Bounds for Algorithmic Mutual Information and a Unifilar Order Estimator

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Abstract
Inspired by Hilberg’s hypothesis, which states that mutual information between blocks for natural language grows like a power law, we seek for links between power-law growth rate of algorithmic mutual information and of some estimator of the unifilar order, i.e., the number of hidden states in the generating stationary ergodic source in its minimal unifilar hidden Markov representation. We consider an order estimator which returns the smallest order for which the maximum likelihood is larger than a weakly penalized universal probability. This order estimator is intractable and follows the ideas by Merhav, Gutman, and Ziv (1989) and by Ziv and Merhav (1992) but in its exact form seems overlooked despite attractive theoretical properties. In particular, we can prove both strong consistency of this order estimator and an upper bound of algorithmic mutual information in terms of it. Using both results, we show that all (also uncomputable) sources of a finite unifilar order exhibit sub-power-law growth of algorithmic mutual information and of the unifilar order estimator. In contrast, we also exhibit an example of unifilar processes of a countably infinite order and an algorithmically random oracle, for which the mentioned two quantities grow as a power law with the same exponent. We also relate our results to natural language research.

Keywords: algorithmic mutual information; unifilar hidden Markov processes; universal coding; order estimation; power laws
1 Introduction

Let $K(w)$ be the prefix-free Kolmogorov complexity and let $J(u; w) := K(u) + K(w) - K(u, w)$ be the algorithmic mutual information for strings $u$ and $w$ [1, 2, 3] in contrast to Shannon entropy $H(X)$ and mutual information $I(X; Y) := H(X) + H(Y) - H(X,Y)$ for random variables $X$ and $Y$. Although Kolmogorov complexity is in general uncomputable, there is a hypothesis stemming from the paper by Hilberg [4], see also [5, 6, 7, 8], that the algorithmic mutual information $J(x_1^n; x_{n+1}^2n)$ between two blocks of a text in natural language grows roughly like a power of the block length,

$$J(x_1^n; x_{n+1}^2n) \propto n^\beta, \quad \beta > 0. \quad (1)$$

(Notation $x_j^k$ denotes string $x_jx_{j+1}...x_k$.) When extrapolated to infinite texts $(x_i)_{i \in \mathbb{Z}}$ and assuming that a computable probability model for natural language exists (which need not be obvious), Hilberg’s hypothesis implies that the respective excess entropy of the computable model, i.e., the Shannon mutual information between the infinite past and future [8], is infinite. Consequently, natural language could not be optimally modeled by computable finite-state hidden Markov processes, since they have finite excess entropy by the data-processing inequality. Recent progress of neural language models, such as much publicized GPT-2 [9] and GPT-3 [10], corroborates this suboptimality of finite-state language models. As for Hilberg’s hypothesis, quite suggestive upper bounds for the power-law growth of mutual information and some partial evidence for divergent excess entropy can be also provided by recent large scale computational experiments [11, 12, 13, 14].

We have been interested in Hilberg’s hypothesis long before the advent of these computational experiments and, over years, we have formulated a mathematical explanation thereof that links abstract semantics, ergodic decomposition, algorithmic randomness, and information theory [15, 16, 17]—for the most detailed exposition see book [18]. This paper subscribes to this research line making further connections and solving an open problem stated in book [18]. The aims are fourfold:

- The main goal of the present paper is to derive an upper bound for algorithmic mutual information in terms of a strongly consistent estimator of the unifilar order, i.e., the number of hidden states in the generating source in its minimal unifilar hidden Markov representation. What is interesting, this bound holds true not only for sources of a finite unifilar order but also for other stationary ergodic sources, where the estimator diverges to infinity or the process distribution is uncomputable. A similar bound for algorithmic mutual information but in terms of a probably inconsistent estimator of the Markov order, i.e., the minimal length of a sufficient context in the generating source, was derived in [17]. Another upper bound for mutual information in terms of the number of distinct non-terminals in the shortest grammar-based compression [19, 20, 21] was demonstrated in [16] and linked there to Herdan-Heaps’ law of power-law growth of vocabulary for natural language [22, 23, 24, 25].

- The unifilar order estimator which we apply is a modification of estimators of the Markov order and the hidden Markov order proposed by
Merhav, Gutman, and Ziv [26] and by Ziv and Merhav [27] respectively. As in [26, 27], the estimator returns the smallest order for which the maximum likelihood is larger than a penalized universal probability. But what is interesting, our penalty is sublinear rather than linear which results in both no underestimation and no overestimation, whereas the estimators of [26, 27] tend to underestimate. In the literature of Markov order estimation [26, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] this kind of sublinear penalty can be traced in [31, 33, 37], whereas it seems to have been overlooked in the literature of hidden Markov order estimation [39, 27, 40, 41, 42, 43, 44, 45, 46, 47] although most of these papers entertain quite similar ideas and prove strong consistency of related estimators. Thus, proving strong consistency of our unifilar order estimator is another goal of this paper. This turns out to be the most technically complex achievement thereof since we apply theory of asymptotically mean stationary channels [48, 49, 50]. Since our estimator requires computing exact maximum likelihood and normalized maximum likelihood, it is intractable and impractical. We discuss it only because it has the desired theoretical property of being both strongly consistent and bounding algorithmic mutual information from above. Some practical estimators of the hidden Markov order can be found in [45, 47].

• The third goal of the paper is to exhibit clear examples of processes for which algorithmic mutual information and the unifilar order estimator grow slower than a power law (first class) or at least as fast as a power law (second class). By the consistency of the unifilar order estimator and the upper bound for algorithmic mutual information, all (also uncomputable) sources of a finite unifilar order belong to the first class. In contrast, in the second class we can find all perigraphic processes defined in [17]. The defining property of perigraphic processes is that they describe a fixed algorithmically random infinite sequence effectively at a power-law rate. As a corollary of our separation, the classes of finite-state hidden Markov processes and of perigraphic processes are disjoint, which solves one of open problems stated in the conclusion of our book [18].

• Looking for simple examples of perigraphic processes, we find them in the class of sources of a countably infinite unifilar order. Our examples are called Oracle processes. The Oracle processes introduced in this paper constitute an encoding of perigraphic Santa Fe processes introduced in [51, 16] into a finite alphabet but much simpler than discussed in [52] and additionally satisfying the condition of unifilarity. In a nutshell, Oracle processes repeatedly emit random binary strings $y$, which uniquely represent natural numbers $\phi(y)$, followed by a comma symbol and by the corresponding bit $z_{\phi(y)}$ read off from an algorithmically random sequence $(z_k)_{k \in \mathbb{N}}$. These processes can achieve an arbitrarily large rate of power-law growth of algorithmic mutual information. We also show that our unifilar order estimator grows at the same rate for these sources, so in this case the bounds for algorithmic mutual information are tight.

Showing that finite-state hidden Markov processes cannot be perigraphic sheds another beam of light onto the old discussion of inadequacy of finite-state models in theoretical linguistics started by Skinner and Chomsky [53, 54, 55,
Chomsky’s idea was that context-free syntax of natural language is incompatible with finite-state models advocated by Skinner. In contrast, our idea is that human language serves for efficient description of a potentially unboundedly complex reality in a repetitive way, therefore a reasonable ergodic language model should be perigraphic. Whereas context-free syntax does not seem directly related to process being perigraphic, yet not only Hilberg’s hypothesis but also Herdan-Heaps’ law of power-law growth of vocabulary [22, 23, 24, 25] would be a corollary of the latter property [16, 19, 20, 21].

The topics discussed in this paper have also connections with standard topics in information theory such as universal codes and distributions in the style of Ryabko [58, 59], normalized maximum likelihood [60, 61], and asymptotically mean stationary sources and channels [48, 49, 50]. Consequently, there may be natural extensions of presented ideas to model selection for other finite-parameter families [61] but we decidedly focus on the unifilar hidden Markov family. To be honest, the present paper provides constructions analogical to our preprint [62], which discussed a strongly consistent estimator of the Markov order and its links with mutual information. That manuscript was rejected in peer review as not novel enough. Whereas the guiding ideas are similar in both papers, the negative review inspired a few simplifying generalizations, a few more technical steps, and a few more overt comments in this paper.

This paper adopts a linear progression of exposition, without appendices. The first three further sections are short and discuss preliminaries. Section 2 sets up familiar concepts: the unifilar hidden Markov family, respective (normalized) maximum likelihood, and the Ryabko mixture. Section 3 names several obvious bounds for these, including a uniform bound for statistical complexity. Section 4 proves universality of the respective Ryabko mixture, which seems also of a preliminary character. In contrast, Section 5 contains a novel proof of strong consistency of the unifilar order estimator. Section 6 discusses lower and upper bounds for power-law growth of mutual information. Section 7 contains the definition of Oracle processes and a few calculations for them. Throughout the paper, \( \ln x \) denotes the natural logarithm, in contrast to the binary logarithm \( \log_x \).

