Acoustic geometry through perturbation of mass accretion rate - axisymmetric flow in static spacetimes

Deepika B Ananda*
Indian Institute of Science Education and Research, Pashan, Pune-411008, INDIA and
Harish-Chandra research Institute, Chhatnag Road, Jhunsi, Allahabad-211019, INDIA †

Sourav Bhattacharya‡ and Tapas K Das§
Harish-Chandra research Institute, Chhatnag Road, Jhunsi, Allahabad-211019, INDIA
(Dated: August 26, 2014)

This is the second of our series of papers devoted to the study of the stability analysis of the stationary transonic integral solutions for accretion flow onto a static compact object, using the acoustic geometry. Precisely, we consider accretion of an axisymmetric, inviscid and irrotational fluid in a general static axisymmetric spacetime and study the perturbation of the mass accretion rate, and demonstrate the natural emergence of the general relativistic acoustic geometry. In other words, the astrophysical accretion process has a natural interpretation as an example of the acoustic analogue gravity phenomenon. We also discuss two explicit examples of the Schwarzschild and the Rindler spacetimes. For the later, in particular, we demonstrate that for smooth flow fields there can be no sonic point.

PACS numbers: 04.25.-g, 04.30.Db, 97.60.Jd, 11.10.Gh
Keywords: black hole accretion, perturbative stability, acoustic geometry

A. Introduction

To obtain reliable spectral signatures of astrophysical black holes from a set of stationary transonic accretion solutions, it is necessary to ensure that such integral solutions are stable under perturbation [1], at least for an astrophysically relevant time scale. In our previous work [2], we demonstrated that a suitable stability analysis scheme can be developed using the acoustic geometry to ensure the stability of spherically symmetric accretion onto astrophysical black holes under linear perturbation. In the present paper we shall extend our earlier perturbation scheme to accommodate low angular momentum axisymmetric flows, assuming the fluid to be inviscid, irrotational and non self-gravitating.

On the analytical front, stationary flow solutions for low angular momentum inviscid accretion has extensively been studied in the literature, see, e.g. [3–26] and references therein. Numerical works have also been reported for such flow configurations in [27–31] and also in the references therein.

In this context it is to be emphasized that the concept of low angular momentum advective flow (where the inviscid assumption is justified) is not a theoretical abstraction and sub-Keplerian flows are observed in nature in reality. For example, such flow configurations may be observed for detached binary systems fed by accretion from OB stellar winds [32, 33], semi-detached low-mass non-magnetic binaries [34], and supermassive black holes fed by accretion from slowly rotating central stellar clusters [35, 36] (see also the references therein). Even for a standard Keplerian accretion disc, turbulence may produce such low angular momentum flow [see, e.g. 37, and references therein]. Moreover, given the background spacetime, there exists a critical value of the specific angular momentum of the flow, $\lambda$, below which there would not be any Keplerian orbits.

Stability properties of the stationary black hole accretion solutions have been studied by several authors, see, e.g. [38–53] and references therein.

However, our present work is different from the existing literature in several aspects. Firstly and most importantly, we address this problem from the point of view of the acoustic analogue geometry (see e.g. [54–58], and references therein). As of now, the acoustic geometry associated with usual analogue gravity models has been obtained by perturbing the corresponding velocity potential of the background fluid flow. Instead, we shall derive here the acoustic geometry via the perturbation of the mass accretion rate. The reason behind this is clear: the mass accretion rate is an astrophysically relevant and measurable quantity. After accomplishing this, we demonstrate that the analysis of those perturbations becomes rather simpler to deal with. Secondly, we provide a metric independent formalism which works for any static axisymmetric metric. Such metric may be relevant near a static assemblage of self-gravitating attractors, which need not be spherically symmetric.

For low angular momentum axisymmetric accretion onto black holes, infalling matter can have three different geometric configurations – the conical, the constant height and the vertical equilibrium models (see, e.g. [51] and references therein). In subsequent sections, we shall derive the general relativistic acoustic geometry

---

*Current affiliation
Electronic address: anandadeepika[AT]hri.res.in
†Electronic address: physourav[AT]gmail.com
‡Electronic address: sourabhhatta[AT]hri.res.in
§Electronic address: tapas[AT]hri.res.in
through which the perturbation of the mass accretion rate propagates, for all these three models. Next, using this we address the stability properties of the axisymmetric matter flows. We then provide specific results for the Schwarzschild and the Rindler spacetimes.

We shall use mostly positive signature for the metric and will set $c = 1 = G$ throughout.

**B. The preliminaries**

We shall briefly mention here the basic ingredients and assumptions necessary for our calculations.

