Degeneracy of Multi-Component Quantum Hall States Satisfying Periodic Boundary Conditions

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Abstract

In systems subject to periodic boundary conditions, Haldane has shown that states at arbitrary filling fraction possess a degeneracy with respect to center of mass translations. An analysis is carried out for multi-component electron systems and extra degeneracies are shown to exist. Their application to numerical studies is discussed.
I. INTRODUCTION

Following Laughlin’s original work [1], Halperin [2] proposed an extension to the original Laughlin-Jastrow wavefunction incorporating the possibility that the electrons have some further degree of freedom in addition to their two-dimensional coordinate, which in Halperin’s original observation was taken to be spin. Recent experiments [3] on electron systems where the extra degree of freedom can be seen to indicate the state of the electron in the third direction seem to suggest the existence of new universality classes of states, as first proposed by Haldane and Rezayi [4] and analyzed by Yoshioka, MacDonald, and Girvin [5]. The possibilities raised by this extra degree of freedom have been analyzed in various contexts [8–19].

In the standard treatment of the problem of electrons in a magnetic field subject to periodic boundary conditions (PBCs), Haldane [7] has shown that every eigenstate of a translationally invariant Hamiltonian has a degeneracy with respect to center of mass translations. For a state at filling fraction $p/q$ the degeneracy is simply $q$ fold. Haldane and Rezayi [6] further constructed the explicit generalization of Laughlin’s wavefunction to the PBCs, again showing the $q$ fold degeneracy inherent to states in this geometry. In light of the interesting possibilities raised by the double layer systems, it is natural to generalize the wavefunction construction to multi-component systems subject to periodic boundary conditions.

The remainder of the paper is organized as follows. In the next section we review the construction of the single layer generalization of Laughlin’s wavefunction subject to periodic boundary conditions. We then extend this construction to multi-component systems, showing interesting new degeneracies distinctive to these systems. Finally, we discuss the quantum numbers of these states with regard to numerical studies and their use in distinguishing between possible ground states.
II. LAUGHLIN WAVEFUNCTION IN PBC

We shall construct the Laughlin state subject to periodic boundary conditions (PBC’s), closely following [6]. In a system subject to PBCs in a magnetic field, the translation operator takes the form

\[ t(a) = \exp(a \cdot (\nabla - \frac{ie}{\hbar} A) - i \frac{a \times r}{l^2}) \]  

(1)

and obeys the non-commutative algebra

\[ t(a)t(a') = t(a + a')e^{i\frac{a \times a'}{2l^2}} \]  

(2)

where \( l = \sqrt{\frac{\hbar}{eB}} \) is the magnetic length. We wish to impose generalized boundary conditions by requiring that all physical quantities be invariant under translation of any particle by the set of translations \( L_{mn} = mL_1 + nL_2 \) where

\[ |L_1 \times L_2| = 2\pi N_\phi l^2 \]  

(3)

and \( N_\phi \) is the number of flux quanta. We impose the general boundary conditions on the wavefunction for any particle \( i \)

\[ t_i(L_{mn})\Psi = (\eta_{mn})^{N_\phi}e^{i\Phi_0 L_{mn}}\Psi \]  

(4)

where \( \eta_{mn} = (-1)^{m+n+mn} \) and we will choose \( \Phi_0 = 0 \) as our boundary condition. The physical region under consideration can be seen to be defined by four points \( z = \frac{1}{2}L_1(\pm 1 \pm \tau) \) where \( \tau = \frac{\mu}{L_1}e^{i\theta} \). We shall use the symmetric gauge \( A = (-By/2, Bx/2) \). The single particle wavefunctions in the lowest Landau level are then given by

\[ \psi(x, y) = e^{-\frac{1}{4\pi|z|^2}}f(z) \quad z = x + iy \]  

(5)

where \( f(z) \) is an analytic function with \( N_\phi \) zeros in the principal region. Applying the boundary conditions to (5) we find

\[ \frac{f(z + L_{mn})}{f(z)} = (\eta_{mn})^{N_\phi}\exp\left(\frac{L_{mn}^*L_{mn} + 2L_{mn}^*z}{4l^2}\right) \]  

