Zassenhaus Conjecture on Torsion Units Holds for \(\text{PSL}(2, p)\) with \(p\) a Fermat or Mersenne Prime

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Abstract. H.J. Zassenhaus conjectured that any unit of finite order in the integral group ring \(\mathbb{Z}G\) of a finite group \(G\) is conjugate in the rational group algebra \(\mathbb{Q}G\) to an element of the form \(\pm g\) with \(g \in G\). Though known for some series of solvable groups, the conjecture has been proved only for thirteen non-abelian simple groups. We prove the Zassenhaus Conjecture for the groups \(\text{PSL}(2, p)\), where \(p\) is a Fermat or Mersenne prime. This increases the list of non-abelian simple groups for which the conjecture is known by probably infinitely many, but at least by 49, groups. Our result is an easy consequence of known results and our main theorem which states that the Zassenhaus Conjecture holds for a unit in \(\mathbb{Z}\text{PSL}(2, q)\) of order coprime with \(2q\), for some prime power \(q\).

1. Introduction

One of the most famous open problems regarding the unit group of an integral group ring \(\mathbb{Z}G\) of a finite group \(G\) is the Zassenhaus Conjecture which was stated by H.J. Zassenhaus [Zas74]:

**Zassenhaus Conjecture**: If \(G\) is a finite group and \(u\) is a unit of finite order in the integral group ring \(\mathbb{Z}G\), then there exists a unit \(x\) in the rational group algebra \(\mathbb{Q}G\) and an element \(g \in G\) such that \(x^{-1}ux = \pm g\).

If for a given \(u\) such \(x\) and \(g\) exist, one says that \(u\) and \(\pm g\) are rationally conjugate. The Zassenhaus Conjecture found much attention and was proved for many series of solvable groups, e.g., for nilpotent groups [Wei91], groups possessing a normal Sylow subgroup with abelian complement [Her06] or cyclic-by-abelian groups [CMdR13]. Regarding non-solvable groups, however, the conjecture is only known for very few groups. The proofs of the results for solvable groups mentioned above often argue by induction on the order of the group. In this way one may assume that the conjecture holds for proper quotients of the original group. The first step in a similar argument for non-solvable groups should consist in proving the conjecture for simple groups. Although this has been studied by some authors, see e.g., [LP89] [Her97] [Her98] [BKL08] [Sal13] [BM17] [BC17], the conjecture is still only known for exactly thirteen non-abelian simple groups all being isomorphic to some \(\text{PSL}(2, q)\) for some particular small prime power \(q\) (see [BM18] Theorem C for an overview). Our aim in this paper is to extend this knowledge by proving the following theorem.

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1 After this paper was submitted a metabelian counterexample to the Zassenhaus Conjecture was announced in [EM17]. Still no simple counterexample is known.
Theorem 1.1. Let $G = \text{PSL}(2, q)$ for some prime power $q$. Then any torsion unit of $\mathbb{Z}G$ of order coprime with $2q$ is rationally conjugate to an element of $G$.

We prove this result employing a variation of a well known method which uses characters of a finite group $G$ to obtain restrictions on the possible torsion units in $\mathbb{Z}G$. The idea of the method was introduced for ordinary characters by Luthar and Passi [LP89] and extended to Brauer characters by Hertweck [Her07]. Today this method is often called the HeLP (Hertweck-Luthar-Passi) Method. In fact to prove our results we do not use the HeLP Method in the classical sense, since this would imply too many case distinctions. For this reason we vary the method in a way suitable for the character theory of $\text{PSL}(2, q)$. Theorem 1.1 can be regarded as a generalization of [Mar16, Theorem 1].

As a direct application of Theorem 1.1 and known facts about the units of $\mathbb{Z}\text{PSL}(2, q)$ collected in Theorem 2.2, we obtain the result which gives name to this paper:

Theorem 1.2. Let $p$ be a Fermat or Mersenne prime. Then the Zassenhaus Conjecture holds for $\text{PSL}(2, p)$.

This result increases the number of simple groups for which the Zassenhaus Conjecture is known from thirteen to sixty-two: The groups $\text{PSL}(2, q)$ with $q \in \{8, 9, 11, 13, 16, 19, 23, 25, 32\}$ or one of the four known Fermat primes different from 3 or one of the forty-nine known Mersenne primes different from 3 [Calb]. Actually, Theorem 1.2 proves the conjecture for probably infinitely many simple groups because, based on heuristic evidences, it has been conjectured that there are infinitely many Mersenne primes [Calb]. Lenstra, Pomerance and Wagstaff have proposed independently a conjecture on the growth of the number of Mersenne primes smaller than a given integer [Pom81, Wag83].

It has been shown in [dRS17] that a result as in Theorem 1.1 can not be achieved using solemnly the HeLP Method if the unit has order $2p$, where $2p$ is coprime with $q$ and $p$ a prime bigger than 3. Looking on the orders of elements in $\text{PSL}(2, q)$, cf. Theorem 2.2, one should not expect a better result for the Zassenhaus Conjecture for $\text{PSL}(2, q)$ when applying only this method. Thus, as so often in Arithmetics and Group Theory, the prime 2 behaves very differently than the other primes.

We collect in Section 2 the notation and known results which will be used during the proof of Theorem 1.1. In Section 3 we prove several number theoretical results which are essential for our arguments and introduce some more notation. Finally in Section 4 we prove Theorem 1.1.

2. Preliminaries

Let $G$ be a finite group. If $g \in G$, then $|g|$ denotes the order of $g$, the cyclic group generated by $g$ is denoted by $\langle g \rangle$ and $g^G$ denotes the conjugacy class of $g$ in $G$. If $R$ is a ring then $RG$ denotes the group ring of $G$ with coefficients in $R$. Denote by $V(\mathbb{Z}G)$ the group of normalized units (i.e. units of augmentation 1) in $\mathbb{Z}G$. As mentioned above, we say that two elements of $\mathbb{Z}G$ are rationally conjugate if they are conjugate in the units of $\mathbb{Q}G$.

The main notion to study rational conjugacy of torsion units in $\mathbb{Z}G$ are the so called partial augmentations. If $\alpha = \sum_{g \in G} \alpha_g g$ is an element of a group ring $\mathbb{Z}G$, with each $\alpha_g \in \mathbb{Z}$, then the partial augmentation of $\alpha$ at $g$ is defined as

$$\varepsilon_g(\alpha) = \sum_{h \in g^G} \alpha_h.$$
The relevance of partial augmentations for the study of the Zassenhaus Conjecture is provided by a result of Marciniak, Ritter, Sehgal and Weiss. The following theorem states this result and collects some known information about partial augmentations.

**Theorem 2.1.** Let $G$ be a finite group and let $u$ be an element of order $n$ in $V(ZG)$.

1. [MRSW87, Theorem 2.5] $u$ is rationally conjugate to element in $G$ if and only if $\varepsilon_g(u^d) \geq 0$ for all $g \in G$ and all divisors $d$ of $n$.
2. [JdR16, Proposition 1.5.1] (Berman-Higman Theorem) If $u \neq 1$ then $\varepsilon_1(u) = 0$.
3. [Her07, Theorem 2.3] If $\varepsilon_g(u) \neq 0$ then $|g|$ divides $n$.
4. [Her07, Theorem 3.2] Let $p$ be a prime not dividing $n$ and let $\chi$ be a $p$-Brauer character of $G$ associated to a modular representation $G \rightarrow M_n(k)$ for a suitable $p$-modular system $(K, R, k)$. Then $\chi$ extends to a $p$-Brauer character defined on the $p$-regular torsion units of $ZG$, associated to the natural algebra homomorphism $RG \rightarrow M_n(k)$. Moreover, if $g_1, \ldots, g_k$ are representatives of the $p$-regular conjugacy classes of $G$ then

\[
\chi(u) = \sum_{i=1}^{k} \varepsilon_{g_i}(u)\chi(g_i).
\]

We collect the group theoretical properties of $PSL(2, q)$ and its integral group ring relevant for us.

