The supersymmetric index in four dimensions

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Abstract

We review the calculation and properties of the supersymmetric index for four-dimensional $\mathcal{N} = 1$ theories, illustrating its physical significance in several examples.

Keywords: supersymmetric index, four dimensional, sigma models

(Some figures may appear in colour only in the online journal)

1. Introduction

The technique of supersymmetric localization allows one to compute the partition functions of several supersymmetric field theories on certain compact manifolds preserving some of the supersymmetry. In favorable cases, the procedure of localization reduces the computation of the infinite dimensional path integral to a finite dimensional integral or to a discrete sum. Many of the computable supersymmetric partition functions in dimension $d \leq 4$ are related to one another, see figure 1. The relations take two forms. First, different partition functions might be related by taking various limits of their parameters. For example, a partition function on a compact manifold can depend on the relative size of different components—sending that size to zero corresponds to computing a partition function of a theory in a lower dimension. Such limits are represented by solid lines in the figure. Second, partition functions on compact manifolds can sometimes be computed by gluing together partition functions on non-compact manifolds with prescribed boundary conditions at infinity. Different patterns of gluing of the same non-compact partition functions can lead to two different compact partition functions. For example, both the $S^2 \times S^1$ partition function (the three-dimensional supersymmetric index) and the $S^3$ partition function are obtained by gluing partition functions on $\mathbb{C} \times S^1$. 

* This is a contribution to the review issue ‘Localization techniques in quantum field theories’ (ed Pestun and Zabzine) which contains 17 chapters available at [1].
Such relations are denoted by dashed lines in the figure. Some of the relations indicated in the
figure are well studied while for others only partial understanding is available.

The main focus of this review article will be the $S^3 \times S^1$ partition function, also known
as the four-dimensional supersymmetric index, because it can be understood as the Witten
index of the theory quantized on $S^3 \times R$, refined by fugacities that keep track of the relevant
quantum numbers. This is the simplest and arguably the most important observable in the
network of partition functions shown in figure 1. For theories that admit a Lagrangian descrip-
tion, the four-dimensional index can be obtained by solving a simple counting problem: one
e numerates (with signs) local gauge invariant operators built from elementary fields in the
four-dimensional theory, in the limit of vanishing coupling. By contrast, the supersymmetric
index in other dimensions gains contributions from more complicated objects, such as instan-
tons in five dimensions, monopoles in three dimensions, and local supersymmetric defects in
two dimensions. The four-dimensional counting problem is efficiently encoded by a simple
matrix integral, which could be equivalently obtained by applying the recipe of supersymmet-
ric localization to the $S^3 \times S^1$ partition function. While the four-dimensional index is compu-
tationally simpler than other partition functions, its properties and the technology needed to
extract physical information from it are more universal applicable.

This review is organized as follows. In section 2 we discuss the definition of the super-
symmetric index and the prescription to compute it in any Lagrangian theory. In section 3
properties of the index of theories built from chiral fields with superpotential interactions are
reviewed. In section 4 we discuss basic properties of the index of gauge theories. In section 5
we review superconformal representation theory and the way different multiplets are encoded
in the index. In particular we review how to extract easily the spectrum of relevant and exactly
marginal deformations. In section 6 we discuss briefly some of the mathematical properties
of indices. In particular we review symmetries of the index and identities between indices of
different looking theories related by dualities. In section 7 we review different physically

\[ \text{Figure 1. Different supersymmetric partition functions in dimensions } 4, 3, 2 \text{ are}
\text{related by limits of parameters (solid lines) and block decompositions (dashed lines).}
\text{The } S^3 \times S^1 \text{ partition function (also known as the four-dimensional index) is one of}
\text{the simplest and most useful partition functions. We denoted a sphere with a flux for}
\text{R-symmetry through it by } \tilde{S}^2. \]

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3 This picture could be extended to a larger network of relations starting from higher dimensional theories—the $S^4$
partition function [2] (see contribution [3]), notably absent in figure 1, which would be part of such an extended
picture.

4 This is not to be confused with the Witten index in flat space defined as partition function on $T^4$. 

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important limits of the index. Finally, in section 8 we mention several topics not covered in detail in this review.

2. Definition of the index

There are two equivalent ways to define the supersymmetric index. It can be defined as the supersymmetric partition function on $S^3 \times S^1$, which depends holomorphically on two complex structure moduli (conventionally denoted $p$ and $q$) and on holonomies for background gauge fields coupling to the flavor symmetries of the theory. Alternatively, it is given by an appropriately weighted trace over the states of the theory quantized on $S^3 \times \mathbb{R}$. If the theory is conformal, one can use the state/operator map to interpret these states as local operators. Only in such cases it is appropriate to refer to the index as the superconformal index. There are many important examples of $\mathcal{N} = 1$ superconformal field theories that can be reached as infrared fixed points of renormalization group (RG) flows starting from weakly-coupled Lagrangian theories. One of the most useful properties of the index (most easily argued using its definition as a partition function) is its invariance under RG flow. This provides a powerful way to obtain the index of an IR fixed point, by performing a simple calculation in the UV.

2.1. Index as a trace

The index of a 4d super conformal field theory is defined as the Witten index of the theory in radial quantization. Let $\mathcal{Q}$ be one of the Poincaré supercharges, and $\mathcal{Q}^i = \mathcal{S}$ the conjugate conformal supercharge. Schematically, the index is defined as

$$I(\mu_i) = \text{Tr} \left( (-1)^F e^{-\beta \delta} e^{-\mu_i M_i} \right),$$

where the trace is over the Hilbert space of the theory quantized on $S^3$, $\delta \equiv \frac{1}{2} \{ \mathcal{Q}, \mathcal{Q}^i \}$, $M_i$ are $\mathcal{Q}$-closed conserved charges and $\mu_i$ the associated chemical potentials. Since states with $\delta > 0$ come in boson/fermion pairs, only the $\delta = 0$ states contribute, and the index is independent of $\beta$. There are infinitely many states with $\delta = 0$—this is true even for a single short irreducible representation of the superconformal algebra, because some of the non-compact generators (some of the spacetime derivatives) have $\delta = 0$. The introduction of the chemical potentials serves both to regulate this divergence and to achieve a more refined counting.

For $\mathcal{N} = 1$, the supercharges are $\{ \mathcal{Q}_\alpha, \mathcal{S}^\alpha \equiv \mathcal{Q}^{\dagger, \alpha}, \tilde{\mathcal{Q}}_\dot{\alpha}, \tilde{\mathcal{S}}^\dot{\alpha} \equiv \tilde{\mathcal{Q}}^{\dagger, \dot{\alpha}} \}$, where $\alpha = \pm$ and $\dot{\alpha} = \pm$ are respectively $SU(2)_1$ and $SU(2)_2$ indices, with $SU(2)_1 \times SU(2)_2 = \text{Spin}(4)$ the isometry group of the $S^3$. The relevant anticommutators are

$$\{ \mathcal{Q}_\alpha, \mathcal{Q}^{\dagger, \beta} \} = E + 2M^\alpha_\beta + \frac{3}{2} r,$$

$$\{ \tilde{\mathcal{Q}}_{\dot{\alpha}}, \tilde{\mathcal{Q}}^{\dagger, \dot{\beta}} \} = E + 2\tilde{M}^\dot{\alpha}_{\dot{\beta}} - \frac{3}{2} r,$$

where $E$ is the conformal Hamiltonian, $M^\alpha_\beta$ and $\tilde{M}^\dot{\alpha}_{\dot{\beta}}$ the $SU(2)_1$ and $SU(2)_2$ generators, and $r$ the generator of the $U(1)_r$ R-symmetry. In our conventions, the $\mathcal{Q}$s have $r = -1$ and $\tilde{\mathcal{Q}}$s have $r = +1$, and of course the dagger operation flips the sign of $r$.

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5 We follow here very closely the discussion of [7].
One can define two inequivalent indices, a ‘left-handed’ index $I^L(t, y)$ and a ‘right-handed’ index $I^R(t, y)$. For the left-handed index, we pick say $Q \equiv Q_-$,

$$I^L(p, q) \equiv \text{Tr} \left( (-1)^F p^{3(E+1)} q^{2(E+1) - j_2} \right) = \text{Tr} \left( (-1)^F p^{j_1 + j_2 - \frac{1}{2}} q^{j_1 - j_2 - \frac{1}{2}} \right), \quad \delta = E - 2j_1 + \frac{3}{2}r,$$

(2.4)

where $j_1$ and $j_2$ are the Cartan generators of $SU(2)_1$ and $SU(2)_2$. The two ways of writing the exponent of $p$ and $q$ are equivalent since they differ by a $Q$-exact term. For the right-handed index, we pick say $Q \equiv Q_+$,

$$I^R(p, q) \equiv \text{Tr} \left( (-1)^F p^{3(E+1)} q^{2(E+1) - j_1} \right) = \text{Tr} \left( (-1)^F p^{j_1 + j_2 + \frac{1}{2}} q^{j_1 - j_2 + \frac{1}{2}} \right), \quad \delta = E - 2j_2 - \frac{3}{2}r.$$

(2.5)

One may also introduce chemical potentials for global symmetries of the theory which commute with the supersymmetry algebra and thus conserve the index property of the trace. Such fugacities can be turned on for continuous and/or discrete symmetries as we will see in what follows.

If the theory is not conformal, and is described instead by an RG flow from a free UV fixed point to an IR fixed point, one can still define the index from (2.1), evaluating the trace over the local operators at the UV fixed point, but making sure that the allowed symmetries are preserved along the flow. (In particular, the R-charge assignments must correspond to a non-anomalous R symmetry). Since the index is an RG invariant, this gives a recipe to evaluate the superconformal index of the IR fixed point. At intermediate scales on the flow, the index is interpreted as the partition function on $S^3 \times S^1$, or equivalently, as the trace over the states of the theory quantized on $S^3$.

2.2. Index as a partition function

Alternatively, the index can be defined as the supersymmetric partition function on $S^3 \times S^1_\tau$. As was argued in [9] (see also [10]) any $\mathcal{N} = 1$ supersymmetric theory can be put in a supersymmetric way on $S^3 \times S^1_\tau$, provided it possesses anomaly-free $U(1)_r$ R symmetry. We refer to contribution [11] for a detailed treatment and mention here only some of the salient points.

The $S^3 \times S^1_\tau$ partition function depends holomorphically on the complex structure moduli $p$ and $q$, and on the holonomies associated to flavor symmetries. It does not depend on gauge and superpotential couplings, and is invariant under RG flow (contribution [11]). The partition function can be evaluated by localization techniques [12, 13], and the result is the same matrix integral that we will obtain in the next section by enumeration of gauge invariant operators.

The precise equivalence of the trace formula for the index and the computation of the partition function requires a bit of care. When computing the index using the trace formula we implicitly normalize it so the vacuum (assuming it is unique) contributes $+1$ to the index. In particular in the large radius limit, $\tau \to \infty$ the index computed as a counting problem receives only contributions from the vacua, and assuming there is a unique vacuum (preserving certain global symmetries) the index in the limit is 1. However, while computing the partition function in the large radius limit one finds a contribution coming from the Casimir energy of the theory.

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6 Picking $Q \equiv Q_+$ would amount to the replacement $j_1 \leftrightarrow -j_2$, which is an equivalent choice because of $SU(2)_1$ symmetry. The same consideration applies to the right-handed index, which can be defined either choosing $Q_- \equiv Q_-$ or $Q_+$.  
7 One can consider additional generalizations of the index such as introduction of charge conjugation [8] to the trace but we will refrain from doing so here.  
8 But of course, the presence of a superpotential restricts the possible R charge assignments.
Table 1. The ‘letters’ of an $\mathcal{N} = 1$ chiral multiplet and their contributions to the index. Here $\delta^e = E - 2j_1 + \frac{1}{2}r_{UV}$ and $\delta^o_{UV} = E - 2j_2 - \frac{1}{2}r_{UV}$. A priori we have to take into account the free equations of motion $\partial^\phi = 0$ and $\Box \phi = 0$, which imply constraints on the possible words, but we see that in this case equations of motions have $\delta^e_{UV} \neq 0$ so they do not change the index. Finally there are two spacetime derivatives contributing to the index, and their multiple action on the fields is responsible for the denominator of the index, $(1 - p(1 - q)) = \sum_{n,m=0}^\infty p^nq^m$.

| Letters | $E_{UV}$ | $j_1$ | $j_2$ | $r_{UV}$ | $r_{BR}$ | $\delta^e_{UV}$ | $\delta^o_{UV}$ | $\mathcal{T}^L$ | $\mathcal{T}^R$ |
|---------|---------|-------|-------|-------|-------|---------------|---------------|---------------|---------------|
| $\phi$  | 1       | 0     | 0     | $\frac{2}{3}$ | $r$   | 2             | 0             | $(pq)^{\frac{1}{2}}$ |                |
| $\psi$  | $\frac{1}{2}$ | $\pm\frac{1}{2}$ | 0 | $-\frac{1}{3}$ | $r - 1$ | 0 &smash{$^+\frac{2}{3}$} | $(pq)^{\frac{1}{2}}$ | 2             |                |
| $\partial^\phi$ | $\frac{3}{2}$ | 0     | 0     | $\frac{2}{3}$ | $r$   | 4             | 2             | $4^+, 2^-$    |                |
| $\Box \phi$ | 3       | 0     | 0     | $\frac{2}{3}$ | $r$   | 4             | 2             | $(pq)^{\frac{1}{2}}$ |                |
| $\partial_\pm$ | 1       | $\pm\frac{1}{2}$ | $\pm\frac{1}{2}$ | 0 | 0 | $0^{\pm\pm}, 2^{\pm-}$ | $p, q$ | $0^{\pm\pm}, 2^{\pm-}$ | $p, q$          |

\[
\lim_{r \to \infty} Z_{\mathcal{G}^1 \times \mathcal{G}_L} \sim e^{-\tau E_{\text{Casimir}}}. \tag{2.6}
\]

The trace formulation of the index and the partition function formulation thus differ by the multiplicative factor $e^{-\tau E_{\text{Casimir}}}$. The Casimir energy can be computed from the trace formulation of the index [14–18],

\[
E_{\text{Casimir}} = \frac{2}{3}(a - c)(\omega_1 + \omega_2) + \frac{2}{27}(3c - 2a)(\frac{\omega_1 + \omega_2}{\omega_1 \omega_2})^3, \tag{2.7}
\]

where $a$ and $c$ are the Weyl anomaly coefficients. We also defined $p = e^{-\tau \omega_1}$, $q = e^{-\tau \omega_2}$.

2.3. Computation of the index

By the state/operator correspondence the computation of the index of a conformal gauge theory proceeds by listing all the possible operators we can construct from modes of the fields and projecting out gauge non-invariant ones. The different modes of the fields are usually called ‘letters’ and the operators are words constructed using this alphabet.

