A New Reduction from Search SVP to Optimization SVP

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Abstract
It is well known that search SVP is equivalent to optimization SVP. However, the former reduction from search SVP to optimization SVP by Kannan needs polynomial times calls to the oracle that solves the optimization SVP. In this paper, a new rank-preserving reduction is presented with only one call to the optimization SVP oracle. It is obvious that the new reduction needs the least calls, and improves Kannan’s classical result. What’s more, the idea also leads a similar direct reduction from search CVP to optimization CVP with only one call to the oracle.

Keywords: Search SVP, Optimization SVP, Lattice, Reduction.

1 Introduction
Given a matrix $B = (b_{ij}) \in \mathbb{R}^{m \times n}$ with rank $n$, the lattice $L(B)$ spanned by the columns of $B$ is

$$L(B) = \{ \sum_{i=1}^{n} x_i b_i | x_i \in \mathbb{Z} \},$$

where $b_i$ is the $i$-th column of $B$. Lattice has many important applications in cryptography. The shortest vector problem (SVP) and the closest vector problem (CVP) are two of the most famous problems of lattice.

SVP refers to find the shortest non-zero vector in a given lattice. There are three different variants of SVP:

1. Search SVP: Given a lattice basis $B \in \mathbb{Z}^{m \times n}$, find $v \in L(B)$ such that $\|v\| = \lambda_1(L(B))$, where $\lambda_1(L(B))$ is the length of the shortest non-zero vector in $L(B)$.

2. Optimization SVP: Given a lattice basis $B \in \mathbb{Z}^{m \times n}$, find $\lambda_1(L(B))$.

3. Decisional SVP: Given a lattice basis $B \in \mathbb{Z}^{m \times n}$ and a rational $r \in \mathbb{Q}$, decide whether $\lambda_1(L(B)) \leq r$ or not.
It has been proved that the three problems are equivalent to each other (see [2]). It is easy to check that the decisional SVP is as hard as the optimization SVP and the optimization variant can be reduced to the search variant.

In 1987, Kannan [1] also showed that the search variant can be reduced to the optimization variant. The basic idea of his reduction is to recover the integer coefficients of some shortest vector under the given lattice basis by introducing small errors to the original lattice basis. However, his reduction is a bit complex. It needs to call polynomial times optimization SVP oracle, since it could not determine the signs of the shortest vector’s entries at one time. It also needs oracle to solve optimization SVP for some lattices with lower rank besides with the same rank as the original lattice.

In this paper, we propose a new rank-preserving reduction which can solve the search SVP with only one call to the optimization SVP oracle. It is obvious that there is no reduction with less calls than ours. Instead of recovering the shortest vector directly as in [1], we first recover the integer coefficients of some shortest vector under the given lattice basis, then recover the shortest vector.

A similar direct reduction from search CVP to optimization CVP with only one call also holds whereas some popular reductions [2, 3] usually takes decisional CVP to bridge the search CVP and optimization CVP. The former reduction from decisional CVP to optimization CVP needs one call to the optimization CVP oracle, but it needs polynomial times calls to the decisional CVP oracle to reduce search CVP to decisional CVP.

2 The New Reduction

For simplicity, we just give the new reduction for the full rank lattice, i.e. $n = m$, as in [1]. It is easy to general the new reduction for the lattices with rank $n < m$.

2.1 Some Notations

Given a lattice basis $B = (b_{ij}) \in \mathbb{R}^{n \times n}$, let $M(B) = \max |b_{ij}|$. For lattice $L(B)$, we define its SVP solution set $S_B$ as:

$$S_B = \{ x \in \mathbb{Z}^n \mid \|Bx\| = \lambda_1(L(B)) \}$$

Denote by $\text{poly}(n)$ the polynomial in $n$.

2.2 Some Lemmas

We need some lemmas to prove our main theorem.

Lemma 1. For every positive integer $n$, there exist $n$ positive integers $a_1 < a_2 < \ldots < a_n$ s.t. all the $a_i + a_j (i \leq j)$’s are distinct and $a_n$ is bounded by $\text{poly}(n)$. 