**Theorem 2.2.** Let $G = PSL(2, q)$ where $q = t^f$ for some prime $t$ and let $d = \gcd(2, q)$.

1. [Hup67, Hauptsatz 8.27] The following properties hold.
   - The order of $G$ is $(q-1)q(q+1)/d$.
   - The orders of elements in $G$ are exactly $t$ and the divisors of $(q+1)/d$ and $(q-1)/d$.
   - Two cyclic subgroups of $G$ are conjugate in $G$ if and only if they have the same order.
   - If $g, h \in G$ with $|g|$ coprime with $t$ and multiple of $|h|$ then $h$ is conjugate in $G$ to an element $h_1$ of $\langle g \rangle$ and the only elements of $\langle g \rangle$ conjugate to $h$ in $G$ are $h_1$ and $h_1^{-1}$.
   In particular a conjugacy class of elements of order coprime with $t$ is a real conjugacy class.
2. If $u$ is a torsion element of $V(ZG)$ of order coprime with $t$, $\zeta$ is root of unity in an arbitrary field $F$ and $\Theta$ is an $F$-representation of $G$ then $\zeta$ and $\zeta^{-1}$ have the same multiplicity as eigenvalues of $\Theta(u)$. This follows from [1] and the formulas for multiplicities of eigenvalues of torsion units as presented in [Her07, Section 4].
3. Let $u \in V(ZG)$ of order $n$.
   - If $\gcd(n, q) = 1$ then $G$ has an element of order $n$ [Her07, Proposition 6.7].
   - If $n$ is a prime power not divisible by $t$, then $u$ is rationally conjugate to an element of $G$ [Mar16, Theorem 1].
   - If moreover $f = 1$ and $n$ is divisible by $t$, then $n = t$ and $u$ is also rationally conjugate to an element of $G$ [Her07, Propositions 6.1, 6.3].
4. [Mar16, Lemma 1.2] Let $n$ be a positive integer coprime with $t$ and let $g \in G$ be an element of order $n$. There exists a primitive $n$-th root of unity $\alpha$ in a field of characteristic $t$ such that for every positive integer $m$, there is a $t$-modular representation $\Theta_m$ of $G$ of degree $1 + 2m$
such that
\[ \Theta_m(g) \text{ is conjugate to diag } (1, \alpha, \alpha^{-1}, \alpha^2, \alpha^{-2}, \ldots, \alpha^m, \alpha^{-m}). \]

We denote by \( \psi_m \) the Brauer character associated with \( \Theta_m \).

As mentioned in the introduction, we actually do not use the HeLP Method in its classical setting. We neither compute many inequalities involving traces as for example in the proofs of [Her07 Proposition 6.5] or [BKL08 Mar16], since these formulas turn out to be too complicated in our setting. Nor do we apply the standard equations obtained from character values on one side and possible eigenvalues on the other side as e.g. in the proofs of [Her07 Propositions 6.4, 6.7], [Her08] or [BM17 Lemma 2.2], since there are too many possibilities for these possible eigenvalues. Still this second strategy is closer to our approach.

3. Number theoretic results

In this section we prove two number theoretical results which are essential for our arguments and might be of independent interest. Our first proof of Proposition 3.2 below was very long. We include a proof which was given to us by Hendrik Lenstra. We are very thankful to him for his simple and nice proof.

For a prime integer \( p \) and a non-zero integer \( n \) let \( v_p(n) \) denote the valuation of \( n \) at \( p \), i.e. the maximal non-negative integer \( m \) with \( p^m \mid n \). If, moreover, \( n > 0 \) then \( \zeta_n \) denotes a complex primitive \( n \)-th root of unity and \( \Phi_n \) denotes the \( n \)-th cyclotomic polynomial, i.e. the minimal polynomial of \( \zeta_n \) over \( \mathbb{Q} \).

**Lemma 3.1.** If \( n \) and \( m \) are positive integers and \( p \) is a prime integer then \( \Phi_{np^m}(\zeta_n) \in p \mathbb{Z}[\zeta_n] \).

**Proof.** We argue by induction on \( v_p(n) \). Suppose first that \( p \nmid n \) and let \( S \) denote the set of primitive \( p^m \)-th roots of unity. Then \( \zeta_n \xi \) is a root of \( \Phi_{np^m}(X) \) for every \( \xi \in S \) and hence \( \prod_{\xi \in S}(X - \zeta_n \xi) \) divides \( \Phi_{np^m}(X) \) in \( \mathbb{Z}[\zeta_n][X] \). Therefore
\[
\Phi_{np^m}(\zeta_n) \in \prod_{\xi \in S}(\zeta_n - \zeta_n \xi) \mathbb{Z}[\zeta_n] = \prod_{\xi \in S}(1 - \xi) \mathbb{Z}[\zeta_n] = \Phi_p(1) \mathbb{Z}[\zeta_n] = p \mathbb{Z}[\zeta_n].
\]

Suppose that \( p \mid n \) and assume that the lemma holds with \( n \) replaced by \( \frac{n}{p} \). Then \( \Phi_{np^{m-1}}(\zeta_n^p) = \Phi_{np^m}(\zeta_n^p) \in p \mathbb{Z}[\zeta_n^p] \subseteq p \mathbb{Z}[\zeta_n] \). As \( \Phi_{np^m}(X) = \Phi_{np^{m-1}}(X^p) \) and \( \zeta_n^p \) is a primitive \( \frac{n}{p} \)-th root of unity, we have \( \Phi_{np^m}(\zeta_n) = \Phi_{np^{m-1}}(\zeta_n^p) \in p \mathbb{Z}[\zeta_n] \). \( \square \)

**Proposition 3.2.** Let \( n \) be a positive integer. Let \( A_0, A_1, \ldots, A_{n-1} \) be integers and for every positive integer \( i \) set
\[ \omega_i = \sum_{j=0}^{n-1} A_j \zeta_n^{ij}. \]

Let \( d \) be a divisor of \( n \) such that \( \omega_d \equiv 0 \) for every prime power \( q \) dividing \( d \) with \( q \neq 1 \). Then \( \omega_d \in d \mathbb{Z}[\zeta_n] \).

**Proof.** Let \( k = \frac{n}{d} \) and consider the polynomial \( f(X) = \sum_{j=0}^{n-1} A_j X^j \). We can take \( \zeta_k = \zeta_n^d \), so that \( \omega_d = f(\zeta_k) \). By hypothesis, for every prime \( p \) and every positive integer \( m \) with \( p^m \) dividing \( d \) we
have \( f(\zeta_{kp^n}) = 0 \), or equivalently \( \Phi_{kp^n}(X) \) divides \( f(X) \) in \( \mathbb{Z}[X] \). Thus \( \prod_{p|d} \prod_{m=1}^{v_p(d)} \Phi_{kp^n}(X) \) divides \( f(X) \) in \( \mathbb{Z}[X] \). Therefore \( \omega_d = f(\zeta_k) \in \prod_{p|d} \prod_{m=1}^{v_p(d)} \Phi_{kp^n}(\zeta_k) \mathbb{Z}[\zeta_k] \). By Lemma 3.1 each \( \Phi_{kp^n}(\zeta_k) \) belongs to \( p \mathbb{Z}[\zeta_n] \). As \( d = \prod_{p|d} \prod_{m=1}^{v_p(d)} p \) we deduce that \( \omega_d \in d \mathbb{Z}[\zeta_n] \), as desired.

