CARATHÉODORY–FEJÉR INTERPOLATION AND RELATED TOPICS IN LOCALLY CONVEX SPACES

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Abstract. We study Carathéodory-Herglotz functions whose values are continuous operators from a locally convex topological space which admits the factorization property into its conjugate dual space. We show how this case can be reduced to the case of functions whose values are bounded operators from a Hilbert space into itself.

1. Introduction

In the present paper we pursue our study of Carathéodory-Herglotz functions in the case of operator valued functions. In [3] we studied Carathéodory-Herglotz functions whose values are bounded operators from a Banach space \( B \) into its conjugate dual space \( B^* \) (that is, into the space of its anti-linear continuous functionals). The present work is set in the framework of locally convex vector spaces which have the factorization property. The methods and focus in [3] are different than those in the present paper. In [3] we used in an essential way the theory of linear relations in Hilbert spaces, and we obtained realization theorems for Carathéodory-Herglotz functions; see Theorem 1.1 below. Here the factorization property for positive operators in certain locally convex topological vector spaces plays a central role, and the main point is the reduction to the Hilbert space case; see Theorem 4.2 below.

Let \( \mathcal{V} \) be a locally convex topological vector space, let \( \mathcal{V}^* \) denote the space of its antilinear continuous functionals and let

\[
\langle \cdot, \cdot \rangle_{\mathcal{V}}
\]

denote the duality between \( \mathcal{V} \) and \( \mathcal{V}^* \). We shall denote by \( \mathbf{L}(\mathcal{V}, \mathcal{V}^*) \) the space of continuous operators from \( \mathcal{V} \) into \( \mathcal{V}^* \) and, more generally, by \( \mathbf{L}(\mathcal{V}, \mathcal{W}) \) the space of continuous operators from \( \mathcal{V} \) into another locally convex topological vector space \( \mathcal{W} \). For a Hilbert space \( \mathcal{H} \), \( \mathbf{L}(\mathcal{H}) \) stands for the space of bounded operators from \( \mathcal{H} \) into itself.

An operator \( A \in \mathbf{L}(\mathcal{V}, \mathcal{V}^*) \) is said to be positive (notation: \( A \geq 0 \)) if

\[
\langle Av, v \rangle_{\mathcal{V}} \geq 0, \quad \forall v \in \mathcal{V}.
\]

The space \( \mathcal{V} \) is said to possess the factorization property if every \( A \geq 0 \) in \( \mathbf{L}(\mathcal{V}, \mathcal{V}^*) \) admits a factorization of the form

\[
A = T^*T,
\]

1 We used the terminology Carathéodory function in [3].
where $T$ is a continuous operator from $\mathcal{V}$ into some Hilbert space (depending on $A$). It was proved by J. Gorniak and A. Weran (see [7, Theorem 2.13 p. 240]) that $\mathcal{V}$ has the factorization property if and only if for every $A \geq 0$ in $\mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ the map

$$v \mapsto \langle A v, v \rangle_{\mathcal{V}}$$

is continuous. We remark that the results of [3] are in fact valid in any locally convex vector space which has the factorization property, and this is the setting in which we consider the present work.

An $\mathcal{L}(\mathcal{V}, \mathcal{V}^*)$-valued function $\Phi$ defined in the open unit disk $\mathbb{D}$ (but a priori without any analyticity property) is called a Carathéodory-Herglotz function if the kernel

$$K_\Phi(z, w) = \frac{\Phi(z) + \Phi(w)^*}{1 - z \overline{w}}$$

is positive in $\mathbb{D}$. We denote by $\mathcal{C}(\mathcal{V}, \mathcal{V}^*)$ the family of $\mathcal{L}(\mathcal{V}, \mathcal{V}^*)$-valued Carathéodory-Herglotz functions. In the case when $\mathcal{V}$ is a Hilbert space $\mathcal{H}$ we use the notation $\mathcal{C}(\mathcal{H})$.

