PROPERTIES OF 3-MANIFOLDS FOR RELATIVISTS

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Abstract

In canonical quantum gravity certain topological properties of 3-manifolds are of interest. This article gives an account of those properties which have so far received sufficient attention, especially those concerning the diffeomorphism groups of 3-manifolds. We give a summary of these properties and list some old and new results concerning them. The appendix contains a discussion of the group of large diffeomorphisms of the $l$-handle 3-manifold.

Introduction

In the canonical formulation of general relativity and in particular in all approaches to canonical quantum gravity it emerges that the diffeomorphism groups of closed 3-manifolds are of particular interest. Here one is interested in a variety of questions concerning either the whole diffeomorphism group or certain subgroups thereof. More precisely, one is e.g. interested in whether the 3-manifold admits orientation reversing self-diffeomorphisms [So] (following this reference we will call a manifold chiral, iff it admits no such diffeomorphism) or what diffeomorphisms not connected to the identity there are in the subgroup fixing a frame [FS][Is][So][Wi1]. Other topological invariants of this latter subgroup are also argued to be of interest in quantum gravity [Gi1]. In the sum over histories approach one is also interested in whether a given 3-manifold can be considered as the spatial boundary of a spin-Lorentz 4-manifold [GH][Gi2] (following ref. [Gi2] such 3-manifolds will be called nuclear). Motivations for studying these questions may be found in the cited literature and references therein.

The purpose of this paper is to present an account of results (partially new) on these questions in a comprehensive way. A simple but non-trivial example on which a

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typical diffeomorphisms group can be studied is presented in the appendix. Throughout the paper we shall take the word 3-manifold to also imply closed, connected and oriented unless stated otherwise. Since 3-manifolds are not classified, we follow the standard route by presenting the results for some prime (defined below) 3-manifolds and indicate how the quantity in question behaves under taking connected sums (defined below). Let us now recall some basic facts from the subject of 3-manifolds. A standard textbook is [He], which also contains all the relevant references which we do not list separately.

Definition 1. Let $\Sigma, \Sigma_1$ and $\Sigma_2$ be 3-manifolds. We say that $\Sigma$ is the connected sum of $\Sigma_1$ and $\Sigma_2$, in symbols: $\Sigma = \Sigma_1 \# \Sigma_2$, if the following conditions are satisfied: There exist two open 3-balls, $B_i \subset \Sigma_i$, and orientation preserving embeddings, $h_i : R_i \to \Sigma$, where $R_i = \Sigma_i - B_i$, such that $h_1(R_1) \cap h_2(R_2) = h_1(\partial R_1) = h_2(\partial R_2)$ and $h_1(R_1) \cup h_2(R_2) = \Sigma$.

The operation of taking the connected sum is well defined, commutative and associative. Taking the connected sum of any 3-manifold $\Sigma$ with the 3-sphere results in a 3-manifold diffeomorphic to $\Sigma$.

Definition 2. A 3-manifold is called a prime 3-manifold (or simply prime), iff it is not the connected sum of two 3-manifolds none of which is diffeomorphic to the 3-sphere.

Theorem 1 (Kneser). Every 3-manifold $\Sigma$ is diffeomorphic to a finite connected sum of prime 3-manifolds $\Pi_1, ..., \Pi_n$:

$$\Sigma = \biguplus_{i=1}^{n} \Pi_i .$$

Theorem 2 (Milnor). This decomposition is unique, in the sense that for any other decomposition of $\Sigma$ into prime 3-manifolds $\{\Pi'_1, ..., \Pi'_m\}$ it follows that $n = m$ and that there exist orientation-preserving diffeomorphisms $\phi_i : \Pi_i \to \Pi'_{\sigma(i)}$, $i = 1, ..., n$, for some permutation $\sigma$.

Note that since the manifolds are all oriented, a prime-manifold, $\Pi$, and its oppositely oriented version, $(-\Pi)$, form different primes (in the sense of Theorem 2), iff $\Pi_i$ is chiral.

A 3-manifold $\Sigma$ is called irreducible, if every embedded 2-sphere in $\Sigma$ bounds a 3-ball. Clearly, an irreducible 3-manifold is prime. The converse is almost true with only one exception: The “handle” 3-manifold, $S^1 \times S^2$, is the only non-irreducible prime. Irreducibility also implies that the 3-manifold has trivial second homotopy group. (Proof: The so called sphere-theorem (see e.g. [He]) implies for a non-trivial second homotopy group the existence of an element different from the identity which can be represented by an embedded sphere. However, irreducibility enforces each embedded sphere to be contractible.)
Section 1: Properties of the Diffeomorphism Group

Associated to a manifold $\Sigma$ we introduce the group of $C^\infty$-diffeomorphisms and several subgroups thereof. To define these, let $\infty \in \Sigma$ denote a fixed, preselected point. We define:

$$D(\Sigma) := \{ \phi = C^\infty - \text{diffeomorphism of } \Sigma \} \quad (1.1a)$$
$$D_\infty(\Sigma) := \{ \phi \in D(\Sigma) / \phi(\infty) = \infty \} \quad (1.1b)$$
$$D_F(\Sigma) := \{ \phi \in D_\infty(\Sigma) / \phi_*|_{\infty} = id \}, \quad (1.1c)$$

where $\phi_*|_{\infty} : T_\infty(\Sigma) \to T_\infty(\Sigma)$ is the differential at $\infty$ of the map $\phi$. In words, $D$ contains all diffeomorphisms, $D_\infty$ those fixing $\infty$, and $D_F$ those which in addition fix the frames at $\infty$. There are clearly no orientation-reversing diffeomorphisms in $D_F$ but we take $D$ and $D_\infty$ to include orientation-reversing diffeomorphisms, should they exist. Additional superscripts + or 0 may then refer to the normal subgroup of orientation-preserving diffeomorphisms or the identity component respectively. Note also that $D_F$ is a normal subgroup of $D_\infty$, but $D_\infty$ is no normal subgroup of $D$.

More subgroups of $D$ may be introduced in the following way: Let $N \subset \Sigma$ be a closed subset. Then

$$D(\Sigma, rel N) = \{ \phi \in D(\Sigma) / \phi |_N = id \} \quad (1.1d)$$

denotes the diffeomorphisms fixing $N$ (thus generalizing (1.1b)). If $B_\epsilon$ denotes any closed 3-disc neighbourhood of $\infty$ with diameter $\epsilon$ (in some fiducial metric) then clearly $D(\Sigma, rel B_\epsilon) \subset D_F(\Sigma)$ $\forall \epsilon$ and one may, loosely speaking, regard $D_F(\Sigma)$ as a limit of $D(\Sigma, rel B_\epsilon)$ for $\epsilon \to 0$. If one is only interested in the topological features of the diffeomorphism groups, as we are, one may indeed replace any $D(\Sigma, rel B_\epsilon)$ with $D_F(\Sigma)$, or vice versa. Now, it is in fact $D_F(\Sigma)$ (or, equivalently, $D(\Sigma, rel B_\epsilon)$) which is of primary interest in General Relativity (we refer to [Gi1] for a deeper discussion of this point) and to the study of which we now turn.

