Fast (Multi-)Evaluation of Linearly Recurrent Sequences: Improvements and Applications

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Abstract. For a linearly recurrent sequence \( P_{n+1} = A(n) \cdot P_n \), consider the problem of calculating either the \( n \)-th term \( P_n \) or \( \ell \leq n \) arbitrary terms \( P_{n_1}, \ldots, P_{n_\ell} \), both for the case of constant coefficients \( A(n) \equiv A \) and for a matrix \( A(N) \) with entries polynomial in \( N \).

We improve and extend known algorithms for this problem and present new applications for it. Specifically it turns out that for instance

- any family \( (p_n) \) of classical orthogonal polynomials admits evaluation at given \( x \) within \( O(\sqrt{n} \cdot \log n) \) operations independent of the family \( (p_n) \) under consideration.
- For any \( \ell \) indices \( n_1, \ldots, n_\ell \leq n \), the values \( p_{n_i}(x) \) can be calculated simultaneously using \( O(\sqrt{n} \cdot \log n + \ell \cdot \log \frac{\ell}{n}) \) arithmetic operations; again this running time bound holds uniformly.
- Every hypergeometric (or, more generally, holonomic) function admits approximate evaluation up to absolute error \( \epsilon > 0 \) within \( O(\sqrt{\log \frac{1}{\epsilon} \cdot \log \log \frac{1}{\epsilon}}) \) arithmetic steps.
- Given \( m \in \mathbb{N} \) and a polynomial \( p \) of degree \( d \) over a field of characteristic zero, the coefficient of \( p^m \) to term \( X^n \) can be computed within \( O(d^2 \cdot M(\sqrt{n})) \) steps where \( M(n) \) denotes the cost of multiplying two degree-\( n \) polynomials.
- The same time bound holds for the joint calculation of any \( \ell \leq \sqrt{n} \) desired coefficients of \( p^m \) to terms \( X^{n_1}, n_1, \ldots, n_\ell \leq n \).

1 Introduction

The naive way of calculating the \( n \)-th factorial \( P_{n+1} = (n+1) \cdot P_n \) uses \( O(n) \) arithmetic operations over \( \mathbb{Z} \). During its course, all lower factorials \( 1, 2, 3!, \ldots, (n-1)! \) are generated as well which might or might not be desirable. In the latter case, most of the intermediate factorials can in fact be bypassed and \( n! \) itself be calculated using only \( O(\sqrt{n} \cdot \log n \cdot \log \log n) \) integer operations. This had been observed by Strassen [27, Abschnitt 6] and is based on fast fourier transforms and polynomial multipoint evaluation. A generalization to the computation of the \( n \)-th element \( P_n \) of a recursively defined sequence of vectors

\[
P_{n+1} = A(n) \cdot P_n
\]

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with a matrix $A$ of polynomials in $n$ has been suggested in [9, SECTION 6], further improved in [3] and extended in [4, Theorem 5] to the joint computation of several (say, $\ell$, not necessarily consecutive) vectors $P_n$. This result has yielded better upper complexity bounds for deterministic integer factorization and for computation with hyperelliptic curves [34].

The present work reveals the fast multi-evaluation of linearly recurrent sequences to be in fact fundamental for several other problems as well; specifically to the evaluation of orthogonal polynomials and to the computation of specific coefficients of very high degree polynomials. Efficient handling of polynomials is itself a basic ingredient to many fast algorithms with a vast range of applications and, as a matter of fact, plays in turn a major role in the fast evaluation of recurrent sequences.

We first review and extend the previously known algorithms for linearly recurrent sequences with both constant and with polynomial coefficients (Section 2). These are then applied to other problems as follows: Section 3 deduces a roughly radial upper algebraic complexity bound [7] uniformly on all orthogonal polynomials; and Section 4 presents computer algebra algorithms [13] for determining specific coefficients of polynomials. This is of particular benefit in cases where the result has high degree $n$ but only few (say, $\ell \ll n$) terms are desired at output-sensitive cost.

In fact, all obtained running times are optimal with respect to $\ell$ in the following sense: As $\ell \to n$ (that is towards the classical case of computing all entries of the sequence or coefficients of the polynomial) and with other parameters fixed, it converges to the asymptotic running time of the respective best known classical algorithm including logarithmic factors. Our new algorithms are thus true generalizations of the latter at cost increased by at most constant factors.

## 2 Fast Evaluation of Linear Recurrences

Recurrence equations like (1) are ubiquitous in mathematics as well as computer science. Many (if not most) have no closed-form solution; and even if one does, it might not induce an efficient algorithm — compare $n!$ above.

In order to explicitly calculate the $n$-th term $P_n$, the naive approach suggested by Equation (1) iteratively proceeds from $P_0$ to $P_1, P_2, \ldots, P_{n-1}, P_n$ and thus has running time proportional to $n$. However, being interested in $P_n$ only, this might be out-performed by other methods which avoid computing all intermediate terms. For instance if $k \times k$-matrix $A$ does not depend on $N$, then repeated squaring yields $A^n$ within $O(k^3 \cdot \log n)$ steps. This is already optimal with respect to $n$ [7, Theorem 13.14]; whereas in terms of $k$, the running time has been further improved in [12] to time $O(k \cdot \text{polylog } k \cdot \log n)$ for computing the solution to

$$P_n = a_1 \cdot P_{n-1} + a_2 \cdot P_{n-2} + \ldots + a_k \cdot P_{n-k} \quad . \quad (2)$$

Notice that (2) is indeed a special case of (1) with constant matrix $A$ of companion form [3] and $P_n := (P_n, P_{n-1}, \ldots, P_{n-k+1})^T$. 
Theorem 1. Let $\mathcal{R}$ denote a commutative ring with 1 supporting multiplication of two polynomials of degree $< n$ at cost at most $M(n) \geq n$.

a) Given $a_1, \ldots, a_k \in \mathcal{R}$, $P_0, \ldots, P_{k-1} \in \mathcal{R}$, and $n \in \mathbb{N}$, $\ell$ consecutive elements $P_n, \ldots, P_{n+\ell-1}$ defined by (\ref{eq:2}) can be computed using $O(M(k) \cdot \log n + \ell \cdot k)$ arithmetic operations in $\mathcal{R}$.

b) Given a constant companion matrix $A \in \mathbb{R}^{k \times k}$ and $P_0 \in \mathbb{R}^k$, vectors $P_{n_1}, P_{n_2}, \ldots, P_{n_\ell}$ defined by (\ref{eq:7}) can be computed simultaneously using $O(\ell \cdot k^{\omega-1} \cdot \log \frac{n}{\ell})$ arithmetic operations in $\mathcal{R}$ where $k \leq \ell$ and $n_1 < n_2 < \ldots < n_\ell = n$.

Here, $\omega \geq 2$ denotes any feasible exponent for matrix multiplication \cite{13,7}, e.g., $\omega = 2.38$. For $\ell \geq k^2$, one may even choose $\omega = 2.34$.

Proof. Claim a) for $\ell = 1$ is \cite{12} PROPOSITION 3.2. Specifically, $P_n$ is obtained as thecalar product of $(P_0, \ldots, P_{k-1})$ with the coefficients of the polynomial $X^n \bmod f$ where $f = X^k - (a_1 + a_2X + \cdots + a_kX^{k-1})$. \cite{12} THEOREM 3.1. Therefore, once $X^n \bmod f$ is known, we can calculate $X^n \bmod f = (X^n \bmod f)$ mod $f$ and $P_{n+1}$ using an additional number $O(k)$ of operations. Iteration thus establishes the case $\ell > 1$.

