REMARK ON THE WELL-POSEDNESS OF WEAKLY DISPERSIVE EQUATIONS

JIAO HE and YOUSEF MAMMERI

Abstract. We improve the results about the well-posedness of the regularized fractional dispersive equation $(1 + D^\alpha_x)u_t + u_x + uu_x = 0$ when $0 < \alpha \leq 1$. When $\alpha < 1$, the existence and uniqueness of vanishing viscosity solution is proved.

INTRODUCTION

Fractional Benjamin-Bona-Mahony (fBBM) equation

$$u_t + u_x + uu_x + D^\alpha_x u_t = 0$$

was introduced by Linares, Pilod and Saut [16] to investigate the role of weak dispersion ($0 < \alpha < 1$) on the solution of the Burgers equation

$$u_t + u_x + uu_x = 0.$$ 

They showed the local in time well-posedness in Sobolev spaces using energy estimates. Nevertheless, this method does not provide the uniqueness. Indeed, the difference $w = u - v$ between two solutions satisfies (see inequality (4.31) of [16])

$$\frac{d}{dt} ||w||^2_{H^{s+\frac{\alpha}{2}}} \leq \int_{\mathbb{R}} |(-\Delta)^{s/2}(u + v)w_x(-\Delta)^{s/2}w|^2 dx + \frac{1}{2} \int_{\mathbb{R}} |(u + v)_x(-\Delta)^{s/2}w|^2 dx$$

$$+ \int_{\mathbb{R}} |R(-\Delta)^{s/2}w|^2 dx + \int_{\mathbb{R}} |(-\Delta)^{s/2}(u + v)_x)w(-\Delta)^{s/2}w|^2 dx,$$

where $\int_{\mathbb{R}} |R(-\Delta)^{s/2}w|^2 dx \leq ||u + v||_{H^{s+\frac{\alpha}{2}}} ||w||^2_{H^{s+\frac{\alpha}{2}}}$. But the last term can not be uniformly controlled by the $H^{s+\frac{\alpha}{2}}$-norm if $0 < \alpha < 1$.

When $0 < \alpha < 1$, it seems difficult to obtain the global well-posedness. It has been shown by Bona and Saut [3] that the linearization around 0 has blow-up solution and Klein, Saut numerically observe blow-up when $0 < \alpha < \frac{3}{2}$ [15]. While $\alpha = 1$, global well-posedness can be obtained thanks to Brezis-Gallouët estimates [5,18].

The paper is organized as follows. In Section 2, we improve the regularity of the global well-posedness when $\alpha = 1$. In Section 3, we deal with the uniqueness of vanishing viscosity solution for $0 < \alpha < 1$.
1. The regularized Benjamin-Ono equation

When $\alpha = 1$, the equation can be rewritten as the Benjamin-Ono equation under the form

$$u_t + H(u_x)_t + u_x + uu_x = 0,$$

where $H$ is the Hilbert defined by its Fourier symbol $\widehat{H(u)}(\xi) = -i \text{sgn}(\xi)u(\xi)$.

**Theorem 1.1.** Let $\alpha = 1$ and $u_0 \in H^{\frac{1}{2}}(\mathbb{R})$. There exists a unique solution $u \in C(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{R}))$ of the initial value problem

$$\begin{cases}
    u_t + u_x + uu_x + D_x u_t = 0, \\
    u(x, 0) = u_0(x).
\end{cases}$$

Moreover, for all $t \in \mathbb{R}$

$$\|u(t)\|_{H^{\frac{1}{2}}} = \|u_0\|_{H^{\frac{1}{2}}},$$

and the map $u_0 \in H^{\frac{1}{2}}(\mathbb{R}) \to u \in C(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{R}))$ is continuous.

The proof is done in two steps: first, a compactness argument is used to obtain a weak solution, then the uniqueness of the weak solution provides the strong continuity of the weak solution.

Let $(u_n, 0)_{n \in \mathbb{N}}$ be a sequence of $H^1(\mathbb{R})$ such that $u_{n, 0} \to u_0$ in $H^{\frac{1}{2}}(\mathbb{R})$. We denote by $u_n(t) \in C(\mathbb{R}; H^1(\mathbb{R}))$ the solution of the initial value problem associated with the initial datum $u_{n, 0}$. Then it is proved that the energy [18]

$$E(t) = \int_{\mathbb{R}} u_n^2 + |D_x |^2 u_n|^2 dx$$

is preserved with respect to time, i.e.

$$\|u_n\|^2_{H^{\frac{1}{2}}} = \|u_{n,0}\|^2_{H^{\frac{1}{2}}} \leq C,$$

and the sequence $(u_n)_{n \in \mathbb{N}}$ is bounded in $C(\mathbb{R}; H^{\frac{1}{2}}(\mathbb{R}))$.

