The stability of homogeneous pulsating vertical seepage of mixture through porous media with solute immobilization

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Abstract. The paper is devoted to the investigation of the linear stability of homogeneous pulsating vertical seepage of mixture through porous media with solute immobilization. The flow through porous media is modelled within the standard Darcy law and immobilization is described by the linear mobile/immobile media (MIM) model. It is known that the immobilization leads to oscillatory mode of convective instability due to transition of solute from the mobile phase to immobile. It means that immobilization effect to the flow pulsations should be non-trivial. The equations for the convection with immobilization is derived, the regime of homogeneous filtration is obtained and its stabilization is investigated. The neutral curves into the space of parameters of investigated system is plotted. The effect of homogeneous filtration stabilization under intensification of external seepage and dynamics interphase exchange is studied.

1. Introduction

The spatial structure of real porous media is rather complicated. The transport of solute particles in such a medium is not always governed by the Gaussian distribution law since the particles can stick to a solid matrix (immobilized), which slows down the solute transport through the porous medium.

In the present work, the immobilization effect is described within the MIM (mobile-immobile medium) model with linear sorption kinetics [1, 2].

The MIM approach is based on the assumption that solute can be divided to two phases: immobile (adsorbed or trapped solutes) and mobile (drifting with the flow). The evolution of concentration of solute in mobile phase is described by the classical diffusion equation including the term, which describes the transfer of solute into the immobile phase. The dependence of the rate of solute transfer into the immobile phase on the solute concentrations in two phases is linear and describes the kinetics of the “phase transition” [1, 2]. This type of MIM model is used for description of the effects at the small solute concentrations. Some problems where some additional effects are important is described in [3, 4, 5, 6, 7].

In our work, we study the stability of vertical seepage of a liquid mixture through a closed porous domain. The average difference of solute concentration is prescribed between the upper and lower boundaries. Thus difference initiates a convective motion in the gravity field. The development of convection in a closed box of porous medium is studied firstly in [8] where the
stability of mechanical equilibrium is investigated. It is found that the lower level of instability is doubly degenerated. The dynamic properties of the convection in a porous medium in case of weak vertical homogeneous seepage is studied in [9, 10]. It was shown that vertical seepage leads to a monotonic instability mode realization. Into the work [11] the immobilization of solutes takes into account within the same problem. It is found that the transfer of the solute between phases provokes the oscillatory instability.

The objective of the present paper is to investigate the influence of modulation of vertical seepage to the stability of homogeneous filtration regime under the effect of solute immobilization. The paper consists of introduction, four Sections and conclusion. The introduction contains the motivation and short review of relevant scientific literature. In the first Section, the governing equations for the solutal convection within the linear MIM model is derived and the mathematical statement of problem is formulated. The second Section is devoted to the obtaining of base homogeneous solution and the discussion about its properties. In the third Section the equations for the small perturbations of the regime of homogeneous filtration is obtained and discussed. The fourth Section contains the numerical results of stability analysis. The conclusion summarizes the main findings of the present study.

2. Mathematical statement of problem

The two-dimensional flow of a mixture through square porous domain (of size $L$) is considered. Solute concentrations at the upper and lower boundaries of the domain oscillates near average values $C_+$ (upper) and $C_-$ (lower) such that $C_- < C_+$ (see Figure 1). The lateral walls of the domain are assumed to be impermeable. The filtration velocity on the horizontal boundaries is prescribed $V = (0, V \cos(\omega t))$.

![Figure 1. The problem configuration.](image_url)

The described problem can be written within the Darcy-Boussinesq model with account of
linear MIM model for solute immobilization as [2, 12]

\[
\phi \frac{\partial}{\partial t} (c + q) = -\mathbf{V} \cdot \nabla c + \phi D \nabla^2 c,
\]

\[
\frac{\partial q}{\partial t} = \alpha(c - K_d q),
\]

\[
-\nabla p = \frac{\eta}{\kappa} \mathbf{V} + \rho_l \beta_c c g j,
\]

\[
\nabla \cdot \mathbf{V} = 0,
\]

\[
\mathbf{V} \cdot |y=0,L = V \cos(\omega t), \quad \mathbf{V} \cdot |x=0,L = 0,
\]

\[
\frac{\partial}{\partial x} \bigg|_{x=0,L} = 0, \quad c|_{y=0} = C_+, \quad c|_{y=L} = C_-.
\]

Here \( c \) is the volumic solute concentration in the mobile phase, \( q \) is the volumic solute concentration in the immobile phase, \( D \) is the effective diffusivity, \( \alpha \) is the mass transport coefficient, \( K_d \) is solute distribution coefficient, \( V \) is the vector of filtration velocity, \( \omega \) is the frequency of flow pulsations, \( \kappa \) is the permeability of the porous medium, \( \eta \) is the dynamic viscosity, \( \rho \) is the fluid density, \( \beta_c \) is the coefficient of concentration expansion, \( p \) is the deviation of pressure from the hydrostatic one, \( g \) is the gravity acceleration, \( j \) is the unit vertical vector and \( \phi \) is the porosity of the medium.

