MORE ON CLOSED NON-VANISHING IDEALS IN $C_B(X)$

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ABSTRACT. Let $X$ be a completely regular topological space. For each closed non-vanishing ideal $H$ of $C_B(X)$, the normed algebra of all bounded continuous scalar-valued mappings on $X$ equipped with pointwise addition and multiplication and the supremum norm, we study its spectrum, denoted by $\text{sp}(H)$. We make a correspondence between algebraic properties of $H$ and topological properties of $\text{sp}(H)$. This continues some previous studies, in which topological properties of $\text{sp}(H)$ such as the Lindelöf property, paracompactness, $\sigma$-compactness and countable compactness have been made into correspondence with algebraic properties of $H$. We study here other compactness properties of $\text{sp}(H)$ such as weak paracompactness, sequential compactness and pseudocompactness. We also study the ideal isomorphisms between two non-vanishing closed ideals of $C_B(X)$.

1. Introduction

Throughout this article by a space, we mean a completely regular topological space; Completely regular spaces are assumed to be Hausdorff. The field of scalars which is fixed throughout our discussion is assumed to be either the complex field $\mathbb{C}$ or the real field $\mathbb{R}$ and it is denoted by $\mathbb{F}$. For a space $X$, let $C_B(X)$ be the normed algebra of all bounded continuous scalar-valued mappings on $X$ equipped with pointwise addition and multiplication and the supremum norm. The normed subalgebra of $C_B(X)$ consisting of mappings which vanish at infinity is denoted by $C_0(X)$.

In [12], the author has constructed for each closed non-vanishing ideal $H$ of $C_B(X)$ a unique locally compact Hausdorff space $Y$ (as a subspace of the Stone-Čech compactification of $X$) such that $H$ and $C_0(Y)$ are isometrically isomorphic. It has been shown that $Y$ and $\text{sp}(H)$, the spectrum of $H$, coincide. In [7], motivated by the results in [12] (see also [4], [5], [9], [10], [11] and [13]) the authors have studied topological properties of $\text{sp}(H)$, by finding equivalent algebraic properties for $H$. In particular, connectedness properties of $\text{sp}(H)$ such as local connectedness, total disconnectedness, zero-dimensionality, strong zero-dimensionality, total separatedness and extremal disconnectedness have been taken into consideration. The study in [7] has been continued in [8] in which the authors have considered compactness properties of $\text{sp}(H)$, by finding necessary and sufficient conditions for $H$ such that the spectrum of $H$ satisfies properties such as the Lindelöf property, $\sigma$-compactness, countable compactness, pseudocompactness and paracompactness. In this article, we study some other compactness properties of $\text{sp}(H)$ such as weak paracompactness and sequential compactness. Also, we reconsider pseudocompactness and find further more necessary and sufficient conditions for a non-vanishing closed ideal $H$ of $C_B(X)$ such that $\text{sp}(H)$ is pseudocompact.

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In [7], for a closed non-vanishing ideal $H$ of $\mathcal{C}_B(X)$, the authors have established a one to one correspondence between the set of all closed subideals of $H$ and the set of all open subspaces of the spectrum of $H$. The analogous natural question of whether there is such a correspondence provided that we identify (algebraically) isomorphic subideals of $H$ and (topologically) homeomorphic open subspaces of $\text{sp}(H)$ remained unsettled. The purpose of the second part of this article is to provide a (partial) answer to this question.

The theory of the Stone-Čech compactification is an essential tools in our study. We state the basic definitions here and refer the reader to [3], [6] and [14] for further details.

The Stone-Čech compactification

Let $X$ be a space. By a compactification of $X$, we mean a compact Hausdorff space $\alpha X$ which contains $X$ as a dense subspace. The “largest” compactification of $X$ (which exists) is called the Stone-Čech compactification of $X$ and is denoted by $\beta X$. The Stone-Čech compactification of $X$ is characterized by the property that every bounded continuous mapping $f : X \to \mathbb{F}$ is extendible to a continuous mapping $F : \beta X \to \mathbb{F}$. For a bounded continuous mapping $f : X \to \mathbb{F}$ we denote by $f^\beta$ the (unique) continuous extension of $f$ to $\beta X$.

