Letter

Controlling the quantum speed limit time for unital maps via filtering operations

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Abstract

The minimum time a system needs to change from an initial state to a final orthogonal state is called the quantum speed limit time. The quantum speed limit time can be used to quantify the speed of the quantum evolution. The speed of the quantum evolution will increase, if the quantum speed limit time decreases. In this work, we will use a relative purity-based bound for the quantum speed limit time. This work is applicable for any arbitrary initial state. Here, we investigate the effects of a filtering operation on the quantum speed limit time. It will be observed that for some intervals of the filtering operation parameter, the quantum speed limit time is decreased by increasing the filtering operation parameter and for some other intervals it is decreased by decreasing the filtering operation parameter.

Keywords: quantum speed limit time, filtering operation, unital maps, unital maps

(Some figures may appear in colour only in the online journal)

1. Introduction

The dynamic speed of a quantum evolution is one of the most important concepts in quantum theory. It also has wide applications in quantum theory, such as quantum communication [1, 2], quantum metrology [3–5], optimal control [6] and so on. Quantum theory sets a bound on the speed of the evolution of quantum systems. The minimal time it takes for a quantum system to transform from an initial state to a target state is known as the quantum speed limit (QSL) time. The QSL time determines the maximum speed of dynamic evolution. There have been many attempts to introduce a comprehensive bound for the QSL time. In reference [7], Mandelstam and Tamm introduced a bound for QSL time in closed quantum systems, which reads

\[ \tau \geq \tau_{QSL} = \frac{\pi \hbar}{2\Delta E}, \]

where \( \Delta E = \sqrt{\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2} \) is the variation of the energy of the initial state and \( \hat{H} \) is a time-independent Hamiltonian describing the dynamics of the quantum system. This bound is called the Mandelstam—Tamm (MT) bound. In reference [8], Margolus and Levitin introduced the bound for a closed quantum system based on the mean energy \( E = \langle \hat{H} \rangle \) as

\[ \tau \geq \tau_{QSL} = \frac{\pi \hbar}{2E}, \]

this bound is called the Margolus—Levitin (ML) bound. Using these two bounds, one can obtain a comprehensive bound, as follows [9]

\[ \tau \geq \tau_{QSL} = \max \left\{ \frac{\pi \hbar}{2\Delta E}, \frac{\pi \hbar}{2E} \right\} . \]

In references [10–13], the generalizations of the MT and ML bounds to nonorthogonal states and to driven systems have been determined. The QSL time for the dynamics of open systems is also investigated in references [14–17]. A unified bound for QSL time including both the MT and ML types for non-Markovian dynamics has been formulated in reference [16]. However, this bound is used for an initial pure state and it is not applicable for mixed initial states.
the authors introduce a bound for QSL time which can be used for arbitrary initial states. They obtained a QSL time for mixed initial states by introducing relative purity as a distance measure, which can define the speed of evolution starting from an arbitrary initial state in the dynamics of open quantum systems.

The QSL time is inversely related to the speed of the quantum evolution. This means that when the QSL time is shortened, the speed of the quantum evolution will increase. Too much effort has been expended with the aim of obtaining a short QSL time [19–22]. It has been shown that memory effects in the non-Markovian dynamics of open quantum systems can reduce the QSL time [23]. It has been also shown that an external classical driving field can speed up quantum evolution [20]. In reference [21], the authors used dynamic decoupling pulses to increase the speed of the quantum evolution. In this work, we will show how the application of a filtering operation can affect the QSL time for the case of unital noise. The filtering operation is defined by a non-trace-preserving map which can increase the entanglement with some probability. However, it is shown that the filtering operation is a very effective scheme for suppressing decoherence [24]. We will show that for unital noise, the quantum speed limit increases with an increase of the filtering parameter. This work is organized as follows: in section 2, we give a brief introduction to the relative purity-based QSL time for open quantum systems; the results and discussion are provided in section 3; finally, the paper is closed with a brief conclusion in section 4.

