Complexity results for $k$-domination and $\alpha$-domination problems and their variants

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Abstract. Let $G = (V, E)$ be a simple and undirected graph. For some integer $k \geq 1$, a set $D \subseteq V$ is said to be a $k$-dominating set in $G$ if every vertex $v$ of $G$ outside $D$ has at least $k$ neighbors in $D$. Furthermore, for some real number $\alpha$ with $0 < \alpha \leq 1$, a set $D \subseteq V$ is called an $\alpha$-dominating set in $G$ if every vertex $v$ of $G$ outside $D$ has at least $\alpha \times d_v$ neighbors in $D$, where $d_v$ is the degree of $v$ in $G$. The cardinality of a minimum $k$-dominating set and a minimum $\alpha$-dominating set in $G$ is said to be the $k$-domination number and the $\alpha$-domination number of $G$, respectively. In this paper, we present some approximability and inapproximability results on the problem of finding of $k$-domination number and $\alpha$-domination number of some classes of graphs. Moreover, we introduce a generalization of $\alpha$-dominating set which we call $f$-$\alpha$-dominating set. Given a function $f : N \rightarrow \mathbb{R}$, where $N = \{1, 2, 3, \ldots\}$, a set $D \subseteq V$ is said to be an $f$-$\alpha$-dominating set in $G$ if every vertex $v$ of $G$ outside $D$ has at least $f(d_v)$ neighbors in $D$. We prove NP-hardness of the problem of finding of a minimum $f$-$\alpha$-dominating set in $G$, for a large family of functions $f$.

Keywords: $f$-Domination, $\alpha$-domination, $k$-domination, approximation.

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1 Introduction

Let $G = (V, E)$ be an undirected and simple graph. A set $D \subseteq V$ is called a dominating set in $G$ if every vertex of $G$ outside $D$ has at least one neighbor in $D$, or equivalently $|N(v) \cap D| \geq 1$, where $N(v)$ is the set of all neighbors of $v$ in $G$. The cardinality of a minimum dominating set in $G$ is called the domination number of $G$ denoted by $\gamma(G)$. In the past three decades, wide researches have been done on the domination number of graphs and related problems. For a survey of the area of domination in graphs and its applications we refer the reader to [15,16]. In 1985, Fink and Jacobson [12,13] introduced the concept of a $k$-dominating set. Let $k$ be a real number with $k \geq 1$. A set $D \subseteq V$ is called a $k$-dominating set in $G$ if for every vertex $v$ outside $D$, $|N(v) \cap D| \geq k$. The cardinality of a $k$-minimum dominating set is called the $k$-domination number of $G$ and denoted by $\gamma_k(G)$. In 2000, Dunbar et al. [10], introduced the concept of $\alpha$-domination. Let $\alpha$ be a real number with $0 < \alpha \leq 1$. A set $D \subseteq V$ is called an $\alpha$-dominating set in $G$ if for every vertex $v$ of $G$ outside $D$, $|N(v) \cap D| \geq \alpha \times d_v$, where $d_v := |N(v)|$ is the degree of $v$. The cardinality of a minimum $\alpha$-dominating set is called the $\alpha$-domination number of $G$ denoted by $\gamma_{\alpha}(G)$.

In this paper, we show that for any integer $k \geq 1$, the problem of finding a minimum $k$-dominating set in a given graph of maximum degree $k + 2$ is APX-complete (that is, there is no PTAS for the problem unless $P = NP$). Furthermore, we present some approximability and inapproximability results on the problem of finding of $k$-domination number and $\alpha$-domination number of some classes of graphs. Note that in this paper, we consider only graphs with no isolated vertices. We can easily extend the results for the graphs with isolated vertices.

Another interesting problem that we consider in this paper is that of approximating the $k$-domination number and $\alpha$-domination number of $p$-claw free graphs, which are graphs without complete bipartite graph $K_{1,p}$ as induced subgraph. We propose an approximation algorithm for
this problem. We will show that our approximation algorithm has an approximation ratio better than previously known values when the maximum degree of input graph satisfies some special conditions.

Now consider the definition of \( \alpha \)-domination. One generalization of this concept is that instead of having at least \( \alpha \times d_v \) neighbors in \( D \) for each vertex \( v \notin D \), we have at least \( f(d_v) \) neighbors in \( D \), for some function \( f \). By selecting \( f(x) = \alpha x \), the definition matches that of \( \alpha \)-domination. Hence, in this paper, we define the notion of an \( f \)-dominating set.

In this section, we discuss approximation hardness of \( f \)-dominating set. A backbone is a part of a computer network that interconnects various pieces of network that provides a path for exchange of information between subnetworks. From a practical point of view, designing a backbone which is fault-tolerant is a major problem in designing computer networks. The fault-tolerance means the networks retain its role, even if some parts (vertices or edges) fail. A dominating set can be used as a backbone of the network, but a small fault can destroy the dominance of the backbone. If one uses a \( k \)-dominating set as the backbone, it remains as a dominating set until \( k \) neighbors of a vertex in the backbone fails. The value of \( k \) is constant and it does not depend on vertex degrees. Now, consider an \( \alpha \)-dominating set as a backbone of a computer network. Actually, the backbone retains its job as far as some portion of neighbors of each vertex in the backbone works. Clearly, using an \( \alpha \)-dominating set as the backbone is more useful than using a \( k \)-dominating set. Now, assume that we use \( f \)-dominating set as a backbone of a network. Actually, since in \( f \)-dominating set the number of neighbors of each vertex inside \( f \)-dominating set is some depends on the value of function \( f \) which is more general than the \( \alpha \)-dominating set, this gives more flexibility and in some applications, \( f \)-dominating set for some special functions \( f \) may work much better. Since the \( f \)-dominating set is a generalization of previous definitions of dominating sets (except for vector dominating set; see [7]), in any application of dominating sets, one can use \( f \)-dominating set, instead.

