A CONSTRUCTIVE PROOF OF A GENERAL WIGNER’S THEOREM

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Abstract. This paper presents a constructive proof of a non-bijective, non-separable form of Wigner’s theorem, that uses only a few basic facts about Hilbert spaces, including the existence of orthonormal bases and the Fourier decomposition of a vector. Our proof is based on a proof by Steven Weinberg from 1995, but improves on that proof with greater simplicity and generality. Furthermore, our proof fills in a few holes where Weinberg is not completely mathematically rigorous.

1. Introduction

Wigner’s theorem is a fundamental result in quantum mechanics that allows one to represent symmetry transformations of physical system by unitary or antiunitary operators on a Hilbert space, where the symmetry transformation is given by an isometry of the projective Hilbert space. Wigner’s original proof [1] is incomplete, but many complete proofs have emerged since then, see for example [2, 3, 4]. The proof presented in this paper is heavily based on a proof by Steven Weinberg [5], but simplifies that proof and generalizes it to the case where the symmetry transformation is not surjective. In addition, our proof does not assume that the underlying Hilbert space is separable. Weinberg’s proof is also not completely mathematically rigorous, and includes a possible division by zero which our proof circumvents. Thus, our proof has the advantage of being both simple and general.

Let us present the necessary definitions and facts that will be used in the proof. Given a complex Hilbert space $H$, we define the projective Hilbert space

$$\mathbb{P}H = \{ C\psi : \psi \in H \}.$$ 

Elements of $\mathbb{P}H$ are called rays and represent physical states. We define the ray product $\langle \cdot , \cdot \rangle : \mathbb{P}H \times \mathbb{P}H \rightarrow \mathbb{R}$ as

$$\langle C\psi_1, C\psi_2 \rangle = \frac{|\langle \psi_1, \psi_2 \rangle|}{|\psi_1||\psi_2|},$$

where the angle brackets on the right represent the inner product on $H$. Physically, this represents a transition amplitude between two states. It is clear that this definition is independent of the representatives $\psi_1$ and $\psi_2$ of the rays. If $H$ and $H'$ are two Hilbert spaces, we define an isometry of projective Hilbert spaces as a map $\mathbb{P}H \rightarrow \mathbb{P}H'$, denoted $R \mapsto R'$, which preserves the ray product, i.e. $\langle R_1, R_2 \rangle = \langle R'_1, R'_2 \rangle$ for any rays $R_1, R_2 \in \mathbb{P}H$. In particular, if we take normalized vectors $\psi_i \in R_i$ and $\psi_i' \in R'_i$ for $i = 1, 2$, then

$$|\langle \psi_1, \psi_2 \rangle| = |\langle \psi_1', \psi_2' \rangle|.$$ \hfill (1)
Symmetry transformations are implemented in quantum mechanics by bijective isometries.

Given an isometry of projective Hilbert spaces, Wigner’s theorem provides a linear and unitary or else antilinear and antiunitary operator \( U : H \to H' \). We define these terms as follows: \( U \)

- **antilinear** if \( U(c_1\psi_1 + c_2\psi_2) = c_1^*U\psi_1 + c_2^*U\psi_2, \)
- **unitary** if \( \langle U\psi_1, U\psi_2 \rangle = \langle \psi_1, \psi_2 \rangle, \)
- **antiunitary** if \( \langle U\psi_1, U\psi_2 \rangle = \langle \psi_1, \psi_2 \rangle^* = \langle \psi_2, \psi_1 \rangle, \)

for all \( \psi_1, \psi_2 \in H, c_1, c_2 \in \mathbb{C}. \)

Finally, our proof will use Bessel’s inequality

\[
\sum_\alpha |\langle \psi', \psi \rangle|^2 \leq \|\psi\|^2, \tag{2}
\]

where \( \{\psi_\alpha\} \subset H \) is an orthonormal set and the sum is written in the sense of the convergence of the net of finite partial sums. Note that our inner product is linear in the second argument, in accordance with the physics convention. Equality holds in (2) if and only if

\[
\psi = \sum_\alpha \langle \psi', \psi \rangle \psi_\alpha. \tag{3}
\]

We will also use the fact that every Hilbert space has a orthonormal basis \( \{\psi_\alpha\} \) and that any vector \( \psi \in H \) may be expanded in the Fourier series (3) in this basis. Proofs of these facts can be found in many analysis texts, see for example [6]. We are now ready to state and prove Wigner’s theorem.

