Prime ends and mappings on Riemann surfaces

V. Ryazanov, S. Volkov

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Abstract

It is proved criteria for continuous and homeomorphic extension to the boundary of mappings with finite distortion between domains on the Riemann surfaces by prime ends of Caratheodory.

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1 Introduction

The theory of the boundary behavior in the prime ends for the mappings with finite distortion has been developed in [12] for the plane domains and in [15] for the spatial domains. The pointwise boundary behavior of the mappings with finite distortion in regular domains on Riemann surfaces was recently studied by us in [31]. Moreover, the problem was investigated in regular domains on the Riemann manifolds for $n \geq 3$ as well as in metric spaces, see e.g. [1] and [34]. It is necessary to mention also that the theory of the boundary behavior of Sobolev’s mappings has significant applications to the boundary value problems for the Beltrami equations and for analogs of the Laplace equation in anisotropic and inhomogeneous media, see e.g. [3, 8–11, 13, 14, 21, 24, 27] and relevant references therein.

For basic definitions and notations, discussions and historic comments in the mapping theory on the Riemann surfaces, see our previous papers [30–32].
2 Definition of the prime ends and preliminary remarks

First recall the necessary definitions of some general notions. Given a topological space $T$, a path in $T$ is a continuous map $\gamma : [a, b] \rightarrow T$. Given $A, B, C \subseteq T$, $\Delta(A, B, C)$ denotes a collection of all paths $\gamma$ joining $A$ and $B$ in $C$, i.e., $\gamma(a) \in A$, $\gamma(b) \in B$ and $\gamma(t) \in C$ for all $t \in (a, b)$. In what follows, $|\gamma|$ denotes the locus of $\gamma$, i.e. the image $\gamma([a, b])$.

We act similarly to Caratheodory [5] under the definition of the prime ends of domains on a Riemann surface $S$, see Chapter 9 in [6]. First of all, recall that a continuous mapping $\sigma : \mathbb{I} \rightarrow S$, $\mathbb{I} = (0, 1)$, is called a Jordan arc in $S$ if $\sigma(t_1) \neq \sigma(t_2)$ for $t_1 \neq t_2$. We also use the notations $\sigma$, $\overline{\sigma}$ and $\partial \sigma$ for $\sigma(\mathbb{I})$, $\overline{\sigma(\mathbb{I})}$ and $\sigma(\mathbb{I}) \setminus \sigma(\mathbb{I})$, correspondingly. A cross–cut of a domain $D \subset S$ is either a closed Jordan curve or a Jordan arc $\sigma$ in the domain $D$ with both ends on $\partial D$ splitting $D$.

A sequence $\sigma_1, \ldots, \sigma_m, \ldots$ of cross-cuts of $D$ is called a chain in $D$ if:

(i) $\overline{\sigma_i} \cap \overline{\sigma_j} = \emptyset$ for every $i \neq j$, $i, j = 1, 2, \ldots$;

(ii) $\sigma_m$ splits $D$ into 2 domains one of which contains $\sigma_{m+1}$ and another one $\sigma_{m-1}$ for every $m > 1$;

(iii) $\delta(\sigma_m) \rightarrow 0$ as $m \rightarrow \infty$.

Here $\delta(E) = \sup_{p_1, p_2 \in S} \delta(p_1, p_2)$ denotes the diameter of a set $E$ in $S$ with respect to an arbitrary metric $\delta$ in $S$ agreed with its topology, see [30]–[31].

Correspondingly to the definition, a chain of cross-cuts $\sigma_m$ generates a sequence of domains $d_m \subset D$ such that $d_1 \supset d_2 \supset \ldots \supset d_m \supset \ldots$ and $D \cap \partial d_m = \sigma_m$. Two chains of cross-cuts $\{\sigma_m\}$ and $\{\sigma'_k\}$ are called equivalent if, for every $m = 1, 2, \ldots$, the domain $d_m$ contains all domains $d'_k$ except a finite number and, for every $k = 1, 2, \ldots$, the domain $d'_k$ contains all domains $d_m$ except a finite number, too. A prime end $P$ of the domain $D$ is an equivalence class of chains of cross-cuts of $D$. Later on, $E_D$ denote the collection of all prime ends of a domain $D$ and $\overline{D}_P = D \cup E_D$ is its completion by prime ends.

Next, we say that a sequence of points $p_l \in D$ is convergent to a prime end $P$ of $D$ if, for a chain of cross-cuts $\{\sigma_m\}$ in $P$, for every $m = 1, 2, \ldots$, the domain $d_m$ contains all points $p_l$ except their finite collection. Further, we say
that a sequence of prime ends \( P_i \) converge to a prime end \( P \) if, for a chain of cross-cuts \( \{ \sigma_m \} \) in \( P \), for every \( m = 1, 2, \ldots \), the domain \( d_m \) contains chains of cross-cuts \( \{ \sigma'_k \} \) in all prime ends \( P_i \) except their finite collection.

Now, let \( D \) be a domain in the compactification \( \overline{S} \) of a Riemann surface \( S \) by Kerekjarto-Stoilow, see a discussion in [30]–[31]. Open neighborhoods of points in \( D \) is induced by the topology of \( \overline{S} \). A basis of neighborhoods of a prime end \( P \) of \( D \) can be defined in the following way. Let \( d \) be an arbitrary domain from a chain in \( P \). Denote by \( d^* \) the union of \( d \) and all prime ends of \( D \) having some chains in \( d \). Just all such \( d^* \) form a basis of open neighborhoods of the prime end \( P \). The corresponding topology on \( \overline{D}_P \) is called the **topology of prime ends**.

Let \( P \) be a prime end of \( D \) on a Riemann surface \( S \), \( \{ \sigma_m \} \) and \( \{ \sigma'_m \} \) be two chains in \( P \), \( d_m \) and \( d'_m \) be domains corresponding to \( \sigma_m \) and \( \sigma'_m \). Then

\[
\bigcap_{m=1}^{\infty} d_m \subseteq \bigcap_{m=1}^{\infty} d'_m \subset \bigcap_{m=1}^{\infty} \overline{d_m},
\]

and, thus,

\[
\bigcap_{m=1}^{\infty} \overline{d_m} = \bigcap_{m=1}^{\infty} \overline{d'_m},
\]

i.e. the set named by a **body of the prime end** \( P \)

\[
I(P) := \bigcap_{m=1}^{\infty} \overline{d_m}
\]

(2.1)

depends only on \( P \) but not on a choice of a chain of cross-cuts \( \{ \sigma_m \} \) in \( P \).

It is necessary to note also that, for any chain \( \{ \sigma_m \} \) in the prime end \( P \),

\[
\Omega := \bigcap_{m=1}^{\infty} d_m = \emptyset.
\]

(2.2)

Indeed, every point \( p \) in \( \Omega \) belongs to \( D \). Moreover, some open neighborhood of \( p \) in \( D \) should belong to \( \Omega \). In the contrary case each neighborhood of \( p \) should have a point in some \( \sigma_m \). However, in view of condition (iii) then \( p \in \partial D \) that should contradict the inclusion \( p \in D \). Thus, \( \Omega \) is an open set and if \( \Omega \) would be not empty, then the connectedness of \( D \) would be broken because \( D = \Omega \cup \Omega^* \) with the open set \( \Omega^* := D \setminus I(P) \).
In view of conditions (i) and (ii), we have by (2.2) that

\[ I(P) = \bigcap_{m=1}^{\infty} (\partial d_m \cap \partial D) = \partial D \cap \bigcap_{m=1}^{\infty} \partial d_m. \]

Thus, we obtain the following statement.

**Proposition 2.1** For each prime end \( P \) of a domain \( D \) on a Riemann surface,

\[ I(P) \subseteq \partial D. \]  \hspace{1cm} (2.3)

**Remark 2.1** If \( D \) is a domain in \( \mathbb{S} \) with \( \partial D \subset \mathbb{S} \), then \( I(P) \) is a continuum, i.e. it is a connected compact set, see e.g. I(9.12) in [37], see also I.9.3 in [4], and \( I(P) \) belongs to only one (connected) component \( \Gamma \) of \( \partial D \). Hence we say that the component \( \Gamma \) is **associated with the prime end** \( P \).

Moreover, every prime end of \( D \) in the case contains a **convergent chain** \( \{\sigma_m\} \), i.e., that is contracted to a point \( p_0 \in \partial D \). Furthermore, each prime end \( P \) contains a **spherical chain** \( \{\sigma_m\} \) lying on circles \( S(p_0, r_m) = \{p \in \mathbb{S} : \delta(p, p_0) = r_m\} \) with \( p_0 \in \partial D \) and \( r_m \to 0 \) as \( m \to \infty \). The proof is perfectly similar to Lemma 1 in [15] after the replacement of metrics, see also Theorem 7.1 in [23], and hence we omit it. Note by the way that the condition (iii) does not depend in the case on the choice of the metric \( \delta \) agreed with the topology of \( \mathbb{S} \) because \( \partial D \) has a compact neighborhood.

It is known that the conformal modulus \( M \) of the family of all paths joining a pair of the opposite sides of a rectangle is equal to the ratio of lengths of other pair of opposite sides and their own, see e.g. I.4.3 in [20]. This simple fact gives a series of useful consequences.

