On the maximal multiplicity of long zero-sum free sequences over $C_p \oplus C_p$

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Abstract

In this paper, we point out that the method used in [Acta Arith. 128(2007) 245-279] can be modified slightly to obtain the following result. Let $\varepsilon \in (0, \frac{1}{4}]$ and $c > 0$, and let $p$ be a sufficiently large prime depending on $\varepsilon$ and $c$. Then every zero-sumfree sequence $S$ over $C_p \oplus C_p$ of length $|S| \geq 2p - c \sqrt{p}$ contains some element at least $\lfloor p^{1-\varepsilon} \rfloor$ times.

Keywords: zero-sumfree, multiplicity.

1. Introduction

The structure of long zero-sumfree sequences over a finite cyclic group has been well studied since 1975 (See [1], [7], [14], [15] and [11]). For example, it has been proved by Savchev and Chen [14], and by Yuan [15] independently, that every zero-sumfree sequence over $C_n$ of length at least $\frac{n^2}{4} + 1$ is a partition (up to an integer factor co-prime to $n$) of a positive integer smaller than $n$. But for the group $G = C_n \oplus C_n$, the structure of zero-sumfree sequences $S$ over $G$ has been determined so far only for the case that $S$ is of the maximal length $2n - 2$. In 1969, Emde Boas and Kruyswijk [3] conjectured that every minimal zero-sum sequence over $C_p \oplus C_p$ of length $2p - 1$ contains some element $p - 1$ times, and in 1999, Gao and Geroldinger [6] conjectured that the same result holds true for any group $C_n \oplus C_n$. It is easy to see that the above conjecture is equivalent to that every zero-sum-free sequence $S$ over $G$ of length $2n - 2$ contains some element at least $n - 2$
Our notation and terminology are consistent with [9]. We briefly gather some key notions and fix the notations concerning sequences over finite abelian groups. Let \( \mathbb{N} \) denote the set of positive integers, \( \mathbb{P} \) the set of prime integers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For any two integers \( a, b \in \mathbb{N} \), we set \( [a, b] = \{ x \in \mathbb{N} : a \leq x \leq b \} \). Throughout this paper, all abelian groups will be written additively, and for \( n, r \in \mathbb{N} \), we denote by \( C_n \) the cyclic group of order \( n \), and denote by \( C_r \) the direct sum of \( r \) copies of \( C_n \).

Let \( G \) be a finite abelian group and \( \exp(G) \) its exponent. Let \( F(G) \) be the free abelian monoid, multiplicatively written, with basis \( G \). The elements of \( F(G) \) are called sequences over \( G \). We write sequences \( S \in F(G) \) in the form

\[
S = \prod_{g \in G} g^{v_g(S)}, \text{ with } v_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.
\]

We call \( v_g(G) \) the multiplicity of \( g \) in \( S \), and we say that \( S \) contains \( g \) if \( v_g(S) > 0 \). Further, \( S \) is called squarefree if \( v_g(S) \leq 1 \) for all \( g \in G \). The unit element \( 1 \in F(G) \) is called the empty sequence. A sequence \( S_1 \) is called a subsequence of \( S \) if \( S_1 | S \in F(G) \). For a subset \( A \) of \( G \) we denote \( S_A = \prod_{g \in A} g^{v_g(S)} \). If a sequence \( S \in F(G) \) is written in the form \( S = g_1 \cdot \ldots \cdot g_l \), we tacitly assume that \( l \in \mathbb{N}_0 \) and \( g_1, \ldots, g_l \in G \).
For a sequence 

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g_{v(g)} \in \mathcal{F}(G),$$

we call

- $|S| = l = \sum_{g \in G} v_g(G) \in \mathbb{N}_0$ the length of $S$,
- $h(S) = \max\{v_g(S) | g \in G\} \in [0, |S|]$ the maximum of the multiplicities of $S$,
- $\text{supp}(S) = \{g \in G | v_g(S) > 0\} \subset G$ the support of $S$,
- $\sigma(S) = \sum_{i=1}^l g_i = \sum_{g \in G} v_g(S)g \in G$ the sum of $S$,
- $\sum_k(S) = \{\sum_{i \in I} g_i | I \subset [1, l] \text{ with } |I| = k\}$ the set of $k$-term subsums of $S$, for all $k \in \mathbb{N}$,
- $\sum_{\leq k}(S) = \bigcup_{j \leq k} \sum_j(S)$, $\sum_{\geq k}(S) = \bigcup_{j \geq k} \sum_j(S)$,
- $\sum(S) = \sum_{\leq 1}(S)$ the set of all subsums of $S$.

