High energy bounds on wave operators

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June 15, 2021

Abstract

The wave operators $W_{\pm}(H_1, H_0)$ of two selfadjoint operators $H_0$ and $H_1$ are analyzed at asymptotic spectral values. Sufficient conditions for $\|(W_{\pm}(H_1, H_0) - P_{0\text{ac}}^{\pm}P_{0\text{ac}})f(H_0)\| < \infty$ are given, where $P_{0\text{ac}}$ projects onto the subspace of absolutely continuous spectrum of $H_1$ and $f$ is an unbounded function (f-boundedness), both in the case of trace-class perturbations and in terms of the high-energy behaviour of the boundary values of the resolvent of $H_0$ (smooth method). Examples include f-boundedness for the perturbed polyharmonic operator and for Schrödinger operators with matrix-valued potentials. We discuss an application to the problem of quantum backflow.

1 Introduction

The purpose of mathematical scattering theory is to compare two selfadjoint operators $H_0$, $H_1$ acting on a common Hilbert space $\mathcal{H}$ via their wave operators (or Møller operators), defined as the strong limits

$$W_{\pm}(H_1, H_0) := \lim_{t \to \pm \infty} e^{itH_1} e^{-itH_0} P_{0\text{ac}}^{\pm},$$

where $P_{0\text{ac}}^{\pm}$ is the projection onto the subspace $P_{0\text{ac}}^{\pm}\mathcal{H}$ of absolutely continuous spectrum of $H_0$. When these operators exist, they define isometries $P_{0\text{ac}}^{\pm}\mathcal{H} \to P_{1\text{ac}}^{\pm}\mathcal{H}$ that intertwine the absolutely continuous parts of $H_0$ and $H_1$ and can therefore be used to obtain information about $H_1$ based on information about $H_0$. This has many applications, in particular in quantum physics, where $H_0$ plays the role of a “free” Hamiltonian that can be investigated directly, and $H_1$ an “interacting”, more complicated operator, that cannot be analyzed directly. The wave operators are then used to characterize asymptotic properties of the dynamics given by the unitary group $e^{-itH_1}$ in terms of the “free” dynamics $e^{-itH_0}$.

Many classical theorems in the field concentrate on establishing conditions on $H_0$ and $H_1$ that ensure existence of the wave operators $W_{\pm}(H_1, H_0)$ and $W_{\pm}(H_0, H_1)$ (see the monographs [Yaf92, Yaf10, RS79] for a thorough presentation of the subject). In contrast, we are here interested in more quantitative questions at asymptotic spectral values $\lambda$ of the operators (meaning the limit $|\lambda| \to \infty$). Namely, one would expect that in typical situations $W_{\pm}(H_1, H_0)$ approximates the identity on spectral subspaces for asymptotic spectral values. Borrowing terminology from applications to Hamiltonians, we also refer to this as the “high energy” behaviour.

One indication for this is as follows: Consider the scattering operator $S := W_{+}(H_1, H_0) W_{-}(H_1, H_0)$, which commutes with $H_0$ and hence in a diagonalization of $H_0$ acts by multiplication with an operator-valued function $\lambda \mapsto S(\lambda)$, where $\lambda$ are the spectral values of $H_0$. In relevant examples, $S(\lambda) - 1$ is of Hilbert-Schmidt class (“finite total cross section” [ES80]) and its Hilbert-Schmidt norm decays as $|\lambda| \to \infty$ [Jen80, SY86]. Related results are also important in the context of the Aharonov-Bohm effect [BW09].

In the present paper, we investigate a different and more direct question: we study the behaviour of $W_{\pm}(H_1, H_0) - 1$ at asymptotic spectral values of $H_1$ and $H_0$. Since the wave operator contains information only on the absolutely continuous parts of the $H_j$, it is better to replace the identity 1

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with the product of projections $P_1^a P_0^c$, and we define a pair $(H_1, H_0)$ of selfadjoint operators to be $f$-bounded (Def. 2.1) if their wave operators exist and satisfy

$$\|(W_\pm(H_1, H_0) - P_1^a P_0^c)f(H_0)\| < \infty \quad (1.1)$$

for some unbounded continuous function $f : \mathbb{R} \to \mathbb{R}$ ("high energy bounds").

This question is partially motivated by our previous work on backflow, a surprising quantum mechanical effect which describes the situation where the probability current of a quantum particle in one dimension can flow in the direction opposite to its momentum. To quantify this effect in a mechanical effect which describes the situation where the probability current of a quantum particle

triple $f_\beta$ or resolvents of $H_0$ and $H_1$ at their spectra, taken in a suitable norm $\| \cdot \|_{X, X^*}$, defined by a Gelfand triple $X \subset H \subset X^*$ (see, for example, [BA11] for a review of this technique). We pay particular attention to deriving sufficient conditions for (1.1) that can be expressed exclusively in terms of "free" data (referring to $H_0$, its resolvent $R_0$, etc.), as required for applications. Our main result is that $f_\beta$-boundedness is essentially implied by the high-energy behaviour of the resolvent of $H_0$: if $R_0$ is locally H"older continuous and $\|R_0(\lambda \pm i0)\|_{X, X^*} = O(\|\lambda\|^{-\beta})$, then $f_\beta$-boundedness follows (Theorem 4.9). This allows us, in particular, to treat the case where $H_0 = (-\Delta)^{\ell/2}$ is a fractional power of the Laplacian: For suitable $V$, we find that $f_\beta$-boundedness holds for $(H_0 + V, H_0)$ whenever $0 < \beta \leq 1 - \frac{1}{\ell}$ (Example 4.10), and this bound is sharp in general (Example 4.12).

We also ask whether this situation is stable under tensor product constructions, i.e., we consider $H_0 = H_A \otimes 1 + 1 \otimes H_B$ where $H_A$ is of the same type as before, and $H_B$ has only point spectrum. Under certain conditions our results transfer to this situation (Corollary 4.15). In the case where $H_A$ is the negative Laplacian—in particular in applications to quantum physics—these $H_0$, and $H_0 + V$ for suitable $V$, are known as matrix-valued Schr"odinger operators or Schr"odinger operators with matrix-valued potentials (see, e.g., [GKM02, FLS07, KR08, CJLS16]), although we allow the matrices to become infinite-dimensional (Example 4.16). A particular problem occurs here for low-dimensional Laplacians if $H_B$ is of infinite rank; we discuss this in Example 4.17.
2 General setting

Throughout this article, our general setup will be as follows. We consider two selfadjoint operators $H_0$ and $H_1$ on a common complex separable Hilbert space $\mathcal{H}$. In our notation, we use an index 0 or 1 to distinguish between the spectral resolutions and subspaces related to $H_0$ or $H_1$, e.g., $P_0^{ac}$ is the projection onto the subspace of absolutely continuous spectrum of $H_0$, and $E_1$ is the spectral resolution of $H_1$. As the main quantity of interest, we consider the strong limits

$$W_\pm(H_1, H_0) := \lim_{t \to \pm \infty} e^{itH_1} e^{-itH_0} P_0^{ac}$$

and call them wave operators if they exist. We will use the following well-known results about wave operators (see, for example, [Yaf92, RS79]), and refer to them as (W1)–(W3).

(W1) $W_\pm(H_1, H_0)$ are partial isometries with initial space $\mathcal{H}_0^{ac} = H_0^{ac} \mathcal{H}$ and final space contained in $\mathcal{H}_1^{ac}$. In case their final spaces coincide with $\mathcal{H}_1^{ac}$, the wave operators are called complete. This is equivalent to the existence of $W_\pm(H_0, H_1)$, and implies $W_\pm(H_1, H_0)^* = W_\pm(H_0, H_1)$.

(W2) A chain rule holds: If $H_0, H_1, H_2$ are selfadjoint such that $W_\pm(H_1, H_0)$ and $W_\pm(H_2, H_1)$ exist, then also $W_\pm(H_2, H_0)$ exists, and

$$W_\pm(H_2, H_0) = W_\pm(H_2, H_1)W_\pm(H_1, H_0).$$

(W3) The intertwiner property holds: For an arbitrary bounded Borel function $\varphi : \mathbb{R} \to \mathbb{C}$, one has

$$\varphi(H_1)W_\pm(H_1, H_0) = W_\pm(H_1, H_0)\varphi(H_0).$$

In many situations, $W_\pm(H_1, H_0)$ approximates the identity, or rather the operator $P_0^{ac} P_0^{ac}$ in the presence of point spectrum, when restricted to spectral subspaces of $H_0$ for asymptotic spectral values. More specifically, one may find that $(W_\pm(H_1, H_0) - P_0^{ac} P_0^{ac}) f(H_0)$ is bounded despite the function $f$ being unbounded on the spectrum of $H_0$. This motivates the following definition; here and throughout the paper, $C(\mathbb{R})$ denotes the space of continuous real-valued functions on $\mathbb{R}$.

**Definition 2.1.** Let $H_0$ and $H_1$ be two selfadjoint operators on a Hilbert space $\mathcal{H}$ such that their wave operators $W_\pm(H_1, H_0)$ exist, and let $f \in C(\mathbb{R})$. Then the pair $(H_1, H_0)$ is called $f$-bounded if $(W_\pm(H_1, H_0) - P_0^{ac} P_0^{ac}) f(H_0)$ is bounded. If both $(H_1, H_0)$ and $(H_0, H_1)$ are $f$-bounded, then we call the operators mutually $f$-bounded.

Clearly one is interested here in functions $f$ that grow as $\lambda \to \pm \infty$; a typical choice would be $f_\beta(\lambda) := (1 + \lambda^2)^{\beta/2}$, $\beta > 0$. This raises the question which rate of growth of $f(\lambda)$ as $\lambda \to \pm \infty$ is still compatible with $f$-boundedness, to be investigated in later sections.

Heuristically, $f$-boundedness should be determined by the behaviour of the wave operator at asymptotic spectral values of $H_0$ and $H_1$. In the following lemma, we show that under the additional restriction $f(H_1) - f(H_0) \in \mathfrak{B}(\mathcal{H})$, this can in fact be made precise. This condition holds for a large class of $f$ in case $H_1 - H_0$ is bounded, see Lemma B.1 in Appendix B.

**Lemma 2.2.** Let $H_0, H_1$ be two selfadjoint operators, and $f \in C(\mathbb{R})$. Suppose that $W := W_\pm(H_1, H_0)$ exists and is complete, and that $f(H_1) - f(H_0)$ is bounded. Then the following statements are equivalent:

(i) $(W - P_1^{ac} P_0^{ac}) f(H_0) \in \mathfrak{B}(\mathcal{H})$;

(ii) $E_1(-\lambda, \lambda)^\perp (W - P_1^{ac} P_0^{ac}) f(H_0) E_0(-\lambda, \lambda)^\perp \in \mathfrak{B}(\mathcal{H})$ for some $\lambda > 0$;
(iii) \( E_1(-\lambda, \lambda)^+ (W - P_1^a P_0^a) f(H_0) E_0(-\lambda, \lambda)^+ \in \mathfrak{B}(\mathcal{H}) \) for all \( \lambda > 0 \).

Here \( E_j(-\lambda, \lambda)^+ := 1 - E_j(-\lambda, \lambda) \).

Proof. It is clear that (i) \( \Rightarrow \) (iii) \( \Rightarrow \) (ii). For (ii) \( \Rightarrow \) (i), since \( f(H_0) E_0(-\lambda, \lambda) \) is bounded, it only remains to show that \( E_1(-\lambda, \lambda)^+ (W - P_1^a P_0^a) f(H_0) \) belongs to \( \mathfrak{B}(\mathcal{H}) \). To that end, we consider a compact set \( \Delta \subseteq \mathbb{R} \) and define the bounded Borel function \( f_\Delta := f \cdot \chi_\Delta \). Using the intertwining property (W3) of \( W \), we get

\[
E_1(-\lambda, \lambda)^+ (W - P_1^a P_0^a) f_\Delta(H_0) = \quad E_1(-\lambda, \lambda) f_\Delta(H_1)(W - P_1^a P_0^a) + E_1(-\lambda, \lambda) P_1^a (f_\Delta(H_1) - f_\Delta(H_0)) P_0^a.
\]

As \( \Delta \to \mathbb{R} \), this operator remains bounded by assumption. \( \square \)

As an aside, we mention that in the situation of two mutually \( f \)-bounded operators \( H_0, H_1 \), one has \( W_\pm(H_0, H_1)^\ast = W_\pm(H_1, H_0) \) by (W1), with which it is easy to show that \( P_1^a (f(H_1) - f(H_0)) P_0^a \) is bounded. This shows that the assumption used in Lemma 2.2 is quite natural.

Under some extra restrictions on the selfadjoint operators involved, we can strengthen (mutual) \( f \)-boundedness to an equivalence relation.

**Definition 2.3.** Let \( \text{AAC}(\mathcal{H}) \) (the “almost absolutely continuous” operators) be the set of densely defined selfadjoint operators \( H \) on the space \( \mathcal{H} \) such that \( (1 - P^{ac}) H \) is bounded.

Fix \( f \in C(\mathbb{R}) \). For \( H_0, H_1 \in \text{AAC}(\mathcal{H}) \), we say that \( H_1 \) and \( H_0 \) are \( f \)-equivalent (written as \( H_1 \sim_f H_0 \)) if \( (H_1, H_0) \) is \( f \)-bounded, \( W_\pm(H_1, H_0) \) are complete, and \( f(H_0) - f(H_1) \) is bounded.

The name “\( f \)-equivalent” is justified because of

**Proposition 2.4.** Let \( f \in C(\mathbb{R}) \).

(i) If \( H_1, H_0 \in \text{AAC}(\mathcal{H}) \) such that \( H_1 \sim_f H_0 \), then \( (W_\pm(H_1, H_0) - 1) f(H_0) \) are bounded.

(ii) \( \sim_f \) is an equivalence relation on \( \text{AAC}(\mathcal{H}) \).

