Solving equations with Hodge theory

Kefeng Liu and Shengmao Zhu

Abstract. We treat two quite different problems related to changes of complex structures on Kähler manifolds by using global geometric method. First, by using operators from Hodge theory on compact Kähler manifold, we present a closed explicit extension formula for holomorphic canonical forms in different complex structures. As applications, we give a closed explicit formula for certain canonical sections of Hodge bundles on marked and polarized moduli spaces of projective manifolds, and provide a closed explicit extension formula for holomorphic pluricanonical forms under certain natural conditions. Second, by using the operators in $L^2$-Hodge theory on Poincaré disk, we present a simple and unified method to solve the Beltrami equations with measurable coefficients for quasi-conformal maps.

1. Introduction

Let $M$ be a complex manifold of complex dimension $\dim_{\mathbb{C}} M = n$, and $\varphi$ is a Beltrami differential which is a tangent bundle valued $(0,1)$-form in $A^{0,1}(M,T^{1,0} M)$. See Section 2.2 for a detailed discussion of complex structures and Beltrami differentials. If the Beltrami differential $\varphi$ is integrable in the sense that

$$\partial \varphi = \frac{1}{2} [\varphi, \varphi],$$

then $\varphi$ determines a new complex structure on $M$, denoted by $M_\varphi$.

Given any differential form $\sigma$ on $M$, let us denote by $\varphi \wedge \sigma = i_\varphi \sigma = \varphi \sigma$ the natural contraction morphism throughout this paper, and we define a map

$$\rho_\varphi(\sigma) = e^{i\varphi} \sigma = \sum_{k \geq 0} \frac{1}{k!} i_\varphi^k \sigma.$$ (1.1)

Then $\rho_\varphi$ gives a bijection from $A^{n,0}(M)$ to $A^{n,0}(M_\varphi)$. Therefore, given a smooth $(n,0)$-form $\Omega$ on complex manifold $M$, $\rho_\varphi(\Omega)$ is a $(n,0)$-form on $M_\varphi$. We will show that $\rho_\varphi(\Omega) = e^{i\varphi} \Omega$ is holomorphic on $M_\varphi$ if and only if

$$\overline{\partial} \Omega + \partial (\varphi \wedge \Omega) = 0.$$ (1.2)

In this paper we present a global geometric method to study the changes of complex structures on Kähler manifolds and discuss the related geometric and
analytic applications without using the local deformation theory as developed by Kodaira-Spencer. The following is a brief outline of the method and main results of the paper.

1.1. Extension formula for holomorphic canonical forms on compact Kähler manifold. We consider compact Kähler manifold \((M, \omega)\) of complex dimension \(n\) with Kähler form \(\omega\). Let \(\varphi \in A^{0,1}(M, T^{1,0}M)\) be an integrable Beltrami differential. Given a holomorphic \((n, 0)\)-form \(\Omega_0\) on \(M\), i.e. a holomorphic section of the canonical bundle \(K_M\), our first goal is to construct a holomorphic \((n, 0)\)-form \(\Omega(\varphi)\) on \(M_\varphi\), such that \(\Omega(0) = \Omega_0\). We construct such \(\Omega(\varphi)\) from the solution of extension equation \((1.2)\), which will be solved by using Hodge theory on \((M, \omega)\).

More precisely, we introduce the operator

\[
T = \overline{\partial}^*G\partial
\]

where \(d = \partial + \overline{\partial}\), with \(\partial\) and \(\overline{\partial}\) the \((1, 0)\) and \((0, 1)\) differentials, \(\partial^*\) and \(\overline{\partial}^*\) the corresponding adjoint operators, and \(G\) denotes the Green operator of the Laplacian operator \(\square\). Then we will show that, for any \(g \in A^{p,q}(M)\),

\[
\|\overline{\partial}^*G\partial g\|^2 \leq \|g\|^2, \tag{1.3}
\]

where \(\| \cdot \|\) denotes the \(L^2\)-norm induced by the Kähler metric \(\omega\). \((1.3)\) implies that \(T\) is an operator of norm \(\|T\| \leq 1\) on the Hilbert space of \(L^2\)-forms. Let \(\varphi\) be a Beltrami differential with \(L^\infty\)-norm \(\|\varphi\|_\infty < 1\), then as a corollary we see that the operator \(I + T\varphi\) is invertible, where \(\varphi\) is considered as a contraction operator on the Hilbert space of \(L^2\)-forms.

We will show in Section 4 that

\[
\Omega = (I + T\varphi)^{-1}\Omega_0
\]

is a solution to the equation \((1.2)\). Then, we obtain

**Theorem 1.1.** Given an integrable Beltrami differential \(\varphi\) such that the \(L^\infty\)-norm \(\|\varphi\|_\infty < 1\), and any holomorphic \((n, 0)\)-form \(\Omega_0\) on \(M\), then

\[
\Omega(\varphi) = \rho_\varphi((I + T\varphi)^{-1}\Omega_0) \tag{1.4}
\]

is a holomorphic \((n, 0)\)-form on \(M_\varphi\). In particular, \(\Omega(0) = \Omega_0\).

Therefore, \((1.4)\) is the explicit closed extension formula which we are looking for. Note that the above construction is global in the sense that it does not depend on the local deformation theory of Kodaira-Spencer and Kuranishi. On
the other hand, Theorem 1.1 can be applied to the integrable Beltrami differential \( \varphi(t) \) constructed from the Kodaira- Spencer-Kuranishi deformation theory with \( |t| \leq \varepsilon \) small, such that \( \| \varphi(t) \|_\infty < 1 \).

**Corollary 1.2.** Let \( \pi : \mathcal{X} \to \Delta_\varepsilon \subset \mathbb{C}^m \) be the Kuranishi family of compact Kähler manifolds with \( M_t = \pi^{-1}(t) = M_{\varphi(t)} \), where \( t \in \Delta_\varepsilon \). Given any holomorphic \((n,0)\)-form \( \Omega_0 \in A^{n,0}(M_0) \), we have that

\[
\Omega(t) = \rho_t((I + T\varphi(t))^{-1}\Omega_0)
\]

is a holomorphic \((n,0)\)-form on \( M_t \), where we denote by \( \rho_t = \rho_{\varphi(t)} \). In particular, \( \Omega(0) = \Omega_0 \).

**Remark 1.3.** By the construction of the integrable Beltrami differential \( \varphi(t) \) in Kodaira-Spencer-Kuranishi deformation theory \([18]\), \( \varphi(t) = \sum_{\mu \geq 1} \varphi_{\mu}(t) \), with \( \varphi_1(t) = \sum_{i=1}^m \eta_i t_i \), where \( \{\eta_i\} \) is a basis for the harmonic space \( H^1(M, T^{1,0}M) \). Now the holomorphic \((n,0)\) form on \( M_t \) is given by \( \Omega(t) \), whose first two terms are

\[
\Omega(t) = \Omega_0 + \sum_{i=1}^m (\eta_i \cdot \Omega_0 - T(\eta_i \cdot \Omega_0)) t_i + O(t^2).
\]  

(1.5)

From (1.5), one can easily derive the curvature formula of the \( L^2 \) metric on the corresponding Hodge bundle.

**1.2. Closed formula for canonical section of Hodge bundle.** We then consider certain global canonical sections of holomorphic forms on the moduli spaces of marked and polarized projective manifolds of complex dimension \( n \). Fixing a base point \( M \) in the moduli space \( \mathcal{M} \) of marked and polarized projective manifolds of complex dimension \( n \), let \( \omega_0 \) be a Kähler form on \( M \). Here we only consider the connected component of the moduli space containing \( M \). Note that the de Rham cohomology group gives us a trivial bundle \( H^n(M) \) on \( \mathcal{M} \) induced by the markings.

We will show that if the complex structure of any other point \( M_1 \) in \( \mathcal{M} \) can be tamed by the same Kähler form \( \omega_0 \) considered as a symplectic form, then there is a natural construction of Beltrami differential \( \varphi \) on \( M \) with its \( L_\infty \)-norm \( \| \varphi \|_\infty < 1 \), such that \( M_1 = M_{\varphi} \). By using a theorem of Moser \([19]\) and the above operator \( T \) from Hodge theory on \( M \), we can deduce the following result which gives a closed formula of a canonical section of the Hodge bundle of holomorphic \( n \)-forms on \( \mathcal{M} \).

**Theorem 1.4.** Given a point \( (M, \omega_0) \) in the marked and polarized moduli space \( \mathcal{M} \) and a holomorphic \( n \)-form \( s_0 \) on \( M \), there is a canonical section \( s \) of the
Hodge bundle $\mathcal{H}^{n,0}$, such that for any point $M_1$ in $M$, the de Rham cohomology class of $s$ in $H^n(M_1)$ is represented by
\[ s = \rho \varphi ((I + T\varphi)^{-1}s_0). \]
where $\varphi$ is the Beltrami differential associated to $M_1$, and $T = \overline{\partial} G\partial$ is the operator of the Hodge theory on $M$ with Kähler metric $\omega_0$.

1.3. Extension formula for pluricanonical form. Next we generalize the previous method to construct the extensions of pluricanonical forms. Let $(M,\omega)$ be a compact Kähler manifold of complex dimension $\dim_C M = n$ with Kähler form $\omega$, and $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential. Let $m \geq 2$ be a fixed integer. We consider a pluricanonical form $\sigma_0$ which is a holomorphic section of $K^{\otimes m}_M$ over $M$, where $K_M$ denotes the canonical line bundle of $M$. An important question is how to construct the pluricanonical forms $\sigma(\varphi)$ on $M_\varphi$, such that $\sigma(0) = \sigma_0$. In fact, for projective manifolds, the existence of extension was proved by Y.-T. Siu. In general there is a famous conjecture due to Siu [24], about the invariance of plurigenera for compact Kähler manifolds.

In our approach, Siu’s conjecture is reduced to solving the extension equation (6.5). By using Hodge theory, we provide a closed explicit formula for the solution of this extension equation under certain conditions.

More precisely, let $(\mathcal{L}, h)$ be an Hermitian holomorphic line bundle over $(M,\omega)$. Let $\nabla = \nabla' + \overline{\partial}$ be the Chern connection of $(\mathcal{L}, h)$ with curvature $\Theta$. We introduce the operator
\[ T\nabla' = \overline{\partial} G\nabla' \]
where $G$ is the Green operator associated to the Laplacian $\square = \overline{\partial}\partial + \overline{\partial}\partial$.

In particular, we consider the holomorphic line bundle $\mathcal{L}_M = K_M^{\otimes (m-1)}$ over $(M,\omega)$ with the induced Hermitian metric $h_\omega = \det(g)^{-(m-1)}$, where $g$ denotes the Kähler metric matrix associated to the Kähler form $\omega$.

We establish the following result in Section 6.

**Theorem 1.5.** Suppose $(\mathcal{L}_M, h_\omega)$ is a positive line bundle over a compact Kähler manifold $(M,\omega)$ with curvature $\sqrt{-1}\Theta = \rho \omega$ for a constant $\rho > 0$, let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential satisfying two conditions $\text{div} \varphi = 0$ and $L_\infty$-norm $\|\varphi\|_\infty < 1$. Then, for any holomorphic pluricanonical form $\sigma_0 \in A^{n,0}(M, \mathcal{L}_M)$,
\[ \sigma(\varphi) = \rho \varphi ((I + T\nabla')^{-1}\sigma_0) \]
is a holomorphic pluricanonical form in $A^{n,0}(M_\varphi, \mathcal{L}_{M_\varphi})$. 

4
Note that Theorem 1.5 is global in the sense that it does not depend on the local deformation family. Theorem 1.5 can be used to construct the closed extension formula for pluricanonical forms of Kähler-Einstein manifold of general type, see Definition 6.12.

Let \( \pi : X \to B \subset \mathbb{C} \) be a holomorphic family of compact Kähler-Einstein manifolds of general type. For \( t \in B \), we assume \( M_t = \pi^{-1}(t) = M_{\varphi(t)} \), where \( \varphi(t) \in A^{0,1}(M_0, T^{1,0}M_0) \) denotes an integrable Beltrami differential satisfying the Kuranishi gauge \( \overline{\partial} \varphi(t) = 0 \).

As an application of Theorem 1.5, we obtain

**Corollary 1.6.** Given any holomorphic pluricanonical form \( \sigma_0 \in A^{n,0}(M_0, \mathcal{L}_{M_0}) \), then

\[
\sigma(t) = \rho_t((I + T^\varphi(t))^{-1}\sigma_0)
\]

is a holomorphic pluricanonical form in \( A^{n,0}(M_t, \mathcal{L}_{M_t}) \) with \( \sigma(0) = \sigma_0 \), where \( \rho_t = \rho_{\varphi(t)} \).

Corollary 1.6 implies the invariance of plurigenera for Kähler-Einstein manifolds of general type, which has been obtained in [25]. Formula (1.6) provides a simple closed explicit formula for the extension of pluricanonical form.

