Monopoles and Gluon Fields in QCD in the Maximally Abelian Gauge

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Abstract

We study monopoles and gluon fields in QCD in the maximally abelian (MA) gauge in the context of the dual superconductor picture for confinement. In the abelian gauge, unit-charge magnetic monopoles appear, but multi-charge monopoles do not in general cases. The appearance of the monopole is studied using the gauge-connection formalism in relation to the SU($N_c$) singular gauge transformation. The relevant role of off-diagonal gluons is found for the appearance of monopoles in the abelian gauge in QCD. We study the gluon-field properties around the monopole in the MA gauge in terms of the action density using the lattice QCD. The monopole provides infinitely large field fluctuations in the abelian sector. In the MA gauge, off-diagonal gluons are strongly suppressed but largely remain around the monopole, which indicates the effective size and the structure of monopoles. We find the large cancellation between the abelian part and the off-diagonal part of the action density around the monopole in the MA gauge. Owing to this cancellation, the monopole can appear in QCD without large cost of the QCD action. Finally, we generalize the framework of the abelian projection, i.e. the extraction of the abelian gauge manifold from QCD, by introducing the ‘gluonic Higgs field’ $\vec{\phi}_D[A_\mu(x)]$ defined from the SU($N_c$) covariant derivative $\hat{D}_\mu$. By way of $\vec{\phi}_D[A_\mu(x)]$, the maximally abelian projection can be performed in the gauge-covariant manner without the notion of gauge fixing in principle.

Key words: QCD, Monopole, Maximally abelian gauge, Abelian projection

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1 Monopoles and Confinement in QCD

In the classical and quantum field theories [1,2], there occasionally appears the topological object as the interesting collective degrees of freedom, reflecting the nontrivial topology of the fiber-bundle. For instance, the Abrikosov vortex [3] is experimentally observed in the type-II superconductor, and the instanton [4,5] in the Euclidean Yang-Mills theory is observed in the lattice QCD simulation using the cooling method[6,7]. The magnetic monopole [8,9] is also the interesting topological object predicted in the grand unified theory (GUT). Here, the magnetic monopole was firstly introduced by Dirac more than 50 years ago from the consideration of the duality of the Maxwell equation, and the Dirac monopole [10] can naturally explain the electric-charge quantization. However, the Dirac monopole cannot be an extended object so as to make the Dirac string invisible, and such a point-like monopole is not allowed in QED, because it provides the divergence of the QED action. In 1974, however, the magnetic monopole was well formulated as the ’t Hooft-Polyakov monopole[8,9], which is the topological object in the nonabelian Higgs theory with the compact and at most semi-simple group.

Also in the $N = 2$ supersymmetric (SUSY) QCD, the soliton-like monopole is recognized as an essential degrees of freedom in the strong-coupling region, and its condensation in the infrared region provides the dual-superconductor picture for confinement in SUSY-QCD[11]. As for QCD, however, it seems difficult to introduce the (color-)magnetic monopole because of the absence of the Higgs field. Nevertheless, the introduction of the monopole degrees of freedom is desired for the physical interpretation of the confinement phenomena in QCD.

In 1970’s, Nambu, ’t Hooft and Mandelstam proposed an interesting idea of the electric confinement by magnetic-monopole condensation, and tried the physical interpretation of quark confinement using the dual version of the superconductivity[12–14]. In the ordinary superconductor, Cooper-pair condensation leads to the Meissner effect, and the magnetic flux is excluded or squeezed like a quasi-one-dimensional tube as the Abrikosov vortex. On the other hand, from the Regge trajectory of hadrons and the lattice QCD[15], the confinement force between the color-electric charge is characterized by the universal physical quantity of the string tension $\sigma \simeq 1\text{GeV}/\text{fm}$, and is brought by one-dimensional squeezing of the color-electric flux [16] in the QCD vacuum. Hence, from the above similarity on the one-dimensional flux squeezing, the QCD vacuum was regarded as the dual version of the superconductor. In this dual-superconductor picture for the QCD vacuum, the squeezing of the color-electric flux between quarks is realized by the dual Meissner effect as the result of condensation of color-magnetic monopoles. However, there are two large gaps between QCD and the dual superconductor picture.
(1) This picture is based on the abelian gauge theory subject to the Maxwell-type equations, where electro-magnetic duality is manifest, while QCD is a nonabelian gauge theory, described with the electric variables (quarks and gluons).

(2) The dual-superconductor scenario requires condensation of magnetic monopoles as key concept, while QCD does not have such a monopole as the elementary degrees of freedom.

As the connection between QCD and the dual superconductor scenario, ’t Hooft proposed concept of the abelian gauge fixing[17], the partial gauge fixing which is defined by diagonalizing a suitable gauge-dependent variable as $\phi[A_\mu(x)]$. The abelian gauge fixing reduces QCD into an abelian gauge theory, where the off-diagonal element of the gluon field behaves as a charged matter field. As a remarkable fact in the abelian gauge, color-magnetic monopoles appear as topological objects corresponding to the nontrivial homotopy group $\Pi_2(SU(N_c)/U(1)^{N_c-1}) = \mathbb{Z}_{N_c-1}^\infty$. Here, assuming abelian dominance [18], which means that the only abelian gauge fields with monopoles would be essential for the description of the nonperturbative QCD, the off-diagonal gluon elements are dropped off, which is called the abelian projection. Thus, by the abelian gauge fixing and the abelian projection, QCD is reduced into abelian projected QCD (AP-QCD), which is an abelian gauge theory including monopoles. If the monopole condenses, the scenario of color confinement by the dual Meissner effect would be a realistic picture for confinement in QCD [19–27].

Recent lattice QCD simulations show strong evidence on this dual Higgs theory for the nonperturbative QCD in the maximally abelian (MA) gauge[28,29]. The MA gauge is the abelian gauge where the off-diagonal gluon is minimized by the gauge transformation. In the MA gauge, the physical information of the gauge configuration is concentrated into the diagonal components as well as possible. The lattice QCD studies indicate abelian dominance [18,30,31] that the string tension[32–34] and the chiral condensate [35,36] are almost described only by abelian variables in the MA gauge [37,38]. In the lattice QCD, monopole dominance is also observed such that only the monopole part in the abelian variable contributes to the nonperturbative QCD in the MA gauge [34,35]. Thus, the lattice QCD studies also suggest the dominant role of abelian variables and monopoles for the nonperturbative QCD in the MA gauge.

In the MA gauge, AP-QCD neglecting the off-diagonal gluon component almost reproduces the essence of the nonperturbative QCD, although AP-QCD is an abelian gauge theory like QED. One may speculate that the strong-coupling nature leads to the similarity between AP-QCD and QCD, because the gauge coupling $e$ in AP-QCD [27] is the same as that in QCD in the lattice simulation. However, the strong-coupling nature would not be enough to explain the nonperturbative feature, because, if monopoles are eliminated from
AP-QCD, nonperturbative features are lost in the remaining system called as the photon part, although the gauge coupling $e$ is the same as that in QCD.

For further understanding, we compare the theoretical structure of QCD, AP-QCD, and QED in terms of the gauge symmetry and the relevant degrees of freedom, as shown in Fig.1. As for the interaction, the linear confinement potential arises both in QCD and both in AP-QCD, while only the Coulomb potential appears in QED. On the symmetry, QCD has nonabelian gauge symmetry, while both AP-QCD and QED have abelian gauge symmetry. The obvious difference between QCD and QED is existence of off-diagonal gluons. On the other hand, the difference between AP-QCD and QED is existence of the monopole, since the magnetic monopole does not exist in QED because of the Bianchi identity. This indicates the close relation between monopoles and off-diagonal gluons. In particular, off-diagonal gluon components play a crucial role for existence of the monopole in QCD as shown below, and the monopole itself is expected to play an alternative role of off-diagonal gluons for the confinement.

Here, we consider what is the QCD-monopole in comparison with the Cooper pair in the superconductivity. In the field theoretical aspect, the essence of the superconductivity is understood as the ordinary Higgs mechanism by Cooper-pair condensation, although the underlying electron-phonon interaction plays relevant role for the creation of the Cooper pair. The composite Cooper-pair field plays the role of the Higgs field and is the essential degrees of freedom for the superconductivity. Similarly, the monopole field to be condensed in the nonperturbative QCD vacuum can be regarded as a kind of composite or collective degrees of freedom relevant for the nonperturbative phenomena in QCD, since QCD includes quarks and gluons as the elementary field only. Different from the simple compositicy of the Cooper pair, the QCD-monopole appears as a topological object relating to the singularity of the gauge field in the abelian gauge, and would be described as a complicated field composed by gluons. Then, the feature of the structure of the monopole in QCD is to be clarified using the field-theoretical framework including the lattice QCD.

In this paper, we study the properties of the monopole appearing in QCD in the MA gauge in terms of the gluon field around it both in the analytical framework and in the lattice QCD. In section 2, we study the general argument of the abelian gauge fixing considering the global Weyl transformation. In the abelian gauge, the monopole appears from the hedgehog configuration of the gluonic Higgs field through the SU($N_c$) singular gauge transformation. We clarify the appearance of monopoles in terms of the gauge connection with respect to the singularity of the SU($N_c$) gauge transformation. We extract the abelian gauge field and the monopole current in the lattice formalism. In section 3, we study the MA gauge fixing in detail in terms of the abelian projection rate, and propose the transparent definition of the MA gauge using
the covariant derivative. The generalization of the MA gauge is also attempted. In section 4, we examine the gluon properties around the monopoles in the MA gauge using the lattice QCD simulation, with considering the role of the off-diagonal gluons. In section 5, we introduce the 'gluonic Higgs field' extracted from the SU(Nc) gauge connection $\hat{D}_\mu$, and formulate the abelian projection in the gauge-covariant manner without explicit use of gauge fixing. Section 6 is devoted to summary and concluding remarks.

2 Appearance of Monopoles in the Abelian Gauge

2.1 Abelian Gauge Fixing and Residual Symmetries

The dual superconductor picture for confinement phenomena is based on the abelian gauge theory including monopoles, and the 't Hooft abelian gauge fixing[17] is key concept for the connection from QCD to such an abelian gauge theory. In this section, we study the abelian gauge fixing considering the role of the global Weyl symmetry and show the appearance of monopoles in the singular SU(Nc) gauge transformation.

The abelian gauge fixing is the partial gauge fixing which remains the abelian gauge symmetry, and is realized by the diagonalization of an SU(Nc) gauge-dependent variable as $\phi[A_\mu(x)] = \phi^a T^a \in su(Nc)$ using the SU(Nc) gauge transformation. In the abelian gauge, $\phi[A_\mu(x)]$ plays the similar role of the Higgs field [39] in the determination of the gauge fixing, and then we call $\phi[A_\mu(x)]$ as the 'gluonic Higgs field'.

Without loss of generality, we consider the case of the hermite variable $\phi[A_\mu(x)]$ which obeys the adjoint gauge transformation. Then, $\phi(x)$ is transformed as

\[ \phi(x) = \phi^a(x) T^a \rightarrow \phi^\Omega(x) = \Omega(x) \phi(x) \Omega^\dagger(x) \]
\[ \equiv \text{diag}(\lambda^1(x), \cdots, \lambda^{Nc}(x)) \equiv H \cdot \tilde{\lambda}(x) \tag{1} \]

using a suitable SU(Nc) gauge function $\Omega(x) = \exp\{i\xi^a(x) T^a\} \in SU(Nc)$. Here, each diagonal component $\lambda^i(x) (i=1, \cdots, Nc)$ is to be real for the hermite variable $\phi[A_\mu(x)]$. The space-time point $x$ satisfying $\lambda^i(x) = \lambda^j(x)$ is called as the ‘degeneracy point’ and reflects the singular structure of $\phi(x)$. Particularly for the SU(2) case, one finds

\[ \phi(x) \equiv \phi^a(x) \frac{\tau^a}{2} \rightarrow \phi^\Omega(x) = \Omega(x) \phi^\dagger(x) = \lambda(x) \frac{\tau^3}{2} \tag{2} \]
with $\lambda(x) = \pm \left\{ \phi^1(x)^2 + \phi^2(x)^2 + \phi^3(x)^2 \right\}^{1/2}$. In the abelian gauge, the SU($N_c$) gauge symmetry is reduced into the $U(1)^{N_c-1}$ gauge symmetry, which corresponds to the gauge-fixing ambiguity. The variable $\phi(x)$ is diagonalized to $\vec{H} \cdot \vec{\lambda}(x)$ also by the gauge function $\Omega(x)$ with $\omega(x) = e^{-i \vec{H} \cdot \vec{\phi}} \in U(1)^{N_c-1}$.

$$
\omega(x) \rightarrow \Omega_{\omega}(x) \omega(x) \Omega_{\omega}^\dagger(x) = \omega(x) \vec{H} \cdot \vec{\lambda}(x) \omega_{\omega}^\dagger(x) = \vec{H} \cdot \vec{\lambda}(x),
$$

and therefore $U(1)^{N_c-1}$ abelian gauge symmetry remains in the abelian gauge. Hence, the diagonal and off-diagonal gluon components play the different role in the abelian gauge; the diagonal gluon remains to be the abelian gauge field, while the off-diagonal gluon behaves as the charged matter like $W_\pm^\mu$ in the Standard Model. In the continuum theory, the abelian projection, the extraction of the abelian gauge manifold, is defined by the simple replacement as

$$
A_\mu \equiv A_\mu^a T^a \in su(N_c) \rightarrow A_\mu \equiv \text{tr}(A_\mu \vec{H}) \cdot \vec{H} \in u(1)^{N_c-1}
$$

after the abelian gauge fixing.

In the abelian gauge, there also remains the global Weyl symmetry as a ‘relic’ of the nonabelian theory [30,40,41]. The Weyl symmetry is the permutation group $P_{N_c}$ corresponding to the permutation of the $N_c$ bases in the fundamental representation. For the SU(2) case, the Weyl transformation is expressed as the constant off-diagonal SU(2) matrix

$$
W \equiv e^{i \left( \frac{\tau_1}{2} \cos \alpha + \frac{\tau_2}{2} \sin \alpha \right) \pi} = i (\tau_1 \cos \alpha + \tau_2 \sin \alpha)
$$

$$
= i \left( \begin{array}{cc} 0 & e^{-i \alpha} \\ e^{i \alpha} & 0 \end{array} \right) \in P_{N_c=2} \subset SU(2)
$$

with $\alpha \in \mathbb{R}$ fixed. In the abelian gauge, the variable $\phi(x)$ is also diagonalized by using $\Omega_W(x) \equiv W \Omega(x)$,

$$
\phi(x) \rightarrow \Omega_W(x) \phi(x) \Omega_W^\dagger(x) = W \lambda(x) \frac{\tau_3}{2} W^\dagger = -\lambda(x) \frac{\tau_3}{2}.
$$

Here, the sign of $\lambda(x)$, or the order of the diagonal component $\lambda^i(x)$, is globally changed by the Weyl transformation. It is noted that the sign of the U(1)$_3$ gauge field $A_\mu \equiv A_\mu^{\frac{\tau_3}{2}}$ is also globally changed under the Weyl transformation,

$$
A_\mu \rightarrow A_\mu^W = W A_\mu^{\frac{\tau_3}{2}} W^\dagger = -A_\mu^{\frac{\tau_3}{2}} = -A_\mu.
$$
Therefore, all the sign of the abelian field strength $F_{\mu\nu}$ as $(\partial \wedge A)_{\mu\nu}$, electric and magnetic charges are also globally changed:

$$F_{\mu\nu} \equiv F_{\mu\nu} \frac{\tau_3}{2} \rightarrow F_{\mu\nu}^W = W F_{\mu\nu} W^+ = -F_{\mu\nu},$$

$$j_\mu \equiv \partial^\alpha F_{\alpha\mu} \rightarrow j_\mu^W = -j_\mu,$$

$$k_\mu \equiv \partial^{\alpha*} F_{\alpha\mu} \rightarrow k_\mu^W = -k_\mu. \quad (8)$$

In the abelian gauge, the absolute sign of the electric and the magnetic charges is settled, only when the Weyl symmetry is fixed by the additional condition. When $\phi[A_\mu(x)]$ obeys the simple adjoint gauge transformation like the non-abelian Higgs field, the global Weyl symmetry can be easily fixed by imposing the additional gauge-fixing condition as $\lambda(x) \geq 0$ for SU(2), or the ordering condition of the diagonal components $\lambda^i(x)$ in $\vec{H} \cdot \vec{\lambda}$ as $\lambda^1(x) \geq \ldots \geq \lambda^{N_c}(x)$ for the SU($N_c$) case. As for the appearance of monopoles in the abelian gauge, the global Weyl symmetry $P_{N_c}$ is not relevant, because the nontriviality of the homotopy group is not affected by the global Weyl symmetry. However, the definition of the magnetic monopole charge, which is expressed by the nontrivial dual root of SU($N_c$)$_{\text{dual}}$ [18], is globally changed by the Weyl transformation.

2.2 Monopoles and the Hedgehog Configuration of Gluonic Higgs Field

The abelian gauge fixing, which reduces QCD into an abelian gauge theory, is realized by the diagonalization of a gluonic Higgs field $\phi[A_\mu(x)]$. In the continuum theory of QCD, the continuous field $A_\mu(x)$ can be taken to be regular everywhere in a suitable gauge as the Landau gauge, and then $\phi[A_\mu(x)]$ is expected to be a regular function almost everywhere. In the abelian gauge, however, there appears the singular point, where the gauge function to diagonalize $\phi[A_\mu(x)]$ is not uniquely determined even for the off-diagonal part, and such a singular point leads to the appearance of the monopole. Here, let us consider the appearance of QCD-monopoles in the abelian gauge in terms of the singularity in the gauge transformation[20]. For the gluonic Higgs field $\phi(x)$ obeying the adjoint transformation, the monopole appears at the ‘degeneracy point’ of the diagonal elements of $\vec{H} \cdot \vec{\lambda}(x) = \text{diag}(\lambda^1(x), \lambda^2(x), \ldots, \lambda^{N_c}(x))$ after the abelian gauge fixing; $(i, j)$-monopole appears at the point satisfying $\lambda^i(x) = \lambda^j(x)$. For the $(i, j)$-monopole, the SU(2) subspace relating to $i$ and $j$ is enough to consider, so that the essential feature of the monopole can be understood in the SU(2) case without loss of generality. Then, we consider the SU(2) case for simplicity. For the SU(2) case, the diagonalized element of $\phi(x)$ are given by $\lambda = \pm(\phi_1^2 + \phi_2^2 + \phi_3^2)^{1/2}$, and hence the ‘degeneracy point’ satisfies the condition $\phi(x) = 0$, which is SU(2) gauge invariant. This gauge-invariant condition $\phi(x) = 0$ can be regarded as the singularity condition
on \( \hat{\phi}(x) \equiv \phi(x)/|\phi(x)| \) with \( |\phi(x)| \equiv (\phi^a(x)\phi^a(x))^{1/2} \). In fact, the ‘degeneracy point’ in the abelian gauge appears as the singular point of \( \hat{\phi}(x) \) like the center of the hedgehog configuration as shown in Fig.2(b) before the abelian gauge fixing.

