Transport Equations and Spin-Charge Propagating Mode in the Two Dimensional Hole Gas

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We find that the spin-charge motion in a strongly confined two-dimensional hole gas (2DHG) supports a propagating mode of cubic dispersion apart from the diffusive mode due to momentum scattering. Propagating modes seem to be a generic property of systems with spin-orbit coupling. Through a rigorous Keldysh approach, we obtain the transport equations for the 2DHG, we analyze the behavior of the hole spin relaxation time, the diffusion coefficients, and the spin-charge coupled motion.

The correct description of the motion of charge and spin packets in materials such as semiconductors requires appropriate transport equations. While transport equations for the Fermi liquid usually describe separate diffusion processes for spin and charge densities, spin-orbit (SO) coupling alters their form dramatically. For example, the presence of coupling between spin and charge, a direct result of SO coupling, leads to the possibility of controlling the spin-orientation by electric fields. Another example is the recently proposed spin-charge mode propagating with a velocity equal to the Rashba SO coupling. This mode has no counterpart in usual transport and is intimately related to a non-equilibrium version of the spin-current in the ground-state of the system.

In the case of the two-dimensional electron gas (2DEG) with Rashba spin-orbit interaction, diffusive transport equations are known. A similar analysis for the two-dimensional hole gas (2DHG), where the spin-orbit coupling is much stronger, has so far been lacking. In this letter we obtain the transport equations for very thin p-doped quantum wells in which only the heavy-hole band is populated. We start with a general Keldysh equation for spin-1/2 systems with generic SO coupling, and then particularize to the case of the heavy hole 2DHG. The same approach can be applied other types of SO couplings, such as the 3D Dresselhauss with little or no modification. The equations obtained here form the basis for studies of transport and spin-orbit effects in 2DHG systems. We find that, unlike in the case of 2DEG, the spin orbit effects do not produce first order derivative terms in the transport equations, but appear only in second and even third order derivative terms, which now have to be taken into consideration. As such, the 2DHG does not exhibit the uniform spin polarization under an applied electric field, or the spin-galvanic effect that its 2DEG counterpart does. The diffusion coefficient becomes a tensor with components for spin and charge which depend on the spin-orbit coupling. Its spin component turns out to usually be smaller than the charge component, which might provide an alternate explanation for the recent experimental observations attributed to the spin-coulomb drag. The Dyakonov-Perel relaxation time for the heavy holes’ spin is calculated as well. We discuss the relation of the previously calculated spin Hall conductivity in the ballistic regime and this formalism. In addition, the transport equations support a spin-charge propagating mode, analogous to the one predicted in Ref. 3. However, its dispersion relation is changed because of the different form of the SO interaction, and is cubic rather than linear in momentum. The effects of Coulomb repulsion on the spectrum of the propagating mode are also accounted for in the present treatment.

The most general single electron Hamiltonian with spin-orbit coupling for spin 1/2 systems is:

\[ H = \frac{k^2}{2m} + \lambda_i(k)\sigma_i, \quad i = 1, 2, 3 \]

with \( \lambda_i(k) = -\lambda_i(-k) \) due to time reversal invariance. The energy eigenvalues are \( E_{\pm}(k) = k^2/2m \pm |\lambda(k)| \) where \( |\lambda| = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2} \). The retarded and advanced Green’s functions of the clean 2DHG have the form:

\[ G^{R,A}(k, \epsilon) = \frac{1}{2} \sum_{s=\pm 1} \frac{1 - s\lambda_i\sigma_i / |\lambda|}{\epsilon + s\lambda \pm i/(2\pi)} \]

where \( \epsilon = \epsilon + k^2/2m - k^2/2m \) and \( \epsilon \) is measured from the Fermi energy \( \epsilon_F = k_F^2/2m \). The density matrix \( g = g(k, r, t) \) is a \( 2 \times 2 \) matrix in spin indices, whose integral over the momentum space yields the reduced density matrix \( \rho = \rho(r, t) \):

