HITTING-TIME LIMITS FOR SOME EXCEPTIONAL RARE EVENTS OF ERGODIC MAPS

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Abstract. We discuss limit distributions for hitting-time functions of certain “exceptional” families of asymptotically rare events for ergodic probability preserving transformations. The abstract core is an inducing argument. The latter applies, for example, to shrinking intervals around periodic points (both uniformly expanding and neutral) of certain finite measure preserving interval maps. In particular, we give a complete answer to a question raised in [FFTV].

1. Introduction

Hitting- and return-time statistics for asymptotically rare events in ergodic dynamical systems have been the subject of intense research over the last years. For a wide variety of probability preserving systems with some hyperbolicity it has been shown that for natural families of rare events, like cylinders or $\varepsilon$-balls shrinking to a given distinguished point $x^*$, convergence (after normalization by the measure of the sets) to a standard exponential random variable $E$ (with $\Pr[E > t] = e^{-t}$ for $t \geq 0$) is typical in that it holds for almost every point $x^*$. Nonetheless, there are often exceptional points at which a different asymptotic behaviour is observed. It is not hard to understand that this should be so in the case of periodic points, when a definite proportion of the set returns after a fixed number of steps, thus giving rise to a point mass at the origin for the limit of scaled return-times. (For some basic classes of systems dichotomy results have been established which confirm that there are no other exceptional points.) There has also been some interest in neutral repellers, which lead to a trivial limit under the usual normalization, but may in fact give rise to a nice limit law when a different scale is used.

The purpose of the present note is to communicate an abstract inducing argument which can be used to clarify the asymptotics of such exceptional rare events once it is known that the standard exponential limit arises in certain situations. It can lead to straightforward and quick proofs. This will be illustrated in the setup of simple prototypical piecewise invertible expanding interval maps. We focus on the basic case of fixed points. Extending the arguments to periodic points only requires routine arguments, and hardly gives new insights.

First, we show that the (well-known) exceptional hitting-time limit $\theta^{-1}E$ with expectation $\theta^{-1} > 1$ at a repelling hyperbolic fixed point of a well-behaved map can be obtained using this approach. Next, we demonstrate that the term “well-behaved” in the previous sentence is there for a reason. We construct a map which looks nice enough (a uniformly expanding piecewise affine Markov map) but
nonetheless admits a hyperbolic fixed point at which the limit variable is a standard exponential $E$ and not the exceptional $\theta^{-1}E$ suggested by the first scenario.

Finally, we turn to the main application of our abstract inducing principle and consider probability preserving maps with neutral fixed points. Here we answer a question raised in [FFTV] by clarifying the asymptotic hitting-time behaviour of neighbourhoods of an indifferent fixed point (with rather general local behaviour).

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2. Preparations

General setup. Throughout the paper, all measures are understood to be finite. We study (possibly non-invertible) measure preserving transformations $T$ on $(X, \mathcal{A}, \mu)$, i.e. measurable maps $T : X \to X$ for which $\mu \circ T^{-1} = \mu$. The transformation $T$ will also be ergodic. For such a system $(X, \mathcal{A}, \mu, T)$, and any $Y \in \mathcal{A}$ with $\mu(Y) > 0$, define the first hitting time function of $Y$, $\varphi_Y : X \to \mathbb{N} \cup \{\infty\}$ by $\varphi_Y(x) := \min\{n \geq 1 : T^n x \in Y\}$, $x \in X$, which is finite a.e. by ergodicity and the Poincaré recurrence theorem. Set $T_Y x := T^{\varphi(x)} x$, $x \in X$. When restricted to $Y$, $\varphi_Y$ is called the first return time of $Y$. If we let $Y \cap \mathcal{A} := \{Y \cap A : A \in \mathcal{A}\}$ denote the trace of $\mathcal{A}$ in $Y$, then $\mu |_{Y \cap \mathcal{A}}$ is invariant under the first return map $T_Y$ restricted to $Y$. It is natural to regard $\varphi_Y$ as a random variable on the probability space $(X, \mathcal{A}, \mu_Y)$, where $\mu_Y(E) := \mu(Y)^{-1} \mu(Y \cap E)$. By Kac' formula, it has expectation $\int \varphi_Y d\mu_Y = \mu(Y)^{-1}$.

In this setup, a sequence $(E_k)_{k \geq 1}$ in $\mathcal{A}$ with $\mu(E_k) > 0$ and $\mu(E_k) \to 0$ will be referred to as a sequence of asymptotically rare events. Asking for an asymptotic hitting-time distribution or hitting-time statistics (HTS) means to look for normalizing constants $\gamma_k > 0$ and a nontrivial random variable $R$ taking values in $[0, \infty]$ such that

\begin{equation}
\gamma_k \cdot \varphi_{E_k} \leq t \implies \Pr[R \leq t] \text{ as } k \to \infty.
\end{equation}

(Here, of course, the symbol $\implies$ means that convergence takes place at continuity points $t$ of the respective limit distribution function). By Kac’ formula, a canonical candidate for $\gamma_k$ is given by $\mu(E_k)$.

