Ordering the smallest claim amounts from two sets of interdependent heterogeneous portfolios

Hossein Nadeb, Hamzeh Torabi, Ali Dolati
Department of Statistics, Yazd University, Yazd, Iran,

Abstract

Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be a set of dependent and non-negative random variables share a survival copula and let $Y_i = I_{p_i} X_{\lambda_i}$, $i = 1, \ldots, n$, where $I_{p_1}, \ldots, I_{p_n}$ be independent Bernoulli random variables independent of $X_{\lambda_i}$'s, with $E[I_{p_i}] = p_i$, $i = 1, \ldots, n$. In actuarial sciences, $Y_i$ corresponds to the claim amount in a portfolio of risks. This paper considers comparing the smallest claim amounts from two sets of interdependent portfolios, in the sense of usual and likelihood ratio orders, when the variables in one set have the parameters $\lambda_1, \ldots, \lambda_n$ and $p_1, \ldots, p_n$ and the variables in the other set have the parameters $\lambda_1^*, \ldots, \lambda_n^*$ and $p_1^*, \ldots, p_n^*$. Also, we present some bounds for survival function of the smallest claim amount in a portfolio. To illustrate validity of the results, we serve some applicable models.

Keywords Copula, Majorization, Smallest claim amount, Stochastic order.

1 Introduction

Suppose that $X_{\lambda_1}, \ldots, X_{\lambda_n}$, assuming $X_{\lambda_i}$ has the survival function $F(x; \lambda_i)$, are non-negative random variables denoting the total random severities of $n$ policyholders in an insurance period. Further, let $I_{p_1}, \ldots, I_{p_n}$ be a set of independent Bernoulli random variables, with $I_{p_i}$ is corresponding to $X_{\lambda_i}$, such that $I_{p_i} = 1$ whenever the $i$th policyholder makes random claim amount $X_{\lambda_i}$ and $I_{p_i} = 0$ whenever does not make a claim. In this notation, $Y_i = I_{p_i} X_{\lambda_i}$ is the claim amount related to $i$th policyholder and $(Y_1, \ldots, Y_n)$ is said to be a portfolio of risks. Further, consider another portfolio of risks $(Y_1^*, \ldots, Y_n^*)$ with the parameter vectors $(\lambda_1^*, \ldots, \lambda_n^*)$ and $(p_1^*, \ldots, p_n^*)$.

The annual premium is the amount received by the insurer which is the primary cost to accept the risk. Determining the annual premium is very important problem for the insurance companies. Therefore, deriving preferences between random future gains or losses is an appealing topic for the actuaries. For this purpose, stochastic orders are very helpful. Stochastic orders have been extensively used in the areas management science, financial economics, insurance, actuarial
science, operation research, reliability theory, queuing theory and survival analysis. For a comprehensive discussions on stochastic orders, one may refer to Müller and Stoyan (2002), Shaked and Shanthikumar (2007) and Li and Li (2013).

The problem of orderings of some statistics in the portfolios \((Y_1, \ldots, Y_n)\) and \((Y^*_1, \ldots, Y^*_n)\), such as the number of claims, \(\sum_{i=1}^n I_{p_i}\), the aggregate claim amounts, \(\sum_{i=1}^n Y_i\), the smallest, \(Y_{1:n} = \min(Y_1, \ldots, Y_n)\), and the largest claim amounts, \(Y_{n:n} = \max(Y_1, \ldots, Y_n)\), have been discussed in many researches; see, e.g., Karlin and Novikoff (1963), Ma (2000), Frostig (2001), Hu and Ruan (2004), Demit and Frostig (2006), Khaledi and Ahmad (2008), Zhang and Zhao (2015), Barmalzan et al. (2015), Li and Li (2016), Barmalzan et al. (2018), Barmalzan and Najafabadi (2015), Barmalzan et al. (2016), Barmalzan et al. (2017), Balakrishnan et al. (2018) and Li and Li (2018).

The most of published articles consider the case that the severities are independent, while sometimes this assumption is not satisfied and many of policies are simultaneously at risk, such as when earthquakes or epidemics occur. Here, the severities have a positive dependence.

In this paper, it is assumed that \(X_{\lambda_1}, \ldots, X_{\lambda_n}\) are non-negative and continuous random variables with the joint survival function \(\bar{H}(x_1, \ldots, x_n)\), marginal survival functions \(\bar{F}(x; \lambda_1), \ldots, \bar{F}(x; \lambda_n)\), and the survival copula \(C_R\) through the relation \(\bar{H}(x_1, \ldots, x_n) = C_R(\bar{F}(x_1; \lambda_1), \ldots, \bar{F}(x_n; \lambda_n))\) in the view of the Sklar’s Theorem; see Nelsen (2007). Here, we compare the smallest claim amounts arising from two sets of interdependent heterogeneous portfolios and then mainly focus on presenting some bounds for the survival function of the smallest claim amount in a set of interdependent heterogeneous portfolio.

The rest of the paper is organized as follows. In Section 2, we recall some definitions and lemmas which will be used in the sequel. Subsection 3.1 provides orderings of the smallest claim amounts from two interdependent heterogeneous portfolios of risks for a general model in the sense of the usual stochastic order. Also, it considers the proportional hazard rate model and provides some characterizations on the likelihood ratio order of the smallest claim amounts under some certain conditions. Subsection 3.2 presents some useful lower and upper bounds for the survival function of the smallest claim amount and it establishes some numerical examples to illustrate the validity of the shown results.

2 The basic definitions and some prerequisites

In this section, we state some notions of stochastic orders, majorization, weak majorization and some lemmas which are needed to prove our main results. Throughout the paper, we use the notations \(\mathbb{R} = (-\infty, +\infty), \mathbb{R}_+ = [0, +\infty)\) and \(\bar{x} = \frac{1}{n}\sum_{i=1}^n x_i\). Also, we use the notion of increasingness for a function \(g : \mathcal{A} \rightarrow \mathbb{R}, \mathcal{A} \subseteq \mathbb{R}^n\), if it is non-decreasing in each argument. Similarly, the notion of decreasingness is used, when \(g\) is non-increasing in each argument.
Let $X$ and $Y$ be two non-negative random variables with the distribution functions $F$ and $G$, the survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, the density functions $f$ and $g$ and the hazard rate functions $r_X = f/\bar{F}$ and $r_Y = g/\bar{G}$, respectively.

