Total positivity of Narayana matrices

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Abstract

We prove the total positivity of the Narayana triangles of type $A$ and type $B$, and thus affirmatively confirm a conjecture of Chen, Liang and Wang and a conjecture of Pan and Zeng. We also prove the strict total positivity of the Narayana squares of type $A$ and type $B$.

Keywords: Totally positive matrices, the Narayana triangle of type $A$, the Narayana triangle of type $B$, the Narayana square of type $A$, the Narayana square of type $B$.

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1. Introduction

Let $M$ be a (finite or infinite) matrix of real numbers. We say that $M$ is totally positive (TP) if all its minors are nonnegative, and we say that it is strictly totally positive (STP) if all its minors are positive. Total positivity is an important and powerful concept and arises often in analysis, algebra, statistics and probability, as well as in combinatorics. See \cite{1, 4, 7, 9, 10, 13, 14, 18} for instance.

Let $C(n, k) = \binom{n}{k}$. It is well known \cite{14, P. 137} that the Pascal triangle

\[
P = [C(n, k)]_{n, k \geq 0} = \begin{bmatrix}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\vdots & \ddots
\end{bmatrix}
\]

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is totally positive. Let

\[ P_{n,k} = \binom{C(n+k,k)}{n,k} \]

be the Pascal square. Then \( P^r = PP^T \) by the Vandermonde convolution formula

\[ \binom{n+k}{k} = \sum_i \binom{n}{i} \binom{k}{i}. \]

Note that the transpose and the product of matrices preserve total positivity. Hence \( P^r \) is also TP.

The main objective of this note is to prove the following two conjectures on the total positivity of the Narayana triangles. Let 

\[ NA(n,k) = \frac{1}{k+1} \binom{n+1}{k} \binom{n}{k}, \]

which are commonly known as the Narayana numbers. Let

\[ N_A = [NA(n,k)]_{n,k \geq 0} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \\ 1 & 4 & 10 & 20 & \cdots \\ \vdots & & & & \ddots \end{bmatrix}. \]

The Narayana numbers \( NA(n,k) \) have many combinatorial interpretations. An interesting one is that they appear as the rank numbers of the poset of noncrossing partitions associated to a Coxeter group of type \( A \), see Armstrong [2, Chapter 4]. For this reason, we call \( N_A \) the Narayana triangle of type \( A \). Chen, Liang and Wang [10] proposed the following conjecture.

**Conjecture 1.1** ([10, Conjecture 3.3]). The Narayana triangle \( N_A \) is TP.

Let \( NB(n,k) = \binom{n}{k}^2 \), and let

\[ N_B = [NB(n,k)]_{n,k \geq 0} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 4 & 1 & \cdots \\ 1 & 9 & 9 & 1 & \cdots \\ 1 & 16 & 36 & 16 & 1 & \cdots \\ \vdots & & & & \ddots \end{bmatrix}. \]
We call \( N_B \) the Narayana triangle of type \( B \) since the numbers \( NB(n, k) \) can be interpreted as the rank numbers of the poset of noncrossing partitions associated to a Coxeter group of type \( B \), see also Armstrong [2, Chapter 4] and references therein. Pan and Zeng [16] proposed the following conjecture.

**Conjecture 1.2** ([16, Conjecture 4.1]). The Narayana triangle \( N_B \) is TP.

In this note, we will prove that the Narayana triangles \( N_A \) and \( N_B \) are TP just like the Pascal triangle in a unified approach. We also prove that the corresponding Narayana squares

\[
N_A^r = [NA(n + k, k)]_{n,k \geq 0} = \begin{pmatrix}
1 & 1 & 1 & 1 & \cdots \\
1 & 3 & 6 & 10 & & \\
1 & 6 & 20 & 50 & & \\
1 & 10 & 50 & 175 & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
\end{pmatrix}
\]

and

\[
N_B^r = [NB(n + k, k)]_{n,k \geq 0} = \begin{pmatrix}
1 & 1 & 1 & 1 & \cdots \\
1 & 4 & 9 & 16 & & \\
1 & 9 & 36 & 100 & & \\
1 & 16 & 100 & 400 & & \\
\vdots & \vdots & \vdots & \vdots & \ddots & \\
\end{pmatrix}
\]

are STP, as well as the Pascal square.

2. The Narayana triangles

The main aim of this section is to prove the total positivity of the Narayana triangles \( N_A \) and \( N_B \).

