Harnack Inequalities for $G$-SDEs with Multiplicative Noise *

Fen-Fen Yang
Center for Applied Mathematics, Tianjin University, Tianjin 300072, China
yangfenfen@tju.edu.cn
March 18, 2022

Abstract

The Harnack and log Harnack inequalities for stochastic differential equation driven by $G$-Brownian motion with multiplicative noise are derived by means of coupling by change of measure, which extend the corresponding results derived in [20] under the linear expectations. Moreover, we generalize the gradient estimate under nonlinear expectations appeared in [14].

Keywords: Harnack inequality; gradient estimate; multiplicative noise; $G$-Brownian motion; SDEs.

1 Introduction

For the extensive applications in strong Feller property, uniqueness of invariant probability measures, functional inequalities, and heat kernel estimates, Wang’s Harnack inequality has been developed [20]. To establish Harnack inequality, Wang introduced the coupling by change of measures, see [1, 18, 19] and references within for details. However, up to now, most of these papers only focus on the case of linear expectation spaces. Song [14] firstly derived the gradient estimates for nonlinear diffusion semigroups by using the method of Wang’s coupling by change of measure, after Peng [10, 11] established the systematic theory of $G$-expectation theory, $G$-Brownian motion and stochastic differential equations driven by $G$-Brownian motion ($G$-SDEs, in short). Subsequently, Yang [21] generalized the theory of Wang’s Harnack inequality and its applications to nonlinear expectation framework, where the noise is additive. Moreover, Wang’s Harnack inequality and gradient estimates are also

*Supported in part by NNSFC (11801403, 11801406).
proved for the degenerate (functional) case in [6]. An interesting question is whether it can be
generalized to the form of multiplicative noise. The answer is positive as some of the results
are showed in [14], whereas neither the form of G-SDEs with the term of \( d\langle B^i, B^j \rangle_t \)
(or the Harnack inequality studied, where \( B_t \) is a \( d \)-dimensional G-Brownian motion, and \( \langle B^i, B^j \rangle_t \)
stands for the mutual variation process of the \( i \)-th component \( B^i_t \) and the \( j \)-th component \( B^j_t \). In this paper, we will improve and extend the above assertions to the multiplicative
noise. Consider the following G-SDE

\[
(1.1) \quad dX_t = b(t, X_t)dt + \sum_{i,j=1}^{d} h_{ij}(t, X_t) d\langle B^i, B^j \rangle_t + \sum_{i=1}^{d} \sigma_i(t, X_t) dB^i_t,
\]

where \( b, h_{ij} = h_{ji} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d \). We aim to establish the
Harnack inequality for the G-SDE (1.1). In addition, we also prove the gradient estimate.
To this end, we firstly recall some basic facts on the G-expectation and G-Brownian motion.

For a positive integer \( d \), let \((\mathbb{R}^d, \langle \cdot, \cdot \rangle, | \cdot |)\) be the \( d \)-dimensional Euclidean space, \( \mathbb{S}^d \) the
collection of all symmetric \( d \times d \)-matrices. For any fixed \( T > 0 \),

\[
\Omega_T = \{ \omega | [0, T] \ni t \mapsto \omega_t \in \mathbb{R}^d \text{ is continuous with } \omega(0) = 0 \}
\]

endowed with the uniform form. Let \( B_t(\omega) = \omega_t, \omega \in \Omega_T, \) be the canonical process. Set

\[
L_{lip}(\Omega_T) = \{ \varphi(B_{t_1}, \ldots, B_{t_n}), n \in \mathbb{N}, t_1, \ldots, t_n \in [0, T], \varphi \in C_{b,lip}(\mathbb{R}^d \otimes \mathbb{R}^n) \},
\]

where \( C_{b,lip}(\mathbb{R}^d \otimes \mathbb{R}^n) \) denotes the set of bounded Lipschitz functions. Let \( G : \mathbb{S}^d \to \mathbb{R} \)
be a monotonic, sublinear and homogeneous function; see e.g. [12, p16]. Now we give the
construction of G-expectation which is also used in [13]. For any \( \xi \in L_{lip}(\Omega_T), \) i.e.,

\[
\xi(\omega) = \varphi(\omega(t_1), \ldots, \omega(t_n)), \quad 0 = t_0 < t_1 < \cdots < t_n = T,
\]

the conditional G-expectation is defined by

\[
\mathbb{E}_t[\xi] := u_k(t, \omega(t); \omega(t_1), \ldots, \omega(t_{k-1})), \quad \xi \in L_{lip}(\Omega_T), \quad t \in [t_{k-1}, t_k), \quad k = 1, \ldots, n,
\]

where \((t, x) \mapsto u_k(t, x; x_1, \ldots, x_{k-1}), \) \( k = 1, \ldots, n, \) solves the following G-heat equation

\[
(1.2) \quad \begin{cases}
\partial_t u_k + G(\partial^2_x u_k) = 0, \quad (t, x) \in [t_{k-1}, t_k) \times \mathbb{R}^d, \quad k = 1, \ldots, n, \\
u_k(t_k, x; x_1, \ldots, x_{k-1}) = u_{k+1}(t_k, x; x_1, \ldots, x_{k-1}, x_k), \quad k = 1, \ldots, n-1, \\
u_n(t_n, x; x_1, \ldots, x_{n-1}) = \varphi(x_1, \ldots, x_{n-1}, x), \quad k = n.
\end{cases}
\]

The corresponding G-expectation of \( \xi \) is defined by \( \mathbb{E}[\xi] = \mathbb{E}_0[\xi] \).

According to [12], there exists a bounded, convex, and closed subset \( \Gamma \subset \mathbb{S}^d_+ \) such that

\[
(1.3) \quad G(A) = \frac{1}{2} \sup_{Q \in \Gamma} \text{trace}[AQ], \quad A \in \mathbb{S}^d.
\]
In particular, fix $\sigma, \overline{\sigma} \in S^d_+ \cap S^d_-$ with $\sigma < \overline{\sigma}$, let $\Gamma = [\sigma^2, \overline{\sigma}^2]$, then

\begin{equation}
G(A) = \frac{1}{2} \sup_{\gamma \in [2, \overline{\sigma}^2]} \text{trace}(\gamma^2 A), \ A \in S^d.
\end{equation}

Denote $L^p_G(\Omega_T)$ be the completion of $L^p(\Omega_T)$ under the norm $(\overline{E}[\cdot])^{\frac{1}{p}}$, $p \geq 1$.

**Theorem 1.1. ([3, 12])** There exists a weakly compact subset $P \subset M_1(\Omega_T)$, the set of probability measures on $(\Omega_T, \mathcal{B}(\Omega_T))$, such that

\begin{equation}
\overline{E}[\xi] = \sup_{P \in P} E_P[\xi] \text{ for all } \xi \in L^1_G(\Omega_T).
\end{equation}

$P$ is called a set that represents $\overline{E}$.

Let $P$ be a weakly compact set that represents $\overline{E}$. For this $P$, we define capacity

\begin{equation}
c(A) = \sup_{P \in P} P(A), \ A \in \mathcal{B}(\Omega_T).
\end{equation}

c defined here is independent of the choice of $P$.

**Remark 1.2.**

(i) Let $(\Omega^0, \mathcal{F}^0, P^0)$ be a probability space and $\{W_t\}$ be a $d$-dimensional Brownian motion under $P^0$. Let $F^0 = \{\mathcal{F}^0_t\}_{t \geq 0}$ be the augmented filtration generated by $W$. [3] proved that $P_M := \{P_h \mid P_h = P^0 \circ X^{-1}, X_t = \int_0^t h_s dW_s, h_s \in L^2_{F^0}([0,T]; \Gamma^1_\sigma)\}$ is a set that represents $\overline{E}$, where $\Gamma^1_\sigma := \{\gamma^1 \mid \gamma \in \Gamma\}$, is the set in the representation of $G(\cdot)$ in the formula (1.3) and $L^2_{F^0}([0,T]; \Gamma^1_\sigma)$ is the set of $F^0$-progressive measurable processes with values in $\Gamma^1_\sigma$.

(ii) For the 1-dimensional case, $L^2_{F^0}([0,T]; \Gamma^1_\sigma)$ reduces to the form below:

\{h \mid h \text{ is an progressive measurable process w.r.t. } F^0 \text{ and } \underline{\sigma} \leq |h_s| \leq \overline{\sigma}\}.

**Definition 1.1.** We say a set $A \subset \Omega_T$ is $c$-polar if $c(A) = 0$. A property holds quasi-surely ($c$-q.s. for short) if it holds outside a $c$-polar set.

**Definition 1.2.**

(1) We say that a map $\xi(\cdot) : \Omega_T \to \mathbb{R}$ is quasi-continuous if for all $\epsilon > 0$, there exists an open set $G$ with $c(G) < \epsilon$ such that $\xi(\cdot)$ is continuous on $G^c$.

(2) We say that a process $M(\cdot) : \Omega_T \times [0,T] \to \mathbb{R}$ is quasi-continuous if for all $\epsilon > 0$, there exists an open set $G$ with $c(G) < \epsilon$ such that $M(\cdot)$ is continuous on $G^c \times [0,T]$.

(3) We say that a random variable $X : \Omega_T \to \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega_T \to \mathbb{R}$ such that $X = Y$, $c$-q.s.
Remark 1.3. Note that a quasi-continuous process defined here is different from [5].

According to [3],

\[ L^p_G(\Omega_T) = \{ X \in L^0(\Omega_T) \mid \lim_{N \to \infty} \mathbb{E}[|X|^p|s_N|] = 0 \text{ and } X \text{ has a quasi-continuous version} \}, \]

where \( L^0(\Omega_T) \) denotes the space of all \( \mathcal{B}(\Omega_T) \)-measurable real function.

In the paper, we discuss the property of distribution for the solution \( X_t \) in (1.1), a polar set does not affect the result, so in the following parts, we did not distinguish the quasi-continuous version and itself any more.

Theorem 1.4. (Monotone Convergence Theorem) [3, Theorem 10, Theorem 31] Let \( \mathcal{P} \) be weakly compact that represents \( \mathbb{E} \).

