ON ARENS REGULARITY OF PROJECTIVE TENSOR PRODUCT OF SCHATTEN SPACES

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Abstract. We discuss Arens regularity of the projective tensor product of Banach algebras consisting of Schatten class operators. More precisely, exploiting Ūlger’s biregularity criterion, we prove that for any Hilbert space $H$, the Banach algebras $S_p(H) \otimes \gamma S_q(H)$ and $B(S_2(H)) \otimes \gamma S_2(H)$ are not Arens regular for every pair $1 \leq p, q \leq 2$; whereas, when $H$ is separable and $S_2(H)$ is equipped with the Schur product, then $S_2(H) \otimes \gamma S_2(H)$ turns out to be Arens regular.

1. Introduction

For any normed algebra $A$, Arens (in [1]) defined two products $\square$ and $\diamond$ on $A^{**}$ such that each product makes $A^{**}$ into a Banach algebra and the canonical isometric inclusion $J : A \to A^{**}$ becomes a homomorphism with respect to both the products. The normed algebra $A$ is said to be Arens regular if the two products $\square$ and $\diamond$ agree, i.e. $f \square g = f \diamond g$ for all $f, g \in A^{**}$. Over the years, people have witnessed some significant connection between Arens regularity and certain geometric properties of Banach spaces - see, for instance, [9], [10] and the references therein.

It is known that operator algebras and reflexive Banach algebras are all Arens regular - see [11] and [12, 1.4.2]. In particular, $S_p(H)$, the Banach algebra consisting of Schatten $p$-class operators, being reflexive for $1 < p < \infty$, is Arens regular. And, even though $S_1(H)$ is not reflexive, it is still Arens regular - a proof can be found in [2].

Given the importance of tensor products in the theory of Banach algebras, among the various algebraic properties that people consider, it is quite natural to analyze the Arens regularity of the projective tensor product of Banach algebras. Some important results in this direction appear, for instance, in [14], [7], [10], [8].

In fact, very recently, in [10], Neufang settled a 40 years old problem regarding Arens regularity of the Varapolous algebra $C(X) \otimes \gamma C(Y)$. Actually, along with some other important results, he proved, more generally, that for $C^*$-algebras $A$ and $B$, $A \otimes \gamma B$ is Arens regular if and only if $A$ or $B$ has the Phillips property. Prior to Neufang’s result, Úlger, in his pioneering work [14] in this direction, had shown that when $\ell^p$ is equipped with pointwise multiplication, then $\ell^p \otimes \gamma \ell^q$ is Arens regular for $1 < p, q < \infty$. It was this setup that motivated us to ask the same question for the non-commutative analogue of $\ell^p$ spaces, namely, the Banach algebra $S_p(H)$ consisting of Schatten $p$-class operators on a Hilbert space $H$. Unlike the $\ell^p$ spaces, it turns out that the projective tensor product of Schatten spaces is not Arens regular. We achieve this by exploiting one of Úlger’s technique from [14] to show that $B(K) \otimes \gamma K$ and $S_p(H) \otimes \gamma S_q(H)$ are not Arens regular for $1 \leq p, q \leq 2$, where $K := S_2(H)$ for any Hilbert space $H$.

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Further, keeping one leg as a $C^*$-algebra, we show in Theorem 4.10 that the Banach algebra $B(S_2(H)) \otimes^\gamma S_2(H)$ is not Arens regular. And then we deduce further that the Banach algebras, $B_0(S_2(H)) \otimes^\gamma S_2(H), B(K) \otimes^\gamma S_p(H)$ and $B_0(K) \otimes^\gamma S_p(H)$ and are all Arens irregular for $1 \leq p \leq 2$.

On the other hand, when $H$ is separable and $S_2(H)$ is equipped with the Schur product, then, in the final section, we show that the commutative Banach algebra $S_p(H) \otimes^\gamma S_q(H)$ is Arens regular.

2. Preliminaries

2.1. Schatten spaces.

Let $H$ be a Hilbert space and $T \in B(H)$. For $1 \leq p < \infty$, the Schatten $p$-norm of $T$ is given by

$$||T||_p = \text{Tr}(|T|^p),$$

where $\text{Tr}$ denotes the canonical semi-finite positive trace on $B(H)_+$. The Schatten $p$-class operators on $H$ are those $T \in B(H)$ for which $||T||_p < \infty$ and

$$S_p(H) := \{T \in B(H) : ||T||_p < \infty\}.$$ 

$S_p(H)$ is known to be an ideal in $B(H)$ and $|| \cdot ||_p$ is a norm on $S_p(H)$ which makes it a Banach $\ast$-algebra (with canonical adjoint involution). Whenever $1 \leq p < q$, we have the inequality $|| \cdot ||_q \leq || \cdot ||_p$, and hence the containment $S_p(H) \subset S_q(H)$. Operators in $S_p(H)$ are compact and $S_p(H)$ contains all finite rank operators. Detailed discussion about these facts can be found in [12].

2.2. Arens regularity. Let $A$ be a normed algebra. For the sake of convenience, we quickly recall the definitions of the two products $\square$ and $\circ$ mentioned in the Introduction. For $a \in A$, $\omega \in A^*$, $f \in A^{**}$, consider the functionals $\omega_a, a\omega \in A^*$, $\omega_f, f\omega \in A^{**}$ given by $w_a = (L_a)\omega$, $a\omega = (R_a)^*\omega$; $\omega_f(a) = f(\omega_a)$ and $f\omega(a) = f(\omega_a)$. Then, for $f, g \in A^{**}$ the operations $\square$ and $\circ$ are given by $(f \square g)(\omega) = f(\omega)g$ and $(f \circ g)(\omega) = (f \omega)g$ for all $\omega \in A^*$. As recalled in the Introduction, $A$ is said to be Arens regular if the products $\square$ and $\circ$ are same on $A^{**}$.

