HODGE THEORY OF THE TURAEV COBRACKET AND THE KASHIWARA–VERGNE PROBLEM

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Abstract. In this paper we show that, after completing in the $I$-adic topology, the Turaev cobracket on the vector space freely generated by the closed geodesics on a smooth, complex algebraic curve $X$ with an algebraic framing is a morphism of mixed Hodge structure. We combine this with results of a previous paper on the Goldman bracket to construct torsors of solutions of the Kashiwara–Vergne problem in all genera. The solutions so constructed form a torsor under a prounipotent group that depends only on the topology of the framed surface. We give a partial presentation of these groups. Along the way, we give a homological description of the Turaev cobracket.

1. Introduction

Denote the set of free homotopy classes of maps $S^1 \to X$ in a topological space $X$ by $\lambda(X)$ and the free $\mathbb{R}$-module it generates by $R\lambda(X)$. When $X$ is an oriented surface with a nowhere vanishing vector field $\xi$, there is a map

$$\delta_\xi : R\lambda(X) \to R\lambda(X) \otimes R\lambda(X),$$

called the Turaev cobracket, that gives $R\lambda(X)$ the structure of a Lie coalgebra. The cobracket was first defined by Turaev [30] on $R\lambda(M)/R$ (with no framing) and lifted to $R\lambda(M)$ for framed surfaces in [31, §18] and [3]. The cobracket $\delta_\xi$ and the Goldman bracket [9]

$$\{ , \} : R\lambda(X) \otimes R\lambda(X) \to R\lambda(X)$$

endow $R\lambda(X)$ with the structure of an involutive Lie bialgebra [31, 6, 25].

The value of the cobracket on a loop $a \in \lambda(X)$ is obtained by representing it by an immersed circle $\alpha : S^1 \to X$ with transverse self intersections and trivial winding number relative to $\xi$. Each double point $P$ of $\alpha$ divides it into two loops based at $P$, which we denote by $\alpha'_P$ and $\alpha''_P$. Let $\epsilon_P = \pm 1$ be the intersection number of the initial arcs of $\alpha'_P$ and $\alpha''_P$. The cobracket of $a$ is then defined by

$$(1.1) \quad \delta_\xi(a) = \sum_P \epsilon_P (a'_P \otimes a''_P - a''_P \otimes a'_P),$$

where $a'_P$ and $a''_P$ are the classes of $\alpha'_P$ and $\alpha''_P$, respectively.

The powers of the augmentation ideal $I$ of $R\pi_1(M, x)$ define the $I$-adic topology on it and induce a topology on $R\lambda(X)$. Kawazumi and Kuno [25] showed that $\delta_\xi$...
is continuous in the $I$-adic topology and thus induces a map

$$
\delta_\xi : R\lambda(X)^\wedge \to R\lambda(X)^\wedge \otimes R\lambda(X)^\wedge
$$
on $I$-adic completions. This and the completed Goldman bracket give $R\lambda(X)^\wedge$ the structure of an involutive completed Lie bialgebra \[25\].

Now suppose that $X$ is a smooth affine curve over $\mathbb{C}$ or, equivalently, the complement of a finite set $D$ in a compact Riemann surface $\overline{X}$. In this case $Q\lambda(X)^\wedge$ has a canonical mixed Hodge structure \[10\]. Our first main result is that the Turaev cobracket is compatible with this structure.

**Theorem 1.** If $\xi$ is a nowhere vanishing holomorphic vector field on $X$ that is meromorphic on $\overline{X}$, then

$$
\delta_\xi : Q\lambda(X)^\wedge \otimes Q(-1) \to Q\lambda(X)^\wedge \otimes Q(1)
$$
is a morphism of pro-mixed Hodge structures, so that $Q\lambda(X)^\wedge \otimes Q(1)$ is a complete Lie coalgebra in the category of pro-mixed Hodge structures.

We will call such a framing $\xi$ an algebraic framing. The main result of \[17\] asserts that

$$
\{ \, , \} : Q\lambda(X)^\wedge \otimes Q\lambda(X)^\wedge \to Q\lambda(X)^\wedge \otimes Q(1)
$$
is a morphism of mixed Hodge structure (MHS), so that $Q\lambda(X)^\wedge \otimes Q(-1)$ is a complete Lie algebra in the category of pro-mixed Hodge structures.

**Corollary 2.** If $\xi$ is an algebraic framing of $X$, then $(Q\lambda(X)^\wedge, \{ \, , \}, \delta_\xi)$ is a “twisted” completed Lie bialgebra in the category of pro-mixed Hodge structures.

By “twisted” we mean that one has to twist both the bracket and cobracket by $Q(\pm 1)$ to make them morphisms of MHS. There is no one twist of $Q\lambda(X)$ that makes them simultaneously morphisms of MHS.

Let $\vec{v}$ be a non-zero tangent vector of $\overline{X}$ at a point of $D$. Standard results in Hodge theory (see \[17, \S 10.2\]) imply:

**Corollary 3.** Hodge theory determines torsors of compatible isomorphisms

(1.2) $$(Q\lambda(X)^\wedge, \{ \, , \}, \delta_\xi) \xrightarrow{\sim} \prod_{m \geq 0} \text{Gr}^{-m} W Q\lambda(X)^\wedge, \text{Gr}^{m} \{ \, , \}, \text{Gr}^{-m} \delta_\xi$$
of the Goldman–Turaev Lie bialgebra with the associated weight graded Lie bialgebra and of the completed Hopf algebras

(1.3) $$(Q\pi_1(X, \vec{v})^\wedge \xrightarrow{\sim} \prod_{m \geq 0} \text{Gr}^{-m} W Q\pi_1(X, \vec{v})^\wedge.$$These isomorphisms are torsors under the prounipotent radical $U_{X, \vec{v}}^{\text{MT}}$ of the Mumford–Tate group of the MHS on $Q\pi_1(X, \vec{v})^\wedge$.

Such splitting of the weight filtration are called “Goldman–Turaev” formality isomorphisms in \[3\].

One can ask whether a fixed affine curve $X$ has an algebraic framing $\xi$ and, if so, how many it has. Such questions can be answered using algebraic geometry. Specifying $\xi$ is equivalent to specifying the meromorphic 1-form $\omega$ on $\overline{X}$ that takes the value 1 on it. This 1-form is holomorphic and nowhere vanishing on $X$; its divisor is supported on $D := \overline{X} - X$. So, up to rescaling, algebraic framings of $X$
correspond to canonical divisors of $\overline{X}$ supported on $D$. These (if they exist) form a principal homogeneous space over
\[ \Gamma H^1(X; \mathbb{Z}(1)) := \text{Hom}_{\text{MHS}}(\mathbb{Z}(-1), H^1(X; \mathbb{Z})) = \{ df/f : f \in H^0(X, O_X^{\times})^{\text{alg}} \}. \]
For each $\overline{X}$ and general $D \subset X$, this group is trivial, so that an algebraic framing of a general curve $X$ is unique if it exists at all.

As for existence, when $D$ consists of a single point, we can take $\overline{X}$ to be hyperelliptic and $D$ to consist of a single Weierstrass point. More generally, a result of Kontsevich and Zorich [26] implies that the locus of curves $X$ in the moduli space of curves $\mathcal{M}_g$ that possess a holomorphic differential $\omega$ whose divisor is supported on a set $D$ has codimension $\max(g-1-\#D,0)$. Similarly in the meromorphic case [4].

Our main application is to the Kashiwara–Vergne problem [3]. Solutions of the Kashiwara–Vergne problem of type $(g,n+1)$ are automorphisms $\Phi$ the complete Hopf algebra
\[ \mathbb{Q}\langle \langle x_1, \ldots, x_g, y_1, \ldots, y_g, z_1, \ldots, z_n \rangle \rangle \]
that solve the Kashiwara–Vergne equations. In [3] it is shown that the Kashiwara–Vergne problem admits solutions for all framed surfaces of genus $g \neq 1$ and for surfaces of genus 1 with certain, but not all, framings. (See [3, Thm. 6.1].)

Solutions of the Kashiwara–Vergne problem correspond to isomorphisms $\Phi$ above that induces isomorphisms of the Goldman–Turaev Lie bialgebra with the completion of its associated weight graded. Corollary [3] and [2, Thm. 5] imply that the automorphism $\Phi$ constructed from a Hodge splitting of $Q\pi_1(X, \vec{v})^\wedge$ in [17, §13.4] solves the KV equations. This constructs solutions of the Kashiwara–Vergne problem for all affine curves with an algebraic framing. To link with the results of [3], we show (Proposition 8.2) that the framings in [3] for which the Kashiwara–Vergne problem can be solved are precisely those that can be realized by an algebraic framing of an affine curve. This gives a second and independent proof of their main result and clarifies their conditions on a framing for which the KV problem has a solution\(^1\).

Solutions of the Kashiwara–Vergne (KV) problem for a framed surface of type $(g,n+1)$ form a torsor under a prounipotent group, denoted $\mathcal{K}\mathcal{R}V_{g,n+1}^d$ in [3], where $d \in \mathbb{Z}^{n+1}$ is the vector of local degrees of $\xi$ at the punctures $x_j \in D$. This group depends only on the local degrees $d$ and not on other topological invariants of $\xi$. The Hodge theoretic construction of solutions to the KV problem has the feature that it produces large torsors of solutions. The solutions of the KV problem of type $(g,n+1,d)$ constructed in this paper form a principal homogeneous space under a group $\tilde{U}_{g,n+1}^d$, which is described explicitly below. It is a subgroup of $\mathcal{K}\mathcal{R}V_{g,n+1}^d$ and we conjecture that it is equal to $\mathcal{K}\mathcal{R}V_{g,n+1}^{d^d}$. Equivalently, we conjecture that all solutions of the Kashiwara–Vergne problem arise from the Hodge theoretic constructions in this paper.

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\(^1\)To compare the two statements, one should note that if $\gamma_j$ is the boundary of sufficiently small disk in $\overline{X}$, centered at $x_j$ and, then $d_j + \text{rot}_x \gamma_j = 1$. Note that the boundary orientation conventions used in [1, 2, 3] differ from those used in this paper. Their adapted framing has the property that $d_0 = 2 - 2g$ and $d_j = 0$ for all $j \geq 1$.

\(^2\)If $d_j$ is the local degree of $\xi$ at $x_j$, then $d = (d_0, \ldots, d_n)$. The Hopf Index Theorem implies that $\sum_j d_j = 2 - 2g$. 

Let $S$ be a closed oriented surface of genus $g$ and $P = \{x_0, \ldots, x_n\}$ a finite subset. Suppose that $v_0$ is a non-zero tangent vector of $S$ anchored at $x_0$. Set $S = S - P$ and suppose that $\xi_o$ is a framing of $S$. Each our solutions to the KV problem arises by choosing a complex structure on $(S, P, v_0, \xi_o)$ and choosing a lifting of the canonical central cocharacter $\mathbb{G}_m \to \pi_1(MHS^e)$. (See [27] §10.2.) The Lie algebra $\mathfrak{u}_{g,n+1}^d$ of $\mathfrak{U}_{g,n+1}$ has a natural grading for each of our solutions of the KV problem, and the action

$$\hat{\mathfrak{u}}_{g,n+1}^d \to \text{End}(\mathbb{Q}\lambda(S)^\wedge, \{\ , \ \}, \delta)$$

is compatible with the splittings of the weight filtrations.

In order to state the next theorem, we need to introduce several prounipotent groups. Denote the category of mixed Tate motives unramified over $\mathbb{Z}$ by $\text{MTM}(\mathbb{Z})$. Denote the Lie algebra of the prounipotent radical of its tannakian fundamental group $\pi_1(\text{MTM}, \omega^B)$ (with respect to the Betti realization $\omega^B$) by $\mathbb{K}$. Its Lie algebra $\mathfrak{k}$ is non-canonically isomorphic to the free Lie algebra

$$\mathfrak{k} \cong \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \ldots)^\wedge.$$

Denote the relative completion of the mapping class group of $(S, P, \tilde{v}_0)$ by $\mathfrak{U}_{g,n+1}$ and its prounipotent radical by $\mathfrak{U}_{g,n+1}^d$. (See [12] for definitions.) These act on $\mathbb{Q}\pi_1(S, \tilde{v}_0)^\wedge$. Denote the image of $\mathfrak{U}_{g,n+1}^d$ in $\text{Aut} \mathbb{Q}\pi_1(S, \tilde{v}_0)^\wedge$ by $\mathfrak{U}_{g,n+1}^d$. The vector field $\xi_o$ determines a homomorphism $\mathfrak{U}_{g,n+1}^d \to H_1(S)$ that depends only on the vector $d$ of local degrees of $\xi$. Denote its kernel by $\mathfrak{U}_{g,n+1}$.

Denote by $\mathfrak{U}_{g,n+1}^d$ the subgroup of $\mathfrak{K} \mathfrak{R}_{g,n+1}$ generated by $\mathfrak{U}_{g,n+1}$ and $U_{X,\tilde{v}}^\text{MT}$. It is not immediately clear that this group is independent of the algebraic structure on $(X, \tilde{v})$ as the group $\text{MTM}^e_{X,\tilde{v}}$ depends non-trivially on the algebraic structure.

**Theorem 4.** If $2g + n > 1$ (i.e., $S$ is hyperbolic), then the group $\mathfrak{U}_{g,n+1}^d$ does not depend on the choice of an algebraic structure $(X, D, \tilde{v}, \xi)$ on $(S, P, v_0, \xi_o)$. The group $\mathfrak{U}_{g,n+1}^d$ is normal in $\mathfrak{U}_{g,n+1}$, and there is a canonical surjective group homomorphism $\mathfrak{K} \to \mathfrak{U}_{g,n+1}^d$, where $\mathfrak{K}$ denotes the prounipotent radical of $\pi_1(\text{MTM})$.

This result follows from a more general result, Theorem 8.12, which is proved in Section 10. We expect the homomorphism $\mathfrak{K} \to \mathfrak{U}_{g,n+1}^d$ to be an isomorphism. The injectivity of this homomorphism is closely related to Oda’s Conjecture [27] (proved in [29]) and should follow from it.

In genus 1, the associated graded Lie algebra $\text{Gr}^W_{g,1} \mathfrak{U}_{1,\tilde{v}}$ of the Lie algebra of $\mathfrak{U}_{1,\tilde{v}}$ contains contains the derivations $\delta_{2n}$ (denoted $\epsilon_{2n}$ in [20]). This implies [3] Thm. 1.5).

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3Conjecturally, the homomorphism $\mathfrak{U}_{g,n+1}^d \to \text{Aut} \mathbb{Q}\pi_1(S, \tilde{v}_0)^\wedge$ is injective, which would imply that $\mathfrak{U}_{g,n+1}^d = \mathfrak{U}_{g,n+1}$.

4Explicit presentations of the Lie algebras of the $\mathfrak{U}_{g,n+1}^d$ are known for all $n \geq 0$ when $g \neq 2$ [12, 15, 20], partial presentations (e.g., generating sets) are known when $g = 2$, [32]. Presentations of the $\mathfrak{U}_{g,n+1}^d$ can be deduced easily from these.
Conjecture 1.1. The inclusion $\hat{U}^{d}_{g,n+\mathbb{I}} \to KRV^{d}_{g,n+\mathbb{I}}$ is an isomorphism if and only if the inclusion of $\pi_1(MTM)$ into the Grothendieck–Teichmüller group is an isomorphism. In this case, $KRV^{d}_{g,n+\mathbb{I}}$ should be a split extension $1 \to U^{d}_{g,n+\mathbb{I}} \to KRV^{d}_{g,n+\mathbb{I}} \to K \to 1$.

As when proving that the Goldman bracket is a morphism of MHS [17], the proof of Theorem 1 consists in:

(i) Finding a homological description of the cobracket $\delta_\xi$ analogous to the homological description of the Goldman bracket given by Kawazumi–Kuno [24, §3]. This description gives a factorization of the cobracket.

(ii) Giving a de Rham description of the continuous dual of each map in this factorization.

(iii) Proving that, for each complex structure on $(\mathcal{S}, P, v_0, \xi_0)$, each map in this factorization of the dual cobracket is a morphism of MHS.