### 1.4. Solving the Beltrami equation

Beltrami equation is very important in the development of complex analysis and moduli theory of Riemann surfaces. It also has many important applications in other subjects. See, for examples [1], [4] and [8].

Given a measurable function \( \mu_0 \) on the unit disc \( D \subset \mathbb{C} \), suppose \( \sup |\mu_0| < 1 \), let \( \mu = \mu_0 \frac{\partial}{\partial z} \otimes d\overline{z} \) be a Beltrami differential on \( D \) with coordinate \( z \). Recall that solving the Beltrami equation is to find a function \( f \) on the unit disc \( D \), such that

\[
\overline{\partial} f = \mu \partial f.
\]

Our observation is that the Beltrami equation can be solved by using the \( L^2 \)-Hodge theory. We will show in Section 7 that the \( L^2 \)-Hodge theory holds on disk \( D \) with the Poincaré metric \( \omega_p \). So we also have the operator \( T = \overline{\partial} G \partial \) with norm \( ||T|| \leq 1 \).

Note that the \( L_\infty \)-norm of \( \mu \) is independent of Hermitian metric on \( D \) and is equal to \( \sup |\mu_0| \), i.e. \( \|\mu\|_\infty < 1 \). Similarly, we show that for a holomorphic one form \( h_0 \) on \( D \), the equation

\[
\overline{\partial} h = -\partial \mu h
\]

has a solution

\[
h = (I + T\mu)^{-1}h_0.
\]
As a corollary we can directly get the a solution of the Beltrami equation for any measurable $\mu_0$. In particular we have,

**Theorem 1.7.** Assume that $||\mu||_\infty = \sup |\mu_0| < 1$, if $\mu_0$ is of regularity $C^k$, then the Beltrami equation

$$ \bar{\partial} f = \mu \partial f $$

has a solution $w(z)$ of regularity $C^{k+1}$.

The rest of this paper is organized as follows. In Section 2, we will first review the basics of operators on differential forms, Beltrami differentials and extension equations which are needed for our discussions. Then in Section 3, we briefly review the Hodge theory on compact Kähler manifold, introduce the operator $T$ and discuss the quasi-isometry formula. In Section 4 we write down and prove a closed formula for extension of holomorphic canonical form on compact Kähler manifold in new complex structures. In Section 5 we present a closed formula for a global canonical section of the Hodge bundle of holomorphic form on moduli space.

In Section 6, we generalize the methods in Sections 3, 4 to construct the closed explicit formula for extension of holomorphic pluricanonical forms under certain conditions.

In Section 7 we review and prove a few basic facts of $L^2$-Hodge decomposition theory. In particular we will present a general result from [9] that $L^2$-Hodge theory holds on the universal cover of a Kähler hyperbolic manifold which is possibly known and of independent interest for other applications, although in this paper we only need the case when the universal cover is the unit ball with standard Poincaré hyperbolic metric. In this section we will also discuss briefly the relationship between the $L^2$-Hodge theory and the $L^2$-estimate of Hörmander. We show that $L^2$-Hodge theory is more general and implies the $L^2$-estimate. In Section 8 we apply the results in Section 7 to solve the Beltrami equations. In Section 9 we discuss various applications and extensions of our method.

**Acknowledgements.** This paper grew out of several lectures the first author presented in the complex geometry seminar we organized in School of Mathematical Science, Capital Normal University during the academic year 2016-2017. The first author would like to thank all the participants of the seminar for their interest. The research of the first author is supported by NSFC (Grant No. 11531012) and NSF. The second author would like to thank CSC to support his visiting in UCLA.
2. Extension equations

In this section, we first review some basic results about the operators on differential forms following [15]. Then we introduce the definitions of Beltrami differentials and integrability condition. We derive the extension equation for constructing the holomorphic canonical form on the new complex manifold $M_\varphi$, which is determined by an integrable Beltrami differential $\varphi$.

2.1. Generalized Cartan formulas. Let $M$ be a complex manifold of dimension $n$. Let $\varphi \in A^{0,k}(M,T^{1,0}M)$ be a $T^{1,0}M$-value $(0,k)$-form. We introduce the contraction operator

\[ i_\varphi : A^{p,q}(M) \to A^{p-1,q+k}(M) \]

as in [15]. If we write $\varphi = \eta \otimes Y$ with $\eta \in A^{0,k}(M)$ and $Y \in C^\infty(T^{1,0}M)$, then for $\sigma \in A^{p,q}(M)$,

\[ i_\varphi(\sigma) = \eta \wedge i_Y \sigma. \]

Sometimes, we also use the notations $\varphi \downarrow \eta = \varphi \eta$ to denote the contraction alternatively. By definition, we have

\[ i_\varphi i_\varphi' = (-1)^{(k+1)(k'+1)} i_{\varphi'} i_\varphi \]

if $\varphi \in A^{0,k}(M)$ and $\varphi' \in A^{0,k'}(M)$. The Lie derivation of $\varphi$ is defined by

\[ L_\varphi = (-1)^k d \circ i_\varphi + i_\varphi \circ d \]

which can be decomposed into the sum of two parts

\[ L^{0,1}_\varphi = (-1)^k \partial \circ i_\varphi + i_\varphi \circ \partial, \quad L^{0,0}_\varphi = (-1)^k \bar{\partial} \circ i_\varphi + i_\varphi \circ \bar{\partial}. \]

The Lie bracket of $\varphi$ and $\varphi'$ is defined by

\[ [\varphi, \varphi'] = \sum_{i,j=1}^n \left( \varphi^i \wedge \partial_i \varphi'^j - (-1)^{kk'} \varphi'^i \wedge \partial_i \varphi^j \right) \otimes \partial_j, \]

if $\varphi = \sum_i \varphi^i \partial_i \in A^{0,k}(M,T^{1,0}M)$ and $\varphi' = \sum_i \varphi'^i \partial_i \in A^{0,k'}(M,T^{1,0}M)$.

We have the following generalized Cartan formula [13, 15] which can be proved by direct computations.

Lemma 2.1. For any $\varphi, \varphi' \in A^{0,1}(M,T^{1,0}_M)$, then on $A^{*,*}(M)$,

\[ i_{[\varphi, \varphi']} = L_\varphi \circ i_{\varphi'} - i_{\varphi'} \circ L_\varphi, \]

Let $\sigma \in A^{*,*}(M)$. By applying the formula (2.1) to $\sigma$ and considering the types, we immediately obtain

\[ [\varphi, \varphi] \downarrow \sigma = 2 \varphi \downarrow \partial \varphi \downarrow \sigma - \partial(\varphi \downarrow \varphi \downarrow \sigma) - \varphi \downarrow \varphi \downarrow \partial \sigma. \]
2.2. Beltrami differentials. In this section, $M$ is a complex manifold with \( \dim_M M = n \), and we denote by $X$ the underlying real manifold of $M$ of real dimension $2n$. The associated almost complex structure of the complex manifold $M$ gives a direct sum decomposition of the complexified tangent bundle,

\[
T_C X = T^{1,0} M \oplus T^{0,1} M.
\]

Let $J$ be another almost complex structure on $X$. Then, $J$ gives another direct sum decomposition,

\[
T_C X = T^{1,0} M_J \oplus T^{0,1} M_J.
\]

Denote by \( \iota_1 : T_C X \to T^{1,0} M, \iota_2 : T_C X \to T^{0,1} M \), the two projection maps.

**Definition 2.2** (cf. Definition 4.2 [10]). Let $J$ be an almost complex structure on $X$, we say that $J$ is of finite distance from the given complex structure $M$ on $X$, if the restriction map \( \iota_1 \mid_{T^{1,0} M_J} : T^{1,0} M_J \to T^{1,0} M \) is an isomorphism.

Therefore, if $J$ is of finite distance from $M$, one can define a map \( \varphi : T^{1,0} M \to T^{0,1} M \) by setting

\[
\overline{\varphi}(v) = -\iota_2 \circ (\iota_1 \mid_{T^{1,0} M_J})^{-1}(v).
\]

This map is well-defined since \( \iota_1 \mid_{T^{1,0} M_J} \) is an isomorphism. It is clear that

\[
T^{1,0} M_J = \{ v - \overline{\varphi}(v) \mid v \in T^{1,0} M \}, \quad T^{0,1} M_J = \{ v - \overline{\varphi}(v) \mid v \in T^{0,1} M \},
\]

and their corresponding dual spaces are

\[
(2.3) \quad \Lambda^{1,0} M_J = \{ w + \varphi(w) \mid w \in \Lambda^{1,0} M \}, \quad \Lambda^{0,1} M_J = \{ w + \overline{\varphi}(w) \mid w \in \Lambda^{0,1} M \}.
\]

In this way, $\varphi$ determines a $T^{1,0} M$-valued $(0,1)$-form which is also denoted by $\varphi \in A^{0,1}(M, T^{1,0} M)$ for convenience. By the condition

\[
T^{1,0} M \oplus T^{0,1} M = T_C X = T^{1,0} M_J \oplus T^{0,1} M_J,
\]

the transformation matrix

\[
\begin{pmatrix}
I_n & -\overline{\varphi} \\
-\varphi & I_n
\end{pmatrix}
\]
from a basis of $T^{1,0}M \oplus T^{0,1}M$ to a basis of $T^{1,0}M_J \oplus T^{0,1}M_J$ must be nondegenerate. Therefore $\det(I_n - \varphi \overline{\varphi}) \neq 0$. In fact, we have

**Proposition 2.3** (cf. Proposition 4.3 [10]). There is a bijective correspondence between the set of almost complex structures of finite distance from $M$ and the set of all $\varphi \in A^{0,1}(M,T^{1,0}M)$ such that, at each point $p \in X$, the map $\varphi \overline{\varphi}$ does not have eigenvalue 1.

**Definition 2.4.** If $\varphi \in A^{0,1}(M,T^{1,0}M)$ satisfies the condition in Proposition 2.3, we say that $\varphi$ is a Beltrami differential. If $\varphi$ satisfies the integrability condition

$$\overline{\partial} \varphi = \frac{1}{2}[\varphi, \varphi],$$

we call $\varphi$ an integrable Beltrami differential.

So a Beltrami differential $\varphi$ determines an almost complex structure of finite distance from $M$. We denote the corresponding almost complex structure (i.e. almost complex manifold) by $M_\varphi$. An integrable Beltrami differential $\varphi$ gives a new complex structure on $X$ by the Newlander-Nirenberg theorem [20], the corresponding complex manifold is denoted by $M_{\varphi}$.

**2.3. Extension equations.** Given a Beltrami differential $\varphi$, for any $x \in X$, we can pick a local holomorphic coordinate $(U, z^1, ..., z^n)$ near $x$. Then by (2.3),

$$\Lambda^1_x(M_\varphi) = \text{Span}_C \{dz^1 + \varphi dz^1, ..., dz^n + \varphi dz^n\},$$

and for any $1 \leq p \leq n$,

$$\Lambda^p_x(M_\varphi) = \text{Span}_C \{(dz^{i_1} + \varphi dz^{i_1}) \wedge \cdots \wedge (dz^{i_p} + \varphi dz^{i_p}) | 1 \leq i_1 < \cdots < i_p \leq n\}.$$

Considering the operator $e^{i\varphi}$ defined by formula (1.1), through a straightforward computation we obtain

$$e^{i\varphi}(dz^{i_1} \wedge \cdots \wedge dz^{i_p}) = (dz^{i_1} + \varphi dz^{i_1}) \wedge \cdots \wedge (dz^{i_p} + \varphi dz^{i_p}) \in \Lambda^p_x(M_\varphi).$$

Therefore, $e^{i\varphi}$ is a map from $A^{p,0}(M)$ to $A^{p,0}(M_\varphi)$. Moreover, it is easy to check that

$$e^{i\varphi} : A^{p,0}(M) \to A^{p,0}(M_\varphi)$$

is a bijection.