On the other hand, multiplying (1) by $\partial_t u_n$ and integrating over space gives

$$\int_{\mathbb{R}} |\partial_t u_n|^2 dx = - \int_{\mathbb{R}} \partial_t u_n \partial_x (1 + D_x)^{-1}(u_n + \frac{u_n^2}{2}) dx,$$

and Young's inequality provides, for $\varepsilon > 0$.

$$\|\partial_t u_n\|^2_{L^2} = \int_{\mathbb{R}} |\partial_t u_n|^2 dx \leq \int_{\mathbb{R}} |\partial_t u_n \partial_x (1 + D_x)^{-1} u_n|^2 dx + \int_{\mathbb{R}} |\partial_t u_n \partial_x (1 + D_x)^{-1} \frac{u_n^2}{2}| dx$$

$$\leq \frac{\varepsilon}{2} \int_{\mathbb{R}} |\partial_t u_n|^2 dx + \frac{1}{2\varepsilon} \int_{\mathbb{R}} |\partial_x (1 + D_x)^{-1} u_n|^2 dx + \frac{1}{2\varepsilon} \int_{\mathbb{R}} |\partial_x (1 + D_x)^{-1} \frac{u_n^2}{2}|^2 dx$$

$$\leq \frac{\varepsilon}{2} \|\partial_t u_n\|^2_{L^2} + \frac{1}{4\varepsilon} \|u_n\|^2_{L^2} + \|u_n^4\|^2_{L^4}$$

Taking $\varepsilon = \frac{1}{2}$, it gets

$$\|\partial_t u_n\|_{L^2} \leq \|u_n\|^2_{H^{\frac{1}{2}}} \leq \|u_{n,0}\|^2_{H^{\frac{1}{2}}} \leq C.$$
Thus,
\[ \|u_n(t) - u_n(s)\|_{L^2} = \left\| \int_s^t \partial_t u_n(r) \, dr \right\|_{L^2} \leq \int_s^t \|\partial_t u_n(r)\|_{L^2} \, dr \leq C|t - s|, \]
and the sequence \((u_n)_{n \in \mathbb{N}}\) is uniformly bounded and equicontinuous.

We deduce according to the Rellich theorem for all \(T > 0\), there exists \(u \in C_w([0, T]; H^\frac{1}{2}(\mathbb{R})) \cap C([0, T]; L^2(\mathbb{R}))\) and a subsequence \((u_{n_k})_{k \in \mathbb{N}}\) such that \(\forall t \in [0, T]\)
\[ u_{n_k}(t) \to u(t) \quad \text{in} \quad H^\frac{1}{2}(\mathbb{R}) \quad \text{and} \quad u_{n_k}(t) \to u(t) \quad \text{in} \quad L^2(\mathbb{R}). \]

Since the subsequence \((u_{n_k})_{k \in \mathbb{N}}\) satisfies for all \(v \in C_c^\infty(\mathbb{R})\)
\[ \int_0^T \int_\mathbb{R} v \partial_t u_{n_k} + v \partial_x D_x u_{n_k} + v \partial_x \left( u_{n_k} + \frac{u_{n_k}^2}{2} \right) \, dx \, dt = 0, \]
the limit verifies
\[ \int_0^T \int_\mathbb{R} \partial_t v + \partial_x D_x v + \left( u + \frac{u^2}{2} \right) \partial_x v \, dx \, dt = 0. \]
The function \(u \in C_w([0, T]; H^\frac{1}{2}(\mathbb{R})) \cap C([0, T]; L^2(\mathbb{R}))\) is a weak solution of the equation (1) and
\[ \|u(t)\|_{H^\frac{1}{2}} \leq \liminf_{k \to \infty} \|u_{n_k}(t)\|_{H^\frac{1}{2}} \leq \|u_0\|_{H^\frac{1}{2}}. \quad (2) \]

Suppose now that the weak solution of the Cauchy problem \(u \in C_w([0, T]; H^\frac{1}{2}(\mathbb{R})) \cap C([0, T]; L^2(\mathbb{R}))\) is unique. By reversing time in the regularized Benjamin-Ono equation (1), we have
\[ \|u(-t)\|_{H^\frac{1}{2}} \leq \|u(0)\|_{H^\frac{1}{2}} \quad \text{or} \quad \|u(0)\|_{H^\frac{1}{2}} \leq \|u(t)\|_{H^\frac{1}{2}}. \]

