Alexander quandles of order 16

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Abstract
Isomorphism classes of Alexander quandles of order 16 are determined, and classes of connected quandles are identified. This paper extends the list of distinct connected finite Alexander quandles.

Keywords: Alexander quandles, finite quandles
2000 MSC: 57M27

1 Introduction
A quandle is a set $Q$ with a non-associative binary operation $\triangleright: Q \times Q \to Q$ satisfying

(i) for every $a \in Q$, we have $a \triangleright a = a$,
(ii) for every pair $a, b \in Q$ there is a unique $c \in Q$ such that $a = c \triangleright b$, and
(iii) for every $a, b, c \in Q$, we have $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$.

The three quandle axioms essentially form an algebraic distillation of the the three Reidemeister moves, which naturally makes quandles useful for defining invariants of knots and links. In [1], the fundamental quandle of a topological space is defined, and a Wirtinger-style presentation by generators and relations is given for the fundamental quandle of a knot or link complement. As with groups, distinguishing quandles defined by generators and relations is a non-trivial problem itself, but various techniques exist for using quandles to distinguish knots, such as the 2-cocycle invariants defined in [2].

For the purpose of computing knot invariants using quandles, it is useful to compute isomorphism classes of finite quandles, particularly those of finite connected quandles. Every group is a quandle with quandle operation given by conjugation, i.e. $a \triangleright b = b^{-1}ab$. Indeed, any union of conjugacy classes in a group forms a quandle.

One important class of quandles is the category of Alexander quandles. Let $M$ be any module over the ring $A = \mathbb{Z}[t^\pm]$ of Laurent polynomials in one variable. Then $M$ is a quandle with quandle operation given by

$$a \triangleright b = ta + (1 - t)b.$$  

A quandle is connected if it has a single orbit under $\triangleright$; for Alexander quandles, this is equivalent to $(1 - t)M = M$. Connected quandles are of particular importance in applications of quandle theory to knot theory since all knot quandles are connected.

In [3], a method was given for determining all distinct isomorphism classes of Alexander quandles of a given finite order $n$, and the numbers of distinct isomorphism classes were listed for values of $n$ up to 15. In this paper, we compute all distinct isomorphism classes of Alexander quandles with 16 elements and identify which of these are connected.

2 Computations
An abelian group $M$ may be given the structure of a $A$-module, and hence an Alexander quandle, by defining $tm = \phi(m)$ where $\phi \in \text{Aut}_A(M)$ for each $m \in M$. Note that $\phi$ must be an automorphism in order to define multiplication by $t^{-1}$ as $\phi^{-1}$.
We will use the following theorem, proved in [3]:

**Theorem 1** Let $M$ and $M'$ be Alexander quandles with finite cardinality $|M| = |M'|$. Then there is an isomorphism of Alexander quandles $f : M \rightarrow M'$ iff there is an isomorphism of $\Lambda$-modules $h : (1-t)M \rightarrow (1-t)M'$.

That is, we can compare Alexander quandles by comparing their $\Lambda$-submodules $\text{Im}(1-t)$. It is also useful to note the following lemma:

**Lemma 2** Let $t_1, t_2 \in \text{Aut}_Z(M)$ be $Z$-automorphisms of $M$. Then the $\Lambda$-module structures $(M, t_1)$ and $(M, t_2)$ determined by $t_1$ and $t_2$ are isomorphic iff $t_1$ is conjugate to $t_2$.

**Proof.** Let $f : (M, t_1) \rightarrow (M, t_2)$ be an isomorphism of $\Lambda$-modules. Then $f(t_1m) = t_2f(m)$ says that $t_1 = f^{-1}t_2f \in \text{Aut}_Z(M)$. Conversely, if there is an $f \in \text{Aut}_Z(M)$ with $t_1 = f^{-1}t_2f$, then we have $f(t_1m) = t_2f(m)$ and $f$ is a $Z$-automorphism which takes the action of $t_1$ to the action of $t_2$ on $M$; that is, $f$ is an isomorphism of $\Lambda$-modules. $\square$

