SOME NEW SEPARATION AXIOMS VIA $\beta$-$I$-OPEN SETS

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Abstract. In this paper, $\beta$-$I$-open sets are used to define some weak separation axioms and to study some of their basic properties. The implications of these axioms among themselves and with the known axioms are investigated.

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1. Introduction

The subject of ideals in topological spaces has been introduced and studied by Kuratowski [7] and Vaidyanathasamy [12]. An ideal $I$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space $(X, \tau)$ with an ideal $I$ on $X$ and if $P(X)$ is the set of all subsets of $X$, a set operator $(.)^*: P(X) \rightarrow P(X)$, called the local function ([12]) of $A$ with respect to $\tau$ and $I$, is defined as follows: For $A \subset X$, $A^*(\tau, I) = \{x \in X|U \cap A \notin I \text{ for every open neighbourhood } U \text{ of } x\}$. A Kuratowski closure operator $\text{Cl}^*(.)$ for a topology $\tau^*(\tau, I)$ called the $*$-topology, finer than $\tau$ is defined by $\text{Cl}^*(A) = A \cup A^*(\tau, I)$ where there is no chance of confusion, $A^*(I)$ is denoted by $A^*$. If $I$ is an ideal on $X$, then $(X, \tau, I)$ is called an ideal topological space. In this paper, $\beta$-$I$-open sets are used to define some weak separation axioms and to study some of their basic properties. The implications of these axioms among themselves and with the known axioms are investigated.
2. Preliminaries

Let $A$ be a subset of a topological space $(X, \tau)$. We denote the closure of $A$ and the interior of $A$ by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset $A$ of $X$ is called $\beta$-open ([1]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$. This notion has been studied extensively in recent years by many topologists (see [2, 3, 11]) because $\beta$-open sets are only natural generalization of open sets. More importantly, they also suggest several new properties of topological spaces. A subset $S$ of an ideal topological space $(X, \tau, I)$ is called $\beta$-$\alpha$-open ([6]) (resp. $\alpha$-$\alpha$-open ([6])) if $S \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(S)))$ (resp. $S \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S)))$). The complement of a $\beta$-$\alpha$-open set is called $\beta$-$\alpha$-closed ([6]). The intersection of all $\beta$-$\alpha$-closed sets containing $S$ is called the $\beta$-$\alpha$-closure of $S$ and is denoted by $\beta\alpha\mathrm{Cl}(S)$. The $\beta$-$\alpha$-Interior of $S$ is defined by the union of all $\beta$-$\alpha$-open sets contained in $S$ and is denoted by $\beta\alpha\mathrm{Int}(S)$. The set of all $\beta$-$\alpha$-open sets of $(X, \tau, I)$ is denoted by $\beta\alpha\mathrm{O}(X)$. The set of all $\beta$-$\alpha$-open sets of $(X, \tau, I)$ containing a point $x \in X$ is denoted by $\beta\alpha\mathrm{O}(X, x)$. A subset $S$ of an ideal topological space $(X, \tau, I)$ is said to be $\beta$-$\alpha$-closed and $\beta$-$\alpha$-open. A point $x \in X$ is called the $\beta$-$\alpha$-cluster point of $S$ if $\beta\alpha\mathrm{Cl}(U) \cap S \neq \emptyset$ for every $\beta$-$\alpha$-open set $U$ of $(X, \tau, I)$ containing $x$. The set of all $\beta$-$\alpha$-cluster points of $S$ is called the $\beta$-$\alpha$-closure of $S$ and is denoted by $\beta\alpha\mathrm{Cl}(S)$. A subset $S$ is said to be $\beta$-$\alpha$-closed set is said to be $\beta$-$\alpha$-open. A point $x \in X$ is called $\beta$-$\alpha$-interior point of $S$ if there exists a $\beta$-$\alpha$-open set $U$ containing $x$ such that $x \in U \subset S$. The set of all $\beta$-$\alpha$-interior points of $S$ and is denoted by $\beta\alpha\mathrm{Int}(S)$. A subset $A$ of an ideal topological space $(X, \tau, I)$ is said to be $\beta$-$\alpha$-open if $A = \beta\alpha\mathrm{Int}(A)$. Equivalently, the complement of $\beta$-$\alpha$-open set is $\beta$-$\alpha$-open. A subset $U_x$ of $(X, \tau, I)$ is said to be $\beta$-$\alpha$-neighbourhood of a point $x \in X$ if and only if there exists a $\beta$-$\alpha$-open set $G$ such that $x \in G \subset U_x$.

**Definition 2.1** ([6]). A function $f: (X, \tau, I) \rightarrow (Y, \sigma, J)$ is said to be $\beta$-$\alpha$-continuous (resp. $\beta$-$\alpha$-irresolute) if the inverse image of every open (resp. $\beta$-$\alpha$-open) set in $Y$ is $\beta$-$\alpha$-open in $X$.

