On the physical relevance of random walks: an example of random walks on a randomly oriented lattice

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Abstract. Random walks on general graphs play an important role in the understanding of the general theory of stochastic processes. Beyond their fundamental interest in probability theory, they arise also as simple models of physical systems. A brief survey of the physical relevance of the notion of random walk on both undirected and directed graphs is given followed by the exposition of some recent results on random walks on randomly oriented lattices. It is worth noticing that general undirected graphs are associated with (not necessarily Abelian) groups while directed graphs are associated with (not necessarily Abelian) $C^*$-algebras. Since quantum mechanics is naturally formulated in terms of $C^*$-algebras, the study of random walks on directed lattices has been motivated lately by the development of the new field of quantum information and communication.

1. Introduction

Random walks are mathematical objects thoroughly studied nowadays by probabilists for their own interest but also for shedding new light into a variety of mathematical problems like diffusions on manifolds, harmonic analysis, infinite graph theory, group theory, etc. The other contributions in this volume give an overview of the most recent developments of random walks in connection with most of these mathematical disciplines.

An interesting class of problems concerns random walks in random environments. The present contribution intends to present some recent results on a particular kind of random environment defined by the orientation of some edges of...
the graph on which the random walk evolves. However, we felt that instead of merely announcing these results and reproducing the proofs — that can anyway be found elsewhere — it should be useful for the mathematical community to enlarge this report by giving a short overview on the physical relevance of random walks both on undirected lattices and directed lattices (based on some recent connections between them and quantum evolution, established recently in [28] and in [34]). Therefore, this contribution is organised as follows. In section 2 we give the notation, definitions, and our main results on the probabilistic problem we have studied, in section 3 we give a short overview of the usefulness of random walks on undirected lattices as representations of physical quantities like Green’s functions, in section 4 we recall an algebraic construction of (partially) oriented walks on undirected lattices as representations of physical quantities like Green’s functions, in section 5 we sketch the proof of our results; for detailed proofs the reader can consult [9].

2. Notation, definitions, and main results

We denote \( \mathbb{N} = \{0,1,2,\ldots\} \) and \( \mathbb{N}_+ = \{1,2,3,\ldots\} \). An oriented (or equivalently directed) graph \( \mathbb{G} \) is the quadruple \( \mathbb{G} = (G^0,G^1,r,s) \), where \( G^0 \) and \( G^1 \) are denumerable sets called respectively vertex and edge sets and \( r,s \) are mappings \( G^1 \to G^0 \) called respectively range and source functions. For every \( v \in G^0 \), the set \( N^+_v = s^{-1}(\{v\}) \subseteq G^1 \) represents the set of outgoing edges from vertex \( v \) and its cardinality \( d^+_v = \text{card} N^+_v \) is called the outwards degree of \( v \). We assume that our graphs are row-finite i.e. \( d^+_v < \infty, \forall v \in G^0 \). Similarly, we denote \( N^-_v = r^{-1}(\{v\}) \subseteq G^1 \) the set of incoming edges at vertex \( v \) and \( d^-_v = \text{card} N^-_v \) the inwards degree of \( v \). We assume that our graphs are locally finite i.e. \( d^+_v < \infty, \forall v \in G^0 \). Finally, we denote \( N_v = N^+_v \cup N^-_v \) and \( d_v = \text{card} N_v \). The sets \( G^0 \) and \( G^1 \) are primitive objects for the graph \( \mathbb{G} \). For every \( n \in \mathbb{N}_+ \), we define a sequence of derived objects

\[ G^n = \{ \alpha = (a_1,\ldots,a_n) : a_i \in G^1, i = 1,\ldots,n \text{ and } r(a_i) = s(a_{i+1}), i = 1,\ldots,n-1 \}, \]

called oriented paths of length \( n \), i.e. composed of \( n \) edges of \( G^1 \). For \( \alpha \in G^n \), we denote \( |\alpha| = n \) its length. The mappings \( r,s \), initially defined on \( G^1 \) are naturally extended to \( G^n \): for \( \alpha = a_1\ldots a_n \), \( r(\alpha) \equiv r(a_n) \) and \( s(\alpha) \equiv s(a_1) \). The vertices \( v \in G^0 \) are considered as paths of length 0 and if \( \alpha \in G^0 \) then \( r(\alpha) = s(\alpha) = \alpha \).

The set \( G^* = \cup_{n \in \mathbb{N}} G^n \) represents the set of oriented paths of arbitrary (finite) length and

\[ \partial G^* \equiv G^\infty = \{ \alpha = (a_i)_{i=0}^\infty : s(a_{i+1}) = r(a_i), i \in \mathbb{N} \} \]

represents the set of infinite paths. Notice that thus the set of oriented paths of \( \mathbb{G} \) acquires a natural tree structure.
All graphs considered here are assumed transitive, i.e. for every \( u, v \in G^0 \), there is an \( \alpha \in G^1 \) such that \( s(\alpha) = u \) and \( r(\alpha) = v \). Transitivity implies in particular the no-sink condition \( d_u^{-} \geq 1 \), for all \( v \in G^0 \).

**Remark 2.1.** Notice that the above definition of \( G \) is quite general.

- It does not exclude multiple edges since no conditions are imposed on the mappings \( r, s \). If both functions are injective then all edges are simple and moreover each edge can be thought as a pair of vertices, i.e. in that case \( G^1 \subseteq G^0 \times G^0 \).

- It does not exclude loops (of length 1) since it is not excluded that \( r(\alpha) = s(\alpha) \) for some \( \alpha \in G^1 \). We say that the graph has no loops if \( r(\alpha) \neq s(\alpha) \) for all \( \alpha \in G^1 \).

- Finally, the existence of unoriented edges is not excluded since an unoriented edge can be thought as a pair of oriented edges \( \alpha, \beta \in G^1 \) with \( r(\alpha) = s(\beta) \) and \( r(\beta) = s(\alpha) \).

