Tradeoffs between Convergence Speed and Reconstruction Accuracy in Inverse Problems

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Abstract

Solving inverse problems with iterative algorithms such as stochastic gradient descent is a popular technique, especially for large data. In applications, due to time constraints, the number of iterations one may apply is usually limited, consequently limiting the accuracy achievable by certain methods. Given a reconstruction error one is willing to tolerate, an important question is whether it is possible to modify the original iterations to obtain a faster convergence to a minimizer with the allowed error. Relying on recent recovery techniques developed for settings in which the desired signal belongs to some low-dimensional set, we show that using a coarse estimate of this set leads to faster convergence to an error related to the accuracy of the set approximation. Our theory ties to recent advances in sparse recovery, compressed sensing and deep learning. In particular, it provides an explanation for the successful approximation of the ISTA solution by neural networks with layers representing iterations.

I. INTRODUCTION

Consider the setting in which we want to recover a vector $x \in \mathbb{R}^d$ from a given set of its linear measurements

$$y = Mx + e \in \mathbb{R}^m,$$

where $M \in \mathbb{R}^{m \times d}$ is the measurement matrix and $e \in \mathbb{R}^d$ is additive noise. This setup appears in many fields including statistics (e.g., regression), image processing (e.g., deblurring and super-resolution), and medical imaging (e.g., CT and MRI), to name just a few.

When $M$ is ill-posed, e.g., in the case $m < n$, it is impossible to recover $x$ from $y$ without introducing additional assumptions on the structure of $x$. A popular strategy is to assume that $x$ resides in a low dimensional set $\mathcal{K}$, e.g., sparse vectors $[6]$ or a Gaussian Mixture Model (GMM) $[30]$. A natural by-product minimization problem then becomes

$$\min_x \|y - Mx\|_2^2 \text{ s.t. } x \in \mathcal{K}. \tag{2}$$

This can be reformulated in an unconstrained form as

$$\min_x \|y - Mx\|_2^2 + \lambda s(x), \tag{3}$$

where $\lambda$ is the regularization weight parameter and $s(\cdot)$ is a penalty function related to the set $\mathcal{K}$. For example, if $\mathcal{K} = \{x \in \mathbb{R}^d : \|x\|_0 \leq k\}$ is the set of $k$-sparse vectors, then a natural selection would be $s(\cdot) = \|\cdot\|_0$ or its convex relaxation $s(\cdot) = \|\cdot\|_1$.

For a very large data dimension $d$, a popular technique for solving (2) and (3) is using iterative methods such as iterative shrinkage algorithms [2], [11], which are applied for the case $s(\cdot) = \|\cdot\|_1$, or the alternating direction method of multipliers (ADMM) [5], [10] that are applicable in more general cases.

Many applications impose some time constraints, which limit us in the number of computations that can be performed to recover $x$ from the measurements. One way to minimize time and computations is to reduce the number of iterations without increasing the computational cost of each iteration. Another alternative is to keep the number of iterations fixed while reducing the cost of each iteration. For example, since the complexity of iterative methods rely, among other things, on $m$, a common technique used by practitioners to save computations is to sub-sample the measurements $y$, removing “redundant information,” to an amount that still allows reconstruction of $x$. A series of recent works [7], [9], [24] suggested that by obtaining more measurements one can benefit from simple efficient methods that cannot be applied with a smaller number of measurements.

While these works studied a tradeoff between the convergence speed and the number of available measurements, we take another route in this paper. Consider the case in which due to time constraints we need to stop before we achieve the desired reconstruction accuracy. An important question is whether we can modify the original iterations (e.g., those dictated by the shrinkage or ADMM algorithms), such that the algorithm convergences to an improved solution with less iterations. This
introduces a tradeoff between the reconstruction error we are willing to absorb and the computational cost we are constrained to. As we shall demonstrate, this goes beyond the trivial relationship between the approximation error and the number of iterations that exist for different iterative methods\cite{ISTA}.

