The Penrose inequality for nonmaximal perturbations of the Schwarzschild initial data

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Abstract
We show that the Penrose inequality is satisfied for generic (in some sense) conformally flat axially symmetric nonmaximal perturbations of the Schwarzschild data. A role of horizon is played by a marginally outer trapped surface which does not have to be minimal. It has a spherical shape with respect to flat metric.

Keywords: Penrose inequality, conformal method, black holes

1. Introduction

The Penrose–Hawking [1, 2] theorems predict a development of singularities in solutions of the Einstein equations if a marginally trapped surface is admitted. According to the cosmic censorship conjecture of Penrose [3] these singularities should be hidden behind event horizons of black holes. A heuristic argument supporting this statement relied on a consideration of collapsing null shells and led to the inequality between the ADM mass $M$ and the area of an apparent horizon $|S_h|$:

$$M \geq \sqrt{\frac{|S_h|}{16\pi}},$$

(1)
called the Penrose inequality. Currently, it is expected that (1) should hold for any initial surface $S$ with an inner boundary $S_b$ which represents surface of a black hole. The Penrose inequality was first proved for the generic apparent horizon in the case of spherical symmetry [4]. A mathematically complete proof for a wide class of data was given by Huisken and Ilmanen [5]. They established the so-called Riemannian Penrose inequality in the case of horizon represented by a minimal surface. This approach utilized a scenario proposed by Jang and Wald [6], based on the monotonicity of the Geroch quasi-local mass [7] under the inverse mean curvature flow (IMCF). A generalization to multi-connected horizons was found by Bray [8]. Some
analytical methods have been proposed later to prove (1) in general case, but technical complexity of these approaches prevents them from being conclusive [9, 10].

In the case of axially symmetric data the inequality (1) can be replaced by its stronger version,

$$E^2 - p^2 \geq \frac{|S_h|}{16\pi} + \frac{4\pi}{|S_h|}J^2,$$

where $E$, $p$ and $J$ are, respectively, energy, momentum and angular momentum of initial data. In paper [11] we studied inequality (2) in the case of axially symmetric conformally flat maximal data on $\mathbb{R}^3$ with removed ball of radius $m/2$. The internal boundary was a marginally outer trapped surface (MOTS). Assuming that data can be expanded with respect to a small parameter $\epsilon$ we proved (2) up to leading terms in $\epsilon$ (typically up to $\epsilon^2$). This means that in the considered class any initial data in a neighbourhood of the Schwarzschild data satisfy the Penrose inequality.

In this paper we continue our approach from [11], but now we admit nonvanishing mean curvature of data (nonmaximal data). This change makes constraint equations more complicated and adds an additional free function in data. Using a slightly new method we are still able to prove (2) up to $\epsilon^2$ in a generic case.

The main value of this result lies in a relaxation of typical assumptions in almost all other proofs ($H = 0$ and a minimal surface as an MOTS, see [12] for an exception). The main disadvantage of our approach is that the considered class of data is relatively small (three free functions of two coordinates). Moreover, we do not know if the inner MOTS is outermost (see a discussion in section 1 in [11]) and we are not able to treat nongeneric case (see definition 3.1).

The remainder of this paper is organized as follows. In section 2 we shortly describe the conformal approach to constraint equations with a boundary condition which guarantees that the internal boundary is MOTS. We also perform expansions in $\epsilon$ and give a perturbative formula for an expression $P_I$ which is crucial for the Penrose inequality. In section 3 we find axially symmetric solutions of the momentum constraint in the leading order in $\epsilon$ and estimate function $P_I$ from below. The main result is formulated in section 4.

## 2. A perturbative formulation of the Penrose inequality

Initial data $g'_{ij}, K'_{ij}$ on a surface $S$ are constrained by equations

$$\nabla_j (K'_{ij} - H'\delta^j_i) = 0,$$

$$R' + H'^2 - K'^2 = 0,$$

where $H' = K'_{ii}, K'^2 = K'_{ij}K'^{ij}$ and $R'$ is the Ricci scalar of $g'_{ij}$. We are interested in data which are asymptotically flat and admit a marginally outer trapped surface (MOTS) $S_h$ with vanishing expansion of outer null rays,

$$H' = K'_{nn} + h = 0 \quad \text{on} \quad S_h,$$

where $K'_{nn} = n'^{i}n'^{i}K'_{ij}, n'^{i}$ is the outer unit normal vector and $h = \nabla' n'^{i}$ is the mean curvature of $S_h$.

