IRREDUCIBLE REPRESENTATIONS OF THE SYMMETRIC GROUPS FROM
SLASH HOMOLOGIES OF \( p \)-COMPLEXES

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ABSTRACT. In the 40s, Mayer introduced a construction of (simplicial) \( p \)-complex by using the unsign ed boundary map and taking coefficients of chains modulo \( p \). We look at such a \( p \)-complex associated to an \((n-1)\)-simplex; in which case, this is also a \( p \)-complex of representations of the symmetric group of rank \( n \) - specifically, of permutation modules associated to two-row compositions. In this article, we calculate the so-called slash homology - a homology theory introduced by Khovanov and Qi - of such a \( p \)-complex. We show that every non-trivial slash homology group appears as an irreducible representation associated two-row partitions, and how this calculation leads to a basis of these irreducible representations given by the so-called \( p \)-standard tableaux.

1. Introduction

The modern notion of \( p \)-complexes first appear in the works of Mayer [M1, M2] back in the 40s. A \( p \)-complex is a sequence of groups \((C_k)_{k \in \mathbb{Z}}\) along with maps (called \( p \)-differentials) \((\partial_k: C_k \to C_{k-1})_{k \in \mathbb{Z}}\) such that any composition of \( p \) consecutive such maps is zero. A simple example is to take \( C_k = \mathbb{Z}\) for all \( 0 \leq k \leq p \), \( \partial_k = \text{id} \) for all \( 0 < k \leq p \), and \( C_k = 0 = \partial_k \) for all other \( k \)'s.

In [M1], Mayer defined a homology theory on a \( p \)-complex \( C_\bullet = (C_k, \partial_k)_{k \in \mathbb{Z}}\) given by the groups \( aH_k := \ker(\partial^a)/\text{im}(\partial^{p-a}) \), with \( a \in \{1, 2, \ldots, p-1\} \). Moreover, the \( p \)-differential \( \partial_k \) naturally induces a map \( \partial^a \) on \( C_k \to C_{k+1} \) for all \( a \), which gives the total homology \( \bigoplus_{a=0}^{p-1} H_k \) a structure of a \((p-1)\)-complex. However, a result of Spanier states that these homology groups can be determined by the classical simplicial homology groups, and so \( p \)-complexes never catch much attention to topologists.

Recently, the Khovanov school advocates that \( p \)-complexes should provide a foundational tool for categorifying quantum groups at primitive \( p \)-th root of unity. As a part of this programme, in [KQ], Khovanov and Qi gave a more detailed account of two alternative homology theories for \( p \)-complexes (which are dual to each other). These homology theories are called slash and backslash homology, and have better compatibility with the technology of differential graded categories than Mayer’s version.

1.1. The slash homology

This article studies the slash homology of a \( p \)-complex associated to an \((n-1)\)-simplex. Let us be more precise about the construction of this \( p \)-complex. Fix an integer \( n \) and for each \( k \in \{0, 1, \ldots, n\} \), let \( \Omega_k \) be the set of \( k \)-subsets, i.e. subsets of \( \{1, 2, \ldots, n\} \) of size \( k \). Fix a field \( \mathbb{F} \) of prime characteristic \( p > 0 \) and let \( \mathbb{F}\Omega_k \) be the \( \mathbb{F} \)-vector space with basis \( \Omega_k \). Let \( \varphi_k : \mathbb{F}\Omega_k \to \mathbb{F}\Omega_{k-1} \) be the \( \mathbb{F} \)-linear map that sends a \( k \)-subset \( \omega \) to the formal sum of all \((k-1)\)-subsets contained in \( \omega \), i.e.

\[
\varphi_k : \omega \mapsto \sum_{\omega' \subset \omega, |\omega'| = k-1} \omega'.
\]

It is an easy exercise to check that \( \varphi_k \varphi_{k-1} \cdots \varphi_1 = 0 \) for any \( k \) (or simply \( \varphi^p = 0 \) by omitting subscripts). Hence, we have a \( p \)-complex of the form

\[
\mathbb{F}\Omega_\bullet := (0 \to \mathbb{F}\Omega_n \xrightarrow{\varphi_n} \mathbb{F}\Omega_{n-1} \xrightarrow{\varphi_{n-1}} \cdots \xrightarrow{\varphi_2} \mathbb{F}\Omega_1 \xrightarrow{\varphi_1} \mathbb{F}\Omega_0 \to 0).
\]

It should not be difficult to see that in the case of \( p = 2 \), \( \mathbb{F}\Omega_\bullet \) is the canonical simplicial chain complex associated to an \((n-1)\)-simplex.

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In fact, $\mathbb{F} \Omega_\bullet$ is not just a $p$-complex of $\mathbb{F}$-vector spaces, it is also a $p$-complex of $G_n$ representations, where $G_n$ is the symmetric group of rank $n$. Indeed, $\mathbb{F} \Omega_k$ is isomorphic to the module induced from the trivial $\mathbb{F}(G_{n-k} \times G_k)$-module, i.e. the permutation module $M^{(n-k,k)}$ of the two-part composition $(n-k, k)$ of $n$, and $\varphi_k$’s are also $\mathbb{F} G_n$-linear.

Our first major result is the complete description of the (total) slash homology of $\mathbb{F} \Omega_\bullet$.

**Theorem.** (Theorem 4.16) For any prime $p$ and any positive integer $n$, the slash homology $(p-1)$-complexes $H^j_k$ at degree $k \in \{0, 1, \ldots, n\}$ of $\mathbb{F} \Omega_\bullet$ is non-vanishing if, and only if, $p \neq 2$ and $n-2k \in [0, p-2]$. Moreover, in the case when $H^j_k$ is non-vanishing, it takes the form

$$H^j_k \simeq (0 \to H^{n-2k}_{n-k} \sim H^{n-2k-1}_{n-k-1} \sim \cdots \sim H^0_{k+1} \sim H^0_k \to 0) \sim (0 \to D^{(n-k,k)} \sim D^{(n-k,k)} \sim \cdots \sim D^{(n-k,k)} \sim D^{(n-k,k)} \to 0),$$

where $D^{(n-k,k)}$ is the simple $\mathbb{F} G_n$-module corresponding to the partition $(n-k, k)$.

We remark that knowing the slash homology is enough to determine the backslash homology and Mayer’s version of homology of $\mathbb{F} \Omega_\bullet$ as well; see subsection 2.2 and subsection 2.3. Note also that our condition means that $H^j_k$ vanishes when $2k \leq n - (p - 1)$. In particular, all slash homology groups vanish when $p = 2$. This is just the classical result which says that the chain complex of an $(n-1)$-simplex has zero homology, as every student taking a first course in algebraic topology would know.

Let us give some remark on the technique we use to achieve our goal. Inspire by the “$\alpha$-sequence” combinatorics used by James in studying the permutation modules [J1], we consider a new statistic, called density, on two-row tableaux (which are equivalent to subsets) of $n$. In fact, if $2k \leq n$ (meaning that $(n-k,k)$ is a two-row partition), then density records when a column in the row-standard tableau is standard. Then we define threshold to allow us carving an induction machine to show that “most” elements in $\ker(\varphi)$ lies in $\text{Im}(\varphi^{p-1})$.

One particular step in proving the above theorem is to show that $H^j_k^{0}$ is a quotient of the Specht module $S^{(n-k,k)}$ when it is non-zero. Recall that every Specht module $S^\lambda$ admits a basis given by the polytabloids $e_t$ associated to standard tableaux $t$ of shape $\lambda$. It turns out the projecting this basis onto $H^j_k^{0}$ results in a basis that is indexed by what Kleshchev calls $p$-standard tableaux in [Kl] (see Definition 5.1, its following remark, and Lemma 5.11).

**Theorem.** (Theorem 5.2) The simple $\mathbb{F} G_n$-module $D^{(n-k,k)}$, where $k$ satisfies $n - (p - 1) < 2k \leq n$ admits a basis of the form

$$\{ e_t + \text{rad} S^{(n-k,k)} \mid t \text{ is a } p \text{-standard tableau} \},$$

We remark that it is shown in [Kl] that $\dim_\mathbb{F} D^{(n-k,k)}$ is the same as the number of $p$-standard tableaux of shape $(n-k,k)$, but no explicit description of such a basis is given. Moreover, Kleshchev’s starting point is to look at restriction of simple $\mathbb{F} G_n$-modules to $\mathbb{F} G_{n-1}$, whereas our deduction comes from understanding the map $\varphi_k$ on permutation modules and the classic proof of the standard polytabloid basis of Specht modules.

**Literature.** After we have finished writing up this script, we discovered that the $p$-homology groups of $\mathbb{F} \Omega_\bullet$ is completely determined in [BJS]. As we have mentioned, the $p$-homology groups contain an equivalent amount of information as the back/slash homology groups via the exact sequences (2.2.1). Nevertheless, the combinatorics we employ to determine the slash homology groups are different and rooted from those that are used by James in [J1, J2].

**Structure.** The preliminary Section 2 presents the necessary material for our investigation - primarily elementary facts about $\mathbb{F} \Omega_k$’s and $p$-complexes. Section 3 is devoted to establish the notion of density and threshold, as well as the tools and techniques that come with them. In Section 4, we calculate the dimension of all slash homology groups and proves the first main theorem. The final Section 5 is devoted to proving Theorem 5.2.

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2. Preliminaries

Throughout, $F$ is a fixed field of positive characteristic $p > 0$. We will mostly follow notations in [Wil]. In particular, by module we mean a finitely generated right modules. Maps are also applied to the right of a module.

Unless otherwise stated, we fix a positive integer $n$ throughout the article, and define $m := \lfloor n/2 \rfloor + 1$.

Define $[a, b] := \{a, a + 1, \ldots, b\}$ for $a < b \in \mathbb{Z}$, and $[a] := \{1, a\}$. For a set $S$, we denote by $|S|$ its cardinality. We use $\mathcal{S} \setminus \mathcal{T}$ to denote the complement of the subset $\mathcal{T}$ of $\mathcal{S}$.

For a finite set $\Psi$, we denote by $F\Psi$ the $F$-vector space with basis $\Psi$. If $\Psi$ is a finite subset of a module over some algebra, then we denote by $F\text{-span}\Psi$ the set of elements spanned by elements of $\Psi$ over $F$ - we do not assume $\Psi$ is a basis of $F\text{-span}\Psi$ in this case, unless otherwise stated.

For a finite set $X$, denote by $\mathcal{S}_X$ the group of symmetries on $X$. Clearly, $\mathcal{S}_X$ is isomorphic to the symmetric group $\mathcal{S}_r$ of rank $r := |X|$.

For a set $X$ acted upon by a group $G$, and a subset $Y \subset X$, the stabiliser (subgroup) of $Y$ in $G$, or just $Y$-stabiliser if the context is clear, is given by the set of $g \in G$ such that $(Y)g = Y$ (instead of $(y)g = y$ for all $y \in Y$).

2.1. The permutation modules $F\Omega_k$. Denote by $\Omega_k$ the set of all $k$-subsets of $\llbracket n \rrbracket$. For $\omega \in \Omega_k$, we denote by $\omega^c$ the complement $\llbracket n \rrbracket \setminus \omega$ of $\omega$ in $\llbracket n \rrbracket$. For $\omega \in \Omega_k$, we label its elements in the way so that $\omega = \{\omega(1) < \omega(2) \cdots < \omega(k)\}$.

The symmetric group $\mathcal{S}_n$ acts naturally on $\Omega_k$ by permuting elements of $\omega \in \Omega_k$; hence $F\Omega_k$ is a $F\mathcal{S}_n$-module. In the language of symmetric group representations, this module is the same as the permutation module $M^{(n-k,k)}$ associated to the composition $(n-k,k)$ of $n$. For $x \in F\Omega_k$, the support of $x$, denoted by $\text{Supp}(x)$, is the set of $k$-subsets $\omega$ that has non-zero coefficient in $x$.

Recall from [Wil] that there is a bilinear operation on $\bigoplus_{k=0}^n F\Omega_k$ given by bilinearly extending

$$u \cdot v := \begin{cases} u \cup v, & \text{if } u \cap v = \emptyset; \\ 0, & \text{otherwise} \end{cases}$$

for $u \in \Omega_k$ and $v \in \Omega_l$. Note that $\omega \in \text{Supp}(x)$ is equivalent to $x \cdot \omega^c \neq 0$.

Recall from [Wil] the $a$-step boundary map $\varphi^{(a)}_k : F\Omega_k \to F\Omega_{k-a}$ with $a \in \{0, 1, \ldots, k\}$ and $k \in \{0, 1, \ldots, n\}$. It will be convenient to define $\varphi^{(a)}_k$ to be the zero map whenever $k \notin \llbracket 0, n \rrbracket$. We will omit the lower script so long as the context is clear. By $\varphi$ we simply mean $\varphi^{(1)}$.

For $a \in \llbracket p-1 \rrbracket$, $a!$ is invertible in $F$, so induction yields

$$(2.1.1) \quad a! \varphi^{(a)} = \varphi^{a} \quad \forall a \in \llbracket p-1 \rrbracket.$$ 

Note that $\varphi^{p} = 0$ but $\varphi^{(p)} \neq 0$.

The following fact will be useful throughout the article.

Lemma 2.1 (Splitting rule). [Wil, Lemma 3.5] For $v \in F\Omega_k$ and $w \in F\Omega_l$ such that $v \cap w = \emptyset$ for all $v \in \text{Supp}(v)$ and $w \in \text{Supp}(w)$, we have

$$(vw)\varphi^{(t)}_k = \sum_{i=0}^{t} ((v)\varphi^{(i)}_k) \cdot ((w)\varphi^{(t-i)}_l).$$

2.2. $p$-complexes and homologies. Within this subsection, $F$ is an arbitrary field, and $p$ a prime number. Suppose $A$ is an $F$-algebra. Let $C_\bullet$ denote the data $(C_k, \partial_k : C_k \to C_{k-1})_{k \in \mathbb{Z}}$, where $C_k$’s are $A$-modules and $\partial_k$’s are $A$-module maps. For a positive integer $r \in \mathbb{Z}_+$, we use the notation $\partial^r_k$ to denote the composition $\partial_k \circ \partial_{k-1} \circ \cdots \circ \partial_0$ - we also regard $\partial^r_k$ as the identity map on $C_\bullet$. We say that $C_\bullet$ is a $p$-complex if $\partial^r_k = 0$ for all $k \in \mathbb{Z}$ in which case, $\partial_k$’s are called the $p$-differentials of $C_\bullet$.

Note that we use chain convention in this article instead of the usual cochain setup in [KQ]; hence, the exchange of subscripts and superscripts in the indices.

A very natural generalisation of the homology theory of ordinary complexes in the setting of $p$-complexes is the following.
Definition 2.2. Let $C_\bullet := (C_k, \partial_k)_{k \in \mathbb{Z}}$ be a $p$-complex of $A$-modules. The $r$-th $p$-homology at degree $k \in \mathbb{Z}$ of $C_\bullet$, where $r \in \llbracket p-1 \rrbracket$, is defined as the $A$-module
\[ rH_k(C_\bullet) := \text{Ker}(\partial_k^r)/\text{Im}(\partial_{k+p-r}^r). \]

It turns out that, at least in the setting of this article, another (equivalent) homology theory of $p$-complexes is much easier to calculate than $p$-homologies.