**Definition 2.2** ([14]). An ideal topological space $(X, \tau, I)$ is said to be $\beta$-$\alpha$-regular if and only if for each closed set $F$ of $X$ and each point $x \in X \setminus F$, there exist disjoint $\beta$-$\alpha$-open sets $U$ and $V$ such that $F \subset U$ and $x \in V$.

**Definition 2.3.** A topological space $(X, \tau)$ is said to be:

(i) $\beta$-$T_0$ ([8]) (resp. semi-$T_0$ ([9])) if to each pair of distinct points $x, y$ of
X there exists a \( \beta \)-open (resp. semiopen) set \( A \) containing \( x \) but not \( y \) or a \( \beta \)-open (resp. semiopen) set \( B \) containing \( y \) but not \( x \);

(ii) \( \beta\mathcal{T}_1 \) ([8]) (resp. semi-\( \beta\mathcal{T}_1 \)) if to each pair of distinct points \( x, y \) of \( X \), there exists a pair of \( \beta \)-open (resp. semiopen) sets, one containing \( x \) but not \( y \) and the other containing \( y \) but not \( x \);

(ii) \( \beta\mathcal{T}_2 \) ([8]) (resp. semi-\( \beta\mathcal{T}_2 \)) if to each pair of distinct points \( x, y \) of \( X \), there exists a pair of disjoint \( \beta \)-open (resp. semiopen) sets, one containing \( x \) and the other containing \( y \).

3. \( \beta\mathcal{I}\mathcal{T}_0 \) spaces

**Definition 3.1.** An ideal topological space \( (X, \tau, \mathcal{I}) \) is \( \beta\mathcal{I}\mathcal{T}_0 \) if for any distinct pair of points in \( X \), there is a \( \beta\mathcal{I} \)-open set containing one of the points but not the other.

**Theorem 3.2.** An ideal topological space \( (X, \tau, \mathcal{I}) \) is \( \beta\mathcal{I}\mathcal{T}_0 \) if and only if for each pair of distinct points \( x, y \) of \( X \), \( \beta\mathcal{I} \text{Cl}(\{x\}) \neq \beta\mathcal{I} \text{Cl}(\{y\}) \).

**Proof. Necessity.** Let \( (X, \tau, \mathcal{I}) \) be an \( \beta\mathcal{I}\mathcal{T}_0 \) space and \( x, y \) be any two distinct points of \( X \). There exists a \( \beta\mathcal{I} \)-open set \( G \) containing \( x \) or \( y \), say, \( x \) but not \( y \). Then \( X \setminus G \) is a \( \beta\mathcal{I} \)-closed set which does not contain \( x \) but contains \( y \). Since \( \in \beta\mathcal{I} \text{Cl}(\{y\}) \) is the smallest \( \beta\mathcal{I} \)-closed set containing \( y \), \( \beta\mathcal{I} \text{Cl}(\{y\}) \subset X - G \), and so \( x \notin \beta\mathcal{I} \text{Cl}(\{y\}) \). Consequently, \( \beta\mathcal{I} \text{Cl}(\{x\}) \neq \beta\mathcal{I} \text{Cl}(\{y\}) \).

**Sufficiency.** Let \( x, y \in X \), \( x \neq y \) and \( \beta\mathcal{I} \text{Cl}(\{x\}) \neq \beta\mathcal{I} \text{Cl}(\{y\}) \). Then there exists a point \( z \in X \) such that \( z \) belongs to one of the two sets, say, \( \beta\mathcal{I} \text{Cl}(\{x\}) \) but not to \( \beta\mathcal{I} \text{Cl}(\{y\}) \). If we suppose that \( x \in \beta\mathcal{I} \text{Cl}(\{y\}) \), then \( z \in \beta\mathcal{I} \text{Cl}(\{x\}) \subset \beta\mathcal{I} \text{Cl}(\{y\}) \), which is a contradiction. So \( x \in X \setminus \beta\mathcal{I} \text{Cl}(\{y\}) \), where \( X \setminus \beta\mathcal{I} \text{Cl}(\{y\}) \) is \( \beta\mathcal{I} \)-open and does not contain \( y \). This shows that \( (X, \tau, \mathcal{I}) \) is \( \beta\mathcal{I}\mathcal{T}_0 \). \( \square \)

**Definition 3.3** ([4]). Let \( A \) and \( X_0 \) be subsets of an ideal topological space \( (X, \tau, \mathcal{I}) \) such that \( A \subset X_0 \subset X \). Then \( (X_0, \tau_{|X_0}, \mathcal{I}_{|X_0}) \) is an ideal topological space with an ideal \( \mathcal{I}_{|X_0} = \{I \in \mathcal{I} | I \subset X_0\} = \{I \cap X_0 | I \in \mathcal{I}\} \).

**Lemma 3.4** ([14]). Let \( A \) and \( X_0 \) be subsets of an ideal topological space \( (X, \tau, \mathcal{I}) \). Then,
(i) If $A \in \beta IO(X)$ and $X_0$ is $\alpha$-$\mathcal{I}$-open in $(X, \tau, \mathcal{I})$, then $A \cap X_0 \in \beta IO(X_0)$;

(ii) If $A \in \beta IO(X)$ and $X_0$ is open in $(X, \tau, \mathcal{I})$, then $A \cap X_0 \in \beta IO(X_0)$;

(iii) If $A \in \beta IO(X_0)$ and $X_0 \in \beta IO(X)$, then $A \in \beta IO(X)$.