**Definition 2.1.** $X$ is said to be smaller than $Y$ in the

(i) usual stochastic order, denoted by $X \leq_{st} Y$, if $\bar{F}(x) \leq \bar{G}(x)$ for all $x \in \mathbb{R}$;

(ii) hazard rate order, denoted by $X \leq_{hr} Y$, if $r_Y(x) \leq r_X(x)$ for all $x \in \mathbb{R}$;

(iii) likelihood ratio order, denoted by $X \leq_{lr} Y$, if $g(x) \frac{f}{g} \frac{g}{x}$ is increasing in $x \in \mathbb{R}_+$.

For a comprehensive discussion on various stochastic orders, we refer to Müller and Stoyan (2002), Li and Li (2013) and Shaked and Shanthikumar (2007).

The concepts of majorization of vectors and Schur-convexity and Schur-concavity of functions are also needed. For a comprehensive discussion of these topics we refer to Marshall et al. (2011).

We use the notation $x_1:n \leq x_2:n \leq \ldots \leq x_n:n$ to denote the increasing arrangement of components of the vector $x = (x_1, \ldots, x_n)$.

**Definition 2.2.** The vector $x$ is said to be

(i) weakly submajorized by the vector $y$ (denoted by $x \preceq_w y$) if $\sum_{i=j} x_{i:n} \leq \sum_{i=j} y_{i:n}$ for all $j = 1, \ldots, n$,

(ii) weakly supermajorized by the vector $y$ (denoted by $x \succeq_w y$) if $\sum_{i=1} x_{i:n} \geq \sum_{i=1} y_{i:n}$ for all $j = 1, \ldots, n$,

(iii) majorized by the vector $y$ (denoted by $x \preceq_m y$) if $\sum_{i=1} x_i = \sum_{i=1} y_i$ and $\sum_{i=1} x_{i:n} \geq \sum_{i=1} y_{i:n}$ for all $j = 1, \ldots, n - 1$.

**Definition 2.3.** A real valued function $\phi$ defined on a set $\mathcal{A} \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on $\mathcal{A}$ if

$$x \preceq_m y \text{ on } \mathcal{A} \Rightarrow \phi(x) \leq (\geq) \phi(y).$$

**Lemma 2.1** (Marshall et al. (2011), Theorem 3.A.8). A real valued function $\phi$ defined on a set $\mathcal{A} \subseteq \mathbb{R}^n$ satisfies

$$\phi(x) \leq \phi(y) \text{ whenever } x \succeq_w y \text{ on } \mathcal{A},$$

if, and only if, $\phi$ is decreasing and Schur-convex on $\mathcal{A}$.

**Lemma 2.2** (Marshall et al. (2011), 3.B.2). Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a decreasing and Schur-convex function and $g : \mathbb{R} \to \mathbb{R}$ be an increasing and concave function. Then, the function $\psi(x) = \phi(g(x_1), \ldots, g(x_n))$ is decreasing and Schur-convex.
One of the needed concepts in this paper is the Archimedean copula. The class of Archimedean copulas having a wide range of dependence structures including the independent copula. In the following, we state some useful definitions and lemmas related to copulas.

**Definition 2.4.** A copula $C$ is called Archimedean if it is of the form $C(u_1, \ldots, u_n) = \phi^{-1}\left(\sum_{i=1}^{n} \phi(u_i)\right)$, for $(u_1, \ldots, u_n) \in [0, 1]^n$, which $\phi : [0, 1] \to [0, \infty]$ is a strictly decreasing function, $\phi(0) = \infty$, $\phi(1) = 0$ and $(-1)^k \frac{d^k \phi(x)}{dx^k} \geq 0$, for $k \geq 0$, where $\phi^{-1}$ is the inverse of the function $\phi$. The function $\phi$ is called generator of the copula $C$.

We state the following lemma from Durante (2006) and Dolati and Dehghan Nezhad (2014) related to Schur-concavity of Archimedean copulas.

**Lemma 2.3.** Every Archimedean copula is Schur-concave.

**Definition 2.5.** A survival copula $C_R$ is positively upper orthant dependent (PUOD), if for all $u \in [0, 1]^n$, $C_R(u) \geq \prod_{i=1}^{n} u_i$.

**Definition 2.6.** Let $C_R$ and $D_R$ be two survival copulas. $C_R$ is less PUOD than $D_R$, denoted by $C_R \prec D_R$, if for all $u \in [0, 1]^n$, $C_R(u) \leq D_R(u)$.

**Definition 2.7.** Let $C$ be a copula. The main diagonal section of $C$ is the function $\delta_C : [0, 1] \to [0, 1]$ defined by $\delta_C(u) = C(u, \ldots, u)$.

**Lemma 2.4.** For any copula $C$ and for all $u \in [0, 1]^n$,

$$\max\left(\sum_{i=1}^{n} u_i - n + 1, 0\right) \leq C(u) \leq \min(u_1, \ldots, u_n),$$

which, the bounds are called the Fréchet-Hoeffding bounds.

For a comprehensive discussion in the topic of copula and the different types of dependency, one may refer to Nelsen (2007).

### 3 Main results

This section consists of two subsections. In Subsection 3.1, we compare the smallest claim amounts from two interdependent heterogeneous portfolios of risks in the sense of the usual and the likelihood ratio orders. In subsection 3.2, some bounds for the survival function of the smallest claim amount are presented and some examples are established to illustrate the validity of the results.
3.1 Stochastic comparison of the smallest claim amounts

The following theorem provides the usual stochastic order between the smallest claim amounts in two heterogeneous portfolios of risks with the common parameter vectors $\lambda$ and $p$ and different associated copulas.

**Theorem 3.1.** Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be non-negative random variables with $X_{\lambda_i} \sim \bar{F}(x; \lambda_i)$, $i = 1, \ldots, n$, and the associated copula $C_R$. Further, suppose that $I_{p_1}, \ldots, I_{p_n}$ is a set of independent Bernoulli random variables, independent of the $X_{\lambda_i}$’s, with $E[I_{p_i}] = p_i$, $i = 1, \ldots, n$. Let $Y_i = I_{p_i}X_{\lambda_i}$, $i = 1, \ldots, n$. Then we have

$$C_R^* \prec C_R \implies Y_{1:n}^* \leq_{st} Y_{1:n},$$

where $Y_{1:n}$ and $Y_{1:n}^*$ are the smallest order statistics of $(Y_1, \ldots, Y_n)$ under the copula structures $C_R$ and $C_R^*$, respectively.