Since the singular point on \( \hat{\phi}(x) \) is to satisfy three conditions \( \phi^1(x) = \phi^2(x) = \phi^3(x) = 0 \) simultaneously, the set of the singular point forms the point-like manifold in \( \mathbb{R}^3 \) or the line-like manifold in \( \mathbb{R}^4 \). We investigate the topological nature near the singular point \((x_0, t)\) of \( \hat{\phi}(x) \) for fixed \( t \), i.e., \( \phi(x_0, t) = 0 \).[20]

Using the Taylor expansion, one finds

\[
\phi(x, t) = \phi^a(x, t)\frac{r^a}{2} \simeq \tau^a C^{ab}(x-x_0)^b, \tag{9}
\]

with \( C^{ab} \equiv \frac{1}{2} \partial^b \phi^a(x_0, t) \). In the general case, one can expect \( \text{det}C \neq 0 \), i.e., \( \text{det}C > 0 \) or \( \text{det}C < 0 \), and the fiber-bundle \( \phi^a(x) \) can be deformed into the (anti-)hedgehog configuration \( \phi(\tilde{x}) \simeq \pm \tau^a \tilde{x}^a \) around the singular point \( x_0 \) by using the continuous modification on the spatial coordinate \( x^a \rightarrow \tilde{x}^a \equiv \text{sgn} (\text{det}C) \cdot C^{ab}(x-x_0)^b \). The linear transformation matrix \( C \) can be written by a combination of the rotation \( R \) and the dilatation of each axis \( \lambda = \text{diag} (\lambda^1, \lambda^2, \lambda^3) \) with \( \lambda^i > 0 \) as \( C = \text{sgn} (\text{det}C) R \lambda \). Here, topological nature is never changed by such a continuous deformation. For \( \text{det}C > 0 \), the configuration \( \phi(x) \) can be continuously deformed into the hedgehog configuration around \( x_0 \), \( \phi(\tilde{x}) \simeq \tau^a \tilde{x}^a \), while, for \( \text{det}C < 0 \), \( \phi(x) \) can be continuously deformed into the anti-hedgehog configuration, \( \phi(\tilde{x}) \simeq -\tau^a \tilde{x}^a \). Since \( \text{det}C = 0 \) is the exceptionally special case and \( \text{det}C < 0 \) is similar to \( \text{det}C > 0 \), we have only to consider the hedgehog configuration. This hedgehog configuration around the singular point of \( \hat{\phi}(x) \) corresponds to the simplest nontrivial topology of the nontrivial homotopy group \( \Pi_2 (\text{SU(2)/U(1)}_3) = \mathbb{Z}_\infty \), and the abelian gauge field has the singularity as the monopole appearing from the hedgehog configuration.

Using the polar coordinate \((r, \theta, \varphi)\) of \( \tilde{x} \), the hedgehog configuration is expressed as

\[
\phi = \tau^a \tilde{x}^a = r \sin \theta \cos \varphi \cdot \tau_1 + r \sin \theta \sin \varphi \cdot \tau_2 + r \cos \theta \cdot \tau_3
= r \begin{pmatrix}
\cos \theta & e^{-i\varphi} \sin \theta \\
e^{i\varphi} \sin \theta & -\cos \theta
\end{pmatrix}, \tag{10}
\]

and \( \phi \) can be diagonalized by the gauge transformation with

\[
\Omega^H = \begin{pmatrix}
e^{i\varphi} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\
-\sin \frac{\theta}{2} & e^{-i\varphi} \cos \frac{\theta}{2}
\end{pmatrix}, \tag{11}
\]
where θ, ϕ denote the polar and the azimuthal angles, respectively. Here, on the z-axis (θ = 0 or θ = π), ϕ is the ‘fake parameter’, and the unique description does not allow the ϕ-dependence on the z-axis. However, at the positive region of z-axis, θ = 0, Ω_H depends on ϕ and is multi-valued as

$$\Omega^H = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}.$$  \hfill (12)

Such a multi-valuedness of Ω^H leads to the divergence in the derivative $\partial_\mu \Omega^H$ at θ = 0. In fact, $\partial_\mu \Omega^H$ includes the singular part as cos $\frac{\theta}{2}$ $\partial_\mu \varphi = \frac{1}{r \sin \frac{\theta}{2}}$, which diverges at θ = 0. By the gauge transformation with Ω^H, the variable φ becomes $\phi^\Omega = \Omega \phi \Omega^\dagger = r \tau^3$, and the gauge field is transformed as

$$A_\mu \rightarrow A_\mu^\Omega = \Omega (A_\mu - \frac{i}{e} \partial_\mu) \Omega^\dagger.$$  \hfill (13)

For regular $A_\mu$, the first term $\Omega A_\mu \Omega^\dagger$ is regular, while $A_\mu^{\text{sing}} \equiv -\frac{i}{e} \partial_\mu \Omega^{\dagger}$ is singular and the monopole appears in the abelian sector originating from the singularity of $A_\mu^{\text{sing}}$ [20]. To examine the appearance of the monopole at the origin $\tilde{x} = 0$, we consider the magnetic flux $\Phi^{\text{flux}}(\theta)$ which penetrates the area inside the closed contour $c(r, \theta) \equiv \{(r, \theta, \phi)|0 \leq \phi < 2\pi\}$. One finds that

$$\Phi^{\text{flux}}(\theta) = \int_c d\mathbf{x} \cdot A^{\text{sing}} = -\frac{i}{e} \int_c d\mathbf{x} \Omega \nabla \Omega^\dagger = -\frac{4\pi}{e} \cdot \frac{1 + \cos \theta \tau^3}{2}.$$  \hfill (14)

which denotes the magnetic flux of the monopole with the unit-magnetic charge $g = \frac{4\pi}{e}$ with the Dirac string [20]. Here, the direction of the Dirac string from the monopole can be arbitrary changed by the singular U_3(1) gauge transformation, which can move $e^{i\phi}$ in $\Omega^H$ from the $\tau_3$-sector to the off-diagonal sector. In fact, the multi-valuedness of Ω is not necessary to be fixed in $\tau_3$-direction. Nevertheless, the singularity in $\Omega \partial_\mu \Omega^\dagger$ appears only in the $\tau_3$-sector, and $\tau_3$-direction becomes special in the abelian gauge fixing.

The anti-hedgehog configuration of $\phi(\tilde{x}) = -\tau^a \tilde{x}^a$ provides a monopole with the opposite magnetic charge, because anti-hedgehog configuration is transformed to the hedgehog configuration by the Weyl transformation. Thus, the only unit-charge magnetic monopole appears in the general case of $\det C \neq 0$. In principle, the multi-charge monopole can also appear when $\det C = 0$, however, the condition is scarcely satisfied in general, because this exceptional case is realized only when four conditions $\phi^1 = \phi^2 = \phi^3 = \det C = 0$ are
simultaneously satisfied. To summarize, in the abelian gauge, the unit-charge magnetic monopoles appear from the singular points of $\phi(x)$, however, multi-charge monopoles do not appear in general cases.

In this way, by the singular SU(2) gauge transformation, there appears the monopole with the Dirac string. Here, we consider the role of the off-diagonal component in the SU(2) gauge function $\Omega^H$ to appearance of the monopole, by comparing with the U(1)$_3$ gauge transformation. Let us consider the singular gauge transformation $\Omega^{U(1)} = e^{i\varphi \tau_3} \in U(1)_3$ instead of $\Omega^H$. This U(1)$_3$ gauge function $\Omega^{U(1)}$ is multi-valued on the whole region of the $z$ axis ($\theta = 0$ and $\theta = \pi$), and $A^{\text{sing}}_u \equiv -\frac{i}{e} \Omega^{U(1)} \partial_\mu \Omega^{U(1)\dagger}$ also has a singularity. The magnetic flux which penetrates the area inside the closed contour $c(r, \theta) = \{ r, \theta, \varphi | 0 \leq \varphi < 2\pi \}$ is found to be

$$
\Phi^{\text{flux}}(\theta) = \int_c d\mathbf{x} \cdot \mathbf{A}^{\text{sing}} = -\frac{4\pi}{e} \frac{\tau_3}{2},
$$

which corresponds to the endless Dirac string along the $z$-axis. It is noted that the singular U(1)$_3$ gauge transformation can provide the endless Dirac string, however, it never creates the monopole.

The monopole is created not by above singular U(1)$_3$ gauge transformation but by a singular SU(2) gauge transformation. Since the multi-valuedness of $\Omega^H$ is originated from the $\varphi$-dependence at $\theta = 0$ or $\theta = \pi$, we separate the SU(2) gauge function (11) as

$$
\Omega = \cos \frac{\theta}{2} e^{i\varphi \tau_3} + (\varphi\text{-independent term}).
$$

At $\theta = 0$ or the positive side of $z$ axis, $\Omega^H$ coincides with $\Omega^{U(1)} \equiv e^{i\varphi \tau_3}$ and is multi-valued like $\Omega^{U(1)}$. Therefore the Dirac string is created at $\theta = 0$ by the gauge transformation $\Omega^H$. On the other hand, at $\theta = \pi$ or the negative side of $z$-axis, $\varphi$-dependent part of $\Omega$ vanishes due to $\cos \frac{\theta}{2} = 0$, so that the Dirac string never appears in $\Omega \partial_\mu \Omega^\dagger$ at $\theta = \pi$. Thus, by the SU(2) singular gauge transformation $\Omega^H$, the Dirac string is generated only on the positive side of the $z$-axis and terminates at the origin $r = 0$, and hence the monopole appears at the end of the Dirac string. Around the origin $\tilde{x} = 0$, the factor $\cos \frac{\theta}{2}$ varies from unity to zero continuously with the polar angle $\theta$, and this makes the Dirac string terminated. Such a variation of the norm of the diagonal component $\cos \frac{\theta}{2} e^{i\varphi}$ cannot be realized in the U(1)$_3$ gauge transformation with $\Omega^{U(1)}$. In the SU(2) gauge transformation with $\Omega^H$, the norm of the diagonal component can be changed owing to existence of the off-diagonal component of $\Omega^H$, and the difference of the multi-valuedness between $\theta = 0$ and $\theta = \pi$ leads to the terminated Dirac string and the monopole. In this
In the abelian gauge, the monopole or the Dirac string appears as the result of QCD in the abelian gauge in detail using the gauge connection formalism.

2.3 Appearance of Monopole in the Gauge-connection Formalism

In this subsection, we study the appearance of monopoles in the abelian sector of QCD in the abelian gauge in detail using the gauge connection formalism. In the abelian gauge, the monopole or the Dirac string appears as the result of the SU(Nc) singular gauge transformation from a regular (continuous) gauge configuration. For the careful description of the singular gauge transformation, we formulate the gauge theory in terms of the gauge connection, described by the covariant-derivative operator $\hat{D}_\mu$ and $\hat{D}_\mu \equiv \hat{\partial}_\mu + ieA_\mu(x)$, where $\hat{\partial}_\mu$ is the derivative operator satisfying $[\hat{\partial}_\mu, f(x)] = \partial_\mu f(x)$.

To begin with, let us consider the system holding the local difference of the internal-space coordinate frame. We attention the neighbor of the real space-time $x_\mu$, and denote by $|q(x)\rangle$ the basis of the internal-coordinate frame. At the neighboring point $x_\mu + \varepsilon_\mu$, we express the difference of the internal-coordinate frame as $|q(x + \varepsilon)\rangle = R_\varepsilon(x)|q(x)\rangle$ with $R_\varepsilon(x) = e^{i\varepsilon_\mu A^\mu(x)} \in G$ being the ‘rotational matrix’ of the internal space. We require the ‘local superposition’ on $r_\varepsilon$ as $r_{\varepsilon_1 + \varepsilon_2} = r_{\varepsilon_1} + r_{\varepsilon_2}$ up to $O(\varepsilon)$, and then we can express $r_\varepsilon(x) = -ie\varepsilon_\mu A^\mu(x)$ using a $\varepsilon$-independent local variable $A_\mu(x) \in G : |q(x + \varepsilon)\rangle = e^{-ie\varepsilon_\mu A^\mu(x)}|q(x)\rangle$.

Then, the ‘observed difference’ of the internal space coordinate depends on the real space-time $x_\mu$, the observed difference of the local operator $O(x)$ between neighboring points, $x_\mu$ and $x_\mu + \varepsilon_\mu$, is given by

$$
\langle q(x + \varepsilon)|O(x + \varepsilon)|q(x + \varepsilon)\rangle - \langle q(x)|O(x)|q(x)\rangle = \langle q(x)|e^{i\varepsilon_\mu A^\mu(x)}O(x + \varepsilon)e^{-i\varepsilon_\mu A^\mu(x)}|q(x)\rangle - \langle q(x)|O(x)|q(x)\rangle \\
\simeq \varepsilon_\mu \langle q(x)|\{\partial_\mu O(x) + ie[A^\mu(x), O(x)]\}|q(x)\rangle \\
= \varepsilon_\mu \langle q(x)|\{[\hat{\partial}_\mu + ieA^\mu(x), O(x)]\}|q(x)\rangle \equiv \varepsilon_\mu \langle q(x)||\hat{D}_\mu, O(x)||q(x)\rangle. \quad (16)
$$

Here, one finds natural appearance of the covariant derivative operator, $\hat{D}_\mu \equiv \hat{\partial}_\mu + ieA_\mu(x)$. The gauge transformation is simply defined by the arbitrary internal-space rotation as $|q(x)\rangle \rightarrow \Omega(x)|q(x)\rangle$ with $\Omega(x) \in G$, and therefore the covariant derivative operator is transformed as $\hat{D}_\mu \rightarrow \hat{D}_\mu^\Omega = \Omega(x)\hat{D}_\mu^\dagger \Omega^\dagger(x)$ with $\Omega(x) \in G$, which is consistent with $A_\mu \rightarrow A_\mu^\Omega = \Omega(A_\mu - \frac{i}{e}\partial_\mu)\Omega^\dagger$.

In the general system including singularities such as the Dirac string, the gauge field and the field strength are defined as the difference between the gauge connection and the derivative connection,
\[ A_\mu \equiv \frac{1}{ie} (\hat{D}_\mu - \hat{\partial}_\mu) \]  
\[ G_{\mu\nu} \equiv \frac{1}{ie} ([\hat{D}_\mu, \hat{D}_\nu] - [\hat{\partial}_\mu, \hat{\partial}_\nu]). \]  

This expression of \( G_{\mu\nu} \) is returned to the standard definition \( G_{\mu\nu} = \frac{1}{ie} [\hat{D}_\mu, \hat{D}_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ie [A_\mu, A_\nu] \) in the regular system. By the general gauge transformation with the gauge function \( \Omega \), the field strength \( G_{\mu\nu} \) is transformed as

\[
G_{\mu\nu} \rightarrow G^\Omega_{\mu\nu} = \Omega G_{\mu\nu} \Omega^\dagger = \frac{1}{ie} ([\hat{D}^\Omega_\mu, \hat{D}^\Omega_\nu] - \Omega [\hat{\partial}_\mu, \hat{\partial}_\nu] \Omega^\dagger) \\
= \partial_\mu A^\Omega_\nu - \partial_\nu A^\Omega_\mu + ie [A^\Omega_\mu, A^\Omega_\nu] + \frac{i}{e} (\Omega [\hat{\partial}_\mu, \hat{\partial}_\nu] \Omega^\dagger - [\hat{\partial}_\mu, \hat{\partial}_\nu]) \\
= (\partial_\mu A^\Omega_\nu - \partial_\nu A^\Omega_\mu) + ie [A^\Omega_\mu, A^\Omega_\nu] + \frac{i}{e} \Omega [\partial_\mu, \partial_\nu] \Omega^\dagger \\
\equiv G^\text{linear}_{\mu\nu} + G^\text{bilinear}_{\mu\nu} + G^\text{sing}_{\mu\nu}. \tag{19}
\]

The last term remains only for the singular gauge transformation on \( \Omega^H \) and \( \Omega^{U(1)} \), and can provide the Dirac string.

Figure 3 shows the SU(2) field strength \( G^\text{linear}_{\mu\nu}, G^\text{bilinear}_{\mu\nu} \) and \( G^\text{sing}_{\mu\nu} \) in the abelian gauge provided by \( \Omega^H \) in Eq.(11). The linear term \( G^\text{linear}_{\mu\nu} \equiv (\partial_\mu A^\Omega_\nu - \partial_\nu A^\Omega_\mu) \) includes in the abelian sector the singular gauge configuration of the monopole with the Dirac string, which supplies the magnetic flux from infinity. Since each component satisfies the Bianchi identity \( \partial^\alpha G^\alpha_{\mu\nu} = \partial^\alpha (\partial \cdot A^\Omega)_{\alpha\mu} = 0 \), the abelian magnetic flux is conserved. The abelian part of \( G^\text{bilinear}_{\mu\nu} \equiv ie [A^\Omega_\mu, A^\Omega_\nu] \), \( (G^\text{bilinear})^3 = -e(A_{\mu}^\Omega A_{\nu}^\Omega - A_{\nu}^\Omega A_{\mu}^\Omega) \), includes the effect of off-diagonal components, and it is dropped by the abelian projection. The last term \( G^\text{sing}_{\mu\nu} \equiv \frac{i}{e} \Omega [\partial_\mu, \partial_\nu] \Omega^\dagger \) appears from the singularity of the gauge function \( \Omega \), and it plays the important role of the appearance of the magnetic monopole in the abelian sector.

