CONTINUUM LIMIT OF TOTAL VARIATION ON POINT CLOUDS

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ABSTRACT. We consider point clouds obtained as random samples of a measure on a Euclidean domain. A graph representing the point cloud is obtained by assigning weights to edges based on the distance between the points they connect. Our goal is to develop mathematical tools needed to study the consistency, as the number of available data points increases, of graph-based machine learning algorithms for tasks such as clustering. In particular, we study when is the cut capacity, and more generally total variation, on these graphs a good approximation of the perimeter (total variation) in the continuum setting. We address this question in the setting of \(\Gamma\)-convergence. We obtain almost optimal conditions on the scaling, as number of points increases, of the size of the neighborhood over which the points are connected by an edge for the \(\Gamma\)-convergence to hold. Taking the limit is enabled by a new metric which allows to suitably compare functionals defined on different point clouds.

1. INTRODUCTION

Our goal is to develop mathematical tools to rigorously study limits of variational problems defined on random samples of a measure, as the number of data points goes to infinity. The main application is to establishing consistency of machine learning algorithms for tasks such as clustering and classification. These tasks are of fundamental importance for statistical analysis of randomly sampled data, yet few results on their consistency are available. In particular it is largely open to determine when do the minimizers of graph-based tasks converge, as the number of available data increases, to a minimizer of a limiting functional in the continuum setting. Here we introduce the mathematical setup needed to address such questions.

To analyze the structure of a data cloud one defines a weighted graph to represent it. Points become vertices and are connected by edges if sufficiently close. The edges are assigned weights based on the distances between points. How the graph is constructed is important: for lower computational complexity one seeks to have fewer edges, but below some threshold the graph no longer contains the desired information on the geometry of the point cloud. The machine learning tasks, such as classification and clustering, can often be given in terms of minimizing a functional on the graph representing the point cloud. Some of the fundamental approaches are based on minimizing graph cuts (graph perimeter) and related functionals (normalized cut, ratio cut, balanced cut), and more generally total variation on graphs \([7, 11, 13, 16, 17, 18, 20, 30, 31, 35, 41, 43, 46, 47]\). We focus on total variation on graphs (of which graph cuts are a special case). The techniques we introduce are applicable to rather broad range of functionals, in particular those where total variation is combined with lower-order terms, or those where total variation is replaced by Dirichlet energy.

The graph perimeter (a.k.a. cut size, cut capacity) of a set of vertices is the sum of the weights of edges between the set and its complement. Our goal is to understand for what constructions of graphs from data is the cut capacity a good notion of a perimeter. We pose this question in terms of consistency as the number of data points increases: \(n \to \infty\). We assume that the data points are random independent samples of an underlying measure \(\nu\) with density \(\rho\) supported in a set \(D\) in \(\mathbb{R}^d\). The question is if the...
graph perimeter on the point cloud is a good approximation of the perimeter on $D$ (weighted by $\rho^2$). Since machine learning tasks involve minimizing appropriate functionals on graphs, the most relevant question is if the minimizers of functionals on graphs involving graph perimeter converge to minimizers of corresponding limiting functionals in continuum setting, as $n \to \infty$. Such convergence is implied by the variational notion of convergence called the $\Gamma$-convergence, which we focus on. The notion of $\Gamma$-convergence has been used extensively in the calculus of variations, in particular in homogenization theory, phase transitions, image processing, and material science. We show how the $\Gamma$-convergence can be applied to establishing consistency of data-analysis algorithms.

1.1. Setting and the main results. Consider a point cloud $V = \{X_1, \ldots, X_n\}$. Let $\eta$ be a kernel, that is, let $\eta : \mathbb{R}^d \to [0, \infty)$ be a radially symmetric, radially decreasing, function decaying to zero sufficiently fast. Typically the kernel is appropriately rescaled to take into account data density. In particular, let $\eta\epsilon$ depend on a length scale $\epsilon$ so that significant weight is given to edges connecting points up to distance $\epsilon$. We assign for $i, j \in \{1, \ldots, n\}$ the weights by

\begin{equation}
W_{i,j} = \eta\epsilon(X_i - X_j)
\end{equation}

and define the graph perimeter of $A \subset V$ to be

\begin{equation}
G\text{Per}(A) = 2 \sum_{X_i \in A, X_j \notin V \setminus A} W_{i,j}.
\end{equation}

The graph perimeter (i.e. cut size, cut capacity), can be effectively used as a term in functionals which give a variational description to classification and clustering [11, 16, 13, 18, 20, 19, 17, 30, 31, 35, 41, 49, 57].

The total variation of a function $u$ defined on the point cloud is typically given as

\begin{equation}
\sum_{i,j} W_{i,j} |u(X_i) - u(X_j)|.
\end{equation}

We note that the total variation is a generalization of perimeter since the perimeter of a set of vertices $A \subset V$ is the total variation of the characteristic function of $A$.

In this paper we focus on point clouds that are obtained as samples from a given distribution $\nu$. Specifically, consider an open, bounded, and connected set $D \subset \mathbb{R}^d$ with Lipschitz boundary and a probability measure $\nu$ supported on $D$. Suppose that $\nu$ has density $\rho$, which is continuous and bounded above and below by positive constants on $D$. Assume $n$ data points $X_1, \ldots, X_n$ (i.i.d. random vectors) are chosen according to the distribution $\nu$. We consider a graph with vertices $V = \{X_1, \ldots, X_n\}$ and edge weights $W_{i,j}$ given by (1), where $\eta\epsilon$ to be defined by $\eta\epsilon(z) := \frac{1}{\epsilon^d} \eta\left(\frac{z}{\epsilon}\right)$. Note that significant weight is given to edges connecting points up to distance of order $\epsilon$.

Having limits as $n \to \infty$ in mind, we define the graph total variation to be the following rescaled form of (3):

\begin{equation}
GTV_{n,\epsilon}(u) := \frac{1}{\epsilon} \frac{1}{n^2} \sum_{i,j} W_{i,j} |u(X_i) - u(X_j)|.
\end{equation}

For a given scaling of $\epsilon$ with respect to $n$, we study the limiting behavior of $GTV_{n,\epsilon(n)}$ as the number of points $n \to \infty$. The limit is considered in the variational sense of $\Gamma$-convergence.

A key contribution of our work is in identifying the proper topology with respect to which the $\Gamma$-convergence takes place. As one is considering functions supported on the graphs, the issue is how to compare them with functions in the continuum setting, and how to compare functions defined on different graphs. Let us denote by $\nu_n$ the empirical measure associated to the $n$ data points:

\begin{equation}
\nu_n := \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i},
\end{equation}
The issue is then how to compare functions in $L^1(\nu_n)$ with those in $L^1(\nu)$. More generally we consider how to compare functions in $L^p(\mu)$ with those in $L^p(\theta)$ for arbitrary probability measures $\mu$, $\theta$ on $D$ and arbitrary $p \in [1, \infty)$. We set

$$TL^p(D) := \{ (\mu, f) : \mu \in \mathcal{P}(D), f \in L^p(D, \mu) \},$$

where $\mathcal{P}(D)$ denotes the set of Borel probability measures on $D$. For $(\mu, f)$ and $(\nu, g)$ in $TL^p$ we define the distance

$$d_{TL^p}(\mu, f)(\nu, g) = \inf_{\pi \in \Gamma(\mu, \nu)} \left( \int_D \int_D |x-y|^p d\pi(x,y) \right)^{\frac{1}{p}} + \left( \int_D \int_D |f(x) - g(y)|^p d\pi(x,y) \right)^{\frac{1}{p}},$$

where $\Gamma(\mu, \theta)$ is the set of all couplings (or transportation plans) between $\mu$ and $\theta$, that is, the set of all Borel probability measures on $D \times D$ for which the marginal on the first variable is $\mu$ and the marginal on the second variable is $\theta$.

The $TL^p$ topology provides a general and versatile way to compare functions in a discrete setting with functions in a continuum setting. It is a generalization of the weak convergence of measures and of $L^p$ convergence of functions. By this we mean that $\{\mu_n\}_{n \in \mathbb{N}}$ in $\mathcal{P}(D)$ converges weakly to $\mu \in \mathcal{P}(D)$ if and only if $(\mu_n, 1) \xrightarrow{TL^p} (\mu, 1)$ as $n \to \infty$, and that for $\mu \in \mathcal{P}(D)$ a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $L^p(\mu)$ converges in $L^p(\mu)$ to $f$ if and only if $(\mu, f_n) \xrightarrow{TL^p} (\mu, f)$ as $n \to \infty$. The fact is established in Proposition 3.6.

Furthermore if one considers functions defined on a regular grid, then the standard way [21, 15], to compare them is to identify them with piecewise constant functions, whose value on the grid cells is equal to the value at the appropriate grid point, and then compare the extended functions using the $L^p$ metric. $TL^p$ metric restricted to regular grids gives the same topology.

The kernels $\eta$ we consider are assumed to be isotropic, and thus can be defined as $\eta(x) := |x|$ where $\eta : [0, \infty) \to [0, \infty)$ is the radial profile. We assume:

(K1) $\eta(0) > 0$ and $\eta$ is continuous at 0.

(K2) $\eta$ is non-increasing.

(K3) The integral $\int_0^r \eta(r)^2 dr$ is finite.

We note that the class of admissible kernels is broad and includes both Gaussian kernels and discontinuous kernels like one defined by $\eta$ of the form $\eta = 1$ for $r \leq 1$ and $\eta = 0$ for $r > 1$. We remark that the assumption (K3) is equivalent to imposing that the surface tension

$$\sigma_\eta = \int_{\mathbb{R}^d} |\nabla h_1| dh,$$

where $h_1$ is the first coordinate of vector $h$, is finite and also that one can replace $h_1$ in the above expression by $h \cdot e$ for any fixed $e \in \mathbb{R}^d$ with norm one; this, given that $\eta$ is radially symmetric.

The weighted total variation in continuum setting (with weight $\rho^2$), $TV(\cdot, \rho^2) : L^1(D, \nu) \to [0, \infty]$, is given by

$$TV(u, \rho^2) = \sup \left\{ \int_D u \text{div}(\phi) dx : |\phi(x)| \leq \rho^2(x) \quad \forall x \in D, \phi \in C_0^\infty(D, \mathbb{R}^d) \right\}$$

if the right-hand side is finite and is set to equal infinity otherwise. Here and in the rest of the paper we use $|\cdot|$ to denote the euclidean norm in $\mathbb{R}^d$. Note that if $u$ is smooth enough then the weighted total variation can be written as $TV(u, \rho^2) = \int_D |\nabla u|^2 dx$.

The main result of the paper is:

**Theorem 1.1 (Γ-convergence).** Let $D \subset \mathbb{R}^d$, $d \geq 2$ be an open set with Lipschitz boundary which is homeomorphic via a bi-Lipschitz mapping to $(0, 1)^d$. Let $\nu$ be a probability measure on $D$ with continuous density $\rho$, which is bounded from below and above by positive constants. Let $X_1, \ldots, X_n, \ldots$
be a sequence of i.i.d. random vectors chosen according to distribution \( \nu \) on \( D \). Let \( \{ \varepsilon_n \}_{n \in \mathbb{N}} \) be a sequence of positive numbers converging to 0 and satisfying
\[
\lim_{n \to \infty} \frac{(\log n)^{3/4}}{n^{1/2}} \frac{1}{\varepsilon_n} = 0 \quad \text{if } d = 2, \\
\lim_{n \to \infty} \frac{(\log n)^{1/d}}{n^{1/d}} \frac{1}{\varepsilon_n} = 0 \quad \text{if } d \geq 3.
\]
Assume the kernel \( \eta \) satisfies conditions (K1)-(K3). Then, \( \text{GTV}_{n, \varepsilon_n} \), defined by (4), \( \Gamma \)-converges to \( \sigma_{\eta} \text{TV}(\cdot, \rho^2) \) as \( n \to \infty \) in the \( TL^1 \) sense, where \( \sigma_{\eta} \) is given by (6) and \( \text{TV}(\cdot, \rho^2) \) is the weighted total variation functional defined in (7).

The notion of \( \Gamma \)-convergence in deterministic setting is recalled in Subsection 2.4 where we also extend it to the probabilistic setting in Definition 2.11. The fact that the density in the limit is \( \rho^2 \) essentially follows from the fact that graph total variation is a double sum (and becomes more apparent in Section 5 when we write the graph total variation in form (60)).

The following compactness result shows that the \( TL^1 \) topology is indeed a good topology for the \( \Gamma \)-convergence (in the light of Proposition 2.10).

**Theorem 1.2 (Compactness).** Under the assumptions of the theorem above, consider a sequence of functions \( u_n \in L^1(D, \nu_n) \), where \( \nu_n \) is given by (5). If \( \{ u_n \}_{n \in \mathbb{N}} \) have uniformly bounded \( L^1(D, \nu_n) \) norms and graph total variations, \( \text{GTV}_{n, \varepsilon_n} \), then the sequence is precompact in \( TL^1 \). More precisely if
\[
\sup_{n \in \mathbb{N}} \| u_n \|_{L^1(D, \nu_n)} < \infty,
\]
and
\[
\sup_{n \in \mathbb{N}} \text{GTV}_{n, \varepsilon_n}(u_n) < \infty,
\]
then \( \{ u_n \}_{n \in \mathbb{N}} \) is \( TL^1 \)-precompact.

When \( A_n \) is a subset of \( \{ X_1, \ldots, X_n \} \), it holds that \( \text{GTV}_{n, \varepsilon_n}(\chi_{A_n}) = \frac{1}{n \varepsilon_n} \text{GPer}(A_n) \), where \( \text{GPer}(A_n) \) was defined in (3). The proof of Theorem 1.1 allows us to consider the variational convergence of the perimeter on the graphs to the weighted perimeter in the domain \( D \), defined by \( \text{Per}(E : D, \rho^2) = \text{TV}(\chi_E, \rho^2) \).

**Corollary 1.3 (\( \Gamma \)-convergence of perimeter).** Under the hypothesis of Theorem 1.7, the conclusions hold when all of the functionals are restricted to characteristic functions of sets. That is, the (scaled) graph perimeters \( \Gamma \)-converge to the continuum (weighted) perimeter \( \text{Per}(\cdot : D, \rho^2) \).

The proofs of the theorems and of the corollary are presented in Section 5. We remark that the Corollary 1.3 is not an immediate consequence of Theorem 1.1 since in general \( \Gamma \)-convergence may not carry over when a (closed) subspace of a metric space is considered. The proof of Corollary 1.3 is nevertheless straightforward.

**Remark 1.4.** When one considers \( \rho \) to be constant in Theorem 1.1, the points \( X_1, \ldots, X_n \) are uniformly distributed on \( D \). In this particular case, the theorem implies that the graph total variation converges to the usual total variation on \( D \) (appropriately scaled by \( 1 / \text{Vol}(D)^2 \)). Corollary 1.3 implies that the graph perimeter converges to the usual perimeter (appropriately scaled).

**Remark 1.5.** The notion of \( \Gamma \)-convergence is different from the notion of pointwise convergence, but often the proof of \( \Gamma \) convergence implies the pointwise convergence. The pointwise convergence of the graph perimeter to continuum perimeter is the statement that for any set \( A \subset D \) of finite perimeter, with probability one:
\[
\lim_{n \to \infty} \text{GTV}_{n, \varepsilon_n}(\chi_A) = \text{Per}(A : D, \rho^2).
\]
In the case that $D$ is smooth, the points $X_1, \ldots, X_n$ are uniformly distributed on $D$ and $A$ is smooth, the pointwise convergence of the graph perimeter can be obtained from the results in [34] and in [6] when $\varepsilon_n$ is converging to zero so that $\frac{(\log n)^{1/(d+1)}}{\varepsilon_n^{1/d}} \to 0$ as $n \to \infty$. In Remark 5.1 we point out that our proof of $\Gamma$-convergence implies that pointwise convergence also holds, with same scaling for $\varepsilon_n$ as in Theorem 1.1 which slightly improves the rate of pointwise convergence in [6]. Note that pointwise convergence does not follow directly from the $\Gamma$-convergence.

Remark 1.6. Theorem 1.2 implies that the probability that the weighted graph, with vertices $X_1, \ldots, X_n$ and edge weights $W_{i,j} = \eta_{\varepsilon_n}(X_i - X_j)$ is connected, converges to 1 as $n \to \infty$. Otherwise there is a sequence $n_k \to \infty$ as $k \to \infty$ such that with positive probability, the graph above is not connected for all $k$. We can assume that $n_k = k$ for all $k$. Consider a connected component $A_n \subset \{X_1, \ldots, X_n\}$ such that $\sharp A_n \leq n/2$. Define function $u_n = \frac{1}{|A_n|} \chi_{A_n}$. Note that $\|u_n\|_{L^1(v)} = 1$ and that $GTV_{\nu, \varepsilon_n}(u_n) = 0$. By compactness, along a subsequence (not relabeled), $u_n$ converges in $TL^1$ to a function $u \in L^1(v)$. Thus $\|u\|_{L^1(v)} = 1$. By lower-semicontinuity which follows from $\Gamma$-convergence of Theorem 1.1 it follows that $TV(u) = 0$ and thus $u = 1$ on $D$. But since the values of $u_n$ are either 0 or greater or equal to 2, it is not possible that $u_n$ converges to $u$ in $TL^1$. This is a contradiction.

1.2. Optimal scaling of $\varepsilon(n)$. If $d \geq 3$ then the rate presented in (8) is sharp in terms of scaling. To illustrate, suppose that the data points are uniformly distributed on $D$ and $\eta$ has compact support. It is known from graph theory (see [33, 27, 28]) that there exists a constant $\lambda > 0$ such that if $\varepsilon_n < \frac{\lambda (\log n)^{1/d}}{n^{1/d}}$ then the weighted graph associated to $X_1, \ldots, X_n$ is disconnected with high probability. Therefore, in the light of Remark 1.6, the compactness property cannot hold if $\varepsilon_n < \frac{\lambda (\log n)^{1/d}}{n^{1/d}}$. It is of course, not surprising that if the graph is disconnected, the functionals describing clustering tasks may have minimizers which are rather different than the minimizers of the continuum functional.

While the above example shows the optimality of our results in some sense, we caution that there still may be settings relevant to machine learning in which the convergence of minimizers of appropriate functionals may hold even when $\frac{1}{n^{1/d}} \ll \varepsilon_n < \frac{\lambda (\log n)^{1/d}}{n^{1/d}}$.

Finally, we remark that in the case $d = 2$, the rate presented in (8) is different from the connectivity rate in dimension $d = 2$ which is $\frac{\lambda (\log n)^{1/2}}{n^{1/2}}$. An interesting open problem is to determine what happens to the graph total variation as $n \to \infty$, when one considers $\frac{\lambda (\log n)^{1/2}}{n^{1/2}} \ll \varepsilon_n \leq \frac{\lambda (\log n)^{3/4}}{n^{1/4}}$.

1.3. Related work. Background on $\Gamma$-convergence of functionals related to perimeter. The notion of $\Gamma$-convergence was introduced by De Giorgi in the 70’s and represents a standard notion of variational convergence. With compactness it ensures that minimizers of approximate functionals converge (along a subsequence) to a minimizer of the limiting functional. For extensive exposition of the properties of $\Gamma$-convergence see the books by Braides [14] and Dal Maso [22].

A classical example of $\Gamma$-convergence of functionals to perimeter is the Modica and Mortola theorem (36) that shows the $\Gamma$-convergence of Allen-Cahn (Cahn-Hilliard) free energy to perimeter.

There is a number of results considering nonlocal functionals converging to the perimeter or to total variation. In [3], Alberti and Bellettini study a nonlocal model for phase transitions where the energies do not have a gradient term as in the setting of Modica and Mortola, but a nonlocal term. In [42], Savin and Valdinoci consider a related energy involving more general kernels.

Esedo˘glu and Otto, [24] consider nonlocal total-variation based functionals in multiphase systems and show their $\Gamma$-convergence to perimeter. Brezis, Bourgain, and Mironescu [12] considered nonlocal functionals in order to give new characterizations of Sobolev and BV spaces. Ponce [40] extended their work and showed the $\Gamma$-convergence of the nonlocal functionals studied to local ones. In our work we adopt the approach of Ponce to show $\Gamma$-convergence as it is conceptually clear and efficient.

In the discrete setting, works related to the $\Gamma$-convergence of functionals to continuous functionals involving perimeter include [15], [53] and [21]. The results by Braides and Yip [15], can be interpreted...
as the analogous results in a discrete setting to the ones obtained by Modica and Mortola. They give
the description of the limiting functional (in the sense of $\Gamma$-convergence) after appropriately rescaling
the energies. In the discretized version considered, they work on a regular grid and the gradient term
gets replaced by a finite-difference approximation that depends on the mesh size $\delta$. Van Gennip and
Bertozzi \cite{Bertozzi2009} consider a similar problem and obtain analogous results. In \cite{Chambolle2009},
Chambolle, Giacomini and Lussardi consider a very general class of anisotropic perimeters defined on
discrete subsets of a finite lattice of the form $\delta \mathbb{Z}^N$. They prove the $\Gamma$-convergence of the functionals
as $\delta \to 0$ to an anisotropic perimeter defined on a given domain in $\mathbb{R}^d$.

