DIAMETER THEOREMS ON KÄHLER AND QUATERNIONIC KÄHLER MANIFOLDS UNDER A POSITIVE LOWER CURVATURE BOUND

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ABSTRACT. We define the orthogonal Bakry-Émery tensor as a generalization of the orthogonal Ricci curvature, and then study diameter theorems on Kähler and quaternionic Kähler manifolds under positivity assumption on the orthogonal Bakry-Émery tensor. Moreover, under such assumptions on the orthogonal Bakry-Émery tensor and the holomorphic or quaternionic sectional curvature on a Kähler manifold or a quaternionic Kähler manifold respectively, the Bonnet-Myers type diameter bounds are sharper than in the Riemannian case.

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1. Introduction

The Bonnet–Myers theorem in [17] is a fundamental theorem connecting compactness and upper bounds on the diameter of a complete Riemannian manifold. A typical result relies on the assumption of a positive lower bound of the Ricci curvature, or a positivity assumption of the Bakry-Émery Ricci...
tensor such as \[1, 2, 9, 13, 14, 20, 23\]. Such results connecting Ricci curvature or its generalizations and Bonnet-Myers type theorems have been studied for Riemannian manifolds, and on sub-Riemannian manifolds in \[3, 4\]. Our focus is on the setting of Kähler geometry and quaternionic Kähler geometry for which such results have not been considered so far.

We will rely on the decomposition of Ricci curvature on Kähler manifolds into orthogonal Ricci curvature and holomorphic sectional curvature as follows.

\[
\text{Ric}(X, \overline{X}) = \text{Ric}^\perp(X, \overline{X}) + \frac{R(X, \overline{X}, X, \overline{X})}{|X|^4},
\]

where \(X\) is a \((1, 0)\)-tangent vector of the holomorphic tangent bundle on a Kähler manifold \(M^n\). A similar decomposition holds on a quaternionic Kähler geometry. Precise definitions are given by (2.1) and (2.3). Using such a decomposition, it is natural to consider a Bakry-Émery tensor in both Kähler geometry and quaternionic Kähler geometry similarly to how the Ricci curvature is replaced by a Bakry-Émery tensor in Riemannian geometry. The goal of this paper is to study diameter theorems for compact Kähler manifolds and quaternionic Kähler manifolds under various notions of positive lower bounds on the orthogonal Bakry-Émery type tensor corresponding to the orthogonal Ricci curvature.

First, we note that replacing the positivity of the orthogonal Ricci curvature with a weaker notion of positivity is justified by the following observation. Indeed, the class of complete Kähler manifolds with a positive orthogonal Ricci curvature is rather small although an orthogonal Ricci curvature bound is usually weaker than a Ricci curvature bound. For example, a complete Kähler manifold \(M^n, n \geq 2\) with a positive lower bound on the orthogonal Ricci curvature must be compact and always projective \([18, \text{Theorem 1.7}]\). Moreover, for \(n = 2\), a compact \(M^2\) which admits a Kähler metric with \(\text{Ric}^\perp > 0\) must be biholomorphic to the two-dimensional complex projective space \(\mathbb{P}^2\), and for \(n = 3\), a compact Kähler manifold under \(\text{Ric}^\perp > 0\) must be biholomorphic to either \(\mathbb{P}_C^3\) or the smooth quadratic hypersurface in \(\mathbb{P}_C^4\) as pointed out by \([12, \text{Theorem 1.8}]\). On the other hand, as there is an example of complete non-compact Riemannian manifold with a non-negative Ricci curvature lower bound, there are also complete non-compact examples of Kähler manifolds with \(\text{Ric}^\perp > 0\) \([18, \text{p151}]\). Therefore in order to consider the complete Kähler manifolds of the wide class rather than the complete Kähler manifolds of the limited classes possible under the positive orthogonal Ricci curvature, it is reasonable to make at least weaker the positivity condition than the orthogonal Ricci curvature. On the other hand, similarly to the Bakry-Émery tensor in the Riemannian case, a complete Kähler manifold need not be compact even under if the orthogonal Bakry-Émery tensor satisfies a positive lower bound as shown in Example 1.

As the Ricci curvature can be written as a sum of orthogonal Ricci curvature and holomorphic (respectively, quaternionic) sectional curvature for Kähler (respectively, quaternionic Kähler) manifolds, we consider two versions of an orthogonal Bakry-Émery tensor corresponding to the orthogonal Ricci curvature. In the first case we consider the orthogonal Bakry-Émery tensor \(\text{Ric}^\perp + \text{Hess}(\phi)\), where \(\phi\) is a real-valued smooth function on \(M\) and \(\text{Hess}(\phi)\) is the Riemannian Hessian. That is, we omit the holomorphic (respectively, quaternionic) sectional curvature. Note that here we only consider the Bakry-Émery tensor of a gradient type \(\text{Hess}(\phi)\). Similarly to how compactness and diameter have been treated on complete Riemannian manifolds in \([13, 14]\), here too, additional smoothness assumptions are needed to obtain the results.
Another approach to compactness and diameter bounds results using Ricci curvature or Bakry-Émery tensor assumptions on a Riemannian manifold is to rely on Bochner’s formula. We are exploring such an approach for complete Kähler manifolds with an orthogonal Ricci curvature bound or its generalization. For this purpose, we derive a new Bochner’s formula in Proposition 6 for the orthogonal Ricci curvature, and then establish such results under the assumptions which are compatible with this Bochner’s formula. The second case is to consider a non-gradient type Bakry-Émery type tensor $\text{Ric}^{\perp}_{m,Z}$ defined by (4.1) with a vector field $Z$ and an additional assumption on the holomorphic sectional curvature (quaternionic sectional curvature in the case of quaternionic Kähler manifolds). In the previous case, the second-order differential operator in Bochner’s formula for the orthogonal Ricci curvature is not hypoelliptic in general, whereas in the second case, as in the Riemannian case, we use the Laplace-Beltrami operator making it possible to use a weaker positivity assumption than in the first case when we replaced the orthogonal Ricci curvature by a Bakry-Émery tensor. In order to show the diameter upper bound, we follow Kuwada’s approach in [11] and consider a stochastic process with this operator as a generator that might be non-symmetric. We then prove an upper bound on the diameter which is sharper than the diameter upper bound in the Riemannian case.

This paper is organized as follows. In Section 2 we introduce basic definitions and properties of Kähler manifolds and quaternionic Kähler manifolds, and in particular, how these structures are connected to their Riemannian structures. In Section 3, diameter theorems are covered under the Bakry-Émery orthogonal Ricci tensor of the gradient type. In Section 4 we prove diameter theorems for a non-gradient Bakry-Émery tensor under the additional assumption on the holomorphic (quaternionic) sectional curvature.

2. Preliminaries on Kähler and quaternionic Kähler manifolds

We start by reviewing basics of Kähler and quaternionic manifolds.

2.1. Kähler manifolds. Let $M$ be an $n$-dimensional complex manifold equipped with a complex structure $J$ and a Hermitian metric $g$. The complex structure $J : T_R M \to T_R M$ is a real linear endomorphism that satisfies for every $x \in M$, and $X, Y \in T_{R,x} M$, $g_x(J_x X, Y) = -g_x(X, J_x Y)$, and for every $x \in M$, $J_x^2 = -\text{Id}_{T_x M}$. We decompose the complexified tangent bundle $T_R M \otimes_R \mathbb{C} = T'M \oplus T'M$, where $T'M$ is the eigenspace of $J$ with respect to the eigenvalue $\sqrt{-1}$ and $T'M$ is the eigenspace of $J$ with respect to the eigenvalue $-\sqrt{-1}$. We can identify $v, w$ as real tangent vectors, and $\eta, \xi$ as corresponding holomorphic $(1,0)$ tangent vectors under the $\mathbb{R}$-linear isomorphism $T_R M \to T'M$, i.e. $\eta = \frac{1}{\sqrt{2}}(v - \sqrt{-1} Jv), \xi = \frac{1}{\sqrt{2}}(w - \sqrt{-1} Jw)$.

