Anisotropic Lifshitz Point at $O(\epsilon_L^2)$

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Abstract

We present the critical exponents $\nu_L$, $\eta_L$, and $\gamma_L$ for an m-axial Lifshitz point at second order in an $\epsilon_L$ expansion. We introduced a constraint involving the loop momenta along the m-dimensional subspace in order to perform two- and three-loop integrals. The results are valid in the range $0 \leq m < d$. The case $m = 0$ corresponds to the usual Ising-like critical behavior.

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Lifshitz multicritical points appear at the confluence of a disordered phase, a uniformly ordered phase and a modulated ordered phase \([1, 2]\). The spatially modulated phase is characterized by a fixed equilibrium wave vector \(\vec{k}_0\). In this phase, \(\vec{k}_0\) goes continuously to zero as the system approaches the Lifshitz point. If this wavevector has \(m\)-components, the critical system under consideration presents an \(m\)-fold Lifshitz critical behavior. This sort of critical behavior is present in a variety of real physical systems including high-\(T_c\) superconductors \([3–5]\), ferroelectric liquid crystals \([6, 7]\), magnetic compounds and alloys \([8–10]\), among others.

In magnetic systems \([11]\), the \(m\)-fold Lifshitz point can be described by a spin-\(\frac{1}{2}\) Ising model on a \(d\)-dimensional lattice with nearest-neighbor ferromagnetic interactions as well as next-nearest-neighbor competing antiferromagnetic couplings along \(m\) directions. This system can be described in a field-theoretic setting using a modified \(\phi^4\) theory with higher order derivative terms, which arises as an effect of the competition along the \(m\)-directions. The Lifshitz universality class is defined by the parameters \((N, d, m)\), where \(N\) is the number of components of the order parameter, \(d\) is the space dimension of the system, and \(m\) is the number of competing directions.

Other examples of field theories containing higher derivative terms have been investigated in different physical scenarios. In cosmology, the recently proposed model known as “\(k\)-inflation” describes inflation driven by higher order kinetic terms for the inflaton scalar field \([12]\). Another instance which arises in quantum field theory in curved spacetime, is the quantization of scalar fields with high frequency dispersion relation around a classical gravitational background \([13]\). In that case, the higher order term accounts for deviations from Lorentz invariance. The modified dispersion relation might arise from an unspecified modification of the short distance structure of spacetime. A further generalization of this idea is to modify the large distance structure of spacetime allowing higher derivative terms, breaking Lorentz invariance in the infrared regime as well \([14]\). Thus a better comprehension of how to calculate arbitrary loop corrections for the Lifshitz critical behavior should give a clue about the proper perturbative treatment needed for a general higher order field theory.
In this work we generalize the method recently developed for the $m = 1$ case \cite{15, 16} to calculate the critical exponents $\eta_{L2}$, $\nu_{L2}$ and $\gamma_{L}$ using renormalization group techniques and the $\epsilon_L$-expansion up to $O(\epsilon^2_L)$, where $\epsilon_L = 4 + \frac{m}{2} - d$ is the expansion parameter in the perturbative analysis. We recover the results for the $m = 1$ case obtained in \cite{16} and show for the first time that the Lifshitz critical behavior reduces to the Ising-like one for $m = 0$. Thus, the Ising-like universality class $(N, d)$ is contained in a nontrivial way into the Lifshitz’s $(N, d, m)$.

We start with the bare Lagrangian associated with the Lifshitz critical behavior. It can be written as a modified $\phi^4$ field theory expressed in the following form:

\[
L = \frac{1}{2} | \nabla_m \phi_0 |^2 + \frac{1}{2} | \nabla_{(d-m)} \phi_0 |^2 + \delta_0 \frac{1}{2} | \nabla_m \phi_0 |^2 + \frac{1}{2} t_0 \phi_0^2 + \frac{1}{4!} \lambda_0 \phi_0^4. \tag{1}
\]

