Inverse Problems for Subdiffusion from Observation at an Unknown Terminal Time

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Abstract

Inverse problems of recovering space-dependent parameters, e.g., initial condition, space-dependent source or potential coefficient, in a subdiffusion model from the terminal observation have been extensively studied in recent years. However, all existing studies have assumed that the terminal time at which one takes the observation is exactly known. In this work, we present uniqueness and stability results for three canonical inverse problems, e.g., backward problem, inverse source and inverse potential problems, from the terminal observation at an unknown time. The subdiffusive nature of the problem indicates that one can simultaneously determine the terminal time and space-dependent parameter. The analysis is based on explicit solution representations, asymptotic behavior of the Mittag-Leffler function, and mild regularity conditions on the problem data. Further, we present several one- and two-dimensional numerical experiments to illustrate the feasibility of the approach.

Key words: backward subdiffusion, inverse source problem, inverse potential problem, subdiffusion, unknown terminal time

1 Introduction

Let Ω ⊂ Rd (d = 1, 2, 3) be an open bounded smooth domain with a boundary ∂Ω. Consider the following initial-boundary value problem with α ∈ (0, 1) for the subdiffusion model:

\[
\begin{align*}
\partial_t^\alpha u - \Delta u + qu &= f, \quad \text{in } \Omega \times (0, \infty), \\
u &= 0, \quad \text{on } \partial \Omega \times (0, \infty), \\
u(0) &= u_0, \quad \text{in } \Omega,
\end{align*}
\]

where T > 0 is a fixed final time, f ∈ L∞(0, T; L2(Ω)) and u0 ∈ L2(Ω) are given source term and initial data, respectively, the nonnegative function q ∈ L∞(Ω) is a spatially dependent potential, and ∆ denotes the Laplace operator in space. The notation ∂t^α u(t) denotes the Djrbashian-Caputo fractional derivative in time t of order α ∈ (0, 1) ([10, p. 70] or [4, Section 2.3.2])

\[
\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds,
\]

where Γ(z) is the Gamma function defined by Γ(z) = \int_0^\infty s^{z-1}e^{-s}ds, for \Re(z) > 0. Note that the fractional derivative ∂t^α u recovers the usual first-order derivative u'(t) as α → 1⁻ if u is sufficiently smooth. Thus the model (1.1) can be viewed as a fractional analogue of the classical parabolic equation.
The model (1.1) arises naturally in the study of anomalously slow diffusion processes, which encompasses a broad range of important applications in engineering, physics and biology. The list of successful applications include thermal diffusion in fractal media [21], dispersion in heterogeneous aquifer [1], ion dispersion in column experiments [2], and protein transport in membranes [11], to name just a few. Thus its mathematical theory has received immense attention in recent years; see the monographs [12, 4] for detailed discussions on the solution theory. Related inverse problems have also been extensively studied [7, 17, 15]. The surveys [17] and [15] cover many inverse source problems and coefficient identification problems, respectively.

The observation $g(x) = u(x, T)$, $x \in \Omega$, at a terminal time $T$ is a popular choice for the measurement data in practice. There is a very extensive literature on inverse problems using terminal data, e.g., backward subdiffusion [22, 28], inverse source problem [7, 17, 3], and inverse potential problem [27, 9, 8, 26], where the references are rather incomplete but we refer to the reviews [7, 17, 15] for further references. Notably, several uniqueness and stability results have been proved. For example, backward subdiffusion is only mildly ill-posed, and enjoys (conditional) Lipschitz stability [22, Theorem 4.1], cf. (2.9) below. In all these existing studies, the terminal time $T$ at which one collects the measurement has always been assumed to be fully known. Nonetheless, in practice, the terminal time $T$ might be known only imprecisely. Therefore, it is natural to ask whether one can still recover some information about the concerned parameter(s). The missing knowledge of $T$ introduces additional challenges since the associated forward map is not fully known then. In this work, we address this question in the affirmative both theoretically and numerically, and study the inverse problem of identifying one of the following three parameters: (i) initial condition $u_0$, (ii) space-dependent source component $\psi$, and (iii) space-dependent potential $q$, from the observation $u(T)$ at an unknown terminal time $T$.

For each inverse problem, we shall establish the unique recovery of the space-dependent parameter and the terminal time $T$ simultaneously from the terminal observation, as well as conditional stability estimates, under suitable a priori regularity assumptions on the initial data $u_0$ and the source $f$; see Theorems 2.2, 3.2 and 4.3 for the precise statements. The analysis relies heavily on explicit solution representations via Mittag-Leffler functions (see, e.g., [22], [4, Section 6.2]). The essence of the argument is that the regularity difference leads to distinct decay behavior of the Fourier coefficients of $u_0$ and $f$. This combined with distinct polynomial decay behaviour of Mittag–Leffler function $E_{\alpha,1}(z)$ (on the negative real axis) allows uniquely determining the terminal time $T$. Note that the polynomial decay holds only for $E_{\alpha,1}(z)$ with a order $\alpha \in (0, 1)$, and does not hold in the integer case (i.e., $\alpha = 1$). Thus, the unique determination of $T$ does not hold for normal diffusion. Once the terminal time $T$ is determined, the unique determination of the space-dependent parameter follows. The proof of the stability results relies on smoothing properties of the solution operators. In addition, we present several numerical experiments to illustrate the feasibility of numerical recovery. The numerical reconstructions are obtained using the Levenberg-Marquadt method [13, 19]. Numerically, by choosing the hyperparameters in the method properly, both space-dependent parameter and terminal time can be accurately recovered. To the best of our knowledge, this work presents the first uniqueness and stability results for inverse problems from terminal data at an unknown time.

The rest of the paper is organized as follows. In Section 2, we present uniqueness and stability results for the backward problem, which are then extended to the inverse source problem in Section 3. In Section 4, we discuss the inverse potential problem, which requires several new technical estimates on the solution regularity and asymptotic decay. Finally, some numerical results for one- and two-dimensional problems are given in Section 5. Throughout, we denote by $u = u(v)$ and $\tilde{u} = u(\tilde{v})$ the solutions to problem (1.1) with the space dependent parameter $v$ and $\tilde{v}$, respectively. We often write a function $f(x, t) : \Omega \times (0, T) \to \mathbb{R}$ as $f(t)$ a vector-valued function on $(0, T)$. The notation $c$ denotes a generic constant which may differ at each occurrence, but it is always independent of the concerned parameter and terminal time $T$.

## 2 Backward problem

In this section, we investigate the backward problem (BP): recover the initial data $u_0 = u(0)$ from the solution profile $u(T)$ to problem (1.1) at an unknown terminal time $T$. 
2.1 Solution representation

First we recall the solution representation for problem (1.1), which plays a key role in the analysis below. For any $s \geq 0$, we denote by $H^s(\Omega) \subset L^2(\Omega)$ the Hilbert space induced by the norm:

$$
\|v\|_{H^s(\Omega)} = \left( \sum_{j=1}^{\infty} \lambda_j^s (v, \varphi_j)^2 \right)^{\frac{1}{2}},
$$

(2.1)

with $\{\lambda_j\}_{j=1}^{\infty}$ and $\{\varphi_j\}_{j=1}^{\infty}$ being respectively the eigenvalues (with multiplicity counted) and eigenfunctions of the operator $A = -\Delta + qI$ on the domain $\Omega$ with a zero Dirichlet boundary condition. Then $\{\varphi_j\}_{j=1}^{\infty}$ can be taken to form an orthonormal basis in $L^2(\Omega)$. Further, $\|v\|_{H_0^s(\Omega)}$ is the norm in $L^2(\Omega)$, $\|v\|_{H^s(\Omega)}$ is the norm in $H_0^s(\Omega)$, and $\|v\|_{H^{-s}(\Omega)} = \|\Delta v\|_{L^2(\Omega)}$ is equivalent to the norm in $H^2(\Omega) \cap H_0^s(\Omega)$ [25, Section 3.1]. For $s < 0$, $\dot{H}^s(\Omega)$ denotes the dual space of $\dot{H}^{-s}(\Omega)$. Throughout, $(\cdot, \cdot)$ denotes both duality pairing between $\dot{H}^{-s}(\Omega)$ and $\dot{H}^s(\Omega)$ and the $L^2(\Omega)$ inner product.

Now we represent the solution $u$ to problem (1.1) using the eigenpairs $\{(\lambda_j, \varphi_j)\}_{j=1}^{\infty}$, following [22] and [4, Section 6.2]. Specifically, we define two solution operators $F(t)$ and $E(t)$ by

$$
F(t)v = \sum_{j=1}^{\infty} E_{\alpha,1}(-\lambda_j t^\alpha)(v, \varphi_j)\varphi_j \quad \text{and} \quad E(t)v = \sum_{j=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_j t^\alpha)(v, \varphi_j)\varphi_j,
$$

(2.2)

where $E_{\alpha,\beta}(z)$ is the Mittag-Leffler function defined by ([10, Section 1.8, pp. 40-45] or [4, Section 3.1])

$$
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \forall z \in \mathbb{C}.
$$

Then the solution $u$ of problem (1.1) can be written as

$$
u(t) = F(t)u_0 + \int_0^t E(t-s)f(s)ds.
$$

(2.3)

The function $E_{\alpha,\beta}(z)$ generalizes the familiar exponential function $e^z$. The following decay estimates of $E_{\alpha,\beta}(z)$ are crucial in the analysis below; See e.g., [10, equation (1.8.28), p. 43] and [4, Theorem 3.2] for the first estimate, and [24, Theorem 4] or [4, Theorem 3.6] for the second estimate.

