Further results on the neutrix composition of distributions involving the delta function and the function $\cosh^{-1}(x^{1/r} + 1)$

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Abstract: The neutrix composition $F(f(x))$ of a distribution $F(x)$ and a locally summable function $f(x)$ is said to exist and be equal to the distribution $h(x)$ if the neutrix limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. The function $\cosh^{-1}(x + 1)$ is defined by

$$\cosh^{-1}(x + 1) = H(x) \cosh^{-1}(|x| + 1),$$

where $H(x)$ denotes Heaviside’s function. It is then proved that the neutrix composition $\delta^{(s)}[\cosh^{-1}(x^{1/r} + 1)]$ exists and

$$\delta^{(s)}[\cosh^{-1}(x^{1/r} + 1)] = \sum_{k=0}^{s-1} \sum_{j=0}^{kr+r-1} \sum_{i=0}^{j} \frac{(-1)^{kr+s+j-1}}{2^{j+2}} \binom{kr+r-1}{j} \binom{j}{i} [(j - 2i + 1)^s - (j - 2i - 1)^s] \delta^{(k)}(x),$$

for $r, s = 1, 2, \ldots$. Further results are also proved.

Our results improve, extend and generalize the main theorem of [Fisher B., Al-Sirehy F., Some results on the neutrix composition of distributions involving the delta function and the function $\cosh^{-1}(x + 1)$, Appl. Math. Sci. (Ruse), 2014, 8(153), 7629–7640].

Keywords: distribution, Dirac-delta function, composition of distributions, neutrix, neutrix limit

MSC: Primary 46F10; Secondary 33B10

1 Introduction

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions $\varphi$ with compact support and let $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$. We let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$ and let $\mathcal{D}'[a, b]$ be the space of distributions defined on $\mathcal{D}[a, b]$.

A sequence of functions $\{f_n\}$ is called regular [1] if

(i) $\{f_n\}$ is infinitely differentiable for all $n$;
(ii) the sequence $\{\langle f_n, \varphi \rangle\}$ converges to a limit $\langle f, \varphi \rangle$ for every $\varphi \in \mathcal{D}$;
(iii) $\langle f, \varphi \rangle$ is continuous in $\varphi$ in the sense that $\lim_{n \to \infty} \langle f, \varphi_n \rangle = 0$ for every sequence $\varphi_n \to 0$ in $\mathcal{D}$.

There are several methods for constructing a sequence of regular functions which converges to $\delta(x)$. For example, let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

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Gel’fand and Shilov defined the distribution and the resulting finite value extracted from divergent integral is referred to as the Hadamard finite part, see [4]. The technique of neglecting appropriately defined infinite quantities was devised by Hadamard of unwanted infinite quantities from asymptotic expansions and this has been exploited in context of distributions. Fisher gave a general principle, by using the neutrix calculus developed by Van der Corput [3], for the discarding the definition of Antosik, it is not possible to define the compositions for many pairs of distributions. Fisher defined the composition for all functions which converge to zero in the usual sense as the Hadamard finite part of \( \langle f(x), \varphi(x) \rangle \) for all \( \varphi \in \mathcal{D}[a, b] \), where \( F_n(x) = F(x) \ast \delta_n(x) \) for \( n = 1, 2, \ldots \) and \( N \) is the neutrix, see [3], having domain \( N' \) the positive integers and range \( N'' \) the real numbers, with negligible functions which are finite linear sums of the functions

\[
n^\lambda \ln^{r-1} n, \quad \ln^r n : \quad \lambda > 0, \quad r = 1, 2, \ldots
\]

and all functions which converge to zero in the usual sense as \( n \) tends to infinity.

In particular, we say that the composition \( F(f(x)) \) exists and is equal to \( h \) on the open interval \( (a, b) \) if

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x) \rangle
\]

for all \( \varphi \in \mathcal{D}[a, b] \), where \( F_n(x) = F(x) \ast \delta_n(x) \) for \( n = 1, 2, \ldots \) and \( N \) is the neutrix, see [3], having domain \( N' \) the positive integers and range \( N'' \) the real numbers, with negligible functions which are finite linear sums of the functions

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\]

for all \( \varphi \in \mathcal{D}[a, b] \).

Note that taking the neutrix limit of a function \( f(n) \), is equivalent to taking the usual limit of Hadamard’s finite part of \( f(n) \).

It was proved in [7] that if the composition \( F(f(x)) \) exists by Gel’fand and Shilov’s definition, then it exists by Definition 1 and the two are equivalent.

