Measures of macroscopicity for quantum spin systems

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Abstract. We investigate the notion of ‘macroscopicity’ in the case of quantum spin systems and provide two main results. Firstly, we motivate the quantum Fisher information as a measure of the macroscopicity of quantum states. Secondly, we make a comparison with the existing literature on this topic. We report on a hierarchy among the measures and we conclude that one should carefully distinguish between ‘macroscopic quantum states’ and ‘macroscopic superpositions’, which is a strict subclass of the former.

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1. Introduction

Quantum mechanics is probably the most successful and fascinating physical theory of the last century. In the first place, it provides a deep understanding of atoms and their interaction with light. In recent years, the prospects of improved technologies have been intensively investigated. This success comes with the price of a difficult interpretation of the theory, which was intensively discussed from a philosophical point of view. As long as we consider only microscopic systems on the scale of an atomic radius, objections to quantum mechanics are nevertheless rare, mainly because of the overwhelming experimental evidence. When it comes to macroscopic systems, many things are not clear anymore. Already in 1935, Schrödinger pointed out in his seminal paper [1] that quantum mechanics in principle allows superpositions of macroscopic states, like a cat that is alive and dead at the same time. This gedanken experiment is deeply connected with the so-called quantum measurement problem and was discussed by generations of physicists.

Besides the interest in the foundations of quantum mechanics, the question of macroscopic quantum mechanics also has practical aspects. Proposed architectures for quantum computers are based on a large number of qubits. Computational tasks that can overcome classical algorithms may require long-range quantum correlations [2, 3]. A further application, quantum metrology, uses a certain kind of multipartite entanglement among many particles for an increased sensitivity in phase estimation protocols [4].

In 1980, Leggett [5] gave an important impulse to the topic of macroscopic quantum mechanics. He asked for a clear definition of the phrases ‘macroscopic quantum phenomenon’ and ‘macroscopic superposition’. He realized that one should distinguish between quantum effects that originate on a microscopic level from ‘true’ macroscopic quantum effects. Among other examples, he highlights the specific heat of insulators. Classical statistical mechanics...
predicts a specific heat that is constant with respect to the temperature $T$. On the other hand, the quantum mechanical Debye model correctly predicts $T^3$ behaviour for small temperatures. Many physicists considered this as an example of a macroscopic quantum phenomenon, since this law is valid even for large insulators. Leggett argued that the phrase macroscopic in this context is not justified, because the interactions that cause this effect are on an atomic scale and thus microscopic. In the following, he demanded a distinction between classical and microscopic quantum effects on the one hand, and macroscopic quantum effects on the other hand. Only the latter allow us to verify quantum mechanics (against classical theories) on a macroscopic scale, as in the example of Schrödinger’s cat.

Consequently, we call quantum states that are capable of inducing macroscopic quantum effects the ‘macroscopic quantum states’. The question at issue is: which properties of a many-body quantum state are appropriate for such a characterization? It is clear that the number of particles is an important but not sufficient criterion. If we consider superpositions of semi-classical quantum states, it seems to be crucial that they be ‘macroscopically distinct’, as Leggett [5] phrased it. However, a straightforward mathematical formulation of this intuitive characterization does not exist. Furthermore, there may be quantum states that do not exhibit a superposition of two semi-classical states, but are superpositions of a large number of those. These and further concerns led to various proposals of macroscopic quantum states [5–12].

The goal of this work is twofold. Firstly, we motivate and introduce another aspect of macroscopic quantum effects in discrete systems. After basic considerations in section 2, we propose to use the so-called quantum Fisher information as a measure of ‘macroscopicity’. The quantum Fisher information originally appeared in the context of phase estimation [13], but also gives insight into the geometry of the space of density operators [14] and their entanglement properties [4, 15, 16]. In section 3, we argue that the quantum Fisher information is an appropriate measure to distinguish macroscopic from microscopic quantum effects in discrete quantum systems. Furthermore, it is a key property to judge the usability of quantum states to overcome classical limits in parameter estimation protocols. In addition, we present a further characterization of superpositions of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|\psi_0\rangle + |\psi_1\rangle),$$

where $|\psi_0\rangle$ and $|\psi_1\rangle$ are orthogonal quantum states. This proposal is called the ‘relative Fisher information’ and compares how macroscopic $|\psi\rangle$ is in relation to the macroscopicity of $|\psi_0\rangle$ and $|\psi_1\rangle$. It is necessary for the latter to be semi-classical quantum states in order to form a macroscopic superposition due to the relative Fisher information. The basic motivation for this proposal is to identify the class of macroscopic superpositions in the spirit of Schrödinger’s cat. In contrast to other proposals, it is unambiguously defined for all superpositions $|\psi\rangle$.

Secondly, we compare the quantum Fisher information and the relative Fisher information as macro-measures with the other measures for qubit systems that were proposed recently [7–11]. Therefore we give a short review of these measures in section 4. There exist two classes of measures. The first class considers arbitrary quantum states, while the second class focuses on superposition states as in equation (1). In section 5, we identify a hierarchy among those proposals. We show that the set of macroscopic superpositions (1)—classified by [8, 10, 11] and the relative Fisher information—is a strict subset of general macroscopic quantum states according to the work of Shimizu and Miyadera [7] and the quantum Fisher information.

We summarize the paper and conclude in section 6.
2. The setting and basic considerations

We exclusively focus on $N$ discrete two-level systems (qubits), defined on the Hilbert space $\mathbb{C}^{2 \otimes N}$. We restrict our discussion to qubit systems, because, on the one hand, they are valid representations of a large class of physical systems. On the other hand, the mathematical treatment of problems for qubit systems is often much simpler than that for other frameworks, which can lead to insight that is useful for the entire quantum theory. However, it is clear that qubit systems are a priori an abstract concept that include neither spatial distances between particles nor masses or energy scales. Therefore, the only parameter that gives rise to a qualification of macroscopic quantum states is the particle number $N$. In general, we assume $N$ to be a ‘large number’, that is, $N \gg 1$. The boundary at which we call $N$ large is not strict. For theoretical considerations as in this paper, it is often more convenient to study the scaling of a certain property like the macroscopicity of quantum states. Therefore, we focus on quantum states that are scalable (i.e. there exists a recipe how to define these states for any $N$) and ask whether these states are macroscopic in the limit of large but finite $N$.

As mentioned by other authors [8, 10, 17], the topic of macroscopic quantum mechanics is ultimately subjective. This is not only because different authors consider different properties of a quantum system as essential for the characterization of a macro-system. (However, we will see in section 5 that different proposals lead to similar measures.) The notion of locality plays an essential role in all works that we are going to discuss in the course of this paper. This means that quantum states are macroscopic with respect to a given concept of locality. The motivation is physical rather than mathematical. It comes from the observation that all interactions we encounter in nature are of finite range. For macroscopic qubit systems, we therefore assume that all Hamiltonians and all measurements are sums of local terms. Since in this paper, we are concerned with the scaling of certain properties, we demand that the range of the addends is independent of the system size (i.e. $O(1)$). For simplicity, we only consider local operators that act non-trivially on distinct groups of qubits.

3. Quantum Fisher information as a measure of macroscopicity

In this section, we define the quantum Fisher information as a measure of macroscopic quantum states. First, we begin with working definitions to motivate our proposal. These starting points are more or less vague characterizations of macroscopic quantum states, since they are built on vaguely defined terms. From these, we try to infer a more stringent definition, which we formulate in section 3.2.

3.1. Starting points for a macro-measure

As already mentioned, there are two important aspects of macroscopic quantum mechanics. Originally, physicists have been interested in the foundations of quantum mechanics. Does a macroscopic object like a cat obey the Schrödinger equation in principle? If not, where is the border between classical and quantum mechanical theory and what kind of theory could unify both realms? Those questions led, for example, to collapse theories such as [18], which state that the Schrödinger equation has to be altered in order to describe the physics of heavy

\[ f(N) = O(N^x) : \Leftrightarrow \lim_{N \to \infty} (f(N)/N^x) > 0 \quad \text{and} \quad f(N) = o(N^x) : \Leftrightarrow \lim_{N \to \infty} f(N)/N^x = 0. \]

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objects correctly. A different viewpoint is given by the decoherence theory. It assumes the general validity of quantum mechanics on macroscopic scales. Generally, a quantum system interacts with its environment locally, which destroys non-local quantum correlation within the system. Larger objects interact with the environment more intensively and therefore lose these correlations more rapidly. In decoherence theory, this is the reason why we never experience macroscopic quantum effects in our daily life. There is a way to provide experimental evidence in favour of the decoherence theory. Suppose that we study a collapse model that forbids quantum mechanics on a certain mass scale. If we observe behaviour of a macroscopic system at this scale that cannot be explained by classical theories, nor by accumulative microscopic quantum effects (such as the Debye-model for the specific heat), this collapse model is falsified. In the spirit of Leggett [5, 17], we take this as a starting point and note

**Working definition 1.** A quantum system is called macroscopic if it is capable of showing behaviour that is neither a classical nor an accumulated microscopic quantum effect.

