Novel Geometrical Models of Relativistic Stars.
II. Incompressible Stars and Heavy Black Dwarfs

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In a series of articles we describe a novel class of geometrical models of relativistic stars. Our approach to the static spherically symmetric solutions of Einstein equations is based on a careful physical analysis of radial gauge conditions.

It turns out that there exist heavy black dwarfs: relativistic stars with arbitrary large mass, which are to have arbitrary small radius and arbitrary small luminosity. In the present article we mathematically prove this new phenomena, using a detailed consideration of incompressible GR stars. We study the whole two parameter family of solutions of extended TOV equations for incompressible stars. This example is used to illustrate most of the basic features of the new geometrical models of relativistic stars. Comparison with newest observational data is discussed.

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I. INTRODUCTION

This is the second of series articles in which we describe a new geometrical models of general relativistic stars (GRS). One can find the general scheme, notations, basic equations, additional conditions and basic principles and properties of these models in the first article of this series – [1]. The essentially new features of GRS in these new models are not based on the critics or revision of very general relativity (GR). They are a result of more deep understanding of its applications, and on solution of some open problems in this theory, like the physical justification of the choice of GR gauges. The preliminary knowledge of the article [1] is highly recommended. It is essential for the right understanding of the present one.

Here we consider the simplest specific example of application of the general scheme, described in [1]: GRS, made of an incompressible matter with equation of state (EOS):

\[ \varepsilon(p) = \text{const} = \varepsilon. \]  (I.1)

In Eq. (I.1) \( \varepsilon \) is the energy density and \( p \) is the pressure of stelar matter. We are using units \( c = G_N = 1 \). Further on the letter ”C”, as an index of different quantities, denotes their values at the center of the star, the sign ”*” – the values at the edge of the star.

Such simple model has a limited physical significance, because it leads to an infinite speed of the sound in the fluid [2]. Nevertheless, its consideration is useful, because:

• This simple model can be used as a good physical approximation for description of neutron stars with matter density \( \approx 2.85 \times 10^{14} \text{g/cm}^3 \) and \( p_C \lesssim 5 \times 10^{35} \text{erg/cm}^3 \), because under these conditions the nuclear matter behaves much like incompressible fluid [2].

• The EOS (I.1) has the advantage that it yields an analytically solvable model of stars. The simplest degenerate set of solutions with luminosity variable \( p_C = 0 \) was at first found in the Schwarzschild pioneer article [3].

• It is a basic example of application of GR to the stelar physics [2].

Because of the overestimation of the role of the luminosity variable \( \rho \), up to now the considerations of GRS structure were based, as a rule, on the Hilbert gauge (HG): \( \rho(r) = r \), in which the condition \( p_C = 0 \) seems to be natural. Here \( r \) is the proper radial variable of the spherically symmetric problem at hand [1].

To the best of our knowledge, the solutions with \( p_C > 0 \) have not been studied and do not have a proper physical interpretation. We intend to fill this gap in the present article. The general solution of this problem, described here, turns to be much more complicated and more interesting then the degenerate one, considered by Schwarzschild in [3].

The new models of GRS recover an essentially new and unexpected relativistic physics. In particular, in these models GRS with arbitrary large mass \( m_* \) are allowed. These are to have arbitrary small geometrical radius \( R_* \) and arbitrary small luminosity. We refer to such amazing relativistic objects as heavy black dwarfs (HBD).

In the present article we give a mathematical proves of the existence of HBD in the specific case of incompressible matter. Taking into account that the EOS has a quite week influence on the proper radial gauge \( \rho(r) \) [1], one may expect this result to be general.

Pure GR reasons, which are independent of EOS, can not yield restrictions on the maximal mass of stars. Constraints of that kind may arise only due to quantum

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The general solutions of the equations widely accepted GR models with extra condition general, there exist no stiff functional relation, as in the freely parameter one can consider, for example, the mass \( m_\ast \) and the radius \( R_\ast \) [1]. Between these quantities, in general, there exist no stiff functional relation, as in the widely accepted GR models with extra condition \( \rho_C = 0 \) [2, 5]. Actually, just this assumption is responsible for existence of the well known mass-radii relations [1]. In a framework of a given theory of gravity (Newtonian, GR, etc) their specific form depends on EOS, but their common origin is in the condition \( \rho_C = 0 \).

The very recent observational data (see J. Madej et al., 2004 in [5]) seem to confirm the existence of two-parameter family of white dwarfs with free parameters \( m_\ast \) and \( R_\ast \) in proper domain, thus rejecting the existence of stiff mass-radii relations in Nature. Here we give a preliminary comparison of these data with our new models of GRS, using the results, obtained for the simple case of incompressible stars.

Making use of the simplest EOS (I.1) we will be able to illustrate in most transparent way the differences between our approach to the GRS structure and the commonly accepted one with extra condition \( \rho_C = 0 \).

II. GENERAL SOLUTION OF ETOV EQUATIONS FOR INCOMPRESSIBLE MATTER IN HILBERT GAUGE

For EOS (I.1) the equation

\[
\frac{dm}{d\rho} = 4\pi\varepsilon\rho^2 > 0,
\]

(II.1)

\( m(\rho_C, \rho_C) = 0, \quad m(\rho_\ast, \rho_C) = m_\ast \)

splits from the whole ETOV system of differential equations and has a simple general solution:

\[
m(\rho, \rho_C) = \frac{4}{3} \pi\varepsilon \left( \rho^3 - \rho_C^3 \right).
\]

(II.2)

The general solution of the equation

\[
\frac{dp}{d\rho} = -\frac{(p + \varepsilon)(m + 4\pi\rho^3\varepsilon)}{\rho(\rho - 2m)} < 0,
\]

(II.3)

\( p(\rho_C, \rho_C) = p_C, \quad p(\rho_\ast, \rho_C) = 0 \)

can be written in the form

\[
p(\rho, \rho_C) + \varepsilon = \frac{\sqrt{-g_{pp}(\rho, \rho_C)}(p_C + \varepsilon)}{1 + (p_C/\varepsilon + 1)\chi(\rho, \rho_C)}.
\]

(II.4)

The general solutions of the equations

\[
\frac{dm_0}{d\rho} = 4\pi\varepsilon\rho^2\sqrt{-g_{pp}} > 0,
\]

(II.5)

\( m_0(\rho_C, \rho_C) = 0, \quad m_0(\rho_\ast, \rho_C) = m_0\ast \),

and

\[
\frac{d\varphi}{d\rho} = m + 4\pi\rho^3\varepsilon \rho(\rho - 2m) > 0,
\]

(II.6)

