Geometric algebra of projective lines

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Abstract. The projective line over a field carries structure of a groupoid with a certain correspondence between objects and arrows. We discuss to what extent the field can be reconstructed from the groupoid.

We consider a transitive groupoid \( \mathbb{L} \) where for any two different objects \( A \) and \( B \), there is given a bijective correspondence between the set \( \text{hom}(A, B) \) and the set of objects \( C \) with \( C \neq A \) and \( C \neq B \). We say that \( C \) is label for the arrow \( A \to B \). Note that \( \mathbb{L} \) has at least three objects (unless it is empty or a group; we exclude these cases).

Transitive groupoids with this structure, we call \textit{projective line candidates}. If \( K \) is a field, the projective line \( P(K^2) \) over \( K \) gives rise to such structure (cf. [2], [4]): the objects are the points of \( P(K^2) \), i.e. the 1-dimensional linear subspaces of \( K^2 \); the arrows are the linear isomorphisms between such. Such linear isomorphism \( A \to B \), for \( A \neq B \), is projection in the direction of a specific direction \( C \) with \( C \neq A, B \), and this determines a bijective correspondence of the kind postulated by the “projective line candidate” notion.

The result of the present note is that \textit{if a projective line candidate enjoys certain properties of geometric character, then it is isomorphic to a } \( P(K^2) \), \textit{with } \( K \) \textit{an essentially unique field.} (The notion of \textit{isomorphism} of projective line candidates is evident: an isomorphism of groupoids, compatible with the assumed labelling of arrows by objects.)

\textbf{Remark.} It is possible for the present purpose to replace the basic structure of groupoid with a weaker notion of “near-groupoid”, which is like a groupoid, except that no endo-arrows \( A \to A \) are assumed. This reduction has here the advantage that \textit{all} non-endo arrows in \( \mathbb{L} \) “are geometric”, i.e. can be drawn as actual projections (or “perspectivities”), as in [2], [1], [4]. An example of such a geometric picture is given in the Appendix.

If \( C \) is label for an arrow \( A \to B \), we write

\[ C : A \to B \quad \text{or} \quad A \xrightarrow{C} B \]

and similar standard diagrammatic notation. We compose from left to right.
The following is an attempt to construct the field $K$ from the groupoid, essentially by providing the set $\text{hom}(A,A) \cup \{0\}$ (for some arbitrarily chosen object $A$) with the structure of field, retaining as multiplication the composition operation on $\text{hom}(A,A)$, as given by the groupoid structure. This is also the reason we shall talk about arrows $A \to A$ as scalars (another reasonable terminology is: pure quantities, cf. [3]).

We assume the following basic equations (other assumptions will be called for later):

1. \[ (A \xrightarrow{C} B \xrightarrow{C} A) = 1_A \]  
2. \[ (A \xrightarrow{C} B \xrightarrow{C} D) = (A \xrightarrow{C} D) \]

We assume that the vertex groups $\text{hom}(A,A)$ of $L$ are commutative. (This can be stated in a way which does not involve any endo-arrow, i.e. it can be stated as a property of near-groupoids, in the sense of the Remark above; namely, for any three parallel arrows $f_1: A \to B$ (with $A \neq B$),

\[ f_1 \cdot f_2^{-1} \cdot f_3 = f_3 \cdot f_2^{-1} \cdot f_1 : A \to B. \]

If one draws these two three-fold composites in the projective line $P(K^2)$, the geometric figure that arises is the Pappus configuration. So for a projective line candidate, commutativity = validity of Pappus’ Axiom.)

So $\text{hom}(A,A)$ is canonically isomorphic to $\text{hom}(B,B)$, by conjugation by some, hence any, $A \to B$ (arrows $A \to B$ exist, since we assume $L$ transitive).

If $\mu \in \text{hom}(A,A)$, we say that $\mu$ is a scalar at $A$; if $\mu' \in \text{hom}(B,B)$ corresponds to it under the conjugation correspondence, we write $\mu \equiv \mu'$.

