Layer Potential Methods
for Elliptic Homogenization Problems

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Abstract

In this paper we use the method of layer potentials to study L^2 boundary value problems in a bounded Lipschitz domain Ω for a family of second order elliptic systems with rapidly oscillating periodic coefficients, arising in the theory of homogenization. Let \( L_\varepsilon = -\text{div}(A(\varepsilon^{-1}X)\nabla) \). Under the assumption that \( A(X) \) is elliptic, symmetric, periodic and Hölder continuous, we establish the solvability of the \( L^2 \) Dirichlet, regularity, and Neumann problems for \( L_\varepsilon(u_\varepsilon) = 0 \) in \( Ω \) with optimal estimates uniform in \( \varepsilon > 0 \).

1 Introduction

This paper continues the study in [20] of elliptic homogenization problems in Lipschitz domains. Let \( Ω \) be a bounded Lipschitz domain in \( \mathbb{R}^d \), \( d \geq 3 \). Consider a family of second order elliptic systems \( L_\varepsilon(u_\varepsilon) = 0 \) in \( Ω \), where \( u_\varepsilon = (u_1^\varepsilon, \ldots, u_m^\varepsilon) \) and

\[
L_\varepsilon = -\frac{\partial}{\partial x_i} \left[ a_{ij}^{\alpha\beta}(X) \frac{\partial}{\partial x_j} \right] = -\text{div} \left[ A(X) \nabla \right], \quad \varepsilon > 0. \tag{1.1}
\]

We will assume that the coefficient matrix \( A(X) = (a_{ij}^{\alpha\beta}(X)) \) is real and satisfies the ellipticity condition,

\[
\mu |\xi|^2 \leq a_{ij}^{\alpha\beta}(X)\xi_i^\alpha\xi_j^\beta \leq \frac{1}{\mu} |\xi|^2 \quad \text{for} \; X \in \mathbb{R}^d \; \text{and} \; \xi = (\xi_i^\alpha) \in \mathbb{R}^{dm}, \tag{1.2}
\]

where \( \mu > 0 \), and the periodicity condition,

\[
A(X + Z) = A(X) \quad \text{for} \; X \in \mathbb{R}^d \; \text{and} \; Z \in \mathbb{Z}^d. \tag{1.3}
\]

We shall also impose the smoothness condition,

\[
|A(X) - A(Y)| \leq \tau |X - Y|^\lambda \quad \text{for some} \; \lambda \in (0, 1) \; \text{and} \; \tau \geq 0, \tag{1.4}
\]

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and the symmetry condition $A^* = A$, i.e.,
\[ a_{ij}^{\alpha\beta}(X) = a_{ij}^{\beta\alpha}(X) \quad \text{for } 1 \leq i, j \leq d \text{ and } 1 \leq \alpha, \beta \leq m. \] (1.5)

Under these conditions, we establish the solvability of the $L^2$ Dirichlet, regularity and Neumann problems for $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega$ with optimal estimates that are uniform in the parameter $\varepsilon > 0$.

We say $A \in \Lambda(\mu, \lambda, \tau)$ if it satisfies conditions (1.2), (1.3) and (1.4). The following are the main results of this paper.

**Theorem 1.1.** Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$, $d \geq 3$ with connected boundary. Let $L_\varepsilon = -\text{div}(A(\varepsilon^{-1}X)\nabla)$ with $A \in \Lambda(\mu, \lambda, \tau)$ and $A^* = A$. Then for any $f \in L^2(\partial\Omega, \mathbb{R}^m)$, there exists a unique $u_\varepsilon$ such that $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega$, $(\nabla u_\varepsilon)^* \in L^2(\partial\Omega)$ and $u_\varepsilon = f$ n.t. on $\partial\Omega$. Moreover, the solution $u_\varepsilon$ satisfies the estimate $\|(u_\varepsilon)^*\|_2 \leq C\|f\|_2$ with constant $C$ independent of $\varepsilon > 0$. Furthermore, $u_\varepsilon$ may be represented by a double layer potential with density $g_\varepsilon \in L^2(\partial\Omega, \mathbb{R}^m)$ and $\|g_\varepsilon\|_2 \leq C\|f\|_2$.

Here $(u_\varepsilon)^*$ denotes the usual nontangential maximal function of $u_\varepsilon$ and $\| \cdot \|_p$ the norm in $L^p(\partial\Omega)$. By $u_\varepsilon = f$ n.t. on $\partial\Omega$, we mean that $u$ converges to $f$ nontangentially.

**Theorem 1.2.** Suppose that $\Omega$ and $A$ satisfy the same conditions as in Theorem 1.1. Then for any $f \in W^{1,2}(\partial\Omega, \mathbb{R}^m)$, there exists a unique $u_\varepsilon$ such that $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega$, $(\nabla u_\varepsilon)^* \in L^2(\partial\Omega)$ and $u_\varepsilon = f$ n.t. on $\partial\Omega$. Moreover, the solution $u_\varepsilon$ satisfies the estimate $\|(\nabla u_\varepsilon)^*\|_2 \leq C\|\nabla_{\text{tan}} f\|_2$ with $C$ independent of $\varepsilon > 0$. Furthermore, $\nabla u_\varepsilon$ exists n.t. on $\partial\Omega$ and $u_\varepsilon$ may be represented by a single layer potential with density $g_\varepsilon \in L^2(\partial\Omega, \mathbb{R}^m)$ and $\|g_\varepsilon\|_2 \leq C\{\|\nabla_{\text{tan}} f\|_2 + \|\sigma(\partial\Omega)\|_{1,\infty}\|f\|_2\}$. Let $\nabla_{\text{tan}} u_\varepsilon$ denote the conormal derivative associated with the operator $L_\varepsilon$. We will use $L^p_0(\partial\Omega, \mathbb{R}^m)$ to denote the subspace of functions in $L^p(\partial\Omega, \mathbb{R}^m)$ with mean value zero.

**Theorem 1.3.** Suppose that $\Omega$ and $A$ satisfy the same conditions as in Theorem 1.1. Then for any $f \in L^2_0(\partial\Omega, \mathbb{R}^m)$, there exists a unique $u_\varepsilon$, unique up to constants, such that $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega$, $(\nabla u_\varepsilon)^* \in L^2(\partial\Omega)$ and $\frac{\partial u_\varepsilon}{\partial \nu} = f$ n.t. on $\partial\Omega$. Moreover, the solution $u_\varepsilon$ satisfies the estimate $\|(\nabla u_\varepsilon)^*\|_2 \leq C\|f\|_2$ with $C$ independent of $\varepsilon > 0$. Furthermore, $\nabla u_\varepsilon$ exists n.t. on $\partial\Omega$ and $u_\varepsilon$ may be represented by a single layer potential with density $g_\varepsilon \in L^2_0(\partial\Omega, \mathbb{R}^m)$ and $\|g_\varepsilon\|_2 \leq C\|f\|_2$.

A few remarks are in order.

**Remark 1.4.** In the case of $m = 1$, the $L^p$ Dirichlet problem for $L_\varepsilon(u_\varepsilon) = 0$ in Lipschitz domains with uniform estimate $\|(u_\varepsilon)^*\|_p \leq C\|u_\varepsilon\|_p$ was solved for $2 - \delta < p < \infty$ by Dahlberg [8], who extended an earlier work of Avellaneda and Lin [2] for domains satisfying the uniform exterior ball condition. Recently the authors initiated the study of the $L^p$ Neumann and regularity problems for $L_\varepsilon(u_\varepsilon) = 0$ with uniform estimates on $\|(\nabla u_\varepsilon)^*\|_p$ in [20]. Under the assumption that $A$ is elliptic, symmetric, periodic and satisfies a certain square-Dini condition, we solve the $L^p$ Neumann and regularity problems in Lipschitz domains for the sharp range $1 < p < 2 + \delta$. A new proof of Dahlberg’s theorem on the Dirichlet problem is also given in [20]. We mention that $L^p$ Neumann and regularity problems for a general second order elliptic equation were formulated and studied in [18, 19] (see [17] for references on related work on the $L^p$ boundary value problems with minimal smoothness assumptions).
Remark 1.5. Theorems 1.1, 1.2 and 1.3 extend the analogous results for the second order elliptic systems with constant coefficients satisfying (1.2) in Lipschitz domains [27, 10, 12, 13, 14] (in the constant coefficient case, some results also hold when (1.2) is relaxed to the so-called Legendre-Hadamard ellipticity condition; see [10, 28, 14]). As in the case of elliptic systems with constant coefficients, our results for elliptic systems with periodic coefficients are established by the method of layer potentials - the classical method of integral equations. We point out that the use of layer potentials in the periodic setting relies on two crucial developments. The first one is the proof of Coifman-McIntosh-Meyer [7] of the $L^p$ boundedness of the Cauchy integrals on Lipschitz curves. By the method of rotation this gives the $L^p$ boundedness of layer potentials on Lipschitz surfaces for elliptic systems with constants coefficients. Using the method of freezing coefficients, one also obtains the $L^p$ boundedness of layer potentials on Lipschitz surfaces, in small scales, for systems with variable coefficients (see e.g. [21, 22] on the use of layer potentials in the periodic setting).

Remark 1.6. Note that the estimates in terms of nontangential maximal functions,

$$
\| (u_\varepsilon)^* \|_2 \leq C \| u_\varepsilon \|_2, \quad \| (\nabla u_\varepsilon)^* \|_2 \leq C \| \nabla_{\text{tan}} u_\varepsilon \|_2, \quad \| (\nabla_\varepsilon)^* \|_2 \leq C \| \partial u_\varepsilon \|_{\varepsilon} \|_{\varepsilon} \|_2
$$

in Theorems 1.1, 1.2 and 1.3 are scale-invariant. Thus by a simple rescaling argument, one may reduce the proof of Theorems 1.1, 1.2 and 1.3 to the case $\varepsilon = 1$, provided that in this special case one can show that the constant $C$ in (1.6) depends only on $d$, $m$, $\mu$, $\lambda$, $\tau$ and the Lipschitz character of $\Omega$.

With the availability of the method of layer potentials and following the approach for elliptic systems with constant coefficients in Lipschitz domains, we are led to the Rellich estimates $\| \nabla u \|_2 \leq C \| \nabla_{\text{tan}} u \|_2$ and $\| \nabla u \|_2 \leq C \| \partial u \|_{\varepsilon} \|_{\varepsilon} \|_2$ for suitable solutions of $\mathcal{L}(u) = 0$, where $\mathcal{L} = \mathcal{L}_1$. Let $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a Lipschitz function such that $\psi(0) = 0$ and $\| \nabla \psi \|_\infty \leq M$. By localization techniques we may further reduce the problem to proving the following $a$-priori estimates for solutions of $\mathcal{L}(u) = 0$ in $D(2r)$,

$$
\int_{\Delta(r)} |\nabla u|^2 \, d\sigma \leq C \int_{\Delta(2r)} |\partial u|_{\varepsilon}^2 \, d\sigma + \frac{C}{r} \int_{D(2r)} |\nabla u|^2 \, dX,
$$

$$
\int_{\Delta(r)} |\nabla u|^2 \, d\sigma \leq C \int_{\Delta(2r)} |\nabla_{\text{tan}} u|^2 \, d\sigma + \frac{C}{r} \int_{D(2r)} |\nabla u|^2 \, dX,
$$

where $\Delta(r) = \{(x', \psi(x')) \in \mathbb{R}^d : |x'| < r\}$, $D(r) = \{(x', x_d) : |x'| < r$ and $\psi(x') < x_d < 10\sqrt{d}(M + 1)r\}$, and the constant $C$ depends only on $d$, $m$, $\mu$, $\lambda$, $\tau$ and $M$. 

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The proof of (1.7) is divided into two parts. Part one deals with the small-scale case \(0 \leq r \leq 1\). We mention that in this case, if \(A \in C^1(\mathbb{R}^d)\), the desired estimates follow readily from the same Rellich-Necas-Payne-Weinberger formulas as in the case of constant coefficients, with constant \(C\) depending on \(\|\nabla A\|_{\infty}\). Our proof of (1.7) in small scales for Hölder continuous coefficients, which also uses the Rellich formulas, involves a delicate three-step approximation argument (see Sections 6 and 7). This is needed to obtain the correct dependence on the constant \(C\) in (1.7), in contrast to arguments in [22] where the dependence of the constants are not clear. Part two, which is given in Section 8, treats the large-scale case \(r > 1\) and uses the periodicity assumption on \(A\). Here we first apply the small-scale estimates and reduce the problem to the control of the integral of \(|\nabla u|^2\) on a boundary layer \(\{(x', x_d) : |x'| < r \text{ and } \psi(x') < x_d < \psi(x') + 1\}\). The desired estimates of the integral over the boundary layer follow from certain integral identities we developed in [20] for elliptic operators with periodic coefficients. These identities may be regarded as Rellich type identities for operators with \(x_d\)-periodic coefficients. We point out that in the case of constant coefficients, Rellich identities are usually derived by using integration by parts on some forms involving \(\partial/\partial x_d\). The basic insight here is to replace the \(x_d\) derivative of \(u\) by the difference \(Q(u)(x', x_d) = u(x', x_d + 1) - u(x', x_d)\). The periodicity of \(A\) is used in the fact that \(Q(u)\) is a solution whenever \(u\) is a solution. In [20] the approach outlined above was used to solve the \(L^p\) boundary value problems for elliptic equations with periodic coefficients by the method of \(L\)-harmonic measures (see Remark 1.4). With the method of layer potentials, the approach works equally well for elliptic systems with periodic coefficients, at least in the case \(p = 2\). It is worth mentioning that the symmetry assumption (1.5), although not needed for the uniform boundedness of layer potentials, is essential for the Rellich estimates both in small and large scales, which are needed for the uniform invertibility.

By the stability of Fredholm properties of operators on a complex interpolation scale, the \(L^2\) results in Theorems 1.1, 1.2 and 1.3 extend easily to the \(L^p\) setting for \(p \in (2 - \delta, 2 + \delta)\), where \(\delta > 0\) depends only on \(d, m, \mu, \lambda, \tau\) and the Lipschitz character of \(\Omega\). Using the \(L^p\) techniques developed in [9, 24, 25] for constant coefficients, we may further extend the results for the Dirichlet problem with \(L^p\) boundary data to the range \(2 < p \leq \infty\) if \(d = 3\), and to \(2 < p < \frac{2(d-1)}{d-3} + \delta\) if \(d \geq 4\). Similarly, the \(L^p\) Neumann and regularity problems for \(\mathcal{L}_\epsilon(u_\epsilon) = 0\) may be solved for \(1 < p < 2\) if \(d = 3\), and for \(\frac{2(d-1)}{d+1} - \delta < p < 2\) if \(d \geq 4\). These results, as well as uniform Sobolev and Besov estimates for \(\mathcal{L}_\epsilon\) in nonsmooth domains (see [26] for uniform \(W^{1,p}\) estimates in the case \(m = 1\)) will appear elsewhere.

We end this section with a few notations and definitions that will be used throughout the paper.

For a ball \(B = B(X, r)\) in \(\mathbb{R}^d\) with center \(X\) and radius \(r\), we will denote \(B(X, tr)\) by \(tB\). If \(0 < \lambda < 1\), then \(\|f\|_{C^{0,\lambda}(\Omega)} = \inf\{M : |f(X) - f(Y)| \leq M|X - Y|^\lambda\text{ for }X, Y \in \Omega\}\). We will let \(\|f\|_{C^{0,\lambda}(\mathbb{R}^d)} = \|f\|_{L^\infty(\mathbb{R}^d)} + \|f\|_{C^{0,\lambda}(\mathbb{R}^d)}\).

Let \(\Omega\) be a bounded Lipschitz domain. We say \(\Omega \in \mathcal{M}(M, N)\) for some \(M > 0\) and \(N > 10\), if there exist \(r > 0\) and \(\{P_i : i = 1, \ldots, N\} \subset \partial\Omega\) such that \(\partial\Omega \subset \bigcup_i B(P_i, r)\) and for each \(i\), there exists a coordinate system, obtained from the standard Euclidean system through translation and rotation, so that \(P_i = (0, 0)\) and

\[
B(P_i, C_Mr) \cap \Omega = B(P_i, C_Mr) \cap \{(x', x_d) \in \mathbb{R}^d : x' \in \mathbb{R}^{d-1} \text{ and } x_d > \psi_i(x')\},
\]

where \(C_M = 10(M + 1), \psi_i : \mathbb{R}^{d-1} \rightarrow \mathbb{R}\) is a Lipschitz function, \(\psi_i(0) = 0\) and \(\|\nabla \psi_i\|_{\infty} \leq M\).
Note that if $Ω ∈ Π(M, N)$, then its dilation $\varepsilon Ω = \{ \varepsilon X : X ∈ Ω \} ∈ Π(M, N)$ for any $\varepsilon > 0$. A constant $C$ is said to depend on the Lipschitz character of $Ω$ if there exist $M$ and $N$ such that $Ω ∈ Π(M, N)$ and the constant can be made uniform for any Lipschitz domain in $Π(M, N)$. We will call $C$ a “good” constant if it depends at most on $d$, $m$, $µ$ in (1.2), $λ$ and $τ$ in (1.4), and the Lipschitz character of $Ω$.

Finally the summation convention will be used throughout this paper.

2 Matrix of fundamental solutions

Let $L = L^A = -\text{div}(A(X)\nabla)$. Under the ellipticity condition (1.2) and Hölder condition (1.4) on $A(X)$, it is well known that the gradients of weak solutions to $L(u) = \text{div}(f)$ are locally Hölder continuous, provided that $f$ is Hölder continuous. More precisely, let $B = B(X_0, R)$ for some $X_0 ∈ \mathbb{R}^d$ and $0 < R ≤ 1$. There exists $C = C(d, m, µ, λ, τ) > 0$ such that if $u ∈ W^{1,2}(2B)$ is a weak solution to $L(u) = \text{div}(f)$ in $2B$, then

$$\|\nabla u\|_{C^{0, λ}(2B)} ≤ \frac{C}{R^{1+λ}} \left\{ \frac{1}{|2B|} \int_{2B} |u|^2 \, dX \right\}^{1/2} + C\|f\|_{C^{0, λ}(2B)}$$  \hspace{1cm} (2.1)

(see e.g. [13], p.88). Also, the gradient $\nabla u$ is locally bounded and

$$\|\nabla u\|_{L^{∞}(B)} ≤ \frac{C}{R} \left\{ \frac{1}{|2B|} \int_{2B} |u|^2 \, dX \right\}^{1/2} + C R^λ\|f\|_{C^{0, λ}(2B)}.$$  \hspace{1cm} (2.2)

If $R > 1$, estimates (2.1)-(2.2) still hold. However the constant $C$ may depend on $R$. With the additional periodicity condition (1.3), Avellaneda and Lin, among other things, were able to establish the following global gradient estimate in [1] (p. 826).