\( \varphi(\rho_C, \rho_C) = \varphi_C, \quad \varphi(\rho_\ast, \rho_C) = \varphi_\ast \)

have the form

\[
m_0(\rho, \rho_C) = 4\pi\varepsilon \int_{\rho_C}^{\rho} \rho^2 \sqrt{-g_{pp}(\rho, \rho_C)} \, d\rho,
\]

(II.7)

\[
\varphi(\rho, \rho_C) = \ln \left( \frac{1 + (w_C + 1)\chi(\rho, \rho_C)}{\sqrt{-g_{pp}(\rho, \rho_C)}} \right) + \varphi_C.
\]

(II.8)

In the above formulas

\[
\chi(\rho, \rho_C) = 4\pi\varepsilon \int_{\rho_C}^{\rho} \rho \left( \sqrt{-g_{pp}(\rho, \rho_C)} \right)^3 \, d\rho.
\]

(II.9)

The solution (II.4) can be rewritten in a more convenient form

\[
\chi(\rho, \rho_C) = \frac{\sqrt{-g_{pp}}}{w + 1} \left( \sqrt{-g_{pp}} \right)^{\frac{3}{w + 1}}.
\]

(II.10)

making use of the scale-invariant ratio \( w = p/\varepsilon[1] \).

Then the metric coefficients can be written in the form:

\[
g_{tt}(\rho, \rho_C) = e^{2\varphi(\rho, \rho_C)},
\]

\[
g_{pp}(\rho, \rho_C) = \frac{-1}{1 - 2m(\rho, \rho_C)/\rho}.
\]

(II.11)

III. SCALE PROPERTIES OF THE PROBLEM AND HG SCALE INVARIANT QUANTITIES FOR INCOMPRESSIBLE STARS

The EOS (I.1) is not scale-invariant. Since the scale factor is \( \lambda = \text{const} \), the scale transformations:

\[
\rho \rightarrow \lambda\rho, \quad m \rightarrow \lambda m, \quad m_0 \rightarrow \lambda m_0, \quad \varepsilon \rightarrow \lambda^{-2}\varepsilon, \quad p \rightarrow \lambda^{-2}p.
\]

(III.1)

described in [1, 6] in more details, map the solution of ETOV system with a fixed value of the constant density \( \varepsilon \) onto another solution of the same system with a constant energy density \( \varepsilon' = \lambda^{-2}\varepsilon \). Utilizing this property one can consider only the case \( \varepsilon = 1 \). Instead, we prefer to use a scale invariant variables. In contrast to other authors [6], we avoid the \( a \text{ priori} \) choice of such variables and will find them during the very course of the solution of ETOV system with EOS (I.1).

In the case of incompressible star one can use the representation

\[
g_{pp}(\rho, \rho_C) = \frac{3}{8\pi\varepsilon} \rho^2 / P_4(\rho, \rho_C),
\]

(III.2)
where
\[ P_4(\rho, \rho_C) := \rho \left( \rho^3 - \frac{3}{8\pi \varepsilon} \rho - \rho_C^3 \right) = \prod_{i=1}^{4}(\rho - \rho_i) \quad (\text{III.3}) \]
is a naturally normalized polynomial of fourth degree, which roots are \( \rho_{1,2,3,4} \). It is easy to find these roots in the form
\[
\rho_1 = \frac{1}{\sqrt{2\pi \varepsilon (1 + \sigma^2/3)}} > \rho_2 = 0 > \\
\rho_3 = \frac{-\frac{1}{3}(1 - \sigma)}{\sqrt{2\pi \varepsilon (1 + \sigma^2/3)}} > \rho_4 = \frac{-\frac{1}{3}(1 + \sigma)}{\sqrt{2\pi \varepsilon (1 + \sigma^2/3)}} \quad (\text{III.4})
\]
where the \( \lambda \)-invariant parameter \( \sigma \) is defined via the substitution
\[ \rho_C = \frac{1}{\sqrt{2\pi \varepsilon (1 + \sigma^2/3)}} \left( \frac{\rho - \rho_1}{\rho_1} \right)^{1/3} \quad [10]. \]
Now it becomes clear that instead of the luminosity variable \( \rho \) it is better to use the \( \lambda \)-invariant one, \( \xi \):
\[
\xi := \sqrt{2\pi \varepsilon (1 + \sigma^2/3)} \rho = \sqrt{\frac{8\pi \varepsilon}{3}} \left( 1 - \xi_C^3 \right) \rho, \quad (\text{III.5})
\]
where \( \xi_C \in (0,1) \) is the only positive root of the cubic equation \( \xi_C^3 + \frac{3}{8\pi \varepsilon \sigma^2} C^3 - 1 = 0 \Rightarrow \sigma = \sqrt{1 - 4C^3} \).

As a result we obtain
\[ g_{\rho \rho}(\xi, \xi_C) = (1 - \xi_C^3) \frac{\xi^2}{P_4(\xi, \xi_C)} \quad (\text{III.6}) \]
The naturally re-normalized \( \lambda \)-invariant polynomial
\[ P_4(\xi, \xi_C) = \prod_{i=1}^{4}(\xi - \xi_i) = \xi(\xi - 1)\left( \xi^2 + \xi + \xi_C^3 \right) \quad (\text{III.7}) \]
has the following re-scaled roots:
\[
\xi_1 = 1 > \xi_2 = 0 > \\
\xi_3 = -1 - \sqrt{1 - 4C^3} > \xi_4 = 1 + \sqrt{1 - 4C^3} \quad (\text{III.8})
\]
and
\[
\xi_1 = 1 > \xi_2 = 0 > \\
\xi_3 = -1 - i\sqrt{1 - 4C^3} > \xi_4 = 1 + i\sqrt{1 - 4C^3} \quad (\text{III.9})
\]
There exist two degenerate cases:
a) Schwarzschild degenerate case: \( \xi_C = 0 \Rightarrow \xi_2 = \xi_3 = 0 \) is a double root and \( P_4(\xi, 0) = -\xi^2(1 - \xi^2) \);
b) A new degenerate case: \( \xi_C = \sqrt{1/4} \Rightarrow \xi_3 = \xi_4 = -\frac{1}{2} \)
is a double root and \( P_4(\xi, 0) = -\xi(1 - \xi)(\xi - 1/2)^2 \).

In both cases the corresponding elliptic integrals, needed for an explicit solution of the problem, are reduced to elementary functions – see Appendix A.