Thanks to Ulger, there is a very useful equivalent characterization of Arens regularity in terms of some properties of bilinear maps.

A bounded bilinear form $m : X \times Y \rightarrow Z$, where $X, Y$ and $Z$ are normed spaces, is called Arens regular if the induced bilinear forms $m^{**} : X^* \times Y^* \rightarrow Z^{**}$ and $m^{**\ast} : X^* \times Y^* \rightarrow Z^{**}$ are same, where $m^* : Z^* \times X \rightarrow Y^*$ is given by $m^*(f, x)(y) = (f, m(x, y))$, $m^r : Y \times X \rightarrow Z$ is given by $m^r(y, x) = m(x, y)$, $m^{**} := (m^*)^*$ and so on.

A normed algebra $A$ is known to be Arens regular if and only if the multiplication $A \times A \rightarrow A$ is Arens regular in the above sense.

2.3. Projective tensor product and its Arens regularity. Let $A$ and $B$ be Banach algebras. There is a natural multiplication on their algebraic tensor product $A \otimes B$ given by $(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1x_2 \otimes y_1y_2$ and extended appropriately to all tensors. And there are various ways to impose a normed algebra structure on $A \otimes B$.

We will be primarily interested in the Banach space projective tensor product $\otimes^\gamma$.

For $u \in A \otimes Y$, its projective norm is given by

$$||u||_\gamma = \inf \left\{ \sum_{i=1}^{n} ||x_i||_Y : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}.$$ 

This norm turns out to be a cross norm, i.e., $||x \otimes y||_\gamma = ||x||_X ||y||_Y$ and the completion of the normed algebra $A \otimes B$ with respect to this norm is a Banach algebra and is denoted by $A \otimes^\gamma B$. 
Ulger gave a very useful characterization of Arens regularity of the projective tensor product of Banach algebras in terms of biregularity of bilinear forms.

Recall that, for Banach algebras $A$ and $B$, a bilinear form $m : A \times B \to \mathbb{C}$ is said to be biregular if for any two pairs of sequences $(a_i), (\hat{a}_j)$ and $(b_i), (\hat{b}_j)$ in the closed unit balls of $A$ and $B$, respectively, one has

$$\lim_{i} \lim_{j} m(a_i \hat{a}_j, b_i \hat{b}_j) = \lim_{j} \lim_{i} m(a_i \hat{a}_j, b_i \hat{b}_j)$$

provided that these limits exist - see [14, Definition 3.1]. Our results will depend heavily on the following characterization provided by Ulger.

**Theorem 2.1.** [14] Let $A$ and $B$ be Banach algebras. Then, their projective tensor product $A \otimes B$ is Arens regular if and only if every bilinear form $m : A \times B \to \mathbb{C}$ is biregular.

### 3. Arens Regularity of Some Bilinear Maps

We will see ahead that $A \otimes^\gamma B$ is Arens regular if $A$ is so and $B$ is finite dimensional, but, prior to that, we observe that, for the space of Hilbert-Schmidt operators, some specific bilinear maps behave well when one leg is finite dimensional, as in:

**Proposition 3.1.** Let $K_1 = S_2(H_1)$ and $K_2 = S_2(H_2)$, where $H_1$ and $H_2$ are two Hilbert spaces. Suppose $H_1$ or $H_2$ is finite dimensional and $m : K_1 \times K_2 \to \mathbb{C}$ is a bilinear form. Then, for every pair of sequences $\{S_i\}, \{\hat{S}_j\}$ and $\{T_i\}, \{\hat{T}_j\}$ in the unit balls of $K_1$ and $K_2$, respectively, if the iterated double limits (as on both sides of Equation (1)) exist then they are equal to $m(S \hat{S}, TT)$, where $S, \hat{S}$ and $T, \hat{T}$ are some weak limits of the corresponding sequences.

In particular, every bilinear form $m : K_1 \times K_2 \to \mathbb{C}$ is biregular.

**Proof.** First, note that, due to Reisz Representation Theorem, there exists a conjugate linear continuous map $\varphi : K_1 \to K_2$ such that $m(S, T) = \langle T, \varphi(S) \rangle$ for all $S, T$.

Suppose that $H_2$ is a finite dimensional Hilbert space with an orthonormal basis $\{e_r\}_{r=1}^n$. Let $\{S_i\}, \{\hat{S}_j\}$ and $\{T_i\}, \{\hat{T}_j\}$ be a two pairs of sequences in the unit balls of $K_1$ and $K_2$, respectively, such that the iterated limits $\lim_i \lim_j m(S_i \hat{S}_j, T_i \hat{T}_j)$ and $\lim_j \lim_i m(S_i \hat{S}_j, T_i \hat{T}_j)$ exist. Then, we have

$$\lim_{i} \lim_{j} m(S_i \hat{S}_j, T_i \hat{T}_j) = \lim_{j} \lim_{i} \langle T_i \hat{T}_j, \varphi(S_i \hat{S}_j) \rangle$$

$$= \lim_{i} \lim_{j} \text{Tr} \left( \varphi(S_i \hat{S}_j)^* T_i \hat{T}_j \right)$$

$$= \lim_{i} \lim_{j} \sum_{r=1}^n \langle T_i \hat{T}_j e_r, \varphi(S_i \hat{S}_j) e_r \rangle.$$  

