Modelling Selforganization and Innovation Processes in Networks

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Abstract. In this paper we develop a theory to describe innovation processes in a network of interacting units. We introduce a stochastic picture that allows for the clarification of the role of fluctuations for the survival of innovations in such a non-linear system. We refer to the theory of complex networks and introduce the notion of sensitive networks. Sensitive networks are networks in which the introduction or the removal of a node/vertex dramatically changes the dynamic structure of the system. As an application we consider interaction networks of firms and technologies and describe technological innovation as a specific dynamic process. Random graph theory, percolation, master equation formalism and the theory of birth and death processes are the mathematical instruments used in this paper.

1 Introduction

This paper is devoted to the interdisciplinary theory of selforganization processes, paying particular attention to stochastic effects connected with innovations in network systems. In our understanding “selforganization” is the spontaneous formation of structures\cite{27,28,45}. An “innovation”, in a general system-theoretical understanding, is the appearance of a new species, of a new mode of behavior, of a new technology, of a new product or of a new idea etc.\cite{35,17,18,16,34}.

Technological innovations are considered as the basic driving process for economic evolution and growth. In economics, innovation networks\cite{50,72} and networks economies\cite{86} have been discussed widely in the last decade. However, a unified understanding of socio-economic networks is still missing\cite{100}. “For many economists the study of networks is limited to the analysis of the functioning of physical networks such as the railway, the telephone system or the internet for example”\cite{69}. Although networks are thought to be constituted by sets of actors and by links\cite{100}, the very nature of these actors and the links between them varies among different authors. Nations, institutions, firms, products, or individuals may represent the actors. The
links can be defined quite differently. Innovation in such a context is mostly seen as an outcome of complex networks with heterogeneous actors.

In this paper, we define innovation not as a product of a network activity but as a specific process in the formation of a complex network. Thus, innovation is a process in the structure formation of a complex networks. We describe innovation as a process which dramatically and decisively changes the composition of a network. Accordingly, the network will behave differently depending on which innovation has been introduced. More specifically, we propose a network description for dynamic processes in a system. By means of this system-theoretical perspective the appearance of innovations is related to certain mechanisms in the growth and change of networks. So, we create a network picture for dynamic processes as the emergence, survival or extinction of innovations which have been also considered in the mathematical theory of biological evolution, in population biology and other fields of complexity theory.

In this paper we will follow a special approach to describe innovations in evolutionary systems. We start in section 1 with an abstract definition of an innovation. We will then show that our approach is related to recent developments in statistical physics which best can be described as an emerging field of complex networks theory [105]. Thus we will first give a survey of network approaches ranging from chemical networks [120,46] to biochemical webs [46], protein webs [64], food webs [24], to the structure of the internet [43] and the world wide web [3] up to general approaches from statistical mechanics [2].

In the second part of the paper we reflect upon structure and relations of socio-economic networks in a static (purely structural) picture. We will discuss some earlier results of random graph theory and percolation theory. We derive statements about connectivity and strongly connected components.

In the third part of the paper, and in connection with our special interest in innovation processes, we will develop a dynamic network theory and in particular the theory of “sensitive” networks in more detail. The term “sensitive” networks denotes networks which are sensitive to the introduction or removal of one or few nodes or edges, or in a more general context, to the occupation or leaving of a node. Specifically, sensitivity is linked to the question of whether a node (a species, a mode of behavior, an idea, a technology) is occupied by at least one individual or not. We will show that this problem is of relevance for the modeling of innovation processes. The dynamics of innovation processes may be described by stochastic equations, which can be formally treated within the framework of statistical physics. Some generalizations are possible.

1.1 Innovation – a System Theoretical Approach

In economics, innovation is mainly understood as technological innovation describing the introduction of new technologies, products and production
processes. The differentiation between invention and innovation relates innovation to the economic exploitation of new ideas. However, it is also possible to look at innovations from a more general, evolutionary point of view. Ziman gives one example for such an approach when he writes “Go to a technological museum and look at the bicycles. Then go to a museum of archaeology and look at the prehistoric stone axes. Finally, go to a natural history museum and look at fossil horses. In each case, you will see a sequence, ordered in time of changing but somewhat similar objects.” [131], p. 3. However, not every change is an innovation. In this paper, we follow a system theoretical approach to innovation. In this framework, innovation is something new to the system and most essentially the emergence of an innovation changes the state of the system dramatically. In other words, the actual state of the system becomes unstable and a transition to a new state occurs. To define an innovation we first have to define the state of the system. Here, we again choose a very specific approach. We represent the state of the system as a point in the high dimensional occupation number space [27,35]. In this space a coordinate axis is attached to a certain type \(i\) of elements (with \(i = 1, 2, \ldots, s\), natural numbers). The occupation numbers are represented on this axis.

To describe technological innovation we have to ask for a re-specification of this abstract concept. For socio-economic systems, the axes of the state space refer to different possible taxonomies. For instance, an axis \(i\) can represent a certain technology from a set \(s\), of different technologies present in the system. With such a technological taxonomy, competition processes between technologies can be described [102,16]. The carriers of this competition process are firms using different technologies and competing with their products on a market. This way, we link back to an economic understanding of an innovation process that “requires insight into system dynamics grounded in a variety of firm competencies and behaviour and a variety of demand” [103], p. 5. Let us note that the state space concept can be applied to quite different processes. The type \(i\) might also stand for the size class that a certain firm belongs to. Then, growth processes of firms are in the focus of the description. Moreover, the type \(i\) may represent a certain group in society. Formation of political opinions [129] or emergence of norms and violence in groups [85] are then considered. Innovation in these cases covers new forms of collective behavior.

We can find each type \(i\) in \(N_i\) exemplars of elements in the system. The exemplars may be individuals, but also organizational and institutional units like firms and groups. \(N_i\), the occupation numbers, are functions of time. They are positive or zero. A complete set of occupation numbers \(N_1, N_2, \ldots, N_s\) at a fixed time, characterizes the occupation state of this system. The time dependent change of the occupation numbers is described by the movement of this point in the space. The whole motion takes place on the non-negative cone \(K\) of the space. In this picture we can describe the case, that a type \(i\) is not present in the system at time \(t\). That means the type \(i\) is occupied
with the number zero \( (N_i = 0) \). We call a system an under-occupied system if we can make the assumption that the sum of the occupation numbers is essentially smaller than the total number of the possible elements \[35\].

In this picture, an innovation is an occupation of a non-occupied type. In deterministic systems, zero occupation can only be achieved in the limit of infinite time \( t \to \infty \), if a sort died out (zero can be a stable stationary state). For finite times \( t > 0 \) types cannot arise, if they are not in the system at time \( t = 0 \), and present types cannot die out. The situation is different if we use the stochastic picture. In stochastic systems the zero state can be reached in finite times \( t \). A stochastic description offers the advantage that at finite times new sorts (innovations) can arise or die out. Therefore, the stochastic description is especially suited for evolutionary processes and in particular for innovation processes.

Let us note here that any innovation will change the taxonomy of types in the system \[5\]. Innovation has to do with uncertainty and its prediction is impossible. With the notion of an under-occupied system we escape the problem of determining \textit{a priori} the place or kind of an innovation. Instead, we equip the system with a reservoir of possible innovations. Which of these possibilities turns into a realization remains uncertain. In some respects this is a trick to avoid the problem with a changing taxonomy. There are other possibilities to escape this problem, e.g., so-called continuous models operating on a characteristic space as we discussed elsewhere \[34,33,30\]. However, the discrete approach has, in our respect, certain advantages, as we will discuss later.

In an under-occupied system most elements have, at a given time \( t \), the occupation number zero. So we can pass from the high-dimensional cone \( K \) to a lower-dimensional cone \( K^+ \). Accordingly, the time dependent variation of the system can be described as a switching of the state point on the edges of the cone \( K \). If \( K^+ \) is an element of the set of all possible cones, we observe a switch from one sub-cone to another. As the process is discrete, it is a hopping on the edges of different positive cones.

In figure \[1\] we visualize such a process for three dimensions. At any point in time, the state of the system is represented by a certain vector \( \mathbf{N}(t) \). In a stationary state, the endpoint of this vector defines a positive cone. In our example, the vector moves in the plane spanned by \( N_1 \) and \( N_2 \). An innovation opens up a new dimension of the system. In our example, a new third type is introduced into the system. After the innovation, the vector is moving in the space defined by \( N_1, N_2 \) and \( N_3 \). In general, we can assume that the system operates in a multidimensional space where the cone can have a very complicated shape, and the vector \( \mathbf{N}(t) \) jumps between the edges of this cone.

The hopping process visualizes the transition between one stable stationary state and another stable stationary state. In this sense, innovation is the outcome of a process of destabilization. In the framework we propose in this
In this paper, innovations are seen as stochastic instabilities. The changes occurring in the occupation number space result from interactions of the different types present in the system. These interactions can be visualized as graphs or networks where the nodes represent the types, and links between them represent different forms of interactions. In the network picture, an innovation corresponds to the appearance of a new node and the activation of a link to this node. The models we will present in section 3 allow us to differentiate between different processes which finally introduce such a new node.

The conceptualization of types as elements (nodes) of a network represents a graph theoretical approach to the dynamics of the system. Therefore, other network approaches are of particular relevance to develop our theoretical approach further. We will use the following part of section 1 to introduce the new emerging specialty of complex network theory and to place our approach in this field.

1.2 Innovations, the Emergence of the Field of Complex Network Theory and Sensitive Networks

In the last years, complex systems in nature and society have been carefully investigated. Already in the 70s, theories of self-organization were used to build a bridge between social and natural systems investigations. As part of this development complex networks have been investigated. Recently, as a new branch in complexity theory complex networks have been reconsidered and extensively studied. They seem to be particularly relevant for the study of innovation processes.
In the context of complexity theory, the concept of networks has not only been used as an easy-to-use metaphor. As Bornholdt and Schuster note: “Recent advances in the theory of complex networks indicate that this notion may be more than just a philosophical term. Triggered by recently available data on large real world networks (e.g., on the structure of the internet or on the molecular networks in the living cell) combined with fast computer power on the scientist’s desktop, an avalanche of quantitative research on network structure and dynamics currently stimulates diverse scientific fields.”[13], p. V.

Social networks form one important area of application of complex networks theory. The structures found cover networks of collaboration [89,7], networks of recognition (citation networks) [123] and networks of corporate directors [119]. In economic theories, innovation is more and more understood as the outcome of the interaction between scientific, economic and political systems [95]. Instead of considering an innovation as a singular event the network character of innovations is stressed. Innovation networks seem to be a new organizational form of knowledge production.

The structural analysis of systems is of great interest. Albert and Barabási [2] give a very good presentation of this subject and its development. In the very beginning, investigations of large complex systems were done by random graphs. More and more it became possible to analyze real complex systems and large systems too. With the development of computer capacity, the amount of empirical data increases. It becomes possible to compare the theoretical results by random graph theory with that of the real data analysis. Obviously more than pure randomness exists. Organization principles and rules of system evolution play a decisive role, leading to small-world behaviour and scale-free networks. Our world is not a random world. Other evolutionary principles are of great interest. In addition, it is evident that a theory of evolving networks may give a more realistic approach to real systems. This is why we give special consideration to evolving networks here.

From the analysis of empirical data we learn that many real-networks have a small-world character. The small-world concept describes the fact that despite their often large size, in most networks there is a relatively short path between any two nodes. The small-world property characterizes most complex networks. For example, the chemicals in a cell are typically separated by only three reactions, or in a more exotic case, the actors in Hollywood are on average within three co-stars from each other. The small-world concept corresponds to our observations; it is a structural, not an organizing principle [127,20].

Not all nodes in a network have the same number of edges (the same node degree). The spread in node degrees is characterized by a distribution function $P(k)$, which gives the probability that a randomly selected node has exactly $k$ edges. Since in a random graph the edges are placed randomly, the majority of nodes have approximately the same degree, close to the average degree of the network. The degree-distribution of a random graph is a Poisson distribution.
with the peak over the average degree \[2\]. In real networks the distributions of the edges are more complicated. Important results were obtained by the analysis of large real systems: the degree distribution deviates significantly from a Poisson distribution and follows general structural rules in many cases. Many large networks are scale-free, that is, their degree distribution follows a power law. In addition, even for those networks for which \( P(k) \) has an exponential character, the degree distribution significantly deviates from a Poisson distribution achieved by random graph theory for such systems \[2,6\].

