COFINITE SUBMODULE CLOSED CATEGORIES AND THE WEYL GROUP

APOLONIA GOTTWALD

ABSTRACT. We consider hereditary Artin algebras over arbitrary fields and prove that there is a natural bijection between the Weyl groups and the sets of full additive cofinite submodule closed subcategories of the module categories. While Oppermann, Reiten and Thomas have shown this for algebraically closed fields and finite fields, we provide a different method of proof that holds independently of the field.

In particular, we show a relatively simple way to construct all modules that contain a given preinjective module as a submodule.

1. Introduction

While submodule closed subcategories have not yet been extensively studied, there are many connections to different parts of representation theory. For example, if $A$ is a finite dimensional algebra, then every infinite submodule closed subcategory of $\text{mod } A$ contains a minimal infinite submodule closed category, see [9].

Submodule closed subcategories can also be used to prove that there is a filtration of the Ziegler spectrum that is indexed by the Gabriel-Roiter filtration, see [10].

In this paper, let $A$ be a hereditary Artin algebra over an arbitrary field.

We aim to prove that there is a natural bijection between the Weyl group and the set of full additive cofinite submodule closed subcategories of the module category. Oppermann, Reiten and Thomas have shown this in [8] for algebraically closed fields and finite fields. While we use the same bijection, we will give a completely different method of proof that does not depend on the field.

Our paper is organized in the following way:

Date: October 17, 2017.
First of all, we regard the Weyl group as a Coxeter group, see Section 2. This allows us to regard the Weyl group elements as equivalence classes of words. In Section 3 we define a total order on these words and call the smallest word of each equivalence class leftmost. Then we collect some results about this order.

We conclude Section 3 by stating the bijection, which is induced by a map between words of Weyl group elements and sets of preinjective modules. In Section 6 to Section 8 we will prove that a cofinite, full additive subcategory is submodule closed if and only if a leftmost word is mapped to its complement. Since we can assign a unique leftmost word to every element of the Weyl group, this gives a bijection between the full additive cofinite submodule closed subcategories and the Weyl group.

For this proof, we will use the results of Section 4, which is devoted to monomorphisms between preinjective modules. In particular, we give a way to construct all modules that contain a given preinjective module as a submodule. This allows us to draw some lemmas in Section 5 about the structure of full additive cofinite subcategories and how they are related to the words of Weyl group elements.

In the sections 6 to 8 we use this to prove inductively that the proposed bijection exists. Finally, we conclude this chapter with some corollaries. A submodule closed subcategory will in the following always denote a full additive submodule closed subcategory of $\text{mod } A$.

2. The Weyl group as a Coxeter group

We define words following [7], pp. 1-4:

**Definition 2.1.** Let $S$ be a set. We call $S$ an alphabet and its elements letters. A word over the alphabet $S$ is a finite sequence $(s_1, s_2, \ldots, s_n)$, $s_i \in S$.

The product of two words is just the concatenation of the sequences. This product is associative and by identifying a letter $s \in S$ with the sequence $(s)$, we can write the word $(s_1, s_2, \ldots, s_n)$ as the product $s_1 s_2 \ldots s_n$. The neutral element for this product is the empty word, which we accordingly denote as 1. Thus, the set of words over $S$ together with the concatenation forms a monoid $S^*$.

If $w := s_1 s_2 \ldots s_n$ is a word over $S$, then $l(w) := n$ is called the length of $w$. Furthermore, a word of the form $v = s_{i_1} s_{i_2} \ldots s_{i_m}$ with

$$1 \leq i_1 < i_2 < \cdots < i_m \leq n$$

and $m \leq n$ is a subword of $w$.

If $v = s_1 s_2 \ldots s_m$ with $m \leq n$, then we say that $v$ is an initial subword of $w$.

An introduction into Coxeter groups can be found in [3]. First, we need the definition of a Coxeter group, see pp. 1-2:

**Definition 2.2.** Let $S$ be a set and $W$ a group generated by $S$. Then $W$ is called a Coxeter group if all relations have the form $(ss')^m(s,s') = 1$ with $s, s' \in S$ so that

1. $m(s, s') = 1$ if and only if $s = s'$.
2. If $m(s, s')$ exists, then $m(s', s)$ also exists and $m(s, s') = m(s', s)$.

If there is no relation between $s$ and $s'$, then we write $m(s, s') = m(s', s) = \infty$.

We can describe the Coxeter group $W$ through the monoid $S^*$, see [3], p. 3:
Proposition 2.3. Let $S$ be a set and $S^*$ the monoid of words over $S$. Let $W$ be a Coxeter group generated by $S$ with relations $(ss')^{m(s,s')} = 1$.

Set $\equiv$ to be the equivalence relation on $S^*$ which is generated by allowing the insertion or deletion of words of the form
\[(ss')^{m(s,s')} = \underbrace{ss'\ldots ss'}_{2m(s,s')} \text{ letters},\]
for all $m(s,s') < \infty$. Then $S^*/\equiv$ is isomorphic to $W$.

We will use the following notation:

Definition 2.4. Set $\{ss'\}^a := \underbrace{ss'\ldots ss'}_{a \text{ letters}}$.

The next lemma makes it easier to work with the relations, see [3], p. 2:

Lemma 2.5. Let $S, W$ and $\equiv$ be as in Proposition 2.3 and $s, s' \in S$. The equivalence of words $\{s's\}^a \equiv \{ss'\}^a$ holds if and only if $m(s,s')$ is a factor of $a$.

Let $S_1, \ldots, S_n$ with $n \in \mathbb{N}$ be a complete list of non-isomorphic simple modules of the Artin algebra $A$.

We can associate to $A$ a Cartan matrix, as in [2], pp 69, 241 and 288:

Definition 2.6. To a hereditary Artin algebra $A$ we associate the Cartan matrix $C = (c_{ij})_{nn}$ of the underlying graph of the quiver $A^{op}$.

That is, we set $c_{ii} = 2$. If $i \neq j$ and $\text{Ext}^1(S_i, S_j) = \text{Ext}^1(S_j, S_i) = 0$, then $c_{ij} = c_{ji} = 0$. Finally, if $\text{Ext}^1(S_i, S_j) \neq 0$, set
\[c_{ij} = -\dim_{\text{End}_A(S_i)^{op}} \text{Ext}^1(S_i, S_j)\]
and
\[c_{ji} = -\dim_{\text{End}_A(S_j)} \text{Ext}^1(S_i, S_j)\]

A description of the Weyl group as a Coxeter group can be found in [5], Proposition 3.13:

Proposition 2.7. The Weyl group associated to $A$ with the Cartan matrix $(c_{ij})_{nn}$ is a Coxeter group generated by the reflections $s_1, s_2, \ldots, s_n$ with relations $s_i^2 = 1$ for all $1 \leq i \leq n$ and $(s_is_j)^{m_{ij}} = 1$ for all $i \neq j$, where $m_{ij}$ depends on $c_{ij}c_{ji}$ in the following way:

\[
\begin{array}{cccc}
  c_{ij}c_{ji} & 0 & 1 & 2 & 3 & \geq 4 \\
  m_{ij} & 2 & 3 & 4 & 6 & \infty \\
\end{array}
\]

We can write all relations as $(ss_j)^{m_{ij}}$ if we set $m_{ij} := 1$ for $i = j$.

Every element of the Weyl group is the equivalence class of several different words over the alphabet $S := \{s_1, s_2, \ldots, s_n\}$. To distinguish between the elements of the Weyl group and the words over $S$, we will always use underlined letters to denote words and normal letters for Weyl group elements.

Remark 2.8. For $A = kQ$ with a field $k$ and a quiver $Q$ without oriented cycles, the relations depend only on the edges in the underlying graph of $Q$, see e.g. [8], p. 570:

We have $m_{ij} = 2$ if there is no edge between the vertices $i$ and $j$ and $m_{ij} = 3$ if there is exactly one edge between $i$ and $j$. If there are two or more edges between $i$ and $j$, then $m_{ij} = \infty$. 
Example 2.9. Let $Q$ be the quiver

```
1 → 3.
4 ← 2
```

The Weyl group of $A = kQ$ is a Coxeter group with the relations $(s_is_j)^{m_{ij}} = 1$ and the following values for $m_{ij}$:

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 3 |
| 2 | 3 | 1 | 2 | 3 |
| 3 | 3 | 3 | 1 | 2 |
| 4 | 3 | 2 | 1 |   |

3. Leftmost words

Let $A$ be a hereditary Artin algebra and $\mod A$ the category of finitely presented modules over $A$. Furthermore, let $\mathcal{I}$ be the subcategory of $\mod A$ consisting of all preinjective modules.

Let $W$ be the Weyl group of $A$ and $S := \{s_1, s_2, \ldots, s_n\}$ be the set of generators of $W$. Furthermore, let the simple modules $S_1, \ldots, S_n$ of $A$ with injective envelopes $I_1, \ldots, I_n$ be ordered in such a way that $\Hom(I_i, I_j) = 0$ if $i < j$. This is possible because $A$ is a hereditary Artin algebra, see [2], Chapter VIII, Proposition 1.5.

Definition 3.1. Consider $N = (\mathbb{N}_0 \times \{1, 2, \ldots, n\}, <)$, where $<$ is the lexicographic order: for pairs $(r, i), (r', j) \in N$, we have $(r, i) < (r', j)$ if and only if one of the following holds:

1. $r < r'$
2. $r = r'$ and $i < j$.

Let $\underline{w} = s_{i_1}s_{i_2}\ldots s_{i_m}$ be a word over the alphabet $S$ and $0 = r_1 \leq r_2 \leq \cdots \leq r_m \in \mathbb{N}_0$ the smallest non-negative integers so that

$$(r_1, i_1) < (r_2, i_2) < \cdots < (r_m, i_m)$$

is fulfilled. Then we define

$$\rho(\underline{w}) := (r_1, i_1)(r_2, i_2)\ldots (r_m, i_m).$$

Example 3.2. Consider the Weyl group of the quiver $Q$ from Example 2.9. If we set $\underline{w} := s_2s_3s_1s_3s_4s_1$ then

$$\rho(\underline{w}) = (0, 2)(0, 3)(1, 1)(1, 3)(1, 4)(2, 1).$$

Now we can define a total order $<_l$ on the words of $W$; this is again a lexicographic order:

Definition 3.3. Consider two words $\underline{w}, \underline{w}'$ with

$$\rho(\underline{w}) = (r_1, i_1)(r_2, i_2)\ldots (r_m, i_m)$$

and

$$\rho(\underline{w}') = (r'_1, i'_1)(r'_2, i'_2)\ldots (r'_m, i'_m).$$

We write $\underline{w} <_l \underline{w}'$ if one of the following holds:

1. $m < m'$
2. $m = m'$ and there is a $j \in \mathbb{N}$ so that

$$(r_1, i_1) = (r'_1, i'_1), (r_2, i_2) = (r'_2, i'_2), \ldots, (r_{j-1}, i_{j-1}) = (r'_{j-1}, i'_{j-1})$$

and

$$(r_j, i_j) < (r'_j, i'_j).$$
Now we define the leftmost word; this definition can be found for example in [1], p. 411:

**Definition 3.4.** We call a word $w'$ for $w \in W$ **leftmost** if for every other word $w''$ for $w$ the inequality $w <_l w''$ holds.

**Example 3.5.** For the Weyl group from Example 2.9, the words

$$s_3 <_l s_2 s_3 <_l s_3 s_2 <_l s_2 s_3 s_2$$

are all leftmost words and

$$s_2 s_3 s_2 <_l s_3 s_2 s_3 <_l s_2 s_3 s_1 s_2 s_1$$

are all words for the same element of the Weyl group.

Since $<_l$ is a total order, every element $w \in W$ has a unique leftmost word. Obviously, the leftmost word is reduced, that is, it has the smallest possible length for a word of $w$.

We follow with a Lemma about the order $<_l$ and the relations:

**Lemma 3.6.** Suppose that $w_1 = u(s_is_j)^{m_{ij}}$. Set

$$\rho(w_1) = \rho(u)(p, i)(q, j)(p + 1, i) \ldots \rho_1$$

for some $p, q \in \mathbb{N}_0$ and a sequence of pairs $\rho_1$. Set

$$w_2 = u(s_is_j)^{m_{ij}}.$$ 

Then $w_2 <_l w_1$, if and only if both of the following conditions are fulfilled:

1. $1 \leq q$.
2. Let $(r, k)$ be a pair in $\rho(w_1)$. Then $(r, k) < (q - 1, j)$.

**Proof.** Let $\rho(w_2) = \rho(u)(q', j)(q', i) \ldots \rho_2$ for some sequence of pairs $\rho_2$.

Suppose that $w_2 <_l w_1$. Then $(q', j) < (p, i)$ by Definition 3.3 and thus $q = q'+1$. So the first condition is fulfilled.

Now consider a pair $(r, k)$ in $\rho(w_1)$. Then $(r, k) < (q', j) = (q - 1, j)$ and the second condition is fulfilled.

On the other hand, suppose that the conditions 1 and 2 are fulfilled. Then $q'$ is the smallest integer so that $(q', j)$ is bigger than all $(r, k)$ in $\rho(w_1)$. By the second condition, $(q', j) \leq (q - 1, j)$.

Furthermore, $q$ is the smallest integer so that $(p, i) < (q, j)$. It follows that $(q - 1, j) < (p, i)$, since $i \neq j$.

Together, $(q', j) \leq (q - 1, j) < (p, i)$ and by Definition 3.3, we have $w_2 <_l w_1$. \qed

The following lemmas are important for the induction with which we prove the main theorem of this chapter:

**Lemma 3.7.** Let $x, x', y$ be words and $s_i, s_j$ reflections. We suppose that the words $w = x s_i y$ and $w' = x' s_i y$ are equivalent, $x$ is leftmost and $w <_l w'$. Let $z$ be the longest initial subword that $w$ and $w'$ share. If there is no $z'' \equiv z'$ that shares an initial subword with $w$ which is longer than $z$, then there are pairs $(r, h), (s, i), (t, j)$ and series of pairs $\rho_1, \rho_2, \rho_3, \rho_4$ so that $\rho(z') = \rho_1(r, h) \rho_2$

$$\rho(w) = \rho_1(r, h) \rho_2(s, i) \rho_3.$$
and there is some word $w'' = w$ with

$$\rho(w'') = \rho_{1} \rho_{2}(s, i)(t, j) \rho_{4}.$$ 

Either $\rho_{3} = \rho_{4}$, or a pair $(q, g)$ is in $\rho_{4}$ if and only if $(q − 1, g)$ is in $\rho_{3}$.

If $w$ is the initial subword of $w$ with $\rho(w) = \rho_{1}$, then no relation on reflections in $w$ is needed to transform $w$ into $w''$.

**Proof.** We prove this inductively and assume without loss of generality that $m_{ij}$ is odd. If $m_{ij}$ is even, we only need to relabel $s_{i}$ and $s_{j}$ in the arguments below.

If there is some word $x_{i}$ so that

$$w = x_{1}\{s_{i}s_{j}\}^{m_{ij}} y < x_{1}\{s_{j}s_{i}\}^{m_{ij}} y$$

then $w'' := x_{1}\{s_{j}s_{i}\}^{m_{ij}} y$ fulfils the assertions by Lemma 3.6.

On the other hand, we get a similar result if

$$w'' := x_{1}\{s_{j}s_{i}\}^{m_{ij}} y < x_{1}\{s_{i}s_{j}\}^{m_{ij}} y :$$

Then

$$\rho(x_{1}\{s_{j}s_{i}\}^{m_{ij}}) = \rho_{1} \rho_{2}(s − 1, i)(t − 1, j) \rho_{4},$$

where a pair $(q − 1, g)$ is in $\rho_{2}'$ if and only if $(q, g)$ is in $\rho_{4}'$. Furthermore, either $\rho_{3}' = \rho_{3}$ or a pair $(q − 1, g)$ is in $\rho_{3}'$ if and only if $(q, g)$ is in $\rho_{3}$.

Now suppose that $w = u\{s_{k}s_{l}\}^{m_{kl}−1}$ is not of the form in equation (2), but the assertion is true for $w' = u\{s_{k}s_{l}\}^{m_{kl}}$. If $w'' < w'$, then there is some word $w''$ with

$$\rho(w''_{1}) = \rho_{1}'(r', h') \rho_{2}'(s, i) \rho_{3}'.$$

and

$$\rho(w''_{1}) = \rho_{1}' \rho_{2}'(s, i)(t, j) \rho_{4}'$$

for some pair $(r', h')$ and series of pairs $\rho_{1}', \rho_{2}', \rho_{3}', \rho_{4}'$. Either $\rho_{3}' = \rho_{4}'$, or a pair $(q, g)$ is in $\rho_{3}'$ if and only if $(q − 1, g)$ is in $\rho_{3}'$.

There is some $w'$ so that $\rho_{1}' = \rho(w')$.