(6)
where we write a two-dimensional vector \( \mathbf{a} \) as \( a_x + ia_y \). The basic building block that we shall use is the quasi periodic function \( w(z) \) which obeys

\[
\frac{w(z + L_{mn})}{w(z)} = \eta_{mn} \exp\left(\frac{2L_{mn}^* z + L_{mn}^* L_{mn}}{4N_\phi l^2}\right).
\]

(7)

An explicit representation of the function \( w(z) \) is given by

\[
w(z) = \exp\left(\frac{z^2}{4N_\phi l^2}\right) \Theta_1(\kappa z|\tau)
\]

(8)

where \( L_{mn} = \kappa^{-1} \pi (m + n \tau) \) and \( \Theta_1(\kappa z|\tau) \) is the odd elliptic theta function. We can therefore construct the single particle wavefunctions

\[
\Psi_{\{a_\alpha\}}(\mathbf{r}) = \left(\prod_{\alpha=1}^{N_\phi} \varphi(\mathbf{r} - a_\alpha) \exp\left(\frac{i\bar{a} \times \mathbf{r}}{2N_\phi l^2}\right)\right)
\]

(9)

where

\[
\bar{a} = \frac{\sum_\alpha a_\alpha}{N_\phi}
\]

(10)

and

\[
\varphi(\mathbf{r}) = w(z) \exp\left[-\left(\frac{z^* z}{4N_\phi l^2}\right)\right]
\]

(11)

The number of linearly independent solutions of (6) can be discerned in the following way. Inserting the above form into the boundary conditions yields the constraint

\[
\bar{a} = \frac{L_{pr}}{N_\phi}
\]

(12)

where \( L_{pr} \) is restricted to be a primitive translation. There are therefore \((N_\phi)^2\) possible values for \( \bar{a} \). One way in which to resolve the degeneracy is to form a superposition of states where every zero is shifted by the same amount in some primitive direction. This is equivalent to constraining the wavefunction to be invariant under a translation of the electron coordinate in this direction. Given one set of zeros \( \{a_\alpha\} \) which satisfies the boundary condition (12), another set can be generated by shifting the zeros

\[
a_\alpha \rightarrow a_\alpha + \frac{L'_{pr}}{N_\phi}
\]

(13)
where $L'_{pr}$ is also a primitive translation. Let us then form a linear superposition of states with shifted zeros

$$
\Psi(r) = \sum_{\gamma=1}^{N_\phi} \Psi(a_{\alpha,\gamma})
$$

(14)

where

$$
a_{\alpha,\gamma+1} = a_{\alpha,\gamma} + \frac{K_1}{N_\phi}
$$

(15)

where $K_1$ is a primitive translation. We can operate on the above wavefunction by performing an overall shift of the zeros

$$
a'_{\alpha,\gamma} = a_{\alpha,\gamma} + n_k \frac{K_2}{N_\phi}
$$

(16)

where $K_2$ is a primitive translation obeying

$$
|K_1 \times K_2| = 2\pi N_\phi l^2.
$$

(17)

We can therefore see that $n_k = 1, ..., N_\phi$ using (12). The number of linearly independent solutions is given by $N_\phi$. It is important to note that this is one program which generates a space of linearly independent solutions but not the only one.

We now consider the many particle Laughlin wavefunction at Landau level filling $\nu = p/q = 1/m$ on the plane

$$
\Psi(\{z_i\}) = \prod_{i<j} (z_i - z_j)^m \prod_i \exp\left[-\left(\frac{z_i^* z_i}{4l^2}\right)\right].
$$

(18)

In the following, we shall denote $N_e = p\bar{N}$ and $N_\phi = q\bar{N}$. When the system is subjected to PBCs, this wavefunction generalizes to

$$
\Psi(\{r_i\}) = \left(\prod_{i<j} \varphi(r_i - r_j)^m\right) \\
\times \prod_{\alpha=1}^{m} \left(\varphi(\mathbf{R}_\alpha - \bar{N}\mathbf{a}_\alpha) \exp\left(i\mathbf{a}_\alpha \times \frac{\mathbf{R}}{2\bar{N}_\phi l^2}\right)\right)
$$

(19)

where
\[ R = \sum_i r_i \]  
(20)

and \( \varphi(\mathbf{r}) \) is defined in (11). If we apply the translation operator to any of the coordinates and apply the boundary conditions, we find

\[ \bar{\mathbf{a}} = \frac{\sum_\alpha a_\alpha}{m} = \frac{\mathbf{L}_{pr}}{m} \]  
(21)

where \( \mathbf{L}_{pr} \) is restricted to be a primitive translation. The center of mass portion of the wavefunction can be seen to be formally equivalent to the single particle wavefunction for a particle of charge \( eN_e \) which sees \( m \) flux quanta. By analogy with the single particle case, we therefore conclude that the Laughlin state on the torus has an \( m \) fold degeneracy related to the action of the center of mass translation operator. Very generally, this degeneracy is inherent to every eigenstate, as was shown by Haldane [7].

