Approximation by linear combinations of translates of a single function

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Abstract

We study approximation of periodic functions by arbitrary linear combinations of \( n \) translates of a single function. We construct some linear methods of this approximation for univariate functions in the class induced by the convolution with a single function, and prove upper bounds of the \( L^p \)-approximation convergence rate by these methods, when \( n \to \infty \), for \( 1 \leq p \leq \infty \). We also generalize these results to classes of multivariate functions defined as the convolution with the tensor product of a single function. In the case \( p = 2 \), for this class, we also prove a lower bound of the quantity characterizing best approximation of by arbitrary linear combinations of \( n \) translates of arbitrary function.

Keywords: Function spaces induced by the convolution with a given function ; Approximation by arbitrary linear combinations of \( n \) translates of a single function.

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1 Introduction

The present paper continues investigating the problem of function approximation by arbitrary linear combinations of \( n \) translates of a single function which has been studied in \cite{1, 3}. In the last papers, some linear methods were constructed for approximation of periodic functions in a class induced by the convolution with a given function, and prove upper bounds of the \( L^p \)-approximation convergence rate by these methods, when \( n \to \infty \), for the case \( 1 < p < \infty \). The main technique of the proofs of the results is based on Fourier analysis, in particular, the multiplier theory. However, this technique cannot be extended to the two important cases \( p = 1 \) and \( p = \infty \). In the present paper, we aim at this approximation problem for the cases \( p = 1 \) and \( p = \infty \) by using a different technique. For convenience of presentation we will do this for \( 1 \leq p \leq \infty \).

We shall begin our discussion here by introducing notation used throughout the paper. In this regard, we merely follow closely the presentation in \cite{1, 3}. The \( d \)-dimensional torus denoted by \( \mathbb{T}^d \) is
the cross product of \( d \) copies of the interval \([0, 2\pi]\) with the identification of the end points. When \( d = 1 \), we merely denote the \( d \)-torus by \( T \). Functions on \( \mathbb{T}^d \) are identified with functions on \( \mathbb{R}^d \) which are \( 2\pi \) periodic in each variable. Denote by \( L^p(\mathbb{T}^d) \), \( 1 \leq p \leq \infty \), the space of integrable functions on \( \mathbb{T}^d \) equipped with the norm

\[
\|f\|_p := \begin{cases} 
(2\pi)^{-d/p} \left( \int_{\mathbb{T}^d} |f(x)|^p \, dx \right)^{1/p}, & 1 \leq p < \infty, \\
\text{ess sup}_{x \in \mathbb{T}^d} |f(x)|, & p = \infty.
\end{cases}
\]

We will consider only real valued functions on \( \mathbb{T}^d \). However, all the results in this paper are true for the complex setting. Also, we will use Fourier series of a real valued function in complex form.

Here, we use the notation \( \mathbb{N}_m \) for the set \( \{1, 2, \ldots, m\} \). For vectors \( x := (x_l : l \in \mathbb{N}_d) \) and \( y := (y_l : l \in \mathbb{N}_d) \) in \( \mathbb{T}^d \) we use \((x, y) := \sum_{l \in \mathbb{N}_d} x_ly_l\) for the inner product of \( x \) with \( y \). Also, for notational convenience we allow \( \mathbb{N}_0 \) and \( \mathbb{Z}_0 \) to stand for the empty set. Given any integrable function \( f \) on \( \mathbb{T}^d \) and any lattice vector \( j = (j_l : l \in \mathbb{N}_d) \in \mathbb{Z}^d \), we let \( \hat{f}(j) \) denote the \( j \)-th Fourier coefficient of \( f \) defined by the equation

\[
\hat{f}(j) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(x) e^{-i(j,x)} \, dx.
\]

Frequently, we use the superscript notation \( \mathbb{B}^d \) to denote the cross product of \( d \) copies of a given set \( \mathbb{B} \) in \( \mathbb{R}^d \).

Let \( S'(\mathbb{T}^d) \) be the space of distributions on \( \mathbb{T}^d \). Every \( f \in S'(\mathbb{T}^d) \) can be identified with the formal Fourier series

\[
f = \sum_{j \in \mathbb{Z}^d} \hat{f}(j) e^{i(j,x)},
\]

where the sequence \( (\hat{f}(j) : j \in \mathbb{Z}^d) \) forms a tempered sequence.

Let \( \lambda : \mathbb{R} \to \mathbb{R} \setminus \{0\} \) be a bounded function. With the univariate \( \lambda \) we associate the multivariate tensor product function \( \lambda_d \) given by

\[
\lambda_d(x) := \prod_{l=1}^d \lambda(x_l), \quad x = (x_l : l \in \mathbb{N}_d),
\]

and introduce the function \( \varphi_{\lambda,d} \), defined on \( \mathbb{T}^d \) by the equation

\[
\varphi_{\lambda,d}(x) := \sum_{j \in \mathbb{Z}^d} \lambda_d(j) e^{i(j,x)}. \tag{1.1}
\]

Moreover, in the case that \( d = 1 \) we merely write \( \varphi_{\lambda} \) for the univariate function \( \varphi_{\lambda,1} \). We introduce a subspace of \( L^p(\mathbb{T}^d) \) defined as

\[
\mathcal{H}_{\lambda,p}(\mathbb{T}^d) := \left\{ f : f = \varphi_{\lambda,d} \ast g, \ g \in L^p(\mathbb{T}^d) \right\},
\]

with norm

\[
\|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)} := \|g\|_p,
\]

where \( f_1 \ast f_2 \) is the convolution of two functions \( f_1 \) and \( f_2 \) on \( \mathbb{T}^d \).
As in [1, 3], we are concerned with the following concept. Let $W$ be a prescribed subset of $L^p(\mathbb{T}^d)$ and $\psi \in L^p(\mathbb{T}^d)$ be a given function. We are interested in the approximation in $L^p(\mathbb{T}^d)$-norm of all functions $f \in W$ by arbitrary linear combinations of $n$ translates of the function $\psi$, that is, by the functions in the set \[ \{ \psi(\cdot - y_l) : y_l \in \mathbb{T}^d, l \in \mathbb{N}_n \} \]
and measure the error in terms of the quantity
\[ M_n(W, \psi)_p := \sup_{f \in W} \inf_{c_l} \left\{ \| f - \sum_{l \in \mathbb{N}_n} c_l \psi(\cdot - y_l) \|_p : c_l \in \mathbb{R}, y_l \in \mathbb{T}^d \right\}. \]

The aim of the present paper is to investigate the convergence rate, when $n \to \infty$, of $M_n(U_{\lambda,p}(\mathbb{T}^d), \psi)_p$ for $1 \leq p \leq \infty$, where $U_{\lambda,p}(\mathbb{T}^d) := \{ f \in H_{\lambda,p}(\mathbb{T}^d) : \| f \|_{H_{\lambda,p}(\mathbb{T}^d)} \leq 1 \}$ is the unit ball in $H_{\lambda,p}(\mathbb{T}^d)$. We shall also obtain a lower bound for the convergence rate as $n \to \infty$ of the quantity
\[ M_n(U_{\lambda,2}(\mathbb{T}^d))_2 := \inf \left\{ M_n(U_{\lambda,2}(\mathbb{T}^d), \psi)_2 : \psi \in L^2(\mathbb{T}^d) \right\}, \]
which gives information about the best choice of $\psi$.

