TAIL SPACES ESTIMATES ON HAMMING CUBE AND BERNSTEIN–MARKOV INEQUALITY

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Abstract. This note contains some estimates for tail spaces on Hamming cube. We use analytic paraproduct operator for that purpose. We also show several types of Bernstein–Markov inequalities for Banach space valued functions on Hamming cube. Here the novelty is in getting rid of some irritating logarithms and in proving Bernstein–Markov inequalities for $|\nabla f|_X$ rather than for $\Delta^{1/2} f$ for $X$-valued polynomials on Hamming cube.

1. Introduction

We are interested in tail spaces $T_d(X)$ of functions

$$f = \sum_{|S| > d} \hat{f}(S) \varepsilon^S$$

defined on the Hamming cube $\Omega_n = \{ \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n \}$. Here $\hat{f}(S)$ are coefficients belonging to Banach space $X$.

Later we will also need the space of polynomials $P_d(X)$:

$$f = \sum_{|S| \leq d} \hat{f}(S) \varepsilon^S$$

defined on the Hamming cube $\Omega_n$. Here $\hat{f}(S)$ are coefficients belonging to Banach space $X$, which, in particular can be just $\mathbb{R}$.

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2. Tail spaces estimates and dependence on \( K \)-convexity and \( d \)

We first cite the problem. In [MN], Mendel and Naor asked if for every \( K \)-convex Banach space \( \mathcal{X} \) and \( p \in (1, \infty) \) there exists a finite positive \( c(p, \mathcal{X}) \) such that for every \( n \) and \( d < n \), every function \( f : \Omega_n \to \mathcal{X} \) in the \( \mathcal{T}_d(\mathcal{X}) \) satisfies the estimate

\[
\|\Delta f\|_{L^p(\mathcal{X})} \geq c(p, \mathcal{X})d\|f\|_{L^p(\mathcal{X})}.
\]

Moreover, they formulated the tail smoothing conjecture: for every \( K \)-convex Banach space \( \mathcal{X} \) and \( p \in (1, \infty) \) there exists a finite positive \( c(p, \mathcal{X}) \), \( C(p, \mathcal{X}) \) such that for every \( n \) and \( d < n \), every function \( f : \Omega_n \to \mathcal{X} \) in the \( \mathcal{T}_d(\mathcal{X}) \) satisfies the estimate

\[
\|P_t f\|_{L^p(\mathcal{X})} \leq c(p, \mathcal{X})e^{-C(p, \mathcal{X})d\min(t, t^{1/2}A(p, X))}\|f\|_{L^p(\mathcal{X})},
\]

where \( P_t = e^{-t\Delta} \) is the heat semi-group on Hamming cube. The estimate for \( \|\Delta f\|_{L^p(\mathcal{X})} \) would follow just by integrating from 0 to \( \infty \). In Theorem 5.1 of [MN] they proved that there exists \( A(p, \mathcal{X}) \geq 1 \) such that

\[
\|P_t f\|_{L^p(\mathcal{X})} \leq c(p, \mathcal{X})e^{-C(p, \mathcal{X})d\min(t, t^{1/2}A(p, X))}\|f\|_{L^p(\mathcal{X})}.
\]

Before continuing let us cite a crucial result of Pisier [P1].

**Theorem 2.1** (Pisier). Let \( \mathcal{X} \) be a \( K \)-convex space. Then for any \( p \in (1, \infty) \) there exists angle \( \alpha \pi \in (0, \pi) \) such that operators \( e^{-z\Delta} \) are well defined and uniformly bounded in \( \mathcal{A}_\alpha := \{z \in \mathcal{C} : |\arg z| \leq \frac{\alpha \pi}{2}\} \).

In [EI1] the following results were proved:

**Theorem 2.2** (Eskenazis–Ivanisvili). Let \( \mathcal{X} \) be a \( K \)-convex Banach space and let its angle be provided by Theorem 2.1. Then for every \( \varepsilon > 0 \) we have

\[
\|P_t f\|_{L^p(\mathcal{X})} \leq c(p, \mathcal{X}, \varepsilon)e^{-C(p, \mathcal{X}, \varepsilon)d}\min(t, t^{1/2}A(p, X))\|f\|_{L^p(\mathcal{X})},
\]

\[
\|\Delta f\|_{L^p(\mathcal{X})} \geq c(p, \mathcal{X}, \varepsilon)d^{\alpha-\varepsilon}\|f\|_{L^p(\mathcal{X})} \quad \forall \varepsilon > 0.
\]

**Remark 2.3.** In [MN] it was proved that in fact

\[
\|P_t f\|_{L^p(\mathcal{X})} \leq c(p, \mathcal{X})e^{-C(p, \mathcal{X})d\min(t, t^{1/2}A(p, X))}\|f\|_{L^p(\mathcal{X})},
\]

\[
\|\Delta f\|_{L^p(\mathcal{X})} \geq c(p, \mathcal{X})d^{\alpha}\|f\|_{L^p(\mathcal{X})}.
\]

We will prove differently the second inequality in Theorem 2.5 below. In the last section we explain how one can slightly change the reasoning of [EI1] to get rid of \( \varepsilon \) as well.


Theorem 2.4 (Eskenazis–Ivanisvili). Let $f \in \mathcal{P}_{d+m}(X) \cap \mathcal{T}_d(X)$. Let $X$ be a $K$-convex Banach space. Then

$$\|\Delta f\|_{L^p(X)} \geq c(p, X) \frac{d}{m} \|f\|_{L^p(X)}.$$ 

Moreover,

$$\|\Delta^{1/2} f\|_{L^p(X)} \geq c(p, X) \left(\frac{d}{m}\right)^{1/2} \|f\|_{L^p(X)}.$$ 

It is well known that the second inequality above implies the first one, just because $[FLP]$

$$\|\Delta^\beta f\|_{L^p(X)} \leq 4\|\Delta \|_{L^p(X)} \cdot \|f\|^{1-\beta}_{L^p(X)}.$$ 

Now we will prove the result that falls a bit short of the conjecture, but at least it gets rid of $1/m$ in the Theorem 2.4 above and gets rid of $\varepsilon$ in Theorem 2.2.

Theorem 2.5. Let $X$ be a $K$-convex space, $p \in (1, \infty)$, and $\alpha$ from Pisier theorem be in $(0, 1]$. Then

$$\|\Delta f\|_{L^p(X)} \geq c(p, X) d^\alpha \|f\|_{L^p(X)}.$$ 

Remark 2.6. This theorem is not new, see [MN] Theorem 5.1. But the proof is different and it uses the so-called analytic para product operators, about which a lot is known. Also in the last section we will show how to get rid of $\varepsilon$ easily in Theorem 2.2 above (=Theorem 6 of [EI1]).

Proof. Given $f$ on $\Omega_n$ from the tail space $\mathcal{T}_d(X)$ let us introduce a new function $F$ of one more variable:

$$F(w, \varepsilon) := \sum_{|S| > d} w^{|S|} \hat{f}(S) \varepsilon^S,$$

and let us recognize that it is a bounded $L^p(X)$-valued function of $w$ in 2-gone $G_\alpha := \{w : w = e^{-z}, z \in A_\alpha\}$. This is just reformulation of Pisier’s theorem. Moreover it is not only bounded but also holomorphic in $G_\alpha,$

$$F \in H^\infty(G_\alpha; L^p(X)).$$

Notice that the same works for $-G_\alpha$, as the flip $w \rightarrow -w$ can be absorbed by the flip $\varepsilon \rightarrow -\varepsilon$. Let us consider a domain $O_\alpha := G_\alpha \cup -G_\alpha$. It is easy to see that its boundary is smooth at all points except $-1$ and $1$, where $O_\alpha$ forms angle $\pi \alpha$ by two real analytic curves (actually
its boundary is real analytic except for \( \pm 1 \), where real analytic curves form an angle \( \alpha \pi \). Operator

\[
\sum_{|S|} \hat{f}(S) \varepsilon^S \to \sum_{|S|} w^{|S|} \hat{f}(S) \varepsilon^S
\]

is uniformly bounded from \( L^p(X) \) to itself by Theorem 2.1.

Notice that \( wF'(w) = \sum_S w^{|S|} |S| \hat{f}(S) \varepsilon^S \) and \( \Delta f = \sum_S |S| \hat{f}(S) \varepsilon^S \). So \( wF'(w) \) is obtained by applying Pisier’s Fourier multiplier operator to \( \Delta f \). Therefore we have

\[
\|F'(w)\|_{H^\infty(O_\alpha, L^p(X))} \leq C \|wF'(w)\|_{H^\infty(O_\alpha, L^p(X))} \leq M \|\Delta f\|_{L^p(X)},
\]

(2.1)

The first inequality is a trivial maximal principle for holomorphic functions, the second one is again the same Pisier’s theorem.

By the next step we want the estimate of the following type (for \( F'(w) \) built by \( f \in T_d(X) \))

\[
\|F(w)\|_{H^\infty(O_\alpha, L^p(X))} \leq \varepsilon_d \|F'(w)\|_{H^\infty(O_\alpha, L^p(X))}
\]

(2.2)

with a small \( \varepsilon_d \) for large \( d \).