The quartic dependence on the momenta along the $m$-directions will be manifest in the free propagator. Here we will consider the system at the Lifshitz critical point, defined by the values $\delta_0 = t_0 = 0$. In order to compute the critical exponents, we need to calculate some Feynman diagrams, namely $I_2, I_3, I_4,$ and $I_5$ \cite{15, 16}. Setting $t_0 = \delta_0 = 0$,

\[
I_2 = \int \frac{d^{d-m}q d^m k}{[((k + K')^2 + (q + P)^2)((k^2)^2 + q^2)} \tag{2}
\]

is the one-loop integral contributing to the four-point function,

\[
I_3 = \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^m k_1 d^m k_2}{((q_1^2 + (k_1^2)^2)(q_2^2 + (k_2^2)^2))((q_1 + q_2 + p)^2 + ((k_1 + k_2 + k')^2)^2)} \tag{3}
\]

is the two-loop “sunset” Feynman diagram of the two-point function,

\[
I_4 = \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^m k_1 d^m k_2}{((q_1^2 + (k_1^2)^2)((P - q_1)^2 + ((K' - k_1)^2)^2)((q_2^2 + (k_2^2)^2))}
\times \frac{1}{(q_1 - q_2 + p_3)^2 + ((k_1 - k_2 + k_3)^2)^2} \tag{4}
\]

is one of the two-loop graphs which will contribute to the fixed-point, and

\[
I_5 = \int \frac{d^{d-m}q_1 d^{d-m}q_2 d^{d-m}q_3 d^m k_1 d^m k_2 d^m k_3}{((q_1^2 + (k_1^2)^2)(q_2^2 + (k_2^2)^2)(q_3^2 + (k_3^2)^2))((q_1 + q_2 - p)^2 + ((k_1 + k_2 - k')^2)^2)}
\times \frac{1}{(q_1 + q_3 - p)^2 + ((k_1 + k_3 - k')^2)^2} \tag{5}
\]
is the three-loop diagram contributing to the two-point vertex function. We then choose a special symmetry point in order to simplify the integrals. We set the external momenta at the quartic directions equal to zero, i.e. \( k' = k'_1 = k'_2 = k'_3 = 0 \), and \( K' = k'_1 + k'_2 \). In addition, for the four-point vertex, the external momenta along the quadratic directions are chosen as \( p_i.p_j = \frac{\kappa^2}{4}(4\delta_{ij} - 1) \), where \( p_1, p_2, p_3 \) are the independent external momenta, and \( P = p_1 + p_2 \). We fix the momentum scale of the two-point function through \( p^2 = \kappa^2 = 1 \). We shall use normalization conditions for the vertex functions along with dimensional regularization for the calculation of the Feynman diagrams.

Let us find out the one-loop integral \( I_2 \). With our choice of the symmetry point, and introducing two Schwinger’s parameters we obtain for \( I_2 \):

\[
\int \frac{d^{d-m}q d^m k}{((k^2)^2 + (q + P)^2)((k^2)^2 + q^2)} = \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \left( \int d^m k \exp(- (\alpha_1 + \alpha_2)(k^2)^2) \right) 
\times \int d^{d-m}q \exp(- (\alpha_1 + \alpha_2)q^2 - 2\alpha_2 q.P - \alpha_2 P^2). \tag{6}
\]

The \( \vec{q} \) integral can be performed to give

\[
\int d^{d-m}q \exp(- (\alpha_1 + \alpha_2)q^2 - 2\alpha_2 q.P - \alpha_2 P^2)
= \frac{1}{2} S_{d-m} \Gamma\left( \frac{d-m}{2} \right) (\alpha_1 + \alpha_2)^{- \frac{d-m}{2}} \exp\left( - \frac{\alpha_1 \alpha_2 P^2}{\alpha_1 + \alpha_2} \right). \tag{7}
\]

For the \( \vec{k} \) integral we perform the change of variables \( r^2 = k_1^2 + \ldots + k_m^2 \). Now take \( z = r^4 \).

The integral turns out to be:

\[
\int d^m k \exp(- (\alpha_1 + \alpha_2)(k^2)^2) = \left( \frac{1}{4} S_m \right) \Gamma\left( \frac{m}{4} \right) (\alpha_1 + \alpha_2)^{- \frac{m}{4}}. \tag{8}
\]

Using Eqs. (7) and (8), \( I_2 \) reads

\[
I_2 = \frac{1}{2} S_{d-m} \left( \frac{1}{4} S_m \right) \Gamma\left( \frac{d-m}{2} \right) \Gamma\left( \frac{m}{4} \right) 
\times \int_0^\infty \int_0^\infty d\alpha_1 d\alpha_2 \exp\left( - \frac{\alpha_1 \alpha_2 P^2}{\alpha_1 + \alpha_2} \right) (\alpha_1 + \alpha_2)^{- \left( \frac{d}{4} - \frac{m}{4} \right)} \tag{9}
\]

