The text is already in a natural form and does not require any changes.
problem reduces further to the exactly solvable Heisenberg XXX spin $s = 0$
chain ([3], [4], [5]). However, despite the richness of mathematical structures
involved (global $SL(2, \mathbb{C})$-invariance and abundance of integral of motions)
as well as diversity of approaches to the problem (Yang-Baxter equations, Bethe
ansatz([4], [5], [6], [8]) quasiclassical approximation[9]), the explicit
expression for the intercept of the odderon has not been found yet.

In this letter we show that the odderon possesses (albeit in a somewhat hidden
form) the modular symmetry, ubiquitous in conformal field theories (CFT)
and string theory. In the next section we recall the theory of the odderon, then,
following Lipatov, the consequences of global $SL(2, \mathbb{C})$-invariance. As new
results, we analyse the role of cyclic symmetry in this framework and we explicitly
demonstrate the link to modular invariance through two alternative descriptions
of elliptic curves. This link may lead to the effective, two-dimensional string
theory for QCD in the moduli space of elliptic curves with fixed “parity”, corre-
sponding to the transformations of the torus through the even number of Dehn
twists.

2 The Odderon

The Regge limit of QCD is defined as the kinematical region

$$s \gg -t \approx M^2$$

where $M$ is the hadron mass scale, or, in the case of Deep Inelastic Scattering, as
the small $x = Q^2/s$ limit. Here we sketch, following [7], the equivalence between
the Regge intercept of amplitudes and energy levels of a two-body Hamiltonian.

The aim is to find the Regge behaviour of the amplitude $A(s, t) \sim s^{\omega_0+1}$.
Mellin transformation leads to

$$A(s, t) = is \int_{\delta - i\infty}^{\delta + i\infty} \frac{d\omega}{2\pi i} \left( \frac{s}{M^2} \right)^\omega A(\omega, t)$$

so the Regge behavior corresponds to finding the poles of the amplitude $A(\omega, t)$. Rewriting this amplitude as the convolution of “hadron” wave functions $\Phi_{A,B}$
and a kernel $T(\{k_i\}, \{k_j\}, \omega)$ we get:

$$A(\omega, t) = \int d^2k_i \int d^2k_j \Phi_A(\{k_i\})T(\{k_i\}, \{k_j\}, \omega)\Phi_B(\{k_j\})$$

where $\{k_i\}$ and $\{k_j\}$ are the transverse momenta of the $N$ exchanged reggeons
( in the case of odderon $N = 3$ ). The next step amounts to writing the Bethe-
Salpeter equations for the kernel $T$:

$$\omega T(\omega) = T_0 + \mathcal{H}T(\omega)$$

2
where $T_0$ is the free propagator and $\mathcal{H}$ is the operator corresponding to the insertion of single gluonic interactions between all pairs of reggeons. This equation can be formally solved:

$$T(\omega) = \frac{T_0}{\omega - \mathcal{H}}$$

(5)

Therefore the poles of the amplitude correspond to the eigenvalues of the hamiltonian operator $\mathcal{H}$. After performing Fourier transformation ($k_i \rightarrow b_i$) and using the complex notation $z_j := x_j + iy_j$, the Hamiltonian splits into a sum of a holomorphic part and an antiholomorphic part. In the large $N_c$ limit the two commute. It is therefore sufficient to consider only the holomorphic part, which in the case of odderon reads

$$(H(z_1, z_2) + H(z_2, z_3) + H(z_3, z_1))\Psi(z_1, z_2, z_3) = E\Psi(z_1, z_2, z_3)$$

(6)

where

$$H(z_1, z_2) = \sum_{l=0}^{\infty} \frac{2l + 1}{l(l + 1) - L_{12}^2} - \frac{2}{l + 1}$$

(7)

with

$$L_{12}^2 := -\frac{1}{\sqrt{2}} \frac{d}{dz_1} \frac{d}{dz_2}$$

(8)

being the holomorphic Casimir operator of the group $SL(2, \mathbb{C})$. The eigenvalue $E$ of the holomorphic hamiltonian and the corresponding eigenvalue $\bar{E}$ of the antiholomorphic one are related to the Regge intercept by the formula:

$$\omega_0 = \frac{\alpha_s N_c}{4\pi}(E + \bar{E})$$

(9)

