ON THE TATE CONJECTURE FOR THE FANO SURFACES OF CUBIC THREEFOLDS

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Abstract. A Fano surface of a smooth cubic threefold \( X \hookrightarrow \mathbb{P}^4 \) parametrizes the lines on \( X \). In this note, we prove that a Fano surface satisfies the Tate conjecture over a field of finite type over the prime field and characteristic not 2.

1. Introduction.

Let \( k \) be a field of finite type over the prime field and let \( \ell \) be a prime integer, prime to the characteristic. We denote by \( \overline{k} \) an algebraic closure of \( k \) and by \( G := \text{Gal}(\overline{k}/k) \) the Galois group. Let \( X \) be a geometrically connected smooth projective variety over \( k \) and \( \overline{X} := X \times_k \overline{k} \).

We denote by \( A^1(X) \) the \( \mathbb{Q}_\ell \)-span of the images of the divisor classes defined over \( k \) in the (twisted) étale cohomology group \( H^2(\overline{X}, \mathbb{Q}_\ell(1)) \). The group \( G \) acts on \( H^2(\overline{X}, \mathbb{Q}_\ell(1)) \) and fixes the subspace \( A^1(X) \). The Tate conjecture for divisors is:

\[ \text{Tate Conjecture \cite{13}} \]

We have \( A^1(X) = H^2(\overline{X}, \mathbb{Q}_\ell(1))^G \).

The Tate conjecture for Abelian varieties has been proved by Tate \cite{12} for finite fields, by Zarhin \cite{14} in characteristic \( > 2 \), by Mori \cite{9} in characteristic 2, and by Faltings \cite{7} for fields of characteristic 0. We know a few cases of surfaces that satisfy the Tate conjecture (K3 surfaces, product of two curves, some Picard modular surfaces...)

In this note, we prove the Tate conjecture for another family of surfaces. Let us suppose that the field \( k \) has moreover characteristic \( \neq 2 \) and let \( X \hookrightarrow \mathbb{P}^4 \) be a smooth cubic hypersurface. The variety that parametrizes the lines on \( X \) is a smooth projective surface defined over \( k \) called the Fano surface of lines of \( X \). This surface \( S \) is minimal of general type and has invariants:

\[ c^2 = 45, \quad c_2 = 27, \quad b_1 = 10, \quad b_2 = 45. \]

We obtain the following result:

**Theorem 1.** The Tate conjecture holds for the surface \( S \).

For the proof we use the fact that the Fano surface is contained in its 5 dimensional Albanese variety \( A \) and has class \( \frac{1}{3} \Theta^3 \) where \( \Theta \) is a principal polarization. Using then the Hard Lefschetz Theorem, the Poincaré Duality, and the equality \( b_2(\overline{S}) = b_2(\overline{A}) \) of Betti numbers, we obtain that the natural map

\[ H^2(\overline{A}, \mathbb{Q}_\ell(1)) \rightarrow H^2(\overline{S}, \mathbb{Q}_\ell(1)) \]

is an isomorphism. Therefore the second étale cohomology group of \( S \) is essentially the same as the second étale cohomology group of the Abelian variety \( A \), for which the Tate conjecture is known.

To the knowledge of the author, Abelian surfaces and Fano surfaces are the only known surfaces \( S \) such that there is an isomorphism between the second étale cohomology groups of \( S \) and its Albanese variety.

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2. THE PROOF

Let $k$ be a field finitely generated over its prime field, and let $\bar{k}$ be the algebraic closure of $k$. Recall [5] Theorem 11.1 that for a smooth $n$-dimensional projective variety $Z$ over $\bar{k}$, there exists a canonical isomorphism $\eta_Z : H^{2n}(Z, \mathbb{Q}_\ell(n)) \rightarrow \mathbb{Q}_\ell$ sending the class of a closed point to 1. Let $A$ be an Abelian variety of dimension $n \geq 2$ defined over $k$.

