Experimental study of numerical methods for the solution of gas dynamics problems with shock waves

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Abstract. The work is devoted to some important questions that come with the solution of gas dynamics equations using standard Godunov scheme with the corrections of A V Safronov. The numerical solution is succeeded by intrinsic differential realization of energy conservation law. It has been found experimentally that in all computational cells the entropy nondecreasing is provided. The fact makes it possible to model the entropy rising on shock waves. Besides the experiments described in the paper gives the intrinsic explanation of the reasons for the appearance of the zones with decreased accuracy order in the results. The influence of the computational grid parameters (Courant number) on the plots of grid structures of shock waves is also studied.

1. Introduction

For the numerical simulations of some effects of high-intensive fast physical and chemical process the Euler equations of gas dynamics are used. The explosive welding can be mentioned as an example. The process of explosive welding was modeled in [1] using Godunov scheme on moving computational grids. Although the scheme was developed in the late 1950-s there is still a number of unresolved questions up to now [2].

To date for the theoretical investigations of the properties of the numerical methods for the solutions of gas dynamics equations the necessity of the consideration of the governing system of equations in the form of mass, momentum and entropy conservation laws is established:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0, \quad (1)
\]

\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial}{\partial x} (\rho u^2 + p) = 0, \quad (2)
\]

\[
\frac{\partial (\rho S)}{\partial t} + \frac{\partial}{\partial x} (\rho u S) = 0. \quad (3)
\]

On smooth solutions one more auxiliary conservation law is valid:

\[
\frac{\partial}{\partial t} [\rho (E + u^2/2)] + \frac{\partial}{\partial x} [\rho u (E + p/\rho + u^2/2)] = 0. \quad (4)
\]
Here $\rho$ is the density, $p$ is the pressure, $u$ is the velocity, $S$ is the entropy, $E = E(\rho, S)$ is the internal energy of the gas per the unit of mass. Here we also restrict ourselves by ideal gas equation of state $E = e^S \rho^{\gamma-1}/(\gamma - 1)$.

It is turned out that after the following choice of variables:

$$
q_0 = E + \rho E_\rho - S E_S - u^2/2, \\
q_1 = u, \\
q_2 = E_S = T,
$$

and the usage of pressure $p = L(q_0, q_1, q_2)$ as the potential the equations (1)–(3) are written as:

$$
\frac{\partial L_{q_i}}{\partial t} + \frac{\partial K_{q_i}}{\partial x} = 0, \quad K = uL,
$$

or in quasilinear form:

$$
L_{q_i} \frac{\partial L_{q_i}}{\partial t} + K_{q_i} \frac{\partial K_{q_i}}{\partial x} = 0.
$$

The system is symmetric hyperbolic with symmetric matrixes of coefficients if the matrix is positive definite (the same as the matrix is convex).

The paper is dedicated to the description of some important questions that go with the solution of system (1)–(3) using standard Godunov scheme with the corrections of A V Safronov [3]. The corrections result in approximate linearized solution of the Riemann problem that nevertheless provides the entropy nondecreasing condition. The numerical solution is succeeded by intrinsic differential realization of energy conservation law:

$$
\frac{\partial}{\partial t}(\rho S) + \frac{\partial}{\partial x}(\rho u S) \geq 0.
$$

It has been found experimentally that in all computational cells the entropy nondecreasing is provided:

$$
\frac{\partial L_{q_2}}{\partial t} + \frac{\partial K_{q_2}}{\partial x} \geq 0,
$$

where $q_2$ has been chosen as in (5) and corresponds to the inequality:

$$
\frac{\partial (\rho S)}{\partial t} + \frac{\partial (\rho u S)}{\partial x} \geq 0
$$

from hydrodynamics system. The fact makes it possible to model the entropy rising on shock waves.