For Claim b), use \cite{12} PROPOSITION 2.4 to compute all binary powers of $A$ up to $n$ within $O(k^2 \cdot \log n)$. Therefore, each $P_{n_i}$ is the product of $P_0$ with $J := O(\log n)$ of these pre-calculated matrices $A^{2^j}$, $j = 1, \ldots, J$. In order to improve the induced naive running time of $O(\ell \cdot k^2 \cdot \log n)$ for the joint computation of $P_{n_1}, P_{n_2}, \ldots, P_{n_\ell}$, batch the matrix-vector products into matrix-matrix products as follows: For each $j = 0, \ldots, J$, collect all $i = 1, \ldots, \ell$ for which $P_{n_i}$ involves $A^{2^j}$ in the above mentioned product; put the corresponding vectors to be multiplied to $A^{2^j}$ as columns into a $k \times \ell$ matrix and multiply that to $A^{2^j}$ using $O(k^\omega \cdot \left\lceil \frac{J}{\ell} \right\rceil)$ operations. Here, $\omega = 2.38$ is feasible due to \cite{10}, alternatively, one can use $O(k^\omega \cdot \left\lceil \frac{J}{\ell} \right\rceil)$ operations with $\omega = 3.34$ \cite{16}. Since $\ell \geq k$ (or $\ell \geq k^2$), this yields running time $O(\ell \cdot k^{\omega-1} \cdot \log n)$.

More careful analysis reveals that it suffices to multiply only one vector (namely $P_0$) to $A^{2^j}$ in time $O(k^2)$; and two vectors (namely $P_0$ and $A^{2^0} \cdot P_0$) to $A^{2^j}$ in twice the time; and, similarly on, to multiply in phase no.$j$ only $2^j$ vectors to $A^{2^j}$ as long as $j \leq \log(k)/\log(\omega - 2)$ with $O(k^{\omega-2})$ vectors multiplied using a total of $O(k^\omega)$ operations dominated by the last phase. From $j \geq \log(k)/\log(\omega - 2)$ on, switch to fast matrix multiplication. For $j \leq \log k$, this involves only one batch and will thus take time $O(k^\omega)$ per phase, that is, a total of $O(k^\omega \cdot \log k)$. Each phase no.$j$ with $\log k \leq j \leq \log \ell$ gives rise to $2^j$ vectors multiplied to $A^{2^j}$, grouped to $2^j/k$ batches and therefore taking $O(2^j \cdot k^{\omega-1})$ operations, again dominated by the last one with duration $O(\ell \cdot k^{\omega-1})$. The final phases $j = \log \ell \ldots \log n$ do not further increase the number of vectors multiplied to $A^{2^j}$ because we are looking for only $\ell$ different results $P_{n_1}, \ldots, P_{n_\ell}$. They thus induce total cost $O(\ell \cdot k^{\omega-1})$ each; times the number $\log \frac{\ell}{2}$ of final phases and added to the aforementioned $O(k^\omega \cdot \log k)$ yields the claim. \hfill $\Box$
For further improvement and regarding the very last paragraph of [12], it seems
worth while to attack the following

Problem 2. Given \(a_1, \ldots, a_k\) and \(P_0, \ldots, P_{k-1}\), compute \(P_k, \ldots, P_{2k-1}\) according to [2] in time \(o(k^2)\).

Let us now relax the condition on \(A\) to be constant and consider matrices...

### 2.1 . . . with Polynomial Coefficients

This case involves not matrix powers but matrix factorials like \(A(n) \cdot A(n - 1) \cdots A(2) \cdot A(1) =: \prod_{j=1}^{n} A(j)\). While the naive iterative approach leads to running time proportional to \(n\), CHUDNOVSKY&CHUDNOVSKY have improved that to cost roughly radical in \(n\) [9, Section 6]:

**Fact 3** Let \(R\) denote a commutative ring permitting multiplication of two polynomials of degree \(< n\) at cost at most \(M(n)\) where \(M\) satisfies some standard regularity conditions [6, bottom of p.582]. Consider a \(k \times k\) matrix \(A(N)\) with polynomial entries \(a_{ij}(N) \in R[N]\) in \(N\) of degree \(< d\). Given the (coefficients of) \(A\) and \(n \in \mathbb{N}\), one can calculate the matrix \(\prod_{j=1}^{n} A(j)\) using

\[O(k^\omega \cdot M(\sqrt{n}d) \cdot \log n)\]

operations in \(R\).

**Proof.** Let \(\nu := \lceil \sqrt{n} \rceil\) and consider the Baby-Step/Giant-Step approach of

i) determining the (coefficients of the) polynomial matrix \(C(N) := A(N + \nu) \cdot A(N + \nu - 1) \cdots A(N + 2) \cdot A(N + 1) \in R[N]^{k \times k}\);

ii) multi-evaluating \(C\) at 0, \(2\nu, \ldots, [n/\nu] \cdot \nu =: \tilde{n}\);

iii) calculating \(\prod_{j=1}^{\tilde{n}} A(j)\) by iterative multiplication of the matrices \(C(0), C(\nu), \ldots, C(\tilde{n} - \nu)\) obtained in ii);

iv) finally computing \(\prod_{j=1}^{n} A(j)\) by iterative multiplication of the result from iii) with the matrices \(A(\tilde{n}), A(\tilde{n} + 1), \ldots, A(n - 1)\).

\(\Box\)

The possibility for further improvement of Fact 3 to, say, \(O(\text{polylog } n)\) for fixed \((k, d)\) is unknown already in the case of the scalar factorial \(n!\) and related to deep open class separation problems in complexity theory [8,21]. For rings of characteristic 0 and fixed \(d\) however, improvements in particular in terms of the size \(k\) of the matrix \(A\) have been obtained by BOSTAN&GAUDRY&SCHOST [4, Theorem 5] as well as a generalization to the simultaneous computation of \(\ell \leq O(n^{1/2-\epsilon})\) matrix factorials \(\prod_{j=1}^{n} A(j), i = 1, \ldots, \ell\).

The present section reviews this result, presented with a new proof and including in its analysis the running time’s dependence on the degree \(d\) of the polynomials in \(A\) as well as on the number \(\ell\) of elements of the sequence to be computed non-trivially extended beyond \(\sqrt{n}\) (Theorem 3b). Further claims deal with a generalization (Theorem 4a) and improvements for the frequent case that \(A\) has companion form (Theorem 4b+d). In the sequel, capital letters \(X\) and \(N\) denote formal indeterminates of polynomials whereas lower case \(x\) and \(n\) refer to variables with values.
Theorem 4. Consider a $k \times k$ matrix $A(N)$ with polynomial entries $a_{ij}(N) \in \mathbb{R}[N]$ of degree $< d$.

a) Given (the coefficients of) $A$ as well as $\ell$ pairs of integers $(m_i, n_i)$ with $0 \leq m_i \leq n_i$, one can simultaneously calculate the $\ell$ matrix products $B_i := \prod_{j=m_i}^{n_i} A(j)$, $i = 1, \ldots, \ell$, using

$$O\left(k^\omega \cdot \left(\sqrt{nd} + \sqrt{d \log \ell} + k^2 \cdot M\left(\sqrt{nd}\right) + k^2 \cdot \frac{M(n\ell/\ell)}{nd/\ell}\right)\right)$$

operations in $\mathbb{R}$ where $n := \max n_i \geq d \cdot \log^2 d$.

b) If $m_i \equiv 1$ and, instead of the matrices $B_i$ themselves, the $\ell$ matrix–vector products $P_i = B_i \cdot P_0$ for a given $P_0 \in \mathbb{R}^k$ are desired, this can be accomplished using

$$O\left(k^\omega \cdot \min\{\sqrt{nd}, nd/\ell\} + k^2 \cdot M\left(\sqrt{nd}\right) + k^2 \cdot \frac{M(n\ell/\ell)}{nd/\ell}\right)$$

operations in $\mathbb{R}$.

c) In case that the matrix $A(n)$ is of companion form and invertible in $\mathbb{R}^{k \times k}$ for all integers $n$ exceeding a given $m \in \mathbb{N}$, then the $\ell$ vectors $B_i \cdot P_0$, $i = 1, \ldots, \ell$, can be computed using

$$O\left(k^2 \cdot M\left(\sqrt{nd}\right) + k^2 \cdot \frac{M(n\ell/\ell)}{nd/\ell}\right)$$

operations in $\mathbb{R}$.

d) If additionally $n \geq k^2$ and the polynomials constituting $A(N)$ obey the restricted degree condition $\deg(a_{1j}) \leq j$, the running time further reduces to

$$O\left(k^2 \cdot \left(\frac{\ell}{\sqrt{n}} + 1\right) \cdot M(\sqrt{n})\right)$$

The algorithms are uniform and — except for the roots of unity $\exp(2\pi i/n)$ employed in the FFT when $M(n) = O(n \log n)$ — free of constants.