**Corollary 2.1** Let \( S \) be the open sector of the ring \( A = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\} \), \( z_0 \in \mathbb{C} \), between the rays \( R_k = \{z \in \mathbb{C} : z = z_0 + re^{i\alpha_k}, \ r \in (0, \infty)\} \), \( k = 1, 2 \), \( 0 \leq \alpha_1 < \alpha_2 \leq 2\pi \). Then

\[ M(\Delta(R_1, R_2, S)) = \frac{\log \frac{r_2}{r_1}}{\alpha_2 - \alpha_1}, \quad M(\Delta(C_1, C_2, S)) = \frac{\alpha_2 - \alpha_1}{\log \frac{r_2}{r_1}} \] \hspace{1cm} (2.4)

where \( C_k \) are the boundary circles \( \{z \in \mathbb{C} : |z - z_0| = r_k\} \), \( k = 1, 2 \), of the ring \( A \).
Indeed, the conclusion follows from the invariance of the modulus $M$ under conformal mappings because the sector $S$ is mapped by $\log(z - z_0)$ onto the rectangle $R = \{\zeta = \xi + i\eta \in \mathbb{C} : \log r_1 < \xi < \log r_2, \alpha_1 < \eta < \alpha_2\}$.

**Corollary 2.2** Under notations of Corollary 2.1 and $\alpha_2 - \alpha_1 = \Delta$, the modulus of all Jordan arcs joining the rays $R_1$ and $R_2$ in the sector $S$ is greater or equal to the number $\frac{1}{\Delta} \log \frac{r_2}{r_1}$.

Indeed, every path $\gamma : [a, b] \to \mathbb{C}$ in $\Delta(R_1, R_2, S)$ has a countable collection of loops because its preimage (without the corresponding point of cusp in $\mathbb{C}$) is open in $(a, b)$. Thus, numbering its loops and removing them by induction, we come to a Jordan arc $\gamma_*$ in $\Delta(R_1, R_2, S)$ with its locus $|\gamma_*| \subseteq |\gamma|$.

### 3 Some general topological lemmas

The following statement is an analog of Proposition 2.3 in [26], see also Proposition 13.3 in [21].

**Proposition 3.1** Let $T$ be a topological space. Suppose that $E_1$ and $E_2$ are sets in $T$ with $\overline{E_1} \cap \overline{E_2} = \emptyset$. Then

$$\Delta(E_1, E_2, T) > \Delta(\partial E_1, \partial E_2, T \setminus (\overline{E_1} \cup \overline{E_2})) \ . \ (3.1)$$

**Proof.** Indeed, let $\gamma \in \Delta(E_1, E_2, T)$, i.e. the path $\gamma : [a, b] \to T$ is such that $\gamma(a) \in E_1$ and $\gamma(b) \in E_2$. Note that the set $\alpha := \gamma^{-1}(\overline{E_1})$ is a closed subset of the segment $[a, b]$ because $\gamma$ is continuous, see e.g. Theorem 1 in Section I.2.1 of [21]. Consequently, $\alpha$ is compact because $[a, b]$ is a compact space, see e.g. I.9.3 in [21]. Then there is $a_* := \max_{t \in \alpha} t < b$ because $\gamma(b) \in E_2$ and by the hypothesis of the proposition $\overline{E_1} \cap \overline{E_2} = \emptyset$. Thus, $\gamma' := \gamma|_{[a, b]}$ belongs to $\Delta(\partial E_1, E_2, T \setminus \overline{E_1})$ because $\gamma$ is continuous and hence $\gamma'(a_*)$ cannot be an inner point of $E_1$.

Arguing similarly in the space $T' = T \setminus E_1$ with $E'_1 := E_2$ and $E'_2 := \partial E_1$, we obtain that there is $b_* := \min_{\gamma'(t) \in E_2} t > a_*$. Thus, by the given construction $\gamma_* := \gamma|_{[a_*, b_*]}$ just belongs to $\Delta(\partial E_1, \partial E_2, T \setminus (\overline{E_1} \cup \overline{E_2}))$. $\Box$
Lemma 3.1 In addition to the hypothesis of Proposition 3.1, let $T$ be a subspace of a metric space $(M, \rho)$. Suppose that

$$\partial E_1 \subseteq C_1 := \{ p \in M : \rho(p, p_0) = R_1 \}, \quad \partial E_2 \subseteq C_2 := \{ p \in M : \rho(p, p_0) = R_2 \}$$

with $p_0 \in M \setminus T$ and $R_1 < R_2$. Then

$$\Delta (E_1, E_2, T) > \Delta (C_1, C_2, A) \quad (3.2)$$

where

$$A = A(p_0, R_1, R_2) := \{ p \in M : R_1 < \rho(p, p_0) < R_2 \}.$$  

Note that here, generally speaking, $C_1 \cap T \neq E_1$ and $C_2 \cap T \neq E_2$ as well as $\gamma_*$ in the proof of Proposition 3.1 is not in $R$.

Proof. First of all, note that by the continuity of $\gamma_*$ the set $\omega := \gamma_*^{-1}(R)$ is open in $[a_*, b_*]$ and $\omega$ is the union of a countable collection of disjoint intervals $(a_1, b_1), (a_2, b_2), \ldots$ with ends in $\Gamma := \gamma_*^{-1}(\partial R)$. If there is a pair $a_k$ and $b_k$ in the different sets $\Gamma_i := \gamma_*^{-1}(C_i), i = 1, 2, \Gamma = \Gamma_1 \cup \Gamma_2, \Gamma_1 \cap \Gamma_2 = \emptyset$, then the proof is complete.

Let us assume that such a pair is absent. Then the given collection is split into 2 collections of disjoint intervals $(a_l', b_l')$ and $(a_l'', b_l'')$ with ends $a_l', b_l' \in \Gamma_1$ and $a_l'', b_l'' \in \Gamma_2, l = 1, 2, \ldots$. Set $\alpha_1 = \bigcup_{l} (a_l', b_l')$ and $\alpha_2 = \bigcup_{l} (a_l'', b_l'')$.

Arguing by contradiction, it is easy to show that $\gamma_* : [a_*, b_*] \to (M, \rho)$ is uniformly continuous because $[a_*, b_*]$ is a compact space. Indeed, let us assume that there is $\varepsilon > 0$ and a sequence of pairs $a_n^* \in [a_*, b_*], n = 1, 2, \ldots$, such that $|b_n^* - a_n^*| \to 0$ as $n \to \infty$ and simultaneously $\rho(\gamma_*(a_n^*), \gamma_*(b_n^*)) \geq \varepsilon$. However, by compactness of $[a_*, b_*]$ there is a subsequence $a_{n_k}^* \to a_0 \in [a_*, b_*]$ and then also $b_{n_k}^* \to a_0$ as $k \to \infty$. Hence by the continuity of $\gamma_*$ it should be $\rho(\gamma_*(a_{n_k}^*), \gamma_*(a_0)) \to 0$ as well as $\rho(\gamma_*(b_{n_k}^*), \gamma_*(a_0)) \to 0$ and then by the triangle inequality also $\rho(\gamma_*(a_{n_k}^*), \gamma_*(b_{n_k}^*)) \to 0$ as $k \to \infty$. The contradiction disproves the assumption.

Note that $b_l' - a_l' \to 0$ as $l \to \infty$ and by the uniform continuity of $\gamma_*$ on $[a_*, b_*]$ we have that $|\gamma_{l'}| \to C_1$ in the sense that

$$\sup_{p \in |\gamma_{l'}|} \inf_{q \in C_1} \rho(p, q) \to 0 \quad \text{as} \quad l \to \infty$$
where \( \gamma'_l := \gamma_*|_{[a'_l, b'_l]}, l = 1, 2, \ldots \). Thus, there is \( R'_2 \in (R_1, R_2) \) such that the set \( L_1 := \bigcup_l |\gamma'_l| \) lies outside of \( B_2 := \{ p \in M : \rho(p, p_0) > R'_2 \} \).

Arguing similarly, we obtain that there is \( R'_1 \in (R_1, R'_2) \) such that the set \( L_2 := \bigcup_l |\gamma''_l| \) lies outside of \( B_1 := \{ p \in M : \rho(p, p_0) < R'_1 \} \). Remark that the sets \( \beta_1 := \gamma^{-1}_*(B_1) \) and \( \beta_2 := \gamma^{-1}_*(B_2) \) are open in \( [a_*, b_*] \) because \( \gamma_* \) is continuous and by the construction \( \delta_1 := \alpha_1 \cup \beta_1 \) and \( \delta_2 := \alpha_2 \cup \beta_2 \) are open, mutually disjoint and together cover the segment \( [a_*, b_*] \). The latter contradicts to connectedness of the segment and, thus, disproves the above assumption. \( \square \)

4 The main lemma

**Lemma 4.1** Let \( S \) be a Riemann surface, \( D \) be a domain in \( \overline{S} \) with \( \partial D \subset S \) and let \( \Gamma \) be a isolated component of \( \partial D \). Then \( \Gamma \) has a neighborhood \( U \) with a conformal mapping \( H \) of \( U^* := U \cap D \) onto a ring \( R = \{ z \in \mathbb{C} : 0 \leq r < |z| < 1 \} \) where \( \gamma := \partial U^* \cap D \) is a closed Jordan curve,

\[
C(\gamma, H) = \{ z \in \mathbb{C} : |z| = 1 \}, \quad C(\Gamma, H) = \{ z \in \mathbb{C} : |z| = r \}
\]

and \( r = 0 \) if and only if \( \Gamma \) is degenerated to a point. Furthermore, the mapping \( H \) can be extended to a homeomorphism \( \bar{H} \) of \( \overline{U^*} \) onto \( \overline{R} \).