2. Generalized solutions. The problem of uniqueness of solution. The law of the entropy nondecreasing

It is almost conventional point of view to date that the “generalized” solutions in gas dynamics comply with the conservation laws (1), (2), (4) and inequality:

$$
\frac{\partial}{\partial t}(\rho S) + \frac{\partial}{\partial x}(\rho u S) \geq 0.
$$

The last one is the law of entropy nondecreasing. The problem of entropy in numerical solution of hyperbolic equations is of great importance. The law of entropy nondecreasing provides the unique solution of the problem. The solution of the entropy problem was first proposed by John von Neumann [4] by means of introduction of artificial viscosity in the equations that where written in the form of conservation laws. Artificial viscosity appears in the gradient zones of the flow and “smears” the discontinuities.

In the numerical solution of gas dynamics problems by Godunov method the role of artificial viscosity plays the scheme viscosity [1]. The Godunov method of the first approximation order based on the exact iterative solution of the Riemann problem of the disintegration of
discontinuities is characterized by minimal scheme viscosity. So it provides the shock-capturing calculation process for complex flows with discontinuities but very computationally expensive. Besides in some cases the theoretical solution of the Riemann problem is difficult. Therefore the methods based on the approximate solution of the Riemann problem became very widespread. Such class of methods contains well-known schemes of Lax–Friedrichs [5, 6], Rusanov [7], Kholodov [8], Harten–Lax–Van Leer (HLL) [9], Roe [10], Engquist–Osher (EO) [11, 12], HLLC scheme [13,14]. In the case of linear advection equation for the entropy condition to be satisfied the scheme viscosity of the methods based on the approximate solution of the Riemann problem should be larger than for the Godunov method.

The principal solution of the entropy problem for the numerical solution of hyperbolic equations is the usage of kinetic relaxation method, see for example [15–19]. In the kinetic variant the non-linear hyperbolic system of equations in consideration is transformed to the linear one with the relaxation source term. The kinetic interpretation of the numerical methods simplifies its entropy analysis [17–19].

The comparative analysis of the numerical methods for the gas dynamics problems that are based on the solution of the Riemann problem and other approaches could be found elsewhere [20–23]. The intensive development of Godunov method is connected with the increase of the approximation order on the basis of interpolation procedures to the faces of the computational cell with limiters that provides the monotonicity of the scheme. For the first time such limiter was proposed by V P Kolgan [24]. The theory treatment and the review of different limiters could be found elsewhere [20–22,25,26]. Note that the approximation order increase diminishes the difference between the methods based on the exact and approximate solution of the Riemann problem. After the interpolation (or reconstruction) of the variables to the cells faces the parameters on the faces for the high approximation order schemes are calculated as for the first order schemes on the basis of Riemann problem solution. So first of all the condition of entropy nondecreasing should be valid for the first order schemes.

3. Godunov method with the linear disintegration of discontinuities

Consider gas dynamics equations in one-dimensional geometry:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \quad (7)
\]

\[
\frac{\partial (\rho u)}{\partial t} + \frac{\partial (P + \rho u^2)}{\partial x} = 0, \quad (8)
\]

\[
\frac{\partial (\rho \sigma(S))}{\partial t} + \frac{\partial (\rho \sigma(S) u)}{\partial x} = 0, \quad (9)
\]

\[
\frac{\partial W}{\partial t} + \frac{\partial (W u + Pu)}{\partial x} = 0, \quad (10)
\]

where \(\rho = \rho(x,t)\)—density, \(V = V(x,t) = \rho^{-1}\)—specific volume, \(u = u(x,t)\)—velocity, \(S = S(x,t)\)—entropy, \(E = E(\rho, S) = \sigma(S)\rho^{\gamma-1}/(\gamma-1)\)—inner energy of the gas, \(\sigma(S) = e^S\)—entropy function, \(\gamma\)—adiabatic index, \(p = \sigma(S)\rho^{\gamma}\)—pressure, \(W = \rho(E + u^2/2)\)—full energy of the gas.