By the generalized Cartan formula (2.1), it follows that

**Proposition 2.5.** [5, 15] Let $\varphi \in A^{0,1}(M,T^{1,0}M)$, then on $A^{*,*}(M)$, we have

$$(2.4) \quad e^{-i\varphi} \circ d \circ e^{i\varphi} = d - \mathcal{L}_{\varphi} - i_{\frac{1}{2}[\varphi, \overline{\varphi}]}.$$
Proof. Let $\eta \in A^{*,*}(M)$, formula (2.1) implies
\begin{equation}
(2.5) \quad di^2 \eta = 2i_\varphi di_\varphi \eta - i^2_\varphi d\eta - i_{[\varphi,\varphi]} \eta.
\end{equation}
Substituting $\eta$ with $i_\varphi \eta$ in (2.5), we obtain
\begin{equation}
d(i^3_\varphi \eta) = 2i_\varphi d(i^2_\varphi \eta) - i^2_\varphi d\eta - i_{[\varphi,\varphi]} i_\varphi \eta = 3i^2_\varphi d(i_\varphi \eta) - 3i_\varphi i_{[\varphi,\varphi]} \eta.
\end{equation}
where we have used $i_\varphi i_{[\varphi,\varphi]} = i_{[\varphi,\varphi]} i_\varphi$. Then by induction, we immediately have
\begin{equation}
d(i^k_\varphi \eta) = ki^{k-1}_\varphi d(i_\varphi \eta) - (k - 1)i^k_\varphi d\eta - \frac{k(k-1)}{2} i^{k-2}_\varphi i_{[\varphi,\varphi]} \eta
\end{equation}
for $k \geq 2$. Through a straightforward computation, it is easy to show that
\begin{equation}
d(e^{i_\varphi \eta}) = e^{i_\varphi} \left(d\eta - \mathcal{L}_\varphi \eta - i_{\frac{1}{2}[\varphi,\varphi]} \eta\right).
\end{equation}
The proof of (2.4) is completed. \qed

Corollary 2.6. If $\varphi \in A^{0,1}(M, T^{1,0}M)$ is integrable, then on $A^{*,*}(M)$, we have
\begin{equation}
(2.6) \quad e^{-i_\varphi} \circ d \circ e^{i_\varphi} = d - \mathcal{L}^{1,0}_\varphi = d + \partial i_\varphi - i_\varphi \partial.
\end{equation}
Proof. By a straightforward computation, we have $\overline{\partial} i_\varphi - i_\varphi \overline{\partial} = i_{\frac{1}{2}[\varphi,\varphi]}$. Then Corollary 2.6 follows directly from the integrability of $\varphi$, i.e. $\overline{\partial} \varphi = \frac{1}{2}[\varphi, \varphi]$. \qed

In particular, we have

Corollary 2.7. Given an integrable Beltrami differential $\varphi \in A^{0,1}(M, T^{1,0}M)$, for any $(n,0)$-form $\Omega$ on $M$, the corresponding $(n,0)$-form $\rho_\varphi(\Omega) = e^{i_\varphi}(\Omega)$ on $M_\varphi$ is holomorphic, if and only if
\begin{equation}
(2.7) \quad \overline{\partial} \Omega = -\partial(\varphi \Omega).
\end{equation}
Proof. Let $d = \overline{\partial}_\varphi + \partial_\varphi$ be the corresponding $\overline{\partial}$ and $\partial$ operators on $M_\varphi$. Since $\Omega \in A^{n,0}(M)$, we have
\begin{equation}
(d + \partial i_\varphi - i_\varphi \partial)\Omega = \overline{\partial} \Omega + \partial(\varphi \Omega).
\end{equation}
Since $e^{i_\varphi}(\Omega) \in A^{n,0}(M_\varphi)$, we have $d \circ e^{i_\varphi} = \overline{\partial}_\varphi(e^{i_\varphi}(\Omega))$. Therefore, $e^{i_\varphi}(\Omega)$ is holomorphic on $M_\varphi$ if and only if
\begin{equation}
\overline{\partial} \Omega + \partial(\varphi \Omega) = 0.
\end{equation}
\qed

Remark 2.8. Equation (2.7) is called the extension equation whose solution can be used to construct the extensions of holomorphic $(n,0)$-forms from $M$ to $M_\varphi$, which will be discussed in detail in Section 4.
3. Hodge theory and the operator $T$ on compact Kähler manifolds

Let $(M, \omega)$ be an $n$-dimensional compact Kähler manifold with Kähler metric $\omega$, and $\| \cdot \|$ be the $L^2$-norm on smooth differential forms $A^{p,q}(M)$ induced by the metric $\omega$. Denote by $L^2_{p,q}(M)$ the $L^2$-completion of $A^{p,q}(M)$. On $A^{p,q}(M)$, we have the equality of the Laplacians

$$\Box = \Box = \frac{1}{2} \Delta_d.$$ 

Let $H$ denotes the orthogonal projection from $A^{p,q}(M)$ to the harmonic space $\mathbb{H}^{p,q}(M) = \ker \Box$, We have

**Proposition 3.1** (cf. Pages 157-158 [18]). There exists a bounded operator $G$ on $A^{p,q}(M)$, called Green operator such that

$$\Box G = G \Box = \text{Id} - H, \quad \overline{\partial} G = G \overline{\partial}, \quad \overline{\partial}^* G = G \overline{\partial}^*,$$

$$HG = GH = 0.$$ 

Moreover, $\overline{\partial} H = H \overline{\partial} = 0, \quad \overline{\partial} H = H \overline{\partial} = 0$. These formulas also hold if we replace the operator $\partial$ with $\overline{\partial}$.

We have the following quasi-isometry formula from [15].

**Theorem 3.2** (cf. Theorem 1.1(3) [15]). For any $g \in A^{p,q}(M)$, we have

$$\| \overline{\partial}^* G \partial g \| \leq \| g \|^2. \quad (3.1)$$

For reader’s convenience, we provide the proof of Theorem 3.2 here.

**Proof.** The proof follows from the following straightforward computation based on the Hodge theory and the formulas in Proposition 3.1. More precisely, we have

$$\| \overline{\partial} G \partial g \|^2 = \langle \overline{\partial} G \partial g, \overline{\partial} G \partial g \rangle = \langle \Box G \partial g, G \partial g \rangle$$

$$= \langle \Box G \partial g, G \partial g \rangle - \langle \overline{\partial} \overline{\partial} G \partial g, G \partial g \rangle$$

$$= \langle \partial g, G \partial g \rangle - \langle \overline{\partial} G \partial g, \overline{\partial} G \partial g \rangle$$

$$= \langle g, G \partial g \rangle - \langle g, \partial \partial^* G g \rangle - \| \overline{\partial} G \partial g \|$$

$$= \langle g, g - H g \rangle - \langle \partial \partial^* G g \rangle - \| \overline{\partial} G \partial g \|$$

$$= \| g \|^2 - \| H g \|^2 - \langle \partial \partial^* G g \rangle - \| \overline{\partial} G \partial g \|$$

$$\leq \| g \|^2.$$

The last inequality holds since the Green operator $G$ is a non-negative operator. \qed
Now we consider the operator

\[ T = \bar{\partial}^* G \partial. \]

The inequality (3.1) implies that \( T \) is an operator of norm less than or equal to 1 in the Hilbert space of \( L^2 \) forms. So we have

**Corollary 3.3.** Given a compact Kähler manifold \((M, \omega)\), let \( \varphi \in A^{0,1}(M, T^{1,0}M) \) be a Beltrami differential acting on the Hilbert space of \( L^2 \) forms by contraction such that its \( L^\infty \)-norm \( \| \varphi \|_\infty < 1 \), then the operator \( I + T \varphi \) is invertible on the Hilbert space of \( L^2 \) forms.

4. Extension of holomorphic canonical form

Let \((M, \omega)\) be a compact Kähler manifold with \( \dim_{\mathbb{C}} M = n \). From the discussions in Section 2.3, we know that, given an integrable Beltrami differential \( \varphi \) on \( M \), in order to find an \((n, 0)\)-form \( \Omega \) on \( M \) such that the corresponding \((n, 0)\)-form \( \rho_{\varphi}(\Omega) = e^{i\varphi} \Omega \) is holomorphic on \( M_{\varphi} \), we only need to find an \((n, 0)\)-form \( \Omega \) on \( M \) such that \( \Omega \) satisfies the extension equation

\[ \bar{\partial} \Omega = -\partial (\varphi \Omega). \]

In this section, we show that the equation (4.1) can be solved by using the Hodge theory on \((M, \omega)\) reviewed in Section 3.

**Proposition 4.1.** Let \( \varphi \) be an integrable Beltrami differential of \( M \) with \( L^\infty \)-norm \( \| \varphi \|_\infty < 1 \). Given a holomorphic \((n, 0)\)-form \( \Omega_0 \) on \( M \), if \( \Omega \) is a solution of the equation

\[ \Omega = \Omega_0 - \bar{\partial}^* G \partial (\varphi \Omega) = \Omega_0 - T \varphi \Omega, \]

then \( \Omega \) is the solution of the equation (4.1).

**Proof.** We first assume that \( \Omega \) satisfies the equation \( \Omega = \Omega_0 - \bar{\partial} G \partial (\varphi \Omega) \). We need to show that

\[ \bar{\partial} \Omega = -\partial (\varphi \Omega). \]

In fact, from the formulae in Proposition 3.1, it follows that

\[
\begin{align*}
\bar{\partial} \Omega &= -\bar{\partial}^* G \partial (\varphi \Omega) \\
&= (\bar{\partial}^* \partial - \Box_{\eta}) G \partial (\varphi \Omega) \\
&= (\bar{\partial}^* G - I + H) \partial (\varphi \Omega) \\
&= -\partial (\varphi \Omega) + \bar{\partial}^* \bar{\partial} G \partial (\varphi \Omega).
\end{align*}
\]
Let
\[ \Phi = \overline{\partial} \Omega + \partial (\varphi \Omega) . \]

Then we have
\[
\Phi = \overline{\partial} \Omega + \partial (\varphi \Omega) \\
= \overline{\partial}^* G \partial (\varphi \Omega) \\
= -\overline{\partial}^* G \partial (\varphi \Omega) \\
= -\overline{\partial}^* G \partial ((\overline{\partial} \varphi) \Omega + \varphi \overline{\partial} \Omega) \\
= -\overline{\partial}^* G \partial \left( \frac{1}{2} \langle \varphi, \varphi \rangle \Omega + \varphi \partial (\varphi \Omega) \right) \\
= -\overline{\partial}^* G \partial (\varphi \Omega) ,
\]

where in the last equality, we have used (2.2), \( \partial \Omega = 0 \) and \( \partial^2 = 0 \).

By Theorem 3.2 and the condition \( ||\varphi||_\infty < 1 \), we have
\[
||\Phi||^2 \leq ||\varphi \Phi||^2 \leq ||\varphi||_\infty ||\Phi||^2 < ||\Phi||^2 .
\]
Then we get the contradiction \( ||\Phi||^2 < ||\Phi||^2 \) unless \( \Phi = 0 \). Hence,
\[ \overline{\partial} \Omega = -\partial (\varphi \Omega) . \]

Our method in the above proof of Proposition 4.1 is also used in [14], see Remark 4.6 of [14].

Conversely, we have

**Proposition 4.2.** If the \((n,0)\)-form \( \Omega \) satisfies the equation (4.1), then there exists a unique holomorphic \((n,0)\)-form \( \Omega_0 \), such that \( \Omega \) satisfies the equation (4.2).

**Proof.** Applying the operator \( \overline{\partial}^* G \) to (4.1), we obtain
\[ \overline{\partial}^* G \partial \Omega = -\overline{\partial}^* G \partial (\varphi \Omega) . \]

From the formulas in Proposition 3.1, it follows that
\[ \overline{\partial}^* G \partial \Omega = \overline{\partial}^* G \partial \Omega = \square_{\overline{\partial}} G \Omega = \Omega - H \Omega . \]

Let \( \Omega_0 = H \Omega \), which is a harmonic \((n,0)\)-form on \( M \), it immediately implies that
\[ \Omega = \Omega_0 - \overline{\partial}^* G \partial (\varphi \Omega) . \]
Furthermore, it is easy to show that the equation (4.2) has a unique solution. Indeed, if we assume that the equation (4.2) has two different solutions \( \Omega \) and \( \Omega' \), i.e. \( \Omega - \Omega' \neq 0 \). Then

\[
\Omega - \Omega' = -T\varphi(\Omega - \Omega').
\]

By Theorem 3.2, we have

\[
\|\Omega - \Omega'\| = \|T\varphi(\Omega - \Omega')\| \leq \|\varphi(\Omega - \Omega')\| \leq \|\varphi\|_{\infty}\|\Omega - \Omega\| < \|\Omega - \Omega'\|.
\]

which contradicts to \( \Omega - \Omega' \neq 0 \).

By Corollary 3.3, this unique solution of the equation (4.2) is given by

\[
\Omega = (I + T\varphi)^{-1}\Omega_0,
\]

which is a smooth \((n,0)\)-form since \( \Omega_0 \) is holomorphic.

In conclusion, we have

**Theorem 4.3.** Given any integrable Beltrami differential \( \varphi \) with \( \|\varphi\|_{\infty} < 1 \), and any holomorphic \((n,0)\)-form \( \Omega_0 \) on \( M \), we have that

\[
\Omega(\varphi) = \rho_\varphi((I + T\varphi)^{-1}\Omega_0)
\]

is a holomorphic \((n,0)\)-form on \( M_\varphi \) with \( \Omega(0) = \Omega_0 \).

