ON THE ASYMPTOTIC FORMULA OF $L'(1, \chi)$

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Abstract. Let $\chi$ be a quadratic Dirichlet character. In some literatures, various asymptotic formulae of $L'(1, \chi)$, under the assumption that $L(1, \chi)$ takes a small value, were derived. In this paper, we will give a new treatment unified for the odd and even cases, not depending on Kronecker limit formula. For imaginary quadratic fields, our result coincides with Proposition 22.10 in [1].

1. Introduction

Siegel zeros (if possible) cause many strange phenomena. For a good survey, we refer to [2]. In [3,4] and Chapter 22 of [1], some asymptotic formulae were derived. And for imaginary quadratic fields, our following main result is the same as Proposition 22.10 in [1] essentially.

Theorem 1.1. Let $\chi$ be a primitive quadratic Dirichlet character modulo $q$. If $L(1, \chi) \ll (\log q)^{-26}$, then we have

$$L'(1, \chi) = \frac{\pi^2}{6} \prod_{p|q} \left(1 + \frac{1}{p}\right) \prod_{\substack{p \nmid q \chi(p)=1}} \left(1 - \frac{1}{p}\right)^{-1} \cdot \left(1 + O\left((\log q)^{-1/10}\right)\right).$$

2. The Proof of the Main Theorem

We need some preliminary results before proving it.

Lemma 2.1. Suppose $f(m, n)$ is an arithmetic function in two variables, and $x > u \geq 1$. Then

$$\sum_{k<\sqrt{ux}} \sum_{u<n \leq \frac{x}{k}} f(k^2, n) = \sum_{d \leq u} \sum_{u<n \leq \frac{x}{d}} \sum \lambda(d) f(dr, n) + \sum_{u < n \leq \frac{x}{d}} \sum \lambda(d) f(m, n),$$

where $\lambda(n) = (-1)^{\Omega(n)}$.

Proof: Using the following property of the function $\lambda(n)$:

$$\sum_{d|m} \lambda(d) = \begin{cases} 1, & m \text{ is a square,} \\ 0, & \text{otherwise.} \end{cases}$$
we get that
\[
\sum_{k<\sqrt{x}} \sum_{u<n \leq \frac{x}{k^2}} f(k^2, n) = \sum_{u<n \leq \frac{x}{k^2}} \sum_{u<n} f(m, n) \sum_{d|n} \lambda(d) \\
= \sum_{u<n \leq \frac{x}{k^2}} \sum_{u<n} f(m, n) \sum_{d|n} \lambda(d) + \sum_{u<n \leq \frac{x}{k^2}} \sum_{u<n \leq \frac{x}{k^2}} f(m, n) \sum_{d|n} \lambda(d) \\
= \sum_{d \leq u} \sum_{u<n \leq \frac{x}{k^2}} \sum_{r \leq \frac{x}{d}} \lambda(d) f(dr, n) \\
+ \sum_{u<n \leq \frac{x}{k^2}} \sum_{u<n \leq \frac{x}{k^2}} \sum_{d|n} \lambda(d) f(m, n).
\]

**Lemma 2.2.** Suppose \( \chi \) is a primitive real Dirichlet character modulo \( q \) and \( q < x \). Let \( \psi_u(z, \chi) = \sum_{u<n \leq x} \Lambda(n) \chi(n), \) for \( z > u \). we have
\[
\sum_{k<\sqrt{x}} \sum_{u<n \leq \frac{x}{k^2}} f(k^2, n) = \sum_{u<n \leq \frac{x}{k^2}} \rho_u(m) \chi(m) \psi_u \left( \frac{x}{m} \right) \\
+ \mathcal{O} \left( \left( q \sqrt{q} + \frac{x}{\sqrt{q}} + u^2 \sqrt{q} \right) \log^2 x \right).
\]

where \( \rho_u(m) = \sum_{d|n} \lambda(d). \)

Proof: Let \( \alpha = a/q \) with \( (a, q) = 1 \), and \( e(t) = \exp(2\pi \alpha t) \).

