Spin Solitons in Magnetized Pair Plasmas

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A set of fluid equations, taking into account the spin properties of the electrons and positrons in a magnetoplasma, are derived. The magnetohydrodynamic limit of the pair plasma is investigated. It is shown that the microscopic spin properties of the electrons and positrons can lead to interesting macroscopic and collective effects in strongly magnetized plasmas. In particular, it is found that new Alfvénic solitary structures, governed by a modified Korteweg–de Vries equation, are allowed in such plasmas. These solitary structures vanish if the quantum spin effects are neglected. Our results should be of relevance for astrophysical plasmas, e.g. in pulsar magnetospheres, as well as low-temperature laboratory plasmas.

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I. INTRODUCTION

The magnetoplasma, first discussed by Alfvén [1], is today a mature research topic, with a wide range of applications. Recently, so called quantum plasmas in which the quantum properties of the plasma particles are taken into account, have received attention (see e.g. [2, 3, 4] for an up-to-date set of references). The collective motion of quantum particles in magnetic fields thus gives a natural extension to the classical theory of magnetohydrodynamics (MHD) in terms of so called quantum magnetoplasmas. The discussion of quantum plasmas has shown that quantum collective effects can have interesting consequences both in laboratory and astrophysical plasmas has shown that quantum collective effects can have interesting consequences both in laboratory and astrophysical environments [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], and part of the literature has indeed been influenced by experimental progress and techniques (see e.g. [19, 20, 21, 22, 23, 24, 25], where also strong field effects are discussed). From the laboratory perspective, the motion of particles with quantum spin in strong fields, using e.g. intense lasers, has attracted interest as a probe of quantum physical phenomena [26, 27, 28, 29, 30, 31]. These studies are however mainly focused on single particle properties. In Refs. [32, 33] however, the kinetic properties of spin plasmas was investigated. Strong fields appear in pulsar and magnetar environments [34, 35, 36]. Discussions of quantum plasmas in such environments can be found in Refs. [15, 16, 17, 18]. Moreover, studies taking both certain quantum electrodynamical effects, such as photon splitting, as well as collective particle effects have been made [37, 38].

In the present paper, starting from the basic set of equations presented in Refs. [3] and [4], we derive the governing MHD equations for a pair plasma. These MHD equations takes into account quantum properties, such as spin, of the electrons and positrons. It is found that the nonlinear propagation of Alfvén waves in such quantum MHD (QMHD) pair plasmas is governed by a modified Korteweg–de Vries equation. This equation is known to support solitary structures. Moreover, if the quantum spin effects are neglected, the solitary structures vanish, making them true quantum solitons. The results should be of interest for both laboratory and astrophysical plasmas.

II. GOVERNING EQUATIONS

In Refs. [3] and [4], it was shown that the evolution of spin plasma particles in a strongly magnetized environment is governed by the equations for the particle densities $n_j$ and velocities $V_j$

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j V_j) = 0,$$

and

$$mn_j \left( \frac{\partial}{\partial V_j} + \nabla \right) V_j = q_j n_j (E + V_j \times B) - \nabla P_j + \mathcal{C}_j + F_{\Omega,j},$$

where $q_j$ stands for the charge of the electrons ($q_e = -e$) or positrons ($q_p = e$), $m = m_e = m_p$ is the electron (positron) mass, $\mathcal{C}_j$ is the collisional contribution between species $j$ and the second species, and $F_{\Omega,j}$ is the total quantum force density (for a complete expression of this force density, see Refs. [3] and [4]). Here we will make use of the following expression for the quantum force on the electrons/positrons

$$F_{\Omega,j} = n_j \left[ \nabla \left( \frac{\hbar^2}{2mn_j^{1/2}} \nabla^2 n_j^{1/2} \right) + \tanh \left( \frac{\mu_B B}{T_j} \right) \mu_B \nabla B \right],$$

where the first term is the gradient of the so called Bohm potential, the second term comes from the spin and $B = |B|$. We note that $\tanh(x) = B_{1/2}(x)$, where $B_{1/2}$ is the Brillouin function with argument $1/2$ describing particles of spin $1/2$. The temperature $T_j$ is measured in units of energy. Furthermore, we have introduced the Bohr magneton $\mu_B = e\hbar/2m$, where $\hbar$ is Planck’s constant divided by $2\pi$.