**Lemma 2.1.** Let $\alpha \in (0, 2)$, $\beta \in \mathbb{R}$, and $\varphi \in (\frac{\alpha \pi}{2}, \min(\pi, \alpha \pi))$, and $N \in \mathbb{N}$. Then for $\varphi \leq |\arg z| < \pi$ with $|z| \to \infty$

$$
E_{\alpha,\beta}(z) = -\sum_{k=1}^{N} \frac{1}{\Gamma(\beta - \alpha k)} \frac{1}{z} + O\left( \frac{1}{z^{N+1}} \right).
$$

For $0 < \alpha_0 < \alpha < \alpha_1 < 1$, there exist constants $c_0, c_1 > 0$ depending only on $\alpha_0$ and $\alpha_1$ such that

$$
c_0(1-x)^{-1} \leq E_{\alpha,1}(x) \leq c_1(1-x)^{-1}, \quad \forall x \leq 0.
$$

2.2 Uniqueness and stability

Now we study uniqueness and stability for BP, when the source $f$ is time-independent, i.e., $f(x, t) \equiv f(x)$. The key idea in proving uniqueness is to distinguish decay rates of Fourier coefficients of the initial data $u_0$ (with respect to the eigenfunctions $\{\varphi_j\}_{j=1}^{\infty}$) and the source $f$. We use the set $S_{\gamma}, \gamma \in [-1, \infty)$, defined by

$$
S_{\gamma} = \left\{ v \in \dot{H}^{-2}(\Omega) : \lim_{n \to \infty} \lambda_n^{\gamma}(v, \varphi_n) = 0 \right\}.
$$

(2.4)

Clearly, for any $\gamma \geq 0$, $\dot{H}^{-\gamma}(\Omega) \subset S_{\gamma}$. If $v \in \dot{H}^{-2}(\Omega) \setminus S_{\gamma}$, the sequence $\{\lambda_n^{\gamma}(v, \varphi_n)\}_{n=1}^{\infty}$ contains a subsequence that is bounded away from zero, i.e., there exists $c_v > 0$ and a sequence $\{n_\ell\}_{\ell=1}^{\infty}$ such that $\lim_{\ell \to \infty} n_\ell = \infty$ and $\lambda_n^{\gamma}(v, \varphi_{n_\ell}) \geq c_v$, for all $\ell \in \mathbb{N}$. 

3
**Theorem 2.1.** Let $f \in \dot{H}^{-2}(\Omega) \setminus S_{\gamma}$ for some $\gamma \geq 0$. If BP has two solutions $(T, u_0)$ and $(\tilde{T}, \tilde{u}_0)$ in the set $\mathbb{R}_+ \times S_{\gamma+1}$ with the observational data $u(T)$ and $\tilde{u}(\tilde{T})$, respectively, then $T = \tilde{T}$ and $u_0 = \tilde{u}_0$.

**Proof.** Using the solution representation (2.3) and noting the identity

$$
\frac{d}{dt} E_{\alpha,1}(-\lambda_n t^\alpha) = -\lambda_n t^{\alpha-1} E_{\alpha,1}(-\lambda_n t^\alpha),
$$

(2.5)
since $f$ is time-independent, the solution $u$ to problem (1.1) can be written as

$$
u(x, t) = \sum_{n=1}^{\infty} \left[ E_{\alpha,1}(-\lambda_n t^\alpha) (u_0, \varphi_n) + \frac{1 - E_{\alpha,1}(-\lambda_n t^\alpha)}{\lambda_n} (f, \varphi_n) \right] \varphi_n(x).
$$

Let $\mathbb{K} = \{ k \in \mathbb{N} : (f, \varphi_n) \neq 0 \}$, which under the condition $f \in \dot{H}^{-2}(\Omega) \setminus S_{\gamma}$ satisfies $|\mathbb{K}| = \infty$. For any $n \in \mathbb{K}$, taking inner product (or duality pairing) with $\frac{\lambda_n \varphi_n}{(f, \varphi_n)}$ on both sides of the identity gives

$$
\frac{\lambda_n (u(t), \varphi_n)}{(f, \varphi_n)} = \lambda_n E_{\alpha,1}(-\lambda_n t^\alpha) \frac{(u_0, \varphi_n)}{(f, \varphi_n)} + 1 - E_{\alpha,1}(-\lambda_n t^\alpha).
$$

Then setting $t = T$ and rearranging the terms lead to

$$
\lambda_n \left( 1 - \frac{\lambda_n (u(t), \varphi_n)}{(f, \varphi_n)} \right) = -\lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) \frac{\lambda_n (u_0, \varphi_n)}{(f, \varphi_n)} + \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha).
$$

(2.6)

By assumption, the source $f \in \dot{H}^{-2}(\Omega) \setminus S_{\gamma}$ and the initial condition $u_0 \in S_{\gamma+1}$, and hence we deduce

$$
\lim_{n \in \mathbb{K}, n \to \infty} \frac{\lambda_n (u_0, \varphi_n)}{(f, \varphi_n)} = \lim_{n \in \mathbb{K}, n \to \infty} \frac{\lambda_n^{\gamma+1} (u_0, \varphi_n)}{\lambda_n (f, \varphi_n)} = 0.
$$

(2.7)

Then, by letting $n \to \infty$ and $t = T$, the relation (2.6) implies

$$
\lim_{n \in \mathbb{K}, n \to \infty} \lambda_n \left( 1 - \frac{\lambda_n (u(t), \varphi_n)}{(f, \varphi_n)} \right) = \lim_{n \in \mathbb{K}, n \to \infty} \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) = \frac{1}{\Gamma(1-\alpha)T^\alpha}.
$$

(2.8)

The last identity follows from the fact that $\lambda_n(q) \to \infty$ and the asymptotic behavior of $E_{\alpha,1}(z)$ in Lemma 2.1. Note that $\frac{1}{\Gamma(1-\alpha)T^\alpha}$ is strictly decreasing in the terminal time $T$. Therefore, $T$ can be uniquely determined from the observation $u(T)$. Finally, the unique determination of $u_0$ follows from [22, Theorem 4.1].

**Remark 2.1.** The validity of Theorem 2.1 relies crucially on the regularity difference between the initial data $u_0$ and source $f$, so that the limit (2.7) holds.

**Remark 2.2.** Theorem 2.1 shows the unique determination of the terminal time $T$ in problem (1.1) from the observation $u(T)$. This interesting phenomenon is due to the distinct asymptotic behaviour of Mittag–Leffler functions and different smoothness of the initial data $u_0$ and source $f$. It sharply contrasts with the backward problem of normal diffusion ($\alpha = 1$): analogous to (2.8)

$$
\lim_{n \in \mathbb{K}, n \to \infty} \lambda_n \left( 1 - \frac{\lambda_n (u(x, T), \varphi_n)}{(f, \varphi_n)} \right) = \lim_{n \in \mathbb{K}, n \to \infty} \lambda_n e^{-\lambda_n T} = 0,
$$

and thus it is impossible to determine both the terminal time $T$ and initial data $u_0$ from the data $u(T)$. This shows one distinct feature of anomalous slow diffusion processes, when compared with standard diffusion.

Next, we establish a stability estimate for BP with an approximately given $T$. When the terminal time $T$ is exactly given, it recovers the following well-known estimate [22, Theorem 4.1] (or [4, Theorem 6.28])

$$
\|u_0 - \tilde{u}_0\|_{L^2(\Omega)} \leq c\|u(T) - \tilde{u}(T)\|_{H^2(\Omega)}.
$$

(2.9)
Theorem 2.2. Let $u_0$ and $\tilde{u}_0$ be the solutions of BP with observations $u(T)$ and $\tilde{u}(\tilde{T})$ with $T < \tilde{T}$, respectively. Then the following conditional stability estimate holds

$$\|u_0 - \tilde{u}_0\|_{L^2(\Omega)} \leq c(1 + c\tilde{T}^\alpha)(\|A(u(T) - u(\tilde{T}))\|_{L^2(\Omega)} + c|T - \tilde{T}|T^{1-\alpha}\|u_0 - A^{-1}f\|_{L^2(\Omega)}).$$

Proof. By the solution representation (2.3) and the identity (2.5), we have

$$u(T) = F(T)u_0 + A^{-1}(I - F(T))f \quad \text{and} \quad \tilde{u}(\tilde{T}) = F(\tilde{T})\tilde{u}_0 + A^{-1}(I - F(\tilde{T}))f.$$ 

Subtracting these two identities leads to

$$u(T) - \tilde{u}(\tilde{T}) = (F(T)u_0 - F(\tilde{T})\tilde{u}_0) + A^{-1}(F(\tilde{T}) - F(T))(I - A^{-1}f).$$

Next we bound the terms separately. For any $v \in L^2(\Omega)$, by Lemma 2.1, we derive

$$\|A^{-1}F(\tilde{T})^{-1}v\|_{L^2(\Omega)}^2 = \sum_{n=1}^\infty \frac{\lambda_n^{-2}E_n\lambda_n^{-\alpha}}{\lambda_n} \leq c \sum_{n=1}^\infty \frac{1}{\lambda_n}(v, \varphi_n)^2 \leq c|1 + \tilde{T}^\alpha|^2\|v\|_{L^2(\Omega)}^2.$$ 

Next we bound $F(\tilde{T})^{-1}(F(T) - F(\tilde{T})) = A^{-1}F(\tilde{T})^{-1} \int_T^{\tilde{T}} AF'(s)\, ds$. Note that for any $s \in (T, \tilde{T})$, there holds [4, Theorem 6.4]

$$\|AF'(s)v\|_{L^2(\Omega)} \leq c\|v\|_{L^2(\Omega)} \leq cT^{-1-\alpha}\|v\|_{L^2(\Omega)}.$$ 

This estimate implies

$$\|A(F(T) - F(\tilde{T}))v\|_{L^2(\Omega)} \leq cT^{-1-\alpha}|\tilde{T} - T|\|v\|_{L^2(\Omega)}. \quad (2.10)$$

Consequently, we obtain

$$\|F(\tilde{T})^{-1}(F(T) - F(\tilde{T}))v\|_{L^2(\Omega)} \leq \|A^{-1}F(\tilde{T})\|A(F(T) - F(\tilde{T}))v\|_{L^2(\Omega)} \leq c(1 + c\tilde{T}^\alpha)|T - \tilde{T}|T^{1-\alpha}\|v\|_{L^2(\Omega)}.$$ 

Then the desired result follows from immediately the preceding estimates. \qed

The next result bounds the terminal time $T$ in terms of data perturbation.