Scale-free networks express a hierarchy between the nodes. Not every node is important at the same level. Accordingly, not each of the links between the types have the same importance. There are very sensitive relations or elements too. As mentioned previously, we consider networks as “sensitive” if their properties depend strongly on the introduction or removal of one or a few nodes or edges, or on changes of the occupation of nodes. We will show that for the evolutionary character, the description of the time-behaviour by master equations on occupation number spaces is an appropriate tool. The discrete character of occupation number description allows for an appropriate description of the introduction respectively of removal of relations, edges etc.. We will analyze not only the steady states of our stochastic systems but also the time evolution. Albert and Barabási \[2\] also refer to approaches with master and rate equations. They write that in addition, these methods, not using a continuum assumption, appear more suitable for obtaining exact results in more challenging network models. In addition they mentioned that the functional form of the degree distribution, \( P(k) \), cannot be guessed until the microscopic details of the network evolution are fully understood. According to our point of view, the method of master equations is an excellent tool to use in the investigation of many open questions and is able to bring much more light to bear on this subject. For example, by using this discrete approach, we have the chance to get statements about the kind of fluctuations. Evidently this is one of the most important questions.

Let us come back now to the question of the distribution of the graph. We remember that random graph theory leads to a Poisson degree-distribution. Albert and Barabási \[2\] give a near exhaustive survey of empirical data sets for real complex networks and show that the real degree-distributions are not Poisson-distributions, but scale-free-distributions, or exponential distributions. These authors write: “The high interest in scale-free networks might give the impression that all complex networks in nature have power-law degree distributions. ... It is true that several complex networks of high interest for scientific community, such as the world wide web, cell networks, the internet, some social networks, and the citation network are scale-free. However, others, such as the power grid or the neural networks of c. elegans, appear to be exponential ... Evolving networks can develop both power-law and exponential degree distributions. While the power-law regime appears to be robust, sublinear preferential attachment, aging effects, and growth con-
The nature of the elements does not play any role with respect to the structure. Opposite to this, the nature of the relations between elements determines the specificity of a structure. Structure here means the manifold of interactions between the elements.

Let us note here, that for many descriptions of socio-economic networks as in sociology and economics, it is just the nature of the elements and the
relations which are relevant. In this paper, we follow a more abstract and
general approach. By using such a general level, we create methodologically
the task of re-specifying the definition of both elements and relations for any
application area that one might think of. On the other hand, the high level
of generalization we use here keeps the application areas open, as we intend
to do.

The idea of “structure” is of great importance in our life – both in reality
and in sciences. In particular, it is obvious by analyzing social structures.
What does “structure” mean in the original sense? On the one hand we have
our conventional understanding of structure – the understanding of structure
in our real life. On the other we may formulate a precise idea of structure in
terms of mathematics, in system theory \[21\] and in the theory of selforgani-
zation \[29\].

“We understand under a “structure” the composition of elements and the set
of relations respectively operations, which connect the elements.”

Kröber \[73\] writes about the idea of structure in real systems: “Each system
consists of elements that are arranged in a certain way and are linked to
each other by relations. We understand by ‘structure of a system’ the kind of
arrangement and connections of their elements . . . In this respect, we do not
consider what kind the elements are. If we speak about structure, we are not
interested in the elements of the structure. We only consider the manifold of
relations. In this respect the structure of a system is a well-defined connection
between the elements of the system. These elements, which are arranged
in a determined manner and connected by determined connections can be
necessary or randomly, universally or uniquely, relevant or irrelevant”.

Moles \[82\] writes to this subject: “The surrounding objects of the material
world, artifical and natural organism in the wide sense of this world are signed
by two main aspects; by their structural and functional properties.”

With the famous book series “The Elements of Mathematics” the group of
scientists, Bourbaki, gave an example for constructing systematically math-
ematics as a science of such “structures”. In the following section we give a
short description of several important concepts of this mathematical theory
of structures. In particular, we give a summary of the theory of relation,
graphs and matrices in the amount we need here. Our purpose is to apply
this abstract theory to socio-economic networks.

In a socio-economic system, the elements of structure are individuals or
groups of individuals in different institutional and organizational forms, e.g.,
firms. The socio-economic connections between these elements are relations
in the sense of this abstract theory of structure. The description of elements
and relations can be given graphically by a system of vertices (nodes), which
model the elements (individuals, groups, firms) and of edges (arcs), which
describe the relations (connections). Nodes and arcs can be weighted. Arcs
can have a direction. This way we can include quantitative aspects. The
The most important aspect in the selforganization of networks is the formation of new connections, which generates new structures. In the following, we want to show that the instruments of the mathematical theory of structure in connection with the ideas and concepts of the theory of selforganization contribute to the description of the connections in complex socio-economic networks.

The idea of “structure” stands in a close connection to order (disorder) respectively information (entropy). The theory of selforganization shows how the generation of structures is connected with the decrease of entropy [29].

Socio-economic systems are complex systems, which consist of many connections between the elements. Therefore, complexity is a further concept to be defined. Ebeling, Freund and Schweitzer [29] write:

“As complex we describe holistic structures consisting of many components, which are connected by many (hierarchically ordered) relations respectively operations. The complexity of a structure can be seen in the number of equal respectively distinct elements, in the number of equal respectively distinct relations and operations, as well as in the number of hierarchical levels. In the stricter sense, complexity requires that the number of elements becomes very large (practically infinite).”

We are especially interested here in the origin of complex structures, and in the development of order (information). In the end we have to answer the question which parameter-relations (order parameters) determine the qualitative behaviour of the system. Ilya Prigogine [91] in collaboration with his coworkers, did pioneering work in the investigation of selforganizing systems. Further important work has been done in this field by Manfred Eigen [38] on the selforganization of macromolecules and by Manfred Eigen and Peter Schuster [38,40] on the hypercycle model. The mechanisms of selforganization are clearly worked out by Nicolis and Prigogine [90]. These authors give a stringent physical and mathematical formulation for these processes, in particular with respect to the energetic and entropic aspects. A somewhat different view on this was developed in the formulation of the synergetics by Haken [52]. The investigation of such systems shows that the formation of order in complex systems can be allocated to physical processes which play a role far from equilibrium [27]. We underline that biological, just as socio-economic processes, can be investigated with the help of the theory of selforganization because they obey the valid physical and chemical laws. However, processes which include the real life (biological and socio-economic systems) also obey other rules and laws, which are not solely determined by physics. This is already evident from the very general character of the structures we consider here. As said above, we formulate the idea of structure mathematically and keep away from how the structure appears in different systems. Then, graph theory will provide us with the level of abstraction we need in order to describe real systems.
2.2 Introduction to the Theory of Relations, Matrices and Graphs

By formulating our ideas in the mathematical language, we have the advantage of having access to the great mathematical potential which is available in this field. An important basic assumption is to start from the theory of sets. As pointed out above, the connection between element and set is the first and most important aspect of a structure. Furthermore, we introduce relations and operations. The concept of structure reflects abstract properties of a system. Due to the abstract character of the concept, results can be translated to other systems and comparisons are possible.

In general we can differ between local, temporal, causal and functional structures. To illustrate the structures we will use graphs which represent the elements and their connections by geometrical symbols. An example of a graph representing an economic network with 4 levels is given in Fig. 2. This graph represents the flow of materials and outcomes in a production process.

In economics, the relations between economic agents can be represented in a network form. In this case information flows, e.g., price signals between market participants, are exchanged. Then, the structure of the network describes a situation with local interaction (not every agent is informed about all other agents but the agents also do not act independently). Socio-economic networks can also describe a variety of different agents’ actions that influence other agents. The diffusion of technologies over firms can be described as a network of actions from formation of a company (with a certain technology) over knowledge transfer between companies (in form of imitation or merging) to the exit of companies due to technological competition. We will come back to such a network interpretation in section 3.

It may be worthwhile to underline that it is not possible to give a complete structural description of living objects by binary relations as graphs only. Graphs are just a device for analyzing these systems, albeit a very useful tool. Of course, we cannot describe complex objects only by graphs because their binary relations are ambiguous. Nevertheless, graphs are very useful for the representation of complex structures. Before the elements of the theory of graphs are explained it is necessary to introduce the two notions; set and relation.

What we understand by a set is nearly the same as in common language. It is always abstract and determined by its elements. A set $M$ is given, which consists of the elements $a_1, a_2, a_3, \cdots$; symbolically $M = \{a_1, a_2, a_3, \cdots\}$. The elements $a_i$ ($i = 1, 2, 3, \cdots$) describe individuals or groups equipped with different attributes and features and belonging to different types. The number of elements determines whether we have a finite or infinite set. Our investigations here are related to finite sets. In the following “sets” means finite sets. Graphically the elements of a set can be described by vertices (nodes) in a
Fig. 2. Graph of an economic network with 4 levels: (i) raw materials such as minerals, fossils, plants etc., (ii) intermediate products as steel, coal, corn, etc., (iii) producing final goods, (iv) level of consumers.

n-dimensional space. This way we assume that a clear relation exists between the vertices and the elements of the set. An example is given in Fig. 3.

\[ M = \{a_1, a_2, a_3\} \]

Fig. 3. A set of three elements (for instance, plants or firms) which are represented in a one dimensional space.

The elements of a set can be ordered to pairs. Let us consider for example a set \( M = \{a_1, a_2, \ldots, a_5\} \) with the pairs \([a_1, a_3]\) and \([a_2, a_5]\). \(a_1\) and \(a_2\) describe the first element and \(a_3\) and \(a_5\) the second one. The pair \([a_3, a_2]\) describes another relation between the elements \(a_5\) and \(a_2\) as \([a_2, a_5]\). In general the elements \(a_i\) and \(a_j\) in an ordered pair \([a_i, a_j]\) are not exchangeable with each other without changing the kind relations of the elements \(a_i\) and \(a_j\) to each other.

As an example we consider a set of firms with a certain economic structure between them. The structure can be determined either in the form of...
trade relations, financial transactions or information flows. Usually, relations in an economy are directed. The set $M$ consists of three plants $a_1$, $a_2$ and $a_3$. There are given the following relationships: $a_1$ provides certain products to $a_2$ and to $a_3$; $a_2$ delivers products to $a_3$. The ordered pairs for these relation follows to: $[a_1, a_2]$, $[a_1, a_3]$ and $[a_2, a_3]$. We can represent the relationship geometrically. This representation is called a directed graph (Fig. 4).

![Directed graph](image)

Fig. 4. Directed graph

A graph consists of the elements of the set and the relations. In such a way a graph can give determined relationships between elements of a set. In general a graph is a structure model which mirrors aspects of the structure of the investigated object. In order to obtain this structure we have to consider the relations between the corresponding partial objects. Let us note, that both the notion "vertex" (vertices) and "node" is used to describe the elements. Their relations are represented by "edges" or "arcs". Sometimes, the notion of an "arc" is used for an directed arc only. Other synonyms are link or line.

We consider a system to be a certain representation of a real phenomenon where certain boundaries between the system and its environment can be defined. A system is, in general, defined by the set of its elements and the set of relations between the elements. Obviously, with regard to the original phenomenon, different degrees of adjusting the model to the origin are possible. If a unique map exists between the structure of the original phenomenon and the structure of the model, the model is called homeomorph. The border line case of maximum adjustment of the model to the original in respect to the structure is called isomorph.

### 2.3 Structures with Random Relations and Their Application to Socio-Economic Groups

In the previous subsection we represented the individual units (e.g., plants or firms) as elements (vertices) of a graph. The relation of the units are shown by the edges. Let us introduce several simplifications. First we exclude the relation of a node to itself (self-reflexibility) from our consideration, as this leads to a special type of graph. Further we only consider graphs with distinguishable vertices and indistinguishable directed edges. A directed graph is also called a digraph. We exclude loops and parallel edges. We concentrate
in this section on random graphs. The capacity of a network (graph) – the number of its elements – is one of its important functions. Of relevance for the structure is also the connectivity of a network. Already 1970 Gardner and Ashby investigated the probability of the stability of great networks, in dependence on capacity and connectivity. May performed similar investigations. Computer simulations and some statements of probability about the structural behaviour of networks may be found also in papers of Sonntag, Feistel, Ebeling and Sonntag. We now introduce several important terms that characterize a network, following in part the work mentioned above.