The ‘letters’ of an $\mathcal{N} = 1$ chiral multiplet are enumerated in table 1. We assume that in the IR the $U(1)$, charge of the lowest component of the multiplet $\phi$ is some arbitrary $r_{BR} = r$ (determined in a concrete theory by anomaly cancellation and in subtle cases $a$-maximization). According to the prescription we have just reviewed, the index receives contributions from the letters with $\delta^e_{UV} = 0$, and each letter contributes as $(-1)^{p_{h+i} + \frac{1}{2}pa}q^{j-i-\frac{1}{2}ra}$ to the left-handed index and as $(-1)^{p_{h+i} + \frac{1}{2}pa}q^{j-i+\frac{1}{2}ra}$ to the right-handed index. To keep track of the gauge and flavor quantum numbers, we introduce characters. We assume that the chiral multiplet transforms in the representation $R$ of the gauge $\times$ flavor group, and denote by $\chi_R(U, V)$, $\chi_R(U, V)$ the characters of $R$ and and of the conjugate representation $\bar{R}$, with $U$ and $V$ gauge and flavor group matrices respectively. All in all, the single-letter left- and right-handed indices for a chiral multiplet are [19]

\[
\lim_{r \to \infty} Z_{\mathcal{G}^1 \times \mathcal{G}_L} \sim e^{-\tau E_{\text{Casimir}}}. \tag{2.6}
\]
\[ \hat{\delta}^k_{\chi(p)}(p, q, U, V) = \frac{(pq)^{1/r} \chi_R(U, V) - (pq)^{1/r^2} \chi_R(U, V)}{(1 - p)(1 - q)} \]  

(2.8)

\[ \hat{\delta}^k_{\chi(p)}(p, q, U, V) = \frac{(pq)^{1/r} \chi_R(U, V) - (pq)^{1/r^2} \chi_R(U, V)}{(1 - p)(1 - q)} \]  

(2.9)

The denominators encode the action of the two spacetime derivatives with \( \delta = 0 \). Note that the left-handed and right-handed indices differ by conjugation of the gauge and flavor quantum numbers. As a basic consistency check [6], consider a single free massive chiral multiplet (no gauge or flavor indices). In the UV, we neglect the mass deformation and as always numbers. As a basic consistency check [6], consider a single free massive chiral multiplet.

Finding the contribution to the index of an \( \mathcal{N} = 1 \) vector multiplet is even easier, since the \( R \)-charge of a vector superfield \( W_{\alpha} \) is fixed at the canonical value +1 all along the flow. For both left- and the right-handed index, the single-letter index of a vector multiplet is

\[ iv(p, q, U) = \frac{2pq - p - q}{(1 - p)(1 - q)} \chi_{ad}(U). \]  

(2.10)

Armed with the single-letter indices, the full index is obtained by enumerating all the words and then projecting onto gauge-singlets by integrating over the Haar measure of the gauge group. Schematically,

\[ \mathcal{I}(q, p, V) = \int [dU] \prod_k \text{PE}[i_k(p, q, U, V)], \]  

(2.11)

where \( k \) labels the different supermultiplets, and \( \text{PE}[i_k] \) is the plethystic exponential of the single-letter index of the \( k \)th multiplet. The plethystic exponential,

\[ \text{PE}[i_k(q, p, U, V)] \equiv \exp \left\{ \sum_{m=1}^{\infty} \frac{1}{m} i_k(p^m, q^m, V^m) \chi_R(U^m, V^m) \right\}, \]  

(2.12)

implements the combinatorics of symmetrization of the single letters, see e.g. [20, 21]. As usual, one can gauge fix the integral over the gauge group and reduce it to an integral over the maximal torus, with the usual extra factor arising of the van der Monde determinant.

The multi-letter contribution to the index of a chiral multiplet (the plethystic exponential of its single-letter index) can be elegantly written as a product of elliptic Gamma functions [19].

For a chiral superfield in the fundamental representation \( \Box \) of \( SU(N_c) \), and with IR \( R \)-charge equal to \( r \), one has

\[ \text{PE}[i_k(p, q, U)] \equiv \prod_{i=1}^{N_c} \Gamma((pq)^{1/r^2} z; p, q), \]

\[ \Gamma(z; p, q) \equiv \prod_{k, m=0}^{\infty} \frac{1 - p^{k+1}q^{m+1}/z}{1 - p^k q^m z}. \]  

(2.13)

Here \( \{z_k\}, k = \{1, \ldots N_c\} \) are complex numbers of unit modulus, obeying \( \prod_{k=1}^{N_c} z_k = 1 \), which parametrize the Cartan subalgebra of \( SU(N_c) \).

Similarly, the multi-letter contribution of a vector multiplet in the adjoint of \( SU(N) \) combines with the \( SU(N) \) Haar measure to give the compact expression [19, 22].
\[
\frac{\kappa^{N-1}}{N!} \int_{\mathbb{C}^{N-1}} \prod_{i=1}^{N-1} \frac{dz_i}{2\pi i z_i} \prod_{k \neq \ell} \frac{1}{\Gamma(z_i/z_\ell; p, q)} \cdots \tag{2.14}
\]

The dots indicate that this is to be understood as a building block of the full matrix integral. Here \(\kappa\) is taken to be

\[
\kappa \equiv (p; p)(q; q) \tag{2.15}
\]

where \((a; b) \equiv \prod_{k=0}^{\infty} (1 - ab^k).\) Note that \(\kappa\) is the index of \(U(1)\) free vector multiplet and we will sometimes denote \(\kappa = \mathcal{I}_V.\) We will often leave implicit the \(q\) and \(p\) dependence of the elliptic gamma functions, \(\Gamma(z; p, q) \to \Gamma(z).\) Also, we will often use the shorthand notation

\[
\Gamma(A z^\pm) \equiv \Gamma(z) \Gamma(A z^{-1}). \tag{2.16}
\]

If the gauge group of the theory has abelian factors, one can turn on the Fayet–Iliopoulos (FI) terms. On \(S^3 \times S^1\) such FI terms should be quantized [23]. Indeed, on \(S^3 \times \mathbb{R}\) with sphere of radius \(r_3\) the FI parameter \(\zeta\) appears in the action as,

\[
\zeta \int d^4x \sqrt{g} (D - \frac{2i}{r_3} A_4), \tag{2.17}
\]

where \(A_4\) is the component of the gauge field along \(\mathbb{R}\) and \(D\) is the auxiliary field of the \(\mathcal{N} = 1\) vector multiplet. Upon compactification of \(\mathbb{R}\) to \(S^1\) we have to insure that this term is invariant under large gauge transformations, \(A_4 \to A_4 + \frac{1}{r_3}.\) Under such a transformation,

\[
\zeta \int d^4x \sqrt{g} (D - \frac{2i}{r_3} A_4) \to \zeta \int d^4x \sqrt{g} (D - \frac{2i}{r_3} A_4) + 8\pi^3 i \zeta r_3^2, \tag{2.18}
\]

which implies that \(\zeta = \frac{1}{4\pi^2 r_3^2} n\) with \(n \in \mathbb{Z}.\) The FI parameter for the \(U(1)_u\) gauge factor will introduce the term \(u^2\) in the matrix integral that computes the index.

The index does not depend on any continuous coupling of the theory. However, the functional form of the superpotential restricts the possible global symmetries and hence the fugacities that the index can depend on. In turning on a certain set of fugacities, we are computing the index for all possible choices of superpotentials consistent with the symmetries associated to those fugacities.

### 3. Index of sigma models

We now turn to discuss basic properties of the index of some of the simplest \(\mathcal{N} = 1\) theories: sigma models built from chiral fields with no gauge interactions.

#### 3.1. Mass terms

Invariance along the RG flow is a basic property of the index. A simple implication is that the index for a massive theory with a single supersymmetric vacuum must be equal to 1. Let us check this fact in the theory of two chiral fields with a superpotential mass term

\[
W = m Q_a Q_b. \tag{3.1}
\]

As the superpotential has R-charge two, the R-charges of the two fields satisfy

\[
r_a + r_b = 2. \tag{3.2}
\]
Moreover there is one $U(1)$ symmetry under which the two fields are oppositely charged. Let us turn on a fugacity $u$ for this symmetry and assign charge $+1$ to field $a$. From our general rules, the index of this theory is

$$
\Gamma((pq)^{\frac{1}{2}} u)^{(pq)^{\frac{1}{2}} (2-\epsilon) u^{-1}} = \prod_{i,j=0}^{\infty} \frac{1 - (pq)^{1 - \frac{1}{2} p^i q^j u^{-1}}}{1 - (pq)^{\frac{1}{2} (2-\epsilon) u^{-1}}} = 1,
$$

(3.3)
as expected.

### 3.2. F-term supersymmetry breaking

As another degenerate example, consider the theory of a chiral field with linear superpotential, $W = \eta Q$, the Polonyi model. This model has no supersymmetric vacuum and thus breaks supersymmetry spontaneously. The field $Q$ has R-charge 2 and is not charged under any global symmetry. The index is

$$
\Gamma(pq) = 0,
$$

(3.4)
consistent with the absence of a supersymmetric vacuum. The vanishing the index can be traced to the presence of a fermionic letter that contributes $-1$ (see table 1): this mode should be interpreted as the Goldstino of supersymmetry breaking. In general, models with spontaneous supersymmetry breaking of the $O'Raifeartaigh$ type will involve fields with R-charge two neutral under all global symmetries—resulting in a vanishing index.

### 3.3. Runaway vacuum

We can consider a slight modification of the above model to restore the supersymmetric vacuum but at infinity in field space. We take

$$
W = \eta Q + \frac{1}{2} \lambda Q^2 S.
$$

(3.5)
The potential of this model has a minimum at zero as $S$ goes to infinity—a runaway behavior. Indeed, the F-term equations read

$$
\eta + \lambda Q S = 0, \quad Q^2 = 0.
$$

(3.6)
The vacuum is reached by taking the limit

$$
Q \to 0, \quad S \to \infty, \quad QS = -\frac{\eta}{\lambda}.
$$

(3.7)
The field $Q$ has R-charge $+2$ and contributes zero to the index (because of the fermionic zero mode mentioned above), while $S$ has R-charge $-2$ and contributes infinity, making the index of this model ill-defined. The divergence in the index of the $S$ field can be traced to the existence of a bosonic zero mode, namely $\partial_{-+} \phi$, which contributes in the plethystic exponential with weight 1 (see table 1). As we will soon discuss, divergences in the index signal the appearance of flat directions. In this example, the vacuum at infinity has a flat direction since the F-term equations are projective—it is this flat direction that gives rise to the divergent contribution.

### 3.4. Non-trivial chiral ring

Next, let us consider a superpotential of the form $W = \lambda Q^{h+1}$ for some integer $h$. This model has a chiral ring relation $Q^h \sim 0$. The field $Q$ has R-charge $\frac{2}{h+1}$, it is not charged under any
continuous global symmetries, but can carry charge under $\mathbb{Z}_{h+1}$. Let us denote by $g (g^{h+1} = 1)$ the fugacity for $\mathbb{Z}_{h+1}$ and write the index of this model as

$$
\Gamma((pq)^{1/2} g) = \text{PE} \left[ \frac{((pq)^{1/2} g)^h}{(1 - p)(1 - q)} \right].
$$

(3.8)

Recall that the numerator in the plethystic exponential of a chiral field comes from the bosonic mode $\phi$ and a fermionic mode $\bar{\psi}$, while the denominator comes from the derivatives, $\partial_{\bar{\psi}, \phi}$. Note that $\bar{\psi}$ contributes to the index the $h$th power of the contribution of $\phi$ with an opposite sign. This implies that the contribution of $\phi^h$ is cancelled by the contribution of $\bar{\psi}$, in accordance with the chiral ring relation discussed above.

4. Index of gauge theories

4.1. D-term supersymmetry breaking

Let us first discuss the simplest gauge theory, $U(1)$ theory with an FI parameter $\zeta$, which as we discussed should be an integer. The index of this model is given by

$$
\kappa \oint \frac{dz}{2\pi i} z^\zeta = \kappa \delta_{\zeta, 0}.
$$

(4.1)

For a non-zero FI parameter the index vanishes, signalling the D-term supersymmetry breaking. As we discussed in the previous section, pairs of chiral fields with a mass term superpotential do not affect the index. The index (4.1) can then be interpreted as the index of a $U(1)$ gauge theory with any number of such pairs. Although the details of the dynamics of the model may depend on the existence of such fields and on the relative values of the gauge coupling/FI term and masses, the index is always zero, capturing only the fact that supersymmetry is broken.

4.2. IR duality

$\mathcal{N} = 1$ gauge theories in four dimensions exhibit a variety of remarkable properties one, of which is the ubiquity of IR dualities first discussed by Seiberg [24]. A basic example is $\mathcal{N} = 1$ $SU(2)$ gauge theory with three flavors of fundamental and anti-fundamental quarks. This theory flows in the IR to a free theory in which is given by a sigma model of the collection of the mesonic and baryonic fields. The index of this gauge theory is given by

$$
I_{\text{gauge}} = \kappa \oint \frac{dz}{4\pi i} \frac{1}{\Gamma(z^2)} \prod_{i=1}^{3} \Gamma((pq)^{1/2} b u_i z^x_{-1}) \Gamma((pq)^{1/2} b^{-1} v_i z^x_{+1})
$$

(4.2)

Here $\prod_{i=1}^{3} u_i = \prod_{i=1}^{3} v_i = 1$, with these fugacities parametrizing the $SU(3)_u \times SU(3)_u$ flavor symmetry rotating the fundamental and anti-fundamental quarks, while $b$ parametrizes the baryonic $U(1)_b$. The distinction between fundamental and anti-fundamental matter here is artificial because of the pseudo-reality of the representations and is motivated by higher rank generalizations. In particular the $SU(3)_u \times SU(3)_u \times U(1)_b$ flavor symmetry enhances to $SU(6)_u$ with $\{ t_i \} = \{ u_i, b^{-1} v_i \}$. The index of the free mesons and baryons is given by

$$
I_{\text{sigma}} = \prod_{i<j} \Gamma((pq)^{1/2} t_i t_j)
$$

(4.3)
If the index is to be independent of the RG flow $I_{\text{gauge}}$ should be equal to $I_{\text{sigma}}$, which is indeed a proven mathematical fact. This identity is known as Spiridonov’s beta function identity in math literature [25]. On the sigma model side we have fifteen chiral fields but the flavor symmetry has only rank five. The remaining symmetries rotating the chiral fields are broken by the superpotential which is is the Pfaffian of the antisymmetric matrix one can build from these fields. This superpotential is encoded in the index through the restriction of the fugacities to the ones of the $SU(6)$ symmetry.

In evaluating the index, we have used the anomaly free R-charges for the quarks, $R = \frac{1}{3}$. Mathematically, the anomaly free condition translates into a constraint on the arguments of the Gamma functions appearing in the numerator of the integrand. In this case we have,

$$
\prod_{i=1}^{6} \Gamma((pq)^{i} b_{i}) = pq.
$$

Such constraints are called balancing conditions in the math literature [26].