Consider four objects $A, B, C, D$, with $A, B, C$ mutually distinct, and $A, B, D$ mutually distinct. If $A \xrightarrow{C} B$ and $A \xrightarrow{D} B$, we write $(A; B; C, D)$ or \[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \] for the scalar at $A$ given as the composite

\[ A \xrightarrow{C} B \xrightarrow{D} A; \]

\[ ^1 \text{or more canonically, take } K \text{ as a quotient of the disjoint union of all the } \text{hom}(A,A) \cup \{0\} \text{ as } A \text{ ranges over all objects} \]
this is the classical *cross ratio* or *bi-rapport* of \( A, B, C, D \), see \([2], [1], [4]\). The cross ratio is thus, in display, the composite

\[
\begin{array}{cccc}
  & A & B \\
  & C & D & \\
A & & & B \\
\end{array}
\]

Note that \((A, B; C, C) = 1\), by \([1]\). The columns in the matrix displayed may be interchanged, modulo \(\equiv\); for, consider the diagram

\[
\begin{array}{cccc}
  & A & C \\
  & B & D & \\
A & & & B \\
\end{array}
\]

The two triangles commute by definition; so the commutativity of the total quadrangle says that \( C : A \rightarrow B \) conjugates the scalar \((A, B; C, D)\) at \( A \) to the scalar \((B, A; D, C)\) at \( B \), or

\[
\begin{bmatrix}
  A & B \\
  C & D \\
\end{bmatrix} \equiv \begin{bmatrix}
  B & A \\
  D & C \\
\end{bmatrix}.
\]

The interchange of rows is less trivial, since it involves change of names of objects into labels for arrows; we impose as an axiom, the “hexagon axiom”, which says that, for \( A, B, C, D \) mutually distinct, the (outer) diagram

\[
\begin{array}{cccc}
  & A & C \\
  & B & D & \\
A & & & B \\
\end{array}
\]

commutes, thus \( B : A \rightarrow C \) conjugates the scalar \((A, B; C, D)\) at \( A \) to the scalar \((C, D; A, B)\) at \( C \), i.e.

\[
\begin{bmatrix}
  A & B \\
  C & D \\
\end{bmatrix} \equiv \begin{bmatrix}
  C & D \\
  A & B \\
\end{bmatrix}.
\]
Note that the two vertical arrows both are $B : A \to C$, but they may be replaced jointly by any other $X : A \to C$ – the conjugation relation is the same, by commutativity of the groups of scalars. But note that the advantage of putting $B$ is that then the hexagon can be reduced to a pentagon (the “inner” pentagon in the diagram), because two arrows with label $B$ in the right hand of the diagram may be replaced by one single $B : D \to C$, by Axiom (2). Similarly if we put the label $D$ on both vertical arrows.

It follows that cross ratios are invariant under the action of the four-group, which is the rationale for the “matrix” notation employed. Therefore also, any cross ratio in which the letter $A$ occurs, may be replaced (up to $\equiv$) by one in which the letter $A$ occurs in the the upper left hand corner, without changing the cross ratio (mod $\equiv$).

Note that $(A, B; C, D) = (A, B; C, D')$ implies $D = D'$.

We now consider the effects of permuting the four vertices by a permutation which is not one of the four-group permutations. Since one of the letters, say $A$, may always be brought to the upper left corner, it suffices to consider the permutations of the remaining three entries $B, C, D$ (assumed mutually distinct, and distinct from $A$). For cross ratios in projective lines over a field, this gives a classical list (cf. e.g. [5] I-4); it is reproduced here: let $\mu$ denote $(A, B; C, D)$ (this is recorded as the first equation in the list). Then

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \mu \\
\begin{bmatrix}
A & B \\
D & C
\end{bmatrix} = \mu^{-1} \\
\begin{bmatrix}
A & C \\
B & D
\end{bmatrix} = 1 - \mu \\
\begin{bmatrix}
A & C \\
D & B
\end{bmatrix} = (1 - \mu)^{-1} \\
\begin{bmatrix}
A & D \\
B & C
\end{bmatrix} = 1 - \mu^{-1} \\
\begin{bmatrix}
A & D \\
C & B
\end{bmatrix} = (1 - \mu^{-1})^{-1}
\]

Equation (5) makes sense and is easy to prove in our context, using (1). But the rest make no sense as they stand, because we have not assumed any further algebraic structure on the vertex group $\text{hom}(A, A)$ to justify the minus signs. The
crucial point is to give meaning to the right hand side of (6); the rest follow by 
combining (5) and (6). What is meant by $1 - \mu$?

