ELEMENTARY MOVES ON LATTICE POLYTOPES

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ABSTRACT
We introduce a graph structure on Euclidean polytopes. The vertices of this graph are the \(d\)-dimensional polytopes contained in \(\mathbb{R}^d\) and its edges connect any two polytopes that can be obtained from one another by either inserting or deleting a vertex, while keeping their vertex sets otherwise unaffected. We prove several results on the connectivity of this graph, and on a number of its subgraphs. We are especially interested in several families of subgraphs induced by lattice polytopes, such as the subgraphs induced by the lattice polytopes with \(n\) or \(n+1\) vertices, that turn out to exhibit intriguing properties.

1. Introduction

In this paper, we introduce a graph structure on the Euclidean polytopes. The vertices of this graph are the \(d\)-dimensional polytopes contained in \(\mathbb{R}^d\) and its edges connect two polytopes when they can be obtained from one another by a transforming move, that we want to keep as elementary as possible. We will consider two types of moves, an insertion move and a deletion move. If \(x\) is a point in \(\mathbb{R}^d \setminus P\), an insertion move will transform \(P\) into the convex hull of
$P \cup \{x\}$. Note that $x$ is then necessarily a vertex of the resulting polytope. If $v$ is a vertex of $P$, a deletion move will transform $P$ into the convex hull of $V \setminus \{v\}$, where $V$ is the vertex set of $P$. Without any other requirement, these moves cannot yet be considered elementary as they significantly alter the boundary complex of $P$. For instance, the convex hull of $P \cup \{x\}$ does not necessarily have more vertices that $P$ itself, even though the performed move inserts a new vertex. Indeed, the convex hull of $P \cup \{x\}$ possibly contains a vertex of $P$ in its relative interior. An undesirable consequence is that deletion moves are not inverse to insertion moves because, then, $P$ cannot be recovered by deleting $x$ from the convex hull of $P \cup \{x\}$. A natural way to solve this issue consists in allowing an insertion move only when all the vertices of $P$ remain vertices of the polytope resulting from that insertion.

**Definition 1.1:** Consider a $d$-dimensional polytope $P$ contained in $\mathbb{R}^d$ and denote by $V$ its set of vertices. A point $x \in \mathbb{R}^d$ can be inserted in $P$ if the convex hull of $P \cup \{x\}$ admits $V \cup \{x\}$ as its vertex set. A vertex $v \in V$ can be deleted from $P$ when the convex hull of $V \setminus \{v\}$ is $d$-dimensional.

By this definition, deletion moves and insertion moves are now the inverse of one another. Recall that all the polytopes we consider here are full-dimensional, making the requirement that deletion moves do not decrease the dimension of a polytope necessary. In particular, a vertex $v$ of a polytope $P$ can be deleted from $P$ if and only if $P$ is not a pyramid with apex $v$ over a $(d-1)$-dimensional polytope. Consider the graph whose vertices are the $d$-dimensional polytopes contained in $\mathbb{R}^d$ and whose edges connect two polytopes that can be obtained from one another by an insertion move (or a deletion move). This graph, which we will refer to as $\Gamma(d)$ here, has an uncountable number of vertices and, as soon as $d \geq 2$, its vertices all have uncountable degree. Indeed, consider an arbitrary point $x$ in the boundary of a polytope $P$, distinct from any vertex of $P$. One can insert in $P$ any point outside of $P$ that is close enough to $x$.

The purpose of this paper is to investigate the connectivity of a number of subgraphs of $\Gamma(d)$. It is a priori not obvious whether $\Gamma(d)$ itself is connected. On the way to establishing the connectedness of $\Gamma(d)$, we will prove that the subgraphs induced in $\Gamma(d)$ by the polytopes with $n$ or $n+1$ vertices are connected for all $n \geq d+1$. All of these subgraphs, while infinite, have a finite diameter and we will obtain lower and upper bounds on these diameters. Observe that these graphs provide a metric on polytopes, in a very different spirit than, for
instance the Gromov-Hausdorff distance \[14\]: instead of measuring how far two bodies are from being isometric, we measure how long it takes to build them from one another with elementary operations.

The two families of graphs we mostly focus on are the subgraph \(\Lambda(d)\) induced in \(\Gamma(d)\) by the lattice polytopes and the subgraph \(\Lambda(d,k)\) induced in \(\Lambda(d)\) by the polytopes contained in the hypercube \([0,k]^d\). Here, by a lattice polytope we mean a polytope whose vertices belong to the lattice \(\mathbb{Z}^d\). These polytopes pop up in many places in the mathematical literature as, for instance in combinatorial optimization \[19\] \[20\] \[22\], in discrete geometry \[4\] \[7\] \[18\], or in connection with toric varieties \[6\] \[13\]. Observe that \(\Lambda(d)\) is a highly non-regular graph: it admits both vertices with finite degree and vertices with infinite (but countable) degree. In particular, \(\Lambda(d)\) gathers in a coherent metric structure polytopes with dramatically different behaviors regarding the ambient lattice. For instance, there are lattice polytopes, like the cube \([0,1]^d\), in which no lattice point can be inserted, while for the lattice simplices, no deletion move is possible. The graph \(\Lambda(d,k)\) is particularly relevant to the study of the lattice polytopes contained in \([0,k]^d\), that have attracted significant attention \[1\] \[3\] \[9\] \[10\] \[11\] \[17\]. Here, we will establish that \(\Lambda(d,k)\) is connected whenever \(d \geq 2\) and \(k \geq 1\). In particular, some lattice point in \([0,k]^d\) can always be inserted in a \(d\)-dimensional lattice simplex contained in \([0,k]^d\). As we shall see, the proof of this seemingly straightforward statement alone turns out to be surprisingly involved. Note that this connectedness result allows for the definition of a Markov chain whose states are the \(d\)-dimensional lattice polytopes contained in \([0,k]^d\), and whose stationary distribution is uniform \[3\]. We will obtain the connectedness of \(\Lambda(d)\) in turn, as a consequence of that of \(\Lambda(d,k)\).

Next, we will study the number of possible insertion and deletion moves on a lattice polytope. We will exhibit lattice polytopes with arbitrarily large dimension and number of vertices such that no insertion of a lattice point is possible. As an immediate consequence, the subgraph induced in \(\Lambda(d)\) by the polytopes with \(n\) or \(n+1\) vertices is not connected for all \(d\) and \(n\). For instance, when \(d = 2\), this subgraph is disconnected whenever \(n\) is greater than 3 and distinct from 5. We also describe a family of \(d\)-dimensional lattice polytopes contained in the hypercube \([0,k]^d\), where \(d\) and \(k\) can grow arbitrarily large, such that every lattice point in \([0,k]^d\) can be either inserted or deleted. These polytopes belong to the broader family of empty lattice polytopes, that is also widely studied, and interesting in its own right \[2\] \[5\] \[12\] \[15\] \[16\] \[21\] \[22\].
We conclude the article by asking a number of questions on the graphs $\Gamma(d)$, $\Lambda(d)$, and $\Lambda(d,k)$. Part of these questions arise naturally from our results, and in particular from the intriguing behavior of the subgraphs induced in $\Lambda(d)$ by the polytopes with $n$ or $n+1$ vertices. We will also mention a number of other subgraphs of $\Gamma(d)$, whose study may be interesting.

2. The connectivity of $\Gamma(d)$

In this section we investigate the connectedness of $\Gamma(d)$ itself and of its subgraphs induced by the polytopes with $n$ and $n+1$ vertices, where $n \geq d+1$. We also obtain precise bounds on the diameter of these subgraphs. Note that from now on, it is always assumed that $d$ is not less than 2.