For nondimensionalization of the problem (1), we use the following scales for length, time, velocity, pressure and concentration:

\[
L, \quad \frac{L^2}{D}, \quad \frac{\phi D}{L}, \quad \frac{\phi D \eta}{\kappa}, \quad C_0 = C_+ - C_-.
\]

Problem (2) includes the following dimensionless parameters:

\[
Rp = \frac{C_0 g L \kappa \rho_l \beta_c}{(D \eta \phi)} \quad \text{is the solutal Rayleigh–Darcy number},
\quad Pe = \frac{V L}{D} \text{is the Péclet number},
\quad \Omega = \frac{\omega L^2}{D},
\quad a = \alpha K_d C_0 L^2 / D \quad \text{and} \quad b = \alpha L^2 / D
\]

are dimensionless frequency of flow pulsations, adsorption and desorption rates respectively.

The fields of concentrations \( c \) and \( q \) and velocity \( \mathbf{V} \) as the solution of problem in general case depends on \( x \) and \( y \) coordinates. But in the case of special choice of function \( f(t) \) the solution of (2) is independent on \( x \) coordinate. This solution is named the regime of homogeneous filtration. The next Section is devoted to obtain the solution for homogeneous filtration.

### 3. Homogeneous filtration

The problem (2) allows the solution in the form \( c = c_h = 1 - y + f(t), \quad q = q_h = a/b (1 - y) + g(t), \quad (V) = (V)_0 = (0, Pe \cos(\Omega t)) \).

The equations describing this regime of filtration take the following form
\[ \partial_t (f + g) = Pe \cos(\Omega t), \]
\[ \partial_t g = af - bg, \]
\[ P = -Pe \cos(\Omega t)y + Rp \left( fy + y^2/2 \right), \]

where symbol \( \partial_t \) denotes the derivative with respect to time \( t \), \( P_h \) is the variable describing pressure distribution.

The equations (3) are the system of ordinary differential equations. The solution of (3) can be written in the following form

\[ f = \frac{aPe}{(a + b)^2 + \omega^2} \left[ \cos(\Omega t) - \frac{a + b}{\Omega} \right] + \frac{Pe}{\Omega} \sin(\Omega t), \]
\[ g = \frac{aPe}{(a + b)^2 + \omega^2} \left[ \frac{a + b}{\Omega} - \cos(\Omega t) \right], \]
\[ P_h = -Pe \cos(\Omega t)y + Rp \left( fy + y^2/2 \right). \]

The fields of concentration \( c_h = 1 - y + f(t) \), \( q_h = a/b (1 - y) + g(t) \) for the obtained functions \( f \) and \( g \) (4) is described the regime of homogeneous (in horizontal direction) filtration. The property of this regime is homogeneous pulsations in time of all fields with frequency \( \Omega \). The amplitude of pulsations depends on sorption parameters and it is proportional to the amplitude of external flow pulsations. The linear stability of this regime with respect to small perturbations is investigated in the next Section.

4. Linear stability problem

Let us proceed to the stability analysis of the homogeneous filtration. To this end, we represent the fields of velocity, pressure and solute concentrations in mobile and immobile phases as the sums of basic state fields and small perturbations:

\[ q = q_h + Q, \quad c = c_h + C, \quad \mathbf{V} = \mathbf{V}_p + \mathbf{V}_0 = \mathbf{V}_p, \]
\[ p = P_h + P. \]

We restrict ourselves to two-dimensional perturbations \( \mathbf{V}_p = (u, w, 0) \). In this case, the system of linearized equations for small perturbations of the homogeneous filtration regime can be written as

\[ \partial_t (C + Q) = -Pe \cos(\Omega t) \partial_y C + \partial_x^2 C + \partial_y^2 C, \]
\[ \partial_t Q = aC - bQ, \]
\[ u = -\partial_x P, \quad w + Rp C = -\partial_y P, \]
\[ \partial_x u + \partial_y w = 0. \]

The boundary conditions for perturbations are

\[ u|_{x=0,1} = 0, \quad w|_{y=0,1} = 0, \]
\[ \partial_x C|_{x=0,1} = 0, \quad C|_{y=0,1} = 0. \]

