Global transient dynamics of three-dimensional hydrodynamical disturbances in a thin viscous accretion disk

P. Rebusco, 1 O.M. Umurhan, 2, 3 W. Kluzniak, 4, 5 and O. Regev 6, 7

1Kavli Institute for Astrophysics and Space Research, MIT, Cambridge, MA
2Astronomy Unit, School of Mathematical Sciences, QMUL, Mile End Road, London E1 4NS, UK
3Department of Astronomy, City College San Francisco, CA 94112, USA
4Institute of Astronomy, Zielona Góra University, ul. Lubuska 2, 65-265 Zielona Góra, Poland
5Nicolaus Copernicus Astronomical Center, ul. Bartycka 18, 00-716 Warsaw, Poland
6Department of Physics, Technion-Israel Institute of Technology, 32000 Haifa, Israel
7Dept of Astronomy, Columbia University, New York NY, 10027

(Dated: June 11, 2009)

Thin viscous Keplerian accretion disks are considered asymptotically stable, even though they can show significant dynamic activity on short timescales. In this paper the dynamics of non-axisymmetric hydrodynamical disturbances of disks are investigated analytically building upon the steady state three-dimensional structure and evolution of axisymmetric perturbations explored in previous work. Assuming a polytropic equation of state solutions are found by means of an asymptotic expansion in the small parameter measuring the ratio of the disk thickness to characteristic radius. In-depth analysis shows that every perturbation that disturbs the radial velocity induces significant transient growth in the (acoustic) energy of the evolving disturbance. This effect is most evident in the density and vertical velocity. The transient growth observed is tied to the non-separable nature of the solutions where, in particular, pattern evolution is controlled by a similarity variable composed of the radial coordinate and time. This leads to growing winding perturbations that display successive radial peaks and troughs. We argue that these transient non-axisymmetric structures may precipitate secondary instabilities which, consequently, may be a critical element for a new alternative picture of turbulence arousal in non-magnetized astrophysical disks.

PACS numbers: Valid PACS appear here

I. INTRODUCTION

Since the late 1980s a new perspective has developed in hydrodynamic stability theory. This new paradigm arose from the long-standing problem of linearly stable shear flows that experimentally exhibit transition into turbulence. The conventional approach had been to examine the linear stability of fluid systems via modal analysis (“normal-modes”). In practice this means that the determination of the eigenvalues and eigenfunctions of the linearized perturbation equations of a given flow indicates the time asymptotic behavior of the disturbances and, consequently, helps to determine the long-time stability of the base flow. One of the best expositions of this classical approach is found in the book by Drazin & Reid 1.

The origin of the new perspective can be traced to the fact that linear stability analysis of shear flows in general gives rise to non-normal operators, i.e., linear operators that do not commute with their adjoint. A typical modal analysis of a problem governed by a non-normal operator can lead to an incomplete description of the full breadth of responses possible for the original initial-value problem (IVP). For example, non-normal operators will have eigenfunctions that are non-orthogonal, and/or imply the existence of solutions which are unobtainable analytically (see the discussion in 2 Chapter 8).

Especially critical in this matter is the fact that the non-orthogonality of the eigenmodes can lead to transient dynamics which, in turn, can imply strong transient growth (TG) for suitable initial conditions, e.g., in perturbation energy or enstrophy. The implications of this for a variety of shear flows has been studied in numerous earlier works, e.g., 3, 4, 5. The framework for stability calculations has thus shifted from just focusing on the time-asymptotic behavior of a disturbance towards studying its TG as well. The recent review article by Schmid 6 gives an up-to-date
account on non-modal stability theory and its successes. A detailed exposition of the subject, including its different aspects and possible extensions, can be found in the book by Schmid & Hemmigson [7].

Ioannaou & Kakouris [8] were first to apply this non-modal perspective for an astrophysical setting, namely, for the problem of accretion disks (“ADs” and “AD” for singular usage). Accretion disks are both important and ubiquitous astrophysical objects which are thought to power systems as diverse as young stellar objects, close binary systems and active galactic nuclei. ADs are flattened, swirling flows in a gravitational field of a central compact object, wherein high specific angular momentum fluid accretes onto the central object. In order to reconcile theoretical models with observations, an efficient mechanism is needed that dissipates energy and transports angular momentum because the slow spiraling-in and eventual accretion of fluid depends critically on this process. When “viscous” ADs were theoretically proposed [8, 10], it was recognized that an anomalous dissipation and transport mechanism must be present in ADs since their hydrodynamical Reynolds numbers (Re) are enormous. Because fluid turbulence greatly enhances transport, turbulence has been proposed to operate in ADs as angular momentum can be transported in rotating flows with the help of a turbulent eddy viscosity. To date, a detailed theoretical understanding of turbulence and the transition to it is still lacking. Consequently, the effective viscosity in disks has been approached in a phenomenological way through parameterizing the effective viscosity coefficient with the help of a non-dimensional parameter (α) on the basis of dimensional arguments. This simple approach has been exceptionally fruitful, giving rise to successful interpretations of many basic observational results [11, 12].

Robust hydrodynamical stability criteria like the Rayleigh and Solberg-Holland criteria indicate that thin non-magnetized Keplerian ADs are linearly stable. To date there are no demonstrations of long-time dynamical activity in global hydrodynamical simulations of AD flows with sufficiently high Re. On the other hand, high resolution 3D numerical simulations of local disk sections [13] report that a subcritical transition to turbulence does exist at very high Reynolds numbers despite suggestions that Coriolis effects quench such a transition [14, 15]. However, the efficiency of turbulent transport in such subcritical flows appears to be insufficient to explain the transport implied by the observations of ADs. In this study we use the term global when the calculation includes a sizable portion of the AD. In contrast, a local calculation is one that is based on the so-called shearing box approximation [10, 17, 18] in which calculations are performed on small “Cartesianized” sections of the disk.

In the years following the work of Ioannaou & Kakouris [8] (who had employed a global approach albeit to a two-dimensional configuration) a number of research groups have used the local approximation to study the relevant IVP in various settings [13, 14, 20, 21, 22, 23, 24, 25]. A common conclusion reached by these studies is that TG may be copious and that, under the right conditions, nonlinear interactions may give rise to what is called a bypass transition to turbulence. It is worth mentioning that vortices and spiral waves are singled out in some of these works as being instrumental for such a transition.

This work is intended to further understanding what the non-modal IVP perspective can teach us about the global transient dynamics of thin slightly viscous hydrodynamical Keplerian disks. To be specific, we consider flows with a viscosity (perhaps of turbulent origin) that is insufficient to provide the angular momentum transport implied by observations. We are interested in examining the excitation of global secondary flows that can, in turn, possibly give rise to angular momentum transport and/or create conditions for a secondary instability atop the weakly turbulent state [26, 27, 28, 29, 30].

In the context of ADs one is confronted with a vast system on which there is no experimental control. In lieu of this, Ioannaou & Kakouris [8] reported in their study that stochastic forcing was found to lead to persistent activity with angular momentum transported outward. More recently, Zhuravlev & Shakura [31] applied optimal perturbation strategy to two-dimensional sub-Keplerian toroidal configurations. They found that optimal perturbations giving rise to substantial TG are composed of certain combinations of non-axisymmetric eigenmodes.

Our tactic is to study the transient dynamics of specific three-dimensional perturbations of a Keplerian AD in which the vertical structure is taken into account. The base flow is the Kluzniak-Kita [32] (hereafter KK) analytical solution of a polytropic steady viscous axisymmetric disk. This solution was obtained by representing all dependent functions as an asymptotic series in the small parameter ε which measures the disk’s “thinness” (i.e., the ratio of the disk’s vertical thickness to its typical radial scale). The lowest order solution is the classical Shakura-Sunyaev solution [9], while higher order terms provide the velocities and structure functions in the meridional plane, as well as corrections to the Keplerian angular velocity [33]. This kind of perturbation strategy for ADs had first been introduced in the context of AD inner boundary layer [34]. We wish to reiterate here that implicit in the KK solution is that there is some viscosity on the smallest scales, perhaps due to the subcritical transition discussed previously [13].

Umurhan et al. [35] (hereafter UNRS), extended the analytical KK solution to consider its time dependent axisymmetric response. The analysis of the corresponding IVP indicated that typical initial data naturally exhibits transient growth lasting many rotation periods before ultimately decaying. The lifespan of this response is inversely proportional to the viscosity parameter α, which is consistent for studies of simpler systems [7]. Because UNS examined only axisymmetric initial disturbances and flows, the transiently growing patterns could not induce any
effective radial angular momentum transfer. In addition, no obvious mechanism was identified that could serve as a suitable secondary instability candidate.

In this paper we report on a work based on the same ideas applied to the same slightly viscous system, but allowing for non-axisymmetric disturbances. Consequently the results, which remain analytical as in UNRS, exhibit a far richer structure. The asymptotic expansion procedure used in UNRS is applied in which the same small parameter (i.e., the disk’s ‘thiness’) is exploited resulting in an IVP that is analytically treatable. The expansion procedure employed here develops a finite-amplitude nonlinear solution. Thus, although the higher order terms and time-dependencies are governed by linear operators, the solutions developed are finite-amplitude and they should not be confused as being infinitesimal solutions.

The use of approximation methods and the simplistic polytropic assumption have the obvious advantages that the treatment can be analytical and the responsible physical effects leading to any interesting dynamics may be transparently traced. It is clear, however, that the present analytic analysis ultimately should be complemented with a detailed and uncompromising numerical solution which includes a proper treatment of energy generation and transfer. We also stress that we do not aim to obtain a general solution for the IVP, nor do we intend to itemize all possible solutions. We also do not aim to find the optimal perturbation, which would be a strategy appropriate for a laboratory transition study. The purpose of this study is to show that there exists a solution that gives rise to prominent transient growth in the energy when small but non-infinitesimal disturbances are introduced. We also investigate how the different parameters of the system (i.e., $\alpha$, symmetry) affect this growth. Implicit in our approach is the tacit assumption that the possible perturbation spectrum in a realistic AD is so rich as to allow essentially any initial condition we desire. We are specifically interested in finding a global transiently growing spatial pattern of the density and assessing its physical consequences.

The paper is organized in the following way. In Section II we formulate the problem by stating the assumptions, notation and scaling, leading to the basic set of non-dimensional partial differential equations for the flow. In Section III the asymptotic expansions of the various dependent variables are given and the equations are solved in each order in the small parameter $\epsilon$, up to the second order. As already stated, the lowest order gives the steady Shakura-Sunyaev solution and the higher orders provide analytically the time dependent solution for a given initial disturbance. The most interesting effect—the transient growth in the density and vertical velocity—occurs at the second order (Section III-D). In Section IV and V, the solutions are discussed in some detail and we examine with the help of graphic visualization some of their important physical properties and finally, in Section VI, we summarize the results and their meaning. Since the analytical procedure needed for obtaining the solutions, in various orders, is lengthy and involved we defer some technical details to the Appendices.

II. FORMULATION OF THE PROBLEM

A. Notation, scaling and basic assumptions.

The problem is formulated in cylindrical coordinates $(r, z, \phi)$ with $P$ and $\rho$ denoting the pressure and density functions. The cylindrical components of the velocity are $u, v$ and $r\Omega$, where $\Omega$ is the angular velocity. Additionally, $c_s$ is the sound speed and $\eta$ the dynamic viscosity. The accretion flow is in the Newtonian gravitational field of a central object with point mass $M$.

The coordinate $r$ is scaled by its value at a typical point $r_*$ (we shall denote the scaling dimensional variables by an asterisk) and the density by a typical density value $\rho_*$. The first assumption is a polytropic pressure-density relation $P = K\rho^{1+1/n}$, with $n$ being the polytropic index and $K$ a constant. Consequently the pressure is scaled by $P_* = K\rho_*^{1+1/n}$. This also gives the typical sound speed (squared) $c_s^2 = dP_*/d\rho_*$. We shall scale the meridional velocities with $c_{s*}$ and express the rotational angular velocity in units of its Keplerian value at $r_*$, so that $\Omega_*^2 = GM/r_*^3$. This then allows us to express the vertical coordinate scale, $h_*$, using the second assumption, that the flow is approximately in vertical equilibrium due to thermal pressure support (that is, $h_* = c_{s*}/\Omega_*$). Our third assumption is that the disk flow is cold, or, equivalently, that the azimuthal rotational velocity is highly supersonic. This is equivalent to assuming very efficient cooling, by, e.g., radiative losses from the surface of the disk. As a result, we find that $\epsilon \equiv h_*/r_* = c_{s*}/(\Omega_* r_*) \ll 1$, which happens to also measure the disk’s thinness, as we alluded to in the Introduction. This small number $\epsilon$ will thus appear in the non-dimensional equations characterizing the flow and becomes our natural expansion parameter.
B. Equations

The non-dimensional polytropic relation is \( \rho = \rho^{1+1/n} \) and we define, for convenience, a function \( W = \int dP/\rho \). This gives supplementary polytropic relations, which will ultimately allow to express \( P, c_s \) and \( W \) in terms of the density alone.

\[
W = n \frac{dP}{d\rho} = nc_s^2 = (n + 1)\rho^{1/n} \quad \text{because} \quad c_s^2 = (1 + \frac{1}{n})\rho^{1/n} \quad (1)
\]