III. EXTENSION TO MULTI-COMPONENT SYSTEMS

We can write the planar extension of Laughlin’s original wavefunction to a system with \( N_s \) species of electrons as first expressed by Halperin [2] in the two-component case

\[ \Psi^K[\{z_i\}] = \prod_{i<j}(z_i - z_j)^{K(\sigma_i, \sigma_j)} \prod_i \exp(-\frac{1}{4l^2} |z_i|^2) \]  
(22)

where \( \sigma_i \) is the species quantum number and \( K \) is an \( N_s \times N_s \) symmetric matrix encoding the correlations between the electrons where we impose \( \text{Det}K > 0 \). If we consider the plasma analogy, we find that in order to have a uniform fluid stable against fluctuations, we must choose \( \text{Det}K \geq 0 \). We shall consider the case of \( \text{Det}K = 0 \) separately. The matrix \( K \) also makes an appearance in effective theories of the fractional quantum Hall effect, as the long distance physics of the Hall fluid can be described by the Lagrangian

\[ \mathcal{L} = \frac{1}{4\pi} (\sum_{\sigma, \sigma'} K(\sigma, \sigma') \epsilon^{\mu\nu\lambda} a^\sigma_\mu \partial_\nu a^{\sigma'}_\lambda + 2\epsilon^{\mu\nu\lambda} \sum_\sigma t^\sigma A_\mu \partial_\nu a^\sigma_\lambda) + \text{Maxwell terms} \]  
(23)

where \( a^\sigma \) are \( N_s \) Chern-Simons gauge fields. This Lagrangian has been discussed extensively previously [13]. The two-component case is of special physical interest, where the two species
of electron may represent the two possible physical spin states of an electron or, in double layer systems, which of two layers an electron resides in. We have suppressed the quasi-spin portion of the wavefunction, which in the case of fermions would correctly anti-symmetrize the overall wavefunction.

We write the Halperin multi-component wavefunction subject to PBCs as

\[
\Psi^K[\{z_i\}] = \prod_{i<j} \varphi(z_i - z_j)^{K(\sigma_i, \sigma_j)} \\
\times \prod_{\alpha} \left( \varphi(\sum_i S^\alpha(\sigma_i)z_i - \Gamma_\alpha) \right) \\
\times \prod_{\alpha} \left( \exp\left(i \frac{\Gamma_\alpha \times \sum_i S^\alpha(\sigma_i)r_i}{2N_\phi l^2} \right) \right)
\]

where \(\{S^\alpha(\sigma)\}\) are integers. We also note the constraint that each electron must have \(N_\phi\) zeros in the wavefunction, implying

\[
N_\phi = \sum_j K(\sigma_i, \sigma_j)
\]

which correctly gives us the filling fraction

\[
\nu = \sum_{\sigma, \sigma'} K^{-1}(\sigma, \sigma').
\]

We can apply the PBCs to any of the electrons to get the constraints on the center of mass portion of the wavefunction. Applying the translation operators to any particle and using the boundary conditions, we find

\[
K(\sigma, \sigma') = \sum_{\alpha=1}^{m} S^\alpha(\sigma)S^\alpha(\sigma') \quad m \geq N_s.
\]

and

\[
\sum_{\alpha} \Gamma_\alpha S^\alpha(\sigma_i) = L_{mn}(\sigma_i)
\]

where \(L_{mn}(\sigma)\) is a primitive translation. Therefore, given that \(S^\alpha(\sigma)\) is integer valued, this implies that the matrix \(K(\sigma, \sigma')\) must be positive definite, as is necessary for thermodynamic stability. We can also impose the constraint on the lowest common denominator
\[ \text{l.c.d}\{S^\alpha(\sigma_1), S^\alpha(\sigma_2), \ldots, S^\alpha(\sigma_{N_s})\} = 1 \]  

for all values of \( \alpha \).