Change of measure. It is a fact, both interesting by itself and useful as a technical tool, that the convergence (2.1) automatically carries over to all probabilities $\nu \ll \mu$. Given a sequence $(R_k)_{k \geq 1}$ of measurable functions and a measure $\nu$ on $(X, \mathcal{A})$, we write

\begin{equation}
R_k \xrightarrow{\nu} R
\end{equation}

to indicate convergence in law of the $R_k$, viewed as random variables on the on the probability space $(X, \mathcal{A}, \nu)$, to a variable $R$. For instance, (2.1) can then be expressed as $\gamma_k \cdot \varphi_{E_k} \xrightarrow{\mu} R$. Corollary 5 of [Z2] contains the following
Theorem 2.1 (Strong distributional convergence of hitting times; [22]).
Let \((X, \mathcal{A}, \mu, T)\) be an ergodic probability-preserving system, \((E_k)_{k \geq 1}\) a sequence of asymptotically rare events, \((\gamma_k)\) a sequence in \((0, \infty)\) with \(\gamma_k \to 0\), and \(R\) any random variable with values in \([0, \infty]\). Then
\[
\gamma_k \cdot \varphi_{E_k} \overset{\mu}{\to} R \quad \text{as } k \to \infty
\]
holds for one probability measure \(\nu \ll \mu\) iff it holds for all probabilities \(\nu \ll \mu\).

We record another basic observation regarding changes of measure. Let \(\overset{\nu}{\to}\) denote convergence in measure with respect to \(\nu\).

Proposition 2.1 (Characterizing asymptotically rare sequences). Suppose that \((X, \mathcal{A}, \mu, T)\) is an ergodic probability-preserving system and \((E_k)_{k \geq 1}\) a sequence in \(\mathcal{A}\). Then the following are equivalent:

(a) \(\mu(E_k) \to 0\) as \(k \to \infty\),
(b) \(\nu(E_k) \to 0\) as \(k \to \infty\) for all probabilities \(\nu \ll \mu\),
(c) \(\varphi_{E_k} \overset{\nu}{\to} \infty\) as \(k \to \infty\),
(d) \(\varphi_{E_k} \overset{\nu}{\to} \infty\) as \(k \to \infty\) for all probabilities \(\nu \ll \mu\),
(e) \(\varphi_{E_k} \overset{\nu}{\to} \infty\) as \(k \to \infty\) for some probability \(\nu \ll \mu\).

Proof. (i) Equivalence of (a) and (b) is immediate from the “continuity” characterization of absolute continuity, and so is equivalence of (c) and (d).

To check that (b) entails (d), fix \(\nu \ll \mu\) and some \(N \geq 1\). Then \(\nu(\varphi_{E_k} \leq N) = \nu(\bigcup_{n=1}^N T^{-n} E_k) \leq \sum_{n=1}^N \nu(T^{-n} E_k)\). But for every \(n \geq 1\), we have \(\nu(T^{-n} E_k) \to 0\) as \(k \to \infty\) by (b) because \(\nu \ll \mu\). Hence (d) follows as \(\nu(\varphi_{E_k} \leq N) \to 0\).

(ii) To establish the more interesting fact that (e) implies (b), take \(\nu\) as in (e) and choose another probability \(\tilde{\nu} \ll \mu\). To prove \(\tilde{\nu}(E_k) \to 0\) assume the contrary, meaning that there are \(\delta > 0\) and \(k_j \to \infty\) such that \(\tilde{\nu}(E_{k_j}) \geq \delta\) for all \(j \geq 1\). We show that this contradicts \(\varphi_{E_k} \overset{\nu}{\to} \infty\).

Set \(u := dv/d\mu\) and \(\tilde{u} := d\tilde{\nu}/d\mu\). We first consider the case where \(\|\tilde{u}\|_{\infty} < \infty\). Let \(\tilde{T}\) denote the transfer operator of \(T\) with respect to \(\mu\), so that \(\int (f \circ T) g d\mu = \int f \tilde{T} g d\mu\) whenever \(f \in L(\mu)\) and \(g \in L(\mu)\). Since \(T\) is ergodic and recurrent, we have \(\sum_{n=1}^\infty \tilde{T}^n u = \infty\) \(\mu\)-a.e. on \(X\). Therefore we can choose (and fix) some \(N \geq 1\) such that \(F := \{\sum_{n=1}^N \tilde{T}^n u \geq 1\}\) satisfies \(\tilde{\nu}(F^c) < \delta/2\). Then,
\[
\tilde{\nu}(E_{k_j} \cap F) \geq \delta/2 \quad \text{for } j \geq 1.
\]
Also,
\[
\tilde{\nu}(E_{k_j} \cap F) = \int 1_{E_{k_j}} 1_F \tilde{u} d\mu
\leq \|\tilde{u}\|_{\infty} \int 1_{E_{k_j}} \left(\sum_{n=1}^N \tilde{T}^n u\right) d\mu
= \|\tilde{u}\|_{\infty} \sum_{n=1}^N \nu(T^{-n} E_{k_j}).
\]
Combining this with \([23]\), we see that for every \(j \geq 1\) there is some \(n_j \in \{1, \ldots, N\}\) such that \(\nu(T^{-n_j} E_{k_j}) \geq \delta' := \delta/(2N \|\tilde{u}\|_{\infty}) > 0\). But \(\varphi_{E} \leq N\) on \(T^{-n} E\) if \(1 \leq n \leq N\). Hence we conclude that \(\nu(\varphi_{E_{k_j}} \leq N) \geq \delta'\) for \(j \geq 1\).
Finally, in the case of unbounded \( \nu \), we still have \( \nu \ll \nu \) for some probability \( \nu \ll \nu \) with a bounded density \( \nu := d\nu/d\mu \). By the above, \( \nu(E_k) \to 0 \) which easily implies \( \nu(E_k) \to 0 \) by absolute continuity.

\[\nu \ll \nu \]

Remark 2.1 (A null-preserving \( \sigma \)-finite version of the proposition). The proof above is formulated in such a way that it actually shows the following: Let \((X, \mathcal{A}, \mu)\) be a \( \sigma \)-finite measure space and \( T \) a null-preserving map (measurable with \( \mu \circ T^{-1} \ll \mu \)) which is conservative (that is recurrent, \( A \subseteq \bigcup_{n \geq 1} T^{-n}A \) (mod \( \mu \)) for all \( A \in \mathcal{A} \)) and ergodic (\( A = T^{-1}A \in \mathcal{A} \) implies \( 0 \in \{ \mu(A), \mu(A^c) \} \)). Then, for any sequence \((E_k)_{k \geq 1} \in \mathcal{A} \), statements (b), (d) and (e) above are equivalent.