Applying Theorem 4.3 to the integrable Beltrami differential \( \varphi(t) \) from the local Kodaira-Spencer-Kuranishi deformation theory, one can choose \( t \) small enough such that \( \|\varphi(t)\|_{\infty} < 1 \), we immediately obtain

**Corollary 4.4.** For any holomorphic \((n,0)\)-form \( \Omega_0 \in A^{n,0}(M) \), and the Beltrami differential \( \varphi = \varphi(t) \) with \( |t| < \varepsilon \) small, there is a holomorphic \((n,0)\)-form \( \Omega(t) \) on \( M_t \),

\[
\Omega(t) = \rho_t((I + T\varphi)^{-1}\Omega_0),
\]

where \( \rho_t = \rho_{\varphi(t)} \), with \( \Omega(0) = \Omega_0 \).

5. Closed formulas on marked and polarized moduli spaces

5.1. **The moduli space \( M \) of marked and polarized manifolds.** We refer the reader to [22] for the basic facts on moduli spaces in this section. Let \((M, L)\) be a polarized manifold. The moduli space \( M_0 \) of polarized manifolds is the complex analytic space parameterizing the isomorphism class of polarized manifolds with the isomorphism defined by

\[
(M, L) \sim (M', L') \ni \text{biholomorphic map } f : M \to M' \text{ s.t. } f^*L' = L.
\]
We fix a lattice $\Lambda$ with a pairing $Q_0$, where $\Lambda$ is isomorphic to $H^n(M_0, \mathbb{Z})/\text{Tor}$ for some $M_0$ in $\mathcal{M}_0$ and $Q_0$ is defined by the cup-product. For a polarized manifold $(M, L) \in \mathcal{M}_0$, we define a marking $\gamma$ as an isometry of the lattices
\[ \gamma : (\Lambda, Q_0) \rightarrow (H^n(M, \mathbb{Z})/\text{Tor}, Q) \]
where $Q$ is the Poincaré pairing.

Recall that a polarized and marked projective manifold is a triple $(M, L, \gamma)$, where $M$ is a projective manifold, $L$ is a polarization on $M$, and $\gamma$ is a marking
\[ \gamma : (\Lambda, Q_0) \rightarrow (H^n(M, \mathbb{Z})/\text{Tor}, Q). \]
Two triples $(M, L, \gamma)$ and $(M', L', \gamma')$ are called equivalent if there exists a biholomorphic map $f : M \rightarrow M'$ with
\[ f^*L' = L \text{ and } f^*\gamma' = \gamma, \]
where $f^*\gamma'$ is given by $\gamma' : (\Lambda, Q_0) \rightarrow (H^n(M', \mathbb{Z})/\text{Tor}, Q)$ composed with
\[ f^* : (H^n(M', \mathbb{Z})/\text{Tor}, Q) \rightarrow (H^n(M, \mathbb{Z})/\text{Tor}, Q). \]
We denote by $[M, L, \gamma]$ the isomorphism class of polarized and marked projective manifolds of $(M, L, \gamma)$.

The moduli space $\mathcal{M}$ of marked and polarized manifolds is the complex analytic space parameterizing the isomorphism class of marked and polarized manifolds. Let $X$ denote the underlying real manifold for $M$. From the definition we know that the first Chern class $c_1(L) \in H^2(X)$ is fixed, which we can take to be the Kähler class on $M$. We will only consider the connected component of the marked moduli space containing $M$ which we still denote by $\mathcal{M}$.

5.2. Closed explicit formula for the global section of holomorphic forms. Now we take two distinct points in $\mathcal{M}$ whose corresponding fibers are $M_0$ and $M_1$. Let $X$ be the background smooth manifold. Denote the corresponding Kähler form on $M_0$ and $M_1$ by $\omega_0$ and $\omega_1$ respectively, which we consider as symplectic forms on $X$. Since
\[ c_1(L) = [\omega_0] = [\omega_1] \in H^2(X), \]
by Theorem 2 of Moser in [19], we know that there exists a continuous family of diffeomorphisms $f_t$ of $X$ with $t \in [0, 1]$ and $f_0 = \text{id}$, such that $\omega_t = f_t^*\omega_0$ and $f_t^*\omega_0 = \omega_1$. Therefore by pulling back the complex structure and $\omega_1$ using $f_t^{-1}$, we may assume that, considered as a symplectic form, $M_1$ also has Kähler form $\omega_0$. 

15
Theorem 5.1. Given a point \((M, \omega_0)\) in the marked and polarized moduli space \(M\) and a holomorphic \(n\)-form \(s_0\) on \(M\), there is a canonical section \(s\) of the Hodge bundle \(H^{n,0}\), such that for any point \(M_1\) in \(M\), the de Rham cohomology class of \(s\) in \(H^*(M_1)\) is represented by

\[
s = \rho_\varphi((I + T\varphi)^{-1}s_0).
\]

where \(\varphi\) is the Beltrami differential associated to \(M_1\), and \(T = \overline{\partial}^* G \partial\) is the operator of the Hodge theory on \(M\) with Kähler metric \(\omega_0\).

Proof. First note that on the moduli space with markings, the de Rham cohomology groups \(H^n(M_1)\) is canonically identified to \(H^n(M)\) which is independent of the point \(M_1\) in \(M\). Second, since we will construct the de Rham cohomology classes in \(H^n(M)\) of the holomorphic forms, and the de Rham cohomology class is independent of continuous diffeomorphisms, therefore by the above discussion using Moser’s theorem in [19], we may assume that \(M\) and \(M_1\) have the same symplectic form \(\omega_0\). By the following Lemma 5.2, we know that associated to the complex structure on \(M_1\), there exists the Beltrami differential \(\varphi \in A^{0,1}(M, T^{1,0}M)\) with \(||\varphi||_\infty < 1\), where the norm is taken with respect to the metric \(\omega_0\) on \(M\).

From Corollary 4.3, since \(||\varphi||_\infty < 1\) on \(M\), we deduce that given the holomorphic \(n\)-form \(s_0\), there exists the holomorphic \(n\)-form

\[
s = \rho_\varphi((I + T\varphi)^{-1}s_0)
\]
on \(M_1\). Note that, considered as de Rham cohomology class, the above formula is independent of the continuous diffeomorphisms used in the Moser theorem to identify the Kähler form \(\omega_1\) on \(M_1\) to the Kähler form \(\omega_0\) on \(M\), both considered as symplectic forms. Therefore, as de Rham cohomology classes, the formula holds on the marked and polarized moduli space \(M\). \(\square\)

Lemma 5.2. Let \(M_0, M_1\) be any two points in the moduli space \(M\). If \(M_0\) and \(M_1\) have the same Kähler form \(\omega\), then the integrable almost complex structure on \(M_1\) is of finite distance from \(M_0\). More precisely, there exists a unique Beltrami differential \(\varphi \in A^{0,1}(M_0, T^{1,0}M_0)\) such that the complex structure on \(M_1 = (M_0)_\varphi\). Moreover the \(L_\infty\)-norm of \(\varphi\), \(||\varphi||_\infty < 1\).

Proof. Since \(T^{1,0}M_0 \oplus T^{0,1}M_0 = T_CX = T^{1,0}M_1 \oplus T^{0,1}M_1\), we let

\[
i_1 : T_CX \to T^{1,0}M_0, \quad i_2 : T_CX \to T^{0,1}M_0,
\]
be the corresponding projection maps.
We let \((M, \omega)\) be any Kähler manifold, let \(g\) be the corresponding Kähler metric, let \(J\) be the complex structure and let \(p \in M\) be a point. Then for any vector \(v \in T^{1,0}_p M\) such that \(v \neq 0\), we know that
\[
0 < \|v\|^2 = g(v, \overline{v}) = \omega(v, J(\overline{v})) = -\sqrt{-1} \omega(v, \overline{v}).
\]
Namely, \(-\sqrt{-1} \omega(v, \overline{v}) > 0\). Since \(\omega\) is skew-symmetric, we have that \(-\sqrt{-1} \omega(u, \overline{u}) < 0\) for any nonzero vector \(u \in T^{0,1}_p M\).

Now we pick any \(v \in T^{1,0}_p M_1\) such that \(v \neq 0\). By the above argument we have
\[
-\sqrt{-1} \omega(v, \overline{v}) > 0.
\]
(5.1)

Let \(v_1 = \upsilon_1(v) \in T^{1,0}_p M_0\) and \(v_2 = \upsilon_2(v) \in T^{0,1}_p M_0\). It follows from type considerations that
\[
\omega(v_1, \overline{v}_2) = 0 = \omega(v_2, \overline{v}_1).
\]
If \(v_1 = 0\) then
\[
-\sqrt{-1} \omega(v, \overline{v}) = -\sqrt{-1} \omega(v_2, \overline{v}_2) < 0
\]
which is a contradiction. Thus we know that \(v_1 \neq 0\) which implies that \(\upsilon_1 \big|_{T^{1,0}_p M_1}\) is a linear isomorphism. Thus, the integrable almost complex structure \(M_1\) is of finite distance from \(M_0\). By the discussion in Section 2.2, the integrable almost complex structure \(M_1\) gives an integrable Beltrami differential \(\varphi \in A^{1,0}(M_0, T^{1,0}_M_0)\) determined by the map
\[
\overline{\varphi}(v) = -\upsilon_2 \circ \left(\upsilon_1 \big|_{T^{1,0}_p M_1}\right)^{-1}(v)
\]
for \(v \in T^{1,0}_p M_0\), such that \(M_1 = (M_0)_\varphi\).

Moreover, since \(v = v_1 + v_2\), the inequality (5.1) implies
\[
\|v_1\|_0 > \|v_2\|_0
\]
(5.2)
where the norm is measured with respect to the Kähler metric on \(M_0\). This inequality holds at each point in \(X\).

Then the inequality (5.2) tells us that the norm of \(\overline{\varphi}\) is pointwise less than 1, which implies that the \(L_\infty\)-norm of \(\varphi\),
\[
\|\varphi\|_\infty = \|\overline{\varphi}\|_\infty < 1.
\]
\(\square\)
Remark 5.3. Lemma 5.2 was first discovered in a joint project of the first author with X. Sun, A. Todorov and S.-T. Yau. The closed formula for holomorphic forms in the above Theorem 5.1 is useful to study the global Torelli problems about injectivity of period maps.

6. Extension of pluricanonical forms

In Section 4, we have presented a closed explicit extension formula for the holomorphic \((n,0)\)-form (i.e. canonical form) over compact Kähler manifolds. In this section, we generalize the method used in Section 4 to construct a closed explicit extension formula for the pluricanonical form. Actually, it is closely related to Siu’s conjecture of the invariance of plurigenera for compact Kähler manifolds [24].

The following Sections 6.1, 6.2, 6.3 can be read by comparing with Sections 2, 3, 4 respectively. In Section 6.4, we present a closed explicit formula for the extension of pluricanonical forms over the Kähler-Einstein manifolds of general type.

6.1. Extension equations. Let \((M, \omega)\) be a compact Kähler manifold of dimension \(n\), let \(\varphi \in A^{0,1}(M, T^{1,0}M)\) be an integrable Beltrami differential. \(\varphi\) determines a new complex manifold denoted by \(M_{\varphi}\). Given an integer \(m \geq 2\), we introduce the line bundles \(L_M = K_M^{\otimes (m-1)}\) and \(L_{M_{\varphi}} = K_{M_{\varphi}}^{\otimes (m-1)}\). For a \(L_M\)-valued \((n,0)\)-form \(\sigma\) on \(M\), we can deform it via the complex structures \(\varphi\). We define the map

\[ \rho_\varphi : A^{n,0}(M, L_M) \to A^{n,0}(M_{\varphi}, L_{M_{\varphi}}) \]

as follows. For any \(x \in M\), one can pick a local holomorphic coordinate system \(\{z^1, ..., z^n\}\) near \(x\), we write the integrable Beltrami differential \(\varphi\) as

\[ \varphi = \varphi^i_d\bar{z}^k \otimes \partial_i. \]

Let \(\sigma = f(z)dz^1 \wedge \cdots \wedge dz^n \otimes \epsilon\), where \(\epsilon = (dz^1 \wedge \cdots \wedge dz^n)^{m-1}\), we define

\[ \rho_\varphi(\sigma) = f(z)((dz^1 + \varphi(dz^1)) \wedge \cdots \wedge (dz^n + \varphi(dz^n)))^{\otimes m}. \]

Let \(\{w^1, ..., w^n\}\) be a local holomorphic coordinate system of \(M_{\varphi}\). Then

\[ dw^i = \partial_j w^j dz^j + \partial_j w^j d\bar{z}^j = \partial_j w^i (dz^j + \varphi(dz^j)) \]

by the definition of the Beltrami differential \(\varphi\). Indeed, if we let \(a = (a_{ij}) = (\partial_j w^i)\) and \(a^{-1} = (a^{ij})\), then

\[ \varphi^i_k = a^{ij} \partial_j w^i, \quad \varphi = \varphi^i_k d\bar{z}^k \otimes \partial_i. \]

18
Hence, the formula (6.1) can be rewritten as

\[ \rho_\varphi(\sigma) = \frac{f(z)}{\det(a)^m}(dw^1 \wedge \cdots \wedge dw^n)^\otimes m. \]

**Lemma 6.1.** Given \( \sigma = f(z)dz^1 \wedge \cdots \wedge dz^n \otimes e \in A^{n,0}(M, \mathcal{L}_M) \), then \( \rho_\varphi(\sigma) \) is holomorphic in \( A^{n,0}(M_\varphi, \mathcal{L}_{M_\varphi}) \) if and only if for \( j = 1, \ldots, n \),

\[ \overline{\partial}_j f = \varphi^*_{j} \partial_i f + mf \partial_i \varphi^*_{j}. \] (6.2)

**Proof.** Since a local smooth function \( h \) is holomorphic on \( M_\varphi \) if and only if

\[ \partial h = \varphi_\lambda(\partial h), \]

i.e. for \( j = 1, \ldots, n \),

\[ \overline{\partial}_j h = \varphi^*_{j} \partial_i h. \]

Therefore, \( \rho_\varphi(\sigma) \) is holomorphic, i.e. \( \frac{f(z)}{\det(a)^m} \) is holomorphic on \( M_\varphi \), if and only if

\[ \overline{\partial}_j \left( \frac{f(z)}{\det(a)^m} \right) = \varphi^*_{j} \partial_i \left( \frac{f(z)}{\det(a)^m} \right) \]

which is equivalent to

\[ (\overline{\partial}_j f - mf a^{ik} \overline{\partial}_j a_{ki}) = \left( \varphi^*_{j} \partial_i f - mf \varphi^*_{j} a^{pl} \partial_i a_{lp} \right) \] (6.3)

by a straightforward computation.

