INHOMOGENEOUS BOUNDARY VALUE PROBLEM FOR HARTREE TYPE EQUATION

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Abstract. In this paper, we settle the problem for time-dependent Hartree equation with inhomogeneous boundary condition in a bounded Lipschitz domain in $\mathbb{R}^N$. A global existence result is derived.

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1. Introduction

In this paper, we are concerned with the following Hartree type equation posed on a bounded smooth domain $\Omega$ of $\mathbb{R}^N$:

$$
\begin{cases}
    iu_t = \Delta u - \lambda \frac{|x|}{|x|^2} \ast |u|^2 u, & x \in \Omega, t \in \mathbb{R} \\
    u(x, 0) = \phi(x), & x \in \Omega, t \in \mathbb{R} \\
    u(x, t) = Q(x, t), & (x, t) \in \partial \Omega \times \mathbb{R},
\end{cases}
$$

where $\phi \in H^1(\Omega)$, $\lambda > 0$, and $Q \in C^3(\partial \Omega \times (-\infty, +\infty))$ has compact support and satisfy the compatibility condition on $\partial \Omega$ in the sense of trace. Here we denote by

$$
\frac{1}{|x|} \ast |u|^2 = \int_{\Omega} \frac{1}{|x-y|} |u|^2(y) dy.
$$

We now normalize the constant $\lambda = 1$.

There is a very large literature on the nonlinear Hartree equation in the whole space $\mathbb{R}^N$, for instance, see ([4]—[16]). One of the interesting results concerning existence and uniqueness of $H^1$ global solutions to Cauchy problem is the paper of Chadam and Glassey [6], where they even show that there exists an unique global solution to the Cauchy problem of the Hartree system with an external Coulomb potential. For small data, the scattering of the global solution is proved in [11]. Recently, $H^1(\mathbb{R}^N)$ critical Hartree equation is settled and global well-posedness and scattering results are obtained in [16]. Stationary solutions of these equations are of great interest to people in the past years. We can solve the remaining problem of E.Lieb that the positive solutions to the Hartree equation are minimizers of the corresponding functional in [15]. However, we are aware that very few authors are concerned with the corresponding problems for Hartree equation in a bounded domain in $\mathbb{R}^N$ with inhomogeneous boundary condition. It is well-known that inhomogeneous boundary value problems for Hartree type Schrödinger equations have physical implications. For instance, in one space dimension, such problem are often called forced problem when an external force is applied to the time evolution.
of systems. Frequently the forcing is put in as a boundary condition. This is the motivation for us to consider the problem in this paper.

There are also relatively a few results for non-linear Schrödinger equations in a bounded domain in $\mathbb{R}^N$ with inhomogeneous boundary condition. When space dimension $N > 1$, Strauss and Bu [17] obtained the global existence of a $H^1$ solution to such Schrödinger equation if the nonlinear term contributes a positive term to the energy. In [18], Tsutsumi obtained interesting result for nonlinear Schrödinger equation on exterior domains.

In this paper, we establish a global existence result for (1.1) borrowing an idea from the work [17]. For (1.1), the conservation of mass and energy do not holds. A direct computation shows that the growths of the $L^2$ norm and energy take the forms

$$\frac{d}{dt} \int_\Omega |u|^2 dx = 2 Im \int_{\partial \Omega} \bar{u} \frac{\partial u}{\partial n} dS \quad (1.2)$$

$$\frac{d}{dt} E(u) = Re \int_{\partial \Omega} \bar{u} \frac{\partial u}{\partial n} dS \quad (1.3)$$

where

$$E(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 + \frac{1}{4} \frac{1}{|x|} * |u|^2 |u|^2 dx \quad (1.4)$$

At first it seems not easy to make use of such identities. Of course, reducing the problem into homogeneous one does not work well too. Using more identities derived from the Hartree equation, we can set up the following existence result.

**Theorem 1.** Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$. Assume $\phi \in H^1(\Omega)$, $Q \in C^3(\partial \Omega \times (-\infty, +\infty))$ have compact support and $\phi(x) = Q(x, 0)$ on $\partial \Omega$ in the sense of trace. Then there exists a unique solution $u \in C^{1}((-\infty, +\infty), H^1(\Omega))$ to (1.1).

