On the duality between the hyperbolic Sutherland and the rational Ruijsenaars–Schneider models

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Abstract

We consider two families of commuting Hamiltonians on the cotangent bundle of the group \(GL(n, \mathbb{C})\), and show that upon an appropriate single symplectic reduction they descend to the spectral invariants of the hyperbolic Sutherland and of the rational Ruijsenaars–Schneider Lax matrices, respectively. The duality symplectomorphism between these two integrable models that was constructed by Ruijsenaars using direct methods can then be interpreted geometrically simply as a gauge transformation connecting two cross sections of the orbits of the reduction group.

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1. Introduction

Around 20 years ago, Ruijsenaars [1] undertook a task of constructing action-angle variables for the non-relativistic and relativistic Calogero models of type \(A_n\) (for reviews of these models see, e.g., [2–5]). In each case, he made use of a commutation relation satisfied by the Lax matrix of the model under study and another matrix function of the phase-space variables, which he exhibited directly. By conjugating these matrices so as to make the Lax matrix diagonal, he noted that the conjugate of the other matrix becomes the Lax matrix of another Calogero-type model whose particle-position variables are furnished by the eigenvalues of the original Lax matrix, i.e., the action variables of the original model. The so-obtained duality between model 1 and model 2 is thus characterized by the fact that the action variables of model 1 are the particle-position variables of model 2, and vice versa. This observation was used in [1] to derive integration algorithms for the commuting flows and to calculate the scattering data. The simplest manifestation of the duality occurs in the rational Calogero model, which is actually self-dual [1, 6]. The self-duality of this model admits a nice geometric ‘explanation’
in terms of the symplectic reduction due to Kazhdan, Kostant and Sternberg [7, 2]. As it will serve as a paradigm motivating our considerations, we next outline this explanation in some detail.

In fact, Kazhdan, Kostant and Sternberg reduced the cotangent bundle,

\[ T^*u(n) \cong u(n) \times u(n) = \{(x, y)\}, \]

by means of the adjoint action of the group \( U(n) \), imposing the moment map constraint

\[ [x, y] = i\kappa (I_n - ww^\dagger) = \mu_\kappa, \]

where all \( n \) components of the column vector \( w \) are equal to 1, and \( \kappa \) is a real constant. The evaluation functions \( X(x, y) := ix \) and \( Y(x, y) := iy \) can be viewed as ‘unreduced Lax matrices’ since \( \{\text{tr}(X^k)\} \) and \( \{\text{tr}(Y^k)\} \) form two Abelian subalgebras in the Poisson algebra \( C^\infty(T^*u(n)) \). These Abelian algebras survive the reduction, because their elements are \( U(n) \) invariants. If one describes the reduced phase space in terms of a gauge slice where \( X \) is diagonal, then—by solving the moment map constraint—\( Y \) becomes the Lax matrix of the rational Calogero model whose action variables are the eigenvalues of \( Y \). If one describes the reduced phase space in terms of a gauge slice where \( Y \) is diagonal, then \( X \) becomes the Lax matrix of the ‘dual Calogero model’. The correspondence between the variables of the two Calogero models is obviously a symplectomorphism, as it represents the transformation between two gauge slices realizing the same reduced phase space. The self-duality stems from the symmetrical roles of \( x \) and \( y \), and the commutation relation of the Lax matrices is just the constraint (1.2) in disguise.

Ruijsenaars hinted in [1, 8, 9] that there might exist a similar geometric picture behind the duality in other cases as well, which he left as a problem for ‘the aficionados of Lie theory’. Later Gorsky and coworkers [10–12] (see also [13, 14]) introduced interesting new ideas and confirmed this expectation in several cases. In particular, they derived the local version of the so-called IIIb trigonometric Ruijsenaars–Schneider model [3, 9, 15], by reducing a Hamiltonian system on the magnetic cotangent bundle of the loop group of \( U(n) \) [10]4. The investigations in [10–14] focused on the local aspects and did not touch on the quite tricky global definition of the pertinent gauge slices, which is necessary to obtain complete commuting flows. We believe, however, that the reduction approach works also globally and that it is possible to characterize the Ruijsenaars duality in a finite-dimensional symplectic reduction picture in all cases studied in [1, 8, 9]. We plan to explore this issue systematically in a series of papers, and here we report the first results of our analysis.

In this paper, we study a case of the duality which has not been previously described in the symplectic reduction framework. Namely, we expound the geometric picture that links together the dual pair consisting of the hyperbolic Sutherland model and the rational Ruijsenaars–Schneider model [1]. In section 2, we start with two sets of ‘canonical integrable systems’ on the cotangent bundle of the real Lie group \( GL(n, \mathbb{C}) \), whose Hamiltonians span two commutative families, \( \{H_j\} \) and \( \{\hat{H}_k\} \). By ‘canonical integrability’ we simply mean that one can directly write down the Hamiltonian flows. Then, in section 3, we describe a symplectic reduction of \( T^*GL(n, \mathbb{C}) \) for which our canonical integrable systems descend to the reduced phase space. By using the shifting trick of symplectic reduction, we exhibit two distinguished cross sections of the orbits of the gauge group that define two models of the reduced phase space. In terms of cross section \( S_1 \), the family \( \{H_j\} \) translates into the action variables of the Sutherland model and the family \( \{\hat{H}_k\} \) becomes equivalent to the Sutherland particle coordinates. Cross section \( S_1 \) is described by theorem 1, which summarizes well-known results.