2.2 Proof of Theorem 4

Reconsider the proof of Fact 3 with its four steps, but leave the value of the trade-off parameter $\nu$ open for the moment to be chosen later as an integral power of 2. We also remark that the coefficients of the polynomials arising in Steps i) and ii) may be taken with respect to any common (rather than the standard monomial) basis. As a matter of fact, regarding the hypothesis that $n \geq d \log^2 d$, it pays off to first spend $O\left(k^2 \cdot M(d) \cdot \log d\right)$ operations for converting $A(N)$ to the falling factorial (also called Newton) basis [14, Section 4.2] because that will accelerate evaluation and interpolation on arithmetic progressions by a logarithmic factor [14, Section 4.3]. Specifically exploit that evaluating a degree-$D$ polynomial $p$ simultaneously at $K$ points of an arithmetic progression takes, by simulating $\lceil K/D \rceil$ multipoint evaluations of $p$ at $D = \deg(p)$ points
each, \( O\left( \frac{k^2}{\nu} \cdot M(D) \right) \) operations \[\text{THEOREM 4.24}\]. Step ii) thus succeeds within a total of \( O\left( k^2 \cdot \frac{n/\nu}{m!} + 1 \cdot M(\nu d) \right) \) operations.

Concerning Step i), \[\text{SECTION 6}\] combines fast matrix multiplication with fast polynomial arithmetic and achieves running time \( O(k^2 \cdot M(\nu d) \log \nu) \). \[\text{3}\] has observed that this allows for improvement, provided the characteristic of \( R \) is zero (or larger than \( m + \nu d \)). Their proof is a recursive descend on \( n \) being an integral power of 4 with a complicated consideration for the general case. We obtain a considerable simplification in particular in Sub-Steps \( \alpha \) and \( \gamma \) below by working in the Newton rather than monomial basis:

\( \alpha \) Perform \( k^2 \) separate multipoint evaluations to obtain the matrix values
\[ A(m + 1), A(m + 2), \ldots, A(m + 2\nu) \in R^{k \times k} \] for arbitrary \( m \in \mathbb{N} \). Since the evaluation points form an arithmetic progression this takes, similarly to Step ii), a total of \( O(k^2 \nu \cdot M(d)) \) operations.

\( \beta \) Determine the matrices \( C(m), C(m + 1), \ldots, C(m + \nu d - 1) \in R^{k \times k} \) using \( O(k^2 \nu d) \) analogously to \[\text{PROPOSITION 2}\]. Specifically, compute the \( 2\nu \) products
\[ A(m + \nu - 1) \cdots A(m + \nu - j + 1) \] and
\[ A(m + \nu + j - 1) \cdots A(m + \nu + 1) \] for \( i = 0, 1, \ldots, \nu \) within \( O(k^2 \nu) \) and observe that each \( C(m), \ldots, C(m + \nu d - 1) \) is composed of two such product ranges.

\( \gamma \) Interpolate the \( \nu d \) matrix values from Sub-Step \( \beta \) to determine the (coefficients in the factorial basis of the) matrix polynomial \( C(N) \) of degree \( < \nu d \) at the expense of \( O(k^2 \cdot M(\nu d)) \) operations \[\text{THEOREM 4.26}\].

Since \( M(\nu d) \leq \nu M(d) \), the asymptotic cost of Step ii) above exceeds that of Sub-Step \( \alpha \), Step i) gives rise to an additional running time of \( O(k^2 \nu d + k^2 M(\nu d)) \).

Step iii) uses \( O\left( \frac{(1 + \frac{\pi}{2})}{\nu} \cdot k^2 \right) \) operations and Step iv) another \( O(\nu \cdot k^2) \).

If, instead of the matrix \( \prod_{j=1}^{n} A(j) \) itself, only the vector \( P_n = \prod_{j=1}^{n} A(j) \cdot P_0 \) is to be calculated, we can replace the \( O(k^2) \)-time matrix-matrix products in Steps iii) and iv) with \( O(k^2) \)-time matrix-vector products. If furthermore \( A(n) \) is in companion form and invertible for all integers \( n \geq m \), also Step i\( \beta \) accelerates to \( O(k^2 \nu d) \) by Lemma \[\text{4}\] below.

Towards the multi-evaluation case \( \ell > 1 \), suppose for a start that all \( n_i \) and \( m_i \) are multiples of \( \nu \). We thus seek an algorithm for the following step:

\( \nu \) Simultaneously calculate the \( \ell \) matrices \( \prod_{j=1}^{\tilde{n}_1} A(j) \) (or their respective product with \( P_0 \)) where \( \tilde{n}_i := \lfloor n_i/\nu \rfloor \cdot \nu \) and \( \tilde{m}_i := \lfloor m_i/\nu \rfloor \cdot \nu \), \( i = 1, \ldots, \ell \). For Claims b+c) with \( \tilde{m}_i = 0 \), it suffices to iteratively multiply the matrices \( C(0), C(\nu), C(2\nu), \ldots \) obtained in ii): this yields all products \( \prod_{j=1}^{s} A(j), s = 1, \ldots, \lfloor n/\nu \rfloor =: I \) and takes \( O\left( \frac{2k^2}{\nu} \right) \) steps. For Claim a) with general \( \tilde{m}_i \), we have to calculate \( \ell \) products \( \prod_{j=1}^{\tilde{m}_i} \tilde{C}_j \) of matrices \( \tilde{C}_j := C(j\nu) \) where the ranges \( [r_i, s_i), i = 1, \ldots, \ell \) may be arbitrary integer intervals contained in \([0, I] \). To this end recall the Range Tree from Computational Geometry \[\text{2}\] \[\text{SECTION 5.1}\]. Specifically, consider the set \( S := \{0, r_i, s_i : i = 1, \ldots, \ell \} \) ordered as \( S = \{0 = t_0 < t_1 < \ldots < t_{\tilde{N} - 1}\} \) where \( \tilde{N} \leq \min\{2\ell + 1, I\} \).
Now compute first the $\tilde{N}$ products $\prod_{t \in [t_j, t_{j+1})} \tilde{C}_t$, $j = 0, \ldots, \tilde{N} - 1$, invoking $O(\sum_j |t_j+1 - t_j - 1|) = O(I)$ matrix multiplications; then compose from these results the $\tilde{N}/2$ products $\prod_{t \in [t_2, t_{j+2})} \tilde{C}_t$, $j = 0, \ldots, \tilde{N}/2 - 1$ using further $\tilde{N}/2 \leq I/2$ matrix multiplications; then the $\tilde{N}/4$ products $\prod_{t \in [t_4, t_{j+4})} \tilde{C}_t$, and so on. So after a total of $O(k^\omega I)$ operations, all products ranging over a binary interval are prepared which concludes the initialization of the Range Tree. Now for its application, observe that each interval $[r_i, s_i]$, $i = 1, \ldots, \ell$ is a disjoint union of $O(\log \tilde{N}) \leq O(\log \ell)$ of these binary intervals. This concludes the entire Step v) within time $O(k^\omega (\frac{n}{\nu} + \ell \log \ell))$ in case of Claim a) or $O(k^2 \frac{n}{\nu})$ for Claims b+c).

