Research Article

A New Iterative Method for Suzuki Mappings in Banach Spaces

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In this paper, an efficient new iterative method for approximating the fixed point of Suzuki mappings is proposed. Some important weak and strong convergence results of the proposed iterative method are established in the setting of Banach space. An example illustrates the theoretical outcome.

1. Introduction and Preliminaries

Throughout the present research, we shall write \( \mathbb{N} \) and \( \mathbb{R} \) to denote the set of natural numbers and real numbers set, respectively. We say that a self-map \( F \) of a subset \( M \) of a Banach space \( E = (E, \| \cdot \|) \) is called a contraction map whenever a real constant \( 0 \leq r < 1 \) exists with the following property:

\[
\| Fu - Fv \| \leq r \| u - v \|, \quad \text{for all } u, v \in M. \tag{1}
\]

An element \( q \in M \) is said to be a fixed point of \( F \) provided that \( q = Fq \). In this manuscript, the notation \( \text{Fix}(F) \) will throughout denote the fixed point set of \( F \). The Banach–Caccioppoli fixed point theorem (see, e.g., [1, 2] and others) states that any contraction mapping in the setting of complete metric spaces admits a unique fixed point \( q \), and this \( q \) is, in fact, the limit of all the sequences \( \{u_k\} \) obtained from the Picard iterates [3], that is, \( u_{k+1} = Fu_k \). However, one of the important classes of mappings in fixed point theory is the class of nonexpansive mappings. Notice that, \( F \) is called a nonexpansive mapping whenever (1) holds true for \( r = 1 \). In 1965, Browder [4] and Gohde [5] differently proved the same result concerning the existence of fixed points for nonexpansive mappings. Indeed, they suggested that any self-nonexpansive map \( F \) of \( M \) always admits at least one fixed point whenever \( M \) is assumed to be a bounded convex closed subset of some uniformly convex Banach space (UCBS). Nevertheless, the sequence \( \{u_k\} \) defined by Picard iterates may not have a limit in the fixed point set associated with a nonexpansive map in general as shown in the next example. Let \( M = [0, 1] \) and set \( Fu = 1 - u \); it is easy to see that \( F \) is a self-nonexpansive mapping on \( M \) having a unique fixed point 1/2. However, for any \( u_1 \neq 1/2 \), we obtain the sequence of Picard iterates as follows: \( u, 1 - u, 1 - u, \ldots \) which is an oscillating sequence and, hence, diverges in \( \text{Fix}(F) = \{1/2\} \). For providing comparatively better convergence speed and to overcome such situations, different iterative methods have been suggested by some authors (cf. the works of Mann [6], Ishikawa [7], Noor [8], Agarwal et al. [9], Abbas and Nazir [10], Thakur et al. [11], and references therein).

In 2008, Suzuki [12] gained a big break through introducing an interesting extension of nonexpansive mappings as follows. We recall that a self-map \( F: M \to M \) mapping with \((C)\) property (also called Suzuki mapping) if the following fact is valid:

\[
\frac{1}{2} \| u - Fu \| \leq \| u - v \| \implies \| Fu - Fv \| \leq \| u - v \|, \quad \text{for each two elements } u, v \in M. \tag{2}
\]

One can easily notice that the Suzuki mappings satisfy the nonexpansive requirement for some elements of the domain. Hence, nonexpansive mappings obviously satisfy
(C) property of Suzuki [12]. Interestingly, an example in [12] (see also an example below) nicely shows that there exist many mappings in the class of Suzuki mappings, which do not belong to the class of nonexpansive mappings. Suzuki also extended the celebrated result of Browder [4] and Goehde [5] from the setting of nonexpansive mappings to the framework of Suzuki mappings.

New iterative methods for the investigation of fixed points and solution of functional equations is the busy research topic and has fruitful applications such as in image recovery and signal processing (see, e.g., [13–19] and others). Therefore, it is our purpose to construct a new iterative method for the larger class of nonexpansive mappings called Suzuki mappings. We also show by an example that this new iterative process gives better approximations as compared to other methods. Suppose \( M \) is a closed nonempty convex subset of a given Banach space, and assume further that \( \alpha_k, \beta_k, \gamma_k \in (0, 1), k \in \mathbb{N} \), and \( F \) is a self-map of \( M \).