### 4.3. Higgsing/mass deformations

As discussed above, giving a mass to a pair of chiral fields trivializes their contribution to the index. If the theory has a dual IR description, the mass deformation corresponds to turning on a vacuum expectation value that Higgs the gauge symmetry on the other side of the duality. Let us discuss how this happens at the level of the index in a simple example. We consider theory $A$ to be an $SU(2)$ gauge theory with four flavors. This model has an $SU(4)_{u} \times SU(4)_{v} \times U(1)_{b}$ flavor symmetry. Is index is given

$$
I_{A}(u, v, b) = \kappa \oint \frac{dz}{4\pi i z} \prod_{i=1}^{4} \Gamma((pq)^{i} b_{u} z^{\pm 1}) \Gamma((pq)^{i} b^{-1} v_{i} z^{\pm 1}),
$$

where the fugacities satisfy the $SU(4)$ constraint

$$
\prod_{i=1}^{4} u_{i} = \prod_{i=1}^{4} v_{i} = 1.
$$

This model enjoys an IR duality. The Seiberg dual of it is a gauge theory with same rank and same charged matter content. However the charges of the quarks under global symmetries are different, they are in the conjugate representation of the $SU(4)_{u} \times SU(4)_{v}$ flavor group. There are moreover gauge singlet fields having same charges as the mesons of the theory on side A and coupling to the mesons of the gauge theory on side B through a superpotential. The index of the theory on side B is

$$
I_{B}(u, v, b) = I_{A}(u^{-1}, v^{-1}, b) \prod_{i,j=1}^{4} \Gamma((pq)^{i} b_{i} v_{j}).
$$

The product over the Gamma functions is the product over the singlet fields. Thanks to an identity proved by Rains [27], the indices on side A and side B coincide

$$
I_{A} = I_{B},
$$

as expected from the duality. Again it was important here to use the anomaly free R-charges for the fields.

Let us now consider giving a mass to a pair of quarks on side $A$. This should give us the $SU(2)$ gauge theory with three flavors we discussed in the previous bullet. We break the
flavor symmetry from $SU(4)_u \times SU(4)_v$ down to $SU(3)_u \times SU(3)_v$. This breaking of symmetry through mass terms is encoded in the index by specializing the corresponding fugacities. For example, let us turn on a mass term $mQ_1\tilde{Q}_1$. The weight of the mesonic operator $Q_1\tilde{Q}_1$ in the index before turning on the mass term is $(pq)^{\frac{1}{2}}u_1v_1$. After turning on the mass it should be $pq$ corresponding to R-charge +2 and no other charges. Thus turning on the mass term in the index corresponds to specializing the fugacities to be

$$u_1v_1 = (pq)^{\frac{1}{2}}. \quad (4.9)$$

We now define $u_1 = (pq)^{\frac{1}{2}}a$, $v_1 = (pq)^{\frac{1}{2}}a^{-1}$, and find from (4.6),

$$\prod_{i=2}^{4} u_i = (pq)^{-\frac{1}{2}}a^{-1}, \quad \prod_{i=2}^{4} v_i = (pq)^{-\frac{1}{2}}a. \quad (4.10)$$

Redefining

$$u_i \equiv \tilde{u}_{i-1}(pq)^{-\frac{1}{4}}a^{-\frac{1}{4}}, \quad v_i \equiv \tilde{v}_{i-1}(pq)^{-\frac{1}{4}}a^{\frac{1}{4}}, \quad b = \tilde{b}a^{\frac{1}{2}}, \quad (4.11)$$

we obtain

$$\prod_{i=1}^{3} \tilde{u}_i = \prod_{i=1}^{3} \tilde{v} = 1. \quad (4.12)$$

After mass deformation, the index on side A becomes

$$I_A \to \kappa \oint \frac{dz}{2\pi i z} \frac{1}{\Gamma(z^{\pm 1})} \prod_{i=1}^{3} \Gamma((pq)^{\frac{1}{2}}\tilde{b}_i u_i z^{\pm 1}) \Gamma((pq)^{\frac{1}{2}}\tilde{b}^{-1} v_i z^{\pm 1}), \quad (4.13)$$

which coincides with (4.2) as expected.

Let us now discuss what happens on side B of the duality. Here the physics is more interesting. We gave a mass to the meson $Q_1\tilde{Q}_1$ on side A of the duality. On side B it maps to a singlet field, $M_{11}$, and thus the mass deformation adds a linear term to the superpotential. The superpotential involving the field $M_{11}$ is thus of the form

$$mM_{11} + q_1\tilde{q}_1M_{11}, \quad (4.14)$$

where $q_i$ and $\tilde{q}_i$ are the quarks of the side B of the duality. The F-term equation thus impose a vacuum expectation value for the meson $q_i\tilde{q}_i$. Turning such a vev Higgses the gauge $SU(2)$ gauge group and brings us to the sigma model of the previous bullet. Let us see what happens at the level of the index. The singlet $M_{11}$ contributes to the index as $\Gamma(Pq)^{\frac{1}{2}}\tilde{b}_1 u_1 z^{\pm 1}$ and two poles from $\Gamma((pq)^{\frac{1}{2}}b^{-1} v_1 z^{\pm 1})$ located at

$$z^{\pm 1} = (pq)^{\frac{1}{2}}b u_1, \quad (pq)^{\frac{1}{2}}b v_1. \quad (4.15)$$

Two of these poles are inside the $z$ integration contour and two are outside. Note then that if we specialize the fugacities to satisfy (4.9) these four poles pinch the integration contour pairwise producing a divergence. The leading, divergent, contribution to the integral in the mass limit we consider thus comes only from two poles in the $z$ integral. These two poles are related by Weyl symmetry in the limit and thus give the same residues. The divergence coming from the
pinching is precisely canceled against the zero coming from the meson $M_{11}$ in the mass limit. The index on side B in the limit is given then by

$$I_B(u^{-1}, v^{-1}, b) \to \text{Res}_{z \to (pq)^{1/2}bu, u_1v_1 \to (pq)^{1/2}} \left[ \frac{1}{1_{(z^{\pm})^2}} \prod_{i=1}^{4} \Gamma((pq)^{1/2}bu, z^{\pm}) \Gamma((pq)^{1/2}b^{-1}v_1, z^{\pm}) \prod_{i,j=1}^{4} \Gamma((pq)^{1/2}u_1, v_1) \right]$$

$$\to \prod_{i<j} \Gamma((pq)^{1/2}t_i t_j).$$

(4.16)

where $\{t_i\} = \{b u_i, b^{-1} v_j\}$. We thus rederived the identity for the index following from the duality of $SU(2)$ theory with there flavors to sigma model from the duality of $SU(2)$ theory with four flavors by following the RG flow triggered by mass term on one side of the duality and vev on the other side.

The general lesson to be learned here is that Higgsing gauge symmetries by vevs for gauge invariant operators manifests itself at the level of the index as reducing the number of integrals in the matrix model through the pinching procedure. In general a vev is possible when a flat direction opens up in the field space and this leads the index to have a pole. The index of the theory obtained in the IR of such an RG flow is given by the residue of the pole.

4.4. Spontaneously broken global symmetries

We discussed spontaneous supersymmetry breaking above; here we will study a case of flavor symmetry breaking. The example we consider is $SU(2)$ gauge theory with two flavors, i.e. two fundamental and two anti-fundamental quarks. This theory has an $SU(4)$ flavor symmetry at the classical level rotating the four quarks. However, at the quantum level the model can be described in terms of the six gauge singlet chiral fields $M_{ij} = Q_i Q_j$ with a quadratic constraint $P f M = \Lambda^4$ where $\Lambda$ is the dynamical scale of the gauge theory. This dynamical superpotential breaks the $SU(4)$ symmetry down to $Sp(4)$.

Let us see what happens here at the level of the index. The gauge theory at hand can be obtained from the $SU(2)$ theory with three flavors we already considered by giving a mass to one of the flavors. Let us denote the six quarks by $Q_i$ and rotate them with $SU(6)$ symmetry. We can turn on a mass term of the form $m Q_1 Q_2$. The theory with three flavors has an IR dual in terms of a sigma model and the analysis is simpler to perform on that side of the duality. Here we have a collection of fifteen singlet fields with a superpotential. The field $Q_1 Q_2$ is dual to singlet $M_{12}$. Turning on the mass term the superpotential on the sigma model side involving field $M_{12}$ will become schematically

$$m M_{12} + M_{12} (M_{34} M_{56} + M_{36} M_{45} - M_{35} M_{46}).$$

(4.17)

In particular the F term coming from $M_{12}$ imposes the constraint we discussed above,

$$m \sim M_{34} M_{56} + M_{36} M_{45} - M_{35} M_{46}.$$  

(4.18)

The weight of field $M_{12}$ before turning on the linear superpotential is $(pq)^{1/2} t_1 t_2$ and after turning it on it becomes $pq$. Thus in the index we need to specialize the parameters as

$$t_1 t_2 = (pq)^{1/2}.$$  

(4.19)

We parametrize the fugacities as
\[ t_1 = (pq)^i a, \quad t_2 = (pq)^i a^{-1}, \quad t_{i>2} = (pq)^{-i} t_{i-2}, \quad \prod_{i=1}^{4} t_i = 1. \] (4.20)

Fugacities \( a \) and \( \tilde{t}_i \) parametrize \( u(1)_a \times su(3)_f = su(4) \) classical symmetry of the model. Then after this specification the index of the sigma model becomes

\[ \mathcal{I}_{\text{sigma}} \rightarrow \Gamma(pq) \prod_{i=1}^{4} \Gamma((pq)^i a^{\pm 1} \tilde{t}_i) \prod_{i<j} \Gamma(\tilde{t}_i \tilde{t}_j). \] (4.21)

This expression vanishes for generic values of \( \tilde{t}_j \). In other words, if we insist on turning on fugacities for the classical \( SU(4) \) symmetry the index vanishes indicating that there is no vacuum of the model having this symmetry. On the other hand let us further take \( \tilde{t}_1 = \tilde{t}_2^{-1} \equiv c \). This also implies that \( \tilde{t}_3 = \tilde{t}_4^{-1} \equiv d \). The symmetry we now parametrize is \( su(2)_c \times su(2)_d \subset sp(4) \).

After this specialization the index becomes

\[ \mathcal{I}_{\text{sigma}} \rightarrow \Gamma(pq) \Gamma(1)^2 \Gamma((pq)^i a^{\pm 1} c^{\pm 1}) \Gamma((pq)^i a^{\pm 1} d^{\pm 1}) \Gamma(c^{\pm 1} d^{\pm 1}) = \Gamma(pq) \Gamma(1)^2 \Gamma(c^{\pm 1} d^{\pm 1}). \] (4.22)

Note that the fields charged under \( U(1)_c \) can form mass terms and their contribution to the index trivializes. Since \( \Gamma(z) \) has a simple pole as \( z \to 1 \) and a simple zero as \( z \to pq \), this expression diverges. We can thus summarize that unless we specialize the \( SU(4) \) fugacities to parametrize an \( Sp(4) \) subgroup the index vanishes and diverges otherwise. The residue of the divergence is given by

\[ \Gamma(c^{\pm 1} d^{\pm 1}), \] (4.23)

which is the index of the collection of the chiral fields in any given quantum vacuum of the model.

Let us consider the \( SU(2) \) gauge theory with two flavors with the \( Sp(4) \) flavor quantum symmetry, theory \( A \), and some other theory with an \( SU(2) \), flavor symmetry, which we will call theory \( B \). Let us also assume that we can gauge in anomaly-free fashion the diagonal combination of the \( SU(2)_A \) symmetry of theory \( B \) and the \( SU(2) \) subgroup of the \( Sp(4) \) symmetry of theory \( A \). Note that at a generic point of the moduli space of theory \( A \) operator charged under \( SU(2) \) obtains a vev. This Higgses the \( SU(2)_c \) gauge group. Careful analysis reveals that the theory in the IR is identical to theory \( B \) with an addition of two singlet fields. We denote the index of theory \( B \) by \( \mathcal{I}_B(c) \) where \( c \) is fugacity for the \( SU(2)_c \) symmetry. The index of the combined theory is then

\[ \mathcal{I}(d, g) = \kappa^2 \int \int \frac{dc}{4\pi i c} \frac{1}{\Gamma(c^{\pm 2})} \frac{dz}{4\pi i z} \frac{1}{\Gamma(z^{\pm 2})} \Gamma(g c^{\pm 1} z^{\pm 1}) \Gamma(g^{-1} d^{\pm 1} c^{\pm 1}) \mathcal{I}_B(c). \] (4.24)

One has to be careful here with the contour of integration since the poles of the index coming from the quarks of theory \( A \) sit on the unit circle. The contour can be obtained by carefully taking the mass limit from the theory with three flavors, and we call it \( C \). This contour separates the sequences of poles these Gamma functions have converging to infinity and zero. Computation of this index reveals that it satisfies

\[ \mathcal{I}(d, g) = \Gamma(g^{\pm 1/2}) \mathcal{I}_B(d). \] (4.25)

We have seen that the index of theory \( A \) vanishes except for a subset of fugacities where it diverges, and the above computation reveals that this index can be thought of as a delta
5. Index spectroscopy

The supersymmetric index contains useful information about the protected spectrum of the theory. The index counts (with signs) short multiplets up to the equivalence relation that sets to zero sets of short multiplets that may recombine into long ones. In general, it is not possible to deduce unambiguously from the index the precise spectrum of short multiplets. However, for certain special multiplets corresponding to relevant and marginal operators, useful statements with a direct physical interpretation can be made. We will follow closely the discussion in [30].

A generic long multiplet $A^{\Delta,j_1,j_2}$ of $\mathcal{N}=1$ superconformal algebra is generated by the action of the four Poincaré supercharges $(Q, \tilde{Q}, S, \tilde{S})$ on a superconformal primary state, which by definition is annihilated by superconformal charges $(S, \tilde{S})$. The multiplet is labeled by the charges $(\Delta, j_1, j_2)$ of the primary with respect to the dilatations, R-symmetry, and the two angular momenta respectively. The absence of negative norm states in the multiplet imposes certain inequalities on these quantum numbers,

\begin{align}
\Delta &\geq 2 - 2\delta_{j,0} + 2j_1 - \frac{3}{2} r, \\
\Delta &\geq 2 - 2\delta_{j,0} + 2j_2 + \frac{3}{2} r, \\
\Delta &\not\in \left( -\frac{3}{2} r, 2 - \frac{3}{2} r \right), \quad \text{if } j_1 = 0, \\
\Delta &\not\in \left( \frac{3}{2} r, 2 + \frac{3}{2} r \right), \quad \text{if } j_1 = 0, \\
\Delta &\geq 2 + j_1 + j_2, \quad \text{if } j_1 \neq 0, j_2 \neq 0, \\
\Delta &\geq 1 + j_1 + j_2, \quad \text{if } j_1 = 0 \text{ or } j_2 = 0.
\end{align}

When these inequalities are saturated, some combination of the Poincaré supercharges will annihilate the primary as well, resulting in a shortened multiplet. The relevant property of these short multiplets is that they must always saturate the unitarity bound in order to be free of negative normed states, and so their conformal dimension is fixed in terms of other quantum numbers and is protected against corrections as one changes the parameters of the theory.