Let \( \varphi \) be the conformal map from the unit disc \( D \) onto \( O_\alpha \), \( \varphi(0) = 0 \). Then (2.2) is equivalent to

\[
\|F \circ \varphi\|_{H^\infty(D, L^p(X))} \leq \varepsilon_d \|F' \circ \varphi\|_{H^\infty(D, L^p(X))}
\]

(2.3)

Obviously, for \( z \in D \)

\[
F \circ \varphi(z) = \int_0^z F' \circ \varphi(\zeta) \cdot \varphi'(\zeta) d\zeta.
\]

Denote \( g(\zeta) := F' \circ \varphi(\zeta) \) and introduce the operator (sometimes called \textit{analytic paraproduct with symbol} \( \varphi \))

\[
T_\varphi g := \int_0^z g(\zeta) \cdot \varphi'(\zeta) d\zeta.
\]

Seems like it was Pommerenke who studied this operator first. Then it was widely researched, see e.g. [Po], [CPPR], [SSV] and the references therein. We need to understand the estimate of this operator on the space of (vector-valued) functions with the property that all their Taylor coefficients at 0 vanish till the order \( d - 1 \). In other words we need to understand the norm of the operator \( T_\psi \), where \( \psi = \int_0^z \zeta^{d-1} \varphi'(\zeta) d\zeta \).

Let us write the Taylor expansion of conformal map \( \varphi \):

\[
\varphi(z) = c_1^\alpha z + c_2^\alpha z^2 + \ldots.
\]
Then it is possible to prove

**Lemma 2.7.**

\[ |\varphi'(z)| \asymp \frac{1}{|1 - z^2|^{1-\alpha}}. \]

(2.4)

And \[ |\epsilon^\alpha_n| \asymp n^{-1-\alpha}. \]

See Section 6 for the proof.

We can write

\[ T \varphi g = \int_0^z g(\zeta) \cdot \varphi'(\zeta) d\zeta = c_1^\alpha \int_0^z g d\zeta + 2c_2^\alpha \int_0^z g \cdot \zeta d\zeta + \cdots + mc_m^\alpha \int_0^z g \cdot \zeta^{m-1} d\zeta + \ldots. \]

Function \( g(\zeta) \cdot \zeta^{m-1} \) in our case is from \( \mathcal{T}_k, k \geq d, \) and its \( H^\infty(D) \) norm is exactly the \( H^\infty(D) \) norm of \( g \) itself.

So we need to understand how to estimate the norm of integration operator on \( \mathcal{T}_{d+m}, m \geq 0, \) for all \( m \) in \( H^\infty(D) \) norm (vector-valued, but this will not be important).

**Lemma 2.8.** Let \( k \) be a positive integer. There exists an \( L^1(T) \) function \( s \) with \( L^1(T) \) norm at most \( \frac{C_0}{k} \) such that \( \hat{s}(k+j), j \geq 0, \) is \( \frac{1}{k+j}. \)

Suppose this Lemma is proved. The integration operator maps \( \zeta^{k+j} \) into \( \zeta^{k+j+1} \). The convolution operator \( s(\zeta) \star \zeta^{k+j} \) maps \( \zeta^{k+j} \) into \( \frac{\zeta^{k+j}}{k+j}. \) Thus the integration operator is division by \( \zeta \) composed with convolution with \( s. \) But division on \( \zeta \) does not change the norm on tail spaces of \( H^\infty(D) \) (regardless of whether it is vector valued or scalar valued). Hence, if Lemma 2.8 is proved then we can estimate \( g \in H^\infty(D; L^p(X)), g(0) = 0, g'(0) = 0, \ldots, g^{(d-1)}(0) = 0: \)

\[ \| Tg \|_{H^\infty(D; L^p(X))} \leq C_0 \| g \|_{H^\infty(D; L^p(X))} \sum_{m=1}^\infty m |c_m^\alpha| \frac{1}{d + m - 1}. \]

Using Lemma 2.7 we conclude

\[ \| Tg \|_{H^\infty(D; L^p(X))} \leq C \| g \|_{H^\infty(D; L^p(X))} \sum_{m=1}^\infty \frac{1}{m^\alpha d + m - 1} \leq C_\alpha \| g \|_{H^\infty(D; L^p(X))} d^{-\alpha}. \]

Hence we proved that

\[ \| F \circ \varphi \|_{H^\infty(D; L^p(X))} \leq C_\alpha \| F_w \circ \varphi \|_{H^\infty(D; L^p(X))} d^{-\alpha}. \]

(2.5)

Then we get (2.2):

\[ \| F \|_{H^\infty(O_\alpha; L^p(X))} \leq C_\alpha \| F_w \|_{H^\infty(O_\alpha; L^p(X))} d^{-\alpha}. \]
We can now combine this with (2.1) and obtain
\[ \|F\|_{H^{1,\alpha}(\Omega_{\alpha}, \mathcal{L}^p(\mathcal{X}))} \leq C_\alpha M \|\Delta F\|_{\mathcal{L}^p(\mathcal{X})} d^{-\alpha}. \]
But \( F(1) = f \). Hence we get the proof of the theorem modulo Lemma 2.8:
\[ \|f\|_{\mathcal{L}^p(\mathcal{X})} \leq C_\alpha M \|\Delta F\|_{\mathcal{L}^p(\mathcal{X})} d^{-\alpha}, \quad \forall f \in \mathcal{T}_d(\mathcal{X}). \]

**Proof.** The proof of Lemma 2.8. We assume that \( k \) is even, which is enough for the proof. We consider first \( S(x) = \sum_{j=1}^{\infty} \frac{\sin jx}{j} \), it is bounded and has Fourier coefficients as we wish: \( 1/j, j \neq 0 \). Now we wish to change its Fourier coefficients in the interval \( j \in [-k, k] \) and not change the Fourier coefficients outside this interval, and make the \( L^1(-\pi, \pi) \) norm of modified function to be at most \( C_0/k \). Consider nodes \( x_r := \frac{\pi r}{k+1}, r = -k + 1, \ldots, -3, -1, 1, 3, \ldots, k - 1 \). The number of nodes is \( k \).
Construct the Lagrange trigonometric interpolation polynomial \( L_k(x) \),
\[ L_k(x) = \sum_r S(x_r) \prod_{m \neq r} (\sin x - \sin x_m) \prod_{m \neq r} (\sin x_r - \sin x_m). \]
Clearly \( L_k(x) \) has non-zero Fourier coefficients only on \([-k+1, k-1]\). It is easy to check that it is an odd function. Notice that the sign of \( S(x) - L_k(x) \) alternates, it is a positive function on \([0, \frac{\pi}{k+1}]\), negative on \((\frac{\pi}{k+1}, \frac{3\pi}{k+1}]\), et cetera.
Consider “triangular” \( \cos \), call it \( c(x) \), it is linear on \([-\pi, 0]\), linear on \([0, \pi]\) and \( c(0) = 1, c(\pm \pi) = -1 \). Its integral vanishes. So if we consider \( c((k+1)x) \) its Fourier coefficients vanish on \([-k, k]\). Then \( c'(k+1)x) \) also has its Fourier coefficients vanishing on \([-k, k]\), in particular, the scalar product in \( L^2(-\pi, \pi) \)
\[ \langle L_k(x), c'((k+1)x) \rangle = 0. \]
Therefore,
\[ \langle S(x) - L_k(x), c'((k+1)x) \rangle = \langle S(x), c'((k+1)x) \rangle. \]
But \( c'((k+1)x) = \pm 1 \), moreover the pattern of signs repeats the pattern of signs of \( S(x) - L_k(x) \), namely it is a positive function on \([0, \frac{\pi}{k+1}]\), negative on \((\frac{\pi}{k+1}, \frac{3\pi}{k+1}]\), et cetera. We conclude that
\[ \|S(x) - L_k(x)\|_1 = \langle S(x) - L_k(x), c'((k+1)x) \rangle = \langle S(x), c'((k+1)x) \rangle. \]
But obviously the right hand side here is at most \( C_0/k \) again by noticing that the oscillation of \( S \) on intervals of order \( \asymp 1/k \) is \( c/k \). Lemma 2.8 is proved and thus the tail theorem is proved as well.
\[ \Box \]
Remark 2.9. Lemma 2.8 should be very well known and widely used, but I am grateful to Rostislav Matveev [RM] for this elegant proof of Lemma 2.8.

3. On Bernstein–Markov inequality and the dependence on X and p

By $L^p(X)$ we always mean $L^p(\Omega_n; X)$.

We first cite four theorems from [EI1].

Theorem 3.1 (Eskenazis–Ivanisvili). Let $X$ be an arbitrary Banach space and $p \in [1, \infty]$. Then

$$\|\Delta f\|_{L^p(X)} \leq d^2 \|f\|_{L^p(X)}, \quad \forall f \in \mathcal{P}_d(X).$$

Theorem 3.2 (Eskenazis–Ivanisvili). Let $X$ be a Banach space and $p \in [1, \infty]$. Then if for all $n$

$$\|\Delta f\|_{L^p(X)} \leq (1 - \eta)d^2 \|f\|_{L^p(X)}, \quad \forall f \in \mathcal{P}_d(X),$$

then $X$ is of finite co-type.

