SPACE-TIME STRUCTURE AND ELECTROMAGNETISM

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Abstract

Two Lagrangian functions are used to construct geometric field theories. One of these Lagrangians depends on the curvature of space, while the other depends on curvature and torsion. It is shown that the theory constructed from the first Lagrangian gives rise to pure gravity, while the theory constructed using the second Lagrangian gives rise to both gravity and electromagnetism. The two theories are constructed in a version of absolute parallelism geometry in which both curvature and torsion are, simultaneously, non-vanishing. One single geometric object, W-tensor, reflecting the properties of curvature and torsion, is defined in this version and is used to construct the second theory. The main conclusion is that a necessary condition for geometric representation of electromagnetism is the presence of a non-vanishing torsion in the geometry used.

1 Introduction

In the context of the philosophy of geometerization of physics, any deviation from flat space indicates the presence of a type, or more, of energy which causes this deviation. From the geometric point of view, the mathematical scheme usually used to detect whether a space is flat or not is to study the commutation of tensor derivatives defined in the space. If $A_\mu$ is an arbitrary covariant vector, one can define the tensor derivative of $A_\mu$ as

$$A_\mu|_\nu \overset{\text{def}}{=} A_\mu,\nu - A_\alpha \Gamma^\alpha_{\mu\nu},$$

where the comma (,) denotes ordinary partial differentiation and $\Gamma^\alpha_{\mu\nu}$ is a linear affine connection defined in the space concerned. In order to study the commutation of such type of derivatives one should evaluate the quantity $(A_\mu|_\nu - A_\mu|_\nu)$. If this quantity vanishes then tensor derivatives commute and the space is flat, otherwise it is not flat. A general expression for the above quantity is given by [1]

$$A_\mu|_\nu - A_\mu|_\nu = A_\alpha B^\alpha_{\mu\nu} - A_\mu|_\alpha \Lambda^\alpha_{\mu\nu},$$

where $B^\alpha_{\mu\nu}$ is the "curvature tensor" defined by,

$$B^\alpha_{\mu\nu} \overset{\text{def}}{=} \Gamma^\alpha_{\mu\sigma,\nu} - \Gamma^\alpha_{\mu\nu,\sigma} + \Gamma^\epsilon_{\mu\sigma} \Gamma^\alpha_{\epsilon\nu} - \Gamma^\epsilon_{\mu\nu} \Gamma^\alpha_{\epsilon\sigma}.$$

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and $\Lambda^{\alpha}_{\mu\nu}$ is the "torsion tensor" defined by,

$$\Lambda^{\alpha}_{\mu\nu} \equiv \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}. \quad (1.4)$$

It is clear from (1.2) that both curvature and torsion can be used as indicators for the deviation of a space from the flat case. The simultaneous vanishing of both is the necessary condition for a space to be flat. Consequently, from the physical point of view, this condition indicates the absence of any type of energy from the space.

In constructing geometric field theories, using Lagrangian formalism, one should start with a Lagrangian function. This Lagrangian is usually formed using a true scalar, i.e. a tensor of zero order. As it reflects the energetic contents of the space, the Lagrangian usually constructed from the curvature tensor for spaces with vanishing torsion (e.g. Riemannian space) (cf. [2]), or from the torsion tensor in the case of spaces with vanishing curvature (e.g. Absolute Parallelism space (AP-space)), [3]. What about spaces with, simultaneously, non-vanishing curvature and torsion? The situation, in this case, is somewhat difficult since it is not easy to find one single geometric object that reflects the effects of both curvature and torsion. If such an object is defined, then the commutation relation (1.2) can be written in the form,

$$A_{\mu|\nu\sigma} - A_{\mu|\sigma\nu} = A_{\alpha} W^{\alpha}_{\mu\nu\sigma}. \quad (1.5)$$

In general, the tensor $W^{\alpha}_{\mu\nu\sigma}$ cannot be defined unless the space is torsion-less. However, there is a rare case in which such a tensor could be defined in spaces with simultaneous non-vanishing curvature and torsion. In this case, this tensor will reflect the effects, of both the curvature and torsion, on the structure of space. Since this tensor will represent deviation from flat space and will reduce to the conventional curvature tensor (in case of spaces with vanishing torsion), we are going to call such a tensor the "W-tensor"[4].

It is the aim of the present work to construct and compare field equations resulting from using the curvature tensor (1.3) and the W-tensor (1.5) in building Lagrangian functions. In Section 2 we are going to review a version of Absolute Parallelism (AP) geometry in which both curvature and torsion are simultaneously non-vanishing. Also, we are going to give the definitions for the curvature and the W-tensors. In Section 3 the field equations constructed using a Lagrangian built from the W-tensor are reviewed. A Lagrangian built using the curvature tensor is used to construct a new set of field equations in Section 4. The two sets of field equations are compared and discussed in Section 5.

2 Riemann-Cartan Version of AP-Geometry

The 4-dimensional AP-space is a manifold, each point of which is labelled by 4-independent variables $x^\mu (\mu = 0, 1, 2, 3)$. At each point we define 4-independent contravariant vector $\lambda^\mu (\mu = 0, 1, 2, 3$ stands for the coordinate components) and $i (i = 0, 1, 2, 3$ stands for the vector numbers) $^1$. We use Latin indices for vector number and Greek indices for coor-

\footnote{In the present work, the authors are going to use the notations in which the vector number is always written in a lower position (neither covariant nor contravariant), consequently, such indices are not lowered or raised (for more details see [5])}
coordinate components. Assuming that \( \| \lambda^i \| \neq 0 \), then we can define the tetrad covariant vectors \( \lambda_i^\mu \) as the normalized cofactors of \( \lambda^i \) in the determinant \( \| \lambda^i \| \), such that
\[
\lambda^\mu_i \lambda_i^\nu = \delta^\mu_\nu. \tag{2.1}
\]
Using these vectors, one can define the second order symmetric tensor,
\[
g_{\mu\nu} \overset{\text{def}}{=} \lambda_i^\mu \lambda_i^\nu. \tag{2.2}
\]
This symmetric tensor can play the role of the metric of Riemannian space associated with the AP-space.

