Forces in dry active matter

Ydan Ben Dor
Department of Physics and the Russell Berrie Nanotechnology Institute, Technion – Israel Institute of Technology, Haifa 32000, Israel

Yariv Kafri
Department of Physics, Technion – Israel Institute of Technology, Haifa 32000, Israel

Julien Tailleur
Laboratoire Matiere et Systemes Complexes, UMR 7057 CNRS/P7, Universite Paris Diderot, 10 rue Alice Domon et Leonie Duquet, 75205 Paris cedex 13, France
Contents

1 Forces in dry active matter
   1.1 A short recap of different expressions for pressure 2
   1.2 The mechanical pressure of non-interacting self-propelled particles 5
   1.3 Run-and-Tumble particles in 1D: non-local steady state and equation of state 12
   1.4 Momentum and active impulse 13
   1.5 Objects immersed in an active bath: Currents and forces 21
   1.6 Acknowledgements 26

References 27
Forces in dry active matter

Depending on one’s background, the word pressure brings about different definitions. Mechanics, energetics, or continuum mechanics all lead to equally natural expressions for the pressure. For systems in thermal equilibrium, the choice of definition is immaterial and measuring or calculating the pressure using the different expressions gives identical results. In contrast, active systems are out of equilibrium and it is a priori not obvious that the knowledge gained from equilibrium systems can be applied to active matter. Indeed, in recent years it has become clear that much of the equilibrium intuition cannot be exported to even the simplest classes of active systems (Marchetti et al., 2013; Cates and Tailleur, 2015; Bechinger et al., 2016). In particular, the forces exerted by active systems exhibit features which are very different from those of equilibrium systems (Mallory et al., 2014; Takatori et al., 2014; Yang et al., 2014; Fily et al., 2014; Solon et al., 2015a; Solomon et al., 2015b; Winkler et al., 2015; Smallenburg and Löwen, 2015; Yan and Brady, 2015a; Yan and Brady, 2015b; Ginot et al., 2015; Nikola et al., 2016; Speck and Jack, 2016; Joyeux and Bertin, 2016; Falasco et al., 2016; Fily et al., 2017; Rodenburg et al., 2017; Sandford et al., 2017; Razin et al., 2017a; Razin et al., 2017b; Ginot et al., 2018; Sandford and Grosberg, 2018; Baek et al., 2018; Rohwer et al., 2018).

As we discuss here, this has many implications for systems ranging from shaken granular gases (Junot et al., 2017) to the motion of cells (Poujade et al., 2007). For example, in equilibrium systems, in order to change the force exerted by the system on its container one has to change the bulk properties of the system. This is a direct consequence of the existence of an equation of state, which expresses the pressure solely in terms of bulk quantities of the system. In contrast, for many active systems the forces exerted by the system on the container walls do not obey an equation of state (Solon et al., 2015a). This implies that forces exerted by a system on its container can be changed without changing its bulk properties but instead by altering the surface of the container.

The purpose of these lecture notes is to give a pedagogical introduction to recent developments in the thermomechanics of active systems. Complementary details and more exhaustive treatments can be found in the corresponding publications (Solon et al., 2015a; Nikola et al., 2016; Fily et al., 2017; Baek et al., 2018). The lectures start in Section 1.1 with a short overview of different definitions of pressure and the conditions under which they are equivalent. This allows us to identify a class of systems – commonly referred to as dry active matter – where the simple equilibrium intuition might break. Following this, we turn to discuss simple models of dry active systems in Section 1.2 and show that there is, in general, no equation of state for the forces they
Fig. 1.1 Illustration of a one-dimensional system of length $L$, confined by two potentials at its two ends. The potentials model the presence of a confining vessel.

exert on their container. To better understand the physical origin of the lack of an equation of state, we then focus in Section 1.3 on an exactly solvable one-dimensional model. We rationalize our results in Section 1.4 in terms of an active impulse, which helps us explain why in certain classes of finely tuned models an equation of state is restored. Finally, we go beyond the question of the equation of state and provide a brief discussion of recent developments concerning the forces exerted on objects immersed in active systems in Section 1.5.

1.1 A short recap of different expressions for pressure

We now present three standard definitions of pressure and show that they are equivalent in equilibrium. For simplicity, the discussion will be carried out in one dimension for a system confined between two walls, specified by bounding potentials at its two ends (see Fig. 1.1). Following this, we will argue why things might be different for active systems.

1.1.1 An energetic definition

In thermodynamics, the pressure is directly related to the change in the energy of a system under a change in its length, $L$, through

$$P_E = -\frac{\partial E}{\partial L}_{S,N} \quad (1.1)$$

with $S$ the entropy of the system (implying here no exchange of heat with the environment), and $N$ the number of particles in the system. Of course, other choices of state functions are possible. For example, using the Helmholtz free energy, $F$, gives

$$P_E = -\frac{\partial F}{\partial L}_{T,N} \quad \text{with } T \text{ the temperature at which the system is held fixed as its length varies.}$$

Note that since the energy is extensive, i.e. $E$ is proportional to $L$, this definition directly implies that the details of the interactions of the particles with the confining walls, which in one dimension scales as $L^0$, do not change the pressure. This insensitivity to the details of the wall potential implies that the pressure is a state-function that depends only on intensive bulk quantities. Whether the interaction of the particles with the wall is hard-core or the interaction binds the particles strongly to the wall, the pressure remains unchanged.
1.1.2 A mechanical definition

The second definition of pressure is simply the force per unit area on the walls confining the system. Given a steady-state density of particles at position $x$, $\rho(x) = \sum_{i=1}^{N} \delta(x - x_i)$, with $x_i$ the position of particle $i = 1 \ldots N$, the pressure can be written as

$$P_M = \int_{x^*}^{\infty} dx \langle \rho(x) \rangle \partial_x V(x - L),$$

(1.2)

where $V(x - L)$ is the potential of the confining wall which, say, starts at position $L$ (see Fig. 1.1), the angular brackets denote an average over the steady-state distribution, and $x^*$ is a point deep in the system. By “deep” we mean that the steady-state distribution at $x^*$ is independent of the details of the bounding potential.

In contrast to the energetic definition, the existence of an equation of state (EOS) for the pressure $P_M$, namely the independence on the shape of the confining wall, is not at all obvious – there is no reason for the integral in (1.2) to be independent of $V$ a priori. However, for equilibrium systems, this is easy to verify by proving the equality $P_M = P_E$. The thermodynamic definition, $P_E$, is given by

$$P_E = -\frac{\partial F}{\partial L} \bigg|_{T,N},$$

(1.3)

with

$$F = -\frac{1}{\beta} \ln Z,$$

(1.4)

and

$$Z = \sum_c e^{-\beta[H + \sum_i V(x_i - L)]}.$$  

(1.5)

Here the sum is over micro-states, $\beta$ is the inverse temperature, and $H$ contains all the other interactions in the system. Using the definition of $P_E$, as given in Eq. (1.3), we have

$$P_E = -\frac{1}{Z} \sum_c \sum_j \partial_L V(x_j - L)e^{-\beta[H + \sum_i V(x_i - L)]} = \int dx \langle \rho(x) \rangle \partial_x V(x - L) = P_M$$

(1.6)

where the angular brackets denote a thermal average. Note that the proof only requires that the system is in equilibrium through the Boltzmann distribution. As stated above, however, there is no a priori obvious reason why $P_M$ should in general be independent of $V$.

1.1.3 A momentum-flux-based definition

A third definition arises in continuum mechanics. We consider a momentum-conserving system comprising $N$ particles of mass $m$, positions $x_i$, and velocities $v_i$. Assuming a

\[1\text{In practice, this means that } L - x^* \gg \xi \text{ and } x^* \gg \xi, \text{ where } \xi \text{ is the correlation length, which is assumed to be finite.}\]
Forces in dry active matter

local dynamics, the conservation equation for the momentum field, \( \dot{\mathbf{p}} = \sum_i m v_i \delta(x - x_i) \), reads

\[
\partial_t \dot{\mathbf{p}}(x) = -\partial_x J_{\mathbf{p}}(x) .
\] (1.7)

Here \( J_{\mathbf{p}}(x) \) is the momentum current which is in general a noisy fluctuating quantity. For a non-interacting ideal gas, for example, the momentum flux is given by \( J_{\mathbf{p}} = \sum_i m v_i v_i \delta(x - x_i) \), with \( v_i \) the velocity of particle \( i \) and \( m \) the particle mass. One then identifies the pressure with the steady-state flux of momentum in the bulk of the system when the fluid is static

\[
P_p = \langle J_{\mathbf{p}}(x^*) \rangle = -\sigma ,
\] (1.8)

where we introduced the stress \( \sigma \) and \( x^* \) is again a point deep in the bulk of the system. In higher dimensions, Eq. (1.8) is generalized to the trace of the stress tensor divided by the number of dimensions, \( P_p = -\frac{1}{d} \text{Tr} \sigma \). The equivalence with the mechanical definition is rather intuitive for momentum conserving systems. It can be seen by explicitly including a wall potential in the equations for the momentum flux

\[
\partial_t \dot{\mathbf{p}}(x) = -\partial_x J_{\mathbf{p}}(x) - \rho(x) \partial_x V(x) ,
\] (1.9)

with \( \rho(x) \) the density of particles at point \( x \) as defined above. In the steady state this becomes

\[
-\langle \partial_x J_{\mathbf{p}}(x) \rangle = \langle \rho(x) \partial_x V(x) \rangle ,
\] (1.10)

which states that the change in momentum flux is caused by the force exerted on the particles by the potential \( V \). Integrating over space we obtain

\[
P_p = \langle J_{\mathbf{p}}(x^*) \rangle = \int_{x^*}^{\infty} dx \langle \rho \rangle \partial_x V(x) = P_M .
\] (1.11)

Note that this result implies that any momentum conserving system with local dynamics (whether in equilibrium or not) and whose bulk properties are independent of the bounding potential admits an equation of state\(^2\). Namely, the force exerted on the wall (or the momentum flux as defined above) is independent of the shape of the potential.