For a positive integer \( n \) and a subfield \( F \) of \( \mathbb{Q}(\zeta_n) \), let \( \Gamma_F \) denote a set of representatives of equivalence classes of the following equivalence relation defined on \( \mathbb{Z} \):

\[
x \sim y \quad \text{if and only if} \quad \zeta_n^x \text{ and } \zeta_n^y \text{ are conjugate in } \mathbb{Q}(\zeta_n) \text{ over } F.
\]

Corollary 3.3. Let \( n \) be a positive integer, let \( F \) be a subfield of \( \mathbb{Q}(\zeta_n) \) and let \( R \) be the ring of integers of \( F \). For every \( x \in \Gamma_F \) let \( B_x \) be an integer and for every integer \( i \) define

\[
\omega_i = \sum_{x \in \Gamma_F} B_x \text{Tr}_{\mathbb{Q}(\zeta_n)/F}(\zeta_n^x).
\]

Let \( d \) be a divisor of \( n \) such that \( \omega_{\frac{d}{q}} = 0 \) for every prime power \( q \) dividing \( d \) with \( q \neq 1 \). Then \( \omega_d \in d R \).

Proof. Apply Proposition 3.2 to the integers \( A_x = B_\overline{x} \) with \( \overline{x} \) denoting the class in \( \Gamma_F \) containing \( x \).

In the remainder of this section we reserve the letter \( p \) to denote positive prime integers.

We now introduce some notation for a positive integer \( n \) which will be fixed throughout. First we set

\[
n' = \prod_{p|n} p \quad \text{and} \quad n_p = p^{v_p(n)}.
\]

If moreover \( x \in \mathbb{Z} \) then we set

\[
(x : n) = \text{representative of the class of } x \bmod n \text{ in the interval } \left( -\frac{n}{2}, \frac{n}{2} \right);
\]

\[
|x : n| = \text{the absolute value of } (x : n) \text{ and};
\]

\[
\gamma_n(x) = \prod_{p|n} p \quad \text{if } |x : n| < \frac{n}{2p}.
\]

Next lemma collects two elementary properties involving this notation whose proofs are direct consequences of the definitions.

Lemma 3.4. Let \( p \) be a prime dividing \( n \) and let \( x, y \in \mathbb{Z} \). Then the following conditions hold:

1. If \( p \mid \gamma_n(x) \) then \( \left( x : \frac{n_p}{p} \right) \equiv x \bmod n_p \).
2. Let \( d \mid n' \) such that \( x \equiv y \bmod \frac{n}{d} \). If \( d \) divides both \( \gamma_n(x) \) and \( \gamma_n(y) \) then \( x \equiv y \bmod n \).

For integers \( x \) and \( y \) we define the following equivalence relation on \( \mathbb{Z} \):

\[
x \sim_n y \quad \iff \quad x \equiv \pm y \bmod n.
\]

We denote by \( \Gamma_n \) a set of representatives of these equivalence classes. Without loss of generality one may assume that \( \Gamma_n = \Gamma_{\mathbb{Q}(\zeta_n, \zeta_n^{-1})} \).
In the remainder of the section we assume that \( n \) is odd. For \( x \) and \( y \) integers let
\[
\alpha_{x}^{(n)} = \zeta_{n}^{x} + \zeta_{n}^{-x}, \quad \kappa_{x}^{(n)} = \begin{cases} 
2, & \text{if } x \equiv 0 \mod n; \\
1, & \text{otherwise}; 
\end{cases} \quad \text{and} \quad \delta_{x:y}^{(n)} = \begin{cases} 
1, & \text{if } x \sim_{n} y; \\
0, & \text{otherwise}. 
\end{cases}
\]

Moreover, \( \mathbb{Q} \left( \alpha_{1}^{(n)} \right) = \mathbb{Q}(\zeta_{n} + \zeta_{n}^{-1}) \) is the maximal real subfield of \( \mathbb{Q}(\zeta_{n}) \) and \( \mathbb{Z} \left[ \alpha_{1}^{(n)} \right] = \mathbb{Z}[\zeta_{n} + \zeta_{n}^{-1}] \) is the ring of integers of \( \mathbb{Q} \left( \alpha_{1}^{(n)} \right) \).

Let \( B_{n} = \left\{ x \in \mathbb{Z}/n\mathbb{Z} : |x : n_{p}| > \frac{n_{p}}{2p} \text{ for every } p \mid n \right\} \) and \( B_{n} = \{ \alpha_{b}^{(n)} : b \in B_{n} \}. \)

In the following proposition we prove that \( B_{n} \) is a \( \mathbb{Q} \)-basis of \( \mathbb{Q}[\alpha_{1}^{(n)}] \). For \( x \in \mathbb{Q}[\alpha_{1}^{(n)}] \) and \( b \in B_{n} \), we use
\[
C_{b}(x) = \text{coefficient of } \alpha_{b}^{(n)} \text{ in the expression of } x \text{ in the basis } B_{n}. \]

We denote by \( \mu \) the number theoretical Möbius function.

**Proposition 3.5.** Let \( n \) be a positive odd integer. Then

(1) \( B_{n} \) is a \( \mathbb{Z} \)-basis of \( \mathbb{Z} \left[ \alpha_{1}^{(n)} \right] \) and in particular, a \( \mathbb{Q} \)-basis of \( \mathbb{Q} \left( \alpha_{1}^{(n)} \right) \).

(2) If \( b \in B_{n} \) and \( i \in \mathbb{Z} \) then \( C_{b}(\alpha_{i}^{(n)}) = \kappa_{i}^{(n)} \cdot \mu(\gamma(i)) \cdot \delta_{b,i}^{(n/\gamma(i))} \).

**Proof.** It is easy to see that \( |B_{n}| \leq \frac{\varphi(n)}{2} = |\mathbb{Q} \left( \alpha_{1}^{(n)} \right) : \mathbb{Q}|. \) Thus it is enough to prove the following equality
\[
\alpha_{i}^{(n)} = \kappa_{i}^{(n)} \cdot \mu(\gamma(i)) \sum_{b \in B_{n}, b \sim_{\gamma(i)} i} \alpha_{b}^{(n)}. \]

Actually we will show \( \zeta_{i}^{n} = \mu(\gamma(i)) \sum_{b \in B_{n}, b \equiv i \mod \frac{n}{\gamma(i)}} \zeta_{b}^{n} \), which implies the desired expression of \( \alpha_{i}^{(n)} \). Indeed, for every \( p \mid n \) let \( \zeta_{n_{p}} \) denote the \( p \)-th part of \( \zeta_{n} \), i.e. \( \zeta_{n_{p}} \) is a primitive \( n_{p} \)-th root of unity and \( \zeta_{n} = \prod_{p \mid n} \zeta_{n_{p}} \). Let \( J \) be the set of tuples \((j_{p})_{p \mid \gamma(i)}\) satisfying \( j_{p} \in \{1, \ldots, p - 1\}\) for every \( p \mid \gamma(i) \). For every \( j \in J \) let \( b_{j} \in \mathbb{Z}/n\mathbb{Z} \) given by
\[
b_{j} \equiv \begin{cases} 
i + j_{p} \frac{n_{p}}{p} \mod n_{p}, & \text{if } p \mid \gamma(i); \\
i \mod n_{p}, & \text{otherwise.} \end{cases}
\]

Then \( \{ b_{j} : j \in J \} \) is the set of elements \( b \) in \( B_{n} \) satisfying \( i \equiv b \mod \frac{n}{\gamma(i)} \). From
\[
0 = n_{p} \zeta_{n_{p}}^{i} \left( 1 + \zeta_{n_{p}}^{n_{p}} + \zeta_{n_{p}}^{2n_{p}} + \cdots + \zeta_{n_{p}}^{(p-1)n_{p}} \right)
\]
we obtain \( \zeta_{n_{p}}^{i} = -\sum_{j_{p}=1}^{p-1} \zeta_{n_{p}}^{i+j_{p} \frac{n_{p}}{p}} \). Therefore
\[
\zeta_{n}^{i} = \prod_{p \mid \gamma(i)} \zeta_{n_{p}}^{i} \prod_{p \mid \gamma(i)} \left( -\sum_{j_{p}=1}^{p-1} \zeta_{n_{p}}^{i+j_{p} \frac{n_{p}}{p}} \right) = \mu(\gamma(i)) \sum_{j \in J} b_{j} = \mu(\gamma(i)) \sum_{i \equiv b \mod \frac{n}{\gamma(i)}} b_{i} \zeta_{n_{p}}^{i}.
\]
4. Proof of Theorem 4.1

In this section we prove Theorem 4.1. In the remainder, set $G = \text{PSL}(2, t^f)$ with $t$ a prime. Our goal is to prove that any element $u$ of order $n$ in $V(\mathbb{Z}G)$, where $n$ is greater than 1 and coprime with $2t$, is rationally conjugate to an element of $G$. By Theorem 2.2 (3) we may also assume that $n$ is not a prime power.

