ASYMPTOTICALLY SATURATED TORIC ALGEBRAS

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Abstract. We show the finite generation of certain invariant
graded algebras defined on toric weak log Fano fibrations. These
are the toric version of FGA algebras, recently introduced by Shokurov
in connections to the existence of flips.

Introduction

Shokurov’s new approach for establishing the existence of flips is to
reduce this problem, by induction on dimension, to the finite generation
of graded algebras which are asymptotically saturated with respect to
weak log Fano fibrations ([6]). Shokurov showed that these algebras are
finitely generated in dimension one and two, and conjectured this to
be true in any dimension ([6]). Our main result is the positive answer
to the toric case of this conjecture.

Theorem 1. Let $\pi: X \to S$ be a proper surjective toric morphism
with connected fibers, and let $B$ be an invariant $\mathbb{Q}$-divisor on $X$ such
that $(X, B)$ has Kawamata log terminal singularities and $-(K + B)$ is
$\pi$-nef.

1. Let $\mathcal{L} \subseteq \bigoplus_{i=0}^{\infty} \pi_* \mathcal{O}_X(iD)$ be an invariant graded $\mathcal{O}_S$-subalgebra
   which is asymptotically saturated with respect to $(X/S, B)$,
   where $D$ is an invariant divisor on $X$. Then $\mathcal{L}$ is finitely
generated.

2. The number of rational maps $X \dashrightarrow \text{Proj}(\bar{\mathcal{L}})$, where $\bar{\mathcal{L}}$ is
   the normalization of an $\mathcal{O}_S$-algebra as in (1), is finite up to
   isomorphism.

The toric case of asymptotic saturation, the key property ensuring
finite generation in (1), can be explicitly written down as a Diophan-
tine system (see the proof of Theorem 4.1). To see this in a special
case, let $S$ be a point and let $X = T_N \text{emb}(\Delta)$ be a torus embedding
for some lattice $N$. Let $M$ be the lattice dual to $N$, and consider a

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compact convex set of maximal dimension $\square$ in $M_\mathbb{R}$. This defines a toric graded algebra

$$\mathcal{R}(\square) = \bigoplus_{i=0}^{\infty} \left( \bigoplus_{m \in M \cap i \square} \mathbb{C} \cdot \chi^m \right),$$

which is finitely generated if and only if $\square$ is a rational polytope. On the other hand, the log discrepancies of the log pair $(X, B)$ with respect to toric valuations can be encoded in a positive function $\psi : N_\mathbb{R} \to \mathbb{R}$, and $\psi$ determines a rule to enlarge any convex set in $M_\mathbb{R}$ to an open convex neighborhood. The asymptotic saturation of $\mathcal{R}(\square)$ with respect to $(X, B)$ means that the lattice points of the neighborhood of $j \square$ are already contained in $j \square$, for every sufficiently divisible positive integer $j$. This Diophantine property restricts the way that $\square$ can be approximated with rational points from the outside.

The key technical tool behind Theorem 1 is a known result in Geometry of Numbers, an effective bound on the width of a convex set in terms of the number of lattice points it contains ([3]).

The outline of this paper is as follows. In Section 4 we explicitly describe toric asymptotic saturation and reduce Theorem 1 to its special case when the algebra is normal and associated to a convex set, the equivalent of $\mathcal{R}(\square)$ above. The rest of the paper is devoted to this special case. In Section 1 we collect some elementary results on convex sets and their support functions, and on Diophantine approximation. In Section 2 we characterize asymptotic saturation in geometric terms (Theorem 2.6) and obtain a boundedness result (Theorem 2.7). These are used in Section 3 to prove Theorem 1 by induction on dimension.

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1. **Preliminary**

We collect in this section elementary results on convex sets, toric geometry and Directed Diophantine Approximation, which we will need later. We refer the reader to Oda [4] for basic notions and terminology on toric varieties and convex sets.

Throughout this section, $N$ is a lattice, with dual lattice $M$. We have a duality pairing $\langle \cdot, \cdot \rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}$, defined over $\mathbb{Z}$.

1.1. **Convex sets and support functions.** Fix a convex rational polyhedral cone $\sigma \subseteq N_\mathbb{R}$, that is $\sigma$ is spanned by finitely many elements of $N$. We denote by $S(\sigma)$ the set of all functions $h : \sigma \to \mathbb{R}$ satisfying the following properties:
1) positively homogeneous: $h(te) = t \cdot h(e)$ for $t \geq 0, e \in \sigma$.
2) upper convex: $h(e_1 + e_2) \geq h(e_1) + h(e_2)$ for $e_1, e_2 \in \sigma$.

**Theorem 1.1.**

(i) Every function $h \in S(\sigma)$ is continuous.
(ii) Let $(h_i)_{i \geq 1}$ be a sequence of functions in $S(\sigma)$ which converges pointwise, and set $h(e) = \lim_{i \to \infty} h_i(e)$ for $e \in \sigma$. Then $h \in S(\sigma)$, and the sequence $(h_i)$ converges uniformly to $h$ on compact subsets of $\sigma$.

**Proof.** This is a special case of [5], Theorems 10.1 and 10.8. □

For a function $h: \sigma \to \mathbb{R}$, define

$$\square_h = \{ m \in M_\mathbb{R}; \langle m, e \rangle \geq h(e), \forall e \in \sigma \}.$$ 

A **convex polytope** $K \subset M_\mathbb{R}$ is the convex hull of a finite set in $M_\mathbb{R}$.

A **rational convex polytope** is the convex hull of a finite set in $M_\mathbb{Q}$.

A **rational convex polyhedral set** is the intersection of finitely many rational affine half spaces in $M_\mathbb{R}$. We denote by $C(\sigma^\vee)$ the set of all nonempty closed convex sets $\square \subseteq M_\mathbb{R}$ satisfying the following two properties:

1) $\square + \sigma^\vee = \square$;
2) $\square \subseteq K + \sigma^\vee$, for some convex polytope $K \subset M_\mathbb{R}$.

The **support function** $h_{\square}: \sigma \to \mathbb{R}$ of $\square \in C(\sigma^\vee)$ is defined by

$$h_{\square}(e) = \inf_{m \in \square} \langle m, e \rangle.$$ 

**Theorem 1.2.** The maps $\square \mapsto h_{\square}$ and $h \mapsto \square_h$ are inverse to each other, inducing a bijection $C(\sigma^\vee) \simeq S(\sigma)$. Under this correspondence, the Minkowski sum $\square + \square'$ and a nonnegative scalar multiple $t\square$ correspond to $h_{\square} + h_{\square'}$ and $th_{\square}$, respectively.

We omit the proof of this theorem, being similar to that of [4], Theorem A.18. When $\sigma = N_\mathbb{R}$, this is the usual correspondence between compact convex sets and support functions. Note that $K + \sigma^\vee \in C(\sigma^\vee)$, for every compact convex set $K \subset M_\mathbb{R}$, but not all elements of $C(\sigma^\vee)$ are of this form. Such an example is $\sigma^\vee = \{(x, y) \in \mathbb{R}^2; x, y \geq 0 \}$ and $\square = \{(x, y) \in \sigma^\vee; xy \geq 1 \}$. Nevertheless, we have

**Lemma 1.3.** The following properties are equivalent for $\square \in C(\sigma^\vee)$:

(i) $\square$ is a rational convex polyhedral set.
(ii) $\square = K + \sigma^\vee$ for some rational convex polytope $K$ such that no vertex of $K$ belongs to the Minkowski sum of $\sigma^\vee$ and the convex hull of the other vertices of $K$.

Furthermore, assume $\dim(\sigma) = \dim(N)$. Then $K$ is uniquely determined by $\square$. 

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Example 1.4. If $\sigma = N_\mathbb{R}$, then $K = \Box$. If $\sigma$ is the positive cone in $\mathbb{R}^d$ and $\Box$ is a Newton polytope, then $K$ is the convex hull of the compact faces of $\Box$.

1.2. **Proper toric morphisms with affine base.** Toric morphisms with affine base which are proper, surjective and with connected fibers, are in one to one correspondence with fans having convex support.

Indeed, let $\Delta$ be a fan in a lattice $N$ such that its support $|\Delta| = \bigcup_{\tau \in \Delta} \tau$ is a convex rational polyhedral cone. Let $\tilde{N} = N/(N \cap |\Delta| \cap (-|\Delta|))$ and let $\tilde{\sigma} \subset \tilde{N}_\mathbb{R}$ be the image of $|\Delta|$ under the natural projection. Then $T_N \operatorname{emb}(\Delta) \to T_{\tilde{N}}(\tilde{\sigma})$ is a toric morphism with affine base, which is proper, surjective, with connected fibers.

