Maximal integral simplices with no interior integer points

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Abstract

In this paper, we consider integral maximal lattice-free simplices. Such simplices have integer vertices and contain integer points in the relative interior of each of their facets, but no integer point is allowed in the full interior. In dimension three, we show that any integral maximal lattice-free simplex is equivalent to one of seven simplices up to unimodular transformation. For higher dimensions, we demonstrate that the set of integral maximal lattice-free simplices with vertices lying on the coordinate axes is finite. This gives rise to a conjecture that the total number of integral maximal lattice-free simplices is finite for any dimension.

1 Introduction

In dimension $d \in \mathbb{N}$, a simplex is defined to be the convex hull of $d + 1$ affinely independent points. It is called integral if its vertices have integer coordinates. Integral simplices have been studied in several contexts. In particular, the literature is rich in investigations of integral simplices that contain no other lattice points besides the vertices - neither on the boundary nor in the interior (see e.g. Reznick [8], Scarf [9], Sebő [11]). This notion of lattice-freeness is an interesting concept which has proven to be valuable for some problems in integer programming and combinatorics. Most notably, this notion is used for primal integer programming and the study of neighbors of the origin (see Scarf [9]).

In this paper, we consider a different notion of lattice-freeness. The application we have in mind is to use integral simplices as a tool to generate cutting planes for (mixed) integer linear problems (see e.g. [1, 2, 3, 4, 5, 13]). For this purpose, we employ the notion of latticeness introduced by Lovász [7].

**Definition 1.1** A simplex $S \subseteq \mathbb{R}^d$ is called lattice-free if $\text{int}(S) \cap \mathbb{Z}^d = \emptyset$.

To obtain deep cutting planes, we look for integral lattice-free simplices which are maximal with respect to inclusion and call them integral maximal lattice-free simplices. A well-known result of Lovász [7] is that for such simplices each facet contains an integer point in its relative interior.

The application to cutting plane generation requires however to have an explicit list of integral maximal lattice-free simplices available. The partial knowledge about structural properties of such bodies is definitely not enough.

In dimension two, it can easily be verified that any integral maximal lattice-free simplex can be unimodularly transformed to $\text{conv}((0,0)^T, (2,0)^T, (0,2)^T)$. However, to the best of our knowledge, a characterization of integral maximal lattice-free simplices in higher dimensions is not known. Moreover, it is not known if their number is finite.

We note that a recent paper of Treutlein [12] shows finiteness of integral maximal lattice-free simplices in dimension three. In this paper, we extend on this result: we completely characterize integral maximal lattice-free simplices in dimension three and show that - up to unimodular transformation - only seven different simplices exist. Furthermore, for a special class of integral maximal lattice-free simplices, namely simplices with vertices on the coordinate axes, we argue that their number is finite for any dimension $d \in \mathbb{N}$. This gives rise to the conjecture that the total number of integral maximal lattice-free simplices is finite, in general.

Section 2 is dedicated to the analysis of the three dimensional case. Extensions are considered in Section 3. Sections 4 and 5 contain details of our proof technique.
Let $S \subseteq \mathbb{R}^3$ be an integral maximal lattice-free simplex. We assume that $S = \text{conv}(0, v^1, v^2, v^3)$, where $v^1, v^2,$ and $v^3$ are integer vectors. By computing the Hermite normal form of the matrix $(v^1, v^2, v^3)$, it can be assumed that $v^1 = (a, 0, 0)^T$, $v^2 = (b, c, 0)^T$, and $v^3 = (d, e, f)^T$ such that all coefficients $a, b, c, d, e,$ and $f$ are integer and, in addition, $a, c, f > 0$, $0 \leq b < c$, $0 \leq d < f$, and $0 \leq e < f$ hold (see Schrijver [10]). Furthermore, we can assume that $c \neq 1$ and $f \neq 1$ since for $c = 1$ it follows $b = 0$ and thus the facet spanned by the three points $0, v^1, v^2$ does not contain an interior integer point and therefore $S$ is not maximal lattice-free. On the other hand, for $f = 1$, $S$ is contained in the split $\{x \in \mathbb{R}^3 : 0 \leq x_3 \leq 1\}$ which is a contradiction to its maximality. Hence, we have $c, f \geq 2$. In the remainder of this paper we work with the following inequality representation of $S$:

$$
\begin{pmatrix}
0 & 0 & -1 \\
0 & -f & e \\
-cf & bf & cf \\
ef & f(a-b) & e(b-a) + c(a-d)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\leq
\begin{pmatrix}
0 \\
0 \\
0 \\
acf
\end{pmatrix}.
$$

(1)

Our proof strategy is based on partitioning the set of potential integral maximal lattice-free simplices according to relations among the unknowns $a, b, c, d, e$ and $f$. For each of the subcases we then manage to compute upper bounds on $a, c,$ and $f$. Once this has been established, the integral maximal lattice-free simplices can be computed by enumeration. The enumeration provides a list of simplices which must then be checked for unimodular equivalence.

**Definition 2.1** Two sets $S, T \in \mathbb{R}^d$ are unimodularly equivalent if there exist a unimodular matrix $M \in \mathbb{Z}^{d \times d}$ and a vector $v \in \mathbb{Z}^d$ such that $T = MS + v$, where $MS := \{Ms \in \mathbb{R}^d : s \in S\}$.

For integral maximal lattice-free simplices $S = \text{conv}(s^1, s^2, s^3, s^4)$ and $T = \text{conv}(t^1, t^2, t^3, t^4)$ in $\mathbb{R}^3$ it follows that they are unimodularly equivalent if there exist a matrix $M \in \mathbb{Z}^{3 \times 3}$ with $|\det(M)| = 1$ and a vector $v \in \mathbb{Z}^3$ such that $s^j = Mt^\sigma(j) + v$ for all $j = 1, 2, 3, 4$, where $\sigma(j)$ is a permutation.

We distinguish our analysis into the two major cases $a \geq 2$ and $a = 1$. For $a \geq 2$ there are two integer points which play a key role in our subcase analysis. In the following let

$$k := \left\lfloor \frac{ac + b - a}{c} \right\rfloor - 1.$$

The distinctions in our subcase analysis for $a \geq 2$ are based on the locations of the points $(1, 1, 1)$ and $(k, 1, 1)$ relative to $S$. Geometrically, this can be interpreted as follows: Firstly, we investigate simplices where the point $(1, 1, 1)$ either lies on or violates the fourth facet of $[1]$. Afterwards, we consider the opposite case and divide, secondly, into simplices where the point $(k, 1, 1)$ either lies on or violates the third facet of $[1]$ and simplices where this is not the case.

Here is the structure of the case distinction for $a \geq 2$ with the corresponding bounds on $a, c,$ and $f$:

I) $cf + f(a-b) + e(b-a) + c(a-d) \geq acf$

( means: $(1, 1, 1)$ either lies on or violates the fourth facet of $[1]$ )

1) $b \geq a$ \hspace{1cm} $\Rightarrow$ no integral maximal lattice-free simplex possible

2) $b < a$

i) $(a, c) = (2, 2)$ \hspace{1cm} $\Rightarrow$ $(a, c, f) \leq (2, 2, 4)$

ii) $(a, c) \neq (2, 2)$ \hspace{1cm} $\Rightarrow$ $(a, c, f) \leq (6, 6, 6)$

II) $cf + f(a-b) + e(b-a) + c(a-d) < acf$

( means: $(1, 1, 1)$ strictly satisfies the fourth facet of $[1]$ )

1) $-ck + bf + cd - be \geq 0$

( means: $(k, 1, 1)$ either lies on or violates the third facet of $[1]$ )

i) $(a, c) = (2, 2)$ \hspace{1cm} $\Rightarrow$ $(a, c, f) \leq (2, 2, 8)$

ii) $(a, c) \neq (2, 2)$ \hspace{1cm} $\Rightarrow$ $(a, c, f) \leq (3, 18, 6)$
The analysis of the subcases is technical and tedious, but not complicated, in principle. The complete analysis is given in Section 4. In summary, the analysis shows that any integral maximal lattice-free simplex in \( \mathbb{R}^3 \) with \( a \geq 2 \) satisfies \( a \leq 6, c \leq 18, \) and \( f \leq 8. \)

The case where \( a = 1 \) must be treated differently. Here, the integer point \((1,1,1)\) and the unknown parameter \( e \) play a key role. The structure of the case distinction for \( a = 1 \) with the corresponding bounds on \( c \) and \( f \) is shown below.

I) \( cf + f(1 - b) + e(b - 1) + c(1 - d) \geq cf \)  
( means: \((1,1,1)\) either lies on or violates the fourth facet of \((\text{i})\) )  
\Rightarrow no integral maximal lattice-free simplex possible

II) \( cf + f(1 - b) + e(b - 1) + c(1 - d) < cf \)  
( means: \((1,1,1)\) strictly satisfies the fourth facet of \((\text{i})\) )

1) \( e = 0 \)  
\Rightarrow \((c, f) \leq (8,16)\)

2) \( e > 0 \)
   i) \( c \leq e \)  
\Rightarrow \((c, f) \leq (6,12)\)
   ii) \( c > e \)  
\Rightarrow reducible to the case \( a \geq 2 \)

The complete subcase analysis for \( a = 1 \) is given in Section 5. It shows that any integral maximal lattice-free simplex in \( \mathbb{R}^3 \) with \( a = 1 \) satisfies \( c \leq 8 \) and \( f \leq 16 \) or is unimodularly transformable to a simplex with \( a \geq 2 \). Thus, in both cases \( a \geq 2 \) and \( a = 1 \) there is only a finite number of potential simplices that need to be checked. After ruling out simplices which are equivalent by unimodular transformation only seven different simplices remain.