This extension is of interest in situations with an infinite invariant measure \( \mu \), where assertions (a) and (c) are no longer about probabilities. See [Z3] for asymptotic hitting-time distributions of certain sequences \((E_k)\) with \( \mu(E_k) = \infty \) for all \( k \), but still satisfying (b), (d) and (e).

**Hitting times and inducing.** As with various other assertions, proving a statement like (2.1) is often facilitated by passing to a suitable (nicer) induced map \( T_Y \), and studying the same question for this new system. To this end, let \( \varphi^Y_E : Y \to \mathbb{N} \) denote the hitting time of \( E \in \mathcal{A} \cap Y \) under the first-return map \( T_Y \), that is,

\[\varphi^Y_E(x) := \inf\{j \geq 1 : T^j_Y(x) \in E\}, \quad x \in Y.\]

Then the following inducing principle for hitting-time limits shows that (in the standard case \( \gamma_k = \mu(E_k) \)) it suffices to analyse the distributions of the \( \varphi^Y_E \) on \( Y \).

**Theorem 2.2** (Hitting-time statistics via inducing; [HWZ]). Let \((X, \mathcal{A}, \mu, T)\) be an ergodic probability-preserving system, and \( Y \in \mathcal{A}, \mu(Y) > 0 \). Assume that \((E_k)_{k \geq 1} \) is a sequence of asymptotically rare events in \( \mathcal{A} \cap Y \), and that \( R \) is any random variable with values in \([0, \infty]\). Then

\[\mu_Y(E_k) \varphi^Y_{E_k} \overset{\text{d}}{\to} R \quad \text{as } k \to \infty\]

iff

\[\mu(E_k) \varphi^Y_{E_k} \overset{\mu}{\to} R \quad \text{as } k \to \infty.\]

**3. Inducing hitting-time statistics - revisited**

The core of this paper is a very useful extension of Theorem 2.2 quoted above. It allows us to sometimes replace a target set \( E \), not necessarily contained in \( Y \), by a more convenient set \( E' \) inside \( Y \). We shall say that points of \( Y \) can only reach \( E \) via \( E' \) if for every \( n \geq 0 \) and a.e. \( x \in Y \),

\[T^n x \in E \quad \text{implies that} \quad T^j x \in E' \quad \text{for some } j \in \{0, \ldots, n\},\]

that is, orbits starting in \( Y \) cannot visit the set \( E \) before \( E' \) is visited\(^1\).

\(^1\)In contrast to the definition of the first hitting-time, we also take visits at time zero into account here. We do so in order to obtain a flexible condition which can easily be applied to certain concrete situations.
Example 3.1. a) Given an ergodic probability-preserving system \((X, \mathcal{A}, \mu, T)\), let \(Y \in \mathcal{A}\), \(\mu(Y) > 0\), and consider \(E := Y^c \cap \{\varphi_Y \geq i\}\) for some \(i \geq 1\). It is then immediate that points of \(Y\) can only reach \(E\) via \(E' := Y \cap T^{-1}E = Y \cap \{\varphi_Y > i\}\).

b) More generally, if \(E \subseteq Y^c\) satisfies \(Y^c \cap T^{-1}E \subseteq E\) (mod \(\mu\)), then points of \(Y\) can only reach \(E\) via \(E' := Y \cap T^{-1}E\).

In case \(E' \subseteq E\) it is clear that, starting from \(Y\), the first visits to \(E\) and \(E'\), respectively, must then coincide. We can cover other interesting scenarios if, more generally, we only require that these times do not differ too much. Then the hitting time distributions of \(E\) and \(E'\) will be comparable when the sets are small, as made precise in the following result.

Theorem 3.1 (Hitting-time statistics via inducing; extended version). Let \((X, \mathcal{A}, \mu, T)\) be an ergodic probability-preserving system, and \((E_k)_{k \geq 1}\) a sequence of asymptotically rare events. Suppose that \(Y \in \mathcal{A}\) is a set of positive measure, \((E'_k)_{k \geq 1}\) a sequence in \(\mathcal{A} \cap Y\) such that, for every \(k \geq 1\), points of \(Y\) can only reach \(E_k\) via \(E'_k\), and let \(R\) be any random variable with values in \([0, \infty]\).

a) If
\[
\mu(E'_k)(\varphi_{E_k} - \varphi_{E'_k}) \xrightarrow{\mu} 0 \quad \text{as } k \to \infty,
\]
then \((E'_k)\) is asymptotically rare, and
\[
\mu_Y(E'_k) \varphi_{E'_k} \xrightarrow{\mu_Y} R \quad \text{as } k \to \infty,
\]
holds iff
\[
\mu(E'_k) \varphi_{E_k} \xrightarrow{\mu} R \quad \text{as } k \to \infty.
\]

b) If there exists some constant \(M \geq 0\) such that \(E'_k \subseteq \bigcup_{m=0}^{M} T^{-m}E_k\) for \(k \geq 1\), then \((E'_k)\) satisfies assumption \((3.2)\) of a).

This result contains Theorem 2.2 (Take \(E_k = E'_k \subseteq Y\), and \(M = 0\) in part b.).