What is true for projective lines over a field is the following property of cross 
ratios:

$$\text{if } (A, B; C, D) = (A', B'; C', D'), \text{ then also } (A, C; B, D) = (A', C'; B', D') \quad (10)$$

To the extent this holds in $\mathbb{L}$, we may define an involution $\Phi$ on $\text{hom}(A, A)$: to 
define $\Phi(\mu)$ for a scalar $\mu : A \to A$, we choose $B, C, D$ so that

$$\mu = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

and then we put

$$\Phi(\mu) = \begin{bmatrix} A & C \\ B & D \end{bmatrix}.$$  

For, by the property assumed, the result $\Phi(\mu)$ does not depend on the way $B, C, D$ 
were chosen. Also, we can then prove (using variation of $A$) that the $\Phi$s thus 
defined on each $\text{hom}(A, A)$ is invariant under the (conjugation-) identification of 
$\text{hom}(A, A)$ with $\text{hom}(A', A')$.

Under these circumstances, there is no harm in denoting $\Phi(\mu)$ by $1 - \mu$, and 
this we shall do.

(So we assume the property (10) as an axiom, but it is unfortunately not purely 
equational, which we would prefer. I am still looking for an equational 
formulation.)

There is another unary “minus” operation possible, uniformly on all the $\text{hom}(A, A)$s. 
We put $-\mu := (-1) \cdot \mu$, where $(-1) : A \to A$ is the scalar at $A$ defined as follows. 
We choose $B$ and $C$ (distinct, and distinct from $A$) and let $(-1)_A$ be the scalar at $A$ 
defined as the composite

$$A \xrightarrow{C} B \xrightarrow{A} C \xrightarrow{B} A;$$

this particular composite is in the coordinate situation (or in a projective line 
embedded in a projective plane) (multiplication by) the scalar $-1$, see [2]. It cannot 
be reduced to a cross ratio, and it is a special case of composites, considered in 
[1] under the name “tri-rapport” (where cross ratio = “bi-rapport”), see below.
The independence of $-1$ on the choice of $A, B, C$ again seems to be something we need to impose as an axiom; here, it will follow from a purely equational one, namely commutativity, for all $A, B, C, B', C'$, of the hexagon

\[
\begin{array}{c}
A \xrightarrow{C} B \xrightarrow{A} C \\
\downarrow \quad \downarrow \quad \downarrow \\
C' \xrightarrow{B'} A \xrightarrow{C'} B'
\end{array}
\] (11)

From this follows that for each $A$, we have a well defined scalar $(-1)_A$ (independent of the choice of $B$ and $C$). Then it easily follows that $(-1)_A \equiv (-1)_B$. For, we may pick $C$ so that $(-1)_A$ is represented, as above, by $A, B, C$, and $(-1)_B$ is similarly represented by $B, C, A$. But then $C : A \rightarrow B$ conjugates the chosen expression for $(-1)_A$ to the chosen one for $(-1)_B$. In this sense, $-1$ is a “uniform” scalar.

Let us in (11) take $C' := B$ and $B' := C$. Then we get the equality

\[
\begin{array}{c}
A \xrightarrow{C} B \xrightarrow{A} C \xrightarrow{B} A \\
= A \xrightarrow{B} C \xrightarrow{A} B \xrightarrow{C} A
\end{array}
\]

and therefore that

\[
\begin{array}{c}
A \xrightarrow{C} B \xrightarrow{A} C \xrightarrow{B} A \xrightarrow{C} B \xrightarrow{A}
\end{array}
\]

equals

\[
\begin{array}{c}
A \xrightarrow{B} C \xrightarrow{A} B \xrightarrow{C} A \xrightarrow{B} C \xrightarrow{A}
\end{array}
\]

and this reduces to $1_A$ by three applications of (11). Thus $(-1).(-1) = 1$, partly justifying the notation. Let us record this:

**Proposition 1** The scalar $-1$ has the property that $(-1) \cdot (-1) = 1$.

We shall consider the notion of *tri-rapports*, in analogy with cross ratios, which are also classically called bi-rapports.
For $A, B, C$ mutually distinct, and $D \neq A, B$, $E \neq B, C$, $F \neq A, C$, we have a scalar at $A$ given the composite

$$A \xrightarrow{D} B \xrightarrow{E} C \xrightarrow{F} A;$$

we denote it $(A, B, C; D, E, F)$ or \[
\begin{bmatrix}
      A & B & C \\
      D & E & F
\end{bmatrix}.\]
Note that $-1$ is such a tri-rapport, $(-1)_A = (A, B, C; C, A, B)$, cf. [2].

