ON TOPOLOGICALLY CONTROLLED MODEL REDUCTION
FOR DISCRETE-TIME SYSTEMS

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ABSTRACT. In this document the author proves that several problems in data-driven numerical approximation of dynamical systems in $\mathbb{C}^n$, can be reduced to the computation of a family of constrained matrix representations of elements of the group algebra $\mathbb{C}[\mathbb{Z}/m]$ in $\mathbb{C}^{m \times m}$, factoring through the commutative algebra $\text{Circ}(m)$ of circulant matrices in $\mathbb{C}^{m \times m}$, for some integers $m \leq n$.

The solvability of the previously described matrix representation problems is studied. Some connections of the aforementioned results, with numerical analysis of dynamical systems, are outlined, a prorotypical algorithm for the computation of the matrix representations, and some numerical implementations of the algorithm, will be presented.

1. Introduction

In this document we will study discrete-time dynamical systems determined by the pair $(\Sigma, \{\Theta_t\})$, with $\Sigma \subseteq \mathbb{C}^n$, and where $\{\Theta_t | t \in \mathbb{Z}\}$ is a family of continuous functions from $\Sigma$ to $\Sigma$, such that $\Theta_t \circ \Theta_s = \Theta_{t+s}$ and $\Theta_0 = \text{id}_\Sigma$, for every pair of integers $t, s \geq 0$.

We will prove that several important problems in numerical analysis and data-driven discovery of discrete-time dynamical systems of the form $(\Sigma, \{\Theta_t\})$ in $\mathbb{C}^n$, can be reduced to the computation of a family of discrete-time transition matrices $\{F_t\}_{t=1}^{m-1} \subseteq \rho_m(\mathbb{C}[\mathbb{Z}/m]) \subset \mathbb{C}^{m \times m}$ of rank at most $m$, for some matrix representation of the group algebra $\mathbb{C}[\mathbb{Z}/m]$, together with two matrices $\hat{K}, \hat{T} \in \mathbb{C}^{n \times n}$ of rank at most $m$, that are related to some evolution history data $\{v_1, \ldots, v_m\} \subseteq \Sigma \subseteq \mathbb{C}^n$, (approximately) generated by the dynamical $(\Sigma, \{\Theta_t\})$, by the equations $v_{t+1} = \hat{K}F_t \hat{T}v_t$ for $1 \leq t \leq m - 1$.

We will also show that each variation of the problem corresponding to the computation of the aforementioned transition matrices, can be reduced to solving a constrained matrix representation problem of the group algebra $\mathbb{C}[\mathbb{Z}/m]$ in $\mathbb{C}^{n \times n}$, factoring through the commutative algebra $\text{Circ}(m)$ of circulant matrices in $\mathbb{C}^{m \times m}$, for some integers $m \leq n$.

The motivation for the theoretical and computational machinery presented in this document came from some questions raised by M. H. Freedman along the lines of [2], concerning to the implications in linear algebra and matrix computations of the so called Kirby Torus Trick, presented by R. Kirby in [3].

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We study the solvability of the previously described matrix representation problems. Some connections of the aforementioned results, with numerical analysis of dynamical systems, are outlined, a prortotypical algorithm for the matrix representation computations, and some numerical implementations of the algorithm will be presented.

2. Preliminaries and Notation

Given two positive integers \( p, q \) such that \( p \geq q \), we will write \( p \mod q \) to denote the integer \( r \), such that \( p = mq + r \) for some integer \( m \). We will write \( \mathbb{Z}_q^+ \) to denote the set \( \{ z \in \mathbb{Z} | z \geq 0 \} \).

Given \( k \in \mathbb{Z}_q^+ \), we will write \( \mathbb{Z}/k \) to denote the (additive) cyclic group \( \mathbb{Z}/k\mathbb{Z} = \{ \hat{0}, \hat{1}, \hat{2}, \ldots, \hat{k-1} \} \).

Given any matrix \( X \in \mathbb{C}^{m \times n} \), we will write \( X_{ij} \) to denote the \( ij \)-entry of \( X \), and we will write \( X^* \) to denote its conjugate transpose \( X^\top = (X_{ji}) \in \mathbb{C}^{n \times m} \).

We will identify elements in \( \mathbb{C}^n \) with elements in \( \mathbb{C}^{n \times 1} \). As a consequence of this identification, given \( x, y \in \mathbb{C}^n \), \( y^*x \) will determine the Euclidean inner product \( \langle x, y \rangle \) in \( \mathbb{C}^n \), while \( xy^* \in \mathbb{C}^{n \times n} \) will determine a rank-one matrix in \( \mathbb{C}^{n \times n} \).

In this document we write \( 1_n \) and \( 0_{m \times n} \) to denote the identity and zero matrices in \( \mathbb{C}^{n \times n} \) and \( \mathbb{C}^{m \times n} \), respectively. We will write \( 0_n \) to denote the zero matrix in \( \mathbb{C}^{n \times n} \).

A set of \( m \) elements \( v_1, \ldots, v_m \in \mathbb{C}\setminus\{0\} \) is said to be an orthogonal \( m \)-system if
\[
\langle v_j, v_k \rangle = \delta_{j,k}
\]
for \( 1 \leq j, k \leq m \).

From here on, we will write \( \delta_{k,j} \) to denote the Kronecker delta defined by
\[
\delta_{k,j} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}
\]

We say that the set of vectors \( v_1, \ldots, v_m \in \mathbb{C}\setminus\{0\} \) is an orthonormal \( m \)-system if the vectors \( v_j \) satisfy (2.1) and in addition
\[
\|v_j\|_2 = \sqrt{v_j^*v_j} = 1,
\]
for \( 1 \leq j \leq m \).

We will write \( \hat{e}_{j,n} \) to denote the element in \( \mathbb{C}^{n \times 1} \) represented by the expression.
\[
\hat{e}_{j,n} = \begin{bmatrix} \delta_{1,j} \\ \delta_{2,j} \\ \vdots \\ \delta_{k,j} \\ \vdots \\ \delta_{n,j} \end{bmatrix}
\]

Each \( \hat{e}_{j,n} \) can be interpreted as the \( j \)-column of \( 1_n = [\hat{e}_{1,n} \hat{e}_{2,n} \cdots \hat{e}_{n,n}] \).

A matrix \( B \in \mathbb{C}^{n \times n} \) is said to be normal if \( BB^* = B^*B \), a matrix \( A \in \mathbb{C}^{n \times n} \) is said to be Hermitian if \( X^* = X \), and a hermitian matrix \( P \) is said to be an orthogonal projection or just a projection, if \( P^2 = P = P^* \).

A matrix \( X \in \mathbb{C}^{n \times n} \) is said to be unitary if \( X^*X = XX^* = 1_n \). A matrix \( A \in \mathbb{C}^{n \times m} \) is said to be positive if there is a matrix \( B \in \mathbb{C}^{n \times n} \) such that \( A = B^*B \).
we also write $A \geq 0$ to indicate that $A$ is positive. We will denote by $\mathbb{U}(n)$ and $\mathbb{P}(n)$, the sets of unitaries and positive matrices in $\mathbb{C}^{n \times n}$, respectively.

Given a matrix $W \in \mathbb{C}^{m \times n}$, we write $\text{Ad}[W]$ to denote the linear map from $\mathbb{C}^{n \times n}$ to $\mathbb{C}^{m \times n}$, defined by the operation $\text{Ad}[W](X) = WXW^*$ for any $X \in \mathbb{C}^{n \times n}$.

We say that a matrix $A \in \mathbb{C}^{n \times n}$ is invertible if there is one matrix $B \in \mathbb{C}^{n \times n}$ such that $AB = BA = I_n$. We will write $\mathbb{GL}(n)$ to denote the set of invertible matrices in $\mathbb{C}^{n \times n}$. Given a matrix $X \in \mathbb{C}^{n \times n}$, we will write $\sigma(X)$ to denote the spectrum of $X$, that is the set $\{z \in \mathbb{C} | X - zI_n \notin \mathbb{GL}(n)\}$.

From here on we write $\| \cdot \|_2$ to denote the Euclidean norm in $\mathbb{C}^n$ defined by the operation $\|x\|_2 = \sqrt{x^*x}$ for any $x \in \mathbb{C}^n$. In this document we will write $\| \cdot \|$ to denote the spectral norm in $\mathbb{C}^{n \times n}$ defined by the operation $\|X\| = \sup_{\|x\|=1} \|Ax\|_2$, for any $A \in \mathbb{C}^{n \times n}$.

**Definition 2.1.** A linear map $\varphi : \mathbb{C}^{n \times n} \to \mathbb{C}^{m \times m}$ is said to be a completely positive (CP) linear map if $\varphi(A) \geq 0$ for every positive $A \in \mathbb{C}^{n \times n}$, and if it has a Choi's representation of the form $\varphi = \sum_{j=1}^k \text{Ad}[W_j]$ for some matrices $W_j \in \mathbb{C}^{m \times m}$.

We will write $\text{CP}(n, m)$ to denote the set of completely positive linear maps from $\mathbb{C}^{n \times n}$ to $\mathbb{C}^{m \times m}$.

Given any $X \in \mathbb{C}^{n \times n}$ and any $p \in \mathbb{C}[z]$ determined by the formula $p(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$, we will write $p(X)$ to denote the matrix defined by the expression $p(X) = a_0I_n + a_1X + a_2X^2 + \cdots + a_nX^n$.

Given $X \in \mathbb{C}^{n \times n}$, we will write $\mathbb{C}[X]$ to denote the commutative algebra determined by the set $\{p(X)| p \in \mathbb{C}[z]\} = \text{span}_{\mathbb{C}}\{1_n,X,X^2,\ldots,X^{n-1}\}$, with respect to the usual addition and multiplication operations in $\mathbb{C}^{n \times n}$. We have that in fact $\mathbb{C}[X]$ is an algebra since, $X^m \in \text{span}_{\mathbb{C}}\{1_n,X,X^2,\ldots,X^{n-1}\}$ for every integer $m \geq n$, as a consequence of the Cayley-Hamilton Theorem, and that $\mathbb{C}[X]$ is commutative as a consequence of the identity $(aX^k)(bX^j) = abX^{k+j} = baX^{k+j} = bX^j aX^k$, that holds for each $a, b \in \mathbb{C}$ and for each pair of integers $k, j \geq 1$.