From inequality (2) and from the uniqueness of the weak solution, we obtain
\[ \|u(t)\|_{H^\frac{1}{2}} = \|u_0\|_{H^\frac{1}{2}} \]
and the solution \(u\) belongs to \(C([0, T]; H^\frac{1}{2}(\mathbb{R}))\). Note that we also obtain the continuity with respect to the initial data since we proved that for \(u_{n,0} \to u_0\) in \(H^\frac{1}{2}(\mathbb{R})\), then the respective solution \((u_n)\) verifies \(u_n \to u\) in \(C([0, T]; H^\frac{1}{2}(\mathbb{R}))\).

It remains to prove the uniqueness of the weak solution. We are inspired by the method introduced by Yudovich [21]. We need a Trudinger-type estimates proved by Gérard and Grellier [9] in the torus \(\mathbb{T}\).

**Lemma 1.2.** There exists a constant \(C > 0\) such that for all \(2 < p < \infty\), we have
\[ \|u\|_{L^p(\mathbb{R})} \leq C \sqrt{p} \|u\|_{H^\frac{1}{2}(\mathbb{R})}. \]

**Proof.** The proof is similar to [9] except that \(u\) is split as
\[ u = u_{>\lambda} + u_{<\lambda} = \frac{1}{2\pi} \int_{|\xi| \leq \lambda} e^{i\xi x} \hat{u}(\xi) d\xi + \frac{1}{2\pi} \int_{|\xi| \geq \lambda} e^{i\xi x} \hat{u}(\xi) d\xi. \]
\[ \square \]
Let \( u \) and \( v \in C^0([0,T]; H^{\frac{1}{2}}(\mathbb{R})) \cap C([0,T];L^2(\mathbb{R})) \) be two weak solutions of (1) starting from the same initial datum. Consider the function \( g \) defined as

\[
g(t) = \|u(t) - v(t)\|_{L^2}^2.
\]

Then, for \( w := u - v \) and \( P(D_x) := \partial_x(1 + D_x)^{-1} \), we have

\[
g'(t) = 2 \int (u - v)(u_t - v_t)dx = -2 \int (u - v)P(D_x)\left(u + \frac{u^2}{2} - v - \frac{v^2}{2}\right) dx
\]

\[
= -2 \int wP(D_x)(w) + wP(D_x)(w(u + v)) dx = -2 \int wP(D_x)(w(u + v)) dx
\]

\[
= \int w^2P(D_x)(u + v) dx - \int w(P(D_x)w)(u + v) dx - \int wRdx
\]

where

\[
R = P(D_x)(w(u + v)) - (u + v)P(D_x)w - wP(D_x)(u + v).
\]

Since \( w^2 = w^{2(1 - \frac{1}{p})}w^\frac{2}{p} \), the Hölder inequality provides

\[
|I| = \int w^2P(D_x)(u + v) dx \leq \|w^{2(1 - \frac{1}{p})}\|_{L^{\frac{q}{p-1}}} \|w^\frac{2}{p}P(D_x)(u + v)\|_{L^p}
\]

\[
\lesssim \|w\|_{L^2}^{2(1 - \frac{1}{p})} \left(\|u\|_{L^2}^{\frac{2}{p}} + \|v\|_{L^2}^{\frac{2}{p}}\right) \|P(D_x)(u + v)\|_{L^2}.
\]

We note that the operator \( P(D_x) \) is bounded in \( L^p \), for \( 1 < p < +\infty \) according to the Mikhlin-Hörmander theorem [17,19]. The Sobolev and the Trudinger inequalities imply

\[
|I| \lesssim \|w\|_{L^2}^{2(1 - \frac{1}{p})} \left(\|u\|_{L^{2p}} + \|v\|_{L^{2p}}\right) \leq \sqrt{2pg(t)^{1 - \frac{1}{p}}} \left(\|u\|_{L^{\frac{p}{2}}} + \|v\|_{L^{\frac{p}{2}}}\right) (\|u\|_{H^{\frac{1}{2}}} + \|v\|_{H^{\frac{1}{2}}})
\]

and from inequality (2)

\[
|I| \lesssim \sqrt{pg(t)^{1 - \frac{1}{p}}}.
\]