Lemma 2.1. Let $B = B(X_0, R)$ for some $X_0 ∈ \mathbb{R}^d$ and $R > 0$. Suppose that $u ∈ W^{1,2}(2B)$ is a weak solution to $\text{div}(A \nabla u) = 0$ in $2B$ for some $A ∈ Λ(µ, λ, τ)$. Then

$$\sup_B |\nabla u| ≤ \frac{C}{R} \left\{ \frac{1}{|2B|} \int_{2B} |u|^2 \, dX \right\}^{1/2},$$  \hspace{1cm} (2.3)

where $C$ depends only on $d$, $m$, $µ$, $λ$ and $τ$.

Using Lemma 2.1 one can construct a matrix-valued function $Γ(X, Y) = (Γ^{αβ}(X, Y))_{m×m}$ such that for each $Y ∈ \mathbb{R}^d$, $\nabla_X Γ(X, Y)$ is locally integrable and

$$\phi^γ(Y) = \int_{\mathbb{R}^d} d_{ij}^{αβ}(X) \frac{∂}{∂x_j} Γ^{βγ}(X, Y) \frac{∂}{∂x_i} φ^α(X) \, dX$$  \hspace{1cm} (2.4)

for $φ = (φ^1, \ldots, φ^m) ∈ C^{∞}_0(\mathbb{R}^d, \mathbb{R}^m)$. Moreover, $Γ(X, Y)$ satisfies the estimates

$$|Γ(X, Y)| ≤ C|X - Y|^{2-d},$$

$$|\nabla_X Γ(X, Y)| + |\nabla_Y Γ(X, Y)| ≤ C|X - Y|^{1-d},$$ \hspace{1cm} (2.5)
where \( C \) depends only on \( d, m, \mu, \lambda \) and \( \tau \) (see e.g. [16]). The function \( \Gamma(X, Y) = \Gamma_A(X, Y) \) is called the matrix of fundamental solutions for the operator \( L \) in \( \mathbb{R}^d \), with pole at \( Y \). Note that

\[
(\Gamma_A(X, Y))^* = \Gamma_{A^*}(Y, X),
\]

where \( A^* \) denotes the adjoint matrix of \( A \). Since \( \nabla_Y \Gamma(\cdot, Y) \) is a weak solution in \( \mathbb{R}^d \setminus \{Y\} \), we also have

\[
|\nabla_X \nabla_Y \Gamma(X, Y)| \leq C|X - Y|^{-d}
\]

for any \( X, Y \in \mathbb{R}^d, X \neq Y \). (2.7)

If \( E \) is a real constant matrix satisfying (1.2), we will let \( \Theta(X, Y; E) \) to denote \( \Gamma_E(X, Y) \), the matrix of fundamental solutions for the operator \( -\text{div}(E \nabla) \). Note that \( \Theta(X, Y; E) = \Theta(X - Y, 0; E) = \Theta(Y - X, 0; E) = \Theta(Y, X; E) \). Also, \( \Theta(X, 0; E) \) is homogeneous of degree \( 2 - d \) in \( X \) and

\[
|\nabla_X^N \Theta(X, 0; E)| \leq C|X|^{2-d-N}
\]

for any \( N \geq 0 \), where \( C \) depends only on \( d, m, \mu \) and \( N \) (see [23]). Moreover, if \( E, \tilde{E} \) are two constant matrices satisfying (1.2), then

\[
|\nabla_X^N \Theta(X, 0; E) - \nabla_X^N \Theta(X, 0; \tilde{E})| \leq C\|E - \tilde{E}\||X|^{2-d-N},
\]

where \( C = C(d, m, \mu, N) > 0 \). We remark that estimate (2.9) may be proved by an argument similar to that in the proof of Lemma 2.6.

For a function \( F = F(X, Y, Z) \), we will use the notation

\[
\nabla_1 F(X, Y, Z) = \nabla_X F(X, Y, Z) \text{ and } \nabla_2 F(X, Y, Z) = \nabla_Y F(X, Y, Z).
\]

The following lemma describes the local behavior of \( \Gamma_A(X, Y) \).

**Lemma 2.2.** Let \( A \in \Lambda(\mu, \lambda, \tau) \). Then for any \( X, Y \in \mathbb{R}^d \),

\[
|\Gamma_A(X, Y) - \Theta(X, Y; A(X))| \leq C|X - Y|^{2-d+\lambda},
\]

\[
|\nabla_X \Gamma_A(X, Y) - \nabla_1 \Theta(X, Y; A(X))| \leq C|X - Y|^{1-d+\lambda},
\]

\[
|\nabla_X^2 \Gamma_A(X, Y) - \nabla_1 \Theta(X, Y; A(Y))| \leq C|X - Y|^{1-d+\lambda},
\]

where \( C > 0 \) depends only on \( d, m, \mu, \lambda \) and \( \tau \).

**Proof.** Let \( \tilde{A} \in \Lambda(\mu, \lambda, \tau) \). Then

\[
\Gamma_{\tilde{A}}(X, Y) - \Gamma_A(X, Y) = \int_{\mathbb{R}^d} \frac{\partial}{\partial z_i} \Gamma_{\tilde{A}}(X, Z) \{a_{ij}^{\beta\gamma}(Z) - \tilde{a}_{ij}^{\beta\gamma}(Z)\} \frac{\partial}{\partial z_j} \Gamma_{\tilde{A}}(Z, Y) \, dZ
\]

for any \( X, Y \in \mathbb{R}^d \) (see e.g. [16]). It follows from (2.11) and estimates (2.5) and (2.7) that

\[
|\Gamma_{\tilde{A}}(X, Y) - \Gamma_A(X, Y)| \leq C \int_{\mathbb{R}^d} \frac{|A(Z) - \tilde{A}(Z)|}{|Z - X|^{d-1}|Z - Y|^{d-1}} \, dZ
\]

and

\[
|\nabla_1 \Gamma_{\tilde{A}}(X, Y) - \nabla_1 \Gamma_A(X, Y)| \leq C \int_{\mathbb{R}^d} \frac{|A(Z) - \tilde{A}(Z)|}{|Z - X|^d|Z - Y|^{d-1}} \, dZ.
\]
To show the first inequality in (2.10), we fix $X \in \mathbb{R}^d$ and let $\tilde{A} = A(X)$. Then $\Gamma_\tilde{A}(X, Y) = \Theta(X, Y; A(X))$ and by (2.12),

$$|\Gamma_A(X, Y) - \Theta(X, Y; A(X))| \leq C \int_{\mathbb{R}^d} \frac{|A(Z) - A(X)|}{|Z - X|^d |Z - Y|^d} dZ \leq C \int_{\mathbb{R}^d} \frac{|Z - X|^d |Z - Y|^d}{|Z - X|^d |Z - Y|^d} dZ \leq C |X - Y|^{2d + \lambda}.$$ 

The second inequality in (2.10) follows from (2.13) in the same manner. Note that by (2.9),

$$|\nabla \Theta(X, Y; A(X)) - \nabla \Theta(X, Y; A(Y))| \leq C |X - Y|^{1 - d + \lambda}. \tag{2.14}$$

The third inequality in (2.10) follows from the second and (2.14).

**Remark 2.3.** If we fix $Y \in \mathbb{R}^d$ and let $\tilde{A} = A(Y)$, the same argument as in the proof of Lemma 2.2 yields that

$$|\Gamma_A(X, Y) - \Theta(X, Y; A(Y))| \leq C |X - Y|^{2d + \lambda},$$

$$|\nabla Y \Gamma_A(X, Y) - \nabla Y \Theta(X, Y; A(Y))| \leq C |X - Y|^{1 - d + \lambda}, \tag{2.15}$$

$$|\nabla Y \Gamma_A(X, Y) - \nabla Y \Theta(X, Y; A(Y))| \leq C |X - Y|^{1 - d + \lambda}.$$

To study the behavior of $\Gamma_A(X, Y)$ for $|X - Y| \geq 1$, we need to introduce the matrix of correctors, $\chi = \chi(X) = (\chi_i^{\alpha \beta}(X)), 1 \leq i \leq d, 1 \leq \alpha, \beta \leq m$. Here for each $i$ and $\alpha$, $\chi_i^{\alpha} = (\chi_i^{\alpha 1}, \ldots, \chi_i^{\alpha m})$ is the solution of the following cell problem:

$$\begin{cases}
\mathcal{L}(\chi_i^{\alpha}) = \mathcal{L}(e^{\alpha} x_i) \quad \text{in} \ \mathbb{R}^d, \\
\chi_i^{\alpha} \text{ is periodic with respect to} \ Z^d, \\
\int_{[0,1]^d} \chi_i^{\alpha} dX = 0,
\end{cases} \tag{2.16}$$

where $e^{\alpha} = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^m$ with 1 in the $\alpha^{th}$ position. Note by estimate (2.3), $\|\nabla \chi\|_{\infty} \leq C$ for some $C = C(d, m, \mu, \lambda, \tau)$. Let $\mathcal{L}_0 = -\text{div}(A_0 \nabla)$ denote the homogenized elliptic operator associated with $\{\mathcal{L}_\varepsilon\}$, where $A_0$ is a constant matrix in $\Lambda(\mu, \lambda, \tau)$ (see e.g. [6], p.121 for the explicit formula of $A_0$, given in terms of $a_{ij}^{\alpha \beta}(X)$ and $\chi_i^{\alpha \beta}(X)$).

The following lemma was proved in [5].

**Lemma 2.4.** Let $A \in \Lambda(\mu, \lambda, \tau)$. Then

$$|\Gamma_A^{\alpha \beta}(X, Y) - \Gamma_{A_0}^{\alpha \beta}(X, Y)| \leq C |X - Y|^{2d - \lambda_0},$$

$$\left| \frac{\partial}{\partial x_i} \Gamma_A^{\alpha \beta}(X, Y) - \frac{\partial}{\partial x_i} \Gamma_{A_0}^{\alpha \beta}(X, Y) \right| \leq C |X - Y|^{1 - d - \lambda_0}, \tag{2.17}$$

for any $X, Y \in \mathbb{R}^d$, where $C > 0$ and $\lambda_0 \in (0, 1)$ depend only on $d, m, \mu, \lambda$ and $\tau$. 

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Remark 2.5. Let $I$ denote the identity matrix (or the identity operator). For brevity the second estimate in (2.17) may be written as
\begin{equation}
|\nabla_X \Gamma_A(X, Y) - (I + \nabla \chi(X)) \nabla_X \Gamma_{A_0}(X, Y)| \leq C|X - Y|^{1 - d - \lambda_0}.
\end{equation}
Using (2.6), one may also deduce that
\begin{equation}
|\nabla_Y (\Gamma_A(X, Y))^* - (I + \nabla \chi^*(Y)) \nabla_Y (\Gamma_{A_0}(X, Y))^*| \leq C|X - Y|^{1 - d - \lambda_0},
\end{equation}
where $\chi^*$ is the matrix of correctors for the adjoint operator $\mathcal{L}^* = -\text{div}(A^*(X) \nabla)$.

The rest of this section is devoted to the estimate of $\Gamma_A(X, Y) - \Gamma_{A_0}(X, Y)$ and its derivatives when $A$ is close to $\tilde{A}$ in the space $C^\lambda(\mathbb{R}^d)$. The results on the derivative estimates are local and will be used in an approximation argument for domains $\Omega$ with $\text{diam}(\Omega) \leq 1$.

Lemma 2.6. Let $A, \tilde{A} \in \Lambda(\mu, \lambda, \tau)$. Then for any $X, Y \in \mathbb{R}^d$,
\begin{equation}
|\Gamma_A(X, Y) - \Gamma_{\tilde{A}}(X, Y)| \leq C\|A - \tilde{A}\|_{\infty}|X - Y|^{2 - d},
\end{equation}
where $C = C(d, m, \mu, \lambda, \tau) > 0$. Moreover, for each $R \geq 1$, there exists a constant $C_R$ depending on $d, m, \mu, \lambda, \tau$ and $R$ such that
\begin{equation}
\begin{aligned}
|\nabla_X \Gamma_A(X, Y) - \nabla_X \Gamma_{\tilde{A}}(X, Y)| &\leq C_R\|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)}|X - Y|^{1 - d}, \\
|\nabla_X \nabla_Y \Gamma_A(X, Y) - \nabla_X \nabla_Y \Gamma_{\tilde{A}}(X, Y)| &\leq C_R\|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)}|X - Y|^{-d},
\end{aligned}
\end{equation}
for any $X, Y \in \mathbb{R}^d$ with $|X - Y| \leq R$.

Proof. It follows from estimate (2.12) that
\begin{equation}
|\Gamma_A(X, Y) - \Gamma_{\tilde{A}}(X, Y)| \leq C\|A - \tilde{A}\|_{\infty} \int_{\mathbb{R}^d} \frac{dZ}{|X - Z|^{d - 1}|Z - Y|^{d - 1}} \\
\leq C\|A - \tilde{A}\|_{\infty}|X - Y|^{2 - d}.
\end{equation}

To see (2.21), we fix $X_0, Y_0 \in \mathbb{R}^d$ and consider $u(X) = \Gamma_A(X, Y_0) - \Gamma_{\tilde{A}}(X, Y_0)$ in $2B$, where $B = B(X_0, r/4)$ and $r = |X_0 - Y_0| \leq R$. It follows from (2.20) that $\|u\|_{L^\infty(2B)} \leq Cr^{2 - d}\|A - \tilde{A}\|_{\infty}$. Let $w(X) = \Gamma_{\tilde{A}}(X, Y_0)$. By (2.5), $\|\nabla w\|_{L^\infty(2B)} \leq Cr^{1 - d}$. In view of (2.1), we also have $\|\nabla w\|_{C^{\alpha, \lambda}(2B)} \leq C_Rr^{1 - d - \lambda}$. Since $\mathcal{L}^A(u) = \text{div}(A\nabla w) = \text{div}((A - \tilde{A})\nabla w)$ in $2B$, it follows from (2.2) that
\begin{equation}
\|\nabla u\|_{L^\infty(B)} \leq C r^{-1}\|u\|_{L^\infty(2B)} + Cr^\lambda\|A - \tilde{A}\|_{C^{\alpha, \lambda}(2B)}\|\nabla w\|_{C^{\alpha, \lambda}(2B)} \\
\leq C r^{-1}\|A - \tilde{A}\|_{\infty} + Cr^\lambda\|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)}\|\nabla w\|_{C^\lambda(2B)} \\
\leq C r^{-1}\|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)},
\end{equation}
where $C$ may depend on $R$. This gives the first inequality in (2.21). The second inequality in (2.21) follows in the same manner. Indeed, let $v(X) = \nabla_Y \Gamma_A(X, Y_0) - \nabla_Y \Gamma_{\tilde{A}}(X, Y_0)$ and $g(X) = \nabla_Y \Gamma_{\tilde{A}}(X, Y_0)$. Then $\mathcal{L}^A(v) = \text{div}((A - \tilde{A})\nabla g)$ in $2B$. Thus,
\begin{equation}
\|\nabla v\|_{L^\infty(B)} \leq C r^{-1}\|v\|_{L^\infty(2B)} + Cr^\lambda\|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)}\|\nabla g\|_{C^\lambda(2B)}.
\end{equation}
It follows from (2.6) and the first inequality in (2.21) that \( \|v\|_{L^\infty(B)} \leq C r^{1-d} \|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)}. \)
By (2.1) and (2.5), we see that \( \|\nabla g\|_{C^\lambda(2B)} \leq C r^{\alpha-d}. \) Hence

\[
\|\nabla v\|_{L^\infty(B)} \leq C r^{-d} \|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)},
\]

where \( C \) may depend on \( R. \) This completes the proof. \( \square \)

Define

\[
\Pi_A(X,Y) = \nabla_X \Gamma_A(X,Y) - \nabla_1 \Theta(X,Y; A(X)).
\]

**Lemma 2.7.** Let \( A, \tilde{A} \in \Lambda(\mu, \lambda, \tau) \) and \( R \geq 1. \) Then for \( X, Y \) with \(|X - Y| \leq R\),

\[
|\Pi_A(X,Y) - \Pi_{\tilde{A}}(X,Y)| \leq C_R \|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)} |X - Y|^{1-d+\lambda},
\]

where \( C_R = C(d, m, \mu, \lambda, \tau, R) > 0. \)

**Proof.** Let \( \Omega = B(X_0, 2R). \) For any fixed \( P \in B(X_0, R) \), it follows from integration by parts that

\[
\Gamma^\alpha_{\beta}(X,Y) - \Theta^\alpha_{\beta}(X,Y; A(P)) = \int_{\Omega} \frac{\partial}{\partial z_i} \Gamma^\alpha_{\beta}(X,Z) \{ a_{ij}^\beta(P) - a_{ij}^\beta(Z) \} \frac{\partial}{\partial z_j} \Theta^{\gamma}(Z,Y; A(P)) \, dZ
\]
\[
+ \int_{\partial \Omega} \frac{\partial}{\partial z_i} \Gamma^\alpha_{\beta}(X,Z) a_{ij}^\gamma(Z) n_j(Z) \Theta^{\gamma}(Z,Y; A(P)) \, d\sigma(Z)
\]
\[
- \int_{\partial \Omega} \Gamma^\alpha_{\beta}(X,Z) n_i(Z) a_{ij}^\gamma(P) \frac{\partial}{\partial z_j} \Theta^{\gamma}(Z,Y; A(P)) \, d\sigma(Z),
\]

\[\text{(2.24)}\]

where \( n = (n_1, \ldots, n_d) \) denotes the unit outward normal to \( \partial \Omega. \) We now take the derivative with respect to \( X \) on the both side of (2.24) and then choose \( X = P. \) With abuse of notation we may write

\[
\Pi_A(X,Y) = \int_{\Omega} \nabla_X \nabla_Z \Gamma_A(X,Z) \{ A(X) - A(Z) \} \nabla_1 \Theta(Z,Y; A(X)) \, dZ
\]
\[
+ \int_{\partial \Omega} \nabla_Z \nabla_X \Gamma_A(X,Z) A(Z) n(Z) \Theta(Z,Y; A(X)) \, d\sigma(Z)
\]
\[
- \int_{\partial \Omega} \nabla_X \Gamma_A(X,Z) n(Z) A(X) \nabla_Z \Theta(Z,Y; A(X)) \, d\sigma(Z).
\]

\[\text{(2.25)}\]

To estimate \( \Pi_A(X,Y) - \Pi_{\tilde{A}}(X,Y) \), we split its solid integrals as \( I_1 + I_2 + I_3, \) where

\[
I_1 = \int_{\Omega} \{ \nabla_X \nabla_Z \Gamma_A(X,Z) - \nabla_X \nabla_Z \Gamma_{\tilde{A}}(X,Z) \} \{ A(X) - A(Z) \} \nabla_1 \Theta(Z,Y; A(X)) \, dZ,
\]
\[
I_2 = \int_{\Omega} \{ \nabla_X \nabla_Z \Gamma_{\tilde{A}}(X,Z) \} \{ A(X) - A(Z) - (\tilde{A}(X) - \tilde{A}(Z)) \} \nabla_1 \Theta(Z,Y; A(X)) \, dZ,
\]
\[
I_3 = \int_{\Omega} \{ \nabla_X \nabla_Z \Gamma_{\tilde{A}}(X,Z) \} \{ \tilde{A}(X) - \tilde{A}(Z) \} \{ \nabla_1 \Theta(Z,Y; A(X)) - \nabla_1 \Theta(Z,Y; \tilde{A}(X)) \} \, dZ.
\]
It follows from (2.21) that
\[ |I_1| \leq C_R \|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)} \int_{\Omega} \frac{dZ}{|X - Z|^{d-\lambda}|Z - Y|^{d-1}} \]
\[ \leq C_R \|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)} |X - Y|^{1-d+\lambda}. \]

Similarly, by estimates (2.7) and (2.9),
\[ |I_2| \leq C \|A - \tilde{A}\|_{C^0(\mathbb{R}^d,\lambda)} |X - Y|^{1-d+\lambda}, \]
\[ |I_3| \leq C \|A - \tilde{A}\|_{\infty} |X - Y|^{1-d+\lambda}. \]

Finally we may split the surface integrals in \(\Pi_A(X,Y) - \Pi_{\tilde{A}}(X,Y)\) in a similar fashion to show that they are bounded by \(C_R \|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)}\).