The coupling between the quantum plasma species is mediated by the electromagnetic field. The total magnetic field include both the classical contribution (from currents $j = \sum q_j n_j V_j$) and the spin sources, such that Ampère’s law reads

$$\nabla \times B = \mu_0 (j + j_m) + \frac{1}{c^2} \frac{\partial E}{\partial t},$$

including the magnetization spin current $j_{M,j} = \nabla \times (2q_j n_j \mu_B \mathbf{S} / \hbar |q_j|)$ for each species, where $\mathbf{S}$ is the spin vector.
Furthermore, we need Faraday’s law
\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \] (5)

We note that the spin current is determined by the spin vector \( \mathbf{S} \). In general the spin vector follows a separate evolution equation, see Ref. [3], but in the limit where the time-scales are much longer than the Larmor period, the spin vector can be approximated by
\[ \mathbf{S} = \frac{\hbar}{2} \left[ q_j \right] \tanh \left( \frac{\mu_B B}{T_f} \right) \mathbf{B} \]
where \( \mathbf{B} \) is a unit vector in the direction of the magnetic field. The magnetization spin current is thus given by
\[ \mathbf{j}_M = \sum_j \nabla \times \left[ n_j \mu_B \tanh \left( \frac{\mu_B B}{T_j} \right) \mathbf{B} \right] \] (6)

### III. ELECTRON–POSITRON PLASMA AND THE MHD LIMIT

We now introduce the total mass density \( \rho \equiv m(n_e + n_p) \), the centre-of-mass fluid flow velocity \( \mathbf{V} \equiv m(n_e \mathbf{v}_e + n_p \mathbf{v}_p)/\rho \), and the current density \( \mathbf{j} = -en_e \mathbf{v}_e + en_p \mathbf{v}_p \), and assume that \( T_e, p = T_e, p \). Using these definitions, we immediately obtain
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0. \] (7)

Assuming quasi-neutrality, i.e. \( n_e \approx n_p \), we next add the momentum conservation equations for the electrons and positrons to obtain
\[ \rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \mathbf{j} \times \mathbf{B} - \nabla \cdot \mathbf{\Pi} + \mathbf{F}_Q, \] (8)
where \( \mathbf{\Pi} = \left[ (T_e + T_p)/2m_n \right] + (m^2/\rho) \mathbf{j} \mathbf{j} \otimes \mathbf{j} \) is the total pressure tensor in the centre-of-mass frame and
\[ \mathbf{F}_Q = \mathbf{F}_{Q,e} + \mathbf{F}_{Q,p} = \rho \left[ \nabla \left( \frac{\hbar^2}{2m^2 \rho^{1/2}} \nabla^2 \rho^{1/2} \right) \right. \\
\left. + \sum_j \tanh \left( \frac{\mu_B B}{T_j} \right) \frac{\mu_B}{m} \nabla \mathbf{B} \right]. \] (9)