The paper is organized as follows. In the next section, we introduce some notations and a basic extension result from Sobolev spaces. Section 3 is devoted to apriori estimates of the solution of (1.1). The local and global existence results are concerned in Section 4.

2. Notations and an extension Lemma

In this section, we introduce some notations that will be used throughout the paper.

Assume there exists a smooth real-valued function $\xi$ satisfying

$$\xi_{|\partial \Omega} = (\xi_1, \cdots, \xi_n) = n$$

where $n$ is the standard outer normal vector for $\partial \Omega$.

Denote by

$$\partial_j = \frac{\partial}{\partial x_j},$$

$$\eta = \sum_j \partial_j \xi_j,$$

and

$$P = \nabla u_{|\partial \Omega}.$$
We also let
\[ \text{Re} = \text{the real part}, \quad \text{Im} = \text{the imaginary part}. \]

We recall a well-known extension result which will play an important role for apriori estimates later.

**Lemma 2.** Assume \( \Omega \) is bounded domain in \( \mathbb{R}^N \) such that \( \partial \Omega \) is Lipschitz, then there exists a bounded linear operator
\[ E : H^1(\Omega) \rightarrow H^1(\mathbb{R}^N) \]
such that for each \( u \in H^1(\Omega) \),
\[ E(u) = u \text{ a.e. in } \Omega; \quad \|E(u)\|_{H^1(\mathbb{R}^N)} \leq C \|u\|_{H^1(\Omega)}. \]

For the proof of this result, we refer to the books [1] and [9].

### 3. Apriori Estimates

We now employ the idea [17] to establish apriori estimates for the smooth solution \( u \) of (1.1).

**Lemma 3.** If \( u \) is a smooth solution of (1.1), then \( u \) satisfies the following three identities:

1. \( \frac{d}{dt} \left( \int_\Omega |u|^2 \, dx \right) = 2 \text{Im} \int_{\partial \Omega} \overline{Q} (P \cdot n) \, dS, \)
2. \( \frac{d}{dt} \left( \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{4} \int_\Omega \frac{1}{|x|} \ast |u|^2 |u|^2 \, dx \right) = \text{Re} \int_{\partial \Omega} (P \cdot n) \overline{Q} \, dS, \)
3. \( \frac{d}{dt} \int_\Omega u (\xi \cdot \nabla u) = 2i \text{Re} \sum_{j,k} \int_\Omega \partial_k \xi_j \partial_j \overline{u} \partial_k u + i \int_\Omega \nabla \eta \cdot \nabla u \)
\[ + i \sum_j \text{Re} \int_\Omega \frac{x_j}{|x|^3} \ast |u|^2 \xi_j |u|^2 + i \int_{\partial \Omega} \frac{1}{|x|} \ast |u|^2 |Q|^2 \, dS + i \int_{\partial \Omega} |P|^2 \, dS \]
\[ - 2i \int_{\partial \Omega} |P \cdot n|^2 \, dS + \int_{\partial \Omega} Q \overline{Q} \, dS - i \int_{\partial \Omega} P \cdot n \eta Q \, dS. \]

**Proof:** To prove (3.1), we multiply the equation in (1.1) by \( 2i \) and then take the imaginary part to obtain
\[ \frac{\partial}{\partial t} |u|^2 = 2 \text{Im} \nabla \cdot (\nabla u). \]
Integrating over \( \Omega \) yields to
\[ \frac{d}{dt} \int_\Omega |u|^2 = \int_{\partial \Omega} 2 \text{Im} \nabla u \cdot n \overline{u} \, dS = 2 \text{Im} \int_{\partial \Omega} \overline{Q} (P \cdot n) \, dS. \]
Similarly, multiplying the equation in (1.1) by \( 2i_t \) and then taking the real part, we have
\[ 2 \text{Re} \nabla \cdot (\nabla u \overline{u}_t) - \frac{\partial}{\partial t} |\nabla u|^2 + \frac{1}{|x|} \ast |u|^2 \frac{\partial}{\partial t} |u|^2 = 0. \]
Integrating over \( \Omega \) to obtain
\[ \frac{d}{dt} \int_\Omega |\nabla u|^2 + \int_\Omega \frac{1}{|x|} \ast |u|^2 \frac{\partial}{\partial t} |u|^2 = 2 \text{Re} \int_{\partial \Omega} P \cdot n \overline{Q} \, dS. \]
Since
\[ \int_{\Omega} \frac{1}{|x|} * |u|^2 \frac{\partial}{\partial t} |u|^2 = \frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{1}{|x|} * |u|^2 |u|^2, \]
thus identity (3.2) follows.