Introducing the stream function \( \psi \) related to the velocity components as \( u = -\partial_y \psi \quad w = \partial_x \psi \) and excluding the pressure we obtain

\[ \partial_t (C + Q) = \partial_x^2 C + \partial_y^2 C - Pe \cos(\Omega t) \partial_y C \]
\[ \partial_t Q = aC - bQ, \]
\[ \partial_x^2 \psi + \partial_y^2 \psi - Rp \partial_x C = 0, \]
\[ \psi|_{x=0,1} = 0, \quad \psi|_{y=0,1} = 0, \]
\[ \partial_x C|_{x=0,1} = 0, \quad C|_{y=0,1} = 0. \]
A solution to problem (5) can be found in the form of normal perturbations, which are periodic in the coordinate $x$: $Q, C \sim \cos (n\pi x), \psi \sim \sin (n\pi x)$, where $n$ is the integer number. We restrict to the case of $n = 1$, since it is known [8, 10, 13] that such perturbations correspond to the smallest possible critical values of the solutal Rayleigh–Darcy number. In this case, system (5) is represented as

$$\begin{align*}
\frac{\partial_t}{(C + Q)} &= \partial_y^2 C - \pi^2 C - Pe \cos(\Omega t) \partial_y C, \\
\partial_t Q &= aC - bQ, \\
\partial_y^2 \psi &= \pi^2 \psi + Rp \partial_x C.
\end{align*}$$

(6)

The problem (6) is one dimensional system of partial differential equations. This problem is solved numerically by implicit finite differences method of second order accuracy in space in the first order in time [14]. The stability problem is solved by the Floquet method [15]. The results of stability problem solution are presented in the next Section.

5. Results

The stability of homogeneous filtration regime with respect to two dimensional weak perturbations is investigated numerically by the Floquet method. We obtain the solution of (6) as the fields $C(y, t), Q(y, t)$ and $\psi(y, t)$ for some time moment $t$. After that we calculate the same fields after one period of external flow modulation i.e. for time moment $t + T$ where $T = 2\pi/\Omega$. The fields $C(y, t)$ and $C(y, t)$ is compared for obtaining the multiplicator of perturbations. We use two different ways of multiplicator calculation: first is $\mu = \frac{\int_0^1 C(y, t + T) dy}{\int_0^1 C(y, t) dy}$ and $\mu = \frac{C(0.5, t + T)}{C(0.5, t)}$. If the multiplicator value $|\mu| > 1$ or $|\mu^*| > 1$ then the perturbations growth and they decay in opposite case [15]. As a result we plot the domains in space of system parameters where perturbations growth. The same domains for both ways of multiplicator calculation are identical, but the second way is simpler for realization because of that all presented plots are obtained for $\mu$. All plots are demonstrated the neutral curves (the curve where $|\mu^*| = 1$) as the dependences of Rayleigh–Darcy number on another parameter. Everywhere we obtain the synchronous

![Figure 2](image-url) Figure 2. The neutral curves into the parameter plane $Pe, Rp$. The calculations are performed for the following values of parameters: $b = 5$ for both panels, $a = 5$ right panel, $\Omega = 5$ left panel. The instability of homogeneous filtration regime corresponds to the domain upper than curve. perturbations (the frequency of pulsation for the field $C(y, t)$ equals to $\Omega$). Due to that fact all
Figure 3. The neutral curves into the parameter plane $Pe$, $Rp$. The calculations are performed for the following values of parameters: $b = 5$ for both panels, $\Omega = 5$ left panel, $a = 5$ right panel. The instability of homogeneous filtration regime corresponds to the domain upper than curve.

Figure 4. The neutral curves into the parameter plane $a$, $Rp$. The calculations are performed for the following values of parameters: $b = 5$ for both panels, $Pe = 5$ left panel, $\Omega = 3$ right panel. The instability of homogeneous filtration regime corresponds to the domain upper than curve.

The presented neutral curves corresponds to the multiplicator value $\mu = 1$, the instability is observed into the domain upper neutral curve.

In the Figure 2 the neutral curves into the plane of parameters $Pe$, $Rp$ is plotted. It is seen from Figure 2 that the increasing of pulsations amplitude ($Pe$) leads to the stabilization. This effect can be explained by the removal of perturbations from the porous domain by the external flux. It is well known effect (see for example [16]). The increasing of frequency (see Figure 3) also is cause of the stabilization effect for homogeneous filtration. It is linked to general growth to external flow intensity and additional the transition of solute between phases is induced by the pulsations. The enlargement of frequency speeds up the dynamics of phase transition, the solute accumulates in the immobile phase the field of mobile concentration becomes more homogeneous and as a result the system becomes more stable. The same reasoning can be apply for the explanation of stabilization under increasing of adsorption rate because of the
concentration in immobile phase is proportional to $a$ (see for example equations (4)). The neutral curves in the plane of parameters $a$, $R_p$ is presented in Figure 4.

6. Conclusion
The problem of exciting of convective motion into the close domain of porous media under pulsating seepage of mixture through the media with taking into account immobilization is investigated. The equations for the solutal convection with immobilization are derived and discussed. The solution for concentration fields is obtained for the regime of homogeneous filtration. The equations for small two-dimensional perturbation of homogeneous regime are derived and linear stability problem is formulated. The neutral curves into the space of system parameters are obtained and analyzed. It is found that both the intensification of external seepage and dynamics of interphase exchange lead to the stabilization of homogeneous filtration regime.

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