A Hermitian metric on $M$ is a positive definite Hermitian inner product $g_p : T_p'M \otimes T_p'M \to \mathbb{C}$.
which varies smoothly for each $p \in M$. Here, varying smoothly means that if $z = (z_1, \cdots, z_n)$ are local coordinates around $p$, and $\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}$ is a standard basis for $T_p^\prime M$, the functions

$$g_{i\overline{j}} : U \to \mathbb{C}, p \mapsto g_p(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j})$$

are smooth for all $i, j = 1, \cdots, n$. Locally, a Hermitian metric can be written as

$$g = \sum_{i,j=1}^{n} g_{i\overline{j}} dz_i \otimes d\overline{z_j},$$

where $(g_{i\overline{j}})$ is an $n \times n$ positive definite Hermitian matrix of smooth functions and $dz_1, \cdots, dz_n$ be the dual basis of $\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}$. The metric $g$ can be decomposed into the real part denoted by $\text{Re}(g)$, and the imaginary part, denoted by $\text{Im}(g)$. $\text{Re}(g)$ induces an inner product called the induced Riemannian metric of $g$, an alternating $\mathbb{R}$-differential 2-form. Define the $(1, 1)$-form

$$\omega := -\frac{1}{2} \text{Im}(g),$$

which is called the fundamental $(1, 1)$-form of $g$. In local coordinates this form can written as

$$\omega = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\overline{j}} dz_i \wedge d\overline{z_j}.$$

In this setting we have two natural connections. The Chern connection $\nabla^c$ is compatible with the Hermitian metric $g$ and the complex structure $J$, and the Levi-Civita connection $\nabla$ is a torsion free connection compatible with the induced Riemannian metric. The components of the curvature 4-tensor of the Chern connection associated with the Hermitian metric $g$ are given by

$$R_{ijkl} := R(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_l}) = g \left( \nabla^c \frac{\partial}{\partial z_p}, \nabla^c \frac{\partial}{\partial z_q}, \nabla^c \frac{\partial}{\partial z_r}, \nabla^c \frac{\partial}{\partial z_s} \right) - \frac{\partial^2 g_{i\overline{j}}}{\partial z_k \partial \overline{z_l}} + \sum_{p,q=1}^{n} g_{p\overline{q}} \frac{\partial g_{i\overline{j}}}{\partial z_k} \frac{\partial g_{k\overline{p}}}{\partial \overline{z_l}},$$

where $i, j, k, l \in \{1, \cdots, n\}$.

The Hermitian metric $g$ is called Kähler if $d\omega = 0$, where $d$ is the exterior derivative $d = \partial + \overline{\partial}$, and the Chern and Levi-Civita connections coincide precisely when the Hermitian metric is Kähler. There are several equivalent ways to show that a metric is Kähler, and one of them is that a metric $g$ is Kähler if and only if for any $p \in M$, there exist holomorphic coordinates $(z_1, \cdots, z_n)$ near $p$ such that $g_{\overline{i}j}(p) = \delta_{\overline{i}j}$ and $(dg_{\overline{i}j})(p) = 0$. Such coordinates are called holomorphic normal coordinates.

The holomorphic sectional curvature with the unit direction $\eta$ at $x \in M$ (i.e., $g_\omega(\eta, \eta) = 1$) is defined by

$$H(g)(x, \eta) = R(\eta, \overline{\eta}, \eta, \overline{\eta}) = R(v, Jv, Jv, v),$$

where $v$ is the real tangent vector corresponding to $\eta$. We will often write $H(g)(x, \eta) = H(g)(\eta) = H(\eta)$. 

Following [19] we define the orthogonal Ricci curvature on a Kähler manifold \((M, g, J)\) by
\[
\text{Ric}^\perp(v, v) = \text{Ric}(v, v) - H(v),
\]
where \(v\) is a real vector field and Ric is the Ricci 2-tensor of \((M, g)\). Unlike the Ricci tensor, \(\text{Ric}^\perp\) does not admit polarization, so we never consider \(\text{Ric}^\perp(u, v)\) for \(u \neq v\). For a real vector field \(v\), we can write
\[
\text{Ric}^\perp(v, v) = \sum R(v, E_i, E_i, v),
\]
where \(\{e_i\}\) is any orthonormal frame of \(\{v, Jv\}^\perp\). We will assign index 1, 2 to \(v\) and \(Jv\) in this summation expression for complex \(n\) dimensional Kähler manifold \(M^n\), and use indices from 3 to \(2n\) for orthonormal frames \(\{E_i\}\) of \(\{v, Jv\}^\perp\). Denote by \(F_i = \frac{1}{\sqrt{2}}(E_i - \sqrt{-1}J(E_i))\) a unitary frame such that \(E_1 = v/|v| =: \tilde{v}\) by following the convention \(E_{n+i} = J(E_i)\), then
\[
\frac{1}{|v|^2}\text{Ric}^\perp(v, v) = \text{Ric}^\perp(\tilde{v}, \tilde{v}) = \text{Ric}(\tilde{v}, \tilde{v}) - R(\tilde{v}, J\tilde{v}, \tilde{v}, J\tilde{v})
= \text{Ric}(F_1, \overline{F_1}) - R(F_1, F_1, F_1, F_1) = \sum_{j=2}^n R(F_1, F_1, F_j, F_j).
\]
In particular, we have \(\text{Ric}(F_i, F_i) = \text{Ric}(E_i, E_i)\), \(\text{Ric}^\perp(\tilde{v}, \tilde{v}) = \text{Ric}(F_1, \overline{F_1}) - R_{1\overline{1}},\overline{1}\overline{1}\).

2.2. Quaternionic Kähler manifolds. We start by recalling the following definition of quaternionic Kähler manifold following [6, Proposition 14.36].

**Definition 1.** A Riemannian manifold \((M, g)\) is called a quaternionic Kähler manifold if there exists a covering of \(M\) by open sets \(U_i\) and, for each \(i\) there exist 3 smooth \((1,1)\) tensors \(I, J, K\) on \(U_i\) such that
- For every \(x \in U_i\), and \(X, Y \in T_xM\), \(g_x(I_xX, Y) = -g_x(X, I_xY)\), \(g_x(J_xX, Y) = -g_x(X, J_xY)\), \(g_x(K_xX, Y) = -g_x(X, K_xY)\);
- For every \(x \in U_i\), \(I_x^2 = J_x^2 = K_x^2 = I_xJ_xK_x = -\text{Id}_{T_xM}\);
- For every \(x \in U_i\), and \(X \in T_xM \nabla_XI, \nabla_XJ, \nabla_XK \in \text{span}\{I, J, K\}\);
- For every \(x \in U_i \cap U_j\), the vector space of endomorphisms of \(T_xM\) generated by \(I_x, J_x, K_x\) is the same for \(i\) and \(j\).

Note that in some cases such as the quaternionic projective spaces the tensors \(I, J, K\) may not be defined globally for topological reasons. However, \(\text{span}\{I, J, K\}\) may always be defined globally according to Definition 1.

On quaternionic Kähler manifolds, we will be considering the following curvature tensors. As above, let
\[
R(X, Y, Z, W) = g((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} )Z, W)
\]
be the Riemannian curvature tensor of \((M, g)\). We define the quaternionic sectional curvature of the quaternionic Kähler manifold \((M, g, J)\) as
\[
Q(X) = \frac{R(X, IX, IX, X) + R(X, JX, JX, X) + R(X, KX, KX, X)}{g(X, X)^2}.
\]
Following [5, Section 2.1.2] we define the orthogonal Ricci curvature of the quaternionic Kähler manifold $(M, g, I, J, K)$ by
\[
\text{Ric}^\perp(X, X) = \text{Ric}(X, X) - Q(X),
\]
where \(\text{Ric}\) is the usual Riemannian Ricci tensor of \((M, g)\) and \(X\) is a vector field such that \(g(X, X) = 1\).

Lastly, given a vector field \(V\) on a Riemannian manifold along a geodesic \(\gamma\) defined on \([a, b]\), the index form \(I\) associated to \(V\) is defined as
\[
I(V, V) = \int_a^b (|\dddot{V}(s)|^2 - R(V(s), \dot{\gamma}(s), \dot{\gamma}(s), V(s))) \, ds,
\]
and using polarization the form \(I\) can be extended to a bilinear form on the space of vector fields along the geodesic \(\gamma\).

3. Bakry-Émery orthogonal Ricci tensor of the gradient type

Given a Riemannian manifold \((M, g)\) and a smooth function \(\phi : M \to \mathbb{R}\), we denote the Hessian of \(\phi\) by \(\text{Hess}(\phi)\), i.e., \(\text{Hess}(\phi)(X, Y) = g(\nabla_X \nabla \phi, Y)\) for any real vector fields \(X, Y\). In this section, we define and consider the orthogonal Bakry-Émery tensor \(\text{Ric}^\perp + \text{Hess}(\phi)\) with a smooth function \(\phi\) on either a Kähler manifold or a quaternionic Kähler manifold. On a Kähler manifold \(M^n\) with \(k \in \mathbb{R}\), we say \(\text{Ric}^\perp + \text{Hess}(\phi) \geq (2n - 2)k\) if for any unit vector \(v\)
\[
\text{Ric}^\perp(v, v) + \text{Hess}(\phi)(v, v) \geq (2n - 2)k,
\]
and similarly, we assume \(\text{Ric}^\perp(v, v) + \text{Hess}(\phi)(v, v) \geq (4n - 4)k\) for a quaternionic Kähler manifold. The Bakry-Émery tensor considered in this section is different from such a tensor in Section 4. Indeed, the modified Bochner’s formula in Proposition 6 shows the relationship between orthogonal Laplacian and orthogonal Ricci curvature, without any assumptions on holomorphic (or quaternionic) sectional curvature. With the modified Bochner formula, assumptions on a smooth function \(\phi\) are important when trying to prove diameter theorems.