Here we use the notation of the cluster set of the mapping \( H \) for \( B \subset \partial D \),

\[
C(B, H) := \left\{ z \in \mathbb{C} : z = \lim_{k \to \infty} H(p_k), \ p_k \to p \in B, \ p_k \in D \right\}.
\]

**Proof.** By the Kerekjarto–Stoilow representation of \( S \), \( \Gamma \) has an open neighborhood \( V \) in \( S \) of a finite genus and we may assume that \( \overline{V} \) is a compact subset of \( S \), \( V \) is connected and does not intersect \( \partial D \setminus \Gamma \) because \( \Gamma \) is an isolated component of \( \partial D \). Thus, \( V \cap D \) is a Riemann surface of finite genus with an isolated boundary element \( g \) corresponding to \( \Gamma \). However, a Riemann surface of finite genus has boundary elements only of the first kind, see, e.g., IV.II.6 in [35]. Consequently, \( \Gamma \) has a neighborhood \( U^* \) from the side of \( D \) of genus zero with a closed Jordan curve \( \gamma = \partial U^* \cap D \). The latter means that \( U^* \) is homeomorphic to a plane domain and, consequently, by the general principle
of Koebe, see e.g. Section II.3 in [17], \( U^* \) is conformally equivalent to a plane domain \( D^* \). Note that by the construction \( U^* \) has two nondegenerate boundary components. Hence there is a conformal mapping \( H \) of \( U^* \) onto a ring \( D^* = R = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \) with \( C(\gamma, H) = C_1 := \{ z \in \mathbb{C} : |z| = 1 \} \) and \( C(\Gamma, H) = C_r := \{ z \in \mathbb{C} : |z| = r \} \), see e.g. Proposition 2.5 in [26] or Proposition 13.5 in [21]. Set \( U = U^* \cup (V \setminus D) \).

If \( \Gamma \) is not degenerated into a point, then \( r \neq 0 \). Indeed, in the contrary case the images of the closed Jordan curves around the origin in the punctured disk \( \mathbb{D}_\varepsilon = \{ z \in \mathbb{C} : 0 < |z| < \varepsilon \} \) under the mapping \( H^{-1} \) should be contracted to \( \Gamma \) as \( \varepsilon \to 0 \) and hence their lengths are not less than \( \delta := \text{diam} \Gamma > 0 \) for small enough \( \varepsilon \). However, the latter contradicts to the conformal invariance of the modulus because by Corollary 2.2 the modulus of all such closed Jordan curves is equal to \( \infty \). Inversely, if \( \Gamma \) is degenerated into a point \( p_0 \in S \), then it is obvious that \( r = 0 \) because \( p_0 \) has arbitrarily small neighborhoods that are conformally mapped onto the unit disk in \( \mathbb{C} \). Hence we omit the consideration of this trivial case and restrict ourselves by the case \( r > 0 \).

Now, by the condition (i) in the definition of prime ends and the invariance of \( M \) we have, for every chain \( \{\sigma_m\} \) in a prime end \( P \) associated with \( \Gamma \) and localized in \( U^* \), that

\[
M(\Delta(\sigma_m, \sigma_{m+1}, U^*)) < \infty \quad \forall \ m = 1, 2, \ldots \quad (4.1)
\]

Moreover, by Remark 2.1 \( P \) contains a chain \( \{\sigma_m\} \) lying on circles \( S_m = S(p_0, r_m) = \{ p \in S : \delta(p, p_0) = r_m \} \) with \( p_0 \in \partial D \) and \( r_m \to 0 \) as \( m \to \infty \) for which and any continuum \( C \) in \( U^* \)

\[
\lim_{m \to \infty} M(\Delta(\sigma_m, C, U^*)) \leq \lim_{m \to \infty} M(\Delta(\sigma_m, \sigma_{m_0}, U^*)) = 0. \quad (4.2)
\]

Indeed, for every continuum \( C \) in \( U^* \), there is \( m_0 \) such that \( C \subset D \setminus d_{m_0} \) and the closed ball \( B_0 = B(p_0, r_{m_0}) = \{ p \in S : \delta(p, p_0) \leq r_{m_0} \} \) is compact and lies in a chart \( U_0 \) of \( p_0 \). Then \( \Delta(\sigma_m, C, U^*) \subset \Delta(\sigma_m, D \setminus d_{m_0}, U^*) \), by Proposition 3.1 \( \Delta(\sigma_m, D \setminus d_{m_0}, U^*) > \Delta(\sigma_m, \sigma_{m_0}, U^*) \) and by Lemma ?? \( \Delta(\sigma_m, \sigma_{m_0}, U^*) > \Delta(S_m, S_{m_0}, A) \) where \( A := \{ p \in S : r_m < \delta(p, p_0) < r_{m_0} \} \) belongs to the chart \( U_0 \) of the point \( p_0 \). Note, \( M(\Delta(S_m, S_{m_0}, A)) \leq M(\Delta(S_m, S_{m_0}, U_0)) \to 0 \) as \( m \to \infty \) because \( S_{m_0} \) is a compact set in \( B_0 \setminus \{p_0\} \) and \( S_m \) is contracted to \( p_0 \).
as \( m \to \infty \), see also 7.5 in [36]. Finally, we obtain (4.2) by the minorization principle, see e.g. [7], p. 178. Similarly, it is proved that prime ends associated with \( \gamma \) also satisfy conditions (4.1) and (4.2).

Thus, the prime ends of \( U^* \) in the sense (i)–(iii) and their images in \( R \) are the prime ends in the sense of Section 4 in [22]. By Lemma 3.5 in [22] the prime ends of Näkki in \( R \) coincide with prime ends of Caratheodory. Moreover, the Näkki prime ends in \( R \) has a one-to-one correspondence with the points of \( \partial R \) whose extension to the mapping between \( \overline{R} \) and \( \overline{R}_P \) by the identity in \( R \) is a homeomorphism with respect to the topologies of \( \overline{R} \) and \( \overline{R}_P \) or with respect to convergence of points and prime ends, respectively, see Theorems 4.1 and 4.2 in [22]. Consequently, if \( p_k \) is a sequence of points in \( U^* \) which is convergent to a prime end \( P \) of \( U^* \), then \( H(p_k) \) is convergent to a unique point \( z_0 \in \partial R \) that depends only on \( P \).

Denote by \( \tilde{H} \) the extension of \( H \) to \( \overline{U^*}_P \). It is clear by definitions of prime ends of Näkki and Caratheodory as classes of equivalence that \( \tilde{H}(P_1) \neq \tilde{H}(P_2) \) for every prime ends \( P_1 \neq P_2 \) of the domain \( U^* \). Let us consider the metric \( \rho(P, P^*):=|\tilde{H}(P)−\tilde{H}(P^*)| \) on the space \( \overline{U^*}_P \). It is obvious by definitions that \( \rho(p_k, P_0) \to 0 \) implies that \( P_k \to P_0 \) as \( k \to \infty \). The inverse conclusion follows because of the mapping \( \tilde{H} : \overline{U^*}_P \to \overline{R} \) is continuous. Indeed, let \( P_k \to P_0, k = 1, 2, \ldots, \) be a sequence in \( \overline{U^*}_P \). It is obvious, \( \tilde{H}(P_k) \to \tilde{H}(P_0) \) for \( P_0 \in U^* \). If \( P_0 \in E_{U^*} \), then we are able to choose \( p_k \in U^* \) such that \( |\tilde{H}(P_k)−\tilde{H}(p_k)| < 2^{-k}, k = 1, 2, \ldots, \) and \( p_k \to P_0 \) as \( k \to \infty \). The latter implies that \( \tilde{H}(p_k) \to \tilde{H}(P_0) \) and then the former implies that \( \tilde{H}(P_k) \to \tilde{H}(P_0) \). Thus, the space \( \overline{U^*}_P \) is metrizable with the given metric \( \rho \) and \( \tilde{H} \) is an isometric embedding of \( \overline{U^*}_P \) in \( \overline{R} \). By construction \( \tilde{H}(U^*) = R \) and, by Proposition 2.5 in [26] or Proposition 13.5 in [21], \( \tilde{H}(E_{U^*}) \subseteq \partial R \). Let us show that \( \tilde{H}(E_{U^*}) = \partial R \).

For this goal, fixing \( z_0 \in \partial C_r \) and \( \varepsilon \in (0, 1) \), consider the family \( \mathcal{S} \) of all Jordan arcs in the open disk \( B_\varepsilon = B(z_0, \varepsilon) := \{ z \in \mathbb{C} : |z−z_0| < \varepsilon \} \) joining in \( R \) the two open arcs \( A_1 \) and \( A_2 \) of \( C_r \cap B_\varepsilon \setminus \{ z_0 \} \). By the minorization principle, see e.g. [7], and the invariance of \( M \) (with respect to the conformal mapping consisting of the composition of the inversion with respect to the unit circle and the reflection with respect to the straight line \( L_0 \) passing through the origin and
the point $z_0$) we obtain from Corollary 2.2 that the conformal modulus of the family $\mathcal{F}$ is equal to $\infty$. By the invariance of the modulus under conformal mappings we have that the modulus of the family $\mathcal{F}_\ast = H^{-1}(\mathcal{F})$ is also equal to $\infty$. Consequently, the length of elements of $\mathcal{F}_\ast$ cannot be restricted from below and, by arbitrariness of $\varepsilon$, there is a sequence of mutually disjoint cross-cuts $\sigma_m \in \mathcal{F}$ of $R$ with $\sigma_m(0) \in A_1$ and $\sigma_m(1) \in A_2$ that is contracted to the point $z_0$ such that $\delta(\sigma^*_m) \to 0$ as $m \to \infty$ where $\sigma^*_m = H^{-1}(\sigma_m)$ and, moreover, $\sigma^*_{m+1} \subset d^*_m$ where $d^*_m$ is the corresponding component of $D$ generated by $\sigma^*_m$, $\partial d^*_m \cap U^* = \sigma^*_m$ for all $m = 1, 2, \ldots$. Note that such rectifiable $\sigma^*_m : (0, 1) \to D$ have limits $p_m^{(1)} = \lim_{t \to +0} \sigma^*_m(t)$ and $p_m^{(2)} = \lim_{t \to 1-0} \sigma^*_m(t)$ because $\overline{U^*}$ is a compact subset of $\mathbb{S}$, see e.g. Proposition I.9.3 in [4], cf. also Theorem 1.3.2 in [36], moreover, the points $p_m^{(1)}$ and $p_m^{(2)}$ belongs to $\Gamma$, see e.g. Proposition 2.5 in [26] or Proposition 13.5 in [21].