Set \( f(m, n) = \Lambda(n) e(\alpha mn) \). By Lemma [2.1], we have
\[
\sum_{k<\sqrt{x}} \sum_{u<n \leq \frac{x}{k^2}} \Lambda(n) e(\alpha k^2 n) = \sum_{d|n} \sum_{u<n \leq \frac{x}{d}} \sum_{r \leq \frac{x}{d}} \lambda(d) \Lambda(n) e(\alpha drn) \\
+ \sum_{u<n \leq \frac{x}{k^2}} \sum_{u<n \leq \frac{x}{k^2}} \rho_u(m) \Lambda(n) e(\alpha mn) \\
:= T^4(\alpha) + T^3(\alpha).
\]

Let \( rn = l \),
\[
T^4(\alpha) = \sum_{d|u} \lambda(d) \sum_{l \leq x/d} e(\alpha ll) \sum_{n>u, n|l} \Lambda(n) \\
= \sum_{d|u} \lambda(d) \sum_{l \leq x/d} e(\alpha ll) \log l - \sum_{d|u} \lambda(d) \sum_{l \leq x/d} e(\alpha ll) \sum_{n\leq u, n|l} \Lambda(n) \\
:= T^4_1(\alpha) - T^4_2(\alpha).
\]

Applying the basic estimations
\[
\left| \sum_{1 \leq n \leq N} e(\alpha n) \right| \leq \min \left( N, \frac{1}{2 \| \alpha \|} \right)
\]
and
\[
\sum_{1 \leq n \leq N} \min \left( \frac{x}{n}, \frac{1}{2 \| \alpha n \|} \right) \ll N \log q + \frac{x}{q} \log N + q \log q.
\]
we derive by partial summation that

\[
T_1^1(\alpha) \ll \sum_{d \leq u} \left| \sum_{l \leq x/d} e(\alpha dl) \log l \right| \\
\ll \log x \sum_{d \leq u} \min \left( \frac{x}{d}, \frac{1}{2\|\alpha d\|} \right) \\
\ll (xq^{-1} + u + q) \log^2 x.
\]

For \(T_2^1(\alpha)\), let \(l = rn\), we get

\[
T_2^1(\alpha) \ll \sum_{d \leq u} \sum_{n \leq u} \Lambda(n) \left| \sum_{r \leq u} e(\alpha drn) \right| \\
\ll \sum_{d \leq u} \sum_{n \leq u} \Lambda(n) \min \left( \frac{x}{dn}, \frac{1}{2\|\alpha dn\|} \right) \\
\ll \sum_{h \leq u^2} \min \left( \frac{x}{h}, \frac{1}{2\|\alpha h\|} \right) \sum_{n|h} \Lambda(n) \\
\ll (xq^{-1} + u^2 + q) \log^2 x.
\]

For a real primitive character \(\chi\) modulo \(q\),

\[
\sum_{k < \sqrt{q} \atop (k,q) = 1} \psi_u \left( \frac{x}{k^2}, \chi \right) = \sum_{k < \sqrt{q} \atop u \leq n \leq \frac{x}{k^2}} \Lambda(n) \chi(k^2n) \\
= \frac{1}{\tau(\chi)} \sum_{a = 1}^{q} \chi(a) \sum_{k < \sqrt{q} \atop u \leq n \leq \frac{x}{k^2}} \Lambda(n) e \left( \frac{ak^2 n}{q} \right)
\]

Using (2.1), we get

\[
\sum_{k < \sqrt{q} \atop (k,q) = 1} \psi_u \left( \frac{x}{k^2}, \chi \right) = \frac{1}{\tau(\chi)} \sum_{a = 1}^{q} \chi(a) T^q \left( \frac{a}{q} \right) + \frac{1}{\tau(\chi)} \sum_{a = 1}^{q} \chi(a) T^\nu \left( \frac{a}{q} \right) \\
= \mathcal{O} \left( \left( q\sqrt{q} + \frac{x}{\sqrt{q}} + u^2 \sqrt{q} \right) \log^2 x \right) \\
+ \sum_{u < n \leq x} \rho_u(m) \Lambda(n) \chi(mn) \\
= \mathcal{O} \left( \left( q\sqrt{q} + \frac{x}{\sqrt{q}} + u^2 \sqrt{q} \right) \log^2 x \right) \\
+ \sum_{u < m \leq x} \rho_u(m) \chi(m) \psi_u \left( \frac{x}{m}, \chi \right).
\]

\[\Box\]

The following lemma shows that the absolute value of the means of some suitable multiplicative functions does vary slowly.
Lemma 2.3 (Corollary 3). Let \( f \) be a complex-valued multiplicative function with \( |f(n)| \leq 1 \). Then for \( 1 \leq \omega \leq x/10 \), we have
\[
\frac{1}{x} \left| \sum_{n \leq x} f(n) - \frac{\omega}{x} \sum_{n \leq x/\omega} f(n) \right| \ll \left( \frac{\log(2\omega)}{\log x} \right)^{1-\frac{2}{\omega}} \log \left( \frac{\log x}{\log(2\omega)} \right) + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}},
\]
The implied constant is absolute and computable.

From the above lemma, we can deduce the following corollary.