In this paper, we prove that for a large family of functions \( f \), the following problem is NP-hard: given a graph \( G \) and a positive integer \( k \), decide whether \( G \) has an \( f \)-dominating set \( S \) with \( |S| \leq k \).

2 Approximation hardness and approximability

In this section, we discuss approximation hardness of \( k \)-domination and \( \alpha \)-domination problems in the case when the input instances are restricted to some subclasses of graphs. For brevity, we denote the \( k \)-domination problem by \( \text{Min } k \text{-Dom Set} \) and \( \alpha \)-domination problem by \( \text{Min } \alpha \text{-Dom Set} \). Moreover, in this section we propose an approximation algorithm for approximating of \( \text{Min } k \text{-Dom Set} \) and \( \text{Min } \alpha \text{-Dom Set} \) for \( p \)-claw free graphs.

Recently, the problems \( \text{Min } k \text{-Dom Set} \), \( \text{Min } \alpha \text{-Dom Set} \) and \( \text{Min Dom Set} \) (special case of \( \text{Min } k \text{-Dom Set} \), in which \( k = 1 \)) were considered by some researchers [5]. In the following, we summarize the most relevant results, on which our new results will be based.

**Theorem 1 ([7]).** \( \text{Min } k \text{-Dom Set} \) and \( \text{Min } \alpha \text{-Dom Set} \) can be approximated in polynomial time by a factor of \( \ln(2\Delta(G)) + 1 \), where \( \Delta(G) \) is the maximum degree of \( G \).

Given a collection \( F \) of subsets of \( S = \{1, \ldots, n\} \), \( \text{Set Cover} \) is the problem of selecting as few as possible subsets from \( F \) such that their union covers \( S \) (see [11]). In 2014, Dinur et al. [9] showed the following result.

**Theorem 2 ([9]).** For any constant \( \epsilon > 0 \) there is no polynomial time algorithm approximating \( \text{Set Cover} \) within a factor of \( (1 - \epsilon) \ln n \) unless \( P = NP \), where \( n \) is the cardinality of the ground set.

Now, by applying Theorem 2 to the results of [5] we have:
Theorem 5. For any constant \( \epsilon > 0 \) there is no polynomial time algorithm approximating \( \text{MIN DOM SET} \) within a factor of \((1 - \epsilon) \ln n \) unless \( P = NP \). The same result holds for bipartite graphs and for split graphs (hence also for chordal graphs).

Also, by applying Theorem 2 to the results of 7 we have:

Theorem 4. For every \( k \geq 1 \) and every \( \epsilon > 0 \), there is no polynomial time algorithm approximating \( \text{MIN } k\text{-DOM SET} \) within a factor of \((1 - \epsilon) \ln n \), unless \( P = NP \).

Theorem 5. For every \( \alpha \in (0, 1) \) and every \( \epsilon > 0 \), there is no polynomial time algorithm approximating \( \text{MIN } \alpha\text{-DOM SET} \) within a factor of \((\frac{1}{2} - \epsilon) \ln n \), unless \( P = NP \).

2.1 APX-completeness of \( \text{MIN } k\text{-DOM SET} \) in graphs of degree at most \( k + 2 \)

Now, we use Theorem 4 to prove that the problem \( \text{MIN } k\text{-DOM SET} \) is APX-complete on the graphs of degree bounded by \( k + 2 \) for all constants \( k \geq 2 \). For brevity, we call the restricted problem \( \text{MIN } k\text{-DOM SET} \) to the graphs of degree bounded by \( R \) by \( \text{MIN } k\text{-DOM SET}-R \). Also in the case of \( \text{MIN } k\text{-DOM SET}-R \), when \( k = 1 \), for brevity, we call the related problem by \( \text{MIN } k\text{-DOM SET}-R \).

First, we recall the notion of \( L\text{-reduction} \).

Definition 1. \((L\text{-reduction})\): Given two NP optimization problems \( F \) and \( G \) and a polynomial transformation \( f : \text{Inst}(F) \rightarrow \text{Inst}(G) \), where \( \text{Inst}(F) \) is the set of instances of \( F \); we say that \( f \) is an \( L\text{-reduction} \) if there are two positive constants \( \alpha \) and \( \beta \) such that for every instance \( x \) of \( F \)

1. \( \text{opt}_G(f(x)) \leq \alpha \text{opt}_F(x) \)
2. for every feasible solution \( y \) of \( f(x) \) with objective value \( m_G(f(x), y) = c_2 \), we can (in polynomial time) find a solution \( y' \) of \( x \) with \( m_F(f(x), y') = c_1 \) such that \( |\text{opt}_F(x) - c_1| \leq \beta |\text{opt}_G(f(x)) - c_2| \).

To prove a problem \( F \) is APX-complete, it is sufficient to prove that \( F \in \text{APX} \) and that there is an \( L\text{-reduction} \) from an APX-complete problem to the problem \( F \).

Theorem 6. \( \text{MIN } k\text{-DOM SET}-(k+2) \) is an APX-complete problem for any \( k \geq 1 \).

