A mixed-norm estimate of two-particle reduced density matrix of many-body Schrödinger dynamics

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Abstract

We provide a mixed-norm estimate of two-particle reduced density matrix of the solution of $N$-body Schrödinger equation. Using that we present a new approach to obtain the Vlasov dynamics from the Schrödinger equation through Hartree-Fock dynamics with $\hbar = N^{-1/3}$ as the re-scaled Plank constant. Furthermore, we provide that, in the sense of distribution, the mean-field residue term $R_m$ has higher rate than the semi-classical residue $R_s$, namely, $R_s \sim \hbar^{1/2}$ and $R_m \sim N^{-1/3} N^{-1/3} \sim N^{-1/3} \hbar^{1/3}$. In addition, in the estimates for the residual terms, we update the oscillation estimate parts appeared in \cite{12}.

Keywords: Large fermionic system, Vlasov equation, Husimi measure, Schrödinger equation, mean-field limit, semi-classical limit

1 Introduction

In this study, we consider the following $N$-particle mean-field Schrödinger equation

\begin{equation}
\begin{aligned}
\text{i}\hbar \partial_t \psi_{N,t} &= -\frac{\hbar^2}{2} \sum_{j=1}^{N} \Delta_{x_j} \psi_{N,t} + \frac{1}{2N} \sum_{i \neq j}^{N} V(x_i - x_j) \psi_{N,t} \\
\psi_{N,0} &= \frac{1}{\sqrt{N!}} \det \{e_j(x_i)\}_{i,j=1}^{N},
\end{aligned}
\end{equation}

(1.1)

where $\{e_j\}_{j=1}^{N}$ is a family of orthonormal basis in $L^2(\mathbb{R}^3)$ and $\Delta_{x_j}$ is the Laplacian on $j$-th particle. The initial data in (1.1) is in the form of a Slater determinant, which stays in the antisymmetric subspace $L^2_a(\mathbb{R}^{3N})$ of $L^2(\mathbb{R}^3)$ with $\|\psi_{N,t}\|_2 = 1$, where

$L^2_a(\mathbb{R}^{3N}) := \{ \psi_N \in L^2(\mathbb{R}^{3N}) : \psi_N(x_{\pi(1)}, \ldots, x_{\pi(N)}) = (\text{sign } \pi) \psi_N(x_1, \ldots, x_N) \text{ for all } \pi \in S_N \}$. 

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In the above formulation, $S_N$ is the odd-permutation group.

The justification of such a choice is stemmed from the fact that, due to the Pauli’s exclusion principle, the kinetic energy for $N$-fermionic particles is of order $\hbar^2 N^{5/3}$. Moreover, to make the potential energy to be comparable with the kinetic energy, the coupling constant in front of the interaction is taken to be of the order $\hbar^2 N^{-1/3}$. Then because of average velocity of particles is of order $\hbar N^{-1/3}$, we consider a short time scale, i.e.,

$$h = N^{-1/3}.$$ 

We call such scaling as semi-classical scale. For more details, we refer to [4, 7, 13].

As it is difficult to solve the Schrödinger equation in (1.1) numerically when the number of particle $N$ is large, we aim to derive its corresponding effective evolution equation. In fact, we consider the $k$-particle reduced density matrix where its corresponding integral kernel is given by

$$\gamma^{(k)}_{N,t}(x_1, \ldots, x_k; y_1, \ldots, y_k) = \frac{N!}{(N-k)!} \int \cdots \int N \psi_{N,t}(y_1, \ldots, y_k, x_{k+1}, \ldots, x_N)\psi_{N,t}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_N),$$

where $1 \leq k \leq N$. Moreover, we denote the expectation of the one-particle observable as follows,

$$\langle \psi_{N,t}, O \psi_{N,t} \rangle = \int \cdots \int \psi_{N,t}(y_1, x_2, \ldots, x_N)(O \psi_{N,t})(x_1, x_2, \ldots, x_N),$$

which can be also written as follows:

$$\text{Tr} O \gamma^{(1)}_{N,t} = \langle \psi_{N,t}, O \psi_{N,t} \rangle.$$

The one-particle reduce density matrix of the initial data given in (1.1) is

$$\omega_N = \sum_{j=1}^{N} \langle e_j | e_j \rangle,$$

where its corresponding integral kernel is $\omega_N(x; y) = \sum_{j=1}^{N} e_j(y)e_j(x)$.

**Short review of mean-field limit, $N \to \infty$**

It is well known that the Hartree-Fock equation

$$i\hbar \partial_t \omega_{N,t} = \left[ -\hbar^2 \Delta + (V * \rho_{N,t}) - X_t, \omega_{N,t} \right],$$

$$\rho_{N,t} = \frac{1}{N} \omega_{N,t}(x; x)$$

$$X_t = \frac{1}{N} V(x - y) \omega_{N,t}(x; y)$$

is used to approximate the Schrödinger equation in the mean-field limit. Here we denote $[A, B] := AB - BA$.

The convergence for fixed $\hbar$ has been given in [17] for short time. Under the scaling $\hbar = N^{-1/3}$, the rates of convergence in the trace norm and the Hilbert-Schmidt norm are obtained for arbitrary given time in [7] when the initial data is an approximation of the Slater determinant. Later on, the case with mixed state initial data has been considered in [5, 8]. Furthermore, for Coulomb and Riesz potentials, the rate of convergence is obtained in [38, 39]. We refer more references on this topic to [19, 38, 36, 37] and the references therein.

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Short review of semi-classical limit, \( h \to 0 \)

The Vlasov equation can be obtained via semi-classical limit and from the Hartree or the Hartree–Fock equations. It has been first investigated in [30] by using Wigner measure. Recently, the rates of convergence in the trace norm as well as the Hilbert-Schmidt norm has been studied in [6] with regular assumptions on the initial data. The \( k \)-particle Wigner measure

\[
W^{(k)}_{N,t}(x_1, p_1, \ldots, x_k, p_k) := \left( \frac{N}{k} \right)^{-1} \int (dy)^{\otimes k} \gamma^{(k)}_{N,t} \left( x_1 + \frac{h}{2} y_1, \ldots, x_k - \frac{h}{2} y_k; x_1 - \frac{h}{2} y_1, \ldots, x_k + \frac{h}{2} y_k \right) e^{-i \sum_{i=1}^{k} p_i \cdot y_i},
\]

has been used to study semi-classical limit, (see e.g. in [6, 40]). Here \( \gamma^{(k)}_{N,t} \) is the kernel of the \( k \)-particle reduced density matrix defined in (1.2).

Some of the recent developments in the semi-classical limit are the following: One can find results for the inverse power law potential in [41], for the rate of convergence in the Schatten norm in [29], for the Coulomb potential and mixed states in [40], and for the convergence in the Wasserstein distance in [27, 28]. Relativistic fermionic system has been studied in [16]. Further analyses of the semi-classical limit can be found in [1, 2, 3, 20, 32].

Combined Mean-Field and semi-classical Limits

Narnhofer and Sewell, and Spohn independently derived Vlasov equation (1.8) from the \( N \)-body Schrödinger equation (1.1) with \( h = N^{-1/3} \), in [34, 44]. Without assuming \( h = N^{-1/3} \), a rate of convergence was obtained in [24] in a weak formulation. The rate of convergence of the combined limits was studied in [21, 22, 23] by using the Wasserstein (pseudo-)distance. Under a generalized Husimi measure framework, the authors in [12] obtained the convergence for regular potentials, and then they also considered a cut-offed potential which converges to Coulomb type singularity in the large \( N \) limit in [13]. Recently, the combined limit for the singular potential case was obtained in [14].

It is known that the Wigner measure in (1.4) is not a (proper) probability density, as there might be some point having negative sign. (We refer, e.g., [10, 11, 25, 26, 31, 43] for further references on Wigner measure.) It has been shown that the Husimi measure, the convolution of the Wigner measure with a Gaussian function, is a nonnegative probability measure [15, 18, 45]. In particular, from [18, p.21], given a specific Gaussian coherent state, the relation between the Husimi measure and Wigner measure is given by the following convolution: for any \( 1 \leq k \leq N \),

\[
m^{(k)}_{N,t} := \frac{N(N-1) \cdots (N-k+1)}{N^k} W^{(k)}_{N,t} \ast G^h,
\]

where

\[
G^h(q_1, p_1, \ldots, q_k, p_k) := \frac{1}{(\pi h)^{3k}} \exp \left( -\frac{\sum_{j=1}^{k} q_j^2 + p_j^2}{h} \right).
\]

In this study, we will consider the following generalized \( k \)-particle Husimi measure: For any \( p, q \in \mathbb{R}^3 \) and \( \psi_{N,t} \in L^2_{a}(\mathbb{R}^{3N}) \), the \( k \)-particle Husimi measure is given by

\[
m^{(k)}_{N,t}(q_1, p_1, \ldots, q_k, p_k) = \langle \psi_{N,t}, a^*(f^{h}_{q_1, p_1}) \cdots a^*(f^{h}_{q_k, p_k}) a(f^{h}_{q_1, p_1}) \cdots a(f^{h}_{q_k, p_k}) \psi_{N,t} \rangle.
\]
Here $a^*(f^h_{q,p})$ and $a(f^h_{q,p})$ are standard creation- and annihilation-operator respectively with respect to the coherent state $f^h_{q,p}$ given by

$$f^h_{q,p}(y) := h^{-\frac{3}{4}} f \left( \frac{y - q}{\sqrt{\hbar}} \right) e^{\frac{y^2}{4\hbar}},$$

where $f$ is any given real-valued function satisfying $\|f\|_2 = 1$.

**Remark 1.1.** As stated in [18], the $k$-particle Husimi measure here describes how many fermions are within the $k$-semi-classical boxes of length $\sqrt{\hbar}$ centered at the phase-spaces $(q_1, p_1), \ldots, (q_k, p_k)$.

**Remark 1.2.** If $f(x) = \pi^{-3/4} e^{-|x|^2/2}$, [15] shows that the $k$-particle Husimi measure $m^{(k)}_{N,t}$ coincides with the $m^{(k)}_{N,t}$ in (1.5).

### 1.1 Main Result

Denoting the one-particle Husimi measure as $m_{N,t} := m^{(1)}_{N,t}$ and assume that $\psi_{N,t}$ is the solution to the Schrödinger equation in (1.1), from [12, Proposition 2.1], we obtain the following identity:

$$\partial_t m_{N,t}(q, p) + p \cdot \nabla_q m_{N,t}(q, p) - \nabla_q \cdot \left( C_{[1]} \right) = \frac{1}{(2\pi)^3} \nabla_p \cdot \int dq_1 dq_2 dq_3 dp_1 dp_2 dp_3 \left( f^h_{q,p}(w) f^h_{q,p}(u) \right)^{\otimes 2}$$

$$\left( \int dq \nabla V_N(s u_1 + (1-s) w_1 - w_2) \gamma^{(2)}_{N,t}(u_1, u_2; w_1, w_2) \right),$$

where

$$\left( f^h_{q,p}(w) f^h_{q,p}(u) \right)^{\otimes 2} := f^h_{q,p}(w_1) f^h_{q,p}(u_1) f^h_{q,p}(w_2) f^h_{q,p}(u_2).$$

**Remark 1.3.** The two-particle reduced density matrix in (1.7) stems from (A.1).

Our aim is therefore to obtain the convergence from $m_{N,t}$, in weak sense, to the solution of Vlasov equation $m_t$ as follows:

$$\partial_t m_t(q, p) = -p \cdot \nabla_q m_t(q, p) + \nabla_q \langle V * \varrho_t \rangle(q) \cdot \nabla_p m_t(q, p),$$

$$m_t(q, p) \big|_{t=0} = m_0(q, p),$$

where the initial data $m_0(q, p)$ is the one-particle Husimi measure with Slater determinant as the wavefunction and

$$\varrho_t = \int dp m_t(q, p).$$

We give the following assumptions in this paper.

**Assumption H1.**

1. (Interaction potential) Assume the interaction potential $V \in L^1(\mathbb{R}^3)$ be a real-valued function such that $V(-x) = V(x)$ and $\int dp (1 + |p|^2)|\hat{V}(p)| < \infty$, where $\hat{V}$ is its Fourier transform of $V$.

2. (Coherent state) The real-valued function $f \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ satisfies $\|f\|_2 = 1$, and has compact support in $B_{R_1}$.

\[1\] Definitions of creation- and annihilation-operator is provided in Appendix A.
3. (Initial data) Let $m_N$ be the initial one-particle Husimi measure of the given initial data in (1.1) which converges to $m_0$ weakly in $L^1$. Furthermore, it satisfies
\[
\int dq dp \left( |p|^2 + |q|^2 \right) m_N(q, p) < \infty. \tag{1.9}
\]

4. (Initial data) Let $\omega_N$ be the one-particle density matrix of initial data which satisfies
\[
\sup_{p \in \mathbb{R}^3} \frac{1}{1 + |p|} \| \left[ e^{ipx}, \omega_N \right] \|_{\text{Tr}} \leq CNh,
\]
\[
\| [h \nabla, \omega_N] \|_{\text{Tr}} \leq CNh, \tag{1.10}
\]
where $\| \cdot \|_{\text{Tr}}$ is the trace norm.

**Remark 1.4.** The assumptions in (1.10) can be explained by the nature of the semi-classical structure. More details can be found in [7] where mean field limit has been studied.

With the assumptions presented above, we have the following theorem:

**Theorem 1.1.** Let $m_{N,t}$ be the one-particle Husimi measure defined in (1.6) with $\psi_{N,t}$ the solution of Schrödinger equation (1.1), and suppose the aforementioned assumptions hold and $m_t$ is the solution of Vlasov equation in (1.8). Then, for any given $T > 0$, $(m_{N,t})_{N \in \mathbb{N}}$ converges to $m_t$ weakly (*) in $L^p((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$ for arbitrary $1 \leq p \leq \infty$.

Instead of using BBGKY hierarchy method in [12], we are working only on the equation for one-particle reduced density matrix. The two-particle reduced density matrix term appeared in the aforementioned equation is treated directly by using the mean-field structure via Bogoliubov transformation method. Furthermore, as shown later in the proof, the order (depending on $N$) of the residual terms for the mean-field term is smaller than the one presented in [12].

Lastly, by refraining from using the hierarchy method, we can shed more light on the structure of the residual terms for both mean-field limit and semi-classical limit.

