QUANTUM ALGEBRAIC SYMMETRIES IN NUCLEAR STRUCTURE

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ABSTRACT

Various applications of quantum algebraic techniques in nuclear structure physics, such as the $su_q(2)$ rotator model and its extensions, the use of deformed bosons in the description of pairing correlations, and the construction of deformed exactly soluble models (Interacting Boson Model, Moszkowski model) are briefly reviewed. Emphasis is put in the study of the symmetries of the anisotropic quantum harmonic oscillator with rational ratios of frequencies, which underlie the structure of superdeformed and hyperdeformed nuclei, the Bloch–Brink $\alpha$-cluster model and possibly the shell structure in deformed atomic clusters.

1. Introduction

Quantum algebras $^{1,2}$ (also called quantum groups) are deformed versions of the usual Lie algebras, to which they reduce when the deformation parameter $q$ is set equal to unity. From the mathematical point of view they have the structure of Hopf algebras $^3$. Their use in physics became popular with the introduction $^4$–$^6$ of the $q$-deformed harmonic oscillator as a tool for providing a boson realization of the quantum algebra $su_q(2)$, although similar mathematical structures had already been known $^7,8$. Initially used for solving the quantum Yang–Baxter equation, quantum algebras have subsequently found applications in several branches of physics, as, for example, in the description of spin chains, squeezed states, rotational and vibrational nuclear and molecular spectra, and in conformal field theories. By now several kinds of generalized deformed oscillators $^9$–$^{13}$ and generalized deformed $su(2)$ algebras $^{14}$–$^{20}$ have been introduced.

Here we shall confine ourselves to applications of quantum algebras in nuclear structure physics. A brief description will be given of the $su_q(2)$ rotator model $^{21}$–$^{26}$ and its extensions $^{14,27}$, of the use of deformed oscillators in the description of pairing correlations $^{28}$–$^{30}$, and of the formulation of deformed exactly soluble models (Interacting Boson Model $^{31}$, Moszkowski model $^{32}$–$^{34}$). The purpose of this short review is to provide the reader with references for further reading. Subsequently, the symmetries of the anisotropic quantum harmonic oscillator with rational ratios of frequencies
will be considered in more detail, since they are of current interest in connection with superdeformed and hyperdeformed nuclei, \( \alpha \)-cluster configurations in light nuclei, and possibly with deformed atomic clusters.

2. The \( \text{su}_q(2) \) rotator model

The first application of quantum algebras in nuclear physics was the use of the deformed algebra \( \text{su}_q(2) \) for the description of the rotational spectra of deformed and superdeformed nuclei. The same technique has been used for the description of rotational spectra of diatomic molecules. The Hamiltonian of the \( q \)-deformed rotator is proportional to the second order Casimir operator of the \( \text{su}_q(2) \) algebra. Its Taylor expansion contains powers of \( J(J+1) \) (where \( J \) is the angular momentum), being similar to the expansion provided by the Variable Moment of Inertia (VMI) model. Furthermore, the deformation parameter \( \tau \) (with \( q = e^{i\tau} \)) has been found to correspond to the softness parameter of the VMI model.

\( B(E2) \) transition probabilities have also been described in this framework. In this case the \( q \)-deformed Clebsch–Gordan coefficients are used instead of the normal ones. (It should be noticed that the \( q \)-deformed angular momentum theory has already been much developed.) The model predicts an increase of the \( B(E2) \) values with angular momentum, while the rigid rotator model predicts saturation. Some experimental results supporting this prediction already exist.

3. Extensions of the \( \text{su}_q(2) \) model

The \( \text{su}_q(2) \) model has been successful in describing rotational nuclear spectra. For the description of vibrational and transitional nuclear spectra it has been found that \( J(J+1) \) has to be replaced by \( J(I+c) \). The additional parameter \( c \) allows for the description of nuclear anharmonicities in a way similar to that of the Interacting Boson Model (IBM) and the Generalized Variable Moment of Inertia (GVMI) model.

Another generalization is based on the use of the deformed algebra \( \text{su}_\Phi(2) \), which is characterized by a structure function \( \Phi \). The usual \( \text{su}(2) \) and \( \text{su}_q(2) \) algebras are obtained for specific choices of the structure function \( \Phi \). The \( \text{su}_\Phi(2) \) algebra has been constructed so that its representation theory resembles as much as possible the representation theory of the usual \( \text{su}(2) \) algebra. Using this technique one can construct, for example, a rotator having the same spectrum as the one given by the Holmberg–Lipas formula. In addition to the generalized deformed \( \text{su}(2) \) algebra, generalized deformed oscillators have also been introduced and found useful in many physical applications.

4. Pairing correlations

It has been found that correlated fermion pairs coupled to zero angular momen-
tum in a single-\(j\) shell behave approximately as suitably defined \(q\)-deformed bosons. After performing the same boson mapping to a simple pairing Hamiltonian, one sees that the pairing energies are also correctly reproduced up to the same order. The deformation parameter used \((\tau = \ln q)\) is found to be inversely proportional to the size of the shell, thus serving as a small parameter.

The above mentioned system of correlated fermion pairs can be described exactly by suitably defined generalized deformed bosons \(^{29}\). Then both the commutation relations are satisfied exactly and the pairing energies are reproduced exactly. The spectrum of the appropriate generalized deformed oscillator corresponds, up to first order perturbation theory, to a harmonic oscillator with an \(x^4\) perturbation.

5. \(q\)-deformed versions of nuclear models

A \(q\)-deformed version of a two dimensional toy Interacting Boson Model (IBM) has been developed \(^{31}\), mainly for testing the ways in which spectra and transition probabilities are influenced by the \(q\)-deformation. A \(q\)-deformed version of the full IBM is under development, while a \(q\)-deformed version of the vibron model, which uses the IBM techniques in the case of molecules, has already been developed \(^{48}\).