The Mann iterative method [6] is defined as follows:

\[
\begin{cases}
  u_1 = u \in M, \\
  u_{k+1} = (1 - \alpha_k)u_k + \alpha_k Fu_k.
\end{cases}
\]  

The Ishikawa [7] iterative method is the extension of the Mann method from one step to two steps as follows:

\[
\begin{cases}
  u_1 = u \in M, \\
  v_k = (1 - \beta_k)u_k + \beta_k Fu_k, \\
  u_{k+1} = (1 - \alpha_k)u_k + \alpha_k Fv_k.
\end{cases}
\]  

The Noor [8] iterative method is the extension of both of the Mann and Ishikawa iterative methods as follows:

\[
\begin{cases}
  u_1 = u \in M, \\
  w_k = (1 - \gamma_k)u_k + \gamma_k Fu_k, \\
  v_k = (1 - \beta_k)u_k + \beta_k Fw_k, \\
  u_{k+1} = (1 - \alpha_k)u_k + \alpha_k Fv_k.
\end{cases}
\]  

Agarwal et al.’s [9] method is the slightly modification of the Ishikawa method as follows:

\[
\begin{cases}
  u_1 = u \in M, \\
  v_k = (1 - \beta_k)u_k + \beta_k Fu_k, \\
  u_{k+1} = (1 - \alpha_k)u_k + \alpha_k Fv_k.
\end{cases}
\]  

Abbas and Nazir’s [10] iterative method is a three-step method read as follows:

\[
\begin{cases}
  u_1 = u \in M, \\
  w_k = (1 - \gamma_k)u_k + \gamma_k Fu_k, \\
  v_k = (1 - \beta_k)u_k + \beta_k Fw_k, \\
  x_{k+1} = (1 - \alpha_k)Fv_k + \alpha_k Fu_k.
\end{cases}
\]  

Thakur et al. [11] proposed a new iterative method as follows:

\[
\begin{cases}
  u_1 = u \in M, \\
  w_k = (1 - \beta_k)u_k + \beta_k Fu_k, \\
  v_k = F((1 - \alpha_k)u_k + \alpha_k w_k), \\
  u_{k+1} = Fv_k.
\end{cases}
\]  

Thakur et al. [11] showed that method (8) is better than all of the methods, namely, Mann (3), Ishikawa (4), Noor (5), Agarwal (6), and Abbas and Nazir (7). Here, in the current research, we first suggest an efficient new iterative method and prove that it can be used for computations of fixed points of the larger class of nonexpansive maps called Suzuki maps. Furthermore, we shall provide a novel example of the so-called Suzuki mappings and prove that it exceeds the corresponding class of nonexpansive mappings.

2. Preliminaries

Here, first we present some earlier important definitions, which are needed for our theoretical outcome.

Let \( E \) be a given Banach space, and suppose \( \{u_k\} \subseteq E \) is weakly convergent to \( w \in E \) and satisfies the following:

\[ \lim \inf_{k \to \infty} \|u_k - w\| < \lim \inf_{k \to \infty} \|u_k - z\|, \quad \text{for all } z \in E - \{w\}. \]  

Whenever any weakly convergent sequence in \( E \) has the abovementioned property, \( E \) is called a Banach space endowed with Opial’s property (for details, see [20]). We now recall a property \( I \) introduced by Sentor and Dotson [21] for \( F: M \to M \) (where \( M \) is a nonempty subset of a Banach space). We recall that \( F \) has condition \( I \) [21] if one can find a nondecreasing function, namely, \( P: [0, \infty) \to [0, \infty) \), with the properties \( P(0) = 0 \), \( P(a) > 0 \) for every \( a > 0 \), and \( \|u - Fu\| \geq P(\text{dist}(u, \text{Fix}(F))) \) for all \( u \in M \).