In addition to fundamental considerations, the applications of quantum mechanics are at the focus of interest. As a first starting point for applications analogous to working definition 1, one could demand that a macroscopic quantum state is able to perform a certain task faster or better than any classical device. However, this leads to ambiguities for two reasons. Firstly, there is a considerable range of different applications that are incommensurable. By focusing on a specific task to characterize macroscopic quantum states, we certainly bias the set of macrostates. If we look at resource states for universal quantum computation in a measurement-based setting [19, 20], we encounter different states from quantum states that are useful for quantum metrology. Secondly, the phrase ‘faster or better’ is vaguely defined. In quantum computation, for example, it is not at all clear how much better certain quantum algorithms are.

In this paper, we focus on quantum metrology for two reasons. First, the system size $N$ is a crucial parameter for the performance of a parameter estimation protocol. In addition, the improvement with the help of quantum mechanics compared to classical strategies is mathematically established. In the next section, we shortly review the theory of quantum metrology. For now, we note

**Working definition 2.** A macroscopic quantum state is defined as a resource state that is capable of increasing the sensitivity of a parameter estimation protocol qualitatively.

Before we go on with the implications of working definitions 1 and 2, we introduce the concept of an ‘effective size’ $N_{\text{eff}}$ of the system. This term has been used in the literature, e.g. in [6, 8, 10, 11], to assign a number $N_{\text{eff}} \leq N$ to a system to judge the ‘macroscopicity’ of the system. We try to make this idea more precise with

**Working definition 3.** A given quantum state $\rho$ of an $N$-qubit system may be capable of showing a non-classical phenomenon. The effective size of $\rho$, denoted by $N_{\text{eff}}(\rho)$, is the minimal system size for which one has to assume validity of quantum mechanics in order to explain this phenomenon.

The effective size gives us therefore a quantification of working definitions 1 and 2. Given a certain quantum state $\rho$ of $N$ qubits, its effective size $N_{\text{eff}}(\rho)$ is taken as the basis for a qualification of $\rho$ as a ‘macroscopic quantum state’. A non-trivial step is now to define for which $N_{\text{eff}}(\rho)$ one should call $\rho$ macroscopic. While for a specific experiment the absolute number of $N_{\text{eff}}(\rho)$ may be crucial, for theoretical investigations the scaling of $N_{\text{eff}}(\rho)$ with $N$
is apparently more interesting. The maximal scaling of the effective size is linear in \( N \) due to working definition 3. It is therefore clear that \( \rho \) is called macroscopic if \( N_{\text{eff}}(\rho) = O(N) \). For a scaling that is sub-linear (i.e. \( N_{\text{eff}}(\rho) = o(N) \)), the decision on whether or not \( \rho \) is called macroscopic is subjective and a straightforward answer does not exist. In this paper, we follow a conservative definition and use for all considered macroscopicity measures

**Definition 1.** A quantum state \( \rho \) is called macroscopic (due to a given measure) if its effective size \( N_{\text{eff}}(\rho) \) (due to this measure) is linear in the system size, that is, \( N_{\text{eff}}(\rho) = O(N) \).

As a counter-example, consider a product state of maximally entangled two-qubit states \(|\psi^-\rangle^{\otimes N/2}\), which is used to model Cooper pairs in superconductivity theory. The effective size is \( N_{\text{eff}} = 2 \) and the phenomenon of superconductivity is hence a microscopic quantum effect according with working definitions 3 and 1.

In the following paragraphs, we try to infer from the working definitions mathematical definitions of macroscopic quantum states and effective size.

### 3.2. Mathematical definition of macro-measure

We are going to propose the use of the quantum Fisher information as a measure of macroscopic quantum states. Since in this paper, we exclusively focus on the quantum Fisher information and are not concerned with the (classical) Fisher information, the term Fisher information always refers to quantum Fisher information.

Consider an initial quantum state \( \rho \) that is subject to a unitary time evolution generated by the time-independent Hamiltonian \( H \), \( \rho(t) = e^{-iHt/\hbar} \rho e^{iHt/\hbar} \). The set \( \{ \rho(t) : t \in \mathbb{R} \} \) is a parameterized curve through the space of density operators. If we use the Bures metric \[21\] to measure distances in this space, we define the quantum Fisher information implicitly as \(14, 22\)

\[
(d\mathbf{s})_{\text{Bures}} = \frac{1}{2\hbar} \sqrt{\mathcal{F}_{\rho}(\rho, H)} \, dt.
\]

In general, \( \mathcal{F}_{\rho}(\rho, H) \) depends on \( t \). In our case, \( H \) is time independent and so is \( \mathcal{F}_{\rho}(\rho, H) \) \[14\], for which reason we omit the index \( t \) in the following. Then, the explicit formula of \( \mathcal{F}(\rho, H) \) reads \[14, 23, 24\]

\[
\mathcal{F}(\rho, H) = 2 \sum_{i, j=1}^{2^N} \frac{(\pi_i - \pi_j)^2}{\pi_i + \pi_j} \langle i | H | j \rangle^2,
\]

where we used the spectral decomposition of \( \rho \)

\[
\rho = \sum_{i=1}^{2^N} \pi_i |i\rangle\langle i|.
\]

For pure states \( \rho = |\psi\rangle\langle\psi| \), the Fisher information reduces to the variance \( V_\psi(H) = \langle H^2 \rangle_\psi - \langle H \rangle_\psi^2 \) of \( H \):

\[
\mathcal{F}(\psi, H) = 4V_\psi(H).
\]

As discussed in the introduction, in physical systems of macroscopic size, we generically encounter Hamiltonians that consist of local terms. These describe interactions with external
fields such as a magnetic field or few-body interactions. It has been noted that entangled states can exhibit a much larger Fisher information than separable states. Let us focus on $H$ as a sum of one-particle terms $H = \sum_{i=1}^{N} H^{(i)}$, where $H^{(i)}$ acts non-trivially only on qubit $i$. We fix the operator norm of the single terms to a characteristic energy $\| H^{(i)} \| = e_0$. It is by now well known that separable states $\rho_{\text{sep}}$ obey for every local $H$ Fisher information that scales at most linearly with the system size [4] $\mathcal{F}(\rho_{\text{sep}}, H) \leq 4e_0^2 N$. On the other hand, there exist quantum states like the multipartite Greenberger–Horne–Zeilinger (GHZ) state

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$$

(6)
gives $\mathcal{F}(\text{GHZ}, e_0 \sum_i \sigma_z^{(i)}) = 4e_0^2 N^2$. (The state vectors $|0\rangle$ and $|1\rangle$ denote the eigenstates of the $\sigma_z$ Pauli operators.)

We therefore observe that for local Hamiltonians certain quantum states exhibit much larger Fisher information than any separable state. We use this insight for

**Definition 2(a).** Let $A$ be the set of linear operators $A$ that are sums of one-particle terms $A = \sum_{i=1}^{N} A^{(i)}$, where each term has a unit operator norm, $\| A^{(i)} \| = 1$. We call a quantum state $\rho$ macroscopic if there exists an $A \in A$ such that $\mathcal{F}(\rho, A) = O(N^2)$. The effective size is defined as

$$N_{\text{eff}}^F(\rho) = \max_{A \in A} \mathcal{F}(\rho, A)/(4N).$$

(7)

We denote by $\rho$ a macroscopic quantum state if $N_{\text{eff}}^F(\rho) = O(N)$. The superscript $F$ indicates the Fisher information and eases the comparison with other proposals in sections 3.4 and 4.

For all pure states $|\psi\rangle$, the possible range of the effective size is $1 \leq N_{\text{eff}}^F(\psi) \leq N$. The upper bound comes from the operator norm of $A \in A$, which equals $N$. The lower bound is a consequence of the fact that one can always find an $A \in A$ for which $\mathcal{F}(\psi, A) \geq N$. For general quantum states, one has $0 \leq N_{\text{eff}}^F(\rho) \leq N$. We now discuss why definition 2(a) is a consequence of working definitions 1, 2 and 3.

**Re working definition 1.** Suppose that one measures the time-evolving state $\rho(t)$ by projecting it onto the range of the initial state $\rho(0)$. We denote this projective measurement by $\Pi$. The expectation value $\langle \Pi \rangle_{\rho(t)}$ is the probability that, at time $t$, we find the quantum state in the subspace that is the range of $\rho(0)$, for which reason this quantity is often called survival probability. Trivially, one has $\langle \Pi \rangle_{\rho(0)} = 1$. In [25], the following statement has been proven.

**Proposition 1.** For the interval $0 \leq t \leq \sqrt{\mathcal{F}(\rho, H)}/\hbar \leq \pi$, we find that

$$\langle \Pi \rangle_{\rho(t)} \geq \cos^2 \frac{\sqrt{\mathcal{F}(\rho, H)} t}{2\hbar}.$$  

(8)

This bound gives an upper limit on the ‘speed’ with which the system evolves. Observe that inequality (8) allows certain quantum states, such as the GHZ state, to evolve much faster than any separable state. This means that the observation of a fast time evolution indicates the validity of quantum mechanics. This effect is a true macroscopic quantum phenomenon, because we need long-range quantum correlations within the state $\rho(t)$ to achieve Fisher information that is proportional to $N^2$ (compare also with [26]). This aspect is discussed later in more detail.
We stress that the observation of rapid oscillations apparently verifies macroscopic quantum effects only if we have full knowledge of the unitary dynamics. If the interaction strength of the Hamiltonian is not known, the plain data do not give any conclusion on the ‘macroscopicity’ of the initial quantum state. However, if in the experiment we are able to vary $N$, we can reveal the dependence of the measured frequency of the oscillation. A linear dependence on $N$ is evidence for a macroscopic quantum effect.