Indeed, such a tradeoff is experimentally demonstrated by the success of the learned ISTA (LISTA)\cite{LISTA,LISTA2,LISTA3} for sparse recovery. LISTA learns a neural network with only several layers, where each is simply a modified version of the ISTA iteration\cite{ISTA}, achieving with one to two orders of magnitude less iterations virtually the same accuracy as the original ISTA. However, it is still unclear why this is possible and a proper theoretical justification is lacking.

In this work we provide theoretical foundations elucidating the tradeoff between the allowed minimization error and the computational cost of iterative algorithms for solving inverse problems. We formally show that it is possible to design algorithms with better convergence speed if we allow a certain reconstruction error in the solution. It is interesting to note that such a tradeoff is natural when working with real data, where both the data and the assumed models are already noisy or approximate; searching for the exact solution of an optimization problem where all the variables are affected by measurement or model noise is an unnecessary use of valuable computational resources. We formally prove this relation for iterative projection algorithms. Interestingly, a related tradeoff exists also in the context of sampling theory, where if we allow some error in the reconstruction with better convergence speed if we allow a certain reconstruction error in the solution. It is interesting to note that such a tradeoff is experimentally demonstrated by the success of the learned ISTA (LISTA)\cite{LISTA,LISTA2,LISTA3} for sparse recovery. LISTA learns a neural network with only several layers, where each is simply a modified version of the ISTA iteration\cite{ISTA}, achieving with one to two orders of magnitude less iterations virtually the same accuracy as the original ISTA. However, it is still unclear why this is possible and a proper theoretical justification is lacking.

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This work is organized as follows. In Section II we describe the iterative projected gradient descent (PGD) algorithm and then show how it is possible to tradeoff between its convergence speed and reconstruction accuracy by introducing the inaccurate projected gradient descent (IPGD) method and analyzing its reconstruction error as a function of the iterations. The proof follows the strategy introduced in\cite{IPGD}. Section III shows the relationship between our result and model-based and spectral compressed sensing\cite{spectralCS,CS}. Section IV addresses the relationship between our results and sparse recovery with side information\cite{sideinfo,sideinfo2,sideinfo3,sideinfo4}. Section V connects the approximation of minimization problems studied here with neural networks and deep learning. Section VI concludes the paper.

II. THE INACCURATE PROJECTED GRADIENT DESCENT ALGORITHM

In this work we focus on the powerful projected gradient descent (PGD) algorithm. Each iteration is composed of a gradient step with step size $\mu$ followed by a projection onto the set $K$, and can be written as

$$z_{t+1} = P_K(z_t + \mu M^*(y - Mz_t)),$$

where $M^*$ is the conjugate of $M$ and $z_t$ is an estimate of $x$ at iteration $t$. The step size $\mu$ is assumed to be constant for the sake of simplicity; in practice, it may also vary between each iteration.

PGD is a generalization of the iterative hard thresholding (IHT) algorithm, which was developed for $K$ being the set of sparse vectors\cite{IHT}. This important method has been analyzed in various works. For example, for IHT in\cite{IHT}, for sparsity patterns that belong to a certain model in\cite{model}, for a general union of subspaces in\cite{union}, and more recently in\cite{IPGD} for a set of the form

$$K = \{z \in \mathbb{R}^d : f(z) \leq R\}.$$}

This formulation generalizes the special cases above. For example, if $f(\cdot) = \|\cdot\|_0$ and $R$ is the sparsity level then we have the IHT method from\cite{IHT}; and if $f$ counts the number of non-zeros of only certain sparsity patterns, which are bounded by $R$, then we have the model-based IHT from\cite{model}.

Theorem\cite{IPGD} below provide convergence guarantees on PGD. Before presenting these theoretical guarantees for PGD, we introduce several basic properties of the set $K$ and some basic lemmas.

**Definition 2.1 (Descent set and tangent cone):** The descent set of the function $f$ at a point $x$ is

$$D_f(x) = \{h \in \mathbb{R}^d : f(x + h) \leq f(x)\},$$

and the tangent cone $D_f(x)$ at a point $x$ is the conic hull of $D_f(x)$, i.e., the smallest closed cone $C_f(x)$ satisfying $D_f(x) \subseteq C_f(x)$.

For concise writing, below we denote $D_f(x)$ and $C_f(x)$ as $D$ and $C$, respectively.

