Lasso Guarantees for Time Series Estimation
Under Subgaussian Tails and $\beta$-Mixing

Kam Chung Wong
Department of Statistics
University of Michigan
kamwong@umich.edu

Zifan Li
Departments of Mathematics
University of Michigan
zifanli@umich.edu

Ambuj Tewari
Departments of Statistics and EECS
University of Michigan
tewaria@umich.edu

Abstract

Many theoretical results on estimation of high dimensional time series require specifying an underlying data generating model (DGM). Instead, this paper relies only on (strict) stationarity and $\beta$-mixing condition to establish consistency of the Lasso when data comes from a $\beta$-mixing process with marginals having subgaussian tails. We establish non-asymptotic inequalities for estimation and prediction errors of the Lasso estimate of the best linear predictor in dependent data. Applications of these results potentially extend to non-Gaussian, non-Markovian and non-linear times series models as the examples we provide demonstrate. In order to prove our results, we derive a novel Hanson-Wright type concentration inequality for $\beta$-mixing subgaussian random vectors that may be of independent interest.

1 Introduction

Efficient estimation methods in high dimensional statistics \cite{6, 17} include methods based on convex relaxation (see, e.g., \cite{7, 30}) and methods using iterative optimization techniques (see, e.g., \cite{4, 1, 15}). A lot of work in the past decade has improved our understanding of the theoretical properties of these algorithms. However, the bulk of existing theoretical work focuses on iid samples. The extension of theory and algorithms in high dimensional statistics to time series data is just beginning to occur as we briefly summarize in Section 1.1 below. Note that, in time series applications, dependence among samples is the norm.
rather than the exception. So the development of high dimensional statistical
theory to handle dependence is a pressing concern in time series estimation.

In this paper, we give guarantees for $\ell_1$-regularized least squares estimation, or
Lasso [18], that hold even when there is temporal dependence in data. Recently, [3] took a step forward in providing guarantees for Lasso in the time series
setting. They considered Gaussian VAR models with finite lag (see Example 1)
and defined a measure of stability using the spectral density, which is the Fourier
transform of the autocovariance function of the time series. Then they showed
that one can derive error bounds for Lasso in terms of their measure of stability.
Their bounds are an improvement over previous work [29, 23, 17] that assumed
operator norm bounds on the transition matrix. These operator norm conditions
are restrictive even for VAR models with a lag of 1 and never hold if the lag
is strictly larger than 1! Therefore, the results of [3] are very interesting. But
they do have limitations.

A key limitation is that [3] assume that the VAR model is the true data generating mechanism (DGM). Their proof techniques rely heavily on having the VAR
representation of the stationary process available. The VAR model assumption,
though popular, can be restrictive. The VAR family is not closed under linear
transformations: if $Z_t$ is a VAR process then $CZ_t$ may not expressible as a fi-
nite lag VAR [24]. We later provides an example (Example 3) of VAR processes
where omitting a single variable breaks down the VAR assumption. What if
we do not assume that $Z_t$ is a finite lag VAR process but simply that it is sta-
tionary? Under stationarity (and finite 2nd moment conditions), the best linear
predictor of $Z_t$ in terms of $Z_{t-d}, \ldots, Z_{t-1}$ is well defined even if $Z_t$ is not a lag
d VAR. If we assume that this best linear predictor involves sparse coefficient
matrices, can we still guarantee consistent parameter estimation? Our paper
provides an affirmative answer to this important question.

We provide finite sample parameter estimation and prediction error bounds for
Lasso in stationary processes with subgaussian marginals and geometrically de-
caying $\beta$-mixing coefficients (Corollary 4). It is well known that guarantees
for Lasso follow if one can establish restricted eigenvalue (RE) conditions and
provide deviation bounds (DB) for the correlation of the noise with the regres-
sors (see Theorem 5 in Appendix B below for a precise statement). Therefore,
the bulk of the technical work in this paper boils down to establishing, with
high probability, that the RE and DB conditions hold under the subgaussian
$\beta$-mixing assumptions. (Propositions 2, 3). Note that RE conditions were pre-
viously shown to hold under the iid assumption by [36] for Gaussian random
vectors and by [39] for subgaussian random vectors. Our results rely on novel
centration inequality (Lemma 1) for $\beta$-mixing subgaussian random variables
that may be of independent interest. The inequality is proved by applying a
blocking trick to Bernstein’s concentration inequality for iid random variables.
All proofs are deferred to the appendix.

To illustrate potential applications of our results, we present four examples.
Example 1 considers a vanilla Gaussian VAR. Example 2 considers VAR models
with subgaussian innovations. Examples 3 is concerned with subgaussian VAR models when the model is mis-specified. Lastly, we go beyond linear models and introduce non-linearity in the DGM in Example 4. To summarize, our theory for Lasso in high dimensional time series estimation extends beyond the classical linear Gaussian settings and provides guarantees potentially in the presence of model mis-specification, subgaussian innovations and/or non-linearity in the DGM.

1.1 Recent Work on High Dimensional Time Series

So far we have only mentioned the one work [3] that is most closely related to ours. But we wish to emphasize that several other researchers have recently published work on statistical analysis of high dimensional time series. [41], [48] and [2] give theoretical guarantees assuming that RE conditions hold. As [3] pointed out, it takes a fair bit of work to actually establish RE conditions in the presence of dependence. [10, 11, 12] use high dimensional time series for global macroeconomic modeling. Alternatives to Lasso that have been explored include the Dantzig selector [17], quantile based methods for heavy-tailed data [35], quasi-likelihood approaches [43], and two-stage estimation techniques [13]. Both [43] as well as [20] establish oracle inequalities for the Lasso applied to time series prediction. [43] established finite sample $\ell_1$-regularized (some other penalties were also considered) LS and GLS estimator error bounds for VAR models with generic innovations assuming their 4th moments exist. Their results allow for dimension $p$ to grow at a polynomial rate of sample size. Similar to [3], [20] provided non-asymptotic Lasso error and prediction error bounds for stable Gaussian VARs. Both [40] and [27] considered subexponential designs. [40] studied lasso on iid subexponential designs and provide finite sample bounds. [27] studied adaptive lasso on linear time series and provided sign consistency results. [46] provided theoretical guarantees of lasso in time series setting assuming the dimension holds constant while sample size goes to infinity. [34, 33], [16], and [32] consider structured penalties beyond the $\ell_1$ penalty. [50], [26], [47] and [9] consider estimation of the covariance (or precision) matrix of high dimensional time series. [26] and [28] both highlight that autoregressive (AR) estimation, even in univariate time series, leads to high dimensional parameter estimation problems if the lag is allowed to be unbounded.

2 Preliminaries

Lasso Estimation Procedure for Dependent Data Consider a stochastic process of pairs $(X_t, Y_t)_{t=1}^{\infty}$ where $X_t \in \mathbb{R}^p$, $Y_t \in \mathbb{R}^q$, $\forall t$. One might be interested in predicting $Y_t$ given $X_t$. In particular, given a dependent sequence $(Z_t)_{t=1}^{\infty}$, one might want to forecast the present $Z_t$ using the past $(Z_{t-d}, \ldots, Z_{t-1})$. A linear predictor is a natural choice. To put it in the regression setting, we
identify \( Y_t = Z_t \) and \( X_t = (Z_{t-4}, \ldots, Z_{t-1}) \). The pairs \((X_t, Y_t)\) defined as such are no longer iid. Assuming strict stationarity, the parameter matrix of interest \( \Theta^* \in \mathbb{R}^{p \times q} \) is

\[
\Theta^* = \arg \min_{\Theta \in \mathbb{R}^{p \times q}} \mathbb{E}[\|Y_t - \Theta' X_t\|_2^2].
\]

(2.1)

Note that \( \Theta^* \) is independent of \( t \) owing to stationarity. Because of high dimensionality \((pq \gg T)\), consistent estimation is impossible without regularization. We consider the Lasso procedure. The \( \ell_1 \)-penalized least squares estimator \( \hat{\Theta} \in \mathbb{R}^{p \times q} \) is defined as

\[
\hat{\Theta} = \arg \min_{\Theta \in \mathbb{R}^{p \times q}} \frac{1}{T} \|\text{vec}(Y - X\Theta)\|_2^2 + \lambda_T \|\text{vec}(\Theta)\|_1.
\]

(2.2)

where

\[
Y = (Y_1, Y_2, \ldots, Y_T)' \in \mathbb{R}^{T \times q} \quad X = (X_1, X_2, \ldots, X_T)' \in \mathbb{R}^{T \times p}.
\]

(2.3)

The following matrix of true residuals is not available to an estimator but will appear in our analysis:

\[
W := Y - X\Theta^*.
\]

(2.4)

**Matrix and Vector Notation** For a symmetric matrix \( M \), let \( \lambda_{\max}(M) \) and \( \lambda_{\min}(M) \) denote its maximum and minimum eigenvalues respectively. For any matrix let \( M, r(M), \|M\|_p, \|M\|_\infty, \text{ and } \|M\|_F \) denote its spectral radius \( \max_i \{\lambda_i(M)\} \), operator norm \( \sqrt{\lambda_{\max}(M'M)} \), entrywise \( \ell_\infty \) norm \( \max_{i,j} |M_{i,j}| \), and Frobenius norm \( \sqrt{\text{tr}(M'M)} \) respectively. For any vector \( v \in \mathbb{R}^p \), \( \|v\|_q \) denotes its \( \ell_q \) norm \( (\sum_{i=1}^p |v_i|^q)^{1/q} \). Unless otherwise specified, we shall use \( ||\cdot|| \) to denote the \( \ell_2 \) norm. For any vector \( v \in \mathbb{R}^p \), we use \( \|v\|_0 \) and \( \|v\|_\infty \) to denote \( \sum_{i=1}^p 1\{v_i \neq 0\} \) and \( \max_i |v_i| \) respectively. Similarly, for any matrix \( M \), \( ||M||_0 = ||\text{vec}(M)||_0 \) where \( \text{vec}(M) \) is the vector obtained from \( M \) by concatenating the rows of \( M \). We say that matrix \( M \) (resp. vector \( v \)) is \( s \)-sparse if \( ||M||_0 = s \) (resp. \( \|v\|_0 = s \)). We use \( v' \) and \( M' \) to denote the transposes of \( v \) and \( M \) respectively. When we index a matrix, we adopt the following conventions. For any matrix \( M \in \mathbb{R}^{p \times q} \), for \( 1 \leq i \leq p, 1 \leq j \leq q \), we define \( M[i,j] : = e_i'Me_j \), \( M[i,:) : = e_i'M \) and \( M(:,j) : = M_j := Me_j \) where \( e_i \) is the vector with all 0s except for a 1 in the \( i \)th coordinate. The set of integers is denoted by \( Z \).