Corollary 2.1. Let $f \in \dot{H}^{-2}(\Omega) \setminus S_\gamma$ for some $\gamma \geq 0$. Let $(T, u_0)$, $(\tilde{T}, \tilde{u}_0) \in \mathbb{R}_+ \times S_{\gamma+1}$ be the solutions of BP with observations $u(T)$ and $\tilde{u}(\tilde{T})$, respectively. Then the following stability estimate holds

$$|T - \tilde{T}| \leq c\min(\Lambda, \tilde{\Lambda})^{-\frac{1}{\alpha} - 1}|\Lambda - \tilde{\Lambda}|,$$

with the quantities $\Lambda$ and $\tilde{\Lambda}$ respectively given by

$$\Lambda = \lim_{n \in \mathbb{N}, n \to \infty} \lambda_n \left(1 - \frac{\lambda_n(u(T), \varphi_n)}{(T, \varphi_n)}\right) \quad \text{and} \quad \tilde{\Lambda} = \lim_{n \in \mathbb{N}, n \to \infty} \lambda_n \left(1 - \frac{\lambda_n(\tilde{u}(\tilde{T}), \varphi_n)}{(\tilde{T}, \varphi_n)}\right).$$

In particular, for $\Lambda < \tilde{\Lambda}$, there holds $\|u_0 - \tilde{u}_0\|_{L^2(\Omega)} \leq c(1 + c\tilde{\Lambda}^{-1})(\|A(u(T) - \tilde{u}(\tilde{T}))\|_{L^2(\Omega)} + c|\Lambda - \tilde{\Lambda}|\|u_0 - A^{-1}f\|_{L^2(\Omega)}).$

Proof. It follows from the relation (2.8) that

$$|T - \tilde{T}| = |\Lambda^{-\frac{1}{\alpha}}\Gamma(1-\alpha)^{-\frac{1}{\alpha}} - \tilde{\Lambda}^{-\frac{1}{\alpha}}\Gamma(1-\alpha)^{-\frac{1}{\alpha}}| \leq \Gamma(1-\alpha)^{-\frac{1}{\alpha}}\min(\Lambda, \tilde{\Lambda})^{-\frac{1}{\alpha} - 1}|\Lambda - \tilde{\Lambda}|.$$ 

The assertion follows from Theorem 2.2 and the identities $T = \Lambda^{-\frac{1}{\alpha}}\Gamma(1-\alpha)^{-\frac{1}{\alpha}}$ and $\tilde{T} = \tilde{\Lambda}^{-\frac{1}{\alpha}}\Gamma(1-\alpha)^{-\frac{1}{\alpha}}$. \qed
3 Inverse source problem

In this section, we extend the argument in Section 2 to an inverse source problem of recovering the space dependent component from the observation $u(T)$. Following the standard setup for inverse source problems \cite{17}, we assume that the source $f(x,t)$ is separable and satisfies

$$f(x,t) = g(t)\psi(x), \quad \text{with} \quad g \in L^\infty(0, T), \ g \geq c_g > 0 \ \text{and} \ \psi \in \dot{H}^{-1}(\Omega). \quad (3.1)$$

Then we consider the following inverse source problem (ISP): determine the spatially dependent source component $\psi(x)$ from the solution profile $u(T)$ at a later but unknown time $T$.

First we give an intermediate result.

**Lemma 3.1.** Let $G(T) := \int_0^T E(s)g(T-s) \, ds$. Then under condition (3.1), $G$ is invertible and

$$\|A^{-1}G(T)^{-1}v\|_{L^2(\Omega)} \leq c_g^{-1}(1 - E_{\alpha,1}(-\lambda_1T^\alpha))^{-1}\|v\|_{L^2(\Omega)}.$$  

**Proof.** Note that the function $E_{\alpha,\alpha}(-t) > 0$ for all $0 \leq t < \infty$ (since it is completely monotone and analytic) (see, e.g., \cite{23, 20}, or \cite[Corollary 3.2 or Corollary 3.3]{4}). This, the condition $g \geq c_g > 0$ and the identity (2.5) imply

$$\int_0^T s^{-\alpha}E_{\alpha,\alpha}(-\lambda_n s^\alpha)g(T-s) \, ds \geq c_g \int_0^T s^{-\alpha}E_{\alpha,\alpha}(-\lambda_n s^\alpha) \, ds = c_g \lambda_n^{-1}(1 - E_{\alpha,1}(-\lambda_n T^\alpha)).$$

This implies the invertibility of the operator $G(T)$:

$$G(T)^{-1}v = \sum_{n=1}^\infty \frac{(v, \varphi_n)}{c_g \lambda_n} \int_0^T s^{-\alpha}E_{\alpha,\alpha}(-\lambda_n s^\alpha)g(T-s) \, ds \varphi_n.$$

Consequently, for any $v \in L^2(\Omega)$, we have

$$\|A^{-1}G(T)^{-1}v\|_{L^2(\Omega)}^2 \leq \sum_{n=1}^\infty \left[ \frac{(v, \varphi_n)}{c_g \lambda_n} \int_0^T s^{-\alpha}E_{\alpha,\alpha}(-\lambda_n s^\alpha)g(T-s) \, ds \right]^2 \leq \sum_{n=1}^\infty \left[ \frac{(v, \varphi_n)}{c_g} \int_0^T s^{-\alpha}E_{\alpha,\alpha}(-\lambda_n s^\alpha) \, ds \right]^2 \leq \sum_{n=1}^\infty \left[ \frac{(v, \varphi_n)}{c_g} \right]^2 = c_g^{-2}(1 - E_{\alpha,1}(-\lambda_1 T^\alpha))^{-2}\|v\|_{L^2(\Omega)}^2,$$

where in the last inequality we have used the monotonicity of $E_{\alpha,1}(-t) \in (0, 1]$ for $t \geq 0$.

The next result gives the unique determination of the source $\psi$ and the terminal time $T$.

**Theorem 3.1.** Let $u_0 \in L^2(\Omega) \setminus S_{\gamma}$ for some $\gamma \geq 0$. If ISP has two solutions $(\tilde{T}, \tilde{\psi})$ and $(\tilde{\tilde{T}}, \tilde{\psi})$ in the set $\mathbb{R}_+ \times \mathbb{S}_{\gamma}$, from the observations $u(T)$ and $\tilde{u}(\tilde{\tilde{T}})$, respectively, then $\tilde{T} = \tilde{\tilde{T}}$ and $\tilde{\psi} = \tilde{\psi}$. 

**Proof.** Using the representation (2.3) and separability assumption (3.1), we have

$$u(t) = \sum_{n=0}^\infty \left[ E_{\alpha,1}(-\lambda_n t^\alpha)(u_0, \varphi_n) + \int_0^t s^{-\alpha}E_{\alpha,\alpha}(-\lambda_n s^\alpha)g(t-s) \, ds (\psi, \varphi_n) \right] \varphi_n. \quad (3.2)$$

Define $\mathbb{K} = \{ k \in \mathbb{N} : (u_0, \varphi_n) \neq 0 \}$. For any $n \in \mathbb{K}$, taking inner product (or duality pairing) with $\frac{\lambda_n \varphi_n}{(u_0, \varphi_n)}$ on both sides of the identity (3.2) and setting $t = T$, we have

$$\frac{\lambda_n(u(T), \varphi_n)}{(u_0, \varphi_n)} = \lambda_n E_{\alpha,1}(-\lambda_n T^\alpha) + \int_0^T s^{-\alpha}E_{\alpha,\alpha}(-\lambda_n s^\alpha)g(T-s) \, ds \frac{\lambda_n(\psi, \varphi_n)}{(u_0, \varphi_n)}.$$
By assumption, \( g \in L^\infty(0,T) \) and \( E_{\alpha,\alpha}(-t) > 0 \) for all \( \infty > t \geq 0 \) (since it is completely monotone and analytic) \([4, \text{Corollary 3.3}]\), we have

\[
\left| \int_0^T s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n s^\alpha) g(T-s) \, ds \right| \leq \int_0^T s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n s^\alpha) \, ds \|g\|_{L^\infty(0,T)} = \lambda_n^{-1}[1 - E_{\alpha,1}(-\lambda_n T^n)] \|g\|_{L^\infty(0,T)} \leq \lambda_n^{-1}\|g\|_{L^\infty(0,T)},
\]

where we have used the identity (2.5) and the inequality \( E_{\alpha,1}(-t) \in [0,1] \) for all \( t \geq 0 \). Since \( \psi \in \mathcal{S}_\gamma \) and \( u_0 \in L^2(\Omega) \backslash \mathcal{S}_\gamma \), we have

\[
\lim_{n \in \mathbb{K}, \, n \to \infty} \left| \int_0^T s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n s^\alpha) g(T-s) \, ds \frac{\lambda_n(\psi,\varphi_n)}{(u_0,\varphi_n)} \right| \leq \lim_{n \in \mathbb{K}, \, n \to \infty} \|g\|_{L^\infty(0,T)} \left| \frac{(\psi,\varphi_n)}{(u_0,\varphi_n)} \right| = 0.
\]

Consequently, we have

\[
\lim_{n \in \mathbb{K}, \, n \to \infty} \frac{\lambda_n(u(T),\varphi_n)}{(u_0,\varphi_n)} = \lim_{n \in \mathbb{K}, \, n \to \infty} \lambda_n E_{\alpha,1}(-\lambda_n T^n) = \frac{1}{\Gamma(1-\alpha) T^n}. \tag{3.3}
\]

Note that the function \( \frac{1}{(1-\alpha) T^n} \) is strictly decreasing in \( T \). Hence, the terminal time \( T \) is uniquely determined by \( u(T) \). Finally, the uniqueness of \( \psi \) follows from the representation \( \psi = G(T)^{-1} u(T) - G(T)^{-1} F(T) u_0 \) (with \( G(T) = \int_0^T E(s) g(T-s) \, ds \)), and Lemma 3.1.

\textbf{Remark 3.1.} Note that the representation \( \frac{1}{(1-\alpha) T^n} \) is also decreasing with respect to \( \alpha \) if \( T \) is sufficiently large \([3, \text{Lemma 4}]\). So the terminal data \( u(T) \) can uniquely determine the fractional order \( \alpha \) if \( T \) is a priori known. See some related argument in \([3, 16]\) for the inverse source problem with an unknown order \( \alpha \).

The next theorem gives a stability result for recovering the source \( \psi \).