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4 It is worth noting that besides the IIIb model there exist also other important, physically different real forms [3, 9] of the complex trigonometric Ruijsenaars–Schneider model.
In terms of cross section $S_2$, the family $\{H_j\}$ translates into the coordinate variables of the rational Ruijsenaars–Schneider model and the family $\{\hat{H}_k\}$ gives the action variables of this model. We call cross section $S_2$ the ‘Ruijsenaars gauge slice’. Its characterization by theorem 2 is our principal technical result.

The duality symplectomorphism between the hyperbolic Sutherland and the rational Ruijsenaars–Schneider model will be realized as the gauge transformation between the cross sections $S_1$ and $S_2$ mentioned above. In addition, analogously to the case of the rational Calogero model, the symplectic reduction immediately yields integration algorithms for the commuting flows of the dual pair of models, and allows us to recognize the commutation relations of the Lax matrices used by Ruijsenaars as equivalents to the moment map constraint of the reduction. These consequences of theorem 1 and theorem 2 are developed in section 4.

The self-contained presentation of the relatively simple example of the Ruijsenaars duality that follows may also facilitate the geometric understanding of this remarkable phenomenon in more complicated cases.

2. Canonical integrable systems on $T^*GL(n, \mathbb{C})$

Here we describe the two families of canonical integrable systems and their symmetries that will be used to derive the dual pair of integrable many-body models by symplectic reduction. For a general reference on symplectic reduction, we mention the textbook [16].

Consider the real Lie algebra $\mathcal{G} := gl(n, \mathbb{C})$ and equip it with the invariant bilinear form$
\langle X, Y \rangle := \Re \text{tr}(XY) \quad \forall X, Y \in \mathcal{G},$
which allows us to identify $\mathcal{G}$ with $\mathcal{G}^*$ by the map $j : \mathcal{G}^* \rightarrow \mathcal{G}$ as$\langle j(\alpha), X \rangle = \alpha(X) \quad \forall \alpha \in \mathcal{G}^*, X \in \mathcal{G}.$

Then use left trivialization to obtain a model of the cotangent bundle of the real Lie group $G := GL(n, \mathbb{C})$ as$T^*G \simeq G \times \mathcal{G} = \{(g, J^R) \mid g \in G, J^R \in \mathcal{G}\},$
where $\alpha_g \in T^*_gG$ is represented by $(g, J^R \circ L^*_g(\alpha_g)) \in G \times \mathcal{G}$ with the left translation $L_g \in \text{Diff}(G)$. In terms of this model, the canonical symplectic form $\Omega$ of $T^*G$ takes the form$\Omega = d(J^R, g^{-1} \, dg).$

Next introduce the matrix functions $L_1$ and $L_2$ on $T^*G$ by the definitions
$L_1(g, J^R) := J^R$ and $L_2(g, J^R) := gg^\dagger.$

We may think of these as ‘unreduced Lax matrices’ since they generate the Hamiltonians
$H_j := \frac{1}{j} \Re \text{tr}(L_j^1), \quad j = 1, \ldots, n,$
and
$\hat{H}_k := \frac{1}{2k} \text{tr}(L_k^2), \quad k = \pm 1, \ldots, \pm n,$
so that both $\{H_j\}$ and $\{\hat{H}_k\}$ form commuting sets with respect to the Poisson bracket on the phase space $T^*G$.

We could have admitted also the imaginary part of the trace in (2.6) and the Hamiltonians in (2.7) are not functionally independent, but these are the definitions that will prove convenient for us.
It is easy to determine the flows of the Hamiltonians introduced above, through any initial value \((g(0), J^R(0))\). In fact, the flow belonging to \(H_j(2.6)\) is given by
\[
g(t) = g(0) \exp(t(J^R(0))^{-1}), \quad J^R(t) = J^R(0).
\] (2.8)
The flow generated by \(H_k(2.7)\) reads as
\[
J^R(t) = J^R(0) - t\, (g^\dagger(0)g(0))^k, \quad g(t) = g(0).
\] (2.9)

We are going to reduce the phase space \(T^*G\) by using the symmetry group \(K = U(n)L \times U(n)R\), (2.10)
where the notation reflects the fact that the two \(U(n)\) factors operate by left and right multiplications, respectively. This means that an element \((\eta_L, \eta_R) \in K\) (with \(\eta_{L,R} \in U(n)\)) acts by the symplectomorphism \(\Psi_{\eta_L, \eta_R}\) defined by
\[
\Psi_{\eta_L, \eta_R}(g, JR) := (\eta_L g \eta_R^{-1}, \eta_R J^R \eta_R^1).
\] (2.11)
This action is generated by an equivariant moment map. To describe this map, let us note that every \(X \in G\) can be uniquely decomposed (the Cartan decomposition) as
\[
X = X_+ + X_- \quad \text{with} \quad X_+ \in u(n), \quad X_- \in iu(n),
\] (2.12)
i.e., into anti-Hermitian and Hermitian parts. Identify \(u(n)\) with \(u(n)^*\) by the ‘scalar product’ \((\cdot, \cdot)\) restricted to \(u(n) \subset G\). Then the moment map \(\Phi : T^*G \to u(n)^L \oplus u(n)^R\) reads
\[
\Phi(g, J^R) = ((gJ^Rg^{-1})_+, -J^R).
\] (2.13)

The Hamiltonians \(H_j\) and \(\hat{H}_k\) are invariant under the symmetry group \(K\). Hence the commutative character of the families \(\{H_j\}\) and \(\{\hat{H}_k\}\) survives any symplectic reduction based on this symmetry group. It is also clear that the flows of the reduced Hamiltonians will be provided as projections of the above-given obvious flows to the reduced phase space. However, in general it is a matter of ‘art and good luck’ to find a value of the moment map that leads to interesting reduced systems. In the present case, it is well known [2] that by setting the \(u(n)^R\)-component of the moment map \(\Phi\) to zero, and by setting the \(u(n)^L\)-component equal to the constant \(\mu_\kappa\) defined in (1.2), one obtains the hyperbolic Sutherland model from the Hamiltonian system \((T^*G, \Omega, H_2)\).