As the order $n$ of $u$ is fixed throughout, we simplify the notation of the previous section by setting

$$\gamma = \gamma_n, \quad \alpha_x = \alpha_x^{(n)}, \quad \kappa_x = \kappa_x^{(n)}, \quad B = B_n, \quad B = B_n.$$

We argue by induction on $n$. So we assume that $u^d$ is rationally conjugate to an element of $G$ for every divisor $d$ of $n$ with $d \neq 1$.

We will use the representations $\Theta_m$ and Brauer characters $\psi_m$ introduced in Theorem 2.2 (1). As usual in modular representation theory, a bijection between the complex roots of unity of order $v$ for (4.3) holds. Conversely, assume that (4.3) holds. For $\Theta_a$ a priori. In this sense we will identify the eigenvalues of $\lambda$ if $u$ is fixed throughout, we simplify the notation of the previous section by setting

$$\gamma = \gamma_n, \quad \alpha_x = \alpha_x^{(n)}, \quad \kappa_x = \kappa_x^{(n)}, \quad B = B_n, \quad B = B_n.$$

We argue by induction on $n$. So we assume that $u^d$ is rationally conjugate to an element of $G$ for every divisor $d$ of $n$ with $d \neq 1$.

We will use the representations $\Theta_m$ and Brauer characters $\psi_m$ introduced in Theorem 2.2 (1).

As usual in modular representation theory, a bijection between the complex roots of unity of order coprime with $t$ and the roots of unity of the same order in a field of characteristic $t$ has been fixed a priori. In this sense we will identify the eigenvalues of $\Theta_m$ and the summands in $\psi_m$. Since units of prime order in $V(\mathbb{Z}G)$ are rationally conjugate to elements of $G$ by Theorem 2.2 (3), we know that the kernel of $\Theta_1$ on $\langle u \rangle$ is trivial and hence $\Theta_1(u)$ has order $n$. As the values of $\psi_1$ on $t$-regular elements of $G$ are real, by Theorem 2.2 (1) and Theorem 2.1 (4), the set of eigenvalues $\Theta_1(u)$ is closed under taking inverses (counting multiplicities). Therefore, $\Theta_1(u)$ is conjugate to diag$(1, \zeta, \zeta^{-1})$ for a suitable primitive $n$-th root of unity $\zeta$. Hence by Theorem 2.2 there exists an element $g_0 \in G$ of order $n$ such that $\Theta_1(g_0)$ and $\Theta_1(u)$ are conjugate. From now on we abuse the notation and consider $\zeta$ both as a primitive $n$-th root of unity in a field of characteristic $t$ and as a complex primitive $n$-th root of unity. Then for any positive integer $m$ we have that

$$(4.1) \quad \Theta_m(g_0) \text{ is conjugate to diag } \{1, \zeta, \zeta^{-1}, \zeta^2, \zeta^{-2}, \ldots, \zeta^m, \zeta^{-m}\},$$

and for every integer $i$ we have

$$(4.2) \quad \psi_m(g_0^i) = \sum_{j=-m}^{m} \zeta^{ij} = 1 + \sum_{j=1}^{m} \alpha_{ij}.$$

The element $g_0 \in G$ and the primitive $n$-th root of unity $\zeta$ will be fixed throughout.

By Theorem 2.2 (1), $x \mapsto (g_0^x)^G$ defines a bijection from $\Gamma_n$ to the set of conjugacy classes of $G$ formed by elements of order dividing $n$. For an integer $x$ (or $x \in \Gamma_n$) we set

$$\varepsilon_x = \varepsilon_{g_0}(u) \quad \text{and} \quad \lambda_x = \sum_{i \in \Gamma_n} \varepsilon_i \alpha_{ix}.$$

By Theorem 2.1, $u$ is rationally conjugate to an element of $G$ if and only if $\varepsilon_x \geq 0$ for every $x \in \Gamma_n$.

**Lemma 4.1.** $u$ is rationally conjugate to $g_0$ if and only if

$$(4.3) \quad \lambda_i = a_i, \text{ for any positive integer } i.$$

**Proof.** If $u$ is rationally conjugate to $g_0$, then $\varepsilon_1 = 1$ and $\varepsilon_x = 0$ for any $x \in \Gamma \setminus \{1\}$. Therefore (4.3) holds. Conversely, assume that (4.3) holds. For $v \in V(\mathbb{Z}G)$ of order dividing $n$ let

$$\lambda_i(v) = \sum_{x \in \Gamma_n} \varepsilon_{g_0}(v) \alpha_{xi}. \quad \text{Then } \lambda_i = \lambda_i(u) = \sum_{j=0}^{n-1} \varepsilon_{g_0}(u) \zeta_n^{ij} \text{ and } a_i = \sum_{j=0}^{n-1} \varepsilon_{g_0}(g_0) \zeta_n^{ij}. \quad \text{As the Vandermonde matrix } (\zeta_n^{ij})_{1 \leq i,j \leq n} \text{ is invertible we deduce that } \varepsilon_{g_0}(u) = \varepsilon_{g_0}(g_0) \text{ for every } j \in \Gamma_n. \quad \text{So}$
\[ \varepsilon_j = \varepsilon_{\Psi}^j(u) = \varepsilon_{\Psi}^j(g_0) = 0 \text{ for every } j \in \Gamma_n \setminus \{1\} \text{ and } \varepsilon_1 = 1. \] As we are assuming that if \( d \) is a divisor of \( n \) different from 1 then \( u^d \) is rationally conjugate to an element of \( G \), we also have \( \varepsilon_g(u^d) \geq 0 \) for every \( g \in G \). Thus \( u \) is rationally conjugate to an element of \( g \in G \) by Theorem 2.1 [1]. Then \( \varepsilon_{g_0}(g) = \varepsilon_{g_0}(u) = 1 \) and therefore \( g \) is conjugate to \( g_0 \) in \( G \). We conclude that \( u \) and \( g_0 \) are rationally conjugate.

By Lemma 1.1 in order to achieve our goal it is enough to prove (1.3). We argue by contradiction, so suppose that \( \lambda_d \neq \alpha_d \) for some positive integer \( d \) which we assume to be minimal with this property. Observe that if \( \lambda_i = \alpha_i \) and \( j \) is an integer such that \( \gcd(i, n) = \gcd(j, n) \), then there exists \( \sigma \in \Gal(Q(\alpha_1)/Q) \) such that \( \sigma(\alpha_i) = \alpha_j \) and applying \( \sigma \) to the equation \( \lambda_i = \alpha_i \) we obtain \( \lambda_j = \alpha_j \). This implies that \( d \) divides \( n \). Note that \( \alpha_1 = \lambda_1 \) by our choice of \( g_0 \) and hence \( d \neq 1 \). Moreover, \( d \neq n \) because \( \lambda_n = 2 \sum_{x \in \Gamma_n} \varepsilon_x = 2 = \alpha_n \) as the augmentation of \( u \) is 1.