Consider a $d$-dimensional polytope $P$ contained in $\mathbb{R}^d$. We will denote by $\text{aff}(F)$ the affine hull of a face $F$ of $P$. If $F$ is a facet, then $\text{aff}(F)$ is a hyperplane of $\mathbb{R}^d$ and we denote by $H_F^-(P)$ the closed half-space of $\mathbb{R}^d$ bounded by $\text{aff}(F)$ such that $P \cap H_F^-(P) = F$. For any vertex $v$ of $P$, the set

\[ C_v(P) = \bigcap_{F \in \mathcal{F}} H_F^-(P), \]

where $\mathcal{F}$ is the set of the facets of $P$ incident to $v$, is a $d$-dimensional polyhedral cone pointed at $v$. This cone is exactly the set of the points $x \in \mathbb{R}^d$ such that the convex hull of $P \cup \{x\}$ does not admit $v$ as a vertex. By this remark, we immediately obtain the following lemma.

**Lemma 2.1**: Consider a $d$-dimensional polytope $P$ contained in $\mathbb{R}^d$. A point $x$ of $\mathbb{R}^d$ can be inserted in $P$ if and only if it does not belong to $P$ and, for every vertex $v$ of $P$, it does not belong to $C_v(P)$.

Lemma 2.1 will be instrumental to prove the connectedness of a number of subgraphs of $\Gamma(d)$ in this section and the next. We will also make use of the following technical lemma that describes how the cones $C_v(P)$ are placed relatively to the supporting hyperplanes of the faces of a polytope $P$.

**Lemma 2.2**: Consider a proper face $F$ of a $d$-dimensional polytope $P$ contained in $\mathbb{R}^d$. Let $H$ be a hyperplane such that $F = P \cap H$, and $H^-$ the half-space of $\mathbb{R}^d$ bounded by $H$ and such that $F = P \cap H^-$. For any vertex $v$ of $F$, $C_v$ is a subset of $H^-$ and $C_v \cap H$ is a subset of $\text{aff}(F)$. 
Proof. Consider the intersection

\[ K = \bigcap_{G \in \mathcal{G}} H^{-}_G(P), \]

where \( \mathcal{G} \) is the set of all the facets of \( P \) incident to \( F \). First observe that \( K \) is a subset of \( H^{-} \). In addition \( K \cap H \) is precisely the affine hull of \( F \). Now consider a vertex \( v \) of \( F \). Since \( C_v(P) \) is a subset of \( K \), the result follows.

We need to state another elementary result. By the following lemma, when a \( d \)-dimensional polytope has at least \( d + 2 \) vertices, several of its vertices can be deleted. The proof of this result, by induction on the dimension, relies on the fact that any of the vertices of a polygon with at least four vertices can be deleted. This proof is straightforward and will be omitted.

**Lemma 2.3:** Let \( P \) be a \( d \)-dimensional polytope with \( n \) vertices. If \( n \geq d + 2 \), then at least \( n - d + 2 \) of the vertices of \( P \) can be deleted from it.

We are now ready to investigate the connectivity of \( \Gamma(d) \). We begin by looking at the connectedness of its subgraphs induced by the polytopes with \( n \) or \( n + 1 \) vertices. The following lemma deals with a particularly convenient special case, that will pop up several times thereafter. We say that a subset \( \mathcal{A} \) of \( \mathbb{R}^d \) is in **convex position** if any finite subset of \( \mathcal{A} \) is the vertex set of a polytope.

**Lemma 2.4:** Let \( \mathcal{A} \) be a \( d \)-dimensional subset of \( \mathbb{R}^d \) in convex position. For any \( n \geq d + 1 \), the polytopes with \( n \) or \( n + 1 \) vertices whose vertex set is a subset of \( \mathcal{A} \) induce a connected subgraph of \( \Gamma(d) \) of diameter at most \( 2n + 2 \). Moreover, two polytopes with \( n \) vertices have distance at most \( 2n \) in this subgraph.

**Proof.** Let \( P \) and \( Q \) be two polytopes with \( n \) or \( n + 1 \) vertices such that the vertex sets of \( P \) and \( Q \) are subsets of \( \mathcal{A} \). By Lemma 2.3, we can assume without loss of generality that both \( P \) and \( Q \) have exactly \( n \) vertices. Since \( \mathcal{A} \) is in convex position, the vertices of the convex hull of \( P \cup Q \) are exactly the vertices of \( P \) and the vertices of \( Q \). As a consequence, any vertex of \( Q \) that is not already a vertex of \( P \) can be inserted in \( P \). After such an insertion, Lemma 2.3 makes sure that one can, in turn, delete a vertex distinct from the inserted point. The polytope resulting from this sequence of two moves shares at least one more vertex with \( Q \) than \( P \). Repeating this process therefore builds a path between \( P \) and \( Q \) in the considered subgraph of \( \Gamma(d) \). The length of this path is at most the sum of the number of vertices of \( P \) with the number of vertices of \( Q \).
Taking into account the two deletion moves that have possibly been performed initially to build \( P \) and \( Q \) from polytopes with \( n + 1 \) vertices, we obtain the desired bound on the diameter of that subgraph. □

Lemma 2.4 is instrumental already in the proof of the following theorem.

**Theorem 2.5:** For any \( n \geq d + 1 \), the polytopes with \( n \) or \( n + 1 \) vertices induce a connected subgraph of \( \Gamma(d) \) of diameter at most \( 6n - 2 \). Moreover, two polytopes with \( n \) vertices are distant of at most \( 6n - 4 \) in this subgraph.

**Proof.** Consider two \( d \)-dimensional polytopes \( P \) and \( Q \) contained in \( \mathbb{R}^d \) both with \( n \) or \( n + 1 \) vertices. Since it is always possible to delete some vertex from a polytope with more than \( d + 1 \) vertices, we can assume without loss of generality that both \( P \) and \( Q \) have \( n \) vertices. Denote

\[
\gamma = \min \{ x_1 : x \in P \cup Q \}.
\]

We will assume that some point \( x \) of \( P \) satisfies \( x_1 = \gamma \), which can be done by exchanging \( P \) and \( Q \), if need be. Let \( M \) denote the hyperplane made up of the points \( x \) such that \( x_1 = \gamma \). The intersection of \( P \) and \( M \) is a non-empty face of \( P \). This face, that we denote by \( E \), is sketched on the left of Fig. 1. As illustrated in the figure, if \( E \) is not a facet of \( P \), it is always possible to insert in \( P \) some point in \( M \) that does not belong to the affine hull of \( E \) (for instance, the point colored green on the left of Fig. 1). Indeed, by Lemma 2.2 for any vertex \( v \) of \( E \), the intersection \( C_v(P) \cap M \) is a subset of \( \text{aff}(E) \). Moreover, for any vertex \( v \) of \( P \) that is not incident to \( E \), \( C_v(P) \cap M \) is necessarily disjoint from \( E \). Since \( C_v(P) \cap M \) and \( E \) both are closed sets, it follows from Lemma 2.1 that any point \( x \) in \( M \) that does not belong to \( \text{aff}(E) \) can be inserted in \( P \), provided it is close enough to \( E \). After \( x \) has been inserted in \( P \), any vertex of \( P \) that is not incident to \( E \) can be deleted: if such a vertex were deleted from \( P \), then the resulting polytope would be at least \( (d - 1) \)-dimensional. Therefore, deleting it after \( x \) has been inserted in \( P \) results in a \( d \)-dimensional polytope because \( x \) is not contained in the affine hull of \( E \). Repeating this procedure, one can transform \( P \) into a polytope \( P' \) such that \( P' \cap M \) is a facet of \( P' \), using at most \( 2d - 2 \) moves (at most \( d - 1 \) insertion moves, each followed by a deletion move). We denote \( E' = P' \cap M \). Now call

\[
\delta = \max \{ x_1 : x \in Q \},
\]
and let $N$ be the hyperplane made up of the points $x$ such that $x_1 = \delta$. As above, if $Q \cap N$ is not a facet of $Q$, we can perform a sequence of at most $d - 1$ insertion moves on $Q$, that insert points in $N$, each followed by a deletion move, in order to obtain a polytope $Q'$ such that $Q' \cap N$ is a facet of $Q'$. In the remainder of the proof, we denote that facet by $F'$. Note that, the sketch on the left of Fig. 1 depicts the case when $Q' = Q$.

