Finite groups having an automorphism with a sufficiently large cycle are solvable

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January 28, 2015

Abstract

Using the Classification of Finite Simple Groups as well as results on maximal orders of elements and automorphisms of finite simple groups by Guest, Morris, Praeger and Spiga, we show that an automorphism of a finite Fitting-free group $G$ never has a cycle of length greater than $\frac{1}{10}|G|$ and deduce that any finite group which does have an automorphism with such a cycle is solvable.

2010 Mathematics Subject Classification: 20D05, 20D10, 20D25, 20D45.

Key words and phrases: finite groups, automorphisms, cycle structure, solvable groups, finite simple groups.

1 Motivation and main results

Let us begin by explaining our notation. We denote by $\mathbb{N}$ the set of natural numbers (including 0) and by $\mathbb{N}^+$ the set of positive integers. The symmetric and alternating group of degree $n$ are denoted by $S_n$ and $A_n$ respectively. For a function $f$ and a subset $M$ of the domain of $f$, we denote the pointwise image of $M$ under $f$ by $f[M]$. As for finite simple groups: The notation for the sporadic groups is the standard one used, for instance, by [1]. The classical groups of Lie type, which can be realized as projective versions of groups of matrices preserving an appropriate bilinear form on a finite vector space (and, in the case of orthogonal groups, having spinor norm 1), are denoted accordingly $(\text{PSL}_d(q), \text{PSU}_d(q), \text{PSp}_d(q), \text{PΩ}_{2m+1}(q), \text{PΩ}_{2m}^+(q), \text{PΩ}_{2m}^-(q))$. For twisted Chevalley groups, the primary parameter refers to the order of the fixed field involved (e.g., $^2E_6(q)$ is well-defined for all primary $q$ and has order $\frac{1}{\gcd(3,q+1)}q^{36}\prod_{i=2,5,6,8,9,12}(q^i - (-1)^i)$).

The following concept underlies several objects of study in both theoretical and applied mathematical research:

Definition 1.1. A finite dynamical system (abbreviated henceforth by FDS) is a finite set $X$ together with a so-called endfunction of $X$, i.e., a function $f : X \to X$.
FDSs are special cases of discrete dynamical systems, where the underlying set $X$ need not be finite. When studying an FDS $(X, f)$, one is interested in the behavior of the endofunction $f$ under iteration. For example, for $x \in X$, the orbit of $x$ under $f$, denoted $O_x$, is defined as the set $\{f^n(x) \mid n \in \mathbb{N}\}$ of points reachable from $x$ via repeated application of $f$, and one typical question in the theory of FDSs is to determine or at least give good bounds for the length $|O_x|$ of such an orbit.

This concept has various practical applications, including cryptography, pseudorandom number generation and reverse engineering. For pseudorandom number generation, periodic FDSs (i.e., FDSs $(X, f)$ where $f$ is a permutation of $X$) with large orbits (that is, cycle lengths of the permutation $f$) and other “well-distribution” properties are needed.

Both in theoretical and practical settings, the set $X$ often is endowed with an algebraic structure, and the function $f$ is defined using the arithmetic given by that algebraic structure. For example, a rational FDS is an FDS where $X$ is a Cartesian power $k^n$ of some finite field $k$ and $f$ is a rational map $k^n \rightarrow k^n$ (that is, an $n$-tuple of rational functions $k^n \rightarrow k$). These FDSs play an important role in reverse engineering, see [9]. They are, of course, also of intrinsic interest.

We are interested in studying finite groups as FDSs, starting with what one might consider to be the easiest case for $f$:

**Definition 1.2.** A finite dynamical group (abbreviated henceforth by FDG) is a finite group $G$ together with an endomorphism $\varphi$ of $G$.

Another example of FDSs $(X, f)$ where $f$ is an endomorphism with respect to an algebraic structure on $X$ are linear finite dynamical systems (LFDSs), where $X$ is a finite-dimensional vector space over a finite field. These are well-understood, see [7]. Some results for LFDSs can be generalized to FDGs, as the author showed in [3]. Associated to each FDS $(X, f)$ is its state space $\Gamma_f$, which is the digraph with vertex set $X$ having a directed edge from $x$ to $y$ if and only if $f(x) = y$. One question naturally arising in the study of FDGs is the following: “Which (isomorphism types of) finite digraphs occur as state spaces of FDGs?”. It turns out that each state space of an FDG $(G, \varphi)$ is the graph tensor product of a 1-tree whose cycle is a loop, corresponding to the nilpotent part of $\varphi$, with a disjoint union of cycles, representing the periodic part of $\varphi$. Conversely, if a finite 1-tree $\Gamma_1$ with loop and a finite disjoint union of cycles $\Gamma_2$ can be realized as the state spaces of FDGs $(G_1, \varphi_1)$ and $(G_2, \varphi_2)$ respectively, then $(G_1 \times G_2, \varphi_1 \times \varphi_2)$ realizes the tensor product $\Gamma_1 \times \Gamma_2$. In [3], the author characterized the finite 1-trees occurring as the nilpotent part of the state space of some finite group endomorphism, but the problem of determining the finite disjoint unions of cycles realized as state spaces of automorphisms of finite groups (which is tantamount to determining all possible cycle types of automorphisms of finite groups) still remains wide open.

Also in view of what was said above on properties of FDSs needed for practical applications, the author started to attack the problem by working on the following question: “What can be said about finite groups $G$ having an automorphism with a prominently large cycle?”, where “prominently large” refers to the largest fraction of elements of $G$ lying on one cycle of an appropriate automorphism. For linguistic simplicity, we make the following definition (part of which is from [2]):

**Definition 1.3.** (1) Let $\sigma$ be a permutation of a finite set $X$. Define

$$A(\sigma) := \max_\zeta |\text{supp}(\zeta)|,$$
where $\zeta$ runs through all cycles of $\sigma$ (so $\Lambda(\sigma)$ is the largest length of a cycle of $\sigma$), and

$$\lambda(\sigma) := \frac{1}{|G|} \Lambda(\sigma).$$

(2) For any finite group $G$, define

$$\Lambda(G) := \max_{\alpha \in \text{Aut}(G)} \Lambda(\alpha)$$

and

$$\lambda(G) := \frac{1}{|G|} \Lambda(G).$$

In [2], methods for classifying the finite groups $G$ with $\lambda(G) \geq \frac{1}{2}$ were developed. One of the main results there was that if $G$ is a finite group with $\lambda(G) > \frac{1}{2}$, then $G$ is abelian. Our main goal in this paper is to prove the following:

**Theorem 1.4.** Let $G$ be a finite group such that $\lambda(G) > \frac{1}{10}$. Then $G$ is solvable.

Recall that a group $G$ is called Fitting-free if and only if its Fitting subgroup is trivial, which is equivalent to its solvable radical $\text{Rad}(G)$ (the subgroup generated by all solvable normal subgroups of $G$, which is the unique largest solvable normal subgroup of $G$ if $G$ is finite) being trivial. For example, all finite simple groups are Fitting-free. We now argue that the bound in Theorem 1.4 is optimal in the sense that there exists a finite nonsolvable group $G$ such that $\lambda(G) = \frac{1}{10}$, namely $G = A_5$. This would not be difficult to see by an ad hoc argumentation, but we want to remark that it can also be derived from a result of Horoševskii which is useful for determining the $\lambda$-value of finite Fitting-free groups in general:

**Theorem 1.5.** (Horoševskii) Let $G$ be a finite Fitting-free group. Then for every automorphism $\alpha$ of $G$, we have $\Lambda(\alpha) = \text{ord}(\alpha)$.

**Proof.** See [8, Theorem 1].

Now just note that $\text{Aut}(A_5) \cong S_5$ and that the maximal order of an element of $S_5$ is $6 = \frac{1}{10} |A_5|$. We will need part of the methods developed in [2] (in a slightly stronger form) for the proof of Theorem 1.4. For the readers’ convenience, we give a short overview of what we need in Section 2. Section 3 discusses the strategy for proving Theorem 1.4 (which will involve use of CFSG) and presents some notable results which will be proved “on the way”. Section 4 then carries out said proof in detail.

## 2 Some large cycle theory

Recall that for an automorphism $\alpha$ of a group $G$, a subgroup $H \leq G$ is called $\alpha$-admissible if and only if $\alpha[H] = H$. It is straightforward to prove the following well-known result: If $\alpha$ is an automorphism of a group $G$ and $N$ is an $\alpha$-admissible normal subgroup of $G$, then there exists a unique automorphism $\tilde{\alpha}$ of the quotient group $G/N$ making the following diagram commute:
One also speaks of the automorphism of $G/N$ induced by $\alpha$. The paper [2] introduced the following concept, which proved helpful in the classification of finite groups $G$ with $\lambda(G) = \frac{1}{2}$:

**Definition 2.1.** (Definition 4.1 in [3].) Let $G$ be a group, $\varphi$ an endomorphism of $G$ and $g_0 \in G$. The (left) affine map of $G$ w.r.t. $\varphi$ and $g_0$ is the function $A_{g_0, \varphi} : G \rightarrow G, g \mapsto g_0 \varphi(g)$.