**Connections, Curvatures and Torsion:**

The AP-space admits a non-symmetric linear connection \( \Gamma^\alpha_{\mu\nu} \), which is a consequence of the AP-condition [3], i.e.
\[
\lambda_i^\nu \overset{\text{def}}{=} \lambda_{i,\nu} - \Gamma^\alpha_{\mu\nu} \lambda_i^\alpha = 0, \tag{2.3}
\]
where the stroke denotes tensor differentiation. Equation (2.3) can be solved to give,
\[
\Gamma^\alpha_{\mu\nu} = \lambda_i^\alpha \lambda_i^\mu, \tag{2.4}
\]
Since \( \Gamma^\alpha_{\mu\nu} \) is non-symmetric, then one can define its dual connection as
\[
\tilde{\Gamma}^\alpha_{\mu\nu} \overset{\text{def}}{=} \Gamma^\alpha_{\nu\mu}. \tag{2.5}
\]
The symmetric part of (2.4) is also a connection defined by
\[
\Gamma^\alpha_{(\mu\nu)} \overset{\text{def}}{=} \frac{1}{2}(\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\nu\mu}). \tag{2.6}
\]
Also, Christoffel symbol \( \{^\alpha_{\mu\nu} \} \) could be defined using the metric tensor (2.2). So, in the AP-geometry one could define four linear connections, at least: the non-symmetric connection (2.4), its dual (2.5), its symmetric part and Christoffel symbol.

The curvature tensors, corresponding to the above connections, respectively, are [6]
\[
M^\alpha_{\mu\nu\sigma} \overset{\text{def}}{=} \Gamma^\alpha_{\mu\sigma,\nu} - \Gamma^\alpha_{\mu\nu,\sigma} + \Gamma^\epsilon_{\mu\sigma} \Gamma^\alpha_{\epsilon\nu} - \Gamma^\epsilon_{\mu\nu} \Gamma^\alpha_{\epsilon\sigma}, \tag{2.7}
\]
\[
\tilde{M}^\alpha_{\mu\nu\sigma} \overset{\text{def}}{=} \tilde{\Gamma}^\alpha_{\mu\sigma,\nu} - \tilde{\Gamma}^\alpha_{\mu\nu,\sigma} + \tilde{\Gamma}^\epsilon_{\mu\sigma} \tilde{\Gamma}^\alpha_{\epsilon\nu} - \tilde{\Gamma}^\epsilon_{\mu\nu} \tilde{\Gamma}^\alpha_{\epsilon\sigma}, \tag{2.8}
\]
\[
\bar{M}^\alpha_{\mu\nu\sigma} \overset{\text{def}}{=} \bar{\Gamma}^\alpha_{(\mu\sigma),\nu} - \bar{\Gamma}^\alpha_{(\mu\nu),\sigma} + \Gamma^\epsilon_{(\mu\sigma)} \Gamma^\alpha_{\epsilon(\nu)} - \Gamma^\epsilon_{(\mu\nu)} \Gamma^\alpha_{\epsilon(\sigma)}, \tag{2.9}
\]
\[
R^\alpha_{\mu\nu\sigma} \overset{\text{def}}{=} \{^\alpha_{\mu\sigma} \}_\nu - \{^\alpha_{\mu\nu} \}_\sigma + \{^\alpha_{\mu\sigma} \}_\epsilon \{^\epsilon_{\nu} \}_\sigma - \{^\epsilon_{\nu} \}_\epsilon \{^\alpha_{\mu\sigma} \}_\sigma. \tag{2.10}
\]
From the AP-condition (2.3) the curvature tensor given by (2.7) vanishes identically, while those given by (2.8),(2.9),(2.10) do not vanish.
Using the non-symmetric connection one can define a third order skew tensor, \[ \Lambda^\alpha_{\mu\nu} \overset{\text{def}}{=} \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu} = \tilde{\Gamma}^\alpha_{\mu\nu} - \tilde{\Gamma}^\alpha_{\nu\mu} = -\Lambda^\alpha_{\nu\mu}. \] (2.11)

This tensor is the torsion of AP-space. We can define another third order tensor as, 
\[ \gamma^\alpha_{\mu\nu} \overset{\text{def}}{=} \frac{1}{2} \Lambda^\alpha_{\mu\nu}, \] (2.12)

where (;) is used to denote covariant differentiation using Christoffel symbol. This tensor is called the contortion of the space, using which, it is shown that (cf. [7])
\[ \gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - \{^\alpha_{\mu\nu} \}. \] (2.13)

The tensor \( \gamma_{\mu\nu\alpha} \) is skew-symmetric in its first two indices. A basic vector could be obtained by contraction, using any one of the above third order tensors,
\[ C_\mu \overset{\text{def}}{=} \Lambda^\alpha_{\mu\alpha} = \gamma^\alpha_{\mu\alpha}. \] (2.14)

Using the contortion, one can define the symmetric third order tensor as, 
\[ \Delta^\alpha_{\mu\nu} \overset{\text{def}}{=} \gamma^\alpha_{\mu\nu} + \gamma^\alpha_{\nu\mu}. \] (2.15)

The symmetric and skew-symmetric parts of the tensor \( \gamma^\alpha_{\mu\nu} \) are respectively, 
\[ \gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} = \frac{1}{2} \Lambda^\alpha_{\mu\nu}, \] (2.16)
\[ \gamma^\alpha_{(\mu\nu)} = \frac{1}{2} \Delta^\alpha_{\mu\nu}, \] (2.17)

where the brackets [ ] are used for anti-symmetrization and the parenthesis ( ) are used for symmetrization, of tensors, with respect to the enclosed indices.