**Proof.** First, for any $x \geq 0$, the survival function of $Y_{1:n}$ is obtained as follows:

$$
\bar{G}_{Y_{1:n}}(x) = P(Y_{1:n} > x) = P(I_{p_i}X_{\lambda_i} > x, \forall 1 \leq i \leq n) = P(I_{p_i} = 1, \forall 1 \leq i \leq n)P(I_{p_i}X_{\lambda_i} > x, \forall 1 \leq i \leq n | I_{p_i} = 1, \forall 1 \leq i \leq n) = \left(\prod_{i=1}^{n} p_i\right) P(X_{\lambda_i} > x, \forall 1 \leq i \leq n) = \left(\prod_{i=1}^{n} p_i\right) C_R(\bar{F}(x; \lambda_1), \ldots, \bar{F}(x; \lambda_n)) \tag{1}
$$

Similarly, the survival function of $Y_{1:n}^*$ is given by

$$
\bar{G}_{Y_{1:n}^*}(x) = \left(\prod_{i=1}^{n} p_i\right) C^*(\bar{F}(x; \lambda_1), \ldots, \bar{F}(x; \lambda_n)).
$$

Thus, by Definition 2.6 and comparing the survival functions of $Y_{1:n}$ and $Y_{1:n}^*$ the proof is completed. 

The following theorem provides the usual stochastic order between the smallest claim amounts in two heterogeneous portfolios of risks with the common associated copulas.

**Theorem 3.2.** Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ ($X_{\lambda_1}^*, \ldots, X_{\lambda_n}^*$) be non-negative random variables with $X_{\lambda_i} \sim \bar{F}(x; \lambda_i)$ ($X_{\lambda_i}^* \sim \bar{F}(x; \lambda_i^*)$), $i = 1, \ldots, n$, and the associated copula $C_R$. Further, suppose that $I_{p_1}, \ldots, I_{p_n}$ ($I_{p_1}^*, \ldots, I_{p_n}^*$) is a set of independent Bernoulli random variables, independent of the $X_{\lambda_i}$’s ($X_{\lambda_i}^*$’s), with $E[I_{p_i}] = p_i$ ($E[I_{p_i}^*] = p_i^*$), $i = 1, \ldots, n$. Assume that the following conditions hold:
(i) \( \bar{F}(x; \lambda) \) is increasing and concave in \( \lambda \) for any \( x \in \mathbb{R}_+ \);

(ii) \( C_R \) is Schur-concave.

Then, we have

\[
\prod_{i=1}^{n} p_i^* \leq \prod_{i=1}^{n} p_i, \ (\lambda_1, \ldots, \lambda_n) \leq_{w} (\lambda_1^*, \ldots, \lambda_n^*) \implies Y_{1:n}^* \leq_{st} Y_{1:n}.
\]

Proof. Define \( \Psi(\lambda) = -C_R(\bar{F}(x; \lambda_1), \ldots, \bar{F}(x; \lambda_n)) \). Based on the condition (ii) and the nature of copula, \(-C_R\) is decreasing and Schur-convex. So, the condition (i) and Lemma 2.2 imply that \( \Psi \) is decreasing and Schur-convex in \( \lambda \). Thus, using the Lemma 2.2 \( \lambda \leq_{w} \lambda^* \) implies \( \Psi(\lambda) \leq \Psi(\lambda^*) \).

Hence, the condition \( \prod_{i=1}^{n} p_i^* \leq \prod_{i=1}^{n} p_i \) and the relation (1) complete the proof.

The following theorem provides the ordering of the smallest claim amounts from two heterogeneous portfolios of risks with the different parameter vectors and different associated copulas.

**Theorem 3.3.** Let \( X_{\lambda_1}, \ldots, X_{\lambda_n} \) \((X_{\lambda_1^*}, \ldots, X_{\lambda_n^*})\) be non-negative random variables with \( X_{\lambda_i} \sim \bar{F}(x; \lambda_i) \) \((X_{\lambda_i^*} \sim \bar{F}(x; \lambda_i^*)\)), \( i = 1, \ldots, n \), and associated copula \( C_R \) \((C_R^*)\). Further, suppose that \( I_{p_1}, \ldots, I_{p_n} \) \((I_{p_1^*}, \ldots, I_{p_n^*})\) is a set of independent Bernoulli random variables, independent of the \( X_{\lambda_i}'s \), with \( E[I_{p_i}] = p_i \) \((E[I_{p_i^*}] = p_i^*)\), \( i = 1, \ldots, n \). Assume that the following conditions hold:

(i) \( \bar{F}(x; \lambda) \) is increasing and concave in \( \lambda \) for any \( x \in \mathbb{R}_+ \);

(ii) \( C_R^* \) is Schur-concave.

Then, we have

\[
\prod_{i=1}^{n} p_i^* \leq \prod_{i=1}^{n} p_i, \ (\lambda_1, \ldots, \lambda_n) \leq_{w} (\lambda_1^*, \ldots, \lambda_n^*), \ C_R^* < C_R \implies Y_{1:n}^* \leq_{st} Y_{1:n}.
\]

Proof. Suppose that \( V_{A,P}^{C_R} \) denotes the smallest of the variables \( Y_i = I_{p_i} X_{\lambda_i}, \ i = 1, \ldots, n \), where \((X_{\lambda_1}, \ldots, X_{\lambda_n})\) has the survival copula \( C_R \). It is easily seen that \( Y_{1:n}^* \leq_{st} V_{A,P}^{C_R^*} \) and \( Y_{1:n} \leq_{st} V_{A,P}^{C_R} \).

On the other hand, Theorem 3.1 and Theorem 3.2 imply that \( V_{A,P}^{C_R^*} \leq_{st} V_{A,P}^{C_R} \) and \( V_{A,P}^{C_R^*} \leq_{st} V_{A,P}^{C_R} \), respectively. Hence, the required result is obtained.