Note that an affine map $A_{g_0, \varphi}$ is injective (resp. surjective) if and only if $\varphi$ is injective (resp. surjective), so if $\varphi$ is an automorphism of $G$, then $A_{g_0, \varphi}$ is a permutation of the underlying set. We call such affine maps periodic and denote the set of all periodic affine maps of a given finite group $G$ by $\text{Aff}(G)$. We now introduce the following function:

**Definition 2.2.** For any finite group $G$, define

$$\Lambda_{\text{aff}}(G) := \max_{A \in \text{Aff}(G)} \Lambda(A)$$

and

$$\lambda_{\text{aff}}(G) := \frac{1}{|G|} \Lambda_{\text{aff}}(G).$$

Note that $\lambda_{\text{aff}}(G) \geq \lambda(G)$ as $\text{Aut}(G) \subseteq \text{Aff}(G)$, and this inequality may be strict. For example, all finite cyclic groups $C$ satisfy $\lambda_{\text{aff}}(C) = 1$, but the trivial group is the only finite group $G$ with $\lambda(G) = 1$.

The following observation, which is a stronger version of Lemma 1.11 from [2], often comes in handy when studying large cycle conditions in finite groups:

**Lemma 2.3.** Let $G$ be a finite group, $g_0 \in G$, $\alpha$ an automorphism of $G$, $N$ an $\alpha$-admissible normal subgroup of $G$, $\pi : G \rightarrow G/N$ the canonical projection and let $\tilde{\alpha}$ be the automorphism of $G/N$ induced by $\alpha$. Then

$$\lambda(A_{\pi(g_0), \tilde{\alpha}}) \geq \lambda(A_{g_0, \alpha}).$$

In particular, $\lambda_{\text{aff}}(G/N) \geq \lambda_{\text{aff}}(G)$ and $\lambda(G/N) \geq \lambda(G)$.

**Proof.** Note that the following diagram commutes:

$$\begin{array}{ccc}
G & \xrightarrow{A_{g_0, \alpha}} & G \\
\downarrow{\pi} & & \downarrow{\pi} \\
G/N & \xrightarrow{A_{\pi(g_0), \tilde{\alpha}}} & G/N
\end{array}$$
Fix a point \( g \in G \) whose cycle length under \( A_{g_{0},\alpha} \) equals \( \lambda(A_{g_{0},\alpha}) \). We claim that, denoting by \( l \) the length of the cycle of \( \pi(g) \) under \( A_{\pi(g_{0}),\tilde{\alpha}} \), we have

\[
\frac{l}{|G/N|} \geq \lambda(A_{g_{0},\alpha})
\]

and will be done once we have shown this claim. Now by commutativity of the above diagram, the support of the cycle of \( g \) under \( A_{g_{0},\alpha} \) is contained in the union of the cosets of \( N \) in \( G \) which are the preimages under \( \pi \) of the elements on the cycle of \( \pi(g) \) under \( A_{\pi(g_{0}),\tilde{\alpha}} \). This union has cardinality \( l \cdot |N| \), from which we conclude

\[
\lambda(A_{g_{0},\alpha}) \leq \frac{l \cdot |N|}{|G|} = \frac{l}{|G/N|}.
\]

In [2], we gave an upper bound for the order of periodic affine maps \( A_{x,\alpha} \) of finite abelian groups in terms of the order of \( \alpha \) and the maximal order of a fixed point of \( \alpha \) (see Lemma 4.8 there). A similar argumentation gives an upper bound for general groups in terms of \( \text{ord}(\alpha) \) and the maximal order of an element of \( G \), see Lemma 2.5(2) below. Since we will frequently make use of the bound, we make the following definition (the first part of which is from [6]):

**Definition 2.4.** Let \( G \) be a finite group.

1. Define \( \text{meo}(G) \) to be the maximal order of an element of \( G \).
2. Define \( \text{mao}(G) := \text{meo}(\text{Aut}(G)) \).

Recall that for a group \( G \), the *holomorph* of \( G \), denoted \( \text{Hol}(G) \), is defined as the external semidirect product \( G \rtimes \phi \text{Aut}(G) \), with \( \phi \) the natural action of \( \text{Aut}(G) \) on \( G \). \( \text{Hol}(G) \) also has a “natural” construction as the group of periodic affine maps of \( G \):

**Lemma 2.5.** Let \( G \) be a group.

1. For all automorphisms \( \alpha_{1}, \alpha_{2} \) and all elements \( g_{1}, g_{2} \) of \( G \), we have

\[
A_{g_{1},\alpha_{1}} \circ A_{g_{2},\alpha_{2}} = A_{g_{1}\alpha_{1}(g_{2}),\alpha_{1}\circ\alpha_{2}}.
\]

In particular, \( \text{Aff}(G) \) is a subgroup of \( \mathcal{S}_{G} \), and the map

\[
\psi : \text{Hol}(G) = G \rtimes \text{Aut}(G) \to \text{Aff}(G), (g, \alpha) \mapsto A_{g,\alpha},
\]

is an isomorphism of groups.

2. Assume that \( G \) is finite, and let \( \alpha \) be an automorphism of \( G \) and \( g \in G \). Then

\[
\text{ord}(A_{\alpha, g}) \leq \text{ord}(\alpha) \cdot \text{meo}(G).
\]

3. If \( G \) is finite, the maximal order of a periodic affine map of \( G \) is bounded above by \( \text{meo}(G) \cdot \text{mao}(G) \).

**Proof.** For (1): For any \( x \in G \), we have

\[
(A_{g_{1},\alpha_{1}} \circ A_{g_{2},\alpha_{2}})(x) = A_{g_{1},\alpha_{1}}(g_{2}\alpha_{2}(x)) = g_{1}\alpha_{1}(g_{2})\alpha_{1}(\alpha_{2}(x)) = A_{g_{1}\alpha_{1}(g_{2}),\alpha_{1}\circ\alpha_{2}}(x),
\]
proving the first part of the assertion. It follows immediately from this that \( \text{Aff}(G) \) is closed under \( \circ \), but also under inversion, since one readily verifies that for all \( g \in G \) and \( \alpha \in \text{Aut}(G) \), we have
\[
A_{g,\alpha}^{-1} = A_{\alpha^{-1}(g),\alpha^{-1}}.
\]
Hence \( \text{Aff}(G) \) is a subgroup of \( S_G \). By the first part, it is clear that \( \psi \) is a surjective homomorphism of groups, and it is not difficult to check that the map \( \text{Aff}(G) \to \text{Hol}(G), A \mapsto (A(1_G), \mu_A(1_G \circ A)) \) is a left inverse to \( \psi \), completing the proof that \( \psi \) is an isomorphism of groups.

For (2): By (1), it is sufficient to show that \( \psi^{-1}(A_{g,\alpha}) = (g, \alpha) \in \text{Hol}(G) \) has order bounded above by \( \text{ord}(\alpha) \cdot \text{meo}(G) \), and this is clear, since \( (g, \alpha)^{\text{ord}(\alpha)} \in G \).

For (3): This follows immediately from (2).

We now have an upper bound on orders and hence also on cycle lengths of affine maps of finite groups. Another result yielding an upper bound is given by Lemma 2.6. We say that \( \alpha \) is an automorphism of \( G \) splits. Then \( \lambda(G) \) splits. Then
\[
\lambda(G) \leq \prod_{i=1}^{r} \lambda(G_i) \leq \min_{i=1,\ldots, r} \lambda(G_i).
\]
Furthermore, every periodic affine map of \( G \) splits as a product of periodic affine maps over the factors \( G_i \) and
\[
\lambda_{\text{aff}}(G) \leq \prod_{i=1}^{r} \lambda_{\text{aff}}(G_i) \leq \min_{i=1,\ldots, r} \lambda_{\text{aff}}(G_i).
\]

**Proof.** The statement about splitting of periodic affine maps is immediate and the proof of the upper bound on \( \lambda_{\text{aff}}(G) \) then is similar to the one for \( \lambda(G) \). To see why the upper bound on \( \lambda(G) \) holds, let \( \alpha \) be an automorphism of \( G \) such that \( \lambda(\alpha) = \lambda(G) \). By assumption, \( \alpha = \prod_{i=1}^{r} \alpha_i \) for automorphisms \( \alpha_i \) of the \( G_i \). Let \( (g_1, \ldots, g_r) \in G \) be a point whose cycle length under \( \alpha \) is equal to \( \lambda(\alpha)|G| \) and denote by \( l_i \) the cycle length of \( g_i \) under \( \alpha_i \). Then we have the following:
\[
\lambda(G)|G| = \lambda(\alpha)|G| = \lambda(G) \lambda(G_i)|G| = \prod_{i=1}^{r} |G_i| \lambda(G_i) = \prod_{i=1}^{r} \lambda(G_i) \cdot |G|,
\]
from which the first inequality is immediate, and the second follows since all \( \lambda(G_i) \leq 1 \).