Now, consider a point in $H_{E'}(P') \setminus \text{aff}(E')$ whose orthogonal projection on $\text{aff}(E')$ is contained in the relative interior of $E'$ as, for instance, the points colored green next to $E'$ on the right of Fig. 1. Observe that this point can be inserted in $P'$ provided it is close enough to $E'$. We will perform a sequence of such insertion moves. Note that, while the first insertion move is in the neighborhood of $E'$, the next insertion moves will be made in the neighborhood of one of the facets introduced by the preceding insertion, in such a way that the orthogonal projection on $\text{aff}(E')$ of each inserted point is contained in the relative interior of $E'$. After each insertion move, a deletion move will be performed on a vertex of $P'$ that is not incident to $E'$ (note that any such vertex, colored red on the right of Fig. 1 can be deleted). Doing so, we can transform $P'$ into a polytope $P''$ that admits $E'$ as a facet and whose vertices not incident to $E'$ can be placed arbitrarily close to $E'$. In particular we can require that, for any facet $G$ of $P''$ other than $E'$, $H_{E'}(P'')$ and $F'$ are disjoint. Similarly, we can find a sequence of insertion moves, each followed by a deletion move that transform $Q'$ into a polytope $Q''$ that admits $F'$ as a facet and such that, for any other facet $G$ of $Q''$, $H_{E'}(Q'')$ and $E'$ are disjoint.

By that construction, all the vertices of $P''$ and all the vertices of $Q''$ are vertices of the convex hull of $P'' \cup Q''$. Hence, by Lemma 2.4 one can transform $P''$ into $Q''$ by a sequence of at most $2n$ moves such that each insertion move is
followed by a deletion move. We have done at most $2n - 2d$ moves to transform $P'$ into $P''$ or $Q'$ into $Q''$, and at most $2n$ moves to transform $P''$ into $Q''$. Taking into account the two deletion moves that have possibly been performed initially to build $P$ and $Q$ from polytopes with $n + 1$ vertices, we obtain the desired upper bound on the diameter of the subgraph induced in $\Gamma(d)$ by the polytopes with $n$ or $n + 1$ vertices.

A consequence of Theorem 2.5 is that it is always possible to transform two polytopes into one another using a sequence of elementary moves.

**Corollary 2.6:** $\Gamma(d)$ is connected.

**Proof.** Let $P$ and $Q$ be two $d$-dimensional polytopes contained in $\mathbb{R}^d$. Say that $P$ has $n$ vertices and $Q$ has $m$ vertices. We can assume without loss of generality that $n \leq m$. By Lemma 2.3, it is always possible to delete some vertex from a polytope with more than $d + 1$ vertices, there is a (possibly empty) sequence of deletion moves that transform $Q$ into a $d$-dimensional polytope with $n$ vertices. The result then follows from Theorem 2.5.

In the remainder of the section, we look at the distance between two polytopes in $\Gamma(d)$. According to Theorem 2.5, the diameter of the subgraph induced in $\Gamma(d)$ by polytopes with $n$ or $n + 1$ vertices is at most $6n - 2$. This upper bound is linear in the number $n$ of vertices of the considered polytopes, but it is independent on the dimension, which may be surprising. We obtain a lower bound on that diameter that is reasonably close to our upper bound.

**Lemma 2.7:** For any $n \geq d + 1$, the subgraph of $\Gamma(d)$ induced by the polytopes with $n$ or $n + 1$ vertices has diameter at least $4n - 2d$.

**Proof.** Consider the two polygons $P$ and $Q$ sketched on the left of Fig. 2. We will assume that each of these polygons has $n - d + 2$ vertices. As can be seen on the figure, $P$ and $Q$ are placed in such a way that for any vertex $v$ of $P$ distinct from the vertices of its longest edge, $Q$ is a subset of the cone $C_v(P)$. The intersection of all these cones is shown as a striped surface in the figure. Observe that, for any $d \geq 3$, we can build a $d$-dimensional polytope by considering a pyramid over $P$, and then a pyramid over that pyramid and so on. We will call $P'$ the resulting $d$-dimensional polytope, and $Q'$ the $d$-dimensional polytope obtained using the same procedure but starting from $Q$ instead of $P$. By construction, both $P'$ and $Q'$ have $n$ vertices. We require, which can be
done without loss of generality, that \( P' \) and \( Q' \) do not share a vertex.

Now consider a sequence of insertion moves, each followed by a deletion move that transform \( P' \) into \( Q' \). Consider the first move in that sequence that introduces a vertex of \( Q \). This move is performed on a polytope \( R' \). We claim that, when this move occurs, all except maybe two vertices of \( P \) have been deleted from \( P' \). Indeed, otherwise, the intersection of \( R' \) with the plane that contains \( P \) and \( Q \) is a polygon \( R \) that shares at least three vertices with \( P \). As can be seen on the right of Fig. 2, where the trace of \( P \) is sketched with dotted lines, these three vertices form a triangle \( T \), depicted in green. Note that the vertices of \( R \) are possibly not all vertices of \( R' \): some of them may be the intersection of a higher dimensional face of \( R' \) with the plane that contains \( P \) and \( Q \). However, \( R \) and \( R' \) necessarily share the three vertices of \( T \). Now observe that \( T \) admits at least one vertex \( v \) such that \( Q \) is a subset of the cone \( C_v(T) \). This cone, shown as a striped surface in the figure, is in turn a subset of \( C_v(R) \). Since \( C_v(R) \) is contained in \( C_v(R') \), we obtain the inclusion \( Q \subset C_v(R') \). In particular, no vertex of \( Q \) can be inserted in \( R' \).

Hence, there must have been at least \( n - d \) insertion moves, each followed by a deletion move before \( R' \) is reached from \( P' \). After that, all the vertices of \( Q \) still have to be introduced, which requires at least \( n - d + 2 \) insertion moves and \( n - d + 2 \) deletion moves. Since \( P' \) and \( Q' \) do not share a vertex, we further need to perform at least \( d - 2 \) insertions and \( d - 2 \) deletions to displace the vertices of \( P' \) that are not incident to \( P \). As a consequence, transforming \( P' \) into \( Q' \) requires at least \( 4n - 2d \) moves. \[ \square \]

It follows from Theorem 2.5 and from Lemma 2.7 that simplices play a central role in \( \Gamma(d) \) in the sense that connecting two polytopes with \( n \) vertices in \( \Gamma(d) \) via a simplex can be much shorter than with any path visiting only polytopes.
with $n$ or $n + 1$ vertices. In fact, according to Lemma 2.8, paths via simplices can be at least half as short when $d$ is fixed and $n$ grows large.

**Lemma 2.8:** The distance in $\Gamma(d)$ between a polytopes with $n$ vertices and a polytope with $m$ vertices is at most $n + m + 4d$.

**Proof.** By Lemma 2.3 one can always transform a polytope into a simplex by performing a sequence of deletions. By Theorem 2.5, the distance of two simplices in $\Gamma(d)$ is at most $6d + 2$ and the result follows. ■

### 3. The insertion move for lattice simplices

Connecting two polytopes within $\Lambda(d)$ turns out to be much more complicated than within $\Gamma(d)$. Recall that Lemma 2.1 makes it obvious that an insertion move is always possible on a polytope when the vertices of this polytope are not constrained to belong to a lattice. Indeed, as already mentioned, one can always insert a point in the polytope, provided this point is close enough to the boundary of the polytope but far enough from its vertices. In the case of lattice polytopes, inserting a point in a neighborhood of the polytope is not always possible. Our strategy here is to establish, first, the connectedness of the subgraph induced in $\Lambda(d,k)$ by the simplices and the polytopes with $d + 2$ vertices. This is similar to what we did in Section 2 with the subgraphs induced in $\Gamma(d)$ by the polytopes with $n$ or $n + 1$ vertices, except that here, $n$ has to be equal to $d + 1$. In this section, we give results on the possibility of inserting a lattice point in a lattice simplex. In particular, we show that, for any positive $k$, there is at least one lattice point in the hypercube $[0, k]^d$ that can be inserted in a given $d$-dimensional lattice simplex contained in $[0, k]^d$.

