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HAMILTON-JACOBI THEORY IN $k$-COSYMPELECTIC FIELD THEORIES

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Abstract. In this paper we extend the geometric formalism of the Hamilton-Jacobi theory for time dependent Mechanics to the case of classical field theories in the $k$-cosymplectic framework.

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1. Introduction

The usefulness of Hamilton-Jacobi theory in Classical Mechanics is well-known, giving an alternative procedure to study and, in some cases, to solve the evolution equations [1]. The use of symplectic geometry in the study of Classical Mechanics has permitted to connect the Hamilton-Jacobi theory with the theory of Lagrangian submanifolds and generating functions [3].

At the beginning of the 1900s an analog of Hamilton-Jacobi equation for field theory has been developed [29], but it has not been proved to be as powerful as the theory which is available for mechanics [4, 5, 25, 26, 28, 30].

There are several recent attempts to extend the Hamilton-Jacobi theory for classical field theories in a geometrical setting. For instance in the framework of the so-called multisymplectic formalism [14, 25, 26] (see also [6, 11] for a general description of the multisymplectic setting) or in the $k$-symplectic formalism in [10] (see [23, 27] for a discussion of the relationship between both formulations, see also [10]). Our method is based in that developed by J.F. Cariñena et al. for Classical Mechanics [7, 8] (see also [13, 15]).

In the context of Classical Field Theories, the Hamiltonian is a function $H = H(x^\alpha, q^i, p^\alpha)$, where $(x^\alpha)$ are coordinates in the space-time, $(q^i)$ represent the field...
coordinates and \((p^a_i)\) are the conjugate momenta. In this context, the Hamilton-Jacobi equation is [29]

\[
\frac{\partial W^\alpha}{\partial x^\alpha} + H\left(x^i, q^i, \frac{\partial W^\alpha}{\partial q^i}\right) = 0
\]  

(1.1)

where \(W^1, \ldots, W^k: \mathbb{R}^k \times Q \rightarrow \mathbb{R}\).

The aim of this paper is to extend the Hamilton-Jacobi theory to field theories just in the context of \(k\)-cosymplectic manifolds [19, 20, 21]. The “dynamics” for a given Hamiltonian function \(H\) is interpreted as a family of vector fields (a \(k\)-vector field) on the phase space \(\mathbb{R}^k \times (T^1_k)^* Q\) (or in general on a \(k\)-cosymplectic manifold \((M, \eta^\alpha, \omega^\alpha, V)\)).

The paper is structured as follows. In Sec. 2, we recall the notion of \(k\)-vector field and their integral sections and give a briefly description of the \(k\)-cosymplectic formalism. In Sec. 3 we discuss the Hamilton-Jacobi equation in the \(k\)-cosymplectic context. Finally, an example is discussed in Sec. 4, with the aim to show how the method works.

We shall also adopt the convention that a repeated index implies summation over the range of the index, but, in some cases, to avoid confusions we will explicitly include the summation symbol.

2. The \(k\)-cosymplectic formalism

The \(k\)-cosymplectic formalisms [19, 21] is one of the simplest geometric frameworks for describing first order classical field theories (see [2] for the \(k\)-symplectic case). It is the generalization to field theories of the standard cosymplectic formalism for nonautonomous mechanics and it describes field theories involving the space-time coordinates on the Lagrangian and on the Hamiltonian cases. The foundations of the \(k\)-cosymplectic formalism are the \(k\)-cosymplectic manifolds. In this section we briefly recall this formalism.

2.1. Geometric preliminaries: \(k\)-vector fields and integral sections.

In this section we briefly recall some well-known facts about tangent bundles of \(k\)-velocities (we refer the reader to [17, 18, 22, 23, 24, 27] for more details).

Let \(\tau_M: TM \rightarrow M\) be the tangent bundle of \(M\). Let us denote by \(T^1_k M\) the Whitney sum \(TM \oplus k \cdot TM\) of \(k\) copies of \(TM\), with projection \(\tau: T^1_k M \rightarrow M\), \(\tau(v_1, \ldots, v_k) = p\), where \(v_\alpha \in T_p M\), \(1 \leq \alpha \leq k\). \(T^1_k M\) can be identified with the manifold \(J^1_0(\mathbb{R}^k, M)\) of the \(k\)-velocities of \(M\), that is, 1-jets of maps \(\eta: \mathbb{R}^k \rightarrow M\) with source at 0 \(\in \mathbb{R}^k\), say

\[
\begin{align*}
J^1_0(\mathbb{R}^k, M) & \equiv TM \oplus k \cdot TM \\
J^1_0 \eta & \equiv (v_1, \ldots, v_k)
\end{align*}
\]

where \(x = \eta(0)\), and \(v_\alpha = T\eta(0)(\frac{\partial}{\partial x^\alpha})\). Here \((x^1, \ldots, x^k)\) denote the standard coordinates on \(\mathbb{R}^k\). \(T^1_k M\) is called the tangent bundle of \(k\)-velocities of \(M\) or simply \(k\)-tangent bundle for short, see [23].

Denote by \((x^i, v^i)\) the fibred coordinates in \(TM\) from local coordinates \((x^i)\) on \(M\). Then we have fibred coordinates \((x^i, v^i_\alpha), 1 \leq i \leq m, 1 \leq \alpha \leq k\), on \(T^1_k M\), where \(m = \dim M\).

**Definition 2.1.** A section \(X: M \rightarrow T^1_k M\) of the projection \(\tau\) will be called a \(k\)-vector field on \(M\).
Since $T^1_k M$ is the Whitney sum $TM \oplus \ldots \oplus TM$ of $k$ copies of $TM$, we deduce given a $k$-vector field $X$ is equivalent to giving a family of $k$ vector fields $X_1, \ldots, X_k$ on $M$ by projecting $X$ onto each factor. For this reason we will denote a $k$-vector field by $(X_1, \ldots, X_k)$.

**Definition 2.2.** An integral section of the $k$-vector field $X = (X_1, \ldots, X_k)$, passing through a point $p \in M$, is a map $\psi: U_0 \subset \mathbb{R}^k \rightarrow M$, defined on some neighborhood $U_0 \in \mathbb{R}^k$, such that

$$\psi(0) = p, \; T\psi \left( \frac{\partial}{\partial x^\alpha} \bigg|_x \right) = X_\alpha(\psi(x)), \quad \text{for every} \; x \in U_0, \; 1 \leq \alpha \leq k$$

or, equivalently, $\psi$ satisfies that $X \circ \psi = \psi^{(1)}$, being $\psi^{(1)}$ is the first prolongation of $\psi$ to $T^1_k M$ defined by

$$\psi^{(1)}: \; U_0 \subset \mathbb{R}^k \rightarrow T^1_k M$$

$$x \rightarrow \psi^{(1)}(x) = j^1_0 \psi_x,$$

where $\psi_x(s) = \psi(x + s)$.