Background on analysis of algorithms on point clouds as $n \to \infty$. In the past years a diverse set of
geometrically based methods has been developed to solve different tasks of data analysis like classification,
regression, dimensionality reduction and clustering. One desirable and important property that
one expects from these methods is consistency. That is, it is desirable that as the number of data points
tends to infinity the procedure used “converges” to some “limiting” procedure. Usually this “limiting”
procedure involves a continuum functional defined on a domain in a Euclidean space or more generally
on a manifold.

Most of the available consistency results are about pointwise consistency. Among them are works
of Belkin and Niyogi \cite{Belkin2008}, Giné and Koltchinskii \cite{Gine2010}, Hein, Audibert, von Luxburg \cite{Hein2010}, Singer \cite{Singer2010}
and Ting, Huang, and Jordan \cite{Ting2010}. The works of von Luxburg, Belkin and Bousquet on consistency
of spectral clustering \cite{Castro2009} and Belkin and Niyogi \cite{Belkin2010} on the convergence of Laplacian Eigenmaps, as well as
\cite{Ting2010}, consider spectral convergence and thus convergence of eigenvalues and eigenvectors, which are
relevant for machine learning. An important difference between our work and the spectral convergence
works is that in them, there is no explicit rate at which $\epsilon_n$ is allowed to converge to 0 as $n \to \infty$. Arias-
Castro, Pelletier, and Pudlo \cite{Arias-Castro2010} considered pointwise convergence of Cheeger energy and consequently
of total variation, and obtained a range of scalings of $\epsilon$ on $n$ under which the convergence holds. Maier,
von Luxburg and Hein \cite{Maier2010} considered pointwise convergence for Cheeger and normalized cuts, both
for the geometric and kNN graphs and obtained an analogous range of scalings of graph construction
on $n$ for the convergence to hold. Pollard \cite{Pollard2010} considered the consistency of the $k$-means clustering
algorithm.

1.4. Example: An application to clustering. Many algorithms involving graph cuts, total variation
and related functionals on graphs are in use in data analysis. Here we present an illustration of how the
$\Gamma$-convergence results can be applied in that context. In particular we show the consistency of minimal
bisection considered for example in \cite{Belkin2008} \cite{Belkin2010}. The example we choose is simple and its primary goal is
to give a hint of the possibilities. We intend to investigate the functionals relevant to data analysis in
future works.

Let $D$ be domain satisfying the assumptions of Theorem 1.1 for example the one depicted on Figure 1.
Consider the problem of dividing the domain into two clusters of equal sizes. In the continuum
setting the problem can be posed as finding $A_{\text{min}} \subset D$ such that $F(A) = TV(\chi_A)$, is minimized over all
$A$ such that $\text{Vol}(D) = 2 \text{Vol}(A)$. For the domain of Figure 1 there are exactly two minimizers ($A_{\text{min}}$ and
its complement); illustrated on Figure 2.

In the discrete setting assume that $n$ is even and that $V_n = \{X_1, \ldots, X_n\}$ are independent random
vectors uniformly distributed on $D$. The clustering problem can be described as finding $\hat{A}_n \subset V_n$, which
minimizes

$$F_n(A_n) = \text{GTV}_{n,\eta_n}(\chi_{A_n})$$

among all $A_n \subset V_n$ with $|A_n| = n/2$. We can extend the functionals $F_n$ and $F$ to be equal to $+\infty$ for sets
which do not satisfy the volume constraint.

The kernel we consider for simplicity is the one given by $\eta(x) = 1$ if $|x| < 1$ and $\eta(x) = 0$ otherwise.
While we did not consider the graph total variation with constraints in Theorem 1.1 that extension is
of technical nature. In particular the liminf inequality of the definition of \( \Gamma \)-convergence of Definition 2.6 in the constraint case follows directly, while the limsup inequality follows using the Remark 5.1.

The compactness result implies that if \( \epsilon(n) \) satisfy (8), then along a subsequence, the minimizers \( A_n \) of \( F_n \) converge to \( A \) which minimizes \( F \). Thus our results provide sufficient conditions which guarantee the consistency (convergence) of the scheme as the number of data points increases to infinity.

Here we illustrate the minimizers corresponding to different \( \epsilon \) on a fixed dataset. Figure 4 depicts the discrete minimizer when \( \epsilon \) is taken large enough. Note that this minimizer resembles the one in the continuous setting in Figure 2. In contrast, on Figure 6 we present a minimizer when \( \epsilon \) is taken too small. Note that in this case the energy of such minimizer is zero. The solutions are computed using the code of \([19]\).

1.5. Outline of the approach. The proof of \( \Gamma \)-convergence of the graph total variation \( GTV_{n,\epsilon_n} \) to weighted total variation \( TV(\cdot, \rho^2) \) relies on an intermediate object, the nonlocal functional \( TV_{\epsilon}(\cdot, \rho) : L^1(D, \nu) \to [0, \infty] \) given by:

\[
TV_{\epsilon}(u; \rho) := \frac{1}{\epsilon} \int_D \int_D \eta_{\epsilon}(x-y)|u(x) - u(y)|\rho(x)\rho(y)dxdy.
\]

Note that the argument of \( GTV_{n,\epsilon_n} \), is a function \( u_n \) supported on the data points, while the argument of \( TV_{\epsilon}(\cdot; \rho) \) is an \( L^1(D, \nu) \) function; in particular a function defined on \( D \). Having defined the \( TL^1 \)-metric, the proof of \( \Gamma \)-convergence has two main steps: The first step is to compare the graph total variation \( GTV_{n,\epsilon_n} \), with the nonlocal continuum functional \( TV_{\epsilon}(\cdot, \rho) \). To compare the functionals one
needs an $L^1(D, \nu)$ function which, in $TL^1$ sense, approximates $u_n$. We use transportation maps (i.e. measure preserving maps) between the measure $\nu$ and $\nu_n$ to define $\tilde{u}_n \in L^1(D, \nu)$. More precisely we set $\tilde{u}_n = u_n \circ T_n$ where $T_n$ is the transportation map between $\nu$ and $\nu_n$ constructed in Subsection 2.3 Comparing $GTV_{n,\epsilon}(u_n)$ with $TV_{\epsilon}(\tilde{u}_n; \rho)$ relies on the fact that $T_n$ is chosen in such a way that it transports mass as little as possible. The estimates on how far the mass needs to be moved were known in the literature when $\rho$ is constant. We extended the results to the case when $\rho$ is bounded from below and from above by positive constants.

The second step consists on comparing the continuum nonlocal total variation functionals $\llbracket TV \rrbracket_{\epsilon}$ with the weighted total variation $\llbracket TV \rrbracket_{\epsilon}(\cdot, \rho)$.

The proof on compactness for $GTV_{n,\epsilon}$, depends on an analogous compactness result for the nonlocal continuum functional $\llbracket TV \rrbracket_{\epsilon}(\cdot, \rho)$.

The paper is organized as follows. Section 2 contains the notation and preliminary results from the weighted total variation, transportation theory and $\Gamma$-convergence of functionals on metric spaces. More specifically, in Subsection 2.1 we introduce and present basic facts about weighted total variation. In Subsection 2.2 we introduce the optimal transportation problem and list some of its basic properties. In Subsection 2.3 we review results on optimal matching between the empirical measure $\nu_n$ and $\nu$, when $\nu$ is the Lebesgue measure and $D = (0, 1)^d$ and extend the results to general domains and general densities. Their proofs are in part provided in the Appendix B. In Subsection 2.4 we recall the notion of $\Gamma$-convergence on metric spaces and introduce the appropriate extension to random setting. In Section 3 we define the metric space $TL^p$ and prove some basic results about it. Section 4 contains the proof of the $\Gamma$-convergence of the nonlocal continuum total variation functional $\llbracket TV \rrbracket_{\epsilon}$ to the TV functional. The main result, the $\Gamma$-convergence of the graph TV functionals to the TV functional is proved in Section 5. In Subsection 5.2 we discuss the extension of the main result to the case when $X_1, \ldots, X_n$ are not necessarily independently distributed points.

2. Preliminaries

2.1. Weighted total variation. Let $D$ be an open and bounded subset of $\mathbb{R}^d$ and let $\psi : D \to (0, \infty)$ be a continuous function. Consider the measure $d\nu(x) = \psi(x)dx$. We denote by $L^1(D, \nu)$ the $L^1$-space with respect to $\nu$ and by $\|\cdot\|_{L^1(D, \nu)}$ its corresponding norm; we use $L^1(D)$ in the special case $\psi \equiv 1$ and $\|\cdot\|_{L^1(D)}$ for its corresponding norm. If the context is clear, we omit the set $D$ and write $L^1(\nu)$ and $\|\cdot\|_{L^1(\nu)}$. Also, with a slight abuse of notation, we often replace $\nu$ by $\psi$ in the previous expressions; for example we use $L^1(D, \psi)$ to represent $L^1(D, \nu)$.
Following Baldi, [8], for \( u \in L^1(D, \psi) \) define

\[
TV(u; \psi) = \sup \left\{ \int_D u \text{div} (\phi) dx : (\forall \phi \in C^0_c(D, \mathbb{R}^d)) \right\} 
\]

the weighted total variation of \( u \) in \( D \) with respect to the weight \( \psi \). We denote by \( BV(D; \psi) \) the set of functions \( u \in L^1(D, \psi) \) for which \( TV(u; \psi) < +\infty \). When \( \psi \equiv 1 \) we omit it and write \( BV(D) \) and \( TV(u) \). Finally, for measurable subsets \( E \subset D \), we define the weighted perimeter in \( D \) as the weighted total variation of the characteristic function of the set: \( \text{Per}(E; \psi) = TV(\chi_E; \psi) \).

Throughout the paper we restrict our attention to the case where \( \psi \) is bounded from below and from above by positive constants. Indeed, in applications we consider \( \psi = \rho^2 \), where \( \rho \) is continuous and bounded below and above by positive constants.

**Remark 2.1.** Since \( D \) is a bounded open set and \( \psi \) is bounded from above and below by positive constants, the sets \( L^1(D) \) and \( L^1(D, \psi) \) are equal and the norms \( \| \cdot \|_{L^1(D)} \) and \( \| \cdot \|_{L^1(D, \psi)} \) are equivalent. Also, it is straightforward to see from the definitions that in this case \( BV(D) = BV(D; \psi) \).

**Remark 2.2.** If \( u \in BV(D; \psi) \) is smooth enough (say for example \( u \in C^1(D) \)) then the weighted total variation \( TV(u; \psi) \) can be written as

\[
\int_D |\nabla u(x)| \psi(x) dx.
\]

If \( E \) is a regular subset of \( D \), then \( \text{Per}(E; \psi) \) can be written as the following surface integral,

\[
\text{Per}(E; \psi) = \int_{\partial E \cap D} \psi(x) dS(x).
\]

One useful characterization of \( BV(D; \psi) \) is provided in the next proposition whose proof can be found in [8].

**Proposition 2.3.** Let \( u \in L^1(D, \psi) \), \( u \) belongs to \( BV(D; \psi) \) if and only if there exists a finite positive Radon measure \( |Du|_\psi \) and a \( |Du|_\psi \)-measurable function \( \sigma : D \to \mathbb{R}^d \) with \( |\sigma(x)| = 1 \) for \( |Du|_\psi \)-a.e. \( x \in D \) and such that \( \forall \phi \in C^0_c(D, \mathbb{R}^d) \)

\[
\int_D u \text{div} (\phi) dx = -\int_D \frac{\phi(x) \cdot \sigma(x)}{\psi(x)} d|Du|_\psi(x).
\]

The measure \( |Du|_\psi \) and the function \( \sigma \) are uniquely determined by the previous conditions and the weighted total variation \( TV(u; \psi) \) is equal to \( |Du|_\psi(D) \).

We refer to \( |Du|_\psi \) as the weighted total variation measure (with respect to \( \psi \)) associated to \( u \). In case \( \psi \equiv 1 \), we denote \( |Du|_\psi \) by \( |Du| \) and we call it the total variation measure associated to \( u \).

Using the previous definitions one can check that \( \sigma \) does not depend on \( \psi \) and that the following relation between \( |Du|_\psi \) and \( |Du| \) holds

\[
d|Du|_\psi(x) = \psi(x) d|Du|(x).
\]

In particular,

\[
TV(u; \psi) = \int_D \psi(x) d|Du|(x).
\]

The function \( \sigma(x) \) is the Radon–Nikodym derivative of the distributional derivative of \( u \) (denoted by \( Du \)) with respect to the total variation measure \( |Du| \).

Since the functional \( TV(\cdot; \psi) \) is defined as a supremum of linear continuous functionals in \( L^1(D, \psi) \), we conclude that \( TV(\cdot; \psi) \) is lower semicontinuous with respect to the \( L^1(D, \psi) \)-metric (and thus \( L^1(D) \)-metric given the assumptions on \( \psi \)). That is, if \( u_n \to_{L^1(D, \psi)} u \) as \( n \to \infty \), then

\[
\liminf_{n \to \infty} TV(u_n; \psi) \geq TV(u; \psi).
\]
We finish this section with the following approximation result that we use in the proof of the main theorem of this paper. We give a proof of this result in Appendix A.

**Proposition 2.4.** Let $D$ be a bounded set with Lipschitz boundary and let $\psi : D \to \mathbb{R}$ be a continuous function which is bounded from below and from above by positive constants. Then, for every function $u \in BV(D, \psi)$ there exists a sequence $\{u_n\}_{n \in \mathbb{N}}$ with $u_n \in C_c^\infty(\mathbb{R}^d)$ such that $u_n \to L^1(D) u$ and $\int_D |\nabla u_n| \psi(x) dx \to TV(u; \psi)$ as $n \to \infty$.

2.2. Transportation theory. In this section $D$ is an open and bounded domain in $\mathbb{R}^d$. We denote by $\mathcal{B}(D)$ the Borel $\sigma$-algebra of $D$ and by $\mathcal{P}(D)$ the set of all Borel probability measures on $D$. Given $1 \leq p < \infty$, the $p$-OT distance between $\mu, \tilde{\mu} \in \mathcal{P}(D)$ (denoted by $d_p(\mu, \tilde{\mu})$) is defined by:

$$d_p(\mu, \tilde{\mu}) := \min \left\{ \left( \int_{D \times D} |x-y|^p d\pi(x,y) \right)^{1/p} : \pi \in \Gamma(\mu, \tilde{\mu}) \right\},$$

where $\Gamma(\mu, \tilde{\mu})$ is the set of all couplings between $\mu$ and $\tilde{\mu}$, that is, the set of all Borel probability measures on $D \times D$ for which the marginal on the first variable is $\mu$ and the marginal on the second variable is $\tilde{\mu}$. The elements $\pi \in \Gamma(\mu, \tilde{\mu})$ are also referred as transportation plans between $\mu$ and $\tilde{\mu}$. When $p = 2$ the distance is also known as the Wasserstein distance. The existence of minimizers, which justifies the definition above, is straightforward to show, see [54]. When $p = \infty$

$$d_\infty(\mu, \tilde{\mu}) := \inf \{ \text{esssup}_x \{ |x-y| : (x,y) \in D \times D \} : \pi \in \Gamma(\mu, \tilde{\mu}) \},$$

defines a metric on $\mathcal{P}(D)$, which is called the $\infty$-transportation distance.

It is known that for any $1 \leq p < \infty$, the convergence in OT metric is equivalent to weak convergence of probability measures and uniform integrability of $p$-moments. In our setting, the uniform integrability of $p$-moments is immediate since the domain $D$ is assumed to be bounded, and hence for our purposes, convergence in OT metric is equivalent to weak convergence. For details see for instance [54], [5] and the references within. In particular, $\mu_n \overset{w}{\to} \mu$ (to be read $\mu_n$ converges weakly to $\mu$) if and only if for any $1 \leq p < \infty$ there is a sequence of transportation plans between $\mu_n$ and $\mu$, $\{\pi_n\}_{n \in \mathbb{N}}$, for which:

$$\lim_{n \to \infty} \int_{D \times D} |x-y|^p d\pi_n(x,y) = 0.$$  

Since $D$ is bounded, (16) is equivalent to $\lim_{n \to \infty} \int_{D \times D} |x-y|^p d\pi_n(x,y) = 0$. We say that a sequence of transportation plans, $\{\pi_n\}_{n \in \mathbb{N}}$ with $\pi_n \in \Gamma(\mu, \mu_n)$, is stagnating if it satisfies the condition (16). We remark that, since $D$ is bounded, it is straightforward to show that a sequence of transportation plans is stagnating if and only if $\pi_n$ converges weakly in the space of probability measures on $D \times D$ to $\pi = (id \times id)_\sharp \mu$.

Given a Borel map $T : D \to D$ and $\mu \in \mathcal{P}(D)$ the push-forward of $\mu$ by $T$, denoted by $T_* \mu \in \mathcal{P}(D)$ is given by:

$$T_* \mu(A) := \mu \left( T^{-1}(A) \right), A \in \mathcal{B}(D).$$

Then for any bounded Borel function $\phi : D \to \mathbb{R}$ the following change of variables in the integral holds:

$$\int_D \phi(x) d(T_* \mu)(x) = \int_D \phi(T(x)) d\mu(x).$$

We say that a Borel map $T : D \to D$ is a transportation map between the measures $\mu \in \mathcal{P}(D)$ and $\tilde{\mu} \in \mathcal{P}(D)$ if $\tilde{\mu} = T_* \mu$. In this case, we associate a transportation plan $\pi_T \in \Gamma(\mu, \tilde{\mu})$ to $T$ by:

$$\pi_T := \left( (id \times T)_\sharp \mu \right),$$

where $(id \times T) : D \to D \times D$ is given by $(id \times T)(x) = (x, T(x))$. For any $c \in L^1(D \times D, \mathcal{B}(D \times D), \pi)$

$$\int_{D \times D} c(x,y) d\pi_T(x,y) = \int_D c(x, T(x)) d\mu(x).$$
It is well known that when the measure $\mu \in \mathcal{P}(D)$ is absolutely continuous with respect to the Lebesgue measure, the problem on the right hand side of (14) is equivalent to:

$$
\min \left\{ \left( \int_D |x - T(x)|^p \, d\mu(x) \right)^{1/p} : T_\mathbb{H} = \hat{\mu} \right\},
$$

and when $p$ is strictly greater than 1, the problem (14) has a unique solution which is induced (via (18)) by a transportation map $T$ solving (20) (see [54]). In particular when the measure $\mu$ is absolutely continuous with respect to the Lebesgue measure, $\mu_n \wto \mu$ as $n \to \infty$ is equivalent to the existence of a sequence $\{T_n\}_{n \in \mathbb{N}}$ of transportation maps, $(T_n \mu = \mu_n)$ such that:

$$
\int_D |x - T_n(x)| \, d\mu(x) \to 0, \quad \text{as} \quad n \to \infty.
$$

We say that a sequence of transportation maps $\{T_n\}_{n \in \mathbb{N}}$ is stagnating if it satisfies (21).

We consider now the notion of inverse of transportation plans. For $\pi \in \Gamma(\hat{\mu}, \mu)$, the inverse plan $\pi^{-1} \in \Gamma(\hat{\mu}, \mu)$ of $\pi$ is given by:

$$
\pi^{-1} := s_\pi实例,
$$

where $s : D \times D \to D \times D$ is defined as $s(x,y) = (y,x)$. Note that for any $c \in L^1(D \times D, \pi)$:

$$
\int_{D \times D} c(x,y) \pi(x,y) = \int_{D \times D} c(y,x) d\pi^{-1}(x,y).
$$

Let $\mu, \hat{\mu}, \tilde{\mu} \in \mathcal{P}(D)$. The composition of plans $\pi_{12} \in \Gamma(\mu, \hat{\mu})$ and $\pi_{23} \in \Gamma(\hat{\mu}, \tilde{\mu})$ was discussed in [5][Remark 5.3.3]. In particular there exists a probability measure $\pi$ on $D \times D \times D$ such that the projection of $\pi$ to first two variables is $\pi_{12}$, and to second and third variables is $\pi_{23}$. We consider $\pi_{13}$ to be the projection of $\pi$ to the first and third variables. We will refer $\pi_{13}$ as a composition of $\pi_{12}$ and $\pi_{23}$ and write $\pi_{13} = \pi_{23} \circ \pi_{12}$. Note $\pi_{13} \in \Gamma(\mu, \tilde{\mu})$.

2.3. Optimal matching results. In this section we discuss how to construct the transportation maps which allow us to make the transition from the functions of the data points to continuum functions. To obtain good estimates we want to match the measure $\nu$, out of which the data points are sampled, with the empirical measure of data points while moving the mass as little as possible.