Corollary 2.4. Suppose \( f \) is a real-valued multiplicative function with \( |f(n)| \leq 1 \) for all \( n \). Then for \( 1 \leq \omega \leq \sqrt{x}/2 \), we have
\[
\frac{1}{x} \sum_{n \leq x} f(n) - \frac{\omega}{x} \sum_{n \leq x/\omega} f(n) \ll \left( \frac{\log(2\omega)}{\log x} \right)^{1-\frac{2}{\omega}} \log \left( \frac{\log x}{\log(2\omega)} \right) + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}}.
\]

Proof: By Lemma 2.3 there exists an absolute constant \( C_0 > 1 \) such that
\[
\frac{1}{x} \left| \sum_{n \leq x} f(n) - \frac{\omega}{x} \sum_{n \leq x/\omega} f(n) \right| < C_0 M(x, \omega),
\]
where
\[
M(x, \omega) = \left( \frac{\log(2\omega)}{\log x} \right)^{1-\frac{2}{\omega}} \log \left( \frac{\log x}{\log(2\omega)} \right) + \frac{\log \log x}{(\log x)^{2-\sqrt{3}}}.
\]
If \( \frac{1}{x} \sum_{n \leq x} f(n) < 2C_0 M(x, \omega) \), then \( \frac{\omega}{x} \sum_{n \leq x/\omega} f(n) < 3C_0 M(x, \omega) \). Therefore
\[
\frac{1}{x} \left| \sum_{n \leq x} f(n) - \frac{\omega}{x} \sum_{n \leq x/\omega} f(n) \right| < 5C_0 M(x, \omega).
\]
Otherwise,
\[
\frac{1}{x} \left| \sum_{n \leq x} f(n) \right| \geq 2C_0 M(x, \omega).
\]
Without loss of generality, we may assume
\[
\frac{1}{x} \sum_{n \leq x} f(n) \geq 2C_0 M(x, \omega).
\]
Then
\[
\frac{\omega}{x} \sum_{n \leq x/\omega} f(n) > C_0 M(x, \omega) \quad \text{or} \quad < -C_0 M(x, \omega).
\]
If \( \frac{\omega}{x} \sum_{n \leq x/\omega} f(n) > C_0 M(x, \omega) \), we have
\[
\left| \frac{1}{x} \sum_{n \leq x} f(n) - \frac{\omega}{x} \sum_{n \leq x/\omega} f(n) \right| = \left| \frac{1}{x} \sum_{n \leq x} f(n) - \frac{\omega}{x} \sum_{n \leq x/\omega} f(n) \right| < C_0 M(x, \omega).
\]
For the case \( \frac{\omega}{x} \sum_{n \leq x/\omega} f(n) < -C_0 M(x, \omega) \), we can show that there exists a \( x_0 \in [\frac{x}{2}, x] \) such that
\[
\sum_{n \leq x_0} f(n) \leq \frac{1}{2}, \quad \text{since} \quad \left( \sum_{n \leq x} f(n) \right) \cdot \left( \sum_{n \leq x/\omega} f(n) \right) < 0 \quad \text{and the real-valued function} \ f \ \text{satisfies} \ |f(n)| \leq
A basic observation is that the function
\[
\frac{1}{x} \sum_{n \leq x} f(n) - \frac{1}{x_0} \sum_{n \leq x_0} f(n) > 2C_0 M(x, \omega) - \frac{1}{2x_0} \\
> 2C_0 M(x, \omega) - \frac{\omega}{2x}
\]

On the other hand, from Theorem 2.3, we deduce that
\[
\left| \frac{1}{x} \sum_{n \leq x} f(n) - \frac{1}{x_0} \sum_{n \leq x_0} f(n) \right| < C_0 M(x, x/x_0).
\]

Since \(x/x_0 \leq \omega\), we have
\[
\frac{\log x}{\log(2(x/x_0))} \geq \frac{\log x}{\log(2\omega)}.
\]

A basic observation is that the function
\[
\left( \frac{1}{2} \right)^{1 - \frac{\omega}{2} \log t} \log t
\]
increases in \([1, \exp(\frac{1}{1-2/\pi})]\) and decreases in \(\exp(\frac{1}{1-2/\pi}), +\infty\).