Proof. It is known that \( \text{MIN DOM SET}-3 \) is APX-complete [1]. So we consider the case \( k > 1 \). Clearly, by Theorem 4 if the vertex degrees of the graph are bounded by a constant, then the approximation ratio is constant. Thus the problem \( \text{MIN } k\text{-DOM SET}-(k+2) \) is in APX. Now, we show that there is an \( L\text{-reduction} \) \( f_k \) from \( \text{MIN DOM SET}-3 \) to \( \text{MIN } k\text{-DOM SET}-(k+2) \). Suppose that \( G = (V, E) \) is a graph of maximum degree at most 3. We construct a graph \( G_k = (V_k, E_k) \) of maximum degree at most \( k + 2 \) as follows. For each vertex \( v \in V \), suppose that \( S_v \) is a set of \( k - 1 \) new vertices, in particular, we assume that the sets \( S_v \) are pairwise disjoint and also for each \( v \in V \), \( S_v \cap V = \emptyset \). Now join each vertex \( v \in V \) to each vertex of the set \( S_v \). It is clear that the maximum degree of \( G_k \) is at most \( k + 2 \).

Now, we define the transformation \( f_k \) as \( f_k(G) = G_k \). Let \( D_k \) be a \( k \)-dominating set in \( G_k \). We define the set \( D \) as follows:

\[
D = D_k - \left( \bigcup_{v \in V(G)} S_v \right),
\]

Suppose that \( v \) is a vertex of \( G \) outside \( D \). Clearly \( v \) is outside \( D_k \) too. Since in the graph \( G_k \) the vertex \( v \) is adjacent to the elements of \( S_v \) (recall \( |S_v| = k - 1 \)) and \( v \) is dominated by at least \( k \) vertices in \( D_k \), clearly there must be at least one vertex in \( D \) which dominates \( v \). Hence, \( D \) is a dominating set in \( G \). Now, we have

\[
\gamma(G) \leq |D| = |D_k| - (k - 1)n,
\]

where \( n = |V| \).
On the other hand, let $D$ be any dominating set in $G$. Clearly the set

$$D_k = \left( \bigcup_{v \in V(G)} S_v \right) \cup D$$

is a $k$-dominating set in $G_k$. So

$$\gamma_k(G_k) \leq |D_k| = |D| + (k - 1)n. \quad (2)$$

Now suppose that $D_k^*$ and $D^*$ are optimal $k$-dominating set in $G_k$ and optimal dominating set in $G$, respectively. According to equations (1) and (2), we have $\gamma_k(G_k) = \gamma(G) + (k - 1)n$, which results in $|D| - |D^*| = |D_k| - |D_k^*|$.

Also since the maximum degree of $G$ is bounded by 3, any dominating set $D$ in $G$ contains at least $\frac{n}{3}$ elements (see [15]). Hence, since $\gamma_k(G_k) = \gamma(G) + (k - 1)n$, it is easy to see that $|D_k^*| = |D^*| + (k - 1)n \leq (4k - 3)|D^*|$. Hence, $|D_k^*| \leq (4k - 3)|D^*|$, and $f_k$ is an $L$-reduction with parameters $\alpha = 4k - 3$ and $\beta = 1$. So problem $\text{Min } k\text{-Dom Set-}(k + 2)$ is APX-complete.

Consider the problem $\text{Min } k\text{-Dom Set-R}$ restricted to bipartite graphs, denoted by $\text{Min } k\text{-Dom Set-RB}$. Also, consider the problem $\text{Min } k\text{-Dom Set-R}$ restricted to chordal graphs, denoted by $\text{Min } k\text{-Dom Set-RC}$. For $k = 1$, we call the special cases $\text{Min } k\text{-Dom Set-RB}$ and $\text{Min } k\text{-Dom Set-RC}$ by $\text{Min } k\text{-Dom Set-RB}$ and $\text{Min } k\text{-Dom Set-RC}$ respectively. Chlebík and Chlebíková [9] proved that it is NP-hard to approximate the problem $\text{Min } k\text{-Dom Set-3B}$ within factor $1 + \frac{1}{3k}$. Thus, we can easily conclude that the problem $\text{Min } k\text{-Dom Set-3B}$ is APX-complete. Using the same construction discussed in the proof of Theorem [6], we can show that the problem $\text{Min } k\text{-Dom Set-}(k + 2)B$ is APX-complete. Hence we have the following result.

**Theorem 7.** For any $k \geq 1$, the problem $\text{Min } k\text{-Dom Set-}(k + 2)B$ is APX-complete.

Note that for every positive integer $k$, the $\text{Min } k\text{-Dom Set-kC}$ problem is solvable in polynomial time. This is because a chordal graph of bounded maximum degree has bounded clique number and therefore it has bounded treewidth (see [3]). For any fixed $k$, the property that a subset of vertices is a $k$-dominating set can be straightforwardly expressed by a formula in Monadic Second Order Logic. Therefore, the $k$-domination problem is polynomial time solvable in any class of graphs of bounded treewidth [2], and even more generally, in any class of graphs of bounded clique-width, using the general results from [22]. Polynomial time solvability of the $k$-domination problem in graphs of bounded clique-width follows also from the results in [5], where a dynamic programming solution was developed for a more general problem, which includes the variant of $k$-domination where $k$ is given as input.

