Null Geodesics and Repulsive Behavior of Gravity in \((2 + 1)\)-dimensional Massive Gravity

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abstract

We study the null geodesics in a static circularly symmetric (SCS) black hole spacetime which is a solution in the \((2 + 1)\)-dimensional massive gravity proposed by Bergshoeff, Hohm and Townsend (BHT massive gravity). We obtain analytic solutions for the null geodesic equation in the SCS black hole background and find the explicit form of deflection angles. We see that for various values of the impact parameter, the deflection angle can be positive, negative or even zero in this black hole spacetime. The negative deflection angle indicates the repulsive behavior of the gravity which comes from the gravitational hair parameter that is the most characteristic quantity of the BHT massive gravity.

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1 Introduction

It is well known that in (2 + 1) dimensions, the Riemann tensor can be expressed in terms of the Ricci tensor and the Ricci scalar since the Weyl tensor is identically zero. This fact means that there is no local degrees of freedom. Due to this simpleness, Einstein gravity in (2 + 1) dimensions does not have non-trivial black hole solution except for the BTZ black hole solution [1, 2] that is the solution of the Einstein equation with a negative cosmological constant.

By the way, there are some modifications to introduce the local degrees of freedom to gravity in (2 + 1) dimensions. For example, the topologically massive gravity [3], which has the Lorentz Chern-Simons term in its action in addition to the Einstein-Hilbert action, possesses a single propagating degree of freedom in the linearization level around maximally symmetric spacetime.

Another massive gravity in (2 + 1) dimensions has been suggested by Bergshoeff, Holm and Townsend (BHT) [4, 5]. The BHT massive gravity is a ghost-free theory with quadratic terms of the Ricci tensor and Ricci scalars by adjusting the coefficients appropriately [6]. The BHT massive gravity has some nontrivial black hole solutions [7], including the BTZ black hole as its special case.

Since the BHT massive gravity includes a massive graviton which gives a new mass scale, we can expect that the large-scale interaction of gravity must be different from that of Einstein gravity. The deviation from Einstein gravity appears as a new parameter in the black hole solution, which is called the gravitational hair parameter [7]. In order to clarify the consequence of the deviation, we investigate the null geodesics and the deflection angles of the null geodesics in the black hole spacetime with the gravitational hair parameter.\(^1\) We obtain analytic solutions for the geodesic equation of massless particle and find that the gravity behaves as if a repulsive force acts on the geodesics with some values of parameters. Instead of the effective potential, we calculate the deflection angles of the null geodesics to evaluate this repulsive behavior of the gravity and then we obtain the explicit form of the deflection angles in the black hole background. Also, we show that the origin of the repulsive behavior of the BHT massive gravity is the existence of the gravitational hair parameter. In fact, it is known that the Lifshitz black hole [11], which is another type of black hole spacetime in the BHT massive gravity, does not have the gravitational hair parameter and the geodesics in its black hole background do not show such a repulsive behavior of the gravity. This is consistent with our claim.

\(^1\)For the (2 + 1)-dimensional black hole in Einstein gravity, i.e., the BTZ black hole, the analytic solution for the geodesic equation for massless particles are examined in [8] and for (3 + 1) and higher-dimensional black holes, these are discussed in [9, 10].
This paper is organized as follows. In Section 2, we explain the BHT massive gravity and the static circularly symmetric (SCS) black hole spacetime. In Section 3, we derive the analytic solution of the geodesic equation for massless particles in the SCS black hole spacetime. Also, we discuss that there exist the geodesics whose behaviors seem as if the massless particles receive the repulsive force from the black hole. In Section 4, we introduce the deflection angles of the null geodesics in the SCS black hole spacetime. We derive the explicit form of the deflection angle and reveal that for various values of the impact parameter, the deflection angles can be positive, negative or even zero. The last section is devoted to the conclusion and the discussion.