The possible shortening conditions of the $\mathcal{N}=1$ superconformal algebra are summarized in table 2. Note that $D$ and $\bar{D}$ multiplets correspond to free fields and our general results below will not hold for them.

If the charges of a collection of short multiplets obey certain relations, they can combine to form a long multiplet which is no longer protected. Alternatively, one can understand this recombination in reverse, as a long multiplet decomposing into a collection of some short multiplets as the conformal dimension of its primary hits the BPS bound. This phenomenon plays a crucial role in extracting spectral information about an SCFT from its index because the index counts short multiplets of the theory up to recombination. The collective contributions
to the index from short multiplets that can recombine vanishes. The recombination equations for $N = 1$ superconformal algebra are as follows:

$$\begin{align*}
A^2_{(j_1,j_2)} &\rightarrow C_{r(0,j_1)} \oplus C_{r-1(j_1-\frac{1}{2},j_2)}, \\
A^2_{(j_1,j_2)} &\rightarrow C_{r(0,j_1)} \oplus C_{r+1(j_1+\frac{1}{2},j_2)}, \\
A^2_{(j_1,j_2)} &\rightarrow C_{r(0,j_1)} \oplus C_{r-1(j_1-\frac{1}{2},j_2)} \oplus C_{r+1(j_1+\frac{1}{2},j_2)}.
\end{align*}$$

(5.7)

The $B$ multiplets can be formally treated as a special case of $C$ multiplets with unphysical spin quantum numbers,

$$B_{(0,j_2)} = \tilde{C}_{r+1(0,-\frac{1}{2})}, \quad \tilde{B}_{(j_1,0)} = \tilde{C}_{r-1(0,-\frac{1}{2})}.$$  

(5.8)

Thus the discussion can be phrased entirely in terms of $C$ type multiplets.

An important example of recombination is for the long multiplet $A^2_{(0,0)}$ as $r \rightarrow 0$. The multiplet hits the BPS bound and splits into three short multiplets according to the third rule in (5.7),

$$A^2_{(0,0)} \rightarrow \tilde{C}_{(0,0)} \oplus C_{-1(0,-\frac{1}{2})} \oplus \tilde{C}_{1(0,-\frac{1}{2})} = \tilde{C}_{(0,0)} \oplus (B_{-2(0,0)} \oplus \tilde{B}_{2(0,0)}).$$

(5.9)

The multiplet $\tilde{C}_{(0,0)}$ contains a conserved current, while the multiplet $B_{-2(0,0)}$ contains a chiral primary $\mathcal{O}$ of dimension three and an associated marginal F-term deformation $\int d^2 \theta \mathcal{O}$. The recombination described above demonstrates the fact that a marginal operator can fail to be exactly marginal if and only if it combines with a conserved current corresponding to a broken global symmetry. This particular recombination and its implications for the space of exactly marginal deformations of an SCFT has been studied in detail in [31].

The $C$ ($\tilde{C}$) multiplets contribute only to the left-handed index (right-handed index), while $\tilde{C}$ multiplets contribute to both. We restrict our attention to $T^L$ and treat $\tilde{C}$ as a special case of $C$ with $r = \frac{1}{2}(j_1 - j_2)$. The recombination rules allow us to define equivalence classes of short representations which make identical contributions to the index,
Furthermore, there are no unitary representations in a fixed equivalence class—for fixed \( \tilde{r} \), there is an upper limit on \( r \), given by \( \tilde{r} \geq -\frac{3}{2} + \frac{3}{2} j_2 \). Consequently, there are a finite number of representatives.

The contribution to the left-handed superconformal index from any short multiplet in a given class is given by

\[
\mathcal{I}^L_{[\tilde{r}, j_2]} = (-1)^{2 j_2 + 1} \frac{(pq)^{\frac{1}{2}(r+2)} \chi_2(p/q)}{(1-p)(1-q)}.
\]

We define the net degeneracy for a given choice of \( \tilde{r}, j_2 \),

\[
\text{ND}[\tilde{r}, j_2] := \# [\tilde{r}, j_2]_+ - \# [\tilde{r}, j_2]_-,
\]

and the extractable content of the superconformal index is encapsulated in precisely the integers \( \text{ND}[\tilde{r}, j_2] \). If the index of an \( \mathcal{N} = 1 \) SCFT is known, the net degeneracies can be systematically extracted by means of a sieve algorithm (see for example [30]). The most precise information about actual operators we can extract from the index comes from the equivalence classes with a small number of representatives.

The optimal case is the chiral primary operators that lie in multiplets \( \mathcal{B}_{(0, j_2)} \) and have \(-2 - \frac{3}{2} j_2 < r \leq -\frac{3}{2} + \frac{3}{2} j_2 \). These have \( \tilde{r} \in \left[ -\frac{3}{2} + \frac{3}{2} j_2, \frac{3}{2} j_2 \right) \), and they are the only representatives of the equivalence class \([\tilde{r}, 0]_-\) for this range of \( \tilde{r} \). Furthermore, there are no unitary representations in the corresponding class \([\tilde{r}, 0]_+\). Consequently, we can read off the exact number of such operators from the superconformal index. Specializing to \( j_2 = 0 \), these are precisely the relevant deformations of the SCFT. The number of such deformations is simply the coefficient of \( (pq)^{-\frac{1}{2}(p/q)^0} \) in the index after subtracting out any non-trivial \( SU(2)_2 \) characters at the same power of \( pq \).

The next best case is for \( \tilde{r} \in \left[ \frac{3}{2} j_2, \frac{3}{2} + \frac{3}{2} j_2 \right) \). Both \([\tilde{r}, j_2]_+\) and \([\tilde{r}, j_2]_-\) have only a single representative in this range, and so the index computes the difference in the number of such operators. For \( j_2 = \tilde{r} = 0 \) in particular, the representatives are \( \hat{\mathcal{C}}_{(0,0)} \) and \( \mathcal{B}_{-2(0,0)} \), respectively. The cancellation between these multiplets corresponds to precisely the recombination described in the example above, and we see that the index computes

\[
\text{ND}[0, 0] = \# \mathcal{B}_{-2(0,0)} - \# \hat{\mathcal{C}}_{(0,0)} = \# \text{marginal operators} - \# \text{conserved currents}.
\]

If all global flavor symmetries are broken at a generic point on the conformal manifold, then this net degeneracy will precisely capture the actual dimension of that conformal manifold. However, not all recombinations of the type discussed in the example necessarily take place, and in this case one must account for conserved currents in extracting the dimension of the conformal manifold. Again, this net degeneracy is easily computed by expanding the index to order \( pq \) and subtracting out all nontrivial characters for \( SU(2)_2 \).

For \( \tilde{r} \geq \frac{3}{2} \), there will be several representatives that are indistinguishable to the index, and the cancellations among them do not correspond to any obvious physical phenomenon such as symmetry breaking. Thus, the most immediate spectroscopic use of the index is the analysis of relevant and marginal operators at a fixed point.
5.1. An example

As an example we discuss $SU(N)$ $\mathcal{N} = 4$ SYM. In $\mathcal{N} = 1$ notation we have here three adjoint chiral fields, $\Phi_j$, with R-charge $\frac{2}{3}$ rotated by $SU(3)$, global symmetry. The superconformal R-charge is that of a free field since the conformal manifold passes through the free point. The index is given by

$$I_N(t,p,q) = \frac{1}{N!} e^{N-1} \int \prod_{j=1}^{N-1} dz_j \prod_{j \neq k} \frac{\Gamma((pq)^\frac{1}{3} t_j z_j/z_k) \Gamma((pq)^\frac{1}{3} t_j z_j/z_k) \Gamma((pq)^\frac{1}{3} \frac{1}{N} z_j/z_k)}{\Gamma(z_j/z_k)}.$$  

(5.14)

For $N > 2$, the first few terms in the $p, q$ expansion are

$$I_N(t,p,q) = 1 + 6_3(pq)^\frac{1}{3} + 3_8(p + q)(pq)^\frac{1}{3} + (1 + 10_5 - 8_8) pq + \cdots.$$  

(5.15)

Following the general prescription of this section we read off the relevant operators to be $6_3$, which are the six quadratic operators $\Phi_1^3 \Phi_2 \Phi_3$. We have also operators charged under $j_2$ at order $(p + q)(pq)^\frac{1}{3}$ which do not correspond to relevant operators. At order $pq$ we have the marginal operators. The contribution here is $1 + 10_5 - 8_8$. The generators of the global symmetry form the $8_8$ which is subtracted and the marginal operators are the gauge coupling and the $10_5$, symmetric cubic combinations of the adjoint chiral fields. At a generic point on the conformal manifold the $SU(3)_L$ symmetry is broken and the dimension of it is $1 + 10 - 8 = 3$ as expected. These exactly marginal deformations are the gauge coupling, the $\beta$ deformation (adding $\text{Tr} \Phi_1 \{ \Phi_2, \Phi_3 \}$ to superpotential), and the $\gamma$ deformation (adding also $\text{Tr} (\Phi_1^2 + \Phi_2^2 + \Phi_3^2)$).

The case of $N = 2$ is special and there the expansion of the index coincides with (5.15) except that $10_5$ term is missing. Here the conformal manifold is actually only one dimensional and corresponds to the gauge coupling. On any point of this manifold the $SU(3)_L$ symmetry is unbroken consistently with the index. The reason here two directions are missing is that a general marginal superpotential cubic in the chiral fields can be decomposed as a sum of two terms, in one of which the gauge indices are contracted with $\epsilon_{abc}$ and the other with $d_{abc} = \text{Tr} T_a \{ T_b, T_c \}$. The latter structure is non zero only for $N > 2$.

6. Dualities and identities

Perhaps the most important application of the supersymmetric index as a test of non-perturbative dualities. Since the index is an RG invariant quantity and does not depend on the marginal couplings, it should be the same when computed for two theories flowing to the same fixed point or two different descriptions of the same conformal theory. Physical dualities translate into mathematical identities between elliptic hypergeometric integrals. Such identities are particularly non-trivial and give the strongest checks to date of many dualities. In several cases, these identities have already appeared in the mathematical literature, but in many others they are new—they are undoubtedly true since they can be checked to very high orders in a series expansion, but a rigorous proof is still lacking.

6.1. Symmetries and transformations of the index

Before discussing relations between indices of dual theories, it is useful to pause and consider the symmetry properties of the index of a single theory. The index is a function $I(a_1, a_2, \cdots, a_v, p, q)$. The parameters $a_i$ are fugacities for $U(1)$ global symmetries forming
the maximal torus of the (possibly non-abelian) global symmetry. If the symmetry enhances to a non-abelian symmetry the index should be invariant under the action of the Weyl group acting on the fugacities. For example, if the $a_i$'s parametrize an $SU(s + 1)$ symmetry, the index should be invariant under permutations of the $a_i$'s and under the transformation of any of the $a_j$ as $a_j \rightarrow 1/a_j$.

We can also ask the converse question: what happens if the index is invariant under the action of certain discrete group $W$ on the flavor fugacities? There are two interesting physical possibilities. First, it might be that the flavor symmetry enhances to a non-abelian group such that $W$ serves as its Weyl group. A second physical possibility is that such a discrete symmetry signals self-duality of the theory. An example is $SU(2)$ $\mathcal{N} = 2$ SYM with four flavors. Here the flavour group (in $\mathcal{N} = 2$ language) is rank four, and let us parametrize it by four fugacities $a_i$. In $\mathcal{N} = 1$ language the index is given by

$$\mathcal{I}(a_1, a_2, a_3, a_4) = \frac{\kappa \Gamma((pq)_{1,2}^i r^{-2})}{2 \pi i} \int \frac{dz}{z} \frac{\Gamma((pq)_{1,2}^i r^{-2} z^\pm 2)}{\Gamma(z^\pm 2)} \Gamma((pq)_{1,2}^i z_{1,2}^\pm 1 z_{3,4}^\pm 1).$$

(6.1)

Here $t$ is fugacity for a $U(1)$ symmetry related to the bigger $R$-symmetry of $\mathcal{N} = 2$. The flavor symmetry here enhance to $SO(8)$ and the index is manifestly invariant under the Weyl group of $SO(8)$. This group is generated by $a_i \rightarrow a_i^{-1}$ and by $a_1 \leftrightarrow a_2, a_3 \leftrightarrow a_4$. However, the index is also invariant under exchanging $a_1$ and $a_3$. This is not part of $SO(8)$ Weyl symmetry and is not manifest in the integral above. This discrete symmetry is the manifestation of the self S-duality (or rather triality) that the theory enjoys. This is a strong/weak type of duality relating the same theory with different values of coupling. This invariance property of the index was proven in [32]. In fact the full discrete symmetry of the index, the one coming from Weyl of $SO(8)$ and the one coming from the duality, is the Weyl group of $F_4$. We are not aware of a physical interpretation for the full $F_4$ symmetry—it would be nice to figure out whether there is any.

Another similar example is that of $\mathcal{N} = 1 SU(2)$ theory with four flavors, i.e. the same theory as above but without the adjoint chiral field. The theory has flavor symmetry of rank seven, the $SU(8)$ symmetry rotating the different matter fields. This theory enjoys Seiberg-duality as we already discussed, but in fact there are many more dualities as discussed in [33]. This theory in fact has 72 dual descriptions. The different descriptions correspond to the action of the Weyl group of $E_7$ on the fugacities. In the different duality frames the gauge structure is the same as in the original one but there are additional singlet fields and superpotentials. It was argued in [34] that taking two copies of this theory coupled through a quartic superpotential the theory is exactly self-dual and that there should be a point on the conformal manifold of this theory where the flavor symmetry is actually enhanced to $E_7$.