2. The quantum speed limit time for open quantum systems

The dynamics of an open quantum system can be characterized by a time-dependent master equation, as

\[
\dot{\rho}_t = L_t(\rho_t),
\]

where \( \rho_t \) is the state of the open quantum system at time \( t \) and \( L_t \) is the time-dependent positive generator. The QSL time is the minimum time it takes for a system to evolve from an initial state \( \rho_\tau \) at an initial time \( \tau \) to a final state \( \rho_\tau + \tau_D \) at a time \( \tau + \tau_D \), where \( \tau_D \) is the driving time. In reference [18], the authors used relative purity to introduce the unified bound for QSL time. They showed that this QSL time can be used for arbitrary initial mixed and pure states. One can obtain the relative purity between the initial state, \( \rho_\tau \), and the final state, \( \rho_\tau + \tau_D \), as

\[
f(\tau + \tau_D) = \frac{\text{Tr}(\rho_\tau \rho_\tau + \tau_D)}{\text{Tr}(\rho_\tau^2)}.\]

For an open quantum system, the ML bound state can be obtained from (see reference [18] for more details)

\[
\tau \geq \frac{|f(\tau + \tau_D) - 1| \text{Tr}(\rho_\tau^2)}{\sum_{i=1}^{n} \sigma_i^2 \rho_i^2},
\]

where \( \sigma_i \) and \( \rho_i \) are the singular values of \( \mathcal{L}_t(\rho_i) \) and \( \rho_\tau \), respectively, and \( \mathcal{L}_t = \frac{1}{\tau} \int_{\tau}^{\tau + \tau_D} dt \). The MT bound of the QSL time for open quantum systems can be obtained from

\[
\tau \geq \frac{|f(\tau + \tau_D) - 1| \text{Tr}(\rho_\tau^2)}{\sqrt{\sum_{i=1}^{n} \sigma_i^2 \rho_i^2}}.
\]

By combining the results for the ML and the MT bounds, one can arrive at the following general result for the QSL time

\[
\tau_{QSL} = \max\left\{ \frac{1}{\sum_{i=1}^{n} \sigma_i^2 \rho_i^2}, \frac{1}{\sqrt{\sum_{i=1}^{n} \sigma_i^2 \rho_i^2}} \right\} |f(\tau + \tau_D) - 1| \text{Tr}(\rho_\tau^2). \tag{8}
\]

In reference [18], the authors have shown that the ML bound of the QSLT is tighter than the MT bound for open quantum systems. From equation (8), it is obvious that the QSL time is always smaller than the driving time \( \tau_D \). The QSL time is inversely related to the speed of the quantum evolution. This means that the speed of the quantum evolution increases when the QSL time is shortened and vice versa.

3. Control of the QSL time with a filtering operation

In this section, we aim to study the effects of a filtering operation on the QSL time. First, we will review the notions of filtering operation and unital noise and then give two examples studying the influence of the filtering operation on the QSL time of unital noise.
3.1. The filtering operation

In the following analysis, we assume that the local filtering operation is implemented on an open quantum system at time \( t \). The filtering operation can be written on a computational basis as

\[
F = \begin{pmatrix} \sqrt{1 - k} & 0 \\ 0 & \sqrt{k} \end{pmatrix}
\]  
(9)

where \( k \) is the filtering operation parameter, with \( 0 < k < 1 \). When this operation is performed on the quantum system, the final state can be written as

\[
\rho_f(t) = F \rho_t F^\dagger \text{tr}(F \rho_t F^\dagger).
\]  
(10)

3.2. Unital quantum noise

Any completely positive-trace-preserving (CPTP) noise \( \Phi \) can be represented in Kraus form as

\[
\Phi(\rho) = \sum_k E_k^\dagger \rho E_k
\]  
(11)

where \( \rho \) is the initial state of the open quantum system and \( E_k \)'s are Kraus operators with \( \sum_k E_k E_k^\dagger = I \). A CPTP noise \( \Phi \) is unital iff \( \sum_k E_k^\dagger E_k = I \), i.e. it maps the identity operator to itself in the same space, \( \Phi(I) = I \). Here, we will consider two examples of unital noises, namely the phase damping dynamic model and the dephasing model with colored noise.