Furthermore a \(q\)-deformed version of the Moszkowski model has been developed \(^{32}\) and RPA modes have been studied \(^{33}\) in it. A \(q\)-deformed Moszkowski model with cranking has also been studied \(^{34}\) in the mean-field approximation. It has been seen that the residual interaction simulated by the \(q\)-deformation is felt more strongly by states with large \(J_z\). The possibility of using \(q\)-deformation in assimilating temperature effects is under discussion.

6. Anisotropic quantum harmonic oscillator with rational ratios of frequencies

The symmetries of the 3-dimensional anisotropic quantum harmonic oscillator with rational ratios of frequencies (RHO) are of high current interest in nuclear physics, since they are the basic symmetries \(^{35,36}\) underlying the structure of superdeformed and hyperdeformed nuclei \(^{37,38}\). The 2-dimensional RHO is also of interest, in connection with “pancake” nuclei, i.e. very oblate nuclei \(^{36}\). Cluster configurations in light nuclei can also be described in terms of RHO symmetries \(^{39,40}\), which underlie the geometrical structure of the Bloch–Brink \(\alpha\)-cluster model \(^{41}\). The 3-dim RHO is also of interest for the interpretation of the observed shell structure in atomic clusters \(^{42}\), especially after the realization \(^{43}\) that large deformations can occur in such systems.

The two-dimensional \(^{49–54}\) and three-dimensional \(^{55–61}\) anisotropic harmonic oscillators have been the subject of several investigations, both at the classical and the quantum mechanical level. These oscillators are examples of superintegrable systems \(^{62,63}\). The special cases with frequency ratios 1:2 \(^{64,65}\) and 1:3 \(^{66}\) have also been considered. While at the classical level it is clear that the \(su(N)\) or \(sp(2N,R)\) algebras
can be used for the description of the N-dimensional anisotropic oscillator, the situation at the quantum level, even in the two-dimensional case, is not as simple. The symmetry algebra of the two-dimensional anisotropic quantum harmonic oscillator with rational ratio of frequencies has been identified as a deformation of the $u(2)$ algebra. The finite dimensional representation modules of the algebra have been studied and the energy eigenvalues have been determined using algebraic methods of general applicability to quantum superintegrable systems. For labelling the degenerate states an “angular momentum” operator has been introduced, the eigenvalues of which are roots of appropriate generalized Hermite polynomials. The cases with frequency ratios $1:n$ have been found to correspond to generalized parafermionic oscillators, while in the special case with frequency ratio 2:1 the resulting algebra has been found to correspond to the finite W algebra $W^{(2)}_3$. For more details the reader is referred to.

In this section we are going to prove that a generalized deformed $u(N)$ algebra is the symmetry algebra of the three-dimensional anisotropic quantum harmonic oscillator, which is the oscillator describing the single-particle level spectrum of superdeformed and hyperdeformed nuclei.

6.1. The deformed $u(N)$ algebra for the N-dimensional oscillator

Let us consider the system described by the Hamiltonian:

$$H = \frac{1}{2} \sum_{k=1}^{N} \left( p_k^2 + \frac{x_k^2}{m_k^2} \right),$$

where $m_i$ are natural numbers mutually prime ones, i.e. their great common divisor is $\gcd(m_i, m_j) = 1$ for $i \neq j$ and $i, j = 1, \ldots, N$.

We define the creation and annihilation operators:

$$a_k^\dagger = \frac{1}{\sqrt{2}} \left( \frac{x_k}{m_k} - ip_k \right),$$
$$a_k = \frac{1}{\sqrt{2}} \left( \frac{x_k}{m_k} + ip_k \right),$$
$$U_k = \frac{1}{2} \left( p_k^2 + \frac{x_k^2}{m_k^2} \right) = \frac{1}{2} \{a_k, a_k^\dagger\}, \quad H = \sum_{k=1}^{N} U_k.$$  (2)

These operators satisfy the relations (the indices $k$ have been omitted):

$$a^\dagger U = (U - \frac{1}{m}) a^\dagger \quad \text{or} \quad [U, (a^\dagger)^m] = (a^\dagger)^m,$$
$$a U = (U + \frac{1}{m}) a \quad \text{or} \quad [U, (a)^m] = -(a)^m,$$
$$a^\dagger a = U - \frac{1}{2m}, \quad a a^\dagger = U + \frac{1}{2m},$$
$$[a, a^\dagger] = \frac{1}{m},$$
$$\left(a^\dagger\right)^m (a)^m = F(m, U),$$
$$\left(a\right)^m (a^\dagger)^m = F(m, U + 1),$$  (3)
where the function $F(m, x)$ is defined by:

$$F(m, x) = \prod_{p=1}^{m} \left( x - \frac{2p - 1}{2m} \right),$$ (4)

Using the above relations we can define the enveloping algebra $\mathcal{C}$, defined by the polynomial combinations of the generators \(\{1, H, A_k, A_k^\dagger, U_k\}\) and \(k = 1, \ldots, N - 1\), where:

$$A_k^\dagger = \left( a_k^\dagger \right)^{m_k} (a_N)^{m_N}, \quad A_k = (a_k)^{m_k} \left( a_N^\dagger \right)^{m_N}.$$

These operators correspond to a multidimensional generalization of eq. (2):

$$[H, A_k] = 0, \quad [H, A_k^\dagger] = 0, \quad [H, U_k] = 0, \quad k = 1, \ldots, N - 1.$$ (6)