Let us start with the metric for a general static and axisymmetric spacetime

$$ds^2 = -g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2,$$  

(1)

where the vector fields $(\partial_t)^\mu$ and $(\partial_\phi)^\mu$ are Killing and generates staticity and axisymmetry respectively. Hence none of the metric functions depends upon $t$ and $\phi$. The spacetime could be spherically symmetric as well, but we do not need to assume it at this level.

Let us take an ideal and inviscid fluid with the energy-momentum tensor,

$$T_{\mu\nu} = (\epsilon + p) v_\mu v_\nu + pg_{\mu\nu}. $$  

(2)

where $\epsilon$ and $p$ are respectively the energy density and pressure. $v^\mu$ is the fluid’s four velocity normalized as $v^\mu v_\mu = -1$.

We shall consider only radial and axisymmetric flows, i.e. $v^\theta = 0$. Then the normalization condition gives

$$v^t = \sqrt{\frac{1 + g_{rr}v^2 + g_{\phi\phi}(v^\phi)^2}{g_{tt}}}, $$  

(3)

where we have written $v^r = v$. We assume that the fluid obeys the adiabatic equation of state $p = k\rho^\gamma$, where $\gamma$ is a polytropic index and $\rho$ is the fluid density. The specific enthalpy $h$ of the fluid is given by $h = \frac{\epsilon + p}{\rho}$, so that

$$dh = Td\left(\frac{S}{\rho}\right) + \frac{dp}{\rho}, $$  

(4)

where $T$ and $S$ are respectively the temperature and entropy of the fluid. Under isentropic conditions, we can define the speed of the sound, $c_s$,

$$c_s^2 = \frac{\partial p}{\partial \epsilon} \bigg|_{\rho S = \text{const}}. $$  

(5)

We shall always assume that all physically relevant quantities (i.e. density, velocities etc.) are independent of $\phi$. The conservation of mass is ensured by the equation of continuity, $\nabla_\mu (\rho v^\mu) = 0$. Then we have

$$\partial_t (\rho v^t \sqrt{-g} H_\theta) + \partial_r (\rho v^r \sqrt{-g} H_\theta) = 0, $$  

(6)

where we have used $v^\theta = 0$ and the factor $H_\theta$ depends upon the model of the accretion we are choosing [51]. Precisely, $H_\theta$ is an appropriate weight function required to get the flux of mass infalling into a stellar object, and hence thereby taking care of the thickness of the accretion disk. The weight function is due to the averaging over $\theta = \frac{\pi}{2}$, and hence inclusion of it effectively permits us to work on that plane although simultaneously taking care of the thickness or the off-equatorial flows of the accretion disk.

For example, if the flow of the fluid makes a cone, which creates a constant solid angle at the origin at all radial values, $H_\theta$ will be a constant. For radial flow from all directions, this factor will clearly be unity, and less than unity otherwise. Likewise, for the so called constant height model, we have $H_\theta = \frac{\sqrt{r}}{r}$, where $H$ is a constant. Clearly, this model ensures that the solid angle subtended by the edges of the flow lines will decrease with radial distance. The most non trivial flow geometry is the vertical equilibrium model, in which $H_\theta$ is kept as a function of $r$ and the flow variables, except the radial velocity. It is assumed that $H_\theta$ is not an explicit function of time.

We note that, effectively this can be understood by replacing $g_{\theta\theta}$ in the metric by $H_\theta^2 g_{\theta\theta}$, so that when one integrates the continuity equation, the factor $H_\theta$ sitting in the integration measure takes care of the geometry which the infalling matter creates.

We also have the conservation equation for the energy-momentum tensor, $\nabla_\mu T^{\mu\nu} = 0$, which, with the help of the continuity equation and the thermodynamic relations, can be written as

$$v^\nu \partial_\nu v^\mu + v^\nu v^\lambda \Gamma^\mu_{\nu\lambda} + \frac{c_s^2}{\rho} (v^\mu v^\nu + g^{\mu\nu}) \partial_\nu \rho = 0, $$  

(7)

For $\mu = t, r, \phi$, we get respectively the conservation equations for energy, radial and the orbital angular momenta,

$$v^t \partial_t v^t + \frac{c_s^2}{\rho} ((v^t)^2 - g^{\mu\nu}) \partial_\nu \rho + v \partial_r v^t + g^{tt} (\partial_r g_{tt}) v v^t + \frac{c_s^2}{\rho} v v^t \partial_r \rho = 0, $$  

(8)

$$v^t \partial_t v + \frac{c_s^2}{\rho} v v^t \partial_t \rho + v \partial_r v + \frac{1}{2g_{rr}} [(\partial_r g_{tt}) (v^t)^2 + (\partial_r g_{rr}) v^2 - (\partial_r g_{\phi\phi}) (v^\phi)^2] + \frac{c_s^2}{\rho} [v^2 + g^{rr}] \partial_r \rho = 0, $$  