Generally, Theorem 3.3 considers the comparison of the smallest claim amounts arising from two portfolios, in the sense of the usual stochastic order. But its result can be obtained in the sense of the stronger orders under some particular cases. The proportional hazard rate (PHR) model is an important model in reliability theory, actuarial science and other fields; see for example Cox (1992), Finkelstein (2008), Kumar and Klefsjö (1994) and Balakrishnan et al. (2018). \( X_{\lambda} \) is said to follow PHR model, if its survival function can be expressed as \( \bar{F}(x; \lambda) = [\bar{F}(x)]^\lambda \), where \( \bar{F}(x) \) is the baseline survival function and \( \lambda > 0 \).
Recently, Li and Li \cite{LiLi} compared $Y_{1:n}$ and $Y^*_{1:n}$ in the sense of the hazard rate order, whenever $X_{\lambda_i} \sim \hat{F}(x; \lambda_i) = \hat{F}^{\lambda_i}(x)$ ($X_{\lambda_i}^* \sim \hat{F}(x; \lambda_i^*) = \hat{F}^{\lambda_i^*}(x)$), for $i = 1, \ldots, n$, and they share a common Gumbel-Hougaard survival copula, which first introduced by Gumbel \cite{Gumbel}, of the form

$$C_R(u) = \exp \left(- \left( \sum_{i=1}^{n} (- \log u_i)^{\theta} \right)^{1/\theta} \right),$$

for $\theta \in [1, \infty)$. They presented a characterization on the hazard rate order of $Y_{1:n}$ as the following lemma.

**Lemma 3.1** (Li and Li \cite{LiLi}). Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ ($X_{\lambda_1}^*, \ldots, X_{\lambda_n}^*$) be non-negative random variables with $X_{\lambda_i} \sim \hat{F}(x; \lambda_i) = \hat{F}^{\lambda_i}(x)$ ($X_{\lambda_i}^* \sim \hat{F}(x; \lambda_i^*) = \hat{F}^{\lambda_i^*}(x)$), $i = 1, \ldots, n$, and associated Gumbel-Hougaard copula. Further, suppose that $I_{p_1}, \ldots, I_{p_n}$ ($I_{p_1}^*, \ldots, I_{p_n}^*$) is a set of independent Bernoulli random variables, independent of the $X_{\lambda_i}$’s, with $E[I_{p_i}] = p_i$ ($E[I_{p_i}^*] = p_i^*$), $i = 1, \ldots, n$. Then, we have

$$\prod_{i=1}^{n} p_i^* \leq \prod_{i=1}^{n} p_i, \quad \sum_{i=1}^{n} \lambda_{i}^\theta \leq \sum_{i=1}^{n} \lambda_{i}^* \theta \iff Y_{1:n}^* \preceq_{hr} Y_{1:n}.$$

The likelihood ratio order of $Y_{1:n}$ can also be characterized under some additional assumptions. The following theorems represent this fact.

**Theorem 3.4.** Under the setup of Lemma 3.1, assume that $\prod_{i=1}^{n} p_i = \prod_{i=1}^{n} p_i^*$. Then, we have

$$\sum_{i=1}^{n} \lambda_{i}^\theta = \sum_{i=1}^{n} \lambda_{i}^* \theta \iff Y_{1:n}^* \preceq_{hr} Y_{1:n}.$$

**Proof.** It can be easily verified that the ratio of density functions can be written as follows:

\[
\frac{g_{Y_{1:n}}(x)}{g_{Y_{1:n}^*}(x)} = \frac{1 - \prod_{i=1}^{n} p_i}{1 - \prod_{i=1}^{n} p_i^*} I_{[x=0]} + \prod_{i=1}^{n} p_i \left( \frac{\sum_{i=1}^{n} \lambda_{i}^\theta}{\sum_{i=1}^{n} \lambda_{i}^* \theta} \right)^{1/\theta} \left[ \hat{F}(x) \right]^{1/\theta} I_{[x>0]},
\]

where, $I_A$ denotes the indicator function. Under the assumption $\prod_{i=1}^{n} p_i = \prod_{i=1}^{n} p_i^*$, we have that

\[
\frac{g_{Y_{1:n}}(x)}{g_{Y_{1:n}^*}(x)} = I_{[x=0]} + \left( \frac{\sum_{i=1}^{n} \lambda_{i}^\theta}{\sum_{i=1}^{n} \lambda_{i}^* \theta} \right)^{1/\theta} \left[ \hat{F}(x) \right]^{1/\theta} I_{[x>0]}.
\]

Clearly, $\frac{g_{Y_{1:n}}(x)}{g_{Y_{1:n}^*}(x)}$ is increasing in $x \geq 0$, if and only if $\left( \frac{\sum_{i=1}^{n} \lambda_{i}^\theta}{\sum_{i=1}^{n} \lambda_{i}^* \theta} \right)^{1/\theta} \left[ \hat{F}(0) \right]^{1/\theta} \left( \sum_{i=1}^{n} \lambda_{i}^\theta \right)^{1/\theta} \left( \sum_{i=1}^{n} \lambda_{i}^* \theta \right)^{1/\theta} \geq 1$ and $\left[ \hat{F}(x) \right]^{1/\theta} \left( \sum_{i=1}^{n} \lambda_{i}^\theta \right)^{1/\theta} \left( \sum_{i=1}^{n} \lambda_{i}^* \theta \right)^{1/\theta}$ is increasing in $x \geq 0$. The former is equivalent to $\sum_{i=1}^{n} \lambda_{i}^\theta \geq \sum_{i=1}^{n} \lambda_{i}^* \theta$ and the latter is equivalent to $\sum_{i=1}^{n} \lambda_{i}^\theta \leq \sum_{i=1}^{n} \lambda_{i}^* \theta$.