1. Suppose \( \{X_n\}_{n \geq 1}, X \in L^0(\Omega_T), X_n \uparrow X, c.q.s. \) and \( \mathbb{E}_P[X^+_1] < \infty \) for all \( P \in \mathcal{P} \). Then \( \int \mathbb{E}[X_n] \uparrow \int \mathbb{E}[X] \).

2. Let \( \{X_n\}_{n=1}^\infty \subset L^1_G(\Omega_T) \) be such that \( X_n \downarrow X, c.q.s. \). Then \( \int \mathbb{E}[X_n] \downarrow \int \mathbb{E}[X] \).

Remark 1.5. We stress that in this theorem \( X \) does not necessarily belong to \( L^1_G(\Omega_T) \).

Let

\[ M^{p,0}_G([0,T]) = \left\{ \eta \mid \eta = \sum_{j=0}^{N-1} \xi_i I_{[t_j,t_{j+1})}, \xi_i \in L^p_G(\Omega_{t_j}), N \in \mathbb{N}, 0 = t_0 < t_1 < \cdots < t_N = T \right\}. \]

For \( p \geq 1 \), let \( M^p_G([0,T]) \) and \( H^p_G([0,T]) \) be the completion of \( M^{p,0}_G([0,T]) \) under the following norm

\[ \|\eta\|_{M^p_G([0,T])} = \left[ \mathbb{E} \left( \int_0^T |\eta|^p dt \right) \right]^{\frac{1}{p}}, \quad \|\eta\|_{H^p_G([0,T])} = \left[ \mathbb{E} \left( \int_0^T |\eta|^2 dt \right) \right]^{\frac{1}{2}}, \]

respectively. Denote by \( [M^p_G([0,T])]^d, [H^p_G([0,T])]^d \) all \( d \)-dimensional stochastic processes \( \eta = (\eta_1, \cdots, \eta_N), \xi_t = (\xi_1^1, \cdots, \xi_1^d), t \geq 0 \) with \( \eta_t \in M^p_G([0,T]), \xi_t \in H^p_G([0,T]) \), respectively.

Definition 1.3. A process \( X = \{X_t \mid t \in [0,T]\} \) is called a \( G \)-martingale if for each \( t \in [0,T] \), we have \( X_t \in L^1_G(\Omega_t) \) and

\[ \mathbb{E}_n[X_t] = X_s \text{ in } t \in [0,T]. \]

We call \( X \) a symmetric \( G \)-martingale if both \( X \) and \( -X \) are \( G \)-martingales.

Remark 1.6. For \( \eta \in M^1_G([0,T]), \) it’s easy to see that the process \( \int_0^t \eta_s(\omega)ds \) has a c-quasi continuous version. Also, [15] shows that any \( G \)-martingale has a c-quasi continuous version.

Let \( B_t \) be a \( d \)-dimensional \( G \)-Brownian motion, then \( G(A) = \frac{1}{2} \mathbb{E}[(AB_1, B_1)], A \in \mathbb{S}^d \). In particular, for 1-dimensional \( G \)-Brownian motion \( (B_t)_{t \geq 0} \), one has \( G(a) = (\sigma^2 a^+ - \sigma^2 a^-)/2, a \in \mathbb{R} \), where \( \sigma^2 := \mathbb{E}[B_1^2] \geq -\mathbb{E}[-B_1^2] =: \sigma^2 > 0 \).
Let \( \langle B \rangle_t = (\langle B^i, B^j \rangle_t)_{1 \leq i,j \leq d}, 0 \leq t \leq T \), which is defined by

\[
\langle B \rangle_t = B^i_t B^j_t - \int_0^t B^i_s dB^j_s - \int_0^t B^j_s dB^i_s.
\]

To establish the Wang’s Harnack inequality, G-Girsanov’s transform plays a crucial role, the following results is taken from [9, 22]. For \( \eta \in [M^2_G([0, T])]^d \), let

\[
M_t = \exp \left\{ \int_0^t \langle \eta_s, dB_s \rangle - \frac{1}{2} \int_0^t \langle \eta_s, (d\langle B \rangle_s \eta_s) \rangle \right\},
\]

\[
\hat{B}_t = B_t - \int_0^t (d\langle B \rangle_s \eta_s), \ t \in [0, T],
\]

where \( (d\langle B \rangle_s \eta_s) = \left( \sum_{i=1}^d \eta^i_s d\langle B^i, B^j \rangle_s \right)_{1 \leq i \leq d} \).

**Lemma 1.7.** \([9, 22]\) If \( \eta \in [M^2_G([0, T])]^d \) satisfies G-Novikov’s condition, i.e., for some \( \epsilon_0 > 0 \), it holds that

\[
\mathbb{E} \left[ \exp \left\{ \left( \frac{1}{2} + \epsilon_0 \right) \int_0^T \langle \eta_s, (d\langle B \rangle_s \eta_s) \rangle \right\} \right] < \infty,
\]

then the process \( M \) is a symmetric G-martingale.

**Lemma 1.8.** \([9]\) (G-Girsanov’s formula) Assume that there exists \( \sigma_0 > 0 \) such that

\[
\gamma \geq \sigma_0 I_d \quad \text{for all} \quad \gamma \in \Gamma,
\]

and that \( M \) is a symmetric G-martingale on \((\Omega_T, L^1_G(\Omega_T), \mathbb{E})\). Define a sublinear expectation \( \mathbb{E} \) by

\[
\mathbb{E}[X] = \mathbb{E}[XM_T], \quad X \in \hat{L}_{lip}(\Omega_T),
\]

where \( \hat{L}_{lip}(\Omega_T) := \{ \varphi(\hat{B}_t, \cdots, \hat{B}_{t_n}) : n \in \mathbb{N}, t_1, \cdots, t_n \in [0, T], \varphi \in C_{b,lip}(\mathbb{R}^d \otimes \mathbb{R}^n) \} \). Then \( \hat{B}_t \) is a G-Brownian motion on the sublinear expectation space \((\Omega_T, \hat{L}^1_G(\Omega_T), \mathbb{E})\), where \( \hat{L}^1_G(\Omega_T) \) is the completion of \( \hat{L}_{lip}(\Omega_T) \) under the norm \( \mathbb{E}[|\cdot|] \).

**Remark 1.9.** The Girsanov theorem also appeared in [4, Theorem 5.2].

**Lemma 1.10.** For \( \hat{B} \) in (1.8), then c-q.s., \( \langle \hat{B} \rangle_t = \langle B \rangle_t, \ t \in [0, T] \).

**Proof.** For any \( P \in \mathcal{P} \), it holds that

\[
P\{ \langle \hat{B} \rangle_t \neq \langle B \rangle_t, \ t \in [0, T] \} = 0.
\]

By (1.6), we have

\[
c\{ \langle \hat{B} \rangle_t \neq \langle B \rangle_t, \ t \in [0, T] \} = \sup_{P \in \mathcal{P}} P\{ \langle \hat{B} \rangle_t \neq \langle B \rangle_t, \ t \in [0, T] \} = 0,
\]

which implies c-q.s., \( \langle \hat{B} \rangle_t = \langle B \rangle_t, \ t \in [0, T] \).
We aim to establish the following Harnack-type inequality introduced by Feng-Yu Wang:

\[
\Phi(\bar{P} f(x))) \leq \bar{P}\Phi(f(y))e^{\Psi(x,y)}, \quad x, y \in \mathbb{R}^d, f \in B_+^\mathcal{B}(\mathbb{R}^d),
\]

where \(\Phi\) is a nonnegative convex function on \([0, \infty)\) and \(\Psi\) is a nonnegative function on \(\mathbb{R}^d \times \mathbb{R}^d\). In the setting of \(G\)-SDEs, we establish this type inequality for the associated nonlinear Markov operator \(\bar{P}_T\). For simplicity, we consider the case of \(d = 1\), but our results and methods still hold for the case \(d > 1\). To get our desired results, we give following assumptions on \(b, \sigma,\) and \(h\) in (1.1).

(H1) There exists a constant \(K > 0\), such that
\[
|b(t, x) - b(t, y)| + |h(t, x) - h(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq K|x - y|, \quad x, y \in \mathbb{R}, t > 0.
\]

(H2) There exist \(\kappa_1, \kappa_2\) with \(\kappa_2 \geq \kappa_1 > 0\), such that \(\kappa_1 \leq \sigma(t, x) \leq \kappa_2, \quad x \in \mathbb{R}, t > 0\).

From [12, Theorem 1.2], under the assumption of (H1), for any \(x \in \mathbb{R}\), (1.1) has a unique solution in \(M^2_G([0, T])\). In what follows, for \(T > 0\), we define
\[
\bar{P}_T f(x) = \mathbb{E}f(X^x_T), \quad f \in C^+_b(\mathbb{R}),
\]

where \(X^x_T\) solves (1.1) with initial value \(x\).

Remark 1.11. In order to ensure the term \(f(X^x_T) \in M^2_G([0, T])\), we always assume \(f \in C^+_b(\mathbb{R})\).

The remainder of the paper is organized as follows. In Section 2, we characterize the quasi-continuity of hitting time for processes of certain forms. Finally, in Section 3 we present the Harnack and log Harnack inequalities for \(G\)-SDE (1.1), so that main results in [18, Theorem 3.4.1, Chap.3] are extended to the present \(G\)-setting. Moreover, the gradient estimate is showed in this section.

\section{Main Results}

Now we turn to the main result of this section.

\subsection{Harnack and log-Harnack inequalities}

\textbf{Theorem 2.1.} Assume (H1)-(H2).