For the \( \lambda \)-invariant quantities
\[
\mu(\xi, \xi_C) = \sqrt{\frac{8\pi \varepsilon}{3}} \left( 1 - \xi_C^3 \right) m(\rho, \rho_C), \\
\mu_0(\xi, \xi_C) = \sqrt{\frac{8\pi \varepsilon}{3}} \left( 1 - \xi_C^3 \right) m_0(\rho, \rho_C) \quad (\text{III.10})
\]
from Eq. (II.2), (II.7) we obtain (see Appendix A)
\[
\mu(\xi, \xi_C) = \frac{1}{2 \xi^3 - \xi_C^3}, \\
\mu_0(\xi, \xi_C) = \frac{3}{2} \frac{1}{\sqrt{1 - \xi_C^2}} \int_{\xi}^{\xi_C} \sqrt{-P_4(\xi_C)} \frac{\xi^3 d\xi}{\sqrt{-P_4(\xi_C)}} = 3 \frac{1}{\sqrt{1 - \xi_C^2}} \left( J_{3|1}(\xi, \xi_C) - J_{3|1}(\xi_C, \xi_C) \right). \quad (\text{III.11})
\]

Instead of the local binding energy \( \Delta m(\rho, \rho_C) := m_0 - m \), which is not \( \lambda \)-invariant, one can consider the ratio
\[ g(\rho, \rho_C) := m/m_0 = \mu/\mu_0 = g(\xi, \xi_C) \in (0,1). \quad (\text{III.12}) \]
It measures in a \( \lambda \)-invariant way the local mass defect of the star mater, i.e. the mass defect in the sphere with luminosity radius \( \rho \) and center \( C \).

Another important \( \lambda \)-invariant local (in the above sense) quantity is

\[
f(\rho, \rho_C) = \varrho(\rho, \rho_C) - g(\rho, \rho_C) + 1.\]

In the case at hand it has the form

\[
f(\rho, \rho_C) = \left( \frac{m}{m_0} \right)^2 + \frac{2m}{\rho} = \varrho^2 + \varsigma^2, \tag{III.13}\]

where \( \varsigma^2 = \frac{2m}{\rho} \geq 0 \) is the local compactness of the star. For incompressible stars \( \varsigma^2 = \frac{\xi^2}{(1+\xi)^2} \).

Using the results (A.5) and (A.6) of Appendix A, after some algebra one obtains the expressions:

\[
\varrho(\xi, \xi_C)^{-1} = 3 \sqrt{1 - \frac{\xi^2}{\xi_C^2}} \int_\xi^{\xi_C} \frac{\xi^3 d\xi}{\sqrt{-P_3(\xi, \xi_C)}} = 3 \sqrt{1 - \frac{\xi^2}{\xi_C^2}} \left( J_{3/1}(\xi, \xi_C) - J_{3/1}(\xi_C, \xi_C) \right), \tag{III.14}\]

and

\[
\chi(\xi, \xi_C) = \frac{3}{2} \sqrt{1 - \frac{\xi^2}{\xi_C^2}} \int_\xi^{\xi_C} \frac{\xi^4 d\xi}{\sqrt{-P_3(\xi, \xi_C)}} = \frac{3}{2} \sqrt{1 - \frac{\xi^2}{\xi_C^2}} \left( J_{4/3}(\xi, \xi_C) - J_{4/3}(\xi_C, \xi_C) \right), \tag{III.15}\]

for the basic \( \lambda \)-invariant local quantities \( \varrho \) and \( \chi \).

As one sees, the technical problem of finding the general solution of ETOV system for incompressible matter in the most general case is reduced to the calculation of the elliptic integrals in Eqs. (III.10), (III.14) and (III.15). This calculation is described in the Appendix A.

**IV. RADIUS OF INCOMPRESSIBLE STARS**

Now we are able to prove the existence of a finite coordinate radius \( r_* < \infty \) of incompressible stars. It corresponds to some finite value of the luminosity variable \( \xi_* < \infty \) and to some finite geometrical radius \( R_* = R_*/\sqrt{8\pi\varepsilon(1-\xi^3_0)/3} < \infty \). (Here \( R_* \) is the dimensionless \( \lambda \)-invariant geometrical radius of the star.)

By definition, in \( \lambda \)-invariant terms the edge of the star is defined as a point \( \xi_* \) at which

\[
p(\xi_*, \xi_C) = 0. \tag{IV.1}\]

**Proposition 1:** For all solutions of ETOV equations (II.2)-(II.6) with EOS (I.1) and arbitrary \( \xi_C \geq 0 \) there exist a unique solution \( \xi_* \in (\xi_C, \xi^{crit}_C(\xi_C)) \) of the Eq. (IV.1). Here the value \( \xi^{crit}_C(\xi_C) \in (\xi_C, 1) \) corresponds to the nonphysical limiting solution with infinite value of the pressure at the stellar center: \( p^{crit}_C = \infty \).

**Proof:**

1) Existence of unique solution of Eq. (IV.1):

From Eq. (III.6) and (III.15) one easily obtains in the limit \( \xi \to 1 - 0 \):

\[
\sqrt{-g_{\rho\rho}(\xi, \xi_C)} \propto \sqrt{\frac{1 - \xi^3}{2 + \xi_C^3(1 - \xi)} \to \infty,}
\]

\[
\chi(\xi, \xi_C) \propto 3 \sqrt{\frac{1 - \xi^3}{2 + \xi_C^3(1 - \xi)} \to \infty.} \tag{IV.2}\]

Hence

\[
\lim_{\xi \to 1^-} \frac{\sqrt{-g_{\rho\rho}(\xi, \xi_C)}}{\chi(\xi, \xi_C)} = \frac{2 + \xi_C^3}{3} \in (2/3, 1). \tag{IV.3}\]
As a result
\[
\lim_{\xi \to 1^-} w(\xi, \xi_C) = -\frac{1 - \xi_C^3}{3} < 0, \tag{IV.4}
\]
i.e. the continuous, strictly monotonic function \(w(\xi, \xi_C)\) decreases on the interval \([\xi_C, 1]\) from some value \(w_C = w(\xi_C, \xi_C) > 0\) to the value \(w(1, \xi_C) = -\frac{1 - \xi_C^3}{3} < 0\) (i.e., \(w(\xi, \xi_C) \downarrow [w_C, -(1 - \xi_C^3)/3]\) for \(\xi \in [\xi_C, 1]\)). Then there exist a unique value \(\xi_* \in (\xi_C, 1)\), such that \(w(\xi_*, \xi_C) = 0\). This value \(\xi_*\) defines the radius of the star.

2) Limitations on the \(\lambda\) -invariant radius of star \(\xi_*\):

The Eq. (II.10) permits us to rewrite the definition of the stelar radius (IV.1) in the following explicit form:
\[
w_C = \frac{1}{\sqrt{-g_{pp}(\xi_*\xi_C) - \chi(\xi_*, \xi_C)}} - 1 =: w_C(\xi_*, \xi_C). \tag{IV.5}
\]

It demonstrates the relation between the value \(w_C\) at the center \(C\) of the star and its radius \(\xi_*\). This relation is the main difference between relativistic theory of stars and Newtonian one.

