FIRST EXTENSION GROUPS OF VERMA MODULES AND R-POLYNOMIALS

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Abstract. We study the first extension groups between Verma modules. There was a conjecture which claims that the dimensions of the higher extension groups between Verma modules are the coefficients of R-polynomials defined by Kazhdan-Lusztig. This conjecture was known as the Gabber-Joseph conjecture (although Gabber and Joseph did not state.) However, Boe gives a counterexample to this conjecture. In this paper, we study how far are the dimensions of extension groups from the coefficients of R-polynomials.

1. Introduction

The category $\mathcal{O}$ is introduced by Bernstein-Gelfand-Gelfand and plays an important role in the representation theory. One of the most important objects in $\mathcal{O}$ are the Verma modules and they are deeply investigated.

In this paper, we consider the $\text{Ext}^i$-groups between Verma modules. If $i = 0$, Verma [Ver68] and Bernstein-Gelfand-Gelfand [BGG71] determine the dimension of this group. There are many studies about higher extension groups. One of these studies is a work of Gabber-Joseph. They proved some inequality between the dimensions of extension groups. After that, it is conjectured that this inequality is, in fact, an equality. Although not actually stated in [GJS], this conjecture is known as the Gabber-Joseph conjecture. If this conjecture is true, then the dimension of extension groups are the coefficients of $R$-polynomial. However, Boe gives a counterexample to this conjecture [Boe92].

This conjecture is false even in the case of $i = 1$. In this paper, we consider how far the dimensions of extension groups from the coefficients of $R$-polynomials. Mazorchuk gives a formula of the dimension of the first extension group between Verma modules in a special case [Maz07, Theorem 32]. Our formula is a generalization of his formula.

Now we state our main theorem. Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraic closed field $K$ of characteristic zero. Fix its Borel subalgebra $\mathfrak{b}$ and a Cartan subalgebra $\mathfrak{h}$. Let $\Delta$ be the root system and $\rho$ the half sum of positive roots. Fix a dominant integral element $\lambda \in \mathfrak{h}^*$. (It is sufficient to consider the integral case by a result of Soergel [Soe90, Theorem 11].) Let $M(x\lambda)$ be the Verma module with highest weight $x\lambda - \rho$ for $x \in W$. Then by a result of Verma [Ver68] and Bernstein-Gelfand-Gelfand [BGG71], if $x \geq y$, then there exists the unique (up to nonzero constant multiple) injective homomorphism $M(x\lambda) \rightarrow M(y\lambda)$. Hence we can regard $M(x\lambda)$ as a submodule of $M(\lambda)$. Then $M(w_0\lambda)$ is a submodule of $M(x\lambda)$ for all $x \in W$ where $w_0$ is the longest Weyl element. Hence we have

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the homomorphism \( \text{Ext}^1(M(x\lambda), M(y\lambda)) \to \text{Ext}^1(M(w_0\lambda), M(\lambda)) \). Let \( V_\lambda(x, y) \) be the image of this homomorphism. We denote the unit element of \( W \) by \( e \). Put \( S_\lambda = \{ s \in S \mid s(\lambda) = \lambda \} \). For integral \( \lambda, \mu \in \mathfrak{h}^* \), let \( T^\mu_\lambda \) be the translation functor from \( \lambda \) to \( \mu \).

**Theorem 1.1** (Proposition \[3.14\] Theorem \[4.3\] Theorem \[5.1\]). Let \( \lambda \) be a dominant integral element.

1. If \( \lambda \) is regular, then \( V_\lambda(w_0, e) \) has a structure of \( W \)-module and it is isomorphic to \( \mathfrak{h}^* \). For \( s \in S \), we denote the element in \( V_\lambda(w_0, e) \) corresponding to the simple root whose reflection is \( s \) by \( v_s \in V_\lambda(w_0, e) \).

2. Assume that \( \lambda \) is regular. For \( x, y \in W \) and \( s \in S \) such that \( xs > x \), we have the following formula.
   
   (a) If \( ys \leq y \), then \( V_\lambda(xs, y) = s(V_\lambda(x, ys)) \).
   
   (b) If \( ys > y \), then \( V_\lambda(xs, y) = K v_s + s(V_\lambda(x, y)) \).

3. For general \( \lambda \), the translation functor \( T^\rho_\lambda \) induces a linear map \( V_\rho(w_0, e) \to V_\lambda(w_0, e) \). The kernel of this linear map is \( \sum_{s \in S_\lambda} K v_s \) and \( T^\rho_\lambda(V_\rho(x, y)) = V_\lambda(x, y) \).

Notice that if \( \lambda \) is regular and \( x \not\leq y \), then \( \text{Ext}^1(M(x\lambda), M(y\lambda)) = 0 \). Hence \( V_\lambda(x, y) = 0 \) for all \( x \in W \), we can determine the space \( V_\lambda(x, y) \) inductively via \([2]\) and \([3]\) if \( \lambda \) is regular. Combining \([3]\) we can determine the space \( V_\lambda(x, y) \). In particular, we can calculate the dimension of \( V_\lambda(x, y) \). Namely, we can get the following theorem. For \( x, y \in W \), let \( R_{x,y} \) be the polynomial defined in \([KL79] \) §2. Let \( \ell(x) \) be the length of \( x \in W \).

**Theorem 1.2** (Theorem \[4.3\]). Assume that \( \lambda \) is regular. Then \( \dim V_\lambda(x, y) \) is the coefficient of \( q \) in \( (1)^{\ell(y)} - (1)^{\ell(x) - 1} R_{x,y} (q) \).

In other words, \( V_\lambda(x, y) \) satisfies the Gabber-Joseph conjecture. Notice that the homomorphism \( \text{Ext}^1(M(x\lambda), M(y\lambda)) \to \text{Ext}^1(M(w_0\lambda), M(\lambda)) \) is not injective. It is easy to see that the kernel is isomorphic to \( \text{Hom}(M(x\lambda), M(\lambda) / M(y\lambda)) \) if \( x \geq y \) (Lemma \[2.1\]).

We summarize the contents of this paper. In Section \[2\] we gather preliminaries and prove some easy facts. In Section \[3\] we define a \( W \)-module structure on \( \text{Ext}^1(M(w_0\lambda), M(\lambda)) \) for a dominant integral regular \( \lambda \). We also prove this module is isomorphic to \( \mathfrak{h}^* \). The proof of the main theorem in the regular case is done in Section \[4\]. We finish the proof of the main theorem in Section \[5\].

### 2. Preliminaries

Let \( \mathfrak{g} \) be a semisimple Lie algebra over an algebraic closed field \( K \) of characteristic zero. Fix a Borel subalgebra \( \mathfrak{b} \) and a Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{b} \). These determine the BGG category \( \mathcal{O} \) \([BGG76]\). We denote \( \text{Hom}_{\mathcal{O}} \) (resp. \( \text{Ext}^1_{\mathcal{O}} \)) by \( \text{Hom} \) (resp. \( \text{Ext}^1 \)). Denote the Weyl group of \( \mathfrak{g} \) by \( W \) and let \( S \) be the set of its simple reflections.

For \( w \in W \), let \( \ell(w) \) be its length. For \( \lambda \in \mathfrak{h}^* \), let \( \mathcal{O}_\lambda \) be the full-subcategory of \( \mathcal{O} \) consisting of objects which have a generalized infinitesimal character \( \lambda \).

In the rest of this paper, we only consider the objects which has a integral generalized infinitesimal character. By \([Soc90] \) Theorem 11], it is sufficient to consider only the integral case.