2 BHT massive gravity and static circularly symmetric black hole solution

The BHT massive gravity is characterized by the following action \[4\]
\[
S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left( R - 2\lambda - \frac{1}{m^2} K \right),
\]
where \(K\) is quadratic in the Ricci tensor and the Ricci scalar as
\[
K = R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2.
\]
The source-free field equation can be obtained as
\[
G_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} = 0,
\]
where
\[
K_{\mu\nu} = 2\Box R_{\mu\nu} - \frac{1}{2} (\nabla_{\mu} \nabla_{\nu} R + g_{\mu\nu} \Box R) - 8 R_{\mu\nu} R_{\rho\nu} + \frac{9}{2} R R_{\mu\nu} + g_{\mu\nu} \left( 3R^{\alpha\beta} R_{\alpha\beta} - \frac{13}{8} R^2 \right).
\]
When a spacetime has a constant curvature as \(R_{\mu\nu\rho\sigma} = \Lambda (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})\), \(K_{\mu\nu}\) in Eq.(2.3) is also simplified as \(K_{\mu\nu} = -\frac{1}{2} \Lambda^2 g_{\mu\nu}\) \([4, 7]\), and there can be solutions of constant curvature with two different curvature radii
\[
\Lambda_{\pm} = 2m (m \pm \sqrt{m^2 - \lambda})
\]
from Eq.(2.3). As mentioned in \([7]\), at the special case with \(m^2 = \lambda\), the theory has a unique maximally symmetric solution, since the two curvature constants \(\Lambda_{+}\) and \(\Lambda_{-}\) takes the same value, \(\Lambda_{+} = \Lambda_{-} = 2\lambda = 2m^2\). For simplicity, we focus on this case in the remaining of this paper.

In this spacial case of the BHT massive gravity, we can obtain the following static circularly symmetric (SCS) black hole solution \([7]\),
\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2,
\]
\[
f(r) = -\Lambda r^2 + br - \mu,
\]
(2.6)
where $\mu$ is related to the mass of the black hole and $b$ is called the gravitational hair parameter [12] which is origin of the repulsive behavior of the gravity as we will explain later. When $b = 0$, the solution (2.6) reduces to the non-rotating BTZ black hole solution. As it is well known, the BTZ black hole is the unique, nontrivial black hole solution for Einstein gravity in $(2 + 1)$ dimensions with a negative cosmological constant and $\mu$ can be regarded as the mass parameter in the BTZ black hole solution. The black hole mass $M$ is encoded to

$$M = \frac{\mu + 1}{4G},$$  \hspace{1cm}  \text{(2.7)}$$

with respect to the AdS spacetime ($b = 0$ and $\mu = -1$). $\Lambda$ in Eq.(2.6) works as an effective cosmological constant, while the original cosmological constant $\lambda$ differs as twice as $\Lambda$. Though there is no black hole solution in Einstein gravity when $\Lambda > 0$, there exist a black hole solution in the BHT massive gravity even if $\Lambda > 0$ [7]. In this paper, however, we set $\Lambda < 0$ to compare with the BTZ black hole and consider the case of $b > 0$ and $\mu > 0$.

The black hole spacetime represented by the metric (2.6) has horizon at

$$r_h = \frac{-b + \sqrt{b^2 - 4\Lambda \mu}}{-2\Lambda},$$  \hspace{1cm}  \text{(2.8)}$$

and its the scalar curvature $R$ is calculated as

$$R = 6\Lambda - \frac{2b}{r}.$$  \hspace{1cm}  \text{(2.9)}$$

Eq.(2.9) means that this spacetime is asymptotically AdS and there is the curvature singularity at $r = 0$. When $\Lambda = 0$, the black hole spacetime represented by the metric (2.6) is the asymptotically locally flat [7]. Figure 1 shows the causal structures of the spacetimes denoted by the metric (2.6) with some values of parameters. The causal structures for the other values of parameters are discussed in [12].

The existence of the gravitational hair $b$ is one of the crucial differences between the BHT massive gravity and Einstein gravity. Interestingly, $b$ enables us to find other types of black hole
solutions. For example, there is a black hole solution with radius \( r = \mu / b \) even when \( \Lambda = 0 \) as long as \( b \) and \( \mu \) are positive [12, 13, 14].