First, we consider the singular part \( G^\text{sing}_{\mu\nu} \). In general, \( G^\text{sing}_{\mu\nu} \) disappears in the regular point in \( \Omega \). It is to be noted that \( G^\text{sing}_{\mu\nu} \) is found to be diagonal from the direct calculation with \( \Omega^H \) in Eq.(11),

\[
G^\text{sing}_{\mu\nu} \equiv \frac{i}{e} \Omega^H [\partial_\mu, \partial_\nu] \Omega^{H\dagger} = \frac{i}{e} (g_{\mu_1 \nu_2} - g_{\mu_2 \nu_1}) \cos^2 \frac{\theta}{2} e^{i\varphi_3} [\partial_1, \partial_2] e^{-i\varphi_3} \\
= \frac{1}{e} (g_{\mu_1 \nu_2} - g_{\mu_2 \nu_1}) \frac{1 + \cos \theta}{2} [\partial_1, \partial_2] \varphi \cdot \tau_3 \\
= \frac{4\pi}{e} (g_{\mu_1 \nu_2} - g_{\mu_2 \nu_1}) (\theta(x_3) \delta(x_1) \delta(x_2) \cdot \tau_3), \tag{20}
\]

where we have used relations,
\[ \partial_1 \partial_2 \varphi = -2\pi \delta(x_1) \delta(x_2), \quad \frac{1 + \cos \theta}{2} \delta(x_1) \delta(x_2) = \theta(x_3) \delta(x_1) \delta(x_2). \] (21)

The off-diagonal component of \( \Omega^H[\partial_\mu, \partial_\nu] \Omega^{H\dagger} \) disappears, since the singularity appears only from \( \varphi \)-dependent term. As a remarkable fact, the last expression in Eq. (20) shows the terminated Dirac string, which is placed along the positive \( z \)-axis with the end at the origin. Hence, in the abelian part of the SU(2) field strength, \( G_{\mu \nu}^{\text{sing}} \) leads to the breaking of the U(1)\(_3\) Bianchi identity,

\[ k_\mu = \partial^\alpha G_{\alpha \mu}^{\text{sing}} = \frac{1}{2} \varepsilon_{\alpha \mu \beta \gamma} \partial^\beta G_{\gamma \nu}^{\text{sing}} = \frac{4\pi}{e} \varepsilon_{\alpha \mu \nu \lambda} \partial^\lambda \{ \delta(x_1) \delta(x_2) \theta(x_3) \} \frac{\tau_3}{2}, \]

(22)

which is the expression for the static monopole with the magnetic charge \( g = \frac{4\pi}{e} \) at the origin. Thus, the magnetic current \( k_\mu \) is induced in the abelian sector by the singular gauge transformation with \( \Omega^H \) and the Dirac condition \( eg = 4\pi \) is automatically derived in this gauge-connection formalism.

In the covariant manner, \( G_{\mu \nu}^{\text{sing}} \) is expressed as \( G_{\mu \nu}^{\text{sing}} = \frac{1}{n \cdot \partial}^* (n \wedge k)_{\mu \nu} \) using the monopole current \( k_\mu \) in Eq. (22) and a constant 4-vector \( n_\mu \). Actually, for the above case, one finds for \( n_\mu = g_\mu^3 \)

\[
\frac{1}{n \cdot \partial}^* (n \wedge k)_{\mu \nu} = \int dx_3' \langle x_3' | \frac{1}{n \cdot \partial} | x_3 \rangle \varepsilon_{\mu \nu \lambda \sigma} n^3 \frac{4\pi}{e} \delta(x_1) \delta(x_2) \delta(x_3') \frac{\tau_3}{2} \]

\[
= \frac{4\pi}{e} (g_{\mu_1 \nu_2} - g_{\nu_2 \mu_1}) \delta(x_3') \delta(x_1) \delta(x_2) \frac{\tau_3}{2} \]

\[
= \frac{i}{e} \Omega^H[\partial_\mu, \partial_\nu] \Omega^{H\dagger} = G_{\mu \nu}^{\text{sing}}, \]

(23)

using the relation \( \langle x_n | \frac{1}{n \cdot \partial} | x_n' \rangle = \theta(x_n - x_n') \).

Thus, the last term \( G_{\mu \nu}^{\text{sing}} \) corresponds to the Dirac string terminated at the origin. Since \( G_{\mu \nu}^{\text{linear}} \) shows the configuration of the monopole together with the Dirac string, the sum of \( G_{\mu \nu}^{\text{linear}} + G_{\mu \nu}^{\text{sing}} \) provides the gauge configuration of the monopole without the Dirac string in the abelian sector. Thus, by dropping the off-diagonal gluon element, \( G_{\mu \nu}^{\text{bilinear}} \) vanishes and the remaining part \( (G_{\mu \nu}^{\text{linear}} + G_{\mu \nu}^{\text{sing}})^3 \) describing the abelian projected QCD includes the field strength of monopoles.

Next, we consider the role of off-diagonal gluon components for appearance of the monopole. The gluon field is divided into the regular part \( \Omega A_{\mu} \Omega^{\dagger} \) and the singular part \( -\frac{i}{e} \Omega \partial_\mu \Omega^{\dagger} \). Since we are interested in the behavior of the singularity, we neglect the regular part of the gluon field. Then, \( G_{\mu \nu}^{\text{bilinear}} \) is written as
\[ ie[A^\Omega_{\mu}, A^\Omega_{\nu}] = \frac{1}{ie} [\Omega \partial_{\mu} \Omega^\dagger, \Omega \partial_{\nu} \Omega] \]

\[ = -\frac{1}{ie} \{ (\partial_{\mu} \Omega) \partial_{\nu} \Omega^\dagger - (\partial_{\nu} \Omega) \partial_{\mu} \Omega^\dagger \} \]

\[ = -\frac{1}{ie} \{ \partial_{\mu} (\Omega \partial_{\nu} \Omega^\dagger) - \partial_{\nu} (\Omega \partial_{\mu} \Omega^\dagger) \} - ie \Omega [\partial_{\mu}, \partial_{\nu}] \Omega^\dagger \]

\[ = -(\partial_{\mu} A^\Omega_{\nu} - \partial_{\nu} A^\Omega_{\mu}) - ie \Omega [\partial_{\mu}, \partial_{\nu}] \Omega^\dagger, \quad (24) \]

where the last term appears as the breaking of the Maurer-Cartan equation. In the abelian gauge, the singularity of the monopole appearing in \( G_{\mu\nu}^{\text{linear}} + G_{\mu\nu}^{\text{bilinear}} \) is exactly canceled by that of \( G_{\mu\nu}^{\text{sing}} \). Thus, in the abelian gauge, the off-diagonal gluon combination \( (G_{\mu\nu}^{\text{bilinear}})^3 = -e \{ (A^\Omega_{\mu})^1 (A^\Omega_{\nu})^2 - (A^\Omega_{\nu})^1 (A^\Omega_{\mu})^2 \} \) includes the field strength of the anti-monopole, and hence the off-diagonal gluons \( (A^\Omega_{\mu})^1 \) and \( (A^\Omega_{\mu})^2 \) have to include some singular structure around the monopole.

The abelian projection is defined by dropping the off-diagonal component of the gluon field \( A_{\mu} \).

\[ A^\Omega_{\mu} \equiv A^\Omega_{\mu a} \frac{\tau^a}{2} \rightarrow A_{\mu} \equiv \text{tr}(A^\Omega_{\mu a} \frac{\tau^a}{2} \tau^3) = (A^\Omega_{\mu})^3 \frac{\tau^3}{2}. \quad (25) \]

Accordingly, the SU(2) field strength \( G_{\mu\nu}^\Omega \) is projected to the abelian field strength \( F_{\mu\nu} \equiv F_{\mu\nu}^\tau \).

\[ G_{\mu\nu}^\Omega \equiv (G_{\mu\nu}^\Omega a \frac{\tau^a}{2} = (\partial_{\mu} A^\Omega_{\nu} - \partial_{\nu} A^\Omega_{\mu}) + ie [A^\Omega_{\mu}, A^\Omega_{\nu}] + ie \Omega [\partial_{\mu}, \partial_{\nu}] \Omega^\dagger \]

\[ \rightarrow F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + ie \Omega [\partial_{\mu}, \partial_{\nu}] \Omega^\dagger \]

\[ = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} - F_{\mu\nu}^{\text{sing}}, \quad (26) \]

where \( F_{\mu\nu}^{\text{sing}} \equiv F_{\mu\nu}^\tau \tau^3 \equiv -ie \Omega [\partial_{\mu}, \partial_{\nu}] \Omega^\dagger \) is diagonal and remains. Here, the bilinear term \( ie [A^\Omega_{\mu}, A^\Omega_{\nu}] \) vanishes in AP-QCD because it is projected to \( ie [A_{\mu}, A_{\nu}] = 0 \) by the abelian projection. The appearance of \( F_{\mu\nu}^{\text{sing}} \) leads to the breaking of the abelian Bianchi identity in the U(1)\(_3\) sector,

\[ \partial^\alpha F_{\alpha\mu} = -\partial^\alpha F_{\alpha\mu}^{\text{sing}} = \partial^\alpha \{ ie \Omega [\partial_{\alpha}, \partial_{\mu}] \Omega^\dagger \} = k_{\mu}, \quad (27) \]

where Eq.(23) is used. Thus, the magnetic current \( k_{\mu} \) is induced into the abelian gauge theory through the singularity of the SU(2) gauge transformation.

Here, we compare AP-QCD and QCD in terms of the field strength. The
SU($N_c$) field strength $G_{\mu}^a$ is controlled by the QCD action, $S_{\text{QCD}} = \int d^4x \{ -\frac{1}{2} \text{tr} G_{\mu}^a G_{\mu}^a \}$, so that each component $G_{\mu}^a$ cannot diverge. On the other hand, the field strength $F_{\mu\nu}$ in AP-QCD is not directly controlled by $S_{\text{QCD}}$, since the QCD action includes also off-diagonal components. It should be noted that the point-like monopole appearing in AP-QCD makes the $U(1)_3$ action $S_{\text{Abel}} = \int d^4x \{ -\frac{1}{2} \text{tr} F_{\mu\nu} F_{\mu\nu} \}$ divergent around the monopole, such a divergence in $F$ should cancel exactly with the remaining off-diagonal contribution from $G_{\mu\nu}^{\text{bilinear}}$ to keep the total QCD action finite. Thus, the appearance of monopoles in AP-QCD is supported by the singular contribution of off-diagonal gluons. In this way, abelian projected QCD includes monopoles generally.

2.4 Monopole Current in the Lattice Formalism

For the study of nonperturbative QCD physics, the lattice QCD formalism provides a useful method for the direct calculation of the QCD generating functional $Z_{\text{QCD}}[15]$. In this subsection, we extract the abelian gauge field and the monopole current in the lattice formalism [42].

In the lattice QCD, the system is described by the link-variable $U_\mu(s) \equiv e^{iaeA_\mu(s)} \in \text{SU}(N_c)$ instead of $A_\mu(x)$. Here, $e$ denotes the QCD gauge coupling and $a$ the lattice spacing. The SU(2) link-variable $U_\mu(s)$ can be factorized as

\[
U_\mu(s) = M_\mu(s)u_\mu(s) \quad \in G \\
M_\mu(s) = \exp \left( i\{\tau_1\theta^1_\mu(s) + \tau_2\theta^2_\mu(s)\} \right) \quad \in G/H, \\
u_\mu(s) = \exp \left( i\tau^3\theta^3_\mu(s) \right) \quad \in H
\]

(28)

with respect to the Cartan decomposition of $G = G/H \times H$ into $G/H = \text{SU}(2)/U(1)_3$ and $H = U(1)_3$. In the lattice formalism, such a factorization has an ambiguity relating to the ordering of $M_\mu$ and $u_\mu$ in this factorization. Instead of $U_\mu = M_\mu u_\mu$, another factorization $U_\mu = u_\mu M_\mu$ is equivalently applicable, however, such an ordering is to be fixed through the whole argument. Here, the abelian link variable,

\[
u_\mu(s) = e^{i\tau^3\theta^3_\mu(s)} = \begin{pmatrix} e^{i\theta^3_\mu(s)} & 0 \\ 0 & e^{-i\theta^3_\mu(s)} \end{pmatrix} \quad \in U(1)_3 \subset \text{SU}(2),
\]

(29)

plays the similar role as the SU(2) link-variable $U_\mu(s) \in \text{SU}(2)$ in terms of the residual $U(1)_3$ gauge symmetry in the abelian gauge, and $\theta^3_\mu(s) \in (-\pi, \pi]$ corresponds to the diagonal component of the gluon in the continuum limit. On the other hand, the off-diagonal factor $M_\mu(s) \in \text{SU}(2)/U(1)_3$ is expressed as
\[ M_\mu(s) = \exp \left( i \{ \tau_1 \theta_\mu^1(s) + \tau_2 \theta_\mu^2(s) \} \right) \]

\[ = \left( \begin{array}{cc}
\cos \theta_\mu(s) & -\sin \theta_\mu(s) e^{-i\chi_\mu(s)} \\
\sin \theta_\mu(s) e^{i\chi_\mu(s)} & \cos \theta_\mu(s)
\end{array} \right) \]

\[ = \left( \begin{array}{cc}
\sqrt{1 - |c_\mu(s)|^2} & -c_\mu(s) \\
c_\mu(s) & \sqrt{1 - |c_\mu(s)|^2}
\end{array} \right) \quad (30) \]

with \( \theta_\mu(s) \equiv \text{mod}_{\frac{\pi}{2}} \sqrt{(\theta_\mu^1)^2 + (\theta_\mu^2)^2} \in [0, \frac{\pi}{2}] \) and \( \chi_\mu(s) \in (-\pi, \pi] \). Near the continuum limit, the off-diagonal elements of \( M_\mu(s) \) correspond to the off-diagonal gluon components. Under the residual \( U(1)_3 \) gauge transformation by \( \omega(s) = e^{-i\varphi(s)\frac{\pi}{2}} \in U(1)_3 \), \( u_\mu(s) \) and \( M_\mu(s) \) are transformed as

\[ u_\mu(s) \rightarrow u_\mu'(s) = \omega(s) u_\mu(s) \omega^\dagger(s + \hat{\mu}) \quad \in H \quad (31) \]

\[ M_\mu(s) \rightarrow M_\mu'(s) = \omega(s) M_\mu(s) \omega^\dagger(s) \quad \in G/H \quad (32) \]

so as to keep \( M_\mu'(s) \) belong \( G/H \). Accordingly, \( \theta_\mu^3(s) \) and \( c_\mu(s) \in \mathbb{C} \) are transformed as

\[ \theta_\mu^3(s) \rightarrow \theta_\mu^{3\omega}(s) = \text{mod}_{2\pi} \left[ \theta_\mu^3(s) + \{ \varphi(s + \hat{\mu}) - \varphi(s) \} / 2 \right] \quad (33) \]

\[ c_\mu(s) \rightarrow c_\mu^{\omega}(s) = c_\mu(s) e^{i\varphi(s)}. \quad (34) \]

Thus, on the residual \( U(1)_3 \) gauge symmetry, \( u_\mu(s) \) behaves as the \( U(1)_3 \) lattice gauge field, and \( \theta_\mu^3(s) \) behaves as the \( U(1)_3 \) gauge field in the continuum limit. On the other hand, \( M_\mu(s) \) and \( c_\mu(s) \) behave as the charged matter field in terms of the residual \( U(1)_3 \) gauge symmetry, which is similar to the charged weak boson \( W_\mu^\pm \) in the Standard Model.

The abelian field strength \( \bar{\theta}_{\mu\nu}(s) \) is defined as \( \bar{\theta}_{\mu\nu}(s) \equiv \text{mod}_{2\pi} (\partial \wedge \theta_\mu^3)_{\mu\nu}(s) \in (-\pi, \pi] \), which is \( U(1)_3 \) gauge invariant. In general, the two form of the abelian angle variable \( \theta_\mu^3(s) \) is divided as

\[ \theta_{\mu\nu}(s) \equiv (\partial \wedge \theta_\mu^3)_{\mu\nu}(s) = \bar{\theta}_{\mu\nu}(s) + 2\pi n_{\mu\nu}(s), \quad (35) \]

where \( n_{\mu\nu}(s) \in \mathbb{Z} \) corresponds to the quantized magnetic flux of the ‘Dirac string’ penetrating through the plaquette. Although \( n_{\mu\nu} \neq 0 \) provides the infinite magnetic field is the continuum limit as \( 2\pi n_{\mu\nu}/a \), the term \( 2\pi n_{\mu\nu}(s) \) does not contribute to the abelian plaquette \( \Box_{\mu\nu}^{\text{abel}}(s) \), and it is changed by the singular \( U(1)_3 \) gauge-transformation as \( \theta_\mu^3(s) \rightarrow \theta_\mu^3(s) + \partial_\mu \varphi(s) \) with \( \varphi(s) \) being the azimuthal angle. Thus, \( 2\pi n_{\mu\nu} \) corresponds to the Dirac string as an unphysical object.

The monopole \( k_{\mu}^{\text{lat}}(s) \) is defined on the dual link as [42],
\begin{equation}
\begin{split}
k_{\mu}^{\text{lat}}(s) & \equiv \frac{1}{2\pi} \partial_\alpha^* \bar{\theta}_{\alpha\mu}(s) = -\partial_\alpha^* n_{\alpha\mu}(s),
\end{split}
\end{equation}

using the abelian field strength $\bar{\theta}_{\mu\nu}(s)$. Here, $k_{\mu}^{\text{lat}}(s)$ is defined such that the topological quantization is manifest, $k_{\mu}^{\text{lat}}(s) \in \mathbb{Z}$. In this definition, for instance, one finds $k_{\mu}^{\text{lat}} = \frac{1}{2} \varepsilon_{ijk} \partial_j n_{jk}$ and $k_{i}^{\text{lat}} = 0$ ($i = 1, 2, 3$) for the static monopole. The magnetic charge of the monopole on the dual lattice is determined by the total magnetic flux of the Dirac strings entering the cube around the monopole. (See Fig.4.)

We show in Fig.5 the typical example of the monopole current at a time slice in the lattice QCD at $\beta = 2.4$ in the maximally abelian (MA) gauge. In each gauge configuration, the monopole current appears as a distinct line-like object, and the neighbor of the monopole can be defined on the lattice. However, taking the temporal direction into account, the monopole current forms a global network covering over $\mathbb{R}^4$.

Here, we summarize several relevant properties of $k_{\mu}(s)$.

(1) The monopole current $k_{\mu}$ is topologically quantized and $k_{\mu}^{\text{lat}}(s)$ takes an integer $k_{\mu}^{\text{lat}}(s) \in \mathbb{Z}$ in the definition of Eq.(36). As the result, $k_{\mu}^{\text{lat}}(s)$ forms a line-like object in the space-time $\mathbb{R}^4$, since $k_{\mu}^{\text{lat}}$ is a conserved current as $\partial_{\mu} k_{\mu} = 0$. These features of $k_{\mu}^{\text{lat}}(s) \in \mathbb{Z}$ are quite unique and different from the electric current $j_{\mu}(s) \in \mathbb{R}$, which can spread as a continuous field.