The celebrated BFKL solution ($N = 2$ case) corresponds in this language to finding the maximal eigenvalue of the equation

$$H(z_1, z_2)\Psi(z_1, z_2) = E\Psi(z_1, z_2)$$

(10)

and has the known solution

$$E = -4[\psi(m) - \psi(1)]$$

(11)

where $\psi$ is the derivative of the logarithm of the Euler $\Gamma$ function and $m$ is a conformal weight. The maximum of (11) is achieved at $m = 1/2$ and reproduces the BFKL slope

$$\omega_0^{BFKL} = \frac{\alpha_s N_c}{\pi} 4\ln 2$$

(12)
2.1 Conservation laws

The Hamiltonian $H$ is invariant with respect to the action of $SL(2, \mathbb{C})$ on holomorphic functions given by:

$$(g \cdot \Psi)(z_1, z_2, z_3) = \Psi \left( \frac{az_1 + b}{cz_1 + d}, \frac{az_2 + b}{cz_2 + d}, \frac{az_3 + b}{cz_3 + d} \right) \quad \text{for } g = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2, \mathbb{C})$$

Therefore it commutes with the holomorphic Casimir operator for this representation:

$$\hat{q}_2 := -2z_{12}^2 \frac{d}{dz_1} \frac{d}{dz_2} - 2z_{23}^2 \frac{d}{dz_2} \frac{d}{dz_3} - 2z_{31}^2 \frac{d}{dz_3} \frac{d}{dz_1}$$

This enables us to consider functions transforming under the unitary representations of $SL(2, \mathbb{C})$ labelled by $n \in \mathbb{N}$ and $\nu \in \mathbb{R}$. In this case the eigenvalue $q_2$ is $((1 + n)/2 + i\nu)((-1 + n)/2 + i\nu)$.

It has been shown [10] that the system possesses another integral of motion — an operator $\hat{q}_3$:

$$\hat{q}_3 = z_{12} z_{23} z_{31} \partial_1 \partial_2 \partial_3$$

which commutes with the hamiltonian $H$.

One of the strategies for solving the odderon problem, proposed by Lipatov [10], was to diagonalize the conservation laws $\hat{q}_2$ and $\hat{q}_3$ and to substitute the solution into the Schrödinger equation in order to find the energy eigenvalue.

2.2 Conformal ansatz

Lipatov [11] has chosen an ansatz, which automatically diagonalizes $\hat{q}_2$:

$$\Psi_{z_0}(z_1, z_2, z_3) = \left( \frac{z_{12} z_{23} z_{31}}{z_{10} z_{20} z_{30}} \right)^{m/3} \varphi(\lambda)$$

where $m = 1/2 + i\nu + n/2$, $n$ is an integer and $\nu$ is a real number. Here, $z_0 \in \mathbb{C}$ is just a parameter and $\lambda$ is the anharmonic ratio:

$$\lambda = \frac{z_{12} z_{30}}{z_{13} z_{20}}$$

Lipatov further derived the form of the operator $\hat{q}_3$ within this ansatz. Inserting $\Psi_{z_0}(z_1, z_2, z_3)$ into the equation

$$\hat{q}_3 \Psi(z_1, z_2, z_3) = q_3 \cdot \Psi(z_1, z_2, z_3)$$

and canceling the factor $\ldots)^{m/3}$ he obtained:

$$\nabla_1 \frac{1}{\lambda(1 - \lambda)} \nabla_2 \nabla_3 \varphi(\lambda) = q_3 \varphi(\lambda)$$
where
\[
\begin{align*}
\nabla_1 &= \frac{m}{3} (1 - 2\lambda) + \lambda(1 - \lambda)\partial, \\
\nabla_2 &= \frac{m}{3} (1 + \lambda) + \lambda(1 - \lambda)\partial, \\
\nabla_3 &= -\frac{m}{3} (2 - \lambda) + \lambda(1 - \lambda)\partial,
\end{align*}
\]
(20) (21) (22)

The Hamiltonian (6) has also been rewritten in terms of $\lambda$.

