Coefficient and Fekete-Szegö Problem Estimates for Certain Subclass of Analytic and Bi-Univalent Functions

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Abstract. In this paper, we obtain the Fekete-Szegö problem for the k-th ($k \geq 1$) root transform of the analytic and normalized functions $f$ satisfying the condition

$$1 + \frac{\alpha - \pi}{2 \sin \alpha} < \text{Re} \left( \frac{zf'(z)}{f(z)} \right) < 1 + \frac{\alpha}{2 \sin \alpha} \quad (|z| < 1),$$

where $\alpha \in [\pi/2, \pi)$. Afterwards, by the above two-sided inequality we introduce a certain subclass of analytic and bi-univalent functions in the disk $|z| < 1$ and obtain upper bounds for the first few coefficients and Fekete-Szegö problem for functions $f$ belonging to this class.

1. Introduction

Let $\mathcal{A}$ be the class of functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$  \hspace{1cm} (1)

which are analytic in the open unit disk $\Delta = \{ z \in \mathbb{C} : |z| < 1 \}$ and normalized by the condition $f(0) = f'(0) - 1 = 0$. Also let $\mathcal{P}$ be the class of functions $p$ analytic in $\Delta$ which are of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots + p_n z^n + \cdots,$$

such that $\text{Re}(p(z)) > 0$ for all $z \in \Delta$. The subclass of all functions $f$ in $\mathcal{A}$ which are univalent (one-to-one) in $\Delta$ is denoted by $\mathcal{S}$. An example for the class $\mathcal{S}$ is the well-known Koebe function which has the following form

$$k(z) := \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots + nz^n + \cdots \quad (z \in \Delta).$$

It is known that the Koebe function maps the open unit disk $\Delta$ onto the entire plane minus the interval $(-\infty, -1/4]$. Also, the well-known Koebe One-Quarter Theorem states that the image of the open unit disk $\Delta$
under every function $f \in S$ contains the disk $\{w : |w| < \frac{1}{4}\}$, see [11, Theorem 2.3]. Therefore, according to the above, every function $f$ in the class $S$ has an inverse $f^{-1}$ which satisfies the following conditions:

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f); \ r_0(f) \geq 1/4),$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots =: g(w). \quad (2)$$

We say that a function $f \in A$ is bi-univalent in $\Delta$ if, and only if, both $f$ and $f^{-1}$ are univalent in $\Delta$. We denote by $\Sigma$ the class of all bi-univalent functions in $\Delta$. The following functions $z_1 - z_2, -\log(1 - z)$ and $1/2 \log(1 + z)$, with the corresponding inverse functions, respectively,

$$\frac{w}{1 + w}, \quad \frac{\exp(w) - 1}{\exp(w)} \quad \text{and} \quad \frac{\exp(2w) - 1}{\exp(2w) + 1},$$

belong to the class $\Sigma$. It is clear that the Koebe function is not a member of the class $\Sigma$, also the following functions

$$z - \frac{1}{2}z^2 \quad \text{and} \quad \frac{z}{1 - z^2},$$

do not belong to the class $\Sigma$, see [35].

It should be mentioned here that the pioneering work on the subject by Srivastava et al. [35] actually revived the study of analytic and bi-univalent functions in recent years. In fact, subsequent to this important investigation by Srivastava et al. [35], many authors have introduced and studied various subclasses of analytic and bi-univalent functions (see, for example, [9, 23, 25, 28, 29, 31, 32, 36, 37, 40, 43, 44]).

A function $f \in A$ is called starlike (with respect to 0) if $tw \in f(\Delta)$ whenever $w \in f(\Delta)$ and $t \in [0, 1]$. We denote by $S^\ast$ the class of all starlike functions in $\Delta$. Also, we say that a function $f \in A$ is starlike of order $\gamma$ ($0 \leq \gamma < 1$) if, and only if,

$$\text{Re}\left\{\frac{z f'(z)}{f(z)}\right\} > \gamma \quad (z \in \Delta).$$

The class of the starlike functions of order $\gamma$ in $\Delta$ is denoted by $S^\ast(\gamma)$. As usual we put $S^\ast(0) \equiv S^\ast$.