Definition 2.3. Let $C_\bullet := (C_k, \partial_k)_{k \in \mathbb{Z}}$ be a $p$-complex of $A$-modules. The $a$-th slash homology group and backslash homology group at degree $k \in \mathbb{Z}$ of $C_\bullet$, where $a \in \llbracket 0, p-2 \rrbracket$, are defined as the $A$-modules
\[
\begin{align*}
H^a_k(C_\bullet) &:= \text{Ker}(\partial_k^{a+1})/(\text{Im}(\partial_k^{a-1}) + \text{Ker}(\partial_k^a)), \\
H_a^k(C_\bullet) &:= (\text{Im}(\partial_k^a) \cap \text{Ker}(\partial_k^{a-1})) / \text{Im}(\partial_k^{a-1+a}),
\end{align*}
\]
respectively. The (total) slash homology and backslash homology at degree $k \in \mathbb{Z}$ are the $(p-1)$-complexes
\[
\begin{align*}
\left( H^k(C_\bullet) := \bigoplus_{a=0}^{p-2} H^a_{k+a}, \mathcal{I} \right) \quad \text{and} \quad \left( H^k(C_\bullet) := \bigoplus_{a=0}^{p-2} H^a_{k+a}, \mathcal{I} \right)
\end{align*}
\]
respectively, where $\mathcal{I}$ are naturally induced from the restricted maps $\partial: \text{Ker}(\partial^p) \to \text{Ker}(\partial^{a+1})$ and $\partial: \text{Im}(\partial^p) \cap \text{Ker}(\partial^{p-1-a}) \to \text{Im}(\partial^{a+1}) \cap \text{Ker}(\partial^{p-2-a})$ respectively.

Note that the $(p-1)$-complex $H^k(C_\bullet)$ has terms lying in degree $k + p - 1, k + p - 2, \ldots, k$.

If we view the category of $p$-complexes of vector spaces as the category of graded modules over $\mathbb{F}[[\partial]]/(\partial^p)$, then the slash-a homology group $H^a_k(C_\bullet)$ picks up (“slashes through”$^{1}$) the $a$-th radical layer of any indecomposable non-projective direct summand of $C_\bullet$, occurring at degree $k$, whereas the backslash-a homology group picks up the $a$-th socle layer. In particular, the total slash homology of a complex corresponds to removing projective direct summands. In contrast, $\bigoplus_{a=0}^{p-2} H^a_k(C_\bullet)$ is almost always much larger than the total slash homology. For example, for $p > 2$, $H^k(C_\bullet)$ of $C_\bullet$ is almost concentrated in degrees $1, 0$ is just $C_\bullet$ itself, whereas $\bigoplus_{a=0}^{p-2} H^a_k(C_\bullet) = (C_\bullet)^{\partial_p}$.

Nevertheless, it is clear that $H^{0}_{k, 1} = 0$ and $H^{0}_{k, 0} = [p-1]H_k$. In fact, as remarked in [KQ], the three types of homologies are related by the exact sequence
\[
0 \to H^{p-r}_{k-1}(C_\bullet) \to r^{-1}H_k(C_\bullet) \xrightarrow{\partial} H^r_k(C_\bullet) \to 0,
\]
for all $r \in \llbracket p-1 \rrbracket$, where $\partial$ is induced by the natural inclusion $\iota : \text{Ker}(\partial^{p-r}) \to \text{Ker}(\partial^r)$.

The following general lemma will be useful later.

Lemma 2.4. Let $C_\bullet := (C_k, \partial_k)_{k \in \mathbb{Z}}$ be a $p$-complex of $A$-modules. Then the following holds for any $k \in \mathbb{Z}$.
\begin{enumerate}
\item (Slash shifting) $\partial^a$ induces an injective map $H^a_k(C_\bullet) \to H^0_{k-a}(C_\bullet)$ of $A$-modules for any $a \in \llbracket 0, p-2 \rrbracket$.
\item If $p^{-1}H_{k-1}(C_\bullet) = 0$ for all $i \in \llbracket b \rrbracket$ with $b < p$, then $p^{-1}H_{k-r}(C_\bullet) = 0$ for all $r \in \llbracket b \rrbracket$.
\end{enumerate}

Proof (1) Denote by $B^a_k$ the module $\text{Im}(\partial^a) + \text{Ker}(\partial^a)$ for all $i, k$. We need to show that, for $x \in \text{Ker}(\partial^{p+1})$, $((x)\partial^a) \in \text{Ker}(\partial^a)$ is in $B^0_{k-a}$, then $x \in B^a_k$. Indeed, the assumption on $x$ means that $(x)\partial^a \in \text{Ker}(\partial^a)$, so the condition of this being in $B^0_{k-a} = \text{Im}(\partial^a)$ means that there is some $y$ so that $(x)\partial^a = (y)\partial^a$. Hence, we have $x - (y)\partial^a \equiv 0 \mod \text{Ker}(\partial^a)$, which implies $x \equiv 0 \mod B^a_k$ as required.

(2) Note that the condition $b < p$ is needed just so that $p - r \geq 1$.

We prove the claim by induction on $r$. The case $r = 1$ is precisely the assumption $p^{-1}H_{k-1}(C_\bullet) = 0$.

For $r \in \llbracket 1, b \rrbracket$, consider $v \in \text{Ker}(\partial^{p-r}_{k-r}) \subset \text{Ker}(\partial^{p-r}_{k-r})$. The assumption of $p^{-1}H_{k-r}(C_\bullet) = 0$ means that we have $u \in C_{k-r-1}$ such that $v = (u)\partial_{k-r-1}$. In particular, as
\[
(u)\partial^{p-r-1}_{k-(r-1)} = (u)\partial_{k-(r-1)}\partial^{p-r}_{k-r} = (v)\partial^{p-r}_{k-r} = 0,
\]
we have $u \in \text{Ker}(\partial^{p-r-1}_{k-(r-1)})$.

By induction hypothesis, we have $p^{-1}H_{k-(r-1)}(C_\bullet) = 0$, meaning that there is some $w \in C_k$ such that $(w)\partial^{p-r}_{k-1} = u$. Thus, we have $(w)\partial^{p-1}_{k-r} = v$, i.e. $v \in \text{Im}(\partial^{p-1}_{k-r})$, as required. 

\footnote{This is not true reason why they are called slash, but we find this mnemonic useful.}
2.3. Duality. Denote by $(-)^*$ := $\text{Hom}_F(-, F)$ the $F$-linear dual, which is an exact contravariant functor on $\text{mod}A$ when $A$ is a symmetric algebra. Recall that a module $M$ is self-dual if $M^* \cong M$. Let us recall some elementary facts.

**Lemma 2.5.** The following holds for $A$-modules $L, M, N$.

1. If $f : M \to N$ is an $A$-module map with $M, N$ both being self-dual, then we have
   \[ \text{Coker}(f : M \to N)^* \cong \text{Ker}(f^* : N \to M). \]

2. If $L, N$ are submodules of $M$, then $(L + N)^* \cong M^*/(M/L)^* \cap (M/N)^*$.

Let us now consider the effect of applying $F$-linear duality $(-)^* := \text{Hom}_F(-, F)$ on the $p$-complex $F\Omega$. Note first that $F\Omega_k$ is a self-dual module for all $k \in \llbracket 0, n \rrbracket$. Moreover, there is a canonical isomorphism $F\Omega_k^* \cong F\Omega_{n-k}$ which sends the basis vector $\omega^*$ of $F\Omega_k^*$ dual to $\omega \in \Omega_k$ to the complement subset $\omega^c \in \Omega_{n-k}$, and it yields a commutative diagram

$$\begin{array}{c}
F\Omega_{k-a}^* \xrightarrow{(\varphi_k^{(a)})^*} F\Omega_k^* \\
\downarrow \cong \downarrow \\
F\Omega_{n-k+a}^* \xrightarrow{(\varphi_{n-k-a}^{(a)})^*} F\Omega_{n-k}^*
\end{array}$$

where the vertical isomorphisms are the canonical ones. In particular, we have an isomorphism $F\Omega_* \cong F\Omega_*^*$ of $p$-complexes, where $F\Omega_*^*$ is the $p$-complex whose degree $k$ term is given by $F\Omega_{n-k}^*$ and $p$-differentials given by $\varphi_k^*$.

Using Lemma 2.5, this means that we have

$$\langle H_k^a(F\Omega_*) \rangle^* \cong H_k^{\text{back}}(F\Omega_*) \cong H_{n-k}^{\text{slash}}(F\Omega_*)$$

for all $k \in \llbracket 0, n \rrbracket$ and $a \in \llbracket 0, p - 2 \rrbracket$. We call the existence of isomorphisms $H_k^a(F\Omega_*)^* \cong H_{n-k}^{\text{back}}(F\Omega_*)$ the back/slash duality\(^2\) of $F\Omega_*$. This duality says that, the slash homology of $F\Omega_*$ alone is enough to understand backslash homology, as well as $p$-homology using (2.2.1).

In particular, combining with (2.1.1), the main result (Theorem 4.16) of this article gives an indirect answer to the question raised by Wildon in [Wil] that asks for the structure of $\text{Ker}(\varphi_k)/\text{Im}(\varphi_{k+p-a}) \cong aH_k(F\Omega_*)$ for all $a \in \llbracket p - 1 \rrbracket$.

2.4. Review on a filtration of $F\Omega_k$. In [J1] (see also [J2, Chapter 17]), James shows that a permutation module $M^\mu$ corresponding to a composition $\mu$ admits a filtration $\{S^n_{\mu,-n}\}_{n=\mu}$ over a certain series of partitions $\mu^-$. In this subsection, we will review his construction and some related results in the special case of $\mu = (n-k, k)$ with $k \in \llbracket 0, n \rrbracket$.

Let $t$ be a (Young) tableau. Denote by $t_{i,j}$ the content of the box located at (row, column) = $(i, j)$, with row counting from top to bottom and column counting from left to right. From now on, we will only work with tableaux that have at most two rows.

**Definition 2.6.** Suppose $\ell, k \in \llbracket 0, n \rrbracket$ are non-negative integers with $\ell \leq k$ and $\ell \leq n - k$. For a $(n - k, k)$-tableau $t$ is row-standard (resp. column-standard) if the entries in each row (resp. column) are arranged in increasing order. It is called $\ell$-standard if it is row-standard and $t_{1,j} < t_{2,j}$ for all $j \in [\ell]$ (in other words, having the first $\ell$ columns being standard). A $k$-standard $(n - k, k)$-tableau is just a standard tableau in the usual sense. Denote by $\text{SYT}_n(k; \ell)$ the set of $\ell$-standard Young tableaux of shape $(n - k, k)$.

Retaining the notation on $t, \ell, k$. Denote by $G_{t,\ell} \leq S_n$ the stabiliser of the first $\ell$ columns of $t$, i.e. the direct product of order 2 subgroups $\langle t_{1,j}, t_{2,j} \rangle$ over all $j \in [\ell]$. Note that $G_{t,\ell}$ is just the usual column stabiliser of $t$.

Recall the following relation between row-equivalent classes (a.k.a. tabloid) of $(n - k, k)$-tableaux and $k$-subsets: For a tableau $t$, its associated $k$-subset, denoted by

$$\{t\} := \{t_{2,1}, t_{2,2}, \ldots, t_{2,k}\} \in \Omega_k,$$

consist of all the elements in its second row. Conversely, given a $k$-subset $\omega$, then we have a (row-standard) tableau $t^\omega$ whose first row consists of elements of $\omega^c$ and second row consists of elements

\(^2\)“back/slash” is just an abbreviation of “slash-and-backslash”.

of \( \omega \) in increasing order. Clearly, we have \( \omega = \{t^\omega\} \), and so we call \( t^\omega \) is the tableau associated to a \( k \)-subset.

**Example 2.7.** Take \( n = 8 \), \( k = 3 \), and \( \omega = \{2, 3, 7\} \), then the (row-standard) tableau \( t^\omega \) associated to \( \omega \) is

\[
\begin{array}{cccc}
1 & 4 & 5 & 6 \\
2 & 3 & 7 & 8
\end{array}
\]

**Definition 2.8.** For a subgroup \( G \subseteq S_n \), the signed sum of \( G \) is an element in the group algebra given by the sum of \( \text{sgn}(\sigma)\sigma \) over all \( \sigma \in G \).

Denote by \( k_{\sigma,t} \) the signed sum of \( C_{k,\ell} \). Define \( \ell \)-polytabloid associated to \( t \) to be the element in \( F \Omega_k \) given by \( \{t\}k_{\ell,t} \). We say that \( e_{\sigma,t} \) is \( \ell \)-standard if so is \( t \).

Note that a \( \ell \)-polytabloid is dependent on the defining tableau, i.e. row-equivalent but distinct tableaux can define distinct \( \ell \)-polytabloids.

**Convention.** Whenever we use any of the notations or terminologies above, if \( \ell \) is omitted, then it is assumed that \( \ell = k \). We also replace \( t \) by \( \omega \in \Omega_k \) in the notation whenever we take the canonical tableau \( t^\omega \) associated \( \omega \).

**Example 2.9.** The \( \ell \)-polytabloid associated to \( \omega := \llbracket k \rrbracket \) can be explicitly written as

\[
e_{\llbracket k \rrbracket,t} = [k]k_{\sigma,t} = [k] \sum_{\sigma \in C_{k,\ell}} \text{sgn}(\sigma)\sigma, \text{ where } C_{k,\ell} = \prod_{i=1}^{\ell}((i,k+i)).
\]

For an even more explicit case, say \( n = 8 \), \( k = 3 \), \( \ell = 2 \), then we have

\[
e_{\llbracket 3 \rrbracket,t} = \{1,2,3\} - \{4,2,3\} - \{1,5,3\} + \{4,5,3\}.
\]

We define an \( F S_n \)-submodule of \( F \Omega_k \) by

\[
S^{k,\ell} := F\text{-span}\{e_{t,\ell} \mid t \text{ a } (n-k,k)\text{-tableau}\}.
\]

Note that, if \( t' = t\sigma \) for some \( \sigma \in S_n \), then one can easily see that \( e_{t',\ell} = e_{t,\ell}\sigma = e_{t,\ell}\sigma \). In particular, \( S^{k,\ell} \) is generated (over \( F S_n \)) by any single \( \ell \)-polytabloid. \( S^{k,\ell} \) is the module \( S^{\mu^\sigma,\mu} \) used in [J1] mentioned before, where \( \mu^\sigma = (n-k,\ell) \).

**Theorem 2.10.** ([J1, Theorem 9.1, Corollary 9.2] Suppose \( k, \ell \) are non-negative integers with \( \ell \leq k \leq n \) and \( \ell \leq n - k \). Then the following statement holds

1. \( \text{Ker}(\varphi_{k,\ell}^{(k,\ell)}|_{S^{k,\ell+1}}) = S^{k,\ell+1} \), where \( S^{k,\ell+1} \) is treated as the zero module if \( \ell = \min\{k,n-k\} \).
2. \( \text{Im}(\varphi_{k,\ell}^{(k,\ell)}|_{S^{k,\ell}}) = S^{k,\ell} \).
3. There is a filtration

\[
\Omega_k = S^{k,0} \supset S^{k,1} \supset \cdots \supset S^{k,\min\{k,n-k\}} \supset S^{k,\min\{k,n-k\}+1} = 0.
\]

**Remark 1.** If \( k - \ell < p \), then Theorem 2.10 (1) and (2) hold with \( \varphi_{k,\ell}^{(k,\ell)} \) in place of \( \varphi_{k,\ell} \).

An intermediate step to show Theorem 2.10 (1) and (2) is to find the defining spanning set of \( S^{k,\ell} \) to a basis. This is achieved by putting a total order on the set of tableoids (\( k \)-subsets in our setting) and generalises the strategy of [Pe], who showed that Specht modules admit a basis given by standard polytabloids.

**Definition 2.11.** Denote by \( \omega \preceq_r \omega' \) if \( |\omega \cap [i]| \leq |\omega' \cap [i]| \) for all \( i \in \llbracket k \rrbracket \); this defines a partial order on \( \Omega_k \) called the row-dominance order.

**Remark 2.** The definition in [J1] is stated differently - they use the backward-reading lexicographical order on \( k \)-subsets, i.e. \( \omega <_{lex} \omega' \in \Omega_k \) if there is some \( j \in \llbracket k \rrbracket \) so that \( |\omega \cap [j]| < |\omega' \cap [j]| \) and \( |\omega \cap [i]| = |\omega' \cap [i]| \) for all \( i \in [j+1,k] \). Our formulation is the opposite of the partial order used in [Sag], which is coarser than the one used in [J1]. With respect to the next result, it does not matter which order one prefer to use.