A $k$-vector field $X = (X_1, \ldots, X_k)$ on $M$ is said to be integrable if there is an integral section passing through every point of $M$.

In local coordinates, we have

$$\psi^{(1)}(x^1, \ldots, x^k) = \left( \psi^i(x^1, \ldots, x^k), \frac{\partial \psi^j}{\partial x^\alpha}(x^1, \ldots, x^k) \right),$$

and then $\psi$ is an integral section of $(X_1, \ldots, X_k)$ if and only if the following equations hold:

$$\frac{\partial \psi^j}{\partial x^\alpha} = X^\alpha_\alpha \circ \psi \quad 1 \leq \alpha \leq k, \; 1 \leq i \leq m,$$

being $X_\alpha = X^\alpha_\alpha \frac{\partial}{\partial x^\alpha}$.

Notice that, in case $k = 1$, Definition 2.2 coincides with the definition of integral curve of a vector field.

### 2.2. $k$-cosymplectic manifolds.

Let $Q$ be a differentiable manifold, $\dim Q = n$, and $\pi : T^* Q \rightarrow Q$ its cotangent bundle. Denote by $(T^1_k)^* Q = T^* Q \oplus \ldots \oplus T^* Q$, the Whitney sum of $k$ copies of $T^* Q$. The manifold $(T^1_k)^* Q$ can be identified with the manifold $J^1(Q, \mathbb{R}^k)$ of 1-jets of mappings from $Q$ to $\mathbb{R}^k$ with target at $0 \in \mathbb{R}^k$, the diffeomorphism is given by

$$J^1(Q, \mathbb{R}^k)_0 \equiv (T^* Q \oplus \ldots \oplus T^* Q)$$

$$\equiv (d\sigma^1(q), \ldots, d\sigma^k(q)),$$

where $\sigma^\alpha = \pi^\alpha \circ \sigma : Q \rightarrow \mathbb{R}$ is the $\alpha^{th}$ component of $\sigma$, and $\pi^\alpha : \mathbb{R}^k \rightarrow \mathbb{R}$ is the canonical projection onto the $\alpha^{th}$ component, for $\alpha = 1, \ldots, k$. $(T^1_k)^* Q$ is called the cotangent bundle of $k$-covelocities of the manifold $Q$.

The manifold $J^1 \pi_Q$ of 1-jets of sections of the trivial bundle $\pi_Q : \mathbb{R}^k \times Q \rightarrow Q$ is diffeomorphic to $\mathbb{R}^k \times (T^1_k)^* Q$, via the diffeomorphism given by

$$J^1 \pi_Q \rightarrow \mathbb{R}^k \times (T^1_k)^* Q$$

$$j^1_\phi \phi = j^1_\phi (\phi_{2k}, Id_Q) \rightarrow (\phi_{2k}(q), \nu^1_\phi, \ldots, \nu^k_\phi),$$

where $\phi_{2k} : Q \stackrel{\phi}{\rightarrow} \mathbb{R}^k \times Q \stackrel{\pi_Q}{\rightarrow} \mathbb{R}^k$, and $\nu^\alpha_\phi = d\phi^\alpha_{2k}(q), \; 1 \leq \alpha \leq k$. 
Throughout the paper, we use the following notation for the canonical projections

\[
\mathbb{R}^k \times (T_k^1)^*Q \xrightarrow{\pi_Q^{-1}} \mathbb{R}^k \times Q
\]

where

\[
\pi_Q(x, q) = q, \quad (\pi_Q)_{1,0}(x, \nu_q^1, \ldots, \nu_q^k) = (x, q), \quad (\pi_Q)_1(x, \nu_q^1, \ldots, \nu_q^k) = q,
\]

with \( x \in \mathbb{R}^k, q \in Q \) and \((\nu_q^1, \ldots, \nu_q^k) \in (T^1_q)^*Q \).

If \((q^i)\) are local coordinates on \( U \subseteq Q \), then the induced local coordinates \((q^i, p_i)\), \(1 \leq i \leq n\), on \((\pi_Q)^{-1}(U) = T^*U \subset T^*Q\), are given by

\[
q^i(\nu_q) = q^i(q), \quad p_i(\nu_q) = \nu_q \left( \frac{\partial}{\partial q^i} \bigg|_q \right),
\]

with \( \nu_q \in T^*Q \), and the induced local coordinates \((x^\alpha, q^i, p_i^\alpha)\) on \([(\pi_Q)^{-1}(U) = \mathbb{R}^k \times (T^1_k)^*U\) are given by

\[
x^\alpha(x, \nu_q^1, \ldots, \nu_q^k) = x^\alpha, \quad q^i(x, \nu_q^1, \ldots, \nu_q^k) = q^i(q), \quad p_i^\alpha(x, \nu_q^1, \ldots, \nu_q^k) = \nu_q \left( \frac{\partial}{\partial q^i} \bigg|_q \right),
\]

for \(1 \leq i \leq n\) and \(1 \leq \alpha \leq k\).

On \( \mathbb{R}^k \times (T^1_k)^*Q \), we consider the differential forms

\[
\eta^\alpha = dx^\alpha = (\pi_Q^1)^* dx, \quad \theta^\alpha = (\pi_Q^2)^* \theta, \quad \omega^\alpha = (\pi_Q^2)^* \omega,
\]

where \( \pi_Q^1 : \mathbb{R}^k \times (T^1_k)^*Q \to \mathbb{R} \) and \( \pi_Q^2 : \mathbb{R}^k \times (T^1_k)^*Q \to T^*Q \) are the projections defined by

\[
\pi_Q^1(x, \nu_q^1, \ldots, \nu_q^k) = x^\alpha, \quad \pi_Q^2(x, \nu_q^1, \ldots, \nu_q^k) = \nu_q^\alpha,
\]

\( \omega = -q^i \theta = dq^i \wedge dp_i \) is the canonical symplectic form on \( T^*Q \) and \( \theta = p_i dq^i \) is the Liouville 1-form on \( T^*Q \). Obviously \( \omega^\alpha = -d\theta^\alpha \).

In local coordinates we have

\[
\eta^\alpha = dx^\alpha, \quad \theta^\alpha = p_i^\alpha dq^i, \quad \omega^\alpha = dq^i \wedge dp_i^\alpha. \quad (2.3)
\]

Moreover, let

\[
V = \text{ker } ((\pi_Q)_{1,0}) = \left\{ \frac{\partial}{\partial p_{i_{\alpha}}} \right\}_{i=1,...,n} \quad (2.4)
\]

be the vertical distribution of the bundle \((\pi_Q)_{1,0} : \mathbb{R}^k \times (T^1_k)^*Q \to \mathbb{R}^k \times Q \).