Similarly, we have

\[
|II| = \int w(P(D_x)w)(u + v)dx \leq \|w^{1 - \frac{2}{p}}P(D_x)w\|_{L^{\frac{q}{p-1}}} \|w^\frac{2}{p}(u + v)\|_{L^p}
\]

\[
\lesssim \|w^{1 - \frac{2}{p}}\|_{L^{\frac{q}{p-1}}} \|P(D_x)w\|_{L^2} \left(\|u\|_{L^2}^{\frac{2}{p}} + \|v\|_{L^2}^{\frac{2}{p}}\right) \|u + v\|_{L^{2p}}
\]

\[
\lesssim \|w\|_{L^2}^{1 - \frac{2}{p}} \|w\|_{L^2} \left(\|u\|_{L^{2p}} + \|v\|_{L^{2p}}\right)
\]

\[
\lesssim g(t)^{1 - \frac{1}{p}} \left(\|u\|_{L^{2p}} + \|v\|_{L^{2p}}\right) \lesssim \sqrt{pg(t)^{1 - \frac{1}{p}}} \|u\|_{H^{\frac{1}{2}}} \|v\|_{H^{\frac{1}{2}}}.
\]

We can write thanks to the fractional Leibniz rule [14].

**Lemma 1.3.** We have for \( \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \)

\[
\|P(D)(uw) - vP(D)(u) - uP(D)(v)\|_{L^p} \lesssim \|u\|_{L^q}\|v\|_{L^r}.
\]
We deduce

\[ |\text{III}| = \int_{\mathbb{R}} wRdx \lesssim \|w\|_{L^2} \|u + v\|_{L^4} \|w\|_{L^r} \lesssim \|w\|_{L^2} (\|u\|_{L^4} + \|v\|_{L^4}) \|w\|_{L^r}^{1 - \frac{2}{p}} \|w\|_{L^r}^\frac{2}{p} \]

where

\[ \frac{1}{r} = \frac{1}{2} - \frac{1}{q} \text{ or } \frac{1}{r} = \frac{1}{2} - \frac{1}{p} \left(1 - \frac{2}{p}\right). \]

Thus

\[ |\text{III}| \lesssim \sqrt{q} \|w\|_{L^2}^{\frac{2}{p}} \|w\|_{L^r} \|u\|_{H^\frac{1}{2}} \lesssim \sqrt{q} g(t)^{1 - \frac{1}{p}} (\|u\|_{L^2}^{\frac{2}{p}} + \|v\|_{L^2}^{\frac{2}{p}}) \lesssim p^2 \sqrt{q} g(t)^{1 - \frac{1}{p}} \|u\|_{H^\frac{1}{2}} \lesssim p^2 \sqrt{q} g(t)^{1 - \frac{1}{p}}. \]

Taking \( p > 2 \) large enough so that

\[ |\text{III}| \lesssim p^2 \sqrt{\frac{p}{1 - \frac{2}{p}}} g(t)^{1 - \frac{1}{p}} \lesssim \sqrt{q} g(t)^{1 - \frac{1}{p}} \]

implies

\[ |g'(t)| \leq C \sqrt{q} g(t)^{1 - \frac{1}{p}}, \quad \forall 2 < p < \infty, \]

with \( C = C(\|u_0\|_{H^\frac{1}{2}}, T) \) independent of \( p \). Then

\[ g(t)^{\frac{1}{p}} \leq C \frac{t}{\sqrt{p}}, \]

or

\[ g(t) \leq C \frac{t^p}{p^2} \rightarrow 0, \quad \text{as } p \rightarrow \infty. \]

Finally, \( g(t) \equiv 0 \), and \( u = v \).

2. The weak dispersive equation

Let us come back to the initial value problem, for \( 0 < \alpha < 1 \),

\[
\begin{aligned}
&u_t + uu_x + u u_x + D_x^\alpha \partial_t u = 0, \\
&u(x, 0) = u_0(x).
\end{aligned}
\]

(3)

The existence is proved in [16] using energy estimates.

**Theorem 2.1.** Let \( 0 < \alpha < 1, r > \max \left(1, \frac{3}{2} - \alpha\right) \) and \( u_0 \in H^r(\mathbb{R}) \). Then the Cauchy problem has at least one solution in \( H^r(\mathbb{R}) \).

We briefly remind the proof in order to highlight the loss of uniqueness (see [16] for details).

