Second Order General Slow-Roll Power Spectrum

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Abstract

Recent combined results from the Wilkinson Microwave Anisotropy Probe (WMAP) and Sloan Digital Sky Survey (SDSS) provide a remarkable set of data which requires more accurate and general investigation. Here we derive formulae for the power spectrum \( P(k) \) of the density perturbations produced during inflation in the general slow-roll approximation with second order corrections. Also, using the result, we derive the power spectrum in the standard slow-roll picture with previously unknown third order corrections.

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1 Introduction

It is generally believed that inflation [1] provides the explanation for many of the unexplained features of the standard hot big bang cosmology. Inflation generates the flatness, homogeneity and isotropy of the largest observable scales today. Furthermore, primordial quantum fluctuations, which were stretched during inflation, are thought to be the origin of the small perturbations necessary for the formation of galaxies, clusters, and other rich structures in the universe. The power spectrum of these perturbations, as detected in many cosmic microwave background observations and galaxy surveys [2], is observed to be approximately scale invariant.

The first year data from NASA’s Wilkinson Microwave Anisotropy Probe (WMAP) [3] and the Sloan Digital Sky Survey (SDSS) [4] are constraining the power spectrum and the spectral index with greater accuracy than ever before. Recent estimates from several observations are presented in [5]. Therefore it is necessary to obtain more precise and general estimation for the power spectrum to use the WMAP results fully and prepare for the future study of the power spectrum.

There has been extensive work on the power spectrum for the density perturbations produced during inflation [6, 7, 8, 9, 10], using the standard slow-roll picture. The standard slow-roll approximation assumes the slow-roll parameters,

\[ \epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{d\ln a} \quad \text{and} \quad \delta_1 \equiv \frac{\ddot{\phi}}{H\dot{\phi}} = \frac{d\ln \dot{\phi}}{d\ln a}, \] (1)

are small. It also makes the extra assumption that \( \delta_1 \) is approximately constant. In the general slow-roll scheme [11], we abandon this extra assumption so that we can consider a wider class of inflation models. In this paper, we use the basic formalism presented in [10, 11, 12] and extend previous results [11] to calculate the power spectrum to second order in the general slow-roll expansion.

We set \( c = \hbar = 8\pi G = 1 \) throughout this paper.

2 General slow-roll formulae for the spectrum

In this section, we will derive the formula for the power spectrum of the density perturbations. We briefly review how to proceed from the fundamental equation using the formalism of Refs. [10, 11, 12]. Then, we present our integral equation for the power spectrum in the context of the general slow-roll expansion [11].

We begin with the effective action for the inflaton field \( \phi \) during inflation,

\[ S = \int \left[ -\frac{1}{2} R + \frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) \right] \sqrt{-g} \, d^4x, \] (2)

from which we can derive the action for the scalar perturbations as [7]

\[ S = \int \frac{1}{2} \left[ \left( \frac{\partial \varphi}{\partial \eta} \right)^2 - (\nabla \varphi)^2 + \left( \frac{1}{\varphi} \frac{d^2 \varphi}{d\eta^2} \right) \varphi^2 \right] d\eta \, dx^3, \] (3)
where
\[
\phi = a \left( \delta \phi + \frac{\dot{\phi}}{H} R \right),
\]
(4)

\( R \) is the intrinsic curvature perturbation of the spatial hypersurfaces, and
\[
z = \frac{a \dot{\phi}}{H}
\]
(5)

Hence the intrinsic curvature perturbation of the comoving hypersurfaces is given by
\[
R_c = \frac{\phi}{z}.
\]
(6)

Eq. (3) gives the equation of motion for the Fourier component of \( \phi \)
\[
\frac{d^2 \phi_k}{d \eta^2} + \left( k^2 - \frac{1}{z} \frac{d^2 z}{d \eta^2} \right) \phi_k = 0,
\]
(7)

where \( \phi_k \) satisfies the boundary conditions
\[
\phi_k \rightarrow \begin{cases} \frac{1}{\sqrt{2}k} e^{-ik \eta} & \text{as } -k \eta \rightarrow \infty \\ A_k z & \text{as } -k \eta \rightarrow 0 \end{cases}.
\]
(8)

Defining \( y \equiv \sqrt{2k} \phi_k \), \( x \equiv -k \eta \) and
\[
f(\ln x) = \frac{2\pi xz}{k} = \frac{2\pi a x \dot{\phi}}{H k},
\]
(9)