**THEOREM 2.1** Any integral maximal lattice-free simplex in dimension three can be brought by a unimodular transformation into one of the simplices \( S_1 - S_7. \)

The convex hulls of the columns of the seven matrices listed below represent these integral maximal lattice-free simplices. We remark that six of these simplices are given as examples in [12].

1. Simplices with vertices on the coordinate axes:
   \( S_1 : \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}, \quad S_2 : \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad S_3 : \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \)

2. Other simplices:
   \( S_4 : \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad S_5 : \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \quad S_6 : \begin{pmatrix} 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad S_7 : \begin{pmatrix} 0 & 4 & 1 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}. \)

3 Simplices in higher dimensions

The case distinctions in Sections 4 and 5 are specialized to the three dimensional geometry. In order to provide a characterization of integral maximal lattice-free simplices in higher dimensions, it seems unavoidable to develop a general proof technique. Although we do not have this machinery at hand today, we believe that the number of integral maximal lattice-free simplices is finite for any \( d \in \mathbb{N}. \) As a
Let $T \subseteq \mathbb{R}^d$ be an integral maximal lattice-free simplex with one vertex being 0 and the other vertices lying on the $d$ coordinate axes. Without loss of generality we assume that $T = \text{conv}(0, \lambda_1 e_1, \ldots, \lambda_d e_d)$, where $\lambda_j \in \mathbb{Z}_{\geq 0}$ for all $j = 1, \ldots, d$ and $e_j$ denotes the $j$-th unit vector in $\mathbb{R}^d$. We further assume that $\lambda_1 \leq \cdots \leq \lambda_d$. In particular, we have $2 \leq \lambda_1$, otherwise there exists a facet of $T$ which does not contain an interior integer point. The inequality representation of $T$ is given by the $d$ inequalities $x_j \geq 0$, $j = 1, \ldots, d$, and an additional inequality of the form

$$\alpha_1 x_1 + \cdots + \alpha_d x_d \leq r,$$

where $\alpha_j \in \mathbb{N}_{>0}$ for all $j = 1, \ldots, d$ and $r \in \mathbb{N}_{>0}$. It may be checked that the following 14 inequalities together with nonnegativity define integral maximal simplices in dimension four that are maximal lattice-free.

$$
\begin{align*}
21 x_1 + 14 x_2 + 6 x_3 + x_4 &\leq 42, \\
15 x_1 + 10 x_2 + 3 x_3 + 2 x_4 &\leq 30, \\
12 x_1 + 8 x_2 + 3 x_3 + x_4 &\leq 24, \\
10 x_1 + 5 x_2 + 4 x_3 + x_4 &\leq 20, \\
9 x_1 + 6 x_2 + 2 x_3 + x_4 &\leq 18, \\
6 x_1 + 4 x_2 + x_3 + x_4 &\leq 12, \\
6 x_1 + 3 x_2 + 2 x_3 + x_4 &\leq 12, \\
5 x_1 + 2 x_2 + 3 x_3 + x_4 &\leq 10, \\
4 x_1 + 2 x_2 + 3 x_3 + x_4 &\leq 12, \\
4 x_1 + 3 x_2 + 2 x_3 + 2 x_4 &\leq 12, \\
4 x_1 + 2 x_2 + x_3 + x_4 &\leq 8, \\
3 x_1 + x_2 + x_3 + x_4 &\leq 6, \\
2 x_1 + 2 x_2 + x_3 + x_4 &\leq 6, \\
x_1 + x_2 + x_3 + x_4 &\leq 4.
\end{align*}
$$

**Theorem 3.1** Let $2 \leq \lambda_1 \leq \cdots \leq \lambda_d$ be integers and $T = \text{conv}(0, \lambda_1 e_1, \ldots, \lambda_d e_d) \subseteq \mathbb{R}^d$.

(a) If $T$ is maximal lattice-free, then $\lambda_d$ is bounded by $\lambda_d^*$ which is a solution to the following recursion:

$$
\begin{align*}
\lambda_1^* &= 2, \\
\lambda_j^* &= 1 + \prod_{i=1}^{j-1} \lambda_i^*, \quad \forall j = 2, \ldots, d - 1, \\
\lambda_d^* &= \prod_{i=1}^{d-1} \lambda_i^*.
\end{align*}
$$

(b) For $d = 4$, every integral maximal lattice-free simplex of the form $T$ is defined by nonnegativity and one of the inequalities $T_1 - T_{14}$.

**Proof.** (a): First, consider the inequality (2). We show that $r = \alpha_1 + \cdots + \alpha_d$. Observe that $r \leq \alpha_1 + \cdots + \alpha_d$, otherwise the point $1 := (1, \ldots, 1) \in \mathbb{R}^d$ is in the interior of $T$. For the purpose of deriving a contradiction, assume that $r < \alpha_1 + \cdots + \alpha_d$. Since $T$ is maximal lattice-free there exists an integer point $v = (v_1, \ldots, v_d) \neq 1$ in the relative interior of the facet $T \cap \{x \in \mathbb{R}^d : \alpha_1 x_1 + \cdots + \alpha_d x_d = r\}$. In particular, $v$ satisfies $v_j \geq 1$ for all $j = 1, \ldots, d$ and $v_j > 1$ for at least one $j \in \{1, \ldots, d\}$. However, since $\alpha_j > 0$ for all $j = 1, \ldots, d$ this implies $r = \alpha_1 v_1 + \cdots + \alpha_d v_d > \alpha_1 + \cdots + \alpha_d$ which is a contradiction. Thus, we have $r = \alpha_1 + \cdots + \alpha_d$.

From the definition of $T$, it follows that $\alpha_j \lambda_j = \alpha_1 + \cdots + \alpha_d$ for all $j = 1, \ldots, d$ which implies that $\alpha_1 + \cdots + \alpha_d = \frac{1}{\lambda_1}(\alpha_1 + \cdots + \alpha_d) + \cdots + \frac{1}{\lambda_d}(\alpha_1 + \cdots + \alpha_d)$. Hence, we obtain

$$1 = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_d}. \quad (3)$$
We make frequently use of the following simple observation.

\[ a, b, c, d, e, \text{ and } \] all the restrictions (4) - (7) are satisfied with a strict inequality. Recall, that the unknowns \( \lambda_1, \ldots, \lambda_d - 1 \) are. It follows from an inductive argument that, starting with \( \lambda_1 \), gradually fixing the \( \lambda_j \)'s at their minimal possible value yields the maximal value for \( \lambda_d \). In this way, we can recursively compute \( \lambda_d \). Again, this recursion can be formally shown using an inductive argument. Thus, \( \lambda_d \) is bounded.

(b): For \( d = 4 \) the recursion yields \( \lambda_4 \leq 42 \). By enumeration, we obtain the 14 simplices defined by nonnegativity and one of the inequalities \( T_1 - T_{14} \).

The key element of the proof of Theorem 3.1 is a recursive formula. Next we illustrate this recursion for the case of \( d = 5 \).

**Example 3.1** First, \( \lambda_1 \) is fixed at 2. Then, (4) implies that \( \frac{1}{2} = \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \frac{1}{\lambda_4} + \frac{1}{\lambda_5} \). Thus, the choice \( \lambda_2 = 3 \) is minimal and substituting \( (\lambda_1, \lambda_2) = (2, 3) \) in (4) yields \( \frac{1}{6} = \frac{1}{\lambda_3} + \frac{1}{\lambda_4} + \frac{1}{\lambda_5} \). The minimal possible value for \( \lambda_3 \) is now 7 and from (4) we obtain \( \frac{1}{12} = \frac{1}{\lambda_3} + \frac{1}{\lambda_4} \). The last step is to fix \( \lambda_4 = 43 \) which leads to \( \lambda_5 = 1806 \).

Observe that the above recursion does not only give an upper bound on \( \lambda_d \), but also constructs an integral maximal lattice-free simplex with the largest possible value for \( \lambda_d \). For instance, if \( d = 5 \), \( T = \text{conv}(2e_1, 3e_2, 7e_3, 43e_4, 1806e_5) \) is maximal lattice-free.

As a second indication why in any dimension \( d \) there should only be a finite number of integral maximal lattice-free simplices, we note that the following property holds. For any integral maximal lattice-free simplex \( T \), let us denote by \( F_1, \ldots, F_{d+1} \) its facets. Each facet \( F_i \) contains interior integer points. In fact, we can search for a maximal sublattice fully contained in the interior of \( F_i \) that is unimodularly transformable to some \( \mathbb{Z}^n \) for \( s_i \in \{0, \ldots, d - 1\} \).

