Conch maximal subrings

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ABSTRACT

It is shown that if \( R \) is a ring, \( p \) a prime element of an integral domain \( D \leq R \) with \( \cap_{n=1}^{\infty}p^nD = 0 \) and \( p \in U(R) \), then \( R \) has a conch maximal subring (see [14]). We prove that either a ring \( R \) has a conch maximal subring or \( U(S) = S \cap U(R) \) for each subring \( S \) of \( R \) (i.e., each subring of \( R \) is closed with respect to taking inverse, see [25]). In particular, either \( R \) has a conch maximal subring or \( U(R) \) is integral over the prime subring of \( R \). We observe that if \( R \) is an integral domain with \( |R| = 2^{2\alpha} \), then \( R \) has a maximal subring or \( |\text{Max}(R)| = 2^\alpha \), and in particular if in addition \( \dim(R) = 1 \), then \( R \) has a maximal subring. If \( R \subseteq T \) is an integral ring extension, \( Q \in \text{Spec}(T) \), \( P := Q \cap R \), then we prove that whenever \( R \) has a conch maximal subring \( S \) with \( (S : R) = P \), then \( T \) has a conch maximal subring \( V \) such that \( (V : T) = Q \) and \( V \cap R = S \). It is shown that if \( K \) is an algebraically closed field which is not algebraic over its prime subring and \( R \) is affine integral domain over a field \( K \), then we prove that \( R \) is an integrally closed maximal subring of a ring \( T \) if and only if \( \dim(R) = 1 \) and in particular in this case \( (R : T) = 0 \).

1. Introduction

Following Faith [14], a subring \( V \) of a commutative ring \( T \) is called a conch subring if there exists a unit \( x \in T \) such that \( x^{-1} \in V \) but \( x \not\in V \) and \( V \) is maximal respect to this property (i.e., excluding \( x \) and including \( x^{-1} \)), \( V \) is called x-conch subring or it is said that \( V \) conches \( x \) in \( T \). In other words, \( V \) is a \( x \)-conch subring of a ring \( T \) if and only if \( V \) is maximal in the set \( \{R \mid R \text{ is a subring of } T, Z[x^{-1}] \subseteq R, x \not\in R \} \), where \( Z \) denotes the prime subring of \( R \). But note that clearly, the previous set is nonempty if and only if \( x \not\in Z[x^{-1}] \) which is equivalent to the fact that \( x \) is not integral over \( Z \). Krull proved that an integral domain \( R \) is integrally closed if and only if \( R \) is the intersection of the conch subrings of \( K = Q(R) \) that contains \( R \), and moreover, each conch subring \( V \) of a field \( K \) is a chain ring and therefore is a valuation domain (i.e., for each \( x \in K \), either \( x \in V \) or \( x^{-1} \in V \)), see [22, p.110]. In [14], Faith studied conch subrings for general commutative rings by the use of Manis valuations rings (or maximal pairs). He proved that if a subring \( A \) conches \( x \) in a ring \( Q \), then \( A \) is integrally closed and \( (A, \sqrt{x^{-1}A}) \) is a maximal pair of \( Q \); in particular, \( A \) is a valuation ring. Conversely, a maximal pair \( (A, P) \) comes from a conch subring of \( Q \) if and only if \( P = \sqrt{x^{-1}A} \) for some unit \( x \in Q \) \( (x^{-1} \in A) \) [Conch Ring Theorem]. Moreover, if \( R \) is a subring of a ring \( Q \), then the intersection \( \text{Conch}_Q(R) \) of the conch subrings of \( Q \) containing \( R \) is integrally closed, and every element of \( \text{Conch}_Q(R) := \text{Conch}_Q(R) \cap \text{Max}(Q) \) is a prime ideal of \( Q \)
$U(Q)$ is integral over $R$. More interestingly, in [20], Griffin proved that if $Q$ is VNR or has few zero-divisor (i.e., the set of all zero divisor of $Q$, $\text{zd}(Q)$, is a finite union of prime ideals of $Q$) and $R$ is integrally closed in $Q = Q(R)$, then $R$ is the intersection of valuation subrings of $Q$. In [14, Section 10], Faith studied conch subrings which are maximal subrings, i.e., if $A$ conches $x$ in $Q$, then $A$ is a maximal subring of $Q$ if and only if $Q = A[x]$.

In this paper, motivated by the above facts and the existence of maximal subring, we are interested to prove that if $R$ is a ring then either $R$ has a conch maximal subring or each subring of $R$ is closed with respect to taking inverse, i.e., $U(S) = S \cap U(R)$ for each subring $S$ of $R$, see [25]. In particular, for each ring $R$ either $U(R)$ is integral over $\mathbb{Z}$, or $R$ has a conch maximal subring.

All rings in this note are commutative with $1 \neq 0$. All subrings, ring extensions, homomorphisms and modules are unital. A proper subring $S$ of a ring $R$ is called a maximal subring if $S$ is maximal with respect to inclusion in the set of all proper subrings of $R$. Not every ring possesses maximal subrings (for example the algebraic closure of a finite field has no maximal subrings, see [8, Remark 1.13]; also see [7, Example 2.6] and [10, Example 3.19] for more examples of rings which have no maximal subrings). A ring which possesses a maximal subring is said to be sub-maximal, see [4, 8] and [10]. If $S$ is a maximal subring of a ring $R$, then the extension $S \subseteq R$ is called a minimal ring extension (see [15]) or an adjacent extension too (see [11]). Minimal ring extensions first appears in [18], for studying integral domains with a unique minimal overring. Next in [15], these extensions are fully studied and some essential facts are obtained. The following result, whose proof could be found in [15] is needed. Before presenting it, let us recall that whenever $S$ is a maximal subring of a ring $R$, then one can easily see that either $R$ is integral over $S$ or $S$ is integrally closed in $R$. If $S \subseteq R$ is a ring extension, then the ideal $(S : R) = \{x \in R \mid Rx \subseteq S\}$ is called the conductor of the extension $S \subseteq R$.

**Theorem 1.1.** Let $S$ be a maximal subring of a ring $R$. Then the following statements hold.

1. $(S : R) \in \text{Spec}(S)$.
2. $(S : R) \in \text{Max}(S)$ if and only if $R$ is integral over $S$.
3. If $S$ is integrally closed in $R$, then $(S : R) \in \text{Spec}(R)$. Moreover if $x, y \in R$ and $xy \in S$, then either $x \in S$ or $y \in S$.
4. There exists a unique maximal ideal $M$ of $S$ such that $S_M$ is a maximal subring of $R_M$ and for each $P \in \text{Spec}(S) \setminus \{M\}$, $S_P = R_P$ ($M$ is called the crucial maximal ideal of the extension $S \subseteq R$ and $(S : R) \subseteq M$. Moreover, $S \subseteq R$ is an integral (resp., integrally closed) extension if and only if $S_M \subseteq R_M$ is an integral (resp., integrally closed) extension.
5. If $S$ is an integrally closed maximal subring of $R$, $P = (S : R)$ and $M$ be the crucial maximal ideal of the extension $S \subseteq R$, then $(\frac{R}{P})_{\neq 0}$ is a one dimensional valuation domain.

Note that by $(u, u^{-1})$-Lemma, one can easily see that if $S \subseteq R$ is an integral ring extension then $U(S) = S \cap U(R)$. In particular, if $S$ is a maximal subring of $R$ and $S \subseteq R$ is integral, then $U(S) = S \cap U(R)$. On the other hand, if $S$ is an integrally closed maximal subring of $R$, then by the second part of (3) of Theorem 1.1 (or by the use of $(u, u^{-1})$-Lemma) for each $x \in U(R)$, we infer that either $x \in S$ or $x^{-1} \not\in S$. Therefore, if $S$ is a maximal subring of $R$ such that there exists a unit $x$ in $R$ such that $x \in S$ but $x^{-1} \not\in S$, we immediately conclude that $S$ is integrally closed in $R$. Now let us prove the following fact for conch subrings by the use of minimal ring extension.

**Corollary 1.2.** Let $T$ be a ring and $x \in U(T)$. If $R$ is a $x$-conch subring of $T$, then $x^{-1}$ is contained in exactly one prime (maximal) ideal of $R$, in other words $\sqrt{x^{-1}R} \in \text{Max}(R)$.