Now we have a version of AP-space with, simultaneously, non-vanishing curvature (2.8) and torsion (2.11) corresponding to the dual connection (2.5). In this space we can define a W-tensor, of the type given by (1.5). Since the L.H.S. of (1.5) contains tensor derivatives using different connections, so we are going to define these derivatives in the present version of AP-geometry.

**Tensor Derivatives**

Using the connections mentioned above, we can define the following derivatives [6],
\[ A^\mu_{\nu|\nu} \overset{\text{def}}{=} A^\mu_{\nu,\nu} + \Gamma^\mu_{\alpha\nu} A^\alpha_{\nu}, \] (2.18)
\[ A^\mu_{\nu -|\nu} \overset{\text{def}}{=} A^\mu_{\nu,\nu} + \tilde{\Gamma}^\mu_{\alpha\nu} A^\alpha_{\nu}, \] (2.19)
\[ A^\mu_{0|\nu} \overset{\text{def}}{=} A^\mu_{\nu} + \Gamma^\mu_{(\alpha\nu)} A^\alpha_{\nu}, \] (2.20)
\[ A^\mu_{\nu} \overset{\text{def}}{=} A^\mu_{\nu} + \{^\mu_{(\alpha\nu)}\} A^\alpha_{\nu}. \] (2.21)
As mentioned in section 1, the definition of a W-tensor, of the type given by (1.5), in spaces with curvature and torsion, is difficult. The only way, to overcome this difficulty, is to replace the arbitrary vector $A_\mu$ by the tetrad vectors $\lambda_i^\mu$ in (1.5). The resulting definitions can be written as,

$$
\lambda_{i+\nu\sigma}^\mu - \lambda_{i+\sigma\nu}^\mu = \lambda_\alpha M_{\mu\nu\sigma}^\alpha, \quad (2.22)
$$

$$
\lambda_{i-\nu\sigma}^\mu - \lambda_{i-\sigma\nu}^\mu = \lambda_\alpha W_{\mu\nu\sigma}^\alpha, \quad (2.23)
$$

$$
\lambda_{i0\nu\sigma}^\mu - \lambda_{i0\sigma\nu}^\mu = \lambda_\alpha T_{\mu\nu\sigma}^\alpha, \quad (2.24)
$$

$$
\lambda_{i\mu\nu\sigma}^\alpha - \lambda_{i\mu\sigma\nu}^\alpha = \lambda_\alpha R_{\mu\nu\sigma}^\alpha, \quad (2.25)
$$

The first (2.22), coincides with the curvature (2.7) and is an identically vanishing tensor because of (2.3). The tensor given by (2.25) is identical with Riemannian-Christoffel curvature tensor (2.10) since the connection used to evaluate the L.H.S. of (2.25) is Christoffel symbol. The symmetric connection (2.6) gives the non-vanishing W-tensor (2.24) which is identical to the curvature (2.9). So, we are left with one W-tensor, $W_{\mu\nu\sigma}^\alpha$, which differs from the curvature tensors (2.7) - (2.10).

Now multiplying both sides of (2.23) by $\lambda_\beta^i$, using (2.1), we get [5]

$$
W_{\mu\nu\sigma}^\alpha \overset{\text{def}}{=} \lambda_\alpha \left( \lambda_{i+\nu\sigma}^\mu - \lambda_{i+\sigma\nu}^\mu \right), \quad (2.26)
$$

which gives an explicit definition W-tensor defined in the present version of AP-geometry. Now, this version is characterized by the connection (2.5), the torsion (2.11), the curvature tensor (2.8) and the W-tensor (2.26). All these geometric objects are, in general, simultaneously non-vanishing. Such type of geometry, with simultaneously non-vanishing curvature and torsion, is known as Riemann-Cartan geometry.

The following table is extracted from [7] and contains second order tensors that are used in most applications.
Table 1: Second Order World Tensors [7]

| Skew-Symmetric Tensors | Symmetric Tensors |
|------------------------|-------------------|
| ξ_{μν} \overset{\text{def}}{=} \gamma_{\mu\nu|\sigma}^\alpha \gamma^\alpha_{\sigma} | \phi_{μν} \overset{\text{def}}{=} C_\alpha \Delta^\alpha_{μν} |
| ζ_{μν} \overset{\text{def}}{=} C_\alpha \gamma^\alpha_{μν} | \psi_{μν} \overset{\text{def}}{=} \Delta^\alpha_{μν|+} |
| η_{μν} \overset{\text{def}}{=} C_\alpha \Lambda^\alpha_{μν} | \chi_{μν} \overset{\text{def}}{=} \Lambda^\alpha_{-μν|+} |
| ξ_{μν} \overset{\text{def}}{=} \gamma_{μν|σ} \gamma^\alpha_{σ} | φ_{μν} \overset{\text{def}}{=} C_\alpha \Delta^\alpha_{μν} |
| ζ_{μν} \overset{\text{def}}{=} \gamma^\alpha_{μν} \gamma^\alpha_{σ} | \psi_{μν} \overset{\text{def}}{=} \Delta^\alpha_{μν|+} |
| η_{μν} \overset{\text{def}}{=} \gamma^\alpha_{μν} \gamma^\alpha_{σ} | \chi_{μν} \overset{\text{def}}{=} \Lambda^\alpha_{-μν|+} |
| ζ_{μν} \overset{\text{def}}{=} \gamma^\alpha_{μν} \gamma^\alpha_{σ} | φ_{μν} \overset{\text{def}}{=} C_\alpha \Delta^\alpha_{μν} |

where \( \Lambda^\alpha_{-μν|+} \equiv \Lambda^\alpha_{+|+σ} \). It can be easily shown that there exist an identity between skew-tensors, of this table, which can be written in the form [7]

\[ η_{μν} + ε_{μν} - χ_{μν} \equiv 0. \quad (2.27) \]

We see from the above table that the torsion tensor plays an important role in the structure of AP-space in which all tensors in Table 1 vanish when the torsion tensor vanishes.

