Birger method of variable parameters for the problems of flexible plates

O A Saltykova1, E Yu Krylova2, V A Krysko1

1Mathematic and Modelling Department, Yuri Gagarin State Technical University of Saratov, 410054, Politechnicheskaya str., 77, Saratov, Russia
2Mathematical and Computer Modelling Department, Saratov State University, 410012, Astrahanskaya str., 83, Saratov, Russia

olga_a_saltykova@mail.ru

Abstract. The paper describes the application of the Birger method of variable parameters as applied to solving problems of flexible plates taking into account physical nonlinearity. The load-deflection curves for various values of the geometric and physical-geometric parameters are given. The limiting values of the physic-geometric parameter are calculated at which the calculation should be carried out with one or two non-linearities.

1. Introduction
Physically non-linear flexible plates are widely used in modern structures for critical purposes. There are a large number of methods for solving the problems of the theory of plates and shells [1-3]. In modern literature the finite differences method [4-6] and the finite elements method [7-9] are widely used. However, to solve the problems of the theory of inhomogeneous shells, these methods cannot always be applied.

To solve the problems of the theory of inhomogeneous shells, one can use the Birger variational method [10, 11]. In this case, the choice of coordinate functions that must satisfy the following conditions is very important:
- Belong to the functional domain.
- Must be linearly independent.
- The condition for the completeness of the coordinate system in some linear separable metric space.

Trigonometric, hyperbolo-trigonometric and power polynomials are used as coordinate functions.

2. Formulation of the problem
When deriving the variational relations, it was assumed that the dependences for the Young's modulus $E = E(x, y, z, \varepsilon_0, \varepsilon_f)$ and Poisson's ratio $\nu = \nu(x, y, z, \varepsilon_0, \varepsilon_f)$ are given, i.e., it is considered, according to the method of variable elastic parameters, that the construction has an inhomogeneous structure, where $E$ and $\nu$ are depending on the deformed state at the point.

In the theory of small elasto-plastic strains, the tensile modulus $E$ and the transverse compression coefficient $k$ are related to the shear moduli $G$ and volumetric strain $K$ by the dependences:
Here we assume that $k = k_0 = \text{const}$. \[ E = \frac{9kG}{3k + G}, \quad \nu = \frac{1}{2} \frac{3k + 2G}{3k + G} \] (1)

where $\sigma_i, \varepsilon_i$ - stress and strain intensities, respectively.

For compactness of the record, we write the differential equations in mixed form in the operator form:

\[ \frac{\partial^2 w}{\partial t^2} + \varepsilon \frac{\partial w}{\partial t} = f_1(w, F), \quad f_2(w, F) = 0. \] (3)

We apply the Bubnov method in higher approximations to these equations with respect to spatial coordinates; as a result, we obtain a system of ordinary differential equations and a system of algebraic equations for mixed equations.

At each time step, an iterative process of simple iterations of the method of variable elastic parameters takes place. Integration and differentiation in the reduction to the Cauchy problem was performed numerically (integration - according to the Newton-Cotes formulas, differentiation - according to the formulas of finite differences of the second order of accuracy). First, the volume of the plate should be divided by a rectangular spatial grid and, at each point in the volume, know the stress-strain state. It does not list all the dependencies of this algorithm due to its extreme cumbersomeness [12].

The convergence of the Bubnov method in higher approximations is investigated and it is established that for the problem posed, it is necessary to take into account 8 terms in the approximating functions for a flexible plate.

Let the sheath material obey the ideal elastic-plastic diagram:

\[ \sigma_i = 3G_0\varepsilon_i \text{ for } \varepsilon_i < \varepsilon_s, \]
\[ \sigma_i = \sigma_s \text{ for } \varepsilon_i \geq \varepsilon_s. \] (4)

Where $G_0$ - dimensionless parameter $G = G_0G, E = G_0E$, $\varepsilon_s$ - yield strain, $\sigma_s$ - yield strength, $\nu_0 = 0.28; K = 1.94G_0$.