Previously Riemannian manifolds endowed with a weighted volume measure \(e^{-f} dV_g\) satisfying a lower bound on the standard Bakry-Émery Ricci tensor has been studied in several settings. For example, a Riemannian manifold \((M, g)\) is called a gradient Ricci soliton if there exists a real-valued smooth function \(f\) on \(M\) such that the Ricci curvature and the Hessian of \(f\) satisfy \(\text{Ric} + \text{Hess}(f) = \lambda g\) for some \(\lambda \in \mathbb{R}\). Gradient Ricci solitons play an important role in the theory of Ricci flow as in [8]. Bakry-Émery Ricci tensor plays fundamental role in [23], and it has been extended to metric measure spaces using the Lott-Villani-Sturm theory initiated by [15, 21]. In particular, if \((M, g)\) is a Kähler manifold and it is a gradient Ricci soliton with a real-valued smooth function \(f\), then \(\nabla f\) is a real holomorphic vector field, i.e., its \((1, 0)\)-part is a holomorphic vector field. Moreover, the weighted Hodge Laplacian from considered with respect to the weighted volume measure \(e^{-f} dV_g\) maps the space of smooth \((p, q)\) forms to itself for \(0 < p + q < 2n\) if and only if \(\nabla f\) is a real holomorphic vector field by [16, Proposition 0.1]. Also, the real holomorphic vector field serves as a critical point of the Calabi functional which yields the Calabi’s extremal Kähler metric that can be used to study the existence of the Kähler-Einstein metric on Fano manifolds [7]. Lastly, on a compact Kähler manifold with Ricci curvature bounded below by a
positive constant $k$, if the first nonzero eigenvalue achieves its optimal lower bound $2k$ then the gradient vector field of the corresponding eigenfunction must be real holomorphic by [22].

In our case, we are interested in bounds on $\text{Ric}^+ + \text{Hess}(\phi)$, where $\phi$ is a real-valued smooth function, and the holomorphic sectional curvature is not controlled, hence it is different from the case where bounds on the Bakry-Émery tensor are used.

For compactness of a complete Riemannian manifold with an upper bound on the diameter, one needs additional assumptions on the function whose Hessian is used to define the Bakry-Émery Ricci tensor as pointed out by [23]. The next example indicates that we need to address such an issue for the orthogonal Bakry-Émery tensor as well.

**Examples 1.** Let $M = \mathbb{C}^n$ with the Euclidean metric $g_E$, and $\phi(x) = \frac{1}{2}|x|^2$. Then both the orthogonal Ricci curvature and the holomorphic sectional curvature are zero, and $\text{Hess}(\phi) = \lambda g_E$ and $(\text{Ric}^+ + \text{Hess}(\phi))(v, v) = \lambda g_E(v, v)$ for any real vector field $v$. This example shows that a complete Kähler manifold with positive lower bound of $\text{Ric}^+ + \text{Hess}(\phi)$ is not necessarily compact.

As we observed from the example, to prove that a complete Kähler (and also quaternionic Kähler) manifold with a positive lower bound on the orthogonal Bakry-Émery tensor is compact we need to have additional assumptions on $\phi$, and we will start with the case when $\phi$ is bounded. In this case, we have the following elementary lemma to control such a function $\phi$. This fact has been used in the proof of [14, Theorem 1] for a Bakry-Émery Ricci tensor on Riemannian manifolds.

**Lemma 2.** Let $M$ be a Riemannian manifold with a Riemannian metric $g$ and let $\phi : M \to \mathbb{R}$ be a smooth function satisfying $|\phi| \leq C$ for some $C \geq 0$. Let $\gamma$ be a minimizing unit speed geodesic segment from $p$ to $q$ of length $l$. Then we have

$$\int_0^l f^2 \text{Hess}(\phi)(\dot{\gamma}, \dot{\gamma})(\gamma(t)) \leq 2C \sqrt{l} \left( \int_0^l \left( \frac{d}{dt}(f \dot{f})^2 \right) dt \right)^{1/2},$$

for any smooth function $f \in C^\infty([0, l])$ such that $f(0) = f(l) = 0$. Here $\dot{f}(t)$ means $\frac{d}{dt}f(\gamma(t))$.

**Proof.** From

$$f(t)^2 \text{Hess}(\phi)(\dot{\gamma}, \dot{\gamma})(\gamma(t)) = f(t)^2 \frac{d}{dt} (g(\nabla \phi, \dot{\gamma})) (\gamma(t))$$

$$= -2f(t)f(t) g(\nabla \phi, \dot{\gamma}) + \frac{d}{dt} \left( f(t)^2 g(\nabla \phi, \dot{\gamma}) \right)(\gamma(t))$$

$$= 2\phi(\gamma(t)) \frac{d}{dt} (f(t)f(t'))(\gamma(t)) - 2 \frac{d}{dt} (\phi f(t)f(t')(\gamma(t)) + \frac{d}{dt} (f(t)^2 g(\nabla \phi, \dot{\gamma}))(\gamma(t)),$$

and $f(0) = f(l) = 0$, we have

$$\int_0^l f^2 \text{Hess}(\phi)(\dot{\gamma}, \dot{\gamma})dt = 2 \int_0^l \phi \frac{d}{dt}(f \dot{f}) dt.$$
Now the Cauchy-Schwarz inequality with the assumption $|\phi| \leq C$ implies
\[
\int_0^l f^2 \text{Hess}(\phi) = 2 \int_0^l \frac{d}{dt}(f \dot{f}) dt \leq 2C\sqrt{l} \left( \int_0^l \left( \frac{d}{dt}(f \dot{f})^2 \right) dt \right)^{1/2}.
\]

$\square$

**Proposition 3.** Let $(M^n, g)$ be a complete Kähler manifold with the complex dimension $n \geq 2$. Suppose that for some constant $k > 0$, $\text{Ric}^\perp + \text{Hess}(\phi) \geq (2n-2)k$ and $|\phi| \leq C$ for some $C \geq 0$. Then the diameter $D$ of $M$ has the upper bound
\[
D \leq \frac{\pi}{\sqrt{k}} \sqrt{1 + \frac{\sqrt{2}C}{n-1}}.
\]

**Proof.** Let $p, q \in M$ and let $\gamma$ be a minimizing unit speed geodesic segment from $p$ to $q$ of length $l$. Consider a parallel orthonormal frame
\[
\{E_1 = \dot{\gamma}, E_2 = JE_1, \ldots, E_{2n}\}
\]
along $\gamma$ and a smooth function $f \in C^\infty([0, l])$ such that $f(0) = f(l) = 0$. Here we used the Kähler condition $\nabla J = 0$ and parallel transport to have $E_2 = JE_1$. From the definition of $\text{Ric}^\perp$, we have
\[
\sum_{i=3}^{2n} \mathcal{I}(f E_i, f E_i) = \int_0^l ((2n-2)f^2 - \sum_{i=3}^{2n} R(f E_i, \dot{\gamma}, \dot{\gamma}, f E_i) dt
\]
\[
= \int_0^l ((2n-2)f^2 - f^2 \text{Ric}^\perp(\dot{\gamma}, \dot{\gamma})) dt.
\]
By the assumption on the orthogonal Bakry-Émery tensor,
\[
\sum_{i=3}^{2n} \mathcal{I}(f E_i, f E_i) \leq \int_0^l ((2n-2)(f^2 - kf^2) + f^2 \text{Hess}(\phi)(\dot{\gamma}, \dot{\gamma})) dt,
\]
where $\mathcal{I}$ denotes the index form of $\gamma$. By Lemma 2, we have
\[
\sum_{i=3}^{2n} \mathcal{I}(f E_i, f E_i) \leq \int_0^l ((2n-2)(f^2 - kf^2) dt + 2C\sqrt{l} \left( \int_0^l \left( \frac{d}{dt}(f \dot{f})^2 \right) dt \right)^{1/2}.
\]
Now, take $f$ to be $f(t) = \sin\left(\frac{\pi}{l}t\right)$, then we get
\[
\sum_{i=3}^{2n} \mathcal{I}(f E_i, f E_i) \leq (2n-2) \int_0^l \left( \frac{\pi^2}{l^2} \cos^2\left(\frac{\pi}{l}t\right) - k \sin^2\left(\frac{\pi}{l}t\right) \right) dt
\]
\[
+ \frac{2C\pi^2}{l\sqrt{l}} \left( \int_0^l \cos^2\left(\frac{2\pi}{l}t\right) dt \right)^{1/2},
\]
and therefore
\[
\sum_{i=3}^{2n} \mathcal{I}(f E_i, f E_i) \leq -\frac{l}{l} \left( (n-1)k l^2 - \sqrt{2}C\pi^2 - (n-1)\pi^2 \right).
\]
Now if \((n-1)kl^2 - \sqrt{2C} \pi^2 - (n-1)\pi^2 > 0\), this forces \(\mathcal{I}(fE_m, fE_m) < 0\) for some \(3 \leq m \leq 2n\). On the other hand, since \(\gamma\) is a minimizing geodesic, the index form \(\mathcal{I}\) is positive semi-definite, which is a contradiction. Therefore,

\[
l \leq \frac{\pi}{\sqrt{k}} \sqrt{1 + \frac{\sqrt{2C}}{n-1}}.
\]

\[\square\]

**Remark 1.** By taking \(C = 0\), we can conclude that a complete Kähler manifold \(M\) with a positive lower bound on the orthogonal Ricci curvature implies compactness of \(M\) and thereby implies that its fundamental group is finite by [18, Theorem 3.2].