Furthermore, there is a pair $(q, l)$ and a series of pairs $\rho$ so that

$$\rho(w) = \rho(w')(q, l).$$

If $w' = w$ or $w' = u\{s_{k}s_{l}\}^{m_{kl}−1}$, then we set $(r, h) := (q, l)$. Since $x$ is leftmost , $w < w'$, and by Lemma 3.6 the assertion is true.

Since $x$ is reduced, there is only one other case: the word $w''$ has $\{s_{k}s_{l}\}^{m_{kl}}$ as a subword and the relation $\{s_{k}s_{l}\}^{m_{kl}−1}$ gives a word $w''$ that fulfils (1).

It remains to prove the assertion if $w' < w''$.

Then we can inductively assume that there is some $w''$ so that

$$\rho(w''_{1}) = \rho_{1} \rho_{2}(s − 1, i)(t − 1, j) \rho_{4}'$$

where a pair $(q − 1, g)$ is in $\rho_{2}'$ if and only if $(q, g)$ is in $\rho_{2}'$. Furthermore, either $\rho_{3}' = \rho_{3}$ or a pair $(q − 1, g)$ is in $\rho_{3}'$ if and only if $(q, g)$ is in $\rho_{3}$.

But there is no $w'' \equiv w'$ that shares an initial subword with $w$ which is longer than $w$, the longest initial subword that $w$ and $w'$ share.

Since $w''_{1}$ and $w_{1}$ share the initial subword $w'$ with $\rho(w_{1}) = \rho_{1}$, this is only possible if $w'' = u\{s_{k}s_{l}\}^{m_{kl}−1}$.

Again, we set $(r, h) := (q, l)$. Since $x$ is leftmost , $w < x_{1}$ and by Lemma 3.6 the assertion is true.

Completely analogously, we can prove the following:
Lemma 3.8. Let \( w, w' \) be words and \( s_i, s_j \) reflections. If the words \( w = x s_j y \) and \( w' = x' s_j y \) are equivalent, \( x \) is leftmost and \( w' <_l w \), then there are pairs \((r, h), (s, i)\) and series of pairs \( \rho_1, \rho_2, \rho_3, \rho_4 \) so that \( p(w') = \rho_1 \rho_2 \rho_3 \rho_4 \)
and there is some word \( w'' \equiv w' \) with
\[
\rho(w) = \rho_1 \rho_2(s, i) \rho_3 \rho_4.
\]
Either \( \rho_3 = \rho_4 \), or a pair \((q, g)\) is in \( \rho_4 \) if and only if \((q + 1, g)\) is in \( \rho_3 \).

If \( w \) is the initial subword of \( w' \) with \( p(w) = \rho_1 \), then no relation on reflections in \( w \) is needed to transform \( w \) into \( w'' \).

We get the following corollary:

Corollary 3.9. If \( w \{ s, s_j \}^{m_i j - 1} \) and \( w s_j \) are leftmost, then either \( w \{ s, s_j \}^{m_{ij}} \) is leftmost or \( w s_j < w \).

Proof. If we have \( w \{ s, s_j \}^{m_{ij}} < \rho w \{ s, s_j \}^{m_{ij}} \), then \( w s_j < w s_i \).

Furthermore, if \( w \{ s, s_j \}^{m_{ij}} \) is not leftmost, but \( w \{ s, s_j \}^{m_{ij} - 1} \) is, then we can write \( w = x s_j y \) and there is some \( w' = x' s_j y \) so that \( w' \equiv w \). By Lemma 3.8 there are some words \( x, y \) and a reflection \( s_k \) so that
\[
w = x s_j y
\]
and \( w'' < x' s_j y \).

So \( w s_j \equiv x s_k w') \) and \( w s_j w'' < x' s_j y \).

Remark 3.10. Note that in Lemma 3.9 we do not actually need to assume that \( x \) is leftmost; it is sufficient that \( x \) is reduced and the following holds: let \( w'' \equiv w \) so that \( w'' < x' s_j y \). Furthermore, assume that \( w' \) is the maximal initial subword that \( w' \) and \( x' \) share. Then there is some \( w'' \equiv w' \) with \( w'' < x' s_j y \).

So analogously to 3.9 we see: Suppose that there is some word \( w' \) so that \( w' \equiv w \) and for all \( w' \equiv w \{ s, s_j \}^{m_{ij} - 1} \), we have \( w' < x' s_j y \) if \( m_{ij} = 3 \) and \( w' < x' s_j y \) otherwise. Then \( w s_j \) is not leftmost.

Lemma 3.11. Suppose that \( w \equiv w \{ s, s_j \}^{m_{ij}} \) with \( m_{ij} \geq 3 \). If there is some \( i \neq j \neq k \) with \( w = w s_k \{ s, s_j \}^{m_{ij} - m} \) for some even \( m \geq 2 \) or \( w = w s_k \{ s, s_j \}^{m_{ij} - m} \) for some odd \( m \geq 3 \), then \( m_{ik} = 2 \) or \( m_{jk} = 2 \).

Proof. This is a simple, inductive proof: without loss of generality, we can assume that \( w = w s_k \{ s, s_j \}^{m_{ij} - 2} \) and \( m_{jk} = 3 \). Then there is some \( w' \) so that
\[
w = w s_k s_j s_k \{ s, s_j \}^{m_{ij} - 2} \equiv w s_j s_k \{ s, s_j \}^{m_{ij} - 1}.
\]
So there is some \( u \) so that
\[
u s_j s_k \equiv u \{ s, s_j \}^{m_{ik}}.
\]
If \( m_{ik} \geq 3 \), then we have the same situation as before, only considering a shorter word. Since \( w \) is finite, we see with induction on the length of \( w \) that \( m_{ik} = 2 \) or \( m_{jk} = 2 \).

Similarly, we can prove the following:
Lemma 3.12. Suppose that \( u s_j \) is leftmost, but \( u s_js_i \) is not leftmost for some word \( u \) and some reflections \( s_i, s_j \) with \( m_{ij} \geq 4 \).

Then there are \( s, t \in \mathbb{N} \) so that \( \rho(u s_j s_i) = \rho(u)(t, j)(s, i) \) and \( \rho(u) \) contains the pair \( (s - 1, i) \). If \( m_{ij} = 6 \), then \( \rho(u) \) additionally contains the pairs \( (s - 2, i) \) and \( (t - 1, j) \).

Proof. Suppose that the assertions are not fulfilled.

We can without loss of generality assume that there is some \( u' \) and reflections \( s_{k_1}, \ldots, s_{k_m}, s_{l_1}, \ldots, s_{l_m'} \) so that

\[
\mathcal{U} = u' s_j s_{k_1} \ldots s_{k_m} s_{l_1} \ldots s_{l_m'} \{s_j s_i\}^{m_{ij} - 2} \equiv u'' \{s_j s_i\}^{m_{ij} - 2}.
\]

Then there are \( s', t', t'', q_1, \ldots, q_m, r_1, \ldots, r_{m'} \in \mathbb{N} \) so that

\[
\rho(u'' s_j) = \rho(u')(t', j)(q_1, k_1) \ldots (q_m, k_m)(s', i)(r_1, l_1) \ldots (r_{m'}, l_{m'})(t'', j).
\]

If \( (t'' - 1, j) < (s', i) \), then the assertions of the lemma are fulfilled. Otherwise, we get one of the following cases:

(a) There is some \( 1 \leq o \leq m' \) with \( (t'' - 1, j) < (r_o, l_o) \) with \( m_{o, j} \geq 3 \).

(b) The words \( s_is_{t_1} \ldots s_{t_{m'}} s_j \) and \( u s_{j} \) are not leftmost, contrary to the assumptions.

So we can assume that the first case is fulfilled. Furthermore, without loss of generality, we can assume \( o = m' \).

If \( m_{d_1} = \ldots = m_{d_m} = 2 \), then

\[
u'' = u' s_j s_{k_1} \ldots s_{k_m} s_{l_1} \ldots s_{l_m'} s_i \equiv u''
\]

Either \( u'' < u'' \), contrary to the assumptions, or there is some \( s'' > s' \) so that

\[
\rho(u''') = \rho(u')(t', j)(q_1, k_1) \ldots (q_m, k_m)(s'', i)(r_1, l_1) \ldots (r_{m'}, l_{m'})(s'', i).
\]

Since \( u s_j \) is leftmost, but \( u s_j s_i \) is not, there is some \( v_1 \) so that

\[
u s_j s_i \equiv u_1 \{s_j s_i\}^{m_{ij}}.
\]

Thus,

\[
u'' = u' \{s_j s_{l_m'}\}^{m_{ij} - 2} s_i
\]

and we are in an analogous situation to before, only considering a shorter word.

Since the length of \( u \) is finite, the assertions of the lemma are inductively true under these assumptions.

So we can assume without loss of generality that \( m_{d_1} \geq 3 \). Furthermore, we have \((s' + 1, i) < (t'' - 1, j) < (r_{m'}, l_{m'})\). So there is at least one \( 1 \leq o \leq m' \) so that \( m_{o, o} \geq 3 \), since otherwise \( s_is_{t_1} \ldots s_{t_{m'}} s_{l_1} \) is equivalent to a smaller word (that begins with \( s_j \)), contrary to the assumption that \( u s_j \) is leftmost.

Then there is some \( v_2 \) so that

\[
u' \{s_j s_{l_m'}\} s_i \equiv v_2 \{s_i s_{l_1}\}^{m_{d_1}}.
\]

Because of (3), Lemma 3.7, Lemma 3.8 and \( m_{d_1} \geq 3 \), we get some \( v_3 \) so that

\[
u'' = v_3 s_j s_i s_{l_1} \ldots s_{l_m'} s_i.
\]

Since \( m_{d_m} \geq 3 \) and \( m_{d_{m'}} \geq 3 \), we are in the same situation as before, only considering a word of shorter length. Inductively, the proof is complete. 

Now we can define an assignment which maps the words of the Weyl group to the cofinite full additive subcategories of \( \text{mod} A \). We will show that this map yields a bijection between the Weyl group and the set of cofinite submodule closed subcategories.
Let $\tau = DTr$ be the Auslander-Reiten translation, see [2], p. 106. With [2], p. 259, every indecomposable preinjective module is of the form $\tau^rI_i$ for some $r \in \mathbb{N}$ and $1 \leq i \leq n$.

**Definition 3.13.** We can identify the pairs in $\mathcal{N}$ and the indecomposable preinjective modules by setting $(r, i) = \tau^rI_i$.

Not only does this give us a natural order on the preinjective modules, but this also yields an injective map from the words of the Weyl group to the cofinite full additive subcategories of $\text{mod} A$: If $\rho(w) = (r_1, i_1), (r_2, i_2), \ldots, (r_m, i_m)$, then $w \mapsto C_w$, where $C_w$ is the full additive category with

$\text{ind} C_w = \text{ind} A \setminus \{ (r_1, i_1), (r_2, i_2), \ldots, (r_m, i_m) \}$.

For a Weyl group element $w$ with the leftmost word $w$, define $C_w := C_w$.

**Example 3.14.** Let $A$ be as in Example 2.9 and $w = s_1s_2s_3s_2s_4s_1$. Then $\rho(w) = (0, 1)(0, 2)(0, 3)(1, 2)(1, 4)(2, 1)$ and

$\text{ind} C_w = \text{ind} A \setminus \{ I_1, I_2, I_3, \tau I_2, \tau I_4, \tau^2 I_1 \}$.

We will prove that the restriction of this map on the leftmost words is a bijection between those and the cofinite submodule closed subcategories.

Since every element of the Weyl group has a unique leftmost word, this gives a bijection between the elements of the Weyl group and the cofinite submodule closed subcategories.

The same bijection is used in [8].

4. Monomorphisms between preinjective modules

An observation makes the aim of the chapter much simpler to achieve: the cofinite submodule closed subcategories of the module category correspond naturally to the cofinite submodule closed subcategories of $\mathcal{I}$, the category of the preinjective modules.

Thus we devote this section to preinjective modules. In particular, we give a way to construct all modules $U$ that contain a given preinjective, indecomposable module $M$ as a submodule.

In Section 5 we will use this to show the connection to the Coxeter structure of the Weyl group. In Section 6 to 8, we will use this connection to prove that the bijection that we described exists.

**Proposition 4.1.** There is a bijection between full additive cofinite submodule closed subcategories of $\text{mod} A$ and full additive cofinite submodule closed subcategories of $\mathcal{I}$. It maps the category $C$ to the category $C' = C \cap \mathcal{I}$. Furthermore,

$\text{ind} A \setminus C = \text{ind} \mathcal{I} \setminus C$.

**Proof.** This is completely analogous to [8], Proposition 2.2:

If $A$ is representation finite, then $\text{mod} A = \mathcal{I}$ and there is nothing to prove. Suppose that $A$ is not representation finite. Since $C$ is cofinite, there is some $r \in \mathbb{N}$, so that $\tau^rI_1, \tau^rI_2, \ldots, \tau^rI_n \in C$. Now suppose that $M$ is a preprojective or regular module. Then $\tau^rM$ exists and has an injective envelope $I$. Since $\tau^r$ preserves monomorphisms, $M \subseteq \tau^rI \in C$. So $M \in C$ and $\text{ind} A \setminus C = \text{ind} \mathcal{I} \setminus C$.
Thus the assignment $C \mapsto C \cap I$ is a bijection between the full additive cofinite submodule closed subcategories of $\mod A$ and the full additive cofinite submodule closed subcategories of $I$. \hfill \Box

We start the construction of exact sequences with a lemma that holds for all Artin algebras:

**Lemma 4.2.** Let $A$ be an arbitrary Artin algebra and $M, X \in \mod A$ indecomposable. Let

$$0 \to M \xrightarrow{[f_1 \ f_2]} X \oplus X' \xrightarrow{[g_{11} \ g_{12} \ 0 \ g_{22}]} Y \oplus Y' \to 0$$

be an exact sequence for some $X', Y, Y' \in \mod A$, so that there is some $Z \in \mod A$ and an AR-sequence

$$0 \to X \xrightarrow{[g_{11} \ f_2]} Y \oplus Z \xrightarrow{[g_1' \ g_2' \ 0 \ 0 \text{id}_{Y'}]} \tau^{-1} X \oplus Y' \to 0 .$$

If for some $U \in \mod A$, a monomorphism $h : M \to U$ factors through $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ and $X \nmid U$, then $h$ also factors through $f'' = \begin{bmatrix} -f_2 f_1 \\ f_2 \end{bmatrix}$ and the following sequence is exact:

$$0 \to M \xrightarrow{\begin{bmatrix} -f_2 f_1 \\ f_2 \end{bmatrix}} Z \oplus X' \xrightarrow{\begin{bmatrix} g_1' \ g_2' \ 0 \ 0 \text{id}_{Y'} \end{bmatrix}} \tau^{-1} X \oplus Y' \to 0 .$$

**Proof.** By [2], the sequence

$$0 \to X \xrightarrow{\begin{bmatrix} g_{11} \\ f_2 \end{bmatrix}} Y \oplus Z \oplus Y' \xrightarrow{\begin{bmatrix} g_1' \ g_2' \ 0 \ 0 \text{id}_{Y'} \end{bmatrix}} \tau^{-1} X \oplus Y' \to 0$$

is also exact. By [2], Chapter I, Corollary 5.7, the diagrams

$$\begin{array}{cc}
M & \xrightarrow{f_1} X \\
\downarrow{\begin{bmatrix} g_{11} \\ f_2 \end{bmatrix}} & \downarrow{\begin{bmatrix} 0 \ 0 \end{bmatrix}} \\
X' & \xrightarrow{\begin{bmatrix} g_1' \ g_2' \ 0 \ 0 \text{id}_{Y'} \end{bmatrix}} \tau^{-1} X \oplus Y'
\end{array}$$

and

$$\begin{array}{cc}
X & \xrightarrow{-f_2} Z \\
\downarrow{\begin{bmatrix} g_{11} \\ 0 \end{bmatrix}} & \downarrow{\begin{bmatrix} g_1' \ 0 \ 0 \text{id}_{Y'} \end{bmatrix}} \\
Y \oplus Y' & \xrightarrow{\begin{bmatrix} 0 \ 0 \text{id}_{Y'} \end{bmatrix}} \tau^{-1} X \oplus Y'
\end{array}$$

are both pushouts and pullbacks. So the diagram

$$\begin{array}{cc}
M & \xrightarrow{\begin{bmatrix} -f_2 f_1 \\ f_2 \end{bmatrix}} Z \\
\downarrow{\begin{bmatrix} g_{11} \ g_{12} \ 0 \ g_{22} \end{bmatrix}} & \downarrow{\begin{bmatrix} g_1' \ g_2' \ 0 \ 0 \text{id}_{Y'} \end{bmatrix}} \\
X' & \xrightarrow{\begin{bmatrix} g_1' \ g_2' \ 0 \ 0 \text{id}_{Y'} \end{bmatrix}} \tau^{-1} X \oplus Y'
\end{array}$$
is itself a pushout and a pullback, see [4], p 334-335. By the definition of the Auslander-Reiten sequence (see [2], p. 137 and p. 144), the sequence

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{-f_2 f_1} & Z \oplus X' & \xrightarrow{g_2' g_1' g_1 2} & \tau^{-1} X \oplus Y' & \longrightarrow & 0
\end{array}
\]

is exact. It remains to show that \( h : M \to U \) factors through \( f'' = [-f_2 f_1 f_2] \).