Eq. (7) becomes
\[
\frac{d^2 y}{dx^2} + \left( 1 - \frac{2}{x^2} \right) y = \frac{g(x)}{x^2} y,
\]
(10)

where
\[
g = \frac{f'' - 3f'}{f}
\]
(11)

and \( f' \equiv df/d \ln x \). Using Green’s method, we can present the solution of Eq. (10) as an integral equation
\[
y(x) = y_0(x) + \frac{i}{2} \int_x^\infty \frac{du}{u^2} g(u) [y_u^*(u) y_0(x) - y_u^*(x) y_0(u)] y(u)
\]
\[
\equiv y_0(x) + L(x, u) y(u),
\]
(12)

where
\[
y_0(x) = \left( 1 + \frac{i}{x} \right) e^{ix}
\]
(14)

is the homogeneous solution with correct asymptotic behaviour.

The power spectrum for the curvature perturbation \( P(k) \) is defined by
\[
\frac{2\pi^2}{k^3} P(k) \delta^{(3)}(k - l) = \langle R_c(k) R_c^\dagger(l) \rangle,
\]
(15)
which, using the above results, we can write conveniently as

\[ P(k) = \lim_{x \to 0} \left| \frac{xy}{f} \right|^2. \] (16)

We assume that \( y(x) \) is given approximately by the scale invariant \( y_0(x) \), or equivalently that \( g \) is small. Then, since we are interested in the second order corrections, we iterate Eq. (12) twice, i.e.

\[ y(x) \simeq y_0(x) + L(x, u) y_0(u) + L(x, u, v) y_0(v). \] (17)

Substituting into Eq. (16), and using the method of Ref. [11], i.e. expanding in powers of \( x \) and using the limit \( x \to 0 \), we get

\[
\ln P(k) = \lim_{x \to 0} \left\{ \ln \left( \frac{1}{f^2} \right) + \frac{2}{3} f' + \frac{1}{9} \left( \frac{f'}{f} \right)^2 + \frac{2}{3} \int_x^\infty \frac{du}{u} W(u) g(u) + \frac{2}{9} \left[ \int_x^\infty \frac{du}{u} X(u) g(u) \right]^2 \\
- \frac{2}{3} \int_x^\infty \frac{du}{u} X(u) g(u) \int_x^\infty \frac{dv}{v^2} g(v) - \frac{2}{3} \int_x^\infty \frac{du}{u} X(u) g(u) \int_x^\infty \frac{dv}{v^4} g(v) + \mathcal{O}(g^3) \right\},
\] (18)

where

\[ W(x) = \frac{3 \sin(2x)}{2x^3} - \frac{3 \cos(2x)}{x^2} - \frac{3 \sin(2x)}{2x} \] (19)

and

\[ X(x) = -\frac{3 \cos(2x)}{2x^3} - \frac{3 \sin(2x)}{x^2} + \frac{3 \cos(2x)}{2x} + \frac{3}{2x^3} \left( 1 + x^2 \right). \] (20)

Note that

\[ \lim_{x \to 0} W(x) = 1 + \mathcal{O}(x^2) \quad \text{and} \quad \lim_{x \to 0} X(x) = \frac{1}{3} x^3 + \mathcal{O}(x^5). \] (21)

The right hand side of Eq. (18) as a whole is well defined in the limit \( x \to 0 \), but we do not know how \( f \) behaves in that limit, so the individual terms are not well defined. As a remedy, we pick some reasonable point, e.g. around horizon crossing, to evaluate \( f \) and rearrange to make the individual terms well defined. Using

\[
\frac{1}{f^2} = \frac{1}{f_\star^2} \exp \left[ 2 \ln \left( \frac{f_\star}{f} \right) \right] \\
= \frac{1}{f_\star^2} \left\{ 1 + 2 \int_x^{f_\star} \frac{du}{u} f' + 2 \left( \int_x^{f_\star} \frac{du}{u} \frac{f'}{f} \right)^2 + \mathcal{O} \left[ \left( \frac{f'}{f} \right)^3 \right] \right\}
\] (22)

and

\[ \frac{f'}{f} = \frac{f'}{f_\star} - \int_x^{f_\star} \frac{du}{u} f'' + \int_x^{f_\star} \frac{du}{u} \left( \frac{f'}{f} \right)^2, \] (23)
where subscript $\star$ denotes evaluation at some convenient time around horizon crossing, we can rewrite the power spectrum as