### 2.2 Inapproximability of $\text{Min } k\text{-Dom Set}$ in bipartite graphs and split graphs

In the following, we present some results on the inapproximability of $\text{Min } k\text{-Dom Set}$ when the problem is restricted to some special classes of graphs.

By some modifications of the proof of Theorem [4], we prove that the result in Theorem [4] holds even if the problem is restricted to bipartite graphs.

**Theorem 8.** For every $k \geq 1$ and every $\epsilon > 0$, there is no polynomial time algorithms approximating $\text{Min } k\text{-Dom Set}$ for bipartite graphs within a factor of $(1 - \epsilon) \ln n$, unless $P = NP$.
Proof. It is sufficient that we make the following modifications in the proof of Theorem 4.

We make a reduction from the domination problem on bipartite graphs. Let $G = (V_1, V_2, E)$ be a bipartite graph with $n$ vertices such that $n + 2k - 2 \leq n^{1+\epsilon}$ and $\gamma(G) \geq \frac{2(2k-1)(1+\epsilon)}{\epsilon^2}$. Note that if $\gamma(G) < \frac{2(2k-1)(1+\epsilon)}{\epsilon^2}$, then with a brute-force algorithm in polynomial time we can solve the problem Min $k$-Dom Set. Hence, without loss of generality, we can have the above assumptions. Since $\gamma(G) \geq \frac{2(2k-1)(1+\epsilon)}{\epsilon^2}$, with some algebraic computations, we have

$$\gamma(G) + 2k - 2 \leq \frac{1 + \epsilon + \epsilon^2}{1 + \epsilon} \times \gamma(G).$$

(3)

Now, transform $G$ into a graph $G'$ by adding two new pairwise disjoint sets $K_1$ and $K_2$ to $G$ such that each of sets $K_2$ and $K_1$ have $k-1$ vertices inducing a graph with no edges. Join each vertex of $V_1$ to each vertex of $K_2$ and join each vertex of $V_2$ to each vertex of $K_1$. Note that there is no edges between sets $K_1$ and $K_2$. Obviously graph $G'$ is bipartite. It is not hard to see that if $D$ is a dominating set in $G$, then $D \cup K_1 \cup K_2$ is a $k$-dominating set in $G'$. Thus $\gamma_k(G') \leq \gamma(G) + 2k - 2$.

Now suppose that there is a polynomial time approximation algorithm that computes a $k$-dominating set $D'$ in $G'$ such that $|D'| \leq (1 - \epsilon) \ln(|V(G')|) \gamma_k(G')$. It is not hard to see that $D := D' \cap V(G)$ is a dominating set in $G$. So we have

$$|D| \leq |D'| \leq (1 - \epsilon)(\ln |V(G')|) \gamma_k(G') \leq (1 - \epsilon)(\ln (n + 2k - 2)) \gamma(G) + 2k - 2 \leq (1 - \epsilon)(\ln n^{1+\epsilon}) (\gamma(G) + 2k - 2) \text{ (since } n + 2k - 2 \leq n^{1+\epsilon})$$

$$\leq (1 - \epsilon)(\ln n^{1+\epsilon}) \left( \frac{1 + \epsilon + \epsilon^2}{1 + \epsilon} \right) \gamma(G) \text{ (by Equation 3)}$$

$$= (1 - \epsilon)(\ln n)(1 + \epsilon + \epsilon^2) \gamma(G)$$

$$= (1 - \epsilon')(\ln n) \gamma(G),$$

where $\epsilon' = \epsilon^3 > 0$. So $D$ approximates the domination number in $G$ within factor $(1 - \epsilon') \ln n$. By Theorem 3, this implies that $P=NP$. Hence there is no polynomial time algorithms approximating Min $k$-Dom Set for bipartite graphs within a factor of $(1 - \epsilon) \ln n$, unless $P=NP$.

A split graph is a graph whose vertices can be partitioned into an independent set and a clique, and a chordal graph is a graph such that every cycle with four or more vertices has a chord. It is clear that in the proof of Theorem 4 if $G$ is a split (chordal) graph, then $G'$ is a split (chordal) graph too. Hence the following result holds.

Corollary 1. For every $k \geq 1$, every $\epsilon > 0$, there is no polynomial time algorithm approximating Min $k$-Dom Set for split (chordal) graphs within a factor of $(1 - \epsilon) \ln n$, unless $P = NP$.

2.3 Inapproximability of Min $\alpha$-Dom Set in bipartite graphs and chordal graphs

Here we present some results on inapproximability of Min $\alpha$-Dom Set when the problem is restricted to bipartite graphs and chordal graphs.

In the proof of Theorem 4, Cicalese et al. first showed that for every integer $B > 0$ and for every $\epsilon > 0$, there is no polynomial time algorithm approximating the domination number of an input graph $G$ without isolated vertices and satisfying $\gamma(G) \geq B \Delta(G)$ within a factor of $(\frac{1}{2} - \epsilon) \ln n$, unless NP $\subseteq$ DTIME($n^{O(\log \log n)}$). Looking at the proof of this result and using Theorem 3, it is easy to see that the result holds for bipartite graphs and chordal graphs, and under the weaker assumption $P \neq NP$. Hence we have:

Lemma 1. For every integer $B > 0$ and for every $\epsilon > 0$, there is no polynomial time algorithm approximating domination on input bipartite graphs $G$ without isolated vertices satisfying $\gamma(G) \geq B \Delta(G)$ within a factor of $(\frac{1}{2} - \epsilon) \ln n$, unless $P = NP$. The same result holds for chordal graphs.
With some modifications in the proof of Theorem 5 and using Lemma 1, we obtain the following result.