**Theorem 2.12.** ([J1, 7.4, 9.4] Suppose we have \( 2k \in [n] \).

1. If \( t \) is \( \ell \)-standard, then \( \{t\} \) is minimal in \( \text{Supp}(e_{t,\ell}) \) with respect to the row-dominance order.
2. The set

\[
\{e_{t,\ell} \mid t \in \text{SYT}_n(k;\ell)\}
\]

of \( \ell \)-standard \( \ell \)-polytabloids is a basis of \( S^{k,\ell} \).
Part (2) of the theorem in the case when \( \ell = k \) is shown by Peel in \([Pe]\).

We need a refinement of the duality between \( \mathbb{F} \Omega_k \) and \( \mathbb{F} \Omega_{n-k} \) mentioned in the previous section.

**Lemma 2.13.** Suppose \( k, \ell \) are non-negative integers with \( \ell \leq k \leq n-k \). The duality \( \mathbb{F} \Omega_k \cong \mathbb{F} \Omega_{n-k} \) restricts to an isomorphism \( (S^k, \ell)^* \cong S^{n-k, \ell} \) of \( \mathbb{F} \mathbb{S}_n \)-modules.

**Proof** Recall that the action of \( g \in \mathbb{S}_n \) on the dual of a module is given by the action of \( g^{-1} \) on the original module. Since every \( \sigma \in C_{[k], \ell} \) is a commuting product of transpositions, its action on \( \mathbb{F} \Omega_k \) is the same as the original, i.e. \( e_{[k], \ell}^* = ([k]_{[k]}, \ell) = \left[ (k)!^{n/[k]} \right]_{[k]} \).

Recall that the duality isomorphism \( \mathbb{F} \Omega_k \cong \mathbb{F} \Omega_{n-k} \) is given by \( \omega^* \mapsto \omega^c \) for any \( \omega^* \) which is the dual basis of \( \omega \in \Omega_k \). Hence, \( e_{[k], \ell}^* \) is mapped to \( [k+1, n]_{[k], \ell} \) under this map. Observe that \( C_{[k], \ell} = C_{[n], \ell} \) where \( t \) is the \( (k, n-k) \) tableau given by swapping the two rows of \( t^{[k]} \), which is the same as \( t^{[k+1, n]} \). Thus, \( e_{[k], \ell}^* \) is mapped to \( e_{[k+1, n], \ell}^* \).

Since \( S^{k, \ell} \) is generated by any single \( \ell \)-polytabloid \( e_{\omega, \ell} \) as an \( \mathbb{F} \mathbb{S}_n \)-module (and similarly for \( S^{n-k, \ell} \)), the argument in the previous paragraph implies the required isomorphism. \( \square \)

## 3. Partitioning \( k \)-subsets

Recall that \( n \) is defined to be \( \lfloor n/2 \rfloor + 1 \). Sometimes for the ease of referencing, we say that \( k \in [0, n] \) is low if \( k < n \) (equivalently, \( 2k \leq n \)), respectively high if \( k \geq n \) (equivalently, \( 2k > n \)).

In subsection 3.1, we introduce the key tools - called *density* and *threshold* - of our investigation, and look at some of their elementary properties. Then in subsection 3.2, we present some ideas how one can modify the proof of exactness of \( \mathbb{F} \Omega \), in the case when \( p = 2 \) to higher prime \( p \).

### 3.1. Density and threshold

Roughly speaking, we need a book-keeping device to see how far \( t^\omega \) is away from being \((\ell)-\)standard.

**Definition 3.1.** For \( \omega \in [n] \), the density of \( \omega \) is an \( m \)-tuple \( d^\omega = (d^\omega_1, \ldots, d^\omega_m) \) such that \( d^\omega_i = \lfloor \omega \cap [2i-1] \rfloor \) for \( 1 \leq i \leq m \). The threshold of \( \omega \) is \( \max \{ 0, i, \in [m] \mid d^\omega_i = i \} \). We denote by \( \Omega^k_k \) the subset of \( \Omega_k \) consisting of \( k \)-subsets with threshold \( h \). It is natural to denote by \( \Omega^{\geq h}_k \) and \( \Omega^{< h}_k \) the union of \( \Omega^h_k \) over all \( i \in [h, m] \) and \( i \in [0, h] \) respectively; similarly for \( \Omega^{\geq h}_k \) and \( \Omega^{< h}_k \).

**Example 3.2.** (1) Consider the \( k \)-subset \([k]\) with \( k > 0 \). Then we have \( d_{[k]}^{[k]} = \min \{ k, 2i-1 \} \), for \( 1 \leq i \leq m \), i.e. \( d^{[k]} = (1, 3, \ldots, 2[k/2] -1, k, \ldots, k) \). Hence, the threshold of \([k]\) is \( \left\{ \begin{array}{ll} k, & \text{if } k \leq m; \\
1, & \text{if } k > m. \end{array} \right. \)

(2) Assume \( k < m \) and consider the \( k \)-subsets \( \eta = \{2, 4, \ldots, 2k \} \) and \( \omega = \{1, 3, \ldots, 2k-1 \} \). Then we have \( d^{\eta} = (0, 1, \ldots, k-1, k, \ldots, k) \) and \( d^{\omega} = (1, 2, \ldots, k, k, \ldots, k) \).

Thus, we have \( \eta \in \Omega^0_k \) and \( \omega \in \Omega^k_k \).

The following facts about density will be useful.

**Proposition 3.3.** The following holds for any \( k \)-subset \( \omega \in \Omega_k \).

1. \( \text{The density } d^\omega \text{ satisfies } 0 \leq d^\omega_1 \leq \min \{1, k\} \) and
   \[ 0 \leq d^\omega_1 \leq d^\omega_2 \leq d^\omega_{i-1} + 2 \leq \min \{ 2i - 1, k \}, \]
   for all \( 2 \leq i \leq m \).

2. \( \text{If there are } i < i' \text{ so that } d^\omega_i > i \text{ and } d^\omega_{i'} < i' \text{ then there exists } j < i < i' \text{ such that } d^\omega_j = j. \)

2'. \( \text{If there are } i < i' \text{ so that } d^\omega_i < i \text{ and } d^\omega_{i'} > i' \text{ then there exists } j < i < i' \text{ such that } d^\omega_j = j. \)

**Proof** (1): This is straightforward from the definition.

(2), (2'): We show only (2); the proof of (2') is analogous and left to the keen reader.

It follows from (1) that \( d^\omega_{i+1} \geq d^\omega_i \), which is by our assumption at least \( i + 1 \). Assume \( d^\omega_{i+1} > i + 1 \) (otherwise, we are done). Now we just replace \( i \) in the statement of by \( i+1 \). Note that it is impossible to have \( d^\omega_j > j \) for all \( i < j < i' \); otherwise, we will get \( d^\omega_{i'} \geq d^\omega_{i'-1} \geq i' - 2 \) - a contradiction to our assumption. \( \square \)
This proposition says that $d^\omega$, regarded as a function $[m] \to \mathbb{N}_0$, is increasing and satisfies a discrete version of “rotated” intermediate value theorem.

The following tells us that entries of $d^\omega$ beyond the threshold “stays either above or below the diagonal”.

**Lemma 3.4.** Let $\omega \in \Omega^h_k$ be a $k$-subset whose threshold is $h \in [0, m - 1]$.

(Lo) If $2k \leq n$, then $d^\omega_i < i$ for all $h < i \leq m$.

(Hi) If $2k > n$, then $d^\omega_i > i$ for all $h < i \leq m$.

**Proof** Suppose on the contrary that $d^\omega_i \geq i$ (resp. $d^\omega_i \leq i$) for some $h < i \leq m$ in the case of $2k \leq n$ (resp. $2k > n$). Since the threshold of $\omega$ is $h$ and $h < i$, we must have $d^\omega_i > i$ (resp. $d^\omega_i < i$). Note $\omega \cap [2m - 1]$ is always the same as $\omega \cap [n]$, so $d^\omega_m = k$ always.

If $2k \leq n$, then we have $k < m$, which means that $d^\omega_m = k < m$.

If $2k > n$, we have $d^\omega_m = k \geq m$. Since the threshold is $h$ and $h < m$, we must have $d^\omega_m > m$.

In both cases, the assumption on $i$ implies that we have $i < m$. So we can apply Proposition 3.3 (2) (resp. (2')) and get that $d^\omega_i = j$ for some $i < j < m$. But this contradicts the assumption that the threshold of $\omega$ is $h$ and $h < i$.

Let us look at what values of threshold are never attainable.

**Lemma 3.5.** (Lo) If $2k \leq n$, then $\Omega^h_k = \emptyset$ for all $h \in [k + 1, m]$.

(Hi) If $2k > n$, then $\Omega^h_k = \emptyset$.

**Proof** Let $\omega \in \Omega_h_k$ be a $k$-subset.

(Lo): By Proposition 3.3 (1), we have $d^\omega_i \leq k$ for any $i$, so there is no $\omega \in \Omega_h_k$ with threshold larger than $k$, as required.

(Hi): We need to show that there exists $i \in [n - k + 1]$ so that $d^\omega_i = i$. Recall from Proposition 3.3 (1) that $d^\omega_i \in (0, 1]$. Clearly, if $d^\omega_i = 1$ or $d^\omega_{n - k + 1} = n - k + 1$ holds, then we are done. If $d^\omega_i < 1$ and $d^\omega_{n + k - 1} > n + k - 1$, then it follows from Proposition 3.3 (2') that there is some $i \in [2, n + k - 2]$ so that $d^\omega_i = i$.

The following proposition explain why density and threshold somewhat measures how far $t^\omega$ is from ($\ell$-)standard tableaux.

**Proposition 3.6.** Consider a $k$-subset $\omega \in \Omega^h_k$ with $2k \leq n$. Then the following holds.

1. $\omega \in \Omega^h_k$ if, and only if, $t^\omega$ is standard.
2. If $\omega \in \Omega^h_k$ with $h > 0$, then there is some $\sigma$ that stabilises $\omega$ and $t^\omega \sigma$ is $(k - h)$-standard.

**Proof**

1. Suppose $t^\omega$ is standard. Then we have $\omega^c_i \geq i$ and $\omega^c(i) \geq 2i$ for all $i \in [k]$. This means that $d^\omega_i = |\omega^c(i) \cap [2i - 1]| \geq i > |\omega(i) \cap [2i - 1]| = d^\omega_i$ for all $1 \leq i \leq k$. Since $d^\omega_i = (2i - 1) - d^\omega_i$, we have $2i - 1 > 2d^\omega_i$ for all $1 \leq i \leq k$, which is equivalent to $i > d^\omega_i$ for all $1 \leq i \leq k$. As $k < m$, for all $k < i \leq m$, we have $d^\omega_i = d^\omega_k < k < i$, and it follows that $\omega \in \Omega^h_k$.

Conversely, if $\omega \in \Omega^h_k$, then by Lemma 3.4 (Lo) that $i > d^\omega_i$ for all $i \in [m]$. In particular, we have $2i - 1 > 2d^\omega_i$ for all $1 \leq i \leq k$, which implies that $d^\omega_i = 2i - 1 - d^\omega_i > d^\omega_i$.

It remains to show that this implies $t^\omega$ is standard. Indeed, we can prove by induction on $k$. For $k = 1$, it is clear. Otherwise, take $k$ minimal so that $\omega^c_k > \omega(k)$. By induction hypothesis, the sub-tableau where the second row takes only $k - 1$ entries is standard. This means that $\omega(k - 1) \geq 2(k - 1)$, and so we have $\omega(k) > 2k - 2$, and $\omega^c(k) > 2k - 1$. Hence, $d^\omega_i = |\omega^c \cap [2k - 1]| < k$ and $d^\omega_k \leq k$, which contradicts the assumption $d^\omega_i < d^\omega_k$.

2. For $\omega \in \Omega^h_k$, take $\sigma \in S_n$ that sends $\omega(h+1)$ to $\omega(i)$ for $i \in [k - h]$, and sends $\omega(i)$ to $\omega(k-h+i)$ for $i \in [h]$. The assumption on threshold says that $\omega(h) \leq 2h - 1$ and $\omega(h+1) > 2h + 1$, which then implies that $\omega^c(h+1) \leq 2h + 1 < \omega(h+1)$. Therefore, $(t\sigma)_{2,i} = \omega(h+i) > \omega^c(i) = (t\sigma)_{1,i}$ for all $i \in [k - h]$.

We warn that the sense of “far” here is in general not related the partial orders $<_{r}, <_{lex}$ reviewed in subsection 2.4. Indeed, it is possible that $\omega \in \Omega^h_k$ with $h > 0$ but $t^\omega$ is row-equivalent to a $\ell$-standard tableau with $\ell < k - h$, as one can see from the following example.

**Example 3.7.** Take $n = 5$, $k = 2$, and $\omega = \{1, 3\}$. Then $d^\omega = (1, 2, 2)$, meaning that $\omega \in \Omega^h_k$. However, the tableaux $t^\omega(1, 3)$ is 1-standard.
The reason for this “imperfection” is that threshold is defined to be the last index where a non-increasing column appear in $t^w$. Threshold are designed solely because it allows us to carve an induction mechanism for calculating $\text{Ker}(\varphi)$. There is a refinement of threshold that can be used to give equivalent criteria for $\ell$-standardness; it is given by the $\alpha$-sequences used by James in [J1].

**Proposition 3.8.** $\mathbb{F}\Omega_k = \mathbb{F}\Omega_k^{>0} \oplus S^{k,k}$ as $\mathbb{F}$-vector spaces.

**Proof** Let $M$ be the matrix with values in $\{-1,0,1\}$ whose rows are indexed by $\Omega_k^0$ and columns are indexed by $\Omega_k^0, \Omega_k^{k-1}, \ldots, \Omega_k^0$ from left to right, such that $M[\omega] = [\epsilon_v]$ where $[\epsilon_v]$ denotes the vector corresponding to $v \in \mathbb{F}\Omega_k$ with respect to the basis $\Omega_k$. Recall that $\Omega_k$ is partially ordered by $\llbracket$ (Definition 2.11). It follows from Theorem 2.10 (1) and Proposition 3.6 (1) that we can write $M = [M_{>0}|M_0]$ as block matrix with $M_0$ being the identity $|\Omega_k^0| \times |\Omega_k^0|$-matrix. In particular, $M$ is a lower unitriangular matrix.

Consider now the $(\binom{n}{k}) \times (\binom{n}{k})$-matrix given by stacking the identity matrix on top of $M$. This matrix defines a change of basis matrix from $\mathbb{F}\Omega_k^{>0} + S^{k,k}$ to $\mathbb{F}\Omega_k$. Note that $\mathbb{F}\Omega_k^{>0} \cap S^{k,k} = 0$ since $\text{Supp}(v) \cap \Omega_k^0 \neq \emptyset$ for all $v \in S^{k,k}$ by construction. \qed

**3.2. Homogeneity.** In the classical case, i.e. when $p = 2$, the fact that $\mathbb{F}\Omega_\ast$ is exact can be explained with a one-line proof, namely,

$$(\omega)\delta \varphi + (\omega)\varphi \delta = \omega;$$

for all $\omega \in \Omega_k$, where $\delta: \mathbb{F}\Omega_k \to \mathbb{F}\Omega_{k+1}$ is a $\mathbb{F}$-linear (but not $\mathbb{F}\mathfrak{S}_n$-linear) map given by $\omega \mapsto \omega \cup \{1\}$ for $\omega \cap \{1\} = \emptyset$ and $\omega \mapsto 0$ otherwise. The displayed formula here no longer holds for $p > 2$, but it turns out that the key to understanding $\text{Ker}(\varphi)/\text{Im}(\varphi^{p-1})$ lies in finding a generalisation of this formula - this is the goal of this subsection, and is achieved in Lemma 3.15 and Lemma 3.16. Let us remark for the moment that the key ingredients in finding the correct generalised formula are (1) Consider $\omega$ whose threshold is positive, and (2) use a “partial version” of $\delta$ (see Definition 3.11).

Since we only consider the case of non-zero threshold in this subsection, and the threshold of the empty subset $\emptyset \in \Omega_0$ is 0, throughout this subsection, we will impose the following

**Assumption:** $k > 0$,

unless otherwise specified. Note that we will no longer assume $k$ is low.

Let us start with the definition of the partial version of $\varphi$.