To establish (3.3), noticing
\[ \int_{\Omega} \nabla \cdot (u \xi \bar{u}) = \int_{\partial \Omega} u \xi \partial_n \bar{u} - \bar{u} \cdot \partial_j u \xi_j + \int_{\partial \Omega} \bar{Q} \xi dS + \int_{\Omega} u \xi \cdot \nabla \bar{u} + \int_{\partial \Omega} Q \xi dS, \]
we get
\[ \frac{d}{dt} \int_{\Omega} u \xi \cdot \nabla \bar{u} = \int_{\Omega} u \xi \cdot \nabla \bar{u} + \int_{\Omega} u \xi \cdot \nabla \bar{u} \]
(3.4)
\[ = \int_{\partial \Omega} u \xi \partial_n \bar{u} - \bar{u} \cdot \partial_j u \xi_j + \int_{\partial \Omega} \bar{Q} \xi dS - \int_{\Omega} \eta \bar{u}. \]
Substitute the equation in (1.1) into (3.4), we obtain
\[ \frac{d}{dt} \int_{\Omega} u \xi \cdot \nabla \bar{u} = -2iRe \sum_j \int_{\Omega} \Delta u \partial_j \bar{u} \xi_j + iRe \sum_j \int_{\Omega} \frac{1}{|x|} \circ |u|^2 \xi_j \partial_j |u|^2 \]
(3.5)
\[ + \int_{\partial \Omega} \bar{Q} \xi dS - \int_{\Omega} \eta \bar{u}. \]
Note that
\[ -2iRe \int_{\Omega} \Delta u \partial_j \bar{u} \xi_j = -2iRe \int_{\Omega} \sum_k \partial^2_k u \partial_j \bar{u} \xi_j \]
\[ = 2iRe \sum_k \int_{\Omega} \partial_k (\partial_j \bar{u} \xi_j) \partial_k u - 2iRe \sum_k \int_{\partial \Omega} \partial_j \bar{u} \bar{u} \partial_k u |n_k| \]
\[ = -iRe \int_{\Omega} \partial_j \xi_j |\nabla u|^2 + iRe \int_{\partial \Omega} n^2_j |P|^2 |dS \]
\[ + 2iRe \sum_k \int_{\Omega} \partial_k \xi_j \partial_j P \partial_k u - 2iRe \sum_k \int_{\partial \Omega} \partial_j \bar{u} \partial_k u |n_k| dS. \]
Add \( j = 1, \ldots, N \) to obtain
\[ -2i \sum_j Re \int_{\Omega} \Delta u \partial_j \bar{u} \xi_j = -i \int_{\Omega} \eta |\nabla u|^2 + 2iRe \sum_{j,k} \int_{\Omega} \partial_k \xi_j \partial_j P \partial_k u \]
\[ + i \int_{\partial \Omega} |P|^2 |dS - 2i \int_{\partial \Omega} |P \cdot n|^2 |dS. \]
Similarly,
\[ i \sum_j Re \int_{\Omega} \frac{1}{|x|} \circ |u|^2 \xi_j \partial_j |u|^2 = iRe \sum_j \int_{\Omega} \frac{1}{|x|} \circ |u|^2 \xi_j |u|^2 \]
\[ + i \int_{\partial \Omega} \frac{1}{|x|} \circ |u|^2 |Q|^2 |dS - i \int_{\partial \Omega} \frac{1}{|x|} \circ |u|^2 |u|^2. \]
Taking the conjugate of the equation in (1.1), multiplying by $\eta \overline{\eta}$ and integrating over $\Omega$ yields to
\[
\int_\Omega \eta u \overline{\eta} = \int_\Omega \eta u \Delta \overline{\eta} - \int_\Omega \eta \left| \frac{1}{|x|} \right| * |u|^2 |u|^2
- \int_\Omega \nabla \eta \nabla u - \int_\Omega \nabla u |^2 + i \int_{\partial \Omega} \overline{\Psi} \cdot \eta u \overline{Q} dS - \int_\Omega \eta \left| \frac{1}{|x|} \right| * |u|^2 |u|^2.
\]
Substitute the above results into (3.4), the identity (3.3) follows.