Our goal is to characterize the reduced Hamiltonian systems coming from \((T^*G, \Omega, H_j)\) and from \((T^*G, \Omega, \hat{H}_k)\). To describe the latter systems, it will be technically very convenient to make use of the so-called shifting trick of symplectic reduction. This means that before performing the reduction we extend the phase space by a coadjoint orbit. In the present case, we consider the \(U(n)\) orbit through \(-\mu_\kappa\) given by
\[
O^L_\kappa := \{i\kappa (vv^\dagger - 1_n) | v \in \mathbb{C}^n, |v|^2 = n\}.
\] (2.14)
The orbit carries its own (Kirillov–Kostant–Souriau) symplectic form, which we denote by \(\Omega^\theta\). The vector \(v\) matters only up to the phase and \((O^L_\kappa, \Omega^\theta)\) can be identified as a copy of \(\mathbb{C}P_{n-1}\) endowed with a multiple of the Kähler form defined by the Fubini–Study metric.

3. Symplectic reduction of the extended phase space

The extended phase space to consider now is
\[
T^*G \times O^L_\kappa = \{(g, J^R, \xi)\}
\] (3.1)
with the symplectic form
\[
\Omega^{ext} = \Omega + \Omega^\theta.
\] (3.2)
The symmetry group $K$ acts by the symplectomorphisms $\Psi_{\eta_1, \eta_2}^{\text{ext}}$ given by
\[
\Psi_{\eta_1, \eta_2}^{\text{ext}}(g, J^R, \xi) := (\eta_1 g^{-1} \eta_R^* \eta_1, g^{-1} \eta_R^* \eta_R, \eta_2 \xi),
\]
and the corresponding moment map $\Phi_{\text{ext}}$ is
\[
\Phi_{\text{ext}}(g, J^R, \xi) = ((g J^R g^{-1})^*, \xi, -J^R).
\]
We are going to reduce at the zero value of the extended moment map, i.e., we wish to describe the reduced phase space
\[
T^*G \times O_2^L/\partial K.
\]
In our case, this space of $\Omega_1$ and the corresponding moment map $\Phi_{\text{ext}}$ is
\[
\pi_{\Omega_1}/\pi_{\Phi_1}
\]
where $G/U(n)$ acts by the symplectomorphisms $K$. This means that the obvious action of $T^*G \times O_2^L$ can always do without the shifting trick: in our case we have
\[
\text{the reduced phase space defined by (Marsden–Weinstein) point reduction on the right-hand side of the equality.}
\]
Third, if we define
\[
\xi(v) := i\kappa(v^\dagger - 1_n) \quad \forall v \in \mathbb C^n \quad \text{with} \quad |v|^2 = n,
\]
then we have
\[
\eta \xi(v) \eta^{-1} = \xi(\eta v) \quad \forall \eta \in U(n).
\]
This means that the obvious action of $U(n)$ on $\mathbb C^n$ underlies the coadjoint action on the orbit $O_2^L$, and we shall use this to transform the $\mathbb C^n$-vectors $v$ that correspond to the elements $\xi(v)$. Fourth, we define the extended Hamiltonians $H_j^{\text{ext}}$ and $H_k^{\text{ext}}$ by
\[
H_j^{\text{ext}}(g, J^R, \xi) := H_j(g, J^R), \quad H_k^{\text{ext}}(g, J^R, \xi) := H_k(g, J^R).
\]
The flows on $T^*G \times O_2^L$ are obtained from the flows of the unextended Hamiltonians on $T^*G$ simply by adding the relation $\xi(t) = \xi(0)$. We may also define the matrix functions $L_1^{\text{ext}}$ and $L_2^{\text{ext}}$ on $T^*G \times O_2^L$ by
\[
L_1^{\text{ext}}(g, J^R, \xi) = J^R \quad \text{and} \quad L_2^{\text{ext}}(g, J^R, \xi) = g g^\dagger,
\]
whereby we can write
\[
H_j^{\text{ext}} = \frac{1}{j} \text{tr}((L_1^{\text{ext}})^j) \quad \text{and} \quad H_k^{\text{ext}} = \frac{1}{2k} \text{tr}((L_2^{\text{ext}})^k).
\]
Now we turn to the description of two alternative models of the reduced phase space that will be shown to carry a dual pair of integrable many-body models. As a preparation, we associate with any vector $q \in \mathbb R^n$ the diagonal matrix
\[
q := \text{diag}(q^1, \ldots, q^n).
\]
We let $C$ denote the open domain (Weyl chamber)
\[ C := \{ q \in \mathbb{R}^n \mid q^1 > q^2 > \cdots > q^n \}, \] (3.14)
and equip
\[ T^*C \simeq C \times \mathbb{R}^n \]
(3.15)
with the Darboux form
\[ \Omega_{T^*C}(q, p) := \sum_k dp_k \wedge dq^k. \] (3.16)

3.1. The Sutherland gauge slice $S_1$

Let us define the $\mathfrak{u}(n)$-valued matrix function $L_1$ on $T^*C$ (cf (3.15)) by the formula
\[ L_1(q, p)_{jk} := p_j \delta_{jk} - i(1 - \delta_{jk}) \frac{\kappa}{\sinh(q^j - q^k)}. \] (3.17)
This is just the standard Lax matrix of the hyperbolic Sutherland model \([2, 17, 18]\). The following result is well known \([2, 7]\), but for readability we nevertheless present it together with a proof.