We will write $\text{Circ}(k)$ to denote the commutative algebra of $k \times k$ Circulant matrices that is defined by the expression:

\begin{equation}
\text{Circ}(k) = \mathbb{C}[C_k] = \{p(C_k)| p \in \mathbb{C}[z]\}
\end{equation}

where $C_k$ is the cyclic permutation matrix defined as follows.

\begin{equation}
C_k = \begin{bmatrix}
0_{1 \times (k-1)} & 1 \\
1_{k-1} & 0_{(k-1) \times 1}
\end{bmatrix}
\end{equation}

Given a matrix $X \in \mathbb{C}^{n \times n}$, we write $Z(X)$ to denote the commutant set of $X$ defined by the expression $Z(X) = \{Y \in \mathbb{C}^{n \times n}| XY = YX \}$.

Given a finite group $G$, we will write $\mathbb{C}[G]$ to denote the group algebra over $\mathbb{C}$.

For a finite group $G$ In this document we will focus on group algebra representations of $\mathbb{C}[G]$, determined by algebra homomorphisms of the form $\rho : \mathbb{C}[G] \to \mathbb{C}^{n \times n}$, $\sum_{g \in G} c_g g \mapsto \sum_{g \in G} c_g \rho(g)$, such that $\rho|_G$ is a group representation of $G$ in $\mathbb{GL}(n)$. We will say that an algebra representation $\rho : \mathbb{C}[G] \to \mathbb{C}^{n \times n}$ is unitary if $\rho|_G(G) \subset \mathbb{U}(n)$.

Given a discrete-time dynamical system $(\Sigma, \{\Theta_t\})$, if there is an integer $T > 0$ such that $\Theta_{t+T} = \Theta_t$ for each every $t \in \mathbb{Z}_0^+$, we say that $(\Sigma, \{\Theta_t\})$ is a discrete-time $T$-periodic dynamical system.
Given a discrete-time dynamical system \((\Sigma, \{\Theta_t\})\), a set of vectors \(H[\Sigma, m] = \{v_1, \ldots, v_m\} \subseteq \Sigma\) will be called a \(m\)-system of history vectors for \((\Sigma, \{\Theta_t\})\), if they satisfy the relations \(v_{k+1} = \Theta_t(v_k) = \Theta_k(v_1)\) for \(0 \leq k \leq m - 1\).

Given \(\delta, \varepsilon > 0\), we will say that a discrete-time dynamical system \((\Sigma, \{\Theta_t\})\) is \((T, \delta, \varepsilon)\)-almost-periodic dynamical system, if there is a \(T\)-periodic discrete-time dynamical system \((\Sigma, \{\tilde{\Theta}_t\})\) such that for each \(x \in \Sigma\) there is \(\tilde{x} \in \Sigma\) such that \(|x - \tilde{x}| \leq \delta\), and \(|\tilde{\Theta}_t(x) - \Theta_t(\tilde{x})| \leq \varepsilon\) for each \(t \in \mathbb{Z}_0^+\) and every \((x, \tilde{x}) \in \Sigma \times \Sigma\) such that \(|x - \tilde{x}| \leq \delta\).

3. Topological Control Method (TCM)

3.1. Switched Closed Loop Reduced Order Models SCL-ROM. In this section we will establish the notion of \(S^1\) topological control considered for this study.

Given a discrete-time dynamical system \((\Sigma, \{\Theta_t\})\) with \(\Sigma \subseteq \mathbb{C}^{n \times n}\), and a \(m\)-system of history vectors \(H[\Sigma, m] = \{v_1, \ldots, v_m\}\) for \((\Sigma, \{\Theta_t\})\), we say that \((\Sigma, \{\Theta_t\}, H[\Sigma, m])\) is \(\text{topologically controlled}\) by a topological manifold \(M \subseteq \mathbb{C}\) or \(M\)-\text{controlled}, if there is a matrix \(Z \in \mathbb{C}^{n \times n}\) with \(\sigma(Z) \subseteq M\), an algebra homomorphism \(\varphi : \mathbb{C}[Z] \to \mathbb{C}^{n \times n}\), a family of polynomials \(\{f_0, \ldots, f_{m-1}\}\), and two projections \(K, T \in \mathbb{C}^{n \times n}\) such that \(\Theta_k(v_1) = K\varphi(f_k(Z))T v_1\), for each \(k \geq 0\). We will call the \(6\)-tuple \((M, Z, K, T, \varphi, \{f_i\})\) a topological control for \((\Sigma, \{\Theta_t\}, H[\Sigma, m])\).

Given \(\varepsilon > 0\) and manifold \(M \subseteq \mathbb{C}\), and a \(M\)-controlled discrete-time dynamical system \((\Sigma, \{\Theta_t\})\) with \(\Sigma \subseteq \mathbb{C}^{n \times n}\), and a topological control \((M, Z, K, T, \varphi, \{f_i\})\) for \((\Sigma, \{\Theta_t\}, H[\Sigma, m])\), we say that \((M, Z, K, T, \varphi, \{f_i\})\) is a \(\text{control of order } k\), if there is an integer \(k > 0\), together with maps \(\Pi_k \in \text{Cp}(n, k), \Phi \in \text{CP}(k, n)\), such that \(|\Theta_k(v_1) - K\varphi(f_k(Z))T v_1| \leq \varepsilon\) and \(|\varphi(X) - \Phi \Pi_k(X)| \leq \varepsilon\), for each \(f_k\), each \(X \in \mathbb{C}[Z]\), and some \(\tilde{v} \in \mathbb{C}^n\). In this case we say that \((\Sigma, \{\Theta_t\}, H[\Sigma, m])\) is \((M, k, \varepsilon)\)-\text{controlled}.

Given \(\varepsilon > 0\), a discrete-time dynamical system \((\Sigma, \{\Theta_t\})\) with \(\Sigma \subseteq \mathbb{C}^{n \times n}\), and a \(m\)-system of history vectors \(H[\Sigma, m] = \{v_1, \ldots, v_m\}\) for \((\Sigma, \{\Theta_t\})\), we say that \((\Sigma, \{\Theta_t\}, H[\Sigma, m])\) is \(\varepsilon\)-\text{approximately topologically controlled by } \mathbb{Z}/m\) or \((\mathbb{Z}/m, \varepsilon)\)-\text{controlled}, if there is a unitary representation \(\rho_m : \mathbb{C}[\mathbb{Z}/m] \to \mathbb{C}^{m \times m}\), an algebra homomorphism \(\varphi : \mathbb{C}^{m \times m} \to \mathbb{C}^{n \times n}\), a family of functions \(\{f_0, \ldots, f_{m-1}\}\), and two projections \(K, T \in \mathbb{C}^{n \times n}\) such that \(|\Theta_k(v_1) - K\varphi(\rho_m(f_k(1)))T v_1| \leq \varepsilon\), for each \(k \geq 0\) and some \(\tilde{v} \in \mathbb{C}^n\), with \(1 \in \mathbb{Z}/m\).

For a given discrete-time dynamical system \((\Sigma, \{\Theta_t\})\) with \(\Sigma \subseteq \mathbb{C}^{n \times n}\), and a \(m\)-system of history vectors \(H[\Sigma, m] = \{v_1, \ldots, v_m\}\) for \((\Sigma, \{\Theta_t\})\), we approach the local controllability of \((\Sigma, \{\Theta_t\})\) by computing a switched closed loop control system \((\tilde{\Sigma}, \{\tilde{\Theta}_t\})\) in the sense of [1, §4.2, Example 4.2], determined by the decomposition.

\[
\tilde{\Sigma} : \begin{cases} 
\dot{v}_1 = \alpha \tilde{T} v_1 \\
\dot{v}_{t+1} = \tilde{F}_t \dot{v}_t, t \geq 0 \\
\dot{v}_{t+1} = \beta \tilde{K} \dot{v}_{t+1}
\end{cases}
\]

for some \((\alpha, \beta, \tilde{K}, \{\tilde{F}_t\}, \tilde{T}) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}\) to be determined. Given \(\varepsilon > 0\), the discrete-time system (3.1) is called an \(\varepsilon\)-approximate switched closed-loop reduced order model (SCL-ROM) of \((\Sigma, \{\Theta_t\})\), if \(\|\tilde{v}_1 - \Theta_1(v_1)\|_2 \leq \varepsilon\) for each \(t \in \mathbb{Z}_0^+\).
3.2. Some Connections with Dynamic Mode Decomposition. Given $\varepsilon > 0$, an integer $T > 0$, and a discrete-time system $(\Sigma, \{\Theta_t\})$ in $\mathbb{C}^n$. Let us consider the evolution history determined by the difference equations.

\begin{equation}
\Sigma: \begin{cases} 
    x_{t+1} = \Theta_t(x_t), \\
    x_0 = x 
\end{cases}, t \in \mathbb{Z}_0^+
\end{equation}

for some $x \in \Sigma$.

Given $\varepsilon > 0$. The computation of a SCL-ROM local $\varepsilon$-approximant $\tilde{\Sigma}$ of $\Sigma$, with respect to some sampled-data history $\{x_t\}$ of $\Sigma$, is related to the computations of closed-loop matrix realizations $H_t \in \mathbb{C}^{n \times n}$ and triples $(P_H(t), Q_H(t), F_H(t)) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}$ such that.

\begin{equation}
\begin{bmatrix} 
    \|P_H(t)F_H(t) - H_tP_H(t)Q_H(t)\| & \leq \varepsilon, \\
    \|P_H(t)H_t - H_tP_H(t)\| & = 0, \\
    \|P_H(t)x_t - x_t\| & = 0, \\
    P_H(t)^2 & = P_H(t) = P_H(t)^*, \\
    Q_H(t)^2 & = Q_H(t) = Q_H(t)^*, \\
    F_H(t)^*F_H(t) & = F_H(t)F_H(t)^*
\end{bmatrix}, 0 \leq t \leq T - 1
\end{equation}

The matrices $\{H_t\}$ in (3.3) are determined by the connecting operator $K$ for the sampled-data history $\{x_t\}$, in the sense of [6, §2] and [4, §2.2], that satisfies the equations $x_{t+1} = Kx_t$, $0 \leq t \leq T-1$. In particular we will consider $H_t = K, t \in \mathbb{Z}_0^+$.

The objective of topological control methods is to compute matrix realizations $(F_H(t), K, \hat{T}(t)) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}$ such that:

\[ \|KF_H(t)\hat{T}(t)x_t - y_t\| \leq \varepsilon \]

for $0 \leq t \leq T - 1$.

Given $\varepsilon > 0$, an integer $T > 0$, a family of vectors $\{x_t\}_{t=0}^{T-1}$ in $\mathbb{C}^n$ and matrices $H_t \in \mathbb{C}^{n \times n}$ determined by the connecting operator $K$ for $\{x_t\}$. For any triples $(P_H(t), Q_H(t), F_H(t)) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}$ that satisfy (3.3).