**Proof.** Since $R$ is a $x$-conch subring of $T$, we immediately conclude that $R \subseteq S := R[x]$ is a minimal ring extension. Thus by (4) of Theorem 1.1, let $M$ be the crucial maximal ideal of the minimal ring extension $R \subseteq S$. We prove that $M$ is the only prime ideal of $R$ which contains $x^{-1}$. First note that since $R_M \subseteq S_M$ is a minimal ring extension ($R_M \neq S_M$), we deduce that $x \not\in R_M$.
and therefore $x^{-1} \in M$. Now for each prime ideal $P$ of $R$ with $P \neq M$, we have $R_P = S_P$ by (4) of Theorem 1.1. Therefore $x^{-1}$ is a unit in $R_P$, hence $x^{-1} \notin P$ and we are done. \hfill \Box

Now, let us sketch a brief outline of this paper. Section 2 is devoted to the existence of integrally closed/conch maximal subrings. We prove that if $R$ is a ring, $D$ is an integral domain which is a subring of $R$, $p$ is a prime element of $D$ with $\cap_{i=1}^{\infty} p^n D = 0$ and $p \in U(R)$, then $R$ has an integrally closed/conch maximal subring. In particular, if $R$ is a ring which contains an atomic (or a completely integrally closed) domain $D$ and a prime element of $D$ is invertible in $R$, then $R$ has an integrally closed/conch maximal subring. Next, we show that if $T$ is a ring, then either $T$ has an integrally closed/conch maximal subring or $U(R) = R \cap U(T)$, for each subring $R$ of $T$, which is one of the main result in Section 2. This result has many consequences. In particular, either a ring $T$ has an integrally closed/conch maximal subring or $U(T)$ is integral over $Z$. It is observed that if $T$ is an integral domain which satisfies ACCP (resp. a BFD, see [1]), then either $T$ has a maximal subring or each subring of $T$ satisfies ACCP (resp. a BFD). We show that if $D$ is a GCD-domain which is a subring of a ring $T$, then either $T$ has a maximal subring or $U(T) \cap K = U(D)$, where $K$ is the quotient field of $D$. We prove that a ring $T$ has a conch maximal subring if and only if $Z[U(T)]$ has an integrally closed proper subring. If $T$ is a ring without maximal subring and $Char(T) \neq 0$, then each element of the group $U(T)$ has finite order. In particular, if in addition $U(T)$ is finitely generated, then $U(T)$ is finite. We observe that if $T$ is a residually finite ring with $Char(T) = 0$ and $U(T)$ is finitely generated, then either $T$ has a maximal subring or $T = Z$. We prove that if $T$ is a reduced ring with $|T| = 2^{2^{25}}$, then either $T$ has a maximal subring or $|Max(T)| = 2^{2^{25}}$. In particular, if $T$ is an integral domain with $|T| = 2^{2^{25}}$ and $dim(T) = 1$, then $T$ has a maximal subring. It is observe that if $R = K + R_x$, where $K$ is a field and $x$ is a subring of a ring $R$ and $x \notin U(R) \cup Zd(R)$, then $R$ has a maximal subring. Finally in Section 2, we prove that if $R$ is an atomic integral domain with $|R| = 2^{2^{25}}$ and $M$ is a principal maximal ideal of $R$, then $|R/M| = |R|$ and in particular $R$ has a maximal subring. Section 3, is devoted to the existence of maximal subring with certain conductor. We studied the conductor of integrally closed maximal subrings for integral extensions. In fact, we show that if $R \subseteq T$ is an integral extension, $Q \in Spec(T)$, $P := Q \cap R$, and there exists an integrally closed maximal subring $S$ of $R$ with $U(R/P) \subseteq S/P$ and $(S : R) = P$, then $T$ has an integrally closed maximal subring $V$ with $(V : T) = Q$. Conversely, if $R \subseteq T$ is an integral extension and each maximal ideal of $T$ is a conductor ideal of an integrally closed maximal subring of $T$, then each maximal ideal of $R$ is a conductor ideal of an integrally closed maximal subring of $R$. We show that if $K$ is an algebraically closed field which is not absolutely algebraic the each prime ideal $Q$ of $K[x_1, ..., x_n]$ with $ht(Q) \geq n - 1$ is a conductor ideal of an integrally closed maximal subring of $K[x_1, ..., x_n]$. But for minimal ring extension of $K[x_1, ..., x_n]$, $n \geq 2$, a prime ideal of $K[x_1, ..., x_n]$, is a conductor ideal of a minimal ring extension of $K[x_1, ..., x_n]$ if and only if $P$ is maximal; and for $n = 1$, each prime ideal of $K[x_1]$ is a conductor ideal of a minimal ring extension of $K[x_1]$. We prove some results for the zeroeness of the conductor of the minimal ring extension of the form $R \subseteq R_1$, where $R$ is a certain ring or $u$ is a certain element of $R$.

Finally, let us recall some notation and definitions. As usual, let $Char(R)$, $U(R)$, $Zd(R)$, $N(R)$, $J(R)$, $Max(R)$, $Spec(R)$ and $Min(R)$, denote the characteristic, the set of all units, the set of all zero-divisors, the nil radical ideal, the Jacobson radical ideal, the set of all maximal ideals, the set of all prime ideals and the set of all minimal prime ideals of a ring $R$, respectively. We also call a ring $R$, not necessarily Noetherian, semilocal (resp. local) if $Max(R)$ is finite (resp. $|Max(R)| = 1$). For any ring $R$, let $Z = Z \cdot 1_R = \{ n \cdot 1_R | n \in Z \}$, be the prime subring of $R$. We denote the finite field with $p^n$ elements, where $p$ is prime and $n \in N$, by $F_{p^n}$. Fields which are algebraic over $F_p$ for some prime number $p$, are called absolutely algebraic fields. If $R$ is a ring and $a \in R \setminus N(R)$, then $R_a$ denotes the ring of quotient of $R$ respect to the multiplicatively closed set $C = \{ 1, a, a^2, ..., a^n, ... \}$. If $D$ is an integral domain, then we denote the set of all non-associate irreducible elements of $D$ by $Irr(D)$. Also, we denote the set of all natural prime numbers by $\mathbb{P}$.
Suppose that $D \subseteq R$ is an extension of domains. By Zorn’s Lemma, there exists a maximal (with respect to inclusion) subset $X$ of $R$ which is algebraically independent over $D$. By maximality, $R$ is algebraic over $D[X]$ (thus every integral domain is algebraic over a UFD; this can be seen by taking $D$ to be the prime subring of $R$). If $E$ and $F$ are the quotient fields of $D$ and $R$, respectively, then $X$ can be shown to be a transcendence basis for $F/E$ (that is, $X$ is maximal with respect to the property of being algebraically independent over $E$). The transcendence degree of $F$ over $E$ is the cardinality of a transcendence basis for $F/E$ (it can be shown that any two transcendence bases have the same cardinality). We denote the transcendence degree of $F$ over $E$ by $\text{tr.deg}(F/E)$. We remind that whenever $R \subseteq T$ is an affine ring extension, i.e., $T$ is finitely generated as a ring over $R$ (in particular, if $T$ is a finitely generated $R$-module), then by a natural use of Zorn Lemma, one can easily see that $T$ has a maximal subring $S$ which contains $R$. If $R$ is a proper subring of $T$, then $R$ is a maximal subring of $T$ if and only if for each $x \in T \setminus R$, we have $R[x] = T$. Finally, we refer the reader to [16], for standard definitions of Bezout domains, QR-domains and completely integrally closed domains.

2. Existence of conch maximal subring

We begin this section by the following main result which is a generalization of [4, Theorem 2.2].

**Theorem 2.1.** Let $D$ be an integral domain with a prime element $p$ such that $\cap_{n=1}^{\infty} p^n D = 0$. Assume that $D \subseteq R$ is a ring extension such that $p \in U(R)$. Then $R$ has an integrally closed maximal subring which contains $D$ and conches $\frac{1}{p}$.

**Proof.** We may assume that $R$ is an integral domain which is algebraic over $D$. To see this, first note that $D \setminus \{0\}$ is a multiplicatively closed set in $R$, therefore $R$ has a prime ideal $Q$, with $Q \cap D = 0$. In other words, $D$ can be embedded in $R/Q$, and clearly $p \in U(R/Q)$ (note that if $S/Q$ is an integrally closed maximal subring of $R/Q$ which contains the image of $D$ and conches $(p + Q)^{-1}$ in $R/Q$, then $S$ is an integrally closed maximal of $R$ which contains $D$ and conches $\frac{1}{p}$). Hence we may suppose that $R$ is an integral domain. If $R$ is not algebraic over $D$, then let $X$ be a transcendence base for $R$ over $D$. Clearly $p$ is prime in $D[X]$ and $\cap_{n=1}^{\infty} p^n D[X] = 0$. Hence we can replace $D$ by $D[X]$. Thus assume that $D \subseteq R$ is an algebraic extension of integral domains with quotient fields $K \subseteq E$. It is clear that $E/K$ is an algebraic extension. Now we claim that $D_{(p)}[\frac{1}{p}] = K$. For proof let $x = \frac{a}{b} \in K$, where $a, b \in D$. Since $\cap_{n=1}^{\infty} p^n D = 0$, we conclude that $b = p^n c$, where $c \notin pD$ and $n \geq 0$. Thus $\frac{c}{a} \in D_{(p)}$ and therefore $x = \frac{a/c}{p^n} \in D_{(p)}[\frac{1}{p}]$, i.e., $K = D_{(p)}[\frac{1}{p}]$. Next we prove that $S := D_{(p)}$ is a maximal subring of $K$. To see this, let $x \in K \setminus S$. From $\cap_{n=1}^{\infty} p^n D = 0$ and $x \notin S$, we easily see that $x = \frac{a}{b}$, where $a, b \notin pD$ and $n \in \mathbb{N}$. Hence we infer that $\frac{1}{a} \notin S$ and therefore $\frac{1}{p^n} \in S[x]$. Thus $\frac{1}{p} \in S[x]$, which immediately by the previous part implies that $S[x] = K$, i.e., $S$ is a maximal subring of $K$. Thus by [9, Proposition 2.1], we conclude that $E$ has a maximal subring $V$ such that $V \cap K = S$. Hence $\frac{1}{p} \notin V$ and therefore we deduce that $V$ is an integrally closed maximal subring of $E$. Thus in the extension $R \subseteq E$ we have $U(R) \subseteq V$. Therefore by [10, Theorem 2.19], we infer that $R$ has an integrally closed maximal subring $W$ which contains $V \cap R$ but $\frac{1}{p} \notin W$ i.e., $W$ is a maximal subring which contains $D$ and conches $\frac{1}{p}$ in $R$ (note that a conch maximal subring is integrally closed by the comment preceding Corollary 1.2).