**Theorem 1.1.** Let $\mathcal{V}$ be a locally convex topological vector space with the factorization property, and let $\Phi \in \mathcal{C}(\mathcal{V}, \mathcal{V}^*)$. Then there exist a Hilbert space $\mathcal{H}$ and operators $C \in \mathcal{L}(\mathcal{V}, \mathcal{H})$, $V \in \mathcal{L}(\mathcal{H})$, $D \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$, such that:

1. $V$ is an isometry:
   $$V^* V = I_\mathcal{H}$$
   and $D$ is purely imaginary:
   $$D + D^*|_\mathcal{V} = 0;$$
2. $\Phi(z)$ admits the representation
   $$\Phi(z) = D + C^*(I + zV^*)(I - zV^*)^{-1}C, \quad z \in \mathbb{D}.$$  
   In particular, $\Phi(z)$ admits (in the weak sense) the power series expansion

$$\Phi(z) = M_0 + 2 \sum_{n=1}^{\infty} z^n M_n,$$

where the coefficients $M_n \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ are given by

$$M_n = \begin{cases} D + C^*C, & \text{if } n = 0, \\ C^*V^nC, & \text{if } n \geq 1, \end{cases}$$

and, therefore, $\Phi(z)$ is weakly analytic\(^2\) in the open unit disk $\mathbb{D}$.

**Proof.** The representation (1.3) was originally obtained in [3] in the setting of Banach spaces; the proof in the present setting is absolutely the same. \(\square\)

**Remark 1.2.** In the case when $\mathcal{V}$ is a Banach space, $\mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ is a Banach space, as well, and weak analyticity is equivalent to strong analyticity; see [11, Theorem VI.4, p. 189]. Then $\Phi$ is strongly analytic, which is also evident from the fact that the coefficients $M_n$ in (1.4) are uniformly bounded.

\(^2\)See [12, p. 80-81].
Remark 1.3. Formulas (1.5) are the operator-theoretic version of formulas appearing in the theory of stationary stochastic processes; see for instance [9, p. 45].

From (1.5) we see that the operator matrices

\[
\begin{pmatrix}
\frac{M_0 + M_1^*}{2} & M_1 & M_2 & \ldots & M_N \\
M_1 & \frac{M_1^* + M_0^*}{2} & M_1 & \ldots & M_{N-1} \\
M_2 & M_1 & \frac{M_2 + M_1^*}{2} & \ldots & M_{N-2} \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
M_N & \ldots & M_2 & M_1 & \frac{M_N + M_0^*}{2}
\end{pmatrix}
\]

(1.6)

= \text{diag}(C^*, \ldots, C^*) \begin{pmatrix} I_{\mathcal{H}} & V^* & \ldots & V^N \end{pmatrix} \begin{pmatrix} I_{\mathcal{H}} & V & \ldots & V^N \end{pmatrix} \text{diag}(C, \ldots, C),

where \( N = 0, 1, \ldots \), are positive elements of \( L(V^{N+1}, (V^*)^{N+1}) \). Post- and premultiplying (1.6) with the permutation matrix

\[
\begin{pmatrix}
0 & \ldots & 0 & I_V \\
0 & I_V & 0 & \ldots \\
\vdots & \vdots & \ldots & \vdots \\
I_V & 0 & \ldots & 0
\end{pmatrix}
\]

and its adjoint, respectively, we conclude that the operator matrices

\[
M_N = \begin{pmatrix}
\frac{M_0 + M_1^*}{2} & M_1 & M_2 & \ldots & M_N \\
M_1 & \frac{M_1^* + M_0^*}{2} & M_1 & \ldots & M_{N-1} \\
M_2 & M_1 & \frac{M_2 + M_1^*}{2} & \ldots & M_{N-2} \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
M_N & \ldots & M_2 & M_1 & \frac{M_N + M_0^*}{2}
\end{pmatrix},
\]

(1.7)

are positive, as well.

As we shall see in the sequel (Theorem 4.2 below), the converse is also true: if all the matrices \( M_N \) are positive, the corresponding power series (1.4) defines an element in \( C(V, V^*) \).

