NOTE ON STABILITY OF AN ABSTRACT COUPLED HYPERBOLIC-PARABOLIC SYSTEM: SINGULAR CASE

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Abstract. In this paper we try to complete the stability analysis for an abstract system of coupled hyperbolic and parabolic equations

\[
\begin{aligned}
&u_{tt} + Au - A^\alpha w = 0, \\
w_t + A^\alpha u_t + A^\beta w = 0,
\end{aligned}
\]

where \( A \) is a self-adjoint, positive definite operator on a complex Hilbert space \( H \), and \((\alpha, \beta) \in [0,1] \times [0,1]\), which is considered in [1], and after, in [4]. Our contribution is to identify a fine scale of polynomial stability of the solution in the region \( S_3 := \{(\alpha, \beta) \in [0,1] \times [0,1]; \beta < 2\alpha - 1\} \) taking into account the presence of a singularity at zero.

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1. Introduction

In this paper, we study the stability of the following system:

(1.1) \[ u_{tt} + Au - A^\alpha w = 0, \]

(1.2) \[ w_t + A^\alpha u_t + A^\beta w = 0, \]

(1.3) \[ u(0) = u_0, u_t(0) = u_1, w(0) = w_0, \]

where \( A \) is a self-adjoint, positive definite operator on a complex Hilbert space \( H \), and \((\alpha, \beta) \in S_3 = \{(a, b) \in [0,1] \times [0,1]; b < 2a - 1\}\).
By denoting $U = (u, u_t, w)^T, U_0 = (u_0, u_1, w_0)^T$, system (1.1)-(1.3) can be written as an abstract linear evolution equation on the space $\mathcal{H} = \mathcal{D}(A^+) \times H \times H$,

\begin{align}
\begin{cases}
\frac{dU}{dt}(t) = A_{\alpha, \beta}U(t), \quad t \geq 0, \\
U(0) = U_0,
\end{cases}
\end{align}

(1.4)

where the operator $A_{\alpha, \beta} : \mathcal{D}(A_{\alpha, \beta}) \subset \mathcal{H} \to \mathcal{H}$ is defined by

$$
A_{\alpha, \beta} = 
\begin{pmatrix}
0 & I & 0 \\
-A & 0 & A^\alpha \\
0 & -A^\alpha & -A^\beta
\end{pmatrix},
$$

with the domain

$$
\mathcal{D}(A_{\alpha, \beta}) = \mathcal{D}(A) \times \mathcal{D}(A^\alpha) \times \mathcal{D}(A^\alpha),
$$

Firstly, we give a detailed review about the well-posedness of the problem (1.4). The operator $A_{\alpha, \beta}$ is densely defined and dissipative, we will prove that its closure generates a $C_0$-semigroup of contractions. Using a Lumer-Phillips theorem [6], it suffices to prove that the adjoint operator $A^*_{\alpha, \beta}$ is also dissipative.

The operator $A^*_{\alpha, \beta}$ is a closed extension of the operator

$$
M_{\alpha, \beta} = 
\begin{pmatrix}
0 & -I & 0 \\
A & 0 & -A^\alpha \\
0 & A^\alpha & -A^\beta
\end{pmatrix},
$$

with domain

$$
\mathcal{D}(M_{\alpha, \beta}) = \mathcal{D}(A_{\alpha, \beta}) = \mathcal{D}(A) \times \mathcal{D}(A^\alpha) \times \mathcal{D}(A^\alpha).
$$

The operator $M_{\alpha, \beta}$ is densely defined and dissipative, then it is closable and its $\overline{M_{\alpha, \beta}}$ is also dissipative. To conclude, it suffices to prove that $A^*_{\alpha, \beta} = M_{\alpha, \beta}$ . For this we use the following lemma [2] [1].