We can also ask whether there are interesting properties of the index involving manipulations of both the flavor fugacities and the superconformal fugacities $p$ and $q$. A simple example is as follows. One can consider assigning different anomaly free $R$-charges to the fields by mixing a given $R$-symmetry with the flavor symmetry. For example given a flavor symmetry $U(1)_a$ we can redefine the $R$-symmetry to be $R \rightarrow R + sq_a$ with $q_a$ being the charge under $U(1)_a$. At the level of the index this transformation corresponds to

$$R \rightarrow R + sq_a \Rightarrow \mathcal{I}(a, p, q) \rightarrow \mathcal{I}((pq)^\tau a, p, q).$$

(6.2)

Let us consider shifting flavor fugacity $a$ to $q'p'a$. When $s$ and $\bar{s}$ are the same this is just a redefinition of the $R$-charge. From the definition of the index the shift in $a$ amounts to

$$\mathcal{I} = \text{Tr}(-1)^F p^{h+j+\frac{1}{2}} q^{h-j+\frac{1}{2}} a^\alpha \rightarrow \text{Tr}(-1)^F p^{h+j+\frac{1}{2} + sq_a} q^{h-j+\frac{1}{2} + s \bar{q}_a} a^\alpha.$$ 

(6.3)
To interpret this expression as an index we can redefine

\[ \hat{r} = r + (s + \tilde{s}) q_a, \quad \hat{j}_1 = j_1 + \frac{s - \tilde{s}}{2} q_a. \]  

(6.4)

In particular for \( s \neq \tilde{s} \) this breaks Lorentz symmetry and does not make sense as a pure 4d index. However, such a transformation might make sense as an index of a coupled 4d–2d system. A simple example is the following important identity of the index of a chiral field,

\[ \mathcal{I}^{(R)}(p a) = \Gamma((p q) \frac{x}{2} p a) = \theta((p q) \frac{x}{2} a; q) \mathcal{I}^{(R)}(a). \]  

(6.5)

The index on the right-hand side can be interpreted as an index of chiral field in four dimensions coupled to a Fermi (0, 2) multiplet in two dimensions. Similarly we have

\[ \mathcal{I}^{(R)}(p^{-1} a) = \Gamma((p q) \frac{x}{2} p^{-1} a) = \frac{1}{\theta((p q) \frac{x}{2} p^{-1} a; q)} \mathcal{I}^{(R)}(a). \]  

(6.6)

Here the right hand side is a chiral field in four dimensions coupled to a chiral (0, 2) field in two dimension. Such a transformation of the index will become important while discussing indices in presence of surface defects [35–37] and we will comment on this more in what follows.

### 6.2. \( \mathcal{N} = 4 \) dualities

A basic example of a duality implying a non-trivial mathematical identity is the S-duality between \( SO(2n + 1) \) \( \mathcal{N} = 4 \) SYM and \( USp(2n) \) \( \mathcal{N} = 4 \) SYM. We use an \( \mathcal{N} = 1 \) language with the three adjoint chiral multiplets having R-charge \( \frac{x}{2} \). Then the index of the \( SO(2n + 1) \) model is given by

\[ \mathcal{I}_o = \kappa^n \prod_{i=1}^3 \Gamma((p q)^i t_i) \frac{1}{2^{n!}} \oint \frac{dz_i}{2\pi i z_i} \prod_{i<k} \prod_{j=1}^n \Gamma(\frac{(p q)^i t_j z_i^{\pm1} z_j^{\pm1}}{z_i z_k}) \frac{1}{\Gamma(z_i^{\pm1})} \prod_{i=1}^n \Gamma((p q)^i t_i z_i^{\pm1}) \Gamma(z_i^{\pm1}) \]  

(6.7)

while for the \( USp(2n) \) model we get

\[ \mathcal{I}_p = \kappa^n \prod_{i=1}^3 \Gamma((p q)^i t_i) \frac{1}{2^{n!}} \oint \frac{dz_i}{2\pi i z_i} \prod_{i<k} \prod_{j=1}^n \Gamma(\frac{(p q)^i t_j z_i^{\pm1} z_j^{\pm1}}{z_i z_k}) \frac{1}{\Gamma(z_i^{\pm1})} \prod_{i=1}^n \Gamma((p q)^i t_i z_i^{\pm2}) \Gamma(z_i^{\pm2}) \]  

(6.8)

Fugacities \( t_i \) parametrize \( SU(3)_t \) symmetry rotating the three adjoint chirals. We have decomposed the \( SU(4) \) R-symmetry of \( \mathcal{N} = 4 \) to \( U(1) \) R-symmetry of \( \mathcal{N} = 1 \) and \( SU(3)_t \). For \( n = 1 \) and \( n = 2 \) the \( SO(2n + 1) \) and \( USp(2n) \) algebras are isomorphic\(^9\) and there is a simple change of integration variables making the two expressions above manifestly the same.\(^{10}\) For \( n > 2 \), one can check that the two expressions coincide to very high orders in a series expansion in fugacities, but no proof is available yet except in certain degeneration limits [38].

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\(^9\)The global form of the gauge group is inessential here—the spectrum of local gauge-invariant operators captured by the index depends only on the gauge algebra.

\(^{10}\)The two root systems define dual lattices in \( n \) dimensions. In \( n = 1, 2 \) there is a linear transformation taking one into the other (line dual to line, and square dual to square), while for \( n > 2 \) there is not.
6.3. Seiberg dualities

Seiberg dualities are the basic examples of IR dualities—two theories flowing to the same fixed point. The simplest example is of an SU(N) gauge theory with Nf on side A being equivalent to SU(Nf - N) gauge theory with Nf flavors, conjugate representation of the flavor group, and a bunch of gauge singlet fields dual to the mesons of side A on side B. The index of side A is given by

$$I^N_{Nf-N} (u, v, b) = \kappa^{N-1} N^! \int \prod_{i=1}^{Nf-1} \frac{dz_i}{2\pi i} \prod_{i \neq j} \Gamma(z_i/z_j) \left( \prod_{j=1}^{Nf} \Gamma((pq)^{Nf-N} b u v_j) \Gamma((pq)^{Nf-N} v^{-1}_j b^{-1}) \right).$$

(6.9)

Here u, v, and b are parametrizing the SU(Nf) x SU(Nf) x U(1)b global symmetry of the theory. On side B we have

$$I^N_{Nf-N} (u, v, b) = \left( \prod_{j=1}^{Nf} \Gamma((pq)^{Nf-N} u v_j) \right) \kappa^{Nf-N-1} \left( \prod_{i=1}^{Nf} \frac{dz_i}{2\pi i} \prod_{i \neq j} \Gamma(z_i/z_j) \right) \left( \prod_{i=1}^{Nf} \prod_{j=1}^{Nf-N} \Gamma((pq)^{Nf-N} b u v_j) \Gamma((pq)^{Nf-N} v^{-1}_j b^{-1}) \right).$$

(6.10)

Duality implies that the two indices above should be equal and indeed it was shown by Rains that they are [27]. The proof is rather non trivial but in section 7 we will discuss a proof for a certain limit of the parameters.

6.4. Kutasov–Schwimmer dualities

Let us give yet another example of duality which implies a mathematical identity of indices yet to be proven rigorously. The example is that of Kutasov–Schwimmer dualities where in addition to Nf flavors of fundamental matter of SU(N) gauge group one introduces two, or one, adjoint fields. The superpotentials for the adjoint fields follow ADE classification,

$$A_k : \quad \text{Tr} X^{k+1},$$

$$D_{k+2} : \quad \text{Tr} X^{k+1} + \text{Tr} XY^2,$$

$$E_6 : \quad \text{Tr} X^4 + \text{Tr} Y^3, \quad E_7 : \quad \text{Tr} X^3 Y + \text{Tr} Y^3, \quad E_8 : \quad \text{Tr} X^5 + \text{Tr} Y^3.$$

(6.11)

These superpotentials fix the R-charge assignments for the adjoint fields. Dual descriptions are known in the A, D, [39–41] and E_7 [42] cases. The dual has gauge group of SU(\alpha Nf - N) with \alpha depending on the superpotential for the adjoint matter,

$$A_k : \quad \alpha = k,$$

$$D_{k+2} : \quad \alpha = 3k,$$

$$E_7 : \quad \alpha = 30.$$

(6.12)

One also has to introduce a variety of singlet fields coupled through a superpotential to gauge singlet operators on the dual side. For details the reader is referred to [43]. One can write down the corresponding identities for the supersymmetric indices, see e.g. [19], and check that they
are true in series expansion in fugacities or in certain limits such as large \( N \). However no proof is known to date.

We have focussed on the simplest representative examples of dualities and there are many more, see the discussion in [44, 45]. The mathematics of these identities is a very active area of research, see e.g. [26, 46–48] for reviews.

7. Limits

In previous sections we have discussed how the index encodes information about four-dimensional physics. Upon taking appropriate limits, the index can also be related to physical quantities in other spacetime dimensions. We will discuss here the two most natural limits of this kind.

7.1. Small \( \tau \) limit, \( S^1 \to 0 \)

We consider taking all the fugacities to 1. This limit in the partition function language corresponds to taking the limit of the size of \( S^1 \) to zero. Since the index and the partition function differ only by the \( e^{-\tau E_{\text{Casimir}}} \) factor the two coincide in the limit. Moreover it was argued on general grounds that in this limit the index has generically the following divergent behavior [49]

\[
I(\tau \to 0) = Z_{S^3 \times S^1} = e^{-\frac{\pi^2}{\tau}(a-c)} \times Z_{S^3}.
\]

(7.1)

This asymptotic behavior can be corrected by subleading power-law terms in \( \tau \) when the theory has moduli spaces on the circle [15, 50].11 Let us discuss how this comes about in detail in a particular example.12

7.1.1. Dimensional reduction of the index of a chiral field. Let us make the relation between the geometry and the index a bit more precise. We compute the partition function on \( S^3 \times S^1 \) with radii \( r_3 \) and \( r_1 \), twisted by fugacities for various global symmetries. Equivalently, after a change of variables it can be thought of as a partition function on \( S^3_b \times \tilde{S}^1 \) with the fugacities responsible for the geometric twisting absorbed in the geometry [53]. Here is \( S^3_b \) is the squashed sphere. We can compute the index as a partition function by first reducing the theory on \( \tilde{S}^1 \) of finite radius, and then computing the 3d partition function of the resulting 3d theory, including all the KK modes on the \( \tilde{S}^1 \). The fugacities corresponding to flavor symmetries can be thought of as couplings to background gauge fields along the \( \tilde{S}^1 \) direction. The gauge fields along the \( S^1 \) have the meaning of real mass parameters for global symmetries in three dimensions. In addition, as we go once around the \( S^1 \), we should rotate the \( S^1_b \) along the Hopf fiber by an angle depending on the fugacities \( p \) and \( q \). This has the effect of changing the geometry.

As discussed in [53], there is a change of coordinates, where the metric becomes that of an \( S^3_b \times \tilde{S}^1 \), where the \( \tilde{S}^1 \) factor is rotated on the \( S^3_b \) base. The parameters are related by

\[
p = e^{-2\pi b^1 \tilde{r}_1} ; \quad q = e^{-2\pi b^{-1} \tilde{r}_1} , \quad \tilde{r}_1 = \frac{2}{b + b^{-1}} r_1.
\]

(7.2)

11 However, in certain non-generic situations even the leading behavior is modified, see [15] for a careful discussion. Perhaps the simplest example that exhibits this non-generic behavior is the ISS model [51] (see also [52] for a discussion of the index of this theory).

12 We follow here the discussion in appendix B of [23].
This procedure leads to the action used in [53] to compute the supersymmetric partition function on $S^4_0$. Then, we can write the $4d$ index as coming from a theory on $S^4_0$, with an infinite tower of KK modes. We refer the reader to the references above and to appendix B of [23] for more details.

For a free chiral field (of R-charge $R$ and charged under a $U(1)_a$ symmetry) we are interested in rewriting the index in the following form,

$$Z^{(R)}_{S^4_0 \times S^1}(p, q, u) \propto \prod_{n=-\infty}^{\infty} Z^{(R)}_{S^4_0}(\omega_1, \omega_2, m + \frac{n}{r_1}). \tag{7.3}$$

where $Z^{(R)}_{S^4_0}$ is the $S^4_0$ partition function of a chiral field depending on the squashing parameter, real mass for $U(1)_a$, and the R-charge,

$$Z^{(R)}_{S^4_0} = \Gamma_{\hbar R} (\omega R + \sum_a m_a e_a; \omega_1, \omega_2),$$

$$\Gamma_{\hbar R}(z; \omega_1, \omega_2) = e^{i \frac{\omega_1}{\omega_2}} \left( e^{2\pi i \omega_1 z} - e^{-2\pi i \omega_1 z} \right) \prod_{\ell=0}^{\infty} \frac{1 - e^{2\pi i \omega_1 z}}{1 - e^{\frac{2\pi i \omega_1 z}{e^{2\pi i \omega_1}}}}. \tag{7.4}$$

The parameters on the two sides in (7.3) are related as

$$u = e^{2\pi i \omega_1 m}, \quad p = e^{2\pi i \omega_1 \omega_1}, \quad q = e^{2\pi i \omega_1 \omega_2}, \quad \omega = \frac{1}{2}(\omega_1 + \omega_2). \tag{7.5}$$

On the left-hand side we have the 4d index of a chiral superfield, and on the right-hand side the product over 3d $S^4_0$ partition functions of the KK modes on $S^4_0$. The inverse radius of $S^4_0$, $1/r_1$, plays the role of a real mass coupled to the KK momentum.