For the final goal, that is to

vi) simultaneously calculate the $\ell$ matrices $\prod_{j=m_i}^{n_i} A(j)$ (or their respective product with $P_0$, $i = 1, \ldots, \ell$, invoke the Range-Tree idea once again. This time, the initialization phase consists in preparing the (coefficients of the) $\nu/2$ polynomial matrices $C_{\nu/2}(\tilde{N}) := A(N + \frac{1}{2}) \cdot A(N + \frac{3}{2} - 1) \cdots A(N + 2) \cdot A(N + 1) \in \mathcal{R}[N]^{k \times k}$, $C_{\nu/4}(\tilde{N})$, $C_{\nu/8}(\tilde{N})$, $\ldots$, $C_2(\tilde{N})$, $C_1(\tilde{N})$. Due to the exponentially decreasing size $\nu$, this will together infer only the same cost as Step i).

In the application phase, first multi-evaluate $C_{\nu/2}(\tilde{N})$ at those $\tilde{n}_i$ whose difference to $n_i$ is at least $\nu/2$ — $O(k^2 \cdot (\frac{\ell}{n/d} + 1) \cdot M(\nu d))$ operations as in Step ii) — and multiply them to the already computed results from Step v) at the expense of another $O(k^\omega \cdot \ell)$ and $O(k^2 \cdot \ell)$ for Claims a) and b+c), respectively. For Claim a) do similarly for those $\tilde{m}_i$ differing from $m_i$ by at least $\nu/2$. Now repeat with multi-evaluating $C_{\nu/4}(\tilde{N})$, then $C_{\nu/8}(\tilde{N})$ and so on. By the same argument as above, this will affect the overall running time by at most a factor of 2 while in the end yielding the desired resulting values.

|   | Case a) | Case b) | Case c) | Case d) |
|---|--------|--------|--------|--------|
| i) | $k^{\nu} \cdot \nu d + k^2 M(\nu d)$ | $k^{\nu} \cdot \nu d + k^2 M(\nu d)$ | $k^{\nu} \cdot \nu d + k^2 M(\nu d)$ | $k^2 M(\nu + k)$ |
| ii) | $k^2 \cdot (\frac{n}{\nu d} + 1) M(\nu d)$ | $k^2 \cdot (\frac{n}{\nu d} + 1) M(\nu d)$ | $k^2 \cdot (\frac{n}{\nu d} + 1) M(\nu d)$ | $k^2 \cdot (\frac{n}{\nu d} + 1) M(\nu + k)$ |
| iii) + iv) | $k^{\nu} (1 + \frac{n}{\nu d})$ | $k^2 (1 + \frac{n}{\nu d})$ | $k^2 (1 + \frac{n}{\nu d})$ | $k^2 (1 + \frac{n}{\nu d})$ |
| v) | $k^{\nu} (\frac{n}{\nu d} + \ell \log \ell)$ | $k^2 \frac{n}{\nu d}$ | $k^2 \frac{n}{\nu d}$ | $k^2 \frac{n}{\nu d}$ |
| vi) | $k^2 (\frac{n}{\nu d} + 1) M(\nu d)$ | $k^2 (\frac{n}{\nu d} + 1) M(\nu d)$ | $k^2 (\frac{n}{\nu d} + 1) M(\nu d)$ | $k^2 (\frac{n}{\nu d} + 1) M(\nu + k)$ |

**Fig. 1.** Big-Oh running times of steps i) to vi) in cases a) to d)

It thus remains to confirm that the costs of the above steps i) to vi) are all covered by the running times claimed in a), b), and c). To this end, choose $\nu$ as (an integral power of 2 close to but not exceeding) $\sqrt{n/d}$ if $\ell \leq \sqrt{n/d}$ and $\nu := n/\ell$ otherwise. We remark that $\nu \leq \sqrt{n/d}$ holds in both cases, so

$$\frac{n}{\nu} + \left(\frac{n}{\nu d} + \frac{\ell}{n/d} + 1\right) M(\nu d) \leq O\left((\frac{n}{\nu} + \ell) \cdot \frac{M(\nu d)}{n/d}\right)$$
which amounts to $O(M(\sqrt{n}d))$ in case $\ell \leq \sqrt{n}d$ and to $O\left(\frac{n^2}{m} \cdot M(\frac{nd}{\ell})\right)$ if $\ell \geq \sqrt{n}d$. Claims a+c) are thus immediate. For Claim b) observe furthermore that $\nu d = \min\{\sqrt{n}d, nd/\ell\}$.

Case d) admits, in addition to Case c), further improvement based on the observation that the degree of the matrix polynomial(s) $C(N)$ involved in Steps i), ii), and vi) reduces from $\nu d$ to $\nu + k$ by virtue of Observation 8b) below. This yields the running times in the last column of Figure 1. Then choose $\nu := \sqrt{n} \geq k$. 

\[ \square \]

\section{A First Application and Some Tools}

Theorem 5\(b) \) includes 5\( \text{Theorem 5} \) by restricting to $\ell \leq O(\sqrt{n})$ and constant $d$. Another consequence, we have the following non-trivial complexity interpolation between, on the one end, STRASSEN’s aforementioned algorithm 27\( \text{Abschnitt 6} \) computing one single factorial $n!$ (that is, the case $\ell = 1$) and, on the other end, the obviously optimal naive iterative $O(n)$ calculation of 1, 2!, 3!, . . ., $n!$ (that is, the case $\ell = n$):

\[ \textbf{Corollary 5.} \text{ Over } \mathcal{R} = \mathbb{Z} \text{ with } M(n) = O(n \log n \log \log n) \text{ 13\( \text{Theorem 8.23} \) and } k = 1 = d, \text{ any } \ell \text{ desired factorials } n_1! < n_2! < \ldots < n_\ell! = n! \text{ can be computed simultaneously using } O(\sqrt{n \log n \log \log n + \ell \log \frac{n}{\ell} \log \log \frac{n}{\ell}}) \text{ arithmetic operations.} \]

Further applications will be given in the sequel. In many of them, the matrix $A$ according to Equation 11 is structured 28. For example a companion matrix as well as its inverse

\[ F = \begin{pmatrix} f_1 & f_2 & f_3 & \cdots & f_{k-1} & f_k \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \quad F^{-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{1}{f_1} & \frac{f_1}{f_k} & \frac{f_2}{f_k} & \cdots & \frac{f_{k-1}}{f_k} \end{pmatrix} \quad (3) \]

is described by the $k$ parameters $(f_1, \ldots, f_k)$ as opposed to the $k^2$ independent entries of a general matrix. Theorem 11\( \text{relies on fast powering of companion matrices, that is, on efficient calculation of iterated products of the same } F. \)

The following tool, employed in the proof of Theorem 11, considers products of several companion matrices and might be of its own interest:

\[ \textbf{Lemma 6.} \text{ Given } m \text{ companion matrices } F_1, \ldots, F_m \in \mathcal{R}^{k \times k}, \]

a) their product $F_1 \cdots F_m$ can be computed in $O(m \cdot k^2)$ steps

b) as well as in $O\left(k^\omega \cdot \left(1 + \frac{m}{k}\right)\right)$ steps.

c) If all $F_1, \ldots, F_m$ are invertible, then the $m - n + 1$ products

\[ F_1 \cdots F_n, \quad F_2 \cdots F_{n+1}, \quad \ldots, \quad F_{m-n+1} \cdots F_m \]

\text{ can be computed simultaneously in } O(m \cdot k^2) \text{ steps.}
Proof.  a) The multiplication of a vector to a companion matrix, from left $F \cdot v$ as well as its transposed from left $v^T \cdot F$, both takes $O(k)$ operations. Therefore the multiplication $A \cdot F_m$ by an arbitrary square matrix like $A = F_1 \cdots F_{m-1}$ takes $O(k^2)$ steps. Iterating establishes the sought $O(m \cdot k^2)$ algorithm.

b) Recall \[17\] Lemma 3.1 the formula

$$F_1 \cdot F_2 \cdots F_{k-1} \cdot F_k = \ (I - L)^{-1} \cdot R$$

where $R$ and $L$ denote (respectively lower and strictly upper triangular) matrices plainly consisting of the $k^2$ joint parameters of $F_1, \ldots, F_k$. Both multiplication with $R$ and the inverse $(I - L)^{-1}$ are feasible within $O(k^\omega)$ \[22\] Proposition 16.6]. This establishes the case $m = k$: the general case now follows by partitioning $m$ into $[m/k]$ blocks of length $k$ each according to the grouping $(F_1 \cdots F_k) \cdot (F_{k+1} \cdots F_{2k}) \cdots (F_{m-k+1} \cdots F_m)$.