The fundamental theorem, first announced in [RS], then elaborated on in [HL] and [HM][McC1], aims to make precise how the diffeomorphisms $D_F(\Sigma)$ for any non-prime 3-manifold $\Sigma$ are built up from those of its prime constituents plus extra ones, and how these extra ones can be generated by certain basic operations which allow for more or less intuitive geometric interpretations. In order to fully appreciate this result, we need to go through some details and preliminary results. At the end, when grouped together, these form what we call the theorem of Rourke de Sá, although the proofs of the claims in [RS],

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(1) The reason for this name is, that in the study of asymptotically isolated (flat) systems, where the underlying 3-manifold is non compact with one asymptotic region, it is convenient to work with the one point compactification by adding a point at infinity.
given in [HL] and [HM], do not seem to have been given in a complete form by the authors of [RS]. In what follows, $D$’s and $S$’s with lower case indices $i, j, k$ denote open 3-disks and 2-spheres respectively. The irreducible primes are notationally separated from the reducible one, $S^1 \times S^2$, and will be denoted by $P_i$.

Let

$$\Sigma = \left( \bigsqcup_{i=1}^{n} P_i \right) \uplus \left( \bigsqcup_{j=1}^{l} S^2 \times S^1 \right)$$

be constructed in the following way: Firstly, take $P_i' = P_i - D_i$, $1 \leq i \leq n$, $\partial P_i' = S_i'$, and $l$ copies of $I \times S^2$, labelled $I \times S_j^2$, $i \leq j \leq l$, such that $\partial (I \times S_j^2) = S_{j,1}' \cup S_{j,2}'$. We call them the factors of $\Sigma$. Secondly, take a closed 3-disc, $D$, ($\partial D = S_0$) and remove $n + 2l$ mutually disjoint open 3-discs:

$$B : = D - \left\{ \bigcup_{i=1}^{n} D_i \cup \bigcup_{(j,k) = (1,1)}^{(l,2)} D_{(j,k)} \right\}$$

$$\partial B = S_0 \cup \bigcup_{i=1}^{n} S_i \cup \bigcup_{(j,k) = (1,1)}^{(l,2)} S_{(j,k)}.$$  

We shall call $B$ the base and $S_0$ the sphere at $\infty$. Thirdly, eliminate all but $S_0$ of the boundary components of $\partial B$ by glueing the boundaries of the factors onto the boundary components of the base by using some identification diffeomorphisms

$$G_i : S_i' \to S_i \quad \text{and} \quad G_{(i,k)} : S_{(i,k)}' \to S_{(i,k)},$$

such that the resulting space carries an orientation which is compatible with the individual orientations given to the factors and the base beforehand. For this, the maps in (1.4) must be orientation-reversing with respect to the induced orientations. This construction is unambiguous since any two diffeomorphisms of 2-spheres which either preserve or reverse orientation are isotopic. Finally, we cap-off $S_0$ with a 3-disk $D_0$ to obtain $\Sigma$. The point $\infty$ may now be taken as the centre of the disc $D_0$ and $D_0$ itself may be identified with one of the $B_\varepsilon$’s mentioned above.

Using this decomposition, we can now define four classes of diffeomorphisms of $\Sigma$ that will suffice to generate all orientation preserving diffeomorphism up to isotopy (theorem 3 below). A proof may be found in [McC1]. To unify the notation we shall commit a slight abuse of language in not always distinguishing between $D_F(\Sigma)$ and $D(\Sigma, relD_0)$ or $D_F(P_i)$ and $D(P_i', relS_i')$.

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$^{(2)}$ Since, in the notation above, it will be taken as the boundary $\partial B_\varepsilon$ of a neighbourhood of $\infty$. 

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1. **Internal diffeomorphisms:** These are diffeomorphism that reduce to the identity when restricted to $B$. The individual supports, $P'_i$ and $I \times S^2_i$, are disjoint in $\Sigma$ and elements with support in two disjoint sets of factors clearly commute. For the handles $S^1 \times S^2$ one can show that $D(I \times S^2_i, \text{rel } \partial I \times S^2_i)$ is homotopy equivalent to the fibre preserving diffeomorphisms not exchanging the boundary components ($I \times S^2$ being viewed as the trivial $S^2$ bundle over $I$), and hence to $\Omega(SO(3))$, the space of based loops of $SO(3)$. We write:

$$D_{\text{int}} = \prod_{i=1}^n D_F(P'_i) \times \prod_{j=1}^l \Omega(SO(3)).$$

(1.5)

2. **Exchange diffeomorphisms:** Given any two diffeomorphic factors $P'_i$, $P'_j$ (resp. any two $I \times S^2_i$, $I \times S^2_j$) and the associated diffeomorphism $\phi_{ji} : P'_i \to P'_j$ (resp. $I \times S^2_i \to I \times S^2_j$). Given also a diffeomorphism $\psi_{ij}$ of $B$, exchanging $S_i$ and $S_j$ (resp. $S_{(i,k)}$ and $S_{(j,k)}$ for $k = 1, 2$) in such a way, that outside some neighbourhood in $B$ containing these but no other boundary spheres $\psi_{ij}$ restricts to the identity. Let it also be adjusted in such a way that it be compatible with the glueing instructions (1.4):

$$G_j \circ \phi_{ji} \bigg|_{S'_i} = \psi_{ji} \bigg|_{S_i} \circ G_i \quad \text{resp.} \quad G_{(j,k)} \circ \phi_{ji} \bigg|_{S'_{(i,k)}} = \psi_{ji} \bigg|_{S_{(i,k)}} \circ G_{(i,k)}.$$

(1.6)

Simultaneously performing $\phi_{ji}$ and $\psi_{ij}$ now defines a diffeomorphism of $\Sigma$ which we call an exchange of $P_i$ and $P_j$ (resp. $I \times S_i$ and $I \times S_j$). Any diffeomorphism generated by the exchanges of the type just described is then called an exchange diffeomorphism of $\Sigma$.