The expression on the right hand side of (7.3) as it stands is divergent and needs to be properly regularized and defined. Moreover one needs to be careful to include the Casimir energy in the definition of the partition function in four dimensions. Concretely, the twisted partition function of the chiral field on $S^4_0 \times S^1$ can be written as

$$Z^{(0)}_{S^4_0 \times S^1}(p, q, u) = e^{Z_0} \Gamma(u; p, q). \tag{7.6}$$

Here we chose to take the R charge to be zero for simplicity with a non trivial R charge easily reintroduced by mixing in the flavor symmetry. The factor $e^{Z_0}$ relates the two different natural normalizations. It is computed in [14],

$$\tilde{r}_1^{-1} I_0 = \frac{1}{4} \left(r^{-1} \frac{d}{dr} \left( r \Gamma_0(e^{2\pi i \omega_1}; e^{2\pi i \omega_1}, e^{2\pi i \omega_2}) \right) \right) \bigg|_{r=0}, \tag{7.7}$$

where $\Gamma_0(z; p, q)$ is the so called single particle index, defined by

$$\Gamma(u; p, q) = \exp \left[ \sum_{a=1}^{\infty} \frac{1}{n} \Gamma_0(z^a; p^a, q^a) \right] \rightarrow \Gamma_0(z; p, q) = \frac{z - pqz^{-1}}{(1 - p)(1 - q)}. \tag{7.8}$$

Using the fact that $\Gamma_0$ has a simple pole at $r = 0$ and a vanishing constant term in the expansion around $r = 0$, equation (7.7) leads to

$$I_0 = \frac{\pi i}{\tilde{r}_1} \left( m - \omega \right) \left( 2m(m - 2\omega) + \omega_1 \omega_2 \right) \frac{\omega_1}{6 \omega_2}. \tag{7.9}$$

Next we compute the right-hand side of (7.3),
\[
\prod_{n=-\infty}^{\infty} \mathcal{Z}_{S^1}^{(0)}(\omega_1, \omega_2, m + \frac{n}{T_1}) = \prod_{n=-\infty}^{\infty} \Gamma_h(m + \frac{n}{T_1}; \omega_1, \omega_2). \tag{7.10}
\]

The infinite product over \( n \) here diverges, since for large \( n \) the hyperbolic Gamma functions approach a divergent exponential behavior,

\[
\log (\Gamma_h(\omega R + \rho(\sigma) + \tau(\mu + s \mu_h))) = \text{sign}(\tau(\mu_h)) \frac{\pi i}{2\omega_1\omega_2} \left( \omega(R - 1) + \rho(\sigma) + \tau(\mu + s \mu_h) \right)^2 - \frac{\omega_1^2 + \omega_2^2}{12} + O(e^{-\alpha R}). \tag{7.11}
\]

We can regularize this divergence using zeta-function regularization \((\sum_{n=1}^{\infty} n^r = \zeta(-s))^{13}\)

\[
\prod_{n=-\infty}^{\infty} e^{-\text{sign}(n) \frac{\pi i}{2\omega_1\omega_2} (m + \frac{n}{T_1} - \omega^2 - \frac{\omega_1^2 + \omega_2^2}{4})} \to \exp \left( \frac{i\pi}{12} \frac{2(1 - 6m\tilde{r}_1 + 1)}{\tilde{r}_1 \omega_1\omega_2} \right).
\]

The precise statement of (7.3) is then the following equality

\[
e^{\Delta} \Gamma(u; p, q) = e^{-\Delta} \prod_{n=-\infty}^{\infty} e^{-\text{sign}(n) \frac{\pi i}{2\omega_1\omega_2} (m + \frac{n}{T_1} - \omega^2 - \frac{\omega_1^2 + \omega_2^2}{4})} \Gamma_h(m + \frac{n}{T_1}; \omega_1, \omega_2). \tag{7.13}
\]

The infinite product on the right-hand side is now well-defined, and in fact by using (7.4) and (2.13) it can be written as a product of two elliptic Gamma functions,

\[
\Gamma(u; p, q) = e^{-\Delta - I_0} \frac{\Gamma(e^{2\pi i\tilde{r}_1}; e^{2\pi i\omega_1}, e^{-2\pi i\omega_1})}{\Gamma(e^{2\pi i\frac{\omega_1^2}{\omega_2}}; e^{2\pi i\omega_1}, e^{-2\pi i\omega_1})}. \tag{7.14}
\]

This equality is discussed in [54]. It is sometimes viewed as an indication of an \( SL(3, \mathbb{Z}) \) structure. Taking the 3d limit by sending \( \tilde{r}_1 \) to zero, we decouple the massive KK modes on the \( S^1 \).

The only term surviving the limit on the right-hand side of (7.13) has \( n = 0 \), and we obtain

\[
\lim_{\tilde{r}_1 \to 0} \left[ \Gamma(e^{2\pi i\tilde{r}_1}; \omega R + m, e^{2\pi i\omega_1}, e^{2\pi i\omega_2}) e^{\pi i \frac{\omega_1^2}{T_1} (m - \omega(1 - R))} \right] = \Gamma_h(\omega R + m; \omega_1, \omega_2). \tag{7.15}
\]

Note that the divergent factor is after turning on flavor fugacity and going to an unsquashed sphere, \( \omega_1 = \omega_2 = \frac{1}{2\pi} i \) and \( \tilde{r}_1 = \tau \),

\[
e^{\pi i \frac{\omega_1^2}{\omega_2} \tau (1 - R)} = e^{-\frac{\omega_2}{\omega_1} (1 - R)} = e^{-\frac{a - c}{16}(1 - R)}, \tag{7.16}
\]
in agreement with (7.1) since the anomalies of the chiral field are given by

\[
a = \frac{1}{32} (9(R - 1)^3 - 3(1 - R)), \quad c = \frac{1}{32} (9(R - 1)^3 - 5(R - 1)), \quad a - c = \frac{1}{16} (R - 1). \tag{7.17}
\]

\(^{13}\) Here we defined \( \text{sign}(n = 0) = -1 \).
7.1.2. Reduction of gauge theories. We can also consider the limit of small $\tau$ for gauge theories. We will not review this in detail here, and only mention the salient features. Up to the divergent factor appearing in (7.1), and its generalization when flavor fugacities are present, the matrix model for the index reduces to the matrix model [55, 56] used to compute $S_3^b$ partition function of the dimensionally reduced theory [23, 57–59]. Two comments are in order. First, the theories in four dimensions might have classical symmetries which are anomalous in the quantum theory. When reducing the theory on a circle a superpotential is produced which explicitly breaks these symmetries [23]. In the partition this manifests itself as a lack of real mass parameter for the symmetry which is anomalous in four dimensions. These superpotentials are extremely important to understand what physics IR dualities in four dimensions reduce to in three dimensions. Second, in certain cases [15, 50] the three-dimensional partition function in (7.1) is by itself divergent. Such examples include reductions of $SO(N)$ gauge theories with $\mathcal{N} = 1$ supersymmetry and $SU(N)$ gauge theories with $\mathcal{N} = 4$ supersymmetry. We refer the reader to contribution [60] for details of the $S_3^b$ partition function.

7.2. Large $\tau$ limit, $S^3 \rightarrow 0$

In this limit the radius of $S^3$ is much smaller than the radius of the circle and we effectively compactify the theory to quantum mechanics on a circle. The supersymmetric index in this limit computes the usual Witten index of the resulting quantum mechanics, that is the number of supersymmetric vacua. More concretely,

$$Z_{S^3 \times S^1} \rightarrow \infty \rightarrow e^{-\tau E_{\text{Casimir}}} \#_{\text{vacua}}.$$ \hspace{1cm} (7.18)

In particular since in the index we strip off the Casimir energy contribution it computes in the limit just the number of supersymmetric vacua. However, often a given theory might have a moduli space of vacua and the limit will diverge. In some examples we can keep some of the flavor fugacities which will regulate this divergence and give a finite result.

For $\tau$ large$^{14}$,

$$p, q \rightarrow 0.$$ \hspace{1cm} (7.19)

We assume implicitly that the index we obtain is finite in the limit because we have enough flavor fugacities to lift the degeneracy of the moduli space (this is not always possible). The fugacities $p$ and $q$ couple to charges $j_2 \pm j_1 + \frac{1}{2}r$. Setting these fugacities to zero is well defined if for all states contributing to the index $j_2 \pm j_1 + \frac{1}{2}r \geq 0$. Let us assume that this is the case and soon we will discuss several examples. Then, the states which contribute to the index satisfy

$$j_2 \pm j_1 + \frac{1}{2}r = 0, \quad \rightarrow \quad j_1 = 0, \quad j_2 = -\frac{1}{2}r.$$ \hspace{1cm} (7.20)

Moreover, since states contributing to the index satisfy $E - 2j_2 - \frac{3}{2}r = 0$ we also get that $E = \frac{1}{2}r$. Now, from unitarity,

$$E \pm 2j_1 + \frac{3}{2}r \geq 0, \quad E \pm 2j_2 - \frac{3}{2}r \geq 0,$$ \hspace{1cm} (7.21)

which imply that the states contributing to the limit we discuss have all charges vanishing,

$$E = r = j_1 = j_2 = 0.$$ \hspace{1cm} (7.22)

$^{14}$This limit has also been considered in [44].
Such states parametrize vacua of the model, i.e. the moduli space, as expected. Again, for the index to be well defined we will have to keep some of the flavor fugacities under which the operators contributing to the limit are charged.

Let us discuss the limit for a free chiral field. The limit is well defined if the R-charge is between zero and two. For R-charge vanishing the index is \( \frac{1}{r^2} \) where \( u \) is fugacity for the \( U(1) \) symmetry rotating the chiral. The index is given just by powers of the scalar component. This is the case when we can give a vacuum expectation value to the scalar parametrizing the moduli space, which will also break the \( U(1)_u \) symmetry. For \( r > 0 \) but less than two the index is 1. For \( r = 2 \) it becomes \( 1 - u^{-1} \). Note that R-charge two is outside the unitarity bounds for free chiral and thus there is no physical meaning for this result. However, such an R-charge would be acceptable for a gauge non-invariant chiral matter field in gauge theory.

We now give a more interesting example. Pure \( SU(N) \) SYM has \( N \) vacua, however it also has a discrete R symmetry. To define the index we need continuous R symmetry and thus we will not discuss this example but rather turn on flavors. Consider \( SU(N) \) SQCD with \( N_f \) flavors. The standard choice of anomaly R-charge is \( \frac{N_f-N}{N} \) for all the matter fields. This choice keeps all the flavor symmetry manifest. Our limit is well defined here. The limit of \( p, q \to 0 \) in this case is trivial, the index is 1 meaning that only the vacuum in the origin of field space satisfies (7.22). However, we can change the choice of R-charges keeping the condition for R-charges to be anomaly free,

\[
\sum R_i + \sum \tilde{R}_i = 2N_f - 2N. \tag{7.23}
\]

For example, let us assign \( N \) quarks and \( N \) anti-quarks R-charge zero, and the remaining matter R-charge one. The anomaly free condition above is satisfied. Taking our limit the index becomes

\[
\mathcal{I}^{(N)}(\{t, \tilde{t}\}_i) = \frac{1}{N!} \int \prod_{i=1}^{N-1} \frac{dz_i}{2\pi i z_i} \prod_{i \neq j} (1 - z_i/z_j) \prod_{j=1}^{N} \frac{1}{1 - t_i z_j} \frac{1}{1 - \tilde{t}_i z_j}. \tag{7.24}
\]

Note that \( N_f \) does not appear here anymore and there is no condition on fugacities \( t_i, \tilde{t}_i \). This integral can be easily computed to give

\[
\mathcal{I}^{(N)}(\{t, \tilde{t}\}_i) = \bigg( 1 - \prod_{i=1}^{N} t_i \tilde{t}_i \bigg) \frac{1}{1 - \prod_{i=1}^{N} t_i} \frac{1}{1 - \prod_{i=1}^{N} (1 - t_i \tilde{t}_j)}. \tag{7.25}
\]

This can be easily understood. The product is the product over the mesonic operators surviving the limit parametrizing a slice of the moduli space. The second and third terms are the baryon and the anti-baryon. The first term is an obvious constraint on this moduli space. We see that the index captures nearly a slice of the moduli space of the theory. This is equivalent to the so called Hilbert series of this slice (for discussion of Hilbert series see for example \([61, 62]\)). We can ask how this limit behaves under Seiberg duality. On side B of the duality we will have \( SU(N_f - N) \) theory with \( N_f \) quarks/anti-quarks and gauge singlets dual to the mesons. The dual quarks in this case have again R-charges zero and one in our case, now \( N \) have R-charge 1 and \( N_f - N \) R-charge zero. The mesons which survive the limit have R-charge zero and R-charges two. Note that as we said above the latter cannot be physical because of the violation of unitarity bounds. The index of the dual theory is

\[
\prod_{j=1}^{N_f} \frac{1}{1 - t_j \tilde{t}_{ij}} \prod_{i=N+1}^{N} (1 - t_i^{-1} \tilde{t}_j^{-1}) \times \mathcal{I}^{(N_f-N)}(\{ \prod_{k=1}^{N_f} \frac{t_k^{N_f}}{t_i}, \prod_{i=1}^{N_f} \tilde{t}_i^{N_f} \}_{i=1}^{N_f}) \tag{7.26}
\]

Note that \( N_f \) does not appear here anymore and there is no condition on fugacities \( t_i, \tilde{t}_i \).
We can now plug in the result from (7.25) for \( I(N_f - N) \) and

\[
\prod_{k=1}^{N_f} t_k \prod_{k=1}^{N} t_k = 1, \tag{7.27}
\]

from anomaly cancelation to obtain that the above is equal to

\[
\left[ \prod_{i,j=1}^{N} \frac{1}{1 - t_i t_j} \prod_{i=N+1}^{N_f} (1 - t_i^{-1} t_j^{-1}) \right] \times \left( \prod_{i=N+1}^{N_f} t_i^{-1} t_j^{-1} \prod_{i,j=1}^{N_f} \tilde{t}_i \tilde{t}_j \prod_{i=1}^{N_f} \frac{1}{1 - t_i^{-1} t_j^{-1}} \right)
\]

\[
\prod_{k=1}^{N_f} t_k \prod_{k=1}^{N} \tilde{t}_k \prod_{i,j=1}^{N_f} (1 - t_i t_j) \prod_{i=1}^{N_f} \frac{1}{1 - t_i^{-1} t_j^{-1}}, \tag{7.28}
\]

in agreement with (7.25).

Note that naively it is important in the gauge theory for the limit to be well defined to have the R-charges of all the chiral fields to be between zero and two. However, even if some of the charges of chirals are outside of this region the limit might be well behaved. Consider for example giving R-charge zero to \( N_f + N \) chiral fields and R-charge two to \( N_f - N \). This is an anomaly free R-charge. Assuming that \( N_f + N \) is even, we might split the choice above equally between the quarks and anti-quarks, that is giving R-charge zero to \( N_f + N/2 \) flavors. In such a case the R-charges of the dual theory are one for \( N_f + N/2 \) flavors and \( -1 \) for \( N_f - N/2 \). Thus although naively the limit of the matter is singular from the duality we know that the limit for the gauge invariant operators has to be well defined.

In summary, the \( p, q \) to zero limit captures protected information associated to a certain submanifold of the moduli space of the theory. The precise submanifold is determined by the choice of the R-charges. One can in principle consider other limits on fugacities coupling to combinations of charges which for a given model are non-negative for states contributing to the index. However since the index gets contributions from fermions and bosons in conjugate representations the index would usually get contributions from both negatively and positively charged objects unless the limit is for an R-symmetry. In certain cases the information captured in this limit is equivalent to the Hilbert series of the moduli space. An example is given by [63] the limit of the index of \( \mathcal{N} = 2 \) theories corresponding to genus zero Riemann surfaces in class \( S \) terminology [64]. See also [65–67] for the 3d variants of such limits.

7.3. Poles and residues

The index is a meromorphic function of the fugacities and in general has numerous poles. Let us assume the index has a behavior of the following form,

\[
\mathcal{I}_0(a_1, a_2, \cdots) = \mathcal{I}_1(a_1, a_2, \cdots) \frac{1}{1 - a_1}, \tag{7.29}
\]

where \( a_i \) are some fugacities and we assume \( \mathcal{I}_1 \) has no zeros or poles at \( a_1 = 1 \). From the trace interpretation of the index we deduce that there is a bosonic operator in the theory, \( \mathcal{O} \), with charges such that it contributes with weight \( a_1 \) to the index. Moreover, any power of this operator also contributes to the index. The pole at \( a_1 = 1 \) corresponds to computing the index
while giving weight 1 to $\mathcal{O}$. Putting it differently, we consider turning on only fugacities for symmetries consistent with giving a vacuum expectation value for $\mathcal{O}$. It is thus natural to interpret the residue of the pole as the index of the theory obtained as the IR fixed point of an RG flow triggered by vacuum expectation value for $\mathcal{O}$.