Conversely, let $\pi: X \to S$ be a proper toric morphism of toric varieties, with $S$ affine. Thus $X = T_N \operatorname{emb}(\Delta)$, $S = T_{N'}(\sigma')$ and $\pi$ corresponds to a lattice homomorphism $\varphi: N \to N'$ such that $\Delta$ is a finite fan in $N$, $\sigma'$ is a strongly convex rational polyhedral cone in $N'$ and $|\Delta| = \varphi^{-1}(\sigma')$. In particular, $|\Delta|$ is a convex rational polyhedral cone. Then $\varphi$ factors through $\tilde{N} = N/(N \cap |\Delta| \cap (-|\Delta|))$ and we have a commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & T_{\tilde{N}}(\tilde{\sigma}) \\
\downarrow \pi & \downarrow j & \\
S & \rightarrow &
\end{array}
\]

where $j$ is finite on its image.

Let now $D = \sum_{e \in \Delta(1)} d_eV(e)$ be an invariant $\mathbb{Q}$-divisor on $X$ which is $\mathbb{Q}$-Cartier. This means that there exists a function $h: |\Delta| \to \mathbb{R}$ such that $h$ is $\Delta$-linear and $h(e) = -d_e$ for every $e \in \Delta(1)$. In particular, $h$ is positively homogeneous. The $\mathbb{Q}$-divisor $D$ is $\pi$-nef if $h$ is upper convex; it is $\pi$-ample if for every maximal cone $\sigma \in \Delta$ there exists $m_\sigma \in \Box_h$ such that $\sigma = \{e \in |\Delta|; h(e) = \langle m_\sigma, e \rangle \}$.  

1.3. **The ample fan of a convex rational polyhedral set.** To each convex rational polyhedral set $\Box \subseteq M_\mathbb{R}$ we associate a fan $\Delta_{\Box}$ in a quotient lattice of $N$, as follows. Assume first that $\dim(\Box) = \dim(M)$. Let $K$ be a rational polytope associated to $\Box$ by Lemma 1.3, with vertices $v_1, \ldots, v_l$. The support function of $\Box$ is $h(e) = \min_{j=1}^l \langle v_j, e \rangle$, and the cones

\[\{e \in |\Delta|; \langle v_j, e \rangle = h(e)\} \ (1 \leq j \leq l)\]

are the maximal dimensional cones of a fan $\Delta_{\Box}$ in $N$. The support $|\Delta_{\Box}|$ is the unique convex cone $\sigma \subseteq N_\mathbb{R}$ such that $\Box \in \mathcal{C}(\sigma^\vee)$. 


If \( \dim(\Box) < \dim(M) \), choose a point \( m_0 \in M_\mathbb{Q} \cap \Box \) and denote \( \Box' = \Box - m_0 \). Define

\[
N' = N/(N \cap \Box' \perp).
\]

If \( M' \) is the dual lattice of \( N' \), then \( M'_\mathbb{R} \) can be identified with the smallest vector subspace of \( M_\mathbb{R} \) which contains \( \Box' \). We have \( \dim(\Box') = \dim(M') \) and let \( \Delta_{\Box'} \) be the fan in \( N' \) defined as above. This fan is independent of the choice of \( m_0 \), and we denote it again by \( \Delta_\Box \). Its support is a convex set.

**Definition 1.5.** \( \Delta_\Box \) is called the *ample fan* of the rational convex polyhedral set \( \Box \subset M_\mathbb{R} \).

Assume now that \( \pi : X \to S \) is a proper toric morphism with affine base \( S \). We may write \( X = T_{N'} \text{emb}(\Delta) \) and \( S = T_{\bar{N}}(\bar{\sigma}) \), and \( \pi \) corresponds to a lattice homomorphism \( \varphi : N \to \bar{N} \) such that \( |\Delta| = \varphi^{-1}(\bar{\sigma}) \).

For a rational polyhedral convex set \( \Box \in \mathcal{C}(|\Delta|^\vee) \), the \( T_N \)-invariant \( \mathcal{O}_S \)-algebra

\[
\mathcal{R}(\Box) = \bigoplus_{i=0}^{\infty} \bigoplus_{m \in M \cap \Box} \mathbb{C} \cdot \chi^m.
\]

is normal and finitely generated. The induced toric rational map

\[
\xymatrix{ X \ar[rr]^-{\Phi} & & \text{Proj}(\mathcal{R}(\Box)) } \quad \xymatrix{ \ar[rd] \quad \ar[ld] \quad S } \quad \text{is defined over } S, \quad \text{and Proj}(\mathcal{R}(\Box)) \text{ is the torus embedding of the ample fan } \Delta_\Box. \text{ If } \dim(\Box) = \dim(M), \text{ then } \Delta_\Box \text{ is a fan in } N \text{ with } |\Delta_\Box| = |\Delta|, \text{ hence } \Phi \text{ is birational in this case. The invariant } \mathbb{Q}-\text{divisor}
\]

\[
\sum_{e \in \Delta_{\Box}(1)} -h(e)V(e)
\]

is ample relative to \( S \). If \( \dim(\Box) < \dim(M) \), \( \Delta_\Box \) is a fan in \( N' \), whose support is the image of \( |\Delta| \) under the natural projection.

**1.4. Directed Diophantine Approximation.** Let \( m \in M_\mathbb{R} \) and let \( e \in N_\mathbb{R} \). Let \( I \) be a positive integer and let \( \| \cdot \| \) be a norm on \( M_\mathbb{R} \).

**Theorem 1.6 (cf. [1]).** For every \( \varepsilon > 0 \), there exists a positive multiple \( k \) of \( I \) and there exists \( \bar{m} \in M \) such that \( \langle \bar{m} - km, e \rangle \in (-\varepsilon, 0] \) and \( \| \bar{m} - km \| < \varepsilon \).

**Proof.** We may find a decomposition \( M = M' \oplus M'' \), with dual decomposition \( N = N' \oplus N'' \), such that \( m = m' + m'' \), \( m', m'' \in M'_\mathbb{Q} \), \( m'' \in M''_\mathbb{R} \).
and \( \{ e'' \in N''; \langle m'', e'' \rangle \in \mathbb{Q} \} = \{ 0 \} \). Let \( k_1 \) be a positive integer such that \( I|k_1 \) and \( k_1m' \in M' \). Since
\[
\{ e'' \in N''; \langle k_1m'', e'' \rangle \in \mathbb{Q} \} = \{ 0 \},
\]
we infer by \[2\], Chapter III, Theorem IV, that the subgroup generated by the class of \( k_1m'' \) is dense in the torus \( M''_\mathbb{R}/M'' \). Equivalently, the set \( \bigcup_{j \geq 1}(M'' + jk_1m'') \) is dense in \( M''_\mathbb{R} \). In particular, the following system has a solution for some \( j \geq 1 \):
\[
\begin{cases}
  m''_j \in M'' \\
  \langle m''_j + jk_1m'', e'' \rangle = \langle m''_j + jk_1m'', e \rangle \in (0, \epsilon) \\
  \| m''_j + jk_1m'' \| < \epsilon.
\end{cases}
\]

Then \( k = jk_1 \) and \( \bar{m} = km' - m''_j \) satisfy the desired properties. \( \square \)

**Theorem 1.7.** Assume that \( e \notin \{ e' \in N; \langle m, e' \rangle \in \mathbb{Q} \} \otimes_{\mathbb{Z}} \mathbb{R} \). Then for every \( \epsilon > 0 \), there exists a positive multiple \( k \) of \( I \) and there exists \( \bar{m} \in M \) such that \( \langle \bar{m} - km, e \rangle \in (-\epsilon, 0) \) and \( \| \bar{m} - km \| < \epsilon \).

**Proof.** We may decompose \( M = M' \oplus M'' \), with dual decomposition \( N = N' \oplus N'' \), such that \( m = m' + m'', m' \in M'_\mathbb{Q}, m'' \in M''_\mathbb{R} \) and \( \{ e'' \in N''; \langle m'', e'' \rangle \in \mathbb{Q} \} = \{ 0 \} \). In particular,
\[
N''_\mathbb{R} = \{ e' \in N; \langle m, e' \rangle \in \mathbb{Q} \} \otimes_{\mathbb{Z}} \mathbb{R}.
\]

Let \( e = e' + e'' \) be the unique decomposition with \( e' \in N'_\mathbb{R} \) and \( e'' \in N''_\mathbb{R} \). Our assumption means that \( e'' \neq 0 \). Let \( k_1 \) be a positive integer such that \( I|k_1 \) and \( k_1m' \in M' \). We have
\[
\{ e'' \in N''_\mathbb{Q}; \langle k_1m'', e'' \rangle \in \mathbb{Q} \} = \{ 0 \}.
\]

By \[2\], Chapter III, Theorem IV, the subgroup generated by the class of \( k_1m'' \) is dense in the torus \( M''_\mathbb{R}/M'' \). Equivalently, the set \( \bigcup_{j \geq 1}(M'' + jk_1m'') \) is dense in \( M''_\mathbb{R} \). Since \( e'' \neq 0 \),
\[
\{ m'' \in M''_\mathbb{R}; \langle m'', e'' \rangle \in (0, \epsilon), \| m'' \| < \epsilon \}
\]
is a non-empty open subset of \( M''_\mathbb{R} \). Therefore the following system has a solution for some \( j \geq 1 \):
\[
\begin{cases}
  \bar{m}''_j \in M'' \\
  \langle \bar{m}''_j + jk_1m'', e'' \rangle \in (0, \epsilon) \\
  \| \bar{m}''_j + jk_1m'' \| < \epsilon.
\end{cases}
\]

The claim holds for \( k = jk_1 \) and \( \bar{m} = km' - \bar{m}''_j \). \( \square \)

**Lemma 1.8.** If \( \langle m, e \rangle \in \mathbb{Q} \), the following properties are equivalent:

(i) \( m \in \{ m' \in M; \langle m', e \rangle \in \mathbb{Q} \} \otimes_{\mathbb{Z}} \mathbb{R} \).