(1) For any nonnegative \(f \in C^+_b(\mathbb{R})\) and \(T > 0\), \(x, y \in \mathbb{R}\), it holds that
\[
(2.1) \quad P_T \log f(y) \leq \log P_T f(x) + \frac{K \left(2 + K + \frac{2}{\kappa^2}\right)|x - y|^2}{2\kappa^2 \left(1 - e^{-\sigma^2 (2 + K + \frac{2}{\kappa^2}) T}\right)}.
\]
(2) For \( p > (1 + \frac{\kappa_2^2 - \kappa_1\kappa_2^2}{\kappa_1^2})^2 \), then

\[
(\bar{P}_T f(y))^p \leq \bar{P}_T f^p(x) \exp \left\{ \frac{\sqrt{p}(\sqrt{p} - 1)K \left(2 + \frac{2}{2} + \frac{2}{2}\right)|x - y|^2}{4(\kappa_2 - \kappa_1)\kappa_1(\sqrt{p} - 1) - C}|1 - e^{-2^2K(2+\frac{2}{2})^2}| \right\}
\]

holds for any \( x, y \in \mathbb{R} \) and \( f \in C_b^+(\mathbb{R}) \).

To make the proof easy to follow, let us divide the proof into the following aspects.

### 2.1.1 Martingale convergence

To apply \( G \)-Girsanov’s formula in Lemma 1.8, we need to check that \( M \) is a symmetric \( G \)-martingale. From Lemma 1.7, we know that \( G \)-Novikov’s condition is a sufficient condition for \( M \) to be a symmetric \( G \)-martingale. However, if we take this for calculation, the assumptions we impose on \( \kappa \) are too strong, thus, we propose the notion of uniform integrability under a nonlinear expectation [2]. We would like to point out [2] discusses the martingale convergence in discrete time, for simplicity, we still use \( \bar{E} \) in this paper instead of the notion in [2].

We define the space \( L^1 \) as the completion under \((\bar{E}[| \cdot |])\) of the set

\[ \{ X \in \mathcal{H} \mid (\bar{E}[| X |]) < \infty \} , \]

where \( \mathcal{H} \) be a vector lattice of real valued functions defined on \( \Omega \), namely \( c \in \mathcal{H} \) for each constant \( c \) and \( |X| \in \mathcal{H} \) if \( X \in \mathcal{H} \).

**Definition 2.1.** Let \( K \subset L^1 \). \( K \) is said to be uniformly integrable (u.i.) if \( \bar{E}(1_{\{|X| \geq c\}} |X|) \) converges to 0 uniformly in \( X \in K \) as \( c \to \infty \).

**Lemma 2.2.** ([2, Corollary 3.1.1]) Let \( K \subset L^1 \). Suppose there is a positive function \( f \) defined on \([0, \infty[\) such that \( \lim_{t \to \infty} t^{-1}f(t) = \infty \) and \( \sup_{X \in K} \bar{E}(f \circ |X|) < \infty \). Then \( K \) is uniformly integrable.

Let

\[ \mathcal{H}^{ext} = \{ X \in mF \mid \min\{E_{Ph}[X^+], E_{Ph}[X^-] < \infty \} \text{ for all } h \in L^2_F([0, T]; \Gamma^1) \} , \]

where \( mF \) is the space of \( F_t \)-measurable \( \mathbb{R} \cup \pm \infty \)-valued functions. According to [2],

\[ L^p_b = \{ X \in L^p(\Omega_T) \mid \lim_{N \to \infty} \bar{E}[|X|^p 1_{|X| \geq N}] = 0 \} . \]

This does not need to restrict our attention to those random variables admitting a quasi-continuous version compared with the structure of \( L^p_G(\Omega_T) \). It’s clear that \( L^p_G(\Omega_T) \subset L^p_b \).

**Lemma 2.3.** ([2, Theorem 3.2]) Suppose \((X_n)_{n \geq 1} \subset L^1_b \), and \( X \in \mathcal{H}^{ext} \). Then \( X_n \) converge in \( L^1 \) norm to \( X \) if and only if the collection \((X_n)_{n \geq 1} \) is uniformly integrable and the \( X_n \) converge in capacity to \( X \). Furthermore, in this case, the collection \((X_n)_{n \geq 1} \cup X \) is also uniformly integrable and \( X \in L^1_b \).
Lemma 2.4. ([2, Theorem 4.4]) Let \((X_n)_{n \geq 1}\) be a \(G\)-submartingale with \(\sup_k \mathbb{E}(|X_k|) < \infty\). Then \(X_n \to X_\infty \in H^\text{ext}, \text{q.s.}\).

Lemma 2.5. ([2, Theorem 4.5]) Let \((X_n)_{n \geq 1}\) be a uniformly integrable \(G\)-submartingale. Then taking \(X_\infty = \lim_{n \to \infty} X_n\), the process \((X_n)_{n \geq 1 \cup \infty}\) is also a uniformly integrable \(G\)-submartingale. In particular, this implies that \(X_\infty \in L^1_b\).

In the following, we aim to extend the convergence theorem for \(G\)-martingale from discrete time to continuous time.

Theorem 2.6. Let \((X_s)_{s \in [0,T]} \subset L^1_G(\Omega_T)\) be a uniformly integrable \(G\)-martingale. Then taking \(X_T = \lim_{t \to T} X_t\), the process \((X_s)_{s \in [0,T]}\) is also a uniformly integrable \(G\)-martingale. In particular, this implies that \(X_T \in L^1_G(\Omega_T)\).

Proof. Since \(\{X_{T-\frac{t}{n}}\}_{n=1}^\infty\) is a sequence of discrete martingale, we have
\[
\mathbb{E}_{T-\frac{t}{n}} X_T = X_{T-\frac{t}{n}}.
\]
For any \(s \in [0,T)\), there exists a \(n \geq 1\), such that \(T - \frac{T}{n} > s\). Moreover,
\[
\mathbb{E}_s X_T = \mathbb{E}_s \mathbb{E}_{T-\frac{t}{n}} X_T \\
= \mathbb{E}_s X_{T-\frac{t}{n}} \\
= X_s,
\]
where the last step by using the fact that \((X_s)_{s \in [0,T]}\) is \(G\)-martingale. This implies that \((X_s)_{s \in [0,T]}\) is \(G\)-martingale. Moreover, the collection \((X_{T-\frac{t}{n}})_{n \geq 1}\) is uniformly integrable and the \(X_{T-\frac{t}{n}}\) converge in capacity to \(X_T\), then the \(X_t\) converge to \(X_T\) in \((\mathbb{E}[|\cdot|])\) norm by Lemma 2.3, which proves that \(X_T \in L^1_G(\Omega_T)\). \(\square\)

To prove Theorem 2.1, we first introduce the construction of coupling by change of measure with multiplicative noise under \(G\)-setting.

2.1.2 Construction of the coupling

In the sequel, we denote \(\hat{\sigma} = \sigma^* (\sigma \sigma^*)^{-1}\). We use the coupling by change of measures as explained in [18]. For \(\alpha \in (0, \frac{2\kappa_2}{\kappa_2})\), let
\[
\lambda_t^{\alpha} = \frac{2\kappa_2^2 - \alpha}{K \left(2 + K + \frac{2}{\sigma^2}\right)} \left(1 - e^{2\kappa_2 K \left(2 + K + \frac{2}{\sigma^2}\right)(t-T)}\right), \ t \in [0,T].
\]
Then \(\lambda_t^{\alpha}\) is smooth and strictly positive on \([0, T]\) such that
\[
\frac{2\kappa_2^2}{\kappa_2} - K \left(2 + K + \frac{2}{\sigma^2}\right) \lambda_t^{\alpha} + \frac{1}{\sigma^2} (\lambda_t^{\alpha})' = \alpha, \ t \in [0,T].
\]
For convenience, we reformulate (1.1) as

\[
(2.5) \quad dX_t = b(t, X_t)dt + h(t, X_t)d\langle B\rangle_t + \sigma(t, X_t)dB_t, \quad X_0 = x.
\]

Consider the equation

\[
(2.6) \quad \begin{cases}
    dY_t = b(t, Y_t)dt + h(t, Y_t)d\langle B\rangle_t + \sigma(t, Y_t)dB_t + \sigma(t, Y_t)g_t d\langle B\rangle_t, \\
    Y_0 = y, \quad t \in (0, T),
\end{cases}
\]

where \( g_t := \frac{1}{\lambda_t} \sigma(t, X_t)(X_t - Y_t). \)

### 2.1.3 Extension of \( Y \) to \( T \)

Let \( s \in [0, T) \) be fixed. By (1.1) and (2.6), \( X_t - Y_t \) satisfies the equation below

\[
(2.7) \quad d(X_t - Y_t) = (b(t, X_t) - b(t, Y_t))dt + (h(t, X_t) - h(t, Y_t))d\langle B\rangle_t \\
+ (\sigma(t, X_t) - \sigma(t, Y_t))dB_t - \sigma(t, Y_t)g_t d\langle B\rangle_t.
\]

Applying Itô’s formula to \( |X_t - Y_t|^2 \), we obtain

\[
(2.8) \quad d|X_t - Y_t|^2 = 2(X_t - Y_t, b(t, X_t) - b(t, Y_t))dt + 2(X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t))dB_t \\
+ 2(X_t - Y_t, h(t, X_t) - h(t, Y_t))d\langle B\rangle_t + |\sigma(t, X_t) - \sigma(t, Y_t)|^2 d\langle B\rangle_t \\
- 2(X_t - Y_t, \sigma(t, Y_t)g_t d\langle B\rangle_t \\
\leq \left(2K + K^2 - \frac{2\kappa_1^2}{\mu^2} \right) |X_t - Y_t|^2 d\langle B\rangle_t \\
+ 2K|X_t - Y_t|^2 dt \\
\leq \left(2K + \frac{2K}{\sigma^2} + K^2 - \frac{2\kappa_1^2}{\mu^2} \right) |X_t - Y_t|^2 d\langle B\rangle_t \\
+ 2(X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t))dB_t.
\]

Combining with the expression (2.4), we have

\[
\frac{d}{\lambda_t^\alpha} \frac{|X_t - Y_t|^2}{\lambda_t^\alpha} \leq -\frac{|X_t - Y_t|^2}{(\mu^2)^2} \left(\frac{2\kappa_1^2}{\mu^2} - 2K\lambda_t^\alpha - \frac{2K}{\sigma^2} \lambda_t^\alpha - K^2 \lambda_t^\alpha + \frac{1}{\sigma^2} (\lambda_t^\alpha)'^2 \right) d\langle B\rangle_t, \\
+ \frac{2}{\lambda_t^\alpha} \langle X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t) \rangle dB_t \\
= -\frac{\alpha}{(\mu^2)^2} |X_t - Y_t|^2 d\langle B\rangle_t + \frac{2}{\lambda_t^\alpha} \langle X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t) \rangle dB_t.
\]

Thus,

\[
(2.9) \quad \int_0^s \frac{|X_t - Y_t|^2}{(\lambda_t^\alpha)^2} d\langle B\rangle_t \leq \int_0^s \frac{2}{\alpha \lambda_t^\alpha} \langle X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t) \rangle dB_t
\]
where $P$.