The following theorems were proved in [8], [9] and [10] respectively.
Theorem 1. The neutrix composition $\delta^{(rs+r-1)}[\cosh^{-1}(x + 1)^{1/r}]$ exists and

$$
\delta^{(rs+r-1)}[\cosh^{-1}(x + 1)^{1/r}] = \sum_{k=0}^{s-1} \sum_{j=0}^{k} \sum_{i=0}^{j} \frac{(-1)^{rs+r-j-1} r}{2^{r+2}} \binom{k}{j} \binom{j}{i} \frac{(j - 2i + 1)^s - (j - 2i - 1)^s}{k!} \delta^{(k)}(x),
$$

for $r, s = 1, 2, \ldots$

In particular,

$$
\delta^{(2r-1)}[\cosh^{-1}(x + 1)]^{1/r} = \frac{(2r-1)!}{2} \delta(x),
$$

for $r = 1, 2, \ldots$

Theorem 2. The neutrix composition $\delta^{(s)}[\sinh^{-1} x_r]^{1/r}$ exists and

$$
\delta^{(s)}[\sinh^{-1} x_r]^{1/r} = \sum_{k=0}^{s-1} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^{i+k} r c_{s,k,i}}{2^{k+1} k!} \delta^{(k)}(x)
$$

for $s = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$, where $M$ is the smallest positive integer greater than $(s - r^2 + 1)/r$ and

$$
c_{s,k,i} = \begin{cases} 
\frac{[(k - 2i + 1)^p + (k - 2i - 1)^p]}{2^p i^s} \frac{(-1)^i}{(k^r r)!}, & p = \frac{s - r + 1}{r} \geq 0, \\
0, & \text{otherwise}.
\end{cases}
$$

In particular, the neutrix composition $\delta^{(s)}[\sinh^{-1} x_r]^{1/r}$ exists and

$$
\delta^{(s)}[\sinh^{-1} x_r]^{1/r} = 0
$$

for $s = 0, 1, 2, \ldots$, $r - 1$ and $r = 2, 3, \ldots$

Theorem 3. The neutrix composition $\delta^{(s)}[\cosh^{-1}(x_1^{1/r} + 1)]$ exists and

$$
\delta^{(s)}[\cosh^{-1}(x_1^{1/r} + 1)] = \sum_{k=0}^{s-1} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^{i+k} r c_{s,k,i}}{(kr + r)!} \delta^{(k)}(x)
$$

for $s = M - 1, M, M + 1, \ldots$ and $r = 1, 2, \ldots$, where

$$
c_{r,s,k} = \sum_{i=0}^{s} \binom{i}{j} \frac{(-1)^{r+s+i} (2j - i)^s}{2^{i+1} r!},
$$

$M$ is the smallest integer for which $s - 2r + 1 < 2Mr$ and $r \leq s/(2M + 2)$.

We now need the following lemma, which can be easily proved by induction:

Lemma 1.

$$
\int_{-1}^{1} t^s p^{(s)}(t) \, dt = \begin{cases} 
0, & 0 \leq i < s, \\
(-1)^i s!, & i = s
\end{cases}
$$

and

$$
\int_{0}^{1} t^s p^{(s)}(t) \, dt = \frac{1}{2} (-1)^s s!
$$

for $s = 0, 1, 2, \ldots$

In the following we define the functions $\cosh^{-1}(x + 1)$ and $\cosh^{-1}(|x| + 1)$ by

$$
\cosh^{-1}(x + 1) = H(x) \cosh^{-1}(|x| + 1), \quad \cosh^{-1}(|x| + 1) = H(-x) \cosh^{-1}(|x| + 1).
$$

It follows that

$$
\cosh^{-1}(|x| + 1) = \cosh^{-1}(x + 1) + \cosh^{-1}(x + 1).
$$
2 Main results

We now prove the following improvement of Theorem 3:

**Theorem 4.** The neutrix composition \( \delta^{(s)}[\cosh^{-1}(x^{1/r} + 1)] \) exists and

\[
\delta^{(s)}[\cosh^{-1}(x^{1/r} + 1)] = \sum_{k=0}^{s-1} \sum_{j=0}^{kr+r-1} \sum_{i=0}^{j} \frac{(-1)^{kr+r+j-i}}{2^{j+2}} \binom{kr+r-1}{j} \binom{j}{i} x((j-2i+1)^2 - (j-2i-1)^2) \delta^{(k)}(x),
\]

(2.1)

for \( r, s = 1, 2, \ldots \).

In particular,

\[
\delta[\cosh^{-1}(x^{1/r} + 1)] = 0
\]

(2.2)

and

\[
\delta'[\cosh^{-1}(x^{1/r} + 1)] = \sum_{j=0}^{r-1} \sum_{i=0}^{j} \frac{(-1)^{j-i}}{2^{j+1}} \binom{j+1}{2} \delta(x),
\]

(2.3)

for \( r = 1, 2, \ldots \).