$$
\ln P(k) = \ln \left( \frac{1}{f_k^2} \right) + \frac{2}{3} f'_k + \frac{1}{9} \left( \frac{f'_k}{f_k} \right)^2 + \frac{2}{3} \int_0^\infty \frac{du}{u} W_\theta(x_*, u) g(u) + \frac{2}{9} \left[ \int_0^\infty \frac{du}{u} X(u) g(u) \right]^2
$$

$$
- \frac{2}{3} \int_0^\infty \frac{du}{u} X(u) g(u) \int_0^\infty \frac{dv}{v^2} g(v) - \frac{2}{3} \int_0^\infty \frac{du}{u} X_\theta(x_*, u) g(u) \int_0^\infty \frac{dv}{v^2} g(v),
$$

(24)

where

$$
W_\theta(x_*, x) \equiv W(x) - \theta(x_* - x)
$$

(25)

and

$$
X_\theta(x_*, x) \equiv X(x) - \frac{x^3}{3} \theta(x_* - x).
$$

(26)

Eq. (24) can be written more compactly by substituting

$$
g = \left( \frac{f'}{f} \right)' - 3 \frac{f'}{f} + \left( \frac{f'}{f} \right)^2,
$$

(27)

integrating by parts, and using the identity

$$
2x^3 W = (3 + x^2) X + x (1 + x^2) X',
$$

(28)

to give

$$
\ln P(k) = \ln \left( \frac{1}{f_k^2} \right) - 2 \int_0^\infty \frac{du}{u} w_\theta(x_*, u) f' + 2 \left[ \int_0^\infty \frac{du}{u} \chi(u) \frac{f''}{f} \right]^2
$$

$$
- 4 \int_0^\infty \frac{du}{u} \chi(u) \frac{f'}{f} \int_u^\infty \frac{dv}{v^2} \frac{f'}{f}
$$

$$
= - \int_0^\infty du W'(u) \left[ \ln \left( \frac{1}{f_k^2} \right) + \frac{2}{3} f' \right] + 2 \left[ \int_0^\infty \frac{du}{u} \chi(u) \frac{f''}{f} \right]^2
$$

$$
- 4 \int_0^\infty \frac{du}{u} \chi(u) \frac{f'}{f} \int_u^\infty \frac{dv}{v^2} \frac{f'}{f}
$$

(29)

where

$$
w(x) \equiv W(x) + \frac{x}{3} W'(x) = \frac{\sin(2x)}{x} - \cos(2x),
$$

(31)

and

$$
\chi(x) \equiv X(x) + \frac{x}{3} X'(x) = 1 - \frac{\cos(2x)}{x} - \sin(2x).
$$

(32)

These functions behave asymptotically as

$$
\lim_{x \to 0} w(x) = 1 + \mathcal{O}(x^2)
$$

and

$$
\lim_{x \to 0} \chi(x) = \frac{2}{3} x^3 + \mathcal{O}(x^5),
$$

(33)

and we define

$$
w_\theta(x_*, x) \equiv w(x) - \theta(x_* - x).
$$

(34)

Eqs. (24) and (29) are our main results. To evaluate these formulae, one needs to determine $f$ and $g$ as functions of $x$. This can be done by solving the background equations for $a$ and $\phi$ as functions of $x$, and using Eqs. (9) and (11).
2.1 Special case of a de Sitter background

In the physically important special case of constant $H$, we have

\[
\frac{1}{f^2} = \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\phi}\right)^2,
\]

\[
\frac{f'}{f} = -\frac{\ddot{\phi}}{H\dot{\phi}},
\]

\[
g = -3V'',
\]

and

\[
\ln \left(\frac{V^3}{12\pi^2 V^2}\right) \simeq \ln \left(\frac{1}{f^2}\right) + \frac{2}{3} \frac{f'}{f} + \frac{1}{9} (\frac{f'}{f})^2.
\]