For \( \lambda \in \mathfrak{h}^* \), let \( M(\lambda) \) be the Verma module with highest weight \( \lambda - \rho \) where \( \rho \) is the half sum of positive roots. Assume that \( \lambda \) is dominant integral. Set
$S_\lambda = S \cap \text{Stab}_W \lambda$. Then $W_{S_\lambda} = \text{Stab}_W \lambda$ is generated by $S_\lambda$. Put $W(S_\lambda) = \{ w \in W \mid ws > w \text{ for all } s \in S_\lambda \}$. This gives a complete representative set of $W/W_{S_\lambda}$. For $x, y \in W(S_\lambda)$, $\text{Hom}(M(x_\lambda), M(y_\lambda)) \neq 0$ if and only if $x \geq y$. Moreover, if $x \geq y$, then $\text{Hom}(M(x_\lambda), M(y_\lambda))$ is one-dimensional and the nonzero homomorphisms from $M(x_\lambda)$ to $M(y_\lambda)$ are injective. Hence we can regard $M(x_\lambda)$ as a submodule of $M(y_\lambda)$. In the rest of this paper, we regard $M(x_\lambda)$ as a submodule of $M(\lambda)$. Then $M(w_0\lambda)$ is a submodule of $M(x_\lambda)$ for all $x \in W$.

Let $\lambda, \mu \in \frakh^*$ be integral dominant elements. Then the translation functor $T^\mu_\lambda : O_\lambda \to O_\mu$ is defined [Jan79].

Assume that $\lambda$ is a regular integral dominant element. Then for $s \in S$, the wall-crossing functor $\theta^\lambda_s$ is defined by $\theta^\lambda_s = T^\mu_\lambda T^\mu_s$ where $\text{Stab}_W \mu = \{ e, s \}$. It is known that this functor is independent of $\mu$. By the definition of $\theta^\lambda_s$, there are natural transformations $\text{Id} \to \theta^\lambda_s$ and $\theta^\lambda_s \to \text{Id}$. Put $C^\lambda_s = \text{Cok}(\text{Id} \to \theta^\lambda_s)$ and $K^\lambda_s = \text{Ker}(\theta^\lambda_s \to \text{Id})$. Then $C^\lambda_s$ (resp. $K^\lambda_s$) gives a right (resp. left) exact functor from $O_\lambda$ to $O_\lambda$. By the self-adjointness of $\theta^\lambda_s$, $(C^\lambda_s, K^\lambda_s)$ is an adjoint pair. Moreover, the derived functor $L^\lambda_s$ gives an auto-equivalence of the derived category $D^b(O_{\lambda})$ and its quasi-inverse is $R K^\lambda_s$. For the action of $C^\lambda_s, K^\lambda_s$ on Verma modules, the following formulas hold: Let $x \in W$ and $s \in S$ such that $xs > x$. Then $C^\lambda_s(M(x)) = M(xs)$, $K^\lambda_s(M(xs)) = M(x)$ and there exists an exact sequence $0 \to M(x)/M(xs) \to C^\lambda_s(M(xs)) \to M(xs) \to 0$. Moreover, we have $L^k C^\lambda_s(M(x)) = L^k C^\lambda_s(M(xs)) = R^k K^\lambda_s(M(xs)) = 0$ for all $k \geq 1$.

Let $\lambda$ be the dominant integral element of $\frakh^*$. Set

$$V_\lambda(x, y) = \text{Im}(\text{Ext}^1(M(x_\lambda), M(y_\lambda)) \to \text{Ext}^1(M(w_0\lambda), M(\lambda)))$$

for $x, y \in W$. Put $\lambda = V_\lambda(w_0, e)$.

(1) of the following lemma is a part of the argument of the proof of [Maz07, Lemma 33]. (2) and (3) follows from (1).

**Lemma 2.1.** Let $\lambda$ be a dominant integral element.

1. Let $P$ be the projective cover of $M(w_0\lambda)$. Let $M_i \subset P$ be the filtration such that $M_i/M_{i-1} \cong \bigoplus_{\ell(w) = i, w \in W(S_\lambda)} M(w_\lambda)$. Then we have

$$\text{Ext}^1(M(x_\lambda), M(\lambda)) \cong \text{Hom}(M(x_\lambda), P/M_0)$$

$$\cong \text{Hom}(M(x_\lambda), M_1/M_0) \cong \bigoplus_{x \in S \setminus S_\lambda} \text{Hom}(M(x_\lambda), M(s_\lambda)).$$

2. We have $\dim V_\lambda = \#(S \setminus S_\lambda)$.

3. For $x, y \in W$ such that $x \geq y$, the kernel of $\text{Ext}^1(M(x_\lambda), M(y_\lambda)) \to \text{Ext}^1(M(w_0\lambda), M(\lambda))$ is isomorphic to $\text{Hom}(M(x_\lambda), M(\lambda)/M(y_\lambda))$.

**Proof.** (1) follows from the proof of [Maz07, Lemma 33]. (2) follows from (1). We prove (3). We have the long exact sequence

$$0 \to \text{Hom}(M(x_\lambda), M(y_\lambda)) \to \text{Hom}(M(x_\lambda), M(\lambda))$$

$$\to \text{Hom}(M(x_\lambda), M(\lambda)/M(y_\lambda)) \to \text{Ext}^1(M(x_\lambda), M(y_\lambda)) \to \text{Ext}^1(M(x_\lambda), M(\lambda))$$

The morphism $\text{Hom}(M(x_\lambda), M(y_\lambda)) \to \text{Hom}(M(x_\lambda), M(\lambda))$ is isomorphic by the classification of homomorphism between Verma modules.
Hence it suffices to prove that \( \text{Ext}^1(M(x\lambda), M(\lambda)) \rightarrow \text{Ext}^1(M(w_0\lambda), M(\lambda)) \) is injective. This follows from \([1]\). \(\square\)

3. **Weyl group action**

In this section, fix a dominant regular integral element \( \lambda \). Put \( V = V_\lambda, V(x, y) = V_\lambda(x, y), \theta_s = \theta_s^\lambda, C_s = C_s^\lambda \) and \( M(x) = M(x\lambda) \). Since \( M(x) \rightarrow \theta_s(M(x)) \rightarrow M(x) \) is zero, we have \( C_s(M(x)) \rightarrow M(x) \).

We begin with the following lemma.

**Lemma 3.1.** For \( M_1, M_2, N_1, N_2 \), take \( a_i \in \text{Ext}^1(M_i, N_i) \) for \( i = 1, 2 \). Take an exact sequence \( 0 \rightarrow N_i \rightarrow X_i \rightarrow M_i \rightarrow 0 \) corresponding to \( a_i \). Then for \( f: N_1 \rightarrow N_2 \) and \( g: M_1 \rightarrow M_2 \), there exists \( \varphi \) such that the following diagram commutes if and only if \( f^*a_1 = g^*a_2 \):

\[
\begin{array}{cccccc}
0 & \rightarrow & N_1 & \rightarrow & X_1 & \rightarrow & M_1 & \rightarrow & 0 \\
& & f & & \varphi & & g & \\
0 & \rightarrow & N_2 & \rightarrow & X_2 & \rightarrow & M_2 & \rightarrow & 0.
\end{array}
\]

**Proof.** First assume that there exists \( \varphi \). Consider the following diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & N_1 & \rightarrow & X_1 & \rightarrow & M_1 & \rightarrow & 0 \\
& & f & & \varphi & & g & \\
0 & \rightarrow & N_2 & \rightarrow & X' & \rightarrow & M_1 & \rightarrow & 0,
\end{array}
\]

where the left square is a push-forward. Then the exact sequence \( 0 \rightarrow N_2 \rightarrow X' \rightarrow M_1 \rightarrow 0 \) corresponds to \( f^*a_1 \) and the existence of \( \varphi \) implies the existence of \( \varphi' \) such that the following diagram commutes:

\[
\begin{array}{cccccc}
0 & \rightarrow & N_2 & \rightarrow & X' & \rightarrow & M_1 & \rightarrow & 0 \\
& & \varphi' & & \varphi' & & g & \\
0 & \rightarrow & N_2 & \rightarrow & X_2 & \rightarrow & M_2 & \rightarrow & 0.
\end{array}
\]