**Theorem 1.** The manifold $S_1$ defined by
\[ S_1 := \{ (e^q, L_1(q, p), -\mu \kappa) \mid (q, p) \in C \times \mathbb{R}^n \} \] (3.18)
is a global cross section of the $K$-orbits in the submanifold $\Phi_{\text{ext}}^{-1}(0)$ of $T^*G \times O^L_\kappa$. If $\iota_1 : S_1 \to T^*G \times O^L_\kappa$ is the obvious injection, then in terms of the coordinates $q, p$ on $S_1$ one has
\[ \iota_1^*(\Omega_{\text{ext}}) = \sum_k dp_k \wedge dq^k. \] (3.19)

Therefore, the symplectic manifold $(S_1, \sum_k dp_k \wedge dq^k) \simeq (T^*C, \Omega_{T^*C})$ is a model of the reduced phase space defined by (3.5).

**Proof.** Our task is to bring every element of $(\Phi_{\text{ext}})^{-1}(0)$ to a unique normal form by the ‘gauge transformations’ provided by the group $K$. For this purpose, let us introduce the submanifold $\mathcal{P}$ of $GL(n, \mathbb{C})$ given by the positive definite Hermitian matrices. Recall that the exponential map from $\mathfrak{u}(n)$ to $\mathcal{P}$ is a diffeomorphism. By the polar (Cartan) decomposition, every $g \in GL(n, \mathbb{C})$ can be uniquely written as
\[ g = g_- g_+ \quad \text{with} \quad g_- \in \mathcal{P}, \quad g_+ \in U(n). \] (3.20)
It is readily seen that $g$ can be transformed by the $K$-action into an element for which $g_+ = 1_n$ and $g_- = e^q$ with $q \in \mathbb{R}^n$, $q^1 > q^2 > \cdots > q^n$. (3.21)
The moment map constraint requires $J^R = 0$ and, by (3.4), for triples of the form $(e^q, J^R, \xi(v))$ we are left with the condition
\[ (e^q J^R e^{-q})_+ + \xi(v) = 0. \] (3.22)
This implies that the diagonal entries of the Hermitian matrix $J^R$ are arbitrary and the diagonal entries of the anti-Hermitian matrix $\xi(v)$ vanish. By (3.8), it follows from $\xi(v)_{jj} = 0$ that
\[ v_j = e^{\theta_j}, \quad \forall j, \] (3.23)
with some phase factors $\theta_j$. The off-diagonal components of the constraint (3.22) are
\[ (J^R)_{jk} \sinh(q^j - q^k) + i \kappa v_j \bar{v}_k = 0, \quad \forall j \neq k. \] (3.24)
Since \( v_j \neq 0 \), we see from (3.21) and (3.24) that \( q \) must belong to the open Weyl chamber \( C \) (3.14). Then the residual gauge transformations permitted by the partial gauge fixing condition (3.21) are given precisely by the maximal torus of \( T_n \subset U(n) \), diagonally embedded into \( K = U(n)^2 \times U(n)^R \). They operate according to
\[
(e^{\theta}, J^R, v) \mapsto (e^{\theta}, \tau J^R \tau^{-1}, \tau v) \quad \forall \tau \in \mathbb{T}_n.
\]
Hence we can completely fix the residual gauge freedom by transforming the vector \( v \) into the representative \( w \), whose components are all equal to 1. At the same time, by (3.24), \( J^R \) becomes equal to \( L_1(q, p) \), where \( p \) is the arbitrary diagonal part of \( J^R \). The calculation of \( \iota_1^*(\Omega^{\text{ext}}) \) as well as the rest of the statements of the theorem is now obvious.

Remark 1. If we spell out the moment map constraint (3.22) for the solution \( J^R = L_1, v = w \) and also multiply this equation both from the left and from the right by \( e^{\theta} \), then we obtain
\[
[e^{\theta}, L_1(q, p)] + 2i \kappa ((e^{\theta}w)(e^{\theta}w)^\dagger - e^{2\vartheta}) = 0.
\]
This is the commutation relation for the Lax matrix (3.17) used in [1].

3.2. The Ruijsenaars gauge slice \( S_2 \)

In this subsection, we denote the elements of \( T^*C = C \times \mathbb{R}^n \) as pairs \((\hat{p}, \hat{q})\). We introduce the \( \mathcal{P} \)-valued matrix function \( L_2 \) on \( T^*C \) by the formula
\[
L_2(\hat{p}, \hat{q}) = u_\lambda(\hat{p}, \hat{q}) \left[ \frac{2i \kappa}{2i \kappa + (\hat{p}) - (\hat{p})^2} \right] u_\lambda(\hat{p}, \hat{q}),
\]
with the \( \mathbb{R}_+ \)-valued functions
\[
u(\hat{p}, \hat{q}) := e^{-\hat{q}_j} \prod_{m \neq j} \left[ 1 + \frac{4 \kappa^2}{(\hat{p} - \hat{p}_m)^2} \right]^\frac{1}{4}.
\]
One can calculate the principal minors of \( L_2 \) with the help of the Cauchy determinant formula, and thereby confirm that \( L_2 \) is indeed a positive definite matrix. Therefore, it admits a unique positive definite square root, and we use it to define the \( \mathbb{R}^n \)-valued function,
\[
u(\hat{p}, \hat{q}) := L_2(\hat{p}, \hat{q})^{-\frac{1}{2}} u(\hat{p}, \hat{q}),
\]
where \( u = (u_1, \ldots, u_n)^T \). It can be verified directly, and can be seen also from the proof below, that \( |\nu(\hat{p}, \hat{q})|^2 = n \). Thus, by using (3.8), we have the \( \mathcal{O}_k^L \)-valued function
\[
\xi(\hat{p}, \hat{q}) := \xi(\nu(\hat{p}, \hat{q})).
\]
The function \( L_2 \) (3.27) is nothing but the standard Lax matrix of the rational Ruijsenaars–Schneider model [19], with variables denoted by somewhat unusual letters. Using the above notations, we now state the main technical result of the present paper.