(9)
We also assume that the flow is irrotational, so that we have
\[ \partial_{\mu} (h v_\mu) - \partial_\nu (h v_\nu) = 0. \tag{11} \]
Choosing the different free indices, we get
\[
\begin{align*}
\partial_t v_r - \partial_r v_t &= \frac{c_s^2}{\rho} [v_t \partial_r \rho - v_r \partial_t \rho], \\
\partial_t v^\phi &= -\frac{v^\phi c_s^2}{\rho} \partial_t \rho,
\end{align*}
\]
\[ v^t \partial_t v^\phi + \frac{c_s^2}{\rho} v^\phi v^t \partial_t \rho + v \partial_r v^\phi + g^{\phi \phi} (\partial_r g_{\phi \phi}) v v^\phi + \frac{c_s^2}{\rho} v v^\phi \partial_r \rho = 0. \tag{10} \]

The stationary state corresponds to the vanishing of the time derivatives. When we perturb the radial equation, we shall include time dependence, but will ignore \( \phi \) dependence throughout.

For our purpose we also need to derive an expression for the derivative of the weight function \( H_\theta \). We note that \( c_s^2, \rho \) and \( p \) are interrelated by the thermodynamic relations. Therefore without any loss of generality, we may take \( H_\theta \equiv H_\theta (r, v^r, \rho) \). Using the chain rule for the partial derivatives and the irrotational conditions (Eq.s (12)), we can write \( H_\theta \)'s derivatives in the form
\[ \frac{1}{H_\theta} \frac{dH_\theta}{dx} = \alpha + \beta \frac{\partial_x \rho}{\rho}, \tag{14} \]

where \( \alpha = \frac{\partial \ln H_\theta}{\partial x} - v^\phi \left( \frac{\partial \ln H_\theta}{\partial v^\phi} \right) \frac{\partial \ln g_{\phi \phi}}{\partial x} \) and \( \beta = \frac{\partial \ln H_\theta}{\partial \rho} - v^\phi c_s^2 \left( \frac{\partial \ln H_\theta}{\partial v^\phi} \right) \) and \( x \) is either \( t \) or \( r \), and \( \partial_t \ln H_\theta = 0 \).

In particular, if we assume following [59] that the vertical equilibrium condition is given by
\[ -\frac{p}{\rho} + \zeta(r) H_\theta^2 v^r = 0, \tag{15} \]
where \( \zeta(r) \) is some arbitrary function of \( r \) independent of the fluid variables, it turns out that the function \( \beta \) appearing in Eq. (14) is always positive. In the following we shall also assume that \( \beta \) is a positive quantity.

With all these ingredients, we are now ready to go into our main course of discussions.

**C. The stationary solutions**

We now look at the stationary solutions, where all the time derivatives are vanishing. Integrating the spatial part of the continuity equation we have
\[ \Omega \rho_0 v_{0t} \sqrt{-g} H_{\theta 0} = -\dot{M} = \text{constant}, \tag{16} \]
where \( \Omega \) is a geometrical factor that appears due to integration over the angular coordinates and the subscript ‘0’ denotes the stationary values. The constant on the right hand side is regarded as the mass accretion rate and the ‘minus’ sign conventionally indicates the infall of the matter.

Setting the time derivatives in the energy conservation equation, Eq. (8), to zero we find
\[ h v_{0t} = -E, \tag{17} \]
where \( E \) is a constant along the flow line and is identified as the specific energy. We also note using Eq.s (12) and (17) that the parameter \( \lambda = -\frac{v_{0\phi}}{v_{0t}} = \frac{l}{E} \) is a constant along the flow line, too. This is regarded as the specific angular momentum of the fluid.

Similarly, setting the time derivatives in the radial equation, Eq. (13) to zero gives
\[ -\frac{2c_s^2}{\rho_0} \partial_r \rho_0 = \left( \frac{g_{rr}}{g_{tt}(v_{0t})^2} \right) \partial_r \left( \frac{g_{tt}(v_{0t})^2}{g_{rr}} \right) + \partial_r (g_{rr} g_{tt}) g_{rr} g_{tt}. \tag{18} \]

Let us now move to the comoving frame of the fluid in which we denote the spatial velocity by \( u_0 \). Let us define
Following [60],

\[
v^t_0 = \sqrt{\frac{g_{\phi\phi}}{g_{tt} (g_{\phi\phi} - g_{tt} \lambda^2)}} \sqrt{\frac{1}{1 - u^2_0}},
\]

\[
v^\phi_0 = \lambda \sqrt{\frac{g_{tt}}{g_{\phi\phi} (g_{\phi\phi} - g_{tt} \lambda^2)}} \sqrt{\frac{1}{1 - u^2_0}},
\]