**Proof.** For \( r = s + \frac{\alpha}{2} \), we define \( J^s = (I - \Delta)^{\frac{\alpha}{2}} \) by its Fourier transform

\[ \hat{J}^s u(\xi) := (I - \Delta)^{\frac{\alpha}{2}} \hat{u}(\xi) = \hat{u}(\xi)(1 + \xi^2)^{\frac{\alpha}{2}}. \]

Applying the operator \( J^s \) to (3), multiplying by \( J^s u \) and integrating by part over space, we obtain

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} |J^s u|^2 + |J^s D_x^\alpha u|^2 dx = \int_{\mathbb{R}} -J^s u J^s (uu_x) dx \]

\[ = \int_{\mathbb{R}} |J^s u|^2 u_x + u J^s u J^s \partial_x u + RJ^s u dx = \frac{1}{2} \int_{\mathbb{R}} |J^s u|^2 u_x + \int_{\mathbb{R}} RJ^s u dx, \]
where, thanks to Leibniz’s rule,
\[ J'(u u_x) = u J^s u_x + u_x J^s u + R, \]
with, for all \( 0 < \epsilon < s \),
\[ \| R \|_{L^{\frac{2}{1+\alpha}}} \lesssim \| J^{s-\epsilon} u \|_{L^{\frac{2}{1+\epsilon}}} \| J^s u_x \|_{L^{\frac{1+\alpha}{2}}} . \]
The Hölder inequality provides
\[ \int_{\mathbb{R}} |(J^s u)^2 u_x| dx \leq \| u_x \|_{L^{\frac{2}{1+\alpha}}} \| J^s u \|_{L^2}^2 \]
and
\[ \int_{\mathbb{R}} |R J^s u| dx \leq \| R \|_{L^{\frac{2}{1+\alpha}}} \| J^s u \|_{L^2} . \]
Using Sobolev's embedding
\[ H^{\frac{2}{\alpha}}(\mathbb{R}) \hookrightarrow L^{\frac{2}{1+\alpha}}, \quad H^{\frac{2}{\alpha}+\epsilon} \hookrightarrow L^{\frac{2}{1+\alpha}}, \quad H^{s+\frac{\epsilon}{2}-1} \hookrightarrow L^{\frac{1}{p}} , \]
we finally find
\[ \frac{d}{dt} \| u \|_{H^{s+\frac{\epsilon}{2}}} \lesssim \| u \|^2_{H^{s+\frac{\epsilon}{2}}} \]
and we conclude with the Gronwall lemma. \( \square \)

Let us try to prove the uniqueness. Let \( u \) and \( v \) be two solutions of (3) and denote \( w = u - v \). We have
\[ w_t + \partial_x (1 + D^0_x)^{-1} \left( w + \frac{1}{2} w (u + v) \right) = 0. \]

Using the fractional Leibniz rule and integrating by parts, it gets
\[ \frac{1}{2} \int_{\mathbb{R}} |J^s w|^2 + |J^{s+\frac{\epsilon}{2}} w|^2 dx \leq \int_{\mathbb{R}} |J^s (u + v) w_x J^s w| dx + \frac{1}{2} \int_{\mathbb{R}} \|(u + v) w_x \| J^s w|^2 dx + \int_{\mathbb{R}} |R J^s w| dx + \int_{\mathbb{R}} |J^s (u + v) x w J^s w| dx . \]

Here, for all \( 0 < \epsilon < s \) and \( \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \),
\[ \| R \|_{L^2} \leq \| J^{s-\epsilon} (u + v) \|_{L^p} \| J^s w \|_{L^q} . \]
Even if the three first terms are bounded by \( \| u + v \|_{H^{s+\frac{\epsilon}{2}}} \| w \|_{H^{s+\frac{\epsilon}{2}}} \), the last one cannot be uniformly controlled by the \( H^{s+\frac{\alpha}{2}} \)-norm when \( 0 < \alpha < 1 \).

To avoid this difficulty, we propose the following regularization [11]
\[ \begin{cases} u_t^\epsilon + u_x^\epsilon + u^\epsilon u_x^\epsilon + D^0_x u_t^\epsilon - \epsilon u_{xx}^\epsilon = 0, \\ u^\epsilon(0, x) = u^\epsilon_0(x) . \end{cases} \tag{4} \]

**Lemma 2.2.** Let \( 0 < \alpha < 1 \) and \( r \geq 0 \). We have
\[ \| \partial_x (1 + D^0_x)^{-1} S_t(uv) \|_{H^r} \leq C(\epsilon, t) \| u \|_{H^r} \| v \|_{H^r} , \]
where
\[ S_t u = F^{-1} \left( e^{-\frac{\epsilon^2 + t^2}{r^2 (1+\alpha)}} \right) \ast u(x), \quad \text{and} \quad C(\epsilon, t) = C(\epsilon t)^{-\frac{\alpha}{2-\alpha}} . \]
Proof. The idea of the proof is introduced in [4, 13]. By duality, it is enough to show that for all function \( w \in \mathcal{S}(\mathbb{R}) \),

\[
\int_{\mathbb{R}} \partial_x (1 + D_x^2)^{-1} S_t (u(x) v(x)) \bar{w}(x) dx \leq C(t) \| u \|_{H^r} \| v \|_{H^r} \| w \|_{H^{-r}}.
\]