In the conformal approach initial data are parameterized in the following way [13] by a preliminary metric $g_{ij},$ symmetric traceless tensor $A_{ij}$ and the mean curvature $H'$.
\[ g'_{ij} = \psi^4 g_{ij}, \quad K'_{ij} = \psi^{-6} A'_{ij} + \frac{1}{3} H' \delta'_{ij}, \quad A'_i = 0. \] (6)

For nonmaximal data \((H' \neq 0)\) constraints (3) and (4) form a coupled system of equations,
\[ \nabla_i A'_j = \frac{2}{3} \psi^6 \nabla_i H', \] (7)
\[ \Delta \psi = \frac{1}{8} R \psi - \frac{1}{8} A_{ij} \psi^{-7} + \frac{1}{12} H^2 \psi^5, \] (8)
and the boundary condition (5) is replaced by
\[ n^i \partial_i \psi + \frac{1}{2} h \psi - \frac{1}{4} A_{ab} \psi^{-3} + \frac{1}{6} H' \psi^3 = 0 \quad \text{on } S_h. \] (9)

Unfortunately, in the asymptotically flat setting there are no existence theorems for the system (7)–(9). We suppose that results of Holst and Meier [14] could be adapted to this case [note that in theorem 3.2 in [14] it is assumed that \(\theta^{-} = 0\) in contrary to our assumption \(\theta^{+} = 0\), represented by (5)]. A review of results on existence theorems in different settings can be found in [15].

As in [11], we assume that \(g_{ij} = \delta_{ij}\) and the initial surface is given by \(S = R^3 \setminus B(0, \frac{m}{2})\), where \(B(0, \frac{m}{2})\) is an open ball with radius \(m/2\) and the spherical boundary \(S_h\). If \(K'_ij = 0\), then solution of (8) and (9) reads
\[ \psi = \psi_0 = 1 + \frac{m}{2r}, \] (10)
and transformation (6) leads to the Schwarzschild initial metric,
\[ g'_0 = \frac{dr^2}{1 - \frac{m}{2r}} + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \] (11)
for which the Penrose inequality is saturated. We are going to investigate it for data with small addition of \(A_{ij}\) and nontrivial \(H'\).

Unlike in the case \(H' = 0\), equation (7) cannot be solved independently of (8). However, this can be done approximately under the assumption that \(\psi, A_{ij} \) and \(H'\) can be expanded in powers of a small parameter \(\epsilon\),
\[ \psi = \psi_0 + \psi_1 + \psi_2 + \cdots \] (12)
\[ A^{ij} = A^{ij}_1 + A^{ij}_2 + \cdots \quad H' = H'_1 + H'_2 + \cdots \] (13)
Here, subscripts \(n = 0, 1, 2, \ldots\) correspond to terms of the order \(\epsilon^n\). We will denote the leading terms in (13) by \(B_{ij}\) and \(B\),
\[ B^{ij} = A^{ij}_1, \quad B = H'_1. \] (14)
In the lowest nontrivial order in \(\epsilon\) the constraints read
\[ \nabla_i B'_j = \frac{2}{3} \psi_0 \nabla_i B, \] (15)
\[ \Delta \psi = -\frac{1}{8} B_{ij} B^{ij} \psi^{-7} + \frac{1}{12} B^2 \psi^5. \] (16)
Thus, one can first solve (15) and then extract information about total energy and area of the MOTS from equation (16). Considerations from section 2 in [11] can be almost literally repeated under the asymptotical flatness conditions of the form

$$\left( B_{ij}, B \right) = O \left( r^{-2} \right), \quad \partial_{p} \left( B_{ij}, B \right) = O \left( r^{-3} \right), \quad \psi \to 1 \text{ if } r \to \infty. \quad (17)$$