We recall that a function $f \in A$ belongs to the class $M(\alpha)$ if $f$ satisfies the following two-sided inequality

$$1 + \frac{\alpha - \pi}{2\sin \alpha} < \text{Re}\left\{\frac{f'(z)}{f(z)}\right\} < 1 + \frac{\alpha}{2\sin \alpha} \quad (z \in \Delta),$$

where $\alpha \in [\pi/2, \pi]$. The class $M(\alpha)$ was introduced by Kargar et al. in [13]. We define the function $\phi$ as follows

$$\phi(\alpha) := 1 + \frac{\alpha - \pi}{2\sin \alpha} \quad (\pi/2 \leq \alpha < \pi).$$

Since

$$2\phi'(\alpha) = [(\pi - \alpha) \cot \alpha + 1] \csc \alpha \quad (\pi/2 \leq \alpha < \pi),$$
therefore for each \( \alpha \in [\pi/2, \pi] \) we see that \( \phi'(\alpha) \neq 0 \). On the other hand, since \( \phi(\pi/2) = 1 - \pi/4 \approx 0.2146 \) and
\[
\lim_{\alpha \to \pi} \phi(\alpha) = \frac{1}{2},
\]
thus the class \( \mathcal{M}(\alpha) \) is a subclass of the starlike functions of order \( \gamma \) where \( 0.2146 \leq \gamma < 0.5 \). By this fact that \( \mathcal{S}(\gamma) \subset \mathcal{S} \) for each \( \gamma \in [0,1) \), thus we conclude that the members of the class \( \mathcal{M}(\alpha) \) are univalent in \( \Delta \).

Now, we recall the following result for the class \( \mathcal{M}(\alpha) \), see [13, Lemma 1.1].

**Lemma 1.1.** Let \( f(z) \in \mathcal{A} \) and \( \alpha \in [\pi/2, \pi] \). Then \( f \in \mathcal{M}(\alpha) \) if, and only if,
\[
\left( \frac{zf'(z)}{f(z)} - 1 \right) \prec \mathcal{B}_\alpha(z) \quad (z \in \Delta),
\]
where
\[
\mathcal{B}_\alpha(z) := \frac{1}{2i \sin \alpha} \log \left( \frac{1 + ze^{i\alpha}}{1 + ze^{-i\alpha}} \right) \quad (z \in \Delta). \tag{3}
\]
Here "\( \prec \)" denotes the well known subordination relation.

The function \( \mathcal{B}_\alpha(z) \) is convex univalent and has the form
\[
\mathcal{B}_\alpha(z) = \sum_{n=1}^\infty A_n z^n \quad (z \in \Delta), \tag{4}
\]
where
\[
A_n := \frac{(-1)^{(n-1)} \sin n \alpha}{n \sin \alpha} \quad (n = 1, 2, \ldots).
\]
Also we have \( \mathcal{B}_\alpha(\Delta) = \Omega_\alpha \) (see [10]) where
\[
\Omega_\alpha := \left\{ \zeta \in \mathbb{C} : \frac{\alpha - \pi}{2 \sin \alpha} < \Re \{\zeta\} < \frac{\alpha}{2 \sin \alpha} \quad : \pi/2 \leq \alpha < \pi \right\}.
\]

Very recently Sun et al. (see [41]) and Kwon and Sim (see [17]) have studied the class \( \mathcal{M}(\alpha) \). Sun et al. showed if the function \( f \) is of the form (1) belongs to the class \( \mathcal{M}(\alpha) \), then \( |a_n| \leq 1 \) while the estimate is not sharp. Subsequently, Kwon and Sim obtained sharp estimates on the initial coefficients \( a_2, a_3, a_4 \) and \( a_5 \) of the functions \( f \) belonging to the class \( \mathcal{M}(\alpha) \). The coefficient estimate problem for each of the Taylor-Maclaurin coefficients \( |a_n| \) \((n = 6, 7, \ldots)\) is still an open question. Also, the logarithmic coefficients of the function \( f \in \mathcal{M}(\alpha) \) were estimated by Kargar, see [12].