For a lag \( l \in \mathbb{Z} \), we define the auto-covariance matrix w.r.t. \((X_t, Y_{t+l})_t \) as \( \Sigma(l) = \Sigma_{(X,Y)}(l) : = \mathbb{E}[(X_t,Y_t)(X_{t+l},Y_{t+l})'] \). Note that \( \Sigma(-l) = \Sigma(l)' \). Similarly, the auto-covariance of lag \( l \) w.r.t. \((X_t)_t \) is \( \Sigma_X(l) : = \mathbb{E}[X_t X_{t+l}'] \), and w.r.t. \((Y_t)_t \) is \( \Sigma_Y(l) : = \mathbb{E}[Y_t Y_{t+l}'] \). The cross-covariance matrix at lag \( l \) is \( \Sigma_{X,Y}(l) : = \mathbb{E}[X_t Y_{t+l}'] \). Note the difference between \( \Sigma_{(X,Y)}(l) \) and \( \Sigma_{X,Y}(l) \): the former is a \((p+q) \times (p+q)\) matrix, the latter is a \( p \times q \) matrix. Thus, \( \Sigma_{(X,Y)}(l) \) is a matrix consisting of four sub-matrices. Using Matlab-like notation, \( \Sigma_{(X,Y)}(l) = [\Sigma_X, \Sigma_{X,Y}; \Sigma_{Y,X}, \Sigma_Y] \). As per our convention, at lag 0, we omit the lag argument \( l \). For example, \( \Sigma_{X,Y} \) denotes \( \Sigma_{X,Y}(0) = \mathbb{E}[X_t Y_t'] \).
A Brief Introduction to the $\beta$-Mixing Condition  

Mixing conditions [6] are well established in the stochastic processes literature as a way to allow for dependence in extending results from the iid case. The general idea is to first define a measure of dependence between two random variables $X,Y$ (that can vector-valued or even take values in a Banach space) with associated sigma algebras $\sigma(X),\sigma(Y)$. In particular,

$$\beta(X,Y) = \sup 2^{-1} \sum_{i=1}^{I} \sum_{j=1}^{J} |P(A_i \cap B_j) - P(A_i)P(B_j)|$$

where the last supremum is over all pairs of partitions $\{A_1, \ldots, A_I\}$ and $\{B_1, \ldots, B_J\}$ of the sample space $\Omega$ such that $A_i \in \sigma(X), B_j \in \sigma(Y)$ for all $i,j$. Then for a stationary stochastic process $(X_t)_{t=-\infty}^{\infty}$, one defines the mixing coefficients, for $l \geq 1$,

$$\beta(l) = \beta(X_{-\infty:t}, X_{t:t+\infty}).$$

The $\beta$-mixing condition has been of interest in statistical learning theory for obtaining finite sample generalization error bounds for empirical risk minimization [45, Sec. 3.4] and boosting [21] for dependent samples. There is also work on estimating $\beta$-mixing coefficients from data [25]. At the same time, many interesting processes such as Markov and hidden Markov processes satisfy a $\beta$-mixing condition [45, Sec. 3.5]. Before we continue, we note an elementary but useful fact about mixing conditions, viz. they persist under arbitrary measurable transformations of the original stochastic process.

**Fact 1.** Suppose a stationary process $\{U_t\}_{t=1}^{T}$ is $\beta$-mixing. Then the stationary sequence $\{f(U_t)\}_{t=1}^{T}$, for any measurable function $f(\cdot)$, also is mixing in the same sense with its mixing coefficients bounded by those of the original sequence.

## 3 Main Results

We start with introducing two well-known sufficient conditions that enable us to provide non-asymptotic guarantees for Lasso estimation and prediction errors – the restricted eigenvalue (RE) and the deviation bound (DB) conditions. Note that in the classical linear model setting (see, e.g., Chap. 2.3 in [19]) where sample size is larger than the dimensions ($n > p$), the conditions for consistency of the ordinary least squares (OLS) estimator are as follows: (a) the empirical covariance matrix $X'X/T \overset{P}{\to} Q$ and $Q$ invertible, i.e., $\lambda_{\min}(Q) > 0$, and (b) the regressors and the noise are asymptotically uncorrelated, i.e., $X'W/T \to 0$.

In high-dimensional regimes, [5], [23] and [30] have established similar consistency conditions for Lasso. The first one is the restricted eigenvalue (RE) condition on $X'X/T$ (which is a special case, when the loss function is the squared loss, of the restricted strong convexity (RSC) condition). The second is the deviation bound (DB) condition on $X'W$. The following lower RE and DB definitions are modified from those given by [23].
Definition 1 (Lower Restricted Eigenvalue). A symmetric matrix $\Gamma \in \mathbb{R}^{p \times p}$ satisfies a lower restricted eigenvalue condition with curvature $\alpha > 0$ and tolerance $\tau(T, p) > 0$ if

$$\forall v \in \mathbb{R}^p, \ v' \Gamma v \geq \alpha \|v\|_2^2 - \tau(T, p) \|v\|_1^2.$$  

Definition 2 (Deviation Bound). Consider the random matrices $X \in \mathbb{R}^{T \times p}$ and $W \in \mathbb{R}^{T \times q}$ defined in (2.3) and (2.4) above. They are said to satisfy the deviation bound condition if there exist a deterministic multiplier function $Q(X, W, \Theta^*)$ and a rate of decay function $R(p, q, T)$ such that:

$$\frac{1}{T} \left\|X'W\right\|_\infty \leq Q(X, W, \Theta^*)R(p, q, T).$$

We will show that, with high probability, the RE and DB conditions hold for dependent data that satisfy Assumptions 1–5 described below. We shall do that without assuming any parametric form of the data generating mechanism. Instead, we will assume a subgaussian tail condition on the random vectors $X_t, Y_t$ and that they satisfy the geometrically $\beta$-mixing condition.

3.1 Assumptions

Assumption 1 (Sparsity). The matrix $\Theta^*$ is $s$-sparse, i.e. $\|\text{vec}(\Theta^*)\|_0 \leq s$.

Assumption 2 (Stationarity). The process $(X_t, Y_t)$ is strictly stationary: i.e.,

$$((X_t, Y_t), \cdots, (X_{t+n}, Y_{t+n})) \overset{d}{=} ((X_{t+\tau}, Y_{t+\tau}), \cdots, (X_{t+\tau+n}, Y_{t+\tau+n})).$$

where “$d$” denotes equality in distribution.

Assumption 3 (Centering). We have, $\forall t$, $E(X_t) = 0_{p \times 1}$, and $E(Y_t) = 0_{q \times 1}$.

The thin tail property of the Gaussian distribution is desirable from the theoretical perspective, so we would like to keep that but at the same time allow for more generality. The subgaussian distributions are a nice family characterized by having tail probabilities of the same as or lower order than the Gaussian. We now focus on subgaussian random vectors and present high probabilistic error bounds with all parameter dependences explicit.

Assumption 4 (Subgaussianity). The subgaussian constants of $X_t$ and $Y_t$ are bounded above by $\sqrt{K_X}$ and $\sqrt{K_Y}$ respectively. (Please see Appendix A for a detailed introduction to subgaussian random vectors.)

Classically, mixing conditions were introduced to generalize classic limit theorems in probability beyond the case of iid random variables [37]. Recent work on high dimensional statistics has established the validity of RE conditions in the
iid Gaussian [36] and iid Subgaussian cases [39]. One of the main contributions of our work is to extend these results in high dimensional statistics from the iid to the mixing case.

**Assumption 5 (β-Mixing).** The process \((X_t, Y_t)\) is geometrically β-mixing, i.e., there exists some constant \(c_\beta > 0\) such that \(\forall l \geq 1, \beta(l) \leq \exp(-c_\beta l)\).

The β-mixing condition allows us to apply the independent block technique developed by [49]. For examples of large classes of Markov and hidden Markov processes that are geometrically β-mixing, see Theorem 3.11 and Theorem 3.12 of [45]. In the independent blocking technique, we construct a new set of independent blocks such that each block has the same distribution as that of the corresponding block from the original sequence. Results of [49] provide upper bounds on the difference between probabilities of events defined using the independent blocks versus the same event defined using the original data. Classical probability theory tools for independent data can then be applied on the constructed independent blocks. In Appendix C, we apply the independent blocking technique to Bernstein’s inequality to get the following concentration inequality for β-mixing random variables.