Theorem 2. The manifold \( S_2 \) defined by
\[
S_2 := \{(L_2(\hat{p}, \hat{q})^{\frac{1}{2}}, \hat{p}, \xi(\hat{p}, \hat{q})) | (\hat{p}, \hat{q}) \in C \times \mathbb{R}^n \}
\]
is a global cross section of the \( K \)-orbits in the submanifold \((\Phi^{\text{ext}})^{-1}(0) \) of \( T^*G \times \mathcal{O}_k^L \). If \( \iota_2 : S_2 \rightarrow T^*G \times \mathcal{O}_k^L \) is the obvious injection, then in terms of the coordinates \( \hat{p}, \hat{q} \) on \( S_2 \) one has
\[
\iota_2^*(\Omega^{\text{ext}}) = \sum_k d\hat{q}_k \wedge d\hat{p}_k.
\]
Therefore, the symplectic manifold \((S_2, \sum_k d\hat{q}_k \wedge d\hat{p}_k) \simeq (T^*C, \Omega_{T^*C}) \) is a model of the reduced phase space defined by (3.5).
Proof. Let us begin by noting that by means of the $K$-action we can transform each element of $(\Phi^\text{ext})^{-1}(0)$ into an element $(g, J^R, \xi(v))$ that satisfies
\[ g = g_- \in \mathcal{P} \quad \text{and} \quad J^R = \hat{\mathbf{p}} \quad \text{with} \quad \hat{\mathbf{p}} \in \mathbb{R}^n, \quad \hat{\mathbf{p}}^1 \geq \hat{\mathbf{p}}^2 \geq \cdots \geq \hat{\mathbf{p}}^n. \]
(3.33)

After this partial gauge fixing the moment map constraint becomes
\[ g^{-1} \hat{\mathbf{p}} g_- - g_- \hat{\mathbf{p}} g_-^{-1} = 2\xi(v). \]
(3.34)

In order to solve this equation, we multiply it both from the left and from the right by $g_-^1$ which gives
\[ [\hat{\mathbf{p}}, g_-^2] = 2i\kappa (u g_-^1 - g_-^2), \]
where we have combined the unknowns $g_-$ and $v$ to define
\[ u := g_- v. \]
(3.35)

This equation then permits us to express $g_-^2$ in terms of $\hat{\mathbf{p}}$ and $u$ as
\[ (g_-^2)_{jk} = u_j \left[ \frac{2i\kappa}{2i\kappa + (\hat{\mathbf{p}}^j - \hat{\mathbf{p}}^k)^2} \right] \tilde{u}_k. \]
(3.37)

By calculating the determinant from the last relation, we obtain
\[ \det(g_-^2) = \left( \prod_m |u_m|^2 \right) \prod_{j<k} \frac{(\hat{\mathbf{p}}^j - \hat{\mathbf{p}}^k)^2}{(\hat{\mathbf{p}}^j - \hat{\mathbf{p}}^k)^2 + 4\kappa^2}. \]
(3.38)

Since $\det(g_-^2) \neq 0$, we must have
\[ \hat{\mathbf{p}}^1 > \hat{\mathbf{p}}^2 > \cdots > \hat{\mathbf{p}}^n \quad \text{and} \quad u_j \neq 0 \quad \forall \, j. \]
(3.39)

In particular, $\hat{\mathbf{p}}$ must belong to the open Weyl chamber $C$. This implies that the residual gauge transformations permitted by our partial gauge fixing are generated by the maximal torus $T_n$ of $U(n)$, diagonally embedded into $K$, which act according to
\[ (g_-, \hat{\mathbf{p}}, v) \mapsto (\tau g_- \tau^{-1}, \hat{\mathbf{p}}, \tau v) \quad \forall \, \tau \in T_n. \]
(3.40)

By applying these gauge transformations to $u$ in (3.36) we have
\[ \tau : u \mapsto \tau u. \]
(3.41)

It follows that we obtain a complete gauge fixing by imposing the conditions
\[ u_j > 0 \quad \forall \, j. \]
(3.42)

For fixed $\hat{\mathbf{p}}$, the positive vector $u$ remains arbitrary, and thus it can be uniquely parametrized by introducing a new variable $\hat{q} \in \mathbb{R}^n$ via equation (3.28).

The outcome of the above discussion is that the manifold $S_2$ given by (3.31) is a global cross section of the $K$-orbits in $(\Phi^\text{ext})^{-1}(0)$, which provides a model of the reduced phase space (3.5). Indeed, formula (3.37) of $g_-^2$ becomes identical to formula (3.27) of $L_2$ if we take into account the gauge fixing conditions (3.2) and the parametrization (3.28). At the same time, the inversion of relation (3.36) yields $v(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ in (3.29).