\textbf{Theorem 3.2.} Fix \( T_0 > 0 \). Let \( \psi \) and \( \tilde{\psi} \) be the solutions of ISP with the data \( u(T) \) and \( \tilde{u}(\tilde{T}) \) with \( T_0 \leq T < \tilde{T} \), respectively. Then for \( g \in C^2[0,T] \), the following conditional stability estimate holds

\[
\|\psi - \tilde{\psi}\|_{L^2(\Omega)} \leq c(\|A(u(T) - \tilde{u}(\tilde{T}))\|_{L^2(\Omega)} + |T - \tilde{T}| T^{-1-\alpha} \|u_0\|_{L^2(\Omega)} + |T - \tilde{T}| (1 + T^{-1} + T)\|\tilde{\psi}\|_{L^2(\Omega)}).
\]

\textbf{Proof.} It follows from the solution representation (2.3) that

\[
u(T) = F(T) u_0 + \int_0^T E(s) g(T-s) \, ds \psi \quad \text{and} \quad \tilde{u}(\tilde{T}) = F(\tilde{T}) u_0 + \int_0^{\tilde{T}} E(s) g(\tilde{T}-s) \, ds \tilde{\psi}.
\]

Then subtracting these two identities leads to

\[
u(T) - \tilde{u}(\tilde{T}) = (F(T) - F(\tilde{T})) u_0 + \int_0^T E(s) g(T-s) \, ds (\psi - \tilde{\psi})
\]

\[
+ \int_0^T E(s)[g(T-s) - g(\tilde{T}-s)] \, ds \tilde{\psi} - \int_T^{\tilde{T}} E(s) g(\tilde{T}-s) \, ds \tilde{\psi}.
\]

Therefore, with \( G(T) := \int_0^T E(s) g(T-s) \, ds \), we arrive at

\[
\psi - \tilde{\psi} = G(T)^{-1} (u(T) - \tilde{u}(\tilde{T})) + G(T)^{-1} (F(T) - F(T)) u_0 \]

\[
+ G(T)^{-1} \int_0^T E(s) [g(T-s) - g(\tilde{T}-s)] \, ds \tilde{\psi} + G(T)^{-1} \int_T^{\tilde{T}} E(s) g(\tilde{T}-s) \, ds \tilde{\psi} = \sum_{j=1}^4 I_j.
\]

Next we bound the four terms separately. First, by Lemma 3.1,

\[
\|I_1\|_{L^2(\Omega)} \leq c_g^{-1} (1 - E_{\alpha,1}(-\lambda_1 T_0^n))^{-1} \|A(u(T) - \tilde{u}(\tilde{T}))\|_{L^2(\Omega)}.
\]
Second, by the estimate (2.10), we can bound the term $I_2$ by

$$
\|I_2\|_{L^2(\Omega)} \leq c \|A(F(\hat{T}) - F(T))u_0\|_{L^2(\Omega)} \leq c|\hat{T} - T|T^{-1-\alpha}\|u_0\|_{L^2(\Omega)}.
$$

Next we bound the term $I_3$. It follows directly from integration by parts and the identities $F'(t) = -AE(t)$ and $F(0) = I$ [4, Lemmas 6.2 and 6.3] that

$$
\int_0^T E(s)g(T-s) \, ds = A^{-1}(g(T)I - g(0)F(T)) - A^{-1}\int_0^T F(s)g'(T-s) \, ds,
$$

$$
\int_0^T E(s)g(\hat{T} - s) \, ds = A^{-1}(g(\hat{T})I - g(\hat{T} - T)F(T)) - A^{-1}\int_0^T F(s)g'(\hat{T} - s) \, ds.
$$

Since $g \in C^2[0,T]$, we have

$$
\|I_3\|_{L^2(\Omega)} \leq c \left\| A \int_0^T E(s)(g(T-s) - g(\hat{T} - s)) \, ds \right\|_{L^2(\Omega)}
\leq c\left( |g(T)| + |g(\hat{T} - T) - g(0)| + \int_0^T |g'(\hat{T} - s) - g'(T-s)| \, ds \right) \|\psi\|_{L^2(\Omega)}
\leq c|\hat{T} - T|(1 + T)\|g\|_{C^2[0,T]}\|\psi\|_{L^2(\Omega)}.
$$

Finally, for the term $I_4$, the estimate $\|AE(s)\| \leq cs^{-1}$ [4, Theorem 6.4] yields

$$
\|I_4\|_{L^2(\Omega)} \leq c \left\| \int_T^\hat{T} AE(s)g(\hat{T} - s) \, ds \varphi \right\|_{L^2(\Omega)}
\leq c\|g\|_{C[0,\hat{T}]} \int_T^\hat{T} s^{-1} \, ds \|\varphi\|_{L^2(\Omega)} \leq c|\hat{T} - T|T^{-1}\|g\|_{C[0,\hat{T}]}\|\varphi\|_{L^2(\Omega)}.
$$

The preceding four estimates together complete the proof of the theorem.

The next result bounds the terminal time $T$ for perturbed data. The proof is identical with that for Corollary 2.1, and hence it is omitted.

**Corollary 3.1.** Fix $T_0 > 0$. Let $u_0 \in L^2(\Omega) \setminus S_{\gamma}$ for some $\gamma \geq 0$, and $(T, \psi), (\hat{T}, \hat{\psi}) \in \mathbb{R}_+ \times S_\gamma$ with $T, \hat{T} \geq T_0$ be the solutions of ISP with observations $u(T)$ and $\hat{u}(\hat{T})$, respectively. Then the following estimate holds

$$
|T - \hat{T}| \leq \Gamma(1 - \alpha)^{-\frac{1}{2}} \min(\Lambda, \hat{\Lambda})^{-\frac{1}{2}} - 1|\Lambda - \hat{\Lambda}|
$$

with the scalars $\Lambda$ and $\hat{\Lambda}$ respectively given by

$$
\Lambda = \lim_{n \to \infty} \frac{\lambda_n(u(T), \varphi_n)}{(u_0, \varphi_n)} \quad \text{and} \quad \hat{\Lambda} = \lim_{n \to \infty} \frac{\lambda_n(\hat{u}(\hat{T}), \varphi_n)}{(u_0, \varphi_n)}.
$$

In particular, for $\Lambda < \hat{\Lambda}$, there holds $\|\psi - \hat{\psi}\|_{L^2(\Omega)} \leq c(\|A(u(T) - \hat{u}(\hat{T}))\|_{L^2(\Omega)} + |\Lambda - \hat{\Lambda}|\Lambda^{-\frac{1}{2}} - 1(1 + \Lambda^\frac{1}{2} + \Lambda^{-\frac{1}{2}})\|\varphi\|_{L^2(\Omega)}).

**4 Inverse potential problem**

In this section, we discuss the identification of the potential $q$ in the model (1.1) from the observation $u(T)$, at an unknown terminal time $T$. Specifically, we consider the domain $\Omega = (0, 1)$ and a nonzero Dirichlet boundary condition:

$$
\begin{align*}
\begin{cases}
\partial_t^2 u - \partial_{xx} u + qu &= f, \quad \text{in } \Omega \times (0, T], \\
u(0, t) = a_0, u(1, t) &= a_1, \quad \text{on } (0, T], \\
u(0) &= u_0, \quad \text{in } \Omega,
\end{cases}
\end{align*}
$$

(4.1)
where the functions \( f > 0 \) and \( u_0 > 0 \) are given spatially dependent source and initial data, respectively, and \( a_0 \) and \( a_1 \) are positive constants. Throughout, the potential \( q \) belongs to the following admissible set

\[
\mathcal{A} = \{ q \in L^\infty(\Omega) : 0 \leq q \leq c_0 \}.
\]

The inverse potential problem (IPP) is to recover the potential \( q \in \mathcal{A} \) from the observation \( u(T) \), for an unknown time \( T \).

Similar to the discussions in Section 2, let \( A_q \) be the realization of the elliptic operator \(-\partial_{xx} + q(x)I\) in \( L^2(\Omega) \), with its domain \( \text{Dom}(A_q) \) given by \( \text{Dom}(A_q) := \{ v \in L^2(\Omega) : -\partial_{xx}v + qv \in L^2(\Omega) \text{ and } v(0) = v(1) = 0 \text{ in } \partial\Omega \} \). Let \( \{(\lambda_n(q), \varphi_n(q))\}_{n=1}^\infty \) be the eigenpairs of the operator \( A_q \), which is not \textit{a priori} known for an unknown \( q \). Note that for any \( q \in \mathcal{A} \), the set \( \{\varphi_n(q)\}_{n=1}^\infty \) can be chosen to form a complete (orthonormal) basis of the space \( L^2(\Omega) \). It is well known that for any \( q \in \mathcal{A} \), the eigenvalues \( \lambda_n(q) \) and eigenfunctions \( \varphi_n(q) \) satisfy the following asymptotics [14, Section 2, Chapter 1]:

\[
\lambda_n(q) = n^2 \pi^2 + O(1) \quad \text{and} \quad \varphi_n(x; q) = \sqrt{2} \sin(n\pi x) + O(n^{-1}).
\] (4.2)

Further, for any \( v \in H^1_0(\Omega) \cap H^2(\Omega) \) and \( q \in \mathcal{A} \), the following two-sided inequality holds

\[
c_1 \|v\|_{H^2(\Omega)} \leq \|A_q v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \leq c_2 \|v\|_{H^2(\Omega)},
\] (4.3)

with constants \( c_1 \) and \( c_2 \) independent of \( q \). Next, we define an auxiliary function \( \phi_q \in H^2(\Omega) \) satisfying

\[
\begin{align*}
-\partial_{xx}\phi + q\phi &= 0, \quad \text{in } \Omega = (0,1), \\
\phi_q(0) &= a_0, \quad \phi_q(1) = a_1.
\end{align*}
\]

It is easy to see \( \phi_q(x) = a_0(1-x) + a_1x \). Then the solution \( u \) of problem (4.1) is given by

\[
u(t) = F_q(t)u_0 + (I - F_q(t))\phi_q + (I - F_q(t))A_q^{-1}f,
\] (4.4)

where \( E_q \) and \( F_q \) denote the solution operators, cf. (2.2), for the elliptic operator \( A_q \), and the subscript \( q \) explicitly indicates the dependence on the potential \( q \).

Next, we show the unique recovery of the terminal time \( T \). Like before, the key is to distinguish the decay rates of Fourier coefficients of \( u_0 \) and \( f \) with respect to the (unknown) eigenfunctions \( \{\varphi_j(q)\}_{j=1}^\infty \).

**Theorem 4.1.** Let \( u_0 \in L^2(\Omega) \setminus \dot{H}^{s'}(\Omega) \), with \( s' \in (0,\frac{1}{2}) \), and \( f \in \dot{H}^s(\Omega) \) and \( q \in \mathcal{A} \cap \dot{H}^s(\Omega) \) with \( s \in (s' + \frac{1}{2}, 1) \). Then in IPP, the terminal time \( T \) is uniquely determined by the observation \( u(T) \).