The following relations are satisfied for \(k \neq \ell\) and \(k, \ell = 1, \ldots, N - 1\):

$$[U_k, A_\ell] = [U_k, A_\ell^\dagger] = [A_k, A_\ell] = [A_k^\dagger, A_\ell^\dagger] = 0,$$ (7)

while

$$U_k A_k^\dagger = A_k^\dagger (U_k + 1),$$

$$U_k A_k = A_k (U_k - 1),$$

$$A_k^\dagger A_k = F(m_k, U_k) F(m_N, H - \sum_{\ell=1}^{N-1} U_\ell + 1),$$ (8)

$$A_k A_k^\dagger = F(m_k, U_k + 1) F(m_N, H - \sum_{\ell=1}^{N-1} U_\ell).$$

One additional relation for \(k \neq \ell\) can be derived:

$$F(m_N, H - \sum_{\ell=1}^{N-1} U_\ell) A_k^\dagger A_\ell = F(m_N, H - \sum_{\ell=1}^{N-1} U_\ell + 1) A_\ell A_k^\dagger.$$ (9)

For illustrative purposes it is useful to work out the algebra in the case of the 3-dim oscillator. Starting from the above definitions one ends up with the commutation relations shown in Table 1, where

$$A_3^\dagger = (a_1^\dagger)^{m_1}(a_2^\dagger)^{m_2}, \quad A_3 = (a_1)^{m_1}(a_2^\dagger)^{m_2},$$ (10)

$$F_i = F(m_i, U_i + 1) - F(m_i, U_i), \quad i = 1, 2, 3,$$ (11)

$$F_{ij} = F(m_i, U_i + 1) F(m_j, U_j) - F(m_i, U_i) F(m_j, U_j + 1), \quad i, j = 1, 2, 3. $$ (12)

In the 1:1:1 case one has \(F_i = 1, i = 1, 2, 3\) and \(F_{ij} = U_j - U_i, i, j = 1, 2, 3\), so that Table 1 gives the usual results for the u(3) algebra.

In the 1:1:2 case the modified commutators read

$$[A_1, A_1^\dagger] = -2U_1 U_3 + U_3^2 + \frac{3}{16}, \quad [A_1, A_2^\dagger] = -2U_3 A_3,$$

$$[A_2, A_2^\dagger] = -2U_2 U_3 + U_3^2 + \frac{3}{16}, \quad [A_2, A_1^\dagger] = -2U_3 A_3^\dagger,$$

Similarly in the 1:1:3 case one has

$$[A_1, A_1^\dagger] = -3U_1 U_2^2 - \frac{5}{36} U_1 + U_3^3 + \frac{23}{36} U_3, \quad [A_1, A_2^\dagger] = -\left( 3U_3^2 + \frac{5}{36} \right) A_3,$$
Table 1  
Commutation relations for the deformed u(3) algebra associated with the $m_1 : m_2 : m_3$ case. The relevant operators are defined in eqs. (2), (5), (10)–(12).

\[
\begin{array}{cccccccc}
U_1 & U_2 & U_3 & A_3^\dagger & A_2^\dagger & A_3 & A_2 & A_1 \\
U_1 & 0 & 0 & 0 & A_3^\dagger & 0 & -A_3^\dagger & 0 & -A_1 \\
U_2 & 0 & 0 & 0 & -A_3 & A_2^\dagger & 0 & A_3 & -A_2 \\
U_3 & 0 & 0 & 0 & -A_2^\dagger & -A_3^\dagger & 0 & A_2 & A_1 \\
A_3^\dagger & -A_3 & A_3^\dagger & 0 & 0 & A_1 F_2 & 0 & -F_{12} & -A_2 F_1 \\
A_2 & 0 & -A_2 & A_2^\dagger & 0 & 0 & 0 & -F_{23} & A_3 F_3 \\
A_1 & -A_1 & 0 & A_1^\dagger & 0 & 0 & 0 & -A_2 F_1 & A_3 F_3 \\
A_3 & A_3 & -A_3 & 0 & F_{12} & 0 & A_1 F_1 & 0 & -A_1 F_2 \\
A_2 & 0 & A_2 & -A_2 & 0 & F_{23} & -A_3 F_3 & A_1 F_2 & 0 \\
A_1 & A_1 & 0 & -A_1 & A_2 F_1 & -A_3 F_3 & F_{13} & 0 & 0 \\
\end{array}
\]

\[ [A_2, A_3^\dagger] = -3 U_2 U_3^2 - \frac{5}{36} U_2 + U_3^3 + \frac{23}{36} U_3, \quad [A_2, A_3^\dagger] = -\left(3 U_3^2 + \frac{5}{36}\right) A_3^\dagger. \]

In the same way in the 1:1:4 case one has

\[ [A_1, A_3^\dagger] = -4 U_1 U_3^3 - \frac{11}{16} U_1 U_3 + U_3^4 + \frac{43}{32} U_3^2 + \frac{105}{4096} U_3, \quad [A_1, A_3^\dagger] = -\left(4 U_3^3 + \frac{11}{16} U_3\right) A_3, \]

\[ [A_2, A_3^\dagger] = -4 U_2 U_3^3 - \frac{11}{16} U_2 U_3 + U_3^4 + \frac{43}{32} U_3^2 + \frac{105}{4096} U_3, \quad [A_2, A_3^\dagger] = -\left(4 U_3^3 + \frac{11}{16} U_3\right) A_3^\dagger. \]

We remark that for the 1:1:m oscillators only four commutators get modified in each case.