**Theorem 9.** For every \( \alpha \in (0, 1) \) and \( \epsilon > 0 \), there is no polynomial time algorithms approximating MIN \( \alpha \)-Dom Set for bipartite graphs within a factor of \( (\frac{1}{2} - \epsilon) \ln n \), unless \( P = NP \).

**Proof.** Let \( 0 < \alpha < 1 \) and \( \epsilon \in (0, 1) \). We define \( N = \left\lceil \frac{\alpha}{1-\alpha} \right\rceil \), \( B = \left\lceil \frac{2N}{\epsilon} \right\rceil \) and \( k = N\Delta(G) \).

The reduction is done from domination in bipartite graphs \( G \) without isolated vertices and with \( n \) vertices such that \( 1 + 2N \leq n^{\epsilon} \) and \( \gamma(G) \geq B\Delta(G) \).

Using the same construction as in the proof of Theorem 5, we transform bipartite graph \( G = (V_1, V_2, E) \) into a graph \( G' \) as follows: consider two sets \( K_1 \) and \( K_2 \) each with \( N\Delta(G) \) extra vertices. We assume that sets \( K_1, K_2 \) and \( V(G) \) are pairwise disjoint. Join each vertex \( v \in V_1 \) to precisely \( k_v \) vertices of \( K_2 \) and join each vertex \( v \in V_2 \) to precisely \( k_v \) vertices of \( K_1 \) where,

\[
k_v = \begin{cases} \left\lceil \frac{\alpha d_G(v)-1}{\epsilon - \alpha} \right\rceil & \text{if } d_G(v) \geq 2 \\ 0 & \text{if } d_G(v) = 1. \end{cases}
\]

Clearly \( 0 \leq k_v \leq N\Delta(G) \) when \( d_G(v) \geq 2 \) and \( k_v = 0 < N\Delta(G) \) when \( d_G(v) = 1 \). Thus, the above transformation can be done.

It is easy to prove that \( \gamma(G) \leq \gamma_{\alpha}(G') \leq \gamma(G) + 2k \) (see [7]). Suppose that there exists a polynomial time algorithm \( A \) that computes an \( \alpha \)-dominating set \( S' \) for \( G' \) such that \( |S'| \leq (\frac{1}{2} - \epsilon)\ln|V(G')|\gamma_{\alpha}(G') \). It is clear that \( |V(G')| = n + 2k = n + 2N\Delta(G) \leq n(1 + 2N) \leq n^{1+\epsilon} \).

Using the inequality \( \gamma(G) \geq \Delta(G)B \), it is easy to see that \( 2k \leq \epsilon\gamma(G) \). Furthermore, it is straightforward that \( S = S' \cap V(G) \) is a dominating set in \( G \) (see [7]). Hence, we have

\[
|S| \leq |S'|
\leq \left( \frac{1}{2} - \epsilon \right)(\ln|V(G')|)\gamma_{\alpha}(G')
\leq \left( \frac{1}{2} - \epsilon \right)(\ln n^{1+\epsilon})\gamma(G) + 2k
\leq \left( \frac{1}{2} - \epsilon \right)(\ln n^{1+\epsilon})\gamma(G) + 2(\epsilon\gamma(G))
= \left( \frac{1}{2} - \epsilon \right)(1 + \epsilon)^2(\ln n)\gamma(G)
= \left( \frac{1}{2} - \epsilon' \right)(\ln n)\gamma(G),
\]

where \( \epsilon' := \epsilon^2(\epsilon + 3/2) \in (0, \frac{1}{2}) \). Thus, using algorithm \( A \), we find a set \( S \) in polynomial time that approximates MIN Dom Set within a factor of \( (\frac{1}{2} - \epsilon') \ln n \), which implies that \( P = NP \). Hence, there is no polynomial time algorithm approximating the problem MIN \( \alpha \)-Dom Set for bipartite graphs within a factor of \( (\frac{1}{2} - \epsilon) \ln n \), unless \( P = NP \).

Using the same construction of the proof of Theorem 5 and using Lemma 1, we can easily prove the following result:

**Theorem 10.** For every \( \alpha \in (0, 1) \), integer \( B > 0 \) and \( \epsilon > 0 \), there is no polynomial time algorithm approximating the MIN \( \alpha \)-Dom Set problem for chordal graphs within a factor of \( (\frac{1}{2} - \epsilon) \ln n \), unless \( P = NP \).

**Proof.** The proof is the same as the proof of Theorem 5. Note that the transformation of chordal graph \( G \) to \( G' \) can be done such that \( G' \) be a chordal graph.
2.4 Approximation algorithm for \textsc{Min $k$-Dom Set} and \textsc{Min $\alpha$-Dom Set} in $p$-claw free graphs

Here we give a simple polynomial time approximation algorithm that gets a $p$-claw free graph $G$ as input and for a constant $k$, computes a $k$-dominating set in $G$ with approximation ratio $\max\{p - 1, k\}$. We also present similar results in the case of \textsc{Min $\alpha$-Dom Set}.

First, we prove a relation between an optimal $k$-dominating set and maximal independent set, denoted by MIS, of a $p$-claw free graph in analogy with Lemma 7 in [13].

**Lemma 2.** Let $D$ be any optimal $k$-dominating set of $G$ and $I$ be any MIS of $G$, where $G$ is a $p$-claw free graph and $k$ is a constant. Then, $|D| \geq \min \left\{ \frac{k}{p-1}, 1 \right\} \cdot |I|$.