**Lemma 2.2:** If $K \subset \mathbb{R}^d$ is a closed set, then for all $v \in \mathbb{R}^d$,

$$P_K(x + v) - x = P_{K - \{x\}}(v) = P_D(v),$$

where $K - \{x\} = \{z - x : z \in K\} = \{z - x : f(z) \leq f(x)\} = \{h \in \mathbb{R}^d : f(h + x) \leq f(x)\} = D$.

**Lemma 2.3 (Lemma 6.4 in\cite{IPGD}):** Let $D$ and $C$ be nonempty and closed set and cone, respectively, such that $0 \in D$ and $D \subset C$. Then for all $v \in \mathbb{R}^d$

$$\|P_D(v)\|_2 \leq \kappa_f \|P_C(v)\|_2,$$

\footnote{ISTA and its variants is one of the most powerful optimization for sparse coding.}
where \( \kappa_f = 1 \) if \( D \) is a convex set and \( \kappa_f = 2 \) otherwise.

We now introduce the convergence rate provided in [24] for PGD. For brevity, we present only its noiseless version.

**Theorem 2.4 (Noiseless version of Theorem 1.2 in [24]):** Let \( x \in \mathbb{R}^d, f : \mathbb{R}^d \to \mathbb{R} \) a proper function, \( C = C_f(x) \) the tangent cone of the function \( f \) at point \( x \), \( M \in \mathbb{R}^{m \times d} \) a random matrix with i.i.d. Gaussian distributed entries \( N(0, 1) \) and let \( y = Mx \) be a vector containing \( m \) linear measurements. Assume we are using PGD with \( K = \{ z \in \mathbb{R}^d : f(z) \leq f(x) \} \) to recover \( x \) from \( y \). Then the estimate \( z_t \) at the \( t \)-th iteration (initialized with \( z_0 = 0 \)) obeys

\[
\|z_t - x\|_2 \leq (\kappa_f \rho(K))^t \|z_0\|_2,
\]

where \( \kappa_f \) is defined in Lemma 2.3 and

\[
\rho(K) = \rho(\mu, M, f, x) = \sup_{u, v \in C \cup \mathbb{B}^d} u^* (I - M^* M) v,
\]

is the convergence rate of PGD.

It has been shown in [24] that the convergence rate \( \rho(K) \) depends essentially on the dimensionality of the set \( (model) \) we assume \( x \) resides in. This dimensionality is measured by the (Gaussian) mean width \( \omega \) of the tangent cone \( C \) (intersected with the unit \( l_2 \)-ball \( \mathbb{B}_2 \) defined as

\[
\omega = \omega(C \cup \mathbb{B}_2) = E[ \sup_{g \in C \cup \mathbb{B}_2} \langle g, v \rangle ], \quad g \sim N(0, I).
\]

This expression is very useful for measuring the “intrinsic dimensionality” of a set as we shall see hereafter. Moreover, it determines the minimal value \( m \) for which PGD iterations converge to \( x \). The smaller \( \omega \), the faster the convergence. More specifically, if \( m \) is very close to \( \omega \) we may apply PGD with a step-size \( \mu = \frac{1}{\sqrt{d + \frac{\omega}{m}}} \approx \frac{1}{\sqrt{m}} \) and have a convergence rate

\[
\rho(K) = 1 - O(\frac{\sqrt{m}}{m + d}).
\]

However, if \( \omega \) is smaller than \( m \) by a certain constant factor then we may apply PGD with a larger step size \( \mu = \frac{1}{m \omega} \), which leads to a better convergence rate \( \rho(K) = O(\frac{\sqrt{m}}{m}) \). More details appear in [24].

It may happen that the function \( f \), and therefore the set \( K \), is too loose for describing \( x \). Instead, we may select a set \( \hat{K} \) that better characterizes \( x \) and therefore leads to a smaller \( \omega \), resulting in faster convergence. This improvement can be very significant as smaller \( \omega \) both improves the convergence rate and allows using a larger step-size.