2 Basic entities

The goal of this section is to introduce familiar concepts such as the unifilar hidden Markov family, respective (normalized) maximum likelihood, and the Ryabko mixture—recast in our notation. To begin, we will consider a family of unifilar hidden Markov chains where the number \( k = 1, 2, 3, \ldots \) of hidden states is finite and the emitted symbols belong to a fixed finite alphabet \( \mathbb{X} \). Unifilarity means that the hidden Markov chain is deterministic in the automata sense, i.e., the next hidden state is a fixed function of the previous hidden state and the previous emitted symbol. That is, for a given sequence of symbols \( x^n \) and states \( y^n \), we consider a family of unifilar hidden Markov distributions

\[
P(x^n, y^n | k, \pi, \tau, \varepsilon) := \pi(y_1)\varepsilon(x_1 | y_1) \prod_{i=2}^{n} 1\{ y_i = \tau(y_{i-1}, x_{i-1}) \} \varepsilon(x_i | y_i),
\]

where \( \pi : \{1, \ldots, k\} \to [0, 1] \) with \( \sum_{y} \pi(y) = 1 \) is the initial hidden state distribution, \( \tau : \{1, \ldots, k\} \times \mathbb{X} \to \{1, \ldots, k\} \) is the transition table, and \( \varepsilon : \mathbb{X} \times \{1, \ldots, k\} \to \)
with \( \sum_x \epsilon(x|y) = 1 \) is the emission matrix. Since so defined probabilities are prequential, i.e.,
\[
\sum_{x_{n+1}, y_{n+1}} P(x_{n+1}, y_{n+1} | k, \pi, \tau, \epsilon) = P(x_n, y_n | k, \pi, \tau, \epsilon),
\]
we can consider a joint stochastic process \((X_i, Y_i)_{i \in \mathbb{N}}\) distributed according to
\[
P(X^n = x_n, Y^n = y_n) = P(x_n, y_n | k, \pi, \tau, \epsilon).
\]
This process is stationary and extendable to a stationary process \((X_i, Y_i)_{i \in \mathbb{Z}}\) if
\[
\sum \pi(y_1) \epsilon(x_1|y_1) 1\{y_2 = \tau(y_1, x_1)\} = \pi(y_2).
\]
In any case, we define the marginal distribution
\[
P(x^n_1 | k, \pi, \tau, \epsilon) := \sum_{y_1} P(x_n, y_n | k, \pi, \tau, \epsilon)
\]
and the conditional distribution
\[
P(x^n_1 | k, y_1, \tau, \epsilon) := \frac{1}{\pi(y_1)} \sum_{y_2} P(x_1, y_1 | k, \pi, \tau, \epsilon).
\]

Subsequently, we define three distributions of the shape well-known in the minimum description length theory [61]: the maximum likelihood (ML)
\[
\hat{P}(x^n_1 | k) := \max_{y, \tau, \epsilon} P(x_n | k, y, \tau, \epsilon),
\]
the normalized maximum likelihood (NML) in the spirit of Shtarkov [60]
\[
P(x^n_1 | k) := \frac{\hat{P}(x^n_1 | k)}{\sum_{x^n_1 \in X^n} \hat{P}(x^n_1 | k)} \leq \hat{P}(x^n_1 | k),
\]
and the Bayesian mixture in the spirit of Ryabko [58, 59]
\[
P(x^n_1) := \sum_{k=1}^{\infty} w_k P(x^n_1 | k), \quad w_k := \frac{1}{k} - \frac{1}{k+1}.
\]

We notice that the maximum likelihood \(\hat{P}(x^n_1 | k)\), the normalized maximum likelihood \(P(x^n_1 | k)\), and the Ryabko mixture \(P(x^n_1)\) are not prequential. Moreover, the maximum likelihood satisfies \(\hat{P}(x^n_1 | k) = 1\) for \(k \geq n\), since having as many hidden states as the string length we can put \(\pi(1) = 1, \tau(i, x_i) = i + 1\), and \(\epsilon(x_i | i) = 1\). Consequently, the normalized maximum likelihood equals \(P(x^n_1 | k) = |X|^n\) for \(k \geq n\) and the Ryabko mixture \(P(x^n_1)\) is a computable function of \(x^n_1\) since the defining infinite series can be truncated. We stress that the maximum likelihood, the normalized maximum likelihood, and the Ryabko mixture are computable in the sense of computability theory, which will suffice for our needs of bounding algorithmic mutual information, but they are computationally intractable since we need to perform exhaustive search over all transition tables \(\tau\) combined with summation over exponentially growing domains \(X^n\).
3 Simple bounds

In this section, we will continue preliminaries and we will discuss some simple inequalities satisfied by the maximum likelihood, the normalized likelihood, and the Ryabko mixture. The proofs of these facts can be easily reconstructed or found in book [61]. Let us introduce the maximizer

$$G(x^n_1) := \arg \max_{k \geq 1} P(x^n_1 | k).$$

As explained in [61], we observe the sandwich bound for the Ryabko mixture

$$- \log \mathbb{P}(x^n_1 | G(x^n_1)) \leq - \log \mathbb{P}(x^n_1) \leq - \log \mathbb{P}(x^n_1 | k) - \log \omega_k.$$  \hspace{1cm} (12)

Moreover, it is easy to see that the maximum log-likelihood is subadditive,

$$- \log \hat{P}(x^n_1 | k) - \log \hat{P}(x^{n+m}_1 | k) + \log \hat{P}(x^{n+m}_1 | k) \leq 0.$$  \hspace{1cm} (13)

The above property probably was not discussed in [61]. It follows by the particular form of family (2) and will be used for the upper bound of algorithmic mutual information in Section 6.

Subsequently, like in [61], we introduce the statistical complexity of the unifilar hidden Markov family

$$C(n | k) := - \log \mathbb{P}(x^n_1 | k) + \log \hat{P}(x^n_1 | k) = \log \sum_{x^n \in \mathcal{X}^n} \hat{P}(x^n_1 | k) \leq n \log | \mathcal{X} |.$$  \hspace{1cm} (14)

It is a rule of thumb that the statistical complexity of a distribution family with exactly \(k\) real parameters is roughly \(k \log n\) and there exist more exact expressions assuming some particular conditions [61]. Here we only need a very rough bound for \(C(n | k)\) but assuming that we have not only a real-parameter emission matrix \(\varepsilon\) but also an integer-parameter transition table \(\tau\). Let us denote the set of distinct maximum likelihood parameters

$$\mathcal{P}_{k,n} := \{(y, \tau, \varepsilon) : \exists x \in \mathcal{X}^n \mathbb{P}(x^n_1 | k, y, \tau, \varepsilon) = \hat{P}(x^n_1 | k)\}.$$  \hspace{1cm} (15)

As explained in [61], we can bound

$$C(n | k) = \log \left( \sum_{x^n \in \mathcal{X}^n} \hat{P}(x^n_1 | k) \right) \leq \log | \mathcal{P}_{k,n} |.$$  \hspace{1cm} (16)

Given fixed \((y, \tau)\), likelihood \(\mathbb{P}(x^n_1 | k, y, \tau, \varepsilon)\) is maximized for the empirical distribution

$$\varepsilon(b | a) = \frac{\sum_{i=1}^{n} \mathbf{1}\{ (y_i, x_i) = (a, b) \}}{\sum_{i=1}^{n} \mathbf{1}\{ y_i = a \}},$$  \hspace{1cm} (17)

where \(y_1 = y\) and \(y_i = \tau(y_{i-1}, x_{i-1})\) for \(i \geq 2\). Since there are \(k \cdot k^{k | \mathcal{X}|}\) possible values of pairs \((y, \tau)\) and given \((y, \tau)\) there are less than \((n + 1)^{k | \mathcal{X}|}\) distinct empirical distributions \(\varepsilon\), we can bound the statistical complexity of the unifilar hidden Markov family as

$$C(n | k) \leq \log | \mathcal{P}_{k,n} | \leq \log \left( k^{k | \mathcal{X}|+1} (n + 1)^{k | \mathcal{X}|} \right) \leq |\mathcal{X}| \log[k(n + 1)].$$  \hspace{1cm} (18)

We can observe a small correction up to the mentioned rule of thumb.
4 Universality

The last relatively preliminary fact that we will present is universality of the Ryabko mixture. For distribution families that contain Markov chain distributions of all orders and whose statistical complexity of each order \( k \) grows sublinearly with the sample size \( n \), the Ryabko mixture is a universal distribution by a reasoning following the ideas of papers [58, 59]. It turns out that this is the case for the unifilar hidden Markov family. For completeness, we will present the full reasoning but we do not claim originality of idea. Also, as we have mentioned, this particular Ryabko mixture is computable in the sense of computability theory but it is intractable and highly impractical as a universal compression procedure. We need it only for further theoretical applications.

Let \( H(X) := \mathbb{E}_P[-\log P(X)] \) be the entropy of random variable \( X \) and let \( H(X|Y) := \mathbb{E}_P[-\log P(X|Y)] \) be the conditional entropy of \( X \) given random variable \( Y \). Let \((X_i)_{i \in \mathbb{Z}}\) be a stationary ergodic process over alphabet \( \mathbb{X} \). We denote the conditional entropies

\[
h^P_k := H(X_0|X_{-1}^{-k}) = \mathbb{E}_P[-\log P(X_0|X_{-1}^{-k})] \tag{19}
\]

and the entropy rate

\[
h^P := \lim_{n \to \infty} \frac{H(X^n)}{n} = \inf_{k \geq 1} h^P_k = H(X_0|X_{-\infty}^{-1}). \tag{20}
\]

The following theorem states universality of the Ryabko mixture.