Theorem 3.3 (Eskenazis–Ivanisvili). Let $X$ be a $K$-convex Banach space and $p \in (1, \infty)$. Then

$$\|\Delta f\|_{L^p(X)} \leq C(p, X)d^{2 - \varepsilon(p, X)} \|f\|_{L^p(X)}, \quad \forall f \in \mathcal{P}_d(X),$$

Theorem 3.4 (Eskenazis–Ivanisvili). Let $X = \mathbb{R}$ and $p \in (1, \infty)$. Then

$$\|\Delta f\|_{L^p} \leq C(p)d^{2 - \frac{2}{p} \arcsin \frac{2d}{\sqrt{p}}} \|f\|_{L^p}, \quad \forall f \in \mathcal{P}_d(X).$$

Now we will prove the following results.

Theorem 3.5. Let $X = \mathbb{R}$, $p \in [2, \infty)$. Then

$$\|\nabla f\|_{L^p} \leq C(p)d^{1 - \frac{1}{p}} \arcsin \frac{2d}{\sqrt{p}} \|f\|_{L^p}, \quad \forall f \in \mathcal{P}_d(X).$$

If $p \in (1, 2)$ then

$$\|\nabla f\|_{L^p} \leq C(p)d^{2 - \frac{2}{p} \arcsin \frac{2d}{\sqrt{p}}} \|f\|_{L^p}, \quad \forall f \in \mathcal{P}_d(X).$$

Remark 3.6. It is almost Theorem 15 of [EI1], but we get rid of log $d$ term in the estimate (30) of Theorem 15 of [EI1].
We recall the reader that if $1 < p < \infty$ then $\|\Delta^{1/2} f\|_{L^p} \leq C(p)\|\nabla f\|_{L^p}$ for scalar valued functions (the result of E. Ben Efraim and F. Lust-Piquard [BELP]). But the opposite inequality true only for $2 \leq p < \infty$, [BELP]. Morally this means that it is more difficult to estimate from above $|\nabla f|$ than $\Delta^{1/2} f$ (even for scalar valued $f$). Also clearly the power of $d$ doubles up when we pass the estimate from $\|\Delta^{1/2} f\|_{L^p}$ to the estimate of $\|\Delta f\|_{L^p}$.

When we deal with the $\mathcal{P}_d(X)$ and $X^*$ has type 2, it also has finite cotype $r \in [2, \infty)$ by König–Tzafriri theorem (see [HVNVW2], Theorem 7.1.14). Then we have the following result.

**Theorem 3.7.** Let $X^*$ be of type 2 (and automatically of certain cotype $r < \infty$), and $p \in (1, \infty)$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|\nabla f|_{L^p(X)} \leq C(p)d^{2 - \frac{2}{\max(q,r)}} \|f\|_{L^p(X)} \quad \forall f \in \mathcal{P}_d(X),$$

where

$$|\nabla f|_X = \left( \sum_{i=1}^n \|D_i f\|_{L^p(X)}^2 \right)^{1/2}.$$

**Proof.** We prove Theorem 3.5. We recall the formula from [IVHV]:

$$D_j e^{-t\Delta} f(\varepsilon) = \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \mathbf{E}_\varepsilon \left( \delta_j(t) f(\varepsilon_1 \cdot \xi_1(t), \ldots, \varepsilon_n \cdot \xi_n(t)) \right). \quad (3.1)$$

Here

$$\delta_j(t) := \frac{\xi_j(t) - e^{-t}}{(1 - e^{-2t})^{1/2}},$$

where $\xi_j(t)$ are independent random variables having values $\pm 1$ with probabilities $\frac{1 \pm e^{-t}}{2}$.

From (4.1) for every $\varepsilon \in \Omega_n$ we can write

$$|\nabla e^{-t\Delta} f(\varepsilon)| = \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \max_{\lambda : \|\lambda\|_{L^2_n}} \mathbf{E}_\varepsilon \left| \sum_{j=1}^n \lambda_j \delta_j(t) f(\varepsilon \cdot \xi(t)) \right|.$$

Hence,

$$|\nabla e^{-t\Delta} f(\varepsilon)| \leq \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \max_{\lambda : \|\lambda\|_{L^2_n}} \mathbf{E}_\varepsilon \left( \sum_{j=1}^n \lambda_j \delta_j(t) f(\varepsilon \cdot \xi(t)) \right) \leq \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \max_{\lambda : \|\lambda\|_{L^2_n}} \left( \mathbf{E}_\varepsilon \left( \sum_{j=1}^n \lambda_j \delta_j(t) \right)^q \right)^{1/q} \left( \mathbf{E}_\varepsilon \left( f(\varepsilon \cdot \xi)^p \right) \right)^{1/p}.$$
Raise it to the power $p$ and integrate:

$$\|\nabla e^{-t\Delta} f(\varepsilon)\|_p^p \leq \left( \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \right)^p E_{\varepsilon} \mathbb{E}_\xi |f(\varepsilon \cdot \xi)|^p \cdot \max_{\lambda : \|\lambda\|_{\ell_1^n} = 1} \left( E_{\xi} \left| \sum_{j=1}^n \lambda_j \delta_j(t) \right|^q \right)^{p/q} =$$

$$E_{\varepsilon} \mathbb{E}_\xi |f(\varepsilon \cdot \xi)|^p \cdot \left( \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \right)^p \cdot \max_{\lambda : \|\lambda\|_{\ell_1^n} = 1} \left( E_{\xi} \left| \sum_{j=1}^n \lambda_j \delta_j(t) \right|^q \right)^{p/q} =$$

$$\left( \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \right)^p \|f\|_p^p \cdot \max_{\lambda : \|\lambda\|_{\ell_1^n} = 1} \left( E_{\xi} \left| \sum_{j=1}^n \lambda_j \delta_j(t) \right|^q \right)^{p/q}.$$

Consider the case $1 < q \leq 2$. Then we just use

$$\left( E_{\xi} \left| \sum_{j=1}^n \lambda_j \delta_j(t) \right|^q \right)^{p/q} \leq \left( E_{\xi} \left| \sum_{j=1}^n \lambda_j \delta_j(t) \right|^2 \right)^{p/2} = 1,$$

because $\{\delta_j(t)\}_{j=1}^n$ is an orthonormal system and $\|\lambda\|_{\ell_1^n} = 1$.

We will use this later:

$$1 < q \leq 2 \Rightarrow \|\nabla e^{-t\Delta} f(\varepsilon)\|_p^p \leq \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \|f\|_p^p \quad (3.2)$$

Now let us consider the case $q > 2$. In this case we need to estimate

$$\left( E_{\xi} \left| \sum_{j=1}^n \lambda_j \delta_j(t) \right|^q \right)^{1/q}$$

differently. First of all we can replace $\delta_j(t)$ by

$$\tilde{\delta}_j(t) := \xi_j(t) - \xi_j'(t) \quad (1 - e^{-2t})^{1/2}$$

with $\xi_j'(t)$ be an independent copy of $\xi_j(t)$. This is just by Jensen inequality and $E \xi_j'(t) = e^{-t}$. Random variables are symmetric and we use the following result.

The following contraction principle is a classical result of Maurey and Pisier (see, e.g., [P, Proposition 3.2]). We spell out a version with explicit constants.

**Theorem 3.8.** Let $(X, \| \cdot \|)$ be a Banach space of cotype $r < \infty$, let $\tilde{\delta}_1, \ldots, \tilde{\delta}_n$ be i.i.d. symmetric random variables, and let $\varepsilon$ be uniformly distributed on $\{-1, 1\}^n$. Then for any $n \geq 1$, $\lambda_1, \ldots, \lambda_n \in X$, and $1 \leq q < \infty$, we have

$$\left( E \left| \sum_{j=1}^n \lambda_j \tilde{\delta}_j \right|^q \right)^{1/q} \leq L_{r,q} \int_0^\infty P\{\tilde{\delta}(t)_1 > s\} \frac{1}{\max(q,r)} ds \left( E \left| \sum_{j=1}^n \lambda_j \varepsilon_j \right|^q \right)^{1/q}$$

with $L_{r,q} = L C_q(X) \max(1, (r/q)^{1/2})$, where $L$ is a universal constant.
In the current situation \( X = \mathbb{R} \), so \( r = 2 \) and \( \max(q, 2) = q \). Notice also that

\[
\int_0^\infty \mathbf{P}\{|\xi_j(t) - \xi_j'(t)| > s\}^{1/q}ds = 2^{1-1/r}(1 - e^{-2t})^{1/q}.
\]

Therefore,

\[
\left( \mathbf{E}_\xi \left| \sum_{j=1}^n \lambda_j \delta_j(t) \right|^{q} \right)^{1/q} \leq C(1 - e^{-2t})^{1/q-1/2} \left( \mathbf{E}_\xi \left| \sum_{j=1}^n \lambda_j \varepsilon_j \right|^{q} \right)^{1/q} \leq \frac{C(q)}{(1 - e^{-2t})^{1/2-1/q}}
\]

by Khintchine inequality and by \( \|\lambda\|_{\ell^2} = 1 \).