In addition to the described dependence, there are others, for example: the law for an elastoplastic body with linear hardening $\sigma_i = 3G_0\varepsilon_i, \text{ for } \varepsilon_i < \varepsilon_s$, $\sigma_i = 3G_0\varepsilon_s + 3G_1(\varepsilon_i - \varepsilon_s), \text{ for } \varepsilon_i \geq \varepsilon_s$, the dependence of H.B. Bülfinger [13] $\sigma_i = Ae_i^m(0 \leq m \leq 1)$, the cubic dependence [14] $\sigma_i = E\varepsilon_i - m\varepsilon_i^3$, the law of V.V. Sokolovsky [15] $\sigma_i = \frac{Ae_i}{\left[1 + \left(\varepsilon_i m^{-1}\right)^2\right]^{3/2}}$, the Ramberg-Osgood law [16] $\sigma_i = E\varepsilon_i + Ae_i^m$, diagram for pure aluminum [17] $\sigma_i = \sigma_s \left[1 - \exp\left(-\frac{\varepsilon_i}{\varepsilon_s}\right)\right]$ and others.
3. Results

Of great scientific and practical interest are studies on the identification of limiting values \( e_s \) at which the calculation should be carried out with one nonlinearity or both simultaneously. Recently, more and more attention has been paid to tasks of this type.

Let us consider the square in terms of plates made of a material that obeys the ideal elastic-plastic dependence \( \sigma_t(e_s) \) (4), the value of the physical-geometric parameter \( e_s = 0.098; 0.882; 1.56; 2.45; 4.8 \) (Fig. 1). The initial value of the Poisson's ratio \( \nu_0 = 0.28, K = 1.94G_0 \)

The geometric parameter \( \lambda = \frac{a}{2h_0} = 10; 30; 40; 50; 70 \), where \( a = b \) are the length and width of the plate, \( 2h_0 \) is the thickness of the plate in the center.

The boundary conditions correspond to the articulated bearing along the contour, the load is evenly distributed over the entire surface of the plate. The solution is given by the Bubnov method for \( M = N = 8 \) in (5).

\[
\begin{align*}
\mathbf{w} &= \sum_{i,j}^{M,N} A_{ij} \mathbf{X}_i^1(x) \mathbf{Y}_j^2(y), \\
\mathbf{F} &= \sum_{i,j}^{M,N} B_{ij} \mathbf{X}_i^1(x) \mathbf{Y}_j^2(y)
\end{align*}
\]

System of approximating functions

\[
\begin{align*}
\mathbf{X}_i^1(x) &= \mathbf{X}_i^1(x) = \sin \pi x, \\
\mathbf{Y}_j^2(y) &= \mathbf{Y}_j^2(y) = \sin j \pi y
\end{align*}
\]

![Figure 1](image-url)

**Figure 1.** The dependencies \( q(w) \) for \( e_s = 0.098; 0.882; 1.56; 2.45; 4.8 \).
Figure 1 shows the curves $q(w)$ in the center of the plate for the above values of the physical-geometric parameter, the values of which are shown in the figure for the plate, taking into account both physical and geometric nonlinearities. When $e_s = 4.8$ solving taking into account physical and geometric nonlinearities, it completely coincided with the solution obtained taking into account only geometric nonlinearity.

The curves in Fig. 1 received excluding unloading. It can be seen that from the solution obtained as a geometrically nonlinear problem, when $e_s < 4.8$ solutions begin to be separated taking into account only physical nonlinearity and solutions obtained taking into account physical and geometric nonlinearities. Moreover, in the curves (a solution taking into account the physical and geometric nonlinearities), an $e_s \geq 3.526$ inflection point is indicated, which becomes sharply expressed with a decrease, with a further decrease $e_s$ to $e_s = 0.882$ constant load area appears, i.e. as if the ultimate load. This inflection point in the graph $q(w)$ indicates that initially the geometric nonlinearity is insignificant and physical non-linearity plays the main role. After the inflection point at which unloading appears, geometric non-linearity begins to play a significant role.

From fig. 1 it can be seen that the ultimate load at $e_s = 0.882$ obtained when solving the physically nonlinear problem and the physically and geometrically nonlinear problem is practically the same, but the picture of the stress-strain state and the distribution of plasticity zones, as a result, sharply differs from each other.

That is, for the hinged support of square plates under the action of a uniformly distributed load, the calculation can be made: taking into account only physical nonlinearity at $e_s \leq 0.1$; taking into account physical and geometric nonlinearity at $0.1 < e_s < 6$; taking into account only geometric nonlinearity at $e_s \geq 6$.