Remark 3.1 (The normalizing constants). Note that the normalizing factor on the left-hand side of \((3.4)\) really is \(\mu(E'_k)\), and not \(\mu(E_k)\). This is in fact one main point of the result. The relation between \(\mu(E'_k)\) and \(\mu(E_k)\) determines what really happens to \(\varphi_{E_k}\). Let \(\theta := \lim_{k \to \infty} \mu(E'_k)/\mu(E_k)\) in case this limit exists.

a) If \(\theta > 0\), then \((3.4)\) is equivalent to
\[
\mu(E_k) \varphi_{E_k} \xrightarrow{\mu} \frac{1}{\theta} R \quad \text{as } k \to \infty.
\]
This is what typically happens at repelling periodic points, see Theorem 4.1 below.

b) There are also interesting situations in which \(\theta = 0\). In this case the theorem identifies a possibly non-trivial limit variable \(R\) for the hitting-times \(\varphi_{E_k}\) on a scale \(\mu(E'_k)\) essentially different from the canonical scale \(\mu(E_k)\), thus identifying the “correct scale” for these variables. Indeed, this is what happens in the case of neighbourhoods \(E_k\) of indifferent fixed points for probability-preserving intermittent maps, see Theorem 5.1 below.

c) For a sequence of pairs \((E_k, E'_k)\) which satisfy \(E'_k = Y \cap T^{-1}E_k\) (as in Example 3.1 b), we have \(\theta = \lim_{k \to \infty} (1 - \mu(Y^c \cap T^{-1}E_k)/\mu(E_k))\).
Proof of Theorem 3.1. (i) We first show that \((E_k')\) is asymptotically rare. Assume otherwise, then there are \(\delta > 0\) and \(k_j \to \infty\) such that \(\mu(E_k') \geq \delta\) for \(j \geq 1\). Since \(|\varphi_{E_k} - \varphi_{E_k'}| \geq 1\) on \(\{\varphi_{E_k} \neq \varphi_{E_k'}\}\), \((3.2)\) ensures that
\[
\mu_Y(\varphi_{E_k} \neq \varphi_{E_k'}) \to 0 \quad \text{as} \quad j \to \infty.
\]
By assumption \((E_k)\) is asymptotically rare and Proposition 2.1 gives \(\varphi_{E_k} \overset{\mu_Y}{\to} \infty\). Due to \((3.6)\) we then get \(\varphi_{E_k'} \overset{\mu_Y}{\to} \infty\) as well. Hence \(\mu(E_k') \to 0\) by another application of the proposition.

(ii) By Theorem 2.2, the convergence in \((3.3)\) is equivalent to \(\mu(E_k') \varphi_{E_k'} \overset{\mu}{\to} \mathbb{R}\). Due to Theorem 2.1 this, in turn, is equivalent to
\[
\mu(E_k') \varphi_{E_k'} \overset{\mu_Y}{\to} \mathbb{R} \quad \text{as} \quad k \to \infty.
\]
As a consequence of \((3.2)\), we see that \((3.7)\) is equivalent to \(\mu(E_k') \varphi_{E_k'} \overset{\mu_Y}{\to} \mathbb{R}\), and in view of Theorem 2.1 the latter is indeed the same as \((3.3)\).

(iii) As \(T\) preserves \(\mu\), \(E_k' \subseteq \bigcup_{m=0}^M T^{-m} E_k\) entails \(\mu(E_k') \leq (M + 1) \mu(E_k) \to 0\), so that \((E_k')\) is asymptotically rare. Next,
\[
\varphi_{E_k} \leq \varphi_{E_k} \leq \varphi_{E_k'} + M \quad \text{a.e. on} \quad Y \setminus (E_k \cup E_k'),
\]
\[
\text{since points of} \ Y \ \text{can reach} \ E_k \ \text{only via} \ E_k' \ \text{and} \ E_k' \subseteq \bigcup_{m=0}^M T^{-m} E_k. \ (\text{We discard} \ Y \cap (E_k \cup E_k') \ \text{because} \ (3.1) \ \text{allows} \ 0 \ \in \ \{j, n\} \ \text{in which case the time of the first visit is not the first hitting time.} \) \ \text{Therefore,} \ 0 \leq \mu(E_k')(\varphi_{E_k} - \varphi_{E_k'}) \leq M \mu(E_k') \to 0 \ \text{on} \ Y \setminus (E_k \cup E_k'). \ \text{But} \ \mu_Y(Y \cap (E_k \cup E_k')) \to 0 \ \text{since both} \ (E_k) \ \text{and} \ (E_k') \ \text{are asymptotically rare. This implies} \ (3.2) \ \square
\]

4. Application to uniformly expanding interval maps

Piecewise monotone interval maps. A piecewise monotonic system is a triple \((X, T, \xi)\), where \(X\) is a bounded interval, \(\xi\) is a collection of nonempty pairwise disjoint open subintervals \(Z\) of \(X\) with \(\lambda(X \setminus \bigcup_{Z \in \xi} Z) = 0\) (where \(\lambda\) denotes Lebesgue measure), and \(T : X \to X\) is such that each branch of \(T\), i.e. its restriction to any of its cylinders \(Z \in \xi\) is a homeomorphism onto \(T^Z\). The system is Markov if \(T^Z \cap Z' \neq \emptyset\) for \(Z, Z' \in \xi \) implies \(Z' \subseteq T^Z\), and piecewise onto if \(T^Z = X \mod \lambda\) for all \(Z \in \xi\).

We focus on systems with \(C^2\) branches, and call such a system \((X, T, \xi)\) a Folklore map if it is piecewise onto, uniformly expanding (\(\inf |T'| > 1\)), and satisfies Adler’s condition, meaning that \(T^n/(T')^2\) is bounded. It is well known that every Folklore map has a unique absolutely continuous invariant probability measure \(\mu\) the density of which admits a continuous version \(h\) bounded away from 0 and \(\infty\).