Since the unit ball of a Hilbert space is weakly compact and the unit ball of the finite dimensional space $K_2$ is compact, the sequences $\{S_i\}, \{\hat{S}_j\}$ and $\{T_i\}, \{\hat{T}_j\}$ have weakly and norm convergent convergent subsequences, respectively. Let $\{S_{ik}\}, \{\hat{S}_{jk}\}$ and $\{T_{ik}\}, \{\hat{T}_{jk}\}$ be the corresponding pair of subsequences converging to $S, \hat{S}$ and $T, \hat{T}$ in weak* (equivalently, weak) topology and norm topology on $K_1$ and $K_2$, respectively. Then, we obtain

$$\lim_{i} \lim_{j} m(S_i \hat{S}_j, T_i \hat{T}_j) = \lim_{i} \lim_{j} \sum_{r=1}^n \langle T_{ik} \hat{T}_{jk} e_r, \varphi(S_{ik} \hat{S}_{jk}) e_r \rangle.$$  

Hence, without loss of generality, we may assume that $\{S_i\} \overset{w}{\to} S, \{\hat{S}_j\} \overset{w}{\to} \hat{S}$ in the weak topology of $K_1$ and $\{T_i\} \to T, \{\hat{T}_j\} \to \hat{T}$ in the norm (HS) topology of $K_2$. The limits and
summation are interchangeable because summation is over finite index set. Hence, we have

\[
\lim_{i} \lim_{j} m(S_i \tilde{S}_j, T_i \tilde{T}_j) = \sum_{r=1}^{n} \lim_{i} \left( \lim_{j} \left( T_i \tilde{T}_j e_r, \varphi(S_i \tilde{S}_j) e_r \right) \right).
\]

Next, we make few claims, whose proofs will be provided later:

**Claims:**

(C1) \( \tilde{T}_j e_r \to \tilde{T} e_r \) in norm for every \( r \).
(C2) \( T_i \tilde{T}_j e_r \to T_i \tilde{T} e_r \) in norm for every \( i \) and \( r \).
(C3) \( S_i \tilde{S}_j \xrightarrow{w} S_i \tilde{S} \) for every \( i \).
(C4) \( \varphi(S_i \tilde{S}_j) e_r \to \varphi(S_i \tilde{S}) e_r \) in norm for every \( r \) and \( i \).
(C5) \( S_i \tilde{S} \to SS \) weakly.
(C6) \( \varphi(S_i \tilde{S}) e_r \to \varphi(SS) e_r \) in norm for every \( r \).

Using (C2) and (C4) and the fact that every inner product is continuous in both coordinates (with respect to the norm topology), Equation (4) gives us

\[
\lim_{i} \lim_{j} m(S_i \tilde{S}_j, T_i \tilde{T}_j) = \sum_{r=1}^{n} \lim_{i} \left( \lim_{j} \left( T_i \tilde{T}_j e_r, \varphi(S_i \tilde{S}_j) e_r \right) \right)
\]

or

\[
\sum_{r=1}^{n} \left( T_i \tilde{T} e_r, \varphi(S \tilde{S}) e_r \right) = m(SS, TT).
\]

Repeating the same process with limits interchanged, we obtain

\[
\lim_{j} \lim_{i} m(S_i \tilde{S}_j, T_i \tilde{T}_j) = \sum_{r=1}^{n} \left( T \tilde{T} e_r, \varphi(SS) e_r \right) = m(SS, TT).
\]

This also proves that the bilinear form \( m \) is biregular.

We now prove the claims made above:

(C1) Since \( (\tilde{T}_j - \tilde{T}) \to 0 \) in the norm topology of \( K_2 \), it must follow that \( (\tilde{T}_j - \tilde{T}) \to 0 \) in the operator norm of \( B(H_2) \) (because Hilbert-Schmidt norm dominates the operator norm). Hence \( \tilde{T}_j e_\alpha \to \tilde{T} e_\alpha \) for each \( r \).

(C2) Follows from the sequential continuity of \( T_k \) as a bounded operator and C1.

(C3) Since \( \tilde{S}_j \xrightarrow{w} \tilde{S} \), we have \( \langle \tilde{S}_j, P \rangle \to \langle \tilde{S}, P \rangle \) for each \( P \in K_1 \). Hence \( \langle \tilde{S}_j, S_k^* P \rangle \to \langle \tilde{S}, S_k^* P \rangle \); so that \( \langle S_k \tilde{S}_j, P \rangle \to \langle S_k \tilde{S}, P \rangle \) for each \( P \in K_1 \). Hence \( S_k \tilde{S}_j \xrightarrow{w} S_k \tilde{S} \).