**Definition 2.2.** Let \( G = (G^0, G^1, r, s) \) be an oriented, transitive, row-finite graph. A simple random walk on \( G \) is a \( G^0 \)-valued Markov chain \( (M_n)_{n \in \mathbb{N}} \) with stochastic matrix defined by

\[
P_{u,v} = P(M_{n+1} = v | M_n = u) = \begin{cases} \frac{1}{d_u^{-}} & \text{if } \exists \alpha \in G^1 : r(\alpha) = v, s(\alpha) = u \\ 0 & \text{otherwise.} \end{cases}
\]

In the sequel, the graphs we shall consider are always transitive, row-finite, without multiple edges and without loops (of length 1). To simplify notation, we denote \( V \equiv G^0 \) and \( A \equiv G^1 \subseteq V \times V \setminus \text{diag}(V \times V) \). The range and source functions for the edges are then naturally determined and we denote the graph simply \( G = (V, A) \).

**Definition 2.3.** Let \( V = V_1 \times V_2 = \mathbb{Z}^2 \), with \( V_1 \) and \( V_2 \) isomorphic to \( \mathbb{Z} \) and \( \epsilon = (\epsilon_y)_{y \in V_2} \) be a sequence of \( \{-1,1\} \)-valued variables assigned to each ordinate. We call \( \epsilon \)-horizontally oriented lattice \( G = G(V, \epsilon) \), the directed graph with vertex set \( V = \mathbb{Z}^2 \) and edge set \( A \) defined by the condition \((u, v) \in A \) if, and only if, \( u \) and \( v \) are distinct vertices satisfying one of the following conditions:

1. either \( v_1 = u_1 \) and \( v_2 = u_2 \pm 1 \),
2. or \( v_2 = u_2 \) and \( v_1 = u_1 + \epsilon u_2 \).

**Example 2.4 (The Alternate lattice \( L \)).** In that case, \( \epsilon \) is the deterministic sequence \( \epsilon_y = (-1)^y \) for \( y \in V_2 \). The figure 1 depicts a part of this graph.

**Example 2.5 (The half-plane one-way lattice \( H \)).** Here \( \epsilon \) is the deterministic sequence

\[
\epsilon_y = \begin{cases} 1 & \text{if } y \geq 0 \\ -1 & \text{if } y < 0. \end{cases}
\]

The figure 2 depicts a part of this graph.
Figure 1: The alternately directed lattice $\mathbb{L}$ corresponding to the choice $\epsilon_y = (-1)^y$.

Figure 2: The half-plane one-way lattice $\mathbb{H}$ with $\epsilon_y = -1$, if $y < 0$ and $\epsilon_y = 1$, if $y \geq 0$.

Example 2.6 (The lattice with random horizontal orientations $\mathbb{O}_\epsilon$). Here $\epsilon = (\epsilon_y)_{y \in \mathbb{V}_2}$ is a sequence of Rademacher, i.e. $\{-1, 1\}$-valued symmetric Bernoulli random variables, that are independent for different values of $y$. The figure depicts part of a realisation of this graph. The random sequence $\epsilon$ is also termed the environment of random horizontal directions.

Figure 3: The randomly horizontally directed lattice $\mathbb{O}_\epsilon$ with $(\epsilon_y)_{y \in \mathbb{Z}}$ an independent and identically distributed sequence of Rademacher random variables.

Now we can state our main results.

**Theorem 2.7.** The simple random walk on the alternate lattice $\mathbb{L}$ is recurrent.
Remark 2.8. This result can be easily generalised to any lattice with periodically alternating horizontal directions (for every finite period).

Theorem 2.9. The simple random walk on the half-plane one-way lattice $H$ is transient.

Remark 2.10. The result concerning transience in theorem 2.9 is robust. In particular, perturbing the orientation of any finite set of horizontal lines either by reversing the orientation of these lines or by transforming them into two-ways does not change the transient behaviour of the simple random walk. Therefore, the half-plane one-way lattice is so deeply in the transience region that the asymptotic behaviour of the simple random walk cannot be changed by simply modifying the transition probabilities along a lower dimensional manifold as was the case in [31] where the bulk behaviour is on the critical point and it can be changed by lower-dimensional perturbations.

Theorem 2.11. For almost all realisations of the environment $\epsilon$, the simple random walk on the randomly horizontally oriented lattice $O_\epsilon$ is transient and its speed is 0.

3. The physical relevance of random walks on undirected lattices

3.1. A brief history

The original impetus for the study of the continuous time analogue of a random walk, the Brownian motion, was given by the seminal work of Einstein [14] on diffusions.\footnote{A more easily accessible source than the original paper [14] exposing the main ideas in an informal but fascinating style, is the commented scientific biography of Einstein [32].} The discrete time process, we nowadays call simple random walk, was first studied by Pólya [35]. It is remarkable however that, contrary to Brownian motion whose physical motivation lies in the very definition of the model, the intrinsic physical relevance of random walks was discovered much later, with the development of polymer physics [16]. Polymers are long, topologically one dimensional molecules, composed by repeating several times (typically 100–10000) the same structural unit. The structural unit can be viewed as a small straight and rigid segment that can be glued with subsequent segments by loose bonds in such a way that if the first segment is held fixed, the second segment forms with the previous
one a given angle \( \theta \) (depending only on the chemical nature of the molecule) but otherwise its position is arbitrary. Assuming that the structural units have unit length, they merely represent directions \( \hat{x} \in \mathbb{S}^2 \). Hence, denoting \( C(\hat{x}, e) \) the cone of opening \( 2\theta \), having its apex at the end point, \( e \), of a segment and its axis collinear with \( \hat{x} \), the second segment will lie on an arbitrary separatrix of \( C(\hat{x}, e) \). One thus immediately recognises a \( \mathbb{R}^3 \)-valued discrete time random process \( (S_n)_{n \in \mathbb{N}} \) with \( S_0 = 0, S_1 = \xi_1 \), and for \( n \geq 2 \), \( S_n = S_{n-1} + \xi_n \) with \( \xi_1 \) uniformly distributed in \( \mathbb{S}^2(0,1) \), and \( \xi_n \) uniformly distributed in \( \mathbb{S}^2(S_{n-1},1) \cap C(\xi_{n-1},S_{n-1}) \), where \( \mathbb{S}^2(x,r) \) is the sphere of center \( x \) and radius \( r \). Here the “time” \( n \) indexing the process corresponds to the order of appearance of a given monomer inside the macromolecular chain. For any bounded measurable function \( f : \mathbb{R}^3 \to \mathbb{R} \), we have then