**Lemma 1 (Concentration of β-Mixing Subgaussian Random Variables).** Let \(Z = (Z_1, \ldots, Z_T)\) consist of a sequence of mean-zero random variables with exponentially decaying β-mixing coefficients as in 5. Let \(K\) be such that \(\max_{t=1}^T \|Z_t\|_{\psi_2} \leq \sqrt{K}\). Choose a block length \(a_T \geq 1\) and let \(\mu_T = \lfloor T/(2a_T) \rfloor\). We have, for any \(t > 0\),

\[
P\left[ \frac{1}{T} \left\| Z \right\|_2^2 - \mathbb{E}[\|Z\|_2^2] > t \right] \leq 4 \exp \left( -C_B \min \left\{ \frac{t^2 \mu_T}{K^2}, \frac{t \mu_T}{K} \right\} \right) + 2(\mu_T - 1) \exp (-c_{\beta} a_T) + \exp \left( \frac{-2t \mu_T}{K} \right).
\]

In particular, for \(0 < t < K\),

\[
P\left[ \frac{1}{T} \left\| Z \right\|_2^2 - \mathbb{E}[\|Z\|_2^2] > t \right] \leq 4 \exp \left( -C_B \frac{t^2 \mu_T}{K^2} \right) + 2(\mu_T - 1) \exp (-c_{\beta} a_T) + \exp \left( \frac{-2t \mu_T}{K} \right).
\]

Here \(C_B\) is the universal constant appearing in Bernstein’s inequality (Proposition 7).

**Remark 1.** The three terms in the bound above all have interpretations: the first is a concentration term with a rate that depends on the “effective sample size” \(\mu_T\), the number of blocks; the second is a dependence penalty accounting for the fact that the blocks are not exactly independent; and the third is a remainder term coming from the fact that \(2a_T\) may not exactly divide \(T\). The key terms are the first two and exhibit a natural trade-off: increasing \(a_T\) worsens the first term since \(\mu_T\) decreases, but it improves the second term since there is less dependence at larger lags.
3.2 High Probability Guarantees for the Lower Restricted Eigenvalue and Deviation Bound Conditions

We show that both lower RE and DB conditions hold, with high probability, under our assumptions.

**Proposition 2 (RE).** Suppose Assumptions 1–5 hold. Let $C_B$ be the Bernstein’s inequality constant, $C = \min\{C_B, 2\}$, $b = \min\{\frac{1}{5KX}\lambda_{\min}(\Sigma_X), 1\}$ and $c = \frac{1}{b} \max\{c_\beta, Cb^2\}$. Then for $T \geq \left(\frac{1}{2} \log(p)\right)^2$, with probability at least $1 - \exp\left(-CT^{\frac{1}{2}}\right) - 2(T\frac{1}{2} - 1) \exp\left(-c_\beta T^{\frac{1}{2}}\right)$, we have for every vector $v \in \mathbb{R}^p$, $v' \hat{\Gamma} v \geq \alpha_2 \|v\|^2 - \tau_2(T, p) \|v\|_1^2$, where $\alpha_2 = \frac{1}{2} \lambda_{\min}(\Sigma_X)$, and $\tau_2(T, p) = 27bK_X \log(p)/cT^{\frac{1}{2}}$.

**Proposition 3 (Deviation Bound).** Suppose Assumptions 1–5 hold. Let $K = \sqrt{K} + \sqrt{KX (1 + \|\Theta^*\|)}$ and $\xi \in (0, 1)$ be a free parameter. Then, for sample size $T \geq \max\{\left(\log(pq) \max\left\{\frac{K^4}{2C_B}, K^2\right\} \right)^{\frac{1}{1-\xi}}, \left[\frac{2}{c_\beta \log(pq)}\right]^{\frac{1}{\xi}}\}$, we have
\[
P\left[\frac{1}{T} \|X' W\|_\infty \leq Q(X, W, \Theta^*) R(p, q, T)\right] \geq 1 - 15 \exp\left(-\frac{1}{2} \log(pq)\right) - 6(T^{1-\xi} - 1) \exp\left(-\frac{1}{2} c_\beta T^\xi\right)
\]
where
\[Q(X, W, \Theta^*) = \sqrt{2K^4/C_B}, \quad R(p, q, T) = \sqrt{\frac{\log(pq)}{T^{1-\xi}}}.
\]

**Remark 2.** Since $\xi \in (0, 1)$ is a free parameter, we choose it to be arbitrarily close to zero so that $R(p, q, T)$ scales at a rate arbitrarily close to $\sqrt{\log(pq)/T}$. However, there is a price to pay for this: both the initial sample threshold and the success probability worsen as we make $\xi$ very small.

3.3 Estimation and Prediction Errors

The guarantees below follow easily from plugging the RE and DB constants from Propositions 2 and 3 into a “master theorem” (Theorem 5 in Appendix B). The “master theorem”, in various forms, is well-known in the literature (e.g., see [5, 23, 30]).
**Corollary 4** (Lasso Guarantee under Subgaussian Tails and \(\beta\)-Mixing). Suppose Assumptions 1–5 hold. Let \(C_B, C, c, b\) and \(K\) be as defined in Propositions 2 and 3 and \(C := \min\{C, c\_\beta\}\). Let \(\xi \in (0, 1)\) be a free parameter. Then, for sample size

\[
T \geq \max \left\{ \left( \frac{\log(p)}{c} \right)^2 \max \left\{ \left( \frac{1728sbK_X}{\lambda_{\min}(\Sigma_X)} \right)^2, 1 \right\}, \left( \log(pq) \max \left\{ \frac{K^4}{2C_B}, K^2 \right\} \right)^{\frac{1}{2}}, \left[ \frac{2}{c\_\beta \log(pq)} \right]^{\frac{1}{2}} \right\}
\]

we have with probability at least

\[
1 - 15 \exp \left( -\frac{1}{2} \log(pq) \right) - 6(T^{1-\xi} - 1) \exp \left( -\frac{1}{2} c\_\beta T^\xi \right) - 5(T^{\frac{1}{2}} - 1) \exp \left( -\tilde{C}T^{\frac{1}{2}} \right)
\]

the Lasso estimation and (in-sample) prediction error bounds

\[
\|\text{vec}(\hat{\Theta} - \Theta^*)\| \leq 4\sqrt{s}\lambda_T / \alpha, \quad (3.1)
\]

\[
\left\| (\hat{\Theta} - \Theta^*)' \hat{\Gamma} (\hat{\Theta} - \Theta^*) \right\|_F^2 \leq \frac{32\lambda_T^2 s}{\alpha}. \quad (3.2)
\]

hold with

\[
\alpha = \frac{1}{2}\lambda_{\min}(\Sigma_X), \quad \lambda_T = 4Q(X, W, \Theta^*)R(p, q, T)
\]

where

\[
\hat{\Gamma} := \mathbf{X}'\mathbf{X}/T, \quad Q(X, W, \Theta^*) = \sqrt{\frac{2K^4}{C_B}}, \quad R(p, q, T) = \sqrt{\frac{\log(pq)}{T^{1-\xi}}}
\]

**Remark 3.** The condition number of \(\Sigma_X\) plays an important part in the literature of Lasso error guarantees [23, e.g.]. Here, we see that the role of the condition number \(\lambda_{\max}(\Sigma_X)/\lambda_{\min}(\Sigma_X)\) is replaced by \(K_X/\lambda_{\min}(\Sigma_X)\) that now serves as the “effective condition number.”

### 4 Examples

We explore applicability of our theory beyond just linear Gaussian processes using the examples below. In the following examples, we identify \(X_t := Z_t\) and \(Y_t := Z_{t+1}\) for \(t = 1, \ldots, T\). For the specific parameter matrix \(\Theta^*\) in each Example below, we can verify that Assumptions 1–5 hold (see Appendix E) for details. Therefore, Propositions 2 and 3 and Corollary 4 follow. Hence we have all the high probabilistic guarantees for Lasso on data generated from DGM potentially involving subgaussianity, model mis-specification, and/or nonlinearity.
Example 1 (Gaussian VAR). Transition matrix estimation in sparse stable VAR models has been considered by several authors in recent years [14, 17, 41]. The Lasso estimator is a natural choice for the problem.

We state the following convenient fact because it allows us to study any finite order VAR model by considering its equivalent VAR(1) representation. See Appendix E.1 for details.

Fact 2. Every VAR(d) process can be written in VAR(1) form (see e.g. [24, Ch 2.1]).

Therefore, without loss of generality, we can consider VAR(1) model in the ensuing Examples.

Formally a first order Gaussian VAR(1) process is defined as follows. Consider a sequence of serially ordered random vectors \((Z_t), Z_t \in \mathbb{R}^p\) that admits the following auto-regressive representation:

\[
Z_t = AZ_{t-1} + E_t
\]  

(4.1)

where \(A\) is a non-stochastic coefficient matrix in \(\mathbb{R}^{p \times p}\) and innovations \(E_t\) are \(p\)-dimensional random vectors from \(\mathcal{N}(0, \Sigma_e)\) with \(\lambda_{\min}(\Sigma_e) > 0\) and \(\lambda_{\max}(\Sigma_e) < \infty\).

Assume that the VAR(1) process is stable; i.e. det \((I_{p \times p} - Az) \neq 0\), \(\forall |z| \leq 1\). Also, assume \(A\) is \(s\)-sparse. In here, \(\Theta^* = A' \in \mathbb{R}^{p \times p}\).

Example 2 (VAR with Subgaussian Innovations). Consider a VAR(1) model defined as in Example 1 except that we replace the Gaussian white noise innovations with subgaussian ones and assume \(\|A\| < 1\).

For example, take iid random vectors from the uniform distribution; i.e. \(\forall t, E_t \overset{iid}{\sim} U\left([-\sqrt{3}, \sqrt{3}]^p\right)\). These \(E_t\) will be independent centered isotropic subgaussian random vectors, giving us a VAR(1) model with subgaussian innovations. If we take a sequence \((Z_t)_{t=1}^{T+1}\) generated according to the model, each element \(Z_t\) will be a mean zero subgaussian random vector. Note that \(\Theta^* = A'\).

Example 3 (VAR with subgaussian Innovations and Omitted Variable). We will study estimation of a VAR(1) process when there are endogenous variables omitted. This arises naturally when the underlying DGM is high-dimensional but not all variables are available/observable/measurable to the researcher to do estimation/prediction. This also happens when the researcher mis-specifies the scope of the model.