We will use this later: if \( q > 2 \) then

\[
\|
\nabla e^{-t}\Delta f(\varepsilon)\|_p \leq \frac{C(q)e^{-t}}{(1 - e^{-2t})^{1-1/q}}\|f\|_p = \frac{C(q)e^{-t}}{(1 - e^{-2t})^{1/p}}\|f\|_p.
\]

(3.3)

Now let us use (3.2) for \( p \geq 2 \) and (3.3) for \( 1 < p < 2 \) to finish the proof. We can consider \( x = e^{-t} \) and write those inequalities as the estimate of \( p \)-th norm of

\[
F_f(x, \varepsilon) := \sum_S x^{|S|} \hat{f}(S) \varepsilon_S, \quad \text{where } f = \sum_S \hat{f}(S) \varepsilon_S, \ 0 \leq x \leq 1 .
\]

We get

\[
\|
\nabla F(x, \cdot)\|_p \leq \frac{|x|}{(1 - x^2)^{1/2}}\|f\|_p, \quad \forall x \in [-1, 1], p \geq 2
\]

(3.4)

and

\[
\|
\nabla F(x, \cdot)\|_p \leq \frac{|x|}{(1 - x^2)^{1/p}}\|f\|_p, \quad \forall x \in [-1, 1], 1 < p < 2 .
\]

(3.5)

We initially have this estimates only for \( 0 \leq x \leq 1 \) but flipping \( x \rightarrow -x \) is absorbed by flipping \( \varepsilon \rightarrow -\varepsilon \). By other methods these estimates were obtained also in [EI1], see (229) and (203) there.

Now we consider an auxiliary domain of the type considered in in [EI1]. Let us fix \( \beta \in (1, 2) \) to be chosen later. Fix \( r > 1 \). Consider lens domain \( \Omega(r) = \{z : |z - i \sqrt{r^2 - 1}| \leq r, |z + i \sqrt{r^2 - 1}| \leq r\} \). Consider

\[
\Omega(r, \beta) := \left( 1 - \frac{1}{d^\beta} \right) \Omega(r).
\]

Let \( G_{\beta, r} \) denote Green’s function with pole at infinity of \( \mathbb{C} \setminus \Omega(r, \beta) \). It is rather easy to see that

\[
G_{\beta, r}(1) \asymp d^{-\beta \frac{2\pi - 2}{2\pi} \arcsin \frac{\sqrt{r^2 - 1}}{r}}, \quad (3.6)
\]
(notice that $2\pi - 2 \arcsin \frac{2\sqrt{p - 1}}{p}$ is the exterior angle for $\Omega(r, \beta)$ at corner points of the lens).

We choose $\beta$ in (3.6) to have $G_{\beta,r}(1) \asymp \frac{1}{d}$, that is
\[
\beta = 2 - \frac{2}{\pi} \arcsin \frac{2\sqrt{p - 1}}{p}.
\] (3.7)

3.1. **Complex variable.** Now consider a new function in the complex domain:
\[
H(z) := \log \| \nabla F(z, \cdot) \|_p.
\]
Notice that this function is subharmonic in the whole $\mathbb{C}$. To see this one should write the norm of the gradient as the supremum over the dual space $L^q(\Omega_n, \ell^2_n)$. Then we will get that $\| \nabla F(z, \cdot) \|_p$ is the supremum over the unit ball of this dual space of the absolute values of linear combinations of $D_j F(z, \varepsilon)$. Each such term is analytic and logarithm of absolute value of linear combination of such terms is subharmonic. The supremum can be interchanged with logarithm and we get that $H(z)$ is subharmonic.

Let us collect properties of $H$. As $f$ is a polynomial of degree $d$, we get that the growth of $H$ at infinity is majorized by $d \log |z|$.

In the other hand, we can always think that $\| f \|_p = 1$, and then we just saw that on the interval $[-1 + \frac{1}{d\beta}, 1 - \frac{1}{d\beta}]$ function $H(x)$ has the estimate:
\[
F(x)/d^{3/2} \leq 1, \quad \text{if } p \geq 2; \quad F(x)/Cd^{3/p} \leq 1, \quad \text{if } 1 < p < 2.
\]

Then, say, $H(z) - \frac{\beta}{2} \log d$ is non positive on $[-1 + \frac{1}{d\beta}, 1 - \frac{1}{d\beta}]$ and is of order $d \log |z|$ at infinity.

But we can say much more by Weissler [We] and Ivanisvili–Nazarov [IN]. It turns out that then $H(z) - \frac{\beta}{2} \log d$ is non positive on $\mathbb{C} \setminus G_{\beta,r}$. These are the complex hypercontractivity results.

Hence, using Green’s function $G_{\beta,r}$ of $\mathbb{C} \setminus G_{\beta,r}$ with pole at infinity we get that
\[
H(z) - \frac{\beta}{2} \log d \leq dG_{\beta,r}(z)
\]
uniformly in $\mathbb{C} \setminus G_{\beta,r}$. Hence,
\[
\frac{\| \nabla F(z, \cdot) \|_p}{d^{3/2}} \leq e^{dG_{\beta,r}(z)}.
\]

We are interested in this inequality for just one particular $z = 1$. Now we use (3.7) to have $e^{dG_{\beta,r}(1)} \asymp 1$. 

Hence we proved that for \( p \geq 2 \)
\[
\frac{\| |\nabla F(1, \cdot)\|_p}{d^{\beta/2}} \leq C.
\]
Exactly the same reasoning shows that for \( 1 < p < 2 \)
\[
\frac{\| |\nabla F(1, \cdot)\|_p}{d^{\beta/p}} \leq C.
\]
Theorem 3.5 is completely proved just by plugging formula (3.7) for \( \beta \).

**Proof.** The proof of theorem 3.7 follows the same lines, but we need to use Theorem 3.8 for Banach spaces \( X^* \). This is where we use that if \( X^* \) is of type 2 then is of finite co-type by König–Tzafriri theorem 7.1.14 in [HVNVW2]. Type 2 is needed to conclude (using Khintchine–Kahane’s inequality, see e.g. [HVNVW2]):

\[
E_\varepsilon \left\| \sum_j \varepsilon_j \lambda_j \right\|_p \leq C_p E_\varepsilon \left\| \sum_j \varepsilon_j \lambda_j \right\|_2 \leq C \left( \sum \|\lambda_j\|_{X^*}^2 \right)^{1/2} \leq C.
\]

\[ \square \]

### 3.2. Comparison of \( \Delta^{1/2} \) and \( |\nabla|_X \)

The reader can notice that in this paper we are mostly interested in bounding the expressions of the type \( |\nabla f|_X \). It would be interesting to get from this the estimates of the expressions of the type \( \Delta^{1/2} f \). But for Banach space valued functions it is mostly an open task.

For Banach space valued functions \( f : \Omega_n \to X \) it is not quite clear who majorized whom if we deal with \( \| \Delta^{1/2} f \|_{L^p(X)} \) and \( \| |\nabla f|_X \|_p \).

We would like to decide for exactly what class of Banach spaces \( X \)
\[
\| \Delta^{1/2} f \|_{L^p(X)} \leq C_p \| |\nabla f|_X \|_p.
\]

We think that this is the class of spaces of finite co-type. In fact, this is one way to express the boundedness from below of Riesz transform of Hamming cube in spaces \( L^p(\Omega_n, X) \), \( 1 < p < \infty \). For \( X = \mathbb{R} \) this boundedness from below is always true, see [BELP].

On the other hand, the converse inequality, that is the boundedness of Riesz transform of Hamming cube from above,
\[
\| |\nabla f|_X \|_p \leq C_p \| \Delta^{1/2} f \|_{L^p(X)}
\]
does not have a reasonably wide class of Banach spaces for which it holds. For \( p \geq 2 \) and \( X = \mathbb{R} \) this boundedness from above holds, but for \( 1 < p < 2 \) it fails, see [BELP].

It can fail for very nice UMD space \( X \) even for \( p > 2 \).
4. Type 2 Banach spaces in Theorem 3.7

By $L^p(X)$ we always mean $L^p(\Omega_n; X)$, where $\Omega_n$ is Hamming cube. Let $1/q + 1/p = 1$. Let $\mathcal{P}(d, X)$ be the collection of polynomials with coefficients in Banach space $X$ and of degree at most $d$. We prove that Theorem 3.7 can be somewhat strengthened in the sense of the power of $d$. Next theorem deals with $X$ such that $X^\ast$ is of type 2. In particular, $X^\ast$ and $X$ are $K$-convex. Let $\pi\alpha$ denote the angle from Theorem 2.1.

**Theorem 4.1.** If $f \in \mathcal{P}(d, X)$ and $X^\ast$ is of type 2. Then $\|\nabla f\|_{L^p(X)} \leq C d^{2\pi\alpha/p} \|f\|_{L^p(X)}$ for $1 < p < 2$, and $\|\nabla f\|_{L^p(X)} \leq C d^{1-\frac{\pi\alpha}{2}} \|f\|_{L^p(X)}$ for $p \geq 2$.

**Proof.** We recall the formula from [IVHV]:

$$D_j e^{-t\Delta} f(\varepsilon) = \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \sum_{t} \frac{\delta_j(t)}{\lambda_j} f(\varepsilon_1 \cdot \xi_1(t), \ldots, \varepsilon_n \cdot \xi_n(t)). \quad (4.1)$$

Here

$$\delta_j(t) := \frac{\xi_j(t) - e^{-t}}{(1 - e^{-2t})^{1/2}},$$

where $\xi_j(t)$ are independent random variables having values $\pm 1$ with probabilities $\tfrac{1 + e^{-t}}{2}$.

The symmetric counterpart is

$$\delta'_j(t) := \frac{\xi_j(t) - \xi'_j(t)}{(1 - e^{-2t})^{1/2}},$$

where vector $\{\xi'_j(t)\}$ is independent copy of $\{\xi_j(t)\}$.