- For \(\frac{\log x}{\log(2\omega)} \geq \exp(\frac{1}{1-2/\pi})\), combining 2.3 we have \(M(x, x/x_0) \leq M(x, \omega)\). Hence
\[
\left| \frac{1}{x} \sum_{n \leq x} f(n) - \frac{1}{x_0} \sum_{n \leq x_0} f(n) \right| < C_0 M(x, \omega).
\]

Furthermore, from \(\frac{\log x}{\log(2\omega)} > \exp(\frac{1}{1-2/\pi}) > 15\), we have \(2\omega < x^{1/15}\). It follows that for \(x \geq 4\), we have
\[
\frac{\omega}{2x} < \frac{1}{4x^{14/15}} < \frac{\log \log x}{(\log x)^{2 - \sqrt{3}}} < C_0 M(x, \omega), \quad \text{(since } C_0 > 1)\]

Taking the inequality 2.2 into consideration, we get
\[
\left| \frac{1}{x} \sum_{n \leq x} f(n) - \frac{1}{x_0} \sum_{n \leq x_0} f(n) \right| > C_0 M(x, \omega),
\]

This leads to a contradiction!

- If \(\frac{\log x}{\log(2\omega)} < \exp(\frac{1}{1-2/\pi})\), we get \(2 \leq \frac{\log x}{\log(2\omega)} < \exp(\frac{1}{1-2/\pi})\), since \(\omega \leq \sqrt{x}/2\).

Hence \(\left( \frac{\log(2\omega)}{\log x} \right)^{1 - \frac{\omega}{2}} \log \left( \frac{\log x}{\log(2\omega)} \right) \geq \left( \frac{1}{2} \right)^{1 - \frac{\omega}{2} \log 2} \).

In such a case, the result is implied by the trivial estimate:
\[
\frac{1}{x} \sum_{n \leq x} f(n) - \frac{\omega}{x} \sum_{n \leq x/\omega} f(n) \ll \left( \frac{\log(2\omega)}{\log x} \right)^{1 - \frac{\omega}{2}} \log \left( \frac{\log x}{\log(2\omega)} \right).
\]

In conclusion, we always have
\[
\frac{1}{x} \sum_{n \leq x} f(n) - \frac{\omega}{x} \sum_{n \leq x/\omega} f(n) \ll M(x, \omega). \quad \Box
\]
Lemma 2.5. Suppose $2\sqrt{x} < u^2 < x$ and $\chi$ is a non-principle Dirichlet character modulo $q$, we have
\[
\sum_{u < m \leq x/u} \rho_u(m) \chi(m) \left( \frac{x}{m} - u \right) = \left( u \sum_{n \leq \frac{x}{u}} \lambda(n) \chi(n) \right) \left( L(1, \chi) \log \frac{x}{eu^2} + L'(1, \chi) \right)
+ \mathcal{O} \left( \left( u^2 \log \frac{x}{u^2} \sqrt{q} \log q + \frac{x}{u} \right) \right)
+ \mathcal{O} \left( \epsilon(x, u) \cdot x \left( \log q + \left( \log \frac{x}{u^2} \right)^2 \right) \right),
\]
where
\[
\epsilon(x, u) = \left( \log \frac{x}{u} \right)^{3/2} + \left( \frac{\log 2x}{\log u} \right)^{1 - \frac{n}{3}} \cdot \log \log x.
\]
Proof: By partial summation,
\[
\sum_{u < m \leq x/u} \rho_u(m) \chi(m) \left( \frac{x}{m} - u \right) = \sum_{u < m \leq x/u} \frac{x}{u} \sum_{u < m \leq t} \rho_u(m) \chi(m)
+ \left( \frac{x}{[x/u]} - u \right) \sum_{u < m \leq x/u} \rho_u(m) \chi(m).
\]
From the property of $\lambda(n)$, we have
\[
\sum_{u < m \leq x/u} \rho_u(m) \chi(m) = \sum_{u < m \leq x/u} \chi(t^2) - \sum_{u < m \leq x/u} \left( \sum_{d \mid m} \lambda(d) \right) \chi(m)
= \mathcal{O} \left( \sqrt{\frac{x}{u}} - \sum_{d \leq u} \lambda(d) \chi(d) \sum_{\frac{m}{d} \leq \frac{x}{u}} \chi(n) \right)
= \mathcal{O} \left( \sqrt{\frac{x}{u}} + u \sqrt{q} \log q \right),
\]
In the last step, we used the Pólya-Vinogradov estimate.
On the other hand, we have
\[
\sum_{u < m \leq t} \rho_u(m) \chi(m) = \sum_{u < m \leq t} \chi(m) \lambda(m) \sum_{d \mid m} \lambda(d)
= \sum_{u < m \leq t} \rho_u(m) \chi(m) = \sum_{d \leq u} \chi(d) \sum_{u \leq n \leq \frac{x}{u}} \lambda(n) \chi(n).
\]
Combining the above identities, we have
\[
(2.4) \sum_{u < m \leq x/u} \rho_u(m) \chi(m) \left( \frac{x}{m} - u \right) = \sum_{u < t \leq \frac{x}{u}} \frac{x}{u(t+1)} \left( \sum_{d \leq \frac{x}{u}} \chi(d) \sum_{u \leq n \leq \frac{x}{u}} \lambda(n) \chi(n) \right)
+ \mathcal{O} \left( \frac{u^2}{x} \left( \sqrt{\frac{x}{u}} + u \sqrt{q} \log q \right) \right)
:= I_1 - I_2 + \mathcal{O} \left( \sqrt{\frac{x}{u}} + u \sqrt{q} \log q \right),
\]
where