**Definition 2.** We say that a 2-dimensional cycle $W$ on $A$ is non-degenerate if the $\mathbb{Q}_\ell$-bilinear form:

$$Q_W : H^2(\bar{A}, \mathbb{Q}_\ell(1)) \times H^2(\bar{A}, \mathbb{Q}_\ell(1)) \rightarrow \mathbb{Q}_\ell$$

is non-degenerate, where we consider the cycle $W$ in $H^{2n-4}(\bar{A}, \mathbb{Q}_\ell(n-2))$ and $\cdot$ denotes the cup product.

An example of a non-degenerate cycle is:

**Proposition 3.** Let $\Theta$ be an ample divisor on $A$. The cycle $\frac{1}{(n-2)!}\Theta^{n-2}$ is non-degenerate.

**Proof.** By the Hard Lefschetz Theorem of Deligne [5, Théorème 4.1.1], the cup product induced by $\Theta^{n-2}$ induces an isomorphism between $H^2(\bar{A}, \mathbb{Q}_\ell(1))$ and $H^{2n-2}(\bar{A}, \mathbb{Q}_\ell(n-1))$. Moreover, by the Poincaré Duality [6, Chap. VI], the cup-product pairing

$$H^2(\bar{A}, \mathbb{Q}_\ell(1)) \times H^{2n-2}(\bar{A}, \mathbb{Q}_\ell(n-1)) \rightarrow H^{2n}(\bar{A}, \mathbb{Q}_\ell(n)) \cong \mathbb{Q}_\ell$$

is perfect. Combining these two assumptions and the fact that cohomology class $\Theta^{n-2}$ is divisible by $(n-2)!$, we get that the cycle $\frac{1}{(n-2)!}\Theta^{n-2}$ is non-degenerate. \(\square\)

Let $S$ be a smooth surface over $k$ with a $k$-rational point $s_0$. Let $A$ be the Albanese variety of $S$ and let $\vartheta : S \rightarrow A$ be the Albanese map such that $\vartheta(s_0) = 0$.

**Proposition 4.** Suppose that the image $W$ of $\bar{S}$ is a non-degenerate cycle in $\bar{A}$ and $b_2(\bar{S}) = b_2(\bar{A})$. The map

$$\vartheta^* : H^2(\bar{A}, \mathbb{Q}_\ell) \rightarrow H^2(\bar{S}, \mathbb{Q}_\ell)$$

is an isomorphism of Galois modules. The surface $S$ satisfies the Tate conjecture and $\rho_S = \rho_A$, where $\rho_S = \dim_{\mathbb{Q}_\ell} A_1(Z)$ for a geometrically smooth irreducible variety $Z/k$.

**Proof.** Let $f : X \rightarrow Y$ be a proper map of smooth complete separated varieties over an algebraically closed field. Let be $a = \dim(X)$, $d = \dim(Y)$ and $c = d - a$. By [5, Remark 11.6 (d)], there is a linear map

$$f_* : H^r(Y, \mathbb{Z}_\ell) \rightarrow H^{r-2c}(X, \mathbb{Z}_\ell)$$

satisfying the projection formula:

$$f_*(y \cdot f^*x) = f_*y \cdot x, \ x \in H^r(X, \mathbb{Z}_\ell(d)), \ y \in H^s(Y, \mathbb{Z}_\ell).$$