We use the notation $\ell^2_\alpha$ for $\ell^2_\alpha(X^\ast)$ with the norm $(\sum_{j=1}^n \lambda_j^2)^{1/2}$. From (4.1) for every $\varepsilon \in \Omega_n$ we can write

$$|\nabla e^{-t\Delta} f(\varepsilon)| = \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \max_{\lambda: \parallel \lambda \parallel_2^2 = 1} \left( \mathbf{E}_\xi \left\{ \sum_{j=1}^n \delta_j(t) \langle \lambda_j, f(\varepsilon \cdot \xi) \rangle \right\} \right).$$

Hence,

$$|\nabla e^{-t\Delta} f(\varepsilon)| \leq \frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \max_{\lambda: \parallel \lambda \parallel_2^2 = 1} \mathbf{E}_\xi \left\{ \sum_{j=1}^n \delta_j(t) \langle \lambda_j, f(\varepsilon \cdot \xi(t)) \rangle \right\} =$$

$$\frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \max_{\lambda: \parallel \lambda \parallel_2^2 = 1} \left( \mathbf{E}_\xi \left\{ \sum_{j=1}^n \delta_j(t) \lambda_j, f(\varepsilon \cdot \xi) \right\} \right) \leq$$

$$\frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \max_{\lambda: \parallel \lambda \parallel_2^2 = 1} \mathbf{E}_\xi \left( \left\| \sum_{j=1}^n \delta_j(t) \lambda_j \right\|_{X^\ast}, \|f(\varepsilon \cdot \xi)\|_X \right) \leq$$
\[
\frac{e^{-t}}{(1 - e^{-2t})^{1/2}} \max_{\lambda : |\lambda|_{\ell^q} = 1} \left( \mathbb{E}_\xi \left\| \sum_{j=1}^n \lambda_j \delta_j(t) \right\|^q \right)^{1/q} \left( \mathbb{E}_\xi \| f(\varepsilon \cdot \xi) \|^p \right)^{1/p}.
\]

We wish to prove that if \( q > 2 \) then
\[
\|\nabla e^{-t} \Delta f(\varepsilon)\|_p \leq \frac{C(q)e^{-t}}{(1 - e^{-2t})^{1/2}} \| f \|_{L^p(X)}, \tag{4.2}
\]
and if \( 1 \leq q \leq 2 \) then
\[
\|\nabla e^{-t} \Delta f(\varepsilon)\|_p \leq \frac{C(q)e^{-t}}{(1 - e^{-2t})^{1/2}} \| f \|_{L^p(X)}, \tag{4.3}
\]

For that let us work now with the term \( \mathbb{E}_\xi \left\| \sum_{j=1}^n \lambda_j \delta_j(t) \right\|_{X^*}^q \) for a fixed \( \{\lambda_j\} \in \ell^2_n \) of norm 1.

\[
B^q := \mathbb{E}_\xi \left\| \sum_{j=1}^n \lambda_j \delta_j(t) \right\|_{X^*}^q \leq \mathbb{E}_\xi,\xi' \left\| \sum_{j=1}^n \lambda_j \delta_j'(t) \right\|_{X^*}^q = \mathbb{E}_{\xi,\xi'} \mathbb{E}_r \left\| \sum_{j=1}^n r_i \lambda_j \delta_j'(t) \right\|_{X^*}^q,
\]
where \( r_i \) are independent Rademacher random variables.

The next lemma was provided by A. Borichev.

**Lemma 4.2.** Let \( a, b \geq 0 \) and \( Q \geq 2 \) be a large number. Then
\[
(a + b)^Q \leq 6a^Q + Q^2b^Q.
\]

**Proof.** We need to show that for all positive \( t \), \( (t + 1)^Q \leq 6t^Q + Q^2 \). If \( t \leq Q - 1 \) this is immediate. If \( t \geq Q - 1 \geq 1 \), we write
\[
(t + 1)^Q = t^Q \left( \frac{t + 1}{t} \right)^Q \leq t^Q \left( \frac{t + 1}{t} \right)^{t+1} \leq 2t^Q \left( 1 + \frac{1}{t} \right)^t \leq 2et^Q.
\]

We continue to estimate \( B^q \):
\[
B^q \leq \mathbb{E}_{\xi,\xi'} \mathbb{E}_r \left\| \sum_{j=1}^n r_i \lambda_j \delta_j'(t) \right\|_{X^*}^q \leq C_q \mathbb{E}_{\xi,\xi'} \left( \mathbb{E}_r \left\| \sum_{j=1}^n r_i \lambda_j \delta_j'(t) \right\|^2_{X^*} \right)^{q/2} \leq C_q D^{q/2} \mathbb{E}_{\xi,\xi'} \left( \sum_{j=1}^n |\delta_j'(t)|^2 \left\| \lambda_j \right\|_{X^*}^2 \right)^{q/2}.
\]

In the last inequality we used that \( X^* \) is of type 2. The penultimate inequality is Kahane–Khintchine’s inequality, see [HVNVW2].
Notice that if $1 \leq q \leq 2$ then the above inequality gives

$$B^q \leq C_q D^{q/2} \left( \mathbb{E}_{\xi, \xi'} \sum_{j=1}^n \left| \delta_j'(t) \right|^q \| \lambda_j \|_{X^*}^2 \right)^{q/2}$$

Hence,

$$1 \leq q \leq 2 \Rightarrow B \leq C_q D^{q/2} \left( \sum_{j=1}^n \| \lambda_j \|_{X^*}^2 \right)^{1/2} \leq C_q D^{q/2}. \quad (4.4)$$

The estimate in case $2 < q < \infty$ is much more interesting.

Now we will continue by thinking that $q$ is an even integer, $q = 2k$ (it is not important, just convenient). Let us now estimate

$$E := \mathbb{E}_{\xi, \xi'} \left( \sum_{j=1}^n \left| \delta_j'(t) \right|^2 \| \lambda_j \|_2^2 \right)^k \quad (4.5)$$

We denote

$$f_j := \left| \delta_j'(t) \right|^2 \| \lambda_j \|_2^2. \quad (4.6)$$

Below we use Lemma 4.2 with $Q := q/2 - 1 = k - 1$:

$$\mathbb{E}_{\xi, \xi'} \left( \sum_{j=1}^n f_j \right)^k = \mathbb{E}_{\xi, \xi'} \sum_{i=1}^n (\sum_{j=1}^n f_j)^{k-1} f_i =$$

$$(k-1)^{k-1} \mathbb{E}_{\xi, \xi'} \sum_{i=1}^n f_i^k + 6 \mathbb{E}_{\xi, \xi'} \left( \sum_{j=1}^n (f_1 + \ldots + f_i + f_{i+1} + \ldots + f_n) \right)^{k-1} f_i \leq$$

$$(k-1)^{k-1} \mathbb{E}_{\xi, \xi'} \sum_{i=1}^n f_i^k + 6 \mathbb{E}_{\xi, \xi'} (\sum_{j=1}^n f_j)^{k-1} \mathbb{E}_{\xi, \xi'} (\sum_{j=1}^n f_j) \leq$$

$$(k-1)^{k-1} \mathbb{E}_{\xi, \xi'} \sum_{i=1}^n f_i^k + 6(k-2)^{k-2} \mathbb{E}_{\xi, \xi'} \sum_{i=1}^n f_i^{k-1} \mathbb{E}_{\xi, \xi'} (\sum_{j=1}^n f_j) +$$

$$6^k \mathbb{E}_{\xi, \xi'} (\sum_{j=1}^n f_j)^2 \left[ \mathbb{E}_{\xi, \xi'} (\sum_{j=1}^n f_j) \right]^2 \leq$$

$$(k-1)^{k-1} \mathbb{E}_{\xi, \xi'} \sum_{i=1}^n f_i^k + \ldots + 6^k (k-\ell)^{k-\ell} \mathbb{E}_{\xi, \xi'} (\sum_{j=1}^n f_j^{k-\ell}) \left[ \mathbb{E}_{\xi, \xi'} (\sum_{j=1}^n f_j)^\ell \right] + \ldots +$$

$$+ 6^{k-1} \left[ \mathbb{E}_{\xi, \xi'} (\sum_{j=1}^n f_j) \right]^k.$$

We used the fact that $\delta_j'(t)$, $j = 1, \ldots, n$, are independent exactly as this has been done in Rosenthal’s [HR].