\[
\mathbb{E}(f(S_n)|\mathcal{F}_{n-1}) = \int_0^{2\pi} f(S_{n-1} + \sin \theta \cos \phi e_1 + \sin \theta \sin \phi e_2 + \cos \theta e_3) \frac{d\phi}{2\pi},
\]

where \( \mathcal{F}_n = \sigma(S_k, k \leq n) \) and \( e_1, e_2, e_3 \) is the canonical basis of \( \mathbb{R}^3 \). A natural simplification of the model consists in considering \( (\xi_n) \) a sequence of independent and identically distributed random variables, getting thus a simple random walk on \( \mathbb{R}^3 \), an object that has been extensively studied and generalised in various respects and especially on non-commutative groups. It is not the intention of the authors to report further in this direction since it is perfectly well known by the community of probabilists and excellent monographs have been devoted to the subject (see [39], [38], [45] for instance), but to report on some aspects developed mainly by physicists and less known by probabilists.

The model of simple random walk is too naive to realistically model physical polymers: two different atoms cannot occupy the same position. Hence a realistic model must be self-avoiding, spoiling thus the Markovian character of the process. Consider the simplest random walk on \( \mathbb{Z}^d \), the nearest neighbour random walk, i.e. let \( E_d = \{ \pm e_1, \ldots, \pm e_d \} \), where \((e_i)_{i=1,\ldots,d} \) denote the standard basis of \( \mathbb{Z}^d \), and let \((\xi_n)_{n \in \mathbb{N}} \) be an independent and identically uniformly distributed sequence of \( E_d \)-valued random variables. Then the process defined by \( S_0 = 0 \) and \( S_n = S_{n-1} + \xi_n \) for \( n \geq 1 \), provides the Markovian description of the ordinary random walk. If we are interested only to a finite sequence \( (S_n)_{n=0}^N \), an equivalent description is provided by the trajectory space

\[
\Omega_N = \{ \omega : \{0, \ldots, N\} \to \mathbb{Z}^d \mid \omega(0) = 0, \omega(i) - \omega(i-1) \in E_d, i = 1, \ldots, N \},
\]

equipped with the uniform probability measure \( \mu_N(\omega) = 1/c_N \) for all \( \omega \in \Omega_N \) where \( c_n = \text{card} \Omega_N = (2d)^N \). For \( k : 0 \leq k \leq N \), the canonical projection \( S_k(\omega) = \omega_k \) has the same law as the random walk defined by the sum \( \sum_{i=1}^k \xi_i \), showing thus the equivalence of the Markovian and trajectoryal descriptions for simple random walks. Adding the self-avoiding condition can be performed on the trajectoryal description but not on the Markovian one. More precisely, let

\[
\Omega_N^m = \{ \omega \in \Omega_N : \omega(i) \neq \omega(j), \text{ for } 0 \leq i < j \leq N \}
and \( \mu_{\text{sw}}(\omega) = 1/c_{\text{sw}}^N \) for all \( \omega \in \Omega_{\text{sw}} \) with \( c_{\text{sw}}^N = \text{card} \Omega_{\text{sw}}^N \). Notice however that the numerical sequence \( (c_{\text{sw}}^N)_{N \in \mathbb{N}} \) is not explicitly known for \( d \geq 2 \) hence the model of self-avoiding random walk has been so far intractable. Nevertheless, the sequence of probability measures \( (\mu_{\text{sw}}^N)_{N \in \mathbb{N}} \) — the probability \( \mu_{\text{sw}}^N \) being defined on \( \Omega_{\text{sw}}^N \) for every \( N \in \mathbb{N} \) — is perfectly well defined although intractable.

Instead of defining \( \mu_{\text{sw}}^N \) on \( \Omega_{\text{sw}}^N \) we can also define it on \( \Omega_N \) by

\[
\mu_{\text{sw}}^N(d\omega) = \frac{1}{Z_{\text{sw}}^N} \mathbb{1}_{\Omega_{\text{sw}}^N}(\omega) \mu_N(d\omega),
\]

where \( Z_{\text{sw}}^N \) is a normalising factor, in fact \( Z_{\text{sw}}^N = c_{\text{sw}}^N/(2d)^N \). Physicists have introduced various approximations to deal with the untractable measure \( \mu_{\text{sw}}^N \). One of them consists in approximating the indicator appearing in the previous formula and defining

\[
\mu_{N,\beta}(d\omega) = \frac{1}{Z_N(\beta)} \exp(-\beta H_N(\omega)) \mu_N(d\omega)
\]

where \( H_N(\omega) = \text{card}\{k : 2 \leq k \leq N|\exists j < k \text{ with } \omega(j) = \omega(k)\} \) (or some variant of this number) counts the self-intersections of the trajectory \( \omega \in \Omega_N \), the real parameter \( \beta \) is non-negative, and \( Z_N(\beta) \) is a normalising factor, in fact \( Z_N(\beta) = \int_{\Omega_N} \exp(-\beta H_N(\omega)) \mu_N(d\omega) \). The continuous version of this model has been introduced by Edwards in \([13]\). The discrete version, defined in \((3.2)\), is known as discrete Edwards random walk or weakly self-avoiding random walk. The reason for this nomenclature is the following. For fixed \( N \), we have that \( \lim_{\beta \to 0} \mu_{N,\beta} = \mu_N \) and \( \lim_{\beta \to \infty} \mu_{N,\beta} = \mu_{\text{sw}}^N \) so that the weakly self-avoiding walk interpolates between ordinary and self-avoiding random walk. A much more difficult limit to study is \( N \to \infty \) for fixed \( \beta \in [0, \infty[ \); this limit is dimension dependent and several authors have contributed to its study \([40, 7, 43, 43, 27, 26, 19, 29, 18, 42, 3, 14, 23, 12, 21]\).