**Lemma 1.1.** [2] We consider on the Hilbert spaces $G$ and $H_1$ the operators

$$
A : \mathcal{D}(A) \subset G \to G, \quad B : \mathcal{D}(B) \subset H_1 \to G,
$$

$$
B^* : \mathcal{D}(B^*) \subset G \to H_1, \quad C : \mathcal{D}(C) \subset H_1 \to H_1.
$$

and we consider the operator matrix $\mathcal{M}$ on $G \times H_1$ defined by

$$
\mathcal{M} := 
\begin{pmatrix}
A & B \\
-B^* & C
\end{pmatrix},
\quad \mathcal{D}(\mathcal{M}) := (\mathcal{D}(A) \cap \mathcal{D}(B^*)) \times (\mathcal{D}(B) \cap \mathcal{D}(C)).
$$
Assume that $C$ is boundedly invertible, that $B \in \mathcal{L}(D(C), G)$ and that $C^{-1}B^*$ extends to a bounded linear operator (we denote its closure with the same symbol). Then $\mathcal{M}$ is closed if and only if $A + BC^{-1}B^*$ is closed and one has

$$\mathcal{M} = \begin{pmatrix} I & BC^{-1} \\ 0 & I \end{pmatrix} \left( \begin{pmatrix} A + BC^{-1}B^* \\ 0 & C \end{pmatrix} \right) \begin{pmatrix} I & 0 \\ 0 & -C^{-1}B^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$ 

$$\mathcal{D}(\mathcal{M}) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A + BC^{-1}B^*) \times H_1, -C^{-1}B^*u + v \in \mathcal{D}(C) \right\}.$$ 

By taking $G = \mathcal{D}(A^{1/2})$, $H_1 = H \times H$, $A = 0$, $B = (-I \ 0) : \mathcal{D}(B) = \mathcal{D}(A^\alpha) \times \mathcal{D}(A^\alpha) \subset H \times H \to \mathcal{D}(A^{1/2})$, $C = \begin{pmatrix} 0 & -A^\alpha \\ A^\alpha & -A^\beta \end{pmatrix}$, with $\mathcal{D}(C) = \mathcal{D}(A^\alpha) \times \mathcal{D}(A^\alpha)$, it appears that

$$B^* = \begin{pmatrix} -A \\ 0 \end{pmatrix}, \ C^{-1}B^* = \begin{pmatrix} A^{1+\beta-2\alpha} \\ A^{1-\alpha} \end{pmatrix}, A + BC^{-1}B^* = -A^{\beta+1-2\alpha} \ (\text{with domain } \mathcal{D}(A)).$$ 

and that $\mathcal{D}(A + BC^{-1}B^*) = \mathcal{D}(A^{1/2})$. Furthermore, the operators $A$, $B$ and $C$ satisfies the hypothesis in Lemma 1.1, then

$$\mathcal{D}(\mathcal{M}_{\alpha, \beta}) = \left\{ (u, v, \theta)^T \in \mathcal{H}, -A^{1+\beta-2\alpha}u + v \in \mathcal{D}(A^\alpha), -A^{1-\alpha}u + \theta \in \mathcal{D}(A^\alpha) \right\}.$$ 

But a direct calculation gives

$$\mathcal{D}(A_{\alpha, \beta}^*) \subset \left\{ (u, v, \theta)^T \in \mathcal{H}, -A^{1+\beta-2\alpha}u + v \in \mathcal{D}(A^\alpha), -A^{1-\alpha}u + \theta \in \mathcal{D}(A^\alpha) \right\}.$$ 

Hence $A_{\alpha, \beta}^* = \mathcal{M}_{\alpha, \beta}$ and $\mathcal{M}_{\alpha, \beta}$ generates a $C_0$-semigroup of contractions.