2. Statement and Proof

**Wigner’s Theorem.** Let \( H \) and \( H' \) be complex Hilbert spaces and let \( \mathbb{P}H \to \mathbb{P}H', \)

\( R \to R' \) be an isometry. Then there exists an operator \( U : H \to H' \) which is either linear and unitary or else antilinear and antiunitary which respects the isometry in the sense that

\[
\psi \in R \implies U\psi \in R'. \tag{4}
\]

**Proof.** Fix an orthonormal basis \( \{\psi_\alpha\} \), with vectors belonging to rays \( R_\alpha \). Let \( \{\psi'_\alpha\} \) be an arbitrary set of normalized vectors with \( \psi'_\alpha \in R'_\alpha \). By (1), these vectors are orthonormal

\[
|\langle \psi'_\alpha, \psi'_\beta \rangle| = |\langle \psi_\alpha, \psi_\beta \rangle| = \delta_{\alpha\beta}, \tag{5}
\]

Furthermore, if \( \psi \in R \subset H \) and \( \psi' \in R' \) are arbitrary normalized vectors, then by (1) and the saturation condition on Bessel’s inequality we have

\[
||\psi'||^2 = ||\psi||^2 = \sum_\alpha |\langle \psi_\alpha, \psi \rangle|^2 = \sum_\alpha |\langle \psi'_\alpha, \psi' \rangle|^2,
\]

which implies that

\[
\psi' = \sum_\alpha \langle \psi'_\alpha, \psi' \rangle \psi'_\alpha. \tag{6}
\]

Now we begin to construct \( U \). Trivially, we define \( U0 \equiv 0 \). Less trivially, let us single out some index \( \alpha \), call it \( \alpha = 1 \) for convenience, and choose an arbitrary normalized vector \( \psi'_1 \in R'_1 \). We define \( U\psi_1 \equiv \psi'_1 \). If \( \dim H = 1 \), then we define
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$U(c\psi_1) = c'\psi_1'$ or $U(c\psi_1) = c^{*}\psi_1'$ for all $c \in \mathbb{C}$ and conclude the proof. Otherwise, for every other index $\alpha$, we define

$$\xi_{1\alpha} \equiv \xi_{\alpha 1} \equiv \frac{1}{\sqrt{2}}(\psi_1 + \psi_\alpha),$$

and let $S_\alpha$ denote the ray containing $\xi_{1\alpha}$. For any normalized vectors $\psi'_\beta \in R'_\beta$ and $\xi'_{1\alpha} \in S'_\alpha$, we know that

$$|\langle \psi'_\beta, \xi'_{1\alpha} \rangle| = |\langle \psi_\beta, \xi_{1\alpha} \rangle| = \begin{cases} 1/\sqrt{2} & : \beta = 1, \alpha \\ 0 & : \beta \neq 1, \alpha \end{cases}.$$  

There are unique choices for the phase of $\xi'_{1\alpha}$ and $\psi'_\alpha$ such that

$$\langle U\psi_1, \xi'_{1\alpha} \rangle = \langle \psi'_\alpha, \xi'_{1\alpha} \rangle = \frac{1}{\sqrt{2}}.$$  

We define $U\psi_\alpha$ and $U\xi_{1\alpha}$ to be the unique normalized elements of $R'_\alpha$ and $S'_\alpha$ that satisfy the above condition. By (6), we know

$$U\xi_{1\alpha} = \frac{1}{\sqrt{2}}(U\psi_1 + U\psi_\alpha).$$  

We can see the linearity of $U$ beginning to take form. Note that the vectors $\{U\psi_\alpha\}$ are orthonormal by (5).

If dim $H = 2$, then the following paragraph should be skipped. Otherwise, we proceed as usual.

We will continue to define $U$ on several more specialized vectors in order to make defining $U$ on an arbitrary vector as simple as possible. Consider the vectors $\eta_{\alpha\beta} \equiv \frac{1}{\sqrt{3}}(\psi_1 + \psi_\alpha + \psi_\beta)$, $1, \alpha, \beta$ distinct.

For any normalized $\eta'_{\alpha\beta}$ in the transformed ray, we know

$$|\langle U\psi_\gamma, \eta'_{\alpha\beta} \rangle| = |\langle \psi_\gamma, \eta_{\alpha\beta} \rangle| = \begin{cases} 1/\sqrt{3} & : \gamma = 1, \alpha, \beta \\ 0 & : \gamma \neq 1, \alpha, \beta \end{cases}.$$  