**Observation 3.1** We have \( s_i = d - 1 \) for at most one \( i \).

**Proof.** For the purpose of deriving a contradiction, assume that \( s_i = s_j = d - 1 \) for two facets \( F_i \) and \( F_j \) with \( i \neq j \). Then, \( F_i \) and \( F_j \) have each at least \( 2^{d-1} \) integer points of different parity in their interior. If two integer points \( w_i \) and \( w_j \) on \( F_i \) and \( F_j \), respectively, have the same parity, then \( \frac{1}{2}(w_i + w_j) \) is an interior point in \( T \). Hence, we have \( 2^d \) integer points with different parity in the interior of \( F_i \cup F_j \). Now, any interior integer point of a facet different from \( F_i \) and \( F_j \) will lead to a contradiction.

This shows that the possibilities for the sublattice structure in the interior of the facets is somehow limited. Finally, let us remark, that if \( s_i = 0 \) for all \( i \), finiteness follows from a result of Lagarias and Ziegler [5].

## 4 Details on the case distinction for \( a \geq 2 \)

Let \( S \subseteq \mathbb{R}^3 \) be an integral maximal lattice-free simplex with \( a \geq 2 \) given by the following inequality description:

\[
- x_3 \leq 0, \quad (4)
\]

\[
- f x_2 + c x_3 \leq 0, \quad (5)
\]

\[
-c f x_1 + b f x_2 \leq 0, \quad \text{ and } \quad (c d - b e) x_3 \leq 0, \quad (6)
\]

\[
-c f x_1 + f (a - b) x_2 + (e (b - a) + c (a - d)) x_3 \leq a c f. \quad (7)
\]

In this section, we will often state interior integer points as counterexamples and simply prove that for those points all the restrictions (4) - (7) are satisfied with a strict inequality. Recall, that the unknowns \( a, b, c, d, e \), and \( f \) are integer and satisfy the following properties:

\[
2 \leq a, \quad 0 \leq b < c, \quad 2 \leq c, \quad 0 \leq d < f, \quad 2 \leq f, \quad 0 \leq e < f.
\]

We make frequently use of the following simple observation.
Observation 4.1

(a) Let \( 2 \leq l \in \mathbb{N} \). Then, for any integer \( x \geq l \), we have \( \frac{x}{x-1} \leq \frac{l}{l-1} \).

(b) Let \( 2 \leq l_x, l_y \in \mathbb{N} \). Then, for any integers \( x \geq l_x \) and \( y \geq l_y \), we have \( \frac{x y}{x+y} \geq \frac{l_x l_y}{l_x+l_y} \).

(c) Let \( 2 \leq l_x \in \mathbb{N} \) and \( 3 \leq l_y \in \mathbb{N} \). Then, for any integers \( x \geq l_x \) and \( y \geq l_y \), we have \( \frac{x y}{x+y-2} \leq \frac{l_x l_y}{l_x-l_x-l_y} \).

I) \( cf + f(a-b) + e(b-a) + c(a-d) \geq acf \)

1) \( b \geq a \)

In this case, we have \( acf \leq cf + (f-e)(a-b) + c(a-d) \leq ac + cf \) which implies that \( af \leq a + f \). Since \( a \geq 2 \) and \( f \geq 2 \) this inequality is only satisfied for \( (a, f) = (2, 2) \). Substituting this in \( cf + f(a-b) + e(b-a) + c(a-d) \geq acf \) yields \( 4c \leq 2c + 2(b-2) + e(b-2) + c(2-d) \). Since \( b \geq a = 2 \) and \( e < f = 2 \) we obtain \( cd \leq 0 \) and therefore \( d = 0 \). If \( b > 2 \) then \( 0 = cd \leq (2-e)(2-b) < 0 \) is a contradiction. Hence, we have \( b = 2 \). However, \( S \) is now contained in \( \text{conv} \left( (0,0,0)^T, (2,0,0)^T, (0,0,2)^T \right) + \text{span}(e_2) \) which is a contradiction to its maximality.

2) \( b < a \)

i) \((a, c) = (2, 2)\) If \( b = 0 \) then there is no interior integer point in the facet \( 0, v_1^2 \). Hence, \( b = 1 \).

Substituting \((a, b, c) = (2, 1, 2)\) in \( cf + f(a-b) + e(b-a) + c(a-d) \geq acf \) yields \( 4f \leq 2f + f - e + 2(2 - d) \leq 3f + 4 \) and therefore \( f \leq 4 \).

ii) \((a, c) \neq (2, 2)\) Here, we obtain \( acf \leq cf + f(a-b) + e(b-a) + c(a-d) \leq ac + af + cf \) which implies

\[
\frac{f}{ac - a - c} \leq \frac{ac}{ac - a - c} \leq \frac{1}{3}.
\]

From \((a, f) = (2, 2)\) and Observation \((a, c) \neq (2, 2)\), it follows that \( f \leq \frac{ac}{ac - a - c} \leq \frac{1}{3} \).

Now assume \((a, f) = (2, 2)\). Then \( c \geq 3 \) and \( b, d, e \in (0, 1) \). Substituting \((a, f) = (2, 2)\) in \( cf + f(a-b) + e(b-a) + c(a-d) \geq acf \) implies that \( 2c + 2(2 - b) + e(b-2) + c(2-d) \geq 4c \). Since \( b \geq a = 2 \) and \( e = 1 \) it follows \( cd \leq 2 \) and therefore \( d = 0 \). Hence, the facet \( 0, v_1, v_3 \) does not contain an interior integer point. Thus, let \( b = e = 0 \). We must have \( d = 1 \) since otherwise the facet \( 0, v_1, v_3 \) does not contain an interior integer point. This implies \( d = 1 \) and \( c = cd \leq 4 \).

Therefore, we can assume that \((a, f) \neq (2, 2)\). From \((a, f) = (2, 2)\) and Observation \((a, c) \neq (2, 2)\), it follows that \( c \leq \frac{a f}{c f - a - f} \leq 6 \) and it remains to find an upper bound on \( a \). If \((c, f) = (2, 2)\) there is no interior integer point in the facet \( 0, v_3 \). On the other hand, if \((c, f) \neq (2, 2)\) we have \( a \leq \frac{c f}{c f - a - f} \leq 6 \) by \((a, f) = (2, 2)\) and Observation \((a, c) \neq (2, 2)\).

II) \( cf + f(a-b) + e(b-a) + c(a-d) < acf \)

For the purpose of deriving a contradiction assume that \( -cf + bf + cd - be < 0 \). In this case, the point \((1, 1, 1)\) is in the interior of \( S \) as one can easily check by substituting \((1, 1, 1)\) in the inequalities \((4)-(7)\): in all four inequalities, the left hand side is strictly less than the right hand side. We therefore must have

\[
-cf + bf + cd - be \geq 0.
\]

We first show that \( k \geq 1 \). Note that \( ac - a - c + b \leq 0 \) holds true only for \((a, b, c) = (2, 0, 2)\). However, then the facet \( 0, v_1, v_2^3 \) does not contain an interior integer point. Thus, we have \( ac - a - c > 0 \) which implies that \( k \geq 1 \).

1) \( -cfk + bf + cd - be \geq 0 \)

We have already shown that \( k \geq 1 \). Assume \( k \geq 2 \). Then \( -cfk + bf + cd - be \leq -2cf + bf + cd - be = f(b - c) + c(d - f) - be < 0 \) which is a contradiction. Hence, \( k = 1 \) and it follows

\[
1 < \frac{a + b - a}{c} \leq 2.
\]
However, in this case the maximality of would lead to a contradiction in this chain of inequalities. Therefore, we must have \((\cdot)\).

Assume \(a = 4\). Then, \((\cdot)\) implies \(2c + b \leq 4\) which can never hold true for \(c \geq 2\) and \(b > 0\). Thus, \(a \neq 4\). So assume \(a = 3\). Now, \((\cdot)\) implies \(c + b \leq 3\) which is only satisfied for \((b, c) = (1, 2)\). Substituting this in \((\cdot)\) yields \(f \leq 2d - e\). For \(d \geq 4\) the point \((2, 1, 1)\) is in the interior of \(S\): obviously, \((\cdot)\) and \((\cdot)\) are strict;

\[
\begin{align*}
\text{(i)} & : -2cf + bf + cd - be = f(b - c) + c(d - f) - be < 0; \\
\text{(ii)} & : 2cf + f(a - b) + e(b - a) + c(a - d) = 4f + 2f - 2e + 6 - 2d = 6f + 2(3 - d - e) < 6f = acf.
\end{align*}
\]

So we have \(d \leq 3\) which implies \(f \leq 2d - e \leq 6\) and \(c \leq af \leq 18\). It remains to consider the case \(a = 2\).

i) \((a, c) = (2, 2)\) From \(0 < b < c = 2\), it follows that \(b = 1\) and \((\cdot)\) implies \(f \leq 2d - e\). First note that \(e \leq 2\) since otherwise the point \((2, 1, 1)\) is in the interior of \(S\): clearly, \((\cdot)\) and \((\cdot)\) are strict;

\[
\begin{align*}
\text{(i)} & : -2cf + bf + cd - be = f(b - c) + c(d - f) - be < 0; \\
\text{(ii)} & : 2cf + f(a - b) + e(b - a) + c(a - d) = 4f + f - e + 4 - 2d \leq 4f - 2e + 4 < 4f = acf.
\end{align*}
\]