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Hence \(\det(\ell_1)\) for \(k = 1, 2, \ldots, n\). Suppose \(a_i + a_j = a_{i_2} + a_{j_2}\) for some \(i_1, j_1, i_2, j_2\), we get \((i_1 - 1)^2 + (j_1 - 1)^2 + 2n^2((i_1 - 1) + (j_1 - 1)) = (i_2 - 1)^2 + (j_2 - 1)^2 + 2n^2((i_2 - 1) + (j_2 - 1))\). Since \((i_1 - 1)^2 + (j_1 - 1)^2, (i_2 - 1)^2 + (j_2 - 1)^2 < 2n^2\), we have \((i_1 - 1)^2 + (j_1 - 1)^2 = (i_2 - 1)^2 + (j_2 - 1)^2\) and \(i_1 + j_1 = i_2 + j_2\), which leads \(\{i_1, j_1\} = \{i_2, j_2\}\). Hence all the \(a_i + a_j (i \leq j)\)'s are distinct. It is obvious that \(a_n \leq (n^2 + n - 1)^2\).

\[\]

**Lemma 2.** Given positive odd integer \(p > 2\), and any positive integer \(n\), which satisfies \(n = \sum_{i=0}^{k} n_ip^i\) where \(|n_i| \leq \lfloor p/2 \rfloor\), then we can recover the coefficients \(n_i\)'s in polynomial time.

**Proof.** We can recover \(n_0\) by computing \(a \equiv n \mod p\) and choose \(a\) in the interval from \(-\lfloor p/2 \rfloor\) to \(\lfloor p/2 \rfloor\). After obtaining \(n_0\), we get another integer \((n - n_0p^0)/p\). Recursively, we can recover all the coefficients. This can be done in polynomial time obviously.

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**Lemma 3.** For bivariate polynomial \(f(x, y) = xy\), given any lattice basis matrix \(B \in \mathbb{Z}^{n \times n}\), \(\lambda_1(L(B))\) has an upper bound \(f(M, n)\), where \(M = M(B)\). What’s more, for every \(x \in S_B\), \(|x_i| (i = 1, \ldots, n)\) has an upper bound \(f(M^n, n^n)\).

**Proof.** The length of any column of \(B\) is an upper bound of \(\lambda_1(L(B))\), so \(\lambda_1(L(B)) \leq n^{1/2}M \leq nM\).

For \(x \in S_B\), we let \(y = Bx\), then \(\|y\| = \lambda_1(L(B)) \leq \sqrt{n}M\). By Cramer’s rule, we know that

\[
x_i = \frac{\det(B^{(i)})}{\det(B)},
\]

where \(B^{(i)}\) is formed by replacing the \(i\)-th column of \(B\) by \(y\). By Hadamard’s inequality, \(|\det(B^{(i)})| \leq n^{n/2}M^n \leq n^nM^n\). We know \(|\det(B)| \geq 1\) since \(\det(B)\) is a non-zero integer. Hence \(|x_i| \leq n^nM^n\).

\[\]

2.3 The Main Theorem

**Theorem 1.** Assume there exists an oracle \(O\) that can solve the optimization SVP for any lattice \(L(B')\) with basis \(B' \in \mathbb{Z}^{n \times n}\), then there is an algorithm that can solve the search SVP for any lattice \(L(B)\) with basis \(B \in \mathbb{Z}^{n \times n}\) with only one call to \(O\) in \(\text{poly}(\log_2 M, n, \log_2 n)\) time, where \(M = M(B)\).

**Proof.** The main steps of the algorithm are as below:

1. Constructing a new lattice basis \(B_\epsilon \in \mathbb{Z}^{n \times n}\).

   We construct \(B_\epsilon\) from the original lattice \(B\):

   \[
   B_\epsilon = \epsilon_{n+1}B + \begin{pmatrix}
   \epsilon_1 & \epsilon_2 & \ldots & \epsilon_n \\
   0 & 0 & \ldots & 0 \\
   \vdots & \vdots & \ddots & \vdots \\
   0 & 0 & \ldots & 0
   \end{pmatrix}
   \]

   Where \(\epsilon_{n+1}\) is non-zero.