The same argument implies the existence of the following diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & N_2 & \rightarrow & X'' & \rightarrow & M_1 & \rightarrow & 0 \\
& & \varphi'' & & \varphi'' & & g & \\
0 & \rightarrow & N_2 & \rightarrow & X'' & \rightarrow & M_1 & \rightarrow & 0,
\end{array}
\]

where \( 0 \rightarrow N_2 \rightarrow X'' \rightarrow M_1 \rightarrow 0 \) corresponds to \( g^*a_2 \). The morphism \( \varphi'' \) should be isomorphism by 5-Lemma. Hence we have \( f^*a_1 = g^*a_2 \).
Conversely, we assume that \( f_\ast a_1 = g_\ast a_2 \). Consider the following diagram:

\[
\begin{align*}
\text{Hom}(M_1, N_2) & \xrightarrow{k_{2\ast}} \text{Hom}(M_1, X_2) \xrightarrow{p_{2\ast}} \text{Hom}(M_1, M_2) \xrightarrow{\delta_2} \text{Ext}^1(M_1, N_2) \\
\downarrow p_1^\ast & \downarrow p_1^\ast & \downarrow p_1^\ast & \downarrow p_1^\ast \\
\text{Hom}(X_1, N_2) & \xrightarrow{k_{2\ast}} \text{Hom}(X_1, X_2) \xrightarrow{p_{2\ast}} \text{Hom}(X_1, M_2) \xrightarrow{\delta_1} \text{Ext}^1(X_1, N_2) \\
\downarrow k_2 & \downarrow k_2 & \downarrow k_2 & \\
\text{Hom}(N_1, N_2) & \xrightarrow{k_{2\ast}} \text{Hom}(N_1, X_2) \xrightarrow{p_{2\ast}} \text{Hom}(N_1, M_2) \\
\downarrow \delta_1 & \downarrow \delta_1 & \\
\text{Ext}^1(M_1, N_2) & \xrightarrow{k_{2\ast}} \text{Ext}^1(M_1, X_2) \\
\end{align*}
\]

Recall that \( f \in \text{Hom}(N_1, N_2) \) and \( g \in \text{Hom}(M_1, M_2) \). The assumption implies \( \delta_1(f) = \delta_2(g) \). Hence \( \delta_2(\delta_1(f)) = \delta_2(\delta_1(f)) = \delta_2(\delta_2(g)) = 0 \). Hence there exists an element \( \varphi' \in \text{Hom}(X_1, X_2) \) such that \( k_2^\ast(\varphi') = k_2(f) \). Since \( k_2^\ast p_{2\ast}(\varphi') = p_{2\ast} k_2^\ast(\varphi') = p_{2\ast} k_2(f) = 0 \), there exists \( g' \in \text{Hom}(M_1, M_2) \) such that \( p_1^\ast(g') = p_2^\ast(\varphi') \). Then \( p_2^\ast(\varphi') = p_1^\ast(g') \) and \( p_1^\ast(\varphi') = k_2(f) \). From the argument in the first part of this proof, we get \( f_\ast a_1 = (g')_\ast a_2 \), namely, we get \( \delta_2(g') = \delta_1(f) = \delta_2(g) \). Hence there exists \( r \in \text{Hom}(M_1, X_2) \) such that \( p_2^\ast(r) = g - g' \). Set \( \varphi = \varphi' + p_1^\ast(r) \). Then \( p_2^\ast(\varphi) = p_1^\ast(g) \) and \( p_1^\ast(\varphi) = k_2(f) \). This proves the lemma. \( \square \)

The following lemma is well-known. We give a proof for the sake of completeness.

**Lemma 3.2.** Let \( x \in W \) and \( s \in S \) such that \( x < xs \).

1. We have \( \dim \text{Ext}^1(M(xs), M(x)) = 1 \). The basis is given by the exact sequence \( 0 \rightarrow M(x) \rightarrow \theta_s(M(x)) \rightarrow M(xs) \rightarrow 0 \).
2. The homomorphism \( \text{Ext}^1(M(xs), M(x)) \rightarrow \text{Ext}^1(M(w_0), M(e)) \) is injective and its image is independent of \( x \).

**Proof.** Let \( \mu \) be the integral dominant element such that \( \text{Stab}_W(\mu) = \{ e, s \} \). Then \( \theta_s = T_\mu^\lambda T_\lambda^\mu \).

**[1]** We have

\[
\text{Ext}^1(M(xs), \theta_s(M(x))) = \text{Ext}^1(T_\mu^\lambda(M(xs)), T_\lambda^\mu(M(x))) = \text{Ext}^1(M(x\mu), M(x\mu))
\]

It is one-dimensional if \( i = 0 \) and zero if \( i > 0 \). Hence from the exact sequence

\[
0 \rightarrow M(x) \rightarrow \theta_s(M(x)) \rightarrow M(xs) \rightarrow 0,
\]

we have an exact sequence

\[
0 \rightarrow \text{Hom}(M(xs), M(x)) \rightarrow \text{Hom}(M(xs), \theta_s(M(x)))
\]

\[
\rightarrow \text{Hom}(M(xs), M(xs)) \rightarrow \text{Ext}^1(M(xs), M(x)) \rightarrow 0.
\]

By the dimension counting, \( \text{Hom}(M(xs), M(x)) \rightarrow \text{Hom}(M(xs), \theta_s(M(x))) \) is isomorphic. Hence \( \text{Hom}(M(xs), M(xs)) \cong \text{Ext}^1(M(xs), M(x)) \). The left hand side is one-dimensional.

**[2]** First we prove that \( \text{Ext}^1(M(xs), M(x)) \rightarrow \text{Ext}^1(M(w_0), M(e)) \) is injective. The kernel of the homomorphism is \( \text{Hom}(M(xs), M(e)/M(x)) \) by Lemma 2.1.

**[3]** We have an exact sequence \( 0 \rightarrow M(e) \rightarrow \theta_s(M(e)) \rightarrow M(s) \rightarrow 0 \) and \( 0 \rightarrow M(x) \rightarrow \theta_s(M(x)) \rightarrow M(xs) \rightarrow 0 \). Since \( M(xs) \rightarrow M(s) \) is injective,
we have that $M(e)/M(x) \rightarrow \theta_s(M(e)/M(x))$ is injective by the snake lemma. Hence it is sufficient to prove that $\text{Hom}(M(xs), \theta_s(M(e)/M(x))) = 0$. We prove $\text{Hom}(M(x\mu), M(\mu)/M(x\mu)) = 0$. This follows from the following exact sequence

$$0 \rightarrow \text{Hom}(M(xs), \theta_s(M(e)/M(x))) \rightarrow \text{Ext}^1(M(x\mu), M(\mu)) = 0.$$

We prove the independence of $x$. We have the following diagram

$$\begin{array}{ccc}
M(xs) & \rightarrow & M(x) \\
\downarrow & & \downarrow \\
M(s) & \rightarrow & M(e).
\end{array}$$

Hence we get the following diagram

$$\begin{array}{ccc}
0 & \rightarrow & M(x) \rightarrow \theta_s(M(x)) \rightarrow M(xs) \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M(e) \rightarrow \theta_s(M(e)) \rightarrow M(s) \rightarrow 0.
\end{array}$$

Therefore the image of

$$\text{Ext}^1(M(xs), M(x)) \rightarrow \text{Ext}^1(M(xs), M(e))$$

and

$$\text{Ext}^1(M(s), M(e)) \rightarrow \text{Ext}^1(M(xs), M(e))$$

coincide with each other by Lemma 3.1. Therefore, we get (2) □

Take a basis $v_s$ of $V(xs, x)$.

**Lemma 3.3.** The set $\{v_s \mid s \in S\}$ is a basis of $V$.

**Proof.** By Lemma 2.1 we have the following commutative diagram

$$\begin{array}{ccc}
\text{Ext}^1(M(s), M(e)) & \sim & \bigoplus_{s' \in S} \text{Hom}(M(s), M(s')) \\
\downarrow & & \downarrow \\
\text{Ext}^1(M(w_0), M(e)) & \sim & \bigoplus_{s' \in S} \text{Hom}(M(w_0), M(s')).
\end{array}$$

We get the lemma. □

We need the following lemma to define the action of $s \in S$ on $V$.