Computational domain is \((x,t) : a \leq x \leq b, 0 \leq t \leq T\). Initial conditions are given:

\[
\rho(x,0) = \rho_0(x), \quad p(x,0) = p_0(x), \quad u(x,0) = u_0(x), \quad a \leq x \leq b. \quad (11)
\]

As boundary conditions we can choose either:

\[
p(a,t) = p_a(t), \quad p(b,t) = p_b(t), \quad (12)
\]

or

\[
u(a,t) = u_a(t), \quad u(b,t) = u_b(t), \quad (13)
\]
and then, consequently, from Riemann invariants, we compute \( u(a, t) \) and \( u(b, t) \) or \( p(a, t) \) and \( p(b, t) \) depending on condition, (12) or (13), has been put. After that, we compute the last boundary conditions
\[
\rho(a, t) = \rho_0(t), \quad \rho(b, t) = \rho_0(t),
\]
using mechanics, which will be described further in this chapter. Boundary conditions are consistent with initial conditions:
\[
\rho_a(0) = \rho_0(a), \quad \rho_b(0) = \rho_0(b), \quad u_a(0) = u_0(a), \quad u_b(0) = u_0(b).
\]
To solve the equations (7)–(14), the region \([a, b]\) is divided into \(N\) independent intervals of size \(h = (b - a)/N\). Cells are numbered with indices \(i + 1/2\), \(i = 1, 2, \ldots, N - 1\). The interfaces between cells are numbered with indices \(i\), \(i = 1, 2, \ldots, N\). Time interval is divided into \(M\), generally speaking, unequal intervals of size \(\tau_i\), \(i = 0, 1, \ldots, M\) and \(M\) is not known in advance. Moreover, the condition is satisfied:
\[
\sum_{i=0}^{M} \tau_i = T.
\]
From the continuous value of each hydrodynamic function \(f = f(x, t)\) we proceed to discrete values on the \(n\)-th time layer and denote with
\[
f_{i+1/2}^n = f\left(x_{i+1/2}, \sum_{k=0}^{n} \tau_k\right)
\]
the values in the middle of the cells and
\[
F = f\left(x_i, \sum_{k=0}^{n} \tau_k\right).
\]
Thus, the specification of the algorithm has the following form.

The input of the algorithm is:
- spatial \(a \leq x \leq b\) and time \(0 \leq t \leq T\) intervals specified by the values of \(a, b, T\);
- splitting of the spatial interval \(a \leq x \leq b\) into \(N\) equal cells, given by the value of \(N\);
- analytical values of functions, \(\rho_0(x), p_0(x), u_0(x)\); \(\rho_a(x)\) or \(u_a(x)\), \(p_a(x)\) or \(u_b(x)\) implemented by appropriate procedures.

### 3.1. Algorithm

(i) The current time is set to \(t = 0\) and the variable \(n\) is also set to \(n = 0\).
(ii) The given interval \(a \leq x \leq b\) is divided into \(N\) intervals of \(h = (b - a)/N\).
(iii) Using the given functions that determine the initial conditions, the values of density, velocity, and pressure in the middles of the cells at the initial moment of time are calculated:
\[
\rho_{i+1/2}^n, \quad u_{i+1/2}^n, \quad p_{i+1/2}^n,
\]
and, consequently, in the middles of cells we compute the values of the impact of sound speed \(c\) and entropy \(\sigma(S)\):
\[
c_{i+1/2}^n = \frac{\sqrt{\gamma \rho_{i+1/2}^n/p_{i+1/2}^n}}{\rho_{i+1/2}^n}, \quad \sigma(S)_{i+1/2}^n = \frac{p_{i+1/2}^n/\gamma}{\rho_{i+1/2}^n}.\]
(iv) We calculate

\[ u_{\text{max}} = \max_{i=0,1,\ldots,N-1} \left( \left| u_{i+1/2}^n \right| + c_{i+1/2}^n \right) \]

and use Courant’s condition (CFL < 1) to determine time step

\[ \tau_n = \frac{\text{CFL} h}{u_{\text{max}}}. \]

(v) In case the value \( t + \tau_n > T \) we redefine the time step into \( \tau_n = T - t \).