The full (time-dependent, non-axisymmetric) scaled hydrodynamic equations in cylindrical co-ordinates read

\[
\epsilon \{ \partial_t u + v \partial_r u + \Omega \partial_\phi u \} + \epsilon^2 \{ u \partial_r u \} = \Omega^2 r - \frac{1}{r^2} \left( 1 + \epsilon^2 \frac{z^2}{r^2} \right)^{-\frac{2}{3}} + \epsilon \left( \frac{1}{\rho} \right) \{ \partial_z (\eta \partial_z u) \} + 
\]

\[
+ \epsilon^2 \left( \frac{1}{\rho} \right) \left\{ -\rho \partial_r W + \partial_z (\eta \partial_z v) - \frac{2}{3} \partial_r (\eta \partial_z v) + \partial_\phi (\eta \partial_\phi \Omega) - \frac{2n}{r} \partial_\phi \Omega - \frac{2}{3} \partial_r (\eta \partial_\phi \Omega) \right\} + 
\]

\[
+ \epsilon^3 \left( \frac{1}{\rho} \right) \left\{ -\frac{2n}{r^2} + \frac{3}{2} \partial_r (\eta \partial_r u) - \frac{2}{3} \partial_\phi (\eta \partial_\phi u) \right\} \quad (2)
\]

\[
\partial_r \Omega + v \partial_z \Omega + \Omega \partial_\phi \Omega = \left( \frac{1}{\rho} \right) \partial_z (\eta \partial_z \Omega) - \epsilon \left( \frac{4}{r^2} \right) \partial_r (\Omega r^2) 
\]

\[
+ \epsilon^2 \left( \frac{1}{\rho r^2} \right) \left\{ \frac{1}{r} \partial_r (\eta \partial_\phi u) - \frac{2}{3} \partial_\phi (\eta \partial_\phi u) + \frac{4}{3} \partial_\phi (\eta \partial_\phi u) \right\} + 
\]

\[
+ \epsilon^3 \left( \frac{1}{\rho r^2} \right) \left\{ \frac{1}{r} \partial_r (\eta \partial_r u) - \frac{2}{3} \partial_\phi (\eta \partial_\phi u) + \frac{4}{3} \partial_\phi (\eta \partial_\phi u) \right\} \quad (3)
\]

\[
\partial_t v + v \partial_z v + \Omega \partial_\phi v + \epsilon \{ u \partial_r v \} = -\partial_z W - \frac{z}{r^3} \left( 1 + \epsilon^2 \frac{z^2}{r^2} \right)^{-\frac{2}{3}} + \frac{1}{\rho} \left[ \frac{4}{3} \partial_z (\eta \partial_z v) + \partial_\phi (\eta \partial_\phi \Omega) - \frac{2}{3} \partial_r (\eta \partial_\phi \Omega) \right] + 
\]

\[
+ \epsilon \left( \frac{1}{\rho r} \right) \left\{ \partial_r (\eta \partial_r u) - \frac{2}{3} \partial_\phi [\eta \partial_r (ru)] \right\} + \epsilon^2 \left( \frac{1}{\rho r} \right) \left\{ \partial_r (\eta \partial_\phi v) + \frac{1}{r} \partial_\phi (\eta \partial_\phi v) \right\} \quad (4)
\]

\[
\partial_t \rho + \partial_z (\rho v) + \partial_\phi (\rho \Omega) + \epsilon \frac{1}{r} \partial_r (r \rho u) = 0, \quad (5)
\]

where (2-4) are, respectively, the radial, azimuthal and vertical momentum conservation equations, while (5) is the equation of mass conservation. Note that this set of equations is very similar to the one in KK and UNRS. The terms containing angle derivatives are missing in KK and UNRS, because of the axisymmetry assumed there, while the terms containing time derivatives are missing in KK, who considered only a steady state.

We impose a Lagrangian pressure condition on the vertical surfaces (namely, that the pressure on the surface be zero). We also require that the stresses on the vertical surfaces be zero as well. See Appendices C-D for details. The radial boundary conditions require some discussion. We will consider these equations for an extended ring, in which the internal radius is considerably larger than the zero-torque radius of the disk \( r_+ \), that is, \( r_+ \gg r_+ \) (see KK for a more extended discussion) and the external radius is significantly smaller than the disk outer edge. In this way we can avoid the treatment of inner and outer boundaries, which greatly complicates the problem, presumably without changing the substantial result for the bulk of the disk. Still it would be interesting to address in the future the dynamics at the inner edge region in the case of black holes (using a different scaling and possibly matched asymptotic expansion) and in the boundary layer in the case of neutron stars (by means of matched asymptotic expansion). The outer boundary, at which the disk is fed by mass, depends strongly on the astrophysical system in question and is of interest as well, but we defer also this question to future work.

III. ASYMPTOTIC SOLUTIONS

A. General

The asymptotic expansion approach consists of expanding all functions in powers of \( \epsilon \) (e.g., KK, UNRS) as follows

\[
f(r, z, \phi, t) = f_0(r, z) + \epsilon \tilde{f}_1(r, z, \phi, t) + \epsilon^2 \tilde{f}_2(r, z, \phi, t) \ldots \quad (6)
\]
We postulate, similarly to UNRS, that the expansions of all the dependent variables are such that the terms \( \tilde{f}_j \) can be split into a spatial steady part \( f_j \)—the steady base flow—and a time dependent dynamical disturbance \( f'_j \)

\[
\tilde{f}_j(r, z, \phi, t) = f_j(r, z, \phi) + f'_j(r, z, \phi, t)
\]  

This form presupposes therefore that the time dependence is included only as an additive function and from the first order in \( \epsilon \) and on. We remark here at the outset that the viscosity, \( \eta \), is not expanded. Instead, we shall express it, using a standard prescription (as found in the Shakura-Sunyaev solution \[9\]) by the lowest order dependent variables (see section III C).

Without assuming anything additional we obtain what we call the complete set of equations in the first three orders in \( \epsilon \) and on. We remark here at the outset that the viscosity, \( \eta \), is not expanded. Instead, we shall express it, using a standard prescription (as found in the Shakura-Sunyaev solution \[9\]) by the lowest order dependent variables (see section III C).

Before turning to the equations and their solutions in the three lowest orders in \( \epsilon \) we assume, for the sake of simplicity, that \( n = 3/2 \) (as in KK). For our calculations we allow for general values of \( n \) and we agree with UNRS who find that the results are little influenced for reasonable \( n \) values. For \( n = 3/2 \), as we assume henceforth, we get from the non-dimensional polytropic relation and equation (1) that

\[
P_0 = \rho_0^{5/3}, \quad W_0 = \frac{5}{2} \rho_0^{2/3}, \quad c_{s0} = \frac{5}{3} \rho_0^{2/3}.
\]

Finally, we remark that if \( g(\rho) \) is a smooth function of the density alone, and the density is written as \( \rho = \rho_0 + \delta \rho \), where \( \delta \rho \) is a small perturbation atop \( \rho_0 \), then the perturbation in \( g \), i.e., \( \delta g \) is well approximated by \( \delta g = (\partial g/\partial \rho)\delta \rho \).

This result will be used in the orders \( \epsilon \) and \( \epsilon^2 \) below.

### B. Order \( \epsilon^0 \)

The complete equations at this order are simple because there is no time dependence at this order. Since the steady base flow is also axisymmetric, it further significantly simplifies the equations and gives a particularly compact equation set.

\[
\Omega_0^2 = \frac{1}{r^3}
\]

\[
v_0\rho_0 \frac{\partial \Omega_0}{\partial z} = \frac{\partial}{\partial z} \left( \eta \frac{\partial \Omega_0}{\partial z} \right)
\]

\[
v_0 \frac{\partial v_0}{\partial z} = -\frac{z}{r^3} - \frac{\partial W_0}{\partial z} + \frac{4}{3} \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial v_0}{\partial z} \right)
\]

\[
\frac{\partial (\rho_0 v_0)}{\partial z} = 0
\]

#### 1. Solution

Equation (10) guarantees the Keplerian form \( \Omega_0 = r^{-3/2} \), which makes equation (11) trivial, and equation (13) immediately gives

\[
\rho_0 v_0 = f(r).
\]
Since \( v_0 \) is an odd function in \( z \) (and \( \rho_0 \) is even in \( z \)), we get that \( v_0 = 0 \) identically and equation (12) can be easily solved (subject to the boundary condition \( \rho_0(r, h) = 0 \), where \( h(r) \) is the height of the disk) to yield \( W_0(r, z) \) and thus \( P_0(r, z) \) and \( \rho_0(r, z) \). Thus the \( O(0) \) solution is identical to the KK solution at the same order,

\[
\Omega_0 = r^{-3/2}, \quad v_0 = 0, \quad \rho_0(r, z) = \left( \frac{h^2 - z^2}{5r^3} \right)^{3/2},
\]

with \( h = h(r) \) as given in KK and in UNRS,

\[
\frac{h(r)}{r} = \lambda_0 \left( 1 - \sqrt{\frac{r_+}{r}} \right)^{1/6} \quad \text{with} \quad \lambda_0 = \left[ \frac{M}{\alpha} \left( \frac{80}{3\pi} \frac{5}{3} \right) \right]^{1/6}.
\]

Also

\[
P_0(r, z) = \left( \frac{h^2 - z^2}{5r^3} \right)^{5/2}, \quad W_0(r, z) = \frac{5}{2} \left( \frac{h^2 - z^2}{5r^3} \right),
\]

following from (13).

\[\boxed{\text{C. Order } \epsilon^1}\]

The complete \( O(\epsilon) \) system, as given in Append. [32] is significantly simplified, when the axisymmetry of \( \Omega_0 \) and the fact that \( v_0 = 0 \), from the zero-order solution (14), are used.

Taking only the "unperturbed" parts gives rise to the following time independent equation set

\[
-2r\Omega_0\Omega_1 = 1
\]

\[
u_0 \frac{\partial}{\partial z} \left( \frac{\partial u_0}{\partial z} \right)
\]

\[
\frac{u_0}{r^2} \frac{\partial}{\partial r} \left( r^2 \Omega_0 \right) = \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial \Omega_1}{\partial z} \right)
\]

\[
0 = -\frac{\partial W_1}{\partial z} + \frac{4}{3\rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial v_1}{\partial z} \right) - \frac{2}{3\rho_0 r} \frac{\partial}{\partial z} \left( \eta \frac{\partial (u_0 r)}{\partial r} \right) + \frac{1}{\rho_0 r} \frac{\partial}{\partial r} \left( \eta \frac{\partial u_0}{\partial z} \right)
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \rho_0 u_0 \right) + \frac{\partial}{\partial z} \left( \rho_0 v_1 \right) = 0
\]

These equations are for the axisymmetric base flow (identical to KK; UNRS did not consider this order and took all the contributions to be zero). In any case, subtracting this from the complete equation set gives

\[
-2r\Omega_0\Omega_1 = 0
\]

This equation guarantees that \( \Omega_1 = 0 \) and we can use the notation \( \Omega_1 \) for \( \tilde{\Omega}_1 \), because it is time independent in this case. The resulting second equation gives no additional information (time-dependent, and see below) and the remaining time-dependent equations to this order are

\[
\frac{\partial v_1'}{\partial t} + \Omega_0 \frac{\partial v_1'}{\partial \phi} = -\frac{\partial W_1'}{\partial z} + \frac{4}{3\rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial v_1'}{\partial z} \right)
\]

\[
\frac{\partial \rho_1'}{\partial t} + \Omega_0 \frac{\partial \rho_1'}{\partial \phi} = -\frac{\partial (\rho_0 v_1')}{\partial z}
\]

We note the appearance of the operator

\[
D_{\phi t} \equiv \frac{\partial}{\partial t} + \Omega_0 \frac{\partial}{\partial \phi},
\]

which we shall use henceforth, for economy of notation, remembering also that \( \Omega_0(r) = r^{-3/2} \). The subscript "\( \phi t " \) in the definition of this operator has the purpose of reminding that it contains both the time and angular derivatives.
1. Solution

In the previous order the explicit form of the viscosity \( \eta \) was not needed, but now the situation is not as comfortable. Following Shakura and Sunyaev\(^\text{[9]}\), we posit the following form of the \((\phi, r)\) viscous stress tensor component

\[
|\tau_{\phi r}| = \alpha P_0,
\]

where \( \alpha \) is an adimensional parameter. This gives

\[
\eta \left| \frac{d\Omega_0}{dr} \right| = \alpha P_0 \quad \rightarrow \quad \eta = \frac{2}{3} \alpha r^{3/2} \rho_0^{5/3}
\]

Notice that this prescription is vertically dependent, i.e., the vertical distribution of stress is assumed to follow the pressure distribution. We now substitute this \( \eta \) together with the polytropic relation, \( \Omega_0 = r^{-3/2} \) and use \( \partial_z W_0 = - z/r^3 \) (from equation (12)) in equations (17-18), which become, after some straightforward algebra

\[
\frac{2\alpha}{3} \rho_0^2 r^3 \frac{\partial^2 u_0}{\partial z^2} - \frac{2\alpha}{3} \rho_0^2 \frac{z}{r} \frac{\partial u_0}{\partial z} + (2\alpha \Omega_1) = 0
\]

(27)

\[
\frac{2\alpha}{3} \rho_0^2 r^3 \frac{\partial^2 (2\alpha \Omega_1)}{\partial z^2} - \frac{2\alpha}{3} \frac{z}{r} \frac{\partial (2\alpha \Omega_1)}{\partial z} - u_0 = 0
\]

(28)

Now, since we consider only even (in \( z \)) solutions for \( u_0 \) and \( \Omega_1 \), it can be shown (see Appendix A) that the only bounded solutions are the trivial solutions:

\[
u_0(r, z) = \Omega_1(r, z) = 0.
\]

(29)

Equations (17-18) thus become trivial and equations (19-20) assume the form

\[
0 = - \frac{\partial W_1}{\partial z} + \frac{4}{3} \frac{\partial}{\partial \rho_0} \left( \eta \frac{\partial v_1}{\partial z} \right)
\]

(30)

\[
\frac{\partial}{\partial z} \left( \rho_0 v_1 \right) = 0
\]

(31)

Considering equation (31) first, we get \( \rho_0 v_1 = f(r) \), but since \( v_1 \) is an odd function of \( z \) (i.e., it is zero at \( z = 0 \)), this implies \( v_1 = 0 \) identically. Similarly, from equation (30) we get \( W_1 = f(r) \), but at the disk vertical edge (some large enough \( z \)) this function must be zero. Consequently we will have, similarly to UNRS,

\[
v_1(r, z) = W_1(r, z) = 0.
\]

(31)

We are thus left with the equations for the perturbations (22-23). UNRS omitted in their expansion the first order terms of the vertical velocity and density, as well as their time dependent perturbations. Here we shall allow for these perturbations and therefore will have to solve equations (22-23). These equations can be rewritten as

\[
D_{\phi \psi} v'_1 = - \frac{5}{3} \frac{\partial}{\partial z} \left( \rho_0^{-1/3} \rho_1 \right) + \frac{4}{3} \frac{\partial}{\partial \rho_0} \left( \eta \frac{\partial v'_1}{\partial z} \right)
\]

(33)

\[
D_{\phi t} \rho'_1 = - \frac{\partial}{\partial z} \left( \rho_0 \eta v'_1 \right),
\]

(34)

where we have used \( W'_1 = (2/3)(W_0/\rho_0) \rho'_1 = (5/3)\rho_0^{-1/3} \rho'_1 \) (see equation (20)). Applying again \( D_{\phi t} \) to (33) and substituting (34) we get the single equation

\[
D^2_{\phi \psi} v'_1 - \frac{4}{3} \frac{\partial}{\partial z} \left( \eta \frac{\partial v'_1}{\partial z} \right) - \frac{5}{3} \frac{\partial^2}{\partial z^2} \rho_0^{-1/3} \frac{\partial}{\partial z} v'_1 - \frac{25}{6} \frac{\partial}{\partial z} \left( \rho_0^{-1/3} \right) \frac{\partial}{\partial z} v'_1 - \frac{5}{2} \frac{\partial^2}{\partial z^2} \rho_0^{-1/3} v'_1 = 0,
\]

(35)

where \( \eta \) is given in terms of \( \rho_0 \) by equation (20). The above differential equation can be written as

\[
\mathcal{L} v'_1 = 0,
\]

(36)
where the form of the linear differential operator $\mathcal{L}$ can be easily inferred from the above. This form becomes explicit using equation (14)

$$\rho_0(r, z) = (5r^3)^{-3/2}(h^2 - z^2)^{3/2},$$  \hspace{1cm} (37)

the viscosity in terms of the coordinates (and constants)

$$\eta(r, z) = \frac{2}{3} \alpha r^{3/2} \rho_0^{5/3} = \frac{2}{3} \times 5^{5/2} \alpha r^{-6} (h^2 - z^2)^{5/2},$$ \hspace{1cm} (38)

and remembering that $h$ is a known function of $r$ (see KK, UNRS).