Proof. For part (i), one notes that

\[
(1 - P_1^a P_0^a) f(H_0) = P_1^a (1 - P_0^a) f(H_0) + (1 - P_1^a) f(H_1) + (1 - P_1^a) (f(H_0) - f(H_1)),
\]

and all terms on the right-hand side are bounded by assumption.

In (ii), reflexivity is evident. For symmetry, let \( H_1 \sim_f H_0 \). We cut down \( f \) to \( f_\Delta = f \cdot \chi_\Delta \) with a compact \( \Delta \subseteq \mathbb{R} \). (W1) and the intertwining property (W3) then yield

\[
(W_\pm(H_0, H_1) - 1) f_\Delta(H_1) = \left( f_\Delta(H_1)(W_\pm(H_1, H_0) - 1) \right)^* \\
= \left( (W_\pm(H_1, H_0) - 1) f_\Delta(H_0) \right)^* + (f_\Delta(H_0) - f_\Delta(H_1)).
\]

As \( \Delta \to \mathbb{R} \), this expression remains bounded to part (i), and we obtain \( H_0 \sim_f H_1 \) using (2.1). Likewise, transitivity follows using the chain rule (W2). \( \square \)

A pair of selfadjoint operators \( H_0, H_1 \) that have wave operators can exhibit very different behaviour regarding \( f \)-boundedness, as we now demonstrate with two examples: The first example shows that \( f \)-boundedness can fail even for functions \( f \) of arbitrarily slow divergence rate at \( +\infty \), whereas the second example shows that \( f \)-boundedness can hold for every \( f \).

**Example 2.5.** Let \( \mathcal{H} = L^2(\mathbb{R}, dx) \) and \( H_0 := -i \frac{d}{dx} \). Let \( v \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) be non-zero and real, and define \( H_1 := H_0 + V \), where \( V \) is the operator multiplying with \( v \). Then the wave operators \( W_\pm := W_\pm(H_0, H_1) \) and \( W_\pm(H_1, H_0) \) exist. If \( f \in C(\mathbb{R}) \) is such that \( \mathbb{R} \ni \lambda \mapsto |f(\lambda)| \) is monotonically increasing to \( +\infty \), then \( (H_1, H_0) \) is not \( f \)-bounded.
Proof. Note that $H_0$ and $H_1$ are selfadjoint on their natural domains. As shown in [Yaf92, p. 83-84], $H_0$ and $H_1$ are unitarily equivalent and have absolutely continuous simple spectrum covering the full real axis. Furthermore, the wave operators $W_\pm := W_\pm (H_0, H_1)$ exist and are unitary. Explicitly, they act as

$$(W_\pm \psi)(x) = w_\pm (x) \psi(x), \quad w_\pm (x) := \exp i \int_x^{\pm \infty} v(y) \, dy.$$  

This implies in particular that the wave operators are complete, and that $W_\pm (H_1, H_0) = W_\pm^*$ acts by multiplication with $w_\pm$.

We have to show that $(W_\pm - 1) f(H_0) = (f(H_0)(W_\pm - 1))^*$ is unbounded. But in view of the form of $W_\pm$, we find an interval $I$ such that the restricted operator $(W_\pm - 1) : L^2(I) \to L^2(I)$ has a bounded inverse. Hence we only have to show that $f(H_0)$ is unbounded on $L^2(I)$ and for this purpose may assume $I = [0, r]$ for some $r > 0$ because $H_0$ commutes with translations. Let $\psi \in L^2(I)$ be the normalized characteristic function of $I$, and consider the normalized sequence $\psi_n(x) := \sqrt{n} \psi(nx)$ in $L^2(I)$. Then we find by straightforward calculations and estimates

$$\|f(H_0)\psi_n\|^2 = \int_{\mathbb{R}} |f(nk)|^2 |\hat{\psi}(k)|^2 \, dk$$

$$\geq \int_1^\infty |f(nk)|^2 |\hat{\psi}(k)|^2 \, dk \geq |f(n)|^2 \int_1^\infty |\hat{\psi}(k)|^2 \, dk,$$

where we have used the monotonicity assumption on $f$. This property also shows that the last expression goes to infinity as $n \to \infty$, which implies the claim. \qed

Example 2.6. Let $H_0$ be a selfadjoint operator with purely absolutely continuous spectrum, $V$ a selfadjoint bounded operator on the same Hilbert space $\mathcal{H}$, and $H_1 := H_0 + V$. Assume that the wave operators $W_\pm (H_1, H_0)$ exist and that there exists a compact $\Delta \subset \sigma(H_0)$ such that $V\mathcal{H} \subset E_0(\Delta)\mathcal{H}$. Then $(H_1, H_0)$ is $f$-bounded for every $f \in C(\mathbb{R})$.

Proof. With $E_0^\pm := 1 - E_0(\Delta)$, our assumption can be rephrased as $0 = VE_0^\pm = (H_1 - H_0)E_0^\pm$, which implies $H_1 E_0^\pm = H_0 E_0^\pm = E_0^\pm H_0$ on $\text{dom } H_0 = \text{dom } H_1$, and, by selfadjointness, also $E_0^\pm H_1 = E_0^\pm H_0$ on this domain. A power series calculation on analytic vectors for $H_0$ then gives $E_0^\pm e^{-itH_0} = e^{-itH_1}E_0^\pm$, and therefore

$$W_\pm (H_1, H_0) E_0^\pm = s\text{-lim}_{t \to \pm \infty} e^{itH_1} e^{-itH_0} E_0^\pm = s\text{-lim}_{t \to \pm \infty} e^{itH_1} E_0^\pm e^{-itH_0} = E_0^\pm.$$

Thus $(W_\pm (H_1, H_0) - P_1^a c) = P_1^a (W_\pm (H_1, H_0) - 1) = P_1^a (W_\pm (H_1, H_0) - 1) E_0(\Delta)$, and for arbitrary $f \in C(\mathbb{R})$, we have the bound

$$\|(W_\pm (H_1, H_0) - P_1^a c)f(H_0)\| = \|(W_\pm (H_1, H_0) - P_1^a c)f_\Delta(H_0)\|$$

$$\leq 2 \|f\|_\infty < \infty. \qed$$

A concrete realisation of this situation on $\mathcal{H} = L^2(\mathbb{R}, dx)$ is given by $H_0 = -\frac{d^2}{dx^2}$ and $V$ an integral operator such that the Fourier transform of its kernel lies in $C^\infty_0(\mathbb{R} \times \mathbb{R})$. Then $V$ is trace-class, which implies existence and completeness of the wave operators by the Kato-Rosenblum theorem [RS79, Thm. XI.8], and the assumption $V\mathcal{H} \subset E_0(\Delta)\mathcal{H}$ follows from the support of the integral kernel.

3 Trace class perturbations and $f$-boundedness

The Kato-Rosenblum theorem states that if $H_0$ and $H_1$ are selfadjoint and their difference is trace class, then $W_\pm (H_1, H_0)$ exist and are complete. Examples of such trace class perturbations are given by integral operators with suitable kernels, or rank one perturbations as the simplest case.

We now investigate $f$-boundedness in this setting and first recall some relevant notions. For a selfadjoint operator $H_0$ on $\mathcal{H}$ and a vector $\xi \in P_0^a \mathcal{H}$, we define

$$\|\xi\|^2_{H_0} := \text{ess sup}_{\lambda \in \sigma(H_0)} \left| \frac{d \langle \xi, E_0(\lambda) \xi \rangle}{d \lambda} \right| = \text{ess sup}_{\lambda \in \sigma(H_0)} \|\xi_\lambda\|^2_\lambda \in [0, +\infty].$$
In particular, Proposition 3.2. Let \( \Lambda \) denote the set of all \( \sigma \) with finite \( \| \xi \|_0 \) and \( \| \cdot \_ \| \) the norm of \( b(\lambda) \) [Yaf92, p. 32]. The set of all \( \xi \) with finite \( \| \xi \|_0 \) is \( \| \cdot \|_0 \)-dense in \( \mathcal{P}_0^a \mathcal{H} \), and \( \| \cdot \|_0 \) is a norm on it [RS79].

Note that if \( H_0, H_1 \) are two selfadjoint operators with complete wave operators, then \( W_\pm(H_0, H_1) : P_0^a \mathcal{H} \to P_0^a \mathcal{H} \) are unitaries intertwining the absolutely continuous parts of \( H_0 \) and \( H_1 \). This implies that we can identify the direct integral decompositions of \( P_0^a \mathcal{H} \) and \( P_0^a \mathcal{H} \), and \( \| W_\pm(H_0, H_1) \xi \|_0 = \| \xi \|_0 \) in this situation.

The following lemma due to Rosenblum [RS79, Lemma 1, p.23] will be essential in our discussion.

**Lemma 3.1.** Let \( H_0 \) be a selfadjoint operator on \( \mathcal{H} \) and \( \psi, \xi \) vectors in \( \mathcal{H} \), with \( \xi \in P_0^a \mathcal{H} \) such that \( \| \xi \|_0 < \infty \). Then

\[
\int_{-\infty}^{\infty} |\langle \psi, e^{\pm \text{i}tH_0} \xi \rangle|^2 dt \leq 2\pi \| \xi \|_0^2 \| \psi \|_0^2 < \infty.
\]

As in the Kato-Rosenblum theorem, we now consider two selfadjoint operators \( H_0, H_1 \) such that \( V := H_1 - H_0 \) is trace class, and hence has an expansion \( V \psi = \sum_n t_n \langle \xi_n, \psi \rangle \xi_n, \psi \in \mathcal{H} \), where the \( \xi_n \) form an orthonormal basis of \( \mathcal{H} \) and the sequence \( t_n \) is summable, \( \sum_n |t_n| < \infty \).

The idea of the f-boundedness result presented below is to choose \( V \) such that not only \( Vf(H_0) \) is trace-class, but also

\[
\| Vf(H_0) \|_{H_1, H_0}^A := \sum_n |t_n| \| P_1^a E_1(-\Lambda, \Lambda)^{-1} \xi_n \|_{H_1} \| P_0^a E_0(-\Lambda, \Lambda)^{-1} f(H_0) \xi_n \|_{H_0}
\]

is finite, where \( \Lambda \geq 0 \) is arbitrary.

**Proposition 3.2.** Let \( H_0, H_1 \) be selfadjoint with \( V := H_1 - H_0 \) of trace class and \( \| Vf(H_0) \|_{H_1, H_0}^A < \infty \) for some \( \Lambda \geq 0 \). Let \( f \in C(\mathbb{R}) \) such that \( f(H_1) - f(H_0) \) is bounded. Then \( W_\pm(H_1, H_0) \) exist, are complete, and satisfy

\[
\| (W_\pm(H_1, H_0) - P_0^a P_0^a) f(H_0) \| \leq 2\pi \| Vf(H_0) \|_{H_1, H_0}^A < \infty.
\]

In particular, \( (H_1, H_0) \) is f-bounded.

**Proof.** Existence and completeness of \( W := W_\pm(H_1, H_0) \) follow from the Kato-Rosenblum Theorem. To obtain f-boundedness, we first recall that for \( \varphi \in \text{dom} \ H_1, \psi \in \text{dom} \ H_0 \), we have

\[
\frac{d}{dt} \langle \varphi, P_1^a e^{\text{i}tH_1} e^{\text{i}tH_0} P_0^a \psi \rangle = \pm i \langle \varphi, P_1^a e^{\text{i}tH_1} V e^{\text{i}tH_0} P_0^a \psi \rangle.
\]

After integration, this yields

\[
\langle \varphi, (W - P_1^a P_0^a) \psi \rangle = \pm i \int_0^\infty \langle P_1^a \varphi, e^{\text{i}tH_1} V e^{\text{i}tH_0} P_0^a \psi \rangle dt.
\]

The proof is based on this identity, the expansion \( V \psi = \sum_n t_n \langle \xi_n, \psi \rangle \xi_n \) of \( V \), and Lemma 3.1. With \( E_1^A := E_1(-\Lambda, \Lambda)^{-1} \), we find

\[
\left| \langle E_1^A \varphi, (W - P_1^a P_0^a) f(H_0) E_0^\Lambda \psi \rangle \right|
= \left| \int_0^\infty \langle \varphi, e^{\text{i}tH_1} P_1^a E_1^A V E_0^\Lambda e^{\text{i}tH_0} f(H_0) P_0^a \psi \rangle dt \right|
\leq \sum_n |t_n| \left| \int_0^\infty \langle \varphi, e^{\text{i}tH_1} P_1^a E_1^A \xi_n \rangle \langle E_0^\Lambda \xi_n, e^{\text{i}tH_0} f(H_0) P_0^a \psi \rangle dt \right|
\leq \sum_n |t_n| \left( \int_0^\infty |\langle P_1^a E_1^A \xi_n, e^{\text{i}tH_1} \varphi \rangle|^2 dt \right) \left( \int_0^\infty |\langle f(H_0) P_0^a E_0^\Lambda \xi_n, e^{\text{i}tH_0} P_0^a \psi \rangle|^2 dt \right)^{1/2}.
\]
We can now use Lemma 3.1 to estimate the integrals, and arrive at the bound
\[
|\langle \varphi, E^\Lambda f (W - P^0_{\pm}) f (H_0) E^\Lambda \psi \rangle| \\
\leq 2\pi \sum_n |P^0_\pm E^\Lambda_n \xi_n|_{H_1} \|f (H_0) E^\Lambda_0 P^0_{\pm} \xi_n\|_{H_0} \|\varphi\| \|\psi\| \\
= 2\pi \|V f (H_0)\|_{H_1, H_0} \|\varphi\| \|\psi\|.
\]
In view of Lemma 2.2, this finishes the proof. \(\square\)

We note that the assumption \(f (H_1) - f (H_0)\) being bounded was only used for the reference to Lemma 2.2 and can therefore be dropped for \(\Lambda = 0\).