Re working definition 2. The Fisher information appears most prominently in the context of metrology, which we are going to review now. In this theory, an experimenter aims to estimate an unknown parameter $\omega$ that is encoded in a quantum state she or he has access to. The state is described by a density operator $\rho(\omega)$. We measure the system with a general measurement (a so-called positive operator-valued measurement) represented by the measurement operators $\{E_i\}$. The probabilities $p_i(\omega) = \text{Tr}[\rho(\omega)E_i]$ of the outcomes give rise to an estimate of $\omega$. The general difficulty of calculating the error of the estimate $\delta\omega$ is eased by the Cramér–Rao bound [13, 27]

$$\delta\omega \geq \frac{1}{\sqrt{nF(\omega)}},$$

where $n$ is the total number of repetitions of the experiment and $F(\omega)$ is the classical Fisher information [13]

$$F(\omega) = \sum_i p_i(\omega) \left( \frac{d}{d\omega}\log p_i(\omega) \right)^2.$$  

The higher the classical Fisher information, the better the lower bound on $\delta\omega$, which can be attained for unbiased estimates in the limit of large $n$. One can improve the Cramér–Rao bound by varying the measurement one uses. Again, we then end up with the quantum Fisher information, which within this approach can be defined as [14]

$$\mathcal{F}(\omega) = \max_{\{E_i\}} F(\omega).$$  

If the curve $\rho(\omega)$ is generated by an $\omega$-independent self-adjoint operator $A$, we end up with the same formula for the Fisher information $\mathcal{F}(\rho, A)$ from equation (3). The previous discussion on the implications for the measure of macroscopic quantum states is similar for quantum metrology. While separable states lead to a minimal error $\delta\omega \propto 1/\sqrt{N}$, ‘macroscopic’ quantum states as a resource give rise to a quadratic improvement $\delta\omega \propto 1/N$. From working definition 2, we again find that a quantum state is macroscopic if $\mathcal{F}(\rho, H) = O(N^2)$, see also definition 2(a).

Re working definition 3. The assignment $N^F_{\text{eff}}(\rho) = \max_{A \in A} \mathcal{F}(\rho, A)/(4N)$ was defined as the effective size of $\rho$. We now argue why this quantity allows us to distinguish between microscopic and macroscopic quantum effects in spin systems. We therefore divide the Hilbert space into groups of at most $k$ particles $\mathcal{H} = \mathbb{C}^{2^k} \otimes \cdots \otimes \mathbb{C}^{2^k}$, with $\sum_{i=1}^n k_i = N$ and $k_i \leq k$. A pure state $|\psi_k\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$ with $|\phi_j\rangle \in \mathbb{C}^{2^k}$ that is a tensor product with respect to this splitting is called $k$-producible. For local $H$, the variance is additive

$$\mathcal{V}_{\psi_k}(H) = \mathcal{V}_{\phi_1} \left( \sum_{i=1}^{k_1} H^{(i)} \right) + \cdots + \mathcal{V}_{\phi_n} \left( \sum_{i=N-k_n}^{N} H^{(i)} \right).$$  

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For every group $j$ of equation (12), the variance can be estimated by the maximal possible value $k_j^2 \leq k^2$. Then, one can easily see that the Fisher information of $|\psi_k\rangle$ is at most
\[
\mathcal{F}(\psi_k, \frac{H}{2e_0}) \leq k_1^2 + \cdots + k_n^2 \leq k (k_1 + \cdots + k_n) = kN.
\] (13)

Fisher information $\mathcal{F}(\psi_k, H) \propto kN$ for the given splitting $\{k_i\}$ is possible only if quantum mechanics is valid (compared to classical physics) on the scale of $k$ particles.

Very recently, a tighter and generalized estimate was shown for $k$-producible mixed states $\rho_k$, which are defined as incoherent mixtures of $k$-producible pure states with respect to possibly distinct splittings $\{k_i\}$. The authors of [15, 16] proved that
\[
\mathcal{F}(\rho_k, \frac{H}{2e_0}) \leq sk^2 + (N - sk)^2
\] (14)
with $s = \lfloor \frac{N}{k} \rfloor$, which coincides with the result for pure states if $k$ is a factor of $N$.

Therefore, if a quantum state $\rho$ has an effective size $N_{\text{eff}}^F \geq k$, we know that this state is potentially able to demonstrate the existence of quantum correlations of the range of at least $k$ qubits.

3.3. Discussion and basic examples

In this section, we discuss some implications and aspects of definition 2(a) and illustrate them with examples.

The first paradigmatic macroscopic quantum state we discuss is the GHZ state (6). As already mentioned, the maximal variance of a dimensionless local operator equals $N^2$, which means that the effective size (7) (with (5)) is $N_{\text{eff}}^F(\text{GHZ}) = N$. This is in accordance with all other proposals [5–8, 10–12].

Prominent quantum states that do not exhibit long-range quantum correlations are the W state [28] and the cluster states [29, 30]. Therefore, these states are not macroscopic with respect to the Fisher information. Observe, however, that the Fisher information is not an entanglement measure as it can increase under local operations and classical communication (LOCC). For example, from the two-dimensional cluster state with $N$ qubits, one can obtain a GHZ state with $O(N)$ qubits by means of LOCC [29]. Because the Fisher information detects a certain kind of correlation, the Fisher information can increase under LOCC. We conclude that if we allow LOCC to manipulate quantum states before we calculate the Fisher information, we enlarge the set of macroscopic quantum states, for example, by the two-dimensional cluster state, which is a valuable resource, as it can be used for universal measurement-based quantum computation [19, 20]. In this paper, we calculate the Fisher information without previous LOCC manipulation, which is in line with other discussed proposals [5, 7–12]. Note, however, that in [6], quantum states were called macroscopic if one can distill the GHZ state with $O(N)$ particles from the resource state. This clearly uses LOCC, and in this sense, also the cluster states are macroscopic.

Several works on macroscopic quantum states [8, 10–12] have discussed a specific quantum state, which was introduced in [6]. It is an example of a superposition, where the two constituents $|\psi_0\rangle$ and $|\psi_1\rangle$ are not orthogonal but exhibit an overlap that vanishes in the limit of large $N$. 

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Example 2 (Generalized GHZ state). This state, which we call ‘generalized GHZ’ state in the following, is defined as
\[
|\phi_\epsilon\rangle = \frac{1}{\sqrt{2(1 + \cos^N \epsilon)}}(|0\rangle^{\otimes N} + |\epsilon\rangle^{\otimes N})
\]  
with $|\epsilon\rangle = \cos \epsilon |0\rangle + \sin \epsilon |1\rangle$, $\epsilon \in [0, \pi/2]$. The authors of [6, 8, 10–12] conclude unanimously that the effective size $N_{\text{eff}}$ (in the sense of the respective contributions) of the quantum state (15) for $\epsilon \ll 1$ equals $N_{\text{eff}} \approx \epsilon^2 N$, which is the effective size of the generalized GHZ state that compared to the standard GHZ state is reduced by a factor $\epsilon^2$ (see also table 1). To quantify the effective size via the Fisher information, we have to maximize over all local observables $A$ (see also [31]). Due to the symmetry of the quantum state, we optimize the variance $\mathcal{V}_{\phi_\epsilon}(A)$ with the ansatz
\[
A = \sum_{i=1}^{N} \cos \alpha \sigma_x^{(i)} + \sin \alpha \sigma_z^{(i)}.
\]  
A simple maximization over the angle $\alpha$ leads to $\alpha = \epsilon/2$ and hence $\mathcal{V}_{\phi_\epsilon}(A_{\text{max}}) = N^2 \sin^2 \epsilon \frac{1}{1 - \cos^N \epsilon} + N(1 - \sin^2 \epsilon \frac{1}{1 - \cos^N \epsilon})$. With $\cos^N N \approx 0$ for large $N$, one has $N_{\text{eff}}^F(\phi_\epsilon) \approx N \sin^2 \epsilon + \cos^2 \epsilon \approx \epsilon^2 N$. The last approximation is valid for small $\epsilon$ and is therefore in accordance with [6, 8, 10–12].