**Idea of the proof:**

For convenience, we use the Fock space formalism, which will be briefly introduced in Appendix [A]. By using Husimi transform given in (1.7), we transform the Schrödinger equation for $\psi_{N,t}$ into the Vlasov type equation for $m_{N,t}$ with residual terms. To be more precise, from the computations in [12], we have
\[
\partial_t m_{N,t}(q, p) + p \cdot \nabla_q m_{N,t}(q, p) = \frac{1}{(2\pi)^3} \nabla_p \cdot \int dq_2 \nabla V(q - q_2) \theta_{N,t}(q_2)m_{N,t}(q, p) + \nabla_q \cdot \mathcal{R} + \nabla_p \cdot \mathcal{R}, \tag{1.11}
\]

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2The literature on Bogoliubov transformation can be found in [7, 9, 33, 42].
where $\varrho_{N,t}(q) := \int dq \, m_{N,t}(q,p)$. In this paper we give a better estimate for the term $\tilde{R}$, namely $\| \int dq \tilde{R} \|_{L^2(\mathbb{R}^3)} \leq C \hbar$. This result will be listed in Section 4.

Furthermore, as shown later 3, the estimate for the term $\tilde{R}$ can be obtained exactly the same as in [12]. (In the updated arXiv version of [12], the oscillation estimates have been corrected, with which the estimates for the residue terms from BBGKY hierarchy as well as the main result still hold true.)

Under the assumptions of Theorem 1.1 the estimate of the residual term $\tilde{R}$ can be obtained exactly the same as in [12]. In this paper we give a better estimate for the term $\tilde{R}$, namely $\| \int dq \tilde{R} \|_{L^2(\mathbb{R}^3)} \leq C \hbar$. This result will be listed in Section 4.

For the mean-field residue term, we insert the intermediate terms in $\mathcal{R}_m$ as follows:

\[
\begin{align*}
    \gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2) &= \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(u_2; w_2) \\
    &= \gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2) - \omega_{N,t}(u_1; w_1)\omega_{N,t}(u_2; w_2) \\
    &\quad + \left[ \omega_{N,t}(u_1; w_1) - \gamma_{N,t}^{(1)}(u_1; w_1) \right] \omega_{N,t}(u_2; w_2) \\
    &\quad + \gamma_{N,t}^{(1)}(u_1; w_1) \left[ \omega_{N,t}(u_2; w_2) - \gamma_{N,t}^{(1)}(u_2; w_2) \right] \\
    &=: T_1 + T_2 + T_3,
\end{align*}
\]

where $\omega_{N,t}$ is the solution to Hartree-Fock equation.

We observe that $T_2$ and $T_3$ are estimated by the trace norm and Hilbert-Schmidt norm of $\gamma_{N,t}^{(1)} - \omega_{N,t}$, respectively. As shown in Lemma 2.1 the estimate for residue term involving $T_1$ will be controlled by using the fast oscillation effect from the coherent state. In this paper, we provide a bound of

\[
\left( \int dw_1 dw_2 \left| \int dw_1 \gamma_{N,t}^{(2)}(w_1, u_1, w_2; w_1, w_2) - \omega_{N,t}(u_1; w_1)\omega_{N,t}(w_2; w_2) \right|^2 \right)^{1/2}.
\]  (1.14)

In [7], the convergence with respect to the trace norm and Hilbert-Schmidt norm of the difference between $\gamma_{N,t}^{(k)}$ and $\omega_{N,t}^{(k)}$ are obtained separately with the help of Wick’s theorem for $k \geq 2$. However, we do not directly use Wick’s theorem to compute (1.14) as extra effort is needed to estimate the residue term involving $T_1$. This is a different approach for the 2-particle reduced density matrix given in [7]. In particular, we trace the strategies given in [7] to obtain the rate of convergence estimate in the mean-field limit. Then, the estimates are reduced to the expectation of the number operator $\hat{N}$ along the quantum fluctuation which will be bounded under the assumption of Theorem 1.1.
Furthermore, after concrete estimates for the residue terms, it will be proved that in the sense of distribution \( R \sim \hbar^{3-\frac{1}{2}} \) and \( R_m \sim \hbar^{3-\frac{1}{2}} \), from which one can observe that the semi-classical and mean field residue terms are not of the same order in the combined limit \( N^{-1} = \hbar^3 \) argument. The rate \( \hbar^{3-\frac{1}{2}} \) for semi-classical residue term should be of the optimal rate because of the semi-classical scale. The rate for mean-field residue we obtained here is not optimal. Nevertheless, to balance the two orders, one could think of other combined limit, which will be investigated in future projects.

This paper is arranged as follows: we first prove the estimate for semi-classical residue in Section 2, followed by the estimate for mean-field residue in Section 3. Finally, we will conclude the proof of Theorem 1.1 in Section 4. In the appendices, for reader’s convenience, we list some basic notations and known estimates.

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## 2 Estimate for mean-field residue

In this section, we will estimate the mean-field residue by first showing in Lemma 2.1 that the estimate for mean-field residue term can be reduced to the estimate for the term

\[
\text{Tr}^{(1)}\left[\gamma^{(2)}_{N,t} - \gamma^{(1)}_{N,t} \otimes \gamma^{(1)}_{N,t}\right](u_1;w_1)
\]

where we denote

\[
\text{Tr}^{(1)}\left|\gamma^{(2)} - \gamma^{(1)} \otimes \gamma^{(1)}\right| := \int dy|\gamma^{(2)}(x,y;z,y) - \gamma^{(1)}(x;y)\gamma^{(1)}(z;y)|,
\]

Then, we prove the estimate for (2.1) in Proposition 2.1 and finally summary the estimation for the mean-field residue in Corollary 2.1.

**Lemma 2.1.** Let \( \varphi, \phi \in C_0^\infty(\mathbb{R}^3) \). Then, for \( \frac{1}{2} < \alpha_1 < 1 \) and \( s = \left\lceil \frac{3(\alpha_1 - \frac{1}{2})}{2(1-\alpha_1)} \right\rceil \), we have

\[
\left| \int dqdp \varphi(q)\phi(p)\nabla_p \cdot \mathcal{R}_m(q,p) \right| \\
\leq C \|\nabla V\|_{L^\infty} \hbar^{3+\frac{3}{2}(\alpha_1-\frac{1}{2})+\frac{3}{2}} \left( \int dw_1du_1 \left[ \text{Tr}^{(1)}\left[\gamma^{(2)}_{N,t} - \gamma^{(1)}_{N,t} \otimes \gamma^{(1)}_{N,t}\right](u_1;w_1) \right]^2 \right)^{\frac{1}{2}},
\]

where the constant \( C \) depends on \( \|\varphi\|_{\infty}, \|\nabla \phi\|_{W^{s,\infty}}, \supp \phi, \|f\|_{L^\infty \cap H^1}, \supp f \).

**Proof.** Recall that, in (1.12), we defined the mean-field residue such that

\[
\mathcal{R}_m := \frac{1}{(2\pi)^3} \int dw_1du_1dw_2du_2dq_2dp_2 \left( f_{q,p}^{\hbar}(w)f_{q,p}^{\hbar}(u) \right)^{\otimes 2} \\
\nabla V(q - q_2) \left[ \gamma^{(2)}_{N,t}(u_1,u_2;w_1,w_2) - \gamma^{(1)}_{N,t}(u_1;w_1)\gamma^{(1)}_{N,t}(u_2;w_2) \right].
\]
Then one obtains

\[
\left| \int dqdp \varphi(q)\phi(p) \nabla_p \cdot \mathcal{R}_m(q, p) \right|
= \frac{1}{(2\pi)^3} \left| \int (dqdp)^{\otimes 2} (dwdu)^{\otimes 2} \varphi(q) \nabla \phi(p) \cdot \left( f_{q,p}(w)\overline{f_{q,p}(u)} \right)^{\otimes 2} \nabla V(q - q_2) \right|
\]

\[
\left[ \gamma^{(2)}_{N,t}(u_1, u_2; w_1, w_2) - \gamma^{(1)}_{N,t}(u_1; w_1)\gamma^{(1)}_{N,t}(w_2) \right]
\]

\[
= \frac{1}{(2\pi)^3} \left| \int (dqdp)^{\otimes 2} (dwdu)^{\otimes 2} \varphi(q) \nabla \phi(p) \cdot \left( f \left( \frac{w - q}{\sqrt{\hbar}} \right) f \left( \frac{u - q}{\sqrt{\hbar}} \right) e^{\frac{i}{\hbar}p \cdot (w - u)} \right)^{\otimes 2} \nabla V(q - q_2) \right|
\]

\[
\left[ \gamma^{(2)}_{N,t}(u_1, u_2; w_1, w_2) - \gamma^{(1)}_{N,t}(u_1; w_1)\gamma^{(1)}_{N,t}(w_2) \right]
\]

\[
= \frac{1}{(2\pi)^3} \left| \int (dqdwdu)^{\otimes 2} \left( f \left( \frac{w - q}{\sqrt{\hbar}} \right) f \left( \frac{u - q}{\sqrt{\hbar}} \right) \right)^{\otimes 2} \nabla V(q - q_2) \left[ \gamma^{(2)}_{N,t}(u_1, w_2; w_1, w_2) - \gamma^{(1)}_{N,t}(u_1; w_1)\gamma^{(1)}_{N,t}(w_2) \right]
\]

\[
\nabla V(q - q_2) \left[ \gamma^{(2)}_{N,t}(u_1, w_2; w_1, w_2) - \gamma^{(1)}_{N,t}(u_1; w_1)\gamma^{(1)}_{N,t}(w_2) \right]
\]

where we use the weighted Dirac-Delta function in the last equality, i.e.,

\[
\frac{1}{(2\pi)^3} \int dp_2 e^{\frac{i}{\hbar}p_2 \cdot (w_2 - u_2)} = \delta_{w_2}(u_2).
\]
\[(\int dp \chi_{(w_1-u_1) \in (\Omega^c_R)} \nabla \phi(p) e^{\frac{1}{\gamma_0}(w_1-u_1)}) \cdot \nabla V(q-q_2) \left( \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2) \right).\]

By the change of variable \(\sqrt{h} \tilde{q}_2 = w_2 - q_2\), we obtain
\[
\int dq \varphi(q) \int dw_1 du_1 dw_2 \left( \frac{u_1 - q}{\sqrt{h}} \right) f \left( \frac{u_1 - q}{\sqrt{h}} \right) \left( \frac{w_1 - q}{\sqrt{h}} \right) f \left( \frac{w_1 - q}{\sqrt{h}} \right) \left| \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2) \right| \\
\leq C \|\nabla V\|_{L^\infty} h^{\frac{3}{2}} \int dq \varphi(q) \int dw_1 du_1 \left( \frac{w_1 - q}{\sqrt{h}} \right) f \left( \frac{w_1 - q}{\sqrt{h}} \right) \left| \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2) \right| \\
\leq C \|\nabla V\|_{L^\infty} h^{\frac{3}{2}} \int dq \varphi(q) \int dw_1 du_1 \left( \frac{w_1 - q}{\sqrt{h}} \right) f \left( \frac{w_1 - q}{\sqrt{h}} \right) \left| \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2) \right| \\
\leq C \|\nabla V\|_{L^\infty} h^{\frac{3}{2}} \int dq \varphi(q) \int dw_1 du_1 \left( \frac{w_1 - q}{\sqrt{h}} \right) f \left( \frac{w_1 - q}{\sqrt{h}} \right) \left| \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2) \right| \\
\leq C \|\nabla V\|_{L^\infty} h^{\frac{3}{2}} \int dq \varphi(q) \int dw_1 du_1 \left( \frac{w_1 - q}{\sqrt{h}} \right) f \left( \frac{w_1 - q}{\sqrt{h}} \right) \left| \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2) \right| \\
\leq C \|\nabla V\|_{L^\infty} h^{\frac{3}{2}} \int dq \varphi(q) \int dw_1 du_1 \left( \frac{w_1 - q}{\sqrt{h}} \right) f \left( \frac{w_1 - q}{\sqrt{h}} \right) \left| \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2) \right| \\
\leq C \|\nabla V\|_{L^\infty} h^{\frac{3}{2}} \int dq \varphi(q) \int dw_1 du_1 \left( \frac{w_1 - q}{\sqrt{h}} \right) f \left( \frac{w_1 - q}{\sqrt{h}} \right) \left| \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2) \right| \\
\leq C \|\nabla V\|_{L^\infty} h^{\frac{3}{2}} \int dq \varphi(q) \int dw_1 du_1 \left( \frac{w_1 - q}{\sqrt{h}} \right) f \left( \frac{w_1 - q}{\sqrt{h}} \right) \left| \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2) \right| \\
\leq C \|\nabla V\|_{L^\infty} h^{\frac{3}{2}} \int dq \varphi(q) \int dw_1 du_1 \left( \frac{w_1 - q}{\sqrt{h}} \right) f \left( \frac{w_1 - q}{\sqrt{h}} \right) \left| \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2) \right| \\
\leq C \|\nabla V\|_{L^\infty} h^{\frac{3}{2}} \int dq \varphi(q) \int dw_1 du_1 \left( \frac{w_1 - q}{\sqrt{h}} \right) f \left( \frac{w_1 - q}{\sqrt{h}} \right) \left| \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2) \right|
Thus, we have
\[
\text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) = \int dw_2 |\gamma_{N,t}^{(2)}(u_1, w_1; w_2, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2)|.
\]

Thus, we have
\[
J_m \leq C \|\varphi\|_{L^\infty} \|
abla \phi\|_{W^{s,\infty}} \|
abla V\|_{L^\infty} h^{3s + \frac{3}{2}} \left(\int dw_1 du_1 \left[ \text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \right)^{\frac{1}{2}},
\]

where we denote
\[
\text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) = \int dw_2 |\gamma_{N,t}^{(2)}(u_1, w_1; w_2, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2)|.
\]

Now, we focus on \( I_m \)
\[
I_m = (2\pi)^2 \int (dq)^2 \varphi(q) \int dw_1 du_1 dw_2 f \left( \frac{w_1 - q}{\sqrt{\hbar}} \right) f \left( \frac{u_1 - q}{\sqrt{\hbar}} \right) \left| \int dp \chi_{u_1 - u_1} e^{ip(u_1 - u_1)} \nabla \phi(p) \cdot \nabla V(q - q_2) \left[ \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(w_2; w_2) \right] \right|.
\]