**Proof.** For all $u \in I$ and for all $v \in D$, let $x_u = |D \cap N[u]|$, $y_u = |I \cap N[v]|$, $x'_u = |D \cap N(u)|$ and $y'_u = |I \cap N(v)|$, where $N[v] = N(v) \cup \{v\}$. Since $D$ is a $k$-dominating set in $G$, $I$ is a maximal independent set and $G$ is $p$-claw free, we have:

1. If $u \in I - D$, then $x_u \geq k$ and if $u \in I \cap D$ then $x_u \geq 1$. Thus
   \[ \sum_{u \in I} x_u \geq k|I - D| + |I \cap D| \]  \quad (4)

2. If $v \in D - I$ then $y_v \leq p - 1$ and if $v \in I \cap D$, then $y_v = 1$. Thus
   \[ \sum_{v \in D} y_v \leq (p - 1)|D - I| + |I \cap D| \]  \quad (5)

3. If $u \in I - D$, then $x_u = x'_u$ and if $u \in I \cap D$, then $x_u = x'_u + 1$. Thus
   \[ \sum_{u \in I} x_u = \sum_{u \in I} x'_u + |I \cap D| \]  \quad (6)

4. If $v \in D - I$ then $y_v = y'_v$ and if $v \in I \cap D$, then $y_v = y'_v + 1$. Thus
   \[ \sum_{v \in D} y_v = \sum_{v \in D} y'_v + |I \cap D| \]  \quad (7)

By considering the edges between sets $I$ and $D$, it is easy to see that $\sum_{v \in D} y'_v = \sum_{u \in I} x'_u$. Consequently, $\sum_{v \in D} y_v = \sum_{u \in I} x_u$. Hence, according to equations (4) and (5), we have

\[ (p - 1)|D - I| + |D \cap I| \geq k|I - D| + |D \cap I|. \]

With replacing $|D - I|$ by $|D| - |D \cap I|$ and $|I - D|$ by $|I| - |I \cap D|$ in above formula, we have

\[ |D| \geq \frac{k}{p - 1}|I| + \frac{p - k - 1}{p - 1}|D \cap I|. \]

Now if $p \geq k + 1$, then $|D| \geq \frac{k}{p - 1}|I|$ and if $p < k + 1$, since $|D \cap I| \leq |I|$, then

\[ |D| \geq \frac{k}{p - 1}|I| + \frac{p - k - 1}{p - 1}|D \cap I| \geq \frac{k}{p - 1}|I| + \frac{p - k - 1}{p - 1}|I| = |I|. \]

Hence, $|D| \geq \min \left\{ \frac{k}{p-1}, 1 \right\} \cdot |I|$.

Now we present an approximation algorithm.

**Algorithm 2.1:** \textsc{Approximate-$k$-Dom-Claw($G$)}

\begin{algorithm}
\DontPrintSemicolon
\textbf{Input:} A $p$-claw free graph $G$.
\textbf{Output:} A $k$-dominating set $D$ in $G$.
\For{$i := 1$ to $k$}
\algorithmicdo\newcommand{\do}{\algorithmicdo}
\Construct an MIS $I_i$ in $G - I_{i-1}$; \newcommand{\in}{\text{in}}
\enddo
\endalgorithm
\end{algorithm}
Theorem 11. The algorithm APPROXIMATE-\(k\)-DOM-CLAW\((G)\) computes in polynomial time a \(k\)-dominating set \(D\) in a \(p\)-claw free graph \(G\) such that \(|D| \leq \max \{p - 1, k\} \cdot \gamma_k(G)\).

Proof. It is well known that a maximal independent set can be computed in polynomial time in any graph by a simple greedy algorithm. Therefore, the algorithm computes the set \(D\) in polynomial time.

Now we prove that \(D\) is a \(k\)-dominating set in \(G\). Suppose that \(v \notin D\). So \(v \notin I_i\), for \(1 \leq i \leq k\). Since \(I_i\) are maximal independent sets, vertex \(v\) has at least one neighbor in each set \(I_i\) for each \(1 \leq i \leq k\). Thus \(D\) is a \(k\)-dominating set in \(G\).

According to the construction of \(D\) and by applying Lemma 2

\[|D| = \sum_{i=1}^{k} |I_i| \leq \sum_{i=1}^{k} \max \left\{ \frac{p - 1}{k}, 1 \right\} \gamma_k(G) = \max \{p - 1, k\} \cdot \gamma_k(G).\]

\[\square\]

Let \(k\) and \(p\) be two positive integers with \(p \geq k + 1\) and let \(G\) be a \(p\)-claw free graph. If \(\Delta_G < \frac{e_p - 2}{p}\), where \(e\) is the Euler’s number, then clearly \(p - 1 < \ln(2\Delta_G) + 1\). So in this case, the approximation ratio in algorithm APPROXIMATE-\(k\)-DOM-CLAW is better than the approximation ratio given by Theorem 11. Now suppose that \(H\) is a graph with \(\delta_H \geq 3\), where \(\delta_H\) is the minimum degree of \(H\), and suppose that \(G = L(H)\) is the line graph of graph \(H\) (the line graph \(L(H)\) of graph \(H\) is a graph such that each vertex of \(L(H)\) represents an edge of \(H\); two vertices of \(L(H)\) are adjacent if and only if their corresponding edges share a common endpoint). It is easy to see that \(G\) is a \(3\)-claw free graph with \(\Delta_G > \frac{2^k - 2}{k} = \frac{2}{7}\). In the case \(p \geq k + 1\), the algorithm is a 2-approximation for \(k\)-domination, where \(k \in \{1, 2\}\). In the case \(k = 1\), the algorithm is equivalent to computing a maximal matching in \(H\) (where \(G = L(H)\)), which is a well-known 2-approximation algorithm for the minimum edge domination problem in \(H\).