Remark 2.8. Let
\[ \Delta_A(X,Y) = \nabla_Y \Gamma_A(X,Y) - \nabla_2 \Theta(X,Y; A(Y)). \] (2.26)

Let \(A, \tilde{A} \in \Lambda(\mu, \lambda, \tau)\) and \(R \geq 1\). Using (2.6), one may deduce from (2.23) that for \(X, Y \in \mathbb{R}^d\) with \(|X - Y| \leq R\),
\[ |\Delta_A(X,Y) - \Delta_{\tilde{A}}(X,Y)| \leq C_R \|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)} |X - Y|^{1-d+\lambda}, \] (2.27)
where \(C_R\) depends only on \(d, m, \mu, \lambda, \tau\) and \(R\).

3 Singular integral operators on Lipschitz surfaces

Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^d\). Consider two singular integral operators on \(\partial\Omega\),
\[ T_A^1(f)(P) = \text{p.v.} \int_{\partial\Omega} \nabla_1 \Gamma_A(P,Y)f(Y) \, d\sigma(Y) \]
\[ := \lim_{\rho \to 0^+} \int_{Y \in \partial\Omega, |Y-P| > \rho} \nabla_1 \Gamma_A(P,Y)f(Y) \, d\sigma(Y), \] (3.1)
\[ T_A^2(f)(P) = \text{p.v.} \int_{\partial\Omega} \nabla_2 \Gamma_A(P,Y)f(Y) \, d\sigma(Y), \]
where \(A \in \Lambda(\mu, \lambda, \tau)\). We also introduce the maximal singular integral operators \(T_{A}^{1,\ast}\) and \(T_{A}^{2,\ast}\) on \(\partial\Omega\), defined by
\[ T_{A}^{1,\ast}(f)(P) = \sup_{\rho > 0} \left| \int_{Y \in \partial\Omega, |Y-P| > \rho} \nabla_1 \Gamma_A(P,Y)f(Y) \, d\sigma(Y) \right|, \]
\[ T_{A}^{2,\ast}(f)(P) = \sup_{\rho > 0} \left| \int_{Y \in \partial\Omega, |Y-P| > \rho} \nabla_2 \Gamma_A(P,Y)f(Y) \, d\sigma(Y) \right|. \] (3.2)

The main purpose of this section is to establish the following.
**Theorem 3.1.** Let \( f \in L^p(\partial \Omega) \) for some \( 1 < p < \infty \). Then \( T_A^1(f)(P) \) and \( T_A^2(f)(P) \) exist for a.e. \( P \in \partial \Omega \) and

\[
\|T_A^{1*}(f)\|_p + \|T_A^{2*}(f)\|_p \leq C_p\|f\|_p,
\]

where \( C_p \) depends only on \( d, m, \mu, \lambda, \tau, p \) and the Lipschitz character of \( \Omega \).

Let \( \psi : \mathbb{R}^{d-1} \to \mathbb{R} \) be a Lipschitz function with \( \|\nabla \psi\|_{\infty} \leq M \) and

\[
D = \{ (x', x_d) : x' \in \mathbb{R}^{d-1} \text{ and } x_d > \psi(x') \}.
\]

(3.3)

By a partition of unity and rotation of coordinate systems, it suffices to prove Theorem 3.1 when \( \Omega = D \), with constant \( C_p \) depending only on \( d, m, \mu, \lambda, \tau, p \) and \( M \). To this end, the basic idea to treat \( T_A^{1*} \) is to approximate the integral kernel \( \nabla \Gamma_A(P, Y) \) by \( \nabla \Theta(P, Y; A(P)) \) when \( |P - Y| \leq 1 \), and by \((I + \nabla \chi(P))\nabla \Gamma_{A_0}(P, Y) \) when \( |P - Y| \geq 1 \). The operator \( T_A^{2*} \) may be handled in a similar manner.

Let \( \mathcal{M}_{\partial D} \) denote the Hardy-Littlewood maximal operator on \( \partial D \). The proof of Theorem 3.1 relies on the following two lemmas.

**Lemma 3.2.** Let \( \Omega = D \) be given by (3.3). Then for each \( P \in \partial D \),

\[
T_A^{1*}(f)(P) \leq C\mathcal{M}_{\partial D}(f)(P) + 2\sup_{\rho > 0} \int_{Y \in \partial D, |Y - P| > \rho} \nabla \Theta(P, Y; A(P)) f(Y) \, d\sigma(Y)
\]

\[
+ C\sup_{\rho > 0} \int_{Y \in \partial D, |Y - P| > \rho} \nabla \Gamma_{A_0}(P, Y) f(Y) \, d\sigma(Y),
\]

\[
T_A^{2*}(f)(P) \leq C\mathcal{M}_{\partial D}(f)(P) + 2\sup_{\rho > 0} \int_{Y \in \partial D, |Y - P| > \rho} \nabla \Theta(P, Y; A(Y)) f(Y) \, d\sigma(Y)
\]

\[
+ \sup_{\rho > 0} \int_{Y \in \partial D, |Y - P| > \rho} \nabla \Gamma_{A_0}(P, Y) g(Y) \, d\sigma(Y),
\]

where \( C = C(d, m, \mu, \lambda, \tau, M) \) and \( |g| \leq C|f| \) on \( \partial D \).

**Proof.** Fix \( P \in \partial D \) and \( \rho > 0 \). If \( \rho \geq 1 \), we use estimate (2.18) to obtain

\[
|\int_{|Y - P| > \rho} \nabla \Gamma_A(P, Y) f(Y) \, d\sigma(Y)|
\]

\[
\leq C \int_{|Y - P| > \rho} \nabla \Gamma_{A_0}(P, Y) f(Y) \, d\sigma(Y) + C \int_{|Y - P| > \rho} |Y - P|^{1 - d - \lambda_0} |f(Y)| \, d\sigma(Y)
\]

\[
\leq C\sup_{t > 0} \int_{|Y - P| > t} \nabla \Gamma_{A_0}(P, Y) f(Y) \, d\sigma(Y) + C\mathcal{M}_{\partial D}(f)(P).
\]

If \( 0 < \rho < 1 \), we write \( \{||Y - P| > \rho\} = \{||Y - P| > 1\} \cup \{1 \geq |Y - P| > \rho\} \). The integral of \( \nabla \Gamma_A(P, Y) f(Y) \) on \( \{||Y - P| > 1\} \) may be treated as above. To handle the integral on \( \{1 \geq |Y - P| > \rho\} \), we use estimate (2.10) to obtain

\[
|\int_{1 \geq |Y - P| > \rho} \nabla \Gamma_A(P, Y) f(Y) \, d\sigma(Y)|
\]

\[
\leq \int_{1 \geq |Y - P| > \rho} \nabla \Theta(P, Y; A(P)) f(Y) \, d\sigma(Y) + C \int_{|Y - P| \leq 1} |P - Y|^{1 - d - \lambda} |f(Y)| \, d\sigma(Y)
\]

\[
\leq 2\sup_{t > 0} \int_{|Y - P| > t} \nabla \Theta(P, Y; A(P)) f(Y) \, d\sigma(Y) + C\mathcal{M}_{\partial D}(f)(P).
\]

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This gives the desired estimate for $T_{A_0}^{1,*}$. The estimate for $T_{A_0}^{2,*}$ follows from (2.19) and (2.15) in the same manner.

**Lemma 3.3.** Let $K(X,Y)$ be odd in $X$ and homogeneous of degree $1 - d$ in $X$. Assume that for all $0 \leq N \leq N(d)$ where $N(d)$ is sufficiently large, $\nabla_N^2 K(X,Y)$ is continuous on $S^{d-1} \times \mathbb{R}^d$ and $|\nabla_N^2 K(X,Y)| \leq C_0$ for $X \in S^{d-1}$ and $Y \in \mathbb{R}^d$. Let $f \in L^p(\partial D)$ for some $1 < p < \infty$. Define

$$S^1(f)(P) = \text{p.v.} \int_{\partial D} K(P - Y, P)f(Y) \, d\sigma(Y),$$

$$S^2(f)(P) = \text{p.v.} \int_{\partial D} K(P - Y, Y)f(Y) \, d\sigma(Y),$$

$$S^1,(f)(P) = \sup_{\rho > 0} \int_{Y \in \partial D, |Y - P| > \rho} K(P - Y, P)f(Y) \, d\sigma(Y),$$

$$S^2,(f)(P) = \sup_{\rho > 0} \int_{Y \in \partial D, |Y - P| > \rho} K(P - Y, Y)f(Y) \, d\sigma(Y).$$

Then $S^1(f)(P)$ and $S^2(f)(P)$ exist for a.e. $P \in \partial D$ and

$$\|S^1,(f)\|_p + \|S^2,(f)\|_p \leq CC_0\|f\|_p,$$

where $C$ depends only on $d$, $p$ and $M$.

**Proof.** By considering $C_0^{-1}K(X,Y)$, we may clearly assume that $C_0 = 1$. In the special case where the integral kernel $K(X,Y)$ is independent of $Y$, the result is a consequence of [7] on Cauchy integrals on Lipschitz curves. The general case may be deduced from the special case by the spherical harmonic decomposition (see e.g. [21]). Note that only the continuity condition in the variable $Y$ is need for $\nabla_N^2 K(X,Y)$.

We are now in a position to give the proof of Theorem 3.1.

**Proof of Theorem 3.1**

If $A_0 \in \Lambda(\mu, \lambda, \tau)$ is a constant matrix, the boundedness of $T_{A_0}^{1,*}$ and $T_{A_0}^{2,*}$ on $L^p(\partial D)$ for $1 < p < \infty$ follows from [7] and is well known. Thus, in view of Lemma 3.2 we only need to treat the maximal singular integral operators with kernels $\nabla_1\Theta(P, Y; A(P))$ and $\nabla_2\Theta(P, Y; A(Y))$.

Let $K_A(X, Y) = \nabla_1\Theta(X, 0; A(Y))$. We may write

$$\nabla_1\Theta(P, Y; A(P)) = \nabla_1\Theta(P - Y, 0; A(P)) = K_A(P - Y, P). \quad (3.4)$$

Similarly, we have

$$\nabla_2\Theta(P, Y; A(Y)) = \nabla_1\Theta(Y - P, 0; A(Y)) = K_A(Y - P, Y). \quad (3.5)$$

Recall that $\Theta(X, 0; A(Y))$ is the matrix of fundamental solutions for the constant coefficient operator $L^Y = -\text{div}(A(Y)\nabla)$ with pole at the origin. It follows that $K_A(X, Y)$ is odd in $X$ and homogeneous of degree $1 - d$ in $X$. Moreover, for any $N \geq 1$, $\nabla_N^N K_A(X, Y)$ is continuous on $(\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d$ and

$$|\nabla_N^N K_A(X, Y)| \leq C|X|^{1-d-N}. \quad (3.6)$$
where $C = C(d, m, \mu, N) > 0$. In view of (3.4)-(3.5) and Lemma 3.3, we may conclude that the maximal singular integral operators with kernels $\nabla_1 \Theta(P, Y; A(P))$ and $\nabla_2 \Theta(P, Y; A(Y))$ are bounded on $L^p(\partial D)$ and their operator norms are bounded by $C(d, m, \mu, \lambda, p, M)$.

Finally we note that for $f$ with compact support, the existence of $T_A^1(f)(P)$ and $T_A^2(f)(P)$ for a.e. $P \in \partial D$ follows readily from estimates (2.10) and (2.15) and Lemma 3.3. The general case follows from this and the boundedness of $T_A^1$ and $T_A^2$ on $L^p(\partial D)$.

The following theorem will be useful to us in an approximation argument.

**Theorem 3.4.** Let $\Omega$ be a bounded Lipschitz domain. Let $T_A^1$, $T_A^1$, $T_A^2$, $T_A^2$ be defined by (3.7), where $A, \tilde{A} \in \Lambda(\mu, \lambda, \tau)$. Suppose that $\text{diam}(\Omega) \leq R$ for some $R \geq 1$. Then for $1 < p < \infty$,

$$
\|T_A^1(f) - T_A^1(f)\|_p \leq C_R \|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)} \|f\|_{L^p},
$$

$$
\|T_A^2(f) - T_A^2(f)\|_p \leq C_R \|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)} \|f\|_{L^p},
$$

(3.7)

where $C_R$ depends only on $d, m, \mu, \lambda, \tau, p$, the Lipschitz character of $\Omega$ and $R$.

**Proof.** Recall that $\Pi_A(P, Y) = \nabla_1 \Gamma_A(P, Y) - \nabla_1 \Theta(P, Y; A(P))$. Then

$$
\nabla_1 \Gamma_A(P, Y) - \nabla_1 \Gamma_A(P, Y)
= \Pi_A(P, Y) - \Pi_{\tilde{A}}(P, Y) + \{\nabla_1 \Theta(P - Y, 0; A(P)) - \nabla_1 \Theta(P - Y, 0; \tilde{A}(P))\}.
$$

(3.8)

It follows from estimate (2.23) that the norm of the integral operator with kernel $\Pi_A(P, Y) - \Pi_{\tilde{A}}(P, Y)$ on $L^p(\partial \Omega)$ is bounded by $C_R \|A - \tilde{A}\|_{C^\lambda(\mathbb{R}^d)}$ for $1 \leq p \leq \infty$. Note that

$$
|\nabla^N Z \{\Theta(Z, 0; A(P)) - \Theta(Z, 0; \tilde{A}(P))\}| \leq C \|A - \tilde{A}\| \|Z\|^{2-d-N}
$$

(3.9)

for any $N \geq 0$ and any $Z \in \mathbb{R}^d$, where $C$ depends only on $d, m, \mu$ and $N$. We may deduce from Lemma 3.3 that the integral operator with kernel $\nabla_1 \Theta(P - Y, 0; A(P)) - \nabla_1 \Theta(P - Y, 0; \tilde{A}(P))$ on $L^p(\partial \Omega)$ for $1 < p < \infty$ is bounded by $C \|A - \tilde{A}\|$. This gives the desired estimate for $\|T_A^1(f) - T_A^1(f)\|_p$. The estimate for $\|T_A^2(f) - T_A^2(f)\|_p$ follows from Remark 2.8 and Lemma 3.3 in the same manner.

We end this section with a theorem on the nontangential maximal functions. For $f \in L^p(\partial \Omega)$, consider the following two functions

$$
u(X) = \int_{\partial \Omega} \nabla_X \Gamma_A(X, Y)f(Y) \sigma(Y),
$$

$$w(X) = \int_{\partial \Omega} \nabla_Y \Gamma_A(X, Y)f(Y) \sigma(Y),
$$

(3.10)

defined on $\mathbb{R}^d \setminus \partial \Omega$.

**Theorem 3.5.** Let $\Omega$ be a bounded Lipschitz domain. Let $u$ and $w$ be defined by (3.10), where $A \in \Lambda(\mu, \lambda, \tau)$. Then for $1 < p < \infty$, $\|(u)^*\|_p + \|(w)^*\|_p \leq C_p \|f\|_p$, where $(\cdot)^*$ denotes the nontangential maximal function taken with respect to $\Omega_+ = \Omega$ or $\Omega_- = \mathbb{R}^d \setminus \overline{\Omega}$, and $C_p$ depends only on $d, m, \mu, \lambda, \tau, p$ and the Lipschitz character of $\Omega$.
Proof. Let $\mathcal{M}_{\partial \Omega}$ denote the usual Hardy-Littlewood maximal operator on $\partial \Omega$. We claim that for any $P \in \partial \Omega$,

$$(u)^*(P) \leq C \mathcal{M}_{\partial \Omega}(f)(P) + C \sup_{\rho > 0} \left| \int_{Y \in \partial \Omega, |Y - P| > \rho} \nabla_1 \Theta(P, Y; A(P)) f(Y) \, d\sigma(Y) \right|$$

$$+ C \sup_{\rho > 0} \left| \int_{Y \in \partial \Omega, |Y - P| > \rho} \nabla_1 \Gamma_{A_0}(P, Y) f(Y) \, d\sigma(Y) \right|,$$

and

$$(w)^*(P) \leq C \mathcal{M}_{\partial \Omega}(f)(P) + C \sup_{\rho > 0} \left| \int_{Y \in \partial \Omega, |Y - P| > \rho} \nabla_2 \Theta(P, Y; A(Y)) f(Y) \, d\sigma(Y) \right|$$

$$+ C \sup_{\rho > 0} \left| \int_{Y \in \partial \Omega, |Y - P| > \rho} \nabla_2 \Gamma_{A_0}(P, Y) g(Y) \, d\sigma(Y) \right|,

(3.11)$$

where $C$ is a “good” constant and $|g| \leq C|f|$ on $\partial \Omega$. Estimate $\|u|_p + \|w\|_p \leq C_p \|f\|_p$ follows from (3.11), as in the proof of Theorem 3.1. We will give the proof for $(u)^*$. The estimate for $(w)^*$ may be carried out in the same manner.

