On the preconditioned AOR iterative method for Z-matrices

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Abstract

Several preconditioned AOR methods have been proposed to solve system of linear equations \( Ax = b \), where \( A \in \mathbb{R}^{n \times n} \) is a unit Z-matrix. The aim of this paper is to give a comparison result for a class of preconditioners \( P \), where \( P \in \mathbb{R}^{n \times n} \) is nonsingular, nonnegative and has unit diagonal entries. Numerical results for corresponding preconditioned GMRES methods are given to illustrate the theoretical results.

Key words: System of linear equations, Preconditioner, AOR iterative method, Z-matrix, Comparison.

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1. Introduction

Consider the system of linear equations

\[ Ax = b, \tag{1} \]

where \( A = (a_{ij}) \in \mathbb{R}^{n \times n} \) and \( b \in \mathbb{R}^n \). A general stationary iterative method for solving Eq. (1) may be expressed as

\[ x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \ldots, \]

in which \( A = M - N \), where \( M, N \in \mathbb{R}^{n \times n} \) and \( M \) is nonsingular. It is well-known that this iterative method is convergent if and only if \( \rho(M^{-1}N) < 1 \), where \( \rho(X) \) refers to the spectral radius of matrix \( X \). Let \( a_{ii} \neq 0, \quad i = 1, 2, \ldots, n \). Therefore,
without loss of generality we can assume that $a_{ii} = 1$, $i = 1, 2, \ldots, n$. In this case, we split $A$ into

$$A = I - L - U,$$

(2)

where $I$ is the identity matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular matrices, respectively. The accelerated overrelaxation (AOR) iterative method to solve Eq. (1) is defined by \[4, 12\]

$$x^{(k+1)} = L_{\gamma,\omega}x^{(k)} + \omega(I - \gamma L)^{-1}b,$$

in which

$$L_{\gamma,\omega} = (I - \gamma L)^{-1}[(1 - \omega)I + (\omega - \gamma)L + \omega U],$$

where $\omega$ and $\gamma$ are real parameters and with $\omega \neq 0$. For certain values of the parameters $\omega$ and $\gamma$ the AOR iterative method results in the Jacobi, Gauss-Seidel and the SOR methods \[4\].

To improve the convergence rate of an iterative method one may apply it to the preconditioned linear system $PAx = Pb$, where the matrix $P$ is called a preconditioner. Several preconditioners have been presented for the stationary iterative methods by many authors. Recently, Wang and Song in \[13\] have proposed a general preconditioner $P$ which is nonsingular, nonnegative and has unit diagonal entries. They also have investigated the properties of the preconditioners of the form

$$P = (p_{ij}) = (-\alpha_{ij}a_{ij}),$$

(3)

where $0 \leq \alpha_{ij} \leq 1$, for $i \neq j$, and $p_{ii} = 1$ for $i = 1, \ldots, n$. Many preconditioners proposed in the literature are the special cases of such general preconditioner (see for example \[3, 5, 6, 7, 8, 9, 13, 15\]). In this paper, we show that under some conditions, preconditioner \[3\] with $\alpha_{ij} = 1$ for $1 \leq i \neq j \leq n$ is the best one among the preconditioners of the form \[3\].

For convenience, we first present some notations, definitions and preliminaries which will be used in this paper. A matrix $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ is said to be nonnegative and denoted by $A \geq 0$ if $a_{ij} \geq 0$ for all $i$ and $j$ and $A$ is said to positive and denoted by $A \gg 0$ if $a_{ij} > 0$ for all $i$ and $j$. If $A \geq 0$, then by the Perron-Frobenius theory (see for example \[2\]), $\rho(A)$ is an eigenvalue of $A$, and corresponding to $\rho(A)$, $A$ has a nonnegative eigenvector, which we refer to as a Perron vector of $A$.

**Definition 1.1.** A matrix $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is an $Z$-matrix if $a_{ij} \leq 0$ for $i \neq j$.

**Definition 1.2.** A $Z$-matrix $A$ is said to be an $M$-matrix if $A$ is nonsingular and $A^{-1} \geq 0$. 

2
Definition 1.3. Let $A \in \mathbb{R}^{n \times n}$. The representation $A = M - N$ is called a splitting of $A$ if $M$ is nonsingular. The splitting $A = M - N$ is called
(a) convergent if $\rho(M^{-1}N) < 1$;
(b) weak regular if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$;
(c) an M-splitting of $A$ if $M$ is an M-matrix and $N \geq 0$.

