Accelerating expansion of the universe may be caused by inhomogeneities

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We point out that, due to the nonlinearity of the Einstein equations, a homogeneous approximation in cosmology leads to the appearance of an additional term in the Friedmann equation. This new term is associated with the spatial inhomogeneities of the metric and can be expressed in terms of density fluctuations. Although it is not constant, it decays much slower (as $t^{-2}$) than the other terms (like density) which decrease as $t^{-2}$. The presence of the new term leads to a correction in the scale factor that is proportional to $t^{2}$ and may give account of the recently observed accelerating expansion of the universe without introducing a cosmological constant.

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Measurements of the light curves of several hundred type Ia supernovae \cite{1, 2, 3, 4, 5} and other independent observations \cite{6, 7, 8, 9} convincingly demonstrate that the expansion of the universe is accelerating.

This unexpected result stimulated a number of theoretical investigations. Most explanations suggested so far seem to belong to one of three categories: assuming a nonzero cosmological constant \cite{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21}, or assuming new gravitational physics \cite{22, 23, 24, 25, 26, 27}. In other words, inhomogeneities (which, unlike a cosmological constant or quintessence, are unquestionably present) induce a correction term in the Friedmann equation. Such a procedure has been introduced for the case of gravitational radiation in \cite{25}, applied for an ideal fluid in \cite{30} and for the case of a scalar field in \cite{31}. We demonstrate this below in the framework of second order perturbation theory. First order corrections are well known \cite{32, 33, 34}. We write down the spatial average of the second order perturbation equations, which (as they are linear in the second order correction of the metric and the density) enable us to calculate the spatial average of the metric and the density.

The metric is written in the form

$$\bar{g}_{jk} = g_{jk}^{(0)} + g_{jk}^{(1)} + g_{jk}^{(2)},$$

(3)

where the upper, bracketed indices refer to the order of the perturbation. Henceforth we use comoving coordinates. Assuming isotropy, we have

$$\bar{g}_{00}(t) = c^2,$$

(4)

$$\bar{g}_{0\alpha}(t) = 0,$$

(5)

$$\bar{g}_{\alpha\beta}(t) = -R^2(t)\delta_{\alpha\beta},$$

(6)

where the bar over quantities means spatial average (cf. Eq. (1)). Especially, we have

$$\bar{g}_{\alpha\beta}^{(i)} = -\delta_{\alpha\beta} \left( R^2 \right)^{(i)}(t), \quad i = 0, 1, 2.$$  

(7)

The use of comoving coordinates implies that

$$T^{00} = \rho = \rho^{(0)} + \rho^{(1)} + \rho^{(2)},$$

(8)

$$T^{0\alpha} = 0,$$  

(9)

$$T^{\alpha\beta} = 0.$$  

(10)

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Zeroth order quantities are those in a flat Friedmann model with matter domination, i.e.

$$R^{(0)} = \frac{2c}{H_0} \left( \frac{3}{2} H_0 t \right)^{\frac{1}{2}},$$  \hspace{1cm} (11)$$

$$\rho^{(0)} = \frac{1}{6\pi G t^2}.$$  \hspace{1cm} (12)$$

First order corrections are given by [33]

$$g_{\alpha\beta}^{(1)} = \frac{4c^2\eta^4}{H_0^2} \left[ \frac{1}{\eta} \frac{\partial}{\partial \eta} \left( \frac{1}{\eta} D_{\alpha\beta} \right) - 2 \left( \frac{8}{\eta^2} - \frac{\Delta}{\eta} \right) (C_{\alpha\beta} + C_{\beta\alpha}) + \frac{A_{\alpha\beta}}{\eta^3} + \eta_{\alpha\beta} B - \frac{\eta^2}{10} B_{\alpha\beta} \right],$$  \hspace{1cm} (13)$$

$$g_{\alpha\beta}^{(1)} = \frac{4c^2}{H_0} \Delta C_{\alpha},$$  \hspace{1cm} (14)$$

$$\rho^{(1)} = \frac{H_0^2}{32\pi G} \Delta \left( \frac{6A}{\eta^2} - \frac{3B}{5\eta^3} \right)$$  \hspace{1cm} (15)$$

with

$$\eta = \left( \frac{3}{2} H_0 t \right)^{\frac{1}{2}}.$$  \hspace{1cm} (16)$$

Here A, B and Cα are functions depending only on the spatial coordinates. A and B are arbitrary (determined by the initial conditions) while Cα has zero divergence. Finally, Dαβ satisfies a wave equation. Note that the assumption of isotropy implies Cα = 0 and Dαβ = 0. Henceforth we set A = 0, too.

