Some $L^p$ rigidity results for complete manifolds with harmonic curvature

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Abstract

Let $(M^n, g)(n \geq 3)$ be an $n$-dimensional complete Riemannian manifold with harmonic curvature and positive Yamabe constant. Denote by $R$ and $\hat{Rm}$ the scalar curvature and the trace-free Riemannian curvature tensor of $M$, respectively. The main result of this paper states that $\hat{Rm}$ goes to zero uniformly at infinity if for $p \geq \frac{n}{2}$, the $L^p$-norm of $\hat{Rm}$ is finite. Moreover, if $R$ is positive, then $(M^n, g)$ is compact. As applications, we prove that $(M^n, g)$ is isometric to a spherical space form if for $p \geq \frac{n}{2}$, $R$ is positive and the $L^p$-norm of $\hat{Rm}$ is pinched in $[0, C_1)$, where $C_1$ is an explicit positive constant depending only on $n, p, R$ and the Yamabe constant.

In particular, we prove an $L^p(\frac{n}{2} \leq p < \frac{n-2}{2}(1+\sqrt{1-\frac{4}{n}}))$-norm of $\hat{Ric}$ pinching theorem for complete, simply connected, locally conformally flat Riemannian $n(n \geq 6)$-manifolds with constant negative scalar curvature.

We give an isolation theorem of the trace-free Ricci curvature tensor of compact locally conformally flat Riemannian $n$-manifolds with constant positive scalar curvature, which improves Theorem 1.1 and Corollary 1 of E. Hebey and M. Vaugon [18]. This result is sharped, and we can precisely characterize the case of equality.

Keywords: Harmonic curvature, trace-free curvature tensor, constant curvature space
MSC 53C21, 53C20

1. Introduction and main results

Recall that an $n$-dimensional Riemannian manifold $(M^n, g)$ is said to be a manifold with harmonic curvature if the divergence of its Riemannian curvature tensor $Rm$ vanishes, i.e., $\delta Rm = 0$. In view of the second Bianchi identity, we
know that $M$ has harmonic curvature if and only if the Ricci tensor of $M$ is a Codazzi tensor. When $n \geq 3$, by the Bianchi identity, the scalar curvature is constant. Thus, every Riemannian manifold with parallel Ricci tensor has harmonic curvature. Moreover, the constant curvature spaces, Einstein manifolds and the locally conformally flat manifolds with constant scalar curvature are also important examples of manifolds with harmonic curvature, however, the converse does not hold (see [2], for example). According to the decomposition of the Riemannian curvature tensor, the metric with harmonic curvature is a natural candidate for this study since one of the important problems in Riemannian geometry is to understand classes of metrics that are, in some sense, close to being Einstein or having constant curvature. The another reason for this study on the metric with harmonic curvature is the fact that a Riemannian manifold has harmonic curvature if and only if the Riemannian connection is a solution of the Yang-Mills equations on the tangent bundle [4]. In recent years, the complete manifolds with harmonic curvature have been studied in literature (e.g., [3, 8, 14, 15, 18, 20, 21, 22, 26, 27, 28]). In particular, G. Tian and J. Viaclovsky [28], and X. Chen and B. Weber [8] have obtained $\epsilon$-rigidity results for critical metric which relies on a Sobolev inequality and a integral bounds on the curvature in dimension 4 and in higher dimension, respectively. The curvature pinching phenomenon plays an important role in global differential geometry. We are interested in $L^p$ pinching problems for complete Riemannian manifold with harmonic curvature.

We now introduce the definition of the Yamabe constant. Given a complete Riemannian $n$-manifold $M$, the Yamabe constant $Q(M)$ is defined by

$$Q(M) = \inf_{0 \neq u \in C_0^\infty(M)} \frac{\int_M (|\nabla u|^2 + \frac{(n-2)}{4(n-1)} Ru^2)}{\left(\int_M |u|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}}$$

where $R$ is the scalar curvature of $M$. The important works of Schoen, Trudinger and Yamabe showed that the infimum in the above on compact manifolds is always achieved (see [1, 25]). There are complete noncompact Riemannian manifolds of negative scalar curvature with positive Yamabe constant. For example, any simply connected complete locally conformally flat manifold has positive Yamabe constant [24], and $Q(M)$ is always positive if $R$ vanishes [12]. In contrast with the noncompact case, the Yamabe constant of a given compact manifold is determined by the sign of scalar curvature [1].

Throughout this paper, we always assume that $M$ is an $n$-dimensional complete Riemannian manifold with $n \geq 3$. In this note, we obtain the following rigidity theorems.

**Theorem 1.1.** Let $M$ be a complete noncompact Riemannian $n$-manifold with harmonic curvature. Assume that $M$ has the positive Yamabe constant or satisfies the Sobolev inequality

$$\left(\int_M |f|^\frac{2n}{n-2}\right)^{\frac{n-2}{n}} \leq CS \int_M |\nabla f|^2, \forall f \in C_0^\infty(M).$$

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For $p \geq \frac{n}{2}$, if $\int_M |\bar{Rm}|^p < +\infty$, then, given any $\epsilon > 0$ and any $x_0 \in M$ there exists a geodesic ball $B_r(x_0)$ with center $x_0$ and radius $r$ such that $|\bar{Rm}|(x) < \epsilon$ for all $x \in M \setminus B_r(x_0)$.

**Theorem 1.2.** Let $M$ be a complete Riemannian $n$-manifold with harmonic curvature and positive scalar curvature. Assume that $M$ has the positive Yamabe constant. For $p \geq \frac{n}{2}$, if $\int_M |\bar{Rm}|^p < +\infty$, then $M$ must be compact.

**Corollary 1.3.** Let $M$ be a complete noncompact Riemannian $n$-manifold with harmonic curvature and nonnegative scalar curvature. Assume that $M$ has the positive Yamabe constant. For $p \geq \frac{n}{2}$, if $\int_M |\bar{Rm}|^p < +\infty$, then $M$ must be scalar flat.

**Theorem 1.4.** Let $M$ be a complete Riemannian $n$-manifold with harmonic curvature and positive scalar curvature $R$. Assume that $M$ has the positive Yamabe constant. For $p \geq \frac{n}{2}$, if

$$
\left( \int_M |\bar{Rm}|^p \right)^{\frac{1}{p}} < C_1,
$$

where

$$
C_1 = \begin{cases} 
\frac{Q(M)}{C(n)} & 3 \leq n \leq 5 \text{ and } p = \frac{n}{2} \\
\frac{2(6-n)pR}{4(n-1)(2p-n)C(n)} \left( \frac{4(4-n)pQ(M)}{(6-n)nR} \right)^{\frac{1}{p}} & 3 \leq n \leq 5 \text{ and } \frac{n}{2} < p < \frac{2n}{n-2} \\
\frac{R}{(n-1)C(n)} \left( \frac{4(4-n)pQ(M)}{(n-2)R} \right)^{\frac{1}{p}} & 3 \leq n \leq 5 \text{ and } p \geq \frac{2n}{n-2}, \text{ and } n \geq 6,
\end{cases}
$$

and $C(n)$ is defined in Lemma 2.1, then $M$ is isometric to a spherical space form.