Then the power spectrum Eq. (24) becomes

\[
\ln P = \ln \left(\frac{V^3}{12\pi^2 V^2}\right) - 2 \int_0^\infty \frac{du}{u} W(x_*, u) \frac{V''}{V} + 2 \left[ \int_0^\infty \frac{du}{u} X(u) \frac{V''}{V} \right]^2 - 6 \int_0^\infty \frac{du}{u} X(u) \frac{V''}{V} \int_u^\infty \frac{dv}{v^2} \frac{V''}{V} - 6 \int_0^\infty \frac{du}{u} X_\theta(x_*, u) \frac{V''}{V} \int_u^\infty \frac{dv}{v^4} \frac{V''}{V},
\]

while Eq. (29) becomes

\[
\ln P = \ln \left(\frac{H^2}{2\pi}\right) ^2 \left(\frac{H}{\phi_*}\right)^2 + 2 \int_0^\infty \frac{du}{u} w_\theta(x_*, u) \frac{\ddot{\phi}}{H\dot{\phi}} + 2 \left[ \int_0^\infty \frac{du}{u} X(u) \frac{\ddot{\phi}}{H\dot{\phi}} \right]^2 - 4 \int_0^\infty \frac{du}{u} x(u) \frac{\ddot{\phi}}{H\dot{\phi}} \int_u^\infty \frac{dv}{v^2} \frac{\ddot{\phi}}{H\dot{\phi}}.
\]

3 Example

As a specific example, we consider an inflaton $\phi$ rolling down a linear potential with slope changing from $-A$ to $-A - \Delta A$ at $\phi = \phi_0$. The potential is

\[
V(\phi) = V_0 \left\{ 1 - [A + \Theta(\phi - \phi_0) \Delta A] (\phi - \phi_0) \right\}.
\]

Assuming $|A| \ll 1$ so that $H \simeq \sqrt{V_0}/3$, and solving the equation of motion for $\phi$, we obtain

\[
\frac{d\phi}{dN} = A + \Theta(N - N_0) \Delta A \left[ 1 - e^{-3(N-N_0)} \right],
\]

where $N = \int H dt$ and $\phi(N_0) = \phi_0$. Then if $|\Delta A/A| \ll 1$, so that the approximate scale invariance of the spectrum is maintained, we have

\[
\frac{V''}{V} = -\Delta A \delta(\phi - \phi_0) = -\frac{\Delta A}{A} \delta(N - N_0).
\]
Figure 1: Plot of $\ln P$ versus $\ln k$. The solid line is the full result, Eq. (42), and the dotted line is the first order term in $\Delta A/A$ only. $\Delta A/A > 0$ in the upper row and $\Delta A/A < 0$ in the lower row. $|\Delta A/A| = 0.2$ in the left column and $|\Delta A/A| = 1$ in the right column.

Substituting into Eq. (37) and performing the integration, we find the power spectrum

$$\ln P = \ln \left[ \frac{V_0}{12\pi^2(A + \Delta A)^2} \right] + 2 \left( \frac{\Delta A}{A} \right) W(x_0) + 2 \left( \frac{\Delta A}{A} \right)^2 X(x_0) \left[ X(x_0) - \frac{3}{2x_0^3} (1 + x_0^2) \right],$$

(42)

where the $x_\star$ dependent terms in Eq. (37) have been absorbed into the first term, the constant leading result, to show explicitly that the spectrum is independent of the evaluation point $x_\star$. The spectrum $\ln P$ is plotted in Figure 1.

Using the identity

$$W^2 + X^2 = \frac{3}{x^3} (1 + x^2) X,$$

(43)

we can write Eq. (12) as

$$P = \frac{V_0}{12\pi^2(A + \Delta A)^2} \left[ 1 + 2 \left( \frac{\Delta A}{A} \right) W(x_0) + 3 \left( \frac{\Delta A}{A} \right)^2 \frac{1 + x_0^2}{x_0^3} X(x_0) \right],$$

(44)

which is exactly the same as found in Ref. [13].
4 Application to the standard slow-roll expansion

In this section, we apply our general slow-roll formula to the special case of standard slow-roll. Our general slow-roll formula, Eq. (29), can be written as

\[ P = \frac{1}{f^2} \left\{ 1 - 2 \int_0^\infty \frac{du}{u} w_\theta(x_*, u) \frac{f'}{f} + 2 \left[ \int_0^\infty \frac{du}{u} w_\theta(x_*, u) \frac{f''}{f} \right]^2 + 2 \left[ \int_0^\infty \frac{du}{u} \chi(u) \frac{f'}{f} \right]^2 \ight. \