**Corollary 2.2.** Let $D$ be an atomic (or a completely integrally closed) domain and $R$ is a ring extension of $D$. If a prime element of $D$ is invertible in $R$, then $R$ has a conch maximal subring.
Proof. Let $p$ be a prime element of $D$ which is invertible in $R$. We claim that $J := \bigcap_{n=1}^{\infty} p^nD = 0$. If $D$ is completely integrally closed then we are done by [16, Corollary 13.4]. Hence assume that $R$ is atomic and $J \neq 0$, then by [21, Ex.5, Sec.1-1], $J$ is a prime ideal of $R$ which is properly contained in $M := (p)$. Now since $J \neq 0$ and $D$ is an atomic domain, we immediately conclude that $J$ contains an irreducible element $q$ of $R$. Thus $q \in M = (p)$ and therefore $q = p$ which is absurd (see [2] for more interesting result in arbitrary commutative rings). Thus in any cases $J = 0$ and hence we are done by the previous theorem.

Fields which have (no) maximal subrings are completely determined in [8]. We need the following in sequel.

Corollary 2.3. Let $K$ be a field which is not absolutely algebraic. Then $K$ has an integrally closed/conch maximal subring.

Proof. If $\text{Char}(K) = 0$ then $K$ contains $\mathbb{Z}$ and use the previous corollary. If $\text{Char}(K) = p$, where $p \in \mathbb{P}$, then by our assumption there exists $x \in K$ which is not algebraic over $\mathbb{Z}_p$. Therefore $K$ contains the atomic domain $\mathbb{Z}_p[x]$ and again use the previous corollary.

Now we have the following main theorem.

Theorem 2.4. Let $T$ be a ring. Then either $T$ has a conch maximal subring or $U(R) = R \cap U(T)$ for each subring $R$ of $T$. In other words either $T$ has a conch maximal subring or each subring of $T$ is closed respect to taking the inverse.

Proof. Suppose that $T$ has a subring $R$ with $U(R) \subseteq R \cap U(T)$. Hence there exists $x \in R \setminus U(R)$ with $x \in U(T)$. We show that $T$ has a maximal subring conches $x^{-1}$. First we claim that we may assume that $T$ is an integral domain. To see this, take $C := \{1 + xy \mid y \in R\}$, clearly $C$ is a multiplicatively closed subset of $R$ and $0 \notin C$, for $x$ is not a unit in $R$. Hence there exists a prime ideal $P$ of $R$ such that $P \cap C = \emptyset$. We may assume that $P$ is a minimal prime ideal of $R$ and therefore $x \notin P$ (note, $x$ is not a zero-divisor of $R$). Since $P$ is a minimal prime ideal of $R$, we conclude that there exists a (minimal) prime ideal $Q$ of $T$ such that $Q \cap R = P$ (see [21, Exercise 1, p. 41]), therefore we can consider $R/P$ as a subring of $T/Q$. Now note that the image of $x$ is a unit of $T/Q$ but is not a unit of $R/P$; for otherwise there exists $y \in R$, such that $1 - xy \in P$, i.e., $1 - (-y)x \in P \cap C$ which is absurd. Thus the extension $R/P \subseteq T/Q$ is an extension of integral domains and $\bar{x} := x + P$ is not a unit in $R/P$ but is a unit in $T/Q$. Clearly, if $S/Q$ is a maximal subring of $T/Q$ which conches $\bar{x}^{-1}$ in $T/Q$, then $S$ is a maximal subring of $T$ conches $x^{-1}$. Thus we may assume $T$ is an integral domain. Now we have two cases:

1. $\text{Char}(T) = p > 0$, where $p$ is a prime number (note $T$ is an integral domain). Thus we infer that $x$ is not algebraic over the prime subring of $T$, for otherwise $x$ is integral over $\mathbb{Z}_p$, the prime subring of $R$ (or $T$). Therefore by $(u, u^{-1})$-Lemma, $x \in U(R)$ which is absurd. Therefore $x$ is not algebraic over $\mathbb{Z}_p$. Hence $D := \mathbb{Z}_p[x]$ is an atomic domain which is contained in $T$ and the prime element $x$ of $D$ is invertible in $T$. Thus $T$ has a maximal subring that conches $x^{-1}$ by Corollary 2.2.

2. $\text{Char}(T) = 0$. If $x$ is not algebraic over $\mathbb{Z}$, then similar to the proof of (1) and considering the atomic domain $D := \mathbb{Z}[x]$ in $T$, we conclude that $T$ has a maximal subring conches $x^{-1}$, by Corollary 2.2. Thus assume that $x$ is algebraic over $\mathbb{Z}$. Since $x^{-1} \notin R$, we conclude that $x^{-1} \notin \mathbb{Z}_p$, for $\mathbb{Z}_p \subseteq R$. Thus by [16, Theorem 30.9], $\dim(\mathbb{Z}[x]) = 1$ and therefore by [21, Theorem 56], the quotient field of $\mathbb{Z}[x]$, i.e., $\mathbb{Q}(x)$ has a valuation $V$ such that $xV \neq V$ (i.e., $x^{-1} \notin U(V)$). Since $\mathbb{Z}[x]$ is a one dimensional Noetherian integral domain, then by Krull-Akizuki Theorem ([21, Theorem 93]), we infer that $V$ is a one dimensional Noetherian valuation domain. In other words, $V$ is a DVR and therefore $V$ is a UFD. Let $M = (\pi)$ denotes
the maximal ideal of \( V \). Thus \( x = v \pi \) for some \( v \in V \) (note, \( V \) is a valuation and \( x^{-1} \notin V \)).

Let \( K \) be the quotient field of \( T \), then \( V \subseteq \mathbb{Q}(x) \subseteq K \). Therefore by Theorem 2.1, \( K \) has a maximal subring \( W \) such that \( V \subseteq W \) and \( \frac{1}{2} \not\in W \). Thus \( v \in W \) and \( \pi \in N \), where \( N \) is the maximal ideal of \( W \). Therefore \( x \in N \) and thus \( U(T) \not\subseteq W \). Therefore by [10, Theorem 2.19], \( T \) has a maximal subring that conches \( x^{-1} \).

In other words, the previous theorem states that if \( R \subseteq T \) is a ring extension and a non-invertible element \( x \) of \( R \), is invertible in \( T \), then \( T \) has a maximal subring. In particular, if a ring \( T \) has no maximal subring, then \( xR \neq R \) implies \( xT \neq T \), for each subring \( R \) of \( T \) and \( x \in R \). Therefore, either a ring \( T \) has a maximal subring or each principal proper ideal of any subring of \( T \), survives in \( T \). The previous theorem has several conclusions as follows.

**Corollary 2.5.** Let \( T \) be a ring, then \( T \) has a conch maximal subring if and only if \( U(T) \) is not integral over \( \mathbb{Z} \). In particular, if \( T \) has no conch maximal subring then \( \mathbb{Z}[x] = \mathbb{Z}[x^{-1}] \), for each \( x \in U(T) \).

**Proof.** If \( T \) has a subring \( R \) which conches \( x^{-1} \) in \( T \), then by \((u,u^{-1})\)-Lemma, \( x^{-1} \) is not integral over \( R \) and therefore \( x^{-1} \) is not integral over \( \mathbb{Z} \). Conversely, assume that \( T \) has no conch maximal subring, then for each \( x \in U(T) \), by Theorem 2.4, we conclude that \( U(\mathbb{Z}[x]) = \mathbb{Z}[x] \cap U(T) \). Therefore \( x^{-1} \in \mathbb{Z}[x] \). Thus by \((u,u^{-1})\)-Lemma, we deduce that \( x^{-1} \) is integral over \( \mathbb{Z} \). Thus \( U(T) \) is integral over \( \mathbb{Z} \). Hence if \( T \) has no conch maximal subring, then for each \( x \in U(T) \) we obtain that \( x^{-1} \in \mathbb{Z}[x] \) and \( x \in \mathbb{Z}[x^{-1}] \). Therefore \( \mathbb{Z}[x] = \mathbb{Z}[x^{-1}] \).