(2) In the lattice formalism, $k_{\mu}^{\text{lat}} \equiv \frac{1}{2\pi} \partial_\alpha^* \bar{\theta}_{\alpha\mu}$ is defined as a three-form on the dual link. For the use of the forward derivative, $k_{\mu}^{\text{lat}}(s)$ is to be defined on the dual link between $s_{\text{dual}}^{\mu} \equiv s + \frac{\hat{i}}{2} + \frac{\hat{j}}{2} + \frac{\hat{k}}{2} + \frac{\hat{\tau}}{2} \pm \frac{\hat{\mu}}{2}$. For instance, $k_{\mu}^{\text{lat}}(s)$ is placed on the dual link between $(s_x + \frac{1}{2}, s_y + \frac{1}{2}, s_z + \frac{1}{2}, s_t)$ and $(s_x + \frac{1}{2}, s_y + \frac{1}{2}, s_z + \frac{1}{2}, s_t + 1)$. Thus, the monopole is defined to appear at the center of the 3-dimensional cube perpendicular to the monopole-current direction as shown in Fig.4.

(3) Because of $k_{\mu} \equiv \partial_\alpha^* F_{\alpha\mu} = -\frac{1}{2} \varepsilon_{\mu\alpha\beta\gamma} \partial_\alpha F_{\beta\gamma}$, $k_{\mu}$ only affects the perpendicular components to the $\mu$-direction for the ‘electric variable’ as $F_{\alpha\beta}$ in a direct manner. For instance, the static monopole with $k_0 \neq 0$ creates the magnetic field $F_{ij} (i, j = 1, 2, 3)$ around it, but does not bring the electric field $F_{0i}$. Hence, in testing the field around the monopole, one has to consider the difference between such perpendicular components and others.

We now consider the relationship between the lattice variable and the field variable in the continuum theory. The continuous abelian field $A_\mu(x) \equiv A_\mu^0(x) x^\mu$ is expressed as
with the gauge coupling constant $e$ and the lattice spacing $a$. The abelian field strength $F_{\mu\nu}(x) \equiv F_{\mu\nu}(x)|_{\tau^3}^0 \frac{a}{2}$ in the continuum theory is written as

$$
e F_{\mu\nu} \equiv \text{mod}_{2\pi}(\theta_{\mu\nu}) \cdot \frac{2}{a^2} = \tilde{\theta}_{\mu\nu} \cdot \frac{2}{a^2},$$

and $F_{\mu\nu}$ is composed of two parts according to the decomposition (35)

$$F_{\mu\nu} = (\partial \wedge A^3)_{\mu\nu} - F^{\text{sing}}_{\mu\nu}.$$

Thus, in the SU($N_c$)-lattice formalism, the difference between the field strength $F_{\mu\nu}$ and two-form $(\partial \wedge A)_{\mu\nu}$ arises from the periodicity of the angle variable in the compact subgroup $U(1)^{N_c-1}$ embedded in SU($N_c$). Here, the singular Dirac-string part $F^{\text{sing}}_{\mu\nu}$ is directly related to $2\pi n_{\mu\nu}$ and is written by

$$F^{\text{sing}}_{\mu\nu} = 2\pi n_{\mu\nu} \cdot \frac{2}{ea^2} = \frac{4\pi}{e} n_{\mu\nu} \frac{1}{a^2}.$$

Owing to existence of $F^{\text{sing}}_{\mu\nu}$ in Eq.(39), the monopole current $k_\mu(x) \equiv k^3_\mu(x)|_{\tau^3}^0 \equiv \partial_\alpha^* F_{\alpha\mu} \frac{a}{2}$ appears in the continuum theory and is written as

$$k^3_\mu = k^3_{\mu\text{lat}} \cdot \frac{4\pi}{ea^2} = -g n_{a\mu} \frac{1}{a^3},$$

where the magnetic-charge unit $g \equiv \frac{4\pi}{e}$ naturally appears in $k_\mu$.

In the lattice formalism, there also appears the monopole-like configuration as the lattice artifact, when the lattice constant $a$ is relatively large. As $a \to 0$, such a monopole-like configuration turns out to be regular large fluctuation rather than the point-like singularity. Hence, one should use a fine mesh lattice to remove such lattice artifact monopoles.

### 3 Monopoles in the Maximally Abelian Gauge

#### 3.1 Maximally Abelian Gauge and Abelian Projection Rate

The abelian gauge has some arbitrariness corresponding to the choice of the variable $\phi[A_\mu(x)]$ to be diagonalized. Several typical abelian gauges have been
tested on the dual superconductor scenario for the nonperturbative QCD\cite{28,29}. Recent lattice QCD studies show that infrared phenomena such as confinement properties and chiral symmetry breaking are almost reproduced only by the abelian variable in the maximally abelian (MA) gauge \cite{30,32–36}. In this subsection, we study the MA gauge in detail, considering the gluon field properties.

In the SU(2) lattice formalism, the MA gauge is defined so as to maximize 

\[
R_{\text{MA}}[U_{\mu}] \equiv \sum_{s,\mu} \text{tr}\{U_{\mu}(s)\tau_3 U_{\mu}^\dagger(s)\tau_3\} \\
= 2 \sum_{s,\mu} \{U_{\mu}^0(s)^2 + U_{\mu}^3(s)^2 - U_{\mu}^1(s)^2 - U_{\mu}^2(s)^2\} \\
= 2 \sum_{s,\mu} \left[1 - 2\{U_{\mu}^1(s)^2 + U_{\mu}^2(s)^2\}\right] 
\]

by the SU(2) gauge transformation. Here, we denote 

\[U_{\mu}(s) = U_{\mu}^0(s) + i\tau^a U_{\mu}^a(s)\]

with \(U_{\mu}^0(s), U_{\mu}^a(s) \in \mathbb{R}\), obeying \(U_{\mu}^0(s)^2 + U_{\mu}^a(s)^2 = 1\).

The MA gauge is a sort of the abelian gauge which diagonalizes the hermite variable

\[
\Phi[U_{\mu}(s)] \equiv \sum_{\mu, \pm} U_{\pm \mu}(s)\tau_3 U_{\pm \mu}^\dagger(s).
\]

Here, we use the convenient notation \(U_{- \mu}(s) \equiv U_{\mu}^\dagger(s - \mu)\) in this paper. Here, \(\Phi[U_{\mu}(s)]\) is gauge transformed by \(V(s) \in \text{SU}(2)\) as

\[
\Phi(s) \rightarrow \Phi^V(s) = V(s)\{\sum_{\mu, \pm} U_{\pm \mu}(s)U_{\pm \mu}^\dagger(s)\tau_3 V(s \pm \mu)U_{\pm \mu}^{-1}(s)\}V^\dagger(s),
\]

which is not a simple adjoint transformation. In the continuum limit \(a \rightarrow 0\), the link-variable reads \(U_{\mu}(s) = e^{iaeA_{\mu}(s)} = 1 + iaeA_{\mu}(s) + O(a^2)\), and hence the MA gauge condition becomes \(\sum_{\mu}(i\partial_{\mu} - eA_{\mu}^3)A_{\mu}^{\pm} = 0\), which can be regarded as the maximal decoupling condition between the abelian gauge sector and the charged gluon sector.

In the MA gauge, \(\Phi(s)\) is diagonalized as \(\Phi_{\text{diag}}(s) = \lambda(s)\frac{\tau_3}{2}\) with \(\lambda(s) \in \mathbb{R}\), and there remain the local \(U(1)_3\) symmetry and the global Weyl symmetry \cite{40}. After the MA gauge fixing, the global Weyl transformation with \(W\) in Eq.(6) never changes the sign of \(\lambda(s)\) as

\[
\Phi_{\text{diag}}(s) \rightarrow \Phi_{\text{diag}}^W(s) = \sum_{\mu, \pm} WU_{\pm \mu}(s)W^\dagger\tau_3 WU_{\pm \mu}^\dagger(s)W^\dagger
\]
\[
-WU_{\mu}(s)\tau_{3}U_{\mu}^{\dagger}(s)W^{\dagger} = -W\Phi_{\text{diag}}(s)W^{\dagger} = \Phi_{\text{diag}}(s).
\]

Thus, the Weyl symmetry is not fixed in the MA gauge by the simple ordering condition as \(\lambda(s) \geq 0\), unlike the simple adjoint case.

In the MA gauge, the absolute values of the off-diagonal components, \(U_{\mu}^{1}(s)\) and \(U_{\mu}^{2}(s)\), are forced to be small. In the continuum limit \(a \to 0\), the MA gauge is found to minimize the functional

\[
R_{\text{ch}}[A_{\mu}] \equiv \frac{1}{2} e^{2} \int d^{4}x \{A_{\mu}^{1}(x)^{2} + A_{\mu}^{2}(x)^{2}\} = e^{2} \int d^{4}x A_{\mu}^{+}(x)A_{\mu}^{-}(x)
\]

with \(A_{\mu}^{\pm}(x) \equiv \frac{1}{\sqrt{2}} \{A_{\mu}^{1}(x) \pm iA_{\mu}^{2}(x)\}\). Thus, in the MA gauge, the off-diagonal gluon component is globally forced to be small by the gauge transformation, which seems a microscopic origin of abelian dominance for the nonperturbative QCD in the MA gauge [30].

Here, let us consider resemblance of the abelian link variable \(u_{\mu}(s)\) to the SU(2) link variable \(U_{\mu}(s)\) quantitatively. To this end, we introduce the ‘abelian projection rate’ \(R_{\text{Abel}}\) [30,43], which is defined as the overlapping factor as

\[
R_{\text{Abel}}(s, \mu) \equiv \frac{1}{2} \Re \text{tr}\{U_{\mu}(s)u_{\mu}^{\dagger}(s)\} = \frac{1}{2} \Re \text{tr}M_{\mu}(s) = \cos \theta_{\mu}(s) \in [0, 1],
\]

where \(\theta_{\mu}(s)\) is defined to belong \([0, \frac{\pi}{2}]\) in the decomposition of \(U_{\mu}(s)\) into \(M_{\mu}(s)\) and \(u_{\mu}(s)\). For instance, the SU(2) link variable \(U_{\mu}(s)\) becomes completely abelian for \(R_{\text{Abel}}(\mu, \mu) = 1\), while \(U_{\mu}(s)\) becomes completely off-diagonal for \(R_{\text{Abel}}(s, \mu) = 0\). This definition of \(R_{\text{Abel}}\) is inspired by the ordinary ‘distance’ between two matrices \(A, B \in \text{GL}(N, \mathbb{C})\) defined as \(d^{2}(A, B) \equiv \frac{1}{2} \text{tr}\{(A - B)^{\dagger}(A - B)\}\) [44], which leads to \(d^{2}(A, B) = 2 - \text{Re} \text{tr}(AB^\dagger)\) for \(A, B \in \text{SU}(2)\). In fact, the similarity between \(U_{\mu}(s)\) and \(u_{\mu}(s)\) can be quantitatively measured in terms of the ‘distance’ between them. In the strong-coupling limit \((\beta = 0)\), \(\langle R_{\text{Abel}} \rangle_{\beta=0} = \langle \cos \theta_{\mu}(s) \rangle_{\beta=0}\) without gauge fixing is analytically calculable as [23,43]

\[
\langle R_{\text{Abel}}(s, \mu) \rangle_{\beta=0} = \frac{\int dU_{\mu}(s) \cos \theta_{\mu}(s)}{\int dU_{\mu}(s)} = \frac{\int_{0}^{\pi} d\theta_{\mu} \sin \theta_{\mu} \cos^{2} \theta_{\mu}}{\int_{0}^{\pi} d\theta_{\mu} \sin \theta_{\mu} \cos \theta_{\mu}} = \frac{2}{3}.
\]

In the MA gauge, we find \(\langle R_{\text{Abel}} \rangle_{\text{MA}} = \langle \frac{1}{2} \Re \text{tr}(U_{\mu}(s)u_{\mu}^{\dagger}(s)) \rangle \simeq 1\), and the SU(2) link variable is U(1)\(_{3}\)-like as \(U_{\mu}(s) \simeq u_{\mu}(s)\) in the relevant gauge configuration. As a typical example, one obtains \(\langle R_{\text{Abel}} \rangle_{\text{MA}} \simeq 0.926\) on 16\(^{4}\) lattice.
with $\beta = 2.4$. Thus, in the MA gauge, the amplitude of the off-diagonal gluon $A_\mu^\pm(x)$ is strongly suppressed, which can be called as microscopic abelian dominance. On the other hand, the phase degrees of freedom $\tilde{\chi}_\mu(x)$ of $A_\mu^\pm(x)$ is not constrained by the MA gauge-fixing condition at all, and the constraint from the QCD action is also suppressed because of the strong reduction of $|A_\mu^\pm(x)|$ in the MA gauge. Therefore, in the MA gauge, the phase degrees of freedom $\tilde{\chi}_\mu(x)$ of the off-diagonal gluon $A_\mu^\pm(x)$ behaves as a random angle variable approximately, and this phase randomness leads to macroscopic abelian dominance on the confinement force \cite{30}.

### 3.2 Maximally Abelian Gauge in the Connection Formalism

In the gauge theory, the covariant derivative is more fundamental than the gauge field, and therefore the MA gauge fixing in the continuum SU($N_c$) QCD using the SU($N_c$) covariant derivative operator $\hat{D}_\mu \equiv \hat{\partial}_\mu + ieA_\mu$, where $\hat{\partial}_\mu$ is the derivative operator satisfying $[\hat{\partial}_\mu, f(x)] = \partial_\mu f(x)$. In addition, both the derivative operator and the Lie algebra appearing in $\hat{D}_\mu$ are expressed by the infinitesimal transformation of the corresponding group elements, so that they are to be described by way of the commutation relation. Then, the MA gauge is defined so as to minimize

$$ R_{\vec{H}}[A_\mu(\cdot)] \equiv \int d^4x \text{tr}[\hat{D}_\mu, \vec{H}]^\dagger[\hat{D}_\mu, \vec{H}] = c^2 \int d^4x \text{tr}[A_\mu, \vec{H}]^\dagger[A_\mu, \vec{H}] $$

$$ = c^2 \int d^4x \sum_{\alpha,\beta} A_\mu^\alpha A_\mu^\beta \vec{\alpha} \cdot \vec{\beta} \text{tr}(E_\alpha^\dagger E_\beta) = \frac{e^2}{2} \int d^4x \sum_{\alpha=1}^{N_c(N_c-1)} |A_\mu^\alpha|^2 $$

by the gauge transformation in the Euclidean QCD. Here, we have used the Cartan decomposition, $A_\mu \equiv A_\mu^a T^a = \vec{A}_\mu \cdot \vec{H} + \sum_{\alpha=1}^{N_c(N_c-1)} A_\mu^\alpha E_\alpha$, where $\vec{H} \equiv (T_3, T_8, \cdots, T_{N_c^2-1})$ is the Cartan subalgebra, and $E_\alpha(\alpha = 1, 2, \cdots, N_c^2 - N_c)$ denotes the raising or lowering operator. Since $R_{\vec{H}}[A_\mu]$ expresses the total amount of the off-diagonal gluon component, SU($N_c$) gauge connection $\hat{D}_\mu = \hat{\partial}_\mu + ieA_\mu^a T^a$ is mostly close to U(1)$^{N_c-1}$ gauge connection $\hat{D}_\mu^\vec{H} = \hat{\partial}_\mu + ie\vec{A}_\mu \cdot \vec{H}$ in the MA gauge. In our definition (49) using $\hat{D}_\mu$, the gauge transformation property of $R_{\vec{H}}[A_\mu]$ becomes quite transparent, because the SU($N_c$) covariant derivative $\hat{D}_\mu$ obeys the simple adjoint gauge transformation, $\hat{D}_\mu \rightarrow \Omega \hat{D}_\mu \Omega^\dagger$, with the SU($N_c$) gauge function $\Omega \in$ SU($N_c$). By the SU($N_c$) gauge transformation, $R_{\vec{H}}$ is transformed as

$$ R_{\vec{H}} \rightarrow R_{\vec{H}}^\Omega = \int d^4x \text{tr}\left([\Omega \hat{D}_\mu \Omega^\dagger, \vec{H}]^\dagger[\Omega \hat{D}_\mu \Omega^\dagger, \vec{H}]\right) $$
\[ = \int d^4x \text{tr} \left( [\hat{D}_\mu, \Omega^\dagger \check{H} \Omega] [\hat{D}_\mu, \Omega^\dagger \check{H} \Omega] \right), \quad (50) \]

and hence the residual symmetry corresponding to the invariance of \( R_{\check{H}} \) is easily found to be \( U(1)^{N_c-1}_{\text{local}} \times P_{\text{global}}^{N_c} \subset SU(N_c)_{\text{local}} \), where \( P_{\text{global}}^{N_c} \) denotes the global Weyl group relating to the permutation of the \( N_c \) bases in the fundamental representation, and \( N_c! \) elements includes. In fact, one finds \( \omega^\dagger \check{H} \omega = \check{H} \) for \( \omega = e^{-i\vec{\alpha}(x) \cdot \vec{H}} \in U(1)^{N_c-1}_{\text{local}} \), and the global Weyl transformation by \( W \in P_{\text{global}}^{N_c} \) only exchanges the permutation of the nontrivial root \( \vec{\alpha}_j \) and never changes \( R_{\check{H}} \). In the MA gauge, by definition, arbitrary gauge transformation by \( \forall V \in SU(N_c) \) is to increase \( R_{\check{H}} \) as \( R_{\check{V}} \geq R_{\check{H}} \). Considering arbitrary infinitesimal gauge transformation \( V = e^{i\varepsilon} \cong 1 + i\varepsilon \) with \( \varepsilon \in su(N_c) \), one finds

\[ V^\dagger \check{H} V \cong \check{H} + i[\check{H}, [\hat{D}_\mu, \hat{D}_\nu]] + 2i \int d^4x \text{tr} \left( [\hat{D}_\mu, [\hat{D}_\nu, \check{H}]] \right). \quad (51) \]

In the MA gauge, the extremum condition of \( R_{\check{V}} \) on \( \forall \varepsilon \in su(N_c) \) provides

\[ [\check{H}, [\hat{D}_\mu, [\hat{D}_\nu, \check{H}]]] = 0, \quad (52) \]

which leads to \( \sum_\mu (i\partial_\mu \pm eA_\mu^3) A_\mu^\pm = 0 \) for the \( N_c=2 \) case. Thus, the variable to be diagonalized in the MA gauge is easily derived as

\[ \Phi[A] = [\hat{D}_\mu, \hat{D}_\nu, \check{H}] \in su(N_c) \quad (53) \]

in the continuum theory. Here, \( \Phi[A] \) is hermite as \( \Phi^\dagger[A] = \Phi[A] \) because of \( \hat{D}_\mu^\dagger = -\hat{D}_\mu \), and hence the diagonal elements of \( \Phi[A] \) should be real.