**Theorem 3.5.** Under the setup of Lemma 3.1, assume that $\sum_{i=1}^{n} \lambda_{i}^\theta = \sum_{i=1}^{n} \lambda_{i}^* \theta$. Then, we have

$$\prod_{i=1}^{n} p_i^* \leq \prod_{i=1}^{n} p_i \iff Y_{1:n}^* \preceq_{hr} Y_{1:n}.$$
Proof. By the assumption \( \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \lambda_i^\theta \), the relation \( \theta \) can be rewritten as

\[
\frac{g_{Y_{1:n}}(x)}{g_{Y_{1:n}}^\ast(x)} = \frac{1 - \prod_{i=1}^{n} p_i I_{[x=0]}}{1 - \prod_{i=1}^{n} p_i I_{[x>0]}} + \prod_{i=1}^{n} p_i \prod_{i=1}^{n} p_i^\ast I_{[x=0]} + \prod_{i=1}^{n} p_i \prod_{i=1}^{n} p_i^\ast I_{[x>0]}. 
\]

Thus, \( \frac{g_{Y_{1:n}}(x)}{g_{Y_{1:n}}^\ast(x)} \) is increasing in \( x \geq 0 \), if and only if

\[
\prod_{i=1}^{n} p_i \geq \prod_{i=1}^{n} p_i^\ast, 
\]

which is equivalent to \( \prod_{i=1}^{n} p_i \geq \prod_{i=1}^{n} p_i^\ast \).

3.2 Bounds for the survival function of the smallest claim amount

Obtaining some bounds for \( \bar{G}_{Y_{1:n}}(x) \) can be included the important informations for the insurance companies. The following theorem presents useful lower and upper bounds for \( \bar{G}_{Y_{1:n}}(x) \) when the insurance company knows the associated copula, exactly.

**Theorem 3.6.** Let \( X_{\lambda_1}, \ldots, X_{\lambda_n} \) be non-negative random variables with \( X_{\lambda_i} \sim \bar{F}(x; \lambda_i), i = 1, \ldots, n \), and the associated copula \( C_R \). Further, suppose that \( I_{p_1}, \ldots, I_{p_n} \) is a set of independent Bernoulli random variables, independent of the \( X_{\lambda_i} \)'s, with \( \mathbb{E}[I_{p_i}] = p_i, i = 1, \ldots, n \). Assume that the following conditions hold:

(i) \( \bar{F}(x; \lambda) \) is increasing and concave in \( \lambda \) for any \( x \in \mathbb{R}_+ \);

(ii) \( C_R \) is Schur-concave.

Then, we have

\[
\left( \prod_{i=1}^{n} p_i \right) \delta_{C_R}(\bar{F}(x; \lambda_{1:n})) \leq \bar{G}_{Y_{1:n}}(x) \leq \left( \prod_{i=1}^{n} p_i \right) \delta_{C_R}(\bar{F}(x; \lambda)) .
\]

**Proof.** The fact that \( (\bar{\lambda}, \ldots, \bar{\lambda}) \preceq (\lambda_1, \ldots, \lambda_n) \preceq (\lambda_{1:n}, \ldots, \lambda_{1:n}) \) and Theorem 3.3 imply the required result.

Usually, the associated copula is unknown for a company, while the sign of dependency is well-known. Naturally, one may wonder whether presenting the lower and upper bounds for \( \bar{G}_{Y_{1:n}}(x) \) is possible? The following theorem has a positive answer for this question.

**Theorem 3.7.** Under the setup of Theorem 3.6, suppose that the following conditions hold:

(i) \( \bar{F}(x; \lambda) \) is increasing and concave in \( \lambda \) for any \( x \in \mathbb{R}_+ \);

(ii) \( C_R \) is PUOD.

Then, we have

\[
\left( \prod_{i=1}^{n} p_i \right) \bar{F}^n(x; \lambda_{1:n}) \leq \bar{G}_{Y_{1:n}}(x) \leq \left( \prod_{i=1}^{n} p_i \right) \bar{F}(x; \lambda_{1:n}) .
\]
Proof. Let \( C_R(u) = \prod_{i=1}^{n} u_i \) be the independent copula. Since \( C_R^* \) is an Archimedean copula, so Lemma \ref{Lemma2.3} guarantees the Schur-concavity of \( C_R^* \). Thus, using the fact that \( (\lambda_1, \ldots, \lambda_n) \preceq (\lambda_1; n, \ldots, \lambda_1; n) \), Theorem \ref{Theorem3.3} and Definition \ref{Definition2.5} we have

\[
\bar{G}_{Y_{1:n}}(x) = \left( \prod_{i=1}^{n} p_i \right) C_R \left( F(x, \lambda_1), \ldots, F(x, \lambda_n) \right)
\]

\[
\geq \left( \prod_{i=1}^{n} p_i \right) C_R^* \left( F(x, \lambda_1; n), \ldots, F(x, \lambda_1; n) \right)
\]

\[
= \left( \prod_{i=1}^{n} p_i \right) F^n(x; \lambda_1),
\]

which proves the first inequality in \ref{3}. On the other hand, the Fréchet-Hoeffding upper bound in Lemma \ref{Lemma2.4} and increasing property of \( \bar{F}(x; \lambda) \) in \( \lambda \), immediately proves the second inequality in \ref{3}. Hence, the proof is completed.

The proportional reversed hazard rate (PRHR) model, which introduced by Gupta et al. (1998), is a flexible family of distributions existing in reliability theory, and can be used in actuarial science and other fields. \( X_\lambda \) is said to follow PRHR model, if its distribution function can be expressed as \( F(x; \lambda) = F^\lambda(x) \), where \( F(x) \) is the baseline distribution function and \( \lambda > 0 \).

The following corollaries provide a lower and upper bounds for the survival function of the smallest claim amount in a portfolio of risks, whenever the marginal distributions of severities belonging to the PRHR model.

Corollary 3.1. Let \( F(x; \lambda_i) = F^\lambda_i(x) \), for \( i = 1, \ldots, n \). Under the setup of Theorem \ref{Theorem3.6} suppose that \( C_R \) is Schur-concave. Then, we have

\[
\left( \prod_{i=1}^{n} p_i \right) \delta_{C_R} \left( 1 - F^{\lambda_1; n}(x) \right) \leq \bar{G}_{Y_{1:n}}(x) \leq \left( \prod_{i=1}^{n} p_i \right) \delta_{C_R} \left( 1 - F^\lambda(x) \right).
\]

Proof. Clearly, \( \bar{F}(x; \lambda) = 1 - F^\lambda(x) \) and \( C_R \) satisfy the conditions (i) and (ii) of Theorem \ref{Theorem3.6} respectively. Hence, Theorem \ref{Theorem3.6} completes the proof.