We claim that

\[ \lambda_d = \alpha_d + d\tau \text{ for some } \tau \in Z[\alpha_1]. \] (4.4)

Indeed, for any \( x \in \Gamma_n \) let \( B_x = \varepsilon_x - 1 \) if \( x \sim n \) and \( B_x = \varepsilon_x \) otherwise. Then for any integer \( i \) we have \( \lambda_i - \alpha_i = \sum_{x \in \Gamma_n} B_x \mathrm{Tr}_{Q(\zeta)/Q(\alpha_1)} (\zeta^ix) \). Therefore, applying Corollary 3.3 for \( F = Q(\alpha_1) \), \( R = Z[\alpha_1] \) and \( \omega_i = \lambda_i - \alpha_i \), the claim follows.

By (1.2) we have, using Theorem 2.1 [4],

\[ \psi_d(g_0) = 1 + \sum_{i=1}^d \alpha_i \quad \text{and} \quad \psi_d(u) = \sum_{x \in \Gamma_n} \varepsilon_x \psi_d(g_0) = \sum_{x \in \Gamma_n} \varepsilon_x \left( 1 + \sum_{i=1}^d \alpha_{ix} \right) = 1 + \sum_{i=1}^d \lambda_i. \] (4.5)

Combining this with (1.3) and the minimality of \( d \), we obtain \( \psi_d(u) = \psi_d(g_0) + d\tau \). Furthermore, \( \tau \neq 0 \), as \( \lambda_d \neq \alpha_d \). Therefore

\[ C_b(\psi_d(u) - 1) \equiv C_b(\psi_d(g_0) - 1) \mod d \quad \text{for every } b \in B \] (4.6)

and

\[ d \leq |C_{b_0}(\psi_d(u) - 1) - C_{b_0}(\psi_d(g_0) - 1)| \quad \text{for some } b_0 \in B. \] (4.7)

The bulk of our argument relies on an analysis of the eigenvalues of \( \Theta_d(u) \) and the induction hypothesis on \( n \) and \( d \). More precisely, we will use (1.6) and (1.7) to obtain a contradiction by comparing the eigenvalues of \( \Theta_d(g_0) \) and \( \Theta_d(u) \). Of course we do not know the eigenvalues of the latter but we know the eigenvalues of each \( \Theta_d(g_0) \). Moreover, if \( c \) is a divisor of \( n \) with \( c \neq 1 \) then \( u^c \) is rationally conjugate to an element \( g \) of \( G \). Then \( \Theta_1(g) \), \( \Theta_1(u^c) \) and \( \Theta_1(g_0) \) are conjugate in \( M_3(F) \), for a suitable field \( F \), and as \( \Theta_1 \) is injective on \( \langle g_0 \rangle \) and \( g \) is conjugate to an element of \( \langle g_0 \rangle \) we conclude that \( u^c \) is conjugate to \( g_0 \). Thus we know the eigenvalues of \( \Theta_d(u^c) \). This has consequences for the eigenvalues of \( \Theta_d(u) \).

To be more precise we fix \( \nu_1, \ldots, \nu_d \in \Gamma_n \) (with repetitions if needed) such that the eigenvalues of \( \Theta_d(u) \) with multiplicities are \( 1, \zeta^{\pm \nu_1}, \ldots, \zeta^{\pm \nu_d} \). This is possible by the last statement of Theorem 2.2 [1]. By the above paragraph, if \( c \mid n \) with \( c \neq 1 \) then the lists \( (\nu_1)_{1 \leq i \leq d} \) and \( (c\nu_i)_{1 \leq i \leq d} \) represent the same elements in \( \Gamma_n \), up to ordering, and hence \( (\nu_1)_{1 \leq i \leq d} \) and \( (i)_{1 \leq i \leq d} \) represent the
same elements of $\Gamma_{\mathbb{Z}}$, up to ordering. We express this by writing

$$(\nu_i) \sim_{\frac{n}{d}} (i) \quad \text{for every } c \mid n \text{ with } c \neq 1.$$  

This provides restrictions on $d$, $n$ and the $\nu_i$.

Moreover, $C_b(\psi_d(u) - 1)$ and $C_b(\psi_d(g_0) - 1)$ are the coefficients of $\alpha_b$ in the expression in the basis $\mathcal{B}$ of $\alpha_{\nu_1} + \cdots + \alpha_{\nu_d}$ and $\alpha_1 + \cdots + \alpha_d$, respectively. By (4.5) and Proposition 3.5 we obtain for every $b \in \mathbb{B}$ that

$$(4.8) \quad C_b(\psi_d(g_0) - 1) = \sum_{i=1}^{d} \mu(\gamma(i)) \cdot \delta^{(n/\gamma(i))}_{b,i} \quad \text{and} \quad C_b(\psi_d(u) - 1) = \sum_{i=1}^{d} \kappa_{\nu_i} \cdot \mu(\gamma(\nu_i)) \cdot \delta^{(n/\gamma(\nu_i))}_{b,i}.$$  

and so

$$(4.9) \quad C_b(\psi_d(u) - 1) - C_b(\psi_d(g_0) - 1) = \sum_{i=1}^{d} \left( \kappa_{\nu_i} \cdot \mu(\gamma(\nu_i)) \cdot \delta^{(n/\gamma(\nu_i))}_{b,i} - \mu(\gamma(i)) \cdot \delta^{(n/\gamma(i))}_{b,i} \right).$$

**Lemma 4.2.**  
(1) If $\kappa_{\nu_i} \neq 1$ for some $1 \leq i \leq d$ then $\frac{n}{d}$ is the smallest prime dividing $n$ and $\kappa_{\nu_j} = 1$ for every $1 \leq j \leq d$ with $j \neq i$.

(2) If $d > 3$ then $n$ is not divisible by any prime greater than $d$.

**Proof.** Let $p$ denote the smallest prime dividing $n$.

[1] Suppose that $\kappa_{\nu_i} \neq 1$. Then $\nu_i \equiv 0 \mod n$. As $(i) \sim_{\frac{n}{d}} (\nu_i)$ we deduce that $k \equiv 0 \mod \frac{n}{d}$ for some $1 \leq k \leq d$. Therefore $d = k = \frac{n}{p}$ and for every $1 \leq j \leq d$ with $j \neq i$ we have $\nu_j \equiv 0 \mod \frac{n}{p}$. Hence $\kappa_{\nu_j} = 1$.

[2] Suppose that $q$ is a prime divisor of $n$ with $d < q$. Then $\frac{n}{d} \neq p$ and therefore, by [1], $\kappa_{\nu_i} = 1$ for every $1 \leq i \leq d$. Thus, by (4.7) and (4.9) and ignoring the signs provided by the $\mu(\gamma(i))$ and $\mu(\gamma(\nu_i))$, it is enough to show that $\delta^{(n/\gamma(i))}_{b,i} \neq 0$ for at most two $i$'s and $\delta^{(n/\gamma(\nu_i))}_{b,i} \neq 0$ for at most two $\nu_i$'s, since by assumption $d > 3$, i.e. $d \geq 5$. Observe that if $1 \leq i \leq d$ then $q \nmid i$ and hence $\frac{n}{\gamma(i)}$ is multiple of $q$. Moreover, if $1 \leq i, j \leq d$ with $i \neq j$ then $-q < i - j < i + j < 2q$. Therefore $i \not\sim_q j$ unless $j = q - i$. As $(i) \sim_{\frac{n}{p}} (\nu_i)$ and $q \mid \frac{n}{p}$ we have $(i) \sim_{\frac{n}{p}} (\nu_i)$, the lemma follows. \[\square\]

For a non-zero integer $m$ let $P(m)$ denote the number of prime divisors of $m$. We obtain an upper bound for $|C_b(\psi_d(u) - 1) - C_b(\psi_d(g_0) - 1)|$ in terms of $P(d)$.