Notice that the system of the retained set of variables is no longer a finite order VAR (and thus non-Markovian). This example serves to illustrate that our theory is applicable to models beyond the finite order VAR setting.

Consider a VAR(1) process \((Z_t, \Xi_t)_{t=1}^{T+1}\) such that each vector in the sequence is generated by the recursion below:

\[
(Z_t; \Xi_t) = A(Z_{t-1}; \Xi_{t-1}) + (E_{Z,t-1}; E_{\Xi,t-1})
\]
where \( Z_t \in \mathbb{R}^p, \Xi_t \in \mathbb{R}, \mathcal{E}_{Z,t} \in \mathbb{R}^p, \) and \( \mathcal{E}_{\Xi,t} \in \mathbb{R} \) are partitions of the random vectors \((Z_t, \Xi_t)\) and \(E_t\) into \(p\) and \(1\) variables. Also,

\[
A := \begin{bmatrix}
A_{ZZ} & A_{Z\Xi} \\
A_{\Xi Z} & A_{\Xi\Xi}
\end{bmatrix}
\]

is the coefficient matrix of the VAR(1) process with \(A_{Z\Xi}\) 1-sparse, \(A_{ZZ}\) \(p\)-sparse and \(\|A\| < 1\). \(E_t := (\mathcal{E}_{X,t-1}; \mathcal{E}_{Z,t-1})\) for \(t = 1, \ldots, T + 1\) are iid draws from a subgaussian distribution; in particular we consider the subgaussian distribution described in Example 2.

We are interested in the OLS 1-lag estimator of the system restricted to the set of variables in \(Z_t\). Recall that

\[
\Theta^* := \arg \min_{B \in \mathbb{R}^{p \times p}} \mathbb{E} \left( \|Z_t - B'Z_{t-1}\|_2^2 \right)
\]

We show in the appendix that \((\Theta^*)' = A_{ZZ} + A_{Z\Xi} \Sigma_{\Xi Z}(0)(\Sigma_Z)^{-1}\) is sparse.

**Example 4 (Multivariate ARCH).** We will explore the generality of our theory by considering a multivariate nonlinear time series model with subgaussian innovations. A popular nonlinear multivariate time series model in econometrics and finance is the vector autoregressive conditionally heteroscedastic (ARCH) model. We chose the following specific ARCH model for convenient validation of the geometric \(\beta\)-mixing property; it may potentially be applicable to a larger class of multivariate ARCH models. Consider a sequence of random vector \((Z_t)_{t=1}^{T+1}\) generated by the following recursion. For any constants \(c > 0, m \in (0,1), a > 0, \) and \(A\) sparse with \(\|A\| < 1:\)

\[
\begin{align*}
Z_t &= A Z_{t-1} + \Sigma(Z_{t-1}) E_t \\
\Sigma(z) &= c \cdot \text{clip}_{a,b}(\|z\|^m) I_{p \times p}
\end{align*}
\]

where \(E_t\) are iid random vectors from some subgaussian distribution and \(\text{clip}_{a,b}(x)\) clips the argument \(x\) to stay in the interval \([a,b]\). We can take innovations \(E_t\) to be iid random vectors from uniform distribution as described in Example 2. Consequently, each \(Z_t\) will be a mean zero subgaussian random vector. Note that \(\Theta^* = A',\) the transpose of the coefficient matrix \(A\) here.

5 Simulations

Corollary 4 in Section 3 makes a precise prediction for the \(\ell_2\) parameter error \(\|\Theta^* - \hat{\Theta}\|_F\). We report scaling simulations for Examples 1–4 to confirm the sharpness of the bounds.

Sparsity is always \(s = \sqrt{p}\), noise covariance matrix \(\Sigma_r = I_p\), and the operator norm of the driving matrix set to \(\|A\| = 0.9\). The problem dimensions are \(p \in\)
\{50, 100, 200, 300\}. Top left, top right, bottom left and bottom right sub-figures in Figure 1 correspond to simulations of Examples 1, 2, 3 and 4 respectively.

In all combinations of the four dimensions and Examples, the error decreases to zero as the sample size \( n \) increases, showing consistency of the method. In each sub-figure, the \( \ell_2 \) parameter error curves align when plotted against a suitably rescaled sample size \( \frac{T}{\log(p)} \) for different values of dimension \( p \). We see the error scaling agrees nicely with theoretical guarantees provided by Corollary 4.

Figure 1: \( \ell_2 \) estimation error of lasso against rescaled sample size for Examples 1–4.

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A  Sub-Gaussian Constants for Random Vectors

The sub-Gaussian and sub-Exponential constants have various equivalent definitions, we adopt the following from [38].

**Definition 3** (Sub-Gaussian Norm and Random Variables/Vectors). A random variable $U$ is called sub-Gaussian with sub-Gaussian constant $K$ if its sub-Gaussian norm

$$
\|U\|_{\psi_2} := \sup_{p \geq 1} p^{-1/2} (\mathbb{E}|U|^p)^{1/p}
$$

satisfies $\|U\|_{\psi_2} \leq K$.

A random vector $V \in \mathbb{R}^n$ is called sub-Gaussian if all of its one-dimensional projections are sub-Gaussian and we define

$$
\|V\|_{\psi_2} := \sup_{v \in \mathbb{R}^n : \|v\| \leq 1} \|v'V\|_{\psi_2}
$$

**Definition 4** (Sub-exponential Norm and Random Variables/Vectors). A random variable $U$ is called sub-exponential with sub-exponential constant $K$ if its sub-exponential norm

$$
\|U\|_{\psi_1} := \sup_{p \geq 1} p^{-1} (\mathbb{E}|U|^p)^{1/p}
$$

satisfies $\|U\|_{\psi_1} \leq K$.

A random vector $V \in \mathbb{R}^n$ is called sub-exponential if all of its one-dimensional projections are sub-exponential and we define

$$
\|U\|_{\psi_1} := \sup_{v \in \mathbb{R}^n : \|v\| \leq 1} \|v'V\|_{\psi_1}
$$

**Fact 3.** A random variable $U$ is sub-Gaussian iff $U^2$ is sub-exponential with $\|U\|_{\psi_2}^2 = \|U^2\|_{\psi_1}$.

B  Proof of Master Theorem

We present a master theorem that provides guarantees for the $\ell_2$ parameter estimation error and for the (in-sample) prediction error. The proof builds on existing result of the same kind [5, 23, 30] and we make no claims of originality for either the result or for the proof.

**Theorem 5** (Estimation and Prediction Errors). *Consider the Lasso estimator $\hat{\Theta}$ defined in (2.2). Suppose Assumption 1 holds. Further, suppose that $\hat{\Gamma} := X'X/T$ satisfies the lower RE($\alpha, \tau$) condition with $\alpha \geq 32s\tau$ and $X'W$ satisfies*
the deviation bound. Then, for any $\lambda_T \geq 4Q(X, W, \Theta^*) \mathbb{R}(p, q, T)$, we have the following guarantees:

$$\|\text{vec}(\hat{\Theta} - \Theta^*)\| \leq 4\sqrt{s} \lambda_T / \alpha, \quad \text{(B.1)}$$

$$\left\| \left( \hat{\Theta} - \Theta^* \right)' \hat{\Gamma} \left( \hat{\Theta} - \Theta^* \right) \right\|_F^2 \leq \frac{32 s^2 \lambda_T^2}{\alpha}. \quad \text{(B.2)}$$

**Proof of Theorem 5.** We will break down the proof in steps.

1. Since $\hat{\Theta}$ is optimal for 2.2 and $\Theta^*$ is feasible,

$$\frac{1}{T} \left\| Y - X \hat{\Theta} \right\|_F^2 + \lambda_T \left\| \text{vec}(\hat{\Theta}) \right\|_1 \leq \frac{1}{T} \left\| Y - X \Theta^* \right\|_F^2 + \lambda_T \left\| \text{vec}(\Theta^*) \right\|_1$$

2. Let $\hat{\Delta} := \hat{\Theta} - \Theta^* \in \mathbb{R}^{p \times q}$

$$\frac{1}{T} \left\| X \hat{\Delta} \right\|_F^2 \leq \frac{2}{T} \text{tr}(\hat{\Delta}'X'W) + \lambda_T \left( \left\| \text{vec}(\Theta^*) \right\|_1 - \left\| \text{vec}(\hat{\Theta}) \right\|_1 \right)$$

Note that

$$\left\| \text{vec}(\Theta^* + \hat{\Delta}) \right\|_1 - \left\| \text{vec}(\Theta^*) \right\|_1 \geq \left\{ \left\| \text{vec}(\Theta^*_S) \right\|_1 - \left\| \text{vec}(\hat{\Delta}_S) \right\|_1 \right\} + \left\| \text{vec}(\hat{\Delta}_S^c) \right\|_1 - \left\| \text{vec}(\Theta^*) \right\|_1$$

$$= \left\| \text{vec}(\hat{\Delta}_S^c) \right\|_1 - \left\| \text{vec}(\hat{\Delta}_S) \right\|_1$$

where $S$ denote the support of $\Theta^*$. 