Now, concerning the stability of the semigroup $e^{tA_{\alpha, \beta}}$, recall that Ammar-Khodja et al. [1], are firstly proved that for every $\alpha, \beta \geq 0$, $e^{tA_{\alpha, \beta}}$ is exponentially stable if and only if $\max(1 - 2\alpha, 2\alpha - 1) < \beta < 2\alpha$. Later on, a stability analysis has been performed by J. Hao and Z. Liu in [2] for $(\alpha, \beta) \in [0, 1] \times [0, 1]$. They divided the unit square $[0, 1] \times [0, 1]$ into four regions $S$, $S_1$, $S_2$, $S_3$ where

$$S := \{ (\alpha, \beta) \in [0, 1] \times [0, 1]; \max(1 - 2\alpha, 2\alpha - 1) \leq \beta \leq 2\alpha \},$$

$$S_1 := \{ (\alpha, \beta) \in [0, 1] \times [0, 1]; 0 < \beta - 2\alpha, \alpha \geq 0, \frac{1}{2} \leq \beta \leq 1 \},$$

$$S_2 := \{ (a, b) \in [0, 1] \times [0, 1]; \beta < 1 - 2\alpha, \alpha \geq 0, 0 \leq \beta \leq \frac{1}{2} \},$$

and $S_3$ as defined below. We summarize the main results in the following theorem (see also [3]).

**Theorem 1.2.** The semigroup $e^{tA_{\alpha, \beta}}$ has the following stability properties:

(i) In $S$, it is exponentially stable;

(ii) In $S_1$, it is polynomially stable of order $\frac{1}{2(\beta - 2\alpha)}$;

(iii) In $S_2$, it is polynomially stable of order $\frac{1}{2(\beta + 2\alpha)}$. 

(iv) In $S_3$, it is not asymptotically stable.

To justify the non asymptotic stability in region $S_3$, they proved that $0 \in \sigma(A_{\alpha,\beta})$, where $\sigma(A_{\alpha,\beta})$ is the spectrum of $A_{\alpha,\beta}$. Moreover, it can be shown that $\sigma(A_{\alpha,\beta}) \cap i\mathbb{R} = \{0\}$ in the region $S_3$.

The main result of this paper then concerns the precise asymptotic behaviour of the solutions of (1.1)-(1.3) for $(\alpha, \beta)$ in the region $S_3$, with initial condition in a special subspace of $D(A_{\alpha,\beta})$. Precisely, we will estimate $\|e^{tA_{\alpha,\beta}}(I - A_{\alpha,\beta})^{-1}\|$ when $t \to \infty$. Our technique is a special frequency and spectral analysis of the corresponding operator.

2. Stabilization

We will justify that the resolvent has only a singularity (at zero) on the imaginary axis (Lemma 2.3 below), and that $(i\lambda - A_{\alpha,\beta})^{-1}$ is bounded outside a neighborhood of zero in $\mathbb{R}$ (Lemma 2.4 below). Then we apply a result due to Batty, Chill and Tomilov ([3, Theorem 7.6]) which relate the decay of $\|e^{tA_{\alpha,\beta}}(I - A_{\alpha,\beta})^{-1}\|$ to the growth of $(i\lambda - A_{\alpha,\beta})^{-1}$ near zero.

So, first we recall the corresponding result

**Theorem 2.1.** ([3, Theorem 7.6]) Let $(T(t))_{t \geq 0}$ be a bounded $C_0$-semigroup on a Hilbert space $X$, with generator $A$. Assume that $\sigma(A) \cap i\mathbb{R} = \{0\}$, and let $\gamma \geq 1$. The following are equivalent:

(i) $\| (is - A)^{-1} \| = \begin{cases} O(|s|^{-\gamma}), & s \to 0, \\ O(1), & |s| \to \infty, \end{cases}$

(ii) $\| T(t)A'(I - A)^{-\gamma} \| = O\left(\frac{1}{t}\right), \quad t \to \infty,$

(iii) $\| T(t)A(I - A)^{-1} \| = O\left(\frac{1}{t^{1/\gamma}}\right), \quad t \to \infty.$

Hence, our main result is the following

**Theorem 2.2.** We have the following decay in the region $S_3$

(2.1) $\|e^{tA_{\alpha,\beta}}(I - A_{\alpha,\beta})^{-1}\| = O\left(\frac{1}{t}\right), \quad t \to \infty.$

In particular, the decay of $e^{tA_{\alpha,\beta}}x$ to zero is uniform with respect to $x \in D(A_{\alpha,\beta})$.