1.1.4 Discussion

To summarize the above discussion, we make the following points and observations:

- The independence of the mechanical pressure on the shape of the potential is guaranteed for both equilibrium systems (Sec. 1.1.2) and systems which conserve momentum and have local dynamics (Sec. 1.1.3).
- Depending on the system, either one of these might be at play. For example, osmotic pressure allows for an extra mechanism of loss of momentum by a flow of the solvent through the membrane which is not captured by Eq. (1.9). In this case, \( P_M \) is guaranteed to be given by an equation of state only when the system is in

\(^2\)Clearly, a trivial non-homogeneity such as phase separation is allowed.
The mechanical pressure of non-interacting self-propelled particles

equilibrium. Conversely, in a non-equilibrium system which conserves momentum in the bulk, the mechanical pressure exerted on a container will obey an equation of state.

- In active systems which continuously absorb and dissipate energy, the thermodynamical definition of pressure is clearly useless.

In active matter, many experimentally relevant situations involve systems which do not conserve momentum and are out of equilibrium. These can be, for example, active particles moving in 2D next to a surface. Such systems are typically referred to as dry active systems (as opposed to wet ones) \cite{Marchetti2013}. The above discussion suggests that in these systems the existence of an equation of state is not obvious. Note that the same holds for wet systems confined by porous walls. Whether or not (and when) the intuition built-in equilibrium on the statistical properties of forces extend to such active cases is one of the issues that these lectures address. While the discussion will be carried out for a particularly simple class of active systems, the broad results and conclusions, as should be evident, are expected to hold generally. For pedagogical purposes, we focus on deriving results for non-interacting particles. Generalizations to cases with interactions will be commented on and can be found in the literature \cite{Takatori2014,Yang2014,Solon2015a,Solon2015b,Falasco2016,Rodenburg2017,Fily2017}.

1.2 The mechanical pressure of non-interacting self-propelled particles

1.2.1 Active Brownian Particles – a simple model for active systems

In what follows we study a model of non-interacting Active Brownian Particles (ABPs) (all our results directly extend to run-and-tumble particles (RTPs)). We first introduce and discuss the active dynamics in the bulk, and then turn to the effect of the walls.

ABPs are described using their positions, \( \mathbf{r}_i \), and orientations, \( \theta_i \), where \( i \) is the particle index. In the overdamped regime, their dynamics are given by:

\[
\begin{align*}
\dot{\mathbf{r}}_i &= v \mathbf{u}(\theta_i) + \sqrt{2D_t} \eta_i(t), \\
\dot{\theta}_i &= \sqrt{2D_r} \xi_i.
\end{align*}
\] (1.12)

Here \( v \) quantifies the activity, signalling that an energy source is constantly used in order to propel the particle. \( \mathbf{u}(\theta_i) = (\cos(\theta_i), \sin(\theta_i)) \) is a director, \( D_t \) and \( D_r \) are the translational and rotational diffusion coefficients, respectively, \( \eta_i, \xi_i \) are random Gaussian noises of zero mean and correlations \( \langle \eta_k(t) \eta_{\ell}(t') \rangle = \delta_{k\ell} \delta(t-t') \), where \( k \) and \( \ell \) refer to spatial components, and \( \langle \xi_i(t) \xi_{\ell}(t') \rangle = \delta_{i\ell} \delta(t-t') \). It is obvious that this system does not conserve momentum and is therefore a dry active system.

The outcome of these dynamics is that an active Brownian particle roughly follows a straight trajectory over a persistence length \( \ell_r = \frac{v}{D_r} \) before reorienting itself (as depicted in Fig. 1.2). Therefore, on large length scales and long times, an ABP has a diffusive dynamics. The diffusion constant can be obtained through a simple dimensional analysis. It is given by
Forces in dry active matter

Fig. 1.2 A cartoon trajectory of an active Brownian particle. The particle is self-propelled at constant speed \( v \); it undergoes translational and rotational diffusion. The particle reorients itself after a persistence length of \( \ell_r = v/D_r \).

\[
D_{\text{eff}} = \frac{v^2}{dD_r} = \frac{1}{d}v\ell_r
\]

(1.13)

with \( d \) being the spatial dimension of the system.

Note that since each particle is self-propelled in a given direction, looking at a movie showing an ABP moving around is different from looking at its reversed version: running the movie backward will not change the director’s direction. This means that ABPs break time-reversal symmetry in a trivial manner, which shows that this model is out of equilibrium.

We now complement this basic model to include the presence of confining walls and calculate the pressure \( P_M \).

1.2.2 The pressure of ABPs

To calculate \( P_M \), we consider a system of \( N \) particles, \( i = 1, ..., N \) that follow the equations of motion (Solon et al., 2015a)

\[
\dot{r}_i = v u(\theta_i) - \mu \nabla V + \sqrt{2D_r} \eta_i(t) ,
\]

\[
\dot{\theta}_i = \Gamma(r_i, \theta_i) + \sqrt{2D_r} \xi_i .
\]

(1.14)

Here, on top of the dynamics of Eq. (1.12), we introduce a confining wall modelled by a potential \( V \). In what follows, we take the potential to be uniform along the \( \hat{y} \) direction (Fig. 1.3), with periodic boundary conditions. The dynamics of the angles \( \theta_i \) now accounts for torques acting on the particles due to the wall through the term \( \Gamma(r_i, \theta_i) \). Evidently, \( \Gamma \) vanishes in the bulk and, with the exception of circular particles, is non-zero in the vicinity of the wall. Finally, the noise terms are taken identical to

\[\text{Note that the breakdown of time-reversal symmetry is much less apparent if only the position of the particle is recorded, and not its orientation (Fodor et al., 2016; Mandal et al., 2017; Shankar and Marchetti, 2018).}\]

\[\text{Note that for ease of notation the rotational mobility, } \mu_r, \text{ is absorbed into } \Gamma.\]
The mechanical pressure of non-interacting self-propelled particles

Fig. 1.3 An active Brownian particle next to a wall. The particle is confined in the $\hat{x}$ direction whereas periodic boundary condition are used along $\hat{y}$. The wall is modelled by a potential $V(x)$ that leads to a confining force, and may also lead to torques acting on the particle.

the case without a wall – white noises with unit variance. We stress again that this model does not conserve momentum and is not in thermal equilibrium (for $v \neq 0$). This is the class of models where interesting behavior might be expected.

To calculate $P_M$, we first write the Fokker-Planck equation corresponding to Eq. (1.14)

$$
\partial_t \mathcal{P}(\mathbf{r}, \theta) = -\nabla \cdot [v \mathbf{u}(\theta) \mathcal{P}(\mathbf{r}, \theta) - \mu (\nabla V) \mathcal{P}(\mathbf{r}, \theta) - D_t \nabla \mathcal{P}(\mathbf{r}, \theta)]
- \partial_\theta [\Gamma(\mathbf{r}, \theta) \mathcal{P}(\mathbf{r}, \theta) - D_r \partial_\theta \mathcal{P}(\mathbf{r}, \theta)].
$$

(1.15)

Since the particles are non-interacting, $\mathcal{P}(\mathbf{r}, \theta)$ can be identified with the average density of particles at position $\mathbf{r}$ with angle $\theta$. Integrating Eq. (1.15) over $\theta$ and utilizing the translational symmetry along the $\hat{y}$ direction, we obtain

$$
\partial_t \rho(\mathbf{r}) = -\partial_x J_x(\mathbf{r}) ; \quad J_x(\mathbf{r}) = v m_1(\mathbf{r}) - \mu (\partial_x V) \rho(\mathbf{r}) - D_t \partial_x \rho(\mathbf{r}).
$$

(1.16)

Here $J_x(\mathbf{r})$ is the density current along the $\hat{x}$ direction,

$$
m_n(\mathbf{r}) = \int d\theta \cos(n\theta) \mathcal{P}(\mathbf{r}, \theta),
$$

(1.17)

and we denote $\rho(\mathbf{r}) = m_0(\mathbf{r})$. In the steady state, the geometry of the system implies that $J_x(\mathbf{r}) = 0$, which sets

$$
(\partial_x V) \rho(\mathbf{r}) = \frac{1}{\mu} [v m_1(\mathbf{r}) - D_t \partial_x \rho(\mathbf{r})].
$$