Finally, it remains to show that $\overline{\sigma^*_m} \cap \overline{\sigma^*_{m+1}} = \emptyset$, passing in case of need to a suitable subchain of cross-cuts $\sigma_m$ in $R$. First of all, by the above construction we may assume that

$$\delta_m := \inf_{z \in \sigma_m} |z - z_0| > \delta^*_m := \sup_{z \in \sigma^*_{m+1}} |z - z_0| > 0 \quad \forall m = 1, 2, \ldots$$

and also that $\sigma^*_m$ is contracted to a point $p_0 \in \Gamma$ because $\Gamma$ is compact and $\delta(\sigma^*_m) \to 0$. It is clear that the desired subchain exists if $\sigma^*_m(0) \neq p_0 \neq \sigma^*_m(1)$ for all large enough $m$.

In the contrary case, it would exist a subchain $\tilde{\sigma}_k := \sigma_{m_k}$, $k = 1, 2, \ldots$, such that either $\tilde{\sigma}^*_k(0) = p_0 = \tilde{\sigma}^*_{k+1}(0)$ or $\tilde{\sigma}^*_k(1) = p_0 = \tilde{\sigma}^*_{k+1}(1)$ for all $k = 1, 2, \ldots$, where $\tilde{\sigma}^*_k := H^{-1}(\tilde{\sigma}_k)$, $k = 1, 2, \ldots$. In the first case, consider the ring $A = \{ z \in \mathbb{C} : r_1 < |z - z_0| < r_2 \}$ with $0 < \delta^*_{m_k} < r_1 < r_2 < \delta_{m_k}$. As above, by the minorization principle, the invariance of $M$ and Corollary 2.1 the conformal modulus of the family $\tilde{\mathcal{F}}$ of all paths in $A \cap R$ joining the open arc $A_0 := A \cap A_1$ of the circle $C_r$ and the interval $I_0 := A \cap L_0$ of the straight line $L_0$ is not less than $\frac{2}{\pi} \log \frac{r_2}{r_1} > 0$. The modulus of the family $\tilde{\mathcal{F}}_\ast = H^{-1}(\tilde{\mathcal{F}})$ should be the same. However, the modulus of $\tilde{\mathcal{F}}_\ast$ is equal to zero because all paths in $\tilde{\mathcal{F}}_\ast$ are ended at the point $p_0$.

Indeed, denote by I the maximal open interval of $L_0$ containing $I_0$ and not
intersecting \( \tilde{\sigma}_k \) and \( \tilde{\sigma}_{k+1} \), and by \( t_0 \) and \( t_* \) the parameter numbers in \((0,1)\) corresponding to its ends on \( \tilde{\sigma}_k \) and \( \tilde{\sigma}_{k+1} \). Then \( H^{-1}(I), \tilde{\sigma}_k^*((0,t_0]), \tilde{\sigma}_{k+1}^*((0,t_*]) \) and the point \( p_0 \) form a closed Jordan curve in \( \overline{U}^* \) with the only point on \( \partial U^* \). Note that the corresponding Jordan domain contains the family \( \tilde{\mathfrak{F}}_* \) of paths \( \gamma \) that should be ended on \( \Gamma \) and, consequently, at the point \( p_0 \). The second possibility is similarly disproved.

Thus, \( \tilde{H} \) is isometry between \( \overline{U^*}_P \) with the given metric \( \rho \) and \( \overline{R} \). \( \square \)

**Remark 4.1** By the proof we have that \( \overline{U^*}_P \) is a compact space with the metric \( \rho \). Moreover, it follows from the proof that the spaces of prime ends by Caratheodory and N"aki coincide not only in the ring \( R \) but also in \( U^* \) because the N"aki prime ends are invariant under conformal mappings.

Furthermore, if \( D \) be a domain in the Kerekjarto-Stoilow compactification \( \overline{\mathbb{S}} \) of a Riemann surface \( \mathbb{S} \) and \( \partial D \) is a set in \( \mathbb{S} \) with a finite collection of components, then their prime ends by Caratheodory and N"aki also coincide, the whole space \( \overline{D}_P \) can be metrized through the theory of pseudometric spaces, see e.g. Section 2.21.XV in \[18\], and \( \overline{D}_P \) is compact.

Namely, let \( \rho_0 \) be one of the metrics on \( \overline{\mathbb{S}} \) and let \( \rho_1, \ldots, \rho_n \) be the above metrics on \( \overline{U^*}_1 \), \ldots, \( \overline{U^*}_n \) for the corresponding components \( \Gamma_1, \ldots, \Gamma_n \) of \( \partial D \). Here we may assume that the sets \( \overline{U^*_j} \) are mutually disjoint. Then \( \rho_j^* := \rho_j/(1+\rho_j) \leq 1, j = 0,1,\ldots,n \), are also metrics generating the same topologies on \( D_0 := D \setminus \bigcup_j \overline{U^*_j}, \overline{U^*_1}, \ldots, \overline{U^*_n} \), correspondingly, see e.g. Section 2.21.V in \[18\], and the topology of prime ends on \( \overline{D}_P \) is generated by the metric \( \rho = \sum_{j=0}^{n} 2^{-(j+1)} \tilde{\rho}_j \leq 1 \) where the pseudometrics \( \tilde{\rho}_j \) are extensions of \( \rho_j^* \) onto \( \overline{D}_P \) by 1, see e.g. Remark 2 in point 2.21.XV of \[18\]. Note that the space \( \overline{D}_P \) is compact because \( \overline{D}_P = \bigcup \overline{U^*_j} \cup D_0 \) where \( D_0 \) is a compact space as a closed subset of the compact space \( \overline{\mathbb{S}} \), see e.g. Proposition I.9.3 in \[4\].

**Corollary 4.1** Under hypothesis of Lemma 4.1, the space of all prime ends associated with a nondegenerate isolated component of \( \partial D \) is homeomorphic to a circle.
5 On boundary behavior in prime ends of inverse maps

The main base for extending inverse mappings is the following fact.

**Lemma 5.1** Let $S$ and $S'$ be Riemann surfaces, $D$ and $D'$ be domains in $\overline{S}$ and $\overline{S'}$, $\partial D \subset S$ and $\partial D' \subset S'$ have finite collections of components, and let $f : D \rightarrow D^*$ be a homeomorphism of finite distortion with $K_f \in L^1_{\text{loc}}$. Then

\[ C(P_1, f) \cap C(P_2, f) = \emptyset \quad (5.1) \]

for all prime ends $P_1 \neq P_2$ in the domain $D$.

Here we use the notation of the cluster set of the mapping $f$ at $P \in E_D$,

\[ C(P, f) := \left\{ P' \in E_{D'} : P' = \lim_{k \to \infty} f(p_k), \ p_k \to P, \ p_k \in D \right\} \]

As usual, we also assume here that the dilatation $K_f$ of the mapping $f$ is extended by zero outside of the domain $D$.

**Proof.** First of all note that $\overline{S}$ and $\overline{S'}$ are metrizable spaces. Hence their compactness is equivalent to their sequential compactness, see e.g. Remark 41.I.3 in [19], and, consequently, $\partial D$ and $\partial D'$ are compact subsets of $S$ and $S'$, correspondingly, see e.g. Proposition I.9.3 in [4]. Thus, by Lemma 4.1, Remarks 2.1 and 4.1 we may assume that $\overline{D}$ is a compact set in $S$, $K_f \in L^1(D)$, $P_1$ and $P_2$ are associated with the same component $\Gamma$ of $\partial D$ and $D'$ is a ring $R = \{ z \in \mathbb{C} : 0 < r < |z| < 1 \}$ and

\[ A_k := C(P_k, f) , \quad k = 1, 2 \]

are sets of points in the circle $C_r := \{ z \in \mathbb{C} : |z| = r \}$, $\partial D$ consists of 2 components: $\Gamma$ and a closed Jordan curve $\gamma$, $f$ is extended to a homeomorphism of $D \cup \gamma$ onto $D' \cup C_1$, $C(C_r, f^{-1}) = \Gamma$, see also Proposition 2.5 in [26] or Proposition 13.5 in [21]. Note that the sets $A_k$ are continua, i.e. closed arcs of the circle $C_r$, because

\[ A_k = \bigcap_{m=1}^{\infty} f(\frac{d_m^{(k)}}{d_m^{(k)}}) , \quad k = 1, 2 , \]

where $d_m^{(k)}$ are domains corresponding to chains of cross–cuts $\{ \sigma_m^{(k)} \}$ in the prime ends $P_k$, $k = 1, 2$, see e.g. I(9.12) in [37] and also I.9.3 in [4]. In
addition, by Remark 2.1 we may assume also that $\sigma_{m}^{(k)}$ are open arcs of the circles $C_{m}^{(k)} := \{p \in S : h(p, p_{k}) = r_{m}^{(k)}\}$ on $S$ with $p_{k} \in \partial D$ and $r_{m}^{(k)} \to 0$ as $m \to \infty, \ k = 1, 2$.