**Lemma 3.4.** The linear map $\text{Ext}^1(M(w_0), M(e)) \rightarrow \text{Ext}^1(C_s(M(w_0)), M(e))$ is an isomorphism.

**Proof.** By an exact sequence $0 \rightarrow M(w_0s)/M(w_0) \rightarrow C_s(M(w_0)) \rightarrow M(w_0) \rightarrow 0$, we get

$$\begin{align*}
\text{Hom}(M(w_0s)/M(w_0), M(e)) & \rightarrow \text{Ext}^1(M(w_0), M(e)) \\
& \rightarrow \text{Ext}^1(C_s(M(w_0)), M(e)) \rightarrow \text{Ext}^1(M(w_0s)/M(w_0), M(e))
\end{align*}$$
It is sufficient to prove that $\text{Ext}^i(M(w_0s)/M(w_0), M(e)) = 0$ for $i = 0, 1$. We have the following exact sequence

$$0 \to \text{Hom}(M(w_0s)/M(w_0), M(e)) \to \text{Hom}(M(w_0s), M(e))$$

$$\to \text{Hom}(M(w_0), M(e)) \to \text{Ext}^1(M(w_0s)/M(w_0), M(e))$$

$$\to \text{Ext}^1(M(w_0s), M(e)) \to \text{Ext}^1(M(w_0), M(e)).$$

The homomorphism $\text{Hom}(M(w_0s), M(e)) \to \text{Hom}(M(w_0), M(e))$ is an isomorphism. The kernel of $\text{Ext}^1(M(w_0s), M(e)) \to \text{Ext}^1(M(w_0), M(e))$ is isomorphic to $\text{Hom}(M(w_0s), M(e)/M(e)) = 0$ by Lemma 2.1 (6). We get the lemma.

Using this lemma, we consider the following homomorphism:

$$V = \text{Ext}^1(M(w_0), M(e)) \to \text{Ext}^1(LC_s(M(w_0)), LC_s(M(e)))$$

$$\simeq \text{Ext}^1(C_s(M(w_0)), M(s))$$

$$\to \text{Ext}^1(C_s(M(w_0)), M(e))$$

$$\simeq \text{Ext}^1(M(w_0), M(e)) = V.$$

Define an action of $s$ on $V$ by the above homomorphism. In other words, for $v_i \in V$ and the corresponding exact sequences $0 \to M(e) \to X_i \to M(w_0) \to 0$, we have $s(v_1) = v_2$ if and only if there exists the following commutative diagram

$$\begin{array}{cccccc}
0 & \to & C_s(M(e)) & \to & C_s(X_1) & \to & C_s(M(w_0)) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M(e) & \to & X_2 & \to & M(w_0) & \to & 0.
\end{array}$$

Here, $C_s(M(e)) \to M(e)$ and $C_s(M(w_0)) \to M(w_0)$ are canonical homomorphisms.

The aim of this section is to show that this gives a structure of a $W$-module. Since $\{C_s\}$ satisfies the braid relations [MS03 Lemma 5.10], this action satisfies the braid relations.

Take an exact sequence $0 \to M(e) \to X \to M(w_0) \to 0$ and consider the following diagram:

$$\begin{array}{cccccc}
0 & \to & M(e) & \to & X & \to & M(w_0) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \theta_sM(e) & \to & \theta_sX & \to & \theta_sM(w_0) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M(e) & \to & X & \to & M(w_0) & \to & 0,
\end{array}$$

here the vertical maps are natural transformations. The compositions of left and right vertical morphisms are zero. Hence composition of the middle vertical morphisms factors through $X \to M(w_0)$ and $M(e) \to X$. By this way, we get an element of $\text{Hom}(M(w_0), M(e))$. This gives a morphism

$$\alpha_s : V = \text{Ext}^1(M(w_0), M(e)) \to \text{Hom}(M(w_0), M(e)).$$

Since we fix an inclusion $M(w_0) \to M(e)$, we regard $\alpha_s(v) \in K$ for $v \in V$. So we get $\alpha_s \in V^*$. 
Lemma 3.5. If \( v \in \text{Ext}^1(M(w_0), M(e)) \) satisfies \( \alpha_s(v) = 0 \), then \( s(v) = v \).

Proof. Let \( 0 \rightarrow M(e) \rightarrow X \rightarrow M(w_0) \rightarrow 0 \) be the corresponding exact sequence. The assumption means that \( X \rightarrow \theta_s(X) \rightarrow X \) is zero. Hence \( \theta_s(X) \rightarrow X \) factors through \( \theta_s(X) \rightarrow C_s(X) \). Namely, we get the following commutative diagram.

\[
\begin{array}{c}
0 \rightarrow C_s(M(e)) \rightarrow C_s(X) \rightarrow C_s(M(w_0)) \rightarrow 0, \\
0 \rightarrow M(e) \rightarrow X \rightarrow M(w_0) \rightarrow 0.
\end{array}
\]

This means that \( s(v) = v \). \( \square \)

Lemma 3.6. We have \( \alpha_s(v_s) = 2 \) and \( s(v_s) = -v_s \).

Proof. We consider \( v_s \) as an element of \( V(w_0, w_0s) \). So we consider all things in \( \{ M \in \mathcal{O}_\lambda \mid [M : L(x)] = 0 \text{ for } x \neq w_0, w_0s \} \). This category is equivalent to the regular integral block of the BGG category of \( g = s_2(K) \). So we may assume that \( g = s_2(K) \).

Set \( P_0 = M(e), P_1 = \theta_s(M(e)) \). Then \( P_0 \oplus P_1 \) is a projective generator of \( \mathcal{O}_\lambda \).

Set \( A = \text{End}(P_0 \oplus P_1) \). Then \( \mathcal{O}_\lambda \) is equivalent to the category of finitely generated right \( A \)-modules.

We have that \( \dim \text{Hom}(P_0, P_0) = \dim \text{Hom}(P_0, P_1) = \dim \text{Hom}(P_1, P_0) = 1 \), \( \dim \text{Hom}(P_1, P_1) = 2 \). Let \( f : P_0 \rightarrow P_1 \) and \( g : P_1 \rightarrow P_0 \) be the natural transformations. Then \( \text{Hom}(P_0, P_1) = Kf, \text{Hom}(P_1, P_0) = Kg \) and \( \text{Hom}(P_1, P_1) = K \text{id} + Kf g \). Moreover, \( g f = 0 \). Set \( e_i : P \rightarrow P_i \) be the projection. Then \( A = K e_0 + K e_1 + K f + K g + K f g \), here \( f \) stands for \( P_0 \oplus P_1 \rightarrow P_0 \xrightarrow{f} P_1 \rightarrow P_0 \oplus P_1 \).

First, we calculate \( \alpha_s(v_s) \). To calculate it, we consider the following diagram:

\[
\begin{array}{c}
0 \rightarrow P_0 \xrightarrow{i} P_1 \xrightarrow{p} P_1/P_0 \rightarrow 0, \\
0 \rightarrow P_1 \xrightarrow{j} P_1^h \xrightarrow{q} P_1 \rightarrow 0, \\
0 \rightarrow P_0 \xrightarrow{i} P_1 \xrightarrow{p} P_1/P_0 \rightarrow 0,
\end{array}
\]

here \( j \) is an inclusion \( x \mapsto (x, 0) \) and \( q \) is second projection. Notice that we used \( \theta_s(P_0) = \theta_s(P_1/P_0) = P_1 \) and the second exact sequence splits.

We regard \( P_0 \) and \( P_1 \) as right \( A \)-modules. Then \( P_0 = K e_0 + K g, P_1 = K e_1 + K f + K f g \). We regard \( P_0 \) as a submodule of \( P_1 \), namely, \( P_0 = K f + K f g \). We fix a inclusion \( M(w_0) = P_1/P_0 \rightarrow M(e) = P_0 \) by \( e_1 \mapsto f g \), here \( e_1 \) is an image of \( e_1 \).