The formula \((3.2)\) has an interest far beyond its application to the study of self-avoiding random walks since it is reminiscent of ideas at the basis of Gibbs formulation of statistical mechanics and quantum field theory. The Radon-Nikodým derivative \( d\mu_{N,\beta}/d\mu_N = \exp(-\beta H_N(\omega))/Z_N(\beta) \) is interpreted as a Boltzmann factor, making more rare trajectories with many self-intersections in the statistical sample described by the measure \( \mu_N \). The limit \( N \to \infty \) is also a standard procedure in statistical mechanics and quantum field theory known as thermodynamic limit or infrared limit respectively. Thus, this formula is naturally connecting random walks and various other physical theories defined on an infinite graph for which the fundamental quantities can be written as (formal) random walk expansions. In some particular situations, these formal expansions converge; they provide therefore an valuable probabilistic tool for the study of the asymptotic behaviour of physical models.

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\( ^2 \)The day of submission of the present contribution, we learnt about a new result on weakly self-avoiding walks \([3]\).
3.2. An example of random walk expansion

It is immediate to see that the Markov operator for a simple random walk on an undirected graph is essentially the discrete Laplacian on the graph. Random walk expansions can take a very sophisticated formulation; all of them can be seen however as (non-trivial) generalisations of a very simple formula of inversion of the Markov operator that connects the Green’s function of the regularised Laplacian with a power series on random walks of arbitrary length given in the following

Lemma 3.1. Let $\Delta$ be the difference Laplacian on $\mathbb{Z}^d$ and $m \neq 0$ a fixed parameter (the free mass of a quantum field theory). Then for any $x, y \in \mathbb{Z}^d$,

$$\left( m^2 \mathbb{1} - \Delta \right)_x^y = \sum_{\omega \in G^*(x,y)} (2d + m^2)^{-|\omega|},$$

where $G^*(x,y) = \cup_{n=0}^{\infty} G^n(x,y)$ and the set $G^n(x,y)$ is the set of paths of length $n$ on the non-oriented graph $\mathbb{G}$ having $\mathbb{Z}^d$ as vertex set and with nearest neighbours vertices as edge set, that start at point $x$ and end at point $y$.

Proof. Write $\Delta = J - 2d \mathbb{1}$ where

$$J_{xy} = \begin{cases} 1 & \text{if } y - x \in E_d \\ 0 & \text{otherwise} \end{cases}$$

and develop $\left( m^2 + 2d \right) \mathbb{1} - J$ as formal series in powers of $J$. For $m \neq 0$ the series converges defining thus the left-hand side of the formula. □

Consider now a general graph $\mathbb{G} = (G^0, G^1, r, s)$ with $G^1$ a particular subset of $G^0 \times G^0 \setminus \text{diag}(G^0 \times G^0)$ (i.e. the graph is simple and without loops); we assume moreover that the graph is undirected, i.e. if $(u,v) \in G^1$ then $(v,u) \in G^1$. Let $J$ and $L$ be $G^0 \times G^0$ matrices such that $J_{uv} > 0$ if $v \in N_u$ and $J_{uv} = 0$ otherwise and $L_{uv} = \lambda_u \delta_{uv}$, with $\lambda_u > 0$ for all $u \in G^0$.

Lemma 3.2. Suppose that the parameters $(\lambda_u)_{u \in G^0}$ are large enough. Then

1. $$\left( L - J \right)^{-1}_{uv} = \sum_{\omega \in G^*(u,v)} \prod_{a \in \omega} J_a \prod_{v \in G^0} \lambda_v^{-\eta_{\omega}(v,\omega)},$$

where $\eta_{\omega}(v,\omega) = \sum_{k=0}^{|\omega|} |\omega_k| v(\omega_k)$ is the occupation time of the vertex $v$ by the trajectory $\omega$.

2. $$\det(L - J)^{-1} = \left( \prod_{u \in G^0} \lambda_u \right)^{-1} \exp \left( \sum_{\omega \in \mathcal{L}^*} J_{\omega} \left( \prod_{u \in G^0} \lambda_u^{-\eta_{\omega}(u,\omega)} \right) \right),$$

where $J_{\omega} = \prod_{a \in \omega} J_a$ and $\mathcal{L}^*$ is the set of loops of arbitrary length (i.e. equivalence classes of random walks of arbitrary length starting and ending on the same vertex.)
**Proof.** The matrix $L$ is diagonal and invertible. The formulae are again obtained by standard formal power series expansions that converge when $\lambda_n$ are chosen sufficiently large (see [3] for details.)