The sequence $S$ is called

- a zero–sum sequence if $\sigma(S) = 0$.
- zero–sumfree if $0 \notin \sum(S)$.

3. Proof of the main results

Lemma 3.1 [9] Lemma 2.6] Let $G$ be prime cyclic of order $p \in \mathbb{P}$ and $S$ a sequence in $\mathcal{F}(G)$. If $v_0(S) = 0$ and $|S| = p$, then $\sum_{\leq h(S)}(S) = G$.

Lemma 3.2 [9] Let $G$ be prime cyclic of order $p \in \mathbb{P}$, $S \in \mathcal{F}(G)$ a squarefree sequence and $k \in [1, |S|]$.

1. $|\sum_k(S)| \geq \min\{p, k(|S| - k) + 1\}$;
2. If $k = [|S|/2]$, then $|\sum_k(S)| \geq \min\{p, (|S|^2 + 3)/4\}$;
3. If $|S| = [\sqrt{4p - 7}] + 1$ and $k = [|S|/2]$, then $\sum_k(S) = G$.

Lemma 3.3 [9] Lemma 4.2] Let $G = C_p \oplus C_p$ with $p \in \mathbb{P}$, $(e_1, e_2)$ a basis of $G$ and

$$S = \prod_{i=1}^l (a_i e_1 + b_i e_2) \in \mathcal{F}(G),$$

where $a_1, b_1, \ldots, a_l, b_l \in \mathbb{F}_p$,
a zero-sumfree sequence of length $|S| = l \geq p$. Then

$$\left| \left\{ \sum_{i \in I} b_i | \emptyset \neq I \subset [1, l] \text{ with } \sum_{i \in I} a_i = 0 \right\} \right| \geq l - p + 1.$$
Lemma 3.4 Let $\varepsilon \in (0, \frac{1}{2})$, $c > 0$ and $1 < r \in \mathbb{N}$, and let $p$ be a sufficiently large prime depending on $\varepsilon, c$ and $r$. Let $G = C_p$, and let $S$ be a sequence over $G$ of length $|S| \geq p$. Suppose that $|S_{g+H}| \leq [cp^\frac{1}{2}-\varepsilon]$ holds for all subgroups $H$ of order $p^{r-1}$ and all $g \in G$. Then $0 \in \Sigma(S)$.

Proof. Let $p$ be a sufficiently large prime depending on $\varepsilon, c$ and $r$. Assume to the contrary that there exists a zero-sumfree sequence

$$S = \prod_{i=1}^s g_i \in \mathcal{F}(G)$$

of length $|S| = s \geq p$

and such that

$$|S_{g+H}| \leq [cp^\frac{1}{2}-\varepsilon]$$

for any subgroup $H$ of order $p^{r-1}$ and any $g \in G$.

Let $\hat{G} = \text{Hom}(G, \mathbb{C}^\times)$ be the character group of $G$ with complex values, $\chi_0 \in \hat{G}$ the principal character, and for any $\chi \in \hat{G}$ let

$$f(\chi) = \prod_{i=1}^s (1 + \chi(g_i)).$$

Clearly, we have $f(\chi_0) = 2^s$ and

$$f(\chi) = 1 + \sum_{g \in \Sigma(S)} c_g \chi(g),$$

where $c_g = |\{0 \neq I \subset [1, s] : \sum_{i \in I} g_i = g\}|$. Since $S$ is zero-sumfree, we have $0 \notin \Sigma(S)$ and the Orthogonality Relations (see [8], Lemma 5.5.2) imply that

$$\sum_{\chi \in \hat{G}} f(\chi) = \sum_{\chi \in \hat{G}} (1 + \sum_{g \in \Sigma(S)} c_g \chi(g)) = |\hat{G}| + \sum_{g \in \Sigma(S)} c_g \sum_{\chi \in \hat{G}} \chi(g) = |G|.$$

Let $\chi \in \hat{G} \setminus \{\chi_0\}$. We set $M = [cp^\frac{1}{2}-\varepsilon]$ and

$$|S| = (2k - 1)M + q$$

with $q \in [0, 2M - 1]$, and continue with the following assertion:

A1. $|f(\chi)| \leq 2s \exp(-\pi^2v/(2p^2))$ with $v = 2M(1^2 + 2^2 + \cdots + (k - 1)^2) + qk^2$.