Since we have assumed that \( h \) factors through \( f = [f_1 f_2] \), there is a morphism \( s = [s_1 s_2] : X \oplus X' \to U \) so that

\[
h = [s_1 s_2] f_1 = s_1 f_1 + s_2 f_2
\]

By the definition of the Auslander-Reiten sequence (see [2], p. 137 and p. 144), the sequence \( s_1 : X \to U \) factors through \([g_1 g_1'][f_2']\): there is a morphism \( s' = [s_1' s_2'] : Y \oplus Z \to U \) so that

\[
s_1 = [s_1' s_2'] [g_1 g_1'] = s_1' g_1 + s_2' f_2.
\]

So we get

\[
h = s_1' g_1 f_1 + s_2' f_2 f_1 + s_2 f_2.
\]

Since (7) is commutative, we have

\[
h = -s_1' g_1 f_2 + s_2' f_2' f_1 + s_2 f_2 = [-s_2' s_2 s_2'] [-f_2 f_1]
\]

and \( h \) factors through \( f'' \). \( \square \)

We can even say more:

**Lemma 4.3.** Let \( A \) be a hereditary Artin algebra, \( M \in \mathcal{I} \) and \( U \in \text{mod} \ A \). Suppose that the sequences of modules

\[
(X_1, X_2, \ldots, X_m)
\]

\[
(X_1', X_2', \ldots, X'_m)
\]

\[
(Y_1, Y_2, \ldots, Y_m)
\]

fulfil the following conditions:

(S1) There is an Auslander-Reiten sequence

\[
\begin{array}{ccccccc}
0 & \longrightarrow & M & \longrightarrow & X_1 \oplus X_1' & \longrightarrow & Y_1 & \longrightarrow & 0
\end{array}
\]

(S2) For all \( 1 \leq i < m \), there is some \( \alpha_i \in \mathbb{N} \) so that \( X_i^{\alpha_i} \mid X_i \oplus X'_i \), but \( X_i^{\alpha_i} \nmid U \).

(S3) For \( 1 \leq i < m \), there is an Auslander-Reiten sequence of the form

\[
\begin{array}{ccccccc}
0 & \longrightarrow & X_i & \longrightarrow & Z_i & \longrightarrow & \tau^{-1} X_i & \longrightarrow & 0
\end{array}
\]

Let \( Y_i' \) be the maximal module that is a direct summand of both \( Y_i \) and \( Z_i \). Write \( Y_i = Y_i' \oplus Y_i'' \) and \( Z_i = Y_i' \oplus Z_i' \).

If \( \tau^{-1} X_i \mid X_i' \), then let \( X_i'' \) be the module so that \( X_i = \tau^{-1} X_i \oplus X_i'' \) and set \( Y_i''' := 0 \). Otherwise, set \( X_i'' := X_i' \) and \( Y_i''' := \tau^{-1} X_i \).

The following equations hold:

\[
X_{i+1} \oplus X_{i+1}' = X_{i+1}'' \oplus Z_i'
\]

\[
Y_{i+1} = Y_{i+1}'' \oplus Y_i'''
\]
Then for all $1 \leq i \leq m$ there is an exact sequence

$$0 \rightarrow M \rightarrow X_i \oplus X'_i \rightarrow Y_i \rightarrow 0.$$  

Furthermore, if a monomorphism $M \rightarrow U$ exists, then it factors through all $f_i$.

To prove Lemma 4.3, we need the following observation:

**Remark 4.4.** Suppose that $X_i \oplus X'_i = X_i \oplus X_{i+1} \oplus B_i$ and that $C_i$ is the maximal module that is both a direct summand of $Y_i$ and $Z_{i+1}$.

Furthermore, write $Y_i = C_i \oplus C'_i$ and $Z_{i+1} = C_i \oplus D_i$. Set $B_i = B'_i$ and $C''_i = \tau X_i$ if $\tau X_i \nmid B_i$ and $B_i = B'_i \oplus \tau X_i$ and $C''_i = 0$ if $\tau X_i \mid B_i$. Then the sequences of modules

$$(X_1, X_2, \ldots, X_{i-1}, X_i, X_{i+2}, X_{i+3}, \ldots, X_m)$$

$$(X'_1, X'_2, \ldots, X'_{i-1}, X_i \oplus B_i, B'_i \oplus D_i, X'_{i+2}, X'_{i+3}, \ldots, X'_m)$$

$$(Y_1, Y_2, \ldots, Y_{i-1}, Y_i, C'_i \oplus C''_i, Y_{i+2}, Y_{i+3}, \ldots, Y_m)$$

also fulfill the conditions (S1) - (S3).

Note that only the $i$th and $(i + 1)$th elements of these sequences differ from the elements in the original sequences.

We can easily generalize this to the following: If $i < j_1 < j_2 < \cdots < j_l$, there is an irreducible morphism $X_{j_k} \rightarrow X_{j_{k+1}}$ for all $1 \leq k \leq l$ and $X_{j_l} \mid X_i$, then there are two sequences with $X'_m$ and $Y_m$ as their $m$-th elements that together with

$$(X_1, \ldots, X_{i-1}, X_{j_1}, X_{j_2}, \ldots, X_{j_l}, X_{i+1}, \ldots, X_{j_1-1}, X_{j_1+1}, \ldots, X_i)$$

$$(\ldots, X_{j_2-1}, X_{j_2+1}, \ldots, X_{j_l-1}, X_{j_l+1}, \ldots, X_i)$$

fulfill (S1) - (S3).

Furthermore, there are sequences of modules that fulfill the conditions (S1) - (S3) with $X_m, X'_m, Y_m$ as their $m$-th elements so that $X''_m = X'_m$ for all $1 \leq i \leq m$.

By Definition 3.13 if there is a morphism $X_i \rightarrow X_j$, then $X_j \cong X_i$. So we can use the above to get sequences that fulfills (S1) - (S3) with $X_m, X'_m, Y_m$ as their $m$-th elements so that $X_1 \geq X_2 \geq \cdots \geq X_{m-1}$. Then $X \geq \tau^{-1} I_i$ for all $X \mid X'_i$ and $X \mid Z_j$ with $j \leq i$ and thus $X''_m = X'_m$ for all $1 \leq i < m$.

**Proof of Lemma 4.3.** We prove the lemma inductively. By Remark 4.4, it is sufficient to prove the assertion for all sequences so that $X''_m = X'_m$ for all $1 \leq i < m$.

For these sequences, we additionally show the following: If there is an indecomposable direct summand $X$ of $X_m \oplus X'_m$ and $\tau X_i$ of $Y_m$ so that an irreducible morphism $X \rightarrow \tau X_i$ exists, then one of the following holds:

(a) There is a direct summand $X' \cong X$ of $X_m \oplus X'_m$ so that the component $X' \rightarrow Y_m$ of $g_m$ is irreducible and $g_m(X') \subseteq \tau^{-1} X_i$.

(b) Either $X \cong X_j$ for some $i < j < m$ or $X$ is isomorphic to a direct summand of $Y'_i$.

If $m = 1$, the assertion is obvious by definition of the Auslander-Reiten sequence.

Now suppose that it holds for all series of modules of length $m \in \mathbb{N}$ or smaller. We want to show that it also holds for sequences of length $m+1$ by applying Lemma 4.2.

To do this, we need to prove that there is an exact sequence of the form

$$0 \rightarrow M \rightarrow X_m \oplus X'_m \rightarrow Y'_m \oplus Y''_m \rightarrow 0.$$
so that $g_{m1}$ is irreducible.

Suppose that $Y_m'$ has some direct summands $Y_{m1}', Y_{m2}', \ldots, Y_{mk}'$ and $g_m$ has a component

$$\text{diag}(g_{11}, g_{22}, \ldots, g_{kk}) : X_m' \to \bigoplus_{i=1}^{k} Y_i'$$

where $g_{11}, g_{22}, \ldots, g_{kk}$ are irreducible and $\text{diag}(g_{11}, g_{22}, \ldots, g_{kk})$ is the diagonal matrix with entries $g_{11}, g_{22}, \ldots, g_{kk}$. Then there is a copy of $X_m$ on which this restricts to

$$\begin{bmatrix}
g_{11} \\
g_{22} \\
\vdots \\
g_{kk}
\end{bmatrix} : X_m \to \bigoplus_{i=1}^{k} Y_i',$$ 

an irreducible morphism.

By condition (S3) and since $Y_1'' = 0$, every indecomposable direct summand of $Y_m$ has the form $\tau^{-1} X_i$ for some $1 \leq i \leq m$.

If for all $\tau^{-1} X_i \mid Y_m'$, there is some copy $X$ of $X_m$ so that the component $X \to \tau^{-1} Y_i'$ of $g_m$ is irreducible and $g_m(X) \subseteq \tau^{-1} X_i$, then the above and the induction hypothesis mean that we can apply Lemma 4.2.

Suppose that there is some $\tau^{-1} X_i \mid Y_m'$, so that the above is not the case.

Since $Y_m' \mid Z_m$, there is an irreducible morphism between $X_m$ and $\tau^{-1} X_i$. By the inductive hypothesis, one of the following holds:

(a) $X_m \cong X_j$ for some $i < j \leq m$
(b) $X_m \mid Y_i'$.

We show that there are sequences

$$\begin{align*}
(X_1^{(1)}, X_2^{(1)}, \ldots, X_m^{(1)}) \\
(X_1''^{(1)}, X_2''^{(1)}, \ldots, X_m''^{(1)}) \\
(Y_1^{(1)}, Y_2^{(1)}, \ldots, Y_m^{(1)})
\end{align*}$$

(11)

that fulfil (S1)-(S3) and have $X_m, X_m', Y_m$ as their $m$-th elements so that $\tau^{-1} X_i = \tau^{-1} X_i^{(1)}$, but $X_m \not\cong X_i^{(1)}$ for all $i' < j \leq m$ and $X_m \not\mid Y_i''$.

Furthermore, we want to show that $X_l''^{(1)} = X_l'^{(1)}$ for all $1 \leq l < m$. Since $X_m = X_1^{(1)}, X_m' = X_m'^{(1)}, Y_m = Y_m^{(1)}$, this is already clear for $i = m$.

Obviously, we have $X_m \mid X_m \oplus X_m'$, so either $X_m \mid X_i'$ or $X_m \mid Z_k'$ for some $1 \leq k \leq m, k \neq i$.

In the first case, (b) is not possible, since $Y_i$ and $X_i \oplus X_i'$ do not share direct summands. In case (a), Remark 4.4 yields a sequence $(X_1^{(2)}, \ldots, X_m^{(2)})$, where $X_j$ comes before $X_i$.

In the second case, we can get a new sequence where $X_k$ comes before $X_i$, since $Z_k' \mid X_m \oplus X_m'$ (otherwise, $\tau^{-1} X_i$ would not be a direct summand of $Y_m$). In case (b), this sequence is already the one we need; in case (a), we can again get a another sequence by Remark 4.4 where $X_j$ comes before $X_i$.

If we call this new exact sequence $(X_1^{(2)}, \ldots, X_m^{(2)})$, then it is clear by (9) that $X_i''^{(2)} = X_i'^{(2)}$ holds for all $1 \leq l < m$.

Since there are only finitely many $j$ with $i' < j \leq m$, we get sequences of the form (11) after finitely many steps.
The inductive assumption gives us an exact sequence

\[ 0 \rightarrow M \xrightarrow{f_m} X_m \oplus X'_m \xrightarrow{g_m} Y_m \rightarrow 0 \]

where the component \( X_m \rightarrow Y_m \) of \( g'_m \) is irreducible and \( g'_m(X_m) \subseteq \tau^{-1}X_i \).

If \( \tau^{-1}X_i = Y_m \), then it is sufficient to look at the sequence \((12)\) instead of \((13)\):

\[ 0 \rightarrow M \xrightarrow{f_m} X_m \oplus X'_m \xrightarrow{g_m} Y_m \rightarrow 0 \]

If there is some \( \tau^{-1}X_k \) so that \( \tau^{-1}X_i \oplus \tau^{-1}X_k \mid Y_m \), then we can assume that \( g_m \) induces an indecomposable morphism \( X_m \rightarrow \tau^{-1}X_k \) and \( g_m(X_m) \subseteq \tau^{-1}X_k \).

So together \((13)\) and \((12)\) give a new exact sequence

\[ 0 \rightarrow M \xrightarrow{f_m} X_m \oplus X'_m \xrightarrow{g_m} Y_m \rightarrow 0 \]

where the induced morphisms \( X_m \rightarrow \tau^{-1}X_i \) and \( X_m \rightarrow \tau X_k \) of \( g''_m \) are irreducible and \( g_m(X_m) \subseteq \tau^{-1}X_i \oplus \tau^{-1}X_k \).

Inductively, there is an exact sequence of the form \((12)\), where \( g_m \) is irreducible and we can use Lemma 4.2 to get an exact sequence

\[ 0 \rightarrow M \xrightarrow{f_{m+1}} X_{m+1} \oplus X'_{m+1} \xrightarrow{g_{m+1}} Y_{m+1} \rightarrow 0 \]

If there is a monomorphism \( \tau \mapsto U \), then it factors through \( f_{m+1} \).

This gives us not only the assertion of the lemma but also the additional assumptions we have made:

Let

\[ g_m = \begin{bmatrix} g'_{m1} & g'_{m2} \end{bmatrix} : Y'_m \oplus Z'_m \rightarrow \tau^{-1}X_m \]

be the epimorphism of the AR-sequence. By \((10)\) and Lemma 4.2 we get

\[ g_{m+1} = \begin{bmatrix} g'_{m2} & g'_{m1} g_{m2} \\ 0 & g_{m3} \end{bmatrix} : Z'_m \oplus X'_m \rightarrow \tau^{-1}X_{m+1} \oplus Y''_m. \]

Let \( X \) be a direct summand of \( Z'_m \oplus X'_m \) and \( \tau^{-1}X_i \) a direct summand of \( \tau^{-1}X_m \oplus Y''_m \) so that there is an irreducible morphism \( X \rightarrow \tau^{-1}X_i \).

If \( i = m \), then \( X \) is a direct summand of \( Z_m \). If it is also a direct summand of \( Z'_m \), then \( g_{m+1}(X) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) \((X)\) and \((a)\) holds. Otherwise, \( X \) is a direct summand of \( Y''_m \).

If \( i \neq m \) and \( X \) is a direct summand of \( X'_m \), then either \((b)\) holds or

\[ \begin{bmatrix} g'_{m2} \\ g_{m3} \end{bmatrix}(X'_m) \subseteq \tau^{-1}X_i. \]

Thus \( g_{m2}(X'_m) = 0 \) and \( g_{m+1}(X) = \begin{bmatrix} 0 \\ g_{m3} \end{bmatrix} \) \((X)\). So \((a)\) holds.

Finally, suppose that \( i \neq m \) and \( X \) is not a direct summand of \( X'_m \). Because of the irreducible morphism between \( X \) and \( \tau^{-1}X_i \), the former is a direct summand of \( Z_i \). By \((S3)\), either it is a direct summand of \( Y'_i \) or of \( X_j \) for some \( i < j < m \).

A perhaps simpler way to interpret the sequences of modules used in the lemma above is the following:

Remark 4.5. Suppose that \( X_i, X'_i, Y_i \) are the \( i \)-th elements of sequences that fulfill \((S1)\) - \((S3)\). Then \( X_{i+1} \oplus X'_{i+1} \) are defined by taking the exact sequence

\[ 0 \rightarrow M \rightarrow X_i \oplus X'_i \rightarrow Y_i \rightarrow 0 \].
and the Auslander-Reiten sequence

\[ 0 \rightarrow X_i \rightarrow Z_i \rightarrow \tau^{-1} X_i \rightarrow 0. \]

We can add these sequences together and get

\[ 0 \rightarrow M \oplus X_i \rightarrow X_i \oplus X'_i \oplus Z_i \rightarrow Y_i \oplus \tau^{-1} X_i \rightarrow 0. \]

Then \( X_i \) is the maximal module that is a direct summand of both the first and the second term; we still get an exact sequence if we delete it in both terms:

\[ 0 \rightarrow M \rightarrow X'_i \oplus Z_i \rightarrow Y_i \oplus \tau^{-1} X_i \rightarrow 0. \]

The same holds for \( G_i \), the maximal module that is both a direct summand of the middle term and the last term. Deleting this in both terms gives us an exact sequence

\[ 0 \rightarrow M \rightarrow X_{i+1} \oplus X'_{i+1} \rightarrow Y_{i+1} \rightarrow 0. \]

These modules have some interesting properties:

**Corollary 4.6.** If there is a monomorphism \( h : M \rightarrow U \), then for sequences of modules

\[
(X_1, X_2, \ldots, X_m) \\
(X'_1, X'_2, \ldots, X'_m) \\
(Y_1, Y_2, \ldots, Y_m),
\]

which fulfills (S1) - (S3), there is a monomorphism \( X_i \oplus X'_i \rightarrow Y_i \oplus U \) for all \( 1 \leq i < m \).