The proof of Proposition 3 is easily modified to the quaternionic Kähler case.

**Proposition 4.** Let \((M^n, g, I, J, K)\) be a complete quaternionic Kähler manifold of the quaternionic dimension \(n \geq 2\). Suppose that for some constant \(k > 0\), \(\text{Ric}^\perp + \text{Hess}(\phi) \geq (4n-4)k\) and \(|\phi| \leq C\) for some \(C \geq 0\). Then the diameter \(D\) of \(M\) satisfies the upper bound

\[
D \leq \frac{\pi}{\sqrt{k}} \sqrt{1 + \frac{\sqrt{2C}}{2n - 2}}.
\]

**Proof.** We consider an orthonormal frame \(\{X_1(x), \cdots, X_{4m}(x)\}\) around \(x \in M\) such that

\[
X_1(x) = \gamma'(0), \ X_2(x) = I\gamma'(0), \ X_3(x) = J\gamma'(0), \ X_4(x) = K\gamma'(0)
\]

We introduce the function

\[
j(k, t) = \cos \sqrt{kt} + \frac{1 - \cos \sqrt{kt}}{\sin \sqrt{kt}} \sin \sqrt{kt},
\]

and we denote by \(X_1, \cdots, X_{4m}\) vector fields obtained by parallel transporting \(X_1(x), \cdots, X_{4m}(x)\) along \(\gamma\) and consider the vector fields defined along \(\gamma\) by

\[
\tilde{X}_2(\gamma(t)) = j(4k,t)X_2, \ \tilde{X}_3(\gamma(t)) = j(4k, t)X_3, \ \tilde{X}_4(\gamma(t)) = j(4k, t)X_4
\]

and for \(i = 5, \cdots, 4m\) by

\[
\tilde{X}_i(\gamma(t)) = j(k, t)X_i.
\]

Then the result follows by arguments similar to the previous proof. \[\square\]

Using an argument similar to the case of a bounded function \(\phi\) in the summation of the index form in the proof of the Proposition 3, we can prove similar types of diameter theorems based on different assumptions on \(\phi\). Let us just mention one setting that can easily replace Lemma 2. Given a vector field \(V\) on a Riemannian manifold \((M, g)\), we denote the Lie derivative of \(V\) by \(\mathcal{L}_V g\) (see [13]).
Lemma 5. Let $M$ be a Riemannian manifold with a Riemannian metric $g$ and let $V$ be the smooth vector field satisfying $|V| \leq C$ for some $C \geq 0$. Let $\gamma$ be a minimizing unit speed geodesic segment from $p$ to $q$ of length $l$. Then we have

$$\int_0^l f^2 \mathcal{L}_V g(\dot{\gamma}, \dot{\gamma}) dt \leq C \sqrt{\frac{l}{2}} \left( \int_0^l f^2 f^2 dt \right)^{1/2},$$

for any smooth function $f \in C^\infty([0, l])$ such that $f(0) = f(l) = 0$.

Now we will consider different assumptions on smooth function $\phi$ to apply the condition of $\text{Ric}^+ + \text{Hess}(\phi) \geq (2n - 2)k$ differently on both Kähler manifolds or quaternionic Kähler manifolds based on the modified Bochner type formula. The Bochner formula was used to control the Laplace-Beltrami operator under the Bakry-Émery tensor with the certain assumption on $\phi$ [14], and we modify this approach. To do so, we first define the orthogonal Laplacian $\Delta^\perp$ on Kähler manifolds (quaternion Kähler case would be similar) as follows: given any fixed point $p$ on a complete Kähler manifold $(M^n, J)$ of complex dimension $n$. For each real-valued smooth function $f$, consider the holomorphic vector field $Z = \frac{1}{\sqrt{2}}(\nabla f - \sqrt{-1}J(\nabla f))$ corresponding to the real vector field $\nabla f$ and define

$$\Delta^\perp f := \Delta f - \text{Hess}(f)(Z, Z).$$

(also see [18, p.151]) Equivalently, with $(E_i)_{i=1}^{2n}$ be an orthonormal frame with $E_1 = \nabla f$ and $E_2 = JE_1$,

$$\Delta^\perp f = \Delta f - \text{Hess}(f)(E_1, E_1) - \text{Hess}(f)(JE_1, JE_1) = \sum_{i=3}^{2n} \text{Hess}(f)(E_i, E_i).$$

Similarly, the orthogonal Laplacian of a real-valued smooth function $f$ on a quaternionic Kähler manifold $(M, I, J, K)$ is defined by

$$\Delta^\perp f := \Delta f - \text{Hess}(f)(E_1, E_1) - \text{Hess}(f)(JE_1, JE_1) - \text{Hess}(f)(KE_1, KE_1)$$

$$= \sum_{i=5}^{2n} \text{Hess}(f)(E_i, E_i),$$

where $(E_i)_{i=1}^{2n}$ is an orthonormal frame around $p$ with $E_2 = JE_1, E_3 = JE_1, E_4 = KE_1$.

Since the orthogonal Laplacian is neither a self-adjoint operator nor a hypo-elliptic operator in general, it might be difficult to study this operator and its applications. Nevertheless, the modified Bochner type formula corresponding to the orthogonal Ricci curvature can be established. We hope this form may have several applications on the geometric analysis side in the future. In our paper, we use the following modified Bochner’s formula to use in Proposition 8. The idea of the proof is a Bochner’s formula modified to fit the orthogonal Ricci tensor.

**Proposition 6** (Bochner’s formula: Kähler’s case). Let $(M^n, g, J)$ be a Kähler manifold of the complex dimension $n$. Let $f$ be a real-valued smooth function on $M$ and $\Delta^\perp$ be the orthogonal Laplacian. Then
for any orthonormal frame \((E_i)_{i=1}^{2n}\) around \(q \in M\) with \(E_1 = \nabla f, E_2 = JE_1\),

\[
\frac{1}{2} \sum_{i=3}^{2n} E_i E_i g(\nabla f, \nabla f)(q) = \text{Ric}^\perp(\nabla f, \nabla f)(q) + g(\nabla f, \nabla \Delta^\perp f)(q)
\]

\[
+ \sum_{i=3}^{2n} g(\nabla \nabla f, \nabla E_i, E_i)(q) + \sum_{i=3}^{2n} (g(\nabla E_i, \nabla f, \nabla E_i) - 2g(\nabla \nabla f, \nabla E_i, E_i))(q).
\]

**Proof.** Let \(q \in M\). Then at \(q\),

\[
\frac{1}{2} \sum_{i=3}^{2n} E_i E_i g(\nabla f, \nabla f)(q) = \sum_{i=3}^{2n} E_i g(\nabla E_i, \nabla f, \nabla f)(q)
\]

\[
= \sum_{i=3}^{2n} E_i \text{Hess}(f)(E_i, \nabla f)(q) = \sum_{i=3}^{2n} E_i \text{Hess}(f)(\nabla f, E_i)(q)
\]

\[
= \sum_{i=3}^{2n} E_i g(\nabla \nabla f, E_i)(q) = \sum_{i=3}^{2n} g(\nabla E_i, \nabla \nabla f, E_i)(q) + g(\nabla \nabla f, \nabla E_i, E_i)(q).
\]

By using the Riemann curvature tensor defined by (2.2) we see that

\[
\sum_{i=3}^{2n} g(\nabla E_i, \nabla \nabla f, E_i)(q) = \sum_{i=3}^{2n} g(\nabla E_i, \nabla \nabla f, E_i)(q) = \sum_{i=3}^{2n} g(R(E_i, \nabla f) \nabla f, E_i) + g(\nabla \nabla f \nabla E_i, \nabla f, E_i) + g(\nabla E_i, \nabla \nabla f, E_i)(q)
\]

\[
= \text{Ric}^\perp(\nabla f, \nabla f)(q) + \sum_{i=3}^{2n} (g(\nabla \nabla f, E_i, E_i) + g(\nabla E_i, \nabla \nabla f, E_i))(q).
\]

The second term \(\sum_{i=3}^{2n} g(\nabla \nabla f, E_i, E_i)\) in (3.1) can be written as

\[
\sum_{i=3}^{2n} g(\nabla E_i, \nabla f, E_i) - g(\nabla E_i, \nabla f, \nabla \nabla f) E_i)(q)
\]

\[
= \nabla f(\Delta^\perp f) - \sum_{i=3}^{2n} g(\nabla E_i, \nabla f, \nabla \nabla f E_i) = g(\nabla f, \nabla \Delta^\perp f) - \sum_{i=3}^{2n} g(\nabla E_i, \nabla f, \nabla \nabla f E_i).\]
The last term $g(\nabla_{[E_i,\nabla f]} \nabla f, E_i)(q)$ in (3.1) can be written as
\[
\sum_{i=3}^{2n} \text{Hess}(f)([E_i, \nabla f], E_i)(q) = \sum_{i=3}^{2n} \text{Hess}(f)(\nabla_{E_i} \nabla f - \nabla_{\nabla f} E_i, E_i)(q)
\]
\[
= \sum_{i=3}^{2n} (g(\nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f) - g(\nabla_{\nabla f} E_i, \nabla_{E_i} \nabla f))(q).
\]
and we obtain the desired formula. \qed