Let $D$ be a domain on $\mathbb{R}^d$ such that there exists a bi-Lipschitz homeomorphism $\Theta : (0,1)^d \to D$. Let $\nu$ be a measure on $D$ with density $\rho$ which is bounded from below and from above by positive constants. Consider $\Omega$, $\mathcal{F}$, $\mathcal{P}$ a probability space that we assume to be rich enough to support a sequence of $\{\mu_n\}_{n \in \mathbb{N}}$ such that $\mu_n \wto \mu$ as $n \to \infty$ is equivalent to the existence of a sequence $\{T_n\}_{n \in \mathbb{N}}$ of transportation maps, $(T_n \mu = \mu_n)$ such that:

$$
\int_D |x - T_n(x)| \, d\mu(x) \to 0, \quad \text{as} \quad n \to \infty.
$$

We say that a sequence of transportation maps $\{T_n\}_{n \in \mathbb{N}}$ is stagnating if it satisfies (21).

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$$
\pi^{-1} := s_\pi实例,
$$

where $s : D \times D \to D \times D$ is defined as $s(x,y) = (y,x)$. Note that for any $c \in L^1(D \times D, \pi)$:

$$
\int_{D \times D} c(x,y) \pi(x,y) = \int_{D \times D} c(y,x) d\pi^{-1}(x,y).
$$

Let $\mu, \hat{\mu}, \tilde{\mu} \in \mathcal{P}(D)$. The composition of plans $\pi_{12} \in \Gamma(\mu, \hat{\mu})$ and $\pi_{23} \in \Gamma(\hat{\mu}, \tilde{\mu})$ was discussed in [5][Remark 5.3.3]. In particular there exists a probability measure $\pi$ on $D \times D \times D$ such that the projection of $\pi$ to first two variables is $\pi_{12}$, and to second and third variables is $\pi_{23}$. We consider $\pi_{13}$ to be the projection of $\pi$ to the first and third variables. We will refer $\pi_{13}$ as a composition of $\pi_{12}$ and $\pi_{23}$ and write $\pi_{13} = \pi_{23} \circ \pi_{12}$. Note $\pi_{13} \in \Gamma(\mu, \tilde{\mu})$.

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$$
d_{\infty}(\nu, \nu_n) = \inf \{ \|Id - T_n\|_\infty : T_n : D \to D, T_n \nu = \nu_n \}.
$$

It measures what is the least maximal distance that a transportation map $T_n$ between $\nu$ and $\nu_n$ has to move the mass.

If $\nu$ were a discrete measure with $n$ particles, then the infinity transportation distance is the min-max matching distance. There is a rich history of discrete matching results (see [2] [32] [44] [51] [48] [49] [50] and references therein). In fact, let us first consider the case where $D = (0,1)^d$ and $\rho$ is constant, that is, assume the data points are uniformly distributed on $(0,1)^d$. Also, assume for simplicity that $n$ is of the form $n = k^d$ for some $k \in \mathbb{N}$. Consider $P = \{p_1, \ldots, p_n\}$ the set of $n$ points in $(0,1)^d$ of the form $(\frac{i_1}{2^d}, \ldots, \frac{i_n}{2^d})$ for $i_1, \ldots, i_n$ odd integers between 1 and $2k$. The points in $P$ form a regular $k \times \cdots \times k$ array in $(0,1)^d$ and in particular each point in $P$ is the center of a cube with volume $1/n$. As in [32] we call the points in $P$ grid points and the cubes generated by the points in $P$ grid cubes.
In dimension $d = 2$, Leighton and Shor [32] showed that, when $\rho$ is constant, there exist $c > 0$ and $C > 0$ such that with very high probability (meaning probability greater than $1 - n^{-\alpha}$ where $\alpha = c_1(\log n)^{1/2}$ for some constant $c_1 > 0$):

$$\frac{c(\log n)^{3/4}}{n^{1/2}} \leq \min_{\pi} \max_i |p_i - X_{\pi(i)}| \leq \frac{C(\log n)^{3/4}}{n^{1/2}}$$  \tag{23}$$

where $\pi$ ranges over all permutations of \{1, \ldots, n\}. In other words, when $d = 2$, with high probability the $\infty$-transportation distance between the random points and the grid points is of order $\frac{(\log n)^{3/4}}{n^{1/2}}$.

For $d \geq 3$, Shor and Yukich [44] proved the analogous result to (23). They showed that, when $\rho$ is constant, there exist $c > 0$ and $C > 0$ such that with very high probability

$$\frac{c(\log n)^{1/d}}{n^{1/d}} \leq \max_i |p_i - X_{\pi(i)}| \leq \frac{C(\log n)^{1/d}}{n^{1/d}}.$$  \tag{24}$$

The result in dimension $d \geq 3$ is based on the matching algorithm introduced by Ajtai, Komlós, and Tusnády in [2]. It relies on a dyadic decomposition of $(0, 1)^d$ and transporting step by step between levels of the dyadic decomposition. The final matching is obtained as a composition of the matchings between consecutive levels. For $d = 2$ the AKT algorithm still gives an upper bound, but not a sharp one. As remarked in [44], there is a crossover in the nature of the matching when $d = 2$: for $d \geq 3$, the matching length between the random points and the points in the grid is determined by the behavior of the points locally, for $d = 1$ on the other hand, the matching length is determined by the behavior of random points globally, and finally for $d = 2$ the matching length is determined by the behavior of the random points at all scales. At the level of the AKT algorithms this means that for $d \geq 3$ the major source of the transportation distance is at the finest scale, for $d = 1$ at the coarsest scale, while for $d = 2$ distances at all scales are of the same size (in terms of how they scale with $n$). The sharp result in dimension $d = 2$ by Leighton and Shor required a more sophisticated matching procedure. An alternative proof in $d = 2$ was provided by Talagrand [45] who also provided more streamlined and conceptually clear proofs in [49, 50].

These results, can be used to obtain bounds on the transportation distance on the continuum setting. In fact, in order to construct a transportation map between $\nu$ and $\nu_n$ which gives a good upper bound to $d_\infty(\nu, \nu_n)$, one can match every point in a grid cube to its corresponding grid point and then match every grid point to a random point using the optimal discrete matching from (23) or (24). In this way every point in $(0, 1)^d$ gets mapped to a data point, and it is straightforward to see that this mapping defines a transportation map $T_n$ between $\nu$ and $\nu_n$.

In the same spirit as before, one can find upper bounds for $d_\infty(\nu, \nu_n)$ when the density $\rho$ is bounded from below and from above by positive constants. We do this in the Appendix where we extend the results in both $d = 2$ and $d \geq 3$. Moreover, one can extend the same results to more general domains $D$. The proofs rely in dimension $d \geq 3$ on the AKT construction and in dimension $d = 2$ on techniques of [49, 50]. The concrete statement of the results is as follows.

**Theorem 2.5.** Let $D$ be an open and bounded subset of $\mathbb{R}^d$ which is homeomorphic to $(0, 1)^d$ via a bi-Lipschitz mapping. Let $\nu$ be a probability measure on $D$ with density $\rho$ which is bounded from below and from above by positive constants. Let $X_1, \ldots, X_n, \ldots$ be a sequence of independent random vectors distributed on $D$ according to measure $\nu$ and let $\nu_n$ be the associated empirical measures $\hat{\nu}_n$. Then there is a constant $C > 0$ such that for $P$-a.e. $\omega \in \Omega$ there exists a sequence of transportation maps $\{T_n\}_{n \in \mathbb{N}}$ from $\nu$ to $\nu_n$ ($T_{n\omega}\nu = \nu_n$) and such that:

$$\limsup_{n \to \infty} \frac{n^{1/2} \|Id - T_n\|_\infty}{(\log n)^{3/4}} \leq C$$  \tag{25}$$

and if $d \geq 3$

$$\limsup_{n \to \infty} \frac{n^{1/d} \|Id - T_n\|_\infty}{(\log n)^{1/d}} \leq C.$$  \tag{26}$$
We say that the sequence of nonnegative functionals \( F \) is the \( \Gamma \)-limit of the sequence \( \{ F_n \} \) if the following holds: Given \( x \in X \) and every sequence \( \{ x_n \} \) converging to \( x \),

\[
\liminf_{n \to \infty} F_n(x_n) \geq F(x),
\]

\[
\limsup_{n \to \infty} F_n(x_n) \leq F(x).
\]

We say that \( F \) is the \( \Gamma \)-limit of the sequence \( \{ F_n \} \) (with respect to the metric \( d_X \)).

**Remark 2.7.** In most situations one does not prove the limsup inequality for all \( x \in X \) directly. Instead, one proves the inequality for all \( x \) in a dense subset \( X' \) of \( X \) where it is somewhat easier to prove, and then deduce from this that the inequality holds for all \( x \in X \). To be more precise, suppose that the limsup inequality is true for every \( x \) in a subset \( X' \) of \( X \) and the set \( X' \) is such that for every \( x \in X \) there exists a sequence \( \{ x_k \}_{k \in \mathbb{N}} \) in \( X' \) converging to \( x \) and such that \( F(x_k) \to F(x) \) as \( k \to \infty \), then the limsup inequality is true for every \( x \in X \). It is enough to use a diagonal argument to deduce this claim.

**Definition 2.8.** We say that the sequence of nonnegative functionals \( \{ F_n \} \) satisfies the compactness property if the following holds: Given \( \{ n_k \}_{k \in \mathbb{N}} \) an increasing sequence of natural numbers and \( \{ x_k \}_{k \in \mathbb{N}} \) a bounded sequence in \( X \) for which

\[
\sup_{k \in \mathbb{N}} F_{n_k}(x_k) < \infty
\]

\( \{ x_k \}_{k \in \mathbb{N}} \) is precompact in \( X \).

**Remark 2.9.** Note that the boundedness assumption of \( \{ x_k \}_{k \in \mathbb{N}} \) in the previous definition is a necessary condition for precompactness and so it is not restrictive.
The notion of $\Gamma$-convergence is particularly useful when the functionals $\{F_n\}_{n \in \mathbb{N}}$ satisfy the compactness property. This is because it guarantees convergence of minimizers (or approximate minimizers) of $F_n$ to minimizers of $F$ and it also guarantees convergence of the minimum energy of $F_n$ to the minimum energy of $F$ (this statement is made precise in the next proposition). This is the reason why $\Gamma$-convergence is said to be a variational type of convergence.

**Proposition 2.10.** Let $F_n : X \to [0, \infty]$ be a sequence of nonnegative functionals which are not identically equal to $+\infty$, satisfying the compactness property and $\Gamma$-converging to the functional $F : X \to [0, \infty]$ which is not identically equal to $+\infty$. Then,

\[
\lim_{n \to \infty} \inf_{x \in X} F_n(x) = \min_{x \in X} F(x).
\]

Furthermore every bounded sequence $\{x_n\}_{n \in \mathbb{N}}$ in $X$ for which

\[
\lim_{n \to \infty} \left( F_n(x_n) - \inf_{x \in X} F_n(x) \right) = 0
\]

is precompact and each of its cluster points is a minimizer of $F$.

In particular, if $F$ has a unique minimizer, then a sequence $\{x_n\}_{n \in \mathbb{N}}$ satisfying (28) converges to the unique minimizer of $F$.

One can extend the concept of $\Gamma$-convergence to families of functionals indexed by real numbers in a simple way, namely, the family of functionals $\{F_h\}_{h>0}$ is said to $\Gamma$-converge to $F$ as $h \to 0$ if for every sequence $\{h_n\}_{n \in \mathbb{N}}$ with $h_n \to 0$ as $n \to \infty$ the sequence $\{F_{h_n}\}_{n \in \mathbb{N}}$ $\Gamma$-converges to the functional $F$ as $n \to \infty$. Similarly one can define the compactness property for the functionals $\{F_h\}_{h>0}$. For more on the notion of $\Gamma$-convergence see [14] or [22].

Since the functionals we are most interested in depend on data (and hence are random), we need to define what it means for a sequence of random functionals to $\Gamma$-converge to a deterministic functional.

**Definition 2.11.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For $\{F_n\}_{n \in \mathbb{N}}$ a sequence of (random) functionals $F_n : X \times \Omega \to [0, \infty]$ and $F$ a (deterministic) functional $F : X \to [0, \infty]$, we say that the sequence of functionals $\{F_n\}_{n \in \mathbb{N}}$ $\Gamma$-converges (in the $d_X$ metric) to $F$, if for $\mathbb{P}$-almost every $\omega \in \Omega$ the sequence $\{F_n(\cdot; \omega)\}_{n \in \mathbb{N}}$ $\Gamma$-converges to $F$ according to Definition 2.7. Similarly, we say that $\{F_n\}_{n \in \mathbb{N}}$ satisfies the compactness property if for $\mathbb{P}$-almost every $\omega \in \Omega$, $\{F_n(\cdot; \omega)\}_{n \in \mathbb{N}}$ satisfies the compactness property according to Definition 2.8.

We do not explicitly write the dependence of $F_n$ on $\omega$ and we simply write $F_n : X \to [0, \infty]$, understanding that we are always working with a fixed value $\omega \in \Omega$, and hence with deterministic functionals.

3. The space $TLP$

In this section $D$ denotes an open and bounded domain in $\mathbb{R}^d$. Consider the set

$$
TLP(D) := \{ (\mu, f) : \mu \in \mathcal{P}(D), f \in L^p(D, \mu) \}.
$$

For $(\mu, f)$ and $(\nu, g)$ in $TLP$ we define

\[
d_{TLP}(\mu, f), (\nu, g)) = \inf_{\pi \in T(\mu, \nu)} \left( \int_{D \times D} |x-y|^p d\pi(x,y) \right) + \left( \int_{D \times D} |f(x) - g(y)|^p d\pi(x,y) \right)^{\frac{1}{p}}
\]

The next proposition shows that $d_{TLP}$ is a metric. We remark that formally $TLP$ is a fiber bundle over $\mathcal{P}(D)$. Namely if one considers the Finsler (Riemannian for $p = 2$) manifold structure on $\mathcal{P}(D)$ provided by the $p - OT$ metric (see [1] for general $p$ and [5] [37] for $p = 2$) then $TLP$ is, formally, a fiber bundle. We also remark that one could also change the set and consider a metric where the powers
of the terms in (29) would be different (p and q, instead of p and p and the natural name for the space in this case would be $TLP^p$).

Remark 3.1. One can think of the convergence in $TLP^p$ as a generalization of weak convergence of measures and of $L^p$ convergence of functions. That is $\{\mu_n\}_{n \in \mathbb{N}}$ in $\mathcal{P}(D)$ converges weakly to $\mu \in \mathcal{P}(D)$ if and only if $(\mu_n, 1) \xrightarrow{TLP} (\mu, 1)$ as $n \to \infty$ (which follows from the fact that on bounded sets $p$-OT metric metrizes the weak convergence of measures [3]), and that for $\mu \in \mathcal{P}(D)$ a sequence $\{f_n\}_{n \in \mathbb{N}}$ in $L^p(\mu)$ converges in $f$ if and only if $(\mu, f_n) \xrightarrow{TLP} (\mu, f)$ as $n \to \infty$. The last fact is established in Proposition 3.6.

Remark 3.2. If one restricts the attention to measures $\mu$ and $v$ which are absolutely continuous with respect to the Lebesgue measure then

$$\inf_{T : T \mu = v} \left( \int_D |x - T(x)|^p d\mu(x) \right)^{1/p} + \left( \int_D |f(x) - g(T(x))|^p d\mu(x) \right)^{1/p}$$

majorizes $d_{TLP}(\mu, f, (v, g))$ and furthermore provides a metric (on the subset of $TLP^p$) which gives the same topology as $d_{TLP}$. The fact that these topologies are the same follows from Proposition 3.6.

Proposition 3.3. $d_{TLP}$ defines a metric on $TLP^p$.

Proof. To prove that $d_{TLP}(\mu, f, (\mu, f)) = 0$, note that if we consider $\pi = r_\mu$ where $r : D \to D \times D$ is given by $r(x) = (x, x)$, $\pi \in \Gamma(\mu, \mu)$ and $\int_{D \times D} |x - y|^p + |f(x) - g(y)|^p d\pi(x, y) = 0$.

Suppose now that $d_{TLP}(\mu, f, (v, g)) = 0$. In particular it is true that $d_\mu(\mu, v) = 0$ and so $\mu = v$. Thus there exists a sequence $\{\pi_n\}_{n \in \mathbb{N}}$ of transportation plans belonging to $\Gamma(\mu, \mu)$ such that:

$$\lim_{n \to \infty} \int_{D \times D} \epsilon(x, y) = 0$$

Note that the sequence $\{\pi_n\}_{n \in \mathbb{N}}$ is tight in $\mathcal{P}(D \times D)$, and hence it has a subsequence (that we do not relabel) that converges weakly to some $\pi \in \Gamma(\mu, \mu)$. Using the boundedness of $D$ we conclude that:

$$\int_{D \times D} \epsilon(x, y) = \lim_{n \to \infty} \int_{D \times D} \epsilon(x, y) = 0$$

Consequently $\pi = r_\mu$ for $r$ as above. For every $\epsilon > 0$ there exist $\tilde{f}$, $\tilde{g}$ functions defined on $D$ which are continuous and bounded and such that $\|f - \tilde{f}\|_{L^p(\mu)} < \epsilon/2$ and $\|g - \tilde{g}\|_{L^p(\mu)} < \epsilon/2$. Then

$$\left( \int_{D \times D} |\tilde{f}(x) - \tilde{g}(y)|^p d\pi_n(x, y) \right)^{1/p} \leq \left( \int_{D \times D} |f(x) - g(y)|^p d\pi_n(x, y) \right)^{1/p}$$

$$+ \left( \int_{D \times D} |f(x) - \tilde{g}(y)|^p d\pi_n(x, y) \right)^{1/p}$$

$$+ \left( \int_{D \times D} |g(y) - \tilde{f}(y)|^p d\pi_n(x, y) \right)^{1/p}$$

$$< \left( \int_{D \times D} |f(x) - \tilde{g}(y)|^p d\pi_n(x, y) \right)^{1/p} + \epsilon.$$
Therefore:
\[ \| f - g \|_{L^p(\mu)} \leq \| f - \tilde{f} \|_{L^p(\mu)} + \| \tilde{f} - \tilde{g} \|_{L^p(\mu)} + \| \tilde{g} - g \|_{L^p(\mu)} < 3\varepsilon. \]

Since \( \varepsilon \) was arbitrary, we conclude that \( f = g \), and so \( d_{TL^p}(\mu, f), (v, g) = 0 \) implies \( (\mu, f) = (v, g) \).

To prove that \( d_{TL^p}(\mu, f), (v, g) = d_{TL^p}(v, g), (\mu, f) \) simply note that for every \( \pi \in \Gamma(\mu, v) \)
\[
\int_{\mathcal{D} \times \mathcal{D}} |x - y|^p d\pi(x, y) = \int_{\mathcal{D} \times \mathcal{D}} |x - y|^p d\pi^{-1}(x, y),
\]
\[
\int_{\mathcal{D} \times \mathcal{D}} |f(x) - g(y)|^p d\pi(x, y) = \int_{\mathcal{D} \times \mathcal{D}} |g(x) - f(y)|^p d\pi^{-1}(x, y)
\]
where \( \pi^{-1} \in \Gamma(v, \mu) \) is the inverse of \( \pi \) as defined in (22).

Finally, consider \( (\mu, f), (v, g), (\sigma, h) \in TL^p \). Take \( \pi_1 \in \Gamma(\mu, v) \) and \( \pi_2 \in \Gamma(v, \sigma) \). We now use a measure \( \pi \in \mathcal{P}(\mathcal{D} \times \mathcal{D} \times \mathcal{D}) \) to obtain \( \pi_2 \circ \pi_1 \in \Gamma(\mu, \sigma) \) as mentioned at the end of Subsection 2.2. Using Minkowski’s inequality for the measure \( \pi \), one obtains that:
\[
d_{TL^p}(\mu, f), (\sigma, h) \leq \left( \int_{\mathcal{D} \times \mathcal{D}} |x - z|^p d\pi_2 \circ \pi_1(x, z) \right)^{\frac{1}{p}} + \left( \int_{\mathcal{D} \times \mathcal{D}} |f(x) - h(z)|^p d\pi_2 \circ \pi_1(x, z) \right)^{\frac{1}{p}}
\]
\[
\leq \left( \int_{\mathcal{D} \times \mathcal{D}} |x - y|^p d\pi_1(x, y) \right)^{\frac{1}{p}} + \left( \int_{\mathcal{D} \times \mathcal{D}} |f(x) - g(y)|^p d\pi_1(x, y) \right)^{\frac{1}{p}}
\]
\[
+ \left( \int_{\mathcal{D} \times \mathcal{D}} |y - z|^p d\pi_2(y, z) \right)^{\frac{1}{p}} + \left( \int_{\mathcal{D} \times \mathcal{D}} |g(y) - h(z)|^p d\pi_2(y, z) \right)^{\frac{1}{p}}.
\]

Taking infimum over \( \pi_1 \) and over \( \pi_2 \) on the previous expression we deduce that \( d_{TL^p}(\mu, f), (\sigma, h) \) ≤ \( d_{TL^p}(\mu, f), (v, g) \) + \( d_{TL^p}(v, g), (\sigma, h) \).

We wish to establish a simple characterization for the convergence in the space \( TL^p \). For this, we need first the following two lemmas.