2. Compactness properties of the spectrum

In [12] the author has studied closed non-vanishing ideals $H$ of $\mathcal{C}_B(X)$, where $X$ is a completely regular space, by studying their spectrum. (Where by $H$ being non-vanishing or free we mean that for every element $x \in X$ there is an element $h \in H$ such that $h(x) \neq 0$.) $\text{sp}(H)$ has a simple description as a subspace of the Stone-Čech compactification of $X$, as we describe in the following theorem (quoted from [12]).

Recall that for a mapping $f : Y \to \mathbb{F}$ the cozero-set of $f$ is defined as the set of all $y \in Y$ such that $f(y) \neq 0$, and is denoted by $\text{coz}(f)$.

**Theorem and Notation 2.1.** Let $X$ be a space. Let $H$ be a non-vanishing closed ideal in $\mathcal{C}_B(X)$. The spectrum of $H$ is defined as

$$\text{sp}(H) = \bigcup_{h \in H} \text{coz}(h^\beta).$$

So $\text{sp}(H)$ is an open subspace of $\beta X$, and thus, is a locally compact space. Also, $\text{sp}(H)$ contains $X$ as a dense subspace. Further, $H$ and $\mathcal{C}_0(\text{sp}(H))$ are isometrically isomorphic. Therefore, $\text{sp}(H)$ is characterized as the unique locally compact space $Y$ such that $H$ and $\mathcal{C}_0(Y)$ are isometrically isomorphic. In particular, $\text{sp}(H)$ coincides with the Gelfand spectrum of $H$, in the usual sense, when the field of scalars is $\mathbb{C}$.

In [8], the authors have studied non-vanishing closed ideals $H$ of $\mathcal{C}_B(X)$, where $X$ is a completely regular space, by relating algebraic properties of $H$ to compactness properties of $\text{sp}(H)$, such as the Lindelöf property, $\sigma$-compactness, countable compactness, pseudocompactness and paracompactness. In this section we proceed with the study of some other compactness properties of the spectrum such as weak paracompactness and sequential compactness. In the following, first we state a definition which is quoted from [7] Definition 3.2] and [12] Definition 3.2.1], and list some well known lemmas that we need for proving our main results. For proofs of lemmas see e.g. [7] Lemmas 4.2, 4.1, 3.9, 4.11, 3.4, 4.8].
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**Definition 2.2**. Let $X$ be a space. For an ideal $G$ of $C_B(X)$ let

$$\lambda_G X = \bigcup_{g \in G} \text{coz}(g^\beta).$$

Note that by Theorem 2.1 for a non-vanishing closed ideal $H$ of $C_B(X)$ we have $\text{sp}(H) = \lambda_H X$.

**Lemma 2.3**. Let $X$ be a space. Let $G$ be an ideal of $C_B(X)$. Then

$$\overline{\lambda_G X} = \lambda_G X.$$  

Here the bar denotes the closure in $C_B(X)$.

**Lemma 2.4**. Let $X$ be a space. Let $G_1$ and $G_2$ be closed ideals of $C_B(X)$ such that $\lambda_{G_1} X = \lambda_{G_2} X$. Then $G_1 = G_2$.

**Lemma 2.5**. Let $X$ be a space. Let $\{G_i : i \in I\}$ be a collection of ideals of $C_B(X)$. Then

1. $\lambda_{\emptyset} X = \emptyset$.
2. $\lambda_{\bigcup_{i \in I} G_i} X = \bigcup_{i \in I} \lambda_{G_i} X$.
3. $\lambda_{\bigcap_{i \in I} G_i} X = \bigcap_{i \in I} \lambda_{G_i} X$, if $I$ is finite.