The remaining parametric integrals can be solved by a change of variables followed by a rescaling \[18\]. The integral is proportional to \( (P^2)^{- \frac{d}{4}} \). Now we can set \( P^2 = \kappa^2 = 1 \). Using the identity
\[ \Gamma(a + bx) = \Gamma(a) \left[ 1 + bx \psi(a) + O(x^2) \right], \]  
\[ \text{(10)} \]

where \( \psi(z) = \frac{d}{dz} \ln \Gamma(z) \), one is able to perform the \( \epsilon_L \)-expansion when the Gamma functions have non-integer arguments. Altogether, the final result for \( I_2 \) is:

\[ I_2 = \left[ \frac{1}{4} S_m S_{d-m} \Gamma(2 - \frac{m}{4}) \Gamma(\frac{m}{4}) \right] \frac{1}{\epsilon_L} \left( 1 + [i_2]_m \epsilon_L \right), \]
\[ \text{(11)} \]

where \([i_2]_m = 1 + \frac{1}{2} (\psi(1) - \psi(2 - \frac{m}{4}))\). From now on, we shall absorb the factor inside the brackets in Eq. (11) in the definition of the coupling constant \( \epsilon_L \). Then the redefined integral is:

\[ I_2 = \frac{1}{\epsilon_L} \left( 1 + [i_2]_m \epsilon_L \right). \]
\[ \text{(12)} \]

Now we shall discuss the two- and three-loop integrals. We introduce a constraint among the loop momenta in different subdiagrams, along the quartic directions only \[16\]. We wish to highlight this approximation here by calculating the integral \( I_4 \) for \( m \neq 8 \).

After our choice for the external momenta along the quartic directions, we can write \( I_4 \) in the following way:

\[ I_4 = \int \frac{d^{d-m}q_1 d^m k_1}{(q_1^2 + (k_1^2)^2) \left( (P - q_1)^2 + (k_1^2)^2 \right)} \]  
\[ \times \int \frac{d^{d-m}q_2 d^m k_2}{(q_2^2 + (k_2^2)^2) \left[ (q_1 - q_2 + p_3)^2 + ((k_1 + k_2)^2)^2 \right]}, \]
\[ \text{(13)} \]

where we changed variables from \( k_2 \rightarrow -k_2 \). We integrate first over the subdiagram \( q_2, k_2 \). In order to integrate over \( k_2 \) we introduce a constraint relating \( k_1 \) to \( k_2 \) inside this subdiagram, i.e., \( k_1 \) is fixed into the second integral in Eq. (13). If the relation between the two loop momenta is of the form \( k_1 = -\alpha k_2 \) we can solve the integral in terms of a product of Gamma functions and a Hypergeometric function. The value \( \alpha = 2 \) is singled out when we demand that the integral is given in terms of Gamma functions only. This is a natural generalization of the \( m = 1 \) case \[16\]. Using Schwinger’s parameterization and setting \( k_1 = -2k_2 \) in the second integral in Eq. (13) we find
\[ I_4 = I_2 \int \frac{d^{d-m}q_1 d^m k_1}{(q_1^2 + (k_1^2)^2)((P_q)^2 + (k_1^2)^2) \left[ (q_1 + p_3)^2 \right]^{\frac{d-m}{2}}} . \] (14)

Performing the integral over \( k_1 \) we obtain
\[ I_4 = I_2 \int_0^1 dz \int \frac{d^{d-m}q_1}{(q_1^2 - 2zP_q + zP^2)^{\frac{d-m}{2}} \left[ (q_1 + p_3)^2 \right]^{\frac{d-m}{2}}} . \] (15)

Using a Feynman parameter the integral turns out to be
\[ I_4 = \frac{1}{2} I_2 \left( 1 - \frac{\epsilon_L}{2} \psi(2 - \frac{m}{4}) \right) \frac{\Gamma(\epsilon_L)}{\Gamma(\frac{d-m}{2})} \int_0^1 dy y^{1-\frac{d-m}{2}} (1-y)^{\frac{d-m}{2}-1} \]
\[ \times \int_0^1 dz \left[ yz(1-yz)P^2 + y(1-y)p_3^2 - 2yz(1-y)p_3P \right]^{-\epsilon_L} . \] (16)

The integral over \( y \) is singular at \( y = 1 \) when \( \epsilon_L = 0 \). We only need to replace the value \( y = 1 \) inside the integral over \( z \) [L6, L7], and integrate over \( y \) afterwards, obtaining
\[ I_4 = \frac{1}{2 \epsilon_L^2} (1 + 3 [i_2]_m \epsilon_L) . \] (17)