In the remainder of the section, $S$ denotes a $d$-dimensional lattice simplex contained in the hypercube $[0, k]^d$. For any $i \in \{1, \ldots, d\}$, we call

$$\gamma_i^- = \min\{x_i : x \in S\} \text{ and } \gamma_i^+ = \max\{x_i : x \in S\}.$$ 

Note that, for all $i \in \{1, \ldots, d\}$, $\gamma_i^- < \gamma_i^+$ because $S$ is $d$-dimensional. The following polytope is the smallest $d$-dimensional combinatorial hypercube containing $S$, and whose facets are parallel to the facets of $[0, k]^d$:

$$Q = \prod_{i=1}^{d} [\gamma_i^-, \gamma_i^+] .$$
Let $R$ be a facet of $Q$. The intersection of $R$ and $S$ is a non-empty, proper face $F$ of $S$. Since $S$ is a simplex, it admits another non-empty face $F^*$ whose vertices are exactly the vertices of $S$ that do not belong to $F$.

By construction,

$$\dim(F) + \dim(F^*) = d - 1.$$ 

In particular, there exists a vector $c$ that is orthogonal to both $F$ and $F^*$. Consider the hyperplane $Y$ of $\mathbb{R}^d$ that admits $c$ as a normal vector and such that $F^* \subset Y$. The intersection $S \cap Y$ is precisely $F^*$. Denote by $Y^-$ the closed half-space of $\mathbb{R}^d$ bounded by $Y$ that does not contain $F$.

Since all the vertices of $S$ are incident to either $F$ or $F^*$, it is an immediate consequence of Lemmas 2.1 and 2.2 that an insertion move is possible on $S$ for any lattice point in $[0,k]^d$ that does not belong to $S$, to $\text{aff}(F)$, or to $Y^-$. We are now going to search for such lattice points.

Assume, without loss of generality that $c$ is a unit vector and that it points towards $Y^-$. Recall that $R$ is a facet of $Q$ and observe that $\text{aff}(R) \cap [0,k]^d$ is a $(d - 1)$-dimensional cube. Denote

$$\delta = \min\{c \cdot x : x \in \text{aff}(R) \cap [0,k]^d\}.$$ 

The set

$$G = \{x \in \text{aff}(R) \cap [0,k]^d : c \cdot x = \delta\}$$

is a face of $\text{aff}(R) \cap [0,k]^d$. It follows that $G$ is a cube of dimension at most $d - 1$. Recall that $c$ is orthogonal to both $F$ and $F^*$. As a consequence, the map $x \mapsto c \cdot x$ is constant within $F$ and within $F^*$. Call $\varepsilon$ the value of $c \cdot x$ when $x \in F$ and $\varepsilon^*$ the value of $c \cdot x$ when $x \in F^*$. Since $F$ and $Y^-$ are disjoint, $\varepsilon < \varepsilon^*$. Moreover, by (2), $\delta \leq \varepsilon$. Observe that the latter inequality is strict if and only if $F$ is not a subset of $G$. In this case, $F$, $G$, and $Y$ belong to distinct parallel hyperplanes and we immediately obtain the following.

**Lemma 3.1:** If $F \nsubseteq G$ then $G$ is disjoint from both $\text{aff}(F)$ and $Y^-$. 

In other words, any lattice point in $G$ can be inserted in $S$ in this case. If, on the contrary, $\delta$ and $\varepsilon$ coincide, then $F \subseteq G$. This situation is familiar: we are looking at a lattice simplex $F$ contained in a (possibly degenerate) lattice hypercube $G$. If the dimension of $G$ is greater than the dimension of $F$, then the following lemma provides the desired result.
Lemma 3.2: If $k$ is positive and if $P$ is a lattice polytope of dimension less than $d$ contained in $[0,k]^d$, then there exists a lattice point $x$ that belongs to $[0,k]^d$ but that does not belong to the affine hull of $P$.

Proof. If $P$ is a lattice polytope of dimension less than $d$ contained in $[0,k]^d$, then the intersection $I$ of its affine hull with $[0,k]^d$ cannot contain more than $(k + 1)^{d-1}$ lattice points. Indeed, one can always project $I$ orthogonally on a facet of $[0,k]^d$ in such a way that the dimension of the projection is exactly that of $I$. Such a projection induces an injection from the lattice points in $I$ into the lattice points in the facet of $[0,k]^d$ on which the projection is made.

Now observe that $[0,k]^d$ contains $(k + 1)^d$ lattice points. Since $k$ is positive, $(k + 1)^{d-1} < (k + 1)^d$ and the lemma is proven. 

We now have to address the case when, regardless of which facet $R$ of $Q$ is chosen, $F$ is a subset of $G$ and these polytopes have the same dimension. This case is dealt with by the following lemma.

Lemma 3.3: Call $g$ the maximal dimension of $F$ over all the possible choices for $R$ among the facets of $Q$. Assume that, for any choice of $R$ among the facets of $Q$ such that $F$ has dimension $g$, $F$ is a subset of $G$ and the dimensions of $F$ and $G$ coincide. If $g$ is not greater than $d - 2$, then for some choice of $R$ such that $F$ has dimension $g$, there exists a lattice point in $R \setminus \text{aff}(F) \cup Y^−$.

Proof. Consider a facet $R$ of $Q$ such that $F$ has dimension exactly $g$. As $G$ admits $F$ as a subset, $G$ must be a face of $Q$. Taking advantage of the symmetries of $[0,k]^d$, we assume that any facet of $Q$ that contains $G$ is of the form

$$\{x \in Q : x_i = \gamma_i^−\}$$

for some $i \in \{1,\ldots,d\}$. In this case, all the coordinates of the vector $c$ are non-negative, except maybe for the coordinate $c_i$ such that all the points $x$ in $R$ satisfy $x_i = \gamma_i^−$. By the maximality of $g$, the intersection of $S$ with any facet of $Q$ incident to $G$ is precisely $F$. In other words, $G$, $F$, $F^*$, $Y$, and $c$ do not depend on which facet $R$ of $Q$ is chosen, provided this facet is incident to $G$. As a consequence, all the coordinates of $c$ are non-negative.

We will assume that $c_1$ is positive and that

$$R = \{x \in Q : x_1 = \gamma_1^−\}.$$
This can be done without loss of generality by, if needed, permuting the coordinates of $\mathbb{R}^d$. Recall that $F^*$ is non-empty and consider a vertex $v$ of $F^*$. By the definition of $\varepsilon^*$, we have the equality

$$\sum_{i=1}^{d} c_i v_i = \varepsilon^*.$$ 

This equality can be transformed into

$$c_1 \gamma_1^- + \sum_{i=2}^{d} c_i v_i = \varepsilon^* - c_1 (v_1 - \gamma_1^-).$$

In other words, the orthogonal projection $w$ of $v$ on $R$ (whose coordinates coincide with the coordinates of $v$, except for the first coordinate that is equal to $\gamma_1^-$ instead of $v_1$) satisfies $c \cdot w = \varepsilon^* - c_1 (v_1 - \gamma_1^-)$. As $c_1$ is non-zero and as $v_1 > \gamma_1^-$, we obtain $c \cdot w < \varepsilon^*$. It immediately follows that $w \notin Y^-$. Now assume that $g$ is at most $d - 2$. In this case, $G$ cannot be a facet of $Q$ and it is incident to at least one facet of $Q$ distinct from $R$. Since $v$ does not belong to any of the facets of $Q$ that contain $G$, its orthogonal projection $w$ on aff($R$) cannot belong to $G$. As a consequence $w$ does not belong to the affine hull of $F$. By construction, $w$ is a lattice point, and the lemma is proven.