3. **Spin diffeomorphisms:** These are like the exchange diffeomorphisms but concern only the two ends of handles. More precisely, take an orientation preserving diffeomorphism $\phi_i : I \times S^2_i \to I \times S^2_i$, exchanging $S'_i$ and $S''_i$ and a diffeomorphism $\psi_i$ of $B$ exchanging $S_{(i,1)}$ and $S_{(i,2)}$ such that outside some neighbourhood in $B$ containing these but no other boundary spheres $\psi_i$ restricts to the identity. Let it also be adjusted in such a way that it be compatible with the glueing instructions:

$$G_{(i,1)} \circ \phi_i \bigg|_{S'_{(i,2)}} = \psi_i \bigg|_{S_{(i,2)}} \circ G_{(i,2)} \quad \text{and} \quad G_{(i,2)} \circ \phi_i \bigg|_{S'_{(i,1)}} = \psi_i \bigg|_{S_{(i,1)}} \circ G_{(i,1)}.$$

(1.7)

Performing $\phi_i$ and $\psi_i$ simultaneously defines a diffeomorphism of $\Sigma$ which we call a spin of the $i'$th handle. All these generate the spin diffeomorphisms of $\Sigma$.

4. **Slide diffeomorphisms:** The diffeomorphisms mentioned so far leave $B$ invariant as a set. Slides represent those elements that mix the interior of the factors with the base $B$. Consider a fixed factor $P'_i$ (resp. $I \times S^2_i$) and a nonintersecting noncontractible and oriented loop $\gamma$ in the complement of all other prime factors through $p \in B$ ($\gamma$ thus represents a nontrivial element in the obvious subgroup $\pi_1(P'_i)$ (resp. $\pi_1(S^1 \times S^2)$)
of $\pi_1(\Sigma)$. Now choose $j \neq i$, cut $\gamma$ at $p$ and connect the two ends to two different (say antipodal) points of $S_j$ so that the curve is still nonintersecting. A thin closed neighbourhood, $N$, of this ‘curve-attached-to-sphere’ has the topology of a solid 2-torus with an open ball removed from its interior. $N$ has two boundary components, the two-sphere $S_j$ as inner boundary, and a two-torus $T$ as outer boundary. An inner collar neighbourhood, $T'$ of $T$, denoted by $[0, 1] \times T$ such that $1 \times T = T$, can be coordinatized by $(t, \theta, \varphi)$, where the $\varphi$ coordinate lines are running “parallel” to $\gamma$ (i.e. generate $Z = \pi_1(N)$), and $\theta$ runs along the meridians (i.e. they are contractible within $N$). Let $\sigma : [0, 1] \to [0, 1]$ be a $C^\infty$-function, such that $\sigma(0) = 0$, $\sigma(1) = 1$, and with vanishing derivatives at 0 and 1 to all orders. We define the following diffeomorphism on $T'$:

$$s : (t, \theta, \varphi) \mapsto (t', \theta', \varphi') := (t, \theta, \varphi + 2\pi\sigma(t)), \quad (1.8)$$

and continue it to the complement of $T'$ by the identity. This defines a diffeomorphism of $\Sigma$ with support in $T' \subset N$ which we call a slide of $P_j'$ along $\gamma$. Analogously, instead of the sphere $S_j$, we could have taken any of the spheres $S_{(j,k)}$ ($j \neq i$ if $\gamma$ generates $\pi_1(S^1 \times S_i)$). In this case the resulting diffeomorphism is called a slide of the $k$’th end of the $j$’th handle, $I \times S_j^2$, along $\gamma$. We restricted attention to those $\gamma$ that where homotopically non-trivial within $\Sigma$ since it may be shown that slides along contractible $\gamma$ are isotopic to the identity and that slides along the composite loop $\gamma_1 \gamma_2$ are isotopic to the composition of each individual slide. This ends our presentation of the four classes of diffeomorphisms.

Another important class of diffeomorphisms is given by the rotations parallel to spheres which we now define. Given a 3-manifold $\Sigma$, an embedding $E : [0, 1] \times S^2 \to \Sigma$, a smooth non-contractible loop $\lambda : [0, 1] \to SO(3)$ based at the identity, and a function $\sigma$ as just defined for slides. On $[0, 1] \times S^2$ one has the diffeomorphism

$$r : [0, 1] \times S^2 \to [0, 1] \times S^2, \quad (t, x) \mapsto r(t, x) := (t, (\lambda \circ \sigma(t)) \cdot x), \quad (1.9)$$

through which we define a diffeomorphism of $\Sigma$ ($Im = Image$)

$$R : \Sigma \to \Sigma, \quad p \mapsto \begin{cases} E \circ r \circ E^{-1}(p) & \text{for } p \in ImE \\ p & \text{for } p \notin ImE \end{cases} \quad (1.10)$$

which we call a a rotation parallel to the spheres $E(0)$ and $E(1)$. Any other choice of a smooth, non-contractible loop $\lambda$ would give rise to an isotopic diffeomorphisms. If $ImE$ is a collar neighbourhood of a sphere $S$ in $\Sigma$, we may simply speak of a rotation parallel to $S$.

From (1.5) we see that a rotation parallel to a sphere $t \times S^2$ defines a diffeomorphism of the handle-manifold which is not connected to the identity. We call it a belt-twist.
Applied to a particular factor $I \times S_i^2$ of $\Sigma$ it is isotopic to a rotation parallel to $S_{(i,1)}$ or $S_{(i,2)}$, which we consider as internal diffeomorphisms. In the same way, a rotation parallel to the connecting sphere $S_i$ of $P_i$ is considered as internal diffeomorphisms. We call it a rotation of $P_i$. We can also define a rotation of the $i$'th handle by a rotation parallel to a sphere enclosing $S_{(i,1)}$ and $S_{(i,2)}$. However, this is easily seen to be isotopic to two rotations parallel to each one of these spheres (see e.g. [Gi1]) and hence isotopic to the identity within $DF(\Sigma)$.

We say that $\Sigma$ is spinorial, iff a rotation parallel to $S_0$ – which we simply call a rotation of $\Sigma$ – is not connected to the identity in $DF(\Sigma)$. A single handle manifold is thus not spinorial. It is easy to see that a rotation of $\Sigma$ is isotopic to rotating all $P_i$’s. Thus, $\Sigma$ is spinorial iff any of the $P_i$’s is [Gi1].