\[ \rho(r, t) = \int \frac{d^dk}{(2\pi)^{d+1}} g(k, r, t) \equiv n(r) + S_i(r)\sigma_i \]

where \( n(r, t), S_i(r, t) \) are the charge and spin densities, \( d \) is the space-dimension, and \( \nu \) is the \( d \)-dimensional density of states. The charge-spin density \( g \) satisfies a Keldysh-type equation which takes electron scattering into account through an isotropic momentum relaxation time \( \tau \):

\[ \frac{\partial g}{\partial t} + \frac{2}{\tau} + \frac{\partial}{\partial r} [H, g] = \frac{i}{\tau} (G^R \rho - \rho G^A) - \frac{1}{2} \left\{ \frac{\partial H}{\partial k_i}, \frac{\partial g}{\partial r_i} \right\} \]
This quantum Boltzmann equation can be solved iteratively in the case of small spatial gradients $\nabla g$ of the charge-spin density $g$. Fourier transforming with respect to time and defining a linear functional $K(g) = \frac{1}{\tau}(G^\mu \rho - \rho G^\lambda) - \frac{1}{2} \left\{ \frac{\partial g}{\partial \tau}, \frac{\partial g}{\partial \tau} \right\}$, one obtains an equation for $g(k, r, \omega)$:

$$g = i\left(2\lambda^2 - \Omega^2\right) K(g) + 2\lambda \lambda_j \sigma_i K(g) \sigma_j - \Omega \lambda_i \sigma_i K(g) \right) \Omega(4\lambda^2 - \Omega^2)$$

with $\Omega \equiv \omega + i/\tau$. Expanding in gradients $g = g^0 + \delta g^{(1)} + \delta g^{(2)} + \ldots$, a perturbative chain of equations is obtained up to the desired accuracy. Integrating over the momentum and taking the DC limit $\omega \tau \ll 1$ one obtains the transport equations for $\rho$. In the case of the Rashba coupling $\lambda = \alpha \epsilon_{ij} k_j$ this was done in Ref. [6].

For 2D systems with an isotropic Fermi surface, which includes the Rashba model, the 2DHG model, and the 2D Dresselhaus model, in the limit of $\epsilon_F \tau \ll 1$, $\epsilon_\alpha \equiv \lambda(k_F) \ll \epsilon_F$ the continuity equation takes the form:

$$\frac{\partial \rho}{\partial t} = -i \int \frac{d^2 k}{(2\pi)^2 m} \Omega(\delta g^{(1)}) + \ldots$$

where $\delta g^{(1)}$ contains first order spatial derivatives, and subsequent terms contain higher order derivatives. Here $m = 1, 2; m \neq 3$. The first term on the right hand side has zero components in the charge-sector and represents spin-relaxation with characteristic time:

$$\frac{1}{\sigma_m} = \frac{1}{\tau_s} \left[ \frac{2\zeta^2}{4\zeta^2 + 1} \right]$$

where $\zeta \equiv \epsilon_\alpha \tau$. Equation (6) and the relaxation time formula (7) are valid for any spin-orbit coupling in a 2D system with an isotropic Fermi surface. The general expressions for higher order derivative terms tend to be exceedingly long.

We now turn to a particular case: the two-dimensional hole gas. The valence band of type III-V bulk semiconductors is a spin 3/2 band, separated into heavy and light-hole bands which touch at the $\Gamma$ point. In a quantum well a gap opens at the $\Gamma$ point so that the light hole band is moved below the heavy hole band. A sufficiently thin well ensures a gap large enough to place the light-hole band below the Fermi level. Then only the heavy-hole Hamiltonian should be considered. Asymmetry of the quantum well introduces a spin-orbit term of the form $\lambda \sigma_i$ with:

$$\lambda_x = \alpha k_y (3k_x^2 - k_y^2), \quad \lambda_y = \alpha k_x (3k_y^2 - k_x^2)$$