(vi) We add the current step to the current time \( t^{(\text{new})} = t^{(\text{prev})} + \tau_n \).

(vii) We determine the values of density, velocity, and pressure at the interfaces between the cells:

- in case \( u_{i-1/2}^n > c_{i-1/2}^n \) the Riemann problem solution is
  \[ P_i = p_i^n, \quad U_i = u_{i-1/2}^n, \quad R_i = \rho_{i-1/2}^n; \]

- in case \( u_{i+1/2}^n < -c_{i+1/2}^n \) the Riemann problem solution is
  \[ P_i = p_{i+1/2}^n, \quad U_i = u_{i+1/2}^n, \quad R_i = \rho_{i+1/2}^n; \]

- otherwise
  \[ P_i^n = \frac{\rho_{i-1/2}^n e_{i-1/2}^n + p_{i-1/2}^n e_{i-1/2}^n}{\rho_{i-1/2}^n e_{i-1/2}^n + p_{i-1/2}^n e_{i-1/2}^n}, \]
  \[ U_{i}^n = \frac{\rho_{i-1/2}^n e_{i-1/2}^n u_{i-1/2}^n + p_{i-1/2}^n e_{i-1/2}^n u_{i-1/2}^n + p_{i-1/2}^n u_{i-1/2}^n - p_{i-1/2}^n}{\rho_{i-1/2}^n e_{i-1/2}^n + p_{i-1/2}^n e_{i-1/2}^n}, \]

as solution of linear system of equations. The density is computed from linear approximation of equations of gas dynamics (16)—(18), considering in discontinuous case:

\[ R_i^n = \rho_{i-1/2}^n - \frac{\rho_{i-1/2}^n}{e_{i-1/2}^n} \left( U_{j}^n - u_{j-1/2}^n \right) \rightarrow U_{j}^n > 0, \]

\[ R_i^n = \rho_{i-1/2}^n + \frac{\rho_{i+1/2}^n}{e_{i+1/2}^n} \left( U_{j}^n - u_{j+1/2}^n \right) \rightarrow U_{j}^n < 0. \]

(viii) Using obtained values of \( R_i^n, P_i^n, U_i^n \), we calculate the values:

\[ (RU)_i^n = R_i^n U_i^n, \]

\[ (P + RU^2)_i^n = P_i^n + R_i^n (U_i^n)^2, \]

\[ (WU + PU)_i^n = \left( P_i^n + \frac{R_i^n (U_i^n)^2}{2} \right) U_i^n + P_i^n U_i^n. \]

(ix) We recalculate the conservation laws (7), (8) and (10) in all cells \( i = 0, 1, \ldots, N-1 \) according to the equations:

\[ p_{i+1/2}^{n+1} = p_{i+1/2}^{n} - \frac{\tau}{h} \frac{(RU)_i^{n+1} - (RU)_i^{n}}{h}, \]

\[ (pu)_{i+1/2}^{n+1} = (pu)_{i+1/2}^{n} - \frac{\tau}{h} \frac{(P + RU^2)_{i+1}^{n+1} - (P + RU^2)_{i+1}^{n}}{h}, \]

\[ W_{i+1/2}^{n+1} = W_{i+1/2}^{n} - \frac{\tau}{h} \frac{(WU + PU)_{i+1}^{n+1} - (WU + PU)_{i+1}^{n}}{h}. \]
Due to the fact, that density is computed from conservation laws directly, we recalculate the values of the velocity in all cells $i = 0, 1, \ldots, N - 1$ according to the equation:

$$
u_{i+1/2}^{n+1} = \frac{(\rho u)^{n+1}_{i+1/2}}{\rho_{i+1/2}^{n+1}},$$

$$p_{i+1/2}^{n+1} = (\gamma - 1) \left( W_{i+1/2}^{n+1} - \frac{\rho_{i+1/2}^{n+1} (u_{i+1/2}^{n})^2}{2} \right).$$

We increase time step number by one $n^{(\text{new})} = n^{(\text{prev})} + 1$.