Thus, \( \Phi[A] \) can be regarded as a sort of the ‘gluonic Higgs field’ relating to the MA gauge fixing, however, \( \Phi(A) \) does not obey the adjoint gauge transformation, so that correspondence between monopole and \( \Phi[A] \) is still unclear. The deviation of the ‘gluonic Higgs field’ \( \phi[A] \) obeying the adjoint transformation will be discussed in section 5.

In the commutator form, the diagonal part of the variable \( \check{O}[A_\mu(x)] \) is expressed as

\[ \check{O}^R = \check{O} - [\check{H}, [\check{H}, \check{O}]]. \quad (54) \]

For the covariant derivative operator, one finds
\begin{equation}
\hat{D}_\mu = \hat{D}_\mu - [\hat{H}, [\hat{H}, \hat{D}_\mu]] = \partial_\mu + ie \vec{A}_\mu(x) \cdot \vec{H}
\end{equation}

with \( A_\mu(x) = \vec{A}_\mu(x) \cdot \vec{H} + A_\mu^a(x) E^a \). Then, the abelian projection, \( \hat{D}_\mu \rightarrow \hat{D}_\mu^\mu \), is expressed by the simple replacement as \( A_\mu(x) \in su(N_c) \rightarrow \tilde{A}_\mu(x) = \vec{A}_\mu(x) \cdot \vec{H} \in u(1)^{N_c-1} \).

### 3.3 Generalization of the Maximally Abelian Gauge

In the MA gauge, \( R_{\vec{H}}[A_\mu(\cdot)] \) in Eq.(49) is forced to be reduced by the MA gauge transformation \( \Omega_{\text{MA}}(x) \in G/H \), and therefore the gluon field \( A_\mu(x) \) is maximally arranged in the diagonal direction \( \vec{H} \) in the internal SU\((N_c)\) color space. In the definition of the MA gauge, \( \vec{H} \) is the specific color-direction, since \( \vec{H} \) explicitly appears in the MA gauge-fixing condition with \( R_{\vec{H}}[A_\mu(\cdot)] \). On this point of view, the MA gauge can be called as the ‘maximally diagonal gauge’. However, for the extraction of the abelian gauge theory from the nonabelian theory, we need not take the specific direction as \( \vec{H} \) in the internal color-space, although the system becomes transparent when the specific color-direction as \( \vec{H} \) is introduced on the maximal arrangement of the gluon field \( A_\mu(x) \).

In this subsection, we consider the generalization of the framework of the MA gauge and the abelian projection, without explicit use of the specific direction \( \vec{H} \) in the internal color-space on the gauge fixing. (Such an attempt is similar to the generalization of the formalism in the center-of-mass frame to that in the general moving frame.) Instead of the special color-direction \( \vec{H} \), we introduce the ‘Cartan frame field’ \( \vec{\phi}(x) \equiv (\phi_1(x), \phi_2(x), \cdots, \phi_{N_c-1}(x)) \), where \( \phi_i(x) \equiv \phi_i^a(x) T^a \ (\phi_i^a(x) \in \mathbb{R}) \) commutes each other as \( [\phi_i(x), \phi_j(x)] = 0 \), and satisfy the orthonormality condition \( 2\text{tr}(\phi_i(x)\phi_j(x)) = \sum_{a=1}^{N_c-1} \phi_i^a(x)\phi_j^a(x) = \delta_{ij} \). At each point \( x_\mu \), \( \vec{\phi}(x) \) forms the Cartan sub-algebra, and can be expressed as

\begin{equation}
\vec{\phi}(x) = \Omega_C^\dagger(x) \vec{H} \Omega_C(x)
\end{equation}

using \( \Omega_C(x) \in G/H \). For the fixed Cartan frame field \( \vec{\phi}(x) \), we define the generalized maximally abelian (GMA) gauge so as to minimize the functional

\begin{equation}
R_{\vec{\phi}}[A_\mu(\cdot)] \equiv \int d^4x \text{tr}[\hat{D}_\mu, \vec{\phi}(x)]^\dagger[\hat{D}_\mu, \vec{\phi}(x)]
\end{equation}

by the SU\((N_c)\) gauge transformation. Here, the Cartan frame field \( \vec{\phi}(x) \) is defined at each \( x_\mu \) independent of the gluon field like \( \vec{H} \), and never changes under the SU\((N_c)\) gauge transformation. For the special case of \( \vec{\phi}(x) = \vec{H} \), the GMA gauge returns to the usual MA gauge. In the GMA gauge, the SU\((N_c)\)
covariant derivative $\hat{D}_\mu$ is maximally arranged to be ‘parallel’ to the $\bar{\phi}(x)$-direction in the internal color-space using the SU($N_c$) gauge transformation.

In the GMA gauge, the gauge symmetry is reduced from SU($N_c$) into U(1)$_{\bar{\phi}}^{N_c-1}$, and the generalized AP-QCD leads to the monopole in the similar manner to the MA gauge. In the GMA gauge, the remaining U(1)$_{\bar{\phi}}^{N_c-1}$ gauge symmetry corresponds to the invariance of $R_{\bar{\phi}}[A_\mu(\cdot)]$ under the U(1)$_{\bar{\phi}}^{N_c-1}$ gauge transformation by

$$\omega_{\phi}(x) \equiv e^{i\bar{\phi}(x)\cdot \bar{\chi}(x)} \in U(1)^{N_c-1}, \quad \bar{\chi}(x) \in R^{N_c-1}. \quad (58)$$

In fact, using $\omega^i_{\phi}(x)\bar{\phi}(x)\omega_{\phi}(x) = \bar{\phi}(x)$, U(1)$_{\bar{\phi}}^{N_c-1}$ invariance of $R_{\bar{\phi}}[A_\mu(\cdot)]$ is easily confirmed as

$$(R_{\bar{\phi}}[A_\mu])^\mu = \int d^4x [\omega(x)\phi \hat{D}_\mu \omega^\dagger_{\phi}(x), \bar{\phi}(x)] [\omega(x)\phi \hat{D}_\mu \omega^\dagger_{\phi}(x), \bar{\phi}(x)] \quad (59)$$

$$= \int d^4x [\hat{D}_\mu, \omega^\dagger_{\phi}(x)\bar{\phi}(x)\omega_{\phi}(x)] [\hat{D}_\mu, \omega^\dagger_{\phi}(x)\bar{\phi}(x)\omega_{\phi}(x)] = R_{\bar{\phi}}[A_\mu].$$

There also remains the global Weyl symmetry $P_{N_c}$ similarly in the usual MA gauge, although the gauge function takes a complicated from.

Here, we consider the generalized abelian projection to $\bar{\phi}(x)$-direction. Similar to the ‘diagonal part’ in Eq.(54), we define the ‘$\bar{\phi}(x)$-projection’ of the operator $\hat{O}(x)$ as

$$\hat{O}^\phi(x) = \hat{O}(x) - [\bar{\phi}(x), [\bar{\phi}(x), \hat{O}(x)]] , \quad (60)$$

using the commutation relation. For the SU($N_c$) covariant derivative operator $\hat{D}_\mu \equiv \hat{\partial}_\mu + ieA_\mu$, its $\bar{\phi}(x)$-projection is defined as

$$\hat{D}_\mu^\phi \equiv \hat{D}_\mu - [\bar{\phi}(x), [\bar{\phi}(x), \hat{D}_\mu]] = \hat{\partial}_\mu + ieA_\mu^\phi(x) + [\bar{\phi}(x), \partial_\mu\bar{\phi}(x)] \quad (61)$$

with $A_\mu^\phi(x) \equiv \vec{A}_\mu^\phi(x) \cdot \vec{\phi} = 2\text{tr}(\vec{\phi}(x)A_\mu(x)) \cdot \bar{\phi}(x)$. Here, the nontrivial term $[\bar{\phi}(s), \partial_\mu\bar{\phi}(x)]$ appears in $\hat{D}_\mu^\phi$ owing to the $x$-dependence of the Cartan-frame field $\bar{\phi}(x)$. The U(1)$_{\bar{\phi}}^{N_c-1}$ gauge field is defined as the difference between $\hat{D}_\mu^\phi$ and $\hat{\partial}_\mu$,

$$\vec{A}_\mu^\phi(x) \equiv \frac{1}{ie} (\hat{D}_\mu^\phi - \hat{\partial}_\mu) = A_\mu^\phi(x) + \frac{1}{ie} [\bar{\phi}(x), \partial_\mu\bar{\phi}(x)] \in su(N_c). \quad (62)$$

Here, $\vec{A}_\mu^\phi(x)$ includes both the $\bar{\phi}(x)$-component $A_\mu^\phi(x) = 2\text{tr}(A_\mu(x)\bar{\phi}(x)) \cdot \bar{\phi}(x)$ and the non-$\bar{\phi}(x)$-component $\frac{1}{ie}[\bar{\phi}(x), \partial_\mu\bar{\phi}(x)]$, because $[\bar{\phi}(x), \partial_\mu\bar{\phi}(x)]$ does not
include \( \tilde{\phi}(x) \)-component as \( \text{tr} \left( \phi_i(x)[\tilde{\phi}(x), \partial_\mu \tilde{\phi}(x)] \right) = 0 \). Here, the \( \tilde{A}_\mu^\phi(x) \) is the image of \( A_\mu^\phi(x) \) mapped into the \( U(1)^{N_c-1} \)-manifold. The the generalized abelian projection for the variable \( O[A_\mu(x)] \) is defined via the two successive mapping, \( O[A_\mu(x)] \rightarrow O[\tilde{A}_\mu^\phi(x)] \rightarrow \tilde{O}_{AP} \equiv 2\text{tr}(\tilde{\phi}(x)O[\tilde{A}_\mu^\phi(x)]) \), after the GMA gauge fixing.

Under the \( U(1)^{N_c-1} \)-abelian gauge transformation by \( \omega_\phi(x) = e^{i\tilde{\phi}(x)\cdot \vec{x}(x)} \in U(1)^{N_c-1} \), \( \tilde{A}_\mu^\phi(x) \) or \( \tilde{A}_\mu^\phi(x) \) behaves as the \( U(1)^{\phi-1} \) abelian gauge field,

\[
\tilde{A}_\mu^\phi(x) \rightarrow (\tilde{A}_\mu^\phi(x))^\omega = \tilde{A}_\mu^\phi(x) + \frac{1}{\text{i}e} \partial_\mu \tilde{x}(x) \cdot \tilde{\phi}(x). \tag{63}
\]

The \( U(1)^{N_c-1} \)-abelian field strength is defined as \( \tilde{F}_{\mu\nu}^\phi(x) \equiv \frac{1}{\text{i}e} \{ [\hat{D}_\mu, \hat{D}_\nu] - [\hat{\partial}_\mu, \hat{\partial}_\nu] \} \), which generally includes the non-\( \tilde{\phi}(x) \) component as well as \( \tilde{A}_\mu^\phi(x) \).

In the \( U(1)^{N_c-1} \)-manifold, the \( \tilde{\phi}(x) \)-component \( \tilde{F}_{\mu\nu}^\phi(x) \equiv 2\text{tr}(\tilde{F}_{\mu\nu}(x)\tilde{\phi}(x)) \) is observed as the mapped image of \( \tilde{F}_{\mu\nu}^\phi(x) \).

Next, we investigate the properties of the GMA gauge function \( \Omega_{\text{GMA}}(x) \), which brings the GMA gauge. Here, \( \Omega_{\text{GMA}}(x) \) is a complicated function of \( A_\mu(x) \) and is expressed by an element of the coset space \( G/H = SU(N_c)/\{U(1)^{N_c-1} \times \text{Weyl}\} \) as the representative element because of the residual gauge symmetry.

For instance, we impose here

\[
\text{tr}(\Omega_{\text{GMA}}(x)\tilde{\phi}(x)) = 0 \tag{64}
\]

for the selection of \( \Omega_{\text{GMA}} \in G/H \). Similarly to the MA gauge function[30], \( \Omega_{\text{GMA}}[A_\mu] \) obeys the nonlinear transformation as

\[
\Omega_{\text{GMA}}(x) \in G/H \rightarrow (\Omega_{\text{GMA}}(x))^V = d^V(x)\Omega_{\text{GMA}}(x)\Omega_{\text{GMA}}(x)^{\dagger} \in G/H \tag{65}
\]

by the \( SU(N_c) \) gauge transformation with \( V(x) \in G \). Here, \( d^V(x) \in H \equiv U(1)^{N_c-1} \times \text{Weyl} \) appears to keep \( (\Omega_{\text{GMA}})^V \) belonging to \( G/H \). Therefore, the gluon field \( A_\mu^{\text{GMA}} = \Omega_{\text{GMA}}(A_\mu + \frac{1}{\text{i}e} \partial_\mu)\Omega_{\text{GMA}}^\dagger \in g \) in the GMA gauge is transformed as

\[
A_\mu^{\text{GMA}} \rightarrow (A_\mu^{\text{GMA}})^V = \Omega_{\text{GMA}}^V(A_\mu + \frac{1}{\text{i}e} \partial_\mu)\Omega_{\text{GMA}}^{\dagger}(x)
= d^V(x)(A_\mu^{\text{GMA}} + \frac{1}{\text{i}e} \partial_\mu)\Omega_{\text{GMA}}^{\dagger}(x) = (A_\mu^{\text{GMA}})^d^V \tag{66}
\]

by the \( SU(N_c) \) gauge transformation. As a remarkable feature, the \( SU(N_c) \) gauge transformation by \( V(x) \in G \) is mapped as the abelian sub-gauge trans-
formation by \( d \mathbf{V}(x) \in H \) in the GMA gauge: \( (A^GMA)_{\mu} = (A^GMA)^{d \mathbf{V}} \). In particular, for the residual gauge transformation by \( \omega(x) = e^{i \bar{\phi}(x) \cdot \bar{\chi}(x)} \in H \), we find \( d \omega(x) = \omega(x) \) to keep the representative-element condition \( \text{tr}(\Omega_{GMA}(x) \bar{\omega}(x)) = 0 \) imposed above, and then \( A^GMA_{\mu} \) obeys the ordinary \( H \)-gauge transformation

\[
A^GMA_{\mu}(x) \rightarrow (A^GMA_{\mu}(x))^{\omega} = \omega(x)(A^GMA_{\mu} + \frac{1}{ie} \partial_\mu \omega^\dagger(x)). \tag{67}
\]

For the arbitrary variable \( \tilde{O}[A^GMA] = \hat{O}[A^\Omega GMA] \) defined in the GMA gauge, we find \( \hat{O}[A^GMA]^\omega = \hat{O}[A^GMA]^{d \mathbf{V}} \) with \( d \mathbf{V} \in H \) from Eq.(66), and hence we get an useful criterion on the SU(\( N_c \)) gauge invariance: if \( \hat{O}[A] \) is \( H \)-invariant as \( \hat{O}[A]^\omega = \hat{O}[A] \) for \( \forall \omega \in H \), \( \hat{O}[A^GMA] \) is also \( G \)-invariant, because of \( \hat{O}[A^GMA]^V = \hat{O}[A^GMA]^{d \mathbf{V}} = \hat{O}[A^GMA] \) for \( \forall V \in G \). All of the above arguments are also applicable to the usual MA gauge by setting \( \bar{\phi}(x) = \bar{H} \).

For the regular field \( \bar{\phi}(x) \) without any discontinuity, the GMA gauge function \( \Omega_{GMA} \in \text{SU}(N_c)/U(1)_{N_c-1} \) becomes singular like the MA gauge, which was discussed in section 2.3. Then, a nontrivial singular term appears in the field strength as

\[
G^GMA_{\mu\nu} = \partial_\mu A^GMA_\nu - \partial_\nu A^GMA_\mu + ie[A^GMA_\mu, A^GMA_\nu] + \frac{i}{e} \Omega_{GMA}[\partial_\mu, \partial_\nu] \Omega^\dagger_{GMA}. \tag{68}
\]

Similar to the MA gauge, the singularity on \( \Omega_{GMA} \) induces breaking of the \( U(1)_{N_c-1} \) abelian Bianchi identity and the monopole current in the \( U(1)_{N_c-1} \) abelian sector.

The correspondence between \( \Omega_{GMA} \) and \( \Omega_{MA} \) is straightforward. Using \( \Omega_C(x) \in \text{SU}(N_c) \) satisfying \( \bar{\phi}(x) = \Omega_C^\dagger(x) \bar{H} \Omega_C(x) \), \( \Omega_{GMA} \) is expressed as

\[
\Omega^GMA(x) = \Omega_C^\dagger(x) \Omega^MA(x). \tag{69}
\]

Then, for regular \( \bar{\phi}(x) \), \( \Omega_C(x) \) becomes regular, and the singularity of \( \Omega_{MA} \) is directly mapped to that of \( \Omega_{GMA} \). However, if singular \( \bar{\phi}(x) \) is used, the singularity of \( \Omega_{MA} \) can be mapped in \( \bar{\phi}(x) \) or \( \Omega_C(x) \) instead of \( \Omega_{GMA} \). In this case, the gluon field \( A^GMA_\mu \) is kept to be regular, and the Cartan frame field \( \bar{\phi}(x) \) includes the multi-valuedness or the singularity, which leads to the monopole. Such a situation will be discussed in section 5 considering the analogy with the nonabelian Higgs theory.
4 Large Field Fluctuation around Monopoles

In this section, we study the QCD-monopole appearing in the abelian gauge in terms of the gluon field fluctuation\([45]\). For simplicity, we take \(N_c = 2\). In the static frame of the QCD-monopole with the magnetic charge \(g\), a spherical ‘magnetic field’ is created around the monopole in the abelian sector of QCD as

\[
H(r) = \frac{g}{4\pi r^3} r
\]

with \(H_i \equiv \varepsilon_{ijk} \partial_j A_k^3\). Thus, the QCD-monopole inevitably accompanies a large fluctuation of the abelian gluon component \(A_3^\mu\) around it. As was discussed in section 2, in the abelian gauge, the formal action of the abelian projected QCD or the abelian part of the QCD action is given by \(S_{\text{Abel}} \equiv -\frac{1}{4} \int d^4x \left\{ \left( \partial_\mu A_3^\nu - \partial_\nu A_3^\mu \right)^2 - F_{\mu\nu}^{\text{sing}} \right\}\), where \(-F_{\mu\nu}^{\text{sing}}\) appears and eliminates the Dirac-string contribution. In the abelian part, the field energy created around the monopole is estimated as the ordinary electro-magnetic energy,

\[
E(a) = \int_0^\infty d^3x \frac{1}{2} H(r)^2 = \frac{g^2}{8\pi a},
\]

where \(a\) is an ultraviolet cutoff like a lattice mesh. As the ‘mesh’ \(a\) goes to 0, the monopole inevitably accompanies an infinitely large energy-fluctuation in the abelian part and makes \(S_{\text{Abel}}\) divergent.