It follows also that, for every $z \in \text{Ran}(A_{\alpha,\beta})$ we have

$\|e^{tA_{\alpha,\beta}}z\| = O\left(\frac{1}{t}\right), \quad t \to \infty.$

**Proof.** In view of Theorem 2.1, the proof is a direct consequence of the following three lemmas.
Lemma 2.3. In the region $S_3$, the operator $A_{\alpha, \beta}$ satisfies

$$\sigma(A_{\alpha, \beta}) \cap i\mathbb{R} = \{0\}.$$ 

Proof. It has been proved in [5], at the third section that $\{0\} \subset \sigma(A_{\alpha, \beta}) \cap i\mathbb{R}$.

Conversely, to show that there is no nonzero spectrum point on the imaginary axis, we use a contradiction argument. In fact, let $\lambda \in \mathbb{R}$, $\lambda \neq 0$ such that $i\lambda \in \sigma(A_{\alpha, \beta})$. Then, there exists a sequence $(U_n) \subset D(A_{\alpha, \beta})$, with $\|U_n\| = 1$ for all $n$, such that

$$(2.2) \lim_{n \to \infty} \| (i\lambda I - A_{\alpha, \beta}) U_n \| = 0$$

or there exists a sequence $(U_n) \subset D(M_{\alpha, \beta}) = D(A_{\alpha, \beta})$, with $\|U_n\| = 1$ for all $n$, such that

$$(2.3) \lim_{n \to \infty} \| (i\lambda I + M_{\alpha, \beta}) U_n \| = 0.$$ 

Setting $U_n = (u_n, v_n, \theta_n)$, then (2.2) is equivalent to

$$(2.4) i\lambda A^{1/2} u_n - A^{1/2} v_n = o(1), \quad \text{in } H,\)

$$(2.5) i\lambda v_n + Au_n - A^\alpha \theta_n = o(1), \quad \text{in } H,\)

$$(2.6) i\lambda \theta_n + A^\alpha v_n + A^\beta \theta_n = o(1), \quad \text{in } H.$$ 

First, since

$${\text{Re}} \left( \langle (i\lambda I - A_{\alpha, \beta}) U_n, U_n \rangle_H \right) = \|A^{\beta/2} \theta_n\|^2$$

we obtain

$$(2.7) \lim_{n \to \infty} \|A^{\beta/2} \theta_n\| = 0,$$

and in particular

$$(2.8) \lim_{n \to \infty} \|\theta_n\| = 0.$$ 

Second, taking inner product of (2.4) with $\frac{1}{\lambda} A^{1/2} u_n$, (2.5) with $\frac{1}{\lambda} v_n$ and (2.6) with $\frac{1}{\lambda} \theta_n$, taking into account (2.7) and (2.8), we get

$$(2.9) i\|A^{1/2} u_n\|^2 - \frac{1}{\lambda} \langle v_n, Au_n \rangle = o(1),$$

$$(2.10) i\|v_n\|^2 + \frac{1}{\lambda} \langle Au_n, v_n \rangle - \frac{1}{\lambda} \langle A^\alpha \theta_n, v_n \rangle = o(1),$$

$$(2.11) \frac{1}{\lambda} \langle A^\alpha v_n, \theta_n \rangle = o(1).$$

Then, by combining (2.9)-(2.11) and using that $\|U_n\| = 1$, it yields

$$(2.12) \|A^{1/2} u_n\| = \frac{1}{2} + o(1), \quad \|v_n\| = \frac{1}{2} + o(1).$$