**Lemma 2.6**. Let $X$ be a space. Let $H$ be a non-vanishing closed ideal in $C_B(X)$. Then the open subspaces of $\text{sp}(H)$ are exactly those of the form $\lambda_G X$ where $G$ is a closed subideal of $H$. Specifically, for an open subspace $U$ of $\text{sp}(H)$ we have

$$U = \lambda_G X,$$

where

$$G = \{g \in H : g^\beta|_{\beta X \setminus U} = 0\}.$$

**Lemma 2.7**. Let $X$ be a space. Let $H$ be a non-vanishing closed ideal in $C_B(X)$. For a closed subideal $M$ of $H$ the following are equivalent:

1. $M$ is a maximal closed subideal of $H$.
2. There is some $x \in \lambda_H X$ such that

$$\lambda_M X = \lambda_H X \setminus \{x\}.$$  

**Lemma 2.8**. Let $X$ be a space. Let $H_1$ and $H_2$ be two closed ideals in $C_B(X)$. Then $\lambda_{H_1} X \subseteq \lambda_{H_2} X$ if and only if $H_1 \subseteq H_2$.

**Proof**. Suppose that $\lambda_{H_1} X \subseteq \lambda_{H_2} X$. Then

$$\lambda_{H_2} X = \lambda_{H_1} X \cup \lambda_{H_2} X = \lambda_{H_1 + H_2} X = \lambda_{\overline{I_1 + I_2}} X,$$

by Lemmas 2.3 and 2.5. By Lemma 2.4 then $H_2 = \overline{I_1 + I_2}$ which implies that $H_1 \subseteq H_2$. That $\lambda_{H_1} X \subseteq \lambda_{H_2} X$ if $H_1 \subseteq H_2$ is trivial.

Our first result concerns weak paracompactness of the spectrum.

Recall that a space $X$ is called *weakly paracompact* if every open cover of $X$ has a point-finite open refinement. (A (bijectively indexed) collection $\{A_s : s \in S\}$ of subsets of a set $Y$ is called *point-finite* if for each $y \in Y$ the set $\{s \in S : y \in A_s\}$ is finite. Also, for a collection $S = \{H_i : i \in I\}$ of subsets of $Y$ we say that a collection $\{K_j : j \in J\}$ of sets refines $S$, if for each $j \in J$, there exists some $i \in I$ such that $K_j \subseteq H_i$.)

**Theorem 2.9**. Let $X$ be a space and $H$ be a non-vanishing closed ideal in $C_B(X)$. The following are equivalent:

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(1) \( sp(H) \) is weakly paracompact.

(2) If \( S = \{ H_i : i \in I \} \) is a collection of closed subideals of \( H \) such that \( H = \sum_{i \in I} H_i \), then there is collection \( \{ K_j : j \in J \} \) of subideals of \( H \) which refines \( S \), \( H = \sum_{j \in J} K_j \) and such that for each maximal closed subideal \( M \) of \( H \) we have \( K_j \subseteq M \) except for a finite number of indices \( j \in J \).

Proof. (1) implies (2). Let \( S = \{ H_i : i \in I \} \) be a collection of closed subideals of \( H \) such that
\[
H = \sum_{i \in I} H_i.
\]
By Theorem 2.1, Lemmas 2.3 and 2.5, we have
\[
sp(H) = \lambda H X = \lambda \sum_{i \in I} H_i X = \lambda \sum_{i \in I} H_i X = \bigcup_{i \in I} \lambda H_i X.
\]
Therefore \( \{ \lambda H_i X : i \in I \} \) is an open cover for \( sp(H) \) and thus, by our assumption, it has a point-finite open refinement \( T = \{ U_j : j \in J \} \). By Lemma 2.6, for each \( j \in J \) there exists a closed subideal \( K_j \) of \( H \) such that \( U_j = \lambda K_j X \). Arguing as above we have
\[
\lambda H X = sp(H) = \bigcup_{j \in J} U_j = \bigcup_{j \in J} \lambda K_j X = \lambda \sum_{j \in J} K_j X = \lambda \sum_{j \in J} K_j X
\]
which by Lemma 2.4 implies that
\[
H = \sum_{j \in J} K_j.
\]

Note that \( \{ K_j : j \in J \} \) refines \( S \), as for any \( j \in J \), we have \( \lambda K_j X = U_j \subseteq \lambda H_i X \) for some \( i \in I \) which, by Lemma 2.8, implies that \( K_j \subseteq H_i \).