Lemma 2.7. (Young Inequality) We first prove the following Young inequality under $2.1.4$ Several lemmas.

$$\frac{|X_t - Y_t|^2}{\alpha \lambda^2_t} + \frac{|x - y|^2}{\alpha \lambda^2_0}, s \in [0, T).$$

Taking expectation $\mathbb{E}$ on both sides of (2.9), we obtain

$$\mathbb{E} \int_0^T \frac{|X_t - Y_t|^2}{(\lambda^2_t)} dt \leq \frac{|x - y|^2}{\alpha \lambda^2_0}, s \in [0, T).$$

Since $X_t, Y_t \in M^2_\mathbb{C}([0, T])$, for any $s \in (0, T)$, $g_1,1_{[0,s]}(t) \in M^2_\mathbb{C}([0, T])$. Note that, for any $s \in (0, T)$,

$$\mathbb{E} \int_s^T \frac{|X_t - Y_t|^2}{(\lambda^2_t)} dt \leq C_1(s - r),$$

where $C_1$ is a constant.

By the Monotone Convergence Theorem in $[1]$ of Theorem 1.4,

$$\mathbb{E} \int_r^T \frac{|X_t - Y_t|^2}{(\lambda^2_t)} dt = \lim_{s \to T} \mathbb{E} \int_s^T \frac{|X_t - Y_t|^2}{(\lambda^2_t)} dt \leq C_1(T - r).$$

There exists a $\bar{g} \in M^2_\mathbb{C}([0, T])$ such that $\bar{g}_s = g_s, s \in [0, T]$. In fact, let $g^n_t = g_1,1_{[0,T-\frac{1}{n}]}(t) \in M^2_\mathbb{C}([0, T])$, then it holds that

$$\mathbb{E} \int_{[0,T]} |\bar{g}_t - g^n_t|^2 dt = \mathbb{E} \int_{[T-\frac{1}{n}, T]} |\bar{g}_t|^2 dt$$

$$= \mathbb{E} \int_{[T-\frac{1}{n}, T]} |g_t|^2 dt$$

$$\leq \frac{1}{\kappa^2_1} \mathbb{E} \int_{T-\frac{1}{n}}^T \frac{|X_t - Y_t|^2}{(\lambda^2_t)} dt$$

$$\to 0, \ n \to \infty,$$

where the last step uses the fact of [2] in Theorem 1.4.

Let $\bar{Y}_t$ solve the following equation

$$\begin{cases}
\text{d}Y_t = b(t, Y_t) dt + h(t, Y_t) d\langle B \rangle_t + \sigma(t, Y_t) dB_t + \sigma(t, Y_t) \bar{g}_t d\langle B \rangle_t,
\end{cases}
\begin{align}
Y_0 &= y, \ t \in (0, T],

\end{align}$$

Thus, $Y$ can be extended to $[0, T]$ as $\bar{Y}$. In the sequel, we still use $Y$ and $g$ instead $Y$ and $\bar{g}$.

### 2.1.4 Several lemmas

We first prove the following Young inequality under $G$-expectation framework.

**Lemma 2.7. (Young Inequality)** For $g_1, g_2 \in L^1_G(\Omega_T)$ with $g_1, g_2 > 0$ and $\mathbb{E}_P[g_1] = 1$, $P \in \mathcal{P}$, then

$$\mathbb{E}[g_1 g_2] \leq \mathbb{E}[g_1 \log g_1] + \log \mathbb{E}[e^{g_2}],$$

where $\mathcal{P}$ is a weakly compact set that represents $\mathbb{E}$. 

10
Proof. For any $P \in \mathcal{P}$, $\mathbb{E}_P$ is a linear expectation, it holds that

$$\mathbb{E}_P[g_1 g_2] \leq \mathbb{E}_P[g_1 \log g_1] + \log \mathbb{E}_P[e^{g_2}].$$

Since $\overline{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[X]$, $X \in L^1_G(\Omega_T)$, then

$$\mathbb{E}[g_1 g_2] \leq \sup_{P \in \mathcal{P}} \{ \mathbb{E}_P[g_1 \log g_1] + \log \mathbb{E}_P[e^{g_2}] \}$$

where the last step due to the function log is increasing.

Let

$$d\hat{B}_t = dB_t + g_t dB_t,$$

$0 \leq t \leq T$.

Following section 3.2.2, we see that $g \in M^2_G([0, T])$, below we aim to prove

$$M_s := \exp \left\{ -\int_0^s g_t dB_t - \frac{1}{2} \int_0^s |g_t|^2 dB_t \right\},$$

is a uniformly integrable symmetric $G$-martingale for $s \in [0, T]$.

Lemma 2.8. Assume (H1)-(H2). There holds

$$\sup_{s \in [0, T]} \mathbb{E}[M_s \log M_s] \leq \frac{|x - y|^2}{2\alpha_k^2 \lambda_0}.$$

Consequently, $M_T := \lim_{s \to T} M_s$ exists and $\{M_s\}_{s \in [0, T]}$ is a uniformly integrable symmetric $G$-martingale.

Proof. Fix $s \in [0, T)$. Applying Itô’s formula to $|X_t|^2$, we have

$$|X_t|^2 = x^2 + \int_0^t \langle X_t, b(t, X_t) \rangle dt + \int_0^t (\langle X_t, h(t, X_t) \rangle + |\sigma(t, X_t)|^2) dB_t + \int_0^t \langle X_t, \sigma(t, X_t) \rangle dB_t.$$

Let

$$X_t = x + \int_0^t \langle X_t, b(t, X_t) \rangle dt + \int_0^t (\langle X_t, h(t, X_t) \rangle + |\sigma(t, X_t)|^2) dB_t + \int_0^t \langle X_t, \sigma(t, X_t) \rangle dB_t,$$

and

$$X_t = y + \int_0^t \langle Y_t, b(t, Y_t) \rangle dt + \int_0^t (\langle X_t, h(t, X_t) \rangle - \sigma(t, Y_t) g_t) dB_t + \int_0^t \langle Y_t, \sigma(t, Y_t) \rangle dB_t.$$
For any \( n \geq 1 \), let \( \hat{\tau}_n = \inf\{ t \in [0, T] | |\dot{X}_t| + |\dot{Y}_t| \geq n \} \). By Lemma 3.3, \( \hat{\tau}_n \) is quasi-continuous, and \( X_{t \wedge \hat{\tau}_n}, Y_{t \wedge \hat{\tau}_n} \) are bounded, which implies \( g_{t \wedge \hat{\tau}_n} \) is bounded. So for any \( n \geq 1 \) and by the Girsanov theorem in [4, Theorem 5.2], \( (\tilde{B}_t)_{t \in [0,t \wedge \hat{\tau}_n]} \) is a \( G \)-Brownian motion under \( \tilde{\mathbb{E}}_n := \tilde{\mathbb{E}}[\cdot | M_{s \wedge \hat{\tau}_n}] \).

Moreover, Lemma 1.10 implies \( \langle \tilde{B} \rangle_t = \langle B \rangle_t \). Rewrite (2.5) and (2.11) as

\[
\begin{align*}
\text{d}X_t &= b(t, X_t)\text{d}t + h(t, X_t)\text{d}(\tilde{B})_t + \sigma(t, X_t)\text{d}\tilde{B}_t - \frac{X_t - Y_t}{\lambda_t^\alpha}\text{d}\langle B \rangle_t, \quad X_0 = x, \\
\text{d}Y_t &= b(t, Y_t)\text{d}t + h(t, Y_t)\text{d}(\tilde{B})_t + \sigma(t, Y_t)\text{d}\tilde{B}_t, \quad Y_0 = y.
\end{align*}
\]

(2.13)

Substituting \( B_t = \tilde{B}_t - \int_0^t g_s \text{d}\langle B \rangle_s \) in the first equation in (2.8), using the fact of \( \langle \tilde{B} \rangle_t = \langle B \rangle_t \), and repeating procedures in (2.8), which yield

\[
\begin{align*}
\text{d}|X_t - Y_t|^2 &\leq \left( 2K + \frac{2K}{\alpha^2} + K^2 - \frac{2}{\lambda_t^\alpha} \right) |X_t - Y_t|^2 \text{d}\langle \tilde{B} \rangle_t \\
&\quad + 2|X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t)| \text{d}\tilde{B}_t.
\end{align*}
\]

So,

\[
\begin{align*}
\text{d}|X_t - Y_t|^2 &\leq - \frac{|X_t - Y_t|^2}{\lambda_t^\alpha} \left( 2 - 2K\lambda_t^\alpha - \frac{2K}{\alpha^2} \lambda_t^\alpha - K^2\lambda_t^\alpha + \frac{1}{\alpha^2}(\lambda_t^\alpha)' \right) \text{d}\langle \tilde{B} \rangle_t \\
&\quad + \frac{2}{\lambda_t^\alpha} |X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t)| \text{d}\tilde{B}_t.
\end{align*}
\]

From (2.4), we know that

\[
\alpha = \frac{2\kappa^2}{\kappa^2} - \Delta \leq 2 - \Delta,
\]

where \( \Delta := 2K\lambda_t^\alpha + \frac{2K}{\alpha^2} \lambda_t^\alpha + K^2\lambda_t^\alpha - \frac{1}{\alpha^2}(\lambda_t^\alpha)' \).

