Constraints on the Sound Speed of Dynamical Dark Energy

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In this paper we study the sound speed \( c_s^2 \), which is directly related to the classical perturbations of the dynamical dark energy (DE), especially with an equation of state crossing the cosmological constant boundary in details and show its implications on Cosmic Microwave Background (CMB) Anisotropy. With the present observational data of CMB, Type Ia Supernova (SNIa) and galaxy clustering, we perform a global analysis to constrain the sound speed of dark energy, using the Markov Chain Monte Carlo method. We find that the sound speed of dark energy is weakly constrained by current observations thus the futuristic precision measurements of CMB on a very large angular scale (low multipoles) are necessary.

I. INTRODUCTION

The analysis of the redshift-distance relation of SNIa revealed that our Universe is currently accelerating \([1, 2]\) which has been confirmed by the current high quality observations of the CMB \([3]\), the Large Scale Structure (LSS) of galaxies \([4, 5, 6, 7]\), and the SNIa \([8, 9, 10, 11]\). One possible explanation for this phenomenon is that this acceleration is attributed to a new form of energy, dubbed Dark Energy, which recently dominate the energy density of the Universe with negative pressure and almost not clustering. The nature of DE is among the biggest problems in modern physics and has been studied extensively. A cosmological constant, the simplest DE candidate whose equation of state \( w \) remains \(-1\), suffers from the well-known fine-tuning and coincidence problems. Alternatively, dynamical dark energy models with rolling scalar fields have been proposed, such as Quintessence \([12, 13, 14]\), Phantom \([15]\), Quintom \([16]\) and K-essence \([17, 18, 19]\). Given our ignorance of the nature of dark energy, the cosmological observations play a crucial role in our understanding of DE. There are many studies on DE both theoretically and phenomenologically in the literature \([20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50]\).

If we regard the mysterious DE as a cosmic fluid, there will be perturbations when \( w \neq -1 \). In the framework of the linear perturbations theory, the DE perturbations can be fully characterized by its equation of state, \( w \equiv p/\rho \), and its sound speed, \( c_s^2 \equiv \delta p/\delta \rho \). For the DE perturbations it is well-defined in the region of Quintessence, \( w > -1 \), and Phantom, \( w < -1 \). In our previous works \([22, 23]\) based on Quintom dark energy model, we have proposed a new method to handle the DE perturbations in the region closed to \( w = -1 \), assuming the sound speed of Dark Energy is equal to unity, \( c_s^2 = 1 \). We found that the constraints on the DE parameters are relaxed and the parameter space are enlarged dramatically when the DE perturbations are included in the analysis. The effects of perturbations on studying the feature of DE are of great significance.

Since the sound speed of DE merely affects the evolution of perturbation, it has not been investigated extensively in the literature. Recently the observational implications of the sound speed on CMB and LSS of some DE models have been discussed: for example, for K-essence \([51, 52]\), condensation of Dark Matter \([53]\) and the Chaplygin gas \([54, 55, 56, 57, 58]\). In Refs. \([59, 60, 61]\) the authors tried to use the current observational data to constrain the sound speed of DE. However, they just considered the equation of state of DE to be constant. In this paper we consider the dynamical DE model with time varying \( w \) and extend our previous works on the sound speed of DE models and allow an arbitrary constant sound speed in our calculation. Combining the present observational data, such as CMB, LSS and SNIa, we discuss the possible constraints on the sound speed of DE. Our paper is organized as follows: In Section II we analyze the feature of the sound speed of dark energy models paying attention to the case when the equation of state gets across \( w = -1 \); In Section III we describe the method and the datasets we used; In Section IV we present our results derived from a global analysis using Markov Chain Monte Carlo (MCMC) method; Finally, Section V contains a discussion of the results.

II. FEATURE OF SOUND SPEED

A. Single Perfect Fluid

Working in the conformal Newtonian gauge, one can easily describe the DE perturbations as follows \([62]\):

\[
\delta' = -(1 + w)(\theta - 3\Phi') - 3\mathcal{H}(c_s^2 - w_\delta)\delta ,
\] (1)
\[ \theta' = -\mathcal{H}(1 - 3w)\theta - \frac{w'}{1 + w}\theta + k^2(\frac{c_s^2\delta}{1 + w} + \Psi), \]  

where the prime denotes the derivative with respect to conformal time, \( c_s^2 \equiv \delta p/\delta \rho \) denotes the sound speed of DE models, \( \delta \) and \( \theta \) are the density and velocity perturbations of DE models respectively, \( w \equiv p/\rho \) is the equation of state, and \( \Phi \) and \( \Psi \) represent the metric perturbations.