Next, let \( M \) be a maximal closed subideal of \( H \). Then, by Lemma 2.7, there exists some \( x \in \lambda H X \) such that \( \lambda M X = \lambda H X \setminus \{ x \} \). Let \( \mu = \{ j \in J : K_j \nsubseteq M \} \). We show that \( \mu \) is finite. Let \( K_j \nsubseteq M \) for some \( j \in J \). By Lemma 2.8, \( \lambda K_j X \nsubseteq \lambda M X \) and hence \( x \in \lambda K_j X \). The converse is also true. So \( \mu = \{ j \in J : x \in U_j \} \). Since \( T \) is point-finite, \( \mu \) must be finite.

(2) implies (1). Let \( S = \{ U_i : i \in I \} \) be an open cover for \( sp(H) \). By Lemma 2.6, for each \( i \in I \), there exists a closed subideal \( H_i \) of \( H \) such that \( U_i = \lambda H_i X \). Now
\[
\lambda H X = sp(H) = \bigcup_{i \in I} U_i = \bigcup_{i \in I} \lambda H_i X = \lambda \sum_{i \in I} H_i X = \lambda \sum_{i \in I} H_i X,
\]
by Theorem 2.1, Lemmas 2.3 and 2.5, which, by Lemma 2.4, implies that
\[
H = \sum_{i \in I} H_i.
\]

By our assumption, there exists a collection \( \{ K_j : j \in J \} \) of closed subideals of \( H \) which refines \( \{ H_i : i \in I \} \) with properties as stated in (2). In particular \( H = \sum_{j \in J} K_j \) which (as argued above) implies that \( T = \{ \lambda K_j X : j \in J \} \) is an open cover for \( sp(H) \). By (2), for each \( j \in J \) there exists some \( i \in I \) such that \( K_j \subseteq H_i \) and hence \( \lambda K_j X \subseteq \lambda H_i X \). (\( = U_i \)). So, \( T \) refines \( S \). To check that \( T \) is point-finite, let \( x \in sp(H) \). Let \( M \) be a closed subideal of \( H \) such that \( \lambda M X = \lambda H X \setminus \{ x \} \). (which exists, by Lemma 2.6). By Lemma 2.7, \( M \) is a maximal closed subideal of \( H \). Let \( \nu = \{ j \in J : x \in \lambda K_j X \}. If j \in \nu \text{, then } \lambda K_j X \nsubseteq \lambda M X \text{ and therefore } K_j \nsubseteq M \), by Lemma 2.8. By (2), \( \nu \) is then finite. □
In the following we find a necessary and sufficient condition for a non-vanishing closed ideal $H$ of $C_B(X)$ such that $\text{sp}(H)$ is sequentially compact. Recall that a space $Y$ is called sequentially compact if every infinite sequence in $Y$ has a convergent subsequence.

**Theorem 2.10.** Let $X$ be a space and $H$ be a non-vanishing closed ideal in $C_B(X)$. The following are equivalent:

1. $\text{sp}(H)$ is sequentially compact.
2. For any sequence $\{M_n : n \in \mathbb{N}\}$ of maximal closed subideals of $H$ there is a subsequence $\{M_{n_j} : n \in \mathbb{N}\}$ and a maximal closed subideal $M$ of $H$ such that for any closed subideal $I$ of $H$ with $I \not\subseteq M$ there is some $n_0 \in \mathbb{N}$ such that $M_{n_j} + I = H$ for all $n \geq n_0$.