**Theorem 1** For a stationary ergodic process \((X_i)_{i \in \mathbb{Z}}\) over alphabet \( \mathbb{X} \),

\[
\lim_{n \to \infty} \frac{1}{n} \left[ -\log \mathbb{P}(X^n) \right] = h^P \text{ P-a.s.}, \tag{21}
\]

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}_P[-\log \mathbb{P}(X^n)] = h^P. \tag{22}
\]

**Proof:** Letting \( \tau(x^n_k, x_{k+1}) = x^{k+1}_2 \) and \( \varepsilon(x_{k+1}|x^n_1) = P(X_{k+1} = x_{k+1}|X^n_1 = x^n_1) \), we obtain the conditional probability bound

\[
\hat{\mathbb{P}}(X^n_1|X^n_k) \geq \mathbb{P}(X^n_1|X^n_k, X^n_{-k+1}, \tau, \varepsilon) = \prod_{i=1}^{n} P(X_i|X^{i-1}_{-k}). \tag{23}
\]

Hence by the upper bound (18) for the statistical complexity and the Birkhoff ergodic theorem, we obtain

\[
\limsup_{n \to \infty} \frac{1}{n} \left[ -\log \mathbb{P}(X^n_1|X^n_k) \right] = \limsup_{n \to \infty} \frac{1}{n} \left[ -\log \hat{\mathbb{P}}(X^n_1|X^n_k) + C(n) |X^n_k| \right] \leq h^P \text{ P-a.s.} \tag{24}
\]

Thus by the upper bound in (12) and the Barron lemma [63, Theorem 3.1], we obtain (21). Noticing that \( \mathbb{P}(X^n_1) \geq w_n \mathbb{P}(X^n_1|n) = w_n |X^n|^{-n} \), we hence obtain (22) by dominated convergence. \( \square \)
5 Order estimation

We are moving on to more difficult reasonings. In this section, we will construct a simple unifilar order estimator and we will prove that it is strongly consistent and asymptotically unbiased. The estimator is intractable but it leads to simple bounds for algorithmic mutual information to be discussed in Section 6. To prove its strong consistency we will apply universality of the Ryabko mixture demonstrated in Section 4.

Let \((X_i)_{i \in \mathbb{Z}}\) be a stationary ergodic process over alphabet \(X\) as before. We define the unifilar order of the process as

\[
M^P := \inf \left\{ k : \exists \pi, \tau, \epsilon \forall n \geq 1, x^n \mathbb{P}(X^n_1 = x^n_1) = \mathbb{P}(x^n_1 | k, \pi, \tau, \epsilon) \right\}
\]

(25)

with the convention that the infimum of the empty set is infinite. Subsequently, we will consider a unifilar order estimator which is a certain modification of estimators of the Markov order and the hidden Markov order proposed by Merhav, Gutman, and Ziv [26] and by Ziv and Merhav [27] respectively. The idea of [26, 27] is that the estimator returns the smallest order for which the maximum likelihood is larger than a penalized universal probability. Consequently, we will define the unifilar order estimator

\[
M(x^n_1) := \min \left\{ k : \hat{\mathbb{P}}(x^n_1 | k) \geq w_n \mathbb{P}(x^n_1) \right\}, \quad w_n := \frac{1}{n} - \frac{1}{n + 1}.
\]

(26)

We can see that the estimator is nicely bounded by \(M(x^n_1) \leq n\) since \(\hat{\mathbb{P}}(x^n_1 | k) = 1\) for \(k \geq n\). In contrast to the hidden Markov order estimator by Ziv and Merhav, we use much smaller penalty \(-\log w_n = o(n)\), whereas penalty \(-\log w_n = \lambda n\) was used in [27]. As we will see, our choice of \(w_n\) results in strong consistency whereas the estimator by [27] suffered from underestimation.

In the literature of Markov order estimation [26, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38], a kind of sublinear penalty \(-\log w_n = o(n)\) in estimators resembling (26) can be traced in [31, 33, 37], whereas it seems overlooked in variants of estimator (26) in the literature of hidden Markov order estimation [39, 27, 40, 41, 42, 43, 44, 45, 46, 47] although majority of these papers treat quite similar ideas and prove strong consistency of related estimators. Penalty \(-\log w_n = o(n)\) was also considered in the rejected paper [62] for Markov order estimation. In this case, minus logarithm of maximum likelihood equals the empirical conditional entropy and can be quickly computed.

In contrast, the unifilar order estimator (26) is computable in the sense of computability theory but it is intractable since it applies exact maximum likelihood and normalized maximum likelihood. We will need it as is for the next section since it yields the most elegant upper bound for the algorithmic mutual information. Ignoring the question of obtaining this bound for a while, we note that we can make the estimator somewhat computationally simpler while preserving strong consistency if we replace universal distribution \(\mathbb{P}(x^n_1)\) with a simpler universal compression procedure such as the Lempel-Ziv code [64]. This idea was proposed by Merhav, Gutman, and Ziv [26] and by Ziv and Merhav [27] themselves. This substitution, however, breaks the simple upper bound for mutual information while not solving the problem of computing the exact maximum likelihood, which requires combinatorial optimization over all transition tables \(\tau\). In contrast, some practical estimators of the hidden Markov order can be found in [45, 47].
The following theorem states strong consistency of the unifilar order estimator. The proof technique for the impossibility of overestimation can be considered to be transferred from Markov order estimation proof ideas such as [33, 37]. We suppose that the proof of the impossibility of underestimation is more original. Since we apply a result about asymptotically mean stationary channels by Kieffer and Rahe [50], we suspected that Kieffer [41] might have used a similar technique in the context of hidden Markov order estimation but we did not find it there.

**Theorem 2** For a stationary ergodic process \((X_i)_{i \in \mathbb{Z}}\) over alphabet \(X\),

\[
\lim_{n \to \infty} M(X^n_1) = M^P \text{ P-a.s.} \tag{27}
\]

and we have the overestimation bound

\[
P \left( M(X^n_1) > M^P \right) \leq w_n. \tag{28}
\]

**Proof:** Our proof is split into impossibility of overestimation and of underestimation. The bound for the overestimation probability is received by inequality

\[
\hat{P}(X^n_1 | M^P) \geq P(X^n_1 | Y_1),
\]

where \(Y_1\) is the hidden state emitting \(X_1\), and by the Barron lemma [63, Theorem 3.1]. Hence

\[
P \left( M(X^n_1) > M^P \right) \leq P \left( \frac{w_n \hat{P}(X^n_1)}{P(X^n_1 | Y_1)} > 1 \right) \leq w_n. \tag{28}
\]

Since \(\sum_{n=1}^{\infty} w_n = 1\), the impossibility of overestimation follows by the Borel-Cantelli lemma.

Now, we demonstrate the impossibility of underestimation, which is much more difficult to see. Since the Ryabko mixture was shown to be universal in the sense of (21) and the penalty is \(-\log w_n = o(n)\), it is sufficient to show that

\[
\liminf_{n \to \infty} \frac{1}{n} \left[ -\log \hat{P}(X^n_1 | k) \right] > h^P \text{ P-a.s. for } k < M^P. \tag{29}
\]

Our reasoning will go by showing that the left hand side of the above inequality equals almost surely a sort of conditional entropy \(h^P_{[k]}\) which is strictly greater than \(h^P\) if \(k < M^P\).

We observe first that for any finite set \(M\),

\[
\lim_{n \to \infty} \inf_{M \in \mathcal{M}} a_{nm} = \min_{m \in M} \liminf_{n \to \infty} a_{nm}. \tag{30}
\]

It is so since we can take sufficiently large \(n\) on both sides to interchange the infinums. In our case, the set of pairs \((y, \tau)\) for a fixed \(k\) is finite. Hence

\[
\liminf_{n \to \infty} \min_{y, \tau, \varepsilon} \frac{1}{n} \left[ -\log \hat{P}(X^n_1 | k, y, \tau, \varepsilon) \right] = \min_{y, \tau} \liminf_{n \to \infty} \varepsilon \left[ -\log \hat{P}(X^n_1 | k, y, \tau, \varepsilon) \right]. \tag{31}
\]

Now we will apply technically quite a difficult but beautiful result by Kieffer and Rahe [50], which says that an ergodic Markov channel applied to an ergodic asymptotically mean stationary process yields a jointly ergodic asymptotically mean stationary process. Denote \(Y_1^y := y \in \{1, \ldots, k\}\) and \(Y^y_{i+1} := \ldots \)
\( \tau(Y_t^{y,\tau}, X_i) \). We can see that the distribution of \((Y_t^{y,\tau})_{i \in \mathbb{N}} \) given \((X_1)_{i \in \mathbb{N}} \) is an ergodic Markov channel, whereas \((X_i)_{i \in \mathbb{N}} \) is stationary ergodic. Thus, process \((X_i, Y_t^{y,\tau})_{i \in \mathbb{N}} \) is asymptotically mean stationary ergodic. Let process \((X_i, Y_t^{y,\tau})_{i \in \mathbb{N}} \) be distributed according to the stationary mean of \((X_i, Y_t^{y,\tau})_{i \in \mathbb{N}} \).

Since \((X_i)_{i \in \mathbb{N}} \) is stationary, we can assume without loss of generality that \((X_i)_{i \in \mathbb{N}} = (X_1)_{i \in \mathbb{N}} \). Moreover, by definition of the stationary mean, recursion \( \dot{Y}_{i+1}^{y,\tau} = \tau(Y_t^{y,\tau}, X_i) \) holds by recursion \( Y_t^{y,\tau} = \tau(Y_t^{y,\tau}, X_i) \). (Notice, however, that we cannot assume \( Y_t^{y,\tau} = \sigma_{y,\tau}(X_{\tau-1}^{\infty}) \) since there is a simple counterexample: a periodic process \((Y_t^{y,\tau})_{i \in \mathbb{N}} \) with a constant process \((X_i)_{i \in \mathbb{N}} \).