If \( K \) is the number of edges in the graph, and \( S \) is the number of vertices, we denote the considered graph by \( D(S, K) \). As a simple and important measure, the connectivity is defined as:

\[
C = \frac{K}{S}.
\]  

(1)

Isolated, connected parts of the digraph \( D(S, K) \) consisting of \( s \) vertices and \( k \) arcs are called components \( d(s, k) \). Let us note here that the “basic step in the structural description of a network is to identify the number and size of its components” [111], p. 102. Socio-economically interpreted, the existence, number and size of components in a graph stands for the opportunities and obstacles to communicate, to exchange information or/and to interact.

\( S^*_k \) is the number of components with \( s \) vertices and \( k \) arcs. So we can write:

\[
\sum_{s,k} s S^*_k = S; \quad \sum_{s,k} k S^*_k = K
\]

(2)

We denote the mean values of the frequency of the component \( d(s, k) \) by:

\[
H^*_k = \langle S^*_k \rangle = \sum_r r P^*_k, \quad \text{with } P^*_k \text{ being the probability that in a special digraph } D(S, K) S^*_k = r \text{ components } d(s, k) \text{ can occur.}
\]

As mean number of components of the digraph \( D(S, K) \) we use:

\[
\eta = \sum_s \sum_k H^*_k.
\]

(4)

Apart from the trivial components, such as single vertices \((d(1, 0))\) and single arcs \((d(2, 1))\), the digraph contains a "structured part", whose number of components is written as follows:

\[
L = \eta - H^*_0 - H^*_1.
\]

(5)

The probability distribution \( P^*_k \) is not known for the finite digraphs. In the limit \( S \to \infty \) the probabilities for the emergence of different finite large
components are uncorrelated, so we have to work with a Poisson distribution for the possibility of \( r \) components with \( s \) vertices and \( k \) arcs in the whole graph [118]:

\[
\lim_{S \to \infty} P_{k,r}^s = \frac{(H_k^s)^r}{r!} \exp(-H_k^s) \quad \text{for} \quad S \gg s.
\]  

(6)

Fig. 5. The probability distribution \( P_{3,r}^3 \) for \( C = 0.05 \) that \( r \) components \( d(3,2) \) occur

In dependence on \( C \) we can for instance represent \( P_{0,r}^1, P_{1,r}^2, P_{2,r}^3, P_{3,r}^3 \) over \( r \). So we can show the building of components in dependence on the connectivity \( C \) (Fig. 5).

Following Sonntag, Feistel, Ebeling [118] we can achieve further statements. In the limit \( S \to \infty \) the mean number of components \( \eta \) in practice is determined only by semicycleless components (trees). So we can use the
approximation:

\[ \eta = \sum_{s=1}^{S} H_{s-1}^s + O(S^0) \]  

(7)

In that limit \( S \to \infty \) the number of non-trivial components:

\[ L = \eta - S \exp(-2C) - SC \exp(-4C) \]  

(8)

follows.

By a constant number of \( S \) of individuals, firms (vertices) the building of relations between these individuals, firms corresponds to an increasing number \( K \). \( K \) is proportional to \( C \):

\[ C = \frac{K}{S} \sim K; \quad S \to \text{const.} \]  

(9)

In such a way, and intuitively clear, the connectivity \( C \) increases during the development of relations. We will show that firstly the number of small components increases and then decreases with increasing \( C \). Great components become important. In the end all components are “absorbed” into one great component.

In our case, great and strong connected components \( d(s, k) \) are of interest. This means that a great number of relations exists \( k \geq s \), so that the components are strong connected. Strong connected components are characterized by the appearance of cycles. For \( S \to \infty \) the frequency of cycles of the length \( l \) in a graph \( D(S, K) \) is:

\[ H_C^{(l)} = \frac{C^l}{l} \]  

(10)

and for semicycles with the length \( l \)

\[ H_C^{l} = \frac{(2C)^l}{2l} \]  

(11)

The number of vertices of \( D(S, K) \) belonging to cycles is:

\[ Z_C = C^3 (1 - C) \quad \text{for } S \to \infty; \quad C < 1/2 \]  

(12)

and of those belonging to semicycles

\[ Z_0 = 4 C^3 (1 - 2C) \quad \text{for } S \to \infty; \quad C < 1/2. \]  

(13)

The number of vertices of \( D(S, K) \) belonging to one tree in \( D(S, K) \) referred to \( S \) is equal to the value given by Erdős and Rényi [42]:

\[ y = \lim_{S \to \infty} \frac{R_{K,S}}{S} = \begin{cases} 1 & C \leq 1/2 \\ x(C)/2C & C > 1/2 \end{cases} \]  

(14)
with

\[ R_{K,S} = \sum_{s=1}^{S} s \, H_{s}^{2x-1} \] (15)

\[ x \, e^{-x} = 2 \, C \, e^{-2C} \] (16)

\[ x(C) = \sum_{s=1}^{\infty} \frac{s^{x-1}}{s!} \left( 2 \, C \, e^{-2C} \right)^{s} \] (17)

The graphical representation shows analogies with a phase transition of 2nd order – a fact that becomes particularly obvious if such networks are considered by means of the percolation theory [117]. For \( C < 1/2 \) the probability is zero, that from a vertex a cycle goes out. For \( C \geq 1/2 \) the probability increases up to the value one. Opposite for \( C < 1/2 \) a vertex belongs with the probability one to a tree. For \( C \geq 1/2 \) that probability goes to zero.

For \( C > 1/2 \) the graphs \( D(S,K) \) consist of one great component. For \( C > 1/2 \) with growing \( C \) value, one tree after the other is “absorbed” by the great component. With an increasing number of arcs ever more trees are linked up with one great component until, finally, all vertices are linked with each other (Fig. 6).

The number of components in directed graphs also corresponds to the value given by Erdős and Rényi [42] for undirected graphs (Fig. 7):

\[ \bar{\delta} = \lim_{K \rightarrow C} \frac{\delta}{K} = \begin{cases} 1 - C & 0 \leq C \leq 1/2 \\ \frac{1}{2C} \left( x(C) - \frac{x^{2}(C)}{2} \right) & C > 1/2 \end{cases} \] (18)

For the examination of the development of socio-economic connections, it is interesting to inspect the share of the graph, that is structured. If we consider a network build from firms the number of the isolated vertices (firms) can be determined. In particular, they are those which do not belong to the connection net. In the same way, the number of components (without isolated vertices) can be obtained. This is the part of the graph that includes the economic or knowledge exchange network.

If we understand economic connections as a manifold of possibilities, we can arrive at some conclusions. In particular, we will observe the network of interaction between firms over time. We keep the number of firms (vertices) in the network constant and let the connections between them develop, i.e. \( K \), the number of edges, will increase. On the other side \( K \) is proportional to \( C \). In accordance to intuition, the connectivity rises if connections develop. As the calculation shows, at first the number of the little components rises and then it decreases with the further rising of \( C \). The number of the great components rises up to the moment when all are absorbed in one. Inside this great component the degree of connectedness of the network plays a
The relative number of vertices of $D(S, K)$ belonging to a tree ($y$) in dependence on the connectivity $C$. A sign for this strong connectedness is the occurrence of cycle in the graph. As we have shown, the frequency of cycles of the length $l$ is a simple function of the connectivity $C$. Likewise, the probability can be given that a cycle comes from a vertex. For $C < 1/2$ the probability is zero, that means that the number of the vertices is unimportant. By a rising development of connections ($C > 1/2$) this probability rises up to 1.
These findings have consequences for the development of information flows in growing socio-economic networks. The question is if a certain type of developing connections can also be observed in empirical data about information flows. For instance, inside or between organizations of firms. One possible method to trace information inside of organizations is the use of e-mail traffic. In a recent paper Tyler, Wilkinson and Huberman [122] used e-mail data to map the communication network inside an industrial organization. In this case the nodes represent people and the connections between them are given by e-mail correspondence. They identified components of different size and interpreted the smaller components as expressions of communities in practice. As far as they reported, the structure of the e-mail networks shows similarities to the structure of the organization. In addition, leadership roles can be identified this way. To relate such empirical observations to the results repeated above, one would need to look at the temporal evolution of components and connectivity in time.

Another recent approach using percolation theory to describe socio-economic change was proposed by Silverberg and co-authors [113,112]. Starting with a space of discrete technologies which have a certain performance, they looked for spanning connected paths in this technological landscape. In this picture, invention can be visualized as isolated islands located ahead of a technological frontier which moves forward. Under the critical value there will only be finite connected sets (clusters or components) of different technologies, and technological change might come to an end. If we remember that firms are the carriers for different technologies, there is a direct link to the concept of a network of firms exchanging information about different technologies and doing search processes in such an abstract technological space.
Then, the different components would be related to groups of firms sharing certain technological knowledge or being part of one chain of production. The increase in connectivity characterizes the diffusion of a certain technology in a specific industrial sector or across sectors.

If we consider a firm, and do not look at the whole graph, additional aspects can be included: If we understand the concept of the economic connections as a sum of abilities which the firm has acquired, we can write in simple terms:

\[
\text{competence} = \sum \text{of the abilities}
\]

If a firm or plant creates a new economic connection, it can be in interaction with others. In the picture of the graph it will be represented, as mentioned previously, through a chain from \(i\) to \(j\).

![Fig. 8. Edge, which goes from \(i\) to \(j\)]

The firm \(i\) builds a connection to firm \(j\). The connection from \(i\) to \(j\) is described by \(a_{ij}\). The elements \(a_{ij}\) build the adjacency matrix. If the sum of economic connections from \(i = \sum_j a_{ij}\) can be represented in a specific way we speak of firm \(i\) as a source in the network (Fig. 9). The sum of connections corresponds to the sum of arcs which leave the vertex \(i\). This is exactly the sum over the elements of row \(i\) of the adjacency matrix \(A\).

![Fig. 9. Source]

Connections which related a set of other firms to the firm \(i\) can also be illustrated in the graph-picture (Fig. 10). The sum of these connections is equal \(\sum_j a_{ji}\). This corresponds to the sum of the elements of the column \(i\) of the adjacency matrix.
Connections which are based on mutual relations at different times, can be described in the graph-picture by establishing two arcs between two vertices (Fig. 11).

In that picture the sum of these connections is proportional to the sum of the cycles which leave the vertex $i$. So we can get the number of cycles of the length $l$ from the $l$th power of the adjacency matrix $A^l$.

Finally let us underline that this is just a very small part of the network theory which we explained here. We concentrated on topics relevant to our task of describing selforganization and evolution processes.

3 Innovation Processes and Other Stochastic Effects in Networks

3.1 Overview of Stochastic Effects in Networks

3.1.1 Birth and Death Processes A special stochastic process called “birth and death process” is of particular importance to our study of stochastic effects in economic processes, and in particular in innovation processes. A birth process is a random appearance of a new element in a system. A death process is the disappearance of an element. Processes of this kind play a big role in biology, ecology and sociology.

Processes of this type are also relevant to the field of economic processes. Economic growth is characterized by structural changes based on the
introduction of new technologies in the economic world. To describe technological evolution, one has to determine the system, the elements and their interactions. Here we consider plants and technologies. We consider firms composed from different plants. The plants are introduced as elementary units which play the role of decision carriers according to market conditions (choosing a new technology or not). Plants also play the role of users of a particular technology. The technologies are understood as the different types present in the system. The plants are the elements or representatives of these technologies. In this perspective, eventually technologies are competing for plants using them. This perspective differs from the way one usually thinks about technological change, where the firm is central. The underlying process is still a decision made by plants or firms. However, the model approach constructs an inverse perspective on it. Let us note here that this perspective is quite normal for any population dynamic approach which deals with types (groups or species) and elements (individuals). However, in contrast to biological processes, human beings, organizations and firms are not bound to a certain type or group they first belong to. In contrast to individuals of biological species they have the opportunity to change the group they belong to. It is this kind of transition behaviour that makes the model particularly relevant for socio-economic applications. Let us note further that we deliberately use the notion of a plant or production unit as the simplest element in the system. By assuming that firms consist of several plants or production units, growth processes of firms are also covered by the model approach. Technological change is usually considered as a macroeconomic change process. However, in order to describe it as an evolutionary process, one has to consider this process at the microscopic level. This means we have to consider the microeconomic carriers of technological changes. In the framework we present here, these are the plants.