In the 2:2:1 case the modified commutators read

\[ [A_1, A_3^\dagger] = 2 U_1 U_3 - U_1^2 - \frac{3}{16}, \quad [A_1, A_3] = 2 U_1 A_2, \]

\[ [A_2, A_3^\dagger] = 2 U_2 U_3 - U_2^2 - \frac{3}{16}, \quad [A_2, A_3] = 2 U_2 A_1, \]

\[ [A_3, A_3^\dagger] = -(U_1 - U_2) \left(2 U_1 U_2 - \frac{3}{8}\right), \quad [A_3, A_3] = 2 U_1 A_2^\dagger, \]

\[ [A_3, A_2^\dagger] = 2 U_2 A_1^\dagger, \]

while the commutators

\[ [A_1, A_3^\dagger] = -A_3, \quad [A_2, A_3^\dagger] = -A_3^\dagger, \]

are not modified in this case. One can see that for the m:m:1 oscillators seven commutators get modified in each case.
6.2. The representations of the deformed $u(N)$

The algebra defined in the previous subsection accepts a Fock space representation. The elements of the basis $|E, p_1, \ldots, p_{N-1}\rangle$ are characterized by the eigenvalues of the $N$ commuting elements of the algebra $H$ and $U_k$ with $k = 1, \ldots, N-1$. The elements $A_k$ and $A_k^\dagger$ are the corresponding ladder operators of the algebra. The following relations hold:

\begin{align*}
H |E, p_1, \ldots, p_{N-1}\rangle &= E |E, p_1, \ldots, p_{N-1}\rangle, \\
U_k |E, p_1, \ldots, p_{N-1}\rangle &= p_k |E, p_1, \ldots, p_{N-1}\rangle, \\
A_k |E, p_1, \ldots, p_{N-1}\rangle &= \sqrt{F(m_k, p_k)} F(m_N, E - \sum_{\ell=1}^{N-1} p_\ell + 1) |E, p_1, \ldots, p_{N-1}\rangle, \\
A_k^\dagger |E, p_1, \ldots, p_{N-1}\rangle &= \sqrt{F(m_k, p_k + 1)} F(m_N, E - \sum_{\ell=1}^{N-1} p_\ell) |E, p_1, \ldots, p_{N-1}\rangle.
\end{align*}

Let $p_k^{\text{min}}$ be the minimum value of $p_k$ such that

\begin{equation}
A_k |E, p_1, \ldots, p_k^{\text{min}}, \ldots, p_{N-1}\rangle = 0.
\end{equation}

From eq. (13) we find that we must have:

\begin{equation}
F(m_k, p_k^{\text{min}}) = 0.
\end{equation}

Then $p_k$ is one of the roots of the function $F$ defined by eq. (4). The general form of the roots is:

\begin{equation}
p_k^{\text{min}} = \frac{2q_k - 1}{2m_k}, \quad q_k = 1, \ldots, m_k.
\end{equation}

Each root is characterized by a number $q_k$. The numbers $q_k$ also characterize the representations of the algebra, as we shall see.

The elements of the Fock space can be generated by successive applications of the ladder operators $A_k^\dagger$ on the minimum weight element

\begin{equation}
|E, p_1^{\text{min}}, \ldots, p_{N-1}^{\text{min}}\rangle = |E, [0]\rangle = |E, 0, \ldots, 0, q_1, \ldots, q_k, \ldots, q_N\rangle.
\end{equation}

The elements of the basis of the Fock space are given by:

\begin{equation}
|E, [p_k^{\text{min}} + n_k]\rangle = \frac{1}{\sqrt{C_n[q]}} \left( \prod_{k=1}^{N-1} (A_k^\dagger)^{n_k} \right) |E, [0]\rangle,
\end{equation}

where $[n] = (n_1, n_2, \ldots, n_{N-1})$ and $[q] = (q_1, q_2, \ldots, q_N)$, while $C_n[q]$ are normalization coefficients.
The generators of the algebra acting on the base of the Fock space give:

\[ H \left| E, [n] \right\rangle = E \left| E, [n] \right\rangle, \]
\[ U_k \left| E, [n] \right\rangle = \left( n_k + p_k^{\text{min}} \right) \left| E, [n] \right\rangle, \quad k = 1, \ldots, N-1, \] (19)

\[ A_k^\dagger \left| E, [n] \right\rangle = F \left( m_k, n_k + p_k^{\text{min}} + 1 \right) F \left( m_N, E - \sum_{\ell=1}^{N-1} \left( n_\ell + p_\ell^{\text{min}} \right) \right), \] (20)

\[ A_k \left| E, [n] \right\rangle = F \left( m_k, n_k + p_k^{\text{min}} \right) F \left( m_N, E - \sum_{\ell=1}^{N-1} \left( n_\ell + p_\ell^{\text{min}} \right) + 1 \right), \] (21)

The existence of a finite dimensional representation implies that after \( \Sigma \) successive applications of the ladder operators \( A_k^\dagger \) on the minimum weight element one gets zero, so that the following condition is satisfied:

\[ F \left( m_N, E - \Sigma - \sum_{\ell=1}^{N-1} p_\ell^{\text{min}} \right) = 0. \]

Therefore:

\[ E - \Sigma - \sum_{\ell=1}^{N-1} p_\ell^{\text{min}} = p_N^{\text{min}}, \]

where \( p_N^{\text{min}} \) is the root of equation \( F \left( m_N, p_N^{\text{min}} \right) = 0 \). Then

\[ E = \Sigma + \sum_{k=1}^{N} \frac{2q_k - 1}{2m_k}. \] (22)