We now state a theorem, obtained by combining Lemmas 3.1, 3.2, and 3.3, that will be used in the next section to prove the connectedness of $\Lambda(d,k)$.

**Theorem 3.4:** Call $g$ the maximal dimension of $F$ over all the possible choices for $R$ among the facets of $Q$. If $g$ is not greater than $d - 2$, then one can choose $R$ among the facets of $Q$ in such a way that $F$ has dimension $g$ and there exists a lattice point in $R \setminus \text{aff}(F)$ that can be inserted in $S$.

**Proof.** Assume that $g \leq d - 2$. If one can choose $R$ among the facets of $Q$ in such a way that $F$ is $g$-dimensional and $F \not\subset G$, then we consider any such facet for $R$ and pick, for $x$, any lattice point in $R$. By Lemma 3.1, $x$ cannot belong to aff($F$) or to $Y^-$ and, by Lemmas 2.1 and 2.2, it can be inserted in $S$.

Now assume that for any choice of $R$ among the facets of $Q$ such that $F$ has dimension $g$, $F \subset G$ but that for some such choice of $R$, the dimension of $F$ is less than the dimension of $G$. In this case, by Lemma 3.2, there exists a lattice point $x$ in $G$ that does not belong to aff($F$). As in addition, $Y^-$ is disjoint from $G$, it follows from Lemmas 2.1 and 2.2 that $x$ can be inserted in $S$. 

Finally, assume that for any choice of $R$ among the facets of $Q$ such that $F$ has dimension $g$, $F$ is a subset of $G$ and the dimensions of $F$ and $G$ coincide. By Lemma 3.3 one can choose $R$ such that $F$ has dimension $g$ and there exists a lattice point in $R$ that does not belong to $\text{aff}(F)$ or to $Y^-$. In this case, by Lemmas 2.1 and 2.2, $x$ can be inserted in $S$.

The following corollary shows that there is at least one lattice point in $[0,k]^d$ that can be inserted in $S$. The argument in this proof will be used again in the next section, in order to prove that $\Lambda(d,k)$ is always connected.

**Corollary 3.5:** For any positive $k$, an insertion move is possible on $S$ for at least one lattice point contained in the hypercube $[0,k]^d$.

**Proof.** If, for any possible choice of $R$ among the facets of $Q$, the dimension of $F$ is at most $d - 2$, then the result follows from Theorem 3.4. Assume that, for some facet $R$ of $Q$, $F$ has dimension $d - 1$. In this case, $Y$ is parallel to $R$ and $F^*$ is made up of a single vertex, say $v$. By Lemma 2.2, the intersection of $C_v$ with $Y$ is precisely $v$ and, for every vertex $u$ of $F$, $C_u(S)$ is disjoint from $Y$. Hence, by Lemma 2.1, any lattice point distinct from $v$ in $Y \cap [0,k]^d$ can be inserted in $S$. As $k \geq 1$, there exists at least one such lattice point.

### 4. The connectedness of $\Lambda(d)$ and $\Lambda(d,k)$

We first prove in this section that $\Lambda(2,k)$ is a connected graph. This will serve as the base case for the inductive proof that $\Lambda(d,k)$ is connected. In the whole section, we call *corner simplex* of $[0,k]^d$ the simplex whose vertices are the origin (the lattice point whose all coordinates are zero), and the $d$ lattice points in $[0,k]^d$ distant from the origin by exactly 1.

**Lemma 4.1:** For any positive $k$, the subgraph induced in $\Lambda(2,k)$ by the triangles and the quadrilaterals is connected.

**Proof.** Since each vertex of a quadrilateral can be deleted, we only need to show that any two triangles are in the same connected component of the subgraph of $\Lambda(2,k)$ induced by triangles and quadrilaterals. Consider a lattice triangle contained in the square $[0,k]^2$. If this triangle does not have a horizontal or a vertical edge then, by Theorem 3.4, an insertion move can be performed to transform it into a quadrilateral with a horizontal or a vertical edge, say $e$. It
is then possible to delete one of the vertices of this quadrilateral that is not incident to $e$ in order to obtain a triangle $T$ that admits $e$ as an edge. The strategy is then to transform $T$ into the corner triangle of $[0, k]^2$ using the sequence of moves sketched in Fig. 3. This figure shows the case when $e$ is the horizontal edge on the bottom of $T$. In each portion of the figure, the next point for which a move will be performed is colored green or red depending on whether the move is an insertion or a deletion. First observe that a lattice point in the line parallel to $e$ that contains the vertex of $T$ opposite $e$ can be inserted in order to obtain a quadrilateral with three horizontal or vertical edges as shown in the first two portions of Fig. 3. A deletion move for one of the vertices of the quadrilateral then results in a triangle with a horizontal and a vertical edge as shown in the center of Fig. 3. After that, the triangle has a unique oblique edge that faces one of the four vertices of the square $[0, k]^2$. It is always possible to make this edge face the vertex on the bottom left of the square by performing an insertion move to obtain a rectangle and then deleting the bottom-left vertex of the rectangle. This sequence of moves is illustrated in the third and fourth portions of Fig. 3 in the case when the oblique edge initially faces the top-left vertex of $[0, k]^2$. Finally, one can transform the resulting triangle $U$ into the corner triangle of $[0, k]^2$ (whose vertices are colored green on the right of Fig. 3) by inserting the vertices of the corner simplex one by one, and by deleting a vertex of $U$ after each insertion. Here, one just needs to take care to insert the origin of $\mathbb{R}^2$ first, and to delete the top-right vertex of $U$ last, in the case when $U$ has one or two of its vertices with a zero coordinate.

**Theorem 4.2:** For any positive $k$, the subgraph induced by simplices and polytopes with $d + 2$ vertices in $\Lambda(d, k)$ is connected.
Proof. The proof proceeds by induction on \( d \). The base case is provided by Lemma \ref{lem:4.1}. According to Lemma \ref{lem:2.3}, one can always transform a \( d \)-dimensional polytope with \( d+2 \) vertices into a lattice simplex by a deletion move. Therefore, we only need to prove that two simplices always are in the same connected component of the subgraph induced in \( \Lambda(d, k) \) by simplices and polytopes with \( d+2 \) vertices. The strategy will be, again, to transform any simplex in this graph into the corner simplex of \([0, k]^d\). Assume that \( d \geq 3 \). Consider a lattice simplex \( S \) contained in \([0, k]^d\), and call \( V \) the vertex set of \( S \).

As in the previous section, for any \( i \in \{1, ..., d\} \), we call \( \gamma_i^- = \min\{x_i : x \in S\} \) and \( \gamma_i^+ = \max\{x_i : x \in S\} \), and we consider the combinatorial cube

\[
Q = \prod_{i=1}^{d} [\gamma_i^-, \gamma_i^+].
\]

Call \( g \) the maximal dimension of the intersection of \( S \) and a facet of \( Q \). If \( g \) is at most \( d - 2 \) then, by Theorem \ref{thm:3.4} there exists a facet \( R \) of \( Q \) such that \( S \cap R \) is \( g \)-dimensional and a lattice point \( x \in R \setminus \aff(S \cap R) \) that can be inserted in \( S \). Consider the polytope \( P \) obtained by inserting \( x \) in \( S \). The intersection \( P \cap R \) is a simplex because \( x \not\in \aff(S \cap R) \). As a consequence, \( P \cap R \) is a face of at least one \( d \)-dimensional simplex that can be obtained by deleting a vertex from \( P \). The intersection of this simplex with \( R \) is equal to \( P \cap R \) and therefore has dimension \( g + 1 \). Repeating this procedure provides a sequence of insertion and deletion moves that transform \( S \) into a lattice simplex whose intersection \( F \) with a facet \( R \) of \( Q \) has dimension \( d - 1 \).