(1.18)
Integrating this expression from a point deep in the bulk of the system (which we set to be \( x = 0 \)) to infinity gives

\[
P_M = \int_0^\infty dx \, \rho(x) V(x) = \frac{1}{\mu} \left[ D_t \rho_0 + \int_0^\infty dx \, v m_1(x) \right]
\]  

(1.19)

where \( \rho_0 \) is the density of particles in the bulk of the system. To proceed, we solve for \( m_1 \). Multiplying Eq. (1.15) by \( \cos(\theta) \) and integrating over the angle \( \theta \), we find that in the steady state

\[
D_r m_1(r) = -\partial_x \left[ \frac{\rho(r) + m_2(r)}{2} - \mu(\partial_x V) m_1(r) - D_t \partial_x m_1(r) \right]
\]

\[
- \int_0^{2\pi} d\theta \, \Gamma(x, \theta) \sin(\theta) P(x, \theta),
\]  

(1.20)

where we used \( \cos^2(\theta) = \frac{1}{2} [1 + \cos(2\theta)] \). Using this in Eq. (1.19) and noting that the system is isotropic in the bulk so that \( m_n \big|_{x=0} = 0 \), gives (Solon et al., 2015a)

\[
P_M = \rho_0 \left[ \frac{v^2}{2\mu D_r} + \frac{D_t}{\mu} \right] - \frac{v}{\mu D_r} \int_0^\infty dx \int_0^{2\pi} d\theta \, \Gamma(x, \theta) \sin(\theta) P(x, \theta).
\]  

(1.21)

The above result has several implications which we now discuss in detail:

**Equilibrium limit.** In equilibrium, taking \( v = 0 \), the mechanical pressure is, as expected, given by the ideal-gas result

\[
P_M = \rho_0 \frac{D_t}{\mu} = \rho_0 T
\]  

(1.22)

where we set the Boltzmann constant to unity and use the fluctuation-dissipation relation \( D_t = \mu T \).

**Torque-free particles.** In cases where the walls do not exert any torque on the active particles, \( \Gamma = 0 \), the mechanical pressure \( P_M \) is independent of the wall potential \( V(x) \). In this case, the pressure is given by

\[
P_M = \rho_0 \left[ T + \frac{1}{\mu} \frac{v^2}{2D_r} \right].
\]  

(1.23)

The first term is the ideal gas contribution, while the second is given by \( \rho_0 D_{\text{eff}}/\mu \). This second term is what one might guess from dimensional analysis; historically, it was first proposed using continuum mechanics arguments and a virial formula (Mallory et al., 2014; Yang et al., 2014; Takatori et al., 2014; Falasco et al., 2016).

**The impact of torques and the lack of an EOS.** In the presence of torques, \( \Gamma(r, \theta) \neq 0 \) depends explicitly on the functional form of the wall potential. Further-

\footnote{Here we assume that another wall is positioned far away at the left end of the system and ensures the existence of a steady state.}
The mechanical pressure of non-interacting self-propelled particles

The mechanical pressure of non-interacting self-propelled particles

\[ \frac{2\mu D_r P_M}{\rho_0 v^2} = \frac{1}{1 - \exp \left( -\frac{\lambda \kappa}{D_r} \right)} \]

(1.25)

This dependence is shown in Fig. 1.4. Importantly, for both \( \kappa \) positive and negative, there is an order-one change of the magnitude of the pressure as the strength of the harmonic potential \( \lambda \) changes, highlighting the important impact of wall torques on the pressure.

An intuitive illustration of how torques affect the value of \( P_M \) is given in Fig. 1.5. The active contribution to the pressure arises from the particles transferring their
Forces in dry active matter

Fig. 1.5 Impact of wall torques on the active contribution to the mechanical pressure. **Top:** For \( \kappa > 0 \), a torque aligns the particle’s orientation with the tangent to the wall, hence diminishing its contribution to the pressure. **Bottom:** For \( \kappa < 0 \), a torque aligns the particle’s orientation with the normal to the wall, hence increasing the mechanical pressure.

active forces onto the wall. If torques rotate the particles and constrain their motion along the wall, the force acting on the wall is reduced (compared to, say, torque-free particles). Conversely, if torques make particles face the wall, the forces acting on the wall are enhanced.

**Pairwise forces.** Following a more elaborate derivation for a torque-free model of ABPs interacting via pairwise interactions of the form \( U_{ij} = U(r_i - r_j) \), one finds that an EOS exists in this case. The pressure \( P_M \) can then be expressed in terms of bulk correlators (see Solon et al. 2015a, Solon et al. 2015b, Fily et al. 2017 for details) which coincides with the pressure derived using a virial/continuum mechanics approach (Takatori et al., 2014; Yang et al., 2014; Falasco et al., 2016).

**Aligning and quorum-sensing interactions.** For models ignoring the torques exerted by the walls but including either aligning interactions or quorum-sensing interactions (in which the single-particle velocity varies with the local density, \( \rho(r) \), in its vicinity, \( v = v[\rho(r)] \)), one finds that there is no EOS. Here again, \( P_M \) depends explicitly on the form of \( V(r) \).

**Obstacles with asymmetric stiffness.** In the large size limit, changing the wall potential has no effect on the bulk properties of the system. In the presence of an equation of state, \( P_M \) is then independent of the wall potential. On the contrary, when \( P_M \) does not obey an equation of state, changing the potential of the walls alters the forces acting on it. This result has recently been observed experimentally.
The mechanical pressure of non-interacting self-propelled particles

Fig. 1.6 A mobile partition that is stiffer on one side divides the system into two compartments, each initially with equal density. Since the pressure depends on the wall’s stiffness through Eq. (1.25), each side of the partition will experience a different force from the active particles. As a result, the mobile partition moves until the forces on both sides balance.

Fig. 1.7 Numerical simulations corresponding to the setup described in Fig. 1.6 using either torque-free ABPs (left panel) or elliptical ABPs (right panel). The absence of an EOS in the latter case is apparent from the spontaneous compression of one half of the system. Figure adapted from (Solon et al., 2015).

in systems of vibrated, self-propelled grains (Junot et al., 2017). This effect can have striking consequences. Consider a system of homogeneous active particles in the middle of which a mobile partition is inserted (see Fig. 1.6). One side of the partition has a stiffer potential than the other side. The lack of an EOS implies that the partition will move until the densities on both sides are such that the values of the mechanical pressure $P_M$, corresponding to different potentials on each side, are equal (see Fig. 1.7). The new position of the partition can be obtained using the construction illustrated in Fig. 1.8 and, in general, leads to unequal densities on both sides of the partition. In fact, this setup is a natural test for the existence of an EOS.

In the discussion above, we found that in dry active systems, that are out of equilibrium by construction and do not conserve momentum, the pressure $P_M$ may or may not have an EOS. It is interesting to understand what are the conditions needed to ensure the existence of an EOS for $P_M$ and conversely when do these conditions break. This will be the focus of much of the discussion that follows. We will show that when an equation of state exists, it results from a new conservation law which holds statistically in the steady state.

It is obvious from (1.21) that the mechanical pressure depends crucially on the form of the steady-state density in the presence of a potential. To study whether the simple fact that this steady state is not given by a Boltzmann law suffices to explain the lack of EOS, we first look at a simple one-dimensional active system, whose steady-
state distribution can be computed exactly. Much of the conclusions from this simple example carry over to more complicated settings.

1.3 Run-and-Tumble particles in 1D: non-local steady state and equation of state

Consider a one-dimensional system of non-interacting run-and-tumble particles, originally inspired by the motion of E. Coli bacteria (Berg, 1993; Schnitzer, 1993; Tailleur and Cates, 2008). In the absence of an external potential, particles move with velocity $v$ or $-v$, changing between the two with rate $\alpha/2$. Accounting for the action of an external potential $V(x)$, the equations for the average density of right moving particles, $P_+(x,t)$, and left-moving particles, $P_-(x,t)$, are given by

$$\partial_t P_+(x,t) = -\partial_x [v P_+(x,t) - \mu(\partial_x V) P_+(x,t)] - \frac{\alpha}{2} P_+(x,t) + \frac{\alpha}{2} P_-(x,t),$$

$$\partial_t P_-(x,t) = -\partial_x [-v P_-(x,t) - \mu(\partial_x V) P_-(x,t)] - \frac{\alpha}{2} P_-(x,t) + \frac{\alpha}{2} P_+(x,t).$$

(1.26)

Here $\mu$ is the mobility and the translational diffusion of the particles is ignored ($D_t = 0$). In the following we follow the derivation presented in Solon et al. 2015a (See also Van den Broeck and Hänggi 1984 for an earlier derivation). While this implies that the particles cannot cross a potential which exerts a force larger than $v/\mu$, this feature does not change the main points that we report below.

Assuming that the system is confined at its boundaries so that there is no current flowing through the system, the equation for the steady-state density $\rho(x) = P_+(x) + P_-(x)$ reads

$$\partial_x \left[(v^2 - \mu^2(\partial_x V)^2)\rho\right] + \alpha \mu(\partial_x V) \rho = 0,$$

(1.27)
whose solution is given by
\[ \rho(x) = \rho(0)e^{-Q} \] (1.28)
with \( \rho(0) \) the density at \( x = 0 \), and
\[ Q = \ln \left( 1 - \left( \frac{\mu}{v} \right)^2 \left( \partial_x V \right)^2 \right) + \frac{\alpha \mu}{v^2} \int_0^x dx' \frac{\partial_x V(x')}{(\frac{x'}{v})^3} - 1. \] (1.29)

This result was first derived in Van den Broeck and Hänggi 1984 in the context of stochastic flows driven by non-Gaussian noises.