Now coming back to our notation (4.6) we see that as

$$\mathbb{E} |\delta_j'(t)|^2 = 2, \quad \mathbb{E} |\delta_j'(t)|^m \leq \frac{2^{m-1}}{\sqrt{1 - e^{-2t^m}}}. \quad (4.7)$$

$$\mathbb{E}_{\xi, \xi'} \left( \sum_{j=1}^n f_j \right) \leq 2 \| \lambda_j \|_{X^*}^2 \| \lambda_j \|_{X^*}^2,$$
\[ E_{\xi, \xi'}(\sum_{j=1}^{n} f_{j}^{k-\ell}) \leq \frac{2^{2k-2\ell}}{\sqrt{1 - e^{-2t}2k-2\ell}} \sum_{j=1}^{n} \|\lambda_{j}\|^{2k-2\ell}. \]

Therefore, we can estimate \( E \) from (4.5) as follows:

\[ E \leq 24k \sum_{\ell=0}^{k-2} \frac{(k - \ell)^{k-\ell}}{\sqrt{1 - e^{-2t}2k-2\ell}} \sum_{j=1}^{n} \|\lambda_{j}\|^{2k-2\ell} \{\{\lambda_{j}\}\}^{2\ell} + 24k \|\{\lambda_{j}\}\|^{2k}. \]

This obviously gives

\[ E \leq 2(24)^{k} \sum_{\ell=0}^{k-2} (k - \ell)^{k-\ell} \left( \frac{1}{\sqrt{1 - e^{-2t}2k-2\ell}} \right)^{2k-2\ell-2} \left( \sum_{j=1}^{n} \|\lambda_{j}\|^{2} \right)^{k-\ell} \|\{\lambda_{j}\}\|^{2k-2} + 24k \|\{\lambda_{j}\}\|^{2k}. \]

And so,

\[ E \leq C'(q)\|\{\lambda_{j}\}\|^{q} \sum_{\ell=0}^{k-2} (k - \ell)^{k-\ell} \left( \frac{1}{\sqrt{1 - e^{-2t}2k-2\ell}} \right)^{2k-2\ell-2} \left( \sum_{j=1}^{n} \|\lambda_{j}\|^{2} \right)^{k-\ell} \|\{\lambda_{j}\}\|^{2k-2} + 24k \|\{\lambda_{j}\}\|^{2k}. \]

Then

\[ B \leq \frac{C(q)}{(1 - e^{-2t})^{\frac{1}{4} - \frac{q}{4}}} \|\{\lambda_{j}\}\|_{L^{q}(X^{*})} = \frac{C(q)}{(1 - e^{-2t})^{\frac{1}{4} - \frac{q}{4}}}. \quad (4.8) \]

Now let us use (4.2) for \( 1 < p < 2 \) and (4.3) for \( p \geq 2 \) to finish the proof. We can consider \( x = e^{-t} \) and write those inequalities as the estimate of \( p \)-th norm of

\[ F_{f}(x, \varepsilon) := \sum_{S} x^{\text{S}} \hat{f}(S) \varepsilon_{S}, \quad \text{where } f = \sum_{S} \hat{f}(S) \varepsilon_{S}, \quad 0 \leq x \leq 1. \]

We get from (4.8) and (4.4) correspondingly that

\[ |||\nabla F(x, \cdot)|||_{p} \leq \frac{|x|}{(1 - x^2)^{1/p}} \|f\|_{L^{p}(X)}, \quad \forall x \in [-1, 1], 1 < p < 2. \quad (4.9) \]

\[ |||\nabla F(x, \cdot)|||_{p} \leq \frac{|x|}{(1 - x^2)^{1/2}} \|f\|_{L^{p}(X)}, \quad \forall x \in [-1, 1], p \geq 2. \quad (4.10) \]

We initially have this estimates only for \( 0 \leq x \leq 1 \) but flipping \( x \rightarrow -x \) is absorbed by flipping \( \varepsilon \rightarrow -\varepsilon \).

Now consider a new function in the complex domain:

\[ H(z) := \log \|\nabla F(z, \cdot)|_{X}\|_{p}. \]
We repeat verbatim the reasoning of Section 3.1 but instead of domain \( \Omega \setminus \Omega(\beta, r) \) and its Green’s function, we consider domain \( C \setminus [-1 + \frac{1}{d^\beta}, 1 - \frac{1}{d^\beta}] \), whose Green’s function \( G_d \) satisfies

\[
G_d(1) \asymp \frac{1}{d}.
\]

This proves

\[
\left\| \frac{\| \nabla F(1, \cdot) \|_X}{d^{\max(2/p, 1)}} \right\|_p \leq C.
\]

But (4.8) and (4.4) can be used more efficiently if we use Pisier’s Theorem 2.1 again. In fact, it can be used. As \( X^* \) has type 2, it is \( K \)-convex. Then \( X \) is \( K \)-convex. Let us fix \( \beta \) to be chosen later and consider domain

\[
O_{\beta, \alpha} := \left( 1 - \frac{1}{d^\beta} \right) O_\alpha,
\]

where \( O_\alpha \) was introduced in the previous Section.

As \( X \) is \( K \)-concave, so is \( \ell^2(X) \). Consequently (4.8) and (4.4) and Pisier’s Theorem 2.1 applied to \( L^p(\Omega_n, \ell^2(X)) \) show that

\[
\left\| \nabla F(z, \cdot) \right\|_X \leq C d^{\beta/p}, \quad 1 < p < 2, \quad \leq C d^{\beta/2}, \quad p \geq 2, \quad z \in O_{\beta, \alpha}.
\]

Let \( G_{\beta, \alpha} \) denote Green’s function of \( C \setminus \bar{O}_{\beta, \alpha} \).

We repeat verbatim the reasoning of Section 3.1 but instead of domain \( \Omega \setminus \Omega(\beta, r) \) and its Green’s function, we consider domain \( C \setminus \bar{O}_{\beta, \alpha} \) whose Green’s function \( G_{\beta, \alpha} \) satisfies

\[
G_{\beta, \alpha}(1) \asymp \left( \frac{1}{d^\beta} \right)^{\frac{\pi}{2\pi - \pi\alpha}} \asymp \frac{1}{d},
\]

if

\[
\beta = 2 - \alpha.
\]

This proves

\[
\left\| \frac{\| \nabla F(1, \cdot) \|_X}{d^{\max(2/p, \frac{2\pi}{2\pi - \pi\alpha})}} \right\|_p \leq C.
\]

Theorem 4.1 is proved.

**Remark 4.3.** We already mentioned in Section 3.2 that the boundedness of Riesz transform of Hamming cube from above,

\[
\left\| \nabla f \right\|_p \leq C_p \| \Delta^{1/2} f \|_{L^p(X)}
\]

does not have a reasonably wide class of Banach spaces for which it holds. For \( p \geq 2 \) and \( X = \mathbb{R} \) this boundedness from above holds, but for \( 1 < p < 2 \) it fails, see [BELP]. It can fail for very nice UMD
space $X$ even for $p > 2$. Therefore, Theorem 3.3 or other Bernstein–
Markov type estimates of $\Delta^{1/2} f$ in [EI1] for $X$-valued polynomials $f$
with $X$ being $K$-convex cannot help to prove the estimates of the type
of Theorem 4.1 or Theorem 3.7.

5. Non-commutative random variables and
Bernstein–Markov inequalities on Hamming cube

We wish to demonstrate how the technique of non-commutative ran-
dom variables can be used to prove certain Bernstein–Markov inequal-
ities on Hamming cube. The estimates below are not as good as in the
previous Section, and what follows serves only illustrative purpose of
showing a beautiful approach.

To the best of our knowledge this approach was introduced by Fran-
coise Lust-Piquard in [FLP], [BELP]. All results of those papers are
commutative, all methods are non-commutative. And even though
many non-commutative proofs of those papers are by now made com-
mutative (see, e.g. [ILVHV]), still some non-commutative proofs did
not get a commutative analogs up to now.

The non-commutative proof of a certain Bernstein–Markov inequa-
licity below is given not because of its efficiency, but because of its beau-
ty.

We will prove now that for $p \geq 2$

$$f : \Omega_n \to \mathbb{R}, \deg f \leq d \Rightarrow \|\nabla f\|_p \leq C_p d \|f\|_p,$$

which is worse than Theorem 3.5.

Let

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}, \quad U = iPQ,$$

They have anti-commutative relationship

$$QP = -PQ.$$  

Let $Q_j = I \otimes \ldots \otimes Q \otimes I \cdots \otimes I$, $P_j = I \otimes \ldots \otimes P \otimes I \cdots \otimes I$, on $j$-th place. These are independent non-commutative random variables in the sense of $tr = \text{sum of diagonal elements divided by } 2^n$.

Put $Q_A = \Pi_{i \in A} Q_i$, $P_A = \Pi_{i \in A} P_i$.

Now one considers algebra generated by $Q_j, P_j$ (this is algebra of all
matrices $M_{2^n}$). We have a projection $P$ from multi-linear polynomials
in $P_j, Q_j$ (notice $P^2 = I, Q^2 = I$) that kills everything except terms
having only $Q$’s.

Small (really easy) algebra shows (see [BELP]) that $P$ can be written
as $\rho \text{Diag} \rho^*$, where $\rho$ is a conjugation by a unitary operator, and $\text{Diag}$, is an operator on matrices that just kills all matrix elements except
the diagonal. This $\text{Diag}$ is obviously the contraction on Schatten-von Neumann class $S_p$ for any $p \in [1, \infty]$ (obvious for Hilbert-Schmidt, $p = 2$, class and for bounded operators—so interpolation does that).

$$\mathcal{R}(\theta)Q_A = \Pi_{j \in A}(Q_j \cos \theta + P_j \sin \theta), \quad \mathcal{R}(\theta)P_A = \Pi_{j \in A}(P_j \cos \theta - Q_j \sin \theta).$$

One can easily check that the action of $\mathcal{R}(\theta)$ is $\mathcal{R}(\theta)^* \mathcal{T} R(\theta)$ where $R(\theta)$ is a unitary matrix which is $n$-fold tensor product of

$$\rho_\theta = \begin{bmatrix} 1 & 0 \\ 0 & e^{i \theta} \end{bmatrix}$$

Extend it by linearity onto the whole algebra $\mathcal{M}_{2^n}$. Then it is obvious that automorphism $\mathcal{R}(\theta)$ preserves all Schatten-von Neumann $S_p$ norms.