If the current time is equal to $T$, then algorithm proceeds to the next step. Otherwise, we go to step (iv).

The algorithm is complete. The output of the algorithm is the values of the density, pressure and velocity functions determined at the moment $t = T$ in the middles of the cells $x_{i+1/2}, i = 0, 1, \ldots, N - 1$:

$$\rho(x_{i+1/2}, T), p(x_{i+1/2}, T), u(x_{i+1/2}, T).$$

### 3.2. The procedure for solving the Riemann problem

Consider the system (7)–(9) written in the following form:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \rho^{-1} \frac{\partial p}{\partial x} = 0, \quad (16)$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \rho c^2 \frac{\partial u}{\partial x} = 0, \quad (17)$$

$$\frac{\partial \sigma(S)}{\partial t} + u \frac{\partial \sigma(S)}{\partial x} = 0, \quad (18)$$

where $c = \sqrt{\gamma p/\rho}$. In the matrix-vector form it looks like:

$$\begin{pmatrix} u \\ p \\ \sigma(S) \end{pmatrix} + \begin{pmatrix} u & \rho^{-1} & 0 \\ 0 & u & 0 \\ 0 & 0 & u_0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} u \\ p \\ \sigma(S) \end{pmatrix} = 0.$$

We will consider some cell in which the values of the speed of sound $c = c_0$, density $\rho = \rho_0$ and velocity $u = u_0$ are constant. Then the system can be written as a linear system:

$$\begin{pmatrix} u \\ p \\ \sigma(S) \end{pmatrix} + \begin{pmatrix} u_0 & \rho_0^{-1} & 0 \\ 0 & u_0 & 0 \\ 0 & 0 & u_0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} u \\ p \\ \sigma(S) \end{pmatrix} = 0.$$

The last system, in turn, can be rewritten in the characteristic form:

$$\frac{\partial}{\partial t} \begin{pmatrix} u + \frac{p}{\rho_0 c_0} \\ u - \frac{p}{\rho_0 c_0} \\ \sigma(S) \end{pmatrix} + \begin{pmatrix} u_0 - c_0 & 0 & 0 \\ 0 & u_0 + c_0 & 0 \\ 0 & 0 & u_0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} u + \frac{p}{\rho_0 c_0} \\ u - \frac{p}{\rho_0 c_0} \\ \sigma(S) \end{pmatrix} = 0,$$

where

$$u \pm \frac{p}{\rho_0 c_0} = \text{const}, \quad \sigma(S) = \text{const}.$$
are called Riemann invariants, which are constant on characteristics

\[ \frac{dx}{dt} = u_0 \pm c_0, \quad \frac{dx}{dt} = u_0 \]

respectively. Consider two adjacent cells with numbers \((i - 1/2)\) and \((i + 1/2)\). In case of a supersonic flow on the left, i.e. when the condition \(u_{i-1/2} > c_{i-1/2}\) is satisfied, the solution of the problem is:

\[ P_i = P_{i-1/2}, \quad U_i = u_{i-1/2}, \quad R_i = R_{i-1/2}. \]

In case of a supersonic flow on the right, i.e. when the condition \(u_{i+1/2} < -c_{i+1/2}\) is satisfied, the solution of the problem is:

\[ P_i = P_{i+1/2}, \quad U_i = u_{i+1/2}, \quad R_i = R_{i+1/2}. \]

Otherwise, the solution of the problem of the decay of the discontinuity, the velocity of the left wave is calculated from the formula

\[ \frac{dx}{dt} = u_{i-1/2} - c_{i-1/2}, \]

and the following conditions are fulfilled:

\[ (u_{i-1/2} - U_i) + \frac{P_{i-1/2} - P_i}{\rho_{i-1/2} c_{i-1/2}} = 0. \]

The speed of the right wave is calculated by the formula:

\[ \frac{dx}{dt} = u_{i+1/2} - c_{i+1/2}, \]

and the following conditions are fulfilled:

\[ (u_{i+1/2} - U_i) + \frac{P_{i+1/2} - P_i}{\rho_{i+1/2} c_{i+1/2}} = 0. \]