A simple inspection of the expressions in local coordinates, (2.3) and (2.4) show that the forms \( \eta^\alpha \) and \( \omega^\alpha \) are closed, and the following relations hold

(i) \( \eta^1 \wedge \cdots \wedge \eta^k \neq 0, \quad (\eta^\alpha)|_V = 0, \quad (\omega^\alpha)|_V \times V = 0 \),

(ii) \( (\cap_{\alpha=1}^k \ker \eta^\alpha) \cap (\cap_{\alpha=1}^k \ker \omega^\alpha) = \{0\}, \quad \text{dim}(\cap_{\alpha=1}^k \ker \omega^\alpha) = k \).

Inspired by the above geometrical model we introduce the following (see [21]),

**Definition 2.3.** Let \( M \) be a differentiable manifold of dimension \( k(n+1) + n \). A \( k \)-cosymplectic structure on \( M \) is a family \((\eta^\alpha, \omega^\alpha, V; 1 \leq \alpha \leq k)\), where each \( \eta^\alpha \) is a closed 1-form, each \( \omega^\alpha \) is a closed 2-form and \( V \) is an integrable \( nk \)-dimensional distribution on \( M \) satisfying (i) and (ii). \( M \) is said to be a \( k \)-cosymplectic manifold.

The following theorem has been proved in [21]:

...
Theorem 2.4 (Darboux theorem). If $M$ is a $k$-cosymplectic manifold, then around each point of $M$ there exist local coordinates $(x^\alpha, q^i, p_i^\alpha; 1 \leq \alpha \leq k, 1 \leq i \leq n)$ such that

$$
\eta^\alpha = dx^\alpha, \quad \omega^\alpha = dq^i \wedge dp_i^\alpha, \quad V = \left( \frac{\partial}{\partial p_i^1}, \ldots, \frac{\partial}{\partial p_i^k} \right)_{i=1, \ldots, n}.
$$

These coordinates will be called Darboux coordinates. The canonical model for these geometrical structures is $(\mathbb{R}^k \times (T^1_k)^* Q, \eta^\alpha, \omega^\alpha, V)$.

2.3. $k$-cosymplectic Hamiltonian field theory. In this section we introduce the $k$-cosymplectic description of the Hamilton-De Donder-Weyl equations

$$
\frac{\partial \psi^i}{\partial x^x} \bigg|_x = \frac{\partial H}{\partial q^i} \bigg|_{\psi(x)}, \quad \sum_{\alpha=1}^k \frac{\partial \psi^\alpha}{\partial x^\alpha} \bigg|_x = -\frac{\partial H}{\partial p_i^\alpha} \bigg|_{\psi(x)}, \quad (2.5)
$$

where locally $\psi(x) = (x, \psi^i(x), p_i^\alpha(x))$. This approach was firstly introduced by M. de León et al. [21]. We will consider the general case on an arbitrary $k$-cosymplectic manifold $M$ but everything can be particularize for the case of $M = \mathbb{R}^k \times (T^1_k)^* Q$.

Definition 2.5. Let $(M, \eta^\alpha, \omega^\alpha, V)$ be a $k$-cosymplectic manifold and $H: M \to \mathbb{R}$ be a Hamiltonian function. The family $(M, \eta^\alpha, \omega^\alpha, H)$ is called $k$-cosymplectic Hamiltonian system.

Theorem 2.6. Let $(M, \eta^\alpha, \omega^\alpha, H)$ a $k$-cosymplectic Hamiltonian system and $X = (X_1, \ldots, X_k)$ a $k$-vector field on $M$ solution to the system of equations

$$
\eta^\alpha(X_\beta) = \delta_\beta^\alpha, \quad 1 \leq \alpha, \beta \leq k
$$

$$
\sum_{\alpha=1}^k \iota_{X_\alpha} \omega^\alpha = dH - \sum_{\alpha=1}^k R_\alpha(H) \eta^\alpha, \quad (2.6)
$$

where $R_1, \ldots, R_k$ are the Reeb vector fields associated with the $k$-cosymplectic structure on $M$ which are characterized by the conditions

$$
\iota_{R_\alpha} \eta^\beta = \delta_\alpha^\beta, \quad \iota_{R_\alpha} \omega^\beta = 0.
$$

If $\psi: \mathbb{R}^k \to M$, $\psi(x) = (x^\alpha, \psi^i(x), p_i^\alpha(x))$ is an integral section of the $k$-vector field $X$, then $\psi$ is a solution of the Hamilton-De Donder-Weyl equations [25].

Proof. Let $X = (X_1, \ldots, X_k)$ be a $k$-vector on $M$ solution to (2.6). In Darboux coordinates each component $X_\alpha$ of the $k$-vector field $X = (X_1, \ldots, X_k)$ has the following local expression

$$
X_\alpha = (X_\alpha)_\beta \frac{\partial}{\partial x^\beta} + (X_\alpha)^i \frac{\partial}{\partial q^i} + (X_\alpha)_i^\beta \frac{\partial}{\partial p_i^\beta}.
$$

Now, since

$$
dH = \frac{\partial H}{\partial x^\alpha} dx^\alpha + \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i^\alpha} dp_i^\alpha,
$$

and

$$
\eta^\alpha = dx^\alpha, \quad \omega^\alpha = dq^i \wedge dp_i^\alpha, \quad R_\alpha = \frac{\partial}{\partial x^\alpha},
$$

with $1 \leq \alpha \leq k$, we deduce that equation (2.6) is locally equivalent to the following local equations

$$
(X_\alpha)_\beta = \delta_\beta^\alpha, \quad \frac{\partial H}{\partial p_i^\alpha} = (X_\alpha)^i, \quad \frac{\partial H}{\partial q^i} = -\sum_{\alpha=1}^k (X_\alpha)_i^\alpha. \quad (2.7)
$$
Let us suppose now that \( X = (X_1, \ldots, X_k) \) is integrable and
\[
\psi: \mathbb{R}^k \rightarrow M
\]
\[x \mapsto (x^\alpha, \psi^i(x), \psi^\alpha_i(x))\]
is an integral section of \( X \) then (see (2.2))
\[
(X_\alpha)_\beta \circ \psi = \delta_\beta^\alpha, \quad \frac{\partial \psi^i}{\partial x^\alpha} = (X_\alpha)^i \circ \psi, \quad \frac{\partial \psi^\beta_i}{\partial x^\alpha} = (X_\alpha)^\beta_i \circ \psi. \tag{2.8}
\]
Therefore, from (2.7) and (2.8) we obtain that \( \psi(x) = (x, \psi^i(x), \psi^\alpha_i(x)) \) is a solution of the following equations
\[
\frac{\partial H}{\partial q^i}|_{\psi(x)} = -\sum_{\alpha=1}^k \frac{\partial \psi^\alpha}{\partial x^i} |_x, \quad \frac{\partial H}{\partial p^\alpha_i}|_{\psi(x)} = \frac{\partial \psi^\alpha_i}{\partial x^\alpha} |_x,
\]
where 1 \( \leq i \leq n \) and 1 \( \leq \alpha \leq k \), that is, \( \psi \) is a solution to the Hamilton-De Donder-Weyl equations (2.5). \( \square \)

Consequently, the equations (2.6) can be considered as a geometric version of the Hamilton-De Donder-Weyl field equations. From now, we will call these equations (2.6) as \( k \)-cosymplectic Hamiltonian equations.