**Proof.** Let \( \bar{u}_0 = u_0 - \phi_0 \). Since \( u_0 \in L^2(\Omega) \setminus \dot{H}^{s'}(\Omega) \), by the asymptotics (4.2), we obtain

\[
\sum_{n=1}^\infty \lambda_n(q)^{s'}(\bar{u}_0, \sin(n\pi x))^2 = \infty.
\] (4.5)

We claim that for \( \bar{s} = \frac{1}{4} + \frac{s'}{2} + \frac{\epsilon}{4} < \frac{s}{2} \), with a small \( \epsilon \in (0, 2s - 2s' - 1) \), there holds \( \bar{u}_0 \notin S_{\bar{s}} \). Indeed, assuming the contrary, i.e., \( \bar{u}_0 \in S_{\bar{s}} \), the definition of \( S_{\bar{s}} \) in (2.4) implies

\[
\lim_{n \to \infty} \lambda_n(q)^{\bar{s}}(\bar{u}_0, \sin(n\pi x)) = 0,
\]

and thus the sequence \( \{\lambda_n(q)^{\bar{s}}(\bar{u}_0, \sin(n\pi x))\}_{n=1}^\infty \) is uniformly bounded. This and the asymptotics (4.2) lead to

\[
\sum_{n=1}^\infty \lambda_n(q)^{s'}(\bar{u}_0, \sin(n\pi x))^2 = \sum_{n=1}^\infty \lambda_n(q)^{2s}(\bar{u}_0, \sin(n\pi x))^2 \lambda_n(q)^{s'-2s} \leq c \sum_{n=1}^\infty \lambda_n(q)^{s'-2s} \leq c \sum_{n=1}^\infty n^{-1-\epsilon} < \infty.
\]

This contradicts the identity (4.5), and hence the desired claim follows. The claim \( \bar{u}_0 \notin S_{\bar{s}} \) and the asymptotics (4.2) imply that there exists a constant \( c_0 > 0 \), for any \( N > 0 \), we can find \( N > N \) such that 

\[
n^{2\bar{s}}(|\bar{u}_0, \sin(n\pi x)|) \geq c_n.
\]

Let \( K = \{ n \in \mathbb{N} : n^{2\bar{s}}(|\bar{u}_0, \sin(n\pi x)|) \geq c_n \} \). Then we have \( |K| = \infty \). By the asymptotics (4.2), we may assume that \( n^{2\bar{s}}(|\bar{u}_0, \varphi_n(q)|) \geq \sqrt{2}c_n/2 \) for \( n \in K \), and hence \( (\bar{u}_0, \varphi_n(q)) \neq 0 \).
Meanwhile, it follows directly from (2.3) that the solution \( u(t) \) satisfies

\[
\begin{align*}
 u(t) - \phi_q &= \sum_{n=0}^{\infty} \left[ E_{\alpha,1}(-\lambda_n(q)t^\alpha)(u_0 - \phi_q, \varphi_n(q)) + \left( 1 - E_{\alpha,1}(-\lambda_n(q)t^\alpha) \right) \frac{(f, \varphi_n(q))}{\lambda_n(q)} \right] \varphi_n(q). \\
(4.6)
\end{align*}
\]

By taking inner product with \( \varphi_n(q), n \in \mathbb{K}, \) on both sides of the identity (4.6), we obtain

\[
\begin{align*}
 \lambda_n(q)(u(t) - \phi_q, \varphi_n(q)) &= \lambda_n(q)E_{\alpha,1}(-\lambda_n(q)t^\alpha) + \left( 1 - E_{\alpha,1}(-\lambda_n(q)t^\alpha) \right) \frac{(f, \varphi_n(q))}{(u_0, \varphi_n(q))} \\
&
\quad + \left( 1 - E_{\alpha,1}(-\lambda_n(q)t^\alpha) \right) \frac{(\lambda_n(q)q_n - \phi_q, \varphi_n(q))}{(u_0, \varphi_n(q))} := \sum_{i=1}^{\infty} I_i. \quad (4.7)
\end{align*}
\]

We analyze the three terms \( I_i, i = 1, 2, 3, \) separately. Since \( E_{\alpha,1}(-\lambda_n(t^\alpha) \in (0, 1] \) for \( t \geq 0, \) the regularity condition \( f \in \dot{H}^s(\Omega) \) implies \( \lim_{n \to \infty} n^s|f| = 0. \) This and the condition \( s > 2\delta \) yield

\[
0 \leq n^{2\delta}|(f, \varphi_n(q))| \leq n^s|(f, \varphi_n(q))| \to 0, \quad \text{as} \ n \to \infty.
\]

Hence

\[
\lim_{n \in \mathbb{K}, \ n \to \infty} \frac{\lambda_n(q)(u(t) - \phi_q, \varphi_n(q))}{(u_0, \varphi_n(q))} = \lim_{n \in \mathbb{K}, \ n \to \infty} \lambda_n(q)E_{\alpha,1}(-\lambda_n(q)t^\alpha) = \frac{1}{\Gamma(1-\alpha)T^\alpha}, \quad (4.9)
\]

since \( \lambda_n(q) \to \infty, \) cf. (4.2). Note that the function \( \frac{1}{\Gamma(1-\alpha)T^\alpha} \) is decreasing in \( T \) and independent of \( q. \) Last, we show that the left hand side of (4.9) can actually be computed independently of \( q \in \dot{H}^s(\Omega) \cap \mathcal{A}, \) using the asymptotics (4.2). Indeed, since \( u_0 \in L^2(\Omega) \cap \dot{H}^s(\Omega), f \in \dot{H}^s(\Omega) \) and \( q \in \dot{H}^s(\Omega) \cap \mathcal{A} \) with \( s \in (s' + \frac{1}{2}, 1) \) and \( s' \in (0, \frac{1}{2}), \) we have \( A(q)u(T) \in L^2(\Omega) \cap \dot{H}^s(\Omega) \) [22]. This and the assumption \( q \in \dot{H}^s(\Omega) \) imply \( qu(T) \in \dot{H}^s(\Omega), \) and hence \( \partial_{xx}u(T) \in L^2(\Omega) \cap \dot{H}^s(\Omega). \) Meanwhile, integration by parts twice yields

\[
\lambda_n(q)(u(T) - \phi_0, \varphi_n(q)) = (\partial_{xx}u(T), \varphi_n(q)) + (q(u(T) - \phi_0), \varphi_n(q)).
\]

Noting the fact \( q(u(T) - \phi_0) \in \dot{H}^s(\Omega) \) and then repeating the argument for (4.8) yield

\[
\lim_{n \in \mathbb{K}, \ n \to \infty} \frac{(q(u(T) - \phi_0), \varphi_n(q))}{(u_0, \varphi_n(q))} = 0.
\]

Consequently, we derive

\[
\lim_{n \in \mathbb{K}, \ n \to \infty} \frac{\lambda_n(q)(u(T) - \phi_0, \varphi_n(q))}{(u_0, \varphi_n(q))} = \lim_{n \in \mathbb{K}, \ n \to \infty} \frac{-(\partial_{xx}u(T), \varphi_n(q))}{(u_0, \varphi_n(q))}. \quad (4.10)
\]

Now using the asymptotics (4.2) again, we obtain

\[
\begin{align*}
\frac{-(\partial_{xx}u(T), \varphi_n(q))}{(u_0, \varphi_n(q))} &= \frac{-(\partial_{xx}u(T), \sqrt{2}\sin(n\pi x)) + O(n^{-1}) \cdot \|\partial_{xx}u(T)\|_{L^1(\Omega)}}{(u_0, \sqrt{2}\sin(n\pi x)) + O(n^{-1}) \cdot \|u_0\|_{L^1(\Omega)}}.
\end{align*}
\]
The bound on $\partial u_0, \sin(n\pi x)) \geq c_n$ for all $n \in \mathbb{K}$, by the condition $2\delta - 1 < 0$, we obtain

$$
\lim_{n \in \mathbb{K}, n \to \infty} \frac{-(\partial_{xx}u(T), \varphi_n(q))}{(u_0, \varphi_n(q))} = \lim_{n \in \mathbb{K}, n \to \infty} \frac{-n^{2s}(\partial_{xx}u(T), \sqrt{2}\sin(n\pi x)) + O(n^{2s-1}) \cdot \|\partial_{xx}u(T)\|_{L^1(\Omega)}}{n^{2s}(u_0, \sqrt{2}\sin(n\pi x)) + O(n^{2s-1}) \cdot \|u_0\|_{L^1(\Omega)}}.
\tag{4.11}
$$

Since $n^{2s}(u_0, \sin(n\pi x)) \geq c_n$, for all $n \in \mathbb{K}$, by the condition $2\delta - 1 < 0$, we obtain

$$
\lim_{n \in \mathbb{K}, n \to \infty} \frac{-(\partial_{xx}u(T), \varphi_n(q))}{(u_0, \varphi_n(q))} = \lim_{n \in \mathbb{K}, n \to \infty} \frac{-n^{2s}(\partial_{xx}u(T), \sqrt{2}\sin(n\pi x)) + O(n^{2s-1}) \cdot \|\partial_{xx}u(T)\|_{L^1(\Omega)}}{n^{2s}(u_0, \sqrt{2}\sin(n\pi x)) + O(n^{2s-1}) \cdot \|u_0\|_{L^1(\Omega)}}.
$$

Therefore, the terminal time $T$ is uniquely determined by the observation $u(T)$.

**Remark 4.1.** The independence of the limit in (4.9) on the potential $q$ relies on the asymptotics (4.2). This seems valid only in the one-dimensional case, and it represents the main obstacle for the extension to the multi-dimensional case. Nonetheless, the rest of the analysis does not use the estimate (4.2), and all the remaining results hold also for the multi-dimensional case.

Next we determine the potential $q \in \mathcal{A}$ from the observation $u(T)$. First, we give useful smoothing properties of the operators $F_q$ and $E_q$. The notation $\| \cdot \|$ denotes the operator norm on $L^2(\Omega)$.

**Lemma 4.1.** For $q \in \mathcal{A}$, there exists a $c > 0$ independent of $q$ and $t$ such that for any $s = 0, 1$ and $\ell = 0, 1$,

$$
\|F_q(t)\| + t^{1-\alpha} \|E_q(t)\| \leq c \min(1, t^{-\alpha}) \quad \text{and} \quad \|A_q F_q(t)\| \leq ct^{-\alpha-\ell}.
$$

**Proof.** The estimates follow from [4, Theorems 6.4]. Indeed, by [4, Theorem 6.4(iv)] and Lemma 2.1,

$$
\|E_q(t)\| \leq t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_1(q) t^\alpha) \leq c t^{\alpha-1} \min(1, t^{-\alpha})
$$

The bound on $F_q(t)$ follows similarly. For the second estimate, the case $s = 1$, $\ell = 0, 1$, the assertion is contained in [4, Theorem 6.4(iii)], and the case $s = 0$, $\ell = 0$ is direct from the first estimate. The remaining case $s = 0$, $\ell = 1$ follows from Lemma 2.1 (noting $1/\Gamma(0) = 0$):

$$
\|F_q'(t)v\|_{L^2(\Omega)}^2 = \|A_q E_q(t)v\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} \lambda_n(q)^2 t^{2\alpha-2} E_{\alpha,\alpha}(-\lambda_n(q) t^\alpha)^2 (v, \varphi_n(q))^2
\leq ct^{2\alpha-2} \sum_{n=1}^{\infty} \frac{\lambda_n^2}{(1 + (\lambda_n t^\alpha)^2)^2} (v, \varphi_n(q))^2 \leq ct^{-2\alpha} \|v\|_{L^2(\Omega)}^2.
$$

Combining these assertions completes the proof of the lemma.

The next lemma gives a priori estimate on the solution $u$ to problem (4.1).