The $|\phi_\epsilon\rangle$ state is a possible generalization of the standard GHZ state. Another possible extension is to superpose slightly entangled states instead of product states. A state of this kind was discussed in [32, 33]. Consider the one-dimensional cluster states with periodic boundary conditions
\[
|C^{+}\rangle = \prod_{i=1}^{N} C^{(i,i+1)} |+\rangle^{\otimes N} \quad \text{and} \quad |C^{-}\rangle = \prod_{i=1}^{N} C^{(i,i+1)} |\rangle^{\otimes N},
\]  

Table 1. Overview of the examples in this paper for the discussed measures (from equations (22), (28), (33) and (20), respectively; note that $N_{\text{eff}}^B \approx N_{\text{eff}}^F$ and $N_{\text{eff}}^S = N_{\text{eff}}^F$). The examples are divided into three blocks. The first block is examples where all measures qualify the quantum states as macroscopic (with different absolute values). The second block of examples are microscopic due to Björk and Mana and the relative Fisher information, but macroscopic (or indefinite, see sections 4.3 and 4.4) for the other measures. The last example (see also [38]) is a macroscopic quantum state but not a macroscopic superposition. These three blocks correspond to the structure illustrated in figure 1.

| Example                                    | $N_{\text{eff}}^F$ | $N_{\text{eff}}^K$ | $N_{\text{eff}}^M$ | $N_{\text{eff}}^F$ |
|--------------------------------------------|---------------------|---------------------|---------------------|---------------------|
| GHZ (example 1)                            | $N$                 | $N$                 | $N$                 | $N$                 |
| General GHZ (example 2)                    | $N/3$               | $N/3$               | $N/3$               | $N/3$               |
| Cluster–GHZ (example 3)                    | $O(1)$              | ?                   | ?                   | $O(N)$              |
| Quantum/classical (example 4)              | $O(1)$              | $O(N)$              | $O(N)$              | $O(N)$              |
| Cloned superposition (example 6)           | $O(1)$              | $O(1)$              | $O(1)$              | $O(N)$              |
where $C^{(i,i+1)} = |0⟩⟨0|^i \sigma^z - |1⟩⟨1|^i \sigma^z$ is a phase gate, $|±⟩$ are the ±1 eigenstates of the $σ_z$ Pauli operator and $N + 1 \equiv 1$.

**Example 3 (Cluster–GHZ state).** The superposition

$$|CG⟩ = \frac{1}{\sqrt{2}}(|Cl^+⟩ + |Cl^-⟩)$$

was called cluster–GHZ state in [33]. One can see easily that the cluster states (17) do not have long-range quantum correlations and neither does the cluster–GHZ state. According to definition 2(a), one has $N_{eff}^{F} = 1$. On the other hand, we have long-range quantum correlations among local groups of particles. If we consider two groups of three particles each, we see with $S_i = σ_z^{(i−1)}σ_z^{(i)}σ_z^{(i−1)}$ that for all $(i, j)$

$$⟨S_iS_j⟩_{CG} − ⟨S_i⟩_{CG}⟨S_j⟩_{CG} = 1,$$

since $S_i|Cl^±⟩ = ±|Cl^±⟩$. Hence, one has $V_{CG}(∑_{i=2,5,...} S_i) = N^2/3$ and we would intuitively assign $N_{eff}^{F}(CG) = N/3$.

Also other states such as logical GHZ states, where the physical particles in equation (6) are replaced by blocks of qubits [33], may not have long-range quantum correlations between single particles, but between groups of size $O(1)$. Since it seems natural to consider these states also as macroscopic, we have to extend definition 2(a)

**Definition 2(b).** Let the Hilbert space $C^2 \otimes N$ be divided into $n = O(N)$ distinct groups of qubits, each group is of size $O(1)$. Let $A$ be the set of all local operators $A = ∑_{i=1}^n A^{(i)}$ such that every operator $A^{(i)}$ acts non-trivially on the group $i$ and $∥A^{(i)}∥ = 1$. A quantum state $ρ$ is called macroscopic if there exists a grouping such that

$$N_{eff}^{F}(ρ) = \max_{A \in A} F(ρ, A)/(4n) = O(N).$$

In the following, the term ‘macroscopic quantum state’ refers to definitions 2(a) and 2(b), together denoted by definition 2. Although it should be clear from the definition, this measure of macro-states is valid for pure and mixed states. As we see later in section 4, for other measures this is not the case, especially if they are defined for superposition states like in equation (1).

For a fixed grouping with $n$ groups in total, the maximal possible effective size is $N_{eff}^{F} = n$. It is therefore clear that in general we try to maximize $n$.

### 3.4. Relative Fisher information for macroscopic superpositions

The Fisher information proved to be a good candidate for measuring the effective size of macroscopic quantum states. It is well defined for general mixed states and does not require specific superpositions like in equation (1), as other measures [8, 10, 11] do. The famous Schrödinger cat gedanken experiment, however, is an example where two classical states are superposed. The total state is the archetypal macroscopic superposition. In order to have a characterization of the ‘catness’ of $|ψ⟩ = \frac{1}{\sqrt{2}} (|ψ_0⟩ + |ψ_1⟩)$ at hand, we introduce a measure of superpositions that is based on the Fisher information.
**Definition 3.** Let $|\psi\rangle = \frac{1}{\sqrt{2}} (|\psi_0\rangle + |\psi_1\rangle) \in \mathbb{C}^{2 \otimes N}$ be a superposition of two quantum states $|\psi_0\rangle$ and $|\psi_1\rangle$. The quantity

$$N_{\text{eff}}^r(\psi) = \frac{N_{\text{eff}}(\psi)}{\frac{1}{2} N_{\text{eff}}(\psi_0) + \frac{1}{2} N_{\text{eff}}(\psi_1)}$$

is called relative Fisher information. If $N_{\text{eff}}^r = O(N)$, then $|\psi\rangle$ is called a macroscopic superposition.

The measure is such that a superposition state is macroscopic only if the Fisher information of the total state is high and at the same time the constituents $|\psi_0\rangle$ and $|\psi_1\rangle$ have small Fisher information, which means that they are considered to be semi-classical, that is, microscopic.

There are two important differences between $N_{\text{eff}}^r$ and $N_{\text{eff}}$:

1. There are states $|\psi\rangle$ that have a small $N_{\text{eff}}^r(\psi)$ and are nevertheless able to show macroscopic quantum effects demanded in working definition 1, e.g. if $N_{\text{eff}}(\psi) = O(N)$ and $N_{\text{eff}}(\psi_0) = O(N)$ (see example 6 given).

2. In contrast to the Fisher information, the extension of definition 3 to mixed states is not trivially possible. However, if we consider an initially pure state $|\psi\rangle = \frac{1}{\sqrt{2}} (|\psi_0\rangle + |\psi_1\rangle)$ that is subject to noise represented by the map $\mathcal{E}$, the relative Fisher information for a noisy state $\rho = \mathcal{E}(\psi)$ may be defined as

$$N_{\text{eff}}^r(\rho) = \frac{N_{\text{eff}}^r[\mathcal{E}(\psi)]}{\frac{1}{2} N_{\text{eff}}^r[\mathcal{E}(\psi_0)] + \frac{1}{2} N_{\text{eff}}^r[\mathcal{E}(\psi_1)].}$$

### 4. Comparison to existing proposals

In general, the genesis of proposals for macroscopic quantum states is the following. First, one tries to find a property of quantum states that seems to be characteristic of genuine macroscopic states. (For the Fisher information, this step is manifested in working definitions 1–3.) Next, this property is defined in a mathematically rigorous way. Finally, the definition is applied to several examples to check for consistency. In this section, we review and discuss the first two points of several measures [7–11] that are suitable for multipartite qubit systems. We find that there exist two main classes of measures. The first class, which incorporates the Fisher information and [7, 9], tries to quantify general quantum states of macroscopic systems. The second class—consisting of the relative Fisher information and the measures [8, 10, 11]—focuses on superpositions of two orthogonal quantum states as in equation (1). To distinguish these two classes, we call macro-states from the first class ‘macroscopic quantum states’ and those from the second class ‘macroscopic superpositions’. The quantum states that are discussed as examples in the course of this section are summarized in table 1.

Before we start with the review of [7–11], we remark that there are two other proposals [5, 6] for macroscopic superpositions that were important for the discussion of this topic but are nevertheless not general enough to be applicable for general superpositions.

The first one was introduced by Leggett [5]. He demanded that two superposed states $|\psi_0\rangle$ and $|\psi_1\rangle$ should be macroscopically distinct. To translate this into mathematical terms, a superposition state $|\psi\rangle = \frac{1}{\sqrt{2}} (|\psi_0\rangle + |\psi_1\rangle)$ is called macroscopic—according to Leggett—if we need at least an $O(N)$-correlation operator in order to be able to distinguish between $|\psi\rangle$
and the incoherent mixture $|\psi_0\rangle\langle\psi_0| + |\psi_1\rangle\langle\psi_1|$. For the GHZ state (see example 1), we need the correlation operator $\sigma^{\otimes N}$ for this task. Unfortunately, the requirement is too insensitive, as can be seen for the generalized GHZ state (example 2). In this case, any $\epsilon > 0$ leads to a macroscopic quantum state. The idea of ‘macroscopic distinctness’ was nevertheless used in [10, 11] and led to more elaborate measures.