We define $U\eta_{\alpha\beta}$ as the unique $\eta'_{\alpha\beta}$ with phase chosen so that the coefficient of $U\psi_1$ is real and positive; then by (6) we have

$$U\eta_{\alpha\beta} = \frac{1}{\sqrt{3}}(U\psi_1 + c_\alpha U\psi_\alpha + c_\beta U\psi_\beta),$$

where $|c_\alpha| = |c_\beta| = 1$. For $\gamma = \alpha, \beta$, equality of $|\langle U\xi_{1\gamma}, U\eta_{\alpha\beta} \rangle|$ and $|\langle \xi_{1\gamma}, \eta_{\alpha\beta} \rangle|$ then implies

$$|1 + c_\gamma| = 2.$$  

This implies that $c_\gamma = 1$, as can easily be shown with a moment of algebraic or geometric consideration. Thus,

$$U\eta_{\alpha\beta} = \frac{1}{\sqrt{3}}(U\psi_1 + U\psi_\alpha + U\psi_\beta).$$  

Next, we consider the vector

$$\xi_{\alpha\beta} = \frac{1}{\sqrt{2}}(\psi_\alpha + \psi_\beta), \quad \alpha, \beta \text{ distinct.}$$
The case where $\alpha = 1$ or $\beta = 1$ reduces to the case we’ve already defined. Following our previous methods, we define $U\xi_{\alpha\beta}$ to be the unique element of the transformed ray such that

$$U\xi_{\alpha\beta} = \frac{1}{\sqrt{2}}(U\psi_\alpha + cU\psi_\beta),$$

where $|c| = 1$. Then equality of $|\langle U\eta_{\alpha\beta}, U\xi_{\alpha\beta} \rangle|$ and $|\langle \eta_{\alpha\beta}, \xi_{\alpha\beta} \rangle|$ implies $|1 + c| = 2$, which again implies $c = 1$.

We now begin to consider vectors with complex coefficients. Consider the vectors $\varphi_{\alpha\beta} = \frac{1}{\sqrt{2}}(\psi_\alpha + i\psi_\beta)$, $\alpha, \beta$ distinct.

We define $U\varphi_{\alpha\beta}$ to be the unique vector in the transformed ray such that

$$U\varphi_{\alpha\beta} = \frac{1}{\sqrt{2}}(U\psi_\alpha + cU\psi_\beta),$$

where $|c| = 1$. Equality of $|\langle U\xi_{\alpha\beta}, U\varphi_{\alpha\beta} \rangle|$ and $|\langle \xi_{\alpha\beta}, \varphi_{\alpha\beta} \rangle|$ yields

$$|1 + c| = |1 + i|,$$

which can be easily shown to imply either

$$c = i \quad (7a)$$

or $c = -i \quad (7b)$.

The crux of the proof is to show that the same option of $(7a)$ or $(7b)$ must be taken for all $\varphi_{\alpha\beta}$. First, observe that $\langle \varphi_{\alpha\beta}, \varphi_{\beta\alpha} \rangle = 0$ but $|\langle U\varphi_{\alpha\beta}, U\varphi_{\beta\alpha} \rangle| = 1$ if different options are taken for $\varphi_{\alpha\beta}$ and $\varphi_{\beta\alpha}$. Thus, the same option must be taken for $\varphi_{\alpha\beta}$ and $\varphi_{\beta\alpha}$. This is all we must show if $\dim H = 2$.

If $\dim H \geq 2$, we consider next $\varphi_{\alpha\beta}$ and $\varphi_{\gamma\beta}$ where $\alpha, \beta,$ and $\gamma$ are all distinct. Suppose $\varphi_{\alpha\beta}$ obeys $(7a)$ and $\varphi_{\gamma\beta}$ obeys $(7b)$. Consider the vector

$$\psi = \frac{1}{\sqrt{3}}(\psi_\alpha + \psi_\gamma + i\psi_\beta)$$

There exists a unique vector $\psi'$ in the transformed ray such that

$$\psi' = \frac{1}{\sqrt{3}}(U\psi_\alpha + c_\gamma U\psi_\gamma + c_\beta U\psi_\beta),$$

where $|c_\gamma| = |c_\beta| = 1$. By taking inner products with $U\xi_{\alpha\gamma}$ and $U\xi_{\alpha\beta}$ and using the isometry property (1), we can conclude that $c_\gamma = 1$ and $c_\beta = \pm i$. If $c_\beta = i$, then equality of $|\langle \psi', U\varphi_{\gamma\beta} \rangle| = 0$ and $|\langle \psi, \varphi_{\gamma\beta} \rangle| = 2/\sqrt{6}$ gives a contradiction. On the other hand, if $c_\beta = -i$, then equality of $|\langle \psi', U\varphi_{\alpha\beta} \rangle| = 0$ and $|\langle \psi, \varphi_{\alpha\beta} \rangle| = 2/\sqrt{6}$ gives a contradiction, so we get a contradiction either way. Therefore, the same choice between $(7a)$ and $(7b)$ must be made between $\varphi_{\alpha\beta}$ and $\varphi_{\gamma\beta}$.