Fix $P \in \partial \Omega$. Let $X \in \mathbb{R}^d \setminus \partial \Omega$ such that $|X - P| < C_0 \text{dist}(X, \partial \Omega)$. Let $r = |X - P|$. If $r \geq 1$, we may use (2.18) to show that

$$|u(X) - (I + \nabla X(X)) \nabla U(X)| \leq C \int_{\partial \Omega} \frac{|f(Y)| \, d\sigma(Y)}{|X - Y|^{d-1+\lambda_0}} \leq C \mathcal{M}_{\partial \Omega}(f)(P),$$

where $U(X) = \int_{\partial \Omega} \Gamma_{A_0}(X, Y) f(Y) \, d\sigma(Y)$. It follows that well known estimates for $\nabla U$ that

$$|u(X)| \leq C \mathcal{M}_{\partial \Omega}(f)(P) + C(\nabla U)^*(P)$$

$$\leq C \mathcal{M}_{\partial \Omega}(f)(P) + C \sup_{\rho > 0} \left| \int_{|Y - P| > \rho} \nabla_1 \Gamma_{A_0}(P, Y) f(Y) \, d\sigma(Y) \right|.$$

Next suppose that $r = |X - P| < 1$. We write $u(X) = J_1 + J_2 + J_3$, where $J_1$, $J_2$, $J_3$ denote the integrals of $\nabla_1 \Gamma(X, Y) f(Y)$ over $E_1 = \{ Y \in \partial \Omega : |Y - P| < r \}$, $E_2 = \{ Y \in \partial \Omega : r \leq |Y - P| \leq 1 \}$, $E_3 = \{ Y \in \partial \Omega : |Y - P| > 1 \}$, respectively. Clearly, $|J_1| \leq C \mathcal{M}_{\partial \Omega}(f)(P).$ For $J_2$, we use (2.10) to obtain

$$|J_2| \leq \left| \int_{E_2} \nabla_1 \Theta(X, Y; A(Y)) f(Y) \, d\sigma(Y) \right| + C \int_{E_2} \frac{|f(Y)| \, d\sigma(Y)}{|X - Y|^{d-1+\lambda_0}}$$

$$\leq | \int_{E_2} \nabla_1 \Theta(P, Y; A(Y)) f(Y) \, d\sigma(Y) | + C \int_{E_2} \frac{|f(Y)| \, d\sigma(Y)}{|Y - P|^{d-1-\lambda_0}}$$

$$\leq 2 \sup_{\rho > 0} \left| \int_{|Y - P| > \rho} \nabla_1 \Theta(P, Y; A(Y)) f(Y) \, d\sigma(Y) \right| + C \mathcal{M}_{\partial D}(f)(P).$$

In view of (2.18), we have

$$|J_3| \leq C \int_{E_3} \frac{|f(Y)| \, d\sigma(Y)}{|X - Y|^{d-1+\lambda_0}} + C \int_{E_3} \nabla_1 \Gamma_{A_0}(X, Y) f(Y) \, d\sigma(Y)$$

$$\leq C \mathcal{M}_{\partial D}(f)(P) + C \int_{E_3} \nabla_1 \Gamma_{A_0}(P, Y) f(Y) \, d\sigma(Y).$$

This, together with estimates of $J_1$ and $J_2$, yields the desired estimate for $(u)^*(P)$. \qed
4 Method of layer potentials

In this section we fix $A \in \Lambda(\mu, \lambda, \tau)$ and let $L = -\text{div}(A \nabla)$, $\Gamma(X, Y) = \Gamma_A(X, Y)$. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^d$ and $n = (n_1, \ldots, n_d)$ the outward unit normal to $\partial \Omega$. For $f \in L^p(\partial \Omega, \mathbb{R}^m)$, the single layer potential $S(f) = S_A(f) = (u^1, \ldots, u^m)$ is defined by

$$w^\alpha(X) = \int_{\partial \Omega} \Gamma^{\alpha\beta}(X, Y) f^\beta(Y) \, d\sigma(Y),$$

(4.1)

while the double layer potential $D(f) = D_A(f) = (w^1, \ldots, w^m)$ is defined by

$$w^\alpha(X) = \int_{\partial \Omega} n_j(Y)a^\beta_{ij}(Y) \frac{\partial}{\partial y_i} \Gamma^{\alpha\beta}(X, Y) f^\gamma(Y) \, d\sigma(Y)$$

$$= \int_{\partial \Omega} \left( \frac{\partial}{\partial u^\gamma A^\gamma} \Gamma^{\alpha\gamma}(Y, X) \right) f^\gamma(Y) \, d\sigma(Y),$$

(4.2)

where $\Gamma^{\alpha\gamma}_{A^\gamma} = (\Gamma^{1\alpha}_{A^1}, \ldots, \Gamma^{m\alpha}_{A^m})$ and $\Gamma_{A^\gamma}(X, Y)$ is the fundamental solution for $L^* = -\text{div}(A^\gamma \nabla)$, with pole at $Y$. Clearly, both $S(f)$ and $D(f)$ are solutions of $L(u) = 0$ in $\mathbb{R}^d \setminus \partial \Omega$.

The definitions of single and layer potentials are motivated by the following Green’s representation formula.

**Proposition 4.1.** Let $u \in C^1(\overline{\Omega})$. Suppose that $L(u) = 0$ in $\Omega$. Then for any $X \in \Omega$,

$$u(X) = S\left( \frac{\partial u}{\partial \nu} \right)(X) - D(u)(X),$$

(4.3)

where $\frac{\partial u}{\partial \nu}$ denotes the conormal derivative of $u$ on $\partial \Omega$, defined by $(\frac{\partial u}{\partial \nu})^\alpha = n_i a^i_{\alpha\beta} \frac{\partial u^\beta}{\partial x^j}$. 

**Proof.** Fix $X \in \Omega$. Choose $r > 0$ so small that $B(X, 4r) \subset \Omega$. Let $\varphi \in C^\infty_0(B(X, 2r))$ be such that $\varphi = 1$ on $B(X, r)$. It follows from (2.4) that

$$u^\gamma(X) = (u \varphi)^\gamma(X) = \int_{\Omega} a^\beta_{ji}(Y) \frac{\partial}{\partial y_j} \Gamma^{\beta\gamma}_{A^\gamma}(Y, X) \cdot \frac{\partial}{\partial y_i} (u^\alpha \varphi) \, dY$$

$$= \int_{\Omega} a^\beta_{ji}(Y) \frac{\partial}{\partial y_j} \Gamma^{\beta\gamma}_{A^\gamma}(Y, X) \cdot \frac{\partial u^\alpha}{\partial y_i} \, dX$$

$$+ \int_{\Omega} a^\beta_{ji}(Y) \frac{\partial}{\partial y_j} \Gamma^{\beta\gamma}_{A^\gamma}(Y, X) \cdot \frac{\partial}{\partial y_i} \{u^\alpha (\varphi - 1) \} \, dX.$$  

(4.4)

Using integration by parts and $L(u) = 0$ in $\Omega$, we obtain

$$u^\gamma(X) = \int_{\partial \Omega} n_j(Y)a^\beta_{ji}(Y) \frac{\partial u^\alpha}{\partial y_i} \cdot \Gamma^{\beta\gamma}_{A^\gamma}(Y, X) \, d\sigma(Y)$$

$$- \int_{\partial \Omega} n_i(Y)a^\beta_{ji}(Y) \frac{\partial}{\partial y_j} \Gamma^{\beta\gamma}_{A^\gamma}(Y, X) \cdot u^\alpha(Y) \, d\sigma(Y).$$

(4.5)

Since $\Gamma^{\alpha\gamma}_{A^\gamma}(Y, X) = \Gamma^{\beta\alpha}_{A}(X, Y)$, this gives (4.3). $\square$
Remark 4.2. Suppose that $u \in C^1(\Omega)$ and $L(u) = -\frac{\partial f_i}{\partial x_i} + g$, where $f_i, g \in C(\Omega, \mathbb{R}^m)$ and $f_i = 0$ on $\partial \Omega$. Then

$$u(X) = S\left(\frac{\partial u}{\partial v}\right)(X) - D(u)(X) + v(X),$$

where

$$v^\alpha(X) = \int_\Omega \frac{\partial}{\partial y_i} \Gamma^{\alpha\beta}(X, Y) \cdot f^\beta_i(Y) \, dY + \int_\Omega \Gamma^{\alpha\beta}(X, Y) g^\beta(Y) \, dY.$$

Theorem 4.3. Let $1 < p < \infty$. Then

$$\|\nabla S(f)\|^p + \|D(f)\|^p \leq C_p \|f\|^p,$$

where $C_p$ depends only on $d, m, \mu, \lambda, \tau, p$ and the Lipschitz character of $\Omega$.

Proof. This follows readily from Theorem 3.5. \qed

For a function $u$ defined in $\mathbb{R}^d \setminus \partial \Omega$, we will use $u_+$ and $u_-$ to denote its nontangential limits on $\partial \Omega$, taken inside $\Omega$ and outside $\Omega$ respectively.

Theorem 4.4. Let $u = S(f)$ for some $f \in L^p(\partial \Omega)$ and $1 < p < \infty$. Then for a.e. $P \in \partial \Omega$,

$$\left(\frac{\partial u^\alpha}{\partial x_i}\right)_\pm(P) = \pm \frac{1}{2} n_i(P) b^{\alpha\beta}(P) f^\beta(P) + \text{p.v.} \int_{\partial \Omega} \frac{\partial}{\partial P_i} \Gamma^{\alpha\beta}(P, Y) f^\beta(Y) \, d\sigma(Y),$$

where $(b^{\alpha\beta}(P))_{m \times m}$ is the inverse matrix of $(a_{ij}^{\alpha\beta}(P) n_i(P) n_j(P))_{m \times m}$.

Proof. By Theorem 4.3 we may assume that $f$ is a Lipschitz function on $\partial \Omega$. Also it is known that there exists a set $F \subset \partial \Omega$ such that $\sigma(\partial \Omega - F) = 0$ and the trace formula holds for any $P \in F$ and for any $\Gamma(\cdot, y) = \Gamma_E(X, Y)$ with constant matrix $E \in \Lambda(\mu, \lambda, \tau)$ (see e.g. [14, 11, 21]).

Now fix $P \in F$ and

$$v^\alpha(X) = \int_{\partial \Omega} \Theta^{\alpha\beta}(X, Y; A(P)) f^\beta(Y) \, d\sigma(Y)$$

be the single layer potential for the elliptic operator $L^{A(P)} = -\text{div}(A(P) \nabla)$ with constant coefficients. In view of (2.10) and (2.9), we have

$$|\nabla_1 \Gamma_A(X, Y) - \nabla_1 \Theta(X, Y; A(P))|$$

$$\leq |\nabla_1 \Gamma_A(X, Y) - \nabla_1 \Theta(X, Y, A(Y))| + |\nabla_1 \Theta(X, Y; A(Y)) - \nabla_1 \Theta(X, Y; A(P))|$$

$$\leq C|X - Y|^{1-d+\lambda} + C|X - Y|^{1-d} |Y - P|^\lambda$$

$$\leq C|P - Y|^{1-d+\lambda}$$

for any $X \in \gamma(P) = \{Z \in \mathbb{R}^d \setminus \partial \Omega : \text{dist}(Z, \partial \Omega) < C_0|Z - P|\}$ and $Y \in \partial \Omega$. By Lebesgue’s dominated convergence theorem, this implies that

$$\nabla u^\alpha \pm(P) = (\nabla v^\alpha) \pm(P) + \int_{\partial \Omega} \{\nabla_1 \Gamma_A^{\alpha\beta}(P, Y) - \nabla_1 \Theta^{\alpha\beta}(P, Y; A(P))\} f^\beta(Y) \, d\sigma(Y)$$

$$= \pm \frac{1}{2} n(P) b^{\alpha\beta}(P) f^\beta(P) + \text{p.v.} \int_{\partial \Omega} \nabla_1 \Gamma_A^{\alpha\beta}(P, Y) f^\beta(Y) \, d\sigma(Y).$$

This finishes the proof. \qed
It follows from (4.8) that if $u = S(f)$,
\begin{align*}
n_j \left( \frac{\partial u^\alpha}{\partial x_i} \right)_+ - n_i \left( \frac{\partial u^\alpha}{\partial x_j} \right)_+ & = n_j \left( \frac{\partial u^\alpha}{\partial x_i} \right)_- - n_i \left( \frac{\partial u^\alpha}{\partial x_j} \right)_- ; \tag{4.12}
\end{align*}
i.e. $(\nabla_{\text{tan}} u)_+ = (\nabla_{\text{tan}} u)_-$ on $\partial \Omega$. Moreover, let $(\frac{\partial u}{\partial \nu})^\alpha \pm = n_i a_{ij}^\alpha \left( \frac{\partial u}{\partial x_j} \right)_\pm$ on $\partial \Omega$. Then
\begin{align*}
(\frac{\partial u}{\partial \nu})^\alpha \mp & = (\pm \frac{1}{2} I + K_A)(f),
\end{align*}
where
\begin{align*}
(K_A(f)(P))^\alpha & = \text{p.v.} \int_{\partial \Omega} K_A^{\alpha \beta}(P, Y) f^\beta(Y) \, d\sigma(Y) \tag{4.13}
\end{align*}
and
\begin{align*}
K_A^{\alpha \beta}(P, Y) & = n_i(P) a_{ij}^\alpha(P) \frac{\partial}{\partial P_j} \Gamma^{\gamma \beta}_{A}(P, Y). \tag{4.14}
\end{align*}
In particular we have the jump relation
\begin{align*}
f & = \left( \frac{\partial u}{\partial \nu} \right)_+ - \left( \frac{\partial u}{\partial \nu} \right)_- . \tag{4.15}
\end{align*}
We may deduce from Theorem 3.1 that $\|K_A(f)\|_p \leq C_p \|f\|_p$, where $C_p$ depends only on $d$, $m$, $\mu$, $\lambda$, $\tau$, $p$ and the Lipschitz character of $\Omega$.

**Remark 4.5.** Let $u = S_A(f)$ for some $f \in L^p(\partial \Omega, \mathbb{R}^m)$ and $1 < p < \infty$. Then $\int_{\partial \Omega} (\frac{\partial u}{\partial \nu})_+ \, d\sigma = 0$. It follows that $((1/2)I + K_A)(L^p(\partial \Omega, \mathbb{R}^m)) \subset L^p_0(\partial \Omega, \mathbb{R}^m)$. This implies that
\begin{align*}
K_A(L^p_0(\partial \Omega, \mathbb{R}^m)) \subset L^p_0(\partial \Omega, \mathbb{R}^m)
\end{align*}
for $1 < p < \infty$.

Let $W^{1,p}(\partial \Omega, \mathbb{R}^m)$ denote the subspace of functions $f$ in $L^p(\partial \Omega, \mathbb{R}^m)$ with tangential derivatives $\nabla_{\text{tan}} f$ in $L^p(\partial \Omega)$, equipped with the scale-invariant norm
\begin{align*}
\|f\|_{1,p} = \|\nabla_{\text{tan}} f\|_p + [\sigma(\partial \Omega)]^{\frac{1}{1-p}} \|f\|_p. \tag{4.16}
\end{align*}
It follows from Theorem 3.1 that for $1 < p < \infty$,
\begin{align*}
\|S(f)\|_{1,p} \leq C \|f\|_p \tag{4.17}
\end{align*}
where $C$ depends only on $d$, $m$, $\mu$, $\lambda$, $\tau$, $p$ and the Lipschitz character of $\Omega$.

The next theorem gives the trace of the double layer potentials.