Using the fact that
\[
\left| \int dp e^{ip(u_1 - u_1)} \nabla \phi(p) \right| \leq \|\nabla \phi\|_{L^1},
\]
we obtain the following estimate:
\[
I_m \leq C \|\nabla \phi\|_{L^1} \|\nabla V\|_{L^\infty} \int dq |\varphi(q)| \left(\int dw_1 du_1 \chi_{|u_1 - u_1| \leq \hbar^{\alpha_1} \chi_{|w_1 - u_1| \leq 2R_1} \left| f \left( \frac{w_1 - q}{\sqrt{\hbar}} \right) f \left( \frac{u_1 - q}{\sqrt{\hbar}} \right) \right|^2 \right)^{\frac{1}{2}}
\]
\[
= \left( \int dq \left( \int dw_1 du_1 \left[ \text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \chi_{|u_1 - q| \leq R_1} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]
\[
\leq C \|\varphi\|_{L^\infty} \|\nabla \phi\|_{L^1} \|\nabla V\|_{L^\infty} h^{\frac{3}{2}} \left( \int dw_1 du_1 \chi_{|\bar{u}_1 - \bar{u}_1| \leq \hbar^{\alpha_1}} \frac{1}{2} \chi_{|\bar{w}_1 - \bar{u}_1| \leq 2R_1} \left| f(\bar{w}_1) f(\bar{u}_1) \right|^2 \right)^{\frac{1}{2}}
\]
\[
\leq C \|\varphi\|_{L^\infty} \|\nabla \phi\|_{L^1} \|\nabla V\|_{L^\infty} h^{\frac{3}{2}} \left( \int dw_1 du_1 \left[ \text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \chi_{|u_1 - q| \leq R_1} \right)^{\frac{1}{2}}
\]
\[
= \left( \int dq \left( \int dw_1 du_1 \left[ \text{Tr}^{(1)} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} \right| (u_1; w_1) \right]^2 \chi_{|u_1 - q| \leq R_1} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}
\]
\[
\leq C \|\varphi\|_{L^\infty} \|\nabla \phi\|_{L^1} \|\nabla V\|_{L^\infty} h^{\frac{3}{2}} \left( \int dw_1 du_1 \chi_{|\bar{u}_1 - \bar{u}_1| \leq \hbar^{\alpha_1}} \left| f(\bar{w}_1) f(\bar{u}_1) \right|^2 \right)^{\frac{1}{2}},
\]

which together with
\[
\int d\bar{w}_1 \left| f(\bar{w}_1) \right|^2 \int d\bar{u}_1 \chi_{|\bar{u}_1 - \bar{u}_1| \leq \hbar^{\alpha_1 - \frac{1}{2}}} \left| f(\bar{u}_1) \right|^2 \leq \|f\|_{L^2(\mathbb{R}^3)}^2 \|f\|_{L^2(\mathbb{R}^3)} h^{3(\alpha_1 - \frac{1}{2})},
\]

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implies immediately that
\[ I_m \leq C \| \varphi \|_{L^\infty} \| \nabla \phi \|_{L^1} \| \nabla V \|_{L^\infty} h^{3+\frac{3}{2} (\alpha_1 - \frac{1}{2})} + \frac{1}{2} \left( \int dw_1 du_1 \left[ \text{Tr}^{(1)} \left( \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} (u_1; w_1) \right) \right]^2 \right)^{\frac{1}{2}}. \]

To balance the order between \( I_m \) and \( J_m \), \( s \) is chosen to be
\[ s = \left\lceil 3 \left( \alpha_1 - \frac{1}{2} \right) \right\rceil \frac{3}{2 (1 - \alpha_1)}, \]
for \( \alpha_1 \in [0, 1) \). Therefore, we have
\[ \left| \int dq dp \varphi(q) \phi(p) \nabla_p \cdot R_m \right| \leq I_m + J_m \leq C \| \varphi \|_{L^\infty} \| \nabla \phi \|_{W^{s,\infty}} \| \nabla V \|_{L^\infty} h^{3+\frac{3}{2} (\alpha_1 - \frac{1}{2})} + \frac{1}{2} \left( \int dw_1 du_1 \left[ \text{Tr}^{(1)} \left( \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} (u_1; w_1) \right) \right]^2 \right)^{\frac{1}{2}}, \]
and we obtained the desired result.

Next, we want to bound the term with the 'mixed'-norm, i.e.,
\[ \int dw_1 du_1 \left[ \text{Tr}^{(1)} \left( \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)} (u_1; w_1) \right) \right]^2. \tag{2.7} \]

The following proposition provides the estimate of (2.7):

**Proposition 2.1.** Let \( \gamma_{N,t}^{(k)} \) be \( k \)-particle reduced density matrix associated with \( \Psi_{N,t}, \omega_{N,t} \) be the solution of the Hartree-Fock equation in (1.3). Suppose the assumption for Theorem 1.1 holds. Then the following inequalities hold for all \( t \in \mathbb{R} \):
\[ \left\| \gamma_{N,t}^{(1)} - \omega_{N,t} \right\|_{HS} \leq Ce^{|t|}, \tag{2.8} \]
and
\[ \left\| \gamma_{N,t}^{(1)} - \omega_{N,t} \right\|_{\text{Tr}} \leq C \sqrt{N} e^{|t|}. \tag{2.9} \]

Furthermore, it holds that
\[ \left( \int dw_1 du_1 \left[ \text{Tr}^{(1)} \left( \gamma_{N,t}^{(2)} - \omega_{N,t} \otimes \omega_{N,t} \right) \right]^2 (u_1; w_1) \right)^{\frac{1}{2}} \leq CN e^{|t|}, \tag{2.10} \]
where we denote the partial trace \( \text{Tr}^{(1)} \left[ \omega_{N,t}(x_1, \cdot ; y_1, \cdot) \right] := \int dx_2 |\omega_{N,t}(x_1, x_2; y_1, x_2)| \)

The proof of Proposition 2.1 requires the following results from [7], namely:

**Lemma 2.2** (Lemma 3.1 of [7]). Let \( d\Gamma (O) \) be the second quantization of any bounded operator \( O \) on \( L^2(\mathbb{R}^3) \), i.e.
\[ d\Gamma (O) := \int dx dy O(x; y) a_x^* a_y. \]
For any \( \Psi \in \mathcal{F}_a \), the following inequalities hold
\[ \| d\Gamma (O) \Psi \| \leq \| O \| \| \nabla \Psi \|. \tag{2.11} \]
If furthermore $O$ is a Hilbert-Schmidt operator, we have the following bounds:

\[
\|d\Gamma(O)\Psi\| \leq \|O\|_{\text{HS}} \left\| \mathcal{N}^{1/2}\Psi \right\|, \quad (2.12)
\]

\[
\left\| \int dx dy O(x; y) a_x a_y \Psi \right\| \leq \|O\|_{\text{HS}} \left\| \mathcal{N}^{1/2}\Psi \right\|, \quad (2.13)
\]

\[
\left\| \int dx dy O(x; y) a_x^* a_y^* \Psi \right\| \leq 2\|O\|_{\text{HS}} \left\| (\mathcal{N} + 1)^{1/2}\Psi \right\|. \quad (2.14)
\]

Finally, if $O$ is a trace class operator, we obtain

\[
\|d\Gamma(O)\Psi\| \leq 2\|O\|_{\text{Tr}}, \quad (2.15)
\]

\[
\left\| \int dx dy O(x; y) a_x a_y \Psi \right\| \leq 2\|O\|_{\text{Tr}}, \quad (2.16)
\]

\[
\left\| \int dx dy O(x; y) a_x^* a_y^* \Psi \right\| \leq 2\|O\|_{\text{Tr}}, \quad (2.17)
\]

where $\|O\|_{\text{Tr}} := \text{Tr} |O| = \text{Tr} \sqrt{O^*O}$.

**Lemma 2.3** (Proposition 3.4 of [7]). Suppose the assumption for Theorem 1.1 holds. Then, there exist constants $K, c > 0$ depending only on potential $\mathcal{V}$ such that

\[
\sup_{p \in \mathbb{R}^3} \frac{1}{1 + |p|} \left| \text{Tr} \left[ \omega_{N,t}, e^{ip\cdot x} \right] \right| \leq KNhe^{c|t|}
\]

\[
\left| \text{Tr} \left[ \omega_{N,t}, h\nabla \right] \right| \leq KNhe^{c|t|}
\]

for all $p \in \mathbb{R}^3$ and $t \in \mathbb{R}$.

**Lemma 2.4** (Theorem 3.2 of [7]). Let $U_N(t; s)$ be the quantum fluctuation dynamics defined in (A.6) and $\mathcal{N}$ be the number operator. If the assumptions in Lemma 2.3 hold. Then for $\xi_N \in F_a$ with $\langle \xi_N, \mathcal{N}^{1/2}\xi_N \rangle \leq C$ for any $k \geq 1$, we have the following inequality:

\[
\left\| (\mathcal{N} + 1)^k U_N(t; 0)\xi_N \right\| \leq C_t, \quad (2.18)
\]

where $C_t := Ke^{c|t|}$ is a positive constant depending on $t \in \mathbb{R}$, $k$ and potential $\mathcal{V}$.

**Remark 2.1.** Here in this paper we only need the result for initial data $\xi_N = \Omega$.

Now, we are ready to provide the proof of Proposition 2.1.

**Proof of Proposition 2.1.** The proof of the inequalities (2.8) and (2.9) follows by modifying Theorem 2.1 of [7]. In particular, from equation (4.3) in [7], we obtain

\[
\left\| \gamma_{N,t}^{(1)} - \omega_{N,t} \right\|_{\text{HS}} \leq C \left\| \mathcal{N}^{1/2}U_N(t; 0)\xi_N \right\|,
\]

\[
\left\| \gamma_{N,t}^{(1)} - \omega_{N,t} \right\|_{\text{Tr}} \leq C\sqrt{N} \left\| \mathcal{N}U_N(t; 0)\xi_N \right\|,
\]

by choosing the appropriate operator $O$ as discussed in [7]. Our results for (2.8) and (2.9) are obtained by applying Lemma 2.4 and taking the assumption that $\| (\mathcal{N} + 1)\xi_N \| \leq C$. 

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Therefore, it remains to prove for (2.10). As remarked previously, the trace norm and Hilbert-Schmidt norm of the difference between $\gamma_{N,t}^{(k)}$ and $\omega_{N,t}^{(k)}$ are obtained separately with the help of Wick’s theorem for $k \geq 2$ in [7]. For our term, however, we do not directly use Wick’s theorem to compute (2.10) as each term requires similar but still unique method when taking the estimation.

Simplifying the notation $\mathcal{R}_t := \mathcal{R}_{Y_N,t}$, where $\mathcal{R}_{Y_N,t}$ is the Bogoliubov transformation given in the Appendix (A.4), we have, from the definition of a 2-particle reduced density matrix and (A.4), that

$$\gamma_{N,t}^{(2)}(x_1, x_2; y_1, y_2) = \langle \xi_N, \mathcal{U}_N^*(t; 0) \mathcal{R}_t a_{y_1} a_{y_2} a_{x_2} a_{x_1} \mathcal{R}_t \mathcal{U}_N(t; 0) \xi_N \rangle$$

$$= \langle \xi_N, \mathcal{U}_N^*(t; 0) \mathcal{R}_t a_{y_1} \mathcal{R}_t a_{y_2} a_{x_2} \mathcal{R}_t a_{x_1} \mathcal{R}_t \mathcal{U}_N(t; 0) \xi_N \rangle$$

$$= \langle \xi_N, \mathcal{U}_N^*(t; 0) \big( a^*(u_{t,y_1}) + a(\nabla_{t,y_1}) \big) \big( a^*(u_{t,y_2}) + a(\nabla_{t,y_2}) \big) \big( a(u_{t,x_2}) + a^*(\nabla_{t,x_2}) \big) \mathcal{U}_N(t; 0) \xi_N \rangle$$

$$= \langle \xi_N, \mathcal{U}_N^*(t; 0) \left[ a(\nabla_{t,y_1}) a(\nabla_{t,y_2}) a(u_{t,x_2}) a(u_{t,x_1}) + a(\nabla_{t,y_1}) a(\nabla_{t,y_2}) a^*(\nabla_{t,x_2}) a^*(\nabla_{t,x_1}) + a(\nabla_{t,y_1}) a(\nabla_{t,y_2}) a^*(\nabla_{t,x_2}) a^*(\nabla_{t,x_1}) + a^*(u_{t,y_1}) a(u_{t,x_2}) a(u_{t,x_1}) + a^*(u_{t,y_1}) a(\nabla_{t,y_2}) a^*(\nabla_{t,x_2}) a^*(\nabla_{t,x_1}) + a^*(u_{t,y_1}) a(\nabla_{t,y_2}) a(u_{t,x_2}) a(u_{t,x_1}) + a^*(u_{t,y_1}) a(\nabla_{t,y_2}) a(u_{t,x_2}) a(u_{t,x_1}) + a^*(u_{t,y_1}) a(u_{t,x_2}) a(u_{t,x_1}) + a^*(u_{t,y_1}) a^*(u_{t,y_2}) a^*(\nabla_{t,x_2}) a^*(\nabla_{t,x_1}) + a^*(u_{t,y_1}) a^*(u_{t,y_2}) a^*(\nabla_{t,x_2}) a^*(\nabla_{t,x_1}) \right] \mathcal{U}_N(t; 0) \xi_N \rangle$$

(2.19)

where we use (A.4) in the third equality. Note that since $\langle \nabla_{t,x_1}, \nabla_{t,y} \rangle = \omega_{N,t}(y; x)$, it holds that

$$a(\nabla_{t,y_1}) a(\nabla_{t,y_2}) a^*(\nabla_{t,x_2}) a^*(\nabla_{t,x_1})$$

$$= a^*(\nabla_{t,x_1}) a(\nabla_{t,y_1}) a^*(\nabla_{t,x_2}) a^*(\nabla_{t,x_1}) + a(\nabla_{t,y_2}) a(\nabla_{t,y_1}) a^*(\nabla_{t,x_2}) a^*(\nabla_{t,x_1})$$

$$= a^*(\nabla_{t,x_1}) a(\nabla_{t,y_1}) a^*(\nabla_{t,x_2}) a^*(\nabla_{t,x_1}) + a(\nabla_{t,y_2}) a(\nabla_{t,y_1}) a^*(\nabla_{t,x_2}) a^*(\nabla_{t,x_1})$$

$$= a^*(\nabla_{t,x_1}) a(\nabla_{t,y_1}) a^*(\nabla_{t,x_2}) a^*(\nabla_{t,x_1}) + a(\nabla_{t,y_2}) a(\nabla_{t,y_1}) a^*(\nabla_{t,x_2}) a^*(\nabla_{t,x_1})$$