Particular examples of trace-class perturbations that have attracted continued attention are perturbations by rank one operators \(V = \langle \xi, \cdot \rangle \xi\) with some \(\xi \in \mathcal{H}\) [Sim95]. In that case, the bound from the previous proposition is
\[
2\pi \|P^0_{\pm} E_1 (-\Lambda, \Lambda) \xi\|_{H_1} \|P^0_{\pm} E_0 (-\Lambda, \Lambda) \xi\|_{H_0},
\]
and in order to have it finite, we need to control the “spectral norms” of both \(H_0\) and \(H_1\). Whereas the norm coming from \(H_0\) can typically be controlled directly in applications, this is typically not the case for \(H_1\). Let us therefore clarify how an estimate on
\[
\|P^0_{\pm} E_1 (-\Lambda, \Lambda) \xi\|_{H_1} = \text{ess sup}_{|\lambda| \geq \Lambda} \frac{|d\xi, E_1 (\lambda) \xi|}{|d\lambda|}
\]
can be given in terms of \(H_0\): With the resolvents \(R_0, R_1\) of \(H_0, H_1\), one has
\[
\frac{|d\xi, E_1 (\lambda) \xi|}{|d\lambda|} = \frac{1}{\pi} \text{Im} \langle \xi, R_j (\lambda + i0) \xi \rangle,
\]
and for a rank one perturbation \(H_1 = H_0 + \langle \xi, \cdot \rangle \xi\) one moreover has the Aronszajn-Krein formula [Sim95]
\[
\langle \xi, R_1 (\lambda + i0) \xi \rangle = \frac{\langle \xi, R_0 (\lambda + i0) \xi \rangle}{1 + \langle \xi, R_0 (\lambda + i0) \xi \rangle}.
\]
It therefore follows that in case \(\xi\) is such that \(\|\xi\|_{H_0} < \infty\) and the boundary values \(\langle \xi, R_0 (\lambda + i0) \xi \rangle\) converge to \(0\) as \(|\lambda| \to \infty\), then also \(\|\xi\|_{H_1} < \infty\).

As a concrete example, we may take \(\mathcal{H} = L^2 (\mathbb{R}, dx)\) with \(H_0 = -\frac{d^2}{dx^2}\) and \(H_1 = H_0 + \langle \xi, \cdot \rangle \xi\), where \(\xi \in \mathcal{S} (\mathbb{R})\) is a Schwartz function. Then the spectral measure of \(P = \int \frac{d\xi, E_P (\lambda) \xi}{d\lambda}\) is given by \(d\xi, E_P (\lambda) \xi = |\xi (p)|^2 dp\) with \(\xi\) the Fourier transform of \(\xi \in \mathcal{H}\). After substituting \(\lambda = p^2\), this shows that the spectral measure of \(H_0 = P^2\) is \(d\xi, E_0 (\lambda) \xi = \frac{1}{2} \lambda^{-1/2} (|\xi (\sqrt{\lambda})|^2 + |\xi (-\sqrt{\lambda})|^2) d\lambda\). Hence \(E_0 (-\Lambda, \Lambda) \xi\) converges to \(0\) as \(|\lambda| \to \infty\) (see Example 4.10 below). Hence \((H_1, H_0)\) is \(f\)-bounded in this situation.

### 4 Smooth method and \(f\)-boundedness

We now discuss another specific setting of scattering theory, known as the smooth method, which is applicable in particular to cases where one of the operators \(H_j\) is a (pseudo)differential operator.

The idea behind this is as follows. If \(H\) is a selfadjoint operator and \(R (z) = (H - z)^{-1}\) its resolvents, then the operator-valued function \(z \mapsto R (z)\) is certainly analytic on the open half planes \(\mathbb{H}_\pm := \mathbb{R} \pm i \mathbb{R}_+\). One now demands that, in a suitable topology, it extends to the boundaries of the half planes, and that the extended functions are locally H"older continuous on \(\mathbb{H}_\pm\) (possibly with the exception of a null set). In this case \(H\) is called smooth. If both \(H_0\) and \(H_1\) are smooth, then the wave operators \(W_\pm (H_1, H_0)\) automatically exist and are complete. In fact, one can express the wave operators, as well as other relevant quantities, in terms of the boundary values of the resolvents,
$R_1(\lambda \pm i0)$ and $R_0(\lambda \pm i0)$. We recall the basic facts about this setting in Sec. 4.1, mainly in the spirit of [KK71, BA11]; see also [Yaf92].

In the context of this setting, we are interested in mutual $f$-boundedness of two operators $H_1$ and $H_0$, where we restrict to $f_\beta(\lambda) = (1 + \lambda^2)^{\beta/2}$ with some $\beta \in (0, 1)$. It turns out that this is implied by the behaviour of $R_0(\lambda \pm i0)$ at large $\lambda$ alone: If a certain norm of this operator is $O(|\lambda|^{-\beta})$, then mutual $f_\beta$-boundedness or even $f_\beta$-equivalence follows (Theorem 4.9).

We apply this result to examples of pseudodifferential operators (Sec. 4.3) and investigate stability under tensor product constructions (Sec. 4.4).

4.1 Setting of the smooth method

Let $\mathcal{X}$ be a Banach space which is continuously and densely embedded in $H$. The scalar product $\langle \cdot, \cdot \rangle$ on $H$ then provides an embedding $H \subset \mathcal{X}^*$, $\varphi \mapsto \langle \cdot, \varphi \rangle$, where $\mathcal{X}^*$ denotes the conjugate dual of $\mathcal{X}$, yielding a so-called Gelfand triple $\mathcal{X} \subset H \subset \mathcal{X}^*$. We assume that the embedding $H \subset \mathcal{X}^*$ is dense, so that $\langle \cdot, \cdot \rangle$ extends to a dual pairing between $\mathcal{X}$ and $\mathcal{X}^*$, which we denote by the same symbol. (In most applications, $\mathcal{X}$ is actually a Hilbert space, but we stress that in this case, the dual pairing $\langle \cdot, \cdot \rangle$ is normally not the scalar product on $\mathcal{X}$; the latter plays no role in our investigation.) In this setting, let us define the class of operators $H$ of interest.

**Definition 4.1.** Let $H$ be a selfadjoint operator on a dense domain in $H$ and $R(z)$ its resolvents. We call $H$ an $\mathcal{X}$-smooth operator if there exists an open set $U \subset \mathbb{R}$ of full (Lebesgue) measure such that the limits

$$R(\lambda \pm i0) := \lim_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon), \quad \lambda \in U,$$

exist in $\mathcal{B}(\mathcal{X}, \mathcal{X}^*)$, and the extended functions $R : \mathbb{H} \cup U \to \mathcal{B}(\mathcal{X}, \mathcal{X}^*)$ are locally Hölder continuous.

In this situation, it follows that $U \subset \sigma_{ac}(H)$ and for any Borel set $\Delta \subset U$, one has $E(\Delta)H \subset P^{ac}H$. The locally Hölder continuous map $A : U \to \mathcal{B}(\mathcal{X}, \mathcal{X}^*)$ given by

$$A(\lambda) = \frac{1}{2\pi i} \left( R(\lambda + i0) - R(\lambda - i0) \right) \quad (4.1)$$

equals the weak derivative $\frac{dE}{d\lambda}$ where $E$ are the spectral projections of $H$, i.e.,

$$\frac{d}{d\lambda} \langle \varphi, E(\lambda)\psi \rangle = \langle \varphi, A(\lambda)\psi \rangle \quad \text{for all } \varphi, \psi \in \mathcal{X}, \lambda \in U. \quad (4.2)$$

It follows that $A(\lambda)$ diagonalizes the absolutely continuous part of $H$, in the sense that for bounded Borel functions $f$,

$$\langle \varphi, f(H)P^{ac}\psi \rangle = \int d\lambda f(\lambda) \langle \varphi, A(\lambda)\psi \rangle \quad \text{for all } \varphi, \psi \in \mathcal{X}. \quad (4.3)$$

Also, we note that the $A(\lambda)$ are positive as quadratic forms on $\mathcal{X} \times \mathcal{X}$, and therefore one has

$$\forall \varphi \in \mathcal{X} : \quad \|A(\lambda)\varphi\|_\mathcal{X}^2 \leq \|A(\lambda)\|_{\mathcal{X}, \mathcal{X}^*} \langle \varphi, A(\lambda)\varphi \rangle. \quad (4.4)$$

Equally, one can start with the maps $A(\lambda) = \frac{dE}{d\lambda}$ and deduce the properties of the resolvents $R(\lambda \pm i0)$, see [BA11, Sec. 3]. We give the following sufficient criterion:

**Lemma 4.2.** Let $A(\lambda) = \frac{dE}{d\lambda} \in \mathcal{B}(\mathcal{X}, \mathcal{X}^*)$ be locally Hölder continuous in $\lambda \in U$, where $U \subset \mathbb{R}$ is an open set of full measure, and suppose in addition that the function $\lambda \mapsto (1 + |\lambda|)^{-1}\|A(\lambda)\|_{\mathcal{X}, \mathcal{X}^*}$ is integrable over $\mathbb{R}$. Then $H$ is $\mathcal{X}$-smooth, and (4.1) holds.

To see this, one considers the integral $R(z) = \int A(\lambda)(\lambda - z)^{-1}d\lambda$ weakly on $\mathcal{X} \times \mathcal{X}$, where $\text{Im } z \neq 0$, splits the integration region into a small interval $J$ around $\text{Re } z \in U$ and into its complement, then applies the Privalov-Korn theorem (Lemma A.1) to the integral over $J$, and the integrability condition on $\mathbb{R} \setminus J$. See [BA11, Thm. 3.6] for details. We will establish a quantitative version of this result below, in Proposition 4.7.
Passing to the setting of scattering theory, let \( \Gamma_2(\mathcal{X}^*, \mathcal{X}) \subset \mathfrak{B}(\mathcal{X}^*, \mathcal{X}) \) be the space of bounded operators that “factor through a Hilbert space”; that is, \( V \in \Gamma_2(\mathcal{X}^*, \mathcal{X}) \) is of the form \( V = V_1^*V_0 \) where \( V_0, V_1 \in \mathfrak{B}(\mathcal{X}^*, \mathcal{K}) \) with a Hilbert space \( \mathcal{K} \). If \( \mathcal{X} \) is Hilbertisable, then \( \Gamma_2(\mathcal{X}^*, \mathcal{X}) = \mathfrak{B}(\mathcal{X}^*, \mathcal{X}) \); but in general, the inclusion may be proper [Pis86].

**Definition 4.3.** A smooth scattering system \((H_0, H_1, \mathcal{X}, \mathcal{H})\) consists of two selfadjoint operators \( H_0 \) and \( H_1 \) on a common dense domain in the Hilbert space \( \mathcal{H} \) which are both \( \mathcal{X} \)-smooth, such that \( V := H_1 - H_0 \in \Gamma_2(\mathcal{X}^*, \mathcal{X}) \).

We can (and often will) assume in this situation that both \( H_j \) are smooth with respect to the same set \( U \) of full measure. In practical examples, \( \mathcal{X} \)-smoothness of one operator (say, \( H_0 \)) can usually be verified directly, whereas the \( \mathcal{X} \)-smoothness of \( H_1 \) is obtained by perturbation arguments. We give a well-known type of sufficient criterion (cf. [Yaf92, BA11]), and sketch its proof in our context.

**Lemma 4.4.** Let \( H_0 \) be a selfadjoint operator on a dense set \( \mathcal{D} \subset \mathcal{H} \) and \( V = V^* \in \mathfrak{B}(\mathcal{X}^*, \mathcal{X}) \).

Suppose that \( H_0 \) is \( \mathcal{X} \)-smooth and that \( R_0(z) \in \mathcal{FA}(\mathcal{X}, \mathcal{X}^*) \) for all \( z \in \mathbb{C} \setminus \mathbb{R} \). Then, \( H_1 := H_0 + V \) is \( \mathcal{X} \)-smooth.

(Here \( \mathcal{FA}(\mathcal{X}, \mathcal{X}^*) \subset \mathfrak{B}(\mathcal{X}^*, \mathcal{X}) \) denotes the norm closure of the space of finite rank operators. If \( \mathcal{X} \) is a Hilbert space, \( \mathcal{FA}(\mathcal{X}, \mathcal{X}^*) \) equals the space of compact operators.)

**Proof.** Since \( V \upharpoonright \mathcal{D} = (V \upharpoonright \mathcal{H})^* \in \mathfrak{B}(\mathcal{H}) \), also \( H_1 \) is selfadjoint on \( \mathcal{D} \). Now for any \( z \in \mathbb{H}_\pm \), the operator \( 1 + VR_0(z) \) is invertible in \( \mathfrak{B}(\mathcal{X}) \). (Otherwise, since \( VR_0(z) \in \mathcal{FA}(\mathcal{X}, \mathcal{X}) \), the Fredholm alternative yields a \( \psi \in \mathcal{X} \setminus \{0\} \) in the kernel of \( 1 + VR_0(z) \). Then \( \varphi := R_0(z)\psi \in \mathcal{D} \) fulfills \((H_0 + V)\varphi = z\varphi \) with \( \text{Im} z \neq 0 \), contradicting the selfadjointness of \( H_1 \) by analytic Fredholm theory [Sim15, Theorem 3.14.3], the \( \mathfrak{B}(\mathcal{X}) \)-valued function \( F : z \mapsto (1 + VR_0(z))^{-1} \) is analytic in \( \mathbb{H}_\pm \). Further, for \( \lambda \in U \), the norm limit \( VR_0(\lambda \pm i0) \) lies in \( \mathcal{FA}(\mathcal{X}, \mathcal{X}) \), hence \( F(\lambda \pm i0) := (1 + VR_0(\lambda \pm i0))^{-1} \) exists for \( \lambda \) outside a closed null set \( N \), cf. [Yaf92, Sec. 1.8.3]; and (local H"older) continuity of \( VR_0(\cdot) \) on \( \mathbb{H}_\pm \cup U \) translates to (local H"older) continuity of \( F \) there. Finally, the resolvent identity

\[
R_1(z) = R_0(z) - R_1(z)VR_0(z) \quad (4.5)
\]

implies \( R_1(z) = R_0(z)F(z) \), showing that \( H_1 \) is \( \mathcal{X} \)-smooth with respect to \( U \setminus N \) rather than \( U \). \( \square \)

Now for a smooth scattering system, the wave operators are known to exist automatically, and they can be expressed in terms of boundary values of resolvents. We derive a related expansion for the operators \( P_1(W_\pm(H_1, H_0) - 1)P_0^* \) which are of interest to us.