**Theorem 4.6.** Let $w = D(f)$ where $f \in L^p(\partial \Omega)$ and $1 < p < \infty$. Then
\begin{align*}
w_\pm & = (\mp \frac{1}{2} I + K_{A^*})(f) \quad \text{on } \partial \Omega,
\end{align*}
where $K_{A^*}$ is the adjoint operator of $K_A^*$, defined by (4.13) and (4.14).
Proof. Note that if $E$ is a constant matrix in $\Lambda(\mu, \lambda, \tau)$, then
\[
\frac{\partial}{\partial x_i} \Theta(X,Y; E) = -\frac{\partial}{\partial y_i} \Theta(X,Y; E).
\] (4.18)
This, together with (2.10) and (2.15), shows that
\[
|\frac{\partial}{\partial y_i} \Gamma_A^{\alpha\beta}(X,Y) + \frac{\partial}{\partial x_i} \Gamma_A^{\alpha\beta}(X,Y)| \leq C|X-Y|^{1-d+\lambda},
\] (4.19)
for any $X,Y \in \mathbb{R}^d$. Thus, as in the proof of Theorem 4.4 it follows from the Lebesgue’s dominated convergence theorem that
\[
w^\alpha_\pm(P) = -v^\alpha_\pm(P) + \int_{\partial \Omega} n_j(Y) a^{\beta\gamma}_{ij}(Y) \left\{ \frac{\partial}{\partial y_i} \Gamma^{\alpha\beta}(P,Y) + \frac{\partial}{\partial P_i} \Gamma^{\alpha\beta}(P,Y) \right\} f^\gamma(Y) d\sigma(Y),
\]
where
\[
v^\alpha(X) = \frac{\partial}{\partial x_i} \int_{\partial \Omega} n_j(Y) a^{\beta\gamma}_{ij}(Y) \Gamma^{\alpha\beta}(X,Y) f^\gamma(Y) d\sigma(Y).
\] (4.20)
In view of the trace formula (4.18), we have
\[
v^\alpha_\pm(P) = \pm \frac{1}{2} n_i(P) b^{\alpha\beta}(P) \cdot n_j(P) a^{\beta\gamma}_{ij}(P) f^\gamma(P) + \text{p.v.} \int_{\partial \Omega} n_j(Y) a^{\beta\gamma}_{ij}(Y) \frac{\partial}{\partial P_i} \Gamma^{\alpha\beta}(P,Y) \cdot f^\gamma(Y) d\sigma(Y)
\] (4.21)
\[= \pm \frac{1}{2} f^\alpha(P) + \text{p.v.} \int_{\partial \Omega} n_j(Y) a^{\beta\gamma}_{ij}(Y) \frac{\partial}{\partial y_i} \Gamma^{\alpha\beta}(P,Y) \cdot f^\gamma(Y) d\sigma(Y).
\]
Thus
\[
w^\alpha_\pm(P) = \pm \frac{1}{2} f^\alpha(P) + \text{p.v.} \int_{\partial \Omega} n_j(Y) a^{\beta\gamma}_{ij}(Y) \frac{\partial}{\partial y_i} \Gamma^{\alpha\beta}(P,Y) \cdot f^\gamma(Y) d\sigma(Y)
\] (4.22)
\[= \pm \frac{1}{2} f^\alpha(P) + \text{p.v.} \int_{\partial \Omega} K^{\beta\alpha}_{A^*}(Y,P) f^\beta(Y) d\sigma(Y),
\]
where $K^{\beta\alpha}_{A^*}(Y,P)$ is defined by (4.14), but with $A$ replaced by $A^*$. This completes the proof. \qed

In summary, if $1 < p < \infty$ and $f \in L^p(\partial \Omega)$, then $u = S(f)$ is a solution to the $L^p$ Neumann problem in $\Omega$ with boundary data $((1/2)I + K_A)f$, while $w = D(f)$ is a solution to the $L^p$ Dirichlet problem in $\Omega$ with boundary data $(-(1/2)I + K_A^*)f$. Furthermore, $(1/2)I + K_A : L^p_0(\partial \Omega, \mathbb{R}^m) \to L^p_0(\partial \Omega, \mathbb{R}^m)$ and $-(1/2)I + K_A^* : L^p(\partial \Omega, \mathbb{R}^m) \to L^p(\partial \Omega, \mathbb{R}^m)$ are bounded. As a result, one may establish the existence of solutions in the $L^p$ Neumann and Dirichlet problems in $\Omega$ by showing that the operators $(1/2)I + K_A$ and $-(1/2)I + K_A^*$ are invertible on $L^p_0(\partial \Omega, \mathbb{R}^m)$ and $L^p(\partial \Omega, \mathbb{R}^m)$ respectively. This is the so-called method of layer potentials.
In the remaining of this section we discuss the layer potentials for $\mathcal{L}_\varepsilon = -\text{div}(A(\varepsilon^{-1}X)\nabla)$. Let $\Gamma_{\varepsilon}(X,Y) = \Gamma_{\varepsilon,A}(X,Y)$ denote the matrix of fundamental solutions for the operator $\mathcal{L}_\varepsilon$ on $\mathbb{R}^d$, with pole at $Y$. By rescaling we have

$$\Gamma_{\varepsilon}(X,Y) = \varepsilon^{2-d}\Gamma(\varepsilon^{-1}X,\varepsilon^{-1}Y).$$  \hfill (4.23)

Thus, by (2.5),

$$|\Gamma_{\varepsilon}(X,Y)| \leq C|X-Y|^{2-d},$$

$$|\nabla X \Gamma_{\varepsilon}(X,Y)| + |\nabla Y \Gamma_{\varepsilon}(X,Y)| \leq C|X-Y|^{1-d}$$  \hfill (4.24)

for any $X,Y \in \mathbb{R}^d$.

For $f \in L^p(\partial \Omega)$, the single layer potential $S_\varepsilon(f) = (S^1_\varepsilon(f), \ldots, S^m_\varepsilon(f))$ is defined by

$$S^\alpha_\varepsilon(f)(X) = \int_{\partial \Omega} \Gamma^\alpha_\varepsilon(X,Y)f^\beta(Y)\,d\sigma(Y),$$  \hfill (4.25)

while the double layer potential $D_\varepsilon(f) = (D^1_\varepsilon(f), \ldots, D^m_\varepsilon(f))$ is defined by

$$D^\alpha_\varepsilon(f)(X) = \int_{\partial \Omega} n_j(Y)\partial_i^\beta(\varepsilon^{-1}Y)\frac{\partial}{\partial y_i} \Gamma^\alpha_\varepsilon(X,Y)f^\gamma(Y)\,d\sigma(Y).$$  \hfill (4.26)

Clearly, both $S_\varepsilon(f)$ and $D_\varepsilon(f)$ are solutions of $\mathcal{L}_\varepsilon(u) = 0$ in $\mathbb{R}^d \setminus \partial \Omega$.

**Theorem 4.7.** Let $1 < p < \infty$. Then

$$\|\nabla S_\varepsilon(f)\|_p + \|D_\varepsilon(f)\|_p \leq C_p\|f\|_p,$$

where $C_p$ depends only on $d$, $m$, $\mu$, $\lambda$, $\tau$, $p$ and the Lipschitz character of $\Omega$.

**Proof.** Fix $\varepsilon > 0$ and define

$$\Omega_\varepsilon = \{\varepsilon^{-1}X : X \in \Omega\}.$$  \hfill (4.27)

Let $u_\varepsilon = S_\varepsilon(f)$. It follows from (4.23) that $u_\varepsilon(X) = v(\varepsilon^{-1}X)$, where $v(x)$ is the single layer potential on $\partial \Omega_\varepsilon$ for the operator $\mathcal{L}$ with density $g$ given by $g(Y) = f(\varepsilon Y)$. Since $\Omega_\varepsilon$ and $\Omega$ share the same Lipschitz character, $\|\nabla v\|_{L^p(\partial \Omega_\varepsilon)} \leq C\|g\|_{L^p(\partial \Omega)}$, where $C$ depends only on $d$, $m$, $\mu$, $\lambda$, $\tau$, $p$ and the Lipschitz character of $\Omega$, not on $\varepsilon$. By rescaling, this gives the desired estimate for $\nabla S_\varepsilon(f)$. The estimate on $D_\varepsilon(f)$ follows in the same manner. \hfill $\Box$

The next theorem follows readily from Theorems 4.4 and 4.6 by rescaling.

**Theorem 4.8.** Let $1 < p < \infty$ and $f \in L^p(\partial \Omega)$. Let $u_\varepsilon = S_\varepsilon(f)$. Then $(\nabla u_\varepsilon)_\pm(P)$ exists for a.e. $P \in \partial \Omega$ and $(\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon})_\pm = (\pm \frac{1}{2}I + \mathcal{K}_{\varepsilon,A})(f)$, where

$$\mathcal{K}^\alpha_{\varepsilon,A}(f)(P) = \text{p.v.} \int_{\partial \Omega_\varepsilon} K^\alpha_{A}(\varepsilon^{-1}P,Y)f^\beta(\varepsilon Y)\,d\sigma(Y)$$  \hfill (4.28)

and the integral kernel $K^\alpha_{A}(P,Y)$ on $\partial \Omega_\varepsilon \times \partial \Omega_\varepsilon$ is given by (4.14). Similarly, if $w_\varepsilon = D_\varepsilon(f)$, then $(w_\varepsilon)_\pm = (\pm \frac{1}{2}I + \mathcal{K}_{\varepsilon,A'})f$. 

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5 Rellich property

In this section we reduce the solvability of the $L^2$ Neumann, Dirichlet and regularity problems for $\mathcal{L}(u) = 0$ in Lipschitz domains to certain boundary Rellich estimates.

Definition 5.1. Let $\mathcal{L} = -\text{div}(A(X)\nabla)$ and $\Omega$ be a bounded Lipschitz domain with connected boundary. We say that $\mathcal{L}$ has the Rellich property in $\Omega$ with constant $C = C(\Omega)$ if $\|
abla u\|_2 \leq C\|\frac{\partial u}{\partial \nu}\|_2$ and $\|
abla u\|_2 \leq C\|
abla_{\text{tan}} u\|_2$, whenever $u$ is a solution to $\mathcal{L}(u) = 0$ in $\Omega$ such that $(\nabla u)^* \in L^2(\partial\Omega)$ and $\nabla u$ exists n.t. on $\partial\Omega$.

Definition 5.2. We say that the Dirichlet problem $(D)_p$ for $\mathcal{L}(u) = 0$ in $\Omega$ is uniquely solvable with estimate $\|((u)^*)_p \leq C\|u\|_p$, if for any $f \in L^p(\partial\Omega, \mathbb{R}^m)$, there exists a unique solution to $\mathcal{L}(u) = 0$ in $\Omega$ with the property that $(u)^* \in L^p(\partial\Omega)$ and $u = f$ n.t. on $\partial\Omega$, and the solution satisfies $\|((u)^*)_p \leq C\|f\|_p$.

We say that the regularity problem $(R)_p$ for $\mathcal{L}(u) = 0$ in $\Omega$ is uniquely solvable with estimate $\|((\nabla u)^*)_p \leq C\|\nabla u\|_{1,p}$, if for any $f \in W^{1,p}(\partial\Omega, \mathbb{R}^m)$, there exists a unique solution to $\mathcal{L}(u) = 0$ in $\Omega$ with the property that $(\nabla u)^* \in L^p(\partial\Omega)$ and $u = f$ n.t. on $\partial\Omega$, and the solution satisfies $\|((\nabla u)^*)_p \leq C\|f\|_{1,p}$.

We say that the Neumann problem $(N)_p$ for $\mathcal{L}(u) = 0$ in $\Omega$ is uniquely solvable with estimate $\|((\nabla u)^*)_p \leq C\|\frac{\partial u}{\partial \nu}\|_p$, if for any $f \in L^p_0(\partial\Omega, \mathbb{R}^m)$, there exists a solution, unique up to constants, to $\mathcal{L}(u) = 0$ in $\Omega$ with the property that $(\nabla u)^* \in L^p(\partial\Omega)$ and $\frac{\partial u}{\partial \nu} = f$ n.t. on $\partial\Omega$, and the solution satisfies $\|((\nabla u)^*)_p \leq C\|f\|_p$.

The following two theorems are the main results of this section. The first theorem treats the solvability in small scale - the constant $C$ in the nontangential-maximal-function estimates in $\square$ depends on $\text{diam}(\Omega)$, if $\text{diam}(\Omega) \geq 1$. The estimates in the second theorem are scale-invariant. As a result, by rescaling, they leads to uniform estimates in a Lipschitz domain for the family of elliptic operators $\{\mathcal{L}_s\}$.

Theorem 5.3. Let $\mathcal{L} = -\text{div}(A\nabla)$ with $A \in \Lambda(\mu, \lambda, \tau)$ and $A^* = A$. Let $R \geq 1$. Suppose that for any Lipschitz domain $\Omega$ with $\text{diam}(\Omega) \leq (1/4)$ and connected boundary, there exists $C(\Omega)$ depending only on the Lipschitz character of $\Omega$ such that for each $s \in (0,1]$, $\mathcal{L}_s = -\text{div}(sA + (1-s)I\nabla)$ has the Rellich property in $\Omega$ with constant $C(\Omega)$. Then for any Lipschitz domain $\Omega$ with $\text{diam}(\Omega) \leq R$ and connected boundary, $(R)_2$ and $(N)_2$ for $\mathcal{L}(u) = 0$ in $\Omega$ are uniquely solvable and the solutions satisfy the estimates

$$\|(\nabla u)^*\|_2 \leq C\|\frac{\partial u}{\partial \nu}\|_2 \quad \text{and} \quad \|(\nabla u)^*\|_2 \leq C\|\nabla_{\text{tan}} u\|_2,$$

(5.1)

where $C$ depends only on $\mu$, $\lambda$, $\tau$, the Lipschitz character of $\Omega$ and $R$ (if $\text{diam}(\Omega) \geq 1$). Furthermore, the $L^2$ Dirichlet problem in $\Omega$ is uniquely solvable with the estimate $\|(u)^*\|_2 \leq C\|u\|_2$.

Recall that $C$ is called a “good” constant if it depends only on $d$, $m$, $\mu$, $\lambda$, $\tau$ and the Lipschitz character of $\Omega$.

Theorem 5.4. Let $\mathcal{L} = -\text{div}(A\nabla)$ with $A \in \Lambda(\mu, \lambda, \tau)$ and $A^* = A$. Suppose that for any Lipschitz domain $\Omega$ with connected boundary, there exists a “good” constant $C(\Omega)$ such
that for each \( s \in (0, 1] \), \( \mathcal{L}_s = -\text{div}(sA + (1 - s)I) \nabla \) has the Rellich property in \( \Omega \) with constant \( C(\Omega) \). Then for any Lipschitz domain \( \Omega \) with connected boundary, \((R)_2\) and \((N)_2\) for \( \mathcal{L}(u) = 0 \) in \( \Omega \) are uniquely solvable and the solutions satisfy the estimates in \((5.1)\) with a “good” constant \( C \). Furthermore, the \( L^2 \) Dirichlet problem in \( \Omega \) is uniquely solvable with the estimate \( \|(u)^*\|_2 \leq C\|u\|_2 \) for a “good” constant \( C \).

The uniqueness for \((R)_2\) and \((N)_2\) follows readily from the Green’s identity,

\[
\int_{\Omega} \sum_{\alpha \beta} a_{ij} \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\beta}{\partial x_j} \, dX = \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^\alpha u^\alpha \, d\sigma,
\]

by approximating \( \Omega \) from inside. We will use the method of layer potentials to establish the existence of solutions in Theorems \( (5.3) \) and \( (5.4) \) by covering \( \partial \Omega \) with \( \{ \Delta_i \} \), each of which may be obtained from \( \Delta(r) \) by translation and rotation. The proof for the second inequality in \( (5.3) \) is similar.
Remark 5.6. Under the same conditions on $A$ as in Theorem 5.4, the estimates in (5.3) hold with constant $C$ independent of $R$. This is because we may choose $r = c(M)r_0$ for any $\Omega$.

Lemma 5.7. Let $R \geq 1$ and $\Omega$ be a bounded Lipschitz domain with $\text{diam}(\Omega) \leq R$. Under the same conditions on $A$ as in Theorem 5.3, the operators $(1/2)I + K_A : L^2_0(\partial \Omega, \mathbb{R}^m) \to L^2_0(\partial \Omega, \mathbb{R}^m)$ and $-(1/2)I + K_A : L^2(\partial \Omega, \mathbb{R}^m) \to L^2(\partial \Omega, \mathbb{R}^m)$ are invertible and

$$
\left\| \left( \frac{1}{2}I + K_A \right)^{-1} \right\|_{L^2_0 \to L^2_0} \leq C,
$$

$$
\left\| \left( - \frac{1}{2}I + K_A \right)^{-1} \right\|_{L^2 \to L^2} \leq C,
$$

where $C$ depends only on $\mu$, $\lambda$, $\tau$, the Lipschitz character of $\Omega$, and $R$ (if $\text{diam}(\Omega) \geq 1$).

Proof. Let $f \in L^2_0(\partial \Omega, \mathbb{R}^m)$ and $u = S_A(f)$. Then $\mathcal{L}(u) = 0$ in $\mathbb{R}^d \setminus \partial \Omega$, $(\nabla u)^* \in L^2(\partial \Omega)$ and $(\nabla u)_{\pm}$ exists n.t. on $\partial \Omega$. Also recall that $(\nabla_{\text{tan}} u)^+ = (\nabla_{\text{tan}} u)^- \text{ on } \partial \Omega$. Since $|u(X)| + |X||\nabla u(X)| = O(|X|^2)$ as $|X| \to \infty$, it follows from integration by parts that

$$
\int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} dX = -\int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^\alpha u^\alpha d\sigma. \quad (5.8)
$$

By the jump relation (4.15), $\int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^- d\sigma = -\int_{\partial \Omega} f d\sigma = 0$. Using Poincaré’s inequality on $\partial \Omega$, (5.8), together with (5.2), gives

$$
\int_{\Omega^2} |\nabla u|^2 dX \leq C r_0 \left\| \frac{\partial u}{\partial \nu} \right\|_2 \left\| \nabla_{\text{tan}} u \right\|_2. \quad (5.9)
$$

By combining (5.3) with (5.9) and then using the Cauchy inequality with an $\varepsilon > 0$, we see that $\left\| (\nabla u)^\pm \right\|_2 \leq C \left\| \left( \frac{\partial u}{\partial \nu} \right)^\pm \right\|_2$ and $\left\| (\nabla u)^\pm \right\|_2 \leq C \left\| \nabla_{\text{tan}} u \right\|_2$. It follows that

$$
\left\| \left( \frac{\partial u}{\partial \nu} \right)^\pm \right\|_2 \leq C \left\| \nabla_{\text{tan}} u \right\|_2 \leq C \left\| \nabla u \right\|_2 \leq C \left\| \left( \frac{\partial u}{\partial \nu} \right)^\pm \right\|_2. \quad (5.10)
$$

Consequently, by the jump relation, for any $f \in L^2_0(\partial \Omega, \mathbb{R}^m)$,

$$
\left\| f \right\|_2 \leq \left\| \left( \frac{\partial u}{\partial \nu} \right)^+ \right\|_2 + \left\| \left( \frac{\partial u}{\partial \nu} \right)^- \right\|_2 \leq C \left\| \left( \frac{\partial u}{\partial \nu} \right)^\pm \right\|_2 \quad (5.11)
$$

Furthermore, if $f \in L^2(\partial \Omega, \mathbb{R}^m)$ and $g = f - f_{\partial \Omega}$, then

$$
\left\| f \right\|_2 \leq C \left\| (-1/2)I + K_A)g \right\|_2 + \left\| f_{\partial \Omega} \right\|_2 \leq C \left\| (-1/2)I + K_A)g \right\|_2 + C \left\| f_{\partial \Omega} \right\|_2 \leq C \left\| (-1/2)I + K_A)g \right\|_2. \quad (5.12)
$$

We remark that the last inequality in (5.12) follows from the observation that $f_{\partial \Omega}$ is also the mean value of $-(\frac{\partial u}{\partial \nu})^-$ on $\partial \Omega$. Since $(\frac{\partial u}{\partial \nu})^+$ has mean value zero, this is a simple consequence of the jump relation (4.15).
Thus, to complete the proof, we only need to show that the operators $(1/2)I + K_A : L^2_0(\partial \Omega, \mathbb{R}^m) \rightarrow L^2_0(\partial \Omega, \mathbb{R}^m)$ and $-(1/2)I + K_A : L^2(\partial \Omega, \mathbb{R}^m) \rightarrow L^2(\partial \Omega, \mathbb{R}^m)$ are onto. To this end, we consider a family of matrices $A^s = sA + (1-s)I$, where $0 \leq s \leq 1$. Note that by $\ref{27}$, $\pm (1/2)I + K_{A^0}$ are invertible on $L^2_0(\partial \Omega, \mathbb{R}^m)$ and $L^2(\partial \Omega, \mathbb{R}^m)$ respectively. Also observe that for each $s \in [0, 1]$, the matrix $A^s$ satisfies the same conditions as $A$. Hence,

$$
\|f\|_2 \leq C\|((1/2)I + K_{A^s})f\|_2 \
\|f\|_2 \leq C\|(-(1/2)I + K_{A^s})f\|_2
$$

for any $f \in L^2_0(\partial \Omega, \mathbb{R}^m)$, for any $f \in L^2(\partial \Omega, \mathbb{R}^m)$, where $C$ is independent of $s$. Since $\|A^{s_1} - A^{s_2}\|_{C^{\lambda}(\mathbb{R}^d)} \leq |s_1 - s_2|\|A\|_{C^{\lambda}(\mathbb{R}^d)}$, it follows from Theorem $\ref{3.1}$ that $\{(1/2)I + K_{A^s} : 0 \leq s \leq 1\}$ and $\{-(1/2)I + K_{A^s} : 0 \leq s \leq 1\}$ are continuous families of bounded operators on $L^2_0(\partial \Omega, \mathbb{R}^m)$ and $L^2(\partial \Omega, \mathbb{R}^m)$ respectively. This, together with the estimates in $\ref{5.13}$ and the invertibility results for $s = 0$, gives the desired invertibility for $s = 1$. The operator norm estimates in $\ref{5.7}$ follow directly from $\ref{5.11}$ and $\ref{5.12}$.