The following Lemma 2.7 which is a generalization of an idea from the proof of Theorem 1.9 in [2], will help us construct lower bounds.

**Lemma 2.7.** Let \( G \) be a finite group. Then for every proper characteristic subgroup \( N \) of \( G \), we have \( \lambda_{\text{aff}}(N) > \lambda(G) \).

**Proof.** Let \( \alpha \) be an automorphism of \( G \) such that \( \lambda(\alpha) = \lambda(G) \), let \( \tilde{\alpha} \) be the automorphism of \( G/N \) induced by \( \alpha \) and let \( \zeta \) be a cycle of \( \alpha \) of length \( \lambda(\alpha) \cdot |G| \). \( \zeta \) induces a cycle \( \tilde{\zeta} \) of \( \tilde{\alpha} \). Set \( l := |\text{supp}(\tilde{\zeta})| \), and consider the automorphism \( \alpha' \) of \( G \). By choice of \( l \), for any \( x \in \text{supp}(\zeta) \),
the cycle of \( x \) under \( \alpha^l \) is contained in the coset \( Nx \), and if we denote by \( l' \) the length of this cycle, we find that since \( l \) is a divisor of \( |\text{supp}(\zeta)| = \lambda(\alpha)|G| \), we have \( l \cdot l' = \lambda(\alpha)|G| \). Hence
\[
\frac{l'}{|N|} = \frac{\lambda(\alpha) \cdot |G|}{l \cdot |N|} = \frac{|G|}{l \cdot |N|} \cdot \frac{|G|}{|G/N|} \cdot \lambda(\alpha) = \lambda(\alpha) = \lambda(G),
\]
where the inequality is strict because \( G/N \) is nontrivial.

Now denote by \( \beta \) the restriction of \( \alpha^l \) to the coset \( xN \). Writing \( \beta(x) = xn_0 \) for an appropriate element \( n_0 \in N \), we find that for all \( n \in N \), we have \( \beta(xn) = xn_0\beta(n) = x_{n_0}\alpha^l(n) \). Hence under the bijection \( xN \to N, xn \mapsto n, \beta \) corresponds to the affine map \( A_{n_0,\alpha^l} \) on \( N \), and by what we showed above, \( \lambda_{\text{aff}}(N) \geq \lambda(A_{n_0,\alpha^l}) > \lambda(G) \) follows.

\[\square\]

## 3 Proof of Theorem 1.4: The main ideas

The abstract core of our argument is summarized in Lemma 3.1 and Lemma 3.4 below.

**Lemma 3.1.** Let \( \mathcal{C} \) be a class of finite groups closed under isomorphism and with the following two properties:

(i) If \( G \in \mathcal{C} \) and \( N \) is a characteristic subgroup of \( G \), then \( G/N \in \mathcal{C} \).

(ii) No nontrivial finite Fitting-free group lies in \( \mathcal{C} \).

Then \( \mathcal{C} \) is a subclass of the class of finite solvable groups.

**Proof.** We show that every element \( G \) of \( \mathcal{C} \) is solvable by induction on \( |G| \), the induction base \( |G| = 1 \) being clear. If the implication has been shown for all groups of smaller order than \( G \in \mathcal{C} \), then since by (ii), \( \text{Rad}(G) \neq \{1\} \), we can apply the induction hypothesis to the quotient \( G/\text{Rad}(G) \), which we know by (i) to lie in \( \mathcal{C} \), and obtain that \( G/\text{Rad}(G) \) is solvable. Hence \( G \) is an extension of a solvable group by a solvable group and therefore solvable itself.

\[\square\]

Before formulating and proving Lemma 3.4, we quickly cite some facts on finite Fitting-free groups; more details can be found in [14, pp.87ff.]. Recall that a group is called completely reducible or a CR-group if and only if it is a (possibly infinite) direct product of simple groups. The finite centerless CR-groups are precisely the groups of the form \( \prod_{i=1}^r S_i^{n_i} \), where \( S_1, \ldots, S_r \) are finite nonabelian simple groups with \( S_i \neq S_j \) for \( i \neq j \). Note that the factors \( S_i^{n_i} \) are characteristic subgroups of \( \prod_{i=1}^r S_i^{n_i} \).

Every group \( G \) has a unique maximal normal centerless CR-subgroup, called the centerless CR-radical of \( G \). It is not difficult to show that in every finite Fitting-free group \( G \), the centralizer of the centerless CR-radical of \( G \) is trivial, whence the centerless CR-radical of a nontrivial finite Fitting-free group \( G \) is always nontrivial. We have the following classification theorem for finite Fitting-free groups:

**Theorem 3.2.** (1) If \( G \) is a finite Fitting-free group with centerless CR-radical \( R \), then \( G \) is isomorphic to a subgroup of \( \text{Aut}(R) \) containing \( \text{Inn}(R) \). Also, two such subgroups of \( \text{Aut}(R) \) are isomorphic if and only if they are conjugate.

(2) Conversely, if \( R \) is a finite centerless CR-group and \( G \) is such that \( \text{Inn}(R) \leq G \leq \text{Aut}(R) \), then \( G \) is a finite Fitting-free group with centerless CR-radical \( \text{Inn}(R) \cong R \).

\[\square\]

Theorem 3.2 shows that the isomorphism types of finite Fitting-free groups with centerless CR-radical \( R \) are in bijective correspondence with the conjugacy classes of subgroups of \( \text{Out}(R) \). The following theorem on automorphism groups of finite centerless CR-groups will also be helpful:
Theorem 3.3. Let \( R = \prod_{i=1}^{r} S_{n}^{*} \) be a finite centerless CR-group, where the \( S_{n} \) are pairwise nonisomorphic nonabelian finite simple groups. Then \( \text{Aut}(R) \cong \prod_{i=1}^{r} \text{Aut}(S_{n}^{*}) \), and \( \text{Aut}(S_{n}^{*}) \cong \text{Aut}(S_{n}) \setminus S_{n}^{*} \).

Let us now give the second abstract lemma for our argument:

Lemma 3.4. Let \( C \) and \( \mathcal{C} \) be classes of finite groups, both closed under isomorphism. Assume that the following hold:

(i) If \( G \in C \) and \( N \) is a characteristic subgroup of \( G \), then \( N \in \mathcal{C} \).

(ii) Denoting by \( C \) the class of nontrivial finite centerless CR-groups lying in \( \mathcal{C} \): For all \( R \in C \), none of the (finitely many) finite Fitting-free groups \( G \) having centerless CR-radical \( R \) lies in \( C \).

Then \( C \) does not contain any finite Fitting-free groups.

Proof. This is almost tautological: If \( G \) was a finite Fitting-free group lying in \( C \), then the centerless CR-radical \( R \) of \( G \) (which is nontrivial) would be in \( \mathcal{C} \), by (i). But this contradicts (ii).

Of course, Lemma 3.4 is particularly helpful when \( \mathcal{C} \) is “small”. In our application, it will not be empty, but it will only contain four different isomorphism types of finite groups. Let \( \mathcal{G}^{\text{fin}} \) denote the class of all finite groups. Set

\[
C_{1} := \{ G \in \mathcal{G}^{\text{fin}} \mid \lambda(G) > \frac{1}{10} \},
\]

\[
C_{2} := \{ G \in \mathcal{G}^{\text{fin}} \mid \lambda_{\text{aff}}(G) > \frac{1}{10} \}.
\]

We want to apply Lemma 3.4 to \( C_{1} \) to prove Theorem 1.4. By Lemma 2.3, the assumption (i) of Lemma 3.4 for \( C_{1} \) is satisfied. It remains to show that \( C_{1} \) does not contain any nontrivial finite Fitting-free groups. This is achieved with the help of Lemma 3.4 assumption (i) of which is satisfied for the above choice of \( C_{1} \) and \( C_{2} \) by virtue of Lemma 2.7. We have thus reduced the proof Theorem 1.4 to determining \( C := C_{2} \cap \text{CLCR} \), where CLCR denotes the class of nontrivial centerless CR-groups, and checking whether the (hopefully few) elements \( R \in \mathcal{C} \) no finite Fitting-free group with centerless CR-radical \( R \) can have \( \lambda \)-value greater than \( \frac{1}{10} \). The first part of this task is achieved by the following:

Theorem 3.5. Let \( G \) be a nontrivial finite centerless CR-group. Then \( \lambda_{\text{aff}}(G) \leq \frac{1}{10} \), with the only exceptions \( A_{5} \cong \text{PSL}(2) \cong \text{PSL}(2) \), \( A_{6} \cong \text{PSL}(2) \), \( PSL(2) \) and \( PSL(2) \), which have the following \( \lambda_{\text{aff}} \)-values: \( \lambda_{\text{aff}}(A_{5}) = \frac{1}{10} \), \( \lambda_{\text{aff}}(A_{6}) = \frac{1}{10} \), \( \lambda_{\text{aff}}(PSL(2)) = \frac{1}{10} \) and \( \lambda_{\text{aff}}(PSL(2)) = \frac{1}{10} \).