**Lemma 4.3.** For every $b \in \mathbb{B}$ we have

$$|C_b(\psi_d(u) - 1) - C_b(\psi_d(g_0) - 1)| \leq 1 + 2^{P(d)+2}.$$  

Moreover if $\kappa_{\nu_i} = 1$ for every $1 \leq i \leq d$ then

$$|C_b(\psi_d(u) - 1) - C_b(\psi_d(g_0) - 1)| \leq 2^{P(d)+2}.$$  

**Proof.** Using (4.9), and ignoring the signs given by $\mu(\gamma(i))$ and $\mu(\gamma(\nu_i))$, it is enough to prove that

$$\sum_{i=1}^{d} \delta^{(n/\gamma(i))}_{b,i} \leq 2^{P(d)+1} \quad \text{and} \quad \sum_{i=1}^{d} \kappa_{\nu_i} \delta^{(n/\gamma(\nu_i))}_{b,i} \leq 1 + 2^{P(d)+1}.$$
Observe that $\kappa_{\nu_i} = 2$ for at most one $i$ by Lemma 4.2. Recall that $d' = \prod_{p \mid d} p$. Thus the lemma is a consequence of the following inequalities for every $e$ dividing $d'$:

\[
\left\{ 1 \leq i \leq d : \gcd(d, \gamma(i)) = e, \delta_{b, i}^{(n/\gamma(i))} = 1 \right\} \leq 2 \quad \text{and} \quad \left\{ 1 \leq i \leq d : \gcd(d, \gamma(\nu_i)) = e, \delta_{b, \nu_i}^{(n/\gamma(\nu_i))} = 1 \right\} \leq 2,
\]

since the number of divisors of $d'$ is $2^P(d)$ and if $\kappa_{\nu_i} = 2$ for some $\nu_i$ this provides an additional 1. We prove the second inequality, only using that $(\nu_i) \sim_d (i)$. This implies the first inequality by applying the second one to $u = g_0$.

For a fixed $e$ dividing $d'$ let $Y_e = \left\{ 1 \leq i \leq d : \gcd(d, \gamma(\nu_i)) = e, \delta_{b, \nu_i}^{(n/\gamma(\nu_i))} = 1 \right\}$. By changing the sign of some $\nu_i$'s, we may assume without loss of generality that if $\delta_{b, \nu_i}^{(n/\gamma(\nu_i))} = 1$ then $b \equiv \nu_i \mod \frac{n}{\gamma(\nu_i)}$. Thus, if $i \in Y_e$ then $b \equiv \nu_i \mod \frac{n}{\gamma(\nu_i)}$. We claim that if $i, j \in Y_e$ then $\nu_i \equiv \nu_j \mod d$. Indeed, let $p$ be prime divisor of $d$. If $n_p \neq d_p$ then $d_p \leq \left(\frac{n}{\gamma(\nu_i)}\right)_p$, so $\nu_i \equiv \nu_j \mod d_p$. If $p \mid e$ then $n_p = \left(\frac{n}{\gamma(\nu_i)}\right)_p$ and so also $\nu_i \equiv \nu_j \mod d_p$. Otherwise, i.e. if $n_p = d_p$ and $p \mid e$, then $p$ divides both $\gamma(\nu_i)$ and $\gamma(\nu_j)$ and $\nu_i \equiv \nu_j \mod \frac{n_p}{p}$. Therefore $\nu_i \equiv \nu_j \mod n_p$, by Lemma 4.3. As $(\nu_i) \sim_d (i)$ and there are at most two $i$'s with $1 \leq i \leq d$ representing the same class in $\Gamma_d$, we deduce that $|Y_e| \leq 2$, as desired. \[\square\]

We are ready to finish the proof of Theorem 1.1. Recall that we are arguing by contradiction and $n$, and hence also $d$, is odd.

By (4.4) and Lemma 4.3 we have $d \leq 1 + 2P(d)+2$ and this has strong consequences on the possible values of $d$. Indeed if $P(d) \geq 3$ then

\[
1 + 2P(d)+2 \geq d \geq 3 \cdot 5 \cdot 7 \cdot 2P(d)^3 > (105 - 2^5) + 2P(d)+2 = 73 + 2P(d)+2,
\]

a contradiction. Thus, if $P(d) = 2$ then $d = 15$ and if $P(d) = 1$ then $d \in \{3, 5, 7, 9\}$.

However, if $d = 9$ then $|C_{b_0}(\psi_9(u) - 1) - C_{b_0}(\psi_9(g_0) - 1)| = 9$ by Lemma 4.3 and hence $\kappa_{\nu_i} = 2$ for one $1 \leq i \leq d$. This implies, by Lemma 4.2, that $n = 27$ contradicting the assumptions that $n$ is not a prime power. Therefore $d \in \{3, 5, 7, 15\}$. We deal with these cases separately using (4.8) and (4.9). Observe that if $p$ is a prime bigger than $d$ then $p \mid \frac{n}{\gamma(\nu_i)}$ for every $1 \leq i \leq d$ and so also $p \mid \frac{n}{\gamma(\nu_i)}$, since $(i) \sim_p (\nu_i)$.

Assume that $d = 3$. Combining Lemma 4.2 with the assumptions that $n$ is not a prime power, we deduce that $\kappa_{\nu_i} = 1$ for every $1 \leq i \leq 3$. Suppose that there is a prime $p \mid n$ with $p \geq 7$. Then $p \mid \frac{n}{\gamma(\nu_i)}$ and $p \mid \frac{n}{\gamma(\nu_i)}$ for every $1 \leq i \leq 3$. Thus

\[
\left\{ 1 \leq i \leq 3 : \delta_{b, i}^{(n/\gamma(i))} = 1 \right\} \leq 1 \quad \text{and} \quad \left\{ 1 \leq i \leq 3 : \delta_{b, \nu_i}^{(n/\gamma(\nu_i))} = 1 \right\} \leq 1 \quad \text{for every} \quad b \in \mathbb{B}
\]

which implies $|C_{b_0}(\psi_3(u) - 1) - C_{b_0}(\psi_3(g_0) - 1)| \leq 2$, contradicting (4.7). So $n' = 15$.

Moreover, $n_3 = 3$ because otherwise 3 $\mid \frac{n}{\gamma(\nu_i)}$ and $3 \mid \frac{n}{\gamma(\nu_i)}$ and so $15 \mid \frac{n}{\gamma(\nu_i)}$ and $15 \mid \frac{n}{\gamma(\nu_i)}$ for every $1 \leq i \leq 3$. Hence $|C_{b_0}(\psi_3(u) - 1)|$ and $|C_{b_0}(\psi_3(u) - 1)|$ are both at most 1, in contradiction with (4.7). If $5^3 \mid n$, then 25 $\mid \frac{n}{\gamma(\nu_i)}$ and 25 $\mid \frac{n}{\gamma(\nu_i)}$ for every $1 \leq i \leq 3$, which implies $|C_{b_0}(\psi_3(g_0) - 1)| \leq 1$ and $|C_{b_0}(\psi_3(u) - 1)| \leq 1$, again a contradiction. Therefore $n \in \{15, 75\}$. Since $(i) \sim_3 (\nu_i)$, we may assume that $3 \mid \nu_3$ and $3 \nmid \nu_1$ for $i = 1, 2$. 