It is also interesting to look at the polarization of the active particles \( \Delta(x) = P_+(x) - P_-(x) \). This can be checked to obey the differential equation
\[ \Delta = \frac{1}{\alpha} \partial_x (v\rho - \mu(\partial_x V)\Delta) \] (1.30)
whose consequences we discuss below.

The above results imply the following:

- To leading order in \( V \), \( Q(x) = \frac{\alpha \mu}{v^2} V(x) \) — the density is then a local function of the potential \( V(x) \) and behaves as an effective Boltzmann distribution, with an effective inverse temperature \( \beta_{\text{eff}} = \frac{\alpha \mu}{v^2} \).

- At second order in \( V \), the density remains a local function of the potential, but is not of the Boltzmann form and depends explicitly on derivatives of the potential.

- At third order and higher, the density profile is not a local function of the potential \( V(x) \). Namely, the density at point \( x \) can depend, in general, on the shape of the potential everywhere in the system. Indeed, at cubic order in the potential, we have a contribution of the form \( \int_0^x dx' (\partial_x V)^3 \). This has many striking consequences. For example, when an asymmetric, say sawtooth potential is placed in the middle of the system, it is easy to check using the results above that it leads to different densities on both of its sides, with the density difference depending on the exact shape of the asymmetric potential. This is illustrated in Fig. 1.9 and can be thought of as a simple model for the bacterial ratchet experiment described in Galajda et al. 2007. Furthermore, the same setup with periodic boundary conditions instead of walls leads to a non-uniform density profile with a current flowing in the system.

- The expression for \( \Delta \), Eq. (1.30), implies, along with the expression for the density, that particles are polarized inside the potential region. This occurs even to leading order in \( V \), when the system is effectively in equilibrium.

- Using the expression for the density (or methods similar to those of Sec. 1.2) it is easy to verify that, for the model defined above, \( P_M \) admits an equation of state. This is not in contradiction with the non-locality mentioned above: outside the wall region, \( \partial_x V = 0 \) and the steady state is uniform. Furthermore, the existence of an equation of state for \( P_M \) is in line with the results of the previous section, as

---

6For a detailed explanation more closely related to the experiment, which also highlights the role of asymmetric tumbles, see Tailleur and Cates 2009.
Forces in dry active matter

Fig. 1.9 An asymmetric barrier placed in an active system. Due to the asymmetry of the barrier, the steady-state distribution of active particles shows density differences between the two sides of the barrier. This stems from the non-local nature of the distribution, as evident from Eq. (1.29). Imposing periodic boundary conditions would generate a steady-state current through the system.

there is no analog of torques in the model. Torques can be incorporated through a position dependent flipping rate \( \alpha(x) \), which indeed leads to a lack of an equation of state for \( P_M \). It is important to stress that the equation of state arises in the presence of polar order inside the potential \( V(x) \). In other words, there is no relation between the existence of polar order inside the potential and the lack of an equation of state.

The above discussion again highlights that even when the system has features that are manifestly non-equilibrium, an equation of state for \( P_M \) may or may not exist. Along with the results of Sec. 1.2, this suggests that active systems generically have no equation of state, but as we saw, and rather surprisingly, in some cases they do. In what follows, we will see that in systems that possess an EOS, there is a hidden “conserved” quantity in the steady state. While the discussion can be carried out directly for overdamped systems, it is more transparent for an underdamped model of ABPs that we introduce below.

1.4 Momentum and active impulse

We now return to address the question of when and why an equation of state emerges in some cases while in others it does not. As we saw in Sec. 1.1, the existence of bulk momentum conservation in the system and the existence of an equation of state are linked. To this end, in the following section, we consider a model of underdamped ABPs and study its momentum flux. Note that the following discussion is done in two dimensions.

1.4.1 Underdamped active Brownian particles

For \( N \) underdamped, non-interacting active Brownian particles, the position of particle \( i \), located at \( \mathbf{r}_i \), evolves according to (Fily et al., 2017)

\[
\dot{\mathbf{r}}_i = \mathbf{v}_i, \\
\frac{m}{\gamma} \dot{\mathbf{v}}_i = -\nabla V \mathbf{r}_i + \mathbf{f}_i(\theta_i) - \nabla \mathbf{r}_i V + \sqrt{2\gamma^2 D_i} \mathbf{\eta}_i, \\
\dot{\mathbf{v}}_i = v_i,
\]

(1.31)

where \( \gamma \) is the inverse mobility of the active particles, \( \mathbf{f}_i \) their propulsive forces, and \( V \) the potential exerted by the confining walls. \( \mathbf{\eta}_i \)'s are Gaussian white noises satisfying \( \langle \eta_i(\tau) \rangle = 0 \) and \( \langle \eta_i(\tau) \eta_j(\tau') \rangle = \delta_{ij} \delta^{\alpha\beta} \delta(t-t') \), where the angular brackets denote an
average over noise histories. \( \mathbf{u}(\theta_i) \) is a director along the orientation, \( \theta_i \), of particle \( i \) which evolves according to the overdamped dynamics

\[
\dot{\theta}_i = \Gamma_i(\mathbf{r}_i, \theta_i) + \sqrt{2D_t}\zeta_i .
\]  

(1.32)

Here \( \Gamma_i(\mathbf{r}_i, \theta_i) \) is the torque (again, we silently absorb the rotational mobility in the torque) exerted on the particle by the wall, and \( \zeta_i \) is a Gaussian white noise with \( \langle \zeta_i(t) \rangle = 0 \) and \( \langle \zeta_i(t)\zeta_j(t') \rangle = \delta_{ij}\delta(t-t') \). Note that when \( f_i = 0 \), the dynamics satisfy a fluctuation-dissipation relation and the system is in equilibrium.

Clearly, since, for example, every active force \( f_i \mathbf{u}(\theta_i) \) injects momentum into the system, there is no momentum conservation in this model. Nonetheless, as suggested above, it is useful to consider the momentum density field of the model. To this end, we consider both the density, \( \hat{\rho}(\mathbf{r}) \), and momentum density fields, \( \hat{\mathbf{p}}(\mathbf{r}) \), defined through

\[
\hat{\rho}(\mathbf{r}) = \sum_i \delta(\mathbf{r} - \mathbf{r}_i) ,
\]

\[
\hat{\mathbf{p}}(\mathbf{r}) = \sum_i m\mathbf{v}_i\delta(\mathbf{r} - \mathbf{r}_i) ,
\]

(1.33)

and whose averages with respect to noise realizations and initial conditions are denoted by \( \rho(\mathbf{r}) = \langle \hat{\rho}(\mathbf{r}) \rangle \) and \( \mathbf{p}(\mathbf{r}) = \langle \hat{\mathbf{p}}(\mathbf{r}) \rangle \).

To proceed, we consider the dynamics of the density and momentum fields. The density field obeys

\[
\partial_t \hat{\rho}(\mathbf{r}) = \sum_i \dot{\mathbf{r}}_i \cdot \nabla_{\mathbf{r}_i} \delta(\mathbf{r} - \mathbf{r}_i) = -\frac{1}{m} \nabla \cdot \hat{\mathbf{p}}(\mathbf{r}) .
\]

(1.34)

Here and in what follows, the subscript \( \mathbf{r}_i \) indicates that the gradient acts on the coordinates of the particle \( i \). In the absence of a subscript, it acts on \( \mathbf{r} \). The dynamics of the momentum-density field can be obtained by differentiating Eq. (1.33) and using the equations of motion (1.31), which leads to

\[
\partial_t \hat{\mathbf{p}}(\mathbf{r}) = \sum_i \left( -\gamma \mathbf{v}_i - \nabla_{\mathbf{r}_i} V + f_i \mathbf{u}(\theta_i) + \sqrt{2\gamma^2 D_t}\eta_i \right) \delta(\mathbf{r} - \mathbf{r}_i) \\
+ \sum_i m\mathbf{v}_i(\mathbf{v}_i \cdot \nabla_{\mathbf{r}_i})\delta(\mathbf{r} - \mathbf{r}_i) \\
= -\gamma \hat{\mathbf{p}} - \hat{\rho} \nabla V + \sum_i f_i \mathbf{u}(\theta_i)\delta(\mathbf{r} - \mathbf{r}_i) + \sqrt{2\gamma^2 D_t}\hat{\rho} \Lambda - \nabla \cdot [\mathcal{J}] ,
\]

(1.35)

where \( \gamma \equiv \tilde{\gamma}/m \). In the second line we define the tensor \( \mathcal{J} \) through

\[
\mathcal{J}(\mathbf{r}) \equiv \sum_i m\mathbf{v}_i\mathbf{v}_i\delta(\mathbf{r} - \mathbf{r}_i)
\]

(1.36)

with \( \mathbf{v}_i\mathbf{v}_i \) implying a tensor product. The Gaussian white noise is given by \( \sqrt{2\gamma^2 D_t}\hat{\rho} \Lambda \equiv \sum_i \sqrt{2\gamma^2 D_t}\eta_i\delta(\mathbf{r} - \mathbf{r}_i) \) and can be verified to obey \( \langle \Lambda_{\alpha}(\mathbf{r},t)\Lambda_{\beta}(\mathbf{r'},t') \rangle = \delta_{\alpha\beta}\delta(\mathbf{r} - \mathbf{r'}). \)
The different terms in Eq. (1.35) can be interpreted as follows:

(i) The loss of momentum through dissipation, \(-\gamma \dot{p}\).
(ii) The change in momentum due to forces exerted by the walls, \(-\dot{\rho} \nabla x V\).
(iii) The change in momentum due to active forces propelling the particles, \(\sum_i f_i u(\theta_i) \delta (r - r_i)\).
(iv) Fluctuations, \(\sqrt{2D_t \dot{\rho} \Lambda}\).
(v) Advection of momentum through the motion of particles arriving and departing from \(r\), \(-\nabla \cdot [\mathbf{J}]\).