Therefore,

\[
(2.14) \quad \int_0^{s \wedge \hat{\tau}_n} \frac{|X_t - Y_t|^2}{\lambda_t^\alpha} \text{d}\langle \tilde{B} \rangle_t \leq \int_0^{s} \frac{2}{\alpha\lambda_s^\alpha} |X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t)| \text{d}\tilde{B}_t - \frac{|X_s - Y_s|^2}{\alpha\lambda_s^\alpha} + \frac{|x - y|^2}{\alpha\lambda_0^\alpha}, \quad s \in [0, T).
\]

Since \( (\tilde{B}_t)_{t \in [0,t \wedge \hat{\tau}_n]} \) is a \( G \)-Brownian motion under \( \tilde{\mathbb{E}} \), taking expectation \( \tilde{\mathbb{E}} \) on both sides of (2.14), we obtain

\[
(2.15) \quad \tilde{\mathbb{E}} \int_0^{s \wedge \hat{\tau}_n} \frac{|X_t - Y_t|^2}{\lambda_t^\alpha} \text{d}\langle \tilde{B} \rangle_t \leq \frac{|x - y|^2}{\alpha\lambda_0^\alpha}.
\]

From the definition of \( M_t, \tilde{B}_t \) and Lemma 1.10, it holds that

\[
M_{s \wedge \hat{\tau}_n} = \exp \left\{ - \int_0^{s \wedge \hat{\tau}_n} g_t \text{d}\tilde{B}_t + \frac{1}{2} \int_0^{s \wedge \hat{\tau}_n} |g_t|^2 \text{d}\langle B \rangle_t \right\}
\]

12
By (H2), we have

\[(2.16) \quad \log M_{s \wedge \hat{\tau}_n} \leq -\int_0^{s \wedge \hat{\tau}_n} g_t \, d\hat{B}_t + \frac{1}{2} \int_0^{s \wedge \hat{\tau}_n} |g_t|^2 \, d\langle \hat{B} \rangle_t, \quad c - q.s..\]

It follows (2.15) that

\[(2.17) \quad \hat{E}[M_{s \wedge \hat{\tau}_n} \log M_{s \wedge \hat{\tau}_n}] = \hat{E}[\log M_{s \wedge \hat{\tau}_n}] \leq \frac{|x - y|^2}{2\kappa^2 \lambda_0}, \quad s \in [0, T).\]

Applying Itô's formula to \(M_{s \wedge \hat{\tau}_n} = e^{u_{s \wedge \hat{\tau}_n}}\) for the process

\[u_{s \wedge \hat{\tau}_n} = -\int_0^{s \wedge \hat{\tau}_n} g_t \, d\hat{B}_t - \frac{1}{2} \int_0^{s \wedge \hat{\tau}_n} |g_t|^2 \, d\langle \hat{B} \rangle_t,\]

we conclude that

\[dM_{s \wedge \hat{\tau}_n} = -\int_0^{s \wedge \hat{\tau}_n} g_t \, d\hat{B}_t,\]

thus \(\{M_t\}_{t \in [0, s \wedge \hat{\tau}_n]}\) is a symmetric \(G\)-martingale. From (2.17) and Lemma 2.2, \(\{M_{s \wedge \hat{\tau}_n}\}_{s \in [0, T)}\) is a uniformly symmetric \(G\)-martingale, thus \(\hat{E}M_s = \lim_{n \to \infty} \hat{E}M_{s \wedge \hat{\tau}_n} = 1\) by Lemma 2.3. So that \(\{M_t\}_{t \in [0, s]}\) is a symmetric \(G\)-martingale.

Let \(\hat{E} = \hat{E}[M_s]\), \(s \in [0, T)\). Letting \(n \to \infty\), we have \(\hat{\tau}_n \uparrow T\). By the Fatou lemma,

\[
\lim_{n \to \infty} \hat{E}_n[\log M_{s \wedge \hat{\tau}_n}] = \lim_{n \to \infty} \hat{E}[\log M_{s \wedge \hat{\tau}_n}] = \hat{E}[\frac{1}{2} \int_0^{s \wedge \hat{\tau}_n} |g_t|^2 \, d\langle \hat{B} \rangle_t] \\
\geq \hat{E}[\frac{1}{2} \int_0^{s} |g_t|^2 \, d\langle \hat{B} \rangle_t] = \hat{E} \log M_s.
\]

Thus

\[\hat{E}[M_s \log M_s] = \hat{E} \log M_s \leq \lim_{n \to \infty} \hat{E}_n[\log M_{s \wedge \hat{\tau}_n}] \leq \frac{|x - y|^2}{2\kappa^2 \lambda_0}, \quad s \in [0, T].\]

Using Theorem 2.6 once again, \(\{M_s\}_{s \in [0, T]}\) is a uniformly symmetric \(G\)-martingale.

\[\square\]

**Lemma 2.9.** Assume (H1)-(H2). We have \(X_T = Y_T\), \(c - q.s..\)

**Proof.** Let

\[\tau = \inf\{t \in [0, T] \mid X_t = Y_t\}.\]

For any \(P \in \mathcal{P}\), define \(\hat{E}_P = \hat{E}_P[M_T]\), then \(\hat{B}_t\) is a martingale under \(\hat{E}_P\). If there exists a \(\omega \in \Omega\) such that \(\tau(\omega) > T\), then

\[\inf_{t \in [0, T]} |X_t - Y_t|^2(\omega) > 0.\]
holds on the set \( \{ \omega | \tau(\omega) > T \} \), which is a contradiction with (2.15), thus \( \hat{\mathbb{E}}_P \)-a.s., \( \tau(\omega) \leq T \), then

\[
\hat{\mathbb{E}}_P 1_{\{ \omega | X_T \neq Y_T \}} = 0.
\]

Similar analysis with Lemma 1.10, we have

\[
\hat{\mathbb{E}}_1 1_{\{ \omega | X_T \neq Y_T \}} = 0.
\]

Therefore, \( X_T = Y_T \) under \( \hat{\mathbb{E}} \). \( \square \)

**Lemma 2.10.** Assume (H1)-(H2). Then

\[
(2.19) \quad \sup_{s \in [0,T]} \hat{\mathbb{E}} \left[ M_s \exp \left\{ \frac{\alpha^2}{8(\kappa_2 - \kappa_1)^2} \int_0^s \frac{|X_t - Y_t|^2}{(\lambda_t^p)^2} d\langle B \rangle_t \right\} \right] \leq \exp \left\{ \frac{\alpha K (2 + K + \frac{2}{\alpha^2})}{4(\kappa_2 - \kappa_1)^2(2\kappa_2^a - \alpha)} (2\alpha \kappa_1 + 2(\kappa_2 - \kappa_1)) \right\}.
\]

Consequently,

\[
\sup_{s \in [0,T]} \hat{\mathbb{E}} (M_s)^{1+a} \leq \exp \left\{ \frac{\alpha K (2 + K + \frac{2}{\alpha^2})}{4(\kappa_2 - \kappa_1)^2(2\kappa_2^a - \alpha)} (2\alpha \kappa_1 + 2(\kappa_2 - \kappa_1)) \right\}
\]

holds for

\[
a = \frac{\alpha^2 \kappa_2^a}{4(\kappa_2 - \kappa_1)^2 + 4\alpha (\kappa_2 - \kappa_1) \kappa_1}.
\]

**Proof.** Let \( \tau_m = \inf\{ t \in [0,T] | \int_0^t \frac{|X_s - Y_s|^2}{(\lambda_t^p)^2} + 1) d\langle B \rangle_s \geq m \} \). Applying Lemma 3.3 for processes \( Z_s = 0, \eta_s = 0, \) and \( \zeta_s = \frac{|X_s - Y_s|^2}{(\lambda_t^p)^2} + 1 \), we know that \( \tau_m \) is quasi-continuous. From (2.10), we know that \( \lim_{m \to \infty} \tau_m = T \). By (2.9), (H2), and Lemma 3.4-3.5, for some \( \delta > 0 \), we have

\[
\hat{\mathbb{E}} \exp \left\{ \delta \int_0^{s \wedge \tau_m} \frac{|X_t - Y_t|^2}{(\lambda_t^p)^2} d\langle B \rangle_t \right\} \leq \hat{\mathbb{E}} \exp \left\{ \frac{\delta |x - y|^2}{\alpha \lambda_0^p} + \frac{2\delta}{\alpha} \int_0^{s \wedge \tau_m} \frac{1}{\lambda_t^p} (X_t - Y_t, \sigma(t, X_t) - \sigma(t, Y_t)) dB_t \right\} \leq \exp \left\{ \frac{\delta |x - y|^2}{\alpha \lambda_0^p} \right\} \hat{\mathbb{E}} \left( \exp \left\{ \frac{2(\kappa_2 - \kappa_1) \delta}{\alpha} \int_0^{s \wedge \tau_m} \frac{1}{\lambda_t^p} |X_t - Y_t| dB_t \right\} \right) \]
\[\begin{align*}
    &= \exp \left( \frac{\delta |x - y|^2}{\alpha m^2} \right) \left( 8\delta^2 (\kappa_2 - \kappa_1)^2 \int_0^{s \wedge \tau_m} \frac{1}{(\lambda t)^2} |X_t - Y_t|^2 \, d\langle B \rangle_t \right) \frac{1}{m^2}.
\end{align*}\]

Taking \( \delta = \frac{a^2}{8(\kappa_2 - \kappa_1)^2} \), we arrive at

\[\begin{align*}
    &\mathbb{E} \exp \left\{ \frac{\alpha^2}{8(\kappa_2 - \kappa_1)^2} \int_0^{s \wedge \tau_m} \frac{|X_t - Y_t|^2}{(\lambda t)^2} \, d\langle B \rangle_t \right\} \leq \exp \left\{ \frac{\alpha K \left(2 + K + \frac{a^2}{\kappa_2^2}\right) |x - y|^2}{4(\kappa_2 - \kappa_1)^2(\frac{2a^2}{\kappa_2^2} - \alpha) \left(1 - e^{-a^2 K(2+K+\frac{a^2}{\kappa_2^2}) T}\right)} \right\}.
\end{align*}\]

Letting \( m \to \infty \), this implies that

\[\begin{align*}
    &\mathbb{E} \exp \left\{ \frac{\alpha^2}{8(\kappa_2 - \kappa_1)^2} \int_0^{s} \frac{|X_t - Y_t|^2}{(\lambda t)^2} \, d\langle B \rangle_t \right\} \leq \exp \left\{ \frac{\alpha K \left(2 + K + \frac{a^2}{\kappa_2^2}\right) |x - y|^2}{4(\kappa_2 - \kappa_1)^2(\frac{2a^2}{\kappa_2^2} - \alpha) \left(1 - e^{-a^2 K(2+K+\frac{a^2}{\kappa_2^2}) T}\right)} \right\},
\end{align*}\]

which is (2.19).