The following improvement to Lemma \[17\] seems conceivable:

Problem 7. Given $k$ companion matrices of size $k \times k$, compute their product using $O(k^2 \cdot \text{polylog } k)$ operations.

Another ingredient to the proof of Theorem \[11\] is the following

Observation 8  a) Let $A \in \mathcal{R}[X]^{k \times k}$ and $b \in \mathcal{R}[X]^k$ denote a matrix and vector of polynomials of degree $(b_j) \leq m - j$ and degree $(a_{ij}) \leq 1 + j - i$, with the convention $\deg(0) = -\infty$. Then, $c := A \cdot b$ has degree $(c_j) \leq m + 1 - j$.

b) Let $F_1, \ldots, F_m \in \mathcal{R}[X]^{k \times k}$ denote polynomial companion matrices with $(f_{i1}, f_{i2}, \ldots, f_{ik})$ the first row of $F_i$, respectively. If $\deg(f_{ij}) \leq j$, then $B := \prod_{\ell=1}^m F_\ell$ has degree $(a_{ij}) \leq m + j - i$.

$$\deg(A) = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ 0 & 1 & \cdots & k-2 & k-1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 2 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \deg(b) = \begin{pmatrix} m-1 \\ m-2 \\ \vdots \\ m-k+1 \\ m-k \end{pmatrix}$$

Proof.  a) is a straight-forward consequence from $\deg(p \cdot q) \leq \deg(p) + \deg(q)$.

b) follows by induction on $m$, applying a) to each column $b$ of $B$.

3  Fast Evaluation of Orthogonal Polynomials

This section concludes from Theorem \[11\] that any family of classical orthogonal polynomials has at most roughly radical complexity\footnote{This is not to be confused with the Paterson\&Stockmeyer Result \[24\] that every polynomial has complexity $O(\sqrt{n})$ when neglecting operations in the coefficient field.} $O(\sqrt{n} \cdot \log n)$.
Horner’s Method evaluates a fixed degree-$n$ polynomial $p \in \mathbb{R}[X]$ at given $x$ within $O(n)$ arithmetic steps. While this is optimal in the ‘generic’ case \[7\ COROLLARY 5.11\], many specific polynomials do admit faster evaluation; monomials $X^n$ for instance in time $O(\log n)$ by means of repeated squaring. Also Chebyshev’s Polynomials $T_n \in \mathbb{Z}[X]$ have complexity logarithmic in their degree; this can be seen either directly from the quadratic recurrence
\[
T_{2n}(X) = 2T_n^2(X) - 1, \quad T_{2n+1}(X) = 2T_n(X) \cdot T_n(X) - X
\]
or by applying Theorem \[9\] to the linear vector recurrence
\[
\begin{pmatrix}
T_{n+1}(X) \\
T_n(X)
\end{pmatrix} = A \cdot \begin{pmatrix}
T_n(X) \\
T_{n-1}(X)
\end{pmatrix}, \quad A := \begin{pmatrix}
2x & -1 \\
1 & 0
\end{pmatrix}
\]
with matrix $A$ independent of $n$ \[11\ SECTION 4\], \[15\]. Recall that $(T_n)$ forms an orthogonal system on $[-1,1]$ with respect to the weight $\rho(x) = (1 - x^2)^{-1/2}$. Other weights lead to other families of orthogonal polynomials. They are an important tool in Mathematical Physics due to their approximation properties \[22\]. The Legendre Polynomials $P_n(X)$ for instance are orthogonal on $[-1,1]$ with respect to $\rho(x) \equiv 1$.

**Theorem 9.** Every monic family $(P_n) \subseteq \mathbb{R}[X]$ of classical orthogonal polynomials has complexity $O(\sqrt{n} \cdot \log n)$. Any $\ell$ members $P_n, \ldots, P_{n-\ell}$ of such a family have joint complexity $O(\sqrt{n} \cdot \log n + \ell \cdot \log \frac{n}{\ell})$. The constants in the big-Oh notation are independent of the family $(P_n)$.

Observe that, as $\ell \to n$ (that is concerning the problem of evaluating all polynomials $P_1(x), \ldots, P_n(x)$), the running time converges to $O(n)$ which is clearly optimal.

**Proof.** It is well-known that any family $(P_n)$ of classical orthogonal polynomials satisfies a three-term recursion
\[
P_{n+1}(X) = (A_n \cdot X + B_n) \cdot P_n(X) - C_n \cdot P_{n-1}(X) \quad (4)
\]
see, e.g., \[22\ SECTION II.6.3\]. In fact for monic $(P_n)$, $A_n, B_n, C_n$ have turned out as rational functions of $n$ with respective numerator and denominator polynomials $a(N), b(N), c(N), \alpha(N), \beta(N), \gamma(N) \in \mathbb{R}[N]$ of degree at most 4 \[19\ THEOREM 1\]. Rewriting Equation \[11\], we obtain
\[
\frac{\alpha(n)\beta(n)\gamma(n)}{P_n(X)} \cdot \begin{pmatrix}
P_{n+1}(X) \\
P_n(X)
\end{pmatrix} = \begin{pmatrix}
\alpha(n)\beta(n)\gamma(n) \cdot X & -\alpha(n)\beta(n)c(n) \\
\alpha(n)\beta(n)\gamma(n) & 0
\end{pmatrix} \cdot \begin{pmatrix}
P_n(X) \\
P_{n-1}(X)
\end{pmatrix}
\]
a recursion with polynomial coefficients of size $k$ and degree $d$ independent of the family $(P_n)$ under consideration. Now apply Lemma \[10\] below with $M(n) = O(n \cdot \log n)$.

$\square$
Lemma 10. Let $F$ denote a field of characteristic 0 permitting multiplication of two polynomials of degree $< n$ at cost at most $M(n)$. Let $(P_n) \subseteq F[X]^k$ be a sequence of polynomial vectors satisfying

$$s(n+1,X) \cdot P_{n+1}(X) = A(n,X) \cdot P_n(X)$$

with companion matrix polynomial $A \in F[N,X]^{k \times k}$ and $s \in F[N,X]$ both of (total) degree $< d$. Finally suppose that $s(n,x) \neq 0$ for all $n \in \mathbb{N}$ and all $x \in \bar{F}$, the latter denoting an arbitrary subset of $F$. Given $x \in \bar{F}$, $P_0(x) \in F^k$, and (the order $k^2d^2$ coefficients of) both $A$ and $s$, one can simultaneously evaluate $P_{n_1}(x), ..., P_{n_\ell}(x)$ using

$$O\left(k^2 \cdot M\left(\sqrt{nd}\right) + k^2 \cdot \ell \cdot \frac{M(nd/\ell)}{nd/\ell}\right)$$

arithmetic operations over $F$ where $\max\{d^3, n_i\} \leq n$.

The multi-evaluation expressed above refers to the indices $n_1, ..., n_\ell$ of the sequence and should not be confused with multipoint evaluation of a polynomial at several point $x_1, ..., x_n$ as, e.g., in [1].

Proof (Lemma 10). Let $\sigma_n(X) := \prod_{i=1}^{n} s(n,X)$ and consider the sequence $Q_n := \sigma_n \cdot P_n$ obviously satisfying $Q_{n+1}(X) = A(n,X) \cdot Q_n(X)$. After plugging in $x$ into $A$ using $O(k^2d^2)$ arithmetic operations, one arrives thus in the situation of Theorem 4. Indeed, if $a_{1,k}(N,x) \in F[N]$ is the zero polynomial, then we may truncate both the last column and row of $A$ and reduce the dimension $k$ of the recurrence by one; whereas if $a_{1,k}(N,x)$ is not identically zero, it has only finitely many roots and $A(n,x)$ is invertible for all $n \geq m$ with some appropriate $m$ which can easily be found using standard bounds. This yields the joint computation of $Q_{n_1}(x), ..., Q_{n_\ell}(x)$ within the claimed time. Now exploit $\sigma_{n+1}(X) = s(n+1,X) \cdot \sigma_n(X)$ to similarly compute $\sigma_{n_1}(x), ..., \sigma_{n_\ell}(x)$. Since these are units by assumption, another $k\ell$ divisions yield the desired values $P_{n_1}(x) = Q_{n_1}(x)/\sigma_{n_1}(x)$.

$$\square$$

4 Fast Partial Polynomial Arithmetic

The present section applies fast evaluation of linearly recurrent sequences to the problem of computing single or few specific coefficients of a polynomial of large degree.