**Remark 2.** To compare with the usual Bochner’s formula, the formula in Proposition 6 uses the orthogonal Ricci tensor, the orthogonal Laplacian, and \(\text{Hessian norm squared of } f\) instead of the Ricci tensor, the Laplace-Beltrami operator, and the Hessian norm squared of \(f\) respectively. If we choose an orthonormal frame satisfying \(\nabla_{E_i} E_j(q) = 0\) at some fixed point \(q\) for any \(i, j = 1, \ldots, 2n\), then each of the three terms in (3.2) are zero at \(q\). However, this does not imply that the first two terms in (3.2) vanish somewhere. For example, if we take \(f\) to be the geodesic distance \(r\) emanating from \(q \in M\), then outside of the cut-locus of \(q\), one can see that \(\lim_{r \to 0^+} r \Delta^1 r = 2n - 2\), but \(\sum_{i=3}^{2n} g(\nabla_{E_i} r, \nabla_{E_i} r) \geq \frac{1}{2n-2} (\Delta^1 r)^2\) (see the proof of Proposition 8), and by combining these two, \(\sum_{i=3}^{2n} g(\nabla_{E_i} r, \nabla_{E_i} r)\) cannot vanish in a small neighborhood of \(q\).

With similar computations, one can obtain the quaternionic version of the modified Bochner’s formula.

**Proposition 7** (Bochner’s formula: quaternionic case). Let \((M^n, g, I, J, K)\) be a quaternionic Kähler manifold of the quaternionic dimension \(n\). Let \(f\) be a real-valued smooth function on \(M\) and \(\Delta^1\) be the orthogonal Laplacian. Then for any orthonormal frame \((E_i)_{i=1}^{4n}\) around \(q\) with \(E_2 = IE_1, E_3 = JE_1, E_4 = KE_1\),
\[
\frac{1}{2} \Delta^1 |\nabla f|^2(q) = \text{Ric}^1(\nabla f, \nabla f)(q) + g(\nabla f, \nabla \Delta^1 f)(q) + \sum_{i=5}^{4n} g(\nabla_{\nabla f} \nabla f, \nabla_{E_i} E_i)(q)
\]
\[
+ \sum_{i=5}^{4n} (g(\nabla_{E_i} \nabla f, \nabla_{E_i} \nabla f) - g(\nabla_{\nabla f} E_i, \nabla_{E_i} \nabla f))(q).
\]

By Bochner’s formula in Proposition 6 applied to the function \(f\) being equal to the geodesic distance \(r\) emanating from a point \(p \in M\) outside of the cut-locus of \(p\), although all terms \(g(\nabla_{\nabla f} \nabla r, \nabla_{E_i} E_i), i = 3, \ldots, 2n\) vanish, we still need to control the term
\[
-2 \sum_{i=3}^{2n} g(\nabla_{\nabla r} E_i, \nabla_{E_i} \nabla r) = -2 \sum_{i=3}^{2n} \text{Hess}(r)(\nabla_{\nabla r} E_i, E_i).
\]
With this consideration, it would be natural to compensate the averaging effect $\sum_{i=3}^{2n} \text{Hess}(r)(\nabla_{r}E_{i}, E_{i})$ by the Hessian of some function $\phi$, for example,

$$\text{Hess}(\phi)(\nabla_{r}, \nabla_{r}) \geq 2 \sum_{i=3}^{2n} \text{Hess}(r)(\nabla_{r}E_{i}, E_{i}).$$

By adding $\text{Hess}(\phi)$ term to the orthogonal Ricci curvature with certain assumptions on $\phi$, we obtain the following diameter theorem. One can see that the assumptions on $\phi$ in the Proposition below are analogous to the assumptions of Theorem 2 in [14]. If we replace the complete Kähler manifold with $\text{Ric}^{+} + \text{Hess}(\phi) - 2 \sum_{i=3}^{2n} \text{Hess}(r)(\nabla_{r}E_{i}, E_{i})$ by the Riemannian manifold with $\text{Ric} + \text{Hess}(\phi)$.

**Proposition 8.** Let $(M^n, g)$ be a complete Kähler manifold with the complex dimension $n \geq 2$. Take any $p \in M$ and let $r$ be the geodesic distance function from $p$. Suppose that for some constant $k > 0$, there exists a local orthonormal frame $(E_{i})_{i=1}^{2n}$ around $p$ with $E_{1} = \nabla_{r}, E_{2} = J\nabla_{r}$ such that

$$\text{Ric}^{+}(\nabla_{r}, \nabla_{r}) + \text{Hess}(\phi)(\nabla_{r}, \nabla_{r}) - 2 \sum_{i=3}^{2n} \text{Hess}(r)(\nabla_{r}E_{i}, E_{i}) \geq (2n - 2)k$$

outside of the cut-locus of $p$ and $|\nabla \phi|^2 \leq \frac{C}{r(x)}$ for some $C > 0$. Then the diameter $D$ of $M$ satisfies

$$D \leq \frac{\pi}{(n-1)k} \sqrt{2\sqrt{C} + n - 1}.$$

**Proof.** Define the modified orthogonal Laplacian

$$\tilde{\Delta}^{\perp}f := \Delta^{\perp}f - g(\nabla \phi, \nabla f) + F(f),$$

where $f$ is a smooth function on $M$ and $F$ is a real-valued function taking values from $\mathbb{R}$. We will take $f = r$, here $r$ is the distance function from the fixed point $p \in M$. Then outside of the cut-locus of $p$,

$$g(\nabla_{r}, \nabla \tilde{\Delta}^{\perp}r) = g(\nabla_{r}, \nabla \Delta^{\perp}r) - \text{Hess}(\phi)(\nabla_{r}, \nabla r) + F'(r), F'(r) = \frac{d}{dr}F(r).$$

(3.4)

On the other hand, from Proposition 6, since $\frac{1}{2} \Delta^{\perp}|\nabla r|^2 \equiv 0$,

$$0 = \text{Ric}^{+}(\nabla_{r}, \nabla r) + g(\nabla_{r}, \nabla \Delta r) + \sum_{i=3}^{2n} (g(\nabla_{E_{i}}\nabla_{r}, \nabla_{E_{i}}\nabla r) + g(\nabla_{\nabla_{r}}E_{i}, \nabla_{E_{i}}\nabla r)).$$

From the Cauchy-Schwarz inequality, the term $\sum_{i=3}^{2n} g(\nabla_{E_{i}}\nabla_{r}, \nabla_{E_{i}}\nabla r)$ has the following lower bound:

$$\sum_{i=3}^{2n} g(\nabla_{E_{i}}\nabla_{r}, \nabla_{E_{i}}\nabla r) = \sum_{i=3}^{2n} \sum_{j=1}^{2n} g(\nabla_{E_{i}}\nabla_{r}, E_{j})^2 \geq \sum_{i=3}^{2n} g(\nabla_{E_{i}}\nabla_{r}, E_{i})^2$$

$$= \frac{1}{2n-2} \sum_{i=3}^{2n} g(\nabla_{E_{i}}\nabla_{r}, E_{i})^2 \sum_{i=3}^{2n} 1 \geq \frac{1}{2n-2} \left( \sum_{i=3}^{2n} g(\nabla_{E_{i}}\nabla_{r}, E_{i}) \right)^2 = \frac{1}{2n-2} (\Delta^{\perp}r)^2,$$

thus we have

$$0 \geq \text{Ric}^{+}(\nabla_{r}, \nabla r) + g(\nabla_{r}, \nabla \Delta^{\perp}r) + \frac{1}{2n-2}(\Delta^{\perp}r)^2,$$

(3.5)
outside of the cut-locus of $p$.