Next, let \( \tilde{\tau}_n = \inf\{t \in [0, T] \mid \int_0^t (|X_s - Y_s|^2 + 1) \, d\langle \hat{B} \rangle_s \geq n\} \), similar with \( \tau_m \), \( \tilde{\tau}_n \) is quasi-continuous. From (2.15), we know that \( \lim_{n \to \infty} \tilde{\tau}_n = T \). Similar with the process of deducing in (2.20), we have

\[\begin{align*}
    &\mathbb{E} \left[ M_s \exp \left\{ \frac{\alpha^2}{8(\kappa_2 - \kappa_1)^2} \int_0^{s} \frac{|X_t - Y_t|^2}{(\lambda t)^2} \, d\langle B \rangle_t \right\} \right] = \mathbb{E} \exp \left\{ \frac{\alpha^2}{8(\kappa_2 - \kappa_1)^2} \int_0^{s} \frac{|X_t - Y_t|^2}{(\lambda t)^2} \, d\langle B \rangle_t \right\} \leq \exp \left\{ \frac{\alpha K \left(2 + K + \frac{a^2}{\kappa_2^2}\right) |x - y|^2}{4(\kappa_2 - \kappa_1)^2(\frac{2a^2}{\kappa_2^2} - \alpha) \left(1 - e^{-a^2 K(2+K+\frac{a^2}{\kappa_2^2}) T}\right)} \right\}.
\end{align*}\]

Moreover,

\[\begin{align*}
    \mathbb{E}(M_s)^{1+\alpha} &= \mathbb{E}(M_s)^{\alpha}
    = \mathbb{E} \exp \left\{ - a \int_0^s g_t \, d\hat{B}_t + \frac{a}{2} \int_0^s |g_t|^2 \, d\langle \hat{B} \rangle_t \right\}
    = \mathbb{E} \exp \left\{ - a \int_0^s g_t \, d\hat{B}_t - \frac{a^2 q}{2} \int_0^s |g_t|^2 \, d\langle \hat{B} \rangle_t + \frac{a^2 q^2}{2} \int_0^s |g_t|^2 \, d\langle \hat{B} \rangle_t + \frac{a^2 q^2 + 1}{2} \right\}.
\end{align*}\]
\begin{align*}
\leq & \left( \mathbb{E} \exp \left\{-aq \int_0^s g_t d\hat{B}_t - \frac{a^2 q^2}{2} \int_0^s |g_t|^2 d\langle \hat{B} \rangle_t \right\} \right)^{\frac{1}{q}} \\
\times & \left( \mathbb{E} \exp \left\{\frac{aq(aq + 1)}{2(q - 1)} \int_0^s |g_t|^2 d\langle \hat{B} \rangle_t \right\} \right)^{\frac{q - 1}{q}} \\
= & \left( \mathbb{E} \exp \left\{\frac{aq(aq + 1)}{2(q - 1)} \int_0^s |g_t|^2 d\langle \hat{B} \rangle_t \right\} \right)^{\frac{q - 1}{q}}.
\end{align*}

From (H2), we have
\begin{equation}
\mathbb{E}(M_s)^{1+a} \leq \left( \mathbb{E} \exp \left\{\frac{aq(aq + 1)}{2\kappa_1^2(q - 1)} \int_0^s \frac{1}{(\lambda_t^a)^2} |X_t - Y_t|^2 d\langle \hat{B} \rangle_t \right\} \right)^{\frac{q - 1}{q}}.
\end{equation}
Taking \( q = 1 + \sqrt{1 + a^{-1}} \), it holds that
\begin{align*}
\frac{aq(aq + 1)}{2\kappa_1^2(q - 1)} &= \frac{(a + \sqrt{a(a + 1))}(a + 1 + \sqrt{a(a + 1))}}{2\kappa_1^2 \sqrt{1 + a^{-1}}} \\
&= \frac{(a + \sqrt{a(a + 1))}^2}{2\kappa_1^2} \\
&= \frac{a^2}{8(\kappa_2 - \kappa_1)^2}.
\end{align*}

Then,
\[ \frac{q - 1}{q} = \frac{\sqrt{1 + a^{-1}}}{1 + \sqrt{1 + a^{-1}}} = \frac{\alpha \kappa_1 + 2(\kappa_2 - \kappa_1)}{2\alpha \kappa_1 + 2(\kappa_2 - \kappa_1)}. \]

Therefore, by recalling the expressions (2.21) – (2.23), we get
\begin{equation}
\mathbb{E}(M_s)^{1+a} \leq \exp \left\{ \frac{\alpha K \left(2 + K + \frac{3}{2}\right)(\alpha \kappa_1 + 2(\kappa_2 - \kappa_1))|x - y|^2}{4(\kappa_2 - \kappa_1)^2 \left(\frac{2\kappa_1^2}{\kappa_2^2} - \alpha(2\alpha \kappa_1 + 2(\kappa_2 - \kappa_1))(1 - e^{-\frac{a^2 K(2 + K + \frac{3}{2})}{T}}) \right)} \right\},
\end{equation}
this completes the proof.

\( \square \)

### 2.1.5 Proof to Theorem 2.1

(1) Lemma 2.8 ensures that under \( \hat{\mathbb{E}} := \mathbb{E}[\cdot|M_T], \{\hat{B}_t\}_{t \in [0, T]} \) is a \( G \)-Brownian motion, and
\[ \mathbb{E}[M_T \log M_T] \leq \frac{|x - y|^2}{2\alpha \kappa_1^2 \lambda_0^a}. \]
Then by (2.5) and (2.13), the coupling \((X_t, Y_t)\) is well constructed under \(\bar{\mathbb{E}}\) for \(t \in [0, T]\). Moreover, due to Lemma 2.9, \(X_T = Y_T\) holds \(\bar{\mathbb{E}}\)-q.s., which fits well the requirement of coupling by change of measure. Since for all \(P \in \mathcal{P}\), \(\mathbb{E}_P[M_T] = 1\), by Young’s inequality in Lemma 2.7, for any \(f \in C_b^+(\mathbb{R})\), we obtain

\[
\bar{P}_T \log f(y) = \mathbb{E}[\log f(X_T^y)] = \mathbb{E}[\log f(Y_T^y)] = \mathbb{E}[M_T \log f(X_T^y)] \\
\leq \log \mathbb{E}[f(X_T^y)] + \mathbb{E}[M_T \log M_T] \\
= \log \bar{P}_T f(x) + \mathbb{E}[M_T \log M_T] \\
\leq \log \bar{P}_T f(x) + \frac{|x-y|^2}{2\alpha \kappa_1^2 \alpha_0}
\]

\[
= \log \bar{P}_T f(x) + \frac{K \left(2 + K + \frac{2}{\kappa_2}\right) |x-y|^2}{2\alpha \kappa_1^2 \left(\frac{2\kappa_2^2}{\kappa_1^2} - \alpha\right)(1 - e^{-2^2 K \left(2+K+\frac{2}{\kappa_2}\right) T}).
\]

For \(\alpha \in (0, \frac{2\kappa_2^2}{\kappa_1^2})\), taking \(\alpha = \frac{\kappa_2^2}{\kappa_1^2}\), (1) of Theorem 2.1 holds.

(2) Taking \(\alpha = \frac{2(\kappa_2 - \kappa_1)}{\kappa_1(\sqrt{p} - 1)}\) in (2.24) which is in \((0, \frac{2\kappa_2^2}{\kappa_1^2})\) for \(p > (1 + \frac{\kappa_3^2 - \kappa_1^2}{\kappa_1^2})^2\), we have \(\frac{p}{p-1} = 1 + a\), by Lemma 2.10, this leads to

\[
(\bar{\mathbb{E}}M_{T^x}^\frac{p}{p-1})^{p-1} = (\bar{\mathbb{E}}M_{T^x}^{1+a})^{p-1} = (\bar{\mathbb{E}}M_{T^x}^{a})^{p-1}
\]

\[
\leq \exp \left\{ \frac{(p-1)K \left(2 + K + \frac{2}{\kappa_2}\right) \alpha(\alpha \kappa_1 + 2(\kappa_2 - \kappa_1)) |x-y|^2}{4(\kappa_2 - \kappa_1)^2 \left(\frac{2\kappa_2^2}{\kappa_1^2} - \alpha\right)(2\alpha \kappa_1 + 2(\kappa_2 - \kappa_1)) \left(1 - e^{-2^2 K \left(2+K+\frac{2}{\kappa_2}\right) T}\right)} \right\}
\]

\[
= \exp \left\{ \frac{\sqrt{p}(\sqrt{p} - 1)K \left(2 + K + \frac{2}{\kappa_2}\right) |x-y|^2}{4(\kappa_2 - \kappa_1)[\kappa_1(\sqrt{p} - 1) - (\kappa_2 - \kappa_1)] \left(1 - e^{-2^2 K \left(2+K+\frac{2}{\kappa_2}\right) T}\right)} \right\}.
\]

Thus, due to Hölder’s inequality, for any \(f \in C_b^+(\mathbb{R})\),

\[
(\bar{P}_T f)^y(y) = (\bar{\mathbb{E}} f(X_T^y))^p = (\bar{\mathbb{E}} f(Y_T^y))^p = (\bar{\mathbb{E}} f(X_T^y))^p = (\bar{\mathbb{E}} M_T f(X_T^y))^p
\]

\[
\leq (\bar{\mathbb{E}} f^p(X_T^y)) \left( \mathbb{E} \left[M_{T^x}^\frac{p}{p-1} \right] \right)^{p-1}
\]

\[
\leq \bar{P}_T f^p(x) \exp \left\{ \frac{\sqrt{p}(\sqrt{p} - 1)K \left(2 + K + \frac{2}{\kappa_2}\right) |x-y|^2}{4(\kappa_2 - \kappa_1)[\kappa_1(\sqrt{p} - 1) - (\kappa_2 - \kappa_1)] \left(1 - e^{-2^2 K \left(2+K+\frac{2}{\kappa_2}\right) T}\right)} \right\},
\]

which is the result (2) of Theorem 2.1.
2.2 Gradient Estimate

Due to the lack of additivity of $G$-expectation, neither from the Bismut formula [18, (1.8), (1.14)] by coupling by change of measure to get gradient estimate, nor Malliavin calculus in the $G$-SDEs. Instead, we directly to estimate the local Lipschitz constant defined below.