Let \( M \) be any nonempty subset of a general Banach space \( E \), and suppose \( \{u_k\} \) is any given bounded sequence in \( E \). We fix \( u \in E \) and denote

(a) by \( \mathcal{R}(u, \{u_k\}) \), the asymptotic radius of \( \{u_k\} \) at \( u \) given by \( \mathcal{R}(u, \{u_k\}) = \limsup_{k \to \infty} \|u - u_k\| \)

(b) by \( \mathcal{R}(M, \{u_k\}) \), the asymptotic radius associated with \( M \) of \( \{u_k\} \) given by \( \mathcal{R}(M, \{u_k\}) = \inf \{\mathcal{R}(u, \{u_k\}) : u \in M\} \)

(c) by \( \mathcal{Z}(M, \{u_k\}) \), the asymptotic center associated with \( M \) of \( \{u_k\} \) given by \( \mathcal{Z}(M, \{u_k\}) = \{u \in E : \mathcal{R}(u, \{u_k\}) = \mathcal{R}(M, \{u_k\})\} \)

The most well-known fact about the set \( \mathcal{Z}(M, \{u_k\}) \) is that it is always singleton whenever \( X \) is UCBS [22]. The fact that the set \( \mathcal{Z}(M, \{u_k\}) \) is convex nonempty whenever \( M \) is weakly compact and convex is also well known (see, e.g., [23, 24]).

Lemma 1 (See [12]). Assume that \( M \) is any nonempty subset of a Banach space, and suppose \( F: M \to M \). If \( F \) is a Suzuki mapping, then for every element \( u \in M \) and for every element \( q \in \text{Fix}(F) \), the fact \( \|Fu - Fq\| \leq \|u - q\| \) holds.
Lemma 2 (See [12]). Assume that $M$ is any nonempty subset of a Banach space, and suppose $F : M \to M$. If $F$ is a Suzuki mapping, then for every two elements $u, v \in M$, we have the following property:

$$\|u - Fv\| \leq 3\|u - Fu\| + \|u - v\|. \quad (10)$$

The following result is known as the demiclosed principle.

Lemma 3 (See [12]). Assume that $M$ is any nonempty subset of a Banach space having the Opial property, and suppose $F : M \to M$. If $F$ is a Suzuki mapping, then the following condition holds:

$$\{u_k\} \subseteq M, \quad u_k \to w, \|u_k - Fu_k\| \to 0 \Rightarrow Fw = w. \quad (11)$$

The fixed-point set endowed with a Suzuki mapping enjoys the following properties.

Lemma 4 (See [12]). Assume that $M$ is any nonempty subset of a Banach space, and suppose $F : M \to M$. If $F$ is a Suzuki mapping, then the following useful lemma can be found in [25].

Let $\alpha, \beta > 0$, and $\alpha \beta < 1$. Suppose $\{u_k\}$ is a sequence in $M$, and suppose $\lim_{k \to \infty} \|u_k - u\| = 0$. Then,

$$\lim_{k \to \infty} \|u_k - Fu_k\| = 0.$$ 

Lemma 5. Let $0 < u \leq y_k \leq v < 1$ for each $k \in \mathbb{N}$ and $\eta \geq 0$. If $\{y_k\}$ and $\{z_k\}$ are any sequences in a UCBS, $E$ with $\limsup_{k \to \infty} \|y_k\| \leq \eta$, $\limsup_{k \to \infty} \|z_k\| \leq \eta$, and $\lim_{k \to \infty} \|y_k - z_k\| = 0$. Then, $\lim_{k \to \infty} \|y_k - z_k\| = 0$.

3. Main Results

Strongly motivated by those mentioned above, we introduce a new iterative process, namely, JK iteration, as follows:

$$u_1 = u, \quad w_k = (1 - \beta_k)u_k + \beta_k Fu_k, \quad v_k = Fu_k, \quad u_{k+1} = F((1 - \alpha_k)w_k + \alpha_k Fv_k), \quad k \in \mathbb{N},$$

where $\alpha_k, \beta_k \in (0, 1)$.