We claim that

\[ a^{ik} \overline{\partial}_j a_{ki} - \varphi^*_{j} a^{pl} \partial_i a_{lp} = \partial_i \varphi^*_{j}. \] (6.4)

In fact, we have

\[
\begin{align*}
\partial_i \varphi^*_{j} &= \partial_i(a^{ik} \overline{\partial}_j w^k) = \partial_i a^{ik} \overline{\partial}_j w^k + a^{ik} \partial_i \overline{\partial}_j w^k \\
&= -a^{ip} \partial_i a_{pl} a^{ik} \overline{\partial}_j w^k + a^{ik} \overline{\partial}_j a_{ki} \\
&= -a^{ip} \partial_i \varphi^*_{j} + a^{ik} \overline{\partial}_j a_{ki}
\end{align*}
\]

which gives (6.4). Therefore, substituting (6.4) into (6.3), we obtain (6.2). \( \square \)

Let \( D = D' + \overline{\partial} \) be the Chern connection of the holomorphic bundle \( T^{1,0}M \) over \( M \). The connection matrix is given by \( \theta = (\partial gg^{-1}) \), where \( g = (g_\overline{\sigma}) \) denotes the Kähler metric matrix associated to the Kähler form \( \omega \). We define
Proposition 6.2. Given if and only if the holomorphic line bundle $L$ the divergence operator $\text{div}$ we have

$$D'\varphi = \partial(\varphi^i dz^j)\partial_i - \varphi_j^i (\partial g^{-1})_i^p \partial_p$$

$$= \partial_k \varphi_j^i dz^k \wedge d\bar{z}^j \partial_i - \varphi_j^i d\bar{z}^j \partial_k g_{k\bar{l}} \bar{g}^0 dz^k \partial_p.$$ Therefore

$$\text{div}\varphi = tr D'(\varphi) = \partial_i \varphi_j^i dz^j + \varphi_j^i \partial_k g_{k\bar{l}} \bar{g}^0 dz^j$$

$$= (\partial_i \varphi_j^i + \varphi_j^i \partial_k g_{k\bar{l}} \bar{g}^0 ) dz^j$$

$$= (\partial_i \varphi_j^i + \varphi_j^i \partial_i \log det(g)) dz^j$$

where we have used the Kähler condition $\partial_k g_{k\bar{l}} = \partial g_{k\bar{l}}$.

Let $\nabla'$ be the $(1,0)$-component of the naturally induced Chern connection on the holomorphic line bundle $L_M = K_M^{\otimes (m-1)}$. The induced Hermitian metric on $L_M$ is given by $(\det g)^{-(m-1)}$. For a holomorphic section $e$ of $L_M$, we have

$$\nabla' e = \partial ((\det g)^{-(m-1)}) (\det g)^{(m-1)} e$$

$$= -(m-1) \partial_i \log (\det g) dz^i \otimes e.$$ Proposition 6.2. Given $\sigma \in A^{n,0}(M, L_M)$, then $\rho_\varphi(\sigma)$ is holomorphic in $A^{n,0}(M_\varphi, L_{M_\varphi})$ if and only if

$$\overline{\partial}\sigma = -\nabla'(\varphi, \sigma) + (m-1) \text{div}\varphi \wedge \sigma.$$ (6.5)

Proof. Let $\sigma = f dz^1 \wedge \cdots \wedge dz^n \otimes e \in A^{n,0}(M_0, L_{M_0})$, then

$$\varphi, \sigma = (-1)^{n+i} f \varphi_j^i dz^1 \wedge \cdots \wedge \hat{d}z^i \wedge \cdots \wedge dz^n \wedge d\bar{z}^j \otimes e,$$

by a straightforward computation. We have

$$\nabla'(\varphi, \sigma) = (-1)^{n+i} \partial(f \varphi_j^i) dz^1 \wedge \cdots \wedge \hat{d}z^i \wedge \cdots \wedge dz^n \wedge d\bar{z}^j \otimes e$$

$$+ (-1)^i f \varphi_j^i dz^1 \wedge \cdots \wedge \hat{d}z^i \wedge \cdots \wedge dz^n \wedge d\bar{z}^j \wedge \nabla' e$$

$$= -\partial_i (f \varphi_j^i) d\bar{z}^j \wedge dz^1 \wedge \cdots \wedge dz^n \otimes e$$

$$+ (m-1) f \varphi_j^i \partial_i \log (\det g) d\bar{z}^j \wedge dz^1 \wedge \cdots \wedge dz^n \otimes e.$$ We also have

$$(m-1) \text{div}\varphi \wedge \sigma = (m-1) f \left( \partial_i \varphi_j^i + \varphi_j^i \partial_i \log (\det g) \right) d\bar{z}^j \wedge dz^1 \wedge \cdots \wedge dz^n \otimes e,$$

$$\overline{\partial}\sigma = (\overline{\partial}_j f) d\bar{z}^j \wedge dz^1 \wedge \cdots \wedge dz^n \otimes e.$$ Therefore, identity (6.5) follows from Lemma 6.1. $\square$
**Remark 6.3.** Note that $\sigma \in A^{n,0}(M, \mathcal{L}_M)$ can also be regarded as a smooth section of the holomorphic line bundle $K_M^{\otimes m}$ since $A^{n,0}(M, \mathcal{L}_M) = A^{0,0}(M, K_M^{\otimes m})$. The equation (6.5) is called the extension equation of the pluricanonical form, which gives the criteria when the extended pluricanonical form is holomorphic under the new complex structure.

On the other hand side, if we let $\widehat{\nabla}'$ be the $(1,0)$-part of the Chern connection on $K_M^{\otimes m}$, it is easy to see that the extension equation (6.5) is equivalent to the equation

$$\overline{\partial}\sigma = \varphi \Delta \widehat{\nabla}' \sigma + m \text{div}\varphi \wedge \sigma$$

which was first derived in a joint project of the first author with X. Sun, A. Todorov and S.-T. Yau [16].

6.2. **Bundle-valued quasi-isometry over compact Kähler manifold.** In this section, we review the bundle-valued quasi-isometry formula obtained in [15]. Let $(E, h)$ be a Hermitian holomorphic vector bundle over the compact Kähler manifold $(M, \omega)$ and $\nabla = \nabla' + \Theta$ be the Chern connection of $(E, h)$. With respect to metrics on $E$ and $X$, we set

$$\Box = \overline{\partial}\partial + \overline{\partial}' \partial,$$

$$\Box' = \nabla' \nabla'^* + \nabla^* \nabla'.$$

Accordingly, we have the Green operator $G$ (resp. $G'$) and harmonic projection $H$ (resp. $H'$) in the Hodge decomposition corresponding to $\Box$ (resp. $\Box'$).

We have

$$I = H + \Box \circ G, \quad I' = H' + \Box' \circ G'.$$

Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on $M$ and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of $E$. Let $h = (h_{\alpha\overline{\beta}})$ where $h_{\alpha\overline{\beta}} = h(e_\alpha, e_\beta)$, and the inverse matrix $h^{-1} = (h^{\alpha\overline{\beta}})$. By the curvature formula of Chern connection $\Theta = \overline{\partial}(\partial hh^{-1})$, we obtain

$$\Theta^{\alpha\overline{\beta}}_{\overline{\gamma}\alpha} = -\left(\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j}\right) h^{\overline{\gamma}\delta} - \frac{\partial h_{\alpha\overline{\beta}}}{\partial z^i} \frac{\partial h^{\overline{\delta}\beta}}{\partial \overline{z}^j}.$$ 

Let $R_{\overline{\gamma}\overline{\delta}} = \sum_{\alpha=1}^r \Theta^{\alpha\overline{\gamma}}_{\overline{\alpha}\overline{\delta}}$, we define the Chern-Ricci form of $(E, h)$ by

$$\text{Ric}(E, h) = \frac{\sqrt{-1}}{2} R_{\overline{\gamma}\overline{\delta}} dz^i \wedge d\overline{z}^j.$$
In particular, when \( E = T^{1,0}M \), the corresponding Chern-Ricci form is given by

\[
\text{Ric}(\omega) = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log(\det g).
\]

Let \( \Theta_{\alpha \beta} = \Theta_{\gamma \alpha} h_{\gamma \beta} \), we obtain

\[
\Theta_{\alpha \beta} = -\frac{\partial^2 h_{\alpha \beta}}{\partial z^i \partial \bar{z}^j} + h_{\gamma \alpha} \frac{\partial h_{\alpha \beta}}{\partial z^i} \frac{\partial h_{\gamma \beta}}{\partial \bar{z}^j}.
\]

**Definition 6.4.** An Hermitian vector bundle \((E, h)\) is said to be semi-Nakano positive (resp. Nakano-positive), if for any non-zero vector \( u = u^i \alpha \partial_i \otimes e_{\alpha} \),

\[
\sum_{i,j,\alpha,\beta} \Theta_{ij\alpha\beta} u^i \alpha u^j \beta \geq 0 \quad \text{(resp.} > 0)\).

In particular, for a line bundle, we say that it is positive, if it is Nakano-positive.

**Proposition 6.5** (cf. Theorem 1.1(2) in [15]). If \((\mathcal{L}, h)\) is a positive line bundle over a compact Kähler manifold \((M, \omega)\) and \(\sqrt{-1}\Theta = \rho \omega\) for a constant \(\rho > 0\), then for any \(f \in A^{n-1}(M, \mathcal{L})\), we have

\[
\| \partial^* G \nabla' f \| \leq \| f \|.
\]

(6.6)

For reader’s convenience, we provide the proof of Proposition 6.5 here.

**Proof.** By the well-known Bochner-Kodaira-Nakano identity

\[
\square = \square' + [\sqrt{-1}\Theta, \Lambda]\omega,
\]

and \([\omega, \Lambda\omega] = (k - n)I\) on \(A^k(M)\), we have

\[
\square(\nabla' f) = \square' (\nabla' f) + \rho q (\nabla' f) = (\square' + \rho q) (\nabla' f),
\]

for any \(f \in A^{n-1,q}(M, \mathcal{L})\). Hence

\[
\text{Ker} \square \subseteq \text{Ker} \square'
\]

which implies that \(H \nabla' f = 0\). Thus

\[
\square G (\nabla' f) = \nabla' f = \square' G (\nabla' f)
\]

by \(H' (\nabla' f) = 0\) and the Hodge decomposition for \(\nabla' f\). Then

\[
\langle \nabla' f, G(\nabla' f) \rangle = \langle \nabla' f, \square^{-1} (\nabla' f) \rangle
\]

\[
= \langle \nabla' f, (\square' + \rho q)^{-1} (\nabla' f) \rangle
\]

\[
\leq \langle \nabla' f, \square'^{-1} (\nabla' f) \rangle
\]

\[
= \langle \nabla' f, G'(\nabla' f) \rangle.
\]
Therefore,
\[
\|\bar{\partial} G \nabla' f\|^2 = \langle \bar{\partial} G \nabla' f, \bar{\partial} G(\nabla' f) \rangle \\
= \langle G \nabla' f, \bar{\partial}\bar{\partial} G(\nabla' f) \rangle \\
= \langle G \nabla' f, \left(\Box - \bar{\partial}\bar{\partial}\right) G(\nabla' f) \rangle \\
= \langle G \nabla' f, \nabla' f \rangle - \langle \bar{\partial} G \nabla' f, \bar{\partial} G \nabla' f \rangle \\
\leq \langle \nabla' f, G(\nabla' f) \rangle \\
\leq \langle \nabla' f, G'(\nabla' f) \rangle \\
= \langle f, \nabla'' \nabla' G' f \rangle \\
= \langle f, f - H'(f) - \nabla' \nabla'' G' f \rangle \\
= \|f\|^2 - \|H'(f)\|^2 - \langle \nabla'' f, G' \nabla'' f \rangle \\
\leq \|f\|^2.
\]

We introduce the operator
\[
T^{\nabla'} = \bar{\partial} G \nabla'.
\]

The quasi-isometry formula (6.6) implies that \(T^{\nabla'}\) is an operator of norm less than or equal to 1 in the \(L^2\) Hilbert space of the \(L\)-valued forms. So we have

**Corollary 6.6.** Let \((\mathcal{L}, h)\) be a positive line bundle over a compact Kähler manifold \((M, \omega)\), with \(\sqrt{-1} \Theta = \rho \omega\) for a constant \(\rho > 0\). Let \(\varphi \in A^{0,1}(M, T^{1,0} M)\) be a Beltrami differential acting on the Hilbert space \(L^2(X, \mathcal{L})\) by contraction such that its \(L_\infty\)-norm \(\|\varphi\|_\infty < 1\). Then the operator \(I + T^{\nabla'} \varphi\) is invertible.