A common scenario with standard exponential limit. We briefly record an auxiliary observation about a type of asymptotically rare events which often arise through induction. The following is just an easy variant of well-known results.

Lemma 4.1 (Exponential limit for sets containing cylinders). Let \((X, T, \xi)\) be a Folklore map and \(\mu \ll \lambda\) its invariant probability. Let \((E_k')_{k \geq 1}\) be a sequence of intervals shrinking to an endpoint of \(X\), and such that \(E_k' = E_k^\bullet \cup E_k^\Delta \mod \lambda\)
with $E_k^\lambda = \bigcup_{Z \in \xi, Z \subset E_k} Z$ a union of cylinders, and $E_k^{\lambda, c}$ a (possibly empty) interval contained in a cylinder $F_k \in \xi$. If $\lambda(F_k) = O(\lambda(E_k^\lambda))$ as $k \to \infty$, then
\begin{equation}
\mu(E_k') \varphi_{E_k'} \Rightarrow \mathcal{E} \quad \text{as } k \to \infty.
\end{equation}

**Proof.** This follows by routine arguments using, for example, the ideas of [HSV]. (The assumptions ensure $\mu(E_k') \varphi_{E_k'} = 1 \to 0$, and that the densities $\tilde{T}(\mu(E_k')^{-1} E_k')$, $k \geq 1$, where $\tilde{T}$ denotes the transfer operator of $T$ with respect to $\mu$, have uniformly bounded variation on the unit interval.)

Alternatively, one can apply the perturbation theory of [KL], [K] by slightly adapting the argument used for the Gauss map example in [KL]. □

**Remark 4.1.** Even easier, the same conclusion holds whenever the $E_k'$ are sets with $\lambda(E_k') \to 0$, and such that each is a union of cylinders. (In this case the argument from [KL] applies without change.)

**The exceptional behaviour of a Folklore map $T$ at a fixed point.** As a warm-up we now show that it is easy to employ Theorem 3.1 to determine the hitting-time statistics for small neighbourhoods of (uniformly repelling) periodic points of Folklore (or similar) maps. Since this type of result is well known (see e.g. [K] or [FFT]) and our emphasis is on the method rather than the most general version, we focus on the most basic case of a fixed point of a map with two branches.

**Example 4.1 (HTS for fixed points of simple Folklore maps).** Let $(X, T, \xi)$ be a Folklore map on $X = [0, 1]$ with two increasing branches, $\xi = \{(0, c), (c, 1)\}$, and $\mu \ll \lambda$ its invariant probability. Let $(E_k)_{k \geq 1}$ be a sequence of intervals which contain the fixed point $x^* = 0$, and such that $\lambda(E_k) \to 0$. Then,
\begin{equation}
\mu(E_k) \varphi_{E_k} \Rightarrow \theta^{-1} \mathcal{E} \quad \text{as } k \to \infty,
\end{equation}
where $\theta := 1 - 1/T'(0^+) \in (0, 1)$.

**Proof.** We use Theorem 3.1 to quickly derive (4.2) from the standard convergence guaranteed in Lemma 4.1. Let $Y := (c, 1)$ be the right-hand cylinder, and write $E_k' := Y \cap T^{-1} E_k$. The local dynamics at $x^* = 0$, together with continuity of the invariant density $h = d \mu / d \lambda$ give $\mu(Y \cap T^{-1} E_k) \sim \mu(E_k) / T'(0^+) \to \infty$. But as $\mu$ is invariant, we have $\mu(E_k) = \mu(Y \cap T^{-1} E_k) + \mu(E_k')$. Hence,
\begin{equation}
\mu(E_k') \sim \theta \mu(E_k) \quad \text{as } k \to \infty.
\end{equation}

It is a standard fact that the induced system $(Y, T_Y, \xi_Y)$ is a Folklore map with infinitely many cylinders $W_j = Y \cap \{ \xi_Y = j \}$, $j \geq 1$. It is easy to see that $(E_k')$ satisfies the assumptions of Lemma 4.1 for this induced system. Indeed, $E_k'$ is (mod $\lambda$) an interval of the form $(c, c + \delta_k)$, and $\lambda(F_k) = O(\lambda(E_k^\lambda))$ follows from $\lambda(W_j) \sim T'(0^+) \lambda(W_{j+1})$ as $j \to \infty$. Hence
\begin{equation}
\mu(E_k') \varphi_{E_k'} \Rightarrow \mathcal{E} \quad \text{as } k \to \infty.
\end{equation}

In view of Example 3.1 Theorem 3.1 applies. Combined with (4.3) it gives (4.2). □
Remark 4.2 (Reformulation in terms of return-times). In view of the general duality between hitting-time statistics and return-time statistics established in [HLV], (4.2) is equivalent to

\[ \mu(E_k) \varphi_{E_k} \xrightarrow{\mu_k} \tilde{\Omega} := \Theta \cdot \Theta^{-1} \mathcal{E} \quad \text{as} \quad k \to \infty, \]

where the random variable \( \Theta \) with \( \Pr[\Theta = 1] = 1 - \Pr[\Theta = 0] = \theta \) is independent of \( \mathcal{E} \). This is easily understood because a subinterval \( E_k \) of length \( \lambda(E_k) \sim \lambda(E_k)/T(0^+)^{1-\theta} \) re-enters \( E_k \) at once, which accounts for the atomic part \( \Pr[\tilde{\Omega} = 0] = 1 - \theta \) of \( \tilde{\Omega} \). The relations (4.2) and (4.4) can also be rephrased in the language of extreme value statistics, see e.g. [FT]. In that context, \( \theta \) is called the extremal index.