**Remark 1.5.** Some $L^2$ and $L^n$ trace-free Riemannian curvature pinching theorems have been shown by Kim [21], Chu [5], and Fu etc. [14], in which the constant $C$ is not explicit, respectively. When $p = \frac{n}{2}$ or $p = n$, the constant $C_1$ in Theorem 1.4 satisfies

$$
C_1 = \begin{cases} 
\frac{Q(M)}{C(n)} & 3 \leq n \leq 5 \\
\frac{4Q(M)}{(n-2)C(n)} & n \geq 6,
\end{cases}
$$

or

$$
C_1 = \begin{cases} 
\frac{\sqrt{3Q(M)}R}{\sqrt{2C(3)}} & n = 3 \\
\frac{1}{\sqrt{C(n)}} & n \geq 4.
\end{cases}
$$

For compact Einstein manifolds or compact conformally flat manifolds, there are some corresponding results in [7, 16, 18, 20, 26].

**Theorem 1.6.** Let $M^n (n \geq 10)$ be a complete Riemannian $n$-manifold with harmonic curvature and negative scalar scalar curvature $R$. Assume that $M$ has the positive Yamabe constant. If

$$
\int_M |\bar{Rm}|^\gamma < \infty, \text{ for some } \gamma \in \left(0, \frac{n(n-2) + \sqrt{n(n-2)(n^2 - 10n + 8)}}{4(n-1)}\right).
$$
Then there exists a small number $C$ such that if for $p \geq \frac{n}{2}$,

$$\int_M |\hat{Rm}|^p < C,$$

then $M$ is a hyperbolic space form.

**Remark 1.7.** Theorem 1.6 can be considered as a generalization of Theorem 1.4 in [14] and some results in [21].

When $M$ is a complete, simply connected, locally conformally flat Riemannian $n$-manifold, $M$ satisfies the Sobolev inequality (see Corollary 3.2 in [17]). Based on (9), using the same argument as in the proofs of Theorems 1.4 and 1.6, we generalize the result due to [27] and [15].

**Theorem 1.8.** Let $M$ be a complete, simply connected, locally conformally flat Riemannian $n$-manifold with constant positive scalar curvature. For $p \geq \frac{n}{2}$, if

$$\left( \int_M |\hat{\text{Ric}}|^p \right)^{\frac{1}{p}} < C_2,$$

where

$$C_2 = \begin{cases} \frac{3\sqrt{6}}{4} \omega_3, & n = 3 \text{ and } p = \frac{3}{2} \\ \omega_3 \left( \frac{6(2p-3)}{n} \right)^{\frac{1}{p}} \omega_3^{\frac{1}{p}}, & n = 3 \text{ and } \frac{3}{2} < p < 2 \\ \left( \frac{n(n-1)}{R} \right)^{\frac{1}{p}} \frac{R}{\sqrt{n(n-1)}} \omega_3, & n = 3 \text{ and } p \geq 2, \text{ and } n \geq 4, \end{cases}$$

then $M$ is isometric to a sphere.

**Remark 1.9.** Theorem 1.8 improves the $L^{\frac{2n}{2n-3}}(n \geq 6)$ trace-free Ricci curvature pinching theorem given by [27] in dimension. The pinching constant in Theorem 1.8 is better than the one in the $L^n$ trace-free Ricci curvature pinching theorem given by [27].

**Theorem 1.10.** Let $M^n(n \geq 5)$ be a complete, simply connected, locally conformally flat Riemannian $n$-manifold with constant negative scalar curvature. Assume that

$$\int_M |\hat{\text{Ric}}|^{\gamma} < \infty, \text{ for some } \gamma \in (0, \frac{n-2}{2}(1 + \sqrt{1 - \frac{4}{n}})).$$

Then there exists a small number $C$ such that if for $p \geq \frac{n}{2}$,

$$\int_M |\hat{\text{Ric}}|^p < C,$$

then $M$ is a hyperbolic space.
Corollary 1.11. Let $M^n (n \geq 6)$ be a complete, simply connected, locally conformally flat Riemannian $n$-manifold with constant negative scalar curvature. For some $p \in \left[\frac{n}{2}, \frac{n-2}{2}(1 + \sqrt{1 - \frac{4}{n}})\right)$, there exists a small number $C$ such that if

$$\left( \int_M |\text{Ric}|^p \right)^{\frac{1}{p}} < C,$$

then $M$ is a hyperbolic space. In particular, when $p = \frac{n}{2}$, there exists an explicit positive constant $C = \sqrt{n(n-1)\omega_n^2}$.

Remark 1.12. Theorems 1.8 and 1.9 improve the corresponding one in \cite{15, 27}. Corollary 1.11 has been proved in \cite{15}.

Using the same argument as in the proof of Theorem 1.8, we obtain

Theorem 1.13. Let $M$ be a compact locally conformally flat Riemannian $n$-manifold with constant positive scalar curvature. For $p \geq \frac{n}{2}$, if

$$\left( \int_M |\text{Ric}|^p \right)^{\frac{1}{p}} < C_3,$$

where

$$C_3 = \begin{cases} \sqrt{6Q(M)}, & n = 3 \text{ and } p = \frac{3}{2} \\ \left(\frac{8Q(M)}{(n-3)(n-2)}\right)^{\frac{1}{2}}, & n = 3 \text{ and } \frac{3}{2} < p < 2 \\ \left(\frac{8Q(M))}{(n-3)(n-2)}\right)^{\frac{1}{2}}, & n = 3 \text{ and } p \geq 2, \text{ and } n \geq 4, \end{cases}$$

then $M$ is isometric to a sphere form.

Remark 1.14. When $n = 3$ and $p \geq 2$, or $n \geq 4$, the inequality of this theorem is optimal. The critical case is given by the following example. If $(S^1(t) \times S^{n-1}, g_t)$ is the product of the circle of radius $t$ with $S^{n-1}$, and if $g_t$ is the standard product metric normalized such that $\text{Vol}(g_t) = 1$, we have $W = 0$, $g_t$ is a Yamabe metric for small $t$ (see \cite{22}), and

$$\left( \int_M |\text{Ric}|^p \right)^{\frac{1}{p}} = \frac{R(g_t)}{\sqrt{n(n-1)}},$$

which is the critical case of the inequality in Theorem 1.13. We know that $(S^1(t) \times S^{n-1}, g_t)$ is not Einstein.

When $n \geq 5$ and $p = \frac{n}{2}$, Theorem 1.13 reduces to Theorem 1.1 in \cite{18}. Theorem 1.13 improves Theorem 1.1 and Corollary 1 in \cite{18}.

As we mentioned above, Theorem 1.13 is sharped. By this we mean that we can precisely characterize the case of equality:

Theorem 1.15. Let $M^n (n \geq 4)$ be a compact locally conformally flat Riemannian $n$-manifold with constant positive scalar curvature. For $p \geq \frac{n}{2}$, if

$$\left( \int_M |\text{Ric}|^p \right)^{\frac{1}{p}} = C_3,$$
then i) $M$ is covered isometrically by $S^1 \times S^{n-1}$ with the product metric;
ii) $M$ is covered isometrically by $(S^1 \times S^{n-1}, dt^2 + F^2(t)g_{S^{n-1}})$, where $(S^{n-1}, g_{S^{n-1}})$ is a round sphere and $F$ is a non-constant, positive, periodic function satisfying a precise ODE. This metric is called a rotationally symmetric Derdziński metric in [6, 12].

2. Proof of Lemmas

In what follows, we adopt, without further comment, the moving frame notation with respect to a chosen local orthonormal frame.