**Proof.** (1) implies (2). Let $\{M_n : n \in \mathbb{N}\}$ be a sequence of maximal closed subideals of $H$. By Lemma 2.7, for each $n \in \mathbb{N}$ there exists some $x_n \in \text{sp}(H)$ such that $\lambda_{M_n} X = \lambda_H X \setminus \{x_n\}$. Since $\text{sp}(H)$ is sequentially compact, there exists a convergence subsequence $\{x_{n_j} : n \in \mathbb{N}\}$ of $\{x_n : n \in \mathbb{N}\}$. Let $x_{n_j} \to x$ where $x \in \text{sp}(H)$. Now consider the closed subideal $M$ of $H$ such that $\lambda_M X = \lambda_H X \setminus \{x\}$ (which exists, by Lemma 2.6). By Lemma 2.7, $M$ is a maximal closed subideal of $H$. Let $I$ be a closed subideal of $H$ such that $I \not\subseteq M$. Let $f \in I \setminus M$. Then $f^3(x) \neq 0$. (Otherwise, $\text{coz}(f^3) \subseteq \lambda_M X$ which implies that $\lambda_H X \setminus \lambda_M X$. Then $(f) \subseteq M$, by Lemma 2.8, which is not correct.) Therefore $x \in \lambda_I X$. Thus, there is some $n_0 \in \mathbb{N}$ such that $x_{n_j} \in \lambda_I X$ for all $n \geq n_0$, which implies that $I \not\subseteq M_{n_j}$, by Lemma 2.8 (as $\lambda_I X \not\subseteq \lambda_{M_{n_j}} X$). Thus $M_{n_j} \not\subseteq M_{n_j} + I$ and hence $M_{n_j} + I = H$, by maximality of $M_{n_j}$, for all $n \geq n_0$.

(2) implies (1). Let $\{x_n : n \in \mathbb{N}\}$ be a sequence in $\text{sp}(H)$. By Lemma 2.6, for each $n \in \mathbb{N}$ there is a closed subideal $M_n$ of $H$ with $\lambda_{M_n} X = \lambda_H X \setminus \{x_n\}$, and $M_n$ is a maximal closed subideal of $H$, by Lemma 2.7. By our assumption, there exists a subsequence $\{M_{n_j} : n \in \mathbb{N}\}$ of $\{M_n : n \in \mathbb{N}\}$ and a maximal closed subideal $M$ of $H$ with properties as stated in (2). By Lemma 2.7, we have $\lambda_{M_n} X = \lambda_H X \setminus \{x\}$ for some $x \in \text{sp}(H)$. We show that $x_{n_j} \to x$. So, let $U$ be an open neighbourhood of $x$ in $\text{sp}(H)$. By Lemma 2.4, there is a closed subideal $I$ of $H$ such that $\lambda_I X = U$. Note that $\lambda_I X \not\subseteq \lambda_M X$, and hence $I \not\subseteq M$, by Lemma 2.8. By our assumption, there is some $n_0 \in \mathbb{N}$ such that $M_{n_j} + I = H$ for all $n \geq n_0$. This implies that $I \not\subseteq M_{n_j}$ (as $M_{n_j}$ is a proper subideal of $H$) and therefore $\lambda_I X \not\subseteq \lambda_{M_{n_j}} X$, for all $n \geq n_0$, by Lemma 2.8. Then $x_{n_j} \in \lambda_I X$ (= $U$) for all $n \geq n_0$, which concludes the proof. 

In [8], the authors have obtained necessary and sufficient conditions for a non-vanishing closed ideal $H$ of $C_B(X)$ such that $\text{sp}(H)$ is pseudocompact. The purpose of our next theorem is to give one more such a necessary and sufficient condition. Recall that a space $X$ is called pseudocompact if there is no unbounded continuous real-valued mapping on $X$, equivalently, if every locally finite open cover of $X$ is finite.