(ii) \( e \in \{ e' \in N; \langle m', e' \rangle \in \mathbb{Q} \} \otimes_{\mathbb{Z}} \mathbb{R} \).
Proof. Assume that (i) holds. We may find a decomposition $N = N' \oplus N''$, with dual decomposition $M = M' \oplus M''$, such that $e = e' + e''$, $e' \in N'_Q$, $e'' \in N'_R$ and
\[
\{m'' \in M''; \langle m'', e'' \rangle \in \mathbb{Q}\} = \{0\}.
\]
The assumption means that $m \in M'_R$. We may find a decomposition $M' = M'_1 \oplus M'_2$, with dual decomposition $N' = N'_1 \oplus N'_2$, such that $m = m'_1 + m'_2$, $m'_1 \in M'_{1,Q}$, $m'_2 \in M'_{2,R}$ and
\[
\{e'_2 \in N'_2; \langle m'_2, e'_2 \rangle \in \mathbb{Q}\} = \{0\}.
\]
We have
\[
\langle m'_2, e'_1 \rangle = \langle m, e' \rangle - \langle m'_1, e' \rangle = \langle m, e \rangle - \langle m'_1, e' \rangle \in \mathbb{Q}.
\]
Therefore $e' \in N'_{1,Q}$. Let $e'' = \sum_i r_ie''_i$, where $r_i \in \mathbb{R}$ and $\{e''_i\}_i$ is a basis of $N''$. Then $e'_1, e''_i \in N_Q$, $\langle m, e' \rangle \in \mathbb{Q}$ and $\langle m, e''_i \rangle = 0$. Therefore
\[
e = e' + e'' \in \{e' \in N; \langle m, e' \rangle \in \mathbb{Q}\} \otimes \mathbb{R},
\]
that is (i) holds. The statement is symmetric in $m$ and $e$, hence the converse holds as well. \qed

2. Asymptotic saturation

Throughout this section, we fix a lattice $N$ and a convex rational polyhedral cone $\sigma \subseteq N_R$.

**Definition 2.1.** A log discrepancy function is a function $\psi: \sigma \rightarrow \mathbb{R}$ satisfying the following properties:

(i) $\psi$ is positively homogeneous.

(ii) $\psi(e) > 0$ for $e \neq 0$.

(iii) $\psi$ is continuous.

**Example 2.2.** Let $\Delta$ be a fan in $N$ with $|\Delta| = \sigma$. Let $B = \sum_{e \in \Delta(1)} b_e V(e)$ be an invariant $\mathbb{R}$-divisor on $X = T_N \text{emb}(\Delta)$ such that $K + B$ is $\mathbb{R}$-Cartier and the pair $(X, B)$ has Kawamata log terminal singularities. Equivalently, there exists a function $\psi: \sigma \rightarrow \mathbb{R}$ such that $\psi(e) = 1 - b_e > 0$ for every $e \in \Delta(1)$, and $\psi$ is $\Delta$-linear. Then $\psi$ is a log discrepancy function.

The terminology comes from the following property: let $e \in N^{prim} \cap \sigma$ be a primitive lattice point, corresponding to a toric valuation $v_e$ of $X$. Then $\psi(e)$ is the log discrepancy of $(X, B)$ at $v_e$.

**Lemma 2.3.** Let $\psi: \sigma \rightarrow \mathbb{R}$ be a log discrepancy function. Then the set $\{e \in \sigma; \psi(e) \leq 1\}$ is compact.
Proof. Choose a norm $\|\cdot\|$ on $N_{\mathbb{R}}$. Since $\psi$ is a log discrepancy function, the infimum $c_0 = \inf \{\psi(e); e \in \sigma, \|e\| = 1\}$ is a well defined positive real number. We have $\psi(e) \geq c_0\|e\|$, for $e \in \sigma$. In particular,
\[
\{e \in \sigma; \psi(e) \leq 1\} \subseteq \{e \in \sigma; \|e\| \leq c_0^{-1}\}.
\]
The left hand side is a closed set, since $\psi$ is continuous, and the right hand side is a bounded set. Therefore the claim holds. \qed

Definition 2.4. For an arbitrary function $h: \sigma \to \mathbb{R}$, define
\[
\square_h = \{m \in M_{\mathbb{R}}; (m, e) > h(e), \forall e \in \sigma \setminus 0\}.
\]

Definition 2.5. Let $\square \in \mathcal{C}(\sigma^\vee)$ and let $\psi: \sigma \to \mathbb{R}$ be a log discrepancy function. We say that

- $\square$ is $\psi$-saturated if $M \cap \square_{\psi^{-1}} \subseteq \square$, where $\square_{\psi} \in \mathcal{S}(\sigma)$ is the support function of $\square$. Note that $\square = \square_{\psi^{-1}}$.
- $\square$ is asymptotically $\psi$-saturated if there exists a positive integer $I$ such that $j\square$ is $\psi$-saturated, for every $I|j$.

Note that saturation (asymptotic saturation) is invariant under lattice (rational) translations of the convex set.

Theorem 2.6 (Characterization of asymptotic saturation). Let $\square \in \mathcal{C}(\sigma^\vee)$ be a rational polyhedral set and let $\psi: \sigma \to \mathbb{R}$ be a log discrepancy function.

Let $N'' = \{e \in N; \square \ni m \mapsto (m, e) \in \mathbb{R} \text{ constant}\}$, with dual lattice $M''$, and define $\psi'': N''_{\mathbb{R}} \to \mathbb{R}$ by
\[
\psi''(e) = \psi(e).
\]
The ample fan $\Delta_{\square}$ is a fan in $N' = N/N''$ with support $\sigma' = \pi(\sigma)$, where $\pi: N_{\mathbb{R}} \to N'_{\mathbb{R}}$ is the natural projection. Define $\psi': \sigma' \to \mathbb{R}$ by
\[
\psi'(e') = \inf_{e \in \pi^{-1}(e')} \psi(e).
\]
Then $\square$ is asymptotically $\psi$-saturated if and only if the following hold:

1. $M'' \cap \square_{\psi''} = \{0\}$.
2. $\Delta_{\square}(1) \subseteq \{e' \in N'_{\mathbb{R}}; \psi'(e') \leq 1\}$.

Proof. After a rational translation, we may assume $0 \in \square$. In particular, $N'' = N \cap \square^\perp$ and $h(e) = h'(\pi(e))$, where $h \in \mathcal{S}(\sigma)$ and $h' \in \mathcal{S}(\sigma')$ are the support functions of $\square \subseteq M_{\mathbb{R}}$ and $\square \subseteq M_{\mathbb{R}}$, respectively.

Assume that (1) and (2) hold. Fix a positive integer $j$ such that $I|j$ and $jh(N) \subseteq \mathbb{Z}$ and assume that $m \in M$ satisfies
\[
\langle m, e \rangle > (jh - \psi)(e), \forall e \in \sigma \setminus 0.
\]
Choose a decomposition \( M = M' \oplus M'' \), and decompose \( m = m' + m'' \). Since \( h|_{N''} = 0 \), we obtain \( m'' \in M'' \cap Q'' \), hence \( m'' = 0 \) by (1). In particular, we have

\[
\langle m, e' \rangle > jh'(e') - \psi'(e'), \forall e' \in \sigma' \setminus 0.
\]

For every \( e' \in \Delta_{\square}(1) \), we have \( \langle m, e' \rangle \in \mathbb{Z} \), hence (2) gives \( \langle m, e' \rangle \geq jh'(e') \). Since \( h' \) is \( \Delta_{\square} \)-linear, we obtain

\[
\langle m, e' \rangle \geq jh'(e'), \forall e' \in |\Delta_{\square}|,
\]

hence \( m \in \square_{jkh} \). Therefore \( jh \) is \( \psi \)-saturated.