In the present research section, we establish very interesting and important results for the larger class of the so-called Suzuki maps under the newly suggested method (12). We will present a numerical example to show that the JK iterative process is better than the iterative process by Thakur et al. (8). Furthermore, in the last section, a novel example of the so-called Suzuki maps which is not non-expansive shows that Suzuki maps properly include non-expansive maps. The numerical observations suggest that JK iterative method is better than the leading method of Thakur and, hence, many others.

We now state and prove a much needed lemma for our main outcome, which will play a significant role in each result of the sequel.

Lemma 6. Assume that $M$ is any nonempty closed convex subset of a Banach space $X$, and suppose $F : M \to M$ is a Suzuki mapping with $\text{Fix}(F) \neq \emptyset$. Suppose $\{u_k\}$ is a sequence given in (12). Then, $\lim_{k \to \infty} \|u_k - q\|$ exists for every fixed point $q$ of $F$.

Proof. Take $q \in \text{Fix}(F)$. By Lemma 1, we have

$$\|u_k - q\| = \|(1 - \beta_k)u_k + \beta_k Fu_k - q\| \leq (1 - \beta_k)\|u_k - q\| + \beta_k\|Fu_k - q\| \leq \|u_k - q\| + \beta_k\|Fu_k - q\| \leq \|u_k - q\|, (13)$$

$$\|v_k - q\| = \|Fu_k - q\| \leq \|u_k - q\|.$$  

They imply that

$$\|u_{k+1} - q\| = \|F((1 - \alpha_k)w_k + \alpha_k Fv_k) - q\| \leq (1 - \alpha_k)\|w_k - q\| + \alpha_k\|Fv_k - q\| \leq (1 - \alpha_k)\|w_k - q\| + \alpha_k\|v_k - q\| \leq (1 - \alpha_k)\|w_k - q\| + \alpha_k\|w_k - q\| = \|u_k - q\|.$$  

From the equations mentioned above, we conclude that $\|u_k - q\|$ is a bounded and nonincreasing sequence of reals, and hence, $\lim_{k \to \infty} \|u_k - q\|$ exists for every fixed point $q$ of $F$.

Theorem 1. Assume that $M$ is any nonempty closed convex subset of a UCBS, and suppose $F : M \to M$ is a Suzuki mapping. Assume further that $\{u_k\}$ is a sequence given in (12). Then, $\text{Fix}(F) \neq \emptyset$ if and only if $\{u_k\}$ is bounded, and $\lim_{k \to \infty} \|Fu_k - u_k\| = 0$.

Proof. First, we assume that $\{u_k\}$ is bounded and $\lim_{k \to \infty} \|Fu_k - u_k\| = 0$. We shall prove that $\text{Fix}(F) \neq \emptyset$. For this, let $q \in \mathcal{Z}(M, \{u_k\})$. By Lemma 2, we have

$$\mathcal{R}(Fq, \{u_k\}) \leq \lim_{k \to \infty} \|u_k - q\| \leq 3 \lim_{k \to \infty} \|u_k - Fu_k\| + \lim_{k \to \infty} \|u_k - q\| = \lim_{k \to \infty} \|u_k - q\| = \mathcal{R}(q, \{u_k\}).$$  

It follows that $Fq \in \mathcal{Z}(M, \{u_k\})$. Since in UCBS, asymptotic centers are singleton, we have $Fq = q$. Hence, the fixed point is nonempty.
Conversely, we assume that \( \text{Fix}(F) \neq \emptyset \). Conclusions of Lemma 6 provide that \( \{u_k\} \) is bounded and \( \lim_{k \to \infty} \|u_k - q\| \) exists for every fixed point \( q \) of \( F \). Now, if
\[
\lim_{n \to \infty} \|u_n - q\| = \eta, \tag{16}
\]
then by observing the proof of Lemma 6 and keeping (16) in mind, we obtain
\[
\limsup_{k \to \infty} \|u_k - q\| \leq \limsup_{n \to \infty} \|u_n - q\| = \eta. \tag{17}
\]