In other words the previous corollaries states that a ring \( T \) has a subring \( S \) which conches \( x^{-1} \) in \( T \) if and only if \( T \) has a maximal subring which conches \( x^{-1} \) in \( T \).

**Corollary 2.6.** Let \( T \) be a ring with nonzero characteristic. Then either \( T \) has a conch maximal subring or each element of the group \( U(T) \) has finite order.

**Proof.** Assume that \( \text{Char}(T) = n \) and therefore \( \mathbb{Z}_n \) is the prime subring of \( T \). Thus if \( T \) has no maximal subring, then for each \( x \in U(T) \), \( x \) is integral over \( \mathbb{Z}_n \). Hence, \( \mathbb{Z}_n[x] \) is a finitely generated \( \mathbb{Z}_n \)-module. Therefore \( \mathbb{Z}_n[x] \) is finite. This immediately implies that \( x \) has finite order in \( U(T) \).

**Corollary 2.7.** Let \( T \) be a ring without maximal subring and \( S \subseteq R \) be subrings of \( T \). Then \( U(S) = S \cap U(R) \).

**Proof.** Since \( T \) has no maximal subring, then by Theorem 2.4, \( U(R) = R \cap U(T) \) and \( U(S) = S \cap U(T) \). Thus \( U(S) = S \cap U(T) = (S \cap R) \cap U(T) = S \cap (R \cap U(T)) = S \cap U(R) \).

**Corollary 2.8.** Let \( R \) be a ring and \( C \) be a multiplicatively closed subset of regular elements (i.e., non zero-divisors) of \( R \). Then either \( R_C \) has a maximal subring or \( R_C = R \), i.e., \( C \subseteq U(R) \). In particular, if \( R \) is an integral domain, then each proper quotient overring of \( R \) (i.e., \( R \subseteq R_C \)) has a maximal subring.

**Proof.** It suffices to take \( T = R_C \) in Theorem 2.4.

**Corollary 2.9.** Let \( R \) be an integral domain and \( C \) be a multiplicatively closed subset of \( R \) which is not contained in \( U(R) \). Then there exists a proper subring \( S \) of \( R_C \) and \( a \in C \) such that \( R_C = S_a \).

An integral domain \( R \) is called a QR-domain, if each overring of \( R \) is of the form \( R_C \) for some multiplicatively closed subset \( C \) of \( R \). It is well-known that each Bezout domain is a QR-domain.
Corollary 2.10. Let $R$ be a QR-domain (which is not absolutely algebraic field). Then each proper overring of $R$ has a maximal subring.

Corollary 2.11. Let $R$ be a ring. Then either $Q(R)$ has a maximal subring or $Q(R) = R$.

**Proof.** Let $C = R \setminus \mathbb{Zd}(R)$, therefore $Q(R) = R_C$. If for some $x \in C$, we have $x^{-1} \notin R$, then by Theorem 2.4, $Q(R)$ has a maximal subring. Hence if $Q(R)$ has no maximal subring, then for each $x \in C$ we conclude that $x^{-1} \in R$ and therefore $R = Q(R)$. □

Corollary 2.12. Let $R$ be a Noetherian ring. Then either $Q(R)$ has a maximal subring or $R$ is a countable Artinian ring with nonzero characteristic which is integral over its prime subring.

**Proof.** Assume that $Q(R)$ has no maximal subring. Therefore by Corollary 2.11, $R = Q(R)$. Now note that since $R$ is Noetherian ring, then $\text{Ass}(0)$ is finite and $\mathbb{Zd}(R) = \bigcup_{P \in \text{Ass}(0)} P$. This immediately implies that $Q(R)$ is a semilocal ring. Thus by [10, Proposition 3.13], we infer that $R = Q(R)$ is an Artinian ring with nonzero characteristic which is integral over $\mathbb{Z}$. □

We remind that an integral domain $R$ satisfies the ascending chain condition for principal ideals (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of $R$, see [1].

Corollary 2.13. Let $T$ be an integral domain satisfies ACCP. Then either $T$ has a conch maximal subring or each subring of $T$ satisfies ACCP. In particular, if $T$ has no maximal subring, then each subring $R$ of $T$ is atomic and if $(a_1) \supset (a_2) \supset \cdots$ is an infinite strictly descending chain of principal ideals of $R$, then $\cap_{i=1}^\infty (a_i) = 0$. Consequently, for each $a \in R \setminus U(R)$ we have $\cap_{i=1}^\infty (a^i) = 0$.

**Proof.** Assume that $T$ has no maximal subring, then by Theorem 2.4, we conclude that $U(R) = R \cap U(T)$. Therefore, by [19, Proposition 2.1 and Corollary], $R$ has ACCP. The final part is evident for it is well-known that domains with ACCP are atomic. □

We remind that an integral domain $R$ is called a bounded factorization domain (BFD) if $R$ is atomic and for each nonzero nonunit of $R$ there is a bound on the length of factorization into product of irreducible elements, i.e., for each nonzero nonunit $x$ of $R$, there exists a positive integer $N(x)$ such that whenever $x = x_1 \cdots x_n$ as a product of irreducible elements of $R$, then $n \leq N(x)$. One can easily see that a BFD satisfies ACCP, but the converse does not hold, see [1, Example 2.1]. Also a Noetherian domain or a Krull domain is a BFD, see [1, Proposition 2.2].

Corollary 2.14. Let $T$ be a BFD. Then either $T$ has a maximal subring or each subring of $T$ is a BFD.

**Proof.** Assume that $T$ has no maximal subring, then by Theorem 2.4, we conclude that $U(R) = R \cap U(T)$. Therefore, by the comment preceding [1, Proposition 2.6], $R$ is a BFD. □

Corollary 2.15. Let $R$ be a GCD-domain and $T$ a ring extension of $R$. Assume that there exist $a, b \in R$ such that $gcd(a, b) = 1$, $ab \notin U(R)$ and $\frac{a}{b} \in U(T)$. Then $T$ has a maximal subring. In particular, if $T$ has no maximal subring, then $U(T) \cap K = U(R)$, where $K$ is the quotient field of $R$.

**Proof.** Let $x = \frac{a}{b}$. If $T$ has no maximal subring, then $x$ (resp. $x^{-1}$) is integral over the prime subring of $T$ and therefore $x$ (resp. $x^{-1}$) is integral over $R$. Thus $x$ and $x^{-1}$ are in $R$, for a GCD-domain is integrally closed, i.e., $b|a$ and $a|b$ in $R$. Therefore $a$ and $b$ are units for $gcd(a, b) = 1$, which is absurd. Thus $T$ has a maximal subring. The final part is evident. □
Corollary 2.16. Let $T$ be a ring such that the prime subring $Z$ of $T$ is integrally closed in $T$. Then either $T$ has a maximal subring or $U(T) = U(Z)$. In particular, if $T$ has no maximal subring, then $U(T)$ is finite.

Proof. By Theorem 2.4, if $T$ has no maximal subring, then $U(T)$ is integral over $Z$ and therefore $U(T) \subseteq Z$, for $Z$ is integrally closed in $T$. This immediately implies that $U(T) = U(Z)$ and therefore $U(T)$ is finite. \qed

Proposition 2.17. Let $T$ be a ring and $R := Z[U(T)]$. The following are equivalent.

1. $R$ has a proper subring which is integrally closed in $R$.
2. $R$ has an integrally closed maximal subring.
3. $T$ has a maximal subring $A$ and there exists $u^{-1} \in U(T) \setminus A$ with $u \in A$ (i.e., $T$ has a conch maximal subring).
4. $U(T)$ is not integral over $Z$.

Proof. (1) $\Rightarrow$ (2) If $S$ is a proper integrally closed subring of $R$, then clearly $U(T)$ is not integral over $S$. Therefore $U(T) = U(R)$ is not integral over $Z$. Thus by Theorem 2.4, $R$ has an integrally closed maximal subring. (2) $\Rightarrow$ (3) If $A$ is an integrally closed maximal subring of $R$, then $U(R) = U(T)$ is not integral over $A$. Therefore $U(T)$ is not integral over $Z$ and hence we are done by Theorem 2.4. (3) $\Rightarrow$ (4) $U(T)$ is not integral over $A$ and therefore is not integral over $Z$. (4) $\Rightarrow$ (1) By Theorem 2.4, $R$ has an integrally closed maximal subring (which is proper). \qed

Proposition 2.18. Let $T$ be a ring without maximal subrings where $U(T)$ is a finitely generated group. Then

1. If $\text{Char}(T) = n$, then $\mathbb{Z}_n[U(T)]$ is finite. In particular, $U(T)$ is finite.
2. If $\text{Char}(T) = 0$, then $\mathbb{Z}[U(T)]$ is a finitely generated $\mathbb{Z}$-module.
3. If $\text{Char}(T) = 0$ and $T$ is a residually finite ring, then $T = \mathbb{Z}$.