As already mentioned, the generalized GHZ state was introduced in [6] to introduce the notion of an effective size. There, it is shown that the trace norm $^3$ of the coherence term $|0\rangle\langle\epsilon|^{\otimes N}$ decays under local phase noise with a rate

$$\| |0\rangle\langle\epsilon|^{\otimes N} \|_1 \approx e^{-\gamma \epsilon^2 N t}$$

in the case of small $\epsilon$ and $t$. It was argued that this indicates a reduced effective size compared to the GHZ state, whose coherences decay with $e^{-\gamma N t}$. However, this insight cannot be generalized; that is, one cannot conclude from the damping rate of the coherences on the effective size. A counter-example is the so-called ‘logical’ GHZ state. Suppose that we replace every physical particle by an $O(1)$ group of qubits. On this group, one defines two orthogonal states $|0_L\rangle$ and $|1_L\rangle$. The logical GHZ state is then defined as

$$|\text{GHZ}_L\rangle = \frac{1}{\sqrt{2}} \left( |0_L\rangle^{\otimes N} + |1_L\rangle^{\otimes N} \right).$$

The effective size of those logical GHZ states is $N$ due to the Fisher information, but also due to other measures [7–11] that are discussed in this paper. However, there exist examples such as the so-called ‘concatenated GHZ state’ [34] that exhibit much more stable coherences than the standard GHZ state. Therefore, the mere decay rate of the coherences is not a suitable measure of macroscopic quantum states.

In [6], a second criterion for macroscopic superpositions was suggested. From the generalized GHZ state, one can distil a GHZ state of approximately $\epsilon^2 N$ qubits. As already discussed in section 3.3, this approach uses LOCC operations and is therefore in contrast to other proposals discussed in this paper.

Finally, the measure proposed in [12] is defined for continuous variable systems. A direct application to qubit systems is not given and we therefore focus on other proposals in the following.

4.1. Indices $p$ and $q$

In classical statistical mechanics, we normally deal with probability distributions in phase space (here we assume $N$ particles) whose uncertainties with respect to a given observable are small compared to its spectral radius. More quantitatively, the expectation values of extensive variable (like the magnetization $M$ of a ferromagnet) scales linearly with the system size $\mu \equiv \langle M \rangle_{cl} = O(N)$, while for the standard deviation $(\Delta M)_{cl} = \sqrt{\langle (M - \mu)^2 \rangle_{cl}}$ we have $(\Delta M)_{cl} = O(\sqrt{N})$. This is because, for typical classical probability distributions in statistical mechanics, the particles are uncorrelated.

In quantum mechanics, there exist pure states like the GHZ state (6) that exhibits an anomalously large standard deviation. For $M = \sum_{i=1}^{N} \sigma_i^z$ we find that $(\Delta M)_{\text{GHZ}} = N$. Based on this observation, Shimizu and Miyadera [7] define a measure of macroscopic quantum systems, which they call index $p$: we consider a pure state $|\psi\rangle$. If there exists a local operator $A \in A$

$^3$ The definition of the trace norm for a linear operator $M$ reads as $\|M\|_1 = \text{Tr}\sqrt{M^*M}$.
(see definition 2(a)) such that $V_\psi(A) = O(N^p)$ with $p = 2$, then $|\psi\rangle$ is called macroscopic. We comment on several aspects of this definition.

1. Although the index $p$ is motivated differently from the Fisher information, for pure states it is mathematically identical to the first part of definition 2(a), since for pure states, the Fisher information is proportional to the variance, see equation (5).

2. The index $p$ does a priori not give a quantification in the form of an effective size. The discussion of the generalized GHZ state in section 3.3 nevertheless suggests also for the index $p$ an assignment of $N_{\text{eff}}$ as for the Fisher information, see equation (7) and the discussions in section 3.3. We define

$$N_{\text{eff}}^S(\psi) = \max_{A \in A} V_\psi(A)/N.$$  \hspace{1cm} (25)

With this definition, one has $N_{\text{eff}}^S = N_{\text{eff}}^F$.

3. Regarding the Fisher information, the index $p$ in its original definition is insensitive to long-range quantum correlations between blocks rather than between single particles. As already mentioned in [35], it therefore makes sense to extend the index $p$ to sums of local operators that act non-trivially on groups of $O(1)$ qubits, as made precise in definition 2(b).

4. The index $p$ is not appropriate for mixed states and makes intuitively wrong statements for those. For instance, the mixture $\rho = \frac{1}{2} |0\rangle\langle 0|^\otimes N + \frac{1}{2} |1\rangle\langle 1|^\otimes N$ gives the same standard deviation as the GHZ state for the same observable $M$. To have a measure for mixed states at hand, in [9] the so-called index $q$ for general states was proposed. For pure states, the indices $p$ and $q$ qualify the same set of quantum states to be macroscopic. We give here a slightly modified definition, which was published later [36]: consider a quantum state $\rho$ and a local observable $A \in A$. Calculate the trace norm of $C(\rho) = [A, [A, \rho]]$, $\|C(\rho)\|_1 = O(N^q)$. If there exist any $A \in A$ such that $q = 2$, then $\rho$ is called a macroscopic quantum state.

Since the scaling of the Fisher information and the index $q$ coincide for pure states, it is interesting whether they also coincide for mixed states. This question is open at the moment. By means of numerical studies we find that in general the difference of Fisher information for a given $\rho$ and the trace norm of $C(\rho)$ is indefinite; that is, the Fisher information cannot be bounded by the index $q$ and vice versa.

4.2. Relative improvement of interference experiments

The work of Björk and Mana [8] considers superpositions like the quantum state in equation (1). The authors try to give an operational interpretation of their measure. They consider a concrete experiment and a ‘natural’ Hamiltonian $H$ associated with it. The main idea is that the constituting states $|\psi_0\rangle$ and $|\psi_1\rangle$ of equation (1) should be of semi-classical nature, whereas the composed state (1) shows significant advantages for interferometric applications, that is, increased phase sensitivity. In mathematical terms, the state evolves under the time-independent Hamiltonian $H$, $|\psi(t)\rangle = \exp(-iHt/\hbar) |\psi\rangle$. Let us consider the minimal time $\theta(\psi)$ for which the state $|\psi(t)\rangle$ evolves to an orthogonal state. We denote by $\theta$ the orthogonalization time. Björk and Mana demand that a macroscopic superposition $|\psi\rangle$ should exhibit a $\theta(\psi)$ that is
much smaller than the orthogonalization time for the constituting states $|\psi_0\rangle$ and $|\psi_1\rangle$. They therefore define\(^4\)

$$N_{\text{eff}}^B(\psi) = \left( \frac{\theta(\psi_0) + \theta(\psi_1)}{\theta(\psi)} \right)^2$$

(26)

as a measure of the effective size of a superposition (1). It has been shown in [8] that this expression can be approximated to ease the calculation of equation (26), which proves difficult in general:

$$N_{\text{eff}}^B(\psi) \approx \left( \frac{|\langle H \rangle_{\psi_0} - \langle H \rangle_{\psi_1}|}{\sqrt{V_{\psi_0}(H)} + \sqrt{V_{\psi_1}(H)}} \right)^2.$$  

(27)

The expression (26) is well defined only if $|\psi_0\rangle$ and $|\psi_1\rangle$ exhibit finite orthogonalization times. For equation (27), $|\psi_0\rangle$ and $|\psi_1\rangle$ should not be eigenstates of $H$. To circumvent this problem, we suggest to maximize the denominator and the numerator of equation (27) separately over all ‘realistic’ Hamiltonians. As already discussed in section 2, this reduces the set of possible $H$ to sums of local terms, that is, the set $\mathcal{A}$ from definition 2(b). The maximal scaling of $N_{\text{eff}}^B$ is with $O(N)$. Therefore, we define the superposition $|\psi\rangle$ to be macroscopic if $N_{\text{eff}}^B = O(N)$.

### 4.3. Local distinguishability

We now review the work of Korsbakken et al [10]. Their notion of the effective size of a macroscopic superposition is defined as follows: given the superposition of equation (1) with $\langle \psi_0 | \psi_1 \rangle = o(1)$ (which means that the overlap vanishes for large $N$), we divide the particles into a maximal number $n$ of distinct groups such that we can distinguish $|\psi_0\rangle$ from $|\psi_1\rangle$ with ‘high’ probability $P = 1 - \delta$ by measuring only one of those groups. Then, the effective size is defined as the number of such groups

$$N_{\text{eff}}^K(\psi) = n.$$  

(28)

A quantum state (1) is called macroscopic if $n = O(N)$.

In other words, we demand that the state (1) is locally distinguishable. We pick a group $i$ of qubits such that we can distinguish between $|\psi_0\rangle$ and $|\psi_1\rangle$. Then, the ranges of the reduced density operators\(^5\) $\rho_k = \text{Tr}_{N \setminus i} |\psi_k\rangle \langle \psi_k|$ ($k = 0, 1$) should be distinct, that is, $1/2 \parallel \rho_0^i - \rho_1^i \parallel_1 \geq 1 - 2\delta$.

