. Then the set of Dirac probability measures supported on periodic orbits of the flow is dense in the set of all invariant ergodic borel probability measures $\mu$ on $X$.

**Proof:** Let $f_i$ be finitely many bounded Lipschitz functions on $X$, with Lipschitz constants bounded by $K$. From the Poincaré recurrence theorem and the Birkhoff
ergodic theorem, we know that for almost all \( x \in X \), \( x \) is recurrent and the Birkhoff averages of \( f_i \) evaluated at \( x \) converges to \( \int f_i \, d\mu \). Fix \( \varepsilon > 0 \) and choose \( t_0 > 0 \) large enough so that for \( t \geq t_0 \), \( d(\phi_t x, x) < \delta \) and for all \( i \), \( \frac{1}{t} \int_0^t f_i \circ \phi_s(x) \, ds - \int_X f_i \, d\mu \leq \varepsilon \). Let \( x_0 \) be the periodic point of period \( l \) given by the closing lemma. We obtain

\[
\left| \frac{1}{t} \int_0^t f_i \circ \phi_s(x_0) \, ds \right| \leq \frac{1}{t} \int_0^t f_i (\phi_s(x)) \, ds - \frac{1}{t} \int_0^t f_i (\phi_s(x_0)) \, ds + \varepsilon \\
\leq \frac{1}{t} \int_0^t K d(\phi_s x, \phi_s x_0) \, ds + 2|t - l| \| f_i \|_{\infty} + \varepsilon \\
\leq (K + 2\| f_i \|_{\infty} + 1) \varepsilon
\]

This shows that the Dirac measure on that periodic orbit is close to the measure \( \mu \).

\[ \square \]

If the space \( X \) is borel standard, that is, a borel subset of a separable complete metric space, the decomposition theorem of Choquet implies that the convex set generated by the invariant ergodic borel probability measures is dense in the set of all invariant probability measures. We have proven:

**Corollary 2.3** Assume moreover that \( X \) is borel standard. Then the convex subset generated by the normalized Dirac measures on periodic orbits is dense in the set of all invariant borel probability measures \( \mathcal{M}^1(X) \).

There is another path that leads to the previous result: first prove a Lifsic type theorem [Li71]; that is, all functions whose integrals on periodic orbit vanish are coboundaries. For these functions, deduce that the integrals with respect to all invariant probability measures is zero; this requires additional work since the function given by the coboundary equation cannot be expected to be integrable when the underlying space is not compact. Then end the argument by applying a Hahn-Banach theorem adapted to the weak topology on the space of measures.

3 Product structure and Dirac measures

Our next goal is to show that Dirac measures on periodic orbits are in fact dense in the set of all invariant measures. This will require an additional assumption. Let \( X \) be a metric space, \( \phi_t \) a continuous flow on \( X \). The strong stable sets of the flow are defined by:

\[
W^{ss}(x) := \{ y \in X \mid \lim_{t \to -\infty} d(\phi_t(x), \phi_t(y)) = 0 \} \\
W^{s\varepsilon}(x) := \{ y \in W^{ss}(x) \mid d(\phi_t(x), \phi_t(y)) \leq \varepsilon \text{ for all } t \geq 0 \}\}
\]

One also defines the strong unstable sets \( W^{su} \) and \( W^{s\varepsilon} \) of \( \phi_t \); these are the stable sets of \( \phi_{-t} \).

**Definition 3.1** The flow \( \phi_t \) is said to admit a local product structure if all points \( v \in X \) have a neighborhood \( V \) which satisfies: for all \( \varepsilon > 0 \), there exists a positive constant \( \delta \) such that for all \( x, y \in V \) with \( d(x, y) \leq \delta \), there is a point \( < x, y > \in X \), a real number \( t \) with \( |t| \leq \varepsilon \), so that:

\[
< x, y > \in W^{su}(\phi_t(x)) \cap W^{ss}(y).
\]

This property is summarized by the following picture:
This assumption allows us to “glue” orbits together. Note that we can glue an arbitrary finite number of pieces of orbits, just by using the product structure recursively. In the next picture, we first glue the trajectory of \( x_1 \) with the trajectory of \( x_2 \). The resulting dotted trajectory is then glued with \( x_3 \). We end up with an orbit which follows first the orbit of \( x_1 \), then the orbit of \( x_2 \) and finally the orbit of \( x_3 \).