Proof. First note that if $U(T) = \langle u_1, \ldots, u_n \rangle$, then by our assumption and Theorem 2.4, for each $i$, $u_i^{-1}$ is integral over $Z$, and therefore $u_i^{-1} \in Z[u_i] \subseteq Z[u_1, \ldots, u_n]$. Therefore $Z[U(T)] = Z[u_1, \ldots, u_n]$ is integral over $Z$. Thus $Z[U(T)]$ is finitely generated $\mathbb{Z}$-module. In particular, if $\text{Char}(R) = n > 0$ (i.e., $Z = \mathbb{Z}_n$), then $Z[U(T)]$ is finite. Hence (1) and (2) hold. For (3), By [4, Theorem 2.29], $T = \mathbb{Z}[U(T)]$. Therefore by the first part of the proof $T = \mathbb{Z}[u_1, \ldots, u_n]$. Hence if $U(T) \neq U(Z)$, then $T$ is finitely generated as a ring over $Z$ and thus $T$ has a maximal subring which is absurd. Therefore $U(T) = U(Z)$ and thus $T = \mathbb{Z}$, hence we are done. \qed

A ring $R$ is called clean, if each nonzero element of $R$ is a sum of a unit and an idempotent of $R$. Now the following is in order.

Proposition 2.19. Let $R$ be a clean ring. Then either $R$ has a maximal subring or $R$ has nonzero characteristic and is integral over $Z$.

Proof. Assume that $R$ has no maximal subring. Then by Corollary 2.5, $U(R)$ is integral over $Z$. Also note that clearly each idempotent of $R$ is integral over $Z$. Therefore by our assumption each element of $R$ is a sum of two elements of $R$ which are integral over $Z$. Hence $R$ is integral over $Z$. Finally, if $\text{Char}(R) = 0$, then $\text{dim}(R) = 1$ for $R$ is integral over $Z$. Let $P$ be prime ideal of $R$ which is not maximal, then $R/P$ is a local integral domain (note, in clean ring each prime ideal is contained in a unique maximal ideal). Therefore, by [10, Corollary 2.24], $R/P$ and therefore $R$ has a maximal subring which is absurd. Hence $R$ has nonzero characteristic and hence we are done. \qed
Example 2.20.

1. There exists a ring $R$ which has a maximal subring, $R$ is integral over $\mathbb{Z}$ and moreover $U(R)$ is a finitely generated group. To see this, let $K$ be a finite field extension of $\mathbb{Q}$ of degree $n \geq 2$ and $R$ be the integral closure of $\mathbb{Z}$ in $K$. Then it is well-known that $R$ is a free $\mathbb{Z}$-module with rank $n$. Therefore $U(R)$ is integral over $\mathbb{Z}$, and by Dirichlet’s unit theorem $U(R)$ is finitely generated. Finally note that since $R \neq \mathbb{Z}$ is a finitely generated algebra, we deduce that $R$ has a maximal subring, but clearly $R$ has no conch maximal subring, for $U(R)$ is integral over $\mathbb{Z}$.

2. There exists a ring $R$ which has a maximal subring, $R$ is not integral (algebraic) over its prime subring and moreover $U(S) = S \cap U(R)$ for each subring $S$ of $R$. To see this, let $p$ be a prime number and $R = \mathbb{Z}_p[X]$ be the polynomial ring over $\mathbb{Z}_p$. Then clearly $R$ has a maximal subring, and $R$ is not algebraic over $\mathbb{Z}_p$. Finally note that for each subring $S$ of $R$, we have $\mathbb{Z}_p \subseteq S$ and therefore $U(S) = U(R)$.

3. For each infinite cardinal number $\alpha$, there exists a ring $R$ with $|R| = |U(R)| = \alpha$ and $R$ has no maximal subring. In particular, if $\alpha > \aleph_0$, then $U(R)$ is not finitely generated. To see this note that if $K$ is a field with zero characteristic and $|K| = \alpha$, then the idealization $R = \mathbb{Z}(+)K$ has no maximal subring by [9, Example 3.19] (for $K$ has no maximal $\mathbb{Z}$-submodule). It is clear that $1(+)K \subseteq U(R)$ and therefore $|U(R)| = \alpha = |R|$.

4. There exists a conch subring of a ring $T$ which is not a maximal subring of $T$. To see this, let $V$ be a valuation domain with $\dim(V) = n \geq 2$ and quotient field $K$. Suppose $P$ and $Q$ are prime ideals of $V$ with $\text{ht}(Q) = n$ and $\text{ht}(P) = n - 1$. Let $x \in Q \setminus P$. Then $V$ conches $x^{-1}$ in $K$ but $V \nsubseteq V_P \subseteq K$, i.e., $V$ is not a maximal subring of $K$.

Proposition 2.21. Let $T$ be a ring with $|T| > 2^{2^{\aleph_0}}$. Let $R$ be the integral closure of $\mathbb{Z}$ in $T$ and $X$ be a generator set for the group $U(T)$. Then either $T$ has a maximal subring or $|T| = |R| = |X|$. In particular, if $S$ is an integrally closed subring of $T$, then $U(T) = U(S)$ and $|S| = |T|$.

Proof. First note that by [9, Corollary 2.4], either $T$ has a maximal subring or $|U(T)| = |T|$. Now assume that $T$ has no maximal subring, therefore by Corollary 2.5, $U(T)$ is integral over $\mathbb{Z}$. Hence $U(T) \subseteq R$ and therefore $|R| = |T|$. Next, we show $|X| = |T|$. If $X$ is finite, then $U(T)$ is finitely generated and therefore $U(T)$ is countable which is absurd. Hence $X$ is infinite. Similar to the proof of Proposition 2.18, $\mathbb{Z}[U(T)] = \mathbb{Z}[X]$. One can easily see that $|\mathbb{Z}[X]| = |X|$ and therefore $|U(T)| = |X|$. The final part is evident. 

In the next result we prove that whenever $R$ is a local integral domain which is an integrally closed maximal subring of a ring $T$, then $R$ is a conch subring of $T$.

Proposition 2.22. Let $R$ be an integral domain which is an integrally closed maximal subring of a ring $T$. Then

1. If $R$ is a local ring, then $R$ is a conch subring of $T$.
2. If $M$ is the crucial maximal of $R \subseteq T$, then $R_M$ is a conch subring of $T_M$.

Proof. First note that by [23, Theorem 10], $T$ is an integral domain. Let $y \in T \setminus R$. Then by maximality of $R$ we have $R[y] = T$. Now by a similar proof [16, Lemma 19.14], we conclude that $y^{-1} \in R$ and therefore $R$ conches $y$ in $T$. This proves (1). (2) is trivial by (1) and (4) of Theorem 1.1.

In [10, Corollaries 3.6 and 3.7], we proved that if $T$ is a reduced ring with $J(T) \neq 0$ or $|T| > 2^{2^{\aleph_0}}$, then $T$ has a maximal subring. Now the following is in order.
Theorem 2.23. Let \( T \) be a reduced ring with \( |T| = 2^{2^{\aleph_0}} \). Then either \( T \) has a maximal subring or \( |\text{Max}(T)| = 2^{\aleph_0} \) and the intersection of each countable family of maximal ideals of \( T \) is nonzero.

Proof. Assume that \( T \) has no maximal subring, then by [10, Corollary 3.6], \( J(T) = 0 \). Therefore \( T \) can be embedded in \( \prod_{M \in \text{Max}(T)} \frac{T}{M} \). Since \( T \) has no maximal subring, then by [8, Proposition 2.6], we conclude that \( |\text{Max}(T)| \leq 2^{\aleph_0} \), and by [8, Corollary 1.3] for each maximal ideal \( M \) of \( T \) we have \( |R/M| \leq \aleph_0 \). Now if \( |\text{Max}(T)| < 2^{\aleph_0} \), then we deduce that

\[
|T| \leq \prod_{M \in \text{Max}(T)} \frac{T}{M} \leq \aleph_0^\aleph_0
\]

which is absurd. Thus we infer that \( |\text{Max}(T)| = 2^{\aleph_0} \). The proof of the final part is similar.

Now we have the following.

Theorem 2.24. Let \( T \) be an integral domain with \( |T| = 2^{2^{\aleph_0}} \) and \( \dim(T) = 1 \). Then \( T \) has a maximal subring.

Proof. First note that if \( \text{Char}(T) = p \) is a prime number, then \( T \) is not algebraic over \( \mathbb{Z}_p \); thus there exists \( x \in T \) which is not algebraic over the prime subring of \( T \). Now if \( T \) has zero characteristic, then we take \( D = \mathbb{Z} \) and if \( \text{Char}(T) = p > 0 \) (where \( p \) is a prime number), we take \( D = \mathbb{Z}_p[x] \). In any cases \( D \) is a PID with infinitely many non-associate prime elements. Let \( \text{Irr}(D) = \{q_1, q_2, \ldots\} \). We have two cases. If there exists \( i \) such that \( q_i \in U(T) \), then by Theorem 2.1, \( T \) has a maximal subring. Hence suppose that for each \( i \), \( q_i \) is not invertible in \( T \) and therefore \( T \) has a maximal ideal \( M_i \) such that \( q_i \in M_i \). Now by Theorem 2.23, \( N := \cap_{i=1}^\infty M_i \neq 0 \). We claim that \( N \cap D = 0 \). To see this, let \( q \in N \cap D \), then for each \( i \) we have \( q \in M_i \cap D = q_i D \) and therefore \( q = 0 \). Thus \( N \cap C = \emptyset \), where \( C := D \setminus \{0\} \) is a multiplicatively closed subset in \( T \). Therefore \( T \) has a prime ideal \( P \) with \( N \subseteq P \) and \( P \cap C = \emptyset \). Since \( \dim(T) = 1 \), we conclude that \( P \) is a maximal ideal of \( T \). From \( D \cap P = 0 \) we deduce that the field \( T/P \) contains a copy of \( D \). Therefore \( T/P \) is not absolutely algebraic field and hence by Corollary 2.3, \( T/P \) has a maximal subring. Thus \( T \) has a maximal subring and we are done.