This paper is organized in the following manner. In Section 2, we give the necessary background from Fourier analysis and construct a method for approximation of functions in the univariate case. In Section 3, we extend the method of approximation developed in Section 2 to the multivariate case, in particular, prove upper bounds for the approximation error and convergence rate, we also prove a lower bound of $M_n(U_{\lambda,2}(\mathbb{T}^d))_2$.

## 2 Univariate approximation

In this section, we construct a linear method in the form of a linear combination of translates of a function $\varphi_\beta$ defined as in [1, 1] for approximation of univariate functions in $H_{\lambda,p}(\mathbb{T})$. We give upper bounds of the approximation error for various $\lambda$ and $\beta$.

Let $\lambda, \beta, \vartheta : \mathbb{R} \to \mathbb{R}$ be given 2-times continuously differentiable functions and $\vartheta$ be such that
\[ \vartheta(x) := \begin{cases} 1, & \text{if } x \in [-\frac{1}{2}, \frac{1}{2}], \\ 0, & \text{if } x \not\in (-1,1). \end{cases} \]

Corresponding to these functions we define the functions $G$ and $H_m$ as
\[ G(x) := \frac{\lambda(x)}{\beta(x)}, \quad H_m(x) := \sum_{k \in \mathbb{Z}} \vartheta(k/m) G(k) e^{ikx}. \] (2.2)

For a function $f \in H_{\lambda,p}(\mathbb{T})$ represented as $f = \varphi_\lambda * g$, $g \in L^p(\mathbb{T})$, we define the operator
\[ Q_{m,\beta}(f) := \frac{1}{2m+1} \sum_{k=0}^{2m} V_m(g) \left( \frac{k}{2m+1} \right) \varphi_\beta \left( \cdot - \frac{k}{2m+1} \right), \] (2.3)

where $V_m(g) := H_m * g$. Finally, we define for a function $h : \mathbb{R} \to \mathbb{R}$,
\[ \sigma_m(h; f)(x) := \sum_{k \in \mathbb{Z}} h(k/m) \hat{f}_k e^{ikx}. \]
Let us obtain upper estimates for the error of approximating a function \( f \in H_{\lambda,p}(T) \) by the trigonometric polynomial \( Q_{m,\beta}(f) \) a linear combination of \( 2m + 1 \) translates of the function \( \varphi_\beta \).

**Definition 2.1** A 2-times continuously differentiable function \( \psi : \mathbb{R} \to \mathbb{R} \) is called a function of monotone type if there exists a positive constant \( c_0 \) such that

\[
|\psi(x)| \geq c_0 |\psi(y)|, \quad |\psi''(x)| \geq c_0 |\psi''(y)| \quad \text{for all} \quad 2|y| \geq |x| \geq |y|/2.
\]

We put

\[
\varepsilon_m := J_m(\lambda) + \sup_{|x| \in [-m,m]} \left( |g(x)| + m^2 \sup_{|x| \in [-m,m]} |g''(x)| \right) J_m(\beta),
\]

where for a 2-times continuously differentiable function \( \psi \),

\[
J_m(\psi) := \int_{|x| \geq m} \left( \left| \frac{\psi(x)}{m} \right| + \left| x\psi''(x) \right| \right) \, dx.
\]

**Theorem 2.2** Let \( 1 \leq p \leq \infty \). Assume that the functions \( \lambda, \beta \) are of monotone type. Then there exists a positive constant \( c \) such that for all \( f \in H_{\lambda,p}(T) \) and \( m \in \mathbb{N} \),

\[
\|f - Q_{m,\beta}(f)\|_p \leq c \varepsilon_m \|f\|_{H_{\lambda,p}(T)}.
\]

Before we give the proof of the above theorem, we recall a lemma proved in [6], [7].

**Lemma 2.3** Let \( 1 \leq p \leq \infty \), \( f \in L^p(T) \) and \( h : \mathbb{R} \to \mathbb{R} \) be 2-times continuously differentiable function, supported on \([-1,1]\). Then there exists a constant \( c_1 \) independent of \( f, h, m \) such that

\[
\|\sigma_m(h; f)\|_p \leq c_1 \|h''\|_\infty \|f\|_p.
\]

We also need a Landau’s inequality for derivatives [4].

**Lemma 2.4** Let \( f \in L^\infty(\mathbb{R}) \) be 2-times continuously differentiable function. Then

\[
\|f'\|_\infty^2 \leq 4 \|f\|_\infty \|f''\|_\infty.
\]

In particular,

\[
\|f'\|_\infty \leq \|f\|_\infty + \|f''\|_\infty.
\]

**Proof.** (Proof of Theorem 2.2) Let \( f \in H_{\lambda,p}(T) \) be represented as \( \varphi_{\lambda,d} \ast g \) for some \( g \in L^p(T) \). We define the kernel \( P_m(x,t) \) for \( x, t \in T \) as

\[
P_m(x,t) := \frac{1}{2m+1} \sum_{k=0}^{2m} \varphi_\beta \left( x - \frac{k}{2m+1} \right) H_m \left( \frac{k}{2m+1} - t \right).
\]

It is easy to obtain from the definition (2.3) that

\[
Q_{m,\beta}(f)(x) = \frac{1}{2\pi} \int_T P_m(x,t)g(t) \, dt.
\]
We now use equation (1.1), the definition of the trigonometric polynomial $H_m$ given in equation (2.2) and the easily verified fact, for $k, s \in \mathbb{Z}, s \in [-m, m]$, that

$$
\frac{1}{2m+1} \sum_{\ell=0}^{2m} e^{i k (t-(\ell/2m+1))} e^{is((\ell/2m+1)-t)} = \begin{cases} 
0, & \text{if } \frac{k-s}{2m+1} \notin \mathbb{Z}, \\
e^{i(k-k_m)t}, & \text{if } \frac{k-s}{2m+1} \in \mathbb{Z}, 
\end{cases}
$$

to conclude that

$$
P_m(x, t) = \sum_{k \in \mathbb{Z}} \gamma(k)e^{ikx}e^{-ikmt},
$$

where $\gamma(k) = \vartheta(k_m/m)G(k_m)\beta(k)$ and $k_m \in [-m, m]$ satisfy $(k - k_m)/(2m + 1) \in \mathbb{Z}$. Hence,

$$
Q_{m, \beta}(f)(x) = \sum_{k>m} \gamma(k)e^{ikx}\tilde{g}(k_m) + \sum_{k<-m} \gamma(k)e^{ikx}\tilde{g}(k_m) + \sum_{k=-m}^{m} \gamma(k)e^{ikx}\tilde{g}(k_m)
$$

$$
=: A_m(x) + B_m(x) + C_m(x).
$$

Consequently,

$$
\|f - Q_{m, \beta}(f)\|_p \leq \|A_m\|_p + \|B_m\|_p + \|f - C_m\|_p. \quad (2.4)
$$