Maps with exceptionally unexceptional behaviour at a fixed point. Theorem 4.1 can be generalized to repelling hyperbolic periodic points \( x^* \) for other families of interesting systems. In particular, it has been shown in [K], [FFT] that in the context of uniformly expanding interval maps it suffices to assume that \( (X, T, \xi) \) is a Rychlik map (that is, belongs to the class studied in [R]) with the additional assumptions that \( T \) should be piecewise \( C^{1+\varepsilon} \) and that its invariant density \( h = d\mu/d\lambda \) should be bounded away from zero near \( x^* \).

We now show that the latter condition cannot be dropped, even if the map has very nice properties otherwise. To this end, we construct simple examples in which the inducing principle of Theorem 3.1 allows us to show that, in contrast to Example 4.1, neighbourhoods of hyperbolic repelling fixed points of general Rychlik maps may still exhibit standard exponential hitting time statistics.

Example 4.2 (Maps with standard HTS at uniform repellers). There exist uniformly expanding piecewise affine ergodic Rychlik maps \((X, T, \xi)\), which admit a fixed point \( x^* \) and neighbourhoods \( E_k \) of \( x^* \) satisfying \( \lambda(E_k) \to 0 \) and

\[ \mu(E_k) \varphi_{E_k} \xrightarrow{\mu_k} \mathcal{E} \quad \text{as} \quad k \to \infty, \]

where \( \mu \) denotes the unique absolutely continuous invariant probability measure.

Proof. (i) Structure of the map. We are going to define a family of uniformly expanding piecewise affine maps with big images on \( X := [0, 1) \). In particular, each of them is a Rychlik system \((X, T, \xi)\). They will be ergodic w.r.t. Lebesgue measure \( \lambda \), with unique right-continuous invariant probability density \( h = d\mu/d\lambda \) strictly positive on \((0, 1)\), but with \( h(x) \to 0 \) as \( x \searrow x^* \) of length \( \lambda \) shrinking to this fixed point still satisfies

\[ \mu(E_k) \varphi_{E_k} \xrightarrow{\mu_k} \mathcal{E} \quad \text{as} \quad k \to \infty. \]

The basic partition takes the form \( \xi = \{Z_0, Z_1, \ldots\} \) with \( Z_j = [z_j, z_{j+1}) \) for points \( 0 = z_0 < 1/2 = z_1 < z_2 < \ldots < z_j < 1 \). The sequence \( (z_j)_{j \geq 0} \), or equivalently the sequence \( (\lambda_j)_{j \geq 0} \) of lengths \( \lambda_j := z_{j+1} - z_j \), will serve as a parameter which completely determines the system. For \( x \in Z_0 = [0, 1/2) \) set \( Tx := 2x \), and let \( Y := Y_0 := Z_0 = [1/2, 1) \). Then, for \( i \geq 1 \), we see that \( Y_i := [2^{-(i+1)}, 2^{-i}) = Y^c \cap \{\varphi^i = i\} \). Note also that \( E_k := \xi_k(x^*) = \bigcup_{j \geq k} Y_j \) for \( k \geq 1 \). On \( Z_j \), \( j \geq 1 \), we define \( T \) to be decreasing and affine, mapping \( Z_j \) onto \( \bigcup_{i < j} Y_i \subset Y \), so that \( T \) has slope \(-s_j\) on \( Z_j \) where \( s_j := (1 - 2^{-i})/\lambda_j \sim \lambda_j^{-1} \) as \( j \to \infty \).
For any \((z_j)\) this gives a system \((X,T,\xi)\) for which \(Y\) is a sweep-out set (meaning that \(X = \bigcup_{n \geq 0} T^{-n}Y \pmod{\mu}\)). \(T_Y\) is a pcw onto and pcw affine map, hence folklore (and ergodic) with invariant measure \(\mu_Y\) on \(Y\). By standard arguments the original map \(T\) is therefore ergodic on \(X\) w.r.t. \(\lambda\). Also being a Rychlik map, \(T\) has a unique invariant probability density with a right-continuous version \(h\) of \(\lambda\). Note that \(h\) is constant on each element of the partition \(\xi' := \{\ldots Y_2, Y_1, Z_1, Z_2, \ldots\}\). Therefore \(h\) is constant on each element of the partition \(\{Y_j\}_{j \geq 0}^{\infty}\) generated by the image sets \(T Z', Z' \in \xi'\). Hence, \(h = \sum_{j \geq 0} \eta_j 1_{Y_j}\) with \(\eta_j = \mu(Y_j)/\lambda(Y_j) = 2^{j+1}\mu(Y_j)\). Below we give an explicit description of the invariant measure \(\mu\) in terms of \((\lambda_j)_{j \geq 0}\), and show in particular that \(h\) is strictly positive on \((0,1)\).

(ii) The invariant measure. For a probability \(\mu\) with density of the form \(h = \sum_{j \geq 0} \eta_j 1_{Y_j}\) to be \(T\)-invariant, we must first have

\[
\mu(Z_1) = \mu(Y_1 \cap T^{-1}Z_1) + \sum_{j \geq 1} \mu(Z_j \cap T^{-1}Z_1).
\]

Since \(Y_1\) is mapped onto \(Y_0\) without distortion, \(\mu(Y_1 \cap T^{-1}Z_1) = \mu(Y_1)\lambda(Z_1)/\lambda(Y_0) = \eta_1\lambda_1/2\), and as \(Z_j\) is mapped onto \(\bigcup_{i < j} Y_i\) without distortion, \(\mu(Z_j \cap T^{-1}Z_1) = \mu(Z_j)\lambda(Z_1)/\lambda(\bigcup_{i < j} Y_i) = \eta_0 \lambda_1 s_j^{-1}\). Letting \(\sigma(m) := \eta_0 2^{-(m+1)} \sum_{i > m} s_i^{-1}\), \(m \geq 0\), this leads to \(\mu(Y_0) = \mu(Y_1) + \sigma(0)\).