4. Local and Global Existence

Denote by $f(u) = \frac{1}{|x|} * |u|^2 = \int_\Omega \frac{|u|^2(y)}{|x-y|} dy.$

For Hartree equation (1.1), we need the following Lemma dealing with the Hartree nonlinearity term.

Lemma 4. There exists a constant $C > 0$ independent of $\|v\|_{H^1(\Omega)}$ and $\|w\|_{H^1(\Omega)}$ such that
\[
\|f(v) - f(w)\|_{H^1(\Omega)} \leq C(\|v\|^2_{H^1(\Omega)} + \|w\|^2_{H^1(\Omega)}) \|v - w\|_{H^1(\Omega)}
\]

Proof: Apply Hardy’s inequality in $\mathbb{R}^N$
\[
\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \leq C \|\nabla u\|_{L^2(\mathbb{R}^N)}^2, \forall u \in H^1
\]
and the extension lemma to obtain
\[
\|f(u)\|_{L^\infty(\Omega)} = \sup_x \int_\Omega \frac{|u|^2(y)}{|x-y|} dy \leq \int_{\mathbb{R}^N} \frac{|Eu|^2}{|x-y|^2} dy \leq C \|\nabla Eu\|_{L^2(\mathbb{R}^N)} \|Eu\|_{L^2(\mathbb{R}^N)} \leq C \|Eu\|^2_{H^1(\Omega)} \leq C \|u\|^2_{H^1(\Omega)}.
\]

Similarly,
\[
\|\nabla f(u)\|_{L^\infty(\Omega)} \leq \sup_x \int_\Omega \frac{|u(y)|^2}{|x-y|^2} dy \leq \int_{\mathbb{R}^N} \frac{|Eu(y)|^2}{|x-y|^2} dy \leq C \|\nabla Eu\|^2_{L^2(\mathbb{R}^N)} \leq C \|u\|^2_{H^1(\Omega)}
\]
and
\[
\|f(v) - f(w)\|_{L^\infty(\Omega)} \leq \int_\Omega \frac{|v| - |w| |(y)|}{|x-y|} dy \leq C(\|v\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}) \|v - w\|_{H^1(\Omega)}.
\]
For simplicity, we drop the integration domain $\Omega$ hereafter. For $v, w \in H^1(\Omega)$, we have
\[
\|f(v)w - f(w)\|_{H^1} \leq C\|f(v)(v - w) + (f(v) - f(w))w\|_{H^1} \leq C\|f(v)(v - w)\|_{H^1} + \|f(v) - f(w)\|_{H^1} := I + II
\]
We now estimate terms $I$ and $II$ by using the above inequalities.
\[
I = \|f(v)(v - w)\|_{H^1} \leq C\|f(v)\|_{L^\infty} \|v - w\|_{L^2} \leq C\|v\|_{H^1}^2 \|v - w\|_{H^1},
\]
where
\[
I_1 = \|f(v)(v - w)\|_{L^2} \leq \|f(v)\|_{L^\infty} \|v - w\|_{L^2} \leq C\|v\|_{H^1} \|v - w\|_{H^1},
\]
\[
I_2 = \|\nabla f(v)(v - w)\|_{L^2} \leq \|\nabla f(v)\|_{L^\infty} \|v - w\|_{L^2} \leq C\|v\|_{H^1} \|v - w\|_{H^1},
\]
and
\[
I_3 = \|f(v)\nabla(v - w)\|_{L^2} \leq \|f(v)\|_{L^\infty} \|\nabla(v - w)\|_{L^2} \leq C\|v\|_{H^1} \|v - w\|_{H^1}.
\]
Similarly,
\[
II = \|(f(v) - f(w))w\|_{H^1} \leq C\|(f(v) - f(w))w\|_{L^2} + \|\nabla f(v)(v - w)\|_{L^2} + \|(f(v) - f(w))\nabla w\|_{L^2} = C[I_4 + I_5 + I_6],
\]
where
\[
I_4 = \|(f(v) - f(w))w\|_{L^2} \leq \|(f(v) - f(w))\|_{L^\infty} \|w\|_{L^2} \leq C\|v\|_{H^1}^2 + \|w\|_{H^1} \|v - w\|_{H^1},
\]
\[
I_5 = \|\nabla(f(v) - f(w))w\|_{L^2} \leq \|\nabla(f(v) - f(w))\|_{L^\infty} \|w\|_{L^2} \leq C\|v\|_{H^1} + \|w\|_{H^1} \|v - w\|_{H^1} \|w\|_{L^2} \leq C\|v\|_{H^1}^2 + \|w\|_{H^1} \|v - w\|_{H^1},
\]
and
\[
I_6 = \|(f(v) - f(w))\nabla w\|_{L^2} \leq \|(f(v) - f(w))\|_{L^\infty} \|\nabla w\|_{L^2} \leq C\|v\|_{H^1}^2 + \|w\|_{H^1} \|v - w\|_{H^1}.
\]
This completes the proof.