The basic ideas for the modelling of these processes go back to so-called urn models. Already in 1907 the physicists Paul and Tatyana Ehrenfest developed a simple model for the diffusion of $N$ molecules. The Ehrenfests studied two urns, A and B, which were isolated with respect to exchange with their surroundings. With respect to exchange between urns the Ehrenfests assumed permeable connections between A and B. Because of the isolation of the two urns, the total number of molecules in A and B remains constant. At regular time-intervals, a molecule is randomly (that means with the probability $1/N$) chosen and changes from its urn to the other urn.

In 1926 Kohlrausch and Schrödinger gave a continuous diffusion-approximation for such processes. Feller formulated a realistic variant of this model. He used a discrete Markovian process with continuous time. The time between the molecule crossings was exponentially distributed.

Originally developed for molecular processes, the model soon found many applications to biological processes. Surveys of biological applications of birth and death type processes were given by Bartholomay and Eigen.
The model of Ehrenfest represents the prototype for the investigation of decision processes in a group between the possibilities A and B respectively between yes and no [31]. For example, we may consider the decision to accept a new technology or not. Applications to social and economic processes were surveyed by Weidlich [129].

3.1.2 Stochastic Effects in Small and Sensitive Networks Many complex systems display a surprising degree of tolerance to errors. The results indicate a strong correlation between robustness and network topology. In particular, scale-free networks are more robust than random networks against random node failures, but are more vulnerable when the most connected nodes are targeted [4].

In small networks any nodes or edges play a specific role. Their addition or removal drastically changes the properties of the whole system. Another problem where stochastic effects play a big role is the question of how a single new mutant can win the selection process. If one considers networks of web sites and competition processes between them about attracting visitors one would ask how can a new web site become a giant cluster among other already important web sites (clusters)? Is it possible to overcome the “once-forever” selection behaviour by stochastic effects?

The addition or removal of sensitive nodes or edges is a subject of investigation in big networks. With sensitive we mean here elements which play a special role in the network. Let us refer to some examples.

Cellular networks can be subject to random errors as a result of mutations or protein misfolding, as well as harsh external conditions eliminating essential metabolites. Jeong et al. [64] studied the responses of the metabolic networks of several organisms to random and preferential node removal. Removing up to 8% of the substrates, they found that the average path length did not increase when nodes were removed randomly. However, it increases rapidly after the removal of the most-connected nodes and up to 500% when only 8% of the nodes are removed. Similar results have been obtained for the protein network of yeast as well [63, 124].

Sóle and Montoya [115] studied the response of the food webs to the removal of species (nodes) (see also [83]). The results indicate that random species removal causes the fraction of species contained in the largest cluster to decrease linearly. However, when the most connected (keystone) species are successively removed, the relative size of the largest cluster quickly decays.

The error and attack tolerance of the Internet and the World Wide Web was investigated by Albert, Jeong and Barabási [4]. The Internet is occasionally subject to hacker attacks targeting some of the most connected nodes. They show that the average path length on the internet is unaffected by the random removal of as many as 60% of the nodes, while if the most connected nodes are eliminated (attack), the average path length peaks at a very small fraction of removed nodes. Albert, Jeong and Barabási investigated the World
Wide Web [3] and showed that the network survives as a large cluster under high rates of failure, but under attack, the system abruptly crashes. These authors write: “The result is that scale-free networks display a high degree of robustness against random errors, coupled with a susceptibility to attacks.”

Wagner and Fell [125] studied the clustering coefficient, focusing on the energy and biosynthesis metabolism of the Escherichia coli bacterium. They found that in addition to the power-law degree distribution, the undirected version of this substrate graph has a small average path length and a large clustering coefficient.

Bianconi and Barabási [12] showed the existence of a closed link between evolving networks and an equilibrium Bose gas. According to them, the mapping to a Bose gas predicts the existence of two distinct phases as a function of the energy distribution. In the fit-get-rich-phase, the fitter nodes acquire edges at a higher rate than older but less fit nodes. In the end, the fittest node will have the most edges, but the richest node is not an absolute winner, since its share of the edges decays to zero for large system size.

Maurer and Huberman [79] present a dynamic model of web site growth in order to explore the effects of competition among web sites. They show that under general conditions, as the competition between sites increases, the model exhibits a sudden transition from a regime in which many sites thrive simultaneously, to a “winner takes all market” in which a few sites grab almost all the users, while most other sites become nearly extinct. This prediction is in agreement with empirical data measurements on the nature of electronic markets.

Dorogovtsev, Mendes and Samukhin [22] developed a statistical mechanic approach for random networks. They summarize: “Using the traditional formalism of statistical mechanics, we have constructed a set of equilibrium statistical ensembles of random networks without correlation and have found their partition function and main characteristics. We have shown that a “scale-free” state in equilibrium networks without condensate may exist only in a single marginal point, so that in such an event this state is an exhibition.” They underline the important fact that this differs crucially from the situation for growing networks. The latter, while growing, self-organize into scale-free structures in a wide range of parameters without condensation.

3.2 Stochastic Analysis of Innovation Processes

3.2.1 A General Formulation of the Model The stochastic approach given here is based on a model, which was developed in the context of general models of evolutionary processes, and in particular biochemical processes [60, 100, 27, 35]. Later, the model found numerous applications in modelling scientific evolution [32, 18] and technological evolution [15, 16]. In this paper, we first introduce the model framework in a general form and later concentrate on its application to economic innovation and competition processes.
In the stochastic picture we use the ideas developed in subsection 1.1 and in the subsection 3.1.1 about the occupation number space. Opposite to the deterministic models, the stochastic description offers the advantage that at finite times new types (sorts, fields, species, technologies) can arise or “die out”.

Let us introduce a set of types numbered by \( i = 1, 2, \ldots, s \). We denote by \( N_i(t) \) the number of elements belonging to a certain type. For an economic application \( N_i(t) \) represents the number of plants using the technology \( i \) (Note, that a type corresponds to an urn in the Ehrenfest problem formulation.). These numbers are called occupation numbers. They are a function of time. The occupation numbers are positive or zero.

\[
N_i(t) = \{0, 1, 2, \ldots\}. \tag{19}
\]

Now, the state of the system at the time \( t \) can be described by the probability distribution of the occupation numbers

\[
P(N_1, N_2, \ldots, N_s; t) = P(N; t) \tag{20}
\]

We consider as elementary processes – processes during which only one occupation number can change and as transition processes – processes during which at most two occupation numbers can change:

\[
\begin{align*}
(N_i) & \rightarrow (N_i + 1) \\
(N_i) & \rightarrow (N_i - 1) \\
(N_i, N_j) & \rightarrow (N_i - 1, N_j + 1)
\end{align*} \tag{21}
\]

For the time being, we also assume that growth and decline processes of the total number of elements in the system are possible. However, let us note here that the original formulation of the Ehrenfest model only contains transition processes. This is due to the fact that in the Ehrenfest model the total number of elements remains constant and only exchange between the urns occurs. Decision processes in the model occur as transitions of elements between types.

If we assume that all decisions leading to a change of the set of the occupation numbers depend mostly on the present state, we can apply the concept of Markov process. Then, we can describe the dynamics of the system with the help of the master equation. This equation is a balance equation between building and reduction processes:

\[
\frac{\partial P(N; t)}{\partial t} = W(N|N') P(N') - W(N'|N) P(N) \tag{22}
\]

with

\[
N = \{N_1, N_2, \ldots, N_s\}. \tag{23}
\]
The transition probabilities per time unit that the system turns from the state \( N' \) to the state \( N \) or vice versa are expressed by \( W(N|N') \) respectively \( W(N'|N) \).

The transition probabilities are supposed as follows [44, 27, 65, 35, 57].

1. Spontaneous generation

\[
W(\ldots, N_i + 1, \ldots, N_j, \ldots, N_k, \ldots | \ldots, N_i, \ldots, N_j, \ldots, N_k, \ldots) = A^{(0)}_i 
\]

(24)

2. Self-reproduction

\[
W(\ldots, N_i + 1, \ldots, N_j, \ldots, N_k, \ldots | \ldots, N_i, \ldots, N_j, \ldots, N_k, \ldots) = A^{(1)}_{ij} N_j + E^{(1)}_i N_i 
\]

(25)

\[
E^{(1)}_i = A^{(1)}_i + B^{(1)}_{ij} N_j + C^{(1)}_{ijk} N_j N_k 
\]

(26)

3. Decay

\[
W(\ldots, N_i - 1, \ldots, N_j, \ldots, N_k, \ldots | \ldots, N_i, \ldots, N_j, \ldots, N_k, \ldots) = E^{(2)}_i N_i 
\]

(27)

\[
E^{(2)}_i = A^{(2)}_i + B^{(2)}_{ij} N_j 
\]

(28)

4. Conversion/Transition/Exchange/Mutation

\[
W(\ldots, N_i + 1, \ldots, N_j - 1, \ldots, N_k, \ldots | \ldots, N_i, \ldots, N_j, \ldots, N_k, \ldots) = E^{(3)}_j N_j 
\]

(29)

\[
E^{(3)}_j = A^{(3)}_{ij} + B^{(3)}_{ij} N_i + \tilde{B}^{(3)}_{ik} N_k + C^{(3)}_{ijk} N_i N_k 
\]

(30)

with \( j \neq i; k \neq i; j \).

The coefficients can be differently introduced for special cases. For instance, they can considered as constant or as being functions of the total number of elements and other the system parameters. Later approach introduces an additional non-linearity to the system.

The content of the four elementary processes introduced above will be quite different according to the nature of the system under consideration. For instance, for catalytic networks, self-reproduction may appear as result of a process of spontaneous self-reproduction (term related to \( A^{(1)}_i \)), error reproduction (\( B^{(1)}_{ij} \)) or catalytic self-reproduction (\( C^{(1)}_{ijk} \)). The decay can appear in the form of spontaneous decay (\( A^{(2)}_i \)) and of decay related to catalytic help
(\(B_{ij}^{(2)}\)). Transition or conversion processes in catalytic networks correspond to mutation processes with reproduction and ternary reproduction processes, which play a role with regard to processes with constant overall particle number.

In the case of technological evolution, self-reproduction appears as a growth process of firms expanding their number of production units and plants using the same technology. Synergetic effects from the surrounding network of firms are supposed to take place when the transition rate also depends on the number of firms using another technology. Spontaneous generation stands for startups. Decay processes both refer to a decrease of firm size (closing of production units) and a closing down of firms. The most interesting process is related to conversion or transition. Here, the use of another technology by a production unit is described. The use of a technology may be new for the firm only (firm-specific innovation) or for the whole system of firms (system-specific innovation). Triggered by R&D, both invention of a technology and imitation behaviour is covered by this process. Also in this case, other technologies might influence the decision of a firm for a certain technology. The advantage of this type of model is that technological change is considered as the outcome of the development of a network of technologies and firms influencing each other.

The transition probabilities are formulated generally. As a special case we can get from this Ansatz the stochastic equations, which correspond to the deterministic EIGEN-model \[10\] with the condition of constant overall particle number \[65,36,57\].