As a result, the “large” values of velocity \(U_i\) and pressure \(P_i\) at the interface between cells are calculated by the formulas:

\[ P_i^n = \frac{P_{i-1/2}^n + P_{i+1/2}^n + u_{i-1/2}^n - u_{i+1/2}^n}{\rho_{i-1/2} c_{i-1/2} + \rho_{i+1/2} c_{i+1/2}}, \]

\[ U_i^n = \frac{\rho_{i-1/2} c_{i-1/2} u_{i-1/2}^n + \rho_{i+1/2} c_{i+1/2} u_{i+1/2}^n + P_{i-1/2}^n - P_{i+1/2}^n}{\rho_{i-1/2} c_{i-1/2} + \rho_{i+1/2} c_{i+1/2}}. \]

Depending on the contact discontinuity speed \(U\), the density value is selected according to the following rule:

\[ R_i^n = \rho_{i-1/2}^n - \rho_{i-1/2}^{n-1/2} \left( U_j^n - u_{j-1/2}^n \right) \quad \text{if} \quad R_i^n > 0, \]

\[ R_i^n = \rho_{i-1/2}^n + \rho_{i-1/2}^{n+1/2} \left( U_j^n - u_{j+1/2}^n \right) \quad \text{if} \quad R_i^n < 0. \]

It comes from discontinuous form of conservation law of mass: \([\rho] \frac{dx}{dt} - u[\rho] - \rho[u] = 0\), where \(dx/dt = u - c\) if \(0 < u_0 < c_0\), and \(dx/dt = u + c\) if \(-c_0 < u_0 < 0\).
Figure 1. Velocity profiles at the successive time moments, $\alpha = 0$ (a) and 0.22 (b) in (19), CFL = 0.8.

4. Numerical shock wave structure

Numerical experiments were carried out on the problem of a shock wave behind which a contact discontinuity extends, the solution of which, in addition to the waves mentioned, still contains a rarefaction wave. The shock wave is given by discontinuous initial data. Computational area: $a \leq x \leq b$, $0 \leq t \leq T$, $a = 0$, $b = 1$, $T = 0.1$. At $x = x_0$ there is a gap, $x_0 = 0.5$. Discontinuous initial conditions are:

$$
\begin{align*}
\rho(x,0) &= 1.2714, & u(x,0) &= 0.2928, & p(x,0) &= 1.4017, & x < x_0, \\
\rho(x,0) &= 1.0000, & u(x,0) &= 0.0000, & p(x,0) &= 1.0000, & x > x_0.
\end{align*}
$$

The calculation is carried out on a uniform grid of $N$ cells, where $N = 100$. If necessary, the grid can be multiply divided to study the convergence (see the next section).

The Godunov scheme with the linearized solver is used. Fluxes between cells are calculated using an approximate linear solver of the Riemann problem.

Consider the shock wave structure during its motion. Fix the CFL number equal to 0.8. It appears that the shock wave width increases with time and the points number on shock wave $N_{SW}$ has the time dependence $N_{SW} = t^\alpha$. To find the value of $\alpha$ plot the numerical solutions for successive time moments in the shock-attached frame

$$
\bar{x} = (x - Dt - x_0) / t^\alpha,
$$

where $D$ is the shock speed and $x_0$ is the initial location of shock. The shock wave is set by discontinuous initial data, however the finite-difference solution is smeared. Moreover, the process of wave formation is visible, i.e. the waveform changes while wave is spreading.