Definition 1.4. A real matrix $A$ is called monotone if $Ax \geq 0$ implies $x \geq 0$.

Lemma 1.1. [9, Lemma 3.2] Let $A = M - N$ be an M-splitting of $A$. Then $\rho(M^{-1}N) < 1$ if and only if $A$ is an M-matrix.

Lemma 1.2. [13, Lemma 2.2] Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be an M-matrix. Then, there exists $\epsilon_0 > 0$ such that, for any $0 < \epsilon \leq \epsilon_0$, $A(\epsilon) = (a_{ij}(\epsilon))$ is also an M-matrix, where
$$a_{ij}(\epsilon) = \begin{cases} a_{ij}, & \text{if } a_{ij} \neq 0, \\ -\epsilon, & \text{if } a_{ij} = 0. \end{cases}$$

Lemma 1.3. [1, Lemma 6.1] $A$ is monotone if and only if $A$ is nonsingular with $A^{-1} \geq 0$.

Lemma 1.4. [14, Lemma 1.6] Let $A$ be a Z-matrix. Then, $A$ is an M-matrix if and only if there is a positive vector $x$ such that $Ax \gg 0$.

Lemma 1.5. [10, Lemma 2.2] Suppose that $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ are weak regular splittings of the monotone matrices $A_1$ and $A_2$, respectively, such that $M_2^{-1} \geq M_1^{-1}$. If there exists a positive vector $x$ such that $0 \leq A_1 x \leq A_2 x$, then for the monotonic norm associated with $x$,
$$\|M_2^{-1}N_2\|_x \leq \|M_1^{-1}N_1\|_x.$$ 
In particular, if $M_1^{-1}N_1$ has a positive Perron vector, then
$$\rho(M_2^{-1}N_2) \leq \rho(M_1^{-1}N_2).$$

2. Main results

Let $A \in \mathbb{R}^{n \times n}$. We consider the preconditioner $\tilde{P} = (\tilde{p}_{ij})$ for Eq. (1), where
$$\tilde{p}_{ij} = \begin{cases} -\alpha_{ij}a_{ij}, & \text{if } i \neq j, \\ 1, & \text{otherwise,} \end{cases}$$
and $\alpha_{ij} \in \mathbb{R}$ for $i \neq j$. We split $\tilde{P}$ into $\tilde{P} = I + L(\alpha) + U(\alpha)$, where $I$ is the identity matrix and $L(\alpha)$ and $U(\alpha)$ are strictly lower and strictly upper triangular matrices, respectively. Let $\tilde{A} = \tilde{P}A = (I + L(\alpha) + U(\alpha))A$ and

$$L(\alpha)U = G_1(\alpha) + E_1(\alpha) + F_1(\alpha),$$

$$U(\alpha)L = G_2(\alpha) + E_2(\alpha) + F_2(\alpha),$$

where $G_1(\alpha)$ and $G_2(\alpha)$ are diagonal matrices, $F_1(\alpha)$ and $F_2(\alpha)$ are strictly lower triangular matrices and $E_1(\alpha)$ and $E_2(\alpha)$ are strictly upper triangular matrices. In this case, $\tilde{A}$ can be split as $\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}$, where $\tilde{D}$, $\tilde{L}$ and $\tilde{U}$, respectively, are diagonal, strictly lower and strictly upper triangular matrices defined as

$$\tilde{D} = I - E_1(\alpha) - E_2(\alpha),$$

$$\tilde{L} = L - L(\alpha) + L(\alpha)L + F_1(\alpha) + F_2(\alpha),$$

$$\tilde{U} = U + G_1(\alpha) - U(\alpha) + G_2(\alpha) + U(\alpha)U.$$ 

If the matrix $\tilde{D} - r\tilde{L}$ is nonsingular, the AOR iteration matrix to solve the preconditioned system $\tilde{P}Ax = \tilde{P}b$ can be written as

$$\tilde{L}_{\gamma,\omega} = (\tilde{D} - \gamma\tilde{L})^{-1}[(1 - \omega)\tilde{D} + (\omega - \gamma)\tilde{L} + \omega\tilde{U}].$$

**Theorem 2.1.** Let $A$ be a Z-matrix and $\alpha_{ij} \in [0, 1]$ for $1 \leq i \neq j \leq n$. Then, $A$ is an M-matrix if and only if $\tilde{A}$ is an M-matrix.