By substituting Eqs. (19) into Eqs. (17) and (18), one can express the stationary solutions in terms of \(u_0\). Then considering the radial derivative of Eq. (17) and using Eqs. (18), (14), we find

\[
\frac{du_0}{dr} = \frac{u_0 (1 - u^2_0)}{2 (1 + \beta - c^2_{so})} \left[ \partial_r \left( \frac{-g}{g_{rr}} \right) + 2\alpha \right] c^2_{so} - (1 + \beta) \partial_r \left( \frac{g_{tt} g_{\phi\phi}}{g_{\phi\phi} - \lambda^2 g_{tt}} \right).
\]

We note that the denominator of the above equation vanishes when

\[
u^2_{0|c} = \frac{c^2_{so} |c|}{(1 + \beta_c)}.
\]

We call this point a critical point and the subscript ‘c’ stands for this. We may see that for accretion models where disk height depends on the flow variables, \(\beta > 0\) and the critical point is different from the sonic point, \(u^2_0 = c^2_{so}\). In particular, the denominator of Eq. (21) shows that the sonic point is reached when \(u^2_0 (1 + \beta) - c^2_{so} = \beta u^2_0\). There are some numerical techniques to find the sonic point as well, see e.g. [61] and references therein.

For a constant height model, we get \(\beta = 0\), \(\alpha = -\frac{1}{2}\) (Eq. (14)). On the other hand, for the conical model, since \(H_\theta\) is constant, we have \(\beta = 0 = \alpha\). In this case, setting \(\lambda = 0\), we recover the results for the spherical flow derived earlier in [2].

Also, in order to make \(\frac{du_0}{dr}|_c\) finite, we simultaneously set the numerator of Eq. (20) to zero, yielding

\[
\frac{c^2_{so} |c|}{(1 + \beta_c)} = \frac{\partial_r \left( \frac{-g}{g_{rr}} \right)}{\partial_r \left( \frac{-g}{g_{rr}} \right) + 2\alpha_c}.
\]

The above stationary solutions for the three disk models were derived earlier in [62], for the Schwarzschild spacetime. It is easy to check that, when we substitute for the metric explicitly into Eq. (20), our general result recovers the result for the Schwarzschild spacetime.

In the following we shall derive the time dependent perturbation equations and demonstrate the emergence of the acoustic geometrical structure to do the stability analysis of the stationary solutions we have found.

**D. Emergence of the acoustic geometry and the stability of the stationary solutions**

As we have stated earlier, the mass accretion rate is an astrophysically relevant and interesting quantity and hence we shall study its linear perturbation. We have

\[
v(r, t) = v_0(r) + \nu'(r, t),
\]

\[
\nu^\phi(r, t) = \nu^\phi_0(r) + \nu^\phi'(r, t),
\]

\[
\rho(r, t) = \rho_0(r) + \nu'(r, t),
\]

\[
\nu'(r, t) = g_{rr} v_0 \nu' + g_{\phi\phi} v^\phi_0 \nu^\phi'/ g_{tt} v_0^2.
\]

where the subscript ‘0’ stands for the stationary state and in the last equation we have used the normalization of the velocity.

Let us define a variable \(\Psi = \rho v_0 \sqrt{-g} H_\theta\) which, one can see from Eq. (16), coincides with the mass accretion rate at its stationary value (apart from the trivial geometric constant \(\Omega\)). From the definition of \(\Psi\) we see that

\[
\Psi(r, t) = \Psi_0(r) + \Psi'(r, t) = \rho_0 v_0 H_\theta \sqrt{-g} + (\rho' v_0 H_\theta + \rho_0 v' H_\theta + \rho_0 v_0 H'_\theta) \sqrt{-g}.
\]

We recall that for the most general disk model, \(H_\theta\) can be a function of the flow variables, except for the radial velocity [51]. Then the perturbation of the flow variables also induces time dependence on \(H_\theta\). From Eq. (14), we can write

\[
\frac{d \ln H'_\theta}{dt} = \beta \frac{\partial_i \rho'}{\rho_0}.
\]

where \(\beta\) is a function of all the time independent terms, since we are studying a first order perturbation theory. Substituting for the perturbed quantities Eqs. (23)-(28) into the continuity equation (6), we get
- \frac{\partial_t \Psi'}{\Psi_0} = \frac{1}{g_{tt}} \left[ \left( g_{rr} v_0^2\right) \frac{\partial_t \nu'}{v_0} + \left( g_{tt} (v_0')^2 (1 + \beta) - g_{\phi\phi} (v_0')^2 c_{s_0}^2 \right) \frac{\partial_r \rho'}{\rho_0} \right]. \quad (29)