**Theorem 3.5.** Every $\alpha$-$\mathcal{I}$-open subspace of a $\beta$-$\mathcal{I}$-$T_0$ space is $\beta$-$\mathcal{I}$-$T_0$.

**Proof.** Let $Y$ be an $\alpha$-$\mathcal{I}$-open subspace of an $\beta$-$\mathcal{I}$-$T_0$ space $(X, \tau, \mathcal{I})$ and $x, y$ be two distinct points of $Y$. Then there exists an $\beta$-$\mathcal{I}$-open set $A$ in $X$ containing $x$ or $y$, say, $x$ but not $y$. Now by Lemma 3.4, $A \cap Y$ is an $\beta$-$\mathcal{I}$-open set in $Y$ containing $x$ but not $y$. Hence $(Y, \tau_Y, \mathcal{I}_Y)$ is $\beta$-$\mathcal{I}$-$T_0$. □

**Corollary 3.6.** Every open subspace of a $\beta$-$\mathcal{I}$-$T_0$ space is $\beta$-$\mathcal{I}$-$T_0$.

**Definition 3.7.** A function $f: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be point-$\beta$-$\mathcal{I}$-closure one-to-one if and only if $x, y \in X$ such that $\beta \mathcal{I} Cl\{x\} \neq \beta \mathcal{I} Cl\{y\}$, then $\beta \mathcal{I} Cl\{f(x)\} \neq \beta \mathcal{I} Cl\{f(y)\}$. 

**Theorem 3.8.** If $f: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is point-$\beta$-$\mathcal{I}$-closure one-to-one and $(X, \tau, \mathcal{I})$ is $\beta$-$\mathcal{I}$-$T_0$, then $f$ is one-to-one.

**Proof.** Let $x$ and $y$ be any two distinct points of $X$. Since $(X, \tau, \mathcal{I})$ is $\beta$-$\mathcal{I}$-$T_0$, then $\beta \mathcal{I} Cl\{x\} \neq \beta \mathcal{I} Cl\{y\}$ by Theorem 3.2. But $f$ is point-$\beta$-$\mathcal{I}$-closure one-to-one implies that $\beta \mathcal{I} Cl\{f(x)\} \neq \beta \mathcal{I} Cl\{f(y)\}$. Hence $f(x) \neq f(y)$. Thus, $f$ is one-to-one. □

**Theorem 3.9.** Let $f: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ be a function from $\beta$-$\mathcal{I}$-$T_0$ space $(X, \tau, \mathcal{I})$ into a topological space $(Y, \sigma)$. Then $f$ is point-$\beta$-$\mathcal{I}$-closure one-to-one if and only if $f$ is one-to-one.

**Proof.** The proof follows from Theorem 3.8. □

**Theorem 3.10.** Let $f: (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{I})$ be an injective $\beta$-$\mathcal{I}$-irresolute function. If $Y$ is $\beta$-$\mathcal{I}$-$T_0$, then $(X, \tau, \mathcal{I})$ is $\beta$-$\mathcal{I}$-$T_0$.

**Proof.** Let $x, y \in X$ with $x \neq y$. Since $f$ is injective and $Y$ is $\beta$-$\mathcal{I}$-$T_0$, there exists an $\beta$-$\mathcal{I}$-open set $V_x$ in $Y$ such that $f(x) \in V_x$ and $f(y) \notin V_x$ or there exists a $\beta$-$\mathcal{I}$-open set $V_y$ in $Y$ such that $f(y) \in V_y$ and $f(x) \notin V_y$ with $f(x) \neq f(y)$. By $\beta$-$\mathcal{I}$-irresoluteness of $f$, $f^{-1}(V_x)$ is $\beta$-$\mathcal{I}$-open set in $(X, \tau, \mathcal{I})$ such that $x \in f^{-1}(V_x)$ and $y \notin f^{-1}(V_x)$ or $f^{-1}(V_y)$ is $\beta$-$\mathcal{I}$-open set in $(X, \tau, \mathcal{I})$ such that $y \in f^{-1}(V_y)$ and $x \notin f^{-1}(V_y)$. This shows that $(X, \tau, \mathcal{I})$ is $\beta$-$\mathcal{I}$-$T_0$. □
4. \(\beta-I\)-\(T_1\) spaces

**Definition 4.1.** An ideal topological space \((X, \tau, I)\) is \(\beta-I\)-\(T_1\) if to each pair of distinct points \(x, y\) of \(X\), there exists a pair of \(\beta-I\)-open sets, one containing \(x\) but not \(y\) and the other containing \(y\) but not \(x\).