Finally, consider $\varphi_{\alpha\beta}$ and $\varphi_{\gamma\delta}$ for arbitrary indices $\alpha, \beta, \gamma, \delta$. We know the same choice must be made between $\varphi_{\alpha\beta}$ and $\varphi_{\gamma\beta}$, as well as between $\varphi_{\delta\beta}$ and $\varphi_{\beta\delta}$, and also between $\varphi_{\beta\delta}$ and $\varphi_{\gamma\delta}$. Following this chain, we see that the same choice must be made between $\varphi_{\alpha\beta}$ and $\varphi_{\gamma\delta}$, as desired.

The work we’ve done up until now makes defining $U$ appropriately on an arbitrary nonzero vector $\psi$ easy. We expand $\psi$ as

$$\psi = \sum_{\alpha} c_{\alpha} \psi_{\alpha}.$$
Let \( c_\alpha \neq 0 \) for some \( \alpha \). We define \( U\psi \) to be the unique normalized vector in the transformed ray such that the coefficient of \( U\psi_\alpha \) is \( c_\alpha \) if (7a) is obeyed and \( c_\alpha^* \) if (7b) is obeyed. In other words, we define

\[
U\psi \equiv c_\alpha U\psi_\alpha + \sum_{\beta \neq \alpha} c'_\beta U\psi_\beta
\]

if (7a) is obeyed, or

\[
U\psi \equiv c_\alpha^* U\psi_\alpha + \sum_{\beta \neq \alpha} c''_\beta U\psi_\beta
\]

if (7b) is obeyed, where \( |c'_\beta| = |c_\beta| \) for all \( \beta \). This is consistent with the definitions we have made up until now. Now for any nonzero \( c_\beta \), using the square of the isometry property (1) with \( \xi_{\alpha\beta} \) and \( \psi \) leads to

\[
\text{Re}(c_\alpha^*(c_\beta - c_\beta')) = 0,
\]

while using the square of the isometry property with \( \phi_{\alpha\beta} \) and \( \psi \) yields

\[
\text{Im}(c_\alpha^*(c_\beta - c_\beta')) = 0.
\]

Thus, \( c_\alpha^*(c_\beta - c_\beta') = 0 \), which implies that \( c_\beta = c_\beta' \) since \( c_\alpha^* \neq 0 \).

Thus, for every \( \psi \in H \) we see that either

\[
U \left( \sum_\alpha c_\alpha \psi_\alpha \right) = \sum_\alpha c_\alpha U\psi_\alpha \quad \text{or} \quad U \left( \sum_\alpha c_\alpha \psi_\alpha \right) = \sum_\alpha c_\alpha^* U\psi_\alpha,
\]

with the same choice taken across all \( \psi \in H \). This implies that \( U \) is either linear or antilinear. If \( U \) is linear, then for two vectors \( \psi = \sum_\alpha c_\alpha \psi_\alpha \) and \( \varphi = \sum_\alpha d_\alpha \psi_\alpha \), we have

\[
(U\psi, U\varphi) = \left( \sum_\alpha c_\alpha U\psi_\alpha, \sum_\alpha d_\alpha U\psi_\alpha \right) = \sum_\alpha c_\alpha d_\alpha^* = \langle \psi, \varphi \rangle,
\]

so \( \varphi \) is unitary. On the other hand, if \( U \) is antilinear, then

\[
(U\psi, U\varphi) = \left( \sum_\alpha c_\alpha^* U\psi_\alpha, \sum_\alpha d_\alpha^* U\psi_\alpha \right) = \sum_\alpha c_\alpha^* d_\alpha = \langle \varphi, \psi \rangle = \langle \psi, \varphi \rangle^*,
\]

so \( U \) is antiunitary. This concludes the proof.

3. Conclusions

We have presented a simple proof Wigner’s theorem, making as few assumptions as possible. In particular, we have considered an isometry between projective Hilbert spaces that do not necessarily come from the same Hilbert space, we have not assumed the Hilbert spaces to be separable, and we have not assumed the isometry to be bijective. As the proof uses only a few basic facts about Hilbert spaces that should be intuitive for physicists and well-known for mathematicians, we hope that this proof will be useful and accessible to all.

4. Acknowledgments

Thanks to Markus Pflaum for his advice and encouragement regarding this proof. This work was supported by the Department of Energy under Grant No. DE-FG02-91-ER-40672 and by the Center for Theory of Quantum Matter.
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