**Theorem 2.11.** Let $X$ be a space and $H$ be a non-vanishing closed ideal in $C_B(X)$. The following are equivalent:

1. $\text{sp}(H)$ is pseudocompact.
2. If $H = \sum_{i \in I} H_i$ where $\{H_i : i \in I\}$ is a bijectively indexed collection of closed subideals of $H$ such that for every maximal closed subideal $M$ of $H$ there is some $f \in H \setminus M$ such that $(f) \cap H_i = 0$ except for a finite number of indices $i \in I$, then $I$ is finite.
Proof. (1) implies (2). Let $H = \sum_{i \in I} H_i$ where $\{H_i : i \in I\}$ is a bijectively indexed collection of closed subideals of $H$ with properties as stated in (1). By Theorem 2.1, Lemmas 2.3 and 2.5, we have

$$\text{sp}(H) = \lambda_H X = \lambda \sum_{i \in I} H_i X = \lambda \sum_{i \in I} H_i X = \bigcup_{i \in I} \lambda_{H_i} X.$$  

Therefore $\mathcal{W} = \{\lambda_{H_i} X : i \in I\}$ is an open cover for $\text{sp}(H)$. First, we show that $\mathcal{W}$ is locally finite. Let $x \in \text{sp}(H)$. By Lemma 2.6, there is a closed subideal $M$ of $H$ such that $\lambda_M X = \text{sp}(H) \setminus \{x\}$. By Lemma 2.7, $M$ is a maximal closed subideal of $H$. By our assumption, there is some $f \in H \setminus M$ and a finite set $J \subseteq I$ such that $H_i \cap (f) = 0$, and therefore $\lambda_{H_i \cap (f)} X = \emptyset$, for all $i \in I \setminus J$. By Lemma 2.5, $\lambda_{H_i} X \cap \lambda_{(f)} X = \emptyset$ for all $i \in I \setminus J$. Also $\lambda_{(f)} X \notin \lambda_M X$, by Lemma 2.8, as $(f) \notin M$, which implies that $x \in \lambda_{(f)} X$. Therefore $\lambda_{(f)} X = \lambda_{(f)} X$, by Lemma 2.3, is an open neighborhood of $x$ which misses all but a finite number of elements of $\mathcal{W}$. This shows that $\mathcal{W}$ is locally finite, and therefore, by pseudocompactness of $\text{sp}(H)$, it is finite. Lemma 2.4 now implies that $\{H_i : i \in I\}$, and therefore $I$, is finite.

(2) implies (1). Let $\mathcal{W} = \{U_i : i \in I\}$ be a bijectively indexed locally finite open cover for $\text{sp}(H)$. By Lemma 2.6, there is a closed subideal $H_i$ of $H$ such that $U_i = \lambda_{H_i} X$ for all $i \in I$. By Theorem 2.1, Lemmas 2.3 and 2.5, we have

$$\lambda_{H_i} X = \text{sp}(H) = \bigcup_{i \in I} U_i = \bigcup_{i \in I} \lambda_{H_i} X = \lambda \sum_{i \in I} H_i X = \lambda \sum_{i \in I} H_i X,$$

which, by Lemma 2.4, implies that

$$H = \sum_{i \in I} H_i.$$

Note that the collection $\{H_i : i \in I\}$ is bijectively indexed. Let $M$ be a maximal closed subideal of $H$. Then $\lambda_M X = \text{sp}(H) \setminus \{x\}$ for some $x \in \text{sp}(H)$, by Lemma 2.7. By our assumption, there exists an open neighbourhood $U$ of $x$ such that $U \cap U_i = \emptyset$ for all but a finite number of indices $i \in I$. By Lemma 2.6, there exists a closed subideal $G$ of $H$ such that $\lambda_G X = U$. Note that $\lambda_G X \neq \lambda_M X$, and therefore $G \notin M$, by Lemma 2.8. Also $G \cap H_i = \emptyset$ for all but a finite number of indices $i \in I$, by Lemma 2.5, as $\lambda_G X \cap \lambda_{H_i} X = \emptyset$. Let $g \in G \setminus M$. Clearly $(g) \cap H_i = \emptyset$ for all but a finite number of indices $i \in I$. Using our assumption, $I$, and therefore $\mathcal{W}$, is finite. ☐