For the converse, assume that \( jh \) is \( \psi \)-saturated for every \( I|j \). We first check (1). Fix \( m'' \in M'' \cap Q'' \). Let \( \| \cdot \| \) be a norm on \( M_\mathbb{R} \) which is compatible with the decomposition \( M = M' \oplus M'' \). Since \( \psi \) is continuous, there exists \( \epsilon > 0 \) such that

\[
\langle m'', e \rangle + \psi(e) > 0, \forall e \in S(\sigma), \| e' \| < \epsilon.
\]

The rational convex polyhedral set \( \square \) has the same dimension as \( M_\mathbb{R} \). Therefore there exists \( m' \in M' \cap \text{relint}(k\square) \), for some positive integer \( k \). We have

\[
\langle m', e' \rangle > kh'(e'), \forall e' \in \sigma' \setminus 0.
\]

The continuity of \( \psi \) implies that the following number is well defined:

\[
t = - \inf \{ \frac{\langle m'', e \rangle + \psi(e)}{\langle m', e' \rangle - kh'(e')} : e \in S(\sigma), \| e' \| \geq \epsilon \}.
\]

Let \( j \) be a positive multiple of \( I \) such that \( j > t \). The identity

\[
\langle jm' + m'', e \rangle - (jkh - \psi)(e) = j(\langle m', e' \rangle - kh'(e')) + \langle m'', e \rangle + \psi(e)
\]

implies that

\[
\langle jm' + m'', e \rangle > (jkh - \psi)(e), \forall e \in S(\sigma).
\]

Since \( jkh \) is \( \psi \)-saturated, we infer that \( jm' + m'' \in \square_{jkh} \). In particular, \( jm' + m'' \in M' \), hence \( m'' = 0 \). This proves (1).

For (2), fix \( e' \in \Delta_{\square}(1) \) and assume by contradiction that \( \psi'(e') > 1 \). We may find a basis \( e_1, \ldots, e_d \) of \( N \) with \( e_1 = e' \). Let \( \| \cdot \| \) be the absolute value norm on \( N_\mathbb{R} \) with respect to this basis and denote

\[
S(\sigma) = \{ e \in \sigma ; \| e \| = 1 \}.
\]

The face \( \{ m \in \square ; \langle m, e_1 \rangle = h(e_1) \} \) of \( \square \) is a positive dimensional convex polyhedral set, hence there exists a 1-dimensional rational compact convex set \( \square_1 \) with

\[
\square_1 \subset \text{relint}(\{ m \in \square ; \langle m, e_1 \rangle = h(e_1) \}).
\]
It is easy to see that there exists a positive real number \( t_1 \) such that
\[ M \cap t \Box_1 \neq \emptyset \] for \( t > t_1 \).

Consider the following set
\[ C = \{ e \in S(\sigma); \psi(e) \leq \langle e_1^*, e \rangle \} . \]

Since \( \psi \) is continuous, the (possibly empty) set \( C \) is closed. Furthermore, \( e_1 \notin C \) and \( \Box_1 \) is included in the relative interior of the face of \( \Box \) corresponding to \( e_1 \), hence \( \langle m, e \rangle - h(e) > 0 \) for \( e \in C \) and \( m \in \Box_1 \). We infer that the following number is well defined
\[ t_2 = \sup_{m \in \Box, e \in C} \frac{(e_1^* - \psi)(e)}{(m, e) - h(e)} . \]

Let \( j \) be a positive multiple of \( I \) such that \( j > \max(t_1, t_2) \). Since \( j > t_1 \), there exists \( m_j \in M \) such that \( m_j + e_1^* \in j \Box_1 \). We have
\[ \langle m_j, e \rangle - (jh - \psi)(e) = j\left( \frac{m_j + e_1^*}{j}, e \right) - h(e) - (e_1^* - \psi)(e) . \]

Since \( j > t_2 \), we obtain \( m_j \in \Box_{jh - \psi} \). Since \( j \Box \) is \( \psi \)-saturated, we obtain \( m_j \in \Box_{jh} \). This is a contradiction, since
\[ \langle m_j, e_1 \rangle = jh(e_1) - 1 < jh(e_1) . \]

This proves (2).

**Theorem 2.7.** Let \( \psi : N \rightarrow \mathbb{R} \) be a log discrepancy function such that 
\( -\psi \) is upper convex and \( M \cap \Box_{-\psi} = \{0\} \). Then there exists \( e \in N \setminus 0 \) such that
\[ \psi(e) + \psi(-e) \leq C , \]
where \( C \) is a positive constant depending only on \( \dim(N) \).

**Proof.** Let \( \Box = \Box_{-\psi} \). Since \( \psi \) is positive, we have \( 0 \in \Box \subset \Box_{-\psi} \). Then \( \Box \) is a compact convex set, of dimension \( \dim(N) = d \), with support function \( -\frac{\psi}{2} \), such that \( M \cap \Box = \{0\} \). By \([3]\), Theorem 4.1, there exists \( e \in N \setminus 0 \) such that
\[ \max_{m \in \Box} \langle m, e \rangle - \min_{m \in \Box} \langle m, e \rangle \leq c_0 d^2 \left( \sqrt{1 + \#(M \cap \Box)} \right) , \]
where \( c_0 \) is a positive universal constant and \( \#(M \cap \Box) \) is the number of lattice points of \( \Box \). In our case, this means
\[ \psi(e) + \psi(-e) \leq C = 2c_0 d^2 \left( \sqrt{2} \right) \].
**Theorem 2.8 (Toric Asymptotic CCS).** Let $\psi: \sigma \to \mathbb{R}$ be a log discrepancy function. We denote by $\mathcal{M}(\psi)$ the set of rational polyhedral sets $\square \in \mathcal{C}(\sigma')$ such that

1. $\square$ is asymptotically $\psi$-saturated.
2. $h_{\square} - \psi$ is upper convex.

Then the set of ample fans $\{\Delta_{\square}\}_{\square \in \mathcal{M}(\psi)}$ is finite.

**Proof.** Let $\square \in \mathcal{M}(\psi)$, with support function $h$. After a rational translation, we may assume $0 \in \square$. Let $N'' = N \cap \square^\perp$ and let $d = \dim(N'')$. If $d = 0$, that is $\dim(\square) = \dim(M)$, the ample fan $\Delta_{\square}$ is a fan in $N$ with $|\Delta_{\square}| = \sigma$, and by Theorem 2.6 we have

$$\Delta_{\square}(1) \subseteq N'_{\text{prim}} \cap \{e \in N_{\mathbb{R}}; \psi(e) \leq 1\}.$$

The right hand side is a finite set, hence we infer that the number of fans $\Delta_{\square}$ is finite.

Assume now $d > 0$. We will show that $N''$ belongs to a finite set of sublattices of $N$. Since $h|_{N''} = 0$, $-\psi|_{N''_{\mathbb{R}}} = (h - \psi)|_{N''_{\mathbb{R}}}$ is an upper convex function. By assumption, $M'' \cap \square - \psi|_{N''_{\mathbb{R}}} = \{0\}$. By Theorem 2.7, there exists $e_1 \in N'' \setminus 0$ such that $\psi(e_1) + \psi(-e_1) \leq C$. We may assume that $e_1$ is a primitive element of $N$. Consider the lattice $N' = N/(\mathbb{Z} \cdot e_1)$ and let $\pi_{\mathbb{Z}}: N \to N'$ be the induced projection map. There exists $h': \sigma' \to \mathbb{R}$ such that $h = h' \circ \pi$. Define

$$\psi'(e') = \inf_{e \in \pi^{-1}(e')} \psi(e).$$

Then $\psi'$ is a log discrepancy function on $N'_{\mathbb{R}}$ and $\square = \square_{h'} \in \mathcal{M}(\psi')$, by Lemma 2.9. We repeat this argument $d$ times, until we obtain a basis $e_1, \ldots, e_d$ of $N''$ with the following properties:

(i) $\psi(e_1) + \psi(-e_1) \leq C$.
(ii) $\inf(e_{e_k + \sum_{i=1}^{k-1} \mathbb{Z} \cdot e_i}) + \inf(e_{-e_k + \sum_{i=1}^{k-1} \mathbb{Z} \cdot e_i}) \leq C$, for $2 \leq k \leq d$.

By Lemma 2.8, $e_1$ belongs to a finite set. By Lemmas 2.9 and 2.3, $e_k$ belongs to a finite set modulo $\sum_{i=1}^{k-1} \mathbb{Z} \cdot e_i$, for every $k$. Therefore $N''$, the subspace of $N$ generated by $e_1, \ldots, e_d$, belongs to a finite set of sublattices of $N$.

For $N''$ as above, let $N' = N/N''$. There exists $h': \sigma' \to \mathbb{R}$ such that $h = h' \circ \pi$. Define the log discrepancy function $\psi': \sigma' \to \mathbb{R}$ by

$$\psi'(e') = \inf_{e \in \pi^{-1}(e')} \psi(e).$$

By Theorem 2.6 again, we have $\Delta_{\square}(1) \subseteq N'_{\text{prim}} \cap \{e' \in N'_{\mathbb{R}}; \psi'(e') \leq 1\}$. Therefore the ample fans $\Delta_{\square}$ are finitely many.
Since $d \leq \text{dim}(\sigma)$, we conclude that the number of ample fans is finite. \qed

**Lemma 2.9** (Restriction of saturation). Let $\psi: \sigma \to \mathbb{R}$ be a log discrepancy function, let $\square \in \mathcal{C}(\sigma')$ and let $\pi_\mathbb{Z}: N \to N'$ be a quotient lattice. We identify the dual lattice $M'$ with $M \cap \text{Ker}(\pi)^\perp \subset M$. The image $\sigma' = \pi(\sigma)$ is a rational convex polyhedral cone in $N'_R$.