For example, consider the case of a \( k \)-sparse vector \( x \), whose sparsity pattern obeys a tree structure (entry may be non-zero only if its parent node is non-zero). In this case if we ignore the structure in \( x \) and pick \( f \) to be the \( l_1 \) norm then the mean width is \( \omega_K \approx 2k \log(d/k) \). However, if we take this structure into account and us the set

\[
\hat{K} = \{ z \in \mathbb{R}^d : \|z\|_1 \leq \|x\|_1 \text{ and } z \text{ obeys a tree structure} \},
\]

then \( \omega_{\hat{K}} = O(k) \). As mentioned above, this improvement might be very significant especially when \( m \) is very close to \( \omega_K \). Such an approach has been taken in the context of model-based compressed sensing [11], where it is shown that we get faster convergence if we project onto the set of \( k \)-sparse vectors with tree structure instead of the set of \( k \)-sparse vectors. Another study, which is similar in its spirit to ours, is [30], where it is shown that it is enough to use a small number of Gaussians to represent all the patches in natural images instead of using a dictionary that spans a much larger union of subspaces. This work relied on Gaussian Mixture Models (GMM), whose mean width scales proportionally to the number of Gaussians used, which is significantly smaller than the mean width of the sparse model, which scales as \( k \log(n/k) \).

A difficulty often encountered is that the projection onto \( \hat{K} \) is more complex to implement than the projection onto \( K \). In such cases, we may introduce a compromise between the reconstruction error and convergence speed by using the potentially larger set \( \hat{K} \) as it is easier to project onto it, compared to the more complex set \( \hat{K} \), which may even be unknown. In this work we suggest to approximate \( P_{\hat{K}} \), the projection onto the set \( \hat{K} = \{ z \in \mathbb{R}^d : f(z) \leq f(x) \} \), by a simpler inaccurate “projection” that is composed of a simple projection \( p \) (e.g., a linear or an element-wise projection) and the projection onto \( K \), \( P_K \) such that (i) \( p \) introduces only a slight distortion into \( x \):

\[
\|p(x) - x\|_2 \leq \epsilon \|x\|_2;
\]

and (ii) the projection onto the tangent cone \( \hat{C} = C_f(x) \) of \( \hat{f} \) at point \( x \) is well approximated by a projection using \( p(\cdot) \) followed by a projection onto the tangent cone \( C_f(p(x)) \) of \( f \) at point \( p(x) \):

\[
\|P_{\hat{C}}(v) - P_{C_f(p(x))}(p(v))\|_2 \leq \epsilon \|v\|_2, \forall v \in \mathbb{R}^d.
\]

Plugging this inaccurate projection into the PGD step results in the inaccurate PGD (IPGD) iteration (compare to (4))

\[
z_{t+1} = P_K (p(z_t + \mu M^*(y - Mz_t)));
\]

For simplicity of the discussion, we analyze the convergence of this technique only for random Gaussian \( M \) and the noiseless case, i.e., for \( \epsilon = 0 \). The extension to other types of operators and the noisy case is straightforward by arguments similar to those used in [24] for treating the noise term and other types of matrices. The following theorem states our main result. It
provides convergence analysis of IPGD only for the case of linear projection \( p(\cdot) \). The general case will be covered in the longer version of this paper.

**Theorem 2.5:** Let \( x \in \mathbb{R}^d \), \( f : \mathbb{R}^d \to \mathbb{R} \) a proper function, \( p(\cdot) \) a linear projection, \( \hat{C} = C_f(x) \) the tangent cone of the function \( \hat{f} \) at point \( x \), \( \mathcal{K} = \{ z \in \mathbb{R}^d : \hat{f}(z) \leq \hat{f}(x) \} \), \( M \in \mathbb{R}^{m \times d} \) a random matrix with i.i.d. Gaussian distributed entries \( N(0,1) \) and let \( y = Mx \) be a vector containing \( m \) linear measurements. Assume we are using IPGD (Eq. (14)) with \( \mathcal{K} = \{ z \in \mathbb{R}^d : f(z) \leq f(x) \} \) to recover \( x \) from \( y \) and (12) and (13) hold. Then the estimate \( z_t \) at the \( t \)-th iteration (initialized with \( z_0 = 0 \)) obeys