Up to $\epsilon^2$ one obtains

$$E^2 - \frac{|S_h|}{16\pi} = P_I, \quad (18)$$

where

$$P_I = \left( \frac{m^2}{128\pi} \right)^2 \left\langle B_{rr} - \frac{2}{3}B\psi_0^6 \right\rangle - \frac{3m^2}{8\pi} \langle \psi_1^2 \rangle_h + \frac{m}{8\pi} \int_{m/2}^{\infty} r^2(2 - \psi_0) \left\langle B_{ij} B_{ij} \psi_0^7 - \frac{2}{3}B^2 \psi_0^5 \right\rangle dr. \quad (19)$$

Here $\langle \rangle$ denotes integral over the two-dimensional sphere with radius $r$ and the subscript $h$ refers to the sphere $r = m/2$. The function $\psi_1$ is a solution of the flat Laplace equation

$$\triangle \psi_1 = 0, \quad (20)$$

with the boundary conditions

$$\partial_r \psi_1 + \frac{1}{m} \psi_1 = \frac{1}{32} \left( B_{rr} - \frac{2}{3}B\psi_0^6 \right) \quad \text{at } r = m/2, \quad (21)$$

$$\psi_1 = 0 \quad \text{at } r = \infty. \quad (22)$$

The standard Penrose inequality is satisfied if $P_I \geq p^2$. In the next section we consider all axially symmetric solutions of (15) and show that in a generic case $P_I \geq p^2 + \frac{J^2}{4m^2}$, hence inequality (2) is satisfied up to $\epsilon^2$.

The only negative term in (19) is specified by the boundary value on the rhs of equation (21). If it vanishes then $\psi_1 = 0$ and $P_I \geq 0$, in accordance with the result of Huisken and Ilmanen (in that case the inner boundary becomes a minimal surface after the conformal transformation). In order to show that this negative term is always dominated by the other ones we could try to apply some version of the trace Sobolev inequality, but we were not able to find a proper one. Another possibility is to use the York decomposition of the constraint equations [13], but the matrix operator $\Delta_L$ is not easily invertible. Finally, we decided to compute and estimate $P_I$ when solutions $B_{ij}$ of the momentum constraints are expressed explicitly in terms of free functions. This can be done e.g. when initial data are axially symmetric (see the next section).

In this case equations are quite easy to solve (in contrary to the York formulation), but the main problem is to find a parameterization of solutions which satisfies regularity and boundary conditions and allows to find a lower boundary of $P_I$. We obtain it in several steps.

3. Axially symmetric perturbations

Equation (15) with index $\varphi$ yields

$$B_{\varphi r} = \frac{\omega_{\varphi r}}{\sin \theta}, \quad B_{\varphi \varphi} = -\frac{\omega_{\varphi \varphi}}{r^2 \sin \theta}. \quad (23)$$
with an arbitrary function $\omega$, and for other indices we obtain the following system,

$$
(r^3 B_{rr} \sin \theta)_{, r} + (r B_{r, r} \sin \theta)_{, \theta} = \frac{2}{3} r^3 \psi_0^6 B_{, r} \sin \theta,
$$

(24)

$$
(B_{\theta \theta} \sin^2 \theta)_{, \theta} + (r^2 B_{r, r})_{, r} \sin^2 \theta + r^2 B_{rr} \sin \theta \cos \theta = \frac{2}{3} r^3 \psi_0^6 B_{, \theta} \sin^2 \theta.
$$

(25)

In order to solve (24) and (25) let us introduce function $S$ such that

$$
B = \frac{3}{2} S_z,
$$

(26)

where $z = \cos \theta$. Equation (24) leads to the existence of another function $Q$, such that

$$
B_{r, \theta} = \frac{1}{r \sin \theta} (Q_{, r} - \kappa S_{, r}),
$$

(27)

$$
B_{rr} = \frac{1}{r^3} Q_{, z},
$$

(28)

where

$$
\kappa = r^3 \psi_0^6.
$$

(29)

Following [11] we also introduce function $F(r, z)$, given by

$$
B_{\theta \theta} + \frac{1}{2} r^2 B_{rr} = F_{, z}.
$$

(30)

Equation (25) now takes the form

$$
\triangle_s F = (r Q_{, r} - r \kappa S_{, r})_{, r} - \frac{1}{2r} (z^2 - 1)(Q + 2 \kappa S)_{, zz},
$$

(31)

where

$$
\triangle_s F = ((1 - z^2) F_{, z})_{, z}.
$$

(32)

Function $F$ can be easily found if the rhs of equation (31) is expressed in terms of the Legendre polynomials $P_n$. A necessary condition is the absence of $P_0$ in (31), hence

$$
(r Q_{0, r} - r \kappa S_{0, r})_{, r} + \frac{1}{3r} ((Q_{, z})_{1} + 2 \kappa (S_{, z})_{1}) = 0,
$$

(33)

where subscripts 0, 1 denote respective coefficients of decompositions into the Legendre polynomials.