This notion of macroscopic superposition is the only one in this paper that depends on a parameter: $\delta$ is the maximal failure probability measuring one block. The specific choice of $\delta$ influences in general the exact value of $N_{\text{eff}}^K$. For instance, the generalized GHZ state of example 2 gives $N_{\text{eff}}^K(\phi_\epsilon) = N \log(\cos^2(\epsilon)) \approx N \epsilon^2 / (- \log(\delta))$ for small $\epsilon$ and $\delta$. However, in this example, the choice of $\delta$ does not affect much the effective size, since it gives only a logarithmic factor. Even if we demand $\delta = O(N^{-1})$, the effective size would be $N_{\text{eff}}^K(\phi_\epsilon) \approx N \epsilon^2 / \log(N)$. On the other hand, consider

---

\(^4\) Note that the original quantity that was considered in [8] was defined as $\sqrt{N_{\text{eff}}^B}$. However, we will refer to (27) as the effective size due to Björk and Mana for an easier comparison to the other proposals.

\(^5\) Let $\rho$ be an $N$ qubit state and $X$ be a subset of some qubits. The expression $\text{Tr}_X \rho$ denotes the tracing over the qubits of $X$; $\text{Tr}_{N \setminus X} \rho$ means that we trace over all qubits but those of the set $X$. 

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Example 4 (PS + domain wall). We equally superpose the product state (PS) $|\psi_0\rangle = |0\rangle^\otimes N$ and

$$|\psi_1\rangle = \frac{1}{\sqrt{N+1}} \sum_{k=0}^{N} |1\rangle^\otimes k \otimes |0\rangle^\otimes N-k .$$

The state $|\psi_1\rangle$ is called domain wall state. If we want to distinguish between $|\psi_0\rangle$ and $|\psi_1\rangle$ and measure a qubit $k$ by means of $\sigma_z$, we do not improve the success probability $P$ if we enlarge the group by another qubit $j > k$. The best strategy to group the qubits is to calculate $P$ for the first qubit, which gives the highest $P$, and go on with the second, for which $P$ is reduced, and so on. For an arbitrary qubit $k$, one has $P = 1 - \frac{k}{2(N+1)}$. We stop if for a certain qubit $j + 1$, $P$ drops below the threshold $1 - \delta$. So $j$ is the number of groups we find such that measuring any group lets us distinguish between $|\psi_0\rangle$ and $|\psi_1\rangle$ with a probability of at least $P = 1 - \delta$. The effective size is therefore $N_{\text{eff}}^k = 2\delta(N+1)$.

While it is obvious that $\delta$ should be small, that is, $\delta \ll 1$, the actual scaling one should demand is not entirely clear. In example 4, for any choice $\delta = O(1)$ one would call $|\psi_0\rangle + |\psi_1\rangle$ a macroscopic state, whereas $\delta = 1/N$ leads to a microscopic qualification.

We discuss this ambiguity of this measure by speculating about the idea behind this definition. We start with the original gedankenexperiment of Schrödinger’s cat [1] and Leggett’s demand to call a superposition macroscopic if its constituents are macroscopically distinct [5]. On a macroscopic level, the distinction between two states $|\psi_0\rangle$ and $|\psi_1\rangle$ may be limited to sums of local measurements. We demand that the probability $P(\text{error})$ of failing in distinguishing $|\psi_0\rangle$ from $|\psi_1\rangle$ may go to zero in the limit of large $N$; for instance, $P(\text{error}) = 1/O(N)$. In [37], it was shown that $P(\text{error}) \leq \frac{1}{\Delta^2}$ with

$$\Delta = \frac{|\langle A \rangle_{\psi_0} - \langle A \rangle_{\psi_1}|}{\sqrt{\mathcal{V}_{\psi_0}(A)} + \sqrt{\mathcal{V}_{\psi_1}(A)}} .$$

Observe that the quantity $\Delta^2$ is formally very similar to the effective size of Björk and Mana, equation (27), except that there we considered a time evolution under a Hamiltonian $H$ and here, in equation (30), the operator $A$ refers to an observable. In addition, for equation (27) we argued that we should maximize the numerator and the denominator separately in order to avoid divergences of $N_{\text{eff}}^k$. Here, we just need $\Delta^2 \geq O(N)$, in order to meet the requirement of good distinguishability of $|\psi_0\rangle$ and $|\psi_1\rangle$ with respect to $A$.

Now, the local distinctness between $|\psi_0\rangle$ and $|\psi_1\rangle$ as demanded by Korsbakken et al leads for all choices of $\delta < 1/2$ to $|\langle A \rangle_{\psi_0} - \langle A \rangle_{\psi_1}| = O(N)$. To have $\Delta^2 \geq O(N)$, we need $\mathcal{V}_{\psi_i}(A) \leq O(N)$, $i = 0, 1$. This can be achieved in two different ways. Either we have no long-range quantum correlations between the local groups we are measuring on; that is, for almost all pairs of groups $(k, j)$ we have $\langle A^{(k)} A^{(j)} \rangle_{\psi_i} = \langle A^{(k)} \rangle_{\psi_i} \langle A^{(j)} \rangle_{\psi_i} = o(1)$. Then a finite $\delta = O(1)$ is enough to guarantee $P(\text{error}) = 1/O(N)$. Or there exist $O(1)$ for $O(N^2)$ pairs of groups. Then the variances $\mathcal{V}_{\psi_i}(A)$ become large and we need $\delta = 1/O(N)$.

This insight is now applied to example 4. Since $|\psi_1\rangle$ exhibits a large variance under the optimal observable, $\mathcal{V}_{\psi_1}(A) \approx N^2/12$, we would need $\delta = 1/O(N)$ and therefore, with this reasoning, we would conclude that this superposition is microscopic.

4.4. Orthogonality with respect to local rotations

In a similar spirit as Korsbakken et al, the authors Marquard et al [11] try to catch the idea of ‘macroscopic distinctness’. They agree with preliminary works that the mere particle
number of a quantum state is not enough to call the state macroscopic. Two orthogonal states of macroscopic size could differ only in microscopic detail and their superposition should hence not be counted to the set of macroscopic superpositions. The actual ansatz they follow is to count the number of one-particle operators one has to apply to \(|\psi_0\rangle\) on average in order to reach a unity overlap with \(|\psi_1\rangle\). The mathematical definition of the effective size is described by these steps:

1. Start with the one-dimensional subspace \(\mathcal{H}_0\), spanned by \(|\psi_0\rangle\). Then all subspaces \(\mathcal{H}_i\) with \(i > 0\) are generated iteratively.
2. Take \(\mathcal{H}_{i-1}\) and apply all possible one-particle operations on this subspace. This results in a set whose linear span is called \(\mathcal{H}_i\).
3. From this, define \(\mathcal{H}_i = \mathcal{H}_i \setminus \bigoplus_{k=0}^{i-1} \mathcal{H}_k\). We assume that there is a finite number of subspaces \(\mathcal{H}_i\) such that the whole Hilbert space \(\mathcal{H}\) is covered. To every \(\mathcal{H}_i\) we define an orthogonal projection \(P_i\). One has \(\sum_i P_i = 1\).
4. Decompose the second state \(|\psi_1\rangle\) with respect to this division, that is,
   \[
   |\psi_1\rangle = \sum_i v_i |\phi_i\rangle
   \]
   with \(P_i |\psi_1\rangle = v_i |\phi_i\rangle\) such that \(|\phi_i\rangle\) has unity norm.
5. Define the effective size as the expectation value of the operator
   \[
   N = \sum_k k |\phi_k\rangle \langle \phi_k|<
   
   under the second state, so one has
   \[
   N^M_{\text{eff}}(\psi) = \langle N \rangle_{\psi_1}.\tag{33}
   \]
   The state \(|\psi\rangle = \frac{1}{\sqrt{2}} (|\psi_0\rangle + |\psi_1\rangle)\) is called macroscopic if \(N^M_{\text{eff}}(\psi) = O(N)\).

As already noted by the authors, there is ambiguity in definition (33) of macroscopic states. The problem is that there exist superpositions \(|\psi_0\rangle + |\psi_1\rangle\), where the decomposition (31) of \(|\psi_1\rangle\) with respect to the Hilbert space structure based on \(|\psi_0\rangle\) gives a different \(N^M_{\text{eff}}\) than if we calculate the ‘local distance’ of \(|\psi_0\rangle\) with respect to \(|\psi_1\rangle\). In spin systems, this can always happen if the entanglement structures of \(|\psi_0\rangle\) and \(|\psi_1\rangle\) differ. This is because with local projections, which can be written as a sum of identity and a Pauli operator, a highly entangled state can be quickly transformed into a product state. We illustrate this with the state of example 4. To reach \(|\psi_1\rangle\) with one-particle operations from \(|\psi_0\rangle\), we need on average \(\langle N \rangle_{\psi_1} = \frac{N+1}{2}\) operations, since \(|\psi_1\rangle = 1/\sqrt{N+1} \sum_{i=0}^{N} |\phi_i\rangle\). In contrast, we find that \(|\psi_0\rangle = \frac{1}{2} \sqrt{N+1} |0\rangle |0\rangle \otimes \mathbb{1}^{\otimes N-1} |\psi_1\rangle\), which is \(\langle N \rangle_{\psi_0} = 1\). In cases like example 4, the measure is not defined. Whenever the global entanglement structures of \(|\psi_0\rangle\) and \(|\psi_1\rangle\) are identical, we find at least the same order of \(\langle N \rangle_{\psi_0}\) and \(\langle N \rangle_{\psi_1}\). For the comparison in section 5, we assume therefore that there exist local unitary operations \(U_i\) acting on \(O(1)\) qubits such that \(|\psi_1\rangle = U_1 \otimes \cdots \otimes U_n |\psi_0\rangle\).