For a real-valued function $f$ defined on a metric sapce $(H, \rho)$, define

$$|\nabla f(z)| = \limsup_{x \to z} \frac{|f(x) - f(z)|}{\rho(x, z)}, \quad z \in H.$$  (2.25)

Then $|\nabla f(z)|$ is called the local Lipschitz constant of $f$ at point $z \in H$.

**Theorem 2.11.** Assume (H1)-(H2). Then for every $f \in C^+_b(\mathbb{R})$, it holds that

$$\|\nabla \bar{P}_T f\|_{\infty} \leq \|f\|_{\infty} \frac{2}{\kappa_1 \sqrt{\alpha \lambda_0^\alpha}},$$  (2.26)

where $\lambda_0^\alpha$ is defined in (2.3) for $t = 0$.

**Proof.** By the proof of Theorem 2.1, we have

$$|\bar{P}_T f(y) - \bar{P}_T f(x)| = |\hat{E} f(X^y_T) - \hat{E} f(X^x_T)|$$

$$= |\hat{E} M_T f(X^x_T) - \hat{E} f(X^x_T)|$$

$$\leq \|f\|_{\infty} (\hat{E} |M_T - 1|).$$

Noting that $|x - 1| \leq (x + 1)|\log x|$ for any $x > 0$, then

$$|\bar{P}_T f(y) - \bar{P}_T f(x)| \leq \|f\|_{\infty} (\hat{E} (M_T + 1) \log M_T)$$

$$= \|f\|_{\infty} \left( \hat{E} (|\log M_T|) + \hat{E} (|\log M_T|) \right)$$

From (2.16) and (2.17), it holds that

$$\hat{E} (|\log M_T|) \leq \hat{E} \left[ \int_0^T g_t d\hat{B}_t \right] + \hat{E} \left[ \frac{1}{2 \kappa_1^2} \int_0^T \frac{1}{\lambda_0^\alpha} (X_t - Y_t)^2 d\langle \hat{B} \rangle_t \right]$$

$$\leq \hat{E} \left[ \int_0^T \frac{1}{\lambda_0^\alpha} \left| \frac{1}{\kappa_1} (X_t - Y_t)^2 \right| d\langle \hat{B} \rangle_t \right] + \hat{E} \left[ \frac{1}{2 \kappa_1^2} \int_0^T \frac{1}{\lambda_0^\alpha} |X_t - Y_t|^2 d\langle \hat{B} \rangle_t \right]$$

$$\leq \frac{1}{\kappa_1 \sqrt{\alpha \lambda_0^\alpha}} |x - y| + \frac{1}{2 \alpha \kappa_1^2 \lambda_0^\alpha} |x - y|^2.$$

Similarly, we obtain

$$\hat{E} (|\log M_T|) \leq \hat{E} \left[ \int_0^T g_t dB_t \right] + \hat{E} \left[ \frac{1}{2 \kappa_1^2} \int_0^T (X_t - Y_t)^2 d\langle B \rangle_t \right]$$
\[
\begin{align*}
\leq \mathbb{E} \left[ \int_0^T \frac{1}{\lambda_t^2} \left| \frac{1}{\kappa_1} (X_t - Y_t) \right|^2 d\langle B \rangle_t \right] + \mathbb{E} \left[ \frac{1}{2\kappa_1^2} \int_0^T \frac{1}{\lambda_t^2} (|X_t - Y_t|)^2 d\langle B \rangle_t \right] \\
\leq \frac{1}{\kappa_1 \sqrt{\alpha \lambda_0}} |x - y| + \frac{1}{2\alpha \kappa_1^2 \lambda_0^2} |x - y|^2.
\end{align*}
\]

It follows from (2.27) that
\[
(2.28) \quad |\bar{P}_T f(y) - \bar{P}_T f(x)| \leq \|f\|_\infty \left( \frac{2}{\kappa_1 \sqrt{\alpha \lambda_0}} |x - y| + \frac{1}{\alpha \kappa_1^2 \lambda_0^2} |x - y|^2 \right).
\]

This together with (2.25) yields
\[
(2.29) \quad |\nabla \bar{P}_T f(x)| \leq \|f\|_\infty \frac{2}{\kappa_1 \sqrt{\alpha \lambda_0^3}},
\]

which implies (2.26).

\[
\square
\]

3 Appendix—The quasi-continuity of stopping times

This part is essentially from [14, 15]. To make the content self-contained, we cite some results from [14, 15] and restated them as follows.

**Lemma 3.1.** ([15, Lemma 3.3]) Let \( E \) be a metric space and a mapping \( E \times [0, T] \ni (\omega, t) \rightarrow M_t(\omega) \in \mathbb{R} \) be continuous on \( E \times [0, T] \). Define \( \tau_a = \inf \{ t > 0 | M_t > a \} \wedge T \) and \( \tau_a = \inf \{ t > 0 | M_t \geq a \} \wedge T \). Then \( -\tau_a \) and \( \tau_a \) are both lower semi-continuous.

**Lemma 3.2.** ([15, Lemma 3.4]) For any closed set \( F \subset \Omega_T \), we have
\[
c(F) = \inf \{ c(O) | F \subset O, \ O \text{ is open} \},
\]
where \( c \) is the capacity induced by \( \mathbb{E} \).

The following lemma plays a crucial role in studying the quasi-continuity of stopping times under nonlinear expectation space, which is a dramatic different with classic linear expectation space. For reader’s convenience, we give the proof of the lemma.

**Lemma 3.3.** ([14, Lemma 4.3]) Let \( Y_t = \int_0^t \langle Z_s, dB_s \rangle + \int_0^t \eta_s ds + \int_0^t tr[\zeta_s d\langle B \rangle_s] \) with \( Z \in [H^1_G([0, T])]^d \) and \( \eta, \zeta^{i,j} \in M^1_G([0, T]) \). Assume \( \int_0^t \eta_s ds + \int_0^t tr[\zeta_s d\langle B \rangle_s] \) is non-decreasing and
\[
\int_0^t tr[Z_s Z_s^* d\langle B \rangle_s] + \int_0^t \eta_s ds + \int_0^t tr[\zeta_s d\langle B \rangle_s]
\]
is strictly increasing. Then, for \( a > 0, \tau_a := \inf \{ t \geq 0 | Y_t > a \} \wedge T \) is quasi-continuous.
Proof. Let $\tau_a = \inf\{t \geq 0|Y_t \geq a\} \land T$. Since $Y$ is quasi-continuous, then for all $\epsilon > 0$, there exists an open set $O_1$ with $c(O_1) < \frac{\epsilon}{2}$ such that $Y(\cdot)$ is continuous on $O_1^c \times [0, T]$. Define

$$S_a(Y) = \{\omega \in \Omega_T| \text{there exists } (r, s) \in Q, \text{s.t. } Y_t(\omega) = a \text{ for all } t \in [s, r]\},$$

where

$$Q_T = \{(r, s)|T \geq r > s \geq 0, \text{r, s} \in \mathbb{Q}\},$$

and $\mathbb{Q}$ is the totality of rational numbers.

We divide the proof into following five steps.

(1) We first prove $[\tau_a > \tau_a] \subset S_a(Y) \cup \bigcup_{r \in \mathbb{Q} \cap [0, T]} [Y_r \land \tau_a < Y_r \land \tau_a] =: A$.

It is equivalent to prove $[\tau_a > \tau_a] \subset S_a(Y) + A \setminus S_a(Y)$.

For any $\omega \in [\tau_a > \tau_a]$, i.e., for any $\omega$ with $\tau_a(\omega) > \tau_a(\omega)$, if $\omega \in S_a(Y)$, which ends the proof. If $\omega \notin S_a(Y)$, i.e., for any $(r, s) \in Q_T$, there exists a $t \in [s, r]$, s.t. $Y_t(\omega) \neq a$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, and $\tau_a \geq \tau_a$, it's clear that $\omega \in A \setminus S_a(Y)$.

(2) We claim that $c(S_a(Y)) = 0$.

(i) If $Z = 0$, then $Y_t$ is strictly increasing, thus $\tau_a = \tau_a$, which implies $c(S_a(Y)) = 0$.

(ii) If $Z \neq 0$, since $B_t$ with infinite variation, it is impossible for $Y_t = a, t \in [s, r]$, then $c(S_a(Y)) = 0$.

(3) We claim that $c(A) = 0$.