Corollary 3.6. Let \( G \) be a finite Fitting-free group. Then \( \lambda(G) \leq \frac{1}{10} \).

Note that Theorem 3.3 implies that every automorphism of a finite centerless CR-group \( R = \prod_{i=1}^{r} S_{n}^{*} \) splits. In view of this and Lemma 2.6, it will not be difficult to derive Theorem 3.5 from the following formally weaker statement (note that the finite characteristically simple groups are precisely the finite powers of finite simple groups, see [14, 3.3.15, pp. 87f.]):

Theorem 3.7. Let \( G \) be a finite nonabelian characteristically simple group. Then \( \lambda_{\text{aff}}(G) \leq \frac{1}{10} \), with the only exceptions \( A_{5} \cong \text{PSL}(2) \cong \text{PSL}(2) \), \( A_{6} \cong \text{PSL}(2) \), \( PSL(2) \) and \( PSL(2) \), which have the following \( \lambda_{\text{aff}} \)-values: \( \lambda_{\text{aff}}(A_{5}) = \frac{1}{10} \), \( \lambda_{\text{aff}}(A_{6}) = \frac{1}{10} \), \( \lambda_{\text{aff}}(PSL(2)) = \frac{1}{10} \) and \( \lambda_{\text{aff}}(PSL(2)) = \frac{1}{10} \).
We will hence focus on the proof of Theorem 3.7, which will mainly be an application of Lemma 2.5(3). In order to make use of the bound given there, we need suitable upper bounds for meo($S^n$) and mao($S^n$) for nonabelian finite simple groups $S$ and $n \in \mathbb{N}^+$. We will derive these using Theorem 3.3 and results from [6], which gave upper bounds for meo($S$) and mao($S$) for the various finite nonabelian simple groups $S$. In virtue of the CFSG, the proof then is mainly a checking of cases, although some cases require special treatment. This concludes our outline of the main ideas for the proof of Theorem 1.4. For the technical details, the reader is referred to the next and last section of the paper.

4 Proof of Theorem 1.4: The details

The proofs in this section involve some numerical computations which were carried out using Wolfram|Alpha [15]. For the group-theoretic computations, GAP [5] was used. We begin with a known result on the largest orders of permutations on $n$ elements. The function $g : \mathbb{N}^+ \to \mathbb{N}^+$ assigning, to each $n \in \mathbb{N}^+$, the value meo($S_n$), is called Landau’s function. While precise values of $g(n)$ may be hard to compute in reasonable time, the asymptotic growth of $g$ is well-studied: Landau proved in [11] that $\log g(n) \sim \sqrt{n \cdot \log n}$. Later, Massias in [12] gave the following explicit upper bound for Landau’s function:

**Theorem 4.1.** For all $n \in \mathbb{N}^+$, we have that $\log g(n) \leq c \cdot \sqrt{n \cdot \log n}$, where $c = 1.05313\ldots$. □

For our purposes, the following exponential bound on $g(n)$ will be sufficient:

**Corollary 4.2.** For all $n \in \mathbb{N}^+$, we have that $g(n) < \left(\frac{4}{7}\right)^n$.

**Proof.** The inequality in question is equivalent to

$$\log g(n) < n \cdot \log \frac{3}{2},$$

which by Theorem 4.1 is satisfied whenever

$$1.06 \cdot \sqrt{n \cdot \log n} \leq n \cdot \log \frac{3}{2},$$

or

$$\left(\frac{1.06}{\log 1.5}\right)^2 \cdot \log n \leq n.$$

By considering the first derivative of the left-hand side and noting that $(\frac{1.06}{\log 1.5})^2 = 6.83447\ldots$, we see that once this inequality is satisfied for some $n_0 \geq 7$, it will be satisfied for all $n \geq n_0$. By numerical computations, it is satisfied for $n_0 = 21$, so it remains to check the values $1, \ldots, 20$, which we did “by hand”, using the table of values of $g$ from [13]. □

The following lemma gives a useful sufficient condition for $\lambda_{aff}(S^n) \leq \frac{1}{10}$ for nonabelian finite simple groups $S$:

**Lemma 4.3.** Let $S$ be a nonabelian finite simple group. If

$$\lambda_{aff}(S) < \frac{1}{\sqrt{20}},$$

then...
then for all \( n \geq 2 \), we have
\[
\lambda_{\text{aff}}(S^n) < \frac{1}{10}.
\]
In particular, if
\[
10 \cdot \text{meo}(S)^2 \cdot \text{meo}(\text{Out}(S)) \leq |S|,
\]
then for all \( n \in \mathbb{N}^+ \), we have
\[
\lambda_{\text{aff}}(S^n) \leq \frac{1}{10},
\]
and a strict inequality in (1) implies a strict inequality in (2).

Proof. By Theorem 3.3, it is clear that
\[
\lambda_{\text{aff}}(S^n) \leq \lambda_{\text{aff}}(S) \cdot g(n).
\]
For \( n = 2 \), the upper bound is by assumption itself strictly bounded above by
\[
\left(\frac{1}{\sqrt{20}}\right)^2 \cdot g(2) = \frac{1}{20} \cdot 2 = \frac{1}{10},
\]
whereas for \( n \geq 3 \), we use Corollary 4.2 to obtain the upper bound
\[
\left(\frac{1}{\sqrt{20}}\right)^2 \cdot \frac{3}{2}^n = \left(\frac{3}{2\sqrt{20}}\right)^3 \leq \frac{1}{10}.
\]
The “In particular” follows by observing that the upper bound \( \text{meo}(S) \cdot \text{mao}(S) \) for \( \Lambda_{\text{aff}}(S) \) from Lemma 2.5(3) is itself bounded above by \( \text{meo}(S)^2 \cdot \text{meo}(\text{Out}(S)) \). \( \square \)

As a first application of Lemma 4.3 let us prove Theorem 3.7 for the case where \( G \) is a power of a sporadic finite simple group:

Lemma 4.4. Let \( S \) be one of the 26 sporadic finite simple groups. Then for all \( n \in \mathbb{N}^+ \):
\[
\lambda_{\text{aff}}(S^n) < \frac{1}{10}.
\]
Proof. This follows from Lemma 4.3 via Table 1 on the next page. Information on maximal element orders, outer automorphism groups and orders of sporadic finite simple groups was taken from [1]. \( \square \)

Having treated the sporadic simple groups, we continue our investigations with the finite alternating groups \( A_m \), for \( m \geq 7 \) since \( A_5 \cong \text{PSL}_2(4) \cong \text{PSL}_2(5) \) and \( A_6 \cong \text{PSL}_2(9) \) will be discussed in Lemma 4.6.

Lemma 4.5. Let \( (m,n) \in (\mathbb{N}^+)^2 \), \( m \geq 7 \). Then
\[
\lambda_{\text{aff}}(A_m^n) \leq \frac{1}{10}.
\]
Proof. First consider the case \( m \geq 7 \). By Lemma 4.3 it is sufficient to show that
\[
10 \cdot \text{meo}(A_m)^2 \cdot \text{mao}(A_m) \leq \frac{m!}{2},
\]
for which in turn, as \( \text{Aut}(A_m) \cong S_m \), it is sufficient to show that \( 20 \cdot g(m)^2 \leq m! \). Validity of this inequality is easily verified for \( m = 7 \), and by Corollary 4.2 the inequality is weaker than \( 20 \cdot \left(\frac{3}{2}\right)^{2m} \leq m! \), which by an induction argument holds for all \( m \geq 8 \) if and only if it holds for \( m = 8 \), a case in which its validity is readily checked as well. \( \square \)
Table 1: Checking the assumption of Lemma 4.3 for sporadic finite simple groups