Likewise, for every \(j \geq 1\), \(\mu(Y_j) = \mu(Y_{j+1}) + \sum_{i > j} \mu(Z_i \cap T^{-1}Y_j)\), and using the definition of the individual branches this becomes \(\mu(Y_j) = \mu(Y_{j+1}) + \sigma(j)\).

We therefore see that

\[
\mu(Y_j) = \sum_{i \geq j} \sigma(i) \quad \text{for} \quad j \geq 0.
\]

(iii) Hitting-time statistics. We first consider the induced map \(T_Y\) and \(E'_k := Y \cap T^{-1}E_k\), \(k \geq 1\). The latter defines asymptotically rare events in \(Y\), and it is easily seen that each \(E'_k\) is a union of cylinders from \(\xi_Y\) (recall that \(T\) is Markov for \(\xi'\)). Consequently (see Lemma 4.1 and Remark 4.1),

\[
\mu(E'_k) \varphi_{E'_k}^{Y} \xrightarrow{\mu_Y} \mathcal{E} \quad \text{as} \quad k \to \infty.
\]

Due to Theorem 3.1, then

\[
\mu(E'_k) \varphi_{E'_k} \xrightarrow{\mu} \mathcal{E} \quad \text{as} \quad k \to \infty,
\]

and our claim (4.6) follows in case \(\mu(E'_k) \sim \mu(E_k)\) as \(k \to \infty\). However, we are exactly in the situation of Remark 4.1 meaning that the latter is fulfilled iff

\[
\sum_{j > k} \mu(Y_j) = o(\mu(Y_k)) \quad \text{as} \quad k \to \infty,
\]

because \(E_k = \bigcup_{j \geq k} Y_j\). In view of (4.7) and the definition of \(\sigma(m)\) this holds if \(\lambda_j \downarrow 0\) sufficiently fast.

To get specific examples, note that if the \(\lambda_i\) are such that \(\lambda_{i+1} \leq 2^{-(i+3)}\lambda_i\) for \(i \geq k_0\), then (using \(\lambda_i/2 \leq s_i^{-1} \leq \lambda_i\)) we find that \(\mu(Y_{k+1}) \leq 2^{-(k+1)}\mu(Y_k)\) for \(k \geq k_0\), which implies (4.9).

\[
\square
\]

Remark 4.3. In the particular version of the maps we describe in the proof, the \(E_k\) are the one-sided neighbourhoods \([0, 2^{-k})\) of \(x^*\). It is also easy to construct a variant in which the \(E_k\) are symmetric two-sided neighbourhoods of a repeller which lies in the center of some cylinder.
5. A limit theorem for indifferent fixed points

**Previous results.** In a recent paper [FFTV] the authors have studied hitting-time limits for a concrete parametrized family of non-uniformly expanding interval maps $T_p : [0, 1] \to [0, 1]$ possessing an indifferent fixed point at $x^* = 0$,

$$
T_p x := \begin{cases} 
  x + 2^p x^{1+p} & \text{for } x < 1/2, \\
  2x - 1 & \text{for } x > 1/2,
\end{cases}
$$

with parameter $p \in (0, 1)$. In this parameter range, each $T_p$ possesses a unique absolutely continuous (w.r.t. Lebesgue measure $\lambda$) invariant probability measure $\mu_p$ with density $h_p$ strictly positive and continuous on $(0, 1]$. Among other things, the hitting-time distributions of small neighbourhoods $[0, \epsilon]$ of the distinguished point $x^* = 0$ were analysed and shown to converge to a standard exponential random variable $E$, but under a normalization essentially different from the usual fact or $\mu_p([0, \epsilon]) \approx \epsilon^{1-p}$ as $\epsilon \to 0$. More precisely, in [FFTV] this result was established, using extreme value theory, only under the assumption that $p \in (0, \sqrt{5} - 2)$. The authors state the conjecture that the same assertion should be true at least for $p \in (0, 1/2)$.

In the present section we clarify the asymptotics of the hitting-time distributions of these exceptional families of rare events by extending the result of [FFTV] to arbitrary $p \in (0, 1)$ and, in fact, to a more general class of maps. We emphasize that no assumption on the analytical behaviour at the indifferent point akin to regular variation is needed, and that the argument does not use information on the decay of correlations for $T_p$. Instead, we are going to use Theorem 3.1 in a straightforward manner. The significantly more technical results from extreme value theory employed in [FFTV] are not required.

**Remark 5.1 (The infinite measure case, $p \geq 1$).** Asymptotic hitting-time distributions of neighbourhoods of neutral fixed points for infinite measure preserving situations (as encountered when $T = T_p$ as in (5.1) but with parameter $p \geq 1$) have been obtained in [Z3].

For cylinders shrinking to typical points of such a null-recurrent map, hitting-time statistics have been clarified more recently in [PSZ], see also [RZ].

**The behaviour of an intermittent map at the neutral source.** Again we formulate the result in the setup of simple maps with two full branches. It can be extended to more general maps and periodic points via routine arguments.