Applying Lemma 1, we get
\[
\limsup_{k \to \infty} \|Fu_k - q\| \leq \limsup_{k \to \infty} \|u_k - q\| = \eta, \tag{18}
\]
and by observing the proof of Lemma 6, we see
\[
\|u_{k+1} - q\| \leq \|w_k - q\|. \tag{19}
\]
It gives, together with (16),
\[
\eta \leq \liminf_{k \to \infty} \|w_k - q\|. \tag{20}
\]
From (17) and (20), we obtain
\[
\eta = \lim_{k \to \infty} \|w_k - q\|. \tag{21}
\]
From (21), we have
\[
\eta = \lim_{k \to \infty} \|w_k - q\|,
= \lim_{k \to \infty} \|(1 - \beta_k)u_k + \beta_k Fu_k - q\|
= \lim_{k \to \infty} \|(1 - \beta_k)(u_k - q) + \beta_k(Fu_k - q)\|. \tag{22}
\]
Hence,
\[
\eta = \lim_{k \to \infty} \|(1 - \beta_k)(u_k - q) + \beta_k(Fu_k - q)\|. \tag{23}
\]
Now, from (16), (18) and (23) together with Lemma 5, we obtain
\[
\lim_{k \to \infty} \|Fu_k - u_k\| = 0. \tag{24}
\]

Now, we are in the position to prove our weak convergence result.

**Theorem 2.** Assume that \( E \) is a UCBS with Opial’s property and \( M \) is a nonempty convex closed subset of \( E \), and suppose \( F: M \to M \) be a Suzuki mapping with Fix\((F) \neq \emptyset \). Suppose \( \{u_k\} \) is a sequence given in (12). Then, \( \{u_k\} \) converges weakly to a fixed point of \( F \).

**Proof.** By Theorem 1, \( \{u_k\} \) is bounded and \( \lim_{k \to \infty} \|Fu_k - u_k\| = 0 \). Since \( E \) is uniformly convex, \( E \) is reflexive. Hence, one can easily find a subsequence, namely, \( \{u_{k_j}\} \) of \( \{u_k\} \) such that \( u_{k_j} \to u \) for some \( u \in M \). By Lemma 3, \( u \in \text{Fix}(F) \). We shall prove that \( u \) is the weak limit of \( \{u_k\} \). Let \( u \) not be the weak limit of \( \{u_k\} \). Then, one can find another subsequence, namely, \( \{u_{k_m}\} \) of \( \{u_k\} \) such that \( u_{k_m} \to v \) and \( v \neq u \). Again by Lemma 3, \( v \in \text{Fix}(F) \). Now, using Lemma 6 and Opial’s property, we have
\[
\lim_{k \to \infty} \|u_k - u\| \leq \lim_{l \to \infty} \|u_k - u\| < \lim_{n \to \infty} \|u_{k_n} - v\|,
= \lim_{m \to \infty} \|u_{k_m} - v\|
= \lim_{n \to \infty} \|u_k - u\|. \tag{25}
\]
Hence, \( \lim_{k \to \infty} \|u_k - u\| < \lim_{k \to \infty} \|u_k - v\| \), clearly a contradiction, and so we must accept that \( u \) is the only weak limit of \( \{u_k\} \).

Now, we prove the following strong convergence result.

**Theorem 3.** Assume that \( M \) is any nonempty convex compact subset of a UCBS, and suppose \( F: M \to M \) be a Suzuki mapping. Assume further that \( \{u_k\} \) is a sequence given in (12). Then, \( \{u_k\} \) converges strongly to a fixed point of \( F \).

**Proof.** From Theorem in [12], we can write Fix\((F) \neq \emptyset \). By Theorem 1, \( \lim_{k \to \infty} \|Fu_k - u_k\| = 0 \). Since the domain \( M \) is compact, one can easily find a strongly convergent subsequence, namely, \( \{u_{k_j}\} \) of \( \{u_k\} \) having a limit say \( z \). By using Lemma 2, the following holds:
\[
\|u_{k_j} - Fz\| \leq 3 \cdot \|u_{k_j} - Fu_{k_j}\| + \|u_{k_j} - z\|. \tag{26}
\]
Hence, \( u_{k_j} \to Fz \) whenever \( j \to \infty \), so the uniqueness of limits follows \( Fz = z \). By Lemma 6, \( \lim_{k \to \infty} \|u_k - z\| \) exists. Hence, \( z \) is the strong limit of \( \{u_k\} \).