**Proof.** The proof of this theorem is similar to the proof of Lemma 3.3 in [9]. Let $A$ be an M-matrix and $\tilde{A} = (\tilde{a}_{ij})$. Then

$$\tilde{a}_{ij} = \begin{cases} 1 - \sum_{k=1}^{n} \alpha_{ik}\tilde{a}_{ik}\tilde{a}_{ki}, & 1 \leq i = j \leq n, \\ a_{ij} - \sum_{k=1, k \neq j}^{n} \alpha_{ik}a_{ik}a_{kj}, & 1 \leq i \neq j \leq n. \end{cases} \quad (4)$$

Since $A$ is a Z-matrix, we have $\tilde{a}_{ij} \leq 0$, for $i \neq j$. This means that $\tilde{A}$ is also a Z-matrix. By Lemma 1.4 there exists a positive vector $x$ such that $Ax \gg 0$. On the other hand, $\tilde{A} = (I + L(\alpha) + U(\alpha))A \gg 0$. Invoking Lemma 1.4 implies that $\tilde{A}$ is also an M-matrix.

Conversely, let $\tilde{A}$ be an M-matrix. Then, $\tilde{A}^T$ is also an M-matrix. By Lemma 1.4 there exists a positive vector $x$ such that $\tilde{A}^Tx \gg 0$, i.e., $A^T(I + L(\alpha)^T + U(\alpha)^T)x \gg 0$. Let $y = (I + L(\alpha)^T + U(\alpha)^T)x$. Obviously $y \gg 0$. Therefore by Lemma 1.4, $A^T$ is an M-matrix. As a result, $A$ is an M-matrix, as well. \qed
Theorem 2.2. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular Z-matrix, $0 \leq \gamma \leq \omega \leq 1$, $\omega \neq 0$ and $\alpha_{ij} \in [0, 1]$ for $1 \leq i \neq j \leq n$. If $\rho(L_{\gamma, \omega}) < 1$, then $\rho(\hat{L}_{\gamma, \omega}) \leq \rho(\hat{L}_{\gamma, \omega}) < 1$.

Proof. Under the assumptions of the theorem, it is easy to see that

$$A = M - N = \frac{1}{\omega}(I - \gamma L) - \frac{1}{\omega}[(1 - \omega)I + (\omega - \gamma)L + \omega U],$$

is an $M$-splitting of $A$. On the other hand, we have $\rho(M^{-1}N) = \rho(L_{\gamma, \omega}) < 1$. Therefore, from Lemma 1.1 we deduce that $A$ is an $M$-matrix. Now, the result follows immediately from Theorems 2.6 and 2.7 in [13]. □

In the sequel, we show that among the preconditioners of the form $\hat{P} = I + L(\alpha) + U(\alpha)$ with $\alpha_{ij} \in [0, 1]$, the preconditioner $\hat{P} = I + L + U$ is the best one to speed up the convergence rate of the AOR iterative method. We mention that, if we assume $\alpha_{ij} = 1$, for $1 \leq i \neq j \leq n$, then the preconditioner $\hat{P}$ results in the preconditioner $\tilde{P}$. Let the AOR iteration matrix of the preconditioned system $\hat{P}Ax = \hat{P}b$ be

$$\hat{L}_{\gamma, \omega} = (\hat{D} - \gamma \hat{L})^{-1}[(1 - \omega)\hat{D} + (\omega - \gamma)\hat{L} + \omega \hat{U}],$$

where $\hat{A} = \hat{P}A = \hat{D} - \hat{L} - \hat{U}$ in which $\hat{D}, \hat{L}$ and $\hat{U}$ are the diagonal, strictly lower and strictly upper triangular matrices, respectively.

Theorem 2.3. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be a nonsingular Z-matrix. Let also $0 \leq \gamma \leq \omega \leq 1$, $\omega \neq 0$ and $\alpha_{ij} \in [0, 1]$ for $1 \leq i \neq j \leq n$. If $\rho(L_{\gamma, \omega}) < 1$ and

$$(\alpha_{ij} - 1)a_{ij} + \left( \sum_{k=1, k \neq i}^{n} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=1, k \neq i}^{n} a_{ik}a_{kj} \right) + \left( \sum_{k=1}^{i-1} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=1}^{i-1} a_{ik}a_{kj} \right) \leq 0,$$

for $1 \leq j < i \leq n$, then $\rho(\hat{L}_{\gamma, \omega}) \leq \rho(\hat{L}_{\gamma, \omega})$.