For any $f = \sum_{A \subset [n]} \hat{f}(A) \varepsilon_A$, the reasoning of $[\text{BELP}]$ dictates to assign a non-commutative object, a matrix from $\mathcal{M}_{2^n}$ given by

$$T_f = \sum_{A \subset [n]} \hat{f}(A) Q_A.$$

Such matrices form commutative sub-algebra $M_{2^n} \subset \mathcal{M}_{2^n}$. Operators $\partial_j, D_j$ can be considered on $M_{2^n}$, acting in a canonical way. For example,

$$\partial_i Q_A = \begin{cases} Q_{A \setminus i}, & \text{if } i \in A; \\ 0, & \text{if } i \notin A. \end{cases}$$

And $D_i := \varepsilon_i \partial_i$.

Consider now a matrix valued function

$$A_f(\theta) = \mathcal{R}(\theta) T_f.$$

It is a trigonometric polynomial of degree at most $d$ with matrix coefficients. Bernstein–Markov inequality (its proof) works for such matrix valued polynomials in exactly the same way as for scalar polynomials. The easiest way to see that is to prove Bernstein–Markov estimate by convolution with Fejer kernels. Then we get

$$\left\| \frac{d}{d \theta} A_f(\theta) \right\|_{S_p} \leq 2d \left\| A_f(\theta) \right\|_{S_p}, \quad 1 \leq p \leq \infty.$$

On the other hand we can calculate easily $\frac{d}{d \theta} \mathcal{R}(\theta)(Q_A) = - \sum_{j \in A} \Pi_{i \in A, i < j} (\cos \theta Q_i + \sin \theta P_i) (\sin \theta Q_j + \cos \theta P_j) \Pi_{i \in A, i > j} (\cos \theta Q_i + \sin \theta P_i).$
By commutativity relations between $P_j, Q_i$, we observe that this is nothing else but $\mathcal{R}(\theta)(P_j \partial_j Q_A)$. Hence

$$
\frac{d}{d\theta} A_f(\theta) = \frac{d}{d\theta} \mathcal{R}(\theta) T_f = -\mathcal{R}(\theta) \left( \sum_{j=1}^{n} P_j \partial_j T_f \right).
$$

(5.3)

Therefore,

$$
\| \mathcal{R}(\theta) \left( \sum_{j=1}^{n} P_j \partial_j T_f \right) \|_{s_p} \leq 2d \| \mathcal{R}(\theta) T_f \|_{s_p}.
$$

Transformation $\mathcal{R}_\theta$ preserves $S_p$ norms (see above), and so

$$
\| \sum_{j=1}^{n} P_j \partial_j T_f \|_{s_p} \leq 2d \| T_f \|_{s_p}.
$$

(5.4)

Let $\varepsilon_j^{(k)} = -1$ if $j = k$ and = 1 otherwise. Following [BELP] we see that $\| \sum_{j=1}^{n} P_j \partial_j T_f \|_{s_p} = \| \sum_{j=1}^{n} \varepsilon_j^{(k)} P_j \partial_j T_f \|_{s_p}$. This is because

$$
Q_k \left( \sum_{j=1}^{n} P_j \partial_j \right) Q_k = \sum_{j=1}^{n} \varepsilon_j^{(k)} P_j \partial_j
$$

by anti-commutative relation $PQ = -QP$. Hence, for any sequence of signs

$$
\| \sum_{j=1}^{n} P_j \partial_j T_f \|_{s_p} = \| \sum_{j=1}^{n} \varepsilon_j P_j \partial_j T_f \|_{s_p}.
$$

Now one should use a non-commutative Khintchine inequality of Lust-Piquard and Pisier [FLPGP] and $2 \leq p \leq \infty$:

$$
E \varepsilon \left\| \sum_{j=1}^{n} \varepsilon_j P_j \partial_j T_f \right\|_{s_p} \leq_p \left\| \left( \sum_{j=1}^{n} (\partial_j T_f)^* P_j^* P_j \partial_j T_f \right)^{1/2} \right\|_{s_p} +

\left( \sum_{j=1}^{n} P_j \partial_j T_f (\partial_j T_f)^* P_j^* \right)^{1/2} \right\|_{s_p}.
$$

But $P_j^* P_j = P_j^2 = I$, and in the second term $P_j$ and $\partial_j T_f$ commute (as there is identity matrix on the $j$-th place of $\partial_j T_f$). Therefore

$$
\left\| \left( \sum_{j=1}^{n} (\partial_j T_f)^* P_j^* P_j \partial_j T_f \right)^{1/2} \right\|_{s_p} + \left( \sum_{j=1}^{n} P_j \partial_j T_f (\partial_j T_f)^* P_j^* \right)^{1/2} \right\|_{s_p} =

2 \left\| \left( \sum_{j=1}^{n} (\partial_j T_f)^* \partial_j T_f \right)^{1/2} \right\|_{s_p}.
$$
Using (5.4) we conclude that for $p \in [2, \infty)$

$$
\| \left( \sum_{j=1}^{n} (\partial_j T_f)^* \partial_j T_f \right)^{1/2} \|_{S_p} \leq C_p d \| T_f \|_{S_p}.
$$

Both matrices in the left hand side and the right hand side are form commutative algebra $M_n$. They are $T_f$ and $T_{|\nabla f|}$. It is left to notice that for any scalar function $f$ on $\Omega_n$ we have $\| T_f \|_{S_p} = \| f \|_{L^p(\Omega_n)}$. This is just by using the basis of characteristic function of point sets $\{ \varepsilon \}$ on $\Omega_n$ to compute the $S_p$ norm of $T_f$. This basis consist of eigenfunctions of $T_f$ with eigenvalues $f(\varepsilon)$. This is easy, see in [BELP].

We finally proved (5.1) by non-commutative approach of Francoise Lust-Piquard.

6. Addendum 1: Fourier coefficients of conformal map $\varphi$

We consider the domain

$$
O_\alpha := -G_\alpha \cup G_\alpha,
$$

where $G_\alpha = \{ w : w = e^{-z}, | \arg z | \leq \frac{\pi \alpha}{2} \}$. It is not very difficult to write down the boundary of this domain (see Section 7 below, where we partially do this). Then one can notice that it consists of two real analytic curve $\Gamma_+, \Gamma_-$, symmetric with respect to $\mathbb{R}$ and forming angle $\pi \alpha$ at $-1, 1$.

Hence, the conformal map $\varphi^{-1} : O_\alpha \to \mathbb{D}$ can be extended to a slightly wider domain bounded by real analytic curves $\gamma_+, \gamma_-$, such that $\gamma_+$ lies a bit higher than $\Gamma_+$ and meets $\Gamma_+$ at $\pm 1$, and forms angle $\tau \pi$ with $\Gamma_+$ at points $\pm 1$, where $\tau$ is a small strictly positive number. Symmetrically for $\gamma_-, \Gamma_-.$

Then conformal map $\varphi$ is extended to domain $\mathcal{R}$ bounded by two symmetric real analytic curves, intersecting $\mathbf{T}$ only at $\pm 1$ and making angle $(\alpha + \tau) \pi$ with $\mathbf{T}$ at those points.

Then

$$
c_m = \int_{\mathbf{T}} \frac{1}{z^{m+1}} \varphi(z) dz = -\frac{1}{m} \int_{\partial\mathcal{R}} \varphi(z) d\frac{1}{z^m} = \frac{1}{m} \int_{\partial\mathcal{R}} \frac{1}{z^m} \varphi'(z) dz.
$$

Now we us that on $\partial\mathcal{R}, \pm 1$ we have $|\frac{1}{z}| \leq \frac{1}{1+a(\tau)|y|}$ for $z = x + iy$. We get

$$
|c_m| \lesssim \int_0^2 \frac{1}{(1 + ay)^m y^{1-\alpha}} dy \lesssim \int_0^2 e^{-a_1 m y} dy^\alpha = \int_0^{2^\alpha} e^{-b(m \alpha t)^{1/\alpha}} dt.
$$

The last integral is $\leq \frac{1}{m^\alpha} \int_0^\infty e^{-bs^{1/a}} ds \lesssim \frac{1}{m^\alpha}$. 
7. **Addendum 2: boundary of $O_\alpha$ and getting rid of $\varepsilon$ in the proof of Theorem 6 of [EI1]**

We consider two domains

$$
\Omega(r) := \{ z \in \mathbb{C} : \max\{|z - i\sqrt{r^2 - 1}|, |z + i\sqrt{r^2 - 1}|\} < r \},
$$

and

$$
O_\alpha := -G_\alpha \cup G_\alpha,
$$

where $G_\alpha = \{ w : w = e^{-z}, |\arg z| \leq \frac{\pi \alpha}{2} \}$. We would like to compare those two domains for

$$
\pi \alpha = 2 \arcsin \frac{1}{r}.
$$

The choice of $r$ is dictated by the fact that for this choice the angle that the boundaries have at point 1 are the same (and symmetrically at $-1$).

It is not very difficult to write down the boundary of $O_\alpha$, we will do this now for its parts near points $\pm 1$.