| sporadic $S$ | $\text{meo}(S)$ | Out($S$) | $10 \cdot \text{meo}(S)^2 \cdot \text{meo}(\text{Out}(S)) \leq \ldots$ | $|S| \geq \ldots$ |
|--------------|-----------------|----------|-------------------------------------------------|----------------|
| $M_{11}$     | 11              | 1        | 1210                                            | 7920           |
| $M_{12}$     | 11              | $\mathbb{Z}/2\mathbb{Z}$ | 2420                                            | 95040          |
| $M_{22}$     | 11              | $\mathbb{Z}/2\mathbb{Z}$ | 2420                                            | 443520         |
| $M_{23}$     | 23              | 1        | 5290                                            | $1.0 \cdot 10^3$ |
| $M_{24}$     | 23              | 1        | 5290                                            | $2.4 \cdot 10^3$ |
| HS           | 20              | $\mathbb{Z}/2\mathbb{Z}$ | 8000                                            | $4.4 \cdot 10^3$ |
| $J_2$        | 15              | $\mathbb{Z}/2\mathbb{Z}$ | 4450                                            | 604800         |
| Co$_1$       | 60              | 1        | 360000                                          | $4.1 \cdot 10^{16}$ |
| Co$_2$       | 30              | 1        | 9000                                            | $4.2 \cdot 10^{13}$ |
| Co$_3$       | 30              | 1        | 9000                                            | $4.9 \cdot 10^{14}$ |
| McL          | 30              | $\mathbb{Z}/2\mathbb{Z}$ | 18000                                           | $8.9 \cdot 10^3$ |
| Suz          | 24              | $\mathbb{Z}/2\mathbb{Z}$ | 11520                                           | $4.4 \cdot 10^{14}$ |
| He           | 28              | $\mathbb{Z}/2\mathbb{Z}$ | 15680                                           | $4 \cdot 10^9$ |
| HN           | 40              | $\mathbb{Z}/2\mathbb{Z}$ | 32000                                           | $2.7 \cdot 10^{14}$ |
| Th           | 39              | 1        | 15210                                           | $9 \cdot 10^{16}$ |
| Fi$_{22}$    | 30              | $\mathbb{Z}/2\mathbb{Z}$ | 18000                                           | $6.4 \cdot 10^{13}$ |
| Fi$_{23}$    | 60              | 1        | 360000                                          | $4 \cdot 10^{15}$ |
| Fi$_{24}'$   | 84              | $\mathbb{Z}/2\mathbb{Z}$ | 141120                                          | $1.2 \cdot 10^{24}$ |
| B            | 70              | 1        | 49000                                           | $4.1 \cdot 10^{25}$ |
| M            | 119             | 1        | 141610                                          | $8 \cdot 10^{25}$ |
| J$_1$        | 19              | 1        | 3610                                            | 175560         |
| O'N          | 31              | $\mathbb{Z}/2\mathbb{Z}$ | 19220                                           | $4.6 \cdot 10^{14}$ |
| J$_3$        | 19              | $\mathbb{Z}/2\mathbb{Z}$ | 7220                                            | $5.0 \cdot 10^{14}$ |
| Ru           | 29              | 1        | 8410                                            | $1.4 \cdot 10^{14}$ |
| J$_4$        | 66              | 1        | 435600                                          | $8.6 \cdot 10^{19}$ |
| Ly           | 67              | 1        | 44890                                           | $5.1 \cdot 10^{15}$ |

By CFSG, all finite simple groups not treated so far are of Lie type. The maximal element orders of those of odd characteristic were determined in [10], and [6] extended these results by giving upper bounds for maximal element and automorphism orders in all characteristics. Let us begin with the following:

**Lemma 4.6.** Let $(d, n, q) \in (\mathbb{N}^+)^3$ such that $d \geq 2$, $q$ is primary and $(d, q) \neq (2, 2), (2, 3)$. Then

$$\lambda_{\text{aff}}(\text{PSL}_d(q)^n) \leq \frac{1}{10^4},$$

with the only exceptions

$$\text{PSL}_2(4) \cong \text{PSL}_2(5) \cong A_5, \text{PSL}_2(9) \cong A_6, \text{PSL}_2(7) \cong \text{PSL}_3(2) \text{ and } \text{PSL}_2(8),$$

which have the following $\lambda_{\text{aff}}$-values:

$$\lambda_{\text{aff}}(\text{PSL}_2(4)) = \frac{1}{4}, \lambda_{\text{aff}}(\text{PSL}_2(7)) = \frac{1}{6}, \lambda_{\text{aff}}(\text{PSL}_2(8)) = \frac{1}{8} \text{ and } \lambda_{\text{aff}}(\text{PSL}_2(9)) = \frac{1}{9}.$$
Proof. By Lemma 2.3, it suffices to show that $\lambda_{\text{aff}}(\text{PSL}_d(q)) \leq \frac{1}{10}$ except for the groups listed and that $\lambda_{\text{aff}}(\mathcal{A}_5) = \frac{1}{7}$, $\lambda_{\text{aff}}(\mathcal{A}_6^2) \leq \frac{1}{10}$ for $n \geq 2$, $\lambda_{\text{aff}}(\mathcal{A}_6) = \frac{1}{7}$, $\lambda_{\text{aff}}(\text{PSL}_2(7)) = \frac{1}{7}$ and $\lambda_{\text{aff}}(\text{PSL}_2(8)) = \frac{1}{8}$.

Let us start by discussing $\lambda_{\text{aff}}$-values of powers of $\mathcal{A}_5$ and of $\mathcal{A}_6$. Using GAP, it is easy to verify that $\lambda_{\text{aff}}(\mathcal{A}_6) = \frac{1}{7}$. As for $\lambda_{\text{aff}}(\mathcal{A}_6^2)$, assume first that $n \geq 3$. Applying Lemma 2.6(3), we obtain that

$$|\mathcal{A}_5|^n \cdot \lambda_{\text{aff}}(\mathcal{A}_6^n) \leq \text{meo}(\mathcal{A}_6^n) \cdot \text{mao}(\mathcal{A}_6^n) \cdot \exp(\mathcal{A}_5) \cdot \exp(\text{Aut}(\mathcal{A}_5)) \cdot g(n) = 1800 \cdot g(n) < 1800 \left(\frac{3}{2}\right)^n,$$

which yields $\lambda_{\text{aff}}(\mathcal{A}_6^2) < 1800 \cdot (\frac{1}{10})^n$. Since the right-hand side of this inequality is decreasing in $n$, it is sufficient to find that the right-hand side for $n = 3$ is not greater than $\frac{1}{10}$. And indeed, $1800 \cdot (\frac{1}{10})^3 = 0.028 \ldots < \frac{1}{10}$.

Furthermore, it is easy to check with GAP that $\lambda_{\text{aff}}(\mathcal{A}_5) = \frac{1}{7}$. We now argue why $\lambda_{\text{aff}}(\mathcal{A}_5^2) \leq \frac{1}{10}$ (this group apparently is too large for our general algorithm for computing the $\lambda_{\text{aff}}$-value to terminate in reasonable time). Note that since by Theorem 3.3, $\text{Aut}(\mathcal{A}_5^2) \cong \text{Aut}(\mathcal{A}_5) \wr S_2$, an automorphism of $\mathcal{A}_5^2$ is either “twisted” (corresponding to an element from outside the base $(\mathcal{A}_5^2)^2$) or lies in the base, in which case it decomposes as a product of automorphisms over the two factors $\mathcal{A}_5$. However, then any affine map $A$ associated with that automorphism also decomposes as a product of affine maps over the factors so that $\lambda(A) \leq \lambda_{\text{aff}}(A_{m_2})^2 \leq \frac{1}{10} < \frac{1}{10}$. So there only remains the case of an affine map with respect to a “twisted” automorphism $\alpha$ of $\mathcal{A}_5^2$. Observe that then there exist automorphisms $\alpha_1, \alpha_2$ of $\mathcal{A}_5$ such that $\alpha^2$ decomposes as the product of the automorphism $\alpha_1 \alpha_2$ over the first and $\alpha_2 \alpha_1$ over the second factor. Hence $\text{ord}(\alpha)$ is bounded above by the maximum value of $\text{lcm}\{\text{ord}(\alpha_1 \alpha_2), \text{ord}(\alpha_2 \alpha_1)\}$, where $\alpha_1$ and $\alpha_2$ run through all automorphisms of $\mathcal{A}_5$ (i.e., all elements of $S_5$). Using GAP, we find that this maximum is 6, which allows us to conclude using Lemma 2.6(3): The maximum order of an affine map of $\mathcal{A}_5^2$ with respect to a twisted automorphism is bounded above by $2 \cdot 6 \cdot 15 = \frac{1}{60} \cdot 460 = \frac{1}{600}^2$.