Proposition 2.25. Let \( T \) be an integral domain with \( |T| = 2^{2^{\aleph_0}} \) and \( \dim(T) = 2 \). Let \( H_1 \) be the set of all height one prime ideals of \( T \). Then either \( T \) has a maximal subring or \( \cap_{P \in H_1} P = 0 \), \( |H_1| \geq 2^{\aleph_0} \) and \( |R/P| \leq 2^{\aleph_0} \) for each \( P \in H_1 \).

Proof. Assume that \( T \) has no maximal subring. Therefore, by [10, Corollary 2.24], \( J(T) = 0 \). Since \( \dim(T) \) is finite we conclude that each maximal ideal of \( T \) contains a height one prime ideal and clearly each height one prime ideal is contained in a maximal ideal of \( T \). Thus \( \cap_{P \in H_1} P \subseteq J(T) = 0 \). Hence \( T \) embeds in \( \prod_{P \in H_1} R/P \). Therefore, if \( |H_1| < 2^{\aleph_0} \), then there exists \( P \in H_1 \) such that \( |R/P| \geq 2^{2^{\aleph_0}} \). Now either \( R/P \) is a field or \( \dim(R/P) = 1 \). In the former case \( R/P \) has a maximal subring by Corollary 2.3 and in the later case \( R/P \) has a maximal subring by Theorem 2.24, which is a contradiction in any cases. Thus \( |H_1| \geq 2^{\aleph_0} \).

Proposition 2.26. Let \( K \) be a field, \( R \) a ring extension of \( K \), \( x \in R \setminus (U(R) \cup \text{Z}(R)) \). If \( R = K + Rx \), then \( R \) has a maximal subring.

Proof. First we show that \( K + Rx^2 \) is a proper subring of \( R \). It is clear that \( K + Rx^2 \) is a subring of \( R \). Now if \( R = K + Rx^2 \), then there exist \( a \in K \) and \( r \in R \) such that \( x = a + rx^2 \). If \( a \neq 0 \), then we conclude that \( x(1 - rx) = a \in U(R) \) which is absurd. Hence \( a = 0 \) and therefore \( x = rx^2 \), and since \( x \) is not a zero-divisor we deduce that \( 1 = rx \) which is impossible by our assumption.
Thus $K + Rx^2$ is a proper subring of $R$. Now note that $R = K + Rx = K + (K + Rx)x = K + Kx + Rx^2$. Therefore $(K + Rx^2)[x] = R$. Thus $R$ has a maximal subring. 

**Proposition 2.27.** Let $R$ be an uncountable PID, then any ring extension of $R$ has a maximal subring.

**Proof.** Let $T$ be a ring extension of $R$. If $U(R)$ is uncountable then we are done by Theorem 2.4, for in this case $U(R)$ is not algebraic over $Z$ and therefore $U(R)$ is not integral over $Z$. Hence assume that $U(R)$ is countable. Thus by the proof of [4, Theorem 3.1], we infer that $R$ has a prime element $q$ such that $U(R/(q))$ is not algebraic over the prime subring of $R/(q)$. Now we have two cases. If $qT = T$, then by Theorem 2.4, $T$ has a maximal subring. Otherwise $qT$ is a proper ideal of $T$ and clearly $qT \cap R = qR$, for $qR$ is a maximal ideal of $R$. Thus $T/qT$ contains a copy of $R/qR$. Hence $U(T/qT)$ contains a copy of $U(R/qR)$. Therefore $U(T/qT)$ is not integral over the prime subring of $T/qT$. Therefore, $T/qT$ has a maximal subring by Theorem 2.4. Thus $T$ has a maximal subring. 

**Proposition 2.28.** Let $R$ be an atomic (or a completely integrally closed) domain with $|R| = 2^{2^{2^0}}$. If $M$ is a principal maximal ideal of $R$, then $|R/M| = |R|$ and therefore $R$ has a maximal subring.

**Proof.** If $M = 0$, then $R$ is a field and therefore we are done by Corollary 2.3. Thus assume that $M = (p)$ is a nonzero principal maximal ideal of $R$. Hence $p$ is a prime element of $R$. Similar to the proof of Corollary 2.2, we conclude that $\cap_{n=1}^{\infty} (p^n) = 0$. Therefore $R$ can be embedded in $\prod_{n=1}^{\infty} R/(p^n)$. Thus there exists $n$ such that $|R/(p^n)| = 2^{2^{2^0}}$. Hence by [9, Lemma 2.8], $|R/(p)| = 2^{2^{2^0}}$ and therefore $R/(p)$ has a maximal subring, by Corollary 2.3. Thus $R$ has a maximal subring. 

### 3. Conductor of integrally closed maximal subring

In this section, we are interested to show that if $K$ is an algebraically closed field, which is not algebraic over its prime subfield, and $R$ is affine ring over $K$, then for each prime ideal $P$ of $R$ with $ht(P) \geq dim(R) - 1$, there exists an integrally closed maximal subring $S$ of $R$ with $(S : R) = P$. First we begin by the following lemma.

**Lemma 3.1.** Let $R$ be a maximal subring of a ring $T$ with $(R : T) \in \text{Spec}(T) \setminus \text{Max}(T)$. Then $R$ is integrally closed in $T$.

**Proof.** If $T$ is integral over $R$, then by [17, Theorem 2.8], $P := (R : T)$ satisfies exactly one of the following:

1. $P$ is a maximal ideal of $T$.
2. There exists a maximal ideal $M$ of $T$ such that $M^2 \subseteq P \subseteq M$. Therefore, $M = P$, for $P$ is prime.
3. There exist distinct maximal ideals $M_1$ and $M_2$ of $T$ such that $P = M_1 \cap M_2$. Thus either $P = M_1$ or $P = M_2$, for $P$ is prime in $T$.

Hence in any case we conclude that $P$ is a maximal ideal of $T$, which is impossible. Thus $R$ is integrally closed in $T$. 

We remind that if $V$ is an integral domain with quotient field $K \neq V$, then one can easily see that the following are equivalent:
1. $V$ is a maximal subring of $K$
2. $V$ is a one dimensional valuation domain.
3. $V$ is a real (archimedean) valuation domain.
4. $V$ is a completely integrally closed valuation domain.

Now the following is in order, see [6, Proposition 1.5].

**Lemma 3.2.** Let $K$ be a field and $T$ be a ring extension of $K$. If $V$ is a maximal subring of $T$ which is integrally closed in $T$ and $K \subseteq V$, then $V \cap K$ is a maximal subring of $K$. In particular, $V \cap K$ is one dimensional valuation domain.

**Proof.** First note that since $V$ is integrally closed in $T$, then for each $u \in U(T)$ either $u \in V$ or $u^{-1} \in V$, by (3) of Theorem 1.1. Since $K \subseteq V$, we infer that $K \cap V$ is a proper subring of $K$. Now for maximality of $K \cap V$ in $K$, it suffices to show that $(K \cap V)[x] = K$, for each $x \in K \setminus (K \cap V)$. By the first part of the proof note that $x^{-1} \in V$. Since $V$ is a maximal subring of $T$ we conclude that $V[x] = T$. Now suppose $\beta \in K$, thus $\beta \in T = V[x]$, which implies that $\beta = v_0 + v_1 x + \cdots + v_n x^n$ for some $v_i \in V$. Therefore $\beta x^{-n} = v_0 x^{-n} + \cdots + v_n = v \in V$, for $x^{-1} \in V$. Finally note that since $K$ is a field and $x, \beta \in K$ we have $\beta x^{-n} = v \in K \cap V$ and therefore $\beta = vx^n \in (K \cap V)[x]$. Hence $K \cap V$ is a non field maximal subring of $K$ and therefore is a one dimensional valuation domain. \qed

We need the following corollary from [6, Corollary 1.9].

**Corollary 3.3.** Let $K$ be an algebraically closed field which is not absolutely algebraic. Then $K[X]$ has an integrally closed maximal subring with zero conductor.

We remind the reader that if $K$ is an algebraically closed field which is absolutely algebraic, then by [5, Lemma 4.6], one can easily see that $K[X]$ has no integrally closed maximal subring. Now we have the following result which is a lying-over property of conductors of integrally closed maximal subrings in integral extensions.