**Lemma 3.4.** Let \( \mu \in \mathcal{P}(\mathcal{D}) \) and let \( \pi_n \in \Gamma(\mu, \mu) \) for all \( n \in \mathbb{N} \). If \( \{\pi_n\}_{n \in \mathbb{N}} \) is a stagnating sequence of transportation plans, then for any \( u \in L^0(\mu) \)
\[
\lim_{n \to \infty} \int_{\mathcal{D} \times \mathcal{D}} |u(x) - u(y)|^p d\pi_n(x, y) = 0.
\]

**Proof.** We prove the case \( p = 1 \) since the other cases are similar. Let \( u \in L^1(\mu) \) and let \( \{\pi_n\}_{n \in \mathbb{N}} \) be a stagnating sequence of transportation maps with \( \pi_n \in \Gamma(\mu, \mu) \). Since the probability measure \( \mu \) is inner regular, we know that the class of Lipschitz and bounded functions on \( \mathcal{D} \) is dense in \( L^1(\mu) \). Fix \( \varepsilon > 0 \), we know there exists a function \( v : D \to \mathbb{R} \) which is Lipschitz and bounded and for which:
\[
\int_{\mathcal{D}} |u(x) - v(x)| d\mu(x) < \frac{\varepsilon}{3}.
\]

Note that:
\[
\int_{\mathcal{D} \times \mathcal{D}} |v(x) - v(y)| d\pi_n(x, y) \leq \text{Lip}(v) \int_{\mathcal{D} \times \mathcal{D}} |x - y| d\pi_n(x, y) \to 0, \text{ as } n \to \infty.
\]

Hence we can find \( N \in \mathbb{N} \) such that if \( n \geq N \) then \( \int_{\mathcal{D} \times \mathcal{D}} |v(x) - v(y)| d\pi_n(x, y) < \frac{\varepsilon}{3} \). Therefore, for \( n \geq N \), using the triangle inequality, we obtain
\[
\int_{\mathcal{D} \times \mathcal{D}} |u(x) - u(y)| d\pi_n(x, y) \leq \int_{\mathcal{D} \times \mathcal{D}} |u(x) - v(x)| d\pi_n(x, y)
\]
\[
+ \int_{\mathcal{D} \times \mathcal{D}} |v(x) - v(y)| d\pi_n(x, y) + \int_{\mathcal{D} \times \mathcal{D}} |v(y) - u(y)| d\pi_n(x, y)
\]
\[
= 2 \int_{\mathcal{D}} |v(x) - u(x)| d\mu(x) + \int_{\mathcal{D} \times \mathcal{D}} |v(x) - v(y)| d\pi_n(x, y) < \varepsilon.
\]
This proves the result. \hfill \square

**Lemma 3.5.** Suppose that the sequence \( \{\mu_n\}_{n \in \mathbb{N}} \) in \( \mathcal{P}(D) \) converges weakly to \( \mu \in \mathcal{P}(D) \). Let \( \{u_n\}_{n \in \mathbb{N}} \) be a sequence with \( u_n \in L^p(\mu_n) \) and let \( u \in L^p(\mu) \). Consider two sequences of stagnating transportation plans \( \{\pi_n\}_{n \in \mathbb{N}} \) and \( \{\pi_n\}_{n \in \mathbb{N}} \) (with \( \pi_n, \pi_n \in \Gamma(\mu, \mu_n) \)). Then:

\[
\lim_{n \to \infty} \int_{D \times D} |u(x) - u_n(y)|^p \, d\pi_n(x, y) = 0 \quad \Leftrightarrow \quad \lim_{n \to \infty} \int_{D \times D} |u(x) - u_n(y)|^p \, d\hat{\pi}_n(x, y) = 0
\]

**Proof.** We present the details for \( p = 1 \), as the other cases are similar. Take \( \hat{\pi}_n^{-1} \in \Gamma(\mu_n, \mu) \) the inverse of \( \hat{\pi}_n \) defined in (22). We can consider \( \pi_n \in \mathcal{P}(D \times D) \) as the measure mentioned at the end of Subsection 2.2 (taking \( \pi_{23} = \hat{\pi}_n^{-1} \) and \( \pi_{12} = \pi_n \)). In particular \( \hat{\pi}_n^{-1} \circ \pi_n \in \Gamma(\mu, \mu) \). Then

\[
\int_{D \times D} |u_n(z) - u(y)| \, d\hat{\pi}_n(y, z) = \int_{D \times D} |u_n(z) - u(y)| \, d\hat{\pi}_n(y, z) = \int_{D \times D} |u_n(z) - u(y)| \, d\hat{\pi}_n(y, z),
\]

which imply after using the triangle inequality:

\[
\left| \int_{D \times D} |u_n(z) - u(y)| \, d\pi_n(x, y) - \int_{D \times D} |u(z) - u_n(y)| \, d\hat{\pi}_n(y, z) \right| 
\]

\[
\leq \int_{D \times D \times D} |u(z) - u(x)| \, d\pi_n(x, y, z) = \int_{D \times D} |u(z) - u(x)| \, d\hat{\pi}_n(y, z).
\]

Finally note that:

\[
\int_{D \times D} |x - z| \, d\hat{\pi}_n^{-1} \circ \pi_n(x, z) \leq \int_{D \times D} |x - y| \, d\pi_n(x, y) + \int_{D \times D} |y - z| \, d\hat{\pi}_n(y, z),
\]

as \( n \to \infty \). The sequence \( \{\hat{\pi}_n^{-1} \circ \pi_n\}_{n \in \mathbb{N}} \) satisfies the assumptions of Lemma 3.4, so we can deduce that \( \int_{D \times D} |u(z) - u(x)| \, d\hat{\pi}_n^{-1} \circ \pi_n(x, z) \to 0 \) as \( n \to \infty \). By (32) we get that:

\[
\lim_{n \to \infty} \int_{D \times D} |u_n(z) - u(x)| \, d\pi_n(x, y) - \int_{D \times D} |u_n(z) - u(y)| \, d\hat{\pi}_n(y, z) = 0.
\]

This implies the result. \hfill \square

**Proposition 3.6.** Let \((\mu, f) \in TL^p \) and let \( \{\mu_n, f_n\}_{n \in \mathbb{N}} \) be a sequence in \( TL^p \). The following statements are equivalent:

1. \((\mu_n, f_n) \overset{TL^p}{\longrightarrow} (\mu, f) \) as \( n \to \infty \).
2. \( \mu_n \overset{w}{\longrightarrow} \mu \) and for every stagnating sequence of transportation plans \( \{\pi_n\}_{n \in \mathbb{N}} \) (with \( \pi_n \in \Gamma(\mu, \mu_n) \))

\[
\int_{D \times D} |f(x) - f_n(y)|^p \, d\pi_n(x, y) \to 0, \text{ as } n \to \infty.
\]

3. \( \mu_n \overset{w}{\longrightarrow} \mu \) and there exists a stagnating sequence of transportation plans \( \{\pi_n\}_{n \in \mathbb{N}} \) (with \( \pi_n \in \Gamma(\mu, \mu_n) \)) for which (33) holds.

Moreover, if the measure \( \mu \) is absolutely continuous with respect to the Lebesgue measure, the following are equivalent to the previous statements:
4. \(\mu_n \rightarrow^{\text{w}} \mu\) and there exists a stagnating sequence of transportation maps \(\{T_n\}_{n \in \mathbb{N}}\) (with \(T_n \mu = \mu_n\)) such that:

\[
\int_D |f(x) - f_n(T_n(x))|^p \, d\mu(x) \to 0, \text{ as } n \to \infty.
\]

5. \(\mu_n \rightarrow^{\text{w}} \mu\) and for any stagnating sequence of transportation maps \(\{T_n\}_{n \in \mathbb{N}}\) (with \(T_n \mu = \mu_n\)) \[34\] holds.

**Proof.** By Lemma 3.5 claims 2. and 3. are equivalent. In case \(\mu\) is absolutely continuous with respect to the Lebesgue measure, we know that there exists a stagnating sequence of transportation maps \(\{T_n\}_{n \in \mathbb{N}}\) (with \(T_n \mu = \mu_n\)). Considering the sequence of transportation plans \(\{\pi_n\}_{n \in \mathbb{N}}\) (as defined in 18) and using \(19\) we see that 2., 3., 4., and 5. are all equivalent. We prove the equivalence of 1. and 3.

(1. \(\Rightarrow\) 3.) Note that \(d_p(\mu, \mu_n) \leq d_{\mathcal{L}^p}(f, (\mu_n, f_n))\) for every \(n\). Consequently \(d_p(\mu, \mu_n) \to 0\) as \(n \to \infty\) and in particular \(\mu_n \rightarrow^{\text{w}} \mu\) as \(n \to \infty\). Furthermore, since \(d_{\mathcal{L}^p}(f, (\mu_n, f_n)) \to 0\) as \(n \to \infty\), there exists a sequence \(\{\pi_n\}_{n \in \mathbb{N}}\) of transportation plans (with \(\pi_n \in \Gamma(\mu, \mu_n)\)) such that:

\[
\lim_{n \to \infty} \int_{D \times D} |x - y|^p \, d\pi_n(x, y) = 0,
\]

\[
\lim_{n \to \infty} \int_{D \times D} |f(x) - f_n(y)|^p \, d\pi_n(x, y) = 0.
\]

\(\{\pi_n\}_{n \in \mathbb{N}}\) is then a stagnating sequence of transportation plans for which \[33\] holds.

(3. \(\Rightarrow\) 1.) Since \(\mu_n \rightarrow^{\text{w}} \mu\) as \(n \to \infty\) (and since \(D\) is bounded), we know that \(d_p(\mu_n, \mu) \to 0\) as \(n \to \infty\). In particular, we can find a sequence of transportation plans \(\{\pi_n\}_{n \in \mathbb{N}}\) with \(\pi_n \in \Gamma(\mu, \mu_n)\) such that:

\[
\lim_{n \to \infty} \int_{D \times D} |x - y|^p \, d\pi_n(x, y) = 0
\]

\(\{\pi_n\}_{n \in \mathbb{N}}\) is then a stagnating sequence of transportation plans. By the hypothesis we conclude that:

\[
\lim_{n \to \infty} \int_{D \times D} |f(x) - f_n(y)|^p \, d\pi_n(x, y) = 0.
\]

We deduce that \(\lim_{n \to \infty} d_{\mathcal{L}^p}(f, (\mu_n, f_n)) = 0\).

**Definition 3.7.** Suppose \(\{\mu_n\}_{n \in \mathbb{N}}\) in \(\mathcal{P}(D)\) converges weakly to \(\mu \in \mathcal{P}(D)\). We say that the sequence \(\{u_n\}_{n \in \mathbb{N}}\) (with \(u_n \in L^p(\mu_n)\)) converges in the \(\mathcal{L}^p\) sense to \(u \in L^p(\mu)\), if \(\{\mu_n, u_n\}\) \(\mathcal{L}^p\)-converges to \((\mu, u)\) in the \(\mathcal{L}^p\) metric. In this case we use a slight abuse of notation and write \(u_n \mathcal{L}^p \to u\) as \(n \to \infty\). Also, we say the sequence \(\{u_n\}_{n \in \mathbb{N}}\) (with \(u_n \in L^p(\mu_n)\)) is precompact in \(\mathcal{L}^p\) if the sequence \(\{\mu_n, u_n\}\) \(\mathcal{L}^p\)-converges.

**Remark 3.8.** Thanks to Proposition 3.6 when \(\mu\) is absolutely continuous with respect to the Lebesgue measure \(u_n \mathcal{L}^p \to u\) as \(n \to \infty\) if and only if for every \(\{T_n\}_{n \in \mathbb{N}}\) stagnating sequence of transportation maps (with \(T_n \mu = \mu_n\)) it is true that \(u_n \circ T_n \mathcal{L}^p \to u\) as \(n \to \infty\). Also \(\{u_n\}_{n \in \mathbb{N}}\) is precompact in \(\mathcal{L}^p\) if and only if for every \(\{T_n\}_{n \in \mathbb{N}}\) stagnating sequence of transportation maps (with \(T_n \mu = \mu_n\)) it is true that \(\{u_n \circ T_n\}_{n \in \mathbb{N}}\) is precompact in \(L^p(\mu)\).

4. \(\Gamma\)-convergence of \(TV_e(\cdot, \rho)\)

In this section we prove the \(\Gamma\)-convergence of the nonlocal functionals \(TV_e(\cdot, \rho)\) to weighted perimeter.
Theorem 4.1. Consider an open, bounded domain \( D \) in \( \mathbb{R}^d \) with Lipschitz boundary. Let \( \rho : D \to \mathbb{R} \) be continuous and bounded below and above by positive constants. Then, \( \{TV_\varepsilon(\cdot; \rho)\}_{\varepsilon > 0} \) (defined in (9)) \( \Gamma \)-converges with respect to the \( L^1(D, \rho) \)-metric to \( \sigma_2TV(\cdot; \rho^2) \). Moreover, the functionals \( \{TV_\varepsilon(\cdot; \rho)\}_{\varepsilon > 0} \) satisfy the compactness property (Definition 2.8) with respect to the \( L^1(D, \rho) \)-metric.

Part of the proof of this result follows ideas present in the work of Ponce [40]. Specifically, Lemma 4.2 below and the first part of the proof of the liminf inequality are adaptations of results by Ponce. The first part of the proof of the limsup inequality is a careful adaptation of the appendix of a paper by Alberti and Bellettini [3]. We also prove compactness of the functionals \( \{TV_\varepsilon(\cdot; \rho)\}_{\varepsilon > 0} \). This part required new arguments, due to the presence of domain boundary and lack of \( L^\infty \) control. For this result we impose extra regularity assumptions on the domain \( D \). Part of the proof on compactness in [3] is used. As a corollary, we show that if one considers only functions uniformly bounded in \( L^\infty \), the compactness holds for open and bounded domains \( D \) regardless of its boundary.

Since the definition of \( \Gamma \)-convergence for a family of functionals indexed by real numbers is given in terms of sequences, in this section we adopt the following notation:

In this section \( \varepsilon \) is a short-hand notation for \( \varepsilon_n \) where \( \{\varepsilon_n\}_{n \in \mathbb{N}} \) is an arbitrary sequence of positive real numbers converging to zero as \( n \to \infty \). Limits as \( \varepsilon \to 0 \) simply mean limits as \( n \to \infty \) for every such sequence.

Lemma 4.2. Let \( D \) be a bounded open subset of \( \mathbb{R}^d \) and let \( \rho : D \to \mathbb{R} \) be a Lipschitz function that is bounded from below and from above by positive constants. Suppose that \( \{u_\varepsilon\}_{\varepsilon > 0} \) is a sequence of \( C^2 \) functions such that

\[
\sup_{\varepsilon > 0} \left\{ \|\nabla u_\varepsilon\|_{L^\infty(\mathbb{R}^d)} + \|D^2 u_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \right\} < \infty.
\]

If \( \nabla u_\varepsilon \xrightarrow{\varepsilon \to 0} \nabla u \) for some \( u \in C^2(\mathbb{R}^d) \), then

\[
\lim_{\varepsilon \to 0} TV_\varepsilon(u_\varepsilon; \rho) = \sigma_2 \int_D |\nabla u(x)|(\rho(x))^2 \, dx.
\]

Proof. Step 1: For an arbitrary function \( v \in C^2(\mathbb{R}^d) \) we define

\[
H_\varepsilon(v) = \frac{1}{\varepsilon} \int_D \int_D \eta_\varepsilon(x-y) |\nabla v(x) \cdot (y-x)| \rho(x) \rho(y) \, dy \, dx.
\]

First we show that

\[
\lim_{\varepsilon \to 0} |TV_\varepsilon(u_\varepsilon; \rho) - H_\varepsilon(u_\varepsilon)| = 0.
\]

For this purpose, note that by Taylor’s theorem and by (35), for \( x, y \in D \), \( x \neq y \) and \( \varepsilon > 0 \)

\[
\frac{u_\varepsilon(x) - u_\varepsilon(y)}{|x-y|} - \frac{\nabla u_\varepsilon(x) \cdot (y-x)}{|x-y|} \leq \|D^2 u_\varepsilon\|_{L^\infty(\mathbb{R}^d)} |x-y| \leq C|x-y|,
\]

where \( \|D^2 u_\varepsilon\|_{L^\infty(\mathbb{R}^d)} \) denotes the \( L^\infty \) norm of the Hessian matrix of the function \( u_\varepsilon \) and \( C \) is a positive constant independent of \( \varepsilon \). Using this inequality and a simple change of variables we deduce

\[
|TV_\varepsilon(u_\varepsilon; \rho) - H_\varepsilon(u_\varepsilon)| \leq \frac{C \text{Vol}(D)\|\rho\|_{L^\infty(D)}}{\varepsilon} \int_{|h| \leq \gamma} \eta_\varepsilon(h)|h|^2 \, dh
\]

\[
= C \text{Vol}(D)\|\rho\|_{L^\infty(D)} \int_{|\hat{h}| \leq \gamma} \varepsilon \eta(\hat{h})|\hat{h}|^2 \, d\hat{h},
\]

and thus the limit is zero.
where \( \gamma \) denotes the diameter of the set \( D \). Finally, using assumption (K3) on the kernel \( \eta \), it is straightforward to deduce that the last term in the previous expression goes to zero as \( \varepsilon \) goes to zero, and thus we obtain (37).

**Step 2:** Now, for \( v \in C^2([\mathbb{R}^d]) \) consider

\[
(38) \quad \tilde{H}_\varepsilon(v) = \frac{1}{\varepsilon} \int_D \int_{x+h\in D} \eta_h(h) |\nabla v(x) \cdot h| (\rho(x))^2 \, dh \, dx.
\]

We claim that

\[
(39) \quad \lim_{\varepsilon \to 0} |H_\varepsilon(u_\varepsilon) - \tilde{H}_\varepsilon(u_\varepsilon)| = 0.
\]

Indeed, using the fact that \( \rho \) is Lipschitz,

\[
|H_\varepsilon(u_\varepsilon) - \tilde{H}_\varepsilon(u_\varepsilon)| \leq \frac{1}{\varepsilon} \int_D \int_{x+h\in D} \eta_h(h) |\nabla u_\varepsilon(x) \cdot h| |\rho(x+h) - \rho(x)||\rho(x)| \, dh \, dx
\]

\[
\leq ||\nabla u_\varepsilon||_{L^\infty([\mathbb{R}^d])} \text{Lip}(\rho) ||\rho||_{L^\infty(D)} \int_D \int_{x+h\in D} \eta_h(h) |h|^2 \, dh \, dx
\]

\[
\leq ||\nabla u_\varepsilon||_{L^\infty([\mathbb{R}^d])} \text{Lip}(\rho) ||\rho||_{L^\infty(D)} \text{Vol}(D) \int_{|h|<\gamma} \eta_h(h) |h|^2 \, dh,
\]

where as in Step 1 \( \gamma \) denotes the diameter of the set \( D \). The last term in the previous expression goes to zero as \( \varepsilon \) goes to zero (as in Step 1).

**Step 3:** We claim that

\[
(40) \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_D \int_{x+h\in \mathbb{R}^d \setminus D} \eta_h(h) |\nabla u_\varepsilon(x) \cdot h| (\rho(x))^2 \, dh \, dx = 0.
\]

Note that,

\[
\frac{1}{\varepsilon} \int_D \int_{x+h\in \mathbb{R}^d \setminus D} \eta_h(h) |\nabla u_\varepsilon(x) \cdot h| (\rho(x))^2 \, dh \, dx
\]

\[
\leq ||\nabla u_\varepsilon||_{L^\infty([\mathbb{R}^d])} ||\rho||_{L^\infty(D)}^2 \int_D \int_{x+h\in \mathbb{R}^d \setminus D} \eta_h(h) |h| \, dh \, dx.
\]

Using (35) and assumption (K3) on \( \eta \), we deduce that the right hand side of the previous inequality goes to zero as \( \varepsilon \) goes to zero, thus implying (40).

**Step 4:** Using steps 1, 2, and 3 in order to obtain (36) it is enough to prove that

\[
(41) \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_D \int_{\mathbb{R}^d} \eta_h(h) |\nabla u_\varepsilon(x) \cdot h| (\rho(x))^2 \, dh \, dx = \sigma_\eta \int_D |\nabla u| (\rho(x))^2 \, dx.
\]

Note that using the change of variables \( \tilde{h} = \frac{h}{\varepsilon} \) and the isotropy of the kernel \( \eta \), imply

\[
\frac{1}{\varepsilon} \int_D \int_{\mathbb{R}^d} \eta_h(h) |\nabla u_\varepsilon(x) \cdot h| (\rho(x))^2 \, dh \, dx
\]

\[
= \sigma_\eta \int_D |\nabla u_\varepsilon(x)| (\rho(x))^2 \, dx.
\]

Taking \( \varepsilon \) to zero in the previous expression we obtain (41), and consequently (36). \( \square \)

4.1. **Proof of Theorem 4.1: the Liminf Inequality.**

**Proof. Case 1: \( \rho \) is Lipschitz.** Consider an arbitrary \( u \in L^1(\rho) \) and suppose that \( u_\varepsilon \rightharpoonup u \) in \( L^1(\rho) \) as \( \varepsilon \to 0 \). Recall that given the assumptions on \( \rho \) this is equivalent to \( u_\varepsilon \rightharpoonup u \) in \( L^1(D) \) as \( \varepsilon \to 0 \). We want to show that \( \liminf_{\varepsilon \to 0} TV_\varepsilon(u_\varepsilon; \rho) \geq \sigma_\eta TV(u; \rho^2) \). Without the loss of generality we can assume that \( \{ TV_\varepsilon(u_\varepsilon; \rho) \}_{\varepsilon>0} \) is bounded.