(2.20)
Therefore, we obtain
\[
\gamma_{N,t}^{(2)}(x_1, x_2; y_1, y_2) = \omega_{N,t}(x_1; y_1) \omega_{N,t}(x_2; y_2)
\]
\[
= \left\langle \xi_N, U_N^*(t; 0) \left[ a(\nabla_{t,y_1})a(u_{t,x_1})a(\nabla_{t,y_2})a(u_{t,x_2}) + a(\nabla_{t,y_1})a(u_{t,x_1})a^*(u_{t,y_2})a^*(\nabla_{t,x_2}) + a^*(\nabla_{t,x_1})a^*(u_{t,y_1})a^*(u_{t,y_2})a(u_{t,x_2}) + a^*(\nabla_{t,x_1})a^*(u_{t,y_1})a^*(u_{t,y_2})a(u_{t,x_2}) + a^*(u_{t,y_1})a^*(\nabla_{t,y_2})a(u_{t,x_2}) + a^*(\nabla_{t,x_1})a^*(u_{t,y_1})a^*(u_{t,y_2})a(u_{t,x_2}) + a^*(u_{t,y_1})a^*(u_{t,y_2})a^*(\nabla_{t,x_2})a^*(\nabla_{t,y_2})a(u_{t,x_2}) + a^*(u_{t,y_1})a^*(u_{t,y_2})a^*(\nabla_{t,x_2})a^*(\nabla_{t,y_2})a(u_{t,x_2}) - a^*(\nabla_{t,x_1})a^*(\nabla_{t,y_1})a(u_{t,x_2})a(u_{t,y_2}) - a^*(u_{t,y_1})a^*(\nabla_{t,x_1})a^*(u_{t,y_2})a(u_{t,x_2}) - a(u_{t,x_1})a(u_{t,x_2})a(u_{t,y_1})a(u_{t,y_2}) \right] U_N(t; 0) \xi_N \right\rangle,
\]
(2.21)
where we used the fact that $\langle \nabla_{t,x}, u_{t,y} \rangle = 0$, $\langle u_{t,x}, \nabla_{t,y} \rangle = 0$ and CAR.
\[=: \sum_{i=1}^{16} A_i + \sum_{j=1}^{16} B_j + C. \quad (2.22)\]

Using the fact that \(\|u_t\|_{\text{op}}, \|v_t\|_{\text{op}} \leq 1, \|v_t\|_{\text{HS}} \leq \sqrt{N}, \|\omega_{N,t}\|_{T_1} = N\) and the assumption \(\|\xi_N\| \leq 1\), we do the following estimates for the first term from \(\{A_i\}_{i=1}^{16}\) and \(\{B_i\}_{i=1}^{16}\) separately.

\[|A_1| = \left| \int dx_1 dx_2 dz_1 dz_2 \left( \xi_N, U_N^*(t; 0) \right) \int d\eta_1 d\eta'_1 a_{\eta_1} a_{\eta'_1} v_t(\eta_1; x_1) O_1(x_1; z_1) u_t(z_1; \eta'_1) \right.\]

\[\left. \int d\eta_2 d\eta'_2 a_{\eta_2} a_{\eta'_2} v_t(\eta_2; x_2) O_2(x_2; z_2) u_t(z_2; \eta'_2) U_N(t; 0) \xi_N \right|\]

\[= \left| \left( \xi_N, U_N^*(t; 0) \right) \int d\eta_1 d\eta'_1 a_{\eta_1} a_{\eta'_1} (v_t O_1 u_t)(\eta_1; \eta'_1) \int d\eta_2 d\eta'_2 a_{\eta_2} a_{\eta'_2} (v_t O_2 u_t)(\eta_2; \eta'_2) U_N(t; 0) \xi_N \right|\]

\[= \left| \left( \xi_N, U_N^*(t; 0) \right) a_{\eta_1} a_{\eta'_1} (v_t O_1 u_t)(\eta_1; \eta'_1) a_{\eta_2} a_{\eta'_2} (v_t O_2 u_t)(\eta_2; \eta'_2) U_N(t; 0) \xi_N \right|\]

\[\leq \int d\eta_1 \int d\eta'_1 a_{\eta_1} a_{\eta'_1} U_N(t; 0) \xi_N \int d\eta_2 \int d\eta'_2 a_{\eta_2} a_{\eta'_2} U_N(t; 0) \xi_N\]

\[\leq \int d\eta_1 \int d\eta'_1 a_{\eta_1} a_{\eta'_1} U_N(t; 0) \xi_N \int d\eta_2 \int d\eta'_2 a_{\eta_2} a_{\eta'_2} U_N(t; 0) \xi_N\]

\[\leq \sqrt{N} \|v_t O_1 u_t\|_{\text{HS}} \left( \int d\eta_1 \int d\eta'_1 a_{\eta_1} a_{\eta'_1} U_N(t; 0) \xi_N \int d\eta_2 \int d\eta'_2 a_{\eta_2} a_{\eta'_2} U_N(t; 0) \xi_N \right)^{\frac{1}{2}}\]

\[\leq \sqrt{N} \|v_t O_1 u_t\|_{\text{HS}} \left( \int d\eta_1 \int d\eta'_1 a_{\eta_1} a_{\eta'_1} U_N(t; 0) \xi_N \int d\eta_2 \int d\eta'_2 a_{\eta_2} a_{\eta'_2} U_N(t; 0) \xi_N \right)^{\frac{1}{2}}\]

\[\leq \sqrt{N} \|v_t O_1 u_t\|_{\text{HS}} \left( \int d\eta_1 \int d\eta'_1 a_{\eta_1} a_{\eta'_1} U_N(t; 0) \xi_N \int d\eta_2 \int d\eta'_2 a_{\eta_2} a_{\eta'_2} U_N(t; 0) \xi_N \right)^{\frac{1}{2}}\]

\[\leq \sqrt{N} \|v_t O_1 u_t\|_{\text{HS}} \left( \int d\eta_1 \int d\eta'_1 a_{\eta_1} a_{\eta'_1} U_N(t; 0) \xi_N \int d\eta_2 \int d\eta'_2 a_{\eta_2} a_{\eta'_2} U_N(t; 0) \xi_N \right)^{\frac{1}{2}}\]

\[\leq \sqrt{N} \|v_t O_1 u_t\|_{\text{HS}} \left( \int d\eta_1 \int d\eta'_1 a_{\eta_1} a_{\eta'_1} U_N(t; 0) \xi_N \int d\eta_2 \int d\eta'_2 a_{\eta_2} a_{\eta'_2} U_N(t; 0) \xi_N \right)^{\frac{1}{2}}\]

Additionally, we have

\[|B_1| = \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1) O_2(x_2; z_2) \left( \xi_N, U_N^*(t; 0), \langle \nabla_{x_1}, \nabla_{x_2} \rangle a(\nabla_{x_1}) a(u_{x_2}) U_N(t; 0) \xi_N \right) \right|\]

\[= \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1) O_2(x_2; z_2) \left( \xi_N, U_N^*(t; 0), \omega_{N,x}(x_2; z_1) U_N(t; 0) \xi_N \right) \right|\]

\[\int d\eta d\eta' a_{\eta} a_{\eta'} v_t(\eta; x_1) u_t(\eta'; z_2) U_N(t; 0) \xi_N \right|\]
After getting the bound of each terms, we have obtained the following estimates:

\[
= \left| \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 (\xi_N, \mathcal{U}_N(t; 0), \int d\eta d\eta' a_\eta a_{\eta'} \right|
\]

\[
\eta_i(\eta; x_1)O_1(x_1; z_1)\omega_{N,t}(z_1; x_2)O_2(x_2; z_2)u_\eta(z_2; \eta')\mathcal{U}_N(t; 0)\xi_N
\]

\[
= \left| \int d\eta d\eta' (\theta_1 \omega_{N,t} O_2 \eta_\eta') (\eta; \eta')\mathcal{U}_N(t; 0)\xi_N \right|
\]

\[
\leq \| \mathcal{U}_N \|_{\mathcal{U}(t; 0)\xi_N} \| N^{1/2} \mathcal{U}_N(t; 0)\xi_N \|
\]

\[
\leq \| \mathcal{U}_N \|_{\mathcal{U}(t; 0)\xi_N} \| N^{1/2} \mathcal{U}_N(t; 0)\xi_N \|
\]

\[
\leq \| \mathcal{U}_N \|_{\mathcal{U}(t; 0)\xi_N} \| N^{1/2} \mathcal{U}_N(t; 0)\xi_N \|
\]

The estimates for the rest of the terms can be done with similar steps, some of which are given in Appendix C. After getting the bound of each terms, we have obtained the following estimates:

\[
\sum_{i=1}^{16} A_i \leq \| \mathcal{U}_N \|_{\mathcal{U}(t; 0)\xi_N} \| N^{1/2} \mathcal{U}_N(t; 0)\xi_N \|
\]

\[
\sum_{j=1}^{16} B_i \leq \| \mathcal{U}_N \|_{\mathcal{U}(t; 0)\xi_N} \| N^{1/2} \mathcal{U}_N(t; 0)\xi_N \|
\]

Lastly, the final term is estimated as follows:

\[
|C|
\]

\[
= \left| \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \Omega_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, \mathcal{U}_N(t; 0) \rangle \langle \mathcal{U}_N(t; 0) \rangle \cdot \langle \mathcal{U}_N(t; 0) \rangle \right|
\]

\[
= \left| \int d\xi_1 d\xi_2 d\xi_3 d\xi_4 \Omega_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, \mathcal{U}_N(t; 0) \rangle \cdot \langle \mathcal{U}_N(t; 0) \rangle \right|
\]

\[
= \left| \int d\xi_2 \langle \xi_N, \mathcal{U}_N(t; 0) \rangle \left( \int d\xi_3 \langle \mathcal{U}_N(t; 0) \rangle \right) \right|
\]

\[
\leq \| \mathcal{U}_N \|_{\mathcal{U}(t; 0)\xi_N} \| N^{1/2} \mathcal{U}_N(t; 0)\xi_N \|
\]

As a summary, we have

\[
| \text{Tr} O(\gamma^{(2)}_{N,t} - \omega_{N,t} \otimes \omega_{N,t}) | \leq \sum_{i=1}^{16} A_i + \sum_{j=1}^{16} B_j + |C|
\]

\[
\leq N\| \mathcal{U}_N \|_{\mathcal{U}(t; 0)\xi_N} \| N^{1/2} \mathcal{U}_N(t; 0)\xi_N \|
\]

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which implies that, for \( O_1 \) and \( O_2 \) being Hilbert-Schmidt and trace class operators, we get
\[
\left( \int dx_1 dy_1 \left[ \int dx_2 \left| \gamma_{N,t}^{(2)}(x_1, x_2; y_1, x_2) - \omega_{N,t}(x_1; y_1) \omega_{N,t}(x_2; x_2) \right|^2 \right] \right)^{1/2} \leq N \|(N + 1)U_N(t; 0)\xi_N\|
\] (2.25)

Applying Lemma 2.4, we obtain the inequalities in Proposition 2.1 as desired.

Finally, we have the following estimate for the mixed-norm. Since it is one of the main contributions of this paper, we write it as a theorem.

**Theorem 2.1.** Suppose the assumptions given in Proposition 2.1 hold. Then, we have the following estimate
\[
\left( \int dw_1 du_1 \left[ \text{Tr} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)}(u_1; w_1) \right|^2 \right] \right)^{1/2} \leq C_t N,
\] (2.26)
where the constant \( C_t \) depends on potential \( V \) and time \( t \).

**Proof of Theorem 2.1** Inserting the intermediate terms
\[
\begin{align*}
\gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2) - \gamma_{N,t}^{(1)}(u_1; w_1)\gamma_{N,t}^{(1)}(u_2; w_2) &= \gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2) - \omega_{N,t}(u_1; w_1)\omega_{N,t}(u_2; w_2) \\
&\quad + \left[ \omega_{N,t}(u_1; w_1) - \gamma_{N,t}^{(1)}(u_1; w_1) \right] \omega_{N,t}(u_2; w_2) \\
&\quad + \gamma_{N,t}^{(1)}(u_1; w_1) \left[ \omega_{N,t}(u_2; w_2) - \gamma_{N,t}^{(1)}(u_2; w_2) \right] \\
&=: T_1 + T_2 + T_3,
\end{align*}
\] (2.27)
the estimate in (2.26) is then reduced into the estimates of the following terms.

- \( \left( \int dw_1 du_1 \left[ \text{Tr} \left| \gamma_{N,t}^{(2)} - \gamma_{N,t}^{(1)} \otimes \gamma_{N,t}^{(1)}(u_1; w_1) \right|^2 \right] \right)^{1/2} \),

- \( \left\| \omega_{N,t} - \gamma_{N,t}^{(1)} \right\|_{\text{HS}} \left\| \omega_{N,t} \right\|_{\text{Tr}} \),

- \( \left\| \gamma_{N,t}^{(1)} \right\|_{\text{HS}} \left\| \omega_{N,t} - \gamma_{N,t}^{(1)} \right\|_{\text{Tr}} \).

Proposition 2.1 implies immediately (2.26) considering the fact that \( \left\| \omega_{N,t} \right\|_{\text{Tr}} \leq N \).

**Corollary 2.1.** Let \( \varphi, \phi \in C_0^{\infty}(\mathbb{R}^3) \) are test functions, then the following inequality holds:
\[
\left| \int dq dp \varphi(q) \phi(p) \nabla_p \cdot \mathcal{R}_m(q, p) \right| \leq C h^{\frac{3}{2}(\alpha_1 - \frac{1}{2}) + \frac{3}{2}}
\] (2.28)
where \( \frac{1}{2} < \alpha_1 < 1 \), \( C \) depends on \( \phi, \varphi, f \).
3 Estimate for semi-classical residue

In this section, we will estimate the semi-classical residual term under the assumption with $V \in W^{2,\infty}(\mathbb{R}^3)$ in Proposition 3.1 with which give us the insight to compare the rate between semi-classical and mean-field residuals.

**Proposition 3.1.** Assume that $V \in W^{2,\infty}$ and $V(x) = V(-x)$ and let $\phi, \varphi$ are test functions, then the following inequality holds:

$$\left| \int dq dp \varphi(q)\phi(p) \nabla_p \cdot \mathcal{R}_s(q,p) \right| \leq C \hbar^{\frac{1}{4} + 3(\alpha_2 - 1)},$$

(3.1)

where $\frac{1}{2} < \alpha_2 < 1$, and $C$ depends on $\varphi, \phi, f$.