The proof of the following lemma is similar to related one given in [17].

**Lemma 5.** For any $C_0 > 0$, there exists $T_0 > 0$ such that if $\|\phi\|_{H^1} \leq C_0$, then there exists a unique solution $u \in C([0, T_0], H^1(\Omega))$ which solves (1.1).

**Proof:** We choose $\hat{Q}(x, t)$ to be any $C^3$ function on $\overline{\Omega} \times [0, +\infty)$ with compact support in $x$ such that
\[
\begin{cases}
\Delta \hat{Q} = f(\hat{Q})Q - i\hat{Q}_t, & \text{in } \Omega \\
\hat{Q} = Q, & \partial \Omega \times [0, +\infty)
\end{cases}
\]
Set $v = u - \hat{Q}$. Then $v$ solves
\[
\begin{cases}
i v_t = \Delta v + \Delta \hat{Q} - i\hat{Q}_t - f(v + \hat{Q})(v + \hat{Q}), & \text{in } \Omega \\
v(0) = \phi(x) - \hat{Q}(x, 0) := \psi(x), & \partial \Omega \times [0, +\infty)
\end{cases}
\]
with $v = 0$, $\partial \Omega \times [0, +\infty)$. \hfill (4.1)
Given $\psi \in H^1_0$, the problem (4.1) can be written as the integral equation

$$v(t) = e^{-it\Delta} \psi(x) - i \int_0^t e^{-i(t-s)\Delta} (\Delta \tilde{Q} - i \tilde{Q}_t - f(v + \tilde{Q})(v + \tilde{Q})) ds := \mathcal{H}(v)$$

where $e^{-it\Delta}$ is a group of unitary operators on $H^1_0(\Omega)$ to itself, $v(t) \in H^1_0(\Omega)$. Consider the set

$$E = \{ v \in C([0,T_0],H^1_0(\Omega)) : \|v\|_{C([0,T_0],H^1_0(\Omega))} = \max_{0 \leq t \leq T_0} \|v\|_{H^1_0(\Omega)} \leq M \}.$$

Define

$$d(v,w) = \|v - w\|_{C([0,T_0],H^1_0(\Omega))}, \text{ for every } v, w \in E.$$

Thus $(E,d)$ is a Banach space.