In what follows, we are going to examine and compare the consequences of constructing field theories, in the AP-geometry, using the curvature (2.8) and the W-tensor (2.26) to form the corresponding Lagrangian functions. In 1977 Mikhail and Wanas [8] have constructed the Generalized Field Theory GFT using the W-tensor (2.26). In the next section we are going to review briefly this theory, for the sake of the completeness.

### 3 The Use of The W-Tensor

In constructing GFT, Mikhail and Wanas [8] have used the W-tensor, in the AP-geometry, (2.23) which can be written as,

\[ \lambda_{\mu|\nu|σ}^\mu - \lambda_{\mu|σ}^\mu \overset{\text{def}}{=} \lambda_α W^α_{\mu|\nu|σ}. \quad (3.1) \]
As stated, the arbitrary vector $A_\mu$ of (1.5) is replaced by the vectors $\lambda^\mu$ in (3.1). It is impossible to define the $W$-tensor unless we make this replacement. Now, contracting the tensor $W^{\alpha\mu}_{\cdot \nu}$ twice we get the scalar curvature $W$, using which the Lagrangian scalar density can be written as,

$$ L \overset{\text{def}}{=} \lambda^* W, \quad (3.2) $$

where $\lambda^* = \| \lambda_\mu \|$ , $L \overset{\text{def}}{=} g^{\mu\alpha} W_{\mu\alpha}$ and

$$ W_{\mu\alpha} \overset{\text{def}}{=} \Lambda^{\epsilon}_{\mu} \delta_{\alpha}^{\lambda} \epsilon_{\alpha} - C_{\mu} C_{\alpha}. \quad (3.3) $$

It is clear that the tensor $W_{\mu\alpha}$ is a symmetric tensor. The theory corresponding to the scalar density (3.2) is called the GFT which has been derived by using the variational method of ”Dolan and McCrea ” [9]. The field equations of the GFT can be re-derived by using an action principle method [10]. It is shown that the two methods give the same set of field equations, which can be written as,

$$ E_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} W - 2W_{\mu\nu} - 2g_{\mu\alpha} C_{\cdot \nu} |_{\gamma} - 2C_{\mu} C_{\nu} - 2g_{\mu\alpha} C^{\alpha}_{\cdot \nu} + 2C_{\nu} |_{\mu} - 2g^{\gamma\alpha} \Lambda_{\mu\nu\alpha} |_{\gamma} = 0. \quad (3.4) $$

The symmetric part of $E_{\mu\nu}$:

The symmetric part of $E_{\mu\nu}$ is defined by,

$$ E_{(\mu\nu)} \overset{\text{def}}{=} \frac{1}{2} (E_{\mu\nu} + E_{\nu\mu}). \quad (3.5) $$

Using the tensors of Table (1), one can evaluate the definition (3.5) and gets

$$ E_{(\mu\nu)} = g_{\mu\nu} (\sigma - \overline{\sigma}) + g_{\mu\nu} R - 2R_{\mu\nu} - 2\sigma_{\mu\nu} + 2\overline{\sigma}_{\mu\nu}. \quad (3.6) $$

Hence one can write the symmetric part of the field equations (3.4) in the form $E_{(\mu\nu)} = 0$, i.e.

$$ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} g_{\mu\nu} (\sigma - \overline{\sigma}) + \overline{\sigma}_{\mu\nu} - \sigma_{\mu\nu}, \quad (3.7) $$

which can be written in the more compact form,

$$ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}, \quad (3.8) $$

where,

$$ T_{\mu\nu} \overset{\text{def}}{=} g_{\mu\nu} \Lambda + \overline{\sigma}_{\mu\nu} - \sigma_{\mu\nu}, \quad (3.9) $$

$$ \Lambda \overset{\text{def}}{=} \frac{1}{2} (\sigma - \overline{\sigma}), \quad (3.10) $$

$$ \sigma \overset{\text{def}}{=} g^{\mu\nu} \sigma_{\mu\nu}, $$

and,

$$ \overline{\sigma} \overset{\text{def}}{=} g^{\mu\nu} \overline{\sigma}_{\mu\nu}. $$
Consequently, by evaluating the vectorial divergence of both sides of (3.8), it can be easily shown that

\[ T^{\mu\nu} = 0, \]

which gives the conservation of the physical quantities represented by the tensor \( T^{\mu\nu} \). The tensor \( T^{\mu\nu} \) can be used to represent the distribution of matter and energy, i.e., a material-energy tensor. It is to be considered that this tensor is a geometric object, defined in terms of the building blocks of the AP-structure used, and not a phenomenological one, in contrast to the case of GR. In the case of the weak field limit, it has been shown [11] that the symmetric part of the field equations gives rise to Newtonian gravity.

**The Skew Part of \( E_{\mu\nu} \):**

The skew part of the field equations (3.4) can be written as

\[ E_{[\mu\nu]} \overset{\text{def}}{=} \frac{1}{2}(E_{\mu\nu} - E_{\nu\mu}) = 0. \]

Using the skew tensors of Table (1), one can write the above equation in the form,

\[ F_{\mu\nu} = C_{\mu,\nu} - C_{\nu,\mu}, \tag{3.11} \]

where,

\[ F_{\mu\nu} \overset{\text{def}}{=} \zeta_{\mu\nu} + \eta_{\mu\nu} - \xi_{\mu\nu}. \tag{3.12} \]

Consequently, \( F_{\mu\nu} \) will satisfy the relations,

\[ F_{\mu\nu;\sigma} + F_{\sigma\mu;\nu} + F_{\nu\sigma;\mu} = F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} = 0. \tag{3.13} \]

It is clear that equations (3.11) and (3.13) represent a generalization of Maxwell’s field equations, with the skew-symmetric tensor \( F_{\mu\nu} \) representing the electromagnetic field strength, and the vector \( C_{\mu} \) representing the generalized electromagnetic potential. In the case of the weak field limits, it has been shown [11] that the skew part of the field equations (3.4) gives rise to Maxwell’s equation.