Equation (30) can be solved analytically. At this purpose we introduce the similarity variable $T \equiv \Omega_0 t = t/r^{3/2}$ and the Ansatz $v'_1 = \hat{v}_1(r, z, \phi) e^{sT} + c.c.$, where $s$ is an eigenvalue. Note that $v'_1$ must be periodic in $\phi$, with period $2\pi$, and thus it can be Fourier expanded and the Ansatz takes the form

$$v'_1(r, z, \phi, t) = \sum_{m=-\infty}^{\infty} \hat{v}_{1(m)}(r, \zeta) e^{sT + im\phi} + c.c.,$$ \hspace{1cm} (39)

where $\zeta \equiv z/h(r)$ is an another similarity variable and the lower index in parentheses denotes the Fourier components.

Now, for each particular Fourier component, the Ansatz and the resulting eigenvalue equation are like the ones discussed in UNRS (see equation (48) of that paper, which is for the case of axisymmetry, i.e., $m = 0$). The procedure is thus identical.

We summarize here the main result (which will also be used in the discussion of the next order). The ordinary differential equation resulting from the substitution of the above Ansatz, is known as the Gegenbauer (or hyper-spherical) equation and its solutions are known in terms of combinations of the associated Legendre functions (also known as Gegenbauer polynomials), e.g., [37], [38]. These analytical expressions are also included in Wolfram’s Mathematica 6 software, which makes our calculations feasible. Applying the upper (and lower) boundary conditions give a condition on the eigenvalues. All the eigenvalues thus obtained have negative real parts (i.e., they decay in time). For example, the fundamental mode for the Fourier component $m$ is

$$s_{(m)}^\pm = -\frac{4}{9} \alpha - im \pm \sqrt{\frac{16}{81} \alpha^2 - \frac{8}{3}}^{1/2}.$$

(40)

This is correct for any $\alpha \leq 4$, other values of this parameter being un-physical for the problem at hand. So actually the fundamental mode solution is

$$v'_1(r, z, \phi, t) = \sum_{m=-\infty}^{\infty} \hat{v}_{1(m)}(r, \zeta) e^{s_{(m)}^+ T + im\phi} + S_- e^{s_{(m)}^- T} + c.c.,$$ \hspace{1cm} (41)

where $S_+$ and $S_-$ just are (integration) constants. Details can be found in Appendix [C]

Clearly, the first order solutions found here lead to solutions exponentially decaying in time. Since in this work we are interested to find transiently growing solutions, we can reasonably make the choice of initial conditions such that $v'_1 = \rho'_1 = W'_1 = \rho_1 = 0$ at all times (as it was assumed a priori in UNRS). In any case, the decaying solutions, even if not chosen to be zero initially, become quickly negligible when the algebraically transiently growing solution appear (see next order for a discussion). Moreover we stress that, whatever choices are made, they do not affect at all the solution at the next order (see 40 [47] [48] and [49]).

D. Order $\epsilon^2$

The second order equations are complicated (see Appendix [B3], but taking only the “unperturbed” parts (base flow), which are axisymmetric and steady, one gets the following equation set:

$$-2r\Omega_0\Omega_2 = \frac{\partial W_0}{\partial r} + \frac{2z^2}{2r^2} + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial u_1}{\partial z} \right)$$ \hspace{1cm} (42)

$$\frac{u_1}{r^2} \frac{\partial}{\partial r} (\rho^2 \Omega_0) = -\frac{1}{r^3 \rho_0} \frac{\partial}{\partial r} \left( r^2 \eta \frac{\partial \Omega_0}{\partial r} \right) + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial \Omega_2}{\partial z} \right)$$ \hspace{1cm} (43)

$$0 = \frac{\partial W_2}{\partial z} + \frac{3z^3}{2r^5} + \frac{4}{3r^2 \rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial v_2}{\partial z} \right) - \frac{2}{3r \rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial (ru_1)}{\partial r} \right) + \frac{1}{\rho_0} \frac{\partial}{\partial r} \left( \eta \frac{\partial u_1}{\partial z} \right)$$ \hspace{1cm} (44)
0 = \frac{1}{r} \frac{\partial}{\partial r} (r \rho_0 u_1) + \frac{\partial (\rho_0 v_z)}{\partial z} \tag{45}

The above equations \([12, 15]\) are identical to the corresponding steady equation set of KK and UNRS (22-25).

After subtracting from the complete set we are left with the equations for the non-axisymmetric time-dependent perturbations

\[
\begin{align*}
\frac{\partial u_1'}{\partial t} + \frac{\partial}{\partial \phi} \left( \frac{\partial u_1'}{\partial \phi} \right) &= 2r \Omega_0 \Omega_2' + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial u_1'}{\partial z} \right) \tag{46} \\
\frac{\partial \Omega_2'}{\partial t} + \frac{\partial}{\partial \phi} \left( \frac{\partial \Omega_2'}{\partial \phi} \right) &= -\frac{u_1' \frac{\partial}{\partial r}}{r^2} \frac{\partial}{\partial r} \left( r^2 \Omega_0 \right) + \frac{1}{r \rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial \Omega_2'}{\partial z} \right) \tag{47} \\
\frac{\partial v_z'}{\partial t} + \frac{\partial}{\partial \phi} \left( \frac{\partial v_z'}{\partial \phi} \right) &= -\frac{\partial W_2'}{\partial z} + \frac{4}{3} \frac{\partial}{\partial \phi} \left( \frac{\partial v_z'}{\partial \phi} \right) - \frac{2}{3} \frac{\partial}{\partial \phi} \left( \eta \frac{\partial \Omega_2'}{\partial \phi} \right) + \frac{1}{\rho_0} \frac{\partial}{\partial \phi} \left( \eta \frac{\partial \Omega_2'}{\partial \phi} \right) - \frac{2}{3} \frac{\partial}{\partial z} \left( \frac{\partial (ru_1')}{\partial r} \right) + \frac{1}{r \rho_0} \frac{\partial}{\partial r} \left( r \eta \frac{\partial u_1'}{\partial z} \right) \tag{48} \\
\frac{\partial \rho_0'}{\partial t} + \frac{\partial}{\partial \phi} \left( \frac{\partial \rho_0'}{\partial \phi} \right) &= -\frac{1}{r} \frac{\partial}{\partial r} (r \rho_0 u_1') - \frac{\partial (\rho_0 v_z')}{\partial z} + \frac{\partial \Omega_2'}{\partial \phi}. \tag{49}
\end{align*}
\]

Note that the LHS of all the four equations again contains the operator \(D = \partial_t + \Omega_0 \partial_{\phi}\).

\[1. \text{ Reformulation in terms of linear differential operators}\]

We notice that Equations \([46, 47, 49]\) dynamically decouple from Equations \([48, 49]\), as in the axisymmetric case, treated in UNRS. The whole set of equations can thus be be reduced to

\[
\begin{align*}
\mathcal{P} u_1' &= 0 \tag{50} \\
\mathcal{P} \Omega_2' &= 0 \tag{51} \\
\mathcal{L} v_z' &= |\mathcal{D} \mathcal{F} + \mathcal{G}| u_1' + |\mathcal{D} \mathcal{H} + \mathcal{J}| \Omega_2', \tag{52}
\end{align*}
\]

The linear differential operators \(\mathcal{P}, \mathcal{L}, \mathcal{F}, \mathcal{G}\) are identical to those in UNRS, in particular the operator \(\mathcal{L}\) is the one inferred (in the previous order) from equation \([55]\) but here we have, in addition, the non-axisymmetric operators \(\mathcal{J}\) and \(\mathcal{H}\). The operators are given, in terms of the known function \(\rho_0(z), \eta\) (a known function of \(\rho_0, r\) and constants) as follows

\[
\begin{align*}
\mathcal{P} &\equiv \left[ \mathcal{D} \mathcal{F} - \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial}{\partial z} \right) \right]^2 + \Omega_0^2 \\
\mathcal{L} &\equiv \mathcal{D}^2 - \frac{4}{3} \frac{\partial}{\partial \phi} \mathcal{D} \frac{\partial}{\partial z} \left( \eta \frac{\partial}{\partial z} \right) - \frac{5}{3} \rho_0^{2/3} \frac{\partial^2}{\partial z^2} - \frac{25}{6} \frac{\partial (\rho_0^{2/3})}{\partial z} \frac{\partial}{\partial z} - \frac{5}{2} \frac{\partial^2 (\rho_0^{2/3})}{\partial z^2} \\
\mathcal{F} &\equiv -\frac{2}{3} \frac{\partial}{\partial \phi} \frac{\partial}{\partial r} \left( \eta \frac{\partial}{\partial \phi} \right) + \frac{1}{r \rho_0} \frac{\partial}{\partial r} \left( r \eta \frac{\partial}{\partial z} \right); \tag{54} \\
\mathcal{G} &\equiv 2 \frac{\partial}{\partial z} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} \right), \tag{55} \\
\mathcal{H} &\equiv -\frac{2}{3} \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi}, \tag{56} \\
\mathcal{J} &\equiv \frac{5}{3} \frac{\partial}{\partial z} \left( \rho_0^{2/3} \frac{\partial}{\partial \phi} \right). \tag{57}
\end{align*}
\]

where we have used the expressions for the \(n = 3/2\) polytrope, and \(\eta\) is as given in eq. \([58]\).

\[2. \text{ Solution}\]

We first find the eigenfunctions of the operator \(\mathcal{P}\) (as in UNRS 50-51). We then decide to consider just the fundamental modes to determine \(u_1'\) and \(\Omega_2'\). These can be substituted in the inhomogeneous equation \([52]\) to get
$v'_2$, in view of the the similarity between $F$, $G$ in UNRS and $H,F$ here (the analogous solution was called in UNRS \textit{driven acoustics}). We shall discuss here only the case of $u'_1$ (the details can be found in Appendix D), since the case of $\Omega'_c$ is the same.

Mindful of the $2\pi$–periodicity in $\phi$ we use the Ansatz

$$u'_1(r, z, \phi, t) = \sum_{m=-\infty}^{\infty} \hat{u}_{1(m)}(r, z)e^{i\phi T + im\phi} + \text{c.c.}$$  \hspace{1cm} (58)

and

$$\Omega'_2(r, z, \phi, t) = \sum_{m=-\infty}^{\infty} \hat{\Omega}_{2(m)}(r, z)e^{i\phi T + im\phi} + \text{c.c..}$$  \hspace{1cm} (59)

Note that the eigenvalue $p$ is different from $s$, while $T = t/r^{3/2}$ is defined as before.

The substitution of the Ansatz into $P u'_1 = 0$ gives rise, for each $m$, to a differential equation that is somewhat more complicated than the Gegenbauer equation we had in the first order:

$$\left\{ \frac{1}{5} \alpha(1 - \zeta^2) \frac{\partial^2}{\partial \zeta^2} - \alpha \zeta \frac{\partial}{\partial \zeta} - \frac{3}{2} (p + im) \right\} \hat{u}_{1(m)}(r, \zeta) = 0. \hspace{1cm} (60)$$

Still the equation can be tackled analytically, using the series expansion (Frobenius method). As shown in Appendix D the spatial part of the fundamental mode ($k = 0$) has a rather simple structure

$$\hat{u}^{(0)}_{1(m)}(r, \zeta) = A(r) \left( \zeta^2 - \frac{1}{6} \right), \hspace{1cm} (61)$$

and, similarly

$$\hat{\Omega}^{(0)}_{2(m)}(r, \zeta) = C(r) \left( \zeta^2 - \frac{1}{6} \right). \hspace{1cm} (62)$$

The appropriate eigenvalue $p$, following from the surface boundary conditions, can take two values

$$p_{(m)}^\pm = -\frac{8}{5} \alpha - im \pm i. \hspace{1cm} (63)$$

So equation (58) should be replaced by

$$u'_1(r, z, \phi, t) = \sum_{-\infty}^{\infty} \hat{u}_{1(m)}(r, z)e^{im\phi} \left[ P_+ e^{P_{(m)}^+ T} + P_- e^{P_{(m)}^- T} \right] + \text{c.c.,} \hspace{1cm} (64)$$

where $P_{\pm}$ are constants. The same holds for $\Omega'_2$. We note that the homogeneous part of the linear inhomogeneous equation (62) is identical to the equation considered in the first order (i.e., involving the operator $L$) and that the RHS (the inhomogeneous part) can be found using the solutions for $u'_1$ and $\Omega'_2$ that we just discussed. As is well known, the general solution of the inhomogeneous equation $(v'_p)$ is the sum of the general solution to the homogeneous equation (denoted by $v'_h$) and a particular solution to the inhomogeneous one (v'p)

$$v'_2 = v'_h + v'_p. \hspace{1cm} (65)$$

We already know that $v'_h$ is an exponentially time decaying function. Since $v'_2$ has to be $2\pi$–periodic in $\phi$ we make a Fourier expansion. Due to the linearity of the operators, $v'_h, v'_h, u'_1$ and $\Omega'_2$ must have the same $\phi$–dependence. The structure of the operators $L,F,G,H$ and $F$ points to a particular solution in the form

$$v'_p = \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} e^{im\phi} \left[ \hat{v}^{(k)}_{(m)}+ (r, \zeta) e^{P_{(m)}^+ T} + \hat{\Omega}^{(k)}_{(m)}+(r, \zeta) T e^{P_{(m)}^- T} + \hat{v}^{(k)}_{(m)}-(r, \zeta) e^{P_{(m)}^- T} + \hat{\Omega}^{(k)}_{(m)}-(r, \zeta) T e^{P_{(m)}^- T} \right] + \text{c.c.,} \hspace{1cm} (66)$$

with $T$ and $\zeta$ as before and where we include, for generality, also the overtones $k \neq 0$. Inserting the above expression for $v'_p$ in equation (62) and substituting on the RHS the solutions based on equations (58), (59), (61) and (62), we
are led to a rather long and involved expression (see Appendix D, equation [26]). The fundamental mode \((k = 0\) and dropping the vertical mode superscript) has the following form (the superscript \(\pm\) corresponds to the two eigenvalues \(P_{(m)}^\pm\))

\[
v_{(m)}^\pm = a_{3(m)}^\pm(r)\zeta^3 + a_{1(m)}^\pm(r)\zeta, \quad \dot{V}_{(m)}^\pm = b_{3(m)}^\pm(r)\zeta^3 + b_{1(m)}^\pm(r)\zeta. \tag{67}
\]

By setting to zero the coefficients of \(t\) and each power of \(\zeta\) we get \(a_{1(m)}^\pm(r), a_{3(m)}^\pm(r), b_{1(m)}^\pm(r)\) and \(b_{3(m)}^\pm(r)\).