We have encountered an example of the effect of vacuum expectation values while discussing Higgsing in section 4. Let us give several additional examples. First let us consider a sigma model with two chiral fields and a superpotential

$$W = \Phi_1 \Phi_2^2.$$ 

(7.30)

We have one $U(1)_a$ global symmetry preserved by the superpotential and we choose $\Phi_1$ to have charge $-2$ and $\Phi_2$ has charge $+1$. We also assign R-charge $2$ to $\Phi_1$ and $1 - R$ to $\Phi_2$. The index of the model is given by

$$I(a) = \Gamma((pq)^a - 2) \Gamma((pq)^{1 - R} a).$$

(7.31)

Note that the chiral ring here has the relations

$$\Phi_1 \Phi_2 \sim 0, \quad \Phi_2^2 \sim 0.$$ 

(7.32)

In particular, as we already discussed not all powers of the scalar component of $\Phi_2$ contribute to the index, but any power of the scalar from $\Phi_1$ appears. This index has many poles one of which is at $a = (pq)^2$. The operator which leads to the divergence is the scalar component of $\Phi_1$. The residue is given by

$$I(a) \sim \frac{1}{1 - (pq)^2 a - 2} \Gamma((pq)^2) I^{-1} + O(1).$$

(7.33)

The index of $\Phi_2$ becomes $\Gamma((pq)^{1/2} - 1)$, which is the index of a massive fields since the vacuum expectation value for $\Phi_1$ generates a mass term for $\Phi_2$. The index of field $\Phi_1$ stripping off the divergence is $\Gamma(1') = \frac{1}{(pq)^{1/2}} = \mathcal{I}_v^{-1}$ which is the index of the Nambu–Goldstone boson corresponding to the broken $U(1)_a$ symmetry. The residue is thus just given by the index of the Nambu–Goldstone boson as expected as the theory is empty in the IR. It is thus natural to write the general relation

$$\text{Figure 2. An } SU(2) \times SU(2) \text{ quiver gauge theory.}$$

15 One way to see that this is the contribution of the Goldstone boson is to see that this exactly cancels the contribution of the vector multiplets when studying the Higgs mechanism as in section four.
We consider now a more involved example of a gauge theory. The theory we discuss is $SU(2) \times SU(2)$ quiver gauge theory of figure 2. The superpotential is
\[
W = Q_1^a \Phi_1 \tilde{Q}_1 + Q_2^a \Phi_2 \tilde{Q}_2.
\]
We will assign R-charge zero to the $Q_i$ and $\tilde{Q}_i$ fields and R-charge two to $\Phi_j$. This model has three abelian global symmetries which we will denote by $U(1)_T \times U(1)_X \times U(1)_Y$. The different fields have the charges specified in table 3.

We can consider giving a vacuum expectation value to a baryonic operator of the form $B Q = \epsilon \cdot Q_1^a$. This will Higgs one of the $SU(2)$ gauge groups and reduce the rank of the flavor group by one. Let us analyze how this comes about from the index. The index of the model is given by
\[
I = \frac{1}{1-a^2} \mathcal{I}_\mu(a_2, \cdots) \mathcal{I}_{\text{Num.-Gold.}} + O(1).
\]

We can consider giving a vacuum expectation value to a baryonic operator of the form $B Q = \epsilon \cdot Q_1^a$. This will Higgs one of the $SU(2)$ gauge groups and reduce the rank of the flavor group by one. Let us analyze how this comes about from the index. The index of the model is given by
\[
I = \frac{1}{1-a_1} \mathcal{I}_\mu(a_2, \cdots) \mathcal{I}_{\text{Num.-Gold.}} + O(1).
\]

We can consider giving a vacuum expectation value to a baryonic operator of the form $B Q = \epsilon \cdot Q_1^a$. This will Higgs one of the $SU(2)$ gauge groups and reduce the rank of the flavor group by one. Let us analyze how this comes about from the index. The index of the model is given by
\[
I = \frac{1}{1-a_1} \mathcal{I}_\mu(a_2, \cdots) \mathcal{I}_{\text{Num.-Gold.}} + O(1).
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\[
I = \frac{1}{1-a_1} \mathcal{I}_\mu(a_2, \cdots) \mathcal{I}_{\text{Num.-Gold.}} + O(1).
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We can consider giving a vacuum expectation value to a baryonic operator of the form $B Q = \epsilon \cdot Q_1^a$. This will Higgs one of the $SU(2)$ gauge groups and reduce the rank of the flavor group by one. Let us analyze how this comes about from the index. The index of the model is given by
\[
I = \frac{1}{1-a_1} \mathcal{I}_\mu(a_2, \cdots) \mathcal{I}_{\text{Num.-Gold.}} + O(1).
\]
\[
2\kappa \text{Res}_{z=+1} \frac{dz}{z^{u_1} u_2 z^{v_1} u_2} T = \prod_{i=3}^4 \Gamma\left(\frac{u_i}{u_1}\right) \Gamma\left(\frac{u_i}{u_2}\right) \prod_{i=1}^4 \Gamma\left(T^3 v_i u_1\right) \Gamma\left(T^2 v_i u_2\right) \\
\times \kappa \int \frac{dz_2}{4\pi i z_2} \Gamma\left(z_2^{-1}\right) \Gamma\left(\frac{u_1}{u_2}\right) \prod_{i=3}^4 \Gamma\left(\frac{u_i}{u_2}\right) \left(\frac{1}{\sqrt{u_1 u_2}} z_2^{\pm 1}\right) \prod_{i=1}^4 \Gamma\left(T^3 Y^2 v_i^{-1} \sqrt{u_1 u_2} z_2^{\pm 1}\right).
\]

(7.38)

This is the index of \( \mathcal{N} = 1 SU(2) \) SCFT with four flavors and additional singlet fields coupled to the charged matter through a superpotential. This is exactly the matter content one would expect after giving a vacuum expectation value to baryon \( B_Q \).

More general poles correspond to turning on vacuum expectation values to derivatives of operators and thus break explicitly Lorentz invariance. The theory in the IR is expected to have co-dimension two defects. The residue computes then an index of a theory in presence of such defects. Such flows and corresponding defects were discussed in the \( \mathcal{N} = 2 \) context in \[35\] and in \( \mathcal{N} = 1 \) context in \[68\] (see \[69\] for a review). The IR theory here has 4\( d \) degrees of freedom coupled to 2\( d \) ones, and the index is often expressible as some difference operator, shifting flavor fugacities by general powers of \( p \) and \( q \), acting on the four-dimensional index \[35, 37, 68, 70\]. This is reminiscent of the observation below (6.5).

### 7.4. Large \( N \) limit

The matrix models of indices of gauge theories can be simplified and explicitly evaluated in the limit of large number of colors using large \( N \) matrix model techniques (see e.g. \[4, 21, 71\]). Let us here give a general result for the large \( N \) limit of an index of a quiver gauge theory with \( U(N) \) gauge groups. We follow here the discussion and notations of \[7\].

We consider a quiver theory with gauge group \( \prod_{a=1}^N U(N_a) \). Let \( \{e^{\alpha_{ia}}\}_{i=1}^{N_a} \) denote the \( N_a \) eigenvalues of \( u_a \). Then the matrix model integral (2.11) is,

\[
\mathcal{I}(x) = \int \prod_{a,i} [d\alpha_{ia}] \exp\left\{-\sum_{a \neq b} V^a_b(\alpha_{ai} - \alpha_{ib})\right\}.
\]

(7.39)

Here, the potential \( V \) is the following function

\[
V^a_b(\theta) = \delta^a_b (\ln 2) + \sum_{n=1}^\infty \frac{1}{n}[\delta^a_b - \tilde{r}^a_b(x^n)] \cos n \theta,
\]

(4.0)

where \( \tilde{r}^a_b(x) \) is the total single letter index in the representation \( r^a \otimes r_b \) and \( x \) stands for all the fugacities we can turn on. Writing the density of the eigenvalues \( \{e^{\alpha_{ia}}\} \) at the point \( \theta \) on the circle as \( \rho_a(\theta) \), we reduce it to the functional integral problem,

\[
\mathcal{I}(x) = \int \prod_a [d\rho_a] \exp\{-\int d\theta_1 d\theta_2 \sum_{a,b} n_{a,b} n_{a,b} \rho_a(\theta_1) V^a_b(\theta_1 - \theta_2) \rho_b^*(\theta_2)\}.
\]

(7.41)

For large \( N \), we can evaluate this expression with the saddle point approximation,

\[
\mathcal{I}(x) = \prod_k \frac{1}{\det(1 - i(x^k))}.
\]

For \( SU(N) \) gauge groups instead of \( U(N) \), the result is modified as follows,
\[ I(x) = \prod_k \frac{e^{-\frac{i}{n} \text{tr} i(x^k)}}{\det(1 - i(x^k))}. \]  

(7.42)

Here \( i(x) \) is the matrix with entries \( i_a^b(x) \).

The single-trace partition function can be obtained from the full partition function,

\[ I_{\text{s.t.}} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log I(x^n) \]  

(7.43)

\[ = -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log[\det(1 - i(x^k))] - \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \sum_{k=1}^{\infty} \text{tr} \frac{i(x^k)}{k} \]  

(7.44)

\[ = -\sum_{k=1}^{\infty} \frac{\varphi(k)}{k} \log[\det(1 - i(x^k))] - \text{tr} \ i(x). \]  

(7.45)

The second term in the summation would be absent for the \( U(N) \) gauge theories. Here \( \mu(n) \) is the Möbius function (\( \mu(1) \equiv 1, \mu(n) \equiv 0 \) if \( n \) has repeated prime factors and \( \mu(n) \equiv (-1)^k \) if \( n \) is the product of \( k \) distinct primes) and \( \varphi(n) \) is the Euler Phi function, defined as the number of positive integers less than \( n \) that are coprime to \( n \). We have used the properties

\[ \sum_{d|n} \mu(n/d) = \varphi(n), \quad \sum_{d|n} \mu(d) = \delta_{n,1}. \]  

(7.46)

Indices in the large \( N \) limit can be used to check holographic dualities. For example the index of \( \mathcal{N} = 4 \) SYM in this limit can be matched with the spectrum of fields in \( AdS_5 \) computed in supergravity [4]. The large \( N \) indices [7] of a variety of \( Y_{p,q} \) models [72] where also matched with the holographic duals [73]. In general the field theory expressions in the large \( N \) limit are rather simple though the dual holographic computation can be involved, see [73]. For example, the index of \( \mathcal{N} = 2 \) class \( S \) theories [64] of genus \( g \) is explicitly known in large \( N \) limit [63] though that simple result was not yet reproduced from the gravity side [74].

### 8. Other topics and open problems

There are many other interesting related topics that we could review here. We conclude with a brief mention of a few of them:

- **Holomorphic blocks**—The localization procedure leading directly to the trace-formula formulation of the index is the so called Coulomb branch localization. The computation reduces to a matrix integral over the zero modes of the vector field in the direction of \( \mathbb{S}^1_\tau \).

  The name comes from the fact that these components upon reduction to three dimensions become scalar components in the vector multiplet and parametrize the Coulomb branch. However, there is a different localization procedure one can employ [75–77].

  The dimensional reduction of this procedure to three dimensions leads to the so called Higgs branch localization form for the index [78–80]. In this localization procedure the index can be written as a finite sum over vortex/anti-vortex partition functions which are effectively partition functions on \( \mathbb{C} \times T^2 \). This ‘holomorphic block’ factorization of the partition function is extremely powerful since it connects together apriori unrelated partition functions. By gluing differently the blocks one can obtain various geometry and thus relate the supersymmetric index for example to \( \mathbb{S}^2 \times T^2 \) partition function. Let us
mention here only the simplest example of such a factorization in the case of a free chiral field. Here we have
\[ I^{(R)}(a) = \Gamma((pq)^\frac{1}{2}a; p, q) = \Gamma((pq)^\frac{1}{2}a; p, pq)\Gamma((pq)^\frac{1}{2}q; q, pq). \]  
(8.1)

There are many interesting results yet to be uncovered following this direction.

- **Lens space index**—As was mentioned in the introduction the supersymmetric index is a special case of a sequence of partition functions, the lens space indices \( \mathbb{S}^3/\mathbb{Z}_r \times \mathbb{S}^1 \) [81]. As a counting problem the lens index is computed as follows. Since the geometry involves an orbifold projection the lens index receives contributions from local operators consistent with the action of the orbifold. Let us call this sector the ‘untwisted’ one. Let us again here give just an example of the lens index of a free chiral field in the ‘untwisted’ sector,
\[ I^{(R)}_r(a) = \Gamma((pq)^\frac{1}{2}a; p, pq)\Gamma((pq)^\frac{1}{2}q; q, pq). \]  
(8.2)

On the other hand, for \( r > 1 \) the lens space \( \mathbb{S}^3/\mathbb{Z}_r \) has a non-contractable torsion cycle, and upon quantizing the theory on this space one should consider configurations wrapping this cycle. This leads to a finite number, since the cycle is torsion, of ‘twisted’ sectors which receive contributions from extended objects in the theory. Thus although the supersymmetric index, \( r = 1 \), gets contributions only from local operators, the lens index captures a much larger variety of objects. Moreover, the spectrum of the non-local objects is sensitive to the global structure of the gauge groups [82] and not just to the Lie algebras making lens indices a more refined characteristic of the physics. Taking the limit of large \( r \) the non-trivial cycle of the lens space shrinks to zero size and \( \mathbb{S}^3/\mathbb{Z}_r \) becomes \( \mathbb{S}^2 \). In this limit the lens index in four dimensions reduces to the supersymmetric index in three dimensions. The finite sum over the twisted sectors becomes an infinite sum over monopoles sectors in three dimensions. Although there are several works studying the lens index it has been largely neglected and there are many avenues for farther research.

- **Relations to integrable models**—Finally let us mention that the supersymmetric index is closely related to quantum mechanical integrable systems. These relations come in different forms. For example the (lens) index itself can be related to partition function of two dimensional lattice integrable models [83, 84]. On the other hand, as we discussed in the previous sections, computing indices of theories in presence of surface defects amounts to acting on indices without defects with difference operators [35, 68, 85]. Such difference operators are Hamiltonians for well known Ruijsenaars–Schneider integrable systems when the theories are \( \mathcal{N} = 2 \) [35, 70, 86–88], and give rise to novel integrable models when one has \( \mathcal{N} = 1 \) supersymmetry [68, 89, 90]. These relations deserve a much more thorough investigation.