The idea is to reduce the problem to a setting where we can use Lemma 4.2. The plan is to first regularize the functions \( u_\varepsilon \) to obtain a new sequence of functions \( \{ u_{\varepsilon, \delta} \}_{\varepsilon>0} \) (\( \delta > 0 \) is a parameter that
controls the smoothness of the regularized functions). The point is that regularizing does not increase the energy in the limit, while it gains the regularity needed to use Lemma 4.2.

To make this idea precise, consider \( J : \mathbb{R}^d \to [0, \infty) \) a standard mollifier. That is, \( J \) is a smooth radially symmetric function, supported in the closed unit ball \( B(0, 1) \) and is such that \( \int_{\mathbb{R}^d} J(z)dz = 1 \). We set \( J_\delta \) to be \( J_\delta(z) = \frac{1}{\delta^d} J\left( \frac{z}{\delta} \right) \). Note that \( \int_{\mathbb{R}^d} J_\delta(z)dz = 1 \) for every \( \delta > 0 \).

Fix \( D' \) an open domain compactly contained in \( D \). There exists \( \delta' > 0 \) such that \( D'' = \bigcup_{\delta < \delta'} B(x, \delta') \) is contained in \( D \). For \( 0 < \delta < \delta' \) and for a given function \( v \in L^1(D) \) we define the mollified function \( v_\delta \in L^1(\mathbb{R}^d) \) by setting \( v_\delta(x) = \int_{\mathbb{R}^d} J_\delta(x-z)v(z)dz = \int_{\mathbb{R}^d} J_\delta(z)v(x-z)dz \) where we have extended \( v \) to be zero outside of \( D \). The functions \( v_\delta \) are smooth, and satisfy \( v_\delta \to L^1(D') \) \( v \) as \( \delta \to 0 \), see for example [33]. Furthermore

\[
\nabla v_\delta(x) = \int_{\mathbb{R}^d} \nabla J_\delta(z)v(x-z)dz = \frac{1}{\delta} \int_{\mathbb{R}^d} \nabla J\left( \frac{z}{\delta} \right)v(x-z)dz.
\]

By taking the second derivative, it follows that there is a constant \( C > 0 \) (only depending on the mollifier \( J \)) such that

\[
||\nabla v_\delta||_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{\delta} ||v||_{L^1(D)} \quad \text{and} \quad ||D^2 v_\delta||_{L^\infty(\mathbb{R}^d)} \leq \frac{C}{\delta^2} ||v||_{L^1(D)}.
\]

Since \( u_\varepsilon \to L^1(D) \) \( u \) as \( \varepsilon \to 0 \) the norms \( ||u_\varepsilon||_{L^1(D)} \) are uniformly bounded. Therefore, taking \( v = u_\varepsilon \) in inequalities (43) and setting \( u_{\varepsilon, \delta} = (u_\varepsilon)_\delta \), implies

\[
\sup_{\varepsilon > 0} \left\{ ||\nabla u_{\varepsilon, \delta}||_{L^\infty(\mathbb{R}^d)} + ||D^2 u_{\varepsilon, \delta}||_{L^\infty(\mathbb{R}^d)} \right\} < \infty.
\]

Moreover, using (42) to express \( \nabla u_{\varepsilon, \delta} \) and \( \nabla u_\delta \), it is straightforward to deduce that

\[
\int_{D'} |\nabla u_{\varepsilon, \delta}(x) - \nabla u_\delta(x)| dx \leq \frac{C}{\delta} \int_{D'} |u_\varepsilon(x) - u(x)| dx.
\]

for some constant \( C \) independent of \( \varepsilon \). In particular, \( \int_{D'} |\nabla u_{\varepsilon, \delta}(x) - \nabla u_\delta(x)| dx \to 0 \) as \( \varepsilon \to 0 \) and hence we can apply Lemma 4.2 taking \( D \) to be \( D' \) to infer that

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{D'} \int_{D'} \eta_{\varepsilon}(x-y)|u_{\varepsilon, \delta}(x) - u_{\varepsilon, \delta}(y)| \rho(x)\rho(y) dxdy = \sigma_\eta \int_{D'} |\nabla u_\delta(x)||\rho(x)||^2 dxdy.
\]

To measure the approximation error in the energy, we set

\[
a_{\varepsilon, \delta} = \frac{1}{\varepsilon} \int_{D''} \int_{D'} \int_{\mathbb{R}^d} J_\delta(z) \eta_{\varepsilon}(x-y)|u_\varepsilon(x) - u_\varepsilon(y)| (\rho(x)\rho(y) - \rho(x+z)\rho(y+z)) dzdxdy,
\]

and estimate

\[
TV_\varepsilon(u_\varepsilon; \rho) \geq a_{\varepsilon, \delta} + \frac{1}{\varepsilon} \int_{D''} \int_{D'} \int_{\mathbb{R}^d} J_\delta(z) \eta_{\varepsilon}(x-y)|u_\varepsilon(x) - u_\varepsilon(y)| \rho(x)\rho(y) dxdy
\]

\[
= a_{\varepsilon, \delta} + \frac{1}{\varepsilon} \int_{D''} \int_{D'} \int_{\mathbb{R}^d} J_\delta(z) \eta_{\varepsilon}(x-y)|u_\varepsilon(x) - u_\varepsilon(y)| \rho(x)\rho(y) dxdy dxdy
\]

\[
\geq a_{\varepsilon, \delta} + \frac{1}{\varepsilon} \int_{D''} \int_{D'} \int_{\mathbb{R}^d} J_\delta(z) \eta_{\varepsilon}(x-y)|u_\varepsilon(x) - u_\varepsilon(y)| \rho(x)\rho(y) dxdy dxdy
\]

\[
= a_{\varepsilon, \delta} + \frac{1}{\varepsilon} \int_{D''} \int_{D'} \int_{\mathbb{R}^d} J_\delta(z) \eta_{\varepsilon}(x-y)|u_\varepsilon(x) - u_\varepsilon(y)| \rho(x)\rho(y) dxdy dxdy.
\]
where the second inequality is obtained using the change of variables $\hat{x} = x + z$, $\hat{y} = y + z$, $z = z$ together with the choice of $\delta$ and $\delta'$; Jensen’s inequality justifies the third one. This chain of inequalities and (44) imply that

$$\liminf_{\epsilon \to 0} TV_\epsilon(u_\epsilon; \rho) \geq \liminf_{\epsilon \to 0} a_{\epsilon, \delta} + \sigma_\eta \int_{D'} |\nabla u_\delta(x)| (|\rho(x)|^2) dx.$$  (45)

We estimate $a_{\epsilon, \delta}$ as follows

$$|a_{\epsilon, \delta}| \leq \frac{2|\rho| |L|}{\epsilon} \int_{D'} \int_{D'} \int_{\mathbb{R}^d} J_\delta(z) \eta_\epsilon(x-y) |u_\epsilon(x)-u_\epsilon(y)| |\rho(x)-\rho(x+z)| dz dxdy$$

$$\leq \frac{2\delta|\rho| |L| \text{Lip}(\rho)}{\epsilon} \int_{D'} \int_{D'} \int_{\mathbb{R}^d} J_\delta(z) \eta_\epsilon(x-y) |u_\epsilon(x)-u_\epsilon(y)| dz dxdy$$

$$= \frac{2\delta|\rho| |L| \text{Lip}(\rho)}{\epsilon} \int_{D'} \int_{D'} \eta_\epsilon(x-y) |u_\epsilon(x)-u_\epsilon(y)| dxdy.$$

Since we had assumed that $\{TV_\epsilon(u_\epsilon; \rho)\}_{\epsilon > 0}$ is bounded, and also that $\rho$ is bounded from below by a positive constant, we conclude from the previous inequalities that $\liminf_{\delta \to 0} \liminf_{\epsilon \to 0} a_{\epsilon, \delta} = 0$ and thus, by (45),

$$\liminf_{\epsilon \to 0} TV_\epsilon(u_\epsilon; \rho) \geq \sigma_\eta \liminf_{\delta \to 0} \int_{D'} |\nabla u_\delta| (|\rho(x)|^2) dx.$$

Given that $u_\delta \rightharpoonup_{L^1(D')} u$ as $\delta \to 0$, we can use the lower semicontinuity of the weighted total variation, (12), to obtain

$$\liminf_{\epsilon \to 0} TV_\epsilon(u_\epsilon; \rho) \geq \sigma_\eta \liminf_{\delta \to 0} \int_{D'} |\nabla u_\delta| (|\rho(x)|^2) dx \geq \sigma_\eta |Du|_{\text{TV}}(D').$$  (46)

Given that $D'$ was an arbitrary open set compactly contained in $D$, we can take $D' \supset D$ in the previous inequality to obtain the desired result.

**Case 2: $\rho$ is continuous but not necessarily Lipschitz.** The idea is to approximate $\rho$ from below by a family of Lipschitz functions $\{\rho_k\}_{k \in \mathbb{N}}$. Indeed, consider $\rho_k : D \to \mathbb{R}$ given by

$$\rho_k(x) := \inf_{y \in D} \rho(y) + k|x-y|.$$  (47)

The functions $\rho_k$ are Lipschitz functions which are bounded from below and from above by the same constants bounding $\rho$ from below and from above. Moreover, given that $\rho$ is continuous, for every $x \in D$, $\rho_k(x) \rightarrow \rho(x)$ as $k \to \infty$.

Let $u \in L^1(D)$ and suppose that $u_\epsilon \rightharpoonup_{L^1(D)} u$. Since $\rho_k$ is Lipschitz, we can use Case 1 and the fact that $\rho_k \leq \rho$ to conclude that

$$\liminf_{\epsilon \to 0} TV_\epsilon(u_\epsilon; \rho) \geq \liminf_{\epsilon \to 0} TV_\epsilon(u_\epsilon; \rho_k) \geq \sigma_\eta TV(u; \rho_k^2).$$  (48)

Using (12) and the monotone convergence theorem, we see that:

$$\lim_{k \to \infty} TV(u; \rho_k^2) = \lim_{k \to \infty} \int_D \rho_k^2(x) |Du| (x) = \int_D \rho^2(x) |Du| (x) = TV(u; \rho^2).$$

Combining with (48) yields the desired result. \hfill \square

### 4.2. Proof of Theorem 4.1: The Limsup Inequality

**Proof.** **Case 1: $\rho$ is Lipschitz.** We start by noting that since $\rho : D \to \mathbb{R}^d$ is a Lipschitz function, there exists an extension (that we denote by $\rho$ as well) to the entire $\mathbb{R}^d$ which has the same Lipschitz constant as the original $\rho$ and is bounded below by the same positive constant. Indeed, the extended function $\rho : \mathbb{R}^d \to \mathbb{R}$ can be defined by $\rho(x) = \inf_{y \in D} \rho(y) + \text{Lip}(\rho) |x-y|$, where $\text{Lip}(\rho)$ is the Lipschitz constant of $\rho$. 
To prove the limsup inequality we show that for every $u \in L^1(\rho)$:
\begin{equation}
\limsup_{\varepsilon \to 0} TV_\varepsilon (u; \rho) \leq \sigma_\eta TV(u, \rho^2).
\end{equation}

It suffices to show \((49)\) for functions $u \in BV(D)$ (if the right hand side of \((49)\) is $+\infty$ there is nothing to prove). Since $D$ has Lipschitz boundary, for a given $u \in BV(D)$ we use Proposition 3.21 in \([3]\) to obtain an extension $\hat{u} \in BV(\mathbb{R}^d)$ of $u$ to the entire space $\mathbb{R}^d$ with $|D\hat{u}|(\partial D) = 0$. In particular from \((11)\) we obtain
\begin{equation}
(D) = \limsup_{\varepsilon \to 0} TV_\varepsilon (u; \rho) \leq \sigma_\eta TV(u, \rho^2).
\end{equation}

We split the proof of \((49)\) in two cases:

**Step 1:** Suppose that $\eta$ has compact support, i.e. assume there is $\alpha > 0$ such that if $|h| \geq \alpha$ then $\eta(h) = 0$. Let $D_\varepsilon := \{ x \in \mathbb{R}^d : \text{dist}(x, D) < \alpha \varepsilon \}$. For $u \in BV(D)$, Theorem 3.4 in \([8]\) and our assumptions on $\rho$ provide a sequence of functions $\{v_k \}_{k \in \mathbb{N}} \subset C^\infty(D_\varepsilon) \cap BV(D_\varepsilon)$ such that as $k \to \infty$
\begin{equation}
\begin{aligned}
v_k & \to TV_\varepsilon (u; \rho) \quad \text{and} \quad \int_{D_\varepsilon} |\nabla v_k(x)| \rho^2(x) dx \to |D\hat{u}|_{L^p}(D_\varepsilon).
\end{aligned}
\end{equation}

For every $k \in \mathbb{N}$
\begin{align*}
TV_\varepsilon(v_k; \rho) &= \frac{1}{\varepsilon} \int_D \int_{D \cap B(y, \varepsilon)} \eta(x-y)|v_k(x) - v_k(y)| \rho(x) \rho(y) dxdy \\
&= \frac{1}{\varepsilon} \int_D \int_{B(y, \varepsilon)} \eta(x-y) \int_0^1 \nabla v_k(y+t(x-y)) \cdot (x-y) dt |\rho(x)\rho(y)| dxdy \\
&\leq \frac{1}{\varepsilon} \int_D \int_{B(y, \varepsilon)} \eta(x-y) \nabla v_k(y+t(x-y)) \cdot (x-y) |\rho(x)\rho(y)| dt dxdy \\
&\leq \int_{D_\varepsilon} \int_{|h| < \alpha} (\eta(h)|\nabla v_k(z) \cdot h| \rho(z-\varepsilon h) \rho(z+(1-\varepsilon) h) dt dh dz \\
&= \int_{D_\varepsilon} \int_{|h| < \alpha} |\nabla v_k(z) | \rho(z)^2 dh dz + a_{\varepsilon, k} \\
&= \sigma_\eta \int_{D_\varepsilon} |\nabla v_k(z) | \rho(z)^2 dh dz + a_{\varepsilon, k},
\end{align*}

where the last inequality is obtained after using the change of variables $(t,y,x) \to (t,h,z)$, $h = \frac{x-y}{\varepsilon}$ and $z = y + t(x-y)$, noting that the Jacobian of this transformation is equal to $\varepsilon^d$ and that the transformed set $D$ is contained in $D_\varepsilon$. The last equality is obtained thanks to the fact that $\eta$ is radially symmetric. Finally the $a_{\varepsilon, k}$ are given by
\begin{align*}
a_{\varepsilon, k} &= \int_{D_\varepsilon} \int_{|h| < \alpha} \int_0^1 \eta(h)|\nabla v_k(z) \cdot h| \left( \rho(z-\varepsilon h) \rho(z+(1-\varepsilon) h) - \rho(z)^2 \right) dt dh dz.
\end{align*}

Since $\rho : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz and since it is bounded below by a positive constant, it is straightforward to show that there exists a constant $C > 0$ independent of $\varepsilon$ and $k$ for which
\begin{align*}
a_{\varepsilon, k} &\leq C \varepsilon \int_{D_\varepsilon} |\nabla v_k(x)| \rho^2(x) dx.
\end{align*}

Using \((51)\) in particular we obtain that $v_k \to TV_\varepsilon (u; \rho)$ as $k \to \infty$. This together with continuity of $TV_\varepsilon (\cdot; \rho)$ with respect to $L^1$ convergence implies that $TV_\varepsilon(v_k; \rho) \to TV_\varepsilon(u; \rho)$ as $k \to \infty$. Therefore, from the previous chain of inequalities and from \((51)\) we conclude that
\begin{equation}
TV_\varepsilon(u; \rho) \leq \sigma_\eta |D\hat{u}|_{L^p}(D_\varepsilon) + \limsup_{k \to \infty} a_{\varepsilon, k} \leq \sigma_\eta |D\hat{u}|_{L^p}(D_\varepsilon) + Ce|D\hat{u}|_{L^p}(D_\varepsilon).
\end{equation}
Using (50), we deduce \( \lim_{\varepsilon \to 0} |D\hat{u}|_{p^2}(D_\varepsilon) = |D\hat{u}|_{p^2}(D) = |D\hat{u}|_{p^2}(\hat{D}) = TV(u; \rho)^2 < \infty \). Combining with (52) implies the desired estimate, (49).

**Step 2:** Consider \( \eta \) whose support is not compact. The needed control of \( \eta \) at infinity is provided by the condition (K3). For \( \alpha > 0 \) define the kernel \( \eta^\alpha(h) := \eta(h) \chi_{B(0,\alpha)}(h) \), which satisfies the conditions of Step 1. Denote by \( TV^\alpha(\cdot, \rho) \) the nonlocal total variation using the kernel \( \eta^\alpha \). For a given \( u \in BV(D) \)

\[
TV^\varepsilon(u; \rho) = TV^\alpha(u; \rho) + \frac{1}{\varepsilon} \int_D \int_{|x-y| > \varepsilon \alpha} \eta\varepsilon(x-y)|u(x) - u(y)|\rho(x)\rho(y) dxdy.
\]

The second term on the right-hand side satisfies:

\[
\frac{1}{\varepsilon} \int_D \int_{|x-y| > \varepsilon \alpha} \eta\varepsilon(x-y)|u(x) - u(y)|\rho(x)\rho(y) dxdy = \frac{1}{\varepsilon} \int_D \int_{|h| > \alpha} \eta(h)|\hat{u}(x) - \hat{u}(y)|\rho(h)\rho(y) dxdy
\]

\[
\leq ||\rho||_{L^2(\mathbb{R}^d)} \int_{|h| > \alpha} \eta(h)|h| \int_{|y| < |h|} |\hat{u}(y) - \hat{u}(y + \varepsilon h)| dy dh
\]

\[
\leq ||\rho||_{L^2(\mathbb{R}^d)}|D\hat{u}|([\mathbb{R}^d]) \int_{|h| > \alpha} \eta(h)|h| dh,
\]

where the first inequality is obtained using the change of variables \( h = \frac{x-y}{\varepsilon} \) and the second inequality obtained using Lemma 13.33 in [33].

By Step 1 we conclude that:

\[
\limsup_{\varepsilon \to 0} TV^\varepsilon(u; \rho) \leq \limsup_{\varepsilon \to 0} TV^\alpha(u; \rho) + ||\rho||_{L^2(\mathbb{R}^d)}|D\hat{u}|([\mathbb{R}^d]) \int_{|h| > \alpha} \eta(h)|h| dh
\]

\[
\leq \sigma^\alpha TV(u; \rho^2) + ||\rho||_{L^2(\mathbb{R}^d)}|D\hat{u}|([\mathbb{R}^d]) \int_{|h| > \alpha} \eta(h)|h| dh.
\]

Taking \( \alpha \) to infinity and using condition (K3) on \( \eta \) implies (49).

**Case 2:** \( \rho \) is continuous but not necessarily Lipschitz. The idea is to approximate \( \rho \) from above by a family of Lipschitz functions \( \{\rho_k\}_{k \in \mathbb{N}} \). Consider \( \rho_k : D \to \mathbb{R} \) given by

(53)

\[
\rho_k(x) := \sup_{y \in D} \rho(y) - k|x-y|.
\]

The functions \( \rho_k \) are Lipschitz functions which are bounded from below from and above by the same constants bounding \( \rho \) from below and from above. Moreover, given that \( \rho \) is continuous, it is simple to verify that for every \( x \in D, \rho_k(x) \leq \rho(x) \) as \( k \to \infty \).

As in Step 1, it is enough to consider \( u \in BV(D) \) and prove that:

\[
\limsup_{\varepsilon \to 0} TV^\varepsilon(u; \rho) \leq \sigma^\eta TV(u; \rho^2).
\]

The proof of the limsup inequality in Case 1 and the fact that \( \rho \leq \rho_k \) imply that

(54)

\[
\limsup_{\varepsilon \to 0} TV^\varepsilon(u; \rho) \leq \limsup_{\varepsilon \to 0} TV^\varepsilon(u; \rho_k) \leq \sigma^\eta TV(u; \rho_k^2).
\]

By the dominated convergence theorem,

\[
\lim_{k \to \infty} TV(u; \rho_k^2) = \lim_{k \to \infty} \int_D \rho_k^2(x)d|Du|(x) = \int_D \rho^2(x)d|Du|(x) = TV(u; \rho^2).
\]

Combining with (54) provides the desired result. \( \square \)

**Remark 4.3.** Note that using the liminf inequality and the proof of the limsup inequality we deduce the pointwise convergence of the functionals \( TV^\varepsilon(\cdot; \rho) \); namely, for every \( u \in L^1(D; \rho) \):

\[
\lim_{\varepsilon \to 0} TV^\varepsilon(u; \rho) = \sigma^\eta TV(u; \rho^2).
\]
4.3. **Proof of Theorem 4.1: Compactness.** We first establish compactness for regular domains and then extend it to more general ones.