\[ I_1 = \sum_{u < t \leq \frac{x}{4} - 1} \frac{x}{t(t+1)} \left( \sum_{d \leq \frac{x}{u}} \chi(d) \left( \sum_{n \leq \frac{t}{u}} \lambda(n) \chi(n) \right) \right), \]

\[ I_2 = \sum_{u < t \leq \frac{x}{4} - 1} \frac{x}{t(t+1)} \left( \sum_{d \leq \frac{x}{u}} \chi(d) \left( \sum_{n < u} \lambda(n) \chi(n) \right) \right). \]

\[ I_2 = x \sum_{n < u} \lambda(n) \chi(n) \left( \sum_{d \leq \frac{x}{u^2} - 1} \chi(d) \left( \sum_{u < t \leq \frac{x}{4} - 1} \frac{1}{t(t+1)} \right) \right) \]

\[ = x \sum_{n < u} \lambda(n) \chi(n) \left( \sum_{d \leq \frac{x}{u^2} - 1} \chi(d) \left( \frac{1}{du} - \frac{1}{x/u} \right) + O \left( \sum_{d \leq \frac{x}{u^2} - 1} \frac{1}{(du)^2} \right) \right) \]

\[ = x \sum_{n < u} \lambda(n) \chi(n) \left( \frac{1}{u} \sum_{d \leq \frac{x}{u^2} - 1} \chi(d) \frac{1}{d} + O \left( \frac{u^2 \sqrt{q \log q}}{x} \right) + O \left( \frac{1}{u^2} \right) \right) \]

Noticing that

\[ \sum_{d \leq \frac{x}{u^2} - 1} \frac{\chi(d)}{d} = L(1, \chi) + O \left( \frac{u^2 \sqrt{q \log q}}{x} \right), \]

we have

\[ I_2 = \frac{x}{u} L(1, \chi) \sum_{n < u} \lambda(n) \chi(n) + O \left( u^2 \sqrt{q \log q} + \frac{x}{u} \right). \]

Applying Corollary 2.4, we have

\[ I_2 = \left( u \sum_{n \leq \frac{x}{u}} \lambda(n) \chi(n) \right) \cdot L(1, \chi) + O \left( u^2 \sqrt{q \log q} + \frac{x}{u} + \epsilon(x, u) \right). \]

By Corollary 2.4, for \( du \leq t \leq \frac{x}{u} - 1 \), we have

\[ \frac{1}{t/d} \sum_{n \leq \frac{x}{d}} \lambda(n) \chi(n) = \frac{1}{x/u} \sum_{n \leq \frac{x}{u}} \lambda(n) \chi(n) + O \left( \epsilon(x, u) \right), \]

where

\[ \epsilon(x, u) = \left( \log \frac{x}{u} \right)^{\sqrt{3}-2} + \left( \log \frac{2x}{\log x} \right)^{1-\frac{2}{3}} \cdot \log \log x. \]
Due to the identity (2.8), now we can handle the sum $I_1$.

\[
I_1 = \left( \frac{u}{x} \sum_{n \leq \frac{x}{u}} \lambda(n) \chi(n) \right) \left( \sum_{u < t \leq \frac{x}{u} - 1} \frac{x}{t(t + 1)} \sum_{d \leq \frac{x}{t}} \frac{\chi(d)}{d} \right)
\]
\[
+ O \left( \epsilon(x, u) \left( \sum_{u < t \leq \frac{x}{u} - 1} \frac{x}{t(t + 1)} \sum_{d \leq \frac{x}{t}} \frac{t}{d} \right) \right)
\]
\[
= \left( \frac{u}{x} \sum_{n \leq \frac{x}{u}} \lambda(n) \chi(n) \right) \left( \sum_{d \leq \frac{x}{u} - \frac{1}{u}} \frac{\chi(d)}{d} \sum_{d \leq \frac{x}{u} - \frac{1}{u} - 1} \frac{x}{t + 1} \right)
\]
\[
+ O \left( \epsilon(x, u) x \log^2 \left( \frac{x}{u} \right) \right)
\]
\[
= \left( \frac{u}{x} \sum_{n \leq \frac{x}{u}} \lambda(n) \chi(n) \right) \left( x \sum_{d \leq \frac{x}{u} - \frac{1}{u}} \frac{\chi(d)}{d} \left( \log \frac{x}{u^2} - \log d \right) + O \left( \sum_{d \leq \frac{x}{u} - \frac{1}{u}} \frac{x}{d^2 u} \right) \right)
\]
\[
+ O \left( \epsilon(x, u) x \log^2 \left( \frac{x}{u^2} \right) \right)
\]
\[
= \left( \frac{u}{x} \sum_{n \leq \frac{x}{u}} \lambda(n) \chi(n) \right) \left( \log \frac{x}{u^2} \sum_{d \leq \frac{x}{u} - \frac{1}{u}} \frac{\chi(d)}{d} - \sum_{d \leq \frac{x}{u} - \frac{1}{u} - 1} \frac{\chi(d)}{d} \log d \right)
\]
\[
+ O \left( \frac{x}{u} + \epsilon(x, u) x \log^2 \left( \frac{x}{u^2} \right) \right)
\]