**Definition 3.9.** Consider a set $I$. For any $k \in [0,n]$, we define an $\mathbb{F}$-linear map

$$\varphi_I = \varphi_{k,I} : \mathbb{F}\Omega_k \to \mathbb{F}\Omega_{k-1}$$

$$\omega \mapsto \sum_{i \in I \cap \omega} \omega \setminus \{i\},$$

for all $\omega \in \Omega_k$.

Note that if $I \supset [n]$, then $\varphi_I = \varphi_{[n]}$ and $\varphi_I\ast = 0$. Note also that $\varphi_I$ is not necessarily a morphism of $\mathbb{F}\mathfrak{S}_n$-modules.

Let us list some elementary properties about $\varphi_I$.

**Proposition 3.10.** Let $I$ be a subset of $[n]$, and $\omega$ be a $k$-subset of $[n]$ for some positive $k \in [n]$. 
(1) $\varphi = \varphi_I + \varphi_{I\ast}$ and $\text{Supp}((y)\varphi_I) \cap \text{Supp}((y)\varphi_{I\ast}) = \emptyset$.
(2) If $\omega$ and $I$ are disjoint, then $(\omega)\varphi_I = 0$.
(3) If the threshold of the $k$-subset $\omega$ is $k$, then $(\omega)\varphi_{[2k-1]} = (\omega)\varphi$.

**Proof** (1), (2) This is straightforward by definition.
(3) The assumption $\omega \in \Omega_k^0$ implies that $\omega \cap [2k-1] = \omega$, so it follows from (2) that $(\omega)\varphi_{[2k-1]} = 0$. The claim now follows from (1). \qed

**Definition 3.11.** Consider a set $I$. For any $k \in [0,n]$, define an $\mathbb{F}$-linear map

$$\delta_I = \delta_{k,I} : \mathbb{F}\Omega_k \to \mathbb{F}\Omega_{k+1}$$

$$\omega \mapsto \sum_{i \in I \cap [n]} \omega \setminus \{i\}$$

for all $\omega \in \Omega_k$. 
Similar to $\varphi_I$, $\delta_I$ is not necessarily a morphism between $\mathbb{F}\mathcal{S}_n$-modules; also, if $I \supset [n]$, then $\delta_I = \delta_{[n]}$ and $\delta_I = \delta_{\emptyset} = 0$.

The map $\delta_{[n]}$ is the same as $\delta$ in the notation of [Wil, Section 3].

**Definition 3.12.** Let $I$ be a subset of $[n]$ and $h$ be a non-negative integer. A set $\mathcal{S}$ of subsets of $[n]$ is $h$-homogeneous on $I$ if $|S \cap I| = h$ for all $S \in \mathcal{S}$.

**Example 3.13.** (1) Any $\mathcal{S} \subset \Omega_k$ is $k$-homogeneous on $[n]$.

(2) Any non-empty subset of $\Omega_k^0$ is $h$-homogeneous on $[2h - 1]$ when $\Omega_k^h \neq \emptyset$.

While our definition of $\varphi_I, \delta_I$, and homogeneity-on-$I$ are made on arbitrary subset $I$ of $[n]$; these are almost always applied in the setting the of second example above, i.e. $I = [2h - 1]$.

The following lemma roughly says that homogeneity-on-$I$ is preserved by $\delta_I$.

**Lemma 3.14.** Suppose $y \in \mathbb{F}\Omega_k$ has $\text{Supp}(y)$ being $h$-homogeneous on some $I \subset [n]$ with $h > 0$. Consider an integer $a \in [p - 1]$. If $h + a - 1 < |I|$, then $\text{Supp}((y)^{\varphi_I})$ is $(h + a)$-homogeneous on $I$; empty otherwise.

**Proof** It suffices to show for the case $a = 1$; otherwise, replace $y$ by $(y)^{\delta_{I}^{a-1}}$.

If $|I| = h$, then $(y)^{\delta_{I}} = 0$, so $\text{Supp}((y)^{\delta_{I}}) = \emptyset$. Suppose now that $|I| > h$. Take $\omega \in \text{Supp}(y)$, by assumption we have $|\omega \cap I| = h$. So for any $\omega' \in \text{Supp}((\omega)^{\delta_{I}})$, we have $\omega' \cap I = (\omega \cup \{i\}) \cap I$ for some $i \in I \setminus (\omega \cap I)$, and so $|\omega' \cap I| = h + 1$. Hence, $\text{Supp}((\omega)^{\delta_{I}})$ is $(h + 1)$-homogeneous on $I$.

**Lemma 3.15.** Let $y \in \mathbb{F}\Omega_k$ be such that $\text{Supp}(y)$ is $h$-homogeneous, for some $h \geq 0$, on a set $I \subset [n]$ with $|I| = m \geq 0$. Then $(y)^{\delta_{I}} \varphi = (y)^{\varphi \delta_I} - (2h - m)y$.

**Proof** We first note the equation holds if $m < h$, since $y = 0$ by condition. Without loss of generality, we can assume $y = \omega \in \Omega_k$ such that $|\omega \cap I| = h$. Let $I' := I \setminus (\omega \cap I)$. Then we have

$$(\omega)^{\varphi \delta_I} \varphi = \left( \sum_{i \in I'} \omega \cup \{i\} \right) \varphi = \sum_{i \in I'} \left( \sum_{j \in \omega \cup \{i\}} (\omega \cup \{i\}) \setminus \{j\} \right) = |I'| \omega + \sum_{j \in \omega \cup I'} (\omega \cup \{i\}) \setminus \{j\} = (m - h)\omega + \sum_{j \in \omega \cup I'} (\omega \cup \{i\}) \setminus \{j\}.$$ (3.2.1)

On the other hand, we have

$$(\omega)^{\varphi \delta_I} \varphi = \left( \sum_{j \in \omega \setminus I'} \omega \setminus \{j\} \right) \varphi = \sum_{i \in I \setminus (\omega \setminus I')} \sum_{j \in \omega \setminus \{j\} \cup \{i\}} (\omega \setminus \{j\} \cup \{i\}.$$ (3.2.2)

Since $|\omega \cap I| = h$, precisely $h$ terms of the sum on the right is $\omega$, whereas the element $i$ in the other terms is in $I \setminus (\omega \cap I) = I'$, so we have

$$(\omega)^{\varphi \delta_I} \varphi = h\omega + \sum_{i \in I'} \sum_{j \in \omega \setminus I'} (\omega \setminus \{j\} \cup \{i\}.$$ (3.2.2)

Finally, as $I'$ and $\omega$ are disjoint by construction, we have

$$\sum_{i \in I'} \sum_{j \in \omega \setminus I'} (\omega \setminus \{j\} \cup \{i\}) = \sum_{j \in \omega \setminus I'} \sum_{i \in I'} (\omega \setminus \{j\} \cup \{i\}) = \sum_{j \in \omega \setminus I'} (\omega \setminus \{j\} \cup \{i\}).$$

so (3.2.2) minus (3.2.1) gives $y(\varphi \delta_I - \delta_I \varphi) = (2h - m)y$, as required. □

**Lemma 3.16.** Consider $y \in \mathbb{F}\Omega_k^h$ for some $h > 0$. Then there are $b_1, b_2, \ldots, b_{p-1} \in \mathbb{F}$ so that

$$(y)^{\varphi_{[2h - 1]}} \varphi_{p-1} = y + \sum_{i=1}^{p-1} (b_i y)^{\varphi_{[2h - 1]}} \varphi_{i-1}.$$ (3.2.3)

**Proof** Define $y(\ell) := (y)^{\varphi_{[2h - 1]}}$ for all $0 \leq \ell < p$. The assumption on the threshold of $y$ says that it is $h$-homogeneous on $[2h - 1]$. Hence, it follows from Lemma 3.14 that $y(\ell)$ is $(h + \ell)$-homogeneous on
\[ [2h - 1] \] as long as they are within bounds (i.e. \( 2h - 1 > h + \ell - 1 \) and \( 2h - 1 < n \)); otherwise \( y(\ell) = 0 \).

In any case, by Lemma 3.15, we have
\[
(y(\ell))\delta_{[2h-1]}^p = (y(\ell))\varphi \delta_{[2h-1]}^p = (2h + \ell) - (2h - 1)y(\ell)
\]
(3.2.4)

We claim that the following equation holds for all \( a \in \{p - 1\} \):
\[
(y)\delta_{[2h-1]}^p = (y)\varphi \delta_{[2h-1]}^p - a^2(y)\delta_{[2h-1]}^{p-1}.
\]
Indeed, we have the following calculation by repeatedly applying (3.2.4):
\[
(y)\delta_{[2h-1]}^p = (y(a-1))\delta_{[2h-1]}^p
= (y(a-1))\varphi \delta_{[2h-1]}^p - (2a - 1)y(a-1)
= ((y(a-2))\varphi \delta_{[2h-1]}^p - (2a - 3)y(a-2))\delta_{[2h-1]}^p - (2a - 1)y(a-1)
= (y(a-3))\varphi \delta_{[2h-1]}^p - ((2a - 3) + (2a - 1))y(a-1)
= \cdots = (y)\varphi \delta_{[2h-1]}^p - (1 + 3 + \cdots + (2a - 1))y(a-1)
= (y)\varphi \delta_{[2h-1]}^p - a^2y(a-1)
= (y)\varphi \delta_{[2h-1]}^p - (a^2y)\delta_{[2h-1]}^{p-1},
\]
as required. Note that the argument still holds even if \( 2h - 1 \leq h + a - 1 \) or \( 2h - 1 > n \) since (3.2.4) holds in these conditions.

Set \( y[0] := y \) and \( y[\ell] := -(p - \ell)^2y[\ell-1] \) for all \( \ell \geq 1 \). Starting with \( (y)\delta_{[2h-1]}^{p-1} \), we repeatedly apply (3.2.5) to obtain the following.
\[
(y)\delta_{[2h-1]}^{p-1} = (y)\varphi \delta_{[2h-1]}^{p-1} - y[0]
= (y)\varphi \delta_{[2h-1]}^{p-1} - (y[1])\delta_{[2h-1]}^{p-2}
= (y)\varphi \delta_{[2h-1]}^{p-1} - (y[1])\varphi \delta_{[2h-1]}^{p-2} - (y[2])\delta_{[2h-1]}^{p-3}
= \cdots = (y)\varphi \delta_{[2h-1]}^{p-1} - \left( \sum_{\ell=1}^{p-2} (y[\ell])\varphi \delta_{[2h-1]}^{p-1-\ell} \right) + y[p-1].
\]

By Wilson’s theorem, \((-1)^{p-1} = 1 \) and \((p - 1)! = -1 \) in any field of characteristic \( p > 0 \), so we have \( y[p-1] = (-1)^{p-1}(p - 1)^2y = y \). Take \( b_1 := 1 \) and \( b_i \) such that \( y[p-i] = b_iy \) (explicitly, \( b_i = (-1)^{p-i}((p - 1)!(i - 1)!)^2 \) for \( i \in [2, p - 1] \), then the last line in the above calculation is equal to the right-hand side of (3.2.3), as required.

4. Determining the slash homologies

Our goal of this section is to prove the first main result of the article, namely, to calculate the slash homologies of \( \mathbb{F} \Omega_x \). To achieve this, it turns out that knowing a few vanishing conditions for various slash homologies or \( p \)-homologies will be immensely helpful. The key result that allows us to obtain these vanishing conditions is Theorem 4.7, whose proof relies on the notion of threshold for \( k \)-subsets.

Subsection 4.1 will be devoted to proving Theorem 4.7. We will determine the dimension of all slash homology groups in subsection 4.2, and then finally prove the first main theorem (Theorem 4.16) of the article in subsection 4.3.

4.1. Reduction mechanism. Let us start by introducing a notation for ease of exposition.

Notation 4.1. For \( x \in \mathbb{F} \Omega_k \), we define \( x^{(h)} \in \mathbb{F} \Omega_k \) with \( h \in [m] \) to be the projection of \( x \) onto \( \mathbb{F} \Omega_k^h \).

Lemma 4.2. Suppose we have \( y \in \text{Ker}(\varphi_k) \cap \mathbb{F} \Omega_k^{>0} \). If either
\[(L_0) 2k \leq n \text{ and Supp}(y) \subset \Omega_k^{>h}; \]
\[(H_1) 2k > n \text{ and Supp}(y) \subset \Omega_k^{<h}, \]
holds, then the projection \( y^{(h)} \) of \( y \) onto \( \mathbb{F} \Omega_k^h \) satisfies \( (y^{(h)})\varphi_{[2h-1]} = 0 \).
Proof. We have

\[ (4.1.1) \quad 0 = (y) \varphi = (y(h)) \varphi + (y - y(h)) \varphi = (y(h)) \varphi_{[2h-1]} + (y(h)) \varphi_{[2h-1]} + (y - y(h)) \varphi, \]

where the last equality follows from Proposition 3.10 (1).

We claim that

\[ \text{Supp} \left( (y(h)) \varphi_{[2h-1]} \right) \cap \text{Supp} \left( (y - y(h)) \varphi \right) = \emptyset. \]

Indeed, for \( \omega \in \text{Supp}(y(h)) \), it follows from Lemma 3.4 (Lo) that \( d_\omega^h = |\omega \cap [2i-1]| < i \) for all \( h < i \leq m \). This means that the necessary condition for \( \theta \in \text{Supp}((y(h)) \varphi_{[2h-1]}) \) to be in \( \text{Supp}((\omega) \varphi_{[2h-1]}) \subset \text{Supp}((y(h)) \varphi_{[2h-1]}) \) is

\[ (4.1.2) \quad \begin{aligned}
  d_\omega^h &= h - 1, \\
  d_\omega^i &= i - 1, & \text{for all } h < i \leq m.
\end{aligned} \]

So it remains to show that any \( k \)-subset \( \omega' \in \text{Supp}(y(y(h)) \varphi) \) does not satisfy one of these criteria.

Proof of Claim, case 2\( k \leq n \): Consider now \( \omega' \in \text{Supp}(y(y(h)) \varphi) \). Note that if \( h = k \), then the assumption says that \( y = y(h) \) and the claim is trivial. Let us now assume \( h < k \). Then we have \( \omega' \in \Omega_k^h \) for some \( i > h \). By the definition of \( \varphi \), any \( \theta' \in \text{Supp}(\omega') \varphi \) must have \( d_\theta' = d_\omega' - \epsilon \) with \( \epsilon \in \{0, 1\} \). In particular, we have \( d_\omega' \geq i - 1 \). Hence, it follows from (4.1.2) that \( \text{Supp}(\omega') \varphi \) and \( \text{Supp}(y(y(h)) \varphi_{[2h-1]}) \) are disjoint. Since \( \text{Supp}(y(y(h)) \varphi_{[2h-1]}) \subset \bigcup_{\omega' \in \text{Supp}(y(y(h)) \varphi)} \text{Supp}(\omega') \varphi \), the former set must also be disjoint from \( \text{Supp}(y(y(h)) \varphi_{[2h-1]}) \), as required.

Proof of Claim, case 2\( k > n \): Consider \( \omega' \in \text{Supp}(y(y(h)) \varphi) \), then the assumption implies that \( \omega' \in \Omega_k^h \) for all \( h < k \). It then follows from Lemma 3.4 (Hi) that \( d_\omega^h > h \). Using a similar argument as in the case of 2\( k \leq n \), we get that \( d_\omega^i > h - 1 \) for any \( \theta' \in \text{Supp}(\omega') \varphi \). Mutatis mutandis, we get that \( \text{Supp}(y(y(h)) \varphi) \) and \( \text{Supp}(y(y(h)) \varphi_{[2h-1]}) \) are disjoint, as required. This finishes the proof of the claim.