Let us discuss the behavior of the perturbations of single perfect fluid when \( w \) crosses the boundary \( w = -1 \) in the conformal Newtonian gauge, following [60]. Firstly, we consider the DE as a single barotropic fluid. The adiabatic speed of sound, \( c_a^2 \), is purely determined by the equation of state \( w \),

\[ \delta p = \left( \frac{p'}{\rho} \right) \delta \rho \equiv c_a^2 \delta \rho = \left( w - \frac{w'}{3\mathcal{H}(1 + w)} \right) \delta \rho. \]  

Whenever the equation of state closes to the boundary \( w = -1 \) and \( w'|_{w=-1} \neq 0 \), the adiabatic speed of sound, \( c_a^2 \), will be divergent due to the existence of the term \( w'/ (1 + w) \). Apparently, the perturbations of this system become unstable.

However, in non-barotropic fluids, for example in most scalar field models, this simple relation in Eq.(3) between the equation of state and the sound speed breaks down, because of the intrinsic entropy perturbations, and we have the more general relation

\[ c_s^2 \equiv \frac{\delta p}{\delta \rho}. \]  

In this case, if \( c_s^2 \neq c_a^2 \), the intrinsic entropy perturbation \( \Gamma \) will be induced [61, 63]:

\[ w\Gamma \equiv (c_s^2 - c_a^2)\delta = \frac{p'}{\rho} \left( \delta p - \frac{\delta \rho}{\rho'} \right). \]  

Whereas the adiabatic sound speed, \( c_a^2 \), and \( \Gamma \) are scale independent and gauge invariant quantities, while \( c_s^2 \) can be neither. In order to construct the gauge invariant \( c_s^2 \) we use a helpful transformation [63]:

\[ \hat{\delta} = \delta + 3\mathcal{H}(1 + w)\frac{\theta}{k^2}. \]  

This transformation relates the gauge invariant, rest frame density perturbation, \( \hat{\delta} \), to the density and velocity perturbations in a general frame, \( \delta \) and \( \theta \). Using Eqs.(5,16), we can rewrite the pressure perturbation in a general frame, \( \delta p \), in terms of the rest frame sound speed, \( c_s^2 \),

\[ \delta p = \hat{c}_s^2 \delta \rho + 3\mathcal{H}(1 + w)(\hat{c}_s^2 - c_a^2)\rho \frac{\theta}{k^2}. \]  

For example single scalar field models with standard kinetic terms always have \( \hat{c}_s^2 = 1 \). Moreover, the Eqs.(12) can be rewritten as:

\[ \delta' = -(1 + w)(\theta - 3\Phi') - 3\mathcal{H}(\hat{c}_s^2 - w)\delta - 3\mathcal{H}(w' + 3\mathcal{H}(1 + w)(\hat{c}_s^2 - w))\frac{\theta}{k^2}, \]  

\[ \theta' = -\mathcal{H}(1 - 3\hat{c}_s^2)\theta + k^2(\frac{\hat{c}_s^2\delta}{1 + w} + \Psi). \]  

In this case if we require the DE equation of state can cross the boundary \( w = -1 \), the DE perturbations are still divergent (see appendix for details).

**B. Two-field Quintom Dark Energy Models**

As discussed above, the DE models, whose equation of state can cross the boundary \( w = -1 \), need more degrees of freedom to keep the whole system stable [22, 64]. As an example, we consider one type of Quintom DE models with two components, one is Quintessence-like, \(-1 \leq w \leq 1\), and the other is Phantom-like, \( w \leq -1 \) [16, 22, 65] and analyze the perturbations of this DE system.
The most notable advantage, in this case, is that it is unnecessary to let the total equation of state, \( w_{\text{tot}} \), of this Quintom system cross the boundary \( w = -1 \) by forcing the equation of state of each component to cross the boundary. For the Quintessence-like component, \(-1 \leq w \leq 1\), while for the Phantom-like component, \( w \leq -1 \), during the evolution, without any crossing behavior. For each component of the Quintom model the density and velocity perturbations, \( \delta \) and \( \theta \), still satisfy the Eqs. (10-14). Because the equation of state for each component needs not cross the boundary, the perturbations of each component will be stable and all the variables will be well defined. Furthermore, all these corresponding variables of the whole Quintom DE model can be combined with each component:

\[
\begin{align*}
\bar{\rho}_{\text{tot}} &= \bar{\rho}_1 + \bar{\rho}_2 , \\
\tilde{\rho}_{\text{tot}} &= \tilde{\rho}_1 + \tilde{\rho}_2 , \\
\bar{w}_{\text{tot}} &= \frac{\bar{w}_1 \bar{\rho}_1 + \bar{w}_2 \bar{\rho}_2}{\bar{\rho}_1 + \bar{\rho}_2} \\
\delta_{\text{tot}} &= \frac{\bar{\rho}_1 \delta_1 + \bar{\rho}_2 \delta_2}{\bar{\rho}_1 + \bar{\rho}_2} , \\
\bar{\theta}_{\text{tot}} &= \frac{(1 + w_1) \tilde{\rho}_1 \theta_1 + (1 + w_2) \tilde{\rho}_2 \theta_2}{(1 + w_1) \tilde{\rho}_1 + (1 + w_2) \tilde{\rho}_2} .
\end{align*}
\]

Using these Eqs. (10-14), these variables of Quintom DE models are well defined except the velocity perturbations \( \bar{\theta}_{\text{tot}} \).

As long as \( \bar{w}_{\text{tot}} \) gets across the boundary, \( \bar{w}_{\text{tot}} \rightarrow -1 \), the denominator of Eqs. (15) goes to zero, \([1 + w_1] \bar{\rho}_1 + [1 + w_2] \bar{\rho}_2 \rightarrow 0\). Consequently, the total velocity perturbations, \( \bar{\theta}_{\text{tot}} \), become divergent unless these two independent perturbations are equal all the time, \( \bar{\theta}_{\text{tot}} = \theta_1 = \theta_2 \). During the evolution, it is impossible to keep the velocity perturbations of each component equal, \( \theta_1 = \theta_2 \), all the time. However, the physically meaningful velocity perturbation which is relevant to the CMB observations is \( V \equiv (1 + w) \theta \), not \( \theta \) alone. Eqs. (15) can now be written as:

\[
V_{\text{tot}} = \frac{V_1 \bar{\rho}_1 + V_2 \bar{\rho}_2}{\bar{\rho}_1 + \bar{\rho}_2} .
\]

Using this Eq. (16), the divergences disappear when the total equation of state crosses the boundary \( w = -1 \). And then all the variables are well defined and the perturbations of Quintom dark energy system are stable in the whole parameter space of the total equation of state, adding more degrees of freedom.

Let us move to the discussion of the sound speed of the whole Quintom dark energy system. Physically we can use the independent sound speed of each component to describe the whole system. However, the present constraints on the sound speed of DE are so weak that considering to study the two independent sound speed is not justified at present. For simplicity we investigate the effective sound speed, \( c^2_{s,\text{eff}} \):

\[
c^2_{s,\text{eff}} = \frac{c^2_{s,1} (1 + w_1) \bar{\rho}_1 + c^2_{s,2} (1 + w_2) \bar{\rho}_2}{(1 + w_1) \bar{\rho}_1 + (1 + w_2) \bar{\rho}_2} .
\]

It is apparent that this effective sound speed suffers from the same problem with the velocity perturbations \( \bar{\theta}_{\text{tot}} \). In order to settle this divergence of the effective sound speed we fix the sound speed of each component to be equal, \( c^2_{s,1} = c^2_{s,2} = c^2_{s} \). The effective sound speed of the whole Quintom system should be a constant during the evolution:

\[
c^2_{s,\text{eff}} = \frac{c^2_{s} (1 + w_1) \bar{\rho}_1 + c^2_{s} (1 + w_2) \bar{\rho}_2}{(1 + w_1) \bar{\rho}_1 + (1 + w_2) \bar{\rho}_2} = c^2_{s} = c^2_{s,1} = c^2_{s,2} .
\]

The total equation of state of Quintom system has nothing to do with the effective sound speed now.

In our previous works \cite{22,23} we considered the Quintom dark energy model with the effective sound speed \( c^2_{s,\text{eff}} = 1 \). In this paper we relax this limitation and assume an effective arbitrary constant sound speed of Quintom DE model, \( c^2_{s,\text{eff}} = c^2_{s} \). The effects of this effective sound speed on CMB power spectrum will be studied in the next subsection.