Since

\[
- \sum_{d \leq \frac{x}{u} - \frac{1}{u}} \frac{\chi(d)}{d} \log d = L'(1, \chi) + O \left( \frac{u^2 \log \frac{x}{u^2}}{x} \sqrt{q \log q} \right),
\]

combining (2.5), we get

\[
(2.9) \quad I_1 = \left( \frac{u}{x} \sum_{n \leq \frac{x}{u}} \lambda(n) \chi(n) \right) \left( \log \frac{x}{u^2} \cdot L(1, \chi) + L'(1, \chi) \right)
\]
\[
+ O \left( \frac{x}{u} + \left( u^2 \log \frac{x}{u^2} \right) \sqrt{q \log q} + \epsilon(x, u) x \log^2 \left( \frac{x}{u^2} \right) \right).
\]

Finally, combining equations (2.4), (2.9) and (2.7), we get

\[
\sum_{u < m \leq \frac{x}{u}} \rho_u(m) \chi(m) \left( \frac{x}{m} - u \right)
\]
\[
= \left( \frac{u}{x} \sum_{n \leq \frac{x}{u}} \lambda(n) \chi(n) \right) \left( \log \frac{x}{u^2} \cdot L(1, \chi) + L'(1, \chi) \right)
\]
\[
+ O \left( \frac{x}{u} + \left( u^2 \log \frac{x}{u^2} \right) \sqrt{q \log q} + \epsilon(x, u) x \left( \log q + \log^2 \left( \frac{x}{u^2} \right) \right) \right). \quad \square
\]
Lemma 2.6 (see (22.109) in Chapter 22 of [1]). Suppose $\chi$ is a non-principle Dirichlet character modulo $q$ and $x \geq q$, we have

$$\sum_{n \leq x} \frac{\tau(n, \chi)}{n} = L(1, \chi)(\log x + \gamma) + L'(1, \chi) + O\left(q^{1/4}x^{-1/2}\log x\right),$$

where $\tau(n, \chi) = \sum_{d \mid n} \chi(d)$.

The following corollary shows that if $L(1, \chi)$ takes small value, then $\chi(p)$ takes negative value for most of small prime $p$.

Corollary 2.7. Suppose $\chi$ is a quadratic Dirichlet character modulo $q$ and $x \geq q$. Then

$$\sum_{n \leq x} \Lambda(n)\chi(n) = -x + O\left((L(1, \chi) + q^{-1/4}) x \log^2 x + xe^{-c\sqrt{\log x}} + q\right),$$

where $c$ is some positive constant.

Proof: By Lemma 2.6, we have

$$(2.10) \sum_{q < n \leq y} \frac{\tau(n, \chi)}{n} = L(1, \chi) \log \frac{y}{q} + O\left(q^{-1/4}\log q\right).$$

For a real character $\chi$, $\tau(n, \chi) \geq 0$. Then for $y > q$, we have

$$(2.11) \sum_{q < p \leq y} \frac{1 + \chi(p)}{p} \log p \leq \log y \sum_{q < n \leq y} \frac{\tau(n, \chi)}{n} \ll \left(L(1, \chi) + q^{-1/4}\right) \log^2 y.$$ 

Applying partial summation, we have

$$(2.12) \sum_{q < p \leq x} (1 + \chi(p)) \log p \ = \ [x] \cdot \sum_{q < p \leq x} \frac{(1 + \chi(p)) \log p}{p} - \sum_{m \leq x} \sum_{q < m \leq x} \frac{(1 + \chi(p)) \log p}{p} \ll \left(L(1, \chi) + q^{-1/4}\right) x \log^2 x.$$ 