We now will determine the number of linearly independent solutions to these equations, or equivalently, the number of independent ways of arranging the zeros of the center of mass wavefunction. For the purposes of forming sets of zeros such that the corresponding wavefunctions are orthogonal, we may constrain the center of mass portion of the wavefunction to be invariant under independent translations of the individual centers of mass of each species, keeping in mind that we are not constraining the overall wavefunction to be invariant under these translations, only the center of mass wavefunction independent of the relative piece. In order to achieve this, we will form a superposition of states with shifted zeros as we did in the single particle case. We then wish to determine the set of allowed translations \( \{\gamma_j\} \) of the zeros such that

\[ \Gamma_\alpha \rightarrow \Gamma_\alpha + \sum_j S^\alpha(\sigma_j)\gamma_j \]  

is allowed by the boundary conditions. Inserting this into the boundary conditions, we find

\[ \gamma_j = \sum_k K^{-1}(\sigma_j, \sigma_k)L_{mn}(\sigma_k). \]  

By analogy with the single particle Hilbert space, the number of linearly independent solutions is simply given by

\[ \text{Deg} = \text{Det}K. \]  

As an example we consider the two component case where the \( K \) matrix is given by

\[ K = \begin{pmatrix} m_1 & n \\ n & m_2 \end{pmatrix} \]  

and the degeneracy is \( m_1m_2 - n^2 \). This result was noted in the context of Chern-Simons effective field theory approaches to the Hall effect [12,14] but not elaborated upon. The explicit result is crucial in numerical simulations, if one is to try to distinguish between possible ground states. The degeneracy provides another quantum number in addition to
filling fraction to determine the universality class of a given incompressible quantum fluid. It is important to make several comments.

1. Every eigenstate of a Hamiltonian that depends only on the relative position of the particles has a degeneracy of $q$ on the torus if the filling fraction is given by $p/q$. The degeneracy described here is only a feature of the state given by the Laughlin-Halperin form \( (22) \). The overall multi-component degeneracy is due to the independent translation of the zeros of the centers-of-mass of different species, subject to constraints from the correlated part of the wavefunction. This includes $q$ translations of the overall center of mass of the system, as well as translations of the zeros which can be interpreted as relative translations of the centers-of-mass of the different species.

2. We have assumed in the derivation that $\det K > 0$. If $\det K = 0$ and $K(\sigma, \sigma') = m$ the system is equivalent to a single layer $\nu = 1/m$ Laughlin state with an extra degree of freedom. In this case, the state has the usual center of mass degeneracy of $m$. In addition, the system has a residual $U(1)$ degeneracy with respect to the number of particles of each species. While the total number of particles is conserved, the Hamiltonian is invariant under rearrangements of the number of particles of each species. This wavefunction has many other interesting features, as has been investigated in several papers [10,15,17].

**IV. QUANTUM NUMBERS OF MULTI-COMPONENT STATES**

In his analysis of two-dimensional electron systems in magnetic fields subject to PBCs, Haldane [7] constructed a general symmetry formalism which clarified the center-of-mass degeneracy seen previously in finite size studies subject to PBCs. This construction has proven to be very useful in finite size studies on the torus, simplifying the calculations as well as providing a correct classification of states allowing a direct comparison with studies done in other geometries. We shall briefly review this analysis and proceed with its application to the multi-component systems.

As before we shall consider a physical region defined by the vectors $L_1$ and $L_2$ and require
physical quantities to be invariant under translation through $L_{mn}$ where $L_{mn} = mL_1 + nL_2$. We can consider the operator that generates relative translations between the particles while maintaining the center of mass, defined to be

$$\tilde{t}_i(pL_{mn}) = \prod_j t_i(pL_{mn}/N_e)t_j(-pL_{mn}/N_e)$$

where $N_e$ is the number of electrons in the system and the filling fraction is given by $\nu = N_e/N_\phi = p/q$ and the operators $t_i(a)$ are the translation operators acting on particle $i$ defined as before (I). We can rewrite the above operator, using the fact that the physical states that this operator acts upon are subject to the conditions (I) as

$$\tilde{t}_i(pL_{mn}) = t_i(pL_{mn}) \prod_j t_j(-pL_{mn}/N_e)$$

$$= (\eta_{mn})^{pN_e} \exp(ip(\Phi_0 \cdot L_{mn}))T(-\frac{L_{mn}}{N})$$

where

$$T(a) = \prod_i t_i(a)$$

is the center of mass operator. We can therefore classify the eigenstates of a translationally invariant Hamiltonian obeying $[H, T(a)] = 0$ by the quantum number $k$ defined to be

$$T(\frac{L_{mn}}{N})|\Psi\rangle = (\eta_{mn})^{pN_e} \exp\left(i \frac{(k + N_e\Phi_0) \cdot L_{mn}}{N}\right)|\Psi\rangle.$$ 