3 Cyclic invariance

It is easy to see that both the Hamiltonian $H$ and $\hat{q}_3$ are invariant under cyclic permutations of the gluonic coordinates $z_1, z_2, z_3$. We show now how this symmetry manifests itself in the formalism of the preceding section. Under the permutation $z_1 \rightarrow z_2 \rightarrow z_3$ the anharmonic ratio transforms as follows:
\[
\lambda \rightarrow 1 - \frac{1}{\lambda} \rightarrow \frac{1}{1 - \lambda}
\]
(24)

We postulate, that the ground state is symmetric under this transformation and so
\[
\varphi(\lambda) = f(s_1, s_2, s_3, \tilde{j})
\]
(25)
where $s_i$ are the symmetric polynomials in $x_1 = \lambda, x_2 = 1 - 1/\lambda$ and $x_3 = 1/(1 - \lambda)$, and $\tilde{j}$ is the Vandermonde determinant. Namely
\[
\begin{align*}
\ s_1 &= x_1 + x_2 + x_3 = \frac{\lambda^3 - 3\lambda + 1}{\lambda(\lambda - 1)} \\
\ s_2 &= x_1x_2 + x_2x_3 + x_3x_1 = \frac{\lambda^3 - 3\lambda^2 + 1}{\lambda(\lambda - 1)} \\
\ s_3 &= x_1x_2x_3 = -1 \\
\ \tilde{j} &= (x_1 - x_2)(x_2 - x_3)(x_3 - x_1) = \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}
\end{align*}
\]
(26) (27) (28) (29)

It turns out that the only independent quantity is $A = s_1 + s_2 = \frac{(\lambda+1)(2\lambda-1)(\lambda-2)}{\lambda(\lambda-1)}$ related to $\tilde{j}$ by the equation $4\tilde{j} = A^2 + 27$. It is convenient to introduce the notation:
\[
\begin{align*}
\ B &= 8A = \sqrt{j - 1728} \\
\ j &= 256\tilde{j} = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}
\end{align*}
\]
(30) (31)
At this moment we make a refinement of Lipatov’s ansatz, namely

\[ \Psi_{z_0}(z_1, z_2, z_3) = \left( \frac{z_{12} z_{23} z_{31}}{z_{10} z_{20} z_{30}} \right)^{m/3} f(B) \]

\[ = \left( \frac{z_{12} z_{23} z_{31}}{z_{10} z_{20} z_{30}} \right)^{m/3} \frac{f \left( 8 \left( \lambda + 1 \right) \left( 2 \lambda - 1 \right) \left( \lambda - 2 \right) \right)}{\lambda (\lambda - 1)} \] (32)

where \( m = 1/2 + i \nu + n/2 \), \( n \) is an integer and \( \nu \) is a real number. \( \lambda \) is the anharmonic ratio:

\[ \lambda = \frac{z_{12} z_{30}}{z_{13} z_{20}} \] (33)

Now we insert the function \( \varphi(\lambda) = f(B) \) into the conservation law (19). After reexpressing the result in terms of \( j \) and \( B = \sqrt{j - 1728} \) we get:

\[ \left\{ \frac{j^2}{2} \frac{d^3}{dB^3} + 2 \sqrt{j - 1728} j \frac{d^2}{dB^2} + \left( j (1 + \frac{m(1 - m)}{6}) - 3 \cdot 2^8 \right) \frac{d}{dB} + \right. \]

\[ \left. \frac{(m - 3)m^2}{27} \right\} \sqrt{j - 1728 - 8q_3} f(\sqrt{j - 1728}) = 0 \] (34)

The original Hamiltonian (6) expressed by Lipatov in terms of \( \lambda \) can also be recast using the functions \( B = \sqrt{j - 1728} \) (although obtaining an explicit expression seems to be highly non-trivial).

In the next section we will show that the variable \( j \) can indeed be considered as a modular invariant and we give a geometric interpretation of the symmetry considered here.