Set \( h = f g \). Since \( \text{End}(P_0) = K \text{id} + K h, b \) is given by \( b = (\alpha_1 + \beta_1 h, \alpha_2 + \beta_2 h) \) for some \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in K \). Since \( a \) and \( i \) are both natural transformation. So \( a = i = f \). By \( b i = j a \), we have \( \alpha_1 = 1 \). Since \( c \) is given by \( e_1 \mapsto f g \), we get \( \alpha_2 = 0 \) and \( \beta_2 = 1 \) by \( c p = q b \). So we have \( b = (1 + \beta_1 h, h) \).

Next, we consider \( t \). Define \( \gamma_1, \gamma_2, \delta_1, \delta_2 \in K \) by \( t = (\gamma_1 + \delta_1 h, \gamma_2 + \delta_2 h) \). The morphism \( s \) is given by \( e_1 \mapsto f g, f \mapsto 0 \) and \( f g \mapsto 0 \). Hence \( t j = i s \) implies that \( \gamma_1 = 0 \) and \( \delta_1 = 1 \). Since \( u q = p t \), we get \( \gamma_2 = 1 \). Therefore, \( t = (h, 1 + \delta_2 h) \). Hence
the composition \( tb \) is given by \( 2h \). The image of \( e_1 \) under \( P_1 \to \theta_s(P_1) \to P_1 \) is \( 2h \). Since \( P_1/P_0 \) is given by \([e_1] \to h\), this means \( \alpha_s(v_s) = 2 \).

Set \( t' = (h, -1) : \theta_s(P_1) \simeq P_1^{\otimes 2} \to P_1 \). Then we have \( t'b = 0 \) and \( (-u)q = pt' \).

This means that there exists a following diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & C_s(P_0) \\
\downarrow \bar{s} & & \downarrow \bar{v} \\
0 & \longrightarrow & P_0 \\
\end{array} \quad \begin{array}{ccc}
C_s(P_1) & \longrightarrow & C_s(P_1/P_0) & \longrightarrow & 0 \\
\downarrow \bar{v} & & \downarrow \bar{u} & & \downarrow 0 \\
P_1 & \longrightarrow & P_0/P_1 & \longrightarrow & 0.
\end{array}
\]

Here \( \bar{s} \) (resp. \( \bar{v}, \bar{u} \)) is the morphism induced by \( s \) (resp. \( t', -u \)). Hence we have \( s(v_s) = -v_s \).

\[\square\]

**Lemma 3.7.** For \( v \in V \), we have \( s(v) = -\alpha_s(v) v_s \).

**Proof.** Since \( \alpha_s(v_s) \neq 0 \), we have \( V = K v_s + \text{Ker } \alpha_s \). So we may assume that \( v \in \text{Ker } \alpha_s \) or \( v = v_s \). The first one is Lemma 3.5 and the second one is Lemma 3.6. \(\square\)

**Proposition 3.8.** The action of \( s \in S \) on \( V \) defines a representation of \( W \).

**Proof.** We should prove \( s^2 = 1 \). This follows from Lemma 3.7 and Lemma 3.6. \(\square\)

In fact, \( V \) is isomorphic to \( h^* \) as a \( W \)-module. We prove it in the rest of this section.

**Lemma 3.9.** If \( xs > x \) and \( ys > y \), then \( V(xs, ys) = s(V(x, y)) \).

**Proof.** This follows from \( C_s(M(x)) \simeq M(xs) \) and \( C_s(M(y)) \simeq M(ys) \). \(\square\)

**Lemma 3.10.** Let \( m'_{s,s'} \) be an order of \( ss' \in \text{GL}(V) \). Then there exists \( v_s' \in K^x v_s \) such that \( s(v_{s}') = v_s' - 2 \cos(\pi/m_{s,s'}) v_s \) for all \( s, s' \in S \).

**Proof.** Since \( W \) is a finite group, \( V \) is defined over \( \mathbb{C} \). Hence we may assume \( K = \mathbb{C} \). Moreover, by the base change, we may assume that \( K = \mathbb{C} \). Since \( W \) is a finite group, \( V \) has a \( W \)-invariant inner product \( \langle \cdot, \cdot \rangle \). Take \( v_{s}'' \in Cv_s \) such that \( \langle v_{s}'', v_{s}' \rangle = 2 \). By Lemma 3.7, \( s(v'') = v'' - v''(v_s) \alpha_s \) for \( v'' \in V \). Hence the kernel of \( s + 1 : V^* \to V^* \) is \( K \alpha_s \). The linear map \( v \mapsto (v, v'') \) belongs to \( \text{Ker } (s + 1) = K \alpha_s \).

Since \( \langle v_{s}', v_{s}'' \rangle = \alpha_s(v_s) \), we have \( s(v) = v - (v, v'') v'' \) by Lemma 3.7.

Put \( a_{s,s'} = \langle v_{s}', v_{s}' \rangle \). Then we have \( a_{s,s'} = \frac{1}{m_{s,s'}} \). By Lemma 3.7, \( K v'_s + K v'_{s'} \) is stable under the action of \( ss' \). Its characteristic polynomial is \( t^2 + (2 - |a_{s,s'}|^2) t + 1 \). Let \( \alpha, \alpha^{-1} \) be its eigenvalue. Since \( (W, S) \) is a Weyl group, \( m_{s,s'} \leq 6 \). Hence \( \alpha = e^{2\sqrt{-1}/m_{s,s'}} \) or \( \alpha = e^{-2\sqrt{-1}/m_{s,s'}} \). We get \(|a_{s,s'}|^2 = 2 + \alpha + \alpha^{-1} = 2 + 2 \cos(\pi/m_{s,s'}) = (2 \cos(\pi/m_{s,s'}))^2 \). Hence we have \( a_{s,s'} = 2 e^{\sqrt{-1} \theta_{s,s'}} \cos(\pi/m_{s,s'}) \) for some \( \theta_{s,s'} \in \mathbb{R} \). Recall that the Coxeter graph of \( (W, S) \) is a tree. Hence we can choose \( v_{s}' \in e^{\sqrt{-1} \theta_{s,s'}} v_s'' \) such that \( a_{s,s'} = 2 \cos(\pi/m_{s,s'}) \). So we get the lemma. \(\square\)

**Lemma 3.11.** For \( x, y \in W \) such that \( xs > x, x \geq y \) and \( y < ys \), let \( A \) (resp. \( B \)) be the image of \( \text{Ext}^1(M(xs), M(ys)) \) \( \to \text{Ext}^1(M(xs), M(y)) \) (resp. \( \text{Ext}^1(M(xs), M(x)) \) \( \to \text{Ext}^1(M(xs), M(y)) \)). Then \( A \cap B = 0 \) if and only if \( x \not\geq y \).

**Proof.** By Lemma 3.12, \( B \) is one-dimensional. If \( x \geq y \), then we have a homomorphism \( \text{Ext}^1(xs, x) \to \text{Ext}^1(xs, ys) \). So \( B \subset A \).

On the other hand, assume that \( A \cap B \neq 0 \). Let \( \mu \in h^* \) be a dominant integral element such that \( \text{Stab}_{W} (\mu) = \{e, s\} \). Then \( T_{h}^{\mu} \) induces an homomorphism
\[ \text{Ext}^1(M(xs), M(y)) \to \text{Ext}^1(M(xs\mu), M(y\mu)). \] Since \( T^a_{\alpha}(M(xs)) = T^a_{\alpha}(M(x)) = M(x\mu), \) the image of \( B \) under \( T^a_{\alpha} \) is zero. Hence the homomorphism
\[ \text{Ext}^1(M(xs), M(ys)) \to \text{Ext}^1(M(xs\mu), M(y\mu)) \]
has a kernel. This homomorphism is equal to
\[ \text{Ext}^1(M(xs), M(ys)) \to \text{Ext}^1(M(xs), \theta_s(M(y))). \]

From an exact sequence
\[ 0 \to M(ys) \to \theta_s(M(y)) \to C_s(M(ys)) \to 0, \]
we have \( \text{Hom}(M(xs), C_s(M(ys))) \neq 0. \) By the adjointness,
\[ \text{Hom}(M(x), C_s(M(ys))) = \text{Hom}(K_s(M(xs)), M(ys)) = \text{Hom}(M(x), M(ys)). \]
Hence we have \( x \geq y s. \)

**Lemma 3.12.** For \( s, s' \in S, \) let \( \langle s, s' \rangle \) be the group generated by \( \{s, s'\}. \) Then for all \( x, y \in w_0(s, s'), \) \( \text{Ext}^1(M(x), M(y)) \to \text{Ext}^1(M(w_0), M(e)) \) is injective.