Thus, every injective direct summand of \( X_i \oplus X'_i \) is a direct summand of \( U \).

**Proof.** By Lemma 4.5, there is an exact sequence

\[ 0 \rightarrow M \xrightarrow{f_i} X_i \oplus X'_i \xrightarrow{g_i} Y_i \rightarrow 0 \]

for \( 1 \leq i \leq m \) so that \( h \) factors through \( f_i \). Thus, there is some morphism \( h_i \) with \( h = h_i f_i \). So \( h_i \) is a monomorphism on \( \text{Im} f_i \). Since \( \text{Ker} g_i = \text{Im} f_i \), the morphism

\[
\begin{bmatrix} g_i \\ h_i \end{bmatrix} : X_i \oplus X'_i \rightarrow Y_i \oplus U
\]

is a monomorphism.

So every injective direct summand \( I \) of \( X_i \oplus X'_i \) is a direct summand of \( Y_i \oplus U \). Since \( X_i \oplus X'_i \) and \( Y_i \) do not share any direct summands, \( I \) is even a direct summand of \( U \). \( \square \)

We can use the following lemma to show that there is an algorithm that, for given indecomposable, preinjective module \( M \) constructs all \( U \) with \( M \rightarrow U \).

**Lemma 4.7.** Suppose there is an irreducible morphism between \( (s, i) = \tau^s I_i \) and \( (t, j) = \tau^t I_j \). Then either \( s = t \) and \( i > j \) or \( s = t + 1 \) and \( i < j \).

**Proof.** By [2], Chapter VIII, Proposition 1.2 and Lemma 1.8,

\[ s - 1 \leq t \leq s. \]

Furthermore, we have ordered the injective modules so that \( \text{Hom}(I_i, I_j) = 0 \) if \( i < j \).

By [2] Chapter VIII, Corollary 4.2, we get \( \text{Hom}(\tau^s I_i, \tau^t I_j) = 0 \) if \( i < j \) and by [2] Chapter VIII, Corollary 4.3., there is an irreducible morphism \( \tau^s I_i \rightarrow \tau^{s-1} I_j \) if and only if there is an irreducible morphism \( \tau^s I_j \rightarrow \tau^s I_i \).

So if \( s = t \), then \( i > j \) and if \( s = t + 1 \) then \( i < j \). \( \square \)
Proposition 4.8. Let $A$ be a hereditary Artin algebra with $M \in \text{mod } A$ indecomposable and preinjective. Let $U \in \text{mod } A$, so that $M$ is not a direct summand of $U$. There is a monomorphism $M \hookrightarrow U$ if and only if for some $m \in \mathbb{N}$ there are three sequences of modules

$$
(X_1, X_2, \ldots, X_m) \\
(X'_1, X'_2, \ldots, X'_m) \\
(Y_1, Y_2, \ldots, Y_m)
$$

that fulfil the conditions (S1) - (S3) and furthermore

(S4) If for some $1 \leq i \leq m$ the module $X_i \oplus X'_i$ has an injective direct summand $I$, then $I \mid U$.
(S5) $X_m \oplus X'_m$ is a direct summand of $U$.

Proof. To prove this, we use Lemma 4.3: since the sequences fulfil (S1)-(S 3), there are exact sequences of the form

$$
0 \rightarrow M \xrightarrow{f_i} X_i \oplus X'_i \xrightarrow{g_i} Y_i \rightarrow 0
$$

for all $1 \leq i \leq m$. If a monomorphism $M \hookrightarrow U$ exists, it factors through $f_i$ for all $1 \leq i \leq m$.

Thus one direction is obvious: if such sequences of modules exist, $f_m : M \hookrightarrow U$ is a monomorphism.

On the other hand, suppose that no series of modules fulfil (S1) - (S5).

If $M$ is injective, then it cannot be a submodule of $U$. Otherwise, there are series of modules that fulfil (S1) - (S3), since there is an AR-sequence that starts in $M$ and we can set $m = 1$.

If (S4) is not fulfilled, then $M$ cannot be a submodule of $U$ by Corollary 4.6. Otherwise, there is some non-injective $X_{m+1}$ and some $\alpha_{m+1} \in \mathbb{N}$ so that $X_m \oplus X'_m \mid X_{m+1}$, but $X_{m+1} \nmid U$. So we can extend the sequences of modules to

$$
(X_1, X_2, \ldots, X_m, X_{m+1}) \\
(X'_1, X'_2, \ldots, X'_m, X'_{m+1}) \\
(Y_1, Y_2, \ldots, Y_m, Y_{m+1})
$$

so that these series fulfil (S1) - (S3). If these sequences fulfil (S4), we can extend them again to sequences of length $m + 2$.

We have $M = (r, i)$ for some $r \in \mathbb{N}$ and $1 \leq i \leq n$. Every indecomposable direct summand of $X_i \oplus X'_i$ is of the form $(r', j) < (r, i)$ for some $r' \in \mathbb{N}_0$ and $1 \leq j \leq n$. Furthermore, if $X_1 = (r', j)$, then every direct summand of $Z_i$ is of the form $(r'', k) < (r', j)$, and analogously for $X_2, X_3, \ldots$.

So after finitely many steps, either we find sequences that do not fulfil (S4), or there is some $m'$ so that every direct summand of $X_{m} \oplus X'_{m}$ is injective. If (S4) is still fulfilled, then (S5) is also fulfilled, a contradiction to our assumption. □

The proof of Proposition 4.8 shows the following:

Corollary 4.9. Let $M$ and $U$ be preinjective modules over $A$. If $M \subset U$, then all sequences of modules that fulfil (S1) - (S3) can be extended to sequences of modules that fulfil (S1) - (S5).

If $M \not\subset U$, then all sequences of modules that fulfil (S1) - (S3) can be extended to sequences that fulfil (S1) - (S3) so that $X_m \oplus X'_m$ has an injective direct summand that is not a direct summand of $U$.
Remark 4.10. By Corollary 4.9, we can use the proposition as an algorithm that finds out for given indecomposable, preinjective $M$ and modules $U$, if there is a monomorphism $M \hookrightarrow U$. Alternatively, we can use it to construct all $U$ with $M \subseteq U$.

Note that it is very simple to generalize this for arbitrary preinjective $M$:

**Corollary 4.11.** Let $M$ be a preinjective module so that $M = \bigoplus_{i=1}^{m} M_i$ with $M_i$ indecomposable. Let $U$ be some module in $\text{mod} \ A$. Denote the middle term of the Auslander-Reiten sequence that starts in $M_i$ by $N_i$.

Furthermore, order $M_1, \ldots, M_m$ so that there is some $0 \leq k \leq m$ with $M_i \mid U$ if and only if $i \leq k$.

Suppose that the sequences of modules
\[(X_1, X_2, \ldots, X_m)\]
\[(X'_1, X'_2, \ldots, X'_m)\]
\[(Y_1, Y_2, \ldots, Y_m)\]
fulfil (S2), (S3) and
\[(S'1)\] We have
\[X_1 \oplus X'_1 = \bigoplus_{i=1}^{k} M_i \oplus \bigoplus_{i=k+1}^{m} N_i\]
and
\[Y_1 = \bigoplus_{i=k+1}^{m} \tau^{-1} M_i.\]

Then for all $1 \leq i \leq m$, there is an exact sequence
\[0 \to M \overset{f_i}{\to} X_1 \oplus X'_1 \overset{g_i}{\to} Y_1 \to 0.\]

There is a monomorphism $M \hookrightarrow U$ if and only if there is some $m' > m$ and modules $X_{m+1}, \ldots, X_{m'}, X'_{m+1}, \ldots, X'_{m'}, Y_{m+1}, \ldots, Y_{m'}$ so that the sequences
\[(X_1, X_2, \ldots, X_{m'})\]
\[(X'_1, X'_2, \ldots, X'_{m'})\]
\[(Y_1, Y_2, \ldots, Y_{m'})\]
fulfil (S'1) and (S2) - (S5).

Furthermore, if a monomorphism $M \hookrightarrow U$ exists, then it factors through all $f_i$.

**Example 4.12.** Take $A$ as in Example 2.9. A part of the preinjective component of the AR-quiver of $A$ is:

\[
\begin{array}{ccccccc}
\ldots & \tau I_3 & \tau I_1 & I_3 & I_1 & I_3 & I_1 \\
\ldots & \tau I_4 & \tau I_2 & I_4 & I_2 & I_4 & I_2 \\
\end{array}
\]

Suppose that we want to know whether $M = \tau I_3$ is a submodule of, say, $U = I_3 \oplus I_3 \oplus I_4$.

Then by (S1), $X_1 \oplus X'_1 = \tau I_1 \oplus \tau I_2$ and $Y_1 = I_4$. Since neither $\tau I_1$ nor $\tau I_2$ is a direct summand of $U$, we arbitrarily set $X_1 := \tau I_1$.

The AR-sequence
\[0 \to \tau I_1 \to I_3 \oplus I_4 \to I_1 \to 0\]
and (S3) show that $X_2 \oplus X_2' = \tau I_2 \oplus I_1$ and $Y_2 = I_1$. Since $I_4$ is a direct summand of $U$, we set $X_2 := \tau I_2$ to fulfil (S2). Using the AR-sequence

$$
\begin{array}{cccccccc}
0 & \rightarrow & \tau I_2 & \rightarrow & I_3 \oplus I_4 & \rightarrow & I_2 & \rightarrow & 0
\end{array}
$$

we get $X_3 \oplus X_3' = I_3 \oplus I_3$ and $Y_3 = I_1 \oplus I_2$. Since $I_4$ is injective, but not a direct summand of $U$, the condition (S4) is not fulfilled and there is no monomorphism between $M$ and $U$.

We have one more lemma:

**Lemma 4.13.** Let $M$ be an indecomposable, preinjective module and $U \in \text{mod } A$. If the sequences

$$
\begin{align*}
(X_1, X_2, \ldots, X_m) \\
(X_1', X_2', \ldots, X_m') \\
(Y_1, Y_2, \ldots, Y_m)
\end{align*}
$$

fulfil (S1) - (S3), then for every $1 \leq i \leq m$, there is an exact sequence

$$
\begin{array}{cccccccc}
0 & \rightarrow & X_i \oplus X_i' & \rightarrow & Y_i \oplus X_m \oplus X_m' & \rightarrow & Y_m & \rightarrow & 0
\end{array}
$$

Furthermore, if there is an exact sequence

$$
\begin{array}{cccccccc}
0 & \rightarrow & X_i \oplus X_i' & \rightarrow & Y_i \oplus U & \rightarrow & Z & \rightarrow & 0
\end{array}
$$

then there is also an exact sequence

$$
\begin{array}{cccccccc}
0 & \rightarrow & M_0 & \rightarrow & U & \rightarrow & Z & \rightarrow & 0
\end{array}
$$

**Proof.** Let $Y$ be the maximal module so that $Y_i \mid Y_i$ and $Y \mid Y_m$. Furthermore, suppose that $Y_i = Y \oplus Y'$ and $Y_m = Y \oplus Y''$.

We use Corollary [4.14] on $X_i \oplus X_i'$ and $Y \oplus U$. Take $i < j_1 < j_2 \cdots < j_l$ so that $X_{j_k}$ are those modules in the sequence $(X_{i+1}, \ldots X_m)$ which are already a direct summand of $X_i \oplus X_i'$. Then

$$(X_{m+1}, X_{j_1-1}, X_{j_1+1}, \ldots, X_{j_2-1}, X_{j_2+1}, \ldots, X_{j_l-1}, X_{j_l+1}, \ldots, X_m)$$

is part of a triple of sequences that fulfil (S'1) and (S2) - (S5) with respect to $X_i \oplus X_i'$ and $Y \oplus U$.

So the same construction that yields an exact sequence

$$
\begin{array}{cccccccc}
0 & \rightarrow & M & \rightarrow & X_m \oplus X_m' & \rightarrow & Y_m & \rightarrow & 0
\end{array}
$$

also gives an exact sequence

$$
\begin{array}{cccccccc}
0 & \rightarrow & X_i \oplus X_i' & \rightarrow & Y' \oplus X_m \oplus X_m' & \rightarrow & Y'' & \rightarrow & 0
\end{array}
$$

when used on $X_i \oplus X_i'$ and $Y' \oplus U$ instead of $M$ and $U$. Adding $Y$ to both the middle and the last term gives (15).

The exact sequence (15) is given by a sequence of modules that fulfil (S'1) and (S2) - (S5). Together with the sequences (14), this yields the exact sequence (17). □

5. Preinjective modules and the Weyl group

In this section we connect our results about preinjective modules with the relations of the Weyl group.

First we give a Lemma that shows the connection between the AR-sequences and the relations:
Lemma 5.1. Fix two integers $1 \leq i, j \leq n$. Set
\[
\alpha := \max \{ v \mid \exists (s, i), (t, j) : \text{there is an irreducible morphism } (s, i) \to (t, j)^v\}
\]
\[
\beta := \max \{ v \mid \exists (s, i), (t, j) : \text{there is an irreducible morphism } (t, j) \to (s, i)^v\}
\]
Let $(s_i, s_j)^{m_{ij}}$ be the defining relation of the Weyl group as in Lemma 2.7.
Then the value of $m_{ij}$ depends on $\alpha \beta$ in the following way:
\[
\begin{array}{c|cccccc}
\alpha \beta & 0 & 1 & 2 & 3 & 4 & \geq 4 \\
\hline
m_{ij} & 2 & 3 & 4 & 6 & \infty \\
\end{array}
\]
Proof. By [2] VIII, Corollaries 4.2 and 4.3, the integers $\alpha$ and $\beta$ do not depend on $s$. Let $(c_{ij})_{nn}$ be the Cartan matrix. Then by [2], p. 267-268, either $\alpha = c_{ij}$ and $\beta = c_{ji}$ or $\beta = c_{ij}$ and $\alpha = c_{ji}$. Lemma 2.7 gives the stated values for $m_{ij}$. □

Next, we need some notation:

Definition 5.2. For given $\alpha, \beta \in \mathbb{N}$ define a recursion formula by
\[
E(0) = 1
\]
\[
E(1) = \alpha
\]
\[
E(2m) = \beta E(2m - 1) - E(2m - 2)
\]
\[
E(2m + 1) = \alpha E(2m) - E(2m - 1)
\]
for all $m \in \mathbb{N}$.

This recursion is directly linked to the Weyl group:

Lemma 5.3. Let $\alpha, \beta$ be as in Lemma 5.1. Then
\[
E(m) = 0 \iff m \geq m_{ij} - 1.
\]
Proof. If $\alpha \beta < 4$, then we get the following values for $m \leq 6$:

\[
\begin{array}{c|ccccccc}
\alpha \beta & m_{ij} & E(0) & E(1) & E(2) & E(3) & E(4) & E(5) \\
\hline
0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 \\
2 & 4 & 1 & \alpha & 1 & 0 & 0 & 0 \\
3 & 6 & 1 & \alpha & 2 & \alpha & 1 & 0 \\
\end{array}
\]

(18)

Obviously, $E(m) = 0$ if $\alpha \beta < 4$ and $m \geq 6$.
If $\alpha \beta \geq 4$, then $m_{ij} = \infty$ by Lemma 5.1 and we need to show that $E(m) \neq 0$ for all $m \in \mathbb{N}$.
Since $E(2) = \alpha \beta - 1 > E(0) > 0$, we get inductively for $m > 1$:
\[
E(2m) = \beta E(2m - 1) - E(2m - 2)
\]
\[
= (\alpha \beta - 1)E(2m - 2) - \beta E(2m - 3)
\]
\[
= (\alpha \beta - 1)E(2m - 2) - E(2m - 2) - E(2m - 4)
\]
\[
> (\alpha \beta - 3)E(2m - 2)
\]
\[
\geq E(2m - 2).
\]
The proof that $E(2m + 1) > E(2m - 1) > 0$ is completely analogous. □

Next, we need some notation:
Definition 5.4. Fix $s \in \mathbb{N}_0$ and $1 \leq i \neq j \leq n$ and let $M_0 := \tau^s I_i$. If $s \geq 1$ or $j < i$, let $t$ be the integer with $(s - 1, i) < (t, j) < (s, i)$. Denote $M_1 := \tau^t I_j$, $M_2 := \tau^{t-1} I_i$, $M_3 := \tau^{t-1} I_j$, \ldots.