**Proof.** It is clear that \( \delta^{(s)}[\cosh^{-1}(x^{1/r} + 1)] = 0 \) on any interval not containing the origin and so we only need to prove equation (2.1) on the interval \([-1, 1]\). To do this, we will first of all need to evaluate

\[
\int_{-1}^{1} x^k \delta^{(s)}[\cosh^{-1}(x^{1/r} + 1)] \, dx = \int_{0}^{1} x^k \delta^{(s)}[\cosh^{-1}(x^{1/r} + 1)] \, dx + \int_{-1}^{0} x^k \delta^{(s)}(0) \, dx
\]

\[
= n^{s+1} \int_{-1}^{1} x^k \rho^{(s)}(n[\cosh^{-1}(x^{1/r} + 1)]) \, dx + n^{s+1} \int_{-1}^{0} x^k \rho^{(s)}(0) \, dx
\]

\[
= I_1 + I_2.
\]

(2.4)

Making the substitution \( t = n[\cosh^{-1}(x^{1/r} + 1)] \) or \( x = [\cosh(t/n) - 1]' \), we have

\[
I_1 = n^s \int_{0}^{1} \frac{[\cosh(t/n) - 1]^{kr+r-1}}{k!} \sinh(t/n) \rho^{(s)}(t) \, dt
\]

\[
= n^s \sum_{j=0}^{kr+r-1} (-1)^{kr+r+j-i} \binom{kr+r-1}{j} \cosh^j(t/n) \sinh(t/n) \rho^{(s)}(t) \, dt
\]

\[
= n^s \sum_{j=0}^{kr+r-1} \frac{(-1)^{kr+r+j-i}}{2^{j+1}} \binom{kr+r-1}{j} \int_{0}^{1} (e^{t/n} + e^{-t/n})(e^{t/n} - e^{-t/n}) \rho^{(s)}(t) \, dt
\]

\[
= n^s \sum_{j=0}^{kr+r-1} \sum_{i=0}^{j} \frac{(-1)^{kr+r+j-i}}{2^{j+1}} \binom{kr+r-1}{j} \binom{j}{i} \int_{0}^{1} (e^{(j-2i+1)t/n} - e^{(j-2i-1)t/n}) \rho^{(s)}(t) \, dt
\]

and it follows that

\[
N_{-}\lim_{n \to \infty} I_1 = \sum_{j=0}^{kr+r-1} \sum_{i=0}^{j} \frac{(-1)^{kr+r+j-i}}{2^{j+1}} \binom{kr+r-1}{j} \binom{j}{i} \frac{(j-2i+1)^{s} - (j-2i-1)^{s}}{s!} \int_{0}^{1} t^s \rho^{(s)}(t) \, dt
\]

\[
= \sum_{j=0}^{kr+r-1} \sum_{i=0}^{j} \frac{(-1)^{kr+r+j-i}}{2^{j+2}} \binom{kr+r-1}{j} \binom{j}{i} [(j-2i+1)^{s} - (j-2i-1)^{s}].
\]

(2.5)

It is obvious that

\[
N_{-}\lim_{n \to \infty} I_2 = 0
\]

(2.6)
and it now follows from equations (2.4), (2.5) and (2.6) that

\[ N \lim_{n \to \infty} \int_{-1}^{1} x^s \delta_n^{(s)}[\cosh^{-1}(x^{1/r} + 1)] \, dx = \]

\[ = \sum_{j=0}^{kr+r-1} \sum_{i=0}^{j} \frac{(-1)^{kr+r+s-j-1}r}{2^{r+j}} \left( \begin{array}{c} kr + r - 1 \\ j \end{array} \right) \left( \begin{array}{c} j \\ i \end{array} \right) [(j - 2i + 1)s - (j - 2i - 1)] \]  

(2.7)

for \( k = 0, 1, 2, \ldots \).

Next, when \( k = s \), we note that

\[ [\cosh(t/n) - 1]^{s+r-1} \sinh(t/n) = O(n^{-2rs-2s+1}) \]

and it follows that

\[ |I_1| \leq r n^s \int_{0}^{1} [\cosh(t/n) - 1]^{s+r-1} \sinh(t/n) \rho^{(s)}(t) \, dt \]

\[ = O(n^{-2rs-2r+s+1}). \]

Hence, if \( \psi(x) \) is an arbitrary continuous function, then

\[ \lim_{n \to \infty} \int_{-1}^{1} x^s \delta_n^{(s)}[\cosh^{-1}(x^{1/r} + 1)] \psi(x) \, dx = 0, \]  

(2.8)

for \( s = 0, 1, 2, \ldots \).