Assume that $m_0 \in M \cap \square$. The convex set $\square' = (\square - m_0) \cap M'_R$ belongs to $\mathcal{C}(\sigma'^v)$ and its support function $h' \in \mathcal{S}(\sigma')$ is computed as follows

$$h'(e') = \sup\{h(e) - \langle m_0, e \rangle; e \in \sigma \cap \pi^{-1}(e')\}.$$ Define a positively homogeneous function $\psi': \sigma' \to \mathbb{R}$ by

$$\psi'(e') = h'(e') - \sup\{h(e) - \langle m_0, e \rangle - \psi(e); e \in \sigma \cap \pi^{-1}(e')\}.$$ Then the following properties hold:

(i) If $\square$ is $\psi$-saturated, then $\square'$ is $\psi'$-saturated.

(ii) For a positive integer $k$, define $\psi'_k: \sigma' \to \mathbb{R}$ by

$$\psi'_k(e') = k h'(e') - \sup\{kh(e) - \langle m_0, e \rangle - \psi(e); e \in \sigma \cap \pi^{-1}(e')\}.$$ If $\square$ is asymptotically $\psi$-saturated, then $\square'$ is asymptotically $\psi'_k$-saturated.

(iii) If $h - \psi$ is upper convex, then $h' - \psi'$ is upper convex and $\psi'$ is a log discrepancy function.

(iv) $\psi'(e') \geq \inf_{e \in \sigma \cap \pi^{-1}(e')} \psi(e)$.

**Proof.** We may assume $m_0 = 0$ after a translation of $\square$.

(i) The inclusion $\square_{h' - \psi'} \subseteq \square$ is easy to see. Since $\square$ is $\psi$-saturated, $M' \cap \square_{h' - \psi} \subseteq \square$. Therefore $M' \cap \square_{h' - \psi'} \subseteq M' \cap \square = \square'$.

(ii) Note first the following identity:

$$(\psi'_j - \psi'_k)(e') = (j - k) \sup_{\pi(e) = e'} h(e) + \sup_{\pi(e) = e'} (kh - \psi)(e) - \sup_{\pi(e) = e'} (jh - \psi')(e).$$

Therefore $\psi'_k \leq \psi'_j$ for $k \leq j$.

By assumption, there exists a positive integer $I$ such that $jh$ is $\psi$-saturated for every $I|j$. Fix $k \geq 1$ and let $j$ be a common multiple of $I$ and $k$. By (i), $jh'$ is $\psi'_j$-saturated. Since $\psi'_j \geq \psi'_k$, we infer that $jh'$ is also $\psi'_k$-saturated.

(iii) The upper convexity of $h' - \psi'$ follows from the upper convexity of $h - \psi$ and the formula

$$(h' - \psi')(e') = \sup_{\pi(e) = e'} (h - \psi)(e).$$
In particular, $\psi'$ is a continuous function, being the difference of the continuous functions $h'$ and $h' - \psi'$. Furthermore, $\psi'$ is positively homogeneous by its definition. Let $0 \neq e' \in \sigma'$. The restriction $\psi|_{\pi^{-1}(e')}$ is strictly positive, continuous and at least 1 outside some bounded subset, by Lemma 2.3. Therefore
\[ \inf_{\pi(e) = e'} \psi(e) > 0. \]
We conclude from (iv) that $\psi'(e') > 0$.

(iv) This is a direct consequence of the definitions of $h'$ and $\psi'$.

□

3. Rational polyhedral criterion

**Theorem 3.1.** Let $\sigma \subseteq N_\mathbb{R}$ be a rational convex polyhedral cone and let $\square \in C(\sigma')$. Assume that there exists a log discrepancy function $\psi: \sigma \to \mathbb{R}$ such that $\square$ is asymptotically $\psi$-saturated.

Then for every $e_1 \in \sigma \setminus 0$, there exist $m \in M_\mathbb{Q} \cap \square$ and a rational convex polyhedral cone $\sigma_1 \subset \sigma$, with the following properties:

(i) $e_1 \in \text{relint}(\sigma_1)$.

(ii) $\psi(\square)(e) = \langle m, e \rangle$ for $e \in \sigma_1$.

**Proof.** Choose norms $\| \cdot \|$ on $N_\mathbb{R}$ and $M_\mathbb{R}$, defined as the maximum of the absolute values of the components with respect to some basis of $N$ and its dual basis in $M$, respectively. Let
\[ S(\sigma) = \{ e \in \sigma; \| e \| = 1 \}. \]
Define the positive real number $\epsilon(\psi)$ by the formula
\[ -\epsilon(\psi)^{-1} = \min \{ \frac{\langle m, e \rangle}{\psi(e)}; \| m \| = 1, e \in S(\sigma) \}. \]

The restriction of $\psi$ to $S(\sigma)$ is a positive, continuous function, hence $\epsilon(\psi)$ is a well defined. In particular,
\[ \langle m, e \rangle + \psi(e) > 0 \text{ for } 0 \neq e \in \sigma, \| m \| < \epsilon(\psi). \]
Denote by $h \in S(\sigma)$ the support function of $\square$. (1) There exists $m \in M_\mathbb{Q} \cap \square$ such that $\langle m, e_1 \rangle = h(e_1)$.

Indeed, let $\tau$ be the unique face of $\sigma$ which contains $e_1$ in its relative interior. We may find orthogonal decompositions
\[ N = N' \oplus N'', M = M' \oplus M'', \]
where $N' = N \cap (\tau - \tau)$, $M' = M \cap \tau^\perp$ and $M'$, $N'$ and $M''$, $N''$ are dual lattices, respectively. If $N'' \neq 0$, let $\sigma''$ be the image of $\sigma$ under the projection map $N_\mathbb{R} \to N''_\mathbb{R}$. Since $\tau \supseteq \sigma \cap (-\sigma)$, we infer that $\sigma''$ is a strongly rational convex polyhedral cone.
(1a) Since $h$ is the support function of the non-empty convex set $\square_h$, there exists a sequence of points $m_k \in \square_h$ such that
\[
\lim_{k \to \infty} \langle m_k, e_1 \rangle = h(e_1).
\]
If we decompose $m_k = m'_k + m''_k$, we claim that $m'_k$ belongs to a bounded set of $M'_\mathbb{R}$.

Indeed, assume by contradiction that $\lim_{k \to \infty} \|m'_k\| = +\infty$. By the usual compactness argument, we may assume that there exists $m' \in M'_\mathbb{R}$ such that
\[
\lim_{k \to \infty} \frac{m'_k}{\|m'_k\|} = m'.
\]
For every $e \in \tau$, we have
\[
\frac{\langle m'_k, e \rangle}{\|m'_k\|} = \frac{\langle m_k, e \rangle}{\|m'_k\|} \geq \frac{h(e)}{\|m'_k\|}.
\]
Letting $k$ converge to infinity, we obtain $\langle m', e \rangle \geq 0$. Furthermore,
\[
\lim_{k \to \infty} \langle m'_k, e_1 \rangle = \lim_{k \to \infty} \langle m_k, e_1 \rangle = h(e_1),
\]
so a similar argument gives $\langle m', e_1 \rangle = 0$. Therefore $0 \neq m' \in \tau' \cap e_1^\perp$.
Since $e_1$ belongs to the relative interior of $\tau$, we infer that $m' \in \tau^\perp$. This implies $m' = 0$, a contradiction. Therefore the claim holds.

(1b) By (1a), we may replace $(m_k)_k$ by a subsequence so that there exists $m' \in M'_\mathbb{R}$ with
\[
\lim_{k \to \infty} m'_k = m' \text{ and } \langle m', e_1 \rangle = h(e_1).
\]
By Theorem 1.6, there exists a positive multiple $j$ of $I$ such the following system has a solution:
\[
\begin{cases}
  m'_j \in M' \\
  \langle jm', e_1 \rangle - \psi(e_1) < \langle m'_j, e_1 \rangle \leq \langle jm', e_1 \rangle \\
  \|m'_j - jm'\| < \frac{1}{2} \psi(e)
\end{cases}
\]
We now choose $k$ large enough so that
\[
j \|m' - m'_k\| < \frac{1}{2} \psi(e).
\]
Since $\sigma''$ is a strongly rational convex polyhedral cone, the following system has a solution
\[
\begin{cases}
  m''_j \in M'' \\
  m''_j \in jm'_k + \sigma''^\perp
\end{cases}
\]
Set $m_j = m'_j + m''_j \in M$. The following holds for $e \in S(\sigma)$:

$$
\langle m_j, e \rangle - jh(e) + \psi(e) = \langle m_j - jm_k, e \rangle + \psi(e) + j(\langle m_k, e \rangle - h(e)) \\
\geq \langle m_j - jm_k, e \rangle + \psi(e) \\
\geq \langle m'_j - jm'_k, e \rangle + \psi(e) \\
> 0,
$$

where the latter inequality follows from

$$
\|m'_j - jm'_k\| \leq \|m'_j - jm'\| + \|jm' - jm'_k\| < \epsilon(\psi).
$$

Since $jh$ is $\psi$-saturated, we infer that $m_j \in \square_{jh}$. In particular,

$$
\langle m_j, e_1 \rangle \geq jh(e_1).
$$

The opposite inequality holds from construction, hence $\langle m_j, e_1 \rangle = jh(e_1)$. Therefore we obtain

$$
m := \frac{m_j}{j} \in M_\mathbb{Q} \cap \square_h, \langle m, e_1 \rangle = h(e_1).
$$