The collisional contributions have here cancelled due to momentum conservation. Subtracting the momentum equations for the electrons and positrons, assuming \( \mathbf{\Pi}_e = \eta \mathbf{v}_e \mathbf{j} \), where \( \eta \) is the resistivity, we also obtain the generalized Ohm’s law
\[ \frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{j} \otimes \mathbf{V} + \mathbf{V} \otimes \mathbf{j}) = \frac{e^2}{m} \left( \mathbf{E} + \mathbf{V} \times \mathbf{B} - \eta \mathbf{j} \right) \\
- \frac{e}{2m^2} \Delta T \nabla \rho - \frac{e}{m} \Delta \mathbf{F}_Q, \] (10)
where \( \Delta T = T_e - T_p \), and \( \Delta \mathbf{F}_Q = \mathbf{F}_{Q,e} - \mathbf{F}_{Q,p} \). We note that in the limit of quasi-neutrality, the non-spin quantum force contributions to Ohm’s law cancel, thus only leaving the spin quantum force, as long as the species have different temperatures. Eq. (10) is valid during rather general conditions, and accounts for different temperatures of the species, as well as for short scale lengths of the order of the gyro-radius. From now on we will however limit ourselves to an equal temperature plasma, with scale lengths longer than the gyro-radius. Dividing by the density and taking the curl of Eq. (10), we obtain
\[ \frac{\partial \mathbf{B}}{\partial t} \equiv \nabla \times (\mathbf{V} \times \mathbf{B} - \eta \mathbf{j}) \] (11)

For the cases where the Alfvén speed is much smaller than the speed of light, the displacement current in (4) can be neglected, and thus the current can be expressed in terms of the magnetic field. In this case, Eqs. (7), (8) and (11) constitute a closed system for \( \mathbf{B}, \mathbf{V} \) and \( \rho \). But in plasmas with strong magnetic fields, for example close to pulsars and magnetars, the Alfvén speed approaches \( c \), and the displacement current must be included in Ampère’s law. However, we still obtain a closed system for the same equations (7), (8) and (11), simply by eliminating the electric field using Ohm’s law and writing the current as
\[ \mathbf{j} = \frac{1}{\mu_0} \nabla \times \mathbf{B} - c_0^2 \frac{\partial}{\partial t} (\mathbf{V} \times \mathbf{B}) - \mathbf{j}_M \] (12)
where from (6) we write the spin current as
\[ \mathbf{j}_M = \frac{\mu_B}{2m} \nabla \times \left[ \rho \tanh \left( \frac{\mu_B B}{T} \right) \mathbf{B} \right] \] (13)
for a quasineutral electron-positron plasma with equal temperature \( T \). Note that in (12) we have for simplicity omitted dissipative effects (\( \eta \to 0 \)).

### IV. ONE-DIMENSIONAL ALFVÉN WAVES

The spin terms have rather different properties than the standard MHD- terms. As a consequence, the spin force does not necessarily need to be as large as the ordinary \( \mathbf{j} \times \mathbf{B} \)-force, in order to significantly affect the evolution of the system. In order to illustrate this specific property, we consider the weakly nonlinear evolution of shear Alfvén waves propagating at an angle to the magnetic field. In the absence of spin effects, the Alfvén waves propagates with the Alfvén speed \( c_A = [c^2 B_0^2/(\mu_0 \mu_0 + B_0^2)]^{1/2} \), where \( B_0 \) is the magnitude of the unperturbed magnetic field, and \( \rho_0 \) is the unperturbed density. We consider the case with spatial dependence on a single coordinate \( \xi = x \sin \theta + z \cos \theta \). As a prerequisite, we first consider the linearized equations. Letting the unperturbed magnetic field lie along the \( z \)-axis, and combining Eqs. (3), (11) and (12) (with \( \eta \to 0 \)) we, as a first approximation, obtain
\[ \left[ \frac{\partial^2}{\partial t^2} - c_{A,sp}^2 \sin^2 \theta \frac{\partial^2}{\partial \xi^2} \right] v_y = 0 \] (14)
Here \( c_{A,sp} = c_A^2/(1 + \delta_{sp}) \) is the spin-modified Alfvén velocity, with the spin modification determined by \( \delta_{sp} = \hbar \omega_c \tanh(\mu_B B_0/T) / 2m_c^2 \omega_c \), where \( \omega_p = [(e^2/(\hbar n_p + n_e)) / e_i m_i]^{1/2} \) is the plasma frequency of the pair plasma, which comes from the part of the spin current proportional to \((\rho_0 / m) \nabla \times [\tanh(\mu_B B_0/T) \mathbf{B}] \). Here we have introduced the cyclotron frequency \( \omega_c = e B_0 / m \) [45]. For the sake of definiteness we also consider waves propagating in the positive \( \xi \)-direction, such that we can use the expression \( \partial / \partial t = c_{A,sp} \cos \theta \partial / \partial \xi \) in the linear nondispersive approximation. Furthermore, from now on we assume that the spin effects are small in the sense that \( \delta_{sp} \ll 1 \), and also that the spins are weakly aligned with the external magnetic field such that we can approximate \( \tanh(\mu_B B_0/T) \approx \mu_B B_0 / T \).