Noting that $Y_r \land \tau_a \leq Y_r \land \tau_a$, and

$$(3.1) \quad \mathbb{E}[Y_r \land \tau_a - Y_r \land \tau_a] = \mathbb{E} \left[ \int_{r \land \tau_a}^{r \land \tau_a} \langle Z, dB_s \rangle + \int_{r \land \tau_a}^{r \land \tau_a} \eta_s ds + \int_{r \land \tau_a}^{r \land \tau_a} tr[\zeta_s d\langle B \rangle_s] \right] = \mathbb{E} \left[ \int_{r \land \tau_a}^{r \land \tau_a} \eta_s ds + \int_{r \land \tau_a}^{r \land \tau_a} tr[\zeta_s d\langle B \rangle_s] \right].$$

For $r \leq \tau_a$ and $r \geq \tau_a$, it hold that $\mathbb{E}[Y_r \land \tau_a - Y_r \land \tau_a] = 0$. For $\tau_a < r < \tau_a$, by (3.1), we have

$$\mathbb{E}[Y_r \land \tau_a - Y_r \land \tau_a] = \mathbb{E} \left[ \int_{\tau_a}^{r} \eta_s ds + \int_{\tau_a}^{r} tr[\zeta_s d\langle B \rangle_s] \right].$$

From the assumption of non-decreasing for $\int_{0}^{t} \eta_s ds + \int_{0}^{t} tr[\zeta_s d\langle B \rangle_s]$, we derive that $\mathbb{E}[Y_r \land \tau_a - Y_r \land \tau_a] \geq 0$. By the fact that $Y_r \land \tau_a \leq Y_r \land \tau_a$ and $\mathbb{E}[Y_r \land \tau_a - Y_r \land \tau_a] \geq 0$, we know that $Y_r \land \tau_a = Y_r \land \tau_a$, q.s.. Since $\mathbb{Q}$ is countable, then $c(A) = 0$.

(4) $A \cap O_1^c$ is an open set under the topology induced by $O_1^c$.

Since $Y(\cdot)$ is continuous on $O_1^c \times [0, T]$, by Lemma 3.1, $\tau_a$ is lower (upper) semi-continuous on $O_1^c$, then $Y_r \land \tau_a$ is lower (upper) semi-continuous on $O_1^c$, which means that $[Y_r \land \tau_a < Y_r \land \tau_a] \cap O_1^c$ is an open set under the topology induced by $O_1^c$. Since the union of any collection of open sets in $O_1^c$ is open, then we prove it.
(5) $S_a(Y)$ can be covered by countable open sets with capacity small enough.

By the definition of $S_a(Y)$, we have

$$S_a(Y) = \bigcup_{(r,s) \in Q_T} \bigcap_{t \in [s,r]} \{ \omega | Y_t(\omega) = a \}.$$  

Since $Y(\cdot)$ is continuous on $O_1^c \times [0, T]$, \{\omega | Y_t(\omega) = a \} \cap O_1^c$ is a closed set under the topology induced by $O_1^c$ for any $t \in [0, T]$. Moreover, \{\omega | Y_t(\omega) = a \} is a closed set as $O_1^c$ is closed. Then $\bigcap_{t \in [s,r]} \{ \omega | Y_t(\omega) = a \}$ is closed. By Lemma 3.2 and the fact that $c(S_a(Y)) = 0$, for all $\epsilon > 0$, there exists an open set $O_2^{s,r}$ with $0 \leq c(O_2^{s,r}) < \frac{\epsilon}{2^{n+1}}$ such that $\bigcap_{t \in [s,r]} \{ \omega | Y_t(\omega) = a \} \subset O_2^{s,r}$. Let $O_2 = \bigcup_{(r,s) \in Q_T} O_2^{s,r}$, then

$$S_a(Y) \subset O_2, \ c(O_2) < \frac{\epsilon}{2},$$

where $O_2$ is open.

Combining (1)–(5), we know that

$$[\tau_a > \tau_a] \subset O_2 \cup A,$$

where $O_2$ is open under topology induced by $\Omega_T$ and $A \cap O_1^c$ is open under the topology induced by $O_1^c$. So, there exists an open set $O_3 \subset \Omega_T$, such that

$$A \cap O_1^c = O_3 \cap O_1^c \subset O_3.$$

Noting that

$$A = (A \cap O_1) \cup (A \cap O_i^c)$$
$$\subset O_1 \cup (O_3 \cap O_i^c)$$
$$\subset O_1 \cup O_3.$$  

Moreover, $O_3 = (O_3 \cap O_1) \cup (O_3 \cap O_i^c)$, by $c(O_3 \cap O_i^c) = 0$ of (3), we have

$$c(O_3) \leq c(O_3 \cap O_1) + (O_3 \cap O_i^c) < \epsilon.$$  

Therefore,

$$[\tau_a > \tau_a] \subset O_2 \cup O_1 \cup O_3,$$

where $c(O_2 \cup O_1 \cup O_3) \leq c(O_1) + c(O_2) + c(O_3) \leq 2\epsilon$. It is clear that

$$[\tau_a > \tau_a] = [\tau_a \leq \tau_a] = [\tau_a = \tau_a] \supset (O_2 \cup O_1 \cup O_3)^c,$$

thus

$$[\tau_a = \tau_a] \cap O_1^c \supset (O_2 \cup O_1 \cup O_3)^c \cap O_i^c = (O_1 \cup O_2 \cup O_3)^c,$$

By Lemma 3.1, $\tau_a$ is continuous on $[\tau_a = \tau_a] \cap O_1^c$. Therefore, for all $\epsilon > 0$, for the open set, $O_1 \cup O_2 \cup O_3$, with $c(O_1 \cup O_2 \cup O_3) < 2\epsilon$, $\tau_a$ is continuous on $(O_1 \cup O_2 \cup O_3)^c$, which implies that $\tau_a$ is quasi continuous by Definition 1.2.  

\[\square\]
Lemma 3.4. ([8, Proposition 4.10]) Let $\tau \leq T$ be a quasi-continuous stopping time. Then for each $p \geq 1$, we have $I_{[0,\tau]} \in M^p_G([0, T])$.

Lemma 3.5. ([8, Remark 4.12]) Let $\tau \leq T$ be a quasi-continuous stopping time and $\eta \in M^p_G([0, T])$. Then for each $p \geq 1$, we have $\eta I_{[0,\tau]} \in M^p_G([0, T])$.

According to [7], for a stopping time $\tau \leq T$, and $\eta \in M^p_G([0, T])$, it holds that

$$\int_0^\tau \eta_s dB_s = \int_0^T \eta_s I_{[0,\tau]}(s) dB_s.$$

Acknowledgement. The authors are grateful to Professor Feng-Yu Wang for his guidance and helpful comments, as well as Yongsheng Song and Xing Huang for their patient helps, valuable suggestions and corrections.

References

[1] J. Bao, F.-Y. Wang, C. Yuan, Derivative formula and Harnack inequality for degenerate functionals SDEs, Stoch. Dyn. 13 (2013), 1–22.

[2] S. Cohen, S. Ji, and S. Peng, Sublinear expectations and Martingales in discrete time, arXiv:1104.5390.

[3] L. Denis, M. Hu, S. Peng, Function spaces and capacity related to a sublinear expectation: application to $G$-Brownian motion pathes, Potential Anal. 34 (2011) 139–161.

[4] M. Hu, S. Ji, S. Peng, Y. Song, Comparison theorem, Feynman-Kac formula and Girsanov transformation for BSDEs driven by $G$-Brownian motion, Stochastic Process. Appl. 124 (2014), 1170-1195.

[5] M. Hu, F. Wang, G. Zheng, Quasi-continuous random variables and processes under the $G$-expectation framework, Stochastic Process. Appl. 126 (2016), 2367–2387.

[6] X. Huang, F.-F. Yang, Harnack inequality and gradient estimate for (functional) $G$-SDEs with degenerate noise, (2019) arXiv:1812.04290.

[7] X. Li, S. Peng, Stopping times and related Itô’s calculus with $G$-Brownian motion, Stochastic Process, Appl. 121 (2011) 1492–1508.

[8] G. Liu, Exit times for semimartingales under nonlinear expectation, (2019) arXiv:1812.00838.

[9] E. Osuka, Girsanov’s formula for $G$-Brownian motion, Stochastic Process. Appl. 123 (2013) 1301–1318.

[10] S. Peng, $G$-Brownian motion and dynamic risk measures under volatility uncertainty, (2007) arXiv: 0711.2834v1.
[11] S. Peng, G-expectation, G-Brownian motion and related stochastic calculus of Itô type, in: Stochastic Analysis and Applications, in: Abel Symp., vol. 2, Springer, Berlin, 2007, pp.541–567.

[12] S. Peng, Nonlinear expectations and stochastic calculus under uncertainty, (2010), arXiv: 1002.4546v1.

[13] P. Ren, F.-F. Yang, Path independence of additive functionals for stochastic differential equations under G-framework, Front. Math. China 14 (2019) 135–148.

[14] Y. Song, Gradient estimates for nonlinear diffusion semigroups by coupling methods, Sci. China Math., https://doi.org/10.1007/s11425-018-9541-6.

[15] Y. Song, Properties of hitting times for G-martingales and their applications, Stochastic Process. Appl. 121 (2011) 1770–1784.

[16] Y. Song, Some properties on G-evaluation and it’s applications to G-martingale decomposition, Sci. China Math. 54 (2011) 287–300.

[17] F.-Y. Wang, Estimates for invariant probability measures of degenerate SPDEs with singular and path-dependent drifts, Probab. Theory Related Fields 172 (2018), no. 3–4, 1181–1214.

[18] F.-Y. Wang, Harnack inequalities for stochastic partial differential equations, Springer Briefs in Mathematics, Springer, New York, 2013, pp, ISBN: 978–1–4614–7933-8, 978–1–4614–7934–5.

[19] F.-Y. Wang, Harnack inequality for SDE with multiplicative noise and extension to Neumann semigroup on nonconvex manifolds, (English summary) Ann. Probab. 39 (2011), 1449–1467.

[20] F.-Y. Wang, Logarithmic Sobolev inequalities on noncompact Riemannian manifolds, Probab. Theory Related Fields 109 (1997) 417–424.

[21] F.-F. Yang, Harnack inequality and applications for SDEs driven by G-Brownian motion, (2018) arXiv:1808.08712.

[22] J. Xu, H. Shang, B. Zhang, A Girsanov type theorem under G-framework, Stoch. Anal. Appl. 29 (2011) 386–406.