In a similar way, suppose that \(p < k + 1\). If \(\Delta_G > \frac{e_p - 1}{p}\), then clearly \(k < \ln(2\Delta_G) + 1\). So in this case, the approximation ratio given by algorithm APPROXIMATE-\(k\)-DOM-CLAW is better than the approximation ratio given by Theorem 11. In this case, one can find many \(p\)-claw free graphs with \(\Delta_G > \frac{e_p - 1}{p}\).

In the following, we show that Algorithm APPROXIMATE-\(k\)-DOM-CLAW\((G)\) for \(k := \lfloor \alpha \delta_G \rfloor\) computes an \(\alpha\)-dominating set \(D\) for a given \(p\)-claw free graph \(G\) with approximation ratio \(\max\{p - 1, k\}\).

Lemma 3. Let \(D\) be any optimal \(\alpha\)-dominating set in \(G\) and let \(I\) be any maximal independent set of \(G\), where \(G\) is a \(p\)-claw free graph and \(\alpha\) is a constant such that \(0 < \alpha < 1\). Then \(|D| \geq \min \left\{ \frac{\lfloor \alpha \delta_G \rfloor}{p - 1}, 1 \right\} \cdot |I|\).

Proof. Suppose that \(k' = \lfloor \alpha \delta_G \rfloor\). Clearly \(|D| \geq \gamma_{k'}(G)\). So by Lemma 2, we have

\[|D| \geq \gamma_{k'}(G) \geq \min \left\{ \frac{\lfloor \alpha \delta_G \rfloor}{p - 1}, 1 \right\} \cdot |I|\]

\[\square\]

Theorem 12. Algorithm APPROXIMATE-\(k\)-DOM-CLAW\((G)\) for \(k := \lfloor \alpha \delta_G \rfloor\) computes in polynomial time an \(\alpha\)-dominating set \(D\) in a \(p\)-claw free graph \(G\) such that \(|D| \leq \max\{p - 1, k\} \cdot \gamma_\alpha(G)\).

Proof. Proof is similar to the proof of Theorem 11, just use Lemma 3 instead of Lemma 2.
3 NP-completeness result

We recall the definition of an \( f \)-dominating set and of the \( f \)-domination number. Given a function \( f : \mathbb{N} \to \mathbb{R} \), where \( \mathbb{N} = \{1, 2, 3, \ldots\} \), a set \( D \subseteq V \) is called an \( f \)-dominating set in \( G \) if for every vertex \( v \) of \( G \) outside \( D \), \( |N(v) \cap D| \geq f(d_v) \). The cardinality of a minimum \( f \)-dominating set in a graph \( G \) is called the \( f \)-domination number of \( G \), and the problem of finding the \( f \)-domination number of a graph is called the \( f \)-domination problem.

In this section, we prove that the problem of finding the \( f \)-domination number of a graph is NP-complete, for every given function \( f \) with some special properties. It is well known that the following decision problem, denoted by 3-Regular Domination (3RD), is NP-complete [10]: given a 3-regular graph \( G = (V,E) \) and a positive integer \( k \), does \( G \) have a dominating set \( S \) of size at most \( k \)? Now, consider the following decision problem, denoted by \( f \)-Domination (\( f \)DM); given a graph \( G = (V,E) \) without isolated vertices and a positive integer \( k \), does \( G \) have an \( f \)-dominating set \( S \) of size at most \( k \)?

We show that \( f \)DM is NP-complete for some special functions. We extend the proof of NP-completeness of \( \alpha \)-domination problem (see [10]).

**Theorem 13.** Let \( f : \mathbb{N} \to \mathbb{R} \) be a polynomially computable function such that \( \exists x, y \in \mathbb{N} \) satisfying \( x = \lceil f(y) \rceil < y \) and \( x + 1 = \lfloor f(x + 3) \rfloor \). Then, the \( f \)DM problem is NP-complete.

**Proof.** Since \( f \) is a polynomially computable function, the membership in NP is trivial. Now, we prove the NP-hardness.

We make a transformation from 3RD to \( f \)DM. We assume that \( f \) is a fixed function that satisfies the conditions of the theorem. Consider a pair \( x, y \) satisfying the conditions of the theorem. Also, suppose that \( K_{y+1} \) is a complete graph on \( y + 1 \) vertices. Denote the vertex set \( V \) of \( K_{y+1} \) by \( W \) and let \( U \) be a subset of \( W \) with \( x \) elements.

Now assume that \( G \) is a 3-regular graph. We transform graph \( G \) to a graph denoted by \( \hat{G} \) as follows: join each vertex of \( U \) to all vertices of \( G \) (see Figure 1). It is easy to see that since \( f \), \( x \) and \( y \) are fixed, the transformation can be done in polynomial time. Now we prove that \( G \) has a dominating set \( S \) of size of at most \( k \) if and only if \( \hat{G} \) has an \( f \)-dominating set \( D \) of size of at most \( x + k \).