Now assume \(f \geq 9\). If \(f + 4 < 2d + e\) holds true, then the point \((2, 1, 1)\) is in the interior of \(S\): as above, \((\cdot)\), \((\cdot)\), and \((\cdot)\) are strict;

\[
\begin{align*}
\text{(i)} & : 2cf + f(a - b) + e(b - a) + c(a - d) = 4f + f - e + 4 - 2d < 4f = acf.
\end{align*}
\]

If \(f + 4 \geq 2d + e\), the point \((2, 1, 2)\) is in the interior of \(S\): clearly, \((\cdot)\) is strict; \((\cdot)\) is strict since \(f \geq 9\) and \(e \leq 2\);

\[
\begin{align*}
\text{(i)} & : -2cf + bf + 2cd - 2be = -4f + f + 4d - 2e \leq -f + 8 - 4e < 0; \\
\text{(ii)} & : 2cf + f(a - b) + 2e(b - a) + 2c(a - d) = 4f + f - 2e + 8 - 4d \leq 4f - f - 4e + 8 < 4f = acf.
\end{align*}
\]

Hence, we must have \(f \leq 8\).

ii) \((a, c) \neq (2, 2)\) In this case, we have \(c \geq 3\). If \(f \geq 7\) the point \((2, 1, 1)\) is in the interior of \(S\): clearly, \((\cdot)\), \((\cdot)\), and \((\cdot)\) are strict (see above);

\[
\begin{align*}
\text{(i)} & : 2cf + f(a - b) + e(b - a) + c(a - d) = cf + cf - bf - cd + be + 2(e + f - e) \leq cf + 2(e + f - e) < cf + cf = 2cf = acf.
\end{align*}
\]

Here, the strict inequality in the last row follows from the fact that \(c + f - e < \frac{1}{2}cf\) for \(c \geq 3\) and \(f \geq 7\) (use Observation \((\cdot)\)). Therefore, we obtain \(f \leq 6\) and it follows \(c \leq af \leq 12\).

2) \(-cfk + bf + cd - be < 0\)

For the purpose of deriving a contradiction assume \(cfk + f(a - b) + e(b - a) + c(a - d) < acf\). Then, the point \((k, 1, 1)\) is in the interior of \(S\). One can see this by substituting \((k, 1, 1)\) in the inequalities \((\cdot)-(\cdot)\).

Thus, it follows

\[
\begin{align*}
\text{(i)} & : cfk + f(a - b) + e(b - a) + c(a - d) \geq acf.
\end{align*}
\]

We know that \(k \geq 1\). Now assume \(k = 1\). From \((\cdot)\) and \((\cdot)\) it follows that \(acf \leq cf + f(a - b) + e(b - a) + c(a - d) \leq acf - f - e\). Note that \(c + f - e < cf\) holds true for any feasible triple \((c, e, f) \neq (2, 0, 2)\) and would lead to a contradiction in this chain of inequalities. Therefore, we must have \((c, e, f) = (2, 0, 2)\). However, in this case \(S\) is contained in \(\text{conv}\left((0,0,0)^T, (0,2,0)^T, (0,0,2)^T\right) + \text{span}(e_1)\) which contradicts the maximality of \(S\). Thus, we have \(k \geq 2\).
From (11) it follows that \( cd - be \geq f(c - b) > 0 \) and from (11) it follows that \( e(b - a) + c(a - d) \geq f(-ck + ac + b - a) > f(-\frac{ac + b - a}{e} + ac + b - a) = 0 \). Thus, we have

\[
  cd - be > 0
\]

and

\[
e(b - a) + c(a - d) > 0.
\]

i) \( e > 0 \) Here, the point \((d, e, 1)\) is in the interior of \(S\): obviously, (1) is strict; (5) is strict since we have \( e > 0 \);

\[
  (8) : \quad -cf + b(e - c) + cd - be = (1 - f)(cd - be) < 0;
\]
\[
  (7) : \quad cf + f(a - b) + e(b - a) + c(a - d) =
\]
\[
  (1 - f)(-cd + e(b - a)) + ac < (1 - f)(-ac) + ac = acf.
\]

The strict inequalities follow from (12) and (13).

ii) \( e = 0 \) By (12) and (13), we obtain \( 0 < d < a \). Furthermore, (9) and (11) change to

\[
  - cf + b f + cd \geq 0
\]

and

\[
  cfk + f(a - b) + c(a - d) \geq acf.
\]

A) \( a = b \) In this case we have \( k = a - 1 \). Substituting this in (15) yields \( d \leq a - f \) and since \( d > 0 \) this implies that \( f < a \). If \( d \geq 2 \) the point \((2, 1, 1)\) is in the interior of \(S\): evidently, (4) and (5) are strict;

\[
  (6) : \quad -2cf + bf + cd - be = f(b - c) + c(d - f) < 0;
\]
\[
  (7) : \quad 2cf + f(a - b) + e(b - a) + c(a - d) =
\]
\[
  2cf + c(a - d) < 2cf + cf(a - d) = acf + cf(2 - d) \leq acf.
\]

So we have \( d = 1 \). If \( a \geq 4 \) we have \((2, 1, 1)\) in the interior of \(S\): as above, (4), (5), and (6) are strict;

\[
  (7) : \quad 2cf + f(a - b) + e(b - a) + c(a - d) = 2cf + c(a - 1) < 2cf + cf(a - 2) = acf.
\]

The strict inequality follows from Observation 4.1.1: \( \frac{a - 1}{a - 2} \leq \frac{3}{2} < 2 \leq f \) since \( a \geq 4 \). Thus, let \( a \leq 3 \). The chain \( 0 < d < f < a \leq 3 \) implies that \((a, f) = (3, 2)\). Substituting this in (14) yields \( c \leq 6 \).

B) \( a < b \) If \( d \geq 2 \), the point \((2, 1, 1)\) is in the interior of \(S\): as above, (4), (5), and (6) are strict;

\[
  (7) : \quad 2cf + f(a - b) + e(b - a) + c(a - d) =
\]
\[
  2cf + c(a - d) < 2cf + cf(a - d) = acf + cf(2 - d) \leq acf.
\]

Hence, \( d = 1 \). From (14) and Observation 4.1.1, it follows that

\[
  c \leq b \frac{f}{f - 1} \leq 2b.
\]

If \( a \geq 3 \), the point \((2, 1, 1)\) is in the interior of \(S\): as above, (4), (5), and (6) are strict;

\[
  (7) : \quad 2cf + f(a - b) + e(b - a) + c(a - d) < 2cf + c(a - 1) \leq 2cf + cf(a - 2) = acf.
\]

The last inequality follows from Observation 4.1.1: \( \frac{a - 1}{a - 2} \leq \frac{3}{2} < 2 \leq f \) since \( a \geq 3 \). So let \( a = 2 \). This implies \( b \geq 3 \). Furthermore, it follows \( 2 \leq k < \frac{ac + b - a}{e} = \frac{1}{3} + \frac{1}{3} < 3 \) and therefore we obtain that \( k = 2 \). From (15) it follows that \( 2cf + f(2 - b) + c \geq 2cf \) which implies

\[
  c \geq f(b - 2) \geq f.
\]
I) $d = 0$. However, there is no interior integer point in the facet spanned by the three points (0,0,0), (1,0,0), and (b,c,d). This remains to consider the case $f = 2$. Here, (16) and (17) imply $2b - 4 \leq c \leq 2b$. If $b \geq 7$, the point (2,1,1) is in the interior of $S$: clearly, (4) and (5) are strict;

\begin{align*}
(5): & \quad -2cf + 2bf + cd - be = 4b - 3c \leq 4b + 12 - 6b = 12 - 2b < 0; \\
(6): & \quad 2cf + 2f(a - b) + e(b - a) + c(a - d) = 4c + 4(2-b) + c \leq 4c + 8 - 2b < 4c = acf.
\end{align*}

Thus, $b \leq 6$ and it follows $c \leq 2b \leq 12$.

C) $a > b$ First we argue that $a \neq 2$. Assuming that $a = 2$, we obtain that $k \leq a - 1 = 1$. This contradicts $k \geq 2$. Hence, let $a \geq 3$. If $a \geq 7$, the point (2,1,1) is in the interior of $S$: as above, (4), (5), and (6) are strict;

\begin{align*}
(7): & \quad 2cf + f(a - b) + e(b - a) + c(a - d) = cf + ac + af + cf - bf - cd \leq ac + af + cf < acf.
\end{align*}

The strict inequality follows from the fact that $\frac{a}{f-1} \leq \frac{7}{6} < \frac{6}{5} \leq \frac{cf}{a-f}$ by Observation 4.1(a) and (4.2(b)) as $a \geq 7$ and as we cannot have $(c,f) = (2,2)$ since in this case $S$ is contained in $\text{conv}\{(0,0,0)^T, (0,2,0)^T, (0,0,2)^T\} + \text{span}(e_1)$ which contradicts the maximality of $S$. Thus, $a \leq 6$.