   Then we let \(\epsilon_{n+1} = 1\), where \(\lambda_1(L(B_\epsilon)) \leq \sqrt{n}M\) and \(\lambda_1(L(B)) \leq \lambda_1(L(B_\epsilon)) \leq \sqrt{n}M\).
where the $\epsilon_i$ will be determined as below.

For any $x \in \mathbb{Z}^n$, we define $c(x) = \sum_{i=1}^{n} b_i x_i$. For $x \in S_B$, by Lemma 3 $|x_i|$ has an upper bound $f(M^n, n^n)$. Let $M_1 = 2f((M + 1)^n, n^n)$. In addition, $\|Bx\| = \lambda_1(L(B))$ is bounded by $f(M, n)$. Let $M_2 = f(M + 1, n)$. $|c(x)|$ is also bounded by $M_2$ since $|c(x)| \leq \|Bx\|$. We let

$$p = 2 \cdot \max \{ M_2^2, 2M_1M_2, 2M_2^2 \} + 1.$$

By Lemma 1, we can choose $n + 1$ positive integers $a_1 < a_2 < \ldots < a_{n+1}$, such that all the $a_i + a_j (i \leq j)$'s are distinct where $a_{n+1}$ is bounded by poly$(n)$. Let

$$\epsilon_i = \epsilon_i^n.$$

We first show that $|\det (\frac{1}{\epsilon_{n+1}} B_\epsilon)| \geq \frac{1}{2}$, so $B_\epsilon$ is indeed a lattice basis. Notice that

$$\det (\frac{1}{\epsilon_{n+1}} B_\epsilon) = \det (B) + \sum_{i=1}^{n} \epsilon_i \alpha_i \epsilon_{n+1},$$

where $\alpha_i$ is the cofactor of $B_{1i}$ in $B$. Since $\frac{\epsilon_i}{\epsilon_{n+1}} \leq \frac{1}{p^2}$ and $|\alpha_i| \leq M^{n-1} (n-1)^{n-1}$, $\sum_{i=1}^{n} \alpha_i \frac{\epsilon_i}{\epsilon_{n+1}} \leq \frac{1}{p^2} M^{n-1} n^n < \frac{1}{2}$. By the fact $\det(B)$ is a non-zero integer, we get

$$|\det (\frac{1}{\epsilon_{n+1}} B_\epsilon)| \geq \frac{1}{2}. \quad (1)$$

We claim that $S_{B_\epsilon} \subseteq S_B$. Since $S_{B_\epsilon} = S_{\frac{1}{\epsilon_{n+1}} B_\epsilon}$, it is enough to prove $S_{\frac{1}{\epsilon_{n+1}}B_\epsilon} \subseteq S_B$.

For any $x \in S_{\frac{1}{\epsilon_{n+1}}B_\epsilon}$, by (1) and the proof of Lemma 3, we know that $|x_i| \leq M_1$, $|c(x)| \leq M_2$.

By the choice of $p$, $x_i^2, 2c(x)x_i, 2x_i x_j$ are in the interval $[-|p/2|, |p/2|]$. Together with the fact that $\frac{\epsilon_i \epsilon_j}{\epsilon_{n+1}} (i \leq j)$’s are different powers of $p$, we have

$$\lambda_1(L(\frac{1}{\epsilon_{n+1}} B_\epsilon))^2 = \|\frac{1}{\epsilon_{n+1}} B_\epsilon x\|^2$$

$$= \|Bx\|^2 + \sum_{i=1}^{n} x_i^2 (\frac{\epsilon_i}{\epsilon_{n+1}})^2 + \sum_{i=1}^{n} 2c(x)x_i \frac{\epsilon_i}{\epsilon_{n+1}} + \sum_{i<j} 2x_i x_j \frac{\epsilon_i \epsilon_j}{\epsilon_{n+1}}.$$ \quad (2)