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Suppose that $n = 15$. Then, as $(i) \sim_5 (\nu_1)$, we have $\gamma(1) = \gamma(2) = \gamma(\nu_1) = \gamma(\nu_2) = 1$ and $\gamma(3) = \gamma(\nu_3) = 3$. So

\[ C_b(\psi_3(g_0) - 1) = \delta_{b,1}^{15} + \delta_{b,2}^{15} - \delta_{b,3}^{5} \quad \text{and} \quad C_b(\psi_3(u) - 1) = \delta_{b,\nu_1}^{15} + \delta_{b,\nu_2}^{15} - \delta_{b,\nu_3}^{5} \quad \text{for every} \quad b \in \mathbb{B}, \]

implying

\[ C_b(\psi_3(u) - 1) - C_b(\psi_3(g_0) - 1) = \delta_{b,\nu_1}^{15} + \delta_{b,\nu_2}^{15} - \delta_{b,\nu_3}^{5} - \delta_{b,1}^{15} - \delta_{b,2}^{15} + \delta_{b,3}^{5}. \]

Since $1 \sim_5 2$, we must have $C_{b_0}(\psi_3(u) - 1) - C_{b_0}(\psi_3(g_0) - 1) = 3$ and $\nu_3 \sim_5 1$ while $\nu_1 \sim_5 \nu_2 \sim_5 2$.

Then $C_1(\psi_3(u) - 1) - C_1(\psi_3(g_0) - 1) = -2$, contradicting (4.6).

Suppose that $n = 75$. Then $\gamma(1) = \gamma(2) = 5$, $\gamma(3) = 3$ and

\[ C_b(\psi_3(g_0) - 1) = -\delta_{b,1}^{15} - \delta_{b,2}^{15} - \delta_{b,3}^{25} \quad \text{for every} \quad b \in \mathbb{B}. \]

Suppose $\nu_3 \sim_3 25$. Then

\[ C_b(\psi_3(u) - 1) = -\delta_{b,1}^{15} - \delta_{b,2}^{15} - \delta_{b,3}^{25} \quad \text{for every} \quad b \in \mathbb{B}. \]

As $\delta_{b,\nu_3}^{25} = \delta_{b,3}^{25}$, we have $|C_{b_0}(\psi_3(u) - 1) - C_{b_0}(\psi_3(g_0) - 1)| \leq 2$, contradicting (4.7). Thus $\nu_3 \not\sim_3 25$ and we may assume $\nu_1 \sim_3 3$. If $\nu_3 \sim_3 2$ then

\[ C_b(\psi_3(u) - 1) = \delta_{b,\nu_1}^{15} - \delta_{b,\nu_2}^{15} + \delta_{b,\nu_3}^{5}. \]

However $C_{13}(\psi_3(u) - 1) - C_{13}(\psi_3(g_0) - 1) = 2$, contradicting (4.6). So $\nu_3 \sim_3 1$ and arguing as above we obtain $C_{14}(\psi_3(u) - 1) - C_{14}(\psi_3(g_0) - 1) \in \{1, 2\}$, again a contradiction with (4.6).

Assume that $d = 5$. By Lemma 4.2 (2) and the assumptions on $n$, we obtain $n' = 15$. As $(i) \sim_5 (\nu_1)$, we may assume that $5 \mid \nu_5$ and $5 \nmid \nu_i$ for every $1 \leq i \leq 4$. Suppose that $n = 15$. In this case

\[ C_b(\psi_5(g_0) - 1) = \delta_{b,1}^{15} + \delta_{b,2}^{15} - \delta_{b,3}^{5} + \delta_{b,4}^{15} - \delta_{b,5}^{3} \quad \text{for every} \quad b \in \mathbb{B}. \]

If $3 \mid \nu_5$ and $3 \nmid \nu_i$ for every $1 \leq i \leq 4$, then

\[ C_b(\psi_5(u) - 1) = \delta_{b,\nu_1}^{15} + \delta_{b,\nu_2}^{15} + \delta_{b,\nu_3}^{15} + \delta_{b,\nu_4}^{15} + 2 \quad \text{for every} \quad b \in \mathbb{B} \]

and hence

\[ C_1(\psi_5(u) - 1) - C_1(\psi_5(g_0) - 1) = 2 + \delta_{1,\nu_1}^{15} + \delta_{1,\nu_2}^{15} + \delta_{1,\nu_3}^{15} + \delta_{1,\nu_4}^{15} \leq 4, \]

contradicting (4.6). Therefore, as $(i) \sim_3 (\nu_1)$, we may assume that $3 \mid \nu_1$ and $3 \nmid \nu_i$ for every $2 \leq i \leq 5$. This implies

\[ C_b(\psi_5(u) - 1) = -\delta_{b,\nu_1}^{5} + \delta_{b,\nu_2}^{15} + \delta_{b,\nu_3}^{15} + \delta_{b,\nu_4}^{15} - \delta_{b,\nu_5}^{3} \quad \text{for every} \quad b \in \mathbb{B}. \]

As both $|C_{b_0}(\psi_5(u) - 1)|$ and $|C_{b_0}(\psi_5(g_0) - 1)|$ are at most 2, we obtain a contradiction with (4.7). Therefore $n \neq 15$ and $\kappa_{\nu_i} = 1$ for every $1 \leq i \leq 5$ by Lemma 4.2 (1).

If $25 \mid n$ or $27 \mid n$ then it is easy to see that $|C_{b_0}(\psi_5(u) - 1)| \leq 2$ and $|C_{b_0}(\psi_5(g_0) - 1)| \leq 2$, contradicting (4.7). Thus $n = 45$. In this case we have

\[ C_b(\psi_5(g_0) - 1) = -\delta_{b,1}^{15} + \delta_{b,2}^{15} + \delta_{b,3}^{15} + \delta_{b,4}^{45} - \delta_{b,5}^{3} \quad \text{for every} \quad b \in \mathbb{B}. \]

If $\nu_5 \sim_9 1$ then

\[ C_b(\psi_5(u) - 1) = \delta_{b,\nu_1}^{45} + \delta_{b,\nu_2}^{45} + \delta_{b,\nu_3}^{45} + \delta_{b,\nu_4}^{45} + \delta_{b,\nu_5}^{3} \quad \text{for every} \quad b \in \mathbb{B}. \]
As \((i) \sim_{15} (\nu_i)\), we obtain \(|C_{b_0}(\psi_5(u) - 1)| \leq 2\) and \(|C_{b_0}(\psi_5(g_0) - 1)| \leq 2\) contradicting (4.7). If \(\nu_5 \not\sim 9\), then we may assume that \(\nu_1 \sim_{9} 1\). Hence

\[ C_b(\psi_5(u) - 1) = -\delta_{b,1}^{15} + \delta_{b,2}^{45} + \delta_{b,3}^{45} + \delta_{b,4}^{45} - \delta_{b,5}^{9}, \]

for every \(b \in \mathbb{B}\).

Again as \((i) \sim_{15} (\nu_i)\), we have both \(|C_{b_0}(\psi_5(u) - 1)|\) and \(|C_{b_0}(\psi_5(g_0) - 1)|\) at most 2, which yields a contradiction.

Assume that \(d = 7\). As \((i) \sim_{7} (\nu_i)\), we may assume that \(7 \mid \nu_i\) and \(7 \not\mid \nu_i\) for every \(1 \leq i \leq 6\). Thus \(7 \mid \frac{\mathfrak{m}_{\nu_i}}{\mathfrak{m}_{\nu_i}}\) and \(7 \not\mid \frac{\mathfrak{m}_{\nu_i}}{\mathfrak{m}_{\nu_i}}\) for every \(1 \leq i \leq 6\). Hence \(|C_{b_0}(\psi_7(g_0) - 1)| \leq 3\). Moreover, if \(\kappa_{\nu_i} \neq 2\) the we also have \(|C_{b_0}(\psi_7(u) - 1)| \leq 3\) yielding a contradiction with (4.7). Therefore \(\kappa_{\nu_i} = 2\) (i.e. \(n \mid \nu_7\)) and by Lemma 4.12 and the assumptions on \(n\) we deduce that either \(n = 21\) or \(n = 35\).