Given these definitions, we remark that there is a certain ambiguity in the definition of slides of ends of handles or spinorial primes. To see this, note that an alternative choice of the map $s$ in (1.8) would have been to let also $\theta$ wind once around a full range: $\theta' = \theta + 2\pi \sigma(t)$. This would impose an additional rotation parallel to the sphere boundary component $S_j$ of $N$ so that the resulting diffeomorphism differs from the previous one by the rotation parallel to $S_j$ which may be thought of as an internal diffeomorphism and which is isotopic to the identity in $D(P'_j, rel S'_j)$, iff $P_j$ is a non-spinorial prime. In the case we slide the $k$’th end of the $j$’th handle the resulting diffeomorphisms contains an additional rotation parallel to the non-separating sphere $S_{(j,k)}$ ($k = 1$ or 2) which is not isotopic to the identity within $D(I \times S'_j, rel S'_{(j,1)} \cup S'_{(j,2)})$. This in fact exhausts all ambiguities since higher rotation numbers in the $\theta$ coordinate just result in more rotations parallel to the spheres, which, within internal diffeomorphisms, are isotopic to the identity for even rotation numbers. Thus we have exactly two isotopically inequivalent definitions for sliding spinorial primes or ends of handles. They differ by rotations of the primes or belt-twists of the handles. Let us now state the fundamental theorems, the proofs of which may be found in the cited literature.

**Theorem 3.** Every diffeomorphism of $\Sigma$ is isotopic to a finite sequence of diffeomorphisms build from the four types described above.

**Theorem 4 (Rourke de Sá).** Given the prime factorisation (1.2), then there is a homotopy equivalence (denoted by $\sim$; $\Omega(\cdot)$ denotes the space of based loops in $\cdot$)

$$DF(\Sigma) \sim \left( \prod_{i=1}^{n} DF(P_i) \right) \times \left( \prod_{j=1}^{l} \Omega SO(3) \right) \times \Omega C. \quad (1.11)$$

The significance of this result lies in the following: It is non-trivial that $DF$ is a product fibration whose fibres are the internal diffeomorphisms. We express this by
saying that internal diffeomorphisms do not “interact” with external diffeomorphisms represented by $\Omega C$. Generally, one might have expected a weaker result to hold, namely that $DF$ is a non-product fibration. In fact, had one considered all diffeomorphisms and given up the restriction to those fixing a frame, an analogous result would fail to hold. A counterexample is given by a 3-manifold $\Sigma$ which is the connected sum of two spinorial primes $P_1$ and $P_2$. In $D(\Sigma)$ the rotation parallel to the connecting sphere $S_1$ (considered as an internal diffeomorphism) is isotopic to the rotation parallel to the connecting sphere $S_2$. That is, there is a path in $D(\Sigma)$ connecting two elements in $DF(\Sigma)$ which (by spinoriality) cannot be connected by a path running entirely within the internal diffeomorphisms. The fibration can thus not be a product.

An immediate corollary of the product structure (1.11) is:

$$\pi_k(DF(\Sigma)) = \left( \prod_{i=1}^{n} \pi_k(DF(P_i)) \right) \times \left( \prod_{j=1}^{l} \pi_{k+1}(SO(3)) \right) \times \pi_{k+1}(C). \quad (1.12)$$

Here, the only undetermined object is the space $C$. It is called the configuration space of the 3-manifold $\Sigma$, and, in some sense, labels and topologizes the different relative positions of the primes when combined to form $\Sigma$. From theorem 3 it follows that its fundamental group is generated by exchange-, spin-, and slide diffeomorphisms. Furthermore, it has been shown [HM] that $\Omega C \sim F_l^n \times \Omega C_1$ where $F_l$ is the free group of $l$ generators and $F_l^n$ its $n$-fold product. It accounts for the slides of the $n$ irreducible primes $P_i$ through the $l$ handles. It would certainly be desirable to continue the factorisation to $\Omega C_1$ if possible, but generally not much seems to be known about the detailed structure of $C_1$. For more general information on $C_1$ we refer to the literature [HL][HM]. In the appendix we investigate in some more detail the group $\pi_0(DF(\Sigma))$ for the case where $\Sigma$ is given by the connected sum of $l$ handles. There we shall be interested in learning how slides interfere with exchange- and spin-diffeomorphisms by studying small quotient groups with obvious representations. Heuristic arguments in canonical quantum gravity suggest that the different representations of $\pi_0(DF(\Sigma))$ characterize different sectors in quantum gravity [Is]. Usually attention is restricted to one-dimensional representations, but this seems unnecessarily restrictive [Gi3]. However, special proposals for the construction of quantum states might be employed to preclude certain sectors [HW][GiL]. In theorem A of the appendix it is e.g. shown that restriction to abelian sectors implies that for a $l > 2$ handle manifold, spins followed by exchanges are necessarily represented trivially. This is a purely kinematical result and independent of any requirement as to how to construct the quantum states. In contrast, the considerations in [HW][GiL] made essential use of the no-boundary proposal for the construction of quantum states.

We can now introduce the properties of 3-manifolds which we wish to give information about in this article. We indicate how these properties behave under taking
connected sums so that we can eventually restrict attention to prime 3-manifolds. A

table summarizing their properties is then presented at the end of section 2.

• **Chirality**: A manifold is called chiral, iff it does not allow for an orientation

reversing self diffeomorphism. It follows immediately from theorem 2 that a 3-manifold

is non-chiral iff no prime in its prime decomposition is chiral. Chirality for the relevant

spherical primes is nicely demonstrated in [Wi1]. Chirality of the flat 3-manifolds $R^3/G_i$,

$i = 3, 4, 5, 6$, may be shown by inspection [McC2], using results from [LSY]. In the table

+ stands for chiral and – for non-chiral. Chirality is abbreviated by $C$ and is listed in the

fourth column.

• **Spinoriality**: A 3-manifold is called spinorial, iff a rotation parallel to the boundary

sphere of an embedded 3-disc is not in the connected component of $D_F$, where $D_F$

stabilizes a fixed interior point (and all frames at this point) of the disc. It can be shown

(e.g. [Gi1]) that a 3-manifold is non-spinorial, iff no prime in its prime decomposition

is spinorial. In the table + stands for spinorial and – for non-spinorial. Spinoriality is

abbreviated by $S$ and is listed in the third column.

• **Nuclearity**: A 3-manifold is nuclear, iff it is the spacelike boundary of a Lorentz

4-manifold with $SL(2, C)$ spin structure. The necessary and sufficient condition for $\Sigma$

to be nuclear is that the $\{0, 1\}$– valued, so-called Kevaire semi-characteristic, $u(\Sigma)$, is 0

[GH]. It can be shown that a 3-manifold is nuclear, iff the number of nuclear primes in its

prime decomposition is odd. For disconnected $\Sigma$ one has the following simple rule [Gi2]:

$\Sigma$ is nuclear, iff the number of components with even number of nuclear primes in their

decomposition is even. In the table + stands for nuclear $(u = 0)$ and – for non-nuclear

$(u = 1)$. Nuclearity is abbreviated by $N$ and is listed in the fifth column. There, $(−)^p$

stands for: + if $p$ even and − if $p$ odd; a * stands for: has to be decided on a case-by-case

analysis.