The Plancherel identity provides

\[
\int_{\mathbb{R}} \partial_x (1 + D_x^2)^{-1} S_t (u(x) v(x)) \bar{w}(x) dx = \int_{\mathbb{R}} \frac{i \xi}{1 + |\xi|^2} e^{-\frac{i \xi^2}{1 + |\xi|^2}} u(\xi) v(\xi) \bar{w}(\xi) d\xi
\]

The triangle inequality \((\xi)^r \lesssim (\xi^r)(\eta)^{r} \) implies

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\frac{i \xi^2}{1 + |\xi|^2}} \frac{|\xi||\eta|^r}{(\xi^r)(\eta^r)^{r}} \hat{U}(\xi - \eta) \hat{V}(\eta) \bar{W}(\xi) d\xi d\eta \lesssim C(\varepsilon, t) \| u \|_{H^r} \| v \|_{H^r} \| w \|_{H^{-r}}.
\]

Let us denote \((\xi) = (1 + \xi^2)^{\frac{1}{2}}, \hat{U} = (\xi)^{r} \hat{u}, \hat{V} = (\xi)^{r} \hat{v}, \hat{W} = (\xi)^{-r} \hat{w} \), it is necessary to prove

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} e^{-\frac{i \xi^2}{1 + |\xi|^2}} \frac{|\xi||\eta|^r}{(\xi^r)(\eta^r)^{r}} \hat{U}(\xi - \eta) \hat{V}(\eta) \bar{W}(\xi) d\xi d\eta \lesssim C(\varepsilon, t) \| u \|_{H^r} \| v \|_{H^r} \| w \|_{H^{-r}}.
\]

Since for \( x \in \mathbb{R}, \theta \in \mathbb{R} \), we have \( e^{-\frac{x}{t^2}} \leq \frac{1}{2^\frac{1}{2}}, \) we deduce

\[
\left\| (\xi)^{1-\alpha} e^{-\frac{i \xi^2}{1 + |\xi|^2}} \right\|_{L^2} \leq \left\| (\xi)^{1-\alpha} \frac{1}{(\xi^2 + |\xi|^2)^{\frac{1}{2}}} \right\|_{L^2} \leq \frac{1}{(\frac{1}{t})^\theta} \left\| (\xi^{2-\alpha})^{\frac{1}{2}} \right\|_{L^2} \lesssim \frac{1}{(\frac{1}{t})^\theta},
\]

if \((2 - \alpha)\theta - (1 - \alpha) > 1/2, \) i.e. \( \theta > \frac{2 - \alpha}{2 - \alpha}. \)

\[\square\]

**Lemma 2.3.** Let \( 0 < \alpha < 1, r \geq 0 \) and \( u_0^\varepsilon \in H^r(\mathbb{R}) \), then there exist a time \( T_\varepsilon = T(\varepsilon) > 0 \) and a unique solution \( u^\varepsilon \in C([0, T_\varepsilon]; H^r(\mathbb{R})). \)

**Proof.** Thanks to the Duhamel formula, \( u^\varepsilon \) is solution of (4) if and only if \( u^\varepsilon \) is the fixed point of \( \Phi^\varepsilon \) defined as

\[
\Phi^\varepsilon u^\varepsilon(t) := S_t u_0^\varepsilon - \frac{1}{2} \int_0^t S_{t - \tau} ((1 + D_x^2)^{-1} \partial_x u^\varepsilon(\tau))^2 d\tau
\]
where $S_t u = \mathcal{F}^{-1}\left(e^{-\frac{t^2|x|^2}{4}}\right) \ast u(x)$. Let $\mathcal{B}_T$ be the closed ball

$$
\mathcal{B}_T := \{ u \in C([0, T]; H^r(\mathbb{R})) \,|\, \| u \|_{L^\infty([0, T], H^r(\mathbb{R}))} \leq 2\| u_0 \|_{H^r} \}.
$$

We prove that $\Phi^\varepsilon(\mathcal{B}_T) \subseteq \mathcal{B}_T$ and $\Phi^\varepsilon$ is a contraction mapping on $\mathcal{B}_T$. Let $u^\varepsilon \in \mathcal{B}_T$, we have

$$
\| \Phi^\varepsilon u^\varepsilon(t) \|_{H^r} = \left\| S_t u_0 - \frac{1}{2} \int_0^t S_{t-\tau} (1 + D_x^\alpha)^{-1} \partial_x u^\varepsilon(\tau)^2 \, d\tau \right\|_{H^r}.
$$