**Remark 5.8.** Under the same assumptions on $A$ and $\Omega$ as in Lemma $\ref{5.7}$, the operator $S_A : L^2(\partial \Omega, \mathbb{R}^m) \rightarrow W^{1,2}(\partial \Omega, \mathbb{R}^m)$ is invertible and $\|((S_A)^{-1})\|_{W^{1,2}_{\partial \Omega} \rightarrow L^2} \leq C$. To see this, we let $f \in L^2(\partial \Omega, \mathbb{R}^m)$ and $u = S(f)$. It follows from the proof of Lemma $\ref{5.7}$ that

$$
\|(\nabla u)\|_2 \leq C\|\nabla_{\partial \Omega}u\|_2 + Cr_0^{-1}\|u\|_2.
$$

This, together with $\|(\nabla u)\|_2 \leq C\|\nabla_{\partial \Omega}u\|_2$ and the jump relation, gives

$$
\|f\|_2 \leq C\|\nabla_{\partial \Omega}S(f)\|_2 + Cr_0^{-1}\|S(f)\|_2 \leq C\|S(f)\|_{1,2}.
$$

Estimate $\ref{5.15}$ implies that $S : L^2(\partial \Omega, \mathbb{R}^m) \rightarrow W^{1,2}(\partial \Omega, \mathbb{R}^m)$ is one-to-one. A continuity argument similar to that in the proof of Lemma $\ref{5.7}$ shows that the operator is in fact invertible.

**Remark 5.9.** Under the same conditions on $A$ as in Theorem $\ref{5.4}$, the estimates in $\ref{5.7}$ and $\ref{5.15}$ hold with a “good” constant $C$.

We are now in a position to give the proof of Theorems $\ref{5.3}$ and $\ref{5.4}$.

**Proof of Theorems $\ref{5.3}$ and $\ref{5.4}$**

As we mentioned earlier, the uniqueness for the $L^2$ Neumann and regularity problems follows from the Green’s identity $\ref{5.2}$ by approximating $\Omega$ from inside. The existence for the $L^2$ Neumann and regularity problems is a direct consequence of the invertibility of $(1/2)I + K_A$ on $L^2_0(\partial \Omega, \mathbb{R}^m)$ and that of $S_A : L^2(\partial \Omega, \mathbb{R}^m) \rightarrow W^{1,2}(\partial \Omega, \mathbb{R}^m)$ respectively. Since $-(1/2)I + K_A$ is invertible on $L^2(\partial \Omega, \mathbb{R}^m)$, it follows by duality that $-(1/2)I + K_A$ is also invertible on $L^2(\partial \Omega, \mathbb{R}^m)$ and $\|(-(1/2)I + K_A)^{-1}\|_{L^2 \rightarrow L^2} = \|(-(1/2)I + K_A)^{-1}\|_{L^2 \rightarrow L^2}$. This gives the existence for the $L^2$ Dirichlet problem in $\Omega$. Note that under the conditions in Theorem $\ref{5.4}$ the operator norms of $\pm (1/2)I + K_{A^0}$ and $(S_A)^{-1}$ are bounded by a “good” constant $C$. It follows that estimates in $\ref{5.11}$ and $\|(u)^*\|_2 \leq C\|u\|_2$ hold with a “good” constant $C$.

To establish the uniqueness, we construct a matrix of Green’s functions $(G^{\alpha\beta}(X, Y))$ for $\Omega$, where

$$
G^{\alpha\beta}(X, Y) = \Gamma^{\alpha\beta}(X, Y) - W^{\alpha\beta}(X, Y)
$$

(5.16)
and for each $\beta$ and $Y \in \Omega$, $W^\beta(\cdot, Y) = (W_1^\beta(\cdot, Y), \ldots, W_m^\beta(\cdot, Y))$ is the solution to the $L^2$ regularity problem for $\mathcal{L}(u) = 0$ in $\Omega$ with boundary data

$$\Gamma^\beta(\cdot, Y) = (\Gamma_1^\beta(\cdot, Y), \ldots, \Gamma_m^\beta(\cdot, Y))$$ on $\partial \Omega$.

Suppose now that $\mathcal{L}(u) = 0$ in $\Omega$, $(u)^* \in L^2(\partial \Omega)$ and $u = 0$ n.t. on $\partial \Omega$. For $\rho > 0$ small, choose $\varphi = \varphi_\rho$ so that $\varphi = 1$ in $\{X \in \Omega : \text{dist}(X, \partial \Omega) \geq 2\rho\}$, $\varphi = 0$ in $\{X \in \Omega : \text{dist}(X, \partial \Omega) \leq \rho\}$ and $|\nabla \varphi| \leq C\rho^{-1}$. Fix $Y \in \Omega$ so that $\text{dist}(Y, \partial \Omega) \geq 2\rho$. It follows from (2.4) that

$$u^\gamma(Y) = u^\gamma(Y)\varphi(Y) = \int_{\Omega} a_{ij}^\alpha(X) \frac{\partial}{\partial x_j} \{G^\beta(Y, X')\} \frac{\partial}{\partial x_i} (u^\gamma \varphi) dX$$

$$= - \int_{\Omega} a_{ij}^\alpha(X) G^\beta(Y, X') \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial \varphi}{\partial x_j} dX$$

$$+ \int_{\Omega} a_{ij}^\alpha \frac{\partial}{\partial x_j} \{G^\beta(Y, X')\} u^\alpha \frac{\partial \varphi}{\partial x_i} dX,$$

where we have used the integration by parts and $A^* = A$. This gives

$$|u(Y)| \leq C \frac{\rho}{\rho} \int_{F^\rho} |G(X, Y)||\nabla u| dX + C \frac{\rho}{\rho} \int_{F^\rho} |\nabla X G(X, Y)||u| dX,$$

(5.17)

where $F^\rho = \{X \in \Omega : \rho \leq \text{dist}(X, \partial \Omega) \leq 2\rho\}$. Using $G(\cdot, Y) = u = 0$ n.t. on $\partial \Omega$ as well as the gradient estimate (2.3) on $u$, we may deduce from (5.18) that

$$|u(Y)| \leq C \int_{\partial \Omega} (\nabla G(\cdot, Y))^*_{3\rho}(u)^*_{3\rho} d\sigma,$$

(5.19)

where $(u)^*_{3\rho}(P) = \sup\{|\nabla u(X)| : X \in \gamma(P) \text{ and dist}(X, \partial \Omega) < 3\rho\}$. As $(\nabla G(\cdot, Y))^*_{3\rho}(u)^*_{3\rho} \in L^1(\partial \Omega)$, we may conclude from (5.19) by the Lebesgue dominated convergence theorem that $u(Y) = 0$. This completes the proof. \qed

### 6 Solvability for small scales, Part I

The main purpose of this and next sections is to establish the following theorem.

**Theorem 6.1.** Let $A = (a_{ij}^{\alpha\beta})$ be a real matrix satisfying the symmetry condition (1.5), the ellipticity condition (1.2) and the smoothness condition (1.4). Let $R \geq 1$. Then for any bounded Lipschitz domain $\Omega$ with connected boundary and $\text{diam}(\Omega) \leq R$, the $L^2$ Neumann and regularity problems for $\text{div}(A\nabla u) = 0$ in $\Omega$ are uniquely solvable and the solutions satisfy the estimates in (5.17) with constant $C$ depending only on $\mu, \lambda, \tau$, the Lipschitz character of $\Omega$, and $R$ (if $\text{diam}(\Omega) > 1$). Furthermore, the $L^2$ Dirichlet problem for $\text{div}(A\nabla u) = 0$ in $\Omega$ is uniquely solvable with estimate $\|(u)^*\|_2 \leq C\|u\|_2$.

**Remark 6.2.** Note that the periodicity of $A$ is not needed in Theorem 6.1. This is because we may reduce the general case to the case of the periodic coefficients. Indeed, by translation, we may assume that $0 \in \Omega$. If $\text{diam}(\Omega) \leq (1/4)$, we construct $\tilde{A} \in \Lambda(\mu, \lambda, \tau_0)$ so that $\tilde{A} = A$ on...
Let \( \Omega \) be a bounded Lipschitz domain with connected boundary. Suppose that \( \Lambda(\mu, \lambda, \tau) \) holds. Assume that \( \mathcal{L}(u) = 0 \) in \( \Omega \), \( (\nabla u)^* \in L^2(\partial\Omega) \) and \( (\nabla u)_+ \) exists n.t. on \( \partial\Omega \). Then

\[
\int_{\partial\Omega} |\nabla u|^2 \, d\sigma \leq C \int_{\partial\Omega} |\nabla u|^2 \, d\sigma + C \int_{\Omega} (|\nabla A| + r_0^{-1})|\nabla u|^2 \, dX,
\]

\[
\int_{\partial\Omega} |\nabla u|^2 \, d\sigma \leq C \int_{\partial\Omega} |\nabla_{\tan} u|^2 \, d\sigma + C \int_{\Omega} (|\nabla A| + r_0^{-1})|\nabla u|^2 \, dX,
\]

where \( C \) depends only on \( \mu \) and the Lipschitz character of \( \Omega \).

Proof. Let \( h \) be a \( C^1 \) vector field on \( \mathbb{R}^d \) such that \( \text{supp}(h) \subset \{ X : \text{dist}(X, \partial\Omega) < cr_0 \} \), \( |\nabla h| \leq Cr_0^{-1} \) and \( <h, n>_c > c > 0 \) on \( \partial\Omega \). As in the case of constant coefficients, the estimates in (6.2) follow from the so-called Rellich identities,

\[
\int_{\partial\Omega} <h, n>_c a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial u^\beta}{\partial x_j} \, d\sigma = 2 \int_{\partial\Omega} <h, \nabla u^\alpha>_c \left( \frac{\partial u}{\partial\nu} \right)^\alpha \, d\sigma + I_1,
\]

\[
\int_{\partial\Omega} <h, n>_c a_{ij}^{\alpha\beta} \frac{\partial u^\alpha}{\partial x_i} \cdot \frac{\partial u^\beta}{\partial x_j} \, d\sigma = 2 \int_{\partial\Omega} h_k a_{ij}^{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} \left( n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k} \right) u^\alpha \, d\sigma + I_2
\]
where

\[ |I_1| + |I_2| \leq C \int_{\Omega} \{ |\nabla h| + |h| |\nabla A| \} |\nabla u|^2 \, dX \]

and \( C \) depends only on \( \mu \). The proof of (6.3), which uses integration by parts and the assumption that \( A^*_\epsilon = A \), is similar to the case of constant coefficients. The latter may be found in [12]. \( \square \)

**Remark 6.5.** Let \( L(u) = 0 \) in \((-1/2, 1/2)^d \setminus \overline{\Omega}\). Suppose that \((\nabla u)^*_\epsilon \in L^2(\partial \Omega)\) and \((\nabla u)_-\epsilon \) exists n.t. on \( \partial \Omega \). Under the same conditions on \( \Omega \) and \( A \) as in Lemma 6.4, we have

\[
\int_{\partial \Omega} |(\nabla u)_-|^2 \, d\sigma \leq C \int_{\partial \Omega} |(\partial u/\partial \nu)_-|^2 \, d\sigma + C \int_{\Omega \cap \{|(\nabla A) + r_0^{-1}|\nabla u|^2 \} dX},
\]

and

\[
\int_{\partial \Omega} |(\nabla u)_-|^2 \, d\sigma \leq C \int_{\partial \Omega} |(\nabla \tan u)_-|^2 \, d\sigma + C \int_{\Omega \cap \{|(\nabla A) + r_0^{-1}|\nabla u|^2 \} dX},
\]

where \( C \) depends only on \( \mu \) and the Lipschitz character of \( \Omega \). The proof is similar to that of Lemma 6.4.

**Lemma 6.6.** Under the same assumptions as in Lemma 6.4, we have

\[
\int_{\partial \Omega} |\nabla u|^2 \, d\sigma \leq C \{ 1 + r_0^{2\lambda_0 + \rho^{2\lambda_0 - 2}} \} \int_{\partial \Omega} |\partial u/\partial \nu|^2 \, d\sigma + C(\rho r_0)^{\lambda_0} \int_{\partial \Omega} |(\nabla u)^*_\epsilon|^2 \, d\sigma,
\]

and

\[
\int_{\partial \Omega} |\nabla u|^2 \, d\sigma \leq C \{ 1 + r_0^{2\lambda_0 + \rho^{2\lambda_0 - 2}} \} \int_{\partial \Omega} |\nabla \tan u|^2 \, d\sigma + C(\rho r_0)^{\lambda_0} \int_{\partial \Omega} |(\nabla u)^*_\epsilon|^2 \, d\sigma,
\]

where \( 0 < \rho < 1 \) and \( C \) depends only on \( \mu \), the Lipschitz character of \( \Omega \) and \( \lambda_0, C_1 \) in (6.7).

**Proof.** Write \( \Omega = F_1 \cup F_2 \), where \( F_1 = \{ X \in \Omega : \text{dist}(X, \partial \Omega) \leq \rho r_0 \} \) and \( F_2 = \{ X \in \Omega : \text{dist}(X, \partial \Omega) > \rho r_0 \} \). Using the condition (6.1), we obtain

\[
\int_{\Omega} |\nabla A| |\nabla u|^2 \, dX \leq C_1 \int_{F_1} \{ \text{dist}(X, \partial \Omega) \} \lambda_0^{-1} |\nabla u|^2 \, dX + C_1(\rho r_0)^{\lambda_0^{-1}} \int_{F_2} |\nabla u|^2 \, dX
\]

\[
\leq C(\rho r_0)^{\lambda_0} \int_{\partial \Omega} |(\nabla u)^*_\epsilon|^2 \, d\sigma + C_1(\rho r_0)^{\lambda_0^{-1}} \int_{\Omega} |\nabla u|^2 \, dX.
\]

This, together with (6.2) and (5.9) for \( \Omega_+ \), gives

\[
|\nabla u|^2 \leq C \frac{\partial u/\partial \nu}{\partial u/\partial \nu}^2 + C(1 + r_0^{\lambda_0 - \rho^{2\lambda_0 - 1}}) \frac{\nabla t}{\nabla t}^2 + C(\rho r_0)^{\lambda_0} |(\nabla u)^*_\epsilon|^2 + C(\rho r_0)^{\lambda_0} \lambda_0^{-1} \int_{\nabla u}^2 \, dX.
\]

The first inequality in (6.5) follows from (6.7) by the Cauchy inequality with an \( \varepsilon \). The proof of the second inequality in (6.5) is similar. \( \square \)

**Remark 6.7.** Let \( L(u) = 0 \) in \( \Omega_- \). Suppose that \((\nabla u)^*_\epsilon \in L^2(\partial \Omega), (\nabla u)_-\epsilon \) exists n.t. on \( \partial \Omega \), and \(|u(X)| = O(|X|^{2-d}) \) as \( |X| \to \infty \). In view of Remark 6.5 and (5.9) for \( \Omega_- \), the same
argument as in the proof of Lemma 6.6 shows that
\[
\int_{\partial \Omega} |(\nabla u)_-|^2 \, d\sigma \leq C\{1 + r_0^{2\lambda_0} \rho^{2\lambda_0-2}\} \|\left(\frac{\partial u}{\partial \nu}\right)_-\|^2_2 + C(\rho_0)^{\lambda_0}\| (\nabla u)^* \|^2_2 \\
+ C(\rho_0)^{\lambda_0-1}|u_{\partial \Omega}| \left| \int_{\partial \Omega} \left(\frac{\partial u}{\partial \nu}\right)_- \, d\sigma \right|,
\]
(6.8)
for any \(0 < \rho < 1\).

The following theorem completes Step One.

**Theorem 6.8.** Suppose that \(\Omega\) and \(A\) satisfy the conditions in Theorem 6.3. We further assume that \(0 \in \Omega\) and \(A\) satisfies (6.7). Then \((1/2)I + K_A\) and \(-(1/2)I + K_A\) are invertible on \(L^2_{0}(\partial \Omega, \mathbb{R}^m)\) and \(L^2(\partial \Omega, \mathbb{R}^d)\) respectively, and the estimates in (5.7) hold with a constant \(C\) depending only on \(\mu, \lambda, \tau,\) the Lipschitz character of \(\Omega\) and \(C_1, \lambda_0\) in (6.7).