With the help of the s-dimensional generation function

\[F(s_1, s_2, \ldots, s_s; t) = \sum_{N} s_1^{N_1} s_2^{N_2} \ldots s_s^{N_s} P(N_i, t)\quad \text{with } |s_i| < 1 \quad (31)\]

we can write the master equation with the transition probabilities as follows:

\[
\begin{align*}
\dot{F}(s; t) &= \sum_{i \neq j} \left\{ A_{ij}^{(0)} (s_i - 1) F + A_{ij}^{(1)} s_i (s_i - 1) \frac{\partial F}{\partial s_i} \right. \\
&\quad + B_{ij}^{(1)} s_i s_j (s_i - 1) \frac{\partial^2 F}{\partial s_i \partial s_j} \\
&\quad + C_{ijk} s_i s_j s_k (s_i - 1) \frac{\partial^3 F}{\partial s_i \partial s_j \partial s_k} \\
&\quad + A_{ij}^{(2)} (1 - s_i) \frac{\partial F}{\partial s_i} \\
&\quad + B_{ij}^{(2)} s_i (1 - s_i) \frac{\partial^2 F}{\partial s_i \partial s_j} \\
&\quad + A_{ij}^{(3)} (s_i - s_j) \frac{\partial F}{\partial s_j} \\
&\quad + B_{ij}^{(3)} s_i (s_i - s_j) \frac{\partial^2 F}{\partial s_i \partial s_j} + \bar{B}_{ik}^{(3)} s_k (s_i - s_j) \frac{\partial^2 F}{\partial s_k \partial s_j}
\end{align*}
\]
\[ + C_{ijk}^{(3)} s_i s_k (s_i - s_j) \frac{\partial^3 F}{\partial s_i \partial s_j \partial s_k} \] (32)

3.2.2 A Network Representation of the Model In order to relate the model to the idea of sensitive networks we now introduce a network representation of transition probabilities given in (24)–(30). We consider a system with \( s \) interacting types (sorts, fields, plants). This system can be described by a graph in which each element \( i \) corresponds to a vertex of the number \( i \). We mark the transition probabilities for different processes by edges of different type in order to distinguish these probabilities:

1. Spontaneous generation (simple innovation)
   
   \( A_i^{(0)} \)

2. Self-reproduction
   
   \( A_i^{(1)} N_i \)

   Error reproduction
   
   \( A_{ij}^{(1)} N_j \)

   Catalytic self-reproduction (sponsored self-reproduction)
   
   \[ \begin{cases} B_{ij}^{(1)} N_i N_j \\ C_{ijk}^{(1)} N_i N_j N_k \end{cases} \]

3. Spontaneous decay
   
   \( A_i^{(2)} N_i \)

   Catalytic decay
   
   \( B_{ij}^{(2)} N_i N_j \)

4. Mutation (innovation)
   
   \( A_{ij}^{(3)} N_j \)

   Mutation (innovation) with reproduction
   
   \[ \begin{cases} B_{ij}^{(3)} N_i N_j \\ C_{ijk}^{(3)} N_i N_j N_k \end{cases} \]

   Mutation without reproduction
   
   \( \tilde{B}_{ik}^{(3)} N_k N_j \)
For under-occupied systems the transition from a non-occupied \((N_i = 0)\) state to an occupied \((N_i > 0)\) state is of special interest. We will call such a transition an innovation. To describe this in the network picture we distinguish two states for one vertex. The states are marked as follows:

\[
\begin{align*}
\text{o} & \quad \text{non-occupied} & N_i &= 0 \\
\text{•} & \quad \text{occupied} & N_i &> 0
\end{align*}
\]

We also distinguish two states for the edges of the graph. Edges which go out from unoccupied vertices \((N_i = 0)\) are omitted, because they are inactive (they cannot work). Active (working) edges are characterized by the graphic representation shown above. For socio-economic networks, non-occupied vertices stay for new technologies, which have not yet discovered. They can be understood as hidden possibilities. The model does not allow for the prediction of a certain new technology but it can make statements how the system handles the appearance of new technologies in general.

Eventually, we get a graph which describes the whole system at a fixed time \(t\). If the colour of a vertex is changed, new connections (elementary processes) can flare up or former connections can no longer continue. Processes can spread the elements on the non-occupied vertices or they can select them from the occupied vertices. The basic structure of the network is the “maximal” graph (all vertices \(i\) are occupied, all reactions can work).

In a network with a small overall number of elements (individuals, organizations, plants) \((N \ll s)\), a lot of vertices are not occupied. Let us consider a graph with \((N \gg s)\) and reduce the overall number of elements to a state \((N \ll s)\). In general, vertices such as \(i\) and \(j\) become non-occupied by the following processes: \(A^{(2)}_{ij}N_i\), \(B^{(2)}_{ij}N_iN_j\), \(A^{(3)}_{ij}N_j\), \(B^{(3)}_{ij}N_iN_j\), \(\bar{B}^{(3)}_{ik}N_iN_j\), \(C^{(3)}_{ijk}N_iN_jN_k\). These are the processes of decay and the processes of conversion. If only a certain part of the process works because of the small overall number, we get a graph with a lot of components. With a decreasing number of elements (individuals, organizations, plants) the maximal graph with few components develops to a graph with a lot of components \[27\]. The “minimal” graph we can get (maximally decomposed), consists of self-reproduction-processes, spontaneous generation (simple innovations), sponsored innovation-processes and sponsored innovation-processes with self-reproduction (self-reproducing process). Vice versa, vertices can be occupied by the following processes: spontaneous generation (simple innovation) \(A^{(0)}_i\), error reproduction \(A^{(1)}_{ij}N_j\), mutation without reproduction (innovation without self-
reproduction) $A_{ij}^{(3)} N_j$, and mutation without reproduction but with catalytic help (sponsored innovation without self-reproduction) $B_{ik}^{(3)} N_k N_j$.

We assume that the processes through which the vertices can be occupied are very rare. This means spontaneous generations (simple innovations) and mutations have a small probability. After a relatively short time the components are in a local equilibrium (first process). Then, sponsored innovation-processes play a role starting from an occupied component which is in equilibrium. New components (non-occupied) can be occupied (second process). This is a hopping process between components. Certain components die out under selection pressure, others survive and growth.

3.3 Application to Technological Innovations

3.3.1 Technologies with Linear Growth Rates In the following, we consider a system in which the total number of elements is constant. As mentioned above, a useful instrument to model such processes is the theory of stochastic transitions between urns, established by Ehrenfest. Compared with the general model introduced so far in the case of a closed system, all elementary processes appear as transition processes. Further, let us restrict ourselves to the case of technological evolution. Then, the urns stay for different technologies. Symbolic spheres travelling between the urns stand for plants looking for technologies.

$N_i$ is the number of plants, which uses the technology $i$, this means they belong to the urn $i$. This numbers are called occupation numbers:

$$N_i(t) = \{0, 1, 2, \ldots \} .$$

(33)

First, we formulate a simple model for the binary decision process:

$$N = N_1 + N_2 .$$

(34)

We assume that during elementary processes the occupation number only changes by $\pm 1$. This is the so called one-step-process. During transition processes at most two occupation numbers can change in such a way:

$$\left( \begin{array}{c} N_1 \\ N_2 \end{array} \right) \rightarrow \left( \begin{array}{c} N_1 - 1 \\ N_2 + 1 \end{array} \right) .$$

(35)

For instance, we assume that $E_1$ is the growth rate of plants using technology 1. For a new technology 2 the growing rate is $E_2$. We assume:

$$E_2 > E_1 .$$

(36)

In this case the technology 2 has a greater growth potential or will grow faster than technology 1. We will assume that this is an expression for technology 2 to be “better” or more suitable for plants. In the stochastic picture the
transition from urn 1 to urn 2 will simply have a higher probability than the other way around. So, plants will more often change their technology towards technology 2. They will replace the old technology by a better (new) one.

We write the transition probability for this process as follows:

$$W^+ (N_2 + 1 | N_2) = E_2 \frac{N_1}{N} + E_{21} = W_{N_2}^+$$  \hspace{1cm} (37)

In particular, we assume, that the transition probability is proportional to the number of plants that use the old technology:

$$W^+(N_2) \sim N_1$$  \hspace{1cm} (38)

Furthermore, the probability is also proportional to the relative number of plants that use the new technology

$$W^+(N_2) \sim \frac{N_2}{N}$$  \hspace{1cm} (39)

We introduce an additional process of spontaneous change from technology 1 to technology 2 which is described by the coefficient $E_{21}$. The opposite spontaneous transition appears with the coefficient $E_{12}$. In total the opposite transition process has the probability:

$$W^- (N_2 - 1 | N_2) = E_1 \frac{N_1}{N} + E_{12} = W_{N_2}^-$$  \hspace{1cm} (40)

We can formulate the master equation for this discrete process. This equation describes the time-behaviour of the probability distribution of the occupation numbers.

$$P(N_1, N_2; t)$$  \hspace{1cm} (41)

With the mentioned transition probabilities follows:

$$\frac{\partial}{\partial t} P(N_1, N_2; t) = W_{N_2-1}^+ (N_2 | N_2 - 1) P(N_2 - 1; t)$$

$$+ W_{N_2+1}^- (N_2 | N_2 + 1) P(N_2 + 1; t)$$

$$- W_{N_2}^+ (N_2 + 1 | N_2) P(N_2; t)$$

$$- W_{N_2}^- (N_2 - 1 | N_2) P(N_2; t)$$  \hspace{1cm} (42)

and

$$\frac{\partial}{\partial t} P(N_1, N_2; t) = \left[ E_{21} + \frac{E_2}{N} (N_2 - 1)(N_1 + 1) \right] P(N_1 + 1, N_2 - 1; t)$$
\[ P(N_1, N_2; t) = \frac{E_2}{N} (N_2 + 1)(N_1 - 1) \quad P(N_1 - 1, N_2 + 1; t) \]

\[ - [E_{21} + E_{12} + (E_1 + E_2) N_2 (N - N_2)] \quad P(N_1, N_2; t) \quad (43) \]

If we use the relation \( N_1 + N_2 = N \) (i.e., \( N_1 = N - N_2 \)), we can write:

\[
\frac{\partial}{\partial t} P(N_2; t) = \left[ E_{21} + E_2 N (N_2 - 1)(N - N_2 + 1) \right] P(N_2 - 1; t)
\]

\[
+ \left[ E_{12} + \frac{E_1}{N} (N_2 + 1)(N - N_2 - 1) \right] P(N_2 + 1; t)
\]

\[
- \left[ E_{21} + E_{12} + \frac{(E_1 + E_2)}{N} N_2 (N - N_2) \right] P(N_2; t) \quad (44) \]

To obtain statements for the deterministic case, we define the mean value:

\[
\langle N_2(t) \rangle = \sum_{N_2=0}^{\infty} N_2 \ P(N_2; t). \quad (45) \]

By multiplying the master equation with \( N_2 \) and a following summation, we obtain:

\[
\frac{d}{dt} \langle N_2(t) \rangle = \frac{E_2 - E_1}{N} \langle N_2(N - N_2) \rangle + (E_{21} - E_{12}). \quad (46) \]

Using the approximation \( \langle (N_2)^2 \rangle \approx \langle N_2 \rangle^2 \) and the abbreviations

\[
x_2 = \frac{\langle N_2 \rangle}{N}; \quad \alpha = E_2 - E_1; \quad \beta = \frac{E_{21} - E_{12}}{N} \quad (47) \]

we achieve the corresponding deterministic equations:

\[
\frac{dx_2}{dt} = \alpha x_2 (1 - x_2) + \beta \quad (48) \]

Let us compare the stationary behaviour of both the stochastic and the deterministic model. In the case of two technologies we can explicitly derive the stationary solution of the master equation:

\[
P^0(N_2) = \frac{W^+(N_2) W^+(N_2 - 1) \cdots W^+(N - 1)}{W^-(N_2 + 1) W^-(N_2 + 2) \cdots W^-(N)} \quad P^0(N) \quad (49) \]

In the deterministic case the stationary solution for \( E_2 > E_1 \) is:

\[
x_2 = 1, \quad \text{that is} \quad N_2 = N \quad (50) \]
The new (better) technology will replace the old one. If a final stable stationary solution is reached by the system all plants will use the new technology.

For the stochastic case we get a quite different picture. In particular, the probability distribution \( P(N_2) \neq 0 \) for \( N_2 \neq N \). That means, in the stochastic case the old and new technology will coexist, given that the difference between \( E_1 \) and \( E_2 \) is not too large. Otherwise, the old technology will only survive in a small niche.