Proof of A1. Let $ j \in [- (p - 1)/2, (p - 1)/2]$ and $g \in G$ with $\chi(g) = \exp(2\pi ij/p)$. Note that for any real $x$ with $|x| < \pi/2$, we have $\cos x \leq \exp(-x^2/2)$. Thus

$$|1 + \chi(g)| = 2 \cos\left(\frac{\pi j}{p}\right) \leq 2 \exp\left(\frac{-\pi^2j^2}{2p^2}\right).$$

If $H = \text{Ker}(\chi)$, then $|H| = p^{r-1}$ and $g + H = \chi^{-1}(\exp(2\pi ij/p))$. Thus applying

$$|S_{g+H}| \leq M$$

there are at most $M$ elements $h \mid S$ such that $\chi(h) = \exp(2\pi ij/p)$. Consequently, the upper bound for $f(\chi)$, obtained by repeated application of (1), is maximal if the values $0, 1, -1, \ldots, k-1, -(k-1)$ are
accepted $M$ times each and the values $k, -k$ are accepted $q$ times as images of $\chi(g)$ for $g \in \text{supp}(S)$. Therefore
\[
|f(\chi)| \leq 2^s \exp\left(-\pi^2v/(2p^2)\right).
\]
Since $|S| = s = (2k - 1)M + q$, we get $k = \frac{s-q+M}{2M}$ and hence
\[
v = 2M \sum_{j=1}^{k-1} j^2 + qk^2 = 2M \frac{(k-1)(2k-1)k}{6} + qk^2
\]
\[
= \frac{(s-q-M)(s-q+M)(s-q) + 3q(s-q+M)^2}{12M^2}.
\]
Since $q \in [0, 2M-1]$ and $q \leq s$, it follows that
\[
v = \frac{s(s^2-M^2)}{12M^2} + \frac{q(2M-q)(2M+3s-2q)}{12M^2} \geq \frac{s(s^2-M^2)}{12M^2}.
\]
We deduce that (here we need $p$ sufficiently large)
\[
\exp\left(\frac{\pi^2v}{2p^2}\right) = \exp\left(\frac{\pi^2s(s^2-M^2)}{24M^2p^2}\right) > 2p^r,
\]
where the last inequality holds because $s \geq p$ and $p$ is sufficiently large and then $s^2-M^2 > \frac{p^2}{2}$ and
\[
\frac{\pi^2s(s^2-M^2)}{24M^2p^2} > \frac{\pi^2 p^{2\epsilon}}{2 \cdot 24c^2} > \ln(2p^r).
\]
Therefore it follows that
\[
p^r = |G| = \sum_{\chi \in G} f(\chi) \geq f(\chi_0) = \sum_{\chi \neq \chi_0} |f(\chi)|
\]
\[
\geq 2^s(1 - (p^r - 1)\exp\left(-\frac{\pi^2v}{2p^2}\right)) > 2^s(1 - \frac{p^r - 1}{2p^r}) > 2^{s-1} > p^r,
\]
a contradiction.

\[\square\]

**Proof of Theorem 1.1** We may assume that $c > 8$. Let $(e_1, e_2)$ be a basis of $G$ and for $i \in [1, 2]$ let $\varphi_i : G \rightarrow \langle e_i \rangle$ denote the canonical projections. Let $\varepsilon > 0$, and let $p$ be sufficiently large and assume to the contrary that there exists a zero-sumfree sequence
\[
S = \Pi_{i=1}^{|S|}(a_i e_1 + b_i e_2) \in \mathcal{F}(G), \text{ with } a_1, b_1, \ldots, a_s, b_s \in [0, p-1]
\]
of length $|S| = s \geq 2p - c \sqrt{p}$ and with $h(S) \leq p^{\frac{1}{3} - \varepsilon}$. Let $T$ denote a maximal squarefree subsequence of $S$ and set $h_0 = h(\varphi_1(T))$. After renumbering if necessary we may assume that
\[
T = \Pi_{i=1}^{|T|}(a_i e_1 + b_i e_2), \text{ } a_1 = \cdots = a_h = a.
\]
Now we set
\[ W = \prod_{i=1}^{h_0} (ae_1 + b_ie_2), \quad S_1 = SW^{-1} \]
and distinguish three cases.