The following lemma is a key part in the proof that there is a bijection between cofinite, submodule closed subcategories and the elements of the Weyl group:

Lemma 5.5. Let $U$ be a module so that $M_k \nmid U$ for all $M_k \neq 0$ with $0 \leq k \leq m_{ij} - 1$. Then for all $m \geq 1$ with $M_{m+1} \neq 0$ and $E(m - 1) > 0$, there are series of modules that fulfil (S1) - (S3) and yield exact sequences

\begin{equation}
0 \rightarrow M_0 \xrightarrow{f_m} M^{E(m)} \oplus U \rightarrow M^{E(m-1)}_m \rightarrow 0
\end{equation}

so that no $M_k \nmid U_m$ for all $M_k \neq 0$ with $0 \leq k \leq m_{ij} - 1$.

If there is a monomorphism $M_0 \rightarrowtail U$, then it factors through $f_m$ for all $m$.

Proof. If $M_0$ is injective, there is nothing to show. So we can assume that an AR-sequence starts in $M_0$.

Let $\alpha, \beta$ be as in Lemma 5.4. Then there are modules $M', N'$ so that

\begin{equation}
0 \rightarrow M_0 \xrightarrow{f_m} M^{E(m)} \oplus U \rightarrow M^{E(m-1)}_m \rightarrow 0
\end{equation}

and

\begin{equation}
0 \rightarrow M_1 \xrightarrow{f_m} M^{E(m)}_1 \oplus M' \rightarrow M_2 \rightarrow 0
\end{equation}

are AR-sequences. Note that by \cite{2} VIII, Corollaries 4.2 and 4.3, for all non-injective $M_{2m-1}$, $m \in \mathbb{N}_0$, there are AR-sequences of the form

\begin{equation}
0 \rightarrow M_{2m-1} \xrightarrow{f_m} M^{E(m)}_{2m-1} \oplus \tau^m M' \rightarrow M_{2m+1} \rightarrow 0.
\end{equation}

For all non-injective $M_{2m}$, $m \in \mathbb{N}$ they are of the form

\begin{equation}
0 \rightarrow M_{2m} \xrightarrow{f_m} M^{E(m)}_{2m} \oplus \tau^m N' \rightarrow M_{2m+2} \rightarrow 0.
\end{equation}

If we set $U_1 := N'$, the AR-sequence that starts in $M_0$ is the exact sequence

\begin{equation}
0 \rightarrow M_0 \xrightarrow{f_m} M^{E(1)}_1 \oplus U_1 \rightarrow M^{E(0)}_2 \rightarrow 0.
\end{equation}

If $M_1$ is injective, then the proof is complete. So we can assume that an AR-sequence starts in $M_1$ and use Lemma 4.3.

Since $M_1 \nmid U$, we set

\[ X_1 := X_2 := X_3 := \cdots := X_{E(1)} := M_1. \]

Then we get sequences that fulfil conditions (S1) - (S3) by setting

\[ X'_1 := M^{E(1)-1}_1 \oplus N', \]

\[ X'_2 := M^{E(1)-2}_1 \oplus M^{E(2)}_2 \oplus N' \oplus M', \]

\[ \cdots \]

\[ X'_{E(1)} := M^{E(1)-1}_{E(1)} \oplus N' \oplus (M')^{E(1)-1} \]
and
\[ Y_1 := M_2, \]
\[ Y_2 := M_3, \]
\[ Y_3 := M_2^2, \]
\[ \ldots \]
\[ Y_{E(1)} := M_3^{E(1)-1}. \]

We get
\[ X_{E(1)+1} \oplus X_{E(1)+1}' = M_2^{E(1)-1} \oplus N' \oplus (M')^{E(1)} \]
and \( Y_{E(1)+1} = M_3^{E(1)} \). Thus by Lemma 4.3 there is some \( f_1 \) so that the following sequence is exact:
\[ 0 \rightarrow M_0 \xrightarrow{f_1} M_2^{E(1)-1} \oplus N' \oplus (M')^{E(1)} \rightarrow M_3^{E(1)} \rightarrow 0. \]

If there is a monomorphism \( M \rightarrow U \), then it factors through \( f_1 \).
Since \( U_2 := N' \oplus (M')^{E(1)} \), we can write the exact sequence as
\[ 0 \rightarrow M_0 \xrightarrow{f_1} M_2^{E(2)} \oplus U_2 \rightarrow M_3^{E(1)} \rightarrow 0. \]

We show the rest inductively: Suppose that
\[ 0 \rightarrow M_0 \rightarrow M_2^{E(2m-1)} \oplus U_{2m-1} \rightarrow M_2^{E(2m-2)} \rightarrow 0 \]
is an exact sequence and \( E(2m - 1) \neq 0 \). Furthermore, suppose that this exact sequence is yielded by sequences of modules of the length \( m' - 1 \). Then
\[ X_{m'} := M_{2m-1}, X_{m'+1} := M_{2m-1}^{E(2m-1)-1} \oplus U_{2m-1}, Y_{m'} := M_{2m-2}^{E(2m-2)} \]
are elements of sequences that fulfil the condition (S1) - (S3) of Lemma 4.3.

If \( M_{2m-1} \) is injective, then \( M_{2m+1} = \tau M_{2m-1} = 0 \) and there is nothing to prove. If \( M_{2m-1} \) is not injective, then the AR-sequence (21) exists. As above, we set
\[ X_{m'+1} := \cdots := X_{m'+E(2m-1)-1} := M_{2m-1}. \]

This determines \( X_{m'+1}, \ldots, X_{m'+E(2m-1)-1} \) and \( Y_{m'+1}, \ldots, Y_{m'+E(2m-1)-1} \) completely.

Since \( E(2m) = \beta E(2m - 1) - E(2m - 2) \), we get
\[ X_{m'+E(2m-1)} \oplus X_{m'+E(2m-1)}' = M_2^{E(2m)} \oplus U_{2m-1} \oplus (\tau^{m-1} M')^{E(2m-1)} \]
\[ Y_{m'+E(2m-1)} = M_2^{E(2m-1)} \]
Together with \( U_{2m} := U_{2m-1} \oplus (\tau^{m-1} M')^{E(2m-1)} \), this yields an exact sequence
\[ 0 \rightarrow M_0 \xrightarrow{f_{2m}} M_2^{E(2m)} \oplus U_{2m} \rightarrow M_2^{E(2m-1)} \rightarrow 0 \]
for some \( f_{2m} \). By Lemma 4.3 \( M \rightarrow U \) factors through \( f_{2m} \).

Analogously, we can construct
\[ 0 \rightarrow M_0 \xrightarrow{f_{2m+1}} M_2^{E(2m+1)} \oplus U_{2m} \rightarrow M_2^{E(2m+2)} \rightarrow 0 \]
if \( E(2m) \neq 0 \) and \( M_{2m+2} \neq 0 \). \( \square \)
Corollary 5.6. Let $U_m$ be as in Lemma 5.3. If $(r,l) \mid U_m$, then $M_0 > (r,l) > M_{m+1}$.

If $M_0 > (r,l) > M_1$, then $(r,l) \mid U_m$ if and only if $m_{il} \geq 3$.

If $M_1 > (r,l) > M_m$, then $(r,l) \mid U_m$ if and only if $m_{il} + m_{jl} \geq 5$.

If $M_m > (r,l) > M_{m+1}$ and $m$ is even, then $(r,l) \mid U_m$ if and only if $m_{il} \geq 3$. If $m$ is odd, then $(r,l) \mid U_m$ if and only if $m_{jl} \geq 3$.

Proof. This is obvious from the proof of Lemma 5.5.

Remark 5.7. Note that $m_{ij} = m_{ji}$. If we fix $s,t$ as in Definition 5.4, we can set $M_0' := \tau^s I_j, M_1' := \tau^{s-1} I_i, M_2' := \tau^{t-1} I_j, M_3' := \tau^{s-2} I_i, \ldots$ and

$$E'(0) = 1$$
$$E'(1) = \beta$$
$$E'(2m) = \alpha E(2m - 1) - E(2m - 2)$$
$$E'(2m + 1) = \beta E(2m) - E(2m - 1).$$

With this definition, we get analogous results to 5.3, 5.5, and 5.6.

6. Preliminaries for the main theorem

This section collects some preliminaries which are necessary to prove that there is a bijection between the Weyl group and the cofinite submodule closed subcategories. First, we show that every cofinite submodule closed subcategory is of the form $C_w$ for some word $w$.

Then we will prove an auxiliary result that will make the inductions in the next section possible.

Lemma 6.1. If a cofinite, full additive subcategory $C$ of $\text{mod } A$ is submodule closed, then there is a word $w$ over $S = \{s_1, s_2, \ldots, s_n\}$ with $C = C_w$.

Proof. By Lemma 5.11

$$\text{ind } A \setminus C = \text{ind } I \setminus C := \{(r_1, i_1), (r_2, i_2), \ldots (r_m, i_m)\}$$

for some $m \in \mathbb{N}$ and modules $(r_1, i_1) < (r_2, i_2) < \cdots < (r_m, i_m)$.

Suppose that for all words $w$ over $S$

$$\rho(w) \neq (r_1, i_1)(r_2, i_2) \cdots (r_m, i_m).$$

By Definition 5.11 either $r_1 > 0$ or there is some $1 \leq j \leq m - 1$ so that

$$(r_j, i_j) < (r_{j+1}, i_{j+1} - 1).$$

In the first case, $C$ contains the middle term of the AR-sequence that starts in $(r_1, i_1)$ by Lemma 4.17. In the second case, $C$ contains the middle term of the AR-sequence that starts in $(r_{j+1}, i_{j+1})$. In both cases, $C$ is not submodule closed.

So by Definition 5.13 there is some $w$ with

$$\rho(w) = (r_1, i_1)(r_2, i_2) \cdots (r_m, i_m)$$

and $C = C_w$.

Recall that $C_w = C_{\overline{w}}$, where $\overline{w}$ is the leftmost word for $w$. So we need to prove that the word $\overline{w}$ in Lemma 6.1 is leftmost. Furthermore, we need the other direction, namely, that $C_w$ is submodule closed if $w$ is leftmost.

We will use the following lemma for the proofs of both directions:
Lemma 6.2. Suppose that the words $w$ and $w''$ are equivalent and there are pairs $(r, h), (s, i), (t + 1, j)$ and series of pairs $\rho_1, \rho_2, \rho_3$ so that
\[
\rho(w) = \rho_1(r, h)\rho_2(s, i)\rho_3,
\]
and either $\rho_3 = \rho_4$, or a pair $(q, g)$ is in $\rho_4$ if and only if $(q - 1, g)$ is in $\rho_3$.

Furthermore, suppose that the word $z$ with $\rho(z) = \rho_1(r, h)\rho_2$ is reduced and $m_{ij} \geq 3$.

If $M_0 = (s, i), M_1 = (t, j), M_2, \ldots, M_{m_{ij} - 3} \notin C_w$, then there are sequences of modules as in Lemma 5.3 (used on $M_0$ and any $U \in \mathcal{C_w}$) that yield some $U', Y \in \mathcal{I}$ so that
\[
\begin{array}{c}
0 \longrightarrow M_0 \longrightarrow M_{m_{ij} - 2} \oplus U' \longrightarrow Y \longrightarrow 0
\end{array}
\]
is an exact sequence and either $Y \in \mathcal{C_w}$ or both $Y = (r, h)^{E(m_{ij} - 3)}$ and $U' \in \mathcal{C_w}$ hold.

Proof. We show this by induction on the number $m$ of Coxeter relations needed to transform $w$ into $w''$.

Furthermore, we show that a few additional assertions hold, which we need for the inductive proof:

(A1) If $(r', h') \gamma | Y$ and $Z \in \mathcal{C_w}$ is a direct summand of the middle term of the AR-sequence that ends in $(r', h')$, then $Z \gamma | U'$.

(A2) Let $V$ be the maximal direct summand of $U'$, so that for every $(q, g) | Y$ there is some $(r', h') | Y$ and an indecomposable morphism $(q, g) \rightarrow (r', h')$.

Furthermore, let $(q', g')$ be the biggest indecomposable direct summand of $V$ and $(r'', h'')$ the smallest indecomposable direct summand of $Y$.

If there are some $o \in \mathbb{N}_0$ so that $Y, \tau^{-1} Y, \ldots, \tau^{-o} Y \in \mathcal{C_w}$ and $\tau^{-o-1} Y \notin \mathcal{C_w}$, then one of the following holds:

(a) $\tau^{-o-1} Y = (r, h)^{E(m_{ij} - 3)}$ and $\tau^{-1} V, \tau^{-2} V, \ldots, \tau^{-o-1} V \in \mathcal{C_w}$.

(b) Let $V'$ be the maximal direct summand of $X$ so that for every $(q, g) | V'$ there is some $0 \leq k \leq o$ so that $(q - k, g) \notin \mathcal{C_w}$. Then there is some module $Y'$ with an exact sequence
\[
0 \longrightarrow \tau^{-o-1} V' \longrightarrow \tau^{-o-1} Y \oplus U'' \longrightarrow Y' \longrightarrow 0.
\]

Either $Y' \in \mathcal{C_w}$ or both $Y' = (r, h)^{E(m_{ij} - 3)}$ and $U'' \in \mathcal{C'}$ hold, where
\[
\text{ind} \mathcal{C'} = \text{ind} \mathcal{C_w} \setminus \mathcal{M}
\]
with
\[
\mathcal{M} = \left\{ M \in \text{ind} \mathcal{I} : \exists 0 \leq o' \leq o : \begin{cases} \tau^{o'+1} M \notin \mathcal{C_w} \\ (r'', h'') < \tau^{o'+1} M \\ \tau^{o'+1} M < (q' - 1, g') \end{cases} \right\}
\]

Furthermore, (A1) and (A2) still hold if we exchange $U'$, $Y$ and $\mathcal{C_w}$ for $U'', Y'$ and $\mathcal{C'}$ respectively.

If there are some reflections $s_k, s_i$ and words $w, w'$ so that
\[
w = w [s_k s_i]^{m_{ij} + 1} w' \quad \text{and} \quad w'' = w [s_k s_i]^{m_{ij} + 1} w,
\]
then this is the result of Lemma 5.3.

Now suppose that $w = w [s_k s_i]^{m_{ij} + 1} w$ and the lemma, (A1) and (A2) are proved for the word $w_{12} = w [s_k s_i]^{m_{ij} + 1} w$. 

Either there are modules $U_{(1)}', Y_{(1)}$ so that the exact sequence given by the inductive assumptions is
\begin{equation}
0 \longrightarrow M_0 \longrightarrow M_{m_{ij}-2} \oplus U_{(1)}' \longrightarrow Y_{(1)} \longrightarrow 0,
\end{equation}
or we can write the exact sequence as a $\tau$-translate or a $\tau^{-1}$-translate of (24).

Let $w_1 \equiv w''$ with $$\rho(w_1) = \rho_1'(r_1, h_1)\rho_2'(s', i)\rho_3'$$
and $$\rho(w'') = \rho_1'(r_1, h_1)\rho_2'(s', i)(t' + 1, j)\rho_4',$$
so that either $\rho_3' = \rho_4'$, or a pair $(q, g)$ is in $\rho_4'$ if and only if $(q - 1, g)$ is in $\rho_3'$.

Furthermore, we can assume without loss of generality that $m_{kl}$ is even. Then there are $q_1, q_2 \in \mathbb{N}_0$ and a series of pairs $\rho''$ so that
$$\rho(w) = \rho(w)(q_1 - \frac{m_{kl}}{2} + 1, k)(q_2 - \frac{m_{kl}}{2} + 1, l) \cdots (q_1, k)(q_2, l)\rho''.$$ 
We can assume that $(r, h)$ is in the series of pairs $\rho(w)(q_1 - \frac{m_{kl}}{2} + 1, k)(q_2 - \frac{m_{kl}}{2} + 1, l) \cdots (q_1, k)(q_2, l)$, since otherwise there is nothing to show. Analogously, we assume that the pair $(r_1, h_1)$ is in the series of pairs $\rho(w_{s_1s_2})^{m_{kl}-1}$. Furthermore, we can assume $w'' \neq w'''$ and $M_0 > (q_2, l)$.

Analogously to Lemma 4.3, if $m_{kl} \geq 3$, then there is some $X \in C_w$ so that
\begin{equation}
0 \longrightarrow (q_2, l) \longrightarrow (q_1, k) \oplus X \longrightarrow (q_1 - \frac{m_{kl}}{2} + 1, k)
\end{equation}
is an exact sequence.