For each $j \in \mathbb{N}$, we define the functions $\Lambda_{j,m}(x), J_m(x), K_{j,m}(x), D_{j,m}(x)$ and the set $I_{j,m}$ as follows

$$
\Lambda_{j,m}(x) := \beta(mx + j(2m + 1)), \quad J_m(x) := G(mx),
$$

$$
K_{j,m}(x) := \Lambda_{j,m}(x)\vartheta(x)J_m(x), \quad D_{j,m}(x) := \sum_{k \in I_{j,m}} \gamma(k)e^{ikx}\tilde{g}(k_m),
$$

$$
I_{j,m} := \{k \in \mathbb{Z} : (2m + 1)j - m \leq k \leq (2m + 1)j + m\}.
$$

Then we have

$$
A_m(x) = \sum_{j \in \mathbb{N}} \sum_{k \in I_{j,m}} \gamma(k)e^{ikx}\tilde{g}(k_m) = \sum_{j \in \mathbb{N}} D_{j,m}(x), \quad (2.5)
$$

and for all $k \in I_{j,m}$,

$$
\gamma(k) = \beta(k)\vartheta(k_m/m)G(k_m) = \beta(j(2m + 1) + k_m)\vartheta(k_m/m)G(k_m)
$$

$$
= \Lambda_{j,m}(k_m/m)\vartheta(k_m/m)G(k_m) = \Lambda_{j,m}(k_m/m)\vartheta(k_m/m)J_m(k_m/m) = K_{j,m}(k_m/m).
$$

Hence,

$$
D_{j,m}(x) = \sum_{k \in I_{j,m}} \gamma(k)e^{ikx}\tilde{g}(k_m) = \sum_{k_m \in [-m,m]} K_{j,m}(k_m/m)e^{i(j(2m+1)+k_m)x}\tilde{g}(k_m)
$$

$$
= e^{ij(2m+1)x} \sum_{k_m \in [-m,m]} K_{j,m}(k_m/m)e^{ik_m x}\tilde{g}(k_m) = e^{ij(2m+1)x}\sigma_m(K_{j,m}; \tilde{g}).
$$

Therefore, by Lemma 2.3 there exists a constant $c_1$ such that

$$
\|D_{j,m}\|_p \leq c_1\|\gamma(K_{j,m})\|_{\infty}\|g\|_p.
$$
Then it follows from (2.5) that
\[ \|A_m\|_p \leq \sum_{j \in \mathbb{N}} \|D_{j,m}\|_p \leq c_1 \sum_{j \in \mathbb{N}} \|(K_{j,m})''\|_\infty \|g\|_p. \] (2.6)

From the definition of $K_{j,m}$, supp $\vartheta \subset [-1, 1]$, and $\|\vartheta\|_\infty \leq 2\|\vartheta'\|_\infty \leq 4\|\vartheta''\|_\infty$, we deduce that
\[ \|(K_{j,m})''\|_\infty \leq 4\|\vartheta''\|_\infty \sup_{x \in [-1,1]} \left( |\Lambda_{j,m}(x)J_{m}(x)| + |(\Lambda_{j,m}J_m)'(x)| + |(\Lambda_{j,m}J_m)''(x)| \right) \]
\[ \leq 4\|\vartheta''\|_\infty \left( \sup_{x \in I_{j,m}} \left( |\beta(x)| + m|\beta'(x)| + m^2|\beta''(x)| \right) \sup_{x \in [-m,m]} |\mathcal{G}(x)| \right) \]
\[ + m\sup_{x \in I_{j,m}} \left( |\beta(x)| + m|\beta'(x)| \right) \sup_{x \in [-m,m]} |\mathcal{G}'(x)| + m^2 \sup_{x \in I_{j,m}} \sup_{x \in [-m,m]} |\mathcal{G}''(x)| \right). \]

Hence,
\[ \|(K_{j,m})''\|_\infty \leq 4\|\vartheta''\|_\infty \sup_{x \in I_{j,m}} \left( |\beta(x)| + m|\beta'(x)| + m^2|\beta''(x)| \right) \sup_{x \in [-m,m]} \left( |\mathcal{G}(x)| + m|\mathcal{G}'(x)| + m^2|\mathcal{G}''(x)| \right) \]
for all $j \in \mathbb{N}$. Therefore, it follows from (2.6) that
\[ \|A_m\|_p \leq 4c_1\|\vartheta''\|_\infty \sum_{j \in \mathbb{N}} \sup_{x \in I_{j,m}} \left( |\beta(x)| + m|\beta'(x)| + m^2|\beta''(x)| \right) \times \]
\[ \times \sup_{x \in [-m,m]} \left( |\mathcal{G}(x)| + m|\mathcal{G}'(x)| + m^2|\mathcal{G}''(x)| \right) \|g\|_p. \]

So, by Lemma 2.4, we have
\[ \|A_m\|_p \leq 16c_1\|\vartheta''\|_\infty \sum_{j \in \mathbb{N}} \sup_{x \in I_{j,m}} \left( |\beta(x)| + m^2|\beta''(x)| \right) \sup_{x \in [-m,m]} \left( |\mathcal{G}(x)| + m^2|\mathcal{G}''(x)| \right) \|g\|_p. \] (2.7)

Since the function $\alpha, \beta$ is of monotone type, there exists a constant $c_0$ such that
\[ |\alpha(x)| \geq c_0|\alpha(y)|, |\alpha''(x)| \geq c_0|\alpha''(y)|, |\beta(x)| \geq c_0|\beta(y)|, |\beta''(x)| \geq c_0|\beta''(y)| \]
(2.8)
for all $4|y| \geq |x| \geq |y|/4$. Hence,
\[ \sup_{|x| \in I_{j,m}} |\beta(x)| \leq \frac{c_0}{m} \int_{|x| \in I_{j,m}} |\beta(x)|dx, \]
\[ \sup_{|x| \in I_{j,m}} |m^2\beta''(x)| \leq c_0m \int_{|x| \in I_{j,m}} |\beta''(x)|dx. \]

So,
\[ \sum_{j \in \mathbb{N}} \sup_{|x| \in I_{j,m}} \left( |\beta(x)| + m^2|\beta''(x)| \right) \leq c_0 \int_{|x| \geq m} \left( \frac{|\beta(x)|}{m} + m|\beta''(x)| \right) dx \leq c_0 J_m(\beta). \]
Combining this with (2.7), we obtain that
\[ \| A_m \|_p \leq 16c_0c_1}\| \vartheta''\|_\infty c.m\| g\|_p. \] (2.9)

Similarly,
\[ \| B_m \|_p \leq 16c_0c_1}\| \vartheta''\|_\infty c.m\| g\|_p. \] (2.10)