Potentials $S, Q$ define smooth quantities $B_{ij}$ and $B$, provided they are smooth functions of $r, z$ and

$$
(Q_{, r} - \kappa S_{, r}) \sim \sin^2 \theta.
$$

(34)

The asymptotic flatness conditions (17) yield

$$
r^2 S, \frac{1}{r} Q < \infty.
$$

(35)
Function (19) can be written in the form

\[ P_J = \left( \frac{m^2}{128\pi} \right)^2 \left\langle B_{rr} - \frac{2}{3} B \psi_0^0 \right\rangle_{\hbar} - \frac{3m^2}{8\pi} \left\langle \psi_1^2 \right\rangle_{\hbar} + P_J + \tilde{P}_J, \]  

(36)

where

\[ P_J = 2 \int_{\frac{\pi}{2}}^{\infty} d\varrho r \int_{-1}^{1} \left( B_{r\varphi}^2 + r^2 B_{\varphi\varphi}^2 \right) \frac{dz}{1 - z^2}, \]  

(37)

\[ \tilde{P}_J = \int_{\frac{\pi}{2}}^{\infty} d\varrho r \int_{-1}^{1} I \, dz \]  

(38)

and

\[ \varrho = \frac{m \left( 1 - \frac{m^2}{r^2} \right)}{4r^3 \psi_0^0}. \]  

(39)

\[ I = 2r^2 (B_{r\varphi})^2 + 2 \left( B_{\varphi\varphi} + \frac{1}{2} r^2 B_{rr} \right)^2 + \frac{3}{2} r^4 (B_{rr})^2 - \frac{2}{3} r^4 \psi_0^{12}(B)^2. \]  

(40)

Because of (30) and (31) \( I \) contains squares of second derivatives with respect to \( r \). This makes estimating integral (38) rather difficult. In order to reduce order of derivatives in \( I \) we replace \( Q, S \) by new variables \( U, W \), given by

\[ U = Q_r - \kappa S_r, \quad W = Q_z - \kappa S_z. \]  

(41)

In terms of \( U, W \) equations (26)–(28) and (31) take the form

\[ B = \frac{3}{2\kappa_r} (U_z - W_z), \]  

(42)

\[ B_{r\varphi} = \frac{U}{r \sin \vartheta}, \]  

(43)

\[ B_{rr} = \frac{1}{r^3} \left( W + \frac{\kappa}{\kappa_r} (U_z - W_z) \right), \]  

(44)

\[ \triangledown_s F = (rU)_r - \frac{1}{2r} \left( \frac{r^2}{2} - 1 \right) \left( W + \frac{3\kappa}{\kappa_r} (U_z - W_z) \right)_{\, z}. \]  

(45)

Since

\[ \kappa_r = 3\alpha r^2 \psi_0^5, \]  

where

\[ \alpha = 1 - \frac{m}{2r}, \]  

(46)

functions \( B \) and \( B_{rr} \) are finite at \( r = m/2 \) iff

\[ U_z = W_r \text{ at } r = \frac{m}{2}. \]  

(47)
Conditions (33)–(35) transform into
\[ r(rU_0)_r + \frac{1}{3} W_1 + \frac{\kappa}{\kappa_r} ((U_z)_1 - W_1)_r = 0, \quad (48) \]
\[ U, \frac{1}{r} W < \infty, \quad (49) \]
\[ U \sim (z^2 - 1). \quad (50) \]

In terms of \( U, W \) function (40) reads
\[ I = \frac{2U^2}{1 - z^2} + \frac{2F_0^2}{2r^2} W \left( W + \frac{2\kappa}{\kappa_r} (U_z - W_1)_r \right), \quad (51) \]