**Proposition 4.5.** Let \((H_0, H_1, \mathcal{X}, \mathcal{H})\) be a smooth scattering system. Then the wave operators \( W_\pm(H_1, H_0) \) exist and are complete. We have for real H"older continuous functions \( \chi_0, \chi_1 \) with support in compact intervals \( I_0, I_1 \subset U \) and for any \( \varphi_0, \varphi_1 \in \mathcal{X} \),

\[
\langle \chi_1(H_1)\varphi_1, (W_\pm(H_1, H_0) - 1)\chi_0(H_0)\varphi_0 \rangle = \lim_{\epsilon \downarrow 0} \int d\lambda d\mu \frac{\chi_1(\mu)\chi_0(\lambda)}{\lambda - \mu \mp i\epsilon} \langle \varphi_1, A_1(\mu)V A_0(\lambda)\varphi_0 \rangle, \quad (4.6)
\]

where \( V = H_1 - H_0 \).

**Proof.** The existence of wave operators follows from well-known results: If \( I \subset U \) is a compact interval, and \( V = V_1^*V_0 \), then \( \mathcal{X} \)-smoothness of \( H_j \) implies that \( V_j(R_j(\lambda \mp i\epsilon) - R_j(\lambda \mp i\epsilon))V_j^* \) is uniformly bounded in \( \mathfrak{B}(\mathcal{K}) \) for \( \lambda \in I \) and \( |\epsilon| \) sufficiently small. Hence the operators \( V_jE_j(I) \) are Kato-smooth with respect to \( H_j \) [Yaf92, Theorem 4.3.10]. This suffices to show that the wave operators \( W_\pm(H_1, H_0) \) and \( W_\pm(H_0, H_1) \) exist, since \( U \) is of full measure [Yaf92, Corollary 4.5.7].

Given the existence of the wave operators, they have the “stationary” representations [Yaf92, Lemma 2.7.1]

\[
\langle \psi_1, W_\pm(H_1, H_0)\psi_0 \rangle = \lim_{\epsilon \downarrow 0} \int \langle \psi_1, R_1(\lambda \mp i\epsilon)R_0(\lambda \pm i\epsilon)\psi_0 \rangle d\lambda \quad (4.7)
\]
for all \( \psi_j \in P_j^\infty \mathcal{H} \). On the other hand, for every \( \epsilon > 0 \),
\[
\langle \psi_1, \psi_0 \rangle = \frac{\epsilon}{\pi} \int \langle \psi_1, R_0(\lambda \mp i\epsilon) R_0(\lambda \pm i\epsilon) \psi_0 \rangle \, d\lambda. \tag{4.8}
\]
Hence if \( \varphi_j \in \mathcal{X} \) and \( \text{supp} \chi_j \subset I_j \subset U \), then it follows from (4.7), (4.8) and from the resolvent identity (4.5) that
\[
\langle \chi_1(H_1)\varphi_1, (W_\pm(H_1, H_0) - 1)\chi_0(H_0)\varphi_0 \rangle \\
= -\lim_{\epsilon,0} \frac{\epsilon}{\pi} \int \langle \varphi, \chi_1(H_1) R_1(\lambda \mp i\epsilon) V R_0(\lambda \mp i\epsilon) R_0(\lambda \pm i\epsilon) \chi_0(H_0) \varphi_0 \rangle \, d\lambda \\
= -\lim_{\epsilon,0} \int \frac{\chi_1(\mu_1)}{\mu_1 - \lambda \pm i\epsilon} \frac{\chi_0(\mu_0)\epsilon/\pi}{(\mu_0 - \lambda)^2 + \epsilon^2} \langle \varphi, A_1(\mu_1) V A_0(\mu_0) \varphi_0 \rangle \, d\lambda \, d\mu_1 \, d\mu_0 \\
= \lim_{\epsilon,0} \int \langle \Phi_1(\lambda, \epsilon), \Phi_0(\lambda, \epsilon) \rangle_K \, d\lambda,
\]
where (4.3) has been used twice, and where \( \Phi_j \) are the \( K \)-valued functions
\[
\Phi_1(\lambda, \epsilon) = \int \frac{\chi_1(\mu)}{\lambda - \mu \pm i\epsilon} V_1 A_1(\mu) \varphi_1 \, d\mu, \\
\Phi_0(\lambda, \epsilon) = \int \frac{\chi_0(\mu)\epsilon/\pi}{(\mu_0 - \lambda)^2 + \epsilon^2} V_0 A_0(\mu) \varphi_0 \, d\mu.
\]
Note that these integrals exist as they run over the compact intervals \( I_j \) where the integrand is continuous in the norm of \( K \). As \( \epsilon \to 0 \), the first integral \( \Phi_1(\lambda, \epsilon) \) has a limit by the Privalov-Korn theorem (Lemma A.1), whereas the second one evidently satisfies \( \Phi_0(\lambda, \epsilon) \to \chi_0(\lambda) V_0 A_0(\lambda) \varphi_0 \).

Using the estimate from the Privalov-Korn theorem on \( \Phi_1 \) and an elementary estimate on \( \Phi_0 \), it moreover follows that \( ||\Phi_1(\lambda, \epsilon)||_K ||\Phi_0(\lambda, \epsilon)||_K \) has an \( \epsilon \)-independent upper bound that is integrable in \( \lambda \). Hence we may use dominated convergence to conclude the claimed result.

4.2 High energy behaviour

We now analyze the high-energy behaviour of resolvents and wave operators, starting with a single selfadjoint operator \( H \).

Definition 4.6. Let \( \beta \in (0, 1) \). We say that an \( \mathcal{X} \)-smooth operator \( H \) is of high-energy order \( \beta \) if there exist \( \hat{\lambda}, b > 0 \) such that
\[
||R(\lambda \pm i0)||_{\mathcal{X}, \mathcal{X}^*} \leq b|\lambda|^{-\beta} \quad \text{for all} \quad \lambda \in U, \quad |\lambda| \geq \hat{\lambda}. \tag{4.10}
\]
We say that \( H \) is of strict high-energy order \( \beta \) if, additionally, \( (-\infty, -\hat{\lambda}] \cup [\hat{\lambda}, \infty) \) is contained in \( U \).

Note that being of strict high-energy order \( \beta \) implies \( H \in AAC(\mathcal{H}) \). Hence, in a smooth scattering system with such operators \( H_0, H_1 \), mutual \( f_\beta \)-boundedness is the same as \( f_\beta \)-equivalence, since \( f_\beta(H_0) - f_\beta(H_1) \) is always bounded, cf. Lemma B.1. We will return to this topic shortly.

With \( R(\lambda \pm i0) \), also \( A(\lambda) \) fulfill similar bounds at large \( |\lambda| \), see Eq. (4.1). Vice versa, we can deduce the high-energy behaviour of \( R(\lambda \pm i0) \) from that of \( A(\lambda) \), given a uniform Hölder estimate.

Proposition 4.7. Let \( H \) be a selfadjoint operator, \( U \subset \mathbb{R} \) an open set of full measure and \( A : U \to \mathcal{B}(\mathcal{X}, \mathcal{X}^*) \) be such that \( \frac{d}{d\lambda}(\varphi, E(\lambda)\varphi) = \langle \varphi, A(\lambda)\varphi \rangle \) for all \( \varphi \in \mathcal{X} \), \( \lambda \in U \). Suppose that \( \lambda \mapsto ||A(\lambda)||_{\mathcal{X}, \mathcal{X}^*} \) is locally integrable, that \( A(\lambda) \) is locally Hölder continuous, and that there are constants \( c > 0, \beta, \theta \in (0, 1), \hat{\lambda} > 0, q > 1 \) such that \( (-\infty, -\hat{\lambda}] \cup [\hat{\lambda}, \infty) \subset U \) and
\[
||A(\lambda)||_{\mathcal{X}, \mathcal{X}^*} \leq c|\lambda|^{-\beta} \quad \text{whenever} \quad |\lambda| \geq \hat{\lambda}, \\
||A(\lambda) - A(\lambda')||_{\mathcal{X}, \mathcal{X}^*} \leq c|\lambda|^{-\beta-\theta}|\lambda - \lambda'|^\theta \quad \text{whenever} \quad |\lambda| \geq \hat{\lambda}, \quad 1 < \lambda'/\lambda \leq q^2.
\]
Then \( H \) is \( \mathcal{X} \)-smooth and of strict high-energy order \( \beta \).
Proof. $H$ is $X$-smooth by Lemma 4.2. For more quantitative estimates, fix $\lambda \geq q \hat{\lambda}$ (the case $\lambda \leq -q \hat{\lambda}$ is analogous). Let $I = [\lambda/q, q\lambda] \subset U$. For $\epsilon > 0$, we can write in the sense of weak integrals on $X \times X$,

$$R(\lambda \pm i0) = \lim_{\epsilon \downarrow 0} \int I \frac{A(\lambda')d\lambda'}{\lambda' - \lambda - \epsilon} + \int_{|\lambda'| \geq \hat{\lambda}} \frac{A(\lambda')d\lambda'}{\lambda' - \lambda} + E(\hat{\lambda}, \hat{\lambda})(H - \lambda)^{-1}.$$  

To estimate these terms, we note that by our hypothesis,

$$\sup_{\lambda' \in I} \|A'(\lambda')\|_{X, X^*} + \sup_{\lambda' \neq \lambda'' \in I} \frac{\lambda^\beta}{|\lambda' - \lambda''|^\beta} \|A(\lambda') - A(\lambda'')\|_{X, X^*} \leq c_1 \lambda^{-\beta}$$

with a constant $c_1 > 0$. Hence the Privalov-Korn theorem (Lemma A.1) yields the estimate

$$\|J_1(\lambda)\|_{X, X^*} \leq c_1 k_0 \lambda^{-\beta} \quad (4.11)$$

with constants independent of $\lambda$. For estimating $J_2$, we split the integration region further into $(-\infty, -\hat{\lambda}] \cup (-\hat{\lambda}, -\hat{\lambda}) \cup (\hat{\lambda}, \lambda/q] \cup [q\lambda, \infty)$. We have

$$\left\| \int_{-\infty}^{-\hat{\lambda}} d\lambda' \frac{A(\lambda')}{\lambda' - \lambda} \right\|_{X, X^*} \leq c_2 \lambda^{-\beta}$$

with some $c_2 > 0$. A similar estimate holds for the integral over $[q\lambda, \infty)$. Further,

$$\left\| \int_{-\hat{\lambda}}^{-\lambda} d\lambda' \frac{A(\lambda')}{\lambda' - \lambda} \right\|_{X, X^*} \leq \frac{1}{\hat{\lambda}} \int_{-\lambda}^{-\hat{\lambda}} d\lambda' \lambda'^{-\beta} \leq c_3 \lambda^{-\beta} + c_4 \lambda^{-1}$$

with $c_3, c_4 > 0$. The interval $(\hat{\lambda}, \lambda/q)$ is handled similarly. Therefore, one has $\|J_2\|_{X, X^*} \leq c_5 \lambda^{-\beta}$. Finally, it is clear that $\|J_3\|_{X, X^*} \leq c_6 \|J_3\|_{H, H} \leq c_6 (\lambda - \hat{\lambda})^{-1}$.

Combined, we have shown that $R(\lambda \pm i0)$ exists for all $|\lambda| \geq q \hat{\lambda}$ and fulfills $\|R(\lambda \pm i0)\|_{X, X^*} \leq c_7 |\lambda|^{-\beta}$; therefore $H$ is of strict high-energy order $\beta$. □

Now passing to smooth scattering systems, it turns out that both operators $H_j$ are always of the same high-energy order, hence we can meaningfully speak of the system having (strict) high-energy order $\beta$.

Proposition 4.8. Let $(H_0, H_1, X, \mathcal{H})$ be a smooth scattering system, and let $\beta \in (0, 1)$. Then $H_0$ is of (strict) high-energy order $\beta$ if and only if $H_1$ is.

Proof. Let $U_j$ be the open set of full measure pertaining to the $X$-smooth operator $H_j$, and set $V := H_1 - H_0$. Suppose that $H_0$ is of high-energy order $\beta$. After possibly increasing $\hat{\lambda}$, we may assume that $\|R_0(\lambda \pm i0)\|_{X, X^*} < (2\|V\|_{X^*, X})^{-1}$ for $\lambda \in U_0$, $|\lambda| \geq \hat{\lambda}$. Hence for $z = \lambda \pm i0$ in this range and in a small neighbourhood within $\mathbb{H}_\pm$, we have $\|VR_0(z)\|_{X, X} \leq \frac{1}{2}$, and the Neumann series

$$(1 + VR_0(z))^{-1} = \sum_{n=0}^{\infty} (-VR_0(z))^n$$

converges in $\mathcal{B}(X, X')$. That is, $1 + VR_0(z)$ has an inverse in $\mathcal{B}(X, X')$, with norm at most 2, which is locally Hölder continuous in $z$ since $R_0$ is. Now from the resolvent equation (4.5), we obtain

$$R_1(z) = R_0(z)(1 + VR_0(z))^{-1}.$$  

(4.13)

Thus $U_1$ can be chosen to contain $U_0 \cap \{\lambda : |\lambda| \geq \hat{\lambda}\}$, and on that set we have

$$\|R_1(\lambda \pm i0)\|_{X, X^*} \leq 2\|R_0(\lambda \pm i0)\|_{X, X^*} \leq 2b|\lambda|^{-\beta}$$

as claimed. The other direction follows by symmetric arguments. □
We will now turn our attention to the high-energy behaviour of the wave operator in a smooth scattering system, and investigate $f_\beta$-boundedness and $f_\beta$-equivalence (which we write as $H_0 \sim_\beta H_1$): here $f_\beta(\lambda) = (1 + \lambda^2)^{\beta/2}$ with some $\beta \in (0, 1)$, as announced.