**Lemma 4.2.** Let $u_0, f \in L^2(\Omega)$ and $q \in \mathcal{A}$, and $u$ be the solution to problem (4.1). Then there exists $c > 0$ independent of $q$ and $t$ such that

$$
\|\partial_t u(t)\|_{H^2(\Omega)} \leq ct^{-\alpha-1},
\|\partial_t^2 u(t)\|_{L^2(\Omega)} + \|u(t)\|_{H^2(\Omega)} \leq c(1 + t^{-\alpha}).
$$

**Proof.** The proof employs (4.3) and Lemma 4.1. First, by (4.4), we have $\partial_t u(t) = F_q(t)(u_0 - \phi_q - A_q^{-1} f)$. Then from Lemma 4.1 and the norm equivalence (4.3), the first estimate follows

$$
\|\partial_t u(t)\|_{H^2(\Omega)} \leq c(\|A_q F_q(t)\| + \|F_q(t)\|) \|u_0 - \phi_q - A_q^{-1} f\|_{L^2(\Omega)} \leq ct^{-\alpha-1}.
$$

Similarly, by (4.4) and Lemma 4.1, there holds

$$
\|u(t)\|_{L^2(\Omega)} \leq \|F_q(t)u_0 + (1 - F_q(t))\phi_q + (1 - F_q(t))A_q^{-1} f\|_{L^2(\Omega)} \leq c,
\|u(t)\|_{H^2(\Omega)} \leq \|F_q(t)(u_0 - \phi_q) + (1 - F_q(t))A_q^{-1} f\|_{H^2(\Omega)} + \|\phi_q\|_{H^2(\Omega)}
\leq c \min(1, t^{-\alpha}) \|u_0 - \phi_q\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\phi_q\|_{H^2(\Omega)} \leq c(1 + t^{-\alpha}).
$$

The bound on $\partial_t^2 u$ is direct from the identity $\partial_t^2 u(t) = -A_q F_q(t)(u_0 + \phi_q - A_q^{-1} f)$ and Lemma 4.1.
For any $q \in A$, we denote the solution $u$ to problem (4.1) by $u(q)$. The next lemma provides a crucial \textit{a priori} estimate. Like before, we denote by $u$ and $\tilde{u}$ to be $u(q)$ and $u(\tilde{q})$ below.

**Lemma 4.3.** Let $T_* > 0$ be fixed, and $q, \tilde{q} \in A$. Then there exists $c > 0$ independent of $q, \tilde{q}$ and $t$ such that for any $t \geq T_* > 0$,

$$\|\partial^\alpha_\tau (u - \tilde{u})(t)\|_{H^2(\Omega)} \leq c t^{-\alpha} \|q - \tilde{q}\|_{L^2(\Omega)}.$$

**Proof.** Let $w = u - \tilde{u}$. Then $w$ solves

$$\begin{cases}
\partial^\alpha_\tau w - \Delta w + qw = (\tilde{q} - q)\tilde{u}, & \text{in } \Omega \times (0, T], \\
\partial^\alpha_\tau w = 0, & \text{on } \partial \Omega \times (0, T], \\
w(0) = 0, & \text{in } \Omega.
\end{cases}$$

(4.12)

The representation (2.3) implies $w(t) = \int_0^t E_q(t - s)(\tilde{q} - q)\tilde{u}(s) \, ds$. The governing equation for $w$ and the identities $\partial^\alpha_\tau F_q(t) = -A_q F_q(t)$ and $A_q E_q(t) = -F_q'(t)$ [4, Lemma 6.3] lead to

$$\partial^\alpha_\tau w(t) = -A_q \int_0^t E_q(t - s)(\tilde{q} - q)\tilde{u}(s) \, ds + (\tilde{q} - q)\tilde{u}(t)$$

$$= \int_0^t F_q'(t - s)(\tilde{q} - q)\tilde{u}(s) \, ds + (\tilde{q} - q)\tilde{u}(t) = \partial_t \int_0^t F_q(t - s)(\tilde{q} - q)\tilde{u}(s) \, ds.$$

Let $\phi(t) = \int_0^t F_q(t - s)(\tilde{q} - q)\tilde{u}(s) \, ds$. Sobolev embedding theorem and Lemmas 4.1 and 4.2 imply

$$\|A_q \phi(t)\|_{L^2(\Omega)} \leq \|\tilde{q} - q\|_{L^2(\Omega)} \int_0^t \|A_q F_q(t - s)\| \|\tilde{u}(s)\|_{L^\infty(\Omega)} \, ds$$

$$\leq c \|\tilde{q} - q\|_{L^2(\Omega)} \int_0^t (t - s)^{\alpha - 1} \|\tilde{u}(s)\|_{H^2(\Omega)} \, ds$$

$$\leq c (1 - \alpha + t^{-2\alpha}) \|\tilde{q} - q\|_{L^2(\Omega)} \leq c_T t^{1 - \alpha} \|\tilde{q} - q\|_{L^2(\Omega)}.$$

Next, the identity $t A_q \phi(t) = \int_0^t (t - s) A_q F_q(t - s)(\tilde{q} - q)\tilde{u}(s) \, ds + \int_0^t A_q F_q'(t - s)(\tilde{q} - q)(t - s)\tilde{u}(t - s) \, ds$ yields

$$\partial_t (t A_q \phi(t)) = \int_0^t [(t - s) A_q F_q'(t - s) + A_q F_q(t - s)](\tilde{q} - q)\tilde{u}(s) \, ds$$

$$+ \int_0^t A_q F_q'(s)(\tilde{q} - q)\tilde{u}(s) + (t - s)\tilde{u}'(t - s) \, ds =: I_1 + I_2.$$

Next we derive bounds for $I_1$ and $I_2$. First, we bound the term $I_1$ by Lemmas 4.1 and 4.2:

$$\|I_1\|_{L^2(\Omega)} \leq \int_0^t [(t - s) \|A_q F_q'(t - s)\| + \|A_q F_q(t - s)\| \] \|\tilde{q} - q\| \tilde{u}(s) \|_{L^2(\Omega)} \, ds$$

$$\leq c \|\tilde{q} - q\|_{L^2(\Omega)} \int_0^t (t - s)^{\alpha - 1} \|\tilde{u}(s)\|_{L^\infty(\Omega)} \, ds \leq c_T t^{1 - \alpha} \|\tilde{q} - q\|_{L^2(\Omega)}.$$

Similarly, by Lemma 4.1, Sobolev embedding theorem and Lemma 4.2, the term $I_2$ can be bounded as

$$\|I_2\|_{L^2(\Omega)} \leq \int_0^t \|A_q F_q(s)\| \|\tilde{q} - q\|_{L^2(\Omega)} [(t - s) \|\tilde{u}'(t - s)\|_{L^\infty(\Omega)} + \|\tilde{u}(t - s)\|_{L^\infty(\Omega)}] \, ds$$

$$\leq c \|\tilde{q} - q\|_{L^2(\Omega)} \int_0^t s^{-\alpha} [(t - s) \|\tilde{u}'(t - s)\|_{H^2(\Omega)} + \|\tilde{u}(t - s)\|_{H^2(\Omega)}] \, ds$$

$$\leq c \|\tilde{q} - q\|_{L^2(\Omega)} \int_0^t s^{-\alpha} [(t - s)^{-\alpha} + 1] \, ds \leq c_T t^{1 - \alpha} \|\tilde{q} - q\|_{L^2(\Omega)}.$$

Then the triangle inequality yields that for any $t > 0$, there holds

$$t \|A_q \phi'(t)\|_{L^2(\Omega)} \leq \|(t A_q \phi(t))'\|_{L^2(\Omega)} + \|A_q \phi(t)\|_{L^2(\Omega)} \leq c_T t^{1 - \alpha} \|\tilde{q} - q\|_{L^2(\Omega)}.$$

Now the desired inequality follows directly. This completes the proof of the lemma.
Next we give a stability result. It improves a known result [27, 26] by relaxing the regularity assumption.

**Theorem 4.2.** Let $u_0, f \in L^2(\Omega)$, with $u_0, f \geq m$ a.e. in $\Omega$ and $a_0, a_1 \geq m$ for some $m > 0$. Then for $q, \tilde{q} \in A$, and sufficiently large $T$, there exists $c > 0$ independent of $q, \tilde{q}$ and $T$ such that
\[
\|q - \tilde{q}\|_{L^2(\Omega)} \leq c\|u(T) - \tilde{u}(T)\|_{H^2(\Omega)}.
\]

**Proof.** It follows from (4.1) that $q$ can be expressed as
\[
q = [u(T)]^{-1}(f - \partial_\alpha^a u(T) + \partial_{xx} u(T)).
\]

Then we split the difference $q - \tilde{q}$ into
\[
q - \tilde{q} = f \frac{\tilde{u}(T) - u(T)}{u(T)\tilde{u}(T)} + \frac{u(T)\partial^a \tilde{u}(T) - \tilde{u}(T)\partial^a u(T)}{u(T)\tilde{u}(T)} + \frac{\tilde{u}(T)\partial_{xx} u(T) - u(T)\partial_{xx} \tilde{u}(T)}{u(T)\tilde{u}(T)} = \sum_{i=1}^3 I_i.
\]

By the maximum principle of time-fractional diffusion [18], we deduce $u(T) \geq m > 0$. This and the standard Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ (for $d = 1, 2, 3$) imply
\[
\|I_1\|_{L^2(\Omega)} \leq m^{-2}\|f\|_{L^2(\Omega)}\|\tilde{u}(T) - u(T)\|_{H^2(\Omega)}.
\]

By Lemma 4.2 and Sobolev embedding theorem, we have the *a priori* bound $\|u(T)\|_{L^\infty(\Omega)} + \|\partial_\alpha^a u(T)\|_{L^2(\Omega)} \leq cT^{-\alpha}$ for $T \geq T^*$. This and Lemma 4.3 lead to
\[
\|I_2\|_{L^2(\Omega)} \leq c\|u(T)\|_{L^\infty(\Omega)}\|f\|_{L^2(\Omega)} + \|\partial_\alpha^a u(T)\|_{L^2(\Omega)}\|u(T)\|_{L^\infty(\Omega)} + \|\partial_\alpha^a u(T)\|_{L^2(\Omega)}\|\tilde{u}(T)\|_{L^\infty(\Omega)}
\]

Then for sufficiently large $T$, we have
\[
\|q - \tilde{q}\|_{L^2(\Omega)} \leq c(1 - cT^{-\alpha})^{-1}\|u(T) - \tilde{u}(T)\|_{H^2(\Omega)}.
\]

This completes the proof of the theorem. \(\blacksquare\)

The next stability estimate is the main result of this section.