C. Implications on CMB Anisotropy

In Fig. 1 we illustrate how the CMB temperature anisotropies of four different DE models characterized by different equations of state \( w \) change on large scales, for different constant sound speed \( c^2_{s} \). Recently we know that our Universe
is dominated by the dark energy component. This means the sound speed of DE can only affect the CMB power spectrum at the very large observable scale via the late Integrated-Sachs-Wolfe (ISW) effect which describes the perturbations induced by the passage of CMB photons through the time evolving gravitational potential wells during DE domination \[59, 60, 61, 67\]. The explicit effect on CMB power spectra caused by the sound speed of DE appears at the large angular scale \(l < 20\). However, as the behavior of the equation of state is close to the Cosmological Constant boundary \(w = -1\), these differences get smaller and even disappear. The different effects of the two DE models in the bottom panels are much clearer than the ones in the top panels which are closer to the boundary.

![Graphs showing CMB temperature power spectra for different DE models](image)

**FIG. 1:** The effects on CMB temperature power spectra of four different Dark Energy models characterized by different equation of states \(w\). On the left two panels, the bottom Black solid line is for the sound speed of \(c_s^2 = 1\) and the top Magenta dash dot dot line is with \(c_s^2 = 0\). In between the sound speed is increasing from top to down with \(c_s^2 = 0.01, 0.05, 0.2\). On the right two panels, the top Black solid line is for the sound speed of \(c_s^2 = 1\) and the bottom Magenta dash dot dot line is with \(c_s^2 = 0\). In between the sound speed is decreasing from top to down with \(c_s^2 = 0.2, 0.05, 0.01\).

### III. METHOD AND DATASETS

During our calculation we choose the commonly used parametrization of the DE equation of state as \[68\]:

\[
    w_{DE}(a) = w_0 + w_1(1 - a) ,
\]

where \(a = 1/(1 + z)\) is the scale factor and \(w_1 = -dw/da\) characterizes the “running” of the equation of state. We modify the current publicly available codes like CMBFAST \[63\] and CAMB \[70\] to allow the DE equation of state to cross the boundary \(w = -1\). For the parametrization of the equation of state which gets across -1, we introduce a small positive constant \(\epsilon\) to divide the full range of the allowed value of \(w\) into three parts: 1) \(w > -1 + \epsilon\); 2) \(-1 + \epsilon \geq w \geq -1 - \epsilon\); and 3) \(w < -1 - \epsilon\). We neglect the entropy perturbation contributions, for the regions 1) and 3) the equation of state does not get across -1 and perturbations are well defined by solving Eqs. \[112\]. For the case 2), the density perturbation \(\delta\) and velocity perturbation \(\theta\), and the derivatives of \(\delta\) and \(\theta\) are finite and continuous for the realistic Quintom Dark Energy models. However, for the perturbations of the parameterized Quintom there...
is clearly a divergence. In our study for such a regime, we match the perturbation in region 2) to the regions 1) and 3) at the boundary and set:

\[ \delta' = 0, \quad \theta' = 0. \] (20)

In our numerical calculations we have limited the range to be \(|\epsilon| < 10^{-5}\) and we find our method is a very good approximation to the multi-field Quintom DE model. For more details of this method we refer the readers to our previous companion papers [22, 23].

In this study we have implemented the publicly available Markov Chain Monte Carlo package CosmoMC [71], which has been modified to allow for the inclusion of DE perturbation with the equation of state getting across \(-1\). We assume purely adiabatic initial conditions and a flat Universe. Our most general parameter space is:

\[ \mathbf{P} \equiv (\omega_b, \omega_c, \Theta_s, \tau, \hat{c}_s^2, w_0, w_1, n_s, \log[10^{10} A_s]) \] (21)

where \(\omega_b \equiv \Omega_b h^2\) and \(\omega_c \equiv \Omega_c h^2\) are the physical baryon and Cold Dark Matter densities relative to the critical density, \(\Theta_s\) is the ratio (multiplied by 100) of the sound horizon to the angular diameter distance at decoupling, \(\tau\) is the optical depth to re-ionization, \(\hat{c}_s^2\) is the effective sound speed of Dark Energy, \(A_s\) and \(n_s\) characterize the primordial scalar power spectrum. For the pivot of the primordial spectrum we set \(k_s = 0.05\) Mpc\(^{-1}\). Furthermore, we make use of the Hubble Space Telescope (HST) measurement of the Hubble parameter \(H_0 \equiv 100h\) km s\(^{-1}\) Mpc\(^{-1}\) [72] by multiplying the likelihood by a Gaussian likelihood function centered around \(h = 0.72\) and with a standard deviation \(\sigma = 0.08\). We also impose a weak Gaussian prior on the baryon density \(\Omega_b h^2 = 0.022 \pm 0.002 (1\sigma)\) from Big Bang Nucleosynthesis [73]. Simultaneously we will also use a cosmic age tophat prior as 10 Gyr < \(t_0\) < 20 Gyr.