Note that the \((\alpha, \beta)\) component of the \(\mathbf{J}\) tensor, \(J^{\alpha\beta} = \sum_i m v^\alpha_i v^\beta_i \delta (r - r_i)\), is the momentum flux along \(\hat{\alpha}\) of momentum along \(\hat{\beta}\). Also note that only the last term, \(\nabla \cdot [\mathbf{J}]\), has a momentum-conserving form.

1.4.2 Momentum sources and sinks

As before, we study a system with confining walls parallel to the \(\hat{y}\) direction and with periodic boundary conditions along that direction. In addition, for simplicity, we take the system’s length along the \(\hat{y}\) direction to be equal to one. Since this confining potential does not allow for currents in the system, the expectation value of the momentum fields is zero in the steady state, \(\mathbf{p}(r) = 0\). Integrating Eq. (1.35) along the \(y\) coordinate and averaging over the steady-state distribution (denoted by angular brackets), we obtain

\[
0 = -\rho(x) \partial_x V(x) + \left\langle \sum_i f_i \cos \theta_i \delta(x - x_i) \right\rangle - \partial_x \left\langle [J^{xx}(x)] \right\rangle, \quad (1.37)
\]

where we defined \(\rho(x) = \int_0^1 dy \rho(x, y)\). This expression has a simple interpretation: The momentum flux through the system, encoded in \(J^{xx}\), is modulated in space by both the force exerted by the wall and the active forces.

From the expression above we can easily obtain an expression for the mechanical pressure exerted on the wall. Integrating over \(x\), we find

\[
P_M = \int_0^\infty ds \rho \nabla_s V = \langle J^{xx}(0) \rangle + \int_0^\infty ds \left\langle \sum_i f_i \cos \theta_i \delta(s - x_i) \right\rangle, \quad (1.38)
\]

where we set \(x = 0\) to be a point deep in the bulk of the system. The equation implies that the total decrease in the momentum flux, \(J^{xx}\) from its bulk value \(\langle J^{xx}(0) \rangle\) to zero is a result of the total force exerted by the wall and the total active force exerted in the \(x > 0\) region. That is, the pressure is given by the overall momentum flux entering the region \(x > 0\) and the total active force exerted in this region.

At a first glance, it seems that the last term of Eq. (1.38) suggests that there is no EOS for this model. However, in Sec. 1.2.2 we saw that this model, in its overdamped version, does have an equation of state in the absence of wall torques. As the overdamped limit can be taken at this point as well, this system must have an EOS when

\[
\text{Note that there is no mechanism in the model that can possibly lead to a current along the } \hat{y} \text{ direction.}
\]
\( \Gamma(r, \theta) = 0 \). Thus, in this case, the second term of Eq. (1.38) must be expressible as a local bulk quantity.

To re-express this term, we note that

\[
\partial_t \left\langle \sum_i f_i \cos \theta_i \delta(x-x_i) \right\rangle = - \left\langle \sum_i f_i \Gamma_i \sin \theta_i \delta(x-x_i) \right\rangle - D_r \left\langle \sum_i f_i \cos \theta_i \delta(x-x_i) \right\rangle \\
- \partial_x \left( \left\langle \sum_i v_i^r f_i \cos \theta_i \delta(x-x_i) \right\rangle \right)
\]

(1.39)

where we used Eq. (1.32) and the Itô rule for differentiation. In the steady state, this gives

\[
\left\langle \sum_i f_i \cos \theta_i \delta(x-x_i) \right\rangle = - \left\langle \sum_i f_i \Gamma_i \sin \theta_i \delta(x-x_i) \right\rangle - \partial_x \left( \left\langle \sum_i v_i^r f_i \cos \theta_i \delta(x-x_i) \right\rangle \right) \\
- \partial_x \left[ \left\langle \sum_i v_i^r f_i \cos \theta_i \delta(x-x_i) \right\rangle \right] \]

(1.40)

The active force is decomposed into two terms – one which depends explicitly on the torques exerted by the wall and another, divergence-like term, which does not depend explicitly on the torque. The pressure then takes the form \( \text{\cite{Fily2017}} \)

\[
P_M = \langle J_{xx}(0) \rangle + \sum_i \left\langle \frac{v_i^r}{D_r} f_i \sin \theta_i \delta(x-x_i) \right\rangle - \int_0^\infty dx \left\langle \frac{f_i}{D_r} \Gamma_i \sin \theta_i \delta(x-x_i) \right\rangle
\]

(1.41)

where \( x = 0 \), as above, is a point deep in the bulk of the system. The second term written above correlates the velocity in the \( \hat{x} \) direction, \( v_i^r \), with the \( \hat{x} \) projection of the director, \( u_i^r \). This term is in general non-zero even in a uniform isotropic bulk. Importantly, the first two terms depend only on the bulk properties of the system, while the third term depends on the distribution of particles inside the wall and the form of the torque. Evidently, this last term is the one responsible for the general lack of an equation of state for \( P_M \). In its absence \( (\Gamma = 0) \), \( P_M \) clearly has an equation of state.

To understand this further, it is useful to rewrite Eq. (1.37) in the form

\[
\rho(x) \partial_x V(x) + \sum_i \left\langle \frac{f_i}{D_r} \Gamma_i \sin \theta_i \delta(x-x_i) \right\rangle = - \partial_x \left[ \sum_i \left\langle \frac{v_i^r}{D_r} f_i \cos \theta_i \delta(x-x_i) \right\rangle \right] + \langle J_{xx}(x) \rangle
\]

(1.42)

The right-hand side of this equation is a divergence of a local quantity, and hence has the form of a "conserving" piece, where the quotations indicate that the conservation holds only in the steady state. \( \langle J_{xx}(x) \rangle \) accounts for the flow of momentum by the particles and the second term will be discussed shortly. Note that these fluxes are balanced by both the forces exerted by the potential and by a term related to the active forces and torques, see the left-hand-side of Eq. (1.42). These last two terms effectively act as sources or sinks of momentum.

To further illustrate these results, we turn to numerical simulations. We consider the two-dimensional system described above where the system is confined by half-
harmonic potentials on the right $V^R$ and left $V^L$ starting at $x_w$ and $-x_w$ respectively,

$$V^R(x) = \lambda_R \frac{(x - x_w)^2}{2} \Theta(x - x_w) \quad \text{and} \quad V^L(x) = \lambda_L \frac{(x + x_w)^2}{2} \Theta(-x - x_w) ,$$

(1.43)

where $\Theta(x)$ denotes the Heaviside function. We treat the particles as point-like ellipses experiencing a torque of the form

$$\Gamma^R = \lambda_R \kappa \Theta(x - x_w) \sin 2\theta \quad \text{and} \quad \Gamma^L = \lambda_L \kappa \Theta(-x - x_w) \sin 2\theta$$

(1.44)

where $\kappa = \mu_r(a^2 - b^2)$ is a measure of the anisotropy of the particles and $\mu_r$ is a rotational mobility [Solon et al., 2015]. The three contributions to the pressure $P_M$ defined in Eq. (1.41) are shown in the left panel of Fig. 1.10 for walls with $\lambda_R = \lambda_L = \lambda$. The figure shows that only the torque-dependent contributions depend on the wall stiffness $\lambda$. The corresponding wall-dependent sources and sinks $\Delta f_{\text{act}}(x) = -\sum_i \lambda_i \Gamma_i \sin \theta_i \delta(x - x_i)$ are shown in the right panel of Fig. 1.10 for three stiffness values. Clearly, the pressure varies due to the physics in the vicinity of the walls. The torques applied by the walls induce net momentum sources or sinks, which cause the breakdown of the EOS.

From the expression in Eq. (1.41), we see that part of the active force has now been subsumed into a “momentum-conserving”-like term: the second term of the right-hand side is indeed evaluated in the bulk of the system, much like the momentum flux $J^{xx}$. 