Substituting (51) to (38) and integrating by parts the term proportional to \( W, r \) leads to
\[ \tilde{P}_1 = \int_{\frac{m}{2}}^{\infty} dr \rho_r \int_{-1}^{1} \tilde{I} dz + \frac{\pi}{8m^2} \int_{-1}^{1} (W^2)_h dz, \quad (52) \]
where index \( h \) denotes value at \( r = m/2 \) and
\[ \tilde{I} = \frac{2U^2}{1 - z^2} + \frac{2F_0^2}{2r^2} W U_z. \quad (53) \]

In order to satisfy (50) and assure simplicity of (52) if the dependent functions are expanded into the Legendre polynomials it is convenient to define new variables \( X, Y \) by
\[ U = \psi_0^3 (z^2 - 1) Y, \quad W = 2r \psi_0^2 (X - \alpha Y). \quad (54) \]
The regularity conditions imply that they are smooth functions of \( r \) and \( z \), bounded at infinity. Condition (47) yields
\[ Y - \triangle_r Y = \frac{m}{2} X_r \quad \text{at} \quad r = \frac{m}{2}. \quad (55) \]

In terms of \( X, Y \) equations (45) and (52) read
\[ \triangle_r \tilde{F} = \frac{1}{\alpha} (z^2 - 1) \partial_z \left( r X_r + \frac{1}{2} \psi_0 \triangle_r Y + \left( \psi_0 - \frac{3m}{r \psi_0} \right) Y \right), \quad (56) \]
\[ \tilde{P}_1 = \frac{m}{2} \int_{\frac{m}{2}}^{\infty} \frac{dr}{r^2 \psi_0} \int_{-1}^{1} (\alpha F_0^2 - X \triangle_r Y) dz + \frac{1}{4} \int_{-1}^{1} (X^2)_h dz, \quad (57) \]
where function \( \tilde{F} \) is related to \( F \) by
\[ \tilde{F} = \psi_0^{-3} F. \quad (58) \]

Let us expand functions \( X \) and \( Y \) into the Legendre polynomials \( P_n \),
\[ X = \sum_{n=0}^{\infty} X_n P_n, \quad Y = \sum_{n=1}^{\infty} Y_n P_n \quad (59) \]
(note that (54) admits \( Y_0 = 0 \)). Condition (55) yields
\[ X_0_r = 0 \quad \text{at} \quad r = \frac{m}{2}. \quad (60) \]
\[ Y_n = \frac{m}{2(2N+1)} X_{n,r} \quad \text{at } r = \frac{m}{2}, \quad n \geq 1, \] (61)

where
\[ N = \frac{1}{2} n(n+1). \] (62)

The integrability condition of (56), equivalent to (48), leads to relation
\[ Y_1 = \frac{r^2 \psi_0}{3m} X_{1,r}, \] (63)

compatible with (61). From (56) one obtains
\[ \tilde{F}_z = \frac{1}{\alpha} \sum_{n=2} \left( r X_{n,r} - \beta_n Y_n \right) \tilde{P}_n, \] (64)

where
\[ \beta_n = (N-1) \psi_0 + \frac{3m}{r \psi_0} \] (65)

and
\[ \tilde{P}_n = \partial_z \left( \triangle^{-1} ((z^2 - 1) \partial_z P_n) \right) = \frac{1}{n+2} \left( nP_n - \frac{2}{n-1} P_{n-1,z} \right), \quad n \geq 2. \] (66)

Polynomials \( \tilde{P}_n \) are orthogonal,
\[ \int_{-1}^{1} \tilde{P}_k \tilde{P}_n dz = c_n \delta_{kn}, \] (67)

where
\[ c_n = \frac{2n(n+1)}{(n-1)(n+2)(2n+1)}. \] (68)

From (57) and (64) it follows that
\[ \tilde{F}_1 = \frac{2m}{3} I_1 + \frac{m}{2} \sum_{n=2} c_n I_n + \frac{1}{2} \sum_{n=0} \frac{X_{m,n}}{2n+1}, \] (69)

where
\[ I_1 = \int_{\frac{m}{2}}^{\infty} X_1 Y_1 \frac{dr}{r^2 \psi_0}. \] (70)