**Theorem 4.3.** Let $u_0, f \in L^2(\Omega)$, with $u_0, f \geq m$ a.e. in $\Omega$ and $a_0, a_1 \geq m$ for some $m > 0$. Then for $q, \tilde{q} \in A$, and $T_0 \leq T \leq \tilde{T}$ with sufficiently large $T_0$, there exists $c$ independent of $q, \tilde{q}$, $T$ and $\tilde{T}$ such that
\[
\|q - \tilde{q}\|_{L^2(\Omega)} \leq C\|u(T) - \tilde{u}(\tilde{T})\|_{H^2(\Omega)} + T^{-\alpha - 1}\|T - \tilde{T}\|.
\]

**Proof.** In view of the identity (4.13), we have the following splitting
\[
q - \tilde{q} = \left(\frac{f - \partial_\alpha^a u(T) + \partial_{xx} u(T)}{u(T)} - \frac{f - \partial_\alpha^a \tilde{u}(T) + \partial_{xx} \tilde{u}(T)}{\tilde{u}(T)}\right) =: I_1 + I_2.
\]

Theorem 4.2 implies the following estimate on $I_1$:
\[
\|I_1\|_{L^2(\Omega)} \leq c(T^{-\alpha}\|q - \tilde{q}\|_{L^2(\Omega)} + \|u(T) - \tilde{u}(T)\|_{H^2(\Omega)}).
\]

The triangle inequality and Lemma 4.2 with $\gamma = 2$ lead to
\[
\|u(T) - \tilde{u}(T)\|_{H^2(\Omega)} \leq \|u(T) - \tilde{u}(\tilde{T})\|_{H^2(\Omega)} + \|\tilde{u}(T) - \tilde{u}(\tilde{T})\|_{H^2(\Omega)}
\leq \|u(T) - \tilde{u}(\tilde{T})\|_{H^2(\Omega)} + cT^{-\alpha - 1}\|T - \tilde{T}\|,
\]

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By choosing $T$ with the scalars $\tilde{\Lambda}$ respectively. Then the following estimate holds.

**Corollary 4.1.** By the maximum principle [18], we have $u(T), \tilde{u}(\tilde{T}) \geq 0$. This and Lemma 4.2 lead to

$$\|I_{2,1}\|_{L^2(\Omega)} \leq m^{-2}\|f\|_{L^2(\Omega)}\|\tilde{u}(\tilde{T}) - \tilde{u}(T)\|_{H^2(\Omega)} \leq cT^{-\alpha-1}|\tilde{T} - T|.$$  

Meanwhile, by Sobolev embedding theorem and Lemma 4.2, we obtain

$$\|I_{2,2}\|_{L^2(\Omega)} \leq c\|\tilde{u}(\tilde{T})\|_{L^2(\Omega)}\|\tilde{u}(T) - \tilde{u}(\tilde{T})\|_{H^2(\Omega)} + \|\tilde{u}(T)\|_{H^2(\Omega)}\|\tilde{u}(T) - \tilde{u}(\tilde{T})\|_{H^2(\Omega)} \leq cT^{-\alpha-1}|\tilde{T} - T|.$$  

Combining the preceding estimates yields

$$\|q - \tilde{q}\|_{L^2(\Omega)} \leq cT^{-\alpha}\|q\|_{L^2(\Omega)} + c\|u(T) - \tilde{u}(T)\|_{H^2(\Omega)} + cT^{-\alpha-1}|\tilde{T} - T|.$$  

By choosing $T_0$ large enough such that $cT_0^{-\alpha} \leq \frac{1}{2}$, we deduce that for $\tilde{T} \geq T \geq T_0$, the desired estimate holds.

The next corollary of Theorem 4.3 bounds the terminal time $T$ for perturbed data.

**Corollary 4.1.** Suppose that $u_0 \in L^2(\Omega) \setminus \hat{H}^s(\Omega)$, $f \in \hat{H}^s(\Omega)$ and $q \in A \cap \hat{H}^s(\Omega)$, with $s \in (0, \frac{1}{2})$ and $s_1 \in (0, s)$. Let $(T, q), (\tilde{T}, \tilde{q}) \in \mathbb{R}_+ \times A \cap \hat{H}^s(\Omega)$ be the solutions of IPP with observations $u(T)$ and $\tilde{u}(\tilde{T})$, respectively. Then the following estimate holds

$$|T - \tilde{T}| \leq \Gamma(1 - \alpha)^{-\frac{1}{2}} \min(\Lambda, \tilde{\Lambda})^{-\frac{1}{2} - 1}|\Lambda - \tilde{\Lambda}|,$$

with the scalars $\Lambda$ and $\tilde{\Lambda}$ respectively given by

$$\Lambda = \lim_{n \in \mathbb{K}, n \to \infty} -\frac{(\partial_{xx} u(T), \sin(n\pi x))}{(u_0 - \phi_0, \sin(n\pi x))} \quad \text{and} \quad \tilde{\Lambda} = \lim_{n \in \mathbb{K}, n \to \infty} -\frac{(\partial_{xx} \tilde{u}(\tilde{T}), \sin(n\pi x))}{(u_0 - \phi_0, \sin(n\pi x))}.$$  

In particular, for $\Lambda < \tilde{\Lambda}$, there holds $\|q - \tilde{q}\|_{L^2(\Omega)} \leq c\|u(T) - \tilde{u}(\tilde{T})\|_{H^2(\Omega)} + |\Lambda - \tilde{\Lambda}|.$

## 5 Numerical experiments and discussions

In this section we present numerical results to illustrate simultaneous recovery of the spatially dependent parameter and terminal time $T$.

### 5.1 Numerical algorithm

First we describe a numerical algorithm for reconstruction. The inverse problems involve two parameters: unknown time $T$ and space-dependent parameter $v$ ($u_0$, $\psi$ or $q$). Since these two parameters have different influence on the measured data $u(T)$, standard iterative regularization methods, e.g., Landweber method and conjugate gradient method, do not work very well. We employ the Levenberg-Marquardt method [13, 19], which has been shown to be effective for solving related inverse problems [16]. Due to the ill-posedness of the inverse problems, early stopping is required in order to obtain good reconstructions.
Specifically, we define a nonlinear operator \( F : (v, T) \in L^2(\Omega) \times \mathbb{R}_+ \to u(v)(x, T) \in L^2(\Omega) \), where \( u(v) \) solves problem (1.1), with the parameter \( v \). Let \( (v^0, T^0) \) be the initial guess of the unknowns \( (v, T) \). Now given the approximation \((v^k, T^k)\), we find the next approximation \((v^{k+1}, T^{k+1})\) by

\[
(v^{k+1}, T^{k+1}) = \arg \min J_k(v, T),
\]

with the functional \( J_k(v, T) \) at the \( k \)th iteration (based at \((v^k, T^k)\)) given by

\[
J_k(v, T) = \frac{1}{2} \| F(v^k, T^k) - g^\delta + \partial_v F(v^k, T^k)(v - v^k) + \partial_T F(v^k, T^k)(T - T^k) \|^2_{L^2(\Omega)} + \frac{\gamma^k}{2} \| v - v^k \|^2_{L^2(\Omega)} + \frac{\mu^k}{2} \| T - T^k \|^2,
\]

where \( \gamma^k > 0 \) and \( \mu^k > 0 \) are regularizations parameters, and \( \partial_v F(v^k, T^k) \) and \( \partial_T F(v^k, T^k) \) are the derivatives of the forward map \( F \) in \( v \) and \( T \), respectively. We employ two parameters since \( v \) and \( T \) influence the data \( u(T) \) differently. The parameters \( \gamma \) and \( \mu \) are often decreased geometrically with \( \rho \in (0, 1) \): \( \gamma^{k+1} = \rho \gamma^k \) and \( \mu^{k+1} = \rho \mu^k \). The derivative \( \partial_v F(v, T) \) can be evaluate explicitly. For example, for \( \text{BP} \), the (directional) derivative \( w = \partial_v F(\hat{v}, T)[h] \) (in the direction \( h \)) satisfies

\[
\begin{cases}
\partial_v^2 w - \Delta w + qw = 0, & \text{in } \Omega \times (0, T), \\
w = 0, & \text{on } \partial \Omega \times (0, T), \\
w(0) = h, & \text{in } \Omega.
\end{cases}
\]

To approximate the derivative \( \partial_T F(v, T) \), we use the finite difference \( \partial_T F(v, T) \approx (\delta T)^{-1}(F(v, T + \delta T) - F(v, T)) \), where \( \delta T \) is a small number, fixed at \( \delta T = 1 \times 10^{-3} \) below. Due to the quadratic structure of the functional \( J_k(v, T) \), the increments \( \delta v^k := v^{k+1} - v^k \) and \( \delta T^k := T^{k+1} - T^k \) satisfy

\[
\begin{bmatrix}
J_v^* J_v + \mu^{k+1} I & J_v^* J_T \\
J_T^* J_v & J_T^* J_T + \gamma^{k+1}
\end{bmatrix}
\begin{bmatrix}
\delta v^k \\
\delta T^k
\end{bmatrix}
= \begin{bmatrix}
J_v^* (g^\delta - F(v^k, T^k)) \\
J_T^* (g^\delta - F(v^k, T^k))
\end{bmatrix},
\]

where \( * \) denotes the adjoint operator, and \( J_v = \partial_v F(v^k, T^k) \) and \( J_T = \partial_T F(v^k, T^k) \).

### 5.2 Numerical illustrations

Now we present numerical results for the three problems, and with the domain \( \Omega = (0, 1) \) in 1D and \( \Omega = (0, 1)^2 \) in 2D, and the terminal time \( T = 0.5 \). We discretize problem (1.1) using the Galerkin finite element method with continuous piecewise linear functions in space, and \( L^1 \) approximation in time [5, 6]. The accuracy of a reconstruction \( \hat{u} \) relative to the exact one \( u^1 \) is measured by the \( L^2(\Omega) \) error \( e(\hat{u}) = \| \hat{u} - u^1 \|_{L^2(\Omega)} \).

The residual \( r(\hat{v}) \) of the recovered tuple \( (\hat{v}, \hat{T}) \) is computed as \( r(\hat{v}) = \| F(\hat{v}, \hat{T}) - g^\delta \|_{L^2(\Omega)} \). The exact data \( g^1 \) is generated on a fine space-time mesh. The noisy data \( g^\delta \) is generated from \( g^1 \) by \( g^\delta(x) = g^1 + \epsilon \| g^1 \|_{L^\infty(\Omega)} \xi(x) \), where \( \xi(x) \) follows the standard Gaussian noise, and \( \epsilon > 0 \) indicates the noise level.