Since there seems no plausible reason to eliminate such a divergence via renormalization, the monopole seems difficult to appear in the abelian gauge theory controlled by \(S_{\text{Abel}}\). This is the reason why QED does not have the point-like Dirac monopole. Then, why can the QCD-monopole appear in the abelian projected QCD? To answer it, let us consider the division of the total QCD action \(S_{\text{QCD}}\) into the abelian part \(S_{\text{Abel}}\) and the remaining part \(S_{\text{off}} \equiv S_{\text{QCD}} - S_{\text{Abel}}\), which is contribution from the off-diagonal gluon component. While \(S_{\text{QCD}}\) and \(S_{\text{Abel}}\) are positive definite in the Euclidean metric, \(S_{\text{off}}\) is not positive definite and can take a negative value. Then, around the QCD-monopole, the abelian action \(S_{\text{Abel}}\) should be partially canceled by the remaining contribution \(S_{\text{off}}\) from the off-diagonal gluon component, so as to keep the total QCD action \(S_{\text{QCD}}\) finite even for \(a \to 0\). Similar cancellation between the gauge field and the Higgs field fluctuation is also found around the GUT monopole. Thus, we expect large off-diagonal gluon components around the QCD-monopole for its existence as well as a large field fluctuation in the abelian part. Based on this analytical consideration, we study the field fluctuation and monopoles in the MA gauge using the lattice QCD.
To begin with, we study the gluon field configuration around the monopole in the MA gauge in terms of the abelian angle variable $\theta_3^\mu(s)$ and abelian projection rate $R_{\text{Abel}} \equiv \cos \theta_\mu(s)$, which measures the off-diagonal gluon remaining in the MA gauge [30]. For the argument of $\theta_3^\mu(s)$, the $U(1)_3$ gauge degrees of freedom should be also fixed after the MA gauge fixing, because $\theta_3^\mu(s)$ is $U(1)_3$ gauge dependent. Here, we adopt the $U(1)_3$ Landau gauge [46,47] defined by maximizing

$$R[U_\mu] \equiv \sum_{s,\mu} \text{tr} u_\mu(s) = 2 \sum_{s,\mu} \cos \theta_3^\mu(s)$$

by the residual $U(1)_3$ gauge transformation. In the $U(1)_3$ Landau gauge, there remains no local symmetry, and the lattice variable mostly approaches to the continuum field under the constraint of the MA gauge fixing.

Now, let us consider the correlation between the field variables and the monopole in the lattice QCD. For this argument, one has to recall the property of the monopole current shown in section 2.4. In particular, one should note that $k_\mu(s)$ is defined on the dual link and only affects the perpendicular components to the $\hat{\mu}$-direction for the electric variable as $F_{\alpha\beta}$ because of

$$k_\mu \equiv \partial_\alpha F_{\alpha\mu} = -\frac{1}{2} \varepsilon_{\mu\alpha\beta\gamma} \partial_\alpha F_{\beta\gamma}.$$ 

Taking account of these properties, we study the local correlation between the field variables and the monopole current $k_\mu(s)$ in the MA gauge with the $U(1)_3$ Landau gauge using the lattice-QCD Monte-Carlo simulation. We first measure the average of the abelian angle variable $\theta_3^\mu(s)$ over the neighboring links around the dual link (See Fig.4),

$$|\bar{\theta}_3^\mu(s,\hat{\mu})| \equiv \frac{1}{12} \sum_{\alpha\beta\gamma} \sum_{m,n=0}^1 \frac{1}{2} |\varepsilon_{\mu\alpha\beta\gamma}| \cdot |\theta_3^\alpha(s + m\hat{\beta} + n\hat{\gamma})|,$$

which only consists of the perpendicular components considering the above monopole property. Here, the index $\hat{\mu}$ denotes the direction of the dual link, and $|\bar{\theta}_3^\mu(s,\hat{\mu})|$ corresponds to the average over the 12 sides of the 3-dimensional cube perpendicular to the $\hat{\mu}$-direction. We show in Fig.6 the probability distribution $P(|\bar{\theta}_3^\mu|)$ of $|\bar{\theta}_3^\mu(s,\hat{\mu})|$ in the MA gauge with the $U(1)_3$ Landau gauge at $\beta = 2.4$. The solid curve denotes $P(|\bar{\theta}_3^\mu|)$ around the monopole current, while the dashed curve denotes the total distribution on the whole lattice. The abelian angle variable $|\theta_3^\mu(s)|$ takes a large value around the monopole. In other words, the monopole provides the large fluctuation of the abelian gauge field, which would enhance the randomness of the abelian link variable.

Similar to $|\bar{\theta}_3^\mu(s,\hat{\mu})|$, we measure the average $\bar{R}_{\text{Abel}}$ of the abelian projection rate $R_{\text{Abel}}(s,\hat{\mu}) \equiv \cos \theta_\mu(s)$ over the neighboring links around the dual link,
\[
\bar{R}_{\text{Abel}}(s, \hat{\mu}) \equiv \frac{1}{12} \sum_{\alpha\beta\gamma} \sum_{m,n=0}^{1} \frac{1}{2} |\varepsilon_{\mu\alpha\beta\gamma}| \cos \theta_{\alpha}(s + m\hat{\beta} + n\hat{\gamma})
\]

(74)

in the MA gauge to investigate the correlation between off-diagonal gluons and monopoles. As shown in Fig.7(a), \(\bar{R}_{\text{Abel}}\) around the monopole current becomes smaller than the total average of \(\bar{R}_{\text{Abel}}\) and therefore the magnitude of the off-diagonal gluon component becomes larger around the monopole. The \(\beta\) dependence of the abelian projection rate \(\langle R_{\text{Abel}} \rangle\) is shown in Fig.7(b). Although \(\langle R_{\text{Abel}} \rangle\) on the whole lattice approaches to unity as \(\beta \to \infty\), \(\langle R_{\text{Abel}} \rangle\) around the monopole is about 0.88 and is not changed even in the large \(\beta\) region. Thus, the monopole provides the large fluctuation both for the abelian field and for the off-diagonal gluon.

We next study monopoles in terms of the plaquette action density. We define the SU(2), abelian and ‘off-diagonal’ plaquette action densities as

\[
S_{\mu\nu}^{\text{SU(2)}}(s) \equiv 1 - \frac{1}{2} \text{tr} \square_{\mu\nu}^{\text{SU(2)}}(s),
\]

(75)

\[
S_{\mu\nu}^{\text{Abel}}(s) \equiv 1 - \frac{1}{2} \text{tr} \square_{\mu\nu}^{\text{Abel}}(s),
\]

(76)

\[
S_{\mu\nu}^{\text{off}}(s) \equiv S_{\mu\nu}^{\text{SU(2)}}(s) - S_{\mu\nu}^{\text{Abel}}(s),
\]

(77)

where \(\square_{\mu\nu}^{\text{SU(2)}}(s)\) and \(\square_{\mu\nu}^{\text{Abel}}(s)\) denote the SU(2) and the abelian plaquette variables, respectively;

\[
\square_{\mu\nu}^{\text{SU(2)}}(s) \equiv U_{\mu}(s)U_{\nu}(s + \hat{\mu})U_{\mu}^\dagger(s + \hat{\nu})U_{\nu}^\dagger(s),
\]

(78)

\[
\square_{\mu\nu}^{\text{Abel}}(s) \equiv u_{\mu}(s)u_{\nu}(s + \hat{\mu})u_{\mu}^\dagger(s + \hat{\nu})u_{\nu}^\dagger(s).
\]

(79)

Here, all of \(S_{\mu\nu}\) are defined as symmetric tensors, \(S_{\mu\nu} = S_{\nu\mu}\), instead of the Lorentz scalar, considering the above property of the monopole current. In the continuum limit \(a \to 0\), \(S_{\mu\nu}^{\text{SU(2)}}(s)\) and \(S_{\mu\nu}^{\text{Abel}}(s)\) are related to the SU(2) and the abelian action densities as \(S_{\mu\nu}^{\text{SU(2)}}(s) \to \frac{1}{4} a^4 e^2 \text{tr} G_{\mu\nu}^2\) and \(S_{\mu\nu}^{\text{Abel}}(s) \to \frac{1}{4} a^4 e^2 \text{tr} F_{\mu\nu}^2\), and then we call \(S_{\mu\nu}\) as the action density, in spite of the lack of the summation on the Lorentz indices. Here, \(S_{\mu\nu}^{\text{off}}\) corresponds to the contribution of the off-diagonal gluon. While \(S_{\mu\nu}^{\text{SU(2)}}\) and \(S_{\mu\nu}^{\text{Abel}}\) are positive-definite, \(S_{\mu\nu}^{\text{off}}\) is not positive-definite and can take a negative value.

In order to examine the correlation between the action densities and the monopole current defined on the dual link, we measure the average of the action density \(S(s)\) over the neighboring plaquettes around the dual link,

\[
\bar{S}(s, \hat{\mu}) \equiv \frac{1}{6} \sum_{\alpha\beta\gamma} \sum_{m,n=0}^{1} \frac{1}{2} |\varepsilon_{\mu\alpha\beta\gamma}| S_{\alpha\beta}(s + m\hat{\gamma}).
\]

(80)
Here, \( \hat{\mu} \) appearing in \( \hat{S}(s, \hat{\mu}) \) denotes the direction of the dual link, and \( \hat{S}(s, \hat{\mu}) \) corresponds to the average over 6 faces of the 3-dimensional cube perpendicular to the \( \hat{\mu} \)-direction.

We show in Fig.8 the probability distribution \( P(\hat{S}) \) of the action densities \( \hat{S}(s, \hat{\mu}) \) in the SU(2), the abelian and the off-diagonal parts. Before the argument around the monopole current, we show the action densities on the whole lattice in Fig.8 (a). On the whole lattice, most \( \hat{S}^{\text{off}} \) are positive, and both \( \hat{S}^{\text{Abel}} \) and \( \hat{S}^{\text{off}} \) tend to take smaller values than \( \hat{S}^{\text{SU}(2)} = \hat{S}^{\text{Abel}} + \hat{S}^{\text{off}} \). In other words, \( \hat{S}^{\text{Abel}} \) and positive \( \hat{S}^{\text{off}} \) additionally contribute to \( \hat{S}^{\text{SU}(2)} \).

However, such a tendency of the action densities is drastically changed around the monopole as shown in Fig.8(b). We find remarkable features of the action densities around the monopole as follows.

1. Around monopoles, most \( \hat{S}^{\text{off}} \) take negative values, and \( \hat{S}^{\text{Abel}} \) is larger than \( \hat{S}^{\text{SU}(2)} = \hat{S}^{\text{Abel}} + \hat{S}^{\text{off}} \).
2. Due to the cancellation between \( \hat{S}^{\text{Abel}} \) and \( \hat{S}^{\text{off}} \), \( \hat{S}^{\text{SU}(2)} \) does not take an extremely large value around the monopole.

Thus, the large abelian action density \( \hat{S}^{\text{Abel}} \) around the monopole is strongly canceled by the off-diagonal contribution \( \hat{S}^{\text{off}} \) to keep the total QCD action \( \hat{S}^{\text{QCD}} = \hat{S}^{\text{Abel}} + \hat{S}^{\text{off}} \) small. Here, different from \( \hat{S}^{\text{SU}(2)} \), \( \hat{S}^{\text{Abel}} \) itself does not control the system directly, and hence there is no severe constraint from \( \hat{S}^{\text{Abel}} \). However, large \( \hat{S}^{\text{Abel}} \) is still not preferable, because the large-cancellation requirement between \( \hat{S}^{\text{Abel}} \) and \( \hat{S}^{\text{off}} \) leads to a strong constraint on the off-diagonal gluon and brings the strong reduction of the configuration number.

Around the monopole, the abelian action density \( \hat{S}^{\text{Abel}} \) takes a large value, and this value can be estimated from a following simple calculation. Without loss of generality, the monopole-current direction is locally set to be parallel to the temporal direction as \( k_0^{\text{lat}}(s) \equiv \frac{1}{2\pi} \partial_0^* \hat{\theta}_{00}(s) = \pm 1 \). Here, \( k_0^{\text{lat}}(s) \) is expressed as the sum of six plaquette variables \( \hat{\theta}_{ij}(i,j=1,2,3) \) around the monopole, because of \( k_0^{\text{lat}}(s) = -\frac{1}{4\pi} \varepsilon_{ijk} \hat{\theta}_{ij}(s) + \frac{1}{2\pi} \sum_{i,j<k} \varepsilon_{ijk} \{ \hat{\theta}_{jk}(s) - \hat{\theta}_{ij}(s) \} \). Hence, the total sum of six \( |\hat{\theta}_{ij}(s)|(i < j) \) is to exceed \( 2\pi \) to realize \( k_0(s) = \pm 1 \). Since large \( |\hat{\theta}_{ij}(s)| \) accompanying large \( \hat{S}^{\text{Abel}} \) is not preferable, the magnetic field \( |\hat{\theta}_{ij}| \) around the monopole is estimated as \( |\hat{\theta}_{ij}| \simeq 2\pi/6 = \pi/3 \) on the average, using the spherical symmetry of the magnetic field in the vicinity of the monopole. Accordingly, we estimate as \( \hat{S}^{\text{Abel}}_{ij} = 1 - \cos(|\hat{\theta}_{ij}|) \simeq 1 - \cos \frac{\pi}{3} = \frac{1}{2} \) around the monopole on the average. The above argument can be easily generalized to the case with arbitrary monopole-current direction.

Then, existence of monopoles brings a peak around \( \hat{S}^{\text{Abel}} = \frac{1}{2} \) in the distribution \( P(\hat{S}^{\text{Abel}}) \). In fact, the abelian action density \( \hat{S}^{\text{Abel}} \) has two ingredients; one is nontrivial large fluctuation about \( \hat{S}^{\text{Abel}} = 1/2 \) originated from the monopole, and the other is remaining small fluctuations, which is expected to vanish as
$S_{\text{Abel}} \to 0$ as $a \to 0$. As shown in Fig.9, the peak originated from the monopole is almost $\beta$ independent, while the other fluctuation becomes small for large $\beta$. At a glance from this result, the monopole seems hard to exist at the small mesh $a$, since the monopole needs a large abelian action $S_{\text{Abel}}$. Nevertheless, the monopole can exist in QCD even in the large $\beta$ region owing to the contribution of the off-diagonal gluon. As shown in Fig.8(b), the off-diagonal part $S_{\text{off}}$ of the action density around the monopole tends to take a large negative value, and strongly cancels with the large abelian action $S_{\text{Abel}}$ to keep the total SU(2) action $S_{\text{QCD}}$ finite.

Here, we consider the angle variable $\tilde{\chi}_\mu(x)$ of the off-diagonal gluons $A_\mu^\pm(x)$ around the monopole. In the MA gauge, the amplitude of $A_\mu^\pm(x)$ is strongly reduced, and $\tilde{\chi}_\mu(x)$ can be approximated as a random variable on the whole, because $\tilde{\chi}_\mu(x)$ is free from the MA gauge condition entirely and is less constrained from the QCD action due to the small $|A_\mu^\pm(x)|$. However, around the monopole, the off-diagonal gluon $A_\mu^\pm(x)$ inevitably has a large amplitude even in the MA gauge to cancel the large abelian action density. This requirement on the reduction of the total action density severely constrains the randomness of the angle variable $\tilde{\chi}_\mu(x)$ of the off-diagonal gluon $A_\mu^\pm(x)$ around the monopole. As the result, the randomness of $\tilde{\chi}_\mu(x)$ is weaken, and continuity of $\tilde{\chi}_\mu(x)$ or $A_\mu^\pm(x)$ becomes clear in the vicinity of the monopole even in the MA gauge. This continuity of $A_\mu^\pm(x)$ around the monopole ensures the topological stability of the monopole itself as $\Pi_2(\text{SU}(2)/\text{U}(1)) = \mathbb{Z}_\infty$.

To summarize, existence of the monopole inevitably accompanies a large abelian plaquette action $S_{\text{Abel}}$ around it, however, the off-diagonal part $S_{\text{off}}$ takes a large negative value around the monopole and strongly cancels with $S_{\text{Abel}}$ to keep $S_{\text{QCD}}$ not so large. Due to this strong cancellation between $S_{\text{Abel}}$ and $S_{\text{off}}$, monopoles can appear in the abelian sector in QCD without large cost of the QCD action $S_{\text{QCD}}$, which controls the generating probability of the gluon configuration. The extension of the off-diagonal rich region around the monopole can be interpreted as the effective size or the structure of the monopole, because the abelian gauge theory is largely modified inside the QCD-monopole like the ’t Hooft-Polyakov monopole.

Finally, in this section, let us consider the correlation between monopoles and instantons [48] in terms of the gluon-field fluctuation. The instanton is a non-trivial classical solution of the Euclidean Yang-Mills theory, corresponding to the homotopy group $\Pi_3(\text{SU}(N_c)) = \mathbb{Z}_\infty[4,5]$. For the instanton, the SU(2) structure of the gluon field is necessary at least. In spite of the difference on the topological origin, recent studies indicate the strong correlation between monopoles and instantons in the QCD vacuum in the MA gauge[49,50]. What is the origin of the relation between two different topological objects, monopoles and instantons? In the MA gauge, off-diagonal components are forced to be small, and the gluon field configuration seems abelian on the
whole. However, even in the MA gauge, off-diagonal gluons largely remain around the QCD-monopole. The concentration of off-diagonal gluons around monopoles leads to the local correlation between monopoles and instantons: instantons appear around the monopole world-line in the MA gauge, because instantons need full SU(2) gluon components for existence.

5 Gluonic Higgs Field in QCD and Monopoles

QCD in the maximally abelian (MA) gauge has several similarities with the nonabelian Higgs (NAH) theory in terms of the gauge-symmetry reduction and the appearance of monopoles, although the symmetry reduction is realized by the gauge fixing instead of Higgs condensation. In this section, we try to formulate QCD in the MA gauge in the similar manner to the NAH theory. To this end, we introduce the concept of the ‘gluonic Higgs field’ $\tilde{\phi}_D[A_\mu(x)]$ with $\phi_D[A_\mu(x)] \in su(N_c)$, a gluonic composite scalar defined from the SU($N_c$) covariant derivative $\tilde{D}_\mu$ in subsection 5.1. By way of $\tilde{\phi}_D[A_\mu(x)]$, we formulate the abelian projection in QCD without explicit use of the notion of the gauge fixing. In this formalism, the abelian projection resembles the extraction of the photon field in the NAH theory. In subsection 5.2, we study the connection of the gluonic Higgs field with the MA gauge, and examine the correspondence between the monopole in the MA gauge and the hedgehog configuration of $\tilde{\phi}_D[A_\mu(x)]$ in the lattice QCD.