Next, we will estimate \( \| f - C_m \|_p \). Notice that \( \gamma(k) = \vartheta(k/m)\mathcal{G}(k)\beta(k) = \vartheta(k/m)\lambda(k) \) for \( k \in [-m, m] \), and then
\[
\sigma_m(\vartheta; f)(x) = \sum_{k \in \mathbb{Z}} \vartheta(k/m)\hat{f}(k)e^{ikx} = m \sum_{k=-m}^m \vartheta(k/m)\lambda(k)\hat{g}(k)e^{ikx} = \sum_{k=-m}^m \gamma(k)\hat{g}(k)e^{ikx} = C_m(x),
\]
and therefore,
\[ \| f - C_m \|_p = \| f - \sigma_m(\vartheta; f)\|_p. \] (2.11)

We define the functions \( S(x), \Phi_{j,m}(x) \) and \( \Psi_{j,m}(x) \) as
\[
S(x) := \vartheta(x) - \vartheta(x/2), \quad \Phi_{j,m}(x) := \lambda(2^jmx), \quad \Psi_{j,m}(x) := S(x)\Phi_{j,m}(x).
\]

Clearly, we have that
\[
(\vartheta(k/(2^{j+1}m)) - \vartheta(k/(2^j m)))\lambda(k) = S(k/(2^j m))\Phi_{j,m}(k/(2^j m)) = \Psi_{j,m}(k/(2^j m)),
\]
which together with
\[
\sigma_{2^j+1m}(\vartheta; f) - \sigma_{2^jm}(\vartheta; f) = \sum_{k \in \mathbb{Z}} (\vartheta(k/(2^{j+1}m)) - \vartheta(k/(2^j m)))\hat{f}(k)e^{ikx}
\]
\[
= \sum_{k \in \mathbb{Z}} (\vartheta(k/(2^{j+1}m)) - \vartheta(k/(2^j m)))\lambda(k)\hat{g}(k)e^{ikx}
\]
implies that
\[
\sigma_{2^j+1m}(\vartheta; f) - \sigma_{2^jm}(\vartheta; f) = \sum_{k \in \mathbb{Z}} \Psi_{j,m}(k/(2^j m))\hat{g}(k)e^{ikx} = \sigma_{2^jm}(\Psi_{j,m}; g).
\]

Then by Lemma 2.3 \( \| \sigma_{2^j+1m}(\vartheta; f) - \sigma_{2^jm}(\vartheta; f)\|_p \leq c_1\| \Psi_{j,m}''\|_\infty \| g\|_p. \) (2.12)

Moreover, from the definition of \( \Psi_{j,m} \), supp\( S \subset [-2, -1/2] \cup [1/2, 2] \), and \( \| S \|_\infty \leq 2\| S' \|_\infty \leq 4\| S'' \|_\infty \leq 8\| \vartheta'' \|_\infty \), we have that
\[
|\Psi_{j,m}''(x)| = |S''(x)\Phi_{j,m}(x) + 2S'(x)\Phi_{j,m}'(x) + S(x)\Phi_{j,m}''(x)|
\]
\[
\leq 8\| \vartheta'' \|_\infty \sup_{x \in [1/2, 2]} \left( |\Phi_{j,m}(x)| + |\Phi_{j,m}'(x)| + |\Phi_{j,m}''(x)| \right)
\]
\[
\leq 16\| \vartheta'' \|_\infty \sup_{x \in [1/2, 2]} \left( |\Phi_{j,m}(x)| + |\Phi_{j,m}'(x)| \right)
\]
\[
= 16\| \vartheta'' \|_\infty \sup_{x \in [2^{j+1}m, 2^{j+2}m]} \left( |\lambda(x)| + (2^j m)^2|\lambda''(x)| \right)
\]
\[
\leq 64\| \vartheta'' \|_\infty \sup_{x \in [2^{j+1}m, 2^{j+2}m]} \left( |\lambda(x)| + x^2|\lambda''(x)| \right).
\]
Combining this and (2.12), we deduce
\[ \| \sigma_{2^{j+1} m}(\vartheta; f) - \sigma_{2^j m}(\vartheta; f) \|_p \leq 64 c_1 \| \vartheta'' \|_\infty \sup_{|x| \leq 2^{-j-1} m, 2^j+1 m} \left( |\lambda(x)| + |x^2 \lambda''(x)| \right) \|g\|_p. \]
Therefore, by (2.11) and \( \lim_{m \to \infty} \| f - \sigma_{2^j m}(\vartheta; f) \|_p = 0 \), we have that
\[ \| f - C_m \|_p \leq \sum_{j=0}^{\infty} \| \sigma_{2^{j+1} m}(\vartheta; f) - \sigma_{2^j m}(\vartheta; f) \|_p \]
\[ \leq 64 c_1 \| \vartheta'' \|_\infty \sum_{j=0}^{\infty} \sup_{|x| \leq 2^{-j-1} m, 2^j+1 m} \left( |\lambda(x)| + |x^2 \lambda''(x)| \right) \|g\|_p. \] (2.13)

Since (2.8),
\[ \sup_{|x| \leq 2^{-j-1} m, 2^j+1 m} |\lambda(x)| \leq \frac{c_0}{2^j m} \int_{|x| \leq 2^{j+1} m} |\lambda(x)| dx \leq \frac{c_0}{m} \int_{|x| \leq 2^j m} |\lambda(x)| dx, \]
and
\[ \sup_{|x| \leq 2^{-j-1} m, 2^j+1 m} |x^2 \lambda''(x)| \leq 2 c_0 \int_{|x| \leq 2^j m} |x^2 \lambda''(x)| dx. \]
So,
\[ \sum_{j=0}^{\infty} \sup_{|x| \leq 2^{-j-1} m, 2^j+1 m} \left( |\lambda(x)| + |x^2 \lambda''(x)| \right) \leq 2 c_0 \int_{|x| \geq m} \left( \frac{|\lambda(x)|}{m} + |x \lambda''(x)| \right) dx = 2 c_0 J_m(\lambda). \]
Hence, by (2.13), we deduce
\[ \| f - C_m \|_p \leq 128 c_0 c_1 \| \vartheta'' \|_\infty \varepsilon_m \|g\|_p. \] (2.14)
Combining (2.9), (2.10) and (2.13) we have
\[ \| f - Q_{m, \beta}(f) \|_p \leq c \varepsilon_m \| f \|_{\mathcal{H}_{\lambda, p}(\mathbb{T})}. \]

From the above theorem, by letting \( \lambda = \beta \), we obtain the following corollary.

**Corollary 2.5** Let \( 1 \leq p \leq \infty \) and \( \lambda \) be of monotone type. Then there exists a positive constant \( c \) such that for all \( f \in \mathcal{H}_{\lambda, p}(\mathbb{T}) \) and \( m \in \mathbb{N} \),
\[ \| f - Q_{m, \lambda}(f) \|_p \leq c J_m(\lambda) \| f \|_{\mathcal{H}_{\lambda, p}(\mathbb{T})}. \]

**Definition 2.6** Let \( r, \kappa \in \mathbb{R} \). A function \( f : \mathbb{R} \to \mathbb{R} \) will be called a mask of type \( (r, \kappa) \) if \( f \) is an even, 2 times continuously differentiable such that for \( t \geq 1 \), \( f(t) = |t|^{-r} (\log(|t| + 1))^{-\kappa} F(\log |t|) \) for some \( F : \mathbb{R} \to \mathbb{R} \) such that \( |F^{(k)}(t)| \leq a_1 \) for all \( t \geq 1, k = 0, 1, 2 \).