On the other hand,

$$\sum_{n \leq x} \Lambda(n)\chi(n) \ = \ \sum_{n \leq x} \Lambda(n)(1 + \chi(n)) - \sum_{n \leq x} \Lambda(n)$$

$$\ = \ \sum_{p \leq x} (1 + \chi(p)) \log p - x + O\left(xe^{-c\sqrt{\log x}}\right)$$

$$\ = \ -x + \sum_{q \leq x} (1 + \chi(p)) \log p + O\left(q + xe^{-c\sqrt{\log x}}\right).$$

Combining the estimate (2.12), we get the conclusion. □

Lemma 2.8 (Proposition 4.5 in [6] or Proposition 3 in [5]). For any multiplicative function $f$ with $|f(p^k)| \leq 1$ for every prime power $p^k$, let

$$\Theta(f, x) := \prod_{p \leq x} \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \cdots\right) \left(1 - \frac{1}{p}\right),$$

and

$$s(f, x) := \sum_{p \leq x} \frac{|1 - f(p)|}{p}.$$
For any $\varepsilon$ satisfying $1 > \varepsilon \geq \frac{\log 2}{\log x}$, let $g$ be a completely multiplicative function satisfying the following condition:
\[ g(p) = \begin{cases} 1, & p \leq x^\varepsilon; \\ f(p), & p > x^\varepsilon. \end{cases} \]

Then we have
\[ \frac{1}{x} \sum_{n \leq x} f(n) = \Theta(f, x^\varepsilon) \frac{1}{x} \sum_{m \leq x} g(m) + O(\varepsilon \exp(s(f, x))), \]
where the implied constant is absolute.

Similarly as corollary 2.7, we get the following result.

**Corollary 2.9.** Suppose $\chi$ is a quadratic Dirichlet character modulo $q$ and $x \geq q$. Then
\[ \sum_{n \leq x} \chi(n) \lambda(n) = P(q)x + O\left(\left(L(1, \chi) + q^{-1/4}\right)x \log x + x \frac{\log^3 q}{\log x}\right), \]
where
\[ P(q) = \prod_{p \leq q} \left(1 - \frac{1}{p}\right) \left(1 + \frac{\chi(p)}{p}\right)^{-1}. \]

**Proof:** Let the completely multiplicative function $f$ uniquely determined by the following condition:
\[ f(p) = \begin{cases} \chi(p)\lambda(p) = -\chi(p), & p \leq q; \\ 1, & p > q. \end{cases} \]

Denote $E(x) = \{n \leq x \mid n \text{ has at least a prime factor } p \text{ such that } p > q \text{ and } \lambda(p)\chi(p) \neq 1\}$. Obviously, we have
\[ \left| \sum_{n \leq x} \chi(n)\lambda(n) - \sum_{n \leq x} f(n) \right| \leq 2|E(x)| \]
\[ \leq \sum_{q < p \leq x} \frac{1}{x_{\chi(p) \neq 1}} \sum_{n \leq x} \frac{1}{p} \]
\[ \leq x \sum_{q < p \leq x} \frac{1 + \chi(p)}{p}. \]

Similarly as (2.11), from (2.10) we get that
\[ \sum_{q < p \leq x} \frac{1 + \chi(p)}{p} \leq \sum_{q < n \leq x} \frac{\tau(n, \chi)}{n} \ll \left(L(1, \chi) + q^{-1/4}\right) \log x. \]

Hence
\[ (2.13) \sum_{n \leq x} \chi(n)\lambda(n) = \sum_{n \leq x} f(n) + O\left(\left(L(1, \chi) + q^{-1/4}\right)x \log x\right). \]
Now let the completely multiplicative function \( g(n) \equiv 1 \).

Note that \( g(p) = f(p) \) for \( p > q \), by Lemma 2.8 we have
\[
\frac{1}{x} \sum_{n \leq x} f(n) = \Theta(f, q) \frac{1}{x} \sum_{m \leq x} g(m) + O\left( \frac{\log q}{\log x} \exp(s(f, x)) \right),
\]
where
\[
\Theta(f, q) = \prod_{p \leq q} \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{\chi(p)}{p} \right)^{-1},
\]
\[
s(f, x) = \sum_{p \leq x} \frac{1 - f(p)}{p} = \sum_{p \leq q} \frac{1 + \chi(p)}{p}.
\]
Therefore,
\[
\sum_{n \leq x} f(n) = \Theta(f, q) x + O\left( x \frac{\log q}{\log x} \right).
\]
Combining (2.13), we have
\[
\sum_{n \leq x} \chi(n) \lambda(n) = \Theta(f, q) x + O\left( (L(1, \chi) + q^{-1/4}) x \log x + x \frac{\log q}{\log x} \right). \quad \square
\]