The homological description of the cobracket is established in Sections 4 and 5. This description appears to be new. The de Rham descriptions of the factors of the dual cobracket are given in Section 6. The proof of Theorem 1 is completed in Section 7 where it is shown that each map in the factorization of the cobracket is a morphism of MHS.

This paper is a continuation of [17]. We assume familiarity with the sections of that paper on rational $K(\pi, 1)$ spaces, iterated integrals, and Hodge theory.

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2. Notation and Conventions

Suppose that $X$ is a topological space. There are two conventions for multiplying paths. We use the topologist’s convention: The product $\alpha\beta$ of two paths $\alpha, \beta : [0, 1] \to X$ is defined when $\alpha(1) = \beta(0)$. The product path traverses $\alpha$ first, then $\beta$. We will denote the set of homotopy classes of paths from $x$ to $y$ in $X$ by $\pi(X; x, y)$. In particular, $\pi_1(X, x) = \pi(X; x, x)$. The fundamental groupoid of $X$ is the category whose objects are $x \in X$ and where $\text{Hom}(x, y) = \pi(X; x, y)$.

As in [17], we have attempted to denote complex algebraic and analytic varieties by the roman letters $X, Y, \ldots$ and arbitrary smooth manifolds (and differentiable spaces) by the letters $M, N, \ldots$. This is not always possible. The diagonal in $T \times T$ will be denoted $\Delta_T$.

The singular homology of a smooth manifold $M$ will be computed using the complex $C_*(M)$ of smooth singular chains. The complex $C^*(M)$ will denote its dual, the complex of smooth singular cochains. The de Rham complex of $M$ will be denoted by $E^*(M)$. The integration map $E^*(M) \to C^*(M; \mathbb{R})$ is thus a well-defined cochain map.

2.1. Local systems and connections. Here we regard a local system on a manifold $N$ as a locally constant sheaf. We will denote the complex of differential forms on $N$ with values in a local system $V$ of real (or rational) vector spaces by $E^*(N; V)$. As in [17], we denote the flat vector bundle associated to $V$ by $\mathcal{V}$ and
the sheaf of \( j \)-forms on \( N \) with values in \( V \) by \( \delta_N^j \otimes \mathcal{V} \). So \( E^j(N, V) \) is just the space of global sections of \( \delta_N^j \otimes \mathcal{V} \). There are therefore isomorphisms

\[
H^\bullet(E^\bullet(N; V)) \cong H^\bullet(N; V)
\]

The pullback of a local system \( V \) over \( Y \times Y \) along the interchange map \( \tau : Y^2 \to Y^2 \) will be denoted by \( V^{\text{opp}} \).

2.2. Cones. Several homological constructions will use cones. Since signs are important, we fix our conventions. The cone of a map \( \phi : A_\bullet \to B_\bullet \) of chain complexes is defined by

\[
C_\bullet(\phi) := \text{cone}(A_\bullet \to B_\bullet)[-1],
\]

where \( C_j(\phi) = B_j \oplus A_{j-1} \) with differential \( \partial(b, a) = (\partial b - \phi(a), -\partial a) \). The cone of a map \( \psi : B^\bullet \to A^\bullet \) of cochain complexes is defined by

\[
C^\bullet(\psi) := \text{cone}(B^\bullet \to A^\bullet)[-1],
\]

where \( C^j(\psi) := B^j \oplus A^{j-1} \) with differential \( d(\beta, \alpha) = (d\beta, -d\alpha - \psi(\beta)) \). Pairings of complexes

\[
\langle \ , \, \rangle_A : A^\bullet \otimes A_\bullet \to V \text{ and } \langle \ , \, \rangle_B : B^\bullet \otimes B_\bullet \to V
\]

induce the pairing

\[
\langle \ , \, \rangle : C^\bullet(\psi) \otimes C_\bullet(\phi) \to V
\]

defined by \( \langle \beta, \alpha \rangle \otimes (b, a) \mapsto \langle \alpha, a \rangle_A + \langle \beta, b \rangle_B \). It satisfies \( \langle dz, c \rangle = \langle z, dc \rangle \) and thus induces a pairing

\[
\langle \ , \, \rangle : H^\bullet(C^\bullet(\psi)) \otimes H_\bullet(C_\bullet(\phi)) \to V.
\]

3. Preliminaries

We recall and elaborate on notation from [17]. Fix a ring \( k \). Typically, this will be \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) or \( \mathbb{C} \). Suppose that \( M \) is a smooth manifold, possibly with boundary. All paths \( [0, 1] \to M \) will be piecewise smooth unless otherwise noted. Denote the space of paths \( \gamma : [0, 1] \to M \) by \( PM \). This is endowed with the compact open topology. For each \( t \in [0, 1] \), one has the map

\[
p_t : PM \to M
\]
defined by \( p_t(\gamma) = \gamma(t) \). It is a (Hurewicz) fibration.

3.1. Fibrations. The most fundamental path fibration is the map

\[
p_0 \times p_1 : PM \to M \times M.
\]

(3.1)

Its fiber over \((x_0, x_1)\) is the space \( P_{x_0, x_1}M \) of paths in \( M \) from \( x_0 \) to \( x_1 \). When \( x_0 = x_1 = x \), the fiber is the space \( \Lambda_x M \) of loops in \( M \) based at \( x \). The local system whose fiber over \((x_0, x_1)\) is \( H_0(P_{x_0, x_1}M; k) \) will be denoted by \( P_M \).

More generally, for \((t_1, \ldots, t_n) \in [0, 1]^n \) with \( 0 < t_1 \leq t_2 \leq \cdots \leq t_n < 1 \), one has the fibration

\[
\prod_{j=1}^n p_{t_j} : PM \to M^n
\]

whose fiber over \((x_1, \ldots, x_n)\) is

\[
P_{-x_1}M \times P_{x_1, x_2}M \times \cdots \times P_{x_{n-1}, x_n}M \times P_{x_{n-1}}M.
\]
Here $P_{x,M}$ denotes the space of paths terminating at $x \in M$ and $P_{x,-}M$ denotes the space of paths emanating from $x$. Since $P_{x,M}$ and $P_{x,-}M$ are contractible, the fiber of the corresponding local system over $\bar{M}$ is
\[ \pi^* P_M \otimes \pi_{2,3}^* P_M \otimes \cdots \otimes \pi_{n,n}^* P_M, \]
where $\pi_{j,k} : M^n \to M \times M$ denotes the projection onto the product of the $j$th and $k$th factors.

The “pullback path fibration” obtained by pulling back (3.1) along a smooth map $f : N \to M \times M$ will be denoted by $P_f M \to N$. When $f$ is the diagonal map $M \to M \times M$, the pullback is the fibration $p : \Lambda M \to M$ of the free loop space of $M$ over $M$. Its fiber over $x \in M$ is the space $\Lambda x M$ of loops in $M$ based at $x$. The corresponding local system will be denoted by $L_M$. It has fiber $H_0(\Lambda x M ; k)$ over $x \in M$.

3.2. Homology. The following result follows easily from the fact that a non-compact surface is a $K(\pi, 1)$ and has cohomological dimension 1. Cf. [17, Prop. 3.5].

**Proposition 3.1.** If $M$ is a surface and if $M$ is not closed, then $H_j(\Lambda M)$ vanishes (with all coefficients) for all $j > 1$. □

4. Factoring Loops

In this section $M$ is a smooth manifold and $k$ is any commutative ring. Recall from [17, §3.3] the construction of the Chas–Sullivan map $\beta_{CS} : H_0(\Lambda M) \to H_1(M; L_M)$.

It is induced by the map that takes a loop $\alpha : S^1 \to M$ to the horizontal lift $\hat{\alpha} : S^1 \to L_M$ of $\alpha$ defined by $\hat{\alpha}(\theta)(\phi) = \alpha(\phi + \theta)$. We now describe a generalization of the Chas–Sullivan map. It arises from the factorization of a loop into two arcs.

The evaluation map
\[ (4.1) \quad p_0 \times p_{1/2} : \Lambda M \to M \times M \]
is a fibration. Its fiber over $(x, y)$ is $P_{x,y}M \times P_{y,x}M$. The corresponding local system over $M \times M$ is
\[ P_M \otimes P_M^{\text{op}} \]
where $V^{\text{op}}$ denotes the pullback of the local system $V$ on $M \times M$ along the map $(x, y) \mapsto (y, x)$. The restriction of $P_M \otimes P_M^{\text{op}}$ to the diagonal $\Delta_M \cong M$, is $L_M \otimes L_M$.

Composing $\beta_{CS}$ with the maps induced on homology by the two maps $L_M \to L_M \otimes L_M$ defined by $\alpha \mapsto \alpha \otimes 1$ and $\alpha \mapsto 1 \otimes \alpha$,

where 1 denotes the horizontal section of $L_M$ whose value at $x$ is $1_x$, gives two maps $\beta_{CS} \otimes 1$ and $1 \otimes \beta_{CS} : H_0(\Lambda M) \longrightarrow H_1(M; L_M \otimes L_M)$.

Composing these with the diagonal map $\Delta_* : H_1(M; L_M \otimes L_M) \longrightarrow H_1(M^2; P_M \otimes P_M^{\text{op}})$ yields two maps $\Delta_*(\beta_{CS} \otimes 1)$ and $\Delta_*(1 \otimes \beta_{CS}) : H_0(\Lambda M) \longrightarrow H_1(M^2; P_M \otimes P_M^{\text{op}})$. 

Proposition 4.1. These two maps are identical.

Proof. Each loop $\alpha : S^1 \to M$ induces a map $\alpha^2 : S^1 \times S^1 \to M \times M$. This lifts to a horizontal section $s_\alpha$ of $P_M \otimes P_M^{\text{op}}$ defined over $(S^1 \times S^1) - \Delta_{S^1}$. It is defined by

$$s_\alpha(\theta, \phi) = \alpha' \otimes \alpha'',$$

where $\alpha'$ is the restriction of $\alpha$ to the positively oriented arc in $S^1$ from $\theta$ to $\phi$ and $\alpha''$ is its restriction to the arc from $\phi$ to $\theta$. This lift does not extend continuously to $S^1 \times S^1$, except when $\alpha$ is null homotopic.

To extend the lift, we replace $S^1 \times S^1$ by $U := [0, 2\pi] \otimes S^1$. The map

$$U \to S^1 \times S^1, \quad (t, \phi) \mapsto (t + \phi, \phi)$$

is a quotient map that takes the boundary of $U$ onto the diagonal $\Delta_{S^1}$. It induces a homeomorphism $(0, 2\pi) \times S^1 \approx (S^1 \times S^1) - \Delta$ and identifies $(0, \phi)$ with $(2\pi, \phi)$. The horizontal lift $s_\alpha : (S^1 \times S^1) - \Delta_{S^1} \to P_M \otimes P_M^{\text{op}}$ of $\alpha^2$ extends uniquely to a horizontal lift

$$U \to P_M \otimes P_M^{\text{op}},$$

which we will also denote by $s_\alpha$. The boundary of $U$ is $\{2\pi\} \times S^1 - \{0\} \times S^1$. The result follows from the fact that $\partial s_\alpha = 1 \otimes \hat{\alpha} - \hat{\alpha} \otimes 1$.

5. A Homological Description of the Turaev Cobracket

Throughout this section, $M$ will be a smooth oriented surface, possibly with boundary, and $\kappa$ is arbitrary. Denote space of non-zero tangent vectors of $M$ by $\hat{M}$ and the projection by $\pi : \hat{M} \to M$. Denote the composition of the projection $\pi : \hat{M} \to M$ with the diagonal map $\Delta : M \to M \times M$ by $\Delta$.

5.1. The group $H_\ast(M^2, \hat{M}; L_{\hat{M}}) \to P_M \otimes P_M^{\text{op}}$. The homological description of the Turaev cobracket uses a cone construction that arises from the computations in the proof of Proposition 4.1. Define

$$\iota : L_{\hat{M}} \to \Delta(\hat{P}_M \otimes P_M^{\text{op}})$$

(5.1) to be the map whose restriction to the fiber $L_{\hat{M}, v}$ over $v$ is defined by

$$\iota(x) = 1_x \otimes (\pi \circ x) - (\pi \circ x) \otimes 1_x \in H_0(P_{x, x} M) \otimes H_0(P_{x, x} M),$$

where $x, y \in \Lambda_{\hat{M}}$ and $x = \pi(v)$.

The maps $\Delta$ and $\iota$ induce a chain map

$$\Delta \otimes \iota : C_\ast(\hat{M}; L_{\hat{M}}) \to C_\ast(M; P_M \otimes P_M^{\text{op}})$$

of singular chain complexes. We can therefore form the cone

$$C_\ast(\Delta \otimes \iota) := \text{cone} \left( C_\ast(\hat{M}; L_{\hat{M}}) \to C_\ast(M; P_M \otimes P_M^{\text{op}}) \right)[-1]$$

Set

$$H_\ast(M^2, \hat{M}; L_{\hat{M}}) \to P_M \otimes P_M^{\text{op}} := H_\ast(C_\ast(\Delta \otimes \iota)).$$

For each $\alpha \in \Lambda_{\hat{M}}$ we have the 2-cycle $(s_{\pi \circ \alpha}, \hat{\alpha}) \in C_2(\hat{M}; \Delta \otimes \iota)$, where

$$s_{\pi \circ \alpha} : U \to P_M \otimes P_M^{\text{op}}$$

is the section associated to the loop $\pi \circ \alpha \in \Lambda M$ that is defined in the proof of Proposition 4.1.
Proposition 5.1. The map that takes the class of a loop \( \alpha \in \Lambda \hat{M} \) to the class of the cycle \( (s_{\pi \circ o}, \hat{\alpha}) \in C_\bullet(\hat{\Sigma}, \otimes \iota) \) defines a homomorphism
\[
\varphi : H_0(\Lambda \hat{M}) \to H_2(M^2, \hat{M}; L_{\hat{M}} \to P_M \otimes P_M^{op})
\]
whose composition with the map \( H_2(M^2, \hat{M}; L_{\hat{M}} \to P_M \otimes P_M^{op}) \to H_1(\hat{M}; L_{\hat{M}}) \) is the Chas-Sullivan map \( \beta_{CS} \) for \( \hat{M} \).

Remark 5.2. If \( M \) is not \( S^2 \), then \( M \) is a \( K(\pi, 1) \). In this case, after applying the Serre spectral sequence to the fibration \( \hat{\Sigma} \), one sees that
\[
H_2(M \times M; P_M \otimes P_M^{op}) \cong H_2(\Lambda M).
\]
This vanishes when \( M \) is not a closed surface by Proposition 3.1. Plugging this into the long exact sequence of the cone \( C_\bullet(\hat{\Sigma}, \otimes \iota) \), we obtain the commutative diagram
\[
\begin{array}{ccccccc}
H_0(\Lambda \hat{M}) & \xrightarrow{\varphi} & H_2(M^2, \hat{M}; L_{\hat{M}} \to P_M \otimes P_M^{op}) & \xrightarrow{\psi} & H_1(\Lambda \hat{M}) & \xrightarrow{\beta_{CS}} & H_1(\Lambda M) & \cdots \\
0 & & H_1(\Lambda \hat{M}) & & H_1(\Lambda M) & & \cdots \\
\end{array}
\]
whose bottom row is exact for all non-closed surfaces. For future reference, we note that the existence of the lift \( \varphi \) implies that \( \psi \circ \beta_{CS} = 0 \).