Thus, choosing $T < \frac{1}{C_{\varepsilon, r} \| u_0 \|_{H^r}}$ implies

$$
\| \Phi^\varepsilon u^\varepsilon(t) \|_{L^\infty([0, T], H^r)} \leq 2\| u_0 \|_{H^r}.
$$

Let $u_1^\varepsilon, u_2^\varepsilon \in \mathcal{B}_T$, one gets from Lemma 2.2

$$
\| \Phi^\varepsilon u_1^\varepsilon(t) - \Phi^\varepsilon u_2^\varepsilon(t) \|_{H^r} \leq \int_0^t \| S_{t-\tau} (1 + D_x^\alpha)^{-1} \partial_x ((u_1^\varepsilon)^2 - (u_2^\varepsilon)^2) \|_{H^r} \, d\tau
$$

and $\Phi^\varepsilon$ is a contraction mapping on $\mathcal{B}_T$ as soon as $T < \frac{1}{C_{\varepsilon, r} \| u_0 \|_{H^r}}$. Finally, there exists a unique fixed point $u^\varepsilon := \Phi^\varepsilon(u^\varepsilon)$ in $\mathcal{B}_T$ for $T < \frac{1}{C_{\varepsilon, r} \| u_0 \|_{H^r}}$.

To obtain the continuity with respect to the initial data, for $u_0^\varepsilon$ and $v_0^\varepsilon$ in $H^r(\mathbb{R})$ with $\| u_0^\varepsilon \|_{H^r} \leq M$, $\| v_0^\varepsilon \|_{H^r} \leq M$, we consider $u^\varepsilon, v^\varepsilon$ the respective solution. Let $0 \leq t \leq T = \frac{1}{C_{\varepsilon, r} M}$, we find similarly

$$
\| u^\varepsilon(t) - v^\varepsilon(t) \|_{H^r} \leq \| S_t (u_0^\varepsilon - v_0^\varepsilon) \|_{H^r} + \int_0^t \| S_{t-\tau} (1 + D_x^\alpha)^{-1} \partial_x ((u^\varepsilon)^2 - (v^\varepsilon)^2) \|_{H^r} \, d\tau
$$

or in other words, since $1 - C_{\varepsilon, r} TM > 0$,

$$
\| u^\varepsilon(t) - v^\varepsilon(t) \|_{H^r} \leq \frac{1}{1 - C_{\varepsilon, r} TM} \| u_0^\varepsilon - v_0^\varepsilon \|_{H^r}.
$$

\[ \square \]

**Lemma 2.4.** Let $0 < \alpha < 1$, $r \geq 2 - \frac{\alpha}{2}$ and $u_0^\varepsilon \in H^r(\mathbb{R})$, then $T_\varepsilon$ can be chosen independent of $\varepsilon$. |
Proof. Let \( r = s + \frac{\alpha}{2} \). Applying \( J^r \) to (4), multiplying by \( J^s u^\varepsilon \) and integrating by parts over space, it comes
\[
\frac{d}{dt} \left( \int_R (J^s u^\varepsilon)^2 + (J^s D_x^2 u^\varepsilon)^2 dx \right) + 2\varepsilon \int_R (J^s u^\varepsilon)^2 dx = \int_R -J^s u^\varepsilon J^s (u^\varepsilon u^\varepsilon_x) dx.
\]
We obtain from Kato-Ponce commutator estimates [12]
\[
\int_R J^s u^\varepsilon J^s (u^\varepsilon u^\varepsilon_x) dx = \frac{1}{2} \int_R u^\varepsilon (J^s u^\varepsilon)^2 dx + \int_R RJ^s u^\varepsilon dx \leq C \|u^\varepsilon\|^3_{H^{1+\frac{\alpha}{2}}}
\]
It follows that
\[
\frac{d}{dt} \|u^\varepsilon\|^2_{H^{1+\frac{\alpha}{2}}} \leq C \|u^\varepsilon\|^3_{H^{1+\frac{\alpha}{2}}}
\]
or in other words \( \|u^\varepsilon\|^2_{H^{1+\frac{\alpha}{2}}} \leq y(t) \) where \( y(t) \) satisfies the ordinary differential equation
\[
\begin{align*}
\begin{cases}
y'(t) = Cy(t)^2 \\
y(0) = \|u_0^\varepsilon\|^2_{H^{1+\frac{\alpha}{2}}} (\mathbb{R})
\end{cases}
\end{align*}
\]
which solution is given by
\[
y(t) = \frac{y(0)}{(1 - Cy(0)^{1/2})^2} = \frac{\|u_0^\varepsilon\|^2_{H^{1+\frac{\alpha}{2}}} (\mathbb{R})}{(1 - C \|u_0^\varepsilon\|^3_{H^{1+\frac{\alpha}{2}}})^2},
\]
and \( u^\varepsilon \) can be extended until \( T = \frac{1}{2C\|u_0^\varepsilon\|^3_{H^{1+\frac{\alpha}{2}}}} \). \( \square \)