**Definition 2.7.** A \( k \)-vector field \( X = (X_1, \ldots, X_k) \) is called a \( k \)-cosymplectic Hamiltonian \( k \)-vector field for a \( k \)-cosymplectic Hamiltonian system \((M, \eta^\alpha, \omega^\alpha, H)\) if \( X \) is a solution of (2.6). We denote by \( \mathcal{X}_k^H(M) \) the set of \( k \)-vector fields which are solution of (2.6).

**Remark 2.8.** We will discuss here about the existence and uniqueness of solutions of equations (2.6).

First of all, we shall prove the existence of geometric solutions.

Let \((M, \eta^\alpha, \omega^\alpha, V)\) be a \( k \)-cosymplectic manifold; then, we can define the vector bundle morphism
\[
\omega^i: T^1_k M \rightarrow T^* M
\]
\[ (X_1, \ldots, X_k) \mapsto \sum_{\alpha=1}^k \iota_{X_\alpha} \omega^\alpha + \eta^\alpha(X_\alpha) \eta^\alpha. \tag{2.9} \]
and, denoting by \( \mathcal{M}_k(C^\infty(M)) \) the space of matrices of order \( k \) whose entries are functions on \( M \), we also have the vector bundle morphism
\[
\eta^i: T^1_k M \rightarrow \mathcal{M}_k(C^\infty(M))
\]
\[ (X_1, \ldots, X_k) \mapsto \eta^i(X_1, \ldots, X_k) = (\eta^\alpha(X_\beta)). \tag{2.10} \]

From the local conditions (2.7) we can define in a neighborhood of each point \( x \in M \) a \( k \)-vector field that satisfies (2.6). For example we can put
\[
(X_\alpha)_\beta = \delta_\alpha^\beta, \quad (X_1)^i = \frac{\partial H}{\partial q^i}, \quad (X_\alpha)^i = 0 \text{ for } \alpha \neq 1 \neq \beta, \quad (X_\alpha)^i = \frac{\partial H}{\partial p^\alpha_i}.
\]

Now one can construct a global \( k \)-vector field, which is a solution of (2.6), by using a partition of unity in the manifold \( M \) (see [19, 21] for more details).

It should be noticed that, in general, equations (2.6) do not have a unique solution. In fact, the solutions of (2.6) are given by \((X_1, \ldots, X_k) + (\ker \omega^i \cap \ker \eta^i)\) for a particular solution \((X_1, \ldots, X_k)\).
Let us observe that given a $k$-vector field $Y = (Y_1, \ldots, Y_k)$ the condition $Y \in \ker \omega^\sharp \cap \ker \eta^\sharp$ is locally equivalent to
\begin{equation}
(Y_\beta)_\alpha = 0, \quad Y^i_\beta = 0, \quad \sum_{\alpha=1}^k (Y^i_\alpha)^\circ = 0. \tag{2.11}
\end{equation}

\begin{remark}
In the case $k = 1$ with $M = \mathbb{R} \times T^*Q$ the equations (2.6) reduces to the equations of the non-autonomous Hamiltonian Mechanics.
\end{remark}

3. The Hamilton-Jacobi Equation

There are several attempts to extend the Hamilton-Jacobi theory for classical field theories. In [16] we have described this theory in the framework of the so-called $k$-symplectic formalism [2, 9, 17, 18]. In this section we consider the $k$-cosymplectic framework. Another attempts in the framework of the multisymplectic formalism [6, 11] have been discussed in [14, 25, 26].

Along this section we only consider Hamiltonian systems defined on the phase-space $\mathbb{R}^k \times (T^*_k)^*Q$.

In Classical Field Theory the Hamilton-Jacobi equation is [29]
\begin{equation}
\frac{\partial W^\alpha}{\partial x^\alpha} + H\left(x^\beta, q^i, \frac{\partial W^\alpha}{\partial q^i}\right) = 0 \tag{3.1}
\end{equation}
where $W^1, \ldots, W^k : \mathbb{R} \times Q \to \mathbb{R}$.

The classical statement of time-dependent Hamilton-Jacobi equation for analytical mechanics is the following [1]:

**Theorem 3.1.** Let $H : \mathbb{R} \times T^*Q \to \mathbb{R}$ be a Hamiltonian and $T^*Q$ the symplectic manifold with the canonical symplectic structure $\omega = -d\theta$. Let $X_{H_t}$ be a Hamiltonian vector field on $T^*Q$ associated to the Hamiltonian $H_t : T^*Q \to \mathbb{R}$, $H_t(\nu_q) = H(t, \nu_q)$, and $W : \mathbb{R} \times Q \to \mathbb{R}$ be a smooth function. The following two conditions are equivalent:

(i) for every curve $c$ in $Q$ satisfying
\[ c'(t) = (\pi)_*\left(X_{H_t}(dW_t(c(t)))\right) \]
the curve $t \mapsto W_t(c(t))$ is an integral curve of $X_{H_t}$, where $W_t : Q \to \mathbb{R}$, $W_t(q) = W(t, q)$.

(ii) $W$ satisfies the Hamilton-Jacobi equation
\[ H(x, q^i, \frac{\partial W}{\partial q^i}) + \frac{\partial W}{\partial t} = \text{constant on } T^*Q \]
that is,
\[ H_t \circ dW_t + \frac{\partial W}{\partial t} = K(t). \]

In this section we introduce a geometric version of the Hamilton-Jacobi theory based in the $k$-cosymplectic formalism. In the particular case $k = 1$ we recover the above Theorem 3.1 for the time-dependent classical mechanics.