From the study of the symmetric and the skew parts of the field equations (3.4) and from applications of the theory [12], [13], [14], [15], [16], it is clear that the GFT represents a pure geometric theory, unifying gravity and electromagnetism in the framework of the geometerization philosophy.

### 4 The Use of The Curvature Tensor

The curvature tensor (2.8), defined in the context of the structure mentions above, can be written in the form

\[ \tilde{M}^{\beta}_{\mu\nu\sigma} \overset{\text{def}}{=} \Gamma^{\beta}_{\mu,\nu} - \Gamma^{\beta}_{\nu,\mu} + \Gamma^{\sigma}_{\rho,\mu} \Gamma^{\rho}_{\nu,\epsilon} - \Gamma^{\sigma}_{\rho,\nu} \Gamma^{\rho}_{\mu,\epsilon}, \]

which is, in general, non-vanishing tensor. Contracting the above tensor by setting \( \beta = \sigma \) we get,

\[ \tilde{M}_{\mu\nu} = \Gamma^{\beta}_{\mu,\nu} - \Gamma^{\beta}_{\nu,\mu} + \Gamma^{\epsilon}_{\rho,\mu} \Gamma^{\beta}_{\nu,\epsilon} - \Gamma^{\epsilon}_{\rho,\nu} \Gamma^{\beta}_{\mu,\epsilon}. \tag{4.1} \]
Substituting from (2.13) and making some rearrangements, we can write

\[ \tilde{M}_{\mu \nu} = R_{\mu \nu} - \gamma_{\gamma, \mu; \beta} + \gamma_{\gamma, \beta; \nu} \]  

(4.2)

where \( R_{\mu \nu} \) is Ricci tensor defined using Christoffel symbol. The scalar Lagrangian function corresponding to (4.2), can be defined as

\[ \tilde{M} \stackrel{\text{def}}{=} g_{\mu \nu} \tilde{M}_{\mu \nu}. \]  

(4.3)

Substituting from definition (4.2) into definition (4.3) we get,

\[ \tilde{M} \stackrel{\text{def}}{=} g_{\mu \nu} R_{\mu \nu} + C_{\beta, \beta} + \frac{1}{4} g^{\mu \nu} [\gamma_{\gamma, \beta, \mu} \gamma_{\gamma, \beta, \nu} + \gamma_{\beta, \nu} \gamma_{\beta, \mu}], \]  

(4.4)

which can be written, using (2.16) and (2.17), as

\[ \tilde{M} = g_{\mu \nu} R_{\mu \nu} + C_{\beta, \beta} + \frac{1}{4} g_{\mu \nu} [\Delta^{\epsilon, \beta}_{\beta, \mu} \Delta^{\beta}_{\beta, \nu} + \Lambda^{\epsilon, \beta}_{\beta, \mu} \Lambda^{\beta}_{\beta, \nu}]. \]  

(4.5)

The Lagrangian scalar density corresponding to (4.1) can be defined as,

\[ \mathcal{L} = \lambda^* \tilde{M}. \]  

Substituting from (4.5) into the above definition we get,

\[ \mathcal{L} = \lambda^* g_{\mu \nu} R_{\mu \nu} + \frac{1}{4} \lambda^* g^{\mu \nu} [\Delta^{\epsilon, \beta}_{\beta, \mu} \Delta^{\beta}_{\beta, \nu} + \Lambda^{\epsilon, \beta}_{\beta, \mu} \Lambda^{\beta}_{\beta, \nu}] + (\lambda^* C^\beta)_{\beta}. \]  

(4.6)

Since the last term, \((\lambda^* C^\beta)_{\beta}\), gives no contribution to the variation, the Lagrangian density (4.6) can be written in the form

\[ \mathcal{L} = \lambda^* g_{\mu \nu} [R_{\mu \nu} + \frac{1}{4} \Delta^{\epsilon, \beta}_{\beta, \mu} \Delta^{\beta}_{\beta, \nu} + \frac{1}{4} \Lambda^{\epsilon, \beta}_{\beta, \mu} \Lambda^{\beta}_{\beta, \nu}] = \lambda^* \tilde{M}. \]  

(4.7)

From which it is clear that

\[ \tilde{M}_{\mu \nu} \stackrel{\text{def}}{=} R_{\mu \nu} + N_{\mu \nu}, \]  

(4.8)

where,

\[ N_{\mu \nu} \stackrel{\text{def}}{=} \frac{1}{4} \Delta^{\epsilon, \alpha}_{\alpha, \mu} \Delta^{\alpha}_{\alpha, \nu} + \frac{1}{4} \Lambda^{\epsilon, \alpha}_{\alpha, \mu} \Lambda^{\alpha}_{\alpha, \nu}. \]  

(4.9)

To derive the field equations corresponding to the Lagrangian density function (4.7) we are going to use the Dolan-McCrea [9] variational method.