From (3.5) and (3.4),

$$0 \geq \text{Ric}^\perp(\nabla r, \nabla r) + \text{Hess}(\phi)(\nabla r, \nabla r) + g(\nabla r, \nabla \Delta^\perp r) + \frac{1}{2n-2}(\Delta^\perp r)^2; \quad (3.6)$$

Combining with (3.3), (3.6) becomes

$$0 \geq \text{Ric}^\perp(\nabla r, \nabla r) + \text{Hess}(\phi)(\nabla r, \nabla r) + g(\nabla r, \nabla \Delta^\perp r) + \frac{1}{2n-2}(\tilde{\Delta}^\perp r + g(\nabla \phi, \nabla r) - F'(r))^2. \quad (3.7)$$

From an elementary inequality $(a \mp b)^2 \geq \frac{1}{\gamma + 1}a^2 - \frac{1}{\gamma}b^2$ for any real numbers $a, b$ and $\gamma > 0$,

$$(\tilde{\Delta}^\perp r + g(\nabla \phi, \nabla r) - F'(r))^2 \geq \frac{1}{\gamma + 1}(\tilde{\Delta}^\perp r + g(\nabla \phi, \nabla r))^2 - \frac{1}{\gamma}(F(r))^2. \quad (3.8)$$

By using the same inequality applied to $(\tilde{\Delta}^\perp r + g(\nabla \phi, \nabla r))^2$, any $\gamma, \eta > 0$

$$(\tilde{\Delta}^\perp r + g(\nabla \phi, \nabla r))^2 \geq \frac{1}{(\gamma + 1)\eta + \gamma + 1}(\Delta^\perp r)^2 - \frac{1}{\gamma}(F(r))^2 - \frac{1}{(\gamma + 1)\eta}(g(\nabla \phi, \nabla r))^2. \quad (3.9)$$

Inserting (3.8) into (3.7) with $\alpha = (2n - 2)\gamma > 0, \beta = (2n - 2)(\gamma + 1)\eta > 0$,

$$0 \geq \text{Ric}^\perp(\nabla r, \nabla r) + H(\phi)(\nabla r, \nabla r) + g(\nabla r, \nabla \Delta^\perp r) \quad (3.9)$$

$$+ \frac{1}{\alpha + \beta + 2n - 2}(\tilde{\Delta}^\perp r)^2 - \frac{1}{\alpha}(F(r))^2 - \frac{1}{\beta}(g(\nabla \phi, \nabla r))^2. \quad (3.10)$$

By the Cauchy-Schwarz inequality,

$$(g(\nabla \phi, \nabla r))^2 \leq g(\nabla \phi, \nabla \phi)g(\nabla r, \nabla r) = g(\nabla \phi, \nabla \phi),$$

thus (3.9) becomes

$$0 \geq \text{Ric}^\perp(\nabla r, \nabla r) + H(\phi)(\nabla r, \nabla r) + g(\nabla r, \nabla \Delta^\perp r) \quad (3.9)$$

$$+ \frac{1}{\alpha + \beta + 2n - 2}(\tilde{\Delta}^\perp r)^2 - \frac{1}{\alpha}(F(r))^2 - \frac{1}{\beta}(g(\nabla \phi, \nabla \phi)).$$

From the assumption on $(g(\nabla \phi, \nabla \phi))$,

$$0 \geq \text{Ric}^\perp(\nabla r, \nabla r) + H(\phi)(\nabla r, \nabla r) + g(\nabla r, \nabla \Delta^\perp r) \quad (3.9)$$

$$+ \frac{1}{\alpha + \beta + 2n - 2}(\tilde{\Delta}^\perp r)^2 - \frac{1}{\alpha}(F(r))^2 - \frac{C}{\beta^2 r^2}. \quad (3.11)$$

Now we take $\beta = \frac{4C}{\alpha}$ and $F(r) = \frac{\alpha}{r}$, then

$$0 \geq \text{Ric}^\perp(\nabla r, \nabla r) + H(\phi)(\nabla r, \nabla r) + g(\nabla r, \nabla \Delta^\perp r) + \frac{\alpha}{\alpha^2 + (2n - 2)\alpha + 4C(\tilde{\Delta}^\perp r)^2}.$$

By the assumption on $\text{Ric}^\perp + H(\phi)$,

$$0 \geq \partial_r(\tilde{\Delta}^\perp r) + \frac{\alpha}{\alpha^2 + (2n - 2)\alpha + 4C(\tilde{\Delta}^\perp r)^2} + (2n - 2)k.$$
From (3.3),

$$\lim_{r \to 0^+} r \tilde{\Delta} r = \lim_{r \to 0^+} \left( r \Delta r - r g(\nabla \phi, \nabla r) + \frac{\alpha}{2} \right)$$

$$= 2n - 2 + \frac{\alpha}{2} \leq \frac{\alpha^2 + (2n - 2)\alpha + 4C}{\alpha}.$$

Here, we used \(\lim_{r \to 0^+} r \Delta r = \lim_{r \to 0^+} r \tilde{\Delta} r = 2n - 2\). Thus by the Sturm-Liouville comparison argument,

$$\tilde{\Delta} r \leq \sqrt{(2n - 2)k \left( 2n - 2 + \alpha + \frac{4C}{\alpha} \right)} \cot \left( \frac{\sqrt{\alpha(2n - 2)k}}{\sqrt{\alpha^2 + (2n - 2)\alpha + 4C}} \right). \tag{3.11}$$

Now we use the contradictory argument which was used in [14, 24] for \(\tilde{\Delta} r\). Let \(q \in M\) and let \(\gamma\) be a minimizing unit speed geodesic segment from \(p\) to \(q\). Assume that \(d(p, q) > \pi \sqrt{(2n - 2)k \left( 2n - 2 + \alpha + \frac{4C}{\alpha} \right)} \cot \left( \frac{\sqrt{\alpha(2n - 2)k}}{\sqrt{\alpha^2 + (2n - 2)\alpha + 4C}} \right)\).

Then \(\gamma \left( \frac{\pi}{\sqrt{(2n - 2)k}} \sqrt{4\sqrt{C} + 2n - 2} \right)\) must belong to \(M\) outside of the cut-locus of \(p\). In particular, the distance function \(r\) is smooth at this point. Now, the left-hand side of (3.11) is constant, whereas the right-hand side goes to \(-\infty\), which yields the contradiction. Hence the diameter \(D\) of \(M\) must satisfy

$$D \leq \frac{\sqrt{\alpha^2 + (2n - 2)\alpha + 4C}}{\sqrt{\alpha(2n - 2)k}} \pi.$$  

By taking \(\alpha = 2\sqrt{C}\), we have

$$D \leq \frac{\pi}{\sqrt{(n - 1)k}} \sqrt{2\sqrt{C} + n - 1}.$$  

After one obtains (3.5), Proposition (8) can be proven similar to the proof of [14, Theorem 2]. Also the proof can be also generalized to the quaternionic Kähler case.

**Proposition 9.** Let \((M^n, g, I, J, K)\) be a complete quaternion Kähler manifold with the quaternionic dimension \(n \geq 2\). Take any \(p \in M\) and let \(r\) be the geodesic distance function from \(p\). Suppose that for some constant \(k > 0\), there exists a local orthonormal frame \((E_i)_{i=1}^{4n}\) around \(p\) with \(E_1 = \nabla r, E_2 = IE_1, E_3 = JE_1, E_4 = KE_1\) satisfying \(\text{Ric}^{-1}(\nabla \phi, \nabla r) + \text{Hess}(\phi)(\nabla r, \nabla r) - 2 \sum_{i=1}^{4n} \text{Hess}(r)(\nabla \nabla r, E_i) \geq (4n - 4)k\) outside of the cut-locus of \(q\) and \(|\nabla \phi|^2 \leq \frac{C}{r(2)}\) for some \(C \geq 0\). Then the diameter \(D\) of \(M\) has the upper bound

$$D \leq \frac{\pi}{\sqrt{(n - 1)k}} \sqrt{\sqrt{C} + n - 1}.$$
4. Bakry-Émery orthogonal Ricci tensor associated with the possibly non-symmetric generator of a diffusion process

In this section we consider Bakry-Émery orthogonal Ricci tensor for the orthogonal Ricci curvature while imposing an additional assumption on the holomorphic sectional curvature on Kähler manifolds (quaternionic sectional curvature in the case of quaternion Kähler manifolds). Let $M$ be a real $n$-dimensional connected smooth manifold with $n \geq 2$, either equipped with a Kähler structure or a quaternionic Kähler structure. Choose a complete Riemannian metric $g$ which is compatible with underlying Kähler or quaternionic Kähler structure and define $\mathcal{L} := \triangle + Z$, where $\triangle$ is the Laplace-Beltrami operator associated with $g$ and $Z$ a smooth vector field. We denote the Riemannian distance on $M$ associated with $g$ by $r$. Let us define $(0,2)$-symmetric tensor $(\nabla Z)^b$ by

$$\langle (\nabla Z)^b(X,Y) := \frac{1}{2}(\langle \nabla_X Z, Y \rangle + \langle \nabla_Y Z, X \rangle).$$

Given a constant $m \in \mathbb{R}$ with $m \geq n$, we define the Bakry-Émery type orthogonal Ricci tensor $\mathrm{Ric}_{m,Z}^\perp$ by

$$\mathrm{Ric}_{m,Z}^\perp := \mathrm{Ric}^\perp - (\nabla Z)^b - \frac{1}{m-n} Z \otimes Z.$$ (4.1)

In this definition, the convention is that for $m = n$ the vector field $Z \equiv 0$.