Suppose on the contrary that \( (y(h)) \varphi_{[2h-1]} \neq 0 \). Then we have some \( \omega \in \text{Supp}((y(h)) \varphi_{[2h-1]}) \) whose coefficient in \( (y(h)) \varphi_{[2h-1]} \) is \( \lambda \in \mathbb{R}^x \). It follows from (4.1.1) that there must be some \( \omega' \in \text{Supp}((y(h)) \varphi_{[2h-1]}) \cap \text{Supp}(y(y(h)) \varphi) \) whose coefficient is \(-\lambda\). However, by Proposition 3.10 (1) and the above claim, \( \text{Supp}((y(h)) \varphi_{[2h-1]}) \) is disjoint from \( \text{Supp}((y(y(h))) \varphi_{[2h-1]}) \) and from \( \text{Supp}(y(y(h))) \varphi) \) respectively; a contradiction.

\[ \square \]

Example 4.3. Let us go through the proof with a concrete example. Suppose \( p = 3, k = 4, n = 8 \) (so \( k \) is low). Consider the element \( y \in F \Omega_k \) given by the sum of all 4-subsets \( \omega \subset [5] \cup \{8\} \). Since \( y = ([5] \cup \{8\}) \varphi(2) \) and \( p = 3, (y) \varphi = 0 \).

For \( \omega \in \text{Supp}(y) \), we have

\[ d_\omega := \begin{cases} (d_\omega^1, d_\omega^2, 3.3, 4), & \text{if } 8 \in \omega; \\
(d_\omega^1, d_\omega^2, 4, 4, 4), & \text{if } 8 \notin \omega.
\end{cases} \]

So the threshold of \( \omega \) is 3 (resp. 4) if and only if \( 8 \in \omega \) (resp. \( 8 \notin \omega \)). Now we can see that \( y = y(3) + y(4) \), so it satisfies the assumption of Lemma 4.2 by taking \( h = 3 \).

The summand \( y(3) \) is then given by the sum of \( \omega \cup \{8\} \) over all 3-subsets \( \omega \subset [5] \). So any \( \theta \in \text{Supp}((y(3)) \varphi_{[5]}) \) has density \( d_\theta = (d_\theta^1, d_\theta^2, 2, 2, 3) \) (satisfying (4.1.2)). For \( \theta' \in \text{Supp}((y(4)) \varphi_{[5]}) \), following the argument of the proof, we get that \( d_\theta^4 \geq 3 > d_\theta^4 \) for all \( \theta \in \text{Supp}((y(3)) \varphi_{[5]}) \). Hence, we get the disjointness as claimed.

Finally, for the sake of completeness, let us see how \( \varphi_{[5]} \) kills \( y(3) \):

\[ (y(3)) \varphi_{[5]} = \sum_{\omega \subset [5]} (\omega) \varphi_{[5]} \cup \{8\} = \sum_{\omega \subset [5]} \sum_{\theta \cup \{8\} \cap \omega \neq \emptyset} (\theta) \in \text{Supp}((y(3)) \varphi_{[5]}), \]

\[ = \sum_{\omega \subset [5]} \sum_{\theta \cup \{8\} \cap \omega \neq \emptyset} 3 (\theta \cup \{8\}) = 0. \]

Lemma 4.4. Suppose \( \omega \in \Omega_k^h \) is a \( k \)-subset whose threshold is \( h < k \). If for an \( i \in [p - 1] \) the support \( \text{Supp}((\omega) \varphi_{[2h-1]} \delta_1^{[2h-1]} \varphi_j^{[2h-1]}) \) is not empty, then for any \( \theta \) in this support, \( d_\theta^j \geq d_\theta^j \) holds for all \( j \in [h, m] \) with strict inequality in the case when \( j = h \). In particular, the threshold of \( \theta \) cannot be \( h \).
Proof. Note that if $2h-1 \geq n$, then $(\omega)\varphi^{[2h-1]}_{\theta} = 0$; and if $2h-1 \leq h-i-1$, then $(\omega)\varphi^{[2h-1]}_{\theta} \delta^{i}_{[2h-1]} = 0$, so the assumption of the lemma says that $h$ must necessarily satisfy $h-i-1 < 2h-1 < n$.

Consider first $\omega_i \in \text{Supp}(\varphi^{[2h-1]}_{\theta})$. There are $\epsilon_j \in \{0, 1\}$ for $j \in \{h+1, m\}$, so that

$$d^{\omega}_{j} = \begin{cases} d^{\omega}_{j}, & \text{if } j \in \llbracket h \rrbracket; \\ d^{\omega}_{j} - \epsilon_j, & \text{if } j \in \llbracket h+1, m \rrbracket. \end{cases}$$

Hence, we have $d^{\omega}_{j} \geq d^{\omega}_{j} - 1$ for all $j \in \llbracket h \rrbracket$. Moreover, the assumption of $h-i-1 < 2h-1$ guarantees that the inequality is strict in the case when $j = h$.

Next, consider $\omega_j \in \text{Supp}(\varphi^{[2h-1]}_{\theta} \delta^{i}_{[2h-1]})$. Since $2h-1 < n$, $\omega_j$ is obtained by adjoining $i$ new elements from $[2h-1] \setminus ([2h-1] \cap \omega_1)$, so we have $d^{\omega}_{j} = d^{\omega}_{j} + i$ for all $j \in \llbracket h, m \rrbracket$.

Finally, by the definition of $\varphi, \theta$ is obtained by removing any $i-1$ elements from $\omega_j$, so we have

$$(4.1.3) \quad d^{\theta}_{j} \geq d^{\omega}_{j} - (i - 1) = (d^{\omega}_{j} + i) - (i - 1) \geq (d^{\omega}_{j} - 1) + 1 = d^{\omega}_{j},$$

for all $j \in \llbracket h, m \rrbracket$. Moreover, in the case when $j = h$, it follows from above and the assumption of $h < k$ that the last inequality in (4.1.3) is strict.

For the final statement, it follows from the previous part that $h = d^{\omega}_{h} < d^{\omega}_{h}$ when $h < k$, and so $\theta \notin \Omega^{h}_k$. \hfill \Box

Lemma 4.5. Let $y_h \in \text{Ker}(\varphi^{[2h-1]}_{\theta} \cap \mathbb{F}\Omega^{h}_k$ for some $h > 0$. Then

$$y_h = (y_h)\delta^{p-1}_{[2h-1]}\varphi^{p-1} + y',$$

where $y' \in \mathbb{F}\Omega^{h}_k$ satisfies the following condition:

(1) If $h = k$, then $y' = 0$.

(Lo) If $2k < n$, then $\text{Supp}(y') \subset \Omega^{k}_k$.

(Hi) If $2k > n$, then $\text{Supp}(y') \subset \Omega^{k}_k$. Moreover, we have $y' = 0$ when $h = 1$.

Proof. Apply Lemma 3.16 with $y = y_h$ yields $y' := -\sum_{i=1}^{p-1} b_i(y_h)\varphi^{[2h-1]}_{\theta} \delta^{i}_{[2h-1]}$ for some scalars $b_1, \ldots, b_{p-1} \in \mathbb{F}$.

By Proposition 3.10 (1), we can write $(y_h)\varphi$ as $(y_h)\varphi^{[2h-1]}_{\theta} + (y_h)\varphi^{[2h-1]}_{\theta}$. Applying the assumption of $(y_h)\varphi^{[2h-1]}_{\theta} = 0$, we get that

$$y' = -\sum_{i=1}^{p-1} b_i(y_h)\varphi^{[2h-1]}_{\theta} \delta^{i}_{[2h-1]} \varphi^{i-1}.$$  

(1):

For the case when $2k \leq n$, it follows from Lemma 3.5 (Lo) that $\Omega^{k}_k = \emptyset$. Therefore, by Proposition 3.10 (3) we have $(y_h)\varphi = (y_h)\varphi^{[2h-1]}_{\theta}$. This means that $(y_h)\varphi^{[2h-1]}_{\theta}$ is zero, and hence so is $y'$, as required.

For the case when $2k > n$, we have by definition $\varphi^{[2h-1]}_{\theta} = 0$, so it follows that $y' = 0$.

From now on, we can assume $h < k$ since the claims (Lo) and (Hi) in the case when $h = k$ follows immediately from (1).

Denote by $f_i$ the $\mathbb{F}$-linear map $\varphi^{[2h-1]}_{\theta} \delta^{i}_{[2h-1]}$ for each $i \in \llbracket p - 1 \rrbracket$. To prove the claims (Lo) and (Hi), it suffices to look at the cases when $(y_h)f_i \neq 0$.

For a non-zero $(y_h)f_i$, we want to understand the density of the $k$-subsets in its support. To this end, we need to look at how the density of a $k$-subset $\omega \in \text{Supp}(y_h)$ changes upon applying the map $f_i$, and so we can assume $(\omega)f_i$ is non-zero. In particular, we have $[2h-1] \cap \omega \neq \emptyset$. In such a case, note that for any $a \in [2h-1] \cap \omega$ and any $\omega \in \text{Supp}(y_h)$, $[(\omega \setminus \{a\}) \cap [2h-1]] = |\omega \cap [2h-1]| = h$, so the set $\text{Supp}((y_h)\varphi^{[2h-1]}_{\theta})$ is $h$-homogeneous on $[2h-1]$.

(Lo): Since $y_h \in \mathbb{F}\Omega^{k}_k$, it follows from Lemma 4.4 that any $\theta \in \text{Supp}((y_h)\varphi^{[2h-1]}_{\theta})$ satisfies $d^{\theta}_{h} > h$. On the other hand, we have

$$d^{\theta}_{h} = |\theta \cap [2m - 1]| = |\theta \cap [n]| = k < m,$$

where the last inequality follows from the assumption of $k$ being low. Taking $(i, j) = (h, m)$ in Proposition 3.3 (2) yields some $j \in \llbracket h+1, m-1 \rrbracket$ such that $d^{\theta}_{j} = j$, meaning that $\theta \notin \Omega^{\geq h}_k$ as required.

(Hi): As explained above, we only need to consider $\omega \in \text{Supp}(y_h) \subset \Omega^{k}_k$ so that $\text{Supp}((\omega)f_i)$ is non-empty. For such $\omega$, it follows from Lemma 3.4 (Hi) that $d^{\omega}_{j} > j$ for all $h < j \leq m$.

Moreover, Lemma 4.4 implies that, for any $\theta \in \text{Supp}((\omega)f_i)$, we have $d^{\theta}_{j} \geq d^{\omega}_{j} > j$ for all $h < j \leq m$, and $d^{\theta}_{h} > h$. This implies that $\theta \notin \Omega^{\geq h}$, hence $y' \notin \mathbb{F}\Omega^{\geq h}$, and the first part of the claim now
follows. Finally, if we consider the case when $h = 1$, then by Lemma 3.5 (Hi), we have $\Omega_k^0 = \emptyset$, so $\text{Supp}(y') \subset \Omega_k^{< h}$ implies that $y' = 0$.

Now we are ready to give the key inductive mechanism needed.

**Lemma 4.6.** Consider a non-zero $y_h \in \text{Ker}(\varphi_h)$ that satisfies the following condition.

1. If $2k \leq n$, then $\text{Supp}(y_h) \subset \Omega_k^{\geq h}$ with $h > 0$.
2. If $2k > n$, then $\text{Supp}(y_h) \subset \Omega_k^{< h}$.

Then there is some $x \in \text{Ker}(\varphi_h)$ such that $y_h \equiv x \mod (\varphi^{p-1})$ and the following hold.

1. If $2k \leq n$, then $\text{Supp}(x) \subset \Omega_k^{>h}$. In particular, if $h = k$, then $x = 0$.
2. If $2k > n$, then $\text{Supp}(x) \subset \Omega_k^{< h}$. In particular, if $h = 1$, then $x = 0$.

Note that if $2k > n$, then $\Omega_k^{\leq 0} = 0$ by Lemma 3.5 (Hi), so we omitted the condition $h > 0$ when $k$ is high.

**Proof** For ease of exposition, define

$$\Omega_{k,h} := \begin{cases} \Omega_k^{\geq h} & \text{if } 2k \leq n; \\ \Omega_k^{< h} & \text{if } 2k > n, \end{cases} \quad \text{and} \quad \Theta_{k,h} := \begin{cases} \Omega_k^{\geq h} & \text{if } 2k \leq n; \\ \Omega_k^{< h} & \text{if } 2k > n. \end{cases}$$

Using this notation, the assumption says that $\text{Supp}(y_h) \subset \Omega_{k,h}$ and $\text{Supp}(y_h - y(h)) \subset \Theta_{k,h}$.

Since $h$ is necessarily non-zero, it follows from Lemma 4.2 that $y(h) \in \text{Ker}(\varphi_{[2h-1]})$. Hence, Lemma 4.5 tells us that $y(h) = (\delta_{[2h-1]}(y(h))^{p-1} + y' \mod \text{Im}(\varphi^{p-1})$ and $\text{Supp}(y') \subset \Theta_{k,h}$.

Summarising, we have

$$y_h = y_h - y(h) + y' = (y_h - y(h)) + \left(y' + \left(\delta_{[2h-1]}(y(h))^{p-1}\right)_{\varphi_{[2h-1]}^{p-1}}\right),$$

so by taking $x := y_h - y(h) + y'$, we have $x \equiv y_h \mod \text{Im}(\varphi^{p-1})$ and $\text{Supp}(x) \subset \Theta_{k,h}$. Note that since $\text{Im}(\varphi_{[k+p-1]}^{p-1}) \subset \text{Ker}(\varphi_k)$, $x$ must also be in $\text{Ker}(\varphi_k)$.

Finally, when $k$ is low and $h = k$, then $x = 0$ follows from combining $y_h = y(h)$ (which holds by construction) and $y' = 0$ (which is stated in Lemma 4.5 (1)). When $k$ is high and $h = 1$, similarly, we have $y' = 0$ by Lemma 4.5 (Hi) and $y_h = y(h)$ by $\Omega_k^{\leq 1} = \Omega_k^1$ (which in turns follow from Lemma 3.5 (Hi)).

**Theorem 4.7.** Consider an element $y \in \text{Ker}(\varphi_k)$. When $2k \leq n$, we assume in addition that $\text{Supp}(y) \subset \Omega_k^{< 0}$. Then we have $y \in \text{Im}(\varphi_{[k+p-1]}^{p-1})$. In particular, $H_k^{[0]} = 1$ $H_k = 0$ when $2k > n$.

**Proof** Suppose we have $2k \leq n$. We have a chain of congruences

$$y = y_1 \equiv y_2 \equiv \cdots \equiv y_k \equiv 0 \mod \text{Im}(\varphi^{p-1})$$

of elements in $\text{Ker}(\varphi)$ such that $\text{Supp}(y) \subset \Omega_k^{> a}$; here the $i$-th congruence follows by applying Lemma 4.6 with $h = i$ and using $\Theta_{k,h} = 1_{k,h+1}$ (in the notation of Lemma 4.6) for all $i \leq k$.

Suppose we have instead $2k > n$. By Lemma 3.5 (Hi), any $y \in \mathbb{F} \Omega_k$ must have $\text{Supp}(y) \subset \Omega_k^{< a}$ for some $0 < a \leq n - k + 1 < k + 1$. So we have a chain of congruences

$$y = y_0 \equiv y_{a-1} \equiv \cdots \equiv y_1 \equiv 0 \mod \text{Im}(\varphi^{p-1})$$

of elements in $\text{Ker}(\varphi)$ with $\text{Supp}(y) \subset \Omega_k^{> a}$; here the $i$-th congruence follows by applying Lemma 4.6 with $h = a - i + 1$ and using $\Theta_{k,h} = 1_{k,h+1}$ (in the notation of Lemma 4.6) for all $i \in [a]$.

4.2. **Dimensions of homologies.** In this subsection, we determine the dimension of all slash homology groups - along the way, we show various vanishing conditions for some classical $p$-homology groups.

Since we only look at the $p$-complex $\mathbb{F} \Omega_\bullet$, we omit writing $\mathbb{F} \Omega_\bullet$ for the homology groups and just use $H_k^{[a]} \cdot H_k^{[a]}$ instead.

**Lemma 4.8.** Suppose $k$ is high, i.e. $2k > n$. Then for any $a \in [0, p-2]$ satisfying $2k - 2a > n$, we have $H_k^{[a]} = 0$.

**Proof** Since Theorem 4.7 asserts that $H_k^{[0]} = 1$ $H_k = 0$ whenever $2k > n$, the claim follows immediately from Lemma 2.4 (1).