- 4 \int_0^\infty \frac{du}{u} \chi(u) \frac{f'}{f} \int_0^\infty \frac{dv}{v^2} \frac{f'}{f} + O \left( \left( \frac{f'}{f} \right)^3 \right) \right\}. \tag{45} \]

In addition to the general slow-roll assumptions, standard slow-roll assumes

\[ \frac{1}{f} \frac{d^n f}{(d \ln x)^n} = O \left( \left( \frac{f'}{f} \right)^n \right) \tag{46} \]

in which case we can Taylor expand \( f \) in terms of \( \ln(x/x_*) \) where \( x_* \) is some convenient time around horizon crossing

\[ \frac{f'}{f} \equiv \frac{f'}{f_*} + \left( \frac{f'}{f_*} \right) \ln \frac{x}{x_*} + \frac{1}{2} \left( \frac{f''}{f_*} \right) \left( \ln \frac{x}{x_*} \right)^2 + \cdots \\
= \frac{f'}{f_*} + \left[ \frac{f''}{f_*} - \left( \frac{f'}{f_*} \right)^2 \right] \ln \frac{x}{x_*} + \frac{1}{2} \left[ \frac{f'''}{f_*} - 3 \frac{f'}{f_*} \frac{f''}{f} + 2 \left( \frac{f'}{f_*} \right)^3 \right] \left( \ln \frac{x}{x_*} \right)^2 + \cdots. \tag{47} \]

Substituting into Eq. (45) gives

\[ P = \frac{1}{f_*^2} \left\{ 1 - 2 \alpha_* \frac{f'}{f_*} + \left( -\alpha_*^2 + \frac{\pi^2}{12} \right) \frac{f''}{f_*} + \left( 3\alpha_*^2 - 4 + \frac{5\pi^2}{12} \right) \left( \frac{f'}{f_*} \right)^2 \ight. \

+ \left[ -\frac{1}{3} \alpha_*^3 + \frac{\pi^2}{12} \alpha_* - \frac{4}{3} + \frac{2}{3} \zeta(3) \right] \frac{f'''}{f_*} \\
+ \left[ 3\alpha_*^3 - 8\alpha_* + \frac{7}{12} \pi^2 \alpha_* + 4 - 2\zeta(3) \right] \frac{f' f''}{f_*^2} \\
+ A \left( \frac{f'}{f_*} \right)^3 + O \left( \left( \frac{f'}{f_*} \right)^4 \right) \right\} \tag{48} \]

where

\[ \alpha_* \equiv \alpha - \ln x_*, \tag{49} \]

\[ \alpha \equiv 2 - \ln 2 - \gamma \approx 0.729637, \tag{50} \]

\( \gamma \approx 0.577216 \) is the Euler-Mascheroni constant, \( \zeta \) is the Riemann zeta function, and \( A \) is an undetermined coefficient. This is consistent with the second order standard slow-roll result, Eq. (40) in Ref. [10], where the result was evaluated at \( x_* = 1 \). However, because \( A \) is
the only undetermined third order coefficient, we can use the known exact solutions 8 to determine the complete third order standard slow-roll result.1 In fact, if we were sufficiently motivated, we could apply the same method to determine the fourth order standard slow-roll result from Eq. (45).

The simplest exact solution is inflation near a maximum, where the potential is

$$V(\phi) = V_0 \left(1 - \frac{1}{2} \mu^2 \phi^2 + \cdots \right). \tag{51}$$

In this case

$$H = \sqrt{\frac{V_0}{3}}, \quad x = \frac{k}{aH} \tag{52}$$

and

$$f = \frac{3\pi \phi_0}{H^2} \left(\sqrt{1 + \frac{4}{3} \mu^2} - 1\right) \left(\frac{x}{x_0}\right)^{-\frac{3}{2}} \left(-\frac{1}{4} \sqrt{1 + \frac{4}{3} \mu^2} - 1\right) \tag{53}$$

Then, by comparing the exact solution 8 and Eq. (48), we obtain

$$A = -4\alpha_*^3 + 16\alpha_* - \frac{5}{3}\pi^2 \alpha_* - 8 + 6\zeta(3). \tag{54}$$

We can check this result by using the exact solution for power law inflation, where

$$V(\phi) = V_0 \exp \left(-\sqrt{\frac{2}{p}} \phi\right), \quad x = \left(\frac{p}{p-1}\right) \left(\frac{k}{aH}\right) \tag{55}$$

and

$$f = \frac{2\pi \sqrt{2p}}{(p-1)H_0} \left(\frac{x}{x_0}\right)^{-\frac{p-1}{p+1}} \tag{56}$$