Throughout this section, we will assume that with some constant $m, k > 0$, $\mathrm{Ric}_{m,Z}^\perp \geq (2m - 2)k$. This condition means that for a smooth curve $\gamma(t)$

$$\mathrm{Ric}_{m,Z}^\perp(\dot{\gamma}(t), \dot{\gamma}(t)) = \mathrm{Ric}^\perp(\dot{\gamma}(t), \dot{\gamma}(t)) - (\nabla Z)^b(\dot{\gamma}(t), \dot{\gamma}(t)) - \frac{1}{m-n} \langle Z(\gamma(t)), \dot{\gamma}(t) \rangle$$

$$\geq (2m - 2)k.$$

**Proposition 10.** Suppose $(M^n, g, J)$ is a Kähler manifold of the complex dimension $n \geq 2$. We denote by $r$ the Riemannian distance on $M$ from $p$ associated with the metric $g$, and by $\text{Cut}_p$ the cut-locus of $p$. Suppose that for some constant $k > 0$, $\mathrm{Ric}_{m,Z}^\perp \geq (2m - 2)k$ and $H \geq 4k$. Let $x \in M \setminus \text{Cut}_p \cup \{p\}$ with $r(x) < \frac{\pi}{2\sqrt{k}}$. Then

$$\mathcal{L}r(x) \leq (m - 2) \frac{\mathfrak{s}'(k, r(x))}{\mathfrak{s}(k, r(x))} + \frac{\mathfrak{s}'(4k, r(x))}{\mathfrak{s}(4k, r(x))},$$

where $\mathfrak{s}(k, t) := \sin \sqrt{kt}$.

**Proof.** When $m = 2n$, we have $Z \equiv 0$ and we can use the same argument as we use below. Thus without loss of generality, we will assume $m \geq 2n$.

Let $p \in M$ and $x \neq p$ which is not in the cut-locus of $p$. Let $\gamma : [0, r(x)] \to M$ be the unique arclength parameterized geodesic connecting $p$ to $x$. At $x$, we consider an orthonormal frame $\{X_1(x), ..., X_{2n}(x)\}$ such that

$$X_1(x) = \gamma'(r(x)), X_2(x) = JX_1(x).$$
Then
\[ \mathcal{L}r = \triangle r(x) + Zr(x) = \sum_{i=1}^{2n} \nabla^2 r(X_i(x), X_i(x)) + Zr(x). \]

Since \( X_1(x) = \gamma'(r(x)), \) \( \nabla^2 r(X_1(x), X_1(x)) \) is zero. Now we divide the above sum into three parts: \( \nabla^2 r(X_2(x), X_2(x)), \sum_{i=3}^{2n} \nabla^2 r(X_i(x), X_i(x)), \) and \( Zr(x). \)

For \( \nabla^2 r(X_2(x), X_2(x)), \) since \( J \) is parallel and \( \gamma \) is a geodesic, the vector field defined along \( \gamma \) by \( J\gamma' \) is parallel. Define the vector field along \( \gamma \) by
\[ \tilde{X}(\gamma(t)) = \frac{s(4k, t)}{s(4k, r(x))} J\gamma'(t), \]
where \( s(k, t) := \sin \sqrt{kt}. \) From the index lemma,
\[
\begin{align*}
\nabla^2 r(X_2(x), X_2(x)) & \leq \int_0^{r(x)} \left( \langle \nabla_{\gamma'} \tilde{X}, \nabla_{\gamma'} \tilde{X} \rangle - \langle R(\gamma', \tilde{X}) \tilde{X}, \gamma' \rangle \right) dt \\
& = \frac{1}{s(4k, r(x))^2} \int_0^{r(x)} (s'(4k, t)^2 - s(4k, t)^2 \langle R(\gamma', \tilde{X}) \tilde{X}, \gamma' \rangle) dt \\
& \leq \frac{1}{s(4k, r(x))^2} \int_0^{r(x)} (s'(4k, t)^2 - 4ks(4k, t)^2) dt
\end{align*}
\]

Next, let us estimate \( \sum_{i=3}^{2n} \nabla^2 r(X_i(x), X_i(x)). \) We denote by \( \{X_3, \ldots, X_{2n}\} \) the vector fields along \( \gamma \) obtained by parallel transport of \( \{X_3(x), \ldots, X_{2n}(x)\}. \) Define the vector fields along \( \gamma \) by
\[ \tilde{X}_i(\gamma(t)) = \frac{s(k, t)}{s(k, r(x))} X_i(\gamma(t)), i = 3, \ldots, 2m. \]

By the index lemma,
\[
\begin{align*}
\sum_{i=3}^{2n} \nabla^2 r(X_i(x), X_i(x)) & \leq \sum_{i=3}^{2n} \int_0^{r(x)} \left( \langle \nabla_{\gamma'} \tilde{X}_i, \nabla_{\gamma'} \tilde{X}_i \rangle - \langle R(\gamma', \tilde{X}_i) \tilde{X}_i, \gamma' \rangle \right) dt \\
& = \frac{1}{s(k, r(x))^2} \sum_{i=3}^{2n} \int_0^{r(x)} (s'(k, t)^2 - s(k, t)^2 \langle R(\gamma', \tilde{X}_i) \tilde{X}_i, \gamma' \rangle) dt \\
& \leq \frac{1}{s(k, r(x))^2} \int_0^{r(x)} ((2n - 2)s'(k, t)^2 - s(k, t)^2 \text{Ric}^+(\gamma', \gamma')) dt.
\end{align*}
\]
For the last term \( Zr(x) \), we have

\[
Zr(x) = \langle Z(x), \dot{γ}(r(x)) \rangle \frac{s(k, r(x))^2}{s(k, r(x))^2} - \langle Z(p), \dot{γ}(0) \rangle \frac{s(k, 0)^2}{s(k, r(x))^2}
\]

\[
= \int_0^{r(x)} \frac{d}{dt} \left( \langle Z(γ(t)), \dot{γ}(t) \rangle \frac{s(k, t)^2}{s(k, r(x))^2} \right) dt
\]

\[
= \int_0^{r(x)} \left( \nabla Z \dot{γ}(t), \dot{γ}(t) \right) \frac{s(k, t)^2}{s(k, r(x))^2} + 2\langle Z(γ(t)), \dot{γ}(t) \rangle \frac{s^2(k, t)}{s(k, r(x))} \frac{s(k, t)}{s(k, r(x))} \right) dt
\]

\[
\leq \int_0^{r(x)} \left( \nabla Z \dot{γ}(t), \dot{γ}(t) \right) \frac{s(k, t)^2}{s(k, r(x))^2} + \frac{1}{m-2n} \int_0^{r(x)} \left( \langle Z(γ(t)), \dot{γ}(t) \rangle \frac{s(k, t)^2}{s(k, r(x))^2} \right) dt
\]

\[
+ (m-2n) \int_0^{r(x)} \left( \frac{s'(k, t)^2}{s(k, r(x))^2} \right) dt.
\]

Here the last inequality follows from the arithmetic-geometric mean inequality. Since \( \text{Ric}^1_{m,Z}(γ(t), \dot{γ}(t)) = \text{Ric}^1(γ(t), \dot{γ}(t)) - (\nabla Z)^b(γ(t), \dot{γ}(t)) \leq \frac{1}{m-2n} \langle Z(γ(t)), \dot{γ}(t) \rangle \geq (2m-2)k \) for all \( t \in [0, r(x)] \), we obtain

\[
\mathcal{L}r(x) \leq \int_0^{r(x)} (m-2) \left( s'(k, t)^2 \right) \frac{1}{s(k, r(x))^2} dt - \int_0^{r(x)} \text{Ric}^1_{m,Z}(γ(t), \dot{γ}(t)) \frac{s(k, t)^2}{s(k, r(x))^2} dt
\]

\[
+ \frac{1}{s(4k, r(x))^2} \int_0^{r(x)} (s'(4k, t)^2 - 4ks(4k, t)^2) dt
\]

\[
\leq (m-2) \int_0^{r(x)} \left( s'(k, t)^2 \frac{1}{s(k, r(x))^2} - k \frac{s(k, t)^2}{s(k, r(x))^2} \right) du
\]

\[
+ \frac{1}{s(4k, r(x))^2} \int_0^{r(x)} (s'(4k, t)^2 - 4ks(4k, t)^2) dt
\]

\[
= (m-2) \left( s'(k, r(x)) \left. s'(k, r(x)) \right|_{t=0} \right) \frac{s'(k, r(x))}{s(k, r(x))} + \left( \frac{s'(k, r(x))}{s(k, r(x))} \right) \frac{s'(4k, r(x))}{s(4k, r(x))} \right|_{t=0}
\]

\[
= (m-2) \frac{s'(k, r(x))}{s(k, r(x))} + \frac{s'(4k, r(x))}{s(4k, r(x))}.
\]

\[
\square
\]

For the proposition below, let \( \{X_t^x\}_{t \geq 0}, x \in M \) be the diffusion process with the infinitesimal generator \( \mathcal{L} \). Let us define a stopping time \( \sigma_p \) by

\[
\sigma_p := \inf \left\{ t \geq 0 : d_p(X_t) = \frac{\pi}{2\sqrt{k}} \right\}
\]

in the Kähler case

\[
\sigma_p := \inf \left\{ t \geq 0 : d_p(X_t) = \frac{\pi}{2\sqrt{3k}} \right\}
\]

in the quaternionic case.
Proposition 11. Given a Kähler manifold \((M^n, g, J)\) of the complex dimension \(n \geq 2\), suppose that for some constant \(k > 0\), \(\text{Ric}^\perp_{\text{in}, \mathbb{Z}} \geq (2m-2)k\) and \(H \geq 4k\) hold on the open ball of radius \(\frac{\pi}{2\sqrt{k}}\) centered at \(p\). Then \(\sigma_p = \infty\) holds \(\mathbb{P}_q\)-almost surely for any \(q \in M \setminus (\text{Cut}_p \cup \{p\})\) with \(d_p(q) < \frac{\pi}{2\sqrt{k}}\).