Let $M$ be a Riemannian manifold with harmonic curvature. The decomposition of the Riemannian curvature tensor into irreducible components yield

\[ R_{ijkl} = W_{ijkl} + \frac{1}{n-2}(R_{ik}\delta_{jl} - R_{il}\delta_{jk} + R_{jl}\delta_{ik} - R_{jk}\delta_{il}) \]

\[ + \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \]

where $R_{ijkl}$, $W_{ijkl}$, $R_{ij}$ and $\check{R}_{ij}$ denote the components of $Rm$, the Weyl curvature tensor $W$, the Ricci tensor $Ric$ and the trace-free Ricci tensor $\check{Ric} = Ric - \frac{R}{n}g$, respectively, and $R$ is the scalar curvature.

The trace-free Riemannian curvature tensor $\check{R}m$ is

\[ \check{R}_{ijkl} = R_{ijkl} - \frac{R}{n(n-1)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \]  

Then the following equalities are easily obtained from the properties of curvature tensor:

\[ g^{ik}\check{R}_{ijkl} = \check{R}_{jl}, \]  

\[ \check{R}_{ijkl} + \check{R}_{iljk} + \check{R}_{iklj} = 0, \]  

\[ \check{R}_{ijkl} = \check{R}_{klji} = -\check{R}_{jikl} = -\check{R}_{ijk}, \]

\[ |\check{R}m|^2 = |W|^2 + \frac{4}{n-2}|\check{Ric}|^2. \]  

Moreover, by the assumption of harmonic curvature, we compute

\[ \check{R}_{ijkl,m} + \check{R}_{ijm,k} + \check{R}_{ijm,k} = 0, \]
and
\[ \tilde{R}_{ijkl,l} = 0. \]  
(7)

Now, we compute the Laplacian of \( |\tilde{R}m|^2 \).

**Lemma 2.1.** Let \( M \) be a complete Riemannian \( n \)-manifold with harmonic curvature. Then
\[ \triangle |\tilde{R}m|^2 \geq 2|\nabla \tilde{R}m|^2 - 2C(n)|\tilde{R}m|^3 + 2AR|\tilde{R}m|^2, \]  
(8)
where
\[ A = \begin{cases} \frac{1}{n-1}, & R \geq 0 \\ \frac{1}{2}, & R < 0, \end{cases} \]
and \( C(n) = 2[\frac{2(n^2 + n - 4)}{n(n-1)(n+1)(n+2)} + \frac{n^2 - n - 4}{2(n-2)(n-1)n(n+1)} + \frac{(n-2)(n-1)}{4n}] \).

**Remark 2.2.** Although Lemma 2.1 has been proved in [9], we give an explicit coefficient of the term \( |\tilde{R}m|^3 \) in (8). When \( M \) is a complete locally conformally flat Riemannian \( n \)-manifold, it follows from (10) that
\[ \triangle |\tilde{Ric}|^2 \geq 2|\nabla \tilde{Ric}|^2 - 2nR \frac{2}{n(n-1)}|\tilde{Ric}|^3 + 2R \frac{n}{n-1} |\tilde{Ric}|^2. \]

By the Kato inequality \( |\nabla \tilde{Ric}|^2 \geq \frac{n+2}{n}|\nabla |\tilde{Ric}||^2 \), we obtain (see [22, 27])
\[ |\tilde{Ric}| \triangle |\tilde{Ric}| \geq 2n \frac{2}{n}|\nabla |\tilde{Ric}||^2 - 2nR \frac{2}{n(n-1)}|\tilde{Ric}|^3 + R \frac{n}{n-1} |\tilde{Ric}|^2. \]  
(9)
Proof. By the Ricci identities, we obtain from (1)-(7)

\[ \triangle |\tilde{R}m|^2 = 2|\nabla \tilde{R}m|^2 + 2 \langle \tilde{R}m, \triangle \tilde{R}m \rangle = 2|\nabla \tilde{R}m|^2 + 2 \tilde{R}_{ijkl} \tilde{R}_{ijkl, mm} \]

\[ = 2|\nabla \tilde{R}m|^2 + 2 \tilde{R}_{ijkl}(\tilde{R}_{ijkl, lm} + \tilde{R}_{ijkl, km}) \]

\[ = 2|\nabla \tilde{R}m|^2 + 4 \tilde{R}_{ijkl} \tilde{R}_{ijkl, lm} \]

\[ = 2|\nabla \tilde{R}m|^2 + 4 \tilde{R}_{ijkl}(\tilde{R}_{ijkl, ml} + \tilde{R}_{hlkm} \tilde{R}_{hilm} + \tilde{R}_{ijkl, hlm}) \]

\[ = 2|\nabla \tilde{R}m|^2 + 4 \tilde{R}_{ijkl}(\tilde{R}_{hjkm} \tilde{R}_{hilm} + \tilde{R}_{ihkm} \tilde{R}_{hjlm} + \tilde{R}_{ijkl} hilm) \]

\[ + \tilde{R}_{ijkl} \tilde{R}_{hl} - \frac{8R}{n(n-1)} |\text{Ric}|^2 + \frac{4R}{n} |\tilde{R}m|^2 \]

\[ = 2|\nabla \tilde{R}m|^2 + 4 \tilde{R}_{ijkl}(2 \tilde{R}_{hjkm} \tilde{R}_{hilm} + \frac{1}{2} \tilde{R}_{hjkm} \tilde{R}_{hilm}) \]

\[ + \tilde{R}_{ijkl} \tilde{R}_{hl} - \frac{8R}{n(n-1)} |\text{Ric}|^2 + \frac{4R}{n} |\tilde{R}m|^2. \] (10)

We consider \( \tilde{R}m \) as a self adjoint operator on \( \wedge^2 V \) and \( S^2 V \). By the algebraic inequality for \( m \)-trace-free symmetric two-tensors \( T \), i.e., \( tr(T^3) \leq \frac{m^2}{m(m-1)} |T|^3 \), and the eigenvalues \( \lambda_i \) of \( T \) satisfy \( |\lambda_i| \leq \sqrt{\frac{m-1}{m}} |T| \) in [19], we obtain

\[ |\tilde{R}_{ijkl}(2 \tilde{R}_{hjkm} \tilde{R}_{hilm} + \frac{1}{2} \tilde{R}_{hjkm} \tilde{R}_{hilm})| \leq 2 |\tilde{R}_{ijkl} \tilde{R}_{hjkm} \tilde{R}_{hilm}| + \frac{1}{2} |\tilde{R}_{ijkl} \tilde{R}_{hjkm} \tilde{R}_{hilm}| \]

\[ \leq \left[ \frac{2(n^2 + n - 4)}{\sqrt{(n-1)(n+1)(n+2)}} + \frac{n^2 - n - 4}{2 \sqrt{(n-2)(n-1)n(n+1)}} \right] |\tilde{R}m|^3, \] (11)

and

\[ |\tilde{R}_{ijkl} \tilde{R}_{ijkl} \tilde{R}_{hl}| \leq \sqrt{\frac{n-1}{n}} |\text{Ric}| |\tilde{R}m|^2. \] (12)

From (5), we have

\[ |\text{Ric}|^2 \leq \frac{n-2}{4} |\tilde{R}m|^2. \] (13)
Combining with (10)-(13), we obtain that
\[
\|\hat{\nabla} R_m\|^2 + 2\|\hat{\nabla} R_m\|^2 - 4\sqrt{\frac{(n-2)(n-1)}{4n}}
\]
\[
+ \frac{2(n^2+n-4)}{\sqrt{(n-1)n(n+1)(n+2)}} + \frac{n^2-n-4}{2\sqrt{(n-2)(n-1)n(n+1)}} |\hat{R}_m|^3.
\]
This completes the proof of this Lemma. 