Let us consider the structured matrix equations in $\mathbb{C}^{2n \times 2n}$ determined by:

\begin{equation}
\begin{bmatrix} 
    (Q(t)X(t) - X(t)Q(t))P(t) & = 0_{2n}, \\
    Q(t)^4 & = Q(t)^2, \\
    Q(t)^2 & = ZQ(t) = (Q(t)^2)^*
\end{bmatrix}, 0 \leq t \leq T - 1
\end{equation}

where $P(t)$ and $Z$ have the form.

\[ P(t) = \begin{bmatrix} 
1_n - P_H(t) & 0_n \\
0_n & Q_H(t)
\end{bmatrix}, \quad Z = \begin{bmatrix} 
0_n & 1_n \\
1_n & 0_n
\end{bmatrix} \]

If we set.

\[ \hat{Q}(t) = \begin{bmatrix} 
0_n & P_H(t) \\
P_H(t) & 0_n
\end{bmatrix}, \quad \hat{X}(t) = \begin{bmatrix} 
H_t & 0_n \\
0_n & F_H(t)
\end{bmatrix} \]

It can be seen that $(\hat{Q}(t), \hat{X}(t)) \in \mathbb{C}^{2n \times 2n}$ are $\varepsilon$-approximate solvent pairs for (3.4) in the sense that, $\|\hat{Q}(t)\hat{X}(t) - \hat{X}(t)\hat{Q}(t)\| \leq \varepsilon$, $Q(t)^4 = Q(t)^2$ and $Q(t)^2 = ZQ(t) = (Q(t)^2)^*$ for $0 \leq t \leq T - 1$.

Given a positive integer $T$, together with $\varepsilon, \delta > 0$. In this first paper on the subject of topological control of discrete-time systems, we focus on the computation of transition mappings $F_H(t)$, avoiding an explicit computation of the connecting operator $K$, instead we compute the family $\{F_H(t)\}$ by solving some constrained representation problems for $\mathbb{C}[\mathbb{Z}/T]$ for a given integer $T > 0$, restricting our attention to almost $(T, \varepsilon, \delta)$-periodic discrete-time systems.
3.3. Main Objectives. We prove the solvability of the problem of finding a \( \varepsilon \)-approximate SCL-ROM \( \tilde{\Sigma} \) described by (3.1) for a discrete-time system \((\Sigma, \{\Theta_t\})\) with a \( m \)-system of history vectors \( \mathbb{H}[\Sigma, m] = \{v_1, \ldots, v_m\} \subseteq \Sigma \), by computing matrix representations of \( \mathbb{C}[\mathbb{Z}/T] \) such that the switching law of the family \( \{f_t\} \) is controlled by some family \( \{f_t\} \subset \mathbb{C}[\mathbb{Z}/T] \) subject to almost time-periodic constraints on \((\Sigma, \{\Theta_t\})\). We will then design and implement a prototypical numerical algorithm that numerically solves the aforementioned problems.

3.4. Topologically Controlled Model Order Reduction. Let us consider an orthogonal \( m \)-system \( v_1, \ldots, v_m \in \mathbb{C}^{n \times 1} \setminus \{0\} \) with \( m \leq n \). From here on, we will write \( C[v_1|v_m] \) to denote the matrix in \( \mathbb{C}^{n \times n} \) defined by the equation.

\[
C[v_1|v_m] = \frac{1}{v_m^* v_m} v_1 v_m^* + \sum_{j=1}^{m-1} \frac{1}{v_j^* v_j} v_j v_{j+1}^* + 1_n - P[v_1|v_m]
\]

where \( P[v_1|v_m] \) is the matrix in \( \mathbb{C}^{n \times n} \) defined by the equation.

\[
P[v_1|v_m] = \sum_{j=1}^{m} \frac{1}{v_j^* v_j} v_j v_j^*
\]

Lemma 3.1. Given an orthogonal \( m \)-system \( v_1, \ldots, v_m \in \mathbb{C}^{n \times 1} \setminus \{0\} \) with \( m \leq n \). The matrix \( P[v_1|v_m] \) defined by (3.6) is an orthogonal projection such that \( P[v_1|v_m] v_j = v_j, 1 \leq j \leq m \). Moreover, \( 1_n - P[v_1|v_m] \) is an orthogonal projection such that \( P[v_1|v_m](1_n - P[v_1|v_m]) = (1_n - P[v_1|v_m])P[v_1|v_m] = 0_n \)

Proof. Given an orthogonal \( m \)-system \( v_1, \ldots, v_m \in \mathbb{C}^{n \times 1} \setminus \{0\} \) with \( m \leq n \). For each \( 1 \leq j \leq m \), let us set

\[
V_j = \frac{1}{v_j^* v_j} v_j v_j^*
\]

it is clear that each \( V_j \) satisfies the relation,

\[
V_j^* = V_j,
\]

and we will also have that,

\[
V_j^2 = \frac{1}{v_j^* v_j} v_j v_j^* \frac{1}{v_j^* v_j} v_j v_j^* = \frac{v_j^* v_j}{(v_j^* v_j)^2} v_j v_j^* = \frac{1}{v_j^* v_j} v_j v_j^* = V_j.
\]

By (3.8) and (3.9), we will have that \( V_j^* = V_j = V_j^2 \), and this implies that each \( V_j \) is projection, and by orthogonality of the system \( v_1, \ldots, v_m \), we have that

\[
V_j V_k = \frac{1}{v_j^* v_j} v_j v_j^* \frac{1}{v_k^* v_k} v_k v_k^* = \frac{v_j^* v_k}{v_j^* v_j v_k^* v_k} v_j v_k^* = \frac{\delta_{j,k} v_j^* v_k}{v_j^* v_k v_j v_k} v_j v_k^* = \delta_{j,k} V_j V_k.
\]

This implies that the projections \( V_1, \ldots, V_m \) are mutually orthogonal projections, and it can be seen that \((\sum_j V_j)^* = \sum_j V_j = \sum_j V_j\). Moreover,

\[
P[v_1|v_m]^2 = \left( \sum_{j=1}^{m} V_j \right)^2 = \sum_{j=1}^{m} V_j V_k = \sum_{j=1}^{m} V_j^2 = \sum_{j=1}^{m} V_j = P[v_1|v_m].
\]
We also have that.

\[(3.12) \quad (P[v_1|v_m])^* = \left( \sum_{j=1}^{m} V_j \right)^* = \sum_{j=1}^{m} V_j^* = \sum_{j=1}^{m} V_j = P[v_1|v_m].\]

By (3.6) and by orthogonality of the system \(v_1, \ldots, v_m\), we will have that for \(1 \leq j \leq m\),

\[(3.14) \quad (1_n - P[v_1|v_m])^2 = 1_n - 2P[v_1|v_m] + P[v_1|v_m]^2 = 1_n - 2P[v_1|v_m] + P[v_1|v_m] = 1_n - P[v_1|v_m].\]

As a consequence of (3.12) we will also have that.

\[(3.15) \quad (1_n - P[v_1|v_m])^* = 1_n - P[v_1|v_m]^* = 1_n - P[v_1|v_m].\]

By (3.11) we will have that,

\[(3.16) \quad (1_n - P[v_1|v_m])P[v_1|v_m] = P[v_1|v_m] - P[v_1|v_m]^2 = P[v_1|v_m] - P[v_1|v_m] = 0_n\]

and also that.

\[(3.17) \quad P[v_1|v_m](1_n - P[v_1|v_m]) = P[v_1|v_m] - P[v_1|v_m]^2 = P[v_1|v_m] - P[v_1|v_m] = 0_n\]

This completes the proof.

**Lemma 3.2.** Given an orthogonal \(m\)-system \(v_1, \ldots, v_m \in \mathbb{C}^{n \times 1}\setminus\{0\}\) with \(m \leq n\). The matrices \(C[v_1|v_m]\) and \(P[v_1|v_m]\) defined by (3.5) and (3.6), respectively, satisfy the following conditions:

- \(C[v_1|v_m]v_j = v_{j+1}, \ 1 \leq j \leq m - 1,\)
- \(C[v_1|v_m]v_m = v_1,\)
- \(C[v_1|v_m][P[v_1|v_m] = P[v_1|v_m]C[v_1|v_m] = P[v_1|v_m]C[v_1|v_m]P[v_1|v_m],\)
- \(P[v_1|v_m]C[v_1|v_m][P[v_1|v_m] = \frac{1}{v_m^*v_m} v_m^*v_m + \sum_{j=1}^{m} \frac{1}{v_j^*v_j} v_j^*v_j,\)
- \(C[v_1|v_m](1_n - P[v_1|v_m]) = (1_n - P[v_1|v_m])C[v_1|v_m] = 1_n - P[v_1|v_m].\)

**Proof.** Since \(C[v_1|v_m]\) is defined by (3.5), by lemma 3.1, we will have that for \(1 \leq j \leq m\):

\[(3.18) \quad C[v_1|v_m]v_j = \frac{1}{v_m^*v_m} v_1(v_m^*v_j) + \sum_{j=1}^{m-1} \frac{1}{v_k^*v_k} v_{k+1}(v_k^*v_j) + v_j - P[v_1|v_m]v_j = \frac{\delta_{m,j}}{v_m^*v_m} v_1 + \sum_{j=1}^{m-1} \frac{\delta_{k,j}}{v_k^*v_k} v_{k+1} + 0\]

\[= \begin{cases} 
v_{j+1}, & 1 \leq j \leq m - 1, 
\v_1, & j = m. 
\end{cases}\]
By lemma 3.1, and by (3.5) and (3.6), on one hand we will have that,

\[ P[v_1|v_m]C[v_1|v_m] = P[v_1|v_m]\left(\frac{1}{v_m^* v_m} v_1 v_m^* + \sum_{j=1}^{m-1} \frac{1}{v_j^* v_j} v_{j+1} v_j^* + 1_n - P[v_1|v_m]\right) \]

\[ = \frac{1}{v_m^* v_m} (P[v_1|v_m] v_m^* + \sum_{j=1}^{m-1} v_j^* v_j (P[v_1|v_m] v_{j+1}) v_j^* + P[v_1|v_m] (1_n - P[v_1|v_m])) \]

\[ = \frac{1}{v_m^* v_m} v_1 v_m^* + \sum_{j=1}^{m-1} v_j^* v_j v_{j+1} v_j^* + 0_n \]