![Image showing the gluing of orbits]

Geodesic flows on negatively curved manifolds admit a local product structure. In fact, there is even a global product structure on the unit tangent bundle of the universal cover of the manifold.

We are now ready to prove:

**Proposition 3.2** Let \( X \) be a metric space, \( \phi_t \) a transitive flow on \( X \) admitting a local product structure and satisfying the closing lemma. Then the set of normalized Dirac measures on periodic orbits is dense in its convex closure.

From this proposition and corollary 2.3, we deduce that the set of normalized Dirac measures on periodic orbits is dense in the set of all invariant probability measures \( \mathcal{M}^1(X) \).

**Proof:** We show that any convex combination of Dirac measures on periodic orbits can be approached by a Dirac measure on a single periodic orbit. So, let \( x_1, x_3, \ldots, x_{2n-1} \) be periodic points with period \( l_1, l_3, \ldots, l_{2n-1} \), and \( c_1, c_3, \ldots, c_{2n-1} \) positive real numbers with \( \sum c_{2i+1} = 1 \). Let us denote the Dirac measure on a periodic orbit of some point \( x \) by \( \delta_x \). We want to find a periodic point \( x \) such that \( \delta_x \) is close to the sum \( \sum c_{2i+1} \delta_x \). The numbers \( c_{2i+1} \) may be assumed to be rational numbers of the form \( p_{2i+1}/q \).

By transitivity, for each integer \( i \) less than \( n \), we can find a point \( x_{2i} \) close to \( x_{2i-1} \) whose trajectory becomes close to \( x_{2i+1} \), say, after time \( t_{2i} \). We can also find a point \( x_{2n2} \) close to \( x_{2n-1} \) whose trajectory becomes close to \( x_1 \) after some time.

![Image showing the gluing of trajectories]

Now these trajectories are glued together. We fix an integer \( N \), that will be chosen big at the end. First glue the piece of orbit starting from \( x_1 \), of length \( N l_1 p_1 \), together with the orbit of \( x_2 \), of length \( t_2 \). The resulting orbit ends in a neighborhood of \( x_3 \), and that neighborhood does not depend on the value of \( N \). This orbit is glued with the trajectory starting from \( x_3 \), of length \( N l_2 p_2 \), and so on. The reader may want to write down the sequence of \( \varepsilon \) obtained through the successive gluings. This is done in [C04], where a similar argument was used.

We end up with a trajectory starting close to \( x_1 \), doing \( N \) turns around the first periodic orbit, then following the trajectory of \( x_2 \) until it reaches \( x_3 \); then it turns \( N \) times around the second periodic orbit, and so on, until it reaches \( x_{2n} \) and goes back to \( x_1 \).
For sake of clarity, the trajectory only makes one turn around the periodic orbits on the pictures. Finally, we use the closing lemma to obtain the periodic orbit we are looking for.

Integrating a function on that closed orbit gives a result close to what we would have obtained if we integrate first $N_{p_1}$ times around the first closed orbit, then on the piece of orbit leading to $x_3$, followed by the second periodic orbit, $N_{p_2}$ times, and so on. This amounts to integrating with respect to the measure:

$$\frac{\sum N_{p_{2i+1}} \delta_{x_{2i+1}} + \sum \delta_{x_i}}{\sum N_{p_{2i+1}} + \sum t_{2i}}$$

If $N$ is big enough, the contribution of the pieces of orbits associated to the $x_{2i}$ is small, and this measure is close to $\sum \frac{p_{2i+1}}{q} \delta_{x_{2i}}$; so we are done □

Remark :
Under the hypothesis of the previous proposition, transitivity is equivalent to the connectedness of $X$ together with the density of the periodic orbits on $X$ [C04].