**Lemma 4.4.** Let \( D \) be a bounded, open, and connected set in \( \mathbb{R}^d \), with \( C^2 \) boundary. Let \( \{v_\varepsilon\}_{\varepsilon > 0} \) be a sequence in \( L^1(D, \rho) \) such that:

\[
\sup_{\varepsilon > 0} \|v_\varepsilon\|_{L^1(D, \rho)} < \infty,
\]

and

\[
\sup_{\varepsilon > 0} TV_\varepsilon(v_\varepsilon; \rho) < \infty.
\]

Then, \( \{v_\varepsilon\}_{\varepsilon > 0} \) is precompact in \( L^1(D, \rho) \).

**Proof.** Note that thanks to assumption (K1), we can find \( a > 0 \) and \( b > 0 \) such that the function \( \tilde{\eta} : [0, \infty) \to [0, a] \) defined as \( \tilde{\eta}(t) = a \) for \( t < b \) and \( \tilde{\eta}(t) = 0 \) otherwise, is bounded above by \( \eta \). In particular, (55) holds when changing \( \eta \) for \( \tilde{\eta} \) and so there is no loss of generality in assuming that \( \eta \) has the form of \( \tilde{\eta} \). Also, since \( \rho \) is bounded below and above by positive constants, it is enough to consider \( \rho \equiv 1 \).

We first extend each function \( v_\varepsilon \) to \( \mathbb{R}^d \) in a suitable way. Since \( \partial D \) is a compact \( C^2 \) manifold, there exists \( \delta > 0 \) such that for every \( x \in \mathbb{R}^d \) for which \( d(x, \partial D) \leq \delta \) there exists a unique closest point on \( \partial D \). For all \( x \in U := \{x \in \mathbb{R}^d : d(x, D) < \delta\} \) let \( Px \) be the closest point to \( x \) in \( D \). We define the local reflection mapping from \( U \) to \( \overline{D} \) by \( \hat{x} = 2Px - x \). Let \( \xi \) be a smooth cut-off function such that \( \xi(s) = 1 \) if \( s \leq \delta/8 \) and \( \xi(s) = 0 \) if \( s \geq \delta/4 \). We define an auxiliary function \( \tilde{v}_\varepsilon \) on \( U \), by \( \tilde{v}_\varepsilon(x) := v_\varepsilon(\hat{x}) \) and the desired extended function \( \bar{v}_\varepsilon \) on \( \mathbb{R}^d \) by \( \bar{v}_\varepsilon(x) = \xi(|x - Px|)v_\varepsilon(\hat{x}) \).

We claim that:

\[
\sup_{\varepsilon > 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \eta_\varepsilon(x - y)|\bar{v}_\varepsilon(x) - \bar{v}_\varepsilon(y)| < \infty.
\]

To show the claim we first establish the following geometric properties: Let \( W := \{x \in \mathbb{R}^d \setminus D : d(x, D) < \delta/4\} \) and \( V := \{x \in \mathbb{R}^d \setminus D : d(x, D) < \delta/8\} \). For all \( x \in W \) and all \( y \in D \)

\[
|\hat{x} - y| < 2|x - y|.
\]

Since the mapping \( x \mapsto \hat{x} \) is smooth and invertible on \( W \), it is bi-Lipschitz. While this would be enough for our argument, we present an argument which establishes the value of the Lipschitz constant: for all \( x, y \in W \)

\[
\frac{1}{4}|x - y| < |\hat{x} - y| < 4|x - y|.
\]

By definition of \( \delta \) the domain \( D \) satisfies the outside and inside ball conditions with radius \( \delta \). Therefore if \( x \in W \) and \( z \in \overline{D} \)

\[
|z - \left(Px + \frac{x - Px}{|x - Px|}\right)| \geq \delta.
\]

Squaring and straightforward algebra yield

\[
|z - Px|^2 \geq 2\delta(z - Px) \cdot \frac{x - Px}{|x - Px|}.
\]

For \( x \in W \) and \( y \in D \), using (59) we obtain

\[
|y - \hat{x}|^2 - |y - x|^2 = |y - Px + (x - Px)|^2 - |y - Px - (x - Px)|^2
\]

\[
= 4(y - Px) \cdot (x - Px) \leq \frac{2}{\delta}|y - Px|^2 |x - Px|
\]

\[
\leq \frac{1}{2}|y - Px|^2 \leq |y - x|^2 + |x - Px|^2 \leq 2|y - x|^2.
\]
Therefore \(|y - \hat{x}|^2 \leq 3|y - x|^2\), which establishes (57).

For distinct \(x, y \in W\) using (59), with \(z = Py\) and with \(z = Px\), follows

\[
|x - y| \geq (x - y) \cdot \frac{Px - Py}{|Px - Py|} = (x - Px - (y - Py) + Px - Py) \cdot \frac{Px - Py}{|Px - Py|}
\]

\[
\geq |Px - Py| - \frac{1}{2}\delta (|x - Px| |Py - Px| + |y - Py| |Py - Px|)
\]

\[
\geq |Px - Py| \cdot \frac{3}{4}.
\]

Therefore

\[
|x - y| = |2Px - x + 2Py - y| \leq 2|Px - Py| + |x - y| \leq \left(\frac{8}{3} + 1\right) |x - y| \leq 4|x - y|.
\]

Since the roles on \(x, y\) and \(\hat{x}, \hat{y}\) can be reversed it follows that \(|x - y| \leq 4|\hat{x} - \hat{y}|\). These estimates establish (58).

We now return to proving (56). For \(\varepsilon\) small enough,

\[
\frac{1}{\varepsilon} \int_{\mathbb{R}^d \setminus D} \int_{\mathbb{R}^d \setminus \tilde{D}} \eta_e(x - y)|\tilde{v}_e(x) - \tilde{v}_e(y)| dxdy = \frac{1}{\varepsilon} \int_{V} \int_{D} \eta_e(x - y)|\tilde{v}_e(x) - \tilde{v}_e(y)| dxdy
\]

\[
= \frac{1}{\varepsilon} \int_{V} \int_{D} \eta_e(x - y)|v_e(\hat{x}) - v_e(y)| dxdy
\]

\[
\leq \frac{4}{\varepsilon} \int_{V} \int_{D} \eta_{4e}(\hat{x} - y)|v_e(x) - v_e(y)| dxdy
\]

\[
\leq \frac{16}{\varepsilon} \int_{D} \int_{D} \eta_{4e}(z - y)|v_e(x) - v_e(z)| dzdy,
\]

where the first inequality follows from (57) and the second follows from the fact that the change of variables \(x \mapsto \hat{x}\) is bi-Lipschitz as shown in (58). Also,

\[
\frac{1}{\varepsilon} \int_{\mathbb{R}^d \setminus D} \int_{\mathbb{R}^d \setminus \tilde{D}} \eta_e(x - y)|\tilde{v}_e(x) - \tilde{v}_e(y)| dxdy
\]

\[
= \frac{1}{\varepsilon} \int_{W} \int_{W} \eta_e(x - y)|\tilde{v}_e(x) - \tilde{v}_e(y)| dxdy
\]

\[
\leq \frac{1}{\varepsilon} \int_{W} \int_{W} \eta_e(x - y)|\tilde{v}_e(x) - \tilde{v}_e(y)| dxdy
\]

\[
+ \frac{1}{\varepsilon} \int_{W} \int_{W} \eta_e(x - y)|\tilde{v}_e(x) - \tilde{v}_e(y)| dxdy.
\]

Note that for all \(x \neq y\), \(\eta_e(x - y) \leq \frac{b}{|x - y|} \eta_e(x - y)\). Therefore:

\[
\frac{1}{\varepsilon} \int_{W} \int_{W} \eta_e(x - y)|\tilde{v}_e(x) - \tilde{v}_e(y)| dxdy \leq b \int_{W} \int_{W} \eta_e(x - y) \left|\tilde{v}_e(x) - \tilde{v}_e(y)\right| |x - y| dxdy
\]

\[
\leq b \operatorname{Lip}(\tilde{v}_e) \int_{W} \int_{W} \eta_e(x - y)|\tilde{v}_e(x)| dxdy
\]

\[
\leq 4d b \operatorname{Lip}(\tilde{v}_e)||v_e||_{L^1(D)},
\]

where we used (58) and change of variables to establish the last inequality. Also,

\[
\frac{1}{\varepsilon} \int_{W} \int_{W} \eta_e(x - y)|\tilde{v}_e(x) - \tilde{v}_e(y)| \left|\xi(x) - \xi(y)\right| dxdy \leq \frac{4}{\varepsilon} \int_{D} \int_{D} \eta_{4e}(\hat{x} - y)|\tilde{v}_e(x) - \tilde{v}_e(y)| dxdy
\]

\[
\leq \frac{4}{\varepsilon} \int_{D} \int_{D} \eta_{4e}(x - y)|v_e(x) - v_e(y)| dxdy.
\]
The first inequality is obtained thanks to the fact that $|\xi(y)| \leq 1$ and (58), while the second inequality is obtained by a change of variables.

Using that
\[
\int_D \int_D \eta(x-y)|v(x) - v(y)|dxdy \leq 4\int_D \int_D \eta(x-y)|\tilde{v}(x) - \tilde{v}(y)|dxdy
\]
by combining the above inequalities we conclude that
\[
\sup_{\varepsilon > 0} \frac{1}{\varepsilon} \int_D \int_D \eta(x-y)|\tilde{v}(x) - \tilde{v}(y)|dxdy
\leq C \sup_{\varepsilon > 0} \left( \int_D \int_D \eta(x-y)|v(x)-v(y)|dxdy + \|v\|_{L^1(D)} \right) < \infty.
\]

Using the proof of Proposition 3.1 in [3] we deduce that the sequence $\{\tilde{v}_\varepsilon\}_{\varepsilon > 0}$ is precompact in $L^1(\mathbb{R}^d)$ which implies that the sequence $\{v_\varepsilon\}_{\varepsilon > 0}$ is precompact in $L^1(D)$.

**Remark 4.5.** We remark that the difference between the compactness result we proved above and the one proved in Proposition 3.1 in [3] is the fact that we consider functions bounded in $L^1$, instead of bounded in $L^\infty$ as was assumed in [3]. Nevertheless, after extending the functions to the entire $\mathbb{R}^d$ as above, one can directly apply the proof in [3] to obtain the desired compactness result.

**Proposition 4.6.** Let $D$ be a bounded, open, and connected set in $\mathbb{R}^d$, which is homeomorphic to a $C^2$ domain $\tilde{D}$ via a bi-Lipschitz mapping $\Theta$. Suppose that the sequence of functions $\{u_\varepsilon\}_{\varepsilon > 0} \subseteq L^1(D, \rho)$ satisfies:

\[
\sup_{\varepsilon > 0} ||u_\varepsilon||_{L^1(D, \rho)} < \infty, \\
\sup_{\varepsilon > 0} TV_\varepsilon(u_\varepsilon; \rho) < \infty.
\]

Then, $\{u_\varepsilon\}_{\varepsilon > 0}$ is precompact in $L^1(D, \rho)$.

We remark that the proposition applies to any domain which is bi-Lipschitz homeomorphic to $(0, 1)^d$, since $(0, 1)^d$ is bi-Lipschitz homeomorphic to the ball $B(0, 1)$ by, for example, linear rescaling along each ray from the origin.

**Proof.** Suppose $\{u_\varepsilon\}_{\varepsilon > 0} \subseteq L^1(D)$ is as in the statement. As in Lemma 4.4, we can assume that $\rho \equiv 1$. For every $\varepsilon > 0$ consider the function $v_\varepsilon := u_\varepsilon \circ \Theta$ and set $\eta(s) := \eta(\text{Lip}(\Theta)s)$, $s \in \mathbb{R}$.

Since $\Theta$ is bi-Lipschitz we can use a change of variables, to conclude that there exists a constant $C > 0$ (only depending on $\Theta$) such that:
\[
\int_D |v_\varepsilon(x)|dx \leq C \int_D |u_\varepsilon(y)|dy,
\]
and
\[
C \int_D \int_D \eta(x-y)|u_\varepsilon(x) - u_\varepsilon(y)|dxdy \geq \int_D \int_D \eta(\Theta(x) - \Theta(y))|v_\varepsilon(x) - v_\varepsilon(y)|dxdy
\geq \int_D \int_D \eta(x-y)|v_\varepsilon(x) - v_\varepsilon(y)|dxdy.
\]

The second inequality using the fact that $\eta$ is non-increasing (assumption (K2)). We conclude that the sequence $\{v_\varepsilon\}_{\varepsilon > 0} \subseteq L^1(D)$ satisfies the hypothesis of Lemma 4.4 (taking $\eta = \tilde{\eta}$). Therefore, $\{v_\varepsilon\}_{\varepsilon > 0}$ is precompact in $L^1(D)$, which implies that $\{u_\varepsilon\}_{\varepsilon > 0}$ is precompact in $L^1(D)$.
Corollary 4.7. Let $D$ be a bounded, open, and connected set in $\mathbb{R}^d$. Suppose that the sequence of functions $\{u_\varepsilon\}_{\varepsilon > 0} \subseteq L^1(\rho)(D)$ satisfies:

$$\sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^1(\rho)(D)} < \infty,$$
$$\sup_{\varepsilon > 0} \text{TV}_\varepsilon(u_\varepsilon; \rho) < \infty.$$ 

Then, $\{u_\varepsilon\}_{\varepsilon > 0}$ is locally precompact in $L^1_p(D)$.

In particular if

$$\sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^\infty(D)} < \infty,$$

then, $\{u_\varepsilon\}_{\varepsilon > 0}$ is precompact in $L^1_p(D)$.

Proof. If $B$ is a ball compactly contained in $D$ then the pre-compactness of $\{u_\varepsilon\}_{\varepsilon > 0}$ in $L^1_p(B)$ follows from Lemma 4.4. We note that if compactness holds on two sets $D_1$ and $D_2$ compactly contained in $D$, then it holds on their union. Therefore it holds on any set compactly contained in $D$, since it can be covered by finitely many balls contained in $D$.

The compactness in $L^1_p(D)$ under the $L^\infty$ boundedness follows via a diagonal argument. This can be achieved by approximating $D$ by compact subsets: $\overline{D}_k \subset D$, $D = \bigcup_k D_k$, and using the fact that $\lim_{k \to \infty} \sup_{\varepsilon > 0} \|u_\varepsilon\|_{L^1_p(D_k)} = 0$. 

5. Γ-Convergence of Total Variation on Graphs

5.1. Proof of Theorems 1.1 and 1.2. Let $D \subset \mathbb{R}^d$, $d \geq 2$ be an open set with Lipschitz boundary which is homeomorphic via a bi-Lipschitz mapping to $(0,1)^d$. Assume $\nu$ is a probability measure on $D$ with continuous density $\rho$, which is bounded from below and above by positive constants. Let $\{\varepsilon_n\}_{n \in \mathbb{N}}$ be a sequence of positive numbers converging to 0 satisfying assumption (8).

Proof. If $\nu$ is of the form $\nu(t) = a$ for $t < b$ and $\nu = 0$ for $t > b$, where $a,b$ are two positive constants. Note it does not matter what value we give to $\nu$ at $b$. The key idea in the proof is that the estimates of the Section 2.3 on transportation maps imply that the transportation happens on a length scale which is small compared to $\varepsilon_n$. By taking a kernel with slightly smaller ‘radius’ than $\varepsilon_n$ we can then obtain a lower bound, and by taking a slightly larger radius a matching upper bound on the graph total variation.

Liminf inequality: Assume that $u_n \xrightarrow{TV^1} u$ as $n \to \infty$. Since $T_n \nu = \nu_n$, using the change of variables formula (17) it follows that

$$\text{GTV}_{\nu,\varepsilon_n}(u_n) = \frac{1}{\varepsilon_n} \int_{D \times D} \eta_{\varepsilon_n}(T_n(x) - T_n(y)) |u_n \circ T_n(x) - u_n \circ T_n(y)| \rho(x) \rho(y) dxdy.$$

Note that for Lebesgue almost every $(x,y) \in D \times D$

$$|T_n(x) - T_n(y)| > h \varepsilon_n \Rightarrow |x - y| > h \varepsilon_n - 2\|Id - T_n\|_\infty.$$

Thanks to the assumptions on $\{\varepsilon_n\}_{n \in \mathbb{N}}$ (25) and (26) in cases $d = 2$ and $d \geq 3$ respectively, for large enough $n \in \mathbb{N}$:

$$\tilde{\varepsilon}_n := \varepsilon_n - \frac{2}{b} \|Id - T_n\|_\infty > 0.$$
By (61), for large enough \( n \) and for almost every \((x, y) \in D \times D\),
\[
\eta\left(\frac{|x - y|}{\varepsilon_n}\right) \leq \eta\left(\frac{|T_n(x) - T_n(y)|}{\varepsilon_n}\right).
\]
Let \( \tilde{u}_n = u_n \circ T_n \). Thanks to the previous inequality and (60), for large enough \( n \)
\[
\text{GTV}_{n, \varepsilon_n}(u_n) \geq \frac{1}{\varepsilon_n} \int_{D \times D} \eta\left(\frac{|x - y|}{\varepsilon_n}\right) |\tilde{u}_n(x) - \tilde{u}_n(y)| \rho(x)\rho(y)dx dy
\]
\[
= \left(\frac{\varepsilon_n}{\tilde{\varepsilon}_n}\right)^{d+1} TV_{\varepsilon_n}(\tilde{u}_n; \rho).
\]
Note that \( \varepsilon_n \rightarrow 1 \) as \( n \rightarrow \infty \) and that \( u_n \xrightarrow{TV} u \) implies \( \tilde{u}_n \xrightarrow{TV} u \) as \( n \rightarrow \infty \). We deduce from Theorem 4.1 that \( \liminf_{n \rightarrow \infty} TV_{n, \rho}(\tilde{u}_n; \rho) \geq \sigma_\eta TV(u; \rho^2) \) and hence:
\[
\liminf_{n \rightarrow \infty} TV_{n, \varepsilon_n}(u_n) \geq \sigma_\eta TV(u; \rho^2).
\]
**Limsup inequality:** By Remark 2.7 and Proposition 2.4, it is enough to prove the limsup inequality for Lipschitz continuous functions \( u : D \rightarrow \mathbb{R} \). Define \( u_n \) to be the restriction of \( u \) to the first \( n \) data points \( X_1, \ldots, X_n \). Consider \( \varepsilon_n := \varepsilon_n + \frac{\varepsilon_n}{\varepsilon_n} ||Id - T_n||_\infty \) and let \( \tilde{u}_n = u_n \circ T_n \). Then note that for Lebesgue almost every \((x, y) \in D \times D\)
\[
\eta\left(\frac{|T_n(x) - T_n(y)|}{\varepsilon_n}\right) \leq \eta\left(\frac{|x - y|}{\varepsilon_n}\right).
\]
Then for all \( n \)
\[
\frac{1}{\varepsilon_n^{d+1}} \int_{D \times D} \eta\left(\frac{|T_n(x) - T_n(y)|}{\varepsilon_n}\right) |\tilde{u}_n(x) - \tilde{u}_n(y)| \rho(x)\rho(y)dx dy
\]
\[
\leq \frac{1}{\varepsilon_n} \int_{D \times D} \eta\varepsilon_n(x - y) |\tilde{u}_n(x) - \tilde{u}_n(y)| \rho(x)\rho(y)dx dy.
\]
Also
\[
\frac{1}{\varepsilon_n} \left| \int_{D \times D} \eta\varepsilon_n(x - y) |u(x) - u(y)| - |u \circ T_n(x) - u \circ T_n(y)| \rho(x)\rho(y)dx dy \right|
\]
\[
\leq \frac{2}{\varepsilon_n} \int_{D \times D} \eta\varepsilon_n(x - y) |u(x) - u \circ T_n(x)| \rho(x)\rho(y)dx dy
\]
\[
\leq \frac{2C \text{Lip}(u)||\rho||_{L^\infty(D)}}{\varepsilon_n} \int_D |x - T_n(x)| dx,
\]
where \( C = \int_{[0,1]} \eta(h) dh \). The last term of the previous expression goes to 0 as \( n \rightarrow \infty \), yielding
\[
\lim_{n \rightarrow \infty} \frac{1}{\varepsilon_n} \left( \int_{D \times D} \eta\varepsilon_n(x - y) |u(x) - u(y)| \rho(x)\rho(y)dx dy
\]
\[
- \int_{D \times D} \eta\varepsilon_n(x - y) |u \circ T_n(x) - u \circ T_n(y)| \rho(x)\rho(y)dx dy \right) = 0.
\]
Since \( \varepsilon_n \rightarrow 1 \) as \( n \rightarrow \infty \), using (62) we deduce:
\[
\limsup_{n \rightarrow \infty} \text{GTV}_{n, \varepsilon_n}(u_n) = \limsup_{n \rightarrow \infty} \frac{1}{\varepsilon_n^{d+1}} \int_{D \times D} \eta\left(\frac{|T_n(x) - T_n(y)|}{\varepsilon_n}\right) |u \circ T_n(x) - u \circ T_n(y)| \rho(x)\rho(y)dx dy
\]
\[
\leq \limsup_{n \rightarrow \infty} \frac{1}{\varepsilon_n} \int_{D \times D} \eta\varepsilon_n(x - y) |u \circ T_n(x) - u \circ T_n(y)| \rho(x)\rho(y)dx dy
\]
\[
= \limsup_{n \rightarrow \infty} TV_{\varepsilon_n}(u; \rho) \leq \sigma_\eta TV(u; \rho^2),
\]
where the last inequality follows from the proof of Theorem 4.1, specifically inequality (49).