5. The relations among the measures

In this section, we study the relations among macroscopic quantum states due to the respective measures. We find that for qubit systems we can establish a hierarchy among these measures. An
important consequence is that measures [8, 10, 11] and the relative Fisher information, which are defined for superpositions of two orthogonal states (see equation (1)), detect a strict subclass of general macroscopic quantum states as detected by the index \( p \) [7] and the Fisher information (see section 3).

With the Fisher information and the relative Fisher information, we introduced two proposals that are well defined for all quantum states and superposition states, respectively. In contrast, we encountered some ambiguities for the measures of Björk and Mana, Korsbakken et al and Marquardt et al. Therefore, we have to make some assumptions. As suggested in section 4.2, we maximize the numerator and the denominator of equation (27) over all \( A \in \mathcal{A} \) separately. For the local distinguishability in section 4.3, we demand a constant \( \delta = O(1) \) for the success probability and only short-range quantum correlations for the quantum states \( |\psi_0\rangle \) and \( |\psi_1\rangle \) with respect to the local observable \( A \). To circumvent problems of section 4.4, we assume that there exist \( n = O(N) \) distinct groups such that \( |\psi_1\rangle = U_1 \otimes \cdots \otimes U_n |\psi_0\rangle \).

We now summarize these observations. Proposition 2 is illustrated in figure 1. The results are proven at the end of the section.

**Proposition 2.** Let \( |\psi\rangle \) be an \( N \) qubit quantum state. For all measures that rely on the superposition structure of \( |\psi\rangle \), we have \( |\psi\rangle = 1/\sqrt{2}(|\psi_0\rangle + |\psi_1\rangle) \) with \( \langle \psi_0|\psi_1\rangle = o(1) \) (i.e. the states become orthogonal in the limit of large \( N \)). For all discussed measures, a quantum state is considered to be macroscopic if \( N_{\text{eff}}(\psi) = O(N) \). Under the assumptions formulated in the beginning of this section, then one has

\[
\begin{align*}
(i) & \quad N_{\text{eff}}^B(\psi) = O(N) \leftrightarrow N_{\text{eff}}^{RF}(\psi) = O(N) \ (\text{see equations (21) and (27)}), \\
(ii) & \quad N_{\text{eff}}^B(\psi) = O(N) \Rightarrow N_{\text{eff}}^{K}(\psi) = O(N) \ (\text{see equations (27) and (28)}), \\
(iii) & \quad N_{\text{eff}}^{K}(\psi) = O(N) \leftrightarrow N_{\text{eff}}^{M}(\psi) = O(N) \ (\text{see equations (28) and (33)}) \ \text{and} \\
(iv) & \quad N_{\text{eff}}^{K}(\psi) = O(N) \Rightarrow N_{\text{eff}}^{F}(\psi) = O(N) \ (\text{see equations (28) and (20)}).
\end{align*}
\]

We therefore identify three different classes of macroscopic quantum states.
1. **Björk and Mana/relative Fisher information.** Quantum states that are superpositions of two ‘classical’ (i.e. microscopic) states. This is the most restrictive class of macro-states and is closest to the original Schrödinger cat.

2. **Korsbakken et al/Marquardt et al.** Macroscopic superposition of locally distinct quantum states. Apart from the relation between the two states $|\psi_0\rangle$ and $|\psi_1\rangle$ there are no restrictions on them. Hence, these states can show true quantum effects with respect to other criteria (see example 5 given later).

3. **Index p/Fisher information. A priori** no restrictions on $|\psi\rangle$. Quantum states of this category may not have anything in common with the original Schrödinger cat gedanken experiment, except that they can show counter-intuitive, macroscopic quantum effects.

The idea of the relative Fisher information introduced with definition 3 is very similar to the proposal of Björk and Mana. And indeed, the connection between $N_{\text{eff}}^B$ and $N_{\text{eff}}^F$ can be made more rigorous with part 2 of proposition 2. This means that $|\psi\rangle$ is macroscopic due to Björk and Mana if and only if it is macroscopic due to the relative Fisher information. These two definitions therefore detect the same set of macroscopic quantum states.

Next, it is claimed in part 2 that whenever a superposition is macroscopic due to Björk and Mana (and the relative Fisher information, see part 2), it is also macroscopic due to Korsbakken et al and Marquardt et al, see part 2. The difference is that the first group demands that $|\psi_0\rangle$ and $|\psi_1\rangle$ are classical with respect to every possible local measurement $A$. Here, by classical is meant that the standard deviation of $A$ for $|\psi_0\rangle$ and $|\psi_1\rangle$ is small (i.e. $O(\sqrt{N})$) compared to the spectral radius $\sigma$ of $A$. Therefore, in the limit of large $N$, the ratio $\sqrt{\bar{V}_\psi(A)/\sigma(A)} = 1/O(\sqrt{N})$ vanishes.

On the other hand, the measures of Korsbakken et al and Marquardt et al are not so restrictive. There are local measurements for which the total state $|\psi_0\rangle + |\psi_1\rangle$ behaves macroscopically quantum but the single constituents do not. In contrast, there may be local observables for which also $|\psi_0\rangle$ and $|\psi_1\rangle$ show macroscopic behaviour, as in

**Example 5 (Quantum/classical).** We divide the $N$ qubits into distinct groups of two qubits each. We relabel the basis states of one two-qubit block: $|\alpha_0\rangle = |00\rangle$, $|\alpha_1\rangle = |11\rangle$, $|\beta_0\rangle = |10\rangle$ and $|\beta_1\rangle = |01\rangle$. Consider two normalized quantum states

$$
|\psi_0\rangle = \sum_{i_1,...,i_{N/2}=0}^1 c_{i_1,...,i_{N/2}} |\alpha_{i_1}\rangle \otimes \cdots \otimes |\alpha_{i_{N/2}}\rangle,
$$

$$
|\psi_1\rangle = \sum_{i_1,...,i_{N/2}=0}^1 d_{i_1,...,i_{N/2}} |\beta_{i_1}\rangle \otimes \cdots \otimes |\beta_{i_{N/2}}\rangle.
$$

(34)

The superposition $|\psi_0\rangle + |\psi_1\rangle$ is for all choices of coefficients $c_{i_1,...,i_{N/2}} \in \mathbb{C}$ and $d_{i_1,...,i_{N/2}} \in \mathbb{C}$ macroscopic due to Korsbakken et al and Marquardt et al. For the first measure the observable $\sigma_x^{\otimes 2}$ always distinguishes perfectly between $|\psi_0\rangle$ and $|\psi_1\rangle$ on a local level. For the second measure, we need on every local group a unitary operation that maps the $|\alpha_i\rangle$ space to the orthogonal $|\beta_i\rangle$ space. Therefore, the distance between $|\psi_0\rangle$ and $|\psi_1\rangle$ in terms of local operations is of the order of $N$, i.e. the superposition is macroscopic.

In contrast, $|\psi_0\rangle + |\psi_1\rangle$ is in general not macroscopic due to the relative Fisher information, since the single constituents may show non-classical behaviour due to other local operators such as $\sigma_x^{\otimes 2}$.

The third part of proposition 2 states that a superposition is macroscopic due to Korsbakken et al if and only if it is macroscopic due to Marquardt et al under the conditions mentioned in the
beginning of this section. It is clear that the absolute effective sizes can differ for both measures, which is also a consequence of the open parameter $\delta$ in the proposal of Korsbakken et al.

Focusing on a superposition of two states $|\psi_0\rangle$ and $|\psi_1\rangle$ means that we implicitly assume that these states are a natural choice of quantum states with more or less classical behaviour. Dropping this restriction, we have to consider arbitrary macroscopic spin states, which leads to measures like the Fisher information or the index $p$. It is interesting that we find with proposition 2 that all superposition states that are called macroscopic due to the measures of [8, 10, 11] and the relative Fisher information (22) are also macroscopic due to the more general measures [7] and the Fisher information (definition 2). This indicates that—assuming that the last two proposals are well suited for classifying macroscopic quantum states—Schrödinger correctly caught the idea of macroscopic quantum phenomena with his cat-gedankenexperiment. But these phenomena can be demonstrated with much more general quantum states, which are not so illustrative any more. This means that with the help of local interactions within the system and with the environment to measure to systems, there exist other quantum states that do not rely on a superposition of two semi-classical states and can still show macroscopic quantum effects.

A counter-example that the reverse statement of part 5 is not true is discussed extensively in [38]. We summarize it in

**Example 6 (Cloned superposition state).** The eigenstates of $\sum_{i=1}^{N} \sigma_z^{(i)}$ with eigenvalue $N - 2x$ that are completely symmetric under particle exchange are called Dicke-$x$ states $|N, x\rangle$ [39]. For odd $N$, we consider the superposition state $|\psi\rangle = \frac{1}{\sqrt{2}} (|\psi_0\rangle + |\psi_1\rangle)$ with

$$
|\psi_0\rangle = \frac{1}{\sqrt{2}} \left( |N, \frac{N-1}{2}\rangle + |N, \frac{N+1}{2}\rangle \right),
$$

$$
|\psi_1\rangle = \frac{1}{\sqrt{2}} \left( |N, \frac{N-1}{2}\rangle - |N, \frac{N+1}{2}\rangle \right).
$$

This superposition arises if an initial state $|\phi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$ is subject to an optimal phase covariant cloning device [40].