Proof. Let

$$M = \frac{1}{\omega}(I - \gamma L),$$
$$N = \frac{1}{\omega}[(1 - \omega)I + (\omega - \gamma)L + \omega U],$$
$$\tilde{M} = \frac{1}{\omega}(\tilde{D} - \gamma \tilde{L}),$$
$$\tilde{N} = \frac{1}{\omega}[(1 - \omega)\tilde{D} + (\omega - \gamma)\tilde{L} + \omega \tilde{U}],$$
\[ \dot{M} = \frac{1}{\omega}(\dot{D} - \gamma \dot{L}), \]
\[ \dot{N} = \frac{1}{\omega}[(1 - \omega)\dot{D} + (\omega - \gamma)\dot{L} + \omega \dot{U}]. \]

It is easy to see that
\[ A = M - N, \quad \tilde{A} = \dot{M} - \dot{N}, \quad \hat{A} = \dot{M} - \dot{N}. \]

Similar to the proof of Theorem 2.2 we see that the matrix \( A \) is an M-matrix. Let \( x = A^{-1}e \), where \( e = (1, \ldots, 1)^{T} \). We have \( x \gg 0 \), since none of the rows of \( A^{-1} \) can be zero. Therefore,
\[ (\hat{A} - \tilde{A})x = [(L - L(\alpha)) + (U - U(\alpha))]x = [(L - L(\alpha)) + (U - U(\alpha))]e \geq 0. \]

Since \( \rho(L_{\gamma, \omega}) < 1 \), from Theorem 2.2 we have \( \rho(\tilde{L}_{\gamma, \omega}) \leq \rho(L_{\gamma, \omega}) < 1 \). Now, Theorem 2.1 shows that \( \hat{A} \) is an M-matrix. Hence, \( \hat{D} \) is a diagonal matrix with positive diagonal entries. Therefore, \( \hat{M} \) and \( \hat{M} \) are nonsingular matrices and splittings \( \hat{A} = \hat{M} - \hat{N} \) and \( \hat{A} = \hat{M} - \hat{N} \) are M-splitting.

By definition of \( \hat{A} \) and \( \hat{A} \), we have
\[ \hat{D} - \tilde{D} = E_{1}(1) - E_{1}(\alpha) + E_{2}(1) - E_{2}(\alpha) \geq 0. \]

So \( \hat{D} \geq \tilde{D} \). On the other hand, we have
\[ \tilde{L} - \hat{L} = (L - L(\alpha)) + (L(\alpha) - L)L + (F_{2}(\alpha) - F_{2}(1)) + (F_{1}(\alpha) - F_{1}(1)). \]

Hence,
\[ (\tilde{L} - \hat{L})_{ij} = (\alpha_{ij} - 1)a_{ij} + (\sum_{k=1}^{i-1} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=1}^{i-1} a_{ik}a_{kj}) \]
\[ + (\sum_{k=i+1}^{n} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=i+1}^{n} a_{ik}a_{kj}) + (\sum_{k=1}^{i-1} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=1}^{i-1} a_{ik}a_{kj}) \]
\[ = (\alpha_{ij} - 1)a_{ij} + (\sum_{k=1,k\neq i}^{n} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=1,k\neq i}^{n} a_{ik}a_{kj}) \]
\[ + (\sum_{k=1}^{i-1} \alpha_{ik}a_{ik}a_{kj} - \sum_{k=1}^{i-1} a_{ik}a_{kj}) \leq 0, \]

which shows that \( \tilde{L} \leq \hat{L} \). This result together with \( \hat{D} \geq \tilde{D} \) gives \( \hat{D} - \gamma \hat{L} \leq \tilde{D} - \gamma \tilde{L} \).