We may divide the problem of determining isomorphism classes of Alexander quandles with 16 elements into cases depending on which abelian group $M$ of order 16 forms the underlying abelian group. The possibilities are $M = \mathbb{Z}_{16}$ (the linear Alexander quandles), $M = (\mathbb{Z}_2)^4$, $M = \mathbb{Z}_4 \oplus \mathbb{Z}_4$, $M = \mathbb{Z}_8 \oplus \mathbb{Z}_2$, and $M = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

### 2.1 Linear Alexander quandles of order 16

We are able to compute the linear Alexander quandles of order 16 by using Corollary 2.2 in [3], which says:

**Corollary 3** Let $a$ and $b$ be coprime to $n$. Then the Alexander quandles $\Lambda_n/t-a$ and $\Lambda_n/t-b$ are isomorphic iff $N(n, a) = N(n, b)$ and $a \equiv b \mod N(n, a)$, where $N(n, a) = \frac{16}{\gcd(n, 1-a)}$.

To do this we must first compute $N(16, a) = \frac{16}{\gcd(16, 1-a)}$ for each $a \in \mathbb{Z}_{16}$ coprime to 16. This yields four different values for $N$; from the corollary we can tell there are three pairs of linear Alexander quandles which are isomorphic to one another, namely $\Lambda_{16}/t-3 \cong \Lambda_{16}/t-11$, $\Lambda_{16}/t-7 \cong \Lambda_{16}/t-15$, and $\Lambda_{16}/t-5 \cong \Lambda_{16}/t-13$. The quandle $\Lambda_{16}/t-9$ and the trivial quandle $\Lambda_{16}/t-1 \cong T_{16}$ form their own isomorphism classes.

For the purpose of comparing the linear Alexander quandles of order 16 with other Alexander quandles of order 16, we must still compute the submodules $\text{Im}(1-t)$ for each linear quandle. The result are summarized in figure 2.1.

| $M$         | $(1-t)M$ |
|-------------|----------|
| $\Lambda_{16}/t-1$ | $0$      |
| $\Lambda_{16}/t-3$ | $\mathbb{Z}/t-3$ |
| $\Lambda_{16}/t-11$ | $\mathbb{Z}/t-3$ |
| $\Lambda_{16}/t-7$ | $\mathbb{Z}/t-7$ |
| $\Lambda_{16}/t-15$ | $\mathbb{Z}/t-7$ |
| $\Lambda_{16}/t-5$ | $\mathbb{Z}/t+3$ |
| $\Lambda_{16}/t-13$ | $\mathbb{Z}/t+3$ |
| $\Lambda_{16}/t-9$ | $\mathbb{Z}/t+1$ |

Figure 1: $\text{Im}(1-t)$ for linear Alexander quandles of order 16.

### 2.2 Alexander quandles structures on $(\mathbb{Z}_2)^4$

To compute this case, we realize that $(\mathbb{Z}_2)^4$ is not just a $\mathbb{Z}$-module but also a $\mathbb{Z}_2$-module, and since $\mathbb{Z}_2$ is a principal ideal domain, we are able to use the classification theorem for modules over
a PID. We start by listing all $\Lambda$-modules whose underlying abelian group is $(\mathbb{Z}_2)^4$. Specifically, these have the form

$$\bigoplus_{i=1}^n \Lambda_2/h_i,$$

where $h_1|h_2|\ldots|h_n$, and $\sum_{i=1}^n \deg(h_i) = 4$.

With this list and theorem 1, we are able to compare the Alexander quandles by computing $(1-t)/M$ for each module $M = \Lambda_2/h$ where $h \in \Lambda_2$ is a polynomial over $\mathbb{Z}_2$ with lead coefficient 1 and constant term 1. That is, we must multiply every element in each module $M = \Lambda_2/h$ by $(1-t)$ and reduce modulo $h$. Then we must identify the resulting submodule. Note that when $h(a,b,c) = 1 + at + bt^2 + ct^3 + t^4$ with $a,b,c \in \mathbb{Z}_2$ has an even number of terms, the submodules $(1-t)/M/h(a,b,c)$ are equal as sets (though distinct as $\Lambda$-modules), and the same is true for all $h(a,b,c)$ with an odd number of terms.