Using the following properties of the functions \(\sqrt{-g_{pp}(\xi_*\xi_C)}\) and \(\chi(\xi_*, \xi_C)\)

a) \(-g_{pp}(\xi_*, \xi_C) = 1\);

b) \(\sqrt{-g_{pp}(\xi_*, \xi_C)} \searrow [1, \infty), \chi(\xi_*, \xi_C) \nearrow [0, \infty)\) – for \(\xi \in [\xi_C, 1]\);

c) for \(0 < \xi - \xi_C \ll 1\): \(\sqrt{-g_{pp}(\xi_*, \xi_C)} - \chi(\xi_*, \xi_C) > 0\);

d) for \(\xi \to 1\): \(\sqrt{-g_{pp}(\xi_*, \xi_C)} / \chi(\xi_*, \xi_C) \propto (2 + \xi_C^3)/3 \in \left(2/3, 1\right)\), one easily obtains that:

i) \(w_C(\xi_*, \xi_C) = 0\);

ii) \(w_C(1, \xi_C) = -1\);

ii) There exist a unique point \(\xi_{crit}^{\ast} \in (\xi_C, 1)\), such that \(\sqrt{-g_{pp}(\xi_{crit}^{\ast}, \xi_C)} = \chi(\xi_{crit}^{\ast}, \xi_C) \Rightarrow w_C(\xi_{crit}^{\ast} - 0, \xi_C) = +\infty\) and \(w_C(\xi_{crit}^{\ast} + 0, \xi_C) = -\infty\).

This means that there exist a limiting nonphysical solution – an incompressible star with a critical radius \(\xi_{crit}^{\ast} \in (\xi_C, 1)\), which corresponds to an infinite pressure \(p_{crit}^{\ast} \Rightarrow \infty\) at the center \(C\). Hence, the \(\lambda\)-invariant radius of the incompressible stars is constraint in the interval \((\xi_C, \xi_{crit}^{\ast})\). It is obvious that \(\xi_{crit}^{\ast}\) is a function of \(\xi_C\).

At the end of this Section we shall obtain the equation for determining of the function \(\xi_{crit}^{\ast}(\xi_C)\) and study its solution in details.

This completes the proof of our Proposition.

The luminosity radius \(\rho_*\) of the star can be find in the form \(\rho_* = \frac{\xi_*}{\xi_C} \rho_C \geq \rho_C\) and is not bounded from above if \(\varepsilon \to 0\), or/and for \(\xi_C \to 1\).

Having in disposal the \(\lambda\)-invariant radius \(\xi_*\) of the star, we are able to introduce another basic \(\lambda\)-invariant characteristics:

i) The total Keplerian mass \(\mu_* = \mu(\xi_*, \xi_C) \in (0, 1/2)\)\) for \(0 \leq \xi_C \leq \xi_* < 1\), see Fig. 2.

ii) The total proper mass \(\mu_{0*} = \mu_0(\xi_*, \xi_C) > 0\).

iii) The total mass defect ratio \(\rho_* = g(\xi_*, \xi_C) > 0\).

iv) The total compactness \(\kappa_* = s(\xi_*, \xi_C) = 2\mu_*/\rho_* = \frac{\xi_*^3 - \xi_C^3}{\xi_*^3(1 - \xi_C^3)} \in (0, 1), \text{etc.}\)

v) In addition we obtain the relations
\[
\varphi_* = \varphi(\xi_*, \xi_C) = \ln (w_C + 1) + \varphi_C \tag{IV.6}
\]

and (see Appendix A):
\[
R_* = R_*(\xi_*, \xi_C) = \sqrt{1 - \xi_C^3} \int_{\xi_C}^{\xi_*} \frac{\xi d\xi}{\sqrt{-P_{4}(\xi, \xi_C)}} = \\
\sqrt{1 - \xi_C^3} \left(J_{311}(\xi_*, \xi_C) - J_{311}(\xi_C, \xi_C)\right). \tag{IV.7}
\]

It is easy to see that \(R_*(\xi_*, \xi_C) \to \infty\) when \(\xi_C = 0\) and \(\xi_* \to 1 - 0\).

As seen in Fig. 2 the function \(\mu_*(\xi_*, \xi_C)\) has values in the interval \([0, 1/2]\). At the same time we obtain a new

Proposition 2: In the specific limit: \(\xi_* \to 1, \xi_C \to 1, 0 \leq \xi_C \leq \xi_* \leq 1\) we have
\[
\lim_{\xi_C \to 1 - 0} \mu_*(\xi_*, \xi_C) =: \mu_*(1 - 0, 1 - 0) \in [0, 1/2], \tag{IV.8}
\]
i.e., the limit \(\mu_*(1 - 0, 1 - 0)\) is bounded, but not definite and can have any value in the interval \([0, 1/2]\).

The analytical proof becomes obvious from the representation \(\mu_*(\xi_*, \xi_C) = \frac{1}{2} - \xi_* \xi_C \xi_C + \xi_C^3\) and consideration of the limit \(\xi_C \to 1\) of \(\mu_*(\xi_*, \xi_C)\) on the curves \(\xi_* = \xi_C + k(1 - \xi_C)\) with an arbitrary \(k \in [0, 1]\).

Note that for any \(k \in [0, 1]\) and \(n > 1, p > 1\):

i) On the curves \(\xi_* = \xi_C + k(1 - \xi_C)^n\) the limit (IV.8) has an universal value 0.

ii) On the curves \(\xi_* = 1 - k(1 - \xi_C^n)\) the limit (IV.8) has an universal value 1/2.
FIG. 6: A part of the function $R^*(\xi^*, \xi_C)$ and the lines $\xi^* = \text{const}$ and $\xi_C = \text{const}$.

Hence, the typical behavior of the limit (IV.8) is given by the last two cases: i) and ii). The case of linear dependence between $\xi^*$ and $\xi_C$ is an exceptional one.

The simple, but important property of the function $\mu^*(\xi^*, \xi_C)$, described in Proposition 2, will influence further results in the theory of incompressible relativistic stars, when the same limit of other quantities will emerge.

V. THE BASIC $Rm$ MAPPING FOR INCOMPRESSIBLE STARS

The basic mapping

$$\{R_*, \mu_*\} \xrightarrow{Rm} \{\xi^*(R_*, \mu_*), \xi_C(R_*, \mu_*)\}. \quad (V.1)$$

was defined in general case in [1]. Here we study this mapping in the specific case of incompressible GRS in scale-invariant variables:

Substituting in the formula (IV.7) $\xi^* = \xi^*(\mu_*, \xi_C)$, or $\xi_C = \xi_C(\mu_*, \xi^*)$, obtained from the first of the relations (III.11) in the form

$$\xi^*(\mu_*, \xi_C) = \sqrt{2\mu_* + (1 - 2\mu_*)\xi_C^2}, \quad (V.2a)$$

$$\xi_C(\mu_*, \xi^*) = \sqrt{\xi_C^2 - 2\mu_*}/(1 - 2\mu_*), \quad (V.2b)$$

one obtains the functions $R_* = R_*(\mu_*, \xi_C)$ and $R_* = R_*(\mu_*, \xi^*)$.