(3.19)

\[ = \frac{1}{v_m^* v_m} v_1 v_m^* + \sum_{j=1}^{m-1} v_j^* v_j v_{j+1} v_j^* \]

on the other hand we will have that,

\[ C[v_1|v_m]P[v_1|v_m] = (\frac{1}{v_m^* v_m} v_1 v_m^* + \sum_{j=1}^{m-1} v_j^* v_j v_{j+1} v_j^* + 1_n - P[v_1|v_m])P[v_1|v_m]^* \]

\[ = \frac{1}{v_m^* v_m} v_1 (v_m^* P[v_1|v_m]^*) + \sum_{j=1}^{m-1} \frac{1}{v_j^* v_j} v_{j+1} (v_j^* P[v_1|v_m]^*) \]

\[ + (1_n - P[v_1|v_m])P[v_1|v_m]^* \]

\[ = \frac{1}{v_m^* v_m} v_1 (P[v_1|v_m] v_m)^* + \sum_{j=1}^{m-1} \frac{1}{v_j^* v_j} v_{j+1} (P[v_1|v_m] v_j)^* \]

\[ + (1_n - P[v_1|v_m])P[v_1|v_m]^* \]

\[ = \frac{1}{v_m^* v_m} v_1 v_m^* + \sum_{j=1}^{m-1} v_j^* v_j v_{j+1} v_j^* + 0_n \]

(3.20)

\[ = \frac{1}{v_m^* v_m} v_1 v_m^* + \sum_{j=1}^{m-1} v_j^* v_j v_{j+1} v_j^* \]

By combining (3.19) and (3.20) we have that

\[ P[v_1|v_m]C[v_1|v_m] = C[v_1|v_m]P[v_1|v_m] \]

Since \( P[v_1|v_m] \) is an orthogonal projection, we will also have that.

\[ P[v_1|v_m]C[v_1|v_m] = P[v_1|v_m]P[v_1|v_m]C[v_1|v_m] = P[v_1|v_m]C[v_1|v_m]P[v_1|v_m] \]

By (3.22) and (3.19), it can be seen that \( C[v_1|v_m] \) can be represented in the form.

\[ C[v_1|v_m] = P[v_1|v_m]C[v_1|v_m]P[v_1|v_m] + 1_n - P[v_1|v_m] \]

By lemma 3.1, we will have that,

\[ (1_n - P[v_1|v_m])C[v_1|v_m] = (1_n - P[v_1|v_m])P[v_1|v_m]C[v_1|v_m]P[v_1|v_m] \]

\[ + (1_n - P[v_1|v_m])^2 \]

\[ = 0_n C[v_1|v_m]P[v_1|v_m] + 1_n - P[v_1|v_m] \]

(3.24)

\[ = 1_n - P[v_1|v_m] \]
and also that.

\[ C[v_1|v_m](1_n - P[v_1|v_m]) = P[v_1|v_m]C[v_1|v_m]P[v_1|v_m](1_n - P[v_1|v_m]) + (1_n - P[v_1|v_m])^2 \]

\[ = P[v_1|v_m]C[v_1|v_m]0_n + 1_n - P[v_1|v_m] \]

(3.25)

This completes the proof.

Lemma 3.3. Given an orthogonal \( m \)-system \( v_1, \ldots, v_m \in \mathbb{C}^{n \times 1}\) with \( m \leq n \). The matrix \( C[v_1|v_m] \) defined by (3.5) satisfies the equation

\[ C[v_1|v_m] = 1_n. \]

Moreover, \( p(z) = z^m - 1 \) is the minimal polynomial of \( C[v_1|v_m] \).

Proof. Given an orthogonal \( m \)-system \( v_1, \ldots, v_m \in \mathbb{C}^{n \times 1}\) with \( m \leq n \). By (3.5), (3.6), and by iterating on lemma 3.2, we will have that.

\[ C[v_1|v_m]^m = C[v_1|v_m]^{m-1} \left( \frac{1}{v_m^{v_m}} v_1 v_1^* + \sum_{j=1}^{m-1} \frac{1}{v_j^* v_j} v_{j+1} v_j^* + 1_n - P[v_1|v_m] \right) \]

\[ = \frac{1}{v_m^{v_m}} (C[v_1|v_m]^{m-1} v_1) v_1^* + \sum_{j=1}^{m-1} \frac{1}{v_j^* v_j} (C[v_1|v_m]^{m-1} v_{j+1}) v_j^* \]

\[ + C[v_1|v_m]^{m-1} (1_n - P[v_1|v_m]) \]

\[ = \frac{1}{v_m^{v_m}} v_m v_1 v_1^* + \sum_{j=1}^{m-1} \frac{1}{v_j^* v_j} v_{j+1} + 1_n - P[v_1|v_m] \]

\[ = \frac{1}{v_m^{v_m}} v_m v_1 v_1^* + \sum_{j=1}^{m-1} \frac{1}{v_j^* v_j} v_j v_j^* + 1_n - P[v_1|v_m] \]

\[ = \sum_{j=1}^{m} \frac{1}{v_j^* v_j} v_j v_j^* + 1_n - P[v_1|v_m] \]

(3.27)

\[ = P[v_1|v_m] + 1_n - P[v_1|v_m] = 1_n. \]

By orthogonality properties we have that the system \( v_1, \ldots, v_m \) is linearly independent. This fact combined with (3.27) implies that \( p(z) = z^m - 1 \) is the minimal polynomial of \( C[v_1|v_m] \), since if \( C[v_1|v_m]^k = 1_n \) for some \( k < m \), we would have that \( v_1 = v_k \), which contradicts the linear independence of the system.

Lemma 3.4. Given an orthonormal \( m \)-system \( \{v_1, \ldots, v_m\} \in \mathbb{C}^n\) with \( m \leq n \), we will have that the corresponding matrix \( C[v_1|v_m] \) is unitary.

Proof. Given an orthonormal \( m \)-system \( \{v_1, \ldots, v_m\} \in \mathbb{C}^n\), since \( v_j^* v_j = 1 \) for each \( 1 \leq j \leq m \), by (3.5) we will have that the matrix \( C[v_1|v_m] \) satisfies the equation.

\[ C[v_1|v_m] = v_1 v_m^* + \sum_{j=1}^{m-1} v_{j+1} v_j^* + 1_n - P[v_1|v_m] \]

(3.28)
and also that the matrix $P|v_1, v_m|$ satisfies the equation.

$$P|v_1, v_m| = \sum_{j=1}^{m} v_j v_j^*$$

(3.29)

By lemma 3.2 we will have that,

$$C[|v_1, v_m|] (1_n - P|v_1, v_m|) = (1_n - P|v_1, v_m|) C[|v_1, v_m|]^*$$

and also that.

$$C[|v_1, v_m|] P|v_1, v_m| = (P|v_1, v_m| C[|v_1, v_m|])^*$$

(3.30)

and also that.

$$C[|v_1, v_m|] P|v_1, v_m| = (P|v_1, v_m| C[|v_1, v_m|])^*$$

$$= (P|v_1, v_m| C[|v_1, v_m|] P|v_1, v_m|)^*$$

(3.31)

$$= v_m v_1^* + \sum_{j=1}^{m-1} v_j v_{j+1}^*$$

Since

$$C[|v_1, v_m|]^* = C[|v_1, v_m|] (P|v_1, v_m| + 1_n - P|v_1, v_m|)$$

$$= C[|v_1, v_m|] P|v_1, v_m| + 1_n - P|v_1, v_m|$$

and

$$C[|v_1, v_m|] = (P|v_1, v_m| + 1_n - P|v_1, v_m|) C[|v_1, v_m|]$$

$$= P|v_1, v_m| C[|v_1, v_m|] + 1_n - P|v_1, v_m|,$$

by (3.28) and (3.29) we will have that.

$$C[|v_1, v_m|] C[|v_1, v_m|] = C[|v_1, v_m|] P|v_1, v_m| C[|v_1, v_m|] + (1_n - P|v_1, v_m|)^2$$

$$= \left( v_m v_1^* + \sum_{j=1}^{m-1} v_j v_{j+1}^* \right) \left( v_1 v_m^* + \sum_{j=1}^{m-1} v_j v_m^* \right)$$

$$+ 1_n - P|v_1, v_m|$$

(3.32)

$$= P|v_1, v_m| + 1_n - P|v_1, v_m| = 1_n$$

This implies that.

$$C[|v_1, v_m|] C[|v_1, v_m|]^* = (C[|v_1, v_m|] C[|v_1, v_m|])^* = (1_n)^* = 1_n$$

This completes the proof. \qed

**Lemma 3.5.** Given $m$ vectors $v_1, \ldots, v_m \in \mathbb{C}^n$ such that $2m \leq n$, there is an orthonormal $m$-system $\hat{v}_1, \ldots, \hat{v}_m \in \mathbb{C}^n$, two scalars $\rho, \kappa \in \mathbb{C}$, two projections $K$, $T$ and a unitary $U|v_1, v_m|$ in $\mathbb{C}^{n \times n}$ such that:

$$\begin{align*}
Tv_1 &= \rho \hat{v}_1, \\
U|v_1, v_m|^\dagger \hat{v}_j &= \hat{v}_{j+1}, \\
U|v_1, v_m|^\dagger \hat{v}_m &= \hat{v}_1 \\
K \kappa \hat{v}_j &= v_j
\end{align*}$$

(3.34)

for each $1 \leq j \leq m - 1$. 
Proof. Let us consider the matrix \( \mathbb{H}[v_1|v_m] \in \mathbb{C}^{n \times m} \) defined by the expression.

\[
(3.35) \quad \mathbb{H}[v_1|v_m] = \begin{bmatrix} \vert & \vert & \cdots & \vert \\
 v_1 & v_2 & \cdots & v_m \end{bmatrix}
\]

By the singular value decomposition theorem, we have that \( \mathbb{H}[v_1|v_m] \) has a representation of the form.

\[
(3.36) \quad \mathbb{H}[v_1|v_m] = \begin{bmatrix} \vert & \vert & \cdots & \vert \\
 V_1 & V_2 & \cdots & V_m \end{bmatrix} \begin{bmatrix} \vert & \vert & \cdots & \vert \\
 s_1 & 0 & \cdots & 0 \\
 0 & s_2 & \cdots & 0 \\
 \vdots & \vdots & \ddots & 0 \\
 0 & \cdots & 0 & s_m \end{bmatrix} \begin{bmatrix} \vert & \vert & \cdots & \vert \\
 W_1 & W_2 & \cdots & W_m \end{bmatrix}
\]