Second order corrections of the (0,0) component of the Einstein equations have the general structure

(terms linear in $g_{ik}^{(2)}$) + (terms quadratic in $g_{ik}^{(1)}$) = 8\pi G \rho^{(2)}). \hspace{1cm} (17)$$

Calculating the spatial average of this equation we have

$$6 \frac{\dot{R}^{(0)}}{R^{(0)}} \left( \frac{\dot{R}^{(2)}}{R^{(0)}} - \frac{\dot{R}^{(2)}}{(R^{(0)})^2} \right) + \frac{1}{400} H_0^2 \left( \nabla B \right)^2 \frac{1}{\eta^2}$$

$$- \frac{17}{80} H_0^2 \left( \frac{\nabla B}{\eta} \right)^2 \frac{1}{\eta^2} = 8\pi G \rho^{(2)}.$$  \hspace{1cm} (18)$$

Similarly, for the spatial average of the second order corrections of the 0 component of the divergence equation we obtain

$$\frac{1}{(R^{(0)})^2} \left( \dot{\rho}^{(2)} \left( \frac{R^{(0)}}{3} \right)_0 - \frac{1}{2} \rho^{(1)} \left( \frac{g_{\alpha\beta}^{(1)}}{(R^{(0)})^2} \right)_0 \right)$$

$$+ \frac{1}{2} \rho^{(0)} \left\{ 3 \left( \frac{2 R^{(2)}}{R^{(0)}} + \frac{1}{4} \left[ \frac{B}{R^{(0)}} \right]^2 \right)_0 - \frac{1}{(R^{(0)})^2} g_{\alpha\beta}^{(1)} g_{\beta\alpha}^{(1)} \right\} - \frac{2}{(R^{(0)})^2} \left[ \frac{1}{c^2} \left( \frac{g_{\alpha\beta}^{(1)}}{R^{(0)}} \right)^2 - \frac{1}{(R^{(0)})^2} \left( g_{\alpha\beta}^{(1)} \right)^2 \right] \right\} = 0 . \hspace{1cm} (19)$$

Its solution sounds

$$8\pi G \rho^{(2)} = -6\pi G \rho^{(0)} \left( \frac{4 R^{(2)}}{R^{(0)}} + A_1 \right)$$

$$- \frac{3}{20} H_0^2 \left( \nabla B \right)^2 \frac{1}{\eta^2} + \frac{9}{800} H_0^2 \left( \nabla B \right)^2 \frac{1}{\eta^2}$$  \hspace{1cm} (20)$$

where $A_1$ is an integration constant. Putting this into the Einstein equation we get an ordinary differential equation for $R^{(2)}$

$$\dot{R}^{(2)} + \frac{1}{3t} R^{(2)} = - \frac{3}{4} \frac{c A_1}{(\frac{3}{2} H_0 t)^{\frac{3}{2}}} + \frac{1}{48} \frac{c}{(\nabla B)^2} c \left( \frac{3}{2} H_0 t \right)^{\frac{3}{2}}$$

$$+ \frac{7}{1600} \left( \nabla B \right)^2 H_0 c t.$$  \hspace{1cm} (21)$$

with the solution

$$R^{(2)} = A_2 t^{-\frac{3}{2}} - \frac{1}{4} A_1 R^{(0)} + \frac{1}{120} H_0^2 \left( \nabla B \right)^2 \left( \frac{3}{2} H_0 t \right)^{\frac{3}{2}}$$

$$+ \frac{3}{1600} \left( \nabla B \right)^2 H_0 c t^2.$$  \hspace{1cm} (22)$$

The integration constants $A_1$, $A_2$ are to be determined from the initial conditions for the second order correction of the metric and its time derivative. It is interesting to compare Eq. (22) with the perturbation series of a homogeneous open universe, by considering the curvature term as a perturbation in a flat universe:

$$R_o = R^{(0)} + \frac{1}{10} c \frac{3}{2} H_0 t^{\frac{3}{2}} - \frac{27}{5600} H_0 c t^2 + ... \hspace{1cm} (23)$$

In order to check Eq. (22), we determined numerically the second time derivative of the spatial metric from the Einstein equations by using an effective pseudospectral scheme. The relative error of the coefficients (mainly due to third order corrections) is displayed in Fig[11].

For sub-horizon-sized perturbations in the late universe the last term of Eq. (22) is dominant. The complete scale factor up to second order then reads

$$R(t) = R^{(0)} + \frac{3}{1600} \left( \nabla B \right)^2 H_0 c t^2$$

$$= R^{(0)} \left[ 1 + \frac{1}{6} \left( \frac{\delta \rho}{\rho} \right)^2 \right], \hspace{1cm} (24)$$

where $\delta \rho \equiv \rho^{(1)}$. 

The present value of the fractional density fluctuation can be estimated as (provided that linear perturbation theory is applicable, for \( \frac{2c}{\rho} > 1 \) fluctuations grow faster)

\[
\left( \frac{\delta \rho}{\rho} \right)^2_0 = \left( \frac{\delta \rho}{\rho} \right)_{\text{dec}}^2 (1 + z_{\text{dec}})^2 \approx 1...100,
\]

because

\[
1 + z_{\text{dec}} = \frac{R(t_0)}{R(t_{\text{dec}})} \approx 1100
\]

and

\[
\left( \frac{\delta \rho}{\rho} \right)_{\text{dec}} \approx 10^{-2}...10^{-3}
\]