and
\[ I_{n \geq 2} = \int_{\frac{m}{2}}^{\infty} \left[ (r X_{n,r} - \beta_n Y_n)^2 + 2(N-1) \alpha X_n Y_n \right] \frac{dr}{\alpha r^2 \psi_0} \]
\[ = \int_{\frac{m}{2}}^{\infty} \frac{\beta_n^2}{\alpha r^2 \psi_0} \left( Y_n - \frac{r}{\beta_n} X_{n,r} + \frac{(N-1) \alpha}{\beta_n^2} X_n \right)^2 dr - \frac{N-1}{m(2N+1)} X_{m,n}. \] (71)
For a fixed value of $X_{nh}$ functional $I_n \geq 2$ takes its minimum if

$$Y_n = \frac{r}{\beta_n} X_{n,r} - \frac{(N - 1)\alpha}{\beta_n^2} X_n$$  \hspace{1cm} (72)

[note that (72) is compatible with (61)]. Hence

$$I_n \geq -\frac{N - 1}{m(2N + 1)} X_{nh}^2, \quad n \geq 2$$  \hspace{1cm} (73)

and from (63) and (70) one obtains

$$I_1 = \frac{1}{6m} (X_{1,\infty}^2 - X_{1h}^2).$$  \hspace{1cm} (74)

Substituting (73) and (74) into (69) leads to

$$\bar{P}_J \geq \frac{1}{9} X_{1,\infty}^2 + \frac{1}{2} \sum_{n=0}^{\infty} \frac{X_{nh}^2}{(2n + 1)(n^2 + 1)}.$$  \hspace{1cm} (75)

To complete the estimate of the rhs of (36) we still need a harmonic function $\psi_1$ which satisfies the boundary condition

$$\partial_r \psi_1 + \frac{1}{m} \psi_1 = \frac{1}{m} X_h$$ \quad on $S_h$$  \hspace{1cm} (76)

following from (9) and (85). Since $\psi_1$ vanishes at $\infty$ it is given by

$$\psi_1 = \sum_{n=0}^{\infty} a_n \left( \frac{m}{2r} \right)^{n+1} P_n.$$  \hspace{1cm} (77)

Condition (76) imply

$$a_n = -\frac{1}{m(2n + 1)} X_{nh},$$  \hspace{1cm} (78)

hence

$$\langle \psi_1^2 \rangle_h = \frac{4\pi}{m^2} X_{1,\infty}^2 \frac{1}{(2n + 1)^2} X_{nh}^2.$$  \hspace{1cm} (79)

Substituting (75), (79) and (85) into (36) leads to

$$P_J \geq P_J + \frac{1}{9} X_{1,\infty}^2 + \sum_{n=0}^{\infty} \frac{(n - 1)(n + 2) X_{nh}^2}{2(n^2 + n + 1)^2}.$$  \hspace{1cm} (80)

It follows from the ADM formula that the linear momentum (directed along the symmetry axis) is given by

$$p = \frac{1}{3} X_{1,\infty}.$$  \hspace{1cm} (81)

As in [11], the angular momentum $J$ is present in a structure of the function $\omega$ coming from regularity assumption about $A_{ij}$,

$$\omega = f (1 - z^2)^2 + J(z^3 - 3z) + c.$$  \hspace{1cm} (82)
hence
\[ P_J \geq \frac{J^2}{4m^2}. \]  
(83)

Substituting (81), (83) and (85) into (80) yields
\[ E^2 - p^2 \geq \frac{|S_h|}{16\pi} f^2 + \frac{4\pi}{|S_h|} \frac{(n-1)(n+2)}{2(n^2+n+1)(2n+1)^3} X^2_{shh}. \]  
(84)

It follows from (42), (44) and (54) that on \( S_h \) one has
\[ X_h = \frac{m^2}{32} \left( B_{rr} - \frac{2}{3} B_{\psi_0} \right). \]  
(85)

Thanks to (85), inequality (84) for maximal data (\( B = 0 \)) coincides with that in [11]. Moreover, it is saturated if \( f = 0 \) and (72) is satisfied.

For the purpose of this paper let us define generic data in the following way.

**Definition 3.1.** We call data generic if at least one of the following conditions is not satisfied:

(a) The leading terms (of order \( \epsilon \)) in \( A_{\phi\theta} \) and \( A_{\phi r} \) are given by \( B_{\phi\theta} = 0 \) and \( B_{\phi r} = -3Jr^{-2}\sin^2 \theta \).