Where \( V_1, \ldots, V_m \) and \( W_1, \ldots, W_m \) are orthonormal \( m \)-systems in \( \mathbb{C}^n \) and \( \mathbb{C}^m \) respectively, and with \( s_1 \geq s_2 \geq \cdots s_m \geq 0 \). Since \( 2m \leq n \), by the Gram-Schmidt orthonormalization theorem, we will have that there is an orthonormal \( m \)-system \( U_1, \ldots, U_m \in (\text{span}_\mathbb{C}\{V_1, \ldots, V_m\})^\perp \). Since \( \{v_1, \ldots, v_m\} \in \mathbb{C}^n \backslash \{0\} \) we will have that for each \( 1 \leq j \leq m, s_1 \geq s_j > 0 \). Let us set \( t_j = \sqrt{1 - (s_j/s_1)^2} \), for \( 1 \leq j \leq m \). We will have that \( 0 \leq t_j \leq 1 \) and \( (s_j/s_1)^2 + t_j^2 = 1 \) for each \( 1 \leq j \leq m \), since \( s_j/s_1 \leq 1 \) for every \( 1 \leq j \leq m \). Let us define the matrix \( \mathcal{CH}[v_1|v_m] \) by the expression.

\[
(3.37) \quad \mathcal{CH}[v_1|v_m] = \begin{bmatrix} \vert & \vert & \cdots & \vert \\
 V_1 & V_2 & \cdots & V_m \end{bmatrix} \begin{bmatrix} \vert & \vert & \cdots & \vert \\
 1 & 0 & \cdots & 0 \\
 0 & s_2/s_1 & \cdots & 0 \\
 \vdots & \vdots & \ddots & 0 \\
 0 & \cdots & 0 & s_m/s_1 \end{bmatrix} \begin{bmatrix} \vert & \vert & \cdots & \vert \\
 W_1 & W_2 & \cdots & W_m \end{bmatrix}
\]

and

\[
(3.38) \quad \mathcal{SH}[v_1|v_m] = \begin{bmatrix} \vert & \vert & \cdots & \vert \\
 U_1 & U_2 & \cdots & U_m \end{bmatrix} \begin{bmatrix} \vert & \vert & \cdots & \vert \\
 t_1 & 0 & \cdots & 0 \\
 0 & t_2 & \cdots & 0 \\
 \vdots & \vdots & \ddots & 0 \\
 0 & \cdots & 0 & t_m \end{bmatrix} \begin{bmatrix} \vert & \vert & \cdots & \vert \\
 W_1 & W_2 & \cdots & W_m \end{bmatrix},
\]

Let us define the matrix \( \hat{V} = [\hat{v}_1 \cdots \hat{v}_m] \in \mathbb{C}^{n \times n} \) by the expression.

\[
(3.39) \quad \hat{V} = \mathcal{CH}[v_1|v_m] + \mathcal{SH}[v_1|v_m]
\]

Since \( (s_j/s_1)^2 + t_j^2 = 1 \) for each \( 1 \leq j \leq m \), by (3.37) and (3.38) and by orthogonality of the \( 2m \)-system \( V_1, \ldots, V_m, U_1, \ldots, U_m \), we will have that \( \hat{V}^*\hat{V} = 1_m \). This implies that \( \hat{v}_1, \ldots, \hat{v}_m \) is an orthonormal \( m \)-system. By lemma 3.4 we have that \( C[\hat{v}_1|\hat{v}_m] \) is a unitary matrix that satisfies the constraints \( C[\hat{v}_1|\hat{v}_m] \hat{v}_j = \hat{v}_{j+1}, 1 \leq j \leq m - 1 \), and \( C[\hat{v}_1|\hat{v}_m] \hat{v}_m = \hat{v}_1 \).
Let us set.

\[ \kappa = s_1 \]
\[ \rho = \frac{1}{s_1} (v_1^* v_1) \]
\[ K = \begin{bmatrix} V_1 & V_2 & \cdots & V_m \end{bmatrix} \begin{bmatrix} V_1 & V_2 & \cdots & V_m \end{bmatrix}^* \]
\[ T = \hat{v}_1 v_1^* \]

(3.40) \[ U[v_1|v_m] = C[\hat{v}_1|\hat{v}_m] \]

Since \( \hat{V}^* \hat{V} = 1_n \) we will have that \( K^2 = \hat{V}^* \hat{V} \hat{V}^* \hat{V} = \hat{V}^* \hat{V} \) and \( K^* = (\hat{V}^* \hat{V})^* = \hat{V}^* \hat{V} = K \).

Since \( U_1, \ldots, U_m \in \langle \text{span}_C \{V_1, \ldots, V_m\} \rangle \), we will have that \( K\hat{V} = \mathbb{C} \mathbb{H}[v_1|v_m] \), by (3.37) this implies that.

\[ K\hat{e}_j = \kappa K \hat{V} \hat{e}_{j,m} = s_1 \mathbb{C} \mathbb{H}[v_1|v_m] \hat{e}_{j,m} = \mathbb{H}[v_1|v_m] \hat{e}_{j,m} = v_j \]

We will first show that \( \hat{v}_1 v_1 \neq 0 \), in fact, since \( \mathbb{H}[v_1|v_m] = s_1 \mathbb{C} \mathbb{H}[v_1|v_m] \), and \( \mathbb{C} \mathbb{H}[v_1|v_m] \mathbb{C} \mathbb{H}[v_1|v_m] = 0_n \), by orthogonality of \( V_1, \ldots, V_m, U_1, \ldots, U_m \), we will have that.

\[ \hat{v}_1^* v_1 = (\hat{V} \hat{e}_{1,m})^* \mathbb{H}[v_1|v_m] \hat{e}_{1,m} = \hat{e}_{1,m} \hat{V}^* \mathbb{H}[v_1|v_m] \hat{e}_{1,m} = \frac{1}{s_1} \hat{e}_{1,m} \mathbb{H}[v_1|v_m]^* \mathbb{H}[v_1|v_m] \hat{e}_{1,m} = \frac{1}{s_1} (\mathbb{H}[v_1|v_m] \hat{e}_{1,m})^* \mathbb{H}[v_1|v_m] \hat{e}_{1,m} = \frac{1}{s_1} v_1^* v_1 > 0 \]

(3.41)

We will also have that \( T^2 = \hat{v}_1 \hat{v}_1^* \hat{v}_1 \hat{v}_1^* = \hat{v}_1 \hat{v}_1^* = T \) and \( T^* = (\hat{v}_1 \hat{v}_1^*)^* = \hat{v}_1 \hat{v}_1^* = T \).

Since \( \hat{v}_1^* v_1 \neq 0 \), by (3.41) we have that \( Tv_1 = \hat{v}_1 (\hat{v}_1^* v_1) = \frac{1}{s_1} (v_1^* v_1) \hat{v}_1 = \rho v_1 \). This completes the proof. \( \square \)

**Lemma 3.6.** Given an orthonormal \( m \)-system \( \hat{v}_1, \ldots, \hat{v}_m \in \mathbb{C}^n \setminus \{0\} \) with \( m \leq n \).

There is \( \hat{V} \in \mathbb{C}^{n \times m} \) determined by \( \hat{v}_1, \ldots, \hat{v}_m \), such that the map \( \Pi_m = \text{Ad}[\hat{V}^*] \in CP(n,m) \) from \( Z(P[\hat{v}_1|\hat{v}_m]) \) onto \( \mathbb{C}^{m \times m} \) preserves products in \( Z(P[\hat{v}_1|\hat{v}_m]) \), with \( P[\hat{v}_1|\hat{v}_m] \) determined by (3.6). Moreover, we will have that \( \Pi_m(P[\hat{v}_1|\hat{v}_m]) = 1_m \), and \( \Pi_m(X) = \Pi_m(P[\hat{v}_1|\hat{v}_m]X) \) for any \( X \in \mathbb{C}^{n \times n} \).

**Proof.** Let us set.

(3.42) \[ \hat{V} = \sum_{j=1}^m \hat{v}_j \hat{e}_{j,m} = \begin{bmatrix} \hat{v}_1 & \hat{v}_2 & \cdots & \hat{v}_m \end{bmatrix} \]
Since \( \hat{\ell}_1, \ldots, \hat{\ell}_m \in \mathbb{C}^n \) is an orthonormal \( m \)-system, we will have that \( \hat{\mathcal{V}}^* \hat{\mathcal{V}} = \mathbf{1}_m \), by (3.6) we will have that

\[
(3.43) \quad P[\hat{\ell}_1|\hat{\ell}_m] = \sum_{j=1}^{m} \hat{\ell}_j \hat{\ell}_j^* = \hat{\mathcal{V}}^* \hat{\mathcal{V}}
\]

Since \( \hat{\mathcal{V}}^* \hat{\mathcal{V}} = \mathbf{1}_m \), by (3.43) we will have that,

\[
(3.44) \quad \Pi_m(P[\hat{\ell}_1|\hat{\ell}_m]) = \hat{\mathcal{V}}^* \hat{\mathcal{V}}^* \hat{\mathcal{V}} = \mathbf{1}_n^2 = \mathbf{1}_n
\]

we will also have that for any \( X \in \mathbb{C}^{n \times n} \).

\[
(3.45) \quad \Pi_m(P[\hat{\ell}_1|\hat{\ell}_m]X) = \hat{\mathcal{V}}^* \hat{\mathcal{V}} \Pi_m(X) \hat{\mathcal{V}}^* \hat{\mathcal{V}}
\]

By (3.43) and (3.45) we will have that for any two \( X, Y \in Z(P[\hat{\ell}_1|\hat{\ell}_m]) \).

\[
(3.46) \quad \Pi_m(XY) = \Pi_m(P[\hat{\ell}_1|\hat{\ell}_m]XY) = \Pi_m(XP[\hat{\ell}_1|\hat{\ell}_m]Y) = \hat{\mathcal{V}}^* X P[\hat{\ell}_1|\hat{\ell}_m] Y \hat{\mathcal{V}} = \hat{\mathcal{V}}^* X \hat{\mathcal{V}}^* \hat{\mathcal{V}}^* \hat{\mathcal{V}}
\]

By (3.42) and by orthonormality of \( \hat{\ell}_1, \ldots, \hat{\ell}_m \) we will also have that for each \( 1 \leq i, k \leq m \).

\[
(3.47) \quad \Pi_m(\hat{\ell}_i \hat{\ell}_k^* ) = \hat{\mathcal{V}}^* \hat{\ell}_i \hat{\ell}_k^* \hat{\mathcal{V}}^*
\]

By (3.47) we have that \( \{ \hat{\ell}_i, m \hat{\ell}_k^* \}_{i,k=1}^m \subset \Pi_m(Z(P[\hat{\ell}_1|\hat{\ell}_m])) \), since \( \Pi_m = \text{Ad}[\hat{\mathcal{V}}^*] \in CP(m, n) \) and \( \mathbb{C}^{m \times m} = \text{span}_\mathbb{C} \{ \hat{\ell}_i, m \hat{\ell}_k^* \}_{i,k=1}^m \), we have that \( \Pi_m \) is surjective. This completes proof. \( \square \)

**Definition 3.1.** Given \( v_1, \ldots, v_m \in \mathbb{C}^n \setminus \{ \mathbf{0} \} \) with \( n \geq 2m \), the matrix \( U[v_1, v_m] \) whose existence is proved in lemma 3.5 will be called a circular shift factor (CSF) for \( v_1, \ldots, v_m \).