Let $W$ be the image of $S$ in its Albanese variety $A$. Using the projection formula, we have

$$\eta_{\bar{A}}(x \cdot y \cdot \vartheta_* S) = \eta_{\bar{S}}(\vartheta^* x \cdot \vartheta^* y \cdot S) = \eta_{\bar{S}}(\vartheta^* x \cdot \vartheta^* y)$$

for $x, y \in H^2(\bar{A}, \mathbb{Q}_\ell(1))$, and we obtain the following equality:

$$\eta_{\bar{S}}(\vartheta^* x \cdot \vartheta^* y) = (\deg \vartheta)\eta_{\bar{A}}(x.W.y),$$

where $\deg \vartheta \neq 0$ is the degree of $\vartheta$ onto its image $W$. Since $Q_W(x, y) = \eta_{\bar{A}}(x.W.y)$ and $Q_W$ is a non-degenerate pairing, the map

$$\vartheta^* : H^2(\bar{A}, \mathbb{Q}_\ell(1)) \rightarrow H^2(\bar{S}, \mathbb{Q}_\ell(1))$$

is injective. As $b_2(\bar{S}) = b_2(\bar{A})$, the map $\vartheta^*$ is then an isomorphism of Galois modules and $H^2(\bar{S}, \mathbb{Q}_\ell(1)) \cong H^2(\bar{A}, \mathbb{Q}_\ell(1))$. Since the Tate conjecture is satisfied for divisors on Abelian varieties over the field $k$ of finite type over the prime field, we have $\rho_A = \dim H^2(\bar{A}, \mathbb{Q}_\ell(1))^2$. 
Since \( \vartheta^* \) is injective, we have \( \rho_S \geq \rho_A \). On the other hand, for every variety \( X \) over \( k \), we have \( \dim H^2(X, \mathbb{Q}_l(1))^G \geq \rho_X \) therefore:
\[
\dim H^2(\bar{S}, \mathbb{Q}_l(1))^G \geq \rho_S \geq \rho_A = \dim H^2(\bar{A}, \mathbb{Q}_l(1))^G,
\]
we thus obtain \( \rho_S = \dim H^2(\bar{S}, \mathbb{Q}_l(1))^G = \rho_A \) and the Tate conjecture holds for \( S \).
\( \square \)

Let us suppose that the field \( k \) has characteristic not \( 2 \). Let \( X \) be a smooth cubic hypersurface defined over the field \( k \) and let \( S \) be its Fano surface. The surface \( S \) is a smooth geometrically connected variety defined over \( k \) [2 Theorem 1.16 i and (1.12)].

Let us suppose that the cubic \( X \) contains a \( k \)-rational line \( L_0 \) such that for every line \( L' \) (defined over \( k \)) in \( X \) meeting \( L_0 \), the plane containing \( L \) and \( L_0 \) cuts out on \( X \) three distinct lines. Proposition (1.25) in [10] ensures that such a line \( L_0 \) exists on a finite extension of \( k \). Since the Tate conjecture for \( S \) over \( k \) holds if and only if it holds over any finite extension of \( k \) (see [11 Theorem 2.9]), this assumption on the existence of \( L_0 \) is not a restriction.

The Albanese variety \( A \) of \( S \) is defined over \( k \) [1 Lemma 3.1] and is 5 dimensional. Let \( \vartheta : S \to A \) be the Albanese map such that \( \vartheta(s_0) = 0 \), where \( s_0 \) is the point of the Fano surface corresponding to \( L_0 \). Let \( \Theta \) be the (reduced) image of \( S \times S \) by the map \((s_1, s_2) \to \vartheta(s_1) - \vartheta(s_2)\). The variety \( \Theta \) is a divisor on \( A \) defined over \( k \) and \( (A, \Theta) \) is a principally polarized Abelian variety [3 Proposition 5]; we checked that although [3] deals with an algebraically closed field, the assumption on the existence of \( L_0 \) ensures that it remains true for the field \( k \). The Albanese map \( \vartheta : S \to A \) is an embedding and the class of \( S \) in \( A \) is \( \frac{\Theta}{\Theta} \) [3 Corollaire of §4, and Proposition 7]. Moreover, by [4] p. 11, \( b_2(\bar{S}) = b_2(A) = 45 \).

We thus see that the Fano surface \( S \) of \( X \) satisfies the hypothesis of Proposition [4] and therefore Theorem [1] holds.

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