3 Null geodesics of the SCS black hole spacetime in the BHT massive gravity

In this section, we study the null geodesics in the SCS black hole spacetime (2.6). There are two killing vectors associated to the spacetime, \((\partial_t)\mu\) and \((\partial_\phi)\mu\). Thus we can find the constants of motion along the geodesic as follows

\[
E = -g_{\mu t}p^\mu = f(r)\dot{t}, \tag{3.1}
\]

and

\[
L = g_{\mu \phi}p^\mu = r^2 \dot{\phi}, \tag{3.2}
\]

where \( p^\mu \) is the four momentum of a massless particle. The dot denotes the derivative with respect to an affine parameter \( \lambda \). \( E \) and \( L \) corresponds to the energy and the angular momentum of a massless particle, respectively. The geodesic equation reduces to an ordinary differential equation by using the null condition \( p_\mu p^\mu = 0 \) as

\[
\dot{r}^2 = E^2 - f(r)\frac{L^2}{r^2}. \tag{3.3}
\]

Now we define the effective potential \( V_{\text{eff}}^2(r) \) by

\[
V_{\text{eff}}^2(r) = f(r)\frac{L^2}{r^2}. \tag{3.4}
\]

In Figure 2 we depict the effective potentials for some values of the angular momentum \( L \). The effective potential has a maximum that correspond to an unstable circular orbit of a massless particle around the SCS black hole that locates at \( r_a = 2\mu / b \). The maximum value \( V_{\text{max}}^2 \) of the effective potential increases as \( L \) becomes larger.

Combining the geodesic equation (3.3) and \( \dot{\phi} = L / r^2 \), we obtain

\[
\left( \frac{dr}{d\phi} \right)^2 = \frac{r^4}{L^2} \left( E^2 - f(r)\frac{L^2}{r^2} \right) = \frac{E^2}{L^2} r^2 - (-\Lambda r^2 + br - \mu) r^2. \tag{3.5}
\]

We can integrate this equation easily. The solutions are classified into the following three types
by values of the parameters $\bar{D}$, $\beta$ and $\mu$ (see Figure 3),

\[
\begin{align*}
  r_I(\phi) &= \frac{2\mu}{b + 2\mu \kappa_I \sinh(\pm \sqrt{\mu} \phi + \beta)}, \quad \text{(Type I : } E^2 > V_{\text{max}}^2) \quad (3.6) \\
  r_{II}(\phi) &= \frac{2\mu}{b + 2\mu \kappa_{II} \cosh(\pm \sqrt{\mu} \phi + \beta)}, \quad \text{(Type II : } E^2 < V_{\text{max}}^2, \ r_0 < r_a) \quad (3.7) \\
  r_{III}(\phi) &= \frac{2\mu}{b - 2\mu \kappa_{II} \cosh(\pm \sqrt{\mu} \phi + \beta)}, \quad \text{(Type III : } E^2 < V_{\text{max}}^2, \ r_0 > r_a) \quad (3.8)
\end{align*}
\]

where $r_0$ is an initial location of a massless particle. Also, $\kappa_I^2 = (4\mu/\bar{D}^2 - b^2)/4\mu^2$, $\kappa_{II}^2 = (b^2 - 4\mu/\bar{D}^2)/4\mu^2$ and

\[
\bar{D}^2 = \frac{D^2}{1 + D^2 \Lambda}, \quad D = \frac{L}{E}, \quad (3.9)
\]

where $D$ is the impact parameter. For the geodesics of Type I, the effective impact parameter $\bar{D}$ satisfies $4\mu/\bar{D}^2 > b^2$ and for the geodesics of Type II and III, $b^2 > 4\mu/\bar{D}^2$. $\beta$ is an integration constant. The behaviors of the null geodesics of Type I, II and III are summarized as follows:

- Type I: the energy of a particle is larger than the maximum value of the effective potential. The particle can cross the event horizon from the outside and hits the singularity at the center of the black hole.
- Type II: all particles fall to the singularity at the center.
- Type III: starting from outside the horizon, a particle gets close to the black hole and it gets apart from that.
Figure 3: Effective potential for a massless particle in the SCS spacetime. \( r_h \) and \( r_a \) correspond to the radius of the event horizon and the unstable circular orbit of a massless particle, respectively.