**Example 6.7.** We will consider the holomorphic line bundle \(\mathcal{L}_M = K_M^{\otimes(m-1)}\) over the compact Kähler manifold \((M, \omega)\), the corresponding Hermitian metric is given by \(h_\omega = (\det g)^{-(m-1)}\). In this case, the curvature of the Chern connection of \(\mathcal{L}_M\) is given by
\[
\Theta = -(m-1) \bar{\partial}\bar{\partial} \log(\det g).
\]

Therefore
\[
\sqrt{-1} \Theta = -2(m-1) \text{Ric}(\omega).
\]

In particular, if \((M, \omega)\) is a Kähler-Einstein manifold of general type as defined in Definition 6.12, i.e. \(\text{Ric}(\omega) = -\omega\), then we have
\[
\sqrt{-1} \Theta = 2(m-1) \omega.
\]
6.3. Solving the extension equation. As discussed in Section 6.1, in order to construct an extension pluricanonical form over $M_\varphi$, we need to solve the extension equation (6.5).

Before going further, we need the following lemma.

**Lemma 6.8.** Let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential and let $\sigma \in A^{n,0}(M, L_M)$, we set

$$\Psi = \overline{\partial} \sigma + \nabla'(\varphi, \sigma) - (m - 1) \text{div} \varphi \wedge \sigma,$$

then we have the identity:

$$\overline{\partial}(\nabla'(\varphi, \sigma) - (m - 1) \text{div} \varphi \wedge \sigma) = - (\nabla'(\varphi, \sigma) - (m - 1) \text{div} \varphi \wedge \Psi).$$

**Proof.** Locally, we write $\varphi = \varphi_i^j dz^i \otimes \partial_j \in A^{0,1}(M, T^{1,0}M)$, $\sigma = f dz^1 \wedge \cdots \wedge dz^n \otimes e \in A^{n,0}(M, L_M)$ where $e = (dz^1 \wedge \cdots \wedge dz^n) \otimes (m-1)$. Then

$$\text{div} \varphi = (\partial_i \varphi^i_j + \varphi^i_j \partial_i \log \det(g)) dz^i.$$

For brevity, we introduce the notations $dZ = dz^1 \wedge \cdots \wedge dz^n$ and $dZ^{[k]} = dz^1 \wedge \cdots \wedge \hat{dz}^k \wedge \cdots \wedge dz^n$, where the hat indicates that the corresponding term is to be dropped.

By the computations in the proof of Proposition 6.2, we have

$$\nabla'(\varphi, \sigma) - (m - 1) \text{div} \varphi \wedge \sigma = - \left( (\partial_i f) \varphi^i_j + mf \partial_i \varphi^i_j \right) d\xi^j \wedge dZ \otimes e.$$

Therefore

$$\overline{\partial}(\nabla'(\varphi, \sigma) - (m - 1) \text{div} \varphi \wedge \sigma)$$

$$= - \overline{\partial}_l \left( (\partial_i f) \varphi^i_j + mf \partial_i \varphi^i_j \right) d\xi^j \wedge d\xi^l \wedge dZ \otimes e$$

$$= \sum_{1 \leq i < j \leq n} \left( (\overline{\partial}_j f \varphi^i_j - \overline{\partial}_i f \varphi^i_j + \partial_i f (\overline{\partial}_j \varphi^i_j - \overline{\partial}_j \varphi^i_j) + m(\overline{\partial}_j f \partial_i \varphi^i_j - \overline{\partial}_i f \partial_j \varphi^i_j) + m f (\overline{\partial}_j \partial_i \varphi^i_j - \overline{\partial}_i \partial_j \varphi^i_j) \right) d\xi^j \wedge d\xi^l \wedge dZ \otimes e.$$

On the other hand side, since

$$\Psi = \overline{\partial} \sigma + \nabla'(\varphi, \sigma) - (m - 1) \text{div} \varphi \wedge \sigma$$

$$= \left( \overline{\partial}_j f - \varphi^i_j \partial_i f - mf \partial_i \varphi^i_j \right) d\xi^j \wedge dZ \otimes e,$$
we have
\[
\varphi \cdot \Psi = (\varphi_T^k d\bar{\varphi}^l \otimes \partial_k) \cdot \left(\overline{\partial_j} f - \varphi_j^l \partial_i f - mf \partial_i \varphi_j^l\right) d\bar{\varphi}^i \land dZ \land e
\]
\[
= \sum_{k=1}^{n} (-1)^k \varphi_T^k (\overline{\partial_j} f - \varphi_j^l \partial_i f - mf \partial_i \varphi_j^l) d\bar{\varphi}^i \land d\bar{\varphi}^j \land dZ^{[k]} \land e.
\]
Thus
\[
\nabla'(\varphi \cdot \Psi)
\]
\[
= \partial \left( \sum_{k=1}^{n} (-1)^k \varphi_T^k (\overline{\partial_j} f - \varphi_j^l \partial_i f - mf \partial_i \varphi_j^l) \right) d\bar{\varphi}^i \land d\bar{\varphi}^j \land dZ^{[k]} \land e
\]
\[
+ (-1)^{n+1} \sum_{k=1}^{n} (-1)^k \varphi_T^k (\overline{\partial_j} f - \varphi_j^l \partial_i f - mf \partial_i \varphi_j^l) d\bar{\varphi}^i \land d\bar{\varphi}^j \land dZ^{[k]} \land \nabla' e
\]
\[
= -\partial_k \left( \varphi_T^k (\overline{\partial_j} f - \varphi_j^l \partial_i f - mf \partial_i \varphi_j^l) \right) d\bar{\varphi}^i \land d\bar{\varphi}^j \land dZ \land e
\]
\[
+ \varphi_T^k \left( \overline{\partial_j} f - \varphi_j^l \partial_i f - mf \partial_i \varphi_j^l \right) (m-1) \partial_k \log(\det g) d\bar{\varphi}^i \land d\bar{\varphi}^j \land dZ \land e.
\]
We also have
\[
-(m-1) \text{div} \varphi \land \Psi = -(m-1) \left( \partial_i \varphi_T^i + \varphi_T^k \partial_k \log(\det g) \right)
\]
\[
\cdot \left( \overline{\partial_j} f - \varphi_j^l \partial_i f - mf \partial_i \varphi_j^l \right) d\bar{\varphi}^i \land d\bar{\varphi}^j \land dZ \land e.
\]
Therefore
\[
(6.11) \quad -(\nabla'(\varphi \cdot \Psi) - (m-1) \text{div} \varphi \land \Psi)
\]
\[
= m(\partial_k \varphi_T^k) \left( \overline{\partial_j} f - \varphi_j^l \partial_i f - mf \partial_i \varphi_j^l \right) + \varphi_T^k \partial_k \left( \overline{\partial_j} f - \varphi_j^l \partial_i f - mf \partial_i \varphi_j^l \right)
\]
\[
\cdot d\bar{\varphi}^i \land d\bar{\varphi}^j \land dZ \land e
\]
\[
= \sum_{0 \leq i \leq j \leq n} \left( m \left( \partial_k \varphi_T^k \overline{\partial_j} f - \partial_k \varphi_j^l \overline{\partial_i} f \right) + \left( \varphi_T^k \partial_k \overline{\partial_j} f - \varphi_j^l \partial_k \overline{\partial_i} f \right) \right)
\]
\[
+ \left( \varphi_T^k \partial_k \varphi_j^l \partial_i f - \varphi_j^l \partial_k \varphi_T^k \partial_i f \right) + mf \left( \varphi_T^k \partial_k \partial_i \varphi_j^l - \varphi_j^l \partial_k \partial_i \varphi_T^k \right)
\]
\[
\cdot d\bar{\varphi}^i \land d\bar{\varphi}^j \land dZ \land e.
\]
Since \( \varphi \) is integrable, i.e. \( \overline{\partial} \varphi = \frac{1}{2}[\varphi, \varphi] \), we obtain
\[
\varphi_T^k \partial_k \varphi_j^l - \varphi_j^l \partial_k \varphi_T^k = \overline{\partial_j} \varphi_j^l - \overline{\partial_j} \varphi_T^k,
\]
and
\[ \partial_i \left( \overline{\partial}_j \varphi^i_j - \overline{\partial}_j \varphi^i_j \right) = \partial_i \left( \varphi^k_i \partial_k \varphi^i_j - \varphi^k_i \partial_k \varphi^i_j \right) = \varphi^k_i \partial_k \varphi^i_j - \varphi^k_i \partial_k \varphi^i_j. \]

Comparing the two expressions in formulae (6.10) and (6.11), we finally obtain the identity (6.9). □

**Proposition 6.9.** Suppose that \((L, h, \omega)\) is a positive line bundle over a compact Kähler manifold \((M, \omega)\) with \(\sqrt{-1} \Theta = \rho \omega\) for a constant \(\rho > 0\), let \(\varphi \in A^{0,1}(M, T^{1,0} M)\) be an integrable Beltrami differential which satisfies the conditions that \(\text{div} \varphi = 0\) and \(L_\infty\)-norm \(\| \varphi \|_\infty < 1\). Then for any holomorphic \(\sigma_0 \in A^{n,0}(M, L_M)\), a solution of the following equation
\[ \sigma - \sigma_0 = -\overline{\partial}^* G \nabla'(\varphi \sigma), \tag{6.12} \]
is also a solution of the equation
\[ \overline{\partial} \sigma = -\nabla'(\varphi \sigma). \tag{6.13} \]

**Proof.** Suppose that \(\sigma \in A^{n,0}(M, L_M)\) satisfies the equation (6.12). First, by using the positivity condition for \(L_M\), we have
\[
\overline{\partial} \sigma = -\overline{\partial}^* G \nabla'(\varphi \sigma)
= (\overline{\partial}^* \overline{\partial} G - \Box) G \nabla'(\varphi \sigma)
= (\overline{\partial}^* \overline{\partial} G - I + H) \nabla'(\varphi \sigma)
= -\nabla'(\varphi \sigma) + \overline{\partial} \nabla'(\varphi \sigma).
\]
Let \(\Phi = \overline{\partial} \sigma + \nabla'(\varphi \sigma)\), then under the condition \(\text{div} \varphi = 0\), we obtain
\[
\overline{\partial} \nabla'(\varphi \sigma) = -\nabla'(\varphi \Phi)
\]
by Lemma 6.8.

Therefore
\[
\Phi = \overline{\partial} \sigma + \nabla'(\varphi \sigma)
= \overline{\partial}^* \overline{\partial} G \nabla'(\varphi \sigma)
= \overline{\partial}^* G \overline{\partial} \nabla'(\varphi \sigma)
= -\overline{\partial}^* G \nabla'(\varphi \sigma).
\]

By quasi-isometry formula (6.6) and the condition \(\| \varphi \|_\infty < 1\), we have
\[
\| \Phi \|^2 \leq \| \varphi \|_\infty \| \Phi \|^2 \leq \| \varphi \|^2 \| \Phi \|^2 < \| \Phi \|^2,
\]
and
we get the contradiction $\|\Phi\|^2 < \|\Phi\|^2$ unless $\Phi = 0$. Hence,
\[
\overline{\partial}\sigma = -\nabla'(\varphi \cdot \sigma).
\]

Remark 6.10. When $(\mathcal{L}_M, h_\omega)$ is semi-positive, we can obtain the same conclusion as in Proposition 6.9, if we substitute the global condition $\|\varphi\|_\infty < 1$ by requiring that $\varphi$ (under the Hölder norm as in [18]) is small enough. We leave the further discussion of the extension equation (6.5) to another paper.