Set $p_{0} = p_{1}$. By the definition of the topology of the prime ends in the space $\overline{D}_{P}$, we have that $d_{m}^{(1)} \cap d_{m}^{(2)} = \emptyset$ for all large enough $m$ because $P_{1} \neq P_{2}$. For a such $m$, set $R_{1} = r_{m+1}^{(1)} < R_{2} = r_{m}^{(1)}$ and

$$U_{k} = d_{m}^{(k)}, \quad \Sigma_{k} = \sigma_{m}^{(k)}, \quad C_{k} = \{p \in S : h(p, p_{0}) = R_{k}\}, \ k = 1, 2.$$  

Let $K_{1}$ and $K_{2}$ be arbitrary continua in $U_{1}$ and $U_{2}$, correspondingly. Applying Proposition 3.1 and Lemma 3.1 with $T = D, E_{1} = d_{m+1}^{(1)}$ and $E_{2} = D \setminus d_{m}^{(1)}$, and taking into account the inclusion $\Delta(K_{1}, K_{2}, D) \subset \Delta(E_{1}, E_{2}, D)$, we obtain that

$$\Delta(K_{1}, K_{2}, D) > \Delta(C_{1}, C_{2}, A), \quad A := \{p \in S : R_{1} < h(p, p_{0}) < R_{2}\}, \quad (5.2)$$

which means that any path $\alpha : [a, b] \to S$ joining $K_{1}$ and $K_{2}$ in $D$, $\alpha(a) \in K_{1}$, $\alpha(b) \in K_{2}$ and $\alpha(t) \in D, t \in (a, b)$, has a subpath joining $C_{1}$ and $C_{2}$ in $A$. Thus, since $f$ is a homeomorphism, we have also that

$$\Delta(fK_{1}, fK_{2}, fD) > \Delta(fC_{1}, fC_{2}, fA) \quad (5.3)$$

and by the minorization principle, see e.g. [7], p. 178, we obtain that

$$M(\Delta(fK_{1}, fK_{2}, fD)) \leq M(\Delta(fC_{1}, fC_{2}, fA)). \quad (5.4)$$

So, by Lemma 3.1 in [31] we conclude that

$$M(\Delta(fK_{1}, fK_{2}, fD)) \leq \int_{A} K_{f}(p) \cdot \xi^{2}(h(p, p_{0})) \, dh(p) \quad (5.5)$$

for all measurable functions $\xi : (R_{1}, R_{2}) \to [0, \infty]$ such that

$$\int_{R_{1}}^{R_{2}} \xi(R) \, dR \geq 1. \quad (5.6)$$

In particular, for $\xi(R) \equiv 1/\delta$, $\delta = R_{2} - R_{1} > 0$, we get from here that

$$M(\Delta(fK_{1}, fK_{2}, fD)) \leq M_{0} := \frac{1}{\delta} \int_{D} K_{f}(p) \, dh(p) < \infty. \quad (5.7)$$
Since \( f \) is a homeomorphism, (5.7) means that
\[
M(\Delta(K_1, K_2, D')) \leq M_0 < \infty \quad (5.8)
\]
for all continua \( K_1 \) and \( K_2 \) in the domains \( V_1 = fU_1 \) and \( V_2 = fU_2 \), correspondingly.

Let us assume that \( A_1 \cap A_2 \neq \emptyset \). Then by the construction there is \( p_0 \in \partial R \cap \partial V_1 \cap \partial V_2 \). However, the latter contradicts (5.8) because the ring \( R \) is a QED (quasiextremal distance) domains, see e.g. Theorem 3.2 in [21], see also Theorem 10.12 in [36]. \( \Box \)

**Theorem 5.1** Let \( S \) and \( S' \) be Riemann surfaces, \( D \) and \( D' \) be domains in \( S \) and \( S' \), correspondingly, \( \partial D \subset S \) and \( \partial D' \subset S' \) have finite collections of nondegenerate components, and let \( f : D \to D' \) be a homeomorphism of finite distortion with \( K_f \in L^1_{loc} \). Then the inverse mapping \( g = f^{-1} : D' \to D \) can be extended to a continuous mapping \( \tilde{g} \) of \( \overline{D'} \) onto \( \overline{D} \).

**Proof.** Recall that by Remark 4.1 the spaces \( \overline{D} \) and \( \overline{D'} \) are compact and metrizable with metrics \( \rho \) and \( \rho' \). Let a sequence \( p_n \in D' \) converges as \( n \to \infty \) to a prime end \( P' \in E_{D'} \). Then any subsequence of \( p^*_n := g(p_n) \) has a convergent subsequence by compactness of \( \overline{D} \). By Lemma 5.1 any such convergent subsequence should have the same limit. Thus, the sequence \( p^*_n \) is convergent, see e.g. Theorem 2 of Section 2.20.II in [18]. Note that \( p^*_n \) cannot converge to an inner point of \( D \) because \( I(P) \subset \partial D \) by Proposition 2.1 and, consequently, \( p_n \) is convergent to \( \partial D' \), see e.g. Proposition 2.5 in [26] or Proposition 13.5 in [21]. Thus, \( E_{D'} \) is mapped into \( E_D \) under this extension \( \tilde{g} \) of \( g \). In fact, \( \tilde{g} \) maps \( E_{D'} \) onto \( E_D \) because \( p_n = f(p^*_n) \) has a convergent subsequence for every sequence \( p^*_n \in D \) that is convergent to a prime end \( P \) of the domain \( D \) because \( \overline{D'} \) is compact. The map \( \tilde{g} \) is continuous. Indeed, let a sequence \( P'_n \in \overline{D'} \) be convergent to \( P' \in \overline{D'} \). Then there is a sequence \( p_n \in D' \) such that \( \rho'(P'_n, p_n) < 2^{-n} \) and \( \rho(p^*_n, P^*_n) < 2^{-n} \) where \( p^*_n := g(p_n) \), \( P^*_n := \tilde{g}(P_n) \) and \( P^* = \tilde{g}(P') \). Then \( p_n \to P' \) and by the above \( p^*_n \to P^* \) as well as \( P^*_n \to P^* \) as \( n \to \infty \). \( \Box \)
Lemma 6.1 Under the hypothesis of Theorem 5.1, let in addition
\[ \int_{R(p_0, \varepsilon, \varepsilon_0)} K_f(p) \cdot \psi_{p_0, \varepsilon, \varepsilon_0}^2(h(p, p_0)) \, dh(p) = o \left( I_{p_0, \varepsilon_0}^2(\varepsilon) \right) \quad \forall \ p_0 \in \partial D \quad (6.1) \]
as \( \varepsilon \to 0 \) for all \( \varepsilon_0 < \delta(p_0) \) where \( R(p_0, \varepsilon, \varepsilon_0) = \{ p \in \mathbb{S} : \varepsilon < h(p, p_0) < \varepsilon_0 \} \) and \( \psi_{p_0, \varepsilon, \varepsilon_0}(t) : (0, \infty) \to [0, \infty], \varepsilon \in (0, \varepsilon_0), \) is a family of measurable functions such that
\[ 0 < I_{p_0, \varepsilon_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{p_0, \varepsilon, \varepsilon_0}(t) \, dt < \infty \quad \forall \ \varepsilon \in (0, \varepsilon_0) . \]

Then \( f \) can be extended to a continuous mapping \( \tilde{f} \) of \( \overline{D}_P \) onto \( \overline{D'}_P \).

We assume here that the function \( K_f \) is extended by zero outside of \( D \).

Proof. By and Lemma 4.1, Remarks 2.1 and 4.1, arguing as in the beginning of the proof of Lemma 5.1, we may assume that \( \overline{D} \) is a compact set in \( \mathbb{S} \), \( \partial D \) consists of 2 components: a closed Jordan curve \( \gamma \) and one more nondegenerate component \( \Gamma \), \( D' \) is a ring \( R = \{ z \in \mathbb{C} : 0 < r < |z| < 1 \} \), \( \overline{D'}_P = \overline{R} \),
\[ C(\Gamma, f) = C_r := \{ z \in \mathbb{C} : |z| = r \}, \quad C(\gamma, f) = C_* := \{ z \in \mathbb{C} : |z| = 1 \} \]
and that \( f \) is extended to a homeomorphism of \( D \cup \gamma \) onto \( D' \cup C_* \).

Let us first prove that the set \( L := C(P, f) \) consists of a single point of \( C_r \) for a prime end \( P \) of the domain \( D \) associated with \( \Gamma \). Note that \( L \neq \emptyset \) by compactness of the set \( \overline{R} \) and, moreover, \( L \subseteq C_r \) by Proposition 2.1.