We pose the following Carathéodory-Fejér interpolation problem:

**Problem 1.4.** Given \( N \in \mathbb{N} \) and operators \( M_0, \ldots, M_N \in L(V, V^*) \), such that the Toeplitz operator matrix \( M_N \) is positive, find a function \( \Phi \in C(V, V^*) \) such that

\[
\Phi(z) - (M_0 + 2 \sum_{n=1}^{N} z^n M_n) = O(z^{N+1}).
\]

We shall prove (see Theorem 4.1 below) that Problem 1.4 is always solvable. Furthermore, it turns out that \( \Phi \in C(V, V^*) \) if and only if

\[
\Phi(z) = D + T^* \varphi(z) T,
\]

where \( D \in L(V, V^*) \) is purely imaginary, \( T \in L(V, \mathcal{H}) \) for some Hilbert space \( \mathcal{H} \) and \( \text{ran}(T) = \mathcal{H} \), \( \varphi \in C(\mathcal{H}) \). Moreover, given \( \Phi \), the operator \( D \) is determined uniquely.
while the operator $T$ and the space $H$ are determined up to unitary mappings. Hence we can reduce interpolation problems for which the value of the function $\Phi \in \mathcal{C}(V, V^*)$ is pre-assigned at the origin to interpolation problems for functions $\varphi \in \mathcal{C}(H).$

This last point explains also why formula (1.8) is much more precise and useful than the seemingly similar formula (1.3). In (1.3) the operator $C$ need not have dense range (in fact $C$ can be chosen to be the adjoint of point evaluation at the origin), and the reduction mentioned above does not apply for (1.3).

The original motivation for [3] and for the present work originates with works on extrapolation stationary stochastic processes with values in a Banach space, or more generally in a linear space; see for instance [8], [6], [10]. Our aim is to develop the theory of de Branges-Rovnyak spaces, and associated problems such as inverse scattering (see [1], [2]) in the linear space case.

The paper consists of four sections besides the introduction. In the second section we review some facts on locally vector spaces which have the factorization property.

2. Preliminaries: spaces with the factorization property

Let $V$ be a locally convex topological vector space with the factorization property and let $A \in \mathcal{L}(V, V^*)$ be positive. We shall say that the factorization $A = T^*T$, where $T \in \mathcal{L}(V, H)$ and $H$ is a Hilbert space, is minimal if the range of $T$ is dense in $H$.

**Proposition 2.1.** Let $V$ be a locally convex topological vector space with the factorization property and let $A \in \mathcal{L}(V, V^*)$ be positive. Let

\begin{align*}
A &= T^*T, \\
A &= (T')^*T'
\end{align*}

be two factorizations via Hilbert spaces $H$ and $H'$ and assume that the factorization (2.1) is minimal. Then there exists an isometry $V \in \mathcal{L}(H, H')$ such that $T' = VT$. Moreover, if the factorization (2.2) is also minimal then $V$ is unitary.

**Proof.** The proof follows the argument in [7]. The formula

$$\mathcal{R} = \{(Tv, T'v), \quad v \in V\}.$$ 

define a relation on $H \times H'$, which is isometric and has a dense domain; it is therefore the graph of an isometric operator $V$. If the range of $\mathcal{R}$ is dense, as well, then $V$ must be unitary. \qed

**Proposition 2.2.** Assume that $V$ admits the factorization property. Then so does $V^N$.

**Proof.** Let $A$ be a positive continuous operator from $V^N$ into $(V^*)^N$. We consider the matrix representation $A = (A_{ij})_{i,j=1,...,N}$, where $A_{ij} \in \mathcal{L}(V, V^*)$. We want to
prove that the map

\[(v_1, \ldots, v_n) \mapsto \left\langle A \begin{pmatrix} v_1 \\ \vdots \\ v_N \\ v_N \end{pmatrix}, \begin{pmatrix} v_1 \\ \vdots \\ v_N \\ v_N \end{pmatrix} \right\rangle_{V^N} \]

is continuous. The positivity of $A$ implies that the operators $A_{\ell \ell}$ are positive. Since $V$ has the factorization property, the maps

$v \mapsto \langle A_{\ell \ell}v, v \rangle_{V}, \ \ell = 1, \ldots, N,$

are continuous. Thus it suffices to show that the maps

\[(v, w) \mapsto \langle A_{\ell j}w, v \rangle_{V} \]

are continuous for $\ell \neq j$.