3. With RE constant $\alpha$ and tolerance $\tau$, deviation bound constant $Q(\Sigma_X, \Sigma_W)$ and $\lambda_T \geq 2Q(\Sigma_X, \Sigma_W)\sqrt{\frac{\log(n)}{T}}$, we have

$$\alpha \|\hat{\Delta}\|_F^2 - \tau \|\text{vec}(\hat{\Delta})\|_1^2$$

$$\leq \frac{1}{T} \|X\hat{\Delta}\|_F^2$$

$$\leq \frac{2}{T} \text{tr}(\hat{\Delta}X'W) + \lambda_T \{\|\text{vec}(\hat{\Delta}_S)\|_1 - \|\text{vec}(\hat{\Delta}_{Sc})\|_1\}$$

$$\leq \frac{2}{T} \sum_{k=1}^q \|\hat{\Delta}_k\|_1 \|X'(W):k\|_\infty + \lambda_T \{\|\text{vec}(\hat{\Delta}_S)\|_1 - \|\text{vec}(\hat{\Delta}_{Sc})\|_1\}$$

$$\leq \frac{2}{T} \|\text{vec}(\hat{\Delta})\|_1 \|X'W\|_\infty + \lambda_T \{\|\text{vec}(\hat{\Delta}_S)\|_1 - \|\text{vec}(\hat{\Delta}_{Sc})\|_1\}$$

$$\leq \frac{3\lambda_T}{2} \|\text{vec}(\hat{\Delta}_S)\|_1 - \frac{\lambda_T}{2} \|\text{vec}(\hat{\Delta}_{Sc})\|_1$$

$$\leq 2\lambda_T \|\text{vec}(\hat{\Delta})\|_1$$

4. In particular, this says that $3 \|\text{vec}(\hat{\Delta}_S)\|_1 \geq \|\text{vec}(\hat{\Delta}_{Sc})\|_1$

So $\|\text{vec}(\hat{\Delta})\|_1 \leq 4 \|\text{vec}(\hat{\Delta}_S)\|_1 \leq 4\sqrt{s} \|\text{vec}(\hat{\Delta})\|$

5. Finally, with $\alpha \geq 32s\tau$,

$$\frac{\alpha}{2} \|\text{vec}(\hat{\Delta})\|_F^2 \leq (\alpha - 16s\tau) \|\text{vec}(\hat{\Delta})\|_F^2$$

$$\leq \alpha \|\text{vec}(\hat{\Delta})\|_F^2 - \tau \|\text{vec}(\hat{\Delta})\|_1^2$$

$$\leq 2\lambda_T \|\text{vec}(\hat{\Delta})\|_1$$

$$\leq 2\sqrt{s}\lambda_T \|\hat{\Delta}\|_F$$

6. $\|\text{vec}(\hat{\Delta})\|_F \leq \frac{4\lambda_T\sqrt{s}}{\alpha}$

7. From step 4, we have

$$\frac{1}{T} \|X\hat{\Delta}\|_F^2 \leq 8\lambda_T\sqrt{s} \|\text{vec}(\hat{\Delta})\|$$

Then, from step 6

$$\frac{1}{T} \|X\hat{\Delta}\|_F^2 \leq 8\lambda_T\sqrt{s} \|\text{vec}(\hat{\Delta})\| \leq 32\lambda_T^2s/\alpha$$
C Proofs for Sub-Gaussian Random Vectors under $\beta$-Mixing

Proof of Lemma 1.
Following the description in [49], we divide the stationary sequence of real valued random variables $\{Z_t\}_{t=1}^T$ into $2\mu_T$ blocks of size $a_T$ with a remainder block of length $T - 2\mu_T a_T$. Let $H$ and $T$ be sets that denote the indices in the odd and even blocks respectively, and let $Re$ to denote the indices in the remainder block. To be specific,

$$
O = \bigcup_{j=1}^{\mu_T} O_j \text{ where } O_j := \{ i : 2(j - 1)a_T + 1 \leq i \leq (2j - 1)a_T \}, \forall j
$$

$$
E := \bigcup_{j=1}^{\mu_T} E_j \text{ where } E_j := \{ i : (2j - 1)a_T + 1 \leq i \leq (2j)a_T \}, \forall j
$$

Let $Z_O := \{ Z_t : t \in O \}$ be a collection of the random vectors in the odd blocks. Similarly, $Z_e := \{ Z_t : t \in E \}$ is a collection of the random vectors in the even blocks, and $Z_{re} := \{ Z_t : t \in Re \}$ a collection of the random vectors in the remainder block. Lastly, $Z := Z_O \cup Z_e \cup Z_{re}$.

Now, take a sequence of i.i.d. blocks $\{Z_{O_j} : j = 1, \cdots, \mu_i \}$ such that each $Z_{O_j}$ is independent of $\{Z_t\}_{t=1}^T$ and each $Z_{O_j}$ has the same distribution as the corresponding block from the original sequence $\{Z_t : j \in O_j \}$. We construct the even and remainder blocks in a similar way and denote them $\{Z_{E_j} : j = 1, \cdots, \mu_i \}$ and $Z_{re}$ respectively.

$\hat{Z}_O := \bigcup_{j=1}^{\mu_T} \hat{Z}_{O_j}$, $\hat{Z}_E := \bigcup_{j=1}^{\mu_T} \hat{Z}_{E_j}$ denote the union of the odd(even) blocks.

For the odd blocks: $\forall t > 0,$

$$
\Pr \left( \frac{1}{T} \left\| Z_o \right\|_2^2 - \mathbb{E}(\|Z_o\|_2^2) \geq t \right)
= \mathbb{E}[\mathbb{1}\left\{ \frac{1}{T} \|Z_o\|_2^2 - \mathbb{E}(\|Z_o\|_2^2) \geq t \right\}]
\leq \mathbb{E}[\mathbb{1}\left\{ \frac{1}{T} \|Z_o\|_2^2 - \mathbb{E}(\|Z_o\|_2^2) \geq t \right\}] + (\mu_T - 1)\beta(a_T)
= \mathbb{P}\left[ \frac{1}{T} \|Z_o\|_2^2 - \mathbb{E}(\|Z_o\|_2^2) > t \right] + (\mu_T - 1)\beta(a_T)
= \mathbb{P}\left[ \frac{1}{\mu_T} \sum_{i=1}^{\mu_T} \|Z_{O_i}\|_2^2 - \mathbb{E}(\|Z_{O_i}\|_2^2) > ta_T \right] + (\mu_T - 1)\beta(a_T)
\leq 2 \exp \left\{ -C_B \min \left\{ \frac{t^2 \mu_T}{K^2}, \frac{t \mu_T}{K} \right\} \right\} + (\mu_T - 1)\beta(a_T)
$$

Where the first inequality follows from [49, Lemma 4.1] with $M = 1$. By Fact (3), the corresponding sub-exponential constant of each $\|Z_{O_i}\|_2^2 \leq a_T K$ where $K$
is the sub-exponential norm because of fact 3. With this, the second inequality follows from the Bernstein’s inequality (Proposition (7)) with some constant $C_B > 0$.

Then

$$2 \exp \left\{ -C_B \min \left\{ \frac{t^2 \mu T}{K^2}, \frac{t \mu T}{K} \right\} \right\} + (\mu_T - 1) \beta(a_T)$$

$$\leq 2 \exp \left\{ -C_B \min \left\{ \frac{t^2 \mu T}{K^2}, \frac{t \mu T}{K} \right\} \right\} + (\mu_T - 1) \exp \left\{ -c \beta \alpha_T \right\}$$

So,

$$\mathbb{P}\left[ \frac{2}{T} \|Z_o\|_2^2 - \mathbb{E}(\|Z_o\|_2^2) > t \right| \leq 2 \exp \left\{ -C_B \min \left\{ \frac{t^2 \mu T}{K^2}, \frac{t \mu T}{K} \right\} \right\} + (\mu_T - 1) \exp \left\{ -c \beta \alpha_T \right\}$$

Taking the union bound over the odd and even blocks,

$$\mathbb{P}\left[ \frac{1}{T} \|Z\|_2^2 - \mathbb{E}(\|Z\|_2^2) > t \right| \leq 4 \exp \left\{ -C_B \min \left\{ \frac{t^2 \mu T}{K^2}, \frac{t \mu T}{K} \right\} \right\} + 2(\mu_T - 1) \exp \left\{ -c \beta \alpha_T \right\}$$

For $0 < t < K$, it reduces to

$$\mathbb{P}\left[ \frac{1}{T} \|Z\|_2^2 - \mathbb{E}(\|Z\|_2^2) > t \right| \leq 4 \exp \left\{ -C_B \frac{t^2 \mu T}{K^2} \right\} + 2(\mu_T - 1) \exp \left\{ -c \beta \alpha_T \right\}$$

For the remainder block, since $\|Z_r\|_2^2$ has sub-exponential constant at most $\alpha_T K \leq KT/(2 \mu_T)$, we have

$$\mathbb{P}\left[ \frac{1}{T} \|Z_r\|_2^2 - \mathbb{E}(\|Z_r\|_2^2) > t \right| \leq \exp \left\{ -\frac{t T}{\alpha_T K} \right\} \leq \exp \left\{ -\frac{2 t \mu T}{K} \right\}$$

Together, by union bound

$$\mathbb{P}\left[ \frac{1}{T} \|Z\|_2^2 - \mathbb{E}(\|Z\|_2^2) > t \right| \leq 4 \exp \left\{ -C_B \frac{t^2 \mu_T}{K^2} \right\} + 2(\mu_T - 1) \exp \left\{ -c \beta \alpha_T \right\} + \exp \left\{ -\frac{2 t \mu T}{K} \right\}$$

\[\square\]

**Proof of Proposition 2.** Recall that the sequence $X_1, \cdots, X_T \in \mathbb{R}^p$ form a $\beta$-mixing and stationary sequence.

Now, fix a unit vector $v \in \mathbb{R}^p$, $\|v\|^2 = 1$.

Define real valued random variables $Z_t = X_t^\prime v$, $t = 1, \cdots, T$. Note that the $\beta$ mixing rate of $\{Z_t\}_{t=1}^T$ is bounded by the same of $\{X_t\}_{t=1}^T$ by Fact 1. We suppress the $X$ subscript of the sub-Gaussian constant $\sqrt{K_X}$ here, and refer it
as $\sqrt{K}$.