3. Isomorphism theorems

It is clear that each ideal of $C_B(X)$ is also a subalgebra of $C_B(X)$. So one may define two kinds of homomorphisms between ideals of $C_B(X)$. One preserve the structures as ideals and the other preserves the structures as algebras. To avoid ambiguity, in the first case we use the term “ideal homomorphism” and in the second case we use the term “algebra homomorphism.” More precisely, let $H_1$ and $H_2$ be two ideals of $C_B(X)$ and $\rho: H_1 \to H_2$ be a map. If for each $f \in C_B(X)$ and $g \in H$ we have $\rho(fg) = f \rho(g)$, then $\rho$ is an ideal homomorphism and $H_1$ is said to be homomorphic to $H_2$ as an ideal. However, if $\rho(fg) = \rho(f)\rho(g)$ for all $f, g \in H_1$, then $\rho$ is an algebra homomorphism and $H_1$ is said to be homomorphic to $H_2$ as an algebra. Of course if $\rho$ is also bijective, then it is called an isomorphism. Moreover, if $\rho$ is an ideal (algebra) isomorphism and preserves the norm (note that every ideal of $C_B(X)$ is a normed space with the normed it inherits from $C_B(X)$), then it is called an isometric ideal (algebra) isomorphism and the ideals $H_1$ and $H_2$ are called isometrically isomorphic as ideals (algebras).
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A version of the Banach-Stone theorem states that the topology of a locally compact space $Y$ determines and is determined by the normed algebraic properties of $C_0(Y)$. (Indeed, even the algebraic structure of $C_0(Y)$ suffices to determine the topology of $Y$; see [1].) More precisely, for locally compact spaces $Y$ and $Z$, the normed algebras $C_0(Y)$ and $C_0(Z)$ are isometrically isomorphic if and only if the spaces $Y$ and $Z$ are homeomorphic. (See [2] Theorem 7.1.)

We have the following proposition.

**Proposition 3.1.** Let $X$ be a space and $H_1$ and $H_2$ be two closed non-vanishing ideals in $C_B(X)$. Then $H_1$ is isometrically isomorphic to $H_2$ (as algebras) if and only if $\text{sp}(H_1)$ is homeomorphic to $\text{sp}(H_2)$.

**Proof.** Note that (by Theorem 2.1) $H_1$ and $H_2$ are isometrically isomorphic to $C_0(\text{sp}(H_1))$ and $C_0(\text{sp}(H_2))$ (as algebras), respectively. Therefore $H_1$ and $H_2$ are isometrically isomorphic (as algebras) if and only if $C_0(\text{sp}(H_1))$ and $C_0(\text{sp}(H_2))$ are isometrically isomorphic (as algebras) if and only if $\text{sp}(H_1)$ and $\text{sp}(H_2)$ are homeomorphic. $\square$

The above proposition motivates to ask when two non-vanishing closed ideals in $C_B(X)$ are isometrically isomorphic as ideals. The following theorem is to provide a partial answer to this.

Recall that for a space $X$, an extension $Y$ of $X$ is defined as a space which contains $X$ as a dense subspace. Two extensions $Y_1$ and $Y_2$ of a space $X$ are said to be equivalent if there is a homeomorphism between them which fixes $X$ pointwise. This indeed defines an equivalence relation on the class of all extensions of a space $X$. We identify equivalence classes with individuals. (See [14] Section 4.1 for more details.)

**Theorem 3.2.** Let $X$ be a space and $H_1$ and $H_2$ be two non-vanishing closed ideals in $C_B(X)$. Then, if $H_1$ and $H_2$ are isometrically isomorphic as ideals, then $\text{sp}(H_1)$ and $\text{sp}(H_2)$ are equivalent extensions of $X$.