• **Homology groups of $\Sigma$**: The homology groups are merely listed for reasons of

completeness. Let $A$ denote the operation of abelianizing a group and $F$ that of taking the

free part of a finitely generated abelian group. The homology, $H_*$ (abbreviating the zeroth
to third homology group as a row-quartuple), is then given in terms of the fundamental
group: $H_* = (Z, A\pi_1, FA\pi_1, Z)$. The fundamental group of a connected sum of two 3-

manifolds, $\Sigma_1$ and $\Sigma_2$, is the free product (denoted by *) of the individual ones:

$\pi_1(\Sigma_1 \uplus \Sigma_2) = \pi_1(\Sigma_1) \ast \pi_1(\Sigma_2)$. It is infinite if neither of the two manifolds is simply connected.

For the homology this implies: $H_*(\Sigma_1 \uplus \Sigma_2) = (Z, H_1(\Sigma_1) \times H_1(\Sigma_2), H_2(\Sigma_1) \times H_2(\Sigma_2), Z)$. Since $H_2 = FH_1$, it is enough to list $H_1$ which is done in the sixth column. There, the symbol $A\pi$

denotes the abelianisation of the group represented by $\pi$.

• **Homotopy Groups of $D_F(\Sigma)$**: In general relativity, the classical configuration

space, $Q$, satisfies $\pi_k(D_F) = \pi_{k+1}(Q)$ so that we obtain information on its topology by
studying the topology of $D_F$. It is explained in [Gi1] in what sense this is equally valid for the configuration space of closed and open universes. Theorem 4 shows how far these groups are fixed by the corresponding ones for the primes. It tells us that the latter ones are contained as sub- and factor groups. The zeroth and first homotopy group of $D_F(S)$ (resp. the first and second homotopy group of $Q(S)$) are listed in the seventh and eighth column, and the higher ones $\pi_k(D_F(S))$ for $k \geq 2$ (resp. $\pi_{k+1}(Q(S))$) are reduced to those of spheres of dimension two and three in the nineth column. Calculations for $\pi_0(D_F(\Pi))$, where $\Pi$ is a spherical prime, where first presented in [Wi1]. Details for the other calculations are given in [Gi1]. In the seventh column the symbols $\text{Aut}^Z_2(G)$ are interpreted as follows: By $\text{Aut}(G)$ (where $G$ is the fundamental group of the prime $P$, which we identify with $\pi_1(\infty, P)$) we denote the automorphism group of $G$. It is generated by $D_\infty$ via its action on the fundamental group. $\text{Aut}_+(G)$ denotes the subgroup of index 2 generated by the orientation preserving diffeomorphisms $D_\infty^+(P)$. Finally, $\text{Aut}_Z^2(G)$ denotes a central $Z_2$ extension thereof (due to spinoriality of $P$), where the extending $Z_2$ is generated by a rotation parallel to a sphere whose bounding disc contains $\infty$. $\text{St}(3, Z)$ is a so-called Steinberg group (see e.g. paragraph 10 in [Mi] for more information about $\text{St}(n, Z)$) which is a central $Z_2$-extension of $\text{SL}(3, Z)$. It is a perfect group (i.e. its own commutator subgroup). Finally, a $\ast$ means: has to be decided on a case by case analysis.

- **Validity of Hatcher Conjecture**: For the particular class of spherical primes (explained below), the information given about $\pi_k(D_F)$ depends on the validity of the so-called Hatcher conjecture. This motivates to indicate its status for the relevant primes within the list. It states that the diffeomorphism group for a spherical prime is (as a topological space) homotopy equivalent to the space of isometries (of the obvious metric). A weak implication thereof (called the weak conjecture) is that these spaces have isomorphic zeroth homotopy group. The calculation of $\pi_0(D_F)$ only depends on the validity of the weak conjecture, whereas those for $\pi_k(D_F)$, $k \geq 1$, depend on the full conjecture. A $+$ indicates validity of the full conjecture, a $w$ validity the weak form and a $?$ that we do not have any information. The Hatcher conjecture is abreviated by $HC$ and is listed for spherical primes in the second column.

**Section 2: The Prime 3-Manifolds**

In the table presented at the end of this section we list the relevant data for all known prime 3-manifolds except the non-sufficiently large $K(\pi, 1)$. The first column contains their conventional names as already used in the physics literature (e.g. [Wi1]). The top line names the columns as outlined above. Below this line, the table is divided into three disconnected parts (framed by bold lines) which we call subtables, the first and third of
which are again subdivided by bold lines into so-called blocks. The first subtable contains the known prime 3-manifolds with finite fundamental group (which is clearly identical to the set of all known 3-manifolds with finite fundamental group). They are all of the form $S^3/G$ where $G$ is a finite subgroup of $SO(4)$ with free action on $S^3$. They are also called spherical primes. The first two blocks contain those $S^3/G$ with $G$ non-cyclic, the third $G = Z_p$ for $p > 2$, and the fourth $G = Z_p$ for $p \leq 2$, i.e. the real projective 3-space, $RP^3 = S^3/Z_2$, and the 3-sphere itself.

The first block has $G \subset SU(2)$ and the resulting 3-manifolds are homogeneous. The indexing integer has range $n \geq 1$. In the second block $G$ is not contained in any $SU(2)$ and the manifolds are not homogeneous. The order of $Z_p$, i.e. $p$, is coprime to the order of the group the $Z_p$ is multiplied with and $p > 1$ in the first six cases. In the remaining two cases $p = 1$ is also an allowed value. The other indexing integers have ranges $n \geq 1$, $m \geq 2$, and $k \geq 3$. The third block contains the so-called lens spaces. Since $Z_p$ can act in different, non-equivalent (i.e. not conjugate by a diffeomorphism) ways, they are labelled by $L(p,q)$ with an additional integer $q$ coprime to $p$. Here, $q_1$ stands for $q = \pm 1 \mod p$, $q_2$ for $q \notin \pm 1 \mod p$ and $q^2 = 1 \mod p$, $q_3$ for $q^2 = -1 \mod p$, and $q_4$ for the remaining cases. Amongst all $L(p,q)$ are those of the form $L(4n,2n-1)$, $n \geq 2$. For those the + is valid in the $HC$-column and $w$ for the remanining cases. The $L(p,q_1)$’s are the only homogeneous lens spaces. Finally, we note that it is a still open conjecture (involving some subconjectures) that this list comprises all 3-manifolds of finite fundamental group (see e.g. [Th]). Presentations of the finite fundamental groups occurring here may be found in [O] or [Wi1].