5.2. The groups \( H^\bullet_{\Delta}(M^2, N) \). Denote the singular cochain complex of a pair \((Y, Z)\) with coefficients in \( k \) by \( C^\bullet(Y, Z) \). For a continuous map \( h : T \to M^2 \), define
\[
C^\bullet_{\Delta}(M^2, T) := \text{cone} \left( C^\bullet(M^2, M^2 - \Delta_M) \xrightarrow{h^*} C^\bullet(T) \right)[-1]
\]
where \( j : M^2 - \Delta_M \to M^2 \) is the inclusion. Denote its cohomology groups by \( H^\bullet_{\Delta}(M^2, T) \). They can also be computed by the complex
\[
\text{cone} \left( C^\bullet(M^2) \xrightarrow{j^* \circ h^*} C^\bullet(M^2 - \Delta_M) \oplus C^\bullet(T) \right)[-1].
\]

Lemma 5.3. There is a long exact sequence
\[
\cdots \to H^{j-1}(T) \to H^j_{\Delta}(M^2, T) \to H^j_{\Delta}(M^2) \to H^j(T) \to \cdots.
\]

Proof. The long exact sequence comes from the short exact sequence
\[
0 \to C^\bullet(T)[-1] \to C^\bullet_{\Delta}(M^2, T) \to C^\bullet_{\Delta}(M^2) \to 0
\]
of complexes.

We are interested in the 3 cases: \( T \) is empty: \( T = \Delta_M \) and \( h \) is the inclusion; \( T = \hat{M} \) and \( h \) is the composition of the projection \( \pi \) with the diagonal map. When \( T \) is empty, the Thom isomorphism gives an isomorphism \( H^j(M) \cong H^j_{\Delta}(M^2) \). We'll consider the case \( T = \hat{M} \) in the next section. Here we consider the case \( T = \Delta_M \).

We will suppose that \( \xi \) is a nowhere vanishing vector field on \( M \). The normal bundle of the diagonal \( \Delta_M \) in \( M^2 \) is isomorphic to the tangent bundle \( TM \) of \( M \). The exponential map induces a diffeomorphism of a closed disk bundle in \( TM \) with a regular neighbourhood \( N \) of \( \Delta_M \) in \( M^2 \). By rescalling \( \xi \), we may assume that
exp\(\xi\) is mapped into \(\partial N\). We will henceforth regard \(\xi\) as the section \(\exp \xi\) of \(\partial N\). Denote the closed unit ball in \(\mathbb{R}^2\) by \(B\). We can choose a trivialization
\[
\pi \times q : V \xrightarrow{\simeq} \Delta_M \times B
\]
such that \(q \circ \xi : M \to B - \{0\}\) is null homotopic. This condition determines the homotopy class of the trivialization.

The inclusion \((N, \partial N) \hookrightarrow (M^2, M^2 - \Delta_M)\) induces an isomorphism
\[
H^*_\Delta(M^2, \Delta_M) \xrightarrow{\cong} H^*(N, \Delta_M \cup \partial N).
\]
The K"unneth Theorem implies that \(q^* : H^2(B, \partial B) \to H^2(N, \partial N)\) is an isomorphism.

**Proposition 5.4.** There is a short exact sequence
\[
0 \to H^1(\Delta_M) \to H^2(N, \Delta_M \cup \partial N) \to H^2(N, \partial N) \to 0.
\]

**Proof.** This is part of the long exact sequence of the triple \((N, \Delta_M \cup \partial N)\). Exactness on the left follows from the K"unneth Theorem (or the Thom Isomorphism Theorem); exactness on the right follows as \(\Delta_M \hookrightarrow N \xrightarrow{q} B\) is the constant map 0.

The projection \(q : N \to B\) induces an isomorphism
\[
q^* : H^2(B, \partial B) \cong H^2(B, \{0\} \cup \partial B) \to H^2(N, \Delta_M \cup \partial N).
\]
This map depends on the homotopy class of the trivialization \(\xi\). Denote the positive integral generator of \(H^2(B, \partial B)\) by \(\tau_B\). Define
\[
\tau_\xi := q^* \tau_B \in H^2(N, \Delta_M \cup \partial N).
\]
The image of \(\tau_\xi\) in \(H^2(N, \partial N)\) is the Thom class \(\tau_M\) of the tangent bundle of \(M\).

To better understand \(\tau_\xi\), suppose that \(\gamma : S^1 \to \partial N \cong \Delta_M \times \partial B\). Define the rotation number \(\text{rot}_\xi(\gamma)\) of \(\gamma\) with respect to \(\xi\) to be the rotation number of \(q \circ \gamma\) about 0 \(\in B\). Let \(\Gamma_\gamma\) to be the relative 2-cycle
\[
\Gamma_\gamma : (I \times S^1, \partial I \times S^2) \to (N, \Delta_M \cup \partial N)
\]
that corresponds to the map
\[
(I \times S^1, \partial I \times S^1) \to (B, \partial B), \quad (t, \theta) \mapsto t\gamma(\theta).
\]
Give \(I \times S^1\) has the product orientation.

**Lemma 5.5.** We have \(\langle q^* \tau_B, \Gamma_\gamma \rangle = \text{rot}_\xi(\gamma)\).

**Proof.** Write \(\tau_B = d\eta_B\) in \(C^2(B)\). Observe that \(\text{rot}_\xi(\gamma) = \langle \eta_B, \gamma \rangle\). Since \(\partial \Gamma_\gamma = \gamma - c_0\), where \(c_0\) denotes the constant map \(S^1 \to B\) with value 0,
\[
\langle q^* \tau_B, \Gamma_\gamma \rangle = \langle \tau_B, q_* \Gamma_\gamma \rangle = \langle d\eta_B, q_* \Gamma_\gamma \rangle = \langle \eta_B, q_* \partial \Gamma_\gamma \rangle = \langle \eta_B, \gamma \rangle = \text{rot}_\xi(\gamma).
\]
\[
\square
\]
5.3. **The class** $c_\xi$. In this section, we show that each non-vanishing vector field $\xi$ determines a class $c_\xi \in H^2_\Delta(M^2, \wedge M)$. Pairing with this class corresponds to intersecting with the diagonal and is a key component of the homological description of $\delta_\xi$.

**Lemma 5.6.** Each section $\xi$ of $\wedge M \to M$ determines a class $f_\xi \in H^1(\wedge M; \mathbb{Z})$ whose pullback $\xi^*f_\xi$ to $M$ vanishes and whose restriction to each fiber $\wedge M_x$ is the positive integral generator of $H^1(\wedge M_x; \mathbb{Z})$. It is characterized by these properties.

**Proof.** This follows from the Künneth Theorem and the fact that the section $\xi$ is a trivialization of $H_\beta(\wedge M_x; \mathbb{Z})$. It is unique up to homotopy. Take $f_\xi$ to be the pullback of the positive generator of $H^1(S^3; \mathbb{Z})$ under the projection $\wedge M \to \{x \in (\mathbb{R}^2 - \{0\}) \to \mathbb{R}^2 - \{0\} \to S^1$.

**Lemma 5.7.** When $\wedge M \to M$ is a trivial bundle, there is a short exact sequence

$$
\begin{array}{cccc}
0 & \longrightarrow & H^1(\wedge M) & \longrightarrow & H^2_\Delta(M^2, \wedge M) & \longrightarrow & H^2_\Delta(M^2) & \longrightarrow & 0.
\end{array}
$$

Each framing $\xi$ of $M$ induces a natural splitting $s_\xi : H^2_\Delta(M^2) \to H^2_\Delta(M^2, \wedge M)$ which depends only on the homotopy class of $\xi$.

**Proof.** This is part of the long exact sequence in Lemma 5.3. Exactness of the sequence follows from the Thom isomorphism $H^1(\Delta M) \cong H^2_\Delta(M^2)$, which implies that $H^1_\Delta(M^2) = 0$. The triviality of the tangent bundle of $M$ implies that the normal bundle of the diagonal in $M^2$ is trivial, which gives the exactness on the right.

Since $H^2_\Delta(M^2)$ is freely generated by the Thom class $\tau_M$ of $M$, to construct the lift, it suffices to lift $\tau_M$ to $H^2_\Delta(M^2, \wedge M)$. To do this, note that $\pi : \wedge M \to \Delta M$ induces a map

$$
\pi^* : H^2_\Delta(M^2, \Delta M) \to H^2_\Delta(M^2, \wedge M)
$$

and recall that $H^2_\Delta(M^2, \Delta M) \cong H^2(N, \Delta M \cup \partial N)$. Define $s_\xi(\tau_M) = \pi^*\tau_\xi$.

**Definition 5.8.** Define $c_\xi := \pi^*\tau_\xi + f_\xi \in H^2_\Delta(M^2, \wedge M)$, where $f_\xi \in H^2(\wedge M)$ is identified with its image in $H^2_\Delta(M^2, \wedge M)$.

5.4. **The pairing.** Here we define a pairing and compute the pairing of $c_\xi$ and $s_\alpha$ whose value is close to being the value $\delta_\xi(\alpha)$ of the cobracket.

**Proposition 5.9.** There is a well-defined pairing

$$
\langle \; , \; \rangle : H_2(M^2, \wedge M; \hat{L}_M \to P_M \otimes P^p_M) \otimes H^2_\Delta(M^2, \wedge M) \to H_0(M; L_M \otimes L_M).
$$

**Proof.** We continue with the notation above. Let $U = N - \partial N$. Let $r : U \to \Delta M$ be a retraction. Let $\mathcal{U} = \{M^2 - \Delta M, U\}$. It is an open cover of $M^2$. We can compute the product using $\Delta$-small chains $C^\bullet_\Delta$ and cochains $C^\bullet_\Delta$ via the pairing

$$
\text{cone}(C^*_\Delta(\wedge M)) \to C^*_\Delta(\wedge M^2; P_M \otimes P^p_M)[-1] \otimes \text{cone}(C^*_\Delta(\wedge M^2, M^2 - \Delta M) \to C^*_\Delta(\wedge M)[-1]
$$

defined by $(s, u) \otimes (\zeta, \eta) \mapsto (\zeta, s) + (\eta, u)$. It takes values in $H_0(U, (P_M \otimes P^p_M))$. This group is naturally isomorphic to $H_0(M; L_M \otimes L_M)$ as the homotopy equivalence $r : U \to M$ induces a natural isomorphism $r^*(L_M \otimes L_M) \cong (P_M \otimes P^p_M)|_U$.

Recall from the introduction (or the next proof) the notation for $\epsilon_P, \alpha_P$, and $\alpha'_P$. 

Lemma 5.10. If $\alpha : S^1 \rightarrow M$ is an immersed circle with transverse self intersections, then

$$\langle \tau_\xi, s_\alpha \rangle = \text{rot}_\xi(\alpha)(\alpha \otimes 1 - 1 \otimes \alpha) - \sum P' \epsilon_P(\alpha'_P \otimes \alpha''_P - \alpha''_P \otimes \alpha'_P)$$

where $P$ ranges over the double points of $\alpha$.

Proof. We use the notation of Section 5.2. Since the map $\alpha^2 : S^1 \times S^1 \rightarrow M^2$ maps the diagonal $\Delta_{S^1}$ in $S^1 \times S^1$ to the diagonal in $M^2$, $\alpha^2$ cannot be transverse to $\Delta_M$. However, by shrinking $N$ if necessary, we may assume that $\alpha$ is transverse to $\partial N_r$ for all $0 < r \leq 1$, where $N_r$ denotes disk sub-bundle of $N$ of radius $r$, where $N = N_1$. In this case, the inverse image of $N$ under $\alpha^2$ is a disjoint union

$$\Gamma \cup \bigcup (\theta, \phi) \bigcup \bigcup \theta, \phi$$

where $\Gamma$ is a neighbourhood of $\Delta_{S^1}$ diffeomorphic to $[-1,1] \times S^1$, and where $U_{\theta, \phi}$ is a disk about the point $(\theta, \phi) \in (S^1 \times S^1) - \Delta_{S^1}$ that corresponds to a double point of $\alpha$.

Each double point $P$ of $\alpha$ determines a pair of points $(\theta, \phi)$ and $(\phi, \theta)$ in $S^1 \times S^1 - \Delta_{S^1}$, where $\alpha(\theta) = \alpha(\phi) = P$. As in the introduction, $\alpha'_P$ denotes the restriction of $\alpha$ to the positively oriented arc in $S^1$ from $\theta$ to $\phi$, and $\alpha''_P$ denotes its restriction to the arc from $\phi$ to $\theta$. Denote the initial tangent vectors of $\alpha'_P$ and $\alpha''_P$ by $\vec{v}$ and $\vec{v}'$. The intersection number $\epsilon_P$ is defined by

$$\vec{v} \wedge \vec{v}' \in \epsilon_P \times (\text{a positive number}) \times (\text{the orientation of } M \text{ at } P).$$

An elementary computation shows that the intersection number of $\alpha^2 : S^1 \times S^1 \rightarrow M^2$ with $\Delta_M$ at $(\theta, \phi)$ is $-\epsilon_P$, and is $\epsilon_P$ at $(\phi, \theta)$. Consequently,

$$\langle \tau_\xi, Z' \rangle = -\epsilon_P \text{ and } \langle \tau_\xi, Z'' \rangle = \epsilon_P$$

where $U'$ (resp. $U''$) denotes $U_{\theta, \phi}$ (resp. $U_{\phi, \theta}$) and $Z'$ (resp. $Z''$) is the positive generator of $H_2(U', \partial U'; \mathbb{Z})$ (resp. $H_2(U'', \partial U''; \mathbb{Z})$).

The contribution of the double point $P$ to $\langle \tau_\xi, s_\alpha \rangle$ is thus

$$\langle \tau_\xi, Z' \rangle \alpha'_P \otimes \alpha''_P + \langle \tau_\xi, Z'' \rangle \alpha''_P \otimes \alpha'_P = -\epsilon_P(\alpha'_P \otimes \alpha''_P - \alpha''_P \otimes \alpha'_P).$$

(5.3) It remains to compute the contribution of the strip $\Gamma$ to $\langle \tau_\xi, s_\alpha \rangle$. The derivative $\dot{\alpha} : M \rightarrow TM$ of $\alpha$ corresponds to a section of the circle bundle $\partial N \rightarrow \Delta_M$, unique up to homotopy. By the construction preceding Lemma 5.5, this determines a relative chain $\Gamma_\dot{\alpha}$ in $(N, \Delta_M \cup \partial N)$.

The inverse image of $\Gamma$ in $[0,1] \times S^1$ under the map (1.2) is the disjoint union of two strips, $\Gamma_0$, a regular neighbourhood of $0 \times S^1$, and $\Gamma_{2\pi}$, a regular neighbourhood of $2\pi \times S^1$.

Give $\Gamma_0$ and $\Gamma_{2\pi}$ the orientation induced from $S^1 \times S^1$. Then, as classes in $H_2(N, \Delta_M \cup \partial N)$, we have

$$[\Gamma_0] = [\Gamma_\alpha] \text{ and } [\Gamma_{2\pi}] = -[\Gamma_\alpha].$$

As observed in the proof of Proposition 4.1, the restriction of $s_\alpha$ to $\Gamma_{2\pi}$ is homotopic to $1 \otimes \alpha$, and to $\Gamma_0$ is homotopic to $\alpha \otimes 1$. 
Lemma 5.5 now implies that the contribution to $\langle \tau_\xi, s_\alpha \rangle$ from $\Gamma$ is

\begin{equation}
\langle \tau_\xi, \Gamma \rangle = \langle \tau_\xi, \Gamma_{2\pi} \rangle 1 \otimes \alpha + \langle \tau_\xi, \Gamma_0 \rangle \alpha \otimes 1 = -\langle \tau_\xi, \Gamma_0 \rangle 1 \otimes \alpha + \langle \tau_\xi, \Gamma_\alpha \rangle \alpha \otimes 1
= \rot_\xi(\alpha)(\alpha \otimes 1 - 1 \otimes \alpha).
\end{equation}

The result follows by adding the contribution of the strip (5.4) to the sum of the contributions of the double points $P$. \qed

Remark 5.11. By an elementary case of a theorem of Hirsch [21] (that goes back to Whitney [33]), regular homotopy classes of immersed loops in $M$ correspond to homotopy classes of loops in $\tilde{M}$. As shown in [25], the expression for $\langle \tau_\xi, s_\alpha \rangle$ in Lemma 5.10 is constant on regular homotopy classes of immersed circles in $M$ and thus defines a map

$$H_0(\Lambda \tilde{M}) \to H_0(\Lambda M) \otimes H_0(\Lambda M).$$

5.5. A homological description of $\delta_\xi$. We can now give a homological description of the Turaev cobracket. Recall that, when $V$ is a local system over $M$, then $H_0(M; V)$ is the maximal trivial quotient of $V$. Applying this when $V = L_M \otimes L_M$, we see that there is a canonical map

$$H_0(M; L_M \otimes L_M) \to H_0(M; L_M) \otimes H_0(M; L_M) \cong H_0(\Lambda M) \otimes H_0(\Lambda M).$$

For a section $\xi : \tilde{M} \to M$, define

$$p_\xi : H_2(M^2, \tilde{M}; L_{\tilde{M}}M) \to P_M \otimes P_{\Lambda M} \to H_0(M; L_M) \otimes H_0(M; L_M) \cong H_0(\Lambda M) \otimes H_0(\Lambda M).$$

to be the composite

$$H_2(M^2, \tilde{M}; L_{\tilde{M}}M) \to P_M \otimes P_{\Lambda M} \xrightarrow{(\Lambda \xi)_*} H_0(\Lambda M) \otimes H_0(\Lambda M).$$

Each section $\xi : M \to \tilde{M}$ of $\pi$ induces a map $\Lambda \xi : \Lambda M \to \Lambda \tilde{M}$ and thus a homomorphism

$$(\Lambda \xi)_* : H_0(\Lambda M) \to H_0(\Lambda \tilde{M}).$$

It is injective as its composition with $(\Lambda \pi)_*$ is the identity. The image of a free homotopy class of $f : S^1 \to M$ corresponds to the regular homotopy class of an immersed circle $\alpha$ with $\rot_\xi(\alpha) = 0$ that is freely homotopic to $f$.