(see e.g. \[24\]). Here 0 and \( \text{dec} \) stand for present value and value at decoupling, respectively. Thus, the final results are expressed in terms of the (large) relative density fluctuations. Eq.\[24\] demonstrates that taking into account of density fluctuations is essential in the late universe. A perturbative treatment may not even be sufficient. It is also seen that the additional term due to inhomogeneities is a quadratic function of the time, thus for a long time after the Big Bang, accelerating expansion occurs. This can be quantified in terms of the deceleration parameter which reads

\[
q = -\frac{\ddot{R} R}{R^2} = \left[ 1 - \frac{7}{3} \left( \frac{\delta \rho}{\rho} \right)^2 \right].
\]

Thus, according to second order perturbation theory the deceleration parameter becomes negative (acceleration) if \( \delta \rho/\rho > \sqrt{3/7} \approx 0.65 \). Assuming that \( \langle \delta \rho/\rho \rangle_0 \approx 1 \), this happens in our approximation at

\[
z = \sqrt{7/3} - 1 \approx 0.53.
\]

It is instructive to calculate the effective pressure as well. In a homogeneous universe with the scale factor \[24\] we obtain

\[
p = -\frac{7c^2}{1200} \frac{H_0^3 c^2}{8\pi G} \left( \frac{3}{2} \frac{H_0 t}{M} \right)^{-\frac{2}{3}} \langle \Delta B \rangle^2.
\]

This negative pressure is related to the gravitational attraction of the underlying inhomogeneities rather than a cosmological constant. Expressing it in terms of the density we get up to second order accuracy the equation of state

\[
p = -\frac{7c^2}{4800} \frac{H_0^3}{3(\pi G)^2} \langle \Delta B \rangle^2 \rho^\frac{2}{3}.
\]

In order to have a tentative comparison with supernova measurements, we calculate the luminosity distance \( d_L \) versus the redshift \( z \) (Hubble diagram) by approximating the inhomogeneous universe with a homogeneous one, which expands according to the corrected scale factor. Differences in distance modulus \( m - M = 5 \log_{10}(d_L/10 pc) \) (compared to an empty universe, \( d_L(\text{empty}) = z + z^2/2 \)) are displayed in Fig.\[2\] This is to be compared with Figs.10-11. in \[1\] or Figs.4-5. in \[2\]. Note that in the applied approximation curves belonging to higher values of relative density fluctuations converge to an upper bound which is roughly identical with the uppermost curve displayed.
coordinate distance

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where the second term indicates accelerating expansion.

As for the Hubble diagram, more precise calculation
would take into account the inhomogeneities of the met-
ic along light trajectories, too, not only in the global
scale factor. Such a procedure would need the complete
second order perturbation term, including the spatial de-
pendence. Note that a nonperturbative approach to this
problem has been published in Ref. [30]. As the spatial
average of the second order term has proved to be large,
it may happen (if the variance is also large) that the ac-
tual Hubble diagram broadens to a strip. This would
mean that the rather scattered measurement points fol-
low a systematic and are not due simply to experimental
error.

In conclusion, we have demonstrated that inhom-
geneties essentially influence the time evolution of the
late universe. Our results may even need further corre-
cisions, since the terms treated perturbatively proved to be
quite large. Nevertheless, it is remarkable that accelerating
expansion may result from inhomogeneities, without
assuming a cosmological constant, quintessence, or mod-
ified Einstein equations.

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Assuming $(\delta \rho/\rho_0) \approx 1$ it yields $t_0 = \frac{2}{3H_0} \left( \frac{\delta \rho}{\rho_0} \right)_0 \approx 1.3 \times \frac{1}{3H_0}$.

Some remarks are in order. The averaging procedure
we used is not quite unambiguous. We might have in-
cluded e.g. the square root of the spatial metric. In this
special case we obtain a further positive contribution to
the coefficient of the $t^2$ term in the scale factor. Never-
thless, it is advisable to consider directly the physical
distance between two points as a more significant quan-
tity. For points that are not very far from each other,
one has $l = \sqrt{-\frac{\delta \rho}{\rho}} \Delta x_\alpha \Delta x_\beta$, or, inserting the previous expansions for $\frac{\delta \rho}{\rho}$,

\[
l \approx l_0 - \frac{1}{2H_0} \left( \frac{\delta \rho}{\rho_0} \right) \Delta x_\alpha \Delta x_\beta
- \frac{1}{2H_0} \left( \frac{\delta \rho}{\rho_0} \right)^2 \Delta x_\alpha \Delta x_\beta
= R^{(0)} \sqrt{\Delta x^2}.
\]

with $l_0 = \sqrt{-\frac{\delta \rho}{\rho}} \Delta x_\alpha \Delta x_\beta = R^{(0)} \sqrt{\Delta x^2}$. By fixing the
coordinate distance $d_0 = \sqrt{\Delta x^2}$ and averaging over both
points one obtains for sub-horizon-sized perturbations

\[
l = d_0 \left( R^{(0)} + \frac{3}{400} \left( \frac{\delta \rho}{\rho_0} \right)^2 \frac{H_0}{c} t^2 \right)
\]

where the second term indicates accelerating expansion.

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