An interesting case occurs, if the spontaneous rates are \( E_{21} = E_{12} = 0 \). In this case the states \( N_1 = 0, N_2 = 0 \) are called absorber states. This means, the system can not leave these states. For the stationary solution of the master equation we get:

\[
P^0(N_2) = \sigma_1 \delta_{0,N_2} + \sigma_2 \delta_{N,N_2} : \sigma_1 + \sigma_2 = 1 \tag{51}
\]

\( \sigma_i \) is a real number between zero and one, From the absorber states \( N_2 = 0 \) and \( N_1 = 0 (N_2 = N) \) the other states can not be reached.

From the initial state follows, which stationary solution is occupied by the system. After some calculations we can give for \( \sigma_2 \) in the limit \( N \gg 1 \) and \( N_2(t = 0) \) the following equation:

\[
\sigma_2 = \begin{cases} 
0 & \text{for } E_2 < E_1 \\
1 - \left( \frac{E_1}{E_2} \right)^{N_2(0)} & \text{for } E_2 > E_1 
\end{cases} \tag{52}
\]

with \( N_2(0) = N_2(t = 0) \) the initial state of the system. Is \( N_2(0) \) the number of users at time \( t = 0 \) of the technology 2, then \( \sigma_2 \) is the probability, that for \( t \to \infty \) \( N_2 = 0 \) users changes to technology 2. \( \sigma_1 \) is the probability that for \( t \to \infty \) \( N_2 = 0 \) (i.e., \( N_1 = N \)) users changes to technology 2, this means the new technology has not survived. In general, \( \sigma_i \) is the survival probability of technology \( i \).

If a small number of plants \( N_2(0) \) use a new technology, this technology will disappear if its growth rate \( E_2 \) is smaller than that of the old one. If the growth rate \( E_2 \) of the new technology is considerable larger, the new technology will be successful with the probability.

\[
1 - \left( \frac{E_1}{E_2} \right)^{N_2(0)} \tag{53}
\]

Then, with the small probability:

\[
\left( \frac{E_1}{E_2} \right)^{N_2(0)} \tag{54}
\]

the old technology will still be used in the system.

In the deterministic case a new technology with higher growth rates is always successful. This is the case of pure Darwinian selection where the fittest and only the fittest survives at the end of the process.
What we observe in socio-economic system is usually a variety of technologies. This can be either explained due to the action of mutation processes and error reproduction or due to the presence of stochastic processes. As we showed above in the stochastic situation the new better technology will only survive with a certain probability. Empirical studies of growth processes in ensembles of technologies might shed light on which growth mechanism is present in a certain system.

3.3.2 Technologies with Quadratic Growth Rates

In the following we consider again the case with absorber states, that means we neglect the terms $E_{21}, E_{12}$. As an extension to the model introduced above, instead of the linear terms ($E_1, E_2$) we introduce now non-linear components ($b_1, b_2, V$). $V$ is a parameter which in biochemical applications stays for the spatial volume of the system. In general terms it is related to the size of the system. Remember that we still have a system with a constant total number of elements (plants). So all processes appear as transition processes. The non-linearities might stay for processes of catalytic self-reproduction as well as for transition or mutation processes influences by other types (technologies). We can write the transition probabilities as:

$$W^+(N_1) = \frac{b_1}{NV} N_2 N_1^2$$  \hspace{1cm} (55)

$$W^-(N_1) = \frac{b_2}{NV} N_2^2 N_1$$  \hspace{1cm} (56)

(compare [36] in detail for an application to biochemical systems). We assume that the sum of the occupation numbers remains constant:

$$N_1 + N_2 = N = \text{const.}$$  \hspace{1cm} (57)

The probability distribution of the occupation numbers is:

$$P(N_1, N_2; t)$$  \hspace{1cm} (58)

Analogous to the above chapter we can formulate the master equation as follows:

$$\frac{\partial}{\partial t} P(N_1; t) = W^+(N_1 - 1) P(N_1 - 1; t) + W^-(N_1 + 1) P(N_1 + 1; t)$$

$$- \left[ W^+(N_1) + W^-(N_1) \right] P(N_1; t)$$  \hspace{1cm} (59)

By multiplying the master equation with $N_k/V$ and summing over all occupation numbers, we can get after factorization of the mean values the deterministic equation:

$$\langle N_k \rangle/V = x_k \quad \text{with} \quad x_1 + x_2 = \frac{N}{V} = C = \text{const.}$$

$$\dot{x}_i = b_i x_i^2 - \varphi x_i ; \quad i = 1, 2$$  \hspace{1cm} (60)
with

$$\varphi = \frac{b_1 x_1^2 + b_2 x_2^2}{C}$$

Equations of the same form have been derived for so-called hypercyclic system to describe the evolution of macromolecules [40]. Their behaviour is very well understood. As a result of the quadratic terms in the growth rates the phase space is split into two regions separated by a separatrix $S_i$:

$$s x_i = 0 \quad s S_i \quad s x_i = C$$

Fig. 12. Phase space in the deterministic case

Let us also note here that because of the $N$ equal constant condition the dimensionality of the system reduces. Due to the nature of the transition process, the occupation numbers in the system change along a diagonal of the two-dimensional phase space. In the deterministic picture, the selection behaviour depends on the initial conditions $x_i(t = 0)$ of the system. A certain technology $i$ only can win ($x_i = C$ for $t \to \infty$), if the initial condition places the system beyond the point of the separatrix:

$$x_i(0) > S_i; \quad S_i = \frac{C b_j}{b_i + b_j}$$

This way it is possible that a new technology $i$, even when having a greater growth rate, will not win because at the beginning the number of users (plants) is too small.

$$x_i(t = 0) < S_i$$

Such a situation has been called once-forever selection or hyperselection. In the case of macromolecular evolution this feature was used to explain the uniqueness of the genetic code. For technological evolution we have argued elsewhere that hyperselection is an alternative explanation for so-called lock-in phenomena of technologies [1516]. The situation changes again if we look at the stochastic picture. Here the phase space looks as follows:

$$N_2 = 0 \quad S_2 V \quad N_2 = N = CV$$

Fig. 13. Phase space in the stochastic case
The two absorber states for the stochastic case are again:

\[ N_2 = 0, N_1 = N, \quad (64) \]
\[ N_2 = N, N_1 = 0. \quad (65) \]

The initial state is \( N_2(t) = N_2(t = 0) \):

\[ P(N_2; 0) = \delta_{N_2 N_2(t=0)}. \quad (66) \]

The final state is:

\[ P(N_2; t = \infty) = \sigma_1 \delta_0 N_2 + \sigma_2 \delta_{N_2}. \quad (67) \]

After some calculations we get for the case \( N_2(0) = 1 \) (one user of technology 2 occurs at \( t = 0 \)):

\[ \sigma_2 = \frac{1}{\left(1 + \frac{b_1}{b_2}\right)^{N-1}}. \quad (68) \]

\( \sigma_2 \) is the probability, that the new technology wins the competition process.

![Figure 14](image)

**Fig. 14.** Survival probability of a new technology in systems of different size (fat line \( N = 10 \), dashed line \( N = 40 \), dotted line \( N = 70 \), fine line \( N = 100 \))

In the stochastic case, the separatrix \( S \) is penetrated with a certain probability. The once-forever behaviour disappears. The better technology can win the selection process with certain probability even if it starts only with
one user at the beginning. This probability increases rapidly for small overall numbers of users, as follows from the equation and as it can be seen in Fig. 14. In the stochastic description, the presence of fluctuations is responsible to help the technology to cross the barrier for entry in the market.

This effect is size dependent. In smaller systems the probability for survival of new technologies even in a hyperselection situation increases. This can be also interpreted in economic terms. A small size of the system refers to the size of the market in which the competition between the old and the new technology takes place. If a new technology is protected by certain mechanisms it can win the process in a market of limited competition. This way it can obtain a significant number of early adopters and then go to a more open and bigger market with an improved chance of survival. In other words, in small niches, even in a lock-in situation, a new technology can replace the old one. By travelling from one niche to another or in growing niches (dynamic niche), the new technology would even be able to get control over the whole market. There is some empirical evidence that such processes take place in technological change (for a discussion see [19]).

3.3.3 Technologies with Mixed Growth Rates In the following, we consider the general case where the growth rates of a certain technology contain both linear as well as non-linear terms. The transition probabilities for this case are:

\[
W^+(N_1) = \frac{E_1}{N} N_1 N_2 + \frac{b_1}{NV} N_2 N_1^2 \quad (69)
\]

\[
W^-(N_1) = \frac{E_2}{N} N_1 N_2 + \frac{b_2}{NV} N_1 N_2^2 \quad (70)
\]

Again here we assume that \(E_{12} = E_{21} = 0\), so we have absorber states. Further, the number of elements (plants) in the system is constant: \(N_1 + N_2 = N = \text{const.}\). In analogy to the case studies above we can write the master equation:

\[
\frac{\partial}{\partial t} P(N_1; t) = W^+(N_1 - 1) P(N_1 - 1; t) + W^-(N_1 + 1) P(N_1 + 1; t) - [W^+(N_1) + W^-(N_1)] P(N_1; t) \quad (71)
\]

In the same way as describes above we can derive the deterministic counterpart to the stochastic dynamics.
For the general case the survival probability for a new technology $\sigma_{N_2(0),N}$ can be calculated:

$$
\sigma_{N_2(0),N} = \frac{1 + \sum_{j=1}^{N_2(0)-1} \prod_{i=1}^{j} \frac{E_1 + b_1 N - i}{E_2 + b_2 i V} \sum_{i=1}^{N-1} \prod_{i=1}^{j} \frac{E_1 + b_1 N - i}{E_2 + b_2 i V}}{1 + \sum_{j=1}^{N-1} \prod_{i=1}^{j} \frac{E_1 + b_1 N - i}{E_2 + b_2 i V}}
$$

(72)

This result was first obtained in [36] and extensively applied to socio-economic problems in [16]. As can be seen from the formula above, the survival probability for a new technology $\sigma$ not only depends on the selection advantage (the relation of the parameters $E_i, b_i$), but also on the size of the system $N$ (the overall user number) and the initial number of users.

3.3.4 Summary – General Aspects of the Survival Probability of a New Technology

The stochastic analysis of competition processes of technologies in a market where the number of overall users is restricted shows remarkable differences to the results known from a deterministic description. Let us note here that with the restriction to markets with a fixed number of economic agents, we restrict ourselves to technological substitution processes. However, using the approach of under-occupied systems this restriction does not represent a limitation for the innovation process. A variety of future possible innovative types is included in the system all times. The population of agents (fixed in size) can unlimited travel over the field of innovations. Already with this restriction the stochastic analysis compared with the deterministic one represents an increase in mathematical complexity at the descriptive level. In the multidimensional general case an analytical solution of the master equation is not available. Some results can be obtained using computer simulations [106, 57, 18]. However, what we showed so far is how statements about the survival probability in the long run can be derived. Results are available for certain special cases as we presented above. Let us summarize the results for the special case of two-dimensional systems with $N_1 + N_2 = N = \text{const.}$ and with the two absorber states $N_2 = 0, N_2 = N$. Absorber states means here that once the states $N_2 = 0$ or $N_2 = N$ are reached, they cannot be left anymore.

As mentioned above, this case can be treated exactly [106, 57, 18]. According to the absorber character we may assume that the stationary probability which is the target of evolution has a delta-character and can be written as:

$$
P(N_2; t = \infty) = \sigma \delta_{N,N_2} + (1 - \sigma) \delta_{0,N_2}.
$$

(73)

Here, $\sigma$ is the survival probability of the second technology which is supposed to be the new one entering the market. An expression for the survival prob-
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ability $\sigma$ can be calculated with the help of one constant of motion for the general case [36]:

$$
\sigma_{N_2(0),N} = \frac{1 + \sum_{j=1}^{N_2(0)-1} \prod_{i=1}^{j} \frac{W_i^-}{W_i^+}}{1 + \sum_{j=1}^{N-1} \prod_{i=1}^{j} \frac{W_i^-}{W_i^+}} \quad \text{for} \quad 0 < N_2(0) < N \quad (74)
$$

and

$$
\sigma_{N_2(0),N} = 1 \quad \text{for} \quad N_2(0) = N \quad (75)
$$

As we introduced in subsection 3.2.1, the transition probabilities $W^+$ and $W^-$ represent quite a variety of different processes which all are linked to certain parameters. Further, the initial conditions and the system size $N$ are important for the survival of a new technology (see for details [16]).