It is interesting to mention this subject that Brannan and Taha [7] introduced certain subclass of the bi-univalent function class \( \Sigma \), denoted by \( \mathcal{S}_\gamma(\gamma) \) similar to the class of the starlike functions of order \( \gamma \) \((0 \leq \gamma < 1)\). For each function \( f \in \mathcal{S}_\gamma(\gamma) \) they found non-sharp estimates for the initial Taylor-Maclaurin coefficients. Recently, motivated by the Brannan and Taha’s work, many authors investigated the coefficient bounds for various subclasses of the bi-univalent function class \( \Sigma \), see for instance [8, 21, 22, 26, 27, 35, 38, 39].

In this paper, motivated by the aforementioned works, we introduce and investigate a certain subclass of \( \Sigma \) similar to the class \( \mathcal{M}(\alpha) \) as follows.

**Definition 1.2.** Let \( \alpha \in [\pi/2, \pi] \). A function \( f \in \Sigma \) is in the class \( \mathcal{M}_\Sigma(\alpha) \), if the following inequalities hold:
\[
1 + \frac{\alpha - \pi}{2 \sin \alpha} < \Re \left\{ \frac{zf'(z)}{f(z)} \right\} < 1 + \frac{\alpha}{2 \sin \alpha} \quad (z \in \Delta)
\]
and
\[
1 + \frac{\alpha - \pi}{2 \sin \alpha} < \Re \left\{ \frac{wg'(w)}{g(w)} \right\} < 1 + \frac{\alpha}{2 \sin \alpha} \quad (w \in \Delta),
\]
where \( g \) is defined by (2).
Remark 1.3. Upon letting $\alpha \to \pi^-$ it is readily seen that a function $f \in \Sigma$ is in the class $M_\Sigma(1/2)$ if the following inequalities are satisfied:

$$\text{Re} \left( \frac{z f'(z)}{f(z)} \right) > \frac{1}{2} (z \in \Delta)$$

and

$$\text{Re} \left( \frac{w g'(w)}{g(w)} \right) > \frac{1}{2} (w \in \Delta),$$

where $g$ is defined by (2).

The following lemma will be useful.

Lemma 1.4. (see [19]) Let the function $p$ be of the form $p(z) = 1 + az + \sum_{n=2}^{\infty} a_n z^n$. Then for any complex number $\mu$ we have

$$|p_2 - \mu p_1^2| \leq \left\{ \begin{array}{ll}
-4\mu + 2, & \text{if } \mu \leq 0; \\
2, & \text{if } 0 \leq \mu \leq 1; \\
4\mu - 2, & \text{if } \mu \geq 1.
\end{array} \right.$$  

The result is sharp for the cases $\mu < 0$ or $\mu > 1$ if and only if $p(z) = \frac{1+z}{1-z}$ or one of its rotations. If $0 < \mu < 1$, then the equality holds if and only if $p(z) = \frac{1+\mu z}{1-\mu z}$ or one of its rotations. For the case $\mu = 0$, the equality holds if and only if $p(z) = 1 + \frac{1}{2} (1 + \nu) z + \sum_{n=2}^{\infty} a_n z^n$ for $0 \leq \nu \leq 1$, or one of its rotations. If $\mu = 1$, the equality holds if and only if $p(z) = 1 + \frac{1}{2} (1 + \nu) z + \sum_{n=2}^{\infty} a_n z^n$ for $0 \leq \nu \leq 1$, or one of its rotations.

This paper is organized as follows. In Section 2 we derive the Fekete-Szegö coefficient functional associated with the $k$-th root transform for functions in the class $M(\alpha)$. In Section 3 we propose to find the estimates on the Taylor-Maclaurin coefficients $|a_2|, |a_3|$ and Fekete-Szegö problem for functions in the class $M_\Sigma(\alpha)$ which we introduced in Definition 1.2.