(C4) Any bounded linear operator between Hilbert spaces is weak-weak continuous and weak topology on \( K_2 \) is nothing but norm topology (due to \( K_2 \) being finite dimensional), hence due
to C3 we obtain \( \varphi(S_{ik} \tilde{S}_{jk}) \rightarrow \varphi(S_{ik} \tilde{S}) \) in norm of \( \mathcal{K}_2 \). And hence we have the following norm convergence in \( \mathcal{H}_2 \).

\[
\varphi(S_{ik} \tilde{S}_{jk})e_\alpha \rightarrow \varphi(S_{ik} \tilde{S})e_\alpha
\]

(C5) Consider \( \left< S_i \tilde{S}, P \right> = \text{Tr}(P^* S_i \tilde{S}) \) for some \( P \in \mathcal{K}_1 \). We know that product of two Hilbert Schmidt operators is a trace class operator. Hence \( P^* S_i \in \mathcal{S}_1(\mathcal{H}_1) \). Also for any \( A \in \mathcal{S}_1(\mathcal{H}) \) and \( B \in \mathcal{B}(\mathcal{H}) \), we have \( \text{Tr}(AB) = \text{Tr}(BA) \). So,

\[
\left< S_i \tilde{S}, P \right> = \text{Tr}(P^* S_i \tilde{S}) = \text{Tr}(\tilde{S} P^* S_i) \left< S_i, P \tilde{S}^* \right>.
\]

And, since \( S_i \xrightarrow{w} S \), we see that

\[
\left< S_i, P \tilde{S}^* \right> \rightarrow \left< S, P \tilde{S}^* \right> = \text{Tr}(P^* S) = \text{Tr}(P^* S \tilde{S}) = \left< SS, P \right>.
\]

Hence, \( \left< S_i \tilde{S}, P \right> \rightarrow \left< SS, P \right> \) for each \( P \in \mathcal{K}_1 \), i.e., \( S_i \tilde{S} \xrightarrow{w} SS \).

(C6) This again follows from (C5) and the weak-norm continuity of \( \varphi \).

The following type of Bilinear forms play an important role in upcoming results in the next section. We first note in the next theorem that these maps are Arens regular. Later on we will show that when \( \mathcal{H} \) is replaced by Hilbert space of Hilbert-Schmidt operators, then such forms, though Arens regular, are not ”Biregular” in general.

**Proposition 3.2.** Let \( \mathcal{H} \) be a Hilbert space and \( l : B(\mathcal{H}) \times \mathcal{H} \to \mathcal{H} \) be the bounded bilinear map given by \( l(T, \zeta) = T(\zeta) \). Then, \( l \) is Arens regular. Also, the bilinear form \( m : B(\mathcal{H}) \times \mathcal{H} \to \mathbb{C} \), defined as \( m(T, \zeta) = \langle T(\zeta), \beta \rangle \) for some \( \beta \in \mathcal{H} \), is Arens regular.

**Proof.** Let \( V \in \mathcal{B}(\mathcal{H})^{**} \) and \( F \in \mathcal{H}^{**} \). Since Hilbert spaces are reflexive, \( F = J_\xi \) for some \( \xi \in \mathcal{H} \). We need to show that \( l^{***(V, F)} = l^{****}(V, F) \). Let \( f \in \mathcal{H}^* \). Then, by Reisz representation Theorem, \( f = \langle \cdot, \eta \rangle \) for a unique \( \eta \in \mathcal{H} \). Further, we have

\[
l^{***(V, F)}(f) = \langle V, l^{**}(F, f) \rangle.
\]

Note that \( l^{**}(F, f)(T) = \langle F, l^*(f, T) \rangle \) and \( l^*(f, T)(\zeta) = f(T\zeta) = \langle l(T, \zeta), \eta \rangle \) for all \( T \in \mathcal{H}^{**} \) and \( \zeta \in \mathcal{H} \); so,

\[
l^{**}(F, f)(T) = \langle F, l(l(T, \cdot ), \eta) \rangle
\]

\[
= J_\xi (l(l(T, \cdot ), \eta))
\]

\[
= \langle l(T, \xi), \eta \rangle.
\]

for all \( T \in \mathcal{H}^{**} \); so, \( l^{**}(F, f) = \langle l(\cdot, \xi), \eta \rangle \). And, similarly, we obtain

\[
l^{****}(V, F)(f) = l^{***(V, F)}(f) = \langle F, l^{**}(V, f) \rangle = l^{**}(V, f)(\xi)
\]

and \( l^{**}(V, f)(\xi) = \langle V, l^{**}(f, \xi) \rangle \). Also, note that

\[
l^{**}(f, g)(S) = f(l(S, \xi)) = \langle l(S, \xi), \eta \rangle
\]

for all \( S \in \mathcal{B}(\mathcal{H}) \); so,

\[
l^{***(V, f)}(\xi) = \langle V, l(\cdot, \xi), \eta \rangle.
\]

Thus, from Equations (6), (7), (8) and (5), we conclude that

\[
l^{***(V, F)}(f) = l^{****}(V, F)(f) = \langle V, l(\cdot, \xi), \eta \rangle
\]

for all \( f \in \mathcal{H}^* \). Thus, \( l^{***(V, F)} = l^{****} \). We conclude that \( l^{***} = l^{****} \) and hence \( l \) is Arens regular. Arens regularity of \( m \) can be proved in similar fashion. \(\square\)
4. Arens regularity of projective tensor product of Schatten spaces

We will be particularly interested in $S_2(\mathcal{H})$, the space of Hilbert-Schmidt operators on $\mathcal{H}$. It is known to be a Hilbert space with respect to the inner product $\langle A, B \rangle := \text{Tr}(B^*A)$, and is also denoted by $\mathcal{K}$. Being reflexive, $\mathcal{K}$ is an Arens regular Banach algebra.