As this operator commutes with the center of mass translation operator $T(\frac{L_{mn}}{N_\phi})$, each eigenstate of a translationally invariant Hamiltonian, labeled by $k$, will have $q$ fold center of mass degeneracy. In the following we shall specialize to the boundary conditions with $\Phi_0 = 0$.

We can explicitly define eigenstates for the above operator, useful for numerical work, in the following way. We can denote a fundamental set of translations as

$$t(\frac{L_1}{N_\phi}) = t_1 \quad t(\frac{L_2}{N_\phi}) = t_2$$

and using the relation

$$t_1t_2 = t_2t_1 e^{-i \frac{2\pi}{N_\phi}}$$

(39)
we can define a basis as
\[ t_1 |m\rangle = e^{-\frac{2\pi m}{N_N} |m\rangle} \quad t_2 |m\rangle = |m + 1\rangle \] (40)
where \(|m + N_N\rangle = |m\rangle\). This is the smallest set of translations consistent with a given set of boundary conditions. If we denote the many particle occupation number state as
\[ |n_1, n_2, ..., n_{N_N}\rangle \] (41)
we can write the eigenstate of (37) in the following fashion
\[ |\Psi\rangle = \sum_k |n_1 + q_k, n_2 + q_k, ...\rangle e^{\frac{2\pi i}{N_N} \tilde{j}_y} \] (42)
If we act with the operator (37) on this state, we find
\[ T(\frac{L_{mn}}{N}) |\Psi\rangle = (-1)^{mnpq} e^{-\frac{2\pi i}{N_N} (\tilde{j}_x m + \tilde{j}_y n)} |\Psi\rangle \] (43)
where
\[ \tilde{j}_x = \sum_{t=1}^{N_N} n_t \cdot t. \] (44)
Therefore, to set up the Hilbert space, we group sets of states into their \(\tilde{j}_x\) value. We then construct the full Hilbert space by forming linear superpositions of these states, with their occupation numbers shifted, multiplied by a phase factor. These are our fundamental basis states. We note that \(\tilde{j}_x\) and \(\tilde{j}_y\) are integers defined modulo \(\bar{N}\). If we then define two reciprocal lattice vectors as
\[ \mathbf{G}_\alpha \cdot \mathbf{L}_\beta = 2\pi \delta_{\alpha\beta} \] (45)
one can see that the Brillouin zone pseudomomentum \(\mathbf{k}\) is given by
\[ \mathbf{k} = -(j_x \mathbf{G}_1 + j_y \mathbf{G}_2) \] (46)
where
\[ (j_x, j_y) = (\tilde{j}_x, \tilde{j}_y) - \left( \frac{N}{2}, \frac{N}{2} \right) \quad \text{pq = odd} \]
\[ = (\bar{j}_x, \bar{j}_y) \quad \text{pq = even} \] (47)
The most symmetrical Bravais lattice that we can consider is the hexagonal PBCs, invariant under a $\pi/3$ rotation. We have defined the $k = 0$ point in (37) to be the unique point invariant under a $\pi/3$ rotation, where $L_1 \to L_2$ and $L_2 \to L_2 - L_1$. Therefore, the invariant point must be the solution to the equation

$$T\left(\frac{L_1}{N}\right) = T\left(-\frac{L_1}{N}\right) = T\left(\frac{L_2}{N}\right) = T\left(\frac{L_2 - L_1}{N}\right).$$

(48)

If we separate the last term and solve, we conclude

$$T\left(\frac{L_{mn}}{N}\right)|\Psi\{k = 0\}\rangle = (\eta_{mn})^{pq}|\Psi\{k = 0\}\rangle.$$  

(49)

This defines the $k = 0$ point. The significance of these comments for numerical work is in the fact that in the thermodynamic limit the $k = 0$ point becomes rotationally invariant and therefore the signal for a QHE fluid is a $k = 0$ ground state with an energy gap to any excited states.