We can apply Lemma 1 on $Z := \{Z_t\}_{t=1}^t$. Set $t = bK$. We have,

$$\Pr\left[ \frac{1}{t} \|Z\|^2 - \mathbb{E}\|Z\|^2 > bK \right] \leq 4 \exp \left\{ -C_B b^2 \mu T \right\} + 2(\mu_t - 1) \exp \{ -c_\beta a_t \} + \exp \{ -b \mu_T \}$$

$$\leq 5 \exp \{ - \min \{C_B, 2\} b^2 \mu T \} + 2(\mu_t - 1) \exp \{ -c_\beta a_t \}$$

Using Lemma F.2 in [3], we extend the inequality to hold for all vectors $\mathbb{J}(2k)$, the set of unit norm 2s-sparse vectors. We have

$$\Pr\left[ \sup_{s \in \mathbb{J}(2k)} \frac{1}{t} \|Z\|^2 - \mathbb{E}\|Z\|^2 > bK \right] \leq 5 \exp \{ -C_B b^2 \mu T + 3k \log(p) \} + 2(\mu_t - 1) \exp \{ -c_\beta a_t + 3k \log(p) \}$$

The constant $C$ is defined as $C := \min \{C_B, 2\}$.

Recall $\hat{\Gamma} := \frac{X'X}{K}$, the above concentration can be equivalently expressed as

$$\Pr\left[ \sup_{v \in \mathbb{J}(2k)} \left| v' \left( \hat{\Gamma} - \Sigma_X(0) \right) v \right| > bK \right] \geq 1 - 5 \exp \{ -C_B^2 \mu_T + 3k \log(p) \} - 2(\mu_t - 1) \exp \{ -c_\beta a_t + 3k \log(p) \}$$

Finally, we will extend the concentration to all $v \in \mathbb{R}^p$ to establish the lower-RE result. By Lemma 12 of [23], for parameter $k \geq 1$, w.p. at least

$$1 - 5 \exp \{ -C_b^2 \mu_T + 3k \log(p) \} - 2(\mu_t - 1) \exp \{ -c_\beta a_t + 3k \log(p) \}$$

we have

$$\left| v' \left( \hat{\Gamma} - \Sigma_X(0) \right) v \right| \leq 27Kb \left[ \|v\|^2 + \frac{1}{K} \|v\|_1^2 \right]$$

This implies that

$$v' \hat{\Gamma} v \geq \|v\|^2 \left[ \lambda_{\min}(\Sigma_X(0)) - 27bK \right] - \frac{27bK}{K} \|v\|_1^2$$

w.p. $1 - 5 \exp \{ -C_b^2 \mu_T + 3k \log(p) \} - 2(\mu_t - 1) \exp \{ -c_\beta a_t + 3k \log(p) \}$.

Now, choose set $k = \frac{1}{\log(p)} \min \{ C_b^2 \mu_t, c_\beta a_T \}$. Let’s choose that, for some $\xi \in (0, 1)$, $a_t = T^\xi$ and $\mu_T = T^{1-\xi}$. Then,

$$k = c \frac{1}{\log(p)} \min \{ a_t, \mu_T \} = c \frac{1}{\log(p)} \min \{ T^\xi, T^{1-\xi} \}$$

Where $c = \frac{1}{b} \max \{ c_\beta, C_b^2 \}$. To ensure $k \geq 1$, we require $T \geq \left( \frac{1}{c} \log(p) \right)^{\min \{ \frac{1}{\xi}, \frac{1}{1-\xi} \}}$

With these specifications, We have for probability at least

$$1 - 5 \exp \{ -C_b^2 T^{\frac{1}{2}} \} - 2(T^{\frac{1}{2}} - 1) \exp \{ -c_\beta T^{\frac{1}{2}} / 2 \}$$
By first order optimality of the optimization problem in (2.1), we have
\[
\mathbf{v}^\top \mathbf{v} \geq \|\mathbf{v}\|^2 \left[ \lambda_{\min}(\Sigma_X(0)) - 27bK \right] - \frac{27bK \log(p)}{e \min\{T, T^{-1} - \xi\}} \|\mathbf{v}\|_1^2.
\]

Now, choose \( \xi = \frac{1}{2} \) since it optimizes the rate of decay in the tolerance parameter. Also, choose \( b = \min\{\frac{1}{\log K} \lambda_{\min}(\Sigma_X(0)), 1\} \); this ensures that \( \lambda_{\min}(\Sigma_X(0)) - 27bK \geq \frac{1}{2} \lambda_{\min}(\Sigma_X(0)) \).

In all, for \( T \geq \left( \frac{1}{b} \log(p) \right)^2 \) w.p. at least
\[
1 - 5 \exp\{-C\bar{b}^2T^{\frac{1}{2}}\} - 2(T^{\frac{1}{2}} - 1) \exp\{-c_\beta T^{\frac{1}{2}}/2\}
\]
\[
\mathbf{v}^\top \mathbf{v} \geq \|\mathbf{v}\|^2 \frac{1}{2} \lambda_{\min}(\Sigma_X(0)) - \frac{27bK \log(p)}{eT^{\frac{1}{2}}} \|\mathbf{v}\|_1^2.
\]

\[\square\]

**Proof of Proposition 3.**

Recall \( \|\mathbf{X}'\mathbf{W}\|_\infty = \max_{1 \leq i < p, 1 \leq j \leq q} |\mathbf{X}'\mathbf{W}|_{i,j} = \max_{1 \leq i \leq p, 1 \leq j \leq q} |\mathbf{X}_i'\mathbf{W}_{i,j}| \).

By lemma condition (3), we have
\[
\mathbf{E}\mathbf{X}_i = 0, \forall i \quad \text{and} \quad \mathbf{E}\mathbf{Y}_{j} = 0, \forall j
\]

By first order optimality of the optimization problem in (2.1), we have
\[
\mathbf{E}\mathbf{X}_i'(\mathbf{Y} - \mathbf{X}\Theta^*) = 0, \forall i \Rightarrow \mathbb{E}\mathbf{X}_i'\mathbf{W}_{i,j} = 0, \forall i, j
\]

We know \( \forall i, j \)
\[
|\mathbf{X}_i'\mathbf{W}_{i,j}| = |\mathbf{X}_i'\mathbf{W}_{i,j} - \mathbb{E}[\mathbf{X}_i'\mathbf{W}_{i,j}]|
\]
\[
= \frac{1}{2} \left( \|\mathbf{X}_i + \mathbf{W}_{i,j}\|^2 - \mathbb{E}[\|\mathbf{X}_i + \mathbf{W}_{i,j}\|^2] - \left( \|\mathbf{X}_i\|^2 - \mathbb{E}[\|\mathbf{X}_i\|^2] \right) - \left( \|\mathbf{W}_{i,j}\|^2 - \mathbb{E}[\|\mathbf{W}_{i,j}\|^2] \right) \right)
\]
\[
\leq \frac{1}{2} \left( \|\mathbf{X}_i + \mathbf{W}_{i,j}\|^2 - \mathbb{E}[\|\mathbf{X}_i + \mathbf{W}_{i,j}\|^2] + \frac{1}{2} \|\mathbf{X}_i\|^2 - \mathbb{E}[\|\mathbf{X}_i\|^2] \right) + \frac{1}{2} \left( \|\mathbf{W}_{i,j}\|^2 - \mathbb{E}[\|\mathbf{W}_{i,j}\|^2] \right)
\]

Therefore,
\[
\mathbb{P} \left( \frac{1}{T} |\mathbf{X}_i'\mathbf{W}_{i,j}| > 3t \right)
\]
\[
\leq \mathbb{P} \left( \frac{1}{2T} \|\mathbf{X}_i + \mathbf{W}_{i,j}\|^2 - \mathbb{E}[\|\mathbf{X}_i + \mathbf{W}_{i,j}\|^2] > t \right) + \mathbb{P} \left( \frac{1}{2T} \|\mathbf{X}_i\|^2 - \mathbb{E}[\|\mathbf{X}_i\|^2] > t \right)
\]
\[
+ \mathbb{P} \left( \frac{1}{2T} \|\mathbf{W}_{i,j}\|^2 - \mathbb{E}[\|\mathbf{W}_{i,j}\|^2] > t \right)
\]

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This suggests proof strategy via controlling tail probability on each of the terms $\|X_i\|^2 - \mathbb{E}\|X_i\|^2$, $\|W_{ij}\|^2 - \mathbb{E}\|W_{ij}\|^2$ and $\|X_{i} + W_{ij}\|^2 - \mathbb{E}\|X_{i} + W_{ij}\|^2$. Assuming the conditions in lemma 3, we can apply lemma 1 on each of them. We have to figure out their sub-Gaussian constants.

Let’s define $K_W := \sup_{1 \leq t \leq T, 1 \leq j \leq q} \|W_{tj}\|_{\psi_2}$ and $K_{X+W} := \sup_{1 \leq t \leq T, 1 \leq i \leq p} \|X_{it} + W_{tj}\|_{\psi_2}$.