In the case of finite dimensional representations only the energies given by eq. (22) are permitted and the elements of the Fock space can be described by using \( \Sigma \) instead of \( E \). The action of the generators on the Fock space is described by the following relations:

\[ H \left| \Sigma, [n] \right\rangle = \left( \Sigma + \sum_{k=1}^{N} \frac{2q_k - 1}{2m_k} \right) \left| \Sigma, [n] \right\rangle, \]
\[ U_k \left| \Sigma, [n] \right\rangle = \left( n_k + \frac{2q_k - 1}{2m_k} \right) \left| \Sigma, [n] \right\rangle, \quad k = 1, \ldots, N-1, \] (23)
\[ A_k^\dagger \begin{bmatrix} \Sigma, \{n\} \\ \{q\} \end{bmatrix} = \sqrt{F(m_k, n_k + \frac{2q_k - 1}{2m_k} + 1) \cdot F(m_N, \Sigma - \sum_{\ell=1}^{N-1} n_\ell + \frac{2q_N - 1}{2m_N})}, \]
\[ A_k \begin{bmatrix} \Sigma, \{n\} \\ \{q\} \end{bmatrix} = \sqrt{F(m_k, n_k + \frac{2q_k - 1}{2m_k}) \cdot F(m_N, \Sigma - \sum_{\ell=1}^{N-1} n_\ell + \frac{2q_N - 1}{2m_N} + 1)}. \]

The dimension of the representation is given by
\[ d = \left( \frac{\Sigma + N - 1}{\Sigma} \right) = \frac{(\Sigma + 1)(\Sigma + 2) \cdots (\Sigma + N - 1)}{(N - 1)!}. \]
It is clear that to each value of \( \Sigma \) correspond \( m_1 m_2 \ldots m_N \) energy eigenvalues, each eigenvalue having degeneracy \( d \).

### 6.3. Connection to the Cartesian basis

Using eqs (3) we can prove that the algebra generated by the generators \( a_\ell^\dagger, a_\ell, N_\ell = m_\ell U_\ell - 1/2 \) is an oscillator algebra with structure function \( 9,12 \)
\[ \Phi_\ell(x) = x/m_\ell, \]
i.e. with
\[ \Phi_\ell(N_\ell) = U_\ell - \frac{1}{2m_\ell}. \]
This oscillator algebra is characterized by the commutation relations:
\[ [N_\ell, a_\ell^\dagger] = a_\ell^\dagger, \quad [N_\ell, a_\ell] = -a_\ell, \quad a_\ell^\dagger a_\ell = \Phi_\ell(N_\ell), \quad a_\ell a_\ell^\dagger = \Phi_\ell(N_\ell + 1). \]

There are in total \( N \) different oscillators of this type, uncoupled to each other. The Fock space corresponding to these oscillators defines an infinite dimensional representation of the algebra defined eqs (4-9). In order to see the connection of the present basis to the usual Cartesian basis, one can use for the latter the symbol \( [r] = (r_1, r_2, \ldots, r_N) \). One then has
\[ a_\ell^\dagger [r] = \sqrt{\Phi(r_\ell + 1)} [r_1, \ldots, r_\ell + 1, \ldots, r_N], \]
\[ a_\ell [r] = \sqrt{\Phi(r_\ell)} [r_1, \ldots, r_\ell - 1, \ldots, r_N], \]
\[ N_\ell [r] = r_\ell [r]. \]
The connection between the above basis and the basis defined by eqs (23-25) is given by:

$$\left| r \right\rangle = \left| \sum_{\ell=1}^{N} [r_{\ell}/m_{\ell}] n_{\ell} [q_{\ell}] \right\rangle$$

where $[x]$ means the integer part of the number $x$.

Using the correspondence between the present basis and the usual Cartesian basis, given in eq. (27), the action of the operators $a_{k}^{\dagger}$ on the present basis can be calculated for $k = 1, \ldots, N - 1$:

$$a_{k}^{\dagger} \left| \sum_{\ell=1}^{N} [n_{\ell}] [q_{\ell}] \right\rangle = \sqrt{n_{k} + q_{k}/m_{k}} \left| \sum'_{\ell=1}^{N} [n'_{\ell}] [q'_{\ell}] \right\rangle,$$

where:

- $n'_{\ell} = n_{\ell}$, $q'_{\ell} = q_{\ell}$ for $\ell \neq k$,
- $n'_{k} = n_{k} + [q_{k}/m_{k}]$, $\Sigma' = \Sigma + [q_{k}/m_{k}]$, $q'_{k} = \text{mod} (q_{k}, m_{k}) + 1$,

while for the operator $a_{N}^{\dagger}$ one has

$$a_{N}^{\dagger} \left| \sum_{\ell=1}^{N} [n_{\ell}] [q_{\ell}] \right\rangle = \sqrt{\Sigma N - \sum_{k=1}^{N-1} n_k + q_N/m_N} \left| \sum'_{\ell=1}^{N} [n'_{\ell}] [q'_{\ell}] \right\rangle,$$

where:

- $n'_{k} = n_{k}$, $q'_{k} = q_{k}$, for $k = 1, \ldots, N - 1$,
- $\Sigma' = \Sigma + [q_{N}/m_{N}]$, $q'_{N} = \text{mod} (q_{N}, m_{N}) + 1$.