Without loss of generality, we assume that $\ell < j$. The positivity of $A$ implies that the operator

\[\begin{pmatrix} A_{\ell \ell} & A_{\ell j} \\ A_{j \ell} & A_{jj} \end{pmatrix} \]

is positive from $V^2$ into $(V^*)^2$. It follows that for every choice of $v, w \in V$ and $\alpha, \beta \in \mathbb{C}$

\[\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \begin{pmatrix} \langle A_{\ell \ell}v, v \rangle_{V} & \langle A_{\ell j}w, v \rangle_{V} \\ \langle A_{j \ell}v, w \rangle_{V} & \langle A_{jj}w, w \rangle_{V} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \geq 0.\]

This inequality implies first that

\[\langle A_{\ell j}v, w \rangle_{V} = \langle A_{j \ell}w, v \rangle_{V},\]

and so

\[A_{\ell j}^*|_{V} = A_{j \ell}.\]

Moreover the matrix

\[\begin{pmatrix} \langle A_{\ell \ell}v, v \rangle_{V} & \langle A_{\ell j}w, v \rangle_{V} \\ \langle A_{j \ell}v, w \rangle_{V} & \langle AA_{jj}w, w \rangle_{V} \end{pmatrix} \]

is positive, and hence so is its determinant, that is:

\[\left|\langle A_{\ell j}w, v \rangle_{V}\right|^2 \leq \langle A_{\ell \ell}v, v \rangle_{V} \langle AA_{jj}w, w \rangle_{V}.\]

We deduce that the map \[2.4\] is continuous. The continuity of the map \[2.3\] follows.

3. The Carathéodory–Fejér extension problem in the Hilbert space case

We recall a result on the Carathéodory–Fejér extension problem in the Hilbert space case. We give a proof for completeness, using a one step extension procedure, as in for instance \[13\]. Note that the spaces are not assumed to be separable.

We shall use the following formula (see \[5\] (0.3) p. 3)): let

\[G = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \]
be such that $A$ is invertible, then
\begin{equation}
G = \begin{pmatrix}
I & 0 \\
CA^{-1} & I
\end{pmatrix}
\begin{pmatrix}
A & 0 \\
0 & D^x
\end{pmatrix}
\begin{pmatrix}
I & A^{-1}B \\
0 & I
\end{pmatrix},
\end{equation}
where $D^x = D - CA^{-1}B$ is called the Schur complement of $D$.

**Theorem 3.1.** If $\mathcal{V} = \mathcal{H}$ is a Hilbert space then Problem 1.4 is solvable.

**Proof.** We replace $\mathbb{M}_N$ by $\epsilon I_{\mathcal{H}^{N+1}} + \mathbb{M}_N$, where $\epsilon > 0$. This last operator is strictly positive (that is, positive and boundedly invertible):
\[ \epsilon I_{\mathcal{H}^{N+1}} + \mathbb{M}_N > 0. \]
We set
\[
(\epsilon I_{\mathcal{H}^{N+1}} + \mathbb{M}_N)^{-1} = \begin{pmatrix}
\alpha_N & \beta_N^* \\
\beta_N & \delta_N
\end{pmatrix}
\text{ and } (M_N \ldots M_1) = \gamma_N,
\]
where, to lighten the notation, we do not stress for the moment the dependence on $\epsilon$. Let $X \in L(\mathcal{H})$. In view of (3.1), we see that the Toeplitz matrix
\[
\begin{pmatrix}
\epsilon I_{\mathcal{H}^{N+1}} + \mathbb{M}_N & X^* \\
X & \gamma_N
\end{pmatrix}
\]
is strictly positive if and only if
\[
\epsilon I_\mathcal{H} + M_0 > (X \gamma_N) (\epsilon I_{\mathcal{H}^{N+1}} + \mathbb{M}_N)^{-1} (X^* \gamma_N^*).
\]
Using (3.1), the last inequality can be rewritten as
\begin{equation}
\epsilon I_\mathcal{H} + M_0 - \gamma_N \delta_N^* \gamma_N^* > (X + \gamma_N \beta_N \alpha_N^{-1}) \alpha_N (X + \gamma_N \beta_N \alpha_N^{-1})^*.
\end{equation}
But the operator on the left hand side of (3.2) is the Schur complement of $\epsilon I_\mathcal{H} + M_0$ in $\epsilon I_{\mathcal{H}^{N+1}} + \mathbb{M}_N$, which is strictly positive. Hence the operator ball defined by (3.2) is non-empty. Setting $M_{N+1}(\epsilon) = X$, where $X$ satisfies (3.2), we obtain $\mathbb{M}_{N+1} > 0$.