**Variational Derivatives**

In this method, the Lagrangian function is assumed to be a function of \( \lambda_{\mu} \) and its first and second derivatives, i.e.

\[ \mathcal{L} \equiv \mathcal{L}(\lambda_{\mu}, \lambda_{\mu, \nu}, \lambda_{\mu, \nu, \sigma}). \]  

(4.10)

\(^2\)Since Dolan-McCrea method is not published, so the reader may refer to [8] for more details about its use in the AP-geometry.
Consider now the new function,

\[ \mathcal{L}_\eta \equiv \mathcal{L}(\lambda_i + \eta h_i, \lambda_{i\mu} + \eta h_{i\mu}, \lambda_{i\mu\nu} + \eta h_{i\mu\nu}), \]  

(4.11)

where \( \eta \) is a small parameter and \( h_i \) is a vector field, which will be defined later, in the AP-space. Then one can write,

\[ I_\eta - I = \int_\Omega (\mathcal{L}_\eta - \mathcal{L}) dx(4), \]  

(4.12)

where 

\[ dx(4) \overset{\text{def}}{=} dx^0 dx^1 dx^2 dx^3. \]

and \( \Omega \) is a region in the 4-space. Substituting from (4.10) and (4.11) into (4.12) and integrating by parts one gets,

\[ I_\eta - I = \eta \int_\Omega (\delta L/\delta \lambda_i \lambda_{i\mu}) h_i \lambda^* dx(4) + O(\eta^2), \]  

(4.13)

where

\[ \lambda^* \overset{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial \lambda_i \lambda_{i\mu}}, \]

(4.14)

Using (4.14), we can define the new object,

\[ S^\mu_{\nu} \overset{\text{def}}{=} \frac{\delta L}{\delta \lambda_i \lambda_{i\mu}} \lambda_{\mu\nu} = \frac{1}{\lambda^*} \frac{\delta L}{\delta \lambda_i \lambda_{i\mu}} \lambda_{\mu\nu}, \]  

(3.15)

where \( \mathcal{L} = \lambda^* L \). It has been shown [8] that this new object is a tensor of the character indicated by its indices .

**An Integral Identity**

We are going to use the infinitesimal transformation

\[ \bar{x}^\mu = x^\mu + \eta \xi^\mu, \]  

(4.16)

where \( \eta \) is the small parameter, mentioned above, whose square and higher orders can be neglected. Now, we calculate \( \bar{\lambda}_i \) in terms of \( \lambda_i \) using two methods. The first method, is by using the tensor properties, we have

\[ \bar{\lambda}_i(\bar{x}) = \frac{\partial x^\nu}{\partial \bar{x}^\mu} \lambda_{\mu\nu}(x). \]

Substituting from (4.16) into the above equation we get,

\[ \bar{\lambda}_i(\bar{x}) = \lambda_i(x) + \eta \lambda_{i\mu} z^\mu + O(\eta^2). \]  

\[(i)\]

The second is by using Taylor’s expansion, we can write

\[ \bar{\lambda}_i(\bar{x}) = \bar{\lambda}_i(x - \eta z) = \bar{\lambda}_i(x) - \eta \lambda_{i\mu} a^\mu + O(\eta^2). \]  

\[(ii)\]
From (i) and (ii), we can write
\[ \tilde{X}_\mu(x) - \lambda_\mu(x) = \eta h_\mu + O(\eta^2), \]  
(4.17a)

where,
\[ h_\mu \overset{\text{def}}{=} \lambda_\nu z^{\nu,\mu} + \lambda_\mu,\nu z^{\nu} = \tilde{\lambda}_\mu. \]  
(4.17b)

Also, one can treat \( \bar{\xi}(\bar{x}) \) in a similar manner, in the following:
First, we have
\[ \bar{\xi}(\bar{x}) = \bar{\xi}(x - \eta z) = \bar{\xi}(x) - \eta \bar{\xi}(x)_{,a} z^a + O(\eta^2). \]  
(iii)

The function \( \bar{\xi}(x) \) stands for
\[ \bar{\xi}(x) = \bar{\xi}(\tilde{\lambda}_\mu(x), \lambda_\mu,\nu(x), \lambda_\mu,\nu,\sigma(x)). \]  
(iv)

Secondly, we know that the function \( L \) is a scalar so,
\[ \bar{L}(\bar{x}) = L(x), \]
thus,
\[ \tilde{\lambda}^* \bar{L}(\bar{x}) = \tilde{\lambda}^* L(x) = \lambda^* L(x) \frac{\partial(x)}{\partial(\bar{x})}. \]  
(v)

Using equation (4.16), one can write the equation (v) in the form,
\[ \bar{\xi}(\bar{x}) = \xi(x) + \eta \xi(x)_{,a} z^a + O(\eta^2), \]  
(vi)

comparing (iii) and (vi), one can write
\[ \int_{\Omega} (\bar{\xi}(\bar{x}) - \xi(x)) d\bar{x}(4) = \eta \int_{\Omega} (\xi(x)_{,a} z^a) d\bar{x}(4) + O(\eta^2), \]  
(vii)

since,
\[ \int_{\Omega} (\xi(x)_{,a} z^a) d\bar{x}(4) = \int_{\Sigma} \xi(x) n_a z^a d\Sigma \equiv 0. \]

Gauss’s theorem has been used to convert the volume integral to a surface integral in which \( n_a \) is a unit vector normal to hypersurface \( \Sigma \). The integral vanishes because \( z^a \) vanishes at all points of \( \Sigma \). Then we can write equation (vii) in the form,
\[ \int_{\Omega} (\bar{\xi}(\bar{x}) - \xi(x)) d\bar{x}(4) = O(\eta^2). \]

If we substitute for \( \tilde{\lambda}_\mu(x) \) from (4.17a) into (iv) using (4.11), we get
\[ \bar{\xi}(x) = \xi(x) + O(\eta^2). \]  
(viii)

Substituting from (viii) into (vii) we get,
\[ \int_{\Omega} (\eta \xi - \xi) d\bar{x}(4) = \theta(\eta^2). \]  
(4.18)
Comparing the orders of magnitude of different terms on the R.H.S. of (4.12) and (4.18), we get the identity
\[ \int_{\Omega} \left( \frac{\delta L}{\delta \lambda_\mu} \right) h_\mu \lambda^\ast dx(4) \equiv 0. \] (4.19)
Substituting from (4.17b) and (4.15) into (4.19) we get the differential identity,
\[ S^\mu_\nu \equiv 0. \] (4.20)