**Lemma 3.7.** Given \( v_1, \ldots, v_m \in \mathbb{C}^n \setminus \{ \mathbf{0} \} \) with \( n \geq 2m \), there is an algebra homomorphism \( \pi_m \) from \( \mathbb{C}[U[v_1|v_m]] \) onto \( \text{Circ}(m) \).

**Proof.** By lemma 3.5 we have that there is an orthonormal \( m \)-system \( \hat{\ell}_1, \ldots, \hat{\ell}_m \in \mathbb{C}^n \) together with a CSF \( U[v_1|v_m] = C[\hat{\ell}_1|\hat{\ell}_m] \). By lemma 3.5 we have that
Definition 3.2. Given \( C[\hat{v}_1|\hat{v}_m] \in Z(P[\hat{v}_1|\hat{v}_m]) \) with \( P[\hat{v}_1|\hat{v}_m] \) determined by (3.6), this in turn implies that \( \mathbb{C}[U[v_1|v_m]] = \mathbb{C}C[\hat{v}_1|\hat{v}_m] \subseteq Z(P[\hat{v}_1|\hat{v}_m]) \).

By lemma 3.6 there is \( V \in \mathbb{C}^{n \times m} \) such that \( \Pi_m = \text{Ad}[V^*] \) is \( CP \) map from \( \mathbb{C}^{n \times m} \) onto \( \mathbb{C}^{n \times m} \) that preserves products in \( Z(P[\hat{v}_1|\hat{v}_m]) \). Let us set \( \pi_m = \Pi_m|_{\mathbb{C}[U[v_1|v_m]]} \).

It is clear that \( \pi_m \in CP(n, m) \).

By (3.45) and (3.47) we will have that.

\[
\pi_m(U[v_1|v_m]) = \Pi_m(P[\hat{v}_1|\hat{v}_m]C[\hat{v}_1|\hat{v}_m]) = \Pi_m(\hat{v}_1\hat{v}_m^* + \sum_{j=1}^{m-1} \hat{v}_{j+1}\hat{v}_j^*).
\]

(3.48)

\[
= \Pi_m(\hat{v}_1\hat{v}_m^*) + \sum_{j=1}^{m-1} \Pi_m(\hat{v}_{j+1}\hat{v}_j^*) = \hat{e}_{1,m}\hat{e}_{m,m}^* + \sum_{j=1}^{m-1} \hat{e}_{j+1,m}\hat{e}_{j,m}^*.
\]

(3.49)

\[
= \begin{bmatrix}
0_{1\times(m-1)} & 1 \\
0_{(m-1)\times1} & 1_{m-1}
\end{bmatrix} = C_m
\]

The identity (3.45) also implies that.

\[
\pi_m(1_n) = \Pi_m(1_n) = \Pi_m(P[\hat{v}_1|\hat{v}_m]) = 1_m
\]

Since \( \Pi_m \) preserves products in \( Z(P[\hat{v}_1|\hat{v}_m]) \), for any two integers \( j, k \geq 1 \) we will have that,

\[
\pi_m(U[v_1|v_m]^jU[v_1|v_m]^k) = \Pi_m(U[v_1|v_m]^jU[v_1|v_m]^k) = \Pi_m(U[v_1|v_m]^j)\Pi_m(U[v_1|v_m]^k) = \pi_m(U[v_1|v_m]^j)\pi_m(U[v_1|v_m]^k).
\]

(3.51)

and also that.

\[
\pi_m(U[v_1|v_m]^j) = \pi_m(U[v_1|v_m]^j)\pi_m(U[v_1|v_m])^j = \pi_m(U[v_1|v_m])^j = C_m^j
\]

(3.52)

By (3.50), (3.51) and (3.52), we will have that the map \( \pi_m \in CP(n, m) \) determines an algebra homomorphism from \( \mathbb{C}[U[v_1|v_m]] \) onto \( \text{Circ}(k) \).

\[
\text{Definition 3.2.} \text{ Given } v_1, \ldots, v_m \in \mathbb{C}^n \setminus \{0\} \text{ with } n \geq 2m, \text{ with corresponding CSF } U[v_1, v_m] \in \mathbb{U}(n). \text{ The algebra homomorphism } \pi_m \text{ whose existence is warranted by lemma 3.7, will be called a Circulant representation (CR) for } \mathbb{C}[U[v_1|v_m]].
\]

\[
\text{Theorem 3.1.} \text{ Given } v_1, \ldots, v_m \in \mathbb{C}^n \setminus \{0\} \text{ with } n \geq 2m, \text{ there is a projection } P \in \mathbb{C}^{m \times m} \text{ together two maps } \varphi \in CP(n, n) \text{ and } \Phi \in CP(m, m), \text{ such that the following diagram commutes,}
\]

\[
\begin{array}{ccc}
\text{Circ}(m) & \xrightarrow{\pi_m} & \mathbb{C}[U[v_1|v_m]] \\
\downarrow \Phi & & \downarrow \varphi \\
& \xrightarrow{\varphi} & Z(P)
\end{array}
\]

\[
\text{(3.53)}
\]
where \( \pi_m \) is a CR of \( \mathbb{C}[U[v_1|v_m]] \). Moreover, \( \Phi \) preserves products on \( \mathbb{C}^{n \times m} \) and \( PU[v_1|v_m] = U[v_1|v_m]P \).

Proof. By 3.5 there is an orthonormal \( m \)-system \( \dot{v}_1, \ldots, \dot{v}_m \in \mathbb{C}^n \) together with projection \( P[\dot{v}_1|\dot{v}_m] \) such that \( P[\dot{v}_1|\dot{v}_m]\dot{v}_j = \dot{v}_j \) for each \( 1 \leq j \leq m \). As a consequence of the argument implemented in the proof of lemma 3.7, we have that by lemma 3.6, there is a matrix \( \tilde{V} \in \mathbb{C}^{n \times m} \) such that \( \pi_m = Ad[\tilde{V}^*]|_{\mathbb{C}[U[v_1|v_m]]} \) and \( P[\dot{v}_1|\dot{v}_m] = \tilde{V}^* \).

Let us set

\[
\begin{cases}
    P = P[\dot{v}_1|\dot{v}_m] \\
    \varphi = Ad[P] \\
    \Phi = Ad[\tilde{V}]
\end{cases}
\]

(3.54)

It is clear that \( \varphi \in CP(n,n) \) and \( \Phi \in CP(m,n) \). Given \( X \in \mathbb{C}^{n \times n} \), we will have that

\[
\varphi(X) = P[\dot{v}_1|\dot{v}_m]XP[\dot{v}_1|\dot{v}_m] = \tilde{V}^*X\tilde{V} = \tilde{V}\Pi_m(X)\tilde{V}^* = \Phi(\Pi_m(X))
\]

(3.55)

Since \( \tilde{V}^* = 1_m \) we will have that for any two \( X,Y \in \mathbb{C}^{n \times m} \).

\[
\Phi(XY) = \tilde{V}XY\tilde{V}^* = \tilde{V}X\tilde{V}^*Y = \Phi(X)\Phi(Y)
\]

(3.56)

Since \( P = P[\dot{v}_1|\dot{v}_m] \) is a projection, for any \( X \in \mathbb{C}^{n \times n} \), we will have that \( P\varphi(X) = P^2X = PX = \varphi(X)P \). This implies that \( \varphi(\mathbb{C}^{n \times n}) \subseteq Z(P) \).

By (3.2) we have that \( PU[v_1|v_m] = P[\dot{v}_1|\dot{v}_m]C[\dot{v}_1|\dot{v}_m] = C[\dot{v}_1|\dot{v}_m]P[\dot{v}_1|\dot{v}_m] = U[v_1|v_m]P \).

By (3.55) we will have that \( \varphi \) has a representation of the form

\[
\varphi = \Phi \circ \Pi_m
\]

(3.57)

By (3.57) we will have that

\[
\varphi|_{\mathbb{C}[U[v_1|v_m]]} = \Phi \circ \Pi_m|_{\mathbb{C}[U[v_1|v_m]]} = \Phi \circ \pi_m
\]

(3.58)

This completes the proof. \( \square \)

Given a discrete-time dynamical system \( (\Sigma, \{\Theta_t\}) \) with \( \Sigma \subseteq \mathbb{C}^{n \times n} \), and a \( m \)-system of (history) vectors \( \mathbb{H}[\Sigma, m] = \{v_1, \ldots, v_m\} \) for \( (\Sigma, \{\Theta_t\}) \), the matrix \( \mathbb{V} \in \mathbb{C}^{n \times m} \) defined by the formula (3.39) for \( \{v_1, \ldots, v_m\} \), will be called the orthonormal history factor (OHF) of \( \mathbb{H}[\Sigma, m] \).

**Theorem 3.2.** Given \( \varepsilon > 0 \), an integer \( T > 0 \), and a discrete-time \( T \)-periodic dynamical system \( (\Sigma, \{\Theta_t\}) \) with \( \Sigma \subseteq \mathbb{C}^{n \times n} \), then \( (\Sigma, \{\Theta_t\}, \mathbb{H}[\Sigma, m]) \) is \((S^1, T, \varepsilon)\)-controlled, for every \( \varepsilon > 0 \) and each \( m \in \mathbb{Z} \) such that \( 2T \leq 2m \leq n \).

Proof. Given \( \varepsilon > 0 \), and any vector history \( \mathbb{H}[\Sigma, m] = \{v_1, \ldots, v_m\} \subseteq \mathbb{C}^n \) for \( (\Sigma, \{\Theta_t\}) \), since \( 2T \leq 2m \leq n \), we can apply lemma 3.5 to compute an orthonormal \( T \)-system \( \{\hat{v}_1, \ldots, \hat{v}_T\} \in \mathbb{C}^n \), scalars \( \kappa, \rho \), a CSF \( U[v_1|v_T] \in \mathbb{C}^{n \times n} \) and two projections \( \hat{K}, \hat{T} \in \mathbb{C}^{n \times n} \), that satisfy (3.34).