The second subtable consists of a single member, namely the only non-irreducible prime: $S^2 \times S^3$. The third subtable comprises the irreducible primes with infinite fundamental group which in addition are sufficiently large (SL). They all fall into the class of $K(\pi,1)$ spaces (Eilenberg-MacLane spaces), that is spaces whose only non-vanishing homotopy group is the first. The condition of being sufficiently large means that these 3-manifolds allow an embedding of a closed Riemannian surface such that the induced homomorphism on the fundamental groups is injective. Physically speaking, a non-contractible loop on the embedded surface is also not contractible within the ambient 3-manifold. In particular, the fundamental group of a SL 3-manifolds contains as subgroup the fundamental group of a Riemannian surface. A sufficient condition for for an irreducible manifold to be SL is that the first Homology group (which is the abelianisation of the fundamental group) is infinite. The reason why we restrict to the subclass of sufficiently large ones is simply that not much seems to be known for general $K(\pi,1)$’s. Now, the first block contains all 3-manifolds of the form $R^3/G$, where $G$ is discrete subgroup of the affine group in 3 dimensions that acts freely and properly discontinously on $R^3$. They comprise all flat 3-manifolds (i.e. admitting a flat metric). $G_1$ is equal to
$Z \times Z \times Z$ (i.e. $R^3/G_1$ is just the 3-torus), and $G_2, \ldots, G_6$ are certain extensions of the groups $Z_2, Z_3, Z_4, Z_6$ and $Z_2 \times Z_2$ respectively by $G_1$ (i.e. $R^3/G_2, \ldots, R^3/G_6$ are further quotients of the 3-torus). The second block contains manifolds of the form $S^1 \times R_g$ where $R_g$ denotes a Riemannian surface of genus $g$. The third block represents all other sufficiently large $K(\pi, 1)$ primes. For relativists, an interesting property of $K(\pi, 1)$ primes is that no connected sum containing at least one of them admits a Riemannian metric of everywhere positive scalar curvature. Moreover, if it admits a nowhere negative scalar curvature metric, it must in fact be flat, i.e. the 3-manifold must be one of the six ones listed in the first block. This has been proven in [GL], and its significance for General Relativity pointed out in [Wi2].
# Properties of Prime 3-Manifolds

| Prime II         | HC | S   | C   | N   | $H_1(\Pi)$ | $\pi_0(D_F(\Pi))$ | $\pi_1(D_F(\Pi))$ | $\pi_k(D_F(\Pi))$ |
|------------------|----|-----|-----|-----|-------------|-------------------|-------------------|-------------------|
| $S^3/D_8^*$      | +  | +   | +   | −   | $Z_2 \times Z_2$ | $O^*$             | 0                 | $\pi_k(S^3)$     |
| $S^3/D_{8n}$     | +  | +   | +   | −   | $Z_2 \times Z_2$ | $D_{16n}^*$       | 0                 | $\pi_k(S^3)$     |
| $S^3/D_4^{2(n+1)}$ | +  | +   | +   | +   | $Z_4$        | $D_{8(2n+1)}^*$   | 0                 | $\pi_k(S^3)$     |
| $S^3/T^*$        | ?  | +   | +   | −   | $Z_3$        | $O^*$             | 0                 | $\pi_k(S^3)$     |
| $S^3/O^*$        | w  | +   | +   | +   | $Z_2$        | $O^*$             | 0                 | $\pi_k(S^3)$     |
| $S^3/I^*$        | ?  | +   | +   | −   | 0            | $I^*$             | 0                 | $\pi_k(S^3)$     |
| $S^3/D_8^* \times Z_p$ | + | +   | +   | −   | $Z_2 \times Z_{2p}$ | $Z_2 \times O^*$ | $Z$           | $\pi_k(S^3) \times \pi_k(S^3)$ |
| $S^3/D_{8n} \times Z_p$ | + | +   | +   | −   | $Z_2 \times Z_{2p}$ | $Z_2 \times D_{16n}^*$ | $Z$           | $\pi_k(S^3) \times \pi_k(S^3)$ |
| $S^3/D_{4(2n+1)}^* \times Z_p$ | + | +   | +   | +   | $Z_{4p}$ | $Z_2 \times D_{8(2n+1)}^*$ | $Z$           | $\pi_k(S^3) \times \pi_k(S^3)$ |
| $S^3/T^* \times Z_p$ | ? | +   | +   | −   | $Z_{3p}$ | $Z_2 \times O^*$ | $Z$           | $\pi_k(S^3) \times \pi_k(S^3)$ |
| $S^3/O^* \times Z_p$ | w | +   | +   | +   | $Z_{2p}$ | $Z_2 \times O^*$ | $Z$           | $\pi_k(S^3) \times \pi_k(S^3)$ |
| $S^3/I^* \times Z_p$ | ? | +   | +   | −   | $Z_p$ | $Z_2 \times I^*$ | $Z$           | $\pi_k(S^3) \times \pi_k(S^3)$ |
| $S^3/D_{2k(2n+1)}^* \times Z_p$ | + | +   | +   | +   | $Z_p \times Z_{2k}$ | $Z_2 \times D_{8(2n+1)}^*$ | $Z$           | $\pi_k(S^3) \times \pi_k(S^3)$ |
| $S^3/T_{3,3m}^* \times Z_p$ | ? | +   | +   | −   | $Z_p \times Z_{3m}$ | $O^*$ | $Z$           | $\pi_k(S^3) \times \pi_k(S^3)$ |
| $L(p,q_1)$       | w  | −   | +   | (−)p | $Z_p$ | $Z_2$ | $Z$           | $\pi_k(S^3)$ |
| $L(p,q_2)$       | w+ | −   | +   | (−)p | $Z_p$ | $Z_2 \times Z_2$ | $Z \times Z$ | $\pi_k(S^3) \times \pi_k(S^3)$ |
| $L(p,q_3)$       | w  | −   | −   | (−)p | $Z_p$ | $Z_2$ | $Z \times Z$ | $\pi_k(S^3) \times \pi_k(S^3)$ |
| $L(p,q_4)$       | w  | −   | +   | (−)p | $Z_p$ | $Z_2$ | $Z \times Z$ | $\pi_k(S^3) \times \pi_k(S^3)$ |
| $RP^3$           | +  | −   | −   | +   | $Z_2$ | 1     | 0            | 0                 |
| $S^3$            | +  | −   | −   | −   | 1     | 1     | 0            | 0                 |