One may follow a similar procedure for the density perturbation, using

\[
\rho' = \rho_h + \rho_p', \tag{68}
\]

Following the argument outlined in Appendix D, we get that \(\rho_h' \equiv 0\). The Ansatz

\[
\rho_p' = \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} e^{im\phi} \left[ \hat{\rho}_{(m)}^k(r, \zeta) e^{P_{(m)}^k T} + \hat{\tilde{R}}_{(m)}^k (r, \zeta) T e^{P_{(m)}^k T} + \hat{\tilde{\rho}}_{(m)}^k (r, \zeta) e^{P_{(m)}^k T} + \hat{\tilde{\tilde{R}}}_{(m)}^k (r, \zeta) T e^{P_{(m)}^k T} \right] \quad \text{c.c.} \tag{69}
\]

can be substituted in equation (49) to find for the fundamental mode

\[
\hat{\rho}_{(m)}^\pm = (1 - \zeta^2)^{\frac{1}{2}} \left[ c_{4(m)}^\pm \zeta^4 + c_{2(m)}^\pm \zeta^2 + c_{0(m)}^\pm \right],
\]

\[
\hat{\tilde{R}}_{(m)}^\pm = (1 - \zeta^2)^{\frac{1}{2}} \left[ d_{4(m)}^\pm \zeta^4 + d_{2(m)}^\pm \zeta^2 + d_{0(m)}^\pm \right]. \tag{70}
\]

The functions \(a_{1(m)}^\pm, b_{1(m)}^\pm, c_{4(m)}^\pm\) and \(d_{4(m)}^\pm\) are known analytical, generally complex, functions of \(r\) and depend also on the parameter \(\alpha\). Being very complicated expressions, we handle them using the symbolic tools in Wolfram’s Mathematica 6, but we do not write them in the paper explicitly. Finally we can rewrite, in a more compact way, the fundamental driven (by the horizontal velocity perturbations) vertical acoustic modes as

\[
\begin{bmatrix}
\hat{v}_{(m)}^\pm \\
\hat{V}_{(m)}^\pm
\end{bmatrix} = \begin{bmatrix} a_{1(m)}^\pm \\
b_{1(m)}^\pm \\
\end{bmatrix} \zeta + \begin{bmatrix} c_{4(m)}^\pm \\
c_{2(m)}^\pm \\
\end{bmatrix} \zeta^3, \tag{71}
\]

and

\[
\begin{bmatrix}
\hat{\rho}_{(m)}^\pm \\
\hat{\tilde{R}}_{(m)}^\pm
\end{bmatrix} = (1 - \zeta^2)^{\frac{1}{2}} \left\{ \begin{bmatrix} d_{4(m)}^\pm \\
d_{2(m)}^\pm \\
\end{bmatrix} \zeta^3 + \begin{bmatrix} c_{4(m)}^\pm \\
c_{2(m)}^\pm \\
\end{bmatrix} \zeta^6 \right\}. \tag{72}
\]

IV. TEMPORAL EVOLUTION OF THE PERTURBATION ENERGY

Armed with analytical expressions for the dynamical variables of the disk, that are solutions of an IVP in which the initial conditions are all small perturbations on a steady base flow, we may now consider possible physical implications of our analysis.

The solutions we found point to an exponential time decay of most variables, but there is also a transient fast algebraic growth (before an ultimate, rather slow, exponential decay) of disturbances of order \(\epsilon^2\) in two variables—the vertical velocity \(v_p'\) and density \(\rho_p'\). These quantities can be naturally interpreted as sound waves, which were called in UNRS driven acoustics. UNRS study was axisymmetric, while here we have allowed for azimuthal dependence. Exploiting the obvious \(2\pi\) periodicity in \(\phi\), however, we took recourse to Fourier series expansions in this angle variable and thus were able to consider separately the various Fourier modes.

As in UNRS, we perceive the acoustic energy as a relevant variable to follow, but here we consider the time dependence of the energy contained in various Fourier modes as well as the effect of different values of \(\alpha\) and of the (free) radial form of the initial conditions. Following Rayleigh’s book [39] (Chapter XI) we define the acoustic energy density as the sum of the kinetic and potential energy densities. The potential term is the work gained/lost during expansion/compression and can be easily found using \(P' dV\) and \(P' = \rho_0 \epsilon^2 \partial \phi'\). Hence, the acoustic energy volume-density in the \(m\)-th Fourier component is

\[
\varepsilon_{(m)}(r, z, \phi, t, \phi; \alpha) = \frac{1}{2} \rho_0 \dot{c}_0^2 + \frac{1}{2} \frac{\epsilon^2 \dot{\rho}_{(m)}^2(r)}{\rho_0}. \tag{73}
\]
The total acoustic energy volume-density is

\[
\varepsilon(r, z, \phi, t; \alpha) = \frac{1}{2} \rho_0 \dot{v}^2 + \frac{1}{2} \frac{c_0^2 \dot{\rho}^2}{\rho_0},
\]

and we shall consider the fundamental \( k = 0 \) mode only. The zeroth order vertical velocity is zero, as well as \( v_1(r, z, \phi) \) and \( \rho_1(r, z, \phi) \). Thus remembering that \( \tilde{f} = f(r, z, \phi) + f'(r, z, \phi, t) \), \( v'_2 = v'_h + v'_p \) and \( \rho'_2 = \rho'_p \), we get

\[
\tilde{v} = \epsilon v'_1 + \epsilon^2 (v_2 + v'_h + v'_p), \quad \tilde{\rho} = \epsilon \rho'_1 + \epsilon^2 (\rho_2 + \rho'_p),
\]

where (we repeat all solutions here, for completeness)

\[
v'_1 = \sum_{m=-\infty}^{\infty} e^{im\phi} \hat{v}_{1(m)}(r, \zeta) \left[ S^+ e^{s^+(m)}e^{T} + S^- e^{s^-(m)}e^{T} \right] + \text{c.c.}
\]

\[
v'_h = \sum_{m=-\infty}^{\infty} e^{im\phi} \hat{v}_{h(m)}(r, \zeta) \left[ S^+ e^{s^+(m)}e^{T} + S^- e^{s^-(m)}e^{T} \right] + \text{c.c.}
\]

\[
v'_p = \sum_{m=-\infty}^{\infty} e^{im\phi} \left[ \hat{v}^+(m)(r, \zeta)e^{s^+(m)}e^{T} + \hat{V}^+(m)(r, \zeta) Te^{s^+(m)}e^{T} + \hat{v}^-(m)(r, \zeta)e^{s^-(m)}e^{T} + \hat{V}^-(m)(r, \zeta) Te^{s^-(m)}e^{T} \right] + \text{c.c.}
\]

\[
\rho'_1 = \sum_{m=-\infty}^{\infty} e^{im\phi} \hat{\rho}_{1(m)}(r, \zeta) \left[ Z^+ e^{s^+(m)}e^{T} + Z^- e^{s^-(m)}e^{T} \right] + \text{c.c.}
\]

\[
\rho'_p = \sum_{m=-\infty}^{\infty} e^{im\phi} \left[ \hat{\rho}^+(m)(r, \zeta)e^{s^+(m)}e^{T} + \hat{R}^+(m)(r, \zeta) Te^{s^+(m)}e^{T} + \hat{\rho}^-(m)(r, \zeta)e^{s^-(m)}e^{T} + \hat{R}^-(m)(r, \zeta) Te^{s^-(m)}e^{T} \right] + \text{c.c.},
\]

where \( S^\pm, S^\pm, Z^\pm \) are (integration) constants, and where

\[
T = \Omega_0 t = t/r^{3/2}, \quad s^\pm(m) = -\frac{4}{9} \alpha - im \pm i \left( \frac{16}{81} \alpha^2 - \frac{8}{3} \right)^{1/2}, \quad \rho^\pm(m) = -\frac{8}{9} \alpha - im \pm i.
\]

We recall that \( \hat{v}_{1(m)} \), \( \hat{v}_{h(m)} \) are the appropriate Gegenbauer polynomials, while \( \hat{v}^\pm(m) \), \( \hat{V}^\pm(m) \) are odd polynomials of \( \zeta \) and \( \hat{\rho}^\pm(m) \), \( \hat{R}^\pm(m) \) are even polynomials of \( \zeta \). The terms \( v'_1 \) and \( v'_h \) exponentially decay while \( v'_p \sim Te^{PT} \) grows linearly, so that the algebraically growing terms (like \( \sim T \)) dominate the particular solution. Moreover, when its eventual decay takes over, the former decaying terms are negligible (for the relevant very small values of \( \alpha \)). So, for simplicity we have decided to take the initial conditions such that \( v'_1 \) and \( v'_h \) are zero and thus remain so for all times. The same reasoning is valid for \( \rho' \). Therefore when we calculate the acoustic energy we can take just \( \tilde{v} = v'_p \) and \( \tilde{\rho} = \rho'_p \). In order to study the temporal behavior of the acoustic energy, we define two integral quantities the radius-dependent—i.e., averaged over a ring—energy per unit area of the disk \( E_r \) and the total acoustic energy of the fluctuation \( E_a \)

\[
E_r(r; t; \alpha, m) \equiv \int_{-h(r)}^{h(r)} \int_0^{2\pi} \mathcal{E}(r, z, t, \phi; \alpha, m) dz d\phi, \quad E_a(t; \alpha, m) \equiv \int_{r_{\text{min}}}^{r_{\text{max}}} r \mathcal{E}_r(r) dr.
\]

**A. The azimuthally averaged energy surface density**

We obtain \( E_r \) for a particular Fourier mode \( m \) in the form

\[
\mathcal{E}_r(r; t; \alpha, m) = e^{-\frac{16}{9} \alpha T} F(r; \alpha, m, \cos 2T, \sin 2T),
\]

where the function \( F \) is known analytically, but the expression is extremely long and we shall not write it out here explicitly. The derivation is outlined in Appendix F. Instead, we display the results graphically.
FIG. 1: Time evolution of the surface-density of the acoustic energy in the fundamental $k = 0$ mode at $r = 1$, $\alpha = 0.001$ and $A(r) = C(r) = e^{i\pi/4}$. The three analytical curves are for $m = 0$ (solid lower black line), $m = 1$ (black dotted line), $m = 10$ (solid upper red line) and are shown in a log-log plot. $\mathcal{E}_r(T)$ is scaled to its corresponding value at $T = 0$. Notice that $\mathcal{E}_r$ is modulated by fast oscillations.

FIG. 2: Same as in Figure 1 but for a single Fourier component ($m = 3$) and different values of $\alpha$. Four curves, for $\alpha = 0.0001, 0.001, 0.01, 0.1$ (from the top to the bottom), are shown.

In the first three figures the temporal behavior of the normalized acoustic energy surface-density contained in a particular Fourier component, $\mathcal{E}_r(r, t; m, \alpha)/\mathcal{E}_r(r, 0; m, \alpha)$, is shown. This quantity is shown as a function of the similarity variable $T$, defined before and for the fundamental $k = 0$ mode, in a log-log plot. In all the figures the free radial functions are set to $A(r) = C(r) = e^{i\pi/4}$, which means that the initial perturbation is taken, for simplicity, to be $r$-independent.

In Figure 1 $\alpha = 0.001$ and $r = 1$ are fixed and the various curves are for different Fourier components. The most
FIG. 3: Same as in Figure 2 but for one fixed $\alpha = 0.001$ and at different radial positions. Four curves for $r = 1, 3, 5, 10$ (from the top to the bottom) are shown.

significant feature of the behavior displayed in this figure (as well as in all subsequent ones) is the strong transient growth (by orders of magnitude) of the perturbation energy, which typically takes place for a rather significant time. The non-axisymmetric modes display maximal growth. All the mode have fast oscillations, atop the much slower, secular growth. From this Figure it appears that the maximum is attained at $t \approx 1000$ (since $r = 1$, we have here $t = T$).