The slow-roll parameters are defined as

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2} \left(\frac{\dot{\phi}}{\dot{\phi}}\right) \quad \text{and} \quad \delta_n = \frac{1}{H^n \dot{\phi}} \frac{d^n \dot{\phi}}{dt^n}, \tag{57}$$

where in standard slow-roll

$$\epsilon = \mathcal{O}(\xi) \quad \text{and} \quad \delta_n = \mathcal{O}(\xi^n) \tag{58}$$

for some small parameter $\xi$. Then

$$\frac{1}{f^2} = \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\dot{\phi}}\right)^2 \left[1 - 2\epsilon - 3\epsilon^2 - 4\epsilon \delta_1 - 4\epsilon \delta_2 - 16\epsilon^3 - 28\epsilon^2 \delta_1 - 4\epsilon \delta_1^2 + \mathcal{O}(\xi^4)\right],$$

$$\frac{f'}{f} = -2\epsilon - \delta_1 - 4\epsilon^2 - 3\epsilon \delta_1 - 2\epsilon \delta_2 - 18\epsilon^3 - 25\epsilon^2 \delta_1 - 4\epsilon \delta_1^2 + \mathcal{O}(\xi^4),$$

$$\frac{f''}{f} = \delta_2 + 8\epsilon^2 + 9\epsilon \delta_1 + 4\epsilon \delta_2 + 36\epsilon^3 + 50\epsilon^2 \delta_1 + 8\epsilon \delta_1^2 + \mathcal{O}(\xi^4),$$

$$\frac{f'''}{f} = -\delta_3 - 13\epsilon \delta_2 - 48\epsilon^3 - 85\epsilon^2 \delta_1 - 18\epsilon \delta_1^2 + \mathcal{O}(\xi^4), \tag{59}$$

1It was brought to our attention that the authors of Ref. 14 used these exact solutions to determine the third order corrections valid under some special conditions. Note that we use them to help determine the completely general case.
where now the standard slow-roll conditions, Eq. (46), are manifest in terms of the slow-roll parameters. Substituting into Eq. (48) and using Eq. (54), we obtain

\[ P = \left(\frac{H_\star}{2\pi}\right)^2 \left(\frac{H_\star}{\dot{\phi}_\star}\right)^2 \left\{ 1 + (4\alpha_\star - 2)\epsilon_\star + 2\alpha_\star \delta_{1\star} + \left(-\alpha_\star^2 + \frac{\pi^2}{12}\right) \delta_{2\star} \right. \]
\[ + \left(4\alpha_\star^2 - 19 + \frac{7\pi^2}{3}\right) \epsilon_\star^2 + \left(3\alpha_\star^2 + 2\alpha_\star - 20 + \frac{29\pi^2}{12}\right) \epsilon_\star \delta_{1\star} + \left(3\alpha_\star^2 - 4 + \frac{5\pi^2}{12}\right) \delta_{1\star}^2 \]
\[ + \left[ \frac{1}{3}\alpha_\star^3 - \frac{\pi^2}{12}\alpha_\star + \frac{4}{3} - \frac{2}{3}\zeta(3) \right] \delta_{3\star} \]
\[ + \left[ -\frac{5}{3}\alpha_\star^3 - 2\alpha_\star^2 + 20\alpha_\star - \frac{9}{4}\pi^2\alpha_\star + \frac{16}{3} + \frac{\pi^2}{6} - \frac{14}{3}\zeta(3) \right] \epsilon_\star \delta_{2\star} \]
\[ + \left[ -3\alpha_\star^3 + 8\alpha_\star - \frac{7}{12}\pi^2\alpha_\star - 4 + 2\zeta(3) \right] \delta_{1\star} \delta_{2\star} \]
\[ + \left[ 4\alpha_\star^2 + 8\alpha_\star + 16 + 5\pi^2 - 48\zeta(3) \right] \epsilon_\star^3 \]
\[ + \left[ -\frac{5}{3}\alpha_\star^3 + 4\alpha_\star^2 + 32\alpha_\star - \frac{9}{4}\pi^2\alpha_\star + \frac{88}{3} + \frac{23}{3}\pi^2 - \frac{230}{3}\zeta(3) \right] \epsilon_\star^2 \delta_{1\star} \]
\[ + \left[ 3\alpha_\star^3 + 4\alpha_\star^2 - 24\alpha_\star + \frac{13}{4}\pi^2\alpha_\star + 16 + \frac{7}{3}\pi^2 - 30\zeta(3) \right] \epsilon_\star^3 \delta_{1\star}^2 \]
\[ + \left[ 4\alpha_\star^3 - 16\alpha_\star + \frac{5}{3}\pi^2\alpha_\star + 8 - 6\zeta(3) \right] \delta_{1\star}^3 \}. \quad (60) \]