**Lemma 4.9.** If $k$ is low, i.e. $2k \leq n$, then the following holds.

1. $H_k^{[0]} \cong S^{k,k} / \text{Im}(\varphi^{p-1}) \cap S^{k,k}$. 

(2) \( p^{-1}H_k = 0 \) if \( 2k \neq n \); in particular, \( p^{-r}H_{k-r} = 0 \) for all \( r \in \lfloor \min\{k, p-1\} \rfloor \).

**Proof.** (1) For \( v \in \text{Ker}(\varphi_k) \), it follows from Proposition 3.8 that we can write \( v = y + z_k \) with \( z_k \in S^{k,k} \) and \( \text{Supp}(y) \subset \Omega_k^0 \).

Since \( 2k \leq n \), we have \( S^{k,k} \subset \text{Ker}(\varphi_k) \) by Theorem 2.10 (1), i.e. \( (z_k)\varphi_k = 0 \). This implies that \( (y)\varphi_k = 0 \), and so we can apply Theorem 4.7 to get \( y = (w)\varphi_{k+p-1}^{-1} \) for some \( w \in \mathbb{F}\Omega_{k+p-1} \). In other words, we have

\[
v = z_k + (w)\varphi_{k+p-1}^{-1} \in \text{Ker}(\varphi_k).
\]

This means that \( v + \text{Im}(\varphi_{k+p-1}^{-1}) = z_k + \text{Im}(\varphi_{k+p-1}^{-1}) \), as required.

(2) By Lemma 4.8, we have \( H_i^{a-j} = 0 \) whenever \( n-j > n/2 \). In particular, we have

\[
p^{-1}H_j = H_j^{a} = 0
\]

for all integers \( 0 \leq j < n/2 \), where the isomorphism follows from the back/slash duality (2.3.2); the first part of the claim follows.

Since we have \( p^{-1}H_{k-r} = 0 \) for all integers \( r \in \lfloor \min\{k, p-1\} \rfloor \), the last part follows immediately from Lemma 2.4 (2). □

**Lemma 4.10.** For an integer \( k \) satisfying \( 0 \leq 2k \leq n - (p-1) \), the following hold.

(1) \( H_{k-a}^i = 0 \) for all \( a \in [k+1, p-2] \).

(2) \( H_{k-1}^i = 0 \).

(3) \( H_{k+1}^i = 0 \) for all \( a \in \lfloor 0, p-2 \rfloor \).

In particular, we have \( p^{-1}H_{k} = H_{k}^{a} = 0 \) for all \( \ell \in [(n+p-1)/2, n] \).

**Proof.** (1) By the assumption on \( a \), we have \( \text{Ker}(\varphi^a) = \mathbb{F}\Omega_k \); the claim now follows from the definition of \( H_{k}^i \).

(2) By Theorem 2.10, we have \( \text{Im}(\varphi_{k+a}^a|_{S^{k+1} \rightarrow \ldots}) = S^{k-a,k-a} \) if \( a' < p \) and \( k' < n \). Take \( a' = p-1 \) and \( k' = k + p - 1 \), then we can now see that \( \text{Im}(\varphi_{k+p-1}^{-1}|_{S^{k+a+1}}) = S^{k,k} \). Indeed, first observe from the assumption on \( k \) that \( n = 2k + 1 \) and \( (k')^{p-1} = (k)^{p-1} < p \), which means that \( (k)^{p-1} \) is a submodule of \( S^{k,k} \). On the other hand, we can apply \( \mathbb{F} \)-linear dual to \( \text{Ker}(\varphi_{n-k}^{n-2k+1}) \) which yields

\[
\left( \text{Ker}(\varphi_{n-k}^{n-2k+1}) \right)^* \cong \text{Coker}(\varphi_{n-k}^{n-2k+1})^* \cong \text{Coker}(\varphi_{n-k}^{n-2k+1})^*,
\]

where the first isomorphism follows from Lemma 2.5 (1) and the second isomorphism follows from the commutative diagram (2.3.1).

Suppose on the contrary that there is a non-zero element of \( S^{k,k} \) taking the form \( (x)\varphi_{n-k}^{n-2k+1} \) for some \( x \in \mathbb{F}\Omega_{k+1} \). For \( m \in \sum_{\omega \in \Omega_k} \lambda_{\omega}^{a} \omega^* \in \mathbb{F}\Omega_k \), denote by \( m^* \) the dual element of \( m \) in \( \mathbb{F}\Omega_k^* \), i.e. \( m^* = \sum_{\omega \in \Omega_k} \lambda_{\omega}^{a} \omega^* \). By Lemma 2.13, we have \( ((x)\varphi_{n-k}^{n-2k+1})^* \in \left( S^{k,k} \right)^* \cong S^{n-k,k} \), which is a submodule of \( \text{Ker}(\varphi_{n-k}^{n-2k+1}) \) as shown in the previous paragraph. Hence, the dual element of \( (x)\varphi_{n-k}^{n-2k+1} \) is just \( (x)\varphi_{n-k}^{n-2k+1} \). But (4.2.1) tells us that \( (\text{Ker}(\varphi_{n-k}^{n-2k+1})^* \cong \text{Coker}(\varphi_{n-k}^{n-2k+1}) \), so we get that \( (x)\varphi_{n-k}^{n-2k+1} \notin \text{Im}(\varphi_{n-k}^{n-2k+1}) \), which is absurd. This finishes the proof of the claim.
Recall that \( \varphi_{n-k+1}^{n-2k+1} \) induces a map on the slash-cohomologies which sends the (slash-)homology class \([x] \in H_{n-k+1}^{n-2k+1} \) of \( x \in \Omega_{n-k+1} \) to the homology class of \([(x)\varphi_{n-k+1}^{n-2k+1}] \) in \( H_{k}^{0} \). Recall also from slash shifting Lemma 2.4 (1) this induced map is injective. Therefore, it suffices to show that this induced map is zero in order to finish the proof. Indeed, by Lemma 4.9 (1) we have that \( H_{k}^{0} \) is a quotient of \( S^{k,k} \), so for any \( x \in \Omega_{n-k+1} \), if \([(x)\varphi_{n-k+1}^{n-2k+1}] \) is non-zero in \( H_{k}^{0} \), then \((x)\varphi_{n-k+1}^{n-2k+1} \) must also be non-zero in \( S^{k,k} \) - which is impossible as shown in the claim above.

For the remaining slash homology groups, it turns out they are the same as the slash-0 terms.

**Lemma 4.12.** For an integer \( k \) satisfying \( n - (p - 1) < 2k \leq n \), and \( a \in [0, n-2k] \), \( \varphi^{a} \) induces an isomorphism \( H_{k+a}^{a} \cong H_{k}^{0} \).

**Proof.** We know from slash-shifting Lemma 2.4 (1) that \( \varphi^{a} \) induces an injection \( H_{k+a}^{a} \cong H_{k}^{0} \) for \( a \in [0, p-2] \). It remains to show that these are surjective for \( a \in [0, n-2k] \). Indeed, it follows from Lemma 4.9 (1) that \( H_{k}^{0} \) is a quotient of \( S^{k,k} \), so let us consider \( x \in S^{k,k} \). Since \( n - 2k < p - 1 \) by the assumption on \( k \), we have \( \text{Im}(\varphi^{a}) = \text{Im}(\varphi^{a}) \). So it follows from Theorem 2.10 (2), which says that \( S^{k,k} = \text{Im}(\varphi^{a} \cdot S^{k,k}) \), that \( \varphi^{a} \) is surjective onto \( S^{k,k} \), as required.

Our next goal is to show that \( H_{k}^{0} \) is non-vanishing when \( n - (p - 1) < 2k \leq n \) and \( p \neq 2 \). For this purpose, we consider certain \((2-1)\)-complexes obtained by “contracting \( F_{r} \)’s.”

For any \( k \in [0, n] \) and \( a \in [p-1] \), consider the \((2-)\)-complex \( C(k,a) \) as follows:

\[
C(k,a) = \left( \bigoplus_{j \in \mathbb{Z}} \mathbb{F} \Omega_{pj+k} \oplus \mathbb{F} \Omega_{pj+k-a} \right) \oplus \sum_{j \in \mathbb{Z}} \mathbb{F} \Omega_{pj+k} + \mathbb{F} \Omega_{pj+k-a}
\]

Note that \( \mathbb{F} \Omega_{r} \), \( \varphi_{r} \) are regarded as zero whenever the index lies outside \([0, n]\). Denote by \( H_{m}(k,a) \) the homology of \( C(k,a) \) at the degree where \( \mathbb{F} \Omega_{m} \) appears. Note that, by (2.1.1), \( H_{m}(k,a) \) is the same as the homology of the complex given by replacing all \( \varphi^{b} \) by \( \varphi^{(b)} \) (as we work over a field).

By construction, we have

\[
H_{m}(k,a) = \begin{cases} H_{m}, & \text{if } m \in k + p \mathbb{Z}; \\ p-aH_{m}, & \text{if } m \in (k-1) + p \mathbb{Z}. \end{cases}
\]

Hence, counting the dimensions yields

\[
\sum_{j \in \mathbb{Z}} \dim^{a}H_{pj+k} - \dim^{p-a}H_{pj+k-a} = \sum_{j \in \mathbb{Z}} \left( \begin{array}{c} n \\ pj+k \end{array} \right) - \left( \begin{array}{c} n \\ pj+k-a \end{array} \right).
\]

Note that the binomial coefficient \( \left( \begin{array}{c} n \\ r \end{array} \right) \) is zero whenever the integer \( r \) is not in \([0, n]\).

Readers well-versed in combinatorics may have already realised that the summation on the right-hand side enumerates certain prefixes of Dyck paths. For completeness, let us give more detail to this.

**Definition 4.13.** Let \( n \geq 0 \) be a non-negative integer. An \( n \)-step lattice path is a sequence \( P \) of \( n+1 \) points \( p_{0} = (0,0), p_{1}, \ldots, p_{n} \) in \( \mathbb{Z}^{2} \), so that, for all \( 1 \leq i \leq n \), the \( i \)-th step \( s_{i} := p_{i} - p_{i-1} \) is either \((0,1)\) or \((1,0)\). We also consider such a path as a 1-dimensional object in \( \mathbb{R}^{2} \) by taking the union of all intervals \([p_{i-1}, p_{i}] \subset \mathbb{R}^{2} \) over \( 1 \leq i \leq n \).

Consider \( k \in [n] \) and \( t \in [0, n+1] \). A lattice path is \((n-k,k)\)-open if it has \( n \) steps and \( k \) of which are \((1,0)\). Note that such a path necessarily ends on \((n-k,k)\).

For integers \( s, t \in \mathbb{Z}_{\geq 0} \cup \{\infty\} \), we say that a lattice path is \( s \)-bounded-above and \( t \)-bounded-below if all the points \((x,y)\) in the path satisfies \(-t < y - x < s\).

Denote by \( \mathcal{L}^{n-k,k}_{s,t} \) to be the set of all \((n-k,k)\)-open lattice path that is \( s \)-bounded-above and \( t \)-bounded-below.

Enumeration of lattice paths is a well-studied subject, see, for example, [Moh, Theorem 2], which gives

\[
|\mathcal{L}^{s,t}_{n-k,k}| = \sum_{j \in \mathbb{Z}} \left( \begin{array}{c} n \\ k+j(t+s) \end{array} \right) - \left( \begin{array}{c} n \\ k+j(t+s)+t \end{array} \right).
\]
Lemma 4.14. For any $k \in [n]$, we have
\begin{equation}
\sum_{j \in \mathbb{Z}} \dim a H_{pj+k} - \dim n-a H_{pj+k-a} = [\mathcal{L}_{n-k,k}^{a,p-a}].
\end{equation}
In particular, if $2k \leq n - (p-a)$ or $2k \geq n+a$, then the summation on the left vanishes.

**Proof.** It is straightforward to see that, by taking $(s,t) = (a,p-a)$, the right-hand side of (4.2.3) coincides with that of (4.2.2); combining the two gives (4.2.4).

For the last statement, it is clear that $\mathcal{L}_{n-k,k}^{a} = \emptyset$ when $(n-k,k)$ lies beyond $-s < x - y < t$. Hence, in the setting of $(s,t) = (a,p-a)$, $\mathcal{L}_{n-k,k}^{a} = \emptyset$ when the condition $-a < n - 2k < p-a$, equivalently $n - (p-a) < 2k < n+a$, does not hold. In other words, there is no lattice path whenever $2k \leq n - (p-a)$ or $2k \geq n+a$. The claim now follows by combining this condition with (4.2.3). \qed

Lemma 4.15. For $n - (p-1) < 2k \leq n$, the homology $H_{m}(k,1)$ of $\mathcal{C}(k,1)$ is zero for all $m \neq k$. In particular, $H_{k}(k,1) = H_{k}^{0}$ is non-zero if, and only if, $n - (p-1) < 2k \leq n$ and $p \neq 2$; in which case, its dimension is given by $[\mathcal{L}_{n-k,k}^{a,p-1}]$, i.e.
\[ \dim H_{k}^{0} = \dim H_{k}^{a,p-1} = \sum_{j \in \mathbb{Z}} \left( \frac{n}{pj+k} - \frac{n}{pj+k-1} \right). \]

**Proof.** For all negative $j$ (resp. positive $j$), it follows from the assumption on $k$ that $2(pj+k) \in \{0, n - (p-1)\}$ (resp. $2(pj+k) \geq 2n$), so we have $H_{pj+k}(k,1) = H_{pj+k}^{0} = 0$ by Lemma 4.10 (2) (resp. Lemma 4.8).

For all non-positive $j$, it follows from Lemma 4.9 (2) that $H_{pj+k-1}(k,1) = p^{-1}H_{pj+k-1} = 0$; whereas for the case when $j$ is positive, we have $pj + k - 1 > (n-p+1)/2$, which means that $p^{-1}H_{pj+k-1} = 0$ by Lemma 4.10.

For the last statement, we know already from Lemma 4.8 and Lemma 4.10 (2) that $H_{k}^{0} = 0$ whenever $n - (p-1) \leq 2k$ or $2k > n$. For the other cases, it follows from the previous part of the lemma and Lemma 4.14 that $\dim H_{k}(k,1) = [\mathcal{L}_{n-k,k}^{a,p-1}]$. Observe that the lattice path that starts with a $(1,0)$-step and changes direction in every step is always in $\mathcal{L}_{n-k,k}^{a,p-1}$ with the exception of $p = 2$. The claim now follows. \qed

4.3. Main result and examples. Now we can combine all of the information above to obtain the main result of this article.

Before we do that, recall that the Specht module $S^{\lambda}$ of a two-row partition $\lambda = (n-k,k)$ has a simple top $D^{k}$ whenever $p \neq 2$.

**Theorem 4.16.** For any prime $p$ and any positive integer $n$, the slash-homology $(p-1)$-complexes $H_{k}^{i}$ at degree $k \in \{0, n\}$ of $\mathcal{F}_{n}$ is non-vanishing if, and only if, $p \neq 2$ and $n - 2k \in \{0, p-2\}$. Moreover, in the case when $H_{k}^{i}$ is non-vanishing, it takes the form
\[ H_{k}^{i} \cong \left( 0 \to H_{n-k}^{i-2k} \to H_{n-k-1}^{i-2k-1} \to \cdots \to H_{k+1}^{i-2k-1} \to H_{k}^{i-1} \to H_{k}^{i} \to 0 \right) \to \left( 0 \to D^{(n-k,k)} \to \cdots \to D^{(n-k,k)} \to D^{(n-k,k)} \to 0 \right). \]

**Proof.** If $n - 2k \notin \{0, p-2\}$, then $H_{k}^{i} = 0$ follows from combining Lemma 4.8, 4.10 (3), 4.11. All other cases (non-)vanishing condition, including the form of the non-vanishing slash homologies, follows from combining Lemma 4.15 and Lemma 4.12.

Let us now show that $H_{k}^{0} \cong D^{(n-k,k)}$ when it is non-zero. By Lemma 4.9 (1), we know that $H_{k}^{0}$ is a quotient of $S^{k,k}$, which is a Specht module corresponding to the partition $(n,k)$. Hence, by the fact that $S^{k,k} \cong S^{(n-k,k)}$ has a simple top, it suffices to show that $\dim H_{k}^{0} = \dim D^{(n-k,k)}$.