5.1 Gluonic Higgs Field for the Relevant Abelian Submanifold in QCD

The abelian dominance in the MA gauge observed in the lattice QCD indicates the existence of the infrared-relevant abelian gauge submanifold embedded in QCD, and we call it the ‘relevant abelian submanifold’ in QCD. The ordinary abelian projection in the MA gauge can be interpreted as an concrete procedure to extract this abelian manifold in QCD. Here, we extend the concept of the ‘abelian projection’ as the extraction of this relevant abelian submanifold in QCD. In the NAH theory, the extraction of the photon field corresponds to the abelian projection, and the Higgs field $\phi(x)$ indicates the ‘abelian direction’ in the nonabelian gauge manifold. Based on the similarity with the NAH theory, we introduce the ‘gluonic Higgs field’ $\phi[A_\mu(x)]$ to extract the relevant abelian submanifold in QCD, referring the abelian projection in the MA gauge. As was shown in section 2, if $\phi[A_\mu(x)]$ obeys the adjoint transformation, the monopole in the abelian gauge can be understood as the topological defect on $\phi[A_\mu(x)]$, and the abelian-projection scheme in QCD becomes analogous to the NAH theory, by regarding $\phi[A_\mu(x)]$ as the Higgs field.
The variable $\Phi[A_\mu(x)]$ in Eq.(43) appearing in the MA gauge seems a candidate of the gluonic Higgs field $\phi[A_\mu(x)]$, however, it does not obey the adjoint transformation, and correspondence between $\Phi[A_\mu(x)]$ and the location of the monopole is unclear. In general, when a variable $O(x)$ is diagonalized, all the functions $f(O(x))$ are also diagonalized. Hence, there appears the ambiguity to choose $\phi[A_\mu(x)]$. Then, we require the following properties for the gluonic Higgs field $\phi[A_\mu(x)]$ in QCD.

1. To extract $(N_c-1)$ abelian gauge fields $\vec{A}_\mu(x)$ corresponding to $U(1)^{N_c-1} \subset SU(N_c)$, the gluonic Higgs field $\vec{\phi}[A_\mu(x)]$ consists of $(N_c-1)$ components $\phi_j[A_\mu(x)]$ ($j = 1, \ldots, N_c-1$). Each $\phi_j$ is defined to be an hermite gluonic composite scalar as $\phi_j[A_\mu(x)] = \phi_j^a T^a \in su(N_c)$ with $\phi_j^a \in \mathbb{R}$.

2. Similar to the Higgs field in the NAH theory, $\vec{\phi}[A_\mu(x)]$ obeys the adjoint gauge transformation as $\vec{\phi} \rightarrow \vec{\phi}^\Omega = \Omega \vec{\phi}^\dagger \Omega^\dagger$ by the SU($N_c$) gauge transformation $\Omega(x) \in SU(N_c)$.

3. Corresponding to the direct product of $U(1)^{N_c-1}$, the $(N_c-1)$ components $\phi_j[A_\mu(x)] \in su(N_c)$ are to be commutable each other as $[\phi_i(x), \phi_j(x)] = 0$, and are normalized as $\text{tr}[\phi_i(x)\phi_j(x)] = \frac{1}{2}\delta_{ij}$. This means that $\vec{\phi}[A_\mu(x)]$ forms the Cartan subalgebra at each local point $x_\mu$, and $\vec{\phi}[A_\mu(x)]$ is required to be written as $\vec{\phi}[A_\mu(x)] = \Omega_C \vec{H} \Omega_C^\dagger$, with a suitable $\Omega_C[A_\mu(x)] \in SU(N_c)$.

4. When the gluonic Higgs field $\vec{\phi}[A_\mu(x)]$ is diagonalized by the SU($N_c$) gauge transformation, the diagonal gluon component $\vec{A}_\mu(x)$ is required to provide the relevant abelian submanifold, which corresponds to the abelian projected QCD in the MA gauge.

Inspired from the argument of the GMA gauge, the gluonic Higgs field $\vec{\phi}_D[A_\mu(x)]$ satisfying the above requirements is found as

$$
\vec{\phi}_D[A_\mu(x)] \quad \text{s.t.} \quad \text{Min}_{\vec{\phi}(x) \in C} R_\vec{\phi}[A_\mu(x)] = \text{Min}_{\vec{\phi}(x) \in C} \int d^4x \text{tr}[\vec{D}_\mu, \vec{\phi}(x)]^\dagger [\vec{D}_\mu, \vec{\phi}(x)],
$$

where $\text{Min}_{\vec{\phi}(x) \in C} F[\vec{\phi}(x)]$ means the minimization of $F[\vec{\phi}(x)]$ by taking a suitable $\vec{\phi}(x) \in C$. Here, $C$ denotes the set of the Cartan-frame field $\vec{\phi}(x)$ introduced in section 3.3, and $\vec{\phi}(x)$ can be described as $\vec{\phi}(x) = \Omega_C \vec{H} \Omega_C^\dagger$ with $\Omega_C(x) \in SU(N_c)$.

The gluonic Higgs field $\vec{\phi}_D[A_\mu(x)] \in su(N_c)$ in Eq.(82) is determined directly from the gluon configuration $A_\mu(x)$, without the notion of the gauge fixing. From Eq.(82), the gluonic Higgs field $\vec{\phi}_D[A_\mu(x)]$ is interpreted as the local ‘color-direction’ averaged over the four $SU(N_c)$ covariant derivatives $\vec{D}_\mu$, and is a complicated function of the gluon field $A_\mu(x)$. The local form of the definition
of $\vec{\phi}_D[A_\mu(x)]$ is obtained from the extremum condition of $R^D_\phi[A_\mu(x)]$ on $\vec{\phi}(x)$ as

$$[\vec{\phi}_D(x), [\hat{D}_\mu, [\hat{D}_\mu, \vec{\phi}_D(x)]]] = 0.$$  \hspace{1cm} (82)

Hence, $\vec{\phi}_D[A_\mu(x)]$ is derived from $\hat{D}_\mu$ as the solution of the eigenvalue equation,

$$[\hat{D}_\mu^\dagger, [\hat{D}_\mu, \phi_{Dj}(x)]] = \lambda_j(x)\phi_{Dj}(x),$$  \hspace{1cm} (83)

where the eigenvalue $\lambda_j(x)$ is non-negative due to $\hat{D}_\mu^\dagger = -\hat{D}_\mu$ and satisfies $R^D_\phi[A_\mu(x)] = \frac{1}{2}[d^4x \sum_{j=1}^{N_c} \lambda_j(x)]$.

As a relevant property, $\vec{\phi}_D(x)$ obeys the adjoint transformation

$$\vec{\phi}_D[A_\mu(x)] \rightarrow (\vec{\phi}_D[A_\mu(x)])^V = V(x)\vec{\phi}_D[A_\mu(x)]V^\dagger(x)$$  \hspace{1cm} (84)

by the SU($N_c$) gauge transformation with $V \in$ SU($N_c$). In fact, $R^D_\phi[A_\mu]$ is transformed as

$$R^D_\phi[A_\mu] \rightarrow (R^D_\phi[A_\mu])^V = \int d^4x \text{tr} \left([V\hat{D}_\mu V^\dagger, \vec{\phi}_D^V][V\hat{D}_\mu^\dagger V, \vec{\phi}_D^V]\right)$$  \hspace{1cm} (85)

and $(R^\phi)V$ is minimized for $\vec{\phi}_D^\dagger = V(x)\vec{\phi}_D(x)V^\dagger(x)$, whose uniqueness can be proved by considering the infinite product of the infinitesimal gauge transformation. In particular, $\vec{\phi}_D(x)$ is $U(1)^{N_c-1}_{\phi}$ gauge invariant, because of $\omega_\phi(x)\vec{\phi}_D(x)\omega_\phi^\dagger(x) = \vec{\phi}_D(x)$ with $\omega_\phi(x) \equiv e^{i\vec{\phi}_D(x) \vec{\chi}(x)} \in U(1)^{N_c-1}_{\phi}$.

The gluonic Higgs field $\vec{\phi}_D[A_\mu(x)]$ is introduced to indicate the color-direction to be projected and to extract the abelian gauge manifold. In this respect, $\vec{\phi}_D[A_\mu(x)]$ plays the similar role to the Higgs field in the NAH theory, although there are two following differences.

1. $\vec{\phi}_D[A_\mu(x)]$ is a composite field of the gluon $A_\mu(x)$, and is not an elementary degrees of freedom.
2. $\vec{\phi}_D[A_\mu(x)]$ only has the color-direction degrees of freedom and does not have the amplitude degrees of freedom.

Now, we consider the projection of the operator $\hat{O}(x)$ to $\vec{\phi}_D$-direction. In the continuum QCD, interesting operators consist of the derivative operator and the Lie algebra, which are described by the infinitesimal transformation of the corresponding group elements, and therefore they are to be expressed with the
commutation relation. Using the commutation relation with the gluonic Higgs field \( \tilde{\phi}_D[A_\mu(x)] \), we define \( \phi_D \)-projection of the local infinitesimal operator \( \hat{O}(x) \) as

\[
\hat{O}^\phi(x) \equiv \hat{O}(x) - [\tilde{\phi}_D(x), [\tilde{\phi}_D(x), \hat{O}(x)]],
\]

which is a generalized version of Eq.(54). When \( \hat{O}(x) \) does not include the derivative operator \( \hat{\partial}_\mu \), this definition is trivial as \( \hat{O}^\phi(x) = 2\text{tr}\{\hat{O}(x)\tilde{\phi}_D(x)\} \cdot \tilde{\phi}_D(x) \) for \( \hat{O}(x) = \hat{O}(x)T^a \in su(N_c) \).

The construction of the relevant abelian gauge submanifold in QCD is performed with \( \tilde{\phi}_D \)-projection of the SU\( (N_c) \) covariant derivative operator \( \hat{D}_\mu \equiv \hat{\partial}_\mu + ieA_\mu \),

\[
\hat{D}_\mu^\phi \equiv \hat{D}_\mu - [\tilde{\phi}_D, [\tilde{\phi}_D, \hat{D}_\mu]] = \hat{\partial}_\mu + ieA_\mu^\phi + [\tilde{\phi}_D, \partial_\mu \tilde{\phi}_D] \tag{87}
\]

with \( A_\mu^\phi(x) = 2\text{tr}\{A_\mu(x)\tilde{\phi}_D(x)\} \cdot \tilde{\phi}_D(x) \). Here, the nontrivial term \([\tilde{\phi}_D, \partial_\mu \tilde{\phi}_D]\) appears in \( \hat{D}_\mu^\phi \) owing to the \( x \)-dependence of the Cartan-frame field \( \tilde{\phi}_D(x) \). It is to be noted that \([\tilde{\phi}_D, \partial_\mu \tilde{\phi}_D]\) does not include \( \tilde{\phi}_D \)-component as

\[
\text{tr} \left( \partial_\mu [\tilde{\phi}_D(x), \partial_\mu \tilde{\phi}_D(x)] \right) = \text{tr} \left( \partial_\mu [\tilde{\phi}_D(x), \tilde{\phi}_D(x)] \right) = 0. \tag{88}
\]

In this formalism, the abelian projection is defined by the replacement of \( \hat{D}_\mu \) by \( \hat{D}_\mu^\phi \). Accordingly, the abelian-projected gluon \( \tilde{A}_\mu^\phi(x) \) is defined as the difference between \( \hat{D}_\mu^\phi \) and \( \hat{\partial}_\mu \),

\[
\tilde{A}_\mu^\phi(x) \equiv \frac{1}{ie}(\hat{D}_\mu^\phi - \hat{\partial}_\mu) = A_\mu^\phi(x) + \frac{1}{ie}[\tilde{\phi}_D(x), \partial_\mu \tilde{\phi}_D(x)] \in su(N_c), \tag{89}
\]

and the abelian projection is expressed by the mapping of \( A_\mu(x) \rightarrow \tilde{A}_\mu^\phi(x) \). It is remarkable that \( \tilde{A}_\mu^\phi(x) \) includes both the \( \tilde{\phi}_D \)-component \( A_\mu^\phi(x) = 2\text{tr}(A_\mu(x)\tilde{\phi}_D(x)) \cdot \tilde{\phi}_D(x) \) and the non-\( \phi_D \)-component \( \frac{1}{ie}[\tilde{\phi}_D(x), \partial_\mu \tilde{\phi}_D(x)] \). The abelian gauge field is defined by the \( \tilde{\phi}_D \)-component of \( \tilde{A}_\mu^\phi(x) \),

\[
\tilde{A}_\mu^\phi(x) \equiv 2\text{tr}(\tilde{A}_\mu^\phi(x)\tilde{\phi}_D(x)) = 2\text{tr}(A_\mu^\phi(x)\tilde{\phi}_D(x)). \tag{90}
\]

Here, \( \tilde{A}_\mu^\phi(x) \) is the image of \( \tilde{A}_\mu^\phi(x) \) projected into the U(1)\(^{N_c-1}\) gauge manifold, and corresponds to the photon field in the NAH theory.
As for the gauge symmetry, both $\tilde{A}_\mu(x)$ and $\bar{A}_\mu(x)$ surely behave as the abelian gauge field under the $U(1)^{N_c-1}$ abelian gauge-transformation by $\omega(x) = e^{i\tilde{\phi}_D(x) \cdot \tilde{x}(x)} \in U(1)^{N_c-1}$. In fact, $\bar{A}_\mu(x)$ is gauge-transformed as

$$
\bar{A}_\mu(x) \rightarrow (\bar{A}_\mu(x))^\omega = 2\text{tr}(A_\mu(x)\tilde{\phi}_D(x)) = 2\text{tr}(\omega[A_\mu(x) + \frac{1}{ie} \partial_\mu] \omega^\dagger(x) \cdot \tilde{\phi}_D(x))
$$

$$
= 2\text{tr}(A_\mu(x)\tilde{\phi}_D(x)) + \frac{2}{e} \text{tr}\{\partial_\mu(\chi_i(x)\phi_Di(x)) \cdot \tilde{\phi}_D(x)\}
$$

$$
= \bar{A}_\mu(x) + \frac{1}{e} \partial_\mu \tilde{x}(x),
$$

where we have used

$$
\text{tr}\{\partial_\mu(\phi_{Di}(x)\phi_{Dj}(x))\} = \text{tr}\{\partial_\mu(\Omega_C H_i \Omega_C^\dagger) \cdot (\Omega_C H_j \Omega_C^\dagger)\}
$$

$$
= \text{tr}(\Omega_C^\dagger \partial_\mu \Omega_C H_i H_j + \partial_\mu \Omega_C^\dagger \Omega_C H_j H_i) = 0.
$$

Then, $\tilde{A}_\mu(x)$ is gauge-transformed as

$$
\tilde{A}_\mu(x) \rightarrow (\tilde{A}_\mu(x))^\omega = \tilde{A}_\mu(x) + \frac{1}{e} \partial_\mu \tilde{x}(x) \cdot \tilde{\phi}_D(x).
$$

Next, we study the abelian field strength and the monopole current. The abelian field-strength matrix is defined as

$$
\tilde{F}_{\mu\nu}(x) \equiv \frac{1}{ie} \left( [\hat{D}_\mu, \hat{D}_\nu] - [\hat{\partial}_\mu, \hat{\partial}_\nu] \right)
$$

$$
= \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x) + ie[\tilde{A}_\mu(x), \tilde{A}_\nu(x)],
$$

which generally includes the non-$\tilde{\phi}_D$-component as well as $\tilde{A}_\mu(x)$. The $\tilde{\phi}_D$-component of $\tilde{F}_{\mu\nu}(x)$ is the image of $\tilde{F}_{\mu\nu}(x)$ projected into the $U(1)^{N_c-1}$ gauge manifold, and is observed as the ‘real abelian field-strength’ in the abelian-projected gauge theory. The explicit form of $\tilde{F}_{\mu\nu}(x)$ is derived as

$$
\tilde{F}_{\mu\nu}(x) \equiv 2\text{tr} \left( \tilde{F}_{\mu\nu}(x) \tilde{\phi}_D(x) \right)
$$

$$
= \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x) + \frac{4}{ie} \text{tr}(\tilde{\phi}_D(x)[\partial_\mu \phi_{Di}(x), \partial_\nu \phi_{Di}(x)])
$$

$$
= \partial_\mu \tilde{A}_\nu(x) - \partial_\nu \tilde{A}_\mu(x) + \frac{2}{e} f_{abc} \tilde{\phi}_D \partial_\mu \phi_D^a \partial_\nu \phi_D^c,
$$

where the last term breaks the abelian Bianchi identity and provides the monopole current. The magnetic monopole current is derived as
\[\vec{k}_\mu^\phi(x) \equiv \partial^{\alpha*} \vec{F}_{\alpha\mu}(x) = -\frac{1}{e} \epsilon_{\mu\alpha\beta\gamma} f_{abc} \vec{\phi}_D^\alpha(x) \partial^\beta \phi_{D_i}^b(x) \partial^\gamma \phi_{D_j}^c(x),\]  

(97)

which is the topological current induced by \(\vec{\phi}_D[A_\mu(x)]\). Hence, the monopole appears from the center of the hedgehog configuration of \(\vec{\phi}_D[A_\mu(x)]\) as shown in Fig.2 in the SU(2) case.

The gluonic Higgs field \(\vec{\phi}_D[A_\mu(x)]\) in QCD plays the similar role to the Higgs field in the NAH theory on the extraction of the abelian gauge manifold and the appearance of the monopole current. In principle, the abelian projection can be performed in QCD in the gauge-covariant manner using the gluonic Higgs field \(\vec{\phi}_D[A_\mu(x)]\) without the notion of the gauge fixing.