Let us assume that there is at least two points \( \zeta_0 \) and \( \zeta_* \in L \). Set \( U = \{ \zeta \in \mathbb{C} : |\zeta - \zeta_0| < \rho_0 \} \) where \( 0 < \rho_0 < |\zeta_* - \zeta_0| \).
Let $\sigma_k$, $k = 1, 2, \ldots$, be a chain in the prime end $P$ from Remark 2.1 lying on the circles $S_k := \{p \in \mathbb{S} : h(p, p_0) = r_k\}$ where $p_0 \in \Gamma$ and $r_k \to 0$ as $k \to \infty$. Let $d_k$ be the domains associated with $\sigma_k$. Then there exist points $\zeta_k$ and $\zeta_k^*$ in the domains $d'_k = f(d_k) \subset R$ such that $|\zeta_0 - \zeta_k| < \rho_0$ and $|\zeta_0 - \zeta_k^*| > \rho_0$ and, moreover, $\zeta_k \to \zeta_0$ and $\zeta_k^* \to \zeta_0^*$ as $k \to \infty$. Let $\gamma_k$ be paths joining $\zeta_k$ and $\zeta_k^*$ in $d'_k$. Note that by the construction $\partial U \cap \gamma_k \neq \emptyset$, $k = 1, 2, \ldots$.

By the condition of strong accessibility of the point $\zeta_0$ in the ring $R$, there is a continuum $E \subset R$ and a number $\delta > 0$ such that

$$M(\Delta(E, \gamma_k; R)) \geq \delta$$

(6.2)

for all large enough $k$. Note that $C = f^{-1}(E)$ is a compact subset of $D$ and hence $h(p_0, C) > 0$. Let $\varepsilon_0 \in (0, \delta_0)$ where $\delta_0 := \min(\delta(p_0), h(p_0, C))$. Without loss of generality, we may assume that $r_k < \varepsilon_0$ and that (6.2) holds for all $k = 1, 2, \ldots$.

Let $\Gamma_m$ be the family of paths joining the circle $S_0 := \{p \in \mathbb{S} : h(p, p_0) = \varepsilon_0\}$ and $\sigma_m$, $m = 1, 2, \ldots$, in the intersection of $D \setminus d_m$ and the ring $R_m := \{p \in \mathbb{S} : r_m < h(p, p_0) < \varepsilon_0\}$. Applying Proposition 3.1 and Lemma 3.1 with $T = D$, $E_1 = d_m$ and $E_2 = B_0 := \{p \in \mathbb{S} : h(p, p_0) > \varepsilon_0\}$, and taking into account the inclusion $\Delta(C, C_k, D) \subset \Delta(E_1, E_2, D) = \Delta(B_0, d_m, D)$ where $C_k = f^{-1}(\gamma_k)$, we have that $\Delta(C, C_k, D) > \Gamma_m$ for all $k \geq m$ because by the construction $C_k \subset d_k \subset d_m$. Thus, since $f$ is a homeomorphism, we have also that $\Delta(E, \gamma_k, D) > f \Gamma_m$ for all $k \geq m$, and by the principle of minorization, see e.g. [7], p. 178, we obtain that $M(f(\Gamma_m)) \geq \delta$ for all $m = 1, 2, \ldots$.

On the other hand, every function $\xi(t) = \xi_m(t) := \psi_{p_0, r_m, \varepsilon_0}(t)/I_{p_0, \varepsilon_0}(r_m)$, $m = 1, 2, \ldots$, satisfies the condition (5.6) and by Lemma 3.1 in [31]

$$M(f \Gamma_m) \leq \int_{R_m} K_f(p) \cdot \xi_m^2(h(p, p_0)) \, dh(p),$$

i.e., $M(f \Gamma_m) \to 0$ as $m \to \infty$ in view of (6.1).

The obtained contradiction disproves the assumption that the cluster set $C(P, f)$ consists of more than one point.

Thus, we have the extension $\bar{f}$ of $f$ to $\overline{D_P}$ such that $\bar{f}(E_D) \subseteq E_{D'}$. In fact, $\bar{f}(E_D) = E_{D'}$. Indeed, if $\zeta_0 \in D'$, then there is a sequence $\zeta_n$ in $D'$ that is
convergent to $\zeta_0$. We may assume with no loss of generality that $f^{-1}(\zeta_n) \to P_0 \in \overline{D_P}$ because $\overline{D_P}$ is compact, see Remark 4.1. Hence $\zeta_0 \in E_D$ because $\zeta_0 \notin D$, see e.g. Proposition 2.5 in [26] or Proposition 13.5 in [21].

Finally, let us show that the extended mapping $\tilde{f} : \overline{D_P} \to \overline{D'_P}$ is continuous. Indeed, let $P_n \to P_0$ in $\overline{D_P}$. The statement is obvious for $P_0 \in D$. If $P_0 \in E_D$, then by the last item we are able to choose $P_n^* \in D$ such that $\rho(P_n, P_n^*) < 2^{-n}$ and $\rho'(\tilde{f}(P_n), \tilde{f}(P_n^*)) < 2^{-n}$ where $\rho$ and $\rho'$ are some metrics on $\overline{D_P}$ and $\overline{D'_P}$, correspondingly, see Remark 4.1. Note that by the first part of the proof $f(P_n^*) \to f(P_0)$ because $P_n^* \to P_0$. Consequently, $\tilde{f}(P_n) \to \tilde{f}(P_0)$, too. \hfill \Box

Remark 6.1 Note that condition (6.1) holds, in particular, if

$$\int_{D(p_0, \varepsilon_0)} K_f(p) \cdot \psi^2(h(p, p_0)) \, dh(p) < \infty \quad \forall \ p_0 \in \partial D \quad (6.3)$$

where $D(p_0, \varepsilon_0) = \{p \in \mathbb{S} : h(p, p_0) < \varepsilon_0\}$ and where $\psi(t) : (0, \infty) \to [0, \infty]$ is a locally integrable function such that $I_{p_0, \varepsilon_0}(\varepsilon) \to \infty$ as $\varepsilon \to 0$. In other words, for the extendability of $f$ to a continuous mapping of $\overline{D_P}$ onto $\overline{D'_P}$, it suffices for the integrals in (6.3) to be convergent for some nonnegative function $\psi(t)$ that is locally integrable on $(0, \infty)$ but that has a non-integrable singularity at zero.

7 On the homeomorphic extension to the boundary

Combining Lemma 6.1 and Theorem 5.1, we obtain the significant conclusion:

Lemma 7.1 Under the hypothesis of Lemma 6.1, the homeomorphism $f : D \to D'$ can be extended to a homeomorphism $\tilde{f} : \overline{D_P} \to \overline{D'_P}$.

Proof. Indeed, by Lemma 5.1 the mapping $\tilde{f} : \overline{D_P} \to \overline{D'_P}$ from Lemma 6.1 is injective and hence it has the well defined inverse mapping $\tilde{f}^{-1} : \overline{D'_P} \to \overline{D_P}$ and the latter coincides with the mapping $\tilde{g} : \overline{D'_P} \to \overline{D_P}$ from Theorem 5.1 because a limit under a metric convergence is unique. The continuity of the mappings $\tilde{g}$ and $\tilde{f}$ follows from Theorem 5.1 and Lemma 6.1, respectively. \hfill \Box

We assume everywhere in this section that the function $K_f$ is extended by zero outside of $D$. 
Theorem 7.1 Under the hypothesis of Theorem 5.1, let in addition
\[
\int_0^{\varepsilon_0} \frac{dr}{||K_f||(p_0, r)} = \infty \quad \forall \ p_0 \in \partial D, \quad \varepsilon_0 < \delta(p_0) \quad (7.1)
\]
where
\[
||K_f||(p_0, r) := \int_{S(p_0, r)} K_f(p) \ ds_h(p). \quad (7.2)
\]
Then \( f \) can be extended to a homeomorphism of \( \overline{D}_P \) onto \( \overline{D'}_P \).

Here \( S(p_0, r) \) denotes the circle \( \{ p \in \mathbb{S} : h(p, p_0) = r \} \).

Proof. Indeed, for the functions
\[
\psi_{p_0, \varepsilon_0}(t) := \begin{cases} 
1/||K_f||(p_0, t), & t \in (0, \varepsilon_0), \\
0, & t \in [\varepsilon_0, \infty),
\end{cases} \quad (7.3)
\]
we have by the Fubini theorem that
\[
\int_{R(p_0, \varepsilon, \varepsilon_0)} K_f(p) \cdot \psi_{p_0, \varepsilon_0}^2(h(p, p_0)) \ dh(p) = \int_{\varepsilon}^{\varepsilon_0} \frac{dr}{||K_f||(p_0, r)} \quad (7.4)
\]
where \( R(p_0, \varepsilon, \varepsilon_0) \) denotes the ring \( \{ p \in \mathbb{S} : \varepsilon < h(p, p_0) < \varepsilon_0 \} \) and, consequently, condition (6.1) holds by (7.1) for all \( p_0 \in \partial D \) and \( \varepsilon_0 \in (0, \varepsilon(p_0)) \).

Here we have used the standard conventions in the integral theory that \( a/\infty = 0 \) for \( a \neq \infty \) and \( 0 \cdot \infty = 0 \), see, e.g., Section I.3 in [33].

Thus, Theorem 7.1 follows immediately from Lemma 6.1. \( \square \)

Corollary 7.1 In particular, the conclusion of Theorem 7.1 holds if
\[
k_{p_0}(r) = O \left( \log \frac{1}{r} \right) \quad \forall \ p_0 \in \partial D \quad (7.5)
\]
as \( r \to 0 \) where \( k_{p_0}(r) \) is the average of \( K_f \) over the infinitesimal circle \( S(p_0, r) \).