4 Ergodic measures with full support

We first prove that there are always probability measures with full support, as soon as there are enough periodic orbits. Recall that a polish space is a topological space homeomorphic to a separable complete metric space. Locally compact separable metric spaces are polish and the set of borel probability measures on a polish space is again a polish space with respect to the weak topology. Hence, the Baire property holds in that context.

Lemma 4.1 Let $X$ be a polish space, $\phi_t$ a flow defined on $X$. We assume that the periodic orbits of the flow form a dense subset in $X$. Then the set of borel probability measures of full support is a dense $G_\delta$ subset of the set of all invariant borel probability measures.

Proof : Let $\{U_i\}_{i \in \mathbb{N}}$ be a countable basis of open sets for the topology of $X$. Let $F_i$ be the closed set of all probability measures which are zero on $U_i$. The complement of the union of all the $F_i$ is exactly the set of borel probability measures with full support. Let us show that the $F_i$ have empty interiors. Take $\nu \in F_i$ and $x_0$ a periodic point in $U_i$; the Dirac measure on the orbit of $x_0$ is denoted by $\delta_{x_0}$. Then the sequence $(1 - \frac{1}{n}) \nu + \frac{1}{n} \delta_{x_0}$ is not in $F_i$ and converges to $\nu$. The Baire theorem can now be applied and the lemma is proven. □

We can now prove a genericity result concerning ergodic invariant measures with full support.
Theorem 4.2 Let $X$ be a polish space, $\phi_t$ a transitive continuous flow on $X$ admitting a local product structure and satisfying the closing lemma. Then the set of invariant ergodic borel probability measures with full support is a dense $G_\delta$ subset of the set of all invariant borel probability measures.

Theorem 1.1 is a particular case of this result.

Proof : As noted above, the periodic orbits are dense in $X$. From the previous lemma, we know that the set of probability measures with full support is a dense $G_\delta$ subset of the set of all invariant borel probability measures $\mathcal{M}^1(X)$. The proposition shows that the set of ergodic invariant probability measures is also dense in $\mathcal{M}^1(X)$. It remains to check that this set is a $G_\delta$ set.

Let $\{f_i\}_{i\in\mathbb{N}}$ be some countable algebra of Lipschitz bounded functions which separate points on $X$. This set is dense in $L^1(m)$, for all borel probability measure $m$; this fact goes back to Rohlin; we refer to [C02] for a short proof. We can now write the complement of the set of ergodic probability measures as the union of the following closed sets:

$$F_{k,l,i} = \{ m \in \mathcal{M}^1(X) \mid \exists \mu, \nu \in \mathcal{M}^1(X), \alpha \in \left[\frac{1}{k},1-\frac{1}{k}\right], \text{ s.t. } m = \alpha \mu + (1-\alpha)\nu \text{ and } \int f_id\mu \geq \int f_id\nu + \frac{1}{l}\}$$

The condition $\int f_id\mu \geq \int f_id\nu + \frac{1}{l}$ insures that the two measures $\mu$ and $\nu$ are not equal. This is a closed condition. Now let $m_n$ be a sequence of measures that can be written as barycenters $\alpha_n\mu_n + (1-\alpha_n)\nu_n$. If the sequence $m_n$ is converging in $\mathcal{M}^1(X)$, then it is tight. Up to a subsequence, we can assume that $\mu_n$ and $\nu_n$ converge to probability measures $\mu$ and $\nu$ of mass $\leq 1$, and $\alpha_n \to \alpha \in \left[\frac{1}{k},1-\frac{1}{k}\right]$. The relation $m = \alpha \mu + (1-\alpha)\nu$ implies that the limit $m$ of $m_n$ is a barycenter, and $F_{k,l,i}$ is closed. This ends the proof.

5 Non-positively curved manifolds

We have just proven that a geodesic flow on a negatively curved manifold always admits an ergodic invariant measure fully supported on the non-wandering set of the flow. We now give generalisations of this result to non-positively curved manifolds.