Call \( v \) the unique vertex of the lattice simplex that does not belong to \( R \). Observe that, in this case, any sequence of insertion and deletion moves that can be performed on \( F \) within the cube \( \aff(R) \cap [0, k]^d \) can also be performed within \([0, k]^d\) for the pyramid with apex \( v \) over \( F \). By induction, one can transform \( F \) into any lattice simplex contained in the intersection \( \aff(R) \cap [0, k]^d \) by carrying out an alternating sequence of insertion and deletion moves in this intersection. This sequence of moves can therefore be performed in order to transform \( S \) into the \( d \)-dimensional lattice simplex \( S' \) whose vertex set is made up of \( v \), of the lattice point \( w \) in \( \aff(R) \cap [0, k]^d \) with a unique non-zero coordinate, and of the \( d - 1 \) lattice points in \( \aff(R) \cap [0, k]^d \) distant by exactly 1 from \( w \).
Now observe that one can perform an insertion move on any lattice point distinct from $v$ in the intersection of $[0,k]^d$ with the hyperplane parallel to $R$ that contains $v$. We proceed by inserting the lattice point in this intersection whose orthogonal projection on $R$ is $w$ and then, by deleting $v$. Calling $v'$ any of the $d-1$ lattice points in $\text{aff}(R) \cap [0,k]^d$ distant from $w$ by exactly 1, the simplex that results from the latter deletion is a pyramid with apex $v'$ over a $(d-1)$-dimensional simplex $F'$ such that, for some $i \in \{1, \ldots, d\}$, $v'$ satisfies $v'_i = 1$ and every vertex $u$ of $F'$ satisfies $u_i = 0$. Call $R'$ the facet of $[0,k]^d$ made up of the points $x$ such that $x_i = 0$. By induction, one can transform $F'$ within $R'$ into the corner simplex of $R'$. From there, one can perform an insertion move on any lattice vertex distinct from $v'$ in the intersection of $[0,k]^d$ with the hyperplane parallel to $R'$ that contains $v'$. We insert the lattice point in this intersection whose orthogonal projection on $R'$ is the origin. Since $v'_i = 1$, the $i$-th coordinate of the inserted point is 1, and its other coordinates are all equal to 0. Hence, after a last deletion move on $v'$, the resulting simplex is the corner simplex of $[0,k]^d$, which completes the proof.

Combining Lemma 2.3 and Theorem 4.2, we get the following.

**Corollary 4.3:** For any positive $k$, $\Lambda(d,k)$ is connected.

We now turn our attention to the connectedness of $\Lambda(d)$.

**Theorem 4.4:** For any $d \geq 2$, $\Lambda(d)$ is connected.

**Proof.** Note that the translations of $Z^d$ by lattice vectors induce automorphisms of $\Lambda(d)$. Since two $d$-dimensional lattice polytopes contained in $\mathbb{R}^d$ can always be displaced into the hypercube $[0,k]^d$ for some large enough $k$ by a such a translation, they both belong to a subgraph of $\Lambda(d)$ isomorphic to $\Lambda(d,k)$. The result therefore follows from Corollary 4.3.

5. The number of possible insertion and deletion moves

One of the main purposes of this section is to study the subgraphs induced in $\Lambda(d)$ by the polytopes with $n$ and $n+1$ vertices when $n > d+1$. In Section 2, we have shown that the corresponding subgraphs of $\Gamma(d)$ are always connected, and in Section 4 that the subgraph induced in $\Lambda(d)$ by the simplices and the polytopes with $d+2$ vertices is also always connected. Here, we will exhibit a
family of polytopes whose dimension and number of vertices can be arbitrarily large, but in which no lattice point can be inserted. As a consequence, there is no hope for the subgraphs induced in \( \Lambda(d) \) by the polytopes with \( n \) and \( n + 1 \) vertices to be connected for all \( d \) and \( n \). We will also study the vertex degrees in \( \Lambda(d, k) \). These degrees are bounded above by \((k + 1)^d\), the number of lattice points in \([0, k]^d\). This bound is obviously sharp when \( k = 1 \) since all the lattice points in \([0, 1]^d\) can be deleted from the hypercube itself. We will show that this bound is also sharp when both \( d \) and \( k \) grow large by exhibiting an extensive family of \( d \)-dimensional lattice polytopes contained in \([0, k]^d\) such that every lattice point in \([0, k]^d\) can either be inserted in or deleted from these polytopes.

We first prove the following about \( \Lambda(2) \). Thereafter, by a unit square we mean the square \([0, 1]^2\) or any of its translates by a lattice vector.

**Lemma 5.1:** For any \( n > 3 \) such that \( n \neq 5 \), there exists a lattice polygon \( P \subset \mathbb{R}^2 \) with \( n \) vertices such that no point of \( \mathbb{Z}^2 \) can be inserted in \( P \).

**Proof.** First observe that if \( P \) is a unit square, then every point in the lattice \( \mathbb{Z}^2 \) is contained in the cone \( C_v(P) \), where \( v \) is one of the four vertices of \( P \). It then follows from Lemma 2.1 that no point of \( \mathbb{Z}^2 \) can be inserted in \( P \), which proves the lemma when \( n = 4 \).

Now assume that \( n \geq 6 \) and consider the map

\[ f : x \mapsto x(x - 1)/2. \]

Let \( A \) be the set of the points \( x \) in \( \mathbb{Z}^2 \) such that \( x_2 = f(x_1) \). Note that these points are the vertices of a convex polygonal line. We are going to build a polygon \( P_n \) from this polygonal line.

First assume that \( n \) is even and consider the point \( a \) satisfying

\[ a_1 = \frac{n}{2} - 1 \quad \text{and} \quad a_2 = f(a_1) + 1. \]

Let \( P_n \) denote the polygon whose vertices are the elements \( x \) in \( A \) such that \( 0 \leq x_1 < n/2 \) and their symmetric with respect to the point \( a/2 \). This polygon is depicted in Fig. 4 when \( n \) is equal to 6 (left), 8 (center), and 10 (right). By construction, \( P_n \) is centrally-symmetric and its centroid is \( a/2 \). Note that it has \( n \) vertices, half of whose belong to \( A \). Further note that \( a \) and the point \( b \) such that \( b_1 = a_1 - 1 \) and \( b_2 = a_2 \) are the two vertices of an horizontal edge of \( P_n \).

In the figure, the portion of \( \mathbb{R}^2 \) covered by the cones \( C_v(P_n) \), where \( v \) ranges over the vertices of \( P_n \), is colored red. Observe that the portion of \( \mathbb{R}^2 \) that is
not covered by $P_n$ or by any of the cones $C_v(P_n)$, where $v$ is a vertex of $P_n$, is the union of the interiors of a set of triangles colored white in the figure. By construction, each of these triangles is contained in the region of $\mathbb{R}^d$ made up of the points $x$ such that $i \leq x_1 \leq i + 1$, for some integer $i$. Hence, the interiors of these triangles entirely avoid the lattice $\mathbb{Z}^2$. It then follows from Lemma 2.1 that no point in $\mathbb{Z}^2$ can be inserted in $P_n$.