**Theorem 4.9.** Let $(H_0, H_1, \mathcal{X}, \mathcal{H})$ be a smooth scattering system of high-energy order $\beta \in (0, 1)$. Then $H_0$ and $H_1$ are mutually $f_\beta$-bounded. If the system is of strict high-energy order $\beta$, then $H_0 \sim_\beta H_1$.

**Proof.** First, let $\varphi_0, \varphi_1 \in \mathcal{X}$, and let $\chi_0, \chi_1 : \mathbb{R}_+ \to [0, 1]$ be Hölder continuous such that $\text{supp} \chi_j \subset \bigcup_k I_k$, where $I_k$ are finitely many disjoint compact intervals and $I_k \subset U \cap [\bar{\lambda}, \infty)$. By Proposition 4.5, we have

$$\langle (\chi_0(H_1)\varphi_1, (W_\pm(H_1, H_0) - 1)H_0^\beta \chi_0(H_0)\varphi_0) \rangle = \lim_{\epsilon \downarrow 0} \int_0^\infty d\lambda d\mu \frac{\chi_1(\mu)\chi_0(\lambda)}{\lambda - \mu \mp i\epsilon} \langle \varphi_1, A_1(\mu)V A_0(\lambda)\varphi_0 \rangle$$

where $V = V_1^* V_0$ with $V_j : \mathcal{X}^* \to \mathcal{K}$, and $\Phi_j$ are the $\mathcal{K}$-valued functions on $\mathbb{R}_+$ given by

$$\Phi_j(\lambda) = \chi_j(\lambda)\lambda^{\beta/2} V_j A_j(\lambda) \varphi_j.$$ 

From our assumption on the high-energy order of the $H_j$, we have the estimate $\|A_j(\lambda)\|_{\mathcal{X}, \mathcal{X}^*} \leq \frac{b}{1} \lambda^{-\beta}$ for all $\lambda \in U$ with $|\lambda| \geq \bar{\lambda}$. Hence, using (4.4),

$$\int_0^\infty d\lambda \|\Phi_j(\lambda)\|_{\mathcal{K}}^2 \leq \int_0^\infty d\lambda \chi_j(\lambda)^2 \lambda^{\beta} \|V_j\|_{\mathcal{X}^*, \mathcal{K}}^2 \|A_j(\lambda)\|_{\mathcal{X}, \mathcal{X}^*}^2 \langle \varphi_j, A_j(\lambda) \varphi_j \rangle \leq \frac{b}{\pi} \|V_j\|_{\mathcal{X}^*, \mathcal{K}}^2 \langle \varphi_j, A_j(\lambda) \varphi_j \rangle \leq \frac{b}{\pi} \|\varphi_j\|_{\mathcal{X}^*, \mathcal{K}}^2 \langle \varphi_j, A_j(\lambda) \varphi_j \rangle$$

In other words, the $\Phi_j$ are elements of $L^2(\mathbb{R}_+, \mathcal{K})$, and the constant $b > 0$ in their norm is independent of our choice of $\chi_j$ under the given constraints. Now in (4.14), $K(\lambda, \mu) = (\lambda/\mu)^{\beta/2}(\lambda - \mu \mp i\epsilon)^{-1}$ is the kernel of a bounded operator $T$ on $L^2(\mathbb{R}_+)$ by Lemma C.1. Hence also $T \otimes 1_{\mathcal{K}}$ is bounded on $L^2(\mathbb{R}_+, \mathcal{K})$. This yields

$$\langle (\varphi_1, \chi_1(H_1)(W_\pm(H_1, H_0) - 1)\chi_0(H_0)\varphi_0) \rangle \leq c \|\Phi_0\|_{L^2(\mathbb{R}_+, \mathcal{K})} \|\Phi_1\|_{L^2(\mathbb{R}_+, \mathcal{K})}$$

with a universal $c > 0$ (depending only on $\beta$).

Now we can choose a sequence of $\chi_j$ of the form stated above such that $\chi_j(H_j)$ converges strongly to $P_{\mathcal{K}} E_j(\hat{\lambda}, \infty)$. Thus (4.15) yields, considering that $E_{0}(\hat{\lambda}, \infty)(H_0^\beta - f_\beta(H_0))$ is bounded,

$$E_{1}(\hat{\lambda}, \infty) P_{\mathcal{X}}(W_\pm(H_1, H_0) - 1)f_\beta(H_0)P_{0} \in \mathfrak{B}(\mathcal{H}).$$

Similar arguments show that the analogous expressions with one or both of the $E_{j}(\hat{\lambda}, \infty)$ swapped for $E_{j}(\infty, -\hat{\lambda})$ are bounded. (This requires boundedness of the integral operator with kernel $K'(\lambda, \mu) = (\lambda/\mu)^{\beta/2}(\mu + \lambda)^{-1}$, see again Lemma C.1.) Moreover, as Lemma B.1 shows, the boundedness of $H_0 - H_1$ implies that also $f_\beta(H_0) - f_\beta(H_1)$ is bounded. Hence Lemma 2.2 is applicable, and we obtain that $P_{0}^\mathcal{X}(W_\pm(H_1, H_0) - 1)f_\beta(H_0)P_{0}^\mathcal{X}$ is bounded. The statement with $H_1$ and $H_0$ exchanged follows symmetrically; thus $H_1$ and $H_0$ are mutually $f_\beta$-bounded. In the case of strict high energy order, we have $H_j \in \mathcal{A}(\mathcal{H})$ and hence it follows that $H_1 \sim_\beta H_0$.  \(\square\)
4.3 Pseudo-differential operators

Our results in the smooth method can be applied to a wide range of examples where $H_0$ is a differential or pseudo-differential operator and the perturbation $V = H_1 - H_0$ is a multiplication operator. Here we treat $f_\beta$-equivalence for the perturbed polyharmonic operator, i.e., where $H_0$ is a fractional power of the Laplace operator, using familiar techniques for the Schrödinger operator ($\ell = 2$ below); see, e.g., [Yaf10].

Example 4.10. Let $H_0 = (-\Delta)^{\ell/2}$ act on its natural domain of selfadjointness in $H = L^2(\mathbb{R}^n)$, where $\ell \in (1, \infty)$, $n \in \mathbb{N}$. Let $v \in L^\infty(\mathbb{R}^n)$ such that $\sup_x(1 + |x|^2)^\alpha|v(x)| < \infty$ with some $\alpha > \frac{1}{2}$, and let $V \in \mathfrak{B}(H)$ be the multiplication with $v$. Then $H_0$ and $H_1 := H_0 + V$ are $f_\beta$-equivalent for any $0 < \beta \leq 1 - \frac{1}{2\ell}$.

Proof. Let $\langle \cdot \rangle$ be the multiplication operator by $(1 + |x|^2)^{\ell/2}$. We define $\mathcal{X} \subset H$ as the completion of $\mathcal{S}(\mathbb{R}^n)$ in the norm $\|f\|_{\mathcal{X}} := \|\langle \cdot \rangle^\alpha f\|_H$. To show that $H_0$ is $\mathcal{X}$-smooth, let us introduce for fixed $\lambda > 0$ the map $\Gamma(\lambda) : \mathcal{S}(\mathbb{R}^n) \to L^2(S^{n-1})$ given by

$$
(\Gamma(\lambda)f)(\omega) = (2\pi)^{-n/2} \int d^n x \ e^{i\lambda \omega \cdot x} f(x) = \tilde{f}(\lambda \omega).
$$

By [Yaf10, Theorem 1.1.4] it extends to a bounded operator $\Gamma(\lambda) : \mathcal{X} \to L^2(S^{n-1})$ with norm bound

$$
\|\Gamma(\lambda)\|_{\mathcal{X}, L^2(S^{n-1})} \leq C\lambda^{-\frac{\alpha}{2}}
$$

(4.16)

for all $\lambda > 0$, where $C$ is independent of $\lambda$. It also follows from [Yaf10, Theorem 1.1.5] that $\Gamma(\lambda)$ is locally Hölder continuous, in the sense that

$$
\|\Gamma(\lambda) - \Gamma(\lambda')\|_{\mathcal{X}, L^2(S^{n-1})} \leq C' |\lambda - \lambda'|^\theta
$$

(4.17)

for some $\theta \in (0, 1)$ (in fact $\theta = \alpha - \frac{1}{2} \frac{\ell}{\beta}$ if $\frac{1}{\beta} < \alpha < \frac{1}{\beta}$), where $C'$ can be chosen uniformly for all $\lambda, \lambda'$ in a fixed compact interval in the open half line $\mathbb{R}_+$. The derivative of the spectral measure of $H_0$ is now given by

$$
\frac{d}{d\lambda} \langle \varphi, E_0(\lambda) \psi \rangle = \frac{1}{\ell} \lambda^{\frac{n}{2}-1} \langle \varphi, \Gamma(\lambda^{1/\ell})^* \Gamma(\lambda^{1/\ell}) \psi \rangle =: \langle \varphi, A_0(\lambda) \psi \rangle
$$

for $\varphi, \psi \in \mathcal{X}$ and $\lambda > 0$, and $A_0(\lambda) = 0$ for $\lambda < 0$. As a consequence of (4.16), $A_0(\lambda)$ is a bounded operator from $\mathcal{X}$ to $\mathcal{X}^*$ with norm bounded by

$$
\|A_0(\lambda)\|_{\mathcal{X}, \mathcal{X}^*} \leq \hat{C}\lambda^{\frac{n}{2}-1+\frac{1}{\beta}}, \quad \lambda > 0,
$$

(4.18)

and (4.17) implies that $A_0(\lambda)$ is locally Hölder continuous on $\mathbb{R}_+$ as a composition of Hölder continuous functions. Therefore all the requirements of Lemma 4.2 are satisfied with $U = \mathbb{R}\setminus\{0\}$, and we can conclude that $H_0$ is $\mathcal{X}$-smooth.

By Lemma 4.4, also $H_1$ is then $\mathcal{X}$-smooth if we can show that $R_0(z) \in \mathfrak{B}(\mathcal{X}, \mathcal{X}^*)$ is compact for $\text{Im} \ z \neq 0$ (note that $\mathcal{X}$ is Hilbertisable). But this is equivalent to compactness of $\langle x \rangle^{-\alpha} R_0(z) \langle x \rangle^{-\alpha}$ in $\mathfrak{B}(H)$, which follows since this operator is a product of suitable multiplication and convolution operators [Yaf92, Lemma 1.6.5].

For analyzing the high-energy behaviour of $H_0$, we define the unitary dilation operators on $H$, 

$$
(D(\tau)\varphi)(x) = \tau^{-n/2} \varphi(\tau^{-1} x), \quad \varphi \in \mathcal{H}, \ \tau > 0.
$$

(4.19)

Considering them as operators from $\mathcal{X}$ to $\mathcal{X}$, or from $\mathcal{X}^*$ to $\mathcal{X}^*$, one finds that

$$
\|D(\tau)\|_{\mathcal{X}, \mathcal{X}} \leq C \tau^\alpha \quad \text{and} \quad \|D(\tau^{-1})\|_{\mathcal{X}^*, \mathcal{X}^*} \leq C \tau^\alpha \quad \text{for all } \tau \geq 1
$$

(4.20)

with some $C > 0$. One also computes that

$$
D(\tau^{-1}) R_0(z) D(\tau) = \tau^\ell R_0(\tau^\ell z).
$$

(4.21)
Together with (4.20), we then find
\[ ||R_0(\lambda \pm i0)||_{X, X^*} \leq C^2 ||R_0(1 \pm i0)||_{X, X^*} \lambda^{-1+\frac{2\beta}{\ell}} \]
and conclude that \( H_0 \) is of strict high-energy order \( \beta = 1 - \frac{2\beta}{\ell} \), provided this number is positive. By Theorem 4.9, we then have \( H_0 \sim_\beta H_1 \). Since \( \alpha > \frac{1}{2} \) was arbitrary and the \( X \)-norm becomes stronger with increasing \( \alpha \), we have thus shown our claim for all \( 0 < \beta < 1 - \frac{1}{2} \).

(Alternatively, we may deduce this as follows: Pick an interval \([1, q^2]\) where \( A_0(\lambda) \) is uniformly Hölder continuous. Using \( D(\tau^{-1})A_0(\lambda)D(\tau) = \tau^\epsilon A_0(\tau^\epsilon \lambda) \) and (4.20), a scaling argument like above shows that the hypothesis of Proposition 4.7 is satisfied, yielding the result.)

Proving the claim for \( \beta = 1 - \frac{1}{2} \) requires more effort; we sketch the argument. We make use of the Agmon-Hörmander space \( B \subset H \); see [Yaf10, Secs. 6.3 and 7.1] for its definition and properties. Here we need only that \( X \subset B \subset H \) are continuous dense inclusions, and that, in some improvement over (4.20),
\[ ||D(\tau)||_{B, B^*} \leq C^{1/2} \quad \text{and} \quad ||D(\tau^{-1})||_{B^*, B^*} \leq C^{1/2} \quad \text{for all} \ \tau \geq 1. \] (4.22)
We will show below that \( R_0(\lambda \pm i0) \) is bounded from \( B \) to \( B^* \) for each fixed \( \lambda > 0 \). A scaling argument as above then yields
\[ ||R_0(\lambda \pm i0)||_{B, B^*} \leq C^2 ||R_0(1 \pm i0)||_{B, B^*} \lambda^{-1+\frac{1}{2}} \]
for all \( \lambda \geq 1 \). An analogous estimate holds for \( ||R_0(\lambda \pm i0)||_{X, X^*} \), since the inclusion \( X \subset B \) is continuous. Hence \( H_0 \) is of strict high-energy order \( \beta = 1 - \frac{1}{2}, \) and \( H_0 \sim_\beta H_1 \).