5.1 Closing lemma and local product

Let $M$ be a riemannian manifold and $v$ a vector belonging to the unit tangent bundle $T^1M$ of $M$. The vector $v$ is a rank one vector, if the only parallel Jacobi fields along the geodesic generated by $v$ are proportional to the generator of the geodesic flow. A connected complete non-positively curved manifold is said to be a rank one manifold if its tangent bundle admits a rank one vector. In that case, the set of rank one vectors is an open subset of $T^1M$. Rank one vectors generating closed geodesics are precisely the hyperbolic periodic points of the geodesic flow.

Negatively curved manifolds are rank one manifolds. There are also examples of rank one manifolds for which the sectional curvatures vanish on an open subset of the manifold. The concept of rank one manifold was introduced by W. Ballmann, M. Brin, P. Eberlein [BBES85]; we refer to the survey of G. Knieper [Kn02] for an overview of the properties of rank one manifolds.

The goal is now to find a big subset of the non wandering set $\Omega$, so that the geodesic flow $\phi_t$ has a local product structure and satisfies the closing lemma in
restriction to that subset. Let us consider the set $\Omega_1 \subset T^1M$ of non-wandering rank one vectors satisfying the following condition:

*If $w \in T^1M$ is such that the distance $d(\phi_t(w), \phi_t(v))$ stays bounded for $t \geq 0$, then there is a $t_0 \in \mathbb{R}$ such that $\phi_{t_0}(w) \in W^s(v)$.***

In other words, any non-wandering half-geodesic staying at a bounded distance from a geodesic in $\Omega_1$ is in fact asymptotic to that geodesic. Typical examples of vectors which are not in $\Omega_1$ are vectors whose associated geodesic is asymptotic to a closed geodesic belonging to a flat half-cylinder.

**Proposition 5.1** Let $M$ be a rank one manifold. Then the restriction of the geodesic flow to $\Omega_1$ admits a local product structure and satisfies the closing lemma.

*Proof:* The geodesic flow, in restriction to the set of non-wandering rank one vectors, satisfies the closing lemma. This is proven by P. Eberlein in [Eb96], prop 4.5.15, in the general setting of non-positive curvature. We recall the proof in an appendix at the end of the paper, since the rank one hypothesis allows for some simplification.

We also know that recurrent rank one vectors belong to $\Omega_1$ [Kn02], prop. 4.4. Hence, rank one periodic orbits are in $\Omega_1$. Now, if we apply the closing lemma to a non-wandering rank one vector, the closed orbit obtained is close to the rank one vector. Hence, as the set of rank one vectors is open, it is a closed rank one orbit and belongs to $\Omega_1$. So the closing lemma is satisfied in restriction to $\Omega_1$.

Let us consider two vectors $v, w$ in $T^1M$ which are close to each other. Then there is a point $w'$ close to $v$ and $w$, such that $d(\phi_t(w), \phi_t(w'))$ is bounded for negative times, and $d(\phi_t(v), \phi_t(w'))$ is bounded for positive times [Kn02], lemma 1.3. So, if $v$ and $w$ belong to $\Omega_1$, then $w'$ is a rank one vector whose associated geodesic is positively asymptotic trajectory of $v$, and negatively asymptotic to the trajectory of $w$. This shows that $w'$ is in $\Omega_1$ and that there is a local product structure in restriction to $\Omega_1$. □

**Proposition 5.2** Let $M$ be a rank one manifold. Then the restriction of the geodesic flow to $\Omega_1$ is transitive.

*Proof:* Let $U_1$ and $U_2$ be two open sets containing points in $\Omega_1$. We show that there is a trajectory in $\Omega_1$ that starts from $U_1$ and ends in $U_2$. This will prove transitivity on $\Omega_1$. From the previous proposition, we have seen that rank one periodic vectors are dense in $\Omega_1$. So there exists rank one periodic vectors $v_1 \in U_1$ and $v_2 \in U_2$. There is also a rank one vector $v_3$ whose trajectory is negatively asymptotic to the trajectory of $v_1$ and positively asymptotic to the trajectory of $v_2$, cf [Kn02] lemma 1.4 ff.