The proof of the following theorem is elementary and, therefore, omitted.

**Theorem 4.** Assume that \( M \) is any nonempty closed convex subset of a UCBS, and suppose \( F: M \to M \) be a Suzuki mapping. If Fix\((F) \neq \emptyset \) and \( \liminf_{k \to \infty} \text{dist}(u_k, \text{Fix}(F)) = 0 \), where \( \{u_k\} \) is a sequence given in (12), then \( \{u_k\} \) converges strongly to a fixed point of \( F \).

We finish this section with a strong convergence theorem under the condition I.

**Theorem 5.** Assume that \( M \) is any nonempty convex closed subset of a UCBS, and suppose \( F: M \to M \) be a Suzuki mapping with Fix\((F) \neq \emptyset \). Assume further that \( \{u_k\} \) is a sequence given in (12). If \( F \) fulfils condition (I), then \( \{u_k\} \) converges strongly to a fixed point of \( F \).

**Proof.** In view of Theorem 1, we can conclude that \( \liminf_{k \to \infty} \|Fu_k - u_k\| = 0 \). Since \( F \) fulfils condition (I), one has \( \liminf_{k \to \infty} \text{dist}(u_k, \text{Fix}(F)) = 0 \). The conclusions are now clear from Theorem 4.
Table 1: Sequences defined by JK, Thakur, and Agarwal iterative methods for mapping $F$ defined in Example 1.

| $k$ | JK (12)          | Thakur (8)      | Agarwal (6)    |
|-----|------------------|-----------------|----------------|
| 1   | 0.9              | 0.9             | 0.9            |
| 2   | 0.9982071573     | 0.9965059644    | 0.9825298221   |
| 3   | 0.9999814317     | 0.9998861029    | 0.9971525733   |
| 4   | 0.9999999265     | 0.9999964422    | 0.9995552826   |
| 5   | 1                | 0.999998920     | 0.9999325257   |
| 6   | 1                | 0.999999967     | 0.9999899750   |
| 7   | 1                | 0.999999999     | 0.9999985340   |
| 8   | 1                | 1               | 0.999997883    |
| 9   | 1                | 1               | 0.999999697    |
| 10  | 1                | 1               | 0.999999957    |
| 11  | 1                | 1               | 0.999999994    |
| 12  | 1                | 1               | 0.999999999    |
| 13  | 1                | 1               | 1              |

Figure 1: Convergence behavior of JK (12), Thakur (8), and Agarwal (6) iterates to converge to $q = 1$, a unique fixed point of the self-map $F$ in Example 1 where $u_1 = 0.9$.

Table 2: $\alpha_k = ((k + 1)/(5k + 1))^{1/11}$ and $\beta_k = k/\sqrt{2k + 3}$.

| $u_1$ | $S$ | Thakur | JK |
|-------|-----|--------|----|
| 0.10  | 20  | 11     | 8  |
| 0.15  | 20  | 11     | 8  |
| 0.20  | 20  | 11     | 8  |
| 0.50  | 20  | 10     | 8  |
| 0.80  | 19  | 10     | 8  |
| 0.90  | 19  | 10     | 7  |

Table 3: $\alpha_k = k/(6k + 7)$ and $\beta_k = 1/(2k + 3)^{1/5}$.

| $u_1$ | $S$ | Thakur | JK |
|-------|-----|--------|----|
| 0.10  | 21  | 11     | 9  |
| 0.15  | 21  | 11     | 9  |
| 0.20  | 21  | 11     | 9  |
| 0.50  | 21  | 11     | 9  |
| 0.80  | 21  | 11     | 9  |
| 0.90  | 20  | 10     | 9  |
Select Case 2.