The integrals \( I'_3 \) and \( I'_5 \) can be solved using a similar reasoning. They are given by
\[ I'_3 = -\frac{1}{8-m} \frac{1}{\epsilon_L} \left[ 1 + \left( [i_2]_m + \frac{3}{4 - \frac{m}{2}} \right) \epsilon_L \right] , \] (18)
\[ I'_5 = -\frac{1}{3(2 - \frac{m}{4})} \frac{1}{\epsilon_L^2} \left[ 1 + 2 \left( [i_2]_m + \frac{1}{2 - \frac{m}{4}} \right) \epsilon_L \right] . \] (19)

Note that the leading singularities for \( I_2, I_4 \) are the same as their analogous integrals in the pure \( \phi^4 \) theory. However, \( I'_3 \) and \( I'_5 \) do not have the same leading singularities for they include a factor of \( \frac{1}{(2 - \frac{m}{4})} \). We then introduce a weight factor for \( I'_3 \) and \( I'_5 \), namely \( (1 - \frac{m}{8}) \), so that they have the same leading singularities as in the pure \( \phi^4 \) theory. This has the advantage of allowing a smooth transition to the Ising-like case \( (m = 0) \) from the general Lifshitz anisotropic critical behavior \( (m \neq 8) \) as we shall see next.

The fixed point at two-loop level is given by:
\[ u^* = \frac{6}{8+N} \epsilon_L \left( 1 + \epsilon_L \left[ \left( \frac{4(5N + 22)}{(8+N)^2} - 1 \right) [i_2]_m - \frac{(2+N)}{(8+N)^2} \right] \right) . \] (20)
With this fixed-point one readily obtains the critical exponents $\eta_{L2}$ and $\nu_{L2}$:

$$\eta_{L2} = \frac{1}{2} \epsilon_L \frac{2 + N}{(8 + N)^2}$$

$$+ \epsilon_L^3 \frac{(2 + N)}{(8 + N)^2} \left[ \left( \frac{4(5N + 22)}{(8 + N)^2} - \frac{1}{2} \right) [i_2]_m + \frac{1}{8 - m} - \frac{2 + N}{(8 + N)^2} \right], \quad (21)$$

$$\nu_{L2} = \frac{1}{2} + \frac{1}{4} \epsilon_L \frac{2 + N}{8 + N}$$

$$+ \frac{1}{8} \frac{(2 + N)}{(8 + N)^3} \left[ 2(14N + 40) [i_2]_m - 2(2 + N) + (8 + N)(3 + N) \right] \epsilon_L^2. \quad (22)$$

Using the scaling law $\gamma_L = \nu_{L2}(2 - \eta_{L2})$, the exponent $\gamma_L$ is

$$\gamma_L = 1 + \frac{1}{2} \epsilon_L \frac{2 + N}{8 + N}$$

$$+ \frac{1}{4} \frac{(2 + N)}{(8 + N)^3} \left[ 12 + 8N + N^2 + 4 [i_2]_m (20 + 7N) \right] \epsilon_L^2. \quad (23)$$

It should be emphasized that $[i_2]_m$ is a universal amount, for the dependence on $m$ is encoded in such quantity. The parameter $m$ only appears in a explicit way at the $O(\epsilon_L^3)$ contribution to the index $\eta_{L2}$. To our knowledge, the explicit dependence on $m$ is obtained for the first time at $O(\epsilon_L^3)$ for $\eta_{L2}$. When setting $(m = 1)$ in the formulae above, we recover the exponents previously reported in reference [16]. As discussed there, the two-loop calculation ($N = 1$) in three dimensions yields $\gamma_L = 1.45$, in a nice agreement with the numerical Monte Carlo simulation $\gamma_L = 1.4 \pm 0.06$.

The amazing fact obtained using the method outlined here is that the critical exponents reduce to the Ising-like ones when $m = 0$, for $\epsilon_L \rightarrow \epsilon = 4 - d$. This means that the universality class for the $m$–fold Lifshitz point includes the Ising-like one for this particular value of $m$ in a nontrivial way. This provides a unified description of the anisotropic Lifshitz critical behavior ($m \neq 8$, $d \neq m$). This is the first time that an isotropic behavior ($m = 0$) can be recovered from the most general anisotropic Lifshitz criticality.
Note that our result for the exponent $\eta_{L^2}$ is in agreement with Mukamel’s \cite{13} at $O(\epsilon_L^2)$ and is independent on $m$ at this order. It should not be surprising that the approach fails to describe the $d = m = 8$ case, for the exponent $\eta_{L^2}$ is divergent as can be seen from Eq.\((21)\). The approximation made is not suitable for general isotropic cases $d = m \neq 8$ as well, since there is no preferred directions any longer. Another treatment should be employed to obtain information along the $m$-dimensional competition axes, since the symmetry point used here is not suitable to find out quantities along the competing directions.