The following factorization of $\delta_\xi$ follows directly from Lemma 5.10

**Theorem 5.12.** If $\xi$ is a section of $\pi : \tilde{M} \to M$, then the diagram

$$H_0(\Lambda M) \xrightarrow{(\Lambda \xi)_*} H_0(\Lambda \tilde{M}) \xrightarrow{\varphi} H_2(M^2, \tilde{M}; L_{\tilde{M}}M) \to P_M \otimes P_{\Lambda M} \xrightarrow{-p_\xi} H_0(\Lambda M) \otimes H_0(\Lambda M)$$

commutes.

6. De Rham Aspects

In this section, in preparation for applying the machinery of Hodge theory in Section 4 we construct de Rham versions of the continuous duals of the maps used in the homological description of the Turaev cobracket given in Section 5.
6.1. Preliminaries. Suppose that $N$ is a smooth manifold with finite first Betti number and that $\mathbb{k}$ is a field of characteristic zero. We are especially interested in the case where $N$ is a rational $K(\pi, 1)$ space.

Recall from [17, §7] that $H_0(P_{x_0,x_1}; N; \mathbb{k})$ and $H_0(\Lambda M; \mathbb{k})$ have natural topologies and that their continuous duals are denoted

$$
\hat{H}^0(P_{x_0,x_1}; N; \mathbb{k}) := \text{Hom}^{cts}_\mathbb{k}(H_0(P_{x_0,x_1}; N), \mathbb{k})
$$

and

$$
\hat{H}^0(\Lambda N; \mathbb{k}) := \text{Hom}^{cts}_\mathbb{k}(H_0(\Lambda N), \mathbb{k}).
$$

Recall from [17, §8] that $\hat{L}_N$ denotes the continuous dual of the local system $L_N$. There is a natural isomorphism [17, Thm. 6.9].

$$
\hat{H}^0(\Lambda N; \mathbb{k}) \cong H^0(N; \hat{L}_N).
$$

Denote the local system over $N \times N$ whose fiber over $(x_0, x_1)$ is $\hat{H}^0(P_{x_0,x_1}; N; \mathbb{k})$ by $\hat{P}_N$ and its pullback along the interchange map $N^2 \to N^2$ by $\hat{P}_N^{op}$.

**Lemma 6.1.** Let $p : N \times N \to N$ be projection onto the first factor. If $N$ is a rational $K(\pi, 1)$, then there is a natural isomorphism of locally constant sheaves

$$
R^k p_* (\hat{P}_N \otimes \hat{P}_N^{op}) \cong \begin{cases} 
\hat{L}_N & k = 0, \\
0 & k \neq 0
\end{cases}
$$

over $N$.

**Proof.** This follows directly from [17, Cor. 9.2].

**Corollary 6.2.** If $N$ is a rational $K(\pi, 1)$, then there is a natural isomorphism

$$
H^j(N^2; \hat{P}_N \otimes \hat{P}_N^{op}) \cong H^j(N; \hat{L}_N).
$$

**Proof.** Apply the Leray spectral sequence of the projection $p : N \times N \to N$. The previous result and the fact that $N$ is a rational $K(\pi, 1)$ imply that

$$
E_2^{k,j} \cong \begin{cases} 
H^j(N; \hat{L}_N) & k = 0, \\
0 & k > 0
\end{cases}
$$

so that the spectral sequence collapses at $E_2$. 

6.1.1. Differential forms. Now $\mathbb{k}$ will be $\mathbb{R}$ or $\mathbb{C}$. We regard a local system on $N$ as a locally constant sheaf. We will denote the complex of differential forms on $N$ with values in a local system $V$ of real (or rational) vector spaces by $E^\bullet(N; V)$. In [17], we denoted the flat vector bundle associated to $V$ by $\mathcal{V}$ and the sheaf of $j$-forms on $N$ with values in $V$ by $\mathcal{E}_j^V \otimes \mathcal{V}$. So $E^j(N, V)$ is just the space of global sections of $\mathcal{E}_j^V \otimes \mathcal{V}$. There are therefore isomorphisms

$$
H^\bullet(E^\bullet(N; V)) \cong H^\bullet(N; V)
$$

To connect with [17], we point out that the flat vector bundle associated to $\hat{L}_N$ is denoted by $\hat{\mathcal{L}}_N$, and the flat vector bundle associated to $\hat{P}_N$ by $\hat{\mathcal{P}}_N$.

6.2. Continuous DR duals. In this section, $M$ is an oriented surface of non-positive Euler characteristic and $\pi : \hat{M} \to M$ is the bundle of non-zero tangent vectors of $M$. Both $M$ and $\hat{M}$ are rational $K(\pi, 1)$ spaces.  

\[\text{For } M \text{ this is proved in [17] \S5.1}. \text{ That } \hat{M} \text{ is also a rational } K(\pi, 1) \text{ follows from this using the fact that an oriented circle bundle over a rational } K(\pi, 1) \text{ is a rational } K(\pi, 1).\]
6.2.1. The continuous dual of $H_\bullet(M^2, \hat{M}; L_{\overline{M}} \to P_M \otimes P_M^{op})$. As in Section 5 we denote the composition of the projection $\pi$ with the diagonal map $M \to M^2$ by $\overline{\pi}$. There is a natural restriction mapping

$$\iota^* : \overline{\pi}^*(P_M \otimes P_M^{op}) \to \hat{L}_{\overline{M}}$$

dual to the map $[6.1]$. Its restriction

$$\hat{H}^0(\Lambda_x M) \otimes \hat{H}^0(\Lambda_x M) \to \hat{H}^0(\Lambda_x \hat{M})$$

to the fiber over $v \in \hat{M}$, where $x = \pi(v)$, is

$$f \otimes g \mapsto (\pi^* f) \otimes g(1_x) - f(1_x) \otimes (\pi^* g).$$

Since $\hat{P}_M$ and $L_{\overline{M}}$ are local systems of algebras, $\overline{\pi}$ and $\iota$ induce a DGA homomorphism

$$\overline{\pi} \otimes \iota^* : E^\bullet(M^2; \hat{P}_M \otimes \hat{P}_M^{op}) \to E^\bullet(\hat{M}, \hat{L}_{\overline{M}}).$$

Define

$$E^\bullet(M^2, \hat{M}; \hat{P}_M \otimes \hat{P}_M^{op} \to \hat{L}_{\overline{M}}) := \text{cone}(E^\bullet(M^2; \hat{P}_M \otimes \hat{P}_M^{op}) \overline{\pi} \otimes \iota^* E^\bullet(\hat{M}, \hat{L}_{\overline{M}}) [-1].$$

Denote its cohomology groups by

$$H^\bullet(M^2, \hat{M}; \hat{P}_M \otimes \hat{P}_M^{op} \to \hat{L}_{\overline{M}}).$$

**Proposition 6.3.** If $M$ is not a closed surface, then there is an exact sequence

$$\cdots \to H^1(M, L_M) \to H^1(\hat{M}, \hat{L}_{\overline{M}}) \to H^2(M^2, \hat{M}; \hat{P}_M \otimes \hat{P}_M^{op} \to \hat{L}_{\overline{M}}) \to 0.$$  

where $\hat{\psi}$ is dual to the connecting homomorphism $\psi$ in Remark 6.2.

**Proof.** The cohomology long exact sequence of the cone is

$$\cdots \to H^1(M^2, \hat{M}; \hat{P}_M \otimes \hat{P}_M^{op}) \to H^1(\hat{M}, \hat{L}_{\overline{M}}) \to H^2(M^2, \hat{M}; \hat{P}_M \otimes \hat{P}_M^{op} \to \hat{L}_{\overline{M}}) \to H^2(M^2, \hat{P}_M \otimes \hat{P}_M^{op}) \to \cdots$$

Since $M$ is not closed, it is homotopy equivalent to a wedge of circles and therefore a rational $K(\pi, 1)$ of cohomological dimension 1. In particular, $H^2(M^2, \hat{P}_M \otimes \hat{P}_M^{op})$ vanishes. Finally, Corollary 6.2 gives an isomorphism $H^1(M^2, \hat{P}_M \otimes \hat{P}_M^{op}) \cong H^1(M; L_M)$.

The cohomology of this cone is dual to the homology of the cone defined in Section 5.

**Proposition 6.4.** The pairing

$$\langle , \rangle : E^\bullet(M^2, \hat{M}; \hat{P}_M \otimes \hat{P}_M^{op} \otimes \overline{\pi}^*(\overline{\pi} \otimes \iota) \to k$$

$$\langle (\omega, \xi), (s, u) \rangle = \int_s \omega + \int_u \xi$$

defined using integration and the pairings

$$\hat{P}_M \otimes \hat{P}_M \to k$$

and

$$\hat{L}_{\overline{M}} \otimes \hat{L}_{\overline{M}} \to k$$

respects the differentials and thus induces a pairing

$$\langle , \rangle : H^\bullet(M^2, \hat{M}; \hat{P}_M \otimes \hat{P}_M^{op} \to \hat{L}_{\overline{M}}) \otimes H_\bullet(M^2, \hat{M}; L_{\overline{M}} \to P_M \otimes P_M^{op}) \to k.$$  

□
6.2.2. The continuous dual of $\varphi$. Recall [17 Prop. 8.1] that there is a map
\[ \hat{\beta}_{CS} : H^1(\tilde{M}; \tilde{L}_M) \to \tilde{H}^0(\tilde{M}). \]
dual to the dual Chas–Sullivan map
\[ \beta_{CS} : H_0(\Lambda \tilde{M}) \to H_1(\tilde{M}; L_{\tilde{M}}). \]

**Proposition 6.5.** There is a (necessarily unique) map
\[ \hat{\varphi} : H^2(M^2, \tilde{M}; P_M \otimes \tilde{P}_M \to \tilde{L}_M) \to \tilde{H}^0(\Lambda \tilde{M}) \]
that makes the diagram
\[
\begin{array}{ccc}
H^1(\tilde{M}; L_{\tilde{M}}) & \xrightarrow{\hat{\varphi}} & H^2(M^2, \tilde{M}; P_M \otimes \tilde{P}_M \to \tilde{L}_M) \\
\downarrow{\hat{\beta}_{CS}} & & \downarrow{\beta_{CS}} \\
\tilde{H}^0(\Lambda \tilde{M}) & & 0
\end{array}
\]
commute. It is defined over $\mathbb{k} = \mathbb{Q}$ and is dual to the map $\varphi$ in the sense that
\[ \langle \Omega, \varphi(z) \rangle = \langle \hat{\varphi}(\Omega), z \rangle \]
whenever $z \in H_0(\Lambda \tilde{M})$ and $\Omega \in H^2(M^2, \tilde{M}; P_M \otimes \tilde{P}_M \to \tilde{L}_M)$.

**Proof.** Since (6.1) is exact, it suffices to show that
\[ H^1(M, \tilde{L}_M) \xrightarrow{\hat{\varphi}} H^1(\tilde{M}; L_{\tilde{M}}) \xrightarrow{\hat{\beta}_{CS}} \tilde{H}(\Lambda \tilde{M}) \]
is zero, where $\hat{\psi}$ is the connecting homomorphism in the long exact sequence (6.1). Since the diagram
\[
\begin{array}{ccc}
H^1(M, \tilde{L}_M) & \xrightarrow{\hat{\varphi}} & H^1(\tilde{M}; L_{\tilde{M}}) \\
\downarrow{\beta_{CS}} & & \downarrow{\beta_{CS}} \\
H_1(M; L_M)^* & \xrightarrow{\psi^*} & H_1(\tilde{M}; L_{\tilde{M}})^* \\
\downarrow{\hat{\beta}_{CS}} & & \downarrow{\beta_{CS}} \\
H_0(\Lambda \tilde{M})^* & & 0
\end{array}
\]
commutes, where $(\ )^*$ denotes $\text{Hom}_k(\_, k)$, and since the right-hand vertical map is injective, it suffices to show that $\psi^* \circ \hat{\beta}_{CS} = 0$. But this follows from the commutative diagram in Remark 6.2 as noted there. \hfill $\square$

6.2.3. The cup product. The de Rham incarnation of the complex $C^*_\Delta(M^2, \tilde{M})$ defined in Section 5.3 is
\[ E^*_\Delta(M^2, \tilde{M}) := \text{cone}(E^*(M^2) \to E^*(M^2 - \Delta) \oplus E^*(\tilde{M}))[\![-1]\!] \]
De Rham’s Theorem and the 5-lemma imply that it computes $H^*_\Delta(M^2, \tilde{M}; k)$.

**Lemma 6.6.** There is a well-defined product
\[ \cup : H^0(M; \tilde{L}_M \otimes L_M) \otimes H^2(\tilde{M}; \tilde{P}_M) \to H^2(M^2, \tilde{M}; P_M \otimes \tilde{P}_M \to \tilde{L}_M). \]
It is dual to the pairing
\[ \langle \_, \_ \rangle : H_2(M^2, \tilde{M}; L_{\tilde{M}} \to P_M \otimes \tilde{P}_M \otimes H^2(\tilde{M}, \tilde{M}) \to H_0(M; L_M \otimes L_M). \]
of Proposition 5.9 in the sense that
\[ \langle f \cup c, z \rangle = \langle f, \langle z, c \rangle \rangle \]
for all $f \in H^0(M; \mathcal{L}_M \otimes \mathcal{L}_M)$, $c \in H^2_\Delta(M^2, \mathcal{M})$, $z \in H_2(M^2, \mathcal{M}; L_{\mathcal{M}} \to P_M \otimes P_{M}^{\text{op}})$. 

**Proof.** This result can be proved using differential forms or singular cochains. We will use differential forms. The proof using singular cochains is similar.