Consider always a simple undirected graph without loops $G = (G^0, G^1, r, s)$ and let $f_v : \mathbb{R} \rightarrow \mathbb{R}$ be a family of maps indexed by $v \in G^0$ such that $\lim_{t \rightarrow \infty} f_v(t) \exp(ct) = 0$ for some $c > 0$. Denote $X = \{x : G^0 \rightarrow \mathbb{R}^v\} \simeq (\mathbb{R}^v)^{G^0}$ the configuration space, so that each configuration $x \in X$ is the collection $(x_v)_{v \in G^0}$ with $x_v \in \mathbb{R}^v$. The space $\mathbb{R}^v$ is equipped with its Borel $\sigma$-algebra, $\mathcal{B}(\mathbb{R}^v)$ and let $\kappa_n$ be a family of continuous measures on $(\mathbb{R}^v, \mathcal{B}(\mathbb{R}^v))$ defined to have a density with respect to the $\nu$-dimensional Lebesgue measure $\kappa_n(dx_u) = f_u(x^2)^d x_u = f_u(x^2_{u,1} + \ldots x^2_{u,v})dx_{u,1} \ldots dx_{u,v}$. The configuration space can be naturally equipped with a product measure structure $\prod_{u \in G^0} \kappa_u$. However, a product measure structure is not very interesting for physical purposes since it corresponds to an infinite system of non-interacting components. To introduce some interaction, let $J$ be an infinite $G^0 \times G^0$ matrix with $J_{uv} = J_{vu} > 0$ when $v \in N_u$ and $J_{uv} = 0$ otherwise. Since the matrix is symmetric, it defines a quadratic form on $X$, known as (formal) Hamiltonian $H(x) = -\frac{1}{2} \sum_{u,v \in G^0} (x_u, J_{uv} x_v) = -\frac{1}{2} \sum_{u,v \in G^0} \sum_{\alpha=1}^{\nu} x_{u,\alpha} J_{uv} x_{v,\alpha}$. We can then define (formally) a probability $\mu$ on $(X, \mathcal{F})$, where $\mathcal{F}$ is the natural $\sigma$-algebra, by

$$\mu(A) = \frac{1}{Z} \int_A \exp(-H(x)) \prod_{v \in G^0} (f_v(x^2_v)dx_v),$$

where $Z$ is a normalising factor. There are various standard procedures to give a mathematical meaning to the above expressions. One of them is the following: suppose that the graph $G$ is isometrically embedded in $\mathbb{R}^d$ for some $d$. Let $\Lambda_n = [-n,n]^d \cap G^0$ be the set of vertices of the graph inside an hypercubic box of size $2n + 1$. Then, we define the finite volume Hamiltonian

$$H_n(x) = -\frac{1}{2} \sum_{u,v \in \Lambda_n} (x_u, J_{uv} x_v)$$

and the finite volume probability

$$\mu_n(A) = \frac{1}{Z_n} \int_A \exp(-H_n(x)) \prod_{v \in \Lambda_n} (f_v(x^2_v)dx_v),$$

where $Z_n$ is the corresponding normalising factor. The sequence $(\mu_n)$ is a perfectly well defined sequence of probability measures. When this sequence converges weakly, we call the weak limit infinite volume Gibbs measure associated with the Hamiltonian $H$. Notice however that the existence of the weak limit is highly non-trivial and it is granted only for some cases (see [10, 3, 17, 30] for instance).

**Theorem 3.3 (Symanzik).** Assume that the Hamiltonian is such that the weak limit $\mu$ exists. For $p = 1, 2, \ldots$ let $\{v_1, \ldots, v_{2p}\}$ be a given set of vertices; partition this set into $p$ disjoint pairs. For each such pair of vertices let $\omega^{(i)}$ be a random walk of arbitrary length starting in one vertex and ending to the other vertex of
the pair. Then
\[
\mu(x_{v_1,\alpha_1} \cdots x_{v_2,\alpha_2}) = \int x_{v_1,\alpha_1} \cdots x_{v_2,\alpha_2} \mu(dx) = \sum_{\omega(1), \ldots, \omega(p)} \frac{W(\omega(1), \ldots, \omega(p))}{Z},
\]
where the sum extends over all partitions of the set of vertices and all random walk defined on the \(p\) pairs and
\[
W(\omega(1), \ldots, \omega(p)) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\nu}{2}\right)^n \sum_{\gamma(1), \ldots, \gamma(n) \in \mathcal{L}^*} \prod_{m=1}^{n} J_{\gamma(m)} \prod_{l=1}^{p} J_{\omega(l)} \exp(-U(\gamma(1), \ldots, \gamma(n), \omega(1), \ldots, \omega(p))),
\]
where \(\mathcal{L}^*\) denotes the set of loops of arbitrary length. Moreover, the normalising factor reads
\[
Z = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\nu}{2}\right)^n \sum_{\gamma(1), \ldots, \gamma(n) \in \mathcal{L}^*} \prod_{m=1}^{n} J_{\gamma(m)} \exp(-U(\gamma(1), \ldots, \gamma(n))),
\]
and the mapping \(U\) defined on an arbitrary Cartesian product of random walks is given by
\[
\exp(-U(\omega(1), \ldots, \omega(k))) = \prod_{l=1}^{k} \int_{\Gamma} \hat{f}_v(z_v)(2\pi z_v)^{-h_v(\omega(1), \ldots, \omega(k))} dz_v,
\]
\(\hat{f}\) being the Fourier transform of \(f\), \(h_v(\omega(1), \ldots, \omega(k)) = \sum_{l=1}^{k} \eta_{|\omega(l)|}(v, \omega(l)) + \frac{\nu}{2}\), and \(\Gamma\) an appropriately chosen integration contour.

Remark: The previous theorem looks formidable. It is worth noticing however that it is nothing more than a clever combinatorial recombination of terms appearing in the power series expansion of the exponential and that far-reaching results in quantum field theory \([1, 15]\) — impossible or much more difficult to obtain otherwise — are obtained by the random walk representation it provides. Variants of this representation — known under the generic name of cluster expansions or abstract polymer models \([30, 17]\) — are used in many different contexts, like statistical mechanics, disordered systems etc. and in the cases it can be rigorously applied it provides a very powerful probabilistic tool for the study of covariance properties of limit Gibbs measures.