**Lemma 2.3.** Let $M$ be a complete noncompact Riemannian $n$-manifold satisfying a Sobolev inequality of the following form:

\[
\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq D_1 \int_M |\nabla f|^2 + F_1 \int_M |f|^2, \forall f \in C^\infty_0(M).
\]

(14)

If a non-negative function $u \in C^\infty(M)$ satisfies $\int_M u^\frac{n}{n-2} < +\infty$ and

\[
\Delta u \geq au^2 + bu
\]

for some constants $a$ and $b$. Then, given any $\epsilon > 0$ and any $x_0 \in M$ there exists a geodesic ball $B_r(x_0)$ with center $x_0$ and radius $r$ such that $u(x) < \epsilon$ for all $x \in M \setminus B_r(x_0)$.

**Remark 2.4.** We can carry out the proof of Lemma 2.3 by suitable modification to the proof of Theorem 1.1 in [3].

**Proof.** Let us fix a point $x_0 \in M$, consider the following open domains:

\[
E(T) = \{x \in M | d(x_0, x) > T\}, A(T, S) = \{x \in M | T < d(x_0, x) < S\}.
\]

First, take a cut-off function $\varphi \in C^\infty_0(M)$ with the properties $\text{Supp}(\varphi) \subset A(r, s + 2) \subset E(r), r + 2 < s$ and $\varphi = 1$ on $A(r + 2, s)$. Using Cauchy-Schwarz inequality, multiplying (15) by $\varphi^2 u^\frac{n}{n-1}$ and integrating, we have

\[
\frac{8(n-2)}{n^2} \int_M \varphi^2 |\nabla u^\frac{n}{n-2}|^2 \leq -a \int_M \varphi^2 u^\frac{n}{n-2} - b \int_M \varphi^2 u^\frac{n}{n-2}
\]

\[
+ \frac{4}{n^2} \int_M \varphi^2 |\nabla u^\frac{n}{n-2}| + 4 \int_M u^\frac{n}{n-2} |\nabla \varphi|^2,
\]

which gives

\[
\int_M \varphi^2 |\nabla u^\frac{n}{n-2}|^2 \leq -na \int_M \varphi^2 u^\frac{n}{n-2} - n(b-4) \int_{\text{Supp}(\varphi)} \varphi^2 u^\frac{n}{n-2}.
\]

(16)

We now apply Sobolev inequality (14) to the function $f = \varphi u^\frac{n}{n-2}$ and obtain

\[
\left(\int_M |\varphi u^\frac{n}{n-2}|^{\frac{n-2}{n}}\right)^{\frac{n-2}{n}} \leq 2D_1 \int_M \varphi^2 |\nabla u^\frac{n}{n-2}|^2 + 2D_1 \int_M u^\frac{n}{n-2} |\nabla \varphi|^2 + F_1 \int_M \varphi^2 u^\frac{n}{n-2}.
\]

(17)
Plugging (16) into (17), we obtain
\[
\left(\int_M \varphi^{\frac{2n}{n-2}} u \frac{\varphi^2 n}{n-2} \right)^\frac{n-2}{n} \leq -2naD_1 \int_M \varphi^2 u^{\frac{n+1}{2}} + F_2 \int_{\text{Supp}(\varphi)} u^\frac{n}{2}.
\] (18)

Using Hölder inequality, we find that
\[
\int_M \varphi^2 u^{\frac{n+1}{2}} \leq \left(\int_{\text{Supp}(\varphi)} u^\frac{n}{2}\right)^\frac{2}{n} \left(\int_M \varphi^{\frac{2n}{n-2}} u \frac{\varphi^2 n}{n-2} \right)^\frac{n-2}{n} \leq -2naD_1 \int_M \varphi^2 u^{\frac{n+1}{2}} + F_2 \int_{\text{Supp}(\varphi)} u^\frac{n}{2}.
\] (19)

and we deduce from (18) that
\[
\left(1 + 2naD_1 \left(\int_{\text{Supp}(\varphi)} u^\frac{n}{2}\right)^\frac{2}{n} \right) \left(\int_M \varphi^{\frac{2n}{n-2}} u \frac{\varphi^2 n}{n-2} \right)^\frac{n-2}{n} \leq F_2 \int_{\text{Supp}(\varphi)} u^\frac{n}{2}.
\] (20)

If we choose \(r\) sufficiently big such that \(1 + 2naD_1 \left(\int_{\text{Supp}(\varphi)} u^\frac{n}{2}\right)^\frac{2}{n} \geq \frac{1}{2}\), then we get
\[
\left(\int_{E(r+2)} u^\frac{n}{2(n-2)} \right)^\frac{2(n-2)}{n} \leq F_3 \left(\int_{E(r)} u^\frac{n}{2}\right)^\frac{2}{n}.
\] (21)

Second, we multiply (15) by \(\varphi^2 u^{k-1}\), for \(k \geq \frac{3}{2}\), and integrate to find
\[
\frac{4(k-1)}{k^2} \int_M \varphi^2 \left|\nabla u^\frac{k}{2}\right|^2 \leq -a \int_M \varphi^2 u^{k+1} - b \int_M \varphi^2 u^k + \frac{4}{k} \int_M u^\frac{k}{2} |\nabla \varphi||\nabla u^\frac{k}{2}|,
\]
which gives
\[
\frac{4k-5}{k^2} \int_M \varphi^2 \left|\nabla u^\frac{k}{2}\right|^2 \leq -a \int_M \varphi^2 u^{k+1} - b \int_M \varphi^2 u^k + 4 \int_M u^k |\nabla \varphi|^2.
\] (22)

Applying Sobolev inequality (14) to the function \(f = \varphi u^\frac{k}{2}\), we have
\[
\left(\int_M \left|\varphi u^\frac{k}{2}\right|^\frac{2(n-2)}{n} \right)^\frac{n}{n-2} \leq 2D_1 \int_M \varphi^2 \left|\nabla u^\frac{k}{2}\right|^2 + 2D_1 \int_M u^k |\nabla \varphi|^2 + F_1 \int_M |\varphi u^\frac{k}{2}|^2.
\] (23)

Plugging (22) into (23), we get
\[
\left(\int_M \left|\varphi u^\frac{k}{2}\right|^\frac{2(n-2)}{n} \right)^\frac{n}{n-2} \leq D_2 k \left(\int_M \varphi^2 u^{k+1} + \int_M u^k |\nabla \varphi|^2 + \int_M \varphi^2 u^k\right).
\] (24)