and $\lambda_z = 0$. This model has an isotropic Fermi surface with $|\lambda| = \alpha k^3$. The limit of validity of our calculations is $\hbar/\tau$, $\alpha k_F^3 \ll k_F^2/2m_{hh}$ where $m_{hh}$ is the heavy-hole mass. For a compact notation, we define the charge-spin four-vector $N^\mu = (n, S)$, $\mu = 0, 1, 2, 3$. The generic continuity equation expanded up to the third order in spatial derivatives has the form:

$$\frac{\partial N^\mu}{\partial t} + \frac{N^\mu}{\tau_s} (1 - \delta_{\mu,0}) = C^\mu_0 \partial_i N^\nu + D^\mu_0 \partial_i \partial_j N^\nu + H^\mu_0 \partial_i \partial_j \partial_k N^\nu$$

where $i, j, k = \{x, y\}$. We first concentrate on the coefficients $C^\mu_0$ and $D^\mu_0$. In the 2DHG case we find $C^\mu_0 = 0$ due to the symmetries of the Hamiltonian. This is in contrast with the linear Rashba model of the 2DEG, where nonzero linear terms were calculated in Ref. [3]. The diffusion coefficient tensor $D^\mu_0$ has only three distinct elements due to the $x \approx y$ symmetry of the 2DHG Hamiltonian. As we have already mentioned the general forms for the transport coefficients are very cumbersome so we assume small SO coupling ($\epsilon_\alpha \ll \epsilon_F$) and expand up to linear order in $\frac{\epsilon_\alpha}{\epsilon_F}$ in all coefficients ($D$'s and $H$'s) except for the charge diffusion coefficient whose first correction due to SO coupling is at order ($\frac{\epsilon_\alpha}{\epsilon_F}$)². We find:

$$D_{xx}^{00} = D_0 \left[ 1 + \frac{9}{8} \left( \frac{\epsilon_\alpha}{\epsilon_F} \right)^2 \left( 1 + \frac{2}{1 + 4\zeta^2} \right) \right]$$

$$D_{xy}^{11} = D_0 \left( 1 + 24 \frac{\epsilon_\alpha^4 + 32 \epsilon_\alpha^6}{(1 + 4\zeta^2)^3} \right)$$

$$D_{xx}^{33} = D_0 \left( 1 - 12 \frac{\epsilon_\alpha^2}{(1 + 4\zeta^2)^3} \right)$$

$$D_{xy}^{03} = D_0 \left( 3 \frac{\epsilon_\alpha^2 (4\zeta^2 - 1)}{(4\zeta^2 + 1)^2} \right)$$

where $D_0 = \frac{\epsilon_F^2}{2\epsilon_\alpha}$, and with symmetries $D^\mu_{xx} \to D^\mu_{yy}$, $D^\mu_{yy} \to D^\mu_{xx}$. Another notable symmetry is $D^\mu_{xy} = -D^\mu_{yx}$ so the off-diagonal terms in the sum $D^\mu_{ij} \partial_i \partial_j N^\nu$ cancel. The expressions above are valid in the limit when the spin-orbit band splitting and momentum relaxation energy are assumed to be smaller than the Fermi energy $i.e.$ $\epsilon_\alpha, \frac{1}{\tau_s} << \epsilon_F$. However, the value of $\zeta$ is unrestricted and we find for some common physical parameters that $0 < \zeta < 1$. The transport equation above with these diffusion coefficients represents one of the main results of this paper.