Before proceeding to the proof, let us first note a simple property of totally positive matrices. Let \( X = [x_{n,k}] \) and \( Y = [y_{n,k}] \) be two matrices. If there exist positive numbers \( a_n \) and \( b_k \) such that \( y_{n,k} = a_n b_k x_{n,k} \) for all \( n \) and \( k \), then we denote \( x_{n,k} \sim y_{n,k} \) and \( X \sim Y \). The following result is direct by definition.

**Proposition 2.1.** Suppose that \( X \sim Y \). Then the matrix \( X \) is TP (resp. STP) if and only if the matrix \( Y \) is TP (resp. STP).

Our proof of Conjectures 1.1 and 1.2 is based on the Pólya frequency property of certain sequences. Let \((a_n)_{n \geq 0}\) be an infinite sequence of real
numbers, and define its Toeplitz matrix as
\[
[a_{n-k}]_{n,k \geq 0} = \begin{bmatrix}
a_0 & a_0 & a_0 & & \\
a_1 & a_0 & & a_0 & \\
a_2 & a_1 & a_0 & & \\
a_3 & a_2 & a_1 & a_0 & \\
& \ddots & \ddots & \ddots & \ddots
\end{bmatrix}.
\]

Recall that \((a_n)_{n \geq 0}\) is said to be a Pólya frequency (PF) sequence if its Toeplitz matrix is TP. The following is the fundamental representation theorem for PF sequences, see Karlin [14, p. 412] for instance.

**Schoenberg-Edrei Theorem.** A nonnegative sequence \((a_0 = 1, a_1, a_2, \ldots)\) is PF if and only if its generating function has the form
\[
\sum_{n \geq 0} a_n x^n = \prod_j (1 + \alpha_j x) \prod_j (1 - \beta_j x) e^\gamma x,
\]
where \(\alpha_j, \beta_j, \gamma \geq 0\) and \(\sum_j (\alpha_j + \beta_j) < +\infty\).

Clearly, the sequence \((1/n!)_{n \geq 0}\) is PF by Schoenberg-Edrei Theorem, which implies that the corresponding Toeplitz matrix \([a_{n-k}] = [1/(n-k)!]\) is TP. Also, note that
\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} \sim \frac{1}{(n-k)!}.
\]
Hence the Pascal triangle \(P\) is TP by Proposition 2.1.

We are now in a position to prove Conjectures 1.1 and 1.2.

**Theorem 2.2.** The Narayana triangles \(N_A\) and \(N_B\) are TP.

**Proof.** We have
\[
N_A(n, k) = \frac{n!(n+1)!}{k!(k+1)!(n-k)!(n-k+1)!} \sim \frac{1}{(n-k)!(n-k+1)!}
\]
and
\[
N_B(n, k) = \frac{n!^2}{k!(n-k)!^2} \sim \frac{1}{(n-k)!^2}.
\]
So, to show that the Narayana triangles \(N_A\) and \(N_B\) are TP, it suffices to show that the sequences \((1/(n!(n+1)!))_{n \geq 0}\) and \((1/(n!)^2)_{n \geq 0}\) are PF. Based on a classic result of Laguerre on multiplier sequences, Chen, Ren and Yang [8,
Proof of Conjecture 1.1] already proved that the sequence \((1/(t_n n!))_{n\geq 0}\) is PF for any \(t > 0\), where \((t_n) = t(t+1)\cdots(t+n-1)\). Letting \(t = 2\) (resp. \(t = 1\)), we obtain the PF property of \((1/(n!(n+1)!))_{n\geq 0}\) (resp. \((1/(n!^2))_{n\geq 0}\)), as desired.

The method used here applies equally well to the triangle composed of \(m\)-Narayana numbers, which we will recall below. Fix an integer \(m \geq 0\). For any \(n \geq m\) and \(0 \leq k \leq n - m\), the \(m\)-Narayana number \(NA_{(m)}(n, k)\) is given by

\[
NA_{(m)}(n, k) = \frac{m+1}{n+2} \binom{n+2}{k+1} \binom{n-m}{k}.
\]

(2.1)

When \(m = 0\) we get the usual Narayana numbers \(NA(n, k)\). For more information on the numbers \(NA_{(m)}(n, k)\), see [20]. It is easy to show that the Narayana triangle \(NA\) is symmetric: \(NA(n, k) = NA(n, n - k)\), but

\[
NA_{(m)} = [NA_{(m)}(n, k)]_{n\geq m, 0\leq k\leq n-m}
\]

and

\[
\widehat{NA}_{(m)} = [NA_{(m)}(n, n - m - k)]_{n\geq m, 0\leq k\leq n-m}
\]

are two different triangles for \(m \geq 1\). The proof of Theorem 2.2 carries over directly to the following more general result.