Based on FFT-methods, many algorithms have been devised which yield fast solutions to many problems in polynomial arithmetic [13, Part II]. These tend to be optimal in running time up to poly-logarithmic factors, simply by comparison with the sizes of the input and output. However the operations of

composition: given $p, q \in F[X]$, determine $p \circ q$;

powering: given $p \in F[X]$ and $n \in \mathbb{N}$, determine $p^n$;

inversion: given $p \in F[X]$ with $p(0) \neq 0$ and $n \in \mathbb{N}$, determine $q := 1/p \mod X^n \in F[X]$. 

generate results of degree significantly larger than the input: quadratic in the first case, unbounded\(^2\) in the second and third. This leaves room for improved algorithms in cases where only one or few terms of the power or inverse are desired — preferably with output-sensitive running times proportional to the number of terms desired. For instance, [11] Corollary 2.3] accelerates polynomial multiplication when some coefficients of the result are already known. Our interest lies in situations where coefficients are not known nor of interest anyway, that is, in the partial calculation of polynomials. In this spirit, [11] presents improved algorithms for computing the lowest \(\ell\) coefficients of the result where \(\ell\) coincides with the degree \(d\) of the input [14] Corollary 2.33, Theorem 2.34], for composition for instance in time \(O(d^{3/2} \cdot \text{polylog } d)\). Our result deals with determining either the \(\ell\) most significant as well as arbitrary coefficients.

**Theorem 11.** Let \(\mathcal{F}\) denote a field permitting multiplication of two polynomials of degree \(< n\) at cost at most \(M(n)\).

- a) Given \(p \in \mathcal{F}[X]\) of degree \(d\) and \(n \in \mathbb{N}\), the \(\ell \geq d\) most significant coefficients of the power \(p^n \in \mathcal{F}[X]\) can be computed in time \(O(M(\ell) + \log \frac{n\ell}{d})\).
- b) Given \(p \in \mathcal{F}[X]\) of degree \(d\) with \(p(0) \neq 0\) and \(n \in \mathbb{N}\), the \(\ell\) most significant coefficients of \(q := 1/p \mod X^n\) can be computed in time \(O(M(\ell \cdot \log \frac{d}{\ell}) \cdot \log \frac{n\ell}{d})\) where \(d \leq \ell \leq n\).
- c) Let \(\mathcal{F}\) have characteristic zero. Given \(m \in \mathbb{N}, p \in \mathcal{F}[X]\) of degree \(d\), and \(n_1, \ldots, n_\ell \in \mathbb{N}\), the coefficients of \(p^m\) to the terms \(X^{n_i}, i = 1, \ldots, \ell\), can be computed simultaneously in time

\[
O\left( d^2 \cdot \left( \frac{d}{n} + 1 \right) \cdot M\left( \sqrt{n} \right) \right)
\]

where \(d \leq \ell \leq n\) and \(n_1, \ldots, n_\ell \leq n\) and \(n \geq d^2\).
- d) Given \(p \in \mathcal{F}[X]\) of degree \(d\) with \(p(0) \neq 0\) and \(n \in \mathbb{N}\), the \(n\)-th to \((n - 1 + \ell)\)-th coefficients of \(q := 1/p \mod X^n\), can be computed simultaneously in time \(O\left( M(d) \cdot \log n + \ell d \right)\) where \(d \leq n\).

**Proof.** a) is easy based on the observation that the \(\ell\) top-most coefficients of \(p \cdot q\) depend only on the \(\ell\) top-most coefficients of both \(p\) and \(q\). More formally, using the convenient notation of [5] Section 2], it holds

\[
[p]_{\text{deg}(p) - \ell} = \left\lfloor \text{rev} \left( \text{deg}(p), p \right) \right\rfloor^{\ell + 1}, \quad [p \cdot q]^\ell = \left\lfloor [p]^\ell \cdot [q]^\ell \right\rfloor^\ell,
\]

and

\[
\text{rev} \left( \text{deg}(p) + \text{deg}(q), p \cdot q \right) = \text{rev} \left( \text{deg}(p), p \right) \cdot \text{rev} \left( \text{deg}(q), q \right)
\]

where \(\text{rev}(N, \sum_{n=0}^{N} a_nX^n) := \sum_{n=0}^{N} a_{N-n}X^n\) and \(\sum_{n=0}^{\infty} a_nX^n)^\ell := \sum_{\ell=0}^{\infty} a_nX^n, \quad \sum_{n=0}^{\infty} a_nX^n)^\ell := \sum_{n=0}^{\infty} a_{n+\ell}X^n\).

Now calculate first \(q := \frac{p^\ell}{d}\) (w.l.o.g. \(\ell/d\) integral) of degree \(\ell\) by repeated squaring within time \(O(\ell)\); and then obtain from that the \(\ell\) top-most coefficients of \(q^{nd/\ell}\) as the \(\ell\) least ones of \(\text{rev} \left( q^{nd/\ell} \right) = \text{rev} \left( q^{nd/\ell} \right)\) based on Equation (6) and [11] Corollary 2.33 within \(O\left( M(\ell) + \log \frac{n\ell}{d} \right)\).

\(^2\)We refer to the algebraic size of course; in terms of the bit size of \(n\), the output is of exponential degree — still too large.
b) Consider the classical Newton iteration

\[ \tilde{q} \mapsto 2\tilde{q} - \tilde{q}^2 \cdot p \mod X^{2\deg(\tilde{q})} \]  

(7)

which yields a sequence of ‘approximations’ \( \tilde{q}_j \) of doubling degrees such that \( \tilde{q} \equiv q \mod X^{\deg(\tilde{q})} \). In particular, \( q \) itself of degree \( n \) (w.l.o.g. a power of 2) is obtained after \( J := \log n \) iterations with the running time governed by the cost of the polynomial multiplications in Equation (7) and thus dominated, due to the exponentially growing degree of \( \tilde{q} \), by the last step [7, Section 9.1].

Let us analyze Newton's iteration backwards regarding which coefficients of \( q = \tilde{q}_J \)'s predecessors \( \tilde{q}_{J-i} \) the \( \ell \) top-most coefficients of \( q \) depend on. To this end observe that Equation (7) turns some \( \tilde{q}_i \) of degree \( m \) first into the polynomial \( 2\tilde{q}_i - \tilde{q}_i^2 \cdot p \) of degree \( 2m + d \) and then cuts off its \( d \) top-most coefficients in order to obtain \( \tilde{q}_{i+1} \) of degree \( 2m \). By Equation (8), the \( k \) top-most coefficients of \( \tilde{q}_{J-i} \) having degree \( m \) thus depend on and can be computed in time \( O(M(k)) \) from the \( k + d \) top-most terms of \( \tilde{q}_{J-i-1} \) having degree \( m/2 \). In particular for the sought \( \ell \) top-most coefficients of \( q \) having degree \( n \) to be calculated efficiently, it suffices to know the \( \ell + dI \) top-most ones of \( \tilde{q}_{J-1} \) having degree \( n/2^I \) as long as \( \ell + dI \leq n/2^I \). Since \( d \leq \ell \), the algorithm may choose \( I := \lceil \log \frac{\ell}{2} - \log \frac{n}{2} \rceil \) and first perform the classical Newton iteration from \( \tilde{q}_0 \) to \( \tilde{q}_{J-1} \); from this polynomial of degree \( O(\ell \cdot \log \frac{n}{2}) \) continue via steps \( J - (I - 1) \rightarrow J \), keeping at stage no. \( (J - i) \) only the \( \ell + dI \) top-most coefficients. This yields the claimed overall running time.