The beauty of asymptotically mean stationary processes lies in the fact that we have a generalization of the Birkhoff ergodic theorem [48]. The claim is that the Cesàro averages converge almost surely to expectations with respect to the stationary mean. Hence, by the application of the Birkhoff ergodic theorem to empirical counts in the most likely distribution \( \varepsilon \) given \((y, \tau) \), we obtain

\[
\min_{\varepsilon} \lim_{n \to \infty} \inf \frac{1}{n} \left[ \log p(X_1^n | k, y, \tau, \varepsilon) \right] = \min_{\varepsilon} \lim_{n \to \infty} \inf \frac{1}{n} \sum_{i=1}^{n} \left[ \log p(X_i | Y_i^{y,\tau}) \right] = h^P_{[k]} := \min_{\varepsilon} \mathbb{E}_P \left[ - \log p(X_n | \tilde{Y}_i^{y,\tau}) \right] \quad \text{P-a.s.} \tag{32}
\]

Let \( y \) and \( \tau \) be some minimizing parameters and let us abbreviate \( \tilde{Y}_i := Y_i^{y,\tau} \). What happens if \( h^P_{[k]} := H(X_i | \tilde{Y}_i) = h^P \)? Since \( \tilde{Y}_{i+1} = \tau(\tilde{Y}_i, X_i) \), we can write

\[
H(X_i | \tilde{Y}_i) = H(X_i | X_i^{1-1}, \tilde{Y}_i) = I(X_i; X_i^{1-1}, \tilde{Y}_i^{j-1}) \). \tag{33}
\]

Hence by stationarity of \((X_i, \tilde{Y}_i^{y,\tau})_{i \in \mathbb{Z}} \),

\[
\sum_{i=1}^{n} H(X_i | \tilde{Y}_i) = H(X_i^n | \tilde{Y}_1) + \sum_{i=0}^{n-1} I(X_j; X_j^{1-1}, \tilde{Y}_j^{j-1}) \). \tag{34}
\]

Dividing by \( n \) and letting \( n \to \infty \) yields

\[
h^P_{[k]} := H(X_i | \tilde{Y}_i) = h^P + I(X_j; X_j^{1-1}, \tilde{Y}_j^{j-1}) \), \tag{35}
\]

where we freely apply Shannon information measures for arbitrary, also infinite \( \sigma \)-fields, whose properties were described in [65, 66]. That is, if \( h^P_{[k]} = h^P \) then \( I(X_i; X_i^{1-1}, \tilde{Y}_j^{j-1} | \tilde{Y}_j) = 0 \). Since also \( \tilde{Y}_{i+1} = \tau(\tilde{Y}_i, X_i) \) then \((X_i)_{i \in \mathbb{N}} \) is a unifilar hidden Markov process with \( \leq k \) hidden states distributed according to \( \tilde{Y}_i \). Consequently, we have \( M^P \leq k \). \( \square \)

The above result implies that the estimator is asymptotically unbiased, which we will need in the next section.

**Theorem 3** For a stationary ergodic process \((X_i)_{i \in \mathbb{Z}} \) over alphabet \( \mathcal{X} \),

\[
\lim_{n \to \infty} \mathbb{E}_P M(X^n_1) = M^P. \tag{36}
\]

**Proof:** By \( M(x^n_1) \leq n \) and by the overestimation bound in Theorem 2 we have

\[
\mathbb{E}_P M(X^n_1) \leq M^P + n P(M(x^n_1) > M^P) = M^P + \frac{1}{n+1}. \tag{37}
\]
On the other hand, by the Fatou lemma,
\[ \lambda P = \mathbb{E}_P \liminf_{n \to \infty} \mathbb{M}(X^n_1) \leq \liminf_{n \to \infty} \mathbb{E}_P \mathbb{M}(X^n_1). \] (38)
Hence the claim follows. \( \square \)

6 Mutual information

In this section, we will present the culmination of this paper, namely, various bounds for the asymptotic power-law growth of algorithmic mutual information and related quantities such as the hidden Markov order estimator introduced in the previous section. For this aim we will apply the basic bounds presented in Section 3, universality of the Ryabko mixture from Section 4, and strong consistency of the hidden Markov order estimator from Section 5.

For the sake of further considerations concerning the power-law growth of various quantities, let us introduce so called Hilberg exponents
\[ \tilde{\text{hilb}}_{n \to \infty}s(n) := \limsup_{n \to \infty} \frac{\log \max \{1, s(n)\}}{\log n}. \] (39)
for real functions \( s(n) \) of natural numbers, cf. [67, 17, 18], where we gradually approached the above definition. The Hilberg exponents capture the asymptotic power-law growth of the respective functions, such as \( \tilde{\text{hilb}}_{n \to \infty} n^\beta = \beta \) for \( \beta \geq 0 \).

We will begin with a strengthening of a simple observation from [17, 18]. Our improvement is also very simple and it consists in replacing condition \( J(n) \geq -C \) with \( S(n) - ns \geq -C \) as sufficient for equality of the respective Hilberg exponents. It is curious that we have not noticed this earlier.

**Theorem 4 (cf. [17, 18])** For a function \( \mathcal{S} : \mathbb{N} \to \mathbb{R} \), define \( J(n) := 2\mathcal{S}(n) - \mathcal{S}(2n) \). If \( \lim_{n \to \infty} \mathcal{S}(n)/n = s \) for a \( s \in \mathbb{R} \) then
\[ \tilde{\text{hilb}}_{n \to \infty}(\mathcal{S}(n) - ns) \leq \tilde{\text{hilb}}_{n \to \infty}\mathcal{J}(n) \] (40)
with an equality if \( \mathcal{S}(n) - ns \geq -C \) for all but finitely many \( n \) and some \( C > 0 \).

**Proof:** Write \( \delta = \tilde{\text{hilb}}_{n \to \infty} \mathcal{J}(n) \). The proof of \( \tilde{\text{hilb}}_{n \to \infty}(\mathcal{S}(n) - ns) \leq \delta \) can be found in [17, 18]. Now assume that \( \mathcal{S}(n) - ns \geq -C \) for all but finitely many \( n \). We have then
\[ \mathcal{S}(n) - ns = \frac{\mathcal{J}(n)}{2} + \frac{\mathcal{S}(2n) - 2ns}{2} \geq \frac{\mathcal{J}(n) - C}{2} \] (41)
for sufficiently large \( n \). Hence \( \delta \leq \tilde{\text{hilb}}_{n \to \infty}(\mathcal{S}(n) - ns) \). Thus we obtain the equality in (40). \( \square \)

Let \((X_i)_{i \in \mathbb{Z}}\) be a stationary ergodic process over alphabet \( X \). As discussed in [8, 18], we can equivalently define the excess entropy
\[ E^P := \lim_{n \to \infty} [H(X^n_1) - nh^P] = \lim_{n \to \infty} I(X^n_1; X^{2n}_{n+1}). \] (42)
By the data-processing inequality for the Shannon mutual information, we obtain \( E^P \leq \log \lambda P \). That is, the excess entropy is finite for finite-state
hidden Markov processes (also non-unifilar ones). In the next turn, we may ask how fast the defining expressions diverge to infinity. By Theorem 4 and
\[ I(X_1^n; X_{2n}^{2n+1}) = 2H(X_1^n) - H(X_{2n}^{2n+1}) \] from stationarity, we can define the Hilberg exponent
\[ \beta_1^P := \text{hilb} \lim_{n \to \infty} \left[ H(X_1^n) - nh^P \right] = \text{hilb} \lim_{n \to \infty} I(X_1^n; X_{2n}^{2n+1}) \leq 1. \] (43)

Let us compare this result with Kolmogorov complexity and algorithmic mutual information. By the theorem of Brudno [68], the prefix-free Kolmogorov complexity is the length of a universal code,
\[ \lim_{n \to \infty} K(X_1^n) = h^P \text{ P-a.s.}, \] (44)
\[ \lim_{n \to \infty} \frac{E_P K(X_1^n)}{n} = h^P. \] (45)

Since \( E_P K(X_1^n) \geq H(X_1^n) \) by the prefix-free property, Theorem 4 yields
\[ \beta_1^P \leq \beta_2^P := \text{hilb} \lim_{n \to \infty} E_P \left[ K(X_1^n) - nh^P \right] = \text{hilb} \lim_{n \to \infty} E_P J(X_1^n; X_{2n}^{2n+1}) \leq 1. \] (46)

Similarly, by universality of the Ryabko mixture proved in Theorem 1 and inequality
\[ K(x_1^n) \leq -\log P(X_1^n) - \log w_n + K(P) \] (47)
from computability of the Ryabko mixture and Shannon-Fano coding, we obtain
\[ \beta_1^P \leq \beta_2^P \leq \beta_3^P := \text{hilb} \lim_{n \to \infty} E_P \left[ -\log P(X_1^n) - nh^P \right] \]
\[ = \text{hilb} \lim_{n \to \infty} E_P \left[ -\log P(X_1^n) - \log P(X_{2n}^{2n+1}) + \log P(X_2^n) \right] \leq 1. \] (48)

The above bounds besides the second equality in (48) (stated as inequality in [17]) were observed in the context of universal coding in [17]. Consequently in [17], we have bounded Hilberg exponents \( \beta_1^P, \beta_2^P, \) and \( \beta_3^P \) in terms of so called theorems about facts and words. These mathematical propositions state that under some formalization, the number of independent binary facts predictable from a finite text is roughly less than the mutual information between two halves of the text and this is roughly less than the number of distinct words detectable in the text. Namely, the theorems about facts and words take form of inequalities
\[ \text{hilb} \lim_{n \to \infty} E_P U(X_1^n) \leq \beta_1^P \leq \text{hilb} \lim_{n \to \infty} E_P V(X_1^n), \] (49)
where \( U(x_1^n) \) is approximately the number of independent binary facts effectively predictable from a fixed text \( x_1^n \) and \( V(x_1^n) \) is approximately the number of distinct words detectable in a fixed text \( x_1^n \). On a conceptual level, one may suppose that inequalities (49) are echoing to a certain extent inequalities between mutual information and the common information functions by Gács and Körner [69] and by Wyner [70]. The exact formulation of the theorems about facts and words rests on a few particular modeling assumptions and allows for some freedom with respect to the definition of \( U(x_1^n) \) and \( V(x_1^n) \), cf. [16, 17].
Whereas $V(x^n)$ can be actually taken as a number of entities that resemble orthographic words in the ordinary sense, cf. [16, 19, 20, 21], we can look for other possibilities. Following [17] and [62], where $V(x^n)$ was taken as the number of distinct substrings of the length of a Markov order estimator (inconsistent in [17] and strongly consistent in [62]), we can suspect that one can also take $V(x^n)$ to be a hidden Markov order estimator. This is exactly the case. The following reasoning squeezes to a triviality our earlier ideas for proving the right inequality in (49), cf. [16, 17].