For $v \in E$, $\|\psi\|_{H^1_0} \leq c_0$ and each $T_0 > 0$, there exist constants $C_{T_0}$ and $\tilde{C}_{T_0}$ such that

$$\|\mathcal{H}(v)\|_{H^1_0} \leq \|\psi\|_{H^1_0} + \int_0^t \|\Delta \tilde{Q} - i \tilde{Q}_t - f(v + \tilde{Q})(v + \tilde{Q})\|_{H^1_0} dS,$$

$$\leq \|\psi\|_{H^1_0} + C \int_0^t (\|v + \tilde{Q}\|^2_{H^1} + \|v + \tilde{Q}\|_{H^1}) dS + C_{T_0},$$

$$\leq \|\psi\|_{H^1_0} + C_{T_0} M^3 + \tilde{C}_{T_0}.$$

Similarly,

$$\|\mathcal{H}(v) - \mathcal{H}(w)\|_{H^1_0} \leq \int_0^t e^{-i(t-s)\Delta} (f(v + \tilde{Q})(v + \tilde{Q}) - f(w + \tilde{Q})(w + \tilde{Q})) ds \|_{H^1_0}$$

$$\leq \int_0^t \|f(v + \tilde{Q})(v + \tilde{Q}) - f(w + \tilde{Q})(w + \tilde{Q})\|_{H^1_0} dS,$$

$$\leq \int_0^t (\|v + \tilde{Q}\|^2_{H^1} + \|w + \tilde{Q}\|^2_{H^1}) \|v - w\|_{H^1_0} dS$$

$$\leq 2(M + C)T_0 d(v,w).$$

Take $M = 2(c_0 + \tilde{C}_{T_0})$ and choose suitable small $T_0 > 0$ to satisfy $CM^2T_0 < \frac{1}{8}$ and $2(M + C)T_0 < \frac{1}{4}$. Then we conclude that $\mathcal{H}$ is a strict contraction in $(E,d)$. By Banach fixed point theorem, for any $c_0 > 0$, there exists $T_0 > 0$ such that there is a unique solution $v \in E$ to (4.1). Thus $u = v + \tilde{Q}$ is the unique solution in $C([0,T_0],H^1)$ to (1.1).

Next we use the above results to prove Theorem 1. In fact, once establishing the following lemma, Theorem 1 follows.

**Lemma 6.** Let $T > 0$ be fixed. Let $u$ be the solution of (1.1) in the space $C([0,T],H^1(\Omega))$. Then there exists a constant $C_T > 0$ such that $\|u\|_{H^1} \leq C_T$ for all $0 \leq t \leq T$.

**Proof:** We remark that we have set up the identities (3.1), (3.2), (3.3) for the smooth solution of (1.1) with smooth initial and boundary datum. By the approximations and a subsequence passage to the limit, we can set up the existence of the unique solution $u \in C([0,T],H^1)$ for general data. Hence we shall always consider smooth data.

At each point on $\partial \Omega$, we can write

$$|P|^2 = |P \cdot n|^2 + |A \cdot \nabla Q|^2$$
where $A \cdot \nabla Q$ denotes the tangential component of $P$. Substituting the above identity into (3.3), we find

$$i \int_{\partial \Omega} |P \cdot n|^2 dS + i \int_{\partial \Omega} \overline{P} \cdot n \eta Q dS = -\partial_t \int_{\Omega} u(\xi \cdot \nabla \overline{\n}) + 2i \text{Re} \sum_{j,k} \int_{\Omega} \partial_k \xi_j \overline{\n} u_k$$

$$+ i \sum_{j} \text{Re} \int_{\Omega} \frac{x_j}{|x|^2} * |u_j|^2 |u|^2 + i \int_{\Omega} \n \eta \cdot \n u + \int_{\partial \Omega} Q \overline{Q} dS$$

$$+ i \int_{\partial \Omega} \frac{1}{|x|} * |u|^2 |Q|^2 dS + i \int_{\partial \Omega} |A \cdot \nabla Q|^2 dS.$$