Using Hölder inequality, from (24) we get
\[
\left(\int_M \left|\varphi u^\frac{k}{2}\right|^\frac{2(n-2)}{n} \right)^\frac{n}{n-2} \leq D_2 k \left(\int_M \left|\varphi u^\frac{k}{2}\right|^\frac{2(n-2)}{n} \right)^\frac{n}{n-2} \left(\int_{\text{Supp}(\varphi)} u^\frac{n}{2}\right)^\frac{2}{n} \leq D_2 k \left(\int_{\text{Supp}(\varphi)} u^\frac{n}{2}\right)^\frac{2}{n}
\]
\[
+ \int_M u^k |\nabla \varphi|^2 + \int_M \varphi^2 u^k,
\]
where $t = \frac{n^2}{n-2}$. By (21) and the above inequality, we conclude that

$$
\left( \int_M |\varphi u^\frac{n}{n-2}|^{\frac{n-2}{n}} \right)^{\frac{n-2}{n}} \leq D_3k \left[ \left( \int_M |\varphi u^\frac{n}{n-2}|^{\frac{n-2}{n}} \right)^{\frac{n-2}{n}} + \int_M u^k |\nabla \varphi|^2 + \int_M \varphi^2 u^k \right]. \quad (25)
$$

Since $2 < \frac{2t}{t-2} < \frac{2n}{n-2}$, we have, by interpolation,

$$
\left( \int_M |\varphi u^\frac{n}{n-2}|^{\frac{n-2}{n}} \right)^{\frac{n-2}{n}} \leq 2 \epsilon \left( \int_M |\varphi u^\frac{n}{n-2}|^{\frac{n-2}{n}} \right)^{\frac{n-2}{n}} + 2\epsilon \frac{\alpha}{n} \int_M \varphi^2 u^k.
$$

We choose $\epsilon$ such that $4D_3k\epsilon = 1$ and plug the above inequality into (25) to obtain

$$
\left( \int_M |\varphi u^\frac{n}{n-2}|^{\frac{n-2}{n}} \right)^{\frac{n-2}{n}} \leq D_4k \left[ (1 + k\frac{\alpha}{n}) \int_M \varphi^2 u^k + \int_M u^k |\nabla \varphi|^2 \right]. \quad (26)
$$

Finally, we now perform the iteration method under (26). Define $T_0 = T, T_{i+1} = T_i + 2^{-\frac{i+1}{2}}T$, and $S_{i+1} = S_i - 2^{-\frac{i}{2}}S$. Choose $\varphi_i \in C_0^\infty(M)$ such that $0 \leq \varphi_i \leq 1, |\nabla \varphi_i| \leq 5^i, \sup \varphi_i \subset A_i \equiv A(T_i, S_i)$ and $\varphi_i | A_{i+1} = 1$. Let $\chi_i = 1_{\sup \varphi_i}$. By (26), we get

$$
\left( \int_M |\varphi_i u^\frac{n}{n-2}|^{\frac{n-2}{n}} \right)^{\frac{n-2}{n}} \leq D_4k \left[ (1 + k\frac{\alpha}{n}) \int_M \chi_i^2 u^k + 25^i \int_M u^k |\nabla \varphi|^2 \right] \leq D_4k(26^i + k\alpha) \int_M u^k \chi_i^2,
$$

where $\beta = \frac{(n-2)^2}{2n}$. We can rewrite the above inequality as

$$
\left( \int_{A_{i+1}} u^{k_i} \right)^{\frac{1}{k_i}} \leq D_4k(26^i + k\alpha) \int_{A_i} u^k.
$$

Take $k = \frac{1}{k_i} = \gamma \alpha^i$, for $i \geq 0$, then with obvious notations

$$
\left( \int_{A_{i+1}} u^{k_{i+1}} \right)^{\frac{1}{k_{i+1}}} \leq D_4\gamma \alpha^i (26^i + (\gamma \alpha^i)\alpha^i) \frac{1}{\gamma \alpha^i} \left( \int_{A_i} u^{k_i} \right)^{\frac{1}{k_i}} \leq D_4\gamma (1 + \gamma \alpha^i) \frac{1}{\gamma \alpha^i} \left( \alpha^i \max\{26, \alpha^i \} \right) \frac{1}{\gamma \alpha^i} \left( \int_{A_i} u^{k_i} \right)^{\frac{1}{k_i}} \leq D_5e^{D_6\alpha^{-1}} \left( \int_{A_i} u^{k_i} \right)^{\frac{1}{k_i}}.
$$

Therefore,

$$
\|u\|_{\infty, A_{i+1}} = \|u\|_{\infty, A(T, 2S)} \leq D_5e^{D_6\sum_{i=0}^{\infty} \frac{i\alpha^{-1}}{i}} \left( \int_{A(T, 2S)} u^{\gamma} \right)^{\frac{1}{\gamma}}.
$$

Taking $\gamma = \frac{n}{4}$, we conclude that $\|u\|_{\infty, A(T, 2S)} \leq A_M \|u\|_{n, E(T)}$ and the assertion in Lemma 2.3 follows by letting $S$ tend to infinity. \qed
3. Proof of Theorems

Proof of Theorem 1.1. From (8), by the Kato inequality $|\nabla \tilde{R}m|^2 \geq |\nabla |\tilde{R}m||^2$, we obtain

$$|\tilde{R}m|\triangle |\tilde{R}m| = \frac{1}{2} |\triangle |\tilde{R}m||^2 - |\nabla |\tilde{R}m||^2 \geq -C(n)|\tilde{R}m|^3 + AR|\tilde{R}m|^2. \quad (27)$$

Let $u = |\tilde{R}m|$. By (27), we compute

$$u^\alpha \triangle u^\alpha = u^\alpha \left( (\alpha - 1)u^{\alpha - 2}|\nabla u|^2 + \alpha u^{\alpha - 1} \triangle u \right)$$

$$= \frac{\alpha - 1}{\alpha} |\nabla u|^2 + \alpha u^{2\alpha - 2} u \triangle u$$

$$\geq \frac{\alpha - 1}{\alpha} |\nabla u|^2 - C(n)\alpha u^{2\alpha + 1} + \alpha ARu^{2\alpha}, \quad (28)$$

where $\alpha$ is a positive constant. Taking $\alpha = \frac{2p}{n} \geq 1$, using the Young’s inequality, from (28) we obtain

$$u^\alpha \triangle u^\alpha \geq au^{3\alpha} + bu^{2\alpha}. \quad (29)$$

where $a$ and $b$ are two constants depending only on $n, \alpha$ and $R$. Setting $w = u^\alpha$, we can rewrite (29) as

$$\triangle w \geq aw^2 + bw. \quad (30)$$

Since $M$ has the positive Yamabe constant or satisfies the Sobolev inequality, combining with (30), by Lemma 2.3, we can prove Theorem 1.1.

Proof of Theorem 1.2. By (1), we have

$$R_{ijij} = \tilde{R}_{ijij} + \frac{R}{n(n-1)}. \quad (31)$$

Note that $R$ is positive. From (31), we see from Theorem 1.1 that there is a positive constant $\delta$ such that $R_{ijij} > \delta$ in $M \setminus \Omega$ for some compact set $\Omega$. This implies that the Ricci curvature is bounded from below by a positive constant outside some geodesic sphere, hence the manifold is compact (for detail, see Lemma 3.5 of [27]).