2. Fekete-Szegö problem for the class $M(\alpha)$

Recently, many authors have obtained the Fekete-Szegö coefficient functional associated with the $k$-th root transform for certain subclasses of analytic functions, see for instance [5, 14, 15]. In this section, we investigate this problem for the class $M(\alpha)$. At first, we recall that for a univalent function $f$ is of the form (1), the $k$-th root transform is defined by

$$F_k(z) := (f(z))^k = 1 + \sum_{n=1}^{m} b_{kn+1} z^{kn+1} (z \in \Delta, k \geq 1).$$  

For $f$ given by (1), we have

$$(f(z))^k = 1 + \frac{1}{k} a_2 z^{k+1} + \left( \frac{1}{k} a_3 - \frac{1}{2} \frac{k-1}{k^2} a_2^2 \right) z^{2k+1} + \cdots.$$  

Equating the coefficients of (5) and (6) yields

$$b_{k+1} = \frac{1}{k} a_2 \text{ and } b_{2k+1} = \frac{1}{k} a_3 - \frac{1}{2} \frac{k-1}{k^2} a_2^2.$$  

Now we have the following.
Theorem 2.1. Let \( \alpha \in [\pi/2, \pi) \) and \( f \in M(\alpha) \). If \( F \) is the \( k \)-th (\( k \geq 1 \)) root transform of the function \( f \) defined by (5), then for any complex number \( \mu \) we have

\[
|b_{2k+1} - \mu b_{k+1}^2| \leq \begin{cases} 
\frac{1}{2\pi} \left( 1 - \cos \alpha - \frac{2\mu + k - 1}{k} \right), & \text{if } \mu \leq \delta_1; \\
\frac{1}{2\pi}, & \text{if } \delta_1 \leq \mu \leq \delta_2; \\
\frac{1}{2\pi} \left( \cos \alpha + \frac{2\mu + k - 1}{k} - 1 \right), & \text{if } \mu \geq \delta_2,
\end{cases}
\]

(8)

where \( \delta_1 := (1 - k(1 + \cos \alpha))/2 \), \( \delta_2 := (1 + k(1 - \cos \alpha))/2 \) and \( b_{2k+1} \) and \( b_{k+1} \) are defined by (7). The result is sharp.

Proof. Let \( \alpha \in [\pi/2, \pi) \). If \( f \in M(\alpha) \), then by Lemma 1.1 and by definition of subordination, there exists a Schwarz function \( w : \Delta \rightarrow \Delta := \{ z : |z| \leq 1 \} \) with the following properties

\[ w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta), \]

such that

\[
\frac{z f'(z)}{f(z)} = 1 + \mathcal{B}_w(w(z)) \quad (z \in \Delta),
\]

(9)

where \( \mathcal{B}_w \) is defined by (3). We define

\[ p(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \Delta). \]

(10)

It is clear that \( p(0) = 1 \) and \( p \in \mathcal{P} \). Relationships (4) and (10) give us

\[
1 + \mathcal{B}_w(w(z)) = 1 + \frac{1}{2} A_1 p_1 z + \left( \frac{1}{4} A_2 p_2^2 + \frac{1}{2} A_1 \left( p_2 - \frac{1}{2} p_1^2 \right) \right) z^2 + \cdots,
\]

where \( A_1 = 1 \) and \( A_2 = -\cos \alpha \). If we equate the coefficients of \( z \) and \( z^2 \) on both sides of (9), then we get

\[ a_2 = \frac{1}{2} p_1 \]

(11)

and

\[ a_3 = \frac{1}{4} \left( p_2 - \frac{1}{2} \cos \alpha p_1^2 \right). \]

(12)

From (7), (11) and (12), we get

\[ b_{k+1} = \frac{p_1}{2k}, \]

and

\[
b_{2k+1} = \frac{1}{4k} \left( p_2 - \frac{1}{2} \left( \cos \alpha + \frac{k-1}{k} \right) p_1^2 \right),
\]

where \( k \geq 1 \). Therefore

\[
b_{2k+1} - \mu b_{k+1}^2 = \frac{1}{4k} \left( p_2 - \frac{1}{2} \left( \cos \alpha + \frac{2\mu + k - 1}{k} \right) p_1^2 \right) \quad (\mu \in \mathbb{C}).
\]

If we apply the Lemma 1.4 and letting

\[ \mu' := \frac{1}{2} \left( \cos \alpha + \frac{2\mu + k - 1}{k} \right), \]
then we get the desired inequality (8).