4. The physical relevance of random walks on oriented lattices
For any simple directed graph without loops $G = (G^0, G^1, r, s)$ we define two operators \[ D : \ell^2(G^0) \to \ell^2(G^1) \] by $(D f)(a) = f(s(a)) - f(r(a))$ for every $a \in G^1$ and its adjoint $D^* : \ell^2(G^1) \to \ell^2(G^0)$ by $(D^* \phi)(v) = \sum_{a \in s^{-1}(v)} \phi(a) - \sum_{a \in r^{-1}(v)} \phi(a)$ for every $v \in G^0$. Then $(-D^* D f)(v) = -d_v f(v) + \sum_{u \in N_v} f(u) \equiv (\Delta f)(v)$. For a simple random walk on $G$, defined by its stochastic matrix $(P(u,v))_{u,v \in G^0}$, the Markov operator $M$ defined on bounded functions $f$ by $M f(u) = \sum_{v \in G^0} P(u,v) f(v) - f(u)$ — contrary to the case of unoriented graphs where it is expressed in terms of the Laplacian — cannot be expressed in terms of the Laplacian since $M f(u) = \frac{1}{d_u} \sum_{a \in s^{-1}(u)} D f(a) \neq \frac{1}{d_u} (\Delta f)(u)$. As a matter of fact, $M$ is (roughly) reminiscent of the Dirac operator on the graph, providing thus the first hint that random walks on oriented lattices are relevant for non-commutative geometry.

### 4.1. A $C^*$-algebraic description of oriented lattices

With every oriented graph we can associate a $C^*$-algebra of operators, known as the Cuntz-Krieger algebra \[ \mathcal{O}_A \] of the graph \[ G^1 \].

Let $(V_i)_{i \in I}$ be a finite or denumerable family of non-zero partial isometries and $A$ a $I \times I \{0,1\}$-valued matrix whose rows contain a finite number of ones. The Cuntz-Krieger algebra, $\mathcal{O}_A$, associated with the matrix $A$ is the $C^*$-algebra, defined up to isomorphisms, by the relations

$$V_i^* V_i = \sum_{j \in I} A_{ij} V_j V_j^*, \ i \in I.$$ 

The connection of the $\mathcal{O}_A$ algebra with oriented graphs is done as follows. Let $G = (G^0, G^1, r, s)$ be a row-finite, locally finite graph and consider the corresponding path space $G^*$ defined in section \[ 3 \]. Let $(P_v)_{v \in G^0}$ be a set of mutually orthogonal projections and $(V_a)_{a \in G^1}$ a set of non-zero partial isometries satisfying

$$V_a^* V_a = P_{r(a)}, \ \forall a \in G^1 \text{ and } P_v = \sum_{a \in s^{-1}(v)} V_a V_a^*, \ \forall v \in G^0.$$ 

Define the edge-matrix $(A_G(a,b))_{a,b \in G^1}$ of the graph $G$ by

$$A_G(a,b) = \begin{cases} 1 & \text{if } r(a) = s(b) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$P_{r(a)} = V_a^* V_a = \sum_{b \in G^1, r(a) = s(b)} V_b V_b^* = \sum_{b \in G^1} A_G(a,b) V_b V_b^*.$$ 

The $G^1$ indexed Cuntz-Krieger $C^*$-algebra $\mathcal{O}_{A_G}$ is called the $C^*$-algebra of the graph $G$ and is denoted $C^*(G)$. The partial isometries $V_a$ defined on $G^1$ are naturally extended on the path space $G^*$ for every $\alpha \in G^*$ by

$$V_\alpha = V_{a_1} \cdots V_{a_{n(\alpha)}}.$$
Then we have the

**Theorem 4.1 (Kumjian, Pask, and Raeburn).** Let \( \{P_v, V_a; v \in G^0, a \in G^1\} \) be the Cuntz-Krieger algebra associated with \( G \) and let \( \beta \) and \( \gamma \) be arbitrary paths of \( G^* \). Then

\[
V_{\beta}^* V_{\gamma} = \begin{cases} 
V_{\gamma}' & \text{if } \gamma = \beta \gamma', \gamma' \ni G^0 \\
P_{\gamma}'(\gamma) & \text{if } \gamma = \beta \\
V_{\beta}' & \text{if } \beta = \gamma \beta', \beta' \neq G^0 \\
0 & \text{otherwise.}
\end{cases}
\]

This construction associates with every path in \( G^* \) an operator of \( C^*(G) \). Therefore the \( C^* \)-algebra \( C^*(G) \) can be thought as the non-commutative analogue of the subshift space \( \mathbb{G} \), corresponding to the matrix \( A_G \). It is worth noticing that beyond directed lattices, the Cuntz-Krieger algebras are ultimately connected with various other topics, like wavelets \( \mathbb{W} \), tilings \( \mathbb{T} \), generalised sub-shifts \( \mathbb{F} \), non-commutative geometry \( \mathbb{N} \), etc.

### 4.2. Non-reversible evolution of quantum states

It is convincingly argued lately that the progress in semiconductor technology will reach the limits of applicability of classical reasoning used in information theory and computer science because quantum effects will start to be determining \( \mathbb{Q} \), \( \mathbb{S} \), \( \mathbb{T} \), \( \mathbb{X} \).

In classical physics a microscopic state of a multi-component system is described as an element of the Cartesian product of state spaces of individual components; for instance to determine the microstate of a litre of gas, it is necessary to know the precise positions and momenta of all its molecules. The set of all microstates is called *configuration space*. Macroscopic states are classically probability measures on the configuration space and physical observables are bounded measurable real-valued functions on the configuration space. Time evolution is implemented as a Markov semi-group acting on macrostates; it can be reversible when the stochastic kernel of the semigroup is deterministic, or irreversible in general.