Ülger had proved in [14, Corr. 4.7] that $\ell^2 \otimes \gamma \ell^2$ is Arens regular (where $\ell^2$ is equipped with the pointwise product). One would guess that the same should hold for $\mathcal{K} \otimes \gamma \mathcal{K}$ as well. However, we have the following:

**Theorem 4.1.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two infinite dimensional Hilbert spaces. Let $\mathcal{K}_1 := S_2(\mathcal{H}_1)$ and $\mathcal{K}_2 := S_2(\mathcal{H}_2)$. Then, the Banach algebra $\mathcal{K}_1 \otimes \gamma \mathcal{K}_2$ is not Arens regular.

**Proof.** Without the loss of generality, we can assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are separable infinite dimensional Hilbert spaces. Fix any two orthonormal bases $\{e_i\}_{i \in \mathbb{N}}$ and $\{f_j\}_{j \in \mathbb{N}}$ for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Consider the pair of sequences $\{S_i\}$ and $\{\tilde{S}_j\}$ in $\mathcal{K}_1$ given by $S_i = e_i \otimes e_i$ and $\tilde{S}_j = e_1 \otimes e_j$. Note that $S_i \tilde{S}_j = e_i \otimes e_j$ for all $i, j \in \mathbb{N}$. Recall that the association

$$\mathcal{H} \otimes \mathcal{H} \ni \xi \otimes \eta \mapsto \theta_{\xi, \eta} \in S_2(\mathcal{H})$$

extends to a unitary from $\mathcal{H} \otimes \mathcal{H}$ onto $S_2(\mathcal{H})$ for any Hilbert space $\mathcal{H}$. We will take the liberty to use this identification without any further mention. In particular, the set $\{S_i \tilde{S}_j\}_{i,j \in \mathbb{N}}$ thus forms an orthonormal basis for $\mathcal{K}_1$.

Now, let $F_{i,j} := f_i \otimes f_j$ whenever $j \leq i$ and 0 otherwise. Define $\varphi : \mathcal{K}_1 \to \mathcal{K}_2$ by

$$\varphi\left(\sum c_{i,j} S_i \tilde{S}_j\right) = \sum c_{i,j} F_{i,j}, \quad \sum_{i,j} |c_{i,j}|^2 < \infty.$$ 

Clearly, $\varphi$ is a conjugate linear (contractive) continuous map. Thus, the map $m : \mathcal{K}_1 \times \mathcal{K}_2 \to \mathbb{C}$ given by $m(S, T) = \langle T, \varphi(S) \rangle$ is a bounded bilinear form. We assert that it is not biregular.

Towards this end, let $T_i := f_i \otimes f_1$ and $\tilde{T}_j := f_1 \otimes f_j$ for all $i, j \in \mathbb{N}$. Note that both the pairs of sequences $\{T_i\}$, $\{\tilde{T}_j\}$ and $\{S_i\}$, $\{\tilde{S}_j\}$ are bounded sequences in the unit balls of $\mathcal{K}_2$ and $\mathcal{K}_1$, respectively. Observe that

$$m(S_i \tilde{S}_j, T_i \tilde{T}_j) = \left\langle T_i \tilde{T}_j, \varphi(S_i \tilde{S}_j) \right\rangle = \text{Tr} \left(\varphi(S_i \tilde{S}_j) T_i \tilde{T}_j \right)$$

$$= \sum_{r \in \mathbb{N}} \left\langle T_i \tilde{T}_j(f_r), \varphi(S_i \tilde{S}_j)(f_r) \right\rangle$$

for all $i, j \in \mathbb{N}$. Thus,

$$\lim_i \lim_j m(S_i \tilde{S}_j, T_i \tilde{T}_j) = \lim_i \lim_j \sum_{r \in \mathbb{N}} \left\langle T_i \tilde{T}_j(f_r), \varphi(S_i \tilde{S}_j)(f_r) \right\rangle = \lim_i \lim_j \langle f_i, F_{i,j}(f_j) \rangle = 0; \text{ and}$$

$$\lim_j \lim_i m(S_i \tilde{S}_j, T_i \tilde{T}_j) = \lim_j \lim_i \sum_{r \in \mathbb{N}} \left\langle T_i \tilde{T}_j(f_r), \varphi(S_i \tilde{S}_j)(f_r) \right\rangle = \lim_j \lim_i \langle f_i, F_{i,j}(f_j) \rangle = 1.$$

Thus, the bilinear form $m$ is not biregular and hence, by Theorem 2.1, $\mathcal{K}_1 \otimes \gamma \mathcal{K}_2$ is not Arens regular. $\square$
Corollary 4.2. For $1 \leq p, q \leq 2$, the Banach algebra $S_p(\mathcal{H}_1) \otimes^\gamma S_q(\mathcal{H}_2)$ is not Arens regular.

Proof. Since $S_p(\mathcal{H}) \subset S_2(\mathcal{H})$ for every $1 \leq p \leq 2$, the bilinear form $m : S_p(\mathcal{H}_1) \times S_q(\mathcal{H}_2) \to \mathbb{C}$ defined as $m(S, T) = \langle T, \varphi(S) \rangle$ for some conjugate linear bounded map $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ is still bounded because

$$|m(S, T)| \leq ||T||_2 ||S||_2 \leq ||\varphi||_p ||S||_p ||T||_q$$

for all $(S, T) \in S_p(\mathcal{H}_1) \times S_q(\mathcal{H}_2)$. For the same choices of pairs of sequences and the conjugate linear map $\varphi$ as in previous theorem, $m$ is not biregular. □

We can thus deduce the following well known result:

Corollary 4.3. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Then, the Banach algebra $S_p(\mathcal{H}_1) \otimes^\gamma S_q(\mathcal{H}_2)$ is not reflexive for every pair $1 \leq p, q \leq 2$.