Further,

\[ \int_{-1}^{1} x^s \delta_n^{(s)}(0) \psi(x) \, dx = n^{s+1} \int_{-1}^{1} x^s \rho^{(s)}(0) \psi(x) \, dx \]

and it follows that

\[ N \lim_{n \to \infty} \int_{-1}^{1} x^s \delta_n^{(s)}(0) \psi(x) \, dx = 0. \]  

(2.9)

Now let \( \varphi(x) \) be an arbitrary function in \( \mathcal{D}[-1, 1] \). By Taylor’s Theorem we have

\[ \varphi(x) = \sum_{k=0}^{s-1} \frac{x^k}{k!} \varphi^{(k)}(0) + \frac{x^s}{s!} \varphi^{(s)}(\xi x), \]

where \( 0 < \xi < 1 \). Then with \( s \geq 1 \), we have

\[ \langle \delta_n^{(s)}[\cosh^{-1}(x^{1/r} + 1)], \varphi(x) \rangle = \int_{-1}^{1} \delta_n^{(s)}[\cosh^{-1}(x^{1/r} + 1)] \varphi(x) \, dx \]

\[ = \sum_{k=0}^{s-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} x^k \delta_n^{(s)}[\cosh^{-1}(x^{1/r} + 1)] \, dx \]

\[ + \frac{1}{s!} \int_{-1}^{1} x^s \delta_n^{(s)}[\cosh^{-1}(x^{1/r} + 1)] \varphi^{(s)}(\xi x) \, dx. \]

It now follows from equations (2.7), (2.8) and (2.9) that

\[ N \lim_{n \to \infty} \langle \delta_n^{(s)}[\cosh^{-1}(x^{1/r} + 1)], \varphi(x) \rangle = \]
and equation (2.1) follows.

Equations (2.2) and (2.3) follow immediately from equation (2.1).

Replacing \( x \) by \( -x \) in Theorem 4, we get

**Corollary 4.1** The neutrix composition \( \delta^{(s)}[\cosh^{-1}(|x|^{1/r} + 1)] \) exists and

\[
\delta^{(s)}[\cosh^{-1}(|x|^{1/r} + 1)] = \sum_{k=0}^{s-1} \sum_{j=0}^{k+r-1} \sum_{i=0}^{j} \frac{(-1)^{kr+rs-k-j-1}r}{2j^2} \begin{pmatrix} kr + r - 1 \\ j \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} x[(j-2i+1)^s - (j-2i-1)^s] \delta^{(k)}(x),
\]

(2.10)

for \( r, s = 1, 2, \ldots. \)

In particular,

\[
\delta[\cosh^{-1}(|x|^{1/r} + 1)] = 0
\]

(2.11)

and

\[
\delta'[\cosh^{-1}(|x|^{1/r} + 1)] = \sum_{j=0}^{r-1} \sum_{i=0}^{j} \frac{(-1)^{r-j}r}{2j^2} \begin{pmatrix} r - 1 \\ j \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} \delta(x),
\]

(2.12)

for \( r = 1, 2, \ldots. \)

**Corollary 4.2** The neutrix composition \( \delta^{(s)}[\cosh^{-1}(|x|^{1/r} + 1)] \) exists and

\[
\delta^{(s)}[\cosh^{-1}(|x|^{1/r} + 1)] = \sum_{k=0}^{s-1} \sum_{j=0}^{k+r-1} \sum_{i=0}^{j} \frac{(-1)^{kr+rs-k-j-1}r[1 - (-1)^s]}{2j^2} \begin{pmatrix} kr + r - 1 \\ j \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} x[(j-2i+1)^s - (j-2i-1)^s] \delta^{(k)}(x),
\]

(2.13)

for \( r, s = 1, 2, \ldots. \)

In particular,

\[
\delta[\cosh^{-1}(|x|^{1/r} + 1)] = 0
\]

(2.14)

and

\[
\delta'[\cosh^{-1}(|x|^{1/r} + 1)] = \sum_{j=0}^{r-1} \sum_{i=0}^{j} \frac{(-1)^{r-j}r[1 - (-1)^s]}{2j^2} \begin{pmatrix} r - 1 \\ j \end{pmatrix} \begin{pmatrix} j \\ i \end{pmatrix} \delta(x),
\]

(2.15)

for \( r = 1, 2, \ldots. \)

**Proof.** Equation (2.13) follows on noting that

\[
\delta^{(s)}[\cosh^{-1}(|x|^{1/r} + 1)] = \delta^{(s)}[\cosh_{-1}(x^{1/r} + 1)] + \delta^{(s)}[\cosh^{-1}(|x|^{1/r} + 1)].
\]

Equations (2.14) and (2.15) follow immediately.

For further related results, see [11], [7], [12], [13] and [14].
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