**Fig. 1.10** Simulations of self-propelled ellipses for $x_w = 10$, $v_0 = 1$, $D_r = 0.1$, $D_t = 0$, $\tilde{\gamma} = 1$, $m = 1$, $\kappa = 1$, $\rho_0 = 1$. **Left** Plot of the various contributions to the pressure listed in Eq. (1.41) as the wall stiffness varies. Only the torque-dependent term depends on the wall stiffness, leading to a breakdown of the EOS. $P$ is measured from its definition in Eq. (1.38). **Right** The wall-dependent sources and sinks, shown for three different wall stiffnesses, are localized in the vicinity of the walls. Figure adapted from Fily et al. 2017.
As the active particles keep pumping momentum into the system, it is not clear what is the origin of this fact. In particular, in analogy with $J^{xx}$ describing the flow of momentum, it is natural to ask what are the quantities whose flow is described by the second term of Eq. (1.41).

1.4.3 The active impulse

To answer this question, we consider the torque-free version of the model above, where an equation of state for $P_M$ exists. All the statements to follow generalize to cases in which the dynamics of the active force $f_i u(\theta_i)$ is decoupled from other degrees of freedom (Fily et al., 2017). As noted above, the contribution of the activity to the pressure stems from the momentum transfer to the particles through the active forces. Hence, it is natural to look at how much momentum the active particle will receive, on average, from its active force in the future. This is directly quantified by the active impulse:

$$\Delta p^a_i(t) = \int_t^\infty f_i u[\theta_i(s)] ds = \frac{f_i D_r}{u[\theta_i(t)]} \quad (1.45)$$

where the over-bar denotes an average with respect to histories of the system in the time interval $[t, +\infty)$ for a fixed value of $\theta_i(t)$. In (1.45), the active impulse simply depends on the initial angle $\theta_i(t)$ because the dynamics of the active force $f_i u(\theta_i)$ is independent of all other degrees of freedom, and is randomized by the rotational noise after a time $1/D_r$, even where the potential $V(r)$ is non-zero. Note that, by construction, the active impulse obeys

$$\partial_t \Delta p^a_i(t) = -f_i u[\theta_i(t)]. \quad (1.46)$$

Turning from the single-particle description to a many-body description, we define an active impulse field, $\langle \Delta p^a(x) \rangle = \langle \sum_i \Delta p^a_i \delta(r-r_i) \rangle$. Using Eq. (1.46) in the steady state, this field satisfies

$$0 = \partial_t \langle \Delta p^a(x) \rangle = -\left\langle \sum_i f_i u[\theta_i(s)] \delta(r-r_i) \right\rangle - \nabla \cdot \left\langle \sum_i v_i \Delta p^a_i \delta(r-r_i) \right\rangle, \quad (1.47)$$

where the divergence $\nabla$ is contracted with the velocities $v_i$. Note that the first term on the right-hand side of this equation represents the decay of $\Delta p^a$ due to the production of a non-zero mean active force; in the steady state, it has to be balanced by the second term which is the influx of $\Delta p^a$. Re-expressing the active impulse using Eq. (1.45), we find

$$\left\langle \sum_i f_i u[\theta_i(s)] \delta(r-r_i) \right\rangle = -\nabla \cdot \left\langle \sum_i v_i \Delta p^a_i \delta(r-r_i) \right\rangle = -\nabla \cdot \left\langle \sum_i v_i \frac{f_i}{D_r} u[\theta_i] \delta(r-r_i) \right\rangle. \quad (1.48)$$

The above relation shows that, despite the fact that active forces inject momentum into the system, in the steady state, their average contribution takes the form of a
Forces in dry active matter

**Fig. 1.11** Left: Particles enter the right half of the system carrying their momentum $m v_i$. The corresponding flux of momentum, $J^{xx}$, will fall to zero outside the container, since there are no particles there. This decrease of the momentum flux is due to the conjugated effect of the wall force $F_w$ and the active forces (see Eq. (1.38)). Center: An active particle randomizes its force after a typical time $D_r^{-1}$. Particles deep in the right side of the system thus do not receive any momentum from their active forces on average. The total momentum the active particles exchange with the substrate in the right side of the system is therefore given by the active impulse they bring with them as they cross the boundary at $x = 0$. See Eq. (1.50). Right: The force exerted by the wall is thus balanced by the sum of the incoming fluxes of momentum and active impulse, which leads to an EOS for the pressure in the absence of torques (1.51).

...divergence of a local quantity. We refer to this quantity as the active-impulse flux tensor, $G(r)$, whose definition is given by

$$G(r) = \sum_i v_i f_i u(\theta_i) \delta(r - r_i). \quad (1.49)$$

Equation (1.48) can be understood as follows: since $\theta_i$ undergoes rotational diffusion, the contribution of an active particle entering a given volume decays to zero with time. The only way for the mean local active force to sustain its non-zero value is thus by incoming fluxes of particles entering the volume, each carrying its own active force (see Fig. 1.11). As $v_i$ and $u_i$ are correlated, such fluxes are non-vanishing.

Inserting the definition of the flux of the active impulse tensor $G$ into Eq. (1.42), and setting $\Gamma = 0$, we find

$$\rho(x) \partial_x V(x) = -\partial_x \left[ \sum_i \left( \frac{v_i}{D_r} f_i \cos \theta_i \delta(x - x_i) \right) + \langle J^{xx}(x) \rangle \right]$$

$$= -\partial_x \left( \langle G^{xx}(x) \rangle + \langle J^{xx}(x) \rangle \right), \quad (1.50)$$

and so the pressure takes the form

$$P_M = \langle J^{xx}(0) \rangle + \langle G^{xx}(0) \rangle. \quad (1.51)$$

With this result, we make the following comments:

- Using Itô formula, one directly shows that
\[ \partial_t \left( \sum_i r_i f_i u(\theta_i) \delta(r - r_i) \right) = \left\langle \sum_i v_i f_i u(\theta_i) \delta(r - r_i) \right\rangle - D_r \left\langle \sum_i r_i f_i u(\theta_i) \delta(r - r_i) \right\rangle - \nabla \cdot \left\langle \sum_i r_i f_i u(\theta_i) \delta(r - r_i) \right\rangle. \] (1.52)

The steady-state average of \( G \) in a homogeneous system can thus be identified with the “swim pressure”, introduced by Takatory and Brady (Takatori et al., 2014) and by Yang, Manning and Marchetti (Yang et al., 2014):

\[ \langle G(r) \rangle = \left\langle \sum_i \delta(r - r_i) r_i f_i u(\theta_i) \right\rangle. \] (1.53)

Given the lack of an explicit solvent in our description, we prefer the term “active pressure” for this contribution. This validates the virial-based approaches in this context (Winkler et al., 2015; Falasco et al., 2016). The above discussion shows that the swim-pressure is the flux of active impulse. Note, however, that it should now be clear that this expression does not, in general, provide the expression for \( P_M \), when \( \Gamma(r, \theta) \neq 0 \) for example. Furthermore, Eq. (1.53) is only valid in the steady state of homogeneous systems.

- Recall that all of the discussion above is done in the steady state. Averaging over Eq. (1.35) in the steady state and using the results presented above gives

\[ 0 = \partial_t \langle p \rangle = -\gamma \langle p \rangle - \partial_t \langle \Delta p^a \rangle - \langle \rho \rangle \nabla V - \nabla \cdot \left[ \langle G + J \rangle \right], \] (1.54)

or alternatively

\[ 0 = \partial_t \langle p + \Delta p^a \rangle = -\gamma \langle p \rangle - \langle \rho \rangle \nabla V - \nabla \cdot \left[ \langle G + J \rangle \right]. \] (1.55)

We therefore find that in a flux-free steady state, and outside the wall, \( \langle p \rangle + \langle \Delta p^a \rangle \) is conserved. This is the hidden “conserved” quantity which is responsible for the appearance of an equation of state for \( P_M \). Note that this “conservation” is not guaranteed for other cases – outside of the steady state and even for curved walls, which are known to generate currents, with \( \langle p \rangle \neq 0 \).

- Generalizing this treatment to include interactions does not modify the important points of the discussion above. We refer the reader to Fily et al. (2017) for details.

Finally, note that even in cases in which an EOS exists (\( \Gamma = 0 \)), the “conservation” of \( \langle p \rangle + \langle \Delta p^a \rangle \) rests on the fact that the expectation value of the momentum field vanishes - \( \langle p \rangle = 0 \). But, as seen in Sec. 1.3, the one-dimensional configuration illustrated in Fig. 1.9 generates currents in the system, with \( \langle p \rangle \neq 0 \). It is rather generic for currents to exist in an active system and it is thus natural to ask how currents modify the picture described above.

1.5 Objects immersed in an active bath: Currents and forces

Another situation in which the statistics of active forces play an important role is when passive objects are immersed in an active bath (Kikuchi et al., 2009; Maitra et al., 2014).
Forces in dry active matter

Fig. 1.12 An asymmetric object, here a semi-circle, is placed in an active medium. As discussed above, such an object induces a current in the system, which, in turn, exerts a force back on the object. This force is opposite in direction to the total current, as active particles accumulate on its concave side of the semi-circle and glide off its convex side.