We have

$$\begin{bmatrix}
A & B & C \\
D & E & F
\end{bmatrix} \equiv \begin{bmatrix}
B & C & A \\
E & F & D
\end{bmatrix}$$

(cyclic permutation of columns); this is clear: $D : A \to B$ will conjugate the composite defining the left hand side to the one defining the right hand side, just by associativity of composition. (This is essentially the same argument as the argument given previously for interchangability of columns in bi-rapports (= cross ratios), and it generalizes to “multi-rapports”, as considered in [1].)

For tri-rapports, it is not true that the two rows of the matrix can be interchanged.

The following equation is trivial, by repeated use of (1):

$$\begin{bmatrix}
A & B & C \\
D & E & F
\end{bmatrix}^{-1} = \begin{bmatrix}
A & C & B \\
F & E & D
\end{bmatrix}
\quad \text{(12)}$$

Not all tri-rapports can be expressed as bi-rapports with the same entries, but every bi-rapport can be expressed as a tri-rapport:

**Proposition 2** We have

$$\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
A & C & D \\
B & A & B
\end{bmatrix}. \quad \text{(13)}$$

**Proof.** This is just a re-interpretation of the commutative diagram (3); Here, the triangle commutes by (1). Hence the inner pentagon commutes. The upper composite in it is the bi-rapport considered; the lower composite is the tri-rapport considered.

We rewrite the classical “cross ratio” list, augmenting it with the expression of the respective cross ratios (= bi-rapports) in terms of tri-rapports, using Proposition [2].
With the $-1$ available as a “uniform” scalar, the six $\mu$-expressions in the “classical list” above may be augmented by the six further ones, obtained by putting minus sign on the right hand sides. The scalars thus defined cannot in general be expressed as cross ratios (bi-rapport) of four points, but can, by Proposition 2 be expressed as *tri-rapports* of four points. First, we have

**Proposition 3** We have

\[-\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B & D \\ C & A & B \end{bmatrix} = \begin{bmatrix} A & C & B \\ B & A & D \end{bmatrix}\]

**Proof.** To prove the first equality, consider the diagram
The triangle commutes, by definition of \((A,B;C,D)\); the square commutes by (a variation of) the definition of \(-1\). The clockwise composite is the tri-rapport \((A,B,D,C,A,B)\). So we get that this tri-rapport is \((A,B;C,D) \cdot (-1)\), proving the first equality. Next, consider the diagram

```
   A ---- C ---- B ---- D ---- A
     |        \   |        \   |
     |          \   |        \   |
     |           \   |        \   |
     B ---- -1 ---- C ---- -1 ---- A
     |        /   |        /   |
     |          /   |        /   |
     |           /   |        /   |
     C ---- A ---- B ---- D ---- A
```

The clockwise composite is again \(- (A,B;C,D)\), the counterclockwise is \((A,C,B;B,A,D)\).

Having the expressions in right hand column of the above table, we can give tri-rapport expressions for the “additive inverses” of the six scalars listed, using Proposition 3 and substitution instances thereof. We refrain from using arithmetic reductions like \(- (1 - \mu) = \mu - 1\), because validity of arithmetic has not been assumed. We do, however, implicitly use that the involutions \(x \mapsto x^{-1}\) and \(x \mapsto -x\) (\(:= (-1) \cdot x\)) do commute; this follows from \((-1)^{-1} = -1\) (Proposition I).

\[
-\mu = \begin{bmatrix} A & B & D \\ C & A & B \end{bmatrix} = \begin{bmatrix} A & C & B \\ B & A & D \end{bmatrix} \quad (20)
-\mu^{-1} = \begin{bmatrix} A & B & C \\ D & A & B \end{bmatrix} = \begin{bmatrix} A & D & B \\ B & A & C \end{bmatrix} \quad (21)
-(1 - \mu)^{-1} = \begin{bmatrix} A & C & D \\ B & A & C \end{bmatrix} = \begin{bmatrix} A & B & C \\ C & A & D \end{bmatrix} \quad (22)
-(1 - \mu)^{-1} = \begin{bmatrix} A & C & B \\ D & A & C \end{bmatrix} = \begin{bmatrix} A & D & C \\ C & A & B \end{bmatrix} \quad (23)
-(1 - \mu^{-1}) = \begin{bmatrix} A & D & C \\ B & A & D \end{bmatrix} = \begin{bmatrix} A & B & D \\ D & A & C \end{bmatrix} \quad (24)
-(1 - \mu^{-1})^{-1} = \begin{bmatrix} A & D & B \\ B & A & D \end{bmatrix} = \begin{bmatrix} A & C & D \\ D & A & B \end{bmatrix} \quad (25)
\]