\[
\| z_t - x \|_2 \leq \left( \kappa_f \left( \rho(\hat{C}) + \epsilon \gamma \right) \right)^t + \frac{1 - \left( \kappa_f \left( \rho(\hat{C}) + \epsilon \gamma \right) \right)^t}{1 - \kappa_f \left( \rho(\hat{C}) + \epsilon \gamma \right)} \| x \|_2, \tag{15}
\]

where \( \kappa_f \) is defined in Lemma 2.3 \( \gamma = \| I - \eta M^* M \|_2 \) and \( \rho(\hat{C}) = \rho(M, \hat{f}, x) = \sup_{u, v \in \mathcal{C}, \lambda \in [0, d]} \| u^* (I - M^* M) v \|_2 \).

The theorem implies that if \( \epsilon \) is small enough then IPGD attains approximately the same convergence rate as if we would have used PGD with \( \mathcal{K} \) at the cost of an additional error of order \( \epsilon \) in the solution. Note that \( \gamma \approx \frac{m}{d} + 2 \sqrt{\frac{m}{d}} \) if \( \mu = \frac{1}{d} \) and \( \gamma \approx \frac{d}{m} + 2 \sqrt{\frac{d}{m}} \) if \( m = \frac{1}{d} \). Therefore, for \( \mu = \frac{1}{m} \) we have stricter condition on the value of \( \epsilon \) as \( m \leq d \). In both cases we need to pick \( \gamma \epsilon < \rho(\hat{C}) \) in order to get convergence of order \( \rho(\hat{C}) \).

**Proof:** The IPGD error at iteration \( t + 1 \) reads as,

\[
\| z_{t+1} - x \|_2 = \| \mathcal{P}_\mathcal{K}(p(z_t + M^*(y - Mz_t))) - x \|_2
\]

\[
\leq \| \mathcal{P}_\mathcal{K}(p(z_t + M^*(y - Mz_t))) - p(x) \|_2 + \| p(x) - x \|_2
\]

\[
\leq \| \mathcal{P}_\mathcal{K}((C_f(p(x))) (p((I - \mu M^* M)(z_t - x)))\|_2 + \| p(x) - x \|_2
\]

\[
\leq \kappa_f \| \mathcal{P}_\mathcal{K}((C_f(p(x))) (p((I - \mu M^* M)(z_t - x)))\|_2 + \| p(x) - x \|_2
\]

\[
\leq \kappa_f \| \mathcal{P}_\mathcal{K}((C_f(p(x))) (p((I - \mu M^* M)(z_t - x)))\|_2 + \| p(x) - x \|_2
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\]

\[
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\]

Step (a) follows from the triangle inequality and the fact that at the noiseless case \( y = Mx \); step (b) from Lemma 2.2; step (c) from Lemma 2.3; step (d) from (13); and step (e) by using the norm inequality with the definition of \( \gamma \) for the first term and the steps used in Eq. (6.9) in the proof of Theorem 1.2 in [24]. Applying the inequality in (16) recursively leads to the desired result. \( \square \)

Theorem 2.5 allows us to tradeoff approximation error of the set \( \hat{K} \) and improved convergence. The error term at iteration \( t \) comprises two components. The first goes to zero as \( t \) increases while the second increases with iterations and is as the order of \( \epsilon \). Therefore, the fewer iterations we perform the larger \( \epsilon \) we may allow. On the other hand, the larger the error we can tolerate the less iterations we need to take.

To demonstrate our theorem we generate a \( k \)-sparse vector \( x \in \mathbb{R}^{127} \) with \( k = 13 \) and a sparsity pattern that obeys a tree structure. Moreover, we generate the non-zero entries in \( x \) independently from a Gaussian distribution with zero mean and variance \( \sigma^2 = 1 \) if they are at the first two levels of the tree, \( \sigma^2 = 0.1^2 \) if they are at the second level, and \( \sigma^2 = 0.01^2 \) for the rest of the levels. Note that it would be best to recover this vector with projections onto the set \( \hat{K} \) in (11). As the projection onto this set requires some additional computations at each iteration (see (11)), we approximate it by a simple linear projection onto the first levels of the tree followed by a projection onto \( \hat{K} = \{ z : \| z \|_1 \leq \| x \|_1 \} \). Note that the more levels we add in the projection \( p \), the smaller the approximation error \( \epsilon \) turns to be.