Reiterating this procedure we can choose a sequence of operators
\[ M_{N+1}(\epsilon), M_{N+2}(\epsilon), \ldots \]
such that all the corresponding Toeplitz matrices
\[ \mathbb{M}_{N+1}(\epsilon), \mathbb{M}_{N+2}(\epsilon), \ldots \]
are strictly positive. The corresponding function
\[
\Phi_{\epsilon}(z) = M_0 + 2 \sum_{k=1}^{N} M_k z^k + 2 \sum_{k=N+1}^{\infty} M_k(\epsilon) z^k
\]
is in the Carathéodory-Herglotz class since
\begin{equation}
\frac{\Phi_{\epsilon}(z) + \Phi_{\epsilon}(w)^*}{1 - zw}
= 2 \lim_{n \to \infty} (z^n I_{\mathcal{H}} \ z^{n-1} I_{\mathcal{H}} \ldots \ I_{\mathcal{H}}) \mathbb{M}_n(\epsilon) (w^n I_{\mathcal{H}} \ w^{n-1} I_{\mathcal{H}} \ldots \ I_{\mathcal{H}})^*.
\end{equation}
The positivity of the matrices $\mathbb{M}_n(X)$ implies, much in the same way as in the proof of Proposition 2.2, that the operators $M_n(\epsilon)$ are uniformly bounded in norm for $\epsilon \leq 1$. We now want to let $\epsilon \to 0$. We use the fact (see for instance [4] Chapter
5) that \( L(H) \) is the dual space of the trace class operators in \( L(H) \), and hence, by Banach-Alaoglu theorem (see [4] Lemma 2.3, p. 102) the closed unit ball of the space \( L(H) \) is weakly compact. Let now \( \epsilon_k, k = 0, 1, \ldots \) be a sequence of numbers decreasing to 0. Since rank one operators are trace class, we can find a subsequence \( \epsilon_{1,k} \) and an operator \( M_{N+1} \in L(H) \) such that

\[
    \lim_{k \to \infty} \langle M_{N+1}(\epsilon_{1,k})x, y \rangle_H = \langle M_{N+1}x, y \rangle_H, \quad \forall x, y \in H.
\]

Similarly there exists a subsequence \( \epsilon_{2,k} \) of the sequence \( \epsilon_{1,k} \) and an operator \( M_{N+2} \in L(H) \) such that

\[
    \lim_{k \to \infty} \langle M_{N+2}(\epsilon_{2,k})x, y \rangle_H = \langle M_{N+2}x, y \rangle_H, \quad \forall x, y \in H.
\]

Using the diagonal argument, we find a sequence of bounded operators \( M_n, n > N \) and a sequence of numbers \( \eta_k = \epsilon_{k,k} \) decreasing to 0 such that

\[
    \lim_{k \to \infty} \langle \Phi_{\eta_k}(z)v, w \rangle = \langle \Phi(z)v, w \rangle,
\]

where \( \Phi(z) = M_0 + 2 \sum_{n=1}^{\infty} z^n M_n \). The function \( \Phi \) is a Carathéodory-Herglotz function. Indeed, for every \( k \) the kernel \( \Phi_{\eta_k}(z, w) \) is positive. Thus for every choice of \( n \in \mathbb{N}, z_1, \ldots, z_n \in D \) and \( h_1, \ldots, h_n \in H \) the \( n \times n \) matrix with \( \ell_j \) entry equal to

\[
    \langle K_{\Phi_{\eta_k}}(z_{\ell}, z_j)h_{\ell}, h_j \rangle_H
\]

is positive. Letting \( k \to \infty \) we get that the kernel \( K_{\Phi}(z, w) \) is positive in \( D \). \( \Box \)

4. Characterization of a Carathéodory-Herglotz function in terms of its power series

As already noticed, the space \( V^N \) has the factorization property when \( V \) has. As a consequence we have:

Theorem 4.1. Let \( N \in \mathbb{N} \) and let \( M_0, \ldots, M_N \) be continuous operators from \( V \) into \( V^* \). Assume that the Toeplitz block operator matrix \( M_N \) given by (1.7) is a positive operator from \( V^{N+1} \) into \( (V^*)^{N+1} \). Let

\[
    \frac{M_0 + M_0^*}{2} = T_0^* T_0,
\]

where \( T_0 \) is a continuous operator from \( V \) into a Hilbert space \( \mathcal{H}_0 \) with dense range. Then there exist a unitary operator \( U \) from \( \mathcal{H}_0 \) into itself and a bounded operator \( C \) from \( \mathcal{H}_0 \) into itself such that

\[
    (4.1) \quad \frac{M_0 + M_0^*}{2} = T_0^* C T_0 \quad \text{and} \quad M_j = T_0^* C^* U^j C T_0, \quad j = 1, \ldots, N.
\]