We examine the detail of the orbits of these solutions by the particles’ energy and initial positions as shown in Figure 4. The null geodesics of Type I and II always fall into the black hole across the horizon. In particular, the null geodesics of Type III is worthy to be focused on. The null geodesics of Type III get close to the horizon and can escape to infinity, reflected by the wall of the effective potential. However, there are two subclasses of the orbits of this type. One is the orbit which goes around the black hole (Type III-1 in Figure 4) and another one is the orbit that is bounced off from the black hole (Type III-2 in Figure 4). So we can classify all types of geodesics as follows:

\[
\begin{align*}
\text{Energy} & \quad \left\{ \begin{array}{l}
\text{lager that the maximum of } V_{\text{eff}}^2 (I) \\
\text{smaller that the maximum of } V_{\text{eff}}^2 (II) \\
\text{cannot escape to infinity (II)} \\
\text{can escape to infinity (III)} \\
\text{go around (III-1)} \\
\text{be bounced off (III-2)}
\end{array} \right.
\end{align*}
\]

This fact means that the gravity works repulsively on massless particles when the energy takes a value in some range that corresponds to the geodesics of Type III-2. We will show it explicitly in next section. Such behaviors of the geodesics in black hole spacetimes cannot be seen in Einstein gravity where the gravitational force always works attractively.

4 Deflection angle

In this section we explicitly calculate the deflection angles of the null geodesics in the SCS black hole spacetime represented by the metric (2.6) to characterize the repulsive behavior of
the gravity. We can rewrite the geodesic equation for massless particles (3.5) as

\[ \left( \frac{d\theta}{d\phi} \right)^2 = \frac{1}{\tilde{D}}^2 r^4 - br^3 + \mu r^2, \]

(4.1)

where \( \tilde{D} \) is defined by Eq.(3.9). As we showed in the previous section, we should classify the null geodesics not only by the energy of a particle but also the impact parameter \( D \). Here, let \( D_c \) denote the value of \( D \) corresponding to the unstable circular orbit, which is given by

\[ D_c = \sqrt{4\mu/b^2 - 4\mu\Lambda}. \]

The geodesics with \( D < D_c \) fall into the black hole, which corresponds to the geodesics of Type I. On the other hand, the geodesics with \( D > D_c \) are bounced by the wall of the effective potential, which corresponds to the geodesics of Type III.

To characterize the repulsive behavior of the gravity that acts on the geodesics of Type III, we calculate the deflection angle. Introducing a new variable \( u \) by \( u = 1/r \), the geodesic equation

![Figure 4: Behaviors of the geodesics for various parameters.](image-url)
Figure 5: Schematic picture of the deflection angle $\alpha$ and the impact parameter $D$.

becomes

$$\left( \frac{du}{d\phi} \right)^2 = \frac{1}{D^2} - bu + \mu u^2 = F(u). \quad (4.2)$$

We define the deflection angle $\alpha$ as

$$\alpha = 2 \int_0^{1/R_0} \frac{du}{\sqrt{F(u)}} - \pi, \quad (4.3)$$

where $R_0$ is the closest approach radius that is given by

$$R_0 = \frac{bD^2 + \bar{D} \sqrt{b^2D^2 - 4\mu}}{2}. \quad (4.4)$$

For $F(u)$ in Eq.(4.2) we obtain the deflection angle as

$$\alpha = \frac{1}{\sqrt{\mu}} \log \left| \frac{b\bar{D} + 2\sqrt{\mu}}{b\bar{D} - 2\sqrt{\mu}} \right| - \pi. \quad (4.5)$$

Figure 5 shows a schematic picture of the deflection angle and the impact parameter. In Einstein gravity, the deflection angle is always positive for black holes since the gravity works attractively. However, as we have seen in the previous section, the gravity works repulsively for some values of parameters in the SCS black hole spacetime represented by the metric (2.6). This repulsive behavior of the gravity appears as a negative deflection angle. Here, we have to note that in the spacetime (2.6) there also exists the region where the geodesics receive the attractive force as we can see the geodesic of Type III-1 in Figure 4. Actually, we can define the critical value $\bar{D}_{\alpha=0}$ that corresponds to the border of the repulsive and attractive force, which is calculated as

$$\bar{D}_{\alpha=0} = \frac{2\sqrt{\mu}}{b \tanh \frac{\pi \sqrt{\mu}}{2}}. \quad (4.6)$$
Figure 6: The geodesics for massless particles for various values of the impact parameter. Red line corresponds to the $D_c$ orbit, which takes the unstable circular orbit. Orange line is the one for which the gravity works attractively. Blue line receives no net gravitational force. Purple line corresponds to the orbit with the negative deflection angle.