From now, we shall show that the result is sharp. For the sharpness of the first and third cases of (8), i.e. \( \mu \leq \delta_1 \) and \( \mu \geq \delta_2 \), respectively, consider the function

\[
f_1(z) := z \exp \left\{ \int_0^z \frac{\mathcal{B}_\alpha(\xi) - 1}{\xi} \, d\xi \right\} \quad (z \in \Delta)
\]

\[
= z + z^2 + \frac{1}{2} (1 - \cos \alpha) z^3 + \frac{1}{18} (1 - 9 \cos \alpha + 8 \cos^2 \alpha) z^4 + \cdots,
\]
or one of its rotations. It is easy to see that \( f_1 \) belongs to the class \( M(\alpha) \) and

\[
(f_1(z^k))^{1/k} = z + \frac{1}{k} z^{k+1} + \left( \frac{1}{2k} (1 - \cos \alpha) - \frac{1 - 1}{2k^2} \right) z^{2k+1} + \cdots.
\]

The last equation shows that these inequalities are sharp. For the sharpness of the second inequality, we consider the function

\[
f_2(z) := z^2 \exp \left\{ \int_0^z \frac{\mathcal{B}_\alpha(\xi^2) - 1}{\xi} \, d\xi \right\} = z + \frac{1}{2} z^3 + \cdots \quad (z \in \Delta).
\]

A simple calculation gives that

\[
(f_2(z^k))^{1/k} = z + \frac{1}{2k} z^{2k+1} + \cdots.
\]

Therefore the equality in the second inequality (8) holds for the \( k \)-th root transform of the above function \( f_2 \). This completes the proof of Theorem 2.1.

The problem of finding sharp upper bounds for the coefficient functional \( |a_3 - \mu a_2^2| \) for different subclasses of the normalized analytic function class \( A \) is known as the Fekete-Szegö problem. In the recent years, many scholars have investigated the Fekete-Szegö problem for some certain subclasses of analytic functions, see for example [16, 24, 30, 33, 34, 42].

Letting \( k = 1 \) in the Theorem 2.1 we get the Fekete-Szegö inequality for the class \( M(\alpha) \) which we give in the following corollary.

**Corollary 2.2.** Let \( \alpha \in [\pi/2, \pi) \) and \( f \in M(\alpha) \). Then for any complex number \( \mu \) we have

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{1}{2} (1 - \cos \alpha) - \mu, & \text{if } \mu \leq -\frac{1}{2} \cos \alpha; \\
\frac{1}{2}, & \text{if } -\frac{1}{2} \cos \alpha \leq \mu \leq 1 - \frac{1}{2} \cos \alpha; \\
\frac{1}{2} (\cos \alpha - 1) + \mu, & \text{if } \mu \geq 1 - \frac{1}{2} \cos \alpha.
\end{cases}
\]

The result is sharp.

Putting \( \alpha = \pi/2 \) in the Corollary 2.2 we get the following.

**Corollary 2.3.** Let the function \( f \) be given by (1) satisfies the inequality

\[
\left| \frac{\text{Re} \left\{ z f'(z) \right\}}{f(z)} - 1 \right| < \frac{\pi}{4} \quad (z \in \Delta).
\]

Then for any complex number \( \mu \in \mathbb{C} \) we have the following sharp inequalities

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{1}{2} - \mu, & \text{if } \mu \leq 0; \\
\frac{1}{2}, & \text{if } 0 \leq \mu \leq 1; \\
\mu - \frac{1}{2}, & \text{if } \mu \geq 1.
\end{cases}
\]
If we let \( \alpha \to \pi^- \) in the Corollary 2.2, then we have:

**Corollary 2.4.** If the function \( f \) is of the form (1) is starlike of order \( 1/2 \), then for any complex number \( \mu \in \mathbb{C} \) the following sharp inequalities hold true.

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
1 - \mu, & \text{if } \mu \leq \frac{1}{2}; \\
\frac{1}{2}, & \frac{1}{2} \leq \mu \leq \frac{3}{2}; \\
\mu - 1, & \text{if } \mu \geq \frac{3}{2}.
\end{cases}
\]

From (11) and (12) and the first case of the Lemma 1.4 we get:

**Corollary 2.5.** If a function \( f \in \mathcal{A} \) is of the form (1) belongs to the class \( M(\alpha) \) \((\pi/2 \leq \alpha < \pi)\), then the following sharp inequalities hold.

\[
|a_2| \leq 1 \quad \text{and} \quad |a_3| \leq \frac{1}{2}(1 - \cos \alpha).
\]