Also, taking the time derivative of Eq. (27) and using Eq. (28), we obtain

\frac{\partial_t \Psi'}{\Psi_0} = (1 + \beta) \frac{\partial_t \rho'}{\rho_0} + \frac{\partial_t \nu'}{v_0}. \quad (30)

Solving Eqs. (29) and (30), we can express the derivatives of \( \rho' \) and \( \nu' \) solely in terms of \( \Psi' \),

\begin{align*}
\frac{\partial_t \nu'}{v_0} &= \frac{1}{\Lambda} \left[ \frac{\partial_t \Psi'}{\Psi_0} + g_{tt} v_0 v_0' (1 + \beta) \frac{\partial_r \Psi'}{\Psi_0} \right], \quad (31) \\
\frac{\partial_t \rho'}{\rho_0} &= -\frac{1}{\Lambda} \left[ g_{rr} v_0^2 \frac{\partial_t \Psi'}{\Psi_0} + g_{tt} v_0 v_0' \frac{\partial_r \Psi'}{\Psi_0} \right], \quad (32)
\end{align*}

where

\Lambda = (1 + \beta) + (1 + \beta - c_{s_0}^2) g_{\phi\phi} (v_0')^2. \quad (33)

We now substitute for the perturbed quantities into the radial equation, Eq. (13), to get

\begin{align*}
\left( g_{rr} v_0 \right) \frac{\partial_t \nu'}{v_0} + \left( g_{rr} v_0 c_{s_0}^2 \frac{\partial_t \rho'}{\rho_0} \right) + \partial_r \left( c_{s_0}^2 \frac{\rho'}{\rho_0} + \frac{v_0'}{v_0} \right) = 0. \quad (34)
\end{align*}

Taking the time derivative of the above equation and substituting for the time derivatives of \( \rho' \) and \( \nu' \) from Eqs. (31) and (32), we obtain

\begin{align*}
\partial_t \left[ g_{rr} v_0 \frac{c_{s_0}^2 + \{(1 + \beta) - c_{s_0}^2\} g_{tt} (v_0')^2}{g_{tt}} \right] \frac{\partial_t \Psi'}{\Psi_0} + \partial_t \left[ g_{rr} v_0 \frac{(v_0')^2 (1 + \beta) - c_{s_0}^2}{g_{tt}} \right] \frac{\partial_r \Psi'}{\Psi_0} + \partial_r \left[ g_{rr} v_0 \frac{c_{s_0}^2 + \{(1 + \beta) - c_{s_0}^2\} g_{rr} v_0^2}{g_{rr}} \right] \frac{\partial_r \Psi'}{\Psi_0} = 0, \quad (35)
\end{align*}

so that we can readily identify a symmetric tensor \( f^{\mu\nu} \),

\begin{align*}
f^{\mu\nu} &= \left( \frac{g_{rr} v_0 c_{s_0}^2}{v_0^2 \Lambda} \right) \left[ -g^{tt} + \left( 1 - \frac{1 + \beta}{c_{s_0}^2} \right) (v_0')^2 - v_0 v_0' \left( 1 - \frac{1 + \beta}{c_{s_0}^2} \right) \right] \left( v_0 v_0' \left( 1 + \frac{1 + \beta}{c_{s_0}^2} \right) \right),
\end{align*}

such that the equation for the perturbed mass accretion rate can be written in a compact form,

\begin{align*}
\partial_\mu ( f^{\mu\nu} \partial_\nu \Psi' ) = 0. \quad (36)
\end{align*}
\[ f_{\mu \nu} = - \frac{g_{\mu \nu} v_0^t}{v_0} \left[ g^{rr} + \left( 1 - \frac{1 + \beta}{c_{s_0}} \right) v_0^2 - v_0 v_0^t \left( 1 - \frac{1 + \beta}{c_{s_0}} \right) \right]. \]

The non relativistic limit of \( f_{\mu \nu} \) is obtained by setting the metric elements to unity and \( v_0^t \to 1, c_{s_0} \ll 1. \)

\[ f_{\mu \nu} \big|_{NR} = \frac{(1 + \beta)}{v_0 c_{s_0}^2} \left[ v_0^2 - \frac{c_{s_0}^2}{1 + \beta} - v_0 \right]. \]

If we set \( \beta = 0 \) in the above equations, we recover the results for the conical and the constant height accretion disk models, where the weight function \( H_\theta \) appearing in the continuity equation is not a function of the flow variables.