As argued above, currents naturally arise when a non-symmetric potential – a passive asymmetric object – is placed inside an active fluid. If the interaction between the active particles and the object lead to a particle current resulting from a non-zero net force, Newton’s third law implies that the active particles will in turn exert a force back on the object. This is illustrated in Fig. 1.12 where, as an example, a semi-circular object is placed in a two-dimensional fluid of active particles. Due to the persistence of the active particles, we expect them to accumulate on the concave side (inside the semi-circle), and to glide along the convex side. This in turn generates a current in the system whose direction is opposite to the force exerted on the semi-circle. As we now show, this intuitive connection between currents and forces can be shown to manifest itself for a class of systems, including ABPs, through a simple relation between the current surrounding an object and the force acting on it

\[ \mathcal{J} = -\mu \mathbf{F}^{(\text{tot})}. \]  

(1.56)

Here \( \mathbf{F}^{(\text{tot})} = \int \, d\mathbf{r} \rho(\mathbf{r}) \nabla V(\mathbf{r}) \) is the total active force on the object, \( \mu \) is the mobility of the active particles, \( \mathcal{J} = \int \, d\mathbf{r} \, J(\mathbf{r}) \) is the total current flowing through the system, and \( J(\mathbf{r}) \) is the active particle current density.

To derive this relation we use the overdamped model of non-interacting ABPs
from Sec. 1.2.2 in two dimensions in the absence of torques. Note that the following treatment can be generalized to models with pairwise interactions. In the absence of torques, the Fokker-Planck equation for this model takes the form

$$\partial_t P(r, \theta) = -\nabla \cdot [v u(\theta) P(r, \theta) - \mu \nabla V P(r, \theta)] + D_r \partial_\theta^2 P(r, \theta)$$  \hspace{1cm} \text{(1.57)}$$

Integrating over $\theta$, we obtain

$$\partial_t \rho(r) = -\nabla \cdot [v m_{x,1} \hat{x} + v m_{y,1} \hat{y} - \mu \nabla V \rho(r) - D_r \nabla \rho] \equiv -\nabla \cdot J$$  \hspace{1cm} \text{(1.58)}$$

where similarly to before, we define the moments $m_{\alpha,n}$ as

$$m_{x,n}(r) = \int d\theta \cos(n\theta) P(r, \theta),$$  \hspace{1cm} \text{(1.59)}$$

$$m_{y,n}(r) = \int d\theta \sin(n\theta) P(r, \theta),$$  \hspace{1cm} \text{(1.59)}$$

and $\rho(r) = m_{x,0}$. Integrating the current over a surface $s$ within the system containing the object, we find

$$J = \int_s d^2 r J$$

$$= -\mu \int_s d^2 r \rho \nabla V + \int_s d^2 r [v m_{x,1} \hat{x} + v m_{y,1} \hat{y} - D_r \nabla \rho] .$$  \hspace{1cm} \text{(1.60)}$$

Note that the first term on the right-hand side in Eq. (1.60) is the total force $-\mathbf{F}^{(\text{tot})}$ acting on particles located inside $s$ times the mobility $\mu$. To simplify further, we multiply Eq. (1.57) by $\cos \theta$ and integrate over $\theta$, to find

$$m_{x,1} = -\frac{1}{D_r} \nabla \cdot \left[ \frac{v}{2} (\rho + m_{x,2}) \hat{x} + \frac{v}{2} m_{y,2} \hat{y} - \mu m_{x,1} \nabla V - D_r \nabla m_{x,1} \right]$$

$$\equiv -\nabla \cdot \mathbf{M}_{x,1} ,$$  \hspace{1cm} \text{(1.61)}$$

for a wall parallel to the $\hat{y}$ direction. Similarly, one can obtain an equation for $m_{y,1}$ after proper permutations of indices

$$m_{y,1} = -\nabla \cdot \mathbf{M}_{y,1} .$$  \hspace{1cm} \text{(1.62)}$$

Evidently, both $m_{x,1}$ and $m_{y,1}$ can be written as the divergence of a local quantity. Using Stokes’ theorem, with the surface $s$ being large enough so that its boundaries are far from the object in the homogeneous bulk, we find

$$\int_s d^2 r m_{x,1} \hat{x} = -\oint_{\partial s} d\hat{l} \cdot \mathbf{n} \cdot \mathbf{M}_{x,1} = 0 ,$$  \hspace{1cm} \text{(1.63)}$$

where $\mathbf{n}$ is a normal to the surface $s$. The vanishing of the last integral simply stems from the fact that, in the bulk, $\mathbf{M}_{x,1} = \text{const.}$ Similarly, the surface integrals of $m_{y,1}$
and $\nabla \rho$ vanish. Using all this in Eq. (1.60) proves Eq. (1.56).

Several comments are in place:

- The derivation can be generalized to models with pairwise interactions between active particles, yielding the same result (Nikola et al., 2016).
- One can show that to leading order in the far-field limit, the density is given by (Baek et al., 2018)

$$\rho(r) = \rho_{\text{bulk}} - \frac{\mu}{2\pi D_{\text{eff}}} \frac{r \cdot F^{(\text{tot})}}{r^2} + O(r^{-2})$$  (1.64)

with the force exerted on the particles, $F^{(\text{tot})}$, given again by

$$F^{(\text{tot})} = \int d^3r' \rho \nabla V.$$  (1.65)

The density profile induces a diffusive current $J$ whose expression reads

$$J(r) \simeq -\frac{\mu}{2\pi} \left[ \frac{F^{(\text{tot})}}{r^2} - \frac{2(r \cdot F^{(\text{tot})})r}{r^4} \right].$$  (1.66)

This has the functional form of a dipolar field. Again we see that the density profile is a non-local function of the potential.

- If the object is mobile, the force exerted by the active particles will make it move. This was observed experimentally, with a passive wedge emersed in a $B. \text{subtilis}$ suspension (Kaiser et al., 2014).

- The discussion above shows that asymmetric objects induce currents, and one could thus wonder about the impact of structured walls. There are two different cases to consider here.

  * Walls with “up-down” symmetry: Numerical measurements of the density and currents are shown in Fig. 1.13 for a wall of periodicity $L_p$ that is up-down symmetric. In such cases, one can define an average pressure in the $\hat{x}$ direction

$$\langle P_x \rangle = \frac{1}{L_p} \int_0^{L_p} dy \, P_x(y)$$  (1.67)

using the following definition

$$P_x(y) = \int_0^\infty dx \, \rho \partial_x V.$$  (1.68)

One then finds that despite the fact that $P_x(y)$ depends both on the details of the potential and the exact value of $y$, $\langle P_x \rangle$ obeys the same EOS as it would obey if the walls were flat. As before, the $\hat{y}$ direction was taken as the periodic direction and $x = 0$ is a point in the bulk. As one may expect, the pressure is not uniform along the wall and depends on the shape of the wall.
Objects immersed in an active bath: Currents and forces

Fig. 1.13 Density (a) and current (b) of non-interacting ABPs near the right edge of the system with a sinusoidal hard wall potential of spatial periodicity $L_p = 1$, spatial amplitude of $A = 1$. The red dashed curve corresponds to the position of the wall. In the left panel, the color encodes the density, which falls quickly to zero in the wall region. Figure adapted from Nikola et al. 2016, where more details about the simulations are provided.

Fig. 1.14 Pressure normal to the wall, normalized by $P_{th} = \rho_0 \left[ \frac{\nu^2}{2\mu D_r} + \frac{D_r}{\mu} \right]$, the pressure of spherical ABPs near a flat wall (1.21). $\lambda$ is the stiffness of the wall. The pressure is displayed as a function of $y$ along the wall, in the hard wall regime. Figure adapted from Nikola et al. 2016.

potential (see Fig. 1.14). This result holds for non-interacting and pairwise interacting models.

* Walls with no “up-down” symmetry: If the up-down symmetry is broken, as illustrated in Fig. 1.15 the discussions above suggest that a current will be generated along the walls. This phenomenon is indeed observed numerically in Fig. 1.15. These currents, via the discussion above, are associated with shearing forces acting on the walls. Indeed, if the wall was allowed to move, it would do so. This can be thought of as a toy model for the observed rotations of a ratchet wheel in a bacterial bath (Di Leonardo et al., 2010; Sokolov et al., 2010). We note that in this case, even though $\langle P_x \rangle$ as defined in Eq. (1.67) still obeys an EOS, the force in the $\hat{y}$ direction does not and is non-zero. Thus, there is no EOS describing the stresses in this system.
Forces in dry active matter

Fig. 1.15 Left: Equipotential line of an asymmetric confining potential. As the walls are asymmetric, currents run along them. Since currents on one side will flow in the opposite direction of the current on the other wall, shear forces will be exerted on the system as a whole. Right: Ratchet current and shear stress are functions of $x$, for such asymmetric walls. The relation between the total force and the overall current (1.56) is verified numerically within 1%. Figures adapted from Nikola et al. 2016.

1.6 Acknowledgements

In these lecture notes, we have presented works which have been done in collaboration with a number of colleagues that we thank warmly: Y. Baek, A. Baskaran, M. E. Cates, Y. Fily, M. Kardar, N. Nikola, A. P. Solon, J. Stenhammar, A. Turner, R. Voituriez, R. Wittkowski, X. Xu. YK acknowledge support from I-CORE Program of the Planning and Budgeting Committee of the Israel Science Foundation and an Israel Science Foundation grant. JT is funded by ANR Bactterns. JT & YK acknowledge support from the MLB center for theoretical physics and a joint CNRS-MOST grant.
References

Baek, Y, Solon, AP, Xu, X, Nikola, N, and Kafri, Y (2018). Generic long-range interactions between passive bodies in an active fluid. *Physical review letters*, **120**(5), 058002.