Corollary 3.2. Let \( F(x; \lambda_i) = F^\lambda_i(x) \), for \( i = 1, \ldots, n \). Under the setup of Theorem \ref{Theorem3.6} suppose that \( C_R \) is PUOD. Then, we have

\[
\left( \prod_{i=1}^{n} p_i \right) \left( 1 - F^{\lambda_1; n}(x) \right) \leq \bar{G}_{Y_{1:n}}(x) \leq \left( \prod_{i=1}^{n} p_i \right) \left( 1 - F^\lambda(x) \right).
\]

Proof. It is clear that the conditions (i) and (ii) of Theorem \ref{Theorem3.7} are satisfied. Hence, Theorem \ref{Theorem3.7} implies the required result.

The following example provides a numerical example to illustrate the validity of corollaries \ref{Corollary3.1} and \ref{Corollary3.2}.
Example 3.1. Let $X_i \sim F(x; \lambda_i) = (1 - e^{-x})^{\lambda_i}$, for $i = 1, 2, 3$, with the associated Frank copula, which introduced by Frank (1979), of the form

$$C_R(u_1, u_2, u_3) = -\frac{1}{\theta} \log \left( 1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)(e^{-\theta u_3} - 1)}{(e^{-\theta} - 1)^2} \right),$$

where $\theta \in (0, \infty)$. Further, suppose that $I_{p_1}, I_{p_2}, I_{p_3}$ is a set of independent Bernoulli random variables, independent of the $X_i$’s, with $E[I_{p_i}] = p_i$, for $i = 1, 2, 3$. We take $(\lambda_1, \lambda_2, \lambda_3) = (3, 6, 2)$, $(p_1, p_2, p_3) = (0.5, 0.6, 0.1)$ and $\theta = 5$. According to Nelsen (2007), $C_R$ is an Archimedean copula and according to Lemma 2.3 is Schur-concave. Also it is a PUOD copula. Thus, the conditions of Theorems 3.1 and 3.2 are satisfied. Figure 1 represents the plots of the survival function of the smallest claim amounts and the proposed bounds in corollaries 3.1 and 3.2.

![Figure 1: Plots of the survival function of the smallest claim amounts and the proposed bounds in Corollary 3.1 (left) and Corollary 3.2 (right) for Example 3.1.](image)

Recently, Barmalzan et al. (2018) considered the Marshall-Olkin extended exponential distribution as the claim amount distribution in portfolios of risks and compared aggregate claim amounts in two heterogeneous portfolios. Marshall-Olkin distribution, which introduced by Marshall and Olkin (1997) is a special case of a wide range family of distributions, called Harris family. Aly and Benkherouf (2011) used the Harris distribution introduced by Harris (1948), and generated the Harris family. $X_\lambda$ is said to follow the Harris family, if its survival function is given by

$$\bar{F}(x; \lambda, \theta) = \left( \frac{\lambda \bar{F}^\theta(x)}{1 - (1 - \lambda) \bar{F}^\theta(x)} \right)^{1/\theta}, \quad \lambda > 0, \theta > 0,$$

where, $\bar{F}(x)$ is the baseline survival function.

The following corollaries provide a lower and upper bounds for the survival function of the smallest claim amount in a portfolio of risks, whenever the marginal distributions of severities belonging to the Harris family.
Corollary 3.3. Let $\bar{F}(x; \lambda_i, \theta) = \left(\frac{\lambda_i F^\theta(x)}{1 - (1 - \lambda_i) F^\theta(x)}\right)^{1/\theta}$, for $\theta \geq 1$ and $i = 1, \ldots, n$. Under the setup of Theorem 3.6, suppose that $C_R$ is Schur-concave. Then, we have
\[
\left(\prod_{i=1}^n p_i\right) \delta_{C_R} \left(\frac{\lambda_{1:n} \bar{F}^\theta(x)}{1 - (1 - \lambda_{1:n}) F^\theta(x)}\right)^{1/\theta} \leq \bar{G}_{Y_{1:n}}(x) \leq \left(\prod_{i=1}^n p_i\right) \delta_{C_R} \left(\frac{\lambda \bar{F}^\theta(x)}{1 - (1 - \lambda) F^\theta(x)}\right)^{1/\theta}.
\]

Proof. By simplification, the first and second partial derivatives of $\bar{F}(x; \lambda, \theta) = \left(\frac{\lambda F^\theta(x)}{1 - (1 - \lambda) F^\theta(x)}\right)^{1/\theta}$ are obtained as follows:
\[
\frac{\partial \bar{F}(x; \lambda, \theta)}{\partial \lambda} = \frac{\bar{F}(x; \lambda, \theta)}{\lambda (1 - (1 - \lambda) F^\theta(x))} \geq 0,
\]
\[
\frac{\partial^2 \bar{F}(x; \lambda, \theta)}{\partial \lambda^2} = \frac{1 - \bar{F}(x)}{\theta} \frac{\bar{F}(x; \lambda, \theta)}{\lambda^2 (1 - (1 - \lambda) F^\theta(x))} \left(\left(1 - \bar{F}(x)\right) \left(\frac{1}{\theta} - 1\right) - 2\lambda \bar{F}(x)\right) \leq 0,
\]

which, the first inequality is clear and the second is due to the assumption $\theta \geq 1$. Thus, $\bar{F}(x; \lambda, \theta)$ is increasing and concave in $\lambda$. Hence, in the view of Theorem 3.6, the desired result is obtained.

Corollary 3.4. Let $\bar{F}(x; \lambda_i, \theta) = \left(\frac{\lambda_i F^\theta(x)}{1 - (1 - \lambda_i) F^\theta(x)}\right)^{1/\theta}$, for $\theta \geq 1$ and $i = 1, \ldots, n$. Under the setup of Theorem 3.6, suppose that $C_R$ is PUOD. Then, we have
\[
\left(\prod_{i=1}^n p_i\right) \left(\frac{\lambda_{1:n} \bar{F}^\theta(x)}{1 - (1 - \lambda_{1:n}) F^\theta(x)}\right)^{n/\theta} \leq \bar{G}_{Y_{1:n}}(x) \leq \left(\prod_{i=1}^n p_i\right) \left(\frac{\lambda \bar{F}^\theta(x)}{1 - (1 - \lambda) F^\theta(x)}\right)^{1/\theta}.
\]

Proof. Obviously, the conditions (i) and (ii) of Theorem 3.6 are satisfied. Hence, Theorem 3.7 completes the proof.