In our calculations we have taken the total likelihood to be the products of the separate likelihoods of CMB, LSS and SNIa. In the computation of CMB we have included the three-year WMAP (WMAP3) data with the routine for computing the likelihood supplied by the WMAP team [3]. For LSS information, we have used the 3D power spectrum of galaxies from the SDSS [5] and 2dFGRS [4]. To be conservative but more robust, in the fittings to the 3D power spectrum of galaxies from the SDSS, we have used the first 14 bins only, 0.015 < \(k_{\text{eff}}\) < 0.1, which are supposed to be well within the linear regime [6]. In the calculation of the likelihood from SNIa we have marginalized over the nuisance parameter [74]. The supernova data we used are the “gold” set of 157 SNIa published by Riess et al in Ref.[9].

For each regular calculation, we run 8 independent chains comprising of \(150,000 - 300,000\) chain elements and spend thousands of CPU hours to calculate on a supercomputer. The average acceptance rate is about 40%. We test the convergence of the chains by Gelman and Rubin criteria [75] and find that \(R - 1\) is of order 0.01 which is more conservative than the recommended value \(R - 1 < 0.1\).

![Constraints in the (\(w_0, \hat{c}_s^2\)) and (\(w_1, \hat{c}_s^2\)) planes at 68% (dark) and 95% (light) C.L. from a combined analysis of CMB, LSS and SNIa observational data together.](image-url)
IV. GLOBAL FITTING RESULTS

In this section we mainly present our global fitting results of the dark energy parameters $w_0$, $w_1$ and the effective sound speed of dark energy $c_s^2$. Firstly, in Fig. 3 we show the constraints on the effective sound speed $c_s^2$ from a combined analysis of the CMB, LSS and SNIa observational data. The likelihood contours of $(w_0, c_s^2)$ and $(w_1, c_s^2)$ at 68% and 95% C.L. are nearly vertical lines. The dark energy parameters, as well as other cosmological parameters which we do not list here, are almost independent of the effective sound speed of DE. There is nearly no constraint on the effective sound speed from the present astronomical data, namely, the current observations are still not sensitive to $c_s^2$. Thus the futuristic precision measurements of CMB on a very large angular scale (low multipoles) are necessary.

On the other hand, we consider the effect of the effective sound speed on the constraints of DE parameters. In Fig. 3 we compare the constraints on the dark energy parameters $w_0$, $w_1$ with $c_s^2 = 1$ (Black Dashed lines) and with arbitrary $c_s^2$ (Blue Solid lines). As expected, the constraints on $w_0$, $w_1$ almost unchange. For the model with $c_s^2 = 1$, we find that $w_0 = -1.05^{+0.16}_{-0.15} - 0.30$ and $w_1 = 0.53^{+0.53+0.75}_{-0.55-1.54}$, meanwhile, $w_0 = -1.06^{+0.17+0.37}_{-0.16-0.28}$ and $w_1 = 0.54^{+0.52+0.77}_{-0.55-1.52}$ for the $c_s^2$ varying models. We cut the parameter space of $w_0 - w_1$ plane into six parts by the line of $w_0 = -1$, $w_0 + w_1 = -1$ and $w_1 = 0$. Part III is for quintessence-like models, namely, the equation of state remains greater than -1 regardless of cosmic time, say, $w > -1$ for past, present and future. Correspondingly, part VI is for Phantom-like models. Part I, II, V and IV are all for Quintom-like models. For the models lie within part I and IV, their equations of state have crossed over -1 till now while the EoS of the DE models in part II and V will cross -1 in future.

V. SUMMARY

In this paper we have studied the features of the sound speed of DE in detail and have used the present observational data to constrain the effective sound speed of dark energy $c_s^2$ and the equations of state $w_0$, $w_1$.

If the sound speed is smaller than zero the system is unstable, due to the divergent classical perturbations. If we regard the DE models as single barotropic fluid or non-barotropic fluid, when the equation of state $w$ is close to the boundary $w = -1$ and $w'|_{w=-1} \neq 0$, the classical perturbation of the DE system will be divergent. In order to settle this problem it is necessary to add more degrees of freedom. As an example, we analyze the stability of Quintom DE models with two components. We assume that the two components have the same sound speed to avoid the intuitive divergence.

Using the Markov Chain Monte Carlo method, we preform a global analysis of the $c_s^2$ and $w_0$, $w_1$. We find that the current astronomical data have nearly nothing to do with constraining the effective sound speed of Dark Energy system. The constraint on the sound speed of DE is very weak. The futuristic precision measurements of CMB on a very large angular scale (low multipoles) are necessary.
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APPENDIX: THE PROOF OF THE NO-GO THEOREM

In this appendix we present the detailed proof of the “No-Go” Theorem which forbids the equation of state parameter of a single perfect fluid or a single scalar field to cross the $-1$ boundary.