Now, since in the region $S_3$, $1 - \alpha < \frac{1}{2}$ and $\frac{3}{2} - 2\alpha + \beta < \frac{1}{2}$, then using the boundedness of $\|A^{1/2} u_n\|$ we get, by interpolation,

$$(2.13) \|A^{1-\alpha} u_n\| = O(1), \quad \|A^{1-2\alpha+\beta} u_n\| = O(1).$$
Then, replacing \( v_n \) by \( \lambda u_n \) in (2.20), due to (2.21) and interpolation, next taking the inner product of the obtained equation with \( \frac{1}{\lambda} A^{1-2\alpha+\beta} u_n \) to get

\[
\lambda \| A^{\frac{1}{2}-\alpha+\frac{\beta}{2}} u_n \|^2 + \frac{1}{\lambda} \| A^{1-\alpha+\frac{\beta}{2}} v_n \|^2 - \frac{1}{\lambda} \langle A^\alpha \theta_n, A^{1-2\alpha+\beta} u_n \rangle = o(1).
\]

Taking the inner product of (2.24) with \( \frac{1}{\lambda} A^{1-\alpha} u_n \), we get

\[
\lambda \| A^{\frac{1}{2}-\alpha+\frac{\beta}{2}} u_n \|^2 + \frac{1}{\lambda} \| A^{1-\alpha+\frac{\beta}{2}} v_n \|^2 + \frac{1}{\lambda} \langle A^\alpha \theta_n, A^{1-2\alpha+\beta} u_n \rangle = o(1).
\]

By (2.28) and (2.13), the first term in (2.14) converge to zero. Moreover, using (2.4), we can replace \( \frac{1}{\lambda} A^{1/2} v_n \) in the second term in (2.15) by \( i A^{1/2} u_n \). Consequently, the sum of (2.14) and (2.15) yields

\[
\lambda \| A^{\frac{1}{2}-\alpha+\frac{\beta}{2}} u_n \|^2 + \frac{1}{\lambda} \| A^{1-\alpha+\frac{\beta}{2}} u_n \|^2 + i \| A^{\frac{1}{2}} u_n \|^2 = o(1).
\]

Hence, \( \| A^{\frac{1}{2}} u_n \|^2 = o(1) \), which contradict the first estimate in (2.12).

The same approach applied to (2.3) leads to the same conclusion without any difficulty.

**Lemma 2.4.** *In the region \( S_3 \),*

\[
\| (is - A_{\alpha,\beta})^{-1} \| = O(1), \quad s \to \infty.
\]

**Proof.** It is a direct consequence of Theorem 2.3 in [5], since \( \limsup_{\lambda \in \mathbb{R}, \lambda \to \infty} |\lambda|^\frac{\alpha}{2} \| (i \lambda - A_{\alpha,\beta})^{-1} \| < \infty \).

**Lemma 2.5.** *In the region \( S_3 \),*

\[
\| (is - A_{\alpha,\beta})^{-1} \| = O(|s|^{-1}), \quad s \to 0.
\]

**Proof.** By contradiction, suppose that \( \limsup_{s \in \mathbb{R}, s \to 0} \| s (is - A_{\alpha,\beta})^{-1} \| = \infty \).

Putting \( s = \frac{1}{w} \), this is equivalent to \( \limsup_{w \in \mathbb{R}, |w| \to \infty} \| w^{-1} (iw^{-1} I - A_{\alpha,\beta})^{-1} \| = \infty \). Then, there exists a sequence \( (w_n) \) of real numbers with \( |w_n| \to \infty \), and a sequence \( (U_n) \subset D(A_{\alpha,\beta}) \), with \( \| U_n \| = 1 \) for all \( n \) such that

\[
\lim_{n \to \infty} |w_n| \| (iw_n^{-1} I - A_{\alpha,\beta}) U_n \| = 0.
\]