**Theorem 5.1 (HTS for neutral fixed points of simple maps).** Let $(X, T, \xi)$ be piecewise increasing with $X = [0, 1]$ and $\xi = \{(0, c), (c, 1)\}$, mapping each $Z \in \xi$ onto $(0, 1)$. Assume that $T \big|_{(c, 1)}$ admits a uniformly expanding $C^2$ extension to $[c, 1]$, while $T \big|_{(0, c)}$ extends to a $C^2$ map on $(0, c]$ and is expanding except for an indifferent fixed point at $x^* = 0$: for every $\varepsilon > 0$ there is some $\rho(\varepsilon) > 1$ such that $T' \geq \rho(\varepsilon)$ on $[\varepsilon, c]$, while $T0 = 0$ and $\lim_{x \to 0} T'x = 1$ with $T'$ increasing on some $(0, \delta)$. Suppose also that

there is a continuous decreasing function $g$ on $(0, c]$ with

$$
\int_0^c g(x) \, dx < \infty \quad \text{and} \quad |T''| \leq g \text{ on } (0, c].
$$

(5.2)
Let \((E_k)_{k \geq 1}\) be a sequence of intervals which contain the fixed point \(x^*\), and such that \(\lambda(E_k) \to 0\). Then,

\[
\mu(E_k) \varphi_{E_k} \xrightarrow[k \to \infty]{\mu} \infty,
\]

where \(\mu \ll \lambda\) is the unique invariant probability measure of \(T\). Much more precisely,

\[
\frac{h(c)}{T'(c^+)} \lambda(E_k) \varphi_{E_k} \xrightarrow[k \to \infty]{\mu} \mathcal{E},
\]

where \(h\) is the continuous version of the invariant density \(d\mu/d\lambda\).

**Proof.** Let \(E'_k := Y \cap T^{-1}E_k\) with \(Y := (c, 1) \in \xi\) the right-hand cylinder. Then

\[
T_Y \text{ is a Folklore map.}
\]

Specifically, with \(c_0 := 0\), \(c_1 := c\), and \(c_{j+1}\) such that \(c_{j+1} < Tc_{j+1} = c_j\) for \(j \geq 1\), the sets \(V_j := (c_j, c_{j-1})\) accumulate at \(x^*\) and satisfy \(\lambda(V_j) \sim \lambda(V_j)\) since \(V_j = TV_j + 1\) and \(\lim_{x \to x^*} T'x = 1\). In particular, \(\lambda(V_j) = o(\lambda(\bigcup_{j \geq 1} V_j))\) as \(j \to \infty\). Since \(V_1 = Y\), the cylinders of \(T_Y\) are the sets \(W_j := Y \cap T^{-1}V_j = Y \cap \{\varphi Y = j\}\), \(j \geq 1\), which accumulate at \(c^+\) and satisfy \(\lambda(W_j) \sim \lambda(V_j) / T'c^+\). Hence,

\[
\lambda(W_j) = o(\lambda(\bigcup_{j \geq 1} W_i)) \quad \text{as} \quad j \to \infty.
\]

Adler’s condition for \(T_Y\) follows from assumption \([5.2]\) by an analytic argument which goes back to \([Z1]\), see for example §3 of \([Z1]\) or §4 of \([T2]\).

As observed in Example 3.1 points of \(Y\) can enter \(E_k\) only via \(E'_k\), and Lemma 4.1 for the induced system yields \(\mu_Y(E'_k) \varphi_{E'_k} \xrightarrow[\mu_Y]{} \mathcal{E}\). Indeed, \(E'_k\) is a one-sided neighborhood of \(c\), so that (up to endpoints of cylinders) \(\bigcup_{i > j(k)} W_i \subseteq E'_k \subseteq \bigcup_{i > j(k)} W_i\) for suitable \(j(k) \to \infty\). Consequently, using the notation of the lemma and \((5.6)\) above, \(E_k^\star = \bigcup_{i > j(k)} W_i\) and \(F_k = W_{j(k)}\) satisfy \(\lambda(F_k) = o(\lambda(E'_k))\).

Now Theorem 3.1 shows that

\[
\mu(E'_k) \varphi_{E_k} \xrightarrow[k \to \infty]{\mu} \mathcal{E}.
\]

Straightforward calculation, using continuity of \(h\) on \(Y\), shows that

\[
\mu(E'_k) \sim \frac{h(c)}{T'(c^+)} \lambda(E_k) \quad \text{as} \quad k \to \infty,
\]

proving \((5.4)\). But then \((5.3)\) follows at once, since in the present situation, \(\mu(E'_k) = \mu(E_k) - \mu(Y^c \cap T^{-1}E_k)\) satisfies \(\mu(E'_k) = o(\mu(E_k))\). This is because \(\mu(Y^c \cap T^{-1}E_k) \sim \mu(E_k) / T'(c^+) \sim \mu(E_k)\) as \(k \to \infty\).

**Remark 5.2.** Under the assumptions of the theorem, the return-time distributions (law of \(\varphi_{E_k}\) under \(\mu_{E_k}\)) are of limited interest, since \(\mu_{E_k} (\varphi_{E_k} = 1) \to 1\).

**Remark 5.3.** In the statement of the theorem, condition \((5.2)\) can be replaced by its consequence \((5.5)\) because the proof only depends on the latter property.

**Example 5.1.** Each map \(T = T_p\) from \((5.1)\) with \(p \in (0, 1)\) trivially satisfies \((5.2)\).

Therefore, whenever \(E_k = [0, \epsilon_k]\) with \(\epsilon_k \to 0\), the theorem shows that

\[
\frac{h_p(\frac{\epsilon}{2})}{2} \epsilon_k \varphi_{E_k} \xrightarrow[k \to \infty]{\mu} \mathcal{E} \quad \text{as} \quad k \to \infty.
\]

Consequently, we also have (generalizing \([FFTV]\))

\[
\frac{h_p(\epsilon)}{2} \epsilon \varphi_{[0, \epsilon]} \xrightarrow[k \to \infty]{\mu} \mathcal{E} \quad \text{as} \quad \epsilon \to 0.
\]

(Otherwise there would be some sequence \((\epsilon_k)\) violating \((5.8)\).
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