**Theorem 2.7** Let \( 1 \leq p \leq \infty \), \( 1 < r < \infty \), \( \kappa \in \mathbb{R} \) and the function \( \lambda \) be a mask of type \( (r, \kappa) \). Then there exists a positive constant \( c \) such that for all \( f \in \mathcal{H}_{\lambda, p}(\mathbb{T}) \) and \( m \in \mathbb{N} \),
\[ \| f - Q_{m, \lambda}(f) \|_p \leq c m^{-r} (\log m)^{-\kappa} \| f \|_{\mathcal{H}_{\lambda, p}(\mathbb{T})}. \]
Proof. Since the function \( \lambda \) be a mask of type \((r, \kappa)\) and \( r > 1 \),
\[
\int_{|x| \geq m} \frac{\lambda(x)}{m} \, dx \leq a_1 \int_{|x| \geq m} \frac{|x|^{-r} (\log(|x| + 1))^{-\kappa}}{m} \, dx \leq a_2 m^{-r} (\log(m + 1))^{-\kappa} \quad \forall m \in \mathbb{N}. \tag{2.15}
\]
On the other hand,
\[
\int_{|x| \geq m} |x\lambda''(x)| \, dx \leq \int_{|x| \geq m} |x| \left( (|x|^{-r} (\log(|x| + 1))^{-\kappa})'' |F(\log |x|)| + 2(|x|^{-r} (\log(|x| + 1))^{-\kappa})' |F'(\log |x|)| / |x| + (|x|^{-r} (\log(|x| + 1))^{-\kappa})'' |F''(\log |x|) - F'(\log |x|)/x^2 \right) \, dx
\]
\[
\leq a_1 \int_{|x| \geq m} |x| \left( (|x|^{-r} (\log(|x| + 1))^{-\kappa})'' + 2(|x|^{-r} (\log(|x| + 1))^{-\kappa})' / |x| + 2(|x|^{-r} (\log(|x| + 1))^{-\kappa}) / x^2 \right) \, dx
\]
\[
\leq a_3 m^{-r} (\log(m + 1))^{-\kappa}.
\]
Hence, by (2.15), we deduce
\[
J_m(\lambda) \leq a_4 m^{-r} (\log(m + 1))^{-\kappa}.
\]
From this and Corollary 2.5 we complete the proof. \( \square \)

**Corollary 2.8** For \( 1 \leq p \leq \infty, 1 < r < \infty \) and \( \lambda(x) = \beta(x) = x^{-r} \) for \( x \neq 0 \), \( \mathcal{H}_{\lambda,p}(\mathbb{T}) \) becomes the Korobov space \( K_r^p(\mathbb{T}) \). Then we have the estimate as in \([1]\):
\[
M_n(U_{\lambda,p}(\mathbb{T}), \kappa_r)_p \leq cm^{-r}
\]
where \( \kappa_r \) is the Korobov function.

**Definition 2.9** A function \( f : \mathbb{R} \to \mathbb{R} \) is called a function of exponent type if \( f \) is 2 times continuously differentiable and there exists a positive constant \( s \) such that \( f(t) = e^{-s|t|} |F(|t|)| \) for some decreasing function \( F : [0, +\infty) \to (0, +\infty) \).

**Theorem 2.10** Let \( 1 \leq p \leq \infty, 1 < r < \infty, \kappa \in \mathbb{Z} \), the function \( \lambda \) be a mask of type \((r, \kappa)\), the function \( \beta \) of exponent type. Then there exists a positive constant \( c \) such that for all \( f \in \mathcal{H}_{\lambda,p}(\mathbb{T}) \) and \( m \in \mathbb{N} \), we have
\[
\|f - Q_{m,\beta}(f)\|_p \leq cm^{-r} (\log(m + 1))^{-\kappa} \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T})}.
\]

**Proof.** We will use the notation in the proof of Theorem 2.2. For \( k \in I_{j,m} \) we have \( k_m = k - j(2m + 1) \) and then
\[
|\gamma(k)| = \left| \beta(k_m + j(2m + 1)) \vartheta(k_m/m) \frac{\lambda(k_m)}{\beta(k_m)} \right|
\]
\[
= e^{-sj(2m+1)} \frac{\lambda(k_m)F(k_m + j(2m + 1))}{|F(k_m)|} \leq b_1 e^{-sj(2m+1)}.
\]
Hence,
\[
\left\| \sum_{k \in I_{j,m}} \gamma(k) e^{ikx} \tilde{g}(k_m) \right\|_p \leq 3b_1 me^{-sj(2m+1)} \|g\|_p.
\]
This implies that

\[ \|A_m\|_p = \left\| \sum_{j \in \mathbb{N}} \sum_{k \in I_{j,m}} \gamma(k) e^{ikx^\ast \hat{g}(k_m)} \right\|_p \leq 3b_1 \sum_{j \in \mathbb{N}} me^{-s j(2m+1)} \|g\|_p \leq b_2 m^{-r} (\log(m + 1))^{-\kappa} \|g\|_p. \]  

(2.16)

Similarly,

\[ \|B_m\|_p \leq b_2 m^{-r} (\log(m + 1))^{-\kappa} \|g\|_p. \]  

(2.17)

We also known that in the proof of Theorem 2.2 that

\[ \|f - C_m\|_p \leq b_3 \sum_{j=0}^{\infty} \sup_{|x| \in [2^{j-1}m, 2^j m]} \left( |\lambda(x)| + |x^2 \lambda''(x)| \right) \|g\|_p. \]  

(2.18)

We see that

\[
\sup_{|x| \in [2^{j-1}m, 2^j m]} |\lambda(x)| \leq b_4 \int_{|x| \in [2^{j-1}m, 2^j m]} \frac{|\lambda(x)|}{|x|} dx
\]

\[
\sup_{|x| \in [2^{j-1}m, 2^j m]} |x^2 \lambda''(x)| \leq b_4 \int_{|x| \in [2^{j-1}m, 2^j m]} |x \lambda''(x)| dx.
\]

So,

\[
\sum_{j=0}^{\infty} \sup_{|x| \in [2^{j-1}m, 2^j m]} \left( |\lambda(x)| + |x^2 \lambda''(x)| \right) \leq b_4 \int_{|x| \geq m} \left( \frac{|\lambda(x)|}{|x|} + |x \lambda''(x)| \right) dx.
\]

Hence, by (2.18), we deduce that

\[ \|f - C_m\|_p \leq b_3 b_4 \|g\|_p \int_{|x| \geq m} \left( \frac{|\lambda(x)|}{|x|} + |x \lambda''(x)| \right) dx \leq b_5 m^{-r} (\log(m + 1))^{-\kappa} \|g\|_p. \]

Combining this, (2.16), (2.17) and (2.4), we complete the proof.

\[ \square \]

3 Multivariate approximation

In this section, we make use of the univariate operators \( Q_{m,\lambda} \) to construct multivariate operators on sparse Smolyak grids for approximation of functions from \( \mathcal{H}_{\lambda,p}(\mathbb{T}^d) \). Based on this approximation with certain restriction on the function \( \lambda \) we prove an upper bound of \( M_n(U_{\lambda,p}(\mathbb{T}^d), \varphi_{\lambda,d})_p \) for \( 1 \leq p \leq \infty \) as well as a lower bound of \( M_n(U_{\lambda,2}(\mathbb{T}^d))_2 \). The results obtained in this section generalize some results in [1, 2].