**Theorem:** For theory of dark energy (DE) in the Friedmann-Robertson-Walker (FRW) universe described by a single perfect fluid or a single scalar field $\phi$ with a lagrangian of $L = L(\phi, \partial_{\mu}\phi\partial^{\mu}\phi)$, which minimally couples to Einstein Gravity, its equation of state $w$ cannot cross over the cosmological constant boundary.

**Proof:** Let us consider the case of fluid firstly. Generally, a perfect fluid, without viscosity and cannot conduct heat, can be described by parameters such as pressure $p$, density $\rho$ and entropy $S$, satisfying the equation of state $p = p(\rho, S)$. According to the properties of fluid, single perfect fluid can be classified into two kinds of form, dubbed as barotropic and non-barotropic.

If the fluid is barotropic, the iso-pressure surface is identical with the iso-density surface, thus the pressure only depends on its density in form of $p = p(\rho)$. From Eq.(3), we can see that the sound speed of a single perfect fluid is apparently divergent when $w$ crosses $-1$, which leads to instability in Dark Energy perturbation.

If the fluid is non-barotropic, the pressure generally depends both on its density and entropy, $p = p(\rho, S)$. The simple form of the sound speed defined in Eq.(3) is not well-defined. From Eq.(4) while taking gravitational gauge invariance into consideration, we can obtain a more general relationship between the pressure and the energy density as follows,

$$\delta \hat{p} = \hat{c}_s^2 \delta \rho,$$

where the hat denotes the gauge invariance. Eq.(6) can be written in a more explicit form as:

$$\delta \hat{p} = \delta p + 3H(\bar{\rho} + \bar{p}) \frac{\theta}{k^2},$$

and correspondingly, gauge-invariant perturbation of pressure turns out to be

$$\delta \hat{p} = \delta p + 3Hc_a^2(\bar{\rho} + \bar{p}) \frac{\theta}{k^2},$$

where $\bar{\rho}$ and $\bar{p}$ are background energy density and pressure respectively, while $\theta$ is the perturbation of velocity as defined in the main text. The gauge-invariant intrinsic entropy perturbation $\Gamma$ can be described as,

$$\Gamma = \frac{1}{w\bar{\rho}}(\delta p - c_a^2 \delta \rho) = \frac{1}{w\bar{\rho}}(\delta \hat{p} - c_a^2 \delta \hat{\rho}).$$

Combining Eqs.(22,23), we obtain the following expression,

$$\delta p = \hat{c}_s^2 \delta \rho + \frac{3H\bar{\rho}(1 + w)(c_a^2 - \hat{c}_s^2)}{k^2}$$

$$= \hat{c}_s^2 \delta \rho + \frac{3H\bar{\rho}(1 + w)}{k^2} - \frac{3H\bar{\rho}(1 + w)}{k^2}w + \bar{\rho}w'.$$

From the Dark Energy perturbation equation Eq.(2), one can see that $\theta$ will be divergent when $w$ crosses $-1$, unless $\theta$ satisfies the condition

$$\theta w' = k^2 \frac{\delta p}{\bar{\rho}}.$$

So we check what will occur when the condition is satisfied.
By substituting the definition of adiabatic sound speed \( c_a^2 \) and the condition Eq. (27) into Eq. (24), we obtain \( \delta \dot{\rho} = 0 \). Note that \( \Gamma = \frac{1}{\epsilon \rho} (\delta \dot{\rho} - c_a^2 \delta \dot{\rho}) \), it is obvious that due to the divergence of \( c_a^2 \) at the crossing point, we have to require \( \delta \dot{\rho} = 0 \) to maintain a finite \( \Gamma \). So we come to the last possibility, that is, \( \delta \dot{\rho} = 0 \) and \( \delta \dot{p} = 0 \). From Eqs. (23, 24), this case requires that \( \delta p = -c_a^2 \frac{3H(1+w)}{k^2} \) and \( \delta \dot{p} = - \frac{3H(1+w)}{k^2} \), and thus \( \delta p = c_a^2 \delta \dot{p} \). It returns to the case of adiabatic perturbation, which is divergent as mentioned above.

Finally, from the analysis of classical stability, we demonstrate that there is no possibility for a single perfect fluid to realize \( w \) crossing \(-1\). For other proofs, see [64, 65].