For each $x = (x^1, \ldots, x^k) \in \mathbb{R}^k$ we consider the following mappings
\begin{align*}
    i_x : \quad & Q \to \mathbb{R}^k \times Q & \text{and} & \quad j_x : \quad (T^*_k)^*Q \to \mathbb{R}^k \times (T^*_k)^*Q \\
    q \mapsto (x, q) & & (\nu^1_q, \ldots, \nu^k_q) \mapsto (x, \nu^1_q, \ldots, \nu^k_q)
\end{align*}
Let $\gamma: \mathbb{R}^k \times Q \to \mathbb{R}^k \times (T^*_k)^*Q$ be a section of $(\pi_Q)_{1,0}$. Let us observe that given a section $\gamma$ is equivalent to giving a mapping $\tilde{\gamma}: \mathbb{R}^k \times Q \to (T^*_k)^*Q$. If fact, given $\gamma$ we define $\tilde{\gamma} = \pi_2 \circ \gamma$ where $\pi_2$ is the canonical projection $\pi_2: \mathbb{R}^k \times (T^*_k)^*Q \to (T^*_k)^*Q$; conversely, given $\tilde{\gamma}$ we define $\gamma$ as the composition $\gamma(x, q) = (j_x \circ \tilde{\gamma})(x, q)$. Now, since $(T^*_k)^*Q$ is the Whitney sum of $k$ copies of the cotangent bundle, to give $\gamma$ is equivalent to give a family $(\tilde{\gamma}^1, \ldots, \tilde{\gamma}^k)$ of 1-forms along the map $\pi_Q: \mathbb{R}^k \times Q \to Q$.

If we consider local coordinates $(x^\alpha, q^i, p^\alpha_i)$ we have the following local expressions:

\[
\begin{align*}
\gamma(x^\alpha, q^i) &= (x^\alpha, q^i, \gamma^\beta(x^\alpha, q^i)), \\
\tilde{\gamma}^\alpha(x, q) &= \tilde{\gamma}^\alpha(x, q) dq^i(q).
\end{align*}
\]

Moreover, along this section we suppose that each $\tilde{\gamma}^\alpha$ satisfies that its exterior differential $d\tilde{\gamma}^\alpha$ vanishes over two $\pi_2$-vertical vector fields. In local coordinates, using the local expressions (3.2), this condition implies that

\[
\frac{\partial \gamma^\alpha}{\partial q^i} = \frac{\partial \tilde{\gamma}^\alpha}{\partial q^i}.
\]

Now, let $Z = (Z_1, \ldots, Z_k)$ be a $k$-vector field on $\mathbb{R}^k \times (T^*_k)^*Q$. Using $\gamma$ we can construct a $k$-vector field $Z^\gamma = (Z^\gamma_1, \ldots, Z^\gamma_k)$ on $\mathbb{R}^k \times Q$ such that the following diagram is commutative

\[
\begin{array}{ccc}
\mathbb{R}^k \times (T^*_k)^*Q & \xrightarrow{Z} & T^*_k(\mathbb{R}^k \times (T^*_k)^*Q) \\
\downarrow{\gamma} & & \downarrow{(\pi_Q)_{1,0}} \\
\mathbb{R}^k \times Q & \xrightarrow{Z^\gamma} & T^*_k(\mathbb{R}^k \times Q) \\
\end{array}
\]

that is,

\[
Z^\gamma := T^*_k(\pi_Q)_{1,0} \circ Z \circ \gamma.
\]

Let us recall that for an arbitrary differentiable map $f: M \to N$, the induced map $T^*_k f: T^*_k M \to T^*_k N$ of $f$ is defined by

\[
T^*_k f(v_{1x}, \ldots, v_{kx}) = (f_1(x)(v_{1x}), \ldots, f_k(x)(v_{kx})),
\]

where $v_{1x}, \ldots, v_{kx} \in T_x M$, $x \in M$.

Let us observe that if $Z$ is integrable then $Z^\gamma$ is also integrable.

In local coordinates, if each $Z^\alpha$ is locally given by

\[
Z^\alpha = (Z^\alpha)(x) \frac{\partial}{\partial x^\alpha} + Z^\alpha_i \frac{\partial}{\partial q^i} + (Z^\alpha)_i \frac{\partial}{\partial p^i}
\]

then $Z^\gamma_\alpha$ has the following local expression:

\[
Z^\gamma_\alpha = ((Z^\alpha) \circ \gamma(\cdot)) \frac{\partial}{\partial x^\alpha} + (Z^\gamma_\alpha(\cdot)) \frac{\partial}{\partial q^i}.
\]

In particular, if we consider the $k$-vector field $R = (R^1_\gamma, \ldots, R^k_\gamma)$ given by the Reeb vector fields, we obtain, by a similar procedure, a $k$-vector field $(R^1_\gamma, \ldots, R^k_\gamma)$ on $\mathbb{R}^k \times Q$. In local coordinates, since $R^\alpha_\gamma = \partial / \partial x^\alpha$ we have

\[
R^\gamma_\alpha = \frac{\partial}{\partial x^\alpha}.
\]
Next, we consider a Hamiltonian function $H: \mathbb{R}^k \times (T^*_k)^*Q \rightarrow \mathbb{R}$, and the corresponding Hamiltonian system on $\mathbb{R}^k \times (T^*_k)^*Q$. Notice that if $Z$ satisfies the Hamilton-De Donder-Weyl equations \((2.4)\), then we have

$$(Z_\alpha)_\beta = \delta_{\alpha\beta}.$$  

**Theorem 3.2 (Hamilton-Jacobi theorem).** Let $Z \in X^*_\mathbb{H} (\mathbb{R}^k \times (T^*_k)^*Q)$ be a $k$-vector field solution to the $k$-cosymplectic Hamiltonian equation \((2.7)\) and $\gamma: \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k \times (T^*_k)^*Q$ be a section of $(\pi_Q)_0$ with the property described above. If $Z$ is integrable then the following statements are equivalent:

(i) If a section $\psi: U \subset \mathbb{R}^k \rightarrow \mathbb{R}^k \times Q$ of $\pi_\mathbb{H}: \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k$ is an integral section of $Z$, then $\gamma \circ \psi$ is a solution of the Hamilton-De Donder-Weyl equations \((2.3)\):

(ii) $(\pi_Q)^*[d(H \circ \gamma \circ i_x)] + \sum \alpha \gamma_\alpha^* d\gamma^\alpha = 0$ for any $x \in \mathbb{R}^k$.