**Proof.** Let \(f \in L^2_{0}(\partial \Omega, \mathbb{R}^m)\) and \(u = S(f)\) be the single layer potential. In view of (6.8), we obtain
\[
\|(\nabla u)_-\|_2 \leq C\rho_1^{\lambda_0-1}\|(\nabla u)_+\|_2 + C\rho_1^{\lambda_0/2}\|f\|_2,
\]
(6.9)
for any \(0 < \rho_1 < 1\), where we also used the fact that \((\nabla u)_- = (\nabla u)_+\) and \(\|(\nabla u)^*\|_2 \leq C\|f\|_2\). Similarly, by (6.5),
\[
\|(\nabla u)_+\|_2 \leq C\rho_2^{\lambda_0-1}\|(\partial u)/(\partial \nu)_+\|_2 + C\rho_2^{\lambda_0/2}\|f\|_2,
\]
(6.10)
for any \(0 < \rho_2 < 1\). It follows from the jump relation (4.15), (6.9) and (6.10) that
\[
\|f\|_2 \leq \|\left(\frac{\partial u}{\partial \nu}\right)_+\|_2 + \|\left(\frac{\partial u}{\partial \nu}\right)_-\|_2 \\
\leq C\rho_1^{\lambda_0-1}\rho_2^{\lambda_0-1}\|\left(\frac{\partial u}{\partial \nu}\right)_+\|_2 + C\{\rho_1^{\lambda_0-1}\rho_2^{\lambda_0/2} + \rho_1^{\lambda_0/2}\}\|f\|_2.
\]
(6.11)
We now choose \(\rho_1 \in (0, 1)\) and then \(\rho_2 \in (0, 1)\) so that \(C\{\rho_1^{\lambda_0-1}\rho_2^{\lambda_0/2} + \rho_1^{\lambda_0/2}\} \leq (1/2)\). This gives
\[
\|f\|_2 \leq C\|\left(\frac{\partial u}{\partial \nu}\right)_+\|_2 = C\|((1/2)I + K_A)f\|_2,
\]
(6.12)
for any \(f \in L^2_{0}(\partial \Omega, \mathbb{R}^m)\). The same argument also shows that for any \(f \in L^2_{0}(\partial \Omega, \mathbb{R}^m)\),
\[
\|f\|_2 \leq C\|\left(\frac{\partial u}{\partial \nu}\right)_-\|_2 = C\|(-(1/2)I + K_A)f\|_2.
\]
(6.13)
The rest of the proof is the same as that of Lemma 5.7. \(\square\)
Remark 6.9. Let \( f \in L^2(\partial \Omega) \) and \( u = S(f) \). It follows from (6.3) and (6.8) that
\[
\|(\nabla u)_+\|_2 \leq C \rho_1^{\lambda_0 - 1} \|\nabla \tan u\|_2 + C \rho_1^{\lambda_0/2} \|f\|_2, \\
\|(\nabla u)_-\|_2 \leq C \rho_2^{\lambda_0 - 1} \|\nabla \tan u\|_2 + C \rho_2^{\lambda_0/2} \|f\|_2 + Cr_0^{-1}\|u\|_2,
\]
for any \( \rho_1, \rho_2 \in (0, 1) \). This, together with the jump relation, implies
\[
\|f\|_2 \leq C\|\nabla \tan S(f)\|_2 + Cr_0^{-1}\|S(f)\|_2 \leq C\|S(f)\|_{1,2}.
\]
Thus \( S : L^2(\partial \Omega, \mathbb{R}^m) \to W^{1,2}(\partial \Omega, \mathbb{R}^m) \) is one-to-one. A continuity argument similar to that in the proof of Lemma 6.7 shows that the operator is in fact invertible.

7 Solvability for small scales, Part II

In this section we complete the second and third steps in the proof of Theorems 6.3.

Step Two: Given any \( A \in \Lambda(\mu, \lambda, \tau) \) and \( \Omega \) such that \( A^* = A \), \( 0 \in \Omega \) and \( r_0 = \text{diam}(\Omega) \leq (1/4) \), construct \( \tilde{A} \in \Lambda(\mu, \lambda_0, \tau_0) \) with \( \lambda_0 \) and \( \tau_0 \) depending only on \( \mu, \lambda, \tau \) and the Lipschitz character of \( \Omega \), such that
\[
\tilde{A}(X) = A(X) \quad \text{if dist}(X, \partial \Omega) \leq c r_0,
\]
and such that the operators
\[
(1/2)I + K_{\tilde{A}} : L^2(\partial \Omega, \mathbb{R}^m) \to L^2(\partial \Omega, \mathbb{R}^m), \\
-(1/2)I + K_{\tilde{A}} : L^2(\partial \Omega, \mathbb{R}^m) \to L^2(\partial \Omega, \mathbb{R}^m),
\]

are invertible and the operator norms of their inverses are bounded by a “good” constant.

Lemma 7.1. Given \( A \in \Lambda(\mu, \lambda, \tau) \) and a Lipschitz domain \( \Omega \) such that \( \text{diam}(\Omega) \leq (1/4) \) and \( 0 \in \Omega \). There exists \( \tilde{A} \in \Lambda(\mu, \lambda_0, \tau_0) \) such that \( \tilde{A} = A \) on \( \partial \Omega \) and \( \tilde{A} \) satisfies the condition (6.7), where \( \lambda_0 \in (0, \lambda] \), \( \tau_0 \) and \( C_1 \) in (6.7) depend only on \( \mu, \lambda, \tau \) and the Lipschitz character of \( \Omega \). In addition, \( \tilde{(A)^*} = \tilde{A} \) if \( A^* = A \).

Proof. By periodicity it suffices to define \( \tilde{A} = (\tilde{a}_{ij}^{\alpha \beta}) \) on \([-1/2, 1/2]^d \). This is done as follows. On \( \Omega \) we define \( \tilde{A} \) to be the Poisson extension of \( A \) on \( \partial \Omega \); i.e., \( \tilde{a}_{ij}^{\alpha \beta} \) is harmonic in \( \Omega \) and \( \tilde{a}_{ij}^{\alpha \beta} = a_{ij}^{\alpha \beta} \) on \( \partial \Omega \), for each \( i, j, \alpha, \beta \). On \([-1/2, 1/2]^d \setminus \Omega \), we define \( \tilde{A} \) to be the harmonic function in \((-1/2, 1/2)^d \setminus \Omega\) with boundary data \( \tilde{A} = A \) on \( \partial \Omega \) and \( \tilde{A} = I \) on \( \partial[-1/2, 1/2]^d \). Note that the latter boundary condition allows us to extend \( \tilde{A} \) to \( \mathbb{R}^d \) by periodicity.

Since \( a_{ij}^{\alpha \beta} \xi_i^\alpha \xi_j^\beta \) is harmonic in \((-1/2, 1/2)^d \setminus \partial \Omega\), the ellipticity condition (1.2) for \( \tilde{A} \) follows readily from the maximum principle. By the solvability of Laplace’s equation in Lipschitz domains with Hölder continuous data (see e.g. [17]), there exists \( \lambda_1 \in (0, 1) \), depending only on the Lipschitz character of \( \Omega \), such that \( \tilde{A} \in C^{\lambda_1}(\Omega) \) and \( \tilde{A} \in C^{\lambda_1}([-1/2, 1/2]^d \setminus \Omega) \), where \( \lambda_0 = \lambda \) if \( \lambda < \lambda_1 \), and \( \lambda_0 = \lambda_1 \) if \( \lambda \geq \lambda_1 \). It follows that \( \tilde{A} \in C^{\lambda_0}(\mathbb{R}^d) \). Using the well known interior estimates for harmonic functions, one may also show that \( |\nabla \tilde{A}(X)| \leq C_1 \{ \text{dist}(X, \partial \Omega) \}^{\lambda_0 - 1} \) for \( X \in [-3/4, 3/4]^d \setminus \partial \Omega \), where \( C_1 \) depends only on \( \mu, \lambda, \tau \) and the Lipschitz character of \( \Omega \). Thus we have proved that \( \tilde{A} \in \Lambda(\mu, \lambda_0, \tau) \) and satisfies the condition (6.1). Clearly, \( \tilde{(A)^*} = \tilde{A} \) if \( A^* = A \).
Let $\theta \in C^\infty_0(-1/2, 1/2)$ such that $0 \leq \theta \leq 1$ and $\theta = 1$ on $(-1/4, 1/4)$. Given $A \in \Lambda(\mu, \lambda, \tau)$ with $A^* = A$, define

$$A^\theta(X) = \theta \left(\frac{\delta(X)}{\rho}\right) A(X) + \left[1 - \theta \left(\frac{\delta(X)}{\rho}\right)\right] \bar{A}(X)$$

(7.3)

for $X \in [-1/2, 1/2]^d$, where $\rho \in (0, 1/8)$, $\delta(X) = \text{dist}(X, \partial \Omega)$ and $\bar{A}(X)$ is the matrix constructed in Lemma [7.1]. Extend $A^\rho$ to $\mathbb{R}^d$ by periodicity. Clearly, $A^\rho$ satisfies the ellipticity condition [1.2] and $(A^\rho)^* = A^\rho$.

**Lemma 7.2.** Let $A^\rho$ be defined by (7.3). Then

$$\|A^\rho - \bar{A}\|_\infty \leq C\rho^\lambda_0 \quad \text{and} \quad \|A^\rho - \bar{A}\|_{C^{0, \lambda_0}(\mathbb{R}^d)} \leq C,$$

(7.4)

where $C$ depends only on $\mu$, $\lambda$, $\tau$ and the Lipschitz character of $\Omega$.

**Proof.** Let $H^\rho = A^\rho - \bar{A}$. Given $X \in [-1/2, 1/2]^d$, let $P \in \partial \Omega$ such that $|X - P| = \delta(X)$. Since $A(P) = \bar{A}(P)$, we have

$$|A(X) - \bar{A}(X)| \leq |A(X) - A(P)| + |\bar{A}(P) - \bar{A}(X)| \leq C|X - P|^\lambda_0 = C\{\delta(X)\}^\lambda_0.$$

It follows that

$$|H^\rho(X)| \leq C\theta(\rho^{-1}\delta(X))\{\delta(X)\}^\lambda_0 = C\theta(\rho^{-1}\delta(X))\{\rho^{-1}\delta(X)\}^\lambda_0 \rho^\lambda_0 \leq C\rho^\lambda_0.$$

This gives $\|A^\rho - A\|_\infty \leq C\rho^\lambda_0$.

Next we show $|H^\rho(X) - H^\rho(Y)| \leq C|X - Y|^\lambda_0$ for any $X, Y \in \mathbb{R}^d$. Since $\|H^\rho\|_\infty \leq C\rho^\lambda_0$, we may assume that $|X - Y| \leq \rho$. Note that $H^\rho = 0$ on $[-1/2, 1/2]^d \setminus [-3/8, 3/8]^d$. Thus it is enough to consider the case where $X, Y \in [-1/2, 1/2]^d$. We may further assume that $\delta(X) \leq \rho$ or $\delta(Y) \leq \rho$. For otherwise, $H^\rho(X) = H^\rho(Y) = 0$ and there is nothing to show. Finally, suppose that $\delta(Y) \leq \rho$. Then

$$|H^\rho(X) - H^\rho(Y)| \leq \theta(\rho^{-1}\delta(X))\{\rho^{-1}\delta(X)\}|(A(X) - \bar{A}(X)) - (A(Y) - \bar{A}(Y))|$$

$$+ |A(Y) - \bar{A}(Y)||\theta(\rho^{-1}\delta(X)) - \theta(\rho^{-1}\delta(Y))||$$

$$\leq C|X - Y|^\lambda_0 + C\{\delta(Y)\}^\lambda_0 |X - Y| \cdot \rho^{-1}$$

$$\leq C|X - Y|^\lambda_0 + C\rho^{\lambda_0-1}|X - Y|$$

$$\leq C|X - Y|^\lambda_0.$$

The proof for the case $\delta(X) \leq \rho$ is the same. \qed

It follows from Lemma [7.2] that for $\rho \in (0, 1/4),$

$$\|A^\rho - \bar{A}\|_{C^{\lambda_0/2}(\mathbb{R}^d)} \leq C\rho^{\lambda_0/2}.$$  

(7.5)

Since $A^\rho = A = \bar{A}$ on $\partial \Omega$, we may deduce from Theorem [3.4] that

$$\|\mathcal{K}_{A^\rho} - \mathcal{K}_{\bar{A}}\|_{L^2 \rightarrow L^2} \leq C\|A^\rho - \bar{A}\|_{C^{\lambda_0/2}(\mathbb{R}^d)} \leq C\rho^{\lambda_0/2}.$$

(7.6)
for any $\rho \in (0, 1/4)$. Note that by Lemma 7.4 and Theorem 6.8, the operator $(1/2)I + K_A$ is invertible on $L^2_0(\partial \Omega, \mathbb{R}^m)$ and $\|(1/2)I + K_A\|_{L^2_0 - L^2_0} \leq C$. Write

$$(1/2)I + K_{A^\rho} = (1/2)I + K_A + (K_{A^\rho} - K_A).$$

In view of (7.6), one may choose $\rho > 0$ depending only on $\mu, \lambda, \tau$ and the Lipschitz character of $\Omega$ so that

$$\|(1/2)I + K_A\|_{L^2_0 - L^2_0} \leq 1/2.$$ 

It follows that $(1/2)I + K_{A^\rho}$ is invertible on $L^2_0(\partial \Omega, \mathbb{R}^m)$ and

$$\|(1/2)I + K_{A^\rho}\|_{L^2_0 - L^2_0} \leq 2\|(1/2)I + K_A\|_{L^2_0 - L^2_0} \leq 2C.$$ 

Similar arguments show that it is possible to choose $\rho$ depending only on $\mu, \lambda, \tau$ and the Lipschitz character of $\Omega$ such that $-(1/2)I + K_{A^\rho} : L^2(\partial \Omega, \mathbb{R}^m) \to L^2(\partial \Omega, \mathbb{R}^m)$ and $S_{A^\rho} : L^2(\partial \Omega, \mathbb{R}^m) \to W^{1,2}(\partial \Omega, \mathbb{R}^m)$ are invertible and the operator norms of their inverses are bounded by a “good” constant. Let $\bar{A} = A^\rho$. Note that if $\text{dist}(X, \partial \Omega) \leq (1/4)\rho$, $A^\rho(X) = A(X)$. This completes Step Two.

Step Three, which is given in the following lemma, involves a perturbation argument.

**Lemma 7.3.** Let $A^0 = (a_{ij}^{0\alpha})$, $A^1 = (b_{ij}^{0\beta}) \in \Lambda(\mu, \lambda, \tau)$. Let $\Omega$ be a bounded Lipschitz domain. Suppose that $A^0 = A^1$ in $\{X \in \Omega : \text{dist}(X, \partial \Omega) \leq c_0r_0\}$ for some $c_0 > 0$, where $r_0 = \text{diam}(\Omega)$. Assume that $\mathcal{L}^0 = -\text{div}(A^0\nabla)$ has the Rellich property in $\Omega$ with constant $C_0$. Then $\mathcal{L}^1 = -\text{div}(A^1\nabla)$ has the Rellich property in $\Omega$ with constant $C_1$, where $C_1$ depends only on $d, m, \mu, \lambda, \tau, c_0, C_0$ and the Lipschitz character of $\Omega$.

**Proof.** Suppose that $\mathcal{L}^1(u) = 0$ in $\Omega$, $(\nabla u)^* \in L^2(\partial \Omega)$ and $\nabla u$ exists n.t. on $\partial \Omega$. Let $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $|\nabla \varphi| \leq Cr_0^{-1}$, $\varphi = 1$ on $\{X \in \mathbb{R}^d : \text{dist}(X, \partial \Omega) \leq (1/4)c_0r_0\}$ and $\varphi = 0$ on $\{X \in \mathbb{R}^d : \text{dist}(X, \partial \Omega) \geq (1/2)c_0r_0\}$. Let $\bar{u} = \varphi(u - E)$, where $E = u_{\Omega}$ is the average of $u$ over $\Omega$. Note that

$$\mathcal{L}^0(\bar{u}) = -\partial_i\{a_{ij}^{0\alpha}(\partial_j\varphi)(u - E)_{\beta}\} - a_{ij}^{0\beta}(\partial_i\varphi)(\partial_ju),$$

where we have used the fact that $\mathcal{L}^0(u) = \mathcal{L}^1(u) = 0$ on $\{X \in \Omega : \text{dist}(X, \partial \Omega) < c_0r_0\}$. It follows from (4.6) that

$$\bar{u}(X) = S_{A^0} \left( \frac{\partial \bar{u}}{\partial \nu_{A^0}} \right) - D_{A^0}(\bar{u}) + v(X) = w(X) + v(X),$$

where $v$ satisfies

$$|\nabla v(X)| \leq C \int_\Omega |\nabla_X \nabla_Y \Gamma_{A^0}(X, Y)||\nabla \varphi||u - E|dY$$

$$+ C \int_\Omega |\nabla_X \Gamma_{A^0}(X, Y)||\nabla \varphi||\nabla u|dY.$$ 

This, together with (2.5) and (2.7), implies that if $X \in \Omega$ and $\text{dist}(X, \partial \Omega) \leq (1/5)c_0r_0$,

$$|\nabla v(X)| \leq \frac{C}{r_0^d} \int_\Omega |\nabla u|^2 dY \leq Cr_0^{-d} \parallel \frac{\partial u}{\partial \nu_{A^1}} \parallel_2 \parallel \nabla_{\tan} u \parallel_2,$$

(7.9)
where we have used \((7.9)\) for the last inequality.