We now turn our attention to the multi-component system. In this case, the same analysis carries through as above with the occupation basis being expanded to include the pseudospin quantum number. If one acts with the operator (34) on the single layer wavefunction at $\nu = 1/m$, one finds the Laughlin state to be a $k = 0$ eigenstate. In the following we shall denote $\text{Det} K = qN'$. If one acts with the operator (34) on the wavefunction $\Psi^K[\{z_i\}]$ one can determine the quantum numbers

$$j_x = p\bar{N} \sum_i \frac{N_i}{N_e} p_i \pmod{\bar{N}}$$

(50)

and

$$j_y = p\bar{N} \sum_i \frac{N_i}{N_e} p'_i \pmod{\bar{N}}$$

(51)

where $p_i$ and $p'_i$ are specified integers and $N_i$ is the number of electrons of species $i$. One can therefore see that in the multi-component systems, the Laughlin-Halperin like ground state will have an overall degeneracy given by $\text{Det} K$ with a center of mass degeneracy given by $q$ as well as a $N'$ fold degeneracy related to the different translations of the centers of
mass of the different species. There are four points in k space that have parity invariance, 
\{j_x, j_y\} = \{0, 0\}, which corresponds to k = 0 and three states that lie on the zone boundaries 
\{\bar{N}/2, 0\}, \{0, \bar{N}/2\} and \{\bar{N}/2, \bar{N}/2\}. The relative portion of the wavefunction is an eigen-
state of parity so the k vector of the states representing a Laughlin-Halperin like ground state can only lie on one of these four points. We can therefore write

$$\text{Det} K = q(N_0 + 3N_B)$$  \hspace{1cm} (52)

where $N_0$ is the number of k = 0 states and $N_B$ is the number of zone boundary multiplets. 
There are only two possibilities

$$\frac{N}{N_0 P} \text{ = even integer} \quad N_0 = N_B = N'/4$$ \hspace{1cm} (53)

for some species $i$, otherwise

$$N_0 = N', N_B = 0.$$ \hspace{1cm} (54)

As an example, we examine the (3, 3, 1) state, represented by the matrix

$$K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$ \hspace{1cm} (55)

recently shown \[3\] to be a physically realizable state. In this case, as the filling fraction of 
the state is $\nu = 1/2$, we have a two fold center of mass degeneracy. From the above analysis, 
we also find that there should be degenerate states possessing $j_x = 0, \bar{N}/2$ and $j_y = 0, \bar{N}/2$. 
There are then four degenerate states corresponding to \{j_x, j_y\} = \{0, 0\}, a k = 0 state, 
\{\bar{N}/2, \bar{N}/2\}, \{\bar{N}/2, 0\}, and \{0, \bar{N}/2\} for an overall degeneracy of eight as expected. This 
result has been borne out by numerical studies \[19\]. It should be noted that this degeneracy

is a feature of the exact $\Psi^K[\{z_i\}]$ state, which can be generated numerically by choosing a truncated Hamiltonian possessing the correct pseudopotentials to make the $\Psi^K[\{z_i\}]$ state the unique ground state.
V. CONCLUSION

In this paper it has been shown that the degeneracy of a multi-component quantum Hall state on the torus, denoted in the standard fashion by the matrix $K$ is given by $\text{Det}K$. The quantum numbers of these multi-component states in Haldane’s symmetry analysis have also been determined. These predicted quantum numbers provide a powerful topological invariant with which one can distinguish possible ground states. One example of this is in the $\nu = 5/2$ system. There have been two wavefunctions suggested to explain the observed anomaly \cite{21,22}, both possessing the correct filling fraction. On the torus, however, in addition to their center of mass degeneracies, the Pfaffian state possesses a three fold degeneracy \cite{22} while the spin-singlet state has a five fold degeneracy \cite{23}. While this issue is not yet resolved, it is clear that the ground state degeneracy is a useful characteristic in distinguishing between these two states numerically. Lastly, there is a remarkable transition in the $\nu = 2/3$ double layer system between two different ground states that occur at the same filling fraction but with different degeneracies on the torus, as seen in numerical studies \cite{20}. It provides a fascinating example of the importance of degeneracy considerations in numerical studies on the torus.

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