\textbf{Theorem 2.3.} For any \(m \geq 0\), both \(NA_{(m)}\) and \(\widehat{NA}_{(m)}\) are TP.

3. The Narayana squares

The object of this section is to prove the total positivity of the Narayana squares \(NA^r_A\) and \(NA^r_B\). Our proof is based on the theory of Stieltjes moment sequences.

Given an infinite sequence \((a_n)_{n\geq 0}\) of real numbers, define its Hankel matrix as

\[
[a_{n+k}]_{n,k\geq 0} = \begin{bmatrix}
a_0 & a_1 & a_2 & a_3 & \cdots \\
a_1 & a_2 & a_3 & a_4 & \cdots \\
a_2 & a_3 & a_4 & a_5 & \cdots \\
a_3 & a_4 & a_5 & a_6 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

We say that \((a_n)_{n\geq 0}\) is a \textit{Stieltjes moment} (SM) sequence if it has the form

\[
a_n = \int_0^{+\infty} x^n d\mu(x),
\]
where \( \mu \) is a non-negative measure on \([0, +\infty)\). The following is a classic characterization for Stieltjes moment sequences (see [18, Theorem 4.4] for instance).

**Lemma 3.1.** A sequence \((a_n)_{n \geq 0}\) is SM if and only if

(i) the Hankel matrix \([a_{i+j}]\) is STP; or

(ii) both \([a_{i+j}]_{0 \leq i,j \leq n}\) and \([a_{i+j+1}]_{0 \leq i,j \leq n}\) are positive definite.

Many well-known counting coefficients are Stieltjes moment sequences, see [15]. For example, the sequence \((n!)_{n \geq 0}\) is a Stieltjes moment sequence since

\[
n! = \int_0^{+\infty} x^ne^{-x}dx = \int_0^{+\infty} x^n d\left(1-e^{-x}\right).
\]

Thus the corresponding Hankel matrix \([ (n+k)! ]\) is STP. Note that

\[
\binom{n+k}{k} = \frac{(n+k)!}{n!k!} \sim (n+k)!
\]

Hence the Pascal square \(P^r\) is also STP. The main result of this section is as follows.

**Theorem 3.2.** The Narayana squares \(N_A^r\) and \(N_B^r\) are STP.

**Proof.** We have

\[
NA(n+k,k) = \frac{(n+k)!(n+k+1)!}{k!(k+1)!n!(n+1)!} \sim (n+k)!(n+k+1)!
\]

and

\[
NB(n+k,k) = \frac{(n+k)!^2}{n!k!^2} \sim (n+k)!^2.
\]

So, to show that the Narayana squares \(N_A^r\) and \(N_B^r\) are STP, it suffices to show that the sequences \((n!(n+1)!)_{n \geq 0}\) and \(((n!)^2)_{n \geq 0}\) are SM.

Note that the submatrix of a STP matrix is still STP. Hence if the sequence \((a_n)_{n \geq 0}\) is SM, then so is its shifted sequence \((a_{n+1})_{n \geq 0}\) by Lemma 3.1(i). Now the sequence \((n!)_{n \geq 0}\) is SM, so is the sequence \(((n+1)!)_{n \geq 0}\). On the other hand, the famous Schur product theorem states that the Hadamard product \([a_{i,j}b_{i,j}]\) of two positive definite matrices \([a_{i,j}]\) and \([b_{i,j}]\) is still positive definite. As a result, if both \((a_n)_{n \geq 0}\) and \((b_n)_{n \geq 0}\) are SM, then so is \((a_nb_n)_{n \geq 0}\) by Lemma 3.1(ii). We refer the reader to [18, §4.10.4] for details. Thus we conclude that both \((n!(n+1)!)_{n \geq 0}\) and \(((n!)^2)_{n \geq 0}\) are SM, as required. \(\square\)
We can also consider the strict total positivity of the \(m\)-th Narayana square:
\[
N^r_{A,(m)} = \left[ N_A(m)(n+k,k) \right]_{n\geq m,k\geq 0},
\]
where \(N_A(m)(n,k)\) is given by (2.1). The following result can be proved in the same way as above.