Let us now see how \( f_{\mu \nu} \) recovers the causal structure of the acoustics described in the previous section. We substitute Eq.s (19) into the expression for \( f_{\mu \nu} \). In particular, we have

\[ f_{tt} = \left( \frac{g_{\mu \mu} g_{\phi \phi}}{g_{rr} (g_{\phi \phi} - \lambda^2 g_{tt})} \right)^{1/2} \frac{1}{u_0 (1 - u_0^2)} \left[ (1 + \beta) \frac{u_0^2}{c_{s_0}^2} - 1 \right]. \]

It is now easy to recover the critical point and the sonic point conditions discussed after Eq. (21).

One may proceed to define an acoustic metric from \( f_{\mu \nu} \) following exactly what is done in the velocity potential approach. But one then gets an overall divergent factor. This difficulty is usually attributed to the conformal invariance in two dimensions [58, 63]. We therefore do not attempt to define the so called acoustic metric from \( f_{\mu \nu} \), but nevertheless its inverse, \( f_{\mu \nu} \), gives the desired causal properties of the acoustic perturbation, as we have already seen above.

We shall now briefly address the stability issues below. The details of the construction of the various equations are essentially the same as the radial flow, which can be seen in e.g. [2]. We start by taking a trial solution,

\[ \Psi' = p_\omega(r) \exp(-i \omega t), \] (38)

\[ (37) \] where the spatial part \( p_\omega(r) \) satisfies

\[ \omega^2 p_\omega(r) f^{tt} + i \omega \left[ \partial_r (p_\omega(r) f^{rr}) + f^{tt} \partial_r p_\omega(r) - [\partial_r (f^{rr} \partial_r p_\omega(r))] \right] = 0. \] (39)

We shall first consider the stability issues for standing waves, which requires \( p_\omega(r) \) to be vanishing at two different radii, \( r_1 \) and \( r_2 \) at all times. Multiplying Eq. (39) with \( p_\omega(r) \) and integrating between the boundaries, we have

\[ \omega^2 \int p_\omega^2(r) f^{tt} dr + i \omega \int [\partial_r (p_\omega^2 f^{rr})] dr - \int p_\omega(r) [\partial_r (f^{rr} \partial_r p_\omega(r))] dr = 0. \] (40)

Using \( p_\omega(r_1) = 0 = p_\omega(r_2) \), we get

\[ \omega^2 = - \frac{\int f^{rr} (\partial_r p_\omega(r))^2}{\int f^{tt} p_\omega^2(r)}. \] (41)

Since \( f^{tt} < 0 \) always, the denominator stays positive. So the nature of \( \omega \) (i.e. whether real or imaginary) depends on the sign of \( f^{rr} \). It is easy to check that \( f^{rr} > 0 \) outside the critical point. Usually the standing waves correspond to the subsonic flows [41] and we take the outer boundary \( r = r_2 \) to be outside the critical point. Thus if the inner boundary \( r = r_1 \) is outside the critical point too, we have \( f^{rr} > 0 \) and hence \( \omega \) has two real roots confirming that the stationary solutions are stable. But it is interesting to note that if the inner boundary lies inside the critical point (but outside the sonic point), \( \omega^2 \) can be negative and there can be either instability or damping effects,
even though the flow is subsonic there (we recall that we have taken $\beta$ to be positive).

Clearly, this happens only when $\beta \neq 0$. For any accretion disk model in which the weight function $H_\theta$ is not a function of the flow variables, we have $\beta = 0$, and the critical and sonic points are coincident then (see the discussions after Eq. (21)). In that case there will be no such damping or instability effects, because the inner boundary is located outside the sonic point.

Let us now come to the travelling waves. We use the trial power series solution similar to the flat spacetime [41],

$$p_\omega(r) = \exp \left[ \sum_{n=-1}^{\infty} \frac{k_n(r)}{\omega^n} \right].$$  \hspace{1cm} (42)

In order to let the wave propagate to large radial distances, it is always necessary to have $\omega$ values to be large. In that case the advantage of making the above ansatz is that, it converges sufficiently rapidly. Thus the above ansatz looks like the WKB like solutions.

Substituting Eq. (42) into Eq. (39) and setting the coefficients of individual powers of $\omega$ to zero gives the leading coefficients in the power series, the first two of them being

$$k_{-1} = I \int \frac{f^{tr} \pm \sqrt{(f^{tr})^2 - f^{tt} f^{rr}}}{f^{tr}} dr,$$

$$k_0 = -\frac{1}{2} \ln(\sqrt{(f^{tr})^2 - f^{tt} f^{rr}}).$$  \hspace{1cm} (43)

It can be explicitly checked from the expression of $f^{\mu\nu}$ that $k_{-1}$ is purely imaginary. Then the leading behaviour for the amplitude of $\Psi'$ is given by

$$|\Psi'| \sim \left[ \frac{\Lambda (1 + g_{rr} v_0^2)}{g_{rr} v_0^2 c_s^2 (1 - \frac{\lambda^2}{g_{\phi\phi} g_{tt}})} \right]^{\frac{1}{2}},$$  \hspace{1cm} (44)

where $\lambda = -\frac{v_0}{c_s}$ is the conserved specific angular momentum of the fluid.