We can now focus on those projective special linear groups which are not isomorphic to any alternating group. That is, we may assume $(d, q) \neq (2, 4), (2, 5), (2, 9)$ and also $(d, q) \neq (3, 2)$ because of $\text{PSL}_3(2) \cong \text{PSL}_2(7)$. For the sake of simplicity, we will work with the inequality $\Lambda_{\text{aff}}(\text{PSL}_d(q)) \leq \text{mao}(\text{PSL}_d(q))^2$, which will be good enough for our purposes. Note that

$$|\text{PSL}_d(q)| = \frac{1}{\gcd(d, q - 1)} q^{\frac{d(d-1)}{2}} \prod_{i=2}^{d} (q^i - 1).$$

By [8, Table 3], we have

$$\text{mao}(\text{PSL}_d(q)) = \frac{q^d - 1}{q - 1},$$

which gives us the following:

$$\Lambda_{\text{aff}}(\text{PSL}_d(q)) \leq \text{mao}(\text{PSL}_d(q))^2 = \left(\frac{q^d - 1}{q - 1}\right)^2.$$

First consider the case $d = 2$. Then $|\text{PSL}_d(q)| \geq \frac{1}{2} q(q^2 - 1)$, so we can refute those cases where

$$\left(\frac{q^2 - 1}{q - 1}\right)^2 \leq \frac{1}{20} q(q^2 - 1),$$

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which, as is easy to see, is equivalent to

\[ q^3 - 22q^2 + q + 20 \geq 0, \]

and this is easily seen to be true for \( q \geq 23 \). For \( q = 7, 8, 11, 13, 16, 17, 19 \), we used GAP to explicitly compute the \( \lambda_{\text{aff}} \)-value. The results are summarized in Table 2 on the next page, concluding examination of this case.

| \( q \) | \( \lambda_{\text{aff}}(\text{PSL}_2(q)) \) |
|--------|----------------|
| 7      | 1/6            |
| 8      | 1/8            |
| 11     | 1/10           |
| 13     | 1/12           |
| 16     | 1/16           |
| 17     | 1/16           |
| 19     | 1/18           |

Table 2: \( \lambda_{\text{aff}} \)-values of some \( \text{PSL}_2(q) \)

For \( d \geq 3 \), we use the lower bound

\[ |\text{PSL}_d(q)| \geq \frac{1}{q-1} q^{\frac{d(d-1)}{2}} \prod_{i=2}^{d} (q^i - 1), \]

by which in order to refute the case, it is sufficient to have

\[ \left( \frac{q^d - 1}{q - 1} \right)^2 \leq \frac{1}{10(q-1)} q^{\frac{d(d-1)}{2}} \prod_{i=2}^{d} (q^i - 1), \]

or equivalently

\[ \frac{q^d - 1}{q^{d-1} - 1} \leq \frac{1}{10} q^{\frac{d(d-1)}{2}} \prod_{i=1}^{d-2} (q^i - 1). \]

Since the left-hand side of this inequality is bounded above by \( q^2 \), it is hence sufficient to have

\[ 10 \leq q^{\frac{d(d-1)}{2} - 2} \prod_{i=1}^{d-2} (q^i - 1). \]

For \( d = 3 \) (where we may assume \( q \geq 3 \)), this turns into \( 10 \leq q(q-1) \), which is satisfied for \( q \geq 4 \). Using GAP, one verifies that \( \lambda_{\text{aff}}(\text{PSL}_3(3)) = \frac{1}{14} < \frac{1}{10} \). Finally, since the right-hand side of the above sufficient inequality is increasing both in \( d \) and \( q \), it suffices to refute all \( q \) for \( d = 4 \) to actually refute the case \( d \geq 4 \) entirely. For \( d = 4 \), in turn, the inequality turns into \( 10 \leq q^4(q-1)(q^2 - 1) \), which certainly is satisfied for all \( q \geq 2 \).

From now on, in accordance with Theorem 3.7 there will be no further examples of \( \lambda_{\text{aff}} \)-value larger than \( \frac{1}{10} \). By Lemma 4.3 it is sufficient to prove this for the respective simple groups themselves, not general powers, whence for the sake of simplicity, we henceforth state and
prove the corresponding assertions without using a general exponent \( n \) as before, but setting \( n := 1 \) for the rest of the discussion. We next discuss special unitary groups \( \text{PSU}_d(q) \), with \( d \geq 3 \) and \( (d, q) \neq (3, 2) \) (since \( \text{PSU}_3(2) \) is not simple).

**Lemma 4.7.** Let \( d \geq 3 \) and \( q \) be primary such that \( (d, q) \neq (3, 2) \). Then
\[
\lambda_{\text{aff}}(\text{PSU}_d(q)) \leq \frac{1}{10}.
\]

**Proof.** We begin by noting that
\[
|\text{PSU}_d(q)| = \frac{1}{\gcd(d, q + 1)} q^{\frac{d(d-1)}{2}} \prod_{i=2}^{d} q^i - (-1)^i.
\]

For this proof, we follow the case distinction from [6, Table 3], each case giving a different formula for \( \text{mao}(\text{PSU}_d(q)) \).

**Case 1:** \( 2 \nmid d, q \) is composite and \( (d, q) \neq (3, 4) \). Then \( \text{mao}(\text{PSU}_d(q)) = q^{d-1} - 1 \). Hence it is sufficient to show
\[
(q^{d-1} - 1)^2 \leq \frac{1}{10} \cdot \frac{1}{\gcd(d, q + 1)} q^{\frac{d(d-1)}{2}} (q^d + 1)(q^{d-1} - 1),
\]
or equivalently
\[
q^{d-1} - 1 \leq \frac{1}{10 \gcd(d, q + 1)} q^{\frac{d(d-1)}{2}} (q^d + 1).
\]

To see this, we argue that
\[
\frac{1}{10 \cdot \gcd(d, q + 1)} q^{\frac{d(d-1)}{2}} \geq 1.
\]

For \( d = 3 \), we have
\[
\frac{1}{10 \cdot \gcd(d, q + 1)} q^{\frac{d(d-1)}{2}} \geq \frac{1}{30} q^3 \geq \frac{1}{30} 3^3 \geq 1,
\]
and for \( d \geq 4 \), we find that
\[
\frac{1}{10 \cdot \gcd(d, q + 1)} q^{\frac{d(d-1)}{2}} \geq \frac{1}{10(q + 1)} q^6 \geq \frac{1}{10} \cdot q^4 \geq \frac{1}{10} 4^4 > 1.
\]

**Case 2:** \( (d, q) = (3, 4) \). We have \( \text{mao}(\text{PSU}_3(4)) = 16 \), so it is sufficient to have \( 16^2 = 256 \leq \frac{1}{10}|\text{PSU}_3(4)| \), which is satisfied as \( |\text{PSU}_3(4)| = 62400 \).

**Case 3:** \( 2 \nmid d, q \) is a prime \( p \) and \( (d, q) \neq (5, 2) \). In this case, \( \text{mao}(\text{PSU}_d(q)) = (p^{d-2} + 1)p \).

Hence it suffices to check that
\[
(p^{d-2} + 1)^2 p^2 \leq \frac{1}{10 \gcd(d, p + 1)} p^{\frac{d(d-1)}{2}} (p^{d} + 1)(p^{d-1} - 1),
\]
which is equivalent to
\[
10 \leq \frac{1}{\gcd(d, p + 1)} p^{\frac{d(d-1)}{2} - 2} \cdot p^d + 1 \cdot \frac{p^{d-1} - 1}{p^{d-2} + 1}.
\]

For \( d = 3 \), since \( p \geq 3 \), we have
\[
\frac{1}{\gcd(d, p + 1)} p^{\frac{d(d-1)}{2} - 2} \cdot p^d + 1 \cdot \frac{p^{d-1} - 1}{p^{d-2} + 1} \geq \frac{1}{3} \cdot 2p \cdot (p - 1) \geq \frac{1}{3} \cdot 3 \cdot 2 = 12 > 10.
\]
And for \( d \geq 4 \), we find that
\[
\frac{1}{\gcd(d, p + 1)} p^{\frac{d(d-1)}{2}} \cdot \frac{p^d + 1}{p^{d-2} + 1} \cdot \frac{p^{d-1} - 1}{p^{d-2} + 1} \geq \frac{1}{p + 1} p^4 \cdot (p + 1) \cdot 1 > 10.
\]

**Case 4**: \((d, q) = (5, 2)\): We have \( \text{mao}(\text{PSU}_5(2)) = 24 \), which settles this case since
\[
|\text{PSU}_5(2)| = 13685760 > 5760 = 10 \cdot 24^2.
\]

**Case 5**: \(2 \mid d\) and \( q > 2\). In this case, \( \text{mao}(\text{PSU}_d(q)) = q^{d-1} + 1\). Since \( d \geq 4\), it is sufficient to check that
\[
(q^{d-1} + 1)^2 \leq \frac{1}{10(q + 1)} q^6 \cdot (q^d - 1)(q^{d-1} + 1)(q^{d-2} - 1),
\]
or equivalently
\[
10 \leq q^6 \cdot \frac{q^d - 1}{q^{d-1} + 1} \cdot \frac{q^{d-2} - 1}{q + 1},
\]
which is obviously true.