**Theorem 2.5.** Let \( 0 < \alpha < 1, r \geq 2 - \frac{\alpha}{2} \) and \( u_0 \in H^r(\mathbb{R}) \). Then there exist a time \( T > 0 \) and a unique vanishing viscosity solution \( u \in C([0,T];H^r(\mathbb{R})) \) of the initial value problem (3). In other words, the solution \( u^\varepsilon \) of the initial value problem (4) converges, when \( \varepsilon \to 0 \), uniformly to \( u \) solution of (3) in \( C([0,T];H^r(\mathbb{R})) \).

**Proof.** We prove that \( (u^\varepsilon)_{\varepsilon > 0} \) is a Cauchy sequence in the Sobolev space \( H^r(\mathbb{R}) \). Let \( u^\varepsilon \) and \( v^\delta \) be two solutions of (4). The difference \( w = u^\varepsilon - v^\delta \) satisfies
\[
w_1 + w_x + D_x^2 \partial_t w - \delta w_{xx} + (u^\varepsilon w - \frac{w^2}{2})_x = (\varepsilon - \delta) u^\varepsilon_{xx}.
\]
(5)

One multiplies (5) by \( w \) and integrates over space to obtain
\[
\frac{d}{dt} \int_{-\infty}^{\infty} w^2 + (D_x^2 w)^2 dx = \int_{-\infty}^{\infty} -2\delta w_x w_x^\varepsilon + 2(\varepsilon - \delta) u^\varepsilon_{xx} w dx \leq \int_{-\infty}^{\infty} -w_x^2 w^2 + 2(\varepsilon - \delta) u^\varepsilon_{xx} w dx.
\]
and Sobolev’s inequality provide
\[
\frac{d}{dt} \int_{-\infty}^{\infty} w^2 + (D_x^2 w)^2 dx \leq C \|w\|_{H^{2-\frac{\alpha}{2}}} \|w\|_{H_{\frac{\alpha}{2}}}^2 + C(\varepsilon - \delta) \|u\|_{H^{2-\frac{\alpha}{2}}} \|w\|_{H_{\frac{\alpha}{2}}}^2.
\]
Finally
\[
\frac{d}{dt} \|w\|_{H^{\frac{\alpha}{2}}}^2 \leq C \|w\|_{H_{\frac{\alpha}{2}}}^2 + C(\varepsilon - \delta) \|w\|_{H_{\frac{\alpha}{2}}}^2
\]
and the Gronwall Lemma offers a limit of \( (u^\varepsilon)_{\varepsilon > 0} \) in \( H^r(\mathbb{R}) \).
From the preceding lemma, the map \( t \in [0,T] \to u^\varepsilon(t) \) is continuous and uniformly bounded. In particular, the sequence \( (u^\varepsilon(t))_{\varepsilon > 0} \) is weakly convergent in \( H^s(\mathbb{R}), s \geq \frac{\alpha}{2} \) to \( u(t) \) a weakly continuous and uniformly bounded function. We deduce
\[
t \in [0,T] \to (1 + D_x^2)^{-1} (\partial_x u + w\partial_x u) \in H^{s+1-\alpha}(\mathbb{R})
\]
is weakly continuous and
\[ u(t) = u_0 - \int_0^t (1 + D^\alpha_x)^{-1}(\partial_x u + u\partial_x u) \quad \text{in} \quad H^{s+1-\alpha}(\mathbb{R}) \]
is unique. Indeed, let \( v \in H^{s+1-\alpha}(\mathbb{R}) \) defined by
\[ v(t) = v_0 - \int_0^t (1 + D^\alpha_x)^{-1}(\partial_x v + v\partial_x v), \]
we obtain with similar computations
\[ \frac{d}{dt} \|u - v\|_{H^\alpha}^2 \leq C \|u - v\|_{H^\alpha}^2 \]
and the Gronwall lemma allows to conclude.

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