It remains to show that the variables $(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ that parametrize $S_2$ are Darboux coordinates on the reduced phase space. Direct substitution into the symplectic form $\Omega^\text{ext}$ (3.2) is now cumbersome, since we do not have $L_2(\hat{\mathbf{p}}, \hat{\mathbf{q}})$ explicitly. We circumvent this problem by proceeding as follows. We define the smooth functions $F^a$ and $E^a$ on $T^*G \times O^L_2$ by
\[ F^a(g, J^R, \xi) := \frac{1}{d} \text{tr}[(J^R)^a] \quad \text{and} \quad E^a(g, J^R, \xi) := \frac{1}{d} \text{tr}[(gg^1)^a], \quad a = 1, \ldots, n. \]
(3.43)
We also define the functions $F^a$ and $E^a$ on the gauge slice $S_2$ by
\[
F^a(\hat{p}, \hat{q}) := \frac{1}{a} \text{tr}[\hat{p}^a] \quad \text{and} \quad E^a(\hat{p}, \hat{q}) := \frac{1}{a} \text{tr}[L_2(\hat{p}, \hat{q})^a], \quad a = 1, \ldots, n. \tag{3.44}
\]
It is clear that $F^a$ and $E^a$ are $K$-invariant functions, and by means of the injection map $\iota_2 : S_2 \to T^* G \times O^L_k$ we have
\[
\iota_2^* F^a = F^a, \quad \iota_2^* E^a = E^a \quad \forall a = 1, \ldots, n. \tag{3.45}
\]
Therefore, we can determine the induced Poisson brackets of these functions by two methods. First, denote the Poisson bracket on the extended phase space by $[\ldots]^{\text{ext}}$. The Poisson bracket of $K$-invariant functions is again $K$-invariant and a straightforward calculation gives (for any $1 \leq a, b \leq n$)
\[
[\iota_2^* E^a, \iota_2^* E^b]^{\text{ext}} = \iota_2^* [F^a, F^b]^{\text{ext}} = 0, \quad \iota_2^* [E^a, F^b]^{\text{ext}} = 2\text{tr}[\hat{p}^{b-1} L_2^a]. \tag{3.46}
\]
Second, denote by $[\ldots]^{\text{red}}$ the Poisson bracket on $C^\infty(S_2)$ induced by the reduced symplectic structure. We wish to show that
\[
[\hat{p}^j, \hat{p}^k]^{\text{red}} = [\hat{q}_j, \hat{q}_k]^{\text{red}} = 0 \quad \text{and} \quad [\hat{p}^j, \hat{q}_k]^{\text{red}} = \delta^j_k. \tag{3.47}
\]
Now, if we assume that the last relation holds, then it can be verified by direct calculation\(^6\) (for any $1 \leq a, b \leq n$) that
\[
[E^a, E^b]^{\text{red}} = [F^a, F^b]^{\text{red}} = 0, \quad [E^a, F^b]^{\text{red}} = 2\text{tr}[\hat{p}^{b-1} L_2^a]. \tag{3.48}
\]
From general principles, the restriction of the Poisson bracket of $K$-invariant functions to a gauge slice always yields the induced Poisson bracket of the restricted functions. By taking into account (3.45), we conclude from the comparison of equations (3.46) and (3.48) that the Poisson bracket on $C^\infty(S_2)$ that arises from the symplectic reduction coincides with the Poisson bracket given in coordinates by (3.47), at least if we restrict our attention to the collection of the functions $F^a, E^a$. Thus it remains to prove that the functions $F^a, E^a$ can serve as local coordinates around any point from a dense open submanifold of $S_2$. Indeed, the symplectic form on this dense submanifold could then be written as $\sum_k d\hat{q}_k \wedge d\hat{p}^k$. The smoothness of the symplectic form would then allow us to conclude the same on the whole of $S_2$.

The map from $C \times \mathbb{R}^n$ to $C \times (\mathbb{R}^n)^n$ given by $(\hat{p}, \hat{q}) \mapsto (\hat{p}, u(\hat{p}, \hat{q}))$ is clearly a diffeomorphism, and thus we can use $\hat{p}^k, u_k$ ($k = 1, \ldots, n$) as coordinates on $S_2$. When expressed in these new coordinates, we denote our functions of interest as $\tilde{F}^a, \tilde{E}^a$ and $\tilde{L}_2$:
\[
\tilde{F}^a(\hat{p}, u) = F^a(\hat{p}, \hat{q}), \quad \tilde{E}^a(\hat{p}, u) = E^a(\hat{p}, \hat{q}), \quad \tilde{L}_2(\hat{p}, u) = L_2(\hat{p}, \hat{q}) \tag{3.49}
\]
if $u = u(\hat{p}, \hat{q})$. To finish the proof, it is sufficient to show that the Jacobian determinant of the map $C \times (\mathbb{R}^n)^n \to \mathbb{R}^n \times \mathbb{R}^n$ given in coordinates by $\hat{p}^k, u_k \mapsto \tilde{F}^a(\hat{p}, u), \tilde{E}^a(\hat{p}, u)$ is non-vanishing on a dense open subset of $C \times (\mathbb{R}^n)^n$. For this, note from (3.27) and (3.44) that $\tilde{F}^a$ and $\tilde{E}^a$ are rational functions of the variables $\hat{p}^k, u_k$, and thus the Jacobian,
\[
\frac{\partial (\tilde{F}^a, \tilde{E}^b)}{\partial (\hat{p}^j, u_k)}, \tag{3.50}
\]
is also a rational function of the same variables (that is, the quotient of two polynomials in the $2n$-variables $\hat{p}^k, u_k$). This implies that the Jacobian (3.50) either vanishes identically or is non-vanishing on a dense open subset of $C \times (\mathbb{R}^n)^n \simeq S_2$.

