Free vibration analysis on axially graded beam resting on variable Pasternak foundation

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Abstract. Free vibration analysis is conducted on axially functionally graded Euler-Bernoulli beam resting on variable Pasternak foundation. The material properties of the beam and the stiffness of the foundation are considered to be varying linearly along the axial direction. Two types of boundary conditions namely; clamped and simply supported are used in the analysis. The problem is formulated using Rayleigh-Ritz method and governing equations are derived with the help of Hamilton’s principle. The numerical results are generated for different material gradation parameter, foundation parameter and boundary conditions and the effect of these parameters on the free vibration behaviour of the beam is discussed.

Keywords - Functionally graded material, elastic foundation, free vibration, natural frequency

1. Introduction

Functionally graded material (FGM) structures have gained popularity due to their inherent advantages over traditional composite materials. Recently, researchers are investigating the behaviour of these structures in different conditions. Structures resting on elastic foundation is one such condition. Fallah and Aghdam [1] analytically solved the nonlinear dynamic problem of FGM beams using He’s variational method. Kanani et al. [2] conducted nonlinear vibration analysis on FGM beams using Galerkin method where the solution was obtained by variable iteration method. Calim [3, 4] carried out vibration analysis on FGM beams using complementary function method to study the influence of different parameters on its dynamic behaviour. Mirzabeigi and Madoliat [5] performed nonlinear dynamic analysis on FGM beams using energy method and Hamilton’s principle. The Hencky bar-chain model was used by Zhang et al. [6] to carry out buckling and free vibration of FGM beams resting on variable elastic foundation. Yas et al. [7] used differential quadrature method to investigate similar beams. Deng et al. [8] investigated double functionally graded beams using Hamilton’s principle and Wittrick-William algorithm. Avcar and Mohammed [9] carried out free vibration analysis on FGM beams resting on two parameter elastic foundation using classical beam theory and separation of variables technique. Mohamed et al. [10] examined the nonlinear dynamic characteristics of buckled FGM beam using differential-integral quadrature method where the solution of the governing equation was obtained by Newton’s method. Eisen [11] conducted vibration analysis on FGM beams subjected to moving mass of variable velocity using a modified finite element method. It was concluded that, the vibration can be controlled by adjusting the size of foundation parameters. Lohar et al. [12] conducted nonlinear forced vibration analysis using energy method and Hamilton’s principle. M. A. Kumbhalkar et. al. [13] conducted free and forced vibration numerical analysis of non-linear curved beam. Hadji and Bernard [14] used Hamilton’s principle to obtain Navier type analytical solutions of bending and free vibration of FGM beam. Eisen et al. [15] investigated symmetric and sigmoid FGM beams considering
the effect of a moving mass using Hamilton’s method and finite element method where Newmark method was used to obtain the solutions.

From the above literature it is evident that investigating the dynamic aspects of FGM beams on elastic foundation is building up ample interest among the researchers. As such, in this paper, axially functionally graded material beams resting on variable two parameter Pasternak elastic foundation are investigated for free vibration.

2. Mathematical Formulation
A functionally graded beam with axial material gradation resting on two parameter elastic foundation with its geometrical dimensions is shown in Figure 1.

**Mathematical Formulation**

A functionally graded beam with axial material gradation resting on two parameter elastic foundation with its geometrical dimensions is shown in Figure 1.

The material properties (elastic modulus and material density) are considered to be varying along the x direction in the beam as per the following expression,

\[ E(x) = E_0 \left(1 - \frac{x}{L}\right), \rho(x) = \rho_0 \left(1 - \frac{x}{L}\right) \]  

(1)

where, \(E_0\) and \(\rho_0\) are elastic modulus and material density at the left end of the beam respectively.

The beam is supported on an elastic foundation which contains a shear layer and a Winkler layer (group of linear springs). The stiffness of the shear layer (\(K_s\)) is kept constant whereas the stiffness of the Winkler layer (\(K_w\)) is varying linearly along the x-axis as per the following expression,

\[ K_w(x) = K_0 \left(1 + s \left(\frac{x}{L}\right)\right) \]  

(2)

where, \(K_0\) is the stiffness of the Winkler layer at the left end and \(s\) is the foundation variation parameter.

The displacement fields \(u(x)\) and \(v(y)\) for the beam are represented as linear combinations of unknown parameters \((d_i)\) and orthogonal admissible functions \((\alpha_i)\) and \((\beta_i)\) as per the Rayleigh-Ritz method.