Choose regular neighbourhoods $U$ and $V$ of the diagonal $\Delta$ in $M^2$, where $V \subset U$, $V$ is closed and $U$ is open. Since $\Delta \to U$ is a homotopy equivalence, every flat section of $\mathcal{L}_M \otimes \mathcal{L}_M$ over the diagonal extends uniquely to a flat section of $P_M \otimes P_M^{\text{op}}$ over $U$. It follows that restriction to the diagonal induces a quasi-isomorphism

$$E^\bullet(U; P_M \otimes P_{M}^{\text{op}}) \to E^\bullet(M, \mathcal{L}_M \otimes \mathcal{L}_M).$$

Since the inclusion $\Delta \to V$ is a homotopy equivalence, the map

$$E^\bullet_\Delta(M^2) \to E^\bullet_V(M^2) := \text{cone} \left( E^\bullet(M^2) \to E^\bullet(M^2 - V) \right)[-1]$$

is a quasi-isomorphism. Denote the complex of forms of $M^2$ that vanish on $M^2 - V$ by $E^\bullet(M^2, M^2 - V)$. The 5-lemma implies that the cochain map

$$E^\bullet(M^2, M^2 - V) \to E^\bullet_V(M^2)$$

that takes $\omega$ to $[\omega, 0]$ is a quasi-isomorphism. Together these imply that $E^\bullet_\Delta(M^2, \mathcal{M})$ is quasi-isomorphic to the complex

$$\text{cone} \left( E^\bullet(M^2, M^2 - V) \to E^\bullet(\mathcal{M}) \right)[-1].$$

The cup product pairing (6.2) is induced by the map of complexes

$$E^\bullet(U, P_M \otimes P_{M}^{\text{op}}) \otimes \text{cone} \left( E^\bullet(M^2, M^2 - V) \to E^\bullet(\mathcal{M}) \right)[-1] \to E^\bullet(M^2, \mathcal{M}; P_M \otimes P_{M}^{\text{op}} \to L_{\mathcal{M}})$$

defined by $F \otimes [\omega, \eta] \mapsto [F \wedge \omega, (-1)^{|F|}(\pi^* F) \wedge \eta]$. This is a chain map according to the conventions in Section 2.2.

To prove the remaining assertion, suppose that $z$ is represented by $[s, u]$ in $C^2_\Delta(M^2, \mathcal{M}; L_{\mathcal{M}} \to P_M \otimes P_{M}^{\text{op}})$, $f$ is represented by $F \in E^0(U; P_M \otimes P_{M}^{\text{op}})$, and $c$ is represented by $[\omega, \eta] \in \text{cone} \left( E^\bullet(M^2, M^2 - V) \to E^\bullet(\mathcal{M}) \right)[-1]$. Then, $f \smile c$ is represented by $[F\omega, \pi^* F \cdot \eta]$, and

$$\langle f \smile c, z \rangle = \langle [F\omega, \pi^* F \cdot \eta], [s, u] \rangle = \int_s F\omega + \int_u F\eta.$$

On the other hand, since $F$ is locally constant,

$$\langle f, \langle z, c \rangle \rangle = \langle f \langle [\omega, \eta], [s, u] \rangle \rangle = \langle f, \int_s \omega + \int_u \eta \rangle = \int_s F\omega + \int_u F\eta.$$

$\square$

### 6.3. Factorization of the continuous dual of the Turaev cobracket

Define

$$\tilde{\delta}_\xi : \hat{H}^0(\Lambda M) \otimes \hat{H}^0(\Lambda M) \to \hat{H}^0(\Lambda M)$$

so that the diagram

$$\begin{array}{ccc}
\hat{H}^0(\Lambda M) \otimes \hat{H}^0(\Lambda M) & \xrightarrow{\hat{\delta}_\xi} & \hat{H}^0(M, \mathcal{L}_M) \otimes \hat{H}^0(M, \mathcal{L}_M) \\
\downarrow & & \downarrow \\
\hat{H}^0(\Lambda M) \otimes \hat{H}^0(\Lambda M) & \xrightarrow{(\Lambda \xi)^*} & \hat{H}^0(M, \mathcal{L}_M) \otimes \hat{H}^0(M, \mathcal{L}_M) \\
\downarrow & & \downarrow \\
\hat{H}^0(\Lambda M) & \xrightarrow{\hat{\delta}_\xi} & \hat{H}^2(M^2, \mathcal{M}; P_M \otimes P_{M}^{\text{op}} \to L_{\mathcal{M}}) \\
\end{array}$$
commutes. The next result follows directly from Theorem 5.12 and the results in Section 6.2.

**Proposition 6.7.** The map \( \tilde{\delta} \xi \) is the continuous dual of \( \delta \xi \) in the sense that
\[
\langle \tilde{\delta} \xi (f \otimes g), \alpha \rangle = \langle f \otimes g, \delta \xi (\alpha) \rangle
\]
for all \( f, g \in \check{H}^0(\Lambda M) \) and \( \alpha \in \Lambda M \).

7. Proof of Theorem 1

In this section, \( k \) will be \( \mathbb{Q} \), \( \mathbb{R} \) or \( \mathbb{C} \), as appropriate, and \( X \) will be a smooth affine curve over \( \mathbb{C} \). Equivalently, \( X \) is the complement \( X \setminus D \) of a finite subset \( D \) of a compact Riemann surface \( \overline{X} \). We will also assume that \( \xi \) is an algebraic framing of \( X \); that is, \( \xi \) is a meromorphic vector field on \( \overline{X} \) whose restriction to \( X \) is a nowhere vanishing holomorphic vector field. We prove the theorem by showing that each group in the factorization of
\[
\tilde{\delta} \xi : \check{H}^0(\Lambda X) \otimes \check{H}^0(\Lambda X) \rightarrow \check{H}^0(\Lambda X) \otimes \mathbb{Q}(-1)
\]
given in Section 6.3 has a mixed Hodge structure (MHS) and that each morphism in the factorization is a morphism of MHS. The twist by \( \mathbb{Q}(-1) \) occurs in the map \( \tilde{c} \xi \). Note that the topological factorization of \( \delta \xi \) in Section 5 implies that all of the maps in the factorization of \( \tilde{\delta} \xi \) in Section 6.3 are also defined over \( \mathbb{Q} \).

Since \( X \) and \( \check{X} \) are smooth varieties, \( \check{H}^0(\Lambda \check{X}) \) and \( \check{H}^0(\Lambda X) \) have natural MHS by [17, Cor. 10.7]. The naturality of the MHS and that fact that \( \xi \) is meromorphic on \( \overline{X} \) imply that
\[
(\Lambda \xi)^* : \check{H}^0(\Lambda \check{X}) \rightarrow \check{H}^0(\Lambda X)
\]
is a morphism of MHS. Since the map \( \check{L}_X \rightarrow \check{L}_X \otimes \check{L}_X \) is a direct limit of morphisms of admissible variations of MHS over \( X \), the Theorem of the Fixed Part (alternatively, by a direct argument that uses the construction of these MHS) implies that
\[
\text{mult} : \check{H}^0(X, \check{L}_X) \otimes^2 \rightarrow \check{H}^0(X, \check{L}_X^\otimes 2)
\]
is a morphism of MHS.

To prove that the remaining groups have natural MHS and that the maps between them are morphisms, we need to recall the following standard fact about cones of mixed Hodge complexes, which is implicit in [17].

**Lemma 7.1.** The cone \( C^\bullet(\phi) \) of a morphism \( \phi : B^\bullet \rightarrow A^\bullet \) of mixed Hodge complexes is a mixed Hodge complex, and the corresponding long exact sequence
\[
\cdots \rightarrow H^{j-1}(A^\bullet) \rightarrow H^j(C^\bullet(\phi)) \rightarrow H^j(B^\bullet) \rightarrow H^j(A^\bullet) \rightarrow \cdots
\]
is a long exact of MHS. \( \square \)

**Proposition 7.2.** Each group in the diagram
\[
\begin{array}{ccc}
\check{H}^0(\Lambda \check{X}) & \xrightarrow{\tilde{\beta}_{\text{cs}}} & \check{H}^1(\check{X}, \check{L}_X) \\
\| & & \downarrow \tilde{\varphi} \\
H^1(\check{X}, \check{L}_X) & \rightarrow & H^2(\check{X}^2, \check{X}, \check{P}_X \otimes \check{P}_X^{\text{op}}) \rightarrow \check{L}_X \rightarrow 0
\end{array}
\]
has a natural MHS, and each map is a morphism of MHS.
Proof. The work of Saito [28] implies that if $V$ is an admissible variation of MHS over the complement of a divisor $W$ with normal crossings in a smooth variety $Z$, then the complex $E^*(Z \log W; V)$ of smooth forms on $Z$ with values in the canonical extension of $V$ to $Z$ and log poles along $W$ is part of a mixed Hodge complex and is naturally quasi-isomorphic to $E^*(Z - W; V)$. In particular, it computes $H^*(Z - W; V) \otimes \mathbb{C}$, together with its Hodge and weight filtrations.

The compactification $P = \mathbb{P}(T\overline{X} \oplus \mathcal{O}_\overline{X})$ of the tangent bundle $T\overline{X}$ of $\overline{X}$ is a compactification of $\hat{X}$ whose complement $W$ is a divisor with normal crossings. The cone

$$\text{cone} \left( E^*(\overline{X}^2, \log((\overline{X} \times D) \cup (D \times \overline{X})); \hat{P}_X \otimes \hat{P}_X^{\text{op}}, E^*(P \log W; L_\overline{X}) \right)[-1]$$

is quasi-isomorphic to $E^*(X^2, \hat{X}; \hat{P}_X \otimes \hat{P}_X^{\text{op}} \rightarrow \hat{L}_\overline{X})$. Lemma [7.1] implies that it is the complex part of a mixed Hodge complex and that the bottom row of the diagram is an exact sequence of MHS.

The map $\beta_{CS}$ is morphism of MHS by Lemma [17] Lem. 11.1. The fact that the category of MHS is abelian implies that $\hat{\phi}$ is a morphism of MHS. □

Proposition 7.3. The group $H^\bullet_{\Delta}(X^2, \hat{X})$ has a natural mixed Hodge structure and $c_\xi$ is a Hodge class of type $(1, 1)$.

Proof. Let $Y$ be the blow up $\overline{X} \times \overline{X}$ at $\Delta_D$. Then $X^2 - \Delta$ is the complement of a normal crossing divisor $E$ in $Y$. Write $E = E' + \Delta_\overline{X}$. The restriction of $E'$ to the diagonal $\Delta_\overline{X}$ is $\Delta_D$.

Let $Z$ be the normal crossings compactification of $\hat{X}$ constructed in the proof of Proposition [7.2]. The commutative diagram of morphisms of complex algebraic maps

$$\begin{array}{ccc}
\hat{X} & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \\
Z - W & \xrightarrow{\Delta} & X^2 - \Delta_X \\
\downarrow & & \downarrow \\
Y - E' & = & Y - E
\end{array}$$

induces a commutative diagram

$$\begin{array}{c}
E^*(Y \log E) \xrightarrow{\Delta'} E^*(X^2) \xrightarrow{\pi^*} E^*(\hat{X}) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
E^*(X^2 - \Delta_X) \xrightarrow{\Delta^*} E^*(X) \xrightarrow{\pi^*} E^*(\hat{X}) \xrightarrow{\pi^*} E^*(Z \log W)
\end{array}$$

of DGAs in which each vertical map is a quasi-morphism. Each DGA in this diagram is the complex part of the natural mixed Hodge complex associated to the corresponding variety. The Five Lemma implies that the complex $E^\bullet_{\Delta}(X^2, \overline{X})$ is naturally quasi-isomorphic to

$$(7.1) \quad \text{cone} \left( E^*(Y \log E') \rightarrow E^*(Y \log E) \oplus E^*(Z \log W) \right)[-1]$$

Lemma [7.1] implies that it is the complex part of a mixed Hodge complex. It follows that $H^\bullet_{\Delta}(X, \hat{X})$ has a natural MHS and that the exact sequence of Lemma [5.7]

$$0 \rightarrow H^1(\hat{X}) \rightarrow H^2_{\Delta}(X^2, \hat{X}) \rightarrow H^2_{\Delta}(X^2) \rightarrow 0$$

is an exact sequence of mixed Hodge structures.
It remains to show that \( c_\xi \) is a Hodge class that spans a copy of \( \mathbb{Q}(-1) \). Recall the notation and the construction of \( c_\xi \) from Section 5.3. In particular, \( c_\xi = \pi^* \tau_\xi + f_\xi \). The topological constructions in that section imply that \( \pi^* \tau_\xi \) and \( f_\xi \) are both defined over \( \mathbb{Q} \). We first show that \( f_\xi \) is a Hodge class.

The framing \( \xi \) induces an algebraic isomorphism
\[
q : X \times \mathbb{C}^* \rightarrow \hat{X}, \quad (x, t) \mapsto t\xi(x).
\]
Denote the corresponding projection \( \hat{X} \rightarrow \mathbb{C}^* \) by \( r \). Then \( r^* dt/t \) spans a copy of \( \mathbb{Z}(1) \) in \( H^1(\hat{X}) \). Since the natural inclusion \( H^1(\hat{X}) \rightarrow H^2_\Delta(X^2, \hat{X}) \) is a morphism of MHS, \( f_\xi \) is a Hodge class.

Since \( c_\xi \) is defined over \( \mathbb{Q} \), to prove that it is a Hodge class, it suffices to show that it is a real Hodge class. To do this, we use the fact that the MHS on \( H^*_\Delta(X^2, \Delta_X) \) depends only on \( X \) and the normal bundle of \( \Delta_X \) in \( X^2 \), which is just the (holomorphic) tangent bundle \( TX \) of \( X \). This follows from the construction of a (real) mixed Hodge complex for the punctured neighbourhood of one variety in another that was constructed in [8]. That construction implies that the natural isomorphism
\[
H^*_X(TX, X) \cong H^*_\Delta(X^2, \hat{X})
\]
that is constructed using topology, is an isomorphism of real MHS. Here \( H^*_X(TX, X) \) is defined to be the homology of the complex
\[
\text{cone}(C^*(TX, \hat{X}) \rightarrow C^*(X))[−1],
\]
where the map is restriction to the zero section. The trivialization (7.2) induces a MHS morphism \( q^*: H^2(\mathbb{C}, C^*) \rightarrow H^*_\Delta(TX, \hat{X}) \). The class \( c_\xi \) is the image of the positive generator \( \tau_B \) of \( H^2(\mathbb{C}, C^*) \cong \mathbb{Z}(-1) \) under the sequence
\[
H^2(\mathbb{C}, C^*) \xrightarrow{q^*} H^*_\Delta(TX, \hat{X}) \xrightarrow{\sim} H^*_\Delta(X^2, \Delta_X).
\]
It follows that \( c_\xi \) is a real (and therefore rational) Hodge class. The final observation is that \( \pi^*: H^*_\Delta(X^2, \Delta_X) \rightarrow H^*_\Delta(X^2, \hat{X}) \) is a morphism of MHS, from which it follows that \( \pi^* c_\xi \) is a Hodge class.

\[\boxdot\]

**Corollary 7.4.** The cup product (6.2) is a morphism of MHS. Consequently, cupping with \( c_\xi \)
\[
\sim c_\xi : H^0(X; \hat{L}_X \otimes \hat{L}_X) \rightarrow H^2(X^2, \hat{X}; P_X \otimes \hat{P}_X^{\mathbb{Q}} \rightarrow \hat{L}_X) \otimes \mathbb{Q}(-1)
\]
is a morphism of MHS

8. **Torsors of Splittings of the Goldman–Turaev Lie Bialgebra**

Suppose that \( 2g + n > 1 \), where \( g \) and \( n \) are non-negative integers. Suppose that \( S \) is an \( (n + 1) \)-punctured surface of genus \( g \). Write \( S = \overline{S} - P \), where \( P = \{x_0, \ldots, x_n\} \) is a subset of \( \overline{S} \). Fix a vector \( d = (d_0, \ldots, d_n) \in \mathbb{Z}^{n+1} \). Suppose that \( \mathfrak{v}_n \) is a non-zero tangent vector of \( \overline{S} \) anchored at the point \( x_0 \) and that \( \xi_o \) is a nowhere vanishing vector field on \( S \) with local degree \( d_j \) at \( x_j \).

**Definition 8.1.** A complex structure on \( (\overline{S}, P, \mathfrak{v}_o, \xi_o) \) is an orientation preserving diffeomorphism
\[
\phi : (\overline{S}, P, \mathfrak{v}_o, \xi_o) \xrightarrow{\sim} (\overline{X}, D, \mathfrak{v}, \xi)
\]
where \( \overline{X} \) is a compact Riemann surface and \( \xi \) is a meromorphic vector field.
Complex structures on \((\overline{S}, P, \xi_o)\) and \((\overline{S}, P)\) are defined similarly.