Here we briefly show how an interesting new phenomenon, the spin Hall effect, emerges from the above equations. Under an applied electric field along the $\hat{x}$ axis we expect out-of-plane spins to be transported in the $\hat{y}$ direction[11, 12]. We take into account the effects of an electric field per the substitution $\vec{V} n \to \hat{\nabla} n - e \vec{E}$. From the transport equations above we see that $E^x$ generates:

$$J_y = (D^0_{xy} e \nu) E^x$$

which is in the preferred form[11, 12] and where the symmetry $D^\mu_{xy} = -D^\mu_{yx}$ gives us the antisymmetric tensor structure $J_i^x = \sigma_{\mu \nu} \epsilon_{ijx} E^x$. In the low impurity limit i.e.
as $\tau \to \infty$ we find a value of

$$D^{03}_{xy} = \frac{3}{4m}. \quad (12)$$

This gives us: $J_y = \frac{3aE^x}{8\pi}$. When we take into account, like \(\ref{3}\) that we are using spin-1/2 matrices to represent a spin-3/2 system, we account for an additional factor of 3 which yields:

$$\sigma_{SH} = \frac{9e}{8\pi} \quad (13)$$

in agreement with \(\ref{8}\)\(\text{ and }\ref{9}\). This value for the conductivity matches the clean-limit and supports the previously known fact that vertex corrections vanish for this model. \(\text{ Ref.}\ref{8}\)

Nontrivial dynamics, in the absence of applied fields, appears in the third order derivative terms of the transport equations. The sum \(H^{\mu\nu}_{ijk}\partial_j\partial_k N^\nu\) only depends on coefficients \(H^{\mu\nu}_{zz}, H^{\mu\nu}_{yy}\) and symmetrized combinations: \(H^{\mu\nu} = H^{\nu\mu}_{xx} + H^{\nu\mu}_{zy} + H^{\nu\mu}_{yz},\) and \(H^{\mu\nu} = H^{\nu\mu}_{yy} + H^{\nu\mu}_{yy} + H^{\nu\mu}_{yy} + H^{\nu\mu}_{yy}\). The number of distinct coefficients is further reduced to just four by the exact symmetries of the Hamiltonian:

\[
\begin{align*}
H^{02}_{zzz} &= H^{01}_{yy} = -\frac{1}{3}H^{03}_{xx} = -\frac{1}{3}H^{01}_{x}, \\
H^{20}_{zzz} &= H^{00}_{yy} = -\frac{1}{3}H^{10}_{xx} = -\frac{1}{3}H^{10}_{x}, \\
H^{13}_{zzz} &= -H^{23}_{yy} = -\frac{1}{3}H^{13}_{yy} = \frac{1}{3}H^{23}_{x}, \\
H^{31}_{zzz} &= -H^{32}_{yy} = -\frac{1}{3}H^{31}_{yy} = \frac{1}{3}H^{32}_{x}.
\end{align*}
\]

For the third-order coefficients we find:

\[
\begin{align*}
H^{02}_{zzz} &= H_0 \left( \frac{3(3 + 16\gamma^2 + 72\gamma^4 + 24\gamma^6)}{2(1 + 4\gamma^2)^3} \right), \\
H^{13}_{zzz} &= H_0 \left( \frac{4\gamma \tau \xi C (4\xi^2 - 1)}{(1 + 4\gamma^2)^4} \right).
\end{align*}
\]

where $H_0 = \frac{\mu m}{2\pi}$ and up to this order we have the additional symmetries: $H^{02}_{zzz} = H^{20}_{zzz}$ and $H^{13}_{zzz} = -H^{31}_{zzz}$.

It was found in Ref.\(\text{[8]}\) that in 2DEG systems the SO interaction can produce a mode of coupled spin and charge oscillations (fractionalized spin packets mode, or spin-galvanic mode) which can be weakly damped even in the diffusive regime. It was later found that the Coulomb interactions of the charge component do not suppress this mode. Here we study the analogue of such mode in the 2DHG system using the ansatz $\delta n(x,t) = \delta n \exp[iq x - i\omega t]$, $S_y(x,t) = \mu \exp[iq x - i\omega t]$, $S_x, S_y = 0$ to find a solution of Eqs.\(\text{[9]}\) in the absence of external fields. Following the procedure of Ref.\(\text{[10]}\) to account for Coulomb interactions, we obtain the spectrum of the mode