The corresponding inverse functions $\xi_* = \xi_*\bigl(\mu_*, R_*\bigr)$ and $\xi_C = \xi_C\bigl(\mu_*, R_*\bigr)$ define the basic mapping $Rm$ (V.1) in $\lambda$-invariant terms. This mapping is illustrated in figures 8 and 9.

As a byproduct from Eq. (V.2b), $2\mu_* \leq 1$ and $\xi_C \geq 0$ we obtain a new inequality $\xi^*_C \geq 2\mu_*$. Then, since $0 < \xi^*_C < 1$, we have

$$0 < \frac{2\mu_*}{\xi^*_C} < \frac{2\mu_*}{\xi^*_C} \leq 1. \quad (V.3)$$

For incompressible stars the relations, which determine the domain $D_{\xi^*, \xi_C}$ [1], acquire the following specific form (For the used notations see Appendix A.):

$$0 \leq \xi_C \leq \xi_* < 1, \quad (V.4a)$$

$$-\xi_* (1 + \xi_*) \leq 0 \leq \xi_C^2, \quad (V.4b)$$

$$\sqrt{\xi^*} \left(1 - \xi_*\right) \left(\xi_C^2 + \xi_*^2 + \xi_*\right) \sqrt{1 - \xi_C^2} < \frac{1}{\left(1 + \xi_*\right)} \left[J_{4/3}(\xi_*, \xi_C) - J_{4/3}(\xi_C, \xi_C)\right] < \frac{\xi_*}{\left(1 - \xi_*\right)\left(\xi_C^2 + \xi_*^2 + \xi_*\right)} \quad (V.4c)$$

FIG. 7: The function $\xi_C = \xi_C(\mu_*, \xi^*)$ (V.2b) for different fixed values of parameter $\mu_* \in (0, 1/2)$.

FIG. 8: The function $\xi^*(\mu_*, R_*) \in [0, 1]$. The behavior of this function in the limit $\mu_* \to 0, R_* \to 0$ is similar to the one, described in Proposition 2 for the function $\mu_*\bigl(\xi^*, \xi_C\bigr)$. 

FIG. 9: The function $\xi^*(\mu_*, R_*)$ for different fixed values of parameter $\mu_*$. The domain $D_{\xi^*, \xi_C}$ (V.1) illustrated for the case $\xi^*_C \geq 2\mu_*$. The behavior of this function in the limit $\mu_* \to 0, R_* \to 0$ is similar to the one, described in Proposition 2 for the function $\mu_*\bigl(\xi^*, \xi_C\bigr)$.
FIG. 9: The function $\xi_C(\mu_*; R_*) \in [0, 1]$. The behavior of this function in the limit $\mu_* \to 0, R_* \to 0$ is similar to the one, described in Proposition 2 for the function $\mu_*(\xi_*, \xi_C)$.

As a result of the first of the relations (V.4) the condition (V.4b) is fulfilled.

Parts of the corresponding domains $D^{(2)}_{\mu_*; R_*}$ are shown in Figs. 10 and 11, where the condition (V.4c) is still not taken into account.

FIG. 10: The domain $D^{(2)}_{\mu_*; R_*}$ for the function $\xi_C(\mu_*; R_*)$.

VI. THE CRITICAL RADIUS AND CRITICAL MASS OF INCOMPRESSIBLE STARS

The limiting case of Eq. (V.4c) yields the following equation for the critical value $\xi^{crit}_* = \xi^{crit}_*(\xi_C)$ of the dimensionless luminosity variable:

$$J_{4/3}(\xi^{crit}_*, \xi_C) - J_{4/3}(\xi_C, \xi_C) = \frac{2\xi^{crit}_*}{3\sqrt{-P_4(\xi^{crit}_*, \xi_C)}}.$$  \hspace{1cm} (VI.1)

It is obvious that this equation defines a new function $\xi^{crit}_*(\xi_C)$, which describes the basic difference between GR and Newtonian incompressible stars. There are no any physical parameters in the Eq. (VI.1). Hence, the function $\xi^{crit}_*(\xi_C)$ is an universal mathematical one.

In the degenerated Schwarzschild case when $\xi_C = 0$ we have $-P_4(\xi, \xi_C) = \xi^2(1 - \xi^2)$ and the integral in Eq. (VI.1) can be easily done. This gives (see Appendix A)

$$\frac{1}{\sqrt{1 - (\xi^{crit}_*)^2}} - 1 = \frac{2}{3\sqrt{1 - (\xi^{crit}_*)^2}}.$$  

Hence,

$$\xi^{crit}_*(\xi_C = 0) = \sqrt{8/9} \approx 0.942809.$$  \hspace{1cm} (VI.2)

Thus we have obtained the exact value of the new function $\xi^{crit}_*(\xi_C)$ at the point $\xi_C = 0$. It is clear that $\xi^{crit}_*(\xi_C = 1) = 1$.

The shape of function $\xi^{crit}_*(\xi_C)$ is shown in Fig. 12. Some of its basic values are given in Appendix B. As seen, the values of the function $\xi^{crit}_*(\xi_C)$ are in the interval $[\sqrt{8/9}, 1]$. Hence, for a given value of $\lambda$-invariant luminosity variable $\xi_C > 0$, the value of $\xi_*$ varies in the interval $[\xi_C, \xi^{crit}_*(\xi_C)]$, with upper limit $\xi^{crit}_*(\xi_C) > \sqrt{8/9}$.

This changes radically our understanding of the relativistic theory of stars, because now in the domain $0 \leq \xi_C \leq \xi_* \leq 1$:

$$\frac{2m_*}{\rho_*} = s_* = \frac{\xi^{3}_* - \xi^{3}_C}{\xi_*(1 - \xi^{3}_C)} \leq \xi_*^2 < \left(\xi^{crit}_*(\xi_C)\right)^2.$$  \hspace{1cm} (VI.3)
As a result of the last inequality one obtains the familiar relativistic restriction for the Schwarzschild degenerate case:

$$\left(2m_*/\rho_*\right)_{\xi_C=0} = \xi_*^{\text{crit}}_{\xi_C=0} = \xi_*^2 < 8/9.$$ 

This constraint yields the following restrictions on the luminosity variable and Keplerian mass of Schwarzschild incompressible stars:

$$\rho_*|_{\xi_C=0} < \frac{1}{\sqrt{3\pi\varepsilon}}, \quad m_*|_{\xi_C=0} < \frac{4}{9\sqrt{3\pi\varepsilon}}. \quad (\text{VI.4})$$

Hence, in the relativistic model with $\rho_C = 0$ we are not able to describe stars with fixed density $\varepsilon = \text{const}$ and with arbitrary large mass $m_*$. In contrast, in our general geometrical model of incompressible relativistic stars with arbitrary fixed value of luminosity variable $\xi_C \in [0, 1]$ we have the restrictions:

$$\rho_* < \rho_*^{\text{crit}} := \frac{1}{\sqrt{8\pi\varepsilon/3}} \frac{\xi_*^{\text{crit}}(\xi_C)}{\sqrt{1-\xi_C^2}} \to \infty : \text{ for } \xi_C \to 1-0,$$

$$m_* < m_*^{\text{crit}} := \frac{1}{2} \frac{1}{\sqrt{8\pi\varepsilon/3}} \frac{(\xi_*^{\text{crit}}(\xi_C))^3-\xi_C^3}{(\sqrt{1-\xi_C^2})^3} \to \infty : \quad (\text{VI.5})$$

for $\xi_C \to 1-0$.