**E. Two explicit examples**

We shall now discuss our general results obtained in the earlier sections for two specific examples. First we consider the Schwarzschild spacetime,

$$g_{tt} = g^{-1}_{rr} = f(r) = 1 - \frac{2M}{r}, \quad g_{\phi\phi} = \frac{g_{tt}}{\sin^2 \theta} = r^2.$$  \hspace{1cm} (45)

The explicit expression for the symmetric tensor $f^{\mu\nu}$ and its inverse $f_{\mu\nu}$ can be read off from the general expression given in Section D, for $\theta = \frac{\pi}{2}$.

The issues regarding the standing waves clearly remains the same as discussed for the general case.

The amplitude of the travelling wave (Eq. (44)) is given explicitly by

$$|\Psi'| \sim \left[ \frac{\Lambda (1 - \frac{2M}{r} + v_0^2)}{v_0^2 c_s^2 (1 - \frac{\lambda^2}{g_{\phi\phi} (1 - \frac{2M}{r})})} \right]^{\frac{1}{2}},$$  \hspace{1cm} (46)

where $\lambda = -\frac{v_0}{c_s}$ is the conserved specific angular momentum of the fluid and $\Lambda$ is given by Eq. (33). Using the normalization condition for the velocity, it is easy to check that the quantity in parenthesis in the denominator is always greater than unity. Secondly, $v_0$ can never be vanishing for our case, since we have assumed the angular momentum to be ‘sufficiently low’ so that there is no turning point. Then since $c_s$ is never vanishing, the above amplitude remains bounded everywhere.

We note that $\Lambda$ here stands for the most general height model. If we set $\beta = 0$ in Eq. (33), we recover the results for constant height and conical models.

Setting $\lambda = 0 = \Lambda$ in Eq. (46) recovers the result for the spherical flow, derived earlier in [2, 48].

It is also interesting to note that the effect of $\lambda$ or the azimuthal flow may be to magnify the amplitude of the wave. This can be understood as the centrifugal effect of the angular momentum. Although it seems intuitively obvious, it is nevertheless essential to quantify such statement, as $c_s$ may depend upon $\lambda$. Numerical analysis seems a suitable tool to address this issue.

Before we finish, we shall address here the Rindler spacetime. Let us first write down the Rindler spacetime in the cylindrical coordinates

$$ds^2 = -a^2 x^2 dt^2 + dx^2 + dp^2 + \rho^2 d\phi^2.$$  \hspace{1cm} (47)

The above metric formally looks the same as Eq. (1), with the formal analogy $x \equiv 0$ and $\phi \equiv \rho$. Clearly, the above is an example of a static and axisymmetric spacetime.

We shall consider fluid motion in this spacetime with $\rho = $ constant, and will not consider any accretion disk model. This means that we are simply restricting our attention to flow of fluid in the $x - \phi$ plane. This implies $\alpha = 0 = \beta$ in Eq. (14), for this case.

The normalization condition $v_\rho v^\rho = -1$ gives

$$a^2 x^2 (v_t)^2 \left[ 1 - \frac{a^2 x^2 \lambda^2}{\rho^2} \right] = 1 + v_t^2,$$  \hspace{1cm} (48)

where we have used $\lambda = -\frac{v_0}{c_s}$. We may study acoustic perturbation in this spacetime and derive the symmetric tensor $f_{\mu\nu}$ following exactly the similar procedure described in Section D. The components of $f_{\mu\nu}$ can be readily read off by substituting for the metric functions and setting $\beta = 0$.

It was demonstrated in [2] that there can be no sonic point in the Rindler spacetime with $\lambda = 0$, for a smooth flow. In the following we wish to show the similar conclusion holds if we take angular momentum as well, based on the fact that sound speed can never reach the speed of light.
In order to do this, we substitute for the metric functions (47) into Eq.s (19), and then substitute the result into Eq. (20), to finally get
\[
\frac{du_0}{dx} = \frac{u_0(1 - u_0^2)}{x(u_0^2 - c_{s_0}^2)} \left[ c_{s_0}^2 - 1 - \frac{\lambda^2 a^2 x^2}{\rho^2 - \lambda^2 a^2 x^2} \right]. \tag{49}
\]
At a sonic point, we must have \(c_{s_0}^2 = u_0^2\). Then the above equation shows, for \(x > 0\), to make \(\frac{du_0}{dx}\) finite at the sonic point we must set the numerator to be vanishing as well. This means either \(u_0^2 = 1\) or \(c_{s_0}^2 = 1 + \frac{\lambda^2 a^2 x^2}{\rho^2 - \lambda^2 a^2 x^2}\). None of them can be possible for the following reason. First, if \(u_0^2 = 1\), we must have \(c_{s_0}^2 = 1\) as well, which is impossible.