5.2 Relation among the Gluonic Higgs Field, the MA Gauge Function and Monopoles

In this subsection, we consider the correspondence among the gluonic Higgs field \(\vec{\phi}_D[A_\mu(x)]\) and the MA gauge function \(\Omega_{\text{MA}}[A_\mu(x)] \in G/H\), and QCD-monopoles. As the relation between \(\Omega_{\text{MA}}\) and \(\vec{\phi}_D\), the minimization condition for \(R_{\vec{H}}[\vec{\phi}_D[A_\mu(x)]]\) by the gauge degrees of freedom can be equivalently rewritten into the minimization of \(R_{\vec{H}}[\vec{\phi}[A_\mu(x)]]\) in terms of \(\vec{\phi}(x) \in C\). Then, the gluonic Higgs field \(\vec{\phi}_D[A_\mu(x)] \in C\) directly corresponds to the MA gauge function \(\Omega_{\text{MA}}(x) \in G/H\) as

\[\vec{\phi}_D(x) = \Omega_{\text{MA}}^\dagger(x) \vec{H} \Omega_{\text{MA}}(x).\]  

(98)

Then, if the MA gauge is uniquely determined beside \(U(1)_{\text{local}} \times \text{Weyl}_{\text{global}}\), \(\vec{\phi}_D[A_\mu(x)]\) is also uniquely determined beside the global Weyl symmetry, because of the \(U(1)^{N_c-1}\) gauge invariance of \(\vec{\phi}_D[A_\mu(x)]\).

Also from Eq.(98), we can derive the adjoint gauge-transformation property of the gluonic Higgs field \(\vec{\phi}_D[A_\mu(x)]\) again. In the arbitrary \(SU(N_c)\) gauge transformation with \(V(x) \in SU(N_c)\), the MA gauge function \(\Omega_{\text{MA}} \in G/H\) obeys the nonlinear gauge transformation as

\[\Omega_{\text{MA}}(x) \rightarrow \Omega_{\text{MA}}^V(x) = d^V(x) \Omega_{\text{MA}}(x) V^\dagger(x).\]  

(99)

Here, \(d^V(x) \in H\) appears so as to keep \(\Omega_{\text{MA}}^V(x)\) belonging to the coset space \(G/H\), i.e., \(\Omega_{\text{MA}}^V(x) \in G/H\). Then, \(\vec{\phi}_D(x)\) is transformed by \(V(x) \in SU(N_c)\) as

\[\vec{\phi}_D \rightarrow \vec{\phi}_D^V = \Omega_{\text{MA}}^{V\dagger} \vec{H} \Omega_{\text{MA}}^V = V \Omega_{\text{MA}}^{V\dagger} \vec{H} d^V \Omega_{\text{MA}} V^\dagger = V \vec{\phi}_D V^\dagger,\]  

(100)
which is nothing but the SU($N_c$) adjoint gauge transformation.

Here, we consider the singularity relating to the monopole appearing in the abelian gauge manifold of QCD. In a suitable gauge like the Landau gauge, the gluon field can be taken as a regular field, however, the gluonic Higgs field $\vec{\phi}_D[A_\mu(x)]$ generally includes the singularity like the hedgehog configuration as shown in Fig.2 (b), and therefore the relevant abelian manifold described by $\vec{A}_\mu \equiv A_\mu^a \vec{\phi}_D^a$ holds the monopole singularity at the hedgehog center of $\vec{\phi}_D(x)$. On the other hand, in the MA gauge, the gluonic Higgs field $\vec{\phi}_D[A_\mu(x)]$ is arranged into $\vec{H}$-direction by the SU($N_c$) gauge transformation like the unitary gauge in the NAH theory. Then, $(\vec{\phi}_D)^{\Omega_{MA}} = \Omega_{MA} \vec{\phi}_D \Omega_{MA}^\dagger = \vec{H}$ becomes trivially regular as shown in Fig.2(a). Instead, the gluon field $A_\mu^{MA}(x) \equiv \Omega_{MA}^\dagger(A_\mu + \partial_\mu)\Omega_{MA}(x)$ includes the singularity as monopoles in the abelian sector. Such a movement of the singularity from the Higgs field to the gauge field is also seen during the unitary gauge fixing around the 't Hooft-Polyakov monopole in the NAH theory.

Now, we examine the local correlation between the monopole current and the gluonic Higgs field $\vec{\phi}_D[A_\mu(x)]$ in the SU(2) lattice simulation. In the lattice QCD simulation, the gluon configuration generated on the lattice is far from the continuous field, because of the random appearance of the gauge degrees of freedom on each cite. On the other hand, to see the topology of $\vec{\phi}_D[A_\mu(x)]$, the gluon field $A_\mu(x)$ is desired to be continuous, because discontinuity of $A_\mu(x)$ as the lattice artifact inevitably breaks the continuity of $\vec{\phi}_D(x)$ via Eq.(82) and provides ‘fake singularities’ of $\vec{\phi}_D(x)$. Then, we remove unphysical discontinuity on the gauge degrees of freedom from the lattice-QCD gluon configuration by a suitable gauge transformation. To this end, we impose the SU($N_c$) Landau gauge fixing[46], which is defined by maximizing

$$R_L[U_\mu] \equiv \text{Re} \sum_{s,\mu} \text{tr} U_\mu(s) = N_c \text{Re} \sum_{s,\mu} U^0_\mu(s)$$

using the SU($N_c$) gauge transformation. In the SU($N_c$) Landau gauge, all the gluon components on the lattice become mostly continuous owing to the suppression of their fluctuation around $U_\mu(s) = 1$. In the continuum limit, this gauge-fixing condition coincides the ordinary SU($N_c$) Landau gauge condition, $\partial_\mu A_\mu = 0$. Thus, we first prepare the continuous gluon configuration by the SU(2) Landau gauge fixing, and then the MA gauge fixing is performed by the MA gauge function $\Omega_{MA}(x)$. The gluonic Higgs field $\vec{\phi}_D(x)$ is also obtained using Eq.(98).

We show in Fig.10 the local correlation between the gluonic Higgs field $\phi_D[U_\mu(s)]$ and the monopole in the MA gauge in the SU(2) lattice QCD with $\beta=2.4$ and $16^4$. The gluonic Higgs field $\phi_D(s) = \phi^a_D(s)T^a$ is expressed by the arrows.
$(\phi_D, \phi_D^2, \phi_D^3)$ in the SU(2) internal space. The tendency of the local correspondence is found between the hedgehog configuration of the gluonic Higgs field $\vec{\phi}_D(x)$ and the monopole in the MA gauge.

Finally, we consider the physical meaning of the gluonic Higgs field $\vec{\phi}_D[A_\mu(x)]$ and relevant abelian manifold embedded in QCD. Here, the gluonic Higgs field $\vec{\phi}_D[A_\mu(x)]$ can be obtained in the gauge-covariant manner using Eq. (98) from the gluon field $A_\mu(x)$ in QCD, and the abelian projection can be performed with $\vec{\phi}_D(x)$ without the notion of the gauge fixing. Physically, $\vec{\phi}_D[A_\mu(x)]$ means the local color-direction which is determined so as to minimize the difference between the SU($N_c$) gauge connection $\hat{D}_\mu$ and the abelian gauge connection $\hat{D}_\mu^\phi$ along $\vec{\phi}_D(x)$. In terms of the maximal similarity of $\hat{D}_\mu^\phi$ with $\hat{D}_\mu$, the gluonic Higgs field $\vec{\phi}_D[A_\mu(x)]$ indicates a relevant color-direction peculiar to the gluon field $A_\mu(x)$, and the projection into the color-direction $\vec{\phi}_D(x)$ provides the extraction of a relevant abelian gauge manifold embedded in QCD in the gauge-covariant manner.

Since the gluonic Higgs field $\vec{\phi}_D[A_\mu(x)]$ obeys the adjoint gauge transformation, it has the direct similarity to the Higgs field in the NAH theory. Several parallel arguments to the NAH theory are applicable for abelian-projected QCD in terms of $\vec{\phi}_D[A_\mu(x)]$ on the extraction of the abelian gauge manifold and appearance of the monopole from the hedgehog configuration. In terms of the ‘gluonic Higgs theory’ with $\vec{\phi}_D(x)$, the MA gauge fixing directly corresponds to the unitary gauge fixing in the NAH theory. In particular, abelian dominance in the MA gauge observed in the lattice QCD indicates that only the $\phi_D$-component gluon remains at the long-distance scale like the photon field in the NAH theory, and the other gluon component perpendicular to $\vec{\phi}_D$ becomes infrared-irrelevant like the charged massive vector field in the NAH theory. In other words, the abelian gauge submanifold projected to $\vec{\phi}_D[A_\mu(x)]$ in QCD is considered to hold essence of the whole nonabelian gauge manifold in the infrared region.

6 Summary and Concluding Remarks

On the basis of the dual Higgs picture for confinement, we have studied the properties of monopoles and gluon fields in QCD in the maximally abelian (MA) gauge both in the analytical framework and in the lattice QCD calculation. In the dual Higgs theory, color confinement is realized by the onedimensional squeezing of the color-electric flux in the QCD vacuum through the dual Meissner effect caused by monopole condensation. The extraction of the abelian gauge theory and the appearance of monopoles in QCD can be carried out by taking the ’t Hooft abelian gauge, which is defined by diagonalizing
the ‘gluonic Higgs field’ $\phi[A_\mu(x)]$.

In the abelian gauge, SU($N_c$) gauge theory is reduced into U(1)$^{N_c-1}$ gauge theory including the monopole, which topologically appears corresponding to the nontrivial homotopy group, $\Pi_2(SU(N_c)/U(1)^{N_c-1}) = \mathbb{Z}^{N_c-1}$. In the abelian gauge, the diagonal gluon component behaves as the U(1)$^{N_c-1}$ gauge field, while the off-diagonal gluon behaves as the charged matter field in terms of the residual gauge symmetry. For $\phi[A_\mu(x)]$ obeying the adjoint gauge transformation, the hedgehog configuration of $\phi[A_\mu(x)]$ leads to the unit-charge magnetic monopole. In the abelian gauge, multi-charge monopoles do not appear in $\mathbb{R}^3$ in general cases, because of the over condition.

Using the gauge-connection formalism, the appearance of the Dirac string and the monopole has been studied in relation with the SU($N_c$) singular gauge transformation. The appearance of the Dirac string is originated from the multi-valuedness of the gauge function $\Omega(x)$, which leads to the divergence of $\Omega^\dagger \partial_\mu \Omega$. In the singular SU(2) gauge transformation, the multi-valued point of $\Omega(x)$ terminates at the hedgehog center of $\phi[A_\mu(x)]$, which leads to the appearance of the monopole. We have shown the relevant role of off-diagonal gluons for the appearance of monopoles.

The maximally abelian (MA) gauge has been well formulated in terms of the gauge connection. To remove the explicit use of the specific direction as $\vec{H}$, we have formulated the generalized maximally abelian (GMA) gauge by introducing the Cartan-frame field $\vec{\phi}(x)$, which is the local Cartan sub-algebra defined at each point. The generalization of the abelian projection has been defined based on the commutation relation. We have investigated the gauge-transformation properties of the GMA gauge function $\Omega_{\text{GMA}}$, and have derived the criterion on the SU($N_c$) gauge invariance for the variable $\hat{O}[A_\mu(x)]\Omega_{\text{GMA}}$ defined in the GMA gauge. This criterion is also applicable for the MA gauge.

We have studied the gluon-field properties around the monopole in QCD in the MA gauge in terms of the field strength and the action density both in the analytical consideration and in the lattice QCD simulation. The monopole provides large field fluctuations in the abelian sector: both the abelian gauge field and the abelian action density are largely fluctuated around monopoles. The large fluctuation of off-diagonal gluons has been also found around the monopole in the MA gauge, and the off-diagonal rich region indicates the effective size and the structure of monopoles, which is similar to the ’t Hooft-Polyakov monopole. Since the instanton needs the full SU(2) structure, it is expected to appear in the off-diagonal rich region around the monopole in the MA gauge, which leads to the local correlation between monopoles and instantons. We have found the large cancellation between abelian part and the off-diagonal part of the action in the MA gauge. Owing to this cancellation, the monopole can appear in the abelian sector in QCD without large cost.
of the QCD action, although existence of monopoles inevitably enlarges the abelian action. In other words, the off-diagonal gluon is necessary for existence of the monopole in the short-distance scale, and the effective monopole size relating to the off-diagonal gluon can be regarded as the critical scale of the abelian projected QCD, in the similar sense of the correspondence between the GUT monopole size and the GUT scale.

Finally, the abelian projection in QCD has been formulated without the notion of gauge fixing in the similar manner to the extraction of the photon field in the nonabelian Higgs theory, by introducing the ‘gluonic Higgs field’ \( \vec{\phi}_D[A_\mu(x)] \) defined from the gluon field. Here, the color-direction \( \vec{\phi}_D[A_\mu(x)] \) is determined so as to minimize the difference between the SU\((N_c)\) gauge connection and the projected abelian gauge connection along \( \vec{\phi}_D[A_\mu(x)] \). The gluonic Higgs field \( \vec{\phi}_D[A_\mu(x)] \) obeys the adjoint transformation by the SU\((N_c)\) gauge transformation, and the monopole current appears at the center of the hedgehog configuration of \( \vec{\phi}_D[A_\mu(x)] \) in the SU\((2)\) case.

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Fig. 1. Comparison among QCD, abelian projected QCD (AP-QCD) and QED in terms of the gauge symmetry and essential degrees of freedom.

Fig. 2. Topological structure of the gluonic Higgs field $\phi[A_\mu(x)]$ in the abelian gauge fixing in the SU(2) QCD. In the abelian gauge, the monopole appears at the singular point of $\tilde{\phi}(x) \equiv \phi/|\phi|$ with $|\phi| \equiv (\phi^a\phi^a)^{1/2}$. (a) For the regular (trivial) configuration of $\tilde{\phi}[A_\mu(x)]$, no monopole appears in the abelian gauge. (b) For the hedgehog configuration of $\tilde{\phi}[A_\mu(x)]$, the unit-charge monopole appears in the abelian gauge.

Fig. 3. Appearance of monopoles in abelian projected QCD (AP-QCD). After the abelian gauge fixing, monopole with the Dirac string appears from $G^\text{linear}_{\mu\nu}$ in Eq. (19) and the ‘anti-Dirac string’ appears in the singular part $G^\text{sing}_{\mu\nu}$. The off-diagonal contribution $G^\text{bilinear}_{\mu\nu} = ie[A_\mu, A_\nu]$ forms the anti-monopole configuration and compensates to the singularity of the other parts. As the result, the monopole without the Dirac string appears in the abelian field strength $F_{\mu\nu}$ in AP-QCD.

Fig. 4. The (static) monopole defined on the dual lattice is equivalent to the total magnetic flux of the Dirac string, and therefore the magnetic charge of the QCD-monopole is quantized.

Fig. 5. The typical example of the 3-dimensional time-slice of the monopole current in the MA gauge in the lattice QCD with $\beta = 2.4$ on $16^4$.

Fig. 6. The solid curve denotes the probability distribution $P(|\tilde{\theta}^3|)$ of the averaged abelian angle variable $|\tilde{\theta}^3(s, \hat{\mu})|$ around the monopole current in the MA gauge with the U(1)$_3$ Landau gauge fixing. Here, $|\tilde{\theta}^3(s, \hat{\mu})|$ is the average of $|\theta^3_\alpha(s)|$ over the neighboring links around the dual link, and data are obtained from the SU(2) lattice QCD with $\beta = 2.4$ and $16^4$. For comparison, the total distribution $P$ on the whole lattice is also added by the dashed curve. Around the monopole, $|\tilde{\theta}^3|$ corresponding to the abelian gluon component takes a large value.

Fig. 7. (a) The solid curve denotes the probability distribution $P(\bar{R}_{\text{Abel}})$ of the averaged abelian projection rate $\bar{R}_{\text{Abel}}(s, \hat{\mu})$ around the monopole current in the SU(2) lattice QCD with $\beta = 2.4$ on $16^4$. For comparison, the total distribution $P$ on the whole lattice is also added by the dashed curve. (b) The solid curve denotes abelian projection rate $\langle \bar{R}_{\text{Abel}} \rangle$ around the monopole current in the MA gauge as the function of $\beta$. The dashed curve denotes $\langle \bar{R}_{\text{Abel}} \rangle$ on the whole lattice.

Fig. 8. (a) The probability distribution $P(\bar{S})$ of density $\bar{S}(s, \hat{\mu})$ on the whole lattice in the MA gauge at $\beta = 2.4$ on $16^4$ lattice. (b) The probability distribution $P_k(\bar{S})$ of the action density $\bar{S}(s, \hat{\mu})$ around the monopole current $k_\mu$. The dotted and the solid curves denote $P(\bar{S}^\text{SU(2)})$ and $P(\bar{S}^\text{Abel})$, respectively. The dashed curve denotes $P(\bar{S}^\text{off})$ for the off-diagonal part $S^\text{off}$ of the action density.
Fig. 9. The action density as the function of $\beta$ in the MA gauge in the SU(2) lattice QCD. The closed symbols denote the action densities $\langle S \rangle$ around the monopole current, while the open symbols denote those on the whole lattice. The square, circle and rhombus denote $\langle S^{\text{SU}(2)} \rangle$, $\langle S^{\text{Abel}} \rangle$ and $\langle S^{\text{off}} \rangle$, respectively. The monopole accompanies a large U(1)$_3$ plaquette action, however, such a large U(1)$_3$ action is strongly canceled by the off-diagonal part.

Fig. 10. The local correlation between the gluonic Higgs field $\phi_D[A_\mu(x)]$ and the monopole in the MA gauge in the SU(2) lattice QCD with $\beta = 2.4$ and $16^4$. The gluonic Higgs field $\phi_D[A_\mu(x)] \equiv \phi_D^a \frac{\tau^a}{2}$ on a 2-dimensional plane is expressed by the arrow, which denotes its color-direction in the SU(2) internal space.
SU(2) gauge symmetry
degrees of freedom
SU(2) gauge field

SU(2) QCD

Abelian Projection

SU(2) gauge symmetry
degrees of freedom
SU(2) gauge field

Abelian Projected QCD

U(1) gauge symmetry
degrees of freedom
U(1) gauge field
with monopole

U(1) gauge theory

Confinement Potential

QED

U(1) gauge symmetry
degrees of freedom
U(1) gauge field

difference
monopole

difference
off-diagonal elements

Coulomb Potential

U(1) gauge theory
Fig. 3

\[ G_{\mu\nu}^\Omega = \partial_\mu A_\nu^\Omega - \partial_\nu A_\mu^\Omega + \frac{i}{\epsilon} \Omega[\partial_\mu, \partial_\nu] \Omega^\dagger + ie[A_\mu^\Omega, A_\nu^\Omega] \]

Abelian Projected QCD
Fig. 6

![Graph showing the probability density function of $|\theta^3|$ over different values of $\theta$.]
Fig. 7

(a) 

(b)
Fig. 8

(a) $P(S)$ vs $S(\mu,s)$

(b) $P_k(S)$ vs $S(\mu,s)$

Legend:
- Dashed line: SU(2)
- Solid line: Abelian
- Dotted line: Off-diagonal
Fig. 9
Fig. 10