Figure 2 shows the dependence of the transient growth on the value of the $\alpha$ parameter, for the single Fourier component $m = 3$ (other modes display a similar behavior). The radius is fixed, as before, at $r = 1$. We clearly see that the lower is the value of $\alpha$, the higher is the maximum, and the later it occurs. While $\alpha = 0.1$ does not give rise to any growth at all, for $\alpha = 10^{-4}$ the growth is enormous—by a factor of a few times $10^5$. Since the growth occurs when taking into account the $\epsilon^2$ terms of our expansion and the energy is composed of squares of these terms, the validity of the expansion is marginal for such a growth (assuming $\epsilon \sim 10^{-2}$). However for $\alpha \sim 10^{-3}$ (close to a “realistic” value, as found numerically in the sub-critical hydrodynamic transition [13]), the growth is somewhat less than $10^4$ and the asymptotic expansion reasonably holds.

In Figure 3 $\alpha = 0.001$ and the Fourier component $m = 3$ are fixed, while we show the behavior at different radii (from the top to the bottom $r = 1, 3, 5, 10$). The growth is higher and it reaches its maximum earlier at smaller radii: clearly at large radii the perturbation becomes negligible.

Before moving on to the description of the behavior of the total acoustic energy, we would like to remark that although our choice of the radial functions $A(r)$ and $C(r)$ may seem non-physical, because it is difficult to imagine an $r$-independent perturbation, it is still meaningful. Indeed, since so far we have dealt with the angle averaged $r$-dependent surface density of the energy, it is obvious what to expect when an $r$-localized perturbation is considered instead. The results can be simply scaled, depending on the relative value of the perturbation at the particular radius where they are sought (e.g., see Figure 3). In the next subsection, when we shall consider the total (also $r$-integrated) energy in an extended ring, we will use perturbations that are $r$-localized (in the form of a Gaussian).

### B. The total acoustic energy in a ring

The total acoustic energy of the perturbation is evaluated as an integral over $r$ of the afore considered function

$$E_a(t; \alpha, m) \equiv \int_{r_{\min}}^{r_{\max}} \mathcal{E}_r(r, t; \alpha, m) \, r \, dr. \quad (80)$$

The ring we consider is between $r_{\min} = 1$ and $r_{\max} = 10$, far from the inner and outer edge of the disk.
V. SURFACE DENSITY SPATIO-TEMPORAL BEHAVIOR

The formalism developed in UNRS and this paper can be used to follow the time evolution (and, in particular, the transient growth) of various small perturbations, which are included in the initial conditions of the appropriate IVP. We have already shown the copious transient growth of the acoustic surface energy (averaged over the azimuthal angle) as well as of the total acoustic energy in a finite ring of the disk. As suggestive as these results may be, for the possible disk energetics, they do not contain explicit spatio-temporal dynamical information. The possibilities to gain the latter are rather abundant, and it would be outside the scope of a single paper to examine a great many of them, in detail. Thus, we have decided to conclude here, by calculating and presenting just one of the important dynamical variables—the surface density. Other information that can be extracted from our three-dimensional analytical solution, found in this paper, will be presented in later works.
Expression for the leading terms

In Section IV we have already estimated that the algebraically growing terms will be the dominant ones. We shall repeat now this argument applied to the density. Focusing on the fundamental vertical mode \( k = 0 \) (thus dropping the superscripts) and taking only one, the \( m \)-th, say, Fourier component, we get for the particular solution for the density perturbation, appearing in second order in \( \epsilon \), the following expression, resulting from equation (77),

\[
\rho'_p = \left[ \tilde{h}^\pm_i(z, r)e^{p_i T + im\phi} + \tilde{h}^+_{(m)}(z, r)Te^{p_i T + im\phi} \right] + \left[ \tilde{h}^\pm_i(z, r)e^{p_i T + im\phi} + \tilde{h}^-_{(m)}(r, \zeta)Te^{p_i T + im\phi} \right] + c.c., \quad (83)
\]

where \( T \equiv tr^{3/2}, \zeta \equiv z/h(r) \) and \( p_i^\pm_{(m)} = -8\alpha/5 - i(m \mp 1) \).

The full asymptotic series for the density includes, however, more terms and is rewritten here, up to the second order in \( \epsilon \)

\[
\rho(r, \zeta, \phi, t) = \rho_0(r, \zeta) + \epsilon [p_1(r, \zeta, \phi) + \rho'_1(r, \zeta, \phi, t)] + \epsilon^2 \left[ p_2(r, \zeta, \phi) + \rho'_2(r, \zeta, \phi, t) + \rho'_p(r, \zeta, \phi, t) \right]. \quad (84)
\]

It was shown before that the first order steady term, that is \( \rho_1 \), can actually be set to zero. We can approximately ignore the steady second order term \( \rho_2 \), as well, because even if it is not zero, its magnitude is of the order \( \epsilon^2 \) as compared to the zeroth order steady term. Substituting also the relevant terms in (78) from the explicit formulae for \( \rho'_1 \) and \( \rho'_2 \), given by formulae (77), we get a complicated expression for the density, including the steady base state and the \( m \)th Fourier mode perturbation. Rather than presenting these expressions, we recall (see the discussion following equations (77)), that after some time only the transiently growing terms (those proportional to \( T \)) will be dominant. These terms will be large for a rather long time, of the order of \( 1/\alpha \). Thus, in this example we shall only examine the spatio-temporal behavior of these terms. Thus we shall consider

\[
\rho(r, \zeta, \phi, t) = \rho_0(r, \zeta) + \epsilon \left\{ T e^{im\phi} \left[ \tilde{h}^+_{(m)}(r, \zeta)e^{p_i T} + \tilde{h}^-_{(m)}(r, \zeta)e^{p_i T} \right] + c.c. \right\}, \quad (85)
\]

that is,

\[
\rho(r, \zeta, \phi, t) = \rho_0(r, \zeta) + \epsilon^2 \left\{ T \exp \left( -\frac{8}{5} \alpha T \right) \left[ \tilde{h}^+_{(m)}(r, \zeta)e^{-i(m-1)T-m\phi} + \tilde{h}^-_{(m)}(r, \zeta)e^{-i(m+1)T-m\phi} \right] + c.c. \right\}, \quad (86)
\]

To obtain the surface density one has to integrate over the disk thickness, thus

\[
\Sigma(r, \phi, t) = \Sigma_0(r) + \epsilon^2 \left\{ T \exp \left( -\frac{8}{5} \alpha T \right) \left[ e^{-i\theta_m} R^+_{m}(r) + e^{-i\theta_m} R^-_{m}(r) + c.c. \right] \right\}, \quad (87)
\]

with \( R^+_{m}(r) \equiv \int_{-1}^{1} \tilde{h}^+_{(m)}(r, \zeta) d\zeta \) and \( \theta^m_{\pm} \equiv (m \mp 1)T - m\phi \).

Using the expression for \( \rho_0 \), as in equation (14) of the paper, we can easily get

\[
\Sigma_0(r) = 5^{-3/2}h^3r^{-9/2} \int_{-1}^{1} (1 - \zeta^2)^{3/2} d\zeta \approx 0.1h^3(r)r^{-9/2}, \quad (88)
\]

where \( h \approx h_1 r \) (see UNRS) and \( h_1 \) is a constant of order unity, very weakly dependent on the mass transfer rate and \( \alpha \). This is a particularly good approximation for \( r \gg r_+ \) (as we assume). In the example considered here, we shall take \( h_1 = 1 \) for the sake of simplicity (the mass transfer rate can always be chosen accordingly). Thus we have

\[
\Sigma_0(r) \approx 0.1 r^{-3/2}, \quad (89)
\]

so that the surface density of the unperturbed disk increases significantly for small values of \( r \).

Using equation (72) we get

\[
\tilde{h}^\pm_{(m)}(r, \zeta) = (1 - \zeta^2)^{1/2} \left[ d^\pm_0(r) + d^\pm_{2(m)}(r) \zeta^2 + d^\pm_{4(m)}(r) \zeta^4 \right], \quad (90)
\]

where, as explained before, the radial function are very complicated, albeit analytically known expressions. Integrating over \( \zeta \) we get

\[
R^\pm_{m}(r, \alpha) \equiv \int_{-1}^{1} \tilde{h}^\pm_{(m)}(r, \zeta) d\zeta = \pi \left[ d^\pm_0(r) + \frac{1}{4} d^\pm_{2(m)}(r) + \frac{1}{8} d^\pm_{4(m)}(r) \right], \quad (91)
\]
where we have explicitly reminded the $\alpha$ dependence of this quantity.

Thus the total surface density (base flow + perturbation) after a sufficiently long time, when the algebraic term dominates (i.e., for $t \gtrsim 100$ time units) and before the overall exponential decay takes over (i.e., for $t \lesssim 1/\alpha$) can be well approximated by

$$
\Sigma(r, \phi, t) = 0.1 r^{-3/2} + 2e^{2} \left\{ T \exp\left(- \frac{8}{5} \alpha T \right) \left[ \cos \theta_m \Re(R_m^+ \sigma) + \sin \theta_m \Im(R_m^+ \sigma) + \cos \theta_m \Re(R_m^- \sigma) + \sin \theta_m \Im(R_m^- \sigma) \right] \right\}. \quad (92)
$$

With the definitions of $R_m^+$ and $\theta_m$ as above, we can now calculate the spatio-temporal evolution of the surface density in the time interval where our approximations hold.

Results—example of a pattern evolution

In what follows we shall present graphically three time snapshots of the ratio between the perturbation of the surface density (i.e., the $\epsilon^2$ term of the above equation) and the unperturbed surface density, $\Sigma_0$. It should be remarked that the radial functions include, in principle, two functions, $A(r)$ and $C(r)$, which are technically arbitrary, and can be only determined by the initial conditions. In the case of the density perturbation only $A(r)$ is needed. We take it to be real, for simplicity, and to consist of a Gaussian peak, centered around some radius in the disk, $r_0$, in the region we wish to consider.

$$
A(r) = g_0 e^{-(r-g)^2/\Delta^2}, \quad (93)
$$

where the width $\Delta$ is chosen appropriately. This is done in an effort to mimic a perturbation localized in $r$. The parameter $g_0$ determines how large the perturbation is at its peak value. Because this expression is included in the $\epsilon^2$ term, the resulting initial perturbation is very small, if $g_0$ is kept to be $\lesssim 1$.

In this example, we follow a disk ring and are thus far enough away from the zero torque radius, which our scaling ensures is much smaller than 1. As mentioned before, we consider an appropriate ring, because both the disk unperturbed surface density $\Sigma_0(r)$ and the radial function $R(r)$ exhibit a power-law decay—the former $\propto r^{-1.5}$ and the latter $\propto r^{-2}$. Thus the perturbation becomes relatively less important, as compared to the steady quantity, for very large radii, while for small radii it may be too big. For the sake of a clear demonstration of the transient growth we choose, as before, $\alpha = 10^{-3}$.

In the example, for which the surface density evolution is displayed in the figures, we chose only one Fourier mode $(m = 2)$ for simplicity. The perturbation was introduced with a real value of $A(r)$, as given above, with $g_0 = 0.5$ (other choices of this parameter do not change the result significantly, as long as the parameters remain of order 1) and with $r_0 = 5$, $\Delta = 1$.

As it can be seen in the in Figure [V] which displays the relative surface density perturbation, that is,

$$
\sigma(r, \phi, t) = \frac{\Sigma(r, \phi, t) - \Sigma_0(r)}{\Sigma_0(r)}, \quad (94)
$$

at time $t = 10$, in units of $\Omega_0^{-1}(r_*)$, the perturbation is still very small. Even though we cannot be sure that at such an early time our neglecting of other terms is justified, we are confident that the transiently growing term is still insignificant. We also verify that it has the typical $m = 2$ Fourier component form.

As mentioned before, we can follow the time-evolution of the surface density perturbation by calculating, with the help of Mathematica 6, the appropriate analytical expressions. The snapshot figures are also produced by that software. The results for $t = 100$ and $t = 200$ are displayed in Figure [V] and [W] Two basic features are immediately apparent

1. The absolute value of the perturbation grows with time, as it should according to the non-modal transient growth process (see above, in the body of the paper). The growth, at these times, before the exponential decay takes over, is approximately algebraic with $T$.

2. Since the time variable always appears in the similarity variable $T = \Omega_0 t = r^{-3/2} t$, the initial perturbation pattern (having an $m = 2$ Fourier angle dependence) is being wound-up by the close to Keplerian flow, producing successive peaks and troughs in the surface density.

The winding of the perturbation pattern becomes rather strong for later times, giving rise to a very narrow “radial wavelength” of the basic pattern. The overall pattern becomes close to an axially symmetric one by $t = 100$. Note that $\sigma$ relates to the total time-dependent disk surface density according to,

$$
\Sigma(r, \phi, t) = \Sigma_0(r) \left[ 1 + \sigma(r, \phi, t) \right]. \quad (95)
$$
FIG. 5: The relative perturbation in surface density, $\sigma(r, \phi, t)$, as a function of position, calculated in a ring $1 \leq r \leq 7$ of the disk, at a short time after the initial condition ($t = 10$). For details see the text.

FIG. 6: The relative perturbation in surface density, $\sigma(r, \phi, t)$, as a function of position, calculated in a ring $1 \leq r \leq 7$ of the disk, at time $t = 100$. The magnitude of the relative perturbation grows and its pattern becomes more complex. For details see the text.

Thus, as long as $|\sigma| < 1$, negative densities are not encountered. Since the pattern at $t = 200$ is almost axially symmetric we can display the total surface density variations along a radial cut, that is to say, on a two dimensional plot with respect to radius for a fixed value of the azimuthal angle. In Figure 8 we present such a cut through the disk at $\phi = 0$. The total surface density displays a pattern resembling cylindrically symmetric “waves”, with an amplitude of the order of the unperturbed surface density itself (also shown). By $t = 200$, the variations in the surface density are of the same order as the unperturbed surface density, approximately $0.5 \Sigma_0(r) \lesssim \Sigma(r, \phi, t) \lesssim 1.5\Sigma_0(r)$. 
VI. SUMMARY AND DISCUSSION—A NEW DIRECTION TOWARD DISK TURBULENCE?

In this work we considered the approximate nonlinear dynamics of a disturbed hydrodynamical viscous thin disk. The base flow is a Keplerian polytropic accretion disk with vertical structure (the KK analytical solution). By means of an asymptotic expansion in the small parameter $\epsilon$ (the ratio of the characteristic height to radius of the disk) we find the temporal evolution of global non-axisymmetric perturbations. While in the first order all the variables decay, in the second order the perturbed density and vertical velocity display a strong transient growth. In a short time successive peaks and troughs appear in the surface density, similar to what was observed in the axisymmetric study of UNRS. The fact that these structures appear for general non-axisymmetric disturbances promotes the conjecture that
this phenomenon could be commonplace in non-magnetized ADs and this, in turn, has very interesting consequences for dynamics on the small scales.