We have two different cases to consider:

First, assume that $w$ is also an initial subword of $w'''$. Then $w_{s_1s_2}^{m_{kl}-1}$ is an initial subword of $w''$, since $\rho(x) = \rho_1(r, h)\rho_2$ is reduced. Furthermore, $w <_{\mathcal{C}} w$ and $(q_1 - \frac{m_{kl}}{2}) + 1, k) = (r, h)$.

If $(r_1, h_1) \in C_w$, then we can set $Y := Y_{(1)}$ and we have $(q_1, k) = (r_1 - 1, h_1)$.

(A1) holds by the inductive assumption, (A2) holds by (24), (A1) and Lemma 4.13.

On the other hand, if $(r_1, h_1) \notin C_w$, then $(q_1, k) = (r_1, h_1)$. Furthermore, if we have $Y_{(1)} = (r_1, h_1)E^{(m_{ij}-3)}$, then (A1), (25) and Lemma 4.13 give an exact sequence of the form (22) with $Y = (r, h)E^{(m_{ij}-3)}$. (A1) holds obviously.

If $Y_{(1)} \neq (r_1, h_1)E^{(m_{ij}-3)}$, then we set $Y := Y_{(1)}$ and we only need to prove that (A2) holds. Analogously to above, this is the result of Lemma 4.13 and (25).

It remains to prove the assumption in the case that $w$ is not an initial subword of $w'''$.

If $w <_{\mathcal{C}} w_1$, then $\rho(w_1)$ contains the series of pairs
$$\rho(w_1)(q_2 - \frac{m_{kl}}{2} + 1, l) \cdots (q_1, k)(q_2, l)(q_1 + 1, k)$$
and the exact sequence given by the induction is either (24) or the $\tau$-translate of (24). We can assume without loss of generality, that some indecomposable direct summand of $Y_{(1)}$ or $\tau Y_{(1)}$ respectively is smaller than $(q_1 + 1, k)$. Otherwise, the arguments below hold analogously for an exact sequence given by (A2).

By Proposition 4.3, the exact sequence yielded by the induction is given by sequences of modules that fulfil (S1)-(S3). By Remark 4.13, we can assume that $X''_i = X'_i$, for all $X'_i$.

In the following we begin with the case where this exact sequence is (24).
By Lemma 4.3 these series of modules yield an exact sequence

\[(26) 0 \rightarrow M_0 \rightarrow X_\gamma \oplus X'_\gamma \rightarrow Y_\gamma \rightarrow 0\]

so that for every \((r'_1, h'_1) \mid Y_\gamma\) the inequality \((q_1, k) < (r'_1, h'_1)\) holds, since \(M_0 > (q_1 + 1, k)\). We can even assume that \((q_1, k) < (r'_1, h'_1) \leq (q_1 + 1, k)\) for all \((r'_1, h'_1) \notin C_w\).

Furthermore, there is an irreducible morphism \(X_\gamma \rightarrow (r'_1, h'_1)\) and \((q_1 + 1, k) \leq X_\gamma \notin C_w\). Analogously to Lemma 5.3 if \(mkl \geq 3\), there is some \(X \in C_w\) so that

\[(27) 0 \rightarrow (q_1 + 1, k) \rightarrow (q_2, l) \oplus X \rightarrow (q_1 - \frac{mkl}{2}, 1, k)\]

with \(V \in C_w\). Together with the \(\tau\)-translate of the exact sequence \((25)\), this shows that \(X_\gamma \neq (q_2 + 1, l)\) and \(X_\gamma \neq (q_1 + 1, k)\): Otherwise, by Lemma 4.13 we would get an exact sequence where either \((q_1 - \frac{mkl}{2} + 1, k)\) or \((q_2 - \frac{mkl}{2} + 1, l)\) is a direct summand of the last term, but every direct summand of the middle term is in \(C_w\). This is a contradiction to the inductive assumption.

Since \((q_1 + 2, k) > X_\gamma > (q_1 + 1, k)\), we have \(\tau^{-1}X_\gamma \in C_w\). Inductively, \(Y_{(1)} \in C_w\) and \((q_1, k) < (r'_1, h'_1) \mid Y_{(1)}\).

If \((q_1 + 1, k) \mid Y_\gamma\) for any exact sequence of the form \((26)\), then there is such a sequence so that \(Y_\gamma \in C_w\), but \(Y_\gamma \notin C_w\).

Otherwise, the sequence \((24)\) is already of the form \((22)\).

In the latter case, it is easily seen that this sequence fulfills (A1) and (A2): the former holds by the inductive assumption. Define \(V, V', C'\) as in (A2) and let \(V_{(1)}, V'_{(1)}\) be the corresponding modules, \(C'_{(1)}\) the corresponding category for the sequence \((24)\). Then \(V = V_{(1)}, V' = V'_{(1)}\) and \(C' = C'_{(1)}\). Thus, assertion (2) also holds.

So assume that there is some \(\gamma\) with \((q_1 + 1, k) \mid Y_\gamma\). This sequence is of the form \((22)\) and (A1) holds. Let \(\alpha \in \mathbb{N}\) be the maximal exponent so that \((q_1 + 1, k)^\alpha \mid Y_\gamma\)

We can write \(X_\gamma \oplus X'_\gamma = B_1 \oplus B'_1 \oplus M_{m_{ij} - 2}\) so that \(B_1\) is the maximal direct summand of \(X_\gamma \oplus X'_\gamma\) with an irreducible morphism \(B_1 \rightarrow (q_1 + 1, k)\). Then there is an exact sequence

\[0 \rightarrow B_1 \rightarrow C_1 \oplus (q_1 + 1, k)^\alpha \rightarrow \tau^{-1}B_1.\]

By Remark 4.3 and Lemma 4.13, \(U_{(1)} = B'_1 \oplus C_1\). If \(Y_\gamma = D \oplus (q_1 + 1, k)^\alpha\), then \(Y_{(1)} = D \oplus \tau^{-1}B_1\).

So we can write \(X'_{(1)} = B''_1 \oplus C''_1\) with \(B''_1 \mid B'_1\) and \(C''_1 \mid C_1\). We get \(X' = B''_1 \oplus B_1\).

By Proposition 4.8 Lemma 4.1.3 and the inductive assumption, assertion (2) is fulfilled.

If we still have \(w <_{\mathcal{I}} w_{-1}\), but the exact sequence given by the inductive assumption is the \(\tau\)-translate of \((24)\), then analogously we have \(\tau Y_{(1)} \in C_w\), and \((q_1, k) < (r'_1, h'_1)\) for all direct summands \((r'_1, h'_1)\) of \(\tau Y_{(1)}\). Suppose that we have \(\tau Y_{(1)}, Y_{(1)}, \ldots, \tau^{o-1}Y_{(1)} \in C_w\) and \(\tau^{-o}Y \notin C_w\). If \(o > 0\), then \((24)\) is of the form \((22)\). By the inductive hypothesis, (A2) is fulfilled.

If \(o = 0\), then we use assertion (A2) of the inductive hypothesis: by Lemma 4.13 there is an exact sequence of the form \((22)\) that fulfills (A1) and (A2).
It only remains to regard what happens if \( w_1 <_l w \). Then \( \rho(w_1) \) contains the series of pairs
\[
\rho(w_1) = (q_2 - \frac{mk_l}{2}, l) \cdots (q_2 - 1, l)(q_1, k).
\]
The exact sequence given by the inductive assumption is either (24) or the \( \tau^{-1} \)-translate. In the first case, we show analogously to the above that \( Y(1) \in \mathcal{C}_w \) and \((q_1, k) < (r_1^I, h_1^I)\) for all \((r_1^I, h_1^I) \in Y(1)\). If \((q_2, l) \in Y(1)\), then the sequence (24) is already of the form (22) and the assertions hold.

Otherwise, \( \tau^{-1} Y(1) \notin \mathcal{C}_w \). Analogously to before, \((q_1, k) \notin \tau^{-1} X(1)\) by Lemma 5.5 and (A2): If \((q_1, k) \notin \tau^{-1} X(1)\) we use Lemma 4.12 and get an exact sequence where either \((q_1 - \frac{mk_l}{2}, k)\) or \((q_2 - \frac{mk_l}{2}, l)\) is a direct summand of the last term, but every direct summand of the middle term is in \( \mathcal{C}_w \). This is a contradiction to the inductive assumption.

So \( \tau X(1) \in \mathcal{C}_w \) which means that no direct summand of \( X(1) \) is in \( \mathcal{C}_w \).

As before, the assertion (A2) of the inductive hypothesis and Lemma 4.13 show that there is an exact sequence of the form (22) and that (A1) and (A2) are fulfilled.

If the exact sequence yielded by the inductive hypothesis is the \( \tau^{-1} \)-translate of (24), then similar to the arguments above we get an exact sequence (22) so that \( Y \in \mathcal{C}_w \) and \((q_1, k) < (r_1^I, h_1^I)\) for every direct summand \((r_1^I, h_1^I)\) of \( Y \). This exact sequence fulfills (A1) and (A2).

\[\square\]

7. The First Direction

In this section we show inductively that for every \( w \in W \), the category \( \mathcal{C}_w \) is submodule closed. Afterwards, it only remains to show that every cofinite, submodule closed category is of the form \( \mathcal{C}_w \).

We begin with the basis of the induction:

**Lemma 7.1.** Let \( m_{ij} < \infty \) and \( U_1, \ldots, U_{m_{ij} - 1} \) be as in Lemma 5.6. If we have \( U_1, \ldots, U_{m_{ij} - 1} \in \mathcal{C}_w \) and \( M_0 \notin \mathcal{C}_w \), then \( \mathcal{C}_w \) is not submodule closed and \( w \) is not leftmost.

**Proof.** Since \( M_0 \subseteq U_{m_{ij} - 1} \), the category \( \mathcal{C}_w \) is not submodule closed. Let \( u \) be a word for the element \( w \in W \). By Definition 5.4, \( M_0 = (s, i) \) and \( M_1 = (t, j) \).

Suppose that \( M_1 \in \mathcal{C}_w \). Let \( u \) be the initial subword of \( w \) that is defined through the inequality \((r, k) \leq (s - 1, i)\) for every pair \((r, k)\) in \( \rho(u) \).

Then there are reflections \( s_{k_1}, s_{k_2}, \ldots, s_{k_m} \) and a word \( w_u \) so that
\[
w = u_{s_{k_1}s_{k_2} \cdots s_{k_m}w_u}.
\]
Since \( U_1 \in \mathcal{C}_w \), we have
\[
m_{k_1} = n_{k_2} = \cdots = n_{k_m} = 2
\]
by Lemmas 5.1 and 4.7. So
\[
w' = u_{s_{k_1}s_{k_2} \cdots s_{k_m}w_u}
\]
is equivalent to \( w_u \) and thus a word for \( w \). Since \((r, k) \leq (s - 1, i)\) for all reflections \((r, k)\) in \( \rho(u) \), we see that either \( w' <_l w \) or \( w \) is not reduced.

Clearly, the same argument holds if \( M_m \in \mathcal{C}_w \) for some \( 1 \leq m \leq m_{ij} - 1 \).

It remains to prove that \( \mathcal{C}_w \) is not submodule closed if \( M_0, \ldots, M_{m_{ij} - 1} \notin \mathcal{C}_w \). Without loss of generality, we can assume that \( M_{m_{ij} - 1} = (p, i) \) for some \( p \in \mathbb{N} \) and \( M_{m_{ij} - 2} = (q, j) \) for some \( q \in \mathbb{N} \). Suppose that \((r, k)\) is a pair in \( w \). If
(q - 1, j) < (r, k) < (t, j) then we use that \( U_1, U_2, \ldots, U_{m_{ij} - 1} \in C_w \) and \( m_{jk} = 2 \) by \( \text{[5.6]} \). If \((p, i) < (r, k) < (s, i)\) then \( m_{ik} = 2 \).

Let \( u' \) be the initial subword of \( u \) that is defined through the inequality \((r, k) \leq (q - 1, j)\) for every pair \((r, k)\) in \( \rho(u) \).

Then there are reflections \( s_{k_1}, \ldots, s_{k_m} \) with

\[
m_{k_1} = m_{k_2} = \ldots = m_{k_m} = 2
\]

so that \( u \equiv u' \) for

\[
(28) \quad w' = w's_{k_1} \ldots s_{k_m} \{s_is_j\}^{m_{ij}}w'.
\]

So \( w \) is also equivalent to

\[
(29) \quad w'' = w's_{j} s_{k_1} \ldots s_{k_m} \{s_is_j\}^{m_{ij} - 1}w'
\]

and either \( w'' <_I w \) or \( w \) is not reduced.

This proof even shows the following:

**Corollary 7.2.** If \( m_{ij} < \infty \), \( M_0, M_1, \ldots, M_{m_{ij} - 1} \notin C_w \) and \( w \) is reduced, then there is a word \( w' \equiv w \) with \( w' <_I w \), pairs \((r, h) = M_{m_{ij} - 1}, (r', h') \neq (r, h)\) and series of pairs \( p_1, p_2, p_3 \) so that

\[
\rho(w') = \rho_1(r, h) p_2 \quad \text{and} \quad \rho(w) = \rho_1(r', h') p_3.
\]

For the inductive step, we still need some lemmas:

**Lemma 7.3.** Suppose that for some \( w \), we have \( M_0 \notin C_w \) and \( M_0 \) is a submodule of \( U \in C \). Let \( U_{m_{ij} - 1} \) be as in Lemma \( \text{[5.6]} \) with modules \((r_k, l_k) \notin C_w \) for \( 1 \leq k \leq a \) so that \( \bigoplus_{k=1}^a (r_k, l_k) | U_{m_{ij} - 1} \).

Let

\[
(30) \quad \begin{align*}
(X_1, X_2, \ldots, X_m) \\
(X'_1, X'_2, \ldots, X'_m) \\
(Y_1, Y_2, \ldots, Y_m)
\end{align*}
\]

be the sequences of modules that yield the exact sequences

\[
\eta_k : 0 \to M_0 \xrightarrow{f_k} M_k^{E(k)} \oplus U_k \to M_k^{E(k - 1)} \to 0
\]

for all \( 1 \leq k \leq m_{ij} - 1 \).

Then one of the following holds:

(a) There is some \( 1 \leq m' \leq m_{ij} - 1 \) and some \( U' \in C_w \) with a monomorphism \( M_0 \to U' \).

(b) If \((X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m'})\) is part of a triple of sequences that fulfils (S1) - (S5), there is some \( 1 \leq k \leq m' \) so that \( M_{m_{ij}} \mid X_k \oplus X_k' \). Furthermore, for \( l \in \mathbb{N}, 1 \leq k \leq a \) and \( M_{m_{ij} - 1} < (r_k - l, l_k) \), we have \( (r_k - l, l_k) \notin C_w \).

(c) \( a = 1, m_{i1} + m_{j1} = 5 \) and there is no indecomposable morphism \( M_{m''} \to (r_1, l_1) \) for \( m'' < m_{ij} - 3 \).

(d) \( a = 1 \) and \((r_1, l_1) < m_{ij} - 1 \).

If \((r_k - l, l_k) \in C_w \) for some \( 1 \leq k \leq a \), then (a) holds.

**Proof.** By Corollary \( \text{[5.6]} \) for all \( 1 \leq k \leq a \), there are some \( X \in \mathcal{I}, \beta_k \in \mathbb{N} \) and \( 2 \leq m_k \leq m_{ij} \) so that the AR-sequence that starts in \((r_k, l_k)\) is of the form

\[
0 \to (r_k, l_k) \to M_0^{m_k} \oplus X \to \to 0.
\]
First, suppose that \(a > 1\). By Lemma 7.3 and Lemma 7.4, there is an exact sequence

\[
0 \rightarrow M_0 \xrightarrow{r_{m_{ij}-1}} M_{m_{ij}} \xrightarrow{r} 0 .
\]

and we can assume that \(X_m \oplus X_m' = U_{m_{ij}-1}\) and \(Y_m = M_{m_{ij}}\). By Corollary 4.9, we set \(X_m := (r_1,r_1')\) and \(Y_{m+1} := (r_2,r_2')\). If \(M_{m_1} = M_{m_2} = M_{m_{ij}}\), then \(M_{m_{ij}+1} \oplus X_{m+2} \oplus X_{m+2}'\). Thus (b) is fulfilled.

Otherwise, set \(\gamma = m + m_{ij} + 2 - m_1\), \(\delta = \gamma + m_{ij} - m_2\).

\[
X_{m+3} := M_{m_1}, X_{m+4} := M_{m_1+1}, \ldots , X_{\gamma} := M_{m_{ij}-1}
\]

and

\[
X_{\gamma+1} := M_{m_{ij}+1} \oplus X_{\delta+2} := M_{m_{ij}+1+1} \oplus X_{\delta} := M_{m_{ij}-1}.
\]

Then \(M_{m_{ij}} \mid X_{\delta+1} \oplus X_{\delta+2}\) and (b) holds.