| $S^2 \times S^1$ | /  | −   | −   | +   | $Z$ | $Z_2 \times Z_2$ | $Z$ | $\pi_k(S^3) \times \pi_k(S^2)$ |
| $R^3/G_1$        | /  | +   | −   | +   | $Z \times Z \times Z$ | $St(3, Z)$ | 0 | $\pi_k(S^3)$ |
| $R^3/G_2$        | /  | +   | −   | +   | $Z \times Z_2 \times Z_2$ | $\text{Aut}^2(G_2)$ | 0 | $\pi_k(S^3)$ |
| $R^3/G_3$        | /  | +   | +   | +   | $Z \times Z_3$ | $\text{Aut}^2(G_3)$ | 0 | $\pi_k(S^3)$ |
| $R^3/G_4$        | /  | +   | +   | −   | $Z \times Z_2$ | $\text{Aut}^2(G_4)$ | 0 | $\pi_k(S^3)$ |
| $R^3/G_5$        | /  | +   | +   | +   | $Z$ | $\text{Aut}^2(G_5)$ | 0 | $\pi_k(S^3)$ |
| $R^3/G_6$        | /  | +   | +   | −   | $Z_4 \times Z_4$ | $\text{Aut}^2(G_5)$ | 0 | $\pi_k(S^3)$ |
| $S^1 \times R_g$ | /  | +   | −   | −   | $Z \times Z_{2g}$ | $\text{Aut}^2(Z \times F_g)$ | 0 | $\pi_k(S^3)$ |
| $K(\pi,1)_{sl}$  | /  | +   | *   | *   | $\text{Aut}^2(\pi)$ | $\text{Aut}^2(\pi)$ | 0 | $\pi_k(S^3)$ |
Appendix:

In this appendix we investigate in some more detail the diffeomorphisms of the \( l \)-fold connected sum of handles:

\[
\Sigma = \bigsqcup_{i=1}^l S^1 \times S^2.
\]

(A1)

Its fundamental group is given by the \( l \)-fold free product of \( \mathbb{Z} \):

\[
\pi_1(\Sigma) = \bigstar_{i=1}^l \mathbb{Z} =: F_l,
\]

(A2)

where \( F_l \) just stands for the free group on \( l \) generators. We visualize the generator \( g_i \) of the \( i \)’th \( \mathbb{Z} \) as a smooth, nonintersecting and oriented curve which starts from some basepoint, enters the \( i \)’th handle through \( S_{(i,1)} \), leaves it through \( S_{(i,2)} \), and returns to the basepoint. The direction defined by \( g_i \) is called positive. By \( g_i g_j \) we denote a curve that first traverses the \( i \)’th and then the \( j \)’th handle in positive directions.

According to eq. (1.12) and the following remarks, one has [McC1]

\[
\pi_0(DF(\Sigma)) = \left( \bigoplus_{i=1}^l Z_2 \right) \times \pi_1(C_1),
\]

(A3)

where the \( i \)’th \( Z_2 \) is generated by a belt-twist of the \( i \)’th handle.

As remarked earlier, \( \pi_1(C_1) \) is generated by slides, exchanges, and spins. In paragraph 4.3 of ref. [La] it is shown that the following sequence is exact:

\[
0 \rightarrow \bigoplus_{i=1}^l Z_2 \rightarrow \pi_0(D^+_\infty(\Sigma)) \rightarrow \text{Aut}(F_l) \rightarrow 1,
\]

(A4)

where \( \bigoplus_{i=1}^l Z_2 \) is the same as above (i.e. the \( i \)’th \( Z_2 \) is generated by a belt-twist of the \( i \)’th handle). Now, the handle manifold \( S^1 \times S^2 \) is non-spinorial so that \( \Sigma \) is also non-spinorial. But for non-spinorial 3-manifolds one has [Gi1][Wi1]

\[
\pi_0(DF) \cong \pi_0(D^+_\infty).
\]

(A5)

Together (A3-5) imply

\[
\pi_0(DF(\Sigma)) = \left( \bigoplus_{i=1}^l Z_2 \right) \times \text{Aut}(F_l),
\]

(A6)

that is \( \pi_1(C_1) \cong \text{Aut}(F_l) \).

(A7)

In quantum gravity one is e.g. interested in some of the representation properties of \( \pi_0(DF) \) which we can now investigate. We are interested in the question of how slides
interact with the other operations, in particular the exchanges. The reason being that slides
generate those diffeomorphisms which mix the interior and exterior of primes (as explained
in section 1) and are thus harder to interpret physically, at least in an approximation where
the primes are treated as individual particle like entities (Geons). In view of this, one might
e.g. be interested in the following question: Under what conditions may we forget about
slides? A first simple answer for the example considered here is provided by theorem A
below.

Since internal diffeomorphisms (here the belt twists) can be given any representation
independent of the rest (product structure of (A3)) we may restrict attention to the group
\( \Gamma_l := Aut(F_l) \). We will follow chapter 7 of ref. [CM] in our description of the presentations
for \( \Gamma_l \). \( \Gamma_l \) can be generated by four generators: \( P, O, Q \) and \( U \), whose action on the \( g_i \) is
given by

\[
P : [g_1, g_2, g_3, \ldots, g_l] \mapsto [g_2, g_1, g_3, \ldots, g_l] \quad \text{(A8a)}
\]

\[
Q : [g_1, g_2, g_3, \ldots, g_l] \mapsto [g_2, g_3, g_4, \ldots, g_1] \quad \text{(A8b)}
\]

\[
O : [g_1, g_2, g_3, \ldots, g_l] \mapsto [g_1^{-1}, g_2, g_3, \ldots, g_l] \quad \text{(A8c)}
\]

\[
U : [g_1, g_2, g_3, \ldots, g_l] \mapsto [g_1 g_2, g_2, g_3, \ldots, g_l] \quad \text{(A8d)}
\]

Their “physical” interpretation is as follows: \( P \) exchanges handle no 1 and handle no 2, \( Q \)
exchanges all \( l \) handles in cyclic order, \( O \) spins handle no 1, and \( U \) slides the second end of
the first handle through the second handle in a positive direction. \( P \) and \( Q \) alone generate
the permutation group of \( l \) objects which is a sub- but no factor group of \( \Gamma_l \), and \( P, Q, O \)
generate an order \( 2^l! \) subgroup with obvious interpretation (it may be characterized as the
group of rearrangements of \( l \) books in a shelf with no orientations given to the backs). The
element \((OU)^2 \) \((= (UO)^2)\) represents a slide of the whole of the first handle through handle
two. If, in addition to the relations given below, one imposes the relation \((OU)^2 = E\), one
obtains a presentation of the group \( GL(l, Z) \) which is a factor group of \( \Gamma_l \) [CM].

We shall now give the relations for the generators \( P, Q, O, U \) where \( E \) denotes the
identity. Given them, we then study some quotient groups by imposing additional relations
\( R_i(P, Q, O, U) = E \) by hand which gives us a presentation of \( \Gamma_l/N_R \), where \( N_R \) is the
smallest normal subgroup in \( \Gamma_l \) containing the elements \( R_i(P, Q, O, U) \). The reason
for this is that a representation \( \rho : \Gamma_l \to GL(l, C) \) satisfying \( R_i(\rho(P), \rho(Q), \rho(O), \rho(U)) = 1 \)
comes from a representation of \( \Gamma_l/N_R \), which, in the cases we choose, is very simple. We
can thus immediately give all the representations of \( \Gamma_l \) which satisfy these relations.