**Proposition 8.2.** If \(g \neq 1\), or if \(g = 1, n > 0\), and at least one local degree \(d_j\) is non-zero, then every framing \((\overline{S}, P, \xi_o)\) of \(S\) has a complex structure.

Note that a once punctured genus \(g = 1\) has only one algebraic framing \(\xi\). This is the restriction of the unique translation invariant framing. It is characterized by the property that \(\text{rot}_\xi(\gamma) = 0\) for every simple closed curve \(\gamma\) in the surface.

**Proof.** This is elementary when \(g = 0\). Suppose now that \(g \geq 1\). Note that \(d = 0\) except possibly when \(g = 1\) as \(\sum_j d_j = 2 - 2g\). Regard the tangent bundle \(T\) of \(\overline{S}\) as a \(\mathbb{C}^\infty\) complex line bundle. The section \(\xi_o\) trivializes it over \(S\). We can extend it to the topological analogue of a meromorphic section of \(T\). This section will be homotopic to the section \(z \mapsto zd_j\) in a punctured neighbourhood of \(x_j\). For each complex structure \((\overline{S}, D)\) on \((\overline{S}, P)\), we have the holomorphic line bundle \(L\) over \(\overline{S}\) whose sheaf of sections is \(\mathcal{O}_{\overline{S}}(\sum d_jx_j)\). It has a meromorphic section \(s\) with divisor \(\sum d_jx_j\). When \(d \neq 0\), this section trivializes \(L\) over \(\overline{S} = \overline{X} - D\).

There is thus a homotopy class of isomorphisms

\[
\begin{array}{ccc}
T & \longrightarrow & L \\
\downarrow & & \downarrow \\
\overline{S} & \longrightarrow & \overline{X}
\end{array}
\]

under which the restriction of \(s\) to \(X\) is homotopic to \(\xi_o\). The framing \(\xi_o\) of \(S\) is algebraic on \(X\) if and only if \(L\) is isomorphic to \(T_X\), the holomorphic tangent bundle of \(X\). Equivalently, it is algebraic if and only if the divisor \(\sum d_jx_j\) is an anti-canonical divisor on \(X\).

To prove the result, we have to prove that there is a point \((\overline{X}, D)\) in the moduli space \(\mathcal{M}_{g,n+1}\) where this is the case. To this end, consider the section

\[F_d : (\overline{X}, D) \mapsto K_{\overline{X}} + \sum_{j=0}^n d_jx_j \in \text{Jac} \overline{X}\]

of the universal jacobian \(J\) over \(\mathcal{M}_{g,n}\). We have to show that it vanishes at some point of \(\mathcal{M}_{g,n+1}\).

When \(g \geq 2\), the most direct way to do this is to appeal to [25] when all \(d_j \geq 0\), and [4] in the general case, which implies that there are curves \((\overline{X}, D)\) for which \(\sum d_jx_j\) is anti-canonical.

To prove the genus 1 case we need to assume that \(d \neq 0\). In this case, we will show that for every \(\overline{X}\), there are distinct points \(\{x_0, \ldots, x_n\}\) in \(\overline{X}\) such that \(\sum d_jx_j\) is linearly equivalent to 0. Write \(\overline{X} = \mathbb{C}/\Lambda\), where \(\Lambda\) is a lattice in \(\mathbb{C}\). We may assume that each \(d_j \neq 0\). We may also take \(x_0\) to be the identity. Consider the map \(\overline{F_d} : \mathbb{C}^n \to \mathbb{C}\) defined by

\[\overline{F_d}(x_1, \ldots, x_n) = \sum_{j=1}^n d_jx_j.\]

Since all components of \(d\) are non-zero, \(\ker \overline{F_d}\) is not contained in any diagonal \(x_j = x_k\) \((j \neq k)\) or any coordinate hyperplane \(x_j = 0\). This implies that there

\[\text{If } g = 1 \text{ and } d = 0, L \text{ is the trivial holomorphic line bundle over } \overline{X} \text{ and } \xi_o \text{ is homotopic to a holomorphic section if and only if } \text{rot}_{\xi_o}(\gamma) = 0 \text{ for every simple closed curve } \gamma \text{ in } \overline{X}.\]
are solutions \( x = (x_1, \ldots, x_n) \) of \( \tilde{F}_d(x) = 0 \) with the \( x_j \) distinct and non-zero in every neighbourhood of 0. The result follows as the quotient map \( \mathbb{C}^n \to E^n \) is a biholomorphism in a neighbourhood of 0.

**Remark 8.3.** This result implies that the framings that occur in \([3, \text{Thm. 6.1}]\) are precisely those that admit a complex structure. To compare the two statements, one should note that if \( \gamma_j \) is the boundary of sufficiently small disk in \( \mathbb{S} \), centered at \( x_j \) and, then

\[
  d_j + \text{rot}_\xi \gamma_j = 1.
\]

Note that the boundary orientation conventions used in \([1, 2, 3]\) differ from those used in this paper. Their adapted framing has the property that \( d_0 = 2 - 2g \) and \( d_j = 0 \) for all \( j \geq 1 \).

**Proposition 8.4.** Each homotopy class of complex structures \([8.1]\) on \((\mathbb{S}, P, \vec{v}, \xi)\) gives a torsor of splittings \((1.2)\) and \((1.3)\). This implies that they give torsors of solutions to the Kashiwara–Vergne problem \( KV_d^{(g,n+1)} \), as defined in \([1]\). These form a torsor under the prounipotent radical \( U_{MT}^{X, \vec{v}} \) of the Mumford–Tate group of \( \mathbb{Q}\pi_1(X, \vec{v})^\wedge \).

**Proof.** By \([17, \text{Thm. 6}]\), the MHS on \( \mathbb{Q}\pi_1(X, \vec{v})^\wedge \) determines a torsor of isomorphisms

\[
  \mathbb{Q}\pi_1(X, \vec{v})^\wedge \to \prod_{m \leq 0} \text{Gr}_m^W \mathbb{Q}\pi_1(X, \vec{v})^\wedge
\]

each of which solves the problem \( KV^{(g,n+1)} \) as defined in \([1, \text{Def. 4}]\). These are a torsor under \( U_{MT}^{X, \vec{v}} \). Corollary \([2]\) implies (via the discussion in \([17, \S 10.2]\)) that the induced isomorphism

\[
  \mathbb{Q}\lambda(X)^\wedge \cong \prod_{m \leq 0} \text{Gr}_m^W \mathbb{Q}\lambda(X)^\wedge
\]

is an isomorphism of Lie bialgebras. By \([1, \text{Thm. 5}]\) each isomorphism \([8.2]\) given by Hodge theory is a solution of the higher genus Kashiwara–Vergne problem \( KV_d^{(g,n+1)} \). \( \square \)

An immediate consequence of \([2, \text{Thm. 5}]\) is that the automorphism \( \Phi \) of

\[
  \mathbb{Q}\langle (x_1, \ldots, x_g, y_1, \ldots, y_g, z_1, \ldots, z_n) \rangle
\]

constructed from the choice of lifting \( \tilde{\chi} \) of the canonical central cocharacter \( \chi : \mathbb{G}_m \to \pi_1(\text{MHS}^\wedge) \) in \([17, \S 13.4]\) is a solution of the Kashiwara–Vergne problem.

**Corollary 8.5.** Each homotopy class of complex structures \([8.7]\) on \((\mathbb{S}, P, \vec{v}, \xi)\) gives a torsor of solutions to the Kashiwara–Vergne problem \( KV_d^{(g,n+1)} \), as defined in \([1]\). These solutions form a torsor under the prounipotent radical \( U_{MT}^{X, \vec{v}} \) of the Mumford–Tate group of \( \mathbb{Q}\pi_1(X, \vec{v})^\wedge \).

Solutions of \( KV_d^{(g,n+1)} \) that arise from Hodge theory will be called *motivic solutions* as they arise from an algebraic structure on \((\mathbb{S}, P, \vec{v}, \xi)\). For a given complex structure \( \phi \), the motivic solutions will be a torsor under \( U_{MT}^{X, \vec{v}} \). All solutions of \( KV_d^{(g,n+1)} \) \([3, \text{Def. 7.1}]\) for the definition) comprise a torsor under a prounipotent group subgroup \( KRV_d^{(g,n+1)} \) of \( \text{Aut} \mathbb{Q}\pi_1(S, \vec{v})^\wedge \). For each complex structure \( \phi \)
on \((\mathcal{S}, P, \tilde{\nu}_o, \xi_o)\), there is an inclusion \(\phi_s : \mathcal{U}^\text{vol}_{X,D} \hookrightarrow \mathcal{K} \mathcal{R} \mathcal{V}^d_{g,n+1}\). These homomorphisms depend non-trivially on \(\phi\) and are, in general, not surjective.

### 8.1. Geometric automorphisms

We can also generate solutions of \(\mathcal{K} \mathcal{V}^d_{g,n+1}\) by composing the complex structure \(\phi\) with homotopy classes of topological symmetries of \((\mathcal{S}, P, \xi_o)\). These are elements of the mapping class group\(^7\)

\[\Gamma_{g,n+1} := \pi_0 \text{Diff}^+ (\mathcal{S}, P, \tilde{\nu}_o),\]

the group of isotopy classes of orientation preserving diffeomorphisms of \(\mathcal{S}\) that fix \(P\) (pointwise) and \(\tilde{\nu}_o\). It acts on \(\mathbb{Q}\pi_1(\mathcal{S}, \tilde{\nu}_o)\), the Goldman Lie algebra \((\mathbb{Q}\lambda(S), \{ , \})\) and on their \(I\)-adic completions.

**Remark 8.6.** The classification \([20]\) Thm. 1, §2.3] of strata of abelian differentials of genus \(g\) curves given by Kontsevich and Zorich implies that, when \(g \geq 3\), there can be more than one \(\Gamma_{g,n+1}\)-orbit of triples \((\mathcal{S}, P, \xi)\) with \(\xi\) having local degree vector \(d\). A topological classification of orbits of framings is given in \([23]\).

Set \(H_k = H_1(\mathcal{S}; \mathcal{k})\), where \(\mathcal{k}\) is a commutative ring. The intersection pairing is a unimodular symplectic form on \(H_Z\) and thus gives an isomorphism (Poincaré duality) of \(H_Z\) with its dual \(\text{Hom}_Z(H_Z, Z)\). There is an affine group \(\text{Sp}(H)\) whose group of \(k\)-rational points is \(\text{Sp}(H_k)\). The action of \(\Gamma_{g,n+1}\) on \(\tilde{\mathcal{S}}\) induces a surjective homomorphism

\[\rho : \Gamma_{g,n+1} \to \text{Sp}(H_Z).\]

The Torelli group \(T_{g,n+1}\) is, by definition, the kernel of \(\rho\).

**Lemma 8.7.** The action of \(\Gamma_{g,n+1}\) on \((\tilde{\mathcal{S}}, P, \xi_o)\) defines a homomorphism

\[\tau_d : \Gamma_{g,n+1} \to \text{Sp}(H_Z) \ltimes H_Z\]

whose kernel \(T^d_{g,n+1}\) is the stabilizer of \((\tilde{\mathcal{S}}, P, \xi_o)\). Consequently, there is a homomorphism \(T^d_{g,n+1} \to \mathcal{K} \mathcal{R} \mathcal{V}^d_{g,n+1}(\mathbb{Q})\).

**Proof.** The group \(\text{Diff}^+(\tilde{\mathcal{S}}, P, \tilde{\nu}_o)\) acts on \(\tilde{\mathcal{S}}\), the space of non-zero tangent vectors of \(\mathcal{S}\). This implies that

\[0 \to Z \to H_1(\tilde{\mathcal{S}}; Z) \to H_1(S; Z) \to 0\]

is an extension of \(\Gamma_{g,n+1}\)-modules. The vector field \(\xi_o\) induces a splitting \(s_o\) of this sequence, and thus an isomorphism \(H_1(\tilde{\mathcal{S}}; Z) \cong H_1(S; Z) \oplus Z\). There is therefore a homomorphism \(\Gamma_{g,n+1} \to \text{Aut} H_1(S; Z) \ltimes H^1(S; Z)\) whose kernel is the stabilizer of \(\xi_o\).

For each \(x \in P\), let \(\gamma_x\) be an oriented loop in \(S\) that bounds a small disk in \(\mathcal{S}\) centered at \(x\). Observe that \(H_1(\mathcal{S}) = H_1(S)/\langle \gamma_x : x \in P \rangle\). Since \(\Gamma_{g,n+1}\) fixes \(x \in P\), it acts trivially on \(s_o(\gamma_x)\). This implies that the image of the action lies in the subgroup \(\text{Aut} H_1(\tilde{\mathcal{S}}; Z) \ltimes H_Z\). Since the mapping class group respects the intersection pairing on \(H_Z\), the image of \(\Gamma_{g,n+1}\) lies in the subgroup \(\text{Sp}(H_Z) \ltimes H_Z\).

\(\square\)

\(\)More generally, \(\Gamma_{g,n+r}\) is the mapping class group \(\pi_0 \text{Diff}^+(\mathcal{S}, P, V)\), where \(P\) is a finite subset of \(S\) with \(\#P = n\), and \(V\) is a set of \(r\) non-zero tangent vectors (or \(r\) boundary components) that are anchored at \(r\) distinct points, none of which are in \(P\).
One might hope that $\mathcal{KRV}^d_{g,n+\Gamma}$ is the subgroup of $\text{Aut} \, \mathbb{Q}_{\pi_1}(X, \tilde{\nu}_o)^\wedge$ generated by $\mathcal{U}^d_{X,\xi}$ and the Zariski closure of the image of $T^d_{g,n+\Gamma}$. While this might be true when $g \neq 1$, it cannot be true when $g = 1$. (See Remark 8.8) To circumvent this issue, and to allow the use of Hodge theory, it is better to replace mapping class groups by their relative unipotent completions.

Denote the completion of $\Gamma_{g,m+r}$ relative to $\rho : \Gamma_{g,m+r} \to \text{Sp}(H_\mathbb{Q})$ by $\mathcal{G}_{g,m+r}$. This is an affine $\mathbb{Q}$-group that is an extension

$$1 \to \mathcal{U}_{g,m+r} \to \mathcal{G}_{g,m+r} \to \text{Sp}(H) \to 1$$

where $\mathcal{U}_{g,m+r}$ is pro-unipotent. There is a Zariski dense homomorphism $\tilde{\rho} : \Gamma_{g,m+r} \to \mathcal{G}_{g,m+r}(\mathbb{Q})$ whose composition with the homomorphism $\mathcal{G}_{g,m+r}(\mathbb{Q}) \to \text{Sp}(H_\mathbb{Q})$ is $\rho$.

The cohomology groups of $\mathcal{U}_{g,m+r}$ are ind-objects of the category of $\text{Sp}(H_\mathbb{Q})$-modules. The homology groups of $\mathcal{U}_{g,m+r}$ are their hom duals and are pro-objects of the category of $\text{Sp}(H_\mathbb{Q})$-modules.

Remark 8.8. The homomorphism $T_{g,m+r} \to \mathcal{U}_{g,m+r}(\mathbb{Q})$ induced by $\tilde{\rho}$ has Zariski dense image when $g > 1$. This follows from the right exactness of relative completion [13, Thm. 3.11] and the vanishing of $H^1(\text{Sp}_g(\mathbb{Z}), V)$ for all $V$ when $g \neq 1$. (See [13, Thm. 4.3]) When $g = 1$, $T_{1,n+\Gamma} \to \mathcal{U}_{1,n+\Gamma}(\mathbb{Q})$ is not Zariski dense. For example, $T_{1,1} \cong \mathbb{Z}$, while the Lie algebra of $\mathcal{U}_{1,1}$ is freely topologically generated by an infinite dimensional vector space, [12, Rem. 3.9, 7.2]. See also [20].

The universal property of relative completion implies that the homomorphism $\tau_d : \Gamma_{g,n+\Gamma} \to \text{Sp}(H_2) \times H_2$ induces a homomorphism $\mathcal{G}_{g,n+\Gamma} \to \text{Sp}(H) \times H$. It factors through the homomorphism $\Gamma_{g,n+\Gamma} \to \Gamma_{g,n+1}$. Denote its kernel by $\mathcal{U}^d_{g,n+\Gamma}$. The restriction of $\tau_d$ to $\mathcal{U}_{g,n+\Gamma}$ induces an $\text{Sp}(H)$-equivariant homomorphism $\mathcal{H}_1(\mathcal{U}_{g,n+\Gamma}) \to H$.