These upper limits $\rho_*^{\text{crit}}(\xi_C)$ and $m_*^{\text{crit}}(\xi_C)$ are exact, i.e., the corresponding quantities $\rho_*$ and $m_*$ can become arbitrary close to them for a given value of $\xi_C$.

The calculation of the limit of $\rho_*^{\text{crit}}(\xi_C)$ is obvious from $\xi_*^{\text{crit}}(\xi_C = 1 - 0) = 1$, see Fig. 12. The analytical proof of the above limit for $m_*^{\text{crit}}(\xi_C)$ when $\xi_C \to 1-0$ follows immediately from our Proposition 3 (see Section IV, C) and Eq.(III.10). The behavior of the critical mass $m_*^{\text{crit}}(\xi_C)$ as a function of the central value $\xi_C$ of luminosity variable is shown in Fig. 13 for density $32\pi\varepsilon/3 = 1$ [11].

Thus we have arrived at the following

\textbf{Proposition 3:} For any fixed density $\varepsilon = \text{const}$ in our essentially non-Euclidean general model of relativistic stars there exist incompressible stars with arbitrary large mass $m_*$. Obviously, stars with arbitrary large mass $m_*$ exist only for large enough values of luminosity variable $\xi_C$.

From Eq. (IV.7) one obtains the following critical value for the geometrical radius of our relativistic incompressible stars:

$$R_*^{\text{crit}}(\xi_C) = \frac{J_{111}(\xi_*^{\text{crit}}(\xi_C), \xi_C) - J_{111}(\xi_C, \xi_C)}{\sqrt{8\pi\varepsilon/3}}. \quad (\text{VI.6})$$

The behavior of the critical geometrical radius $R_*^{\text{crit}}(\xi_C)$ as a function of the central value $\xi_C$ of luminosity variable is shown in Fig. 14 for density $32\pi\varepsilon/3 = 1$.
of strong gravity, their geometrical radius \( R_* \) and luminosity \( L_* \sim 1/\rho_*^2 \) are arbitrary small, when the mass \( m_* \) is large enough. At the same time in presence of such objects the geometry of the space-time is regular everywhere. Horizons of any type do not exist.

It seems reasonable to refer to objects as to heavy black dwarfs. As seen in Fig. 13, the existence of HBD is impossible under widely accepted extra condition \( \rho_* = 0 \).

The critical proper mass \( m_{\text{crit}}^\rho(\xi_C) \) is bigger then \( m_{\text{crit}}^\rho(\xi_C) \). Hence, it goes to infinity in the limit \( \xi_C \to \infty \) together with \( m_{\text{crit}}^\rho(\xi_C) \).

More interesting characteristic is the \( \rho \)-invariant critical mass defect ratio \( \varrho_\rho(\xi_C) \), obtained from Eq. (III.14) in the form:

\[
\varrho_\rho(\xi_C) = \frac{(\xi_C^{\text{crit}}(\xi_C))^3 - \xi_C^3}{3(J_{3/1}(\xi_C^{\text{crit}}(\xi_C), \xi_C) - J_{3/1}(\xi_C, \xi_C))} \sqrt{1 - \xi_C^2}.
\]

This function is represented in Fig. 15.

As seen, the critical value of the mass ratio \( \varrho_\rho(\xi_C) \) decreases monotonically from

\[
\varrho_\rho(0) = \frac{16}{9} \left( 1 - 3 \sqrt{9/8 \arcsin \left( \sqrt{8/9} \right) } \right)^{-1} \approx .609477
\]

to 1/2 when the variable \( \xi_C \) increases from 0 to 1.

Hence, in the case of incompressible mater the extremely strong gravitational field, which arise in the limit \( R_* \to 0, m_* \to \infty \) is able to extract no more then one halve of the initial bare mass \( m_{\text{bare}} \) of the stelar matter, although the relative part of gravitational energy increases in this limit, together with the mass defect. It can be shown that this limitation is EOS-dependent.

One can understand the above behavior of the critical mass ratio \( \varrho_\rho(\xi_C) \) analyzing the form of the functions \( \varrho^\rho(\xi_C) \) – Fig. 16 and \( \varrho_{\mu}(\xi_C) \) – Fig. 17.

A more deep understanding of the mass defect one can obtain looking on 3D surface of the function, \( g(\xi_*, \xi_C) = \)

\[
\text{FIG. 16: The critical function } \mu_{\rho}(\xi_C) \text{ as a boundary of the physical domain of the variable } \mu_{\rho} \in (0, \mu_{\rho}(\xi_C)) \text{. As seen, on the specific curve } \xi_* = \xi_{\rho}(\xi_C) \text{ the function } \mu_{\rho}(\xi_C) \text{ goes to } 1/2, \text{ when } \xi_C \to 1. \text{ (Compare this specific result with Proposition 2 and with comments after it.)}
\]

\[
\text{FIG. 17: The critical function } \mu_{\rho}(\xi_C) \text{ as a boundary of the physical domain of the variable } \mu_{\rho} \in (0, \mu_{\rho}(\xi_C)) \text{.}
\]

\[
\text{FIG. 18: The function } g(\xi_*, \xi_C) = \mu(\xi_*, \xi_C)/\mu_0(\xi_*, \xi_C) \geq 4/3\pi \text{ for incompressible star. The shadowed triangle is a part of the horizontal plane } 4/3\pi \approx 1.24413.
\]
\( \mu(\xi_*, \xi_C)/\mu_0(\xi_*, \xi_C) \geq 4/3\pi, \) shown in Fig. 18. The last limit from below of the mass ratio \( \rho(\xi_*, \xi_C) \) originates from EOS for incompressible stars Eq. (I.1) and will be different for relativistic stars with other EOS. It reflects the form of the right border of the 3D surface:

\[
\rho_*(\xi_*, 0) = \frac{2}{3} \frac{\xi_*^3}{\arcsin \xi_* - \xi_* \sqrt{1 - \xi_*^2}} \quad (\text{VI.8})
\]

calculated for the Schwarzschild degenerate case of incompressible stars. An analogous simple expression can be drawn for \( \rho_*(\xi_*, \sqrt{1/4}), \) using results, given in Appendix A.