Similarly for the operators $a_{k}$ one can find for $k = 1, \ldots, N - 1$:

$$a_{k} \left| \sum_{\ell=1}^{N} [n_{\ell}] [q_{\ell}] \right\rangle = \sqrt{n_{k} + (q_{k} - 1)/m_{k}} \left| \sum'_{\ell=1}^{N} [n'_{\ell}] [q'_{\ell}] \right\rangle,$$

where:

- $n'_{\ell} = n_{\ell}$, $q'_{\ell} = q_{\ell}$, for $\ell \neq k$,
- $n'_{k} = n_{k} + [(q_{k} - 2)/m_{k}]$, $\Sigma' = \Sigma + [(q_{k} - 2)/m_{k}]$, $q'_{k} = \text{mod} (q_{k} - 2, m_{k}) + 1$, 

with $r_{\ell} = n_{\ell} m_{\ell} + \text{mod} (r_{\ell}, m_{\ell})$ and $\ell = 1, \ldots, N$.
while for the operator $a_N$ one has

$$a_N \left| \Sigma, [n] \right\rangle = \sqrt{N - \sum_{k=1}^{N-1} n_k + (q_N - 1)/m_N} \left| \Sigma', [n'] \right\rangle,$$

where

$$n'_k = n_k, \quad q'_k = q_k, \quad \text{for } k = 1, \ldots, N - 1,$$

$$\Sigma' = \Sigma + [(q_N - 2)/m_N],$$

$$q'_N = \text{mod}(q_N - 2, m_N) + 1.$$ 

For illustrative purposes, we shall discuss a few cases in some detail.

i) In the 1:1:1 case (isotropic 3-dim oscillator) the only allowed $[q_1 q_2 q_3]$ set is $[111]$. In the Cartesian notation $|r_1 r_2 r_3>$ the lowest energy corresponds to the state $|000>$, the next energy level corresponds to the three states $|100>$, $|010>$ and $|001>$, while the next energy level corresponds to the six states $|200>$, $|020>$, $|002>$, $|110>$, $|101>$, $|011>$. In the basis of eqs. (23)-(25) these states are written as

$$|000> = \left| \begin{array}{c} 0,0,0 \\ 111 \end{array} \right>, \quad |100> = \left| \begin{array}{c} 1,1,0 \\ 111 \end{array} \right>, \quad |010> = \left| \begin{array}{c} 1,1,1 \\ 111 \end{array} \right>, \quad |001> = \left| \begin{array}{c} 1,0,0 \\ 111 \end{array} \right>,$$

$$|200> = \left| \begin{array}{c} 2,2,0 \\ 111 \end{array} \right>, \quad |020> = \left| \begin{array}{c} 2,1,0 \\ 111 \end{array} \right>, \quad |002> = \left| \begin{array}{c} 1,2,0 \\ 111 \end{array} \right>,$$

$$|110> = \left| \begin{array}{c} 2,1,1 \\ 111 \end{array} \right>, \quad |101> = \left| \begin{array}{c} 2,1,0 \\ 111 \end{array} \right>, \quad |011> = \left| \begin{array}{c} 2,0,1 \\ 111 \end{array} \right>.$$

It is clear that the irreps are characterized by the quantum numbers $\Sigma$ and $[q_1 q_2 q_3]$, while $[n_1 n_2]$ enumerate the degenerate states within each irrep. The lowest irrep, characterized by $\Sigma = 0$ and $[q_1 q_2 q_3] = [111]$, has dimension $d = 1$, the next irrep is characterized by $\Sigma = 1$ and $[q_1 q_2 q_3] = [111]$ and has dimension $d = 3$, while the next irrep has $\Sigma = 2$, $[q_1 q_2 q_3] = [111]$ and dimension $d = 6$. According to eq. (26) the dimensions of the first few $u(3)$ irreps are 1, 3, 6, 10, 15, 21, . . . .

ii) In the 1:1:2 case the allowed $[q_1 q_2 q_3]$ sets are [111] and [112]. The lowest irrep is characterized by $\Sigma = 0$, $[q_1 q_2 q_3] = [111]$, has dimension 1 and contains the state $|000>$. The second irrep has $\Sigma = 0$, $[q_1 q_2 q_3] = [112]$, $d = 1$ and contains the state $|001>$. The third irrep has $\Sigma = 1$, $[q_1 q_2 q_3] = [111]$, $d = 3$, and contains the states $|100>$, $|010>$, $|002>$. The fourth irrep is characterized by $\Sigma = 1$, $[q_1 q_2 q_3] = [112]$, has dimension $d = 3$ and contains the states $|101>$, $|011>$, $|003>$. The states are listed in both notations below:

$$|000> = \left| \begin{array}{c} 0,0,0 \\ 111 \end{array} \right>, \quad |001> = \left| \begin{array}{c} 0,0,0 \\ 112 \end{array} \right>,$$
\[ |100> = \begin{bmatrix} 1,10 \\ 111 \end{bmatrix}, \quad |010> = \begin{bmatrix} 1,01 \\ 111 \end{bmatrix}, \quad |002> = \begin{bmatrix} 1,00 \\ 111 \end{bmatrix}, \]
\[ |101> = \begin{bmatrix} 1,10 \\ 112 \end{bmatrix}, \quad |011> = \begin{bmatrix} 1,01 \\ 112 \end{bmatrix}, \quad |003> = \begin{bmatrix} 1,00 \\ 112 \end{bmatrix}. \]

iii) In the 2:2:1 case the allowed \([q_1 q_2 q_3]\) sets are \([111], [211], [121], [221]\). The lowest energy level is characterized by \(\Sigma = 0\), \([q_1 q_2 q_3] = [111]\), has dimension \(d = 1\) and contains the state \(|000>\). The next energy level has \(d = 2\) and is containing the 1-dim irrep with \(\Sigma = 0\), \([q_1 q_2 q_3] = [211]\), i.e. the state \(|100>\), and the 1-dim irrep with \(\Sigma = 0\), \([q_1 q_2 q_3] = [121]\), i.e. the state \(|010>\). The next energy level has \(d = 4\) and contains two irreps: the 1-dim irrep with \(\Sigma = 0\), \([q_1 q_2 q_3] = [221]\) (state \(|110>\)), and the 3-dim irrep with \(\Sigma = 1\), \([q_1 q_2 q_3] = [111]\) (states \(|001>, |200>, |020>\)). The next energy level has \(d = 6\) and contains two irreps: the 3-dim irrep with \(\Sigma = 1\), \([q_1 q_2 q_3] = [211]\) (states \(|101>, |300>, |120>\)), and the 3-dim irrep with \(\Sigma = 1\), \([q_1 q_2 q_3] = [121]\) (states \(|011>, |030>, |210>\)). These states are listed in both notations below:

\[ |000> = \begin{bmatrix} 0,00 \\ 111 \end{bmatrix}, \quad |100> = \begin{bmatrix} 0,10 \\ 211 \end{bmatrix}, \quad |010> = \begin{bmatrix} 0,01 \\ 121 \end{bmatrix}, \]
\[ |110> = \begin{bmatrix} 0,11 \\ 221 \end{bmatrix}, \quad |001> = \begin{bmatrix} 1,00 \\ 111 \end{bmatrix}, \]
\[ |200> = \begin{bmatrix} 1,20 \\ 111 \end{bmatrix}, \quad |020> = \begin{bmatrix} 1,02 \\ 111 \end{bmatrix}, \]
\[ |101> = \begin{bmatrix} 1,10 \\ 211 \end{bmatrix}, \quad |300> = \begin{bmatrix} 1,30 \\ 211 \end{bmatrix}, \quad |120> = \begin{bmatrix} 1,12 \\ 211 \end{bmatrix}, \]
\[ |011> = \begin{bmatrix} 1,01 \\ 121 \end{bmatrix}, \quad |030> = \begin{bmatrix} 1,03 \\ 121 \end{bmatrix}, \quad |210> = \begin{bmatrix} 1,21 \\ 121 \end{bmatrix}. \]

The following comments can now be made:

i) In the basis described by eqs (23)-(25) it is a trivial matter to distinguish the states belonging to the same irrep for any \(m_1 : m_2 : m_3\) ratios, while in the Cartesian basis this is true only in the 1:1 case.

ii) In the 1:1:2 case we see that the irreps have degeneracies 1, 1, 3, 3, 6, 6, 10, 10, . . . , i.e. “two copies” of the \(u(3)\) degeneracies 1, 3, 6, 10, . . . are obtained.

iii) It can be easily seen that the 1:1:n case corresponds to “n copies” of the \(u(3)\) degeneracies. In the case 1:1:3, for example, the degeneracies are 1, 1, 1, 3, 3, 3, 6, 6, 6, . . ..

iv) In the 2:2:1 case the energy levels do not correspond to a single irrep each, but some of them correspond to sums of irreps, i.e. to reducible representations (rreps) of the deformed \(u(3)\) algebra. The same is true for the m:m:1 case. In the 2:2:1 case, in particular, the degeneracies are 1, 2, 4, 6, 9, 12, 16, . . . , which correspond to the dimensions of the irreps of \(O(4)\) 76.
v) It can be easily seen that the condition for each energy eigenvalue to correspond to one irrep of the deformed algebra is that $m_1$, $m_2$, $m_3$ are mutually prime numbers. If two of them possess a common divisor other than 1, then some energy eigenvalues will correspond to sums of irreps, i.e. to reps.

vi) Cases where reps appear, can be approximated by cases where only irreps appear. For example, 2:2:1 can be approximated by 21:19:10 or 201:199:100. (See \textsuperscript{52} for more details.)

vii) The difference between the formalism used here and the one used in \textsuperscript{55,57,58,61} is that in the latter case for given $m_1$, $m_2$, $m_3$, appropriate operators have to be introduced separately for each set of $[q_1,q_2,q_3]$ values, while in the present case only one set of operators is introduced. It should be noticed that (s) of refs \textsuperscript{55,57,58} ($\{\lambda\}$ of ref. \textsuperscript{61}) is analogous to the $[\tilde{q}]$ used in the present work.

6.4. The 3-dimensional oscillator and relation to the Nilsson model

In this subsection the 3-dim case will be studied in more detail, because of its relevance for the description of superdeformed nuclei and of nuclear and atomic clusters. The 3-dim anisotropic oscillator is the basic ingredient of the Nilsson model \textsuperscript{77}, which in addition contains a term proportional to the square of the angular momentum operator, as well as a spin-orbit coupling term, the relevant Hamiltonian being

$$ H_{\text{Nilsson}} = H_{\text{osc}} - 2k\vec{L} \cdot \vec{S} - k\nu \vec{L}^2, $$

where $k$, $\nu$ are constants. The spin-orbit term is not needed in the case of atomic clusters, while in the case of nuclei it can be effectively removed through a unitary transformation, both in the case of the spherical Nilsson model \textsuperscript{78–80} and of the axially symmetric one \textsuperscript{81,82}. An alternative way to effectively remove the spin-orbit term in the spherical Nilsson model is the $q$-deformation of the relevant algebra \textsuperscript{83}. It should also be noticed that the spherical Nilsson Hamiltonian is known to possess an osp(1|2) supersymmetry \textsuperscript{84}.

In the case of the 3-dim oscillator the relevant operators of eq. (2) form a nonlinear generalization of the algebra $\mathfrak{u}(3)$, the $q$-deformed version of which can be found in \textsuperscript{31,85,86}.