The following must be a folklore. We include a proof because of lack of knowledge of a proper reference.

Proposition 4.4. Let $A$ be an Arens regular Banach algebra and $B$ be a finite dimensional Banach algebra. Then, $A \otimes^\gamma B$ is Arens regular.

Proof. Since $B$ is finite dimensional, for each $b'' \in B^{**}$ the maps $\tau_{b''} : B^{***} \to B^{***}$ and $\varphi_{b''}$ defined as $x'' \mapsto x''b''$ and $x'' \mapsto b''x''$ are compact. Hence, by [13 Th. 4.5], each bilinear form $m : A \times B \to \mathbb{C}$ is biregular; so, by Theorem 2.1, $A \otimes B$ is Arens regular.

Corollary 4.5. Let $K_1 = S_2(\mathcal{H}_1)$ and $K_2 = S_2(\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces. Then, $K_1 \otimes^\gamma K_2$ is Arens regular if and only if either $K_1$ or $K_2$ is finite dimensional.

Proof. Necessity follows from Theorem 4.1. We saw a proof of sufficiency in Proposition 5.1 and it also follows from Proposition 4.4. Alternately, if $\mathcal{H}_2$ is finite dimensional then every map $K_1 \to K_2$ is compact. This implies that $K_1 \otimes K_2$ is reflexive (by [13 Th. 4.21]) and hence Arens regular.

Being a $C^*$-algebra, $B(\mathcal{H})$ is Arens regular for any Hilbert space $\mathcal{H}$. One would guess that $B(K) \otimes^\gamma K$ should also be Arens regular, but it turns out to be the opposite.

Theorem 4.6. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $K$ denote the Banach algebra $S_2(\mathcal{H})$. Then, the Banach algebra $B(K) \otimes^\gamma K$ is not Arens regular.

Proof. For $\xi, \eta \in \mathcal{H}$, let $\theta_{\xi, \eta}$ be the rank-one operator on $\mathcal{H}$ given by $\theta_{\xi, \eta}(\gamma) = \langle \gamma, \eta \rangle \xi$ for $\gamma \in \mathcal{H}$. Clearly,

$$\theta_{\xi, \eta} \circ \theta_{\zeta, \delta} = \langle \zeta, \eta \rangle \theta_{\xi, \delta}.$$ 

Fix a unit vector $\xi_0 \in \mathcal{H}$ and define a bilinear form $m : B(K) \times K \to \mathbb{C}$ as

$$m(T, A) = \langle T(A), \theta_{\xi_0, \xi_0} \rangle, \quad T \in B(K), A \in K.$$ 

Clearly, $m$ is bounded. We show that $m$ is not biregular, which will then, by Theorem 2.1, imply that $B(K) \otimes^\gamma K$ is not Arens regular.

Let $(e_i)$ be an orthonormal sequence in $\mathcal{H}$ with $e_1 = \xi_0$. Set $S_i = \theta_{e_1, e_i} \in B(\mathcal{H})$ and let $R_j$ denote the orthogonal projection onto the span of $\{e_1, e_2, \ldots, e_j\}$. Then,

$$\lim_{i \to 0} \langle S_i R_j (e_i), e_1 \rangle = \lim_{i \to 0} \langle S_i (e_i), e_1 \rangle = \lim_{i \to 0} \langle e_1, e_i \rangle = 1.$$ 

And, on the other hand, we have

$$\lim_{j \to 0} \lim_{i \to 0} \langle S_i R_j (e_i), e_1 \rangle = \lim_{j \to 0} \lim_{i \to 0} \langle R_j (e_i), e_1 \rangle \langle e_i, e_1 \rangle = 0.$$ 

Now, let $\tilde{A}_j := \theta_{e_1, e_i}$ and $A_i := \theta_{e_i, e_i}$; so that $A_i \tilde{A}_j = A_i$ for all $i, j \in \mathbb{N}$. Clearly, $\|A_i\|, \|\tilde{A}_j\| \leq 1$ for all $i, j \in \mathbb{N}$. 
Further, for each \( i, j \in \mathbb{N} \), define \( \tilde{T}_j, T_i : S_2(\mathcal{H}) \to S_2(\mathcal{H}) \) by
\[
T_i(A) = \theta_{S_i(A(e_i)), e_i}, \quad \tilde{T}_j(A) = \theta_{R_j(A(e_j)), e_j}
\]
for all \( i, j \in \mathbb{N} \) and \( j \in \mathbb{N} \). Then, we have
\[
\tilde{T}_j(A)(e_1) = \theta_{R_j(A(e_1)), e_1}(e_1) = R_j(A(e_1)) \quad \text{for all } A \in S_2(\mathcal{H}).
\]
One can easily check that \( \|T_i\|, \|\tilde{T}_j\| \leq 1 \) for all \( i, j \in \mathbb{N} \). Then, we have
\[
\tilde{T}_j(A)(e_1) = \theta_{R_j(A(e_1)), e_1}(e_1) = R_j(A(e_1)) \quad \text{for all } A \in S_2(\mathcal{H}).
\]
In particular, \( T_i \tilde{T}_j(A) = \theta_{S_i, R_j(A(e_1)), e_1} \) for all \( A \in S_2(\mathcal{H}) \), which then yields \( T_i \tilde{T}_j(A, \tilde{A}_j) = T_i \tilde{T}_j(A, \tilde{A}_j) = \theta_{S_i, R_j(A(e_1)), e_1} \) for all \( i, j \in \mathbb{N} \). In particular, we have
\[
m(T_i \tilde{T}_j, A_1, \tilde{A}_1) = \langle T_i \tilde{T}_j(A_1, \tilde{A}_1), D \rangle = \text{Tr} \left( \theta_{S_i, R_j(A(e_1), e_1)} \right) = \langle S_i, R_j(A(e_1), e_1) \rangle
\]
for all \( i, j \in \mathbb{N} \). Thus, Equations 10, 11 and 11 tell us that \( m \) is not biregular. \( \square \)