The real limit from below on \( \rho(\xi_*, \xi_C) \) in the problem at hands is \( 1/2. \) It reflects the existence of the curve \( \xi_*^{crit}(\xi_C), \) which is not shown in Fig. 18.

**VII. MASS-RADII RELATIONS FOR WHITE DWARFS AND THE NEW GEOMETRICAL MODELS OF GRS**

Our previous consideration [1] showed that there exist two different approaches to the GRS with stiff mass-radii functional dependence:

1. The standard one, based on the extra condition \( \rho(0) = \rho_C = 0, \) and

2. A new one, with extra condition: \( F - \text{fixed}. \) It arises naturally in basic regular radial gauge [1]. Here \( F \in (-\infty, \infty) \) is the stelar crust parameter. It defines the jump of the derivative of luminosity variable \( \rho(r) \) as a function of the radial one, \( r: \rho'(r_* - 0) = e^{-F} \rho'(r_* + 0), \) at the edge of the star \( r_* \).

At the same time one can consider a geometrical models of GRS without any additional condition of that type. In this most general case stiff mass-radii relations, based only on GR considerations, i.e. independent on EOS, do not exist [1]. In this case \( m_* \) and \( R_* \) are free parameters, constrained in some 2D domain, the precise form of which depends on the EOS.

The physical domain of variables \( m_* \) and \( R_* \) for incompressible GRS is shown in Fig. 19. It is obvious that this physical domain is very similar to the one, recently observed for real white dwarfs, see Fig. 2 in the article by J. Madej, M. Należyty and L. G. Althaus in [5]. This demonstrates how we are able to reproduce the real observational data in a qualitatively right way, using the simplest new geometrical model of incompressible GRS.

The most important consequence of our analysis of the observational data is the conclusion that actually we do not observe stiff mass-radii in Nature. Instead, these data demonstrate a clear indication that \( m_* \) and \( R_* \) are filling a 2D domain with a sharp boundary of the type, similar to the one, shown in our Fig. 19. This corresponds to our general models of GRS without any extra conditions [1]. Therefore we will skip the consideration of such relations in the present article, although they may be of some interest, too.

For a more precise qualitative treatment of the observational data one need to obtain new geometrical models of the GRS with realistic EOS. This may give a basis for explanation of the mass distributions of special type of stars, like white dwarfs, or neutron stars, independently of theory of gravity. We will present the corresponding results elsewhere.

**VIII. CONCLUDING REMARKS**

In conclusion several remarks have to be made:

1. In the present article we have recovered an interesting new GR phenomena, studying the most general solutions of ETOV equations for incompressible matter. We were able to find proper physical meaning of all solutions of these equations, overcoming the requirement for regularity of solutions at the zero value of the luminosity variable \( \rho. \) As we have seen, this way we obtain a two parameters family of solutions.

2. We wish to comment once more the most important result of present article: the proof of the existence of new GR objects – heavy black dwarfs (HBD).

From physical point of view this becomes possible, because the fulfillment of the condition \( \rho \geq \rho_C > \rho_G \) [1], where \( \rho_G = 2m_* \) is the Schwarzschild radius, is equivalent to the introduction of non-permeable potential wall in the space of the luminosity variable \( \rho. \) Because of the Gauss theorem, this introduces in the energy balance of the body an effective "potential energy" with an infinite jump.

Then, following the original Landau energetic consideration of the problem [9], we conclude that the presence of any additional energy terms, due to EOS, can not destroy the existence of static equilibrium of the body, at least in the case of non-ultra-relativistic matter. This "p-space picture" explains in a more usual language the consequences of the new boundary conditions at the real
center of the star, under which we are solving Einstein equations. See for details [1].

In terms of the proper radial variable $r$ no wall exist at all. Thus we see once more, that one is to give up the idea to consider the luminosity variable $\rho$ as a measure of distances. It is responsible only for determination of geometrical area and luminosity of physical objects.

3. The results of the present article raise many new questions and problems.

For example, we obtain a new real alternative to the black holes (BH) for explanation of nature of the massive compact dark objects, observed in astrophysics. Hence, we need a new criteria to distinguish BH and HBD.

An obvious problem is the possible relation of HBD with the observed could dark matter in the universe, as well as the relation of HBD with gravitational collapse of usual bodies. It is clear, that the present series of articles calls for a novel consideration of the collapse, too.

For a comparison with observational data, we obviously need consideration of new geometrical models of GRS with more realistic EOS for stellar matter.

We intend to address all these questions in the forthcoming articles in this series.

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APPENDIX A: CALCULATION OF THE ELLIPTIC INTEGRALS

Using standard incomplete elliptic integrals of first, second and third kind ($F$, $E$ and $\Pi$) [7], we can write down the integrals

$$J_{n|m}(\xi, \xi_C) := \int d\xi \xi^m / \left(\sqrt{-P_4(\xi, \xi_C)}\right)^m,$$  \hspace{1cm} (A.1)

which are needed for the relativistic incompressible star problem, in the following form:

I. $n = 0, m = 1$:

$$J_{0|1}(\xi, \xi_C) = g \frac{1}{2} F(i\xi, k),$$  \hspace{1cm} (A.2)

In the generate cases one obtains:

a) $J_{0|1}(\xi, 0) = \frac{1}{2} \ln \left(\frac{1 - \sqrt{1 - \xi^2}}{1 + \sqrt{1 - \xi^2}}\right)$;
b) $J_{0|1}(\xi, \sqrt{1/4}) = \frac{2}{\sqrt{3} \cdot \tan(i\xi)} - \arcsin(1 - 2\xi)$.

II. $n = 1, m = 1$:

$$J_{1|1}(\xi, \xi_C) = g \frac{1}{2} F(i\xi, k) + \frac{1}{2} \Pi(i\xi, \alpha^2, k),$$  \hspace{1cm} (A.3)

In the generate cases one obtains:

a) $J_{1|1}(\xi, 0) = \arcsin(\xi)$;
b) $J_{1|1}(\xi, \sqrt{1/4}) = \frac{1}{\sqrt{3}} \cdot \arctan(\frac{\sqrt{3}(1 - 4\xi)}{6\sqrt{1 - \xi} \cdot \xi}) - \arcsin(1 - 2\xi)$.