**Proof.** First, recall that

$$\mathcal{R}_s := \frac{1}{(2\pi)^3} \int dw_1 dw_2 dw_3 dw_4 \left( f_{q,p}^h(w) f_{q,p}^h(u) \right)^{\otimes 2} \left[ \int_0^1 ds \nabla V(su_1 + (1-s)w_1 - w_2) - \nabla V(q - q_2) \right] \gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2).$$

(3.2)

Since $\phi(q), \varphi(p)$ are test functions, we estimate

$$\left| \int dq dp \phi(q)\varphi(p) \nabla_p \cdot \mathcal{R}_s(q,p) \right|$$

$$= \frac{1}{(2\pi)^3} \int (dq dp)^{\otimes 2} \phi(q) \nabla_p \varphi(p) \cdot \left( f_{q,p}^h(w) f_{q,p}^h(u) \right)^{\otimes 2} \left[ \int_0^1 ds \nabla V(su_1 + (1-s)w_1 - w_2) - \nabla V(q - q_2) \right] \gamma_{N,t}^{(2)}(u_1, u_2; w_1, w_2)$$

$$= \frac{1}{(2\pi)^3} \int (dq)^{\otimes 2} dp dw_1 dw_2 \phi(q) \nabla_p \varphi(p) f \left( \frac{w_1 - q}{\sqrt{\hbar}} \right) f \left( \frac{u_1 - q}{\sqrt{\hbar}} \right)$$

$$e^{ip \cdot (w_1 - u_1)} \int dw_2 dp_2 e^{ip_2 \cdot (w_2 - u_2)} f \left( \frac{w_2 - q_2}{\sqrt{\hbar}} \right) f \left( \frac{u_2 - q_2}{\sqrt{\hbar}} \right)$$

$$\left[ \int_0^1 ds \nabla V(su_1 + (1-s)w_1 - w_2) - \nabla V(q - q_2) \right] \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2)$$

$$= \left| \int (dq)^{\otimes 2} dp dw_1 dw_2 \phi(q) \nabla_p \varphi(p) f \left( \frac{w_1 - q}{\sqrt{\hbar}} \right) f \left( \frac{u_1 - q}{\sqrt{\hbar}} \right)$$

$$e^{ip \cdot (w_1 - u_1)} f \left( \frac{w_2 - q_2}{\sqrt{\hbar}} \right)^2 \left[ \int_0^1 ds \nabla V(su_1 + (1-s)w_1 - w_2) - \nabla V(q - q_2) \right]$$

$$\gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2),$$

where we applied the fact that $(2\pi\hbar)^3 \delta_x(y) = \int e^{ip \cdot (x-y)} dp$. Then, inserting $\pm \nabla V(q - w_2)$ and we have

$$\leq \left| \int (dq)^{\otimes 2} dp dw_1 dw_2 \phi(q) \nabla \varphi(p) \cdot \left( \frac{w_1 - q}{\sqrt{\hbar}} \right) f \left( \frac{u_1 - q}{\sqrt{\hbar}} \right)$$
\[ e^{\frac{w}{2} - \frac{q}{2}} \left| f \left( \frac{w - q}{\sqrt{\hbar}} \right) \right|^2 \left[ \int_0^1 ds \nabla V \left( su_1 + (1 - s)w_1 - w_2 \right) - \nabla V \left( q - w_2 \right) \right] \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \]
\[ + \int (dq)^{(2)} dp dw_1 dw_2 \phi(q) \nabla \varphi(p) \cdot f \left( \frac{w - q}{\sqrt{\hbar}} \right) f \left( \frac{u_1 - q}{\sqrt{\hbar}} \right) \]
\[ e^{\frac{w}{2} - \frac{q}{2}} \left| f \left( \frac{w - q}{\sqrt{\hbar}} \right) \right|^2 \left[ \nabla V \left( q - w_2 \right) - \nabla V \left( q - q_2 \right) \right] \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \]
\[ =: I_s + J_s \]

where we used integration by part in the second to last equality.

Before advancing, recalling (B.4), we split the integral and obtain the following estimate, \( \forall \alpha_2 \in \left( \frac{1}{2}, 1 \right) \),
\[
\left| \int dp \nabla \varphi(p) e^{\frac{w}{2} - \frac{q}{2}} \right| = \left| \int dp \left( \chi_{(w_1 - u_1) \in \Omega^*_{\alpha_2}} + \chi_{(w_1 - u_1) \in (\Omega^*_{\alpha_2})^c} \right) \nabla \varphi(p) e^{\frac{w}{2} - \frac{q}{2}} \right|
\[
\leq \tilde{C} \left( \chi_{(w_1 - u_1) \in \Omega^*_{\alpha_2}} + \hbar^{(1 - \alpha_2)s} \right),
\]

where \( \tilde{C} \) depends on \( \| \phi \|_{W^{s+1, \infty}} \) and \( \text{supp} \phi \).

Now we want to estimate the term \( I_s \) and \( J_s \) separately. We begin by estimating \( I_s \),
\[
I_s = \hbar^{\frac{3}{2}} \left| \int dq dp dw_1 dw_2 \phi(q) \nabla \varphi(p) f \left( \frac{w_1 - q}{\sqrt{\hbar}} \right) f \left( \frac{u_1 - q}{\sqrt{\hbar}} \right) e^{\frac{w}{2} - \frac{q}{2}} \right|
\[
\left( \int dq \left| f(q) \right|^2 \right) \left[ \int_0^1 ds \nabla V \left( su_1 + (1 - s)w_1 - w_2 \right) - \nabla V \left( q - w_2 \right) \right] \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \]

Using (3.3), we have
\[
I_s \leq \| D^2 V \|_{L^\infty} \hbar^{\frac{3}{2}} \int dq \left| \phi(q) \right| \int dw_1 dw_2 \left( \chi_{(w_1 - u_1) \in \Omega^*_{\alpha_2}} + \hbar^{(1 - \alpha_2)s} \right) f \left( \frac{w_1 - q}{\sqrt{\hbar}} \right) f \left( \frac{u_1 - q}{\sqrt{\hbar}} \right) \]
\[
\left( \| u_1 - q \| + \| w_1 - q \| \right) \gamma_{N,t}^{(2)}(u_1, w_2; w_1, w_2) \]
\[
\leq C \hbar^{\frac{3}{2}} \int dq \left| \phi(q) \right| \int dw_1 \left( \chi_{(w_1 - u_1) \in \Omega^*_{\alpha_2}} + \hbar^{(1 - \alpha_2)s} \right) f \left( \frac{w_1 - q}{\sqrt{\hbar}} \right) f \left( \frac{u_1 - q}{\sqrt{\hbar}} \right) \]
\[
\left( \| u_1 - q \| + \| w_1 - q \| \right) \int dw_2 \gamma_{N,t}^{(2)}(w_1, w_2; u_1, w_2) \]
\[
= C \hbar^{\frac{3}{2}} \int dq \left| \phi(q) \right| \int dw_1 \left( a_{w_2} a_{w_1} \psi_{N,t}, a_{w_2} a_{u_1} \psi_{N,t} \right) \]
\[
\left( \| u_1 - q \| + \| w_1 - q \| \right) \int dw_2 \gamma_{N,t}^{(2)}(w_1, w_2; u_1, w_2) \]
\[
\leq C \hbar^{\frac{3}{2}} \int dq \left| \phi(q) \right| \int dw_1 \left( \chi_{(w_1 - u_1) \in \Omega^*_{\alpha_2}} + \hbar^{(1 - \alpha_2)s} \right) f \left( \frac{w_1 - q}{\sqrt{\hbar}} \right) f \left( \frac{u_1 - q}{\sqrt{\hbar}} \right) \]
\[
\left( \| u_1 - q \| + \| w_1 - q \| \right) \int dw_2 \left| a_{w_2} a_{w_1} \psi_{N,t} \right| \left| a_{w_2} a_{u_1} \psi_{N,t} \right| \]
\[
\| a_{w_2} a_{w_1} \psi_{N,t} \| \| a_{w_2} a_{u_1} \psi_{N,t} \| \]

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\[
\leq C h^{\frac{3}{2}} \int dq |\phi(q)| \int dw_1 dw_2 \chi_{(w_1-u_1) \in \Omega_h} \left( f \left( \frac{w_1-q}{\sqrt{h}} \right) f \left( \frac{u_1-q}{\sqrt{h}} \right) \right) \cdot |u_1 - q| \int dw_2 \| a_{w_2} a_{w_1} \psi_{N,t} \|^2
\]
\[
= C \left[ i_{s,1} + i_{s,2} \right],
\]
where we use \( i_{s,1} \) to be the term with \( \chi_{(w_1-u_1) \in \Omega_h} \), and \( i_{s,2} \) to be the other one. Due to the symmetric property, we can reduce the estimate for \( i_{s,1} \) into the following
\[
i_{s,1} \leq 2 C h^{\frac{3}{2}} \int dq |\phi(q)| \int dw_1 dw_1 \chi_{(w_1-u_1) \in \Omega_h} \left( f \left( \frac{w_1-q}{\sqrt{h}} \right) f \left( \frac{u_1-q}{\sqrt{h}} \right) \right) \cdot |u_1 - q| \int dw_2 \| a_{w_2} a_{w_1} \psi_{N,t} \|^2
\]
\[
\leq C h^{\frac{3}{2}} \int dq |\phi(q)| \int dw_1 dw_1 \chi_{(w_1-u_1) \in \Omega_h} \left( f \left( \frac{w_1-q}{\sqrt{h}} \right) f \left( \frac{u_1-q}{\sqrt{h}} \right) \right) \cdot |u_1 - q| \langle \psi_{N,t}, a_{w_1}^* N a_{w_1} \psi_{N,t} \rangle
\]
\[
\leq C h^{\frac{3}{2}} R_1 h^{\frac{3}{2}} h^{\frac{3}{2}} h^{-\frac{3}{2}} \int dw_1 \langle \psi_{N,t}, a_{w_1}^* N a_{w_1} \psi_{N,t} \rangle
\]
\[
\leq C h^{3+3+\frac{3}{2} - 6} \tag{3.4}
\]
This above estimate shows that the error is of order \( h^{\frac{3}{2}} - which should be the optimal rate for semi-classical limit. Jinyeop: The optimal rate for semi-classical limit expected to be \( h \).

Using similar steps with \( i_{s,1} \), noticing that \( \phi(q) \) has compact support, one obtains the estimate for the term \( i_{s,2} \),
\[
i_{s,2} \leq 2 C h^{\frac{3}{2}} \int dq |\phi(q)| \int dw_1 dw_1 h^{(1-\alpha_2) s} \left( f \left( \frac{w_1-q}{\sqrt{h}} \right) f \left( \frac{u_1-q}{\sqrt{h}} \right) \right) \cdot |u_1 - q| \int dw_2 \| a_{w_2} a_{w_1} \psi_{N,t} \|^2
\]
\[
\leq 2 C h^{\frac{3}{2} + \frac{3}{2} + \frac{3}{2} + \frac{1}{2}} h^{(1-\alpha_2) s} h^{\frac{3}{2} h^{-6}} \leq C h^{(1-\alpha_2) s - 1}
\]
To balance the order between \( i_{s,1} \) and \( i_{s,2} \), the term \( s \) is chosen to be
\[
s = \left[ \frac{3(\alpha_2 - \frac{3}{2})}{1-\alpha_2} \right],
\]
where \( \alpha_2 \in (\frac{1}{2}, 1) \). Therefore, we have
\[
I_s \leq C h^{\frac{1}{2} + 3(\alpha_2 - 1)}. \tag{3.5}
\]
Now, to estimate \( J_s \), we recall the estimate in (3.3) and obtain
\[
J_s = \int (dq)^{\otimes 2} dp dw_1 dw_2 \phi(q) \nabla \varphi(p) \cdot f \left( \frac{w_1-q}{\sqrt{h}} \right) f \left( \frac{u_1-q}{\sqrt{h}} \right)
\]
\[
eq C^{i_{P}(w_1-u_1)} \left[ \right]^{2} \left[ \nabla V(q-w_2) - \nabla V(q-q_2) \right]_{N,t}(u_1, w_2; w_1, w_2) \right]^{(2)}
\]
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In this section, we prove the main theorem. As have been mentioned in the introduction, we give first the

\[ \text{Proposition 4.1.} \]

Remark 3.1. The key step in the estimates of semi-classical residue is in (3.4), with which the computational bugs appeared in [12, 13] can both be fixed by the same technique.

Remark 3.2. The convergence rate of the semi-classical residue \( R_s \) and mean-field residue \( R_m \) are of the following orders

\[ R_s \sim h^\frac{1}{2}, \quad R_m \sim N^{-\frac{1}{2}} h^{\frac{1}{2}}. \]

Notice that the rates that we obtained for the semi-classical residue should be optimal, while for the mean field residue it is not optimal, nevertheless one can observe that the semi-classical and mean field residue terms are not of the same order in the combined limit argument.

\section{Proof of Theorem 1.1}

In this section, we prove the main theorem. As have been mentioned in the introduction, we give first the estimate for \( \tilde{R} \).

\[ \text{Proposition 4.1.} \]

\[ \left\| \int dp \tilde{R}(p, \cdot) \right\|_{L^\infty(\mathbb{R}^3)} \leq C h. \]

Proof. Noticing that

\[ \left| \int dp \tilde{R} \right| \leq h \int dp \left\| \nabla_q a(f^{h \phi}(q, p))\Psi_{N,t} \right\| \left\| a(f^{h \phi}(q, p))\Psi_{N,t} \right\| \]

\[ \leq \left[ h^2 \int dp \left( \nabla_q a(f^{h \phi}(q, p))\Psi_{N,t}, \nabla_q a(f^{h \phi}(q, p))\Psi_{N,t} \right) \right]^\frac{1}{2} \left[ \int dp m_{N,t}(q, p) \right]^\frac{1}{2} \]
where $C$ together with (4.3), we arrive at

\[
\rho(t) = \Theta_{N,t}^\frac{1}{2}
\]

Proof. The estimates in Appendix B imply

\[
\rho(t) = \Theta_{N,t}^\frac{1}{2}
\]

which implies

\[
\int dq \int dp \frac{\hat{R}}{R} \leq \left[ h^2 \int dq dp \left( \nabla_q a(f_{q,p}) \Psi_{N,t}, \nabla_q a(f_{q,p}) \Psi_{N,t} \right) \right]^{\frac{1}{2}} \left( \int dq \hat{R}_{N,t} \right)^{\frac{1}{2}} \leq h^\frac{5}{2} C.
\]

In the above estimate, we have used the fact that $\|\rho_{N,t}\|_{L^\infty(0,T;L^2(R^3))} \leq C$, which is a direct result from Appendix B.