Next we show that it is impossible for $c$ to be greater or equal to 7. If $c \geq 7$, then the point (2,1,1) would be in the interior of $S$: as above, (4), (5), and (6) are strict;

\begin{align*}
(7): & \quad 2cf + f(a - b) + e(b - a) + c(a - d) = cf + ac + af + cf - bf - cd \leq ac + af + cf < acf.
\end{align*}

Here, the strict inequality follows from the fact that $\frac{c-f}{a-1} \leq \frac{7}{6} < \frac{6}{5} \leq \frac{af}{a-f}$ by Observation 4.1(a) and (4.2(b)) as $c \geq 7$ and as $(a,f) \geq (3,2)$. Thus, $c \leq 6$. Similarly, it can be verified that we must have $f \leq 6$ as otherwise the point (2,1,1) is in the interior of $S$.

5 Details on the case distinction for $a = 1$

Let $S \subseteq \mathbb{R}^3$ be an integral maximal lattice-free simplex with $a = 1$ given by the following inequality description:

\begin{align*}
-x_3 & \leq 0, & \text{(18)} \\
-fx_2 & + ex_3 \leq 0, & \text{(19)} \\
-cfx_1 & + bfsx_2 & + (cd-be)x_3 \leq 0, & \text{(20)} \\
cfx_1 & + f(1-b)x_2 & + (e(b-1) + c(1-d))x_3 \leq cf. & \text{(21)}
\end{align*}

As in Section 4, we will often state interior integer points as counterexamples by proving that for those points all the restrictions $\text{(18)} - \text{(21)}$ are satisfied with a strict inequality. Recall, that the unknowns $b, c, d, e,$ and $f$ are integer and satisfy the following properties:

\begin{align*}
2 & \leq c, & 0 & \leq b < c, \\
2 & \leq f, & 0 & \leq d < f, \\
0 & \leq e < f.
\end{align*}

First note that $b > 1$ since otherwise there is no interior integer point in the facet spanned by the three points $(0,0,0)^T$, $(1,0,0)^T$, and $(b,c,d)^T$. This implies $b \geq 2$ and $c \geq 3$.

I) $cf + f(1-b) + e(b-1) + c(1-d) \geq cf$

From $cf + f(1-b) + e(b-1) + c(1-d) \geq cf$, it follows that $c(d-1) \leq (f-e)(1-b) < 0$. Thus, we obtain $d = 0$. However, there is no interior integer point in the facet spanned by the three points $(0,0,0)^T$, $(1,0,0)^T$, and $(0,e,f)^T$ which is a contradiction.
II) \( cf + f(1 - b) + e(b - 1) + c(1 - d) < cf \)

In this case, we must have \(-cf + bf + cd - be \geq 0\) since otherwise the point \((1,1,1)\) is in the interior of \(S\) as one easily checks by substituting \((1,1,1)\) in the inequalities \((18)-(21)\). Thus, we obtain

\[-cf + bf + cd - be \geq 0.\] (22)

For the purpose of deriving a contradiction assume that \( cf + f(1 - b) + e(b - 1) + c(1 - d) < 0 \). Then, the point \((2,1,1)\) is in the interior of \(S\): clearly, \((18),\ (19),\ \text{and}\ (21)\) are strict.

\((20)\ : \ -2cf + bf + cd - be = f(b - c) + c(d - f) - be < 0.\)

Hence, it follows

\[ cf + f(1 - b) + e(b - 1) + c(1 - d) \geq 0.\] (23)

Assume \( d \leq e \). Using \((22)\) yields \( 0 \leq -cf + bf + cd - be \leq -cf + bf + ce - be = (b - c)(f - e) < 0 \) which is a contradiction. Therefore, it holds \( d > e \).

We now construct a sequence of points which helps to derive conditions for the unknown variables. These conditions are used later in the subcase analysis. For the moment assume \( c - e \leq f \) and consider the sequence of points \((\theta, \theta, 1)\), where \( \theta \geq 1 \). By equation \((22)\) and the relation \( c > b \), we obtain that there exists some \( \Theta \geq 2 \) such that

\[-cf(\Theta - 1) + bf(\Theta - 1) + cd - be \geq 0,\] (24)

\[-cf + bf + cd - be < 0.\] (25)

Since the point \((\Theta, \Theta, 1)\) satisfies \((18),\ (19),\ \text{and}\ (20)\) strictly, we must have that

\[ cf\Theta + f(1 - b)\Theta + e(b - 1) + c(1 - d) \geq cf \] (26)

since otherwise the point \((\Theta, \Theta, 1)\) is in the interior of \(S\). Adding \((24)\) and \((20)\) yields

\[ f(\Theta - b) + c - e \geq 0.\] (27)

Using \((23)\) and \((24)\) together with our assumption \( c - e \leq f \), we obtain \( f(\Theta - 1)(c - b) \leq cd - be \leq f(c - b) + f + c - e \leq f(c - b) + 2f \) and hence \( f(\Theta - 1)(c - b) \leq f(c - b) + 2f \Leftrightarrow \Theta \leq 2 + \frac{2}{c - b} \). We infer

\[ \Theta \leq \begin{cases} 
4, & \text{if } c = b + 1 \\
3, & \text{if } c = b + 2 \\
2, & \text{if } c \geq b + 3,
\end{cases} \] (28)

This shows that \( 2 \leq b \leq 5 \) whenever \( c - e \leq f \) holds true.

1) \( e = 0 \)

Since \( e = 0 \) the inequalities \((22)\) and \((23)\) change to

\[-cf + bf + cd \geq 0\] (29)

and

\[ cf + f(1 - b) + c(1 - d) \geq 0.\] (30)

Furthermore, we can assume without loss of generality that \( c \leq f \). Otherwise, if \( c > f \), we switch coordinates by applying the unimodular transformation to \(S\) with \( M \) being the identity matrix in \( \mathbb{R}^3 \) where the last two columns are interchanged and \( v \) being 0 (see Definition \([21]\)).

By assumption, \( c - e \leq f \) is now satisfied and thus, by \((23)\), the variable \( b \) is bounded. In addition, \( c \) is bounded for \( b = 4 \) and \( b = 5 \). We first find an upper bound on \( c \) for the remaining cases where \( b = 2 \) and \( b = 3 \).

Let \( b = 2 \). We show that \( c \leq 7 \). So assume \( c \geq 8 \). If \( f(2c - 3) + c(1 - d) < cf \) holds true, then the point \((2,3,1)\) is in the interior of \(S\): clearly, \((18)\) and \((19)\) are strict;

\[ (20) : \ -2cf + 3bf + cd - be = -2cf + 6f + cd = f(6 - c) + c(d - f) < 0; \]

\[ (21) : \ 2cf - 3f + c(1 - d) = f(2c - 3) + c(1 - d) < cf. \]

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Thus, let $f(2c - 3) + c(1 - d) \geq cf \iff f \geq \frac{c(d-1)}{c-3}$. From (29), it follows that $f \leq \frac{cd}{c^2 - 2}$. Putting this together we have $\frac{cd}{c^2 - 2} \leq \frac{cd}{c-2} \iff c \geq d + 2$. We infer that the point $(2,4,1)$ is in the interior of $S$: clearly, (18) and (19) are strict;

$$
\begin{align*}
(20) & : \quad -2cf + 4bf + cd - be = -2cf + 6f + cd = f(8 - c) + c(d - f) < 0; \\
(21) & : \quad 2cf + 4f(1 - b) + e(b - 1) + c(1 - d) = 2cf - 4f + c(1 - d) = 2f(c - 2) + c(1 - d) \leq 2cd + c(1 - d) = c(d + 1) \leq c(c - 1) \leq c(f - 1) < cf.
\end{align*}
$$

Here, the inequalities in the last row follow from (29), $d + 2 \leq c$, and $c \leq f$. Hence, we must have $c \leq 7$.

Let $b = 3$. We show that $c \leq 8$. So assume $c \geq 9$. If $2f(c - 2) + c(1 - d) < cf$ holds true, then the point $(2,2,1)$ is in the interior of $S$: clearly, (18) and (19) are strict;

$$
\begin{align*}
(20) & : \quad -2cf + 2bf + cd - be = -2cf + 6f + cd = f(6 - c) + c(d - f) < 0; \\
(21) & : \quad 2cf + 2f(1 - b) + e(b - 1) + c(1 - d) = 2cf - 4f + c(1 - d) = 2f(c - 2) + c(1 - d) < cf.
\end{align*}
$$

Thus, let $2f(c - 2) + c(1 - d) \geq cf \iff f \geq \frac{cd}{c^2 - 2}$. From (29), it follows that $f \leq \frac{cd}{c^2 - 2}$. Putting this together we have $\frac{cd}{c^2 - 2} \leq \frac{cd}{c-1} \iff c \geq d + 3$. We infer that the point $(2,3,1)$ is in the interior of $S$: clearly, (18) and (19) are strict;

$$
\begin{align*}
(20) & : \quad -2cf + 3bf + cd - be = -2cf + 9f + cd = f(9 - c) + c(d - f) < 0; \\
(21) & : \quad 2cf + 3f(1 - b) + e(b - 1) + c(1 - d) = 2cf - 6f + c(1 - d) = 2f(c - 3) + c(1 - d) \leq 2cd + c(1 - d) = c(d + 1) \leq c(c - 2) \leq c(f - 2) < cf.
\end{align*}
$$

Here, the inequalities in the last row follow from (29), $d + 3 \leq c$, and $c \leq f$. Hence, we must have $c \leq 8$.