**Theorem 3.4.** Let $R \subseteq T$ be an integral extension of rings, $Q \in \text{Spec}(T)$ and $P = Q \cap R$. Assume that $R$ has an integrally closed maximal subring $S$ with $(S : R) = P$ and $U(R/P) \subseteq S/P$. Then $T$ has an integrally closed maximal subring $V$ with $(V : T) = Q$.

**Proof.** It is clear that $A := R/P \subseteq B := T/Q$ is an integral extension and $C := S/P$ is an integrally closed maximal subring of $A$ with $(C : A) = 0$ and $U(A) \subseteq C$. Hence by (3) of Theorem 1.1, assume that $x \in U(A)$ such that $x \in C$ but $x^{-1} \notin C$. Thus $C[x^{-1}] = A$, by maximality of $C$. Let $D$ be the integral closure of $C$ in $B$, then $x^{-1} \notin D$ and one can easily see that $D[x^{-1}] = B$ (similar to the proof of Lemma 3.2). Therefore $B$ has a maximal subring $E$ which contains $D$ but $x^{-1} \notin E$ (see [9, Proposition 2.1]). Clearly, $E$ is integrally closed in $B$ and $E \cap A = C$. Let $V$ be a subring of $T$ such that $E = V/Q$ and $Q_1 = (V : T)$. Thus $P \subseteq Q_1 \cap R$, for $Q \subseteq Q_1$. Now let $x \in Q_1 \cap R$, then $x + Q \in (V/Q) \cap (R/P) = S/P$, i.e., $x \in S$. Therefore $Q_1 \cap R \subseteq S$. Thus $P \subseteq Q_1 \cap R \subseteq (S : R) = P$ which immediately implies that $Q_1 \cap R = P$. Now since $R \subseteq T$ is an integral extension (and therefore INC holds) and $Q \subseteq Q_1$ are primes ideals of $T$ with a same contraction in $R$, we conclude that $Q = Q_1$. Therefore $(V : T) = Q$ and we are done. \qed

The following is the main result in this section.

**Theorem 3.5.** Let $K$ be an algebraically closed field which is not absolutely algebraic and $X_1, \ldots, X_n$ be indeterminates over $K$. Then for each prime ideal $Q$ of $T = K[X_1, \ldots, X_n]$ with $n - 1 \leq \text{ht}(Q) \leq n$, there exists an integrally closed maximal subring $S$ of $R$ with $(S : R) = Q$. 
Proof. We have two cases. First assume that \( n = 1 \). If \( Q = 0 \) then we are done by Corollary 3.3, hence suppose that \( Q \) is a maximal ideal of \( T \) and therefore \( T/Q \cong K \). By Corollary 2.3, \( T/Q \) has an integrally closed maximal subring \( V/Q \) and \( (V/Q : T/Q) = 0 \). Thus \( V \) is an integrally closed maximal subring of \( T \) which contains \( Q \) and therefore \( Q = (V : T) \), by maximality of \( Q \). Now suppose that \( n \geq 2 \). If \( ht(Q) = n \), then \( Q \) is a maximal ideal of \( T \) and similar to the case \( n = 1 \) we are done. Hence assume that \( ht(Q) = n - 1 \). Thus \( \text{dim}(T) = 1 \). Therefore by Noether’s Normalization Theorem (see [13, Theorem A1 p.221 or Theorem 13.3]), there exist \( Y_1, ..., Y_n \) in \( T \) such that \( T \) is integral over \( R := K[Y_1, ..., Y_n] \) and \( P := Q \cap R = (Y_2, ..., Y_n) \). Thus \( T/Q \) is integral over \( K[Y_1] \). Now by Corollary 3.3, \( K[Y_1] \) has an integrally closed maximal subring with zero conductor which does not contain \( K \) (see [5, Lemma 4.6]). Hence \( T/Q \) has an integrally closed maximal subring with zero conductor by Theorem 3.4. Thus \( T \) has an integrally closed maximal subring with conductor \( Q \).

\[ \square \]

Corollary 3.6. Let \( K \) be an algebraically closed field which is not absolutely algebraic and \( R \) be an affine ring over \( K \). Then for each prime ideal \( Q \) of \( R \) with \( \text{ht}(Q) \geq \text{dim}(R) - 1 \), there exists an integrally closed maximal subring \( S \) of \( R \) with \( (S : R) = Q \).

Proposition 3.7. Let \( K \) be a field which is not absolutely algebraic. The following are equivalent.

1. For each \( n \geq 0 \), the ring \( K[X_1, ..., X_n] \) has an integrally closed maximal subring \( S \) with zero conductor and \( K \subseteq S \).
2. For each \( n \geq 0 \) and each prime ideal \( Q \) of \( R = K[X_1, ..., X_n] \), there exists an integrally closed maximal subring \( S \) with \( (S : R) = Q \) and \( K \subseteq S \).
3. Each affine ring over \( K \) has an integrally closed maximal subring \( S \) with zero conductor and \( K \subseteq S \).

Proof. It suffices to prove \( (1) \Rightarrow (2) \). We may assume that \( n \geq 2 \) (note that for \( n = 0 \) use Corollary 2.3, and for \( n = 1 \) use Theorem 3.5). Let \( Q \) be a nonzero prime ideal of \( R \), and \( \text{ht}(Q) = m \). Hence \( 1 \leq m \leq n \). If \( m = n \), then \( Q \) is a maximal ideal of \( R \), therefore the field \( R/Q \) contains a copy of \( K \). Thus \( R/Q \) is not absolutely algebraic field. Therefore by Corollary 2.3, \( R/Q \) has an integrally closed maximal subring. Hence \( R \) has an integrally closed maximal subring \( S \) which contains \( Q \). Thus \( Q \subseteq (S : R) \) and therefore \( (S : R) = Q \), for \( Q \) is a maximal ideal of \( R \). Hence assume that \( 1 \leq m \leq n - 1 \). Thus \( d := \text{dim}(\frac{R}{Q}) = n - m \geq 1 \). Therefore by Noether’s Normalization Theorem (see [13, Theorem A1 p.221 or Theorem 13.3]), there exist \( Y_1, ..., Y_n \) in \( R \) such that \( R \) is integral over \( T := K[Y_1, ..., Y_n] \) and \( P := Q \cap T = (Y_{d+1}, ..., Y_n) \). Thus \( B := R/Q \) is integral over \( A := K[Y_1, ..., Y_d] \), Now by (1), \( A \) has an integrally closed maximal subring \( S \) with zero conductor and \( K \subseteq S \). Thus by Theorem 3.4, \( B \) has an integrally closed maximal subring \( V \) with zero conductor \( K \subseteq V \). Hence \( R \) has an integrally closed maximal subring with conductor \( Q \), and we are done.

\[ \square \]

Let \( T \) be a ring, then we denote the set of all integrally closed maximal subrings of \( T \) by \( \text{X}^{i.c.}(T) \) and also define \( \text{Spec}(\text{X}^{i.c.}(T)) := \{ (S : T) \mid S \in \text{X}^{i.c.}(T) \} \). Note that (3) of Theorem 1.1, \( \text{Spec}(\text{X}^{i.c.}(T)) \subseteq \text{Spec}(T) \). Therefore \( P \in \text{Spec}(\text{X}^{i.c.}(R)) \) if and only if \( T \) has an integrally closed maximal subring \( S \) with \( (S : T) = P \). Now we have the following.

Theorem 3.8. Let \( R \subseteq T \) be an integral extension of rings. If \( \text{Max}(T) \subseteq \text{Spec}(\text{X}^{i.c.}(T)) \), then \( \text{Max}(R) \subseteq \text{Spec}(\text{X}^{i.c.}(R)) \).

Proof. Let \( P \) be a maximal ideal of \( R \), then there exists a maximal ideal \( Q \) of \( T \) such that \( Q \cap R = P \). Therefore \( R/P \subseteq T/Q \) is an integral extension of fields. Now by our assumption there exists an integrally closed maximal subring \( V \) of \( T \) with \( (V : T) = Q \). Hence \( V/Q \) is an integrally closed
maximal subring of \( T / Q \). Therefore \( R / P \not\subseteq V / Q \). Thus by Lemma 3.2, \( V / Q \cap R / P \) is an integrally closed maximal subring of \( R / P \). Hence \( R \) has an integrally closed maximal subring \( W \) which contains \( P \) and therefore \( (W : R) = P \).

**Proposition 3.9.** Let \( R \) be an integrally closed maximal subring of an integral domain \( T \) with \( U(T) \not\subseteq R \). If \( R \) is Noetherian, then \( (R : T) = 0 \). In particular, \( R_M \) is a DVR, where \( M \) is the crucial maximal ideal of the extension \( R \subseteq T \).

**Proof.** Assume that \( t^{-1} \in U(T) \setminus R \). Therefore \( t \in R \) and \( R[S] = T \). Now if \( P = (R : T) \ne 0 \), then \( tP = P \) for \( t \in U(T) \). Therefore \( t^{-1}P = P \). This immediately implies that \( t^{-1} \) is integral over \( R \) which is absurd. The final part is evident by the fact that \( R \) is Noetherian and (5) of Theorem 1.1.