In order to show that \( R_0(\lambda \pm i0) \in \mathfrak{B}(B, B^*) \), let us define for \( \epsilon > 0 \),
\[ S_{\pm \epsilon} := LR_0^{(2)}(\lambda^{2/\ell} \pm i\epsilon) \in \mathfrak{B}(H), \] (4.23)
where \( R_0^{(2)}(\cdot) \) is the resolvent of \( -\Delta \), and \( L \) is the multiplication operator in Fourier space by the function
\[ \hat{\ell}(\xi) := \frac{|\xi|^2 - \lambda^{2/\ell}}{|\xi|^\ell - \lambda}. \]
One notices that \( \langle \varphi, S_{\pm \epsilon} \psi \rangle \to \langle \varphi, R_0(\lambda \pm i0) \psi \rangle \) for each \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \subset B \) as \( \epsilon \to 0 \). On the other hand, [Yaf10, Theorem 6.3.3] yields that \( ||S_{\pm \epsilon}||_{B, B^*} \) is uniformly bounded for all \( \epsilon \leq 1 \). Hence, \( ||\varphi, R_0(\lambda \pm i0) \psi || \leq c||\varphi||_B ||\psi||_B \) for all \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \), and \( R_0(\lambda \pm i0) \) extends to a bounded operator from \( B \) to \( B^* \).

Remark. Theorem 6.3.3 in [Yaf10] assumes that the function \( \hat{\ell} \) is smooth everywhere, but this is not essential for its proof; it suffices that, as in our case, the function is smooth outside the origin \( \xi = 0 \), and bounded in a neighbourhood of the origin.

The estimates on the high-energy order, \( 0 < \beta \leq 1 - \frac{1}{\ell} \), cannot be improved in general, as the following special case shows.

**Example 4.12.** In Example 4.10, let \( n = 1, \ell = 2, \) and suppose that \( v \neq 0 \) is compactly supported and nonnegative. Then \( (H_1, H_0) \) is not \( f_\beta \)-bounded for any \( \beta > \frac{1}{2} \). In particular, \( H_1 \not\sim_\beta H_0 \).

**Proof.** First, note that \( P_0^{\text{ac}} = 1 \); also, since \( v \geq 0 \), we know that \( H_1 \geq 0 \) and hence \( H_1 \not\sim_\beta H_0 \).

Now let \( v \) be supported in the compact interval \([a, b] \). We choose \( \varphi \in C_0^\infty((b, \infty) \) and \( \psi \in L^2(\mathbb{R}) \) with its Fourier transform \( \check{\psi} \in C_0^\infty(\mathbb{R}_+) \) such that \( \langle \varphi, \check{\psi} \rangle \neq 0 \). For \( n \in \mathbb{N} \), define \( \varphi_n(x) := e^{inx} \varphi(x) \), \( \psi_n(x) := e^{inx} \psi(x) \). The wave operator \( W := W_{2}(H_1, H_0) \) can in our case be written as
\[ (W\psi)(x) = \frac{1}{\sqrt{2\pi}} \int dk \, m(x, k) T(k) e^{ikx} \check{\psi}(k) \]
with a complex-valued function \( T \) and a certain integral kernel \( m \), for which \( x > b \) is given by \( m(x, k) = 1 \) [DT79, Sec. 2]. Hence we have
\[ \langle \varphi_n, (W - 1) f_\beta(H_0) \psi_n \rangle = \int dx \, \bar{\varphi}_n(x) \int dk \, (m(x, k) T(k) - 1) e^{ikx} (1 + k^4)^{\beta/2} \check{\psi}_n(k) \]
(4.24)
\[ = \int dk \, \bar{\varphi}(k) (T(k + n) - 1) (1 + (k + n)^4)^{\beta/2} \check{\psi}(k). \]
Now by [DT79, Proof of Thm. 1.IV], $T$ has the asymptotics
\[ T(k) = 1 + \frac{\int dx \, v(x)}{2k} + O(k^{-2}) \quad \text{as } k \to \infty, \]
where $\int dx \, v(x) \neq 0$ by hypothesis. Since $\beta > 1/2$, we find that (4.24) diverges as $n \to \infty$, while $\|\varphi_n\|$ and $\|\psi_n\|$ are independent of $n$.

### 4.4 Tensor products

We now ask whether the high-energy order of an operator is stable under taking tensor products, in the following sense: Let $H_A$ be a selfadjoint operator on $\mathcal{H}_A$ which is smooth with respect to some Gelfand triple $\mathcal{X}_A \subset H_A \subset \mathcal{X}_A^*$. Let $H_B$ be another selfadjoint operator on a Hilbert space $\mathcal{H}_B$, assumed to have purely discrete spectrum. On $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$, consider $H := H_A \otimes 1 + 1 \otimes H_B$.

In the following, we will always denote the resolvent of $H_A$ as $R_A(z)$, etc. We note that, if $H_B = \sum_j \lambda_j P_j$ is the spectral decomposition of $H_B$, then
\[ R(z) = \sum_j R_A(z - \lambda_j) \otimes P_j. \quad (4.25) \]
at least weakly on $\mathcal{H} \times \mathcal{H}$; cf. [BA11, Sec. 5.1]. The same relation then holds with $z$ replaced with $\lambda \pm i0$ as long as $\lambda - \lambda_j \in U_A$ for all $j$, and at least in the sense of matrix elements between vectors of the form $\psi_A \otimes \psi_B$ where $\psi_A \in \mathcal{X}_A$ and $\psi_B$ is an eigenvector of $H_B$. (We will clarify below when the limit exists in the norm sense.)

For simplicity, we first treat the case of a finite-dimensional space $\mathcal{H}_B$.

**Proposition 4.13.** Let $H_A$ be a selfadjoint operator on a separable Hilbert space $\mathcal{H}_A$ which is $\mathcal{X}_A$-smooth and of strict high-energy order $\beta \in (0,1)$. Let $H_B$ be a selfadjoint operator on a finite-dimensional Hilbert space $\mathcal{H}_B$.

Set $\mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{X} := \mathcal{X}_A \otimes \mathcal{H}_B \subset \mathcal{H}$. Then $H := H_A \otimes 1 + 1 \otimes H_B$ is $\mathcal{X}$-smooth and of strict high-energy order $\beta$.

**Proof.** Set $N := R \setminus U_A$ and $S := \sigma(H_B) + N$ (both closed null sets), and let $U := R \setminus S$. We will show that $R$ is locally Hölder continuous on $U \pm i[0,\infty)$; clearly it suffices to show this in a neighbourhood of each real point.

Since $U$ is open, we can for any given $\lambda \in U$ find a neighbourhood $V_\lambda = [\lambda - \delta, \lambda + \delta] \pm i[0,\epsilon]$ ($\delta, \epsilon > 0$) such that $V_\lambda - \lambda_j \subset U_A \pm i[0,\infty)$ for all $j$. We now employ (4.25) and use local Hölder continuity of $R_A$ to estimate
\[ ||R(z') - R(z'')||_{\mathcal{X},\mathcal{X}^*} \leq \sum_j ||R_A(z' - \lambda_j) - R_A(z'' - \lambda_j)||_{\mathcal{X}_A,\mathcal{X}_A^*} \]
\[ \quad \leq \sum_j c_j |z' - z''|^\theta_j \quad (4.26) \]
for all $z', z'' \in V_\lambda$, with constants $c_j > 0$, $\theta_j \in (0,1)$. Since the sum is finite, this proves local Hölder continuity of $R$. In particular, the limits $R(\lambda \pm i0)$ exist in $\mathfrak{B}(\mathcal{X},\mathcal{X}^*)$ for all $\lambda \in U$. Hence $H$ is $\mathcal{X}$-smooth.

For the high-energy behaviour of $R$, we note that $R(\lambda \pm i0)$ exists for sufficiently large $|\lambda|$, and we can estimate using the high-energy order of $H_A$,
\[ ||R(\lambda \pm i0)||_{\mathcal{X},\mathcal{X}^*} \leq \sum_j ||R_A(\lambda - \lambda_j \pm i0)||_{\mathcal{X}_A,\mathcal{X}_A^*} \]
\[ \quad \leq \sum_j c_j |\lambda - \lambda_j|^{-\beta} \leq c' |\lambda|^{-\beta} \quad (4.27) \]
which shows that $H$ is also of strict high-energy order $\beta$. \qed
We now aim at a similar result for infinite-dimensional spaces \( \mathcal{H}_B \), which requires more care. We will need stronger uniformity assumptions on the bounds on \( R_A \), as well as some restrictions on the spectrum of \( H_B \). In order to avoid technical complications with the tensor product, we also assume that \( \mathcal{X}_A \) is a Hilbert space. The result will be weaker inasmuch as the high-energy order of the sum operator is no longer known to be strict.

**Theorem 4.14.** Let \( \mathcal{X}_A \subset \mathcal{H}_A \subset \mathcal{X}_A^* \) a Gelfand triple with a Hilbert space \( \mathcal{X}_A \), and \( H_A \) a selfadjoint operator on \( \mathcal{H}_A \). Let \( H_B \) be another selfadjoint operator on a Hilbert space \( \mathcal{H}_B \). Suppose that:

(a) \( H_A \) is \( \mathcal{X}_A \)-smooth; moreover, there exist \( \Lambda > 0, \theta \in (0, 1) \), \( c > 0 \) and \( \epsilon > 0 \) such that

\[
\| R_A(z) - R_A(z') \|_{\mathcal{X}_A, \mathcal{X}_A^*} \leq c|z - z'|^{\theta}
\]

whenever \( \Re z^{(t)} \geq \Lambda, \pm \Im z^{(t)} \in [0, \epsilon], \) and \( |z - z'| \leq 1 \).

(b) \( H_A \) is of strict high-energy order \( \beta \in (0, 1) \); moreover, there exists \( c > 0 \) such that for all \( \lambda \in U_A \),

\[
\| R_A(\lambda \pm i0) \|_{\mathcal{X}_A, \mathcal{X}_A^*} \leq c(1 + \lambda^2)^{-\beta/2}.
\]

(c) There exists \( \gamma > 0 \) such that \((1 + H_B^2)^{-\gamma + \beta/2}\) is of trace class.

Set \( \mathcal{H} := \mathcal{H}_A \otimes \mathcal{H}_B \) and let \( \mathcal{X} \subset \mathcal{H} \) be the Hilbert space with the following norm:

\[
\| \cdot \|_\mathcal{X} = \| \cdot \|_{\mathcal{X}_A} \otimes \| (1 + H_B^2)^{\gamma/2} \cdot \|_{\mathcal{H}_B}.
\]

(4.28)

Then \( H := H_A \otimes 1 + 1 \otimes H_B \) is \( \mathcal{X} \)-smooth and of high-energy order \( \beta \).

**Proof.** First note that \( |\lambda_j| \rightarrow \infty \) due to (c), hence \( \sigma(H_B) \) is locally finite; and \( \mathbb{R}\backslash U_A \) is closed and bounded by (b). Hence \( U \) as defined in the proof of Prop. 4.13 is still open, and for given \( \lambda \in U \) we can choose a compact complex neighbourhood \( V_\lambda \) such that \( V_\lambda - \lambda_j \subset U_A \pm i[0, \infty) \) for all \( j \). Analogous to (4.26), but now with the modified norm (4.28), we obtain the estimate for \( z', z'' \in V_\lambda \),

\[
\| R(z') - R(z'') \|_{\mathcal{X}, \mathcal{X}^*} \leq \sum_j (1 + \lambda_j^2)^{-\gamma} \| R_A(z' - \lambda_j) - R_A(z'' - \lambda_j) \|_{\mathcal{X}_A, \mathcal{X}_A^*}.
\]

(4.29)

We split the sum into those \( j \) where \( |\lambda - \lambda_j| \leq \Lambda + 1 \) (with \( \Lambda \) as in condition (a)) and their complement. Since \( |\lambda_j| \rightarrow \infty \), the first mentioned sum is finite and can be estimated as in (4.26). For the remaining sum, we use (a) to show that \( V_\lambda \) was chosen sufficiently small so that \( |z' - z''| \leq 1 \),

\[
\sum_{j: |\lambda - \lambda_j| \geq \Lambda + 1} (1 + \lambda_j^2)^{-\gamma} \| R_A(z' - \lambda_j) - R_A(z'' - \lambda_j) \|_{\mathcal{X}_A, \mathcal{X}_A^*} \leq \sum_j (1 + \lambda_j^2)^{-\gamma} \epsilon |z' - z''|^{\theta} \leq \epsilon' |z' - z''|^{\theta}
\]

(4.30)

with a finite \( \epsilon' > 0 \), since \((1 + H_B^2)^{-\gamma}\) is trace class. In conclusion, \( R(z) \) is locally Hölder continuous (also at the boundary \( U \pm i0 \)), hence \( H \) is \( \mathcal{X} \)-smooth.

For the high-energy order, we estimate for \( \lambda \in U \) using condition (b),

\[
\| R(\lambda \pm i0) \|_{\mathcal{X}_A, \mathcal{X}_A^*} \leq \sum_j (1 + \lambda_j^2)^{-\gamma} \| R_A(\lambda - \lambda_j \pm i0) \|_{\mathcal{X}_A, \mathcal{X}_A^*} \leq c \sum_j (1 + (\lambda - \lambda_j)^2)^{-\beta/2} (1 + \lambda_j^2)^{-\gamma} \leq \epsilon' (1 + \lambda^2)^{-\beta/2} \sum_j (1 + \lambda_j^2)^{-\gamma + \beta/2}.
\]

(We have used the inequality \( \frac{1}{1 + (x-y)^2} \leq 2 \frac{1}{1 + (x+y)^2} \) for \( x, y \in \mathbb{R} \).) The series here is convergent due to (c). Thus \( H \) is of high-energy order \( \beta \). \( \square \)
As usual, the detailed estimates in the previous theorem can be explicitly verified only in very simple examples. However, a perturbation argument as in Lemma 4.4 allows us to extend them:

**Corollary 4.15.** In the situation of Proposition 4.13 or Theorem 4.14, suppose that $R_A(z) ∈ \mathcal{FA}(\mathcal{X},\mathcal{X}^*)$ for every $z ∈ \mathbb{C}\setminus \mathbb{R}$. If $V = V^* ∈ \Gamma_2(\mathcal{X}^*,\mathcal{X})$, then the tuple $(H, H + V, \mathcal{X}, \mathcal{H})$ is a smooth scattering system of high-energy order $\beta$. In particular, $H$ and $H + V$ are mutually $f_\beta$-bounded. In the situation of Proposition 4.13, the system is of strict high-energy order $\beta$ and $H \sim_\beta H + V$.