(b) On the MOTS \( (B_{rr} - \frac{2}{3} B_{\psi_0}) = b_1 + b_2 \cos \theta \), where \( b_i \) are constants.

(c) All moments \( Y_n \) with \( n \geq 2 \) are defined by \( X_n \), according to (72).

Condition (a) corresponds to \( f = 0 \) in \( \omega \). It represents a minimal assumption leading to a nonvanishing angular momentum \( J \). Condition (b) is equivalent to \( X_n \geq 2 \) on MOTS. Concerning condition (c), let us note that (72) is also satisfied if \( n = 1 \) [see (63)]. Since \(-n(n+1)\) are eigenvalues of \( \Delta_s \), equation (72) multiplied by \( \beta_n^2 \) is equivalent to
\[ \psi_0^2 \left( \triangle_s + 2 - \frac{6m}{r\psi_0} \right)^2 Y = -2r\psi_0 \left( \triangle_s + 2 - \frac{6m}{r\psi_0} \right) X_s + 2\alpha(\Delta_s + 2)X + f(r), \]  
(86)

where function \( f \) can be adjusted to \( X_0 \). Action of \( \Delta_s \) on (86) yields the following equation for original leading terms in \( K_{ij} \)
\[ \left( \triangle_s + 2 - \frac{6m}{r\psi_0} \right) \left( B_{i\theta} + \frac{1}{2} r^2 B_{i\phi} \right) \sin^2 \theta \_z = (\Delta_s + 2) \left( B_{i\theta} + \frac{1}{2} r^2 B_{i\phi} \right) \psi_0 - \frac{1}{2} r^2 \Delta_s \left( B_{i\phi} - \frac{2}{3} B_{\psi_0} \right). \]  
(87)

Thus, condition (c) is equivalent to the additional constraint relating values of \( B_{ij} \) and \( B \) on each sphere \( r = \text{const.} \) separately. Equation (87) combined with condition (b) and regularity of \( B_{ij} \) gives
\[ B_{i\theta} + \frac{1}{2} r^2 B_{i\phi} = 0 \quad \text{on} \ S_h. \]  
(88)

Geometric interpretations of (87) and (88) are unclear.
4. Summary

Let us formulate the main result of this paper in a way similar to that in [11] for maximal data.

**Theorem 4.1.** Let \( S = \mathbb{R}^3 \setminus B \left( 0, \frac{m}{2} \right) \) be an initial surface bounded by the sphere \( S_h \) with radius \( m/2 \) and \( g_{ij} \) be flat metric on \( S \). Assume that constraints (7)–(9) admit axially symmetric asymptotically flat solution depending on a parameter \( \epsilon \) in agreement with expansions (12) and (13).

Then, in generic case (see definition 3.1), initial data (6) satisfy the sharp Penrose inequality (2) up to the second order in \( \epsilon \).

Data covered by this theorem depend on three free functions of two variables, which is the maximal number in the axisymmetric case with conformally flat metric. Unlike in [11], we were not able to prove the Penrose inequality for nongeneric perturbations. It follows from conditions (a)–(c) of definition 3.1 that they depend on one free function \( X \) and not on several constants like in [11]. In order to include terms \( \epsilon^3 \) and \( \epsilon^4 \) in \( P_I \) we would have to solve (7) for any \( X \) up to the order of \( \epsilon^2 \) and calculate a lower bound of \( P_I \). We could not solve these technical difficulties.

Admitting nonmaximal data has some advantages. In this case the minimalization of \( P_I \) could be obtained in an algebraic way after an appropriate change of potentials. For maximal data we had to solve differential equations and there was no guarantee that it would be possible.

Can one reconstruct results on generic maximal data in [11] from theorem 4.1? To this end one should show that there is a function \( X \) which together with \( Y \) given by (72) leads to \( B = 0 \). This condition is equivalent to a system of second order linear differential equations on \( X_{\varphi}(r) \) with a singularity on MOTS. We did not prove existence of its solutions but coincidence between inequality (84) for \( B = 0 \) with inequality (102) in [11] indicates that such solutions do exist.

Concerning generalizations of our results, admitting data depending on the azimuthal angle \( \varphi \) seems feasible, but a change of shape of the MOTS or admitting metrics which are not conformally flat would require a new approach, not based on expansions into the Legendre polynomials.

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