On the other hand, suppose that there is an indecomposable morphism \(M_{m''} \rightarrow (r_{1},l_1)\) for some \(m'' < m_{ij} - 3\). By Corollary 5.6, we have \((r_1,l_1) \mid U_{m_{ij}-1}\). By the construction of \(\eta_{m''+1}\) from \(\eta_{m''+1}\), there must be a module \(U_{m''+1}\) so that we can write \(\eta_{m''+1}\) as the following:

\[
0 \rightarrow M_0 \xrightarrow{r_{m''+1}} M_{m''+1} E^{(m''+1)} \oplus (r_1, l_1) E^{(m''+1)} \oplus U' \xrightarrow{r_{m''+1}} M_{m''+2} E^{(m''+1)} \rightarrow 0 .
\]

So there is a module \(U'\) with an exact sequence

\[
0 \rightarrow M_0 \xrightarrow{r_{m''+1}} M_{m''+1} E^{(m''+1)} \oplus U' \xrightarrow{(r_1, l_1) E^{(m''+1)}} (r_{m''+1}) E^{(m''+1)} \rightarrow 0 .
\]

By Corollary 4.6 and the monomorphism \(M_0 \rightarrow U\), we get a monomorphism

\[
M_{m_{ij}-1} \rightarrow U \oplus (r_1, l_1).
\]

So either (a) is fulfilled or \((r_1, l_1) \notin C_w\). In this case, \(a > 1\) and (b) is fulfilled.

Finally, suppose that \(m_{ij} \geq 4\) or \(m_{ij} \geq a - 1\). By Lemma 5.1, \(\beta_1 > 1\) if either \(m_{ij} \geq 4\) or \(m_{ij} \geq 4\). We can set \(X_{m+1} := (r_1, l_1)\) and \(X_{m+2} := X_{m+3} := M_{m_1}\). Analogously to the case \(a > 1\), this yields some \(m' \in \mathbb{N}\) so that \(M_{m_{ij}} \mid X_{m'} \oplus X_{m''}\).

If \(m_{ij} = m_{ij} = 3\), then either (d) holds, or we can write the AR-sequence that starts in \((r_1, l_1)\) as

\[
0 \rightarrow (r_1, l_1) \rightarrow M_{m_1} \oplus M_{m+1} \oplus X' \rightarrow (r_1-1, l_1) \rightarrow 0
\]

for some \(X' \in \mathcal{U}\) and some \(1 \leq m_1 \leq m_{ij}-1\). Since \(X_{m+1} := (r_1, l_1)\), \(X_{m+2} := X_{m+3} := M_{m_1}\) and \(X_{m+3} = M_{m+1+1}\), we see that (b) holds.

\hfill \Box

We can generalize this lemma in the following way, which we will need for an induction:

**Remark 7.4.** Let us assume that the assumptions of Lemma \(7.3\) hold, except that \(M_0\) is not a submodule of some \(U \in \mathcal{C}_w\) but of \(X \oplus U\) and \(X \notin \mathcal{C}_w\). If \(X = M_k\) for some \(1 \leq k \leq m_{ij} - 1\), we alter the assumptions on the sequences \(40\) accordingly so that they fulfill (S1) - (S3) with respect to \(M_0\) and \(X \oplus U\).

Furthermore, we assume that \(X^2 \mid X_k \oplus X'_k\) for some \(1 \leq k \leq m\) then for every sequence \((X_1, \ldots , X_m, X_{m+1}, \ldots , X_{m'})\) which is part of a triple of sequences that fulfills (S1) - (S5), we have some \(1 \leq k \leq m'\) so that \(X_k = X\).

So every argument in the proof of \(7.3\) still holds and we still get that one of the conditions (a) - (d) must be fulfilled.
There is some \( N < M \)

In these calculations, we construct exact sequences which contain modules of the form \( X_k \oplus X_k' \) for some \( 1 \leq k \leq m' \), there must be some \( k' \) so that \( M_{m_{ij}} \mid X_k \oplus X_k' \).

We still need a result about the case \( a > 1 \):

**Lemma 7.5.** Suppose that for some \( w \), we have \( M_0 \not\in \mathcal{C}_w \) and \( M_0 \) is a submodule of \( U \in C \). Let \( U_{m_{ij}-1} \) be as in Lemma 5.3 with modules \( (r_k, l_k) \not\in \mathcal{C}_w \) for \( 1 \leq k \leq a \) so that \( \bigoplus_{k=1}^a (r_k, l_k) \mid U_{m_{ij}-1} \).

Furthermore, suppose that for all \( 1 \leq k \leq a \), there is an irreducible morphism \( M_m \rightarrow (r_k, l_k) \) for some \( 0 \leq m < m_{ij} - 2 \). If \( a > 1 \), then one of the following holds:

(a) There is some \( N < M \), \( N \not\in \mathcal{C}_w \) that is a submodule of some \( U' \in \mathcal{C}_w \).

(b) We have \( a = 2 \), \( m_{i2} = 2 \), \( m_{i1} = 3 \), \( m_{j2} = 2 \) for \( 1 \leq k \leq 2 \) and \( m_{ij} = 3 \).

(c) We have \( a = 2 \), \( l_1 = l_2 \), \( m_{i1} = 3 \) and \( m_{ij} = 3 \).

In case (b),

\[
M_{m_{ij}}, M_{m_{ij}+1}, M_{m_{ij}+2}, M_{m_{ij}+3} \not\in \mathcal{C}_w.
\]

and

\[
(r_1 - 1, l_1), (r_2 - 1, l_2), (r_1 - 2, l_1), (r_2 - 2, l_2) \not\in \mathcal{C}_w.
\]

**Proof.** The proof is analogous to that of Lemma 5.3. First note that if \( m_{ij} = 3 \), we get \( m_{i1} \geq 3 \) for \( 1 \leq k \leq a \), since there is an irreducible morphism \( M_m \rightarrow (r_k, l_k) \) for some \( 0 \leq m < m_{ij} - 2 \).

If \( m_{i1} \neq 2 \), we can exchange \( j \) and \( l_2 \) in the calculations below.

Define \( \alpha_k, \beta_k, \alpha_{kj}, \beta_{kj} \) analogous to \( \alpha \) and \( \beta \) with \( i, l_k \) and \( j, l_k \) instead of \( i, j \). If \( m_{i1} = 2 \), but (b) is not fulfilled, then

\[
\alpha \beta + \sum_{k=1}^a (\alpha_{ki} + \alpha_{kj})(\beta_{ki} + \beta_{kj}) \geq 4.
\]

By Corollary 4.11 we can do completely analogous calculations to the case \( \alpha \beta = 4 \).

In these calculations, we construct exact sequences which contain modules of the form \( M_{\alpha}, (\alpha', l_1) \) and \( (\alpha', l_2) \) for some \( \alpha, \alpha' \in \mathbb{N} \).

It remains to show that (a) is fulfilled if any of these modules is in \( \mathcal{C}_w \).

By Lemma 7.3 if \( (r_k - 1, l_k) \in \mathcal{C}_w \) for some \( 1 \leq k \leq a \), then (a) holds. The rest follows inductively with the same argument as in Lemma 7.3.

Now we can show the following, which is the last lemma that we need to prove the first direction of the main theorem:

**Lemma 7.6.** Let \( w \) be a word so that \( (s, i) = M_0, M_1, \ldots, M_{m_{ij}-1} \not\in \mathcal{C}_w \). Then there are words \( w, w' \) so that \( w = w_1 w_2 \) and there is some \( \rho \) with \( \rho(w) = \rho(w_1)(s, i) \rho \).

Suppose that there is some \( U \in \mathcal{C}_w \) with a monomorphism \( M_0 \rightarrow U \) and for every \( X < M_0 \) with some \( U' \in \mathcal{C}_w \) and a monomorphism \( X \rightarrow U' \), we have \( X \in \mathcal{C}_w \).

Then there exists some \( w \) so that

\[
w = w_1 \{s, s_j\}^{m_{ij}} w_2.
\]
Proof. By Lemma 5.3 and 5.5 Proposition 4.8 yields an exact sequence
\[ 0 \rightarrow M_0 \rightarrow U_{m_{ij}-1} \rightarrow M_{m_{ij}} \rightarrow 0 . \]
If \((r, l) \in \mathcal{C}_w\) for all \((r, l) \mid U_{m_{ij}-1}\) with \((r, l) > M_{m_{ij}-1}\), then (33) is obvious by Corollary 6.6.

Otherwise, we get \(M_{m_{ij}} \notin \mathcal{C}_w\), since there is a monomorphism \(U_{m_{ij}-1} \rightarrow M_{m_{ij}} \oplus U\) by Lemma 1.6.

There are direct summands \((r_1, l_1) \ldots (r_a, l_a) \notin \mathcal{C}_w\) of \(U_{m_{ij}-1}\) so that \(M_{m_{ij}-1} < (r_k, l_k)\) for all \(1 \leq k \leq a\).

We can assume without loss of generality that \(m_{ij}\) is odd; otherwise we only need to exchange \(s_j\) and \(s_i\) in the arguments below.

If there is no morphism \(M_m \rightarrow (r_k, l_m)\) for some \(0 \leq m < m_{ij}-2\) for all \(1 \leq k \leq a\), then \(m_{ij} = 2\) and for some word \(w\) we have
\[ w = w s_i s_1 \ldots s_k \{ s_j s_i \}^{m_{ij}-1} \equiv w s_i s_1 \ldots s_k \{ s_i s_j \}^{m_{ij}-1}. \]

So we can assume that for \((r_k, l_k)\), there is a morphism \(M_m \rightarrow (r_k, l_m)\) for some \(0 \leq m < m_{ij}-2\).

By Lemma 7.3 one of the following cases hold:
(a) \(a = 1, m_{ij} = m_{j1} = 5\) and there is no indecomposable morphism \(M_{m_{ij}} \rightarrow (r_1, l_1)\) for \(m'' < m_{ij} - 3\).
(b) There is some \(m'\) so that \(M_{m_{ij}} \mid X_{m'} \oplus X_{m'}'\).

In case (a) there are words \(w'', w\) so that either
\[ w = w'' s_i s_1 \{ s_j s_i \}^{m_{ij}-1} w \]

or both \(m_{ij} = 2\) and
\[ w = w'' s_i s_j s_1 \{ s_i s_j \}^{m_{ij}-2} w. \]

Then there is some word \(w_4\) so that \(w \equiv w_4 s_i s_1 \{ s_i s_j \}^{m_{ij}} w_4\). If \(m_{i1} = 2\), there is nothing to show. If \(m_{i1} = 2\), then we use that \(M_{m_{ij}}\) is of the form \((q, j)\) for some \(q\) and there is no morphism \(U_{m_{ij}-1} \rightarrow M_{m_{ij}} \oplus U\). Let \(w_4\) be the subword of \(w_4\) that does not contain the reflection that corresponds to \((q, j)\). By the inductive assumption, \(w_4 s_i s_1\) is equivalent to a smaller word, since \(C_{w_4 s_i s_1}\); thus it is equivalent to a word \(w_4 \{ s_i s_1 \}^{m_{i1}}\).

Now we go back to looking at \(w_4\), not \(w_2\). Since \(m_{i1} = 3\) and \(m_{j1}\), there is some word \(w_4\) with \(w_4 s_i s_1 \equiv w_4 \{ s_i s_1 \}^{m_{i1}}\) and thus
\[ w \equiv w'' s_i s_j s_1 \{ s_i s_j \}^{m_{ij}} w \]

for some word \(w''\).

On the other hand, suppose that
\[ M_{m_{ij}} \mid X_{m'} \oplus X_{m'}'. \]
Since we have \(\bigoplus_{k=1}^{a} (r_k, l_k) \mid U_{m_{ij}}\), there is a monomorphism
\[ \bigoplus_{k=1}^{a} (r_k, l_k) \rightarrow U \oplus M_{m_{ij}}. \]

Assume that \((r_1, l_1) \leq (r_2, l_2) \leq \cdots \leq (r_a, l_a)\).

Let the AR-sequence that starts in \((r_a, l_a)\) be
\[ 0 \rightarrow (r_a, l_a) \rightarrow M_{m_1} \oplus Z_1 \rightarrow (r_a - 1, l_1) \rightarrow 0. \]

If \(a = 1\), then by Lemma 7.3 \(w\) must have the form (34) or (35). The only difference to case (a) is that \(m_{i1} + m_{j1} > 5\).
We can use case (a) as the basis of an induction: Instead of the modules $M_0, M_1, (r_1, l_1)$, we take $(r_1, l_1), M_{m_1}, M_{m_1+1}$ and use the same arguments as before. Since $M_{m_1} \mid X_{m_1} \oplus X'_{m_1}$, we can use Remark 7.4 and either we get the analogue to case (a) above or the analogue to the case (b). In the first case, we get $m_{il_1} + m_{ij} = 5$ or $m_{ij} + m_{il_1} = 5$ and there is some $w'$ so that $w$ is equivalent to a word with the subword $\{s_i, s_j\}^{m_{ij}}$ if $m_1$ is even and $\{s_i, s_j\}^{m_{ij}}$ is odd. Thus we also get $w \equiv w' \{s_i, s_j\}^{m_{ij}}$ for some words $w', w$. In the case (b), we continue this inductively.

After finitely many steps we get

$$w \equiv w' \{s_i, s_j\}^{m_{ij}}.$$

If $a \neq 1$, then $a = 2$ by Lemma 7.5. If $l_1 = l_2$, then $l_1 = l_2, m_{il_1} = 3$ and $m_{ij} = 3$. We can exchange $j$ and $l_1$ to get the case $a = 1$. Otherwise, $m_{i_1} = 2, m_{il_2} = 3, m_{ij} = 2$ for $1 \leq k \leq 2$ and $m_{ij} = 3$. Furthermore,

$$M_{m_{ij}}, M_{m_{ij}+1}, M_{m_{ij}+2}, M_{m_{ij}+3} \not\in \mathcal{C}_w,$$

and

$$(r_1 - 1, l_1), (r_2 - 2, l_2), (r_1 - 2, l_1), (r_2 - 2, l_2) \not\in \mathcal{C}_w.$$

Analogously to before, we see inductively that $w$ is equivalent to a word with the subword

$$s_{l_1}, s_{l_2}, s_{j}, s_{l_1}, s_{l_2}, s_{j}, s_{l_1}, s_{l_2}, s_{j}, s_{i}.$$

(For the purpose of this induction, we can treat the word above completely analogously to a word of the form $\{s_i, s_j\}^{m_{ij}}$ with $m_{ij} = 6$. As in 7.5 all calculations are the same by Corollary 1.11)

We have the following equivalences, where bold reflections denote those which differ from the reflections in the word above:

$$s_{l_1}, s_{l_2}, s_{j}, s_{l_1}, s_{l_2}, s_{j}, s_{l_2}, s_{l_1}, s_{l_2}, s_{j}, s_{i}$$

$\equiv s_{l_1}, s_{l_2}, s_{j}, s_{l_1}, s_{l_2}, s_{l_1}, s_{l_2}, s_{j}, s_{j}$$

$\equiv s_{l_1}, s_{l_2}, s_{j}, s_{l_1}, s_{l_2}, s_{l_1}, s_{l_1}, s_{l_2}, s_{j}$$

$\equiv s_{l_1}, s_{l_2}, s_{j}, s_{l_1}, s_{l_1}, s_{l_2}, s_{j}, s_{i}$$

$\equiv s_{l_1}, s_{l_2}, s_{j}, s_{l_1}, s_{l_1}, s_{l_2}, s_{i}, s_{l_2}, s_{j}$$

$\equiv s_{l_1}, s_{l_2}, s_{j}, s_{l_1}, s_{l_1}, s_{l_1}, s_{l_2}, s_{j}, s_{i}$$

$\equiv s_{l_1}, s_{l_2}, s_{j}, s_{l_1}, s_{l_1}, s_{l_2}, s_{i}, s_{l_2}, s_{j}$$

$\equiv s_{l_1}, s_{l_2}, s_{j}, s_{l_1}, s_{l_1}, s_{l_2}, s_{i}, s_{l_2}, s_{i}$$

So $w$ is equivalent to a word with the subword $s_i, s_j, s_i$ and the assertion is true. \(\square\)

Finally, we can prove the first direction of our main result:

**Lemma 7.7.** If $w$ is a leftmost word, then $\mathcal{C}_w$ is submodule closed.

**Proof.** Suppose that $\mathcal{C}_w$ is not submodule closed. Then there is some $M_0 \in \text{ind} T \setminus \mathcal{C}_w$ and some $U \in \mathcal{C}_w$ with a monomorphism $M_0 \to U$. Furthermore, we can assume that for every $X < M_0$ with some $U' \in \mathcal{C}_w$ and a monomorphism $X \to U'$, we have $X \in \mathcal{C}_w$.