To be more concrete, we start by observing that according to \( \text{[57]} \) the surface density has the functional form \( \sim T e^{-\alpha T} \sin \theta_m^\pm \), where the similarity variable \( T = tr^{-3/2} \) and the radial “wavelength” \( \theta_m^\pm \sim (m \mp 1)T \). We note here two points. Firstly, the amplitude of surface density patterns increase steadily during the algebraic phase of the structure’s response before viscosity finally gains importance and the disturbance subsequently dies away. The steady shortening of the acoustic pattern’s radial wavelength is a direct consequence of the non-normal nature of the operator governing the pattern’s response and the non-separable nature of these solutions. Secondly, the temporal decrease of the radial wavelength is enhanced for larger values of \( m \), and we interpret this as the pattern getting “wound up”.

Thus, we remark that as the pattern winds and the crenelation deepens, the amplitude of the surface density pattern grows with time. This latter effect is causally tied to the growing response in the vertical velocity which comes from shear energy being converted into vertical mechanical motions.

The pathways by which this energy is fed into vertical motions can be ascertained by studying the driving terms in the inhomogeneous operator \( \text{[52]} \). Referring to the definitions given in \( \text{[54-57]} \), we see that the effects responsible for bringing about the algebraic time dependence derive from the operators \( \mathcal{F} \) and \( \mathcal{G} \) which represent, respectively, the work done by radial compression of the radial velocity \( u_1 \) and the radial gradient of \( rz \) viscous stresses. Technically speaking, the operators \( \mathcal{F} \) and \( \mathcal{G} \), each having gradients with respect to \( r \), bring about factors of \( T(t) \) when they act on the solution to \( u_1 \) \( \text{[64]} \). We note, however, that as the viscosity parameter \( \alpha \) decreases, the algebraic behavior is more strongly tied to the compressional work rather than the work extracted from the viscous stresses, as the latter is scaled by \( \alpha \). Nonetheless, all perturbations that include disturbances in the angular velocity and/or radial velocities (i.e., \( u'_1, \Omega'_1 \)) will give rise to this algebraic behavior. However, and although the point may be academic, we remark also that the azimuthal compressional work of the angular velocity \( \Omega'_1 \) does not contribute to the algebraic growth. This growth specifically comes from the radial compression effect \( \partial_r u'_1 \).

The development of structure on finer scales as time goes forth also means that terms in the asymptotic expansions formally start to break order. This comes about because the very same radial derivatives discussed above no longer remain an order \( \epsilon \) smaller than the \( z \)-derivatives when the algebraic part of the growth becomes substantial and this is physically related to the steady development of radial structure \( \text{[10]} \). This situation becomes especially severe for smaller \( \alpha \), as the algebraic growth persists for timescales which are inversely proportional to the viscosity. The breakdown of the asymptotic expansion means that the equations of motion must be re-expanded in order to handle the evolution of these highly wound structures with fine radial structure. The resulting equations will be something like the shearing box equations \( \text{[16, 18]} \) or some other appropriate model set \( \text{[41, 42, 43, 44, 45]} \).

Fortunately, there are preliminary indications of what may happen under these circumstances. There is recent literature devoted to exploring what occurs on small disk scales when there is a sizable deviation from a Keplerian flow which, in turn, is related to significant radial variations of the surface density. One of the effects we expect to happen from the solution scheme employed in this work is that the fourth order correction to the angular velocity, \( \Omega_4 \), will be algebraically forced in \( T \) by \( \rho_2 \) via the radial pressure gradient. This can be found in the \( \epsilon^2 \) order expansion of \( \text{[2]} \). As the pattern continues to wind and the crenelation deepens, the correction \( \Omega_4 \) will algebraically grow with \( T \), so strongly that one of several things could possibly happen

1. Aside from breaking its asymptotic ordering, \( \Omega_4 \) would also eventually provide corrections so strong that the composite angular rotation profile will satisfy the Rayleigh criterion for axisymmetric instability in many sections of the disk. In other words, for a given small scale disk section under examination, the composite rotation profile could conceivably develop a \( 1/r^2 \) profile (i.e., the inviscid “Rayleigh Line”) or steeper. This would, presumably, result in the termination of the algebraic growth and replace it with radial transport arising from the axisymmetric instability. A transition may occur even before the Rayleigh line is crossed, as Lesur and Longaretti \( \text{[13]} \) showed that a subcritical transition into a turbulent state does exist and that the turbulent activity becomes more vigorous as the Rayleigh line is approached.

2. Li et al. \( \text{[40]} \) consider the fate of linearized infinitesimal disturbances in a vertically integrated disk model in which there are strong radial variations of the surface density. They find that Rossby wave instability occurs when the surface density “bumps” are at least twenty percent above the mean. Non-linear simulations show that these non-axisymmetric instabilities turn into long-lived vortex trains which transport significant amounts of angular momentum \( \text{[47]} \). Prior to the results obtained in this current work, a criticism of this proposed scenario would have been to say that it is unrealistic to suppose that disks are spotted with seemingly arbitrary bumps of surface density. However, the calculation we have performed in this work shows that surface density variations are not only common, but they can grow to large amplitude with sufficient time. A reference to Figure \( \text{[8]} \) demonstrates how a small perturbation can develop into sizable fluctuations of the surface density—easily meeting the rough twenty percent minimum requirement needed to trigger a Rossby wave instability. Furthermore, this transition is reported to occur as the local flow profile approaches the Rayleigh line from the
Keplerian state (see Figure 9 of Li et al. [46]). Preliminary calculations performed in a quasi-3D annular model of a disk[42] indicate that these trends are robust. We shall detail these results in a forthcoming work.

We feel that the Rossby wave instability, which is a non-axisymmetric shear instability, is likely to be strong and pronounced under these circumstances. Of course, this assertion must be verified by further study. Whatever the outcome may be for the smallest scales, the influence of the largest scales upon the smallest ones ought not be ignored in the study of ADs. The importance of this could be deeper than previously realized and it may be one of the reasons why ADs are such perplexing structures: while there is an obvious separation of scales in a disk, the solutions developed here indicate that, if one waits long enough, the dynamics originating on large scales invariably generates power on the smallest scales too. Usually this downscale cascade of power is rationalized as happening because of nonlinear mode-mode interactions. However, in this case it comes about due to the fundamental inseparability of the solutions to the lowest order dynamical response. Nonetheless, might it be incomplete to examine small scale disk behavior without representing the dynamical influence precipitating from the large scales?

If secondary instabilities do develop as a result of one or more of these processes, then we venture to say that such radial variations of the surface density could be either maintained or replenished due to the anomalous activity generated by them. This is because such activity could conceivably generate power back onto the largest scales and, consequently, become the seed disturbances for the large scale dynamics elucidated in this work. This would complete a dynamical cycle describing sustained disk activity. Even if such small-to-large-scale causal connections are either absent or insignificant in disks, one can also envisage that bursts of activity and transport can occur in some disks simply due to random perturbations from outside in the way discussed in Ioannou and Kakouris [8]. Disks certainly do not sit in isolation and periodic disturbances of them by stars passing nearby them is likely, especially for disks found in crowded environments like young star-forming regions.

The outcome of our work can be summarized as follows:

• Although hydrodynamical thin accretion disks are linearly stable, we find that the transient dynamics of initial 3D non-axisymmetric perturbations can give rise to substantial growth. This confirms that this transient growth is not restricted to axisymmetric disturbances like investigated in UNRS.

• In particular, every perturbation that disturbs the radial velocity, leads to an evolutionary phase in which there is algebraic growth of the density and vertical velocity. After a longer time this temporal response gives way to an exponentially decaying phase wherein viscosity dominates and the perturbed quantities eventually go to zero.

• Due to the non-normal nature of the linear operators involved, the evolution of the perturbation patterns are controlled by a similarity variable $T$, a non-separable combination of the radial coordinate and time. This leads to a winding of the perturbations, producing successive peaks and troughs. This directly contributes to the resulting algebraic growth of the other quantities discussed above as compressional work converts energy contained in the wound pattern into vertical mechanical motions.

• For a given particular Fourier mode, the acoustic energy associated with such a perturbation grows more for higher azimuthal number $m$ and for smaller viscosity $\alpha$.

• We conjecture that as the perturbed surface density continues to wind up and deepen, secondary instabilities could arise (e.g., Rayleigh or Rossby wave instabilities). The development of such instabilities and their interplay with the large scale perturbed disk structures could lead to sustained turbulent activity. These processes could contribute to the enhanced transport of angular momentum needed to match observations of ADs.

The aforementioned results should be ubiquitous for general non-magnetized accretion disks. Therefore they are relevant for the understanding of accretion disks around compact objects, whether they be active galactic nuclei or close binary systems, as well as for protoplanetary disks and circumstellar disks around Be stars. In the future we plan to investigate the possible development of secondary instabilities and the observational consequences of our findings in the different contexts.

Acknowledgements P.R. is supported by the Pappalardo Postdoctoral Fellowship in Physics at MIT. Partial support by KBN grant N N203 381436 is acknowledged. We are grateful to Marek Abramowicz for organizing the JPL birthday conference where this work was initiated. We wish to thank Michael Mond for critical technical suggestions. We also like to thank Ed Bertschinger, Saul Rappaport and Bruno Coppi for valuable discussions.
APPENDIX A: THE VANISHING OF $\Omega_1$ AND $u_0$

Rewriting the first order equations (27) and (28) with the definitions $U \equiv u_0$ and $V \equiv 2r\Omega_1$ gives

$$\left[\left(\frac{2\alpha}{3}\right)r^3\rho_0^{2/3}\right] \frac{\partial^2 U}{\partial z^2} - \left(\frac{2\alpha}{3}\right)z \frac{\partial U}{\partial z} + V = 0 \quad (A1)$$

$$\left[\left(\frac{2\alpha}{3}\right)r^3\rho_0^{2/3}\right] \frac{\partial^2 V}{\partial z^2} - \left(\frac{2\alpha}{3}\right)z \frac{\partial V}{\partial z} - U = 0. \quad (A2)$$

Substituting now the zeroth order solution $\rho_0^{2/3} = (h^2 - z^2)/(5r^3)$ and rearranging leads to

$$U_{zz} - \frac{5z}{h^2 - z^2} U_z + \frac{a}{h^2 - z^2} V = 0 \quad (A3)$$

$$V_{zz} - \frac{5z}{h^2 - z^2} V_z - \frac{a}{h^2 - z^2} U = 0, \quad (A4)$$

where the subscripts $z$ denote here the differentiation with respect to $z$ and $a \equiv 15/(2\alpha)$ is a constant.

Using now $Q(z) \equiv (h^2 - z^2)^{5/2}$ as an integrating factor for the first two terms in both of the above equations, we see that

$$\frac{\partial}{\partial z} \left[ Q \frac{\partial U}{\partial z} \right] = -a(h^2 - z^2)^{3/2} V \quad (A5)$$

$$\frac{\partial}{\partial z} \left[ Q \frac{\partial V}{\partial z} \right] = a(h^2 - z^2)^{3/2} U. \quad (A6)$$

Multiplying the first equation by $U$, the second by $V$, adding and integrating over the domain $[-h, h]$, gives, after dropping the integrated parts,

$$\int_{-h}^{h} \left[ \left( \frac{\partial U}{\partial z} \right)^2 + \left( \frac{\partial V}{\partial z} \right)^2 \right] Q(z) dz = 0. \quad (A7)$$

Because $Q(z) \neq 0$, except at $z = \pm h$ and the functions $U, V$ are bound, they must be equal to constants. Thus, it follows from equations (A5, A6), that $U = V = 0$, except perhaps at $z = \pm h$. However, since they are bound and constant in all the domain, they (and hence $u_0$ and $\Omega_1$) must be zero identically.

APPENDIX B: COMPLETE $O(1)$, $O(\epsilon)$ AND $O(\epsilon^2)$ SYSTEMS

The complete equation sets in these orders will be given here. Complete means here that it is only assumed that the zeroth order functions and the non-perturbed portions of higher-order functions are time independent and nothing is assumed about the axisymmetry of the solutions.

We shall also use here the notation for the full function at $i$-th order, $\tilde{f}_i = f_i(r, z, \phi) + f'_i(r, z, \phi, t)$.