(2) Since $m$ is rational, we may replace $\square$ by $\square - m$, or equivalently, we replace $h$ by $h - m$. Thus we may assume that $0 \in \square_h$ and $e_1 \in \sigma_0$, where

$$
\sigma_0 = \{ e \in \sigma; h(e) = 0 \}.
$$

There exists a decomposition $N = N' \oplus N''$, with dual decomposition $M = M' \oplus M''$, such that $e_1 = e'_1 + e''_1$, $e'_1 \in N'_\mathbb{Q}, e''_1 \in N''_\mathbb{R}$ and

$$
\{ m'' \in M''; \langle m'', e''_1 \rangle \in \mathbb{Q} \} = \{ 0 \}.
$$

If $e''_1 = 0$, then $e_1 \in N_\mathbb{Q}$ and the theorem holds for $\sigma_1 = \mathbb{R}_{\geq 0} \cdot e_1$ and $m = 0$.

(2a) Assume that $e''_1 \neq 0$. We claim that the following equality holds

$$
\sigma''_0 \cap e''_1 = \sigma''_0 \cap M''_\mathbb{R} \cap e''_1.
$$

We only have to prove the direct inclusion. Fix $m \in \sigma''_0 \cap e''_1$. We have to show that $m'' = 0$, where $m = m' + m''$ is the decomposition in $M'_\mathbb{R} \oplus M''_\mathbb{R}$. Assume by contradiction that $m'' \neq 0$. Since $\langle m, e_1 \rangle \in \mathbb{Q}$, we infer by Lemma 18 and Theorem 17 that there exist a positive integer $k$ and $m_1 \in M$ such that $-\psi(e_1) < \langle m_1 - km, e_1 \rangle < 0$ and $\|m_1 - km\| < \epsilon(\psi)$. Since $\langle m, e_1 \rangle = 0$, we obtain

$$
-\psi(e_1) < \langle m_1, e_1 \rangle < 0, \|m_1 - km\| < \epsilon(\psi).
$$

We consider the following set

$$
C = \{ e \in S(\sigma); \langle m_1, e \rangle + \psi(e) \leq 0 \}.
$$

Since $0 \in \square_h$, we have $h \leq 0$. If the set $C$ is empty, then

$$
\langle m_1, e \rangle - jh(e) + \psi(e) \geq \langle m_1, e \rangle + \psi(e) > 0, \forall e \in S(\sigma).
$$
Therefore $m_1 \in M \cap \circ_{jh-\psi}$, hence $m_1 \in \square_{jh}$ by saturation. In particular, $\langle m_1, e_1 \rangle \geq 0$, which contradicts the choice of $m_1$.

Therefore the set $C$ is non-empty. Since $\psi$ is continuous, $C$ is also compact. If $C \cap \sigma_0 = \emptyset$, then there exists a positive integer $j$ with

$$j > \sup_{e \in C} \frac{\langle m_1, e \rangle + \psi(e)}{h(e)}.$$

Then $m_1 \in M \cap \circ_{jh-\psi}$, and saturation implies that $m_1 \in \square_{jh}$, hence $\langle m_1, e_1 \rangle \geq 0$, which contradicts the choice of $m_1$.

Therefore there exists $e \in C \cap \sigma_0$. In particular,

$$\langle m, e \rangle < \frac{\langle m_1, e \rangle + \psi(e)}{k} \leq 0.$$

Therefore $\langle m, e \rangle < 0$, contradicting the assumption $e \in \sigma_0, m \in \sigma_0^\vee$.

(2b) The function $h$ is continuous, being upper convex \((\text{[5], Theorem 10.1})\). Therefore $\sigma_0$ is a closed convex cone in $N_{\mathbb{R}}$. By duality (cf. \([4], \text{Theorem A.1})$, (2a) is equivalent to

$$\sigma_0 + \mathbb{R} \cdot e_1 + N''_{\mathbb{R}} = \sigma_0 + \mathbb{R} \cdot e_1.$$

In particular, there exists an open neighborhood $U''$ of $e''_1$ in $N''_{\mathbb{R}}$ such that

$$e'_1 + U'' \subset \sigma_0.$$

Since $\dim(U'') = \dim(N'')$, there exist $\bar{e}_1, \ldots, \bar{e}_{n+1} \in U'' \cap N''_{\mathbb{Q}}$, where $n = \dim(N'')$, and there exists $\lambda_i \in (0, 1)$ such that $\sum_{i=1}^{n+1} \lambda_i = 1$ and

$$e''_1 = \sum_{i=1}^{n+1} \lambda_i \bar{e}_i.$$

Let $\sigma_1$ be the rational polyhedral cone spanned by $e'_1 + \bar{e}_1, \ldots, e'_1 + \bar{e}_{n+1}$. It is clear that $\sigma_1 \subset \sigma, e_1 \in \text{relint}(\sigma_1)$ and $h|_{\sigma_1} = 0$. \(\square\)

**Theorem 3.2.** Let $\sigma \subseteq N_{\mathbb{R}}$ be a rational convex polyhedral cone and let $\square \in C(\sigma^\vee)$, with support function $h \in S(\sigma)$. Assume that there exists a log discrepancy function $\psi: \sigma \rightarrow \mathbb{R}$ such that

(i) $\square$ is asymptotically $\psi$-saturated;

(ii) $h - \psi$ is upper convex.

Then $\square$ is a rational convex polyhedral set.

**Proof.** We prove the result by induction on $\dim(N)$. If $\dim(N) = 1$, then $\square$ is either a point or an interval of the form $[a, b]$ or $[a, +\infty)$. Its endpoints are rational by Theorem 3.1, hence $\square_h$ is a rational convex polyhedral set.
Assume now that \( \dim(N) > 1 \) and the theorem holds for smaller dimensional lattices \( N \). We prove the theorem in three steps.

1. Assume 0 ∈ □ and \( \sigma_1 \subset \sigma \) is a rational convex polyhedral cone such that \( h|_{\sigma_1} = 0 \). Then there exists a rational polyhedral cone \( \sigma_2 \subset \sigma \) such that \( \text{relint}(\sigma_1) \subset \text{relint}(\sigma_2) \), and one of the following two properties holds:
   (a) \( \dim(\sigma_2) = \dim(\sigma_1) + 1 \), and \( h|_{\sigma_2} = 0 \).
   (b) \( \dim(\sigma_2) = \dim(\sigma) \) and there exist finitely many rational points \( m_1, \ldots, m_n \in M_Q \cap \square \) such that for every \( e \in \sigma_2 \) there exists some \( i \) with \( \langle m_i, e \rangle = h(e) \).

Proof. Let \( N' = N/(N \cap (\sigma_1 - \sigma_1)) \), with dual lattice \( M' = M \cap \sigma_1^\perp \). If \( \dim(\sigma_1) = \dim(N) \), we are in case (1b). Assume now \( \dim(\sigma_1) < \dim(N) \), so that \( 0 < \dim(N') < \dim(N) \). With the notations of Lemma 2.9, we have a projection homomorphism

\[
\pi_Z: N \to N', \sigma' = \pi(\sigma),
\]

the support function \( h': \sigma' \to \mathbb{R} \) of \( \square \cap M'_R \) and the log discrepancy functions \( \psi'_k: \sigma' \to \mathbb{R} \), for \( k \geq 1 \). By Lemma 2.9 \( \square_{h'} \) is asymptotically \( \psi'_k \)-saturated and \( kh' - \psi'_k \) is upper convex. The inductive assumption implies that there exists a finite set \( \{m'_i\}_{i \in I} \subset M'_Q \cap \square \) such that for every \( e' \in \sigma' \),

\[
h'(e') = \langle m'_i, e' \rangle \text{ for some } i \in I.
\]

We distinguish two cases, depending on whether the convex set \( \square_{h'} \subset M'_R \) is maximal dimensional or not.

(a) Assume \( \dim(\square_{h'}) < \dim(M') \). Equivalently, the lattice \( N'' = N' \cap \square_{h'}^\perp \) is non-zero. Let \( \psi'' = \psi'|_{N''} \) and let \( M'' \) be the dual lattice of \( N'' \). By Theorem 2.6 \( M'' \cap \square_{-\psi''} = \{0\} \).