**Theorem 4.2.** For an ideal topological space \((X, \tau, I)\), each of the following statements are equivalent:

1. \((X, \tau, I)\) is \(\beta-I\)-\(T_1\);
2. Each one point set is \(\beta-I\)-closed in \(X\);
3. Each subset of \(X\) is the intersection of all \(\beta-I\)-open sets containing it;
4. The intersection of all \(\beta-I\)-open sets containing the point \(x \in X\) is the set \(\{x\}\).

**Proof.** (1)\(\Rightarrow\)(2): Let \(x \in X\). Then by (1), for any \(y \in X\), \(y \neq x\), there exists an \(\beta-I\)-open set \(V_y\) containing \(y\) but not \(x\). Hence \(y \in V_y \subset X\setminus \{x\}\). Now varying \(y\) over \(X\setminus \{x\}\) we get \(X\setminus \{x\} = \bigcup \{V_y:\ y \in X\setminus \{x\}\}\). Hence \(X\setminus \{x\}\) being a union of \(\beta-I\)-open set. Accordingly \(\{x\}\) is \(\beta-I\)-closed.

(2)\(\Rightarrow\)(1): Let \(x, y \in X\) and \(x \neq y\). Then by (2), \(\{x\}\) and \(\{y\}\) are \(\beta-I\)-closed sets. Hence \(X\setminus \{x\}\) is a \(\beta-I\)-open set containing \(y\) but not \(x\) and \(X\setminus \{y\}\) is an \(\beta-I\)-open set containing \(x\) but not \(y\). Therefore, \((X, \tau, I)\) is \(\beta-I\)-\(T_1\).

(2)\(\Rightarrow\)(3): If \(A \subset X\), then for each point \(y \notin A\), there exists a set \(X\setminus \{y\}\) such that \(A \subset X\setminus \{y\}\) and each of these sets \(X\setminus \{y\}\) is \(\beta-I\)-open. Hence \(A = \bigcap \{X\setminus \{y\}: y \in X\setminus A\}\) so that the intersection of all \(\beta-I\)-open sets containing \(A\) is the set \(A\) itself.

(3)\(\Rightarrow\)(4): Obvious.

(4)\(\Rightarrow\)(1): Let \(x, y \in X\) and \(x \neq y\). Hence there exists a \(\beta-I\)-open set \(U_x\) such that \(x \in U_x\) and \(y \notin U_x\). Similarly, there exists a \(\beta-I\)-open set \(U_y\) such that \(y \in U_y\) and \(x \notin U_y\). Hence \((X, \tau, I)\) is \(\beta-I\)-\(T_1\). \(\square\)

**Theorem 4.3.** Every \(\alpha-I\)-open subspace of a \(\beta-I\)-\(T_1\) space is \(\beta-I\)-\(T_1\).

**Proof.** Let \(A\) be an \(\alpha-I\)-open subspace of a \(\beta-I\)-\(T_1\) space \((X, \tau, I)\). Let \(x \in A\). Since \((X, \tau, I)\) is \(\beta-I\)-\(T_1\), \(X\setminus \{x\}\) is \(\beta-I\)-open in \((X, \tau, I)\). Now, \(A\) being open, \(A \cap (X\setminus \{x\}) = A\setminus \{x\}\) is \(\beta-I\)-open in \(A\) by Lemma 3.4. Consequently, \(\{x\}\) is \(\beta-I\)-closed in \(A\). Hence by Theorem 4.2, \(A\) is \(\beta-I\)-\(T_1\). \(\square\)
Corollary 4.4. Every open subspace of a $\betaI-T_1$ space is $\betaI-T_1$.

Theorem 4.5. Let $X$ be a $T_1$ space and $f: (X, \tau) \rightarrow (Y, \sigma, I)$ be a $\betaI$-closed surjective function. Then $(Y, \sigma, I)$ is $\betaI-T_1$.

Proof. Suppose $y \in Y$. Since $f$ is surjective, there exists a point $x \in X$ such that $y = f(x)$. Since $X$ is $T_1$, $\{x\}$ is closed in $X$. Again by hypothesis, $f(\{x\}) = \{y\}$ is $\betaI$-closed in $Y$. Hence by Theorem 4.2, $Y$ is $\betaI-T_1$. □

Definition 4.6. A point $x \in X$ is said to be a $\betaI$-limit point of $A$ if and only if for each $V \in \betaI(X)$, $U \cap (A \setminus \{x\}) \neq \emptyset$ and the set of all $\betaI$-limit points of $A$ is called the $\betaI$-derived set of $A$ and is denoted by $\betaId(A)$.

Theorem 4.7. If $(X, \tau, I)$ is $\betaI-T_1$ and $x \in \betaId(A)$ for some $A \subset X$, then every $\betaI$-neighbourhood of $x$ contains infinitely many points of $A$.

Proof. Suppose $U$ is a $\betaI$-neighbourhood of $x$ such that $U \cap A$ is finite. Let $U \cap A = \{x_1, x_2, \ldots, x_n\} = B$. Clearly $B$ is a $\betaI$-closed set. Hence $V = (U \cap A) \setminus (B \setminus \{x\})$ is a $\betaI$-neighbourhood of point $x$ and $V \cap (A \setminus \{x\}) = \emptyset$, which implies that $x \in \betaId(A)$, which contradicts our assumption. Therefore, the given statement in the theorem is true. □

Theorem 4.8. In an $\betaI-T_1$ space $(X, \tau, I)$, $\betaId(A)$ is $\betaI$-closed for any subset $A$ of $X$.