Proof. By Itô’s formula for the radial process \(d_p(X_t)\) together with Proposition 10, we have

\[
d_p(X_t) \leq d_p(q) + \sqrt{2}\beta_t + \int_0^t \mathcal{L}d_p(X_s)ds
\]

\[
\leq d_p(q) + \sqrt{2}\beta_t + \int_0^t (m - 2) \frac{\rho'(k, r(X_s))}{\rho(k, r(X_s))} + \frac{\rho'(4k, r(X_s))}{\rho(4k, r(X_s))}ds
\]

for \(t < \sigma^p\), where \(\beta_t\) is a 1-dimensional standard Brownian motion. Let us define \(\rho_t\) as the solution to the following stochastic differential equation

\[
d\rho_t = \sqrt{2}d\beta_t + \left((m - 2) \frac{\rho'(k, r(\rho))}{\rho(k, r(\rho))} + \frac{\rho'(4k, r(\rho))}{\rho(4k, r(\rho))}\right)dt
\]

with \(\rho_0 = d_p(q)\). (see for example [10, Theorem 3.5.3]). Thus it suffices to show that \(\rho_t\) never hit \(\frac{\pi}{2\sqrt{k}}\). Since

\[
\frac{\rho'(4k, r(\rho))}{\rho(4k, r(\rho))} = \frac{1}{u - \frac{\pi}{2\sqrt{k}}} + o(1)
\]

as \(u \uparrow \frac{\pi}{2\sqrt{k}}\) and \(m \geq 2n \geq 2\), a general theory of 1-dimensional diffusion processes yields that \(\rho_t\) cannot hit \(\frac{\pi}{2\sqrt{k}}\) (see e.g. [10, Proposition 4.2.2]).

By using Proposition above, we can easily show the Bonnet-Myers theorem.

Corollary 12. Given a Kähler manifold \((M^n, g, J)\) of the complex dimension \(n \geq 2\), suppose that for some constant \(k > 0\), \(\text{Ric}^\perp_{\text{in}, \mathbb{Z}} \geq (2m-2)k\) and \(H \geq 4k\) hold on \(M\). Then the diameter of \(M\) is less than equal to \(\frac{\pi}{2\sqrt{k}}\).

Proof. Suppose that there are \(p, q \in M\) such that \(d(p, q) > \frac{\pi}{2\sqrt{k}}\). We may assume that \(M\) is compact and that \(\text{Ric}^\perp_{\text{in}, \mathbb{Z}} \geq (2m-2)k\), holds on the open ball of radius \(\frac{\pi}{2\sqrt{k}}\) centered at \(p\) by modifying outside of a ball of large radius. Then there is an open neighborhood \(G\) of \(q\) such that \(d(p, y) > \frac{\pi}{2\sqrt{k}}\) for all \(y \in G\). Take \(p'\) from a small neighborhood of \(p\). Then Proposition yields that \(\mathbb{P}_{p'}[\sigma_p = \infty] = 1\). It implies \(\mathbb{P}_{p'}[X_t \in G] = 0\) for any \(t > 0\). This is absurd since the law of \(X_t\) has a strictly positive density with respect to the Riemannian volume measure for \(t > 0\). 

The similar propositions of the quaternion Kähler case can be obtained by repeating the proofs of the Kähler case. We follow the structure of the proof in [5, Theorem 3.2].

Proposition 13. Given a quaternionic Kähler manifold \((M^n, g, I, J, K)\) of the quaternionic dimension \(n \geq 2\) and we denote the Riemannian distance on \(M\) from \(p\) associated with \(g\) by \(r\) and \(\text{Cut}_p\) the cut-locus
of $p$. Suppose that for some constant $k > 0$, $\text{Ric}_{m, Z} \geq (4m - 4)k$ and $Q \geq 12k$. Let $x \in M \setminus \text{Cut}_p \cup \{p\}$ with $r(x) < \frac{\pi}{2\sqrt{3}k}$. Then

$$\mathcal{L}r(x) \leq (m - 4) \frac{\mathcal{s}'(k, r(x))}{\mathcal{s}(k, r(x))} + \frac{\mathcal{s}'(12k, r(x))}{\mathcal{s}(12k, r(x))},$$

where $\mathcal{s}(k, t) \equiv \sin \sqrt{kt}$.

**Proof.** When $m = 4n$, we just need to put $Z = 0$ and proceed the same argument that we provide below. Thus without loss of generality, we will assume $m > 4n$.

Let $p \in M$ and $x \neq p$ which is not in the cut-locus of $p$. Let $\gamma : [0, r(x)] \to M$ be the unique arclength parameterized geodesic connecting $p$ to $x$. At $x$, we consider an orthonormal frame $\{X_1(x), \ldots, X_{4n}(x)\}$ such that

$$X_1(x) = \gamma'(r(x)), X_2(x) = IX_1(x), X_3(x) = JX_1(x), X_2(x) = KX_1(x).$$

Then

$$\mathcal{L}r = \triangle r(x) + Zr(x) = \sum_{i=1}^{4n} \nabla^2 r(X_i(x), X_i(x)) + Zr(x).$$

Since $X_1(x) = \gamma'(r(x))$, $\nabla^2 r(X_1(x), X_1(x))$ is zero. Now we divide the above sum into three parts: $\sum_{i=2}^{4} \nabla^2 r(X_i(x), X_i(x))$, $\sum_{i=3}^{4n} \nabla^2 r(X_i(x), X_i(x))$, and $Zr(x)$.

For $\sum_{i=2}^{4} \nabla^2 r(X_i(x), X_i(x))$, note that vectors $IX_1, JX_1, KX_1$ might not be parallel along $\gamma$. Denote $X_2, X_3, X_4$ the vector fields along $\gamma$ obtained by parallel transport along $\gamma$ of $X_2(x), X_3(x)$ and $X_4(x)$. Then one can deduce that

$$R(\gamma', X_2, X_2, \gamma') + R(\gamma', X_3, X_3, \gamma') + R(\gamma', X_4, X_4, \gamma')$$

$$= R(\gamma', I\gamma', I\gamma', \gamma') + R(\gamma', J\gamma', J\gamma', \gamma') + R(\gamma', K\gamma', K\gamma', \gamma') = Q(\gamma')$$

(see [5, Theorem 3.2] for more details). Define the vector field along $\gamma$ by

$$\tilde{X}_i(\gamma(t)) = \frac{\mathcal{s}(12k, t)}{\mathcal{s}(12k, r(x))} J\gamma'(t), i = 2, 3, 4,$$

where $\mathcal{s}(k, t) \equiv \sin \sqrt{kt}$, we obtain by the same computation as in the Proposition 10,

$$\sum_{i=2}^{4} \nabla^2 r(X_i(x), X_i(x)) \leq \frac{1}{\mathcal{s}(12k, r(x))^2} \int_0^{r(x)} (\mathcal{s}'(12k, t)^2 - 12k\mathcal{s}(4k, t)^2) dt$$

The rest steps are similar as in the Proposition 10. Consequently,

$$\mathcal{L}r(x) = (m - 4) \frac{\mathcal{s}'(k, r(x))}{\mathcal{s}(k, r(x))} + \frac{\mathcal{s}'(12k, r(x))}{\mathcal{s}(12k, r(x))},$$

Combining the proposition above with Proposition 13, we obtained:
Proposition 14. Given a quaternionic Kähler manifold \((M^n, g, I, J, K)\) of the quaternionic dimension \(n \geq 2\), suppose that for some constant \(k > 0\), \(\text{Ric}^\perp_{m,Z} \geq (4m - 4)k\) and \(Q \geq 12k\) hold on the open ball of radius \(\frac{\pi}{2\sqrt{k}}\) centered at \(p\). Then \(\sigma_p = \infty\) holds \(\mathbb{P}_q\)-almost surely for any \(q \in M \setminus (\text{Cut}_p \cup \{p\})\) with \(d_p(q) < \frac{\pi}{2\sqrt{3k}}\).

Corollary 15. Given a quaternionic Kähler manifold \((M^n, g, I, J, K)\) of the quaternionic dimension \(n \geq 2\), suppose that for some constant \(k > 0\), \(\text{Ric}^\perp_{m,Z} \geq (4m - 4)k\) and \(Q \geq 12k\) hold on \(M\). Then the diameter of \(M\) is less than equal to \(\frac{\pi}{2\sqrt{3k}}\).

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