Now, 

$$
\sup_{1 \leq t \leq T, 1 \leq i \leq p, 1 \leq j \leq q} \|X_{it} + W_{tj}\|_{\psi_2} \leq \sup_{1 \leq t \leq T, 1 \leq i \leq p} \|X_{it}\|_{\psi_2} + \sup_{1 \leq j \leq q, 1 \leq t \leq T} \|W_{tj}\|_{\psi_2} 
$$

by definition of sub-Gaussian random vector

$$
= \|W_1\|_{\psi_2} \quad \text{by stationarity}
$$

Let’s figure out $\|W_1\|_{\psi_2}$.

$$
W_1 := Y_1 - (X^* \Theta)_1 = Y_1 - X^1 \Theta
$$

Thus, 

$$
\|W_1\|_{\psi_2} \leq \|Y_1\|_{\psi_2} + \|X^1 \Theta^*\|_{\psi_2} \quad \text{since } \|\cdot\|_{\psi_2} \text{ is a norm}
$$

$$
\leq \|Y_1\|_{\psi_2} + \|X^1\|_{\psi_2} \|\Theta^*\| \quad \text{by lemma 6}
$$

$$
= \sqrt{K_Y} + \|\Theta^*\| \sqrt{K_X} \quad \text{by stationarity}
$$

Therefore, 

$$
K_W \leq \sqrt{K_Y} + \|\Theta^*\| \sqrt{K_X} \quad \text{(C.1)}
$$

Similarly, 

$$
\sup_{1 \leq i \leq p, 1 \leq j \leq q, 1 \leq t \leq T} \|X_{it} + W_{tj}\|_{\psi_2} \leq \sup_{1 \leq i \leq p, 1 \leq t \leq T} \|X_{it}\|_{\psi_2} + \sup_{1 \leq j \leq q, 1 \leq t \leq T} \|W_{tj}\|_{\psi_2}
$$

$$
\leq \|X_1\|_{\psi_2} + \|W_1\|_{\psi_2}
$$

$$
\leq \sqrt{K_Y} + \sqrt{K_X} (1 + \|\Theta^*\|) \quad \text{by equation (C.1)}
$$

Therefore, 

$$
K_{X+W} \leq \sqrt{K_Y} + \sqrt{K_X} (1 + \|\Theta^*\|) \quad \text{(C.2)}
$$

Take 

$$
K := \max\{K_X, K_W, K_{X+W}\} \leq \sqrt{K_Y} + \sqrt{K_X} (1 + \|\Theta^*\|) \quad \text{(C.3)}
$$
For $\xi \in [0,1]$, set $a_T = T^\xi$ and $\mu_T = T^{1-\xi}$. Applying lemma 1 three times with sub-Gaussian constant $K$, we have

$$P \left( \frac{1}{T} \left| X'_i W_j \right| > 3t \right) \leq P \left( \frac{1}{2T} \left| \| X_i + W_j \|^2 - E[\| X_i + W_j \|^2] \right| > t \right) + P \left( \frac{1}{2T} \left| \| W_j \|^2 - E[\| W_j \|^2] \right| > t \right)$$

$$\leq 4 \exp \left\{ -C_B \frac{4t^2 T^{1-\xi}}{K^4} \right\} + 2(T^{1-\xi} - 1) \exp \left\{ -c_B T^\xi \right\} + \exp \left\{ -\frac{2}{K^2} tT^{1-\xi} \right\}$$

By union bound,

$$P \left[ \frac{1}{T} \left\| X'W \right\|_\infty > 3t \right] = P \left[ \max_{1 \leq i \leq p, 1 \leq j \leq q} \frac{1}{T} \left| X'_i W_j \right| > 3t \right]$$

$$\leq 3pq \left\{ 4 \exp \left\{ -C_B \frac{4t^2 T^{1-\xi}}{K^4} \right\} + 2(T^{1-\xi} - 1) \exp \left\{ -c_B T^\xi \right\} + \exp \left\{ -\frac{2}{K^2} tT^{1-\xi} \right\} \right\}$$

To ensure proper decay in the probability, we require

$$T \geq \max \left\{ \left( \log(pq) \max \left\{ \frac{K^4}{2C_B}, K^2 \right\} \right)^{\frac{1}{1+\xi}}, \left[ \frac{2}{c_B} \log(pq) \right]^{\frac{1}{1+\xi}} \right\}$$

With

$$t := \sqrt{\frac{K^4 \log(pq)}{2T^{1-\xi} C_B}}$$

$$P \left[ \frac{1}{T} \left\| X'W \right\|_\infty > \sqrt{72K^4 \log(pq)} \right] \leq 15 \exp \left\{ -\frac{1}{2} \log(pq) \right\} + 6(T^{1-\xi} - 1) \exp \left\{ -\frac{1}{2} c_B T^\xi \right\}$$

where $K = \sqrt{K_X} + \sqrt{K_Y} (1 + \| \Theta^* \|)$

□

**Lemma 6.** For any sub-Gaussian random vector $X$ and non-stochastic matrix $A$. We have

$$\| AX \|_{\psi_2} \leq \| A \| \| X \|_{\psi_2}$$
Proof. We have,

\[
\|AX\|_{\psi 2} = \sup_{\|v\|_2 \leq 1} \|v^TAX\|_{\psi 2} \\
= \sup_{\|v\|_2 \leq 1} \|(A^Tv)^T X\|_{\psi 2} \\
\leq \sup_{\|u\|_2 \leq \|A\|} \|u^T X\|_{\psi 2} \\
= \|A\| \sup_{\|u\|_2 \leq 1} \|u^T X\|_{\psi 2} \\
= \|A\| \|X\|_{\psi 2}.
\]

\[\square\]

### D Bernstein’s Concentration Inequality

We state the Bernstein’s inequality [44, Proposition 5.16] below for completeness.

**Proposition 7** (Bernstein’s Inequality). Let \(X_1, \ldots, X_N\) be independent centered sub-exponential random variables, and \(K = \max_i \|X_i\|_{\psi 1}\). Then for every \(a = (a_1, \ldots, a_N) \in \mathbb{R}^N\) and every \(t \geq 0\), we have

\[
P \left\{ \left| \sum_{i=1}^{N} a_i X_i \right| \geq t \right\} \leq 2 \exp \left[ -C_B \min \left( \frac{t^2}{K^2 \|a\|_2^2}, \frac{t}{K \|a\|_\infty} \right) \right]
\]

where \(C_B > 0\) is an absolute constant.

### E Verification of Assumptions for the Examples

#### E.1 VAR

Formally a finite order Gaussian VAR(\(d\)) process is defined as follows. Consider a sequence of serially ordered random vectors \((Z_t)_{t=1}^{T+d}, Z_t \in \mathbb{R}^p\) that admits the following auto-regressive representation:

\[
Z_t = A_1 Z_{t-1} + \cdots + A_d Z_{t-d} + \mathcal{E}_t
\]

where each \(A_k\), \(k = 1, \ldots, d\) is a non-stochastic coefficient matrix in \(\mathbb{R}^{p \times p}\) and innovations \(\mathcal{E}_t\) are \(p\)-dimensional random vectors from \(\mathcal{N}(0, \Sigma)\). Assume \(\lambda_{\min}(\Sigma) > 0\) and \(\lambda_{\max}(\Sigma) < \infty\).

Note that every VAR(\(d\)) process has an equivalent VAR(1) representation (see e.g. [24, Ch 2.1]) as

\[
\tilde{Z}_t = \tilde{A} \tilde{Z}_{t-1} + \tilde{\mathcal{E}}_t
\]
where

\[
\tilde{Z}_t := \begin{bmatrix} Z_t \\ Z_{t-1} \\ \vdots \\ Z_{t-d+1} \end{bmatrix}_{(pd \times 1)}, \quad \tilde{\xi}_t := \begin{bmatrix} \xi_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(pd \times 1)} \quad \text{and} \quad \tilde{A} := \begin{bmatrix} A_1 & A_2 & \cdots & A_{d-1} & A_d \\ I_p & 0 & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I_p & 0 \end{bmatrix}_{(dp \times dp)}
\]

(E.3)

Because of this equivalence, justification of Assumption 5 will operate through this corresponding augmented VAR(1) representation.

For both Gaussian and sub-Gaussian VARs, Assumption 3 is true since the sequences \((Z_t)\) is centered. Second, \(\Theta^* = (A_1, \cdots, A_d)\). So Assumption 1 follows from construction.

For the remaining Assumptions, we will consider the Gaussian and sub-Gaussian cases separately.

**Gaussian VAR** \((Z_t)\) satisfies Assumption 4 by model assumption.

To show that \((Z_t)\) is \(\beta\)-mixing with geometrically decaying coefficients, we use the following facts together with the equivalence between \((Z_t)\) and \((\tilde{Z}_t)\) and Fact 1.

Since \((\tilde{Z}_t)\) is stable, the spectral radius of \(\tilde{A}, r(\tilde{A}) < 1\), hence Assumption 2 holds. Also the innovations \(\tilde{\xi}\) has finite first absolute moment and positive support everywhere. Then, according to Theorem 4.4 in [42], \((\tilde{Z}_t)\) is geometrically ergodic. Note here that Gaussianity is not required here. Hence, it also applies to innovations from mixture of Gaussians.

Next, we present a standard result (see e.g. [22, Proposition 2]).

**Fact 4.** A stationary Markov chain \(\{Z_t\}\) is geometrically ergodic implies \(\{Z_t\}\) is absolutely regular (a.k.a. \(\beta\)-mixing) with

\[
\beta(n) = O(\gamma^n), \quad \gamma^n \in (0, 1)
\]

So, Assumption 5 holds.

**Sub-Gaussian VAR** When the innovations are random vectors from the uniform distribution, they are sub-Gaussian. That \((Z_t)\) are sub-Gaussian follows from arguments as in Appendix E.3 with \(\Sigma(\cdot)\) set to be the identity operator in this case. So, Assumption 4 holds.

To show that \((Z_t)\) satisfies Assumptions 2 and 5, we establish that \((Z_t)\) is geometrically ergodic. To show the latter, we use Propositions 1 and 2 in [22] together with the equivalence between \((Z_t)\) and \((\tilde{Z}_t)\) and Fact 1.
To apply Proposition 1 in [22], we check the three conditions one by one. Condition (i) is immediate with $m = 1$, $E = \mathbb{R}^p$, and $\mu$ is the Lebesgue measure. For condition (ii), we set $E = \mathbb{R}^p$, $\mu$ to be the Lebesgue measure, and $\bar{m} = \inf_{u \in C, v \in A} \|u - v\|_2$ the minimum “distance” between the sets $C$ and $A$. Because $C$ is bounded and $A$ Borel, $\bar{m}$ is finite. Lastly, for condition (iii), we again let $E = \mathbb{R}^p$, $\mu$ to be the Lebesgue measure, and now the function $Q(\cdot) = \|\cdot\|$ and the set $K = \{x \in \mathbb{R}^p : \|x\| \leq \frac{2E\|\tilde{E}\|}{c\epsilon}\}$ where $c = 1 - \|\tilde{A}\|$.