In the following part we will discuss the case of a single scalar field. The analysis is an extension of the discussion in [22]. The action of the field is given by

\[
S = \int d^4x \sqrt{-g} \mathcal{L}(\phi, \partial_{\mu} \phi \partial^{\mu} \phi),
\]

where \( g \) is the determinant of the metric \( g_{\mu\nu} \). To study the equation of state (EOS) of DE, we firstly write down its energy-momentum tensor. By definition that \( \delta p \equiv \rho \), we define

\[
\delta S = \int d^4x \sqrt{-g} T_{\mu\nu} \delta g^{\mu\nu},
\]

where we define \( X \equiv \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi \). In FRW universe, we have the metric in form of \( g_{\mu\nu} = diag(1, -a^2, -a^2, -a^2) \). In the framework of Einstein Gravity, we can neglect the spatial derivatives of \( \phi \) and rewrite \( X = \frac{1}{2} \dot{\phi}^2 \), where the overdot denotes the derivative with respect to cosmic time \( t \). So \( T_{\mu\nu} \) has the matrix form of \( diag(\rho, p, p, p) \), comparing with the fluid definition: \( T_{\mu\nu} = (\rho + p)U^\mu U^\nu - pg^{\mu\nu} \) (or \( T_{\mu\nu}^0 = (\rho + p)U^\mu U^\nu - pg^{\mu\nu} \)) where \( U^\mu \) is the four-velocity of the fluid. Thus one can get:

\[
\begin{align*}
p &= -T_i^i = \mathcal{L}, \\
\rho &= T_0^0 = -\delta_0^0 \mathcal{L} + \frac{\partial \mathcal{L}}{\partial X} \partial^0 \phi \partial^0 \phi, \\
&= 2Xp, X < 0,
\end{align*}
\]

where \( \cdot \cdot \cdot \) stands for \( \frac{\partial}{\partial X} \).

Using the formulae above, the equation of state \( w \) is given by:

\[
w = \frac{p}{\rho} = \frac{p}{2Xp, X < 0} = -1 + \frac{2Xp, X < 0}{2Xp, X < 0}.
\]

This means that, at the crossing point \( t^* \), \( Xp, X|_{t^*} = 0 \) must vanish. Since \( w \) needs to cross \(-1\), it is required that \( Xp, X \) changes sign before and after the crossing point. That is, in the neighborhood of \( t^* \), \((t^* - \epsilon, t^* + \epsilon)\), we have

\[
Xp, X|_{t^* - \epsilon} \cdot Xp, X|_{t^* + \epsilon} < 0.
\]

Since \( X = \frac{1}{2} \dot{\phi}^2 \) is non-negative, the Eq. (33) can be simplified as \( p, X|_{t^* - \epsilon} \cdot p, X|_{t^* + \epsilon} < 0 \). Due to the continuity of perturbation during the crossing epoch, we obtain \( p, X|_{t^*} = 0 \).

Here we consider the perturbation of the field. We calculate the perturbation equation with respect to conformal time \( \eta \) in form of:

\[
u'' - c_a^2 \nabla^2 u - \left[ \frac{z''}{z} + 3c_a^2 (H' - H^2) \right] u = 0,
\]

where we define

\[
u \equiv aX \frac{\delta \phi}{\delta \eta}, \quad z \equiv \sqrt{\phi^2 |\rho, X|},
\]

and prime denotes \( \frac{d}{d\eta} \) and \( H = \frac{da}{d\eta} \).

When expanding the perturbation function \( u \) by Fourier transformation, one can easily obtain the dispersion relation:

\[
\omega^2 = c_a^2 k^2 - \frac{z''}{z} - 3c_a^2 (H' - H^2),
\]
with $g_2^2$ defined as $\frac{p_X}{\rho_X}$. To make the system stable, we need $g_2^2 > 0$. Note that at the crossing point we require that $\rho_X = 0$ and $p_X$ will change its sign during crossing, one can always find a small region where $g_2^2 < 0$ unless $\rho_X$ also becomes zero at the crossing point with the similar behavior of $p_X$. Therefore, the parameter $z$ will vanish when crossing.

At the crossing point, $H' - H^2$ in the last term in the expression of $\omega^2$ is finite, and when assuming that the universe is fulfilled by that scalar field, it turns out to be zero.