1. Zeroth order system

The complete equations at this order are relatively simple because there is no time dependence at this order.

$$\frac{\Omega_0^2}{r^3} = \frac{1}{r^3} \quad (B1)$$

$$v_0 \rho_0 \frac{\partial \Omega_0}{\partial z} + \rho_0 \Omega_0 \frac{\partial \Omega_0}{\partial \phi} = \frac{\partial}{\partial z} \left( \frac{\eta \Omega_0}{\partial z} \right) \quad (B2)$$

$$v_0 \frac{\partial v_0}{\partial z} + \Omega_0 \frac{\partial v_0}{\partial \phi} = -\frac{z}{r^3} - \frac{\partial W_0}{\partial z} + \frac{4}{3} \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\eta \partial v_0}{\partial z} \right) + \frac{1}{\rho_0} \frac{\partial}{\partial \phi} \left( \frac{\eta \partial \Omega_0}{\partial \phi} \right) - \frac{2}{3} \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \frac{\eta \partial \Omega_0}{\partial \phi} \right) \quad (B3)$$

$$\frac{\partial (\rho_0 v_0)}{\partial z} + \frac{\partial (\rho_0 \Omega_0)}{\partial \phi} = 0 \quad (B4)$$
2. First order system

\[ v_0 \frac{\partial u_0}{\partial z} + \Omega_0 \frac{\partial u_0}{\partial \phi} = 2r \Omega_0 \tilde{\Omega}_1 + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial u_0}{\partial z} \right) \]  
(B5)

\[ \frac{\partial \tilde{\Omega}_1}{\partial t} + (\rho_0 \tilde{v}_1 + v_0 \tilde{\rho}_1) \frac{\partial \Omega_0}{\partial z} + (\rho_0 \tilde{\Omega}_1 + \Omega_0 \tilde{\rho}_1) \frac{\partial \Omega_0}{\partial \phi} + \rho_0 v_0 \frac{\partial \tilde{\Omega}_1}{\partial z} + \rho_0 \Omega_0 \frac{\partial \tilde{\Omega}_1}{\partial \phi} = \frac{\partial}{\partial z} \left( \eta \frac{\partial \tilde{\Omega}_1}{\partial z} \right) - \frac{u_0 \rho_0}{r^2} \frac{\partial}{\partial r} \left( r^2 \Omega_0 \right) \]  
(B6)

\[ \frac{\partial v'}{\partial t} + u_0 \frac{\partial v_0}{\partial r} + v_0 \frac{\partial v_0}{\partial z} + \Omega_0 \frac{\partial v_0}{\partial \phi} + v_0 \frac{\partial \tilde{\Omega}_1}{\partial z} + \Omega_0 \frac{\partial \tilde{\Omega}_1}{\partial \phi} = -\frac{\partial W_1}{\partial z} - \frac{2 \Omega_0}{3 r \rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial \theta_0}{\partial z} \right) + \frac{2 \rho_1}{3 \rho_0^2} \frac{\partial}{\partial z} \left( \eta \frac{\partial \Omega_0}{\partial \phi} \right) - \frac{2 \rho_1}{3 \rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial \Omega_0}{\partial \phi} \right) \]  
(B7)

\[ \frac{\partial \rho'}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho_0 u_0) + \frac{\partial}{\partial z} (\rho_0 \tilde{v}_1) + \frac{\partial}{\partial z} (v_0 \tilde{\rho}_1) + \frac{\partial}{\partial \phi} \left( \rho_0 \tilde{\Omega}_1 \right) + \frac{\partial}{\partial \phi} (\Omega_0 \tilde{\rho}_1) = 0 \]  
(B8)

3. Second order system

\[ \frac{\partial u'_1}{\partial t} + \Omega_0 \frac{\partial \tilde{u}_1}{\partial \phi} - 2r \Omega_0 \tilde{\Omega}_2 = -\frac{\partial W_0}{\partial r} + \frac{3 z^2}{2 r^4} + \frac{1}{\rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial \tilde{u}_1}{\partial z} \right) \]  
(B9)

\[ \frac{\partial \tilde{\Omega}'_2}{\partial t} + \Omega_0 \frac{\partial \tilde{\Omega}_2}{\partial \phi} + \tilde{u}_1 \frac{\partial \Omega_0}{\partial r} (r^2 \Omega_0) = -\frac{1}{r^3 \rho_0} \frac{\partial}{\partial r} \left( r^3 \eta \frac{\partial \Omega_0}{\partial r} \right) + \frac{1}{r^2 \rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial \tilde{\Omega}_2}{\partial z} \right) \]  
(B10)

\[ \frac{\partial v'_2}{\partial t} + \Omega_0 \frac{\partial \tilde{v}_2}{\partial \phi} = -\frac{\partial \tilde{W}_2}{\partial z} + \frac{3 z^3}{2 r^5} + \frac{4}{3 \rho_0} \frac{\partial}{\partial z} \left( \eta \frac{\partial \tilde{v}_2}{\partial z} \right) - \frac{2}{3 \rho_0} \frac{\partial}{\partial \phi} \left( \eta \frac{\partial \tilde{\Omega}_2}{\partial \phi} \right) + \frac{1}{r^2 \rho_0} \frac{\partial}{\partial \phi} \left( \eta \frac{\partial \tilde{\Omega}_2}{\partial \phi} \right) \]  
(B11)

\[ \frac{\partial \rho'_2}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho_0 \tilde{u}_1) + \frac{\partial (\rho_0 \tilde{v}_2)}{\partial z} + \frac{\partial}{\partial \phi} \left( \rho_0 \tilde{\Omega}_2 + \Omega_0 \tilde{\rho}_2 \right) = 0 \]  
(B12)

APPENDIX C: FIRST ORDER SOLUTION, DETAILS

Solution

Substituting

\[ v'_1(r, z, \phi, t) = \sum_{-\infty}^{\infty} \hat{v}_{1(m)}(r, \zeta) e^{i \Omega_0 t + i m \phi} + \text{c.c.,} \]  
(C1)

i.e., equation (50) in the linear differential equation (36), gives the ordinary differential equation

\[ -\left( 1 + \frac{8}{15} \alpha_s \right) \frac{\partial^2}{\partial \zeta^2} - 5 \zeta \frac{\partial}{\partial \zeta} - 3 \frac{1 + \frac{8}{15} \alpha_s}{1 + \frac{8}{15} \alpha_s} \hat{v}_{1(m)}(r, \zeta) = 0, \]  
(C2)
where \(s_\ast = s + im\). The previous equation (which is similar to B.3 in UNRS, who derived it for \(m = 0\)) is valid for any \(m\) and \(n = 3/2\). We can find the solution with Wolfram’s Mathematica 6 (or see [37, 38]). Using the initial condition \(\hat{v}_1(m)(r, 0) = 0\), one gets

\[
\hat{v}_1(m)(r, \zeta) = \frac{A(r)}{(1 - \zeta^2)^{3/4}} \cdot \left[ P^{3/2}_\nu(\zeta) Q^{3/2}_\nu(0) - Q^{3/2}_\nu(\zeta) P^{3/2}_\nu(0) \right],
\]

(C3)

where \(P^{\mu}_\nu(\zeta)\) and \(Q^{\mu}_\nu(\zeta)\) are the associated Legendre functions of the first and second kind, respectively. \(A(r)\) is the constant (in \(\zeta\)) of integration and is free, depending on the initial conditions. The parameter \(\nu\) relates to the parameters of the system via,

\[
\nu(\alpha, s_\ast) = \frac{-\sqrt{22.5 + 12\alpha s_\ast} + \sqrt{90 - 270s_\ast^2 + 192\alpha s_\ast}}{(2\sqrt{22.5 + 12\alpha s_\ast})}.
\]

In addition, in order for the solutions above not to be singular at the boundaries \(\zeta = \pm 1\), \(\nu\) must be equal to half integers (see next section). The general solution to (34) is the sum of the general solution to the homogeneous part and a particular solution to the inhomogeneous. The homogeneous solution is simply a wave—i.e., any function in the form \(\zeta(\phi - \Omega_0t)\). However, if we consider the complete system (33-34), then we get that \(\zeta(\phi - \Omega_0t)\) must be zero. Therefore \(\rho^0_1\) is only equal to the particular solution. It can be found by substituting \(v_1^0\) resulting from (C3) in (34) and using the Ansatz

\[
\rho^0_1(r, \phi, t) = \sum_{m=-\infty}^{\infty} \rho^0_{1(m)}(r, \zeta)e^{i\Omega_0t + im\phi} + c.c.,
\]

(C4)

Then, for a given \(m\), we find

\[
\rho^0_{1(m)}(r, \zeta) = \frac{1}{10\sqrt{5}^{3/2}s_\ast(-1 + \zeta^2)^{3/4}(2\nu - 1)A(r)h(r) \times \sqrt{1 - \zeta^2}} \frac{h(r)^2}{r^3} \left[ \zeta P^{3/2}_\nu(\zeta) Q^{3/2}_\nu(0) - P^{3/2}_{\nu+1}(\zeta) Q^{3/2}_\nu(0) + P^{3/2}_\nu(0) \left(-\zeta Q^{3/2}_\nu(\zeta) + Q^{3/2}_{\nu+1}(\zeta) \right) \right].
\]

The eigenvalues, \(s_\ast\), for any \(m\), will follow from the boundary conditions on \(\zeta\) (see below).

**Vertical boundary conditions**

We assume that the lagrangian pressure perturbation vanishes at \(z = \pm h(r)\)

\[
\frac{dP}{dt} = 0 = (\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla})P = 0, \text{ at } z = \pm h(r).
\]

(C6)

Using the polytropic equation and the continuity equation, (C6) can be simplified to

\[
P\vec{\nabla} \cdot \vec{v} = 0, \text{ at } z = \pm h(r)
\]

(C7)

At this order, since \(u_0 = \Omega_1 = 0\), it reduces to

\[
P_0(r, z) \frac{\partial v'_1}{\partial z} = 0, \text{ at } z = \pm h(r).
\]

(C8)

We substitute the Ansatz for \(v'_1\) and the solution (C8) into (C8) and consider the limit for \(\zeta \to \pm 1\). Therefore the boundary conditions (for \(n > 0\) an integer or half odd integer) resulting from regularity conditions at the disk upper and lower edges are satisfied for \(\nu = 2k + 5/2\), where \(k \geq 0\) is an integer or \(k = -3/2\). Following the discussion in UNRS B.1 we can consider the fundamental mode (\(k=0\)) and so find the two fundamental eigenvalues \((s_{(m)}^\pm)\), that lead to a typical oscillatory decay. The values of \(\alpha\) which we consider in this work (\(\alpha \ll 1\)) do not allow a non-oscillatory decay—see [40]. The actual solution for the fundamental mode (\(k = 0\)) becomes
\[
v_1'(r, z, \phi, t) = \sum_{-\infty}^{\infty} \hat{v}_{1(m)}(r, \zeta)(S_+ e^{s_+^{s(m)}T+im\phi} + S_- e^{s_-^{s(m)}T+im\phi}) + c.c., \tag{C9}
\]

and

\[
\rho_1'(r, z, \phi, t) = \sum_{-\infty}^{\infty} \hat{\rho}_{1(m)}(r, \zeta)(S_{\rho} e^{s_{\rho}^{s(m)}T+im\phi} + S_{\rho} e^{s_{\rho}^{s(m)}T+im\phi}) + c.c. \tag{C10}
\]

**Surface stress conditions**

It is easy to see that the vanishing stress conditions are satisfied on the surface

\[
\lim_{z \to h} \eta \hat{n} \cdot \vec{\nabla} u \to 0 \quad \text{and} \quad \lim_{z \to h} \eta \hat{n} \cdot \vec{\nabla} \Omega \to 0, \tag{C11}
\]

in which \( \hat{n} = \hat{z} \) to lowest order.

**APPENDIX D: SECOND ORDER SOLUTION—DETAILS**

As detailed in the text, \( (60) \) holds for the \( m \)-th Fourier component of the function \( u_1'(r, z, \phi) \), see \( (58) \). We rewrite it here for convenience.

\[
\left\{ \frac{1}{5} \alpha(1 - \zeta^2) \frac{\partial^2}{\partial \zeta^2} - \alpha \zeta 
\frac{\partial}{\partial \zeta} - \frac{3}{2} p_* \right\} \hat{u}_{1(m)}(r, \zeta) = 0, \tag{D1}
\]

where \( \zeta = z/h(r) \) and \( p_* = p + im \) is an eigenvalue.

A similar equation is also satisfied by the Fourier components of \( \Omega_2'(r, z, \phi) \), see \( (59) \). To solve the above equation we assume a truncated series expansion, which for a general \( k \geq 0 \) reads

\[
\hat{u}_{1(m)} = \hat{u}_{1(m)}^{(k)} = \sum_{j=0}^{k+1} A_j(r) \zeta^{2(k+1)-2j}, \tag{D2}
\]

We consider the fundamental mode \( k = 0 \) and the relative eigenvalue \( p \), as above, and obtain

\[
\hat{u}_1 = \hat{u}_1^{(0)} = A(r) \zeta^2 + B(r), \tag{D3}
\]

where we have dropped the \( (m) \) subscript for economy of notation.

Inserting \( \text{(D3)} \) into \( \text{(D1)} \) leads to

\[
\zeta^2 \left( 225 \left(1 + p_*^2 \right) + 144\alpha (5p_* + 4\alpha) \right) A(r) + (225 \left(1 + p_*^2 \right) B(r) - 24\alpha (5p_* + 4\alpha) A(r) \right) = 0. \tag{D4}
\]

We set to zero the coefficient of \( \zeta^2 \) to find \( p_* \), and therefore \( p \), and the coefficient of \( \zeta^0 \) to find \( B(r) \)

\[
p_+^{k(m)} = -\frac{8}{5} \alpha - im \pm i, \quad \frac{B(r)}{A(r)} = -\frac{1}{6}. \tag{D5}
\]

These modes are decaying oscillations. We then substitute \( v_p', z = \zeta h, \hat{u}_1 \) as in \( \text{(D3)} \) and the fundamental mode \( \hat{\Omega}_2 = \hat{\Omega}_2^{(0)} = Q(r) (z^2/h^2 - 1/6) \) in \( \text{(62)} \) we get \( (n = 3/2) \)
\[ \dot{v}^{(0)}_{(m)} + (p + im)^2 v^{(0)}_{(m)} + 2(p + im) \dot{V}^{(0)}_{(m)} + \frac{5}{3} \frac{\partial \dot{v}^{(0)}_{(m)}}{\partial \zeta} - \frac{1}{3} \frac{\partial^2 v^{(0)}_{(m)}}{\partial \zeta^2} + \frac{1}{3} \frac{\partial^2 \dot{v}^{(0)}_{(m)}}{\partial \zeta^2} + \]

\[ t_{\nu}^{-3/2} \left( \dot{V}^{(0)}_{(m)} + (p + im)^2 \dot{V}^{(0)}_{(m)} + \frac{5}{3} \frac{\partial \dot{V}^{(0)}_{(m)}}{\partial \zeta} - \frac{1}{3} \frac{\partial^2 \dot{V}^{(0)}_{(m)}}{\partial \zeta^2} + \frac{1}{3} \frac{\partial^2 \ddot{V}^{(0)}_{(m)}}{\partial \zeta^2} \right) = \]

\[ \text{At the second order (C7) becomes} \]

\[ P_0 r (u'_1 + r \frac{\partial u'_1}{\partial r}) + P_0 (\frac{\partial u'_2}{\partial z} + \frac{\partial Q'_2}{\partial \phi}) + P'_0 \frac{\partial u'_1}{\partial z} = 0, \quad \text{at} \quad z = \pm h(r). \quad \text{(D7)} \]

The first two terms satisfy this condition since \( P_0 = \left( \frac{\hat{h}^2 - z^2}{2 \pi r} \right)^{5/2} \) is zero at \( z = \pm h \) and \( u'_1, v'_2 \) and \( Q'_2 \) and their derivatives are finite. The last term vanishes as well, for \( k = -3/2 \) or integer \( \geq 0 \). Therefore the BC is satisfied also at this order.