By Corollary 6.6, it is easy to see that the equation
\[
\sigma - \sigma_0 = -\overline{\partial} G\nabla' (\varphi \cdot \sigma) = T^{\nabla'} \varphi \cdot \sigma
\]
has a unique solution given by
\[
\sigma = (I + T^{\nabla'})^{-1} \sigma_0.
\]

In conclusion, we obtain

**Theorem 6.11.** Suppose that $(\mathcal{L}_M, h_\omega)$ is a positive line bundle over a compact Kähler manifold $(M, \omega)$ with curvature $\sqrt{-1}\Theta = \rho \omega$ for a constant $\rho > 0$, let $\varphi \in A^{0,1}(M, T^{1,0}M)$ be an integrable Beltrami differential satisfying the two conditions div$\varphi = 0$ and $L_\infty$-norm $\|\varphi\|_\infty < 1$. Then, given any holomorphic pluricanonical form $\sigma_0 \in A^{n,0}(M, \mathcal{L}_M)$, we have that
\[
\sigma(\varphi) = \rho \varphi ((I + T^{\nabla'})^{-1} \sigma_0)
\]
is a holomorphic pluricanonical form in $A^{n,0}(M, \mathcal{L}_M)$.

Theorem 6.11 gives a closed formula for the extension of pluricanonical forms. Note that the above construction is global in the sense that it does not depend on the local deformation theory of Kodaira-Spencer and Kuranishi [18]. The application of Theorem 6.11 will be discussed in the following section.

6.4. Extension of pluricanonical form over Kähler-Einstein manifold of general type. The invariance of plurigenera for Kähler-Einstein manifold of general type has already been known, see for example [25]. Here we derive an explicit and closed formula as a direct application of Theorem 6.11.

Let $(M, \omega)$ be a Kähler manifold. Denote the associated Kähler form by $\omega = \sqrt{-1} \frac{1}{2} g_{ij} dz^i \wedge d\bar{z}^j$. The corresponding Chern-Ricci form $\text{Ric}(\omega)$ is given by
\[
\text{Ric}(\omega) = \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log(\det g).
\]
**Definition 6.12.** We say $(M,\omega)$ is a Kähler-Einstein manifold of general type if $\text{Ric}(\omega) = -\omega$.

In the following discussion, we assume that $(M,\omega)$ is a Kähler-Einstein manifold of general type.

**Proposition 6.13** (cf. Theorem 1.1 in [25]). Let $\varphi \in A^{0,1}(M,T^{1,0}M)$ be an integrable Beltrami differential, then $\bar{\partial} \varphi = 0$ if and only if $\text{div}\varphi = 0$.

Now we consider the deformation of complex structures of Kähler-Einstein manifolds of general type. Let

$$
\pi : M \to B
$$

be a holomorphic family of Kähler-Einstein manifolds of general type. Here $B = B_\epsilon \subset \mathbb{C}$ is the open disk of radius $\epsilon$. Let $t$ be the holomorphic coordinate on $B$. For $t \in B$, we let $M_t = \pi^{-1}(t)$ be the fiber with the complex structure induced by the integrable Beltrami differential $\varphi(t) \in A^{0,1}(M,T^{1,0}M)$, which satisfies

$$
\left\{
\begin{array}{l}
\bar{\partial} \varphi(t) = \frac{1}{2} [\varphi(t), \varphi(t)] \\
\bar{\partial}^* \varphi(t) = 0.
\end{array}
\right.
$$

where $\partial, \bar{\partial}^*$ are the operators on $M_0$ and $\bar{\partial}^*$ is defined with respect to the Kähler-Einstein metric $g_0$. We can choose $\epsilon$ small enough, such that $\|\varphi(t)\|_\infty < 1$.

Let $\mathcal{L}_{M_0} = K_{M_0}^{\otimes (m-1)}$ be the holomorphic line bundle over $M_0$, and the corresponding Hermitian metric be given by $h_0 = (\det g_0)^{-(m-1)}$. Let $\nabla = \nabla' + \bar{\partial}$ be the Chern connection of $(\mathcal{L}_{M_0}, h_0)$. We have that $\sqrt{-1} \Theta = -2(m-1)\text{Ric}(\omega_0)$ by formula (6.7). Recall the operator $T^{\nabla'} = \bar{\partial}^* G \nabla'$ we introduced.

**Corollary 6.14.** Given any holomorphic pluricanonical form $\sigma_0 \in A^{n,0}(M_0, \mathcal{L}_{M_0})$, we have that

$$
\sigma(t) = \rho_t ((I + T^{\nabla'} \varphi(t))^{-1} \sigma_0)
$$

is a holomorphic pluricanonical form in $A^{n,0}(M_t, \mathcal{L}_{M_t})$ with $\sigma(0) = \sigma_0$, where $\rho_t = \rho_{\varphi(t)}$.

**Proof.** Since $(M_0, \omega_0)$ is Kähler-Einstein of general type, i.e. $\text{Ric}(\omega_0) = -\omega_0$. Hence $\sqrt{-1} \Theta = 2(m-1)\omega_0$. Thus $(\mathcal{L}_{M_0}, h_0)$ is a line bundle which satisfies the conditions in Theorem 6.11. Then Corollary 6.14 followed by Theorem 6.11 and Proposition 6.13.  \(\square\)
By using Corollary 6.14, one can write down the curvature formula of the induced $L^2$ metric on the generalized Hodge bundle over the base $B$ with fiber $H^0(M_t, K_{M_t}^{\otimes m})$ as shown in [25].

7. $L^2$-Hodge theory and quasi-isometry

We will first review abstract Hodge theory, then we derive from [9] that $L^2$-Hodge decomposition theory holds on the universal cover of a Kähler hyperbolic manifold. Then we show the quasi-isometry formula in $L^2$-Hodge theory, and discuss the relationship between $L^2$-Hodge theory and the $L^2$-estimate of Hörmander.

7.1. Abstract Hodge theory. For the basics of the $L^2$-Hodge theory, we refer the reader to Chapter 2 in [21] or Appendix C in [2]. Let $H_1, H_2$ and $H_3$ be three Hilbert spaces. Let $T : H_1 \to H_2$ and $S : H_2 \to H_3$ be densely defined closed operators. We assume

$$\text{Im}(T) \subset \text{Dom}(S)$$

and

$$STx = 0 \text{ for } x \in \text{Dom}(T).$$

Then their Hilbert space adjoint operators $S^* : H_3 \to H_2$ and $T^* : H_2 \to H_1$ satisfy the same conditions, namely,

$$\text{Im}(S^*) \subset \text{Dom}(T^*),$$

and

$$T^*S^*x = 0 \text{ for } x \in \text{Dom}(S^*).$$

If we define $\Delta : H_2 \to H_2$ by

$$\text{Dom}(\Delta) = \{x \in \text{Dom}(S) \cap \text{Dom}(T^*); T^*x \in \text{Dom}(T), Sx \in \text{Dom}(S^*)\},$$

$$\Delta s = S^*Sx + TT^*x \text{ for } x \in \text{Dom}(\Delta).$$

Then we have the following existence theorem for the Green operator of the operator $\Delta$, as given in [11]. We summarize it here for reader’s convenience.

**Proposition 7.1.** Let $S, T$ and $\Delta$ be as above, and suppose that both $\text{Im}(S)$ and $\text{Im}(T)$ are closed. Denoting $\text{Ker}(\Delta)$ by $\mathcal{H}$, we have the following:

1. $\Delta$ is self-adjoint, that is $\Delta = \Delta^*$ holds.
2. $\mathcal{H} = \text{Ker}(\Delta) = \text{Ker}(T^*) \cap \text{Ker}(S)$ and $\mathcal{H}^\perp = \text{Im}(\Delta).$
Denoting by $P_H$ the projection operator: $H_2 \to H$, the Green operator

$$G = (\Delta|_{H^+})^{-1}(I - P_H)$$

is well-defined and bounded.

7.2. Kähler hyperbolicity and $L^2$-Hodge theory. Let $M$ be a compact Kähler manifold with a Kähler form $\omega$. Let $\pi: \tilde{M} \to M$ be the universal cover of $M$. Then $M$ is called Kähler hyperbolic as defined in [9], if

$$\pi^*\omega = d\alpha$$

with $L_\infty$-norm $\|\alpha\|_\infty$ finite under the metric $\pi^*\omega$ on $\tilde{M}$.

We are interested in the $L^2$-cohomology on $\tilde{M}$ with the metric $\tilde{\omega} = \pi^*\omega$.

**Lemma 7.2.** There exists $L^2$-Hodge theory on $\tilde{M}$ with the complete Kähler metric $\tilde{\omega} = \pi^*\omega$. More precisely complete Hodge decompositions holds on $\tilde{M}$ with Kähler form $\tilde{\omega}$.

**Proof.** This follows from the estimates in Theorem 1.4.A in [9]. Indeed, let $\Delta$ denote one of the three Laplacians

$$\Box = \overline{\partial} \partial^* + \partial \overline{\partial}^*, \quad \Box_\partial = \partial \partial^* + \partial^* \partial, \quad \Delta = dd^* + d^*d.$$  

Then the estimates in Theorem 1.4.A in [9] gives us that, for any $L^2$-form $\psi$ that is orthogonal to the harmonic forms, we have

$$\langle \psi, \Delta \psi \rangle \geq \lambda_0^2 \langle \psi, \psi \rangle$$

where $\lambda_0$ is a strictly positive constant which only depends on the dimension of $M$ and the bound on $\alpha$.

Since $\tilde{\omega}$ is a complete metric, as discussed in Appendix C in [2], we know that

$$\ker \Box = \ker \overline{\partial} \cap \ker \overline{\partial}^*,$$

similarly for the operators $\partial$ and $d$. By Theorem 2.1 and Theorem 2.2 in Chapter 2.1.2 of [21], we know that the images of the operators $\overline{\partial}$, $\partial$ and $d$ are closed. Therefore from Proposition 7.1, we know that the abstract Hodge theory holds for these operators.

On the other hand, since $\tilde{\omega}$ is Kähler, we have the equalities for the Laplacians

$$\Box = \Box_\partial = \frac{1}{2} \Delta$$

which implies that complete Hodge decomposition theory holds for the $L^2$ cohomology on $\tilde{M}$ with the metric $\tilde{\omega} = \pi^*\omega$. \hfill $\Box$
Let $H$ denote the orthogonal projection from the $L^2$-completion of smooth $(p, q)$-forms on $\tilde{M}$, $L^{p,q}_{2}(\tilde{M}, \tilde{\omega})$, to the harmonic space $\mathbb{H} = \text{ker} \, \square_{\tilde{\omega}}$. As a corollary of the abstract Hodge decomposition in Proposition 7.1, we have

**Corollary 7.3.** On the universal cover $\tilde{M}$ of the Kähler hyperbolic manifold $M$ with complete Kähler metric $\tilde{\omega}$, there exists a bounded operator $G$ on $L^{p,q}_{2}(\tilde{M}, \tilde{\omega})$, called the Green operator such that

$$\square_{\tilde{\omega}} G = G \square_{\tilde{\omega}} = \text{Id} - H, H G = G H = 0,$$

Therefore we have the following quasi-isometry formula in $L^2$-Hodge theory. Its proof is completely the same as in the case for compact Kähler manifold as given in Section 3.

**Corollary 7.4.** For $g \in \text{Dom}(\partial) \subset L^{p,q}_{2}(\tilde{M}, \tilde{\omega})$, we have that

$$\| \overline{\partial} \overline{\partial}^* G g \|^2 \leq \| g \|^2.$$

We consider the operator

$$T = \overline{\partial} \overline{\partial}^* G \partial$$

in $L^2$ Hodge theory. Then Corollary 7.4 tells us that $T$ is an operator of norm less than or equal to 1, in the Hilbert space of $L^2$-forms. So we have the following

**Corollary 7.5.** On the complete Kähler manifold $(\tilde{M}, \tilde{\omega})$ discussed above, let $\varphi$ be a Beltrami differential acting on the Hilbert space of $L^2$-forms by contraction such that its $L^\infty$-norm $\| \varphi \|_\infty < 1$, then the operator $I + T \varphi$ is invertible.

### 7.3. Relation to $L^2$-estimates

We give a brief discussion of the results in this section to the $L^2$-estimates of Hörmander. We believe that this point of view is more geometric and will be useful in studying the optimal constants in $L^2$-estimate problems. This subsection is independent of the rest of this article.

We consider a complete Kähler manifold $M$ with a holomorphic vector bundle $E$. First note that the same argument of this section, applied to a bundle version of $L^2$-Hodge theory, shows that

$$\| \overline{\partial} G g \|^2 = \langle \overline{\partial} G g, \overline{\partial} G g \rangle = \langle \overline{\partial} \overline{\partial}^* G g, G g \rangle \leq \langle G g, G g \rangle.$$

See [15] for the case of compact Kähler manifold. Here we have used the identity

$$\langle \overline{\partial} \overline{\partial}^* G g, G g \rangle = \langle \square_{\tilde{\omega}} G g, g \rangle - \langle \overline{\partial} G g, g \rangle = \langle G g, g \rangle - \langle H g, H g \rangle - \langle G \overline{\partial} g, G \overline{\partial} g \rangle$$

to derive the above inequality. From this we see that if $\overline{\partial} g = 0$ and $H g = 0$, then

$$f = \overline{\partial} G g$$
solves the equation $\overline{\partial}f = g$ with $L^2$-estimate. From these we can see the interesting relation to the $L^2$-estimate of Hörmander in solving the $\overline{\partial}$-equation $\overline{\partial}f = g$.