Consider a closed convex subset of a Banach space. We suggest that the novel JK scheme is far better than the earlier iterative methods using this example. Since we are using Suzuki maps in our work, the provided outcome holds simultaneously for nonexpansive maps as well.

4. Numerical Example

This section introduces a novel example of self-Suzuki maps on a closed convex bounded subset of a Banach space. We suggest many different cases that the novel JK scheme is far better than the earlier iterative methods using this example. Since we are using Suzuki maps in our work, the provided outcome holds simultaneously for nonexpansive maps as well.

Example 1. Consider a closed convex $M = [0, 1]$ of a Banach space $\mathbb{R}$. Set a self-map $F: M \rightarrow M$ as follows:

$$Fu = \begin{cases} 1 - u, & \text{if } u < \frac{1}{8} \\ \frac{u + 4}{5}, & \text{if } u \geq \frac{1}{8} \end{cases}$$

(27)

One can conclude that $F$ is a Suzuki mapping and, however, not nonexpansive by studying the computations given below. Select $u = 8/65$ and $v = 1/8$, and observe that

$$\|Fu - Fv\| = \|Fu - Fv\| = |u - v| = \frac{1}{65} - \frac{33}{40},$$

(28)

which proves that $F$ is not a nonexpansive on $M$.

Next, we suggest the proof of the Suzuki property of $F$ on $M$. The proof can be divided as given below.

Case 1. Select $u \in [0, (1/8))$; then, $(1/2)\|u - v\| = ((1 - 2u)/2) \in ((3/8), (1/2))$. For $(1/2)\|u - Fu\| \leq \|u - v\|$, one has

$$((1 - 2u)/2) \leq v - u, \quad i.e., \quad (1/2) \leq v; \quad \text{therefore}, \quad v \in [(1/2), 1].$$

So, one has

$$\|Fu - Fv\| = \left|\frac{v + 4}{5} - (1 - u)\right| = \frac{v + 4u - 1}{5} \leq \frac{1}{8},$$

(29)

$$\|u - v\| = |u - v| = \frac{1}{8} - \frac{1}{2} = \frac{3}{8},$$

Hence, $(1/2)\|u - Fu\| \leq \|u - v\| \Rightarrow \|Fu - Fv\| \leq \|u - v\|.$

Case 2. Select $u \in [(1/8), 1]$; then, $(1/2)\|u - Fu\| = (1/2)((u + 4)/5) - u = ((4 - 4u)/10) \in [0, (7/20)].$

For $(1/2)\|u - Fu\| \leq \|u - v\|$, one has

$$((4 - 4u)/10) \leq |v - u|,$$

and so the possible following situations occur:

(a) Whenever $u < v$, $((4 - 4u)/10) \leq v - u \Rightarrow v \geq ((4 + 6u)/10) \Rightarrow v \in [(19/40), 1] \subset [(1/8), 1]$. So,

$$\|Fu - Fv\| = \left|\frac{u + 4}{5} - v - 4\right| = \frac{1}{5}|u - v| \leq \|u - v\|.$$

(30)

Therefore,

$$(1/2)\|u - Fu\| \leq \|u - v\| \Rightarrow \|Fu - Fv\| \leq \|u - v\|.$$

(b) Whenever $u > v$, $((4 - 4u)/10) \leq u - v \Rightarrow v \leq ((4 - 4u)/10) = ((14u - 4)/10) \Rightarrow u \in [-9/40, 1]$. Since $v \in [0, 1], v \leq ((14u - 4)/10) \Rightarrow u \in [4/14, 1].$ So, the case is $u \in [4/14, 1]$ and $v \in [0, 1]$. When $u \in [4/14, 1]$ and $v \in [(1/8), 1]$ are already included in (a), we assume $u \in [4/14, 1]$ and $v \in [0, (1/8)];$

then,

$$\|Fu - Fv\| = \left|\frac{u + 4}{5} - (1 - v)\right| = \frac{|u + 5v - 1|}{5}.$$
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