The proof of Theorem 1.1
Now we compute the sum
\[
\sum_{u < m \leq x/u} \rho_u(m) \chi(m) \psi_u\left( \frac{x}{m}, \chi \right)
\]
in two different ways under our assumption, which leads to the main result.
Let \( T = \exp\left( (\log q)^8 \right) \). We choose \( x = T \) and \( u = \sqrt{T/q} \). For sufficiently large \( q \), we have \( u \geq q \) and \( u^2 > 2\sqrt{x} \). From Lemma 2.2 and Corollary 2.7 it follows that
\[
\sum_{k < (qT)^{1/4}} \left( -\frac{T}{k^2} + O\left( \sqrt{T/q} \right) \right)
\]
\[
+ O\left( \sum_{k < (qT)^{1/4}} \left( L(1, \chi) + q^{-1/4} \right) \frac{T}{k^2} \log^2 T + \frac{T}{k^2} e^{-c\sqrt{\log(T/k^2)}} + q \right)
\]
\[
= - \sum_{\sqrt{T/q} < m \leq (qT)^{1/2}} \rho_u(m) \chi(m) \left( \frac{T}{m} - \sqrt{T/q} \right) + O\left( \frac{T}{\sqrt{q}} \log^2 T \right)
\]
\[
+ O\left( \sum_{\sqrt{T/q} < m \leq (qT)^{1/2}} d(m) \left( L(1, \chi) + q^{-1/4} \right) \frac{T}{m} \log^2 T + \frac{T}{m} e^{-c\sqrt{\log(T/m)}} + q \right).
\]
Since \( \sum_{m \leq y} d(m) \ll y \log y \), by partial summation we have
\[
\sum_{\sqrt{T/q} < m \leq (qT)^{1/2}} \frac{d(m)}{m} \ll \log T \log q,
\]
and
\[ \sqrt{\frac{q}{T}} \sum_{m \leq (qT)^{1/2}} \frac{d(m)}{m} e^{-c\sqrt{\log(T/m)}} \ll (\log T)^2 e^{-c\sqrt{\log(T/q)}}. \]

Hence
\[ (2.14) \quad T \sum_{k \leq (qT)^{1/4}} \frac{1}{k^2} = \sum_{\sqrt{T/q} \leq m \leq (qT)^{1/2}} \rho_u(m) \chi(m) \left( \frac{T}{m} - \sqrt{T/q} \right) \]
\[ + O \left( (L(1, \chi) + q^{-1/4}) T (\log T)^3 \log q \right). \]

On the other hand, it follows from Lemma 2.5 and Corollary 2.9 that
\[ \sqrt{T/q} \sum_{m \leq (qT)^{1/2}} \rho_u(m) \chi(m) \left( \frac{T}{m} - \sqrt{T/q} \right) \]
\[ = \left( P(q)T + O \left( (L(1, \chi) + q^{-1/4}) T \log T + T \frac{\log q}{\log T} \right) \right) \]
\[ \times (L(1, \chi) + O(L(1, \chi) \log q)) \]
\[ + O \left( \frac{T}{\sqrt{q}} \log^2 q + T \log^2 q \left( (\log T)^{3 - 2} \log \log T \right)^{1 - \frac{2}{3}} \cdot \log \log T \right). \]

Note that \( \log T = (\log q)^{10} \) and \( P(q) \gg (\log q)^{-2} \), under the assumption of \( L(1, \chi) \ll (\log q)^{-26} \), we have
\[ (2.15) \quad \sum_{\sqrt{T/q} \leq m \leq (qT)^{1/2}} \rho_u(m) \chi(m) \left( \frac{T}{m} - \sqrt{T/q} \right) = P(q) L'(1, \chi) T + O \left( T (\log q)^{-1/10} \right). \]

Besides,
\[ (2.16) \quad \sum_{k \leq (qT)^{1/4}} \frac{1}{k^2} = \frac{\pi^2}{6} \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right) + O \left( (qT)^{-1/4} \right). \]

Inserting (2.14) and (2.16) into (2.14) gives
\[ P(q) L'(1, \chi) = \frac{\pi^2}{6} \prod_{p \mid q} \left( 1 - \frac{1}{p^2} \right) + O \left( (\log q)^{-1/10} \right). \]

It follows that
\[ L'(1, \chi) = \frac{\pi^2}{6} \prod_{p \mid q} \left( 1 + \frac{1}{p} \right) \prod_{p \mid q \atop x(p) = 1} \left( 1 + \frac{1}{p} \right)^{-1} \left( 1 - \frac{1}{p} \right)^{-1} \left( 1 + O \left( (\log q)^{-1/10} \right) \right). \]

\[ \Box \]

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