4 Elliptic curves

According to one of the many possible definitions (see e.g. [12]), an elliptic curve is a complex curve of genus one. There are two alternative descriptions of these objects. The first one is the Weierstrass parametrization which labels each elliptic curve by a complex number \( \lambda \in \mathbb{C} \). The curve given by \( \lambda \) is given by the equation

\[ y^2 = x(x - 1)(x - \lambda) \] (35)

where \( x \) and \( y \) are complex coordinates. Two such curves are conformally isomorphic if and only if their \( j \)-invariants coincide. The \( j \)-invariant is given by the formula:

\[ j = 2^8 \cdot \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2} \] (36)
Note that this expression is identical to the Vandermonde determinant considered before (31). The only problem that we encounter here in providing a geometric interpretation to our variables (30) and (31) is the fact that \( \sqrt{j - 1728} \) may differ in sign for isomorphic elliptic curves. Therefore one must consider elliptic curves with some additional structure, which we will define after presenting the other description of genus one curves.

We see here that, since the Hamiltonian can be expressed through the invariants \( \sqrt{j - 1728} \), it can be identified in a natural way with an operator acting on the moduli space of elliptic curves with that additional structure.

Alternatively one can view elliptic curves as complex tori \( \mathbb{C}/(\mathbb{Z} + \mathbb{Z} \tau) \) parametrized by \( \tau \in \mathbb{C} \) in the upper half-plane. Another way of looking at this quotient space is to consider it as the torus obtained by identifying opposite edges in the parallelogram bounded by 0,1,\( \tau \) and 1 + \( \tau \). The \( j \)-invariant is now a transcendental function of \( \tau \). This description is linked to the preceding one by the correspondence [13]:

\[
\lambda(\tau) = \left( \frac{\Theta_2(0; \tau)}{\Theta_3(0; \tau)} \right)^4
\]

where \( \Theta_2(0; \tau) \) and \( \Theta_3(0; \tau) \) are the Jacobi theta functions. Moreover, the symmetry which leaves \( j \) invariant corresponds in this description to modular invariance in the \( \tau \)-plane i.e.

\[
j(\tau) = j(\tau') \iff \tau' = \left( \begin{array}{cc} a \tau + b \\ c \tau + d \end{array} \right) \text{ for } \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL(2, \mathbb{Z})
\]

In our case, the symmetry \( \lambda \rightarrow 1 - \frac{a}{b} \rightarrow \frac{1}{1 - \frac{a}{b}} \) corresponds to modular transformations belonging to \( \Gamma^2 \) — the unique normal subgroup of \( SL(2, \mathbb{Z}) \) of index 2 [13]. This is an infinite group generated by the matrices \( \left( \begin{array}{cc} 1 & 2 \\ 0 & 1 \end{array} \right) \) and \( \left( \begin{array}{cc} 1 & -1 \\ 1 & 0 \end{array} \right) \). We still have to define the additional geometric structure on the torus which is left invariant by the subgroup \( \Gamma^2 \).

The two elementary Dehn twists, which generate the full modular group, are associated to the two noncontractible loops of winding number one. The “Dehn twist” operation consists of cutting the torus along the chosen loop, twisting one boundary by \( 2\pi \), and gluing it back (see e.g. [14]). Although the number of Dehn twists \( (n_D) \) is not well defined, given an isomorphism corresponding to a modular transformation between equivalent tori, the parity of number of Dehn twists \( ((-1)^{n_D}) \) is well defined. The subgroup \( \Gamma^2 \) corresponds precisely to the transformations of the torus through an even number of Dehn twists. We may therefore attach a kind of “sign” to each torus.

Using the correspondence between \( \lambda \) and \( \tau \) one can reexpress the Hamiltonian and the integral of motion in terms of \( \tau \).
5 Conclusions

In this letter we have shown that the odderon problem possesses a new symmetry - i.e. modular symmetry with respect to $\Gamma^2$ — an index 2 normal subgroup of $SL(2, \mathbb{Z})$. Expressing the conservation law (19) through modular invariants leads in a natural way to the new methods of solving 3rd order Fuchsian differential equations proposed by B.H. Lian and S-T Yau’s in the framework of mirror symmetry. This analogy may be an aid in carrying out Lipatov’s strategy mentioned in section 2.1 and may lead to the analytical solution of the odderon problem.

Apart from the practical applications of this symmetry, we hope that it may lead to deeper understanding of the Regge limit of QCD. The modular invariance of the odderon leads to a natural interpretation of all the operators as acting on the moduli space of genus one curves with fixed ‘sign’, i.e. the even-parity of the number of Dehn twists. In particular this may be a further step in establishing the relation between QCD and effective string theory.

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