3.2.1. The phase damping dynamic model.

Here we consider a two-level quantum system which interacts with a bosonic environment. The dynamics of the quantum system can be described by the following Hamiltonian

\[
H = H_S + H_B + H_{SB},
\]  
(12)

where \( H_S = \frac{\omega_0}{2} \sigma_3 \) is the Hamiltonian of the system, \( H_B = \sum_k \omega_k b_k^\dagger b_k \) is the Hamiltonian of the bath and \( H_{SB} = \sigma_3 \sum_k (g_k b_k^\dagger + g_k^* b_k) \) is the interaction Hamiltonian between the system and the bath. With \( \sigma_3 \) as the Pauli operator in the \( z \) direction, \( \omega_0 \) represents the two-level system frequency, and \( b_k \) and \( b_k^\dagger \) are the annihilation and creation operators, respectively. \( g_k \) is the coupling constant between the system and the bosonic environment. In this model, the dynamics of the quantum system are characterized by the following time-dependent master equation \([25]\)

\[
\mathcal{L}(\rho(t)) = \gamma(t) (\sigma_3 \rho(t) \sigma_3 - \rho(t)),
\]  
(13)

where \( \gamma(t) \) is the time-dependent dephasing rate. In this model, the off-diagonal elements of the density matrix of quantum system decay with the decoherence factor are \( e^{-\Gamma(t)} \), while the diagonal elements remain unchanged. For the case in which the temperature of the environment is zero, \( \Gamma(t) \) is given by

\[
\Gamma(t) = 4 \int d\omega J(\omega) \frac{1 - \cos \omega t}{\omega^2},
\]  
(14)

where \( J(\omega) \) is the spectral density of the environment \([25]\). Here, we consider the Ohmic-like spectral density for the environment

\[
J(\omega) = \omega_c^{1-i} \omega^\alpha e^{-\omega/c},
\]  
(15)

\[0 < \omega_c < \infty, \quad 0 < \alpha < 1 \]
where \( \omega_c \) is the cutoff frequency and \( s \) is the Ohmicity parameter. Based on the value of the Ohmicity parameter, the environment is sub-Ohmic when \( s < 1 \), Ohmic when \( s = 1 \) and super-Ohmic when \( s > 1 \). In this model, the dynamic is non-Markovian when \( s \in [2.5, 5.5] \) [26]. The model can be characterized by the following Kraus operators

\[
\rho_f(t) = \left( \begin{array}{cc}
\frac{1-k}{2} p_x \sqrt{1-k} \sqrt{k} & \frac{1}{2} p_y \sqrt{1-k} \sqrt{k} \\
\frac{1}{2} p_y \sqrt{1-k} \sqrt{k} & \frac{1}{2} p_z \sqrt{1-k} \sqrt{k}
\end{array} \right),
\]

where \( p_i = e^{-\Gamma(t)} \).

From equation (8), the QSL time is obtained as

\[
\tau_{QSL} = \sqrt{\frac{k(1-k)}{2} \left( p_x (p_y + p_z - p_x) \right)},
\]

In figure 1, the QSL time is plotted as a function of the initial time \( \tau \) for sub Ohmic environment \( s = 0.5 \) for different values of the filtering operation parameter \( k \) when deriving time \( \tau_D = 1 \). It is obvious that increasing the filtering operation parameter in the region \( 0.5 < k < 1 \) leads to a quantum speedup of the quantum evolution. On the other hand, increasing the filtering operation parameter in the region \( 0 < k < 0.5 \) slows down the quantum evolution. Figure 2 represents the QSL time for an Ohmic environment \( (s = 1) \) with different values of the filtering operation parameter when deriving time \( \tau_D = 1 \). As can be seen, the QSL time is decreased by an increase of the filtering operation parameter in the region \( 0.5 < k < 1 \). So, in this region increasing the filtering operation parameter speeds up the quantum evolution. We also observe that the QSL time is increased by increasing the filtering operation in the region \( 0 < k < 0.5 \). Figure 3 shows the QSL time for a super-Ohmic environment \( (s = 3.5) \) with various values of the filtering operation parameters when deriving time \( \tau_D = 1 \). As can be seen, the QSL time is increased by increasing the filtering operation parameter in the region \( 0.5 < k < 1 \). We also observe that the QSL time is increased by increasing the filtering operation parameter in the region \( 0 < k < 0.5 \). The fluctuation observed in the QSL time is due to the non-Markovian property of the noise. The valley in figure 3 is due to the non-Markovian feature of the quantum evolution for \( s = 3.5 \).