Next, we will show that, for any $T > 0$, the sequence $m_{N,t}$ is weakly compact and that any accumulation point $m_t$ is exactly the solution of the Vlasov equation. For this purpose, we need the following lemma.

Lemma 4.1. Let $m_{N,t}$ be weak solution of the reformulated Schrödinger equation (1.11) and $\rho_{N,t}(q) := \int dp m_{N,t}(p, q)$. Then there exists a subsequence of $m_{N,t}$ (without relabeling for convenience) and function $m_t$ such that as $N \to \infty$

\[
m_{N,t} \xrightarrow{\ast} m_t \quad \text{in} \quad L^\infty(0,T;L^s(R^3 \times R^3)), \quad s \in [1, \infty],
\]

\[
\nabla V \ast \rho_{N,t} \rightharpoonup \nabla V \ast \rho_t \quad \text{in} \quad L^r(0,T;L^r(R^3)), \quad r \in (1, \infty),
\]

where $\rho_t := \int dp m_t$.

Proof. The estimates in Appendix B imply

\[
\|m_{N,t}\|_{L^\infty(0,T;L^s(R^3 \times R^3))} + \|m_{N,t}\|_{L^\infty(0,T;L^{\infty}(R^3 \times R^3))} \leq C,
\]

where $C$ appeared in this section denotes a positive constant independent of $N$. And combining interpolation inequality, we have

\[
\|m_{N,t}\|_{L^\infty(0,T;L^s(R^3 \times R^3))} \leq C, \quad s \in [1, \infty].
\]

Therefore, (4.1) is a direct consequence of the above inequality and the moment estimates in Proposition B.1

Furthermore Proposition B.1

\[
\|\rho_{N,t}\|_{L^\infty(0,T;L^1(R^3 \times R^3))} \leq C,
\]

together with (4.3), we arrive at

\[
\|\rho_{N,t}\|_{L^\infty(0,T;L^1(R^3))} \leq C, \quad s \in \left[1, \frac{5}{3}\right],
\]

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This implies that

\[ \rho_{N,t} \overset{s}{\to} \rho_t \text{ in } L^\infty(0,T;L^s(\mathbb{R}^3)), \quad s \in \left[ 1, \frac{5}{3} \right]. \]

Owing to \( V \in W^{2,\infty}(\mathbb{R}^3) \) and Young’s convolution inequality, we have for a.e. \( t \in (0,T) \),

\[ \| \nabla^2 V * \rho_{N,t} \|_{L^\infty(\mathbb{R}^3)} \leq \| \nabla^2 V \|_{L^\infty(\mathbb{R}^3)} \| \rho_{N,t} \|_{L^1(\mathbb{R}^3)} \leq C. \]

Similarly, we obtain that

\[ \| \nabla V * \rho_{N,t} \|_{L^\infty(0,T;L^1(\mathbb{R}^3))} \leq C. \] (4.4)

By means of (4.4), we get

\[ \partial_t \rho_{N,t} = \partial_t \int dp \, m_{N,t}(p,q) = -\nabla_q \cdot \int dp \, p \, m_{N,t}(p,q) + \nabla_q \cdot \int dp \, \tilde{R}. \]

It is easy to see

\[ \partial_t (\nabla V * \rho_{N,t}) = -\nabla_q \cdot \left( \nabla V \otimes \int dp \, p \, m_{N,t}(p,q) \right) + \nabla_q \cdot \left( \nabla V \otimes \int dp \, \tilde{R} \right), \]

where \((u \otimes v)_t = u_t \ast v_t\) for \((u,v) \in \mathbb{R}^3 \times \mathbb{R}^3\). Noticing that

\[ \left\| \int dp \, p \, m_{N,t}(p,q) \right\|_{L^\infty(0,T;L^s(\mathbb{R}^3))} \leq C, \quad s \in \left[ 1, \frac{5}{4} \right], \]

we derive for any test function \( \tilde{\varphi}(q) \in W^{1,3}(\mathbb{R}^3) \) and a.e. \( t \in (0,T) \),

\[ \left| \int_{\mathbb{R}^3} dq \, \nabla_q \cdot \left( \nabla V \otimes \int dp \, m_{N,t}(p,q) \right) \tilde{\varphi}(q) \right| \leq C(V) \left\| \int dp \, m_{N,t}(p,q) \right\|_{L^1(\mathbb{R}^3)} \left\| \nabla \tilde{\varphi}(q) \right\|_{L^3(\mathbb{R}^3)}. \]

For the second term, applying Proposition 4.1, we have

\[ \left| \int dq \, \nabla_q \cdot \left( \nabla V \otimes \int dp \, \tilde{R} \right) \tilde{\varphi}(q) \right| \leq C(V) \left\| \int dq \, \nabla_q \cdot \left( \nabla V \otimes \int dp \, \tilde{R} \right) \tilde{\varphi}(q) \right\|_{L^\infty(\mathbb{R}^3)} \leq C \left\| \nabla \varphi(q) \right\|_{L^3(\mathbb{R}^3)}. \]

The estimates above show that

\[ \| \partial_t (\nabla V * \rho_{N,t}) \|_{L^\infty(0,T;W^{-1,3}(\mathbb{R}^3))} \leq C. \] (4.5)

The inequalities (4.4) and (4.5) allow us to apply Aubin-Lions lemma to infer that (4.2). We mention here that the application of Aubin-Lions lemma is proceeded in a sequence of growing balls, and the convergent subsequence is obtained through diagonal rule.
Proof of Theorem 1.1. With the help of (4.1), (4.2), Proposition 4.1, Proposition 3.1 and Corollary 2.1 we can take limit $N \to \infty$ in the weak formulation of the reformulated Schrödinger equation (1.11). More precisely, for any $\phi, \varphi \in C_0^\infty(\mathbb{R}^3)$ and $\eta \in C_0^\infty(0, T)$, $m_{N,t}$ satisfies the following equation

$$
\int_0^T dt \int dq dp m_{N,t}(p,q) \left[ \partial_t \eta \varphi(q) \phi(p) + p \cdot \eta(t) \nabla_q \varphi(q) \phi(p) - \frac{1}{(2\pi)^3} \nabla V * \rho_{N,t}(t) \eta(t) \varphi(q) \cdot \nabla_p \phi(p) \right] \\
+ \eta(0) \int dq dp \varphi(q) \phi(p) m_{M,0} \\
= - \int_0^T dt \eta(t) \int dq dp \varphi(q) \phi(p) (\nabla_q \cdot \tilde{R} + \nabla_p \cdot \tilde{R}).
$$

Since the sums and products of functions of the form $\eta(t) \varphi(q) \phi(p)$ are dense in $C_0^\infty([0, T) \times \mathbb{R}^3 \times \mathbb{R}^3)$, we have showed that the limit of the subsequence is a weak solution of the Vlasov equation. On the other hand, the assumption $V \in W^{2,\infty}$ implies that the Vlasov equation has a unique weak solution. Therefore, the whole sequence $m_{N,t}$ converges weakly. Hence the proof of Theorem 1.1 is completed. 

\[\blacksquare\]
Appendices

A Second Quantization

The Fock space formalism and some results from Bogoliubov theorem for the proof of this paper are listed in the following. In particular, as in [9], we will introduce the fermionic Fock space over Hilbert spaces as the following direct sum:

\[ \mathcal{F}_a := \mathbb{C} \bigoplus_{n \geq 1} L^2_\alpha(\mathbb{R}^{3n}). \]

By convention, we say that the vacuum state, denoted as \( \Omega = \{1, 0, 0, \ldots \} \), belongs to \( \mathbb{C} \). For all \( \Psi = \{\psi^{(n)}\}_{n \in \mathbb{N}} \in \mathcal{F}_a \) and \( \psi^{(n)} \in L^2_\alpha(\mathbb{R}^{3n}) \), we define the number of particle operator on the \( n \)-th sector by \( (\mathcal{N}\Psi)^{(n)} = n\psi^{(n)} \).

As in [9], the creation and annihilation operators acting on \( \Psi \in \mathcal{F}_a \) is defined as follows: for any \( f \in L^2(\mathbb{R}^3) \)

\[
\begin{align*}
(a^*(f)\Psi)^{(n)}(x_1, \ldots, x_n) := & \sum_{j=1}^n \frac{(-1)^j}{\sqrt{n}} f(x_j)\psi^{(n-1)}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n), \\
(a(f)\Psi)^{(n)}(x_1, \ldots, x_n) := & \sqrt{n+1} \int dx f(x)\psi^{(n+1)}(x, x_1, \ldots, x_n),
\end{align*}
\]

where \( \psi^{(n)} \in L^2(\mathbb{R}^{3n}) \) for any \( n \in \mathbb{N} \). Additionally, for convenient purposes, the creation and annihilation operators will be represented by its operator-value distribution, \( a^*_x \) and \( a_x \) respectively, so that

\[
a^*(f) = \int dx f(x)a_x, \quad a(f) = \int dx \overline{f(x)}a_x.
\]

Therefore, the canonical anticommutator relation (CAR) is written as

\[
\{a^*_x, a_y\} = \delta_{x=y}, \quad \{a^*_x, a^*_y\} = \{a_x, a_y\} = 0,
\]

for any \( x, y \in \mathbb{R}^3 \).

Observe that for given any \( \Psi, \Phi \in \mathcal{F}_a \), it holds that

\[
\langle \Psi, \mathcal{N}\Phi \rangle = \int dx \langle a_x \Psi, a_x \Phi \rangle.
\]

Therefore, we write the number of particles operator as \( \mathcal{N} = \int dx a^*_x a_x \). Similarly, the integral kernel of the \( k \)-particle reduced density matrix is written as follows:

\[
\gamma^{(k)}(x_1, \ldots, x_k; y_1, \ldots, y_k) = \langle \Psi, a^*_{y_1} \cdots a^*_{y_k} a_{x_k} \cdots a_{x_1} \Psi \rangle. \tag{A.1}
\]

Moreover, the Hamiltonian acting on \( \Psi \in \mathcal{F}_a \) can be written as

\[
\mathcal{H}_N := \frac{\hbar^2}{2} \int dx \nabla_x a^*_x \nabla_x a_x + \frac{1}{2N} \int dx \int dy V(x - y)a^*_x a^*_y a_y a_x, \tag{A.2}
\]

\[^3\text{See [33] for more pedagogic treatment on the topics.}\]
where \( V \) is the interaction potential. We will denote the operator of the kinetic term as

\[
\mathcal{K} = \hbar^2 \int \, dx \nabla_x a_x^* \nabla_x a_x.
\]  

(A.3)

As presented in \([7, 42]\), for any \( t \geq 0 \), there exists a unitary transformation \( \mathcal{R}_{V_{N,t}} : \mathcal{F}_a \to \mathcal{F}_a \) such that

\[
\mathcal{R}_{V_{N,t}}^* a_x \mathcal{R}_{V_{N,t}} = a(u_{x,t}) + a^*(v_{x,t}),
\]

\[
\mathcal{R}_{V_{N,t}}^* a_x^* \mathcal{R}_{V_{N,t}} = a^*(u_{x,t}) + a(v_{x,t}),
\]

(A.4)

where \( v_{x,t} := \sum_{j=1}^{N} |e_{j,t}\rangle \langle e_{j,t}| \) and \( u_{x,t} := 1 - \sum_{j=1}^{N} |e_{j,t}\rangle \langle e_{j,t}| \), for any orthonormal basis \( \{e_{j,t}\}_{j=1}^{N} \subset L^2(\mathbb{R}^3) \).

Then, for \( t \geq 0 \), the solution of the Schrödinger equation is given as

\[
\Psi_{N,t} = e^{-\frac{i}{\hbar} H_{N,t}} \mathcal{R}_{V_{N,0}} \Omega = \mathcal{R}_{V_{N,t}} U_N(t; 0) \Omega,
\]

(A.5)

where \( \mathcal{R}_{V_{N,t}} \) is a unitary Bogoliubov mapping and \( U_N \) is the quantum fluctuation dynamics defined as follows,

\[
U_N(t; s) := R_{V_{N,t}}^* e^{-\frac{i}{\hbar} H_{N}(t-s)} R_{V_{N,s}}.
\]

(A.6)
B  A priori estimates

In this appendix, we present in this section a sequence of estimates from \cite{12} that will prove useful to our calculation. First, we have the following properties of \(k\)-particle Husimi measures from \cite[(12, Lemma 2.2)]{12}.

**Lemma B.1.** Let \(m_{N,t}^{(k)}(q,p,...,q_k,p_k)\) be the \(k\)-particle Husimi measure as defined in (1.5). Then, the following properties hold:

1. \(m_{N,t}^{(k)}(q,p,...,q_k,p_k)\) is symmetric,
2. \[\frac{1}{(2\pi \hbar)^3} \int dq dp \cdots dq dp m_{N,t}^{(k)}(q,p,...,q_k,p_k) = \frac{N(N-1)\cdots(N-k+1)}{N^k},\]
3. \[\frac{1}{(2\pi \hbar)^3} \int dq dp m_{N,t}^{(k)}(q,p,...,q_k,p_k) = (N-k+1)m_{N,t}^{(k-1)}(q,p,...,q_{k-1},p_{k-1}),\]
4. \(0 \leq m_{N,t}^{(k)}(q,p,...,q_k,p_k) \leq 1 \text{ a.e.},\)

where \(1 \leq k \leq N\).

From \cite[(12, Lemma 2.6)]{12} and \cite[(12, Proposition 2.3)]{12}, we have the following estimate for the kinetic energy as well as the moment estimate of the \(1\)-particle Husimi measure respectively:

**Lemma B.2.** Assume \(V \in W^{1,\infty}(\mathbb{R}^3)\), then the kinetic energy is bounded as follows:

\[
\langle \Psi_{N,t}, \frac{K_{N}}{N} \Psi_{N,t} \rangle \leq \langle \Psi_{N}, K\Psi_{N} \rangle + Ct^2, \tag{B.1}
\]

where \(K\) is defined in (A.3) and the constant \(C\) depends on \(\|\nabla V\|_{\infty}\).

**Proposition B.1.** For \(t \geq 0\), we have the following finite moments:

\[
\int dq dp (|q| + |p|^2)m_{N,t}(q,p) \leq C(1 + t^3), \tag{B.2}
\]

where \(C > 0\) is a constant that depends on initial data \(\int dq dp (|q| + |p|^2)m_{N}(q,p)\).