By theorem 3.1 we will have that there are a projection \( P \in \mathbb{C}^{n \times n}, \varphi \in CP(n,n) \) and a product preserver \( \Phi \in CP(T,n) \) such that \( \varphi = \Phi \circ \pi_T \) and \( \varphi|_{\mathbb{C}[U[v_1|v_T]]} \subseteq Z(P) \).

Since \( \pi_T \) and \( \Phi \) are linear product preservers in \( \mathbb{C}[U[v_1|v_T]] \) and \( CT \times T \), respectively, we will have that for any \( p \in \mathbb{C}[z] \).
(3.59) \[ \varphi(p(U[v_1|v_T])) = \Phi(\pi_m(p(U[v_1|v_T]))) = p(\Phi(\pi_m(U[v_1|v_T]))) = p(\varphi(U[v_1|v_T])) \]

By theorem 3.1 we will have that,
\begin{equation}
(3.60) \quad ||\varphi(X) - \Phi \circ \pi_T(p(U[v_1|v_T]))|| = 0 < \varepsilon
\end{equation}

for each \( X \in \mathbb{C}[U[v_1|v_T]] \).

By (3.54) we have that \( P = P[\hat{v}_1|\hat{v}_T] \), with \( P[\hat{v}_1|\hat{v}_T] \) defined by equation (3.6), by lemma 3.1 we will have that,
\begin{equation}
(3.61) \quad P\hat{v}_1 = P[\hat{v}_1|\hat{v}_T]\hat{v}_1 = \hat{v}_1
\end{equation}

Let \( p_k(z) = \frac{z}{p} z^k \), \( 0 \leq k \leq m - 1 \), this implies that \( p_k \in \mathbb{C}[z] \). By lemma 3.5 and by (3.61), for each \( 0 \leq k \leq m - 1 \) we will have that,
\begin{align}
\hat{K} \varphi(p_k(U[v_1|v_T]))\hat{T}v_1 &= \hat{K} \frac{K}{\rho} U[v_1|v_T]^k \rho \hat{v}_1 \\
&= \hat{K} K U[v_1|v_T]^k \hat{v}_1 \\
&= \hat{K} K^{\ell} \hat{v}_{1+k} = \Theta_k(v_1)
\end{align}

By (3.60) and (3.62), we will have that \( (\Sigma, \{\Theta_i\}, \mathbb{H}[\Sigma, m]) \) is \((\mathbb{S}^1, T, \varepsilon)\)-controlled by \((\mathbb{S}^1, U[v_1|v_T], \hat{K}, T, \varphi, \{p_1\})\).

**Theorem 3.3.** Given \( \varepsilon > 0 \), an integer \( T > 0 \), and a discrete-time \( T \)-periodic dynamical system \( (\Sigma, \{\Theta_i\}) \) with \( \Sigma \subseteq \mathbb{C}^{n \times n} \), then \( (\Sigma, \{\Theta_i\}, \mathbb{H}[\Sigma, m]) \) is \((\mathbb{Z}/T, \varepsilon)\)-controlled, for every \( \varepsilon > 0 \) and each \( m \in \mathbb{Z} \) such that \( 2T \leq 2m \leq n \).

**Proof.** Given \( \varepsilon > 0 \), by theorem 3.2, we will have that \( (\Sigma, \{\Theta_i\}, \mathbb{H}[\tilde{\Sigma}, m]) \) is \((\mathbb{S}^1, T, \varepsilon)\)-controlled by some control \((\mathbb{S}^1, U[v_1|v_T], \hat{K}, T, \varphi, \{p_1\})\).

We have that theorem 3.2 also implies that there is an algebra homomorphism \( \Phi: \mathbb{C}^{n \times m} \to \mathbb{C}^{n \times n} \) such that \( ||\varphi(X) - \Phi \circ \pi_T(X)|| \leq \varepsilon \), for each \( X \in \mathbb{C}[U[v_1|v_T]] \).

Since \( U[v_1|v_T] \in \mathbb{U}(m) \), we will have that \( C_T = \pi_m(U[v_1|v_T]) \in \mathbb{U}(m) \) and also that,
\[ C_T^{-} = \pi_T(U[v_1|v_m])^{-} = \pi_T(U[v_1|v_m]) = \pi_T(1_m) = 1_m \]

By universality of \( \mathbb{C}[\mathbb{Z}/T] \) there is a representation \( \rho_T: \mathbb{C}[\mathbb{Z}/T] \to \mathbb{U}(T) \), determined by the assignment \( \hat{1} \mapsto C_T \) for \( \hat{1} \in \mathbb{Z}/T \).

Applying universality of \( \mathbb{C}[\mathbb{Z}/T] \) to \( \pi_T(\mathbb{C}[U[v_1|v_T]]) \) we have that,
\[ \pi_T(p_t(U[v_1|v_T])) = p_t(\pi_T(U[v_1|v_T])) = p_t(C_T) = p_t(\rho_T(\hat{1})) = \rho_T(p_t(\hat{1})) \]

for each \( 0 \leq t \leq m - 1 \). This completes the proof.

**Theorem 3.4.** Given \( \delta, \varepsilon > 0 \), every \((T, \delta, \varepsilon)\)-almost-periodic dynamical system \((\Sigma, \{\Theta_i\}) \) with \( \Sigma \subseteq \mathbb{C}^n \) is \((\mathbb{S}, T, \varepsilon)\)-controlled, whenever \( 2T \leq 2m \leq n \).

**Proof.** Let \( x \in \Sigma \). Since \((\Sigma, \{\Theta_i\}) \) is \((T, \delta, \varepsilon)\)-almost-periodic, we will have that there is a discrete-time \( T \)-periodical dynamical system \((\tilde{\Sigma}, \tilde{\Theta}_i) \) such that there is \( \tilde{x} \in \tilde{\Sigma} \) such that \( ||x - \tilde{x}|| \leq \delta \), and \( ||\Theta_i(x) - \tilde{\Theta}_i(\tilde{x})||_2 \leq \varepsilon \) for each \( t \in \mathbb{Z}^+_0 \) and every \( (x, \tilde{x}) \in \Sigma \times \tilde{\Sigma} \).
By theorem 3.2 we will have that \((\hat{\Sigma}, \{\hat{\Theta}_t\}, \mathbb{H}[\hat{\Sigma}, T])\) is \((S^1, T, 0)\)-controlled by some control \((S^1, U[v_1|v_T], K, \hat{T}, \varphi, \{p_t\})\), for some \(T\)-system of history vectors \(\mathbb{H}[\hat{\Sigma}, T] = \{v_1, \ldots, v_T\}\) with \(v_1 = \hat{x}\), this implies that for each \(0 \leq t \leq T\).

\[
\|\Theta_t(x) - \hat{\Theta}_t(\hat{x})\|_2 \leq \|\Theta_t(x) - \hat{\Theta}_t(\hat{x})\|_2 + \|\hat{\Theta}_t(\hat{x}) - \hat{\Theta}_t(\hat{x})\|_2 \leq \varepsilon
\]

(3.63)

By (3.63) we have that \((\Sigma, \{\Theta_t\})\) is \((S^1, T, \varepsilon)\)-controlled by \((S^1, U[v_1|v_T], K, \hat{T}, \varphi, \{p_t\})\). This completes the proof. \(\square\)

**Theorem 3.5.** Given \(\delta, \varepsilon > 0\), every \((T, \delta, \varepsilon)\)-almost-periodic dynamical system \((\Sigma, \{\Theta_t\})\) with \(\Sigma \subseteq \mathbb{C}^n\) is \((\mathbb{Z}/T, \varepsilon)\)-controlled, whenever \(2T \leq n\).

**Proof.** Let \(x \in \Sigma\). Since \((\Sigma, \{\Theta_t\})\) is \((T, \delta, \varepsilon)\)-almost-periodic, we will have that there is a discrete-time \(T\)-periodic dynamical system \((\hat{\Sigma}, \hat{\Theta}_t)\) such that there is \(\hat{x} \in \hat{\Sigma}\) such that \(\|x - \hat{x}\| \leq \delta\), and \(\|\Theta_t(x) - \hat{\Theta}_t(\hat{x})\|_2 \leq \varepsilon\) for each \(t \in \mathbb{Z}_0^\delta\) and every \((x, \hat{x}) \in \Sigma \times \hat{\Sigma}\).

By theorem 3.3 we will have that \((\hat{\Sigma}, \{\hat{\Theta}_t\}, \mathbb{H}[\Sigma, T])\) is \((\mathbb{Z}/T, 0)\)-controlled, this implies that for some \(T\)-system of history vectors \(\mathbb{H}[\Sigma, T] = \{v_1, \ldots, v_T\}\) with \(v_1 = \hat{x}\), there is a unitary representation \(\rho_T : \mathbb{C}[\mathbb{Z}/T] \rightarrow \mathbb{C}^{T \times T}\), an algebra homomorphism \(\varphi : \mathbb{C}^{T \times T} \rightarrow \mathbb{C}^{n \times n}\), a family of functions \(\{f_0, \ldots, f_T-1\}\), and two projections \(\hat{K}, \hat{T} \in \mathbb{C}^{n \times n}\) such that for \(1 \in \mathbb{Z}/T\) and for each \(t \geq 0\).

\[
\|\Theta_t(x) - \hat{\Theta}_t(\hat{x})\|_2 \leq \|\Theta_t(x) - \hat{\Theta}_t(\hat{x})\|_2 + \|\hat{\Theta}_t(\hat{x}) - \hat{\Theta}_t(\hat{x})\|_2 \leq \varepsilon
\]

(3.64)

By (3.63) we have that \((\Sigma, \{\Theta_t\})\) is \((\mathbb{Z}/T, \varepsilon)\)-controlled. This completes the proof. \(\square\)
4. Algorithm

The techniques and computations used to prove the previous results, can be implemented to derive a prototypical algorithm described by the following block diagram.

![Block Diagram](image)

(4.1)

The proofs of lemma 3.5 and theorem 3.2 provide a computational procedure that is sketched in algorithm 1, and can be used to compute the elements $F_t \in Z_{T-1}$ in diagram (4.1), where each matrix $F_t$ has a representation $F_t = \hat{K}\hat{H}_t\hat{T}$, for some matrices $\hat{K}, \hat{H}_t, \hat{T}$ to be determined by algorithm 1.