**APPENDIX E: MASS ACCRETION RATE**

We want

\[ r \int_0^{2\pi} d\phi \int_{-h}^{h} p u dz = -\dot{M} = \text{const.} \quad \text{(E1)} \]

If we expand \( \rho \) and \( u \), we get different equations at different orders. The time-independent parts are

\[ \int_0^{2\pi} d\phi \int_{-h}^{h} \rho_0 u_1 dz = -\dot{M}_1 = \text{const.} \quad \int_0^{2\pi} d\phi \int_{-h}^{h} \rho_2 u_1 dz = -\dot{M}_2 = \text{const.} \quad \text{(E2)} \]

Therefore \( \dot{M} = \epsilon \dot{M}_1 + \epsilon^3 \dot{M}_2 \).

For the time-dependent part we get

\[ \int_0^{2\pi} d\phi \int_{-h}^{h} \rho_0 u'_1 dz = 0, \quad \int_0^{2\pi} d\phi \int_{-h}^{h} \rho'_1 u'_1 dz = 0. \quad \text{(E3)} \]

\[ \int_0^{2\pi} d\phi \int_{-h}^{h} \rho'_2 u'_1 dz = 0, \quad \int_0^{2\pi} d\phi \int_{-h}^{h} \rho''_2 u'_1 dz = 0. \quad \text{(E4)} \]
However
\[ \int_0^{2\pi} d\phi \int_{-h}^{h} r \rho'_f u'_f dz \neq 0, \quad \int_0^{2\pi} d\phi \int_{-h}^{h} r \rho_i u'_i dz \neq 0. \] (E5)

Indeed in both cases the function that has to be integrated is even in \( z \) and is not \( 2\pi \)-periodic in \( \phi \). We can satisfy the first equation by setting \( \rho'_f = 0 \), but the second equation does not vanish and gives a correction to \( \dot{M} \) that grows and subsequently decays like \( \sim \epsilon^3 T e^{-\frac{4\alpha}{5} T} \). This is a fluctuation of the order of \( \epsilon^3 \) on a quantity (\( \dot{M} \)) of the order of \( \epsilon \). It can be neglected for \( \epsilon \) sufficiently small, that is for \( \alpha \) not too small.

**APPENDIX F: ENERGY-DETAILS**

The particular solutions \( v'_p \) and \( \rho'_p \) are both in the form \( f'_p = A_m e^{r \eta} + B_m e^{G_m + c.c., \text{ see [60]}} \). Let now \( A_m = A_R + i A_I \) and, likewise, \( \Gamma_m = \Gamma_R + i \Gamma_I \), \( B_m = B_R + i B_I \), \( G_m = G_R + i G_I \), where \( A_R, A_I, \Gamma_R, \Gamma_I, B_R, B_I, G_R \) and \( G_I \) are all real quantities. Then we have

\[ \text{Re}(f'_p)^2 = e^{2T \eta} (\mathcal{F}_a + \mathcal{F}_b + \mathcal{F}_c), \] (F1)

where
\[ \mathcal{F}_a = 2A_R^2 + 2A_I^2 + 2B_R^2 + 2B_I^2 + (4A_R B_I + 4A_I B_R) \cos (2T) + (-4A_I B_I + 4A_R B_R) \cos (\Gamma + G), \]
\[ \mathcal{F}_b \equiv -2(B_R^2 + B_I^2) \cos (2G) + (-2A_R^2 + 2A_I^2) \cos (2 \Gamma) + 4(A_R B_I - 4A_I B_R) \sin (2T), \]
\[ \mathcal{F}_c \equiv -4(A_R B_I + A_I B_R) \sin (\Gamma + G) - 4B_I B_R \sin (2G) - 4A_I A_R \sin (2 \Gamma). \]

\( v'_p \) and \( \rho'_p \) have the same phases, i.e., \( \Gamma_R = \Gamma_R = -\left(\frac{8}{5}\right) \alpha T \), \( \Gamma_I = \Gamma_I = T(1 - m) + m \phi \) and \( \Gamma_R = T(1 + m) + m \phi \). The \( \phi \) integral of the quantity in (F2) can be simplified using trigonometric relations

\[ \int_0^{2\pi} \frac{1}{2} \text{Re}(f'_p)^2 d\phi = 2\pi e^{-\frac{4\alpha}{5} T} \left[ A_R^2 + A_I^2 + B_R^2 + B_I^2 + 2(A_R B_I + A_I B_R) \cos (2T) + 2(A_R B_I + A_I B_R) \sin (2T) \right]. \] (F2)

For \( m = 0 \) the above equation reduces to

\[ \int_0^{2\pi} \frac{1}{2} \text{Re}(f'_p)^2 d\phi = 4\pi e^{-\frac{16\alpha}{5} T} \left[ (A_R + B_R) \cos (T) - (A_I - B_I) \sin (T) \right]^2 \] (F3)

We now write equation [60] in the form (F1) and proceed with the vertical integration, remembering that in the fundamental mode \( (k = 0) \) we had for the radial and angular perturbations \( \hat{u}_{1(m)}(r, \zeta) = A(r) \left( \zeta^2 - \frac{1}{T} \right) \) and \( \hat{\Omega}_{2(m)}(r, \zeta) = C(r) \left( \zeta^2 - \frac{1}{r} \right) \), with the radial functions \( A(r) \) and \( C(r) \) free.

Moreover, we notice that \( h(r) \rightarrow (2\Delta)^{1/6} r = c_1 r \) for \( r >> r_* \), with \( c_1 \) depending on \( \alpha \) and on the mass flux. However it multiplies every coefficient and therefore we can set it to one without loss of generality. After the integration in the vertical direction, we obtain \( \mathcal{E}_r \) in the form

\[ \mathcal{E}_r (r, T; \alpha, m) = e^{-\frac{16\alpha}{5} T} F(r; \alpha, m, \cos 2T, \sin 2T). \] (F4)

\( F \) is a known analytical function: we shall not write it out explicitly for space considerations.

[1] P.G. Drazin and W.H. Reid, *Hydrodynamic Stability*, (Cambridge Univ. Press, Cambridge, 1981).
[2] W.O. Criminale, T.L. Jackson and R.D. Joslin, Theory and Computation of Hydrodynamic Stability, (Cambridge Univ. Press, Cambridge, 2003).
[3] L. Boberg and U. Borsa, “Onset of turbulence in a pipe,” Z. Naturforsch. Teil A 43, 697 (1988).
[4] H. Gustavsson, “Energy growth of three-dimensional disturbances in plane Poiseuille flow,” J. Fluid Mech. 224, 241 (1991).
[5] K.M. Butler and B.F. Farrell, “Three-dimensional optimal perturbations in viscous shear,” Phys. Fluids A 4, 1637, (1992).
[6] P.J. Schmid, “Nonmodal Stability Theory,” Ann. Rev. Fluid Mech 39, 129 (2007).
[7] P.J. Schmid and D.S. Henningson, Stability and Transition in Shear Flows, (Springer, New York, 2001).
[8] O.M. Umurhan and O. Regev, “Asymptotic models of meridional flows in thin viscous accretion disks,” Astron. Astrophys. J. 550, 931 (2001).
[9] N.I. Shakura and R.A. Sunyaev, “Black holes in binary systems. Observational appearance,” Astron. Astrophys. 24, 337 (1973). (SS)
[10] D. Lynden-Bell and J.E. Pringle, “The evolution of viscous discs and the origin of the nebular variables,” Mon. Not. R. Ast. Soc. 168, 603 (1974).
[11] D.N.C. Lin and J.C.B. Papaloizou, “Theory of Accretion Disks II: Application to Observed Systems,” Ann. Rev. Astron. Astrophysics. 34, 703 (1996).
[12] J. Frank, A.R. King and D.J. Lin, Accretion Power in Astrophysics, (Cambridge Univ. Press, Cambridge, 2002).
[13] G. Lesur and P.-Y. Longaretti, “On the relevance of subcritical hydrodynamic turbulence to accretion disk transport,” Astron. Astrophys. 444, 25 (2005).
[14] S.A. Balbus, J.F. Hawley and J.M. Stone, “Nonlinear Stability, Hydrodynamical Turbulence, and Transport in Disks,” Astrophys. J. 467, 76 (1996).
[15] J.F. Hawley, S.A. Balbus, and W.F. Winters, “Local Hydrodynamical Stability of Accretion Disks,” Astrophys. J. 518, 394, (1999).
[16] P. Goldreich and D. Lynden-Bell, “II. Spiral arms as sheared gravitational instabilities,” Mon. Not. R. Astr. Soc. 130, 125 (1965).
[17] S.A. Balbus and J.F. Hawley, “A powerful local shear instability in weakly magnetized disks. I—Linear analysis,” Astrophys. J. 376, 214 (1991).
[18] O.M. Umurhan and O. Regev, “Hydrodynamic stability of rotationally supported flows: Linear and nonlinear 2D shearing box results,” Astron. Astrophys. 427, 855 (2004).
[19] P.A. Yecko, “Accretion disk instability revisited. Transient dynamics of rotating shear flow,” Astron. Astrophys. 425, 385 (2004).
[20] G.D. Chageshvili, J.-P. Zahn, A.G. Tevzadze and J.G. Lominadze, “On hydrodynamic shear turbulence in Keplerian disks: Via transient growth to bypass transition,” Astron. Astrophys. 402, 401 (2003).
[21] A.G. Tevzadze, G.D. Chageshvili, J.-P. Zahn, R. Chanshivili and J.G. Lominadze, “On hydrodynamic shear turbulence in stratified Keplerian disks: Transient growth of small-scale 3D vortex mode perturbations,” Astron. Astrophys. 407, 779 (2003).
[22] A.G. Tevzadze, G.D. Chageshvili and J.-P. Zahn, “Hydrodynamic stability and mode coupling in Keplerian flows: local strato-rotational analysis,” Astron. Astrophys. 478, 9 (2008).
[23] N. Afshordi, B. Mukhopadhyay and R. Narayan, “Bypass to turbulence in hydrodynamic accretion: Lagrangian analysis of energy growth,” Astrophys. J. 629, 373 (2005).
[24] B. Mukhopadhyay, N. Afshordi and R. Narayan, “Bypass to turbulence in hydrodynamic accretion disks: an eigenvalue approach,” Astrophys. J. 629, 383 (2005).
[25] A. Sternberg, O.M. Umurhan, Y. Gil and O. Regev, “Hydrodynamic response of rotationally supported flows in the small shearing box model,” Astron. Astrophys. 486, 341 (2008).
[26] S. Kato, “Pulsational instability of accretion disks to axially symmetric oscillations,” Mon. Not. R. Astr. Soc., 185, 629 (1978).
[27] R. Kleiber and W. Glatzel, “On the stability of viscous accretion tori,” Mon. Not. R. Astr. Soc., 303, 107 (1999)
[28] L.N. Latter and G.I. Ogilvie, “Viscous overstability and eccentricity evolution in three-dimensional gaseous discs,” Mon. Not. R. Astr. Soc., 372, 1829 (2006)
[29] E. Kersalé, D.W. Hughes, G.I. Ogilvie, S.M. Tobias and N.O. Weiss, “Global magnetorotational instability with inflow. I. Linear theory and the role of boundary conditions,” Astro. Phys. J., 602, 892 (2004)
[30] O.M. Umurhan and G. Shaviv, “Linear dynamics of weakly viscous accretion disks: A disk analog of Tollmien-Schlichting waves,” Astron. Astrophys. 497, 1 (2009).
[31] V.V. Zhuravlev and N.I. Shakura, “Temporal behaviour of global perturbations in compressible axisymmetric flows with free boundaries,” Astr. Nach. 330, 84 (2009).
[32] W. Kluzniak and D. Kita, “Three-dimensional structure of an alpha accretion disk,” arXiv:astro-ph/0006266v1 (2000). (KK)
[33] The quality of the KK solution has been shown to be virtually unaffected if one relaxes the polytropic assumption and use, instead, a more realistic model for the disk’s thermal structure [34].
[34] O. Regev, “The disk-star boundary layer and its effect on the accretion disk structure,” Astron. Astrophys. 126, 146 (1981).
[35] O.M. Umurhan, A. Nemirovsky, O. Regev and G. Shaviv, “Global axisymmetric dynamics of thin viscous accretion disks,” Astron. Astrophys. 446, 1 (2006). (UNRS)
[36] O. Regev and L. Gitelman, “Asymptotic models of meridional flows in thin viscous accretion disks,” Astron. Astrophys. 396, 623 (2002).
[37] E.W. Weisstein, “Gegenbauer Differential Equation.” From MathWorld—A Wolfram Web Resource, Wolfram Research Inc,
[38] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1972)

[39] J.W.S. Rayleigh, *The Theory of Sound, vol. II*, (Dover, New York, 1945)

[40] In other words, since the operator $\partial_r$ includes $Tr^{-5/2} \partial_T$ as one of its terms, the term acted on by it produces a term which will eventually break order.

[41] J. A. Barranco and P. S. Marcus, “Three-dimensional Vortices in Stratified Protoplanetary Disks,” Astr. Phys. J. 623, 1157 (2005).

[42] O. M. Umurhan, “A shallow-water theory for annular sections of Keplerian disks,” Astron. Astrophys. 489, 953 (2008).

[43] H. H. Klahr and P. Bodenheimer, “Turbulence in accretion disks: vorticity generation and angular momentum transport via the global baroclinic instability,” Astrophys. J. 582, 869 (2003).

[44] M. R. Petersen, K. Julien and G. R. Stewart, “Baroclinic vorticity production in protoplanetary disks. I. vortex formation,” Astrophys. J. 658, 1236 (2007).

[45] M. R. Petersen, G. R. Stewart and K. Julien, “Baroclinic vorticity production in protoplanetary disks. II. vortex growth and longevity” Astrophys. J. 658, 1252 (2007).

[46] H. Li, J. M. Finn, R. V. E. Lovelace and S. A. Colgate, “Rossby Wave Instability of Thin Accretion Disks. II. Detailed Linear Theory,” Astrophys. J. 533, 1023 (2000).

[47] H. Li, S. A. Colgate, B. Wendroff and R. Liska, “Rossby wave instability of thin accretion disks III. Nonlinear simulations,” Astrophys. J. 551, 874 (2001).