\(^6\) Only the relation $[E^a, E^b]^{\text{red}} = 0$ requires non-trivial effort, but this was established in [19] as well as in the more recent papers dealing with the dynamical $r$-matrix structure of the Ruijsenaars–Schneider models [20].
Now we show that the Jacobian (3.50) does not vanish identically. First of all, from the fact that \( \hat{F}^a \) does not depend on \( u_k \), we see easily that

\[
\det \left( \frac{\partial (\hat{F}^a, \hat{E}^b)}{\partial (\hat{p}^i, u_k)} \right) = \det \left[ \frac{\partial \hat{F}^a}{\partial \hat{p}^i} \right] \det \left[ \frac{\partial \hat{E}^b}{\partial u_k} \right].
\]  
(3.51)

The first determinant on the rhs is the Vandermonde one:

\[
\det \left[ \frac{\partial \hat{F}^a}{\partial \hat{p}^i} \right] = \prod_{i<m} (\hat{p}^m - \hat{p}^i),
\]  
(3.52)

which never vanishes on \( S_2 \) due to (3.39). The second determinant on the rhs of (3.51) parametrically depends on \( \hat{p} \). In particular, if we take a real parameter \( s > 0 \) and consider the curve \( \hat{p}(s) \) defined by \( \hat{p}^i(s) := e^{(i s + 1) j} \) for \( j = 1, \ldots, n \), then we observe that in the limit \( s \to \infty \) the matrix \( \hat{L}_2 \) (3.49) becomes diagonal:

\[
\lim_{s \to \infty} \hat{L}_2(\hat{p}(s), u) = \text{diag}(u_1^2, \ldots, u_n^2).
\]  
(3.53)

Hence, in the limit \( s \to \infty \), we encounter again a Vandermonde determinant:

\[
\lim_{s \to \infty} \det \left[ \frac{\partial \hat{E}^b}{\partial u_k} \right](\hat{p}(s), u) = 2^n \left( \prod_{j} u_j \right) \prod_{i<m} (u_m^2 - u_i^2).
\]  
(3.54)

Obviously, we can choose \( u \) in such a way that the rhs of equation (3.54) does not vanish. Then the Jacobian determinant (3.50) does not vanish at \((\hat{p}(s), u)\) for large enough \( s \). □

**Remark 2.** The consequence (3.35) of the moment map constraint yields the commutation relation satisfied by the Lax matrix \( L_2 \) (3.27) and \( u(\hat{p}, \hat{q}) \) (3.28):

\[
[L_2(\hat{p}, \hat{q}), \hat{p}] + 2i \hbar (u(\hat{p}, \hat{q}) u(\hat{p}, \hat{q})^\dagger - L_2(\hat{p}, \hat{q})) = 0,
\]  
(3.55)

which played a crucial role in the analysis in [1].

### 4. The dual pair of many-body models

We now enumerate important consequences of the results presented in the preceding sections.

(i) Since \( S_1 \) (3.18) and \( S_2 \) (3.31) are two models of the same reduced phase space (3.5), there exists a natural symplectomorphism between these two models:

\[
\left( S_1, \sum_k d p_k \wedge dq^k \right) \equiv (T^* G \times \mathcal{O}_G^C//\theta K, \Omega^\text{red}) \equiv \left( S_2, \sum_k dq_k \wedge d \hat{p}^k \right).
\]  
(4.1)

By definition, a point of \( S_1 \) is related to that point of \( S_2 \) which represents the same element of the reduced phase space.

(ii) The pullbacks of the ‘unreduced Lax matrices’ (3.11) to \( S_1 \) and \( S_2 \) satisfy, respectively,

\[
i_1^* L_1^\text{ext} = L_1 \quad \text{and} \quad i_2^* L_2^\text{ext} = L_2.
\]  
(4.2)

By the symplectic reduction, the \( K \)-invariant Hamiltonians (3.12) descend to the families of Hamiltonians defined on \((S_1, \sum_k d p_k \wedge dq^k)\) and on \((S_2, \sum_k dq_k \wedge d \hat{p}^k)\), respectively, by

\[
H^\text{red}_j = \frac{1}{j} \text{tr}(L^j_1), \quad (j = 1, \ldots, n) \quad \text{and}
\]

\[
\hat{H}^\text{red}_k = \frac{1}{2k} \text{tr}(L^k_2), \quad (k = \pm 1, \ldots, \pm n).
\]  
(4.3)
The commutativity of \( \{ H^\text{red}_j \} \) and \( \{ H^\text{red}_k \} \) (where the elements with fixed sign of \( k \) form a complete set) is inherited from the commutativity of the unreduced Hamiltonians in (3.12). (Note that \( \text{tr}(L_1) = \text{tr}(L_1^\dagger) \) since \( L_1 (3.17) \) is a Hermitian matrix.)