**Proof.** Let us suppose that a section $\psi: U \subset \mathbb{R}^k \rightarrow \mathbb{R}^k \times Q$ is an integral section of $Z$. In local coordinates that means that if $\psi(x) = (x^\alpha, \psi^\alpha(x))$, then

$$[Z_\alpha)_\beta (\psi(x)) = \delta_{\alpha\beta}, \quad (Z_\alpha)_\beta (\psi(x)) = \frac{\partial \psi^\alpha}{\partial x^\beta}.$$  

Now by hypothesis, $\gamma \circ \psi: U \subset \mathbb{R}^k \rightarrow \mathbb{R}^k \times (T^*_k)^*Q$ is a solution of the Hamilton-De Donder-Weyl equation for $H$. In local coordinates, if $\psi(x) = (x, \psi^i(x))$, then $\gamma \circ \psi(x) = (x, \psi^i(x), \gamma_\alpha^i(\psi(x)))$ and as it is a solution of the Hamilton-De Donder-Weyl equations for $H$, we have

$$\frac{\partial \psi^i}{\partial x^\alpha} = \frac{\partial H}{\partial p^\alpha} \gamma_\alpha^i(\psi(x)) \quad \text{and} \quad \sum \alpha \gamma_\alpha^* d\gamma^\alpha.$$  

Now, if we compute the differential of the function $H \circ \gamma \circ i_x: Q \rightarrow \mathbb{R}$, we obtain that:

$$(\pi_Q)^*[d(H \circ \gamma \circ i_x)] + \sum \alpha \gamma_\alpha^* d\gamma^\alpha = \left( \frac{\partial H}{\partial q^i} \gamma_\alpha^i \circ i_x + \frac{\partial H}{\partial p^i} \gamma_\alpha^i \circ i_x \right) dq^i \quad \text{for} \quad i = 1, \ldots, k.$$  

(3.7)

Therefore from \((3.3), 3.6\) and \((3.7)\) and taking into account that one can write $\psi(x) = (i_x \circ \pi_Q \circ \psi)(x)$, where $\pi_Q: \mathbb{R}^k \times Q \rightarrow Q$ is the canonical projection, we obtain

$$\left( \frac{\partial H}{\partial q^i} \gamma_\alpha^i \circ i_x + \frac{\partial H}{\partial p^i} \gamma_\alpha^i \circ i_x \right) dq^i = 0.$$  

As we have mentioned above, since $Z$ is integrable, the $k$-vector field $Z'$ is also integrable, and then for each point $(x, q) \in \mathbb{R}^k \times Q$ we have an integral section...
\[ \psi : U \subset \mathbb{R}^k \to \mathbb{R}^k \times Q \] passing through this point. Therefore, for any \( x \in \mathbb{R}^k \), we get
\[ (\pi_Q)^* [d(H \circ \gamma \circ i_x)] + \sum_{\alpha} \iota_{R^\alpha_i} d\gamma^\alpha = 0. \]

Conversely, let us suppose that \( (\pi_Q)^* [d(H \circ \gamma \circ i_x)] + \sum_{\alpha} \iota_{R^\alpha_i} d\gamma^\alpha = 0 \) and take \( \psi \) an integral section of \( Z^\gamma \). We now will prove that \( \gamma \circ \psi \) is a solution to the Hamilton-De Donder-Weyl field equations for \( H \).

Since \( (\pi_Q)^* [d(H \circ \gamma \circ i_x)] + \sum_{\alpha} \iota_{R^\alpha_i} d\gamma^\alpha = 0 \) from (3.7) we obtain
\[ \frac{\partial H}{\partial q^i} \circ \gamma \circ i_x + \left( \frac{\partial H}{\partial p^j} \circ \gamma \circ i_x \right) \left( \frac{\partial \gamma^\alpha}{\partial x^\alpha} \circ i_x \right) = 0. \quad (3.8) \]

From (2.7) and (3.5) we know that
\[ Z^\gamma = \frac{\partial}{\partial x^\alpha} + \left( \frac{\partial H}{\partial p^j} \circ \gamma \right) \frac{\partial}{\partial q^j}, \quad (3.9) \]
and then since \( \psi(x, q) = (x, \psi^i(x, q)) \) is an integral section of \( Z^\gamma \) we deduce that
\[ \frac{\partial \psi^i}{\partial x^\alpha} = \frac{\partial H}{\partial p^j} \circ \gamma \circ \psi. \quad (3.10) \]

On the other hand, from (3.3), (3.8) and (3.10) we get
\[ \sum_{\alpha=1}^k \left( \frac{\partial (\gamma^\alpha \circ \psi)}{\partial x^\alpha} \right) \bigg|_{x} = \sum_{\alpha=1}^k \left( \frac{\partial \gamma^\alpha}{\partial x^\alpha} \bigg|_{\psi(x)} + \frac{\partial \gamma^\alpha}{\partial p^j} \bigg|_{\psi(x)} \frac{\partial \psi^j}{\partial x^\alpha} \right) = 0, \]
\[ = \sum_{\alpha=1}^k \left( \frac{\partial \gamma^\alpha}{\partial x^\alpha} \bigg|_{\psi(x)} + \frac{\partial H}{\partial q^j} \bigg|_{\psi(x)} \frac{\partial \psi^j}{\partial x^\alpha} \right) = - \frac{\partial H}{\partial q} \bigg|_{\gamma(\psi(x))} \]
and thus we have proved that \( \gamma \circ \psi \) is a solution to the Hamilton-de Donder-Weyl equations.

**Theorem 3.3.** Let \( Z \in \mathcal{X}_{\mathfrak{h}}(\mathbb{R}^k \times (T^k_i)^* Q) \) be a \( k \)-vector field solution to the \( k \)-cosymplectic Hamiltonian equation (2.2) and \( \gamma : \mathbb{R}^k \times Q \to \mathbb{R}^k \times (T^k_i)^* Q \) be a section of \( (\pi_Q)_{1,0} \) satisfying the same conditions of the above theorem. Then, the following statements are equivalent:

1. \( Z|_{\mathfrak{m} \gamma} - T_{1} \gamma(Z^\gamma) \in \ker \omega^i \cap \ker \eta^i \), being \( \omega^i \) and \( \eta^i \) the vector bundle morphism defined in Remark 2.8.
2. \( (\pi_Q)^* [d(H \circ \gamma \circ i_x)] + \sum_{\alpha} \iota_{R^\alpha_i} d\gamma^\alpha = 0. \)

**Proof.** A direct computation shows that \( Z_{\alpha}|_{\mathfrak{m} \gamma} - T_{1} \gamma(Z^\gamma_{\alpha}) \) has the following local expression
\[ \left( (Z_{\alpha})^\beta \circ \gamma - \frac{\partial \gamma^\beta}{\partial x^\alpha} \right) \frac{\partial}{\partial p^j} \bigg|_{\gamma(\psi(x))}. \]
Thus from (2.11) we know that \( Z|_{\mathfrak{m} \gamma} - T_{1} \gamma(Z^\gamma) \in \ker \omega^i \cap \ker \eta^i \) if and only if
\[ \sum_{\alpha=1}^k \left( (Z_{\alpha})^\alpha \circ \gamma - \frac{\partial \gamma^\alpha}{\partial x^\alpha} - (Z^\gamma_{\alpha} \circ \gamma) \frac{\partial \gamma^\alpha}{\partial q^j} \right) = 0. \quad (3.11) \]