We use induction on the length $m$ of the sequences of modules in Proposition 4.3 applied on $M_0$ and $U$. If $m = 1$, then $w$ is not leftmost by Lemma 7.4.

Now suppose that $w$ is not leftmost if the sequences have the length $m$ or smaller. We prove that this is also the case if they have length $m + 1$:

We can assume without loss of generality that $M_1 \not\in \mathcal{C}_w$, since $m + 1 > 1$. On the other hand, by Lemma 5.3 the sequences of modules induce an exact sequence

$$0 \to M_0 \to M_1^{E(1)} \oplus U_1 \to M_2 \to 0.$$
So by Corollary 4.6 there is a monomorphism $M_1^{E(1)} \rightarrow M_2 \oplus U$. Since $M_1 < M_0$, our assumptions yield $M_2 \notin \mathcal{C}_w$.

By the same argument, we can see that $M_3, M_4, \ldots, M_{m_j-1} \notin \mathcal{C}_w$ and by (S4) this means $m_j < \infty$.

By Lemma 7.6 if we choose $u$ and $v$ so that $w = u \cdot v$ and there is some $\rho$ so that $\rho(u) = \rho(\phi(s, i))$, then $w \equiv \rho(s_i s_j)^{m_j} u$ for some word $\phi$.

We still need to show that $\rho(s_i s_j)^{m_j} u$ is equivalent to a word which is smaller than $w$.

To do this, we use Lemma 8.7. Either there is nothing to show, or there are $\rho_1, \ldots, \rho_4$ and a pair $(r, h)$ so that
\begin{equation}
\rho(u) = \rho_1(r, h)\rho_2(s, i)\rho_3 \tag{38}
\end{equation}
and there is some $w'' \equiv w$ with
\begin{equation}
\rho(w'') = \rho_1\rho_2(s, i)(t + 1, j)\rho_3. \tag{39}
\end{equation}
We can assume that the word $x$ with $\rho(x) = \rho_1(r, h)\rho_2$ is reduced, because otherwise there is nothing to prove. So by Lemma 6.2 there are some sequences of modules as in Lemma 4.3 that yield some $U' \in \mathcal{I}$ and an exact sequence
\[
0 \rightarrow M_0 \rightarrow M_{m_j-2} \oplus U' \rightarrow Y \rightarrow 0
\]
with either $Y \in \mathcal{C}_w$ or $Y = (r, h)^{E(m_j-3)}$. By Corollary 4.6 there is a monomorphism $M_{m_j-2} \rightarrow U \oplus Y \in C_w$.

By 13 and the induction hypothesis, $w''$ is not leftmost.

So there is some $w_3 \equiv w''$ with $w_3 <_l w''$. We still need to show that $w_3 <_l w$.

If $w''$ is not reduced, this is obvious. If $Y \in \mathcal{C}_w$, we can use the inductive assumption.

So suppose that $w''$ is reduced and $Y \equiv (r, h)^{E(m_j-3)}$. We denote the sequences of modules that fulfil (S1) - (S5) with respect to $M_0$ and $U$ by
\[
(X_1, X_2, \ldots, X_m)
\]
\[
(X'_1, X'_2, \ldots, X'_m)
\]
\[
(Y_1, Y_2, \ldots, Y_m).
\]
There are sequences of modules
\[
(1X_1, 1X_2, \ldots, 1X_m')
\]
\[
(1X'_1, 1X'_2, \ldots, 1X'_m')
\]
\[
(1Y_1, 1Y_2, \ldots, 1Y_m')
\]
that fulfil (S1) - (S5) with respect to $M_{m_j-2}$ and $U \oplus (r, h)^{E(m_j-3)}$. Let $(r', h') < (r, h)$. Then $(r', h') < (r, h)$.

If there is a pair $(r'', h'') \neq (r', h')$ and series of pairs $\rho_1', \rho_2', \rho_3'$ so that we can write
\[
\rho(w_3) = \rho_1'(r', h')\rho_2' \quad \text{and} \quad \rho(w) = \rho_1'(r'', h'')\rho_3',
\]
then $w_3 <_l w''$ implies $w_3 <_l w$.

A simple induction on $m'$ shows that this is indeed the case. If $m' = 1$, then $(r, h) = M_{m_j-1}$, $(r', h') = M_{m_j}$ and the assertion is true by Corollary 7.2.

By 13 the smallest direct summand of $1Y_{m'}$ is also the smallest direct summand of $Y_m$ and the inductive step is obvious.

So $w$ is not leftmost and the proof is complete. \qed
8. The other direction

In this section we finally conclude the proof that the map between words and full additive cofinite subcategories of $\text{mod} \ A$ introduced in Definition 3.13 gives rise to a bijection between the leftmost words and the cofinite submodule closed subcategories. Since every element of the Weyl group has a unique leftmost element, this gives a bijection between the Weyl group elements and the cofinite, submodule closed subcategories.

Again, we start with the basis of an induction:

**Lemma 8.1.** (a) Let $w := u \{ s_j s_i \}^{m_{ij}} v$. If $C_w$ is submodule closed, then $w < l u \{ s_j s_i \}^{m_{ij}} v$.

(b) The category $C_{u s_i s_i v}$ is not submodule closed.

(c) Let $w' := u \{ s_j s_i \}^{m_{ij} + 1} v$. Then $C_{w'}$ is not submodule closed.

**Proof.** We prove (a) by contraposition. By Definition 3.13, $\text{ind} I \setminus C_w$ consists of the modules which correspond to the reflections in $w$.

Assume that $u \{ s_j s_i \}^{m_{ij}} v < l u \{ s_j s_i \}^{m_{ij}} v = w$ and $\rho(w) = \rho(u)(p, i)(q, j)(p + 1, i)\ldots$ for a sequence of pairs $\rho_1$.

By Lemma 3.6, the module $(q - 1, j)$ exists and by Definition 3.13, $C_w$ contains all indecomposable, preinjective modules $M$ with $(q - 1, j) < M < (p, i)$ or $(p, i) < M < (q, j)$.

First, suppose that $m_{ij} = 2$. In this case, $C_w$ contains the middle term of the AR-sequence that starts in $(q, j)$ by Lemmas 4.7 and 5.1. Since $(q, j) \notin C_w$, the subcategory is not submodule closed.

Now let $m_{ij} \geq 3$. In 5.4 we defined $M_0 := (s, i), M_1 := (t, j), \ldots$ for some arbitrary, fixed $s, t$. By Remark 5.7 we can assume without loss of generality that $m_{ij}$ is odd and we can choose $s, t$ so that $M_{m_{ij} - 1} = (p, i)$.

Then $M_{m_{ij}} = (q - 1, j) \neq 0$ and by Lemma 5.3 $E(m_{ij} - 2) \neq 0$. By Lemma 5.3 there is an exact sequence

$$0 \rightarrow M_0 \rightarrow (M_{m_{ij} - 1})^{E(m_{ij} - 1)} \oplus U_{m_{ij} - 1} \rightarrow M^{E(m_{ij} - 2)} \rightarrow 0$$

so that no $M_0, M_1, \ldots, M_{m_{ij} - 1}$ is a direct summand of $U_{m_{ij} - 1}$. By Lemma 5.3 we have $E(m_{ij} - 1) = 0$, so there is a monomorphism $M_0 \rightarrow U_{m_{ij} - 1}$.

It remains to show that $U_{m_{ij} - 1} \in C_w$, by Corollary 5.6. If $X$ is a direct summand of $U_{m_{ij} - 1}$, then $M_{m_{ij}} < X < M_0$ and thus $X \in C_w$.

By Lemma 4.7, part (b) is obvious.

The proof of (c) is completely analogous to the proof of (a).

The proof of (a) is completed.

Finally, we are prepared to prove that Definition 3.13 gives a bijection. Recall that $C_w = C_{w'}$ if $w$ is the leftmost word for $w$.

**Theorem 8.2.** The map $w \mapsto C_w$ is a bijection between the elements of the Weyl group of $A$ and the cofinite submodule closed subcategories of $\text{mod} \ A$. 
Proof. The map is well defined by Lemma 8.1, and obviously injective. It remains to prove that it is surjective, i.e. that for all cofinite submodule closed subcategories \( C \) of \( \text{mod} \ A \), there is a \( w \in W \), so that \( C = C_w \).

We already know that \( C = C_w \) for some word from Lemma 6.1, so we only need to show that \( w \) is leftmost.

Assume that the word \( w \) for the element \( w \in W \) is not leftmost. We show that \( C_w \) is not submodule closed by induction on the number of Coxeter relations that are needed to transform \( w \) into a smaller word.

If only one relation is needed, then the theorem is the result of Lemma 8.1. Now suppose that the assertion is true if we need \( m \) or less relations and that we need \( m + 1 \) relations to transform \( w \) into a smaller word.

Then there are some \( 1 \leq i, j \leq n \) and some words \( x, x' \), so that (40)

\[
 w = x s_j y \equiv x' s_j y = w',
\]

and

\[
 w' \leq_i w
\]

with \( i \neq j \).

Thus, there are some words \( w'', w''' \) so that

\[
 w = w'' \equiv w''' = x'' \{ s_i s_j \}^{m_{ij}} y.
\]

Because of the inductive assumption, we can suppose that \( w \leq_i w'' \) and that \( x'' \) is leftmost. Obviously, we can choose \( x'' \) to be leftmost.

Let the reflection \( s_i \) in (10) correspond to \( M_0 \). We can assume that \( m_{ij} \geq 3 \), since there is nothing to show if the middle term of the Auslander-Reiten sequence that starts in \( M_0 \) is contained in \( C_w \). We can also assume that there is some word \( x'' \) so that \( x = x'' s_j \); If there are \( s_{k_1}, \ldots, s_{k_m} \) with \( m_{k_1,i} = \cdots = m_{k_m,i} \) and \( x = x'' s_j s_{k_1} \ldots s_{k_m} \), then \( x s_i = x'' s_j s_{k_1} \ldots s_{k_m} \) and if there is some \( U \in C_{x'' s_j s_{k_1} \ldots s_{k_m}} \) with a monomorphism \( M_0 \to U \), then \( U \in C_w \).

Without loss of generality, we can assume that \( m_{ij} \) is odd; otherwise we relabel \( i \) and \( j \) and get the same arguments by Remark 5.7.

By Lemma 3.12, we can suppose that \( M_0, M_1, \ldots, M_{m_{ij} - 3}, M_{m_{ij} - 2} \in C_w \). We show that there is some \( U \in C_w \) with a monomorphism \( M_0 \to U \).

Let \( (q, j) := M_{m_{ij} - 2} \). We use Lemma 3.4. Because \( m > 1 \), there is some\( w_3 = w \) with series of pairs \( r_1, \ldots, r_4 \) and a pair \( (r, h) \) so that

\[
\rho(w) = r_1(r, h) r_2(q, j) r_3(s, i) r_4
\]

and

\[
\rho(w_3) = r_1 r_2(q, j) r_3(s, i)(t + 1) r_4.
\]

By Lemma 6.2 if \( m_{ij} \geq 3 \), then there is an exact sequence

\[
0 \to M_0 \to M_{m_{ij} - 2} \oplus U' \to Y \to 0
\]

so that either \( Y \in C_w \) or both \( Y = (r, h) E_{(m_{ij} - 3)} \) and \( U' \in C_w \) hold.

We want to show that there is some \( U'' \in C_w \) and a monomorphism

\[
M_{m_{ij} - 2} = (q, j) \to U'' \oplus Y.
\]

We prove this inductively: First, note that the word \( x'' s_j \) is not leftmost Lemma 3.9 and 3.10. If \( w = w'' \), then \( (r, h) = M_{m_{ij} - 1} \). So by the inductive hypothesis, there is a monomorphism \( (q, j) \to U'' \oplus (r, h) \).
Since $E(1) = E(m_{ij} - 3)$ by table (18), we obviously get $Y = (r, h)^{E(m_{ij} - 3)}$ and $\gamma = E(m_{ij} - 3)$.

The inductive step is completely analogous to the one in Lemma 6.2.

By our assumptions, $x$ is leftmost and thus $Y \notin C_w$ by Lemma 7.7. So $U' \in C_w$.

By Lemma 4.13 there is a monomorphism $M_0 \hookrightarrow U' \oplus U'' \in C_w$ and $C_w$ is not submodule closed. □

9. Some consequences

We conclude the chapter with a generalization and a corollary:

As in [8], Section 8 we can extend the notion of leftmost words:

**Definition 9.1.** Define infinite words analogously to words, only as infinite instead of finite sequences. We say that an (infinite) word is leftmost if any initial subword of finite length is leftmost.

Analogously to [8], Theorem 8.1, we get the following:

**Theorem 9.2.** There is a bijection between the (finite and infinite) leftmost words over $S = \{s_1, s_2, \ldots, s_n\}$ and the submodule closed subcategories of $I$, the preinjective component of $\text{mod}
A$.

**Proof.** This is completely analogous to [8], 8.1:

Let $C$ be a submodule closed subcategory of $I$. Since $I$ contains at most a countable number of indecomposable modules, we can set

$$\text{ind}
I \setminus C =: \{(r_1, i_1), (r_2, i_2), \ldots\}$$

and $(r_1, i_1) < (r_2, i_2) < \ldots$. Then the subcategory $C_m$ with

$$\text{ind}
I \setminus C_m = \{(r_1, i_1), (r_2, i_2), \ldots, (r_m, i_m)\}$$

is submodule closed for all $m \in \mathbb{N}$. By Lemma 4.1 and Theorem 8.2, the words $w_m$ with

$$\rho(w_m) = (r_1, i_1)(r_2, i_2)\ldots(r_m, i_m)$$

are leftmost for all $m \in \mathbb{N}$. By Definition 9.1 the (infinite) word $w$ with

$$\rho(w) = (r_1, i_1)(r_2, i_2)\ldots$$

is leftmost and $C = C_w$.

On the other hand assume that the (infinite) word $w$ with

$$\rho(w) = (r_1, i_1)(r_2, i_2)\ldots$$

is leftmost. Then the words with

$$\rho(w_m) = (r_1, i_1)(r_2, i_2)\ldots(r_m, i_m)$$

are leftmost for all $m \in \mathbb{N}$. By 8.2 the categories $C_w$ are submodule closed. Thus $C_w$ is also submodule closed: if there was a module $M \notin C_w$ and some module $U \in C_w$ with a monomorphism $M \hookrightarrow U$, then $U \in C_w$ for all $m \in \mathbb{N}$ and there is some $m \in \mathbb{N}$ so that $M \notin C_w$, since $M$ is finitely generated. □

We can draw a further corollary. Let $A'$ be a hereditary and let the module category $\text{mod}
A'$ be equivalent to the subcategory of $\text{mod}
A$ with the simple modules $S_j$, $j \in J$ for some $J \subseteq \{1, 2, \ldots, n\}$. Let $\mathcal{I}'$ be the subcategory of $\text{mod}
A'$ consisting of all preinjective modules.

**Corollary 9.3.** There is a bijection between the submodule closed subcategories of $\mathcal{I}'$ and the submodule closed subcategories of $\mathcal{I}$ which contain all $\tau^rI_i$ with $r \in \mathbb{N}_0$ and $i \in \mathbb{N} \setminus J$.
Proof. The words in the Weyl group of $A'$ are exactly the words in the Weyl group of $A$ which only consist of reflections $s_j$ with $j \in J$. \hfill \square

Acknowledgements

The author would like to thank Henning Krause for his suggestions and insights and Hugh Thomas for his comments concerning the exposition.

References

[1] D. Armstrong, The sorting order on a Coxeter group, J. Combin. Th. Series A 116 (2009), 1285-1305.
[2] M. Auslander, I. Reiten, S. Smalø, Representation Theory of Artin Algebras, first paperback edition, Cambridge University Press, New York 1995.
[3] A. Björner, F. Brenti, Combinatorics of Coxeter Groups. Graduate texts in mathematics, Springer, New York 2005.
[4] M. Grandis, Homological algebra. The Interplay of Homology with Distributive Lattices and Orthodox Semigroups, World Scientific, Hackensack, NJ [u.a.] 2012.
[5] V. G. Kac, Infinite dimensional Lie algebras, third edition, Cambridge Univ. Press, Cambridge 1990.
[6] H. Krause, M. Prest The Gabriel-Roiter Filtration of the Ziegler Spectrum, The Quarterly Journal of Mathematics, Vol. 64 Issue 3 (2013), 891-901.
[7] M. Lothaire, Combinatorics on words, reissued edition, Cambridge Univ. Press, Cambridge 1997.
[8] S. Oppermann, I. Reiten, H. Thomas, Quotient closed subcategories of quiver representations, Compositio Mathematica, 151 (2015), 568-602.
[9] C. M. Ringel, Minimal Infinite Submodule-closed Subcategories, Bulletin des Sciences Mathématiques 136 (2012), 820-830.

Apolonia Gottwald, Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany
E-mail address: alogisma@gmx.de