Indeed, since $n - (p-1) < 2k \leq n$, the formula for $\dim D^{(n-k,k)}$ can be looked up from [Erd, Example 5.3(3)], as we present below. Note that we replace the notations $(n,s,t,d,\delta)$ in loc. cit. by $(k, n-2k, 0, n-2k+1, p - (n-2k+1))$ in our setting for the convenience of the readers.

\[ \dim D^{(n-k,k)} = \sum_{j=0}^{\lfloor k/p \rfloor} \left( \binom{n}{k-pj} - \binom{n}{k-pj-1} \right) - \left( \binom{n}{n-k-p(j+1)} - \binom{n}{n-k-p(j+1)} \right). \]
Using \( \binom{n}{r} = \binom{n}{n-r} \) and reindexing \( j \)'s, the last two binomial coefficients can be written as \( \binom{n}{pj+k} - \binom{n}{pj+k-1} \) with \( j \) varying over all positive integers. Hence, the right-hand side is precisely

\[
\sum_{j \in \mathbb{Z}} \binom{n}{pj+k} - \binom{n}{pj+k-1},
\]

which is exactly the same formula for \( \dim H^i_k \) by Lemma 4.15, as required. \( \square \)

We remark that in the special case of \( p = 3, 5 \), the description of \( H^i_k \) verifies [Wil, Conjecture 7.5, 7.6].

Example 4.17. \( \dim H^i_k \) for \( \mathbb{F} \Omega \) in the case when \( p = 7 \) and \( n = 12, 13 \) are shown in Figure 1. Note that the arrow represents the isomorphisms (Lemma 4.12) induced by \( \varphi \), i.e. the \((p-1)\)-differential of the non-vanishing \( H^i_k \).

Figure 1. \( \dim H^i_k \) when \( p = 7 \) and \( n = 12, 13 \).
Corollary 4.18. The following equality holds.
\[ \sum_{k=0}^{n} \zeta^k \binom{n}{k} = \sum_{k=0}^{n} \sum_{a=0}^{n-2k} \zeta^{k+a} |L_{n-k,k}^{1,p-1}| \]
\[ = \sum_{k=0}^{n} \sum_{a=0}^{n-2k} \zeta^{k+a} \left( \sum_{j \in \mathbb{Z}} \left( \frac{n}{pj+k} \right) - \left( \frac{n}{pj+k-1} \right) \right), \]
where $\zeta$ is the primitive $p$-th root of unity $e^{2\pi \sqrt{-1}/p}$.

Note that for $2k \leq n - (p-1)$ or $2k > n$, $|L_{n-k,k}^{1,p-1}| = 0$ as we have explained in Lemma 4.14, so one can also write $k$ so that $n - (p-1) < 2k \leq n$ instead of $k \in [0,n]$ in the right-hand side.

Proof Similar to dimension counting for (2-) complexes, doing so for $p$-complexes require multiplying the dimension by $\zeta$, since the Grothendieck ring of the stable category of graded modules over $\mathbb{F}[x]/(x^p)$ (the underlying category for slash homologies) is the ring of cyclotomic integers $\mathbb{Z}[\zeta]$; c.f. [KQ, Qi].

Then the left-hand side of the claim is just dimension counting for $\mathbb{F} \Omega_*$, and the right-hand side is dimension counting for the slash homology. \qed

5. Basis for the slash-0 homologies

Throughout this section, we impose the following

Assumption: $n - (p-1) < 2k \leq n$,

unless otherwise stated. Recall from Theorem 2.12 that the set $\text{SYT}_n(k)$ of standard Young $(n-k,k)$-tableaux is an indexing set of a basis (given by standard polytabloid) of $S^{k,k}$, and from Lemma 4.9 that $H_k^0$ is a quotient of $S^{k,k}$. Moreover, there is also a well-known bijection between $L_{n-k,k}^{1,\infty}$ with $\text{SYT}_n(k)$, so given that $\dim H_k^0$ coincide with $|L_{n-k,k}^{1,p-1}|$ as shown in the previous section, it is natural to expect the bijection induces a basis for $H_k^0$. The aim of this subsection is to show that this is indeed the case.

Let us start with the correspondence between $\text{SYT}_n(k)$ with $L_{n-k,k}^{1,\infty}$.

Definition 5.1. Consider an $n$-step lattice path $P \in L_{n-k,k}^{1,\infty}$. The $k$-subset of $[n]$ associated to $P$ is given by
\[ \omega_P := \{ a \in [n] \mid \text{the } a\text{-th step in } P \text{ is } (0,1) \} \in \Omega_k. \]

In particular, $P \mapsto t^{\omega_P}$ defines a bijection $L_{n-k,k}^{1,\infty} \leftrightarrow \text{SYT}_n(k)$. A standard (two-row) tableau $t \in \text{SYT}_n(k)$ is called $p$-standard if $t = t^{\omega_P}$ for some $\omega_P \in L_{n-k,k}^{1,p-1}$.

Remark 3. The notion of $p$-standard tableau is introduced in [Kl]. In the case of a tableau $t$ of shape $\lambda = (n-k,k)$, it is defined as a standard tableau satisfying $t_{2,j} < t_{1,j+p-2}$ for all $j \in \{1,2,\ldots,k\}$, and $n - 2k < p - 1$. We show in Lemma 5.11 that this is equivalent to the formulation given above.

To simplify notation, we denote by $t^P$ and $e_P$ the standard tableau $t^{\omega_P}$ and standard polytabloid $e^{\omega_P}$. Likewise, by the (standard) tableau and polytabloid associated to $P$, we mean $t^P$ and $e^P$ respectively.

From the above correspondence between $L_{n-k,k}^{1,\infty}$ with standard tableaux, it is natural to expect the following.

Theorem 5.2. Suppose we have $H_k^0 \neq 0$, i.e. $k$ is an integer satisfying $n - (p-1) < 2k \leq n$. Then a basis of $H_k^0$ is given by
\[ \{ [e_t] \mid \text{t is a } p\text{-standard tableau of shape } (n-k,k) \}, \]
where $[v]$ denotes $v + \text{Im}(\varphi_{k+p-1}) \in H_k^0$ for $v \in \text{Ker}(\varphi_k)$. In particular, the simple top $D^{(n-k,k)}$ of $S^{(n-k,k)}$ (with $n - (p-1) < 2k \leq n$) admits the same basis where $[v]$ represents $v + \text{rad}S^{(n-k,k)}$.

Our aim from now on is to prove this theorem. Note that the final statement follows from combining Theorem 4.16 with Lemma 4.9, which asserts that the radical $\text{rad}S^{(n-k,k)}$ of the Specht module $S^{(n-k,k)} = S^{k,k}$ is the intersection $\text{Im}(\varphi_{p-1}) \cap S^{k,k}$ (as a submodule of the permutation module $M^{(n-k,k)} = F \Omega_k$).

The strategy to prove Theorem 5.2 is to modify the straightening rule - the induction process that is used to show that a Specht module admits a basis given by standard polytabloids in [Pe]. The
modification are designed so that we can keep track of how “far” away the tableaux appearing in a Garnir relation is from being a standard tableau associated to a path in $L_{n-k,k}^{1,p-1}$.  

\subsection{Preliminaries}

In this subsection, we present the definition of the partially ordered set that allows us to prove Theorem 5.2 by induction, and also a few techniques that we will use frequently.

We also define

$$L_{n-k,k}^{\geq p} := L_{n-k,k}^{1,\infty} \setminus L_{n-k,k}^{1,p-1}.$$  

The lattice paths in $L_{n-k,k}^{1,p-1}$ are said to be \textit{good}, and those in $L_{n-k,k}^{\geq p}$ \textit{bad}; it is then natural to use the same adjectives for the associated standard tableau $t^p$ and standard polytabloid $e_P = e_{t^p}$.

\textbf{Definition 5.3.} Let $t$ be a $(n-k,k)$-tableau.

(1) We say that $t$ \textit{almost standard} if all of the following hold.

\begin{itemize}
    \item $t$ is column-standard (i.e. entries increase as we go down each column);
    \item $t$ is second-row-standard (i.e. entries in the second row increases as we go right).
\end{itemize}

Denote by $\text{aST} := \text{aST}_p(k)$ the set of almost standard $(n-k,k)$-tableaux.

(2) Define a partial order $\preceq$ on $\text{aST}$ by declaring $t \preceq t'$ if one of the following conditions are satisfied.

\begin{itemize}
    \item[(i)] $t = t'$.
    \item[(ii)] $| \{ t \cap [i] \} | \leq | \{ t' \cap [i] \} |$ for all $i \in [n]$ with at least one of the inequalities being strict.
    \item[(iii)] $\{ t \} = \{ t' \}$ and for $\omega := \{ t \}$, there is some $j$ such that $\omega(j)$ lies in the same position in both $t, t'$ for all $i < j$, and $\omega(j) = t_{1,a} = t'_{1,b}$ with $a > b$.
\end{itemize}

This defines a partial order on $\text{aST}$.

The second condition on $\preceq$ is just the row-dominance order of Definition 2.11; it should be regarded as a refinement of the notion of density in Section 3 which gives us a measure on how far a standard tableau is from being $t^p$ with $P \in L_{n-k,k}^{1,p-1}$. The last condition is used to measure how far an almost standard tableau from being standard, it is the partial order used in the proof of standard polytabloid basis of Specht modules in [Pe, J1] “restricted to the first row”.

Note that the standard $(n-k,k)$-tableaux are totally ordered by the row-dominance order, and so combining with condition (iii), we can see that $(\text{aST}, \preceq)$ is a totally ordered set.

\textbf{Example 5.4.} For $(n,k) = (5,2)$, then $(\text{aST}, \preceq)$ is as follows, where the standard ones are shown in the top row.

\begin{equation*}
\begin{array}{cccc}
    1 & 3 & 5 & \succ \\
    2 & 4 & \succ & \\
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{cccc}
    1 & 3 & 4 & \succ \\
    2 & 5 & \succ & \\
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{cccc}
    1 & 4 & 3 & \succ \\
    2 & 5 & \succ & \\
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{cccc}
    1 & 4 & 2 & \succ \\
    3 & 4 & \succ & \\
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{cccc}
    1 & 4 & 2 & \succ \\
    3 & 5 & \succ & \\
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{cccc}
    2 & 1 & 4 & \succ \\
    3 & 5 & \succ & \\
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{cccc}
    2 & 4 & 1 & \succ \\
    3 & 5 & \succ & \\
\end{array}
\end{equation*}

\begin{equation*}
\begin{array}{cccc}
    1 & 3 & 2 & \succ \\
    4 & 5 & \succ & \\
\end{array}
\end{equation*}

Note that only the largest tableau $t_{2,4}$ is good when $p = 3$; for any prime $p > 3$, all standard tableaux are good.

Recall the following Garnir relation that is used to straighten a $(n-k,k)$-tableaux (with non-standard first row); cf. [J2, Section 7, 8].

\textbf{Lemma 5.5.} Let $t$ be a $(n-k,k)$-tableau. For a positive integer $j \leq k$, the following equation holds.

$$e_t = e_t \tau_{t,j} = e_t \sigma_{t,j},$$  

where $\tau_{t,j} := (t_{1,j}, t_{1,j+1})$ and $\sigma_{t,j} := (t_{1,j}, t_{1,j+1}, t_{2,j})$.

\textbf{Lemma 5.6.} The following hold for a $(n-k,k)$-tableau $t$.

(1) Take $i < j \in [k]$ and define the following column reordering permutation.

$$\rho_{t,i,j} := (t_{1,j}, t_{1,j-1}, \ldots, t_{1,i+1})(t_{2,j}, t_{2,j-1}, \ldots, t_{2,i+1})$$

in $S_n$, then $e_t = e_t \rho_{t,i,j}$.

(2) If $(t_{1,i}) \sigma = t_{i,j}$ for all $i \in \{ 1, 2 \}$ and $j \in [k]$, then $e_t = e_t \sigma$

\textbf{Proof} Both of them follow immediately from the definition of polytabloid. 

\textbf{Lemma 5.7.} Consider almost standard tableaux $t, t' \in \text{aST}$. If $\{ t \}$ and $\{ t' \}$ differ only by one element, say $\{ t \} = S \cup \{ a \}$ and $\{ t' \} = S \cup \{ b \}$, and $a > b$, then $t \prec t'$.

\textbf{Proof} Condition (ii) in Definition 5.3 holds by assumption.
Colloquially, Lemma 5.7 says that if we replace an entry in the second row of an almost standard tableau by a smaller value, then the new almost standard tableau (after an appropriate column reordering if needed) is larger than the original one, in the $\succeq$-order.

**Lemma 5.8.** If $t \in \text{aST}$ is a non-standard almost standard tableau, then $e_t \in \mathbb{F}\text{-span}\{e_{t'} \mid t' \triangleright t\}$.

**Proof** By non-standardness, we have an integer $j \in [n-k]$ so that $t_{1,j} > t_{1,j+1}$. If $j \geq k$, then it follows from Lemma 5.6 (2) that $e_t = e_t(t_{1,j}, t_{1,j+1})$. But $t(t_{1,j}, t_{1,j+1}) \triangleright t$ by Definition 5.3 (2) (iii), so the claim follows.

Now we can assume $j \leq k$. By Lemma 5.5, we have $e_t = e_u - e_v$, where $u = t\tau_{t,j}$ and $v = t\sigma_{t,j}$. It is easy to see that $u$ is almost standard and $u \triangleright t$.

Let $i$ be the unique integer so that $t_{2,i} < t_{1,j} < t_{2,i+1}$. Note that from the conditions we have $t_{2,i} < t_{1,j} < t_{2,i+1}$, so $i < j$ as $t$ has standard second-row. By Lemma 5.6, we have $e_v = e_v\rho_{v,i,j}$. Hence, we have $e_t = e_u - e_v\rho_{v,i,j}$, so it remains to show that $v := v\rho_{v,i,j} \triangleright t$.

By the choice of $\sigma_{t,j}$, $v$ is column-standard, and so the choice of $i$, $j$ now ensures that $v \in \text{aST}$. Note also that swapping columns (by $\rho_{v,i,j}$) does not change the underlying set $\{\nu\}$, i.e. $\{\nu\} = \{\nu\}$. Since $t_{2,j} > t_{1,j}$ (by column-standardness of $t$), $\{\nu\} = \{\nu\}$ is obtained from replacing $t_{1,j}$ in $\{t\}$ by the smaller $t_{2,j}$. So it follows from Lemma 5.7 that $v \triangleright t$ as required.

The motivation for the previous lemma should be natural: we want to do induction on $(\text{aST}_t, \succeq)$. In contrast, it may be unclear what the motivation of the forthcoming lemma is; we hope the reader can bear with us for the while as its use will become clear in the next subsection (specifically, Lemma 5.15).

For a standard tableau $t \in \text{SYT}_n(k)$, we define

$$\text{aST}_t := \{t' \in \text{aST} \mid \{t'\} = \{t\}\} \subset \text{aST}.$$ 

In particular, $t$ is the only standard tableau in $\text{aST}_t$ and is also the maximum in $(\text{aST}_t, \succeq)$.

**Lemma 5.9.** For any $s \in \text{aST}_t$, we have

$$e_s \in e_t + \mathbb{F}\text{-span}\{e_{t'} \mid t < t' \in \text{aST}\}.$$ 

**Proof** We prove this by induction (from large to small) on $(\text{aST}_t, \succeq)$. The claim is trivial for the case when $s = t$, which is the maximum of $(\text{aST}_t, \succeq)$. Suppose $s \neq t$. We can assume that $s_{1,k+1} < s_{1,k+2} < \cdots < s_{1,n-k}$ by Lemma 5.6 (2), which means that there exists $j \in [k]$ so that $s_{1,j} > s_{1,j+1}$.

Following the same argument and notation as in the proof of Lemma 5.8 yields $e_s = e_u - e_v$, where $u := s\tau_{s,j}$ and $v := s\sigma_{s,j} \rho$ with $\rho$ the column reordering permutation that ensures $u$ is almost standard.