Proof of Theorem 1.4. When $R > 0$, we see from Theorem 1.2 that $M$ is compact. Taking $\alpha = \frac{2p}{n} \geq 1$. From (28), using the Young’s inequality, we have

$$u^\alpha \triangle u^\alpha \geq \frac{\alpha - 1}{\alpha} |\nabla u|^2 - C(n)\epsilon^{1-\alpha} u^{3\alpha} - [C(n)\alpha - 1]\epsilon - \alpha ARu^{2\alpha}. \quad (32)$$

Setting $w = u^\alpha$, we can rewrite (32) as

$$w \triangle w \geq \frac{\alpha - 1}{\alpha} |\nabla w|^2 - C(n)\epsilon^{1-\alpha} w^3 - [C(n)\alpha - 1]\epsilon - \alpha ARw^2. \quad (33)$$
From (33), we derive
\[ w^\beta \Delta w^\beta \geq (1 - \frac{1}{\omega \beta}) |\nabla w|^{2} - C(n)\beta \epsilon^{1-\alpha} w^{2\beta+1} - \beta [C(n)(\alpha - 1)\epsilon - \alpha AR] w^{2\beta}, \] (34)

where \( \beta \) is a positive constant. Integrating by parts (34), we get
\[ (2 - \frac{1}{\alpha \beta}) \int_{M} |\nabla w^{\beta}|^{2} - C(n)\beta \epsilon^{1-\alpha} \int_{M} w^{2\beta+1} - \beta [C(n)(\alpha - 1)\epsilon - \frac{R \alpha}{n-1}] \int_{M} w^{2\beta} \leq 0. \] (35)

From (35), using the H"older inequality, we have
\[ (2 - \frac{1}{\alpha \beta}) \int_{M} |\nabla w^{\beta}|^{2} - C(n)\beta \epsilon^{1-\alpha} \int_{M} w^{2\beta+1} - \beta [C(n)(\alpha - 1)\epsilon - \frac{R \alpha}{n-1}] \int_{M} w^{2\beta} \leq 0. \] (36)

Case 1. When \( 3 \leq n \leq 5 \) and \( 1 \leq \alpha < \frac{4}{n-2} \), if \( \alpha > 1 \), set \( \epsilon = \frac{(6-n)\alpha R}{4(\alpha-1)c(n)} \); if \( \alpha = 1 \), set \( \epsilon = 1 \). Take \( \beta = \frac{1}{\alpha} \). By the definition of Yamabe constant \( Q(M) \), from (36) we get
\[ \left[ Q(M) - \frac{C(n)\epsilon^{1-\alpha}}{\alpha} \left( \int_{M} |\tilde{R}\tilde{m}|^{\frac{2}{\alpha}} \right) \right] \left( \int_{M} w^{\frac{2\alpha}{n-2}} \right)^{\frac{n-2}{n}} \leq 0. \] (37)

We choose \( \left( \int_{M} |\tilde{R}\tilde{m}|^{\frac{2}{\alpha}} \right) < C_{1} \) such that (37) implies \( \left( \int_{M} w^{\frac{2\alpha}{n-2}} \right)^{\frac{n-2}{n}} = 0 \), that is, \( |\tilde{R}\tilde{m}| = 0 \), i.e., \( M \) is Einstein manifold and locally conformally flat manifold. Hence \( M \) is isometric to a spherical space form.

Case 2. When \( 3 \leq n \leq 5 \) and \( \alpha \geq \frac{4}{n-2} \), and \( n \geq 6 \), set \( \epsilon = \frac{R}{(n-1)c(n)} \) and \( \frac{1}{\alpha} = 1 + \frac{1}{(n-2)\alpha} \). We also get
\[ \left[ (2 - \frac{1}{\alpha \beta})Q(M) - C(n)\beta \epsilon^{1-\alpha} \left( \int_{M} |\tilde{R}\tilde{m}|^{\frac{2}{\alpha}} \right) \right] \left( \int_{M} w^{\frac{2\alpha}{n-2}} \right)^{\frac{n-2}{n}} \leq 0. \] (38)

We choose \( \left( \int_{M} |\tilde{R}\tilde{m}|^{\frac{2}{\alpha}} \right) < C_{1} \) such that (38) implies \( \left( \int_{M} w^{\frac{2\alpha}{n-2}} \right)^{\frac{n-2}{n}} = 0 \), that is, \( |\tilde{R}\tilde{m}| = 0 \), i.e., \( M \) is Einstein manifold and locally conformally flat manifold. Hence \( M \) is isometric to a spherical space form.

**Proof of Theorem 1.6.** Let \( \phi \) be a smooth compactly supported function on
Using the Cauchy-Schwarz inequality, we can rewrite (39) as

\[
1 - \alpha \beta \int_M |\nabla w^\beta|^2 \phi^2 \leq C(n) \beta \epsilon^{1-\alpha} \int_M w^{2\beta+1} \phi^2 + \int_M w^\beta \phi^2 \nabla^2 w^\beta \\
+ \beta[C(n)(\alpha - 1)\epsilon - \frac{2R\alpha}{n}] \int_M w^{2\beta} \phi^2
\]

which gives

\[
(2 - \frac{1}{\alpha \beta} - \epsilon) \int_M |\nabla w^\beta|^2 \phi^2 \leq C(n) \beta \epsilon^{1-\alpha} \int_M w^{2\beta+1} \phi^2 + \frac{1}{\epsilon} \int_M w^\beta |\nabla \phi|^2 \\
+ \beta[C(n)(\alpha - 1)\epsilon - \frac{2R\alpha}{n}] \int_M w^{2\beta} \phi^2.
\]

Using the Cauchy-Schwarz inequality, we can rewrite (39) as

\[
\left(2 - \frac{1}{\alpha \beta} - \epsilon\right) \int_M |\nabla w^\beta|^2 \phi^2 \leq C(n) \beta \epsilon^{1-\alpha} \int_M w^{2\beta+1} \phi^2 + \frac{1}{\epsilon} \int_M w^\beta |\nabla \phi|^2 \\
+ \beta[C(n)(\alpha - 1)\epsilon - \frac{2R\alpha}{n}] \int_M w^{2\beta} \phi^2,
\]

for the positive constant \(\epsilon\). By the definition of Yamabe constant \(Q(M)\) and (40), we have

\[
Q(M) \left( \int_M (\phi w^\beta)^{\frac{n}{n-2}} \right)^\frac{n-2}{n} \leq \int_M \left( |\nabla (\phi w^\beta)|^2 + \frac{(n-2)R w^{2\beta} \phi^2}{4(n-1)} \right)
\]

\[
= \int_M (w^{2\beta} |\nabla \phi|^2 + \phi^2 |\nabla w^\beta|^2 + 2\phi w^\beta \langle \nabla \phi, \nabla w^\beta \rangle + \frac{(n-2)R w^{2\beta} \phi^2}{4(n-1)})
\]

\[
\leq \left(1 + \frac{1}{\eta}\right) \int_M w^{2\beta} |\nabla \phi|^2 + (1 + \eta) \int_M \phi^2 |\nabla w^\beta|^2 + \int_M \frac{(n-2)R w^{2\beta} \phi^2}{4(n-1)}
\]

\[
\leq G \int_M w^{2\beta} |\nabla \phi|^2 + H \int_M w^{2\beta+1} \phi^2 + I \int_M w^{2\beta} \phi^2,
\]

where

\[
G = 1 + \frac{1}{\eta} + \frac{1 + \eta}{\epsilon(2 - \frac{1}{\alpha \beta} - \epsilon)},
\]

\[
H = \frac{(1 + \eta)C(n)\beta \epsilon^{1-\alpha}}{(2 - \frac{1}{\alpha \beta} - \epsilon)},
\]