Next, we will present the oscillation estimate from \cite[(12, Lemma 2.5)]{12} which will be used frequently in our proof:

**Lemma B.3 (Bound on localized number operator).** Let \(\psi_{N} \in \mathcal{F}^{(N)}\) such that \(\|\psi_{N}\| = 1\), and \(R\) be the radius of a ball such that the volume is 1. Then we have

\[
\int dq dx \langle \psi_{N}, \chi_{|x-q| \leq \sqrt{\hbar} R^{\alpha} a_{x}^* a_{x} \Psi_{N} \rangle \leq C(R)\hbar^{-\frac{3}{2}},
\]

where \(\chi\) is a characteristic function.

**Lemma B.4 (Estimate of oscillation).** For \(\varphi \in C^\infty_0(\mathbb{R}^3)\) and

\[
\Omega_{h}^\alpha := \{ x \in \mathbb{R}^3 ; \max_{1 \leq j \leq 3} |x_j| \leq h^\alpha \}, \tag{B.3}
\]

it holds for every \(\alpha \in (0,1), s \in \mathbb{N}\), and \(x \in \mathbb{R}^3 \setminus \Omega_{h}^\alpha\),

\[
\left| \int_{\mathbb{R}^3} dp e^{ip \cdot x} \varphi(p) \right| \leq C h^{(1-\alpha)s}, \tag{B.4}
\]

where \(C\) depends on the compact support and the \(C^s\)-norm of \(\varphi\).
C Rest of Proof of Proposition 2.1

As each of the estimation of the terms in (2.22) is slightly different, for readers who needs more detail, the rest of the estimation will be presented here in this appendix:

\[ |A_2| = \left| \int dx_1dx_2dz_1dz_2 \, O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, \mathcal{U}_N^*(t; 0) a(\nabla_{t,x}) a(u_{t,z}) a^*(u_{t,z}) a(u_{t,z}) \mathcal{U}_N(t; 0)\xi_N \rangle \right| \]

\[ = \left| \int dx_1dx_2dz_1dz_2 \, \langle \xi_N, \mathcal{U}_N^*(t; 0) O_1(x_1; z_1)O_2(x_2; z_2) \int d\eta_1 d\eta_1' a_{\eta_1} a_{\eta_1'} v_t(\eta_1; x_1)u_t(\eta_1; z_1) \right| \]

\[ \left| \int d\eta_2 d\eta_2' a_{\eta_2} a_{\eta_2'} u_{\eta_2}(z_2)u_{\eta_2'}(z_2) \mathcal{U}_N(t; 0)\xi_N \rangle \right| \]

\[ = \left| \int dx_1dx_2dz_1dz_2 \, \langle \xi_N, \mathcal{U}_N^*(t; 0) \int d\eta_1 d\eta_1' a_{\eta_1} a_{\eta_1'} v_t(\eta_1; x_1)O_1(x_1; z_1)u_t(z_1; \eta_1') \right| \]

\[ \left| \int d\eta_2 d\eta_2' a_{\eta_2} a_{\eta_2'} u_{\eta_2}(z_2)O_2(x_2; z_2)u_t(z_2; \eta_2') \mathcal{U}_N(t; 0)\xi_N \rangle \right| \]

\[ = \left| \int d\eta_1 a_{\eta_1} (v_t O_1 u_t)(\eta_1; \eta_1') \int d\eta_2 d\eta_2' a_{\eta_2} a_{\eta_2'} (u_{t} O_2 u_t)(\eta_2; \eta_2') \mathcal{U}_N(t; 0)\xi_N \right| \]

\[ \leq \int d\eta_1 \| v_t O_1 u_t \|_2 \| a_{\eta_1}^* \mathcal{U}_N(t; 0)\xi_N \| || d\Gamma (u_t O_2 u_t) \mathcal{U}_N(t; 0)\xi_N \|
\]

\[ \leq \| v_t \|_{op} \| O_1 \|_{HS} \| u_t \|_{op} \| O_2 \|_{op} \| u_t \|_{op} \| \mathcal{U}_N(t; 0)\xi_N \| \| \mathcal{U}_N(t; 0)\xi_N \|
\]

\[ \leq \| O_1 \|_{HS} \| O_2 \|_{op} \| \mathcal{U}_N(t; 0)\xi_N \|
\]

\[ \leq \sqrt{N} \| O_1 \|_{HS} \| O_2 \|_{op} \| \mathcal{U}_N(t; 0)\xi_N \|
\]

\[ |A_3| = \int dx_1dx_2dz_1dz_2 \, O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, \mathcal{U}_N^*(t; 0) a(\nabla_{t,x}) a(u_{t,z}) a^*(u_{t,z}) a^*(\nabla_{t,z}) \mathcal{U}_N(t; 0)\xi_N \rangle \]

\[ \leq \| v_t \|_{op} \| u_t \|_{op} \| \mathcal{U}_N(t; 0)\xi_N \| \| O_1 \|_{HS} \| O_2 \|_{op} \| (N + 1) \mathcal{U}_N(t; 0)\xi_N \|
\]

\[ \leq \sqrt{N} \| O_1 \|_{HS} \| O_2 \|_{op} \| (N + 1) \mathcal{U}_N(t; 0)\xi_N \|
\]

\[ |A_4| = \int dx_1dx_2dz_1dz_2 \, O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, \mathcal{U}_N^*(t; 0) a^*(\nabla_{t,x}) a(\nabla_{t,x}) a^*(u_{t,z}) a(u_{t,z}) \mathcal{U}_N(t; 0)\xi_N \rangle \]
\[\begin{align*}
|A_5| &= \int dx_1 dx_2 dz_1 dz_2 \langle \xi, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N (N + 1) U_N(t; 0) \xi_N \\
&\leq \|O_1\|_{\text{HS}} \|O_2\|_{\text{op}} \langle \xi_N, \xi_N \rangle_N (N + 1) U_N(t; 0) \xi_N \\
&\leq N \|O_1\|_{\text{HS}} \|O_2\|_{\text{op}} (N + 1) U_N(t; 0) \xi_N. \\

|A_6| &= \int dx_1 dx_2 dz_1 dz_2 \langle \xi, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N (N + 1) U_N(t; 0) \xi_N \\
&\leq \|u_t O_1 u_t\|_{\text{op}} \|v_t O_2 u_t\|_{\text{HS}} \langle \xi_N, \xi_N \rangle_N (N + 1) U_N(t; 0) \xi_N \\
&\leq N \|O_1\|_{\text{HS}} \|O_2\|_{\text{op}} \|U_N(t; 0) \xi_N\|^2. \\

|A_7| &= \int dx_1 dx_2 dz_1 dz_2 \langle \xi, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N (N + 1) U_N(t; 0) \xi_N \\
&\leq \|u_t O_1 u_t\|_{\text{op}} \|v_t O_2 u_t\|_{\text{op}} \langle \xi_N, \xi_N \rangle_N (N + 1) U_N(t; 0) \xi_N \\
&\leq \|O_1\|_{\text{HS}} \|O_2\|_{\text{op}} \|U_N(t; 0) \xi_N\|^2. \\

|A_8| &= \int dx_1 dx_2 dz_1 dz_2 \langle \xi, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N (N + 1) U_N(t; 0) \xi_N \\
&\leq \|u_t O_1 v_t\|_{\text{HS}} \|v_t O_2 u_t\|_{\text{HS}} \langle \xi_N, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N (N + 1) U_N(t; 0) \xi_N \\
&\leq \sqrt{N} \|O_1\|_{\text{HS}} \|O_2\|_{\text{op}} \|U_N(t; 0) \xi_N\|^2. \\

|A_9| &= \int dx_1 dx_2 dz_1 dz_2 \langle \xi, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N (N + 1) U_N(t; 0) \xi_N \\
&\leq \|u_t O_1 v_t\|_{\text{HS}} \|v_t O_2 u_t\|_{\text{HS}} \langle \xi_N, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N \langle \xi_N, \xi_N \rangle_N (N + 1) U_N(t; 0) \xi_N \\
&\leq \sqrt{N} \|O_1\|_{\text{HS}} \|O_2\|_{\text{op}} \|U_N(t; 0) \xi_N\|^2.
\end{align*}\]
\[
\begin{align*}
|A_{10}| &= \left| \int dx_1 dx_2 dz_1 dz_2 \ O_1(x_1; z_1) O_2(x_2; z_2) \langle \xi_N, U_N^*(t; 0) a^* (u_{l,x_1}) a^* (u_{l,x_2}) U_N(t; 0) \xi_N \rangle \right| \\
&\leq \| \nabla \Omega_{l,x_1} \|_{\mathrm{HS}} \| \nabla \Omega_{l,x_2} \|_{\mathrm{op}} \langle \xi_N, U_N^*(t; 0) N U_N(t; 0) \xi_N \rangle^{\frac{1}{2}} \langle \xi_N, U_N^*(t; 0) N^2 U_N(t; 0) \xi_N \rangle^{\frac{1}{2}} \\
&\leq N \| O_1 \|_{\mathrm{HS}} \| O_2 \|_{\mathrm{op}} \| N U_N(t; 0) \xi_N \|. \\
|A_{11}| &= \left| \int dx_1 dx_2 dz_1 dz_2 \ O_1(x_1; z_1) O_2(x_2; z_2) \langle \xi_N, U_N^*(t; 0) a^* (u_{l,x_1}) a(u_{l,z_1}) a^* (u_{l,z_2}) U_N(t; 0) \xi_N \rangle \right| \\
&\leq \| u_{l} O_1 u_{l} \|_{\mathrm{op}} \| u_{l} O_2 u_{l} \|_{\mathrm{op}} \langle \xi_N, U_N^*(t; 0) N^2 U_N(t; 0) \xi_N \rangle \\
&\leq N \| O_1 \|_{\mathrm{HS}} \| O_2 \|_{\mathrm{op}} \| N U_N(t; 0) \xi_N \|. \\
|A_{12}| &= \left| \int dx_1 dx_2 dz_1 dz_2 \ O_1(x_1; z_1) O_2(x_2; z_2) \langle \xi_N, U_N^*(t; 0) a^* (u_{l,x_1}) a(u_{l,z_1}) a^* (u_{l,x_2}) U_N(t; 0) \xi_N \rangle \right| \\
&\leq \| u_{l} O_{l} u_{l} \|_{\mathrm{HS}} \| u_{l} O_{l} u_{l} \|_{\mathrm{op}} \langle \xi_N, U_N^*(t; 0) N U_N(t; 0) \xi_N \rangle \\
&\leq N \| O_1 \|_{\mathrm{HS}} \| O_2 \|_{\mathrm{op}} \langle \xi_N, U_N^*(t; 0) (N + 1) U_N(t; 0) \xi_N \rangle \\
&\leq N \| O_1 \|_{\mathrm{HS}} \| O_2 \|_{\mathrm{op}} \| (N + 1)^{\frac{1}{2}} U_N(t; 0) \xi_N \|. \\
|A_{13}| &= \left| \int dx_1 dx_2 dz_1 dz_2 \ O_1(x_1; z_1) O_2(x_2; z_2) \langle \xi_N, U_N^*(t; 0) a^* (u_{l,x_1}) a(u_{l,z_1}) a^* (u_{l,z_2}) U_N(t; 0) \xi_N \rangle \right| \\
&\leq \| u_{l} O_{l} u_{l} \|_{\mathrm{op}} \| u_{l} O_{l} u_{l} \|_{\mathrm{op}} \langle \xi_N, U_N^*(t; 0) N^2 U_N(t; 0) \xi_N \rangle \\
&\leq N \| O_1 \|_{\mathrm{HS}} \| O_2 \|_{\mathrm{op}} \| N U_N(t; 0) \xi_N \|. \\
\end{align*}
\]
\[ |A_{14}| = \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1, z_1)O_2(x_2, z_2)\langle \xi_N, \mathcal{U}_N(t; 0) a^*(\nabla_{t, z_1}) a(\nabla_{t, x_1}) a(\nabla_{t, x_2}) \mathcal{U}_N(t; 0) \xi_N \rangle \right| \\
\leq \left| \nabla_t \nabla_z O_1 \nabla_t \nabla_z O_2 \right|_{HS} \langle \xi_N, \mathcal{U}_N(t; 0) \mathcal{U}_N(t; 0) \xi_N \rangle^{\frac{1}{2}} \langle \xi_N, \mathcal{U}_N(t; 0) \mathcal{U}_N(t; 0) \xi_N \rangle^{\frac{1}{2}} \\
\leq \sqrt{N} \| O_1 \|_{HS} \| O_2 \|_{op} \langle \xi_N, \mathcal{U}_N(t; 0) \mathcal{U}_N(t; 0) \xi_N \rangle \\
\leq N \| O_1 \|_{HS} \| O_2 \|_{op} \| \mathcal{U}_N(t; 0) \xi_N \|. \\
\]

\[ |A_{15}| = \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1, z_1)O_2(x_2, z_2)\langle \xi_N, \mathcal{U}_N(t; 0) a^*(u_{t, z_1}) a^*(u_{t, x_1}) a^*(u_{t, x_2}) \mathcal{U}_N(t; 0) \xi_N \rangle \right| \\
\leq \| u_t O_1 \|_{HS} \| u_t O_2 u_t \|_{op} \langle \xi_N, \mathcal{U}_N(t; 0) \mathcal{U}_N(t; 0) \xi_N \rangle^{\frac{1}{2}} \langle \xi_N, \mathcal{U}_N(t; 0) \mathcal{U}_N(t; 0) \xi_N \rangle^{\frac{1}{2}} \\
\leq \| O_1 \|_{HS} \| O_2 \|_{op} \langle \xi_N, \mathcal{U}_N(t; 0) \mathcal{U}_N(t; 0) \xi_N \rangle \\
\leq N \| O_1 \|_{HS} \| O_2 \|_{op} \| \mathcal{U}_N(t; 0) \xi_N \|. \\
\]