d) Recall Leibniz' Rule for higher derivatives of a product

\[ (f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \cdot f^{(k)} \cdot g^{(n-k)} \]  

(8)

Applied to \( f := p \) and \( g := 1/(n! \cdot p) \), we obtain for \( n \geq d \geq 1 \):

\[ 0 = \left( p \cdot \frac{1}{n! \cdot p} \right)^{(n)} = \sum_{k=0}^{d} \frac{1}{n!} \cdot p^{(k)} \cdot \left( \frac{1}{(n-k)! \cdot p} \right)^{(n-k)} \]

since \( p^{(k)} \equiv 0 \) for \( k > d = \deg(p) \). Evaluated at \( x = 0 \) and normalized, this constitutes a linear recurrence like [2] of depth \( d-1 \) with constant coefficients \( a_k := \frac{p^{(k)}(0)}{k! \cdot p(0)} \). a recurrence [5, p.41 Eq.(*)] for \( P_n := \left( \frac{1}{n! \cdot p} \right)^{(n)}(0) \), that is, the \( n \)-th coefficient of \( q = 1/p =: \sum_{n=0}^{\infty} P_n X^n \). Now use Theorem [11].

c) W.l.o.g. \( p(0) \neq 0 \), otherwise consider \( p/X \). Apply Equation (8) to \( D^n p^{m+1} := (p^{m+1})^{(n)} \) in two ways\(^3\) where \( D : p \mapsto p' \) denotes the differential operator:

\[ D^n p^{m+1} = D^{n-1} ((m+1) \cdot p' \cdot p^m) = (m+1) \cdot \sum_{k=1}^{d} \binom{n-1}{k-1} \cdot p^{(k)} \cdot D^{n-k} p^m \]

\[ D^n p^{m+1} = D^n (p \cdot p^m) = \sum_{k=0}^{d} \binom{n}{k} \cdot p^{(k)} \cdot D^{n-k} p^m \]

\(^3\) inspired by [28, p.134]
because derivatives of \( p \) higher than \( d \) vanish. Equating right sides yields

\[
p \cdot D^np^m = \sum_{k=1}^{d} \left( (m + 1) \binom{n - 1}{k - 1} - \binom{n}{k} \right) \cdot p^{(k)} \cdot D^{n-k} g^m \quad (9)
\]

which, evaluated at \( x = 0 \) and normalized by \( p(0) \), establishes a linear recurrence for \( P_n := D^n p^m(0) \), that is, the \( n \)-th coefficient of \( p^m \) (up to a factor \( n! \)). This recurrence has depth \( d \) and involves coefficients polynomial in \( n \) of \( \deg(a_k) \leq k \). Now apply Theorem 4d). \( \square \)

Problem 12. Does \( 1/p^m \), that is the concatenation of powering and inversion, also admit fast partial computation?

The related question concerning the product of powered and inverted polynomials is the subject of the following section:

5 Closure Properties

Classical algorithms for fast polynomial arithmetic have all significant coefficients as input and output; they are thus obviously closed under composition and can be combined to solve more advanced problems \( \{26\} \). For partial polynomial arithmetic, on the other hand, the output of two algorithms calculating few coefficients of two respective high-degree polynomials \( p \) and \( q \) cannot simply be fed into a third algorithm in order to obtain merely one coefficient of, say, the product \( p \cdot q \). Instead, we refer to the framework of

5.1 Holonomic Functions and Recurrences

Definition 13. A function \( f(x) \) of one variable \( x \) is holonomic of depth \( k \) if it satisfies a linear ordinary differential equation of order \( k \)

\[
a_0(x) \cdot f^{(k)}(x) + a_1(x) \cdot f^{(k-1)}(x) + \cdots + a_{k-1}(x) \cdot f'(x) + a_k(x) \cdot f(x) = 0 \quad \forall x
\]

where the \( a_i \) are required to be polynomials.

A sequence \( (P_n)_n \) in the field \( \mathcal{F} \) is holonomic of depth \( k \) and degree \( d \) if it satisfies a linear recurrence

\[
a_0(n) \cdot P_{n+k} + a_1(n) \cdot P_{n+k-1} + \cdots + a_{k-1}(n) \cdot P_{n+1} + a_k(n) \cdot P_n = 0 \quad (10)
\]

for all \( n \in \mathbb{N} \) where \( a_i \in \mathcal{F}[\mathbb{N}] \) must be polynomials of degree at most \( d \).

By Theorem 4 holonomic sequences admit multi-evaluation in roughly radical time. This was exploited in Theorem 11c+d) whose proof reveals the following

Example 14. Let \( p \in \mathcal{F}[X] \) denote a polynomial of degree \( d \) with \( p(0) \neq 0 \).
a) The sequence of coefficients of \(1/p\) is holonomic of depth \(d\) and degree 0 (i.e., with constant coefficients).

b) For arbitrary \(n \in \mathbb{N}\), the (finite) sequence of coefficients of \(p^n\) is holonomic of depth \(d + 1\) and degree \(d\).

It is known that a power series represents a holonomic function if its coefficients form a holonomic sequence; see e.g. [20, p.3]. The vast and important classes of hypergeometric [15, Section 5.5] and generalized hypergeometric functions [22] for instance strictly include the holonomic ones. Theorem 4 also yields a roughly quadratic acceleration for their approximation:

**Corollary 15.** Fix a real or complex holonomic power series \(f(x) = \sum_{n=0}^{\infty} c_n x^n\). Then the polynomial given by \(f\)'s first \(N\) terms, that is, \(p_N(x) := \sum_{n=0}^{N-1} c_n x^n\), can be evaluated at given \(x\) using \(O(\sqrt{N} \cdot \log N)\) arithmetic operations.

**Proof.** Let \((c_n)_n\) denote the holonomic recurrence satisfied by \((c_n)_n\); for simplicity with leading coefficient \(a_0 \equiv 1\) — otherwise rescale as in the proof of Lemma 10. Then the sequence of values \(\left(p_N(x)\right)_N\) satisfies

\[
\begin{pmatrix}
  p_N(x) \\
  c_{N+1} \\
  c_N \\
  \vdots \\
  c_{N-k+1}
\end{pmatrix}
= \begin{pmatrix}
  1 & x & 0 & \ldots & 0 \\
  0 & a_k & 0 & \ldots & a_0 \\
  0 & 1 & 0 & \ldots & 0 \\
  0 & 0 & 1 & \ldots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\begin{pmatrix}
  p_{N-1}(x) \\
  c_N \\
  c_{N-1} \\
  \vdots \\
  c_{N-k}
\end{pmatrix}
\]

that is, a recurrence of the form (11) and thus supporting evaluation in the claimed time by virtue of Theorem 4. To choose \(N\), fix \(\rho \in (r, R)\). Then \(|c_n| \leq M \cdot \rho^{-n}\) for all \(n\) with some appropriate \(M \in \mathbb{N}\) — Cauchy’s Estimate. Thus \(f(x)\) differs from \(p_N\) by at most

\[
\sum_{n \geq N} M \cdot \left(\frac{|x|}{\rho}\right)^n = M \cdot \left(\frac{|x|}{\rho}\right)^N \cdot \frac{1}{1 - \frac{|x|}{\rho}}
\]

which drops below \(\epsilon\) for some \(N \leq O\left(\log \frac{1}{\epsilon}\right)\).

**5.2 Product of Fast Partially Computable Polynomials**

The class of holonomic sequences is closed under addition, multiplication, and convolution [25, Theorem 2.1]. Careful inspection of the latter proofs reveals bounds not only on the resulting depth but also on its degree.