| Connected | $M$ | $(1-t)M$ |
|-----------|-----|----------|
| | $\Lambda_2/t^4 + t + 1$ | $\Lambda_2/t^4 + t + 1$ |
| | $\Lambda_2/t^4 + t^2 + 1$ | $\Lambda_2/t^4 + t^2 + 1$ |
| | $\Lambda_2/t^4 + t^3 + 1$ | $\Lambda_2/t^4 + t^3 + 1$ |
| * | $\Lambda_2/t^4 + t^3 + t^2 + t + 1$ | $\Lambda_2/t^4 + t^3 + t^2 + t + 1$ |
| | $\Lambda_2/t^4 + 1$ | $\Lambda_2/t^4 + 1$ |
| | $\Lambda_2/t^4 + t^2 + t + 1$ | $\Lambda_2/t^4 + t^2 + 1$ |
| | $\Lambda_2/t^4 + t^3 + t^2 + 1$ | $\Lambda_2/t^4 + t^3 + t + 1$ |
| | $\Lambda_2/t^4 + t^3 + t + 1$ | $\Lambda_2/t^4 + t^3 + 1$ |
| * | $(\Lambda_2/t^4 + t + 1)^2$ | 0 |
| | $(\Lambda_2/t + 1)^2 \oplus \Lambda_2/t^2 + 1$ | $\Lambda_2/t + 1$ |
| | $(\Lambda_2/t^2 + 1)^2$ | $(\Lambda_2/t + 1)^2$ |
| | $(\Lambda_2/t^2 + t + 1)^2$ | $(\Lambda_2/t^2 + t + 1)^2$ |
| | $\Lambda_2/t + 1 \oplus \Lambda_2/t^3 + 1$ | $\Lambda_2/t^2 + t + 1$ |
| | $\Lambda_2/t + 1 \oplus \Lambda_2/t^3 + t^2 + t + 1$ | $\Lambda_2/t^2 + 1$ |

Figure 2: Im(1−t) for Alexander quandles with abelian group (Z2)4

2.3 Alexander quandles defined by $\mathbb{Z}$-automorphisms

In light of lemma 2, it is sufficient to consider only a single representative from each conjugacy class. Using a maple program, we first compute the $\mathbb{Z}$-automorphism group of $M$. We represent an element $\phi \in \text{Aut}_\mathbb{Z}(M)$ by listing an image for each element of a generating set. The program checks each such set of images to determine whether the linear map thus defined is an automorphism. The program then compares the automorphisms pairwise for conjugacy and deletes redundant automorphisms, yielding a single representative for each conjugacy class. We then compute the $\Lambda$-submodule Im(Id − $\phi$) for each representative automorphism $\phi$.

We applied this procedure for $\mathbb{Z}_6 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The results are collected in table 2.

Two of the classes of linear quandles are isomorphic to quandles listed in the table, namely $\Lambda_4/|t-9| \cong (\Lambda_2/t + 1)^5 \oplus \Lambda_2/t^2 + 1$ and $\Lambda_4/t - 5 \cong (\mathbb{Z}_4 \oplus \mathbb{Z}_4, \phi(1,0) = (0,1), \phi(0,1) = (3,2))$.

Together with the results from the previous two sections, we have our main result, namely:

**Theorem 4** There are a total of 23 distinct isomorphism classes of Alexander quandles with 16 elements. Of these, eight are connected, including five quandles with underlying abelian group $(\mathbb{Z}_2)^4$ and three quandles with underlying abelian group $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.