Now we are ready to prove the result.
Assume that \((i)\) holds, then from (3.3) and (3.11) we obtain
\[
0 = \sum_{\alpha=1}^{k} \left((Z_{\alpha})_{\gamma} \circ \gamma - \frac{\partial \gamma_{\alpha}}{\partial x^\alpha} = (Z_{\alpha} \circ \gamma) \frac{\partial \gamma_{\alpha}}{\partial q^\alpha}\right)
\]
\[
= - \left(\left(\frac{\partial H}{\partial q^\alpha} \circ \gamma\right) + \sum_{\alpha} \frac{\partial \gamma_{\alpha}}{\partial x^\alpha} + \left(\frac{\partial H}{\partial p^\alpha} \circ \gamma\right) \frac{\partial \gamma_{\alpha}}{\partial q^\alpha}\right)
\]
\[
= - \left(\left(\frac{\partial H}{\partial q^\alpha} \circ \gamma\right) + \sum_{\alpha} \frac{\partial \gamma_{\alpha}}{\partial x^\alpha} + \left(\frac{\partial H}{\partial p^\alpha} \circ \gamma\right) \frac{\partial \gamma_{\alpha}}{\partial q^\alpha}\right) .
\]
Therefore \((\pi_Q)^*[d(H \circ \gamma \circ i_x)] + \iota_{R_2} d\tilde{\gamma}^\alpha = 0\) (see (3.7)).

The converse is proved in a similar way by reversing the arguments. \(\Box\)

**Corollary 3.4.** Let \(Z \in \mathcal{X}_H^k(\mathbb{R}^k \times (T_k^1)^*Q)\) be a solution of (2.6) and \(\gamma: \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k \times (T_k^1)^*Q\) be a section of \((\pi_Q)_{1,0}\) as in the above theorem. If \(Z\) is integrable then the following statements are equivalent:

\begin{enumerate}
  \item \((Z|_{\text{ker} \omega^k} - T_k^1 \gamma(Z^\gamma)) \in \ker \omega^k \cap \ker \eta^k;\)
  \item \((\pi_Q)^*[d(H \circ \gamma \circ i_x)] + \sum_{\alpha} \iota_{R_2} d\tilde{\gamma}^\alpha = 0;\)
  \item If a section \(\psi: U \subset \mathbb{R}^k \rightarrow \mathbb{R}^k \times Q\) of \(\pi_{R^k}: \mathbb{R}^k \times Q \rightarrow \mathbb{R}^k\) is an integral section of \(Z^\gamma\) then \(\gamma \circ \psi\) is a solution of the Hamilton-De Donder-Weyl equations (2.3).
\end{enumerate}

Let us observe that there exist \(k\) local functions \(W^\alpha\) such that \(\tilde{\gamma}^\alpha = dW^\alpha\) being \(W^\alpha\) the function defined by \(W^\alpha_x(q) = W^\alpha(x, q)\). Thus \(\gamma^\alpha = \frac{\partial W^\alpha}{\partial q^\alpha}\) (see [12]). Therefore, the condition
\[
(\pi_Q)^*[d(H \circ \gamma \circ i_x)] + \sum_{\alpha} \iota_{R_2} d\tilde{\gamma}^\alpha = 0
\]
can be equivalently written as
\[
\frac{\partial}{\partial q^\alpha} \left(\frac{\partial W^\alpha}{\partial x^\alpha} + H(x^\beta, q^\gamma, \frac{\partial W^\alpha}{\partial q^\gamma})\right) = 0.
\]

The above expressions mean that
\[
\frac{\partial W^\alpha}{\partial x^\alpha} + H(x^\beta, q^\gamma, \frac{\partial W^\alpha}{\partial q^\gamma}) = K(x^\beta)
\]
so that if we put \(\tilde{H} = H - K\) we deduce the standard form of the Hamilton-Jacobi equation (since \(H\) and \(\tilde{H}\) give the same Hamilton-De Donder-Weyl equations).
\[
\frac{\partial W^\alpha}{\partial x^\alpha} + \tilde{H}(x^\beta, q^\gamma, \frac{\partial W^\alpha}{\partial q^\gamma}) = 0 .
\]
Therefore the equation
\[
(\pi_Q)^*[d(H \circ \gamma \circ i_x)] + \sum_{\alpha} \iota_{R_2} d\tilde{\gamma}^\alpha = 0
\]
can be considered as a geometric version of the Hamilton-Jacobi equation for \(k\)-cosymplectic field theories.

4. **AN EXAMPLE**

In this section we will apply our method to a particular example in classical field theories.

The equation of a scalar field \(\phi\) (for instance the gravitational field) which acts on the 4-dimensional space-time is (see [11]):
\[
(\Box + m^2)\phi = F'(\phi) ,
\]
where $m$ is the mass of the particle over which the fields acts, $F$ is a scalar function such that $F(\phi) - \frac{1}{2}m^2\phi^2$ is the potential energy of the particle of mass $m$, and $\Box$ is the Laplace-Beltrami operator given by

$$\Box\phi := \text{div} \, \text{grad} \phi = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \left( \sqrt{-g} g^{\alpha\beta} \frac{\partial \phi}{\partial x^\beta} \right),$$

($g_{\alpha\beta}$) being a pseudo-Riemannian metric tensor in the 4-dimensional space-time of signature $(-+++)$, and $\sqrt{-g} = \sqrt{-\det g_{\alpha\beta}}$.

We consider the Lagrangian

$$L(x^1, x^2, x^3, x^4, q, v_1, v_2, v_3, v_4) = \sqrt{-g} \left( F(q) - \frac{1}{2}m^2q^2 + \frac{1}{2}g^{\alpha\beta}v_\alpha v_\beta \right),$$

where $q$ denotes the scalar field $\phi$ and $v_\alpha$ the partial derivative $\partial q/\partial x^\alpha$. Then the equation (4.1) is just the Euler-Lagrange equation associated to $L$.