As we have already seen, to each $\Sigma$ value correspond $m_1m_2m_3$ energy eigenvalues, each eigenvalue having degeneracy $(\Sigma + 1)(\Sigma + 2)/2$. In order to distinguish the degenerate eigenvalues, we are going to introduce some generalized angular momentum operators, $L_i$ ($i = 1, 2, 3$), defined by:

$$ L_k = i\epsilon_{ijk} \left( (a_i)^{m_i} (a_j^\dagger)^{m_j} - (a_i^\dagger)^{m_i} (a_j)^{m_j} \right). \quad (28) $$

One can prove that

$$ L_1 = i \left( A_2 - A_2^\dagger \right), \quad L_2 = i \left( A_1^\dagger - A_1 \right). $$

The following commutation relations can be verified:

$$ [L_i, L_j] = i\epsilon_{ijk} \left( F(m_k, U_k + 1) - F(m_k, U_k) \right) L_k. \quad (29) $$
It is worth noticing that in the case of \( m_1 = m_2 = m_3 = 1 \) one has \( F(1, x) = x - 1/2 \), so that the above equation gives the usual angular momentum commutation relations.

The operators defined in eq. (28) commute with the oscillator Hamiltonian \( H \) and therefore conserve the number \( \Sigma \) which characterizes the dimension of the representation. Also these operators do not change the numbers \( q_1, q_2, q_3 \) as we can see from eqs (24-25). The eigenvalues of these operators can be calculated using Hermite function techniques.

Let us consider in particular the generalized angular momentum projection:

\[
L_3 = i \left( (a_1)^m_1 (a_2)^m_2 - (a_1)^{m_1} (a_2)^{m_2} \right).
\] (30)

This acts on the basis as follows

\[
L_3 \left| \Sigma, [n] \right\rangle = i \left( \sqrt{F(m_1, n_1 + \frac{2q_1-1}{2m_1}) F(m_2, n_2 + \frac{2q_2-1}{2m_2} + 1)} \left| \Sigma, n_1 - 1, n_2 + 1 \right\rangle - \right.
\]

\[
\left. \sqrt{F(m_1, n_1 + \frac{2q_1-1}{2m_1} + 1) F(m_2, n_2 + \frac{2q_2-1}{2m_2})} \left| \Sigma, n_1 + 1, n_2 - 1 \right\rangle \right).
\] (31)

This operator conserves the quantum number

\[
j = \frac{n_1 + n_2}{2}, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots
\]

In addition one can introduce the quantum number

\[
m = \frac{n_1 - n_2}{2}.
\]

One can then replace the quantum numbers \( n_1, n_2 \) by the quantum numbers \( j, \mu \), where \( \mu \) is the eigenvalue of the \( L_3 \) operator. The new representation basis one can label as

\[
L_3 \left| \Sigma, j, \mu \right\rangle = \mu \left| \Sigma, j, \mu \right\rangle.
\] (32)

This basis is connected to the basis of the previous section as follows

\[
\left| \Sigma, j, \mu \right\rangle = \sum_{m=-j}^{j} \frac{c[j, m, \mu]}{\sqrt{[j+m]_1! [j-m]_2!}} \left| \Sigma, j + m, j - m \right\rangle,
\] (33)

where

\[
[0]_k! = 1, \quad [n]_k! = [n]_k [n-1]_k!, \quad [n]_k = \frac{F(m_k, n + \frac{2q_k - 1}{2m_k})}{},
\]
and the coefficients $c[j, m, \mu]$ in eq. (33) satisfy the recurrence relation:

$$\mu c[j, m, \mu] = i ([j - m]_2 c[j, m + 1, \mu] - [j + m]_1 c[j, m - 1, \mu]).$$

These relations can be satisfied only for special values of the parameter $\mu$, corresponding to the eigenvalues of $L_3$. It is worth noticing that in the case of $m_1 = m_2$, which corresponds to axially symmetric oscillators, the possible values turn out to be $\mu = -2j, -2(j - 1), \ldots, 2(j - 1), 2j$. In nuclear physics the quantum numbers $n_\perp = n_1 + n_2$ and $\Lambda = \pm n_\perp, \pm (n_\perp - 2), \ldots, \pm 1$ or 0 are used. From the above definitions it is clear that $j = n_\perp/2$ and $\mu = \Lambda$. Therefore in the case of $m_1 = m_2$, which includes axially symmetric prolate nuclei with $m_1 : m_2 : m_3 = 1 : 1 : m$, as well as axially symmetric oblate nuclei with $m_1 : m_2 : m_3 = m : m : 1$, the correspondence between the present scheme and the Nilsson model is clear.

### 6.5. Summary

The symmetry algebra of the N-dim anisotropic quantum harmonic oscillator with rational ratios of frequencies has been constructed by a method of general applicability in constructing finite-dimensional representations of quantum superintegrable systems. The case of the 3-dim oscillator has been considered in detail, because of its relevance to the single particle level spectrum of superdeformed and hyperdeformed nuclei, to the underlying geometrical structure of the Bloch-Brink $\alpha$-cluster model, and possibly to the shell structure of atomic clusters at large deformations. The symmetry algebra in this case is a nonlinear generalization of the $u(3)$ algebra. For labeling the degenerate states, generalized angular momentum operators are introduced, clarifying the connection of the present approach to the Nilsson model.

In the case of the 2-dim oscillator with ratio of frequencies $2 : 1$ ($m_1 = 1, m_2 = 2$) it has been shown that the relevant nonlinear generalized $u(2)$ algebra can be identified as the finite W algebra $W_3^{(2)}$. In the case of the 3-dim axially symmetric oblate oscillator with frequency ratio $1 : 2$ (which corresponds to the case $m_1 = m_2 = 2, m_3 = 1$) the relevant symmetry is related to $O(4)$. The search for further symmetries, related to specific frequency ratios, hidden in the general nonlinear algebraic framework given in this work is an interesting problem.

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