As is wellknown, a weakly nonlinear evolution typically leads to wave steepening effects, and the necessity to include dispersion on an equal footing. Following Ref. [39], but with the inclusion of spin terms, treating the Hall current, i.e., the first term of equation (10), as a small correction, weakly dispersive effects are thus kept. Considering only the positive propagating waves, the linear wave operator for shear Alfvén waves can then be generalized to

\[
\left[ \frac{\partial}{\partial t} + c_{A,sp} \cos \theta \frac{\partial}{\partial \xi} + \frac{c_A^3 \cos^3 \theta}{2 \omega_c^2 \sin^2 \theta} \frac{\partial^3}{\partial \xi^3} \right] v_y = 0 \quad (15)
\]

where small spin-contribution to the dispersive term in Eq. (15) has been neglected.

Next, including nonlinear terms, we note that the lowest order terms (i.e., those proportional to \( B_i^2 \) and \( v_x^2 \)) do not contribute directly to the nonlinear coupling, and thus we must include nonlinear terms of higher order in both the amplitude and the \( 1 / m_c \) expansion. We thereby obtain

\[
\left[ \frac{\partial}{\partial t} + c_{A,sp} \cos \theta \frac{\partial}{\partial \xi} + \frac{c_A^3 \cos^3 \theta}{2 \omega_c^2 \sin^2 \theta} \frac{\partial^3}{\partial \xi^3} \right] v_y = \frac{c_A \cos \theta}{2 \omega_c} \frac{\partial v_y}{\partial \xi} \left[ \frac{v_y}{\sin \theta} \frac{\partial v_y}{\partial \xi} + \omega_A \rho_1 / \rho_0 \right] - \frac{\mu_B B_0 v_y^2}{2 m T c_A^2} \frac{\partial^3 v_y}{\partial \xi^3} \frac{\partial}{\partial \xi} \quad (16)
\]

The linear relation between the velocity and density perturbations reads \( \rho_1 = (\rho_0 / \sin \theta) \partial v_y / \partial \xi \), i.e. Eq. (16) reduces to

\[
\left[ \frac{\partial}{\partial t} + c_{A,sp} \cos \theta \frac{\partial}{\partial \xi} + \frac{c_A^3 \cos^3 \theta}{2 \omega_c^2 \sin^2 \theta} \frac{\partial^3}{\partial \xi^3} \right] v_y = - \frac{(\mu_B B_0 v_y^2)}{2 m T c_A^2} \frac{\partial^3 v_y}{\partial \xi^3} \frac{\partial}{\partial \xi} \quad (17)
\]

which is a modified Korteweg de Vries (MKdV) equation with a focusing type of nonlinearity. As is wellknown (see e.g. Ref. [40]), this equation admits sech-shaped soliton solutions where the product of the amplitude and the width is a constant, according to

\[
v_y = v_y(0) \text{sech} \left\{ \frac{1}{2} \left( \frac{(\mu_B B_0 v_y^2)}{3 m T c_A^2} \frac{\sin^2 \theta}{\cos^3 \theta} \right) \right\} \left( \frac{1}{2} \left( \frac{(\mu_B B_0 v_y^2)}{12 m T c_A^2} \right) \right)^{1/2} \quad (18)
\]