\[
I = \frac{(n-2)R}{4(n-1)} - \frac{2(1 + \eta)\alpha \beta R}{n(2 - \frac{1}{\alpha \beta} - \epsilon)} + \frac{(1 + \eta)C(n)(\alpha - 1)\beta \epsilon}{(2 - \frac{1}{\alpha \beta} - \epsilon)}.
\]
We first consider the case of \( \gamma \in (1, \frac{n(n-2) + \sqrt{n(n-2)(n^2-10n+8)}}{4(n-1)}) \). When \( n \geq 10 \), noting that \( \epsilon, \varepsilon \) and \( \eta \) are sufficiently small, we choose \( \frac{1}{2} < \alpha \beta < \frac{n(n-2) + \sqrt{n(n-2)(n^2-10n+8)}}{8(n-1)} \) such that \( I \leq 0 \). Thus from (41) we have

\[
Q(M) \left( \int_M (\phi u^\beta)^{\frac{n-2}{2}} \right)^{\frac{n-2}{n}} \leq G \int_M w^{2\beta} |\nabla \phi|^2 + H \int_M w^{2\beta+1} \phi^2
\]

\[
\leq G \int_M w^{2\beta} |\nabla \phi|^2 + H \left( \int_M (\phi u^\beta)^{\frac{n-2}{2}} \right)^{\frac{n-2}{n}} \left( \int_M w^\frac{2}{n} \right)^{\frac{n-2}{n}}.
\]

Since \( \int_M w^\frac{2}{n} = \int_M u^\beta \) is sufficiently small, the second term in the right-hand side of the above can be absorbed in the left-hand side. Therefore, there exists a constant \( J > 0 \), such that

\[
J \left( \int_M (\phi u^\beta)^{\frac{n-2}{2}} \right)^{\frac{n-2}{n}} \leq G \int_M u^{2\alpha \beta} |\nabla \phi|^2. \quad (42)
\]

Let us choose a cutoff function \( \phi \) satisfying the properties that

\[
\phi(x) = \begin{cases} 
1 & \text{on } B(r) \\
0 & \text{on } M \setminus B(2r),
\end{cases}
\]

and \( |\nabla \phi| \leq \frac{2}{r} \). In particular, if \( M \) is compact, and if \( r > d \), where \( d \) is the diameter of \( M \), then \( \phi = 1 \) on \( M \). From (42), we get

\[
J \left( \int_{B_r} u^{\frac{2\alpha \beta}{n-2}} \right)^{\frac{n-2}{n}} \leq \frac{4}{r^2} B \int_M u^{2\alpha \beta}. \quad (43)
\]

Let \( r \to +\infty \), by assumption that \( \int_M u^{2\alpha \beta} < \infty \), from (43), we have \( u = 0 \), i.e., \( M \) is Einstein manifold and locally conformally flat manifold. Hence \( M \) is isometric to a hyperbolic space form.

In the case of \( \gamma \in (0, 1] \), since \( \int_M |\hat{Rm}|^p < C \), by Theorem 1.1, \( |\hat{Rm}| \) is bounded. Hence \( \int_M |\hat{Rm}|^{\gamma+1} < \infty \) for \( \int_M |\hat{Rm}|^{\gamma} < \infty \). For \( \gamma + 1 \in (1, \frac{n(n-2) + \sqrt{n(n-2)(n^2-10n+8)}}{4(n-1)}) \), we apply the above result to prove Theorem 1.6.

**Proof of Corollary 1.11.** Taking \( \alpha = \frac{2n}{n} \geq 1 \). From (9), proceeding as in the proof of (34), we have

\[
w^{\beta} \Delta w^{\beta} \geq (1 - \frac{n-2}{n\alpha \beta}) |\nabla w^{\beta}|^2 - \frac{n\beta \epsilon^{1-\alpha}}{\sqrt{n(n-1)}} w^{2\beta+1}
\]

\[
- \beta \left[ \frac{n(\alpha - 1)}{\sqrt{n(n-1)}} - \frac{R\alpha}{n-1} \right] w^{2\beta}. \quad (44)
\]
where $\beta$ is a positive constant. Based on (44), proceeding as in the proof of (40), we obtain

$$
\left(2 - \frac{n - 2}{n\alpha\beta} - \varepsilon\right) \int_M |\nabla w^\beta|^2 \phi^2 \leq \frac{n\beta^{1-\alpha}}{\sqrt{n(n-1)}} \int_M w^{2\beta+1} \phi^2 \\
+ \frac{1}{\varepsilon} \int_M w^{2\beta} |\nabla \phi|^2 + \beta \frac{n(\alpha - 1)\varepsilon}{\sqrt{n(n-1)}} - \frac{R\alpha}{n - 1} \int_M w^{2\beta} \phi^2,
$$

(45)

for the positive constant $\varepsilon$.

On the other hand, when $M$ is a complete, simply connected, locally conformally flat Riemannian $n$-manifold, $M$ satisfies the Sobolev inequality (see Corollary 3.2 in [17]):

$$
\left(\int_M |f|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \frac{4}{n(n-2)\omega_n^\frac{2}{n}} \int_M \left(|\nabla f|^2 + \frac{(n - 2)}{4(n - 1)} Rf^2\right), f \in C^\infty_0(M).
$$

Combining the above inequality with (45), we obtain

$$
\frac{n(n-2)\omega_n^{\frac{2}{n}}}{4} \left(\int_M (\phi w^\beta)^{\frac{n-2}{n}}\right)^{\frac{n-2}{n}} \leq \int_M \left(|\nabla (\phi w^\beta)|^2 + \frac{(n - 2)Rw^{2\beta} \phi^2}{4(n - 1)}\right)\\
= \int_M (w^{2\beta} |\nabla \phi|^2 + \phi^2 |\nabla w|^2 + 2\phi w^\beta (\nabla \phi, \nabla w^\beta) + \frac{(n - 2)Rw^{2\beta} \phi^2}{4(n - 1)})\\
\leq (1 + \frac{1}{\eta}) \int_M w^{2\beta} |\nabla \phi|^2 + (1 + \eta) \int_M \phi^2 |\nabla w|^2 + \int_M \frac{(n - 2)Rw^{2\beta} \phi^2}{4(n - 1)}\\
\leq K \int_M w^{2\beta} |\nabla \phi|^2 + L \int_M w^{2\beta+1} \phi^2 + M \int_M w^{2\beta} \phi^2,
$$

(46)

where

$$
K = 1 + \frac{1}{\eta} + \frac{1 + \eta}{\varepsilon (2 - \frac{n - 2}{n\alpha\beta} - \varepsilon)},
$$

$$
L = \frac{(1 + \eta)n\beta^{1-\alpha}}{\sqrt{n(n-1)(2 - \frac{n - 2}{n\alpha\beta} - \varepsilon)}},
$$

$$
M = \frac{(n - 2)R}{4(n - 1)} - \frac{(1 + \eta)\alpha\beta R}{(n - 1)(2 - \frac{n - 2}{n\alpha\beta} - \varepsilon)} + \frac{(1 + \eta)n(\alpha - 1)\beta \varepsilon}{\sqrt{n(n-1)(2 - \frac{n - 2}{n\alpha\beta} - \varepsilon)}}.
$$

When $n \geq 6$, noting that $\eta$, $\varepsilon$ and $\eta$ are sufficiently small, we choose $\frac{1}{2} \leq \alpha\beta < \frac{(n-2)(1+\sqrt{1-\frac{2}{n-2}})}{4}$ such that $M \leq 0$. Thus from (46) we have

$$
\left(\frac{n(n-2)\omega_n^{\frac{2}{n}}}{4} - L \left(\int_M |\tilde{R}m|^p\right)^{\frac{2}{p}}\right) \left(\int_M (\phi w^\beta)^{\frac{n-2}{n}}\right)^{\frac{n-2}{n}} \leq K \int_M |\tilde{R}m|^{2\alpha\beta} |\nabla \phi|^2.\n$$

(47)

So we choose $\left(\int_M |\tilde{R}m|^p\right)^{\frac{2}{p}}$ small enough such that $N = \frac{n(n-2)\omega_n^{\frac{2}{n}}}{4} - L \left(\int_M |\tilde{R}m|^p\right)^{\frac{2}{p}} > 0$. 