*Proof.* Denote the irreducible quotient of \( M(x) \) by \( L(x). \) Let \( w \in \langle s, s' \rangle \) be the longest element. Put
\[ \mathcal{O}' = \{ M \in \mathcal{O}_\lambda \ | \ |M : L(z)| = 0 \text{ for } z \notin w_0(s, s') \}. \]
Then \( \mathcal{O}' \) is equivalent to the regular integral block of the BGG category of semisimple Lie algebra of rank 2. Applying Lemma \ref{2.1}[3] to this category, the kernel of
\[ \text{Ext}^1(M(x), M(y)) \to \text{Ext}^1(M(w_0), M(w_0)) \]
is isomorphic to \( \text{Hom}(M(x), M(w_0)/M(ys)). \) However, we have that \( [M(w_0) : L(x)] = [M(y) : L(x)] = 1. \) Hence this space is zero. By Lemma \ref{2.1}[3] the kernel of
\[ \text{Ext}^1(M(w_0), M(ys)) \to \text{Ext}^1(M(w_0), M(e)) \]
is isomorphic to \( \text{Hom}(M(w_0)/M(w_0)), M(e)). \) It is zero since \( [M(e) : L(w_0)] = [M(w_0) : L(w_0)] = 1. \)

**Remark 3.13.** By Lemma \ref{3.1} and Lemma \ref{3.12} we have the following. For \( x, y \in w_0(s, s') \) such that \( xs > x \geq y < ys, v_s \in V(xs, ys) \) if and only if \( x \geq y s. \) Since \( s(v_s) = -v_s \) and \( V(xs, ys) = \langle V(x, y) \rangle \) (Lemma \ref{3.9} under the same conditions, we have \( v_s \in V(x, y) \) if and only if \( x \geq y s. \)

**Proposition 3.14.** The \( W \)-representation \( V \) is isomorphic to the geometric representation defined in [Bon02].

*Proof.* Let \( w' \in \langle s, s' \rangle \) be the longest element and put \( w = w_0 w'. \) Let \( m_{s, s'} \) be an order of \( ss' \in W. \) It is sufficient to prove that \( m_{s, s'} \) is an order of \( ss' \in \text{GL}(V). \)

Fix \( s, s' \in S \) and set \( m = m_{s, s'}. \) Put \( V' = Kv_s + Kv_{s'} = V(w_0, w). \) Then \( V' \) is \( \langle s, s' \rangle \)-stable. Let \( n \) be an order of \( ss' \in \text{GL}(V'). \) Then \( n|m. \) We prove \( n = m. \) Since \( (W, S) \) is a Weyl group, \( m = 2, 3, 4, 6. \) If \( m = 2, 3, \) then there is nothing to prove.

Assume that \( n = 2 \) and \( m \neq 2. \) Then by Lemma \ref{3.10}, \( s'v'_s = v'_s. \) Therefore, \( v_s \in s'V(ws, w) = V(ws's', ws) \) by Lemma \ref{3.9}. By Remark \ref{3.13} \( ss' \geq s's. \) This is a contradiction since \( m \neq 2. \)

Assume that \( (n, m) = (3, 6). \) Then we have \( s's(v'_s) = -v'_s. \) Hence, \( v_s \in s'sV(ws', ws) = V(ws's's', ws's). \) By Remark \ref{3.13} \( ss's \geq ss's. \) This is a contradiction since \( m = 6. \)
This proposition says that $V \simeq h^*$ as $W$-module. Let $\alpha$ be a simple root. By the proof, $v_\alpha$ corresponds to $\alpha$ up to constant multiple.

4. Regular case

As in the previous section, fix a regular integral dominant element $\lambda$. We continue to use the notation in the previous section. In this section, we determine $V(x, y) \subset V$.

We need the graded BGG category. By a result of Beilinson-Ginzburg-Soergel [BGS96], there exists a graded algebra $A = \bigoplus_{i \geq 0} A_i$ such that $O_{x}$ is equivalent to the category of $A$-modules. Let $\tilde{O}_x$ be the category of graded $A$-modules. Then we have the forgetful functor $\tilde{O}_x \rightarrow O_x$. For $M = \bigoplus_{i} M_i \in \tilde{O}_{x}$, define $M(n) \in \tilde{O}_x$ by $(M(n))_i = M_{i-n}$. There exists a module $\tilde{M}(x) \in \tilde{O}_x$ such that its image in $O_x$ is isomorphic to $M(x)$ and it is unique up to grading shift [BGS96, 3.11]. Take a grading of $\tilde{M}(x)$ such that its top is in degree 0. Set $\tilde{M}^0(x) = \tilde{M}(x)(\ell(x))$. Then by the proof of [Maz07, Corollary 23], if $x \geq y$, then there exists a degree zero injective homomorphism $\tilde{M}^0(x) \rightarrow \tilde{M}^0(y)$.

For $M, N \in \tilde{O}_{x}$, let $\text{Hom}(M, N)_0$ be the space of homomorphisms of degree zero and $\text{Ext}^1(M, N)_0$ its derived functors. The following lemma is proved in [Maz07, Theorem 32]. However, the author thinks, although his proof is correct, the shift in the statement of the theorem is wrong. So we give a proof (it is the same as in the proof in [Maz07]).

**Lemma 4.1.** We have $\text{Ext}^1(\tilde{M}^0(x), \tilde{M}^0(\ell(i)))_0 = 0$ if $i \neq 2$. Moreover, if $i = 2$, its dimension is equal to $\# \{s \in S \mid s \leq x \}$.

**Proof.** We use the argument in the proof of Lemma 2.1 with a grading. Let $\tilde{P}$ be the projective cover of $\tilde{M}(w_0)$. Then it has a Verma flag $M_i$ such that $M_i/M_{i-1}$ is isomorphic to $\bigoplus_{r(x)=i} M(x)$ (here, we ignore the grading). The by Lemma 2.1 (1) we have $\text{Hom}(\tilde{M}^0(x), M_i/M_0) \simeq \text{Ext}^1(\tilde{M}^0(x \lambda), M_0)$. We consider the grading of $M_0$ and $M_1/M_0$. Take $k \in \mathbb{Z}$ such that $M_0 \simeq \tilde{M}(e)(k)$. Then the multiplicity of $\tilde{M}(w_0)(k)$ in $\tilde{M}(e)$ is 1 by [BGS96, Theorem 3.11.4]. Since $\tilde{M}(w_0)(\ell(w_0)) \subset \tilde{M}(e)$, we have $k = \ell(w_0)$. By the same argument, we have $M_1/M_0 \simeq \bigoplus_{s} \tilde{M}(s)(\ell(w_0) - 1)$. So we have

$$\text{Ext}^1(\tilde{M}^0(x), \tilde{M}^0(\ell(i))) \simeq \text{Hom} \left( \tilde{M}^0(x), \bigoplus_{s \in S} \tilde{M}(s)(\ell - 2) \right).$$

We get the lemma. \hfill \Box

We use the following abbreviations.

- $E^i(x(k), y(l)) = \text{Ext}^i(\tilde{M}^0(x)(k), \tilde{M}^0(y)(l))_0$.
- $E^i(x(k), y/l(\ell)) = \text{Ext}^i(\tilde{M}^0(x)(k), \tilde{M}^0(y)/\tilde{M}^0(z)(l))_0$.