3.2.2. The dephasing model with colored noise. We now consider the interaction between a two-level quantum system with an environment which has the property of random telegraph signal noise. The dynamics of the quantum system are characterized by a time-dependent Hamiltonian

\[
H(t) = \sum_{m=1}^{3} \Gamma_m(t) \sigma_m,
\]

where \( \sigma_m \)'s are the Pauli operators in the \((x, y, z)\) directions, \( \Gamma_m(t) \)'s are random variables which follow the statistics of a random telegraph signal, \( \Gamma_m(t) \) depends on the random variable \( n_m(t) \) as \( \Gamma_m(t) = \alpha_m n_m(t) \), where \( n_m(t) \) has a Poisson distribution with an average value equal to \( t/2\tau_m \) and \( \alpha_m \)'s are coin-flip random variables that can randomly have values of \( \pm \alpha_m \). Here, we consider the dephasing model with colored noise, with \( \alpha_1 = \alpha_2 = 0 \) and \( \alpha_3 = \alpha \). In this model, the dynamics can be defined via the following Kraus operators

\[
\rho_f(t) = \left( \begin{array}{cc}
\frac{1-k}{2} p_x \sqrt{1-k} \sqrt{k} & \frac{1}{2} p_y \sqrt{1-k} \sqrt{k} \\
\frac{1}{2} p_y \sqrt{1-k} \sqrt{k} & \frac{1}{2} p_z \sqrt{1-k} \sqrt{k}
\end{array} \right),
\]

where

\[
\Gamma(t) = \frac{1}{2} \Lambda_{\tau} \sigma_0, \quad E_2(t) = \frac{1}{2} \Lambda_{\tau} \sigma_3,
\]

where \( \Lambda_{\tau} = e^{-t/2\Delta} [\cos(\mu t/2\Delta) + \sin(\mu t/2\Delta)/\mu] \), \( \mu = \sqrt{(4\alpha \Delta)^2 - 1} \). The dynamic is non-Markovian for \( \alpha \Delta \geq 1/2 \). After applying the filtering operation to the evolved density matrix, the final density matrix is obtained as

\[
\rho_f(t) = \left( \begin{array}{cc}
\frac{1-k}{2} \Lambda_{\tau} \sqrt{1-k} \sqrt{k} & \frac{1}{2} \Lambda_{\tau} \sqrt{1-k} \sqrt{k} \\
\frac{1}{2} \Lambda_{\tau} \sqrt{1-k} \sqrt{k} & \frac{1}{2} \Lambda_{\tau} \sqrt{1-k} \sqrt{k}
\end{array} \right).
\]

From equation (8), the QSL time is obtained as

\[
\tau_{QSL} = \sqrt{\frac{k(1-k) \Lambda_{\tau}}{2} \left( \Lambda_{\tau} - \Lambda_{\tau} \right)} \frac{1}{2} \Lambda_{\tau} \sqrt{1-k} \sqrt{k} dt.
\]

In figure 4, the QSL time is plotted as a function of initial time \( \tau \) for \( \alpha \Delta = 1/5 \) with different values of the filtering operation parameter \( k \) when deriving time \( \tau_D = 1 \). It is obvious that...
an increase of the filtering operation parameter in the region 0.5 < k < 1 leads to a quantum speedup of the quantum evolution. Conversely, increasing the filtering operation parameter in the region 0 < k < 0.5 leads to a slowdown of the quantum evolution. Figure 5 represents the QSL time as a function of the initial time for αΔ = 2 with different values of the filtering operation parameters when deriving time τD = 1. As can be seen, the QSL time is increased by an increase of the filtering operation parameter in the region 0.5 < k < 1. Conversely, the QSL time is increased by increasing the filtering operation parameter in the region 0 < k < 0.5. The fluctuation observed in the QSL time is due to the non-Markovian property of the noise.

4. Conclusion

In this work, we investigated the effects of a filtering operation on the QSL time. We observed that the QSL time is decreased by increasing the filtering operation parameter in the region 0.5 < k < 1. In other words, in this region one can speed up quantum evolution by increasing the filtering operation parameters. We also observed that the quantum speed limit time was decreased by decreasing the filtering operation parameter in the region 0 < k < 0.5. In other words, in this region one can slow down the quantum evolution by increasing the filtering operation parameters.

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