In quantum physics the configuration space of microstates is a complex separable Hilbert space \( \mathcal{H} \) (in general infinite dimensional but finite dimensional spaces cases or also of interest.) The macrostates are self-adjoint, positive operators with unit trace that are projections. Such operators are usually called *density matrices*. Observables are self-adjoint bounded operators acting on \( \mathcal{H} \). Time evolution is implemented by a completely positive transformation on the set of density matrices \( \rho \rightarrow \sum_{i \in I} T_i \rho T_{i}^* \). Such evolutions can be reversible — when \( I = \{0\} \) is a singleton and \( T_0 \) is unitary — or irreversible — when we only require that \( \sum_{i \in I} T_i T_{i}^* \leq \text{id}_\mathcal{H} \). One immediately remarks that \( C^* \)-algebras provide a unified approach to both classical (Abelian ones) and quantum (non Abelian ones) systems.
In the context of applications in quantum information and communication, unitary transformations correspond to quantum logical gates while irreversible ones to noisy transmission through quantum channels or to measurements. When a quantum macrostate \( \rho \) is transmitted through a noisy channel, different operators from the family \((T_i)_{i \in I}\) will sequentially act on \( \rho \) to get \( \rho \rightarrow \hat{\rho}_n = \sum_{i_1, \ldots, i_n} T_{i_n} \cdots T_{i_1} \rho T_{i_1}^* \cdots T_{i_n}^* \). It is thus clear that products of operators will appear in the form of products of partial isometries along paths of oriented graphs. The main difference is that we have not required here the evolution to be implemented by partial isometries but by general non-commuting operators satisfying only \( \sum_{i \in I} T_i T_i^* \leq \text{id}_H \). Nevertheless the analogy can be made complete by virtue of the following result [36].

**Theorem 4.2 (Popescu).** For every sequence (finite or denumerable) \((T_i)_{i \in I}\) of non-commuting operators acting on a Hilbert space \( \mathcal{H} \), such that \( \sum_{i \in I} T_i T_i^* \leq \text{id}_H \), there exists a minimal isometric dilation into operators \((V_i)_{i \in I}\) acting on a Hilbert space \( \mathcal{K} \supset \mathcal{H} \), uniquely determined up to isomorphisms, such that the family \((V_i)_{i \in I}\) is composed by non-zero partial isometries on \( \mathcal{K} \).

Now it is evident that random walks on a (partially) directed lattice induce a random walk on the space of density matrices; recovering information from the perturbed macrostate \( \hat{\rho}_n \) will be better when \( \hat{\rho}_n \) is close (with respect to an appropriate topology) to \( \rho \). This remark is made for the first time in [28] and gives a practical physical and technological relevance to questions of recurrence of the random walk on a randomly oriented lattices.

## 5. Sketch of the proofs

The idea of the proof of theorems [2.7, 2.9, and 2.11] is to decompose the Markov chain \((M_n)\) into a vertical skeleton \((Y_n)\) and an horizontal embedded random walk \((X_n)\) that — when sampled on a particular sequence of random times defined in terms of the vertical skeleton — has the same recurrence/transience properties as the original random walk \((M_n)\).

Let \((\psi_n)_{n \in \mathbb{N}_+}\) be a sequence of independent, identically distributed, \([-1, 1]\)-valued symmetric Bernoulli variables and define \(Y_0 = 0\) and

\[
Y_n = \sum_{k=1}^{n} \psi_k, \quad n = 1, 2, \ldots ,
\]

the simple \(\mathbb{V}_2\)-valued symmetric one-dimensional random walk. We call the process \((Y_n)_{n \in \mathbb{N}}\) the **vertical skeleton**. We denote by

\[
\eta_n(y) = \sum_{k=0}^{n} \mathbb{1}_{\{Y_k = y\}}, \quad n \in \mathbb{N}, y \in \mathbb{V}_2
\]
its occupation time at level \( y \). If \( \mathcal{F}_n = \sigma(\psi_i, i \leq n) \) then \( \eta_n(y) \) is obviously \( \mathcal{F}_n \)-measurable.

Define \( \sigma_0 = 0 \) and recursively, for \( n = 1, 2, \ldots, \sigma_n = \inf\{k \geq \sigma_{n-1} : Y_k = 0\} \geq \sigma_{n-1} \), the \( n \text{th} \) return to the origin for the vertical skeleton. It is known by the standard theory of simple symmetric one-dimensional random walk that almost surely \( \sigma_n < \infty \) for all \( n \).

To define the horizontal embedded random walk, let \((\xi_y^n)_{n \in \mathbb{N}, y \in \mathcal{V}}\) be a doubly infinite sequence of independent identically distributed \( \mathbb{N} \)-valued geometric random variables of parameters \( p = 2/3 \) and \( q = 1 - p = 1/3 \), i.e. \( \mathbb{P}(\xi_y^n = \ell) = pq^\ell, \ \ell = 0, 1, 2, \ldots \). Let moreover

\[
T_n = n + \sum_{y \in \mathcal{V}_2} \sum_{i=1}^{\eta_{n-1}(y)} \xi_i(y)
\]

be the instant just after the random walk \((M_k)\) has performed its \( n \text{th} \) vertical move (with the convention that the sum \( \sum_{i=1}^{\eta_{n-1}(y)} \) vanishes whenever \( \eta_{n-1}(y) = 0 \).) Then

\[
M_{T_n} = (X_n, Y_n),
\]

where \((Y_n)\) is the vertical skeleton and

\[
X_n = \sum_{y \in \mathcal{V}_2} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i(y)
\]

represents the total horizontal displacement when the random walk \((M_k)\) has completed exactly \( n \) vertical moves. Notice also that the horizontal embedded random walk \((X_n)\) can be viewed as a random walk with unbounded jumps in a random scenery and that \( M_{T_{\sigma_n}} = (X_{\sigma_n}, 0) \). Now \((M_n)\) can return to the origin if and only if both vertical and horizontal components vanish simultaneously. Since the vertical component can vanish only at instants \( \sigma_n, n \in \mathbb{N} \), we prove in [9] the following

**Lemma 5.1.**

1. \( \sum_{n=0}^{\infty} \mathbb{P}(X_{\sigma_n} = 0) = \infty \) if and only if the random walk \((M_t)\) is recurrent.

2. \( \sum_{n=0}^{\infty} \mathbb{P}(X_{\sigma_n} = 0) < \infty \) if and only if the random walk \((M_t)\) is transient.