Proof. As the proof of the theorem is easy, it is omitted. □

Theorem 4.9. Let $f: (X, \tau, I) \rightarrow (Y, \sigma, I)$ be an injective and $\betaI$-irresolute function. If $(Y, \sigma, I)$ is $\betaI-T_1$, then $(X, \tau, I)$ is $\betaI-T_1$.

Proof. Proof is similar to Theorem 3.10 □

Definition 4.10. An ideal topological space $(X, \tau, I)$ is said to be $\betaI-R_0$ (\cite{5}) if and only if for every $\betaI$-open sets contains the $\betaI$-closure of each of its singletons.

Theorem 4.11. An ideal topological space $(X, \tau, I)$ is $\betaI-T_1$ if and only if it is $\betaI-T_0$ and $\betaI-R_0$.

Proof. Let $(X, \tau, I)$ be a $\betaI-T_1$ space. Then by definition and as every $\betaI-T_1$ space is $\betaI-R_0$, it is clear that $(X, \tau, I)$ is $\betaI-T_0$ and $\betaI-R_0$. 

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space. Conversely, suppose that \((X, \tau, \mathcal{I})\) is both \(\beta\mathcal{I}\)-\(T_0\) and \(\beta\mathcal{I}\)-\(R_0\). Now, we show that \((X, \tau, \mathcal{I})\) is \(\beta\mathcal{I}\)-\(T_1\) space. Let \(x, y \in X\) be any pair of distinct points. Since \((X, \tau, \mathcal{I})\) is \(\beta\mathcal{I}\)-\(T_0\), there exists a \(\beta\mathcal{I}\)-open set \(G\) such that \(x \in G\) and \(y \notin G\) or there exists a \(\beta\mathcal{I}\)-open set \(H\) such that \(y \in H\) and \(x \notin H\). Suppose \(x \in G\) and \(y \notin G\). As \(x \in G\) implies the \(\beta\mathcal{I}\) \(\text{Cl} \{x\} \subset G\). As \(y \notin G\), \(y \notin \beta\mathcal{I}\) \(\text{Cl} \{x\}\). Hence \(y \in H = X \\setminus \beta\mathcal{I}\) \(\text{Cl} \{x\}\) and it is clear that \(x \notin H\). Hence, it follows that there exist \(\beta\mathcal{I}\)-open sets \(G\) and \(H\) containing \(x\) and \(y\) respectively such that \(y \notin G\) and \(x \notin H\). This implies that \((X, \tau, \mathcal{I})\) is \(\beta\mathcal{I}\)-\(T_1\).

5. \(\beta\mathcal{I}\)-\(T_2\) spaces

Definition 5.1. An ideal topological space \((X, \tau, \mathcal{I})\) is said to be \(\beta\mathcal{I}\)-\(T_2\) \(((14))\) if to each pair of distinct points \(x, y\) of \(X\), there exists a pair of disjoint \(\beta\mathcal{I}\)-open sets, one containing \(x\) and the other containing \(y\).

Theorem 5.2. For an ideal topological space \((X, \tau, \mathcal{I})\), the following statements are equivalent:

(i) \((X, \tau, \mathcal{I})\) is \(\beta\mathcal{I}\)-\(T_2\);

(ii) Let \(x \in X\). For each \(y \neq x\), there exists \(U \in \beta\mathcal{I}\text{O}(X, x)\) and \(y \in \beta\mathcal{I} \text{Cl}(U)\).

(iii) For each \(x \in X\), \(\bigcap \{\beta\mathcal{I}\text{Cl}(U)_x : U\text{ is a }\beta\mathcal{I}\text{-neighbourhood of }x\} = \{x\}\).

(iv) The diagonal \(\Delta = \{(x, x) : x \in X\}\) is \(\beta\mathcal{I}\text{-closed in }X \times X\).

Proof. (i)\(\rightarrow\)(ii): Let \(x \in X\) and \(y \neq x\). Then there are disjoint \(\beta\mathcal{I}\text{-open sets }U\text{ and }V\text{ such that }x \in U\text{ and }y \in V\). Clearly, \(X \setminus V\) is \(\beta\mathcal{I}\text{-closed, }\beta\mathcal{I}\text{Cl}(U) \subset X \setminus V\text{ and therefore }y \notin \beta\mathcal{I}\text{Cl}(U)\).

(ii)\(\rightarrow\)(iii): If \(y \neq x\), then there exists \(U \in \beta\mathcal{I}\text{O}(X, x)\) and \(y \notin \beta\mathcal{I}\text{Cl}(U)\). So \(y \notin \bigcap \{\beta\mathcal{I}\text{Cl}(U)_x : U \in \beta\mathcal{I}\text{O}(X, x)\}\).