Similarly, for any $y \in S_B$, we have

$$\|\frac{1}{\epsilon_{n+1}} B_\epsilon y\|^2 = \|By\|^2 + \sum_{i=1}^{n} y_i^2 (\frac{\epsilon_i}{\epsilon_{n+1}})^2 + \sum_{i=1}^{n} 2c(y)y_i \frac{\epsilon_i}{\epsilon_{n+1}} + \sum_{i<j} 2y_i y_j \frac{\epsilon_i \epsilon_j}{\epsilon_{n+1}}$$

$$< \lambda_1(L(B))^2 + (|p/2| + 1) \frac{\epsilon_n}{\epsilon_{n+1}}. \quad (3)$$

Next, we prove $S_{\frac{1}{\epsilon_{n+1}}B_\epsilon} \subseteq S_B$. Suppose there exists $x \in S_{\frac{1}{\epsilon_{n+1}}B_\epsilon}$ but $x \notin S_B$, then

$$\|Bx\|^2 \geq \lambda_1(L(B))^2 + 1. \quad (4)$$
Notice that $\frac{\varepsilon_n}{\varepsilon_{n+1}} < \frac{1}{p^2}$, we have $0 < ([p/2] + 1)\frac{\varepsilon_n}{\varepsilon_{n+1}} < \frac{1}{2}$. Together with (2), (3) and (4), we have

$$\lambda_1(L(\frac{1}{\varepsilon_{n+1}}B_e))^2 > \|Bx\|^2 - ([p/2] + 1)\frac{\varepsilon_n}{\varepsilon_{n+1}} \geq \lambda_1(L(B))^2 + 1 - ([p/2] + 1)\frac{\varepsilon_n}{\varepsilon_{n+1}} > \|\frac{1}{\varepsilon_{n+1}}B_y\|^2,$$

which is a contradiction, since $\frac{1}{\varepsilon_{n+1}}B_y \in L(\frac{1}{\varepsilon_{n+1}}B_e)$. Hence $S_{B_e} \subseteq S_B$.

(2) Querying the oracle $O$ with $B_e$ once, we get $\lambda_1(L(B))$.

So there exists $x = (x_1, \ldots, x_n)^T \in S_{B_e} \subseteq S_B$, such that

$$\|Bx\|^2\varepsilon_{n+1}^2 + \sum_{i=1}^nx_i^2\varepsilon_i^2 + \sum_{i=1}^n2c(x)x_i\varepsilon_{n+1}\varepsilon_i + \sum_{i<j}2x_ix_j\varepsilon_i\varepsilon_j = \lambda_1(L(B_e))^2$$

(3) Recovering all the $x_i$'s and output $Bx$.

Since $x \in S_B$, every coefficient $\|Bx\|^2, x_i^2, 2c(x)x_i, 2x_ix_j$ is in the interval $[-[p/2], [p/2]]$ and $\varepsilon_i\varepsilon_j (i \leq j)$'s are different powers of $p$. Hence, $\log_2 (\lambda_1(L(B_e)))$ is bounded by $\text{poly}(\log_2 M, n, \log_2 n)$. Furthermore, by Lemma 2, we can recover all the coefficients in $\text{poly}(\log_2 M, n, \log_2 n)$ time. Especially, we can recover all $x_i^2$ and $x_ix_j(i \neq j)$. Let $k = \min\{i| x_i \neq 0\}$. We fix $x_k = \sqrt{x_k^2} > 0$, and can recover all the remaining $x_j = \text{sign}(x_kx_j)\sqrt{x_j^2}$ according to $x_j^2$ and $x_kx_j(k \neq j)$.

It is easy to check that the complexity of every step is bounded by $\text{poly}(\log_2 M, n, \log_2 n)$. □

**Remark 1.** For any search CVP instant $(B, t)$, given an oracle which can solve the optimization CVP, we can call the oracle with $(B_e, \varepsilon_{n+1}t)$ only once to solve the search CVP similarly.

### 3 Conclusions

In this paper, we give a new reduction from search SVP to optimization SVP with only one call, which is the least, to the optimization SVP oracle. A similar result for CVP also holds. However, it seems hard to apply the idea for GapSVP or GapCVP, since the new reduction is also sensitive to the error.

### References

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