The second condition gives, using Eq. (48), that \(c_{s_0}^2 \geq 1\), which is impossible.

As we approach the Rindler horizon \(x \to 0\), it is easy to check that \(u_0 \to 1\). Then the Rindler horizon itself cannot be sonic point as well because otherwise we shall require \(c_{s_0}^2 = 1\) on the horizon.

Using the L'Hospital's rule, we also find on the Rindler horizon,
\[
\lim_{x \to 0} \frac{du_0}{dx} = 0, \tag{50}
\]
which can be interpreted as the smoothness of the near horizon geometry.

This interesting result can in fact be generalized for flow along all four directions as the following. Let us replace the circular part \((d\rho^2 + \rho^2 d\phi^2)\) in Eq. (47) by the Euclidean 2-planes: \((dy^2 + dz^2)\). Since we have isometries along \(\partial_y\) and \(\partial_z\), we can define conserved specific momenta along those directions: \(k_y = -\frac{v_y}{v_t}\) and \(k_z = -\frac{v_z}{v_t}\). Then the normalization condition for the fluid's four-velocity becomes

\[
a^2 x^2 (v^t)^2 \left[ 1 - a^2 x^2 k^2 \right] = 1 + v^2, \tag{51}
\]
where we have written \(k^2 = k_y^2 + k_z^2\). Next we define the transformations for a local frame

\[
v^t_0 = \frac{1}{\sqrt{a^2 x^2 (1 - a^2 x^2 k^2)}}, \quad v^0_0 = \frac{1}{\sqrt{1 - u_0^2}}, \quad v^y_0 = \frac{ax k_y}{\sqrt{(1 - a^2 x^2 k^2)}}, \quad v^z_0 = \frac{ax k_z}{\sqrt{(1 - a^2 x^2 k^2)}},
\]

Then we find after a little algebra
\[
\frac{du_0}{dx} = \frac{u_0(1 - u_0^2)}{x(u_0^2 - c_{s_0}^2)} \left[ c_{s_0}^2 - 1 - \frac{1}{k^2 a^2 x^2} \right]. \tag{53}
\]
The normalization condition, Eq. (51), shows that we have always \(a^2 x^2 k^2 < 1\). Then the similar procedure as earlier proves that there can be no sonic point, \(u_0^2 = c_{s_0}^2\).

It is easy to see that Eq. (50) holds in this case as well.

**F. Discussions**

In this work we have investigated the perturbation of the mass accretion rate for axisymmetric flows in static spacetimes. We have demonstrated the natural emergence of the effective acoustic geometry through which the acoustic perturbation propagates, using general model for the thickness of the accretion disk. So far the acoustic geometry or acoustic analogue of gravity has been studied via the perturbation of the velocity potential, see e.g. [57] and references therein. The emergence of the acoustic geometry via perturbation of the mass accretion rate seems to put a thread of connection between accretion astrophysics and acoustic analogue gravity. We also note that the mass accretion rate is an observable and in principle measurable quantity.

We also note that we have ignored any axisymmetric perturbation (i.e. \(\phi\)-dependence of \(\Psi'\)) throughout. This seems perfectly reasonable in astrophysical perspective, but nevertheless it would be interesting to extend the present analysis to include \(\phi\)-dependence. We hope to return to this issue sometime in future.

We have discussed stability issues for the standing and the travelling waves. For the standing waves, in particular, we have demonstrated that for nontrivial disk geometry, there can be either damping or instability. It will also be interesting to investigate this effect in details. Possibly, one may numerically investigate this phenomenon by using the components of \(f^{\mu\nu}\). We have also provided a formal expression for the amplitude of the travelling waves (Eq.s 44, 46). For the Schwarzschild spacetime, in particular, we have argued that the amplitude can never become divergent.

We have also proved for the Rindler spacetime that there can be no sonic point, provided the velocity of the fluid is smooth.

In a forthcoming paper, we shall further generalize most of these results for accretion flows in stationary axisymmetric spacetimes.

**Acknowledgements**

Long term visit of the DBA at HRI has been supported by the planned project fund of the Cosmology and the High Energy Astrophysics subproject of HRI.