Choosing in (6.1) \( \psi(t) := \frac{1}{t \log 1/t} \), we obtain by Lemma 6.1 the next result, see also Lemma 4.1 in [26] or Lemma 13.2 in [21].
Theorem 7.2 Under the hypothesis of Theorem 5.1, let $K_f$ have a dominant $Q_{p_0}$ in a neighborhood of each point $p_0 \in \partial D$ with finite mean oscillation at $p_0$. Then $f$ can be extended to a homeomorphism $\tilde{f} : \overline{D}_P \to \overline{D}'_P$.

By Corollary 4.1 in [26] or Corollary 13.3 in [21] we obtain the following.

Corollary 7.2 In particular, the conclusion of Theorem 7.2 holds if

$$\lim_{\varepsilon \to 0} \int_{D(p_0, \varepsilon)} K_f(p) \, dh(p) < \infty \quad \forall \ p_0 \in \partial D \quad (7.6)$$

where $D(p_0, \varepsilon)$ is the infinitesimal disk $\{ p \in \mathbb{S} : h(p, p_0) < \varepsilon \}$.

Corollary 7.3 The conclusion of Theorem 7.2 holds if every point $p_0 \in \partial D$ is a Lebesgue point of the function $K_f$ or its dominant $Q_{p_0}$.

The next statement also follows from Lemma 6.1 under the choice $\psi(t) = 1/t$.

Theorem 7.3 Under the hypothesis of Theorem 5.1, let, for some $\varepsilon_0 > 0$,

$$\int_{\varepsilon < h(p, p_0) < \varepsilon_0} K_f(p) \frac{dh(p)}{h^2(p, p_0)} = o \left( \left[ \log \frac{1}{\varepsilon} \right]^2 \right) \quad \text{as} \ \varepsilon \to 0 \ \forall \ p_0 \in \partial D \quad (7.7)$$

Then $f$ can be extended to a homeomorphism of $\overline{D}_P$ onto $\overline{D}'_P$.

Remark 7.1 Choosing in Lemma 6.1 the function $\psi(t) = 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$, (7.7) can be replaced by the more weak condition

$$\int_{\varepsilon < h(p, p_0) < \varepsilon_0} \frac{K_f(p) \, dh(p)}{\left( h(p, p_0) \log \frac{1}{h(p, p_0)} \right)^2} = o \left( \left[ \log \log \frac{1}{\varepsilon} \right]^2 \right) \quad (7.8)$$

and (7.5) by the condition

$$k_{p_0}(r) = o \left( \log \frac{1}{\varepsilon} \log \log \frac{1}{\varepsilon} \right) \quad (7.9)$$

Of course, we could give here the whole scale of the corresponding condition of the logarithmic type using suitable functions $\psi(t)$. 

8 On interconnections between integral conditions

For every non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$, the inverse function $\Phi^{-1}$ can be well defined by setting

$$\Phi^{-1}(\tau) = \inf_{\Phi(t) \geq \tau} t . \quad (8.1)$$

As usual, here $\inf$ is equal to $\infty$ if the set of $t \in [0, \infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function $\Phi^{-1}$ is non-decreasing, too.

**Remark 8.1** Immediately by the definition it is evident that

$$\Phi^{-1}(\Phi(t)) \leq t \quad \forall \ t \in [0, \infty] \quad (8.2)$$

with the equality in (8.2) except intervals of constancy of the function $\Phi(t)$.

Recall that a function $\Phi : [0, \infty] \to [0, \infty]$ is called convex if

$$\Phi(\lambda t_1 + (1 - \lambda) t_2) \leq \lambda \Phi(t_1) + (1 - \lambda) \Phi(t_2)$$

for all $t_1, t_2 \in [0, \infty]$ and $\lambda \in [0, 1]$.

In what follows, $\mathbb{H}(R)$ denotes the hyperbolic disk centered at the origin with the hyperbolic radius $R = \log (1 + r)/(1 - r)$, $r \in (0, 1)$ is its Euclidean radius:

$$\mathbb{H}(R) = \{ z \in \mathbb{C} : h(z, 0) < R \} , \quad R \in (0, \infty) . \quad (8.3)$$

Further we also use the notation of the **hyperbolic sine**: $\sinh t := (e^t - e^{-t})/2$.

The following statement is an analog of Lemma 3.1 in [29] adopted to the hyperbolic geometry in the unit disk $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$.

**Lemma 8.1** Let $Q : \mathbb{H}(\varepsilon) \to [0, \infty]$, $\varepsilon \in (0, 1)$, be a measurable function and $\Phi : [0, \infty] \to (0, \infty]$ be a non-decreasing convex function with a finite mean integral value $M(\varepsilon)$ of the function $\Phi \circ Q$ on $\mathbb{H}(\varepsilon)$. Then

$$\int_0^\varepsilon \frac{d\rho}{\rho q(\rho)} \geq \frac{1}{2} \int_0^\infty \frac{d\tau}{\tau [\Phi^{-1}(\tau)]} \quad (8.4)$$

where $q(\rho)$ is the average of $Q$ on the circle $S(\rho) = \{ z \in \mathbb{D} : h(z, 0) = \rho \}$ and

$$\delta(\varepsilon) = \exp \left( 4 \sinh^2 \frac{\varepsilon}{2} \right) \cdot \frac{M(\varepsilon)}{\varepsilon^2} > \tau_0 := \Phi(0) > 0 . \quad (8.5)$$
Proof. Since \( M(\varepsilon) < \infty \) we may assume with no loss of generality that \( \Phi(t) < \infty \) for all \( t \in [0, \infty) \) because in the contrary case \( Q \in L^\infty \) and then the left-hand side in (8.4) is equal to \( \infty \). Moreover, we may assume that \( \Phi(t) \) is not constant because in the contrary case \( \Phi^{-1}(\tau) \equiv \infty \) for all \( \tau > \tau_0 \) and hence the right-hand side in (8.4) is equal to 0. Note also that \( \Phi(\tau) \) is (strictly) increasing, convex and continuous in the segment \([t_*, \infty)\) and

\[
\Phi(t) \equiv \tau_0 \quad \forall \ t \in [0, t_*] \quad \text{where} \quad t_* := \sup_{\Phi(t)=\tau_0} t . \quad (8.6)
\]

Setting \( H(t) := \log \Phi(t) \), we see that \( H^{-1}(\eta) = \Phi^{-1}(e^\eta) \), \( \Phi^{-1}(\tau) = H^{-1}(\log \tau) \). Thus, we obtain that

\[
q(\rho) = H^{-1} \left( \log \left( \frac{h(\rho)}{\rho^2} \right) \right) = H^{-1} \left( 2 \log \frac{1}{\rho} + \log h(\rho) \right) \quad \forall \ \rho \in R_* \quad (8.7)
\]

where \( h(\rho) := \rho^2 \Phi(q(\rho)) \) and \( R_* = \{ \rho \in (0, \varepsilon) : q(\rho) > t_* \} \). Then also

\[
q(e^{-s}) = H^{-1} \left( 2s + \log h(e^{-s}) \right) \quad \forall \ s \in S_* \quad (8.8)
\]

where \( S_* = \{ s \in (\log \frac{1}{\varepsilon}, \infty) : q(e^{-s}) > t_* \} \).

Now, by the Jensen inequality, see e.g. Theorem 2.6.2 in [25], we have that

\[
\int_{\log \frac{1}{\varepsilon}}^{\varepsilon} h(e^{-s}) \, ds = \int_{0}^{\varepsilon} h(\rho) \frac{d\rho}{\rho} = \int_{0}^{\varepsilon} \Phi(q(\rho)) \, \rho \, d\rho \leq 2 \sinh^2 \frac{\varepsilon}{2} \cdot M(\varepsilon) \quad (8.9)
\]

because \( \mathbb{H}(\varepsilon) \) has the hyperbolic area \( A(\varepsilon) = 4\pi \sinh^2 \frac{\varepsilon}{2} \) and \( S(\rho) \) has the hyperbolic length \( L(\rho) = 2\pi \sinh \rho \), see e.g. Theorem 7.2.2 in [2], and, moreover, \( \sinh \rho \geq \rho \) by the Taylor expansion. Then arguing by contradiction it is easy to see for the set \( T := \{ s \in (\log \frac{1}{\varepsilon}, \infty) : h(e^{-s}) > M(\varepsilon) \} \) that its length

\[
|T| = \int_{T} ds \leq 2 \sinh^2 \frac{\varepsilon}{2} . \quad (8.10)
\]

Next, let us show for \( T_* := T \cap S_* \) that

\[
q(e^{-s}) \leq H^{-1} \left( 2s + \log M(\varepsilon) \right) \quad \forall \ s \in \left( \log \frac{1}{\varepsilon}, \infty \right) \setminus T_* . \quad (8.11)
\]
Indeed, note that $(\log \frac{1}{\varepsilon}, \infty) \setminus T_* = \left[(\log \frac{1}{\varepsilon}, \infty) \setminus S_* \right] \cup \left[(\log \frac{1}{\varepsilon}, \infty) \setminus T \right] = \left[(\log \frac{1}{\varepsilon}, \infty) \setminus S_* \right] \cup [S_* \setminus T]$. The inequality (8.11) holds for $s \in S_* \setminus T$ by (8.8) because $H^{-1}$ is a non-decreasing function. Note also that

$$e^{2s}M(\varepsilon) > \Phi(0) = \tau_0 \quad \forall \ s \in \left(\log \frac{1}{\varepsilon}, \infty\right)$$

and then

$$t_* < \Phi^{-1}(e^{2s}M(\varepsilon)) = H^{-1}(2s + \log M(\varepsilon)) \quad \forall \ s \in \left(\log \frac{1}{\varepsilon}, \infty\right)$$

Consequently, (8.11) holds for all $s \in (\log \frac{1}{\varepsilon}, \infty) \setminus S_*$, too.