Figure 1 presents the signal reconstruction error (\( \| x - z_t \|_2 \)) and objective value (\( \| y - Mz_t \|_2 \)) as a function of the number of iterations for PGD and IPGD with step size \( \mu = \frac{1}{\sqrt{\sigma^2 + \sigma^2}} \) and \( p \) that projects onto a different number of levels (1-5) of the tree. It is interesting to note that if \( p \) projects only onto the first layer, the algorithm does not converge as the resulting approximation error \( \epsilon \) is too large. However, starting from the second layer, we get a faster convergence at the first iterations with \( p \) that project onto a coarser set that yields a smaller \( \rho \). Yet, as the number of iterations increases, the more accurate projections attain a lower reconstruction error. Clearly, as PGD does not introduce an error in its projection, it gets to the smallest reconstruction error but only after hundreds of iterations. This demonstrates the tradeoff between the approximation error we may allow and the convergence speed. For example, if we are satisfied with a reconstruction error 0.1, then it would be best to use IPGD with \( p \) that projects onto the first two levels of the tree, as it is the fastest to converge to this error (less than 20 iterations compared to PGD that requires 60). On the other hand, if we aim at reaching the smallest possible
In Fig. 1, we see the reconstruction error (left) and objective value (right) as a function of the iterations for sparse recovery in a scenario where a sparse vector has a tree structure. This demonstrates the convergence rate of PGA compared to IPGD with $p$ that projects onto a varying number of levels of the tree.

In Fig. 2, we again plot the reconstruction error (left) and objective value (right) as a function of the iterations for sparse recovery in a scenario with highly coherent dictionary between neighboring atoms. This demonstrates the convergence rate of PGA compared to IPGD. Note that though IPGD does not converge to zero with respect to the objective value (right), it does converge faster and to a lower value with respect to the reconstruction error.

Looking at the plot in Fig. 1, it is interesting to see that, by contrast to the reconstruction error, the reduction in the objective value achieved by PGD is not worse than the one of IPGD. As we have mentioned above, and shall see in the following examples, smaller objective value does not necessarily imply a smaller reconstruction error.

While in the case of the tree sparsity model it is easy to define $\hat{K}$, we present now several examples for which it is hard to set $\hat{K}$ accurately. Yet, we may still find a projection $p$ that helps to approximate $\hat{K}$. Therefore, even in this case IPGA with this $p$ gets better convergence compared to PGD.

### III. Relation to Model-Based and Spectral Compressed Sensing

We now show the connection of our work to spectral compressed sensing [12], in which one wants to recover a sparse representation in a dictionary that has high local coherence. It has been shown that if the non-zeros in the representation are far from each other then it is easier to obtain good recovery [8].

To demonstrate how our technique can be helpful for this scenario, we generate a $k$-sparse vector, with $k = 20$, of dimension $d = 128$ in a four times redundant DCT dictionary such that the minimal distance between neighboring atoms in this vector is greater than 5. The value in each coefficient is generated from the normal distribution. Then we put random Gaussian values at the neighboring coefficients of each active atom in the representation, with zero mean and variances $\sigma^2 = 0.1^2$ and $\sigma^2 = 0.01^2$ for the neighbors at distance 1 and 2, respectively. We compare the recovery with PGD and the proposed IPGD that uses a...
projection \( p(\cdot) \) that zeros neighboring entries in a given representation. While this projection causes an error in the model, it improves the reconstruction as can be seen in Fig. 2.