It follows, that 13 choices of $(b,c)$ are left: $(2,3)$, $(3,4)$, $(4,5)$, $(5,6)$, $(2,4)$, $(3,5)$, $(4,6)$, $(2,5)$, $(2,6)$, $(2,7)$, $(3,6)$, $(3,7)$, $(3,8)$. In the following we will prove upper bounds on $f$ for each of the 13 possibilities.

• Let $(b,c) = (2,3)$. We show that $f \leq 9$. So assume $f \geq 10$. From (29) and (30) we obtain $f \leq 3d$ and $2f \geq 3(d - 1)$. If $d < \frac{1}{2}f$ holds true, then the point $(2,1,3)$ is in the interior of $S$: clearly, (18) and (19) are strict;

$$
\begin{align*}
(20) & : \quad -2cf + bf + 3(cd - be) = -6f + 2f + 9d < -4f + 4f = 0; \\
(21) & : \quad 2cf + f(1 - b) + 3(e(b - 1) + c(1 - d)) = 6f - f + 9(1 - d) < 3f + 2f + 9 - 3f < 3f.
\end{align*}
$$

Thus, let $d \geq \frac{1}{3}f$. If $d < \frac{1}{3}f$ holds true, then the point $(2,1,2)$ is in the interior of $S$: clearly, (18) and (19) are strict;

$$
\begin{align*}
(20) & : \quad -2cf + bf + 2(cd - be) = -6f + 2f + 6d < -4f + 4f = 0; \\
(21) & : \quad 2cf + f(1 - b) + 2(e(b - 1) + c(1 - d)) = 6f - f + 6(1 - d) \leq 3f + 2f + 6 - \frac{4}{9}f = 3f + \frac{2}{3}(9f - f) < 3f.
\end{align*}
$$

Thus, let $d \geq \frac{2}{3}f$. The point $(3,1,3)$ is now in the interior of $S$: clearly, (18) and (19) are strict;

$$
\begin{align*}
(20) & : \quad -3cf + bf + 3(cd - be) = -9f + 2f + 9d \leq -7f + 9(\frac{2}{3}f + 1) = 9 - f = 0; \\
(21) & : \quad 3cf + f(1 - b) + 3(e(b - 1) + c(1 - d)) = 9f - f + 9(1 - d) \leq 3f + 5f + 9 - 6f = 3f + 9 - f < 3f.
\end{align*}
$$

• Let $(b,c) = (3,4)$. We show that $f \leq 8$. So assume $f \geq 9$. From (29) and (30) we obtain $f \leq 4d$ and $f \geq 2(d - 1)$. If $2d < f$ holds true, then the point $(2,2,1)$ is in the interior of $S$: clearly, (18) and (19) are strict;

$$
\begin{align*}
(20) & : \quad -2cf + 2bf + cd - be = -8f + 6f + 4d = 2(2d - f) < 0; \\
(21) & : \quad 2cf + 2f(1 - b) + e(b - 1) + c(1 - d) = 8f - 4f + 4(1 - d) < 4f + 4 - f < 4f.
\end{align*}
$$
Thus, let $2d \geq f$. The point $(2, 1, 2)$ is now in the interior of $S$: clearly, \ref{18} and \ref{19} are strict;

\begin{align*}
\text{(20)} & : \quad -2cf + bf + 2(cf - be) = -8f + 3f + 8d \leq -8f + 8(\frac{1}{2}f + 1) = 8 - f < 0; \\
\text{(21)} & : \quad 2cf + f(1 - b) + 2(e(b - 1) + c(1 - d)) = \\
& \quad 8f - 2f + 8(1 - d) \leq 4f + 2f + 8 - 4f = 4f + 2(4 - f) < 4f.
\end{align*}

Let $(b, c) = (4, 5)$. We show that $f \leq 5$. So assume $f \geq 6$. From \ref{29} and \ref{30} we obtain $f \leq 5d$ and $2f \geq 5(d - 1)$. If $5d < 2f$ holds true, then the point $(2, 2, 1)$ is in the interior of $S$: clearly, \ref{18} and \ref{19} are strict;

\begin{align*}
\text{(20)} & : \quad -2cf + 2bf + cd - be = -10f + 8f + 5d = -2f + 5d < 0; \\
\text{(21)} & : \quad 2cf + 2f(1 - b) + e(b - 1) + c(1 - d) = \\
& \quad 10f - 6f + 5(1 - d) = 5f - f + 5(1 - d) < 5f.
\end{align*}

Thus, let $5d \geq 2f$. The point $(2, 1, 2)$ is now in the interior of $S$: clearly, \ref{18} and \ref{19} are strict;

\begin{align*}
\text{(20)} & : \quad -2cf + bf + 2(cf - be) = -10f + 4f + 10d \leq -6f + 10(\frac{2}{3}f + 1) = 2(5 - f) < 0; \\
\text{(21)} & : \quad 2cf + f(1 - b) + 2(e(b - 1) + c(1 - d)) = \\
& \quad 10f - 3f + 10(1 - d) \leq 5f + 2f + 10 - 4f = 5f + 2(5 - f) < 5f.
\end{align*}

Let $(b, c) = (5, 6)$. We show that $f \leq 6$. So assume $f \geq 7$. From \ref{29} and \ref{30} we obtain $f \leq 6d$ and $f \geq 3(d - 1)$. If $3d < f$ holds true, then the point $(2, 2, 1)$ is in the interior of $S$: clearly, \ref{18} and \ref{19} are strict;

\begin{align*}
\text{(20)} & : \quad -2cf + 2bf + cd - be = -12f + 10f + 6d = 2(3d - f) < 0; \\
\text{(21)} & : \quad 2cf + 2f(1 - b) + e(b - 1) + c(1 - d) = \\
& \quad 12f - 8f + 6(1 - d) = 6f - 2f + 6 - f < 6f.
\end{align*}

Thus, let $3d \geq f$. The point $(2, 1, 2)$ is now in the interior of $S$: clearly, \ref{18} and \ref{19} are strict;

\begin{align*}
\text{(20)} & : \quad -2cf + bf + 2(cf - be) = -12f + 5f + 12d \leq -7f + 12(\frac{1}{3}f + 1) = 3(4 - f) < 0; \\
\text{(21)} & : \quad 2cf + f(1 - b) + 2(e(b - 1) + c(1 - d)) = \\
& \quad 12f - 4f + 12(1 - d) \leq 6f + 2f + 12 - 4f = 6f + 2(6 - f) < 6f.
\end{align*}

Let $(b, c) = (2, 4)$. We show that $f \leq 12$. So assume $f \geq 13$. From \ref{29} and \ref{30} we obtain $f \leq 2d$ and $3f \geq 4(d - 1)$. If $4d < 3f$ holds true, then the point $(2, 1, 2)$ is in the interior of $S$: clearly, \ref{18} and \ref{19} are strict;

\begin{align*}
\text{(20)} & : \quad -2cf + bf + 2(cf - be) = -8f + 2f + 8d = 2(4d - 3f) < 0; \\
\text{(21)} & : \quad 2cf + f(1 - b) + 2(e(b - 1) + c(1 - d)) = \\
& \quad 8f - f + 8(1 - d) \leq 4f + 3f + 8 - 4f = 4f + 8 - f < 4f.
\end{align*}

Thus, let $4d \geq 3f$. The point $(3, 1, 3)$ is now in the interior of $S$: clearly, \ref{18} and \ref{19} are strict;

\begin{align*}
\text{(20)} & : \quad -3cf + bf + 3(cf - be) = -12f + 2f + 12d \leq -10f + 12(\frac{3}{4}f + 1) = 12 - f < 0; \\
\text{(21)} & : \quad 3cf + f(1 - b) + 3(e(b - 1) + c(1 - d)) = \\
& \quad 12f - f + 12(1 - d) \leq 4f + 7f + 12 - 9f = 4f + 2(6 - f) < 4f.
\end{align*}

Let $(b, c) = (3, 5)$. We show that $f \leq 10$. So assume $f \geq 11$. From \ref{29} and \ref{30} we obtain $2f \leq 5d$ and $3f \geq 5(d - 1)$. The point $(2, 1, 2)$ is now in the interior of $S$: clearly, \ref{18} and \ref{19} are strict;

\begin{align*}
\text{(20)} & : \quad -2cf + bf + 2(cf - be) = -10f + 3f + 10d \leq -7f + 10(\frac{3}{5}f + 1) = 10 - f < 0; \\
\text{(21)} & : \quad 2cf + f(1 - b) + 2(e(b - 1) + c(1 - d)) = \\
& \quad 10f - 2f + 10(1 - d) \leq 5f + 3f + 10 - 4f = 5f + 10 - f < 5f.
\end{align*}
Let \((b, c) = (4, 6)\). We show that \(f \leq 12\). So assume \(f \geq 13\). From (29) and (30) we obtain \(f \leq 3d + 1\) and \(f \geq 2(d - 1)\). The point \((2, 1, 2)\) is now in the interior of \(S\): clearly, (18) and (19) are strict;

\[
\begin{align*}
(20) & : \quad -2cf + bf + 2(cf - be) = -12f + 4f + 12d \leq -8f + 12 + \left(\frac{1}{2} f + 1\right) = 2(6 - f) < 0; \\
(21) & : \quad 2cf + f(1 - b) + 2(e(b - 1) + c(1 - d)) = 12f - 3f + 12(1 - d) \leq 6f + 3f + 12 - 4f = 6f + 12 - f < 6f.
\end{align*}
\]