For any Hilberts space \( \mathcal{H} \), \( B_0(\mathcal{H}) \), the space of compact operators on \( \mathcal{H} \), being a C*-algebra, is Arens regular. The preceding technique also shows that \( B_0(\mathcal{K}) \otimes \gamma \) \( \mathcal{K} \) is not Arens regular.

**Corollary 4.7.** Let \( \mathcal{H} \) and \( \mathcal{K} \) be as in Theorem 4.6. Then, the Banach algebra \( B_0(\mathcal{K}) \otimes \gamma \mathcal{K} \) is not Arens regular.

**Proof.** Notice that the operators \( T_i \) and \( \tilde{T}_j \) constructed in Theorem 4.6 are finite rank operators on \( \mathcal{K} \) and hence compact. Thus, the same pairs of sequences \( \{T_i\}, \{\tilde{T}_j\} \) and \( \{A_1\}, \{\tilde{A}_j\} \) tell us that the bounded bilinear form \( m : B_0(\mathcal{K}) \times \mathcal{K} \to \mathbb{C} \) defined by \( m(T, A) = \langle T(A), D \rangle \) for some \( D \in \mathcal{K} \), is not biregular. Rest is again taken care of by Theorem 2.1. \( \square \)

**Corollary 4.8.** Let \( \mathcal{H} \) and \( \mathcal{K} \) be as in Theorem 4.6. Then, the Banach algebras \( B(\mathcal{K}) \otimes \gamma S_p(\mathcal{H}) \) and \( B_0(\mathcal{K}) \otimes \gamma S_p(\mathcal{H}) \), for \( 1 \leq p \leq 2 \), are not Arens regular.

**Proof.** Notice that the bilinear form \( m : B(\mathcal{K}) \times S_p(\mathcal{H}) \to \mathbb{C} \) defined as \( m(T, A) = \langle T(A), D \rangle \) is still a well defined bounded bilinear form because \( S_p(\mathcal{H}) \subset S_2(\mathcal{H}) \) (as \( \|\cdot\|_2 \leq \|\cdot\|_p \) for \( p \leq 2 \)). Hence, for the same choice of the pairs of bounded sequences as in the preceding theorem, \( m \) is not biregular. \( \square \)

**Corollary 4.9.** Let \( \mathcal{H} \) and \( \mathcal{K} \) be as in Theorem 4.6. Then, the Banach algebra \( B(\mathcal{H}) \otimes \gamma S_p(\mathcal{H}) \) is not Arens regular for every \( 1 \leq p \leq 2 \).

**Proof.** Note that \( \mathcal{H} \) and \( S_2(\mathcal{H}) \) are isomorphic Hilbert spaces because dimension of \( \mathcal{H} \) and \( S_2(\mathcal{H}) \) are same. (If \( \{e_\alpha\}_{\alpha \in I} \) is an orthonormal basis for \( \mathcal{H} \) then \( \{e_i \otimes e_j\}_{i, j \in I} \) is an orthonormal basis for \( S_2(\mathcal{H}) \) ). Rest follows from Corollary 4.8. \( \square \)

**Remark 4.10.** We have thus observed that if \( \mathcal{A} \) is any of the Banach algebras \( B(\mathcal{K}) \) or \( S_p(\mathcal{H}) \) for \( p \in [1, \infty) \), and \( \mathcal{B} \) is any of the Banach algebras \( S_1(\mathcal{H}) \) or \( S_2(\mathcal{H}) \), then \( \mathcal{A} \otimes \gamma \mathcal{B} \) is not Arens regular.

**Remark 4.11.** Taking \( \ell^p \) with pointwise multiplication, it was shown in \([15, \text{Corollary 4.7}]\) that \( \ell^p \otimes \gamma \mathcal{A} \) is Arens regular if and only if \( \mathcal{A} \) is Arens regular. Even though \( S_p(\mathcal{H}) \) is Arens regular, we have thus observed that a similar characterization does not hold for \( S_p(\mathcal{H}) \otimes \gamma \mathcal{A} \).

We concluded this section with some immediate consequences.

**Remark 4.12.** (1) Let \( \mathcal{H} \) be an infinite dimensional Hilbert space. Although \( \mathcal{K} \) is an operator algebra (see \([9]\)), the Banach algebra \( \mathcal{K} \otimes \gamma \mathcal{K} \) is not an operator algebra (by Theorem 4.11), since operator algebras being a subalgebra of \( B(\mathcal{H}) \) must be Arens regular (subalgebra of a Arens regular Banach algebra is Arens regular).