**Field Equations**

Considering the identity (4.20) as a generalization of contracted Bianchi second identity implying conservation, a set of field equations corresponding to this identity can be taken as,
\[ S^\mu_\nu = 0. \] (4.21)
We can write the field equations resulting from the Lagrangian function (4.7) by using (4.15) as,
\[ S^\beta_\sigma \equiv \frac{\delta \tilde{M}}{\delta \lambda^\beta_j \lambda^\sigma_j}, \] (4.22)
where,
\[ \frac{\delta \tilde{M}}{\delta \lambda^\beta_j} \equiv \frac{1}{\lambda^\ast} \left( \frac{\partial L}{\partial \lambda^\beta_j} - \frac{\partial \partial L}{\partial \lambda^\beta_j \gamma} + \frac{\partial^2}{\partial x^\mu \partial x^\gamma} \frac{\partial L}{\partial \lambda^\beta_j \mu \gamma} \right). \] (4.23)
Now using (4.7), (4.22),(4.23) in (4.21) and performing necessary calculations, we get after some rearrangements,
\[ S^\beta_\sigma \equiv -2G^\beta_\sigma + N\delta^\beta_\sigma - 2N^\beta_\sigma + 2\gamma_\beta^\gamma \gamma^\gamma_\alpha \sigma + 2\gamma_\beta^\gamma \gamma^\gamma_\alpha \sigma - 2C_\alpha \gamma_\alpha^\beta \gamma^\gamma_\alpha \sigma. \] (4.24)
The above equation represents a new set of field equations corresponding to the Lagrangian function (4.7). To extract physical meanings from the geometric equation (4.24), one should study the symmetric and the skew parts of this equation. For this reason, we are going to write (4.24) in the form,
\[ S^\nu_\sigma \equiv -2G_\nu_\sigma + Ng_\nu_\sigma - 2N_\nu_\sigma + 2\gamma_\nu_\sigma \gamma^\gamma_\nu_\sigma + 2\gamma_\nu_\sigma \gamma^\gamma_\nu_\sigma \sigma - 2C_\alpha \gamma_\alpha^\nu_\nu_\sigma = 0. \] (4.25)

**The Symmetric Part of S_\nu_\sigma**
The symmetric part of $S_\nu_\sigma$ is defined as usual by,
\[ S_{(\nu_\sigma)} \equiv \frac{1}{2} (S_\nu_\sigma + S_\sigma_\nu). \]
Substituting from (4.25) into the above definition and using the symmetric tensors of Table (1), we can write

\[ S_{\nu\sigma} = -2G_{\nu\sigma} - g_{\nu\sigma} \omega + \psi_{\nu\sigma} + 2 \omega_{\nu\sigma} - \phi_{\nu\sigma} = 0, \]  

(4.26)

which can be written in the more convenient form,

\[ R_{\nu\sigma} - \frac{1}{2} g_{\nu\sigma} R = \left( \frac{1}{2} \right) (-g_{\nu\sigma} \omega + \psi_{\nu\sigma} + 2 \omega_{\nu\sigma} - \phi_{\nu\sigma}). \]  

(4.27)

If we define the tensor

\[ \tilde{T}_{\nu\sigma} \overset{\text{def}}{=} \left( \frac{1}{2} \right) (-g_{\nu\sigma} \omega + \psi_{\nu\sigma} + 2 \omega_{\nu\sigma} - \phi_{\nu\sigma}), \]  

(4.28)

then we can write (4.27) in the form

\[ R_{\nu\sigma} - \frac{1}{2} g_{\nu\sigma} R = \tilde{T}_{\nu\sigma}, \]

from which it is clear that,

\[ \tilde{T}_{\nu\sigma} = 0. \]  

(4.29)

This implies conservation (since the vectorial divergence of the left hand side of (4.27) vanishes identically). Consequently, \( \tilde{T}_{\nu\sigma} \) can be taken to represent the material-energy distribution in the present theory. In the case of weak field limits, the symmetric part of the field equations will tend to

\[ R^{(1)}_{\nu\sigma} - \frac{1}{2} \delta_{\nu\sigma} R^{(1)} = \psi^{(1)}_{\nu\sigma} \]  

(4.30)

the index (1) is used to indicate linearity. This equation gives rise to a gravitational field (compare with the similar case in GR) within a material distribution, characterized by \( \psi^{(1)}_{\nu\sigma} \).

**The Skew-Symmetric Part of \( S_{\nu\sigma} \)**

To obtain the skew part of the field equations (4.25) we use, as usual the definition,

\[ S_{[\nu\sigma]} \overset{\text{def}}{=} \frac{1}{2} (S_{\nu\sigma} - S_{\sigma\nu}) = 0. \]

Substituting from (4.25) into the above definition and using the skew tensors of Table (1) we get,

\[ S_{[\nu\sigma]} = \chi_{\nu\sigma} - \eta_{\nu\sigma} = 0. \]

This can be written as,

\[ \eta_{\nu\sigma} - \chi_{\nu\sigma} = 0. \]  

(4.31)

Now using the identity (2.27), we get

\[ \epsilon_{\nu\sigma} = 0, \]  

(4.32)
which can be written in the form,

\[ \eta_{\nu\sigma} = C_{\nu,\sigma} - C_{\sigma,\nu}. \]  

(4.33)

We can take the tensor \( \eta_{\nu\sigma} \) to represent the strength of the electromagnetic field whose generalized potential is \( C_\mu \), as usually done in the literature (cf. [8]), then equation (4.33), apparently, indicates the presence of an electromagnetic field. In the case of the weak field limits it can be shown that the skew part of the field equations (4.25) will give rise to the equation,

\[ C^{(1)}_{\nu,\sigma} - C^{(1)}_{\sigma,\nu} = 0, \]  

(4.34)

where \( C^{(1)}_\mu \) denotes the linear part of the vector \( C_\mu \). This equation implies the absence of the electromagnetic field in the weak field limit.