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References

[1] Pestun V and Zabzine M 2017 Localization techniques in quantum field theories J. Phys. A: Math. Theor. 50 440301
[2] Pestun V 2012 Localization of gauge theory on a four-sphere and supersymmetric Wilson loops Commun. Math. Phys. 313 71
[3] Hosomichi K 2017 $\mathcal{N} = 2$ SUSY gauge theories on $S^4$ J. Phys. A: Math. Theor. 50 443010
[4] Kinney J, Maldaena J M, Minwalla S and Raju S 2007 An index for 4 dimensional super conformal theories Commun. Math. Phys. 275 209–54
[5] Romelsberger C 2006 Counting chiral primaries in $N = 1, d = 4$ superconformal field theories Nucl. Phys. B 747 329–53
[6] Romelsberger C 2007 Calculating the superconformal index and seiberg duality (arXiv:0707.3702 [hep-th])
[7] Gadde A, Rastelli L, Razamat S S and Yan W 2011 On the superconformal index of $N = 1$ IR fixed points: a holographic check J. High Energy Phys. JHEP03(2011)041
[8] Pestun V 2012 Charging the superconformal index J. High Energy Phys. JHEP01(2012)116
[9] Festuccia G and Seiberg N 2011 Rigid supersymmetric theories in curved superspace J. High Energy Phys. JHEP06(2011)114
[10] Sen D 1987 Supersymmetry in the space-time $R \times S^3$ Nucl. Phys. B 284 191–5
[11] Dumitrescu T 2017 An introduction to supersymmetric field theories in curved space J. Phys. A: Math. Theor. 50 443005
[12] Closset C and Shamir I 2014 The $\mathcal{N} = 1$ chiral multiplet on $T^2 \times S^2$ and supersymmetric localization J. High Energy Phys. JHEP03(2014)123
[13] Assel B, Cassani D, Martelli D 2015 Localization on Hopf surfaces J. High Energy Phys. JHEP08(2015)123
[14] Kim H-C and Kim S 2012 M5-branes from gauge theories on the 5-sphere (arXiv:1206.6339 [hep-th])
[15] Ardehali A A 2015 High-temperature asymptotics of supersymmetric partition functions (arXiv:1512.03376 [hep-th])
[16] Closset C and Shamir I 2014 The $\mathcal{N} = 1$ chiral multiplet on $\mathbb{T}^2 \times \mathbb{S}^2$ and supersymmetric localization J. High Energy Phys. JHEP03(2014)040
[17] Assel B, Cassani D and Martelli D 2014 Localization on Hopf surfaces J. High Energy Phys. JHEP08(2014)123
[18] Kim H-C and Kim S 2012 M5-branes from gauge theories on the 5-sphere (arXiv:1206.6339 [hep-th])
[19] Ardehali A A 2015 High-temperature asymptotics of supersymmetric partition functions (arXiv:1512.03376 [hep-th])
[20] Benvenuti S, Feng B, Hanany A and He Y-H 2007 Counting BPS operators in gauge theories: quivers, syzygies and plethystics J. High Energy Phys. JHEP11(2007)050
[21] Aharony O, Marsano J, Minwalla S, Papadodimas K and Van Raamsdonk M 2004 The Hagedorn / deconfinement phase transition in weakly coupled large $N$ gauge theories Adv. Theor. Math. Phys. 8 603–96
[22] Gadde A, Pomoni E, Rastelli L and Razamat S S 2010 S-duality and 2d Topological QFT J. High Energy Phys. JHEP03(2010)032
[23] Aharony O, Razamat S S, Seiberg N and Willett B 2013 3d dualities from 4d dualities J. High Energy Phys. JHEP07(2013)149
[24] Seiberg N 1995 Electric—magnetic duality in supersymmetric non-Abelian gauge theories Nucl. Phys. B 435 129–46
[25] Spiridonov V 2001 On the elliptic beta function Russ. Math. Surv. 56 185
[26] Spiridonov V 2008 Essays on the theory of elliptic hypergeometric functions Usp. Mat. Nauk 63 3–72
[27] Rains E M 2003 Transformations of elliptic hypergeometric integrals (math/0309252)
[28] Spiridonov V and Vartanov G 2012 Elliptic hypergeometric integrals and ’t Hooft anomaly matching conditions J. High Energy Phys. JHEP06(2012)016
[29] Spiridonov V P and Warnaar S O 2006 Inversions of integral operators and elliptic beta integrals on root systems Adv. Math. 207 91–132
[30] Beem C and Gadde A 2014 The $N = 1$ superconformal index for class $S$ fixed points J. High Energy Phys. JHEP04(2014)036
[31] Green D, Komargodski Z, Seiberg N, Tachikawa Y and Wecht B 2010 Exactly marginal deformations and global symmetries J. High Energy Phys. JHEP06(2010)106
[32] van de Bult F J 2009 An elliptic hypergeometric integral with $W(F_4)$ symmetry (arXiv:0909.4793)
[33] Spiridonov V P and Vartanov G S 2010 Superconformal indices for $\hat{N} = 1$ theories with multiple duals Nucl. Phys. B 824 192–216
[34] Dimofte T and Gaiotto D 2012 An E7 surprise J. High Energy Phys. JHEP10(2012)129
[35] Gaiotto D, Rastelli L and Razamat S S 2013 Bootstrapping the superconformal index with surface defects J. High Energy Phys. JHEP01(2013)022
[36] Gaiotto D and Kim H-C 2010 Exact results for Wilson loops in superconformal Chern–Simons theories with matter J. High Energy Phys. JHEP03(2010)009
[37] Intriligator K A, Seiberg N and Shenker S H 1995 Proposal for a simple model of dynamical SUSY breaking Phys. Lett. B 351 15–21
[38] Intriligator K A and Wecht B 2004 RG fixed points and flows in SQCD with adjoints Nucl. Phys. B 677 223–72
[39] Kutasov D and Lin J 2014 Exceptional $N = 1$ duality (arXiv:1401.4168 [hep-th])
[40] Kutasov D and Lin J 2014 $N = 1$ duality and the superconformal index (arXiv:1402.5411 [hep-th])
[41] Spiridonov V P and Vartanov G S 2009 Elliptic hypergeometry of supersymmetric dualities (arXiv:0910.5944 [hep-th])
[42] Spiridonov V and Vartanov G 2011 Elliptic hypergeometry of supersymmetric dualities II. Orthogonal groups, knots and vortices (arXiv:1107.5788 [hep-th])
[43] Spiridonov V 2010 Elliptic hypergeometric terms (arXiv:1003.4491 [math.CA])
[44] Spiridonov V 2005 Classical elliptic hypergeometric functions and their applications Rokko Lecture in Mathematics vol 18 (Kobe: Kobe University) pp 253–87
[45] Spiridonov V 2007 Elliptic hypergeometric functions (arXiv:0704.3099)
[46] Closset C, Dumitrescu T T, Festuccia G and Komargodski Z 2013 Supersymmetric field theories on three-manifolds J. High Energy Phys. JHEP05(2013)017
[47] Aharony O, Razamat S S, Seiberg N and Willett B 2013 3d dualities for orthogonal groups J. High Energy Phys. JHEP08(2013)099
[48] Intriligator K A, Seiberg N and Shenker S H 1995 Proposal for a simple model of dynamical SUSY breaking Phys. Lett. B 342 152–4
[49] Vartanov G 2011 On the ISS model of dynamical SUSY breaking Phys. Lett. B 696 288–90
[50] Imamura Y and Yokoyama D 2012 $N = 2$ supersymmetric theories on squashed three-sphere Phys. Rev. D 85 025015
[51] Felder G and Varchenko A 1999 The elliptic gamma function and $SL(3, Z) \times Z^3$ (arXiv mathematics e-prints math/9907061)
[52] Kapustin A, Willett B and Yaakov I 2010 Exact results for Wilson loops in superconformal Chern–Simons theories with matter J. High Energy Phys. JHEP03(2010)009
[53] Hama N, Hosomichi K and Lee S 2011 SUSY Gauge theories on squashed three-spheres J. High Energy Phys. JHEP05(2011)014
[54] Dolan F, Spiridonov V and Vartanov G 2011 From 4d superconformal indices to 3d partition functions (arXiv:1104.1787 [hep-th])
[55] Imamura Y 2011 Relation between the 4d superconformal index and the $S^3$ partition function (arXiv:1104.4482 [hep-th])
[56] Gadde A and Yan W 2012 Reducing the 4d Index to the $S^3$ partition function J. High Energy Phys. JHEP12(2012)003
[57] Willett B 2017 Localization on three-dimensional manifolds J. Phys. A: Math. Theor. 50 443006
[58] Gray J, Hanany A, He Y-H, Jejjala V and Mekareeya N 2008 SQCD: a geometric Apercu J. High Energy Phys. JHEP05(2008)099
[59] Hanany A and Mekareeya N 2008 Counting Gauge invariant operators in SQCD with classical Gauge groups J. High Energy Phys. JHEP10(2008)012
Gadde A, Rastelli L, Razamat S S and Yan W 2013 Gauge theories and Macdonald polynomials
Commun. Math. Phys. 319 147–93

Gaiotto D 2009 $N = 2$ dualities (arXiv:0904.2715 [hep-th])

Razamat S S and Willett B 2014 Down the rabbit hole with theories of class S (arXiv:1403.6107 [hep-th])

Cremonesi S 2015 The Hilbert series of 3d $\mathcal{N} = 2$ Yang–Mills theories with vectorlike matter
J. Phys. A: Math. Theor. 48 455401

Hanany A, Hwang C, Kim H, Park J and Seong R-K 2015 Hilbert series for theories with Aharony duals
J. High Energy Phys. JHEP11(2015)132

Razamat S S and Willett B 2014 Down the rabbit hole with theories of class S (arXiv:1403.6107 [hep-th])

Cremonesi S 2015 The Hilbert series of 3d $\mathcal{N} = 2$ Yang–Mills theories with vectorlike matter
J. Phys. A: Math. Theor. 48 455401

Gaiotto D 2009 $N = 2$ dualities (arXiv:0904.2715 [hep-th])

Razamat S S and Willett B 2014 Down the rabbit hole with theories of class S (arXiv:1403.6107 [hep-th])

Cremonesi S 2015 The Hilbert series of 3d $\mathcal{N} = 2$ Yang–Mills theories with vectorlike matter
J. Phys. A: Math. Theor. 48 455401

Gaiotto D and Razamat S S 2015 $N = 1$ theories of class $\mathcal{S}$
J. High Energy Phys. JHEP07(2015)073

Rastelli L and Razamat S S 2016 The Superconformal Index of Theories of Class $\mathcal{S}$ New Dualities of Supersymmetric Gauge Theories ed J Teschner (Cham: Springer) pp 261–305

Gaiotto D and Maldacena J 2009 The gravity duals of $N = 2$ superconformal field theories
JHEP09(2009)064

Pasquetti S 2012 Factorisation of $N = 2$ theories on the squashed 3-sphere
J. High Energy Phys. JHEP04(2012)120

Beem C, Dimofte T and Pasquetti S 2012 Holomorphic blocks in three dimensions (arXiv:1211.1986 [hep-th])

Benini F and Peelaers W 2014 Higgs branch localization of $\mathcal{N} = 1$ theories on $S^3 \times S^1$
J. High Energy Phys. JHEP08(2014)060

Bullimore M, Fluder M, Hollands L and Richmond P 2014 The superconformal index and an elliptic algebra of surface defects (arXiv:1401.3379 [hep-th])

Razamat S S and Willett B 2015 Global properties of supersymmetric theories and the lens space
Commun. Math. Phys. 334 661–96

Benini F and Peelaers W 2014 Higgs branch localization in three dimensions
J. High Energy Phys. JHEP05(2014)030

Benini F, Nishioka T and Yamazaki M 2011 4d index to 3d index and 2d TQFT (arXiv:1109.0283 [hep-th])

Razamat S S and Willett B 2015 Global properties of supersymmetric theories and the lens space
Commun. Math. Phys. 334 661–96

Spiridonov V P 2012 Elliptic beta integrals and solvable models of statistical mechanics Contemp. Math. 563 181–211

Yamazaki M 2014 New integrable models from the Gauge/YBE Correspondence J. Stat. Phys. 154 895

Gaiotto D and Kim H-C 2014 Surface defects and instanton partition functions (arXiv:1412.2781 [hep-th])

Bullimore M, Fluder M, Hollands L and Richmond P 2014 The superconformal index and an elliptic algebra of surface defects (arXiv:1401.3379 [hep-th])

Alday L F, Bullimore M, Fluder M and Hollands L 2013 Surface defects, the superconformal index and q-deformed Yang–Mills (arXiv:1303.4466 [hep-th])

Alday L F, Bullimore M, Fluder M and Hollands L 2013 Surface defects, the superconformal index and q-deformed Yang–Mills (arXiv:1303.4466 [hep-th])

Benvenuti S, Franco S, Hanany A, Martelli D and Sparks J 2005 An infinite family of superconformal quiver gauge theories with Sasaki–Einstein duals
J. High Energy Phys. JHEP06(2005)064

Eager R, Schmude J and Tachikawa Y 2014 Superconformal indices, Sasaki–Einstein manifolds and cyclic homologies
Adv. Theor. Math. Phys. 18 129–75

Gaiotto D and Razamat S S 2015 $N = 1$ theories of class $\mathcal{S}$
J. High Energy Phys. JHEP07(2015)073

Yoshida Y 2014 Factorization of 4d $\mathcal{N} = 1$ superconformal index (arXiv:1403.0891 [hep-th])

Nieri F and Pasquetti S 2012 Factorisation of $N = 2$ theories on the squashed 3-sphere J. High Energy Phys. JHEP04(2012)120

Beem C, Dimofte T and Pasquetti S 2012 Holomorphic blocks in three dimensions (arXiv:1211.1986 [hep-th])

Benini F and Peelaers W 2014 Higgs branch localization in three dimensions
J. High Energy Phys. JHEP05(2014)030

Benini F, Nishioka T and Yamazaki M 2011 4d index to 3d index and 2d TQFT (arXiv:1109.0283 [hep-th])

Razamat S S and Willett B 2015 Global properties of supersymmetric theories and the lens space
Commun. Math. Phys. 334 661–96

Spiridonov V P 2012 Elliptic beta integrals and solvable models of statistical mechanics Contemp. Math. 563 181–211

Yamazaki M 2014 New integrable models from the Gauge/YBE Correspondence J. Stat. Phys. 154 895

Gaiotto D and Kim H-C 2014 Surface defects and instanton partition functions (arXiv:1412.2781 [hep-th])

Bullimore M, Fluder M, Hollands L and Richmond P 2014 The superconformal index and an elliptic algebra of surface defects (arXiv:1401.3379 [hep-th])

Alday L F, Bullimore M and Fluder M 2013 On $S$-duality of the superconformal index on lens spaces and 2d TQFT (arXiv:1301.7486 [hep-th])

Razamat S S and Yamazaki M 2013 $S$-duality and the $N = 2$ lens space index
J. High Energy Phys. JHEP1310(2013)048

Maruyoshi K and Yagi J 2016 Surface defects as transfer matrices PTEP 2016 113B01

Ito Y and Yoshida Y 2016 Superconformal index with surface defects for class $\mathcal{S}_k$ (arXiv:1606.01653 [hep-th])