**Proof.** Note that the spectral projectors $P_j$ of $H_B$ have finite rank; in the case of Theorem 4.14, this follows from condition (c). Thus every term $R_A(z) ⊗ P_j$ in the series (4.25) lies in $\mathcal{FA}(\mathcal{X},\mathcal{X}^*)$. On the other hand, the series converges absolutely in $\mathcal{B}(\mathcal{X},\mathcal{X}^*)$: In the first situation, this is trivial; in the second situation, note that $\|R_A(z)\|_{\mathcal{X}_A,\mathcal{X}^*_A} ≤ \|R_A(z)\|_{\mathcal{H}_A,\mathcal{H}_A} ≤ |\text{Im } z|^{-1}$, and hence

$$\sum_j \|R_A(z - \lambda_j) ⊗ P_j\|_{\mathcal{X},\mathcal{X}^*} ≤ |\text{Im } z|^{-1} \sum_j (1 + \lambda_j^2)^{-\gamma} < \infty.$$ 

Thus $R(z) ∈ \mathcal{FA}(\mathcal{X},\mathcal{X}^*)$ since this space is norm-closed. The statement now follows from Lemma 4.4 and Theorem 4.9.

This allows for immediate applications in the finite-dimensional case for $\mathcal{H}_B$, for example, if $H_A = -\Delta$ (see Example 4.10) and $H_B$ is some Hermitian matrix. In the context of quantum physics, $H$ would then be the free Schrödinger operator for a particle with inner degrees of freedom, and $V$ a matrix-valued scattering potential.

We will now give some more concrete examples in the infinite-dimensional case, where the conditions on $H_A$ are more delicate.

**Example 4.16.** Let $H_A = -\Delta$ act on its natural domain of selfadjointness in $\mathcal{H}_A = L^2(\mathbb{R}^3)$ and $H_B = -\Delta$ act on $\mathcal{H}_B = L^2(S^2)$, where $S^2$ is the two-dimensional sphere. Let $v : \mathbb{R} → \mathcal{B}(\mathcal{H}_B)$, $v(x) = v(x)^*$, such that

$$\sup_x (1 + |x|^2)^\alpha \|((1 + H_B^2)^{-\gamma/2}v(x)(1 + H_B^2)^{-\gamma/2}\|_{\mathcal{H}_B,\mathcal{H}_B} < \infty \quad (4.31)$$

with some $\alpha > 1$, $\gamma > (\beta + 1)/2$ and $\beta ≤ 1/2$, and let $V ∈ \mathcal{B}(\mathcal{H}_A ⊗ \mathcal{H}_B)$ be the operator multiplying with $v$. Then, with $H := H_A ⊗ 1 + 1 ⊗ H_B$, we have that $H$ and $H + V$ are mutually $f_\beta$-bounded.

**Proof.** We aim to apply Theorem 4.14 and Corollary 4.15; let $\mathcal{X}$ be as defined there, with $\|f\|_{\mathcal{X}_A} = \|(x)^{\alpha}f\|_{\mathcal{H}_A}$.

For condition (a) in the theorem, note that $H_A$ is $\mathcal{X}_A$-smooth by Example 4.10. For the more detailed estimate, it suffices to consider $\alpha < \frac{3}{2}$ and $\text{Im } z^{(0)} ≥ 0$. We use the dilation operators $D(\tau)$ defined in (4.19) and the relation (4.21) to show that for every $\tau ≥ 1$,

$$\|R_A(z) - R_A(z')\|_{\mathcal{X}_A,\mathcal{X}_A} \leq \tau^{-2}\|D(\tau^{-1})\|_{\mathcal{X}_A,\mathcal{X}_A}\|R_A(\tau^{-2}z) - R_A(\tau^{-2}z')\|_{\mathcal{X}_A,\mathcal{X}_A}\|D(\tau)\|_{\mathcal{X}_A,\mathcal{X}_A} \leq C\tau^{2(\alpha-1)}\|R_A(\tau^{-2}z) - R_A(\tau^{-2}z')\|_{\mathcal{X}_A,\mathcal{X}_A}.$$ 

Since however $R_A(z)$ is locally Hölder continuous with exponent $\theta = \alpha - \frac{1}{2}$, it fulfills a uniform Hölder estimate on, say, the compact region $[1, 2] ± i[0, 1]$. Choosing $\tau = (\text{Re } z)^{1/2}$, we thus have

$$\|R_A(z) - R_A(z')\|_{\mathcal{X},\mathcal{X}^*} ≤ C(\text{Re } z)^{\alpha-1-\theta}|z - z'|^\theta \leq C|z - z'|^\theta$$

whenever $1 ≤ \text{Re } z ≤ \text{Re } z' ≤ 2\text{Re } z$ and $0 ≤ \text{Im } z^{(0)} ≤ 1$; likewise for $z$ and $z'$ exchanged. This includes the region $\text{Re } z^{(0)} ≥ 1, 0 ≤ \text{Im } z^{(0)} ≤ 1, |z - z'| ≤ 1$, hence (a) follows.

To show condition (b), note that $\|R_A(λ ± i0)|_{\mathcal{X}_A,\mathcal{X}_A} ≤ C|λ|^{-1/2}$ for $|λ| ≥ 1$ by Example 4.10. For $|λ| ≤ 1$, we use the fact that $\|R_A(λ ± i0)|_{\mathcal{X}_A,\mathcal{X}_A}$ is globally bounded in $λ$ [Yaf10, Proposition 7.1.16]; it enters here that $\alpha > 1$ and that we consider the Laplacian on $\mathbb{R}^3$. 

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Regarding condition (c): Since $\sigma(H_B) = \{\ell(\ell + 1)\}_{\ell \in \mathbb{N}_0}$ with degeneracy $2\ell + 1$, we can compute
\[
\text{tr}(1 + H_B^2)^{-\gamma + \beta/2} = \sum_{\ell \in \mathbb{N}_0} (2\ell + 1)(1 + \ell^2(\ell + 1)^2)^{-\gamma + \beta/2} \\
\leq 2 \sum_{\ell \in \mathbb{N}_0} (1 + \ell)^{-4\gamma + 2\beta + 1},
\]
which converges for $4\gamma > 2\beta + 2$ as in the hypothesis.— In conclusion, Theorem 4.14 applies. Also, we already noted in Example 4.10 that $R_A(z)$ is compact in $\mathfrak{B}(\mathcal{X},\mathcal{X}^\ast)$ for $z \in \mathbb{C}\backslash \mathbb{R}$, and we have $V \in \Gamma_2(\mathcal{X}^\ast,\mathcal{X})$ by assumption (4.31). Hence we can apply Corollary 4.15 and conclude that $H$ and $H + V$ are mutually $f_{\beta}$-bounded. \hfill $\square$

Similar methods would apply to Laplace operators in higher dimensions ($n \geq 3$), but not for $n < 3$, since in that case there is no uniform bound on $\|R_A(z)||_{\mathcal{X}_1,\mathcal{X}_1^\ast}$ near $z = 0$.

Let us focus on the one-dimensional Laplacian here. Instead of the “free” operator $-\Delta$, one can consider $H_A = -\Delta + V_A$ where $V_A$ is multiplication with a nonnegative, sufficiently rapidly decaying function; its resolvent behaves better near $z = 0$, so that we can obtain a result similar to the 3-dimensional case.

**Example 4.17.** Let $H_A = -\Delta + V_A$ acting on $\mathcal{H}_A = L^2(\mathbb{R})$, where $V_A$ is multiplication with a nonnegative, compactly supported function $v_A$ in $L^\infty(\mathbb{R}) \setminus \{0\}$. Let $\beta \in (0, 1/2)$, and let $H_B$ be another selfadjoint operator on a separable Hilbert space $\mathcal{H}_B$ such that $(1 + H_B^2)^{-\gamma/2}$ is of trace class for some $\gamma > 0$. Let $v : \mathbb{R} \to \mathfrak{B}(\mathcal{H}_B)$, $v(x) = (v(x))^*$, such that
\[
\sup_x (1 + x^2)^\alpha \|(1 + H_B^2)^{-\gamma/2}v(x)(1 + H_B^2)^{-\gamma/2}\|_{\mathcal{H}_B,\mathcal{H}_B} < \infty
\]
with some $\alpha > 3/2$; and let $V \in \mathfrak{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be the operator multiplying with $v$. Then $H := H_A \otimes 1 + 1 \otimes H_B$ and $H + V$ are mutually $f_{\beta}$-bounded.

**Proof.** First, as the hypothesis becomes only stronger with increasing $\alpha$, we can assume without loss of generality that $3/2 < \alpha < 2$.

Now let $\mathcal{X}_A \subset \mathcal{H}_A$ be once more defined by its norm $\| \cdot \|_{\mathcal{X}_A} = \|(x)\|_{L^2(\mathbb{R})}$ and let $\mathcal{H}$ and $\mathcal{X} \subset \mathcal{H}$ be as in Eq. (4.28). As before, we aim to verify conditions (a)--(c) of Theorem 4.14 in our situation.

For (a), we know from Example 4.10 that $H_A = -\Delta + V_A$ is $\mathcal{X}_A$-smooth. In fact, since $H_A \geq 0$ cannot have negative eigenvalues, we can choose $U_A = \mathbb{R} \setminus \{0\}$ [Yaf10, Lemma 6.2.1]. Now let $R_0(z)$ be the resolvent of the negative Laplacian on $\mathcal{H}_A$. Like in (4.13), we write $R_A(z) = R_0(z)V(z)$ with $F(z) := (1 - G(z))^{-1}$ and $G(z) := V_AR_0(z)$, for any $z$ where the inverse exists (we will clarify this below). As in Example 4.16, $R_0$ fulfills a uniform Hölder estimate
\[
\|R_0(z) - R_0(z')\|_{\mathcal{X},\mathcal{X}^\ast} \leq C|z|^{\alpha - 1 - \theta}|z - z'|^\theta
\]
whenever $1 \leq \text{Re } z, |z - z'| < 1$.

(4.32)

Since $\alpha > 3/2$, we can choose any $\theta < 1$ here (cf. [Yaf10, Proposition 1.7.1]). An analogous Hölder estimate then holds for $G(z)$ in $\| \cdot \|_{\mathcal{X},\mathcal{X}^\ast}$.

Further, since $\|R_0(z)\|_{\mathcal{X},\mathcal{X}^\ast}$ decays at large $|z|$, we can choose $\Lambda_0 > 0$ such that $\|G(z)\|_{\mathcal{X},\mathcal{X}^\ast} \leq 1/4$ for all $\text{Re } z \geq \Lambda_0$; the inverse $F(z) = (1 - G(z))^{-1}$ then exists as a convergent Neumann series, and $\|F(z)\|_{\mathcal{X},\mathcal{X}^\ast} \leq 1/4$. To obtain Hölder estimates for $F$, we note the identity
\[
F(z) - F(z') = F(z')\left((1 - (G(z) - G(z'))F(z'))^{-1} - 1\right)
\]
\[
= F(z')\sum_{k=1}^\infty \left((G(z) - G(z'))F(z')\right)^k
\]
where the series converges by the above estimates. By taking norms we obtain:
\[
\|F(z) - F(z')\|_{\mathcal{X},\mathcal{X}^\ast} \leq C'\|G(z) - G(z')\|_{\mathcal{X},\mathcal{X}^\ast} \leq C''|z|^{\alpha - 2}|z - z'|
\]
whenever $\Lambda_0 \leq \text{Re } z, \text{Re } z', |z - z'| < 1$, \hfill 18
where the factor $|z|^{\alpha-1-\theta}$ is decreasing (for suitable $\theta$, noting $\alpha < 2$). Hence, as $R_0$ fulfills (4.32) and is bounded in $\|\cdot\|_{X,A}$, in the relevant region, we know that $R_A(z) = R_0(z)F(z)$ fulfills an analogous Hölder estimate, which proves (a).

Regarding condition (b), we know from Example 4.10 that the resolvents $R_A$ fulfill $\|R_A(\lambda \pm i0)\|_{X,A} \leq C|\lambda|^{-1/2}$ for large $|\lambda|$, and from part (a) above that $R_A$ is continuous in this norm on $\mathbb{R}\setminus\{0\}$. Hence it only remains to show that $\|R_A(z)\|_{X,A}$ is bounded in a neighbourhood of $z = 0$.

To that end, recall that the integral kernel of $R_A(z)$ is given by [Yaf10, Ch. 5] as

$$R_A(x,x';z) = R_A(x',x;z) = \frac{\theta_1(x,\sqrt{z})\theta_2(x',\sqrt{z})}{\omega(\sqrt{z})} \quad \text{for } x > x'$$

(4.33)

where the functions $\theta_1,2(x,\zeta)$ with $\text{Im} \zeta \geq 0$ are the solutions of the differential equation $(-\partial^2_x + v_A(x) - \zeta^2)\theta_j(x,\zeta) = 0$ with asymptotics $\theta_j(x,\zeta) = e^{i\alpha x}\zeta + o(1)$, $\partial_x\theta_j(x,\zeta) = \pm i\zeta e^{i\alpha x}(1 + o(1))$ for $x \to \pm\infty$ (here $+\,$ for $j = 1$ and $-\,$ for $j = 2$), and where $\omega$ is the Wronskian of $\theta_1, \theta_2$, a continuous function of $\zeta$, also at $\zeta = 0$. Note that the solutions $\theta_j(x,0)$ are real. In our case, since $v_A(x) \geq 0$, the $\theta_j(x,0)$ must be convex, and not constant as $v_A$ does not vanish identically; hence for $x$ to the right of the support of $v_A$, one has $\theta_1(x,0) = 1$ and $\theta_2(x,0) = cx + d$ with some $c \neq 0$, and the Wronskian $\omega(0)$ does not vanish. Therefore we can choose a neighbourhood $U$ of $0$ such that $|\omega(\sqrt{z})| \geq \epsilon > 0$ there.