3.1 Error estimates for functions in the space \( \mathcal{H}_{\lambda,p}(\mathbb{T}^d) \)

For \( m \in \mathbb{N}^d \), let the multivariate operator \( Q_m \) in \( \mathcal{H}_{\lambda,p}(\mathbb{T}^d) \) be defined by

\[ Q_m := \prod_{j=1}^{d} Q_{m_j,\lambda}, \]  

(3.19)
where the univariate operator \( Q_{m_j,\lambda} \) is applied to the univariate function \( f \) by considering \( f \) as a function of variable \( x_j \) with the other variables held fixed, \( \mathbb{Z}^d_+ := \{ k \in \mathbb{Z}^d : k_j \geq 0, j \in \mathbb{N}_d \} \) and \( k_j \) denotes the \( j \)th coordinate of \( k \).

Set \( \mathbb{Z}^d_{-1} := \{ k \in \mathbb{Z}^d : k_j \geq -1, j \in \mathbb{N}_d \} \). For \( k \in \mathbb{Z}^d_{-1} \), we define the univariate operator \( T_k \) in \( \mathcal{H}_{\lambda,p}(\mathbb{T}) \) by

\[
T_k := I - Q_{2k,\lambda}, \quad k \geq 0, \quad T_{-1} := I,
\]

where \( I \) is the identity operator. If \( \mathbf{k} \in \mathbb{Z}^d_{-1} \), we define the mixed operator \( T_{\mathbf{k}} \) in \( \mathcal{H}_{\lambda,p}(\mathbb{T}^d) \) in the manner of the definition of (3.19) as

\[
T_{\mathbf{k}} := \prod_{i=1}^{d} T_{k_i}.
\]

Set \( |\mathbf{k}| := \sum_{j \in \mathbb{N}_d} |k_j| \) for \( \mathbf{k} \in \mathbb{Z}^d_1 \) and \( k^{-\kappa}_{(2)} := \prod_{j=1}^{d} (k_j + 2)^{-\kappa} \).

**Lemma 3.1** Let \( 1 \leq p \leq \infty, \ 1 < r < \infty, 0 \leq \kappa < \infty \) and the function \( \lambda \) be a mask of type \((r, \kappa)\). Then we have for any \( f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d) \) and \( \mathbf{k} \in \mathbb{Z}^d_{-1} \),

\[
\|T_{\mathbf{k}}(f)\|_p \leq C k^{-\kappa}_{(2)} 2^{-r|\mathbf{k}|} \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)}
\]

with some constant \( C \) independent of \( f \) and \( \mathbf{k} \).

**Proof.** We prove the lemma by induction on \( d \). For \( d = 1 \) it follows from Theorem 2.7. Assume the lemma is true for \( d - 1 \). Set \( \mathbf{x}' := \{ x_j : j \in \mathbb{N}_{d-1} \} \) and \( \mathbf{x} = (\mathbf{x}', x_d) \) for \( \mathbf{x} \in \mathbb{R}^d \). We temporarily denote by \( \|f\|_{p, \mathbf{x}'} \) and \( \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^{d-1}), \mathbf{x}'} \) or \( \|f\|_{p, x_d} \) and \( \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}), x_d} \) the norms applied to the function \( f \) by considering \( f \) as a function of variable \( \mathbf{x}' \) or \( x_d \) with the other variables held fixed, respectively. For \( \mathbf{k} = (\mathbf{k}', k_d) \in \mathbb{Z}^d_{-1} \), we get by Theorems 2.7 and the induction assumption

\[
\|T_{\mathbf{k}}(f)\|_p = \|T_{\mathbf{k}'} T_{k_d}(f)\|_{p, \mathbf{x'}, p, x_d} \leq \|2^{-r|\mathbf{k}'|} k^{-\kappa}_{(2)} T_{k_d}(f)\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^{d-1}), \mathbf{x'}, p, x_d}
\]

\[
= 2^{-r|\mathbf{k}'|} k^{-\kappa}_{(2)} \|T_{k_d}(f)\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^{d-1}), \mathbf{x'}}
\]

\[
\leq 2^{-r|\mathbf{k}'|} k^{-\kappa}_{(2)} 2^{-r k_d (k_d + 2)^{-\kappa}} \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}), x_d} \|\mathcal{H}_{\lambda,p}(\mathbb{T}^{d-1}), \mathbf{x'}
\]

\[
= 2^{-r|\mathbf{k}'|} \prod_{j=1}^{d} (k_j + 2)^{-\kappa} \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)}.
\]

Let the univariate operator \( q_{k} \) be defined for \( k \in \mathbb{Z}^d_+ \), by

\[
q_{k} := Q_{2^k,\lambda} - Q_{2^{k-1},\lambda}, \quad k > 0, \quad q_{0} := Q_{1,\lambda},
\]

and in the manner of the definition of (3.19), the multivariate operator \( q_{\mathbf{k}} \) for \( \mathbf{k} \in \mathbb{Z}^d_+ \), by

\[
q_{\mathbf{k}} := \prod_{j=1}^{d} q_{k_j}.
\]

For \( \mathbf{k} \in \mathbb{Z}^d_+ \), we write \( k \to \infty \) if \( k_j \to \infty \) for each \( j \in \mathbb{N}_d \).
Theorem 3.2 Let \(1 \leq p \leq \infty\), \(1 < r < \infty\), \(0 \leq \kappa < \infty\) and the function \(\lambda\) be a mask of type \((r, \kappa)\). Then every \(f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d)\) can be represented as the series
\[
f = \sum_{k \in \mathbb{Z}_+^d} q_k(f)
\] (3.20)
converging in \(L^p\)-norm, and we have for \(k \in \mathbb{Z}_+^d\),
\[
\|q_k(f)\|_p \leq C 2^{-r|k|} k^{-\kappa}_2 \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)}
\] (3.21)
with some constant \(C\) independent of \(f\) and \(k\).