First, we assume that \( S \) is a dominating set in \( G \) such that \( |S| \leq k \). Consider the set \( D = S \cup U \). Using the conditions of the theorem, it is easy to see that \( D \) is an \( f \)-dominating set in \( \hat{G} \) with \( |D| \leq x + k \).

Now assume that there is an \( f \)-dominating set \( D \) in \( \hat{G} \) with at most \( x + k \) elements. In the following, we show that there is a dominating set \( S \) in \( G \) of size at most \( k \). We consider two cases: \( W - U \not\subseteq D \) and \( W - U \subseteq D \).

First, suppose that \( W - U \not\subseteq D \). We choose \( S := D \cap V(G) \). We prove that \( S \) is a dominating set in \( G \) with \( |S| \leq k \). First, we prove that \( S \) is a dominating set in \( G \). Suppose that \( v \in V(G) - D \). Since \( D \) is an \( f \)-dominating set for \( \hat{G} \) and the degree of vertex \( v \) is \( x + 3 \), \( |N(v) \cap D| \geq f(x + 3) \). Hence, by the condition \( x + 1 = \lfloor f(x + 3) \rfloor \), we have \( |N(v) \cap D| \geq x + 1 \), which implies that \( v \) has a neighbor in \( S \). As a result, \( S \) is a dominating set in \( G \).

Now, we prove that \( |S| \leq k \). Since \( W - U \not\subseteq D \), then there is a vertex \( u \) such that \( u \in W - U \) and \( u \not\in D \). Since \( D \) is an \( f \)-dominating set for \( \hat{G} \), we have \( |N(u) \cap D| \geq f(y) \). Hence, by the assumption \( x = \lceil f(y) \rceil \), \( |N(u) \cap D| \geq x \), and therefore \( |D \cap W| \geq x \), which means that \( |S| \leq k \).

Now, suppose that \( W - U \subseteq D \). We claim that using the set \( D \), we can construct an \( f \)-dominating set \( D' \) for \( G \) such that \( |D'| \leq x + k \) and \( W - U \not\subseteq D' \). Hence, by the similar reasons for the first case \( (W - U \not\subseteq D) \), this will imply that \( S := D' \cap V(G) \) is a dominating set in \( G \) with \( |S| \leq k \).

It remains to prove the claim. We consider two cases: \( W \subseteq D \) and \( W \not\subseteq D \). If \( W \subseteq D \), then let \( p \) be an arbitrary vertex in \( W - U \). Note that by the assumption \( x < y \), there always exists such a vertex \( p \). Now, suppose that \( D' := D \setminus \{p\} \). It is obvious that \( |D'| < |D| \leq x + k \). Now, since \( p \) is not adjacent to any vertex in \( G \) and also since \( K_{y+1} \) is a complete graph, it is not hard to see that \( D' \) is an \( f \)-dominating set in \( \hat{G} \).
Now, suppose that $W \not\subseteq D$. Since we assumed that $W - U \subseteq D$, there exists a vertex $q \in U - D$. Let $p$ be an arbitrary vertex in $W - U$. We choose $D' := (D \setminus \{p\}) \cup \{q\}$. Obviously, $|D'| = |D| \leq x + k$. Again, since $p$ is not connected to any vertex in $G$ and also since $K_{y+1}$ is a complete graph, clearly $D'$ is an $f$-dominating set in $\hat{G}$.

Because 3RDM is NP-complete \cite{13}, $f$DM is also NP-complete for the function $f$, which satisfies the conditions of Theorem 13.

There are many functions that satisfy the conditions of Theorem 13 such as $\frac{x}{2}$, $\sqrt{x} + 1$ and $2\ln(1 + \frac{x}{2})$. In Table 1, we present concrete examples of pairs $(x, y)$ satisfying the conditions of Theorem 13 for these three functions.

| Function     | $(x, y)$       |
|--------------|----------------|
| $\frac{x}{2}$ | $(1, 2)$       |
| $\sqrt{x} + 1$ | $(3, 4)$       |
| $2\ln(1 + \frac{x}{2})$ | $(2, 3)$       |

Table 1. Pairs $(x, y)$ satisfying the conditions of Theorem 13

4 Concluding remarks

In this paper, we introduced the concept of $f$-domination as a generalization of $\alpha$-domination. Furthermore, we presented some approximability and inapproximability results on the problems of finding the $k$-domination number and the $\alpha$-domination number for some classes of graphs. Furthermore, we proved the NP-completeness of $f$-domination problem under mild assumptions on function $f$. It is remarkable that the family of $f$DM problems can be seen as a generalization of the $\alpha$-domination problems and as a special case of the vector domination problem. The vector domination problem is defined as follows. Given a graph $G = (V, E)$ with $n$ vertex and an $n$-dimensional non-negative vector $(k_v; v \in V)$ such that for all $v \in V$, $k_v \in \{0, 1, \ldots, d_v\}$; the vector domination is the problem of finding a minimum $S \subseteq V$ such that every vertex $v \in V \setminus S$ has at least $k_v$ neighbors in $S$. We refer the reader for reading about the vector domination to \cite{7, 17}. The connection between $f$-domination and vector domination problems implies that for every polynomially computable function $f$ and every class of graphs in which the vector domination problem is polynomially solvable, the $f$DM problem is also polynomially solvable.
Finally, we leave open the following problem.
Could we identify some interesting classes of graphs where the $f$DM problem can be efficiently solved for many choices of function $f$?

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