Astrophysical environments may exhibit extreme fields. Neutron stars have surface magnetic field strengths of the order of \( 10^9 - 10^{10} \) T [34], while magnetars field strengths can reach \( 10^{10} - 10^{11} \) T [36], coming close to energy densities corresponding to the Schwinger limit [20]; here, in the vicinity of magnetars, the quantum vacuum becomes fully nonlinear. However, more moderate, but still very strong, fields appear at a distance from the surface of magnetized stars, and often in conjunction with a pair plasma due to cascading processes [35]. Let us therefore consider a specific example of solitons in a pulsar environment. We then take the unperturbed magnetic field as \( B_0 \approx 10^7 T \). For a pulsar period of \( P = 1 \) s, and a multiplicity \( n = 10 \) [41], the Julian-Goldreich expression \( n_{IG} = 7 \times 10^{15} (0.1 s / P) (B / 10^8 T) m^{-3} \) for the pair plasma density gives \( \rho_0 \approx 10^{-15} Kg/m^3 \) [55]. Furthermore, as supported by the work of e.g. Ref. [42], we let the pair plasma temperature be moderately relativistic, i.e. \( T \approx 0.4 m^2 c^2 \). Finally, we let the Alfvén waves propagate almost parallel to the magnetic field, at an angle \( \theta \approx 0.2 \) rad, and have a velocity amplitude \( v_y(0) \approx 10^5 m/s \). We then find that the width of the soliton is \( d = [(\mu_B B_0 \omega_c)] (v_y^2) \sin^2 \theta / 3 m T c_A^2 \cos^3 \theta \approx 10^{-2} m \). We note that variations of the parameters in the expression for the soliton width may change the size significantly, but this may also require other modifications to be made, such as the inclusion of relativistic effects [43]. However, at this stage, we conclude that the existence of localized spin structures in the pair plasmas surrounding pulsars is likely, and further generalizations are left for future studies.

V. SUMMARY AND DISCUSSION

In the present paper we have studied the dynamics of an electron-positron pair plasma in the MHD limit, and included the spin properties of the constituents. A closed set of one-fluid equations have been obtained, resembling the standard MHD equations but including both a standard quantum force (from the so called Bohm potential), as well as a number of new terms related to the particle spins. The spin terms are of particular importance for strongly magnetized plasmas and for low temperature plasmas, when the spins are aligned with the magnetic field. We stress that the spin terms can have rather different properties than the usual terms in the MHD equations. As a consequence, it turns out that the spin force can be of importance even when its magnitude is smaller than the usual \( j \times B \)-force. In order to demonstrate this property, we have studied the special case of weakly nonlinear shear Alfvén waves in the one-dimensional limit. Due to the spin effects, such waves may be governed by a MKDV-equation, leading to wave steepening and subsequent soliton formation. By contrast, the nonlinearity cancels to all orders for such waves, if the spin terms are omitted. The results of the present paper can be of particular significance for astrophysical pair plasmas in the vicinity of pulsars and magnetars [17,18], as well as low-temperature laboratory plasmas [43], and nano-structured systems [44].
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[45] In the case when the contribution to external magnetic field from the spins is large, it is strictly speaking only the external sources (i.e. not the spin sources) that contribute to the magnetic field and thereby the cyclotron frequency in this formula. Thus formally we should replace the full cyclotron frequency $\omega_c$ with the cyclotron frequency due to external sources only, $\omega_{c, ext}$, where the two are related by $\omega_c = \omega_{c, ext} + h\alpha_x^2 \tanh(\mu_B B_0/T)/m c^2$. In our case, however, this difference is negligible.
[46] Using the non-relativistic Pauli equation means that certain effects are omitted, such as the spin-orbit coupling. Furthermore, a moderately relativistic temperature means that a relativistic pressure model would be preferable. However, since the waves considered are only weakly compressional (as associated with the weak dispersion of the Alfvén waves), the main role of the thermal effects is to determine the average orientation of the spins. Thus it is not crucial to use a relativistic pressure model in the example given here. Finally, for the low amplitude solitons considered, there is clearly no need to account for a relativistic quiver velocity.