**Algorithm 1** Data-driven matrix approximation algorithm

**Data:** Real number $\varepsilon > 0$, State History $H[\Sigma, m]: \{x_t\}_{0 \leq t \leq m \leq T}, T \in Z^+$

**Result:** Approximate matrix realizations: $(\hat{K}, \hat{H}_t, \hat{T}) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}$ of $(\hat{\Sigma}, \{\Theta_t\})$

1. Compute state/output sampled-data history $\{v_t\}_{0 \leq t \leq T}$ of $(\Sigma, \{\Theta_t\})$
2. Compute the SVD $VSV^* = \left[v_1 \cdots v_m\right]$ of $\{v_t\}_{0 \leq t \leq m \leq T}$
3. Compute the OHF $\hat{V} = \hat{V}_1 \cdots \hat{V}_m$ for $\{v_t\}_{0 \leq t \leq m \leq T}$
4. Set $\hat{K} = VV^*$
5. Set $\hat{T} = \hat{v}_1\hat{v}_1^*$
6. For $0 \leq t \leq T - 1$:
   a. Compute $p_t \in \mathbb{C}[z]$ such that:
      i. $\|\hat{K}\hat{V}p_t(C_m)\hat{V}^*\hat{T}v_1 - \Theta_t[v_1]\| \leq \varepsilon$
   b. Set $\hat{H}_t = \hat{V}p_t(C_m)\hat{V}^*$

return $\{\hat{K}, \hat{H}_t, \hat{T}(t)\}_{0 \leq t \leq T}$

5. Experimental results

5.1. Materials and Methods. In order to solve the diagram (4.1), a prototypical GNU Octave code that implements some of the core computations on which the proofs of lemma 3.5 and theorem 3.2 are based, has been developed as part of this project, using GNU Octave 4.4.1 on a five node Ubuntu Linux Beowulf Cluster, at the Scientific Computing Innovation Center of UNAH-CU.
5.2. **Numerical Experiments.** We will consider three numerical experiments in this section. In §5.2.1 we will simulate 1D waves, in §5.2.2 we will simulate damped waves on planar material sections, and in §5.2.3 we will simulate vorticity transport.

In §5.2.1 the periodicity appears naturally, while in §5.2.2 and §5.2.3, we will think of the model as a movie, that is being streamed more than once. Each example will be used to illustrate the potential applications of topological control to system identification, and for extraction of (almost) periodic patterns from sampled-data discrete-time industrial systems and plants.

5.2.1. **1D Waves.** As a first application, let us consider a wave equation under Dirichlet boundary conditions of the form:

\[
\begin{align*}
\frac{\partial^2 w}{\partial t^2} - c^2 \frac{\partial}{\partial x} x^2 w &= 0, \\
\frac{w(0, t)}{w(L, t)} &= 0, \\
\frac{w(x, 0)}{w_0(x)} &= 0
\end{align*}
\]

for some suitable data of \(L, c, w_0\).

The simulation is computed using second order finite difference method combined with second order Crank-Nicolson method. The computation is performed using the following commands.

```matlab
>> [x,t,data_wave,Cx,Sx]=CL_ROM WaveDS([0 1],10);
```

The graphical output is shown in fig. 1.

5.2.2. **Cantilever Elastic Plate.** As a second application of algorithm 1, let us consider a computational mechanics problem consisting on the description of the deformation a damped aluminium Cantilever Lamé beam model under planar displacement hypotheses, whose deformation displacement vector \(v\) is described by a Navier dynamical system of the form:

\[
\begin{align*}
\rho \frac{\partial^2 v}{\partial t^2} - N(\lambda, \mu)v &= u(t), \\
BIC(v) &= 0,
\end{align*}
\]

where \(N(\lambda, \mu)\) is the Navier operator defined by the expression,

\[
N(\lambda, \mu) = (\lambda + \mu)\nabla \cdot + \mu \nabla^2
\]

where \(\lambda, \mu\) are the Lamé’s coefficients for generic aluminium, and where \(BIC(v) = 0\) is some system of equations that determines suitable boundary and initial conditions for Cantilever Lamé beam deflection. We will have that the input \(u(t)\) is determined by the expression \(u(t) = c(t)\rho \partial_t v\) for some smooth time dependent coefficient \(c(t)\).

It is important to consider that the dimensions of the beam model have been normalized, and that relative deformation displacement scale is exaggerated for visualization purposes of the corresponding simulation.

In order to create the data corresponding to the beam deformation, we use an Octave m-file function that computes a second order finite difference approximation of the Navier dynamical system (5.2) for sample sizes 50, 30, 10 and 5, as follows.
Figure 1. $\Sigma$ and $\hat{\Sigma}$ output histories

```matlab
>> [Bx,By,data_x,data_y,Yx,Yy]=CL_ROM_BeamDS(1,50,40,-5e9,-10,1);
>> [Bx,By,data_x,data_y,Yx,Yy]=CL_ROM_BeamDS(1,30,40,-5e9,-10,2);
>> [Bx,By,data_x,data_y,Yx,Yy]=CL_ROM_BeamDS(1,10,40,-5e9,-10,3);
>> [Bx,By,data_x,data_y,Yx,Yy]=CL_ROM_BeamDS(1,5,40,-5e9,-10,4);
```

Running simulation:

```
properties =
scalar structure containing the fields:

Emod = 73100000000
```

properties =

```
scalar structure containing the fields:

Emod = 73100000000
Nu = 0.33000
Lx = 4
Ly = 0.40000
T = 40
M = 600
N = 36
```
c2 = 0.35971
delta = -5000000000
m = 5

Elapsed time is 298.816 seconds.
-------------------------------------------------------------
Computing circular matrix representations in $C[U[v1|vm]]$:
-------------------------------------------------------------
Elapsed time is 3.51714 seconds.
-------------------------------------------------------------
Verifying circular mimetic constraints for $C[U[v1|vm]]$:
-------------------------------------------------------------
Verification passed...
max{||Kx Rx Usx^k Tx Ux0-Oxk|| | 1<=k<=m} = 2.7082e-27 <= eps
max{||Ky Ry Usy^k Ty Uy0-Oyk|| | 1<=k<=m} = 7.4542e-13 <= eps
-------------------------------------------------------------
For m = 121
For n = 13357
For eps = 7.4542e-13
Elapsed time is 20.7964 seconds.
-------------------------------------------------------------

This produces the original output history shown in Figure 2.

![Original Output History](image)

**Figure 2.** Original model Σ output history

The output histories of the corresponding SCL-ROM approximants are shown in Figure 3.
5.2.3. Planar vorticity transport. As a third application of local $S^1$-control, let us consider the vorticity transport PDE system of the form:

$$
\begin{align*}
\partial_t \omega &= -u \partial_x \omega - v \partial_y \omega + \frac{1}{Re} \Delta \omega \\
u &= \partial_y \psi, \\
v &= -\partial_x \psi \\
\Delta \psi &= -\omega \\
BIC(\omega, \psi, u, v) &= 0
\end{align*}
$$

For Reynolds number $Re = 400$ and suitable boundary and initial conditions represented by the system of algebraic differential equations $BIC$.

The simulation is computed using second order finite difference method combined with fourth order Runge-Kutta method. The computation is performed using the following commands.

```matlab
>> [x,y,data,Cx,Sx]=CL_ROM_VortexDS(1,10,[0,0.15],[100,750],[1,3]);
```

Running simulation:

```
Computing circular matrix representations in C[U[v1|vm]]:
```

Elapsed time is 2.08992 seconds.

Verifying circular mimetic constraints for C[U[v1|vm]]:

```
max{||Kx Rx Ux-k Tx X0-Yk|| | 1<=k<m} = 8.2096e-13 <= eps
```

For $m = 76$
For n = 34501
For eps = 8.2096e-13
Elapsed time is 111.598 seconds.

The graphical output is shown in fig. 4.

Figure 4. $\Sigma$ and $\hat{\Sigma}$ output histories

6. Discussion

From a topological perspective, the notion of topological control that we propose in this document can be seen as an extension of the Torus Trick presented by R. Kirby in [3] to algebraic matrix sets in the sense of [7]. This extension and the corresponding matrix computations, were partially inspired by some questions raised by M. H. Freedman along the lines of [2].

In order to perform the previously mentioned computations, we start embedding the vector history of a given discrete-time system under study into a manifold, where we can then use elementary tools from matrix analysis and representation theory, to compute matrix analogies of the surgical cuts corresponding to Kirby’s torus trick, these matrix surgical cuts have a direct effect on the spectrum of the CSF corresponding to the vector history of the corresponding embedded system.

Once we perform the previously mentioned surgical cuts on the spectrum of the unitary matrices that model the dynamical behaviour of the embedded system under study, the computation of the transition matrices of the embedded system
can be easily and efficiently computed in terms of the topologically pre-processed matrices.

Another interesting effect of the aforementioned matrix surgical procedures, consists on the reduction of the group action that determines the global behaviour of the system, to a finite group action that can be efficiently computed using algorithm 1, without aditional computational cost due to the additional liftings in the original state (matrix realization) space, whose large dimension can make the standard lifting impossible to compute. This approach was inspired by the work of M. Rieffel in [5], and will be the subject of further study.

6.1. Forthcoming Research. We will further explore the numerical solvability of (3.4) together with some additional contraints.

We will improve the computational implementation of the prototypical algorithm 1, extending the topological control techniques presente in this document to higher dimensional compact manifolds like $S^1 \times S^1$, $S^n$, and so forth.

We will implement AI tools like TensorFlow to take advantage of the model training capabilities predicted by the proofs of theorem 3.4 and theorem 3.5.

Besides setting the bases for algorithm 1, the family $\{f_0, f_2, \ldots, f_T\} \subset \mathbb{C}[z]$ whose existence and computability is guaranteed by theorem 3.5, provides a natural way to compress the mean dynamical behaviour of a given discrete-time system $(\Sigma, \{\Theta_t\})$.

The information compression property of $\{f_0, f_2, \ldots, f_T\}$ provides a natural connection to video streaming, this connection will be the subject of further study and experimentation. We will also explore further connections to classical and quantum finite automata.

7. Conclusions

Given $\varepsilon, \delta > 0$, and a state $X_t$ of of discrete time almost periodic system $(\Sigma, \{\Theta_t\})$ that is $\varepsilon$-approximated by a SCL-ROM $\tilde{\Sigma}$ determined by a topological control $(M, Z, K, T, \varphi, \{f_t\})$ of $(\Sigma, \{\Theta_t\})$, the learning cost of a model update does not exceed the solving cost of the problem:

$$\arg \min_{p \in \mathbb{C}[z]} \|X_t - KP(Z)TX_0\| \mid \deg(p) \leq k$$

for some $X_0 \in \Sigma$, where $k$ is the control order. The application of this training technique to the extraction of (almost) periodic patterns from sampled-data discrete-time industrial systems and plants, will be further explored.

The family $\{f_1, f_2, \ldots, f_T\} \subset C[z]$ derived from the implementation of algorithm 1 provides an effective way to compress the mean dynamical behaviour of a given discrete-time sampled-data system $\Sigma$.

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