**Step 2:** Now consider \( \eta \) to be a piecewise constant function with compact support, satisfying (K1)-(K3). In this case \( \eta = \sum_{k=1}^{l} \eta_k \) for some \( l \) and functions \( \eta_k \) as in Step 1. For this step of the proof we denote by \( GTV_{n, \epsilon_n} \) the total variation function on the graph using \( \eta_k \).

**Limsup inequality:** By Remark 2.7 it is enough to prove the limsup inequality for \( u : D \to \mathbb{R} \) Lipschitz. Consider \( u_n \) as in the proof of the limsup inequality in Step 1. Then

\[
\limsup_{n \to \infty} GTV_{n, \epsilon_n}(u_n) = \limsup_{n \to \infty} \sum_{k=1}^{l} GTV_{n, \epsilon_n}(u_n) \\
\leq \sum_{k=1}^{l} \limsup_{n \to \infty} GTV_{n, \epsilon_n}(u_n) \leq \sum_{k=1}^{l} \sigma_{\eta_k} TV(u; \rho^2) = \sigma_{\eta} TV(u; \rho^2).
\]

**Step 3:** Assume \( \eta \) is compactly supported and satisfies (K1)-(K3).

**Limsup Inequality:** Note that there exists an increasing sequence of piecewise constant functions \( \eta_k : [0, \infty) \to [0, \infty) \) (\( \eta \) from Step 2 is used as \( \eta_k \) here), with \( \eta_k \uparrow \eta \) as \( k \to \infty \) a.e. Denote by \( GTV_{n, \epsilon_n}^{k} \) the graph TV corresponding to \( \eta_k \). If \( u_n \xrightarrow{T_L} u \) as \( n \to \infty \), by Step 2 \( \sigma_{\eta_k} TV(u; \rho^2) \leq \liminf_{n \to \infty} GTV_{n, \epsilon_n}(u_n) \) for every \( k \in \mathbb{N} \). The monotone convergence theorem implies that \( \lim_{n \to \infty} \sigma_{\eta_k} = \sigma_\eta \) and so we conclude that \( \sigma_{\eta} TV(u; \rho^2) \leq \liminf_{n \to \infty} GTV_{n, \epsilon_n}(u_n) \).

**Limsup inequality:** As in Steps 1 and 2 it is enough to prove the limsup inequality for \( u \) Lipschitz. Consider \( u_n \) as in the proof of the limsup inequality in Steps 1 and 2. Analogously to the proof of the liminf inequality, we can find a decreasing sequence of functions \( \eta_k : [0, \infty) \to [0, \infty) \) (of the form considered in Step 2), with \( \eta_k \downarrow \eta \) as \( k \to \infty \) a.e. Proceeding in an analogous way to the way we proceeded in the proof of the liminf inequality we can conclude that \( \limsup_{n \to \infty} GTV_{n, \epsilon_n}^{k}(u_n) \leq \sigma_{\eta} TV(u; \rho^2) \).

**Step 4:** Consider general \( \eta \), satisfying (K1)-(K3). Note that for the liminf inequality we can use the proof given in Step 3. For the limsup inequality, as in the previous steps we can conclude that \( u \) is Lipschitz and we take \( u_n \) as in the previous steps. Let \( \alpha > 0 \) and define \( \eta_{\alpha} : [0, \infty) \to [0, \infty) \) by \( \eta_{\alpha}(t) := \eta(t) \) for \( t \leq \alpha \) and \( \eta_{\alpha}(t) = 0 \) for \( t > \alpha \). We denote by \( GTV_{n, \epsilon_n}^{\alpha} \) the graph TV using \( \eta_{\alpha} \).

Note that:

\[
GTV_{n, \epsilon_n}(u_n) = GTV_{n, \epsilon_n}^{\alpha}(u_n) + \frac{1}{\epsilon_n^{d+1}} \int |T_n(x) - T_n(y)| \eta_{\alpha} \left( \frac{|T_n(x) - T_n(y)|}{\epsilon_n} \right) dx dy.
\]

Let us find bounds on the second term on the right hand side of the previous equality for large \( n \). Indeed since for almost every \((x, y) \in D \times D\) it is true that \( |x - y| \leq |T_n(x) - T_n(y)| + 2 |Id - T_n| \) and \( |T_n(x) - T_n(y)| \leq |x - y| + 2 |Id - T_n| \), we can use the fact that \( \frac{|Id - T_n|}{\epsilon_n} \to 0 \) as \( n \to \infty \) to conclude that for large enough \( n \), for almost every \((x, y) \in D \times D\) for which \( |T_n(x) - T_n(y)| > \alpha \epsilon_n \) it holds that...
\[ |x - y| \leq 2|T_n(x) - T_n(y)| \text{ and } |T_n(x) - T_n(y)| \leq 2|x - y| \]. We conclude that for large enough \( n \)

\[
\frac{1}{\varepsilon_n^{d+1}} \int_{|T_n(x) - T_n(y)| > \varepsilon_n} \eta \left( \frac{|T_n(x) - T_n(y)|}{\varepsilon_n} \right) |u \circ T_n(x) - u \circ T_n(y)| \rho(x) \rho(y) dxdy 
\]

\[ \leq \frac{\|\rho\|_{L^2(D)}}{\varepsilon_n^{d+1}} \int_{|x-y| > \varepsilon_n} \eta \left( \frac{|x-y|}{2\varepsilon_n} \right) |u \circ T_n(x) - u \circ T_n(y)| dxdy \]

\[ \leq 2 \text{Lip}(u) \|\rho\|_{L^2(D)} \frac{\varepsilon_n}{\varepsilon_n^{d+1}} \int_{|x-y| > \varepsilon_n} \eta \left( \frac{|x-y|}{2\varepsilon_n} \right) |x-y| dxdy. \]

To find bounds on the last term of the previous chain of inequalities, consider the change of variables \((x, y) \in D \times D \mapsto (x, h)\) where \( x = x \) and \( h = \frac{x+y}{2\varepsilon_n} \). We deduce that:

\[
\frac{2}{\varepsilon_n^{d+1}} \int_{|y| > \varepsilon_n/2} \eta \left( \frac{|x-y|}{2\varepsilon_n} \right) |x-y| dxdy \leq C \int_{|h| > \varepsilon_n/2} \eta(h) |h| dh,
\]

where \( C \) does not depend on \( n \) or \( \alpha \). The previous inequalities, (64) and Step 3 imply that

\[
\limsup_{n \to \infty} GTV_{n, \varepsilon_n}(u_n) \leq \limsup_{n \to \infty} GTV_{n, \varepsilon_n}(u_n) + \text{Lip}(u) \|\rho\|_{L^2(D)} C \int_{|h| > \varepsilon_n/2} \eta(h) |h| dh
\]

\[ \leq \sigma_{\eta \alpha} TV(u; \rho^2) + \text{Lip}(u) \|\rho\|_{L^2(D)} C \int_{|h| > \varepsilon_n/2} \eta(h) |h| dh. \]

Finally, given assumptions (K3) on \( \eta \), sending \( \alpha \) to infinity we conclude that

\[
\limsup_{n \to \infty} GTV_{n, \varepsilon_n}(u_n) \leq \sigma_{\eta} TV(u; \rho^2).
\]

We now present the proof of Theorem 1.2 on compactness.

**Proof.** Assume that \{\( u_n \)\} \(_{n \in \mathbb{N}}\) is a sequence of functions with \( u_n \in L^1(D, \nu_n) \) satisfying the assumptions of the theorem. As in Lemma 4.4 and Proposition 4.6 without loss of generality we can assume that \( \eta \)

is of the form \( \eta(t) = a \) if \( t < b \) and \( \eta(t) = 0 \) for \( t > b \), for some \( a \) and \( b \) positive constants.

Consider the sequence of transportation maps \{\( T_n \)\} \(_{n \in \mathbb{N}}\) from Section 2.3. Since \{\( \varepsilon_n \)\} \(_{n \in \mathbb{N}}\) satisfies (9), estimates (25) and (27) imply that for Lebesgue a.e. \( z, y \in D \) with \( |T_n(z) - T_n(y)| > b \varepsilon_n \) it holds that \( |z - y| > b \varepsilon_n - 2\|Id - T_n\|_{\infty} > 0 \). For large enough \( n \), we set \( \varepsilon_n := \varepsilon_n - \frac{2\|Id - T_n\|_{\infty}}{b} > 0 \). We conclude that for large \( n \) and Lebesgue a.e. \( z, y \in D \):

\[
\eta \left( \frac{|z-y|}{\varepsilon_n} \right) \leq \eta \left( \frac{|T_n(z) - T_n(y)|}{\varepsilon_n} \right).
\]

Using this, we can conclude that for large enough \( n \):

\[
\frac{1}{\varepsilon_n^{d+1}} \int_D \int_D \eta \left( \frac{|z-y|}{\varepsilon_n} \right) |u_n \circ T_n(z) - u_n \circ T_n(y)| \rho(z) \rho(y) dzdy 
\]

\[ \leq \frac{1}{\varepsilon_n^{d+1}} \int_D \int_D \eta \left( \frac{|T_n(z) - T_n(y)|}{\varepsilon_n} \right) |u_n \circ T_n(z) - u_n \circ T_n(y)| \rho(z) \rho(y) dzdy 
\]

\[ = GTV_{n, \varepsilon_n}(u_n). \]

Thus

\[
\sup_{n \in \mathbb{N}} \frac{1}{\varepsilon_n^{d+1}} \int_D \int_D \eta \left( \frac{|z-y|}{\varepsilon_n} \right) |u_n \circ T_n(z) - u_n \circ T_n(y)| \rho(z) \rho(y) dzdy < \infty.\]

Finally noting that \( \frac{\varepsilon_n}{\varepsilon_n} \to 1 \) as \( n \to \infty \) we deduce that:

\[
\sup_{n \in \mathbb{N}} \frac{1}{\varepsilon_n} \int_D \int_D \eta \left( \frac{|z-y|}{\varepsilon_n} \right) |u_n \circ T_n(z) - u_n \circ T_n(y)| \rho(z) \rho(y) dzdy < \infty.\]
By Proposition 4.6 we conclude that \( \{u_n \circ T_n\}_{n \in \mathbb{N}} \) is precompact in \( L^1(D) \) and hence \( \{u_n\}_{n \in \mathbb{N}} \) is precompact in \( TL^1 \).

We now prove Corollary 1.3 on the \( \Gamma \) convergence of perimeter.

\textit{Proof.} Note that if \( \{A_n\}_{n \in \mathbb{N}} \) is such that \( A_n \subseteq \{X_1, \ldots, X_n\}_{n \in \mathbb{N}} \) and \( \mathcal{X}_{A_n} \xrightarrow{TL^1} \mathcal{X}_A \) as \( n \to \infty \) for some \( A \subseteq D \), then the liminf inequality follows automatically from the liminf inequality in Theorem 1.1. The limsup inequality is not immediate, since we cannot use the density of Lipschitz functions as we did in the proof of Theorem 1.1, given that we restrict our attention to characteristic functions.

We follow the proof of Proposition 3.5 in [21] and take advantage of the coarea formula of the energies \( GTV_{n, \varepsilon} \). Consider a measurable subset \( A \) of \( D \). By the limsup inequality in Theorem 1.1 we know there exists a sequence \( \{u_n\}_{n \in \mathbb{N}} \) with \( u_n \in L^1(D, \nu_n) \) such that \( \limsup_{n \to \infty} GTV_{n, \varepsilon}(u_n) \leq \sigma(TV(\chi_A, \rho^2)) \). It is straightforward to verify that the functionals \( GTV_{n, \varepsilon} \) satisfy the coarea formula:

\[
GTV_{n, \varepsilon}(u_n) = \int_{-\infty}^{\infty} GTV_{n, \varepsilon}(\chi_{\{u_n > t\}})ds.
\]

Fix \( 0 < \delta < \frac{1}{2} \). Then in particular:

\[
\int_{\delta}^{1-\delta} GTV_{n, \varepsilon}(\chi_{\{u_n > t\}})ds \leq GTV_{n, \varepsilon}(u_n).
\]

For every \( n \) there is \( s_n \in (\delta, 1-\delta) \) such that \( GTV_{n, \varepsilon}(\chi_{\{u_n > s_n\}}) \leq \frac{1}{1-2\delta} GTV_{n, \varepsilon}(u_n) \). Define \( A^\delta_n := \{u_n > s_n\} \). It is straightforward to show that \( \mathcal{X}_{A^\delta_n} \xrightarrow{TL^1} \mathcal{X}_A \) as \( n \to \infty \) and that \( \limsup_{n \to \infty} GTV_{n, \varepsilon}(A^\delta_n) \leq \frac{1}{1-2\delta} \sigma(TV(\chi_A, \rho^2)) \). Taking \( \delta \to 0 \) and using a diagonal argument provides sets \( \{A_n\}_{n \in \mathbb{N}} \) such that \( \mathcal{X}_{A_n} \xrightarrow{TL^1} \mathcal{X}_A \) as \( n \to \infty \) and \( \limsup_{n \to \infty} GTV_{n, \varepsilon}(\chi_{A_n}) \leq \sigma(TV(\chi_A, \rho^2)) \). \( \square 

\textit{Remark 5.1.} There is an alternative proof of the limsup inequality above. It is possible to proceed in a similar fashion as in the proof of the limsup inequality in Theorem 1.1. In this case, instead of approximating by Lipschitz functions, one would approximate \( \chi_A \) in \( TL^1 \) topology by characteristic functions of sets of the form \( G = E \cap D \) where \( E \) is a subset of \( \mathbb{R}^d \) with smooth boundary. As in the proof of Theorem 1.1 the key is to show that for step kernels \( \chi_G \) and \( \chi_G \) with smooth boundary.

\[
\lim_{n \to \infty} GTV_{n, \varepsilon}(\chi_G) = TV(\chi_G, \rho^2).
\]

To do so one needs a substitute for estimate (63). The needed estimate follows from the following estimate: For all \( G \) as above, there exists \( \delta_0 \) such that for all \( n \) for which \( \|1d - T_n\|_{\infty} \leq \delta_0 \),

\[
\int_D |\chi_G(x) - \chi_G(T_n(x))|dx \leq 4Per(E) \|1d - T_n\|_{\infty}.
\]

This estimate follows from the fact that if \( \chi_G(x) \neq \chi_G(T_n(x)) \) then \( d(x, \partial E) \leq |x - T_n(x)| \) and the fact that, for \( \delta \) small enough, \( \{x \in \mathbb{R}^d : d(x, \partial E) < \delta \} \leq 4Per(E)\delta \), which follows form Weyl's formula [56] for the volume of the tubular neighborhood. Noting that the perimeter of any set can be approximated by smooth sets (see Remark 3.42 in [4]) and using Remark 2.7 we obtain the limsup inequality for the characteristic function of any measurable set.

We furthermore remark that if one restricts the functional to the class of sets with specified volume (as in Example 1.4) then it is still possible to find for each set in the class, an approximating sequence of smooth sets satisfying the volume constraint. This follows by a careful modification to the density argument of Remark 3.43 in [4].
5.2. Extension to different sets of points. Consider the setting of Theorem 1.1. The only information about the points $X_i$ that the proof requires is the upper bound on the infinity transportation distance between $v$ and the empirical measure $v_n$. When the $X_i$ are i.i.d. distributed according to $v$ then this result is Theorem 2.5. Such randomness assumption is reasonable when modeling randomly obtained data points, but in other settings one may be in a situation where the points are more regularly distributed and/or given deterministically. In such setting, if one is able to obtain tighter bounds on transportation distance this would translate into better bounds on $\varepsilon(n)$ in Theorem 1.1 for which the $\Gamma$-convergence holds.

That is, if $X_1, \ldots, X_n, \ldots$ are the given points, let $v_n$ still be $\frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$. If one can find transportation maps $T_n$ from $v$ to $v_n$ such that

$$\limsup_{n \to \infty} n^{1/d} \|Id - T_n\|_\infty \leq C$$

for some nonnegative function $f : \mathbb{N} \to (0, \infty)$ then Theorem 1.1 would hold if

$$\lim_{n \to \infty} \frac{f(n)}{n^{1/d} \varepsilon_n} = 0.$$ 

We remark that $f$ must be bounded from below, since for any collection $V = \{X_1, \ldots, X_n\}$ in $D$, $\sup_{y \in D} \text{dist}(y, V) \geq cn^{-1/d}$ and thus $n^{1/d} \|Id - T_n\|_\infty \geq c$.

One special case is when $D = (0,1)^d$, $v$ is the Lebesgue measure and $X_1, \ldots, X_n, \ldots$ is a sequence of grid points on diadically refining grids. In this case, (65) holds with $f(n) = 1$ for all $n$ and thus $\Gamma$-convergence holds for $\varepsilon_n \to 0$ such that $\lim_{n \to \infty} \frac{1}{n^{1/d} \varepsilon_n} = 0$. Note that our results imply $\Gamma$-convergence in the $TL^1$ metric, however in this particular case, this is equivalent to the $L^1$-metric considered in 21 and 15 where for a function defined on the grid points we associate a function defined on $D$ by simply setting the function to be constant on the grid cells. This follows from Proposition 3.6.

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Appendix A. Proof of Proposition 2.4

Proof. Using the fact that $D$ has Lipschitz boundary and the fact that $\psi$ is bounded above and below by positive constants, Theorem 10.29 in [33] implies that for any $u \in C^\infty(D) \cap BV(D)$ there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq C^\infty_c(\mathbb{R}^d)$ with $u_n \to L^1(D) u$ and with $\int_D \nabla u - \nabla u_n \psi(x) dx \to 0$ as $n \to \infty$. Using a diagonal argument we conclude that in order to prove Proposition 2.4 it is enough to prove that for every $u \in BV(D)$ there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq C^\infty(D) \cap BV(D)$ with $u_n \to L^1(D) u$ and with $\int_D \nabla u_n \psi(x) dx \to TV(u; \psi)$ as $n \to \infty$.

Step 1: If $\psi$ is Lipschitz this is precisely the content of Theorem 3.4 in [8].

Step 2: If $\psi$ is not necessarily Lipschitz we can find a sequence $\{\psi_k\}_{k \in \mathbb{N}}$ of Lipschitz functions bounded above and below by the same constants bounding $\psi$ and with $\psi_k \searrow \psi$. The functions $\psi_k$ can be defined as in [33] (replacing $\rho$ with $\psi$).

Using Step 1, for a given $u \in BV(D)$ and for every $k \in \mathbb{N}$ we can find a sequence $\{u_{n,k}\}_{n \in \mathbb{N}}$ with $u_{n,k} \to L^1(D) u$ and with $\int_D \nabla u_{n,k} \psi_k(x) dx \to TV(u; \psi_k)$ as $n \to \infty$. By 12 and by the dominated convergence theorem we know that $TV(u; \psi_k) = \int_D \psi_k(x) Du(x) \to \int_D \psi(x) Du(x) = TV(u; \psi)$ as $k \to \infty$. 

Therefore, a diagonal argument allows us to conclude that there exists a sequence \( \{k_n\}_{n \in \mathbb{N}} \) with the property that, \( u_{n,k_n} \to L^1(D) u \) and \( \int_D |\nabla u_n(x)| \psi_{k_n}(x) dx \to TV(u; \psi) \) as \( n \to \infty \). Taking \( u_n := u_{n,k_n} \) and using the fact that that \( \psi \leq \psi_{k_n} \) we obtain:

\[
\limsup_{n \to \infty} \int_D |\nabla u_n(x)| \psi(x) dx \leq \lim_{n \to \infty} \int_D |\nabla u_n(x)| \psi_{k_n}(x) dx = TV(u; \psi).
\]

Since \( u_n \to L^1(D) u \), the lower semicontinuity of \( TV(\cdot, \psi) \) implies that \( \liminf_{n \to \infty} \int_D |\nabla u_n(x)| \psi(x) dx \geq TV(u; \psi) \). The desired result follows. \( \square \)

**APPENDIX B. THE MATCHING RESULTS FOR NON-CONSTANT DENSITIES**

**B.1. The matching results for non-constant densities:** \( d \geq 3 \). Given a box \( \Pi_{i=1}^d [a_i, b_i] \) in \( \mathbb{R}^d \) we define the *aspect ratio* to be the maximal ratio between the lengths of any two of its sides, that is the maximum of \( (b_i - a_i)/(b_j - a_j) \) over \( i, j = 1, \ldots, d \). We say that a collection of boxes partitions \( D \) if all of its boxes are subsets of \( D \), their union is \( D \) and their interiors are mutually disjoint.