Let us show that $v_3$ is non-wandering. First note that there is also a trajectory negatively asymptotic to $v_2$ and positively asymptotic to the trajectory of $v_1$. That is, the two periodic orbits $v_1, v_2$ are heteroclinically related.

This implies that the two connecting orbits are non-wandering: indeed, using the local product structure, we can glue the two connecting orbits to obtain a trajectory that starts close to $v_3$, follows the second connecting orbit, and then follows the orbit of $v_2$, coming back to the vector $v_3$ itself. Hence $v_3$ is in $\Omega$. Since it is asymptotic to $v_2$, it belongs to $\Omega_1$ and we are done. □

Theorem 4.2 applied in restriction to $\Omega_1$ gives the:
Corollary 5.3 Let $M$ be a non-positively curved connected complete riemannian manifold. Then there exists an ergodic finite borel measure on $T^1M$, invariant with respect to the geodesic flow, whose support contains all hyperbolic periodic orbits.

5.2 Rank one manifolds

There are examples of rank one manifolds for which $\Omega_1$ is not dense in the non-wandering set $\Omega$.

For example, let us consider a non-positively curved surface, with a geodesic $\gamma$ bounding an half euclidean cylinder. This corresponds to the right part on the picture above. The remainder of the surface is compact, negatively curved. This is depicted on the left hand side of the picture. There are three types of geodesics on the surface:

- Closed geodesics in the cylinder part, which are parallel to $\gamma$.
- Geodesics which cross $\gamma$. They are associated to rank one wandering vectors.
- Geodesics that stay in the negatively curved part of the surface. They are associated to rank one vectors.

The non-wandering set $\Omega$ extends arbitrarily far on the right whereas the set $\Omega_1$ is contained in the left part of the picture. The only ergodic invariant probability measures giving positive measure to the cylinder part are the Dirac measures on the periodic orbits. This gives a rank one example with no ergodic invariant finite measure supported by $\Omega$.

There is however a simple condition that insures both the density of $\Omega_1$ in $\Omega$ and the transitivity on $\Omega_1$: if $\Omega = T^1M$, then the rank one vectors form an open dense subset of $T^1M$; moreover we have seen that the geodesic flow is transitive; this was first proven by Eberlein, see e.g. [Eb96], prop. 4.7.3 and 4.7.4. This implies the existence of a rank one vector with dense orbit. As mentioned earlier, such a vector belongs to $\Omega_1$; we refer to G. Knieper, [Kn02] prop 4.4. We have established:

Theorem 5.4 Let $M$ be a rank one manifold. We assume that the non-wandering set of the geodesic flow coincides with $T^1M$. Then there exists an ergodic borel probability measure on $T^1M$, which is invariant by the geodesic flow, and of full support.

6 Additional examples

Theorem 4.2 can be applied to other examples:

- Irreducible Markov chains defined on a countable alphabet.
We recover the density of Dirac measures on periodic points, in the set of all probability invariant measures, a result first proven by Oxtoby \[1963\] in the case of a finite alphabet.

- **Skew-products over Markov chains.**

The fiber has to be discrete, so that both the product structure and the closing lemma lift to the skew-product. The transitivity on the skew-product was studied, for example, in \[2004\]. When the fiber is a non-amenable group, the lift of a probability measure on the basis cannot be ergodic; this result is due to Zimmer \[1978\]. So the ergodic measures on the skew-product must be singular with respect to the action of the group extension.

- **Lift of hyperbolic systems to coverings of the base space.**

For these systems, lifts of Gibbs measures are known to be ergodic only if the deck transformation group is virtually \(\mathbb{Z}\) or \(\mathbb{Z}^2\) \[1991\]. From this viewpoint, even the existence of an infinite ergodic measure of full support would have appeared somehow surprising. Since these systems have received much attention recently, we rephrase Theorem 4.2 in that context.

**Theorem 6.1** Let \(\hat{X}\) be a compact manifold, \(\hat{X}\) a covering of \(X\) and \(\hat{\phi}_t\) a transitive flow on \(\hat{X}\) which is the lift of an Anosov flow defined on \(X\). Then the set of invariant ergodic borel probability measures with full support in \(\hat{X}\) is a dense \(G_\delta\)-set in the set of all invariant borel probability measures.