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When \( p = \frac{n}{2} \), i.e., \( \alpha = 1 \). We choose \( \eta \) such that \( M = 0 \), i.e., \( 2 - \frac{n-2}{n \beta} - \epsilon = \frac{4(1+\eta)\beta}{n-2} \). Thus we have

\[
\frac{n(n-2)\omega_n^+}{L} = \sqrt{n(n-1)\omega_n^+}.
\]

So we choose \( \left( \int_M |\tilde{R}m|^{\frac{2}{n}} \right)^{\frac{n}{2}} < \sqrt{n(n-1)\omega_n^+} \) such that \( N = \frac{n(n-2)\omega_n^+}{4} - L \left( \int_M |\tilde{R}m|^{\frac{2}{n}} \right)^{\frac{n}{2}} > 0 \).

Taking \( \beta = \frac{1}{4} \). We rewrite (47) as

\[
\mathcal{N} \left( \int_M (\phi w^2)^{\frac{2\alpha}{n-2}} \right)^{\frac{n-2}{n}} \leq K \int_M |\tilde{R}m|^p |\nabla \phi|^2.
\]

The rest of the proof runs as before. Hence this completes the proof of Corollary 1.10. \( \square \)

**Proof of Theorem 1.13.** Let \( u = |\tilde{R}c| \). By (9), we compute

\[
u^\alpha \Delta u^\alpha \geq (1 - \frac{n-2}{n \alpha}) |\nabla u^\alpha|^2 - \frac{n \alpha}{\sqrt{n(n-1)}} u^{2\alpha+1} + \frac{R \alpha}{n-1} u^{2\alpha}, \tag{48}\]

where \( \alpha \) is a positive constant. Taking \( \alpha = \frac{2p}{n} \geq 1 \). From (48), using the Young’s inequality, we have

\[
u^\alpha \Delta u^\alpha \geq (1 - \frac{n-2}{n \alpha}) |\nabla u^\alpha|^2 - \frac{n e^{1-\alpha}}{\sqrt{n(n-1)}} u^{3\alpha} - \left[ \frac{n(\alpha-1)\epsilon}{\sqrt{n(n-1)}} - \frac{R \alpha}{n-1} \right] u^{2\alpha}. \tag{49}\]

Setting \( w = u^\alpha \). Based on (49), using the same argument as in the proof of (36), we obtain

\[
(2 - \frac{n-2}{n \alpha \beta}) \int_M |\nabla w|^2 - \frac{\beta e^{1-\alpha}}{\sqrt{n(n-1)}} \left( \int_M w^{\frac{2\alpha}{n-2}} \right)^{\frac{n-2}{n}} \left( \int_M \tilde{Rm} \right)^{\frac{n}{2}} - \beta \left[ \frac{n(\alpha-1)\epsilon}{\sqrt{n(n-1)}} - \frac{R \alpha}{n-1} \right] \int_M w^{2\beta} \leq 0. \tag{50}\]

Case 1. When \( n = 3 \) and \( 1 \leq \alpha < \frac{4}{3} \), if \( \alpha > 1 \), set \( \epsilon = \frac{\sqrt{6} \alpha R}{2\sqrt{(\alpha-1)}} \); if \( \alpha = 1 \), set \( \epsilon = 1 \). Take \( \alpha \beta = \frac{1}{3} \). By the definition of Yamabe constant \( Q(M) \), from (50) we get

\[
\left[ Q(M) - \frac{3 e^{1-\alpha}}{\sqrt{6} \alpha} \left( \int_M |\tilde{R}c|^p \right)^{\frac{n}{2}} \right] \left( \int_M w^{\frac{2\alpha}{n-2}} \right)^{\frac{n-2}{n}} \leq 0. \tag{51}\]

We choose \( \left( \int_M |\tilde{R}c|^p \right)^{\frac{1}{2}} < C_3 \) such that (50) implies \( \left( \int_M w^{\frac{2\alpha}{n-2}} \right)^{\frac{n-2}{n}} = 0 \), that is, \( |\tilde{R}c| = 0 \), i.e., \( M \) is Einstein manifold. Since \( M \) is locally conformally flat manifold, \( M \) is isometric to a spherical space form.

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Case 2. When \( n = 3 \) and \( \alpha \geq \frac{4}{3} \), and \( n \geq 4 \), set \( \epsilon = \frac{R}{\sqrt{(n-1)n}} \) and \( \frac{1}{\alpha} = \frac{n}{n-2} (1 + \sqrt{1 - \frac{4}{n\alpha}}) \). We also get

\[
\left[ \frac{2 - \frac{n - 2}{n\alpha \beta}}{Q(M)} - \frac{n\beta e^{1-\alpha}}{\sqrt{n(n-1)}} \left( \int_M |\hat{Ric}|^p \right)^{\frac{1}{p}} \left( \int_M w^{\frac{2n\beta}{n-2}} \right)^{\frac{n-2}{n}} \right] \left( \int_M w^{\frac{2n\beta}{n-2}} \right)^{\frac{n-2}{n}} \leq 0. \tag{52}
\]

We choose \( \left( \int_M |\hat{Ric}|^p \right)^{\frac{1}{p}} < C_3 \) such that (52) implies \( \left( \int_M w^{\frac{2n\beta}{n-2}} \right)^{\frac{n-2}{n}} = 0 \), that is, \( |\hat{Ric}| = 0 \), i.e., \( M \) is Einstein manifold. Since \( M \) is locally conformally flat manifold, \( M \) is isometric to a spherical space form.

\[\Box\]

**Proof of Theorem 1.15.** The pinching condition in Theorem 1.15 implies that the equality holds in (50). So all inequalities leading to (49) become equalities. Hence at every point, either \( \hat{Ric} \) is null, i.e., \( M \) is Einstein, or it has an eigenvalue of multiplicity \((n-1)\) and another of multiplicity 1. But (1) implies that \( M \) is not Einstein. Since \( M \) has harmonic curvature, and by the regularity result of DeTurck and Goldschmidt [13], \( M \) must be real analytic in suitable (harmonic) local coordinates.

Suppose that the Ricci tensor has an eigenvalue of multiplicity \((n-1)\) and another of multiplicity 1. If the Ricci tensor is parallel, by the de Rham decomposition Theorem [11], \( M \) is covered isometrically by the product of Einstein manifolds. We have \( R = \sqrt{n(n-1)}|\hat{Ric}| \). Since \( M \) is conformally flat and has positive scalar curvature, then the only possibility is that \( M \) is covered isometrically by \( S^1 \times S^{n-1} \) with the product metric.

On the other hand, if the Ricci tensor is not parallel, by the classification result of Derdziński (see Theorem 10 of [12], see also Theorem 3.2 of [6]), this concludes the proof of Theorem 1.15.

\[\Box\]

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