For all $m$ and $r$, the group $\mathcal{U}_{g,m+r}$ has a natural weight filtration [12]

$$\mathcal{U}_{g,m+r} = W_{-3} \mathcal{U}_{g,m+r} \supseteq W_{-2} \mathcal{U}_{g,m+r} \supseteq \cdots$$

This induces a weight filtration on its abelianization $\mathcal{H}_1(\mathcal{U}_{g,m+r})$. When $m = n + 1$ and $r = 0$, it has the property that there are $\Sigma_{n+1} \times \text{Sp}(H)$ equivariant isomorphisms

(8.3) $\mathcal{U}_{g,n+1}/W_{-2} = H_1(\mathcal{U}_{g,n+1})/W_{-2} \cong \begin{cases} 0 & g = 0; \\ H^n & g = 1; \\ H^{n+1} & g = 2; \\ H^{n+1} \oplus V & g \geq 3 \end{cases}$

where $V$ denotes the 3rd fundamental representation of $\text{Sp}(H)$ and where $\Sigma_{n+1}$ acts on the mapping class group by permuting the points and on $H^{n+1}$ by permuting the factors. (When $g = 1$, $H^n$ is identified with $(u_0, \ldots, u_n) \in H^{n+1}$ with $u_0 + \cdots + u_n = 0$.)

Proposition 8.9. Suppose that $2g + n > 1$ and that $d \in \mathbb{Z}^{n+1}$ satisfies $\sum_j d_j = 2 - 2g$. If $g = 0$, then $\mathcal{U}_{0,n+\Gamma}^d = \mathcal{U}_{0,n+\Gamma}$. If $g = 1$, then $\mathcal{U}_{1,n+\Gamma}$ is the kernel of

$$\mathcal{H}_1(\mathcal{U}_{1,n+\Gamma}) \xrightarrow{\oplus_{j=1}^n H} H_d \xrightarrow{F_d} H$$
where \( F_d(u_1, \ldots, u_n) = d_1 u_1 + \cdots + d_n u_n \). If \( g > 1 \), then \( U^d_{g,n+1} \) is the kernel of

\[
\begin{array}{ccc}
U_{g,n+1} & \longrightarrow & H_1(U_{g,n+1}) \\
& \oplus^n_{j=0} H & F_d \\
& \longrightarrow & H
\end{array}
\]

where \( F_d(u_0, \ldots, u_n) = d_0 u_0 + \cdots + d_n u_n \).

**Proof.** The map \( \Gamma_{g,n+1} \rightarrow \Gamma_{g,n+1} \) is a central extension with kernel \( \mathbb{Z} \). The corresponding map \( G_{g,n+1} \rightarrow G_{g,n+1} \) is a central extension with kernel \( \mathbb{Q}(1) \). It therefore induces an isomorphism \( H_1(U_{g,n+1})/W_{-2} \rightarrow H_1(U_{g,n+1})/W_{-2} \).

Since \( H_1(\mathbb{P}^1) = 0 \), the homomorphism \( \tau_d \) is trivial when \( g = 0 \). This implies that \( U^d_{0,n+1} = U_{0,n+1} \) for all \( n > 1 \).

Now suppose that \( g > 0 \). By a complex structure on the diagram

\[
\begin{array}{ccc}
S' & \longrightarrow & S \\
\xi & \downarrow & \downarrow \\
X & \longrightarrow & X
\end{array}
\]

we mean a homotopy class of orientation preserving isomorphisms of the previous diagram with

\[
\begin{array}{ccc}
\xi & \downarrow & \downarrow \\
\xi & \downarrow & \downarrow \\
L' & \longrightarrow & L
\end{array}
\]

where \( X = X - \{x_0, \ldots, x_n\} \), \( L \rightarrow X \) is a holomorphic line bundle of degree \( 2 - 2g \), \( L' \) is the associated \( \mathbb{C}^* \) bundle, and \( \xi \) is a meromorphic section of \( L \) over \( \overline{X} \) whose divisor is \( \sum_{j=0}^d d_j x_j \). Such a complex structure gives a point of the relative Picard scheme \( \text{Pic}_{\overline{C}/\mathcal{M}_{g,n+1}} \) over the universal curve \( C \) over \( \mathcal{M}_{g,n+1} \). That is, the choice of a complex structure on the diagram above, determines a section \( F_d \) of the universal Picard scheme. Denote the divisor classes of degree \( d \) by \( \text{Pic}^d \).

Subtracting the section that takes a pointed curve \( (X, D) \) to its canonical divisor class \( K_X \) in \( \text{Pic}^{2g-2} \overline{X} \) gives a canonical isomorphism

\[
\mathcal{J} \cong \text{Pic}^0_{\overline{C}/\mathcal{M}_{g,n+1}} \xrightarrow{\cong} \text{Pic}_{\overline{C}/\mathcal{M}_{g,n+1}}^{2g-2}
\]

of the universal Jacobian scheme. Denote the divisor classes of degree \( d \) by \( \text{Pic}^d \).

Fix a complex structure \( \phi_o : (\overline{X}, D, \xi_o) \rightarrow (\overline{X}_o, D_o, \xi_o) \) on \( (\overline{X}, P, \xi_o) \). Denote the corresponding base point \( (\overline{X}_o, D_o) \) of \( \mathcal{M}_{g,n+1} \) by \( o \). This gives an isomorphism \( \pi_1(\mathcal{M}_{g,n+1}, o) \cong \Gamma_{g,n+1} \). Denote the identity of the Jacobian of \( \overline{X}_o \) by \( z_o \). The fundamental group of the universal Jacobian \( \mathcal{J} \) over \( \mathcal{M}_{g,n+1} \) with base point \( z_o \)

\footnote{That is, the sheaf of sections of \( L \) is isomorphic to \( C_{\overline{X}}(\sum d_j x_j) \). Note that the real bundle underlying \( L \) is isomorphic to the real tangent bundle of \( \overline{X} \); it is isomorphic to the holomorphic tangent bundle of \( \overline{X} \) if and only if the divisor \( \sum d_j x_j \) is canonical.}
is an extension of $\Gamma_{g,n+1}$ by $H_Z$. The identity section induces a splitting of this extension and thus a canonical isomorphism
\[ \pi_1(\mathcal{J}, z_o) \cong \Gamma_{g,n+1} \ltimes H_Z. \]

The canonical representation $\Gamma_{g,n+\bar{1}} \to \text{Sp}(H_Z)$ induces a homomorphism
\[ \pi_1(\mathcal{J}, z_o) \cong \Gamma_{g,n+\bar{1}} \ltimes H_Z \to \text{Sp}(H_Z) \ltimes H_Z. \]

Every section $\mu$ of $\mathcal{J}$ over $\mathcal{M}_{g,n+\bar{1}}$ that vanishes at $o$ induces a homomorphism
\[ \Gamma_{g,n+\bar{1}} = \pi_1(\mathcal{M}_{g,n+\bar{1}}, o) \to \pi_1(\mathcal{J}, z_o) \to \text{Sp}(H_Z) \ltimes H_Z \]
where the right-hand homomorphism is induced by the canonical homomorphism $\Gamma_{g,n+\bar{1}} \to \text{Sp}(H_Z)$. The homologically trivial algebraic cycle $K_{\mathcal{X}} + \sum_j d_j x_j$ in $\mathcal{X}$ induces the section $F_d$ of $\mathcal{J}$. It vanishes at $o$ as $\sum d_j x_j = -K_{\mathcal{X}_o}$ in $\text{Jac} \mathcal{X}_o$. It induces the homomorphism $\kappa_d : \Gamma_{g,n+\bar{1}} \to \text{Sp}(H_Z) \ltimes H_Z$ which is the composite
\[ \Gamma_{g,n+\bar{1}} \xrightarrow{\kappa_d} \text{Sp}(H_Z) \ltimes H_Z \to \text{Sp}(H_Z) \ltimes H_Z, \]
where the second homomorphism is multiplication by $2 - 2g$ on $H_Z$ when $g > 1$ [16 Prop. 11.2] and the identity when $g = 1$ [16 §12].

When $g = 1$, the universal jacobian is just the universal elliptic curve. We can take $x_0$ to be the identity section. Then the normal function of $x_j - x_0$ is the tautological section of the universal elliptic curve corresponding to $x_j$. This section induces projection on to the $j$th factor, $1 \leq j \leq n$ in $\mathcal{X}$. When $g > 1$, the section $\kappa_j : (\mathcal{X}, D) \mapsto K_{\mathcal{X}} - (2g - 2)x_j$ induces the projection onto the $j$th factor in $\mathcal{X}$. The result now follows as
\[ K_{\mathcal{X}} + \sum_{j=0}^n d_j x_j = (2g - 2)x_0 + \sum_{j=1}^n d_j (x_j - x_0) \]
when $g = 1$ as $K_{\mathcal{X}}$ vanishes, and
\[ (2g - 2)K_{\mathcal{X}} + \sum_{j=0}^n d_j x_j = \sum_{j=0}^n d_j ((2g - 2)x_j - K_{\mathcal{X}}) \]
when $g > 1$. 

There is a natural homomorphism
\[ \mathcal{G}_{g,n+\bar{1}} \to \text{Aut} \mathbb{Q}_{\pi_1(S, \bar{v}_o)^\wedge}. \]

Denote the image of $\mathcal{U}_{g,n+\bar{1}}$ under this homomorphism by $\overline{\mathcal{U}}_{g,n+\bar{1}}$ and the image of $\mathcal{U}^{d}_{g,n+\bar{1}}$ by $\overline{\mathcal{U}}^{d}_{g,n+\bar{1}}$. Fix a complex structure [8,1] on $(\mathcal{S}, P, \bar{v}_o, \xi_o)$. This determines a Mumford–Tate group $\text{MT}_{\mathcal{X}, \bar{v}}$. Denote the subgroup of $\text{Aut} \mathbb{Q}_{\pi_1(S, \bar{v}_o)^\wedge}$ generated by $\mathcal{U}^{\text{MT}}_{\mathcal{X}, \bar{v}}$ and $\overline{\mathcal{U}}^{d}_{g,n+\bar{1}}$ by $\overline{\mathcal{U}}^{\text{MT}}_{g,n+\bar{1}}$ and $\overline{\mathcal{U}}^{d}_{g,n+\bar{1}}$.

**Lemma 8.10.** The complex structure $\phi$ on $(\mathcal{S}, P, \bar{v}_o, \xi_o)$ determines pro-MHS on the Lie algebras (and coordinate rings) of $\overline{\mathcal{U}}_{g,n+\bar{1}}$ and $\overline{\mathcal{U}}^{d}_{g,n+\bar{1}}$. The homomorphism $\overline{\mathcal{U}}_{g,n+\bar{1}} \to \text{Aut} \mathbb{Q}_{\pi_1(X, \bar{v})^\wedge}$ is a morphism of MHS.
Recall that a MHS on an affine $\mathbb{Q}$-group $G$ is, by definition, a MHS on its coordinate ring $\mathcal{O}(G)$. Equivalently, a MHS on $G$ is an algebraic action of $\pi_1(MHS)$ on $G$. A homomorphism $G_1 \to G_2$ of affine $\mathbb{Q}$-groups with MHS is a morphism of MHS if it is $\pi_1(MHS)$ equivariant.

**Proof.** The complex structure $\phi$ determines a mixed Hodge structure on $\mathcal{U}_{g,n+1}$. This gives an action of $\pi_1(MHS)$ on it, so that one has the group $\pi_1(MHS) \ltimes \mathcal{U}_{g,n+1}$. The pro-MHS on $\mathcal{Q}\pi_1(X,\bar{v})^\wedge$ corresponds to a homomorphism $\pi_1(MHS) \to \text{Aut} \mathcal{Q}\pi_1(X,\bar{v})^\wedge$. Since the homomorphism $\mathcal{U}_{g,n+1} \to \text{Aut} \mathcal{Q}\pi_1(X,\bar{v})^\wedge$ is a morphism of MHS, it extends to a homomorphism

$$\pi_1(MHS) \ltimes \mathcal{U}_{g,n+1} \to \text{Aut} \mathcal{Q}\pi_1(X,\bar{v})^\wedge.$$ 

Its image is $\hat{\mathcal{U}}_{g,n+1}$. The inner action of $\pi_1(MHS)$ on $\hat{\mathcal{U}}_{g,n+1}$ gives it a MHS. The inclusion $\hat{\mathcal{U}}_{g,n+1} \hookrightarrow \text{Aut} \mathcal{Q}\pi_1(X,\bar{v})^\wedge$ is $\pi_1(MHS)$ invariant, which implies that it is a morphism of MHS. 

**Remark 8.11.** Lest there be any confusion about the Mumford–Tate groups, we point out that the Mumford–Tate group $\text{MT}_{X,\vec{v}}$ of $\mathcal{Q}\pi_1(X,\bar{v})^\wedge$ is canonically isomorphic to the Mumford–Tate group $\text{MT}_{\lambda(X)}$ of $\mathcal{Q}\lambda(X)^\wedge$. This follows from the fact that the Mumford–Tate groups of $\mathcal{Q}\pi_1(X,\bar{v})^\wedge$ and $\text{Der} \mathcal{Q}\pi_1(X,\bar{v})^\wedge$ are isomorphic, and because the Lie algebra homomorphism

$$\mathcal{Q}\lambda(X)^\wedge \otimes \mathcal{Q}(-1) \to \text{Der} \mathcal{Q}\pi_1(X,\bar{v})^\wedge$$

induced by the Kawazumi–Kuno action is a morphism of MHS by [17, Thm. 2].

The following theorem is proved in Section 9. It and the previous lemma imply Theorem 8.1.

**Theorem 8.12.** There is an injective homomorphism $\hat{\mathcal{U}}^d_{g,n+1} \hookrightarrow K\mathcal{R}\mathcal{V}^d_{g,n+1}$ of prounipotent $\mathbb{Q}$-groups whose conjugacy class does not depend on the complex structure $\phi$. The group $\hat{\mathcal{U}}^d_{g,n+1}$ is a normal subgroup of $\hat{\mathcal{U}}^d_{g,n+1}$. There is a canonical surjection $K \to \hat{\mathcal{U}}^d_{g,n+1}/\mathcal{U}^d_{g,n+1}$, where $K$ is the prounipotent radical of $\pi_1(MTM)$.

**Remark 8.13.** The complex structure on $(\overline{\mathcal{S}}, P, \bar{\nu}, \xi_\circ)$ defines a $\mathbb{C}$-point, and thus a geometric point, $p$ of the moduli stack $\mathcal{M}_{g,n+1}/\mathbb{Q}$. The étale fundamental group $\pi_1(\mathcal{M}_{g,n+1}, p)$ is an extension

$$1 \to \Gamma^\wedge_{g,n+1} \to \pi_1^\ell(\mathcal{M}_{g,n+1}, p) \to \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1.$$

where $\Gamma^\wedge_{g,n+1}$ denotes the profinite completion of the mapping class group. For each prime number $\ell$, there is an homomorphism $\pi_1^\ell(\mathcal{M}_{g,n+1}, p) \to \text{Sp}(H_{2\ell}) \ltimes H_{2\ell}$. Denote its kernel by $\pi_1^d(\mathcal{M}_{g,n+1}, p)^d$. There is a homomorphism

$$\phi_\ell : \pi_1^d(\mathcal{M}_{g,n+1}, p)^d \to K\mathcal{R}\mathcal{V}^d_{g,n+1}(\mathbb{Q}_\ell).$$

Using weighted completion [17, §8], one can show that the Zariski closure of the image of $\phi_\ell$ is $\hat{\mathcal{U}}^d_{g,n+1}(\mathbb{Q}_\ell)$.
8.2. Splittings. Each choice of a complex structure on \((S, P, \vec{\nu}, \xi_o)\) plus a lifting of the canonical central cocharacter \(G_m \to \pi_1(MHS^\infty)\) [17] \([10.2]\) gives an isomorphism of the Goldman--Turaev Lie bialgebra with its associated weight graded Lie bialgebra, and thus a solution of the KV problem. The group \(U^d_{g,n+\Gamma}\) acts simply transitively on the set of such splittings. Next we show that this action is graded. Denote the Lie algebra of \(U^d_{g,n+\Gamma}\) by \(\hat{u}^d_{g,n+\Gamma}\).