**Case 6**: \(2 \mid d\) and \( q = 2\). Then \( \text{mao}(\text{PSU}_d(2)) = 4(2^{d-3} + 1)\), so we check that
\[
16(2^{d-3} + 1)^2 \leq \frac{1}{30} \cdot 2^6 \cdot (2^d - 1)(2^{d-1} + 1)(2^{d-2} - 1),
\]
which is equivalent to
\[
480 \leq 2^6(2^d - 1) \cdot \frac{2^{d-2} + 1}{2^{d-3} + 1} \cdot \frac{2^{d-2} - 1}{2^{d-3} + 1},
\]
and this is true since
\[
2^6(2^d - 1) \geq 2^6(2^4 - 1) = 960 > 480.
\]

The remaining finite classical simple groups of Lie type, i.e., \( \text{PSp}_{2m}(q) \) for \( m \geq 2 \) and \((m, q) \neq (2, 2)\) as well as, for odd \( q \), \( \text{PO}^{\pm}_{2m+1}(q) \) for \( m \geq 3 \) and \( \text{PO}^+_q(q) \) for \( m \geq 4 \) do not yield any exceptional cases either. Note that we can exclude the group \( \text{PSp}_4(2)' \), since it is isomorphic to \( A_6 \).

**Lemma 4.8.** (1) Let \( m \geq 2 \) and \( q \) be primary such that \((m, q) \neq (2, 2)\). Then
\[
\lambda_{\text{aff}}(\text{PSp}_{2m}(q)) \leq \frac{1}{10}.
\]

(2) Let \( m \geq 3 \) and \( q \) be an odd primary number. Then
\[
\lambda_{\text{aff}}(\text{PO}_{2m+1}(q)) \leq \frac{1}{10}.
\]

(3) Let \( m \geq 4 \) and \( q \) be an odd primary number. Then
\[
\lambda_{\text{aff}}(\text{PO}^+_q(q)) \leq \frac{1}{10}
\]
and
\[
\lambda_{\text{aff}}(\text{PO}^-_{2m}(q)) \leq \frac{1}{10}.
\]
Proof. First, we note that by Table 3 from [6], the maximal automorphism order of any of the groups listed above is bounded above by \( \frac{q^{m+1}}{q-1} \). Looking at the orders of the groups in question, we also find that both \(|\text{PSp}_{2m}(q)|\) and \(|\text{PΩ}_{2m+1}(q)|\) are bounded below by

\[
\frac{1}{2}q^{m^2} \prod_{i=1}^{m} (q^{2i} - 1) \geq \frac{1}{2}q^{m^2} (q^{2m-1})(q^{2m-2} - 1).
\]

In order to prove both (1) and (2) simultaneously, it is therefore sufficient to prove that for all \( m \geq 2 \) and all primary \( q \) such that \((m, q) \neq (2, 2)\), we have

\[
\frac{q^{2m+2}}{(q-1)^2} \leq \frac{1}{20}q^{m^2} \cdot \prod_{i=1}^{m} (q^{2i} - 1).
\]

For \( m = 2 \), this is equivalent to

\[
20q^2 \leq (q-1)^2(q^4 - 1)(q^2 - 1),
\]

which is easily seen to be true for all \( q \geq 3 \). And for \( m \geq 3 \), we find that

\[
\frac{1}{20}q^{m^2} \cdot \prod_{i=1}^{m} (q^{2i} - 1) \geq \frac{1}{20}q^9(q^{2m-1})(q^{2m-2} - 1)(q^{2m-4} - 1) \geq \frac{q^6}{20} \cdot q^3(q^{2m} - 1) \geq 1 \cdot q^{2m+2} \geq \frac{q^{2m+2}}{(q-1)^2}.
\]

For (3), we use that both \(|\text{PΩ}_{2m}(q)|\) and \(|\text{PΩ}_{2m}(q)|\) are bounded below by

\[
\frac{1}{4}q^{m(m-1)}(q^m - 1)(q^{2m-2} - 1)(q^{2m-4} - 1)
\]

for all \( m \geq 4 \). It is hence sufficient to show that

\[
\frac{q^{2m+2}}{(q-1)^2} \leq \frac{1}{40}q^{m(m-1)}(q^m - 1)(q^{2m-2} - 1)(q^{2m-4} - 1),
\]

which is equivalent to

\[
40q^{2m+2} \leq (q-1)^2q^{m(m-1)}(q^m - 1)(q^{2m-2} - 1)(q^{2m-4} - 1).
\]

And indeed, we find that the right-hand side is bounded below by

\[
q^{12}(q^{2m-2} - 1) = q^7 \cdot q^5(q^{2m-2} - 1) \geq 40 \cdot q^{2m+2}
\]

for all \( q \geq 3 \).

It remains to treat the exceptional simple groups of Lie type. Note that we can omit the group \( ^2G_2(3)' \) since it is isomorphic to PSL\(_2\)(8).

Lemma 4.9. Let \( S \) be a finite exceptional simple group of Lie type, not isomorphic to \( ^2G_2(3)' \). Then

\[
\lambda_{\text{aff}}(S) \leq \frac{1}{10}.
\]
Proof. We begin with those of odd characteristic. It turns out that for all of these groups, easy and brute estimates “do the job”. We present one example in detail. Let $S = 2G_2(3^{2e+1})$ with $e \geq 1$. By [10, Table A.7], we have $\text{meo}(S) = 3^{2e+1} + 3^{e+1} + 1$, which we bound from above by $3 \cdot 3^{2e+1} = 3^{2e+2}$. Also, by [4, Table 5, page xvi], $|\text{Out}(S)| = 2e + 1$, which we bound from above by $3^{e}$. By Lemma 2.5(3), this gives us

$$\Lambda_{\text{aff}}(S) \leq 3^{4e+5}.$$ 

Now $|S| = 3^{6e+3}(3^{6e+3} + 1)(3^{2e+1} - 1)$, which is clearly larger than the tenfold of the upper bound for $\Lambda_{\text{aff}}(S)$.

All the other cases in odd characteristic can be treated in a similar way. The upper part of Table 3 on the next page gives information on the inequalities one has to prove, which were obtained by collecting information on $|S|$, $\text{meo}(S)$ and $|\text{Out}(S)|$ for the groups $S$ listed and applying Lemma 2.5(3). In some cases, occurrences of greatest common divisors were either moved from one side to the other or replaced by appropriate bounds. The reader easily convinces themselves that all of the inequalities can be proved without difficulty.

For the characteristic 2 case, we follow the treatment of those groups from [6]. For $S = 2B_2(2^{2e+1})$ with $e \geq 1$, as is observed there, the precise maximal element order is known, namely $\text{meo}(S) = 2^{2e+1} + 2e + 1$, and also $|\text{Out}(S)| = 2e + 1$. Therefore, we obtain the following as a sufficient inequality, easily checked to hold for all $e \geq 1$:

$$10(2e + 1)(2^{2e+1} + 2e + 1)^2 \leq |S| = 2^{4e+2}(2^{4e+2} + 1)(2^{2e+1} - 1).$$

For the Tits group $T := 2F_4(2)'$, it follows from [4] that $\text{meo}(T) = 16$ and $|\text{Out}(T)| = 2$, so that $\Lambda_{\text{aff}}(T) \leq 2 \cdot 16^2 = 512$, but $|T| = 17971200 \gg 5210$.

For the other cases, Guest, Morris, Praeger and Spiga derived upper bounds for $\text{meo}(S)$; the vital parameters are summarized in [6, Table 5]. For the readers’ convenience, we summarize the sufficient inequalities which can be derived from this information in the lower part of Table 3, except for the groups $2F_4(2^{2e+1})$ for $e \geq 1$ which would make the row too long. Setting $f := 2e + 1$, their sufficient inequality is given by

$$10 \cdot 2^8 f(2^{2e+2} + 2^{3e+1} + 2^{2e+1} + 2e + 1)^2 \leq 2^{12f} (2^{6f} + 1)(2^{4f} - 1)(2^{3f} + 1)(2f - 1).$$

Again, all the inequalities for exceptional Lie type groups in characteristic 2 are easy to prove.

$\Box$

Proof of Theorem 3.7. Invoking CFSG, this follows from Lemmata 4.4, 4.5, 4.6, 4.7, 4.8 and 4.9.

Proof of Theorem 3.3. By Theorem 3.7, among the nonabelian finite characteristically simple groups, there are no other examples of finite centerless CR-groups with $\lambda_{\text{aff}}$ value greater than $\frac{1}{10}$ than the four groups listed here as well. Also, any finite characteristically simple group $C$ satisfies $\lambda_{\text{aff}}(C) \leq \frac{1}{4}$. Therefore, if $R$ is a finite centerless CR-group which is not characteristically simple (i.e., a product of at least two powers of nonabelian finite simple groups with nonisomorphic bases), then by Theorem 3.3 and Lemma 2.6

$$\lambda_{\text{aff}}(R) \leq \left(\frac{1}{4}\right)^2 = \frac{1}{16} < \frac{1}{10},$$

whence there are no additional examples among such $R$.