Introduce now the characteristic function \( \chi(\theta) = \mathbb{E} \exp(i\theta \xi_0) = \frac{p}{1 - q \exp(i\theta)} = r(\theta) \exp(i\alpha(\theta)) \) and observe that \( r \) is an even function of \( \theta \) while \( \alpha \) is odd. Hence,
denoting $\mathcal{F} = \mathcal{F}_\infty$ and $\mathcal{G} = \sigma(\epsilon_y, y \in \mathbb{V}_2)$, we have

$$
E \exp(i\theta X_n) = E \left( E \left( \exp(i\theta \sum_{y \in \mathbb{V}_2} \epsilon_y \sum_{i=1}^{\eta_{n-1}(y)} \xi_i^{(y)} | \mathcal{F} \vee \mathcal{G} \right) \right)
$$

$$
= E \left( \prod_{y \in \mathbb{V}_2} \chi(\theta \epsilon_y) \eta_{n-1}(y) \right)
$$

$$
= E(r(\theta) \sum_{y \in \mathbb{V}_2} \eta_{n-1}(y) \exp(i\alpha(\theta) \Delta_n)) = r(\theta)^n E(\exp(i\alpha(\theta) \Delta_n))
$$

where $\Delta_n = \sum_{y \in \mathbb{V}_2} \epsilon_y \eta_{n-1}(y)$.

**Lemma 5.2.** For the $\mathbb{L}$ lattice, $\Delta_{\sigma_n} = 0$.

**Proof.** This is an elementary combinatorial result whose complete proof is given in [9]. □

The proof of theorem 2.7 follows now immediately since $E \exp(i\theta X_{\sigma_n}) = E(r(\theta)^{\sigma_n}) = \sqrt{1 - r(\theta)^2}^n$. Hence $P(X_{\sigma_n} = 0) = \lim_{\epsilon \to 0} 2 \int_{\epsilon}^{\pi} \frac{1}{\sqrt{1 - r(\theta)^2}} d\theta = \infty$.

**Remark:** Notice that for the random walk on the $\mathbb{L}$ lattice, various other more elegant proofs can be given; we presented the most elementary one.

For the corresponding result on the $\mathbb{H}$ lattice, the proof is based on the following lemma shown in [8].

**Lemma 5.3.** Denote by $(\rho_k)_{k \in \mathbb{N}}$ a sequence of independent identically distributed Rademacher variables and $(\tau_k)_{k \in \mathbb{N}}$ a sequence of independent, identically distributed random variables, independent of the sequence $(\rho_k)_{k \in \mathbb{N}}$, such that $\tau_1 \overset{d}{=} \sigma_1$, i.e. the random variables $\tau_k$ have the same law as the time of first return to the origin for the skeleton random walk. Then

$$
\Delta_{\sigma_n} \overset{d}{=} \sum_{k=1}^{n} \rho_k(\tau_k - 1) + n.
$$

The proof of the theorem 2.9 follows then from the equality $E(\exp(i\theta X_{\sigma_n})) = g(\theta)^n$, where $g(\theta) = \frac{1}{2} \chi(\theta) \left[ 1 - \sqrt{1 - \chi(\theta)^2} \exp(-i\alpha(\theta)) + \left( 1 - \sqrt{1 - \chi(\theta)^2} \right) \exp(i\alpha(\theta)) \right]$. This expression allows an explicit estimate for $\sum_{n=1}^{\infty} P(X_{\sigma_n} = 0)$ that converges in the present case, establishing transience of the random walk.

Finally for the $\mathbb{G}_\epsilon$ lattice, no simple decomposition can be made and we need joint estimates. The idea of the proof is to decompose the probability $P_n = P(X_{2n} = 0, Y_{2n} = 0)$ into $P_n = P_{n,1} + P_{n,2} + P_{n,3}$, where

$$
P_{n,1} = P(I(X_{2n}, -\epsilon_0 Z) \ni 0; Y_{2n} = 0; B_n)
$$

$$
P_{n,2} = P(I(X_{2n}, -\epsilon_0 Z) \ni 0; Y_{2n} = 0; A_n \setminus B_n)
$$

$$
P_{n,3} = P(I(X_{2n}, -\epsilon_0 Z) \ni 0; Y_{2n} = 0; A_n^c),
$$

and $A_n = A_{n,1} \cap A_{n,2}$ with

$$A_{n,1} = \{ \omega \in \Omega : \max_{0 \leq k \leq 2n} |Y_k| < n^{\frac{1}{2} + \delta_1} \} \text{ for some } \delta_1 > 0,$$

$$A_{n,2} = \{ \omega \in \Omega : \max_{y \in V_2} \eta_{2n-1}(y) < n^{\frac{1}{2} + \delta_2} \} \text{ for some } \delta_2 > 0,$$

$$B_n = \{ \omega \in A_n : \left| \sum_{y \in V_2} \epsilon_y \eta_{2n-1}(y) \right| > n^{\frac{1}{2} + \delta_3} \} \text{ for some } \delta_3 > 0$$

for some $\delta_1 > 0, \delta_2 > 0$ and $\delta_3 > 0$. The technical part of the proof consists in showing that $p_{n,1}$ and $p_{n,3}$ are of order $O(\exp(-n^\delta))$ with some $\delta > 0$ for large $n$ while the main part of the mass charging the event $\{X_{2n} = 0, Y_{2n} = 0\}$ is supported by the set $A_n \setminus B_n$. More precisely it is shown in [9] that on the set $A_n \setminus B_n$ we have $P(X_{2n} = 0|\mathcal{F} \vee \mathcal{G}) = O(\sqrt{\frac{\ln n}{n}})$ while $P(A_n \setminus B_n|\mathcal{F} \vee \mathcal{G}) = O(n^{-1/4+\delta_4})$ so that together with the standard estimate $P(Y_{2n} = 0) = O(1/\sqrt{n})$ the series $\sum_n p_n$ is shown to converge.

Notice also that some additional results concerning the mean speed and law of large numbers are presented in [9].

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