Next, note that \(L^0(w) = 0\) in \(\Omega\), where \(w = \bar{u} - v\). Using \((7.9)\) and the assumption 
\((\nabla u)^* \in L^2(\partial \Omega)\) as well as the \(L^\infty\) gradient estimate \((2.3)\), we may deduce that 
\((\nabla w)^* \in L^2(\partial \Omega)\) and \(\nabla w\) exists n.t. on \(\partial \Omega\). Since \(L^0\) has the Rellich property, this implies that

\[
\|\nabla w\|_2 \leq C_0 \|\frac{\partial w}{\partial \nu}\|_2 \leq C \left\{ \|\frac{\partial u}{\partial \nu}\|_2 + \|\nabla v\|_2 \right\} \tag{7.10}
\]

where we used \((7.9)\) in the last inequality. Using \((7.9)\) again, we obtain

\[
\|\nabla u\|_2 \leq C \left\{ \|\nabla w\|_2 + \|\nabla v\|_2 \right\} \leq C \left\{ \|\frac{\partial u}{\partial \nu}\|_2 + \|\frac{\partial u}{\partial \nu}\|_2^{1/2} \right\} \tag{7.11}
\]

The desired estimate \(\|\nabla u\|_2 \leq C\|\frac{\partial u}{\partial \nu}\|_2\) follows readily from \((7.11)\) by the Cauchy inequality with an \(\varepsilon\). The proof of \(\|\nabla u\|_2 \leq C\|\nabla u\|_2\) is similar.

Finally we give the proof of Theorem 6.3

**Proof of Theorem 6.3** By Step Two there exists \(\bar{A} = A\) in \(\{X \in \mathbb{R}^d : \text{dist}(X, \partial \Omega) \leq c\}\) and \((1/2)I + K_{\bar{A}} : L^2(\partial \Omega, \mathbb{R}^m) \rightarrow L^2(\partial \Omega, \mathbb{R}^m), S_{\bar{A}} : L^2(\partial \Omega, \mathbb{R}^m) \rightarrow W^{1,2}(\partial \Omega, \mathbb{R}^m)\) are invertible. Moreover, the operator norms of these inverses are bounded by a “good” constant \(C\). It follows that the \(L^2\) Neumann and regularity problems for \(L^{\bar{A}}(u) = 0\) in \(\Omega\) are uniquely solvable and the solutions satisfy \(\|(\nabla u)^*\|_2 \leq C\|\frac{\partial u}{\partial \nu}\|_2\) and \(\|(\nabla u)^*\|_2 \leq C\|\nabla u\|_2\) with “good” constant \(C\). In particular the operator \(L^{\bar{A}}\) has the Rellich property in \(\Omega\) with a “good” constant \(C\). By Lemma \((7.3)\) this implies that \(L\) has the Rellich property in \(\Omega\) with a “good” constant \(C\). The proof is complete.

8 Rellich estimates for large scales

Let \(\psi\) be a Lipschitz function on \(\mathbb{R}^{d-1}\) such that \(\psi(0) = 0\) and \(\|\nabla \psi\|_\infty \leq M\). Let \(D(r)\) and \(\Delta(r)\) be defined as in \((5.4)\). In this section we establish the following.

**Theorem 8.1.** Let \(L = -\text{div}(A\nabla)\) with \(A \in \Lambda(\mu, \lambda, \tau)\) and \(A^* = A\). Assume that \(A \in C^2(\mathbb{R}^d)\). Suppose that \(L(u) = 0\) in \(D(8r)\) for some \(r > 0\), where \(u \in C^2(D(8r))\), \((\nabla u)^*_{D(8r)} \in L^2(\Delta(6r))\) and \(\nabla u\) exists n.t. on \(\Delta(6r)\). Then

\[
\int_{\Delta(r)} |\nabla u|^2 \, d\sigma \leq C \int_{\Delta(4r)} \left| \frac{\partial u}{\partial \nu} \right|^2 \, d\sigma + \frac{C}{r} \int_{\Delta(8r)} |\nabla u|^2 \, dX, \tag{8.1}
\]

\[
\int_{\Delta(r)} |\nabla u|^2 \, d\sigma \leq C \int_{\Delta(4r)} |\nabla u|^2 \, d\sigma + \frac{C}{r} \int_{\Delta(8r)} |\nabla u|^2 \, dX,
\]

where \(C\) depends only on \(\mu, \lambda, \tau\) and \(M\).
Observe that by Theorem 6.3 and the localization techniques used in the proof of Lemma 5.3, the estimates in (8.1) hold for $0 < r \leq 3$ with $C = C(\mu, \lambda, \tau, M) > 0$. We will rely on the Rellich identities developed in [20] treat the case $r > 3$.

Let $Q$ be the difference operator defined by

$$Q(f)(x', x_d) = f(x', x_d + 1) - f(x', x_d).$$  \hspace{1cm} (8.2)

It is easy to verify that

$$Q(f) - Q(f)Q(g) = fQ(g) + gQ(f)$$  \hspace{1cm} (8.3)

and

$$\int_{D(r)} Q(f) dX = \int_{|x'|<r} f dX - \int_{\psi(x')<x_d<\psi(x') + 1} f dX,$$  \hspace{1cm} (8.4)

where $C = 10\sqrt{d(M + 1)}$. Also note that since $A(x', x_d + 1) = A(x', x_d)$, we have $Q L = L Q$.

In particular, $L(Q(u)) = 0$ whenever $L(u) = 0$.

**Lemma 8.2.** Let $r > 1$. Under the same conditions on $A$ and $u$ as in Theorem 8.1, we have

$$\int_{\psi(x')<x_d<\psi(x') + 1} |\nabla u|^2 dX \leq \int_{\Delta(2r)} |\partial u/\partial \nu|^2 d\sigma + \frac{C}{r^2} \int_{D(3r)} |\nabla u|^2 dX,$$  \hspace{1cm} (8.5)

where $C$ depends only on $\mu$ and $M$.

**Proof.** By approximating the domain $D(3r)$ from inside, we may assume that $u \in C^2(\overline{D(3r)})$. Let $\Omega_\rho = D(\rho)$ for $\rho \in (r, 2r)$. It follows from integration by parts that

$$\int_{\partial \Omega_\rho} \frac{\partial u}{\partial \nu} \cdot Q(u) d\sigma = \int_{\Omega_\rho} a_{ij}^\alpha \frac{\partial u^\beta}{\partial x_j} \cdot Q \left( \frac{\partial u^\alpha}{\partial x_i} \right) dX$$

$$= \frac{1}{2} \int_{\Omega_\rho} a_{ij}^\alpha \left\{ \frac{\partial u^\beta}{\partial x_j} \cdot Q \left( \frac{\partial u^\alpha}{\partial x_i} \right) + Q \left( \frac{\partial u^\beta}{\partial x_j} \right) \cdot \frac{\partial u^\alpha}{\partial x_i} \right\} dX$$

$$= \frac{1}{2} \int_{\Omega_\rho} a_{ij}^\alpha \left\{ Q \left( \frac{\partial u^\beta}{\partial x_j} \cdot \frac{\partial u^\alpha}{\partial x_i} \right) - Q \left( \frac{\partial u^\beta}{\partial x_j} \right) \cdot \frac{\partial u^\alpha}{\partial x_i} \right\} dX$$

$$\leq \frac{1}{2} \int_{\Omega_\rho} a_{ij}^\alpha Q \left( \frac{\partial u^\beta}{\partial x_j} \cdot \frac{\partial u^\alpha}{\partial x_i} \right) dX,$$

where we have used the symmetry condition (1.5), (8.3) and the ellipticity condition (1.2). This, together with the periodicity condition on $A(X)$, gives

$$- \int_{\Omega_\rho} Q \left( a_{ij}^\alpha \frac{\partial u^\beta}{\partial x_j} \cdot \frac{\partial u^\alpha}{\partial x_i} \right) dX \leq -2 \int_{\partial \Omega_\rho} \frac{\partial u}{\partial \nu} \cdot Q(u) d\sigma.$$
In view of (8.4), we obtain
\[ \mu \int_{|x'|<\rho \atop \psi(x)<x_d<\psi(x)+1} |\nabla u|^2 \, dX \]
\[ \leq -2 \int_{\partial B \Delta \rho} \frac{\partial u}{\partial \nu} \cdot Q(u) \, d\sigma + \frac{1}{\mu} \int_{C\rho<x_d<C\rho+1} |\nabla u|^2 \, dX \]
\[ \leq \delta \int_{\Delta(\rho)} |Q(u)|^2 \, d\sigma + \frac{1}{\delta} \int_{\Delta(\rho)} |\nabla \psi|^2 \, d\sigma \]
\[ + C \int_{\partial B \backslash \Delta(\rho)} |\nabla u||Q(u)| \, d\sigma + C \int_{C\rho<x_d<C\rho+1} |\nabla u|^2 \, dX \]  \hspace{1cm} (8.6)

for \( \rho \in (r, 2r) \), where \( \delta \in (0, 1) \) and we have used the Cauchy inequality.

Next, using \(|Q(u)(x',x_d)|^2 \leq \int_{x_d}^{x_d+1} |\frac{\partial u}{\partial s}(x',s)|^2 \, ds \), we see that
\[ \int_{\Delta(\rho)} |Q(u)|^2 \, d\sigma \leq C \int_{\psi(x')<x_d<\psi(x')+1} |\nabla u|^2 \, dX \]  \hspace{1cm} (8.7)

and
\[ \int_{\partial B \backslash \Delta(\rho)} |Q(u)|^2 \, d\sigma \]
\[ \leq C \int_{\partial B(0,\rho)} \int_{\psi(x')}^{\psi(x')+\rho+1} |\nabla u(x',s)|^2 \, ds \, d\sigma + C \int_{C\rho<x_d<C\rho+1} |\nabla u|^2 \, dX, \]  \hspace{1cm} (8.8)

where \( C \) depends only on \( M \).

Finally we choose \( \delta > 0 \) in (8.6) so small that \( \delta C \leq (1/2)\mu \). In view of (8.6), (8.7) and (8.8), we obtain
\[ \int_{\psi(x')<x_d<\psi(x')+1} |\nabla u|^2 \, dX \]
\[ \leq C \int_{\Delta(2r)} |\nabla u|^2 \, d\sigma + C \int_{\partial B(0,\rho)} \int_{\psi(x')}^{\psi(x')+2r+1} |\nabla u(x',s)|^2 \, ds \, d\sigma \]
\[ + C \int_{|x'|<2r} |\nabla u(x',C\rho)|^2 \, dx' + C \int_{C\rho<x_d<C\rho+1} |\nabla u|^2 \, dX, \]  \hspace{1cm} (8.9)

The desired estimate follows by integrating both sides of (8.9) with respect to \( \rho \) over the interval \((r, 2r)\).

**Lemma 8.3.** Let \( r > 1 \). Under the same conditions on \( A \) and \( u \), we have
\[ \int_{\psi(x')<x_d<\psi(x')+1} |\nabla u|^2 \, dX \leq C \int_{\Delta(2r)} |\nabla \tan u|^2 \, d\sigma + \frac{C}{r} \int_{D(3r)} |\nabla u|^2 \, dX, \]  \hspace{1cm} (8.10)

where \( C \) depends only on \( \mu \) and \( M \).
Proof. Let $\Omega_\rho = D(\rho)$ for $\rho \in (r, 2r)$. As in the proof of Lemma 8.2, we have

$$\mu \int_{\psi(x') < x_d < \psi(x') + 1} |\nabla u|^2 \, dx \leq -2 \int_{\partial \Omega_\rho} \frac{\partial u}{\partial \nu} \cdot Q(u) \, d\sigma + \frac{1}{\mu} \int_{C_\rho < x_d < C_{\rho + 1}} |\nabla u|^2 \, dX. \tag{8.11}$$

To estimate the first term in the right-hand side of (8.11), we observe that

$$\int_{\partial \Omega_\rho} \frac{\partial u}{\partial \nu} \cdot Q(u) \, d\sigma = \int_{\partial \Omega_\rho} u \cdot \frac{\partial}{\partial \nu} Q(u) \, d\sigma = \int_{\partial \Omega_\rho} u^\alpha \cdot n_i a_{ij}^\alpha \frac{\partial}{\partial x_j} Q(u) \, d\sigma$$

$$= \int_{\partial \Omega_\rho} u^\alpha \cdot n_i \left\{ \int_0^1 \frac{\partial}{\partial s} \left( a_{ij}^\alpha (x', x_d + s) \frac{\partial u^\beta}{\partial x_j} (x', x_d + s) \right) \right\} \, ds \, d\sigma$$

$$= \int_{\partial \Omega_\rho} u^\alpha \left( n_i \frac{\partial}{\partial x_d} - n_d \frac{\partial}{\partial x_i} \right) \left\{ \int_0^1 a_{ij}^\alpha (x', x_d + s) \frac{\partial u^\beta}{\partial x_j} (x', x_d + s) \right\} \, ds \, d\sigma$$

$$= -\int_{\partial \Omega_\rho} \left( n_i \frac{\partial}{\partial x_d} - n_d \frac{\partial}{\partial x_i} \right) u^\alpha \cdot \right\{ \int_0^1 a_{ij}^\alpha (x', x_d + s) \frac{\partial u^\beta}{\partial x_j} (x', x_d + s) \right\} \, ds \, d\sigma,$$

where we have used the facts that $\mathcal{L}(Q(u)) = 0$ and that $(n_i \frac{\partial}{\partial x_d} - n_d \frac{\partial}{\partial x_i})$ is a tangential derivative. It follows that

$$|\int_{\partial \Omega_\rho} \frac{\partial u}{\partial \nu} \cdot Q(u) \, d\sigma|$$

$$\leq C \int_{\partial \Omega_\rho} |\nabla u| u(X) \left\{ \int_0^1 |\nabla u(x', x_d + s)| \, ds \right\} \, d\sigma$$

$$\leq \delta \int_{\partial \Omega_\rho} \left\{ \int_0^1 |\nabla u(x', x_d + s)|^2 \, ds \right\} \, d\sigma + C_\delta \int_{\partial \Omega_\rho} |\nabla tan u|^2 \, d\sigma$$

$$\leq C_\delta \int_{\psi(x') < x_d < \psi(x') + 1} |\nabla u|^2 \, dX + C_\delta \int_{\Delta(\rho)} |\nabla tan u|^2 \, d\sigma$$

$$+ C \int_{\partial \Omega_\rho \setminus \Delta(\rho)} |\nabla u|^2 \, d\sigma + C \int_{\partial \Omega_\rho \setminus \Delta(\rho)} \left\{ \int_0^1 |\nabla u(x', x_d + s)|^2 \, ds \right\} \, d\sigma,$$

where $\delta \in (0, 1)$ and we have used the Cauchy inequality.

We now choose $\delta$ so that $C_\delta < (1/4)\mu$. In view of (8.11) and (8.13), we obtain

$$\int_{\psi(x') < x_d < \psi(x') + 1} |\nabla u|^2 \, dX$$

$$\leq C \int_{\Delta(2r)} |\nabla tan u|^2 \, d\sigma + C \int_{C_\rho < x_d < C_{\rho + 1}} |\nabla u|^2 \, dX$$

$$+ C \int_{\partial \Omega_\rho \setminus \Delta(\rho)} |\nabla u|^2 \, d\sigma + C \int_{\partial \Omega_\rho \setminus \Delta(\rho)} \left\{ \int_0^1 |\nabla u(x', x_d + s)|^2 \, ds \right\} \, d\sigma, \tag{8.14}$$

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for any \( \rho \in (r, 2r) \). Estimate (8.10) follows by integrating both sides of (8.14) with respect to \( \rho \) over the interval \((r, 2r)\).

We now give the proof of Theorem 8.1.

**Proof of Theorem 8.1.** We may assume that \( r > 3 \). By covering \( \Delta(r) \) with surface balls of small radius \( c(M) \) on \( \{(x', \psi(x')) : x' \in \mathbb{R}^{d-1}\} \) and using the first inequality in (8.1) on each small surface ball, we obtain

\[
\int_{\Delta(r)} |\nabla u|^2 \, dX \leq C \int_{\Delta(2r)} \left| \frac{\partial u}{\partial \nu} \right|^2 \, d\sigma + C \int_{|x'| < r+1} \left| \frac{\partial u}{\partial \nu} \right|^2 \, dX.
\]

This, together with Lemma 8.2, gives the first inequality in (8.1). The second inequality in (8.1) follows from Lemma 8.3 in a similar fashion.

9 Proof of Theorems 1.1, 1.2 and 1.3

Let \( A \in \Lambda(\mu, \lambda, \tau) \) with \( A^* = A \). As we pointed out in the Introduction, by a simple rescaling argument, it suffices to prove Theorems 1.1, 1.2 and 1.3 for \( \varepsilon = 1 \), but with constant \( C \) depending only on \( d, m, \mu, \lambda, \tau \) and the Lipschitz character of \( \Omega \). Moreover, note that the existence and uniqueness as well as representations by layer potentials of solutions to the \( L^2 \) Dirichlet, regularity and Neumann problems for \( L(u) = 0 \) in \( \Omega \) were already given by Theorem 6.1 and its proof. As a result, we only need to show that the operator norms of \( \left( (1/2)I + K_A \right)^{-1} \) on \( L^2(\partial\Omega, \mathbb{R}^m) \) and \( S_A^{-1} : W^{1,2}(\partial\Omega, \mathbb{R}^m) \to L^2(\partial\Omega, \mathbb{R}^m) \) are bounded by a “good” constant.

To this end, for any \( A \in \Lambda(\mu, \lambda, \tau) \), we choose a sequence \( \{A^k\} \subset \Lambda(\mu, \lambda/2, \eta) \), where \( \eta = \eta(\mu, \lambda, \tau) \), such that \( A^k \in C^2(\mathbb{R}^d) \) and \( ||A^k - A||_{C^{\lambda/2}(\mathbb{R}^d)} \to 0 \). By Theorem 5.1 as well as its proof, it follows from Theorem 8.1 by a simple localization argument that \( \mathcal{L}^k = -\text{div}(A^k\nabla) \) has the Rellich property in any Lipschitz domain \( \Omega \) with constant \( C(\Omega) \) depending only on \( d, m, \mu, \lambda, \tau \) and the Lipschitz character of \( \Omega \). This implies that the operator norms of \( (\pm(1/2)I + K_{A^k})^{-1} \) and \( S^{-1}_{A^k} \) are bounded by a “good” constant \( C_0 \). Since \( ||K_{A^k} - K_A||_{L^2 \to L^2} \to 0 \) and \( ||S_{A^k} - S_A||_{L^2 \to W^{1,2}} \to 0 \) by Theorem 3.4, we may conclude that the operator norms of \( (\pm(1/2)I + K_A)^{-1} \) and \( S^{-1}_A \) are bounded by the same “good” constant \( C_0 \). This completes the proof of Theorems 1.1, 1.2 and 1.3.

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