One can show that $|\psi\rangle$ has a variance that is macroscopic: $\mathcal{V}_\phi (\sum_{i=1}^{N} \sigma_z^{(i)}) = O(N^2)$. This state is therefore macroscopic due to the Fisher information and Shimizu and Miyadera (index $p$). In contrast, the states (35) are not locally distinguishable and the superposition is therefore not macroscopic due to Korsbakken et al.

We now come to

**Proof of proposition 2.** (i) We first derive the order of the denominator and the numerator of equation (27) if $N_{\text{eff}}^B(\psi) = O(N)$. The numerator and the denominator are individually optimized over all local Hamiltonians of the set $A$ from definition 2(b). For convenience, the optimal operator is always denoted by $A$ even if it differs for different states. In general, $\sqrt{\mathcal{V}_{\psi_0}(A)} + \sqrt{\mathcal{V}_{\psi_1}(A)} \geq O(\sqrt{N})$, since one can always find a local operator for which the variance is at least the sum of the variances of the addends $A^{(i)}$. Furthermore, we have $\langle A \rangle_{\psi_0} - \langle A \rangle_{\psi_1} \leq O(N)$ due to the spectral radius $\sigma(A) = N$. To fulfill both requirements simultaneously, one has $\sqrt{\mathcal{V}_{\psi_0}(A)} + \sqrt{\mathcal{V}_{\psi_1}(A)} = O(\sqrt{N})$ and $\langle A \rangle_{\psi_0} - \langle A \rangle_{\psi_1} = O(N)$. The same reasoning can be applied for the relative Fisher information. From $N_{\text{eff}}^B(\psi) = O(N)$, it follows that $\mathcal{F}(\psi, A) = O(N^2)$, $\mathcal{F}(\psi_0, A) = O(N)$ and $\mathcal{F}(\psi_1, A) = O(N)$. 

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We now come to the relation between $N^B_{\text{eff}}(\psi)$ and $N^F_{\text{eff}}(\psi)$. The numerators of $N^B_{\text{eff}}(\psi)$ and $N^F_{\text{eff}}(\psi)$ are always of the same order and we only have to look at the denominators. Let us consider $|\psi_i\rangle$ ($i = 0, 1$) in the eigenbasis $|\{a_k\}\rangle$ of $A$, $|\psi_i\rangle = \sum_k c_{ik} |a_k\rangle$. The conditions $\langle A \rangle_{\psi_0} - \langle A \rangle_{\psi_1} = O(N)$ and $\mathcal{V}_\psi(A) = O(N)$ imply that the state $|\psi\rangle$ exhibits at least two regions in the spectrum $\sigma(A)$ with non-vanishing probabilities to measure the corresponding eigenvalues. The variance of $|\psi\rangle$ reads $\mathcal{V}_\psi(A) = \sum_k |c_{0k} + c_{1k}|^2 (a_k - \langle A \rangle_{\psi})^2$ ($a_k$ denotes the eigenvalues of $A$). We see that $\mathcal{V}_\psi(A)$ contains at least two addends $k$ and $j$ with $a_k - a_j = O(N)$ and $|c_{0k} + c_{1k}|^2 = O(1)$ and $|c_{0j} + c_{1j}|^2 = O(1)$, which leads to $\mathcal{V}_\psi(A) = O(N^2)$. On the other hand, $\mathcal{V}_\psi(A) = O(N^2)$ with $\mathcal{V}_\psi(A) = O(N)$ is only possible if $\langle A \rangle_{\psi_0} - \langle A \rangle_{\psi_1} = O(N)$, which establishes the equivalence of both measures in detecting macroscopic superpositions.

(ii) As we have seen in the first part of the proof, it follows from $N^B_{\text{eff}}(\psi) = O(N)$ that there exists a local operator $A \in A$ such that $|\langle A \rangle_{\psi_0} - \langle A \rangle_{\psi_1}| = O(N)$ and $\sqrt{\mathcal{V}_\psi(A)} + \sqrt{\mathcal{V}_\psi(A)} = O(\sqrt{N})$. Without loss of generality, we assume a locality of $A$ with respect to a fixed grouping into $n = O(N)$ blocks, each consisting of $O(1)$ qubits, i.e. $A = \sum_{i=1}^n A^{(i)}$. To fulfill $|\langle A \rangle_{\psi_0} - \langle A \rangle_{\psi_1}| = O(N)$ there exist $O(N)$ blocks with $||\rho_0^{(i)} - \rho_1^{(i)}||_1 = O(1)$. If $P > 1 - \delta$, i.e. the demanded precision is fulfilled, then the actual grouping into $n$ blocks is the effective size of Korsbakken et al., $N^K_{\text{eff}} = n = O(N)$. If not, then we have to measure on a collection of $m = O(1)$ groups to improve the precision. Since the standard deviation of $\sum_{i=1}^m A^{(i)}$ scales at most with $O(\sqrt{m})$, the success probability can be improved to $P \approx 1 - \epsilon/\sqrt{m}$, which is for some $m$ larger than $1 - \delta$. Then, the effective size reads $N^K_{\text{eff}} = n/m = O(N)$.

The implication $N^K_{\text{eff}} = O(N) \Rightarrow N^B_{\text{eff}} = O(N)$ is in general not true (see example 5).

(iii) If $N^K_{\text{eff}} = O(N)$, then we find $n = O(N)$ distinct groups of $O(1)$ qubits, such that for every group $i$ the ranges of the reduced density operators $\rho_k^{(i)} = \text{Tr}_{N\setminus i} |\psi_k\rangle\langle\psi_k|$ ($k = 0, 1$) are almost orthogonal, i.e. $\frac{1}{2} ||\rho_0^{(i)} - \rho_1^{(i)}||_1 = 1 - 2\delta$ with $\delta \ll 1$. Therefore, one has to apply to every group $i$ a non-trivial rotation. On average, this corresponds to $O(1)$ one-particle operations. Since this has to be done for all groups simultaneously, we need in total $O(N)$ one-particle operations to transform $|\psi_0\rangle$ to $|\psi_1\rangle$ and hence we have $N^M_{\text{eff}} = O(N)$.

On the other hand, $N^M_{\text{eff}} = O(N)$ requires that we have to rotate non-trivially on $O(N)$ groups of $|\psi_0\rangle$ to have a large overlap with $|\psi_1\rangle$. This means that there are $O(N)$ groups where the ranges of $\rho_0^{(i)}$ and $\rho_1^{(i)}$ are almost orthogonal. This makes the two states locally distinguishable, hence $N^K_{\text{eff}} = O(N)$.

(iv) Assuming that $N^K_{\text{eff}} = O(N)$, there exists an $A = \sum_{i=1}^n A^{(i)} \in A$ such that $|\langle A \rangle_{\psi_0} - \langle A \rangle_{\psi_1}| = O(N)$ and $\mathcal{V}(\psi)(A) = O(N)$. As in part 2, we conclude that $\mathcal{V}(\psi)(A) = O(N^2)$, i.e. $N^F_{\text{eff}}(\psi) = N^S_{\text{eff}}(\psi) = O(N)$.

\[ \square \]

6. Conclusion

To summarize, we have introduced the concept of the quantum Fisher information as a measure for macroscopic quantum states in multipartite qubit systems. This definition has been motivated by the search for quantum states that potentially show the validity of quantum mechanics on a macroscopic scale under ‘realistic’ conditions, which resulted in the restriction to local Hamiltonians for the time evolution and local measurements. We have seen that large quantum Fisher information allows a rapid oscillation pattern for certain observables which cannot be
reproduced by classical states. Furthermore, this kind of macroscopic quantum state is able to serve as a resource for high-precision measurements.

This measure is well defined for generally mixed states and does not require a special shape of the state, for example, a superposition of two orthogonal quantum states. However, a slight modification allows us to judge the ‘catness’ of macroscopic superpositions. This variation was subsequently called relative Fisher information. With this proposal, we circumvent some ambiguities that other measures of macroscopic superpositions show.

We have contrasted our proposals with other measures of macroscopic quantum states suitable for qubit systems. We have suggested to distinguish carefully between ‘macroscopic quantum states’ and ‘macroscopic superpositions’. We found that among those measures, a hierarchy can be established that classifies three different kinds of macroscopic quantum states: superpositions of orthogonal, semi-classical states; superpositions of a priori general quantum states that are locally orthogonal; and general quantum states that exhibit long-range quantum correlations. Interestingly, it turned out that macroscopic superpositions are a strict subset of general macroscopic quantum states.

This paper has focused on qubit systems. Although not presented explicitly, it is clear that one can generalize our findings to arbitrary spin systems. In future work, it will be interesting to investigate the role of the quantum Fisher information in macroscopicity in other frameworks such as continuous variable systems.

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