Since \( \gamma \tilde{D}^{-1} \tilde{L} \geq 0 \) is an strictly lower triangular matrix and \( \rho(\tilde{D}^{-1}\tilde{L}) < 1 \), we deduce that
\[ (\hat{D} - \gamma \hat{L})^{-1} = (I - \gamma \tilde{D}^{-1}\tilde{L})^{-1}\tilde{D}^{-1} = I + \sum_{j=1}^{\infty}(\gamma \tilde{D}^{-1}\tilde{L})^{j}\tilde{D}^{-1} \geq 0. \]
In the same way, \((\hat{D} - \gamma \hat{L})^{-1} \geq 0\). Therefore, we obtain
\[
(\hat{D} - \gamma \hat{L})^{-1} \leq (\hat{D} - \gamma \hat{L})^{-1},
\]
and this means that
\[
0 \leq \tilde{M}^{-1} \leq \hat{M}^{-1}.
\]
Let \(A\) be an irreducible matrix. Having in mind that the entries of \(\tilde{A}\) are given by (4), we conclude that \(\tilde{A}\) is also an irreducible matrix. We have
\[
\tilde{L}_{\gamma,\omega} = (\tilde{D} - \gamma \tilde{L})^{-1}[(1 - \omega)\tilde{D} + (w - \gamma)\tilde{L} + \omega\tilde{U}]
= (I - \gamma \tilde{D}^{-1} \tilde{L})^{-1}[(1 - \omega)I + (w - \gamma)\tilde{D}^{-1} \tilde{L} + \omega\tilde{D}^{-1} \tilde{U}]
= [I + (\gamma \tilde{D}^{-1} \tilde{L}) + (\gamma \tilde{D}^{-1} \tilde{L})^2 + \cdots ][(1 - \omega)I + (w - \gamma)\tilde{D}^{-1} \tilde{L} + \omega\tilde{D}^{-1} \tilde{U}]
\geq [(1 - \omega)I + (w - \gamma)\tilde{D}^{-1} \tilde{L} + \omega\tilde{D}^{-1} \tilde{U}].
\]
This shows that for every \(0 \leq \gamma < 1\) the matrix \(\tilde{L}_{\gamma,\omega}\) is a nonnegative irreducible matrix. Hence, from Theorem 4.11 in [1], \(\tilde{L}_{\gamma,\omega} = \tilde{M}^{-1} \tilde{N}\) has a positive Perron vector and from Lemma 1.5 we have
\[
\rho(\tilde{L}_{\gamma,\omega}) \leq \rho(\tilde{L}_{\gamma,\omega}).
\]
If \(\gamma = 1\), then \(\omega = \gamma = 1\) and we have
\[
\rho(\tilde{L}_{1,1}) = \lim_{\gamma \to 1^-} \rho(\tilde{L}_{\gamma,1}) \geq \lim_{\gamma \to 1^-} \rho(\tilde{L}_{1,1}) = \rho(\hat{L}_{1,1}).
\]
Now, if \(A\) is a reducible matrix, then by Lemma 1.2, for sufficiently \(\epsilon > 0\) the matrix \(A(\epsilon)\) is an irreducible M-matrix and one can see that
\[
\rho(\tilde{L}_{\gamma,\omega}) = \lim_{\epsilon \to 0^+} \rho(\tilde{L}_{\gamma,\omega}(\epsilon)) \geq \lim_{\epsilon \to 0^+} \rho(\tilde{L}_{\gamma,\omega}(\epsilon)) = \rho(\tilde{L}_{\gamma,\omega}).
\]

3. Numerical experiments

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a Pentium 4 PC, with a 3.06 GHz CPU and 1.00GB of RAM.

**Example 1.** We consider the two dimensional convection-diffusion equation (see [14])
\[
-(u_{xx} + u_{yy}) + u_x + 2u_y = f(x, y), \quad \text{in} \quad \Omega = (0, 1) \times (0, 1),
\]
with the homogeneous Dirichlet boundary conditions. Discretization of this equation on a uniform grid with \(N \times N\) interior nodes \((n = N^2)\), by using the second
order centered differences for the second and first order differentials gives a linear system of equations of order $n$ with $n$ unknowns. The coefficient matrix of the obtained system is of the form

$$A = I \otimes P + Q \otimes I,$$

where $\otimes$ denotes the Kronecker product,

$$P = \text{tridiag}(-\frac{2+h}{8}, 1, -\frac{2-h}{8}) \quad \text{and} \quad Q = \text{tridiag}(-\frac{1+h}{4}, 0, -\frac{1-h}{4}),$$

are $N \times N$ tridiagonal matrices, and the step size is $h = 1/N$. We consider four preconditioners of the form

$$P_0 = I,$$
$$P_1 = I + 0.5L,$$
$$P_2 = I + 0.5L + 0.5U,$$
$$P_3 = I + L(\alpha) + U(\alpha),$$
$$P_4 = I + L + U,$$

where for the preconditioner $P_3$, $\alpha_{ij}$’s are random numbers uniformly distributed in the interval $(0,1)$. We mention that $P_0 = I$ means that no preconditioner is used. In Table 1, the spectral radius of the AOR iterative method applied to the preconditioned systems $P_iAx = Pb$, $i = 0, \ldots, 4$ for different values of $\gamma$, $\omega$ and $n$ are given. As we observe preconditioner $P_4$ is the best one among the chosen preconditioners.