Proof. Let \(f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d)\). In a way similar to the proof of Lemma 3.1, we can show that
\[
\|f - Q_{2k}(f)\|_p \ll \max_{j \in \mathbb{N}_d} 2^{-rk_j} k_j^\kappa \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)},
\]
and therefore,
\[
\|f - Q_{2k}(f)\|_p \to 0, \ k \to \infty,
\]
where \(2^k = (2^j : j \in \mathbb{N}_d)\). On the other hand,
\[
Q_{2k} = \sum_{s_j \leq k_j, j \in \mathbb{N}_d} q_s(f).
\]
This proves (3.20). To prove (3.21) we notice that from the definition it follows that
\[
q_k = \sum_{e \in \mathbb{N}_d} (-1)^{|e|} T_{k^e},
\]
where \(k^e\) is defined by \(k^e_j = k_j\) if \(j \in e\), and \(k^e_j = k_j - 1\) if \(j \notin e\). Hence, by Lemma 3.1
\[
\|q_k(f)\|_p \leq \sum_{e \in \mathbb{N}_d} \|T_{k^e}(f)\|_p \ll \sum_{e \in \mathbb{N}_d} 2^{-r|e|} (k^e_2)^{-\kappa} \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)} \ll 2^{-r|k|} k^{-\kappa}_2 \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)}.
\]

For approximation of \(f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d)\), we introduce the linear operator \(P_m, m \in \mathbb{N}\), by
\[
P_m(f) := \sum_{|k| \leq m} q_k(f).
\] (3.22)

We give an upper bound for the error of the approximation of functions \(f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d)\) by the operator \(P_m\) in the following theorem.

Theorem 3.3 Let \(1 \leq p \leq \infty\), \(1 < r < \infty\), \(0 \leq \kappa < \infty\) and the function \(\lambda\) be a mask of type \((r, \kappa)\). Then, we have for every \(m \in \mathbb{N}\) and \(f \in \mathcal{H}_{\lambda,p}(\mathbb{T}^d)\),
\[
\|f - P_m(f)\|_p \leq C 2^{-r m} m^{d-1-\kappa} \|f\|_{\mathcal{H}_{\lambda,p}(\mathbb{T}^d)}
\]
with some constant \(C\) independent of \(f\) and \(m\).
Proof. From Theorem 3.2 we deduce that

$$
\|f - P_m(f)\|_p = \left\| \sum_{|k| > m} q_k(f) \right\|_p \leq \sum_{|k| > m} \|q_k(f)\|_p
$$

$$
\ll \sum_{|k| > m} 2^{-r|k|} \|f\|_{H_{\lambda, p}(\mathbb{T}^d)} \ll \|f\|_{H_{\lambda, p}(\mathbb{T}^d)} \sum_{|k| > m} 2^{-r|k|} k_{(2)}^{-r}
$$

$$
\ll 2^{-rm} m^{d-1} \|f\|_{H_{\lambda, p}(\mathbb{T}^d)}.
$$

\[ \blacksquare \]

### 3.2 Convergence rate

We choose a positive integer $m \in \mathbb{N}$, a lattice vector $k \in \mathbb{Z}_+^d$ with $|k| \leq m$ and another lattice vector $s = (s_j : j \in \mathbb{N}_d) \in \prod_{j \in \mathbb{N}_d} Z[2^{k_j} + 1]$ to define the vector $y_{k,s} = \left( \frac{2\pi s_j}{2^{k_j} + 1} : j \in \mathbb{N}_d \right)$. The Smolyak grid on $\mathbb{T}^d$ consists of all such vectors and is given as

$$
G^d(m) := \left\{ y_{k,s} : |k| \leq m, s \in \prod_{j \in \mathbb{N}_d} Z[2^{k_j} + 1] \right\}.
$$

A simple computation confirms, for $m \rightarrow \infty$ that

$$
|G^d(m)| = \sum_{|k| \leq m} \prod_{j \in \mathbb{N}_d} (2^{k_j} + 1) \asymp 2^{d} m^{d-1},
$$

so, $G^d(m)$ is a sparse subset of a full grid of cardinality $2^{dm}$. Moreover, by the definition of the linear operator $P_m$ given in equation (3.22) we see that the range of $P_m$ is contained in the subspace

$$
\text{span}\{ \varphi_{\lambda,d}(\cdot - y) : y \in G^d(m) \}.
$$

Other words, $P_m$ defines a multivariate method of approximation by translates of the function $\varphi_{\lambda,d}$ on the sparse Smolyak grid $G^d(m)$. An upper bound for the error of this approximation of functions from $H_{\lambda, p}(\mathbb{T}^d)$ is given in Theorem 3.3.

Now, we are ready to prove the next theorem, thereby establishing an upper bound of $M_n(U_{\lambda, p}, \varphi_{\lambda,d})_p$.

**Theorem 3.4** If $1 \leq p \leq \infty$, $1 < r < \infty$, $0 \leq \kappa < \infty$ and the function $\lambda$ be a mask of type $(r, \kappa)$, then

$$
M_n(U_{\lambda, p}(\mathbb{T}^d), \varphi_{\lambda,d})_p \ll n^{-r} (\log n)^{r(d-1)-\kappa}.
$$

**Proof.** If $n \in \mathbb{N}$ and $m$ is the largest positive integer such that $|G^d(m)| \leq n$, then $n \asymp 2^m m^{d-1}$ and by Theorem 3.3 we have that

$$
M_n(U_{\lambda, p}(\mathbb{T}^d), \varphi_{\lambda,d})_p \leq \sup_{f \in U_{\lambda, p}(\mathbb{T}^d)} \|f - P_m(f)\|_p \ll 2^{-rm} m^{d-1} \kappa \asymp n^{-r} (\log n)^{r(d-1)-\kappa}.
$$

\[ \blacksquare \]
For $p = 2$, we are able to establish a lower bound for $M_n(U_{\lambda,2}(T^d), \varphi_{\lambda,d})$. We prepare some auxiliary results. Let $\mathbb{P}_q(\mathbb{R}^l)$ be the set of algebraic polynomials on $\mathbb{R}^l$ of total degree at most $q$, and

$$E^m := \{ t = (t_j : j \in N_m) : |t_j| = 1, j \in N_m \}.$$ 

We define the polynomial manifold

$$M_{m,l,q} := \left\{ (p_j(u) : j \in N_m) : p_j \in \mathbb{P}_q(\mathbb{R}^l), j \in N_m, u \in \mathbb{R}^l \right\}.$$ 

Denote by $\|x\|_2$ the Euclidean norm of a vector $x$ in $\mathbb{R}^m$. The following lemma was proven in [5].