**Proof.** Let $\phi: H_1 \to H_2$ be an isometric (ideal) isomorphism. Let $x \in \text{sp}(H_1)$. Then

$$M_x = \{ f \in H_1 : f^\beta(x) = 0 \}$$

is a maximal closed subideal of $H_1$, by Lemma 2.7. Since $\phi$ is an isomorphism, $\phi(M_x)$ is a proper subideal of $H_2$ and since $\rho$ is an isometry, $\phi(M_x)$ is a closed subideal of $H_2$. Also, if $L$ is another closed subideal of $H_2$ which contains $\phi(M_x)$ properly, then $\phi^{-1}(L)$ is a closed subideal of $H_1$ which contains $M_x$ properly. By maximality of $M_x$, we have $\phi^{-1}(L) = H_1$ and hence $L = H_2$. This shows that $\phi(M_x)$ is a maximal closed subideal of $H_2$. So, by Lemmas 2.4 and 2.7 there exists a unique element $y_x \in \text{sp}(H_2)$ such that $\phi(M_x) = M_{y_x}$.

Now define the map $\rho$ as follows:

$$\rho: \text{sp}(H_1) \to \text{sp}(H_2)$$

$$\rho(x) = y_x.$$  

We claim that $\rho$ is a homeomorphism.

Clearly $\rho$ is a well-defined bijective function. We need to show that $\rho$ and $\rho^{-1}$ are continuous. We prove that $\rho$ is continuous, the proof for continuity of $\rho^{-1}$ is the same.

Let $U$ be an open subset of $\text{sp}(H_2)$. By Lemma 2.6 there exists a closed subideal $I$ of $H_2$ such that $\lambda_I X = U$. Clearly $\phi^{-1}(I)$ is a closed subideal of $H_1$. We show that $\rho^{-1}(U) = \lambda_{\phi^{-1}(I)} X$ which will imply that $\rho^{-1}(U)$ is an open subset of $\text{sp}(H_1)$.

Let $x \in \rho^{-1}(U)$. If for all $f \in \phi^{-1}(I)$ we have $f^\beta(x) = 0$, then $\phi^{-1}(I) \subseteq M_x$ and hence $I \subseteq \phi(M_x) = M_{\rho(x)}$. Therefore, every element of $I$ vanishes at $\rho(x)$ which shows that $\rho(x) \notin \lambda_I X$, a contradiction. So, there exists some $f \in \phi^{-1}(I)$ such that $f^\beta(x) \neq 0$ and hence $x \in \lambda_{\phi^{-1}(I)} X$.  

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Conversely, if \( x \in \lambda_{\phi^{-1}(I)}X \), then \( f^3(x) \neq 0 \) for some \( f \in \phi^{-1}(I) \) which shows that \( f \notin M_x \), and consequently \( \phi(f) \notin M_{\rho(x)} \). So \( \phi(f^3(\rho(x))) \neq 0 \) and hence \( \rho(x) \in \lambda_I X = U \). Therefore \( x \in \rho^{-1}(U) \).

Next, we claim that \( \rho(x) = x \) for all \( x \in X \). Let \( x \in X \). We have \( \phi(M_x) = M_{\rho(x)} \). Let \( g \) be an arbitrary element of \( M_{\rho(x)} \) and \( f \in M_x \) be such that \( \phi(f) = g \). As \( H_1 \) is a non-vanishing ideal, there exists some \( h \in H_1 \) such that \( h(x) \neq 0 \). Now we have \( hg = h\phi(f) = f\phi(h) \). So \( h(x)g(x) = f(x)\phi(h)(x) = 0 \) and hence \( g(x) = 0 \). From this we have \( M_{\rho(x)} \subseteq M_x \) and by maximality of the closed ideal \( M_{\rho(x)} \), we have \( M_{\rho(x)} = M_x \). By Lemma 2.7 it follows that \( \rho(x) = x \).

**Remark 3.3.** In Theorem 3.2 the condition that \( H_1 \) and \( H_2 \) are isometrically isomorphic can be replaced by the condition that “There exists an isomorphism \( \rho: H_1 \to H_2 \) such that both \( \rho \) and \( \rho^{-1} \) map maximal closed subideals to maximal closed subideals.” Still, we do not know whether the converse of Theorem 3.2 holds. We state this formally as an open question for future possible reference.

**Question 3.4.** In Theorem 3.2 does the converse hold?

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