For more investigation, we apply the GMRES($m$) method [11] with $m = 10$ to solve $P_iAx = Pb$, $i = 0, \ldots, 4$. In all the experiments, vector $b = A(1,1,\ldots,1)^T$ was taken to be the right-hand side of the linear system and a null vector as an initial guess. The stopping criterion used was always

$$\frac{\|b - Ax_k\|_2}{\|b\|_2} < 10^{-10}.$$ 

In Table 2, we report the number of iterations and the CPU time (in parenthesis) for the convergence. As we see the preconditioner $P_4$ is the best.

**Example 2.** We consider the previous example with

$$-(u_{xx} + u_{yy}) + 2e^{x+y}(xu_x + yu_y) = f(x, y), \quad \text{in} \quad \Omega = (0,1) \times (0,1).$$

All of the assumptions are the same as the previous example. In Table 3 the spectral radii of the AOR iterative method and in Table 4 numerical results of
Table 1: Comparison of spectral radii for Example 1.

| N  | (γ, ω)    | P₀   | P₁   | P₂   | P₃   | P₄   |
|----|------------|------|------|------|------|------|
| 5  | (0.7, 0.8) | 0.8317 | 0.7964 | 0.7404 | 0.7486 | 0.6323 |
| 5  | (0.8, 1)   | 0.7739 | 0.7305 | 0.6540 | 0.6444 | 0.5138 |
| 10 | (0.7, 0.8) | 0.9474 | 0.9350 | 0.9125 | 0.9116 | 0.8677 |
| 10 | (0.8, 1)   | 0.9289 | 0.9135 | 0.8821 | 0.8815 | 0.8221 |

Table 2: Number of iterations and the CPU time for the convergence of the GMRES(10) for Example 1.

| N  | P₀    | P₁    | P₂    | P₃    | P₄    |
|----|------|------|------|------|------|
| 50 | 80 (0.34) | 57 (0.31) | 33 (0.22) | 44 (0.74) | 29 (0.17) |
| 100| 326 (5.33) | 130 (2.83) | 132 (3.30) | 110 (2.75) | 78 (1.97) |
| 150| 702 (29.08) | 365 (19.73) | 244 (15.56) | 350 (23.02) | 185 (12.08) |

the GMRES(10) method applied to the preconditioned systems \( P_i A x = P_i b \), \( i = 0, \ldots, 4 \) are given. As we observe preconditioner \( P_4 \) is the best one among the chosen preconditioners.

4. Conclusion

For a class of matrices, we have shown that among the preconditioners of the form \( \hat{P} = I + L(\alpha) + U(\alpha) \) with \( \alpha_{ij} \in [0, 1] \), the preconditioner \( \hat{P} = I + L + U \) is the best one to speed up the convergence rate of the AOR iterative method. Numerical results of the AOR and GMRES(\( m \)) methods applied to different preconditioned systems confirm the presented theoretical results.

Table 3: Comparison of spectral radii for Example 2.

| N  | (γ, ω)    | P₀   | P₁   | P₂   | P₃   | P₄   |
|----|------------|------|------|------|------|------|
| 5  | (0.7, 0.8) | 0.8657 | 0.8358 | 0.7871 | 0.8049 | 0.6907 |
| 5  | (0.8, 1)   | 0.8193 | 0.7823 | 0.7154 | 0.7033 | 0.5891 |
| 10 | (0.7, 0.8) | 0.9581 | 0.9481 | 0.9298 | 0.9306 | 0.8929 |
| 10 | (0.8, 1)   | 0.9434 | 0.9309 | 0.9053 | 0.9045 | 0.8558 |
Table 4: Number of iterations and the CPU time for the convergence of the GMRES(10) for Example 2.

| N  | $P_0$      | $P_1$      | $P_2$      | $P_3$      | $P_4$      |
|----|------------|------------|------------|------------|------------|
| 50 | 57(0.27)   | 46(0.27)   | 28(0.19)   | 36(0.22)   | 23(0.13)   |
| 60 | 85(0.52)   | 57(0.45)   | 34(0.31)   | 49(0.44)   | 29(0.25)   |
| 70 | 92(0.72)   | 79(0.89)   | 49(0.61)   | 54(0.72)   | 37(0.47)   |
| 80 | 111(1.23)  | 84(1.23)   | 52(0.84)   | 94(1.55)   | 45(0.73)   |

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