Lemma 3.5 Let $m,l,q \in \mathbb{N}$ satisfy the inequality $l \log(4emq) \leq m/4$. Then there is a vector $t \in E^m$ and a positive constant $c$ such that

$$\inf \{ \|t - x\|_2 : x \in M_{m,l,q} \} \geq cm^{1/2}.$$ 

Theorem 3.6 If $1 < r < \infty$, $0 \leq \kappa < \infty$ and the function $\lambda$ be a mask of type $(r,\kappa)$, then we have

$$n^{-r}(\log n)^{r(d-2)-d\kappa} \ll M_n(U_{\lambda,2}) \ll n^{-r}(\log n)^{r(d-1)-\kappa}.$$ 

(3.23)

Proof. The upper bound of (3.23) is in Theorem 3.4. Let us prove the lower bound by developing a technique used in the proofs of [5, Theorem 1.1] and [1, Theorem 4.4]. For a positive number $a$ we define a subset $H(a)$ of lattice vectors by

$$H(a) := \left\{ k = (k_j : j \in N_d) \in \mathbb{Z}^d : \prod_{j \in N_d} |k_j| \leq a \right\}.$$ 

Notice that $|H(a)| \asymp a(\log a)^{d-1}$ when $a \to \infty$. To apply Lemma 3.5, for any $n \in \mathbb{N}$, we take $q = [n(\log n)^{-d+2}] + 1$, $m = 5(2d + 1)|n \log n|$ and $l = (2d + 1)n$. With these choices we obtain

$$|H(q)| \asymp m \quad \text{(3.24)}$$ 

and

$$q \asymp m(\log m)^{-d+1} \quad \text{(3.25)}$$

as $n \to \infty$. Moreover, we have that

$$\lim_{n \to \infty} \frac{\log \left( \frac{4emq}{l} \right)}{\frac{l}{m}} = \frac{1}{5},$$

and therefore, the assumption of Lemma 3.5 is satisfied for $n \to \infty$.

Now, let us specify the polynomial manifold $M_{m,l,q}$. To this end, we put $\zeta := q^{-r}m^{-1/2}(\log q)^{-d\kappa}$ and let $Y$ be the set of trigonometric polynomials on $T^d$, defined by

$$Y := \left\{ f = \zeta \sum_{k \in H(q)} a_k t_k : t = (t_k : k \in H(q)) \in E^{|H(q)|} \right\}.$$ 

If $f \in Y$ and

$$f = \zeta \sum_{k \in H(q)} a_k t_k,$$
then \( f = \varphi_{\lambda,d} + g \) for some trigonometric polynomial \( g \) such that
\[
\|g\|_{L^2(\mathbb{T}^d)}^2 \leq \zeta^2 \sum_{k \in \mathbb{H}(q)} |\lambda(k)|^{-2}.
\]
Since
\[
\zeta^2 \sum_{k \in \mathbb{H}(q)} |\lambda(k)|^{-2} \leq \zeta^2 q^{2r} \sum_{k \in \mathbb{H}(q)} \log \left| \prod_{j=1}^d k_j \right|^{2\kappa} 
\]
\[
\leq \zeta^2 q^{2r} \sum_{k \in \mathbb{H}(q)} \sum_{j=1}^n \log k_j \leq \zeta^2 q^{2r} (\log q)^{2\kappa} |\mathbb{H}(q)| = m^{-1} |\mathbb{H}(q)|,
\]
by (3.24) that there is a positive constant \( c \) such that \( \|g\|_{L^2(\mathbb{T}^d)} \leq c \) for all \( n \in \mathbb{N} \). Therefore, we can either adjust functions in \( \mathbb{Y} \) by dividing them by \( c \), or we can assume without loss of generality that \( c = 1 \), and obtain \( \mathbb{Y} \subseteq U_{\lambda,2}(\mathbb{T}^d) \).

We are now ready to prove the lower bound for \( M_n(U_{\lambda,2}(\mathbb{T}^d)) \). We choose any \( \varphi \in L^2(\mathbb{T}^d) \) and let \( v \) be any function formed as a linear combination of \( n \) translates of the function \( \varphi \):
\[
v = \sum_{j \in \mathbb{N}_n} c_j \varphi(\cdot - y_j).
\]
By the well-known Bessel inequality we have for a function
\[
f = \zeta \sum_{k \in \mathbb{H}(q)} a_k t_k \in \mathbb{Y},
\]
that
\[
\|f - v\|_{L^2(\mathbb{T}^d)}^2 \geq \zeta^2 \sum_{k \in \mathbb{H}(q)} \left| t_k - \frac{\hat{\varphi}(k)}{\zeta} \sum_{j \in \mathbb{N}_n} c_j e^{i(y_j,k)} \right|^2.
\]
\[
\text{(3.26)}
\]
We introduce a polynomial manifold so that we can use Lemma 3.5 to get a lower bound for the expressions on the left hand side of inequality (3.26). To this end, we define the vector \( c = (c_j : j \in \mathbb{N}_n) \in \mathbb{R}^n \) and for each \( j \in \mathbb{N}_n \), let \( z_j = (z_{j,l} : l \in \mathbb{N}_d) \) be a vector in \( \mathbb{C}^d \) and then concatenate these vectors to form the vector \( z = (z_j : j \in \mathbb{N}_n) \in \mathbb{C}^{nd} \). We employ the standard multivariate notation
\[
z_j^k = \prod_{l \in \mathbb{N}_d} z_{j,l}^k
\]
and require vectors \( w = (c, z) \in \mathbb{R}^n \times \mathbb{C}^{nd} \) and \( u = (c, \text{Re} z, \text{Im} z) \in \mathbb{R}^l \) to be written in concatenate form. Now, we introduce for each \( k \in \mathbb{H}(q) \) the polynomial \( q_k \) defined at \( w \) as
\[
q_k(w) := \frac{\hat{\varphi}(k)}{\zeta} \sum_{j \in \mathbb{H}(q)} c_j z_j^k.
\]
We only need to consider the real part of \( q_k \), namely, \( p_k = \text{Re} q_k \) since we have that
\[
\inf \left\{ \sum_{k \in \mathbb{H}(q)} \left| t_k - \frac{\hat{\varphi}(k)}{\zeta} \sum_{j \in \mathbb{N}_n} c_j e^{i(y_j,k)} \right|^2 : c_j \in \mathbb{R}, y_j \in \mathbb{T}^d \right\} \geq \inf \left\{ \sum_{k \in \mathbb{H}(q)} |t_k - p_k(u)|^2 : u \in \mathbb{R}^l \right\}.
\]
\[15\]
Therefore, by Lemma 3.5 and (3.25) we conclude there is a vector \( t^0 = (t_k^0 : k \in \mathbb{H}(q)) \in \mathbb{E}^{h_q} \) and the corresponding function
\[
 f^0 = \zeta \sum_{k \in \mathbb{H}(q)} t_k^0 \chi_k \in \mathbb{Y}
\]
for which there is a positive constant \( c \) such that for every \( v \) of the form
\[
 v = \sum_{j \in \mathbb{N}_n} c_j \varphi(\cdot - y_j),
\]
we have that
\[
 \| f^0 - v \|_{L^2(\mathbb{T}^d)} \geq c \zeta m^{\frac{1}{2}} = q^{-r}(\log q)^{-d\kappa} \asymp n^{-r}(\log n)^{r(d-2)-d\kappa}
\]
which proves the lower bound of (3.23).

Similar to the proof of the above theorem, we can prove the following theorem for the case \(-\infty < \kappa < 0\).

**Theorem 3.7** If \( 1 < r < \infty, -\infty < \kappa < 0 \) and the function \( \lambda \) be a mask of type \((r, \kappa)\), then we have that
\[
 n^{-r}(\log n)^{r(d-2)-\kappa} \ll M_n(U_{\lambda, 2}(\mathbb{T}^d))_2 \ll n^{-r}(\log n)^{r(d-1)-d\kappa}.
\]

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