Furthermore, \( kh - \psi_k \) is upper convex and \( h|_{N''} \) is linear, hence \( -\psi'' \) is upper convex. Therefore Theorem 2.7 applies, hence there exists \( 0 \neq e'_k \in N'' \) such that

\[
\psi''(e'_k) + \psi''(-e'_k) \leq C,
\]

where \( C \) is a positive constant depending only on \( \dim(N'') \). By Lemma 2.3 the \( e'_k \)'s belong to a compact set, hence we may assume that \( e'_k = e' \) for infinitely many \( k \)'s. Then there exist \( e^+_k, e^-_k \in \sigma \) such that \( \pi(e^+_k) = e' \), \( \pi(e^-_k) = -e' \) and

\[
k[h'(e') - h(e^+_k)] + h'(e') - h(e^-_k) + \psi(e^+_k) + \psi(e^-_k) \leq C + 1.
\]

In particular,

\[
\psi(e^+_k) + \psi(e^-_k) \leq C + 1.
\]
By Lemma 2.3, the sequences \((e^+_k)_k, (e^-_k)_k\) belong to a compact set, so we may assume that the limits \(e^- = \lim_{k \to \infty} e^-_k, e^+ = \lim_{k \to \infty} e^+_k\) exist. It is clear that \(e^+, e^- \in \sigma\) and \(\pi(e^+) = e', \pi(e^-) = -e'\). The above inequality and the positivity of \(\psi\) implies

\[ h'(e') - h(e^+_k) + h'(-e') - h(e^-_k) \leq \frac{C + 1}{k}. \]

Letting \(k\) converge to infinity, we obtain \(h'(e') = h(e^+), h'(-e') = h(e^-).\) Since \(e' \in N''\), we have \(h'(e') = h'(-e') = 0.\)

We claim that we may assume that \(e^+, e^- \in N_Q.\) Indeed, since \(\pi(e^+) \in N_Q'\) and \(\sigma_1\) is rational, there exists \(f \in \sigma_1\) such that \(e^+ + f \in \sigma_1 \cap N.\) Then \(h(e^+ + f) \geq h(e^+) + h(f) = 0,\) hence \(h(e^+ + f) = 0.\) Also, \(\pi(e^+ + f) = e',\) hence we may replace \(e^+\) by \(e^+ + f.\) A similar argument applies to \(e^-\).

It is easy to verify that the rational convex polyhedral cone

\[ \sigma_2 = \sigma_1 + \mathbb{R}_{\geq 0}e^+ + \mathbb{R}_{\geq 0}e^- \]

satisfies (1a).

(b) Assume \(\dim(\Delta_{h'}) = \dim(M').\) In this case, the ample fan \(\Delta_{h'}\) of \(h'\) is a fan in \(N'\) with \(|\Delta_{h'}| = \sigma'.\)

(b1) For every \(e' \in \Delta_{h'}(1),\) there exists \(e \in \sigma \cap N\) such that \(\pi(e) = e'\) and \(h(e) = h'(e').\)

Indeed, since \(h'\) is rational piecewise linear and asymptotically \(\psi'_k\)-saturated, we obtain by Theorem 2.6 that

\[ \psi'_k(e') \leq 1, \forall k \geq 1. \]

Therefore there exists \(e_k \in \sigma\) such that \(\pi(e_k) = e'\) and

\[ kh'(e') - (kh - \psi)(e_k) \leq 2. \]

In particular, we obtain \(\psi(e_k) \leq 2.\) By Lemma 2.3 the sequence \((e_k)_k\) belongs to a bounded set, so we may assume that the limit \(e = \lim_{k \to \infty} e_k\) exists. We clearly have \(\pi(e) = e'.\) The positivity of \(\psi\) implies

\[ h'(e') - h(e_k) \leq \frac{2}{k}. \]

Letting \(k\) converge to infinity, we obtain \(h'(e') - h(e) \leq 0,\) hence \(h'(e') - h(e) = 0.\) The rationality of \(e\) is obtained the same way as in the proof of (a) above.

(b2) Let \(\tau\) be a maximal dimensional cone of \(\sigma',\) spanned by \(e'_1, \ldots, e'_r \in \Delta_{h'}(1).\) There exists \(i \in I\) such that \(h'(e') = \langle m_i, e' \rangle\) for every \(e' \in \sigma'.\) By (b1), there exist \(e_j \in \sigma \cap N_Q\)
such that $\pi(e_j) = e'_j$ and $h(e_j) = h'(e'_j)$, for $1 \leq j \leq r$. Therefore $h(e) = \langle m_i, e'_j \rangle$ for every $e \in \sigma_1 + \sum_{j=1}^{p} \mathbb{R}_{\geq 0} e_i$. The cone $\sigma_1 + \sum_{j=1}^{p} \mathbb{R}_{\geq 0} e_i \subset \sigma$ has the same dimension as $\sigma$. The union of all these cones, taken after all maximal cones $\tau$ in $\Delta$, contains a cone $\sigma_2$ satisfying (1b) with respect to $\{m'_i\}_{i \in I}$. □

(2) Every non-zero point $e \in \sigma$ has an open polyhedral neighborhood on which $h$ is rational, piecewise linear.

Indeed, fix $e$ as above. By Theorem 3.1 there exists $m_0 \in M_\mathbb{Q} \cap \square_h$ and there exists a rational convex polyhedral cone $\sigma_0 \subset \sigma$ such that $e \in \text{relint}(\sigma_0)$ and $h(e) = \langle m_0, e \rangle$ for every $e \in \sigma_0$.

We may replace $\square_h$ by its rational translate $\square_h - m_0$, so that we may assume that $m_0 = 0$. In particular, $0 \in \square_h$ and $h|_{\sigma_0} = 0$. By (1), either the claim holds, or there exists a $(\dim(\sigma_0) + 1)$-dimensional rational polyhedral cone $\sigma_1 \subset \sigma$ such that $\text{relint}(\sigma_0) \subset \text{relint}(\sigma_1)$ and $h|_{\sigma_1} = 0$. By (1) again, either the claim holds, or there exists a $(\dim(\sigma_1) + 2)$-dimensional cone $\sigma_2 \subset \sigma$ such that $\text{relint}(\sigma_1) \subset \text{relint}(\sigma_2)$ and $h|_{\sigma_2} = 0$. We repeat this procedure for $\sigma_2$ and so on. This procedure clearly stops in a finite number of steps, hence the claim holds.

(3) Fix a norm $\| \cdot \|$ on $N_\mathbb{R}$ and set $S(\sigma) = \{ e \in \sigma; \|e\| = 1 \}$. For each point $e \in S(\sigma)$, we consider the pair $(\sigma(e); \{m_i(e)\}_{i \in I(e)})$ constructed in (2). We obtain an open covering

$$S(\sigma) = \bigcup_{e \in S(\sigma)} S(\sigma) \cap \text{relint}(\sigma(e)).$$

Since $S(\sigma)$ is compact, it may be covered by the relative interiors of the cones corresponding to finitely many points $e_1, \ldots, e_k$. Let $K$ be the convex hull of the finitely many rational points

$$\{m_i(e_1)\}_{i \in I(e_1)} \cup \ldots \cup \{m_i(e_k)\}_{i \in I(e_k)}.$$

Then $\square = K + \sigma^\vee$, i.e. $\square$ is a rational convex polyhedral set. □

4. Toric FGA algebras

**Theorem 4.1.** Let $\pi: X \to S$ be a proper surjective toric morphism with connected fibers, and let $B$ be an invariant $\mathbb{Q}$-divisor on $X$ such
that \((X, B)\) is a log pair with Kawamata log terminal singularities and \(- (K + B)\) is \(\pi\)-nef. Let

\[ \mathcal{L} \subseteq \bigoplus_{i=0}^{\infty} \pi_* \mathcal{O}_X(iD) \]

be an invariant graded \(\mathcal{O}_S\)-subalgebra, where \(D\) is an invariant \(\mathbb{R}\)-divisor on \(X\), such that \(\mathcal{L}\) is asymptotically saturated with respect to \((X/S, B)\).

Then \(\mathcal{L}\) is finitely generated.

**Proof.** We may assume that \(S\) is affine. Then \(X = T_N \text{emb}(\Delta), S = T_N(\bar{\sigma})\), and \(\pi\) corresponds to a map of fans \(\varphi: (N, \Delta) \to (\bar{N}, \bar{\sigma})\) such that \(|\Delta| = \varphi^{-1}(\bar{\sigma})\) is a rational convex set, denoted by \(\sigma\).

We can write \(B = \sum_{e \in \Delta(1)} b_e V(e)\), where \(\Delta(1)\) is the set of primitive vectors on the one dimensional cones of \(\Delta\). The log canonical divisor \(K + B\) is represented by a function \(\psi: \sigma \to \mathbb{R}\) such that \(\psi\) is \(\Delta\)-linear and \(\psi(e) = 1 - b_e\) for every \(e \in \Delta(1)\). Since \((X, B)\) has Kawamata log terminal singularities, \(\psi\) is a log discrepancy function.