Notice the relationship of the proposed strategy to the method of model-based compressed sensing [1], as in both one assumes that the original representation obeys a certain structure, and projects onto it. This also connects to the studies in [17], [18], [20] that use a similar concept of near-optimal projection (compared to [1] that assumes only exact projections). The main difference between these contributions and ours is that these papers focus on specific models, while we present a general framework that is not specific to a certain low-dimensional model. In addition, in these papers the projection is performed to make it possible to recover a vector from a certain low-dimensional set, while in this work the main purpose of our inaccurate projections is to provide an acceleration of the convergence within a limited number of iterations, where it might be that for a larger number of iterations these projections would not lead to a good reconstruction error as we have seen before. The less iterations we have the larger the projection error we may allow. The main purpose of our projections is to provide a tradeoff between a projection error they introduce, which causes a reconstruction error, and the simplification that this error brings into the set onto which we are projecting.

IV. RELATION TO SPARSE RECOVERY WITH SIDE INFORMATION

A possible strategy to improve recovery is to use side information about the recovered signal, e.g. estimates of similar signals. This approach has been applied to improve the quality of MRI [14], [21], [29] and CT [10] scans, and also in the general context of sparse recovery [15], [22], [28].

We demonstrate this approach, in combination with our proposed framework, for the recovery of a sparse vector under the discrete cosine transform (DCT), given information of its representation under the Haar transform. Our sampling matrix is \( M = AD^* \), where \( A \in \mathbb{R}^{700 \times 1024} \) is a random matrix with i.i.d. normally distributed entries, and \( D \) is the DCT dictionary. We use a central patch of size \( 32 \times 32 \), normalized to be with unit \( \ell_2 \) norm, from the standard house image. Note that this patch is not exactly sparse either in the Haar or the DCT domains. We use IPGA with \( p = PDP^T \), where \( P \) is the projection onto the columns of the Haar matrix corresponding to the largest 400 coefficients of this patch under the Haar transform. The result is presented in Fig. 3 comparing between the convergence of PGD and IPGD. Again we may notice that while IPGD converges slower with respect to the objective value, it does converge faster with respect to the reconstruction error. Note that IPGD does not converge to zero due to the error \( \epsilon \) introduced by the projection \( p \).

V. RELATION TO DEEP LEARNING

Often we may not know what type of a simple projection causes \( P_K(p(\cdot)) \) to approximate \( \hat{K} \) in the best possible way. Therefore, a useful strategy is to learn \( p(\cdot) \) for a given dataset. If our target is to approximate the sparse representation of this dataset and we are using the \( \ell_1 \) norm for \( f \), then the projection operation \( P_K \) is simply a soft thresholding with a value that varies depending on the projected vector [13]. If we fix the thresholding value throughout the iterations we obtain the ISTA technique [2], [11]. The improved LISTA acceleration [19], [27] can be explained as a method for learning a linear operator that together with the projection \( P_K \) approximates the projection onto the true unknown set \( \hat{K} \) and therefore leads to much faster convergence.
With this understanding, we may argue that if we simplify the set \( \tilde{K} \) then we can get a faster convergence, i.e., obtain the same approximation error with less LISTA iterations. In order to show this, we cluster our dataset into 18 clusters (inspired by \([30]\)) and train a different LISTA network for each of them. Once we get a new vector, we apply all the networks on it in parallel (and therefore we do not increase the computational time) and chose the one that provides the best approximation. Indeed, we get the same approximation error that is achieved with one LISTA net with 10 layers using multiple LISTA nets with only 3 layers for decoding patches of size \(8 \times 8\) of an image under a DCT dictionary.

VI. Conclusion

In this work we have demonstrated how it is possible to tradeoff between approximation error and convergence speed. We have provided a theory for the convergence of an iterative algorithm with an approximate projection that has this tradeoff. At the cost of an error in the projection one may achieve a faster convergence at the first iterations. The larger the error the smaller the number of iterations that enjoy the fast convergence. This suggests that if we have a budget for only a small number of iterations it would be worthwhile to use very inaccurate projections that may bring us to a worse solution at the long term but are better to use with the given constraints. However, if we have as may computational resources as we wish, it would be better to use projections that are as accurate as possible.

Our theory explains the recent success of neural networks for approximating the solution of certain minimization problems with accuracy similar to iterative technique developed for these problems (e.g., ISTA for \(\ell_1\) minimization) but with much smaller computational cost. Our results provide also a technique for estimating the solution of these minimization problems by using multiple networks but with less layers in each of them.

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