**Theorem 3.3.** For any \(m \geq 0\), the square \(N^r_{A,(m)}\) is STP.

### 4. Remarks

There are various generalizations of classical Narayana numbers, see for instance [2, 5, 11, 12, 17]. As we mentioned before, the numbers \(N_A(n,k)\) (resp. \(N_B(n,k)\)) appear as the rank numbers of the poset of generalized noncrossing partitions associated to a Coxeter group of type \(A\) (resp. \(B\)). These posets are further generalized by Armstrong [2] by introducing the notion of \(m\)-divisible noncrossing partitions for any positive integer \(m\) and any finite Coxeter group. Armstrong also showed that these generalized posets are not lattices but are still graded.

Fixing an integer \(m \geq 1\), for \(n \geq k \geq 0\) set
\[
F_{NA}(m)(n,k) = \frac{1}{n+1} \binom{n+1}{k} \binom{m(n+1)}{n-k},
\]
\[
F_{NB}(m)(n,k) = \binom{n}{k} \binom{mn}{n-k}.
\]
These numbers are called the Fuss-Narayana numbers by Armstrong [2], who proved that \(F_{NA}(m,n,k)\) (resp. \(F_{NB}(m,n,k)\)) are the rank numbers of the poset of \(m\)-divisible noncrossing partitions associated to a Coxeter group of type \(A\) (resp. \(B\)).

Note that, for any \(m \geq 2\), we have
\[
F_{NA}(m)(n,k) \neq F_{NA}(m)(n,n-k), F_{NB}(m)(n,k) \neq F_{NB}(m)(n,n-k).
\]

Now define the Fuss-Narayana triangles
\[
F_{NA}(m) = \left[ F_{NA}(m)(n,k) \right]_{n,k\geq 0}, \quad F_{NA}^r = \left[ F_{NA}(m)(n,n-k) \right]_{n,k\geq 0},
\]
\[
F_{NB}(m) = \left[ F_{NB}(m)(n,k) \right]_{n,k\geq 0}, \quad F_{NB}^r = \left[ F_{NB}(m)(n,n-k) \right]_{n,k\geq 0},
\]
and the Fuss-Narayana squares
\[
F_{NA}^r(m) = \left[ F_{NA}(m)(n+k,k) \right]_{n,k\geq 0},
\]
\[
F_{NB}^r(m) = \left[ F_{NB}(m)(n+k,k) \right]_{n,k\geq 0}.
\]
We proposed the following conjecture.
Conjecture 4.1. For any $m \geq 1$, the Fuss-Narayana triangles are TP and the Fuss-Narayana squares are STP.

There are other symmetric combinatorial triangles, which are TP and the corresponding squares are STP. The Delannoy number $D(n, k)$ is the number of lattice paths from $(0, 0)$ to $(n, k)$ using steps $(1, 0), (0, 1)$ and $(1, 1)$. Clearly,

$$D(n, k) = D(n - 1, k) + D(n - 1, k - 1) + D(n, k - 1),$$

with $D(0, k) = D(k, 0) = 1$. It is well known that the Narayana number $NA(n, k)$ counts the number of Dyck paths (using steps $(1, 1)$ and $(1, -1)$) from $(0, 0)$ to $(2n, 0)$ with $k$ peaks. It is also known that $NA_m(n, k)$ counts the number of Dyck paths of semilength $n$ whose last $m$ steps are $(1, -1)$ with $k$ peaks, see Callan’s note in [20]. Brenti [6, Corollar 5.15] showed that the Delannoy triangle $D = [D(n - k, k)]_{n \geq k \geq 0}$ and and the Delannoy square $D^\leftarrow = [D(n, k)]_{n,k \geq 0}$ are TP by means of lattice path techniques. The following problem naturally arises.

Question 4.2. Whether the total positivity of Narayana matrices can also be obtained by a similar combinatorial approach?

We have seen that the Pascal square has the decomposition $P^\leftarrow = PP^T$. We also have $D^\leftarrow = P\text{diag}(1, 2, 2^2, \ldots )P^T$ since

$$D(n, k) = \sum_j 2^j \binom{k}{j} \binom{n}{j}$$

(see [4] for instance). A natural problem is to find out the explicit (modified) Choleski decomposition of the Narayana squares $N_A^\leftarrow$ and $N_B^\leftarrow$.

Another well-known symmetric triangle is the Eulerian triangle $A = [A(n, k)]_{n,k \geq 1}$ where $A(n, k)$ is the Eulerian number, which counts the number of $n$-permutations with exactly $k - 1$ excedances. Brenti [2, Conjecture 6.10] conjectured that the Eulerian triangle $A$ is TP. Motivated by the strict total positivity of the Narayana squares, we posed the following conjecture.

Conjecture 4.3. The Eulerian square $A^\leftarrow = [A(n + k, k)]$ is STP.

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