In particular we considered the following representations of transition probabilities:

1. For technologies with linear growth rates (linear case)

$$
W_{N_2}^+ = E_2 \frac{N - N_2}{N} N_2 ; W_{N_2}^- = E_1 \frac{N - N_2}{N} N_2 \quad (76)
$$

2. For technologies with quadratic growth rates (quadratic case)

$$
W_{N_2}^+ = b_2 \frac{N - N_2}{NV} N_2^2 ; W_{N_2}^- = b_1 \frac{(N - N_2)^2}{NV} N_2 \quad (77)
$$

3. For technologies with certain mixed growth rates (general case)

$$
W_{N_2}^+ = E_2 \frac{N - N_2}{N} N_2 + b_2 \frac{N - N_2}{NV} N_2^2 \quad (78)
$$

$$
W_{N_2}^- = E_1 \frac{N - N_2}{N} N_2 + b_1 \frac{(N - N_2)^2}{NV} N_2 \quad (79)
$$

We derived certain expressions for the survival probabilities by solving the absorber problem. These expressions can be treated further and certain special cases can be discusses.

1. For the linear case the survival probability has been defined as

$$
\sigma_{N_2(0),N} = \frac{1 - \left( \frac{E_1}{E_2} \right)^{N_2(0)}}{1 - \left( \frac{E_1}{E_2} \right)^{N_1}} \quad (80)
$$
We now consider the special case of very large systems (many possible users). In this case $\sigma$ can be simplified to:

$$\sigma_{N(0), N \to \infty} = 0 \quad \text{for} \quad E_2 < E_1$$  \hspace{1cm} (81)

and

$$\sigma_{N(0), N \to \infty} = 1 - \left( \frac{E_1}{E_2} \right)^{N_2(0)} \quad \text{for} \quad E_2 > E_1 .$$  \hspace{1cm} (82)

If we then consider a certain initial condition, the case that a technology starts with just one user (plant) $N_2(0) = 1$, we can reduce the expression for the survival probability one step further

$$\sigma_{N_2(0)=1, N \to \infty} = 0 \quad \text{for} \quad E_2 < E_1$$  \hspace{1cm} (83)

$$\sigma_{N_2(0)=1, N \to \infty} = 1 - \frac{E_1}{E_2} \quad \text{for} \quad E_2 > E_1 .$$  \hspace{1cm} (84)

2. For the quadratic case the survival probability has been defined as

$$\sigma_{N_2(0), N} = 1 + \sum_{j=1}^{N_2(0)-1} \left( \frac{b_1}{b_2} \right)^j \left( \frac{N - 1}{j} \right) \left( \frac{1 + \frac{b_1}{b_2}}{N_2(0)} \right)^{N_2(0)-1}$$  \hspace{1cm} (85)

This equation can be written down more specified if we consider a new technology starting with just one user $N_2(0)$:

$$\sigma_{N_2(0), N} = \frac{1}{\left(1 + \frac{b_1}{b_2}\right)^{N_2(0)}}$$  \hspace{1cm} (86)

3. In the general case the survival probability has the following form.

$$\sigma_{N_2(0), N} = \frac{1}{\left(1 + \sum_{j=1}^{N_2(0)-1} \prod_{i=1}^{j} \frac{E_1 + b_1}{E_2 + b_2 \frac{V}{i}} \right)^{N_2(0)-1}}$$  \hspace{1cm} (87)

In the special case of a technology starting with just one user $N_2(0) = 1$ we obtain:

$$\sigma_{N_2(0), N} = \frac{1}{\left(1 + \sum_{j=1}^{N-1} \prod_{i=1}^{j} \frac{E_1 + b_1}{E_2 + b_2 \frac{V}{i}} \right)^{N-1}}$$  \hspace{1cm} (88)
In general, the transition probabilities depend on the system size, the system parameters and the initial conditions. In the stochastic case we obviously find a niche effect. In the niche, the sharpness of selection is diminished. In the linear case, good and bad technologies can temporally coexist. In the quadratic case, the once-forever effect of a competition and selection process is countered by the niche effect. The niche represents a possibility for a better technology to win the competition even in the situation that a lock-in effect can be observed. Locally developed niches may play a constructive role in the technological evolution and can be observed in large complex systems, such as economic systems. The description, with the help of the master equation, includes processes, which play a role in these small domains (niches). In a niche, the new technology is protected against extinction for a limited time. After winning the competition in this small area, the new technology can infect the whole system, and may be established at the end.

3.4 Stochastic Analysis of Multiple Decision Processes – the Modelling of Technological Networks

3.4.1 Ehrenfest’s Urn-Models with Higher Correlations

In subsection 3.1.1 we introduced the urn-model of Ehrenfest. The idea of the Ehrenfest urn model was extended towards processes which also change the total number of elements in the system. In the last subsections we restricted ourselves to the Ehrenfest approach by considering transition probabilities only, and derived analytical results for the survival probability for the special case of two competing technologies. In this section we use again the idea of a system with constant size, but extend the dimensionality of the system. Here, we have in mind under-occupied systems where the number of possible different technologies is large. Only the number of searching plants is restricted. Now we will consider the competition between \( s \) different technologies. This corresponds to the existence of \( s \) urns, which are filled with \( N_1, N_2, \ldots, N_s \) spheres. We start a stochastic game where, at random times, spheres are taken out of an urn and put to another urn. Here we consider a game with binary decisions, where the transition from one urn \( j \) to another urn \( i \) is given by the following transition probabilities.

\[
W(N_1, \ldots, N_i + 1, \ldots, N_j - 1, \ldots, N_s|N_1, \ldots, N_i, \ldots, N_j, \ldots, N_s) = A_{ij} N_j + B_{ij} N_i N_j + \sum_k C_{ijk} N_i N_j N_k
\]  

(89)

This is a generalization of the transition rates introduced so far in the case of two technologies. In particular, we have in mind the mutual support or hindering of technologies. As discussed in the economic literature, innovations are often the outcome of an innovation network with different actors involved. We can assume that the decision of a certain plant or firm to implement a
certain technology depends both on other firms using the same technologies and also on other firms using different, but somehow related, technologies. The stochastic game we propose here is a hopping process of plants between technologies.

The mean values of the occupation numbers in the thermodynamic limit approximately follows a differential equation. In the sections above we showed such approximation for the case of two technologies with linear, quadratic and specific non-linear growth rates. The deterministic equations we obtained correspond to Fisher-Eigen-Schuster equations. For the generalization proposed above a set of Lotka-Volterra equation follows in the deterministic limit:

$$\frac{d}{dt}x_i = \sum_j \left( A_{ij} x_j + B_{ij} N x_i x_j + \sum_k C_{ijk} N^2 x_i x_j x_k \right)$$  \hspace{1cm} (90)

with $\frac{\langle N_i \rangle}{N} = x_i$.

The general case can only be handled by computer simulations. The equation above comprises the case of two technologies with non-linear growth rates if we introduce the following correspondence between parameters 31:

$$A_{21} = 0 ; \quad B_{12} = \frac{E_1}{N} ; \quad C_{12k} = \frac{b_1}{N} \delta_{1k} ; \quad k = \{1; 2\}$$  \hspace{1cm} (91)

$$A_{12} = 0 ; \quad B_{21} = \frac{E_2}{N} ; \quad C_{21k} = \frac{b_2}{N} \delta_{2k}$$  \hspace{1cm} (92)

$$W(N_2 + 1, N_1 - 1|N_1, N_2) = E_2 \frac{N_1 N_2}{N} + b_2 \frac{N_2^2 N_1}{N}$$  \hspace{1cm} (93)

### 3.4.2 Decision Processes and the Dynamics of a Network of Technologies

In this section we return to the general dynamics of interacting technologies. In subsection 3.2.1 we gave a short economic interpretation of processes like spontaneous generation, self-reproduction, decay and conversion or transition. All these processes can be interpreted in terms of decision processes by firms or plants related to expansion or shrink and related to the choice of different technologies from a set of technologies available. The case of innovation interpreted as first occupation of a so far unoccupied technological possibility is just one process in a whole set of decision processes made by firms. The substitution case between old and new technologies that we discussed earlier in detail represents a very specific decision. In the following, we will give a detailed interpretation of different possible decision processes (for further economic interpretation please consult also [10]).

We start again with the set of occupation numbers. $N_i$ is the number of plants using a certain technology $i$. We consider a network of $s$ different
technologies competing with each other $i = 1, \ldots, s$. We will no longer stick to the assumption that the total number of plants in the system will remain constant. This way, we also consider growth and decline processes not only of certain used technologies in the system but also of the system, the market as a whole.

We can differentiate between changes which take place inside one type of users (where the type or the group that the users belong to is characterized by the technology they use). For instance, the number of users of the technology $i$ may increase or decrease. This is a stepwise process which only changes the occupation number by one:

\[
\begin{align*}
(N_i) & \rightarrow (N_i + 1) \\
(N_i) & \rightarrow (N_i - 1)
\end{align*}
\]

(94)

Furthermore, changes can take place between different types of users. Mathematically this is expressed by the simultaneous change of two occupation numbers:

\[
\begin{pmatrix} N_i \\ N_j \end{pmatrix} \rightarrow \begin{pmatrix} N_i - 1 \\ N_j + 1 \end{pmatrix}
\]

(95)

Plants can take decisions to leave a certain technology they have been using before, they can develop a new technology, or use a technology already established in the market. They can also develop further a technology they have already used and in this way create a new type of technology. Not only technologies can enter the market; firms also can enter the market. The creation of a new firm might be connected to a established technology, related to the introduction of a new technology. The problem technologies face by their introduction into a market are comparable to the problems firms have by entering a certain market. Here barriers for entry might also occur which hinder a certain technology in entering a market. On the other side, a network of firms might create support for a new firm, or also for a new technology, to enter the market (the system). Coalitions and cooperation are examples of synergetic effects in the introduction of new technologies.

We will now formulate some possible changes in the language of transition probabilities:

1. The number of plants using a certain technology $i$ increases:

\[
N_i \rightarrow N_i + 1
\]

(96)

This change can be the result of different processes which are further differentiable.

- $A_i^0 N_i$ – linear self-reproduction: the number of plants will grow according to the existing number of plants. If these plants belong to one firm, firm growth is modelled. If they belong to different firms, the growth of an industrial sector is considered. The growth rate of this process is linear. Let us remember here that if only this process takes place in the system the technology would increase exponentially.
• $A_1^i N_i^2$ – self-amplification (second order self-reproduction): here the number of plants already using the technology creates a network effect of higher order which speeds up the growth momentum for this specific technology.

• $B_{ij} N_i N_j$ – sponsoring or supporting from other plants: in this case plants using a different technology $j$ are relevant for the growth of the technology $i$. One can think of systems of coupled technologies, one supporting the other, or of production chains where one technology relies on others.

For this case the transition probabilities can be written:

$$W(N_i + 1, N_j|N_i, N_j) = A_0^i N_i + A_1^i N_i^2 + B_{ij} N_i N_j \quad (97)$$

The different coefficients or parameters describe the strength of a certain effect which is acting in the system.

2. Spontaneous formation of a new plant with a certain technology

$$N_i \rightarrow N_i + 1 \quad (98)$$

• $\Phi_0$ – spontaneous formation: here the entry of plants is not connected to the number of those already existing. One can think of a startup. The startup can either start with an already existing technology or be linked to the development of a new technology. In the latter case, $N_i$ would be zero at the beginning. One would usually assume that the number of new plants created spontaneously is relatively low.

$$W(N_i + 1, N_j|N_i, N_j) = \Phi_0 \quad (99)$$

3. Decrease of the number of the plants using technology $i$

$$N_i \rightarrow N_i - 1 \quad (100)$$

• $D_0^i N_i$ – linear decrease: Given that each technology in the market occupies a certain niche in the market, one can assume that the number of plants using a technology which can not survive will depend on the size of the population of all plants using that technology.