**Theorem 5** For a stationary process $(X_i)_{i\in\mathbb{Z}}$ over alphabet $\mathbb{X}$,

$$
\beta^P_1 \leq \beta^P_3 := \lim_{n\to\infty} \exp E_P M(X^n_1).
$$

**Proof:** Denoting $k := M(X^{2n}_1) \leq 2n$, we observe by (12) and (13) that

\[
- \log \mathbb{P}(X^n_1) - \log \mathbb{P}(X^{2n}_n) + \log \mathbb{P}(X^n_1) \\
\leq - \log \mathbb{P}(X^n_1[k]) - \log \mathbb{P}(X^{2n}_n[k]) - 2 \log w_k + \log \hat{\mathbb{P}}(X^{2n}_n[k]) - \log w_{2n} \\
\leq 2C(n)k - 2 \log w_k - \log w_{2n} - \log \mathbb{P}(X^n_1[k]) - \log \hat{\mathbb{P}}(X^{2n}_n[k]) + \log \hat{\mathbb{P}}(X^{2n}_n[k]) \\
\leq 2C(n)k - 2 \log w_k - \log w_{2n}.
\]

(51)

Hence by (18) we obtain

$$
\beta^P_3 \leq \lim_{n\to\infty} \exp E_P C(n|\mathbb{M}(X^{2n}_1)) \leq \lim_{n\to\infty} \exp E_P M(X^n_1).
$$

(52)

\[
\Box
\]

By Theorems 3 and 5, for processes of a finite unifilar order, we have not only $\beta^P_1 = 0$ but also $\beta^P_2 = \beta^P_3 = \beta^P_3 = 0$. Equality $\beta^P_2 = 0$ carries over also to non-unifilar finite-state hidden Markov processes by the data-processing inequality for algorithmic mutual information.

Consequently, we can ask whether there exist processes such that $\beta^P_2$, $\beta^P_3$, and $\beta^P_3$ are arbitrarily close to 1. The answer is given by a particular form of inequalities (49). We have the following proposition.

**Theorem 6** ([17]) Consider an arbitrary computable function $g : \mathbb{N} \times \mathbb{X}^* \to \{0, 1, 2\}$ and an arbitrary fixed algorithmically random binary sequence $z = (z_k)_{k\in\mathbb{N}}$ (in the Martin-Löf sense), i.e., we have $K(z^n_1) \geq n - c$ for a certain constant $c$. Define

$$
U_g(x^n_1|z) := \min \{k \geq 1 : g(k, x^n_1) \neq z_k\}.
$$

(53)

For a stationary process $(X_i)_{i\in\mathbb{Z}}$ over alphabet $\mathbb{X}$,

$$
\beta^P_{g,z} := \lim_{n\to\infty} \exp E_P U_g(X^n_1|z) \leq \beta^P_2.
$$

(54)

The guiding intuition is behind this result is that sequence $z = (z_k)_{k\in\mathbb{N}}$ is the abstract pool of independent binary “facts” $z_k$ and $U_g(x^n_1|z) - 1$ is the number of initial facts that can be correctly predicted given text $x^n_1$ using an effective prediction procedure. From this point of view, quantity $U_g(x^n_1|z) - 1$ can be seen as an approximation of the common information by Gács and Körner which was demonstrated to be not greater than the mutual information in [69].

Let us entertain this definition.
Definition 1 ([17]) A stationary process \((X_i)_{i \in \mathbb{Z}}\) is called perigraphic if inequality \(\beta_{g,z}^P > 0\) holds for Hilberg exponent (54) given some computable function \(g: \mathbb{N} \times X \to \{0, 1, 2\}\) and some algorithmically random binary sequence \(z = (z_k)_{k \in \mathbb{N}}\).

By Theorems 3, 5, and 6, the class of finite-state hidden Markov processes and the class of perigraphic processes are disjoint, which solves an important open problem stated in the conclusion of book [18].

The motivating idea of this open problem was that human language serves for efficient description of a potentially unboundedly complex reality in a repetitive way. If we equate the complex reality with an infinite algorithmically random sequence, then a reasonable statistical language model should be perigraphic.

We wanted to show that for that reason, a reasonable language model cannot be a finite-state hidden Markov process, which is a different cause against finite-state models than context-free syntax postulated by Chomsky [54, 55, 56].

As shown in [17], perigraphic processes have uncomputable distributions and they look as typical ergodic components of some strongly non-ergodic processes.

In the next section, however, we will show that some perigraphic processes lie quite low in the hierarchy of stochastic processes since there are simple perigraphic unifilar processes with a countably infinite number of hidden states, which we call Oracle processes. In particular, Oracle processes do not exhibit hierarchical context-free structures. For processes that exhibit context-free hierarchical structures and are possibly perigraphic (we have not proved it), we refer to [71, 18].

7 Oracle processes

In this section we will construct some simple unifilar processes, called Oracle processes, which are perigraphic. Necessarily, these processes have a countably infinite number of hidden states. We will show that for these processes \(\beta_{g,z}^P = \beta_{2}^P = \beta_{4}^P = \beta_{M}^P = \beta\), where \(\beta\) is an arbitrary number in range \((0, 1)\). That is, the bounds given by Theorems 5 and 6 for the power-law growth of the number of predictable facts, of algorithmic mutual information, and of the unifilar order estimator are tight also in this case. This tightness seems quite a new result in our little theory of perigraphic processes.

To begin, let us recall that given a fixed algorithmically random binary sequence \(z = (z_k)_{k \in \mathbb{N}}\) such as the typical result of unbiased coin-flip or the binary expansion of Chaitin constant \(\Omega = (\Omega_k)_{k \in \mathbb{N}}\) [2], we can construct a very simple example of a perigraphic process, called the Santa Fe process [51, 17].

The Santa Fe process \((X_i)_{i \in \mathbb{Z}}\) is a process over a countably infinite alphabet and it consists of pairs

\[X_i = (K_i, z_{K_i}),\]

where \(K_i\) is a IID process taking values in natural numbers according to Zipf’s law \(P(K_i = k) \propto k^{-\alpha}, \alpha > 1\). It can be easily shown that in this case \(\beta_{g,z}^P = 1/\alpha\) for quite an obvious choice of predictor \(g(k, x^n_i)\) that simply reads off value \(z_k\) from pair \((k, z_k)\) if it appears in block \(x^n_i\) and returns 2 otherwise [17, 18].

A similar perigraphic process over a finite alphabet can be constructed through stationary variable-length coding of the Santa Fe process [72, 52, 18] or more
directly, as a unifilar process with a countably infinite number of hidden states to be introduced in this paper under the name of Oracle processes.

First, we will generalize the concept of a unifilar process to a countable set of hidden states. In particular, the set of hidden states can be exemplified as the set of natural numbers \( \mathbb{N} \) or as the set of strings \( X^* \).

**Definition 2** A possibly non-stationary process \( (X_i)_{i \in \mathbb{N}} \) over an alphabet \( X \) is called unifilar with respect to a process \( (Y_i)_{i \in \mathbb{N}} \) if \( (Y_i)_{i \in \mathbb{N}} \) is a homogeneous first order Markov process over a countable alphabet \( \mathbb{Y} \) such that

1. \( P(Y_1 = y_1) = \pi(y_1) \),
2. \( P(X_i = x_i|Y_i = y_i, X_{i-1} = x_{i-1}) = \varepsilon(x_i|y_i) \),
3. \( Y_{i+1} = \tau(Y_i, X_i) \)

for certain functions \( \pi : \mathbb{Y} \to [0, 1] \), \( \varepsilon : X \times \mathbb{Y} \to [0, 1] \), and \( \tau : \mathbb{Y} \times X \to \mathbb{Y} \).

The following Oracle(\( \theta \)) process is the Santa Fe process in disguise. The idea is that we first emit some random string \( y2 \) uniquely representing the natural number \( \phi(y) \) and then we emit the corresponding bit \( z_{\phi(y)} \) read off from the oracle being an algorithmically random sequence \( z = (z_k)_{k \in \mathbb{N}} \). Once this bit is emitted, we repeat the procedure ad infinitum.

**Definition 3** Let \( \psi : \mathbb{N} \to \{0,1\}^* \) where \( \psi(k) \) is the binary expansion of number \( k \) stripped of the initial digit 1: \( \psi(1) = \lambda \), \( \psi(2) = 0 \), \( \psi(3) = 1 \), \( \psi(4) = 00 \), ... Let \( \phi = \psi^{-1} \) be the inverse function. Let \( z = (z_k)_{k \in \mathbb{N}} \) be an algorithmically random binary sequence. Oracle(\( \theta \)) process with parameter \( \theta \in [0,1] \) is the unifilar process defined by:

- \( X = \{0,1,2\} \),
- \( \mathbb{Y} = \{a,b\} \times \{0,1\}^* \),
- \( \varepsilon(x|ay) = \theta/2 \) and \( \tau(ay,x) = ayx \) for \( x \in \{0,1\} \) and \( y \in \{0,1\}^* \),
- \( \varepsilon(2|ay) = (1-\theta) \) and \( \tau(ay,2) = by \) for \( y \in \{0,1\}^* \),
- \( \varepsilon(z_{\phi(y)}|by) = 1 \) and \( \tau(by,z_{\phi(y)}) = a \) for \( y \in \{0,1\}^* \).