### 3. Coefficient estimate and Fekete-Szegő problem for the class \( M_\Sigma(\alpha) \)

In this section, motivated by the Zaprawa’s work (see [45]) we shall obtain the Fekete-Szegő problem for the class \( M_\Sigma(\alpha) \). Also, we obtain upper bounds for the first coefficients \( |a_2| \) and \( |a_3| \) of the function \( f \) is of the form (1) belonging to the class \( M_\Sigma(\alpha) \). The coefficient estimate problem for each of the coefficients \( |a_n| \) \((n \geq 4)\) is an open question. Here we recall that the initial coefficients estimate of the class of bi-univalent functions \( \Sigma \) was studied by Lewin in 1967 and he obtained the bound \( 1.51 \) for the modulus of the second coefficient \( |a_2| \), see [18]. Afterward, Brannan and Clunie conjectured that \( |a_2| \leq \sqrt{2} \), see [6]. Finally, in 1969, Netanyahu [20] showed that \( \max f \in \Sigma |a_2| = 4/3 \). For the another coefficients \( a_n \) \((n \geq 3)\) the sharp estimate is presumably still an open problem.

Moreover, we apply the same technique as in [4].

**Theorem 3.1.** Let the function \( f \) given by (1) be in the class \( M_\Sigma(\alpha) \) and \( \alpha \in [\pi/2, \pi) \). Then

\[
|a_2| \leq \sqrt{\frac{2}{2 + \cos \alpha}} \quad \text{(13)}
\]

and for any real number \( \mu \) we have

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{1}{2}, & \text{if } |1 - \mu| \leq \frac{1}{2} \left(1 + \frac{1}{2} \cos \alpha\right); \\
\frac{|1 - \mu|}{1 + \frac{1}{2} |\cos \alpha|}, & \text{if } |1 - \mu| \geq \frac{1}{2} \left(1 + \frac{1}{2} \cos \alpha\right).
\end{cases}
\]

**Proof.** Let \( f \in M_\Sigma(\alpha) \) be of the form (1) and \( g = f^{-1} \) be given by (2). Then by Definition 1.2, Lemma 1.1 and definition of subordination there exist two Schwarz functions \( u : \Delta \to \Delta \) and \( v : \Delta \to \Delta \) with the properties \( u(0) = 0 = v(0), |u(z)| < 1 \) and \( |v(z)| < 1 \) such that

\[
\frac{zf''(z)}{f'(z)} = 1 + \mathcal{B}_n(u(z)) \quad (z \in \Delta)
\]

and

\[
\frac{wg'(w)}{g'(w)} = 1 + \mathcal{B}_n(v(z)) \quad (z \in \Delta),
\]
where $B_\alpha$ is defined by (3). Now we define the functions $k$ and $l$, respectively as follows

$$k(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + k_1 z + k_2 z^2 + \cdots \quad (z \in \Delta)$$

and

$$l(z) = \frac{1 + v(z)}{1 - v(z)} = 1 + l_1 z + l_2 z^2 + \cdots \quad (z \in \Delta)$$

or equivalently

$$u(z) = \frac{k(z) - 1}{k(z) + 1} = \frac{1}{2} \left( k_1 z + \left( k_2 - \frac{1}{2} k_1^2 \right) z^2 + \cdots \right)$$

(16)

and

$$v(z) = \frac{l(z) - 1}{l(z) + 1} = \frac{1}{2} \left( l_1 z + \left( l_2 - \frac{1}{2} l_1^2 \right) z^2 + \cdots \right).$$

(17)

It is clear that the functions $k$ and $l$ belong to class $P$ and $|k_i| \leq 2$ and $|l_i| \leq 2 \ (i = 1, 2, \ldots)$. From (4), (14)-(17), we have

$$zf'(z) = f(z) + B_\alpha \left( \frac{k(z) - 1}{k(z) + 1} \right) = 1 + \frac{1}{2} A_1 k_1 z + \frac{1}{2} A_1 \left( k_2 - \frac{1}{2} k_1^2 \right) z^2 + \cdots,$$