**Remark 3.10.** Note that one can prove the first part of the previous proposition by Krull Intersection Theorem and the fact that \( (R : T) = \cap_{n=1}^{\infty} R_{T^n} \).

**Corollary 3.11.** Let \( R \) be a one dimensional Noetherian integral domain. Then for each \( 0 \ne a \in R \setminus U(R) \), the overring \( R_a = R[\frac{1}{a}] \) has a maximal subring with zero conductor. In particular, if \( C \) is a multiplicatively closed subset of \( R \) which is not contained in \( U(R) \), then \( R_C \) has a maximal subring with zero conductor.

**Proof.** Let \( T = R[\frac{1}{a}] \), then clearly \( T \) has a maximal subring \( S \) which contains \( R \) and \( \frac{1}{a} \notin S \). Therefore \( S_a = T \). Now note that by Krull-Akizuki Theorem ([21, Theorem 93]), \( S \) is Noetherian and therefore by Krull Intersection Theorem we conclude that \( (S : T) = \cap_{n=1}^{\infty} S a^n = 0 \). The final part is similar and follows from Corollary 2.9.

**Proposition 3.12.** Let \( R \) be a normal Noetherian integral domain which is an integrally closed maximal subring of a ring \( T \) with crucial maximal ideal \( M \). Then \( (R : T) = 0 \) and \( R_M \) is a DVR.

**Proof.** First note that by [23, Theorem 10], \( T \) is an integral domain and therefore \( T \) is a minimal overrings of \( R \). Hence by [3, Theorem 2.4 and Lemma 2.8], there exists an ideal \( A \) of \( R \) such that \( (R : T) = \cap_{n=1}^{\infty} A^n \). Thus by Krull intersection theorem we infer that \( (R : T) = 0 \) and therefore by (5) of Theorem 1.1, \( R_M \) is a valuation domain. Since \( R \) is Noetherian, we conclude that \( R_M \) is a DVR.

**Proposition 3.13.** Let \( R \) be a completely integrally closed integral domain which is a conch maximal subring of a ring \( T \). Then \( (R : T) = 0 \).

**Proof.** First note that there exists \( a \in R \) such that \( T = R[\frac{1}{a}] \), for \( R \) is a conch maximal subring. Now one can easily see that \( (R : T) = \cap_{n=1}^{\infty} Ra^n \). Thus by [16, Corollary 13.4], we infer that \( (R : T) = 0 \) for \( R \) is completely integrally closed.

**Lemma 3.14.** Let \( R \) be an integral domain with \( 0 \ne (p) \in \text{Spec}(R) \). Then the following hold.

1. \( R \) is a maximal subring of \( R[\frac{1}{p}] \) with zero conductor if and only if \( \cap_{n=1}^{\infty} (p^n) = 0 \) and \( (p) \in \text{Max}(R) \).
2. If \( R \) is atomic, then \( (R : R[\frac{1}{p}]) = 0 \). Moreover, \( R \) is a maximal subring of \( R[\frac{1}{p}] \) if and only if \( (p) \in \text{Max}(R) \).
Proof. Assume that \( T = R[\{x\}]_p \), then one can easily see that \( (R : T) = \cap_{n=1}^{\infty} R p^n \). Hence \( (R : T) = 0 \) if and only if \( \cap_{n=1}^{\infty} R p^n = 0 \). For (1), first suppose that \( \cap_{n=1}^{\infty} R p^n = 0 \) and \( (p) \in \text{Max}(R) \), then we prove that \( R \) is a maximal subring of \( T \). Let \( A \) be a subring of \( T \) which properly contains \( R \). Let \( x \in A \setminus R \), thus we may assume that \( x = \frac{r}{p^n} \), where \( r \in R, n \geq 1 \) and \( r \notin R p \), for \( \cap_{n=1}^{\infty} R p^n = 0 \) and \( x \in A \subseteq T \) but \( x \notin R \). Hence \( \frac{r}{p^n} \in A \). Since \( M = (p) \) is a maximal ideal of \( R \), we conclude that \( ap + br = 1 \), for some \( a, b \in R \). Therefore \( \frac{1}{p} = a + b p \in A \), and hence \( T \subseteq A \). Thus \( R \) is a maximal subring of \( T \). Conversely, assume that \( R \) is a maximal subring of \( T \). Clearly \( R \) contains \( \frac{1}{p} \) in \( T \). Therefore if \( M \) is the crucial maximal ideal of the extension of \( R \subseteq T \), then \( M \) is the unique prime ideal of \( R \) which contains \( p \), by Corollary 1.2. Therefore \( M = (p) \). For (2) note that by the proof of Corollary 2.2, \( \cap_{n=1}^{\infty} R p^n = 0 \) and therefore we are done by part (1). \( \square \)

**Proposition 3.15.** Let \( R \subseteq T \) be an integral extension of integral domains with \( p \in R \) is prime element in \( T \) and \( pR = R \cap pT \). If \( R \) is a maximal subring of \( R[\{x\}]_p \) with zero conductor, then \( T \) is a maximal subring of \( T[\{x\}]_p \) with zero conductor.

Proof. Since \( R \) is a maximal subring of \( R_1 := R[\{x\}]_p \) with zero conductor, we conclude that \( 0 = (R : R_1) = \cap_{n=1}^{\infty} R p^n \). Let \( T_1 = T[\{x\}]_p \), we claim that \( Q := (T : T_1) = \cap_{n=1}^{\infty} T p^n = 0 \). If \( Q \neq 0 \), then \( Q \cap R \neq 0 \) for \( T \) is an integral domain which is integral over \( R \). Let \( 0 \neq x \in Q \cap R \), then for each \( n \geq 1 \), there exists \( t_n \in T \) such that \( x = t_n p^n \). Thus \( \frac{1}{p} = t_n \in R[\{x\}]_p \) is integral over \( R \). Since \( R \) is integrally closed in \( R_1 \) we deduce that \( t_n \in R \) and therefore \( x = \cap_{n=1}^{\infty} R p^n = 0 \), which is absurd. Thus \( Q = 0 \) and hence we are done by Lemma 3.14. \( \square \)

**Remark 3.16.** Theorem 3.5 raises a natural question for the dual concept of maximal subrings, i.e., minimal ring extensions, as follows. Let \( K \) be a field and \( R = K[X_1, \ldots, X_n] \), where \( X_1, \ldots, X_n \) are independent indeterminate over \( K \). Assume that \( P \in \text{Spec}(R) \) is an arbitrary prime ideal. Now the natural question arises: Does there exist a minimal ring extension \( T \) of \( R \) with \( (R : T) = P \)? If \( P \in \text{Max}(R) \) (for arbitrary ring \( R \)), then the answer to this question is positive by [12, Corollary 2.5], and in this case note that by (2) of Theorem 1.1, \( T \) is integral over \( R \). In fact by [12, Corollary 2.5], for each maximal ideal \( M \) of \( R \), the idealization \( T := R(+) \frac{R}{M} \) is a minimal ring extension of \( R \) with \( (R : T) = M \). Now suppose \( P \) is a prime ideal of \( R = K[X_1, \ldots, X_n] \) which is not a maximal ideal of \( R \). First note that by Lemma 3.1, if there exists a minimal ring extension \( T \) of \( R \) with \( (R : T) = P \), then \( R \) is integrally closed in \( T \) and \( \text{ht}(P) = n - 1 \), by (5) of Theorem 1.1. Also by [23, Theorem 10], \( T \) is an integral domain and therefore \( T \) is an overring of \( R \). Now we have two cases. If \( n = 1 \), then \( P = 0 \) and one can easily see that for each irreducible polynomial \( p(\frac{1}{X}, \frac{1}{X}) \in R[X] \), the overring \( T := R[\frac{1}{p(X)}] \) is a minimal ring extension of \( R \) with \( (R : T) = 0 \). But if \( n \geq 2 \), then by [3, Proposition 6.1], \( R[K[X_1, \ldots, X_n]] \) has no minimal overring and therefore \( R \) has no minimal ring extension with conductor \( P \).

In [24], the authors proved that if \( R \) is a one dimensional Noetherian domain, then \( R \) has a minimal overring. In particular, if \( R \) is an affine integral domain over an infinite field \( K \) then \( R \) has a minimal overring if and only if either \( \text{dim}(R) = 1 \) or \( \text{dim}(R) \geq 2 \) and \( R \) has a maximal ideal \( M \) of depth 1. We conclude this article by the following corollary.

**Corollary 3.17.** Let \( R \) be an affine normal integral domain over a field \( K \). Suppose \( R \) is not a field. Then \( R \) has a minimal overring if and only if \( \text{tr.deg}(R/K) = 1 \).

Proof. First note that \( \text{dim}(R) = \text{tr.deg}(R/K) \). Hence if \( \text{tr.deg}(R/K) = 1 \), then we are done by [24, Theorem 3] (even if \( R \) is not normal). Conversely, assume that \( R \) has a minimal overring. Then by Proposition 3.12, \( R \) has a height one maximal ideal. Therefore \( \text{dim}(R) = 1 \) and hence we are done. \( \square \)
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