By the above lemma, $V(x, y)$ is the image of $E^1(x, y(2)) \rightarrow E^1(w_0, e(2))$.

**Lemma 4.2.** If $i \leq 1$, then $\text{Ext}(\tilde{M}^0(x), \tilde{M}^0(\ell(i)))_0 = 0$.

**Proof.** We have $E^1(x, y(i)) = E^0(x, e/y(i))$ by Lemma 2.1 (3) and the above lemma. Let $L$ be the unique irreducible quotient of $\tilde{M}^0(x)$. By [BGS96, Theorem 3.11.4], we have $[\tilde{M}^0(e)/\tilde{M}(y)(i) : L] = 0$. Hence we have $E^0(x, e/y(i)) = 0$. \hfill \Box
There is a graded lift of $\theta_*$ \cite{Str03}. We take a lift $\tilde{\theta}_*$ of $\theta_*$ such that $\tilde{\theta}_*$ is self-dual and there exist degree zero natural transformations $\text{Id}(1) \to \tilde{\theta}_* \to \text{Id}(-1)$. Let $\overset{\sim}{C}_s$ (resp. $\overset{\sim}{K}_s$) be the cokernel (resp. kernel) of $\text{Id}(1) \to \tilde{\theta}_* \to \text{Id}(-1)$, Then $(\overset{\sim}{C}_s, \overset{\sim}{K}_s)$ is an adjoint pair. By \cite{Str03} Theorem 3.6, we have the following formulas for $x \in W$ and $s \in S$ such that $xs > x$:

- $\overset{\sim}{C}_s(\overset{\sim}{M}^0(x)) = \overset{\sim}{M}^0(xs)(-1)$ and $L^k\overset{\sim}{C}_s(\overset{\sim}{M}^0(x)) = 0$ for $k > 0$.
- We have an exact sequence $0 \to (\overset{\sim}{M}^0(x)/\overset{\sim}{M}^0(xs))(1) \to C_s(\overset{\sim}{M}^0(xs)) \to \overset{\sim}{M}^0(xs)(-1) \to 0$.
- $\overset{\sim}{K}_s(\overset{\sim}{M}^0(xs)) = (\overset{\sim}{M}^0(x))(1)$ and $R^k\overset{\sim}{K}_s(\overset{\sim}{M}^0(xs)) = 0$ for $k > 0$.

We also have that $LC\overset{\sim}{\cdot}$ gives an auto-equivalence of $D^b(\mathcal{O}_\lambda)$ and its quasi-inverse functor is $R\overset{\sim}{K}_s$.

Now we prove the main theorem in the regular case. We have already proved \cite[(4)]{Str03} (Lemma 3.9).

\textbf{Theorem 4.3.} Let $x, y \in W$ and $s \in S$ such that $xs > x \geq y$.

1. If $ys < y$, then $V(xs, y) = s(V(x, ys))$.
2. If $ys > y$, then $V(xs, y) = K\overset{\sim}{v}_s + s(V(x, y))$.

\textbf{Proof.} From \cite[(1)]{Str03} we have $s(V(x, y)) = V(xs, ys)$. Since $y < ys$, we have a homomorphism $\text{Ext}^1(M(xs), M(ys)) \to \text{Ext}^1(M(xs), M(y))$. Hence $V(xs, ys) \subset V(x, y)$. We also have $\text{Ext}^1(M(xs), M(x)) \to \text{Ext}^1(M(xs), M(y))$. Since $K\overset{\sim}{v}_s$ is the image of $\text{Ext}^1(M(xs), M(x))$, the right hand side is contained in the left hand side.

From an exact sequence

$$0 \to (\overset{\sim}{M}^0(y)/\overset{\sim}{M}^0(ys))(2) \to C_s(\overset{\sim}{M}^0(ys))(1) \to \overset{\sim}{M}^0(ys) \to 0,$$

we have an exact sequence

$$\text{Ext}^0(\overset{\sim}{M}^0(xs), C_s(\overset{\sim}{M}^0(ys))(1))_0 \to E^0(xs, ys) \to E^1(xs, y/ys(2)) \to \text{Ext}^1(\overset{\sim}{M}^0(xs), C_s(\overset{\sim}{M}^0(ys))(1))_0.$$

Since $R\overset{\sim}{K}_s$ is the quasi-inverse functor of $L\overset{\sim}{C}_s$, for $i \geq 0$, we have

$$\text{Ext}^i(\overset{\sim}{M}^0(xs), C_s(\overset{\sim}{M}^0(ys))(1))_0 = \text{Hom}_{D^b(\mathcal{O}_\lambda)}(\overset{\sim}{M}^0(xs), L\overset{\sim}{C}_s(\overset{\sim}{M}^0(ys))(1)[i])_0 = \text{Hom}_{D^b(\mathcal{O}_\lambda)}(R\overset{\sim}{K}_s(\overset{\sim}{M}^0(xs)), \overset{\sim}{M}^0(ys)(1)[i])_0 = \text{Hom}_{D^b(\mathcal{O}_\lambda)}(\overset{\sim}{M}^0(x)(1), \overset{\sim}{M}^0(ys)(1)[i])_0 = \text{Ext}^i(\overset{\sim}{M}^0(xs), \overset{\sim}{M}^0(ys))_0.$$

If $i = 1$, then this is zero by Lemma \cite[4.2]{Str03}. Hence we get an exact sequence

$$E^0(xs, ys) \to E^0(xs, ys) \to E^1(xs, y/ys(2)) \to 0.$$

Assume that $x \not\geq ys$. Then $E^0(xs, ys) = 0$. Therefore, we have that $\dim E^1(xs, y/ys(2)) = \dim E^0(xs, ys) = 1$. From an exact sequence

$$E^1(xs, ys(2)) \to E^1(xs, y(2)) \to E^1(xs, y/ys(2)),$$

the codimension of the image $A$ of $E^1(xs, ys(2)) \to E^1(xs, y(2))$ is less than or equal to 1. We also have that the image $B$ of $E^1(xs, x(2)) \to E^1(xs, ys(2))$ is
Hence using \([KL79, (2.0.b), (2.0.c)]\), we have \(w(1)\).

**Theorem 5.1.**  
Assume \(\lambda\) is regular. Then \(\dim V_\lambda(x, y)\) is the coefficient of \(q\) in \((-1)^{\ell(y) - \ell(x) - 1}R_{y, x}(q)\).

**Proof.** Put \(n_{y, x} = \dim V(x, y)\). Then by the above theorem and its proof, for \(x, y \in W, s \in S\) such that \(xs > x \geq y\), we have:

- If \(ys < y\), then \(n_{xs, y} = n_{xs, ys}\).
- If \(ys > y\) and \(x \geq ys\), then \(n_{xs, y} = n_{x, y}\).
- If \(ys > y\) and \(x < ys\), then \(n_{xs, y} \leq n_{x, y} + 1\).

Let \(r_{y, x}\) be the coefficient of \(q\) in \((-1)^{\ell(y) - \ell(x) - 1}R_{y, x}(q)\). By \([KL79, (2.0.b), (2.0.c)]\), the constant term of \((-1)^{\ell(y) - \ell(x) - 1}R_{y, x}(q)\) is 0 or 1 and it is 1 if and only if \(y \leq x\). Hence using \([KL79, (2.0.b), (2.0.c)]\), we have:

- If \(ys < y\), then \(r_{xs, y} = r_{x, ys}\).
- If \(ys > y\) and \(x \geq ys\), then \(r_{xs, y} = r_{x, y}\).
- If \(ys > y\) and \(x < ys\), then \(r_{xs, y} = r_{x, y} + 1\).

Hence we get \(n_{y, x} \leq r_{y, x}\). To prove \(n_{y, x} = r_{y, x}\), it is sufficient to prove \(n_{w_0, x} = r_{w_0, x}\). We prove this by backward induction on \(\ell(x)\). By Lemma 4.1 we have \(n_{w_0, x} = \#\{s' \in S \mid w_0s' \geq x\}\).