The first example is \( \text{BP} \), with \( q \equiv 0 \).

**Example 5.1.**

(i) The source \( f = \min(x, 1 - x) \), and the unknown initial condition \( u_0 = \sin(\pi x) \).

(ii) The diffusion coefficient \( a = 1 + \sin(\pi x) y(1 - y) \), the source \( f = \min(x, 1 - x) e^x \sin(2\pi y) \), and the unknown initial condition \( u_0 = \sin(\pi x) \sin(\pi y) \).

The convergence of the Levenberg-Mardquardt method is shown in Fig. 1. For the exact data, the residual \( r \) decreases rapidly to zero, and the error \( e \) also decreases steadily and eventually levels off at 1e-3 (due to the presence of discretization error). For noisy data, the method exhibits a typical semi-convergence behavior: the error \( e \) first decreases and then starts to increase rapidly afterwards, necessitating the use of early stopping. This behavior is also observed for the estimated terminal time \( T \), but it appears to be more resilient to the iteration number \( k \) and it does not change much after a few extra iterations. Nonetheless, the estimate \( \hat{T} \) will eventually drift away when the method is run for too many iterations. Exemplary reconstructions of the initial data \( u_0 \) are shown in Figs. 2 and 3, which has the smallest \( L^2(\Omega) \) error along
the iteration trajectory; see Table 1 for the stopping index \( k^* \). These plots show that the reconstructions are accurate for up to 5% noise in the data. See also Table 1 for quantitative results. Not that the accuracy \( e \) does not depend very much on the order \( \alpha \), and decreases as the noise level \( \epsilon \to 0^+ \). This agrees with the Lipschitz stability estimate in Theorem 2.2. However, the reconstructions for case (ii) tend to be less accurate than case (i), which is attributed to discretization errors.

Figure 1: The convergence of Levenberg-Marquadt method for Example 5.1(i), for \( \alpha = 0.5 \).

Table 1: The numerical results (error \( e \), stopping index \( k^* \) and recovered \( \hat{T} \)) for Example 5.1. For case (i), \( \gamma_0 = 1e-2, \mu_0 = 2.7e-3, 6.3e-3 \) and \( 1.3e-2 \) for \( \alpha = 0.25, 0.50 \) and 0.75, respectively, and \( \rho = 0.80 \); for case (ii), \( \gamma_0 = 1e-3, \mu_0 = 1.1e-4, 2.7e-4 \), and \( 6e-4 \) for \( \alpha = 0.25, 0.50 \) and 0.75, respectively, and \( \rho = 0.80 \).

| case | \( \alpha \)   | \( 0.25 \) | \( 0.50 \) | \( 0.75 \) |
|------|---------------|-----------|-----------|-----------|
|      | \( \epsilon \) | \( e \)    | \( k^* \) | \( \hat{T} \) | \( e \)    | \( k^* \) | \( \hat{T} \) | \( e \)    | \( k^* \) | \( \hat{T} \) |
| (i)  | 0e-3 | 1.221e-3 | 16 | 0.496 | 9.948e-4 | 20 | 0.498 | 5.023e-4 | 31 | 0.500 |
|      | 1e-3 | 1.455e-3 | 8  | 0.496 | 1.210e-3 | 8  | 0.499 | 7.217e-4 | 10 | 0.500 |
|      | 5e-3 | 2.961e-3 | 7  | 0.496 | 2.468e-3 | 7  | 0.499 | 1.852e-3 | 9  | 0.500 |
|      | 1e-2 | 4.793e-3 | 6  | 0.498 | 4.114e-3 | 7  | 0.499 | 3.275e-3 | 9  | 0.500 |
|      | 2e-2 | 8.121e-3 | 6  | 0.498 | 6.872e-3 | 6  | 0.501 | 5.578e-3 | 8  | 0.502 |
|      | 5e-2 | 1.675e-2 | 5  | 0.505 | 1.476e-2 | 6  | 0.502 | 1.230e-2 | 8  | 0.503 |
| (ii) | 0e-3 | 7.368e-2 | 32 | 0.496 | 7.414e-2 | 33 | 0.500 | 7.586e-2 | 36 | 0.498 |
|      | 1e-3 | 7.574e-2 | 15 | 0.497 | 7.622e-2 | 17 | 0.500 | 7.799e-2 | 20 | 0.498 |
|      | 5e-3 | 8.230e-2 | 11 | 0.497 | 8.315e-2 | 13 | 0.500 | 8.648e-2 | 17 | 0.498 |
|      | 1e-2 | 9.119e-2 | 10 | 0.497 | 9.247e-2 | 11 | 0.500 | 9.737e-2 | 15 | 0.498 |
|      | 3e-2 | 1.169e-1 | 6  | 0.497 | 1.186e-1 | 7  | 0.500 | 1.235e-1 | 7  | 0.499 |
|      | 5e-2 | 1.281e-1 | 4  | 0.499 | 1.275e-1 | 4  | 0.503 | 1.300e-1 | 6  | 0.501 |

The next example is about \( \text{ISP} \), with \( q \equiv 0 \).

**Example 5.2.**

(i) The initial condition \( u_0 = \sin(2\pi x) \), and the unknown source \( \psi(x) = \sin(3\pi x) \).

(ii) The known diffusion coefficient \( a = 1 + \sin(\pi x)(y(1-y)) \), initial condition \( u_0 = \sin(\pi x) \sin(\pi y) \), and the unknown source \( \psi = 4x(1-x)e^x \sin(2\pi y) \).

The numerical results for Example 5.2 are shown in Figs. 4, 5 and 6, and Table 2. The convergence plots in Fig. 4 show the semiconvergence phenomenon, and with the chosen parameters, the method converges rapidly to an acceptable solution, and then the error \( e \) starts to increase shortly afterwards. Nonetheless, the estimate of \( T \) converges fairly fast (within 5 iterations), and it is also quite stable during the iteration. We obtain very accurate reconstructions for the noise level \( \epsilon \) up to \( 5e-2 \). Like before, the reconstruction quality
does not depend much on the order $\alpha$, cf. Table 2 and Fig. 5, concurring with the observations for BP. The latter also agrees with the fact that ISP enjoys similar stability as BP, as indicated by Theorems 2.2 and 3.2.

The last example is about IPP in 1D.

**Example 5.3.** The source $f = |\sin(2\pi x)|$, initial condition $u_0 \equiv 1$, and a zero Dirichlet boundary condition. The unknown potential $q = \sin^2(\pi x)$.

The parameter $\gamma_0$ is fixed at 1e-7 and $\mu_0$ at 1e-8, and the decreasing factor $\rho$ is set to 0.5. The numerical results are summarized in Figs. 7 and 8. Note that we can obtain highly accurate reconstructions for exact data, with the $L^2(\Omega)$ error of the recovered potential $\hat{q}$ being 1.318e-3, 1.374e-3 and 1.174e-3 for $\alpha = 0.25$, 0.50 and 0.75, respectively. The estimated terminal time $T = 0.4996, 0.4997$ and 0.4998 for $\alpha = 0.25, 0.50$ and 0.75, respectively, are also fairly accurate. This clearly shows the feasibility of simultaneous recovery. However, for noisy data, the recovery is very challenging. Numerically we observe that the singular value
Table 2: The numerical results (error e, stopping index \( k^* \) and recovered \( \hat{T} \)) for Example 5.2. For case (i), \( \gamma_0 = 1e-4, \mu_0 = 1e-8, 5e-8 \) and \( 1e-7 \) for \( \alpha = 0.25, 0.50 \) and 0.75, respectively, and \( \rho = 0.8 \); and for case (ii), \( \gamma_0 = 1e-4, \mu_0 = 1e-9 \) and \( \rho = 0.8 \).

| case | \( \alpha \) | 0.25  | 0.50  | 0.75  |
|------|------------|------|------|------|
|     | e         | e    | \( k^* \) | \( \hat{T} \) | e    | e    | \( k^* \) | \( \hat{T} \) | e    | e    | \( k^* \) | \( \hat{T} \) |
| (i) | 0e-3      | 2.534e-3 | 6  | 0.507 | 2.709e-3 | 6  | 0.505 | 3.057e-3 | 6  | 0.506 |
|     | 1e-3      | 3.402e-3 | 6  | 0.507 | 3.381e-3 | 6  | 0.504 | 3.417e-3 | 6  | 0.506 |
|     | 5e-3      | 1.998e-2 | 5  | 0.506 | 1.000e-2 | 5  | 0.504 | 8.016e-3 | 6  | 0.505 |
|     | 1e-2      | 2.035e-2 | 5  | 0.505 | 1.825e-2 | 5  | 0.503 | 1.410e-2 | 5  | 0.505 |
|     | 2e-2      | 3.839e-2 | 4  | 0.503 | 3.488e-2 | 4  | 0.502 | 2.669e-2 | 5  | 0.504 |
|     | 5e-2      | 8.668e-2 | 4  | 0.496 | 7.761e-2 | 4  | 0.499 | 5.944e-2 | 4  | 0.501 |
| (ii)| 0e-3      | 2.122e-1 | 35 | 0.514 | 2.121e-1 | 35 | 0.508 | 2.119e-1 | 35 | 0.509 |
|     | 1e-3      | 2.173e-1 | 9  | 0.514 | 2.171e-1 | 9  | 0.508 | 2.166e-1 | 10 | 0.509 |
|     | 5e-3      | 2.222e-1 | 4  | 0.515 | 2.218e-1 | 4  | 0.509 | 2.206e-1 | 5  | 0.509 |
|     | 1e-2      | 2.276e-1 | 3  | 0.516 | 2.267e-1 | 3  | 0.509 | 2.250e-1 | 3  | 0.509 |
|     | 3e-2      | 2.626e-1 | 3  | 0.522 | 2.576e-1 | 3  | 0.512 | 2.477e-1 | 3  | 0.512 |
|     | 5e-2      | 2.843e-1 | 2  | 0.519 | 2.797e-1 | 2  | 0.507 | 2.707e-1 | 2  | 0.500 |

Figure 5: The reconstructions of the space-dependent source \( \psi \) for Example 5.2(i).

The spectrum of the linearized forward operator has many tiny values (and hence we have to use a tiny value for the parameter \( \gamma_0 \)), which precludes applying any realistic amount of noise to the data and renders the recovery from noisy data highly unstable.

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Figure 6: The reconstructions of the space source component $\psi$ for Example 5.2(ii) with $\alpha = 0.5$, $\epsilon = 5\%$.

Figure 7: The convergence of the Levenberg-Marquadt method for Example 5.3 with exact data.

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Figure 8: The reconstructions of the space-dependent potential $q$ for Example 5.3 with exact data.
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