Since we have not discussed the Oracle processes before, as a warm-up, let us compute the stationary distribution and the entropy rate of an Oracle process. There exists a simple expression for the entropy rate of a unifilar process.

**Theorem 7** ([73, 74, 75]) The entropy rate of a process \( (X_i)_{i \in \mathbb{N}} \) unifilar with respect to a stationary process \( (Y_i)_{i \in \mathbb{N}} \) with entropy \( H(Y_i) = -\sum_{y \in \mathbb{Y}} \pi(y) \log \pi(y) < \infty \) equals

\[
H^P = \sum_{y \in \mathbb{Y}} \pi(y) \left[ -\sum_{x \in X} \varepsilon(x|y) \log \varepsilon(x|y) \right]. \tag{56}
\]

The entropy rate of a non-unifilar hidden Markov processes is much more difficult to compute [76, 77, 78, 79].

Knowing that Oracle processes are unifilar, we state the following.
**Theorem 8** The entropy rate of the stationary Oracle(θ) process equals

\[ h^P = \frac{h(\theta) + \theta}{2 - \theta}, \]  

(57)

where \( h(\theta) := -\theta \log \theta - (1 - \theta) \log(1 - \theta) \).

**Proof:** Using equation (5), we can easily determine the stationary distribution as \( \pi(ay) = \pi(a) \left( \frac{\theta}{2} \right)^{|y|} \), \( \pi(by) = \pi(ay)(1 - \theta) \), and \( \pi(a) = (1 - \theta)/(2 - \theta) \). Hence

\[
H(Y_i) = - \sum_{y \in \{a,b\} \times \{0,1\}} \pi(y) \log \pi(y) \\
= \sum_{y \in \{0,1\}} \left[ -(2 - \theta)\pi(ay) \log \pi(ay) - (1 - \theta)\pi(ay) \log(1 - \theta) \right] \\
= \frac{1 - \theta}{2 - \theta} \sum_{k=0}^{\infty} \left[ -(2 - \theta)\theta^k k \log \frac{\theta(1 - \theta)}{2(2 - \theta)} - (1 - \theta)\theta^k \log(1 - \theta) \right] \\
= \frac{(1 - \theta)}{(2 - \theta)} \left[ (2 - \theta)\theta \log \frac{\theta(1 - \theta)}{2(2 - \theta)} - \log(1 - \theta) \right] \\
= \frac{(2 - \theta)\theta [ -\log \theta + 1 + \log(2 - \theta) - \log(1 - \theta)]}{(1 - \theta)(2 - \theta)}. \]  

(58)

Since this entropy is finite, we can compute the entropy rate by Theorem 7 as

\[
h^P = \sum_{y \in \{0,1\}} \pi(ay) \left[ -\theta \log \frac{\theta}{2} - (1 - \theta) \log(1 - \theta) \right] \\
= \pi(a) \sum_{n=0}^{\infty} \theta^n \left[ h(\theta) + \theta \right] = \frac{h(\theta) + \theta}{2 - \theta}. \]  

(59)

□

Now let us proceed to the main result of this section, i.e., computing Hilberg exponents \( \beta_{g,z}^P, \beta_2^P, \beta_3^P, \) and \( \beta_M^P \) for Oracle processes and showing that they are equal and arbitrarily large. To determine Hilberg exponent \( \beta_{g,z} \), we will use quite an obvious predictor

\[
g(k, x^n) := \begin{cases} 
0 & \text{if } 2^k \phi(k)20 \sqsupseteq x^n \quad \text{and} \quad 2^k \phi(k)21 \sqsupset x^n, \\
1 & \text{if } 2^k \phi(k)21 \sqsupseteq x^n \quad \text{and} \quad 2^k \phi(k)20 \sqsupset x^n, \\
2 & \text{else.} 
\end{cases} 
\]  

(60)

In the above definition, symbol ‘\( \sqsupset \)’ matches any symbol.

**Theorem 9** For predictor (60) and the stationary Oracle(θ) process,

\[
\beta_{g,z}^P = \beta_2^P = \beta_3^P = \beta_M^P = \beta := \frac{1}{1 - \log \theta}. \]  

(61)

**Proof:** By Theorems 5 and 6, it suffices to show \( \beta_{g,z} \geq \beta \) and \( \beta_M^P \leq \beta \). The proof of \( \beta_{g,z} \geq \beta \) will apply techniques developed in [17] for Santa Fe processes. The proof of \( \beta_M^P \leq \beta \) will use some ideas from [52] derived also for Santa Fe processes.
processes. For both goals of the proof, we will apply random variables \( N_n \geq 0 \), \( Z_i \in \{0,1\} \), and \( W_i \in \{0,1\}^* \) constructed through parsing
\[
X_1^{N_n} = W_0Z_0W_1Z_1W_2Z_2\ldots W_nZ_n.
\]
(62)

Obviously \( Z_i = z_{\theta(W_i)} \) for the Oracle(\( \theta \)) process. In contrast, by the strong Markov property, random variables \((W_i)_{i \in \mathbb{N}}\) form an IID process, where
\[
P(W_i = y) = (1 - \theta) \left( \frac{\theta}{2} \right)^{|y|}.
\]
(63)

Since \( N_n = \sum_{i=0}^{n} (|W_i| + 2) \) and \((W_i)_{i \in \mathbb{N}}\) is an IID process then by the Hoeffding inequality \([80]\) probabilities \( P(N_n < [\alpha n]) \) and \( P(N_n > [\alpha n]) \) vanish exponentially fast for \( \alpha \)'s respectively less or greater than \( E_P |W_i| + 2 \). We can evaluate for \( i \geq 1 \) that
\[
E_P |W_i| = (1 - \theta) \sum_{y \in \{0,1\}^*} |y| \left( \frac{\theta}{2} \right)^{|y|} = (1 - \theta) \sum_{n=0}^{\infty} n\theta^n = \frac{\theta}{1 - \theta}.
\]
(64)

Hence \( n \leq \frac{E_P N_n}{E_P |W_i|} \leq (n + 1) \). Observe now that \( U_g(X_1^{N_n}|z) \leq n \). Thus
\[
E_P U_g(X_1^{N_n}|z) \leq E_P U_g(X_1^{[\alpha n]}|z) + nP(N_n > [\alpha n]).
\]
(65)

Similarly, \( \mathbb{M}(X_1^{[\alpha n]}) \leq [\alpha n] \). Thus
\[
E_P \mathbb{M}(X_1^{[\alpha n]}) \leq E_P \mathbb{M}(X_1^{N_n}) + [\alpha n] P(N_n < [\alpha n]).
\]
(66)

Hence we obtain
\[
\text{hllb} E_P U_g(X_1^{N_n}|z) \leq \text{hllb} E_P U_g(X_1^n|z) = \beta_{g,z}^P,
\]
(67)

\[
\beta_{g}^P = \text{hllb} E_P \mathbb{M}(X_1^n) \leq \text{hllb} E_P \mathbb{M}(X_1^{N_n}).
\]
(68)

Define now
\[
U_n := \min \{ k \geq 1 : \psi(k) \notin \{ W_i \}_{i=1}^n \}.
\]
(69)

\[
M_n := |W_0| + 2 + \sum_{y \in \{0,1\}^*} \{ |y| + 2 \} 1 \{ y \in \{ W_i \}_{i=1}^n \}.
\]
(70)

We see that \( U_g(X_1^{N_n}|z) = U_n \). Since \( U_n \) is a non-decreasing function of \( n \), we have
\[
\text{hllb} U_n \leq \text{hllb} E_P U_n \leq \beta_{g,z}^P P\text{-almost surely}.
\]
(71)

by Theorem A9 from [17], see also [67, 18]. On the other hand, we see that \( \mathbb{P}(X_1^{N_n}|M_n) \geq \mathbb{P}(X_1^{N_n}|Y_1) \), where \( Y_1 \) is the hidden state emitting \( X_1 \). It is so since we can express probability \( P(X_1^{N_n}|Y_1) \) as probability of a unifilar process with \( M_n \) hidden states. Thus, by the Barron lemma [63, Theorem 3.1], we obtain
\[
P \left( \mathbb{M}(X_1^{N_n}) > M_n \right) \leq P \left( \mathbb{P}(X_1^{N_n}|M_n) < w_{N_n} \mathbb{P}(X_1^{N_n}) \right)
\leq P \left( \frac{w_{N_n} \mathbb{P}(X_1^{N_n})}{\mathbb{P}(X_1^{N_n}|Y_1)} > 1 \right) \leq E_P w_{N_n}.
\]
(72)
In consequence, by the Hoeffding bound for $N_n$, we obtain
\[ \beta^{\text{P}} \leq \text{hilt} \quad \text{E} \quad \text{M}(X_n) \leq \text{hilt} \quad \text{E} \quad M_n. \] (73)

To finish the proof, it suffices to show
\[ \text{hilt} \quad U_n \geq \beta \quad \text{hilt} \quad \text{E} \quad M_n \quad P \text{-almost surely.} \] (74)