Note $\xi$‘s are smooth functions on $\Omega$ and $Q \in C^3$ with compact support in $x$, then

$$|\sum_j \text{Re} \int_{\Omega} \frac{x_j}{|x|^2} * |u_j|^2 |u|^2| \leq \int_{\Omega} \int_{\Omega} \frac{(\sum_j (x_j - y_j)^2)^{\frac{1}{2}}(\sum_j \xi_j^2)^{\frac{1}{2}}}{|x - y|} |u_j|^2 |u|^2 (x) dy dx$$

$$\leq C \int_{\Omega} \int_{\Omega} \frac{|u_j|^2 (y)}{|x - y|} |u|^2 (x) dy dx$$

$$\leq C \| \frac{|u_j|^2 (y)}{|x - y|} \|_{L^\infty(\Omega)} \| u \|_{L^2(\Omega)}^2$$

$$\leq C \| u \|_{H^1(\Omega)}^4,$$

and

$$| \int_{\partial \Omega} \frac{1}{|x|} * |u|^2 |Q|^2 dS | = | \int_{\partial \Omega} \int_{\Omega} \frac{|u_j|^2 (y)}{|x - y|} dy |Q|^2 (x) dS |$$

$$\leq C \| u \|_{L^2(\Omega)} \| u \|_{H^1(\Omega)}$$

$$\leq C \| u \|_{H^1(\Omega)}^2.$$

Combine the above inequalities with the identity (3.2) and integrate over $[0,t]$ to obtain

$$(4.2) \quad \int_0^t \int_{\partial \Omega} |P \cdot n|^2 dS d\tau \leq | \int_{\Omega} u(\xi \cdot \nabla \overline{\n}) dx | + | \int_{\Omega} \phi(\xi \cdot \nabla \overline{\n}) dx |$$

$$+ C \int_0^t \int_{\Omega} |\nabla u|^2 dx d\tau + C \int_0^t \| u \|^4_{H^1(\Omega)} d\tau$$

$$+ C \int_0^t \int_{\Omega} |\nabla u| |u| dx d\tau + \int_0^t \int_{\partial \Omega} |A \cdot \nabla Q|^2 dS d\tau + C \int_0^t \| u \|^2_{H^1(\Omega)} d\tau$$

$$+ \int_0^t \int_{\partial \Omega} |Q \overline{Q}^*|^2 dS d\tau + C \int_0^t \int_{\partial \Omega} |P \cdot n|^2 dS d\tau.$$

Denote by

$$J = ( \int_0^t \int_{\partial \Omega} |P \cdot n|^2 dS d\tau )^{\frac{1}{2}}.$$

Since $Q$ is $C^3$ with compact support, and $\phi \in H^1(\Omega)$, each term in (4.2) involving $\phi, Q$ is bounded. Therefore

$$J^2 \leq C_0 + 2J + C \int_0^t \| u \|^4_{H^1(\Omega)} d\tau + C \int_0^t \| u \|^2_{H^1(\Omega)} d\tau + C \| u \|^2_{H^1(\Omega)}.$$
Recall the identities (3.1), (3.2) in Lemma 3, we have
\[\int_{\Omega} |u|^2 = \|u\|_{L^2(\Omega)}^2 \leq \|u(0)\|_{L^2(\Omega)}^2 + 2Im \int_0^t \int_{\partial\Omega} \bar{P} \cdot \bar{n}\eta dS d\tau \leq c + cJ\]
and
\[\int_{\Omega} |\nabla u|^2 = \|\nabla u\|_{L^2(\Omega)}^2 \leq \int_{\Omega} |\nabla \phi|^2 + 2 \int_0^t \int_{\partial\Omega} (P \cdot n) \overline{\eta \pm \lambda} dS d\tau \leq \tilde{c} + \tilde{c}J.\]
So we get by Gronwall’s inequality that
\[J^2 \leq \tilde{c}_1(T) + \tilde{c}_2(T)J + \tilde{c}_3(T) \int_0^t J^2(s) dS,
\]
which implies that
\[J \leq C(T)\]
for some uniform constant $C(T) > 0$. Therefore $J$ is bounded for every bounded $T > 0$. Thus $\|u\|_{H^1(\Omega)}$ is bounded by $\tilde{C}(T)$. This completes the proof of Lemma 6.

For any bounded interval $[0, T]$, $T = 1, 2, \cdots$, there exists unique solution $u \in C([0, T], H^1(\Omega))$ to (1.1), and the case $-\infty \leq t < 0$ can be proven in the same way. The uniqueness follows from the standard argument, one may see [14]. Thus we have proved Theorem 1.

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