(iii)\(\rightarrow\)(iv): We prove that \(X \setminus \Delta\) is \(\beta\mathcal{I}\text{-open. Let } (x, y) \notin \Delta\). Then \(y \neq x\) and since \(\bigcap \{\beta\mathcal{I}\text{Cl}(U)_x : U \in \beta\mathcal{I}\text{O}(X, x)\} = \{x\}\), there is some \(U \in \beta\mathcal{I}\text{O}(X, x)\) and \(y \notin \beta\mathcal{I}\text{Cl}(U)\). Since \(U \cap X \setminus \beta\mathcal{I}\text{Cl}(U) = \emptyset\), \(U \times (X \setminus \beta\mathcal{I}\text{Cl}(U))\) is \(\beta\mathcal{I}\text{-open set such that } (x, y) \in U \times (X \setminus \beta\mathcal{I}\text{Cl}(U)) \subset X \setminus \Delta\).

(iv)\(\rightarrow\)(v): If \(y \neq x\), then \((x, y) \notin \Delta\) and thus there exist \(U, V \in \beta\mathcal{I}\text{O}(X)\) such that \((x, y) \in U \times V\) and \((U \times V) \cap \Delta = \emptyset\). Clearly, for the \(\beta\mathcal{I}\text{-open sets }U\text{ and }V\text{ we have }x \in U\text{, }y \in V\text{ and }U \cap V = \emptyset\). \(\square\)
Theorem 5.3. For an ideal topological space \((X, \tau, I)\), the following statements are equivalent:

(i) \((X, \tau, I)\) is \(\beta\)-\(I\)-\(T_2\);

(ii) for any two distinct points \(x, y \in X\), there exists a \(\beta\)-\(I\)-\(\theta\)-open set containing \(x\) but not \(y\) and there exists a \(\beta\)-\(I\)-\(\theta\)-open set containing \(y\) but not \(x\);

(iii) for any two distinct points \(x, y \in X\), there exists a \(\beta\)-\(I\)-\(\theta\)-open set containing \(x\) but not \(y\) or there exists a \(\beta\)-\(I\)-\(\theta\)-open set containing \(y\) but not \(x\).

Proof. The proofs of implications (i)\(\rightarrow\)(ii) and (ii)\(\rightarrow\)(iii) are obvious.

(iii)\(\rightarrow\)(i): Let \(x\) and \(y\) be any two distinct points of \(X\). There exists a \(\beta\)-\(I\)-\(\theta\)-open set containing \(y\) but not \(x\). Therefore, there exists a \(\beta\)-\(I\)-open set \(V\) such that \(x \in V \subset_{\beta I} \text{Cl}(V) \subset U\) and we have \(y \in X \setminus U \subset X \setminus_{\beta I} \text{Cl}(V)\). By [14, Theorem 3.1], \(V\) is both \(\beta\)-\(I\)-open and \(\beta\)-\(I\)-closed. Therefore, \((X, \tau, I)\) is \(\beta\)-\(I\)-\(T_2\).\)

Corollary 5.4. An ideal toological space is \(\beta\)-\(I\)-\(T_2\) if and only if each singleton subsets of \(X\) is \(\beta\)-\(I\)-closed.

Corollary 5.5. An ideal toological space is \(\beta\)-\(I\)-\(T_2\) if and only if two distinct points of \(X\) have disjoint \(\beta\)-\(I\)-closure.

Lemma 5.6. The product of two \(\beta\)-\(I\)-open sets is \(\beta\)-\(I\)-open.

Proof. Similarly to the proof of Lemma 3.4 of [16].\)

Theorem 5.7. The product of two \(\beta\)-\(I\)-\(T_2\) spaces is \(\beta\)-\(I\)-\(T_2\).

Proof. Let \((X, \tau, I)\) and \((Y, \sigma, I)\) be \(\beta\)-\(I\)-\(T_2\) spaces and \(x, y \in X \times Y\), such that \(x \neq y\). Let \(x = (a, b)\) and \(y = (c, d)\). Without loss of generality, suppose that \(a \neq c\) and \(b \neq d\). Since \(a\) and \(d\) are distinct points of \(X\), there exist disjoint \(\beta\)-\(I\)-open sets \(U\) and \(V\) of \(X\) such that \(a \in U\) and \(c \in V\). Similarly, let \(G\) and \(H\) are disjoint \(\beta\)-\(I\)-open sets in \(Y\), such that \(b \in G\) and \(d \in H\). Then \(U \times G\) and \(V \times H\) are \(\beta\)-\(I\)-open sets in \(X \times Y\) containing \(x\) and \(y\), respectively. Also \((U \times G) \cap (V \times H) = (U \cap V) \times (G \cap H) = \emptyset\). Hence \(X \times Y\) is \(\beta\)-\(I\)-\(T_2\).\)