Since at the crossing point, we have $\rho_X = 0$, $z = 0$. Consequently, if $z'' = 0$ at the crossing point the term $\frac{z'}{z}$ will be divergent. Note that, even if $z' = 0$ this conclusion is still valid. In that case, $\frac{z'}{z} = \frac{\rho}{\rho - \rho_X}$ due to the L'Hospital theorem. Since $z$ is a non-negative parameter, $z = 0$ is its minimum and at that point $z'$ must vanish, $\frac{z'}{z}$ is either divergent or equal to $\frac{z''}{z'}$ where $z''$ is also equal to zero as we discussed before. Along this way, if we assume the first $(n - 1)$-th derivative of $z$ with respect to $\eta$ vanishes at the crossing point and $z^{(n)} \neq 0$, which can be applied to an arbitrary positive integer $n$, we can always use the L'Hospital theorem until we find that $\frac{z'}{z} = \frac{z''}{z'}$, which will still be divergent. Therefore, the dispersion relation will be divergent at the crossing point as well, and hence the perturbation will also not be stable.

In summary, we have analyzed the most general case of a single scalar field described by a lagrangian in form of $L = L(\phi, \partial_\mu \phi, \partial_\mu \phi^* \phi)$, and have studied different possibilities of $w$ crossing the cosmological constant boundary. We showed that those cases can either cause the effective sound speed $c_s^2$ to be negative, or lead to a divergent dispersion relation, which makes the system unstable.

To conclude, we have proved that in FRW universe, it is impossible for a single perfect fluid or a single scalar field minimally coupled to Einstein Gravity to have its equation of state $w$ crossing the cosmological constant boundary.

To realize $w$ across $-1$, one must introduce extra degrees of freedom or introduce the nonminimal couplings or modify the Einstein gravity. In the recent years there have been a lot of activities in building models with $w$ crossing $-1$.

The simplest Quintom model is to introduce two scalar fields with one being quintessence-like and the other phantom-like. However, this model suffers from the problem of quantum instability which is inherited from phantom-like. This issue could be solved in the effective description of the quintom model with the operator $\phi \Box \phi$ involved. Consider a canonical scalar field with the lagrangian $L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi)$. This model is well defined at quantum level, however, it does not give $w$ crossing $-1$. As an effective theory as we know the lagrangian should include more operators. If these operators are functions of only the scalar field $\phi$ and its first derivative $\partial_\mu \phi$, as we proved above the $w$ still can not cross over $-1$. However as pointed out in Refs. [27, 29], the existence of the operator $\phi \Box \phi$ makes it possible for $w$ to cross over the cosmological constant boundary. Furthermore, at quantum level as an perturbation theory this effective model is well defined. The connection of this type of Quintom-like theory to the string theory has been considered in [82] and [83].

[1] A. G. Riess et al. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998).
[2] S. Perlmutter et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999).
[3] D. N. Spergel et al., arXiv:astro-ph/0603449; L. Page et al., arXiv:astro-ph/0603450; G. Hinshaw et al., arXiv:astro-ph/0603451; N. Jarosik et al., arXiv:astro-ph/0603452.
[4] S. Cole et al. [The 2dFGRS Collaboration], Mon. Not. Roy. Astron. Soc. 362 (2005) 505.
[5] M. Tegmark et al. [SDSS Collaboration], Astrophys. J. 606, 702 (2004).
[6] M. Tegmark et al. [SDSS Collaboration], Phys. Rev. D 69, 103501 (2004).
[7] M. Tegmark et al., arXiv:astro-ph/0608632.
[8] J. L. Tonry et al. [Supernova Search Team Collaboration], Astrophys. J. 594, 1 (2003).
[9] A. G. Riess et al. [Supernova Search Team Collaboration], Astrophys. J. 607, 665 (2004).
[10] A. Clocchiatti et al. [the High Z SN Search Collaboration], Astrophys. J. 642, 1 (2006).
[11] A. G. Riess et al., arXiv:astro-ph/0611572.
[12] R. D. Peccei, J. Sola and C. Wetterich, Phys. Lett. B 195, 183 (1987).
[13] C. Wetterich, Nucl. Phys. B 302, 668 (1988).
[14] E. Ratra and P. J. E. Peebles, Phys. Rev. D 37, 3406 (1988).
[15] R. R. Caldwell, Phys. Lett. B 545, 23 (2002).
[16] B. Feng, X. L. Wang and X. M. Zhang, Phys. Lett. B 607, 35 (2005).
[17] T. Chiba, T. Okabe and M. Yamaguchi, Phys. Rev. D 62 (2000) 023511.
[18] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. Lett. 85, 4438 (2000).
[19] C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. D 63, 103510 (2001).
[83] I. Y. Aref’eva and A. S. Koshelev, arXiv:hep-th/0605085