Now by the Cauchy-Schwarz inequality,

$$\|R_A(z)\|_{X,A} = \|\langle x \rangle^{-\alpha} R_A(z) \langle x \rangle^{-\alpha}\|_{H_A,H_A}$$

$$\leq |\omega(\sqrt{z})|^{-1} \prod_{j=1,2} \left( \int dx \left(1 + x^2\right)^{-\alpha} |\theta_j(x,\sqrt{z})|^2 \right)^{1/2}.$$  

Using [DT79, Lemma 2.1(ii)], we can estimate $|\theta_j(x,\zeta)|^2 \leq c(1 + x^2)$ for all $\zeta$. Hence for $z \in U$,

$$\|R_A(z)\|_{X,A} \leq \frac{c}{\epsilon} \int dx \left(1 + x^2\right)^{-\alpha+1}.$$  

This integral converges since $\alpha > 3/2$. Hence condition (b) holds.

Property (c) follows directly from the hypothesis on $H_B$, hence Theorem 4.14 is applicable. Further, $R_A(z) = R_0(z)F(z)$ is compact for $\text{Im} \, z \neq 0$, and $V$ is bounded in the norm of $\mathbb{B}(X^*,X)$ by assumption, hence mutual $f_{\beta}$-boundedness of $H$ and $H + V$ follows from Corollary 4.15.  

5 Application to semibounded operators and quantum inequalities

In this concluding section, we highlight an application of our results that is of interest in the context of quantum physics. We deal with the following question.

Suppose that $A$ is a selfadjoint, unbounded operator on a (separable) Hilbert space $\mathcal{H}$, and $B \in \mathbb{B}(\mathcal{H})$, such that the compression $B^*AB$ is (semi)-bounded. Does this (semi-)boundedness transfer to the compression $BW^*_A AW_{\pm}B$ or — closely related — to $(W^*_A BW_{\pm})^*A(W^*_A BW_{\pm})$, where $W_{\pm}$ is the wave operator of some scattering situation?

We can give a sufficient criterion for this in our context.

**Theorem 5.1.** Let $H_0, H_1 \in \mathbb{A}(\mathcal{H})$, $f \in \mathcal{C}(\mathbb{R})$ such that $H_0 \sim_\text{f} H_1$. Denote $W_{\pm} = W_{\pm}(H_1,H_0)$. Let $B$ be a bounded operator and $A$ be a selfadjoint (unbounded) operator such that $\|Af(H_j)^{-1}\| < \infty$ for $j = 0,1$. Then $B^*AB$ is bounded above (below) iff $B^*W^*_A AW_{\pm}B$ is bounded above (below).

**Proof.** It suffices to show that $W^*_A AW_{\pm} - A$ is bounded. But as $\|W_{\pm}\| \leq 1$, we have

$$\|W^*_A AW_{\pm} - A\| = \|(W_{\pm} - 1)^* AW_{\pm} + A(W_{\pm} - 1)\|$$

$$\leq 2\|f(H_1)(W_{\pm} - 1)\|\|Af(H_1)^{-1}\|$$

which is finite by Prop. 2.4 and our assumption on $Af(H_1)^{-1}$. 

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This theoretical result is of interest in quantum physics in the following situation [BCL17].

**Example 5.2 (Quantum mechanical backflow bounds).** In Example 4.10, set \( t = 2, n = 1 \). Let \((X\psi)(x) = x\psi(x)\), write \( P = -i\frac{d}{dx} \), and let \( E \) be the spectral projector of \( P \) for the interval \([0, \infty)\). Pick a nonnegative Schwartz class function \( g \) and set \( J(g) := \frac{1}{2}(Pg(X) + g(X)P) \) (the "averaged probability flux operator"). Then \( EW_\pm J(g)W_\pm E \) is bounded below, but unbounded above.

**Proof.** It follows from elementary arguments that \( EJ(g)E \) is bounded below, but unbounded above [BCL17, Theorem 1]. With \( f(\lambda) := (1 + \lambda^2)^{1/4} \), we know from Example 4.10 that \( H_0 \sim f H_1 \). Also, writing \( J(g) = g'(X) + 2g(X)P \), it is clear that \( J(g)f(H_0)^{-1} \) is bounded, hence also \( J(g)f(H_1)^{-1} \) by Eq. (B.2). Further, since \( H_0 \) and \( H_1 \) are of strict high-energy order \( \frac{1}{\lambda} \), they are elements of \( AA(C) \). Thus, Theorem 5.1 with \( A = J(g) \) and \( B = E \) shows that \( EW_\pm J(g)W_\pm E \) is bounded below but unbounded above.

We have thus recovered our results on backflow bounds in [BCL17] for a slightly different class of potentials. However, our present methods should allow to generalize the analysis of "quantum inequalities" in scattering situations to backflow of particles with inner degrees of freedom, as well as to a much larger class of semibounded operators relevant in quantum mechanics, e.g., those established in [EFV05].

### A The Privalov-Korn theorem

We note the following variant, adapted to our purposes, of the Privalov-Korn theorem on singular Cauchy integrals (also known as the Plemelj-Privalov theorem or just as Privalov’s theorem). For the convenience of the reader, we also sketch its proof.

**Lemma A.1.** Let \( \mathcal{E} \) be a Banach space, let \( a < b \in \mathbb{R} \), and let \( B : [a, b] \to \mathcal{E} \). Suppose there exists a constant \( \theta \in (0, 1) \) such that

\[
\sup_{\lambda \in [a, b]} \|B(\lambda)\|_\mathcal{E} + \sup_{\lambda \neq \lambda' \in [a, b]} \left| \frac{b-a}{\lambda - \lambda'} \right|^\theta \|B(\lambda) - B(\lambda')\|_\mathcal{E} =: c < \infty.
\]

Then, the \( \mathcal{E} \)-valued function (with the integral defined in the weak sense)

\[
C(\zeta) := \int_a^b \frac{B(\lambda)d\lambda}{\lambda - \zeta}, \quad \zeta \in \mathbb{C}\setminus \mathbb{R},
\]

(A.1)

has limits \( C(\lambda \pm i0) := \lim_{\epsilon \to 0} C(\lambda \pm i\epsilon) \) for \( a' \leq \lambda \leq b' \), where \( a' := (1 - \delta)a + \delta b, b' := \delta a + (1 - \delta)b \), and \( 0 < \delta < \frac{1}{2} \) is some fixed number; it is locally Hölder continuous on \([a', b'] \pm i[0, \infty)\), and there is a constant \( k_{\theta, \delta} > 0 \) (depending only on \( \theta, \delta \), not on \( a, b, c, B, \lambda, \mathcal{E} \)) such that

\[
\sup_{\lambda \in [a', b']} \|C(\lambda \pm i0)\|_\mathcal{E} + \sup_{\lambda \neq \lambda' \in [a', b']} \left| \frac{b-a}{\lambda - \lambda'} \right|^\theta \|C(\lambda \pm i0) - C(\lambda' \pm i0)\|_\mathcal{E} \leq k_{\theta, \delta} c.
\]

(A.2)

We will often apply the theorem to \( \mathcal{E} = \mathfrak{B}(\mathcal{X}, \mathcal{X}^*) \) with a Banach space \( \mathcal{X} \), where it then also holds with respect to the weak operator topology.

**Proof.** In the case \( a = 0, b = 1, \mathcal{X} = \mathbb{C} \), we obtain the existence of boundary values and the estimate (A.2) with standard arguments, e.g., as in [Mus88, Ch. 2, §19], noting that the constants in the relevant estimates there depend linearly on \( c \), but not directly on \( B \). To show Hölder continuity in the complex domain \([a', b'] \pm i[0, \infty)\), we choose a smooth function \( \chi : \mathbb{R} \to [0, 1] \) such that \( \chi(x) = 1 \) in a neighbourhood of \([a', b']\) and \( \chi(x) = 0 \) in a neighbourhood of \((\infty, a') \cup [b, \infty)\), and set \( B'(\lambda) = \chi(\lambda)B(\lambda), C'(\zeta) = \int B'(\lambda)(\lambda - \zeta)^{-1}d\lambda \). Then \( C'(\zeta) \) is analytic in \( \mathbb{H}_\pm \) and near \([a', b']\), in particular Hölder continuous. Also, \( C'(\lambda \pm i0) \) exists for all \( \lambda \in \mathbb{R} \) by [Mus88], and decays like \( O(|\lambda|^{-1}) \) for large \( |\lambda| \). By a conformal transformation of \( \mathbb{H}_\pm \) to the closed unit circle, [CH62, Supplement to Ch. IV, §14] shows that \( C'(z) \) is locally Hölder continuous with exponent \( \theta \) in all of \( \mathbb{H}_\pm \). Again, one can choose a Hölder constant that depends linearly on \( c \) only, not on further details of \( B \).
For general $\mathcal{E}$, choose $\varphi \in \mathcal{E}^*$ and apply the previous result to the $\mathbb{C}$-valued function $\lambda \mapsto \varphi(B(\lambda))$; the resulting integral $\eqref{eq:proof}$ is of the form $\varphi(C(\lambda \pm i0))$ with $C(\lambda \pm i0) \in \mathcal{E}$ by linearity in $\varphi$ and uniformity of the estimate $\eqref{eq:proof}$ in $\|\varphi\|_{\mathcal{E}^*}$. In particular, $\eqref{eq:proof}$ follows for $C$. Similarly, by the uniformity of H"{o}lder estimates mentioned above, H"{o}lder continuity with respect to $\|\cdot\|_{\mathcal{E}}$ follows for complex arguments.

Finally, for general $a < b \in \mathbb{R}$, consider $\hat{B}(\lambda) := B(a + \lambda(b - a))$, defined for $\lambda \in [0,1]$, and apply the previous results to $\hat{B}$. □

B A lemma about differences of operators

We require estimates of differences $h(A) - h(B)$ where $A, B$ are selfadjoint unbounded operators, and $h$ is a certain function. The following is a special case of results by Birman, Solomyak and others; see \cite{BS03} for a review.

Lemma B.1. Let $A, B$ be two selfadjoint operators on a common dense domain in a Hilbert space $\mathcal{H}$, such that $B - A$ is bounded. Let $h : \mathbb{R} \to \mathbb{R}$ be differentiable such that $h' \in L^p(\mathbb{R})$ for some $p < \infty$, and suppose that $h''$ is (globally) H"{o}lder continuous with some H"{o}lder exponent $\epsilon > 0$. Then $h(B) - h(A)$ is bounded. In particular, for any $\beta \in (0,1)$,

$$\| (1 + B^2)^{\beta/2} - (1 + A^2)^{\beta/2} \| < \infty. \quad \text{\(B.1\)}$$

Proof. The stated conditions on $h$ imply that, in the notation of \cite{BS03}, the function $\phi_h(\mu, \lambda) = (h(\mu) - h(\lambda))/(\mu - \lambda)$ falls into the class $\mathfrak{W} [\text{BS03, Theorem 8.4}]$. Therefore, \cite[Theorem 8.1]{BS03} is applicable with $\mathfrak{S} = \mathfrak{B}(\mathcal{H})$, yielding the relation $h(B) - h(A) = Z_h^{A,B}(B - A)$ with a continuous map $Z_h^{A,B} : \mathfrak{B}(\mathcal{H}) \to \mathfrak{B}(\mathcal{H})$.

In particular, these conditions are fulfilled for $h(x) = (1 + x^2)^{\beta/2}$, since $h'(x) = O(|x|^{\beta-1})$ for large $|x|$, and $h''$ is bounded. □

As a consequence of Lemma B.1, also

$$\| (1 + B^2)^{-\beta/2}(1 + A^2)^{\beta/2} \| < \infty \quad \text{\(B.2\)}$$

by multiplying the bounded operator in $\eqref{eq:proof}$ with $(1 + B^2)^{-\beta/2}$ from the left.

C Some kernels of bounded operators

For our purposes, we need norm estimates of certain (singular) integral operators, which we collect here.

Lemma C.1. Let $0 < \gamma < \frac{1}{2}$. Then the (distributional) kernels

$$K_1(\lambda, \mu) = \frac{(\lambda/\mu)^\gamma}{\lambda - \mu \pm i0}, \quad K_2(\lambda, \mu) = \frac{(\lambda/\mu)^\gamma}{\lambda + \mu}$$

induce bounded operators $T_1, T_2$ on $L^2(\mathbb{R}_+)$.\n
Proof. Consider the unitary $U : L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$, $(Uf)(x) = e^{x/2}f(e^x)$. The operator $\hat{T}_1 := UT_1U^*$ has the kernel

$$\hat{K}_1(x, y) = e^{x/2}K_1(e^x, e^y)e^{y/2} = \frac{2e^{\gamma(x+y)}}{\sinh((2\gamma^2)/(x-y))}. \quad \text{\(C.1\)}$$

This is a kernel of convolution type, hence we only need to show that the Fourier transform (in the sense of distributions) of

$$f_1(z) = \frac{2e^{\gamma z}}{\sinh(z/2 \pm i0)} \quad \text{\(C.2\)}$$
is a bounded function. This can be extracted from the literature [GR07, Sec. 17.23, formula 20–21], or obtained by comparison with a kernel with the same residue but simpler Fourier transform, e.g., \( g(z) = 4i(z+i)^{-1}(z \pm i0)^{-1} \), as \( g - f \) is analytic and \( L^1 \). — Likewise for \( K_2 \), it suffices to show that

\[
f_2(z) = \frac{2e^{\gamma z}}{\cosh(z/2)}
\]

has a bounded Fourier transform, which is clear since \( f_2 \in L^1(\mathbb{R}) \).

**Acknowledgements**

D.C. is supported by the Deutsche Forschungsgemeinschaft (DFG) within the Emmy Noether grant CA1850/1-1. H.B. gratefully acknowledges a London Mathematical Society research in pairs grant, Ref. 41727.

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