Furthermore the curvature condition on the vector bundle $E$ in Hörmander’s $L^2$-estimate and the Bochner-Kodaira-Nakano identity imply that
\[
\langle \square \overline{\partial} g, g \rangle \geq \langle A g, g \rangle
\]
with a constant $A > 0$ which implies that, if the metric is complete, $L^2$-Hodge theory holds by the same argument as the proof of Lemma 7.2, and in the meantime gives us the needed $L^2$-estimate
\[
\langle G g, g \rangle \leq \langle A^{-1} g, g \rangle.
\]
We refer the reader to [7] for details about Hörmander’s $L^2$-estimates.

8. Solving the Beltrami equations

In this section we will apply $L^2$-Hodge theory on the unit disc to solve the classical Beltrami equations for quasi-conformal maps as discussed in [1], [4] and [8]. The problem of solving the Beltrami equations has a long history in geometry and analysis, and it has important applications in complex and analytic geometry. Many methods have been developed for solving such equations. For some of its many important applications, see [3]. We will show that Hodge theory method gives a much simpler way to solve these equations. Our method can be viewed as a more geometric way to treat these equations.

Recall that the Beltrami equation is to solve for a function $f : D \to \mathbb{C}$ where $D$ is a unit disc in the complex plane, such that
\[
f_{\overline{\partial}} = \mu_0 f_{\overline{z}}.
\]
Here $z$ is the variable in $D$ and $\mu_0$ is some function on $D$ with $\sup |\mu_0| < 1$. Let $\mu = \mu_0 \overline{\partial}_{\overline{z}} \otimes dz$, we can rewrite the Beltrami equation in our familiar form,
\[
\overline{\partial} f = \mu \partial f.
\]
We will assume that $\mu$ is measurable, or equivalently $\mu_0$ is a measurable function.

Let us consider the unit disc $D$ with the standard Poincaré metric $\omega_P$ of curvature $-1$, which clearly satisfies the condition in Lemma 7.2. Therefore, $L^2$-Hodge theory holds on $(D, \omega_P)$.

First note that the $L_\infty$-norm of $\mu$ with the Poincaré metric is actually the same as its $L_\infty$-norm with the Euclidean metric on $D$, therefore is the same as $\sup |\mu_0|$ where the supremum is taken on the unit disc $D$. Also note that the $L^2$-norm of a holomorphic one form on $D$ is independent of the Hermitian
metric on $D$. These are crucial for our applications of $L^2$-Hodge theory on $D$. We have the following result for measurable $\mu$ or $\mu_0$.

**Proposition 8.1.** Assume that the $L_\infty$-norm of $\mu$, $|||\mu|||_\infty < 1$. Then given any holomorphic one form $h_0$ on $D$, the equation

\begin{equation}
\overline{\partial}h = -\partial \mu h
\end{equation}

has a solution

\begin{equation}
h = (I + T\mu)^{-1}h_0.\end{equation}

**Proof.** The proof of Proposition 8.1 can be given by using Corollary 7.5 in the same way as in the proof of Proposition 4.1 as shown in Section 4. Here we will give another direct method to show that (8.2) is the solution to equation (8.1).

Plug $h = h_0 + h_1$ into the equation (8.1), then

\begin{equation}
\overline{\partial}h_1 = -\partial \mu h_0 - \partial \mu h_1.
\end{equation}

Here $\mu$ as above is a simplified notation for the contraction operator $\mu$. Applying operator $\overline{\partial} G$ and Hodge theory to both sides, we get

\begin{equation}
h_1 - Hh_1 = -T\mu h_0 - T\mu h_1.
\end{equation}

We consider the equation by assuming $Hh_1 = 0$,

\begin{equation}
(I + T\mu)h_1 = -T\mu h_0.
\end{equation}

By Corollary 7.5, we know that when the $L_\infty$-norm of $\mu$, $|||\mu|||_\infty < 1$, the operator $(I + T\mu)$ is invertible. Let

\begin{equation}
h_1 = -(I + T\mu)^{-1}T\mu h_0.
\end{equation}

Then it is an easy exercise to check directly that $h = h_0 + h_1$ is the same as formula (8.2) which gives a solution of Equation (8.1) which is a closed formula.

Indeed, since $h = h_0 + h_1$ is a $(1,0)$-form, and $(I + T\mu)h_1 = -T\mu h_0$ which is

\begin{equation}
h_1 = -T\mu(h_0 + h_1) = -T\mu h.
\end{equation}

We have

\begin{equation}
\overline{\partial} T\mu h = \overline{\partial}\partial^* G \partial \mu h = \square \sigma G \partial \mu h = (I - H)\partial \mu h = \partial \mu h.
\end{equation}

Then

\begin{equation}
\overline{\partial}h_1 = -\partial \mu(h_0 + h_1) = -\partial \mu h_0 - \partial \mu h_1
\end{equation}

which can be rewritten as

\begin{equation}
\overline{\partial}h = -\partial \mu h.\end{equation}
Theorem 8.2. Assume that $||\mu||_\infty = \sup |\mu_0| < 1$, if $\mu_0$ is of regularity $C^k$, then the Beltrami equation 
\[ \bar{\partial} f = \mu \partial f \]
has a solution $w$ of regularity $C^{k+1}$.

Proof. By Proposition 8.1, we can find a $(1,0)$-form $h$ of regularity $C^k$ on $D$, which satisfies the equation 
\[ \bar{\partial} h = -\partial \mu h, \]
by noting that the Green operator maps a $C^k$ form to a $C^{k+2}$ form.

It follows that 
\[ d(e^{i\mu} h) = 0. \]

According to Poincaré lemma, there is a function $w$ of regularity $C^{k+1}$ on $D$, such that 
\[ e^{i\mu} h = dw = \bar{\partial} w + \partial w. \]

Since 
\[ e^{i\mu} h = h + \mu h, \]
by considering the types, we obtain 
\[ h = \partial w \text{ and } \mu h = \bar{\partial} w. \]

Therefore 
\[ \bar{\partial} w = \mu \partial w. \]

\[ \square \]

9. Generalizations and open questions

Although we only consider holomorphic $(n,0)$-forms in this paper, our method also works for a general $(p,q)$-form $\sigma_0$ with $d\sigma_0 = 0$. In this case, the equation for an extension is 
\[ d\sigma - \mathcal{L}^{1,0}_\phi \sigma = 0 \]
as follows from the formula in Corollary 2.6. Our method reduces the equation to the system of equations,

\[
\begin{cases}
\partial \sigma = 0, \\
\bar{\partial} \sigma = -\partial (\phi \sigma),
\end{cases}
\]

(9.1)
and $\rho(\sigma) = e^{i\varphi} \sigma$ gives an extension of $\sigma_0$ as $d$ closed forms on $M_\varphi$. The closed formula we derive for the extensions of holomorphic forms applies equally well to $(p, q)$-forms. From this one can give a direct proof of the invariance of Hodge numbers for deformations of compact Kähler manifolds, by showing that any $d$-closed $(p, q)$-form always has an extension by using Hodge theory. From this one may also derive a simpler proof of the stability of the deformation of Kähler manifolds. One may compare this approach with [23]. A related challenging problem is to prove the invariance of plurigenera for compact Kähler manifolds [24] by using the method in Section 6, we leave the further discussion of this topic for another paper.

For the case of complete Kähler manifold with Poincaré type metric on a quasi-projective manifold, the results of [6] and [11] tells us that $L^2$ Hodge decomposition theory holds, therefore we also have similar closed explicit formulas for extensions of $(p, q)$-forms similar to the formulas for compact Kähler manifold. In particular for Riemann surface with punctures and Poincaré metric of constant negative curvature $-1$, our method gives closed formulas for extensions and global sections of holomorphic one forms on the Teichmüller or Torelli space of punctured Riemann surfaces.

Since the Green operator is an integral operator, we may also apply the Hodge theory or $L^2$-Hodge theory method as discussed in this paper to study various integral formulas and extension formulas in complex analysis, which will supply a unifying geometric treatment of some classical formulas. This will be another interesting topic we hope to study in the future.

References

[1] Ahlfors V., Lectures on quasiconformal mappings. With supplemental chapters by CJ Earle, I. Kra, M. Shishikura and JH Hubbard. University Lecture Series, 38.” American Mathematical Society, Providence, RI 1.4 (2006): 12.
[2] Ballmann W., Lectures on Kähler manifolds. European mathematical society, 2006.
[3] Bers L., Quasiconformal mappings, with applications to differential equations, function theory and topology. Bulletin of the American Mathematical Society, 1977, 83(6): 1083-1100.
[4] Bojarski B., On the Beltrami equation. once again: 54 years later. Annales Academiae Scientiarum Fennicae Mathematica 35 (2010): 59-73.
[5] Clemens H., Geometry of formal Kuranishi theory. Adv. Math. 198, 311-365 (2005).
[6] Cattani E., Kaplan A. and Schmid W., $L^2$ and intersection cohomologies for a polarizable variation of Hodge structure. Inven. Math., 87(1987), pp. 217-252.
[7] Demailly J.-P., Analytic Methods in Algebraic Geometry, Higher Education Press, Surveys of Modern Mathematics, Vol. 1, 2010
[8] Gutlyanskii V., et al., *The Beltrami equation: a geometric approach*. Vol. 26. Springer Science & Business Media, 2012.

[9] Gromov M., Kähler hyperbolicity and $L^2$-Hodge theory. J. Differential Geom 33.1 (1991): 263-292.

[10] Kiremidjian G., Deformations of complex structures on certain noncompact manifolds, Ann. of Math., 98, No. 3 (1973), 411-426.

[11] Kashiwara M. and Kawai T., The Poincaré lemma for variations of polarized Hodge structure. *Publ. RIMS*, Kyoto Univ., 23(1987), pp. 345-407.

[12] Kodaira K., Harmonic fields in Riemannian manifolds (Generalized potential theory). *Ann. of Math*, 50 (1949), pp. 587-665.

[13] Liu K. and Rao S., Remarks on the Cartan formula and its applications. *Asian J. Math.* 16(2012), pp. 157-169.

[14] Liu K., Rao S. and Wan X., Geometry of logarithmic forms and deformations of complex structures, arXiv:1708.00097.

[15] Liu K., Rao S. and Yang X., Quasi-isometry and deformations of Calabi-Yau manifolds. arXiv:1207.1182. *Invent. Math.* 199 (2015), no. 2, 423-453.

[16] Liu K., Sun X., Todorov A. and Yau S.-T., unpublished note.

[17] Liu K., Sun X. and Yau S.-T., Recent development on the geometry of the Teichmüller and moduli spaces of Riemann surfaces. Surveys in differential geometry, 2009, 14: 221-259.

[18] Morrow J. and Kodaira K., *Complex manifolds*. Hilt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1971.

[19] Moser J., On the volume elements on a manifold. Transactions of the American Mathematical Society, 1965, 120(2): 286-294.

[20] Newlander A. and Nirenberg L., Complex analytic coordinates in almost complex manifolds. Annals of Mathematics (1957): 391-404.

[21] Ohsawa T., $L^2$ approaches in several complex variables. Development of Oka-Cartan theory by $L^2$ estimates for the $\overline{\partial}$-operator. Springer Monographs in Mathematics. Springer, Tokyo (2015).

[22] Popp H., *Moduli theory and classification theory of algebraic varieties*, Lecture Notes in Mathematics, 620, Springer-Verlag, Berlin-New York, (1977).

[23] Rao S., Wan X. and Zhao Q., Power series proofs for local stabilties of Kähler and balanced structures with mild $\partial\bar{\partial}$-lemma. arXiv preprint arXiv:1609.05637, 2016.

[24] Siu Y., Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semipositively twisted plurigenera for manifolds not necessarily of general type. Complex geometry (Göttingen, 2000), 223-277, Springer, Berlin, (2002).

[25] Sun X., Deformation of canonical metrics I, Asian J. Math. 16 (2012), no. 1, 141-155.

Kefeng Liu, School of Mathematics, Capital Normal University, Beijing, 100048, China
Solving equations with Hodge theory  
Kefeng Liu and Shengmao Zhu

Department of Mathematics, University of California at Los Angeles, California 90095
E-mail address: liu@math.ucla.edu

Shengmao Zhu, Center of Mathematical Sciences, Zhejiang University, Hangzhou, 310027, China
E-mail address: szhu@zju.edu.cn, shengmaozhu@126.com