All the results in this work follow from expanding the theory around its upper critical dimension. The constraint introduced along the $m$-dimensional subspace is equivalent to expand around the theory without competition, with $m$ kept fixed. A different field-theoretic method has been proposed, based on the expansion around the number of the $m$ competing directions \cite{20,22}. The $m = 2, 6$ reported cases are in disagreement with our results. This suggests that the two approaches are inequivalent.

To conclude, we have calculated the critical exponents associated to correlations along the $(d - m)$-directions perpendicular to the competition axes. This was possible because we introduced a constraint between the quartic loop momenta appearing in different subdiagrams in higher-loop Feynman graphs. The Lifshitz universality class turns out to reduce to the Ising-like one for the value $m = 0$ at least up to the loop order considered in this work. In principle, the technique can be readily generalized to analyze general anisotropic Lifshitz type critical behavior with arbitrary powers of the Laplacian at the competing directions. The study of the tricritical Lifshitz points using this formalism is also worthwhile.

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REFERENCES

[1] Hornreich R M, Luban M and Shtrikman S 1975 Phys. Rev. Lett. 35 1678

[2] Hornreich R M 1980 Journ. Magn. Magn. Mat. 15-18 387

[3] Hayden S M et al. 1991 Phys. Rev. Lett. 66 821

[4] Keimer B et al. 1991 Phys. Rev. Lett. 67 1930

[5] Sachdev S and Ye J 1992 Phys. Rev. Lett. 69 2411

[6] Rananavare S B, Pisipati V G K M and Wong E W 1994 Phys. Rev. Lett. 72 3558

[7] Zalar B et al 1998 Phys. Rev. Lett. 80 4458

[8] Becerra C C, Shapira, Oliveira Jr. N F, and Chang T S 1980 Phys. Rev. Lett. 44 1692
   Shapira Y, Becerra C C, Oliveira Jr. N F, and Chang T S 1981 Phys. Rev. B 24 2780

[9] Bindilatti V, Becerra C C and Oliveira Jr. N F 1989 Phys. Rev. B 40 9412

[10] Yokoi C S O, Coutinho-Filho M D and Salinas S R 1984 ibid. 29 6341

[11] Selke W 1988 Phys. Rep. 170 213
   Selke W 1998 Phase Transitions and Critical Phenomena, edited by C. Domb and J.
   Lebowitz (Academic Press, London), vol.15

[12] C. Armendariz-Picon, T. Damour and V. Mukhanov, Phys. Lett. B 458, 209 (1999);
   C. Armendariz-Picon, V. Mukhanov and Paul Steinhardt, Phys. Rev. Lett. 85, 4438
   (2000).

[13] S. Corley and T. Jacobson, Phys. Rev. D 54, 1568 (1996); T. Jacobson and D. Mattingly,
   gr-qc/0007031 T. Jacobson and D. Mattingly, Phys. Rev. D 63, 041502 (2001).

[14] J. M. Carmona and J. L. Cortés, hep-th/0012028; J. M. Carmona and J. L. Cortés,
   Phys. Lett B 494, 75 (2000).

[15] Leite M M 2000 Phys. Rev. B 61 14691
[16] de Albuquerque L C and Leite M M 2000 cond-mat/0006462

[17] Amit D J 1984 *Field Theory, the Renormalization Group and Critical Phenomena* (World Scientific, Singapore)

[18] Abdalla E, Abdalla M C B, Dalmazi D, and Zadra A 1994 *2D Gravity in Non-Critical Strings*, Lecture Notes in Physics m20 (Springer-Verlag, Berlin)

[19] Mukamel D 1979 *J. Phys. A* **10** L249

[20] Sak J and Grest G S 1978 *Phys. Rev. B* **17** 3602

[21] Mergulhão Jr C and Carneiro C E I 1998 *Phys. Rev. B* **58** 6047

Mergulhão Jr C and Carneiro C E I 1999 *ibid.* **59** 13954

[22] Diehl H W and Shpot M 2000 *Phys. Rev. B* **62** 12338