• $D_1^i N_i^2$ – non-linear decrease or restricted capacity: this process remains for a network effect of the number of plants using a certain technology which drives plants out of the market. This process is responsible for the existence of a restricted capacity in markets with linear growing technologies. Without the existence of decrease terms of higher order, one would be confronted with infinite, exploding markets which stay in contradiction to empirical observations.

$$W(N_i - 1, N_j|N_i, N_j) = D_0^i N_i + D_1^i N_i^2 \quad (101)$$
4. Origin of a new technology connected to the formation of a new plant
(induced innovation)

\[
\begin{pmatrix}
0 \\
N_j
\end{pmatrix} \rightarrow \begin{pmatrix}
N_i = 1 \\
N_j
\end{pmatrix}
\]  

\( (102) \)

- \( M_{ij} \) \( N_j \) – Here we assume that the creation of a startup with a new technology is not a pure spontaneous process but is related to the number of plants using another (relevant for the new technology) technology.

\[
W(N_i = 1, N_j | N_i = 0, N_j) = M_{ij} N_j
\]  

\( (103) \)

5. Change in the use of a technology (conversion, transition)

\[
\begin{pmatrix}
N_i \\
N_j
\end{pmatrix} \rightarrow \begin{pmatrix}
N_i + 1 \\
N_j - 1
\end{pmatrix}
\]  

\( (104) \)

- \( A_{ij} \) \( N_j \) – simple transition from \( j \) to \( i \): in this case the decision to take over a new technology is only influenced by the number of plants using a certain technology. One can interpret this process in the following way. If the number of plants using the same technology increases, then the competition between these plants also increases and plants might be motivated to look for another technology to increase their chances on the market. In the case where the technology is not yet occupied (not yet invented), the transition will also create an innovation for the system. Let us note here that beside spontaneous generation and the induced innovation, this process is important for the exploration of new areas in the technological space.

- \( B_{ij} \) \( N_i, N_j \) – transition from \( j \) to \( i \), in addition is promoted by \( i \): in general one can assume that plants do not act in isolation. Conversely, the information flows between firms and plants about market conditions and technological change, are an important part of economic processes. Important in the process we discuss here is the decision to use a certain new technology \( j \) (here we use new in the sense that the technology is new for the plant) which is related to the number of firms already using this technology. We can further assume that the number of firms using a technology can be interpreted as a measure of attractiveness of this particular technology. In this case, the process represents one form in which imitation can be modelled.

- \( C_{ijk} \) \( N_i, N_j, N_k \) – transition from \( j \) to \( i \), that in addition is promoted (sponsored) by \( j \) and \( k \): this process represents one possibility to introduce a network effect in the decision process of one firm to use a certain technology. One can imagine that the technologies \( j \) and \( k \) are related in sense of a production chain or that they complement each other.
\[ W(N_i + 1, N_j - 1|N_i, N_j) \]
\[ = A_{ij} N_j + B_{ij} N_i N_j + \sum_k C_{ijk} N_i N_j N_k \]  
(105)

The introduced model is composed in a modular way. Different processes related to the change in the occupation number space have been introduced. We also called these processes elsewhere elementary processes. In the presentation above we tried to give examples of processes relevant in the decision behaviour of firms using different technologies. However, alternative definition and the introduction of further processes are possible within the model framework. The task consists of the definition of processes which can be observed empirically in the economy. The model represents a specific way to operationalize processes of decision making inside firms, the information flows between firms and the interactive pattern between technologies. By relating all these processes to the occupation number space a certain reduction of information takes place. On the other hand, the different parameters allows for the possibility of including further economic information. The advantage of the modular structure of the model is that it puts different processes together and places them in an evolutionary framework. Growth of firms, substitution processes, invention and imitation, startups and firm closing are all part of one model. If we link the model to its deterministic counterpart, the instrumentarium of dynamic systems becomes available. This way, at least, we can hope to get some insights in the analytic structure of the model and possible stationary states as well as their stability behaviour. Using the stochastic model for simulations we can obtain a lot of statements about the systems behaviour. Interesting investigations can be made by variation of the parameters. Different kinds of connections can be analyzed.

Beyond the economic interpretation chosen for the model, the framework can be also applied in quite different contexts. In any of these new application areas types, elements and the network of connections have to be interpreted at new again. Some of the authors applied the model for biochemical processes \[35\], for growth, competition and evolution of scientific specialties \[18\], and for the dynamics of values and competences \[104\].

### 3.4.3 Further Analysis of the Probability Distribution

In the case that we obtain the probability distribution \( P(N_1, \ldots, N_s; t) \) analytically, or by computer simulations, it is possible to get the time-behaviour of the moments (mean values, correlation functions) in dependence on the parameters of the system using the generation function \[57, 35\]. For the partial differential equation given in section 3.2.3 we can get an approximate solution by introducing the transformation \[90\]

\[ F = \exp \left[ \Psi(\eta_i) N_i \right] ; \quad \eta_i = s_i - 1 \]  
(106)
at which the function $\psi(\eta_i)$ is expanded in a Taylor series

$$
\psi = \sum_i a_i \eta_i + \frac{1}{2!} \sum_{i,j} b_{ij} \eta_i \eta_j + \frac{1}{3!} \sum_{i,j,k} c_{ijk} \eta_i \eta_j \eta_k + \ldots
$$

(107)

The coefficients $a_i$ and $b_{ij}$ are in terms of the moments of the probability distribution as follows:

$$
a_i = \langle N_i \rangle / N
$$

(108)

$$
b_{ii} = \frac{1}{N} \left[ \langle N_i^2 \rangle - \langle N_i \rangle \right]
$$

(109)

$$
b_{ij} = \langle N_i \eta_j \rangle
$$

(110)

These equations give us information about the deviation from the Poisson-distribution (Poisson-distribution means for the coefficients $b_{ij} = 0$, $\forall i, j$ and all exponential coefficients of higher order are also zero). From the equations we can get a closed set of differential equations for the mean values $a_i$ and the variances $b_{ij}$. Provided that we can neglect the coefficients higher than second order, we can achieve an approximative solution of the set of differential equations. Especially for processes which satisfy a multi-Poisson distribution, the coefficients higher than first order are zero in the Taylor expansion. This is the case if we get for $F$ or $\Psi$, respectively, a partial differential equation linear in $s_i$ and at most of first order. If we know in this case the mean values $a_i = \langle N_i \rangle / N$ and start at $t = 0$ with a multi-Poisson distribution, we can get an exact solution for the probability distribution. The solution is a multi-Poisson distribution for all times

$$
P(N; t) = \prod_i \frac{\langle N_i(t) \rangle^{N_i}}{N_i!} \exp[-\langle N_i \rangle]
$$

(111)

Now we answer the question, which processes (transition probabilities from subsection 3.24) lead to a multi-Poisson distribution, i.e., which processes have in the generating function at most terms with a first derivative with respect to $s_i$ and are linear in $s_i$. Then, the generating function looks as follows:

$$
\dot{F}(s; t) = \sum_j A_j^{(2)} (1 - s_i) \frac{\partial F}{\partial s_i} + A_j^{(3)} (s_i - s_j) \frac{\partial F}{\partial s_j}
$$

(112)

So we can get

$$
\dot{\psi}(\eta; t) = \sum_i \sum_j \left\{ -A_i^{(2)} \eta_i a_i + A_{ij}^{(3)} (\eta_i - \eta_j) \right\}
$$

(113)

$$
\dot{a}_i = -A_i^{(2)} a_i + \sum_{i \neq j} A_{ij}^{(3)} a_j - a_i \sum_{i \neq j} A_{ij}^{(3)}
$$

(114)
If we start at \( t = 0 \) with a multi-Poisson distribution, the probability distribution remains a multi-Poisson distribution for all times

\[
P(N; t) = \exp \left[ -\langle N(t) \rangle_C \right] \prod_i \frac{\langle N_i(t) \rangle^{N_i}}{N_i!} \tag{115}
\]

The extinction probability of the whole component is

\[
P(0; t) = \exp \left[ -\langle N_C(t) \rangle \right] \tag{116}
\]

This extinction probability depends on the overall number of individuals \( N_C \) of the component and decreases exponentially with \( \langle N_C(t) \rangle \).

A detailed description for the calculation of the moments of the probability distribution \( P(N_1, \ldots, N_s; t) \) and the probability distribution was given in the work of Heinrich and Sonntag [57]. The time-behaviour of the moments could be achieved with the help of generating function. Deviations from the Poisson behaviour were investigated. Fluctuations and their influence on the system structure and behaviour could be analyzed. The analysis of time-dependent correlation functions could also be interesting for the description of socio-economic systems.

4 Summary

This paper is devoted to the study of innovation processes in socio-economic contexts. In particular, we investigate the influence of stochastic effects on processes of self-organization and evolution. We take a special network perspective. Starting with the new emergent field of complex networks theory, we develop our own approach of sensitive networks relevant to the description of an innovation.

Innovation is first introduced on a general level as a specific process that changes the composition and dynamic constitution of a system. We use a discrete representation of the system in terms of a space of occupation numbers. Then, innovation can be described as a hopping process between positive cones. Further, we introduce the notion of an under-occupied system. This way we implement a set of possible future paths of developments in our modelling. We relate this abstract notion of an innovation to the discussion of innovation processes in economics.

In the economic literature, innovation has been understood as the outcome of processes running on networks of different actors. We concentrate on firms and technologies in this paper. We present a network theory of innovation by mapping the dynamic interactions related to the emergence of an innovation in form of a graph.

Figure 15 illustrates our approach. The system is composed of a large set of enumerable types. Each of these types is represented by a node. At a certain point in time, only a small part of these nodes are active. The
pattern of interaction between them (including processes of self-influence) determines the dynamic composition of the system. It is visualized in terms of (active) links between the nodes. This active network will produce a dynamics which has a certain set of stable states. We assume that the activated part of the network is embedded in a much larger network of inactive nodes and links. The inactive nodes represent future possibilities in the evolution of the system. An innovation appears when a unoccupied node becomes occupied for the first time. With this first occupation, the set of links connecting the "new" node with already occupied nodes also becomes activated. It is readily apparent that such an event changes the whole composition of the system. Accordingly, the stable state that the system might have reached already becomes instable and the system searches for a new stable state. If we assume that the interaction between the nodes (types) is a competitive one, the stable state of a certain activated network can also include the deactivation of certain nodes. Types (nodes) which are selected out will transit to a non-occupied inactive status.

We first discuss structural (static) properties of such a relational network. In this way, instruments from random graph theory and percolation theory become relevant. In particular, formulae for the probabilities of the occurrence and the distribution of components and cycles in large networks have been obtained by combinatorial considerations. We show that connectivity is a central measure in the structure of these kinds of networks. Evolving, dynamic networks show different phenomena compared to random graphs. The results better correspond to findings in real data of empirical networks such as the appearance of power law distributions of the degree function.

In the centre of the paper we turn to the discussion towards the dynamic properties of networks. In particular, we derive descriptions for a network of interacting technologies and interacting firms. We call these networks sensitive because innovation processes described in terms of the removal or appearance of a node might change the dynamic behaviour of the system dramatically. We use a stochastic description of such an evolving network and base this on the theory of birth and death processes. We introduce different forms of transition probabilities for closed systems as well as for open, growing and declining systems. We define master equations and non-linear differential equations as their deterministic equivalents. Contrary to deterministic models, the stochastic description offers the advantage that, at finite times, new technologies (types) can arise or "die out". Further, the emergence of an innovation can be treated as a singular stepwise event implying the transition from an under-occupied to an occupied state.

In some special cases the master equation can be solved analytically and a stationary survival probability for an innovation can be derived. We show that the stochastic dynamics differ essentially from the deterministic. Separatrices, which decompose the phase space, cannot be intersected in the deterministic case. In the stochastic case they can be crossed. This way, a
Fig. 15. Illustration of a sensitive network. Embedded in a large network of inactive nodes and links the activated part of the network (upper part of the figure) changes its composition when an innovation is emerging (lower part of the figure).

"once-for-ever" or hyperselection that is known in economics as lock-in phenomena for technologies can be avoided.

The stochastic model of networked dynamic interactions of technologies is further generalized in a multi-dimensional case. Processes representing different non-linearities are discussed in the context of technological change. Although we use technological evolution as main reference point, we also
point to the fact that the modular structure of the model also allows for its application in quite different fields. Applications in fields as biology, population theory and science of science have been presented [45][18].

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