An analogous result holds for Axiom A systems, in restriction to preimages of basic pieces.

Finally, we remark that our results are again true if we look at transformations instead of flows. The proofs apply verbatim.

### 7 Appendix : the rank one closing lemma

We give a proof of the closing lemma for the geodesic flow on a non-positive complete manifold, in restriction to the rank one non-wandering vectors. There are two motivations for providing a proof: first the rank one hypothesis allows for some simplification, as compared with the classical proof given by P. Eberlein \[1996\]. Second, in the context of non-positive curvature, it seems that it is still unknown whether the closing lemma holds for all non-wandering vectors.

**Theorem 7.1 (Eberlein \[1996\])** Let \(\hat{M}\) be a Hadamard rank one manifold, and \(M = \hat{M}/\Gamma\), with \(\Gamma\) a discrete subgroup of \(\text{Isom}(\hat{M})\). Let \(C \subset T^1M\) be a compact set of rank one vectors and \(\varepsilon > 0\). Then there exists \(\delta > 0\) and \(T > 0\) with the following property:

If there exists \(t \geq T\) and \(v \in C\) such that \(D(g^tv, v) < \delta\), then there exists \(t' \in [t-\varepsilon, t+\varepsilon]\) and \(v'\) a rank one vector such that \(g^{t'}v' = v'\) and \(D(g^sv, g^s v') \leq \varepsilon\) for all \(0 \leq s \leq \min(t, t')\).

Here, we denote by \(D\) the natural distance on \(T^1M\) (and \(T^1\hat{M}\)) induced by the riemannian metric on \(M\).

**Proof :** Assume by contradiction that there exists a compact set \(C \subset T^1M\) of rank one vectors, \(\varepsilon > 0\), and a sequence \(\{v_n\}_{n \in \mathbb{N}}\) of vectors of \(C\), together with a sequence \(t_n \to \infty\), such that \(D(v_n, g^{t_n}v_n) \to 0\), without periodic orbits \(w_n \in \text{period } \omega_n \varepsilon\)-close to \(t_n\) staying in their \(\varepsilon\)-neighbourhood during the time \(t_n\). By compactness, we can assume that \(v_n\) converges to some \(v \in C\), as \(n\) goes to infinity.
Lift the situation to $\tilde{M}$. There exists a compact set $\tilde{C} \subset T^1\tilde{M}$, $\varepsilon > 0$, a sequence $(\tilde{v}_n)_{n \in \mathbb{N}} \in (\tilde{C})^\mathbb{N}$ of rank one vectors converging to $\tilde{v}$, $t_n \geq n$ and $\varphi_n \in \Gamma$, such that $D(\tilde{v}_n, \varphi_n \circ g^{t_n}\tilde{v}_n) \leq \frac{\varepsilon}{n}$.

Moreover, there exists no vector $\tilde{w}_n$ such that $D(\tilde{w}_n, \tilde{v}_n) \leq \varepsilon/2$ and $d\varphi_n \circ g^{t_n}\tilde{w}_n = \tilde{w}_n$ for some $\omega_n \in [t_n - \varepsilon, t_n + \varepsilon]$. Here is a key geometric point: the fact that on $T^1\tilde{M}$, the orbits $g^sw_n$, and $g^{t_n}v_n$ are $\varepsilon$-close during the time $t_n$ is expressed by the fact that the isometry $\varphi_n$ which sends $g^{t_n}\tilde{w}_n$ on $\tilde{w}_n$ is the same which sends $g^{t_n}\tilde{v}_n$ close to $\tilde{v}_n$.