Case 1: \( h_0 \geq \lfloor \sqrt{4p - 7} \rfloor + 1 \). We set \( k = \lfloor \sqrt{4p - 7} \rfloor + 1, \ l = \lfloor k/2 \rfloor \) and
\[ S_2 = \prod_{i=k+1}^{i=t} (ae_1 + b_ie_2). \]
By Lemma 3.2(3) we have
\[ \sum_i (\prod_{j=1}^{i} b_ie_2) = \langle e_2 \rangle. \tag{3} \]
Consider the sequence \( \varphi_1(S_2) = \prod_{i=k+1}^{i=t} a_ie_1. \) Let \( v_0(\varphi_1(S_2)) = t \) and after renumbering if necessary we may set
\[ W_1 = \prod_{i=k+1}^{i=t+1} (ae_1 + b_ie_2), \ W_1 \mid S_2. \]
Since \( W_1 \) is zero-sumfree, the sequence \( \varphi_2(W_1) = \prod_{i=k+1}^{i=t+1} b_ie_2 \) is a zero-sumfree sequence over \( C_p \). It follows from Lemma 3.2(3) that \( |\text{supp}(\varphi_2(W_1))| \leq \lfloor \sqrt{4p - 7} \rfloor \). By the contrary hypothesis we have that \( h(\varphi_2(W_1)) = h(W_1) \leq h(S) < p^{1/4} \). Therefore, \( t = |\varphi_2(W_1)| \leq h(\varphi_2(W_1)) \mid \text{supp}(\varphi_2(W_1)) | \leq p^{1/4} \lfloor \sqrt{4p - 7} \rfloor \geq p \).
Thus Lemma 3.1 implies that \( \sum(\varphi_1(S_2)) = \langle e_1 \rangle \). In particular, \( S_2 \) has a non-empty subsequence \( S_3 \) such that \( \sigma(\varphi_1(S_3)) = -lae_1 \). By equation (3) there is a subset \( I \subset [1, k] \) such that \( \sum_{i \in I} b_ie_2 = -\sigma(\varphi_2(S_3)) \) and \( |I| = l \). Therefore, \( S_3 \prod_{i \in I} (ae_1 + b_ie_2) \) is a non-empty zero-sum subsequence of \( S \), a contradiction.

Case 2: \( cp^{1/2} \leq h_0 \leq \lfloor \sqrt{4p - 7} \rfloor \). Setting \( k = \lfloor h_0/2 \rfloor \) and \( h_1 = h(\varphi_1(S_1)) \) then Lemma 3.2(2) implies that
\[ |\sum_i (\prod_{j=1}^{i} b_ie_2) | \geq \frac{h_0^2 + 3}{4} \tag{4} \]
and by the assumption of Case 2 we get
\[ h_1 \leq h(\varphi_1(T))h(S) < h_0p^{1/4}. \]
Therefore,
\[ |\varphi_1(S_1)| - v_0(\varphi_1(S_1)) \geq |S_1| - h_1 > 2p - cp^{1/2} - h_0 - h_0p^{1/4} \geq p - 1, \]
whence Lemma 3.1 implies \( \sum_{i \leq h_1} (\varphi_1(S_1)) = \langle e_1 \rangle \). In particular, \( S_1 \) has a non-empty subsequence \( S_4 \) such that
\[ \sigma(\varphi_1(S_4)) = -kae_1, \ |S_4| \leq h_1. \tag{5} \]
By equations (4) and (5) we infer that
\[ \sigma(S_4) + \sum_\ell (W) \subset \langle e_2 \rangle, \ |\sigma(S_4) + \sum_\ell (W) | \geq \frac{h_0^2 + 3}{4}. \tag{6} \]
Set \( S_5 = S(S_4)W^{-1} \). By Lemma 3.3 we have
\[ |\sum (S_5) \cap \langle e_2 \rangle | \geq |S_5| - p + 1. \]
Therefore, since \( c > 8 \) and \( p \) is sufficiently large,