Take \(s \in S\) such that \(xs > x\). If \(w_0s \geq xs\), then \(r_{w_0, x} = r_{w_0s, x} = r_{w_0s, xs} = n_{w_0s, xs}\). If \(w_0s < xs\), then \(r_{w_0, x} = r_{w_0s, x} + 1 = r_{w_0s, xs} + 1 = n_{w_0s, xs} + 1\). We compare \(X = \{s' \in S \mid w_0s' \geq x\}\) and \(Y = \{s' \in S \mid w_0s' \geq xs\}\). Since \(xs > x\), we have \(Y \subset X\).

Assume \(s' \in Y\) and \(s' \neq s\). Then \(s' \leq w_0xs\). Hence \(s'\) appears in a reduced expression of \(w_0xs\). Since \(s \neq s'\), \(s'\) appears in a reduced expression of \(w_0x\). Hence \(s' \in X\). This implies \(X \cap (S \setminus \{s\}) = Y \cap (S \setminus \{s\})\).

Since \(xs > x\), we have \(w_0s \geq x\). Therefore, \(s \in X\). Hence if \(w_0s \geq xs\), we have \(X = Y\), this implies \(n_{w_0, x} = n_{w_0, xs}\). If \(w_0s \not\geq xs\), we have \(X = Y \cup \{s\}\), this implies \(n_{w_0, x} = n_{w_0, xs} + 1\). Therefore, \(r_{w_0, x} = n_{w_0, x}\). \(\square\)

5. Singular case

In this section, we fix a dominant integral (may be singular) element \(\lambda \in \mathfrak{h}^*\). We also fix a regular integral dominant element \(\lambda_0 \in \mathfrak{h}^*\). Then the translation functor \(T_{\lambda_0}^\lambda\) is defined and it gives \(V_{\lambda_0}(x, y) \rightarrow V_\lambda(x, y)\). Recall the notation \(S_\lambda = \{s \in S \mid s(\lambda) = \lambda\}\).

In this section, we prove the following theorem.

**Theorem 5.1.**  
(1) The homomorphism \(V_{\lambda_0}(x, y) \rightarrow V_\lambda(x, y)\) induced by the translation functor is surjective.

(2) The kernel of \(V_{\lambda_0} \rightarrow V_\lambda\) is \(\sum_{s \in S_\lambda} K_v\).
We use the notation \( \widetilde{O}_\lambda \), \( M(n) \) and \( \text{Hom}(M, N)_0 \) which we use in the previous section. Then using the same argument in \[\text{Str03}, \] \( T_0^\lambda \) and \( T_0^\lambda \) have graded lifts \( T_0^\lambda : \widetilde{O}_{\lambda_0} \to \widetilde{O}_\lambda \) and \( T_0^\lambda : \widetilde{O}_\lambda \to \widetilde{O}_{\lambda_0} \), respectively.

Using the argument in \[\text{Str03}, \] we can prove the following properties. Put \( \tilde{\theta} = \widetilde{T_0^\lambda} \widetilde{O}_{\lambda_0} \). Set \( W_{S_\lambda} = \text{Stab}_{W}(\lambda) \) and let \( w_{\lambda} \in W_{S_\lambda} \) be the longest element. Then we can take \( T_0^\lambda \) and \( T_0^\lambda \) such that \( \tilde{\theta} \) is self-dual and there exists a natural transformation \( \text{Id}(\ell(w_\lambda)) \to \tilde{\theta} \) and \( \tilde{\theta} \to \text{Id}(-\ell(w_\lambda)) \). Define a subset \( W(S_\lambda) \) of \( W \) by \( W(S_\lambda) = \{ x \in W \mid xS > x \text{ for all } s \in S_\lambda \} \). Then for \( x \in W(S_\lambda) \), \( \tilde{\theta}(\tilde{M}_0(x\lambda_0)) \) has a filtration \( M_i \) such that \( M_i/M_{i-1} \) is isomorphic to \( \bigoplus_{x\in W_{S_\lambda}} \tilde{M}_0(x\lambda_0)(\ell(w_\lambda) - 2\ell(w)) \).

**Proof of Theorem 5.1.** We prove \[\{\text{1}\} \] By Theorem 4.3 for \( y \in W(S_\lambda) \) and \( w \in W_{S_\lambda} \), we have \( V(x, yw) \subset V(x, y) \subset V(x, yw) + \sum_{s \in S_\lambda} K_{vs} \). It is easy to see that \( V_{\lambda_0} \) is in the kernel of \( V_{\lambda_0} \to V_{\lambda} \). Hence we may assume that \( y \in W(S_{\lambda}) \).

It is sufficient to prove that \( \text{Ext}^1(\tilde{M}_0(x\lambda_0), \tilde{M}_0(y\lambda_0)(2))_0 \to \text{Ext}^1(\tilde{M}_0(x\lambda_0), \tilde{\theta}(\tilde{M}_0(y\lambda_0))(2 - \ell(w_\lambda)))_0 \) is surjective. Let \( M \) be the cokernel of \( \tilde{M}_0(y\lambda_0)(2) \to \tilde{\theta}(\tilde{M}_0(y\lambda_0))(2 - \ell(w_\lambda)) \).

Then it is sufficient to prove that \( \text{Ext}^1(\tilde{M}_0(x\lambda_0), M)_0 = 0 \). As we mentioned above, \( M \) has a filtration \( \{ M_i' \}_{i \geq 1} \) such that \( M_i'/M_{i-1}' \approx \bigoplus_{x \in W_{S_\lambda}} \tilde{M}_0(yw_\lambda)(2 - 2\ell(w)) \). By Lemma 4.2, we have \( \text{Ext}^1(\tilde{M}_0(x\lambda_0), \tilde{M}_0(yw_\lambda)(2 - \ell(w)))_0 = 0 \) if \( \ell(w) > 0 \). Hence we get \[\{\text{1}\} \] We have \[\{\text{2}\} \] from \[\{\text{1}\} \] and \[\{\text{2}\} \] of Lemma 2.1.

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6. Erratum to “First extension groups of Verma modules and R-polynomials”

Theorem 1.2 in [Abe15] (Theorem 1.2 and 4.4 in the main body) is false. We can only prove the inequality, namely we only have the following theorem.

**Theorem 6.1.** Assume that \( \lambda \) is regular. Then \( \dim V_\lambda(x, y) \) is less than or equal to the coefficient of \( q \) in \( (-1)^{\ell(y)-\ell(x)-1}R_{y,x}(q) \).

When \( x = w_0 \), we have the equality.

**Theorem 6.2.** Assume that \( \lambda \) is regular. Then \( \dim V_\lambda(w_0, x) \) is equal to the coefficient of \( q \) in \( (-1)^{\ell(x)-\ell(w_0)-1}R_{x,w_0}(q) \).

Both theorems follow from the proof of Theorem 4.4. In the proof of Theorem 4.4, we claimed that Theorem 4.4 follows from Theorem 6.2. However, this argument is not correct.

Here is a counterexample of Theorem 4.4. We use the notation in the proof of Theorem 4.4. Let \( g \) be the simple Lie algebra of type \( B_3 \) and we use the standard notation of the root system. In particular, the set of simple roots is \( \{e_1 - e_2, e_2 - e_3, e_3\} \). Let \( s_1 = s_{e_1 - e_2}, s_2 = s_{e_2 - e_3}, s_3 = s_{e_3} \) be simple reflections. Put \( x = s_2s_3s_2s_1s_2s_3 \) and \( y = s_3s_2 \). Then using the formula in the proof of Theorem 4.4, we have

\[
r_{s_2s_3s_2s_1s_2s_3s_2} = r_{s_2s_3s_2s_1s_2s_3s_2} + 1
= r_{s_2s_3s_2s_1s_3} + 1
= r_{s_2s_3s_2s_3} + 2
= r_{s_2s_3s_3} + 3
= r_{s_2s_3} + 4
= r_{e,e} + 4 = 4.
\]

However \( n_{x,y} \leq \dim h = 3 \). Hence \( n_{x,y} < r_{x,y} \).

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