(iii) According to classical results, \( L_1 (3.17) \) is the Lax matrix of the hyperbolic Sutherland model, and \( L_2 (3.27) \) is the Lax matrix of the rational Ruijsenaars–Schneider model. The basic Hamiltonians of these many-body models are indeed reproduced as

\[
H_{\text{hyp–Suth}}(q, p) = \frac{1}{2} \sum_k p_k^2 + \frac{k^2}{2} \sum_{j \neq k} \frac{1}{\sinh^2(q^j - q^k)} = \frac{1}{2} \text{tr}(L_1(q, p)^2),
\]

\[
H_{\text{rat–RS}}(\hat{\mathbf{p}}, \hat{\mathbf{q}}) = \sum_k \cosh(\hat{q}_k) \prod_{j \neq k} \left[ 1 + \frac{4k^2}{(\hat{p}^j - \hat{p}^k)^2} \right] = \frac{1}{2} \text{tr}(L_2(\hat{p}, \hat{q}) + L_2(\hat{p}, \hat{q})^{-1}).
\]

The Lax matrices themselves arose naturally (4.2) by means of the symplectic reduction.

(iv) Consider two points of \( S_1 \) and \( S_2 \) that are related by the symplectomorphism (4.1). Suppose that these two points are parametrized by \( (q, p) \in C \times \mathbb{R}^n \) and by \( (\hat{p}, \hat{q}) \in C \times \mathbb{R}^n \) according to (3.18) and (3.31), respectively. The fact that they lie on the same \( K \)-orbit means, since \( g \in C \times \mathbb{R}^n \), (3.20) is fixed to \( I \), in both gauges, that there exists some \( \eta \in U(n) \) (actually unique up to the center of \( U(n) \)) that acts trivially for which

\[
(\eta e^\mathbf{h} \eta^{-1}, \eta L_1(q, p) \eta^{-1}, -\eta \mu e \eta^{-1}) = (L_2(\hat{p}, \hat{q})^2, \hat{\mathbf{p}}, \xi(\hat{p}, \hat{q})).
\]

This shows that the matrix \( \hat{\mathbf{p}} \), which encodes the coordinate variables of the rational Ruijsenaars–Schneider model, results by diagonalizing the Sutherland Lax matrix \( L_1(q, p) \). The same formula shows that \( e^{2\mathbf{h}} \), which encodes the coordinate variables of the hyperbolic Sutherland model, results by diagonalizing the Ruijsenaars–Schneider Lax matrix \( L_2(\hat{p}, \hat{q}) \). The original, direct construction [1] of the map between the phase spaces of the two many-body models relied on diagonalization of the Lax matrices, but in that approach it was quite difficult to prove the canonicity of the map, which comes for free in the symplectic reduction framework.

(v) It is obvious from the above observations that the two many-body models characterized by the Hamiltonians (4.4) and (4.5) are dual to each other in the sense described in the Introduction. On the one hand, the Ruijsenaars–Schneider particle coordinates \( \hat{p}^1, \ldots, \hat{p}^n \) regarded as functions on \( S_1 \) define action variables for the hyperbolic Sutherland model. On the other, the Sutherland particle coordinates \( q^1, \ldots, q^n \) regarded as functions on \( S_2 \) can serve as action variables for the rational Ruijsenaars–Schneider model.

(vi) The well-known solution algorithms [2, 19] for the commuting Hamiltonians exhibited in (4.3) can be viewed as byproducts of the geometric approach. First of all, it should be noted that all the flows generated by the reduced Hamiltonians are complete, since this is true for the unreduced Hamiltonians whose flows stay in \( (\Phi^{\text{ext}})^{-1}(0) \). By taking an initial value on the gauge slice \( S_1 \) and projecting the flow (2.8) back to \( S_1 \) we obtain that the reduced Hamiltonian \( H^\text{red}_j (4.3) \) generates the following evolution for the Sutherland coordinate-variables:

\[
e^{2\mathbf{h}_j(t)} = D \left[ e^{\mathbf{h}_j(0)} \exp(2 t L_1(0)^{-1}) e^{\mathbf{q}(0)} \right],
\]

where the zero argument refers to the \( t = 0 \) initial value, and \( D \) denotes the operator that brings its Hermitian matrix argument to diagonal form with eigenvalues in non-increasing
order. Similarly, we obtain from (2.9) that the reduced Hamiltonian $\hat{H}^{\text{red}}_{4.3}$ generates the following flow for the Ruijsenaars–Schneider coordinate variables:

$$\hat{p}(t) = D\left[\hat{p}(0) - tL_2(0)^k\right].$$

(4.8)

We here used that $(g^1(0)g(0))^k = L_2(0)^k$ for any initial value on $S_2$.

To summarize, we presented the duality between the hyperbolic Sutherland model (4.4) and the rational Ruijsenaars–Schneider model (4.5) in the framework of symplectic reduction. In this way, we obtained a Lie theoretic understanding of results due to Ruijsenaars [1], who originally discovered and investigated the duality by direct means. Our approach also simplifies a considerable portion of the original technical arguments. The general line of reasoning that we followed may be adapted to explore more complicated cases of the duality in the future, too. For example, it will be demonstrated in [21, 22] that the reduction approach works in a conceptually very similar manner for the dualities concerning trigonometric Ruijsenaars–Schneider models.

Finally, let us mention that the dual pairs of models studied by Ruijsenaars at the classical level [1, 8, 9] are associated at the quantum-mechanical level with so-called bispectral problems [23], as was first conjectured in [3] and later confirmed in several papers. Concerning the bispectral property, and in particular the bispectral interpretation of the duality between the hyperbolic Sutherland and the rational Ruijsenaars–Schneider models, the reader may consult [24, 25] and references therein. We expect that this intriguing phenomenon could be understood also in terms of a quantum Hamiltonian reduction counterpart of our approach.

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