(18)

and

$$wg'(w) = g(w) + B_\alpha \left( \frac{l(z) - 1}{l(z) + 1} \right) = 1 + \frac{1}{2} A_1 l_1 z + \frac{1}{2} A_1 \left( l_2 - \frac{1}{2} l_1^2 \right) z^2 + \cdots.$$  

where $A_1 = 1$ and $A_2 = -\cos \alpha$. Thus, upon comparing the corresponding coefficients in (18) and (19), we obtain

$$a_2 = \frac{1}{2} A_1 k_1 = \frac{1}{2} k_1,$$

(20)

$$2a_3 - a_2^2 = \frac{1}{2} A_1 \left( k_2 - \frac{1}{2} k_1^2 \right) + \frac{1}{4} A_2 k_1^2 = \frac{1}{2} \left( k_2 - \frac{1}{2} k_1^2 \right) - \frac{k_1^2}{4} \cos \alpha,$$

(21)

$$-a_2 = \frac{1}{2} A_1 l_1 = \frac{1}{2} l_1,$$

(22)

and

$$3a_2^2 - 2a_3 = \frac{1}{2} A_1 \left( l_2 - \frac{1}{2} l_1^2 \right) + \frac{1}{4} A_2 l_1^2 = \frac{1}{2} \left( l_2 - \frac{1}{2} l_1^2 \right) - \frac{l_1^2}{4} \cos \alpha.$$

(23)

From equations (20) and (22), we can easily see that

$$k_1 = -l_1$$

(24)

and

$$8a_2^2 = (k_1^2 + l_1^2).$$
If we add (21) to (23), we get
\[ 2\tilde{a}_2 = \frac{1}{2} \left[ (k_2 - \frac{1}{2}l_1^2) + (l_2 - \frac{1}{2}k_1^2) \right] - \frac{1}{4} \cos \alpha \left( k_1^2 + l_1^2 \right). \] (25)

Substituting (20), (22) and (24) into (25), we obtain
\[ k_1^2 = \frac{k_2 + l_2}{2(1 + \cos \alpha /2)}. \] (26)

Now, (20) and (26) imply that
\[ \tilde{a}_2 = \frac{k_2 + l_2}{2(2 + \cos \alpha)}. \] (27)

Since \( |k_2| \leq 2 \) and \( |l_2| \leq 2 \), (27) implies that
\[ |\tilde{a}_2| \leq \sqrt{\frac{2}{2 + \cos \alpha}}, \]
which proves the first assertion (13) of Theorem 3.1. Now, if we subtract (23) from (21) and use of (24), we get
\[ a_3 = a_2^2 + \frac{1}{8}(k_2 - l_2). \] (28)

From (27) and (28) it follows that
\[ a_3 - \mu a_2^2 = \left( \frac{1}{8} + h(\mu) \right) k_2 + \left( h(\mu) - \frac{1}{8} \right) l_2 \quad (\mu \in \mathbb{R}), \]
where
\[ h(\mu) := \frac{1 - \mu}{2(2 + \cos \alpha)} \quad (\mu \in \mathbb{R}). \]

Since \( |k_2| \leq 2 \) and \( |l_2| \leq 2 \), we conclude that
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2}, & \text{if } 0 \leq |h(\mu)| \leq \frac{1}{4}; \\ 4|h(\mu)|, & \text{if } |h(\mu)| \geq \frac{1}{4}. \end{cases} \]

This completes the proof. \( \square \)

Taking \( \mu = 0 \) in the above Theorem 3.1 we get.

**Corollary 3.2.** Let \( f \) of the form (1) be in the class \( \mathcal{M}_\Sigma(\alpha) \). Then
\[ |a_3| \leq \frac{1}{1 + \frac{1}{2} \cos \alpha} \quad (\pi/2 \leq \alpha < \pi). \]

If we let \( \alpha \to \pi^- \) in the Theorem 3.1, we get the following.

**Corollary 3.3.** If the function \( f \) is of the form (1) belongs to the class \( \mathcal{M}_\Sigma(1/2) \), then \( |\tilde{a}_2| \leq 1 \) and
\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2}, & \text{if } |1 - \mu| \leq \frac{1}{4}; \\ 2|1 - \mu|, & \text{if } |1 - \mu| \geq \frac{1}{4}, \end{cases} \]
where \( \mu \) is real.
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