Let \((b, c) = (2, 5)\). We show that \(f \leq 5\). So assume \(f \geq 6\). From (20) and (21) we obtain \(3f \leq 5d + 1\) and \(4f \geq 5(d - 1)\). If \(5d < 4f\) holds true, then the point \((2, 1, 2)\) is in the interior of \(S\): clearly, (18) and (19) are strict;

\[
\begin{align*}
(20) & : \quad -2cf + bf + 2(cf - be) = -10f + 2f + 10d = 2(5d - 4f) < 0; \\
(21) & : \quad 10f - f + 10(1 - d) \leq 5f + 4f + 10 - 6f = 5f + 2(5 - f) < 5f.
\end{align*}
\]

Thus, let \(5d \geq 4f\). The point \((2, 2, 1)\) is now in the interior of \(S\): clearly, (18) and (19) are strict;

\[
\begin{align*}
(20) & : \quad -2cf + bf + 2(cf - be) = -10f + 4f + 5d \leq -6f + 5\left(\frac{4}{3} f + 1\right) = 5 - 2f < 0; \\
(21) & : \quad 10f - 2f + 5(1 - d) \leq 5f + 3f + 5 - 4f = 5f + 5 - f < 5f.
\end{align*}
\]

Let \((b, c) = (2, 6)\). We show that \((2, 1, 2)\) or \((2, 3, 1)\) is in the interior of \(S\). Note that \(f \geq c = 6\), by assumption. From (29) and (30) we obtain \(2f \leq 3d + 5f \geq 6(d - 1)\). If \(6d < 5f\) holds true, then the point \((2, 1, 2)\) is in the interior of \(S\): clearly, (18) and (19) are strict;

\[
\begin{align*}
(20) & : \quad -2cf + bf + 2(cf - be) = -12f + 2f + 12d = 2(6d - 5f) < 0; \\
(21) & : \quad 12f - f + 12(1 - d) \leq 6f + 5f + 12 - 8f = 6f + 3(4 - f) < 6f.
\end{align*}
\]

Thus, let \(6d \geq 5f\). The point \((2, 3, 1)\) is now in the interior of \(S\): clearly, (18) and (19) are strict;

\[
\begin{align*}
(20) & : \quad -2cf + bf + 2(cf - be) = -12f + 6f + 6d = 6(d - f) < 0; \\
(21) & : \quad 12f - 3f + 6(1 - d) \leq 6f + 3f + 6 - 5f = 6f + 2(3 - f) < 6f.
\end{align*}
\]

Let \((b, c) = (2, 7)\). We show that \((2, 1, 2)\) or \((2, 3, 1)\) is in the interior of \(S\). Note that \(f \geq c = 7\), by assumption. From (29) and (30) we obtain \(5f \leq 7d + 6f \geq 7(d - 1)\). If \(7d < 6f\) holds true, then the point \((2, 1, 2)\) is in the interior of \(S\): clearly, (18) and (19) are strict;

\[
\begin{align*}
(20) & : \quad -2cf + bf + 2(cf - be) = -14f + 2f + 14d = 2(7d - 6f) < 0; \\
(21) & : \quad 14f - f + 14(1 - d) \leq 7f + 6f + 14 - 10f = 7f + 14 - 4f < 7f.
\end{align*}
\]

Thus, let \(7d \geq 6f\). The point \((2, 3, 1)\) is now in the interior of \(S\): clearly, (18) and (19) are strict;

\[
\begin{align*}
(20) & : \quad -2cf + bf + 2(cf - be) = -14f + 6f + 7d = 7(d - f) - f < 0; \\
(21) & : \quad 14f - 3f + 7(1 - d) \leq 7f + 4f + 7 - 6f = 7f + 7 - 2f < 7f.
\end{align*}
\]

Let \((b, c) = (3, 6)\). We show that \(f \leq 12\). So assume \(f \geq 13\). From (29) and (30) we obtain \(f \leq 2d + 1\) and \(2f \geq 3(d - 1)\). The point \((2, 1, 2)\) is now in the interior of \(S\): clearly, (18) and (19) are strict;

\[
\begin{align*}
(20) & : \quad -2cf + bf + 2(cf - be) = -12f + 3f + 12d \leq -9f + 12\left(\frac{2}{3} f + 1\right) = 12 - f < 0; \\
(21) & : \quad 12f - 2f + 12(1 - d) \leq 6f + 4f + 12 - 6f = 6f + 2(6 - f) < 6f.
\end{align*}
\]
\[ \begin{align*}
\text{(20)} & : -2cf + bf + 2(cd - be) = -14f + 3f + 14d \leq -11f + 14\left(\frac{5}{7}f + 1\right) = 14 - f < 0; \\
\text{(21)} & : 2cf + f(1 - b) + 2(e(b - 1) + c(1 - d)) = 14f - 2f + 14(1 - d) \leq 7f + 5f + 14 - 8f = 7f + 14 - 3f < 7f.
\end{align*} \]

- Let \( (b, c) = (3, 7) \). We show that \( f \leq 14 \). So assume \( f \geq 15 \). From (20) and (30) we obtain \( 4f \leq 7d \) and \( 5f \geq 7(d - 1) \). The point \( (2, 1, 2) \) is now in the interior of \( S \): clearly, (18) and (19) are strict.

\[ \begin{align*}
\text{(20)} & : -2cf + bf + 2(cd - be) = -16f + 3f + 16d \leq -13f + 16\left(\frac{3}{4}f + 1\right) = 16 - f < 0; \\
\text{(21)} & : 2cf + f(1 - b) + 2(e(b - 1) + c(1 - d)) = 16f - 2f + 16(1 - d) \leq 8f + 6f + 16 - 10f = 8f + 4(4 - f) < 8f.
\end{align*} \]

2) \( e > 0 \)

i) \( c \leq e \) We first show that in this case we must have \( c = e \). For the purpose of deriving a contradiction assume that \( c < e \). Then, the point \( (2, 2, 1) \) is in the interior of \( S \): clearly, (18) and (19) are strict.

\[ \begin{align*}
\text{(20)} & : -2cf + bf + cd - be \leq 2f(b - c) + f(c - b) + f + c - e = f(b + 1 - c) + c - e < 0; \\
\text{(21)} & : 2cf + f(1 - b) + e(b - 1) + c(1 - d) \leq 2f(c + 1 - b) + c - e + f(b - c) = cf + f(2 - b) + c - e < cf.
\end{align*} \]

The inequalities in the first row follow from (20) and the fact that \( b < c < e \), whereas the inequality in the second row follows from (22) and the inequality in the last row follows from \( 2 \leq b \) and \( c < e \). Therefore, we have \( c = e \).

Consider the sequence of points \( (2, \Theta, 1) \), where \( \Theta \geq 1 \). By equation (23) and the fact that \( b \geq 2 \), we obtain that there exists some \( \Theta \geq 2 \) such that

\[ \begin{align*}
2cf + f(1 - b)(\Theta - 1) + e(b - 1) + c(1 - d) & \geq cf, \\
2cf + f(1 - b)\Theta + e(b - 1) + c(1 - d) & < cf.
\end{align*} \]

(31) (32)

Since the point \( (2, \Theta, 1) \) satisfies (18), (19), and (21) strictly, we must have that

\[ -2cf + bf\Theta + cd - be \geq 0 \]

(33)

since otherwise the point \( (2, \Theta, 1) \) is in the interior of \( S \). Adding (31) and (33) yields \( 0 \leq f(\Theta + b - 1 - c) + c - e = f(\Theta + b - 1 - c) \) which implies

\[ \Theta \geq c - b + 1. \]

(34)

Using (22) and (31), we obtain \( f(c - b) \leq cd - be \leq (\Theta - 1)f(1 - b) + cf + c - e = (\Theta - 1)f(1 - b) + cf \) and hence \( f(c - b) \leq (\Theta - 1)f(1 - b) + cf \iff \Theta \leq 1 + \frac{b}{c - 1} \). We infer

\[ \begin{align*}
\Theta & \leq \begin{cases} 3, & \text{if } b = 2 \\ 2, & \text{if } b \geq 3 \end{cases} \\
c & \leq \Theta - b - 1 \leq \begin{cases} 3, & \text{if } b = 2 \\ b + 1, & \text{if } b \geq 3. \end{cases}
\end{align*} \]

(35)

This shows that for \( b = 2 \) only the two cases \( (b, c) = (2, 3) \) and \( (b, c) = (2, 4) \) need to be considered. Since \( b < c \) the case \( b \geq 3 \) leads to \( b + 1 \leq c \leq b + 1 \) and thus only the case \( (b, c) = (c - 1, c) \) is left.