\( \Gamma_1 \) is uninteresting and just given by \( Z_2 \), generated by the single spin \( O \). For \( \Gamma_2 \) one
has \( P = Q \) and the relations for the remaining generators are:

\[
P^2 = O^2 = (PO)^4 = (POP^2)^2 = (POU)^3 = E, \quad (OU)^2 = (UO)^2. \quad \text{(A9)}
\]
Let us look at this case first before going to the general case.

- **Abelian representations**: The presentation of the abelianisation $\Lambda_2$ of $\Gamma_2$ is easily obtained (all generators commute) and given by

$$P^2 = O^2 = U^2 = POU = E \quad (A10)$$

which is just the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $P$ (left $\mathbb{Z}_2$) and $O$ (right $\mathbb{Z}_2$) and where $U$ generates the diagonal $\mathbb{Z}_2$. For an abelian representation this implies that any of the generators $P, O, U$ is non-trivially represented precisely if one of the others is. There is no $P - Q$ correlation unless one imposes $U = E$. This is also true generally since the factor group $U = E$ is abelian as we show below. One easily checks from (A9) that taking any of the generators $P, O$ or $U$ to commute with the other two already implies commutativity of all generators. Moreover, it is in fact sufficient to require exchanges to commute with slides only. Proof: If $P$ and $U$ commute $(POU)^2 = P(OU)^2P = E \Rightarrow (OU)^2 = E$ and $(POU)^3 = POP(UO)^2PU = POU = E \Rightarrow PU = O$, so that $P$ also commutes with $O$.

- **Slides represented trivially**: Setting $U = E$ yields

$$P^2 = O^2 = (PO) = E \quad (A11)$$

which is just $\mathbb{Z}_2$ generated by $P = O$. Thus representing $U$ trivially leads to an abelian representation with $P - O$ correlation.

- **$P - O$ correlating representations**: Setting $P = O$ yields

$$P^2 = U^3 = (PU)^2 = E \quad (A12)$$

which is the presentation of the permutation group $S_3$ of 3 objects (e.g. $P = (12)$ and $U = (123)$) or, equivalently, of $D_6$, the dihedral group of order 6 which describes the symmetries of an equilateral triangle ($P$ generates reflections about a symmetry axis, $U$ a rotation by $2\pi/3$). There are two non-trivial representations, the one-dimensional one, $R_1$, and the two dimensional one, $R_2$:

$$R_1: \quad P \mapsto -1, \quad U \mapsto 1$$

$$R_2: \quad P \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad U \mapsto \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \quad (A13)$$

$R_2$ shows that there are $P - O$ correlating representations which also represent $U$ non-trivially.

- **Spins represented trivially**: Setting $O = E$ yields

$$P^2 = U^2 = (PU)^3 = E \quad (A14)$$

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which reduces to (A12) by replacing $U \to PU$. Correspondingly, the two non-trivial representations follow from (A14). This shows how slides can interact with exchanges without involving spins.

Next we turn to the general case $l > 2$. A complete list of relations is given below (the relations are not independent). $A \leftrightarrow B$ means that $A$ and $B$ commute.

\begin{align*}
    P^2 &= E \quad \text{(A15a)} \\
    (QP)^{n-1} &= Q^n \quad \text{(A15b)} \\
    P &\leftrightarrow Q^{-1}PQ^i \quad \text{for } 2 \leq i \leq \frac{l}{2} \quad \text{(A15c)} \\
    O^2 &= E \quad \text{(A15d)} \\
    O &\leftrightarrow Q^{-1}PQ \quad \text{(A15e)} \\
    O &\leftrightarrow QP \quad \text{(A15f)} \\
    (PO)^4 &= E \quad \text{(A15g)} \\
    U &\leftrightarrow Q^{-2}PQ^2 \quad \text{for } l > 3 \quad \text{(A15h)} \\
    U &\leftrightarrow QPQ^{-1}PQ \quad \text{(A15i)} \\
    U &\leftrightarrow Q^{-2}OQ^2 \quad \text{(A15j)} \\
    U &\leftrightarrow Q^{-2}UQ^2 \quad \text{for } l > 3 \quad \text{(A15k)} \\
    U &\leftrightarrow UOU \quad \text{(A15l)} \\
    U &\leftrightarrow PQ^{-1}OOUQP \quad \text{(A15m)} \\
    U &\leftrightarrow PQ^{-1}PQPUPQ^{-1}PQP \quad \text{(A15n)} \\
    (PQ^{-1}UQ)^2 &= UQ^{-1}UQU^{-1} \quad \text{(A15o)} \\
    U^{-1}PUPOUOPO &= E \quad \text{(A15p)} \\
    (POPU)^2 &= E \quad \text{(A15q)} \\
\end{align*}

• **Abelian representations**: In this case (A15) boils down to the following relations for commuting generators:

\[ P^2 = O^2 = PO = P^{l-1}Q^{-1} = U = E, \quad \text{(A16)} \]

which is the presentation of $Z_2$ generated by $P (= O)$. $Q$ equals $E$ for $l$ odd and $P$ for $l$ even. The difference of this case to $l = 2$ lies in the condition $U = E$ which followed from (o) in view of (a). In fact, (o) implies $U = E$ already from the commutativity of $U$ with $P$ and $Q$. Abelian representations (equivalently, representations for which slides and exchanges commute) thus necessarily represent slides trivially.
• *Slides represented trivially* : If we set $U = E$, (p) and (a) imply $P = O$, (f) says that $P$ and $Q$ commute, and (b) then implies that $P^{l-1} = Q$, i.e. that $Q = E$ for $l$ odd and $Q = P$ for $l$ even. We thus obtain the group $Z_2$ generated by $P$ (=$O$).

• *$P - O$ correlating representations* : If we impose $P = O$, (f) implies $P \leftrightarrow Q$, (i) implies $U \leftrightarrow Q$ and (j) implies $U \leftrightarrow P$ so that all generators commute.

We can summarize these points in the following

**Theorem A.** Let $\rho$ be a representation for $\Gamma_l$, $l \geq 3$. The following conditions on $\rho$ are equivalent:

- $a$: slides and exchanges commute
- $b$: $\rho$ is abelian
- $c$: slides are represented trivially
- $d$: $\rho$ correlates $P$ and $O$, i.e. $\rho(P) = \rho(O)$

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