$$\omega_{\pm}(q) = -i(\phi_1 + \phi_2) \pm \frac{\sqrt{4\gamma^2\phi_3 - (\phi_1 - \phi_2)^2}}{2\tau_s} \quad (16)$$

with

$$\phi_1 = \frac{D^0_{00} D^0_{xx}}{D^0_{xx}} \kappa^2 R, \quad \phi_2 = \frac{D^0_{00} D^0_{yy}}{D^0_{yy}} \kappa^2 + 1, \quad \phi_3 = \kappa^6 R,\quad (17)$$

where $\gamma = q l_s$, $l_s = \sqrt{D^0_{yy}} \tau_s$ is the spin-diffusion length, $R = 1 + 2\pi \kappa_B/\kappa$ and $\kappa_B = \nu e^2l_s$ characterize the strength of the Coulomb interaction, and $\gamma = H^{02}_{zzz} \tau_s/l_s$. As in the 2DEG case, the shape of the spectrum is controlled by

\[
\begin{align*}
\phi_1 &= \frac{D^0_{00} D^0_{xx}}{D^0_{xx}} \kappa^2 R, \\
\phi_2 &= \frac{D^0_{00} D^0_{yy}}{D^0_{yy}} \kappa^2 + 1, \\
\phi_3 &= \kappa^6 R,\quad (17)
\end{align*}
\]
two dimensionless parameters $\gamma$ and $\kappa_B$. The estimate for $\gamma \sim \epsilon_0 \tau_p/k_F l_s$ can be compared with the 2DEG estimate $\gamma_{2DEG} \sim \epsilon_0^{2DEG} \tau_s/k_F l_s$. One observes that the larger SO splitting $\epsilon_0$ associated with 2DHG systems can be compensated by the small factor $\tau_p/\tau_s$. The Coulomb interaction parameter is large, $\kappa_B \sim 10^3$ for $l_s = 100 \text{nm}$.

A sketch of the spectrum is shown in Fig. 3. As in the case of 2DEG, the modes are purely dissipative at $\kappa = 0$, representing the charge diffusion and spin relaxation. Spin and charge motions get coupled at $\kappa > 0$ and two intervals with $\Re(\omega) \neq 0$ emerge. As in the case of 2DEG, the position of these intervals is very sensitive to the strength of the Coulomb interaction $\kappa_B$. However, in addition to that, the $\Re(\omega)/\Im(\omega)$ ratio is also affected by the Coulomb interaction in the 2DHG case. Our calculation shows that the mode is always highly damped in the first interval. The mode can be weakly damped with $\Re(\omega)/\Im(\omega) \gg 1$ in the second window $\kappa > \kappa_s$, however $\kappa_s \sim (\kappa_B/\epsilon_0^2)^{1/3}$ is usually too close to the boundary of the validity of the diffusive approximation which requires $\kappa \ll \sqrt{\tau_s/\tau_p}$. We conclude that observing the fractionalized spin packets propagating mode in the 2DHG requires more fine tuning of parameters than in the 2DEG case. However, the presence of the weakly damped solution at the boundary between the diffusive and ballistic regimes suggests that a search for a ballistic counterpart of the mode has to be performed.

In conclusion, in the present letter we derived the diffusive transport equations for the 2DHG system. Our calculation shows that the components of the diffusion coefficient tensor depend on the magnitude of the SO coupling and that the spin diffusion is generally smaller than the charge diffusion, thus offering an alternate explanation to spin-coulomb drag for this already experimentally observed phenomenon. We find that, in contrast to the 2DEG case, the spin-charge coupling enters into the transport equations starting with the second order derivative terms. Our diffusive equations reproduce the value of the spin Hall coefficient, previously calculated for this system in the ballistic regime, thus supporting the conclusion about the absence of vertex corrections in this 2DHG system. The fractionalized-spin packets propagating mode in the 2DHG is also studied. It is found that the Coulomb interaction suppresses this mode more than in the case of electron gas. The spectrum of the mode suggests that its counterpart may exist for 2DHG systems in the ballistic regime.

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