To accomplish the left inequality in (74), we observe
\[
P(U_n < 2^m) \leq \sum_{k=0}^{2^m-1} P(\psi(k) \notin \{W_i\}_{i=1}^n) = \sum_{y \in \{0,1\}^m} P(y \notin \{W_i\}_{i=1}^n)
\]
\[
= \sum_{k=0}^{m} 2^k \left( 1 - (1 - \theta) \left( \frac{\theta}{2} \right)^k \right) \leq 2^m \left( 1 - (1 - \theta) \left( \frac{\theta}{2} \right)^m \right)
\]
\[
\leq 2^m \exp \left( -(1 - \theta) m \right) = 2^m \exp \left( -(1 - \theta) n 2^{-m/\beta} \right). \]
(75)

Putting $m_n = \beta(1 - \epsilon) \log n$ for an arbitrary $\epsilon > 0$, we obtain
\[
\sum_{n=1}^{\infty} P(U_n < 2^{m_n}) \leq \sum_{n=1}^{\infty} n^{\beta(1-\epsilon)} \exp(-(1 - \theta)m) < \infty. \]
(76)

Hence by the Borel-Cantelli lemma
\[ \beta \leq \text{hilt} \quad U_n \quad P \text{-almost surely.} \] (77)

To accomplish the right inequality in (74), we notice
\[
\text{E} \quad M_n - \frac{2 - \theta}{1 - \theta} = \sum_{y \in \{0,1\}^m} (|y| + 2) P(y \notin \{W_i\}_{i=1}^n)
\]
\[
= \sum_{k=0}^{\infty} (k + 2) 2^k \left( 1 - (1 - \theta) \left( \frac{\theta}{2} \right)^k \right)^n
\]
\[
\leq \sum_{k=0}^{\infty} (k + 2) 2^k \left( 1 - 2^{k/\beta} \right)^n. \]
(78)

Hence, adapting the computations from the proof of Proposition 1 by [52], we obtain up to a small constant
\[
\text{E} \quad M_n \leq \int_0^{\infty} (k + 2) 2^k \left( 1 - \left( 1 - 2^{-k/\beta} \right)^n \right) dk
\]
\[
= \frac{1}{\ln 2} \int_1^{\infty} (\log p + 2) \left( 1 - (1 - p^{-1/\beta})^n \right) dp \quad \{p := 2^k\}
\]
\[
= \frac{\beta^2}{\ln 2} \int_0^{1} (1 - u)(\log(1 - u^{1/n}) - 2)du \quad \{u := (1 - p^{-1/\beta})^n\}
\]
\[
= \frac{\beta^2 n^2}{\ln 2} \int_0^{1} f_n(u)du + \frac{\beta^2 n^2}{\ln 2} \int_0^{1} g_n(u)du, \]
(79)
where we denote functions
\[ f_n(u) := \frac{(1 - u)}{u^{1-1/n}[1 - u^{1/n}]} \beta + 1, \quad g_n(u) := f_n(u) \log[n(1 - u^{1/n})]^{-1}. \] (80)

These functions tend to limits
\[ \lim_{n \to \infty} f_n(u) = f(u) := \frac{(1 - u)}{u(-\ln u)^\beta + 1}, \quad \lim_{n \to \infty} g_n(u) = g(u) := f(u) \log(-\ln u)^{-1}. \] (81)

We notice upper bounds \( f_n(u) \leq f(u) \) and \( g_n(u) \leq g_1(u) \) for \( u \in (0, 1) \). Moreover, functions \( f(u) \) and \( g_1(u) \) are integrable on \( u \in (0, 1) \). Indeed putting \( t := -\ln u \) and integrating by parts yields
\[ \int_0^1 f(u)du = -\int_0^\infty (1 - e^{-t}) t^{-\beta}dt = (1 - e^{-t})(-\beta)^{-1}/0 + \int_0^\infty e^{-t}\beta^{-1}t^{-\beta}dt = \beta^{-1}\Gamma(1 - \beta), \] (82)

whereas putting \( t = 1 - u \) and integrating by parts yields
\[ \int_0^1 g_1(u)du = -\int_0^1 \log t t^{-\beta}dt = -(\log t)(1 - \beta)^{-1}t^{-\beta}/0 + \int_0^1 (1 - \beta)^{-1}t^{-\beta}dt = (1 - \beta)^{-2}. \] (83)

Hence we derive
\[ \text{hilb}_{n \to \infty} \mathbb{E}_P M_n \leq \beta. \] (84)

This completes the proof. \( \square \)

As we can see by the above theorem, Oracle processes can have arbitrary large Hilberg exponents \( \beta_P^x = \beta_P^y = \beta_P^z \in (0, 1) \). In particular, the hidden Markov order estimator can diverge as a power law even for so simple unifilar processes and it diverges at the slowest possible rate prescribed by the bounds in Theorems 5 and 6. That is, these bounds can be non-trivially tight.

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**References**

[1] P. Gács, “On the symmetry of algorithmic information,” *Sov. Math. Dokl.*, vol. 15, pp. 1477–1480, 1974.

[2] G. J. Chaitin, “A theory of program size formally identical to information theory,” *J. ACM*, vol. 22, pp. 329–340, 1975.
[3] M. Li and P. M. B. Vitányi, *An Introduction to Kolmogorov Complexity and Its Applications, 3rd ed.* Springer, 2008.

[4] W. Hilberg, “Der bekannte Grenzwert der redundanzfreien Information in Texten — eine Fehlinterpretation der Shannonschen Experimente?” *Frequenz*, vol. 44, pp. 243–248, 1990.

[5] W. Ebeling and G. Nicolis, “Entropy of symbolic sequences: the role of correlations,” *Europhys. Lett.*, vol. 14, pp. 191–196, 1991.

[6] W. Ebeling and T. Pöschel, “Entropy and long-range correlations in literary English,” *Europhys. Lett.*, vol. 26, pp. 241–246, 1994.

[7] W. Bialek, I. Nemenman, and N. Tishby, “Complexity through nonextensivity,” *Physica A*, vol. 302, pp. 89–99, 2001.

[8] J. P. Crutchfield and D. P. Feldman, “Regularities unseen, randomness observed: The entropy convergence hierarchy,” *Chaos*, vol. 15, pp. 25–54, 2003.

[9] A. Radford, J. Wu, R. Child, D. Luan, D. Amodei, and I. Sutskever, “Language models are unsupervised multitask learners,” 2019, https://openai.com/blog/better-language-models/.

[10] T. B. Brown, B. Mann, N. Ryder, M. Subbiah, J. Kaplan, P. Dhariwal, A. Neelakantan, P. Shyam, G. Sastry, A. Askell, S. Agarwal, G. K. Ariel Herbert-Voss, T. Henighan, R. Child, A. Ramesh, D. M. Ziegler, J. Wu, C. Winter, C. Hesse, M. Chen, M. L. Eric Sigler, S. Gray, B. Chess, J. Clark, C. Berner, S. McCandlish, A. Radford, I. Sutskever, and D. Amodei, “Language models are few-shot learners,” 2020, https://arxiv.org/abs/2005.14165.

[11] R. Takahira, K. Tanaka-Ishii, and Ł. Dębowski, “Entropy rate estimates for natural language—a new extrapolation of compressed large-scale corpora,” *Entropy*, vol. 18, no. 10, p. 364, 2016.

[12] M. Hahn and R. Futrell, “Estimating predictive rate-distortion curves via neural variational inference,” *Entropy*, vol. 21, p. 640, 2019.

[13] M. Braverman, X. Chen, S. M. Kakade, K. Narasimhan, C. Zhang, and Y. Zhang, “Calibration, entropy rates, and memory in language models,” 2019, https://arxiv.org/abs/1906.05664.

[14] J. Kaplan, S. McCandlish, T. Henighan, T. B. Brown, B. Chess, R. Child, S. Gray, A. Radford, J. Wu, and D. Amodei, “Scaling laws for neural language models,” 2020, https://arxiv.org/abs/2001.08361.

[15] Ł. Dębowski, “On Hilberg’s law and its links with Guiraud’s law,” *J. Quantit. Linguist.*, vol. 13, pp. 81–109, 2006.

[16] ——, “On the vocabulary of grammar-based codes and the logical consistency of texts,” *IEEE Trans. Inform. Theory*, vol. 57, pp. 4589–4599, 2011.
[17] ——, “Is natural language a perigraphic process? The theorem about facts and words revisited,” *Entropy*, vol. 20, no. 2, p. 85, 2018.

[18] ——, *Information Theory Meets Power Laws: Stochastic Processes and Language Models*. Wiley, 2021, in press.

[19] C. G. de Marcken, “Unsupervised language acquisition,” Ph.D. dissertation, Massachusetts Institute of Technology, 1996.

[20] J. C. Kieffer and E. Yang, “Grammar-based codes: A new class of universal lossless source codes,” *IEEE Trans. Inform. Theory*, vol. 46, pp. 737–754, 2000.

[21] M. Charikar, E. Lehman, A. Lehman, D. Liu, R. Panigrahy, M. Prabhakaran, A. Sahai, and A. Shelat, “The smallest grammar problem,” *IEEE Trans. Inform. Theory*, vol. 51, pp. 2554–2576, 2005.

[22] W. Kuraszkiewicz and J. Łukaszewicz, “The number of different words as a function of text length,” *Pamiętnik Literacki*, vol. 42(1), pp. 168–182, 1951, in Polish.

[23] P. Guiraud, *Les caractères statistiques du vocabulaire*. Paris: Presses Universitaires de France, 1954.

[24] G. Herdan, *Quantitative Linguistics*. Butterworths, 1964.

[25] H. S. Heaps, *Information Retrieval—Computational and Theoretical Aspects*. Academic Press, 1978.