**Theorem B.1.** Let \( d \geq 3 \). Assume \( \nu \) is a probability measure on \( D = (0, 1)^d \) with density \( \frac{1}{\lambda} \leq \rho \leq \lambda \) for some \( \lambda > 1 \). Let \( X_1, \ldots, X_n \) be i.i.d. random variables with distribution given by the measure \( \nu \) and let \( \nu_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_i} \) be the empirical measure.

Then, there is a constant \( C > 0 \) such that with probability one there exists a sequence of transportation maps \( \{T_n\}_{n \in \mathbb{N}} \) from \( \nu \) to \( \nu_n \) (\( T_n \nu = \nu_n \)) with

\[
\limsup_{n \to \infty} n^{1/d} \frac{\|T_n - Id\|_\infty}{(\log n)^{1/d}} \leq C.
\]

**Proof.** The proof is a straightforward generalization of the proof by Shor and Yukich \[44\] who considered the case when \( \nu \) is the Lebesgue measure. The idea is to apply a series of mappings to the domain \( D \) at finer and finer scales, so that the composition of these mappings, \( \Phi \), pushes \( \nu_n \) to a measure \( \bar{\nu}_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i} \) such that with high probability (that is probability at least \( 1 - \beta n^{-\alpha/2} \) for some \( \beta > 0 \) and \( \alpha > 4 \)) for all \( i = 1, \ldots, n \)

\[
|X_i - \bar{X}_i| \leq C \frac{(\log n)^{1/d}}{n^{1/d}}
\]

and there exists a partition of \( D \) into boxes \( S_j, j = 1, \ldots, n \) such that

\[
\nu(S_j) = \nu_n(S_j) \quad \text{and} \quad \text{diam}(S_j) \leq C \frac{(\log n)^{1/d}}{n^{1/d}}.
\]

For each \( S_j \) consider any transportation plan \( T_j : S_j \rightarrow S_j \) between \( \nu \) restricted to \( S_j \) and \( \bar{\nu}_n \) restricted to \( S_j \). Then, using \( (67) \), \( \max_j \|T_j - Id\|_\infty \leq \text{diam}(S_j) \leq C \frac{(\log n)^{1/d}}{n^{1/d}} \). Consequently the mapping \( T : D \rightarrow D \) defined by \( T(x) = \Phi^{-1}(T_j(x)) \) if \( x \in S_j \) pushes forward \( \nu \) to \( \nu_n \) and is such that \( \|T_n - Id\|_{L^\infty} \leq 2C \frac{(\log n)^{1/d}}{n^{1/d}} \) as desired. The almost sure convergence follows by using the Borel-Cantelli lemma.

The construction of the mapping \( \Phi \) relies on a slight generalization of the AKT (Ajtai-Komlós-Tusnády) algorithm \[2,44\] to random variables drawn from the measure \( \nu \). Here we outline the main ideas. For simplicity we consider the case \( d = 3 \). First divide the domain \( D \) horizontally into two “halves” \( H_1 = [0, c_1] \times [0, 1]^2 \), \( H_2 = (c_1, 1] \times [0, 1]^2 \) of the same \( \nu \) measure. That is, find \( c_1 \) so that \( \nu(H_1) = \frac{1}{2} \). Then linearly transform each “half”, \( H_j, j = 1, 2 \) so that the \( \nu \)-volume of the transformed half is equal to \( \nu_0(H_j) \) (that is \( \frac{1}{n} \) times the number of points \( X_i \) in the half). Repeat the process with each of the two subdomains, but instead of dividing horizontally divide each half vertically. In the next stage repeat the process again, but divide the four subdomains laterally. Repeat the process with “halving” and transforming the boxes until the stage of \( k \sim \log(n/\log n) \), when all of the boxes have \( \nu \) volume of the scale \( \log(n)/n \). Let \( \Phi \) be the composition of the transformations from the first to the \( k \)-th stage and
let $S_1, \ldots, S_m$ be the boxes after the stage $k$. How large of a transformation was needed after halving each box depends on the distribution of random points over the halves, which is a binary random variable (with $p = \frac{1}{2}$ since the boxes were chosen to have the same $\nu$ volume). Shor and Yukich showed that concentration inequalities for the binary random variables yield that with very high probability for all $x \in D$, $|\Phi(x) - x| \leq C \frac{(\log n)^{1/d}}{n^{d/2}}$ and furthermore that the diameter of the $S_j$ is less than $C \frac{(\log n)^{1/d}}{n^{d/2}}$. The estimates in [44], proved for $\nu$ being the Lebesgue measure, hold when general $\nu$ is considered, only that constants now depend on $\lambda$. We note that by construction $\nu(S_j')$ is equal to the number of points in $S_j \cap \{\Phi(X_i) : i = 1, \ldots, n\}$ divided by $n$. That is $\nu(S_j) = \tilde{\nu}_n(S_j)$. \hfill \Box

B.2. **The matching results for non-constant densities:** $d = 2$. Let $D = (0, 1)^2$ and let $\rho : D \to \mathbb{R}$ be a measurable function with values between $\frac{1}{\lambda}$ and $\lambda$ for some $\lambda > 1$ and such that $\int_D \rho(x) dx = 1$. Let $\nu$ be the probability measure with density $\rho$.

We start with a lemma that is true in any dimension.

**Lemma B.2.** Given $\rho$ and $\nu$ be as above. For any $n \in \mathbb{N}$ there exists a collection of boxes $\{Q_i : i = 1, \ldots, n\}$ that partitions $D$ such that the aspect ratio of all boxes is less than $3\lambda^2$ and their volume according to $\nu$ is $1/n$.

**Proof.** The construction is based on a recursion scheme. We start dividing $D$ into smaller and smaller boxes, whose $\nu$-volume is always an integer multiple of $1/n$. The starting partition has only one box, namely $D$. While constructing the algorithm, we show that the aspect ratio remains below $3\lambda^2$.

In each step of the algorithm find the box with the largest $\nu$-volume, $A = \Pi_{i=1}^d [a_i, b_i]$. Let $k/n$ be its volume and let $e_j$ be the direction of its longest side. We can assume without the loss of generality that $j = 1$. Suppose first that $k$ is even. Then, there exists a unique $c \in (a_1, b_1)$ such that the boxes $A_1 = [a_1, c] \times \Pi_{i=2}^d [a_i, b_i]$ and $A_2 = [c, a_2] \times \Pi_{i=2}^d [a_i, b_i]$ have $\nu$ volumes $k/2n$. Furthermore since $\frac{1}{\lambda} \leq \rho \leq \lambda$ it follows that $\frac{1}{2\lambda^2} \leq \frac{|b_j - c|}{|c - a_1|} \leq \lambda^2$. Combining this with the facts that $[a_1, b_1]$ was the longest side of $A$ and that the aspect ratio of $A$ is less than $3\lambda^2$ it follows that the aspect ratio of $A_1$ and $A_2$ is less than $3\lambda^2$.

If $k$ is odd then there is a unique $c \in (a_1, b_1)$ such that for $A_1$ and $A_2$ as above the $\nu$ volumes are $(k + 1)/(2n)$ and $(k - 1)/(2n)$. The rest of the argument is analogous to the case when $k$ is even. \hfill \Box

**Remark B.3.** In dimension $d = 2$, the boxes $Q_i$ from the previous lemma satisfy $\text{diam}(Q_i) \leq \frac{\sqrt{\lambda^2}}{\sqrt{\lambda}}$.

The proof of Leighton and Shor [32] depends on discrepancy estimates over all regions $R$ formed by squares from a suitable regular grid $G'$ defined on $D$. By discrepancy we mean the difference between $\nu(R)$ and $\nu_n(R)$ for a given region $R$. Obtaining a uniform bound on the discrepancy over all regions $R$ can be interpreted as obtaining probabilistic estimates on the supremum of a stochastic process indexed by the class of regions $R$ considered. A conceptually clear and efficient proof of the matching result, based on obtaining upper bounds of stochastic processes, was presented by Talagrand [49, 50]. We follow the framework of Talagrand and start by stating a general result on obtaining bounds on the supremum of more general stochastic processes (Section 1 in [49]).

Consider a metric space $(Y, d)$. For $n \in \mathbb{N}$ define

$$e_n(Y, d) = \inf_{Y_n \in Y} \sup_{y \in Y_n} d(y, Y_n),$$

where the infimum is taken over all subsets $Y_n$ of $Y$ with cardinality less than $2^{2^n}$.

Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of partitions of $Y$. This sequence of partitions is called admissible if it is increasing (in the sense that for every $n$, $A_{n+1}$ is a refinement of $A_n$) and it is such that the cardinality of $A_n$ is no bigger than $2^{2^n}$. For a given $y \in Y$ and $\{A_n\}_{n \in \mathbb{N}}$ admissible, $A_n(y)$ represents the unique set in $A_n$ containing $y$. 

For an $\alpha > 0$, consider

$$\gamma_\alpha(Y, d) = \inf \sup \sum_{y \in Y} 2^{n/\alpha} \text{diam}(A_n(y)),$$

where $\text{diam}(A_n(y))$ represents the diameter of the set $A_n(y)$ (using the distance $d$) and where the infimum is taken over all $\{A_n\}_{n \in \mathbb{N}}$ admissible sequences of partitions of $Y$. With these definitions we can now state Theorem 1.2.9 in [49].

**Theorem B.4.** Let $Y$ be a set and let $d_1, d_2$ be two distance functions defined on $Y$. Let $\{Z_y\}_{y \in Y}$ be a stochastic process satisfying: for all $y, y' \in Y$ and all $u > 0$

$$\mathbb{P}(|Z_y - Z_{y'}| \geq u) \leq 2 \exp \left( -\min \left( \frac{u^2}{d_2(y, y')^2}, \frac{u}{d_1(y, y')} \right) \right),$$

and also $\mathbb{E}[Z_y] = 0$ for all $y \in Y$. Then, there is a constant $L > 0$ large enough, such that for all $u_1, u_2 > 0$

$$\mathbb{P}(\sup_{y \in Y}|Z_y - Z_{y_n}| \geq L(\gamma_1(Y, d_1) + \gamma_2(Y, d_2)) + u_1D_1 + u_2D_2) \leq L \exp(- \min \{u_1^2, u_1\}),$$

where $D_1 = 2\sum_{n \geq 0} e_n(Y, d_1)$ and $D_2 = 2\sum_{n \geq 0} e_n(Y, d_2)$.

One of the consequences of the previous theorem is that in order to prove a tail estimate of the supremum of the stochastic process $\{Z_y\}_{y \in Y}$ like the one in (69), one needs to do two things. First, estimate the quantities $\gamma_1(Y, d_1)$, $\gamma_2(Y, d_2)$, $D_1$ and $D_2$. Note that these quantities depend only on the distances $d_1, d_2$ and hence are not a priori related to the process $\{Z_y\}_{y \in Y}$. Secondly, relate the stochastic process $\{Z_y\}_{y \in Y}$ with the distances $d_1, d_2$ by establishing condition (68).

**Theorem B.5.** Assume $\nu$ is a probability measure on $D = (0, 1)^2$ with density $\frac{1}{x} \leq \rho \leq \lambda$ for some $\lambda > 1$. Let $X_1, \ldots, X_n$ be i.i.d. random variables with distribution given by the measure $\nu$ and let $\nu_n := \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$ be the empirical measure. Then, there exists a constant $L > 0$ such that with probability at least $1 - L \exp(- (\log(n)^{3/2})/L)$ there is a transportation map $T_n : D \to D$ with

$$||T_n - \text{Id}||_{L^1(D)} \leq L (\log(n))^{3/4}/\sqrt{n}.$$  

In particular, there is a constant $L > 0$ such that with probability one there exists a sequence of transportation maps $\{T_n\}_{n \in \mathbb{N}}$ from $\nu$ to $\nu_n$ $(T_n \nu = \nu_n)$ with

$$\limsup_{n \to \infty} \frac{\sqrt{n}||T_n - \text{Id}||_{L^\infty}}{(\log(n))^{3/4}} \leq L.$$  

This result is based on the proof by Talagrand of Leighton and Shor theorem who proved the previous result in the case when $\nu$ is the Lebesgue measure. See Section 3.5 in [49]. We sketch some of the main steps in the proof by Talagrand and give the details on how to generalize it to non-constant densities.

**Proof.** In what follows $L > 0$ may increase from line to line.

**Discrepancy estimates:** Let $l_1$ be the largest integer such that $2^{-l_1} \geq \frac{(\log(n))^{3/4}}{\sqrt{n}}$. Consider $G$ to be the regular grid of mesh $2^{-l_1}$ given by

$$G = \left\{(x_1, x_2) \in [0, 1]^2; 2^{l_1}x_1 \in \mathbb{N} \text{ or } 2^{l_1}x_2 \in \mathbb{N}\right\}.$$  

A vertex of the grid $G$ is a point $(x_1, x_2)$ in $[0, 1]^2$ such that $2^{l_1}x_1 \in \mathbb{N}$ and $2^{l_1}x_2 \in \mathbb{N}$. A square of the grid $G$ is a square of side length equal to $2^{-l_1}$ and whose edges belong to $G$. The edges are included in the squares.
For a given vertex \( w \) of \( G \) and a given integer \( k \), consider \( \mathcal{C}(w,k) \) the set of simple closed curves that lie on \( G \) which contain the vertex \( w \) and have length \( l(C) \leq 2^k \). Note that every closed simple curve \( C \) in \( \mathbb{R}^2 \) divides the space into two regions, one of which is bounded; this latter one is called the interior of the curve \( C \) and is denoted by \( C^\circ \). For \( C, C' \in \mathcal{C}(w,k) \) set \( d_1(C,C') = 1 \) if \( C \neq C' \) and \( d_1(C,C') = 0 \) if \( C = C' \). Also set \( d_2(C,C') = \sqrt{n}\|\chi_{C^\circ} - \chi_{C'^\circ}\|_{L^2(D)} \).

**Claim 1:** For a given \( w \) of \( G \) and a given integer \( k \) with \( k \leq l_1 + 2 \), there exists \( L > 0 \) large enough such that with probability at least \( 1 - L \exp(-\frac{(\log(n))^{3/2}}{L}) \)

\[
\sup_{C \in \mathcal{C}(w,k)} \left| \sum_{i \leq n} (\chi_{C^\circ}(X_i) - v(C^\circ)) \right| \leq L 2^k \sqrt{n}(\log(n))^{3/4}.
\]

To prove the claim, the idea is to study the supremum of the stochastic process \( \{Z_C\}_{C \in \mathcal{C}(w,k)} \) where

\[
Z_C := \frac{1}{L} \sum_{i \leq n} (\chi_{C^\circ}(X_i) - v(C^\circ)).
\]

For fixed \( C, C' \in \mathcal{C}(w,k) \) one can write the difference \( Z_C - Z_{C'} \) as

\[
Z_C - Z_{C'} = \sum_{i \in \mathbb{N}} Z_i,
\]

where \( Z_i = \frac{1}{L} (\chi_{C^\circ}(X_i) - \chi_{C'^\circ}(X_i) - v(C^\circ) + v(C'^\circ)) \). The random variables \( \{Z_i\}_{i \leq n} \) are independent, identically distributed with mean zero. They satisfy \( |Z_i| \leq \frac{2}{L} \) and moreover, their variance \( \sigma^2 \) is bounded by

\[
\sigma^2 \leq \frac{1}{L^2} \mathbb{E} \left[ |\chi_{C^\circ}(X_i) - \chi_{C'^\circ}(X_i)|^2 \right] \leq \frac{\lambda}{L^2} \|\chi_{C^\circ} - \chi_{C'^\circ}\|_{L^2(D)}^2.
\]

Using Bernstein’s inequality and choosing \( L > 0 \) to be large enough, we obtain

\[
\mathbb{P} \left( |Z_C - Z_{C'}| \geq u \right) \leq 2 \exp \left( -\frac{u^2}{n - \|\chi_{C^\circ} - \chi_{C'^\circ}\|_{L^2(D)}^2 + u} \right) = 2 \exp \left( -\min \left( \frac{u^2}{d_2(C, C')^2}, \frac{u}{d_1(C, C')} \right) \right).
\]

The proof of Proposition 3.4.3 in [49], provides the following estimates \( \gamma_1(\mathcal{C}(w,k), d_1) \leq L 2^k \sqrt{n} \), \( \gamma_2(\mathcal{C}(w,k), d_2) \leq L 2^k \sqrt{n}(\log(n))^{3/4} \), \( D_1 \leq 2(k + l_1 + 1) \) and \( D_2 \leq L 2^k \sqrt{n} \). Setting \( u_1 = (\log(n))^{3/2} \) and \( u_2 = (\log(n))^{3/4} \) one can use Theorem B.4 (with \( Y = \mathcal{C}(w,k) \), \( d_1 \), \( d_2 \) as above and \( y_0 = \{w\} \)) to obtain the claim.

Considering all possible vertices \( w \) of \( G \) and all possible integers \( k \) with \(-l_1 \leq k \leq l_1 + 2 \). It is a direct consequence of Claim 1 above that with probability at least \( 1 - L \exp(-\frac{(\log(n))^{3/2}}{L}) \),

\[
\sup_{C} \left| \sum_{i \leq n} (\chi_{C^\circ}(X_i) - v(C^\circ)) \right| \leq L L(\mathcal{C}) \sqrt{n}(\log(n))^{3/4},
\]

where the supremum is taken over all \( C \) closed, simple curves on \( G \). See the proof of Theorem 3.4.2 in [49]. Let us denote by \( \Omega_n \) the event for which (74) holds.

**Enlarging Regions:** Consider an integer \( l_2 \) with \( l_2 < l_1 \). We consider \( G' \) the grid defined as in (72) but with mesh size \( 2^{-l_2} \). Note that in particular \( G' \subseteq G \). Let \( R \) be a union of squares of the grid \( G' \). One can define \( R' \) to be the region formed by taking the union of all the squares in \( G' \) with at least one side contained in \( R \). With no change in the proof of Theorem 3.4.1 in [49], it follows from the discrepancy estimates obtained previously that in the event \( \Omega_n \) one has

\[
\sum_{i \leq n} \chi_{R_i} \geq \# \{ i \leq n : X_i \in R \}
\]

for all regions \( R \) formed with squares from \( G' \), provided that \( 2^{-l_2} \geq \frac{\sqrt{2} \log(n)}{n} \). This is saying that given the discrepancy estimates obtained previously, in the event \( \Omega_n \), for any region \( R \) formed by taking the union of squares in \( G' \), one can enlarge \( R \) a bit to obtain a region \( R' \) in such a way that the area of the enlarged region \( R' \) according to \( v \) is greater than the area of the original region \( R \) according to \( v_n \). It is worth remarking that the restriction to the number \( 2^{-l_2} \) (the
mesh size of $G'$), for this to be possible, coincides with the scaling for the transportation cost we are after.

**Marriage lemma and construction of $T_n$:** We choose $l_2$ to be the largest integer satisfying $2^{-l_2} \geq \frac{2nT}{\sqrt{n}} ((\log(n))^{3/4}$. Consider $\{Q_1, \ldots, Q_n\}$ the boxes constructed in Lemma B.2. For $i \in \{1, \ldots, n\}$ let

$$B_i = \left\{ j \leq n : \text{dist}(X_i, Q_j) \leq 2\sqrt{2} \cdot 2^{-l_2} \right\}.$$

**Claim 2:** In the event $\Omega_n$, there is a bijection $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ with $\pi(i) \in B_i$ for all $i$.

By the Hall marriage lemma, to prove this claim it is enough to prove that for every $I \subseteq \{X_1, \ldots, X_n\}$, the cardinality of $\cup_{j \in I} B_i$ is greater than the cardinality of $I$. Fix $I \subseteq \{1, \ldots, n\}$ and denote by $R_I$ the region formed with the squares of $G'$ that contain at least one of the points $X_i$ with $i \in I$. Now, take

$$J = \left\{ j \leq n : Q_j \cap (R_I)^c \neq \emptyset \right\},$$

then $J \subseteq \cup_{j \in I} B_i$. From the properties of the boxes $Q_i$ and from (75) it follows that $\# \cup_{j \in I} B_i \geq \# I = n \nu(\cup_{j \in I} Q_j) \geq \nu((R_I)^c) \geq \# I$. This proves the claim.

Finally, we construct the map $T_n$. Indeed, for $x$ in the interior of $Q_i$, set $T_n(x) = X_{\pi^{-1}(i)}$ and if $x$ does not belong to the interior of any of the boxes $Q_i$, then $T_n(x) = X_1$. From the properties of the boxes $Q_i$, it is straightforward to check now that $T_n|_{\partial \mathcal{V}} = \nu$, and that $\|T_n - Id\|_{L^\infty(\mathcal{D})} \leq L \frac{((\log(n))^{3/4}}{\sqrt{n}}$ (see remark B.3). The last part of the Proposition follows directly from Borel-Cantelli lemma.

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