**Remark.** The classical way of dealing with the scalar \(-1\), here defined as a tri-
rapport, is in terms of harmonic conjugates: Given $A, B, C, H$. Then

$$(A, B; C, H) = -1$$

iff $H : B \to A$ equals the composite

$$B \rightarrow A \quad C \rightarrow B \quad A$$

For, precomposing the composite with $C : A \to B$ gives $-1$, and precomposing $H : B \to A$ with $C : A \to B$ gives $-1$ iff $(A, B; C, H) = -1$. The classical way of formulating this characterizing property of $H$ is: $H$ is the harmonic conjugate of $C$ w.r.to $A, B$.

The cross-ratios (bi-rapport) and the particular kind of tri-rapport considered in (11) together equip each $K = \text{hom}(A, A) \cup \{0\}$ with enough structure for a field (provided sufficient equations can be secured), namely

- the groupoid structure assumed for $L$ gives the multiplication (together with $0 \cdot x = 0$ for all $x$).
- the cross ratio relation $(A, B; D, C) = (A, B; C, D)^{-1}$ gives the multiplicative inversion (which anyway was given apriori, since every arrow in a groupoid does have an inverse).
- the involution $(A, B; C, D) \mapsto (A, C; B, D)$ gives the (candidate for) $x \mapsto 1 - x$
- the tri-rapport considered in (11) gives the (candidate for) $-1$.

Then the addition $+$ may be defined by

$$x + y := x \cdot (1 - ((-1) \cdot x^{-1} \cdot y),$$

(together with $0 + x = x$).

We can now state the Theorem. We are assuming a projective line candidate $\mathbb{L}$, with commutative vertex groups $\text{hom}(A, A)$, satisfying (1), (2) and the two hexagon conditions (3) and (11) (these conditions are purely equational), as well as the condition (10). Assume that $\mathbb{L}$ satisfies these conditions, and assume finally that $K = \text{hom}(A, A) \cup \{0\}$ carries a field structure, with the field multiplication in $K^*$ (the group of multiplicative units of $K$) equal to the groupoid composition in $\text{hom}(A, A)$, and such that the operation $x \mapsto (1 - x)$ (as given by (16)) equals the operation $x \mapsto 1 - x$ as given by the field structure. (Such a structure is unique.
if it exists, since we argued that the addition $+$ is determined by the remaining projective-line-candidate structure. So the final assumption may possibly be satisfied automatically: it is a matter of the associative law for $+$, and of the distributive law.)

**Theorem 1** Under these circumstances, $\mathbb{L}$ is isomorphic to the projective line $P(K^2)$. More precisely, given three distinct points in $\mathbb{L}$, then there is a unique isomorphism of projective line (-candidates) taking the three given points to $[0 : 1]$, $[1 : 0]$ and $[1 : 1]$, respectively.

**Proof.** Each hom$(A, A)$ is by construction of $K$ identified with $K^*$; and this identification is compatible with $x \mapsto 1 - x$, by assumption. Then the result is a Corollary of the “Fundamental Theorem” for abstract projective lines over $K$, as formulated in [4] §3.

**Appendix**

The “tri-rapport” table has geometric content in the sense that it gives recipes for geometric construction of certain algebraic combinations. As an illustration, we give a geometric (tri-rapport) construction of the scalar $-3$ ($= -(1 - (-2))$) in terms of the scalar $-2$, (presented in terms of a bi-rapport).

![Diagram of the tri-rapport construction](image_url)

We take the groupoid $\mathbb{L}$ to have as objects the lines through a given point in a plane (indicated by a dot in the figure) (this is a standard representation of a projective line: as (unoriented) directions in a plane); the arrows $A \rightarrow B$ are the bijective linear maps between these lines (viewing them as 1-dimensional vector spaces, with the dot as zero). If $A \neq B$ (as in the figure), these linear maps are given by projection in a specific direction; thus $C : A \rightarrow B$ maps the point $a \in A$ to the point $b \in B$. The cross ratio $(A, B; C, D)$ is a linear endo-map of the line $A$, and it takes $a$ to $a'$. This linear endo-map looks like it is “multiplication by the scalar $-2$”. The tri-rapport $(A, B, C; C, A, D)$ takes $a$ to $a''$, and this looks like it is “multiplication by
the scalar $-3^\prime$, in agreement with the tri-rapport formula for $-(1 - \mu) (= \mu - 1)$ in (22).

**References**

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