On one hand, the construction of $u$ means that it is in $\text{aST}_t$ and also satisfies $s \triangleright u$, so it follows from induction hypothesis that

$$e_u \in e_t + \mathbb{F}\text{-span}\{e_{t'} \mid t < t' \in \text{aST}\}.$$ 

On the other hand, the same argument as in the proof of Lemma 5.8 yields $w \triangleright s \triangleright t$ (while $w$ is not in $\text{aST}_t$). Therefore, combining with $e_s = e_u - e_v$, the claim follows.

**5.2. The case of bad tableaux.** The aim of this subsection is to show a similar statement as Lemma 5.8 but with $t$ being a bad standard tableau and with the polytabloids replaced by their cosets in $H_k^{0\ell}$.

We start with the following key lemma which provides a relation between certain polytabloids in $H_k^{0\ell}$.

**Lemma 5.10.** For a $(n-k,k)$-tableau $t$ and an integer $i \in [k-(p-2)]$, consider the $(p-1)$-subset $B := \{t_{1,i}, t_{1,i+1}, \ldots, t_{1,i+p-2}\}$, the following equation holds:

$$e_t \sum_{\tau \in \mathbb{S}_n} \tau = (e_{t'} \cdot K)p^{p-1},$$

where $t'$ is the tableau obtained from $t$ by removing all the columns containing an element of $B$, and $K$ is the set consisting of all the entries in these removed columns.

We remark that Figure 2 will be helpful to understand the statement.

**Proof** Let us first set up some notations. Take $\ell := \lceil K \cap \{t\} \rceil \in [0, p-2]$. In particular, $t'$ is a $(n-k-(p-1), k-\ell)$-tableau. We take $L$ to be the set of entries in the first row that are not in $B$. See Figure 2.

We will prove the lemma by showing the following two equalities:
For clarity, what we need to show is the equality

$$-e_t \sum_{\tau \in \mathfrak{S}_n} \tau = -e_{t'} \cdot (K)\varphi^{(p-1)}.$$ (i) 

$$-e_{t'} \cdot (K)\varphi^{(p-1)} = (e_{t'} \cdot K)\varphi^{p-1}.$$ (ii)

Proof of (i): For clarity, what we need to show is the equality

$$\sum_{\tau \in \mathfrak{S}_n} \sum_{\sigma \in C_t} \text{sgn}(\sigma)\{t\}\sigma \tau = \sum_{\sigma' \in C_{t'}} \text{sgn}(\sigma')\{t'\}\sigma' \cdot (K)\varphi^{(p-1)}. \quad (5.2.1)$$

Consider a $k$-subset $\omega$ that is of the form $\{t\}\sigma\tau$ with $\sigma \in \mathfrak{S}_B$ and $\tau \in C_t$. Let $\sigma' \in C_{t'}$ be the unique element given by the restriction of $\sigma$ on $t'$. Then we can write $\omega = (\{t'\}\sigma') \cdot (K \cap \omega)$. Using Figure 2, one can observe the following. Firstly, the $\ell$-subset $K \cap \omega$, must appear as a unique support of $(K)\varphi_{t+1}^{(p-1)}$. Hence, $\omega$ must arise in the support in the right-hand side of (5.2.1). Secondly, for every $k$-subset in the said support, one can always obtain it in the form of $\{t\}\sigma\tau$.

Now we need to show that the coefficients $\lambda_\omega$ and $\mu_\omega$ of $\omega$ in the left-hand side and right-hand side of $(5.2.1)$, respectively, coincide.

The number of times that $\omega$ is picked up in the summation over $\mathfrak{S}_B$ is the order of the stabiliser subgroup of $\omega$ in $B$. Since this group is just $\mathfrak{S}_{\omega \cap B} \times \mathfrak{S}_{\omega \cap B}$, the number we need is $(p-1-r)!r!$, where $r = |\omega \cap B|$. It follows from Wilson’s theorem that $(p-1-r)!r! \equiv (-1)^{r+1} (\mod p)$, so we have $\lambda_\omega = (-1)^{r+1} \text{sgn}(\sigma)$, for $\omega = \{t\}\sigma\tau$ as in (5.2.1). Now $\text{sgn}(\sigma)$ is determined by the number of elements in $\omega$ that are not in $\{t\}$. This number is given by $|\omega \cap B| + |\omega \cap L|$, where $L = \{t_{1,j} \mid j \in \{n-k \} \setminus [i, i + p - 2]\}$. Thus, the coefficient $\lambda_\omega$ is $(-1)^{2r+1+r'} = (-1)^{r'+1}$, where $r' := |\omega \cap L|$.

On the other hand, since the coefficient of every $k$-subset in $\text{Supp}(\varphi^{(p-1)})$ is 1, combining with $\omega = \{t'\}\sigma' \cdot (K \cap \omega)$, we have $\mu_\omega = -\text{sgn}(\sigma')$. It then follows from the decomposition

$$\{t'\}\sigma' = (\{t\} \cap \omega) \cup (L \cap \omega)$$

that $-\text{sgn}(\sigma') = (-1)^{r'+1} = \lambda_\omega$. This finishes the proof of (i).

Proof of (ii): It is clear from the construction that every $k$-subset in $\text{Supp}(e_{t'})$ is disjoint from $K$, so it follows from the splitting rule (Lemma 2.1) that

$$(e_{t'} \cdot K)\varphi^{(p-1)} = \sum_{i=0}^{p-1} (e_{t'})^i \cdot (K)\varphi^{(p-1-i)}.$$ (ii)

It then follows from Theorem 2.10 that $e_{t'} \in S^{k-r, k-\ell}$ is annihilated by $\varphi^i = i!\varphi^{(i)}$ for any $i > 0$, so we have $(e_{t'} \cdot K)\varphi^{(p-1)} = (e_{t'} \cdot (K)\varphi^{p-1})$. This finishes the proof of (ii).

The required equality now follows from the observation that $\varphi^{(p-1)} = (p-1)!\varphi^{p-1} = -\varphi^{p-1}$, where the last equality follows from Wilson’s theorem.

Before proceeding to the next step, we need an elementary observation on the elements in $\{t'P\}$ that correspond to a $(0, 1)$-step intersecting or lying below the line $y = x - (p-1)$.

**Lemma 5.11.** Let $t := t^P$ be a standard tableau associated to a path $P$, and $(x, y)$ be the starting point of the $i$-th $(0, 1)$-step of $P$. Then $(x, y)$ satisfies $y \leq x - (p-1)$ if, and only if, we have a strictly descending chain

$$t_{2,i} > t_{1,i+p-2} > t_{1,i+p-3} > \cdots > t_{1,i}.$$
In particular, \( P \) is bad if, and only if, there is some \( i \in [k] \) such that \( t_{2,i} \geq 2i - 1 + p - 1 \); in which case, we also have \( i + p - 2 \leq n - k \).

**Proof** The if statement follows from \( t \) being standard and the correspondence between standard \((n-k,k,k)\)-tableau and \( \mathcal{L}_{n-k,k}^{1,\infty} \).

For the only if part, by definition we have \( y = i - 1 \), and so we have \( x \geq i + p - 2 \). Since \( x \) counts the number of entries in the first row of \( t \) that are less than \( t_{2,i} \), so \( x \geq i + p - 2 \) implies that \( t_{2,i} > t_{1,i+p-2} \); the rest of the chain follows from the row-standardness of \( t \).

For the final statement, observe from the condition on the chain that the smallest possible value \( t_{2,i} \) can take is \((i + p - 2) + (i - 1) + 1 = 2i - 1 + p - 1 \), where \( i + p - 2 \) is the number of terms in the chain above that appear after \( t_{2,i} \) (hence the inequality \( i + p - 2 \leq n - k \)), and \( (i - 1) \) is the number of \( t_{2,j} \) satisfying \( t_{2,j} < t_{2,i} \).

Lemma 5.11 allows us to introduce the following terminology.

**Definition 5.12.** For a bad standard tableau \( t \), we say that \( t_{2,i} \) is a **bad entry** if \( t_{2,i} \geq 2i - 1 + p - 1 \). Since such an entry always exists by Lemma 5.11, we can define the following \((p - 1)\)-subset of \([n]\):

\[
B_i := \{ t_{1,i+1} < t_{1,i+2} < \cdots < t_{1,i+p-1} < t_{2,i} \}.
\]

**Lemma 5.13.** Let \( t \) be a bad standard tableau with a bad entry \( t_{2,i} \). Then the element \( \sum_{\tau \in \mathcal{S}_{n_i}} e_{\tau} \in S^{k,k} \) belongs to the submodule \( \text{Im}(\varphi_{p-1}^{k,k}) \).

**Proof** Take \( B := B_i(t_{1,i}, t_{2,i}) \), then we have \( (t_{1,i}, t_{2,i}) \mathcal{S}_{B_i}(t_{1,i}, t_{2,i}) = B \). This implies that

\[
\sum_{\tau \in \mathcal{S}_{n_i}} e_{\tau} = \left( \sum_{\sigma \in \mathcal{S}_B} e_{\sigma}(t_{1,i+1}, t_{2,i}) \right)(t_{1,i}, t_{2,i}) = - \left( \sum_{\sigma \in \mathcal{S}_B} e_{\sigma} \right)(t_{1,i}, t_{2,i}),
\]

where the last equality follows from the observation that \( e_{t} = -e_{t_{1,i+1}, t_{2,i}} \). On the other hand, it follows from Lemma 5.10 that \( \sum_{\sigma \in \mathcal{S}_{B_i}} e_{\sigma} \in \text{Im}(\varphi_{p-1}) \). Hence, the element on the right-hand side is also in \( \text{Im}(\varphi_{p-1}) \), and so is \( \sum_{\tau \in \mathcal{S}_{n_i}} e_{\tau} \).

**Lemma 5.14.** Let \( t \) be a bad standard tableau with a bad entry \( t_{2,i} \). Then \( t \sigma \) is column-standard for any \( \sigma \in B_i \). In particular, \( t \sigma \) is almost standard if \( t_{2,i} \sigma = t_{2,i} \).

**Proof** Column-standardness follows from the existence of the chain in Lemma 5.11. More precisely, for \( j \in [p-2] \) with \( i + j \leq k \), we have \( (t \sigma)_{2,i+j} = t_{2,i+j} > t_{2,i} = t_{1,i+j} \) for all \( l \in [p-2] \) and so \( t_{2,i+j} > (t \sigma)_{1,i+j} \) always. For the \( i \)-th column, again the chain in Lemma 5.11 tells us that \( (t \sigma)_{1,i+1} > t_{1,i+1} = (t \sigma)_{1,i} \). The last statement follows from the fact that \( t \) is standard and \( \sigma \) fixes all entries in the second row.

Before proceeding, recall our notation

\[
[v] := v + \text{Im}(\varphi_{p-1}) \in H_k^{(0)}
\]

for the coset containing \( v \in F\Omega_k \) in the quotient \( H_k^{(0)} \).

**Lemma 5.15.** Suppose \( t \) is a bad standard tableau, then we have \([e_t] \in F\text{-span}[e_{t'} \mid t' \succ t] \subset H_k^{(0)} \).

**Proof** By Lemma 5.13, if \( t_{2,i} \) is a bad entry (which always exists by Lemma 5.11), we have

\[
-[e_t] = \sum_{1 \neq \sigma \in \mathcal{S}_{B_i}} [e_t \sigma]
\]

in \( H_k^{(0)} \). So it suffices to show that \([e_t \sigma] \) with non-identity \( \sigma \in \mathcal{S}_{B_i} \) is in \( F\text{-span}[e_{t'} | t' \succ t] \).

We partition \( \mathcal{S}_{B_i} \setminus \{1\} \) into two disjoint subsets \( S \sqcup T \) so that \( S \) consists of all the non-identity permutations that fix \( t_{2,i} \).

By Lemma 5.9, for any \( \sigma \in S \), we have

\[
e_{t} \sigma \in e_{t} + F\text{-span}[t' \in aST | t' \succ t].
\]

Therefore, we have \( \sum_{\sigma \in S} e_{t} \sigma = |S|e_{t} + v \) for some \( v \in F\text{-span}[e_{t'} | t' \succ t] \). Note that the subgroup \( \{1\} \sqcup S \) of \( \mathcal{S}_{B_i} \) that stabilises \( t_{2,i} \) is of order \((p-2)!\), which is congruent to 1 mod \( p \) by Wilson’s theorem. This means that \(|S|\) is congruent to 0 mod \( p \), which yields

\[
-[e_t] = ([S][e_t] + [v]) + \sum_{\sigma \in T} [e_t \sigma] = [v] + \sum_{\sigma \in T} [e_t \sigma]
\]
with \( v \in \mathbb{F}\text{-span}\{t' \mid t' > t\} \).

Now consider \( \sigma \in T \subseteq \mathcal{S}_B \). It follows from Lemma 5.14 that \( t\sigma \) is column-standard. By Lemma 5.6, we can apply a column reordering permutation \( \rho \) so that \( t\sigma \rho \) is almost standard and \( e_\sigma \sigma = e_\sigma \sigma \rho \).

Since
\[
\{t\sigma \rho \} = \{t\sigma \} = \{(t) \setminus \{t_{2,j}\}\} \cup \{(t_{2,i})\sigma \},
\]
and Lemma 5.11 says that \( (t_{2,j})\sigma < t_{2,j} \), applying Lemma 5.7 yields \( t\sigma \rho > t \).

Thus, the summation \( \sum_{\sigma \in T} [e_\sigma \sigma] \) is in \( \mathbb{F}\text{-span}\{t' \mid t' > t\} \), and the proof is now completed. \( \square \)

All the required ingredients for proving Theorem 5.2 are now ready.

5.3. Proof of Theorem 5.2. We can assume \( \mathcal{L}^{1,p-1}_{n-k,k} \) is non-empty (in particular, \( p \neq 2 \)); otherwise, there is nothing to show as \( H_k^0 = 0 \).

By Lemma 4.9, \( H_k^0 \) is a quotient of the Specht module \( S^{k,k} = \mathbb{F}\text{-span}\{e_t \mid t : \text{any tableau}\} \), which has a basis given by standard polytabloid (Theorem 2.10 (2)). So we can prove the theorem by showing the following result.

Lemma 5.16. For all \( t \in aST_t \), we have
\[
[e_t] \in \mathbb{F}\text{-span}\{[e_t'] \mid t' \text{ good and standard }\}.
\]

Proof We will show this by induction on \((aST, \leq)\). Note that the maximal (in fact, maximum) element of this poset is the standard tableau \( t^* \) associated to \( \omega = \{2, 4, \ldots, 2k\} \), which corresponds to the lattice path that changes direction at every step. This path is in \( \mathcal{L}^{1,p-1}_{n-k,k} \) so long as the set is non-empty. Hence, there is nothing to show in the case \( t = t^\omega \).

Consider now \( t < t^\omega \). If \( t \) is a good standard tableau, then there is nothing to show; otherwise, there are two possible forms of \( t \) - a bad standard tableau or a non-standard tableau.

If \( t \) is a bad standard tableau, then it follows from Lemma 5.15 that \( [e_t] \in \mathbb{F}\text{-span}\{[e_t'] \mid t' > t\} \).

If \( t \) is non-standard, then we can assume that \( t_{1,k+1} < t_{1,k+2} < \cdots < t_{1,n-k} \) by Lemma 5.6 (2). Now, if \( j \) is an integer with \( t_{1,j} > t_{1,j+1} \), then \( j \in [k] \), and so it follows from Lemma 5.8 that \( [e_t] \in \mathbb{F}\text{-span}\{[e_t'] \mid t' > t\} \).

Now we have shown that \( [e_t] \in \mathbb{F}\text{-span}\{[e_t'] \mid t' > t\} \) in the two possible forms of \( t \). By induction hypothesis, for every \( t' > t \) we have \( [e_{t'}] \in \mathbb{F}\text{-span}\{[e_{t'}] \mid t'' \text{ good and standard }\} \), hence it follows immediately that so is \( [e_t] \).

This completes the proof of Theorem 5.2.

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