Bechinger, C, Di Leonardo, R, Löwen, H, Reichhardt, C, Volpe, G, and Volpe, G (2016). Active particles in complex and crowded environments. *Reviews of Modern Physics*, **88**(4), 045006.

Berg, HC (1993). *Random walks in biology*. Princeton University Press.

Cates, ME and Tailleur, J (2015). Motility-induced phase separation. *Annual Review of Condensed Matter Physics*, **6**(1), 219–244.

Di Leonardo, R, Angelani, L, Della Arciprete, D, Ruocco, G, Iebba, V, Schippa, S, Conte, MP, Mecarini, F, De Angelis, F, and Di Fabrizio, E (2010). Bacterial ratchet motors. *Proceedings of the National Academy of Sciences*, **107**(21), 9541–9545.

Falasco, G, Baldovin, F, Kroy, K, and Biaesi, M (2016). Mesoscopic virial equation for nonequilibrium statistical mechanics. *New Journal of Physics*, **18**(9), 093043.

Fily, Y, Baskaran, A, and Hagan, MF (2014). Dynamics of self-propelled particles under strong confinement. *Soft Matter*, **10**(30), 5609.

Fily, Y, Kafri, Y, Solon, AP, Tailleur, J, and Turner, A (2017). Mechanical pressure and momentum conservation in dry active matter. *Journal of Physics A: Mathematical and Theoretical*, **51**(4), 044003.

Fodor, É, Nardini, C, Cates, ME, Tailleur, J, Visco, P, and van Wijland, F (2016). How far from equilibrium is active matter? *Physical Review Letters*, **117**(3), 038103.

Galajda, P, Keymer, J, Chaikin, P, and Austin, R (2007). A wall of funnels concentrates swimming bacteria. *Journal of bacteriology*, **189**(23), 8704–8707.

Ginot, F, Solon, A, Kafri, Y, Ybert, C, Tailleur, J, and Cottin-Bizonne, C (2018). Sedimentation of self-propelled janus colloids: polarization and pressure. *New Journal of Physics*, **20**, 115001.

Ginot, F, Theurkauff, I, Levis, D, Ybert, C, Bocquet, L, Berthier, L, and Cottin-Bizonne, C (2015). Nonequilibrium equation of state in suspensions of active colloids. *Physical Review X*, **5**(1), 011004.

Harder, J, Valeriani, C, and Cacciuto, A (2014). Activity-induced collapse and reexpansion of rigid polymers. *Physical Review E*, **90**, 062312.

Joyeux, M and Bertin, E (2016). Pressure of a gas of underdamped active dumbbells. *Physical Review E*, **93**(3), 032605.

Junot, G, Briend, G, Ledesma-Alonso, R, and Dauchot, O (2017). Active versus passive hard disks against a membrane: mechanical pressure and instability. *Physical Review Letters*, **119**(2), 028002.

Kaiser, A and Löwen, H (2014). Unusual swelling of a polymer in a bacterial bath. *The Journal of Chemical Physics*, **141**(4), 044903.
References

Kaiser, A, Peshkov, A, Sokolov, A, ten Hagen, B, Löwen, H, and Aranson, IS (2014). Transport powered by bacterial turbulence. *Physical Review Letters*, **112**(15), 158101.

Kikuchi, N, Ehrlicher, A, Koch, D, Käs, JA, Ramaswamy, S, and Rao, M (2009). Buckling, stiffening, and negative dissipation in the dynamics of a biopolymer in an active medium. *Proceedings of the National Academy of Sciences*, **106**(47), 19776–19779.

Li, H, Zhang, B, Li, J, Tian, W, and Chen, K (2015). Brush in the bath of active particles: Anomalous stretching of chains and distribution of particles. *The Journal of chemical physics*, **143**(22), 224903.

Maitra, A, Srivastava, P, Rao, M, and Ramaswamy, S (2014, Jun). Activating membranes. *Physical Review Letters*, **112**, 258101.

Mallory, SA, Šarić, A, Valeriani, C, and Cacciuto, A (2014). Anomalous thermomechanical properties of a self-propelled colloidal fluid. *Physical Review E*, **89**(5), 052303.

Mandal, D, Klymko, K, and DeWeese, MR (2017). Entropy production and fluctuation theorems for active matter. *Physical Review Letters*, **119**, 258001.

Marchetti, MC, Joanny, JF, Ramaswamy, S, Liverpool, TB, Prost, J, Rao, M, and Aditi Simha, TB (2013). Hydrodynamics of soft active matter. *Reviews of Modern Physics*, **85**(3), 1143–1189.

Nikola, N, Solon, A P, Kafri, Y, Kardar, M, Tailleur, J, and R, Vouturiez (2016). Active particles with soft and curved walls: Equation of state, ratchets, and instabilities. *Physical Review Letters*, **117**, 098001–5.

Poujade, M, Grasland-Mongrain, E, Hertzog, A, Jouanneau, J, Chavrier, P, Ladoux, B, Buguin, A, and Silberzan, P (2007). Collective migration of an epithelial monolayer in response to a model wound. *Proceedings of the National Academy of Sciences*, **104**(41), 15988–15993.

Razin, N, Voituriez, R, Elgeti, J, and Gov, NS (2017a). Forces in inhomogeneous open active-particle systems. *Physical Review E*, **96**(5), 052409.

Razin, N, Voituriez, R, Elgeti, J, and Gov, NS (2017b). Generalized archimedes’ principle in active fluids. *Physical Review E*, **96**(3), 032606.

Reichhardt, C and Reichhardt, CJO (2013). Active matter ratchets with an external drift. *Physical Review E*, **88**, 063110.

Rodenburg, J, Dijkstra, M, and van Roij, R (2017). Van’t hoff’s law for active suspensions: the role of the solvent chemical potential. *Soft Matter*, **13**, 8957–8963.

Rohwer, CM, Solon, AP, Kardar, M, and Krüger, M (2018). Nonequilibrium forces following quenches in active and thermal matter. *Physical Review E*, **97**(3), 032125.

Sandford, C and Grosberg, AY (2018). Memory effects in active particles with exponentially correlated propulsion. *Physical Review E*, **97**(1), 012602.

Sandford, C, Grosberg, AY, and Joanny, JF (2017). Pressure and flow of exponentially self-correlated active particles. *Physical Review E*, **96**(5), 052605.

Schnitzer, MJ (1993). Theory of continuum random walks and application to chemotaxis. *Physical Review E*, **48**(4), 2553.

Shankar, S and Marchetti, MC (2018). Hidden entropy production and work fluctuations in an ideal active gas. *Physical Review E*, **98**(2), 020604.
Shin, J, Cherstvy, AG, Kim, WK, and Metzler, R (2015). Facilitation of polymer looping and giant polymer diffusivity in crowded solutions of active particles. *New Journal of Physics, 17*(11), 113008.

Smallenburg, F and Löwen, H (2015). Swim pressure on walls with curves and corners. *Physical Review E, 92*(3), 032304.

Sokolov, A, Apodaca, MM, Grzybowski, BA, and Aranson, IS (2010). Swimming bacteria power microscopic gears. *Proceedings of the National Academy of Sciences, 107*(3), 969–974.

Solon, AP, Fily, Y, Baskaran, A, Cates, ME, Kafri, Y, Kardar, M, and Tailleur, J (2015a). Pressure is not a state function for generic active fluids. *Nature Physics, 11*, 673–678.

Solon, AP, Stenhammar, J, Wittkowski, R, Kardar, M, Kafri, Y, Cates, ME, and Tailleur, J (2015b, May). Pressure and phase equilibria in interacting active brownian spheres. *Physical Review Letters, 114*, 198301.

Speck, T and Jack, RL (2016). Ideal bulk pressure of active brownian particles. *Physical Review E, 93*(6), 062605.

Tailleur, J and Cates, ME (2008). Statistical mechanics of interacting run-and-tumble bacteria. *Physical Review Letters, 100*(21), 218103.

Tailleur, J and Cates, ME (2009). Sedimentation, trapping, and rectification of dilute bacteria. *EPL, 86*, 60002.

Takatori, SC, Yan, W, and Brady, JF (2014). Swim Pressure: Stress Generation in Active Matter. *Physical Review Letters, 113*(2), 028103.

Van den Broeck, C and Hänggi, P (1984). Activation rates for nonlinear stochastic flows driven by non-gaussian noise. *Physical Review A, 30*(5), 2730.

Vandebroek, H and Vanderzande, C (2015). Dynamics of a polymer in an active and viscoelastic bath. *Physical Review E, 92*(6), 060601.

Winkler, RG, Wysocki, A, and Gompper, G (2015). Virial pressure in systems of spherical active brownian particles. *Soft Matter, 11*, 6680–6691.

Yan, W and Brady, JF (2015a). The force on a boundary in active matter. *Journal of Fluid Mechanics, 785*, R1.

Yan, W and Brady, JF (2015b). The swim force as a body force. *Soft Matter, 11*(31), 6235–6244.

Yang, X, Manning, ML, and Marchetti, MC (2014). Aggregation and segregation of confined active particles. *Soft Matter, 10*(34), 6477.