\[
|\sigma(S_4) + \sum_k(W)| + |\sum(S_5) \cap \langle e_2 \rangle| \geq \frac{h_0^2 + 3}{4} + |S_5| - p + 1
\]

\[
\geq \frac{h_0^2 + 3}{4} + 2p - cp^{1/2} - h_0p^{1/4} - h_0 - p + 1
\]

\[
\geq \frac{h_0^2 + 3}{4} - h_0(p^{1/4} + 1) - cp^{1/2} + 1 + p
\]

\[
= h_0(\frac{1}{4}h_0 - p^{1/4} - 1) - cp^{1/2} + \frac{7}{4} + p
\]

\[
\geq cp^{1/4}(\frac{c}{4} - 1)p^{1/4} - 1) - cp^{1/2} + \frac{7}{4} + p \geq p.
\]

It follows from the Cauchy-Davenport theorem that

\[
(\sigma(S_4) + \sum_k(W)) + (\sum(S_5) \cap \langle e_2 \rangle) = \langle e_2 \rangle,
\]

whence \( 0 \in \sigma(S_4) + \sum_k(W)) + (\sum(S_5) \cap \langle e_2 \rangle) \subset \sum(S) \), a contradiction.

Case 3: \( h_0 < cp^{1/4} \). Note that \( |\text{supp}(S) \cap (ae_1 + \langle e_2 \rangle)| = h_0 \). Thus we may suppose that, for every subgroup \( H \subset G \) with \( |H| = p \) and every \( g \in G \), we have

\[
|S_{g+H}| \leq h_0h(S) \leq \lfloor cp^{\frac{1}{4} - \varepsilon} \rfloor,
\]

since otherwise we choose a different basis \((e'_1, e'_2)\) of \( G \) and are back to Case 1 or Case 2. Therefore applying Lemma 3.4 with \( r = 2 \) we deduce that \( S \) is not zero-sum-free, a contradiction. \( \square \)

**Lemma 3.5** ([5], Theorem 6.7) Every sequence over \( C_n \oplus C_n \) of length \( 3n - 2 \) contains a zero-sum subsequence of length \( n \) or \( 2n \).

**Proof of Theorem 1.2** Let \( k = 3p - 2 - |S| \). Then,

\[
k \leq \lfloor c \sqrt[p]{p} \rfloor - 1 < p.
\]

Let \( W = 0^kS \). Then, \( W \) is a sequence over \( C_p \oplus C_p \) of length \( |W| = 3p - 2 \). By Lemma 3.6 \( W \) contains a zero-sum sequence \( T \) of length \( p \) or \( 2p \). So, \( T_1 = T0^{-v_1(T)} \) is a nonempty zero-sum subsequence of \( S \). Since \( S \) contains no short zero-sum subsequence, we infer that \( |T_1| > p \) and \( |T| = 2p \), and \( T_1 \) is minimal zero-sum. It follows that \( 2p \geq |T_1| \geq 2p - k \geq 2p - \lfloor c \sqrt[p]{p} \rfloor + 1 \). Take an arbitrary element \( g|T_1 \). Therefore, \( T_1g^{-1} \) is zero-sum free and \( h(T_1g^{-1}) \geq p^{\frac{1}{4} - \varepsilon} \) by Theorem 1.1. \( \square \)

**Lemma 3.6** ([4], Theorem 2) Let \( G \) be a finite abelian group, and let \( S \) be a sequence over \( G \) of length \( |S| = |G| + k \) with \( k \geq 1 \). If \( S \) contains no zero-sum subsequence of length \( |G| \), then there exist a subsequence \( T|S \) of length \( |T| = k + 1 \) and an element \( g \in G \) such that \( g + T \) is zero-sum free.
Proof of Theorem 1.3. By Lemma 3.6, there exist a subsequence $T|S$ and an element $g \in C_p \oplus C_p$ such that $g + T$ is zero-sum free and $|g + T| = |T| = |S| - p^2 + 1 \geq 2p - c \sqrt{p}$. It follows from Theorem 1.1 that $h(S) \geq h(T) = h(g + T) \geq p^{1/4} - \varepsilon$. □

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