Now assume that $n$ is odd. Let $P_n$ be the polygon obtained as the convex hull of $P_{n+1}$ and of the point $c$ such that $c_1 = a_1$ and $c_2 = a_2 + 1$. The polygon $P_n$ is depicted in Fig. 4 when $n$ is equal to 7 (second polygon from the left) and 9 (next-to-last polygon). While $c$ is a vertex of $P_n$, $a$ and $b$ are no longer vertices of it because they are contained in the relative interiors of the edges of $P_n$ incident to $c$. As $P_{n+1}$ shares all its vertices with $P_n$ except for $a$ and $b$, $P_n$ has exactly $n$ vertices (one less than $P_{n+1}$). As above, the portion of $\mathbb{R}^2$ that is not covered by $P_n$ or by any of the cones $C_v(P_n)$, where $v$ is a vertex of $P_n$, is the union of the interiors of a set of triangles. As we have seen above, the interiors of all these triangles are disjoint from $\mathbb{Z}^2$, except possibly for the two triangles incident to $c$, that have been introduced when building $P_n$ from $P_{n+1}$. Among these two triangles, the one depicted on top of the polygon in Fig. 4 does not depend on $n$, and it can be seen in the figure that its interior is disjoint from $\mathbb{Z}^2$. The other triangle depends on $n$. As can be seen in the figure, its interior is disjoint from $\mathbb{Z}^2$ when $n = 7$. When $n \geq 9$, this triangle is contained in the region of $\mathbb{R}^d$ made up of the points $x$ such that $n/2 - 1 \leq x_1 \leq n/2$, and its interior is therefore also disjoint from $\mathbb{Z}^2$. Hence, by Lemma 2.1 no point of the lattice can be inserted in $P_n$. \[\blacksquare\]
As an immediate consequence of Lemma 5.1, the subgraph induced in $\Lambda(2)$ by the polygons with $n$ or $n+1$ vertices is disconnected when $n$ is greater than 3 but distinct from 5. The exception for $n = 5$ is intriguing, and it may be that the subgraph is connected in this case. We will now generalize Lemma 5.1 by showing that there are lattice polytopes of arbitrarily large dimension whose number of vertices is also arbitrarily large such that no lattice point can be inserted. We will first need the following lemma.

**Lemma 5.2:** Let $P$ and $Q$ be two polytopes. If $u$ and $v$ are a vertex of $P$ and a vertex of $Q$, respectively, then $C_{u \times v}(P \times Q)$ coincides with $C_u(P) \times C_v(Q)$.

**Proof.** Consider two polytopes $P$ and $Q$ which, we assume are a $p$-dimensional and a $q$-dimensional polytope contained in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Let $u$ be a vertex of $P$ and $v$ a vertex of $Q$. The facets of $P \times Q$ incident to $u \times v$ are precisely the cartesian products of the form $F \times Q$ where $F$ is a facet of $P$ incident to $u$ and $P \times G$ where $G$ is a facet of $Q$ incident to $v$.

In particular, if $F$ is a facet of $P$ incident to $u$, then

$$H_{F \times Q}(P \times Q) = H_F^{-}(P) \times \mathbb{R}^q.$$  

Similarly, if $G$ is a facet of $Q$ incident to $v$, then

$$H_{P \times F}(P \times Q) = \mathbb{R}^p \times H_G^{-}(Q).$$

As a consequence, for any point $x$ in $C_u(P)$ and any point $y$ in $C_v(Q)$, $x \times y$ is contained both in $H_{F \times Q}^{-}(P \times Q)$ and in $H_{P \times G}^{-}(P \times Q)$. Inversely, if $x$ and $y$ are two points in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, such that $x \times y$ is contained in $C_{u \times v}(P \times Q)$, then $x$ necessarily belongs to $H_F^{-}(P)$ and $y$ to $H_G^{-}(Q)$. Since these two statements hold for any facet $F$ of $P$ incident to $u$ and any facet $G$ incident to $v$, we obtain the desired equality. 

We prove the following result by considering cartesian products of polygons and hypercubes, for which no insertion of a lattice point is possible.

**Theorem 5.3:** For all $n > 3$ such that $n \neq 5$, and for all $d \geq 4$, there exists a $d$-dimensional lattice polytope $P$ contained in $\mathbb{R}^d$ with $n2^{d-2}$ vertices such that no point in the lattice $\mathbb{Z}^d$ can be inserted in $P$.

**Proof.** Let $n$ be an integer greater than 3 and distinct from 5. By Lemma 5.1 there exists a lattice polygon $Q$ with $n$ vertices such that no point of $\mathbb{Z}^2$ can be inserted in $Q$. Now assume that $d \geq 4$, and recall that no point in $\mathbb{Z}^{d-2}$
can be inserted in the hypercube $[0,1]^{d-2}$. By Lemmas 2.1 and 5.2 no point in $\mathbb{Z}^d$ can be inserted in $Q \times [0,1]^{d-2}$. This cartesian product is a $d$-dimensional lattice polytope with $n2^{d-2}$ vertices, as desired.

We now turn our attention to another interesting family of lattice polytopes. These polytopes play a peculiar role in the graph $\Lambda(d,k)$: they are the polytopes $P$ such that all the lattice points in $[0,k]^d$ can either be inserted in $P$ or deleted from it. When $k = 1$, these polytopes admit a straightforward characterization: they are the $d$-dimensional polytopes contained in $[0,1]^d$ that are not pyramids over any of their facets. For this reason, we will assume that $k \geq 2$ in the remainder of the section. The polytopes we are looking for are necessarily empty lattice polytopes in the sense that their intersection with $\mathbb{Z}^d$ is precisely their vertex set. We will build them as cartesian products of empty simplices due to Bárany and Sebő (see [15, 22]), such as the empty tetrahedron depicted in Fig. 5 inside the cube $[0,2]^3$. Note that the property we are investigating here cannot carry over to the whole lattice $\mathbb{Z}^d$. Indeed, given a lattice polytope $P$ there is always a point in $\mathbb{Z}^d$ that cannot be inserted in or deleted from $P$: any lattice point in the affine hull of an edge of $P$ will have this property as soon as it is distinct from the two extremities of this edge.

**Theorem 5.4:** Consider an integer $k \geq 2$. If $k + 1$ is a proper divisor of $d$, then there exists a $d$-dimensional lattice polytope $P$ with $(k+2)^{d/(k+1)}$ vertices contained in $[0,k]^d$ such that, for any lattice point $x$ in $[0,k]^d$, $x$ can either be inserted in $P$ or removed from it.

**Proof.** Consider the following matrix with $k + 2$ columns and $k + 1$ rows:

$$
\begin{bmatrix}
k & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & k & 1 & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & k & \ddots & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & k & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & k & 1
\end{bmatrix}
$$

Denote by $a(i)$ the point of $\mathbb{R}^{k+1}$ whose vector of coordinates is the $i$-th column of this matrix. These points are the vertices of a $(k+1)$-dimensional empty simplex $S$ [15, 22]. When $k$ is equal to 2, $S$ is the empty tetrahedron.
depicted in Fig. 5 inside the cube $[0,2]^3$. Observe that each of the vertices of $S$ is either a vertex of the hypercube $[0,k]^{k+1}$ or contained in an edge of this hypercube: $a(1)$ is a vertex of $[0,k]^{k+1}$ and, when $2 \leq i \leq k+2$, $a(i)$ is contained in the edge of $[0,k]^{k+1}$ whose two vertices are obtained from $a(i)$ by replacing its $(i-1)$-th coordinate by 0 or $k$. Further observe that each of these edges contains exactly one vertex of $S$. In other words, whenever $1 \leq i \leq k+2$, we can find a face $F$ of $[0,k]^{k+1}$ that contains $a(i)$ and no other vertex of $S$. Consider an hyperplane $H$ of $\mathbb{R}^{k+1}$ whose intersection with $[0,k]^{k+1}$ is $F$. Denote by $H^-$ the closed half-space of $\mathbb{R}^{k+1}$ bounded by $H$ such that $[0,k]^{k+1} \cap H^- = F$. Since $a(i)$ is the only vertex of $S$ contained in $F$, then $S \cap H = \{a(i)\}$. By Lemma 2.2, the intersection of $[0,k]^{k+1}$ with $C_{a(i)}(S)$ is therefore precisely $\{a(i)\}$. According to Lemma 2.1 all the lattice points in $[0,k]^{k+1}$ can be inserted in $S$, except for the vertices of $S$.