Since $H^{-1}$ is non-decreasing, we have by (8.10)-(8.11) that, for $\Delta := \log M(\varepsilon)$,

$$\int_0^\varepsilon \frac{dp}{\rho q(r)} = \int_{\log \frac{1}{\varepsilon}}^\infty \frac{ds}{q(e^{-s})} \geq \int_{(\log \frac{1}{\varepsilon}, \infty) \setminus T_*} \frac{ds}{H^{-1}(2s + \Delta)} \geq (8.14)$$

and after the replacement of variables $\eta = \log \tau$, $\tau = e^\eta$, we come to (8.4).

**Theorem 8.1** Let $Q : \mathbb{H}(\varepsilon) \to [0, \infty]$, $\varepsilon \in (0, 1)$, be a measurable function such that

$$\int_{\mathbb{H}(\varepsilon)} \Phi(Q(\rho)) \ d\rho < \infty \quad (8.15)$$

where $\Phi : [0, \infty] \to [0, \infty]$ is a non-decreasing convex function with

$$\int_{\delta_0}^\infty \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \quad (8.16)$$

for some $\delta_0 > \tau_0 := \Phi(0)$. Then

$$\int_0^\varepsilon \frac{d\rho}{\rho q(r)} = \infty, \quad (8.17)$$

where $q(\rho)$ is the average of $Q$ on the hyperbolic circle $h(z, 0) = \rho$. 
Proof. If $\Phi(0) \neq 0$, then Theorem 8.1 directly follows from Lemma 8.1 because $\Phi^{-1}$ is strictly increasing on the interval $(\tau_0, \infty)$ and $\Phi^{-1}(\delta_0) > 0$. In the case $\Phi(0) = 0$, let us fix a number $\delta \in (0, \delta_0)$ and set $\Phi_*(t) = \Phi(t)$, if $\Phi(t) > \delta$, and $\Phi_*(t) = \delta$, if $\Phi(t) \leq \delta$. Then by (8.15) we have that $\int \Phi(Q(z)) dh(z) < \infty$ because $|\Phi_*(t) - \Phi(t)| \leq \delta$ and the measure of $\mathbb{H}(\varepsilon)$ is finite. Moreover, $\Phi_*^{-1}(\tau) = \Phi^{-1}(\tau)$ for $\tau \geq \delta$ and then by (8.16) $\int_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi_*^{-1}(\tau)} = \infty$. Thus, (8.17) holds again by Lemma 8.1. □

Remark 8.2 Note that condition (8.16) implies that
\[
\int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \quad \forall \; \delta \in [0, \infty), \tag{8.18}
\]
but relation (8.18) for some $\delta \in [0, \infty)$, generally speaking, does not imply (8.16). Indeed, (8.16) evidently implies (8.18) for $\delta \in [0, \delta_0)$, and, for $\delta \in (\delta_0, \infty)$, we have that
\[
0 \leq \int_{\delta_0}^{\delta} \frac{d\tau}{\tau \Phi^{-1}(\tau)} \leq \frac{1}{\Phi^{-1}(\delta_0)} \log \frac{\delta}{\delta_0} < \infty \tag{8.19}
\]
because the function $\Phi^{-1}$ is non-decreasing and $\Phi^{-1}(\delta_0) > 0$. Moreover, by the definition of the inverse function $\Phi^{-1}(\tau) \equiv 0$ for all $\tau \in [0, \tau_0)$, $\tau_0 = \Phi(0)$, and hence (8.18) for $\delta \in [0, \tau_0)$, generally speaking, does not imply (8.16). If $\tau_0 > 0$, then
\[
\int_{\delta}^{\tau_0} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \quad \forall \; \delta \in [0, \tau_0) \tag{8.20}
\]
However, relation (8.20) gives no information on the function $Q$ itself and, consequently, (8.18) for $\delta < \Phi(0)$ cannot imply (8.17) at all.

9 Other criteria for homeomorphic extension in prime ends

Theorem 7.1 has a magnitude of other consequences thanking to Theorem 8.1.
Theorem 9.1 Under the hypothesis of Theorem 5.1, let
\[ \int_{D(p_0, \varepsilon_0)} \Phi_{p_0}(K_f(p)) \, dh(p) < \infty \quad \forall \, p_0 \in \partial D \quad (9.1) \]
for \( \varepsilon_0 = \varepsilon(p_0) \) and a nondecreasing convex function \( \Phi_{p_0} : [0, \infty) \to [0, \infty) \) with
\[ \int_{\delta(p_0)}^{\infty} \frac{d\tau}{\tau \Phi_{p_0}^{-1}(\tau)} = \infty \quad (9.2) \]
for \( \delta(p_0) > \Phi_{p_0}(0) \). Then \( f \) is extended to a homeomorphism of \( \overline{D}_P \) onto \( \overline{D}'_P \).

Proof. Indeed, in the case of the hyperbolic Riemann surfaces, (9.1) and (9.2) imply (7.1) by Theorem 8.1 and, after this, Theorem 9.1 becomes a direct consequence of Theorem 7.1. In the more simple case of the elliptic and parabolic Riemann surfaces, we similarly can apply Theorem 3.1 in [29] for the Euclidean plane instead of Theorem 8.1. \( \square \)

Corollary 9.1 In particular, the conclusion of Theorem 9.1 holds if
\[ \int_{D(p_0, \varepsilon_0)} e^{\alpha_0 K_f(p)} \, dh(p) < \infty \quad \forall \, p_0 \in \partial D \quad (9.3) \]
for some \( \varepsilon_0 = \varepsilon(p_0) > 0 \) and \( \alpha_0 = \alpha(p_0) > 0 \).

Remark 9.1 Note that by Theorem 5.1 and Remark 5.1 in [16] condition (9.2) is not only sufficient but also necessary for a continuous extendibility to the boundary of all mappings \( f \) with the integral restriction (9.1).

Note also that by Theorem 2.1 in [29], see also Proposition 2.3 in [28], (9.2) is equivalent to every of the conditions from the following series:
\[ \int_{\delta(p_0)}^{\infty} H_{p_0}'(t) \frac{dt}{t} = \infty \, , \quad \delta(p_0) > 0 \, , \quad (9.4) \]
\[ \int_{\delta(p_0)}^{\infty} \frac{dH_{p_0}(t)}{t} = \infty \, , \quad \delta(p_0) > 0 \, , \quad (9.5) \]
\[
\int_{\delta(p_0)}^{\infty} H_{p_0}(t) \frac{dt}{t^2} = \infty, \quad \delta(p_0) > 0, \quad (9.6)
\]

\[
\Delta(p_0) \int_{0}^{\infty} H_{p_0} \left( \frac{1}{t} \right) dt = \infty, \quad \Delta(p_0) > 0, \quad (9.7)
\]

\[
\int_{\delta_*(p_0)}^{\infty} \frac{d\eta}{H_{p_0}^{-1}(\eta)} = \infty, \quad \delta_*(p_0) > H_{p_0}(0), \quad (9.8)
\]

where

\[
H_{p_0}(t) = \log \Phi_{p_0}(t). \quad (9.9)
\]

Here the integral in (9.5) is understood as the Lebesgue–Stieltjes integral and the integrals in (9.4) and (9.6)–(9.8) as the ordinary Lebesgue integrals.

It is necessary to give one more explanation. From the right hand sides in the conditions (9.4)–(9.8) we have in mind \( +\infty \). If \( \Phi_{p_0}(t) = 0 \) for \( t \in [0, t_*(p_0)] \), then \( H_{p_0}(t) = -\infty \) for \( t \in [0, t_*(p_0)] \) and we complete the definition \( H'_{p_0}(t) = 0 \) for \( t \in [0, t_*(p_0)] \). Note, the conditions (9.5) and (9.6) exclude that \( t_*(p_0) \) belongs to the interval of integrability because in the contrary case the left hand sides in (9.5) and (9.6) are either equal to \( -\infty \) or indeterminate. Hence we may assume in (9.4)–(9.7) that \( \delta(p_0) > t_0 \), correspondingly, \( \Delta(p_0) < 1/t(p_0) \) where \( t(p_0) := \sup_{\Phi_{p_0}(t)=0} t \), set \( t(p_0) = 0 \) if \( \Phi_{p_0}(0) > 0 \).

The most interesting among the above conditions is (9.6), i.e. the condition:

\[
\int_{\delta(p_0)}^{\infty} \log \Phi_{p_0}(t) \frac{dt}{t^2} = +\infty \quad \text{for some} \quad \delta(p_0) > 0. \quad (9.10)
\]

Finally, it is necessary to note that the restriction on nondegeneracy of boundary components of domains in Theorem 5.1 as well as in all other theorems is not essential because this simplest case is included in our previous paper [31].
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**Vladimir Ryazanov, Sergei Volkov,**

Institute of Applied Mathematics and Mechanics,
National Academy of Sciences of Ukraine,
84116, Ukraine, Slavyansk, 19 General Batyuk Str.,
vl.ryazanov1@gmail.com, sergey.v.volkov@mail.ru