We see that this third order result is consistent with the second order result of Eq. (41) of Ref. [10] if we note that the result of Ref. [10] was evaluated at \( a_\star H_\star = k \), in which case we have \( \alpha_\star \equiv \alpha - \ln x_\star = \alpha - \epsilon_\star + O(\xi^2) \).

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References

[1] E. B. Gliner, JETP 22, 378 (1966) ; E. B. Gliner, Sov. Phys. Dokl. 15, 559 (1970) ; E. B. Gliner and I. G. Dymnikova, Sov. Astron. Lett. 1, 93 (1975) ; A. H. Guth, Phys. Rev. D 23, 347 (1981) ; A. D. Linde, Phys. Lett. B 108, 389 (1982) ; A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. 48, 1220 (1982)
[2] C. L. Bennett et al., Astrophys. J. 464, L1 (1996) astro-ph/9601067 ; P. de Bernardis et al., Nature 404, 955 (2000) astro-ph/0004404 ; A. H. Jaffe et al., Phys. Rev. Lett. 86, 3475 (2001) astro-ph/0007333 ; A. T. Lee et al., Astrophys. J. 561, L1 (2001) astro-ph/0104459 ; C. B. Netterfield et al., Astrophys. J. 571, 604 (2002) astro-ph/0104460 ; N. W. Halverson et al., Astrophys. J. 568, 38 (2002) astro-ph/0104489 ; C. Pryke et al., Astrophys. J. 568, 46 (2002) astro-ph/0104490

[3] C. L. Bennett et al., Astrophys. J. Suppl. 148, 1 (2003) astro-ph/0302207 ; D. N. Spergel et al., Astrophys. J. Suppl. 148, 175 (2003) astro-ph/0302209

[4] I. Zehavi et al., Astrophys. J. 571, 172 (2002) astro-ph/0106476 ; M. Tegmark et al., Phys. Rev. D 69, 103501 (2004) astro-ph/0310723 ; M. Tegmark et al., Astrophys. J. 606, 702 (2004) astro-ph/0310725

[5] V. Barger, H.-S. Lee and D. Marfatia, Phys. Lett. B 565, 33 (2003) hep-ph/0302150 ; W. H. Kinney, E. W. Kolb, A. Melchiorri and A. Riotto, hep-ph/0305130 ; S. M. Leach and A. R. Liddle, Phys. Rev. D 68, 123508 (2003) astro-ph/0306305

[6] A. A. Starobinsky, Phys. Lett. B 117, 175 (1982) ; A. Guth and S.-Y. Pi, Phys. Rev. Lett. 49, 1100 (1982) ; J. M. Bardeen, P. J. Steinhardt and M. S. Turner, Phys. Rev. D 28, 679 (1983) ; H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. 78, 1 (1984)

[7] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. 215, 203 (1992)

[8] E. D. Stewart and D. H. Lyth, Phys. Lett. B 302, 171 (1993) gr-qc/9302019

[9] J. Hwang and H. Noh, Phys. Lett. B 495, 277 (2000) astro-ph/0009268

[10] E. D. Stewart and J.-O. Gong, Phys. Lett. B 510, 1 (2001) astro-ph/0101225

[11] S. Dodelson and E. D. Stewart, Phys. Rev. D 65, 101301 (2002) astro-ph/0109354 ; E. D. Stewart, Phys. Rev. D 65, 103508 (2002) astro-ph/0110322

[12] J.-O. Gong and E. D. Stewart, Phys. Lett. B 538, 213 (2002) astro-ph/0202098

[13] A. A. Starobinsky, JETP Lett. 55, 489 (1992)

[14] D. J. Schwarz and C. A. Terrero-Escalante, hep-ph/0403129