II. $n = 2, m = 1$:

$$J_{2|1}(\xi, \xi_C) = \frac{1 - 3\xi^2}{4} g \frac{1}{2} F(i\xi, k) + \frac{1}{2} \Pi(i\xi, \alpha^2, k),$$  \hspace{1cm} (A.4)

In the generate cases one obtains:

a) $J_{2|1}(\xi, 0) = 1 - \sqrt{1 - \xi^2}$;
b) $J_{2|1}(\xi, \sqrt{1/4}) = -\sqrt{\xi(1 - \xi)} - \frac{\sqrt{3}}{6} \cdot \arctan(\frac{\sqrt{3}(1 - 4\xi)}{6\sqrt{1 - \xi} \cdot \xi})$.

II. $n = 3, m = 1$:

$$J_{3|1}(\xi, \xi_C) = \frac{2 - \xi^3}{4} g \frac{1}{2} F(i\xi, k) - \frac{1}{2} \Pi(i\xi, \alpha^2, k) + \frac{1}{2} \sqrt{-P_4(\xi, \xi_C)}$$  \hspace{1cm} (A.5)

In the generate cases one obtains:

a) $J_{3|1}(\xi, 0) = \frac{1}{2} \arcsin(\xi) - \frac{1}{2} \xi \sqrt{1 - \xi^2}$;
b) $J_{3|1}(\xi, \sqrt{1/4}) = \frac{\sqrt{3}}{12} \cdot \arctan(\frac{\sqrt{3}(1 - 4\xi)}{6\sqrt{1 - \xi} \cdot \xi}) - \frac{1}{4} \cdot \arcsin(1 - 2\xi) - \frac{1}{4} (1 + 2\xi) \sqrt{\xi(1 - \xi)}$. 

II. $n = 4$, $m = 3$:

$$J_{4|3}(\xi, \xi_C) = \frac{(1 + \sigma)(\sigma^2 - 6\sigma - 3)}{\sigma(3 + \sigma)(9 - \sigma^2)} g \frac{1}{i} F(i z, k) + \frac{(2 + \sigma^2)}{\sigma(9 - \sigma^2)} \frac{2}{g} \frac{1}{i} E(i z, k) - \frac{\xi (2 + \sigma^2)\xi (1 + \sigma^2)(3 - \sigma^2)}{\sigma^2(9 - \sigma^2)\sqrt{-P_4(\xi, \xi_C)}}. \quad (A.6)$$

In the generate cases one obtains:

a) $J_{4|3}(\xi, 0) = \frac{1}{i} \sqrt{1 - \xi^2} - 1$;

b) $J_{4|3}(\xi, \sqrt{1/4}) = \frac{g}{\sigma} \sqrt{3} \arctan \left( \frac{\sqrt{3}(1 - 4\xi)}{6\xi(1 - \xi)} \right) + \frac{4\xi^2 + 16\xi + 5}{2(\xi - 1)^2} \sqrt{\frac{\xi}{1 - 4\xi}} - \frac{5\pi}{64}$.

Here $\sigma := \sqrt{1 - 4\xi^2}$, $z := \alpha^{-1} \sqrt{\xi/(1 - \xi)}$,

$$g := \sqrt{(1 + \sigma)(3 - \sigma)}, \quad k := \sqrt{\frac{1 - \sigma}{3 - \sigma}}, \quad \alpha := \frac{1 - \sigma}{3 - \sigma}, \quad \text{and} \quad (A.7)$$

$$\frac{1}{i} F(i z, k) := \int_0^z \frac{dx}{\sqrt{(1 + x^2)(1 + k^2x^2)}}, \quad (A.8a)$$

$$\frac{1}{i} E(i z, k) := \int_0^z \frac{1 + k^2x^2}{1 + x^2} dx, \quad (A.8b)$$

$$\frac{1}{i} \Pi(i z, \alpha^2, k) := \int_0^z \frac{dx}{(1 + \alpha^2x^2)\sqrt{(1 + x^2)(1 + k^2x^2)}} \quad (A.8c)$$

define for any values $0 \leq \xi_C \leq \xi < 1$ three basic elliptic integrals in most convenient for us uniform representation.

Note that for $\xi_C \in [0, \sqrt{1/4}]$ the parameters (A.7) have real values: $k \in [0, 1]$, $g \in [2, 4/\sqrt{3}]$, $\alpha \in [0, 1/\sqrt{3}]$.

For $\xi_C \in (\sqrt{1/4}, 1)$ these parameters are complex numbers. Nevertheless, for $0 \leq \xi_C \leq \xi < 1$ the integrals (A.2)-(A.6) have real values in this case, too.

Of course one can use more complicated representations of the standard elliptic integrals [7], such that the integrals $J_{n|m}(\xi, \xi_C)$ will have transparently real values for $\xi_C \in (\sqrt{1/4}, 1)$.

Taking into account that in this case $k = \exp(i\psi)$, $\psi \in \mathbb{R}$, we can perform the transformation of the modulus $k \rightarrow \tilde{k} = \frac{1}{i} \left( \sqrt{k + 1/\sqrt{k}} \right) = \cos(\psi/2) \in [0, 1]$ in all elliptic integrals. For example, $F(z, k) = F(\tilde{z}, \tilde{k})/2\sqrt{k}$, where $\tilde{z}^{-1} = \frac{1}{i} \left( z\sqrt{k + 1/z\sqrt{k}} \right)$. Unfortunately, such representations for $E(z, k)$ and $\Pi(z, \alpha, k)$ are quite complicated.

We will not give them here. They can be obtained composing several transformations of $k$, described in [7].

**APPENDIX B: TABLE OF THE VALUES OF FUNCTION $\xi_{crit}^* (\xi_C)$**

| $\xi_C$  | 0.00 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 | 0.30 |
|----------|------|------|------|------|------|------|------|
| $\xi_{crit}^*$ | .9428 | .9430 | .9436 | .9448 | .9467 | .9493 | .9525 |
| $\xi_C$ | 0.35 | 0.40 | 0.45 | 0.50 | 0.55 | 0.60 | 0.65 |
| $\xi_{crit}^*$ | .9564 | .9608 | .9655 | .9706 | .9756 | .9806 | .9852 |
| $\xi_C$ | 0.70 | 0.75 | 0.80 | 0.85 | 0.90 | 0.95 | 1.00 |
| $\xi_{crit}^*$ | .9894 | .9930 | .9959 | .9980 | .9993 | .9999 | 1.00 |

TABLE I: The values of function $\xi_{crit}^* (\xi_C)$

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[10] Note that in the units $c = G_N = 1$ the energy density $\varepsilon$ has a dimension of $\text{length}^{-2}$.

[11] For the value $\varepsilon = 3/32\pi$ the critical mass $m_{\text{crit}}^*(\xi_C)$ of incompressible relativistic stars becomes an universal mathematical function, like the function $\xi_{\text{crit}}^*(\xi_C)$. 