Theorem 5.8. Every \(\beta\)-\(I\)-regular \(T_0\)-space is \(\beta\)-\(I\)-\(T_2\).
Proof. Let \((X, \tau, \mathcal{I})\) be a \(\beta\mathcal{I}\)-regular \(T_0\) space and \(x, y \in X\) such that \(x \neq y\). Since \(X\) is \(T_0\) there exists an open set \(V\) containing one of the points, say, \(x\) but not \(y\). Then \(y \in X \setminus V\), \(X \setminus V\) is closed and \(x \notin X \setminus V\). By \(\beta\mathcal{I}\)-regularity of \(X\), there exists \(\beta\mathcal{I}\)-open sets \(G\) and \(H\) such that \(x \in G\), \(y \in X \setminus V \subset H\) and \(G \cap H = \emptyset\). Hence \((X, \tau, \mathcal{I})\) is \(\beta\mathcal{I}\)-\(T_2\). \(\square\)

**Theorem 5.9.** Every \(\alpha\mathcal{I}\)-open subspace of a \(\beta\mathcal{I}\)-\(T_2\) space is \(\beta\mathcal{I}\)-\(T_2\).

**Proof.** Proof is similar to Theorem 4.3 \(\square\)

**Corollary 5.10.** Every open subspace of a \(\beta\mathcal{I}\)-\(T_2\) space is \(\beta\mathcal{I}\)-\(T_2\).

**Theorem 5.11.** If \(f: (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})\) is injective, open and \(\beta\mathcal{I}\)-continuous, and \(Y\) is \(\beta\mathcal{I}\)-\(T_2\), then \((X, \tau, \mathcal{I})\) is \(\beta\mathcal{I}\)-\(T_2\).

**Proof.** Since \(f\) is injective, \(f(x) \neq f(y)\) for each \(x, y \in X\) and \(x \neq y\). Now \(Y\) being \(\beta\mathcal{I}\)-\(T_2\), there exists \(\beta\mathcal{I}\)-open sets \(G, H\) in \(Y\) such that \(f(x) \in G\), \(f(y) \in H\) and \(G \cap H = \emptyset\). Let \(U = f^{-1}(G)\) and \(V = f^{-1}(H)\). Then by hypothesis, \(U\) and \(V\) are \(\beta\mathcal{I}\)-open in \(X\). Also \(x \in f^{-1}(G) = U\), \(y \in f^{-1}(H) = V\) and \(U \cap V = f^{-1}(G) \cap f^{-1}(H) = \emptyset\). Hence \((X, \tau, \mathcal{I})\) is \(\beta\mathcal{I}\)-\(T_2\). \(\square\)

The following modification of the Theorem 5.11 where \(f\) is relaxed but \(Y\) is restricted and is also true.

**Theorem 5.12.** If \(f: (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})\) is injective and \(\beta\mathcal{I}\)-closed and \(Y\) is \(T_2\), then \((X, \tau, \mathcal{I})\) is \(\beta\mathcal{I}\)-\(T_1\).

**Proof.** Although the proof is not identical to that of Theorem 5.11 it is quite similar and thus omitted. \(\square\)

**Theorem 5.13.** If \(f: (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})\) is \(\beta\mathcal{I}\)-irresolute and \(Y\) is \(\beta\mathcal{I}\)-\(T_2\). Then the set \(\{(x_1, x_2) | f(x_1) = f(x_2)\}\) is \(\beta\mathcal{I}\)-closed in \(X \times X\).

**Proof.** Let \(A = \{(x_1, x_2) | f(x_1) = f(x_2)\}\). If \((x_1, x_2) \in X \times X \setminus A\), then \(f(x_1) \neq f(x_2)\). Since \(Y\) is \(\beta\mathcal{I}\)-\(T_2\), there exist disjoint \(\beta\mathcal{I}\)-open sets \(V_1\) and \(V_2\) such that \(f(x_j) \in V_j\) for \(j = 1, 2\). Then by \(\beta\mathcal{I}\)-irresoluteness of \(f\), \(f^{-1}(V_j) \in \beta IO(X, x_j)\) for each \(j\). Thus, \((x_1, x_2) \in f^{-1}(V_1) \times f^{-1}(V_2) \in \beta IO(X_1 \times X_2)\) by Lemma 5.6. Therefore, \(f^{-1}(V_1) \times f^{-1}(V_2) \subset (X \times X) \setminus A\). It follows that \(X \times X \setminus A\) is \(\beta\mathcal{I}\)-open and hence \(A\) is \(\beta\mathcal{I}\)-closed set in \(X \times X\). \(\square\)

**Definition 5.14.** A function \(f: (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})\) is called strongly \(\beta\mathcal{I}\)-open if the image of every \(\beta\mathcal{I}\)-open subset of \((X, \tau, \mathcal{I})\) is \(\mathcal{J}\)-o-en in \((Y, \sigma, \mathcal{J})\).
Theorem 5.15. Let \((X, \tau, I)\) be an ideal topological space, \(R\) an equivalence relation in \(X\) and \(p : (X, \tau, I) \to X|R\) the identification function. If \(R \subset (X \times X)\) and \(p\) is a strongly \(\beta-I\)-open function, then \(X|R\) is \(\beta-I\)-T₂.