5 Discussion And Concluding Remarks

In the present work, we have directed the attention to a version of AP-geometry that admits spaces with, simultaneously non-vanishing, curvature and torsion. Such type of spaces is known in the literature as Riemann-Cartan spaces. The main characteristic of a version of this type is, briefly, reviewed in Section 2. The structure of this space depends on the dual connection (2.5), using which a curvature (2.8) the torsion (2.11) and the W-tensor (2.26) are defined. It is clear that these geometric objects are, in general, simultaneous non-vanishing tensors. It is shown that, in this type of spaces, the fourth order tensor (2.26), which is different from the curvature (2.8), the W-tensor has been first defined [5] and used [8] to construct the GFT. It has been classified [11] as a pure geometric attempt unifying gravity and electromagnetism. Several applications of this theory support this classified (cf. [12],[13],[14],[15],[16]). For the sake of comparison, this theory is briefly reviewed in Section 3.

In Section 4 the authors have constructed a new field theory, using the dual curvature tensor (2.8) to form the Lagrangian function of the theory. It can be easily shown that this tensor is a non-vanishing one, since it is defined in terms of the dual connection (2.5) of the AP-space. The variational method used, to obtain the field equations of this theory, is that of Dolan and McCrea. This method depends mainly on the comparison of the same order of magnitude of different terms of an integral, that has been evaluated using two different expansions. This method is, somewhat, different from the action principle method, although the two methods give identical results (compare [8] and [10]).

The number of field equations of the new theory (4.25) are sixteen. Ten of which (4.27) are symmetric and rest (4.33) are anti-symmetric. Conservation of matter and energy is guaranteed from (4.29). The weak field approximation of the antisymmetric part of the field equations indicates the absence of, conventional, electromagnetic fields. So, the theory, at least, can represent pure gravity and in this case the antisymmetric part of the field equations, (4.33), are used to fix the extra six degrees of freedom of the tetrad (since the field variables are sixteen, the tetrad vectors). As it is shown [11] the generalized electromagnetic potential is represented, geometrically, using the basic vector (2.14) which is the trace of the torsion. In the new theory, derived in Section 4, this potential does
not vanish even in the weak field limit. But since this theory is a pure gravity theory as shown, then this potential can be classified among a type of potentials that do not generate electromagnetism and cannot be removed by any transformation [17]. This type of potentials may give rise to an alternative interpretation of the Aharonov-Bohm effect.

The advantage of using such theories, in applications, is its pure geometric object (4.28) that can describe the material-energy distribution. The use of this object may overcome the difficulty of imposing an equation of state, from outside the theory. This may throw more light on the gravitational field and the physical situation within material distributions.

Table 2: Comparison Between GFT and The New Theory

| Criteria                  | GFT                                                                 | The New Theory                                      |
|---------------------------|----------------------------------------------------------------------|-----------------------------------------------------|
| Geometry                  | AP                                                                  | AP                                                  |
| Lagrangian                | $g^{\mu\alpha}(\Lambda^{\beta}_{\delta}\Lambda^{\delta}_{\beta\alpha} - C_\alpha C_\mu)$ (3.2) | $g^{\mu\nu}(R_{\mu\nu} + \frac{1}{4} \Lambda^i_{\alpha \mu} \Lambda^\alpha_{\nu i}) + \frac{1}{4} \Lambda^i_{\alpha \mu} \Lambda^\alpha_{\nu i}$ (4.7) |
| Tensor used               | W-tensor (2.26)                                                     | dual curvature (2.8)                                |
| Field equations           | non-symmetric                                                       | non-symmetric                                       |
| Material-energy tensor    | geometric                                                           | geometric                                           |
| Interactions              | gravitational and electromagnetic                                   | gravity with a geometric matter-tensor             |
| (weak field regime)       |                                                                      |                                                     |
| Identities                | generalized Bianchi                                                 | generalized Bianchi                                 |

Table 2 gives a comparison between the GFT and the new theory derived in the present work. From this Table, it is clear that the use of the curvature tensor in constructing the Lagrangian (4.6) leads to a pure geometric gravity theory, in which conventional electromagnetism is absent. While the use of the W-tensor (3.1), for the same purpose, leads to a theory for gravity and electromagnetism. It is to be considered that torsion does not enter the Lagrangian (3.2) as an added entity, but this Lagrangian contains curvature and torsion, fused as one entity, in the W-tensor given by (3.1). The difference between
the two tensors is that the curvature one reflects the effect of the curvature, only, on the properties of the space; while the W-tensor (3.1) reflects the effects of both curvature and torsion on the structure of the space. This may lead to the conclusion that torsion and electromagnetism are closely related. Further investigations are needed to support this conclusion and to explore the type of relation between torsion and electromagnetism.

To summarize, and as stated in the introduction, we have: On one hand, the energy of any system is represented by its Lagrangian, which is a scalar density. On the other hand, the presence of energy in a space will deviate this space from the flat case. We have examined and compared the consequences of constructing field theories using two different Lagrangian functions:

1- The first is built up using the W-tensor (2.23), Section 3.
2- The second is built up using the dual curvature (2.8), Section 4.

Note that both depend on the same linear connection, the dual connection (2.5). It is shown, in the present work, that the existence of electromagnetism is closely related with the first Lagrangian (3.2), not with the second one (4.7). Since the first Lagrangian implies the effect of both curvature and torsion, on the structure of space time; while the second implies the effect of the curvature only, one can conclude that: **To unify gravity and electromagnetism, in a 4-dimensional space-time, the geometry should admit a non-vanishing torsion that, together with curvature, is to be used to build the Lagrangian of the theory.** This is to guarantee that more types of energy, including the electromagnetic energy, are taken into consideration.

**Acknowledgements**

The authors are indebted to members of the Egyptian Relativity Group (ERG) for many discussions.

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