Now assume that $k+1$ is a divisor of $d$ and denote by $P$ the cartesian product $S^{d/(k+1)}$. It follows from Lemma 5.2 that every lattice point in $[0,k]^d$ can be inserted in $P$ except for the vertices of $P$. Consider a facet $G$ of $P$. This facet is obtained as the cartesian product of $d/(k+1) - 1$ copies of $S$ with a facet $F$ of $S$. If $F$ is the $j$-th term of the product, the vertices of $P$ that are not incident to $G$ are precisely the cartesian products of $d/(k+1) - 1$ (possibly not pairwise distinct) vertices of $S$ such that the $j$-th term in the product is equal to the vertex of $S$ not incident to $F$. Since $k+1$ is a proper divisor of $d$, then there are several such vertices and $P$ cannot be a pyramid over any of its facets. As a consequence, all the vertices of $P$ can be deleted.

Theorem 5.4 does not hold in dimension 2.
Theorem 5.5: Consider a positive integer \( k \geq 2 \). If \( P \) is a lattice polygon contained in \([0, k]^2\), then there exists a lattice point in \([0, k]^2\) that cannot be inserted in \( P \) or a vertex of \( P \) that cannot be deleted from \( P \).

Proof. Consider a lattice polygon \( P \) contained in \([0, k]^2\). Denote by \( a \) and \( b \) two vertices of \( P \) whose distance is the largest possible. Note that the distance of \( a \) and \( b \) is then at least \( \sqrt{2} \). In particular, if \( a_1 = b_1 \) or if \( a_2 = b_2 \), then the convex hull of \( a \) and \( b \) contains at least one lattice point in its interior. This lattice point cannot be inserted in or deleted from \( P \), and the theorem holds in this case. In the following we assume that \( a_1 \neq b_1 \) and \( a_2 \neq b_2 \). Using the symmetries of the lattice, we can assume without loss of generality that \( a_i < b_i \) when \( i \in \{ 1, 2 \} \). Consider the rectangle \([a_1, b_1] \times [a_2, b_2]\).

First assume that \( P \) has a vertex \( c \) outside of the rectangle \([a_1, b_1] \times [a_2, b_2]\). Taking advantage of the symmetries of that rectangle, we can assume without loss of generality that \( c_1 > b_1 \). In this case, \( c_2 \) is necessarily less than \( b_2 \) because \( c \) is at most as distant from \( a \) than \( b \). If \( a_2 \leq c_2 < b_2 \), then the lattice point \( x \) such that \( x_1 = b_1 \) and \( x_2 = c_2 \) is in the triangle with vertices \( a \), \( b \), and \( c \) and it is distinct from its three vertices. Hence, \( x \) cannot be inserted in or deleted from \( P \) because it is contained in \( P \) and it is distinct from all the vertices of \( P \). If \( c_2 < a_2 \), then the lattice point \( x \) such that \( x_1 = b_1 \) and \( x_2 = a_2 \) is in the interior of the triangle with vertices \( a \), \( b \), and \( c \). As above, this point cannot be inserted in \( P \) or deleted from it, proving the theorem in this case.

Now assume that \( P \) has a vertex \( c \) distinct from \( a \) and \( b \) inside the rectangle \([a_1, b_1] \times [a_2, b_2]\). If \( c \) is in some edge of that rectangle, then \( P \) has a horizontal or a vertical edge. None of the lattice points in \([0, k]^2\) that belong to the affine hull of that edge can be inserted in \( P \) and, since \( k \geq 2 \), at least one of these lattice points is not a vertex of \( P \). This point therefore cannot be inserted in or deleted from \( P \), and the theorem holds in this case. Finally, if \( c \) is in \([a_1, b_1] \times [a_2, b_2]\), we can assume without loss of generality that \( c \) is below the affine hull of \( a \) and \( b \). In this case, \( C_c(P) \) contains all the points \( x \) such that \( x_1 \geq c_1 \) and \( x_2 = c_2 \). In particular, the point \( x \) such that \( x_1 = b_1 \) and \( x_2 = c_2 \) belongs to the interior of this cone, which completes the proof.
6. Discussion and open problems

We have introduced a graph structure on the $d$-dimensional polytopes contained in $\mathbb{R}^d$. We have proven, among other things, that this graph is connected, as well as its subgraph induced by lattice polytopes. The distances in this graph provide a measure of dissimilarity on polytopes in terms of how long it is to transform two of them into one another by a sequence of elementary moves. This allows to gather in a coherent metric structure very different objects from both the geometric and the combinatorial point of view.

This structure, and the results we obtained open up several new questions. For instance, recall that the subgraph induced in $\Gamma(d)$ by the polytopes with $n$ or $n+1$ vertices is always connected. We propose to investigate the subgraphs of $\Gamma(d)$ such that moves are allowed when a quantity other than the number of vertices is almost constant. In particular we ask the following.

Question 6.1: Consider a non-trivial interval $I \subset ]0, +\infty[$. Is the subgraph induced in $\Gamma(d)$ by the polytopes whose volume belongs to $I$ connected?

Note that other measures than the volume of the polytope can be considered as well in Question 6.1, and possibly several of them simultaneously (for instance, the volume and the number of vertices).

The main results in this article deal with lattice polytopes. These polytopes are often constrained to be contained in a hypercube $[1, 9, 10, 11, 17, 19]$. In this case, they form a nice (even if elusive) combinatorial class. The connectedness of $\Lambda(d, k)$ makes it possible to define a Markov chain on this combinatorial class whose stationary distribution is uniform $[8]$. Some authors have considered lattice polytopes contained in a ball $[3]$ or in some arbitrary lattice polytope $[23]$. Pursuing this idea, we ask the following.

Question 6.2: For what balls $B$ is the subgraph induced in $\Lambda(d)$ by the polytopes contained in $B$ connected? For what lattice polytopes $P$ is the subgraph induced in $\Lambda(d)$ by the polytopes contained in $P$ connected?

Another graph that we have obtained results on is the subgraph induced in $\Lambda(d)$ by the polytopes with $n$ or $n+1$ vertices. This graph exhibits an intriguing behavior: by Theorem 4.2 it is connected when $n = d+1$ and $d \geq 2$. However, according to Lemma 5.1 and to Theorem 5.3, this graph is disconnected when $d = 2$ and $n \not\in \{3, 5\}$, and for arbitrarily large values of $d$ and $n$. The exception
for $n = 5$ in Lemma 5.1 is intriguing and it seems plausible that the subgraph is connected in this case. We ask the following.

**Question 6.3:** For what values of $d$ and $n$ are the subgraphs induced in $\Lambda(d)$ by the polytopes with $n$ or $n + 1$ vertices connected? Is the subgraph induced by pentagons and hexagons in $\Lambda(2)$ connected?

In order to prove that some subgraphs induced in $\Lambda(d)$ by the polytopes with $n$ or $n + 1$ vertices are disconnected, we have used lattice polytopes with $n$ vertices such that no insertion move is possible. In other words, these polytopes are isolated vertices in the considered subgraph. We ask whether isolated vertices are the single cause for the disconnectedness of these graphs.

**Question 6.4:** Consider an integer $n \geq d + 1$. Is the graph obtained by removing the isolated vertices from the subgraph induced in $\Lambda(d)$ by the polytopes with $n$ or $n + 1$ vertices connected?

We ask two more questions on the structure of our graphs that are not directly related to the results we obtained in this article.

**Question 6.5:** Do the graphs $\Lambda(d, k)$, where either the value of $d$ or that of $k$ is fixed, form a family of expanders?

**Question 6.6:** What are the chromatic numbers of $\Gamma(d)$, $\Lambda(d)$, and $\Lambda(d, k)$?

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