Define $a$ as follows $\tan \frac{\pi \alpha}{2} = \frac{\pi}{a}$. Let us consider $G_\alpha \cap \{\Re z \in [0, \frac{\pi}{2}]\}$. Consider $G_\alpha(a/2) := e^{-G_\alpha \cap \{\Re z \in [0, \frac{\pi}{2}]\}} = \{ w = u + iv = e^{-z}, z \in G_\alpha \cap \Re z \in [0, \frac{\pi}{2}] \}$. It consists of arcs $S_t$ of the circles centered at point $(0, 0)$ of radii $e^{-t}, 0 \leq t \leq \frac{\pi}{2}$, and each arc is symmetric (w.r. to $R$), and has angle $2 \arctan \frac{\pi}{a} t$. In particular, $S_{a/2}$ is a half-circle that intercepts $v$-axis at points $\pm e^{-a/2}$. The boundary of the domain $G_\alpha(a/2)$ consists of $S_{a/2}$ and of two real analytic symmetric (w.r. to $R$) arcs, one of them $\Gamma(a/2)$ (the one in $C_+$) being given by parametric equation:

$$
\Gamma(a/2) : \quad u = e^{-t} \cos \frac{\pi}{a} t, \quad v = e^{-t} \sin \frac{\pi}{a} t, \quad 0 \leq t \leq a/2.
$$

We also have an interesting circle of radius $r := \sqrt{1 + \frac{a^2}{\pi^2} - \frac{1}{\sin(\frac{\pi \alpha}{2})}}$.

Let us check that $\Gamma(a/2)$ lies below the circle, in other words that

$$
\left( e^{-t} \cos \frac{\pi}{a} t \right)^2 + \left( e^{-t} \sin \frac{\pi}{a} t + \frac{a}{\pi} \right)^2 < 1 + \frac{a^2}{\pi^2}, \text{ for small } t > 0.
$$

This is the same as

$$
e^{-2t} + 2e^{-t} \frac{a}{\pi} \sin \frac{\pi}{a} t < 1.
$$

We write

$$1 - 2t + \frac{4t^2}{2} - \frac{8t^3}{6} + \cdots + 2(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \ldots)(t - ct^3 + \ldots) =$$
1 + t^3 - \frac{4}{3} t^3 - 2ct^3 + \cdots < 1,
if \( t \) is small as \( c \) is positive. So the lens domain \( \Omega(r) \) of [EI1] seems to contain \( O_\alpha \), at least it is not contained in it as \( \Gamma(a/2) \) lies inside \( \Omega(r) \).

That represents a small problem for [EI1] because inclusion (103) there is not valid if one chooses \( r \) according to our preferred choice (7.1). In its turn this is reflected in the formulas for conformal mapping one uses around (103). But the formula for conformal mapping of the unit disc onto \( \Omega(r) \) is straightforward.

But if one chooses \( r \) not according to (7.1) but smaller, than the angle of the lens domain at \( \pm 1 \) is smaller than \( \pi\alpha \) and inclusion (103) holds. Thus Theorem 6 of [EI1] reproves the heat smoothing result of [MN] with \( A(p, X) > \frac{1}{ \alpha} \), where \( \alpha \) is the angle from Pisier’s Theorem 2.1. But one can notice that just a small improvement in [EI1] reasoning gives the heat smoothing result with \( A(p, X) = \frac{1}{ \alpha} \).

Let us indicate this small change that should be implemented to get \( A(p, X) = \frac{1}{ \alpha} \) in Theorem 6 of [EI1].

As, in the contrast to (103) of [EI1], we have \( \Omega_\alpha \subset \Omega(r) \) with \( r \) as in (7.1), then one need the estimates of conformal mapping of the disk onto \( O_\alpha \) (the smaller of two domains). Of course the angle that boundary of \( O_\alpha \) form at point \( 1 \) (and \( -1 \)) is just \( \pi\alpha \) (in notations of [EI1] it is \( \theta \)). This angle is the same for \( \Omega(r) \). But this observation is not enough to conclude the same asymptotic for conformal maps on these two domains.

However, this is a not a real problem. It is easy to see that asymptotic is in fact the same. To see that one transforms \( \Omega_\alpha \) and \( \Omega(r) \) to strips by logarithmic map and then one uses Warschawski’s estimate from [W]. It shows that asymptotic is the same because one can easily compute that \( \int_\infty^\infty \Theta'(u)^2/\Theta(u) \, du \) converges, see [W] for the explanation what is \( \Theta(u) \) for strips.

The heat smoothing conjecture of [MN] claims that \( A(p, X) = 1 \) for \( K \)-convex \( X \), but it is still a conjecture. The important time is \( t_0 = \frac{1}{d\alpha} \). The estimate of Theorem 2.5, or slightly strengthened estimate of Theorem 6 of [EI1] or Theorem 5.1 of [MN], all those estimates show that if \( X \) is \( K \) convex, then for \( X \)-valued \( f \) in the \( d \)-tail space

\[
\|e^{-t_0 \Delta} f\|_{L^p(X)} \leq C \|f\|_{L^p(X)}.
\]

This does not give us any interesting information. What the heat smoothing conjecture basically says is the following, let \( \alpha \) be the angle
from Pisier’s Theorem 2.1, then
\[ t_0 = \frac{1}{d^\alpha} \Rightarrow \| e^{-t_0 \Delta} f \|_{L^p(X)} \leq \varepsilon(d) \| f \|_{L^p(X)}, \quad \varepsilon(d) \to 0, \ d \to \infty . \]
This is still open.

References

[BELP] L. Ben-Efraim and F. Lust-Piquard, *Poincaré type inequalities on the discrete cube and in the CAR algebra*. Probab. Theory Related Fields, 141(3-4) 2008 569–602.

[CPR] M. D. Contreras; J. A. Peláez; Ch. Pommerenke; J. Rätyyä, *Integral operators mapping into the space of bounded analytic functions*. J. Funct. Anal. 271 (2016), no. 10, 2899–2943.

[EI] A. Eskenazis, P. Ivanisvili, *Dimension independent Bernstein–Markov inequalities in Gauss space*, arXiv:1808.01273v2, pp. 1–15.

[EII] A. Eskenazis, P. Ivanisvili, *Polynomial inequalities on the Hamming cube*, Probability Theory and Related Fields (2020) 178:235–287 https://doi.org/10.1007/s00440-020-00973-y.

[FLP] F. Lust-Piquard, *Riesz Transforms Associated with the Number Operator on the Walsh System and the Fermions*, Journal of functional analysis 155, 1998, 263–285.

[FLPGP] F. Lust-Piquard, G. Pisier, *Non commutative Khintchine and Paley inequalities*. Arkiv för Math. (1991) 29, 241–260.

[HN] T. Hytönen, A. Naor, *Pisier’s inequality revisited*, Studia Math., 215 (2013), no. 3, 221–235.

[HNVW] T. Hytönen, J. Van Neerven, M. Veraar, L. Weiss, *Analysis in Banach spaces*, vol. I, Martingales and Littlewood–Paley theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, v. 63, Springer, 2016.

[HNVW2] T. Hytönen, J. Van Neerven, M. Veraar, L. Weiss, *Analysis in Banach spaces*, vol. II, Ergebnisse der Mathematik und ihrer Grenzgebiete, v. 67, Springer, 2017.

[IVHV] P. Ivanisvili, R. van Handel, A. Volberg, *Rademacher type and Enflo type coincide*. Annals of Math., (2) 192 (2020), no. 2, 665–678.

[ILVHV] P. Ivanisvili, D. Li, R. van Handel, A. Volberg, *Improving constant in end-point Poincaré inequality on Hamming cube*, arXiv:1811.05584v5, pp. 1-24, https://doi.org/10.48550/arXiv.1811.05584.

[IN] P. Ivanisvili, F. Nazarov, *On Weissler’s conjecture on the Hamming cube*, at arXiv:1907.11359

[MN] M. Mendel, A. Naor, *Nonlinear spectral calculus and superexpanders*, Publ. Math. Inst. Hautes Études Sci. 119 (2014), 1–95.

[RM] R. Matveev, Personal communication.

[N] A. Naor, *An introduction to the Ribe program*. Jpn. J. Math., 7(2):167–233, 2012.

[P1] G. Pisier, *Holomorphic semigroups and the geometry of Banach spaces*. Ann. of Math. 115 (1982), 375–392.

[P] G. Pisier, *Probabilistic methods in the geometry of Banach spaces*. In Probability and analysis (Varenna, 1985), volume 1206 of Lecture Notes in Math., pages 167–241. Springer, Berlin, 1986.
[Po] Ch. Pommerenke, Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation, Comment. Math. Helv. 52 (1977) 591–602.

[HR] H. Rosenthal, On the subspaces of $L^p (p > 2)$ spanned by sequences of independent random variables. Israel J. Math. 8 (1970), 273–303.

[SSV] W. Smith; D. M. Stolyarov; A. Volberg, Uniform approximation of Bloch functions and the boundedness of the integration operator on $H^\infty$. Adv. Math. 314 (2017), 185–202.

[W] S. E. Warschawski, On conformal mapping of infinite strips, Trans. Amer. Math. Soc. 51 (1942), 280–335.

[We] F. B. Weissler, Two-point inequalities, the Hermite semigroup, and the Gauss–Weierstrass semigroup. J. Funct. Anal. 32(1), 102–121 (1979). https://doi.org/10.1016/0022-1236(79)90080-6

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