**Proposition 8.14.** For each choice of a complex structure \(\phi : (S, P, \vec{\nu}, \xi_o) \to (X, D, \vec{\nu}, \xi)\), there is a natural MHS on \(U^d_{g,n+\Gamma}\) and the natural map

\[
U^d_{g,n+\Gamma} \to \text{Aut}(\mathbb{Q}\pi_1(X, \vec{\nu})^\wedge, \{\cdot\}, \delta)\]

is a morphism of MHS. Consequently, each choice of a lift of the central cocharacter \(\chi : G_m \to \pi_1(MHS^\infty)\) gives isomorphisms

\[
u^d_{g,n+\Gamma} \cong \prod_m \text{Gr}_m^W u^d_{g,n+\Gamma} \quad \text{and} \quad \mathbb{Q}\pi_1(X, \vec{\nu})^\wedge \cong \prod_m \text{Gr}_m^W \mathbb{Q}\pi_1(S, \vec{\nu})\]

such that the diagram

\[
\begin{array}{c}
\hat{u}^d_{g,n+\Gamma} \\
\downarrow \cong \downarrow \cong \\
\prod_m \text{Gr}_m^W \nu^d_{g,n+\Gamma} \\
\rightarrow \text{Der} \mathbb{Q}\pi_1(X, \vec{\nu})^\wedge
\end{array}
\]

commutes.

**Proof.** The complex structure determines an action of \(\pi_1(MHS)\) on \(G_{g,n+\Gamma}\) and thus a semi-direct product \(\pi_1(MHS) \ltimes G_{g,n+\Gamma}\). The inner action of \(\pi_1(MHS)\) on this group determines a MHS on its coordinate ring. As pointed out above, the homomorphism \(\pi_1(MHS) \ltimes G_{g,n+\Gamma} \to \text{Aut} \mathbb{Q}\pi_1(S, \vec{\nu})^\wedge\) is a morphism of MHS. This implies that its image \(\hat{G}_{g,n+\Gamma}\) has a MHS and that the inclusion \(\hat{G}_{g,n+\Gamma} \to \text{Aut} \mathbb{Q}\pi_1(S, \vec{\nu})^\wedge\) is a morphism of MHS.

The complex structure \(\phi\) determines a MHS on \(H = H_1(S)\), and thus a MHS on \(\text{Sp}(H) \ltimes H\). The homomorphism \(G_{g,n+\Gamma} \to \text{Sp}(H) \ltimes H\) is a morphism of MHS as it is the monodromy representation associated to the normal function \(\kappa_d\) defined in the proof of Proposition 8.9. So its kernel \(U^d_{g,n+\Gamma}\) has a natural MHS and the homomorphism \(U^d_{g,n+\Gamma} \to \text{Aut} \mathbb{Q}\pi_1(S, \vec{\nu})^\wedge\) is a morphism of MHS. The remaining statement follows from [17] Prop. 10.2. \(\square\)

9. **Proof of Theorem 8.12**

Fix a complex structure \(\phi_o : (S, P, \vec{\nu}_o, \xi_o) \to (X, D, \vec{\nu}, \xi)\) on \((S, P, \vec{\nu}_o, \xi_o)\). This determines an isomorphism \(\Gamma_{g,n+\Gamma} \cong \pi_1(M_{g,n+\Gamma, \phi_o})\) and a mixed Hodge structure on the relative completion \(G_{g,n+\Gamma}\). This MHS corresponds to an action of \(\pi_1(MHS)\) on \(G_{g,n+\Gamma}\). One can therefore form the semi-direct product

\[
\pi_1(MHS) \ltimes G_{g,n+\Gamma}\]

The natural homomorphism \(G_{g,n+\Gamma} \to \text{Aut} \mathbb{Q}\pi_1(X, \vec{\nu}_o)^\wedge\) is a morphism of MHS, [12] Lem. 4.5]. This implies that the monodromy homomorphism extends to a
homomorphism
\[ \pi_1(\text{MHS}) \ltimes \mathcal{G}_{g,n+1} \to \text{Aut} \mathbb{Q}_1(X,\bar{\nu}_o)^\wedge. \]

Denote its image by \( \hat{\mathcal{U}}_{g,n+1} \). It is an extension
\[ 1 \to \hat{\mathcal{U}}_{g,n+1} \to \hat{\mathcal{G}}_{g,n+1} \to \text{GSp}(H) \to 1, \]
where \( \hat{\mathcal{U}}_{g,n+1} \) is prounipotent.

**Remark 9.1.** Since \( \pi_1(\text{MHS}) \) normalizes \( \mathcal{G}_{g,n+1} \) and \( \mathcal{U}_{g,n+1} \), and since the Mumford–Tate group \( \mathcal{M}_{X,\hat{T}} \) is the image of \( \pi_1(\text{MHS}) \), the Mumford–Tate group normalizes \( \mathcal{G}_{g,n+1} \). This implies that \( \hat{\mathcal{G}}_{g,n+1} \) is a quotient of \( \mathcal{U}^{\mathcal{M}_{X,\hat{T}}} \). In particular, \( \hat{\mathcal{U}}_{g,n+1} \) is generated by \( \mathcal{U}^{\mathcal{M}_{X,\hat{T}}} \) and \( \mathcal{M}_{X,\hat{T}} \).

**Remark 9.2.** One can argue as in [19] that, if \( g \geq 3 \), then then \( \mathcal{U}^{\mathcal{M}_{X,\hat{T}}} \to \hat{\mathcal{U}}_{g,n+1} \) is an isomorphism (equivalently, \( \pi_1(\text{MHS}) \to \hat{\mathcal{G}}_{g,n+1} \) is surjective) if and only if the Griffiths invariant \( \nu(X) \in \text{Ext}^1_{\text{MHS}}(\mathbb{Q}, PH^3(\text{Jac} X(2))) \) of the Ceresa cycle in \( \text{Jac} X \) is non-zero, and if the points \( \kappa_j := (2g-2)x_j - K \in (\text{Jac} X)^{\wedge} \otimes \mathbb{Q}, 0 \leq j \leq n \), are linearly independent over \( \mathbb{Q} \). Similarly, one can show that \( \mathcal{U}^{\mathcal{M}_{X,\hat{T}}} \to \hat{\mathcal{G}}_{g,n+1} \) is surjective if and only if \( \nu(X) \neq 0 \) and the only relation between the \( \kappa_j \) in \( (\text{Jac} X)^{\wedge} \otimes \mathbb{Q} \) is \( \sum d_j \kappa_j = 0 \).

**Proposition 9.3.** For each complex structure \( \phi_o : (\mathcal{S}, P, \bar{\nu}_o) \to (X, D, \hat{\nu}) \), the coordinate ring \( \mathcal{O}(\hat{\mathcal{G}}_{g,n+1}/\mathcal{G}_{g,n+1}) \) has a canonical MHS. These form an admissible variation of MHS over \( \mathcal{M}_{g,n+1} \) with trivial monodromy. Consequently, the MHS on \( \mathcal{O}((\hat{\mathcal{G}}_{g,n+1}/\mathcal{G}_{g,n+1}) \) does not depend on the complex structure \( \phi_o \).

**Proof.** The first task is to show that \( \hat{\mathcal{G}}_{g,n+1} \) form a local system over \( \mathcal{M}_{g,n+1} \). This is not immediately clear, as the size of the Mumford–Tate group depends non-trivially on complex structure on \( (\mathcal{S}, P, \bar{V}) \). To this end, let \( x = (\mathcal{X}, D, \bar{\nu}) \) and denote the relative completion of \( \pi_1(\mathcal{M}_{g,n+1}, x) \) by \( \mathcal{G}_x \). Let \( y = (\mathcal{Y}, E, \bar{\nu}') \) be another point of \( \mathcal{M}_{g,n+1} \) and let \( \mathcal{G}_y \) be the relative completion of \( \pi_1(\mathcal{M}_{g,n+1}, y) \). Denote the relative completion of the torsor of paths in \( \mathcal{M}_{g,n+1} \) from \( x \) to \( y \) by \( \mathcal{G}_{x,y} \). Its coordinate ring has a natural MHS and the multiplication map
\[ \mathcal{G}_x \times \mathcal{G}_{x,y} \to \mathcal{G}_y \]
is a morphism of MHS [11]. This is equivalent to the statement that the map
\[ (\pi_1(\text{MHS}) \ltimes \mathcal{G}_x) \times \mathcal{G}_{x,y} \to \pi_1(\text{MHS}) \ltimes \mathcal{G}_y \]
defined by \( (\sigma, \lambda, \gamma) \mapsto (\sigma, \gamma^{-1} \lambda \gamma) \) is a \( \pi_1(\text{MHS}) \)-equivariant surjection, where \( \alpha \in \pi_1(\text{MHS}) \) acts by
\[ \alpha : (\sigma, \lambda, \gamma) \mapsto (\alpha \sigma \alpha^{-1}, \alpha \cdot \lambda, \alpha \cdot \gamma) \text{ and } \alpha : (\sigma, \mu) \mapsto (\alpha \sigma \alpha^{-1}, \alpha \cdot \mu). \]
The diagram
\[ \begin{array}{ccc}
(\pi_1(\text{MHS}) \ltimes \mathcal{G}_x) \times \mathcal{G}_{x,y} & \longrightarrow & \pi_1(\text{MHS}) \ltimes \mathcal{G}_y \\
\downarrow & & \downarrow \\
\text{Aut} \mathbb{Q}_1(X, \bar{\nu})^\wedge \times \mathcal{G}_{x,y} & \longrightarrow & \text{Aut} \mathbb{Q}_1(Y, \bar{\nu}')^\wedge
\end{array} \]
commutes, where \( Y = \mathcal{Y} - E \) and where the bottom arrow is induced by parallel transport in the local system whose fiber over \( x \) is \( \text{Aut} \mathbb{Q} \pi_1(X, \vec{y}) \). This implies that there is a morphism \( \hat{G}_x \times G_{x,y} \to \hat{G}_y \) that is compatible with path multiplication. It follows that the \( G_x \) form a local system over \( \mathcal{M}_{g,n+1} \).

We now prove the remaining assertions. The monodromy action of \( \Gamma_{g,n+1} \) on \( \hat{G}_{g,n+1}/\mathcal{G}_{g,n+1} \) is the composite

\[
\Gamma_{g,n+1} \to \mathcal{G}_{g,n+1}(\mathbb{Q}) \to \text{Aut}(\hat{G}_{g,n+1}/\mathcal{G}_{g,n+1})(\mathbb{Q}),
\]

where the first homomorphism is the canonical map, and the second is induced by conjugation. It is easily seen to be trivial as \( \mathcal{G}_{g,n+1} \) is normal in \( \hat{G}_{g,n+1} \).

The coordinate ring of \( \hat{G}_{g,n+1}/\mathcal{G}_{g,n+1} \) has a MHS as the inclusion \( \mathcal{G}_{g,n+1} \to \hat{G}_{g,n+1} \) is a \( \pi_1(\text{MHS}) \)-equivariant. This variation has no monodromy, and so is constant by the theorem of the fixed part. Since \( \hat{G}_{g,n+1} = \hat{G}_{g,n+1} \cap \mathcal{G}_{g,n+1} \), the map

\[
\hat{U}_{g,n+1}/\mathcal{U}_{g,n+1} \to \hat{G}_{g,n+1}/\mathcal{G}_{g,n+1}
\]

is a \( \pi_1(\text{MHS}) \)-equivariant inclusion. It follows that \( \hat{U}_{g,n+1}/\mathcal{U}_{g,n+1} \) is also a constant variation of MHS over \( \mathcal{M}_{g,n+1} \). \( \square \)

**Proposition 9.4.** There is a canonical surjection \( \pi_1(\text{MTM}) \to \hat{G}_{g,n+1}/\mathcal{G}_{g,n+1} \).

This map should be an isomorphism and should follow from the proalgebraic analogue of the proof of Oda’s Conjecture \( [29] \).

**Sketch of Proof.** Since the variation \( \mathcal{O}(\hat{U}_{g,n+1}/\mathcal{U}_{g,n+1}) \) is constant, it extends over the boundary of \( \mathcal{M}_{g,n+1} \). Since the variation of MHS over \( \mathcal{M}_{g,n+1} \) with fiber \( \mathcal{G}_{g,n+1} \) is admissible, it has a limit MHS at each tangent vector of the boundary divisor \( \Delta \) of \( \mathcal{M}_{g,n+1} \). These tangent vectors correspond to first order smoothings of an \((n+1)\)-pointed stable nodal curve of genus \( g \) together with a tangent vector at the initial point. For each such maximally degenerate stable curve\( \Gamma(X_0, P_0, \vec{v}_0) \) of type \((g, n + 1)\), Ihara and Nakamura \( [22] \) construct a proper flat curve

\[
\mathcal{X} \to \text{Spec} \mathbb{Z}[[q_1, \ldots, q_N]], \quad N = \dim \mathcal{M}_{g,n+1} = 3g + n - 2
\]

with sections \( x_j \), \( 0 \leq j \leq n \) and a non-zero tangent vector field \( \vec{v} \) along \( x_0 \) that specialize to the points of \( P_0 \) and the tangent vector \( \xi_0 \) at \( q = 0 \). The projection is smooth away from the divisor \( q_1q_2\cdots q_N = 0 \). These are higher genus generalizations of the Tate curve in genus 1.

There is a limit MHS each of

\[
\mathbb{Q} \pi_1(X_\vec{q}, \vec{v})^\wedge, \ \mathcal{O}(\hat{U}_{g,n+1}), \ \mathcal{O}(\hat{U}_{g,n+1})
\]

corresponding to the tangent vector \( \vec{q} := \sum_{j=1}^{N} \partial/\partial q_j \) of \( \mathcal{M}_{g,n+1} \) at the point corresponding to \((X_0, P_0, \vec{v}_0)\). These can be thought of as MHSs on the invariants of \((X_\vec{q}, \vec{v})\), where \( X_\vec{q} \) denotes the fiber of \( \mathcal{X} \) over \( \vec{q} \) and \( X_\vec{q} \) the corresponding affine curve.

The main result of \( [13] \) is that these MHS are Hodge realizations of objects of \( \text{MTM} \). This implies that each has an action of \( \pi_1(\text{MTM}) \) and that the action of

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9These correspond to pants decompositions of \((\vec{S}, P, \vec{v})\) and also to the 0-dimensional strata of \( \mathcal{M}_{g,n+1} \).
\[ \pi_1(\text{MHS}) \] on each factors through the canonical surjection \( \pi_1(\text{MHS}) \to \pi_1(\text{MTM}) \). Brown’s result \[5\] asserts that \( \pi_1(\text{MTM}) \) acts faithfully on \( \mathbb{Q}\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{v}_0)^\wedge \). This implies that it also acts faithfully on \( \mathbb{Q}\pi_1(X_{\vec{q}}, \vec{v})^\wedge \) as (by the construction in \[18\]), the unipotent path torsor of \( X_{\vec{q}} \) is built up from the path torsors of copies of \( \mathbb{P}^1 - \{0, 1, \infty\} \) (and is 6 canonical tangent vectors) in \( X_{\vec{q}} \). In other words, \( \text{MT}_{X_{\vec{q}}, \vec{v}} \) is naturally isomorphic to \( \pi_1(\text{MTM}) \). This implies that there is a surjective homomorphism \( h : \pi_1(\text{MTM}) \to \widehat{U}_{g,n+1} / U_{g,n+1} \).

\[ \square \]

**Corollary 9.5.** There is a canonical surjection \( \mathcal{K} \to \widehat{U}_{g,n+1}^d / U_{g,n+1}^d \).

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