The following example provides a numerical example to illustrate the validity of corollaries 3.3 and 3.4.

Example 3.2. Let $X_{\lambda_i} \sim F(x; \lambda_i) = \left(\frac{\lambda_i e^{-x^2}}{1 - (1 - \lambda_i) e^{-x^2}}\right)^{1/3}$, for $i = 1, 2, 3$, with the associated Clayton copula, which introduced by Clayton (1978), of the form
\[
C_R(u_1, u_2, u_3) = (u_1^\theta + u_2^\theta + u_3^\theta - 2)^{-1/\theta},
\]
where $\theta \in (0, \infty)$. Further, suppose that $I_{p_1}, I_{p_2}, I_{p_3}$ is a set of independent Bernoulli random variables, independent of the $X_{\lambda_i}$’s, with $E[I_{p_i}] = p_i$ , for $i = 1, 2, 3$. We take $(\lambda_1, \lambda_2, \lambda_3) = (3, 5, 1)$, $(p_1, p_2, p_3) = (0.2, 0.3, 0.2)$ and $\theta = 3$. According to Nelsen (2004), $C_R$ is an Archimedean copula and according to Lemma 2.3, it is Schur-concave. Also it is a PUOD copula. Thus, the conditions of theorems 3.3 and 3.4 are satisfied. Figure 2 represents the plots of the survival function of the smallest claim amounts and the proposed bounds in corollaries 3.3 and 3.4.

Hami Golzar et al. (2017) proposed the Lomax-exponential distribution, which is a proper model for right-skewed, approximately symmetric or reversed-J shape populations. Due to simplicity and
flexibility, it is a good alternative for positive populations and especially claim amounts in portfolios of risks. Recently, Nadeb and Torabi (2018) discussed some stochastic comparisons of series systems with independent heterogeneous Lomax-exponential components. $X$ has the Lomax-exponential distribution with the positive parameters $\alpha$, $\beta$ and $\lambda$, denoted by $X \sim \text{LE}(\alpha, \beta, \lambda)$, if its survival function is given by

$$\bar{F}(x; \alpha, \beta, \lambda) = \left(\frac{\lambda}{e^{\beta x} + \lambda - 1}\right)^\alpha, \quad x \in \mathbb{R}_+.$$ 

The following corollaries provide some bounds for the survival function of the smallest claim amount in a portfolio of risks, when the marginal distributions of severities are Lomax-exponential.

**Corollary 3.5.** Let $X_{\lambda_i} \sim \text{LE}(\alpha, \beta, \lambda_i)$, for $\alpha \leq 1$, $\beta > 0$ and $i = 1, \ldots, n$. Under the setup of Theorem 3.6, suppose that $C_R$ is Schur-concave. Then, we have

$$\left(\prod_{i=1}^n p_i\right) \delta_{C_R} \left(\left(\frac{\lambda_{1:n}}{e^{\beta x} + \lambda_{1:n} - 1}\right)^\alpha\right) \leq \bar{G}_{Y_{1:n}}(x) \leq \left(\prod_{i=1}^n p_i\right) \delta_{C_R} \left(\left(\frac{\bar{\lambda}}{e^{\beta x} + \bar{\lambda} - 1}\right)^\alpha\right).$$

**Proof.** The first and second partial derivatives of $\bar{F}(x; \alpha, \beta, \lambda) = \left(\frac{\lambda}{e^{\beta x} + \lambda - 1}\right)^\alpha$ are given by

$$\frac{\partial \bar{F}(x; \alpha, \beta, \lambda)}{\partial \lambda} = \frac{\alpha(e^{\beta x} - 1)}{\lambda(e^{\beta x} + \lambda - 1)} \bar{F}(x; \alpha, \beta, \lambda) \geq 0,$$

$$\frac{\partial^2 \bar{F}(x; \alpha, \beta, \lambda)}{\partial \lambda^2} = \frac{\alpha(e^{\beta x} - 1)}{\lambda^2(e^{\beta x} + \lambda - 1)^2} \bar{F}(x; \alpha, \beta, \lambda) \left((\alpha - 1)(e^{\beta x} - 1) - 2\lambda\right) \leq 0,$$

which, the first inequality is clear and the second is due to the assumption $\alpha \leq 1$. Thus, $\bar{F}(x; \lambda, \theta)$ is increasing and concave in $\lambda$. Hence, in the view of Theorem 3.3, the desired result is obtained.
Corollary 3.6. Let $X_{\lambda_i} \sim \text{LE}(\alpha, \beta, \lambda_i)$, for $\alpha \leq 1$, $\beta > 0$ and $i = 1, \ldots, n$. Under the setup of Theorem 3.6, suppose that $C_R$ is PUOD. Then, we have
\[
\left( \prod_{i=1}^{n} p_i \right) \left( \frac{\lambda_{1:n}}{e^{\beta x} + \lambda_{1:n} - 1} \right)^{n\alpha} \leq G_{Y_{1:n}}(x) \leq \left( \frac{\lambda_{1:n}}{e^{\beta x} + \lambda_{1:n} - 1} \right)^{\alpha}.
\]

Proof. Applying Theorem 3.7, the desired result is immediately obtained. \qed

The following example provides a numerical example to illustrate the validity of corollaries 3.5 and 3.6.

Example 3.3. Let $X_{\lambda_i} \sim \text{LE}(0.1, 3, \lambda_i)$, for $i = 1, 2, 3$, with the associated Gumbel-Hougaard copula, of the form
\[
C_R(u_1, u_2, u_3) = \exp \left( - \left[ (-\log u_1)^{\theta} + (-\log u_2)^{\theta} + (-\log u_3)^{\theta} \right]^{1/\theta} \right),
\]
where $\theta \in [1, \infty)$. Further, suppose that $I_{p_1}, I_{p_2}, I_{p_3}$ is a set of independent Bernoulli random variables, independent of the $X_{\lambda_i}$’s, with $E[I_{p_i}] = p_i$, for $i = 1, 2, 3$. We take $(\lambda_1, \lambda_2, \lambda_3) = (0.7, 5, 0.4)$, $(p_1, p_2, p_3) = (0.1, 0.2, 0.8)$ and $\theta = 2$. According to Nelsen (2007), $C_R$ is an Archimedean copula and according to Lemma 2.3, is Schur-concave. Also it is a PUOD copula. Thus, the conditions of corollaries 3.5 and 3.6 are satisfied. Figure 3 represents the plots of the survival function of the smallest claim amounts and the proposed bounds in corollaries 3.5 and 3.6.