Consider the Hamiltonian function $H \in C^\infty(\mathbb{R}^4 \times (T^*_4) \ast \mathbb{R})$ given by

$$H(x^1, x^2, x^3, x^4, p_1, p_2, p_3, p_4) = \frac{1}{2}\sqrt{-g} g^{\alpha\beta}p_\alpha p_\beta - \sqrt{-g} \left( F(q) - \frac{1}{2}m^2q^2 \right),$$

where $(x^1, x^2, x^3, x^4)$ are the coordinates on $\mathbb{R}^4$, $q$ denotes the scalar field $\phi$ and $(x^1, x^2, x^3, x^4, p_1, p_2, p_3, p_4)$ the canonical coordinates on $\mathbb{R}^4 \times (T^*_4) \ast \mathbb{R}$. Let us recall that this Hamiltonian function can be obtained from the Lagrangian $L$ just using the Legendre transformation defined in [21, 22].

Then

$$\frac{\partial H}{\partial q} = -\sqrt{-g} \left( F'(q) - m^2q \right), \quad \frac{\partial H}{\partial p} = \frac{1}{\sqrt{-g}} g^{\alpha\beta} p_\alpha, \quad (4.2)$$

The Hamilton-Jacobi equation becomes

$$-\sqrt{-g} \left( F'(q) - m^2q \right) + \frac{1}{\sqrt{-g}} g^{\alpha\beta} \gamma^\beta \frac{\partial \gamma^\alpha}{\partial q} + \frac{\partial \gamma^\alpha}{\partial x^\alpha} = 0, \quad (4.3)$$

Since our main goal is to show how the method developed in Section 3 works, we will consider, for simplicity, the following particular case:

$$F(q) = \frac{1}{2}m^2q^2,$$

being $(g_{\alpha\beta})$ the Minkowski metric on $\mathbb{R}^4$, i.e. $(g_{\alpha\beta}) = \text{diag}(-1, 1, 1, 1)$.

Let $\gamma: \mathbb{R}^4 \rightarrow \mathbb{R}^4 \times (T^*_4) \ast \mathbb{R}$ be the section of $(\pi_\mathbb{R})_{1,0}$ defined by the family of 4 1-forms along of $\pi_\mathbb{R}: \mathbb{R}^4 \times \mathbb{R} \rightarrow \mathbb{R}$

$$\gamma^\alpha = \frac{1}{2} C_\alpha q^2 dq$$

with $1 \leq \alpha \leq 4$ and where $C_\alpha$ are four constants such that $C_1^2 = C_2^2 + C_3^2 + C_4^2$.

This section $\gamma$ satisfies the Hamilton-Jacobi equation (4.3) that in this particular case is given by

$$-\frac{1}{2} C_1^2 q^3 + \frac{1}{2} C_2^2 q^3 + \frac{4}{2} C_3^2 q^3 + \frac{1}{2} C_4^2 q^3 = 0,$$

therefore, the condition (ii) of the Theorem 3.2 holds.

The 4-vector field $Z^\gamma = (Z_1^\gamma, Z_2^\gamma, Z_3^\gamma, Z_4^\gamma)$ is locally given by

$$Z_\alpha^\gamma = \frac{\partial}{\partial x^\alpha} \frac{1}{2} C_\alpha q^2 \frac{\partial}{\partial q}, \quad Z_\alpha^\gamma = \frac{\partial}{\partial x^\alpha} \frac{1}{2} C_\alpha q^2 \frac{\partial}{\partial q},$$

with $\alpha = 2, 3, 4$. The map $\psi: \mathbb{R}^4 \rightarrow \mathbb{R}^4 \times \mathbb{R}$ defined by

$$\psi(x^1, x^2, x^3, x^4) = \frac{2}{C_1 x^1 - C_2 x^2 - C_3 x^3 - C_4 x^4 + C}, \quad C \in \mathbb{R},$$
is an integral section of the 4-vector field $Z^\gamma$.

By Theorem 3.2 one obtains that the map $\varphi = \gamma \circ \psi$, locally given by
\[
(x^\alpha) \to (x^\alpha, \psi(x^\alpha), \frac{1}{2} C_\alpha(\psi(x^\alpha))^2),
\]
is a solution of the Hamilton-De Donder-Weyl equations associated to $H$, that is,
\[
0 = \sum_{\alpha=1}^{4} \frac{\partial}{\partial x^\alpha} \left( \frac{1}{2} C_\alpha \psi^2 \right),
\]
\[
-\frac{1}{2} C_1 \psi^2 = \frac{\partial \psi}{\partial x^1},
\]
\[
\frac{1}{2} C_a \psi^2 = \frac{\partial \psi}{\partial x^a}, \quad a = 2, 3, 4.
\]

Let us observe that these equations imply that the scalar field $\psi$ is a solution to the 3-dimensional wave equation.

In this particular example the functions $W^\alpha$ are given by
\[
W^\alpha(x, q) = \frac{1}{6} C_\alpha q^3 + h(x),
\]

where $h \in C^\infty(\mathbb{R}^4)$.

In [26, 31], the authors describe an alternative method that can be compared with the above one.

First, we consider the set of functions $W^\alpha: \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}, 1 \leq \alpha \leq 4$ defined by
\[
W^\alpha(x, q) = (q - \frac{1}{2} \phi(x)) \sqrt{-g} g^{\alpha \beta} \frac{\partial \phi}{\partial x^\beta},
\]
where $\phi$ is a solution to the Euler-Lagrange equation (4.1). Using these functions we can consider a section $\gamma$ of $(\pi \mathbb{R})_{1,0}: \mathbb{R}^4 \times (T^*_1 \mathbb{R})^* \to \mathbb{R}^4 \times \mathbb{R}$ with components
\[
\gamma^\alpha = \frac{\partial W^\alpha}{\partial q} = \sqrt{-g} g^{\alpha \beta} \frac{\partial \phi}{\partial x^\beta}.
\]

By a direct computation we obtain that this section $\gamma$ is a solution to the Hamilton-Jacobi equation (3.13).

Now from (3.9) and (4.2) we obtain the 4-vector field $Z^\gamma$ is given by
\[
Z^\gamma_\alpha = \frac{\partial}{\partial x^\alpha} + \frac{\partial \phi}{\partial x^\alpha} \frac{\partial}{\partial q}.
\]

Let us observe that $Z^\gamma$ is an integrable 4-vector field on $\mathbb{R}^4 \times \mathbb{R}$. Using the Hamilton-Jacobi theorem we obtain that $\sigma = (id_{\mathbb{R}^4}, \phi): \mathbb{R}^4 \to \mathbb{R}^4 \times \mathbb{R}$ is an integral section of the 4-vector field $Z^\gamma$ defined by (4.4), then $\gamma \circ \sigma$ is a solution of the Hamilton-De Donder Weyl equation associated with the Hamiltonian of the massive scalar field.

If we now consider the particular case $F(q) = \frac{1}{2} m^2 q^2$, we obtain the Klein-Gordon equation; this is just the case discussed in [26].

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