Setting \( U_n = (u_n, v_n, \theta_n) \), then (2.17) is rewritten explicitly as follows

\[
(iw_n^{-1}) \| w_n |A^{1/2} u_n| - |w_n|A^{1/2} v_n| = o(1), \quad \text{in } H,
\]

\[
(iw_n^{-1}) |w_n|v_n + |w_n|Au_n - |w_n|A^\alpha \theta_n = o(1), \quad \text{in } H,
\]

\[
(iw_n^{-1}) |w_n|\theta_n + |w_n|A^\alpha v_n + |w_n|A^\beta \theta_n = o(1), \quad \text{in } H.
\]

Since \( \text{Re} \left( \| w_n \| (iw_n^{-1} I - A_{\alpha,\beta}) U_n, U_n \right) \) = \(-|w_n|\| A^{\beta/2} \theta_n \|^2 \), it yields

\[
\lim_{n \to \infty} |w_n|^{1/2} \| A^{\beta/2} \theta_n \| = 0,
\]
and in particular
\[(2.22) \lim_{n \to \infty} \|\theta_n\| = 0.\]
Then, applying \(|w_n|^{-1} A^{-\alpha}\) to \((2.20)\), we get, (by taking into account \((2.22)\)),
\[(2.23) \lim_{n \to \infty} \|v_n\| = 0.\]
Now, taking inner product of \((2.18)\) with \(A^{1/2} u_n\), \((2.19)\) with \(v_n\) and \((2.20)\) with \(\theta_n\), taking into account \((2.21)\), \((2.22)\) and \((2.23)\), we have
\[(2.24) i w_n^{-1} \|A^{1/2} u_n\|^2 - |w_n| \langle v_n, Au_n \rangle = o(1),\]
\[(2.25) |w_n| \langle Au_n, v_n \rangle - |w_n| \langle A^{\alpha} \theta_n, v_n \rangle = o(1),\]
\[(2.26) |w_n| \langle A^{\alpha} \theta_n, v_n \rangle = o(1).\]
By combining \((2.24)-(2.26)\), it yields
\[(2.27) \lim_{n \to \infty} \|A^{1/2} u_n\| = 0.\]
The promised contradiction follows from \((2.22)\), \((2.23)\) and \((2.27)\). Thus, the proof of Lemma \((2.5)\) is completed. □

Which ends the proof of theorem. □

Remark 2.6. In \(S_3\), the semigroup \(e^{t A_{\alpha, \beta}}\) is of Gevrey class \(\delta > \frac{\alpha}{\beta}\) and in particular, it is infinitely differentiable \([5]\). Thus, for \(z \in \text{Ran}(A_{\alpha, \beta})\), we not only has polynomial stability, but also have instantaneous smoothness.

3. Application

As application of Theorem \((2.2)\) we consider the following thermoplate system:

\[
\begin{aligned}
\begin{cases}
\dot{u} + \Delta^2 u - (-\Delta)^\alpha w &= 0, \Omega \times (0, +\infty), \\
\dot{w} + (-\Delta)^\alpha \dot{u} - \Delta w &= 0, \Omega \times (0, +\infty), \\
u &= \Delta u = w = 0, \Gamma \times (0, +\infty), \\
u(x, 0) &= u_0(x), u_t(x, 0) = u_1(x), w(x, 0) = w_0(x), \Omega,
\end{cases}
\end{aligned}
\]

where \(\Omega\) be a bounded domain in \(\mathbb{R}^n\) with smooth boundary \(\Gamma\), \(\beta = \frac{1}{2}\) and \(3/4 < \alpha \leq 1\).

Here, \(H = L^2(\Omega), A = \Delta^2\) avec \(D(A) = \{u \in H^2(\Omega) \cap H^1_0(\Omega); \Delta u = 0 \text{ on } \Gamma\}\).

Then, according to Theorem \((2.2)\) the corresponding semigroup satisfies the estimate \((2.1)\) for all \(\alpha \in (3/4, 1]\).
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