For the geodesic with this value, the net deflection angle equals to zero. When $D > D_{\alpha = 0}$, the gravity works repulsively and the deflection angle is negative. By using Eq.(3.9) we can express $D > D_{\alpha = 0}$ in terms of the energy of the particle as

$$E < L \sqrt{\frac{b^2 \tanh^2 \pi \sqrt{\frac{\mu}{4\Lambda}}}{4\mu} - \Lambda}. \quad (4.7)$$

Therefore, the orbit with such energy behaves like the one of geodesics of Type III-2 shown in Figure 4. In Figure 6 we summarize of these orbits for various values of the impact parameter.

It is worthy to mention $b = 0$ and $b < 0$ cases. In both cases, the effective potentials do not have a maximum and they increase monotonically with respect to $r$, which means that there is no potential barrier and all of the ingoing massless particles fall into the black hole. This implies that in these cases, we cannot define the deflection angle. Therefore, the repulsive behavior of the gravity does not appear in the BTZ black hole background ($b = 0$ case) and it appears only in the the BHT massive gravity with $b > 0$.

5 Conclusion and discussion

In this paper we have studied the null geodesics in the static circularly symmetric black hole in the BHT massive gravity. We obtained the analytic solutions for the geodesic equation for massless particles and found that the gravity behaves repulsively for the null geodesics with the parameters corresponding to (4.7). This repulsive behavior of the gravity can not be seen only by the analysis of the effective potential. This is because the effective potential tells us only the
motion in radial direction.

In order to evaluate the repulsive behavior of the gravity in the spacetime represented by Eq.(2.6), we have also investigated the deflection angles of the null geodesics. We obtained the explicit forms of the deflection angles in terms of the impact parameter $D$, and found that for various values of the impact parameter, the deflection angles can be positive, negative or even zero. The negative deflection angle indicates the repulsive behavior of the gravity.

We showed that the gravitational hair parameter $b$, which characterizes the BHT massive gravity, is essential for the repulsive behavior of the gravity. In fact, when we put $b = 0$, the metric (2.6) reduces to that of the BTZ black hole in Einstein gravity and the repulsive behavior of the gravity does not appear in this case.

We should note that the sign of the gravitational hair parameter $b$ is important. When $b < 0$, the effective potential increases monotonically with $r$, so the effective potential does not have a maximum. Therefore, all of the ingoing massless particles fall into the black hole, which means that we cannot define the deflection angle for this case. The repulsive behavior of the gravity appears only when $b > 0$.

Since the BHT massive gravity is a (2 + 1)-dimensional theory, the results we obtained in this paper is not directly related to realistic observations. However, if there is a spacetime whose metric includes the linear term with respect to $r$ such as $br$, it is possibile that the gravity could work repulsively for a particle in any dimensions. For example, there exist the (3+1)-dimensional spacetime solutions that have such linear term in the Weyl conformal gravity theory [15] or in the dRGT massive gravity theory [16, 17]. The deflection angle for the black hole spacetime in the Weyl conformal gravity was investigated in [18, 19] though the repulsive behavior of the gravity was not referred. Similarly to our case, the linear term in the black hole solutions of the Weyl conformal gravity or the dRGT massive gravity, makes the deflection angles smaller than the ones of the Schwarzschild black hole spacetime (for example, see Eq.(12) in [19]). The explicit form of the deflection angles, we obtained in this paper, is applicable to the strong gravitational lensing as well [20, 21, 22]. The possibility of observations of the repulsive behavior of the gravity would be more interesting in (3 + 1) dimensions and it will be further investigated in the forthcoming paper.
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