The strain energy of the beam \((U)\) is the summation of the strain energy stored in the beam, strain energy stored in the shear layer and the strain energy stored in the Winkler layer, and given as,

\[ U = \frac{1}{2} A \int_0^L (u_x^2) E(x) dx + \frac{1}{2} A \int_0^L (v_{xx}^2) E(x) dx + \frac{1}{2} A \int_0^L K_w(x) v^2 dx + \frac{1}{2} A \int_0^L K_s (v_x)^2 dx \]  

(4)

The kinetic energy of the system \((T)\) can be written as,

\[ T = \frac{1}{2} A \int_0^L (\dot{u}^2 + \dot{v}^2) dx \]  

(5)

The displacement fields \(u(x)\) and \(v(y)\) for the beam are represented as linear combinations of unknown parameters \((d_i)\) and orthogonal admissible functions \((\alpha_i)\) and \((\beta_i)\) as per the Rayleigh-Ritz method. These displacement fields are expressed as follows,
\[ u = \sum_{j=1}^{n} d_j \alpha_j e^{i\omega t}, \quad v = \sum_{j=1}^{n} d_j \beta_j e^{i\omega t} \]  \hspace{1cm} (6)

In the above expressions, \( \omega \) is the natural frequency of the system. The governing equation for the free vibration problem is derived from Hamilton’s principle which states that,

\[ \delta \left( \int_{t_1}^{t_2} (T - U) \, dt \right) = 0 \]  \hspace{1cm} (7)

Solving the above equation gives the characteristic equation of free vibration for the problem in the following form,

\[ [K] - \omega^2 [M] = 0 \]  \hspace{1cm} (8)

where, \([K]\) is the stiffness matrix and \([M]\) is the mass matrix of the beam. The natural frequency \( (\omega) \) can be obtained by solving the above equation.

### 3. Results and Discussion

Numerical results for the present problem were generated using Matlab software following the methodology presented above. The results are presented for two boundary conditions namely clamped and simply supported in non-dimensional form. The non-dimensional parameters are as follows,

\[ \bar{\omega} = \omega L^2 \sqrt{\frac{\rho_0 A}{E_0 I}}, \quad \bar{K}_w = \frac{K_0 L^4}{E_0 I}, \quad \bar{K}_p = \frac{K_p L^4}{E_0 I} \]  \hspace{1cm} (9)

Validation study is carried out by comparing the present results with those established in literature and furnished in Table 1.

**Table 1. Validation of present results with literature**

| Boundary Condition | \( \bar{K}_w \) | \( (s = 0.2) \) | Ref. [16] |
|-------------------|-----------------|-----------------|----------|
| CC                | 10              | 4.7424          | 4.7511   |
|                   | \( 10^2 \)      | 4.9219          | 4.9296   |
|                   | \( 10^3 \)      | 6.1132          | 6.1172   |
| SS                | 10              | 3.2118          | 3.2117   |
|                   | \( 10^2 \)      | 3.6999          | 3.6999   |
|                   | \( 10^3 \)      | 5.6185          | 5.6185   |
Figure 2. Effect of foundation stiffness ($\overline{K}_w$) on natural frequencies of FGM beam for different values of foundation variation parameter ($s$).

![Graph showing effect of foundation stiffness on natural frequencies for different values of $s$.](image1)

**Figure 3.** Effect of foundation stiffness ($\overline{K}_w$) on natural frequencies of FGM beam for different values of foundation stiffness ($\overline{K}_p$).

Following the validation study, new results are presented to study the effect of different parameters on the natural frequency of FGM beam. The effect of foundation stiffness ($\overline{K}_w$) on the natural frequencies of FGM beam for different values of foundation variation parameter is shown in figure 2. The value of $\overline{K}_w$ is varied from 0 to 10000 whereas $\overline{K}_p$ is kept constant at 10. The graphs are presented for 5 values of the foundation variation parameter (0, -0.2, -0.4, -0.6, -0.8). Similar results are also presented for different values of $\overline{K}_p$ in figure 3, where the value of $s$ is kept constant at -0.5. The results are furnished for two boundary conditions and it can be seen from the figures that $\overline{K}_w$ significantly affect the natural frequency of the beam.

Figure 4 shows the effect of foundation variation parameter on natural frequencies of FGM beam. Here, the values of $\overline{K}_w$ and $\overline{K}_p$ are kept constant at 1000 and 10 respectively and $s$ is varied from -1 to 1. It is evident from the figure that as the value of $s$ increases the natural frequency also increases. Figure 5 shows the behaviour of FGM beam for the two selected boundary conditions. It can be seen that the natural frequency of SS beam is lower compared to CC beam owing to the less stiffness but they follow similar pattern for foundation stiffness variation.

![Graph showing effect of foundation variation parameter on natural frequencies for different $\overline{K}_w$ values.](image2)

![Graph showing effect of $\overline{K}_p$ on natural frequencies for different values of $s$.](image3)
4. Conclusion
Present study was carried out to investigate the dynamic behaviour of FGM beam resting on variable elastic foundation. An FGM beam with axial material variation is selected for this purpose along with a two parameter Pasternak foundation with variable foundation stiffness. Euler-Bernoulli beam theory was used for the formulation in conjunction with Rayleigh-Ritz method and Hamilton’s principle was used to generate the governing equations. FGM beams with clamped-clamped and simply-supported-simply supported end conditions are investigated. The investigation revealed that the foundation stiffness significantly affects the natural frequency of the beam as it adds to the stiffness of the beam. Also, it was found out that the variation of stiffness greatly influences the dynamic behaviour of the beam.

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