[26] N. Merhav, M. Gutman, and J. Ziv, “On the estimation of the order of a Markov chain and universal data compression,” *IEEE Trans. Inform. Theory*, vol. 35, no. 5, pp. 1014–1019, 1989.

[27] J. Ziv and N. Merhav, “Estimating the number of states of a finite-state source,” *IEEE Trans. Inform. Theory*, vol. 38, no. 1, pp. 61–65, 1992.

[28] I. Csiszar and P. C. Shields, “The consistency of the BIC Markov order estimator,” *Ann. Statist.*, vol. 28, pp. 1601–1619, 2000.

[29] I. Csiszar, “Large-scale typicality of Markov sample paths and consistency of MDL order estimator,” *IEEE Trans. Inform. Theory*, vol. 48, no. 6, pp. 1616–1628, 2002.

[30] G. Morvai and B. Weiss, “Order estimation of Markov chains,” *IEEE Trans. Inform. Theory*, vol. 51, no. 4, pp. 1496–1497, 2005.

[31] Y. Peres and P. Shields, “Two new Markov order estimators,” 2005, https://arxiv.org/abs/math/0506080.

[32] D. Dalevi and D. Dubhashi, “The Peres-Shields order estimator for fixed and variable length Markov models with applications to DNA sequence similarity,” in *Algorithms in Bioinformatics*, R. Casadio and G. Myers, Eds. Springer, 2005, pp. 291–302.

[33] B. Ryabko and J. Astola, “Universal codes as a basis for time series testing,” *Statist. Methodol.*, vol. 3, pp. 375–397, 2006.
[34] I. Csiszar and Z. Talata, “Context tree estimation for not necessarily finite memory processes, via BIC and MDL,” IEEE Trans. Inform. Theory, vol. 52, pp. 1007–1016, 2006.

[35] Z. Talata, “Divergence rates of Markov order estimators and their application to statistical estimation of stationary ergodic processes,” Bernoulli, vol. 19, no. 3, pp. 846–885, 2013.

[36] A. R. Baigorri, C. R. Goncalves, and P. A. A. Resende, “Markov chain order estimation based on the chi-square divergence,” Canad. J. Statist., vol. 42, no. 4, pp. 563–578, 2014.

[37] B. Ryabko, J. Astola, and M. Malyutov, Compression-Based Methods of Statistical Analysis and Prediction of Time Series. Springer, 2016.

[38] M. Papapetrou and D. Kugiumtzis, “Markov chain order estimation with parametric significance tests of conditional mutual information,” Sim. Model. Pract. Theory, vol. 61, pp. 1–13, 2016.

[39] L. Finesso, “Order estimation for functions of Markov chains,” Ph.D. dissertation, University of Maryland, 1990.

[40] M. J. Weinberger, A. Lempel, and J. Ziv, “A sequential algorithm for the universal coding of finite memory sources,” IEEE Trans. Inform. Theory, vol. 38, no. 3, pp. 1002–1014, 1992.

[41] J. C. Kieffer, “Strongly consistent code-based identification and order estimation for constrained finite-state model classes,” IEEE Trans. Inform. Theory, vol. 39, no. 3, pp. 893–902, 1993.

[42] M. J. Weinberger and M. Feder, “Predictive stochastic complexity and model estimation for finite-state processes,” J. Statist. Plan. Infer., vol. 39, pp. 353–372, 1994.

[43] C.-C. Liu and P. Narayan, “Order estimation and sequential universal data compression of a hidden Markov source by the method of mixtures,” IEEE Trans. Inform. Theory, vol. 40, no. 4, pp. 1167–1180, 1994.

[44] E. Gassiat and S. Boucheron, “Optimal error exponents in hidden markov models order estimation,” IEEE Trans. Inform. Theory, vol. 49, no. 4, pp. 964–980, 2003.

[45] L. Lehérylicy, “Consistent order estimation for nonparametric Hidden Markov Models,” Bernoulli, vol. 25, no. 1, pp. 464–498, 2019.

[46] C. R. Shalizi, K. L. Shalizi, and J. P. Crutchfield, “An algorithm for pattern discovery in time series,” 2003, http://www.arxiv.org/abs/cs/0210025.

[47] J. Zheng and J. H. aand Changqing Tong, “The order estimation for hidden markov models,” Physica A, vol. 527, p. 121462, 2019.

[48] R. M. Gray and J. C. Kieffer, “Asymptotically mean stationary measures,” Ann. Probab., vol. 8, pp. 962–973, 1980.
[49] R. Fontana, R. Gray, and J. Kieffer, “Asymptotically mean stationary channels,” *IEEE Trans. Inform. Theory*, vol. 27, pp. 308–316, 1981.

[50] J. C. Kieffer and M. Rahe, “Markov channels are asymptotically mean stationary,” *SIAM J. Math. Anal.*, vol. 12, no. 3, pp. 293–305, 1981.

[51] Ł. Dębowski, “A general definition of conditional information and its application to ergodic decomposition,” *Statist. Probab. Lett.*, vol. 79, pp. 1260–1268, 2009.

[52] ——, “Mixing, ergodic, and nonergodic processes with rapidly growing information between blocks,” *IEEE Trans. Inform. Theory*, vol. 58, pp. 3392–3401, 2012.

[53] B. F. Skinner, *Verbal Behavior*. Prentice Hall, 1957.

[54] N. Chomsky, “Three models for the description of language,” *IRE Trans. Inform. Theory*, vol. 2, no. 3, pp. 113–124, 1956.

[55] ——, *Syntactic Structures*. The Hague: Mouton & Co, 1957.

[56] ——, “A review of B. F. Skinner’s Verbal Behavior,” *Language*, vol. 35, no. 1, pp. 26–58, 1959.

[57] N. Chomsky and G. Miller, “Finite state languages,” *Inform. Control*, vol. 1, pp. 91–112, 1959.

[58] B. Y. Ryabko, “Prediction of random sequences and universal coding,” *Probl. Inform. Transm.*, vol. 24, no. 2, pp. 87–96, 1988.

[59] B. Ryabko, “Compression-based methods for nonparametric density estimation, on-line prediction, regression and classification for time series,” in *2008 IEEE Information Theory Workshop, Porto*, 2008, pp. 271–275.

[60] Y. M. Shtarkov, “Universal sequential coding of single messages,” *Probl. Inform. Transm.*, vol. 23(2), pp. 3–17, 1987.

[61] P. D. Grünwald, *The Minimum Description Length Principle*. The MIT Press, 2007.

[62] Ł. Dębowski, “On a class of Markov order estimators based on PPM and other universal codes,” 2020, https://arxiv.org/abs/2003.04754.

[63] A. R. Barron, “Logically smooth density estimation,” Ph.D. dissertation, Stanford University, 1985.

[64] J. Ziv and A. Lempel, “A universal algorithm for sequential data compression,” *IEEE Trans. Inform. Theory*, vol. 23, pp. 337–343, 1977.

[65] A. D. Wyner, “A definition of conditional mutual information for arbitrary ensembles,” *Inform. Control*, vol. 38, pp. 51–59, 1978.

[66] Ł. Dębowski, “Approximating information measures for fields,” *Entropy*, vol. 22, no. 1, p. 79, 2020.
[67] ——, “Hilberg exponents: New measures of long memory in the process,” *IEEE Trans. Inform. Theory*, vol. 61, pp. 5716–5726, 2015.

[68] A. A. Brudno, “Entropy and the complexity of trajectories of a dynamical system,” *Trans. Mosc. Math. Soc.*, vol. 44, pp. 124–149, 1982.

[69] P. Gács and J. Körner, “Common information is far less than mutual information,” *Probl. Contr. Inform. Theory*, vol. 2, pp. 119–162, 1973.

[70] A. D. Wyner, “The common information of two dependent random variables,” *IEEE Trans. Inform. Theory*, vol. IT-21, pp. 163–179, 1975.

[71] Ł. Dębowski, “Regular Hilberg processes: An example of processes with a vanishing entropy rate,” *IEEE Trans. Inform. Theory*, vol. 63, no. 10, pp. 6538–6546, 2017.

[72] ——, “Variable-length coding of two-sided asymptotically mean stationary measures,” *J. Theor. Probab.*, vol. 23, pp. 237–256, 2010.

[73] N. F. Travers and J. P. Crutchfield, “Exact synchronization for finite-state sources,” *J. Statist. Phys.*, vol. 145, pp. 1181–1201, 2011.

[74] ——, “Asymptotic synchronization for finite-state sources,” *J. Statist. Phys.*, 2011.

[75] ——, “Infinite excess entropy processes with countable-state generators,” *Entropy*, vol. 16, pp. 1396–1413, 2014.

[76] D. Blackwell, “The entropy of functions of finite-state Markov chains,” in *Transactions of the First Prague Conference on Information Theory, Statistical Decision Functions, Random Processes*. Czechoslovak Academy of Sciences, 1957, pp. 13–20.

[77] Y. Ephraim and N. Merhav, “Hidden Markov processes,” *IEEE Trans. Inform. Theory*, vol. 48, pp. 1518–1569, 2002.

[78] G. Han and B. Marcus, “Analyticity of entropy rate of hidden Markov chain,” *IEEE Trans. Inform. Theory*, vol. 52, pp. 5251–5266, 2006.

[79] P. Jacquet, G. Serroussi, and W. Szpankowski, “On the entropy of a hidden Markov process,” *Theor. Comput. Sci.*, vol. 395, no. 2–3, pp. 203–219, 2008.

[80] W. Hoeffding, “Probability inequalities for sums of bounded random variables,” *J. Amer. Statist. Association*, vol. 58, no. 301, pp. 13–30, 1963.