Suppose that \(n = 21\). As \((i) \sim_3 (\nu_i)\) we may assume that \(3 \mid \nu_3\) and \(3 \not\mid \nu_i\) for every \(i \in \{1, 2, 4, 5, 6\}\). This implies for every \(b \in \mathbb{B}\) that

\[ C_b(\psi_7(u) - 1) = \delta_{b,1}^{21}, \delta_{b,2}^{21} - \delta_{b,3}^{7}, \delta_{b,4}^{21}, \delta_{b,5}^{21} + \delta_{b,6}^{21} + 2 \]

and

\[ C_b(\psi_7(g_0) - 1) = \delta_{b,1}^{21} + \delta_{b,2}^{21} - \delta_{b,3}^{7} + \delta_{b,4}^{21} + \delta_{b,5}^{21} - \delta_{b,6}^{7} - \delta_{b,7}^{3} \]

Hence, as \((i) \sim_7 (\nu_i)\), we obtain \(|C_{b_0}(\psi_7(g_0) - 1)| \leq 2\) and \(|C_{b_0}(\psi_7(u) - 1)| \leq 4\) contradicting (4.7).

Suppose that \(n = 35\). As \((i) \sim_5 (\nu_i)\) and \((i) \sim_7 (\nu_i)\), we have for every \(b \in \mathbb{B}\) that

\[ C_b(\psi_7(u) - 1) = \delta_{b,1}^{35} + \delta_{b,2}^{35} + \delta_{b,3}^{35} + \delta_{b,4}^{35} + \delta_{b,5}^{35} + \delta_{b,6}^{35} + 2 \]

and

\[ C_b(\psi_7(g_0) - 1) = \delta_{b,1}^{35} + \delta_{b,2}^{35} + \delta_{b,3}^{35} + \delta_{b,4}^{35} - \delta_{b,5}^{7} + \delta_{b,6}^{35} - \delta_{b,7}^{5} \]

Hence, again \(|C_{b_0}(\psi_7(g_0) - 1)| \leq 2\) and \(|C_{b_0}(\psi_7(u) - 1)| \leq 4\), yielding a contradiction with (4.7).

Finally assume that \(d = 15\). Suppose that \(n = 45\). In this case we have for every \(b \in \mathbb{B}\) that

\[ C_b(\psi_{15}(g_0) - 1) = -\delta_{b,1}^{15} + \delta_{b,2}^{45} + \delta_{b,3}^{45} + \delta_{b,4}^{45} - \delta_{b,5}^{9} + \delta_{b,6}^{15} + \delta_{b,7}^{15} - \delta_{b,8}^{15} + \delta_{b,9}^{15} + \delta_{b,10}^{9} + \delta_{b,11}^{45} + \delta_{b,12}^{45} + \delta_{b,13}^{45} + \delta_{b,14}^{45} - \delta_{b,15}^{9}, \]

which implies that \(|C_{b_0}(\psi_{15}(g_0) - 1)| \leq 4\). Since \((i) \sim_5 (\nu_i)\), we deduce that \(|C_{b_0}(\psi_{15}(u) - 1)| \leq 10\), since at most ten of the \(\mu(\gamma(\nu_i))\) are equal. This yields a contradiction with (4.7). Therefore \(n \neq 45\) and \(\kappa_{\nu_i} = 1\) for every \(1 \leq i \leq 15\) by Lemma 4.12. If there is a prime \(p \mid n\) with \(p \geq 7\) then it is easy to see that \(|C_{b_0}(\psi_{15}(g_0) - 1)| \leq 7\) and \(|C_{b_0}(\psi_{15}(u) - 1)| \leq 7\), in contradiction with (4.7).

Thus \(n' = 15\). If \(25 \mid n\) or \(27 \mid n\) then \(|C_{b_0}(\psi_{15}(g_0) - 1)| \leq 6\) and \(|C_{b_0}(\psi_{15}(u) - 1)| \leq 6\), again a contradiction. As 15 is a proper divisor of \(n\), this implies \(n = 45\) yielding the final contradiction.

References

[BC17] A. Bächle and M. Caicedo. On the prime graph question for almost simple groups with an alternating socle. *Internat. J. Algebra Comput.*, 27(3):333–347, 2017.

[BKL08] V. A. Bovdi, A. B. Konovalov, and S. Linton. Torsion units in integral group ring of the Mathieu simple group \(M_{22}\). *LMS J. Comput. Math.*, 11:28–39, 2008.

[BM17] A. Bächle and L. Margolis. Rational Conjugacy of Torsion Units in Integral Group Rings of Non-Solvable Groups. *Proc. Edimb. Math. Soc.* (2), 60(4):813–830, 2017.

[BM18] A. Bächle and L. Margolis. On the Prime Graph Question for Integral Group Rings of 4-Primary Groups II. *Algebr. Representat. Theor.*, 2018. [https://doi.org/10.1007/s10468-018-9776-6]
[Cala] C. K. Caldwell. Heuristics: Deriving the Wagstaff Mersenne Conjecture. http://primes.utm.edu/mersenne/heuristic.html. Visited April 2018.

[Calb] C. K. Caldwell. Mersenne primes: History, theorems and lists. http://primes.utm.edu/mersenne/. Visited April 2018.

[CMdR13] M. Caicedo, L. Margolis, and Á. del Río. Zassenhaus conjecture for cyclic-by-abelian groups. J. Lond. Math. Soc. (2), 88(1):65–78, 2013.

[dRS17] Á. del Río and M. Serrano. On the torsion units of the integral group ring of finite projective special linear groups. Communications in Algebra, 45(2):5073–5087, 2017.

[EM17] F. Eisele and L. Margolis. A counterexample to the First Zassenhaus Conjecture. preprint, arxiv.org/abs/arXiv:1710.08780, page 32 pages, 2017.

[Her06] M. Hertweck. On the torsion units of some integral group rings. Algebra Colloq., 13(2):329–348, 2006.

[Her07] M. Hertweck. Partial augmentations and Brauer character values of torsion units in group rings. 2007. arXiv:math/0612429 [math.RA], 16 pages.

[Her08] M. Hertweck. Zassenhaus conjecture for $A_6$. Proc. Indian Acad. Sci. Math. Sci., 118(2):189–195, 2008.

[Hup67] B. Huppert. Endliche Gruppen. I. Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin-New York, 1967.

[JdR16] E. Jespers and Á. del Río. Group ring groups. Volume 1: Orders and generic constructions of units. Berlin: De Gruyter, 2016.

[LP89] I. S. Luthar and I. B. S. Passi. Zassenhaus conjecture for $A_5$. Proc. Indian Acad. Sci. Math. Sci., 99(1):1–5, 1989.

[Mar16] L. Margolis. A Sylow theorem for the integral group ring of PSL(2, q). J. Algebra, 445:295–306, 2016.

[MRSW87] Z. Marciniak, J. Ritter, S. K. Sehgal, and A. Weiss. Torsion units in integral group rings of some metabelian groups II. J. Number Theory, 25(3):340–352, 1987.

[Pom81] C. Pomerance. Recent developments in primality testing. Math. Intelligencer, 3(3):97–105, 1980/81.

[Sal13] M. Salim. The prime graph conjecture for integral group rings of some alternating groups. Int. J. Group Theory, 2(1):175–185, 2013.

[Wag83] S. Wagstaff. Divisors of Mersenne numbers. Math. Comp., 40(161):385–397, 1983.

[Wei91] A. Weiss. Torsion units in integral group rings. J. Reine Angew. Math., 415:175–187, 1991.

[Zas74] H. J. Zassenhaus. On the torsion units of finite group rings. In Studies in mathematics (in honor of A. Almeida Costa) (Portuguese), pages 119–126. Instituto de Alta Cultura, Lisbon, 1974.

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