Figure 1 illustrates the dynamics of shock wave motion at the successive time moments. For $\alpha = 0$ the curves at different time moments intersect each other at $\bar{x} = 0$ under different angles. So the shock wave at different time moments is smeared on different point numbers. But for $\alpha = 0.22$ all curves merge into one line and so we get $N_{SW} = t^{0.22}$ dependence. Consider the CFL number 0.4 now. The dependence for shock wave width changes to $N_{SW} = t^{0.35}$ (figure 2). So the shock wave width depends on the Courant number. For the fixed computational grid and the fixed time moment the shock wave is smeared on more points for the less Courant number. In fact for the less Courant number we need twice larger time and so larger time steps number that leads to the increase of shock wave smearing.
Figure 2. Pressure profiles at the successive time moments, $\alpha = 0$ (a) and 0.35 (b) in (19), CFL = 0.4.

5. Approximation order on the discontinuous solutions

The classical definition of the numerical scheme approximation order tells that the scheme is characterized by the $q$-th approximation order if:

$$\|U_{\text{numerical}} - U_{\text{exact}}\| \leq Ch^q, $$

were $U_{\text{numerical}}$ is the vector of numerical solution of the problem, $U_{\text{exact}}$ is the vector of exact solution, $h$ is the time step and $C$ is constant. The classical Godunov scheme [27] is characterized by theoretically first approximation order, $q = 1$. Note however that the definition (20) is valid only for the continuous solutions because (20) is derived with the use of Taylor series expansion of the calculated function. There is a question how to determine the scheme approximation order in case of discontinuous solution. Moreover in many works, see [28] for example, it is shown that for the theoretically high-order approximation scheme the practical approximation order is decreased to the first on the discontinuities. In [2, 28] the approximation order of the classical non-linear Godunov scheme on the discontinuous solution was investigated and the result was 0.5.

In this work the study of the accuracy order of the numerical scheme in consideration (see section 3) is done using calculation of contour integrals from the conservation laws. The integral from the conservation law on each contour both for continuous and discontinuous cases should be equal to zero. However the numerical value of the integral is not equal to zero exactly but tends to zero with mesh refinement. Estimate the convergence rate using Runge approach:

$$q \approx \log_3 \frac{f(3h) - f(h)}{f(h) - f(h/3)}. $$

In fact the proposed approach differs from the common way of the practical approximation order estimation by the choice of function $f$. Function $f$ is the integral from the conservative law (the conservation laws for mass, momentum, energy and entropy are checked) for the contour $[x_1, x_2] \times [t_1, t_2]$ instead of the norm of difference between the grid solution and the projection of the exact solution to the grid (20). For the momentum conservation law for example we get:

$$f = \int_{x_1}^{x_2} (\rho u) (x, t_2) \, dx - \int_{x_1}^{x_2} (\rho u) (x, t_1) \, dx + \int_{t_1}^{t_2} (p + \rho u^2) (x_2, t) \, dt - \int_{t_1}^{t_2} (p + \rho u^2) (x_1, t) \, dt.$$
For the problem about one isolated shock wave described above in section 4 two contours were considered. The first one coincides with the the whole computational domain \([0,1] \times [0,T]\). The second one is \([0.2,1] \times [T/2,T]\) and corresponds to the well formed shock wave. Consideration of the second contour is necessary for the comparison of the scheme accuracy at the initial moments of disintegration and after some time. It was assumed that the difference will be obtained because at the initial time moments after the disintegration the waves pattern formation occurs and the fact affects to the accuracy. So for the correct results some time should be skipped before the contour integral calculation. Indeed as one can see in figure 3 for the first contour that coincides with the whole computational domain the approximation order values tend to 1. But for the second contour the accuracy is slightly higher and the convergence to the value 1.1 is obtained for all conservation laws.

6. Conclusions
On the example of the problem of isolated-shock-wave propagation the structure of the numerical shock wave is investigated. The dependencies of the shock wave width from the Courant number is shown.

New approach to the numerical scheme approximation order estimation for the discontinuous solutions based on the contour integrals is proposed. The hypothesis is made that the accuracy of the numerical solution is directly affected by the presence of zones of stationary solution formation after the disintegration.

New linearized edition of the classical Godunov scheme with non-linear disintegrations of discontinuous is formulated. For the proposed scheme the nondecreasing entropy property is valid.

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