**Proposition 16.** Let \((P_n)\) and \((Q_n)\) denote two holonomic sequences of degree \(d\) and depths \(k\) and \(\ell\), respectively. Then
a) their sum \((P_n+Q_n)_n\) is holonomic of depth \(K \leq k+\ell\) and degree \(D \leq (k+\ell)^2d\);

b) their product \((P_nQ_n)_n\) is holonomic of depth \(K \leq k\ell\) and degree \(D \leq k^2\ell^2d\);

c) their convolution \((\sum_{m \leq n} P_m \cdot Q_{n-m})_n\) is holonomic of depth \(K \leq k \cdot \ell\) and degree \(D \leq k^2\ell^2d\).

As the degree of the resulting holonomic equations is closely related to the running time of the Gaussian Elimination as the most expensive component in the algorithms in [24, Section 2.1], upper bounds on the complexity of the latter emerge by consequence of Proposition 16. Our present interest however is closure under multiplication of fast partially computable polynomials:

**Corollary 17.** Let \(F\) have characteristic 0.

a) Given \(p, q \in F[X]\) of degrees at most \(d\) with \(q(0) \neq 0\) and given \(m \in \mathbb{N}\), one can compute \(\ell\) arbitrary coefficients of \(p^m / q\) within

\[
O\left(d^2 \cdot M(\sqrt{nd^3}) + d^2 \cdot \ell \cdot \frac{M(nd^3/\ell)}{nd^3/\ell}\right)
\]

operations over \(F\).

b) Given \(p_1, p_2 \in F[X]\) of degrees at most \(d\) and given \(m_1, m_2 \in \mathbb{N}\), one can compute \(\ell\) arbitrary coefficients of \(p_1^{m_1} \cdot p_2^{m_2}\) within

\[
O\left(d^2 \cdot M(\sqrt{nd^5}) + d^2 \cdot \ell \cdot \frac{M(nd^5/\ell)}{nd^5/\ell}\right)
\]

operations over \(F\).

**Problem 18.** Theorem 4d) yields improved multi-evaluation of holonomic recurrences with coefficients of decaying degrees \(\deg(a_j) \leq j\). We have already applied that in Theorem 11c) for the partial computation of \(p^m\) based on Equation (9). This raises the question whether also the product/convolution of two holonomic recurrences with decaying degrees is again one of decaying degree.

**Proof (Corollary 17).** Since multiplication of polynomials corresponds to the convolution of their coefficient sequences, combine Proposition 16c) with Example 14 to see that the coefficients of \(p^m / q\) and \(p_1^{m_1} \cdot p_2^{m_2}\) form holonomic sequences of depth \(D \leq O(d^2)\) and degrees \(K \leq O(d^3)\) and \(\tilde{K} \leq O(d^5)\), respectively. Then apply Theorem 4c).

The proof of Proposition 16 uses the following extension of Observation 8

**Observation 19**

a) Let \(A = (a_{ij}) \in F[X]^{k \times k}\) denote a regular \(k \times k\)–matrix of rational functions in one variable \(X\) with both numerator and denominator of \(a_{ij}(X)\) polynomials of degree at most \(d\). Then the rational functions which \(A^{-1}\) consists of have degree at most \(dk^2\).

b) If furthermore the denominators in \(A\) are identical for each column, that is, \(a_{ij}(X) = p_{ij}(X)/q_j(X)\); then \(A^{-1}\) has degree at most \(dk\).

The entries of \(A^{-1}\) can in fact be achieved to have a common denominator while still observing the above degree bounds.
Proof (Observation 14). By multiplying each column with the denominators it contains, Claim a) immediately reduces with \( d = dk \) to b). For the latter, exploit \( k \)-linearity of the determinant in order to obtain \( \prod_{j=1}^{k} q_j(X) \) of degree \( \leq dk \) as the denominator of \( \det(A) \) with numerator \( P := \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{j=1}^{k} p_{\sigma(j),j} \) of degree at most \( dk \) as well. Now \( A^{-1} \) has entries \( \det(A_{ij})/\det(A) \) based on Cramer’s Rule with \( A_{ij} \) a sub-matrix of \( A \) and its determinant thus a rational function of degree at most \( d(k-1) \). More precisely, the denominator of \( \det(A_{ij}) \) is \( \prod_{j \neq j_0} q_j(X) \) and thus cancels out in \( \det(A_{ij})/\det(A) \) against the denominator of \( \det(A) \), leaving \( P \) as common denominator of all entries in \( A^{-1} \).

Proof (Proposition 16). By prerequisite, the \( k \)-shifted sequence \((P_{k+n})_n\) is a linear combination of the original (i.e., 0–shifted), the 1–shifted, \ldots, and \((k-1)\)-shifted one; a linear combination with coefficients being rational functions of degree at most \( d \) and common denominator. By induction on \( m \), also the \((k+m)\)-shifted sequence is a linear combination of the first \( k \) shifts — this time with coefficients of degree at most \( md \) and common denominator.

In particular, the vector space \( U \) (over the field \( \mathcal{F}(N) \) of rational functions in \( N \)) formed by all shifts of \((P_n)_n\) has dimension at most \( k \); similarly, the shifts of \((Q_n)_n\) give rise to a vector space \( V \) of dimension at most \( \ell \). Therefore, the vector space \( U + V \) of all joint shifts is at most \((k + \ell)\)-dimensional, that is, latest the \((k + \ell)\)-shift of \((P_n + Q_n)_n\) is a linear combination of its predecessors: closure of holonomic sequences under addition at depth at most \( k + \ell \).

In order to estimate the degree of the rational coefficients involved in the latter linear combination, express each of the first \( k + \ell + 1 \) shifts of \((P_n + Q_n)_n\) as linear combinations of the first \( k \) shifts of \((P_n)_n\) and the first \( \ell \) shifts of \((Q_n)_n\). By the above remark, this gives rise to a \((k + \ell) \times (k + \ell + 1)\)-matrix \( B \) over \( \mathcal{F}(N) \) with entries of degree at most \( d \cdot \max\{k, \ell\} \) and in each column at most two different denominators — which is easy to turn into degree \( D \leq d \cdot (k + \ell) \) with column-wise single common denominators. The \( k + \ell + 1 \) columns of \( B \) are linearly dependent, either by the above considerations or simply due to its format.

Suppose for simplicity that the first \( K := k + \ell \) columns are independent — otherwise we argue similarly to obtain an even shorter and lower-degree recurrence. Expressing the \((k + \ell + 1)\)-st column by these first ones yields an explicit representation of the \((k + \ell)\)-shift of \((P_n + Q_n)_n\) in terms of its first \( k + \ell \) shifts. Denoting by \( b \) the last column of \( B \) and by \( A \) its first \( k + \ell \) columns, we obtain as the coefficients of this representation the vector \( A^{-1} \cdot b \) which, by virtue of Observation 14, consists of rational functions of degree \( \mathcal{O}(DK) = \mathcal{O}(d(k + \ell)^2) \) with common denominator. We have thus arrived at the desired holonomic recurrence \( 14 \) for \((P_n + Q_n)_n\).

For \((P_n, Q_n)_n\), consider the tensor product vector space \( U \otimes V \) of dimension at most \( k \cdot \ell \) to obtain a recurrence of the claimed depth. A generator of \( U \otimes V \) is the collection of mixedly-shifted product sequences \((P_{n+i} \cdot Q_{n+j})_n\) with \( 0 \leq i < k \) and \( 0 \leq j < \ell \). Therefore, each of the \( K := k \cdot \ell \) (singly but farther) shifted sequences \((P_{n+m} \cdot Q_{n+m})_n\), \( 0 \leq m \leq K \), can be expressed as a linear combination of this
generator; in fact with coefficients being rational functions of degree $D \leq dK$ with common denominator. Putting them into a $(K+1) \times K$-matrix and arguing as above, we obtain a representation of the $K$-shifted sequence $(P_{n+K} \cdot Q_{n+K})_n$ as linear combination of the $m$-shifts, $0 \leq m < K$, with coefficients being rational functions of degree $O(DK) = O(dk^2 \ell^2)$.

The proof for convolution proceeds similarly. \qed

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