$\Box$
Table 3: Checking the assumption of Lemma 4.3(2) for exceptional Lie type groups. $p$ is an odd prime, $I_1 = \{2, 6, 8, 12\}$, $I_2 = \{2, 5, 6, 8, 9, 12\}$, $I_3 = \{2, 6, 8, 10, 12, 14, 18\}$, $I_4 = \{2, 8, 12, 14, 18, 20, 24, 30\}$

| $S$ | sufficient inequality |
|-----|-----------------------|
| $G_2(3^{e+1}), e \geq 1$ | $(20e + 10)(3^{2e+1} + 3^e + 1)^2 \leq 3^{5e+5}(3^{2e+1} + 1) - 1$ |
| $G_2(p^f)$ | $20f(p^{2f} + p^f + 1)^2 \leq p^{5f}(p^{6f} - 1) - 1$ |
| $D_4(p^f)$ | $30f(p^{2f} + 1)^2(p^f + 1)^2 \leq p^{12f}(p^{3f} + 1)(p^{3f} - 1) - 1$ |
| $F_4(p^f)$ | $10p^2(p + 1)^2(p^f + 1)^2 \leq 2^{24f} \prod_{i \in I_1} (p^f - 1)$ |
| $E_6(p^f), f \geq 2$ | $10f(p^{2f} + 1)^2(p^f + 1)^2 \leq 2^{36f} \prod_{i \in I_1} (p^f - 1)$ |
| $E_6(2^f)$ | $10 \cdot 2^{10f}(2^f + 1)^2(2^f + 2)^2(2^f - 1)^2 \leq 2^{36f} \prod_{i \in I_1} (2^f - 1)$ |
| $E_6(2^f)$ | $10 \cdot 2^{10f}(2^f + 1)^2(2^f + 2)^2(2^f - 1)^2 \leq 2^{36f} \prod_{i \in I_1} (2^f - 1)$ |
| $E_6(2^f)$ | $10 \cdot 2^{10f}(2^f + 1)^2(2^f + 2)^2(2^f - 1)^2 \leq 2^{36f} \prod_{i \in I_1} (2^f - 1)$ |
| $E_6(2^f)$ | $10 \cdot 2^{10f}(2^f + 1)^2(2^f + 2)^2(2^f - 1)^2 \leq 2^{36f} \prod_{i \in I_1} (2^f - 1)$ |
| $D_4(2^f)$ | $30 \cdot 2^6f(2^f + 2^f - 2f - 1)^2 \leq 2^{12f}(2^f + 2^f - 1)(2^f - 1)(2^f - 1)$ |
| $E_6(2^f)$ | $10 \cdot 2^{10f}(2^f + 1)^2(2^f + 2)^2(2^f - 1)^2 \leq 2^{36f} \prod_{i \in I_1} (2^f - 1)^2$ |

Before proving the decisive Corollary 3.6, we note the following, which will make the proof more elegant:

**Theorem 4.10.** Let $S$ be a nonabelian simple group. Then $\text{Aut}(S)$ is complete.

*Proof.* See for example [14, 13.5.10, p.414].

**Corollary 4.11.** Let $S$ be a finite nonabelian simple group. Then $\lambda(S) = \frac{\text{mao}(S)}{|S|}$ and $\lambda(\text{Aut}(S)) = \frac{\text{mao}(S)}{|\text{Aut}(S)|} = \frac{1}{|\text{Out}(S)|} \lambda(S)$.

*Proof.* This follows immediately from Theorem 1.3 and Theorem 4.10

**Proof of Corollary 3.6.** We want to apply Lemma 3.3 with

$$C_1 := \{G \in \mathcal{G}^{(\text{fin})} \mid \lambda(G) > \frac{1}{10}\}$$

and

$$C_2 := \{G \in \mathcal{G}^{(\text{fin})} \mid \lambda_{\text{aff}}(G) > \frac{1}{10}\}.$$ 

Assumption (i) is clear by Lemma 2.7. By Theorem 3.3, we have

$$C = \{A_5, A_6, \text{PSL}_2(7), \text{PSL}_2(8)\}.$$
It remains to verify that no finite Fitting-free group with centerless CR-radical isomorphic to one of the four groups from this list has $\lambda$-value greater than $\frac{1}{10}$. This is an application of Theorem 3.2.

For $R = A_5$, we have $\text{Out}(R) \cong \mathbb{Z}/2\mathbb{Z}$. Now $\mathbb{Z}/2\mathbb{Z}$ has two conjugacy classes of subgroups, corresponding to the two isomorphism types of finite Fitting-free groups with centerless CR-radical $A_5$, namely $A_5$ and $S_5$. By Corollary 4.11, we have

$$\lambda(A_5) = \frac{6}{60} = \frac{1}{10},$$

as was already remarked in Section 1 and

$$\lambda(S_5) = \frac{1}{2} \lambda(A_5) = \frac{1}{20}.$$ 

For $R = \text{PSL}_2(7)$, by [4], $\text{Out}(R) \cong \mathbb{Z}/2\mathbb{Z}$ as well. Hence an argument analogous to the one for $A_5$ works in this case: The only finite Fitting-free groups with centerless CR-radical $\text{PSL}_2(7)$ are $\text{PSL}_2(7)$ and $\text{Aut}(\text{PSL}_2(7))$. By [6, Table 3], $\text{mao}(\text{PSL}_2(7)) = 8$, and so

$$\lambda(\text{PSL}_2(7)) = \frac{8}{168} = \frac{1}{21}$$

and

$$\lambda(\text{Aut}(\text{PSL}_2(7))) = \frac{1}{42}.$$ 

For $R = \text{PSL}_2(8)$, by [4], $\text{Out}(R) \cong \mathbb{Z}/3\mathbb{Z}$, which also just has two conjugacy classes of subgroups, so $\text{PSL}_2(8)$ and $\text{Aut}(\text{PSL}_2(8))$ are the only finite Fitting-free groups with centerless CR-radical $\text{PSL}_2(8)$. By [6, Table 3], $\text{mao}(\text{PSL}_2(8)) = 9$, so

$$\lambda(\text{PSL}_2(8)) = \frac{9}{504} = \frac{1}{56}.$$ 

and

$$\lambda(\text{Aut}(\text{PSL}_2(8))) = \frac{1}{168}.$$ 

It remains to discuss the case $R = A_6$, which is slightly more complicated, since $\text{Out}(A_6) \cong (\mathbb{Z}/2\mathbb{Z})^2$, which has five conjugacy classes of subgroups. One of them corresponds to $A_6$ and one to $\text{Out}(A_6) = S_6 \times \mathbb{Z}/2\mathbb{Z}$; we can treat these two as in the previous three cases, using that $\text{mao}(A_6) = \text{mao}(\text{PSL}_2(9)) = 10$ by [6, Table 3]:

$$\lambda(A_6) = \frac{10}{360} = \frac{1}{36}$$

and

$$\lambda(\text{Aut}(A_6)) = \frac{1}{4} \lambda(A_6) = \frac{1}{144}.$$ 

For any of the three isomorphism types of finite Fitting-free groups $G$ with centerless CR-radical $A_6$ corresponding to one of the three two-element subgroups of $\text{Out}(A_6)$, we use the following argumentation. Let $\alpha$ be an automorphism of $G$ such that $\lambda(\alpha) = \lambda(G)$ (or equivalently, $\Lambda(\alpha) = \text{mao}(G)$). Denote by $o$ the order of the restriction of $\alpha$ to the centerless CR-radical $R \cong A_6$ of $G$ and note that $o \leq 10$. Then $\alpha^o$ acts identically on $R$, and by the main idea of the proof of Lemma 2.7, the action of $\alpha^o$ on the other coset $G \setminus R$ of $R$ in $G$
corresponds to the action of an affine map of the form $A_{\sigma, \text{id}}$ for some $\sigma \in A_6$ on $A_6$. By Lemma 2.5(1), we have

$$\text{ord}(A_{\sigma, \text{id}}) = \text{ord}(\sigma) \leq 5,$$

whence we get that $\text{ord}(\alpha) \leq \alpha \cdot 5 \leq 50$, and hence

$$\lambda(G) \leq \frac{50}{720} < \frac{1}{10}. \quad \Box$$

**Proof of Theorem 1.4.** This follows by an application of Lemma 3.1 with

$$\mathcal{C} := \{G \in G^{(\text{fin})} \mid \lambda(G) > \frac{1}{10}\}.$$

Assumption (i) is satisfied by Lemma 2.3, and assumption (ii) by Corollary 3.0. \hfill \Box

## 5 Acknowledgements

The author would like to thank Peter Hellekalek for his helpful comments.

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