In particular, the function

\[
    \Phi(z) = \frac{M_0 - M_0^*}{2} + T_0^* C^* (I + zU)(I - zU)^{-1} C T_0
\]

is a solution of the corresponding Carathéodory-Fejér extension problem.
Proof: Since $V^{N+1}$ has the factorization property, there exists an Hilbert space $\mathcal{H}$ and a continuous operator $T$ from $V^{N+1}$ into $\mathcal{H}$ such that $M_N = F^*F$. Write $F = (F_0 \ F_1 \ \cdots \ F_N)$.

We have in particular

$$M_0 + M_N^*|_V = F_0^*F_0 = F_1^*F_1 = \cdots = F_N^*F_N.$$

By Proposition 2.1 there exist isometries $V_0, \ldots, V_N$ from $\mathcal{H}_0$ into $\mathcal{H}$ such that $F_j = V_j T_0$, $j = 0, \ldots, N$.

Let $j > i$. The equality $M_N = F^*F$ leads to

$$M_j = T_0^* V_0^* V_j T_0 = T_0^* V_1^* V_{j-1} T_0 = \cdots = T_0^* V_i^* V_{j-i} T_0.$$

In view of (4.2) we have:

$$M_N = \begin{pmatrix} T_0^* & 0 & 0 & \cdots & 0 \\ 0 & T_0^* & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & T_0^* \end{pmatrix} \times \begin{pmatrix} I & V_0^* V_1 & \cdots & V_0^* V_N \\ V_1^* V_0 & I & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ V_N^* V_0 & \cdots & I \\ 0 & \cdots & \cdots & T_0 \end{pmatrix}.$$

Furthermore the block Toeplitz operator

$$\begin{pmatrix} I & V_0^* V_1 & \cdots & V_0^* V_N \\ V_1^* V_0 & I & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ V_N^* V_0 & \cdots & I \\ 0 & \cdots & \cdots & T_0 \end{pmatrix} \geq 0$$

since $T_0$ has dense range. The problem is thus reduced to the Hilbert space case, and Theorem 3.1 is applicable. The proof is now easily completed. \qed

We now turn to the main result of this paper:

Theorem 4.2. Let $M_0, M_1, \ldots$ be an infinite sequence of elements of $L(V, V^*)$ such that all the matrices $M_N$ defined by (1.7), $N = 0, 1, \ldots$, are positive. Then for every $z$ in the open unit disk, the series

$$\Phi(z) = M_0 + 2 \sum_{n=1}^{\infty} z^n M_n$$

converges weakly, and defines a Carathéodory-Herglotz function. Furthermore, $\Phi$ can be written as (1.8):

$$\Phi(z) = \frac{\Phi(0) - \Phi(0)^*}{2i} \nu + T_0^* \varphi(z) T_0.$$

Conversely, every function of the form (1.8) belongs to $C(V, V^*)$. 
Proof: From the proof of Theorem 4.1 follows that there is a Hilbert space $\mathcal{H}_0$ and an operator $T_0 \in L(B, \mathcal{H}_0)$ with dense range and such that for every $N \geq 0$,$$
abla_N = \text{diag} \left( T_0^*, \ldots, T_0^* \right) \nabla_N \text{diag} \left( T_0, \ldots, T_0 \right),$$
where $\nabla_N$ is a uniquely defined positive block Toeplitz operator from $\mathcal{H}_0^{N+1}$ into $\mathcal{H}_0^{N+1}$. Since the $\nabla_N$ are uniquely defined, they are of the form
$$\nabla_N = (t_{i-j})_{i,j=0,\ldots,N},$$
where the $t_j$ are bounded operators from $\mathcal{H}_0$ into itself. As in the proof of Theorem 3.1 (see (3.3)), the function
$$\varphi(z) = t_0 + 2 \sum_{n=1}^{\infty} t_n z^n$$
belongs to $C(\mathcal{H}_0)$, and (1.8) holds with $\varphi$.

The converse follows from
$$K_\Phi(z, w) = T_0^* K_\varphi(z, w) T_0.$$