- Let \( (b, c) = (2, 3) \). We show that \( f \leq 9 \). So assume \( f \geq 10 \). From (22) and (23) we obtain \( f \leq 3(3d - 2) \) and \( 2f \geq 3(d - 2) \). If \(-4f + 9(d - 2) < 0 \) holds true, then the point \( (2, 1, 3) \) is in the interior of \( S \): clearly, (18) is strict; (19) is strict since \( c = 3 \) and \( f \geq 10 \);

\[ \begin{align*}
\text{(20)} & : -2cf + bf + 3(cd - be) = -6f + 2f + 3(3d - 6) = -4f + 9(d - 2) < 0; \\
\text{(21)} & : 2cf + f(1 - b) + 3(e(b - 1) + c(1 - d)) = 6f - f + 3(6 - 3d) = 3f + 2f + 18 - 9d \leq 3f + 2f + 18 - 9\left(\frac{1}{3}f + 2\right) = 3f - f < 3f.
\end{align*} \]
Thus, let \(-4f + 9(d - 2) \geq 0\). If \(-4f + 6(d - 2) < 0\) holds true, then the point \((2, 1, 2)\) is in the interior of \(S\): clearly, (18) and (19) are strict;

\[
\begin{align*}
(20) & : -2cf + bf + 2(cd - be) = -6f + 2f + 2(3d - 6) = -4f + 6(d - 2) < 0; \\
(21) & : 2cf + f(1 - b) + 2(e(b - 1) + c(1 - d)) = 6f - f + 2(6 - 3d) = 3f + 2f + 12 - 6d \leq 3f + 2f + 12 - 6\left(\frac{4}{9}f + 2\right) = 3f - \frac{2}{3}f < 3f.
\end{align*}
\]

Thus, let \(-4f + 6(d - 2) \geq 0 \Leftrightarrow 3(d - 2) \geq 2f\). Since also \(3(d - 2) \leq 2f\) holds true we obtain \(3(d - 2) = 2f\). The point \((3, 1, 3)\) is now in the interior of \(S\): clearly, (18) and (19) are strict;

\[
\begin{align*}
(20) & : -3cf + bf + 3(cd - be) = -9f + 2f + 3(3d - 6) = -7f + 6f = -f < 0; \\
(21) & : 3cf + f(1 - b) + 3(e(b - 1) + c(1 - d)) = 9f - f + 3(6 - 3d) = 2f < 3f.
\end{align*}
\]

\(\bullet\) Let \((b, c) = (2, 4)\). We show that \(f \leq 8\). So assume \(f \geq 9\). From (22) and (23) we obtain \(f + 4 \leq 2d\) and \(3f + 8 \geq 4d\). If \(4d < 3f + 8\) holds true, then the point \((2, 1, 2)\) is in the interior of \(S\): clearly, (18) is strict; (19) is strict since \(e = c = 4\) and \(f \geq 9\);

\[
\begin{align*}
(20) & : -2cf + bf + 2(cd - be) = -8f + 2f + 2(4d - 8) = 2(4d - 3f - 8) < 0; \\
(21) & : 2cf + f(1 - b) + 2(e(b - 1) + c(1 - d)) = 8f - f + 2(4f + 4d - 8) = 3f < 4f.
\end{align*}
\]

Thus, using (22) we have \(4d \geq 3f + 8 \geq 4d\) which implies \(4d = 3f + 8\). The point \((2, 2, 1)\) is now in the interior of \(S\): clearly, (18) and (19) are strict;

\[
\begin{align*}
(20) & : -2cf + 2bf + cd - be = -8f + 4f + 4d - 8 = -f < 0; \\
(21) & : 2cf + 2f(1 - b) + e(b - 1) + c(1 - d) = 8f - 2f + 4f + 4d - 8 = 3f < 4f.
\end{align*}
\]

\(\bullet\) Let \((b, c) = (c - 1, c)\) with \(b \geq 3\). Since \(c - e \leq 0 < f\) in this case, it follows from (28) that \(b \leq 5\) and therefore \(4 \leq c = b + 1 \leq 6\). We show that \(f \leq 12\). So assume \(f \geq 13\). From (22) and (23) we obtain \(f \leq c(d + 1 - c)\) and \(2f \geq c(d + 1 - c)\). If \(f < c(d + 1 - c)\) holds true, then the point \((2, 1, 2)\) is in the interior of \(S\): clearly, (18) is strict; (19) is strict since \(c \leq 6\) and \(f \geq 13\);

\[
\begin{align*}
(20) & : -2cf + bf + 2(cd - be) = -2cf + (c - 1)f + 2c(d + 1 - c) \leq -f(c + 1) + 4f = f(3c - c) < 0; \\
(21) & : 2cf + f(1 - b) + 2(e(b - 1) + c(1 - d)) = 2cf + (2 - c)f + 2c(c - d - 1) < (2 + c)f - 2f = cf.
\end{align*}
\]

Thus, using (22) we have \(f \geq c(d + 1 - c) \geq f\) which implies \(f = c(d + 1 - c)\). The point \((2, 2, 1)\) is now in the interior of \(S\): clearly, (18) and (19) are strict;

\[
\begin{align*}
(20) & : -2cf + 2bf + cd - be = -2cf + 2(c - 1)f + c(d + 1 - c) = -f < 0; \\
(21) & : 2cf + 2f(1 - b) + e(b - 1) + c(1 - d) = 2cf + 2(2 - c)f + c(c - d - 1) = 3f < cf.
\end{align*}
\]

**ii) c > e** We show that by using a unimodular transformation this case can be reduced to a case which has already been analyzed. Assume that the vertices of \(S\) are given by the columns of the matrix

\[
\begin{pmatrix}
0 & 1 & b & d' \\
0 & 0 & c & e' \\
0 & 0 & 0 & f'
\end{pmatrix},
\]

where - besides the usual conditions on the unknowns \(b, c, d, e, f\) - in addition \(c\) is chosen such that it is minimal with respect to all such representations. In the following we show that (36) is unimodularly transformable to a simplex where the parameter \(a\) (here: \(a = 1\)) is greater than or equal to 2.

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Let $r := \gcd(e, f)$ and $g := \gcd(r, d)$ where we assume $r, g > 0$. Then, there exist $p, q, s, t \in \mathbb{Z}$ such that $r = pe + qf$ and $g = sr + td$. Apply the following unimodular transformation to (36):

$$
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix}
\mapsto
\begin{pmatrix}
    r & -sp & -sq \\
    0 & r & -tg \\
    0 & 0 & r
\end{pmatrix}
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix}
+ 
\begin{pmatrix}
    g \\
    0 \\
    0
\end{pmatrix}.
$$

The columns of the matrix listed below represent the vertices of the resulting simplex:

$$
\begin{pmatrix}
    0 & g & g-t & g-tb-psc \\
    0 & 0 & \frac{r}{g} & \frac{rb-psd}{g} \\
    0 & 0 & 0 & \frac{c}{g}
\end{pmatrix}.
$$

If $g = 1$, then (37) can be transformed by elementary row operations into Hermit normal form. However, this does not change the diagonal elements and thus leads to a representation (36) where $\frac{r}{g} = r = \gcd(e, f) \leq e < c$. This contradicts the minimal choice of $c$. Hence, let $g \geq 2$. Using elementary row operations (37) can be brought into Hermit normal form with $a = g \geq 2$. Such simplices were analyzed in Section 4.

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References

[1] K. Andersen, Q. Louveaux, and R. Weismantel. An analysis of mixed integer linear sets based on lattice point free convex sets. *Submitted to Mathematics of Operations Research*, 2008.

[2] K. Andersen, Q. Louveaux, R. Weismantel, and L. Wolsey. Inequalities from two rows of a simplex tableau. *IPCO conference 2007, Lecture Notes in Computer Science 4513, Springer*, pages 1–15, 2007.

[3] E. Balas. Intersection cuts - a new type of cutting planes for integer programming. *Operations Research*, 19:19–39, 1971.

[4] G. Cornuéjols and F. Margot. On the facets of mixed integer programs with two integer variables and two constraints. *LATIN conference 2008, Lecture Notes in Computer Science 4957, Springer*, pages 317–328, 2008.

[5] S.S. Dey and L. Wolsey. Lifting integer variables in minimal inequalities corresponding to lattice-free triangles. *IPCO conference 2008, Lecture Notes in Computer Science 5035, Springer*, pages 463–475, 2008.

[6] J.C. Lagarias and G.M. Ziegler. Bounds for lattice polytopes containing a fixed number of interior points in a sublattice. *Canadian Journal of Mathematics*, 43:1022–1035, 1991.

[7] L. Lovász. Geometry of numbers and integer programming. *Mathematical Programming, Recent Developments and Applications*, pages 177–201, 1989.

[8] B. Reznick. Lattice point simplices. *Discrete Mathematics*, 60:219–242, 1986.

[9] H.E. Scarf. Integral polyhedra in three space. *Mathematics of Operations Research*, 10:403–438, 1985.

[10] A. Schrijver. Theory of linear and integer programming. *Wiley, Chichester*, 1986.

[11] A. Sebő. An introduction to empty lattice simplices. *IPCO conference 1999, Lecture Notes in Computer Science 1610, Springer*, pages 400–414, 1999.

[12] J. Treutlein. 3-dimensional lattice polytopes without interior lattice points. *arXiv:0809.1787*, 2008.

[13] G. Zambelli. On degenerate multi-row Gomory cuts. *Operations Research Letters*, 37:21–22, 2009.