Next, the influence of the dependence $\sigma_f(e_f)$ on the stress-strain state of rectangular plates in the plan $\lambda = \frac{a}{b} = 1$ is studied. To do this, square plates were calculated in the plan under the action of a load uniformly distributed over the surface, for the boundary conditions described above $e_s = 0.098; 0.882; 1.56; 2.45; 4.8$ for two types of diagrams: an ideal elastic-plastic body (4) and a diagram for pure aluminum. The diagram for pure aluminum better describes the real diagram $\sigma_f(e_f)$.

Solutions obtained using the exponential dependence give more developed zones of plastic deformations, both in terms of the plate and its thickness, than when using the dependence (4). The nature of the distribution of ductility zones also depends on the type of diagram taken into account.

4. Concluding remarks

The study of the physico-geometric parameter $e_s$ showed that it is possible to choose this parameter in such a way that we get a linear dependence $q(w)$ that is linear, that is, corresponding to the problem without taking into account geometric and physical non-linearities.

Thus, when calculating real structures, one should use a real diagram $\sigma_f(e_f)$, starting with $e_s > 0.2$

Acknowledgement

This work has been supported by the Grant RSF № 16-11-10138

References

[1] E Yu Masalkin, A review of methods for calculating shells under elastic-plastic deformations, Development of science and technology: The mechanism of choice (2017), 23.

[2] V A Krysko, J Awrejcewicz, I V Papkova, O.A.Saltykova, A.V.Krysko, On reliability of chaotic
dynamics of two Euler–Bernoulli beams with a small clearance. International Journal of Non-Linear Mechanics 104 (2018) 8-18.

[3] Y M Grigorenko, V I Gulyaev, Nonlinear problems of shell theory and their solution methods. Soviet applied mechanics, 27(10), (1991) 929-947.

[4] V A Krysko, J Awrejcewicz, I V Papkova, O. A.Saltykova, A.V.Krysko, Chaotic Contact Dynamics of Two Microbeams under Various Kinematic Hypotheses. International Journal of Nonlinear Sciences and Numerical Simulation, 20(3-4), (2019) 373-386.

[5] Ö Civalek, Harmonic differential quadrature-finite differences coupled approaches for geometrically nonlinear static and dynamic analysis of rectangular plates on elastic foundation. Journal of Sound and Vibration, 294(4-5), (2006) 966-980.

[6] C M C Roque, D Cunha, C Shu, A J M Ferreira, A local radial basis functions—Finite differences technique for the analysis of composite plates. Engineering Analysis with Boundary Elements, 35(3), (2011) 363-374.

[7] A V Krysko, J Awrejcewicz, O A Saltykova, S S Vetsel, V A Krysko, Nonlinear dynamics and contact interactions of the structures composed of beam-beam and beam-closed cylindrical shell members. Chaos, Solitons & Fractals, 91, (2016) 622-638.

[8] V A Krysko, M V Zhigalov, O A Saltykova, A S Desatova, Dissipative dynamics of geometrically non-linear Euler-Bernoulli beams. Izvestia RAS MTT (2008).

[9] M F Caliri Jr, A J Ferreira, V Tita, A review on plate and shell theories for laminated and sandwich structures highlighting the Finite Element Method. Composite Structures, 156, (2016) 63-77.

[10] I A Birger Strength of materials. Moscow, Nauka, (1986) 560 p.

[11] L M Kachanov On variational methods for solving problems of the theory of plasticity. PMM, vol. 23b (1959), no. 3.

[12] V A Krysko Nonlinear Statics and Dynamics of Inhomogeneous Shells, Saratov University Press, (1976).

[13] P A Lukash Calculation of gently sloping shells and slabs, taking into account physical and geometric nonlinearities. - In the book Proceedings of the Central Scientific and Research Institute of Cinematography, Publishing House of the Academy of Construction and Architecture of the USSR, (1961), 7.

[14] H M Mushtari, R G Surkin Transverse bending of a square plate with a nonlinear relationship between strains and stresses. Kazan branch of the Academy of Sciences of the USSR, ser. Physics, Mathematics and Mechanics (1966) v.14.

[15] L S Srubshchik Asymptotic method for determining critical loads of stability loss of strictly convex gentle shells of revolution. PMM, vol. 36, (1972), p. 705-715.

[16] W Ramberg, W R Osgood Discriptions of Stress-Strain Curves by Three Parameters NAGA. TN-902, New NASA, 1943.

[17] Y Ohashi, S Murakami The elastoplastic bending of a clamped thin circular plate. - Proc. Eleventh Int. Cong. App. Mech., Munich, (1964).