**Proof.** In light of theorem 1, this is simply a matter of counting distinct submodules Im(1−t). In all, a total of 23 distinct submodules appear; of these, eight are connected, namely $\Lambda_2/t^4+t^2+1$, $\Lambda_2/t^4+t^3+1$, $\Lambda_2/t^4+t^3+t^2+t+1$, $(\Lambda_2/t^4+t^3+t)^2$ and $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ with Alexander quandle structure given by $\phi(1,0) = (1,0)$ and $\phi(0,1) = (1,1), (3,1)$ and $(3,3)$.
Automorphisms of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$

| $\phi((1,0))$ | $\phi((0,1))$ | $\text{Im}(\text{Id} - \phi)$ |
|---------------|---------------|------------------|
| (1,0)         | (0,1)         | $\mathbb{Z}_2/t + 1$ |
| (1,0)         | (2,1)         | $\mathbb{Z}_2/t + 1$ |
| (1,2)         | (2,1)         | $\mathbb{Z}_2/t + 1$ |
| (1,2)         | (2,3)         | $(\mathbb{Z}_2/t + 1)^2$ |
| (3,0)         | (0,3)         | $(\mathbb{Z}_2/t + 1)^2$ |
| (0,1)         | (3,2)         | $\mathbb{A}_4/t + 3$ |
| (0,1)         | (1,0)         | $\mathbb{A}_4/t + 1$ |
| (0,1)         | (3,0)         | $(\mathbb{Z}_4 \oplus \mathbb{Z}_4, \phi')$ |
| (0,1)         | (1,2)         | $(\mathbb{Z}_4 \oplus \mathbb{Z}_2, \phi''')$ |

$\phi'(1,0) = (3,1)$, $\phi'(0,1) = (2,1)$

$\phi''(1,0) = (1,1)$, $\phi''(0,1) = (2,1)$

Automorphisms of $\mathbb{Z}_8 \oplus \mathbb{Z}_2$

| $\phi((1,0))$ | $\phi((0,1))$ | $\text{Im}(\text{Id} - \phi)$ |
|---------------|---------------|------------------|
| (1,0)         | (0,1)         | 0 |
| (1,1)         | (0,1)         | $\mathbb{Z}_2/t + 1$ |
| (1,0)         | (4,1)         | $\mathbb{A}_2/t + 1$ |
| (3,1)         | (0,1)         | $\mathbb{A}_2/t + 1$ |
| (7,0)         | (0,1)         | $\mathbb{A}_2/t + 1$ |
| (3,1)         | (4,1)         | $\mathbb{A}_4/t + 1$ |

Automorphisms of $\mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$

| $\phi((1,0,0))$ | $\phi((0,1,0))$ | $\phi((0,0,1))$ | $\text{Im}(\text{Id} - \phi)$ |
|-----------------|-----------------|-----------------|------------------|
| (1,0,0)         | (0,1,0)         | (0,0,1)         | 0 |
| (1,0,0)         | (0,1,0)         | (2,0,1)         | $\mathbb{A}_2/t + 1$ |
| (0,1,0)         | (0,1,0)         | (0,0,1)         | $\mathbb{A}_2/t + 1$ |
| (3,0,0)         | (0,1,0)         | (0,0,1)         | $\mathbb{A}_2/t + 1$ |
| (1,0,1)         | (2,1,0)         | (0,0,1)         | $(\mathbb{A}_2/t + 1)^2$ |
| (1,0,1)         | (2,1,1)         | (0,0,1)         | $(\mathbb{A}_2/t + 1)^2$ |
| (1,0,0)         | (0,0,1)         | (2,1,0)         | $\mathbb{A}_2/t^2 + 1$ |
| (1,0,1)         | (0,0,1)         | (0,1,0)         | $\mathbb{A}_2/t^2 + 1$ |
| (1,0,1)         | (0,1,0)         | (2,0,1)         | $\mathbb{A}_2/t^2 + 1$ |
| (1,0,0)         | (0,0,1)         | (0,1,1)         | $\mathbb{A}_2/t^2 + t + 1$ |
| (1,0,1)         | (0,0,1)         | (2,1,1)         | $\mathbb{A}_2/t^3 + 1$ |
| (1,0,1)         | (0,0,1)         | (2,1,0)         | $\mathbb{A}_2/t^3 + t + 1$ |

* Connected quandles.

Figure 3: Results of computation of $\text{Im}(\text{Id} - \phi)$ for Alexander quandles given by automorphisms.

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