![Figure 3](image.png)

Figure 3: Plots of the survival function of the smallest claim amounts and the proposed bounds in Corollary 3.5 (left) and Corollary 3.6 (right) for Example 3.3

Conclusion

In this paper, under some certain conditions, we first discussed stochastic comparisons between the smallest claim amounts under the assumption dependency of severities in the sense of usual
and likelihood ratio orders in some general models. Next we present some helpful bounds for the survival function of the smallest claim amount in an interdependent heterogeneous portfolio. Also, some examples are served to illustrate the established results.

References

Aly, E.E.A.A., and Benkherouf, L. (2011). A new family of distributions based on probability generating functions. *Sankhya B*, **73**(1), 70-82.

Balakrishnan, N., Zhang, Y., and Zhao, P. (2018). Ordering the largest claim amounts and ranges from two sets of heterogeneous portfolios. *Scandinavian Actuarial Journal*, **2018**(1), 23-41.

Barmalzan, G., and Najafabadi, A.T.P. (2015). On the convex transform and right-spread orders of smallest claim amounts. *Insurance: Mathematics and Economics*, **64**, 380-384.

Barmalzan, G., Najafabadi, A.T.P. and Balakrishnan, N. (2015). Stochastic comparison of aggregate claim amounts between two heterogeneous portfolios and its applications. *Insurance: Mathematics and Economics*, **61**, 235-241.

Barmalzan, G., Najafabadi, A.T.P., and Balakrishnan, N. (2016). Likelihood ratio and dispersive orders for smallest order statistics and smallest claim amounts from heterogeneous Weibull sample. *Statistics and Probability Letters*, **110**, 1-7.

Barmalzan, G., Najafabadi, A.T.P., and Balakrishnan, N. (2017). Ordering properties of the smallest and largest claim amounts in a general scale model. *Scandinavian Actuarial Journal*, **2017**(2), 105-124.

Barmalzan, G., Najafabadi, A.T.P. and Balakrishnan, N. (2018). Some new results on aggregate claim amounts from two heterogeneous Marshall-Olkin extended exponential portfolios. *Communications in Statistics-Theory and Methods*, **47**(11), 2779-2794.

Clayton, D.G. (1978). A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. *Biometrika*, **65**, 141-151.

Cox, D.R. (1972). Regression Models and Life Tables. *Journal of the Royal Statistical Society, Series B*, **34**(2), 187-220.

Denuit, M., and Frostig, E. (2006). Heterogeneity and the need for capital in the individual model. *Scandinavian Actuarial Journal*, **2006**(1), 42-66.

Dolati, A., and Dehghan Nezhad, A. (2014). Some results on convexity and concavity of multivariate copulas. *Iranian Journal of Mathematical Sciences and Informatics*, **9**(2), 87-100.
Durante, F. (2006). New results on copulas and related concepts. Università degli Studi di Lecce.

Finkelstein, M. (2008). Failure rate modeling for reliability and risk. Springer, London.

Frank, M.J. (1979). On the simultaneous associativity of $F(x,y)$ and $x + y - F(x,y)$. Aequationes mathematicae, 19(1), 194-226.

Frostig, E. (2001). A comparison between homogeneous and heterogeneous portfolios. Insurance: Mathematics and Economics, 29(1), 59-71.

Gumbel, E.J. (1960). Distributions des valeurs extremes en plusiers dimensions. Publications de l'Institut de statistique de l'Université de Paris, 9, 171-173.

Gupta, R.C., Gupta, P.L., and Gupta, R.D. (1998). Modeling failure time data by Lehman alternatives. Communications in Statistics-Theory and Methods, 27(4), 887-904.

Hami Golzar, N., Ganji, M., and Bevrani, H. (2017). The Lomax-exponential distribution, some properties and application. Journal of Statistical Research of Iran, 13(2), 131-153.

Harris, T.E. (1948). Branching processes. The Annals of Mathematical Statistics, 19, 474-494.

Hu, T., and Ruan, L. (2004). A note on multivariate stochastic comparisons of Bernoulli random variables. Journal of Statistical Planning and Inference, 126(1), 281-288.

Karlin, S., and Novikoff, A. (1963). Generalized convex inequalities. Pacific Journal of Mathematics, 13(4), 1251-1279.

Khaledi, B.E., and Ahmadi, S.S. (2008). On stochastic comparison between aggregate claim amounts. Journal of Statistical Planning and Inference, 138(7), 3121-3129.

Kumar, D., and Klefsjö, B. (1994). Proportional hazards model: a review. Reliability Engineering and System Safety, 44(2), 177-188.

Li, C., and Li, X. (2016). Sufficient conditions for ordering aggregate heterogeneous random claim amounts. Insurance: Mathematics and Economics, 70, 406-413.

Li, C., and Li, X. (2018). Stochastic comparisons of parallel and series systems of dependent components equipped with starting devices. Communications in Statistics-Theory and Methods, DOI: 10.1080/03610926.2018.1435806.

Li, H., and Li, X. (2013). Stochastic Orders in Reliability and Risk. Springer, New York.

Ma, C. (2000). Convex orders for linear combinations of random variables. Journal of Statistical Planning and Inference, 84, 11-25.
Marshall, A.W., and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, **84**(3), 641-652.

Marshall, A.W., Olkin, I., and Arnold, B.C. (2011). *Inequalities: Theory of Majorization and its Applications*. Springer, New York.

Müller, A., and Stoyan, D. (2002). *Comparison methods for stochastic models and risks*. John Wiley & Sons, New York.

Nadeb, H., and Torabi, H. (2018). Stochastic comparisons of series systems with independent heterogeneous Lomax-exponential components. *Journal of Statistical Theory and Practice*, **12**(4), 794-812.

Nelsen, R.B. (2007). *An Introduction to Copulas*. Springer Science & Business Media, New York.

Shaked, M., and Shanthikumar, J.G. (2007). *Stochastic Orders*. Springer, New York.

Zhang, Y., and Zhao, P. (2015). Comparisons on aggregate risks from two sets of heterogeneous portfolios. *Insurance: Mathematics and Economics*, **65**, 124-135.