Indeed, as $\omega_n = t_n \pm \varepsilon$ and $\varphi_n$ is an isometry, the distance between $g^{t_n}\tilde{w}_n$ and $g^{t_n}\tilde{v}_n$ is very close from the distance between $\tilde{w}_n$ and $\tilde{v}_n$, which is very small. By convexity of the riemannian distance on the Hadamard manifold $\tilde{M}$, we deduce that $\gamma_{\omega_n}(s)$ and $\gamma_{\omega_n}(s)$ stay close one another for $0 \leq s \leq \min(t_n, s_n)$. It also implies that $D(g^{t_n}\tilde{v}_n, g^{t_n}\tilde{w}_n)$ is small.

Now, the idea of the proof is to show that for $n$ big enough, $\varphi_n$ is an axial isometry, and to find on its axis a periodic vector $\tilde{w}_n$ converging to $\tilde{v}$, in contradiction with the above assumption.

Denote by $\gamma_{\tilde{v}}$ the geodesic determined by $\tilde{v}$. Let $p$ (resp. $p_n, q_n$) be the base point of $\tilde{v}$ (resp. $\tilde{v}_n, g^{t_n}\tilde{v}_n$). As $\varphi_n^{-1}(p_n)$ is close to $\gamma_{\omega_n}(t_n)$, and $p_n \to p$, $v_n \to v$, it is easy to check that $\varphi_n^{-1}(p) \to \gamma_{\varepsilon}(+\infty)$. Similarly, one gets $\varphi_n(p) \to \gamma_{\varepsilon}(-\infty)$.

We need the following lemma.

**Lemma 7.2** [Ba95, lemma 3.1] Let $\tilde{M}$ be a Hadamard rank one manifold, $\tilde{v} \in T^1\tilde{M}$ a regular vector, $x = \gamma_{\varepsilon}(+\infty)$ and $y = \gamma_{\varepsilon}(-\infty)$. For all $\eta > 0$, there exist neighbourhoods $V_\eta(x)$ and $V_\eta(y)$ of $x$ and $y$ respectively, such that for all $x' \in V_\eta(x)$, $y' \in V_\eta(y)$, there exists a geodesic $\gamma'$ joining $y$ to $x$ such that $d(\gamma_{\varepsilon}(0), \gamma') \leq \eta$.

This lemma is easy to prove on negatively curved manifolds with curvature bounded by above by $-a^2 < 0$. A similar result is also true [EbO73] for regular vectors on higher rank symmetric spaces. If $\tilde{v}$ is a singular vector on a rank one manifold, this lemma is probably almost always false. However it does not mean that a closing lemma could not be proven in another way for such vectors.

Let us now finish the proof of the closing lemma. Let $V_k(x)$ and $V_k(y)$ be the neighbourhoods given by the above lemma for $\eta = \frac{\varepsilon}{k}$. We can also assume that they are homeomorphic to balls of $\mathbb{R}^{d-1}$ and their closures are homeomorphic to closed balls of $\mathbb{R}^{d-1}$ (with $d = \text{dim } \tilde{M}$). (Endowed with the cone topology, the boundary $\partial \tilde{M}$ is homeomorphic to the $d-1$-dimensional sphere, [Eb-O73].)

Using the fact that $\varphi_n(p) \to y$ and $\varphi_n^{-1}(p) \to x$ one proves easily that for $n$ big enough we have $\varphi_n^{-1}(V_k(x)) \subset V_k(x)$ and $\varphi_n(V_k(y)) \subset V_k(y)$. By the Brouwer fixed point theorem, we deduce that $\varphi_n$ has two fixed points $x_n \in V_k(x)$ and $y_n \in V_k(y)$.

Consider a geodesic joining $y_n$ to $x_n$ given by the above lemma. It is invariant by $\varphi_n$, which acts by translation on it. Let $w_n$ be the vector of this geodesic minimizing the distance to $\tilde{v}$, and $p_n'$ be its basepoint. This vector is close to $\tilde{v}$, hence close to $\tilde{v}_n$, if $n$ has been chosen big enough.

Denote by $\omega_n = d(p_n', \varphi_n(p_n'))$ the period associated to $w_n$. Since $p_n'$ is close to $p_n$, $\omega_n$ is close to $d(p_n, \varphi_n(p_n))$, and so close to $t_n$ for $n$ big enough. Thus, we get the desired contradiction. \qed
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