Then,

- Recall from model assumption that $\|\tilde{A}\| < 1$; hence,
  \[
  \mathbb{E} \left[ \left\| \bar{Z}_{t+1} \right\| \mid \bar{Z}_t = z \right] < \left\| \tilde{A} \right\| \|z\| + \mathbb{E}(\|\tilde{E}_{t+1}\|) \leq \left( 1 - \frac{c}{2} \right) \|z\| - \epsilon,
  \]
  for all $z \in E \setminus K$

- For all $z \in K$,
  \[
  \mathbb{E} \left[ \left\| \bar{Z}_{t+1} \right\| \mid \bar{Z}_t = z \right] < \left\| \tilde{A} \right\| \|z\| + \mathbb{E}(\|\tilde{E}_{t+1}\|) \leq \left\| \tilde{A} \right\| \frac{2E\|\tilde{E}\|}{c\epsilon}
  \]

- For all $z \in K$,
  \[
  0 \leq \|z\| \leq \frac{2E\|\tilde{E}\|}{c\epsilon}
  \]

Now, by Proposition 1 in [22], $(\bar{Z}_t)$ is geometrically ergodic; hence $(\tilde{Z}_t)$ will be stationary. Once it reaches stationarity, by Proposition 2 in the same paper, the sequence will be $\beta$-mixing with geometrically decaying mixing coefficients. Therefore, Assumptions 2 and 5 hold.

### E.2 VAR with Misspecification

**Assumptions:** Assumption 3 is immediate from model definitions. By the same arguments as in Appendix E.1, $(Z_t, \Xi_t)$ are stationary and so is the sub-process $(\bar{Z}_t)$; Assumption 2 holds. Again, $(\bar{Z}_t, \bar{\Xi}_t)$ satisfy Assumption 5 according to Appendix E.1. By Fact 1, we have the same Assumptions hold for the respective sub-processes $(Z_t)$ in both cases. Assumption 4 holds by the same reasoning as in Appendix E.1.

To show that $(\Theta^*') = A_{ZZ} + A_{Z\Xi} \Sigma_{\Xi Z}(0)(\Sigma_{ZZ}(0))^{-1}$, consider the following arguments. By Assumption 2, we have the auto-covariance matrix of the whole system $(Z_t, \bar{\Xi}_t)$ as

$$
\Sigma_{(Z, \bar{\Xi})} = \begin{bmatrix}
\Sigma_X(0) & \Sigma_{X\bar{\Xi}}(0) \\
\Sigma_{Z\bar{X}}(0) & \Sigma_{\bar{\Xi}}(0)
\end{bmatrix}
$$
Recall our $\Theta^*$ definition from Eq. (2.1)

$$\Theta^* := \arg \min_{B \in \mathbb{R}^{p \times p}} \mathbb{E} \left( \| Z_t - B' Z_{t-1} \|^2_2 \right)$$

Taking derivatives and setting to zero, we obtain

$$(\Theta^*)' = \Sigma_Z (-1) (\Sigma_Z)^{-1}$$ \hspace{1cm} (E.4)

Note that

$$\Sigma_Z (-1) = \Sigma_{(Z, Z)} (-1) [1 : p_1, 1 : p_1]$$

$$= \mathbb{E} (A_{ZZ} Z_{t-1} + A_{Z\Xi} \Xi_{t-1} + \mathcal{E}_{Z,t-1}) Z'_{t-1}$$

$$= \mathbb{E} (A_{ZZ} Z_{t-1} Z'_{t-1} + A_{Z\Xi} \Xi_{t-1} Z'_{t-1} + \mathcal{E}_{Z,t-1} Z'_{t-1})$$

$$= A_{ZZ} \Sigma_Z (0) + A_{Z\Xi} \Xi Z (0)$$

by Assumptions 2 and the fact that the innovations are iid.

Naturally,

$$(\Theta^*)' = A_{ZZ} \Sigma_Z (0) (\Sigma_Z (0))^{-1} + A_{Z\Xi} \Sigma_{Z\Xi} (0) (\Sigma_Z (0))^{-1} = A_{ZZ} + A_{Z\Xi} \Sigma_Z (0) (\Sigma_Z (0))^{-1}$$

**Remark 4.** Notice that $A_{Z\Xi}$ is a column vector and suppose it is 1-sparse, and $A_{ZZ}$ is $p$-sparse, then $\Theta^*$ is at most $2p$-sparse. So Assumption 1 can be built in by model construction.

**Remark 5.** We gave an explicit model here where the left out variable $\Xi$ was univariate. That was only for convenience. In fact, whenever the set of left-out variables $\Xi$ affect only a small set of variables $\Xi$ in the retained system $Z$, the matrix $\Theta^*$ is guaranteed to be sparse. To see that, suppose $\Xi \in \mathbb{R}^q$ and $A_{Z\Xi}$ has at most $s_0$ non-zero rows (and let $A_{ZZ}$ to be $s$-sparse as always), then $\Theta^*$ is at most $(s_0 p + s)$-sparse.

**Remark 6.** Any VAR($d$) process has an equivalent VAR(1) representation (Lutkepohl 2005). Our results extend to any VAR($d$) processes.

### E.3 ARCH

**Verifying the Assumptions.** To show that Assumption 5 holds for a process defined by Eq. (4.2) we leverage on Theorem 2 from [22]. Note that the original ARCH model in [22] assumes the innovations to have positive support everywhere. However, this is just a convenient assumption to establish the first two conditions in Proposition 1 (on which proof of Theorem 2 relies) from the same paper. Our example ARCH model with innovations from the uniform distribution also satisfies the first two conditions of Proposition 1 by the same arguments in the Sub-Gaussian paragraph of Appendix E.1.

Theorem 2 tells us that for our ARCH model, if it satisfies the following conditions, it is guaranteed to be absolutely regular with geometrically decaying $\beta$-coefficients.
• $E_t$ has positive density everywhere on $\mathbb{R}^p$ and has identity covariance by construction.

• $\Sigma(z) = o(\|z\|)$ because $m \in (0, 1)$.

• $\|\Sigma(z)^{-1}\| \leq 1/(ac)$, $|\det(\Sigma(z))| \leq bc$

• $r(A) \leq \|A\| < 1$

So, Assumption 5 is valid here. We check other assumptions next.

Mean 0 is immediate, so we have Assumption 3. When the Markov chain did not start from a stationary distribution, geometric ergodicity implies that the sequence is approaching the stationary distribution exponentially fast. So, after a burning period, we will have Assumption 2 approximately valid here.

The sub-Gaussian constant of $\Sigma(Z_{t-1})E_t$ given $Z_{t-1} = z$ is bounded as follows: for every $z$,

$$
\|\Sigma(z)E_t\|_{\psi_2} \leq \|\Sigma(z)\| \|E_t\|_{\psi_2}
\leq C\|\Sigma(z)\| : \|e'_tE_t\|_{\psi_2}
\leq C\|\Sigma(z)\| : \|U(\sqrt{3}, \sqrt{3})\|_{\psi_2}
\leq C'eb =: K_E
$$

The second inequality follows since $E_t \overset{iid}{\sim} U(\sqrt{3}, \sqrt{3})^p$ and a standard result that

Fact 5. Let $X = (X_1, \cdots, X_p) \in \mathbb{R}^p$ be a random vector with independent, mean zero, sub-Gaussian coordinates $X_i$. Then $X$ is a sub-Gaussian random vector, and there exists a positive constant $C$ for which

$$
\|X\|_{\psi_2} \leq C \cdot \max_{i \leq p} \|X_i\|_{\psi_2}
$$

The forth inequality follows since the sub-Gaussian norm of a bounded random variable is also bounded.

By the recursion for $Z_t$, we have

$$
\|Z_t\|_{\psi_2} \leq \|A\| \|Z_{t-1}\|_{\psi_2} + K_E.
$$

which yields the bound $\|Z_t\|_{\psi_2} \leq K_E/(1 - \|A\|) < \infty$. Hence Assumption 4 holds.

We will show below that $\Theta^* = A'$. Hence, sparsity (Assumption 1) can be built in when we construct our model 4.2.

Recall Eq. E.4 from Appendix E.2 that

$$
\Theta^* = \Sigma_Z(-1)(\Sigma_Z)^{-1}
$$
Now,

\[ \Sigma_Z(-1) = \mathbb{E}Z_t Z'_t \quad \text{by stationarity} \]

\[ = \mathbb{E}(AZ_{t-1} + \Sigma(Z_{t-1})E_t) Z'_t \quad \text{Eq. (4.2)} \]

\[ = A\Sigma_Z + \mathbb{E}(c \text{clip}_{a,b}(\|Z_{t-1}\|^m) E_t Z'_t) \]

\[ = A\Sigma_Z + \mathbb{E} [cE_t Z'_t \text{clip}_{a,b}(\|Z_{t-1}\|^m)] \]

\[ = A\Sigma_Z + c\mathbb{E} \left[ \text{clip}_{a,b}(\|Z_{t-1}\|^m) \right] \text{i.i.d. innovations} \]

\[ = A\Sigma_Z \quad \mathbb{E}_t \text{mean } 0, \]

where \( \text{clip}_{a,b}(x) := \min\{\max\{x, a\}, b\} \) for \( b > a \).

Since \( \Sigma_Z \) is invertible, we have \( (\Theta^*)' = \Sigma_Z(-1)(\Sigma_Z)^{-1} = A \).