(2) The bilinear form \( m \) defined in Theorem 4.6 serves as an example of a form which is Arens regular (see Theorem 5.2) but not biregular (as proved in Theorem 4.6), although one of the algebra is a unital C*-algebra.
5. Arens regularity of tensor product of Hilbert-Schmit spaces with Schur product

When $\mathcal{H}$ is infinite dimensional and separable, then there is another important multiplication on $S_p(\mathcal{H})$ which is given by pointwise multiplication of matrices of operators with respect to a fixed orthonormal basis, say, $\{e_n : n \in \mathbb{N}\}$ and is known as the Schur product (or the Hadamard product). Schur product of two operators $T$ and $S$ is denoted by $T \star S$. It is known that $S_p(\mathcal{H})$ forms a (commutative) Banach algebra with respect to Schur product as well. We will discuss only $S_2(\mathcal{H})$ in this section.

For each $T \in S_2(\mathcal{H})$, $T_{rs} := \langle T(e_s), e_r \rangle$ for all $r, s \in \mathbb{N}$; so that $\|T\|_2^2 = \sum_{r,s=1}^{\infty} |T_{rs}|^2$.

**Lemma 5.1.** Let $\{V^{(i)}\}$ and $\{W^{(i)}\}$ be two bounded sequences converging weakly to $V$ and $W$, respectively, in $S_2(\mathcal{H})$. Then, $V^{(i)} \star U \rightarrow V \star U$ in norm in $S_2(\mathcal{H})$ for every $U \in S_2(\mathcal{H})$.

**Proof.** Since $V^{(i)}$ converges weakly to $V$, we have

$$V^{(i)}_{rs} = \langle V^{(i)}(e_s), e_r \rangle \rightarrow \langle V(e_s), e_r \rangle = V_{rs}$$

for all $r, s \in \mathbb{N}$.

Now, let $U \in S_2(\mathcal{H})$. Then, for any pair $m, n \in \mathbb{N}$, we have

$$\limsup_i \|V^{(i)} \star U - V \star U\|_2^2 = \limsup_i \sum_{r,s} |V^{(i)}_{rs} U_{rs} - V_{rs} U_{rs}|^2$$

$$= \limsup_i \sum_{r,s=1}^{m,n} |V^{(i)}_{rs} U_{rs} - V_{rs} U_{rs}|^2$$

$$+ \limsup_i \sum_{r>m, s>n} |V^{(i)}_{rs} U_{rs} - V_{rs} U_{rs}|^2$$

$$= \limsup_i \sum_{r>m, s>n} |V^{(i)}_{rs} U_{rs} - V_{rs} U_{rs}|^2$$

(by Equation \[12\])

$$\leq \left( \sup_{r>m, s>n} |U_{rs}|^2 \right) \left( \sum_{r>m, s>n} |V^{(i)}_{rs} - V_{rs}|^2 \right).$$

Note that $\sum_{r,s=1}^{\infty} |U_{rs}|^2 < \infty$. So, for every $\epsilon > 0$, there exist $m, n \in \mathbb{N}$ such that $\sup_{r>m, s>n} |U_{rs}|^2 < \epsilon$. And, since

$$\left( \sum_{r>m, s>n} |V^{(i)}_{rs} - V_{rs}|^2 \right) \leq 2 \left( \sup_i \|V^{(i)}\|_2^2 + \|V\|_2^2 \right) < \infty,$$

we conclude that $\limsup_i \|V^{(i)} \star U - V \star U\|_2^2 = 0$. Thus, $V^{(i)} \star U \rightarrow V \star U$ in norm. \qed

**Theorem 5.2.** $S_2(\mathcal{H}) \otimes^\gamma S_2(\mathcal{H})$ is Arens regular if $S_2(\mathcal{H})$ is equipped with Schur product.

**Proof.** Let $m : \mathcal{K} \times \mathcal{K} \rightarrow \mathbb{C}$ be a bounded bilinear form. Then, $m(S, T) = \langle T, \phi(S) \rangle$ for all $T, S \in \mathcal{K}$, for some conjugate linear continuous operator $\phi : \mathcal{K} \rightarrow \mathcal{K}$.

Let $\{S^{(i)}\}, \{\tilde{S}^{(j)}\}$ and $\{T^{(i)}\}, \{\tilde{T}^{(j)}\}$ be two pair of sequences in the unit ball of $\mathcal{K}$ such that both the iterated limits exists. Through a similar reasoning as in \[5.1\] we can assume that these sequences converges weakly to $S, \tilde{S}$ and $T, \tilde{T}$ respectively (in weak topology of $S_2(\mathcal{H})$).

By Lemma \[5.1\] $S^{(i)} \star \tilde{S}^{(j)} \rightarrow S \star \tilde{S}^{(j)}$; so, $\phi(S^{(i)} \star \tilde{S}^{(j)}) \rightarrow \phi(S \star \tilde{S}^{(j)})$ in norm. Hence, $\lim_i m(T^{(i)} \star \tilde{T}^{(j)}, S^{(i)} \star \tilde{S}^{(j)}) = m(T \star \tilde{T}^{(j)}, S \star \tilde{S}^{(j)})$.

Similarly,

$$\lim \lim_j m(T^{(i)} \star \tilde{T}^{(j)}, S^{(i)} \star \tilde{S}^{(j)}) = m(T \star \tilde{T}, S \star \tilde{S})$$.
By a symmetric argument
\[ \lim_i \lim_j m(T^{(i)} \star \tilde{T}^{(j)}, S^{(i)} \star \tilde{S}^{(j)}) = m(T \star \tilde{T}, S \star \tilde{S}). \]
Hence, \( m \) is biregular and (by Theorem 2.1 again) \( S_2(\mathcal{H}) \otimes_{\gamma} S_2(\mathcal{H}) \) is Arens regular (with respect to the Schur product).

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