Proof. Let \(p(x)\) and \(p(y)\) be the distinct members of \(X|R\). Since \(x\) and \(y\) are not related, \(R \subset (X \times X)\) is \(\beta-I\)-closed in \(X \times X\). There are \(\beta-I\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\) and \(U \times V \subset X \setminus R\). Thus, \(p(U)\), \(p(V)\) are disjoint \(\beta-I\)-open sets in \(X|R\) since \(p\) is strongly \(\beta-I\)-open.

Definition 5.16 ([13]). Let \(f : (X, \tau, I) \to (Y, \sigma)\) be a function. The set \(\{(x, f(x)) \mid x \in X\}\) of the product space \(X \times Y\) is called the graph of \(f\) and is denoted by \(G(f)\).

Theorem 5.17. If \(f : (X, \tau, I) \to (Y, \sigma, I)\) is \(\beta-I\)-irresolute and \((Y, \sigma, I)\) is \(\beta-I\)-T₂ space. Then \(G(f)\) is \(\beta-I\)-closed.

Proof. Let \((x, y) \notin G(f)\) and so \(y \neq f(x)\). Now, \((Y, \sigma, I)\) being \(\beta-I\)-T₂, there exist \(\beta-I\)-open sets \(V\) and \(W\) such that \(f(x) \in W\) and \(y \in V\) and \(V \cap W = \emptyset\). Since \(f\) is \(\beta-I\)-irresolute, there exits \(U \in \beta\mathcal{O}(X, x)\) and \(f(U) \subset W\). Therefore, we obtain \((x, y) \in U \times V \subset X \times Y \setminus G(f)\). Now, by Lemma 5.6, \(U \times V \in \beta-I(X \times Y)\). Hence \(X \times Y \setminus G(f)\) is \(\beta-I\)-open in \(X \times Y\). Thus, \(G(f)\) is \(\beta-I\)-closed in \(X \times Y\).

Definition 5.18. An ideal topological space \((X, \tau, I)\) is said to be \(\beta-I-R₁\) if for \(x, y \in X\) with \(\beta_I \text{Cl}(\{x\}) \neq \beta_I \text{Cl}(\{y\})\), there exists disjoint \(\beta-I\)-open sets \(U\) and \(V\) such that \(\beta_I \text{Cl}(\{x\})\) is a subset of \(U\) and \(\beta_I \text{Cl}(\{y\})\) is a subset of \(V\).

Theorem 5.19. The ideal topological space \((X, \tau, I)\) is \(\beta-I-T₂\) if and only if it is \(\beta-I-R₁\) and \(\beta-I-T₀\).

Proof. The proof is similar to Theorem 4.11 and thus omitted.

Remark 5.20. In the following diagram we denote by arrows the implications between the separation axioms which we have introduced and discussed in this paper and examples show that no other implications hold between them.

\[
\begin{align*}
T₂ & \to \beta-I-T₂ & \to \beta-T₂ \\
\downarrow & & \downarrow \quad \downarrow \\
T₁ & \to \beta-I-T₁ & \to \beta-T₂ \\
\downarrow & & \downarrow \quad \downarrow \\
T₀ & \to \beta-I-T₀ & \to \beta-T₀
\end{align*}
\]
Example 5.21. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $(X, \tau, \mathcal{I})$ is $\beta\mathcal{I}$-$T_i$ ($i = 0, 1, 2$) but not $T_i$ ($i = 0, 1, 2$).

Example 5.22. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $(X, \tau, \mathcal{I})$ is $\beta\mathcal{I}$-$T_0$ but not $\beta\mathcal{I}$-$T_1$.

Example 5.23. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{b\} \{c\}, \{b\}\}$. Then $(X, \tau, \mathcal{I})$ is $\beta\mathcal{I}$-$T_i$ ($i = 0, 1, 2$) but not $\beta\mathcal{I}$-$T_i$ ($i = 0, 1, 2$).

Theorem 5.24. (i) An ideal topological space $(X, \tau, \{\emptyset\})$ is $\beta\mathcal{I}$-$T_0$ (resp. $\beta\mathcal{I}$-$T_1$, $\beta\mathcal{I}$-$T_2$) if and only if it is $\beta\mathcal{T}_0$ (resp. $\beta\mathcal{T}_1$, $\beta\mathcal{T}_2$);
(ii) A topological space $(X, \tau, \mathcal{N})$ is $\beta\mathcal{I}$-$T_0$ (resp. $\beta\mathcal{I}$-$T_1$, $\beta\mathcal{I}$-$T_2$) if and only if it is $\beta\mathcal{T}_0$ (resp. $\beta\mathcal{T}_1$, $\beta\mathcal{T}_2$) ($\mathcal{N}$ is the ideal of all nowhere dense sets of $X$);
(iii) A topological space $(X, \tau, \mathcal{P}(X))$ is $\beta\mathcal{I}$-$T_0$ (resp. $\beta\mathcal{I}$-$T_1$, $\beta\mathcal{I}$-$T_2$) if and only if it is semi-$T_0$ (resp. semi-$T_1$, semi-$T_2$).

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