Additionally, we have

\[ |B_2| = \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1, z_1)O_2(x_2, z_2)\langle \xi_N, \mathcal{U}_N(t; 0) a(u_{t, z_1}) a(u_{t, x_1}) \mathcal{U}_N(t; 0) \xi_N \rangle \right| \\
= \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1, z_1)O_2(x_2, z_2)\langle \xi_N, \mathcal{U}_N(t; 0) a(u_{t, x_1}) \mathcal{U}_N(t; 0) \xi_N \rangle \right| \\
= \left| \int dx_1 dx_2 dz_1 dz_2 \omega_{N, t}(x_1, z_1) \int d\eta d\eta' a_\eta a_{\eta'} \nabla_t(\eta; x_2 \nabla_t(\eta'; z_2) \mathcal{U}_N(t; 0) \xi_N) \right| \\
= \left| \int dx_1 dx_2 dz_1 dz_2 \mathcal{U}_N(t; 0), O_1(x_1, z_1) \omega_{N, t}(z_1; x_1) \int d\eta d\eta' a_\eta a_{\eta'} \nabla_t(\eta; x_2) O_2(x_2, z_2) u_t(z_2, \eta') \mathcal{U}_N(t; 0) \xi_N \right| \\
= \left| \int d\eta d\eta' \langle \xi_N, \mathcal{U}_N(t; 0), \left( \int dx_1 (O_1 \omega_{N, t})(x_1; x_1) \right) a_\eta a_{\eta'} (\nabla_t O_2 u_t)(\eta; \eta') \mathcal{U}_N(t; 0) \xi_N \rangle \right| \\
\]
\[ |B_3| = \|O_1 \omega_{N,t} \|_{\text{Tr}} \|v_t O_2 u_t \|_{HS} \| \mathcal{N}_{1/2} U_N(t; 0) \xi_N \| \leq \|O_1 \|_{HS} \|\omega_{N,t} \|_{HS} \|O_2 \|_{op} \| u_t \|_{op} \| \mathcal{N}_{1/2} U_N(t; 0) \xi_N \| \]

where we use the fact that

\[ |\text{Tr} O_1 \omega_{N,t}| \leq \|O_1 \omega_{N,t}\|_{\text{Tr}} \leq \|O_1\|_{HS} \|\omega_{N,t}\|_{HS} \leq \sqrt{N} \|O_1\|_{HS}. \]

\[ |B_4| = \|v_t O_1 u_t O_2 u_t \|_{HS} \| \mathcal{N}_{1/2} U_N(t; 0) \xi_N \| \leq \|O_1\|_{HS} \|O_2\|_{op} \| \mathcal{N}_{1/2} U_N(t; 0) \xi_N \|. \]

\[ |B_5| = \|v_t O_1 u_t O_2 \nabla_2 \|_{HS} \| \mathcal{N}_{1/2} U_N(t; 0) \xi_N \| \leq \|O_1\|_{HS} \|O_2\|_{op} \| \mathcal{N}_{1/2} U_N(t; 0) \xi_N \|. \]
\[ B_6 \]
\[ = \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, U_N^*(t; 0), \langle \nabla_{t,x_1}, \nabla_{t,z_1} \rangle a^*(u_{t,z_2})a(u_{t,z_2})U_N(t; 0)\xi_N \rangle \]
\[ = \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, U_N^*(t; 0), \omega_{N,t}(z_1; x_1) \rangle \]
\[ \leq \|O_1\|_{\text{HS}}\|\omega_{N,t}\|_{\text{HS}} \]
\[ \leq \sqrt{N}\|O_1\|_{\text{HS}}\|O_2\|_{\text{op}}. \]

\[ B_7 \]
\[ = \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, U_N^*(t; 0), \langle \nabla_{t,x_1}, \nabla_{t,z_1} \rangle a^*(u_{t,z_2})a^*(\nabla_{t,z_2})U_N(t; 0)\xi_N \rangle \]
\[ = \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, U_N^*(t; 0), \omega_{N,t}(z_1; x_1) \rangle \]
\[ \leq \|O_1\|_{\text{HS}}\|\omega_{N,t}\|_{\text{HS}}\|u_t O_2 u_t\|_{\text{op}}\|\xi_N\|\|N^{1/2}U_N(t; 0)\xi_N\| \]
\[ \leq \sqrt{N}\|O_1\|_{\text{HS}}\|O_2\|_{\text{op}}\|N^{1/2}U_N(t; 0)\xi_N\|. \]

\[ B_8 \]
\[ = \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, U_N^*(t; 0), \langle \nabla_{t,x_2}, \nabla_{t,z_2} \rangle a^*(u_{t,x_1})a(u_{t,z_2})U_N(t; 0)\xi_N \rangle \]
\[ = \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, U_N^*(t; 0), \omega_{N,t}(z_1; x_2) \rangle \]
\[
\int d\eta d\gamma' a^*_\gamma a_\gamma u_t(\eta; x_1)\overline{u_t(\eta'; z_2)}\mathcal{U}_N(t; 0)\xi_N \leq \|u_t O_1 \omega_N, t O_2 u_t\|_{op}\|\xi\|\|\mathcal{N}\mathcal{U}_N(t; 0)\xi_N\|
\]

\[
\leq \|O_1\|_{op}\|O_2\|_{op}\|\omega_N, t\|_{op}\|\xi\|\|\mathcal{N}\mathcal{U}_N(t; 0)\xi_N\|
\]

\[
\leq \|O_1\|_{HS}\|O_2\|_{op}\|\xi\|\|\mathcal{N}\mathcal{U}_N(t; 0)\xi_N\|
\]

\[
|B_9| = \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, \mathcal{U}_N^*(t; 0), (\nabla_{x_1, z_1}^t, \nabla_{z_1, x_1}) a^*(u_{t,x_1})a^*(\nabla_{t,z_2})\mathcal{U}_N(t; 0)\xi_N \rangle
\]

\[
= \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, \mathcal{U}_N^*(t; 0), \omega_N, t(z_1; x_2) \rangle
\]

\[
\int d\eta d\gamma' a^*_\gamma a_\gamma u_t(\eta; x_1)\overline{v_t(\eta'; z_2)}\mathcal{U}_N(t; 0)\xi_N \leq \|u_t O_1 \overline{\omega_N, t} O_2 \overline{v_t}\|_{HS}\|\xi_N\|\|\mathcal{N}^{1/2}\mathcal{U}_N(t; 0)\xi_N\|
\]

\[
\leq \|O_1\|_{HS}\|O_2\|_{op}\|\mathcal{N}^{1/2}\mathcal{U}_N(t; 0)\xi_N\|
\]

\[
|B_{10}| = \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, \mathcal{U}_N^*(t; 0), (u_{t,z_1}, u_{t,x_2}) a^*(u_{t,x_1})a(u_{t,z_2})\mathcal{U}_N(t; 0)\xi_N \rangle
\]

\[
= \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, \mathcal{U}_N^*(t; 0), u_t(z_1; x_2) \rangle
\]

\[
\int d\eta d\gamma' a^*_\gamma a_\gamma u_t(\eta; x_1)\overline{u_t(\eta'; z_2)}\mathcal{U}_N(t; 0)\xi_N \leq \|u_t O_1 u_t O_2 u_t\|_{HS}\|\xi_N\|\|\mathcal{N}^{1/2}\mathcal{U}_N(t; 0)\xi_N\|
\]

\[
\leq \|O_1\|_{HS}\|O_2\|_{op}\|\mathcal{N}^{1/2}\mathcal{U}_N(t; 0)\xi_N\|
\]

\[
|B_{11}| = \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, \mathcal{U}_N^*(t; 0), (u_{t,z_1}, u_{t,x_2}) a^*(u_{t,x_1})a^*(\nabla_{t,z_2})\mathcal{U}_N(t; 0)\xi_N \rangle
\]

\[
= \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1)O_2(x_2; z_2)\langle \xi_N, \mathcal{U}_N^*(t; 0), u_t(z_1; x_2) \rangle
\]

\[
\int d\eta d\gamma' a^*_\gamma a_\gamma u_t(\eta; x_1)\overline{v_t(\eta'; z_2)}\mathcal{U}_N(t; 0)\xi_N \leq \|u_t O_1 u_t O_2 v_t\|_{HS}\|\xi_N\|\|\mathcal{N}^{1/2}\mathcal{U}_N(t; 0)\xi_N\|
\]

\[
\leq \|O_1\|_{HS}\|O_2\|_{op}\|\mathcal{N}^{1/2}\mathcal{U}_N(t; 0)\xi_N\|
\]
\[|B_{12}|\]
\[
= \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1) O_2(x_2; z_2) \left( \xi_N, \mathcal{U}_N^*(t; 0), \langle \nabla_{t,x_2}, \nabla_{t,z_2} \rangle a^* (\nabla_{t,z_1}) a (\nabla_{t,x_1}) \mathcal{U}_N(t; 0) \xi_N \right) \right|
\]
\[
= \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1) O_2(x_2; z_2) \left( \xi_N, \mathcal{U}_N^*(t; 0), \omega_{N,t}(z_2; x_2) \right. \right.
\]
\[
\left. \left. \int d\eta d\eta' a^*_\eta a^* \eta' \overline{\nabla_t(\eta; z_1)} \overline{\nabla_t(\eta'; x_1)} \mathcal{U}_N(t; 0) \xi_N \right) \right|
\]
\[
\leq \|O_2\|_{\text{op}} \|\omega_{N,t}\|_{\text{Tr}} \left\| \overline{\nabla_t D_{1}^{-1} v_t} \right\|_{\text{op}} \|\xi_N\| \left\| \mathcal{N} \mathcal{U}_N(t; 0) \xi_N \right\|
\]
\[
\leq N \|O_1\|_{\text{HS}} \|O_2\|_{\text{op}} \left\| \mathcal{N} \mathcal{U}_N(t; 0) \xi_N \right\|.
\]

\[|B_{13}|\]
\[
= \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1) O_2(x_2; z_2) \left( \xi_N, \mathcal{U}_N^*(t; 0), \langle \nabla_{t,x_2}, \nabla_{t,z_2} \rangle a^* (\nabla_{t,z_1}) a (\nabla_{t,x_1}) \mathcal{U}_N(t; 0) \xi_N \right) \right|
\]
\[
= \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1) O_2(x_2; z_2) \left( \xi_N, \mathcal{U}_N^*(t; 0), \omega_{N,t}(z_1; x_2) \right. \right.
\]
\[
\left. \left. \int d\eta d\eta' a^*_\eta a^* \eta' \overline{\nabla_t(\eta; z_2)} \overline{\nabla_t(\eta'; x_1)} \mathcal{U}_N(t; 0) \xi_N \right) \right|
\]
\[
\leq \left\| \overline{\nabla_t D_{2}^{-1} \omega_{N,t}} \overline{D_{1}^{-1} v_t} \right\|_{\text{HS}} \|\xi_N\| \left\| \mathcal{N}^{1/2} \mathcal{U}_N(t; 0) \xi_N \right\|
\]
\[
\leq \|O_1\|_{\text{HS}} \|O_2\|_{\text{op}} \left\| \mathcal{N}^{1/2} \mathcal{U}_N(t; 0) \xi_N \right\|.
\]

\[|B_{14}|\]
\[
= \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1) O_2(x_2; z_2) \left( \xi_N, \mathcal{U}_N^*(t; 0), \langle \nabla_{t,x_2}, \nabla_{t,z_2} \rangle a^* (u_{t,x_1}) a (u_{t,z_1}) \xi_N \right) \right|
\]
\[
= \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1) O_2(x_2; z_2) \left( \xi_N, \mathcal{U}_N^*(t; 0), \omega_{N,t}(z_2; x_2) \right. \right.
\]
\[
\left. \left. \int d\eta d\eta' a^*_\eta a^* \eta' u_t(\eta; x_1) \overline{u_t(\eta)} \mathcal{U}_N(t; 0) \xi_N \right) \right|
\]
\[
\leq \left\| \overline{\nabla_t D_{2}^{-1} \omega_{N,t}} \overline{D_{1}^{-1} v_t} \right\|_{\text{HS}} \|\xi_N\| \left\| \mathcal{N}^{1/2} \mathcal{U}_N(t; 0) \xi_N \right\|
\]
\[
\leq \|O_1\|_{\text{HS}} \|O_2\|_{\text{op}} \left\| \mathcal{N}^{1/2} \mathcal{U}_N(t; 0) \xi_N \right\|.
\]

\[|B_{15}|\]
\[
= \left| \int dx_1 dx_2 dz_1 dz_2 O_1(x_1; z_1) O_2(x_2; z_2) \left( \xi_N, \mathcal{U}_N^*(t; 0), \langle \nabla_{t,x_2}, \nabla_{t,z_2} \rangle a^* (\nabla_{t,z_1}) a^* (u_{t,x_1}) \mathcal{U}_N(t; 0) \xi_N \right) \right|
\]
\begin{align*}
&= \left| \int dx_1 dx_2 dz_1 dz_2 \ O_1(x_1; z_1) O_2(x_2; z_2) \langle \xi_N, U_N^*(t; 0), \omega_{N,t}(z_2; x_2) \rangle \\
&\quad \int d\eta d\eta' a_\eta^* a_{\eta'} \overline{v_t(\eta; z_1) u_t(x_1; \eta')} \mathcal{U}_N(t; 0) \xi_N \right| \\
&\leq \|O_2 \omega_{N,t}\|_{\mathcal{T}} \|\mathcal{N} \xi \mathcal{U}_N(t; 0) \xi_N\| \\
&\leq \|O_1\|_{\mathcal{H}S} \|\omega_{N,t}\|_{\mathcal{H}S} \|v_t\|_{\mathcal{H}S} \|O_2\|_{\text{op}} \|\mathcal{N} \xi \mathcal{U}_N(t; 0) \xi_N\| \\
&\leq N \|O_1\|_{\mathcal{H}S} \|O_2\|_{\text{op}} \|\mathcal{N} \xi \mathcal{U}_N(t; 0) \xi_N\|.
\end{align*}

\begin{align*}
&|B_{16}| \\
&= \left| \int dx_1 dx_2 dz_1 dz_2 \ O_1(x_1; z_1) O_2(x_2; z_2) \langle \xi_N, U_N^*(t; 0), \langle \nabla_{t,x_1}, \nabla_{t,z_1}\rangle a^*(\nabla_{t,z_2}) a(\nabla_{t,x_2}) \mathcal{U}_N(t; 0) \xi_N \rangle \\
&\quad \int d\eta d\eta' a_\eta^* a_{\eta'} \overline{v_t(\eta; z_1; x_1)} \mathcal{U}_N(t; 0) \xi_N \right| \\
&\leq \|O_1\|_{\mathcal{H}S} \|\omega_{N,t}\|_{\mathcal{H}S} \|\mathcal{N} \xi \mathcal{U}_N(t; 0) \xi_N\| \\
&\leq \sqrt{N} \|O_1\|_{\mathcal{H}S} \|O_2\|_{\text{op}} \|\mathcal{N} \xi \mathcal{U}_N(t; 0) \xi_N\|.
\end{align*}
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