If \(\mathcal{L} = \mathcal{L}_0\), then \(\mathcal{L}\) is finitely generated. Otherwise, we may replace \(I\) by a multiple so that \(\mathcal{L}_i \neq 0\) for every \(Ii\). Let \(i\) be a positive multiple of \(I\). Since \(\mathcal{L}\) is torus invariant, there exist finitely many lattice points \(m_{i,1}, \ldots, m_{i,n_i} \in M\) such that \(\chi^{m_{i,1}}, \ldots, \chi^{m_{i,n_i}}\) generate the \(\mathcal{O}_S\)-module \(\mathcal{L}_i\). Define \(h_i: \sigma \to \mathbb{R}\) by

\[ h_i(e) = \min_{j=1}^{n_i} (m_{i,j}, e). \]

The support function \(h_i\) is independent of the choice of generators, and the torus invariant \(\mathcal{O}_S\)-algebra

\[ \bar{\mathcal{L}} = \bigoplus_{i=0}^{\infty} \bigoplus_{m \in M \cap \mathbb{R}_{h_i}} \mathbb{C} \cdot \chi^m \]

is the integral closure of \(\mathcal{L}\) in its field of fractions ([6], Proposition 4.15).

Choose a refinement \(\Delta_i\) of the fan \(\Delta\) so that \(\Delta_i\) is a simple fan and \(h_i\) is \(\Delta_i\)-linear. This corresponds to a toric resolution of singularities \(\mu_i: X_i = T_N \text{emb}(\Delta_i) \to X\) such that \(M_i = \sum_{e \in \Delta_i(1)} -h_i(e) V(e)\) is a \(\pi \circ \mu_i\)-free divisor. Since \(X_i\) is nonsingular, the union of its invariant prime divisors \(\sum_{e \in \Delta_i(1)} V(e)\) has simple normal crossings.

In the above set-up, \(\mathcal{L}\) is asymptotically saturated with respect to \((X/S, B)\) if and only if

\[ H^0(X_i, [K_{X_i} - \mu_i^*(K + B) + \frac{i}{j} M_i]) \subseteq H^0(X_j, M_j), \forall Ii, j. \]
Step 1: Asymptotic saturation is equivalent to the following property:

\[ M \cap \mathring{\varnothing}_{h_i - \psi} \subset \mathring{\varnothing}_{h_j}, \forall I \mid j. \]

Indeed, let \( m \in M \). Then \( \chi^m \in H^0(X_i, [K_{X_i} - \mu_i^*(K + B) + \frac{i}{\tau} M_i]) \) if and only if \( \langle m, e \rangle + [\langle 1 + \psi(e) - \frac{i}{\tau} h_i(e) \rangle] \geq 0 \) for every \( e \in \Delta_i(1) \). Since \( \langle m, e \rangle \in \mathbb{Z} \), this is equivalent to \( \langle m, e \rangle > \frac{i}{\tau} h_i(e) - \psi(e) \) for every \( e \in \Delta_i(1) \). Since \( \psi \) and \( h_i \) are \( \Delta_i \)-linear, the latter is equivalent to \( \langle m, e \rangle > \frac{i}{\tau} h_i(e) - \psi(e) \) for every \( e \in \sigma \setminus 0 \). On the other hand, \( \chi^m \in H^0(X_j, M_j) \) if and only if \( m \in \mathring{\varnothing}_{h_j} \).

Step 2: The function \( h = \lim_{i \to \infty} \frac{i}{\tau} h_i : \sigma \to \mathbb{R} \) is a well defined positively homogeneous, upper convex function.

Indeed, we can write \( D = \sum_{e \in \Delta(1)} d_e V(e) \). Let \( \bar{h} : \sigma \to \mathbb{R} \) be the support function of the convex set

\[ \{ m \in M_\mathbb{R} : \langle m, e \rangle \geq -d_e, \forall e \in \Delta(1) \}. \]

Since \( L_i \subseteq H^0(X, iD) \), we obtain \( h_i \geq \bar{h} \). On the other hand, the property \( L_i \cdot L_j \subseteq L_{i+j} \) implies \( h_i + h_j \geq h_{i+j} \). Then it is easy to see that for every \( e \in \sigma \), the sequence \( \frac{i}{\tau} h_i(e) \) is bounded from below and converges to its infimum. Being a limit of positively homogeneous upper convex functions, \( h \) satisfies these two properties too. Note that \( h_i \geq i \bar{h} \) for every \( i \).

Step 3: Asymptotic saturation is equivalent to the following property:

\[ M \cap \mathring{\varnothing}_{jh - \psi} \subset \mathring{\varnothing}_{h_j}, \forall I \mid j. \]

Indeed, fix \( I \mid j \), choose a norm \( \| \cdot \| \) on \( N_\mathbb{R} \) and set \( S(\sigma) = \{ e \in \sigma ; \| e \| = 1 \} \). Let \( m \in M \cap \mathring{\varnothing}_{jh - \psi} \). This means that the function

\[ f : S(\sigma) \to \mathbb{R}, e \mapsto \langle m, e \rangle - jh(e) + \psi(e) \]

is positive. The functions \( \frac{i}{\tau} h_i \) are upper convex, hence they converge uniformly to \( h \) on the compact set \( S(\sigma) \), by Theorem 1.11. Therefore there exists some \( i \) such that \( f - j(\frac{i}{\tau} h_i - h)|_{S(\sigma)} \) is a positive function.

This means that \( m \in M \cap \mathring{\varnothing}_{jh - \psi} \). By Step 1, we obtain \( m \in \mathring{\varnothing}_{h_j} \).

The converse is clear by Step 1, since \( h_i \geq i \bar{h} \).

Step 4: \( nh = h_n \) for some \( I \mid n \).

Indeed, \( -\psi \) is upper convex since \( -(K + B) \) is nef. Therefore \( h - \psi \) is upper convex. By Step 3, the hypothesis of Theorem 3.2 are satisfied, hence \( \mathring{\varnothing}_h \) is a rational polyhedral set. In particular, there exists some \( I \mid n \) such that \( \mathring{\varnothing}_{nh} \) is the convex hull of its lattice points. We have

\[ M \cap \mathring{\varnothing}_{nh} \subseteq M \cap \mathring{\varnothing}_{nh - \psi} \subset \mathring{\varnothing}_{h_n}, \]
hence $\square_{nh}$, the convex hull of $M \cap \square_{nh}$, is included in $\square_{h_n}$. Therefore $h_n \geq nh$. The opposite inclusion holds by construction, hence $nh = h_n$.

Step 5: We have $kh_n \geq h_{kn} \geq knh$ for $k \geq 1$. By Step 4, we obtain $h_{kn} = kh_n$ for every $k \geq 1$. This means that

$$\bigoplus_{k=0}^{\infty} \mathcal{L}_{kn} = \bigoplus_{k=0}^{\infty} (\pi \circ \mu_n)_* \mathcal{O}_{X_n}(kM_n).$$

The right hand side is finitely generated since $M_n$ is a $\pi \circ \mu_n$-free divisor, hence $\bigoplus_{k=0}^{\infty} \mathcal{L}_{kn}$ is also finitely generated. Therefore $\mathcal{L}$ is finitely generated (cf. [6], Theorem 4.6). Since $\mathcal{L}$ is the integral closure of $\mathcal{L}$ in its field of fractions, we conclude that $\mathcal{L}$ is finitely generated.

Proof. (of Theorem 1) The statement is local over $S$, hence we may assume that $S$ is affine. Thus $X = T_X \text{emb}(\Delta)$, $S = T_X(\sigma)$, and $\pi$ corresponds to a map of fans $\varphi_Z: (N, \Delta) \to (\bar{N}, \bar{\sigma})$ such that $|\Delta| = \varphi^{-1}(\bar{\sigma})$ is a rational convex set.

Let $\mathcal{L} \subseteq \bigoplus_{i=0}^{\infty} \pi_* \mathcal{O}_X(iD)$ be an invariant graded $\mathcal{O}_S$-subalgebra, where $D$ is an invariant $\mathbb{R}$-divisor on $X$, such that $\mathcal{L}$ is asymptotically saturated with respect to $(X/S, B)$. Then $\mathcal{L}$ is finitely generated by Theorem 4.1, which proves (1). In particular, there exists a rational convex polyhedral set $\square \in C(|\Delta|^\vee)$ (corresponding to the limit support function $h \in S(|\Delta|)$ in the Step 2 of the proof of Theorem 4.1) and an $S$-isomorphism

$$\text{Proj}(\mathcal{L}) \simeq \text{Proj}(\bigoplus_{i=0}^{\infty} \bigoplus_{m \in M \cap \square} \mathbb{C} \cdot \chi^m).$$

The right hand side is the torus embedding of the ample fan $\Delta_{\square}$. Let $\psi: |\Delta| \to \mathbb{R}$ be the log discrepancy function associated to $(X, B)$ (Example 2.2). Then $\square$ is asymptotically $\psi$-saturated. Since $-(K + B)$ is $\pi$-nef, $-\psi$ is upper convex. Therefore $h_{\square} - \psi$ is upper convex. Theorem 2.8 applies, hence $\Delta_{\square}$ belongs to a finite set of fans associated to $(X/S, B)$. This proves (2).

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